Quantization of Donaldson-Uhlenbeck-Yau theory

S.L. Lyakhovich and A.A. Sharapov

Department of Quantum Field Theory, Tomsk State University,
Tomsk 634050, Russia
e-mail: sll@phys.tsu.ru, sharapov@phys.tsu.ru

Abstract

A covariant path-integral quantization is proposed for the non-Lagrangian gauge theory described by the Donaldson-Uhlenbeck-Yau equation. The corresponding partition function is shown to admit a nice path-integral representation in terms of the gauged G/G Kähler WZW model. A relationship with the $J$-formulation of the anti-self-dual Yang-Mills theory is explored.

1 Introduction

A considerable number of fundamental field-theoretical models do not admit any natural Lagrangian formulation. The list of examples includes self-dual YM fields, chiral bosons, higher-spin gauge theories, Siberg-Witten equations, superconformal theories with extended supersymmetry, etc. The absence of a Lagrangian poses a subtle question about a partition function one should use to path-integral quantize a classical theory defined solely by equations of motion. Obviously, this question cannot be answered in intrinsic terms, one or another extra structure is needed over and above the equations of motion. In the Lagrangian case, it is the action $S$ that plays the role of such a structure and the corresponding partition function is taken to be the Feynman probability amplitude $\Psi = e^{iS}$.

In our recent papers [1], [2], [3], a new concept of a Lagrange anchor was introduced with the aim to extend the usual path-integral quantization technique beyond the scope of Lagrangian mechanics. As a rough guide, the Lagrange anchor is a geometric structure on the configuration space of fields that allows one to construct a generalized Schwinger-Dyson equation for the partition function $\Psi$ under far less restrictive assumptions than the existence of action. Given a partition function and a suitable integration measure, one can compute the quantum averages of physical observables that suffices, in principle, to reconstruct the whole quantum theory [1]. In [3], the

\[\text{1Notice that the notion of a Lagrange anchor has a phase space counterpart allowing one to define an associative }\]

\[\ast\text{-product and quantum equations of motion on the space of physical observables [4].}\]
proposed quantization scheme was exemplified by quantizing the Maxwell electrodynamics with electric and magnetic currents and chiral bosons in various dimensions.

In the present paper, we apply this general method to quantize the nonlinear gauge theory whose classical dynamics are governed by the Donaldson-Uhlenbeck-Yau (DUY) equation [5], [6]. The DUY equation plays a prominent role both in physics and mathematics. On the mathematics side, it provides a differential-geometric tool for describing the moduli space of stable holomorphic vector bundles over a Kähler manifold, the problem that is of considerable interest to algebraic geometers. In physics, this equation is of critical importance for the paradigm of heterotic string compactification on a Calabi-Yau manifold [7]. Also notice the fact that in four dimensions, the DUY equation reduces to the anti-self-dual Yang-Mills equation.

An ad hoc method for quantizing the DUY theory was proposed some while ago by Nair and Schiff [8]. The key step of their approach is the reformulation of the DUY theory as a Lagrangian topological field theory in one higher dimension, the so-called Kähler-Chern-Simons (KCS) theory. The quantum reduction by the first and second class constraints arising in the KCS theory induces then a covariant quantization of the original DUY theory. The work [8] also reveals a deep interrelation between the DUY theory and higher-dimensional counterparts of the WZW model. In the present paper, the DUY model is quantized by the systematic method proposed in [1],[2],[3] for general non-Lagrangian gauge theories. Although technically and methodologically our approach is significantly different from that by Nair and Schiff, the final results seem to agree with each other. In particular, we derive a path-integral representation for the partition function of the quantum DUY theory in terms of a gauged $G/G$ WZW-like model on a Kähler manifold and establish its connection with the so-called $J$-formulation of the DUY theory proposed in [8].

2 A generalized Schwinger-Dyson equation

In this section, we give a short and simplified presentation of general quantization method [2], [3] in the form convenient for applying to the Yang-Mills type models.

In the covariant formulation of quantum field theory one usually studies the path integrals of the form

$$\langle O \rangle = \int [d\varphi] O e^{\frac{-i}{\hbar}S}.$$  \hspace{1cm} (1)

After normalization, the integral defines the quantum average of an observable $O[\varphi]$ in the theory with action $S[\varphi]$. It is believed that evaluating the path integral (1) for various reasonable observables $O$, one can extract all the physically relevant information about the quantum dynamics of the model.

The partition function $\Psi[\varphi] = e^{\frac{i}{\hbar}S}$, weighting the contribution of a particular field configuration $\varphi$ to the quantum average, is known as the Feynman probability amplitude. This amplitude can be
defined as a unique (up to a normalization factor) solution to the Schwinger-Dyson (SD) equation
\[ \left( \frac{\partial S}{\partial \varphi^i} + i\hbar \frac{\partial}{\partial \varphi^i} \right) \Psi[\varphi] = 0. \]  \hfill (2)

Performing the Fourier transform from the fields \( \varphi \) to their sources \( J \), we can bring (2) to a more familiar form
\[ \left( \frac{\partial S}{\partial \varphi^i}(\hat{\varphi}) - J_i \right) Z[J] = 0, \quad \hat{\varphi}^i \equiv i\hbar \frac{\partial}{\partial J_i}, \]  \hfill (3)

where
\[ Z[J] = \int [d\varphi] e^{i(S - J\varphi)} \]  \hfill (4)
is the generating functional of Green’s functions.

To guess how the Schwinger-Dyson equation could be generalized to a theory whose classical equations of motion do not admit a variational formulation, it might be instructive to start with the following simple observations:

(i) Although the Feynman probability amplitude involves an action functional, the SD equation contains solely the equations of motion, not the action as such.

(ii) In the classical limit \( \hbar \to 0 \), the second term in the SD equation (2) vanishes and the Feynman probability amplitude \( \Psi \) turns to the Dirac distribution supported at the classical solutions to the field equations. Formally, \( \Psi[\varphi] \big|_{\hbar \to 0} \sim \delta[\partial_i S] \) and one can think of the last expression as the classical partition function [9].

(iii) It is quite natural to treat the sources \( J \) as the momenta canonically conjugate to the fields \( \varphi \), so that the only non-vanishing Poisson brackets are \( \{ \varphi^i, J_j \} = \delta^i_j \). Then one can regard the SD operators as resulting from the canonical quantization of the first class constraints \( \Theta_i = \partial_i S - J_i \approx 0 \) on the phase space of fields and sources. Upon this interpretation, the Feynman probability amplitude describes a unique physical state of a first-class constrained theory. This state is unique as the “number” of the first class constraints \( \Theta_i \) equals the “dimension” of the configuration space of fields. Quantizing the constrained system in the momentum representation yields the SD equation (3) for the generating functional of Green’s functions.

The above interpretation of the SD equations as operator first class constraints on a physical wave-function suggests a direct way to their generalization. Namely, consider a set of field equations
\[ T_a(\varphi^i) = 0, \]  \hfill (5)
which do not necessarily come from the variational principle. In this case the (discrete parts of) superindices \( a \) and \( i \) may run over completely different sets. Proceeding from the heuristic arguments above, we can take the following ansatz for the \( \varphi J \)-symbols of the Schwinger-Dyson operators:
\[ \Theta_a = T_a(\varphi) - V^i_a(\varphi)J_i + O(J^2). \]  \hfill (6)
The symbols are defined as formal power series in momenta (sources) \( J \) with leading terms being the classical equations of motion. Requiring the Hamiltonian constraints \( \Theta_a \approx 0 \) to be first class, i.e.,

\[
\{\Theta_a, \Theta_b\} = U_{ab}^c \Theta_c, \quad U_{ab}^c(\varphi, J) = C_{ab}^c(\varphi) + O(J),
\]
we obtain an infinite set of relations on the expansion coefficients of \( \Theta_a \). In particular, examining the involution relations (7) to the leading order in \( J \), we find

\[
V_i^a \partial_i T_b = G_{ab} + C_{ab}^c T_c
\]
for some structure functions

\[
G_{ab}(\varphi) = G_{ba}(\varphi), \quad C_{ab}^c(\varphi) = -C_{ba}^c(\varphi).
\]

The value \( V_i^a(\varphi) \), being defined by relation (8), is called the Lagrange anchor. Under the standard regularity conditions on the field equations (5), any first order solution to (7), determined by the Lagrange anchor \( V \), has a prolongation to all orders in \( J \) [2]. The symmetric matrix \( G_{ab} \) is called the generalized Van Vleck matrix.

For variational field equations, \( T_a = \partial_i S \), one can set the Lagrange anchor to be the unit matrix \( V_i^a = \delta_i^a \). This choice results in the standard Schwinger-Dyson operators (2, 3) obeying the abelian involution relations. Generally, the Lagrange anchor may be field-dependent and/or noninvertible. If the Lagrange anchor is invertible, in which case the number of equations must coincide with the number of fields, then the operator \( V^{-1} \) plays the role of integrating multiplier in the inverse problem of calculus of variations. So, the existence of the invertible Lagrange anchor amounts to the existence of action. The other extreme choice, \( V = 0 \), is always possible and corresponds to a pure classical probability amplitude \( \Psi[\varphi] = \delta[T_a(\varphi)] \) supported at classical solutions. Any nonzero Lagrange anchor, be it invertible or not, yields a “fuzzy” partition function describing nontrivial quantum fluctuations in the directions spanned by the vector fields \( V_a = V_i^a \partial_i \).

In the non-Lagrangian case, the constraints (6) are not generally the whole story. The point is that the number of (independent) field equations can happen to be less than the dimension of the configuration space of fields. In that case, the field equations (5) do not specify a unique solution with prescribed boundary conditions or, stated differently, the system enjoys a gauge symmetry generated by some on-shell integrable vector distribution \( R_\alpha = R_i^\alpha(\varphi) \partial_i \). To allow for the gauge invariance at the quantum level, one has to introduce the additional first class constraints in the phase space of fields and sources

\[
R_\alpha = R_i^\alpha(\varphi) J_i + O(J^2) \approx 0.
\]

The leading terms of these constraints coincide with the \( \varphi J \)-symbols of the gauge symmetry generators and the higher orders in \( J \) are determined from the requirement that the whole set of constraints \( \Theta_I = (T_a, R_\alpha) \) to be the first class. With all the gauge symmetries included, the

\footnote{For a Lagrangian gauge theory we have \( T_i = \partial_i S - J_i \) and \( R_\alpha = -R_i^\alpha T_i = R_i^\alpha J_i \). In this case, one may omit the “gauge” constraints \( R_\alpha \approx 0 \) as they are given by linear combinations of the “dynamical” constraints \( T_i \approx 0 \).}
constraint surface \( \Theta_I \approx 0 \) is proved to be a Lagrangian submanifold in the phase space of fields and sources and the gauge invariant partition function is defined as a unique solution to the generalized SD equation

\[
\hat{\Theta}_I \Psi = 0 .
\] (11)

The last formula is just the definition of a physical state in the Dirac quantization method \[10\]. A more systematic treatment of the generalized SD equation within the BFV-BRST formalism can be found in \[1\], \[2\].

In practice, it can be a problem to explicitly derive the probability amplitude from the SD equation (11), especially in nonlinear field theories. In many interesting cases the amplitude \( \Psi[\varphi] \) is given by an essentially nonlocal functional. More precisely, it can be impossible to represent \( \Psi \) as a (smooth) function of any local functional of fields (by analogy with the Feynman probability amplitude \( e^{i\hbar S} \) in a local theory with action \( S \)) even though the SD equations (11) are local. Fortunately, whatever the field equations and Lagrange anchor may be, it is always possible to write down a path-integral representation for \( \Psi \) in terms of some enveloping Lagrangian theory. By now, two such representations are known. The first one, proposed in \[1\], exploits the equivalence between the original dynamical system described by the classical equations of motion \( T_a = 0 \) and the Lagrangian theory with action

\[
S[\varphi, J, \lambda] = \int_0^1 dt (\dot{\varphi}^iJ_i - \lambda^a\Theta_a) .
\] (12)

The latter can be regarded as a Hamiltonian action of topological field theory on the space-time with one more (compact) dimension \( t \in [0, 1] \). The solution to the SD equation (11) can be formally represented by the path integral

\[
\Psi[\varphi_1] = \int \left[ d\varphi \right] [dJ][d\lambda] e^{i\hbar S[\varphi, J, \lambda]} ,
\] (13)

where the sum runs over all trajectories with \( \varphi(1) = \varphi_1 \) and \( J(0) = J(1) = 0 \). In \[3\], we used such a representation to perform a covariant quantization of the chiral bosons in \( d = 4n + 2 \) dimensions in terms of the \( (4n + 3) \)-dimensional Chern-Simons theory.

An alternative approach to constructing a path-integral representation for \( \Psi \) is the augmentation method \[3\]. With this method, one augments the original configuration space of fields \( \varphi^i \) with the new fields \( \xi^a \), called the augmentation fields, and defines the action

\[
S_{\text{aug}}[\varphi, \xi] = \xi^aT_a(\varphi) + G_{ab}(\varphi)\xi^a\xi^b + O(\xi^3) ,
\] (14)

where \( G_{ab} \) is given by \[8\], and the higher orders in \( \xi \) are determined from the condition that the (partially averaged) amplitude

\[
\Psi[\varphi] = \int [d\xi] e^{i\hbar S_{\text{aug}}[\varphi, \xi]}
\] (15)

obeys the SD equation (11). There is also a simple recursive algorithm allowing one to reconstruct (14) up to any order in \( \xi \)’s \[3\]. Notice that unlike the topological model (12), the augmented theory...
is not classically equivalent to the original (non-)Lagrangian theory. So, the augmentation fields should not be confused with a somewhat similar concept of “auxiliary fields” [11]. With the amplitude (15), the quantum average of an observable $O$ can be written as

$$
\langle O \rangle = \int [d\varphi] O[\varphi] \Psi[\varphi] = \int [d\varphi] [d\xi] O[\varphi] e^{\pm S_{\text{aug}}[\varphi, \xi]} .
$$

(16)

It is significant that the action $S_{\text{aug}}$ is given by a local functional whenever the Lagrange anchor and the equations of motion are local. In that case, the integral (16) is similar in structure to (1), so the usual field-theoretical tools of the Lagrangian theory can be still applied to evaluate the quantum averages.

### 3 Lagrange anchor for DUY theory

Let $\mathcal{E} \to M$ be a holomorphic $G$-vector bundle over a $2n$-dimensional Kähler manifold $M$ with the Kähler 2-form $\omega$. We take $G$ to be a compact Lie group and denote by $\mathcal{G}$ its Lie algebra. Consider a linear connection $A$ on $\mathcal{E}$. As any of the 2-forms on a complex manifold, the curvature $F$ of the connection is decomposed into the sum of the $\mathcal{G}$-valued $(2,0)$, $(0,2)$, and $(1,1)$-forms on $M$. The Donaldson-Uhlenbeck-Yau equations read

$$
F^{(2,0)} = 0 , \quad F^{(0,2)} = 0 , \quad \omega^{n-1} \wedge F^{(1,1)} = 0 .
$$

(17)

(18)

The first two equations just mean that the connection is holomorphic and the last condition is equivalent to the stability of the holomorphic vector bundle in algebraic geometry [5], [6].

For $n = 1$ equations (17, 18) reduce to a single zero curvature condition $F = 0$. Setting $n = 2$ one obtains three independent equations that are equivalent to the anti-self-duality condition for the curvature 2-form $F$. Since the solutions to the DUY equations constitute a part of solutions to the corresponding Yang-Mills equations in any dimension, one may regard (17, 18) as a higher-dimensional generalization of the anti-self-dual YM theory in four dimensions.

Note that equations (17, 18), being gauge invariant, are linearly independent, so no Noether identities are possible. This property is a particular manifestation of a non-Lagrangian nature of the DUY equations. In a Lagrangian theory, any gauge symmetry gives rise to a Noether identity and vice versa. Although the DUY equations are not Lagrangian, they admit a good Lagrange anchor that leads, as we will see, to a reasonable quantum theory.

Denote by $\mathcal{A}$ the affine space of all connections on $\mathcal{E}$. Locally, any connection on $\mathcal{E}$ is represented by a pair of $(1,0)$ and $(0,1)$-forms $(A, \bar{A})$ valued in the Lie algebra $\mathcal{G}$. In terms of the gauge potentials $A$ and $\bar{A}$, the homogeneous components of the curvature $F$ read

$$
F^{(2,0)} = \partial A + A \wedge A , \quad F^{(0,2)} = \bar{\partial} \bar{A} + \bar{A} \wedge \bar{A} , \quad F^{(1,1)} = \partial \bar{A} + \bar{\partial} A + A \wedge \bar{A} + \bar{A} \wedge A ,
$$

(19)
where $\partial$ and $\bar{\partial}$ are holomorphic and anti-holomorphic parts of the de Rham differential $d$. The canonical symplectic structure on the cotangent bundle of $\mathcal{A}$ reads

$$\Omega = \int_M \text{Tr}(\delta A \wedge \delta P) + \int_M \text{Tr}(\delta \bar{A} \wedge \delta \bar{P}) ,$$

with $P$ and $\bar{P}$ being, respectively, $(n-1, n)$ and $(n, n-1)$-forms on $M$ with values in $\mathcal{G}$. As we have explained in Sec.2, one may regard the fields $P$ and $\bar{P}$, playing the role of canonical momenta, as the sources to the gauge fields $A$ and $\bar{A}$. Following the general prescription of Sec.2, we introduce the corresponding set of first class constraints (6) on the phase space of fields and sources:

$$T_0 = \omega^{n-1} \wedge F^{(1,1)} + k(DP - \bar{D}\bar{P}) \approx 0 ,$$

$$T_+ = F^{(2,0)} \approx 0 , \quad T_- = F^{(0,2)} \approx 0 ,$$

$$R = DP + \bar{D}\bar{P} \approx 0 .$$

Here $D$ and $\bar{D}$ are the covariant differentials associated with the gauge fields $A$ and $\bar{A}$ and $k$ is a complex parameter.

Let us comment on the structure of the constraints (21). The constraint $T_0 \approx 0$ is just a one-parameter deformation of the classical stability condition (18) by the momenta dependent term. According to our terminology, this term defines (and is defined by) a Langrange anchor compatible with the classical equations of motion (17, 18). The rest of the DUY equations, namely the holomorphy conditions (17), remain intact and define the holonomic constraints $T_\pm \approx 0$ on the phase space of fields and sources. In physical terms, this means that the quantum fluctuations are nontrivial only for that part of classical dynamics which is governed by the stability condition. Finally, the constraint $R$ reflects the presence of gauge symmetries. The Hamiltonian action of $R$ induces the standard gauge transformations on the configuration space of fields $\mathcal{A}$. Taken together, the Hamiltonian constraints (21) define a topological field theory (12) on the cotangent bundle of $\mathcal{A}$, which is found to be classically equivalent to the original non-Lagrangian dynamics (17, 18).

To describe the Poisson algebra of the first class constraints, it is convenient to interpret them as linear functionals (de Rham’s currents) on an appropriate space of $\mathcal{G}$-valued forms. Define

$$T_0(\varepsilon_0) = \int_M \text{Tr}(\varepsilon_0 \wedge T_0) , \quad T_\pm(\varepsilon_\pm) = \int_M \text{Tr}(\varepsilon_\pm \wedge T_\pm) , \quad R(\varepsilon) = \int_M \text{Tr}(\varepsilon \wedge R) ,$$

where $\varepsilon_0$, $\varepsilon_\pm$, and $\varepsilon$ are gauge parameters whose form degrees are complementary to the degrees of corresponding constraints. The Poisson brackets of the constraints read

$$\{T_0(\varepsilon), T_0(\varepsilon')\} = k^2 R([\varepsilon, \varepsilon']) , \quad \{R(\varepsilon), R(\varepsilon')\} = R([\varepsilon, \varepsilon']) ,$$

$$\{R(\varepsilon), T_0(\varepsilon')\} = T_0([\varepsilon, \varepsilon']) , \quad \{R(\varepsilon), T_\pm(\varepsilon')\} = T_\pm([\varepsilon, \varepsilon']) ,$$

$$\{T_0(\varepsilon), T_\pm(\varepsilon')\} = \pm k T_\pm([\varepsilon, \varepsilon']) , \quad \{T_\pm(\varepsilon), T_\pm(\varepsilon')\} = 0 .$$

Upon canonical quantization the first class constraints (21) turn to the Schwinger-Dyson operators

$$\hat{T}_0 = \omega^{n-1} \wedge F^{(1,1)} - ik \left( D \frac{\delta}{\delta A} - \bar{D} \frac{\delta}{\delta A} \right), \quad \hat{R} = -i \left( D \frac{\delta}{\delta A} + \bar{D} \frac{\delta}{\delta A} \right),$$

(24)

$$\hat{T}_+ = F^{(2,0)}, \quad \hat{T}_- = F^{(0,2)}.$$  

(25)

The partition function $\Phi$ on the configuration space of fields $\mathcal{A}$ is now defined as a unique (up to a multiplicative constant) functional annihilated by all the operator constraints (24, 25). Imposing the operators of holonomic constraints (25) yields the following expression for the partition function:

$$\Phi = \delta[T_+] \delta[T_-] \Psi,$$

(26)

where the function $\Psi[A, \bar{A}]$ is annihilated by the residuary constraints (24),

$$\hat{R} \Psi = 0, \quad \hat{T}_0 \Psi = 0.$$

(27)

The first equation just says that $\Psi$, and hence $\Phi$, are gauge invariant functionals of $A$ and $\bar{A}$. The second equation is the quantum counterpart of the stability condition (18); being nonlinear, it is the most challenging equation to solve. One of the complications in solving (27) is that there is no way to represent $\Psi$ as a function of any local functional of the fields $A$ and $\bar{A}$. Nonetheless, by making use the augmentation method, we can construct a path-integral representation for $\Psi$ in terms of a local action functional on an augmented configuration space. By definition, the augmentation fields take values in the space dual to the space of equations of motion. Therefore, we extend the original configuration space of fields $\mathcal{A}$ by introducing the set of new fields $\xi = (\Lambda, \bar{\Lambda}, B)$, where $\Lambda$ and $\bar{\Lambda}$ are $G$-valued $(n - 2, n)$ and $(n, n - 2)$-form fields, respectively, and $B$ is a $G$-valued scalar field on $M$. Then up two the first order in $\xi$’s the action (14) reads

$$S_{\text{aug}} = S[A, \bar{A}, B] + \int_M \text{Tr}(\Lambda \wedge F^{(2,0)} + \bar{\Lambda} \wedge F^{(0,2)}),$$

(28)

where

$$S[A, \bar{A}, B] = \int_M \omega^{n-1} \wedge \text{Tr}(BF^{(1,1)}) + O(B^2).$$

(29)

Note that the fields $\Lambda$ and $\bar{\Lambda}$ enter the action $S_{\text{aug}}$ only linearly because the corresponding constraints are holonomic. The integration over these fields by formula (15) simply reproduces the delta-functions in (26). So we can focus our attention on the action (29). Applying the general procedure from [3] allows one, in principle, to reconstruct (29) up to any order in $B$. As a practical matter, it is better to work in terms of the group valued field $g = e^B$ rather than the Lie algebra valued field $B$. With the field $g$, we are able to present a closed expression for (29), which appears to be nothing but a gauged version of the Kähler WZW model [8], [12] (KWZW model for short). The details of the construction are exposed in the next section.

\[3\]Hereinafter we set $\hbar = 1.$
4 The DUY equation and gauged G/G KWZW model

Our staring point is the action of the KWZW model associated with the Kähler manifold \((M, \omega)\) and the Lie group \(G\). The basic field of the model is a smooth mapping \(g : M \to G\) and the action is

\[
S_0[g] = \frac{1}{2} \int_M \omega^{n-1} \wedge \text{Tr}(g^{-1} \partial g \wedge g^{-1} \bar{\partial} g) - \frac{1}{6} \int_{M \times I} \bar{\omega}^{n-1} \wedge \text{Tr}(\tilde{g}^{-1} \partial \tilde{g} \wedge \tilde{g}^{-1} \bar{\partial} \tilde{g} \wedge \tilde{g}^{-1} \bar{\partial} \tilde{g}).
\] (30)

In this expression, \(I = [0, 1] \subset \mathbb{R}\), \(\tilde{g}\) denotes an extension of \(g\) to the product manifold \(M \times I\), and \(\tilde{\omega}\) is the pull-back of the Kähler form \(\omega\) with respect to the canonical projection \(M \times I \to M\). More precisely, we identify one boundary component of \(M \times I\), say \(M \times \{1\}\), with the original space \(M\) and extend \(g\) to \(M \times I\) in such a way that it tends to some fixed field \(g_0\) on the other component of the boundary \(M \times \{0\}\); in so doing, the field \(g\) falls into the same homotopy class as \(g_0\). Hereafter we assume that \(g\) takes values in any unitary irreducible representation of \(G\) and \(\text{Tr}\) is the ordinary matrix trace.

In case \(n = 1\) the action (30) reduces to the familiar action of the WZW model [14]. Like its two-dimensional prototype, the higher-dimensional KWZW model enjoys the infinite-dimensional symmetry

\[
g \to h_1 g h_2, \tag{31}\]

\(h_{1,2}\) being holomorphic mappings to \(G\), which can be easily seen from the \(2n\)-dimensional analog of the Polyakov-Wiegmann formula

\[
S_0[gh] = S_0[g] + S_0[h] + \int_M \omega^{n-1} \wedge \text{Tr}(g^{-1} \partial g \wedge \bar{\partial} h h^{-1}). \tag{32}\]

In particular, the model is invariant under the adjoint action of \(G\): \(g \to hgh^{-1}, \forall h \in G\).

The KWZW model possesses many other interesting properties and, as we show below, it results as a part of the augmented action for the DUY model. Recall that in our treatment the field \(g\) is identified with the augmentation field for the non-Lagrangian equation (18). The total action (29) is obtained from (30) by “gauging” the adjoint action of \(G\) through introduction of the minimal coupling with the gauge fields \((A, \bar{A})\). The construction is patterned after the much studied two-dimensional case [13] and results in the following action for the gauged G/G KWZW model:

\[
S[g, A, \bar{A}] = S_0[g] + \int_M \omega^{n-1} \wedge \text{Tr}(A \wedge \bar{\partial} g g^{-1} - g^{-1} \partial g \wedge \bar{A} + A \wedge \bar{A} - A \wedge g \bar{A} g^{-1}). \tag{33}\]

One can easily see that the action is invariant under the infinitesimal gauge transformations

\[
\delta g = [u, g], \quad \delta A = -Du = -\partial u - [A, u], \quad \delta \bar{A} = -\bar{D}u = -\bar{\partial} u - [\bar{A}, u]. \tag{34}\]
The total variation of (33) is given by
\[ \delta S = \delta g S + \delta A S + \delta \bar{A} S, \]
\[ \delta g S = \int_M \omega^{n-1} \wedge \text{Tr} \left[ g^{-1} \delta g (F(1,1) + \bar{D}(g^{-1} D g)) \right] = \int_M \omega^{n-1} \wedge \text{Tr} \left[ \delta g g^{-1} (F(1,1) - D(\bar{D} g g^{-1})) \right], \]
\[ \delta A S = \int_M \omega^{n-1} \wedge \text{Tr} (\delta A \wedge \bar{D} g g^{-1}), \quad \delta \bar{A} S = \int_M \omega^{n-1} \wedge \text{Tr} (\delta \bar{A} \wedge g^{-1} D g). \]
(35)

So the classical equations of motion can be written as
\[ \omega^{n-1} \wedge F(1,1) = 0, \quad D g = 0, \quad \bar{D} g = 0. \]
(36)

We see that the dynamics of the fields \( A \) and \( \bar{A} \) are completely decoupled from the dynamics of the augmentation field \( g \) and are governed by the DUY equation (18). Such a structure of equations is typical for an augmented theory [3].

Now we claim that the path integral
\[ \Psi[A, \bar{A}] = \int [dg] e^{i 2k S[g, A, \bar{A}]}, \]
(37)
where \([dg]\) is induced by the Haar measure on \( G \), yields a desired solution to the Schwinger-Dyson equations (27). The statement is proved simply by substituting (37) into (27) and differentiating under the integral sign. We have
\[ \hat{T}_0 \Psi = \frac{1}{2} \int_M [dg] \omega \wedge [2 F^{(1,1)} + D(\bar{D} g g^{-1}) - D(g^{-1} D g)] e^{i 2k S[g, A, \bar{A}]} \]
\[ = \int_M [dg] (V_L + V_R) e^{i 2k S[g, A, \bar{A}]} . \]
(38)

Here \( V_L = t_a V_L^a \) and \( V_R = t_a V_R^a \) are the first-order variational operators associated with the basis \( \{V_L^a\} \) and \( \{V_R^a\} \) of the left- and right-invariant vector fields on the gauge group \( \hat{G} \) (the group of maps from \( M \) to \( G \)). These vector fields are completely specified by the relations
\[ i_{V_L^a} (g^{-1} \delta g) = i_{V_R^a} (\delta g g^{-1}) = t_a, \]
(39)
where \( \{t_a\} \) are the generators of the Lie algebra \( G \) with \( \text{Tr}(t_a t_b) = \delta_{ab} \). Since the integration measure \([dg]\) is formally invariant under the action of \( V_L^a \) and \( V_R^a \), we deduce that the integrand in (38) is a total divergence. Assuming that one can integrate by parts in functional space, the right-hand side of (38) vanishes. Although the gauge invariance of the amplitude \( \Psi \) is obvious, it is instructive to verify it directly:
\[ \hat{R} \Psi = \frac{1}{2k} \int_M [dg] (V_L - V_R) e^{i 2k S[g, A, \bar{A}]} = 0, \]
(40)

\footnote{In accordance with our definition of the KWZW action (30), the sum runs over all fields belonging to a fixed homotopy class \([g_0]\).}
by the same reasons as above.

Given the partition function \textbf{(26, 37)}, the vacuum expectation value of a gauge invariant observable \( \mathcal{O}[A, \bar{A}] \) is defined by

\[
\langle \mathcal{O} \rangle = \frac{1}{\text{vol}(\hat{G})} \int [dg][dA][d\bar{A}] \delta[F^{(2,0)}][F^{(0,2)}] e^{\frac{i}{2\pi} S[g, A, \bar{A}]},
\]

where \([dA][d\bar{A}]\) is the translation-invariant measure on the space of all connections \( \mathcal{A} \). Since \( \Psi \) is gauge invariant it is natural to divide by the volume of the gauge group \( \hat{G} \). A more rigorous treatment of the integral within the BV formalism involves the standard gauge-fixing procedure \textbf{[10]}. The expression \textbf{(41)} is in a sense final if not particularly convenient for perturbative calculations because of delta-function factors in the integrand. To bring the path integral \textbf{(41)} into the usual form \textbf{(1)} one can either replace the action \( S[g, A, \bar{A}] \) with \textbf{(28)} and extend integration over the Lagrange multipliers \( \Lambda, \bar{\Lambda} \), or directly solve the holonomic constraints \( T_\pm \approx 0 \) in terms of some unconstrained fields. Observe that locally any solution to \textbf{(17)} is representable in the form

\[
A = h^{-1} \partial h, \quad \bar{A} = -\bar{\partial} h^\dagger (h^\dagger)^{-1},
\]

for some \( G^C \)-valued field \( h \). And vice versa, for any \( h \in \hat{G}^C \), the gauge potentials \textbf{(12)} satisfy the holomorphy conditions \textbf{(17)}. The representation \textbf{(12)} goes back to the work of Yang \textbf{[15]}, where it was originally introduced in the context of anti-self-dual YM fields. On substituting \textbf{(12)} into \textbf{(33)}, we get the action \( S[g, h] \) which is the functional of the \( G \)-valued field \( g \) and \( G^C \)-valued field \( h \) and which is invariant under the gauge transformations

\[
g \rightarrow ugu^{-1}, \quad h \rightarrow hu^{-1}.
\]

Using the PW formula \textbf{(32)}, we can write this action as the difference of two explicitly gauge invariant terms

\[
S[g, h] = S_0[hgh^\dagger] - S_0[hh^\dagger].
\]

Let \( J = hh^\dagger \). The field \( J \) takes values in positive-definite Hermitian matrices. By making use the polar decomposition of a nondegenerate matrix, we can write \( h = \sqrt{J} u \), where \( \sqrt{J} \) is still Hermitian and positive-definite matrix representing the points of the homogeneous space \( G^C/G \), while \( u \in G \) is unitary. Then the Haar measure on \( \hat{G}^C \) is factorized as \([dh] = [du][dJ]\). The integration over the unitary factor \( u \in \hat{G} \) gives just the volume of the gauge group, \( \int [du] = \text{vol}(\hat{G}) \).

Performing the change of variables \textbf{(12)}, we can rewrite \textbf{(41)} as

\[
\langle \mathcal{O} \rangle = \frac{1}{\text{vol}(\hat{G})} \int [dg][dh] \Delta[h] \mathcal{O} e^{\frac{i}{2\pi} S_0[hgh^\dagger] - S_0[hh^\dagger]} = \frac{C}{\text{vol}(\hat{G})} \int [dh] \Delta[h] \mathcal{O} e^{\frac{i}{2\pi} S_0[hh^\dagger]},
\]

where the local measure \( \Delta'[h] \) is defined formally by the relation

\[
[dA][d\bar{A}] = [dF^{(2,0)}][dF^{(0,2)}][dh] \Delta[h]
\]
and
\[ C = \int [dg] e^{i S_0[hgh^\dagger]} = \int [dg] e^{i S_0[g]} . \]  
(47)

(The last equality is just a formal extension to infinite dimensions of the invariance of the Haar integral \( \int_G dg f(h_1 gh_2) = \int_G dg f(g) \), where \( f \) is analytic on \( G_C \) and \( h_1, h_2 \in G_C \).) In the absence of gauge anomalies, the integrand of (45) is to be invariant under the gauge transformation (43) and it is reasonable to assume that \( \Delta = \Delta'[hh^\dagger] \) and \( \mathcal{O} = \mathcal{O}'[hh^\dagger] \). If \( G \) is abelian, then \( \Delta \) is just an essential constant factor. By making use the polar decomposition \( h = \sqrt{J} u \), we finally obtain
\[ \langle \mathcal{O} \rangle = \frac{C}{\text{vol}(G)} \int [du][dJ] \Delta'[J] \mathcal{O}'[J] e^{-i S_0[J]} = C \int [dJ] \Delta'[J] \mathcal{O}'[J] e^{-i S_0[J]} . \]  
(48)

The last integral expresses the quantum average of a gauge invariant observable \( \mathcal{O} \) in terms of the KWZW-like action \( S_0[J] \) and the local measure \( \Delta'[J] \) associated to the homogeneous space \( \hat{G}_C/G \). By construction, the field \( J \) describes the (local) physical modes of the gauge fields \( A \) and \( \bar{A} \). The extremum points of the action \( S_0[J] \) are defined by the equation
\[ \bar{\partial}(\omega^{n-1} \wedge J^{-1} \partial J) = 0 . \]  
(49)

We could also arrive at this equation by simply substituting the holomorphic potentials (42) into the DUY equation (18). In the special case that \( M \) is a four-dimensional Kähler manifold, equation (49) was intensively studied in the past under the name of \( J \)-formulation for the anti-self-dual YM theory (see e.g. [16], [17], [8]).

Acknowledgments

We wish to thank Petr Kazinski for fruitful collaboration at the early stage of this work. The work was partially supported by the RFBR grant 06-02-17352 and the grant for Support of Russian Scientific Schools 1743.2003.2.

References

[1] P.O. Kazinski, S.L. Lyakhovich and A.A. Sharapov, Lagrange Structure and Quantization, JHEP 07 (2005) 076.
[2] S.L. Lyakhovich and A.A. Sharapov, Schwinger-Dyson equation for non-Lagrangian field theory, JHEP 02 (2006) 007.
[3] S.L. Lyakhovich and A.A. Sharapov, Quantizing non-Lagrangian gauge theories: an augmentation method, JHEP 01 (2007) 047.
[4] S.L. Lyakhovich and A.A. Sharapov, BRST theory without Hamiltonian and Lagrangian, JHEP 03 (2005) 011.
[5] S.K. Donaldson, *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. 50 (1985) 1.

[6] K.K. Uhlenbeck and S.T. Yau, *On the existence of hermitian Yang-Mills connections in stable vector bundles*, Commun. Pure Appl. Math. 39 (1986) 257.

[7] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, vol. 2 (Cambridge University Press, 1987).

[8] V.P. Nair and J. Schiff, *Kähler-Chern-Simons Theory and Quantization of Instanton Moduli Spaces*, Phys. Lett. B246 (1990) 423; *Kähler-Chern-Simons Theory and Symmetries of Anti-Self-Dual Gauge Fields*, Nucl. Phys. B371 (1992) 329.

[9] E. Gozzi, *Hidden BRS invariance in classical mechanics*, Phys. Lett. B201 (1988) 525; E. Gozzi, M. Reuter and W.D. Thacker, *Hidden BRS Invariance in Classical Mechanics. 2*, Phys. Rev. D40 (1989) 3363.

[10] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton U.P., NJ, 1992).

[11] M. Henneaux, *Elimination Of The Auxiliary Fields In The Antifield Formalism*, Phys. Lett. B238 (1990) 299.

[12] T. Inami, H. Kanno and T. Ueno, *Higher dimensional WZW Model on Kähler Manifold and Toroidal Lie Algebra*, Mod. Phys. Lett. A12 (1997) 2757.

[13] K. Gawedzki and A. Kupiainen, *G/H conformal field theory from gauged WZW model*, Phys. Lett. B215 (1988) 119; *Coset construction from functional integrals*, Nucl. Phys. B320 (1989) 625.

[14] E. Witten, *Non-Abelian Bosonization in Two Dimensions*, Commun. Math. Phys. 92 (1984) 455.

[15] C.N. Yang, *Condition of Self-Duality for SU(2) Gauge Fields on Euclidean Four-Dimensional Space*, Phys. Rev. Lett. 38 (1977) 1377.

[16] Y. Brihaye, D.B. Fairlie, J. Nuyts and R.G. Yates, *Properties of the selfdual equations for an SU(n) gauge theory*, J. Math. Phys. 19 (1978) 2528.

[17] L.L. Chau, M.-L. Ge and Y.-S. Wu, *Kac-Moody algebra in the self-dual Yang-Mills equation*, Phys. Rev. D25 (1982) 1086.