TOTAL INTEGRALS OF SOLUTIONS FOR THE PAINLEVÉ II EQUATION AND SINGULARITY FORMATION IN THE VORTEX PATCH DYNAMICS

PIOTR KOKOCKI

Abstract. In this paper, we establish a formula determining the value of the Cauchy integrals of the real and purely imaginary Ablowitz-Segur solutions for the inhomogeneous second Painlevé equation. Our approach relies on the Deift-Zhou steepest descent analysis of the corresponding Riemann-Hilbert problem and the construction of an appropriate parametrix in the neighborhood of the origin. The obtained formula is used to show that an arbitrary logarithmic spiral is a finite time singularity developed by a geometric flow, which approximates the vortex patch dynamics of the 2D Euler equation.

Contents
1. Introduction 1
2. Solution $\Phi(\lambda, x)$ for the RH problem for Painlevé II equation 5
3. Bessel functions and auxiliary RH problems 8
  3.1. Auxiliary RH problem for $x > 0$ 10
  3.2. Auxiliary RH problem for $x < 0$ 11
4. Steepest descent analysis of $\Phi(\lambda, x)$ for $x > 0$ 16
  4.1. Contour deformation 16
  4.2. Representation of the solution and asymptotic behavior 18
5. Steepest descent analysis of $\Phi(\lambda, x)$ for $x < 0$ 24
  5.1. Contour deformation 24
  5.2. Parametrices near the origin and stationary points 30
  5.3. Representation of the solution and asymptotic behavior 33
6. Proof of Theorem 1.1 41
7. Self-similar solutions for the geometric flow 42
8. Proof of Theorem 1.2 43
9. Appendix: solutions for the classical RH problem 46
References 48

1. INTRODUCTION

We are concerned in the inhomogeneous second Painlevé (PII) equation

$$u''(x) = xu(x) + 2u^3(x) - \alpha, \quad x \in \mathbb{C},$$

(1.1)

where $\alpha \in \mathbb{C}$ is a constant such that $\text{Re} \alpha \in (-1/2, 1/2)$. It is known (see [16], [22], [17] Chapter 11) that the solutions of the PII equation are related with Riemann-Hilbert (RH) problems characterized by the Stokes multipliers, that is, the triple

2010 Mathematics Subject Classification. 33E17, 35Q15, 41A60, 53C44.

Key words and phrases. Painlevé II equation, Riemann-Hilbert-problem, asymptotic expansion, contour dynamics, localized induction approximation.

The researches supported by the MNiSW Iuventus Plus Grant no. 0338/IP3/2016/74.
of parameters \((s_1, s_2, s_3) \in \mathbb{C}^3\) satisfying the following constraint condition
\[
s_1 - s_2 + s_3 + s_1s_2s_3 = -2\sin(\pi\alpha). \tag{1.2}
\]
Roughly speaking, any choice of the triple \((s_1, s_2, s_3)\) satisfying (1.2) gives us a solution \(\Phi(\lambda, x)\) of the corresponding RH problem, which is a \(2 \times 2\) matrix valued function sectionally holomorphic in \(\lambda\) and meromorphic with respect to \(x\). Then, the function \(u\) obtained by the limit
\[
u(x) = \lim_{\lambda \to \infty} (2\lambda\Phi(\lambda, x)e^{\theta(\lambda, x)\sigma_3})_{12}, \quad \text{where } \theta(\lambda, x) := i(4\lambda^2/3 + x\lambda) \tag{1.3}
\]
is a solution of the PII equation (1.1). Proceeding in this way, we can define a map
\[
\{(s_1, s_2, s_3) \in \mathbb{C}^3 \text{ satisfying (1.2)}\} \to \{\text{solutions of the PII}\},
\]
which is a bijection between the set of all Stokes multipliers and the set of solutions of the Painlevé II equation (see e.g. [17]). Let us restrict our attention to the solutions of (1.1) corresponding to the following choice of the Stokes data
\[
s_1 = -\sin(\pi\alpha) - ik, \quad s_2 = 0, \quad s_3 = -\sin(\pi\alpha) + ik, \quad k, \alpha \in \mathbb{C}, \tag{1.4}
\]
that, for the brevity, we denote by \(u(\cdot; \alpha, k)\). Among them we can specify the real Ablowitz-Segur (AS) solutions that are determined by (1.4) with
\[
\alpha \in (-1/2, 1/2) \quad \text{and} \quad k \in (-\cos(\pi\alpha), \cos(\pi\alpha)). \tag{1.5}
\]
It is known that the real AS solutions satisfy \(\text{Im } u(x; \alpha, k) = 0\) for \(x \in \mathbb{R}\) (see e.g. [17, Chapter 11]) and furthermore the results of [12] show that the solutions have no poles on the real line. The borderline case of (1.4) with
\[
\alpha \in (-1/2, 1/2) \quad \text{and} \quad k = \pm \cos(\pi\alpha)
\]
corresponds to the Hasting-McLeod solutions that are also pole-free on the real line (see [8]) and take real values \(u(x; \alpha, k)\), whenever \(x \in \mathbb{R}\). Considering the multipliers (1.4) with \(\alpha, k \in i\mathbb{R}\), we obtain the purely imaginary Ablowitz-Segur solutions satisfying \(u(x; \alpha, k) \in i\mathbb{R}\) for \(x \in \mathbb{R}\) (see e.g. [17, Chapter 11]). In this case \(u\) is also pole-free on the real line, which follows from the fact that the residues of the poles of \(u\) are equal to \(\pm 1\).

The goal of this paper is to establish a formula for the value of the integral
\[
\int_{-\infty}^{\infty} u(y; \alpha, k) \, dy := \lim_{x \to \pm \infty} \int_{-x}^{x} u(y; \alpha, k) \, dy, \tag{1.6}
\]
where \(u\) is either a real or purely imaginary Ablowitz-Segur solution of the inhomogeneous PII equation. The problem of finding the value of (1.6), was considered in [3] in the homogeneous case \(\alpha = 0\), where the following formula was derived:
\[
\int_{-\infty}^{\infty} u(y; 0, k) \, dy = \frac{1}{2} \ln \left( \frac{1 + k}{1 - k} \right) \quad \text{if either } k \in (-1, 0) \text{ or } k \in i\mathbb{R}. \tag{1.7}
\]
The integral in (1.7) exists in the Cauchy sense due to the asymptotic behavior of the solution \(u(x; 0, k)\) as \(x \to \pm \infty\). To be more precise, form [1], [2] and [14] it follows that the real and purely imaginary AS solutions decay exponentially to zero as \(x \to +\infty\) and furthermore we have the following asymptotic behavior
\[
u(x; 0, k) = \frac{d}{(-x)^{1/4}} \cos \left( \frac{2}{3}(-x)^{3/2} - \frac{3}{4}d^2\ln(-x) + \phi \right) + O \left( \frac{\log(-x)}{(-x)^{2}} \right), \quad x \to -\infty,
\]
where \(d > 0\) and \(\phi \in \mathbb{R}\) are parameters dependent from \(k\) by so-called connecting formulas (see [9], [14]). In [3] and [4] it is established a version of (1.7) for the Hasting-McLeod solutions, which reads as follows
\[
\int_{c}^{\infty} u(y; 0, \pm 1) \, dy + \int_{-\infty}^{c} \left( u(y; 0, \pm 1) \mp \sqrt{\frac{|y|}{2}} \right) \, dy = \mp \frac{\sqrt{2}}{3} c|c|^{3/2} \pm \frac{1}{2} \log(2). \tag{1.8}
\]
In this formula \( c \) is an arbitrary real number and the form of the left hand side of (1.8) follows from the fact that the Hastings-McLeod solutions exponentially decay to zero as the parameter \( x \) approaches the plus infinity and
\[
    u(x; 0, \pm 1) = \pm \sqrt{-x/2} + O((-x)^{-5/2}), \quad x \to -\infty.
\]
In the inhomogeneous case \( \alpha \neq 0 \), the recent result [26, Theorem 1.3] provides a counterpart of the total integral formula (1.7) for the increasing tritronquée solutions (see [23]) of the PII equation. In this paper we prove the following theorem.

**Theorem 1.1.** If \( u(\cdot; \alpha, k) \) is either a real or purely imaginary Ablowitz-Segur solution for the inhomogeneous second Painlevé equation, then
\[
    \lim_{x \to \pm \infty} \exp \left( \int_{-x}^{x} u(y; \alpha, k) \, dy \right) = \frac{\cos(\pi \alpha) + k}{(\cos^2(\pi \alpha) - k^2)^{1/2}}, \quad (1.9)
\]
In the proof of this theorem we will use the nonlinear steepest descent method from [13], where it was used to study asymptotic expansions of solutions for the modified KdV equation. The method was also applied in [14] to study asymptotics of Ablowitz-Segur solutions for the homogeneous PII equation. In the series of papers [17, 21] and [12], the approach based on the steepest descent analysis was used to provide rigorous asymptotic expansions of the Ablowitz-Segur solutions for the inhomogeneous PII equation (1.1). In particular, the results of the papers [17] and [21] say that
\[
    u(x; \alpha, k) = \alpha x^{-1} + O(x^{-3}) - ikAi(x)(1 + O(x^{-3/4})), \quad x \to +\infty, \quad (1.10)
\]
where \( Ai(x) \) is the standard Airy function. On the other hand, in [12] there was shown that for the real and purely imaginary AS solutions we have the expansion
\[
    u(x; \alpha, k) = \frac{d_0}{(-x)^{\pi}} \cos \left( \frac{2}{3}(-x)^{\frac{\pi}{2}} - \frac{3}{4}d_0^2 \ln(-x) + \phi_0 \right) + O(x^{-1}), \quad (1.11)
\]
as \( x \to -\infty \). Similarly as in the homogeneous case, the parameters \( d_0 \in \mathbb{C} \) and \( \phi_0 \in \mathbb{R} \) are dependent from \( k \) and \( \alpha \) by explicit connecting formulas. Unfortunately, the results from [12] do not allow us to obtain more accurate expansion of the error term \( O(x^{-1}) \) appearing in the formula (1.11) (see discussion in [12, Section 3.5]). Therefore the asymptotics (1.10) and (1.11) are not sufficient to confirm the convergence of the integral (1.6), which is the reason that the formula (1.9) has a weaker exponential form. However if \( u \) is a real AS solution, then we can take the both-side logarithm of (1.9) and obtain
\[
    \int_{-\infty}^{\infty} u(y; \alpha, k) \, dy = \frac{1}{2} \ln \left( \frac{\cos(\pi \alpha) + k}{(\cos^2(\pi \alpha) - k^2)^{1/2}} \right), \quad k \in (-\cos(\pi \alpha), \cos(\pi \alpha)).
\]
In the proof of Theorem 1.1, we follow [12, 17] and [24] to deform the contour of the RH problem associated with \( u(\cdot; \alpha, k) \), to contours consisting of appropriate steepest descent paths of the phase function \( \theta(\lambda, x) \). The main difficulty follows from the fact that, in the inhomogeneous case, an additional singularity appears at the origin of the complex plane, which requires us to construct a new parametrization around zero for the Riemann-Hilbert problem defined on the steepest descent graph.

The solutions of the PII equation that do not admit poles on the real line are important due to their applications in mathematics and physics. In the present paper we provide an application of Theorem 1.1 to study the vortex patch dynamics of the following incompressible 2D Euler equation
\[
    \begin{cases}
        u_t = u \cdot \nabla u + \nabla p = 0 \quad \text{on } \mathbb{R}^2, \\
        \text{div} \, u = 0 \quad \text{on } \mathbb{R}^2,
    \end{cases} \quad (1.12)
\]
where $u(x, t)$ is the velocity field and $p(x, t)$ is the pressure function. To explain the problem more precisely, let us consider the perpendicular gradient $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and define the 2D vorticity function

$$\omega(x, t) = \nabla^\perp \cdot u(x, t).$$

Then the vector equation (1.12) is equivalent to the scalar vorticity equation

$$\omega_t + u \cdot \nabla \omega = 0,$$

(1.13)

where the velocity field $u$ is given by the Biot-Savart integral

$$u(x, t) = \int_{\mathbb{R}^2} K(x - y)\omega(y) \, dy, \quad K(x) = \frac{1}{2\pi} \nabla^\perp \log |x|.$$  

(1.14)

The existence and uniqueness of global weak solutions of (1.13)–(1.14) for the initial vorticity $\omega_0$ from the class $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is a classical result of [29]. Hence we can deduce the existence of the unique global weak solution $\omega$ starting from the initial datum given by

$$\omega_0(x) = p_0 \chi_{\Omega_0}(x), \quad x \in \mathbb{R}^2,$$

where $p_0 \in \mathbb{R}$ and $\chi_{\Omega_0}$ is a characteristic function of a measurable set $\Omega_0$. In [25] Section 8.2.3 it was shown that there is a family of measurable sets $\Omega(t)$ such that

$$\omega(x, t) = p_0 \chi_{\Omega(t)}(x), \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}.$$  

(1.15)

The solution $\omega$ of the form (1.15) is called a vortex patch and is completely determined by the temporal evolution of the boundary $\partial \Omega(t)$, which we call the contour dynamics. If we additionally assume that $\Omega_0$ is a smooth bounded simply connected region, then the results of [7] say that the boundary $\partial \Omega(t)$ remains smooth for all time (see also [6] for the simplified proof of this fact). Furthermore, from [30] and [25] Section 8.3.1 we know that the dynamics is governed by the non-local differential equation of the form

$$X_s(s, t) = -\frac{p_0}{2\pi} \int_0^{2\pi} \ln ||X(s, t) - X(s', t)|| X_s(s', t) \, ds', \quad (1.16)$$

where $X(s, t)$ represents the parametrization of $\partial \Omega(t)$. In [18] the methods of [10] were used to handle with the non-local equation (1.16) by considering the Taylor expansion of the flow $X$ (see also discussion in [11]). As a result the non-local flow was formally expressed in the form of the infinite sum of local flows such that the first element of this sum is a translation and the second one satisfies the equation

$$z_t = -k_n u - \frac{1}{2} k^2 T, \quad t, s \in \mathbb{R}.$$  

(1.17)

In the above equation $z$ is the flow of regular curves living in the complex plane, $s$ is the arc-length parameter, $T$ is the tangent vector field, $n := iT$ is the oriented normal vector field and $k$ is the curvature defined by the equality $T_s = kn$. In this form, the equation (1.17) is known as the localized induction approximation (LIA) and formally approximates the contour dynamics of the 2D Euler equation. We are interested in the solutions of LIA, that develop finite time singularities of the form

$$z_0(s) = \begin{cases} 
\frac{s}{\sqrt{1 + \mu^2}} e^{i(\theta^+ - \mu \ln s)}, & s > 0, \\
\frac{s}{\sqrt{1 + \mu^2}} e^{i(\theta^- - \mu \ln |s|)}, & s < 0,
\end{cases}$$

where $\mu \in \mathbb{R}$ and $\theta^\pm \in [0, 2\pi)$ are such that $|\theta^+ - \theta^-| \neq \pi$. The function $z_0$ defines a logarithmic spiral, which is a structure appearing frequently in the motion of a turbulent flow. In [27], a perturbation argument and ODEs techniques were used...
to show that, if the expression $|\theta^+ - \theta^-| + |\mu|$ is sufficiently close to zero, then there
is a self-similar solution $z$ of the equation (1.17), such that

$$|z(t,s) - z_0(s)| < ct^\frac{1}{4}, \quad s \in \mathbb{R} \setminus \{0\}, \quad t > 0,$$

where $c > 0$ is a constant. Here the solution $z$ is said to be self-similar provided

$$z(t,s) = t^{\frac{1}{4}} e^{-i\frac{3}{2} \ln t} \left[ \int_{0}^{s/t^\frac{1}{4}} \exp \left( \frac{2}{3 \lambda^3} \int_{0}^{s'} u(s''/3^{1/3}) \, ds'' \right) \, ds' + \omega_0 \right], \quad t > 0,$$

where $u$ is a purely imaginary solution of the PII equation (1.1) with $\alpha = -\frac{i\mu}{2}$ and

$$\omega_0 := -\frac{2\sqrt{3}}{1 - i\mu} (u_x(0) - u^2(0)).$$

In this paper we use Theorem 1.1 to drop the assumption concerning the smallness of $|\theta^+ - \theta^-| + |\mu|$ and establish the following result, which says that any logarithmic spiral is a finite time singularity developed by a self-similar solution of (1.17).

**Theorem 1.2.** For any $\theta^+, \theta^- \in [0, 2\pi)$ and $\mu \in \mathbb{R}$ with $|\theta^+ - \theta^-| \neq \pi$, there is a purely imaginary solution $u$ of the PII equation such that the smooth self-similar solution $z$ of the equation (1.17), satisfies

$$|z(t,s) - z_0(s)| < ct^\frac{1}{4}, \quad s \in \mathbb{R} \setminus \{0\}, \quad t > 0,$$

where $c > 0$ is a constant.

The crucial points of the proof of Theorem 1.2 is to use Theorem 1.1 to show that, the profile $u$ of the self-similar solution $z$ that we are looking for, is a purely imaginary Ablowitz-Segur solution of the form $u(\cdot :-i\mu/2, k)$, where $k \in i\mathbb{R}$ is suitably chosen complex number.

**Outline.** In Section 2 we introduce the Riemann-Hilbert problem for the inhomogeneous PII equation and analyze the related Lax system to reduce the proof of the formula (1.3) to finding appropriate asymptotics of the solution $\Phi(\lambda, x)$. In Section 3 we use Bessel functions to define auxiliary RH problems that will be used in the construction of local parametrices around the origin. Sections 4 and 5 are devoted to steepest descent analysis of the RH problem associated with the inhomogeneous PII equation. We recall the deformations leading to the equivalent RH problems defined on the contours consisting of steepest descent paths and we use the results from Section 3 to represent their solutions in the terms of a local parametrix around the origin. Sections 6 is devoted for the proof of Theorem 1.1 whereas in Section 7 we discuss some properties of the profiles of self-similar solutions for the LIA. Finally in Section 8 we provide the proof of Theorem 1.2.

2. Solution $\Phi(\lambda, x)$ for the RH problem for Painlevé II equation

In this section we start with the Riemann-Hilbert problem related to the second Painlevé equation (1.1), where $\alpha \in \mathbb{C}$ is a constant such that $\text{Re} \alpha \in (-1/2, 1/2)$. We assume that $\Sigma := C \cup \rho_{+} \cup \rho_{-} \cup_{k=1}^{6} \gamma_k$ is the contour in the complex $\lambda$-plane, where $C := \{ \lambda \in \mathbb{C} \mid |\lambda| = r \}$ is a clockwise oriented circle of radius $r > 0$,

$$\rho_{\pm} := \{ \lambda \in \mathbb{C} \mid |\lambda| < r, \text{ arg } \lambda = \pm \pi/2 \},$$

are two radii oriented to the origin and

$$\gamma_k := \{ \lambda \in \mathbb{C} \mid |\lambda| > r, \text{ arg } \lambda = \pi/6 + (k - 1)\pi/3 \}, \quad k = 1, 2, \ldots, 6$$

are six rays oriented to the infinity. The contour $\Sigma$ divides the complex plane into regions $\Omega_{\epsilon}$, $\Omega_{t}$ and $\Omega_{k}$ for $k = 1, 2, \ldots, 6$, as it is depicted in Figure 1.
Let us observe that due to the orientation we can naturally distinguish the left (+) and right (−) side of the contour Σ. We consider the triangular matrices

\[ S_k := \begin{pmatrix} 1 & 0 \\ s_k & 1 \end{pmatrix}, \quad k = 1, 3, 5 \quad \text{and} \quad S_k := \begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix}, \quad k = 2, 4, 6, \]

where constants \( s_k \), for \( k = 1, 2, 3 \), satisfy the constrain condition

\[ s_{k+3} = -s_k, \quad s_1 - s_2 + s_3 + s_1s_2s_3 = -2\sin(\pi\alpha). \tag{2.1} \]

Moreover we assume that \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are the usual Pauli matrices

\[ \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

and the matrix \( E \) is such that the following equation takes place

\[ ES_1S_2S_3 = \sigma_2M^{-1}E\sigma_2, \quad \text{where} \quad M := -ie^{i\pi(\alpha - \frac{1}{2})i\sigma_1\sigma_2}. \tag{2.2} \]

The Riemann-Hilbert problem, that we denote by (RH1), is to find a \( 2 \times 2 \) matrix-valued function \( \Phi(\lambda) = \Phi(\lambda, x) \) such that the following conditions are satisfied.

(a) For any \( k = 1, 2, \ldots, 6 \), the restriction \( \Phi_k := \Phi|_{\Omega_k} \) is holomorphic on \( \Omega_k \) and continuous up to the boundaries of \( \Omega_k \).

(b) The restrictions \( \Phi_r := \Phi|_{\Omega_r} \) and \( \Phi_l := \Phi|_{\Omega_l} \) are holomorphic on \( \Omega_r \) and \( \Omega_l \), respectively, and, for any sufficiently small \( \varepsilon > 0 \),

\[ \Phi_r \in C(\Omega_r^\varepsilon), \quad \Phi_l \in C(\Omega_l^\varepsilon), \]

where \( \Omega_r^\varepsilon := \Omega_r \setminus \{ \lambda \in \mathbb{C} \mid |\lambda| < \varepsilon \} \) and \( \Omega_l^\varepsilon := \Omega_l \setminus \{ \lambda \in \mathbb{C} \mid |\lambda| < \varepsilon \} \).

(c) Given \( \lambda \in \Sigma \setminus \{0\} \), if we denote by \( \Phi^+(\lambda) \) and \( \Phi^-(\lambda) \) the limits of \( \Phi(\lambda') \) as \( \lambda' \to \lambda \) from the left and right side of the contour \( \Sigma \), respectively, then the following jump condition is satisfied

\[ \Phi^+(\lambda) = \Phi^-(\lambda)S(\lambda), \quad \lambda \in \Sigma \setminus \{0\}, \]

where the jump matrix \( S(\lambda) \) is constructed as follows. On the rays \( \gamma_k \) the matrix \( S(\lambda) \) is given by the equation

\[ S(\lambda) = S_k, \quad \lambda \in \gamma_k, \quad k = 1, 2, \ldots, 6, \]
while on the circle $C$ the matrix $S(\lambda)$ is obtained by the following jump relations

\begin{align*}
\Phi_+^1(\lambda) &= \Phi_-^1(\lambda)E, & \Phi_+^2(\lambda) &= \Phi_-^2(\lambda)E S_1, \\
\Phi_+^3(\lambda) &= \Phi_-^3(\lambda)\sigma_2 E \sigma_2 S_3^{-1}, & \Phi_+^4(\lambda) &= \Phi_-^4(\lambda)\sigma_2 E \sigma_2,
\end{align*}

Furthermore, on the radii $\rho_{\pm}$, the matrix $S(\lambda)$ is determined by the equations

\begin{align*}
\Phi_-^1(\lambda) &= \Phi_+^1(\lambda)M, & \lambda &\in \rho_- \setminus \{0\}, \\
\Phi_-^2(\lambda) &= \Phi_+^2(\lambda)\sigma_2 M \sigma_2, & \lambda &\in \rho_+ \setminus \{0\}.
\end{align*}

(d) The function $\Phi(\lambda)\lambda^{-\alpha\sigma_3}$ is bounded for $\lambda$ sufficiently close to zero, where the branch $\lambda^{-\alpha}$ is chosen arbitrarily.

(e) As $\lambda \to \infty$, the function $\Phi$ has the following asymptotic behavior

\[ \Phi(\lambda) = (I + O(\lambda^{-1}))e^{-\theta(\lambda)\sigma_3}, \quad \text{where} \quad \theta(\lambda) := \frac{4}{3}\lambda^3 + x\lambda. \]

Following [17] and [21], the RH problem (a)–(e) is uniquely meromorphically (with respect to $x$) solvable for any choice of the Stokes multipliers $(s_1, s_2, s_3)$, that is, there is a countable set of poles $X = \{x_k\}_{k \geq 1}$ such that the function $\Phi$ is holomorphic on $(\mathbb{C} \setminus X) \times (\mathbb{C} \setminus X)$ and meromorphic along $\mathbb{C} \times X$. It is known that $\Phi$ satisfies the following complex equations called the Lax pair

\[ \begin{cases} \\
\partial_\lambda \Phi(\lambda; x) = A(\lambda; x)\Phi(\lambda; x), \\
\partial_x \Phi(\lambda; x) = U(\lambda; x)\Phi(\lambda; x). \end{cases} \quad (2.3) \]

In the above system $A$ and $U$ are $2 \times 2$ matrix functions given by

\[ A(\lambda; x) := -i(4\lambda^2 + x + 2u^2(x))\sigma_3 - (4\lambda u(x) + 3\lambda^{-1})\sigma_2 - 2u_x(x)\sigma_1, \]
\[ U(\lambda; x) := -i\lambda \sigma_3 - u(x)\sigma_2. \]

Furthermore the function $u$, given by the limit

\[ u(x) := \lim_{\lambda \to \infty} (2\lambda \Phi(\lambda)e^{\theta(\lambda)\sigma_3})_{12}, \quad x \in \mathbb{C} \setminus X \]

defines the solution for the PII equation and the map

\[ \{(s_1, s_2, s_3) \in \mathbb{C}^3 \mid \text{the constraints (2.1) holds}\} \to \{\text{solutions of the PII}\} \]

is a bijection between the set of Stokes multipliers and the set of solutions of the Painlevé equation (1.1). In [19] and [20] it was rigorously proved that any solution of the PII equation is meromorphic function and every pole of $u$ is simple with residue $\pm 1$. We say that $u$ is a purely imaginary solution of the PII equation if

\[ \text{Re} \ u(x) = 0, \quad x \in \mathbb{R} \]

and furthermore the solution $u$ is called purely real provided

\[ \text{Im} \ u(x) = 0, \quad x \in \mathbb{R}. \]

From [17], Chapter 11 we know that if $\alpha \in \mathbb{R}$ and the Stokes multipliers $(s_1, s_2, s_3)$ are such that $s_3 = \overline{s_1}$ and $s_2 = \overline{s_2}$, then the corresponding solution of the PII equation is purely real. Similarly, if $\alpha \in i\mathbb{R}$ and the monodromy data $(s_1, s_2, s_3)$ are such that $s_3 = -\overline{s_1}$ and $s_2 = -\overline{s_2}$, then related solution of the PII equation is purely imaginary. From [17], Chapter 11 we also know that there is a function $Z$, holomorphic for $\lambda \in B(0, r)$ such that

\[ \Phi(\lambda) = Z(\lambda)e^{-\theta(\lambda)\sigma_3}\lambda^{\alpha\sigma_3}, \quad \lambda \in \Omega_r. \quad (2.4) \]
Using the second equation of the Lax pair \((2.3)\), we infer that
\[
\frac{\partial Z(\lambda)}{\partial x} = \frac{\partial \Phi(\lambda)}{\partial x} e^{\theta(\lambda)\sigma_3} x^{-\alpha \sigma_3} + i \lambda \Phi(\lambda) \sigma_3 e^{\theta(\lambda)\sigma_3} x^{-\alpha \sigma_3} = (-i \lambda \sigma_3 - u(x) \sigma_2) \Phi(\lambda) e^{\theta(\lambda)\sigma_3} x^{-\alpha \sigma_3} + i \lambda \Phi(\lambda) \sigma_3 e^{\theta(\lambda)\sigma_3} x^{-\alpha \sigma_3} = (-i \lambda \sigma_3 - u(x) \sigma_2) Z(\lambda) + i \lambda Z(\lambda) \sigma_3 = -i \lambda [\sigma_3, Z(\lambda)] - u(x) \sigma_2 Z(\lambda).
\]

Therefore, if we define \(P(x) := Z(0, x)\), then, passing to the limit with \(\lambda \to 0\), gives the following linear equation
\[
\frac{\partial P(x)}{\partial x} = -u(x) \sigma_2 P(x), \quad x \in \mathbb{C} \setminus X. \tag{2.5}
\]

As it was proved in \([12]\) Section 2.2] by a vanishing lemma, if \(u\) is either a real or purely imaginary AS solution, then the set of poles \(X\) does not contain any real number and the function \(\Phi(\lambda, x)\) is defined for all \(x \in \mathbb{R}\). Therefore, if we denote \(v(x_1, x_2) := \int_{x_1}^{x_2} u(y) \, dy\), \(x_1 < x_2\), then the solution of the equation \((2.5)\) is expressed by the following formula
\[
P(x) = \left(\begin{array}{cc}
\frac{i}{2}(e^{v(0, x)} + e^{-v(0, x)}) & \frac{i}{2}(e^{v(0, x)} - e^{-v(0, x)}) \\
-\frac{i}{2}(e^{v(0, x)} - e^{-v(0, x)}) & \frac{i}{2}(e^{v(0, x)} + e^{-v(0, x)})
\end{array}\right) P(0)
= \left(\begin{array}{cc}
cosh(v(0, x)) & i \sinh(v(0, x)) \\
-i \sinh(v(0, x)) & \cosh(v(0, x))
\end{array}\right) P(0),
\]
and furthermore, for any \(x > 0\), we have
\[
P(x)P(-x)^{-1} = \left(\begin{array}{cc}
\frac{i}{2}(e^{v(-x, x)} + e^{-v(-x, x)}) & \frac{i}{2}(e^{v(-x, x)} - e^{-v(-x, x)}) \\
-\frac{i}{2}(e^{v(-x, x)} - e^{-v(-x, x)}) & \frac{i}{2}(e^{v(-x, x)} + e^{-v(-x, x)})
\end{array}\right). \tag{2.6}
\]

Therefore the proof of the integral formula \((1.9)\) reduces to studying asymptotic behavior of the function \(P(x)\) as \(x \to \pm \infty\).

3. Bessel functions and auxiliary RH problems

Let us introduce the auxiliary function \(\hat{\Psi}^0(z)\), given by the formula
\[
\hat{\Psi}^0(z) = B(z) \left(\begin{array}{cc}
v_1(z) & v_2(z) \\
v_1'(z) & v_2'(z)
\end{array}\right), \quad \text{where } B(z) := \frac{1}{2} e^{-\frac{i}{2} \pi \sigma_3} \left(\begin{array}{cc}1 & 1 \\1 & -1/2
\end{array}\right). \tag{3.1}
\]
and the functions \(v_1, v_2\) are defined by
\[
v_1(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + \frac{1}{2}) z^{\alpha + 2k}}{4^k k! \Gamma(\alpha + \frac{1}{2} + k)} = 2^{\alpha - \frac{1}{2}} \Gamma(\alpha + \frac{1}{2}) e^{i \frac{\pi}{2} (\alpha - \frac{1}{2})} z^{\frac{\alpha}{2}} J_{\alpha - \frac{1}{2}}(e^{-\frac{i}{2} \pi / 2} z) \tag{3.2}
\]
and
\[
v_2(z) := \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{2} - \alpha) z^{1 - \alpha + 2k}}{4^k k! \Gamma(\frac{3}{2} - \alpha + k)} = 2^{\frac{1}{2} - \alpha} \Gamma(\frac{3}{2} - \alpha) e^{i \frac{\pi}{2} (\frac{3}{2} - \alpha)} z^{\frac{1}{2}} J_{\frac{1}{2} - \alpha}(e^{-\frac{i}{2} \pi} z), \tag{3.3}
\]
where \(J_{\nu}(z)\) is the classical Bessel function defined on the universal covering of the punctured complex plane \(\mathbb{C} \setminus \{0\}\). It is known (see e.g. \([5], [17]\)) that \(\hat{\Psi}^0\) satisfies the following differential equation
\[
\frac{\partial}{\partial z} \hat{\Psi}^0(z) = (\sigma_3 - \frac{\alpha}{z} \sigma_2) \hat{\Psi}^0(z)
\]
Lemma 3.1. The following convergence holds and the function \( \hat{\Psi}(z)z^{-\alpha\sigma_3} \) is holomorphic on the complex plane. Let us introduce the following matrices

\[
\hat{E} = \frac{\sqrt{\pi}}{2 \cos \pi \alpha} \begin{pmatrix} \Gamma(\frac{1}{2} + \alpha) & 0 \\ \Gamma(\frac{1}{2} - \alpha) & 0 \end{pmatrix} e^{i\pi \sigma_3} \begin{pmatrix} e^{-i\pi \alpha} & i \\ i e^{i\pi \alpha} & 1 \end{pmatrix},
\]

\[
\hat{S}_1 = \begin{pmatrix} 1 & 2 \sin(\pi \alpha) \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_2 = \begin{pmatrix} 1 & 0 \\ -2 \sin(\pi \alpha) & 1 \end{pmatrix}.
\]

If we define the functions

\[
\hat{\Psi}_1(z) := \hat{\Psi}(z)\hat{E}, \quad \hat{\Psi}_2(z) := \hat{\Psi}_1(z)\hat{S}_1, \quad \hat{\Psi}_3(z) := \hat{\Psi}_2(z)\hat{S}_2
\]

then, for any \( 1 \leq k \leq 3 \), we have the asymptotic behavior

\[
\hat{\Psi}_k(z) = (I - i \alpha \frac{1}{2z} \sigma_1 + O(\frac{1}{z^2}))e^{z\sigma_3}, \quad z \to \infty,
\]

with \( \arg z \in (\pi(k - \frac{1}{2}), \pi(k + \frac{1}{2})) \) and the following equality holds

\[
\sigma_2 \hat{\Psi}_k e^{i\pi z} \sigma_2 = \hat{\Psi}_k(z)
\]

for \( k = 1, 2 \) (see e.g. [5], [17]). It is also not difficult to check the useful equality

\[
\hat{E}\hat{S}_1 = DE,
\]

where the matrix \( E \) is given by the formula (4.12) and

\[
D := \frac{\sqrt{\pi} e^{i\pi \sigma_3}}{2 \cos \pi \alpha} \begin{pmatrix} \frac{2}{\Gamma(1/2 + \alpha)} e^{-i\pi \alpha} & 0 \\ 0 & \frac{2}{\Gamma(3/2 - \alpha)} e^{i\pi \alpha} \end{pmatrix}.
\]

Lemma 3.1. The following convergence holds

\[
\lim_{z \to 0} \hat{\Psi}_0(z)z^{-\alpha\sigma_3} = \frac{1}{2} e^{-i\pi \sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - 2\alpha \end{pmatrix}.
\]

Proof. From the definition of \( \hat{\Psi}_0(z) \) it follows that

\[
\hat{\Psi}_0(z)z^{-\alpha\sigma_3} = \frac{1}{2} e^{-i\pi \sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha/z & 1 \end{pmatrix} \begin{pmatrix} v_1(z) & v_2(z) \\ v_1'(z) & v_2'(z) \end{pmatrix} \begin{pmatrix} z^{-\alpha} & 0 \\ 0 & z^\alpha \end{pmatrix}.
\]

On the other hand, we have

\[
C(z) := \begin{pmatrix} 1 & 0 \\ -\alpha/z & 1 \end{pmatrix} \begin{pmatrix} v_1(z) & v_2(z) \\ v_1'(z) & v_2'(z) \end{pmatrix} \begin{pmatrix} z^{-\alpha} & 0 \\ 0 & z^\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\alpha/z & 1 \end{pmatrix} \begin{pmatrix} v_1(z)z^{-\alpha} & v_2(z)z^\alpha \\ v_1'(z)z^{-\alpha} & v_2'(z)z^\alpha \end{pmatrix} = \begin{pmatrix} v_1(z)z^{-\alpha} & v_2(z)z^\alpha \\ -\alpha v_1(z)z^{-\alpha - 1} + v_1'(z)z^{-\alpha} - \alpha v_2(z)z^{-\alpha - 1} + v_2'(z)z^\alpha \end{pmatrix}.
\]

Using the formula (3.2), we have

\[
v_1'(z)z^{-\alpha} = az^{-1} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + \frac{1}{2})z^{2k}}{4^k k! \Gamma(\alpha + \frac{1}{2} + k)} + \sum_{k=1}^{\infty} 2k \Gamma(\alpha + \frac{1}{2})z^{2k-1}
\]

and consequently

\[
-\alpha v_1(z)z^{-\alpha - 1} + v_1'(z)z^{-\alpha} = \sum_{k=1}^{\infty} 2k \Gamma(\alpha + \frac{1}{2})z^{2k-1} \frac{\Gamma(\alpha + \frac{1}{2})z^{-2k}}{4^k k! \Gamma(\alpha + \frac{1}{2} + k)}.
\]

(3.8)
On the other hand, from the formula (3.2), it follows that
\[ v'(z)z^\alpha = \sum_{k=0}^{\infty} \frac{(2k+1-\alpha)\Gamma(\frac{3}{2}-\alpha)z^{2k}}{4^k k! \Gamma(\frac{3}{2}-\alpha+k)}, \]
which gives
\[ -\alpha v(z)z^{\alpha-1} + v'(z)z^\alpha = \sum_{k=0}^{\infty} \frac{(2k+1-2\alpha)\Gamma(\frac{3}{2}-\alpha)z^{2k}}{4^k k! \Gamma(\frac{3}{2}-\alpha+k)}. \] (3.9)
Combining (3.2), (3.3), (3.8) and (3.9), we infer that
\[ \lim_{z \to 0} C(z) = C(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1-2\alpha \end{pmatrix}, \]
which finally gives
\[ \lim_{z \to 0} \hat{\Psi}(z) = \frac{1}{2} e^{-i\frac{\pi}{4}\sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-2\alpha \end{pmatrix}, \]
and the proof of lemma is completed. □

3.1. **Auxiliary RH problem for** \( x > 0 \). Given \( x > 0 \), let us assume that \( z : \mathbb{C} \to \mathbb{C} \) is a function given by
\[ z(\lambda) := -\theta(\lambda) = -i(4\lambda^3/3 + x\lambda), \quad \lambda \in \mathbb{C}. \] (3.10)
If we define \( R := \frac{1}{4}|x|^{1/2} \), then it is not difficult to check that \( z \) is an injective map on \( B(0, R) \). Then the open mapping theorem for holomorphic functions implies that the set \( V := z(B(0, R)) \) is open and the inverse mapping \( z^{-1} : V \to B(0, R) \) is also a holomorphic function. Let us assume that \( 0 < r < \frac{1}{4}x^{1/2} \). We consider, in the complex \( z \)-plane, the contour \( \hat{\Sigma} := \mathbb{R} \cup \hat{C}_+ \cup \hat{C}_- \), where \( \hat{C}_\pm \) is the image of the set \( \{ \lambda \in \mathbb{C} \mid |\lambda| = r, \pm \text{Re} \lambda \geq 0 \} \) under the map \( z \) (see Figure 2). We define \( \hat{\tau} := z(\pm ir) \) to be the intersection points of \( \hat{C} := \hat{C}_+ \cup \hat{C}_- \) with the real axis.

![Figure 2. The contour \( \hat{\Sigma} \) for the auxiliary RH problem.](image)

The contour \( \hat{\Sigma} \) divides the complex \( z \)-plane into four sets \( \hat{\Omega}_d, \hat{\Omega}_u, \hat{\Omega}_2, \hat{\Omega}_3 \) such that the sets \( \hat{\Omega}_d, \hat{\Omega}_u \) lie in the interior of the circle \( \hat{C} \) and the regions \( \hat{\Omega}_2, \hat{\Omega}_3 \) are located outside \( \hat{C} \). We define the function \( \hat{\Psi}(z) \) as follows
\[ \hat{\Psi}(z) = \hat{\Psi}^2(e^{2\pi i}z), \quad z \in \hat{\Omega}_2, \quad \hat{\Psi}(z) = \hat{\Psi}^3(e^{2\pi i}z), \quad z \in \hat{\Omega}_3, \]
\[ \hat{\Psi}(z) = \hat{\Psi}^0(e^{2\pi i}z)D, \quad z \in \hat{\Omega}_d, \quad \hat{\Psi}(z) = \sigma_2 \hat{\Psi}^0(e^{2\pi i}z)D\sigma_2, \quad z \in \hat{\Omega}_u, \]
where we recall that $\hat{\Psi}^k(z)$, for $0 \leq k \leq 3$, are defined on the universal covering of the punctured complex plane $\mathbb{C} \setminus \{0\}$ and the branch cut is chosen such that $\arg z \in (-\pi/2, \pi/2)$ (see (3.1) and (3.5)).

**Lemma 3.2.** For any $z \in \hat{\Omega}_u$ then following equality holds

$$\hat{\Psi}(z)\sigma_2 E\sigma_2 = \hat{\Psi}^0(e^{2\pi i}z)D\hat{S}_2.$$  

**Proof.** Combining the equality (3.7) with the relation (3.6), for any $z \in \hat{\Omega}_u$, gives

$$\hat{\Psi}(z)\sigma_2 E\sigma_2 = \sigma_2 \hat{\Psi}^0(e^{2\pi i}z)D\sigma_2 = \sigma_2 \hat{\Psi}^0(e^{2\pi i}z)\hat{S}_1\sigma_2$$

$$= \sigma_2 \hat{\Psi}^0(e^{2\pi i}z)\hat{S}_1 = \sigma_2 \hat{\Psi}^2(e^{2\pi i}z)\sigma_2 = \hat{\Psi}^3(e^{2\pi i}z)$$

$$= \hat{\Psi}^0(e^{2\pi i}z)\hat{S}_2 = \hat{\Psi}^0(e^{2\pi i}z)D\hat{S}_2$$

and the proof of lemma is completed. □

In the following proposition we formulate an auxiliary Riemann-Hilbert problem that will be used in the construction of a local parametrix for the steepest descent contour around the origin. For the proof we refer the reader to [17, Section 11.6].

**Proposition 3.3.** The function $\hat{\Psi}(z)$ solves the following auxiliary RH problem.

(a) The function $\hat{\Psi}_{\hat{\Omega}_u}(z)z^{-\alpha_3}$ is analytic on the open set confined by the curve $\hat{C}$.  

(b) We have the jump relation $\hat{\Psi}_+(z) = \hat{\Psi}_-(z)\hat{S}(z)$ for $z \in \Sigma$, where

$$\hat{S}(z) := \begin{cases} S_+ := \hat{S}_2, & \text{for } z \in \mathbb{R}, \ z > \hat{r}, \\ S_- := \hat{S}_1^{-1}, & \text{for } z \in \mathbb{R}, \ z < -\hat{r} \end{cases}$$

and furthermore

$$\hat{S}(z) := M, \ z \in \mathbb{R}, \ -\hat{r} < z < 0, \ \hat{S}(z) := \sigma_2 M^{-1}\sigma_2, \ z \in \mathbb{R}, \ 0 < z < \hat{r},$$

$$\hat{S}(z) := E, \ z \in \hat{C}_-, \ \hat{S}(z) := \sigma_2 E\sigma_2, \ z \in \hat{C}_+.$$  

(c) The function $\hat{\Psi}(z)$ has the asymptotic behavior

$$\hat{\Psi}(z) = (1 + O(z^{-1}))e^{\pi i z^3}, \ z \to \infty.$$  

**3.2. Auxiliary RH problem for** $x < 0$. Let us consider the function $\hat{L}(z)$, given by the formulas

$$\hat{L}(z) = \hat{\Psi}(z), \ z \in \hat{\Omega}_2 \cup \hat{\Omega}_3, \ \hat{L}(z) = \hat{\Psi}(z)\sigma_2 E\sigma_2, \ z \in \hat{\Omega}_u,$$

$$\hat{L}(z) = \hat{\Psi}(z)E, \ z \in \hat{\Omega}_d.$$  

We start with the following proposition.

**Proposition 3.4.** The function $\hat{L}(z)$ is a solution of the following RH problem.

(a) The function $\hat{L}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$.  

(b) On the contour $\Sigma_{\hat{L}}$, the function $\hat{L}(z)$ satisfies the following jump relations

$$\hat{L}_+(z) = \hat{L}_-(z)\hat{S}_L(z), \ z \in \mathbb{R},$$

where the jump matrix is given by

$$\hat{S}(z) := S_-, \ z < 0, \ \hat{S}(z) := S_+, \ 0 < z.$$  

(c) The function $\hat{L}(z)$ has the following behaviors

$$\hat{L}(z) = O\left(\begin{array}{cc} |z|^{-\alpha} & -|z|^{-\alpha} \\ |z|^{-\alpha} & |z|^{-\alpha} \end{array}\right), \ z \to 0, \ \text{if } 0 < \text{Re} \alpha < \frac{1}{2}.$$
and furthermore
\[ \hat{L}(z) = O \left( \frac{|z|^\alpha}{|z|^\beta} \right), \quad z \to 0, \quad \text{if} \quad -\frac{1}{2} < \Re \alpha \leq 0. \]

(d) We have the following asymptotic behavior
\[ \hat{L}(z) = (I + O(z^{-1})) e^{z\sigma}, \quad z \to \infty. \]

![Figure 3](image)

**Figure 3.** Left: the contour deformation between \( \hat{\Sigma} \) and \( \Sigma_L \).

Right: the graph \( \Sigma_L \) together with jump matrix \( S_L \).

In the proof of Proposition 3.4, we will use the following lemma.

**Lemma 3.5.** Given \( r > 0 \) and \( \varphi_1 < \varphi_2 \) let us consider the cone \( S_{\varphi_1, \varphi_2}(r) \) consisting of \( z \in \mathbb{C} \) such that \( 0 < |z| < r \) and \( \varphi_1 < \arg z < \varphi_2 \). Assume that \( A(z) \) and \( B(z) \) are functions defined on \( S_{\varphi_1, \varphi_2}(r) \) with values in \( 2 \times 2 \) complex matrices such that
\[ A(z) = O \left( \frac{|z|^\beta}{|z|^\delta} \right) \quad \text{and} \quad B(z) = O \left( \frac{|z|^\delta}{|z|^\beta} \right), \quad z \to 0, \quad z \in S_{\varphi_1, \varphi_2}(r). \]

If \( 0 \leq \Re \gamma < 1/2 \), then we have
\[ A(z) \gamma^{\sigma_3} B(z) = O \left( \frac{|z|^{\beta + \delta - \gamma}}{|z|^{\beta + \delta - \gamma}} \right), \quad z \to 0, \quad z \in S_{\varphi_1, \varphi_2}(r) \]

and furthermore, if \( 1/2 < \Re \gamma < 0 \), then
\[ A(z) \gamma^{\sigma_3} B(z) = O \left( \frac{|z|^{\beta + \delta + \gamma}}{|z|^{\beta + \delta + \gamma}} \right), \quad z \to 0, \quad z \in S_{\varphi_1, \varphi_2}(r), \]

where the branch of the logarithm is taken such that \(-\pi < \arg z < \pi\).

**Proof.** Assume that the functions \( A(z) \) and \( B(z) \) have the following form
\[ A(z) := \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}, \quad B(z) := \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix}, \quad z \in S_{\varphi_1, \varphi_2}(r), \]

where \( a_{kl}(z) = O(|z|^\beta) \) and \( b_{kl}(z) = O(|z|^\beta) \) for \( 1 \leq k, l \leq 2 \). Suppressing the notation \( z \) for brevity, we have
\[ A \gamma^{\sigma_3} B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} z^\gamma & 0 \\ 0 & z^{-\gamma} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}z^\gamma & a_{12}z^{-\gamma} \\ a_{21}z^\gamma & a_{22}z^{-\gamma} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}z^\gamma + a_{12}b_{21}z^{-\gamma} & a_{11}b_{12}z^\gamma + a_{12}b_{22}z^{-\gamma} \\ a_{21}b_{11}z^\gamma + a_{22}b_{21}z^{-\gamma} & a_{21}b_{12}z^\gamma + a_{22}b_{22}z^{-\gamma} \end{pmatrix}, \quad z \in S_{\varphi_1, \varphi_2}(r) \]
Proof of Proposition 3.4. Checking that the function \( \hat{L} \) satisfies the conditions (a), (b) and (d) is a straightforward consequence of Proposition 3.3. To show that \( \hat{L} \) satisfies also the point (c), let us define the following functions

\[
A_1(z) := \hat{\Psi}(z)\sigma_2 M \sigma_2 z^{-\alpha \sigma_3}, \quad B_1(z) := \sigma_2 M^{-1} E \sigma_2, \quad z \in \hat{\Omega}_u,
\]

\[
A_2(z) := \hat{\Psi}(z) z^{-\alpha \sigma_3}, \quad B_2(z) := E, \quad z \in \hat{\Omega}_d.
\]

Then we have the following representations

\[
\hat{L}(z) = A_1(z) z^{\alpha \sigma_3} B_1(z), \quad z \in \hat{\Omega}_u,
\]

\[
\tilde{L}(z) = A_2(z) z^{\alpha \sigma_3} B_2(z), \quad z \in \hat{\Omega}_d.
\]

By the point (a) of Proposition 3.3, the function \( A(z) \) given by the formula

\[
A(z) := A_1(z), \quad z \in \hat{\Omega}_u, \quad A(z) := A_2(z), \quad z \in \hat{\Omega}_d
\]

is holomorphic in a neighborhood of the origin and hence

\[
A_1(z) = O(1), \quad z \to 0, \quad z \in \hat{\Omega}_u,
\]

\[
A_2(z) = O(1), \quad z \to 0, \quad z \in \hat{\Omega}_d.
\]

Therefore, Lemma 3.5 implies

\[
\hat{L}(z) = O \left( \frac{|z|^{-\alpha}}{|z|^{-\alpha}} \right), \quad z \to 0, \quad \text{if} \quad 0 \leq \text{Re} \alpha < 1/2
\]

and furthermore

\[
\tilde{L}(z) = O \left( \frac{|z|^\alpha}{|z|^\alpha} \right), \quad z \to 0, \quad \text{if} \quad -1/2 < \text{Re} \alpha < 0.
\]

Thus the proof of proposition is completed. \( \square \)

Let us consider the function \( \hat{L}(z) \) given by the formula

\[
\hat{L}(z) := \hat{L}(iz), \quad z \in \hat{\Omega}_u \cup \hat{\Omega}_l,
\]

where \( \hat{\Omega}_r := \{ \text{Re} z > 0 \} \) and \( \hat{\Omega}_l := \{ \text{Re} z < 0 \} \). Using Proposition 3.4, we can easily see that \( \hat{L}(z) \) is a solution of the following RH problem on the contour \( \Sigma_{\hat{L}} := i\mathbb{R} \), depicted on the right diagram of Figure 8.

(a) The function \( \hat{L}(z) \) is an analytic function on \( \mathbb{C} \setminus \Sigma_{\hat{L}} \).

(b) We have the jump relation \( \hat{L}_+(z) = \hat{L}_-(z) S_{\hat{L}}(z) \) for \( z \in \Sigma_{\hat{L}} \), where

\[
S_{\hat{L}}(z) := S_{\hat{L}}^{-1} = \hat{S}_1, \quad \text{Im} z > 0, \quad S_{\hat{L}}(z) := S_{\hat{L}} = \hat{S}_2, \quad \text{Im} z < 0
\]

(c) At the point \( z = 0 \) the function \( \tilde{L}(z) \) has the following behaviors

\[
\tilde{L}(z) = O \left( \frac{|z|^{-\alpha}}{|z|^{-\alpha}} \right), \quad \text{if} \quad 0 \leq \text{Re} \alpha < \frac{1}{2}
\]
and furthermore
\[ \tilde{L}(z) = O \begin{pmatrix} |z|^\alpha & |z|^\alpha \\ |z|^\alpha & |z|^\alpha \end{pmatrix}, \] if \( -\frac{1}{2} < \text{Re} \alpha \leq 0. \)

(d) We have the following asymptotic behavior at infinity
\[ \tilde{L}(z) = (I + O(z^{-1}))e^{iz\sigma_3}, \quad z \to \infty. \] (3.11)

In view of (1.2) and (1.4), we have
\[ s_1 + s_3 = -2 \sin(\pi \alpha), \] which implies that
\[ S = \begin{pmatrix} 1 & 0 \\ 0 & 1 - s_1 \end{pmatrix}. \]

The contour \( \Sigma_{\tilde{L}} \) together with the four rays \( \arg z = \pm \frac{\pi}{4} \) and \( \arg z = \pm \frac{3\pi}{4} \) divide the complex plane on six regions as it is shown on the left diagram of Figure 4.

Then we can represent the sets \( \tilde{\Omega}_l \) and \( \tilde{\Omega}_r \) in the form of the following sums
\[ \tilde{\Omega}_l = \tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup \tilde{\Omega}_3, \quad \tilde{\Omega}_r = \tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup \tilde{\Omega}_3. \]

Assume that \( \Sigma_L \) is the contour consisting of four rays \( \arg z = \pm \frac{\pi}{4} \) and \( \arg z = \pm \frac{3\pi}{4} \)

as it is shown on the right diagram of Figure 4. We define the function \( \hat{L}(z) \) by
\[ \hat{L}(z) := \tilde{L}(z) \begin{pmatrix} 1 & -s_3 \\ 0 & 1 \end{pmatrix}, \quad z \in \tilde{\Omega}_1^3, \quad \hat{L}(z) := \tilde{L}(z) \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}^{-1}, \quad z \in \tilde{\Omega}_1^4 \]
\[ \hat{L}(z) := \tilde{L}(z) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}, \quad z \in \tilde{\Omega}_2^3, \quad \hat{L}(z) := \tilde{L}(z) \begin{pmatrix} 1 & s_3 \\ 0 & 1 \end{pmatrix}, \quad z \in \tilde{\Omega}_2^4 \]
\[ \hat{L}(z) = \tilde{L}(z), \quad z \in \tilde{\Omega}_3^2 \cup \tilde{\Omega}_3^4. \]

We proceed to the second auxiliary RH problem that will be applied in the construction of a local parametrix for the steepest descent contour around the origin in the case of \( x < 0. \) The parametrix will allow us to establish asymptotic behavior of the function \( P(x) \) as \( x \to -\infty. \)

**Theorem 3.6.** The function \( \hat{L}(z) \) satisfies the following RH problem.

(a) The function \( \hat{L}(z) \) is an analytic function on \( \mathbb{C} \setminus \Sigma_L; \)
(b) On the contour $\Sigma_L$, the following jump relation is satisfied
\[ \tilde{L}_+(z) = \tilde{L}_-(z)S_L(z), \quad z \in \Sigma_L, \]
where the jump matrix $S_L$ are given on the Figure 4.

(c) If $0 < \Re \alpha < 1/2$, then the function $\tilde{L}(z)$ has the following asymptotic behavior
\[ \tilde{L}(z) = O\left(\frac{1}{|z|^{\alpha}}\right), \quad z \to 0, \]
and furthermore, for $-1/2 < \Re \alpha \leq 0$, we have
\[ \tilde{L}(z) = O\left(\frac{1}{|z|^\alpha}\right), \quad z \to 0. \]

(d) The function $\tilde{L}(z)$ has the following behavior at infinity
\[ \tilde{L}(z) = (I + O\left(z^{-1}\right)) e^{iz\sigma_3}, \quad z \to \infty. \]

Proof. It is not difficult to check that the function $\tilde{L}(z)$ satisfies conditions (a) and (b). On the other hand, applying Lemma 3.5 with $\gamma = \alpha$ and $\beta = \delta = 0$, we infer that the point (c) holds true. To check that (d) is valid, let us observe that the asymptotic condition (3.11), implies that
\[ \tilde{L}(z) = (I + O\left(z^{-1}\right)) e^{iz\sigma_3}, \quad z \to \infty, \quad z \in \Omega_L^1 \cup \Omega_L^2. \]
Furthermore, for $z \in \Omega_L^1 \cup \Omega_L^1$, we have
\[ \tilde{L}(z) = \tilde{L}(z) e^{-iz\sigma_3} = \tilde{L}(z) e^{-iz\sigma_3} \begin{pmatrix} 1 & ce^{2iz} \\ 0 & 1 \end{pmatrix}, \]
where the parameter $c$ is either $s_1$ or $-s_3$. Consequently we can write
\[ \tilde{L}(z) e^{-iz\sigma_3} - I = \left(\tilde{L}(z) e^{-iz\sigma_3} - I\right) \begin{pmatrix} 1 & ce^{2iz} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & ce^{2iz} \\ 0 & 1 \end{pmatrix} - I. \quad (3.12) \]

Considering the polar coordinates $z = |z|e^{i\varphi}$, we infer that the argument $\varphi$ is an element of the interval $(\frac{\pi}{4}, \frac{3\pi}{4})$, whenever $z \in \Omega_L^1 \cup \Omega_L^1$, which implies that
\[ |e^{2iz}| = |e^{2iz}| \leq e^{-2|z|} \leq e^{-\sqrt{2}|z|}, \quad z \in \Omega_L^1 \cup \Omega_L^1. \]

Combining this with (3.12), we deduce that
\[ \tilde{L}(z) e^{-iz\sigma_3} - I = O(z^{-1}), \quad z \to \infty, \quad z \in \Omega_L^1 \cup \Omega_L^1. \]

Arguing in the similar way we can write
\[ \tilde{L}(z) e^{-iz\sigma_3} = \tilde{L}(z) e^{-iz\sigma_3} \begin{pmatrix} 1 & 0 \\ de^{-2iz} & 1 \end{pmatrix}, \quad \lambda \in \Omega_L^1 \cup \Omega_L^1, \]
where the parameter $d$ is equal to either $-s_1$ or $s_3$. Then we can write
\[ \tilde{L}(z) e^{-iz\sigma_3} - I = \left(\tilde{L}(z) e^{-iz\sigma_3} - I\right) \begin{pmatrix} 1 & 0 \\ s_1 e^{-2iz} & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ s_1 e^{-2iz} & 1 \end{pmatrix} - I. \quad (3.13) \]

Considering the parameter $z$ in polar coordinates once again, we have $\varphi \in (-\frac{2\pi}{3}, -\frac{\pi}{3})$ for $z \in \Omega_L^2 \cup \Omega_L^2$, and hence
\[ |e^{-2iz}| = |e^{-2iz}| \leq e^{-2iz} \leq e^{-\sqrt{2}|z|}, \quad z \in \Omega_L^2 \cup \Omega_L^2 \quad (3.14) \]

By (3.14) and (3.13), we obtain
\[ \tilde{L}(z) e^{-iz\sigma_3} - I = O(z^{-1}), \quad z \to \infty, \quad z \in \Omega_L^2 \cup \Omega_L^2. \]
4. Steepest descent analysis of $\Phi(\lambda, x)$ for $x > 0$

4.1. Contour deformation. In this subsection we recall in detail the deformation of the RH problem graph $\Sigma$ for the function $\Phi(\lambda, x)$ to the steepest descent contour, in the case of $x > 0$. The deformation can be found in [17], [21] and is the first step in determining the asymptotic behavior of $P(x)$ as $x \to +\infty$. Without loss of generality we can assume that the radius $r > 0$ of the circle $C$ satisfies the inequality $\frac{1}{4} x^{1/2} > r$. Furthermore, in view of the choice of the monodromy data (1.4), we see that $S_2 = S_5 = I$ and therefore the diagram $\Sigma$ on Figure 1 takes the form depicted on Figure 5, where we denote $\Omega_u := \Omega_2 \cup \Omega_3$ and $\Omega_d := \Omega_5 \cup \Omega_6$. By (2.1) the remaining Stokes multipliers $s_1, s_3$ satisfy the constraint condition

$$s_1 + s_3 = -2 \sin(\pi \alpha) \quad (4.1)$$

and performing direct calculations using (2.2) shows that the connection matrix $E$ has the following form

$$E = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 1 & i e^{-i \pi \alpha} \\ 1 & -i e^{i \pi \alpha} \end{pmatrix}, \quad \text{where} \quad pq = -\frac{1}{2i \cos(\pi \alpha)}. \quad (4.2)$$

We write $\lambda_{\pm} := \pm \frac{i}{2} \sqrt{x}$ for the stationary points of the phase function $\theta(\lambda, x)$.

Considering the equation $\text{Im} \, \theta(z) = \text{Im} \, \theta(\lambda_{\pm}) = 0$, we infer that the steepest descent paths passing through $\lambda_{\pm}$ are either the line $\text{Re} \, \lambda = 0$ or the curves

$$\gamma_{\pm}(t) := t \pm i \left( \frac{t^2}{3} + \frac{x}{4} \right)^{1/2}, \quad t \in \mathbb{R}.$$

Observe that $\gamma_+$ and $\gamma_-$ are asymptotic to the rays $\arg \lambda = \frac{\pi}{6}, \frac{5\pi}{6}$ and $\arg \lambda = -\frac{\pi}{6}, -\frac{5\pi}{6}$, respectively. Assume that $\ell_{\pm}$ is a vertical segment connecting the origin with the stationary point $\lambda_{\pm}$. Let us consider the contour $\Sigma_0$ consisting of the circle $C$, steepest descent paths $\gamma_{\pm}$ and vertical segments $\ell_{\pm}$, as it is depicted on the Figure 5. The contour divides $\Omega_u$ and $\Omega_d$ on the sum of open regions $\Omega_u = \Omega_1^u \cup \Omega_2^u \cup \Omega_3^u$ and $\Omega_d = \Omega_1^d \cup \Omega_2^d \cup \Omega_3^d$, respectively. Let us consider the sets

$$\Omega_e^u := \Omega_1^u \cup \Omega_1^d \cup \Omega_3^d$$

and

$$\Omega_e^d := \Omega_2^u \cup \Omega_4 \cup \Omega_5^d$$

together with the matrices $S_+ := S_1 S_3$ and $S_- := S_6 S_4^{-1}$. Since $s_2 = 0$, from the constraint condition (4.1), it follows that

$$S_+ = \begin{pmatrix} 1 & 0 \\ -2 \sin(\pi \alpha) & 1 \end{pmatrix}, \quad S_- = \begin{pmatrix} 1 & -2 \sin(\pi \alpha) \\ 0 & 1 \end{pmatrix}.$$
Let us consider the function $\Phi^0(\lambda)$ given by the following formulas

\[
\Phi^0(\lambda) = \Phi(\lambda), \quad \lambda \in \Omega_1 \cup \Omega_4 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6,
\]
\[
\Phi^0(\lambda) = \Phi(\lambda)S_{1}^{-1}, \quad \lambda \in \Omega_{u1}, \quad \Phi^0(\lambda) = \Phi(\lambda)S_{0}, \quad \lambda \in \Omega_{d3},
\]
\[
\Phi^0(\lambda) = \Phi(\lambda)S_{3}, \quad \lambda \in \Omega_{u2}, \quad \Phi^0(\lambda) = \Phi(\lambda)S_{1}^{-1}, \quad \lambda \in \Omega_{d4}.
\]

Then the function $\Phi^0(\lambda)$ is a solution of the following RH problem on the contour $\Sigma_0$ (see the right diagram of Figure 6), which we denote by (RH2).

(a) For any $k \in \{u, d\}$ and $l \in \{1, 4\}$, the restriction of the function $\Phi^0(\lambda)$ to the sets $\Omega^k_l$ and $\Omega^l_k$ are holomorphic and continuous up to $\Omega^k_l$ and $\Omega^l_k$, respectively.

(b) The restrictions $\Phi^0_{\Omega^k_h}$ and $\Phi^0_{\Omega^l_h}$ are holomorphic and, for sufficiently small $\varepsilon > 0$, $\Phi^0_{\Omega^k_h} \in C(\overline{\Omega^k_h})$ and $\Phi^0_{\Omega^l_h} \in C(\overline{\Omega^l_h})$, where $\Omega^k_h := \Omega_h \setminus \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\}$ and $\Omega^l_h := \Omega_h \setminus \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\}$.

(c) Given $\lambda \in \Sigma_0$, let $\Phi^0_\pm(\lambda)$ be the limits of the function $\Phi^0(\lambda')$ as the parameter $\lambda'$ approach $\lambda$ from the left and right side of $\Sigma_0$, respectively. Then the following jump relation is satisfied

\[
\Phi^0_+(\lambda) = \Phi^0_-(\lambda)S_0(\lambda), \quad \lambda \in \Sigma_0,
\]

where the jump matrix is such that

\[
S_0(\lambda) = S(\lambda), \quad \lambda \in \rho_{+} \cup \rho_{-}, \quad S_0(\lambda) = S_{\pm}, \quad \lambda \in \ell_{\pm}, \quad |\lambda| > r,
\]

and furthermore, if we define

\[
\gamma^+_{\pm} := \{\lambda \in \gamma_{\pm} \mid \text{Re} \gamma_{\pm} > 0\}, \quad \gamma^-_{\pm} := \{\lambda \in \gamma_{\pm} \mid \text{Re} \gamma_{\pm} < 0\},
\]

then the matrix $S_0$ has the following form:

\[
S_0(\lambda) = S_{3}^{-1}, \quad \lambda \in \gamma^-_{-}, \quad S_0(\lambda) = S_{4}, \quad \lambda \in \gamma^+_{+},
\]
\[
S_0(\lambda) = S_{4}^{-1}, \quad \lambda \in \gamma^-_{+}, \quad S_0(\lambda) = S_{0}, \quad \lambda \in \gamma^+_{-}.
\]

(d) The function $\Phi^0(\lambda)\lambda^{-\alpha}e^{\sigma_3}$ is bounded for $\lambda$ sufficiently close to zero, where the branch of the multifunction $\lambda^{-\alpha}$ is chosen arbitrarily.

(e) As $\lambda \to \infty$, the function $\Phi^0(\lambda)$ has the following asymptotic behavior

\[
\Phi^0(\lambda) = (I + O(\lambda^{-1}))e^{-\sigma_3}.\]
4.2. Representation of the solution and asymptotic behavior. In this section we intend to use the auxiliary RH problem from Section 3.1 to derive Proposition 4.5, which provides a formula expressing the solution of the problem (RH2) for sufficiently large \( x > 0 \), in the terms of a local parametrix around the origin. We start with a reduced RH problem on the contour \( \Sigma_1 := \ell_+ \cup \ell_- \cup C \cup \gamma_+^\pm \cup \gamma_-^\pm \), which consists in finding a \( 2 \times 2 \) matrix-valued function \( \Phi^1(\lambda) = \Phi^1(\lambda, x) \) such that the following conditions are satisfied.

(i) The function \( \Phi^1(\lambda)\lambda^{-\alpha_3} \) is holomorphic on the set \( \Omega_0 := \{ \lambda \in \mathbb{C} \mid |\lambda| < r \} \).

(ii) Given \( \lambda \in \Sigma_1 \), let \( \Phi^1_+(\lambda) \) and \( \Phi^1_-(\lambda) \) be the limits of the function \( \Phi^1(\lambda') \) as the parameter \( \lambda' \) approach \( \lambda \) from the left and right side of \( \Sigma_1 \), respectively. Then the following jump condition is satisfied (see Figure 7)

\[
\Phi^1_+(\lambda) = \Phi^1_-(\lambda) S_1(\lambda), \quad \lambda \in \Sigma_1,
\]

where the jump matrix is such that

\[
S_1(\lambda) = S_0(\lambda), \quad \lambda \in \ell_+ \cup \ell_- \cup C \quad \text{and} \quad S_1(\lambda) = S_\pm, \quad \lambda \in \gamma_\pm^\pm.
\]

(iii) We have the asymptotic behavior \( \Phi^1(\lambda)e^{\theta(\lambda)\sigma_3} \rightarrow I \) as \( \lambda \rightarrow \infty \).

We begin with the proof of the existence and asymptotic behavior of solutions for the reduced RH problem. Following [17, Section 11.6], we consider an oriented contour \( \Sigma_2 \) on a complex plane, depicted on the right diagram of Figure 8, where \( \gamma_\pm^\pm \) are the steepest descent paths defined by (4.3) and \( L \) is the circle of radius \( R = \frac{1}{4}x^{1/2} \), which is divided into two oriented arcs

\[
L_r := \{ \lambda \in \mathbb{C} \mid |\lambda| = R, \ Re \lambda > 0 \}, \quad L_l := \{ \lambda \in \mathbb{C} \mid |\lambda| = R, \ Re \lambda < 0 \}
\]

Furthermore \( L^R_r \) and \( L^R_l \) are the parts of the curves \( \ell_+ \) and \( \ell_- \), respectively, lying outside the circle \( L \). Consider the sectionally holomorphic function \( \bar{\Psi} \), given by

\[
\bar{\Psi}(\lambda) = \begin{cases} 
\hat{\Psi}(z(\lambda)) & \text{for } |\lambda| < R, \\
e^{-\theta(\lambda)\sigma_3} & \text{for } |\lambda| > R,
\end{cases}
\]

where \( z(\lambda) \) is given by (3.10) and \( \hat{\Psi}(\lambda) \) is a solution of the RH problem from Proposition 3.3. We are looking for the functions \( \Phi^1(\lambda) \) of the form

\[
\Phi^1(\lambda) = \chi(\lambda) \bar{\Psi}(\lambda),
\]

where \( \chi(\lambda) \) solves the following RH problem, which we denote by (RH3).

(a) The function \( \chi \) is holomorphic on \( \mathbb{C} \setminus \Sigma_2 \).
(b) We have the jump relation \( \chi_+(\lambda) = \chi_-(\lambda) H(\lambda) \) for \( \lambda \in \Sigma_2 \), where \( H \) is a matrix defined as follows

\[
H(\lambda) := \begin{cases} 
e^{-\theta \sigma_3} S_{\pm} e^{\theta \sigma_3} & \text{for } \lambda \in l^R_\pm \text{ or } \lambda \in \gamma^\pm_\pm, \\
\hat{\Psi}^2(z(\lambda)) e^{\theta(\lambda) \sigma_3} & \text{for } \lambda \in \mathcal{L}_r, \\
\hat{\Psi}^3(z(\lambda)) e^{\theta(\lambda) \sigma_3} & \text{for } \lambda \in \mathcal{L}_l. 
\end{cases}
\]

(c) We have the asymptotic behavior \( \chi(\lambda) \to I \) as \( \lambda \to +\infty \).

**Figure 8.** Left: the contour \( \Sigma_1 \) with circle \( \mathcal{L} \) of radius \( R = \frac{1}{4} x^{1/2} \) enclosing the neighborhood of local parametrix. Right: the contour \( \Sigma_2 \) for the RH problem with the function \( \chi \).

**Lemma 4.1.** Given \( 1 \leq p < +\infty \), we have the following asymptotic behavior

\[
\| H - I \|_{L^p(\Sigma_2)} = O(|x|^{-\frac{3p-1}{2}}), \quad x \to +\infty. \tag{4.4}
\]

Furthermore we have

\[
\| H - I \|_{L^{\infty}(\Sigma_2)} = O(|x|^{-\frac{3}{2}}), \quad x \to +\infty. \tag{4.5}
\]

**Proof.** In [17, Chapter 11.6] it was proved the following inequality

\[
\| H(\lambda) - I \| \leq \begin{cases} c e^{-c|x|} \| \lambda \| & \text{for } \lambda \in l^R_\pm \cup \gamma^\pm_\pm, \\
c R^{-3} & \text{for } \lambda \in \mathcal{L}. \end{cases} \tag{4.6}
\]

where the constant \( c > 0 \) is independent from \( x > 0 \). This implies that

\[
\| H(\lambda) - I \| \leq \begin{cases} c e^{-c|x|/4} \| \lambda \| & \text{for } \lambda \in l^R_\pm \cup \gamma^\pm_\pm, \\
64 c |x|^{-\frac{3}{2}} & \text{for } \lambda \in \mathcal{L}, \end{cases}
\]

and hence (4.5) holds. To show (4.4), let us observe that an easy calculation gives

\[
|\gamma^\pm(t)| \geq \sqrt{3}|t|/3 + x^\frac{3}{2}/4 \quad \text{and} \quad |\dot{\gamma}^\pm(t)| \leq 2 \quad \text{for } t \in \mathbb{R}. \tag{4.7}
\]

Then we use (4.6) to obtain

\[
\int_{\mathcal{L}} \| H(\lambda) - I \|^p |d\lambda| \lesssim \int_{\mathcal{L}} R^{-3p} |d\lambda| \sim |x|^{-\frac{3(p-1)}{2}}. \tag{4.8}
\]

---

(1) We write \( A \lesssim B \) to denote \( A \leq CB \) for some \( C > 0 \). Furthermore we use the notation \( A \sim B \) provided there are constants \( C_1, C_2 > 0 \) such that \( C_1 B \leq A \leq C_2 B \).
Proof. Observe that by Lemma 4.1, there are problem (RH3) has a unique solution \( \chi \), with the property that
\[
\int_{\gamma^+_t} \|H(\lambda) - I\|^{\frac{p}{2}} |d\lambda| \leq \int_{\gamma^+_t} e^{-pc|x||\lambda|} |d\lambda|
\]
\[
= \int_{0}^{\infty} e^{-pc|x|t} |\gamma^+_t(t)| dt \leq \int_{0}^{\infty} e^{-pc|x|((3t^3 + x^1/2)/4)} dt \tag{4.9}
\]
\[
= e^{-pc|x|^3/2/4} \int_{0}^{\infty} e^{-p\sqrt{3}|x|t/3} dt \sim |x|^{-1} e^{-pc|x|^3/2/4}
\]
and furthermore
\[
\int_{c_r} \|H(\lambda) - I\|^{\frac{p}{2}} |d\lambda| \leq \int_{c_r} e^{-pc|x||\lambda|} |d\lambda| = \int_{\frac{1}{4}|x|^{1/2}}^{\frac{1}{2}|x|^{1/2}} e^{-pc|x||t|} dt \tag{4.10}
\]
\[
\sim |x|^{-1} \left( e^{-pc|x|^3/2/4} - e^{-pc|x|^3/2/4} \right) \leq |x|^{-1} e^{-pc|x|^3/2/2}.
\]
Proceeding in the similar way we obtain
\[
\int_{\gamma^-} \|H(\lambda) - I\|^{\frac{p}{2}} |d\lambda| \leq |x|^{-1} e^{-pc|x|^3/2/4}, \tag{4.11}
\]
Combining (4.8), (4.9), (4.10) and (4.11) we obtain the inequality (4.4) and the proof of lemma is completed. \( \square \)

In the following proposition we prove that the reduced RH problem admits a unique solution provided \( x > 0 \) is sufficiently large.

**Proposition 4.2.** There is a constant \( x_1 > 0 \) such that, for any \( x > x_1 \), the problem (RH3) has a unique solution \( \chi \) with the property that
\[
\|\chi(0) - I\| = O(|x|^{-3/2}), \quad x \to +\infty. \tag{4.12}
\]

**Proof.** Observe that by Lemma 4.1 there are \( c_0 > 0 \) and \( x_0 > 0 \) such that
\[
\|H - I\|_{L^2(S_2)} \leq c_0|x|^{-1}, \quad \|H - I\|_{(L^2 \cap L^\infty)}(S_2) \leq c_0|x|^{-\frac{5}{2}}, \quad x > x_0. \tag{4.13}
\]
Let us assume that \( K : L^2(S_2) \to L^2(S_2) \) is a complex linear map given by (2)
\[
K(\rho) := C_-(\rho(H - I)), \quad \rho \in L^2(S_2),
\]
where \( C_- \) is the Cauchy operator on the contour \( S_2 \) (see Section 9). If we take \( \rho \in L^2(S_2) \) with \( \rho = \rho_0 + \rho_\infty \), where \( \rho_0 \in L^2(S_2) \) and \( \rho_\infty \in M_{2 \times 2}(C) \), then
\[
K(\rho) = C_- (\rho_0(H - I)) + C_- (\rho_\infty(H - I)).
\]
Therefore \( K(\rho) \in L^2(S_2) \) and we have the following estimates
\[
\|K(\rho)\|_{L^2(S_2)} \leq \|C_-\|_{L^2(S_2)} \left( \|\rho_0(H - I)\|_{L^2(S_2)} + \|\rho_\infty\| \right) \|H - I\|_{L^2(S_2)}
\]
\[
\leq \|C_-\|_{L^2(S_2)} \|H - I\|_{(L^2 \cap L^\infty)}(S_2) \left( \|\rho_0\|_{L^2(S_2)} + \|\rho_\infty\| \right) \tag{4.14}
\]
\[
\leq \|C_-\|_{L^2(S_2)} \|H - I\|_{(L^2 \cap L^\infty)}(S_2).
\]

Although the contour \( S_2 \) depends on the parameter \( x > 0 \), from [28] Section 2.5.4 we know that the norm of the operator \( C_- \) satisfies the inequality
\[
\|C_-\|_{L^2(S_2)} \leq m, \quad x > 0, \tag{4.15}
\]
where \( m > 0 \) is a constant. By the inequalities (4.13) and (4.14), we have
\[
\|K(\rho)\|_{L^2(S_2)} \leq m \|C_0|x|^{-5/4}\|\rho\|_{L^2(S_2)}, \quad x > x_0, \tag{4.16}
\]
(2) See Section 9 for the definition of the space \( L^2 \).
which, in particular, implies that
\[ \|KI\|_{L^2(\Sigma_2)} \leq 2mc_0|x|^{-5/4}, \quad x > x_0. \] (4.17)
Furthermore (4.16) shows that there is a constant \( x_1 > x_0 \) such that
\[ \|K\|_{L^2(\Sigma_2)} < 1/2, \quad x > x_1, \]
which implies that the equation \( \rho - K\rho = I \) has a unique solution \( \rho \in L^2(\Sigma_2) \), given by the convergent Neumann series:
\[ \rho = \sum_{i=0}^{\infty} K^i I \quad \text{in the space} \quad L^2(\Sigma_2). \]
Therefore \( \rho - I \in L^2(\Sigma_2) \) and the inequalities (4.15), (4.16) and (4.17) yield
\[
\|\rho - I\|_{L^2(\Sigma_2)} \leq \sum_{i=1}^{\infty} \|K^i I\|_{L^2(\Sigma_2)} \leq \|KI\|_{L^2(\Sigma_2)} \sum_{i=0}^{\infty} \|K\|^i \]
\[ = 2\|KI\|_{L^2(\Sigma_2)} \leq 4mc_0|x|^{-5/4}, \quad x > x_1. \]
Using the representation formula for the solutions of the RH problem, we obtain
\[ \chi(\lambda) = I + \frac{1}{2\pi i} \int_{\Sigma_2} \frac{\rho(\xi)(H(\xi) - I)}{\xi - \lambda} \, d\xi, \quad \lambda \not\in \Sigma_2, \]
which together with Hölder inequality implies that
\[
\|\chi(0) - I\| \leq \frac{1}{2\pi} \int_{\Sigma_2} \frac{\|\rho(\xi)(H(\xi) - I)\|}{|\xi|} \, |d\xi| \lesssim |x|^{-\frac{1}{2}} \int_{\Sigma_2} \|\rho(\xi)(H(\xi) - I)\| \, |d\xi|
\leq |x|^{-\frac{1}{2}} \int_{\Sigma_2} \|\rho(\xi) - I\| \|H(\xi) - I\| \, |d\xi| + |x|^{-\frac{1}{2}} \int_{\Sigma_2} \|H(\xi) - I\| \, |d\xi|
\leq |x|^{-\frac{1}{2}} \|\rho - I\|_{L^1(\Sigma_2)} \|H - I\|_{L^1(\Sigma_2)} + |x|^{-\frac{1}{2}} \|H - I\|_{L^1(\Sigma_2)} \quad \text{for} \quad x > x_0.
\]
Combining this with (4.18) and (4.13), gives
\[ \|\chi(0) - I\| \lesssim |x|^{-\frac{1}{2}} |x|^{-\frac{1}{2}} + |x|^{-\frac{1}{2}} |x|^{-1} = |x|^{-3} + |x|^{-\frac{1}{2}} \lesssim |x|^{-\frac{1}{2}}, \quad x > x_1, \]
and hence the proof of (4.12) is completed.

Now we are ready to find a representation for the solutions of the problem (RH2). To this end we are looking for of the function \( \Phi^{0}(\lambda) \) of the following form
\[ \Phi^{0}(\lambda) = \chi(\lambda)\Phi^{1}(\lambda), \]
where \( \Phi^{1} \) is a solution of the reduced RH problem obtained in Proposition 4.2 and the function \( \chi(\lambda) \) satisfies the following RH problem defined on the contour \( \Sigma_3 := \gamma_{+} \cup \gamma_{-} \), that we continue to denote by (RH4).
(a) The function \( \chi(\lambda) \) is holomorphic on \( \mathbb{C} \setminus \Sigma_3 \).
(b) We have the jump relation \( \chi_{+}(\lambda) = \chi_{-}(\lambda)H(\lambda) \) for \( \lambda \in \Sigma_3 \), where
\[ H(\lambda) := \begin{cases} \Phi_{+}^{-1}(\lambda)S_{+}^{-1}[\Phi_{+}^{-1}]^{-1}(\lambda), & \lambda \in \gamma_{+}, \\ \Phi_{+}^{0}(\lambda)S_{0}[\Phi_{+}^{0}]^{-1}(\lambda), & \lambda \in \gamma_{-}. \end{cases} \]
(c) We have the asymptotic behavior \( \chi(\lambda) \to I \) as \( \lambda \to \infty \).

Lemma 4.3. Given \( 1 \leq p < +\infty \), there is constant \( c_p > 0 \) such that
\[ \|\mathcal{H} - I\|_{L^p(\Sigma_3)} \leq c_p|x|^{-\frac{5}{2}}e^{-\frac{1}{4}x^{3/2}}, \quad x > 0. \] (4.19)
Furthermore if \( p = \infty \) then there is a constant \( c_{\infty} > 0 \) such that
\[ \|\mathcal{H} - I\|_{L^\infty(\Sigma_3)} \leq c_{\infty} e^{-\frac{1}{4}x^{3/2}}, \quad x > 0. \] (4.20)
Proof. We start with the following inequality for the jump matrix $H$:

$$\|H(\lambda) - I\| \leq e^{-\frac{2}{3}|\lambda|^3/2}e^{-c|x|^{1/2}|\lambda-\lambda_\pm|^2}, \quad \lambda \in \gamma_\pm, \ x > 0$$  \hfill (4.21)

where $c > 0$ is a constant independent from $x > 0$. For the proof we refer the reader to [17] Chapter 11.6. As a simple consequence of (4.21), we obtain

$$\|H(\lambda) - I\| \leq e^{-\frac{2}{3}|\lambda|^3/2}, \quad \lambda \in \gamma_\pm, \ x > 0,$$

which immediately gives (4.20). For the proof of (4.19), observe that (4.21) yields

$$\|H - I\|_{L^p} = \int_{\gamma_\pm} \|H(\lambda) - I\|^p |d\lambda| \lesssim e^{-\frac{2}{3}|\lambda|^3/2} \int_{\gamma_\pm} e^{-p|x|^{1/2}|\lambda-\lambda_\pm|^2} |d\lambda|$$

$$= e^{-\frac{2}{3}|x|^{3/2}} \int_{\mathbb{R}} e^{-p|x|^{1/2}|\gamma_\pm(t)-\lambda_\pm|^2} |\gamma_\pm(t)| |dt|$$

$$\lesssim e^{-\frac{2}{3}|x|^{3/2}} \int_{\mathbb{R}} e^{-p|x|^{1/2}t^2} dt \sim |x|^{-1/4} e^{-\frac{2}{3}|x|^3/2}$$

and completes the proof of (4.19). \hfill \square

**Proposition 4.4.** There is $x_1 > 0$ such that, for any $x > x_1$, the problem \((RH3)\) has a unique solution $X$ with the property that

$$\|X(0) - I\| = O(|x|^{-3/4}), \quad x \to +\infty.$$  \hfill (4.22)

**Proof.** Applying Lemma 4.3 we obtain the existence of $c_0, x_0 > 0$ such that

$$\|H - I\|_{L^1(\Sigma_3)} \leq c_0|x|^{-\frac{1}{2}}, \quad \|H - I\|_{L^2(\Sigma_3)} \leq c_1|x|^{-\frac{1}{4}}, \quad x > x_0.$$  \hfill (4.23)

Let us consider a complex linear map $K: L^2_\gamma(\Sigma_3) \to L^2_\gamma(\Sigma_3)$ given by

$$K(\rho) := C_-(\rho(H - I)), \quad \rho \in L^2_\gamma(\Sigma_3),$$

where $C_-$ is the Cauchy operator on the contour $\Gamma_0$ (see Section 9). If $\rho \in L^2(\Sigma_3)$ is such that $\rho = \rho_0 + \rho_\infty$, where $\rho_0 \in L^2(\Sigma_3)$ and $\rho_\infty \in M_{2 \times 2}(\mathbb{C})$, then

$$K(\rho) = C_-(\rho_0(H - I)) + C_-(\rho_\infty(H - I)),$$

which implies that $K(\rho) \in L^2(\Sigma_3)$ and the following

$$\|K(\rho)\|_{L^2(\Sigma_3)} \leq \|C_-(\rho_0(H - I))\|_{L^2(\Sigma_3)} + \|\rho_\infty\|\|H - I\|_{L^2(\Sigma_3)}$$

$$\leq \|C_-(\rho_0(H - I))\|_{L^2(\Sigma_3)} + \|\rho_\infty\|\|H - I\|_{L^2(\Sigma_3)} \leq \|C_-(\rho_0(H - I))\|_{L^2(\Sigma_3)} + \|\rho_\infty\|\|H - I\|_{L^2(\Sigma_3)}, \quad x > 0.$$  \hfill (4.24)

By (4.7) we infer that $\gamma_\pm$ are Lipschitz curves such that the constant that does not depending from the parameter $x > 0$. Then, from [28] Section 2.5.4 it follows that the norm of the Cauchy operator $C_-$ satisfies the following inequality

$$\|C_-\|_{L^2(\Sigma_3)} \leq m, \quad x > 0,$$  \hfill (4.25)

where $m > 0$ is a constant independent from the parameter $x > 0$. Using (4.24) together with (4.23) and (4.25), we find that there is $c > 0$ such that

$$\|K(\rho)\|_{L^2(\Sigma_3)} \leq mc|x|^{-\frac{1}{4}}\|\rho\|_{L^2(\Sigma_3)}, \quad x > x_0,$$  \hfill (4.26)

and consequently

$$\|K\|_{L^2(\Sigma_3)} \leq mc|x|^{-\frac{1}{4}}, \quad x > x_0.$$  \hfill (4.26)

Furthermore (4.26) implies that, there is $x_1 > x_0$ such that

$$\|K\|_{L^2(\Sigma_3)} \leq 1/2, \quad x > x_1,$$
Proposition 4.5. There is a formula expressing the matrix \( \Box \) and therefore the element \( \rho \) which together with (4.27) and (4.23) yield

\[
\rho(\xi) = \int_{\Sigma_3} \rho(z')(H(z') - I) d\xi.
\]

Combining this with Hölder inequality implies that

\[
\rho \leq \frac{1}{2\pi i} \int_{\Sigma_3} \rho(z')(H(z') - I) \left| \frac{dz'}{z' - z} \right| d\xi.
\]

Observe that the solution \( X(z) \) of Riemann-Hilbert problem (RH4) is given by

\[
X(z) = I + \frac{1}{2\pi i} \int_{\Sigma_3} \rho(z')(H(z') - I) \left| \frac{dz'}{z' - z} \right| d\xi.
\]

This gives the inequality (4.22) and completes the proof of the proposition. \( \square \)

In the next proposition we prove the main result of this section, which provides a formula expressing the matrix \( P(x) \), whenever \( x > 0 \) is sufficiently large.

Proposition 4.5. There is \( x_+ > 0 \) such that, for any \( x > x_+ \), we have

\[
P(x) = \frac{1}{2} X(0, x) \chi(0, x) e^{-i\frac{\pi}{2} \sigma_3} \begin{pmatrix} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{pmatrix} e^{2\pi i \sigma_3} (-ix)^{\alpha_3} D.
\]

Proof. According to the notation of Section 2 and the equality (2.4), we have

\[
P(x) = \lim_{\lambda \to 0} Z(\lambda, x) = \lim_{\lambda \to 0} \Phi(\lambda, x) e^{\theta(\lambda, x) \sigma_3} \lambda^{-\alpha_3},
\]

where in the above limit the parameter \( \lambda \) belongs to the set \( \Omega \). By Propositions 4.2 and 4.4 there is \( x_+ > 0 \) such that, for any \( x > x_+ \), we have

\[
\Phi(\lambda, x) = X(\lambda, x) \chi(\lambda, x) \Psi(\lambda, x),
\]

where the functions \( \chi \) and \( X \) admit the asymptotic behaviors (4.12) and (4.22), respectively. Therefore, if we confine our attention to the ray \( \arg \lambda = 0 \), then \( z(\lambda) \in \bar{\Omega}_d \) for sufficiently small \( |\lambda| > 0 \) (see left diagram of Figure 3) and

\[
\Phi(\lambda, x) e^{\theta(\lambda, x) \sigma_3} \lambda^{-\alpha_3} = \Phi^0(\lambda, x) e^{\theta(\lambda, x) \sigma_3} \lambda^{-\alpha_3} = \Phi^0(\lambda, x) e^{\theta(\lambda, x) \sigma_3} \lambda^{-\alpha_3}
\]

\[
= X(\lambda, x) \chi(\lambda, x) \Psi(z(\lambda)) e^{\theta(\lambda, x) \sigma_3} \lambda^{-\alpha_3}
\]

\[
= X(\lambda, x) \chi(\lambda, x) \Psi(\Phi(e^{2\pi i z} z, \lambda, x)) D e^{\theta(\lambda, x) \sigma_3} \lambda^{-\alpha_3}
\]

\[
= X(\lambda, x) \chi(\lambda, x) \Psi(\Phi(e^{2\pi i z} z, \lambda, x)) D e^{\theta(\lambda, x) \sigma_3}
\]

where the last equality follows from the fact that \( D \) is a diagonal matrix. Let us define \( R_+ := \{ \lambda \in \mathbb{C} \mid \arg \lambda = 0 \} \) and observe that

\[
\lim_{\lambda \to 0} (e^{2\pi i z} z, \lambda, x)^{\alpha_3} \lambda^{-\alpha_3} = \lim_{\lambda \to 0} e^{2\pi i \sigma_3} [z, \lambda, x] / \lambda^{\alpha_3}
\]

\[
= \lim_{\lambda \to 0} e^{2\pi i \sigma_3} [-i(4\lambda^2/3 + x)]^{\alpha_3} = e^{2\pi i \sigma_3} (-ix)^{\alpha_3}.
\]

TOTAL INTEGRALS OF SOLUTIONS FOR... 23
Combining Lemma 3.1 with (4.29) and the equality
\[ \hat{\Psi}^0(e^{2\pi i z}(\lambda,x))\lambda^{-\alpha \sigma_3} = \hat{\Psi}^0(e^{2\pi i z}(\lambda,x))(e^{2\pi i z}(\lambda,x))^{-\alpha \sigma_3}(e^{2\pi i z}(\lambda,x))^{\alpha \sigma_3}\lambda^{-\alpha \sigma_3} \]
yields the following limit
\[ \lim_{\lambda \to \infty} \hat{\Psi}^0(e^{2\pi i z}(\lambda,x))\lambda^{-\alpha \sigma_3} = \frac{1}{2} e^{-i \frac{\pi i}{2} \sigma_3} \begin{pmatrix} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{pmatrix} e^{2\pi i \alpha \sigma_3}(-ix)^{\alpha \sigma_3}. \quad (4.30) \]
Using (4.30) and the fact that the functions \( X(\lambda) \) and \( \chi(\lambda) \) are holomorphic in a neighborhood of the origin, we pass in (4.28) to the limit with \( \lambda \to 0 \) along the ray \( \arg \lambda = 0 \) and obtain
\[ P(x) = \lim_{\lambda \to 0} X(\lambda,x)\chi(\lambda,x)\hat{\Psi}^0(e^{2\pi i z}(\lambda,x))\lambda^{-\alpha \sigma_3}D e^{\theta(\lambda,x)\sigma_3} \]
\[ = \frac{1}{2} X(0,x)\chi(0,x)e^{-i \frac{\pi i}{2} \sigma_3} \begin{pmatrix} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{pmatrix} e^{2\pi i \alpha \sigma_3}(-ix)^{\alpha \sigma_3} D. \]
Thus the proof of the proposition is completed. \( \square \)

5. Steepest descent analysis of \( \Phi(\lambda,x) \) for \( x < 0 \)

5.1. Contour deformation. We proceed to study the asymptotic behavior of the matrix \( P(x) \) as \( x \to -\infty \). For this purpose we recall with details the deformation of the graph \( \Sigma \) to the contour consisting of steepest descent paths of the phase function \( \theta(\lambda,x) \), in the case of \( x < 0 \) (see e.g. [12]). Let \( \Sigma_4 := \cup_{k=1}^6 \tau_k \) be a contour in the complex \( \lambda \)-plane, consisting of the six rays oriented from zero to infinity
\[ \tau_k : \arg \lambda = \pi/6 + (k-1)\pi/3, \quad k = 1, 2, \ldots, 6. \]
The contour divides the complex plane into six regions as it is shown on the right diagram of Figure 9. We consider the regions \( \Omega^k_r := \Omega_r \cap \Omega_k \) for \( k = 1, 2, 6 \) and \( \Omega^k_l := \Omega_l \cap \Omega_k \) for \( k = 3, 4, 5 \), that are shown on the left diagram of Figure 9. Let \( \Phi^k(\lambda) \) be a \( 2 \times 2 \) matrix valued function, defined as follows:
\[ \Phi^1(\lambda) := \Phi(\lambda)E, \lambda \in \Omega^1_r, \quad \Phi^4(\lambda) := \Phi(\lambda)ES_1, \lambda \in \Omega^2_r, \quad \Phi^4(\lambda) := \Phi(\lambda)ES_6^{-1}, \lambda \in \Omega^6_r \]
\[ \Phi^1(\lambda) := \Phi(\lambda)s_2E \tilde{s}_1^{-1}, \lambda \in \Omega^1_l, \quad \Phi^4(\lambda) := \Phi(\lambda)s_2E, \lambda \in \Omega^4_l \]
\[ \Phi^1(\lambda) := \Phi(\lambda)s_2E\tilde{s}_1s_3, \lambda \in \Omega^2_l, \quad \Phi^4(\lambda) := \Phi(\lambda), \lambda \in \Omega_1 \cup \ldots \cup \Omega_6. \]
At the beginning we prove the following proposition.

**Proposition 5.1.** The function \( \Phi^k(\lambda) \) is a solution of the following RH problem. 
(a) The function \( \Phi^k(\lambda) \) is analytic for \( \lambda \in \mathbb{C} \setminus \Sigma_4 \). 
(b) For any \( 1 \leq k \leq 6 \), we have the jump relation \( \Phi^k_+(\lambda) = \Phi^k_-(\lambda)S_k \), for \( \lambda \in \tau_k \). 
(c) The function \( \Phi^k(\lambda) \) has the following asymptotic behavior
\[ \Phi^k(\lambda) = (I + O(\lambda^{-1}))e^{-\theta(\lambda)\sigma_3}, \quad \lambda \to \infty. \]
(d) If \( 0 < \text{Re} \alpha < 1/2 \) then the function \( \Phi^k(\lambda) \) has the asymptotic behavior
\[ \Phi^k(\lambda) = O \left( \begin{pmatrix} |\lambda|^{-\alpha} & |\lambda|^{-\alpha} \\ |\lambda|^{-\alpha} & |\lambda|^{-\alpha} \end{pmatrix} \right), \quad \lambda \to 0 \]
and furthermore, if \( 1/2 < \text{Re} \alpha \leq 0 \) then
\[ \Phi^k(\lambda) = O \left( \begin{pmatrix} |\lambda|^\alpha & |\lambda|^\alpha \\ |\lambda|^\alpha & |\lambda|^\alpha \end{pmatrix} \right), \quad \lambda \to 0. \]

In the proof of Proposition 5.1 will use the following lemma.
Lemma 5.2. If \( 0 < \Re \alpha < 1/2 \) then the function \( \Phi(\lambda) \) has the following behavior

\[
\Phi(\lambda) = O \left( \frac{|\lambda|^{\alpha}}{|\lambda|^{\alpha}} \right), \quad \lambda \to 0
\]

and furthermore, if \( 1/2 < \Re \alpha \leq 0 \) then

\[
\Phi(\lambda) = O \left( \frac{|\lambda|^{\alpha}}{|\lambda|^{\alpha}} \right), \quad \lambda \to 0.
\]

Proof. If we define the following functions

\[
A_1(\lambda) := \Phi(\lambda) \sigma_2 M \sigma_2 \lambda^{-\alpha}, \quad \lambda \in \Omega_l, \quad A_2(\lambda) := \Phi(\lambda) \lambda^{-\alpha}, \quad \lambda \in \Omega_r.
\]

then we have the representations

\[
\Phi(\lambda) = A_1(\lambda) \lambda^\alpha \sigma_3, \quad \lambda \in \Omega_l, \quad \Phi(\lambda) = A_2(\lambda) \lambda^\alpha, \quad \lambda \in \Omega_r.
\]

From the fact that the function \( \Phi \) satisfies the point (d) of the problem (RH1), it follows that the function \( A_2(\lambda) \) is bounded whenever \( \lambda \in \Omega_r \) is sufficiently close to zero. Furthermore the function \( A_1(\lambda) \) is a holomorphic extension of \( A_2(\lambda) \) over the set \( \Omega_l \) and therefore, using the point (d) of the problem (RH1) once again we infer that the function \( A_1(z) \) is also bounded provided \( \lambda \in \Omega_r \) is close to the origin.

Thus, Lemma 3.5 gives the asymptotics (5.1) and (5.2).

Proof of Proposition 5.1. It is not difficult to check that the function \( \Phi^d(\lambda) \) satisfies conditions (a), (b) and (c). The condition (d) is a consequence of Lemma 5.2 and Lemma 3.5 with \( \beta = \alpha \) and \( \gamma = \delta = 0 \).

Figure 9. Contour deformation between \( \Sigma \) and \( \Sigma_4 \).

Let us consider the following scaling of variables

\[
\lambda(z) = (-x)^{1/2} z, \quad t(x) = (-x)^{3/2}, \quad z \in \mathbb{C}
\]

and define \( \tilde{\theta}(z) := i(\frac{4}{3} z^3 - z) \). Then the function \( \theta(\lambda) \) becomes

\[
\theta(\lambda(z), x) = \theta((-x)^{1/2} z, x) = i(\frac{4}{3} ((-x)^{1/2} z)^3 + x((-x)^{1/2} z^3)
\]

\[
= i(-x)^{3/2} \left( \frac{4}{3} z^3 - z \right) = i \tilde{\theta}(z).
\]

Let us assume that \( U \) is a \( 2 \times 2 \) matrix valued function given by the formula

\[
U(z, t) := \Phi^d(\lambda(z), -t^{2/3}) \exp(it\tilde{\theta}(z)\sigma_3).
\]
and let $G_k$, for $1 \leq k \leq 6$, be the triangular matrices defined by

$$G_{2k} := e^{-\bar{\theta}(z)s_3} S_{2k} e^{\bar{\theta}(z)s_3} = \begin{pmatrix} 1 & e^{-2t\bar{\theta}(z)} s_{2k} \\ 0 & 1 \end{pmatrix}, \quad k = 1, 2, 3,$$

and furthermore

$$G_{2k+1} := e^{-\bar{\theta}(z)s_3} S_{2k+1} e^{\bar{\theta}(z)s_3} = \begin{pmatrix} 1 & 0 \\ e^{2t\bar{\theta}(z)} s_{2k+1} & 1 \end{pmatrix}, \quad k = 0, 1, 2.$$

In view of the choice of the Stokes initial data (1.4), we have $S_2 = S_5 = I$ and hence the contour $\Sigma_4$ reduces to the contour $\Sigma_U$, which is presented on the Figure 10. Furthermore the function $U(z)$ is a solution of the following RH problem.

(a) The function $U(z)$ is holomorphic for $z \in \mathbb{C} \setminus \Sigma_U$.

(b) We have the jump relation

$$U_+(z) = U_-(z) S_U(z), \quad z \in \Sigma_U,$$

where the jump matrix $S_U$ is shown on the Figure 10.

(c) As $z \to \infty$, we have the following asymptotic behavior

$$U(z) = I + O(z^{-1}), \quad z \to \infty.$$  

(d) As $z \to 0$, we have the following behavior

$$U(z) = O \left( \begin{array}{cc} |z|^{-\alpha} & |z|^{-\alpha} \\ |z|^{-\alpha} & |z|^{-\alpha} \end{array} \right), \quad \text{if} \quad 0 < \text{Re} \alpha < \frac{1}{2} \quad (5.3)$$

and furthermore

$$U(z) = O \left( \begin{array}{cc} |z|^\alpha & |z|^\alpha \\ |z|^\alpha & |z|^\alpha \end{array} \right), \quad \text{if} \quad -\frac{1}{2} < \text{Re} \alpha \leq 0. \quad (5.4)$$

![Figure 10. The graph $\Sigma_U$ and the jump matrices for the function $U(z)$.](image)

Observe that $z_\pm = \pm 1/2$ are stationary points for the phase function $\bar{\theta}$ and $\bar{\theta}(\pm 1/2) = \mp i/3$. It is not difficult to check that the set of solutions of $\text{Re} \bar{\theta}(z) = 0$ consists of the real axis and the curves

$$h_\pm(t) := it \pm \left( t^2/3 + 1/4 \right)^{1/2}, \quad t \in \mathbb{R},$$

where $h_+$ and $h_-$ are asymptotic to the rays $\arg \lambda = \pm \pi/3$ and $\arg \lambda = \pm 2\pi/3$, respectively (see Figure 11).
Figure 11. The regions of sign changing of the function \( \text{Re} \tilde{\theta}(z) \). The dashed rays have directions \( \exp(ik\pi/3) \) for \( k = 1, 2, 4, 5 \).

We intend to define the steepest descent contour. To this end we will need two auxiliary graphs that are shown on Figure 12. One of them is \( \Sigma^0_T \) consisting of rays
\[
\{ \lambda \in \mathbb{C} \mid \arg \lambda = \pi/4 + k\pi/2 \}, \quad 0 \leq k \leq 3
\]
and the other one is \( \Sigma^+_T \), which is formed by the curves
\[
\{ \lambda \in \mathbb{C} \mid \arg \lambda = 7\pi/4 \} \quad \text{and} \quad \{ \lambda \in \mathbb{C} \mid \arg \lambda = k\pi/2 \}, \quad 0 \leq k \leq 3.
\]

Let us consider the maps \( \eta(z) \) and \( \zeta(z) \), given by the formulas
\[
\eta(z) := \tilde{\theta}(z) = z - 4z^3/3,
\]
\[
\zeta(z) := 2\sqrt{-\tilde{\theta}(z) + \tilde{\theta}(z_+)} = 4\sqrt{3}e^{\frac{\pi i}{4}} (z - 1/2)(z + 1)^{1/3},
\]
where the branch cut of the square root is taken such that \( \arg (z - 1/2) \in (-\pi, \pi) \). Let us observe that \( \eta(z) \) and \( \zeta(z) \) are holomorphic functions in a neighborhood of the origin and \( z_+ \), respectively. Since \( \eta'(0) \neq 0 \) and \( \zeta'(z_+) \neq 0 \), there is a small \( \delta > 0 \) with the property that the functions \( \eta(z) \) and \( \zeta(z) \) are biholomorphic on the balls \( B(0, 2\delta) \) and \( B(z_+, 2\delta) \), respectively. Let \( C_0 \) and \( C_+ \) be circles with radius \( \delta \) and the centers at the origin and \( z_+ \), respectively. The images \( \eta(C_0) \) and \( \zeta(C_+) \) are closed curves surrounding the origin (see Figure 12).

Figure 12. Left: the contour \( \Sigma^0_T \) and the closed curve \( \eta(C_0) \). Right: the contour \( \Sigma^+_T \) with \( \zeta(C_+) \).

Then the steepest descent contour \( \Sigma_T \) consists of the curves \( \tilde{\gamma}^\pm_k \), where \( 0 \leq k \leq 4 \), such that \( \tilde{\gamma}^\pm_k \) are straight lines joining the origin with the stationary points \( z_\pm \) and furthermore, its part contained in the ball \( B(0, \delta) \) is an inverse image of the set \( \Sigma^0_T \cap \eta(B(0, \delta)) \) under the map \( \eta \) restricted to the ball \( B(0, 2\delta) \). Since \( \eta'(0) = 1 \) it
follows that the angle between the curves \( \tilde{\gamma}_1^+ \) and \( \tilde{\gamma}_0^+ \) is equal to \( \pi/4 \). We require also that the part of the contour \( \Sigma_T \) contained in the ball \( B(z_+, \delta) \) is an inverse image of the set \( \Sigma_T^+ \cap \zeta(B(z_+, \delta)) \) under the map \( \zeta \), restricted to the ball \( B(z_+, 2\delta) \).

On the other hand the part of the contour \( \Sigma_T \) contained in the ball \( B(z_-, \delta) \) is taken such that it is a point reflection across the origin of the set \( \Sigma_T \cap B(z_-, \delta) \). We also choose the unbounded components \( \tilde{\gamma}_2^+ \) and \( \tilde{\gamma}_3^+ \), emanating from the stationary point \( z_+ \), to be asymptotic to the rays \{\arg \lambda = \pi/6\} and \{\arg \lambda = 11\pi/6\}, respectively. Similarly we take the unbounded components \( \tilde{\gamma}_2^- \) and \( \tilde{\gamma}_3^- \) to be asymptotic to the rays \{\arg \lambda = 5\pi/6\} and \{\arg \lambda = 7\pi/6\}, respectively.

![Figure 13. The contour \( \Sigma_T \) and the circles \( C_0, C_\pm \) that are depicted by dashed lines.](image)

To describe the deformation between \( \Sigma_{U^-} \) and the steepest descent contour \( \Sigma_T \) we consider a graph depicted on Figure 14 consisting of the contour \( \Sigma_{U^-} \) and dashed lines representing the segments \( \tilde{\gamma}_1^\pm \) and the unbounded curves \( \tilde{\gamma}_2^\pm, \tilde{\gamma}_3^\pm \). The graph divides the region \( \tilde{\Omega}_l \) on the sets \( \tilde{\Omega}_{l,k} \) for \( l = 1, 2 \) and \( k = 1, 2, 3 \). Let us define the function \( \tilde{U}(z) \) by the formulas

\[
\tilde{U}(z) := U(z), \quad z \in \tilde{\Omega}_1^+ \cup \tilde{\Omega}_2^+ \cup \tilde{\Omega}_3 \cup \tilde{\Omega}_5 \cup \tilde{\Omega}_6
\]

\[
\tilde{U}(z) := U(z)G_1, \quad z \in \tilde{\Omega}_1
\]

\[
\tilde{U}(z) := U(z)G_3^{-1}, \quad z \in \tilde{\Omega}_3
\]

\[
\tilde{U}(z) := U(z)G_4, \quad z \in \tilde{\Omega}_4^+.
\]

![Figure 14. Diagram for the deformation between contours \( \Sigma_U \) and \( \Sigma_{\tilde{U}} \).](image)

Let us write \( \tilde{\Omega}_l := \tilde{\Omega}_1^+ \cup \tilde{\Omega}_2 \cup \tilde{\Omega}_3 \cup \tilde{\Omega}_4 \) and \( \tilde{\Omega}_4 := \tilde{\Omega}_1^+ \cup \tilde{\Omega}_5 \cup \tilde{\Omega}_6 \cup \tilde{\Omega}_7^+ \). Using the sign changing properties of the function \( \text{Re} \tilde{\theta}(z) \) (see Figure 11 and Lemma 3.5) we infer that \( \tilde{U}(z) \) is a solution of the following RH problem, where we denote by \( \Sigma_{\tilde{U}} \) the contour depicted on Figure 15

(a) The function \( \tilde{U}(z) \) is holomorphic for \( z \in \mathbb{C} \setminus \Sigma_{\tilde{U}} \).

(b) The following jump relation holds

\[
\tilde{U}_+(z) = \tilde{U}_-(z)S_{\tilde{U}}(z), \quad z \in \Sigma_{\tilde{U}},
\]

where the contour \( \Sigma_{\tilde{U}} \) and the jump matrix \( S_{\tilde{U}} \) are presented on the Figure 15.
(c) The function $\tilde{U}(z)$ has the following asymptotic behavior

$$\tilde{U}(z) = I + O(z^{-1}), \quad z \to \infty.$$ 

(d) If $0 < \text{Re} \alpha < 1/2$ then we have the following behavior

$$\tilde{U}(z) = O \left( \frac{|z|^\alpha}{|z|^\alpha} \right), \quad z \to 0$$

and furthermore if $-1/2 < \text{Re} \alpha \leq 0$ then

$$\tilde{U}(z) = O \left( \frac{|z|^\alpha}{|z|^\alpha} \right), \quad z \to 0.$$ 

![Figure 15. The contour $\Sigma_{\tilde{U}}$ and the jump matrices for the RH problem satisfied by the function $\tilde{U}(z)$.](image)

We are using the LDU decomposition to express the jump matrices on the segment joining the origin with $z_+$ in the following form

$$\begin{pmatrix} 1 - s_1 s_3 & s_1 e^{-2t \beta} \\ -s_3 e^{2t \beta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -s_3 e^{2t \beta} & 1 \end{pmatrix} \begin{pmatrix} 1 - s_1 s_3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s_1 e^{-2t \beta} \\ 0 & 1 \end{pmatrix} =: S_{L1} S_D S_{U1}.$$ 

Similarly for the segment connecting $z_-$ with the origin, we have

$$\begin{pmatrix} 1 - s_1 s_3 & -s_3 e^{-2t \beta} \\ s_1 e^{2t \beta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_1 e^{2t \beta} & 1 \end{pmatrix} \begin{pmatrix} 1 - s_1 s_3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - s_3 e^{-2t \beta} \\ 0 & 1 \end{pmatrix} =: S_{L2} S_D S_{U2}.$$ 

The curves $\tilde{\gamma}_k$ divide the set $\tilde{\Omega}_u$ on the regions $\tilde{\Omega}_k^u$, where $1 \leq k \leq 3$. Similarly,

![Figure 16. Diagram for the deformation between contours $\Sigma_G$ and $\Sigma_T$.](image)
the set $\hat{\Omega}_d$ is decomposed by the curves $\xi^\pm$ on the regions $\hat{\Omega}^k_d$, where $1 \leq k \leq 3$, as it is shown on Figure 16. Then we define the function $T(z)$ by the formulas

$$T(z) = \hat{U}(z), \ z \in \hat{\Omega}^1_d \cup \hat{\Omega}^2_d \cup \hat{\Omega}^3_d, \ T(z) = \hat{U}(z)S_{U_1}^{-1}, \ z \in \hat{\Omega}^1_d,$$

$$T(z) = \hat{U}(z)S_{U_2}^{-1}, \ z \in \hat{\Omega}^2_d, \ T(z) = \hat{U}(z)S_{L_2}, \ z \in \hat{\Omega}^3_d.$$

Combining Lemma 3.5 together with decompositions (5.5) and (5.6), we infer that the function $T(z)$ satisfies the following RH problem.

(a) The function $T(z)$ is holomorphic for $z \in \mathbb{C} \setminus \Sigma_T$.

(b) We have the jump relation

$$T_+ (z) = T_- (z)S_T(z), \quad z \in \Sigma_T,$$

where the contour $\Sigma_T$ and the jump matrix $S_T$ are presented on the Figure 17.

(c) The function $T(z)$ has the following asymptotic behavior

$$T(z) = I + O(z^{-1}), \quad z \to \infty.$$

(d) If $0 < \text{Re} \alpha < 1/2$ then the function $T$ has the following behavior

$$T(z) = O \left( \frac{1}{|z|^\alpha} \right), \quad z \to 0$$

and furthermore if $-1/2 < \text{Re} \alpha \leq 0$ then

$$T(z) = O \left( \frac{|z|^{\alpha}}{|z|^\alpha} \right), \quad z \to 0.$$

![Figure 17](image-url) The contour $\Sigma_T$ and the associated jump matrices for the RH problem fulfilled by the function $T(z)$.

### 5.2. Parametrices near the origin and stationary points.

The aim of this subsection is proof of Theorem 5.4 which provides local parametrix around the origin for the Riemann-Hilbert problem defined on the steepest descent graph $\Sigma_T$.

Let us assume that $[z_-, z_+]$ is a segment between the stationary points $z_{\pm} = \pm \frac{\pi i}{2}$. Let us write $\nu := - (2\pi i)^{-1} \ln (1 - s_1 s_3)$ and consider the following function

$$N(z) := \left( \frac{z + \frac{1}{2}}{z - \frac{1}{2}} \right)^{\nu s_3}, \quad z \in \mathbb{C} \setminus [z_-, z_+],$$

where the branch cut is taken such that $\arg (z + \frac{1}{2}) \in (-\pi, \pi)$. Then the function $N(z)$ is a solution of the following Riemann-Hilbert problem.

(a) The function $N(z)$ is analytic on $\mathbb{C} \setminus [z_-, z_+]$. 

The solution of the Riemann-Hilbert problem (LP1) is given by

\[ N_+(z) = N_-(z) S_D, \quad z \in [z_-, z_+] \]

(c) We have the asymptotic behavior \( N(z) = I + O(1/z) \) as \( z \to \infty \).

Remark 5.3. A simple calculations show that \( \nu \) is a purely imaginary complex number. Indeed, it is enough to check that \( 1 - s_1s_3 \) is a positive number. To see this, let us observe that under condition (1.4), we have

\[ 1 - s_1s_3 = 1 - (\sin(\pi \alpha) - ik)(\sin(\pi \alpha) + ik) \]
\[ = 1 - (\sin^2(\pi \alpha) + k^2) = \cos^2(\pi \alpha) - k^2. \]

Obviously \( 1 - s_1s_3 > 0 \) provided (1.4) holds. On the other hand, if \( \alpha = i\eta \) and \( k = ik_0 \), for some \( \eta, k_0 \in \mathbb{R} \), and consequently

\[ 1 - s_1s_3 = \cos^2(\pi \alpha) - k^2 = \cosh^2(\pi \eta) + k_0^2 > 0 \]

as desired. \( \square \)

We are looking for a \( 2 \times 2 \) matrix-valued function \( P_0(z) \) defined on the closed ball \( D(0, \delta) \) satisfying the following RH problem, which we denote by (LP1).

(a) The function \( P_0(z) \) is analytic in \( D(0, \delta) \setminus \Sigma_T \).

(b) On the contour \( D(0, \delta) \cap \Sigma_T \) the function \( P_0(z) \) satisfies the jump conditions depicted on the right diagram of Figure 18.

(c) The function \( P_0(z) \) satisfies the asymptotic condition

\[ P_0(z)N(z)^{-1} = I + O(t^{-1}) \quad \text{as} \quad t \to +\infty, \quad (5.7) \]

uniformly for \( z \in \partial D(0, \delta) \).

(d) At \( z = 0 \), the function \( P_0(z) \) has the same behavior as \( U(z) \) in (5.3) and (5.4).

![Figure 18](image)

**Figure 18.** Left: contour for the RH problem satisfied by the local parametrix around the stationary point \( z_+ = \frac{1}{2} \). Right: graph for the RH problem fulfilled by the parametrix around the origin.

Theorem 5.4. The solution of the Riemann-Hilbert problem (LP1) is given by

\[ P_0(z) := \begin{cases} E(z) \bar{L}(\tau(z)) e^{-i\eta(z)\sigma_3} e^{-i\nu\sigma_3}, & \text{Im} \ z > 0, \\ E(z) \bar{L}(\tau(z)) e^{-i\eta(z)\sigma_3} e^{i\nu\sigma_3}, & \text{Im} \ z < 0, \end{cases} \]

(5.8)

where the function \( E(z) \) is defined as follows

\[ E(z) := \begin{cases} N(z) e^{i\nu\sigma_3}, & \text{Im} \ z > 0, \\ N(z) e^{-i\nu\sigma_3}, & \text{Im} \ z < 0. \end{cases} \]
Proof. The fact that the function \( P_0(z) \) satisfies the points (a) and (b) of the problem (LP1) follows directly from the the formula (5.8). We show that \( P_0(z) \) satisfies the condition (d). We assume that \( t > 0 \) is fixed and furthermore \( 0 \leq \Re a < 1/2 \). The argument in the case of \( 1/2 < \Re a < 0 \) is analogous. Using the point (c) of Theorem 3.6 we obtain the existence of \( C_0 > 0 \) and \( \varepsilon_0 \in (0, \delta) \) such that
\[
\|L(\eta)_k\| \leq C_0|\eta|^{-\Re a}, \quad \eta \in B(0, \varepsilon_0), \quad 1 \leq k, l \leq 2
\]
Since \( \eta(0) = 0 \) and \( \eta'(0) = 1 \), there is \( 0 < \varepsilon_1 < \varepsilon_0 \) such that
\[
|\eta(z)|/|z| \geq 1/2 \quad \text{and} \quad |t\eta(z)| \leq \varepsilon_0, \quad |z| \leq \varepsilon_1.
\]
Then, for any \( 1 \leq k, l \leq 2 \) and \( |z| \leq \varepsilon_1 \), we have
\[
|\bar{L}(t\eta(z))_{kl}| \leq C_0|t\eta(z)|^{-\Re a} \leq C_0(2t)^{-\Re a}|z|^{-\Re a}.
\]
Therefore the following inequality holds
\[
\|\bar{L}(t\eta(z))\| \leq C_1|z|^{-\Re a}, \quad |z| \leq \varepsilon_1,
\]
where \( C_1 := 2C_0(2t)^{-\Re a} \) and \( \|\cdot\| \) is the Euclidean matrix norm. On the other hand, the functions \( E(z) \) and \( e^{-it\eta(z)\sigma_3} \) are holomorphic in \( D(0, 2\delta) \) and in particular
\[
\|E(z)^{-1}\| \leq c, \quad \|E(z)\| \leq c \quad \text{and} \quad \|e^{-it\eta(z)\sigma_3}\| \leq c, \quad |z| \leq \delta, \quad (5.9)
\]
where \( c > 0 \) is a constant. Then, for any \( |z| \leq \varepsilon_1 \), we have
\[
\|P_0(z)\| \leq \|E(z)\|\|\bar{L}(t\eta(z))\|\|e^{-it\eta(z)\sigma_3}\| \leq c^2\|\bar{L}(t\eta(z))\| \leq c^2C_1|z|^{-\Re a},
\]
which proves that \( \bar{L}(z) \) satisfies condition (d). It remains to show that the condition (c) holds true. To this end, let us observe that
\[
P_0(z)N(z)^{-1} - I = E(z)(\bar{L}(t\eta(z)))e^{-it\eta(z)\sigma_3} - I)E(z)^{-1}. \quad (5.10)
\]
On the other hand, by the point (d) of Theorem 3.6 there are \( R, K > 0 \) such that
\[
\|\bar{L}(z)e^{-it\sigma_3} - I\| \leq K|z|^{-1}, \quad |z| \geq R. \quad (5.11)
\]
Since the radius \( \delta > 0 \) is chosen so that the function \( \eta(z) \) is biholomorphic on \( B(0, 2\delta) \) (see page 27), we have \( |\eta(z)| > c_0 > 0 \) for \( |z| = \delta \). Hence we can find \( t_0 > 0 \) such that \( |t\eta(z)| \geq R \) for \( t > t_0 \) and \( |z| = \delta \). Therefore, by the equations (5.10), (5.11) and (5.9), for any \( |z| = \delta \) and \( t > t_0 \), we have
\[
\|P_0(z)N(z)^{-1} - I\| \leq \|E(z)\|\|\bar{L}(t\eta(z))e^{-it\eta(z)\sigma_3} - I\|\|E(z)^{-1}\| \leq c^2K|t\eta(z)|^{-1} \leq c_0^{-1}c^2Kt^{-1},
\]
which gives (5.7) and the proof is completed. \( \square \)

The following proposition asserts the existence of a local parametrix around stationary points \( z_+ \). Its proof is well-known and can be found in [17, Section 9.4].

**Proposition 5.5.** There is a \( 2 \times 2 \) matrix-valued function \( P_\tau(z) \), which is defined on the closed ball \( D(z_+, \delta) \) and satisfies the following RH problem.

(a) The function \( P_\tau(z) \) is analytic in \( D(z_+, \delta) \setminus \Sigma_T \).

(b) On the contour \( \Sigma_T^+ = D(z_+, \delta) \cap \Sigma_T \) the function \( P_\tau(z) \) satisfies the same jump conditions as \( T(z) \) (see left diagram of Figure 15).

(c) The function \( P_\tau(z) \) satisfies the asymptotic condition
\[
P_\tau(z)N(z)^{-1} = I + O(t^{-1/2}), \quad t \to +\infty,
\]
uniformly for \( z \in \partial D(z_+, \delta) \).

**Remark 5.6.** If we define \( P_\tau(z) := \sigma_2P_\tau(-z)\sigma_2 \), then from symmetry of the contour \( \Sigma_T \) it follows that \( P_\tau(z) \) satisfies the analogous RH problem to this from Proposition 5.5 with the disk \( D(z_+, \delta) \) replaced by \( D(z_-, \delta) \). \( \square \)
5.3. Representation of the solution and asymptotic behavior. Let us assume that \( R(z) \) is a function given by the formula

\[
R(z) := \begin{cases} 
T(z)P_1(z)^{-1}, & z \in D(z_+, \delta) \setminus \Sigma_T, \\
T(z)P_2(z)^{-1}, & z \in D(z_-, \delta) \setminus \Sigma_T, \\
T(z)P_3(z)^{-1}, & z \in D(0, \delta) \setminus \Sigma_T, \\
T(z)N(z)^{-1}, & z \in \mathbb{C} \setminus (D(z_+, \delta) \cup D(0, \delta) \cup \Sigma_T)
\end{cases}
\]

and let \( \Sigma_R \) be the contour depicted on the Figure 19 which consists of circles \( C_\pm \) and \( C_0 \) of radius \( \delta > 0 \) (see page 27) and the parts \( \gamma_k^\pm \) of the curves \( \gamma_k \) lying outside the set \( C_+ \cup C_- \cup C_0 \) (see Figure 13). Then the function \( R(z) \) is a solution of the following Riemann-Hilbert problem, which we will denote by (RH5).

(a) The function \( R(z) \) is analytic in \( \mathbb{C} \setminus \Sigma_R \).

(b) The following jump condition holds

\[
R_+(z) = R_-(z)S_R(z), \quad z \in \Sigma_R,
\]

where the jump matrix is given by

\[
S_R(z) := \begin{cases} 
P_1(z)N(z)^{-1}, & z \in \partial D(z_+, \delta), \\
P_2(z)N(z)^{-1}, & z \in \partial D(z_-, \delta), \\
P_3(z)N(z)^{-1}, & z \in \partial D(0, \delta), \\
N(z)S_T(z)N(z)^{-1}, & z \in \Sigma_R \setminus (\partial D(z_+, \delta) \cup \partial D(0, \delta)).
\end{cases}
\]

(c) We have the following asymptotic behavior at infinity

\[
R(z) = I + O(1/z), \quad z \to \infty.
\]

![Figure 19. The contour \( \Sigma_R \) for the RH problem satisfied by the function \( R(z) \).](image)

Lemma 5.7. Given \( 1 \leq p < +\infty \), we have the following asymptotic behavior

\[
\|S_R - I\|_{(L^p, L^\infty)(\Sigma_R)} = O(t^{-1/2}), \quad t \to \infty.
\]

Proof. Applying Theorem 5.4, Proposition 5.5 and Remark 5.6 we infer that

\[
S_R(z) := \begin{cases} 
I + O(t^{-1/2}), & z \in \partial D(z_+, \delta), \\
I + O(t^{-1}), & z \in \partial D(0, \delta),
\end{cases}
\]

where the asymptotic behavior is uniform with respect to the parameter \( z \) from the boundary of the respective disk. Therefore, there is \( t_0 > 0 \) such that

\[
\|S_R - I\|_{L^\infty(C_\pm)} \lesssim t^{-1/2}, \quad \|S_R - I\|_{L^\infty(C_0)} \lesssim t^{-1/2}, \quad t \geq t_0.
\]
On the other hand (5.12) implies that, for any $t > t_0$, the following holds

$$\|S_R - I\|_{L^p(C_0)}^p = \int_{C_0} |S_R(z) - I|^p |dz| \lesssim \int_{C_0} t^{-p} |dz| \sim t^{-p} \lesssim t^{-p/2}$$  \hspace{1cm} (5.13)

and furthermore, for any $t > t_0$, we have

$$\|S_R - I\|_{L^p(C_{\pm})}^p = \int_{C_{\pm}} |S_R(z) - I|^p |dz| \lesssim \int_{C_{\pm}} t^{-p/2} |dz| \sim t^{-p/2}. \hspace{1cm} (5.14)$$

Let us denote $\Sigma' := \Sigma_R \setminus [C_+ \cup C_- \cup C_0]$. By the definition of $N(z)$ and the choice of the component curves of $\Sigma_T$, we have

$$\|N(z)\| \leq C \quad \text{and} \quad \|N(z)^{-1}\| \leq C, \quad z \in \Sigma'',$$

where $C > 0$ is a constant. Consequently, for any $z \in \Sigma'',$ we have

$$\|S_R(z) - I\| = \|N(z)[S_T(z) - I]N(z)^{-1}\| \leq \|N(z)\|\|S_T(z) - I\||N(z)^{-1}\| \leq C^2\|S_T(z) - I\|.$$  \hspace{1cm} (5.15)

Since the curve $\gamma_T^+$ is asymptotic to the ray $\{se^{i\pi/6} \mid s > 0\}$, it can be parametrized by the map $\gamma_T^+: [a, +\infty) \to \mathbb{C}$ given by

$$\gamma_T^+(s) := s + ih(s), \quad s \geq a,$$

where $h: [a, +\infty) \to \mathbb{R}$ is a smooth function satisfying asymptotic condition

$$h(s)/s \to \sqrt{3}/3, \quad s \to +\infty.$$

Let us take sufficiently small $\varepsilon_0 > 0$ such that

$$4(\sqrt{3}/3 + \varepsilon_0)^3/3 - 4(\sqrt{3}/3 - \varepsilon_0) < 0 \hspace{1cm} (5.16)$$

and observe that, there is $a_0 > a$ with the property that, for any $s > a_0$, we have

$$\text{Re} \tilde{\theta}(s + ih(s)) = 4h(s)^3/3 - 4s^2h(s) + h(s) \leq \left(4(\sqrt{3}/3 + \varepsilon_0)^3/3 - 4(\sqrt{3}/3 - \varepsilon_0)\right)s^3 + (\sqrt{3}/3 + \varepsilon_0)s.$$

Therefore, in view of (5.16), there is $a_1 > a_0$ such that

$$\text{Re} \tilde{\theta}(\gamma_T^+(s)) = \text{Re} \tilde{\theta}(s + ih(s)) \leq -s, \quad s \geq a_1. \hspace{1cm} (5.17)$$

On the other hand, using the sign change diagram for the function $\text{Re} \tilde{\theta}(z)$, which is depicted on Figure 14, we obtain the existence of constants $m > 0$ such that

$$\text{Re} \tilde{\theta}(\gamma_T^+(s)) = \text{Re} \tilde{\theta}(s + ih(s)) \leq -m, \quad s \in [a_1, a]. \hspace{1cm} (5.18)$$

Combining this inequality with (5.17) we obtain the existence of constant $m_0 > 0$ such that $\text{Re} \tilde{\theta}(\gamma_T^+(s)) \leq -m_0$ for $s \geq a$. This inequality together with (5.15), give

$$\|I - S_R\|_{L^\infty(\gamma_T^+)} \lesssim \|I - S_T\|_{L^\infty(\gamma_T^+)} \lesssim \sup_{s \geq a} e^{2\text{Re} \tilde{\theta}(\gamma_T^+(s))} \leq e^{-2mt}, \quad t > 0. \hspace{1cm} (5.19)$$

On the other hand, using (5.15), (5.17) and (5.18), for any $t > 0$, we have

$$\|I - S_R\|_{L^p(\gamma_T^+)} \lesssim \|I - S_T\|_{L^p(\gamma_T^+)} \lesssim \int_a^{\infty} \left| e^{2\text{Re} \tilde{\theta}(\gamma_T^+(s))}\right|^{p'}(\gamma_T^+)'(s)|ds| \hspace{1cm} (5.20)$$

$$\approx \int_a^{a_1} e^{2pt\text{Re} \tilde{\theta}(\gamma_T^+(s))} ds + \int_a^{\infty} e^{2pt\text{Re} \tilde{\theta}(\gamma_T^+(s))} ds \hspace{1cm} (5.20)$$

$$= \int_a^{a_1} e^{-2pt} ds + \int_a^{\infty} e^{-2pt} ds = (a_1 - a)e^{-2pt} + (2pt)e^{-2pt_{a_1}}.$$

Let $\gamma_T^+: [0, 1] \to \mathbb{C}$ be a parametrization of the curve $\gamma_T^+$. By the sign changing diagram from Figure 14, there is $m_1 > 0$ such that

$$\text{Re} \tilde{\theta}(\gamma_T^+(s)) \leq -m_1, \quad s \in [0, 1].$$
which together with (5.15) imply that
\[ \|I - S_R\|_{L^p(\gamma_1^+)} \lesssim \|I - S_T\|_{L^p(\gamma_1^+)} \lesssim \int_0^1 |e^{2\theta t}(\gamma_1^+(s))|^p |\gamma_1^+(s)'| \, ds \lesssim e^{-2\rho_{n_1} t}, \] (5.21)
for \( t > 0 \) and furthermore
\[ \|I - S_R\|_{L^\infty(\gamma_1^+)} \lesssim \|I - S_T\|_{L^\infty(\gamma_1^+)} \lesssim \sup_{s \geq 0} |e^{2\theta t}(\gamma_1^+(s))| \leq e^{-2\rho_{n_1} t}, \quad t > 0. \] (5.22)

In the similar way we can obtain the estimates (5.19), (5.20), (5.21) and (5.22) for the remaining components \( \gamma_k^\pm \), where \( k = 2, 3, 4 \). This leads to the inequalities
\[ \|S_R - I\|_{L^\infty(\Sigma_k)} \lesssim e^{-ct}, \quad \|S_R - I\|_{L^p(\Sigma_k)} \lesssim e^{-ct}, \quad t > 0, \] (5.23)
where \( c > 0 \) is some constant. Combining (5.12), (5.13), (5.14) and (5.23) yields
\[ \|S_R - I\|_{L^\infty(\Sigma)} \lesssim t^{-1/2}, \quad \|S_R - I\|_{L^p(\Sigma)} \lesssim t^{-1/2}, \quad t > 0 \]
and the proof of the lemma is completed. □

**Proposition 5.8.** There is \( t_1 > 0 \) such that, for any \( t > t_1 \), the problem (RH5) admits a unique solution \( R(z, t) \) with the property that
\[ \|R(0) - I\| = O(t^{-1/2}), \quad t \to +\infty. \] (5.24)

**Proof.** From Lemma 5.7 it follows that there are \( t_0 > 0 \) such that
\[ \|S_R - I\|_{L^2(\Sigma_0)} \lesssim t^{-1/2}, \quad t > t_0. \] (5.25)

Assume that \( R : L^2_2(\Sigma_R) \to L^2_2(\Sigma_R) \) is a complex linear map given by
\[ R(\rho) := C_-((\rho(S_R - I)), \quad \rho \in L^2_2(\Sigma_R), \]
where \( C_- \) is the Cauchy operator on the contour \( \Sigma_R \) (see Section 4). Let us take \( \rho \in L^2_2(\Sigma_R) \) with \( \rho = \rho_0 + \rho_\infty \), where \( \rho_0 \in L^2(\Sigma_R) \) and \( \rho_\infty \in M_{2 \times 2}(\mathbb{C}) \). Then, by the linearity of the Cauchy operator, we have
\[ R(\rho) = C_-((\rho - \rho_0)(S_R - I)) + C_-((\rho_\infty)(S_R - I)). \]

Therefore \( R(\rho) \in L^2(\Sigma_R) \) and the following estimates hold
\[ \|R(\rho)\|_{L^2(\Sigma_R)} \lesssim \|\rho_0(S_R - I)\|_{L^2(\Sigma_R)} + \|\rho_\infty\| \|(S_R - I)\|_{L^2(\Sigma_R)} \lesssim \|S_R - I\|_{L^2(\Sigma_R)} \|(\rho_0\|_{L^2(\Sigma_R)} + \|\rho_\infty\|) \] (5.26)
\[ \lesssim \|S_R - I\|_{L^2(\Sigma_R)} \|(S_R - I)\|_{L^2(\Sigma_R)}. \]

Therefore, by the inequalities (5.25) and (5.26), we have
\[ \|R(\rho)\|_{L^2_2(\Sigma_R)} \leq c_0 t^{-1/2} \|\rho\|_{L^2_2(\Sigma_R)}, \quad t > t_0, \] (5.27)
which, in particular, implies that
\[ \|RI\|_{L^2_2(\Sigma_R)} \leq c_0 \sqrt{t}^{-1/2}, \quad t > t_0. \] (5.28)

Furthermore (5.27) shows that there is \( t_1 > t_0 \) such that
\[ \|R\|_{L^2_2(\Sigma_R)} < 1/2, \quad t > t_1, \]
which implies that the equation \( \rho - R(\rho) = I \) has a unique solution \( \rho \in L^2_2(\Sigma_R) \), given by the convergent Neumann series:
\[ \rho = \sum_{i=0}^{\infty} R^i I \] in the space \( L^2_2(\Sigma_R) \).
Therefore $\rho - I \in L^2(\Sigma_R)$ and the inequalities (5.27) and (5.28) yield
\[
\|\rho - I\|_{L^2(\Sigma_R)} \leq \sum_{i=1}^{\infty} \|\mathcal{R}^1 I\|_{L^2(\Sigma_R)} \leq \|\mathcal{R} I\|_{L^2(\Sigma_R)} \sum_{i=0}^{\infty} \|\mathcal{R}^i I\|_{L^2(\Sigma_R)} 
\leq \|\mathcal{R} I\|_{L^2(\Sigma_R)} \leq c_1 t^{-1/2}, \quad t > t_1.
\] (5.29)

From the representation formula for the solutions of the RH problem

\[
R(\lambda) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{\rho(\xi)(S_R(\xi) - I)}{\xi - \lambda} d\xi, \quad \lambda \notin \Sigma_R,
\]

which together with Hölder inequality implies that
\[
\|R(0) - I\| \leq \frac{1}{2\pi} \int_{\Sigma_R} \|\rho(\xi)(S_R(\xi) - I)\| |d\xi| \leq \delta^{-1} \int_{\Sigma_R} \|\rho(\xi)(S_R(\xi) - I)\| |d\xi|
\leq \int_{\Sigma_R} \|\rho(\xi) - I\| (S_R(\xi) - I)\| |d\xi| + \int_{\Sigma_R} \|S_R(\xi) - I\| |d\xi|
\leq \|\rho - I\|_{L^2(\Sigma_R)} \|S_R - I\|_{L^2(\Sigma_R)} + \|S_R - I\|_{L^1(\Sigma_R)}, \quad t > t_1.
\]

Combining this with (5.25), (5.29) and Lemma 5.7 gives
\[
\|R(0) - I\| \lesssim t^{-1/2} + t^{-1/2} = t^{-1} + t^{-1/2} \lesssim t^{-1/2}, \quad t > t_1
\]
and the proof of the proposition is completed. \qed

**Proposition 5.9.** There is $x_- < 0$ such that, for any $x < x_-$, there is $\varepsilon > 0$ with the property that, for any $\lambda \in B(0, \varepsilon)$ with $\arg \lambda = \pi/12$, we have
\[
Z(\lambda, x) = R(z)E(z)\tilde{\Psi}^0(e^{2\pi^2 t_1 i\eta(z)})DK z^{-\sigma_3} e^{i \theta(z) \sigma_3 t^{-\sigma_2}}
\] (5.30)
where $z = (-x)^{1/2} \lambda$, $t = (-x)^{3/2}$ and the matrix $K$ is defined as
\[
K := (1 - s_1 s_3)^{-1/2} E \hat{S}_2 \begin{pmatrix} 1 & -s_3 \\ -s_1 & 1 \end{pmatrix} E^{-1}.
\] (5.31)

**Proof.** By Proposition 5.8 there is $t_1 > 0$ such that, for any $t > t_1$, we have
\[
T(z, t) = R(z, t)P_0(z, t), \quad z \in \tilde{\Omega}_n^{\delta} \cap B(0, \delta),
\] (5.32)
where the region $\tilde{\Omega}_n^{\delta}$ is depicted on Figure 16. Let us define $x_- := -t_1^{2/3}$ and fix $x < x_-$. Clearly $t = (-x)^{3/2} > t_1$. As we have seen in the construction of the contour $\Sigma_T$ (see Section 5.1), the angle between the curves $\tilde{\gamma}_1^+$ and $\tilde{\gamma}_0^+$ at the point $O$ is equal to $\pi/4$. Hence, there is $\varepsilon > 0$ such that
\[
z = (-x)^{-\frac{1}{2}} \lambda \in \Omega_1^+ \cap \Omega_1^+ \cap \tilde{\Omega}_n^{\delta} \cap B(0, \delta), \quad |\lambda| \leq \varepsilon, \quad \arg \lambda = \pi/12.
\] (5.33)

Since $\eta'(0) = 1$ we can decrease $\varepsilon > 0$ if necessary such that
\[
\eta(z) = \eta(\lambda(-x)^{-\frac{i}{2}}) \in \tilde{\Omega}_n^{\delta} \cap \{z \in \mathbb{C} \mid \Im z > 0\}, \quad |\lambda| \leq \varepsilon, \quad \arg \lambda = \pi/12,
\]
where the set \( \Omega_r^2 \) is shown on Figure 4 [using (5.33)] we thus obtain
\[
Z(\lambda, x) = \Phi(\lambda, x)e^{\theta(\lambda)\sigma_3} = \Phi^4(\lambda, x)E^{1}e^{-\sigma_3}\theta(\lambda)\sigma_3
\]
\[
= \Phi^4(t^{1/3}z, -t^{2/3})E^{-1}z^{-\sigma_3}e^{\theta(z)\sigma_3}t^{-\sigma_3}
\]
\[
= U(z, t)e^{-\theta(z)\sigma_3}E^{-1}z^{-\sigma_3}e^{\theta(z)\sigma_3}t^{-\sigma_3}
\]
\[
= \tilde{U}(z, t) \begin{pmatrix} 1 & 0 \\ -s_1 e^{2i\theta(z)} & 1 \end{pmatrix} E^{-1}z^{-\sigma_3}e^{\theta(z)\sigma_3}t^{-\sigma_3}
\]
\[
= T(z, t) \begin{pmatrix} 1 & 0 \\ -s_1 e^{2i\theta(z)} & 1 \end{pmatrix}
\]
\[
\Rightarrow \lambda, x \rightarrow (\lambda, x) \Phi(z, t)\Phi(\lambda, x) \Phi^4(z, t) \Phi^4(\lambda, x)
\]
In view of (5.33) and (5.8) we obtain
\[
Z(\lambda, x) = R(z)P_0(z)e^{-\tilde{\theta}(z)\sigma_3}(1 - s_1 s_3)E^{-1}z^{-\sigma_3}e^{\tilde{\theta}(z)\sigma_3}t^{-\sigma_3}. \tag{5.34}
\]
In view of (5.33) and (5.8) we obtain
\[
P_0(z) = E(z)\tilde{L}(t\eta(z))e^{-i\tilde{\theta}(z)\sigma_3}E^{-1}z^{-\sigma_3}e^{i\tilde{\theta}(z)\sigma_3}E^{-1}z^{-\sigma_3}e^{i\tilde{\theta}(z)\sigma_3}
\]
Combining this with Lemma 5.2 and Remark 5.3 yields
\[
P_0(z) = E(z)\tilde{\Psi}(e^{2\pi i t\eta(z)})DES_2e^{\tilde{\theta}(z)\sigma_3}e^{-i\nu} \tag{5.35}
\]
Substituting (5.35) into (5.34), we obtain
\[
Z(\lambda, x) = R(z)E(z)\tilde{\Psi}(e^{2\pi i t\eta(z)})DES_2WE^{-1}z^{-\sigma_3}e^{\tilde{\theta}(z)\sigma_3}t^{-\sigma_3}. \tag{5.36}
\]
where we define
\[
W := e^{\tilde{\theta}(z)\sigma_3}(1 - s_1 s_3)E^{-1}z^{-\sigma_3}e^{\tilde{\theta}(z)\sigma_3}t^{-\sigma_3}
\]
On the other hand the following equalities hold
\[
W = \begin{pmatrix} (1 - s_1 s_3)^{\frac{1}{2}} & 0 \\ 0 & (1 - s_1 s_3)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} -s_3 & 1 \\ 0 & -s_1 \end{pmatrix}
\]
Therefore, combining (5.35) with (5.36), we obtain (5.30), which completes the proof of the proposition.

**Proposition 5.10.** The matrix \( K \) defined in (5.31) is expressed by the formula
\[
K = \frac{1}{(1 - s_1 s_3)^{1/2}} \begin{pmatrix} \cos(\pi\alpha) - k & 0 \\ 0 & \cos(\pi\alpha) + k \end{pmatrix} \left( e^{-i\pi\alpha} \right).
\]
In particular the matrix $K$ is diagonal.

Proof. Let us observe that, by the constraint condition (4.1) and (3.4), we have

$$
\hat{S}_2 \begin{pmatrix} 1 & -s_3 \\ -s_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -s_3 \\ -2\sin(\pi \alpha) - s_1 & 1 + 2s_3 \sin(\pi \alpha) \end{pmatrix} = \begin{pmatrix} 1 & -s_3 \\ s_3 & 1 + 2s_3 \sin(\pi \alpha) \end{pmatrix}.
$$

Since the matrix $E$ is given by (4.2), its inverse is of the form

$$
E^{-1} = \begin{pmatrix} 1 & -ie^{i\pi \alpha} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^{-1}, \quad pq = -\frac{1}{2i \cos(\pi \alpha)}.
$$

Therefore, if we define

$$
A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & ie^{i\pi \alpha} \\ 1 & -ie^{i\pi \alpha} \end{pmatrix} \begin{pmatrix} 1 & -s_3 \\ s_3 & 1 + 2s_3 \sin(\pi \alpha) \end{pmatrix} = \begin{pmatrix} -ie^{i\pi \alpha} & -ie^{-i\pi \alpha} \\ -1 & 1 \end{pmatrix},
$$

then the matrix $K$ has the following form

$$
K = -\frac{(1-s_1s_3)}{2i \cos(\pi \alpha)} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^{-1}.
$$

Observe that after multiplication, we obtain

$$
K_0 = \begin{pmatrix} 1 + is_3e^{-i\pi \alpha} & -s_3 + ie^{-i\pi \alpha} + 2is_3e^{-i\pi \alpha} \sin(\pi \alpha) \\ 1 - is_3e^{i\pi \alpha} & -s_3 - ie^{i\pi \alpha} - 2is_3e^{i\pi \alpha} \sin(\pi \alpha) \end{pmatrix} = \begin{pmatrix} -ie^{i\pi \alpha} & -ie^{-i\pi \alpha} \\ -1 & 1 \end{pmatrix},
$$

which in turn implies that the entries are given by

$$
A_{11} = -ie^{i\pi \alpha}(1 + is_3e^{-i\pi \alpha}) + s_3 - ie^{-i\pi \alpha} - 2is_3e^{-i\pi \alpha} \sin(\pi \alpha),
$$
$$
A_{12} = -ie^{-i\pi \alpha}(1 + is_3e^{i\pi \alpha}) - s_3 + ie^{-i\pi \alpha} + 2is_3e^{i\pi \alpha} \sin(\pi \alpha),
$$
$$
A_{21} = -ie^{i\pi \alpha}(1 - is_3e^{i\pi \alpha}) + s_3 + ie^{i\pi \alpha} + 2is_3e^{i\pi \alpha} \sin(\pi \alpha),
$$
$$
A_{22} = -ie^{-i\pi \alpha}(1 - is_3e^{i\pi \alpha}) - s_3 - ie^{i\pi \alpha} - 2is_3e^{i\pi \alpha} \sin(\pi \alpha).
$$

Calculating the coefficient $A_{12}$ gives

$$
A_{12} = -ie^{i\pi \alpha} + s_3e^{i2\pi \alpha} - s_3 + ie^{-i\pi \alpha} + 2is_3e^{-i\pi \alpha} \sin(\pi \alpha)
$$
$$
= s_3(\cos(\pi \alpha) - i \sin(\pi \alpha))^2 - s_3 + 2is_3(\cos(\pi \alpha) - i \sin(\pi \alpha)) \sin(\pi \alpha)
$$
$$
= s_3(\cos^2(\pi \alpha) - \sin^2(\pi \alpha)) - s_3 + 2s_3 \sin^2(\pi \alpha)
$$
$$
= s_3(\cos^2(\pi \alpha) + \sin^2(\pi \alpha)) - s_3 = 0.
$$

and similar computations for $A_{21}$ yields

$$
A_{21} = -ie^{i\pi \alpha} - s_3e^{i2\pi \alpha} + s_3 + ie^{i\pi \alpha} + 2is_3e^{i\pi \alpha} \sin(\pi \alpha)
$$
$$
= -s_3(\cos(\pi \alpha) + i \sin(\pi \alpha))^2 + s_3 + 2is_3(\cos(\pi \alpha) + i \sin(\pi \alpha)) \sin(\pi \alpha)
$$
$$
= -s_3(\cos^2(\pi \alpha) - \sin^2(\pi \alpha)) + s_3 - 2s_3 \sin^2(\pi \alpha)
$$
$$
= -s_3(\cos^2(\pi \alpha) + \sin^2(\pi \alpha)) + s_3 = 0.
$$

For the coefficient $A_{11}$ we have

$$
A_{11} = -ie^{i\pi \alpha} + s_3 + s_3 - ie^{-i\pi \alpha} - 2is_3e^{-i\pi \alpha} \sin(\pi \alpha)
$$
$$
= 2s_3 - 2i \cos(\pi \alpha) - 2is_3 \cos(\pi \alpha) \sin(\pi \alpha) - 2s_3 \sin^2(\pi \alpha)
$$
$$
= 2s_3 \cos^2(\pi \alpha) - 2i \cos(\pi \alpha) - 2is_3 \cos(\pi \alpha) \sin(\pi \alpha)
$$
$$
= 2 \cos(\pi \alpha)(is_3 \cos(\pi \alpha) - i - is_3 \sin(\pi \alpha)).
$$

(5.39)
Since the triple \((s_1, s_2, s_3)\) is given by (1.4), it follows that
\[
s_3 \cos(\pi \alpha) - i - is_3 \sin(\pi \alpha)
= (-\sin(\pi \alpha) + ik) \cos(\pi \alpha) - i - i(-\sin(\pi \alpha) + ik) \sin(\pi \alpha)
= -\sin(\pi \alpha) \cos(\pi \alpha) + ik \cos(\pi \alpha) + k \sin(\pi \alpha) - i + i \sin^2(\pi \alpha)
= -\sin(\pi \alpha) \cos(\pi \alpha) + ik \cos(\pi \alpha) + k \sin(\pi \alpha) - i \cos^2(\pi \alpha)
= -i \cos(\pi \alpha) (\cos(\pi \alpha) - i \sin(\pi \alpha)) + ik (\cos(\pi \alpha) - i \sin(\pi \alpha))
= (\cos(\pi \alpha) - i \sin(\pi \alpha))(-i \cos(\pi \alpha) + ik) = -ie^{-i\pi \alpha} (\cos(\pi \alpha) - k),
\]
which after substitution to (5.39) gives
\[
A_{11} = -2i \cos(\pi \alpha) (\cos(\pi \alpha) - k) e^{-i\pi \alpha}.
\]
It remains to calculate the coefficient \(A_{22}\). To this end let us observe that
\[
\begin{align*}
A_{22} &= -ie^{-i\pi \alpha}(1 - is_3 e^{i\pi \alpha}) - s_3 - ie^{i\pi \alpha} - 2is_3 e^{i\pi \alpha} \sin(\pi \alpha) \\
&= -ie^{-i\pi \alpha} - s_3 - ie^{i\pi \alpha} - 2is_3 e^{i\pi \alpha} \sin(\pi \alpha) \\
&= -2s_3 - 2i \cos(\pi \alpha) - 2is_3 \cos(\pi \alpha) \sin(\pi \alpha) + 2s_3 \sin^2(\pi \alpha) \\
&= -2 \cos(\pi \alpha) (s_3 \cos(\pi \alpha) + i + is_3 \sin(\pi \alpha)).
\end{align*}
\]
Using (1.4) once again we obtain
\[
s_3 \cos(\pi \alpha) + i + is_3 \sin(\pi \alpha)
= (-\sin(\pi \alpha) + ik) \cos(\pi \alpha) + i + i(-\sin(\pi \alpha) + ik) \sin(\pi \alpha)
= -\sin(\pi \alpha) \cos(\pi \alpha) + ik \cos(\pi \alpha) - k \sin(\pi \alpha) + i - i \sin^2(\pi \alpha)
= -\sin(\pi \alpha) \cos(\pi \alpha) + ik \cos(\pi \alpha) - k \sin(\pi \alpha) + i \cos^2(\pi \alpha)
= i \cos(\pi \alpha) (\cos(\pi \alpha) + i \sin(\pi \alpha)) + ik (\cos(\pi \alpha) + i \sin(\pi \alpha))
= (\cos(\pi \alpha) + i \sin(\pi \alpha) + i \cos(\pi \alpha) + ik) = ie^{i\pi \alpha} (\cos(\pi \alpha) + k),
\]
which together with (5.40) provides
\[
A_{22} = -2i \cos(\pi \alpha) (\cos(\pi \alpha) + k) e^{i\pi \alpha}.
\]
Therefore we have
\[
A = -2i \cos(\pi \alpha) \begin{pmatrix} (\cos(\pi \alpha) - k) e^{-i\pi \alpha} & 0 \\ 0 & (\cos(\pi \alpha) + k) e^{i\pi \alpha} \end{pmatrix}.
\]
In view of (5.38) and (5.41), we have
\[
K = \frac{1}{(1 - s_1 s_3)^{1/2}} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} (\cos(\pi \alpha) - k) e^{-i\pi \alpha} & 0 \\ 0 & (\cos(\pi \alpha) + k) e^{i\pi \alpha} \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^{-1}
= \frac{1}{(1 - s_1 s_3)^{1/2}} \begin{pmatrix} \cos(\pi \alpha) - k & 0 \\ 0 & \cos(\pi \alpha) + k \end{pmatrix} e^{-i\pi \alpha \sigma_3}
\]
e^{-i\pi \alpha \sigma_3}
e^{i\pi \alpha \sigma_3}
\]
and the proof is completed.

\[\square\]

**Proposition 5.11.** There is \(x_- < 0\) such that, for any \(x < x_-\), we have
\[
P(x) = \frac{1}{2} R(0, (-x)^{1/2}) e^{-\frac{i}{2} \pi \sigma_3} \begin{pmatrix} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{pmatrix} DK e^{2\pi i \alpha \sigma_3} (-ix)^{\alpha \sigma_3}.
\]
Proof. By Proposition \[5.9\] we obtain the existence of \(x_+ < 0\) such that, for any \(x < x_+\), there is \(\varepsilon > 0\) with the property that

\[
Z(\lambda, x) = R(z)E(z)\hat{\Psi}^0(e^{2\pi i t\eta(z)})DKz^{-\alpha_3}\hat{\theta}(\lambda, \sigma)\rightarrow \frac{e^{2\pi i z}}{z}\]

for \(|\lambda| < \varepsilon\) with \(\arg \lambda = \pi/12\), where the change of variables is given by

\[
\lambda = (-x)^{1/2}, \quad t = (-x)^{3/2}.
\]

By Proposition \[5.10\] the matrix \(DK\) is diagonal, which implies that

\[
Z(\lambda, x) = R(z)E(z)\hat{\Psi}^0(e^{2\pi i t\eta(z)})z^{-\alpha_3}DKt^{-\alpha_3}, \quad |\lambda| < \varepsilon, \quad \arg \lambda = \pi/12. \tag{5.42}
\]

Let us observe that

\[
E(z) = e^{i\pi i \sigma_3} \left( \frac{z + 1/2}{z - 1/2} \right)^{\alpha_3} \rightarrow I, \quad z \rightarrow 0, \quad \text{Im} z > 0. \tag{5.43}
\]

On the other hand, by Lemma \[3.1\], the function \(z \mapsto \hat{\Psi}^0(z)z^{-\alpha_3}\) is holomorphic on the complex plane and the following convergence holds

\[
\lim_{z \to 0} \hat{\Psi}^0(z)z^{-\alpha_3} = \frac{1}{2} e^{-i\frac{\pi}{4} \sigma_3} \left( \begin{array}{cc} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{array} \right). \tag{5.44}
\]

If we define \(\Lambda := \{z \in \mathbb{C} \mid \arg z = \pi/12\}\), then we have

\[
\lim_{z \in \Lambda, z \to 0} e^{2\pi i t\eta(z)}z^{-\alpha_3} = \lim_{z \in \Lambda, z \to 0} e^{2\pi i \alpha_3 [i\eta(z)/z] \alpha_3} = e^{2\pi i \alpha (it) \alpha_3},
\]

which together with \[5.44\] and the equality

\[
\hat{\Psi}^0(e^{2\pi i t\eta(z)})z^{-\alpha_3} = \hat{\Psi}^0(e^{2\pi i t\eta(z)})(e^{2\pi i t\eta(z)})^{-\alpha_3} = (e^{2\pi i t\eta(z)})^{-\alpha_3} z^{-\alpha_3}
\]

give the following limit

\[
\lim_{z \in \Lambda, z \to 0} \hat{\Psi}^0(e^{2\pi i t\eta(z)})z^{-\alpha_3} = \frac{1}{2} e^{-i\frac{\pi}{4} \sigma_3} \left( \begin{array}{cc} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{array} \right) e^{2\pi i \alpha_3 (it) \alpha_3}. \tag{5.45}
\]

Using \[5.43\] and \[5.45\], we pass in the equation \[5.42\] to the limit with \(z \to 0\) along the ray \(\arg z = \pi/12\) and deduce that

\[
P(x) = \lim_{\lambda \in \Lambda, \lambda \to 0} Z(\lambda, x) = \lim_{z \in \Lambda, z \to 0} R(z)E(z)\hat{\Psi}^0(e^{2\pi i t\eta(z)})z^{-\alpha_3} DK e^{\hat{\theta}(\lambda, \sigma)} t^{-\alpha_3}
\]

\[
= \frac{1}{2} R(0, t)e^{-i\frac{\pi}{4} \sigma_3} \left( \begin{array}{cc} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{array} \right) e^{2\pi i \alpha_3 (it) \alpha_3} DK t^{-\alpha_3}. \tag{5.46}
\]

In view of the fact that the matrix \(DK\) we obtain

\[
P(x) = \frac{1}{2} R(0, t)e^{-i\frac{\pi}{4} \sigma_3} \left( \begin{array}{cc} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{array} \right) DK e^{2\pi i \alpha_3 (it) \alpha_3} t^{-\alpha_3}
\]

\[
= \frac{1}{2} R(0, t)e^{-i\frac{\pi}{4} \sigma_3} \left( \begin{array}{cc} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{array} \right) DK e^{2\pi i \alpha_3 (it) \alpha_3} t^{-\alpha_3}
\]

\[
= \frac{1}{2} R(0, (-x)^{3/2})e^{-i\frac{\pi}{4} \sigma_3} \left( \begin{array}{cc} 1 & 1 - 2\alpha \\ -1 & 1 - 2\alpha \end{array} \right) DK e^{2\pi i \alpha_3 (-ix) \alpha_3},
\]

and the proof of the proposition is completed. \(\square\)
6. Proof of Theorem 1.1

Propositions 4.5 and 5.11 says that, there is \( x_0 > 0 \) such that, for \( x > x_0 \), the functions \( P(x) \) and \( P(-x) \) have the following forms

\[
P(x) = \frac{1}{2} X(0,x) \chi(0,x) e^{-i \frac{\pi}{4} \sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - 2\alpha \end{pmatrix} e^{2\pi i \alpha \sigma_3} (ix)^{\alpha \sigma_3} D,
\]

\[
P(-x) = \frac{1}{2} R(0,x^{3/2}) e^{-i \frac{\pi}{4} \sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - 2\alpha \end{pmatrix} DK e^{2\pi i \alpha \sigma_3} (ix)^{\alpha \sigma_3}.
\]

Let us observe that, for any \( x > 0 \), we have

\[
(ix)^{\alpha \sigma_3} (-ix)^{-\alpha \sigma_3} = (|x|e^{i \frac{\pi}{4}})^{\alpha \sigma_3} (|x|e^{-i \frac{\pi}{4}})^{-\alpha \sigma_3} = e^{i \frac{\pi}{8} \alpha \sigma_3} e^{i \frac{\pi}{8} \alpha \sigma_3} = e^{i \pi \alpha},
\]

which together with the fact that matrices \( D \) and \( K \) are diagonal, imply that

\[
P(-x)P(x)^{-1} = R(0,x^{3/2}) H \chi(0,x)^{-1} X(0,x)^{-1}, \quad x > x_0,
\]

where the matrix \( H \), is given by

\[
H := \frac{1}{2} e^{-i \frac{\pi}{4} \sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} Ke^{i \pi \alpha \sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} e^{i \frac{\pi}{8} \sigma_3}.
\]

Applying Proposition 5.10 and writing \( h_\pm := (1 - s_1 s_3)^{-\frac{1}{2}} (\cos(\pi \alpha) \mp k)/2 \), we have

\[
H = e^{-i \frac{\pi}{4} \sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix} e^{i \frac{\pi}{8} \sigma_3} = e^{-i \frac{\pi}{4} \sigma_3} \begin{pmatrix} h_+ + h_- & i(h_+ - h_-) \\ i(h_- - h_+) & h_+ + h_- \end{pmatrix} e^{i \frac{\pi}{8} \sigma_3} = \begin{pmatrix} h_+ + h_- & i(h_+ - h_-) \\ i(h_- - h_+) & h_+ + h_- \end{pmatrix}.
\]

which together with \( (6.1), (4.12), (4.22) \) and \( (5.24) \) gives

\[
\lim_{x \to +\infty} P(x)P(-x)^{-1} = H^{-1} = \begin{pmatrix} h_+ + h_- & i(h_+ - h_-) \\ i(h_- - h_+) & h_+ + h_- \end{pmatrix}.
\]

Let us observe that the formula (26) implies that

\[
\exp \left( \int_{-x}^{x} u(y; \alpha, k) \, dy \right) = [P(x)P(-x)^{-1}]_{11} + i[P(x)P(-x)^{-1}]_{21}, \quad x > 0
\]

and consequently

\[
\lim_{x \to +\infty} \exp \left( \int_{-x}^{x} u(y; \alpha, k) \, dy \right) = \lim_{x \to +\infty} [P(x)P(-x)^{-1}]_{11} + i \lim_{x \to +\infty} [P(x)P(-x)^{-1}]_{21} = 2h_- = (1 - s_1 s_3)^{-\frac{1}{2}} (\cos(\pi \alpha) + k) = (\cos^2(\pi \alpha) - k^2)^{-1/2} (\cos(\pi \alpha) + k),
\]

which completes the proof of Theorem 1.1. \qed

**Remark 6.1.** Let us assume that \( u(\cdot; \alpha, k) \) is a purely imaginary Ablowitz-Segur solution for the PII equation. In view of the equality

\[
\cos(\pi \alpha) = \cosh(i \pi \alpha) = (e^{i \pi \alpha} + e^{-i \pi \alpha}) / 2
\]

and the fact that \( \alpha \in i \mathbb{R} \), it follows that \( \cos(\pi \alpha) \) is a real number, greater that or equal to one. Moreover the formula \( (1.9) \) takes the form

\[
\lim_{x \to +\infty} \exp \left( \int_{-x}^{x} u(y) \, dy \right) = \exp (i \arg (\cos(\pi \alpha) + k)), \quad k \in i \mathbb{R}.
\]
7. Self-similar solutions for the geometric flow

In this section we consider the flow of planar curves, which is governed by the equation (1.17). Developing the normal vector field \( n \) and curvature \( k \) with respect to the parametrization \( z \), it is not difficult to check that the equation can be written in the following form

\[
\begin{align*}
\dot{z}_t &= -z_{ss} + \frac{3}{2}z_{ss}^2, \quad t, s \in \mathbb{R}, \\
|z_s|^2 &= 1, \quad t, s \in \mathbb{R}.
\end{align*}
\]

(7.1)

Remark 7.1. Let us observe that the system (7.1) is time-reversible and invariant under rotations. More precisely, if \( z(s, t) \) is a solution of (7.1) and \( \theta \in \mathbb{R} \), then the functions \( z(-s, -t) \) and \( e^{i\theta}z(s, t) \) also satisfy this system. \( \square \)

Following [27], we look for the self-similar solutions for (7.1) of the form

\[
z(s, t) = t^{1/3}e^{-i\frac{\pi}{3}\ln t} \omega(st^{-1/3}), \quad t > 0, \quad s \in \mathbb{R},
\]

(7.2)

where \( \omega : \mathbb{R} \rightarrow \mathbb{R} \) is a smooth map. Then \( z \) solves the system (7.1) if and only if \( \omega \) satisfies the equation

\[
\begin{align*}
-\frac{i\mu + 1}{3} \omega - \frac{s}{3} \omega_s &= -\omega_{ss} + \frac{3}{2} \omega_s, \\
|\omega_s|^2 &= 1.
\end{align*}
\]

(7.3)

On the other hand, if we consider the function

\[
\omega(s) = \int_0^s \exp \left( \frac{2}{3s^{1/3}} \int_0^s u \left( s''/3^{1/3} \right) ds'' \right) ds' + \omega_0,
\]

(7.4)

where \( u : \mathbb{R} \rightarrow i\mathbb{R} \subset \mathbb{C} \) is a smooth map and

\[
\omega_0 := -\frac{2\sqrt{3}}{1 - i\mu} (u_x(0) - u^2(0)),
\]

then \( \omega \) is a solution of (7.3) if and only if \( u \) satisfies the equation

\[
u''(x) = xu(x) + 2u^3(x) + i\mu/2.
\]

Substituting (7.4) into (7.2), we obtain the following explicit formula on self-similar solutions for the localized induction approximation

\[
z(t, s) = t^{1/3}e^{-i\frac{\pi}{3}\ln t} \int_0^{s/3^{1/3}} \exp \left( \frac{2}{3s^{1/2}} \int_0^{s'} u(s''/3^{1/2}) ds'' \right) ds' + \omega_0,
\]

(7.5)

where the profile function \( u \) is a purely imaginary solution of the inhomogeneous PII equation with the constant \( \alpha = -i\mu/2 \). Before we proceed to the proof of Theorem 1.2, we formulate the following proposition, which describes the asymptotic behavior of \( u(x) \) and \( u_x(x) \) as \( x \rightarrow \pm\infty \).

Proposition 7.2. If \( u \) is a purely imaginary AS solution of the second Painlevé equation, then there is a constant \( C > 0 \) depending from the solution \( u \) such that

\[
\begin{align*}
|u(x)| &\leq C, \\
|u_x(x)| &\leq C, \\
|u(x)| &\leq C, \\
|u_x(x)| &\leq C(1 + |x|^{-1/4}),
\end{align*}
\]

(7.6)

(7.7)

Proof. Let us assume that \( \alpha, k \in i\mathbb{R} \) are such that \( u = u(\cdot; \alpha, k) \). Using the results obtained in [17, Theorem 11.5] and [21, Theorem 2.2], we have the following asymptotic behavior

\[
u(x) = u(x; \alpha, k) = B(\alpha; x) + kAi(x)(1 + O(x^{-3/4})), \quad x \rightarrow +\infty,
\]

(7.8)
where \( Ai(x) \) is the Airy function and \( B(\alpha; x) \) satisfies
\[
B(\alpha; x) = \alpha x^{-1} + O(x^{-4}), \quad x \to +\infty.
\]
Combining this with the fact that the purely imaginary solutions of PII equation do not admit poles on real line, we infer that the function \( u(\cdot; \alpha, k) \) is bounded on \([0, +\infty)\). By (7.8), there are constants \( C_0 > 0 \) and \( x_0 \geq 1 \) such that
\[
|u(x) - \alpha x^{-1}| \leq C_0 |x|^{-4}, \quad x \geq x_0,
\]
which implies that, for any \( x \geq x_0 \), we have
\[
|u(x)| \leq |\alpha| |x|^{-1} + C_0 |x|^{-4} \leq (|\alpha| + C_0) |x|^{-1},
\]
Multiplying the PII equation (1.11), by \( u_x \) we have
\[
\frac{d}{dx}(u_x^2 - xu^2 - u^4 + 2\alpha u) = -u^2, \quad x \in \mathbb{R},
\]
which after integration gives
\[
u_x(x)^2 = xu(x)^2 + u(x)^4 - 2\alpha u(x) + L = \int_{x_0}^x u(y)^2 \, dy, \quad x \geq x_0,
\]
where we define
\[
L := u_x(0)^2 - u(0)^4 + 2\alpha u(0) - \int_{0}^{x_0} u(y)^2 \, dy.
\]
Using (7.9), we find that, for any \( x \geq x_0 \), the following estimates hold
\[
|xu(x)|^2 \leq (|\alpha| + C_0)^2 |x|^{-1} \leq (|\alpha| + C_0)^2 x_0^{-1},
\]
\[
|u(x)^4| \leq (|\alpha| + C_0)^4 |x|^{-4} \leq (|\alpha| + C_0)^4 x_0^{-4},
\]
\[
\int_{x_0}^x |u(y)|^2 \, dy \leq (|\alpha| + C_0)^2 (x_0^{-2} - x^{-2})/2 \leq (|\alpha| + C_0)^2 x_0^{-2}/2.
\]
Combining (7.11) and (7.10) gives
\[
u_x(x) \leq C, \quad x \geq 0,
\]
where the constant \( C > 0 \) depends from the solution \( u \). The inequalities (7.7) are obtained in the proof of Proposition 2.1 from [27].

8. Proof of Theorem 1.2

We begin the proof with the following proposition.

Proposition 8.1. Given \( a \in (-\pi/2, \pi/2) \) and \( \mu \in \mathbb{R} \), there are \( \theta^\pm \in [0, 2\pi) \) with \( \exp(i(\theta^+ - \theta^-)) = \exp(2ia) \) and a purely imaginary solution \( u \) of the PII equation such that the function \( z \), given by the formula (1.5), is a smooth solution of the equation (1.17) satisfying for some \( c > 0 \) the following inequality
\[
|z(t, s) - z_0(s)| < ct^\frac{1}{4}, \quad s \in \mathbb{R} \setminus \{0\}, \ t > 0.
\]

Proof. Let us choose \( k_0 \in i\mathbb{R} \) such that
\[
\frac{\cos(i\pi\mu/2) - k_0}{(\cos^2(i\pi\mu/2) - k_0^2)^{1/2}} = e^{i\alpha}
\]
and denote \( u^{\alpha; \mu} := u(\cdot; -i\mu/2, k_0) \). By Theorem 1.1 we have
\[
\lim_{x \to +\infty} \exp\left( \int_{-x}^x u^{\alpha; \mu}(y) \, dy \right) = \frac{\cos(i\pi\mu/2) - k_0}{(\cos^2(i\pi\mu/2) - k_0^2)^{1/2}} = e^{i\alpha}.
\]
We show that the function
\[
z^{a,\mu}(t, s) := t^{s/2}e^{-i\frac{s}{p} \ln t} \left[ \int_0^{s/2} \exp \left( \frac{2}{3}\int_0^{s'} a(s''/3^{1/3}) \, ds'' \right) \, ds' + \omega_0 \right]
\]
is solution of the equation (1.17) satisfying, for some \(c > 0\), the following inequality
\[
|z(t, s) - z_0(s)| < ct^{\frac{s}{2}}, \quad s \in \mathbb{R} \setminus \{0\}, \ t > 0.
\]
To this end, we define \(v(s) := (2/\sqrt{3})a^{a,\mu}(s/\sqrt{3})\) for \(s \in \mathbb{R}\). Observe that, by Proposition 7.2 there are a constant \(C \geq 1\) such that
\[
|v(s)|^2 \leq C, \quad |v_0(s)| \leq C, \quad s \geq 0, \quad (8.2)
\]
\[
|v(s)|^2 \leq C, \quad |v_0(s)| \leq C(1 + |s|^{1/4}), \quad s \leq 0. \quad (8.3)
\]
Furthermore, it is not difficult to check that we have the following relations
\[
\omega_{ss} = \omega_s v, \quad \omega_{ss} = \omega_s v + \omega_s v^2. \quad (8.4)
\]
If the function \(g\) is defined by the formula
\[
g(s) = e^{i\mu \ln |s|} \omega(s)/s, \quad s \neq 0,
\]
then, by (7.3) and (8.4), we have
\[
\left( \frac{i\mu - 1}{3} \omega + \frac{s}{3} \omega_s = \omega_{ss} - \frac{3}{2} \omega_s \omega_s = \omega_s v + \omega_s v^2 - \frac{3}{2} \omega_s v^2 \right.
\]
\[
= \omega_s v + \omega_s v^2 - \frac{3}{2} \omega_s v^2 = \omega_s v + \frac{1}{2} \omega_s v^2. \quad (8.6)
\]
Differentiating function \(g\) and using (8.6), we have
\[
g_s(s) = e^{i\mu \ln |s|} \left( \frac{i\mu - 1}{3} \omega + \frac{s}{3} \omega_s \right) = \frac{3}{s^2} \left( v_s - \frac{1}{2} v_s^2 \right) \omega_s e^{i\mu \ln |s|}, \quad (8.7)
\]
which together with (8.2) and (8.3) imply that the function \(g'\) is integrable in the neighborhood of \(\pm \infty\) and
\[
s g_s(s) \to 0, \quad s \to \pm \infty. \quad (8.8)
\]
Consequently the limits \(g(\pm \infty)\) are well-defined and satisfy
\[
g(s) = g(+\infty) - 3 \int_s^{+\infty} \frac{1}{r^2} \left( v_r - \frac{1}{2} v_r^2 \right) \omega_r e^{i\mu \ln |r|} \, dr, \quad s > 0, \quad (8.9)
\]
\[
g(s) = g(-\infty) + 3 \int_{-\infty}^s \frac{1}{r^2} \left( v_r - \frac{1}{2} v_r^2 \right) \omega_r e^{i\mu \ln |r|} \, dr, \quad s < 0. \quad (8.10)
\]
Using (8.7) once again, we obtain
\[
s g_s(s) = e^{i\mu \ln |s|}(i\mu - 1)\omega(s)s^{-1} + e^{i\mu \ln |s|}\omega_s(s)
\]
\[
= -(1 - i\mu)g(s) + e^{i\mu \ln |s|}\omega_s(s), \quad (8.11)
\]
which together with (8.8) gives
\[
|g(\pm \infty)| = (1 + \mu^2)^{-1/2}
\]
and hence, there are \(\theta \pm \in [0, 2\pi)\), such that
\[
g(\pm \infty) = (1 + \mu^2)^{-1/2}e^{i\theta \pm}.
\]
Therefore, by (8.11) and (8.8), we have
\[
\lim_{s \to +\infty} \frac{\omega_s(s)}{\omega_s(-s)} = \lim_{s \to +\infty} \frac{s g_s(s) + (1 - i\mu)g(s)}{(-s) g_s(-s) + (1 - i\mu)g(-s)} = \frac{g(+\infty)}{g(-\infty)}.
\]
Therefore, by the equations (7.4) and (8.1), we have
\[
e^{\iota(\theta - \theta^-)} = \frac{g(\pm \infty)}{g(-\infty)} = \lim_{x \to +\infty} \frac{\omega_+(x)}{\omega_-(x)} = \lim_{x \to +\infty} \exp \left( \frac{2}{3} \int_{-x}^{x} u^{a,\mu}(s/3^\frac{2}{3}) \, ds \right)
\]
\[
= \lim_{x \to +\infty} \exp \left( 2 \int_{-x/3^\frac{1}{3}}^{x/3^\frac{1}{3}} u^{a,\mu}(s) \, ds \right) = e^{2\iota a}.
\]

Moreover, by the equation (8.5), we have
\[
sg(st^{-1/3}) = e^{\iota \mu \ln |st^{-1/3}|} t^{1/3} \omega(st^{-1/3}), \quad s \in \mathbb{R} \setminus \{0\}, \quad t > 0,
\]
which implies that, for any \( s \in \mathbb{R}_\pm \) and \( t > 0 \), we have
\[
|z^{\mu,a}(t, s) - z^{\mu,a}_0(s)| = |e^{-\iota \mu \ln t^{1/3}} \omega \left( \frac{s}{t^{1/3}} \right) - \frac{se^\iota \theta}{\sqrt{1 + \mu^2}} e^{-\iota \mu \ln |s|}| = |sg(st^{-\frac{1}{3}}) - sg(\pm \infty)|.
\]

Therefore, if \( s > 0 \) then combining (8.12), (8.9) and (8.2), gives
\[
|z^{\mu,a}(t, s) - z^{\mu,a}_0(s)| \leq 3s \int_{st^{-1/3}}^{\pm \infty} \left| \frac{1}{r^2} \left( v_r - \frac{1}{2} v^2 \right) \right| \, dr
\]
\[
= 6sC \int_{st^{-1/3}}^{\pm \infty} \frac{1}{r^2} \, dr \leq 6C \frac{s}{st^{-1/3}} = 6Ct^{1/3}, \quad t > 0.
\]

Assume that \( s < 0 \) and \( |st^{-1/3}| \leq 1 \). Then, by (8.12), (8.10) and (8.3), we have
\[
|z^{\mu,a}(t, s) - z^{\mu,a}_0(s)| \leq 3|s| \int_{-\infty}^{st^{-1/3}} \left| \frac{1}{r^2} \left( v_r - \frac{1}{2} v^2 \right) \right| \, dr
\]
\[
\leq 3C|s| \int_{-\infty}^{st^{-1/3}} \frac{2}{r^2} \left( 1 + |r|^{-1/4} \right) \, dr = 6C|s| \left( \frac{1}{|s| |t^{-1/3}|} + \frac{4}{3} \frac{1}{|s| |t^{-1/3}|^\frac{3}{2}} \right)
\]
\[
= 6C|s| \left( \frac{1}{|s| |t^{-1/3}|} + \frac{4}{3} \frac{(|s| |t^{-1/3}|)^{\frac{3}{2}}}{|s| |t^{-1/3}|} \right) \leq 18C|s| \frac{1}{|s| |t^{-1/3}|} = 18Ct^{1/3},
\]
as required. On the other hand, if \( s < 0 \) and \( |st^{-1/3}| \geq 1 \), then, integrating by parts and using (8.4), give
\[
\int_{-\infty}^{\infty} \frac{1}{r^2} \left( v_r - \frac{1}{2} v^2 \right) \omega_r e^{\iota \mu \ln |r|} \, dr
\]
\[
= \frac{v(s)}{s^2} \omega_s e^{\iota \mu \ln |s|} + (2 - \iota \mu) \int_{-\infty}^{\infty} \frac{v}{r^3} \omega_r e^{\iota \mu \ln |r|} \, dr - \frac{3}{2} \int_{-\infty}^{\infty} \frac{v^2}{r^2} \omega_r e^{\iota \mu \ln |r|} \, dr.
\]

(3) We use the notion \( \mathbb{R}_\pm \) for the sets of positive and negative real numbers.
Combining this with (8.12), (8.10) and (8.3), yields
\[
|z^{\alpha}(t, s) - z_0^{\alpha}(s)| \leq 3|s| \int_{-\infty}^{st^{-1/3}} \frac{1}{r^2} \left( v_r - \frac{1}{2} r^2 \right) \omega_r e^{i \mu \ln |r|} dr
\]
\[
\leq 3C|s| \left( \frac{1}{(|s|t^{-1/3})^2} + \frac{1}{2} \frac{(4 + \mu^2)^{1/2}}{|s|t^{-1/3}} \right) dr + 2 \int_{-\infty}^{st^{-1/3}} \frac{1}{r^2} dr
\]
\[
= 3C|s| \left( \frac{1}{(|s|t^{-1/3})^2} + \frac{1}{2} \left( \frac{4 + \mu^2}{|s|t^{-1/3}} \right) \right)
\]
\[
\leq 3C|s| \left( \frac{1}{|s|t^{-1/3}} + \frac{1}{2} \frac{(4 + \mu^2)^{1/2}}{|s|t^{-1/3}} + \frac{2}{|s|t^{-1/3}} \right) \leq ct^{1/3},
\]
where we write \( c := 3C(3 + (4 + \mu^2)^{1/2}) \) and the proof is completed. \( \square \)

Let us denote \( a := (\theta^+ - \theta^-)/2 \). Since the system (7.1) is invariant under rotations (see Remark 7.1), without loss of generality we can assume that \( a \in (-\pi/2, \pi/2) \). By Proposition 8.1 there are \( \theta^+, \theta^- \in [0, 2\pi) \) with
\[
e^{i(\theta^+ - \theta^-)} = e^{i\alpha} e^{i(\theta^+ - \theta^-)}
\]
and a smooth solution \( \tilde{z} \) of the equation (1.17) satisfying
\[
|\tilde{z}(t, s) - \tilde{z}_0(s)| < ct^{1/4}, \quad s \in \mathbb{R} \setminus \{0\}, \quad t > 0,
\]
where \( c > 0 \) is a constant and
\[
\tilde{z}_0(s) = \begin{cases} 
\frac{s}{\sqrt{1 + \mu^2}} e^{i(\theta^+ - \mu \ln s)}, & s > 0, \\
\frac{s}{\sqrt{1 + \mu^2}} e^{i(\theta^- - \mu \ln s)}, & s < 0.
\end{cases}
\]
Using rotation invariance of the system (7.1) once again we infer that the function \( z := e^{-i(\theta^+ - \theta^-)} \tilde{z} \) is also a solution of the system (1.17) and (8.13) gives
\[
|z(t, s) - z_0(s)| = |e^{-i(\theta^+ - \theta^-)} \tilde{z}(t, s) - \tilde{z}_0(s)|
\]
\[
= |\tilde{z}(t, s) - \tilde{z}_0(s)| < ct^{1/4}, \quad s \in \mathbb{R} \setminus \{0\}, \quad t > 0,
\]
which completes the proof of Theorem 1.2 \( \square \)

9. Appendix: solutions for the classical RH problem

Let us consider the contour \( \Sigma \) contained in the complex plane, which is a sum of a finite number of possibly unbounded oriented curves that are smooth in the Riemann sphere. Let us assume that the set \( S \), consisting of the intersection points of these curves, has a finite number of elements and furthermore, assume that the completion \( \mathbb{C} \setminus \Sigma \) has a finite number of connected components. Observe that the contour \( \Sigma \) has the natural orientation determined by the orientations of its component curves. Therefore, for any point of the set \( \Sigma \setminus S \), we can define the (+) and (−) sides of the contour \( \Sigma \) in the usual way. Assume that \( M_{2 \times 2}(\mathbb{C}) \) is the linear space of \( 2 \times 2 \) matrices with complex coefficients, equipped with the Euclidean norm \( \| \cdot \| \). Given a measurable function \( f : \Sigma \to M_{2 \times 2}(\mathbb{C}) \), we define its \( L^2 \) norm as
\[
\| f \|_{L^2(\Sigma)} := \left( \int_{\Sigma} \| f(\lambda) \|^2 |d\lambda| \right)^{1/2}
\]
and write \( L^2(\Sigma) \) for the space of measurable functions \( f \) such that \( \| f \|_{L^2(\Sigma)} < \infty \). If the contour \( \Sigma \) is unbounded, then we define the space \( L^2_1(\Sigma) \) consisting of functions \( f : \Sigma \to M_{2 \times 2}(\mathbb{C}) \) with the property that there is \( f(\infty) \in M_{2 \times 2}(\mathbb{C}) \) such that
$f - f(\infty) \in L^2(\Sigma)$. Using the fact that the contour $\Sigma$ is unbounded, we can easily check that the matrix $f(\infty)$ is uniquely determined and we can set the norm

$$
\|f\|_{L^2(\Sigma)} := \left(\|f - f(\infty)\|_{L^2(\Sigma)}^2 + |f(\infty)|^2\right)^{\frac{1}{2}}, \quad f \in L^2(\Sigma).
$$

Assume that $G : \Sigma \setminus S \to M_{2 \times 2}(\mathbb{C})$ is a map with the property that, for any component $\sigma$ of $\Sigma \setminus S$, the function $G$ has a smooth extension to the closure $\Sigma$. The classical Riemann-Hilbert problem is to find a function $Y(\lambda)$ defined on the complex plane, with values in $M_{2 \times 2}(\mathbb{C})$, such that the following conditions are satisfied.

(a) For any connected component $\Omega$ of $\mathbb{C} \setminus S$, the restriction $Y|_{\Omega}$ is holomorphic on $\Omega$ and has a continuous extension to $\partial \Omega$.

(b) Given $\lambda \in \Sigma \setminus S$, let $Y_+ (\lambda)$ and $Y_-(\lambda)$ be the limits of function $Y(\lambda')$ as the parameter $\lambda'$ approaches $\lambda$ from the left and right side of the contour $\Sigma$, respectively. Then the following jump relation holds

$$
Y_+(\lambda) = Y_-(\lambda)G(\lambda), \quad \lambda \in \Sigma \setminus S.
$$

(c) At $\lambda = \infty$ the function $Y(\lambda)$ has the following asymptotic behavior

$$
Y(\lambda) = I + O(1/\lambda), \quad \lambda \to \infty.
$$

From [15], [17], [28] and [31] we know that the classical Riemann-Hilbert problem has a unique solution if the following assumptions are satisfied.

(i) We have $\det G(\lambda) = 1$ for $\lambda \in \Sigma$.

(ii) Given a connected component $\sigma$ of $\Sigma \setminus S$, there is an open neighborhood $U_\sigma$ of $\sigma$ such that the restriction of $G$ to $\sigma$ has a holomorphic continuation to $U_\sigma$.

(iii) If $\sigma$ is an unbounded connected component of $\Sigma \setminus S$, then there are constants $c > 0$ and $R > 0$ such that

$$
\|G(\lambda) - I\| \leq ce^{-c|\lambda|}, \quad \lambda \in \sigma, \quad |\lambda| > R.
$$

(iv) Let us assume that $a \in S$ and $\Sigma_k$ for $1 \leq k \leq q$ are smooth components of $\Sigma$ intersecting at $a$, which are numbered counter-clockwise. We denote by $G_k(\lambda)$, the restriction of $G$ to a component $\Sigma_k$ and let $U_a$ be a neighborhood of the point $a$ such that the $G_k(\lambda)$ is holomorphic in $U_a$. Then the following smoothness condition is satisfied

$$
G_1^{k_1}(\lambda)G_2^{k_2}(\lambda)\ldots G_q^{k_q}(\lambda) = I, \quad \lambda \in U_a,
$$

where in the term $G_k^{k_1}$ the sign $(+)$ is taken if $\Sigma_k$ is oriented outwards from $a$ and the sign $(-)$ is chosen if $\Sigma_k$ is oriented inwards toward $a$.

The study of the problem $(a) - (c)$ reduces to the analysis of the following equation

$$
\rho = I + \mathcal{C}_-(\rho(G - I))
$$

on the space $L^2_\Sigma$, where $\mathcal{C}_-$ represents the Cauchy operators on the contour $\Sigma$:

$$
[C_- f](\lambda) := \lim_{\lambda' \to \lambda \pm} \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi)}{\xi - \lambda'} d\xi, \quad \lambda \in \Sigma. \quad (9.2)
$$

In the limit $[9.2]$, the variable $\lambda'$ tends non-tangentially to $\lambda$ from the $(\pm)$-side of $\Sigma$, respectively. Furthermore, if the function $\rho \in L^2_\Sigma$ satisfies the equation $[9.1]$, then the following integral

$$
Y(\lambda) := I + \frac{1}{2\pi i} \int_{\Sigma} \frac{\rho(\xi)(G(\xi) - I)}{\xi - \lambda} d\xi, \quad \lambda \notin \Sigma,
$$

represents the solution of the classical Riemann-Hilbert problem.
Acknowledgements. The study of the author are supported by the MNiSW Iuventus Plus Grant no. 0338/IP3/2016/74.

REFERENCES
[1] M.J. Ablowitz, H. Segur, *Asymptotic solutions of the Korteweg-de Vries equation*, Studies in Appl. Math. 57 (1976/77), no. 1, 13–44.
[2] M.J. Ablowitz, H. Segur, *Asymptotic solutions of nonlinear evolution equations and a Painlevé transcendent*, Physica 3D, (1981), 165–184.
[3] J. Baik, R. Buckingham, J. DiFranco, *Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function*, Comm. Math. Phys. 280 (2008), no. 2, 463–497.
[4] J. Baik, R. Buckingham, J. DiFranco, A. Its, *Total integrals of global solutions to Painlevé II*, Nonlinearity 22 (2009), no. 5, 1021–1061.
[5] H. Bateman, A. Erdelyi, *Higher Transcendental Functions*, McGraw-Hill, NY, 1953.
[6] A.L. Bertozzi, P. Constantin, *Global regularity for vortex patches*, Comm. Math. Phys. 152 (1993), no. 1, 19–28.
[7] J.Y. Chemin, *Persistance de structures geometriques dans les fluids incompressibles bidimensionnels*, Ann. Sci. Ecole Norm. Sup. (4) 26 (1993), no. 4, 517–542.
[8] T. Claeys, A.B. Kuijlaars, M. Vanlessen, *Multi-critical unitary random matrix ensembles and the general Painlevé II equation*, Ann. of Math. (2) 168 (2008), no. 2, 601–641.
[9] P.A. Clarkson, J.B. McLeod, *A connection formula for the second Painlevé transcendent*, Arch. Rational Mech. Anal. 103 (1988), no. 2, 97–138.
[10] L.S. Da Rios, *On the motion of an unbounded fluid with a vortex filament of any shape*, Rend. Circ. Mat. Palermo 22 (1906), 117-135.
[11] F. de la hoz, *Numerical study of a flow of regular planar curves that develop singularities at finite time*, SIAM J. Appl. Math. 70:1 (2009), 279-301.
[12] D. Dai, W. Hu, *Connection formulas for the Ablowitz-Segur solutions of the inhomogeneous Painlevé II equation*, Nonlinearity 30 (2017), no. 7, 2982–3009.
[13] P. Deift, X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*, Ann. of Math. (2) 137 (1993), no. 2, 295–368.
[14] P. Deift, X. Zhou, *Asymptotics for the Painlevé II equation*, Comm. Pure Appl. Math. 48 (1995), no. 3, 277–337.
[15] P. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant Lecture Notes in Mathematics, 3. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.
[16] H. Flaschka, A.C. Newell, *Monodromy- and spectrum-preserving deformations I*, Comm. Math. Phys. 76 (1980), no. 1, 65–116.
[17] A.S. Fokas, A.R. Its, A. Kapaev, V. Novokshenov, *Painlevé transcendents. The Riemann-Hilbert approach*, Mathematical Surveys and Monographs, 128. American Mathematical Society, Providence, RI, 2006.
[18] R.E. Goldstein, D.M. Petrich, *Solitons, Euler’s equation, and vortex patch dynamics*, Phys. Rev. Lett. 69 (1992), no. 4, 555–558.
[19] Y. Gromak, I. Laine, S. Shimomura, *Painlevé differential equations in the complex plane*, De Gruyter Studies in Mathematics, 28. Walter de Gruyter & Co., Berlin, 2002.
[20] H. Flaschka, A.C. Newell, *Monodromy- and spectrum-preserving deformations I*, Comm. Math. Phys. 76 (1980), no. 1, 65–116.
[21] A.R. Its, A.A. Kapaev, V. Novokshenov, *Painlevé transcendents. The Riemann-Hilbert approach*, Mathematical Surveys and Monographs, 128. American Mathematical Society, Providence, RI, 2006.
[22] M. Jimbo, M. Tetsuji, K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients Physica D (1981) 306–362.
[23] N. Joshi, M. Mazzocco, *Existence and uniqueness of tri-tronquée solutions of the second Painlevé hierarchy*, Nonlinearity, 16 (2003), no. 2, 427439.
[24] A.A. Kapaev, *Quasi-linear stokes phenomenon for the second Painlevé transcendent*, Nonlinearity 16 (2003), no. 1, 363–386.
[25] M. Jimbo, M. Tetsuji, K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients Physica D (1981) 306–362.
[26] N. Joshi, M. Mazzocco, *Existence and uniqueness of tri-tronquée solutions of the second Painlevé hierarchy*, Nonlinearity, 16 (2003), no. 2, 427439.
[27] A.A. Kapaev, *Quasi-linear stokes phenomenon for the Painlevé first equation*, J. Phys. A 37 (2004), no. 46, 11149–11167.
[28] A.J. Majda, A.L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.
[29] P.D. Miller, *On the increasing tri-tronquée solutions of the Painlevé-II equation*, arXiv:1804.00717v, 2018.
[30] G. Perelman, L. Vega, *Self-similar planar curves related to modified Korteweg-de Vries equation*, J. Differential Equations 235 (2007), no. 1, 56–73.
[31] T. Trogdon, S. Olver, *Riemann-Hilbert problems, their numerical solution, and the computation of nonlinear special functions*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016.
[29] V.I. Yudovitch, Non-stationary flow of an ideal incompressible liquid. USSR Comput. Math. Math. Phys. 3 (1963), 1407–1456

[30] N.J. Zabusky, M.H. Hughes, K.Y. Roberts, Contour dynamics for the Euler equations in two dimensions, J. Comput. Phys. 30 (1979), no. 1, 96–106.

[31] X. Zhou, The Riemann-Hilbert problem and inverse scattering SIAM J. Math. Anal. 20 (1989) No 4, 966-986.

Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18, 87-100 Toruń, Poland
E-mail address: pkokocki@mat.umk.pl