Actions of automorphism groups of Lie groups

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Introduction

The aim of this article is to discuss the actions on a connected Lie group by subgroups of its automorphism group. The automorphism groups are themselves Lie groups (not necessarily connected) and the actions have, not surprisingly, played an important role in the study of various topics, including geometry, dynamics, ergodic theory, probability theory on Lie groups, etc.

We begin in §1 with results on the structure of the automorphism groups Aut(G), G a Lie group, generalities about their subgroups, connected components, etc.. The automorphism group of a connected Lie group can be realised as a linear group via association with the corresponding automorphism of the Lie algebra \( \mathfrak{g} \) of \( G \), and §2 is devoted to relating Aut(G) to algebraic subgroups, and more generally “almost algebraic” subgroups, of GL(\( \mathfrak{g} \)); in particular the connected component of the identity in Aut(G) turns out to be almost algebraic and this has found considerable use in the study of various topics discussed in the subsequent sections. In §3 we discuss orbits of various subgroups of Aut(G) on \( G \), and especially conditions for them to be dense in \( G \). In §4 we discuss invariant measures and various ergodic-theoretic aspects of the actions of subgroups of Aut(G) on \( G \), and in §5 some of the topics in topological dynamics in which various special features of Aut(G) play a role. The action of Aut(G) on \( G \) induces in a natural way an action on the space of probability measures on \( G \) and properties of this action have played an important role in various questions in probability theory on Lie groups, in terms of extending certain aspects of classical probability theory to the Lie group setting. These are discussed in §6.

While our main focus will be on automorphism groups of connected Lie groups, at various points along the way generalisations to the case of more general locally compact groups are mentioned with references to the relevant literature.
# Contents

1 Structure of the automorphism groups .................................................. 3
   1.1 Preliminaries ................................................. 3
   1.2 Subgroups of Aut(G) ........................................... 4
   1.3 Connected components of Aut(G) ................................. 5
   1.4 Locally isomorphic Lie groups ................................. 6

2 Almost algebraic automorphism groups .................................................. 7
   2.1 A decomposition ................................................ 8
   2.2 Almost algebraic subgroups of Aut(G) .......................... 8
   2.3 Groups with Aut(G) almost algebraic .......................... 9
   2.4 Automorphisms preserving additional structure .................. 10
   2.5 Linearisation of the Aut(G) action on G ....................... 10
   2.6 Embedding of G in a projective space ........................ 11
   2.7 Embedding in a vector space .................................. 12
   2.8 Algebraicity of stabilizers ..................................... 13

3 Dense orbits ......................................................................................... 13
   3.1 Aut(G)-actions with dense orbits .................................. 14
   3.2 Connected subgroups of Aut(G) with dense orbits ............. 15
   3.3 Automorphisms with dense orbits .................................. 16
   3.4 $Z^d$-actions ....................................................... 17
   3.5 Discrete groups with dense orbits ................................ 18
   3.6 Orbit structure of actions of some discrete groups ............ 19
   3.7 Aut(G)-actions with few orbits ................................... 20

4 Ergodic theory of actions of automorphism groups ............................... 21
   4.1 Preliminaries ..................................................... 21
   4.2 Finite invariant measures ......................................... 22
   4.3 Convolution powers of invariant measures ...................... 23
   4.4 Infinite invariant measures ....................................... 24
   4.5 Quasi-invariant measure and ergodicity ......................... 24
   4.6 Stabilisers of actions of Lie groups ............................ 25

5 Some aspects of dynamics of Aut(G)-actions ....................................... 26
   5.1 Stabilisers of continuous actions .............................. 26
1 Structure of the automorphism groups

In this section we discuss various structural aspects of the automorphisms of connected Lie groups.

1.1 Preliminaries

Let $G$ be a connected Lie group and $\mathfrak{g}$ be the Lie algebra of $G$. We denote by $\text{Aut}(G)$ the group of all Lie automorphisms of $G$ equipped with its usual topology, corresponding to uniform convergence on compact subsets (see [55] Ch. IX for some details). To each $\alpha \in \text{Aut}(G)$ there corresponds a Lie automorphism $d\alpha$ of $\mathfrak{g}$, the derivative of $\alpha$. We may view $\text{Aut}(\mathfrak{g})$ in a natural way as a closed subgroup of $\text{GL}(\mathfrak{g})$, the group of nonsingular linear transformations of $\mathfrak{g}$, considered equipped with its usual topology. Consider the map $d : \text{Aut}(G) \to \text{Aut}(\mathfrak{g})$, given by $\alpha \mapsto d\alpha$, for all $\alpha \in \text{Aut}(G)$. As $G$ is connected, $d\alpha$ uniquely determines $\alpha$, and thus the map is injective.

When $G$ is simply connected the map is also surjective, by Ado’s theorem (cf. [53], Theorem 7.4.1). When $G$ is not simply connected the map is in general not surjective; in this case $G$ has the form $\tilde{G}/\Lambda$, where $\tilde{G}$ is the universal covering group of $G$ and $\Lambda$ is a discrete subgroup contained in the center of $\tilde{G}$, and the image of $\text{Aut}(G)$ in $\text{Aut}(\mathfrak{g})$ consists of those elements for which the corresponding automorphism of $\text{Aut}(\mathfrak{g})$ leaves the subgroup $\Lambda$ invariant (note that $\mathfrak{g}$ may be viewed also, canonically, as the Lie algebra of $\tilde{G}$), and in particular it follows that it is a closed subgroup. This implies in turn that $d$ as above is a topological isomorphism (see also [55], Ch. IX, Theorem 1.2). We shall view $\text{Aut}(G)$ as a subgroup of $\text{Aut}(\mathfrak{g})$, and in turn $\text{GL}(\mathfrak{g})$, via the correspondence.
As a closed subgroup of $GL(G)$, $Aut(G)$ is Lie group, and in particular a locally compact group; it may be mentioned here that the question as to when the automorphism group of a general locally compact group is locally compact is analysed in [90].

We note that $Aut(\mathfrak{g})$ is a real algebraic subgroup of $GL(\mathfrak{g})$, namely the group of $\mathbb{R}$-points of an algebraic subgroup of $GL(\mathfrak{g} \otimes \mathbb{C})$, defined over $\mathbb{R}$. While $Aut(G)$ is in general not a real algebraic subgroup (when viewed as a subgroup of $GL(G)$ as above), various subgroups of $Aut(G)$ being algebraic subgroups plays an important role in various results discussed in the sequel.

1.2 Subgroups of $Aut(G)$

We introduce here certain special classes of automorphisms which play an important role in the discussion in the following sections.

Let $G$ be a connected Lie group. For any subgroup $H$ of $G$ let $Aut_H(G)$ denote the subgroup of $Aut(G)$ consisting of automorphisms leaving $H$ invariant; viz. $\{\tau \in Aut(G) \mid \tau(H) = H\}$. A subgroup $H$ for which $Aut_H(G) = Aut(G)$ is called a characteristic subgroup of $G$. Clearly, for any Lie group $G$ the center of $G$, the commutator subgroup $[G, G]$, the (solvable) radical, the nilradical are some of the characteristic subgroups of $G$. Similarly, together with any characteristic subgroup, its closure, centraliser, normaliser etc. are characteristic subgroups.

For each $g \in G$ we get an automorphism $\sigma_g$ of $G$ defined by $\sigma_g(x) = gxg^{-1}$ for all $x \in G$, called the inner automorphism corresponding to $g$. For a subgroup $H$ of $G$ we shall denote by $Inn(H)$ the subgroup of $Aut(G)$ defined by $\{\sigma_h \mid h \in H\}$. For any Lie subgroup $H$, $Inn(H)$ is a Lie subgroup of $Aut(G)$; it is a normal subgroup of $Aut(G)$ when $H$ is a characteristic subgroup. In particular $Inn(G)$ is a normal Lie subgroup of $Aut(G)$; the group $Aut(G)/Inn(G)$ is known as the group of outer automorphisms of $G$. We note that $Inn(G)$ may in general not be a closed subgroup of $Aut(G)$; this is the case, for example, for the semidirect product of $\mathbb{R}$ with $\mathbb{C}^2$ with respect to the action under which $t \in \mathbb{R}$ acts by $(z_1, z_2) \mapsto (e^{i\alpha t}z_1, e^{i\beta t}z_2)$ for all $z_1, z_2 \in \mathbb{C}$, with $\alpha, \beta$ fixed nonzero real numbers such that $\alpha/\beta$ is irrational.

Let $Z$ denote the center of $G$ and $\varphi: G \to Z$ be a (continuous) homomorphism of $G$ into $Z$. Let $\tau: G \to G$ be defined by $\tau(x) = x\varphi(x)$ for all $x \in G$. It is easy to see that $\tau \in Aut(G)$; we call it the shear automorphism, or more specifically isotropic shear automorphism (as in [20]), associated with $\varphi$. We note that for any continuous homomorphism $\varphi$ as above the subgroup $[G, G]$ is contained in the kernel of $\varphi$, and hence the associated shear automorphism $\tau$ fixes $[G, G]$ pointwise. In particular, if $[G, G] = G$ then there are no nontrivial shear automorphisms. The shear automorphisms form a closed normal abelian subgroup of $Aut(G)$, say $S$. 
We note that if \( A = \{ \tau \in \text{Aut}(G) \mid \tau(z) = z \text{ for all } z \in Z \} \) then \( \text{Aut}(G) \) is the semidirect product of \( A \) and \( S \).

If \( H \) is the subgroup of \( G \) containing \([G, G]\) and such that \( H/[G, G] \) is the maximal compact subgroup of \( G/[G, G] \), then \( G/H \) is a vector group and the set of continuous homomorphisms \( \varphi \) of \( G \) into \( Z \) such that \( H \subset \text{ker} \varphi \) has the natural structure of a vector space, and in turn the same holds for the corresponding set of shear automorphisms; the dimension of the vector space equals the product of the dimensions of \( G/H \) and \( Z \). When \( G/[G, G] \) is a vector group (viz. topologically isomorphic to \( \mathbb{R}^n \) for some \( n \)) \( S \) is a connected algebraic subgroup of \( \text{Aut}(G) \). On the other hand, when \( G/[G, G] \) is compact then there are only countably many distinct continuous homomorphisms of \( G \) into \( Z \), and hence only countably many shear automorphisms.

For a connected semisimple Lie group \( G \), \( \text{Inn}(G) \) is a subgroup of finite index in \( \text{Aut}(G) \); in the case of a simply connected Lie group this follows from the corresponding statement for the associated Lie algebra (cf. [53], Theorem 5.5.14), and the general case follows from the special case, since the inner automorphisms of the simply connected covering group factor to the original group \( G \). On the other hand, for a connected nilpotent Lie group the group of outer automorphisms is always of positive dimension (see [58], Theorem 4). An example of a 3-step simply connected nilpotent Lie group \( G \) for which \( \text{Aut}(G) = \text{Inn}(G) \cdot S \), where \( S \) is the group of all shear automorphisms of \( G \), is given in [26]; in particular \( \text{Aut}(G) \) is nilpotent in this case; a larger class of connected nilpotent Lie groups for which \( \text{Aut}(G) \) is nilpotent is also described in [26].

We recall here that the center of a connected Lie group \( G \) is contained in a connected abelian Lie subgroup of \( G \) (see [53], Theorem 14.2.8, or [55], Ch. 16, Theorem 1.2). In particular the center is a compactly generated abelian group and hence has a unique maximal compact subgroup; we shall denote it by \( C \). Being a compact abelian Lie subgroup, \( C \) is in fact the cartesian product of a torus with a finite abelian group. As the unique maximal compact subgroup of \( G \), \( C \) is \( \text{Aut}(G) \)-invariant and hence we get a continuous homomorphism \( q : \text{Aut}(G) \to \text{Aut}(C) \), by restriction of the automorphisms to \( C \). Since the automorphism group of a compact abelian group is countable, it follows that \( \text{Aut}(G)/\text{ker} \ q \), where \( \text{ker} \ q \) is the kernel of \( q \), is a countable group.

1.3 Connected components of \( \text{Aut}(G) \)

As a closed subgroup of \( \text{GL}(\mathfrak{g}) \), \( \text{Aut}(G) \) is a Lie group with at most countably many connected components (see [54] for an extensions of this to not necessarily connected Lie groups). It is in general not connected; for \( \mathbb{R}^n \), \( n \geq 1 \), the automorphism group, which is topologically isomorphic to \( \text{GL}(n, \mathbb{R}) \), has two connected
components, while for the torus \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \), \( n \geq 2 \), the automorphism group is in fact an infinite (countable) discrete group.

We shall denote by \( \text{Aut}^0(G) \) the connected component of the identity in \( \text{Aut}(G) \); it is an open (and hence also closed) subgroup of \( \text{Aut}(G) \). We denote by \( c(G) \) the group of connected components, namely \( \text{Aut}(G) / \text{Aut}^0(G) \).

For a connected semisimple Lie group, since \( \text{Inn}(G) \) is of finite index in \( \text{Aut}^0(G) \) (see §1.2), \( \text{Aut}^0(G) = \text{Inn}(G) \). Also, for these groups the number of connected components is finite; the number is greater than one in many cases (see [57] and [71] for details on the group of connected components of \( \text{Aut}(G) \) for simply connected semisimple groups \( G \); see also the recent paper [50] where splitting of \( \text{Aut}(G) \) into the connected component and the component group is discussed).

We note that \( \text{Aut}^0(G) \) acts trivially on the unique maximal compact subgroup \( C \) of the center of \( G \), and hence when the image of the homomorphism \( q : \text{Aut}(G) \to \text{Aut}(C) \), as in §1.2, is infinite, \( c(G) \) is infinite.

When \( C \) as above is the circle group, \( \text{Aut}(C) \) is of order two, and the image of \( q \) has at most two elements, but nevertheless \( c(G) \) can be infinite. A natural instance of this can be seen in the following: Suppose \( G \) has a closed normal subgroup \( H \) such that \( G/H \) is a torus of positive dimension; then there is a unique minimal subgroup with the property and it is invariant under the action of \( \text{Aut}(G) \) and hence, modifying notation, we may assume \( H \) to be \( \text{Aut}(G) \)-invariant. Then we have (countably) infinitely many (continuous) homomorphisms \( \varphi : G/H \to C \) and for each such \( \varphi \) we have an isotropic shear automorphism of \( G \) (see §1.2). It can be seen that the shear automorphisms corresponding to distinct homomorphisms belong to distinct connected components of \( \text{Aut}(G) \), and hence \( \text{Aut}(G) \) has infinitely many connected components. Thus \( c(G) \) is infinite in this case also.

The group \( c(G) \) is finite if and only if \( \text{Aut}(G) \) is “almost algebraic” as a subgroup of \( \text{GL}(\mathfrak{g}) \); see §2.2 for a discussion on conditions for \( \text{Aut}(G) \) to be almost algebraic.

### 1.4 Locally isomorphic Lie groups

Any connected Lie group is locally isomorphic to a unique (upto Lie isomorphism) simply connected Lie group, namely its universal covering group. Now let \( G \) be a simply connected Lie group and \( Z \) be the center of \( G \). Then all connected Lie groups locally isomorphic to \( G \) are of the form \( G/D \), where \( D \) is a discrete subgroup of \( Z \). Moreover, for two discrete subgroups \( D_1 \) and \( D_2 \) of \( Z \) the Lie groups \( G/D_1 \) and \( G/D_2 \) are Lie isomorphic if and only if there exists a \( \tau \in \text{Aut}(G) \) such that \( \tau(D_1) = D_2 \). Thus the class of Lie groups (viewed up to isomorphism of Lie groups) locally isomorphic to a given connected Lie group \( G \) is in canonical
one-one correspondence with the orbits of the action of Aut(G) on the class of
discrete subgroups of its center, under the action induced by the Aut(G)-action
on Z, by restriction of the automorphisms to Z.

For $G = \mathbb{R}^n$, $n \geq 1$, all connected Lie groups isomorphic to $G$ are of the form
$\mathbb{R}^m \times \mathbb{T}^{n-m}$, with $0 \leq m \leq n$, thus $n + 1$ of them altogether. For the group $G$ of
upper triangular $n \times n$ unipotent matrices ($n \geq 2$), which is a simply connected
nilpotent Lie group, the center is one-dimensional and all its nontrivial discrete
subgroups are infinite cyclic subgroups that are images of one another under au-
tomorphisms of $G$; thus in this case there are only two non-isomorphic Lie groups
locally isomorphic to $G$ (including the simply connected one), independently of $n$.
On the other hand there are simply connected nilpotent Lie groups $G$ for which
Aut($G$) is a unipotent group (when viewed as a subgroup of Aut($\mathfrak{G}$)) (see [26]);
hence in this case the Aut($G$)-action on the center has uncountably many dis-
tinct orbits, and therefore there are uncountably many mutually non-isomorphic
connected Lie groups that are locally isomorphic to $G$.

Now let $G$ be a connected Lie group with discrete center, say $Z$. Then $Z$
is finitely generated (see § 1.2) and hence $Z$ has only countably many distinct
subgroups. Considering the indices of the subgroups in $Z$ it can also be seen
that when $Z$ is infinite there are infinitely many subgroups belonging to distinct
orbits of the Aut($G$)-action on the class of subgroups; thus in this case there are
countably infinitely many connected Lie groups locally isomorphic to $G$. This
applies in particular when $G$ is the universal covering group of $\text{SL}(2, \mathbb{R})$. When
$Z$ is finite the number of Lie groups locally isomorphic to $G$ is finite, and at
least equal to the number of prime divisors of the order of $Z$. For simple simply
connected Lie groups the orbits of the Aut($G$)-action on the class of subgroups of
the center have been classified completely in [48].

2 Almost algebraic automorphism groups

Let $G$ be a connected Lie group and $\mathfrak{G}$ be the Lie algebra of $G$. Let GL($\mathfrak{G}$) be
realised as $\text{GL}(n, \mathbb{R})$, where $n$ is the dimension of $\mathfrak{G}$, via a (vector space) basis of
$\mathfrak{G}$. For $g \in \text{GL}(\mathfrak{G})$ let $g_{ij}$, $1 \leq i, j \leq n$, denote the matrix entries of $g$ and let det $g$
denote the determinant of $g$. A subgroup $H$ of GL($\mathfrak{G}$) being a real algebraic group
is equivalent to the condition that it can be expressed as the set of zeros (solutions)
of a set of polynomials in $g_{ij}$, $1 \leq i, j \leq n$, and $(\text{det } g)^{-1}$ as the variables; in the
present instance, the field being $\mathbb{R}$, it suffices to consider a single polynomial in
place of the set of polynomials.

We call a subgroup of GL($\mathfrak{G}$) almost algebraic if it is of finite index in a real
algebraic subgroup. A real algebraic subgroup is evidently a closed (Lie) subgroup
of $\text{GL}(\mathfrak{g})$, and since a connected Lie group admits no proper subgroups of finite index it follows that any almost algebraic subgroup is an open subgroup of a real algebraic subgroup, and in particular it is a closed subgroup of $\text{GL}(\mathfrak{g})$. It is known that any real algebraic subgroup has only finitely many connected components (see [10], Corollary 14.5, for a precise result on the number of components), and hence the same holds for any almost algebraic subgroup.

2.1 A decomposition

Let $G$ be a connected Lie group and $\mathfrak{g}$ the Lie algebra of $G$. In what follows we view $\text{Aut}(G)$ as a subgroup of $\text{Aut}(\mathfrak{g})$ and of $\text{GL}(\mathfrak{g})$, via the identification introduced earlier (§1.1). We recall that any connected Lie group admits a maximal torus (a subgroup topologically isomorphic to $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ for some $n \geq 0$), and any two maximal tori are conjugate to each other in $G$ (see [53], Corollary 14.1.4).

**Theorem 2.1.** (cf. [19], [73]) Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $T$ be a maximal torus in $G$. Then there exists a closed connected normal subgroup $H$ of $G$ such that the following conditions hold:

i) $\text{Inn}(H)$ is an almost algebraic subgroup of $\text{Aut}(\mathfrak{g})$, and

ii) $\text{Aut}(G) = \text{Inn}(H) \text{Aut}_T(G)$.

The subgroup $H$ chosen in the proof is in fact invariant under all $\tau \in \text{Aut}(G)$ and consequently $\text{Inn}(H)$ is a normal subgroup of $\text{Aut}(G)$. Since $\text{Inn}(H) \cap \text{Aut}_T(G)$ is trivial this further implies that $\text{Aut}(G)$ is a semidirect product of the two subgroups. We may also mention here that the subgroup $H$ as chosen in [19] contains $[G,G]$, the commutator subgroup of $G$. When $\text{Inn}(G)$ is an almost algebraic subgroup of $\text{Aut}(\mathfrak{g})$, $H$ as in Theorem 2.1 can be chosen to be $G$ itself; this applies in particular when $G$ is an almost algebraic subgroup of $\text{GL}(n, \mathbb{R})$ for some $n \geq 1$.

2.2 Almost algebraic subgroups of $\text{Aut}(G)$

In view of Theorem 2.1 $\text{Aut}(G)$ is almost algebraic if and only if $\text{Aut}_T(G)$ is almost algebraic for a (and hence any) maximal torus $T$ in $G$. In this respect we recall the following specific results.

**Theorem 2.2.** ([19], [88], [73]) Let $G$ be a connected Lie group and $\mathfrak{g}$ be the Lie algebra of $G$. Then the following statements hold.

i) If the center of $G$ does not contain a compact subgroup of positive dimension, then $\text{Aut}(G)$ is an almost algebraic subgroup of $\text{GL}(\mathfrak{g})$.

ii) $\text{Aut}^0(G)$ is an almost algebraic subgroup of $\text{GL}(\mathfrak{g})$;
iii) if $R$ is the solvable radical of $G$ and $T$ is a maximal torus in $G$ then $\text{Aut}(G)$ is almost algebraic if and only if the restrictions of all automorphisms from $\text{Aut}_T(G)$ to $T \cap R$ form a finite group of automorphisms of $T \cap R$; in particular, $\text{Aut}(G)$ is almost algebraic if and only if $\text{Aut}(R)$ is almost algebraic.

Assertions (i) and (ii) above were deduced from Theorem 2.1 in [19]; (ii) was proved earlier by D. Wigner [88], and (iii) is due to W. H. Previts and S. T. Wu [73], where improved proofs were also given for (i) and (ii).

We note in particular that, in the light of Theorem 2.2(ii), $\text{Inn}(G)$ is contained in an almost algebraic subgroup of $\text{Aut}(G)$, namely $\text{Aut}^0(G)$. This turns out to be useful in various contexts on account of certain properties of actions of almost algebraic subgroups (see for example §§4.6 and 5.1).

Let $G$ be a connected Lie group. Let $T$ be the unique maximal torus contained in the center of $G$. Then $G/T$ is a connected Lie group whose center contains no nontrivial compact subgroup of positive dimension, and hence $\text{Aut}(G/T)$ is an almost algebraic subgroup of $\text{Aut}(\mathfrak{g}')$, where $\mathfrak{g}'$ is the Lie algebra of $G/T$. Each $\alpha \in \text{Aut}(G)$ induces an automorphism of $\text{Aut}(G/T)$, say $\bar{\alpha}$. If $\eta: \text{Aut}(\mathfrak{g}) \rightarrow \text{Aut}(\mathfrak{g}')$ is the canonical quotient homomorphism, defined by $\eta((\alpha)) = \bar{\alpha}$ for all $\alpha \in \text{Aut}(G)$, then the image of any almost algebraic subgroup of $\text{Aut}(\mathfrak{g})$ under $\eta$ is almost algebraic (as the map is a restriction of a homomorphism of algebraic groups). In particular $\eta(\text{Aut}^0(G))$ is an almost algebraic subgroup.

2.3 Groups with $\text{Aut}(G)$ almost algebraic

For a class of connected Lie groups the following characterisation, incorporating a partial converse of Theorem 2.2(i) is proved in [11]; there the issue is considered for all Lie groups with finitely many connected components, but we shall restrict here to when $G$ is connected; (the general case involves some technicalities in its formulation).

**Theorem 2.3.** ([11]) Let $G$ be a connected Lie group admitting a faithful finite-dimensional representation. Then $\text{Aut}(G)$ is almost algebraic if and only if the maximal torus contained in the center of $G$ is of dimension at most one, and it is also the maximal torus in the radical of $G$.

Let $G$ be as in Theorem 2.3 and $C$ be the maximal torus of the center of $G$ and suppose that $C$ is one-dimensional. In the context of the examples of Lie groups with $\text{Aut}(G)$ having infinitely many distinct connected components discussed in §1.3 it may be noted that if $C$ coincides with the maximal torus in the radical of $G$, then there does not exist a closed normal subgroup $H$ such that $G/H$ is a torus, as used in the argument there.
It could happen that Aut(G) may be almost algebraic (which in view of Theorem 2.2(ii) is equivalent to c(G) being finite), for a connected Lie group G (not admitting a faithful finite-dimensional representation) even if its center contains a torus of dimension exceeding 1 (in fact of any given dimension). An example of this was indicated in [19] (page 451) and has been discussed in detail in [73]. The main idea involved in the example is that Lie groups G can be constructed such that the central torus, though of higher dimension, is the product of one-dimensional tori, each of which is invariant under a subgroup of finite index in Aut(G). It is not clear whether there could be situations, with c(G) finite, for which this may also fail, namely with no one-dimensional compact subgroups contained in the center and invariant under a subgroup of finite index in Aut(G). It is proved in [73] (Proposition 3.1), however that for a connected Lie group G of the form $H \times \mathbb{T}$, where $H$ is a connected Lie group and $\mathbb{T}$ is the one-dimensional torus, Aut(G) has infinitely many connected components, and hence is not almost algebraic.

### 2.4 Automorphisms preserving additional structure

When the Lie group has additional structure, the group of automorphisms preserving the structure would also be of interest. In this subsection we briefly recall some results in this respect.

By a result of Hochschild and Mostow [56] if G is a connected complex affine algebraic group then the connected component of the identity in the group A of rational automorphisms of G is algebraic (with respect to a canonical structure arising from the associated Hopf algebra), and moreover A is itself algebraic if either the center of G is virtually unipotent (namely if it admits a subgroup of finite index consisting of unipotent elements) or the center of a (and hence any) maximal reductive subgroup of G is of dimension at most 1; the result may be compared with Theorem 2.2 for real Lie groups. It may be noted that the results of [56] are in the framework of automorphism groups of affine algebraic groups over an algebraically closed field of characteristic zero. Some further elaboration on the theme is provided by Dong Hoon Lee [67].

For the connected component of the group of complex analytic automorphisms of a faithfully representable complex analytic group, a result analogous to that of Hochschild and Mostow recalled above was proved by Chen and Wu [12].

### 2.5 Linearisation of the Aut(G) action on G

Under certain conditions, a connected Lie group can be realised as a subset of $\mathbb{R}^n$, or the projective space $\mathbb{P}^{n-1}$, for some $n \geq 2$, in such a way that the automorphisms
of $G$ are restrictions of linear or, respectively, projective transformations. We call this linearisation of the $\text{Aut}(G)$-action on $G$. We next discuss various results in this respect.

Let $G$ be a connected Lie group. By an affine automorphism of $G$ we mean a transformation of the form $T_g \circ \tau$ where $\tau \in \text{Aut}(G)$, and $T_g$ is a left translation by an element $g$ in $G$, namely $T_g(x) = gx$ for all $x \in G$. We denote by $\text{Aff}(G)$ the group of all affine automorphisms. We identify $G$ canonically as a subgroup of $\text{Aff}(G)$, identifying $g \in G$ with $T_g$ as above. Then $\text{Aff}(G)$ is a semidirect product of $\text{Aut}(G)$ with $G$, with $G$ as the normal subgroup, and we shall consider it equipped with the Cartesian product topology. We shall denote by $\text{Aff}^0(G)$ the connected component of the identity in $\text{Aff}(G)$.

Let $A$ and $B$ be the Lie algebras of $\text{Aut}(G)$ and $\text{Aff}(G)$ respectively. Let $\text{Ad} : \text{Aff}(G) \to \text{GL}(B)$ be the adjoint representation of $\text{Aff}(G)$. Let $a$ be the dimensions of $\text{Aut}(G)$ and $V = \wedge^a B$, the vector space of $a$-th exteriors over $B$. Let $\rho : \text{Aff}(G) \to \text{GL}(V)$ be the representation arising as the $a$-th exterior power of $\text{Ad}$; we call $\rho$ the linearising representation for $G$ (for reasons that would become clear below). Let $L$ be the vector subspace $\wedge^a A$ of $V = \wedge^a B$; since $a$ is the dimension of $A$, $L$ is a one-dimensional subspace.

Consider the (linear) action of $\text{Aff}(G)$ on $V$ via the representation $\rho$. Let $P = P(V)$ denote the corresponding projective space, consisting of all lines (one-dimensional subspaces) in $V$, equipped with its usual topology. The $\text{Aff}(G)$-action on $V$ induces an action of $\text{Aff}(G)$ on $P$. We have the corresponding actions of $\text{Aut}(G)$ and $G$ by restriction of the action ($G$ being viewed as the group of translations as above).

### 2.6 Embedding of $G$ in a projective space

Let $p_0$ denote the point of $P$ corresponding to the line $L$ as above. Let $S$ be the stabiliser of $p_0$ under the action of $G$. Consider any $g \in S$. Then $L$ is invariant under the action of $g$ on $V$ and this means that the subspace $A$ is invariant under the action of $g$ on $B$. In turn we get that $\text{Aut}^0(G)$ is normalised by $g$ in $\text{Aff}(G)$, and since $\text{Aff}^0(G)$ is a semidirect product of $\text{Aut}^0(G)$ and $G$ this implies that $g$ is contained in the center of $\text{Aff}^0(G)$. Conversely it is easy to see that every element of the center of $\text{Aff}^0(G)$ fixes $p_0$. Thus $S$ coincides with the center of $\text{Aff}^0(G)$. We note also that

$$S = \{ g \in G \mid \tau(g) = g \text{ for all } \tau \in \text{Aut}^0(G) \}.$$

The orbit map $g \mapsto gp_0$ of $G$ into $P$ induces a canonical continuous bijection $j$ of $G/S$ onto its image in $P$, defined by $j(gS) = gp_0$ for all $g \in G$. When $S$ is trivial $j$ defines a continuous embedding of $G$ into $P$. We note that for any
\( \tau \in \text{Aut}(G) \) and \( g \in G \) we have \( \tau(g)p_0 = \tau g \tau^{-1} p_0 = \tau g p_0 = \tau(gp_0) \). Thus the map \( j \) is equivariant with respect to the actions of \( \text{Aut}(G) \) on \( G/S \) and \( P \), viz. \( j(\tau(gS)) = \tau j(gS) \) for all \( g \in G \) and \( \tau \in \text{Aut}(G) \). Thus when \( S \) is trivial the orbits of the \( \text{Aut}(G) \)-action on \( G/S \) are in one-one correspondence in a natural way with orbits of the \( \text{Aut}(G) \)-action on \( P \) that are contained in the image of \( G/S \).

Recall that \( \text{Aut}^0(G) \) is an almost algebraic subgroup of \( \text{GL}(G) \) (see Theorem 2.2(ii)). From this it can be deduced that the restriction of the representation \( \rho \) as above to \( \text{Aut}^0(G) \) is an algebraic representation, viz. restriction of an algebraic homomorphism of algebraic groups. Hence every orbit of \( \text{Aut}^0(G) \) on \( P \) is locally closed, namely open in its closure (see [72], Lemma 1.22, for instance). Thus the above argument shows the following.

**Theorem 2.4.** Let \( G \) be a connected Lie group and \( S \) be the center of \( \text{Aff}^0(G) \). Then every orbit of the action of \( \text{Aut}^0(G) \) on \( G/S \) is locally closed (viz. open in its closure).

The subgroup \( S \), which is noted to be contained in the center of \( G \), contains the maximal compact subgroup of the center. In particular if \( S \) is trivial then by Theorem 2.2(i) \( \text{Aut}^0(G) \) has finite index in \( \text{Aut}(G) \) and hence the above theorem implies that all orbits of \( \text{Aut}(G) \) on \( G \) are locally closed.

We recall here that the condition of the orbits being locally closed is well studied in a wider context and is equivalent to a variety of other conditions of interest; see [44]; (see also [42] for a more general result). It represents in various ways the opposite extreme of the action being ergodic.

### 2.7 Embedding in a vector space

In analogy with the embedding of \( G \) as a subset of \( P \) (modulo the subgroup \( S \) as above) we can also get an embedding of \( G \) in the vector space \( V \) as defined above, as follows. Let the notation be as above and let \( v_0 \) be a nonzero point of \( L \). We note that \( S \) as above is also the stabilizer of \( v_0 \) under the \( G \)-action on \( V \), since if \( g \) fixes \( v_0 \) it fixes \( p_0 \), and if it fixes \( p_0 \) then we have \( \tau(g) = \tau g \tau^{-1} = g \) for all \( \tau \in \text{Aut}^0(G) \) and hence it fixes \( v_0 \). As \( L \) is invariant under the action of \( \text{Aut}(G) \) we get a continuous homomorphism \( s : \text{Aut}(G) \to \mathbb{R}^* \) such that for all \( \tau \in \text{Aut}(G) \), \( \tau(v_0) = s(\tau)v_0 \). For any \( g \in G \) and \( \tau \in \text{Aut}(G) \) we have \( \tau(g)v_0 = (\tau g \tau^{-1})v_0 = \tau g(\tau^{-1}(v_0)) = s(\tau)^{-1} \tau g v_0 \). Consider the (linear) action of \( \text{Aut}^0(G) \) on \( V \) such that \( \tau \in \text{Aut}^0(G) \) acts by \( v \in V \mapsto s(\tau^{-1}) \tau v \). Then the \( \text{Aut}^0(G) \)-orbits on \( G/S \) are in canonical correspondence with the orbits on \( V \) under this \( \text{Aut}^0(G) \)-action. Under this action also the orbits are locally closed, by algebraic group considerations.
2.8 Algebraicity of stabilizers

Let $G$ be a connected Lie group and $\rho : \text{Aff}(G) \to \text{GL}(V)$ be the corresponding linearising representation as in §2.5. We recall that $\text{Aut}^0(G)$ is an almost algebraic subgroup of $\text{GL}(\mathfrak{G})$, $\mathfrak{G}$ being the Lie algebra of $G$, and that the restriction of $\rho$ to $\text{Aut}^0(G)$ is the restriction of a homomorphism of algebraic groups. In particular, for any $v \in V$ the stabiliser $\{\tau \in \text{Aut}^0(G) \mid \tau(v) = v\}$ is an almost algebraic subgroup of $\text{Aut}^0(G)$ (algebraic subgroup if $\text{Aut}^0(G)$ is algebraic). Let $v_0$ be as in §2.7 and $S$, as before, be the center of $\text{Aff}^0(G)$, which is the stabiliser of $v_0$ under the action as in §2.7. Then by the preceding observation for all $g \in G$ the subgroup $\{\tau \in \text{Aut}^0(G) \mid \tau(gS) = gS\}$ is almost algebraic. In particular for a connected Lie group $G$ such that the center of $\text{Aff}^0(G)$ is trivial, for all $g \in G$ the stabiliser $\{\tau \in \text{Aut}^0(G) \mid \tau(g) = g\}$ of $g$ under the action of $\text{Aut}^0(G)$ is an almost algebraic subgroup; this holds in particular when the center of $G$ is trivial.

Let $G$ be a connected Lie group and for $g \in G$ let $S(g)$ denote the stabiliser $\{\tau \in \text{Aut}(G) \mid \tau(g) = g\}$. It is shown in [41] that when $G$ is a simply connected solvable Lie group, $S(g)$ is an algebraic subgroup for all $g \in G$. In the general case it is shown that the connected component of the identity in $S(g)$ is an almost algebraic subgroup for all $g$ which are of the form $\exp \xi$ for some $\xi$ in the Lie algebra of $G$, $\exp$ being the exponential map associated with $G$. It is noted that $S(g)$ itself need not be an algebraic subgroup, as may be seen in the case when $G$ is the universal covering group of $\text{SL}(2, \mathbb{R})$ and $g$ is one of the generators of the center of $G$ (the latter is an infinite cyclic subgroup); in this case $S(g) = \text{Inn}(G)$, and it is of index 2 in $\text{Aut}(G)$, but not an algebraic subgroup.

For elements $z$ contained in the center of $G$ it is proved in [47] (see also [19]) that the connected component of the identity in $S(z)$ is an almost algebraic subgroup, and if $G$ has no compact central subgroup of positive dimension then $S(z)$ itself is almost algebraic. It is however not true that $S(z)$ is almost algebraic whenever $\text{Aut}(G)$ is almost algebraic; a counterexample in this respect may be found in [73] (p. 432).

We note in particular that if $A$ is the subgroup of $\text{Aut}(G)$ consisting of all automorphisms which fix the center pointwise then the connected component of the identity in $A$ is an almost algebraic subgroup, and if $G$ has no compact central subgroup of positive dimension then $A$ is almost algebraic.

3 Dense orbits

It is well-known that an automorphism $\alpha$ of the $n$-dimensional torus $\mathbb{T}^n$, $n \geq 2$, admits dense orbits (and is ergodic with respect to the Haar measure; see §4
for a discussion on the ergodicity condition) when $d\alpha$ has no eigenvalue which is a root of unity. These automorphisms constitute some of the basic examples in ergodic theory and topological dynamics, and have been much studied for detailed properties from the point of view of the topics mentioned (see [14], [87], for instance; see also [79] and [22] for generalisations). In this section we discuss actions of subgroups of Aut($G$) with a dense orbit on $G$.

### 3.1 Aut($G$)-actions with dense orbits

Let $G$ be a connected abelian Lie group. Then it has the form $\mathbb{R}^m \times \mathbb{T}^n$ for some $m, n \geq 0$ and in this case it is easy to see that the Aut($G$)-action on $G$ has dense orbits, except when $(m, n) = (0, 1)$ (viz. when $G$ is the circle group); if $n = 0$ the complement of 0 in $\mathbb{R}^m$ is a single orbit. If $m \geq 1$ then the action of Aut$^0(G)$ also has a dense orbit on $G$, as may be seen using the isotropic shear automorphisms (see §1.2) associated with homomorphisms of $\mathbb{R}^m$ into $\mathbb{T}^n$.

Next let $G$ be a 2-step connected nilpotent Lie group, namely $[G, G]$ is contained in the center of $G$. Let $Z$ denote the center of $G$. Then $G/Z$ is simply connected, and hence is Lie isomorphic to $\mathbb{R}^n$ for some $n$. The Aut($G$)-action on $G$ factors to an action on $G/Z$. Using shear automorphisms (see §1.2) it can be seen that the Aut($G$)-action on $G$ has a dense orbit on $G$ if and only if the Aut($G$)-action on $G/Z$ has a dense orbit. The latter condition holds in particular if $G$ is a free 2-step simply connected Lie group; it may be recalled that $G$ is called a free 2-step nilpotent Lie group if its Lie algebra is of the form $V \oplus \wedge^2 V$, with $V$ a finite-dimensional vector space over $\mathbb{R}$, and the Lie product is generated by the relations $[u, v] = u \wedge v$ for all $u, v \in V$, and $[u, v \wedge w] = 0$ for all $u, v, w \in V$. When $G$ is a free 2-step simply connected nilpotent Lie group the quotient $G/Z$ as above corresponds to the vector space $V$ and every nonsingular automorphism of $V$ is the factor of a $\tau \in \text{Aut}(G)$, which leads to the observation as above.

For a general simply connected 2-step nilpotent Lie group $G$ the Lie algebra $\mathfrak{g}$ of $G$ can be expressed as $(V \oplus \wedge^2 V)/W$, where $V$ is the vector space $G/Z$ ($Z$ being the center of $G$) and $W$ is a vector subspace of $\wedge^2 V$; $V \oplus \wedge^2 V$ is the free 2-step nilpotent Lie algebra with the structure as above, and any vector subspace of $\wedge^2 V$ is a Lie ideal in the Lie algebra (see [5], where the example is discussed in a different context). It is easy to see that in this case the image of Aut($G$) in GL$(V)$, under the map associating to each automorphism its factor on $V = G/Z$, is the subgroup, say $I(W)$, consisting of $g \in \text{GL}(V)$ such that the corresponding exterior transformation $\wedge^2(g)$ of $\wedge^2 V$ leaves the subspace $W$ invariant. It can be seen that $I(W)$ has an open dense orbit on $V$ for various choices of $W$, and in these cases by the argument as above Aut($G$) has open dense orbit on $G$. For example, if $e_1, \ldots, e_n, n \geq 2$ is a linear basis of $V$, then this is readily seen to hold.
if \( W \) is the subspace spanned by \( e_1 \wedge e_2 \) or, more generally, by sets of the form \( \{e_1 \wedge e_2, e_3 \wedge e_4, \ldots, e_{2k-1} \wedge e_{2k}\} \), for \( n \geq 2k \).

Along the lines of the above arguments it can be seen that for any \( k \geq 2 \) if \( G \) is a free \( k \)-step simply connected Lie group then the action of \( \text{Aut}(G) \) on \( G \) has an open dense orbit. Also, given a simply connected \( k \)-step nilpotent Lie group \( H \) there exists simply connected \( k + 1 \)-step nilpotent Lie group \( G \) such that \( H \) is Lie isomorphic to \( G/Z \), where \( Z \) is the center of \( G \), and every automorphism of \( H \) is a factor of an automorphism of \( G \). Therefore using the examples as above one can also construct examples, for any \( k \geq 2 \), of simply connected \( k \)-step nilpotent Lie groups such that the \( \text{Aut}(G) \)-action on \( G \) has an open dense orbit.

In the converse direction we have the following.

**Theorem 3.1.** Let \( G \) be a connected Lie group. Suppose that there exists \( g \in G \) such that the closure of the \( \text{Aut}(G) \)-orbit of \( g \) has positive Haar measure in \( G \). Then \( G \) is a nilpotent Lie group.

A weaker form of this, in which it was assumed that the orbit is dense in \( G \), was proved in [23] (Theorem 2.1), but the same argument is readily seen to yield the stronger assertion as above. By a process of approximation, Theorem 2.1 in [23] was extended to all finite-dimensional connected locally compact groups, and in the same way one can also get that the assertion in Theorem 3.1 holds also for all finite-dimensional connected locally compact groups.

A nilpotent Lie group \( G \) need not always have dense orbits under the action of \( \text{Aut}(G) \). Examples of 2-step simply connected nilpotent Lie groups with no dense orbits under the \( \text{Aut}(G) \)-action are exhibited in [26]. There are also examples in [26] of 3-step simply connected nilpotent Lie groups for which \( \text{Aut}(G) \) is a unipotent group, namely when \( \text{Aut}(G) \) is viewed a subgroup of \( \text{GL}(\mathcal{E}) \) all its elements are unipotent linear transformation, and consequently all orbits of \( \text{Aut}(G) \) on \( G \) are closed, and hence lower dimensional, submanifolds of \( G \).

There has been a detailed study of groups in which \( \text{Aut}(G) \) has only finitely many, or countably many, orbits, in the broader context of locally compact groups, and also abstract groups. The results in this respect for connected Lie groups will be discussed in §3.7.

### 3.2 Connected subgroups of \( \text{Aut}(G) \) with dense orbits

Theorem 3.1 implies in particular that there is no connected Lie group \( G \) for which \( \text{Aut}(G) \) has a one-parameter subgroup whose action on \( G \) has a dense orbit; by the theorem, a Lie group \( G \) with that property would be nilpotent and since it has to be noncompact, going to a quotient we get that for some \( n \geq 1 \),
GL($n, \mathbb{R}$) has a one-parameter subgroup acting on $\mathbb{R}^n$ with a dense orbit, but simple considerations from linear algebra rule this out.

On $\mathbb{R}^{2n}$ viewed as $\mathbb{C}^n$, $n \geq 2$, we have linear actions of $\mathbb{C}^{n-1} \times \mathbb{C}^*$, which have an open dense orbit: the action of $(z_1, \ldots, z_{n-1}, z)$, where $z_1, \ldots, z_{n-1} \in \mathbb{C}$ and $z \in \mathbb{C}^*$, is defined on the standard basis vectors $\{e_1, \ldots, e_n\}$ by $e_j \mapsto z e_j$ for $j = 1, \ldots, n-1$ and $e_n \mapsto \sum_{j=1}^{n-1} z_j e_j + z e_n$. The same also holds for (outer) cartesian products of such actions, and in particular we have linear actions of $\mathbb{C}^n$ on $\mathbb{R}^{2n}$, with dense orbits. Since for $n \geq 2$ we can realise $\mathbb{R}^{n+1}$ as a dense subgroup of $\mathbb{C}^n$, we get linear actions $\mathbb{R}^{n+1}$ on $\mathbb{R}^{2n}$ admitting dense (but not open) orbits; we note that in analogy with the above we can get linear actions of $\mathbb{R}^n$ on $\mathbb{R}^n$ with open dense orbits, but they do not yield actions of $\mathbb{R}^n$ with dense orbits. It can be seen using the Jordan canonical form that there is no linear action of $\mathbb{R}^2$ admitting dense orbits, and hence there is no action of $\mathbb{R}^2$ on any Lie group admitting dense orbits, by an argument as above, using Theorem 3.1.

On $\mathbb{R}^n$, apart from GL($n, \mathbb{R}$), various proper subgroups act transitively on the complement of $\{0\}$; e.g. $S \cdot O(n, \mathbb{R})$ where $S$ is the subgroup consisting of nonzero scalar matrices and $O(n, \mathbb{R})$ is the orthogonal group, or the symplectic group $\text{Sp}(n, \mathbb{R})$ for even $n$. Also if $O(p, q)$ is the orthogonal group of a quadratic form of signature $(p, q)$ on $\mathbb{R}^n$, $n = p + q$ then the action of $S \cdot O(p, q)$, with $S$ as above, on $\mathbb{R}^n$ has an open dense orbit (the complement consists of the set of zeros of the quadratic form, which is a proper algebraic subvariety of $\mathbb{R}^n$).

Subgroups of GL($n, \mathbb{R}$) acting with an open dense orbit have been a subject of much interest in another context; $\mathbb{R}^n$ together with such a subgroup is called a pre-homogeneous vector space; the reader is referred to [63] for details.

### 3.3 Automorphisms with dense orbits

Analysis of the issue of dense orbits was inspired by a question raised by P. R. Halmos in his classic book on Ergodic Theory ([51], page 29) as to whether a non-compact locally compact group can admit a (continuous) automorphism which is ergodic with respect to the Haar measure of the group, namely such that there is no measurable set invariant under the automorphism such that both the set and its complement have positive Haar measure (see §4.1 for more on ergodicity). An automorphism as in Halmos’ question would have a dense orbit (assuming the group to be second countable), and in particular one may ask whether there exists an automorphism of a noncompact locally compact group with a dense orbit. This question was answered in the negative in a paper of R. Kaufman and M. Rajagopalan [62] and T. S. Wu [89] for connected locally compact groups, and N. Aoki [4] for a general locally compact group (see also [74] for some clarifications on the proof in [4]); it may also be mentioned here that the analogous question is
studied for affine automorphisms of locally compact groups (namely transformations which are composite of a continuous group automorphism with a translation by a group element) in [17] for connected groups, and in [61] in the generality of all locally compact groups.

Though there are nonabelian compact groups admitting automorphisms with dense orbits, such as $C^\mathbb{Z}$ where $C$ is a compact nonabelian group, for which the shift automorphism has dense orbits, a compact connected Lie group admits such an automorphism only if it is a torus of dimension $n \geq 2$. For a compact semisimple Lie group $G$, $\text{Inn}(G)$ is a subgroup of finite index in $\text{Aut}(G)$ and the orbits of $\text{Aut}(G)$ on $G$ are closed submanifolds of dimension less than that of $G$, and in particular not dense in $G$; a general compact connected Lie group has a simple Lie group as a factor and hence the preceding conclusion holds in this generality also.

### 3.4 $\mathbb{Z}^d$-actions

The analogue of Halmos’ question for $\mathbb{Z}^d$-actions, namely the multi-parameter case with $d \geq 1$, and more generally actions of abelian groups of automorphisms, was considered in [24], where the following is proved.

**Theorem 3.2.** Let $G$ be a connected Lie group. Suppose that there exists an abelian subgroup $H$ of $\text{Aut}(G)$ such that the $H$-action on $G$ has a dense orbit. Then there exists a compact subgroup $C$ contained in the center of $G$ such that $G/C$ is topologically isomorphic to $\mathbb{R}^n$ for some $n \geq 0$; in particular $G$ is a two-step nilpotent Lie group. If moreover the $H$-action leaves invariant the Haar measure on $G$ then $G$ is a torus.

Recall that the $n$-dimensional torus $\mathbb{T}^n$, where $n \geq 2$, admits automorphisms with a dense orbit, and one can find such an automorphism contained in a subgroup $A$ of $\text{Aut}(\mathbb{T}^n) \approx \text{GL}(n, \mathbb{Z})$ which is isomorphic to $\mathbb{Z}^d$ for $d \leq n - 1$. This gives examples of subgroups $A$ of $\text{Aut}(\mathbb{T}^n)$ isomorphic to $\mathbb{Z}^d$ for $d \leq n - 1$, acting with a dense orbit. Conversely every subgroup $A$ of $\text{Aut}(\mathbb{T}^n)$ isomorphic to $\mathbb{Z}^d$ and acting with a dense orbit contains an (individual) automorphism which has a dense orbit; see [7]; see also [76] for analogous results in a more general setting of automorphisms of general compact abelian groups.

It is easy to see that we have a $\mathbb{Z}^2$-action on $\mathbb{R}$ with a dense orbit; the automorphisms defined respectively by multiplication by $e^\alpha$ and $-e^\beta$, $\alpha, \beta > 0$, generate such an action when $\alpha/\beta$ is irrational. More generally, $G = \mathbb{R}^n \times \mathbb{T}^m$ admits a $\mathbb{Z}^d$-action with a dense orbit if any only if $m \neq 1$ and $d \geq (n + 2)/2$; see [24] for details. Examples of nonabelian two-step nilpotent Lie groups $G$ admitting $\mathbb{Z}^2$-actions with a dense orbit are given in [24].
It may be mentioned here that Theorem 3.2 is extended in [38] to actions on general locally compact groups $G$, where it is concluded that under the analogous condition there exists a compact normal subgroup $C$ such that the quotient $G/C$ is a (finite) product of locally compact fields of characteristic zero.

In line with the above it would be interesting to know about Lie groups admitting actions by nilpotent or, more generally, solvable groups of automorphisms with a dense orbit.

### 3.5 Discrete groups with dense orbits

Consider a connected Lie group $G$ with a Lie subgroup $H$ of Aut$(G)$ such that the $H$-action on $G$ has an open dense orbit; e.g. $\mathbb{R}^n$, $n \geq 1$, and $H = SL(n, \mathbb{R})$ – see §3.1 for more examples (recall also that such a $G$ is nilpotent). Let $g \in G$ be such that the $H$-orbit is open and dense in $G$. Let $L$ be the stabiliser of $g$ under the $H$-action, viz. $L = \{ \tau \in \text{Aut}(G) \mid \tau(g) = g \}$. Then it is easy to see that for a subgroup $\Gamma$ of $H$ the $\Gamma$-action on $G$ has a dense in $G$ if and only if the $L$-action on $H/\Gamma$ has a dense orbit; this phenomenon is known as “duality” - see for instance [6], [22] for some details). In certain situations, such as when $\Gamma$ is a lattice in $H$ (viz. $H/\Gamma$ admits a finite measure invariant under the action of $H$ on the left), the question of whether the action of a subgroup has dense orbits on $H/\Gamma$ is amenable via techniques of ergodic theory.

For $G = \mathbb{R}^n$, $n \geq 2$, we have Aut$(G) \approx GL(n, \mathbb{R})$, and there exist many discrete subgroups of the latter whose action on $\mathbb{R}^n$ admits dense orbits. Let $H = SL(n, \mathbb{R})$ and $\Gamma$ be a lattice in SL$(n, \mathbb{R})$. We choose $g$ as $e_1$ where $\{e_1, \ldots, e_n\}$ is the coordinate basis of $\mathbb{R}^n$ and let $L$ be its stabiliser. It is known that this subgroup acts ergodically on $H/\Gamma$ and hence as noted above the $\Gamma$-action on $\mathbb{R}^n$ has a dense orbit. This applies in particular to the subgroup SL$(n, \mathbb{Z})$, consisting of integral unimodular matrices, which is indeed a lattice in SL$(n, \mathbb{R})$ (see [70], Ch. 7 or [75], Ch.10). We shall discuss more about the orbits of these in the next section.

There are also natural examples of discrete subgroups of SL$(n, \mathbb{R})$ other than lattices which have dense orbits under the action on $\mathbb{R}^n$. For example if $n$ is even and $\Gamma$ is a lattice in the symplectic group Sp$(n, \mathbb{R})$ then it has dense orbits on $\mathbb{R}^n$; stronger statements analogous to those for lattices in SL$(n, \mathbb{R})$ are possible but we shall not go into the details. There are also other examples, arising from hyperbolic geometry. Let $\Gamma$ be the fundamental group of a surface of constant negative curvature whose associated geodesic flow is ergodic. Then $\Gamma$ may be viewed canonically as a subgroup of PSL$(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}$, where $I$ denotes the identity matrix, and if $\Gamma$ is the lift of $\Gamma$ in SL$(2, \mathbb{R})$, then the action of $\Gamma$ on $\mathbb{R}^2$ has dense orbits; the action is ergodic with respect to the Lebesgue measure (see [6] for some details; see §4 for a discussion on ergodicity). Analogous examples
can also be constructed in higher dimensions.

In place of \( \mathbb{R}^n \) one may consider other nilpotent connected Lie groups \( G \) which admit an open dense orbit under the action of \( \text{Aut}(G) \). It would be interesting to know analogous results for discrete groups of automorphisms of other connected nilpotent Lie groups, which however does not seem to be considered in the literature.

### 3.6 Orbit structure of actions of some discrete groups

In most of the cases considered in the earlier subsections where we conclude existence of a dense orbit, not all orbits may be dense, and in general there is no good description possible of the ones that are not dense. However in certain cases a more complete description is possible.

Expressing \( \mathbb{T}^n \) as \( \mathbb{R}^n / \mathbb{Z}^n \), \( \text{Aut}(\mathbb{T}^n) \) can be realised as \( \text{GL}(n, \mathbb{Z}) \), the group of all \( n \times n \) matrices with integer entries and determinant \( \pm 1 \). For \( v \in \mathbb{R}^n \) whose coordinates with respect to the standard basis (generating \( \mathbb{Z}^n \)) are rational the orbits under the \( \text{Aut}(\mathbb{T}^n) \)-action are easily seen to be finite. It turns out, and not too hard to prove (see [30], for instance) that conversely for any \( v \) at least one of whose coordinates is irrational the \( \text{Aut}(\mathbb{T}^n) \)-orbit is dense.

For the case of \( \mathbb{R}^n \), \( n \geq 1 \) we noted above that if \( \Gamma \) is a lattice in \( \text{SL}(n, \mathbb{R}) \) then there exist dense orbits, by duality and ergodicity considerations. In fact in this case it is possible to describe the dense orbits precisely. If \( \text{SL}(n, \mathbb{R}) / \Gamma \) is compact (viz. if \( \Gamma \) is a “uniform” lattice) then the orbit of every non-zero point in \( \mathbb{R}^n \) is dense in \( \mathbb{R}^n \). When \( \text{SL}(n, \mathbb{R}) / \Gamma \) is noncompact (but a lattice) the set of points whose orbits are not dense is contained in a union of countably many lines (one-dimensional vector subspaces) in \( \mathbb{R}^n \). In the case of \( \text{SL}(n, \mathbb{Z}) \) the exceptional lines involved are precisely those passing through points in \( \mathbb{R}^n \) with rational coordinates (see [35] for more general results in this direction). These results are consequences of the study of flows on homogeneous spaces which has been a much studied topic in the recent decades, thanks to the work of Marina Ratner on invariant measures of unipotent flows. We shall not go into details on the topic here; the interested reader is referred to the expository works [22] and [64] and other references there, for exploration of the topic. The result in the special case of \( \text{SL}(n, \mathbb{Z}) \) recalled above was first proved in [15] (see also [34] for a strengthening in another direction).

As in the case of the issues in the previous section, it would be interesting to know results analogous to the above for discrete groups of automorphisms of other nilpotent connected Lie groups.
3.7 Aut(G)-actions with few orbits

Observe that when \( G = \mathbb{R}^n, n \in \mathbb{N} \), Aut(G) is GL(n, \mathbb{R}) and the action is transitive on the complement of the zero element. Thus the action of Aut(G) on G has only two orbits. The question as to when there can be only finitely many, or countably many, orbits has attracted attention, not only for Lie groups, but in the general context of locally compact groups; we shall indicate some of the results in that generality, giving references, but our focus shall be on Lie groups.

**Theorem 3.3.** Let G be a locally compact group. Suppose that the action of Aut(G) on G has only countably many orbits. Then the connected component \( G^0 \) of the identity in G is a simply connected nilpotent Lie group. Moreover, the number of Aut(G)-orbits in \( G^0 \) is finite and one of them is an open orbit.

The first statement in the theorem was proved in [84]. We note that for Lie groups it can be deduced from Theorem 3.1 (which is a generalised version of a theorem from [23]); since under the condition in the hypothesis at least one of the orbits has to be of positive measure Theorem 3.1 yields that the group is nilpotent, but on the other hand the condition also implies that there is no compact subgroup of positive dimension contained in the center, so G must in fact be a simply connected nilpotent Lie group. It is proved in [85] (Theorem 6.3) that under the condition in the hypothesis one of the orbits is open. Moreover, as G is a simply connected nilpotent Lie group, the Aut(G)-orbits are in canonical one-one correspondence with Aut(\( \mathfrak{g} \))-orbits on \( \mathfrak{g} \) via the exponential map, where \( \mathfrak{g} \) is the Lie algebra of G, and the latter being an action of a real algebraic group, the cardinality of the orbits can be countable only if it is finite and one of the orbits is open.

Now let G be a connected Lie group such that the Aut(G)-action on G has only finitely many orbits. Since the identity element is fixed, there are at least two orbits; the group is said to be homogeneous if there are only two orbits. The groups \( \mathbb{R}^n, n \geq 1 \), are homogeneous, and they are also readily seen to be the only connected abelian Lie groups for which the Aut(G)-action has only finitely many orbits. It turns out that \( \mathbb{R}^n, n \geq 1 \), are in fact the only connected locally compact groups that are homogeneous (see [82], Theorem 6.4); all (not necessarily connected) homogeneous locally compact groups have also been determined in [82]; \( K^n \), where K is the field of p-adic numbers, with p a prime, or \( \mathbb{Q} \) (with the discrete topology), and \( n \in \mathbb{N} \), are some of the other examples of homogeneous groups.

A locally compact group is said to be almost homogeneous if the Aut(G)-action on G has 3 orbits. The class of groups with the property has been studied in [84] and [85]. An almost homogeneous Lie group is a Heisenberg group, namely a
group defined on $V \oplus Z$, where $V$ and $Z$ are vector spaces, with the product defined by $(v, x) \cdot (w, y) = (v + w, x + y + \frac{1}{2}(v, w))$, where $\langle \cdot, \cdot \rangle$ is an alternating non-degenerate bilinear form over $V$ with values in $Z$; (we note that here the group structure is viewed via identification with the corresponding Lie algebra, and the factor $\frac{1}{2}$ is introduced so that $\langle v, w \rangle$ is the Lie bracket of $v$ and $w$). Moreover, the pair of dimensions of the vector spaces $V$ and $Z$ are either of the form $(2n, 1), (4n, 2), (4n, 3)$, with $n \in \mathbb{N}$, or one of $(3, 3), (6, 6), (7, 7), (8, 5), (8, 6)$ or $(8, 7)$, and each of these pairs determines an almost homogeneous Heisenberg group. Automorphism groups of these Heisenberg groups have been discussed in [83].

It is easy to see that if $G$ is a $k$-step simply connected nilpotent Lie group then the number of $\text{Aut}(G)$-orbits on $G$ is at least $k + 1$. If $G$ is one of the Heisenberg groups as above then for any $r \geq 1$, the action of $\text{Aut}(G^r)$ on $G^r$ has $2r + 1$ orbits (cf. [85], Proposition 6.8). Thus there exist nilpotent Lie groups with arbitrarily large finite number of orbits under the action of the respective automorphism group. We refer the reader to [85], and other references there, for further details and also open problems on this theme, for general (not necessarily connected) locally compact groups.

4 Ergodic theory of actions of automorphism groups

In this section we discuss various aspects of ergodic theory of actions on Lie groups by groups of automorphisms.

4.1 Preliminaries

We begin by briefly recalling some definitions and conventions which will be followed throughout. By a measure we shall always mean a $\sigma$-finite measure. Given a measure space $(X, \mathcal{M})$, an automorphism $\tau : X \to X$ is said measurable if $\tau^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{M}$; a measure $\mu$ on $(X, \mathcal{M})$ is said to be invariant under a measurable automorphism $\tau$ if $\mu(\tau^{-1}(E)) = \mu(E)$ for all $E \in \mathcal{M}$, and it is said to be quasi-invariant if, for $E \in \mathcal{M}$, $\mu(\tau^{-1}(E)) = 0$ if and only if $\mu(E) = 0$; a measure is said to be invariant or, respectively, quasi-invariant, under a group of measurable automorphisms if it has the property with respect to the action of each of the automorphisms from the group. A measure which is invariant or quasi-invariant with respect to an action is said to be ergodic with respect to the action if there are no two disjoint measurable subsets invariant under the action, each with positive measure.
Two measures on a measurable space $X$ are said to be equivalent if they have the same sets of measure 0. It is easy to see that given a measure which is quasi-invariant under an action there exists a finite measure equivalent to it, which is also quasi-invariant; in particular, given an infinite invariant measure there exists a finite quasi-invariant measure equivalent to it (which may however not be invariant under the action).

A measure $\mu$ on $G$ is called a probability measure if $\mu(G) = 1$. We denote by $P(G)$ the space of probability measures on $G$. For actions of a locally compact second countable group every quasi-invariant probability measure can be “decomposed as a continuous sum” of ergodic quasi-invariant probability measures in a canonical way, and hence it suffices in many respects to understand the ergodic quasi-invariant measures (see, for instance, [70], Theorem 14.4.3).

In our context the measure space structure will always be with respect to the Borel $\sigma$-algebra of the topological space in question.

### 4.2 Finite invariant measures

When $G$ is the torus $\mathbb{T}^n$, $n \geq 2$, the Haar measure is invariant under the action of $\text{Aut}(G)$, which is an infinite discrete group. More generally, if $G$ is a connected Lie group such that the center contains a torus of positive dimension, then the Haar measure of the maximal torus of the center, viewed canonically as a measure on $G$, is invariant under $\text{Aut}(G)$, which can be a group with infinitely many connected components (see §2.2). It turns out however that when we restrict to almost algebraic subgroups of $\text{Aut}(G)$ the situation is quite in contrast, as will be seen in Theorem 4.1 below.

Let $G$ be a connected Lie group. For $\mu \in P(G)$ we denote by $\text{supp} \mu$ the support of $\mu$, namely the smallest closed subset of $G$ whose complement is of $\mu$-measure 0. Let

$$I(\mu) = \{ \tau \mid \mu \text{ is invariant under the action of } \tau \},$$

and

$$J(\mu) = \{ \tau \in \text{Aut}(G) \mid \tau(g) = g \text{ for all } g \in \text{supp } \mu \}.$$

Then it can be seen that $I(\mu)$ and $J(\mu)$ are both closed subgroups of $\text{Aut}(G)$ and $J(\mu)$ is a normal subgroup of $I(\mu)$. It turns out that when $G$ has no compact subgroup of positive dimension contained in the center, the quotient $I(\mu)/J(\mu)$ is compact. In fact we have the following.

**Theorem 4.1.** (cf. [20]) Let $G$ be a connected Lie group. Let $\mathfrak{A}$ be an almost algebraic subgroup of $\text{Aut}(G)$. Then for any $\mu \in P(G)$, $(I(\mu) \cap \mathfrak{A})/(J(\mu) \cap \mathfrak{A})$ is compact.
Consider a connected Lie group $G$ such that $\text{Aut}(G)$ is almost algebraic (see Theorem 3.1) then $I(\mu)/J(\mu)$ is compact and in turn for any $x \in \text{supp} \mu$ the $I(\mu)$-orbit of $x$ in $G$ is compact; thus $\text{supp} \mu$ can be expressed as a disjoint union of compact orbits of $I(\mu)$.

We recall that if $G$ is a compact Lie group then the Haar measure of $G$ is a finite measure invariant under all automorphisms. On the other hand when $G = \mathbb{R}^n$ for some $n \geq 1$, the only finite invariant measure is the point mass at $0$ (see Corollary 4.2 below). The general situation is, in a sense, a mix of the two kinds of situations.

From Theorem 4.1 one can deduce the following, by going modulo the maximal compact subgroup contained in the center and applying Theorem 3.1 on the quotient.

**Corollary 4.2.** Let $G$ be a connected Lie group and $H$ be a subgroup of $\text{Aut}(G)$. Let $\mu$ be a finite $H$-invariant measure on $G$. Then for any $g \in \text{supp} \mu$ the $H$-orbit of $g$ is contained in a compact subset of $G$. In particular if $H$ does not have an orbit other than that of the identity which is contained in a compact subset, then the point mass at the identity is the only $H$-invariant probability measure on $G$.

For $G = \mathbb{R}^n$, $n \geq 1$, Theorem 4.1 implies in particular the following result, which was proved earlier in [18].

**Theorem 4.3.** For $\mu \in P(\mathbb{R}^n)$, $I(\mu)$ is an algebraic subgroup of $\text{Aut}(\mathbb{R}^n) = \text{GL}(n, \mathbb{R})$.

Analogous results are also proved in [18] for projective transformations. The approach involves the study of non-wandering points of the transformations, which has been discussed in further detail in [28].

### 4.3 Convolution powers of invariant measures

We recall here the following result concerning convolution powers of probability measures invariant under the action of a compact subgroup of positive dimension. For a probability measure $\mu$ and $k \in \mathbb{N}$ we denote by $\mu^k$ the $k$-fold convolution power of $\mu$.

**Theorem 4.4.** Let $\mu$ be a probability measure on $\mathbb{R}^n$, $n \geq 1$. Suppose that $\mu$ is invariant under the action of a compact connected subgroup $K$ of $\text{Aut}(G) = \text{GL}(n, \mathbb{R})$ of positive dimension. Then one of the following holds:

i) $\mu^n$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$;

ii) there exists an affine subspace $W$ of $\mathbb{R}^n$ such that $\mu(W) > 0$. 

23
This is a variation of Theorem 3.2 in [29] whose proof can be read off from that of the latter; it can be seen that Condition (ii) above fails to hold if the $K$-action has no nonzero fixed point and there is no proper vector subspace $U$ with $\mu(U) > 0$ (the subspace $U$ can also be stipulated to be invariant), so under these assumptions $\mu^n$ is absolutely continuous with respect to the Lebesgue measure; this is the formulation of Theorem 3.2 in [29].

It would be interesting to have a suitable analogue of the above theorem for actions on a general connected Lie group $G$ by compact subgroups of positive dimension in $\text{Aut}(G)$, (with the Lebesgue measure replaced by the Haar measure of $G$).

### 4.4 Infinite invariant measures

The Lebesgue measure on $\mathbb{R}^n$ is invariant under the action of a large subgroup of $\text{Aut}(G) = \text{GL}(n, \mathbb{R})$, namely $\text{SL}(n, \mathbb{R})$, the special linear group, but not the whole of $\text{Aut}(G)$. On the other hand if $G$ is a connected semisimple Lie group then $\text{Inn}(G)$ is of finite index in $\text{Aut}(G)$, and as $G$ is unimodular it follows that the Haar measure is invariant under the action of $\text{Aut}(G)$. In many Lie groups of common occurrence it is readily possible to determine whether the Haar measure is invariant under all automorphisms, but there does not seem to be any convenient criterion in the literature to test it.

Let $G$ be a connected Lie group and let $S$ denote the subgroup which is the center of $\text{Aff}^0(G)$, when $G$ is viewed canonically as a subgroup of $\text{Aff}(G)$ (see §2.5). Recall that by Theorem 2.4 all orbits of $\text{Aut}^0(G)$, and more generally of any almost algebraic subgroup $A$ of $\text{Aut}(G)$, on $G/S$ are locally closed. By a theorem of Effros [42] this implies that for every measure on $G$ which is quasi-invariant and ergodic under the action of an almost algebraic subgroup $A$ of $\text{Aut}(G)$, there exists an $A$-orbit $\mathcal{O}$ such that the complement of $\mathcal{O}$ has measure 0 (that is, the measure is “supported” on $\mathcal{O}$, except that the latter is a locally closed subset which may not be closed); in particular this applies to any infinite measure invariant under the action of an almost algebraic subgroup $A$ of $\text{Aut}(G)$ as above.

### 4.5 Quasi-invariant measure and ergodicity

We recall that for an action of a group on a locally compact second countable space, given an ergodic quasi-invariant measure $\mu$, for almost all points in $\text{supp} \mu$, the orbit is dense in $\text{supp} \mu$. Together with the results on existence of dense orbits in §3 this yields the following:

**Theorem 4.5.** Let $G$ be a connected Lie group and let $\lambda$ be a Haar measure of
G. Then \( \lambda \) is quasi-invariant under the action of \( \text{Aut}(G) \). Moreover the following holds:

i) if \( \lambda \) is ergodic with respect to the \( \text{Aut}(G) \)-action on \( G \) then \( G \) is a nilpotent Lie group.

ii) if \( \lambda \) is ergodic under the action of an abelian subgroup of \( \text{Aut}(G) \) then \( G \) is a two-step nilpotent Lie group, with \([G,G]\) compact.

iii) if \( \lambda \) is ergodic under the action of some \( \alpha \in \text{Aut}(G) \) then \( G \) is a torus.

iv) if \( \lambda \) is ergodic under the action of an almost algebraic subgroup \( A \) of \( \text{Aut}(G) \) then the \( A \)-action on \( G/C \), where \( C \) is the maximal compact subgroup of \( G \), has an open dense orbit.

The first three statements are straightforward consequences of results from §3 on the existence of a dense orbit and the observation above. The last assertion follows from the fact that \( \{ \bar{\alpha} \mid \alpha \in A \} \) is an almost algebraic subgroup of \( \text{Aut}(G) \), where \( \mathfrak{g}' \) is the Lie algebra of \( G/C \), and hence its orbits on \( \mathfrak{g}' \) are locally closed; as \( G/C \) is a simply connected nilpotent Lie group the exponential map is a homeomorphism and hence the preceding conclusion implies that the \( A \)-orbits on \( G/C \) are locally closed, and in particular an orbit which is dense is also open in \( G/C \).

4.6 Stabilisers of actions of Lie groups

Let \( G \) be a connected Lie group and consider a (Borel-measurable) action of \( G \) on a standard Borel space \( X \). By the stabiliser of a point \( x \in X \) we mean the subgroup \( \{ g \in G \mid gx = x \} \), and we denote it by \( G_x \). Each \( G_x \) is a closed subgroup of \( G \) (see [86], Corollary 8.8). When the action is transitive, namely when the whole of \( X \) is a single orbit, then the stabilisers of any two points in \( X \) are conjugate to each other. One may wonder to what extent this generalises to a general ergodic action, with respect to a measure which is invariant or quasi-invariant under the action of \( G \). In this respect the following is known from [25]; the results in [25] strengthen those from [45] in the case of Lie groups, while in [45] the issue of conjugacy of the stabilisers is considered in the wider framework of all locally compact groups.

**Theorem 4.6.** Let \( G \) be a connected Lie group acting on a standard Borel space \( X \). Let \( \mu \) be a measure on \( X \) which is quasi-invariant and ergodic with respect to the action of \( G \). Suppose also that for all \( x \in X \) the stabiliser \( G_x \) of \( x \) has only finitely many connected components. Then there exists a subset \( N \) of \( X \) with \( \mu(N) = 0 \) such that for \( x, y \in X \setminus N \) there exists \( \alpha \in \text{Aut}^0(G) \) such that \( \alpha(G_x) = G_y \); in particular \( G_x \) and \( G_y \) are topologically isomorphic to each other.

This is a variation of Corollary 5.2 in [25], where the ergodicity condition involved is formulated in terms of a \( \sigma \)-ideal of null sets, rather than a quasi-
invariant measure. Without the assumption of $G_x$ having only finitely many connected components it is proved that for $x, y$ in the complement of a null set $N$, the connected components of the identity in $G_x$ and $G_y$ are topologically isomorphic to each other ([25], Corollary 5.7). The proofs of these results are based on consideration of the action of Aut$(G)$ on the space of all closed subgroups of $G$, the latter being equipped with the Fell topology, and the orbits or stabilisers $G_x, x \in X$, under the action. The argument depends on the fact Aut$^0(G)$ is an almost algebraic subgroup of GL$(G)$ containing Inn$(G)$, $G$ being the Lie algebra of $G$ (see Theorem 2.2(ii)); the proof shows that automorphisms $\alpha$ as in the conclusion of the theorem may in fact be chosen to be from the smallest almost algebraic subgroup containing Inn$(G)$, and in particular if Inn$(G)$ is an almost algebraic subgroup of GL$(G)$ then stabilisers of almost all points are conjugates to each other. Other conditions under which the stabilisers of almost all points may be concluded to be actually conjugate in $G$, and also examples for which it does not hold, are discussed in [25]. We shall not go into the details of these here.

5 Some aspects of dynamics of Aut$(G)$-actions

In this section we discuss certain results on dynamics in which some features of the actions of Aut$(G)$ on $G$ described in earlier sections play a role.

5.1 Stabilisers of continuous actions

Let $G$ be a connected Lie group and consider a continuous action of $G$ on a compact Hausdorff space $X$. Here we recall a topological analogue of Theorem 4.6. As before we denote by $G_x$ the stabiliser of $x$ under the action in question and we denote by $G^0_x$ the connected component of the identity in $G_x$. We recall that an action is said to be minimal if there is no proper nonempty closed subset invariant under the action. The following is proved in [27], Proposition 3.1.

**Theorem 5.1.** Let $G$ be a connected Lie group acting continuously on a locally compact Hausdorff space $X$. Suppose that there exists $x \in X$ such that the orbit of $x$ is dense in $X$. Then there exists an open dense subset $Y$ of $X$ such that for all $y \in Y$ there exists $\alpha \in \text{Aut}^0(G)$ such that $\alpha(G^0_x) = G^0_y$. If the action is minimal then $G^0_x, x \in X$, are Lie isomorphic to each other.

Further conditions which ensure $G^0_x, x \in X$, being conjugate in $G$ are discussed in [27]. The results of [27] generalise earlier results of C. C. Moore and G. Stuck proved for special classes of Lie groups (see [27] for details).
5.2 Anosov automorphisms

Anosov diffeomorphisms have played an important role in the study of differentiable dynamical systems; we shall not go into much detail here on the issue – the reader is referred to Smale’s expository article [81] for a perspective on the topic. Hyperbolic automorphisms of tori $\mathbb{T}^n$, $n \geq 2$, namely automorphisms $\alpha$ such that $d\alpha$ has no eigenvalue (including complex) of absolute value 1, serve as the simplest examples of Anosov automorphisms. In [81] Smale described an example, due to A. Borel, of a non-toral compact nilmanifold admitting Anosov automorphisms; a nilmanifold is a homogeneous space of the form $N/\Gamma$ where $N$ is a simply connected nilpotent Lie group and $\Gamma$ is discrete subgroup, and an Anosov automorphism of $N/\Gamma$ is the quotient on $N/\Gamma$ of an automorphism $\alpha$ of $N$ such that $\alpha(\Gamma) = \Gamma$ and $d\alpha$ has no eigenvalue of absolute value 1. Such a system can have a nontrivial finite group of symmetries and factoring through them leads to some further examples on what are called infra-nilmanifolds, known as Anosov automorphisms of infra-nilmanifolds. Smale conjectured in [81] that all Anosov diffeomorphisms are topologically equivalent to Anosov automorphisms on infra-nilmanifolds.

A broader class of Anosov automorphisms of nilmanifolds, which includes also the example of Borel referred to above, was introduced by L. Auslander and J. Scheuneman [5]. Their approach involves analysing $\text{Aut}(N)$ for certain simply connected nilpotent Lie groups $N$ to produce examples of Anosov automorphisms of compact homogeneous spaces of $N/\Gamma$ for certain discrete subgroups $\Gamma$. In [16] the approach in [5] was extended using some results from the theory of algebraic groups and arithmetic subgroups, and some new examples of Anosov automorphisms were constructed. A large class of examples of Anosov automorphisms were constructed in [31] by associating nilpotent Lie groups to graphs and studying their automorphism group in relation to the graph. Study of the automorphism groups of nilpotent Lie groups has also been applied to construct examples of nilmanifolds that can not admit Anosov automorphisms (see [16] and [26]).

Subsequently Anosov automorphisms of nilmanifolds have been constructed via other approaches; the reader is referred to [46] and [40] and various references cited there for further details. There has however been no characterisation of nilmanifolds admitting Anosov automorphisms, and it may be hoped that further study of the automorphism groups of nilpotent Lie groups may throw more light on the issue.
5.3 Distal actions

An action of a group $H$ on a topological space $X$ is said to be *distal* if for any pair of distinct points $x, y$ in $X$ the closure of the $H$-orbit $\{(gx, gy) \mid g \in G\}$ of $(x, y)$ under the componentwise action of $G$ on the cartesian product space $X \times X$ does not contain a point on the “diagonal”, namely of the form $(z,z)$ for any $z \in X$. The action on a locally compact group $G$ by a group $H$ of automorphisms of $G$ is distal if and only if under the $H$-action on $G$ the closure the orbit of any nontrivial element $g$ in $G$ does not contain the identity element.

Distality of actions is a classical topic, initiated by D. Hilbert, but the early studies were limited to actions on compact spaces. The question of distality of actions on $\mathbb{R}^n$, $n \geq 1$, by groups of linear transformations was initiated by C. C. Moore [69] and was strengthened by Conze and Guivarc’h [13]; see also H. Abels [1], where the results are extended to actions by affine transformations. It is proved that the action of a group $H$ of linear automorphisms of $\mathbb{R}^n$ is distal if and only if the action of each $h \in H$ (viz. of the cyclic subgroup generated by it) is distal, and that it holds if and only if all (possibly complex) eigenvalues of $h$ are of absolute value 1 (see [13], [1]).

It is proved in [2] that the action of a group $H$ of automorphisms of a connected Lie group $G$ is distal if and only if the associated action of $H$ on the Lie algebra $\mathfrak{g}$ of $G$ is distal (to which the above characterisations would apply).

For a subgroup $H$ of $\text{GL}(n, \mathbb{R})$, the $H$-action on $\mathbb{R}^n$ is distal if and only if the action of its algebraic hull (Zariski closure) in $\text{GL}(n, \mathbb{R})$ is distal, and the action of an algebraic subgroup $H$ of $\text{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ is distal if and only if the unipotent elements in $H$ form a closed subgroup $U$ (which would necessarily be the unipotent radical) such that $H/U$ is compact. (see [1], Corollaries 2.3 and 2.5). Via the above correspondence these results can be applied also to actions of groups of automorphisms of a general connected Lie group $G$.

An action of a group $H$ on a space $X$ is called *MOC* (short for “minimal orbit closure”) if the closures of all orbits are minimal sets (viz. contain no proper nonempty invariant closed subsets). When $X$ is compact, the MOC condition is equivalent to distality. For the action on a topological group $G$ by a group of automorphisms MOC implies distality. The converse is known in various cases, including for actions on connected Lie groups $G$ by groups of automorphisms (see [2]). The reader is also referred to [3], [80] and [78] for some generalisations of this as well as some of the other properties discussed above to more general locally compact groups.

A connected Lie group $G$ is said to be of type $\mathcal{R}$ if for all $g \in G$ (all (possibly complex) eigenvalues of $\text{Ad} g$ are of absolute value 1. Thus in the light of the results noted above $G$ is of type $\mathcal{R}$ if and only if the action of $\text{Inn}(G)$ on $G$ is
5.4 Expansive actions

A homeomorphism $\varphi$ of a compact metric space $(X, d)$ is said to be expansive if there exists $\epsilon > 0$ such that for any pair of distinct points $x, y$ there exists an integer $n$ such that $d(\varphi^n(x), \varphi^n(y)) > \epsilon$; the notion may also be defined in terms of a uniformity in place of a metric. An automorphism $\tau$ of a topological group $G$ is expansive if there exists a neighbourhood $V$ of the identity such that for any nontrivial element $g$ in $G$ there exists an integer $n$ such that $\tau^n(g) \notin V$. More generally the action of a group $\Gamma$ of automorphisms of a topological group $G$ is said to be expansive if there exists a neighbourhood $V$ of the identity in $G$ such that for any nontrivial $g$ in $G$ there exists a $\gamma \in \Gamma$ such that $\gamma(g) \notin V$.

A compact connected topological group admits a group of automorphisms acting expansively only if it is abelian (cf. [65]) and finite dimensional (cf. [66]). For compact abelian groups the expansiveness condition on actions of automorphism groups has been extensively studied using techniques of commutative algebra (see [79] for details).

If $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$, the action of a subgroup $\Gamma$ of $\text{Aut}(G)$ on $G$ is expansive if and only if for the induced action of $\Gamma$ on $\mathfrak{g}$, for any nonzero $\xi \in \mathfrak{g}$ the $\Gamma$-orbit of $\xi$ is unbounded in $\mathfrak{g}$ (cf. [9], where the issue is also considered for semigroups of endomorphisms). It is also deduced in [9] that if $\Gamma$ is a virtually nilpotent Lie group (viz. with a nilpotent subgroup of finite index) of $\text{Aut}(G)$ whose action on $G$ is expansive, then $\Gamma$ contains an element acting expansively on $G$.

6 Orbits on the space of probability measures

The action of $\text{Aut}(G)$ on $G$, where $G$ is a Lie group, induces an action of $\text{Aut}(G)$ on the space of probability measures on $G$ (see below for details). This action plays an important role in many contexts. This section is devoted to recalling various results about the action and their applications.

6.1 Preliminaries

Let $G$ be a connected Lie group and as before let $P(G)$ denote the space of probability measures on $G$. We consider $P(G)$ equipped with the weak∗ topology
with respect to the space of bounded continuous functions; we note that the topology is metrizable, and a sequence \( \{ \mu_j \} \) in \( P(G) \) converges to \( \mu \in P(G) \) if and only if \( \int_G f d\mu_j \to \int_G f d\mu \), as \( j \to \infty \), for all bounded continuous functions \( f \) on \( G \). We recall that the action of Aut\((G)\) on \( G \) induces an action on \( P(G) \), defined, for \( \tau \in \text{Aut}(G) \) and \( \mu \in P(G) \), by \( \tau(\mu)(E) = \mu(\tau^{-1}(E)) \) for all Borel subsets \( E \) of \( G \).

For \( \lambda, \mu \in P(G) \), \( \lambda * \mu \) denotes the convolution product of \( \lambda \) and \( \mu \), and for any \( \mu \in P(G) \) and \( k \in \mathbb{N} \) we denote by \( \mu^k \) the \( k \)-fold convolution product \( \mu * \cdots * \mu \).

Behaviour of orbits of probability measures under actions of various subgroups is of considerable interest in various contexts. One of the issues is to understand conditions under which orbits of automorphism groups \( A \subset \text{Aut}(G) \) are locally closed, namely open in their closures. We recall here that by a result of Effros [42] this condition is equivalent to a variety of other “smoothness” conditions for the action, including that the orbits map being an open (quotient) map onto its image (the latter being considered with respect to the induced topology from \( P(G) \)).

As before for a connected Lie group \( G \) we view \( \text{Aut}(G) \) as a subgroup of \( \text{GL}(\mathfrak{G}) \), where \( \mathfrak{G} \) is the Lie algebra of \( G \), and a subgroup \( A \) of \( \text{Aut}(G) \) is said to be almost algebraic if it is an almost algebraic subgroup of \( \text{GL}(\mathfrak{G}) \) (see §2).

**Theorem 6.1.** (cf. [36], Theorem 3.3) *Let \( G \) be a connected Lie group and \( A \) be an almost algebraic subgroup of \( \text{Aut}(G) \). Let \( C \) be the maximal compact subgroup contained in the center of \( G \). Suppose that for any \( g \in G \), \( \{ g^{-1} \tau(g) \mid \tau \in A \} \cap C \) is finite. Then for any \( \mu \in P(G) \) the \( A \)-orbit \( \{ \tau(\mu) \mid \tau \in A \} \) is open in its closure in \( P(G) \). Moreover, if \( A \) consists of unipotent elements in \( \text{GL}(\mathfrak{G}) \) then the \( A \)-orbit is closed in \( P(G) \).*

The theorem implies in particular that when \( G \) has no compact subgroup of positive dimension contained in its center, for the action of any almost algebraic subgroup \( A \) of \( \text{Aut}(G) \) (which includes also the whole of \( \text{Aut}(G) \) in this case) the orbits of \( A \) are locally closed.

### 6.2 Factor compactness and concentration functions

Let \( \mu \in P(G) \). We denote by \( G(\mu) \) the smallest closed subgroup of \( G \) containing \( \text{supp} \mu \), the support of \( \mu \), by \( N(\mu) \) the normaliser of \( G(\mu) \) in \( G \) and by \( Z(\mu) \) the centraliser of \( \text{supp} \mu \), namely \( \{ g \in G \mid gx = xg \text{ for all } x \in \text{supp} \mu \} \).

A \( \lambda \in P(G) \) is called a factor of \( \mu \) if there exists \( \nu \in P(G) \) such that \( \mu = \lambda * \nu = \nu * \lambda \). It is known that any factor of \( \mu \) has its support contained in \( N(\mu) \), more specifically in a coset of \( G(\mu) \) contained in \( N(\mu) \) (see [32], Proposition 1.1 for the first assertion; the second is an easy consequence of the first).
A question of interest, which is not yet understood in full generality, is whether given \( \mu \in P(G) \) and a sequence \( \{ \lambda_j \} \) of its factors there exists a sequence \( \{ z_j \} \) in \( Z(\mu) \) such that \( \{ \lambda_j z_j \} \) is relatively compact in \( P(G) \) (in place of \( Z(\mu) \) one may also allow in this respect the subgroup \( \{ g \in G \mid g \mu g^{-1} = \mu \} \), which can be bigger, but the distinction turns out to be a rather technical issue, and we shall not concern ourselves with it here). It is known that given a sequence of factors \( \{ \lambda_j \} \) there exists a sequence \( \{ x_j \} \) in \( N(\mu) \) such that \( \{ x_j \mu x_j^{-1} \}, \{ x_j^{-1} \mu x_j \} \) and \( \{ x_j \lambda_j \} \) are relatively compact \([32],[33]\), and it suffices to show that for such a sequence \( \{ x_j \} \) the sequence of cosets \( \{ x_j Z(\mu) \} \) is relatively compact in \( G/Z(\mu) \). It is known that this is true when \( G \) is an almost algebraic Lie group (and also under some weaker, somewhat technical, conditions) \(\text{(see [32], [36])}\); the proofs however fall back on reducing the question to vector space situation and a more direct approach connecting the question to Theorem 6.1 would be desirable.

A similar issue, or rather a more general one, arises in the study of the decay of concentration functions of \( \mu^n \) as \( n \to \infty \), where one would like to know under what conditions a sequence of the form \( \{ x_j \mu x_j^{-1} \} \) is relatively compact. We shall however not go into the details of the concept or the results about it here; a treatment of the topic in this perspective may be found in \([21],[39]\); for a complete resolution of the problem itself, which however involves a somewhat different approach, the reader is referred to \([59]\) and other references cited there.

### 6.3 Tortrat groups

A locally compact group is called a Tortrat group if for any \( \mu \in P(G) \) which is not idempotent (viz. such that \( \mu^2 \neq \mu \)) the closure of \( \{ g \mu g^{-1} \mid g \in G \} \) in \( P(G) \) does not contain any idempotent measure. We note that the issue concerns the closure of the orbits of \( \mu \) in \( P(G) \) under the action of the subgroup \( \text{Im}(G) \); the latter being contained in \( \text{Aut}^0(G) \) which is an almost algebraic subgroup is useful in this respect.

Recall (see §5.3) that a connected Lie group is said to be type \( \mathcal{R} \) if for all \( g \in G \) all eigenvalues of \( \text{Ad} g \) are of absolute value 1. It was shown in \([37]\) that for a Lie group the two properties are equivalent:

**Theorem 6.2.** (cf. \([37]\)) A connected Lie group is a Tortrat group if and only if it is of type \( \mathcal{R} \).

The result implies also that a connected locally compact group is a Tortrat group if and only if it is of polynomial growth (see \([37]\) for details and the related references).
6.4 Convergence of types

Let \( \{ \lambda_j \} \) and \( \{ \mu_j \} \) be two sequences in \( P(G) \) converging to \( \lambda \) and \( \mu \) respectively. Suppose further that there exists a sequence \( \{ \tau_j \} \) in \( \text{Aut}(G) \) such that \( \tau_j(\lambda_j) = \mu_j \) for all \( j \). A question, arising in various contexts in the theory of probability measures on groups, is under what (further) conditions on \( \lambda \) and \( \mu \) can we conclude that there exists a \( \tau \in \text{Aut}(G) \) such that \( \tau(\lambda) = \mu \); the reader can readily convince herself/himself that further conditions are indeed called for. When there exists a \( \tau \) as above we say that convergence of types holds. Though the term “type of a measure” does not seem to make an appearance in literature freely, implicit in the terminology above is the idea that \( \{ \tau(\mu) \mid \tau \in \text{Aut}(G) \} \) constitutes the “type of \( \mu \)”, or that \( \mu \) and \( \tau(\mu) \) are of the same type for any \( \tau \in \text{Aut}(G) \), and the question is if you have pair of sequences of measures with the corresponding measures of the same type, converging to a pair of measures, under what conditions can we conclude the limits to be of the same type.

We now recall some results in this respect; for reasons of simplicity of exposition we shall not strive for full generality (see [20] for more details). As in §2 let \( \text{Aff}(G) \) be the group of affine automorphisms, and let \( \rho : \text{Aff}(G) \to \text{GL}(V) \) be the linearising representation (see §2.5); recall that \( \rho \) is defined over \( V = \wedge^a \mathfrak{B} \), where \( \mathfrak{B} \) is the Lie algebra of \( \text{Aff}(G) \) and \( a \) is the dimension of \( \text{Aut}(G) \).

We say that \( \mu \in P(G) \) is \( \rho \)-full if there does not exist any proper subspace \( U \) of \( V \) that is invariant under \( \rho(g) \) for all \( g \in \text{supp} \mu \); though this condition is rather technical, as it involves the linearising representation, it is shown in [20] that for various classes of groups it holds under simpler conditions; for instance if \( G \) is an almost algebraic subgroup of \( \text{GL}(n, \mathbb{R}) \) then the condition holds for any \( \mu \) for which \( \text{supp} \mu \) is not contained in a proper almost algebraic subgroup of \( G \). The following is a result which is midway between the rather technical Theorem 1.5 and the specialised result in Theorem 1.6 from [20], whose proof can be read off from the proof of Theorem 1.6.

**Theorem 6.3.** Let \( G \) be a connected Lie group. Let \( \{ \lambda_j \} \) and \( \{ \mu_j \} \) be sequences in \( P(G) \) converging to \( \lambda \) and \( \mu \) respectively. Suppose that there exists a sequence \( \{ \tau_j \} \) in \( \text{Aut}^0(G) \) such that \( \mu_j = \tau_j(\lambda_j) \) for all \( j \) and that \( \lambda \) and \( \mu \) are \( \rho \)-full. Then there exist sequences \( \{ \theta_j \} \) and \( \{ \sigma_j \} \) in \( \text{Aut}(G) \) such that \( \{ \theta_j \} \) is contained in a compact subset of \( \text{Aut}(G) \), \( \{ \sigma_j \} \) are isotropic shear automorphisms, and \( \tau_j = \theta_j \sigma_j \) for all \( j \). If moreover \( \text{supp} \lambda \) is not contained in any proper closed normal subgroup \( M \) of \( G \) such that \( G/M \) is a vector space of positive dimension, then \( \{ \sigma_j \} \) is also relatively compact.

We note that when \( \{ \tau_j \} \) as in the theorem is concluded to be relatively compact, convergence of types holds (for the given sequences); with the notation as
in the theorem, if \( \tau \) is an accumulation point of \( \{\tau_j\} \) then \( \tau(\lambda) = \mu \). In the light of the conclusion of the theorem it remains mainly to understand the asymptotic behaviour under sequences of shear automorphisms.

A special case of interest is when \( \{\lambda_j\} \) are all equal, say \( \lambda \). For this case we recall also the following result, for shear automorphisms, proved in [29] (Theorem 4.3 there); the result played an important role in the proof of the main theorem there concerning embeddability in a continuous one-parameter semigroup, for a class of infinitely divisible probability measures; we shall not go into the details of these concepts here. It may be noted that one of the conditions in the hypothesis is as in statement (i) of Theorem 4.4.

**Theorem 6.4.** Let \( G \) be a connected Lie group and \( T \) be a torus contained in the center of \( G \). Let \( H \) be a closed normal subgroup of \( G \) such that \( G/H \) is topologically isomorphic to \( \mathbb{R}^n \) for some \( n \geq 1 \). Let \( W = G/H \) and \( \theta : G \to W \) be the canonical quotient homomorphism. Let \( \lambda \in P(G) \) be such that \( \theta(\mu)^n \) is absolutely continuous with respect to the Lebesgue measure on \( W \). Let \( \{\varphi_j\} \) be a sequence of homomorphisms of \( W \) into \( T \) and for all \( j \) let \( \tau_j \in \text{Aut}(G) \) be the shear automorphism of \( G \) corresponding to \( \varphi_j \). Suppose that the sequence \( \{\tau_j(\lambda)\} \) converges to a measure of the form \( \tau(\lambda) \) for some \( \tau \in \text{Aut}(G) \) fixing \( T \) pointwise. Then there exists a sequence \( \{\sigma_j\} \) in \( \text{Aut}(G) \) such that \( \sigma_j(\mu) = \mu \) and \( \{\tau_j\sigma_j\} \) is relatively compact.

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