Some Chaotic Properties of $G$ – Average Shadowing Property

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Abstract

Let $(M, d)$ be a metric $G$–space and $\phi : M \to M$ be a continuous map. The notion of the $G$-average shadowing property ($G$ ASP) for a continuous map on $G$ – space is introduced and the relation between the $G$ ASP and average shadowing property(ASP) is investigated. We show that if $\phi$ has $G$ ASP, then $\phi^m$ has $G$ ASP for every $m \in \mathbb{N}$. We prove that if a map $\phi$ is pseudo-equivariant with dense set of $G_\phi$–periodic points and has the $G$ ASP, then $\phi$ is weakly $G$–mixing. We also show that if $\phi$ is a $G$–expansive pseudo-equivariant homeomorphism that has the $G$ASP and $\phi$ is topologically $G$–mixing, then $\phi$ has a $G$ -specification. We obtained that the identity map $\phi$ on $M$ has the $G$ ASP if and only if the orbit space $M/G$ is totally disconnected. Finally, we show that if $\phi$ is a pseudo-equivariant map, and the trajectory map $\Psi : M \to M/G$ is a covering map, then $\phi$ has the $G$ASP if and only if the induced map $\phi : M/G \to M/G$ has $G$ASP.

Keywords: Shadowing; Average shadowing; G-average shadowing; Topologically G-mixing; Weakly G-mixing; G-specification.

Introduction

The concept of shadowing property is one of the influential notions in the theory of dynamical systems. In 1967 The shadowing property (SP) was introduced by Anosov [1] and the concept of average shadowing property (ASP) was introduced by Blank for investigating chaotic dynamical systems [2]. In 1960, the notion of $G$ – space was introduced by R. S. Palais [3]. The $G$ –pseudo-
trajectory tracing property on a metric $G$-space ($\text{PTTP}$) was introduced by Shah and Das. They studied various properties of such maps and obtained features for the identity map to have $\text{PTTP}$. Also, they showed that a pseudo-equivariant map $\hat{\phi} : M \rightarrow M$ has $\text{PTTP}$ if and only if the induced map $\hat{\phi} : M/G \rightarrow M/G$ has $\text{PTTP}$ such that $M$ be metric $G$-space and $\hat{\phi}$ is continuous map [4].

The $G$-shadowing property ($G\text{ SP}$) for the map $\phi$ was introduced by Shah who observed through the examples that $G$-shadowing relies on the action of a group $G$ acting on $M$. Also, she studied $G$-shadowing for the shift map on the contrary limit space produced by the map $\phi$ [5].

In section 1 of this paper, we study the ASP for continuous maps on $G$-spaces ($G\text{ ASP}$). In section 2, we prove some similar results on the ASP in the metric space with some chaotic properties and we put sufficient conditions to prove these results on $G$-spaces.

Preliminaries

Let $\mathbb{Z}$ denote the set of integers numbers, $\mathbb{N}$ denotes the set of natural numbers and $\mathcal{N}_0 = \{0\} \cup \mathbb{N}$. A topological group is a triple $(G, T, *)$, where $(G, *)$ is a group and $T$ is a Hausdorff topology on $G$ such that the map $f : G \times G \rightarrow G$ defined by $f(x, y) = xy^{-1}$ is continuous. By a $G$-space, we mean a triple $(M, G, \theta)$, where $M$ is a Hausdorff space, $G$ is a topological group, and $\theta: G \times M \rightarrow M$ is a continuous action of $G$ on $M$ satisfying $\theta(e, m) = m$ and $\theta(g_1, \theta(g_2, m)) = \theta(g_1g_2, m)$, where $e$ is the identity of $G$, $m \in M$, and $g_1, g_2 \in G$. An action $\theta$ of $G$ on $M$ is called trivial if $\theta(g, m) = m$, $\forall g \in G$ and $m \in M$.

For $m \in M$, the set $G(m) = \{\theta(g, m) : g \in G\}$ is called the $G$-trajectory of $m \in M$. We will denote $\theta(g, m)$ by $gm$. For $S \subseteq M$, let $gS = \{gs : s \in S\}$ be a subset $S$ of a $G$-space and $M$ is called $G$-invariant if $\theta(g, x) \in X$ for all $g \in G$ and $x \in X$. An action $\theta$ of $G$ on $M$ is said to be trivial if $\theta(g, m) = m$, $\forall g \in G$ and $m \in M$.

In this paper, we denote the metric $G$-space, on which there is a topological group $G$ with metric $d$, by $(M, d)$. Also, by the map $\phi : M \rightarrow M$, we mean $\phi : (M, d) \rightarrow (M, d)$. By $(M, d)$ being a compact metric $G$-space, we mean a compact metric $G$-space on which there is a compact topological group $G$ with metric $d$. If $A$ and $B$ are two non-empty subsets of $M$, then $N_g(A \cap B) = \{i \in \mathbb{N} : g \phi^i(A) \cap B \neq \emptyset\} = \emptyset, g \in G$.

Definition 2.1. Let $(M, d)$ be a compact metric space and let $\phi : M \rightarrow M$ be a continuous map. A sequence $\{m_i, i \in \mathbb{Z}\}$ is called trajectory of $\phi$, if $\forall i \in \mathbb{Z}$, we have $m_{i+1} = \phi(m_i)$ and we called it a $\delta$-pseudo-trajectory of $\phi$, $\forall i \in \mathbb{Z}$. We have $d(\phi(m_i), m_{i+1}) \leq \delta$, and the map $\phi$ has the shadowing property if $\forall \varepsilon > 0, \exists \delta > 0$, such that every $\delta$-pseudo-trajectory $\{m_i, i \in \mathbb{Z}\}$ is $\varepsilon$-shadowed by the trajectory $\{\phi^i(m), i \in \mathbb{Z}\}$ for some $z \in M$, that is, $\forall i \in \mathbb{Z}$, thus we have $d(\phi^i(z), m_i) \leq \varepsilon$.

A sequence $\{m_i, i \in \mathbb{Z}\}$ in $M$ is called a $\delta$-average pseudo-trajectory of $M$ if $\exists N \in \mathbb{N}$ and $N = N(\delta)$, such that $\forall n \geq N$, and $k \in \mathbb{N}$, then

$$\frac{1}{n} \sum_{i=0}^{n-1} d(\phi^{i+k}(m), m_{i+k+1}) < \delta.$$ 

The map $\phi$ has the ASP if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that every $\delta$-average pseudo-trajectory $\{m_i, i \in \mathbb{Z}\}$ is $\varepsilon$-shadowed in average by the trajectory of some point $z \in M$, that is

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi^i(z), m_i) < \varepsilon.$$
Definition 2.2. [5]

Let \((M, d)\) be metric \(G\)–space and let \(\phi : M \to M\) be continuous map. For a positive real number \(\delta\), a sequence of points \(\{m_i : a < i < b\}\) in \(M\) is called \((\delta, G)\) –pseudo- trajectory for \(\phi\), if for all \(i, a < i < b - 1, \exists g_i \in G\) such that \(d(g_i \phi(m_i), m_{i+1}) < \delta\).

For a given \(\varepsilon > 0\), \((\delta, G)\) –pseudo-trajectory \(\{m_i : a < i < b\}\) for \(\phi\) is called \(\varepsilon\)–shadowed by a point \(m\) of \(M\), if \(\forall i, a < i < b, \exists p_i \in G\) such that \(d(p_i \phi(m_i), p_i m_i) < \varepsilon\). The map \(\phi\) has the \(G\)–shadowing property if \(\forall \varepsilon > 0, \exists \delta > 0\) such that for each \((\delta, G)\) –pseudo- trajectory for \(\phi\) is \(\varepsilon\)–shadowed by a point of \(M\). Note that if \(\phi\) is bijective then we take \(-\infty < a < b < \infty\). Also, when \(\phi\) is not bijective then we take \(0 \leq a < b < \infty\).

Definition 2.3.

Let \((M, d)\) be metric \(G\) –space and let \(\phi : M \to M\) be continuous map. For a positive real number \(\delta\), a sequence of points \(\{m_i : a < i < b\}\) in \(M\) is called \(\delta\)–average pseudo- trajectory for \(\phi\) if \(\forall i, a < i < b - 1, \exists g_i \in G\) and there exists a positive integer \(N = N(\delta)\) such that \(\forall n \geq N, k \in \mathbb{N}\), then

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(g_i \phi(m_{i+k}), m_{i+k+1}) < \delta.
\]

The map \(\phi\) has the \(G\) ASP if \(\forall \varepsilon > 0\) and there is \(\delta > 0\) such that every \((\delta, G)\) –average pseudo-trajectory \(\{m_i : a < i < b\}\) is \(\varepsilon\)–shadowed in \(G\)–average by a point \(m\) of \(M\), if \(\forall i, A \subset M\), \(\exists g_i \in G\) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi^i(m), g_i m_i) < \varepsilon.
\]

Note that if \(\phi\) is bijective then we take \(-\infty < a < b < \infty\). Also, when \(\phi\) is not bijective then we take \(0 \leq a < b < \infty\).

Definition 2.4. [8]

Let \((M, d)\) be metric \(G\) –space and let \(\phi : M \to M\) be continuous map, then \(\phi\) is called \(G\)–transitive if \(\forall A, B \neq \emptyset\), and \(A, B\) are open subsets of \(M\), \(\exists i \in \mathbb{N}\), and \(g \in G\), such that the set \(N_g(A \cap B) = \{i \in \mathbb{N} : g \phi^i(A) \cap B \neq \emptyset\} \neq \emptyset\). We say that a homeomorphism \(\phi\) is totally \(G\)–transitive if \(\phi^i\) is \(G\)–transitive, \(\forall i \geq 1\).

Definition 2.5. [9]

Let \((M, d)\) be metric \(G\)–space and \(\phi : M \to M\) be a homeomorphism map, then \(\phi\) is called topologically \(G\)–mixing if 

\[
\forall A, B \neq \emptyset, A, B\text{ are open subsets of } M, \exists k \in \mathbb{Z}\text{ such that } \forall n \geq k, \exists g \in G\text{ satisfying } g \phi^k(A) \cap B \neq \emptyset.
\]

Definition 2.6. [9]

Let \((M, d)\) be metric \(G\) –space and \(\phi : M \to M\) be a continuous map, then \(\phi\) is called weakly \(G\)–mixing if \(\phi \times \phi\) is \(G \times G\)–transitive, that means, \(\forall A \times B, E \times D \neq \emptyset\) of are open subsets of \(M \times M, \exists (g, p) \in G \times G\) and \(k \in \mathbb{N}\), such that,

\[
((g, p)(\phi \times \phi)^k(A \times B)) \cap (E \times D) \neq \emptyset.
\]

If \(\exists N > 0\), such that \(\forall m, y \in M, \text{ and } \forall n \geq N\), there exists \((\delta, G)\) –pseudo-trajectory from \(m\) to \(y\) of length exactly \(n\), then the map \(\phi\) is \((\delta, G)\) –chain mixing. The map \(\phi\) is chain mixing if it is \(\delta\) –chain mixing for every \(\delta > 0\).

Main Results

Proposition 3.1

Let \((M, d)\) be metric \(G\) –space, and \(\phi : M \to M\) be a continuous map. If \(\phi\) has \(G\) ASP, then \(\phi^m\) has \(G\) ASP for every \(m \in \mathbb{N}\).

Proof:

Let \(m \in \mathbb{N}\), since \(\phi\) has \(G\) ASP, for any \(\varepsilon_m > 0\), \(\exists \delta > 0\), such that every \((\delta, G)\) –average pseudo- trajectory is \(\frac{\varepsilon_m}{m}\)–shadowed in average by some point in \(M\). Assume that \(\{z_i, i \in N_0\}\) is \((\delta, G)\) – average pseudo – trajectory of \(\phi^m\), that is, \(\exists \mu = \mu(\delta) > 0\), such that

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(g_i \phi^m(z_{i+k}), z_{i+k+1}) < \delta, \text{ for all } n \geq \mu, k \in N_0 \text{ and } g_i \in G.
\]
We write \( x_{nm+j} = \Phi^n(z_n) \) for \( 0 \leq j < m \), \( n \in \mathbb{N}_0 \), that is, \( \{ x_i, i \in \mathbb{N}_0 \} = \{ z_0, \Phi(z_0), \ldots, \Phi^{m-1}(z_0), z_1, \Phi(z_1), \ldots, \Phi^{m-1}(z_1), \ldots \} \).

We have \( \frac{1}{n} \sum_{i=0}^{n-1} d(\Phi^m(x_{i+k}), x_{i+k+1}) < \delta \), for all \( n \geq \mu \) and \( k \in \mathbb{Z}_+ \).

Then \( \{ x_i, i \in \mathbb{N}_0 \} \) is \( (\delta, \mathbb{G}) \)-average pseudo-trajectory \( \Phi \). So, \( \exists \omega \in \mathcal{M} \), such that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\Phi^i(\omega), g_i x_i) < \frac{\varepsilon}{m},
\]
Claim: there are infinite \( t \in \mathbb{N} \), such that
\[
\frac{1}{t} \sum_{i=0}^{t-1} d(\Phi^m(\omega), g_i x_i) < \varepsilon.
\]
Proof of Claim: Assume there is \( \mu_0 \in \mathbb{N} \), such that
\[
\frac{1}{t} \sum_{i=0}^{t-1} d(\Phi^m(\omega), g_i x_i) \geq \frac{\varepsilon}{m}, \quad \text{for all } t \geq \mu_0.
\]
Then
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\Phi^i(\omega), g_i x_i) \geq \frac{\varepsilon}{m}.
\]
This contracts with (3-1), then we have:
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\Phi^m(\omega), g_i x_{im}) < \varepsilon,
\]
since
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\Phi^m(\omega), g_i z_i) < \varepsilon.
\]
Thus, have the \( \Phi^m \in \mathbb{G} \) ASP.

**Proposition 3.2.** [9]

Let \( (\mathcal{M}, d) \) be a metric \( \mathbb{G} \) - space, \( \Phi : \mathcal{M} \to \mathcal{M} \) be pseudo-equivariant and totally \( \mathbb{G} \) - transitive with a dense set of \( \mathbb{G} \Phi \) - periodic points, then \( \Phi \) is weakly \( \mathbb{G} \) - mixing.

**Theorem 3.3**

Let \( (\mathcal{M}, d) \) be a compact metric \( \mathbb{G} \) - space and \( \Phi : \mathcal{M} \to \mathcal{M} \) be pseudo-equivariant with dense set of \( \mathbb{G} \Phi \) - periodic points. If \( \Phi \) has the \( \mathbb{G} \) ASP, then \( \Phi \) is weakly \( \mathbb{G} \) - mixing.

**Proof:**

By Proposition 3.1, if \( \Phi \) has the \( \mathbb{G} \) ASP then so does \( \Phi^m \) for every \( m \in \mathbb{N} \). By Proposition 3.2, if \( \Phi^m \) is totally \( \mathbb{G} \) - transitive for every \( m > 0 \), then it is weakly \( \mathbb{G} \) - mixing. Therefore, it is enough to prove that \( \Phi \) is totally \( \mathbb{G} \) - transitive.

We must prove that \( \Phi^m \) is \( \mathbb{G} \) - transitive for some \( m > 1 \). Assume that \( \Phi^m \) is not \( \mathbb{G} \) - transitive for some \( m > 1 \), then \( \exists D \subseteq \mathcal{M} \), such that \( D \neq \emptyset \) proper, closed and \( \mathbb{G} \) - invariant. Also \( \Phi^m(D) \subseteq D \) and hence \( \Phi^m(D) \subseteq D \) for any \( s \geq 1 \) such that \( \text{int}(D) \neq \emptyset \), implies that \( \Phi^{ms} \) is not \( \mathbb{G} \) - transitive for any \( s \geq 1 \). So, \( \forall s \geq 1 \), \( \exists A_s, B_s \) are non-empty open subsets of \( \mathcal{M} \), such that \( \forall p \in \mathbb{G} \) and \( \forall l \geq 1 \). We have \( (p(\Phi^{ms}(A_s))) \cap B_s = \emptyset \). Note that \( A_1, B_1 \) works \( \forall s \). Assume that \( A_1, B_1 \) are nonempty open subsets of \( \mathcal{M} \) such that \( (p(\Phi^{ms}(B))) \cap B = \emptyset \), \( \forall \ p \in \mathbb{G} \) and \( \forall k \geq 1 \). Since \( \Phi \) is pseudo-equivariant, then \( A \cap (p(\Phi^{ms}(B))) = \emptyset \), \( \forall \ p \in \mathbb{G} \) and \( \forall k \geq 1 \). Suppose that \( \Phi \times \Phi \times \cdots \times \Phi \) is not \( \mathbb{G} \times \mathbb{G} \times \cdots \times \mathbb{G} \) - transitive. We take into account that \( B' = B \times \Phi^{-1}(B) \times \cdots \times \Phi^{-(m-1)}(B) \) and \( A' = A \times A \times \cdots \times A \). Then, \( A' \cap ((p_1, p_2, \ldots, p_m) (\Phi \times \Phi \times \cdots \times \Phi)^{-t}(B')) = \emptyset \).
\( \forall (p_1, p_2, ..., p_m) \in \mathbb{G} \times \mathbb{G} \times ... \times \mathbb{G} \) and \( \forall r \geq 1 \), which implies that \( \Phi \times \Phi \times \cdots \times \Phi \) is not \( \mathbb{G} \times \mathbb{G} \times ... \times \mathbb{G} \) - transitive, which implies a contradiction. Thus \( \Phi^m \) is \( \mathbb{G} \) - transitive for every \( m \geq 1 \) and hence \( \phi \) is totally \( \mathbb{G} \) - transitive. Thus by Proposition 3.2, \( \phi \) is weakly \( \mathbb{G} \) - mixing.

**Definition 3.4.** [5]

Let \( (\mathcal{M}, d) \) be a metric \( \mathbb{G} \) - space and \( \phi : \mathcal{M} \rightarrow \mathcal{M} \) be a homeomorphism map that is called positively \( \mathbb{G} \) - expansive. If there exists real number \( \rho > 0 \) such that \( \forall m, y \in \mathcal{M} \) with \( \mathcal{G}(m) \neq \mathcal{G}(y) \), there exists an integer number \( k \geq 0 \) such that \( d(\phi^k(x), \phi^k(y)) > \rho, \forall \ u \in \mathcal{G}(m), \) and \( v \in \mathcal{G}(y) \). \( \rho \) is then called a \( \mathbb{G} \) - expansive constant for \( \phi \).

**Definition 3.5.** [5]

Let \( (\mathcal{M}, d) \) be a compact metric \( \mathbb{G} \) - space and \( \phi : \mathcal{M} \rightarrow \mathcal{M} \) be a homeomorphism map. Then \( \phi \) has \( \mathbb{G} \) - specification if \( \forall \epsilon > 0, \exists \mathcal{D} = \mathcal{D}(\epsilon) > 0 \) such that for each finite sequence of points \( g_1 m_1, g_2 m_2, ..., g_k m_k \in \mathcal{M} \) for some \( g_1, g_2, ..., g_k \in \mathbb{G} \) and for \( 2 \leq k \leq j \), picking any sequence of integers \( a_1 \leq b_1 < a_2 \leq b_2 < ... < a_k \leq b_k \) such that \( a_k - b_{k-1} \geq \mathcal{D}(2 \leq k \leq j) \) and an integer \( \ell \) with \( \ell \geq \mathcal{D}(b_j - a_1) \), \( \exists m \in \mathcal{M} \) with \( \phi^{\ell}(m) = g m, \exists g \in \mathbb{G} \) and hold \( d(\phi^i(m), \phi^i(g m)) < \epsilon \) for some \( \ell_i \in \mathbb{G} \) and for \( a_k \leq i \leq b_k, \ 1 \leq k \leq j \).

**Theorem 3.6**

Let \( (\mathcal{M}, d) \) be a compact metric \( \mathbb{G} \) - space with \( d \) being an invariant metric and let \( \phi : \mathcal{M} \rightarrow \mathcal{M} \) is a \( \mathbb{G} \) - expansive pseudo-equivariant homeomorphism having the \( \mathbb{G} \) ASP. If \( \phi \) is topologically \( \mathbb{G} \) - mixing then \( \phi \) has the \( \mathbb{G} \) - specification.

Proof:

Let \( \rho > 0 \) be a \( \mathbb{G} \) - expansive constant for \( \phi \) and we choose \( \epsilon \) such that \( 0 < \epsilon < \rho / 2 \). Since \( \phi \) has \( \mathbb{G} \) ASP, \( \exists \beta > 0 \) such that every \( (\beta, \mathbb{G}) \) - average pseudo-trajectory for \( \phi \) is \( \epsilon \) - shadowed in \( \mathbb{G} \) - average by the trajectory of some point \( m \in \mathcal{M} \). Let \( \mathcal{F} = \{ A_1, A_2, ..., A_m \} \) be a finite open cover of \( \mathcal{M} \) with \( A_i \neq \emptyset \) and diam \( A_i < \rho / 2, \forall i, i \in \{1,2,...,m\} \). Since \( \phi \) is topologically \( \mathbb{G} \) - mixing, then for each open sets \( A_i, A_j \) there is \( \mathcal{D}_{i,j} > 0 \), such that \( \forall n \geq \mathcal{D}_{i,j} \) and there is \( g_n \in \mathbb{G} \) satisfying \( A_j \cap g_n^{i}(A_i) \neq \emptyset \) (3 - 2).

Let \( \mathcal{D} = \max \{ \mathcal{D}_{i,j} : 1 \leq i, j \leq m \} \) and \( g_1 m_1, g_2 m_2, ..., g_k m_k \in \mathcal{D}, \) for some \( g_1, g_2, ..., g_k \in \mathbb{G} \) and for \( 2 \leq j \leq k \), picking any sequence of integers \( a_1 \leq b_1 < a_2 \leq b_2 < ... < a_k \leq b_k \) such that \( a_k - b_{k-1} \geq \mathcal{D}(2 \leq k \leq j) \) and an integer \( p \) with \( p \geq \mathcal{D}(b_k - a_1) \). We define \( a_{k+1} = b_{k+1} = p + a_1, m_{k+1} = \phi^{a_{k+1}-b_1}(g m_1) \). We denote by \( A(z) \) an open ball in \( \mathcal{F} \) containing \( z \). Since \( a_{j+1} - b_j \geq \mathcal{D}, \) by (3 - 2), \( \exists g'_{j+1-j} \in \mathbb{G}, \) such that \( A(\phi^{a_{j+1}}(g_{j+1} m_{j+1})) \cap g'_{j+1-j} \phi^{a_{j+1}-b_1}(A(\phi^{b_1}(g m_1))) \neq \emptyset \), that is,

\( \exists y_j \in \phi^{a_{j+1}-b_1}(A(\phi^{b_1}(g m_1))) \neq \emptyset \) such that \( \phi^{a_{j+1}-b_1}(y_j) = k_{j+1-j} y_j \). We establish a \( (\beta, \mathbb{G}) \) - average pseudo-trajectory \( \{ \omega_i : i \in \mathbb{Z} \} \) for \( \phi \) in \( \mathcal{M} \), as follows:

\( \omega_i = \phi^i(g m_1) \) if \( a_i \leq i \leq b_i \)

\( \omega_i = \phi^{i-b_1}(y_j) \) if \( b_j \leq i \leq a_{j+1} \)

\( \omega_{i+p} = \omega_i, \forall i \in \mathbb{Z} \)

Since \( \phi \) has the \( \mathbb{G} \) ASP, \( \{ \omega_i : i \in \mathbb{Z} \} \) is \( \epsilon \) - shadowed in \( \mathbb{G} \) - average by the trajectory of some point \( m \in \mathcal{M} \). Therefore, \( \forall i \in \mathbb{Z}, \exists \ell_i, \ell_{i+p} \in \mathbb{G} \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=-n}^{n-1} d(\phi^i(m), \ell_i \omega_i) < \epsilon, \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \sum_{i=-n}^{n-1} d(\phi^{i+p}(m), \ell_{i+p} \omega_{i+p}) < \epsilon,
\]

this implies that \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=-n}^{n-1} d(\phi^i(m), \ell_i \omega_i) < \epsilon, \)
and $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1-n}^{n-1} d(\phi^i(m), \ell_{i+P} \omega_i) < \varepsilon$, which implies that $\forall i \in \mathbb{Z}, \exists \ell_1, \ell_k \in \mathbb{Z}$, satisfying

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1-n}^{n-1} d(\ell_{i+P}^1 \phi^i(m), \ell_{i}^1 \phi^i(m)) < 2 \varepsilon < \varepsilon.$$ 

But $\phi$ is $\mathbb{G}$-expansive homeomorphism. Consequently, $\mathbb{G}(\phi^p(m)) = \mathbb{G}(m)$. Therefore, $\phi^p(m) = gm$, for some $g \in \mathbb{G}$. Also for $a_i \leq j \leq b_i$, $\omega_i = \phi^i(g, m)$. 

So, $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1-n}^{n-1} d(\phi^i(m), \ell_i \omega_i) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1-n}^{n-1} d(\phi^i(m), \ell_i \phi^i(g, m)) < \varepsilon$ and $\phi^p(m) = gm$. Thus, $\phi$ has the $\mathbb{G}$-specification by Definition 3.5.

**Lemma 3.7.** [5]

Let $\mathcal{M}$ be a compact connected Hausdorff metric space that contains more than one point and let $m, y \in \mathcal{M}$. Then for a continuous map $\phi: \mathcal{M} \to \mathcal{M}$ and $\delta > 0$, there exists a $\delta$ – pseudo-trajectory for $\phi$ containing $m, y$ in $\mathcal{M}$.

We recall that the topological space $\mathcal{M}$ is called a **totally disconnected** space if $\forall m, y \in \mathcal{M}$. There are two sets $A, B \subset \mathcal{M}$ that are disconnection such that $m \in A$ and $y \in B$.

**Theorem 3.8**

Let $(\mathcal{M}, d)$ be a compact metric $\mathbb{G}$–space. Then the identity map $\phi: \mathcal{M} \to \mathcal{M}$ has the $\mathbb{G}$ ASP if and only if the orbit space $\mathcal{M}/\mathbb{G}$ of $\mathcal{M}$ is totally disconnected.

**Proof:** ($\Leftarrow$)

Assume that the identity map $\phi: \mathcal{M} \to \mathcal{M}$ has the $\mathbb{G}$ ASP. By hypothesis, $\mathcal{M}/\mathbb{G}$ is compact, then it is enough to prove that $\dim(\mathcal{M}/\mathbb{G}) = 0$. Suppose, conversely, that $\dim(\mathcal{M}/\mathbb{G}) \geq 1$, so there is a closed connected subset $E$ in $\mathcal{M}/\mathbb{G}$ which has a dimension that is at least one. $E$ is a compact subset of $\mathcal{M}/\mathbb{G}$, since $\mathcal{M}/\mathbb{G}$ is compact. So $\exists E(a) \neq E(b) \in E$, such that $\dim E = d_E(G(a), G(b)) = \gamma$. By compactness of $\mathbb{G}$, there are $y_1 \in \mathbb{G}(a)$ and $y_2 \in \mathbb{G}(b)$ such that $r = d(y_1, y_2)$. Let $\varepsilon = \frac{\gamma}{3}$. We get a contradiction by exhibiting that for $\forall \varepsilon > 0$ there is a $(\delta, \mathbb{G})$ – average pseudo-trajectory for $\phi$ which is not $\varepsilon$ - shadowed in $\mathbb{G}$ – average by the trajectory of some point $m \in \mathcal{M}$.

By Lemma 3.7, there is a $(\delta, \mathbb{G})$ – average pseudo-trajectory $\{m_i : i \in \mathbb{Z}\}$ for $\phi$ in $\mathcal{M}$ containing $y_1, y_2$. Such a $(\delta, \mathbb{G})$ – average pseudo-trajectory can be obtained as follows: Since $E$ is a compact connected subset of $\mathcal{M}/\mathbb{G}$ by Lemma 3.7, there is a $\delta$ – pseudo-trajectory $\{G(m_i) : i \in \mathbb{Z}\}$, for $\phi$ containing $G(a)$ and $G(b)$. This implies that $\forall i \in \mathbb{Z}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d_i(\phi^i(m), \mathbb{G}(m_{i+1})) < \varepsilon.$$ 

Since $\mathbb{G}$ is Compact, implies for $\forall i \in \mathbb{Z}$, $\exists \ell_i, u_i \in \mathbb{G}$ such that,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(\ell_i \phi^i(m), u_i m_{i+1}) < \delta$$ which implies $\frac{1}{n} \sum_{i=0}^{n-1} d(g_i \phi^i(m), m_{i+1}) < \delta,$

for some $g_i \in \mathbb{G}$, and hence $\{m_i : i \in \mathbb{Z}\}$ is a $(\delta, \mathbb{G})$ – average pseudo-trajectory for $\phi$. Now, $\{G(m_i) : i \in \mathbb{Z}\}$ contains $G(a)$ and $G(b)$. Therefore, for some $k, p \in \mathbb{Z}, \mathbb{G}(m_k) = G(a)$ and $\mathbb{G}(m_p) = G(b)$. Also, $y_1 \in \mathbb{G}(a)$ and $y_2 \in \mathbb{G}(b)$, implies $g' y_1 = m_k$ and $g'' y_2 = m_p$, for some $g', g'' \in \mathbb{G}$. We take the place of $m_k$ by $g' y_1$ and $m_p$ by $g'' y_2$ in $\{m_i : i \in \mathbb{Z}\}$ and continue to denote the new $(\delta, \mathbb{G})$ – average pseudo-trajectory, containing $y_1$ and $y_2$, by $\{m_i : i \in \mathbb{Z}\}$.

Let $\{m_i : i \in \mathbb{Z}\}$ is shadowed in $\mathbb{G}$ – average by the point $m \in \mathcal{M}$. So, $\forall i \in \mathbb{Z}, \exists p_i \in \mathbb{G}$, such that

$$\frac{1}{n} \sum_{i=0}^{n-1} d(m_i, p_i m_i) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi^i(m_i), p_i m_i) < \varepsilon$$

(3.3)
Since \( \{m_i : i \in \mathbb{Z}\} \) is a \((\delta, \mathcal{G})\)-average pseudo-trajectory for \( \phi \) containing \( y_1 \) and \( y_2, \exists k, n \in \mathbb{Z} \) such that \( m_k = y_1 \) and \( m_n = y_2 \). So, by (3-3) \( \delta(m, p_k, m_k) < \varepsilon \) and \( \delta(m, p_n, m_n) < \varepsilon \), which implies that \( d_1(\mathcal{G}(m), \mathcal{G}(p_k, m_k)) < \varepsilon \) and \( d_1(\mathcal{G}(m), \mathcal{G}(p_n, m_n)) < \varepsilon \), and hence \( d_1(\mathcal{G}(a), \mathcal{G}(b)) \leq d_1(\mathcal{G}(a), \mathcal{G}(m)) + d_1(\mathcal{G}(m), \mathcal{G}(b)) < \varepsilon + \varepsilon = \frac{2\varepsilon}{3} \), which is a contradiction.

This proves that \( \dim(M/\mathcal{G}) = 0 \). Hence, the orbit space \( M/\mathcal{G} \) of \( M \) is totally disconnected.

**Proof:** (\( \Leftarrow \))

Assume that \( M/\mathcal{G} \) is totally disconnected. Then clopen sets form a basis for topology of \( M \). By hypothesis, \( \mathcal{G} \) is compact, then we have the possibility of an invariant metric \( d \) on \( M \) congruous with topology of \( M \). Let \( \varepsilon > 0 \) be given and let \( \{A_1, A_2, \ldots, A_n\} \) be a finite subcover of \( M/\mathcal{G} \) containing clopen sets such that \( A_i \cap A_j = \emptyset \) for \( i \neq j \) and \( \text{diam} A_i < \varepsilon, \forall i \in \{1, 2, \ldots, n\} \).

A set \( B_i = \Psi_i^{-1}(A_i), \forall i \), since \( A_i \) is a closed subset of \( M/\mathcal{G} \) and \( \pi \) is a continuous map, \( B_i = \Psi_i^{-1}(A_i) \) is compact, since \( B_i \subset M \), and \( B_i \) is a closed. So, \( A_i \cap A_j = \emptyset \), implies \( \Psi_i^{-1}(A_i) \cap \Psi_j^{-1}(A_j) = \emptyset \), implies \( B_i \cap B_j = \emptyset \).

Let \( a_{ij} = d(A_i, A_j) \) for \( i \neq j \). Then \( A_i, A_j \) is compact, implies \( a_{ij} > 0 \) for \( i \neq j \). Choose \( \alpha \) such that \( 0 < \alpha < \min \{a_{ij} : 1 \leq i, j \leq n\} \). We must prove that the identity map \( \phi \) has the \( \mathcal{G} \) ASP. We prove that every \((\alpha, \mathcal{G})\)-average pseudo-trajectory for \( \phi \) is \( \varepsilon \)-shadowed in \( \mathcal{G} \)-average by the trajectory of some point \( m \in M \). Let \( S = \{m_i : i \in \mathbb{Z}\} \) be \((\alpha, \mathcal{G})\)-average pseudo-trajectory for \( \phi \). Then for \( \forall i \in \mathbb{Z}, \exists g_i \in \mathcal{G} \) such that

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(g_i, \phi(m_i), m_{i+1}) < \varepsilon \quad \text{implies} \quad \frac{1}{n} \sum_{i=0}^{n-1} d(g_i, m_i, m_{i+1}) < \varepsilon, \quad (3-4)
\]

Note that if \( m_i \in B_k \) then \( m_{i+1} \in B_k \). For if \( m_{i+1} \in B_j, j \neq k \), then \( B_k \) is \( \mathcal{G} \)-invariant \( g_i m_i \in B_k \) and \( m_{i+1} \in B_j \), implies

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(g_i m_i, m_{i+1}) \geq \frac{1}{n} \sum_{i=0}^{n-1} d(B_k, B_j) = \alpha_{ij} > \varepsilon,
\]

This is a contradiction with (3-4). Similarly, if \( m_i \in B_k \), then \( m_{i-1} \in B_k \). For if \( m_{i-1} \in B_j, j \neq k \), then \( B_j \) is \( \mathcal{G} \)-invariant \( g_{i-1} m_{i-1} \in B_j \) and \( m_{i+1} \in B_j \), implies

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(g_{i-1} m_{i-1}, m_i) \geq \frac{1}{n} \sum_{i=0}^{n-1} d(B_k, B_j) = \alpha_{ij} > \varepsilon,
\]

This is a contradiction with (3-4). So, \( \forall i \in \mathbb{Z}, m_i \in B_k \). This implies that \( \mathcal{G}(m_i) \in A_k \) and \( \forall i \in \mathbb{Z}, \)

\[
\frac{1}{n} \sum_{i=0}^{n-1} d_1(\mathcal{G}(m), \mathcal{G}(m_i)) < \varepsilon.
\]

By hypothesis, \( \mathcal{G} \) is compact, so \( \forall i \in \mathbb{Z}, \exists \ell_i, u_i \in \mathcal{G} \), such that

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(\ell_i m_i u_i m_i) < \varepsilon.
\]

Thus \( \forall i \in \mathbb{Z}, \exists g_i \in \mathcal{G} \) such that

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(\phi^i(m_i), g_i m_i) < \varepsilon.
\]

Hence \( S = \{m_i : i \in \mathbb{Z}\} \) is \( \varepsilon \)-shadowed in \( \mathcal{G} \)-average by the trajectory of some point \( m \in M \). Since \( S \) is an arbitrary \((\alpha, \mathcal{G})\)-average pseudo-trajectory for \( \phi \), it follow that every \((\alpha, \mathcal{G})\)-average pseudo-trajectory for \( \phi \) is \( \varepsilon \)-shadowed in \( \mathcal{G} \)-average by the trajectory of some point \( m \in M \). Hence \( \phi \) has the \( \mathcal{G} \) ASP.

**Definition 3.9.** [5]

Let \( M \) and \( Y \) be metric spaces. A continuous onto map \( h: M \rightarrow Y \) is called a covering map, if for each \( y \in Y \), there exists an open neighborhood \( B_y \) of \( y \) in \( Y \) such that \( \phi^{-1}(B_y) = \bigcup_i A_i \),
(i ≠ j, implies $A_i \cap A_j = \emptyset$, where each $A_i$ is open in $\mathcal{M}$ and $h|_{A_i} : A_i \to B_y$ is a homeomorphism).

**Theorem 3.10**

Let $\phi : \mathcal{M} \to \mathcal{M}$ be a pseudo-equivariant map on a compact metric $\mathbb{G}$–space $(\mathcal{M}, d')$ and let the orbit map $\Psi : \mathcal{M} \to \mathcal{M}/\mathbb{G}$ be a covering map, then $\phi$ has the $\mathbb{G}$ ASP iff the induced map $\bar{\phi} : \mathcal{M}/\mathbb{G} \to \mathcal{M}/\mathbb{G}$ has the ASP.

**Proof:** ($\Rightarrow$)

Assume that $\phi$ has the $\mathbb{G}$ ASP. We must prove that $\bar{\phi}$ has the ASP. We choose $\epsilon > 0$. Since $\Psi$ is uniformly continuous, $\exists \gamma > 0$, such that $d'(m, y) < \gamma$, implies $d'_1 (\Psi(m), \Psi(y)) < \epsilon$. Also, $\phi$ has the $\mathbb{G}$ ASP, so $\exists \mu > 0$, such that every $(\mu, \mathbb{G})$ – average pseudo- trajectory for $\phi$ is $\gamma$ - shadowed in $\mathbb{G}$ –average by a point $m \in \mathcal{M}$. Since $\Psi$ is a covering map on a compact space, $\exists \delta > 0$, such that for all $m \in \mathcal{M}$. We find an $\alpha_m$ satisfying $(\Psi|_{A_{\alpha_m}})^{-1}(A_\delta(\Psi(m))) \subset A_\mu(m)$. We must prove that $\bar{\phi}$ has the ASP. We show that every $\delta$ – average pseudo- trajectory for $\bar{\phi}$ is $\epsilon$ - shadowed in average by a point of $\mathcal{M}/\mathbb{G}$. Let $(\mathbb{G}(m_i) : i \in \mathcal{N}_0)$ is an $\delta$ – average pseudo-trajectory for $\bar{\phi}$. Then $\exists \alpha_{m_i+1}$ such that $m_{i+1} \in (\Psi|_{A_{\alpha_{m_i+1}}})^{-1}(A_\delta(\Psi(m_i))) \subset A_\mu(\phi(m_i))$. implies $\{\mathbb{G}(x_i) : i \in \mathcal{N}_0\}$ is an $(\mu, \mathbb{G})$ – average pseudo-trajectory for $\phi$ and so is $\gamma$ – shadowed in average by some point $m \in \mathcal{M}$. Hence, $\forall i \in \mathcal{N}_0, \exists g_i \in \mathbb{G}$, such that:

$\frac{1}{n} \sum_{i=0}^{n-1} d'(g_i m_i, \phi^i(m_i)) < \gamma$.

Moreover, using uniform continuity of the covering map $\Psi$, we get:

$\frac{1}{n} \sum_{i=0}^{n-1} d_1(\mathbb{G}(\phi^i(m)), \mathbb{G}(m_i)) < \epsilon$.

This proves that uniform continuity of the covering map $\Psi$ gives:

$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d_1(\mathbb{G}(\phi^i(m)), \mathbb{G}(m_i)) < \epsilon$.

Hence, $\bar{\phi}$ has the ASP.

**Proof:** ($\Leftarrow$)

Assume that $\bar{\phi}$ has the ASP. We must prove that $\phi$ has the $\mathbb{G}$ ASP. We choose $\epsilon > 0$. Since $\Psi$ is a covering map and $\mathcal{M}$ is compact, then $\exists \delta > 0$ such that for $\forall A_{\alpha} \subset \mathcal{M}$, $\alpha \in A_{\alpha}$, $\alpha \neq \beta$, which leads to $A_{\alpha} \cap A_{\beta} = \emptyset$ and that $\Psi|_{A_{\alpha}} : A_{\alpha} \to A_\delta(\Psi(m))$ is a homeomorphism. For $\epsilon$ –neighborhood $A_\epsilon(m)$ of $m$, consider $A_{\alpha}$ which contains $m$. If diam $A_{\alpha} < \epsilon$, we have $\Psi^{-1}|_{A_{\alpha}} (A_\delta(\Psi(m))) \subset A_\alpha \subset A_\epsilon(m)$. If diam $A_{\alpha} \geq \epsilon$, then choose $A'_{\alpha} \subset A_{\alpha}$ such that diam $A'_{\alpha} < \epsilon$ and $m \in A'_{\alpha}$, we have $\Psi^{-1}|_{A'_{\alpha}} (A_\delta(\Psi(m))) \subset A'_{\alpha} \subset A_\epsilon(x)$. Since $\bar{\phi}$ has the ASP then $\exists \mu > 0$, such that every $\mu$ – average pseudo- trajectory for $\bar{\phi}$ is $\delta$ - shadowed in average by a point of $\mathcal{M}/\mathbb{G}$. Uniform continuity of $\Psi$ implies that $\exists \gamma > 0$ such that $d(m, y) < \gamma$ which leads to $d'_1 (\Psi(m), \Psi(y)) < \gamma$. To prove that $\phi$ has the $\mathbb{G}$ ASP, we show that every $(\gamma, \mathbb{G})$ – average pseudo-trajectory for $\bar{\phi}$ is $\epsilon$ - shadowed in $\mathbb{G}$ –average by a point of $\mathcal{M}$. Let $\{m_i : i \in \mathcal{N}_0\}$ be a $(\gamma, \mathbb{G})$ – average pseudo-trajectory for $\bar{\phi}$.

This implies that $\forall i \in \mathcal{N}_0, \exists p_i \in \mathbb{G}$ such that $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(p_i h(m_i), m_{i+1}) < \gamma$.

Therefore, $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d'_1(\Psi(\phi(m_i)), \Psi(m_{i+1})) < \mu$, and hence we have

$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d_1(\mathbb{G}(\phi(m_i)), \mathbb{G}(m_{i+1})) < \mu$,

which proves that $\{\mathbb{G}(m_i) : i \in \mathcal{N}_0\}$ is an $\mu$ – average pseudo-trajectory for $\bar{\phi}$. Since $\bar{\phi}$ has the ASP, then $\{\mathbb{G}(m_i) : i \in \mathcal{N}_0\}$ is $\epsilon$ - shadowed in average by a point of $\mathcal{M}/\mathbb{G}$.

Suppose that $\mathbb{G}(x)$ and hence $\frac{1}{n} \sum_{i=0}^{n-1} d_1(\mathbb{G}(\phi^i(m)), \mathbb{G}(m_i)) < \delta$, $\forall i \in \mathcal{N}_0$. But this
Hence \( \Phi \) has the \( G \) ASP.

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