Canonical Sequences of Optimal Quantization for Condensation Measures

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Abstract
We consider condensation measures of the form $P := \frac{1}{3} P \circ S_1^{-1} + \frac{1}{3} P \circ S_2^{-1} + \frac{1}{3} \nu$ associated with the system $(\mathcal{S}, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), \nu)$, where $\mathcal{S} = \{S_i\}_{i=1}^2$ are contractions and $\nu$ is a Borel probability measure on $\mathbb{R}$ with compact support. Let $D(\mu)$ denote the quantization dimension of a measure $\mu$ if it exists. In this paper, we study self-similar measures $\nu$ satisfying $D(\nu) > \kappa$, $D(\nu) < \kappa$, and $D(\nu) = \kappa$, respectively, where $\kappa$ is the unique number satisfying $\left[ \frac{1}{3} \left( \frac{1}{5} \right)^2 \right]^2 = \frac{1}{2}$. For each case we construct two sequences $a(n)$ and $F(n)$, which are utilized in determining the optimal sets of $F(n)$-means and the $F(n)$th quantization errors for $P$. We also show that for each measure $\nu$ the quantization dimension $D(P)$ of $P$ exists and satisfies $D(P) = \max\{\kappa, D(\nu)\}$. Moreover, we show that for $D(\nu) > \kappa$, the $D(P)$-dimensional lower and upper quantization coefficients are finite, positive and unequal; and for $D(\nu) \leq \kappa$, the $D(P)$-dimensional lower quantization coefficient is infinity.

Keywords Canonical sequence · Optimal quantizers · Quantization error · Condensation measure · Self-similar measure · Quantization dimension · Quantization coefficient

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1 Introduction

The problem of quantization of probability measures emanated from information theory and has been a subject of intense investigation in the past decades. Rigorous mathematical treatment of basic quantization theory can be found in [12]; for further results and applications one is referred to [5, 8, 9, 16, 17, 20, 21, 27, 29, 32, 39, 40]. Let $P$ denote a Borel probability measure on $\mathbb{R}^d$ with the Euclidean norm $\| \cdot \|$, $d \geq 1$. For a finite set $\alpha \subset \mathbb{R}^d$, the value $V(P; \alpha) := \int \min_{a \in \alpha} \| x - a \|^2 dP(x)$ is often referred to as the cost or distortion error for $\alpha$ with respect to $P$. Then, the $n$-th quantization error for $P$ is defined by

$$V_n := V_n(P) = \inf \{ V(P; \alpha) : \alpha \subset \mathbb{R}^d, \ \text{card}(\alpha) \leq n \}.$$ 

If $\int \| x \|^2 dP(x) < \infty$, then there exists $\alpha \subset \mathbb{R}^d$ for which the infimum is achieved (see [1, 10–12]). A set $\alpha$ for which the infimum is achieved and contains no more than $n$ points, i.e., $V_n = V(P; \alpha)$, is called an optimal set of $n$-means or an optimal set of $n$-quantizers. Indeed, if the support of $P$ is an infinite set, then an optimal set of $n$-means always contains exactly $n$ elements (see Theorem 4.12 [12] and Theorem 2.4 [14]). The limit $D(P) := \lim_{n \to \infty} \frac{-2 \log n}{\log \min \{ V_n(P) \}}$, if exists, is called the quantization dimension of $P$, which measures the speed at which the specified measure of the error tends to zero as $n \to \infty$. It turns out that determining the optimal sets of $n$-means is much more difficult than calculating the quantization dimension of a measure [6, 13, 30, 31, 37]. If $D(P) := s$ exists, we are further concerned with the $s$-dimensional lower and upper quantization coefficients defined by $\lim \inf_{n \to \infty} n^{\frac{s}{2}} V_n(P)$ and $\lim \sup_{n \to \infty} n^{\frac{s}{2}} V_n(P)$, respectively. These two quantities provide more accurate information for the asymptotics of the quantization error than the quantization dimension. Given a finite subset $\alpha \subset \mathbb{R}^d$, the Voronoi region $M(a|\alpha)$ generated by $a \in \alpha$ is defined as the set of points $x \in \mathbb{R}^d$ such that $a$ is the nearest point to $x$ than to all other elements in $\alpha$. If $\alpha$ is an optimal set and $a \in \alpha$, then $a$ is the conditional expectation of the random variable $X$ given that $X$ takes values in the Voronoi region of $a$ [9, 12]. Such a point $a$ is also referred to as the centroid of the Voronoi region $M(a|\alpha)$ with respect to $P$.

Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, 2, \ldots, N$ be contracting similarities, $p = (p_0, p_1, \ldots, p_N)$ be a probability vector, and $v$ be a Borel probability measure on $\mathbb{R}^d$ with compact support. A probability measure $P$ on $\mathbb{R}^d$ such that $P = \sum_{j=1}^{N} p_j P \circ S_j^{-1} + p_0 v$ is called an inhomogeneous self-similar measure associated with the system $(S, p, v)$, where $S = \{ S_i \}_{i=1}^{N}$. For such an inhomogeneous self-similar measure $P$, if $C$ is the support of $v$, then the support of $P$ is equal to the unique nonempty compact set $K := K_C \subset \mathbb{R}^d$ satisfying $K = \bigcup_{j=1}^{N} S_j(K) \cup C$. For details about inhomogeneous self-similar sets and measures one can see [26]. Following [3, 22], we call $(S, p, v)$ a condensation system. The measure $P$ is also called the condensation measure or the attracting measure for $(S, p, v)$, and the set $K$, which is the support of the measure $P$, is called the attractor of the system.

Consider a condensation measure $P$ such that $P = \frac{1}{3} P \circ S_1^{-1} + \frac{1}{3} P \circ S_2^{-1} + \frac{1}{3} v$, where $v$ is a self-similar measure on $\mathbb{R}$ and $S_1, S_2$ are similarity mappings on $\mathbb{R}$ given by $S_1(x) = cx$, $S_2(x) = cx + r$, with $0 < c \leq \frac{1}{3}$ and $c + r = 1$. Let $\kappa$ be the number
satisfying \((\frac{2}{N})^{\frac{1}{N^2}} = \frac{1}{2}\), which we will call as the critical value of the condensation system \((S, p, \nu)\). In this article our aim is twofold. The main goal is to study the quantization for condensation measures associated to the system \((S, p, \nu)\); in particular, we show that there exist two sequences \(\{a(n)\}_{n \geq 1}\) and \(\{F(n)\}_{n \geq 1}\), which we call as canonical sequences, that are instrumental in this study. With the help of canonical sequences, for a variety of self-similar measures \(\nu\), we obtain closed formulas for the optimal sets of \(F(n)\)-means and the \(F(n)\)th quantization errors for the condensation measures \(P\) for all \(n \geq 1\). Once the optimal sets of \(F(n)\)-means are known, we develop a simple method to determine the optimal sets of \(n\)-means for all \(n \in \mathbb{N}\), and calculate quantization dimension \(D(P)\) and the \(D(P)\)-dimensional quantization coefficients for \(P\). Hence, we furnish the complete quantization program for the condensation systems under consideration, which was not done before in the literature. The second aim is to investigate the relationship between the quantization dimension \(D(P)\) and the \(D(P)\)-dimensional quantization coefficients as the measure \(\nu\) changes. It turns out that \(D(P)\) satisfies the relation \(D(P) = \max(\kappa, D(\nu))\). Furthermore, we determine that for \(D(\nu) > \kappa\), the \(D(P)\)-dimensional lower and upper quantization coefficients are finite, positive and unequal. On the other hand, for \(D(\nu) < \kappa\), and \(D(\nu) = \kappa\), the \(D(P)\)-dimensional lower quantization coefficients are infinity.

When \(p_0 = 0\), the inhomogeneous self-similar measure on \(\mathbb{R}^d\) defined above by \(P = \sum_{j=1}^{N} p_j P \circ S_j^{-1} + p_0 \nu\) reduces to the self-similar measure \(P = \sum_{j=1}^{N} p_j P \circ S_j^{-1}\). Quantization dimension for such self-similar measures and self-conformal measures were determined by Graf–Luschgy [15] and Lindsay–Mauldin [23], respectively. Following these, the quantization dimensions were determined for many fractal probability measures [33–36]. The optimal sets of \(n\)-means and the \(n\)th quantization errors for the (standard) Cantor self-similar measure were determined by Graf–Luschgy [13]. Due to their intricate structures, condensation measures which are more general than self-similar measures, were not studied widely in the literature; in particular, this is the case for the features investigated in this article. To the best of the authors’ knowledge, the current article is the first study of the optimal sets of \(n\)-means and the \(n\)th quantization errors for condensation measures, which include all (Cantor) self-similar measures as a special case. Hence, our results are significant generalization of those in [13]. The novelty in obtaining the quantization for condensation measures is the introduction of canonical sequences and the order \(\succ\) for the associated optimal sets. As a by-product we also derive closed formulas for the quantization errors involved at each step, which lead to direct calculation of the quantization dimensions and study the quantization coefficients for the probability distributions involved.

Techniques utilized in the article can be applied or can further be improved to investigate the optimal sets of \(n\)-means and the \(n\)th quantization errors for many other (fractal-based) singular probability measures. All these results are new and, while providing some insight into the behavior of such systems, they also bring up several questions for further inquiry. We outline some of these at the end of the paper. The points in optimal sets, being the centroids of their Voronoi regions, are an evenly spaced distribution of sites in the domain with minimum distortion error with respect to a given probability measure. Such settings frequently surface in many fields, for instance, numerical probability [4, 28], clustering, data compression, optimal mesh
generation, signal processing, cellular biology, optimal quadrature, and geographical optimization [2, 7, 18, 19, 24, 25, 38]. Therefore the result of the paper have potential for being very useful in addressing problems in these fields.

The arrangement of the paper is as follows: In Sect. 2 basic definitions, lemmas and propositions are developed. In order to bring some transparency to the arguments and reveal the connections between the quantization dimension $D(v)$ and the critical value $\kappa$, in Sect. 2.2 four important cases that will be focus of the article are outlined. Section 3 provides a thorough investigation of the quantization for the condensation measure $P$ with self-similar measure $\nu$ satisfying $D(v) > \kappa$. Also, utilizing canonical sequences, we have determined all the optimal sets of $n$-means and the associated $n$th quantization errors, and calculated the quantization dimension of $P$ (Theorem 3.4.1). Furthermore, it is also shown that the quantization coefficient does not exist, and the lower and the upper quantization coefficients are finite and positive (Theorem 3.4.2).

Section 4 is devoted to the investigation of quantization for the condensation measure $P$ with self-similar measure $\nu$ satisfying $D(v) < \kappa$. Using closed formulas obtained for quantizations errors, we calculate the quantization dimension of $P$ (Theorem 4.4.1), and show that the quantization coefficient is infinity (Theorem 4.4.2). In Sect. 5, we have considered two condensation measures: one with $D(v) > \kappa$ and one with $D(v) < \kappa$; and show that the quantization coefficients in both the cases are infinity. The results in all the above sections lead us to some observations and remarks outlined in Concluding Remarks 4.1 which also contains some open problems to be investigated.

In the sequel, all the arguments will be given for $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for simplicity.

2 Preliminaries

2.1 Basic Definitions, Lemmas and Propositions

Let $P$ be the condensation measure associated with the condensation system $(S, \mathbf{p}, \nu)$, where $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $S$ is as defined above. Consider the self-similar measure $\nu$ given by $\nu = \frac{1}{2}\nu \circ T_{1}^{-1} + \frac{1}{2}\nu \circ T_{2}^{-1}$, where $T_{1}(x) = sx + a(1 - s)$ and $T_{2}(x) = sx + b(1 - s)$ for all $x \in \mathbb{R}$, $0 < s \leq \frac{1}{3}$, and $a = \frac{1+c}{3}$, $b = \frac{2-c}{3}$. These values are to ensure that the associated Cantor sets are disjoint.

Let $I = \{1, 2\}$. By a word $\omega$ of length $k$ over the alphabet $I$, we mean $\omega := \omega_{1}\omega_{2}\cdots\omega_{k} \in I^{k}$; a word of length zero is called the empty word and is denoted by $\emptyset$. $I^{\ast}$ will denote the set of all words over the alphabet $I$ including the empty word $\emptyset$. For $\omega, \tau \in I^{\ast}$, their concatenation is denoted by $\omega \tau$; i.e., if $\omega := \omega_{1}\omega_{2}\cdots\omega_{k}$ and $\tau := \tau_{1}\tau_{2}\cdots\tau_{\ell}$, then $\omega \tau := \omega_{1}\cdots\omega_{k}\tau_{1}\cdots\tau_{\ell}$. Set $J := [0, 1]$ and $L := [a, b]$. For $\omega = \omega_{1}\omega_{2}\cdots\omega_{k} \in I^{k}$, set $S_{\omega} := S_{\omega_{1}} \circ \cdots \circ S_{\omega_{k}}$, $T_{\omega} := T_{\omega_{1}} \circ \cdots \circ T_{\omega_{k}}$, $J_{\omega} := S_{\omega}(J)$, and $L_{\omega} := S_{\omega}(L)$. For $\omega = \emptyset$, $S_{\emptyset}$ is the identity map on $\mathbb{R}$, $J_{\emptyset} = J$ and $L_{\emptyset} = L$. If $C$ is the support of $\nu$, then

$$C := \bigcap_{k \geq 0} \bigcup_{\omega \in I^{k}} T_{\omega}([a, b]).$$
Iterating $P = \frac{1}{3} \sum_{j=1}^{2} P \circ S_j^{-1} + \frac{1}{3} \nu$ and $\nu = \frac{1}{2} \nu \circ T_1^{-1} + \frac{1}{2} \nu \circ T_2^{-1}$, we have

$$P = \frac{1}{3^n} \sum_{|\omega|=n} P \circ S_\omega^{-1} + \sum_{k=0}^{n-1} \frac{1}{3^{k+1}} \sum_{|\omega|=k} \nu \circ S_\omega^{-1}, \text{ and}$$

$$\nu = \frac{1}{2^k} \sum_{\omega \in I^k} \nu \circ T_\omega^{-1}, \text{ for all } k \geq 1. \quad (1)$$

The measure $P$ is ‘symmetric’ about the point $\frac{1}{2}$, i.e., if two intervals of equal lengths are equidistant from the point $\frac{1}{2}$, then they have the same $P$-measure. For $n \geq 1$, $\alpha_n := \alpha_n(P)$ will denote an optimal set of $n$-means with respect to $P$, and $\alpha_n(\nu)$ will represent an optimal set of $n$-means with respect to the self-similar measure $\nu$. Similarly, $V_n := V_n(P)$ and $V_n(\nu)$ represent the $n$th quantization error with respect to $P$ and $\nu$, respectively. By $P|_L$, we denote the conditional probability measure on $L$, i.e., for any Borel $B \subset \mathbb{R}$,

$$P|_L(B) = \frac{P(B \cap L)}{P(L)}. \quad (2)$$

Notice that $P|_L = \nu$. Using Eq. (1), we deduce the following lemma.

**Lemma 2.1.1** Let $g : \mathbb{R} \to \mathbb{R}^+$ be Borel measurable and $n \in \mathbb{N}$. Then,

$$\int g dP = \frac{1}{3^n} \sum_{|\omega|=n} \int (g \circ S_\omega) dP + \sum_{k=0}^{n-1} \frac{1}{3^{k+1}} \sum_{|\omega|=k} \int (g \circ S_\omega) d\nu.$$  

**Lemma 2.1.2** Let $K$ be the support of the condensation measure. Then, for any $n \geq 1$,

$$K \subset \left( \bigcup_{\omega \in I^n} J_\omega \right) \bigcup \left( \bigcup_{k=0}^{n-1} \bigcup_{\omega \in I^k} L_\omega \right) \subset J.$$  

**Proof** Notice that $J_1 \cup L \cup J_2 \subset J$, $J_{11} \cup L_1 \cup J_{12} \subset J_1$, and $J_{21} \cup L_2 \cup J_{22} \subset J_2$. In fact, for any $k \geq 1$, if $\omega \in I^k$, then $J_{\omega 1} \cup L_\omega \cup J_{\omega 2} \subset J_\omega$. Again, notice that for any $\omega \in I^*$, $J_{\omega 1} \cup J_{\omega 2} \subset J_\omega$, and the intervals $L_{\omega 1}, L_{\omega 2}, L_\omega$ are disjoint. Thus, it follows that

$$\left( \bigcup_{\omega \in I^n} J_\omega \right) \bigcup \left( \bigcup_{k=0}^{n-1} \bigcup_{\omega \in I^k} L_\omega \right) \subset J.$$  

The sets being disjoint, we have

$$P \left( \bigcup_{\omega \in I^n} J_\omega \right) \bigcup \left( \bigcup_{k=0}^{n-1} \bigcup_{\omega \in I^k} L_\omega \right) = \sum_{\omega \in I^n} P(J_\omega) + \sum_{k=0}^{n-1} \sum_{\omega \in I^k} P(L_\omega) = \sum_{\omega \in I^n} \frac{1}{3^n} + \sum_{k=0}^{n-1} 2^k \cdot \frac{1}{3^{k+1}} = 1.$$
Again, $P(K) = 1$ and $K$ is the support of $P$. Hence, $K \subset \left( \bigcup_{\omega \in I^n} J_\omega \right) \cup \left( \bigcup_{k=0}^{n-1} \bigcup_{\omega \in I^k} L_\omega \right)$.

Let $E(v)$ and $W := V(v)$ represent the expected value and the variance of $v$, respectively.

**Lemma 2.1.3** For the self-similar measure $\nu$ we have

(i) $E(\nu) = \frac{1}{2}$ and $W = \frac{(1-s)(1-2c)^2}{36(1+s)}$,

(ii) for any $x_0 \in \mathbb{R}$, $\int (x-x_0)^2 d\nu = (x_0 - \frac{1}{2})^2 + V(\nu)$.

**Proof** Since $\int x d\nu = \frac{1}{2} \int [sx + a(1-s)] d\nu + \frac{1}{2} \int [sx + b(1-s)] d\nu$ and $a + b = 1$, we have $E(\nu) = \int x d\nu = \frac{1}{2}$. Moreover,

$$\int x^2 d\nu = \frac{1}{2} \int [sx + a(1-s)]^2 d\nu + \frac{1}{2} \int [sx + b(1-s)]^2 d\nu = s^2 \int x^2 d\nu + s(1-s) \int x d\nu + \frac{1}{2}(a^2 + b^2)(1-s)^2 \int d\nu,$$

which implies that $\int x^2 d\nu = \frac{s(s+a^2+b^2)(1-s)}{2(1+s)}$. Hence,

$$W := V(\nu) = \int x^2 d\nu - \left( \int x d\nu \right)^2 = \frac{s + (a^2 + b^2)(1-s)}{2(1+s)} - \frac{1}{4} = \frac{(1-s)(1-2c)^2}{36(1+s)}.$$

For any $x_0 \in \mathbb{R}$, $\int (x-x_0)^2 d\nu = (x_0 - \frac{1}{2})^2 + V(\nu)$ follows from the standard probability arguments. \hfill $\square$

Let $E(P)$ and $V(P)$ represent the expected value and the variance of $P$, respectively.

**Lemma 2.1.4** For the condensation measure $P$, we have

$$E(P) = \frac{1}{2}, \quad V(P) = \frac{(1-c)^2}{2(3-2c^2)} + V(\nu),$$

and, for any $x_0 \in \mathbb{R}$, $\int (x-x_0)^2 dP = (x_0 - \frac{1}{2})^2 + V(P)$.

**Proof** It is straightforward to see that $E(P) = \frac{1}{2}$. Now, using (1), we have

$$\int x^2 dP = \frac{1}{3} \int (s_1(x))^2 dP + \frac{1}{3} \int (s_2(x))^2 dP + \frac{1}{3} \int x^2 d\nu = \frac{1}{3} \int (cx)^2 dP + \frac{1}{3} \int (cx + r)^2 dP + \frac{1}{3} \int x^2 d\nu;$$
hence, \( (1 - \frac{2c^2}{3}) \int x^2dP = \frac{1}{3} \left( cr + r^2 + \int x^2d\nu \right) \). Since \( c + r = 1 \), this implies that
\[
\int x^2dP = \frac{1}{3 - 2c^2} \left( r + \int x^2d\nu \right).
\]

Thus, it follows that
\[
V(P) = \int x^2dP - \frac{1}{4} = \frac{1}{3 - 2c^2} \left( r + \int x^2d\nu \right) - \frac{1}{4}
\]
\[
= \frac{1}{3 - 2c^2} \left( r + V(\nu) + \frac{c^2 - 1}{2} \right) = \frac{(1 - c^2)}{2(3 - 2c^2)} + V(\nu).
\]

For any \( x_0 \in \mathbb{R} \), \( \int (x - x_0)^2dP = (x_0 - \frac{1}{2})^2 + V(P) \) follows from the standard probability arguments. \( \square \)

**Note 2.1.5** Let \( \omega \in I^k \), \( k \geq 0 \), and let \( X \) be the random variable with probability distribution \( P \). Then, by Eq. (1), it follows that \( P(J_\omega) = \frac{1}{3^k} \), \( P(L_\omega) = \frac{1}{3^{k+1}} \).

\[
E(X : X \in J_\omega) = \frac{1}{P(J_\omega)} \int_{J_\omega} xdP = \int xd(P \circ S^{-1}_\omega) = \int S_\omega(x)dP = S_\omega \left( \frac{1}{2} \right) \), and
\[
E(X : X \in L_\omega) = \frac{1}{P(L_\omega)} \int_{L_\omega} xdP = \int xd(v \circ S^{-1}_\omega) = \int S_\omega(x)d\nu = S_\omega \left( \frac{1}{2} \right) .
\]

For any \( x_0 \in \mathbb{R} \),
\[
\left\{ \begin{array}{l}
\int_{J_\omega} (x - x_0)^2dP(x) = \frac{1}{3^k} \left( c^{2k}V + (S_\omega \left( \frac{1}{2} \right) - x_0)^2 \right), \\
\int_{L_\omega} (x - x_0)^2dP(x) = \frac{1}{3^{k+1}} \left( c^{2k}W + (S_\omega \left( \frac{1}{2} \right) - x_0)^2 \right).
\end{array} \right. \tag{3}
\]

On the other hand, for any \( x_0 \in \mathbb{R} \), any \( \omega \in I^k \), \( k \geq 0 \),
\[
\int_{T_{\omega}(L)} (x - x_0)^2d\nu(x) = \frac{1}{2^k} \left( s^{2k}W + \left( T_\omega \left( \frac{1}{2} \right) - x_0 \right)^2 \right) . \tag{4}
\]

**Remark 2.1.6** By Lemma 2.1.4, it follows that the optimal set of one-mean for the condensation measure \( P \) consists of the expected value \( \frac{1}{2} \) and the corresponding quantization error is the variance \( V(P) \) of \( P \), i.e., \( V(P) = V_1(P) \). Notice that by ‘the variance of \( P \’) it is meant the variance of the random variable \( X \) with distribution \( P \).

**Proposition 2.1.7** (see [13]) For \( n \in \mathbb{N} \) with \( n \geq 2 \) let \( \ell(n) \) be the unique natural number with \( 2\ell(n) \leq n < 2\ell(n) + 1 \). Let \( \alpha_n(v) \) be an optimal set of \( n \)-means for \( v \), i.e., \( \alpha_n(v) \in \mathcal{C}_n(v) \). Then,
\[
\alpha_n(v) = \left\{ T_\omega \left( \frac{1}{2} \right) : \omega \in I^{\ell(n)} \setminus \tilde{I} \right\} \cup \left\{ T_{\omega 1} \left( \frac{1}{2} \right) : \omega \in \tilde{I} \right\} \cup \left\{ T_{\omega 2} \left( \frac{1}{2} \right) : \omega \in \tilde{I} \right\}
\]
for some $\tilde{I} \subset I^{\ell(n)}$ with $\text{card}(\tilde{I}) = n - 2^{\ell(n)}$. Moreover,

$$V_n(\nu) = \int \min_{a \in \alpha_n(\nu)} (x - a)^2 d\nu = \left(\frac{s^2}{2}\right)^{\ell(n)} W\left(2^{\ell(n)+1} - n + s^2(n - 2^{\ell(n)})\right).$$

The following lemma is straightforward; hence, we will state it without proof.

**Lemma 2.1.8** Let $\alpha$ be an optimal set of $n$-means for the condensation measure $P$. Then, for any $\omega \in I^*$, the set $S_\omega(\alpha) := \{S_\omega(a) : a \in \alpha\}$ is an optimal set of $n$-means for the image measure $P \circ S_\omega^{-1}$. Conversely, if $\beta$ is an optimal set of $n$-means for the image measure $P \circ S_\omega^{-1}$, then $S_\omega^{-1}(\beta)$ is an optimal set of $n$-means for $P$.

**Lemma 2.1.9** If $\alpha_n(\nu)$ is an optimal set of $n$-means for $\nu$, then, for any $\omega \in I^k, k \geq 0$, $S_\omega(\alpha_n(\nu))$ is an optimal set of $n$-means for the measure $\nu \circ S_\omega^{-1}$. Moreover,

$$\int_{L_\omega} \min_{a \in S_\omega(\alpha_n(\nu))} (x - a)^2 dP = \frac{c^{2k}}{3^k+1} V_n(\nu).$$

**Proof** Let $\alpha_n(\nu)$ be an optimal set of $n$-means for $\nu$. Then, $S_\omega(\alpha_n(\nu))$ is an optimal set of $n$-means for the image measure $\nu \circ S_\omega^{-1}$ follows from Lemma 2.1.8. Now, using (1) and Proposition 2.1.7,

$$\int_{L_\omega} \min_{a \in S_\omega(\alpha_n(\nu))} (x - a)^2 dP = \frac{1}{3^k+1} \int_{L_\omega} \min_{a \in S_\omega(\alpha_n(\nu))} (x - a)^2 d(\nu \circ S_\omega^{-1})$$

$$= \frac{1}{3^k+1} \int_{L} \min_{a \in S_\omega(\alpha_n(\nu))} (S_\omega(x) - a)^2 d\nu$$

$$= \frac{1}{3^k+1} c^{2k} \int_{L} \min_{a \in \alpha_n(\nu)} (x - a)^2 d\nu = \frac{c^{2k}}{3^k+1} V_n(\nu),$$

which completes the proof of the lemma. \(\Box\)

Next, we will determine the optimal sets of 2 and 3 means which will provide the base needed to determine the optimal sets of $F(n)$-means and the $F(n)$th quantization errors for a canonical sequence $\{F(n)\}_{n \geq 1}$.

**Proposition 2.1.10** Let $\alpha := \{a_1, a_2\}$ be an optimal set of two-means with $a_1 < a_2$. Then, $a_1 = \frac{2}{3} \left(S_1\left(\frac{1}{2}\right) + \frac{1}{2} T_1\left(\frac{1}{2}\right)\right), a_2 = \frac{2}{3} \left(S_2\left(\frac{1}{2}\right) + \frac{1}{2} T_2\left(\frac{1}{2}\right)\right)$, and the corresponding quantization error is

$$V_2 = 2 \left(\int_{J_1} (x - a_1)^2 dP + \int_{L_1} (x - a_1)^2 dP\right).$$

**Proof** Due to symmetry of the condensation measure $P$ with respect to the midpoint $\frac{1}{2}$, we can assume that if $\{a_1, a_2\}$ is an optimal set of two-means with $a_1 < a_2$, then
Similarly, due to this set of three points 

\[ a_1 = E(X : X \in [0, \frac{1}{2}]) \text{ and } a_2 = E(X : X \in [\frac{1}{2}, 1]). \] 

Since \( P(J_1 \cup T_1(L)) = P(J_1) + P(T_1(L)) = \frac{1}{3} + \frac{1}{3} \nu(T_1(L)) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \), we have

\[
a_1 = E(X : X \in J_1 \cup T_1(L)) = \frac{1}{P(J_1 \cup T_1(L))} \left( \int_{J_1} x \, dP + \int_{T_1(L)} x \, dP \right) = 2 \left( \frac{1}{3} S_1 \left( \frac{1}{2} \right) + \frac{1}{3} \int_{T_1(L)} \, dP \right) = \frac{2}{3} \left( S_1 \left( \frac{1}{2} \right) + \frac{1}{2} T_1 \left( \frac{1}{2} \right) \right).
\]

Similarly, \( a_2 = \frac{2}{3} \left( S_2 \left( \frac{1}{2} \right) + \frac{1}{2} T_2 \left( \frac{1}{2} \right) \right) = 1 - a_1 \). The corresponding quantization error is given by

\[
V_2 = \int \min_{a \in \omega} (x - a)^2 \, dP = 2 \left( \int_{J_2} (x - a_1)^2 \, dP + \int_{T_2(L)} (x - a_1)^2 \, dP \right),
\]

which completes the proof. \( \square \)

It should be observed that, by symmetry, in the proposition above we also have

\[
V_2 = \int \min_{a \in \omega} (x - a)^2 \, dP = 2 \left( \int_{J_2} (x - a_2)^2 \, dP + \int_{T_2(L)} (x - a_2)^2 \, dP \right).
\]

The proof of the following lemma is straightforward.

**Lemma 2.1.11** Let \( \omega \in I^k \) for \( k \geq 0 \). Then,

\[
\int_{J\omega \cup S_\omega[a, \frac{1}{2}]} (x - S_\omega(a_1))^2 \, dP = \frac{c^{2k}}{3^k} \frac{1}{2} V_2 = \int_{S_\omega[a, \frac{1}{2}, b] \cup J\omega \cup S_\omega(a_2)} (x - S_\omega(a_2))^2 \, dP.
\]

From the above lemma we deduce the following corollary.

**Corollary 2.1.12** Let \( \omega \in I^k \) for \( k \geq 0 \). Then, for any \( a \in \mathbb{R}, \)

\[
\int_{J\omega \cup S_\omega(T_1(L))} (x - a)^2 \, dP = \frac{c^{2k}}{3^k} \left( \frac{1}{2} V_2 + \frac{1}{2} (S_\omega(a_1) - a)^2 \right), \quad \text{and} \quad \int_{S_\omega(T_2(L)) \cup J\omega \cup S_\omega(a_2)} (x - a)^2 \, dP = \frac{c^{2k}}{3^k} \left( \frac{1}{2} V_2 + \frac{1}{2} (S_\omega(a_2) - a)^2 \right).
\]

**Proposition 2.1.13** Let \( \alpha := \{a_1, a_2, a_3\} \) be an optimal set of three-means with \( a_1 < a_2 < a_3 \). Then, \( a_1 = S_1 \left( \frac{1}{2} \right) = \frac{5}{6}, \) \( a_2 = \frac{1}{2}, \) and \( a_3 = S_2 \left( \frac{1}{2} \right) = 1 - \frac{c_2}{2}. \) The corresponding quantization error is \( V_3 = \frac{1}{4} (2c^2 V + W). \)

**Proof** Since both \( \nu \) and \( P \) are symmetric uniform measures with support \( K \), we can assume that \( a_1 \in J_1, \) \( a_2 \in L \) and \( a_3 = 1 - a_1. \) Hence, by (3), the quantization error due to this set of three points \( \beta := \{a_1, a_2, a_3\} \) is
\[
\int_{a \in \beta} \min(x - a)^2 dP \\
= \int_{J_1} (x - a_1)^2 dP + \int_{L} (x - a_2)^2 dP + \int_{J_2} (x - 1 + a_1)^2 dP \\
= \frac{1}{3} \left[ c^2 V + (S_1 \left( \frac{1}{2} \right) - a_1)^2 \right] \\
+ \frac{1}{3} \left[ W + \left( \frac{1}{2} - a_2 \right)^2 \right] + \frac{1}{3} \left[ c^2 V + (S_2 \left( \frac{1}{2} \right) - 1 + a_1)^2 \right] \\
= \frac{1}{3} \left[ 2c^2 V + W + 2 \left( \frac{c}{2} - a_1 \right)^2 + \left( \frac{1}{2} - a_2 \right)^2 \right] \\
= \frac{1}{3} \left( 2c^2 V + W + \frac{c^2}{2} + \frac{1}{4} \right) + \frac{1}{3} \left( a_1^2 - c a_1 - a_2 \right).
\]

The function \( f(a_1, a_2) = a_1^2 + a_2^2 - c a_1 - a_2 \) attains its minimum at \( a_1 = \frac{c}{2} \) and \( a_2 = \frac{1}{2} \) with minimum value \(-\frac{1}{4} (c^2 + \frac{1}{2}) \). Therefore, since \( V_3 \) is the quantization error for three-means, we have \( V_3 = \frac{1}{3} (2c^2 V + W) \). \( \square \)

**Proposition 2.1.14** Let \( \alpha \) be an optimal set of \( n \)-means for \( n \geq 3 \) such that \( \alpha \cap J_1 \neq \emptyset \), \( \alpha \cap J_2 \neq \emptyset \), and \( \alpha \cap L \neq \emptyset \). Then, for \( i = 1, 2 \), the Voronoi region of any point in \( \alpha \cap J_i \) does not contain any point from \( L \), and the Voronoi region of any point in \( \alpha \cap L \) does not contain any point from \( \alpha \cap J_i \).

**Proof** Let \( \alpha := \{a_1, a_2, \ldots, a_n\} \) be an optimal set of \( n \)-means for \( n \geq 3 \) such that \( 0 < a_1 < a_2 < \cdots < a_n < 1 \). Let \( j = \max\{1 \leq i \leq n : a_i \in \alpha \cap J_1\} \). Then, by the hypothesis, \( a_j \leq c \). Suppose that the Voronoi region of \( a_j \) contains points from \( L \). Then, \( \frac{1}{2} (a_j + a_{j+1}) > a \) implying that

\[
a_{j+1} > 2a - a_j \geq 2a - c = b.
\]

This leads to a contradiction because \( \alpha \cap L \neq \emptyset \). Hence, the Voronoi region of any point in \( \alpha \cap J_1 \) does not contain any point from \( L \). The rest of the statements are proved similarly. \( \square \)

### 2.2 Set-up for Optimal Sets of \( n \)-Means for \( n \geq 4 \)

As observed in the previous subsection, the interaction of the measures \( \nu \) and \( P \) leads to rather intricate arguments. In order to bring some transparency to the arguments and reveal the connections between the quantization dimension \( D(\nu) \) and the critical value \( \kappa \) we will proceed by considering some special cases of the measures \( P \) and \( \nu \).

Throughout the rest of the article we will assume that \( c = \frac{1}{3} \). Consequently, \( a = \frac{2}{3}, b = \frac{3}{5} \) and the critical value will be \( \kappa = \frac{2 \log 2}{\log 2 - \log 3} \). Three different cases of the self-similar measure \( \nu \), which will be determined by the values below, are considered.

1. \( s = \frac{1}{3} \), which implies \( D(\nu) = \frac{\log 2}{\log 3} > \kappa \),
(ii) \( s = \frac{1}{7} \), which implies \( D(\nu) = \frac{\log 2}{\log 7} < \kappa \), and

(iii) \( s = \sqrt{\frac{7}{15}} \), which implies \( D(\nu) = \frac{2\log 2}{\log 15} = \kappa \).

In order to provide further insight to the question posed at the end of the article, we will also consider \( s = \frac{1}{5} \), which implies \( D(\nu) = \frac{\log 2}{\log 5} > \kappa \). In each case, we will construct the canonical sequences to investigate the quantization for the associated condensation measures while exhibiting the optimal sets of \( n \)-means for \( n \geq 4 \) and determining the quantization dimensions.

3 Condensation Measure \( P \) with Self-Similar Measure \( \nu \) Satisfying \( D(\nu) > \kappa \)

In this case \( s = \frac{1}{3} \); hence, from the general results obtained in the previous section, we have

- \( E(\nu) = \frac{1}{2} \), \( W = V(\nu) = \frac{1}{200} \); \( E(P) = \frac{1}{2} \), \( V = V(P) = \frac{65}{384} \),
- \( \alpha_1 = \{ 1/2 \} \), with \( V_1 = V(P) \),
- \( \alpha_2 = \{ 19/90, 71/90 \} \) with \( V_2 = \frac{32929}{1182600} \), and
- \( \alpha_3 = \{ 1/10, 9/10 \} \) with \( V_3 = \frac{203}{43800} \).

3.1 Essential Lemmas and Propositions

**Lemma 3.1.1** Let \( \beta := \{ c, 1 \} \), where \( 0 < c < 1 \). Then, \( \int_{a \in \beta} \min(x - a)^2 dP = \frac{1517}{43800} \), and the minimum occurs when \( c = \frac{3}{10} \).

**Proof** Since \( \frac{3}{5} < \frac{1}{2} \left( \frac{3}{10} + 1 \right) = \frac{13}{20} < \frac{4}{5} \), the distortion error due to the set \( \beta := \{ \frac{3}{10}, 1 \} \) is

\[
\int_{a \in \beta} \min(x - a)^2 dP = \int_{J_1 \cup L} \left( x - \frac{3}{10} \right)^2 dP + \int_{J_2} (x - 1)^2 dP = \frac{1517}{43800}.
\]

Let \( \alpha := \{ a, 1 \} \) be an optimal set of two-means for which the minimum in the hypothesis occurs, and \( \tilde{V}_2 \) is the corresponding quantization error. Then, \( \tilde{V}_2 \leq \frac{1517}{43800} \). Suppose that \( a \leq \frac{1}{5} \). Then, since \( \frac{1}{2} \left( \frac{1}{5} + 1 \right) = \frac{3}{5} \), we have the distortion error as

\[
\int_{J_1} \left( x - S_1 \left( \frac{1}{2} \right) \right)^2 dP + \int_{L} \left( x - \frac{1}{5} \right)^2 dP + \int_{J_2} (x - 1)^2 dP = \frac{1663}{43800} > \tilde{V}_2,
\]

which leads to a contradiction. So, we can assume that \( \frac{1}{5} < a \). If \( a \geq \frac{1}{2} \), then

\[
\tilde{V}_2 \geq \int_{J_1} \left( x - \frac{1}{2} \right)^2 dP + \int_{T_1(L)} \left( x - \frac{1}{2} \right)^2 dP = \frac{65}{1168} > \tilde{V}_2,
\]
which is a contradiction. Next, if \( \frac{2}{5} \leq a < \frac{1}{2} \), then, as \( \frac{3}{5} < \frac{1}{2} \left( \frac{1}{2} + 1 \right) = \frac{3}{4} < \frac{4}{5} \), we have

\[
\tilde{V}_2 \geq \int_{J_1} \left( x - \frac{2}{5} \right)^2 \, dP + \int_{T_2(L)} \left( x - \frac{1}{2} \right)^2 \, dP + \int_{J_2} (x - 1)^2 \, dP = \frac{3253}{87600} \geq \tilde{V}_2,
\]

which is also a contradiction. So, we can assume that \( \frac{1}{2} < a < \frac{3}{5} \), and then notice that \( \frac{1}{2}(\frac{1}{2} + 1) = \frac{3}{5} < \frac{1}{2}(a + 1) < \frac{1}{2}(\frac{2}{5} + 1) < \frac{4}{5} \), yielding the fact that \( a = E(X : X \in J_1 \cup L) = \frac{P(J_1 \cup L)}{P(L_1)}(P(J_1)S_1(\frac{1}{2})) + P(L)\frac{1}{2} = \frac{1}{2}(\frac{1}{10} + \frac{1}{2}) = \frac{3}{10} \), and the corresponding quantization error is \( \tilde{V}_2 = \frac{1517}{438000} \).

**Corollary 3.1.2** Let \( \beta := \{c, \frac{1}{2}\} \), where \( 0 < c < \frac{1}{5} \). Then, \( J_1 \min_{a \in \beta} (x - a)^2 \, dP = \frac{1517}{3285000} \), and the minimum occurs when \( c = \frac{3}{50} \).

**Proof** By Lemma 3.1.1, we have

\[
\int_{J_1} \min_{a \in \beta} (x - a)^2 \, dP = \frac{1}{3} \int_{J_1} \min_{a \in \beta} (x - a)^2 \, dP \circ S_1^{-1} = \frac{1}{3} \int \min_{a \in \beta} (S_1(x) - a)^2 \, dP
\]

\[
= \frac{1}{3} \int \min_{a \in S_1^{-1}(\beta)} (S_1(x) - S_1(a))^2 \, dP = \frac{1}{75} \int \min_{a \in S_1^{-1}(\beta)} (x - a)^2 \, dP = \frac{1517}{3285000},
\]

which occurs when \( c = \frac{3}{50} \). \( \square \)

**Lemma 3.1.3** Let \( \alpha \) be an optimal set of four-means. Then, \( \alpha \cap J_1 \neq \emptyset \), \( \alpha \cap J_2 \neq \emptyset \), and \( \alpha \cap L \neq \emptyset \). Moreover, \( \alpha \) does not contain any point from the open intervals \( (\frac{1}{5}, \frac{2}{5}) \) and \( (\frac{3}{5}, \frac{4}{5}) \).

**Proof** Let us first consider the set \( \beta := \{S_1(\frac{1}{2}), T_1(\frac{1}{2}), T_2(\frac{1}{2}), S_2(\frac{1}{2})\} \). Then,

\[
\int \min_{a \in \beta} (x - a)^2 \, dP = 2 \left( \int_{J_1} \left( x - S_1\left( \frac{1}{2} \right) \right)^2 \, dP + \int_{T_2(L)} \left( x - T_1\left( \frac{1}{2} \right) \right)^2 \, dP \right)
\]

\[
= 2 \left( \frac{1}{75} V + \frac{1}{3} W \right) = \frac{1243}{394200}.
\]

Since \( V_4 \) is the quantization error for four-means, we have \( V_4 \leq \frac{1243}{394200} \). Let \( \alpha := \{a_1 < a_2 < a_3 < a_4\} \) be an optimal set of four-means. We first show that \( \alpha \cap J_1 \neq \emptyset \). For the sake of contradiction, assume that \( \alpha \cap J_1 = \emptyset \). Then \( \frac{1}{5} < a_1 \), which yields

\[
V_4 \geq \int_{J_1} \left( x - \frac{1}{5} \right)^2 \, dP = \frac{211}{43800} > V_4,
\]
which is a contradiction. Thus, we can assume that $\alpha \cap J_1 \neq \emptyset$. Similarly, we can show that $\alpha \cap J_2 \neq \emptyset$. We now show that $\alpha \cap L \neq \emptyset$. For the sake of contradiction, assume that $\alpha \cap L = \emptyset$. Suppose that $a_2 > \frac{3}{5}$. Then,

$$V_4 \geq \int_{J_1} (x - a_1)^2 \, dP + \int_{L} \left(x - \frac{3}{5}\right)^2 \, dP$$

$$\geq \int_{J_1} \left(x - S_1 \left(\frac{1}{2}\right)\right)^2 \, dP + \int_{L} \left(x - \frac{3}{5}\right)^2 \, dP$$

$$= \frac{1}{75} V + \frac{1}{3} \left(W + \left(\frac{1}{2} - \frac{3}{5}\right)^2\right) = \frac{71}{10950} > V_4,$$

which leads to a contradiction. So, we can assume that $a_2 < \frac{2}{5}$. Similarly, we have $\frac{3}{5} < a_3$. Due to symmetry of $P$, the following two cases can occur:

- **Case 1.** $\frac{1}{3} \leq a_2 < \frac{2}{5}$ and $\frac{1}{3} < a_3 \leq \frac{2}{3}$.
  In this case, $\frac{1}{3} \leq a_2 < \frac{2}{5}$ implies $\frac{1}{3}(a_1 + a_2) < \frac{1}{3}$ yielding $a_1 < \frac{2}{5} - a_2 \leq \frac{2}{5} - \frac{1}{3} = \frac{1}{15} < \frac{2}{25}$. Thus, due to symmetry, we have

$$V_4 \geq 2 \left(\int_{S_1(L) \cup J_{1.2}} \left(x - \frac{1}{15}\right)^2 \, dP + \int_{T_1(L)} \left(x - \frac{2}{5}\right)^2 \, dP\right) = \frac{110597}{29565000} > V_4,$$

which gives a contradiction.

- **Case 2.** $a_2 \leq \frac{1}{3}$ and $\frac{2}{5} \leq a_3$.
  In this case we have,

$$V_4 \geq 2 \int_{T_1(L)} \left(x - \frac{1}{3}\right)^2 \, dP = \frac{19}{5400} > V_4,$$

which is a contradiction.

By Case 1 and Case 2, we deduce that $\alpha \cap L \neq \emptyset$. We now show that $\alpha$ does not contain any point the open intervals $(\frac{1}{3}, \frac{2}{5})$ and $(\frac{2}{3}, \frac{4}{5})$. Suppose that $a_2 \in (\frac{1}{3}, \frac{2}{5})$.

Then, notice that $a_3 \in L$ and $a_4 \in J_2$. Again, two cases can arise:

- **Case I.** $\frac{1}{3} \leq a_2 < \frac{2}{5}$.
  In this case, $\frac{1}{2}(a_1 + a_2) < \frac{1}{3}$ implying $a_1 < \frac{2}{5} - a_2 \leq \frac{2}{5} - \frac{1}{3} = \frac{1}{15} < \frac{2}{25}$, and so

$$V_4 \geq \int_{S_1(L) \cup J_{1.2}} \left(x - \frac{1}{15}\right)^2 \, dP + \int_{T_1(L)} \left(x - \frac{2}{5}\right)^2 \, dP$$

$$+ \int_{T_2(L)} \left(x - T_2 \left(\frac{1}{2}\right)\right)^2 \, dP + \int_{J_2} \left(x - S_2 \left(\frac{1}{2}\right)\right)^2 \, dP;$$

and hence, $V_4 \geq \frac{101911}{29565000} > V_4$, which is a contradiction.

- **Case II.** $\frac{1}{3} < a_2 < \frac{1}{3}$.
  In this case, $\frac{1}{2}(a_2 + a_3) > \frac{2}{5}$ implying $a_3 > \frac{4}{5} - a_2 \geq \frac{4}{5} - \frac{1}{3} = \frac{7}{15}$. Recall Corollary 3.1.2, and $\int_{J_2} (x - S_2(\frac{1}{2}))^2 \, dP = \frac{1}{75} V$. The following subcases arise:
(i) If \( \frac{7}{15} < a \leq \frac{22}{45} \), then

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{1}(L)} \left( x - \frac{7}{15} \right)^2 dP + \int_{T_{2}(L)} \left( x - \frac{22}{45} \right)^2 dP + \frac{1}{75}
\]

\[
V = \frac{294859}{88695000} > V_4,
\]

(ii) If \( \frac{22}{45} \leq a \leq \frac{1}{2} \), then \( T_{111}(\frac{3}{5}) < \frac{1}{2} \left( \frac{4}{5} + \frac{22}{45} \right) < T_{112}(\frac{2}{3}) \) implying that

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{111}(L)} \left( x - \frac{1}{3} \right)^2 dP + \int_{T_{112}(L) \cup T_{12}(L)} \left( x - \frac{22}{45} \right)^2 dP + \frac{1}{75} V
\]

\[
= \frac{218863}{66521250} > V_4,
\]

(iii) If \( \frac{1}{2} \leq a \leq \frac{91}{180} \), then \( T_{111}(\frac{3}{5}) < \frac{1}{2} \left( \frac{3}{5} + \frac{1}{2} \right) < T_{1122}(\frac{2}{5}) \) implying that

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{111}(L)} \left( x - \frac{1}{3} \right)^2 dP + \int_{T_{1122}(L) \cup T_{12}(L)} \left( x - \frac{1}{2} \right)^2 dP
\]

\[
+ \int_{T_{2}(L)} \left( x - \frac{91}{180} \right)^2 dP + \frac{1}{75} V
\]

\[
= \frac{121358413}{38316240000} > V_4,
\]

(iv) If \( \frac{91}{180} \leq a \leq \frac{23}{45} \), then \( T_{1121}(\frac{3}{5}) < \frac{1}{2} \left( \frac{3}{5} + \frac{91}{180} \right) < T_{1122}(\frac{2}{5}) \) implying that

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{111}(L) \cup T_{1121}(L)} \left( x - \frac{1}{3} \right)^2 dP
\]

\[
+ \int_{T_{1122}(L) \cup T_{12}(L)} \left( x - \frac{91}{180} \right)^2 dP + \int_{T_{2}(L)} \left( x - \frac{23}{45} \right)^2 dP + \frac{1}{75} V
\]

\[
= \frac{83388217}{25544160000} > V_4,
\]

(v) If \( \frac{23}{45} \leq a \leq \frac{31}{60} \), then \( T_{11}(\frac{3}{5}) = \frac{1}{2} \left( \frac{1}{3} + \frac{23}{45} \right) \) implying that

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{11}(L)} \left( x - \frac{1}{3} \right)^2 dP + \int_{T_{12}(L)} \left( x - \frac{23}{45} \right)^2 dP
\]

\[
+ \int_{T_{2}(L)} \left( x - \frac{31}{60} \right)^2 dP + \frac{1}{75} V = \frac{381587}{118260000} > V_4,
\]
(vi) If \( \frac{31}{60} \leq a_3 \leq \frac{21}{40} \), then \( T_{11}(\frac{3}{5}) < \frac{1}{2} (\frac{1}{3} + \frac{31}{60}) < T_{12}(\frac{2}{5}) \) implying

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{11}(L)} (x - \frac{1}{3})^2 \, dP + \int_{T_{12}(L)} (x - \frac{31}{60})^2 \, dP \\
+ \int_{T_{2(L)}} (x - \frac{21}{40})^2 \, dP + \frac{1}{75} V = \frac{1491673}{473040000} > V_4,
\]

(vii) If \( \frac{21}{40} \leq a_3 \leq \frac{127}{240} \), then \( T_{11}(\frac{3}{5}) < \frac{1}{2} (\frac{1}{3} + \frac{21}{40}) < T_{12}(\frac{2}{5}) \) implying

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{11}(L)} (x - \frac{1}{3})^2 \, dP + \int_{T_{12}(L)} (x - \frac{21}{40})^2 \, dP \\
+ \int_{T_2(L)} (x - \frac{127}{240})^2 \, dP + \frac{1}{75} V = \frac{6034217}{1892160000} > V_4,
\]

(viii) If \( \frac{127}{240} \leq a_3 \leq \frac{8}{15} \), then \( T_{11}(\frac{3}{5}) < \frac{1}{2} (\frac{1}{3} + \frac{127}{240}) < T_{12}(\frac{2}{5}) \) implying

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{11}(L)} (x - \frac{1}{3})^2 \, dP + \int_{T_{12}(L)} (x - \frac{127}{240})^2 \, dP \\
+ \int_{T_2(L)} (x - \frac{8}{15})^2 \, dP + \frac{1}{75} V = \frac{12070259}{3784320000} > V_4,
\]

(ix) If \( \frac{8}{15} \leq a_3 \leq \frac{1}{2} (\frac{8}{15} + T_{212}(\frac{2}{5})) = \frac{73}{135} \), then \( T_{11}(\frac{3}{5}) < \frac{1}{2} (\frac{1}{3} + \frac{8}{15}) < T_{12}(\frac{2}{5}) \) implying

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{11}(L)} (x - \frac{1}{3})^2 \, dP + \int_{T_{12}(L)} (x - \frac{8}{15})^2 \, dP \\
+ \int_{T_2(L)} (x - \frac{73}{135})^2 \, dP + \frac{1}{75} V = \frac{10098449}{3193020000} > V_4,
\]

(x) If \( \frac{1}{2} (\frac{8}{15} + T_{212}(\frac{2}{5})) = \frac{73}{135} \leq a_3 \leq \frac{5}{9} = T_{21}(\frac{2}{5}) \), then \( T_{11}(\frac{3}{5}) < \frac{1}{2} (\frac{1}{3} + \frac{73}{135}) < T_{12}(\frac{2}{5}) \) implying

\[
V_4 \geq \frac{1517}{3285000} + \int_{T_{11}(L)} (x - \frac{1}{3})^2 \, dP + \int_{T_{12}(L)} (x - \frac{73}{135})^2 \, dP \\
+ \int_{T_2(L)} (x - \frac{5}{9})^2 \, dP + \frac{1}{75} V = \frac{10098449}{3193020000} > V_4,
\]
(xi) If $T_{21}(\frac{3}{2}) = \frac{5}{9} \leq a_3$, then $\frac{1}{2}(\frac{1}{2} + \frac{5}{9}) = T_{12}(\frac{5}{9})$ implying that

$$V_4 \geq \frac{1517}{3285000} + \int_{T_{11}(L)} \left(x - \frac{1}{3}\right)^2 dP + \int_{T_{12}(L)} \left(x - \frac{5}{9}\right)^2 dP + \frac{1}{75}V$$

$$= \frac{584243}{177390000} > V_4,$$

each of which is a contradiction. Thus, by Case I and Case II, it follows that $a_2 \notin (\frac{1}{3}, \frac{2}{3})$, i.e., $\alpha$ does not contain any point from the open interval $(\frac{1}{3}, \frac{2}{3})$. Reflecting the situation with respect to the point $\frac{1}{2}$, we also deduce that $\alpha$ does not contain any point from the open interval $(\frac{3}{5}, \frac{4}{5})$. Thus, the proof of the lemma is complete. \qed

**Proposition 3.1.4** \{S_1(\frac{1}{2}), T_1(\frac{1}{2}), T_2(\frac{1}{2}), S_2(\frac{1}{2})\} is an optimal set of four-means for $P$ with quantization error $V_4 = \frac{1243}{394200}$.

**Proof** If $\beta := \{S_1(\frac{1}{2}), T_1(\frac{1}{2}), T_2(\frac{1}{2}), S_2(\frac{1}{2})\}$, then

$$\int_{a \in \beta} \min(x - a)^2 dP = 2 \left(\int_{J_1} \left(x - S_1\left(\frac{1}{2}\right)\right)^2 dP + \int_{T_1(L)} \left(x - T_1\left(\frac{1}{2}\right)\right)^2 dP\right)$$

$$= 2 \left(\frac{1}{75}V + \frac{1}{3}18W\right) = \frac{1243}{394200}.$$

Since $V_4$ is the quantization error for four-means, $V_4 \leq \frac{1243}{394200}$. Let $\alpha := \{a_1, a_2, a_3, a_4\}$ be an optimal set of four-means. Since optimal quantizers are the centroids of their own Voronoi regions, without any loss of generality, we can assume that $0 < a_1 < a_2 < a_3 < a_4 < 1$. By Lemma 3.1.3, we see that $\alpha$ contains points from $J_1$, $L$, $J_2$, and $\alpha$ does not contain any point from the open intervals $(\frac{1}{5}, \frac{2}{5}), (\frac{3}{5}, \frac{4}{5})$. We now show that $\text{card}(\alpha \cap L) = 2$. Suppose that $\text{card}(\alpha \cap L) = 1$. Then, without any loss of generality, we can assume that $\text{card}(\alpha \cap J_1) = 2$ and $\text{card}(\alpha \cap J_2) = 1$. Recall Proposition 2.1.14. Then, by Lemma 2.1.8, we have $\alpha \cap J_1 = \{S_1(\frac{19}{20}), S_1(\frac{7}{10})\}$, $\alpha \cap L = \{\frac{1}{2}\}$, and $\alpha \cap J_2 = \{S_2(\frac{5}{6})\}$. Then, we see that the quantization error is $\frac{3123}{38895000} > V_4$, which is a contradiction. Hence, $\text{card}(\alpha \cap L) = 2$, $\text{card}(\alpha \cap J_1) = \text{card}(\alpha \cap J_2) = 1$ implying the fact that $\alpha = \{S_1(\frac{1}{2}), T_1(\frac{1}{2}), T_2(\frac{1}{2}), S_2(\frac{1}{2})\}$ is an optimal set of four-means with quantization error $V_4 = \frac{1243}{394200}$. \qed

**Lemma 3.1.5** Let $\alpha$ be an optimal set of five-means. Then, $\alpha \cap J_1 \neq \emptyset$, $\alpha \cap J_2 \neq \emptyset$, and $\alpha \cap L \neq \emptyset$. Moreover, $\alpha$ does not contain any point from the open intervals $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$. 

Proof First, consider the set $\beta := \{ S_1(\frac{19}{90}), S_1(\frac{71}{90}), T_1(\frac{1}{2}), T_2(\frac{2}{3}), S_2(\frac{1}{2}) \}$. Then,

\[
\int \min_{a \in \beta} (x-a)^2 \, dP = 2 \left( \int_{T_1(L) \cap S_1(T_1(L))} (x - S_1(\frac{19}{90}))^2 \, dP + \int_{T_1(L)} (x - T_1(\frac{1}{2}))^2 \, dP \right) + \int_{T_2} (x - S_2(\frac{1}{2}))^2 \, dP = 2 \left( \frac{1}{75} V + \frac{1}{3} W + \frac{1}{18} W \right) + \frac{1}{75} V = \frac{180979}{88695000}.
\]

Since $V_5$ is the quantization error for five-means, we have $V_5 \leq \frac{180979}{88695000}$. Let $\alpha := \{ a_1 < a_2 < a_3 < a_4 < a_5 \}$ be an optimal set of five-means. Proceeding in the similar way as shown in the proof of Proposition 2.1.13, we have $0 < a_1 < \frac{1}{5}$ and $\frac{4}{5} < a_5 < 1$ implying $\alpha \cap J_1 \neq \emptyset$ and $\alpha \cap J_2 \neq \emptyset$. We now show that $\alpha \cap L \neq \emptyset$. For the sake of contradiction, assume that $\alpha \cap L = \emptyset$.

If $\frac{3}{5} < a_2$. Then,

\[
V_5 \geq \int_{J_1} (x - S_1(\frac{1}{2}))^2 \, dP + \int_{L} (x - \frac{3}{5})^2 \, dP = \frac{1}{75} V + \frac{1}{3} \left( W + \left( \frac{1}{2} - \frac{3}{5} \right)^2 \right) = \frac{71}{10950} > V_5.
\]

If $a_2 < \frac{2}{5} < \frac{3}{5} < a_3$, the following three cases can arise:

(i) $\frac{3}{10} \leq a_2 < \frac{2}{5}$ and $\frac{3}{5} < a_3 \leq \frac{7}{10}$. Then, $\frac{1}{2}(a_1 + a_2) < \frac{1}{5}$ and $\frac{4}{5} < \frac{1}{2}(a_3 + a_4)$ implying $a_1 < \frac{2}{5} - a_2 \leq \frac{1}{10}$ and $a_4 > \frac{8}{5} - a_3 \geq \frac{9}{10}$. Hence, by symmetry,

\[
V_5 \geq \int_{J_1 \cup S_1(T_1(L))} (x - S_1(\frac{19}{90}))^2 \, dP + 2 \int_{S_1(T_2(L)) \cup J_{12}} (x - \frac{1}{10})^2 \, dP + 2 \int_{T_1(L)} (x - \frac{2}{5})^2 \, dP = \frac{394729}{177390000} > V_5.
\]

(ii) $\frac{3}{10} \leq a_2 < \frac{2}{5}$ and $\frac{7}{10} \leq a_3$. Then, $T_{21} \left( \frac{3}{5} \right) < \frac{1}{2} \left( \frac{3}{5} + \frac{7}{10} \right) < T_{21} \left( \frac{2}{3} \right)$, and so

\[
V_5 \geq \int_{J_1 \cup S_1(T_1(L))} (x - S_1(\frac{19}{90}))^2 \, dP + \int_{S_1(T_2(L)) \cup J_{12}} (x - \frac{1}{10})^2 \, dP + \int_{T_1(L)} (x - \frac{2}{5})^2 \, dP + \int_{T_{22}(L)} (x - \frac{7}{10})^2 \, dP = \frac{794483}{354780000} > V_5.
\]

(iii) $a_2 \leq \frac{3}{10}$ and $\frac{7}{10} \leq a_3$. Then, due to symmetry,

\[
V_5 \geq 2 \int_{T_1(L)} (x - \frac{3}{10})^2 \, dP = \frac{11}{1800} > V_5.
\]
In each of the cases we reach a contradiction; hence, we can assume that \( \alpha \cap L \neq \emptyset \).

Next, recall Proposition 2.1.14. If \( \text{card}(\alpha \cap L) = 3 \), then

\[
V_5 \geq \int_{J_1} \left( x - S_1 \left( \frac{1}{2} \right) \right)^2 dP + \int_{J_2} \left( x - S_2 \left( \frac{1}{2} \right) \right)^2 dP
\]

\[
= \frac{2}{75} V = \frac{13}{4380} > V_5,
\]

which gives a contradiction. So, we can assume that \( 1 \leq \text{card}(\alpha \cap L) \leq 2 \). If \( \text{card}(\alpha \cap L) = 1 \), then, due to symmetry, the following two cases can arise:

Case 1. \( a_3 = \frac{1}{2}, \frac{3}{10} \leq a_2 < \frac{2}{5} \) and \( \frac{3}{5} < a_4 \leq \frac{7}{10} \): As shown above, we have \( a_1 < \frac{1}{10} \) and \( a_5 > \frac{9}{10} \). Moreover, \( T_{1212}(\frac{2}{5}) < \frac{1}{2} (\frac{3}{5} + \frac{1}{2}) < T_{1212}(\frac{3}{5}) \). Thus,

\[
V_5 \geq 2 \int_{J_1 \cup S_1(T_1)} \left( x - S_1 \left( \frac{19}{90} \right) \right)^2 dP + 2 \int_{S_1(T_2(L)) \cup J_12} \left( x - \frac{1}{10} \right)^2 dP
\]

\[
+ 2 \int_{T_1(L) \cup T_{1211}} \left( x - \frac{2}{5} \right)^2 + 2 \int_{T_{122}(L)} \left( x - \frac{1}{2} \right)^2 = \frac{40002139}{19158120000} > V_5, \text{ contradiction.}
\]

Case II. \( a_3 = \frac{1}{2}, a_2 \leq \frac{3}{10} \) and \( \frac{7}{10} \leq a_4 \): First, notice that \( \frac{1}{2} (\frac{3}{10} + \frac{1}{2}) = \frac{2}{5} \) and \( \frac{1}{2} (\frac{1}{2} + \frac{7}{10}) = \frac{3}{5} \) implying the fact that the Voronoi regions of \( a_2 \) and \( a_4 \) do not contain any point from \( L \), and thus, we have \( a_1, a_2 \in J_1 \), and \( a_4, a_5 \in J_2 \). Hence,

\[
V_5 \geq \int_{J_1} \min_{a \in [a_1, a_2]} (x - a)^2 dP + \int_L (x - \frac{1}{2})^2 + \int_{J_2} \min_{a \in [a_4, a_5]} (x - a)^2 dP
\]

\[
= \frac{2}{75} V_2 + \frac{1}{3} W + \frac{2}{75} \frac{1}{2} V_2
\]

yielding \( V_5 \geq \frac{213683}{88695000} > V_5 \), which gives a contradiction.

Therefore, we can assume that \( \text{card}(\alpha \cap L) = 2 \). We now show that \( \alpha \) does not contain any point from the open intervals \( (\frac{1}{5}, \frac{2}{5}) \) and \( (\frac{3}{5}, \frac{4}{5}) \). Suppose that \( \alpha \) contains a point from \( (\frac{1}{5}, \frac{2}{5}) \). In that case, \( \alpha \) does not contain any point from \( (\frac{3}{5}, \frac{4}{5}) \) as \( a_1 \in J_1 \) and \( a_3, a_4 \in L \); and \( a_5 \in J_2 \); consequently, the following two possibilities arise:

1. \( \frac{3}{10} \leq a_2 < \frac{2}{5} \): Then, \( a_1 < \frac{1}{10} \), and

\[
V_5 \geq \int_{J_1 \cup S_1(T_1)} \left( x - S_1 \left( \frac{19}{90} \right) \right)^2 dP + \int_{S_1(T_2(L)) \cup J_12} \left( x - \frac{1}{10} \right)^2 dP
\]

\[
+ \int_{T_1(L)} \left( x - \frac{2}{5} \right)^2 dP + \int_{T_{21}(L)} \left( x - T_{21} \left( \frac{1}{2} \right) \right)^2 dP
\]
Lemma 3.1.6 Let $a$ be an optimal set of $n$-means for $n = 6$. Then, $a \cap J_1 \neq \emptyset$, $a \cap J_2 \neq \emptyset$ and $a \cap L \neq \emptyset$. Moreover, $a$ does not contain any point from the open intervals $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{3}{5}, \frac{4}{5})$.

Proof Let $\alpha := \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be an optimal set of six-means. Since optimal quantizers are the centroids of their own Voronoi regions, without any loss of generality, we can assume that $0 < a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < 1$. Now, consider the set of points $\beta := \{S_1(\frac{19}{90}), S_1(\frac{21}{90}), T_1(\frac{1}{2}), T_2(\frac{1}{2}), S_2(\frac{19}{90}), S_2(\frac{21}{90})\}$. Then,

$$\int_{a \in \beta} \min(x-a)^2 dP = 2 \left( \int_{T_1(L) \cup S_1(T_1(L))} (x-S_1(\frac{19}{90}))^2 dP + \int_{J_1 \cup S_1(T_1(L))} (x-S_1(\frac{71}{90}))^2 dP \right) + \int_{T_1(L)} (x-T_1(\frac{1}{2}))^2 dP = 2 \left( \frac{1}{75} \frac{1}{2} V_2 + \frac{1}{75} \frac{1}{2} V_2 + \frac{1}{3} \frac{1}{18} W \right) = \frac{82283}{88695000}.$$

Since $V_6$ is the quantization error for six-means, we have $V_6 < \frac{82283}{88695000}$. Proceeding in the similar way as in the proof of Proposition 2.1.13, we have $0 < a_1 < \frac{1}{3}$ and $\frac{4}{5} < a_4 < 1$ implying $a_1 \neq \emptyset$ and $a_2 \neq \emptyset$. $P$ being symmetric about $\frac{1}{2}$, we can assume that there are three optimal quantizers to the left of $\frac{1}{2}$ and three optimal quantizers to the right of $\frac{1}{2}$. We now show that $a_3 \neq \emptyset$. Suppose that $a_3 \neq \emptyset$. Then, $a_3 < \frac{2}{3}$ and $\frac{3}{5} < a_4$. Due to symmetry, first suppose that $\frac{3}{10} < a_3 < \frac{2}{3}$ and
\[
\frac{3}{7} < a_4 \leq \frac{7}{10}.
\]
Then, \(\frac{1}{3}(a_2 + a_3) < \frac{1}{5}\) and \(\frac{4}{3} < \frac{1}{3}(a_4 + a_5)\) implying \(a_2 < \frac{2}{5} - a_3 \leq \frac{3}{5} - \frac{3}{10} = \frac{9}{10}\) and \(a_5 > \frac{8}{5} - a_4 \geq \frac{8}{5} - \frac{7}{10} = \frac{9}{10}\). Hence, by Corollary 2.1.12, we have
\[
V_6 \geq 2 \left( \int_{S_1(T_2(L)) \cup S_1(J_2)} (x - \frac{1}{10})^2 dP \right) = \frac{13}{8760} > V_6,
\]
which gives a contradiction. Next, suppose that \(a_3 \leq \frac{3}{10}\) and \(\frac{7}{10} \leq a_4\). Then,
\[
V_6 \geq 2 \int_{T_1(L)} (x - \frac{3}{10})^2 dP = \frac{11}{1800} > V_6,
\]
which leads to another contradiction. So, we can assume that \(a \cap L \neq \emptyset\). We now show that \(a\) does not contain any point from the open intervals \((\frac{1}{5}, \frac{2}{5})\) and \((\frac{3}{5}, \frac{4}{5})\). Suppose that \(a\) contains a point from the open interval \((\frac{1}{5}, \frac{2}{5})\). Then, due to symmetry, \(a\) will also contain a point from \((\frac{3}{5}, \frac{4}{5})\), and the only possible case is that \(a_2 \in (\frac{1}{5}, \frac{2}{5})\) and \(a_5 \in (\frac{3}{5}, \frac{4}{5})\). The following two case can happen:

Case 1. \(\frac{3}{10} \leq a_2 < \frac{2}{5}\) and \(\frac{3}{5} < a_5 \leq \frac{7}{10}\): Then,
\[
V_6 \geq 2 \left( \int_{S_1(T_2(L)) \cup S_1(J_2)} (x - \frac{1}{10})^2 dP \right) = \frac{13}{8760} > V_6,
\]
a contradiction.

Case 2. \(\frac{1}{5} < a_2 \leq \frac{3}{10}\) and \(\frac{7}{10} \leq a_5 < \frac{4}{5}\): Then, \(\frac{1}{3}(a_2 + a_3) > \frac{2}{5}\) implies \(a_3 > \frac{4}{5} - a_2 \geq \frac{4}{5} - \frac{3}{10} = \frac{1}{2}\), i.e., \(a_3 > \frac{1}{2}\). Similarly, \(a_4 < \frac{1}{2}\), which is a contradiction, as \(a_3 < a_4\).

Hence, \(a\) does not contain any point from the open intervals \((\frac{1}{5}, \frac{2}{5})\) and \((\frac{3}{5}, \frac{4}{5})\). □

**Proposition 3.1.7** Let \(\alpha_n\) be an optimal set of \(n\)-means for all \(n \geq 3\). Then, \(\alpha_n \cap J_1 \neq \emptyset\), \(\alpha_n \cap J_2 \neq \emptyset\), and \(\alpha_n \cap L \neq \emptyset\). Moreover, \(\alpha_n\) does not contain any point from the open intervals \((\frac{1}{5}, \frac{2}{5})\) and \((\frac{3}{5}, \frac{4}{5})\).

**Proof** By Proposition 2.1.13, Lemmas 3.1.3, 3.1.5, and 3.1.6, it follows that the proposition is true for \(3 \leq n \leq 6\). We now show that the proposition is true for all \(n \geq 7\). Let \(\alpha_n := \{a_1, a_2, a_3, \ldots, a_n\}\) be an optimal set of \(n\)-means for all \(n \geq 7\). Since optimal quantizers are the centroids of their own Voronoi regions, without any loss of generality, we can assume that \(0 < a_1 < a_2 < a_3 < \cdots < a_n < 1\). Consider the set of seven points

\[
\beta_7 = \left\{ S_1 \left( \frac{19}{90} \right), S_1 \left( \frac{71}{90} \right), T_1 \left( \frac{1}{2} \right), T_2 \left( \frac{1}{2} \right), S_{21} \left( \frac{1}{2} \right), S_2 \left( \frac{1}{2} \right), S_{22} \left( \frac{1}{2} \right) \right\}.
\]

Then, \(\int_{\alpha \in \beta_7} (x - a)^2 dP = \frac{10967}{17739000}\). Since \(V_n\) is the quantization error for \(n\)-means for all \(n \geq 7\), we have \(V_n \leq V_7 \leq \frac{10967}{17739000}\). Proceeding in the similar way as in the proof of Proposition 2.1.13, we have \(0 < a_1 < \frac{1}{3}\) and \(\frac{4}{5} < a_n < 1\) implying \(\alpha_n \cap J_1 \neq \emptyset\).
and \( \alpha_n \cap J_2 \neq \emptyset \). We now show that \( \alpha_n \cap L \neq \emptyset \). Suppose that \( \alpha_n \cap L = \emptyset \). Let \( j = \max \{ i : a_i < \frac{2}{5} \text{ for all } 1 \leq i \leq n \} \), and then \( a_j < \frac{2}{5} \). First, assume that \( \frac{3}{10} \leq a_j < \frac{2}{5} \). Then, \( \frac{1}{2}(a_{j-1} + a_j) < \frac{1}{2} \) implying \( a_{j-1} < \frac{2}{5} - a_j \leq \frac{2}{5} - \frac{3}{10} = \frac{1}{10} \), and so

\[
V_n \geq \int_{S_1(T_2(L)) \cup S_1(J_2)} \left( x - \frac{1}{10} \right)^2 dP + \int_{T_1(L)} \left( x - \frac{2}{5} \right)^2 dP + \int_{T_2(L)} \left( x - \frac{3}{5} \right)^2 dP = 0.00129756 > V_n,
\]

which is a contradiction. Next, assume that \( a_j \leq \frac{3}{10} \). Then, notice that \( T_{11}(\frac{3}{5}) = \frac{19}{45} < \frac{11}{2} (\frac{3}{10} + \frac{3}{5}) = \frac{9}{20} \); hence,

\[
V_n \geq \int_{T_{11}(L)} \left( x - \frac{3}{10} \right)^2 dP = \frac{67}{64800} > V_n,
\]

which leads to a contradiction. Thus, we can conclude that \( \alpha_n \cap L \neq \emptyset \). We now prove that \( \alpha_n \) does not contain any point from the open intervals \((\frac{1}{5}, \frac{2}{5})\) and \((\frac{3}{5}, \frac{4}{5})\).

Suppose that \( \alpha_n \) contains a point from the open interval \((\frac{1}{5}, \frac{2}{5})\). Let \( k = \max \{ i : a_i < \frac{2}{5} \text{ for all } 1 \leq i \leq n \} \), and then \( a_k < \frac{2}{5} \).

Case I. \( \frac{3}{10} \leq a_k < \frac{1}{5} \): Then, \( \frac{1}{2}(ak_{-1} + ak) \leq \frac{1}{2} \) implies \( ak_{-1} \leq \frac{1}{10} \) which yields

\[
V_n \geq \int_{S_1(T_2(L)) \cup S_1(J_2)} \left( x - \frac{1}{10} \right)^2 dP = \frac{13}{17520} > V_n, \quad \text{which is a contradiction.}
\]

Case II. \( \frac{1}{5} < a_k \leq \frac{3}{10} \): Then, \( \frac{1}{2}(ak + ak_{+1}) > \frac{2}{5} \) implies \( ak_{+1} > \frac{1}{2} \) which yields

\[
V_n \geq \int_{T_1(L)} \left( x - \frac{1}{2} \right)^2 dP = \frac{1}{1200} > V_n, \quad \text{which also leads to a contradiction.}
\]

By Case I and Case II, we can assume that \( \alpha_n \) does not contain any point from the open interval \((\frac{1}{5}, \frac{2}{5})\). Reflecting the situation with respect to \( \frac{1}{2} \), we can also assume that \( \alpha_n \) does not contain any point from the open interval \((\frac{3}{5}, \frac{4}{5})\). \hfill \square

The following proposition gives a property of the \( n \)-th quantization error.

**Proposition 3.1.8** Let \( \alpha \) be an optimal set of \( n \)-means for \( P \), where \( n \geq 3 \). Set \( \alpha_1 := \alpha \cap J_1 \), \( \alpha_2 := \alpha \cap J_2 \) and \( \alpha_L := \alpha \cap L \). Let \( n_1 = \text{card}(\alpha \cap J_1) \), \( n_2 = \text{card}(\alpha \cap J_2) \) and \( n_L = \text{card}(\alpha \cap L) \). Then,

\[
V_n(P) = \frac{1}{75} (V_{n_1}(P) + V_{n_2}(P)) + \frac{1}{3} V_{n_L}(v).
\]
Proof By Proposition 3.1.7, \( \alpha \) does not contain any point from the open intervals \((\frac{1}{3}, \frac{2}{3})\) and \((\frac{3}{4}, \frac{4}{3})\), and so \( n = n_1 + n_2 + n_L \). By Lemma 2.1.8, we see that \( S_1^{-1}(\alpha_1) \) is an optimal set of \( n_1 \)-means for \( P \), and \( S_2^{-1}(\alpha_2) \) is an optimal set of \( n_2 \)-means for \( P \). In the similar way it can be seen that \( \alpha_L \) is an optimal set of \( n_L \)-means for \( P|_L = v \), where \( P|_L \) is the conditional probability measure of \( P \) on \( L \) as defined by (2). Again, \( P \)-almost surely, the Voronoi region of any point in \( \alpha \cap J_i \) for \( i = 1, 2 \), does not contain any point from the Voronoi region of any point in \( \alpha \cap L \), and vice versa. Thus,

\[
V_n(P) = \int_{J_1} \min_{a \in \alpha_1} (x - a)^2 dP + \int_L \min_{a \in \alpha_L} (x - a)^2 dP + \int_{J_2} \min_{a \in \alpha_2} (x - a)^2 dP
\]

\[
= \frac{1}{75} \int_{\alpha \in S_1^{-1}(\alpha_1)} (x - a)^2 dP + \frac{1}{3} \int_{\alpha \in \alpha_L} (x - a)^2 dP
\]

\[
+ \frac{1}{75} \int_{\alpha \in S_2^{-1}(\alpha_2)} (x - a)^2 dP
\]

\[
= \frac{1}{75} (V_{n_1}(P) + V_{n_2}(P)) + \frac{1}{3} V_{n_L}(v),
\]

which completes the proof. \( \square \)

3.2 Canonical Sequences and Optimal Quantization

In this section, we first define the canonical sequences \( \{a(n)\}_{n \geq 1} \) and \( \{F(n)\}_{n \geq 1} \) associated with the condensation system, then calculate the optimal sets of \( F(n) \)-means, \( F(n) \)-th quantization error, quantization dimension and quantization coefficients.

Observation. Let \( A \) be a Borel subset of \( \mathbb{R} \), \( \alpha_n \) be an optimal set of \( n \)-means for \( P \), and let \( V(P; \alpha_n, A) \) be the quantization error contributed by \( \alpha_n \) on the set \( A \) with respect to \( P \). Using Propositions 2.1.14, 3.1.7, and 3.1.8, it follows that if

\[
V(P; \alpha_n, J_i) \geq \max \{ V(P; \alpha_n, J_j), V(P; L) \},
\]

for \( 1 \leq i \neq j \leq 2 \), then \( \text{card}(\alpha_{n+1} \cap J_i) = \text{card}(\alpha_n \cap J_i) + 1 \), \( \text{card}(\alpha_{n+1} \cap L) = \text{card}(\alpha_n \cap L) \), and \( \text{card}(\alpha_{n+1} \cap J_j) = \text{card}(\alpha_n \cap J_j) \); and if

\[
V(P; \alpha_n, J_i) \geq \max \{ V(P; \alpha_n, J_i) : 1 \leq i \leq 2 \},
\]

then \( \text{card}(\alpha_{n+1} \cap J_i) = \text{card}(\alpha_n \cap J_i) \) for \( i = 1, 2 \), and \( \text{card}(\alpha_{n+1} \cap L) = \text{card}(\alpha_n \cap L) + 1 \).

In order to avoid routine technicalities, we omit the justification of the Observation. Notice that the Observation enables one to construct optimal sets \( \alpha_n \), \( n \geq 3 \), beginning with the optimal set of three-means. For example, by Proposition 2.1.13, the set \( \alpha_3 := \{ S_1(\frac{1}{2}), \frac{1}{2}, S_2(\frac{1}{2}) \} \) is an optimal set of three-means which contains one element from each of \( J_1, L, \) and \( J_2 \). Since \( V(P; \alpha_3, L) = \frac{1}{3}W > \frac{1}{75}V = V(P; \alpha_3, J_1) = V(P; \alpha_3, J_2) \), by the Observation above, an optimal set \( \alpha_4 \) of four-means must contain two elements from \( L \), and one element from each of \( J_1 \) and \( J_2 \) yielding
\( \alpha_4 = \{ S_1(\frac{1}{2}), T_1(\frac{1}{2}), T_2(\frac{1}{2}), S_2(\frac{1}{2}) \} \), which is justified by Proposition 3.1.4. Once \( \alpha_4 \) is known, similarly, one calculates an optimal set \( \alpha_5 \) of five-means, and so forth.

Having optimal sets of \( n \)-means for \( 1 \leq n \leq 6 \), and utilizing the Observation, we define the canonical sequences that are useful in calculating the optimal sets of \( n \)-means and the \( n \)th quantization errors for all \( n \in \mathbb{N} \).

**Definition 3.2.1** The sequences \( \{ a(n) \}_{n \geq 1} \) and \( \{ F(n) \}_{n \geq 1} \), where

\[
\begin{align*}
a(n) &= \frac{1}{4} \left( 6n + (-1)^{n+1} - 7 \right), \quad \text{and} \\
F(n) &= \begin{cases} 
2^n(n + 1) & \text{if } 1 \leq n \leq 4, \\
5 \cdot 2^n + 2n - \frac{7}{4} \sum_{k=5}^{n} 2^{\frac{k}{2} + \frac{1-k+1}{4}} & \text{if } n \geq 5,
\end{cases}
\end{align*}
\]

are the canonical sequences associated with the condensation system \((S, p, v)\).

**Remark 3.2.1** First several terms of the sequences \( \{ a(n) \} \) and \( \{ F(n) \} \) are

\[
\begin{align*}
\{ a(n) \}_{n \geq 1} &= \{ 0, 1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, \ldots \}, \\
\{ F(n) \}_{n \geq 1} &= \{ 4, 12, 32, 80, 224, 576, 1664, 4352, 12800, 33792, \ldots \}.
\end{align*}
\]

Also, it is easy to see that \( a(2k + 1) = 3k \) and \( a(2k) = 3k - 2 \) for all \( k \geq 1 \).

**Lemma 3.2.2** For any positive integer \( k \geq 1 \), we have

(i) \( F(2k + 1) = 2^{k+1}(1 + 3 \cdot 2^{k-1}) = 2 \cdot 2^{2k} + 3 \cdot 2^k \), and

(ii) \( F(2k) = 2^{2k}(1 + 2^k) = 2^{2k} + 2^{3k} \).

In fact, for any \( n \geq 3 \), we have \( F(n) = 2^{a(n)} + 2F(n-1) \), and if \( n = 2 \), then \( F(2) = 2^2 + 2F(1) \).

**Proof** Notice that if \( k = 1 \) and \( k = 2 \), then the relations given by \( F(2k + 1) \) and \( F(2k) \) are clearly true. Let \( n = 2k + 1 \) for some positive integer \( k \geq 2 \). Then,

\[
F(2k + 1) = 5 \cdot 2^{2k+1} + 2^{2k+1-\frac{7}{4}} \sum_{\ell=5}^{2k+1} 2^{\frac{\ell}{2} + \frac{(-1)^{\ell+1}}{4}}
\]

\[
= 5 \cdot 2^{2k+1} + 2^{2k-\frac{3}{4}} \left( \left( 2^{\frac{11}{4}} + 2^{\frac{7}{4}} + \cdots + 2^{\frac{2k+1}{2} + \frac{1}{4}} \right) \\
+ \left( 2^{\frac{7}{4} - \frac{1}{4}} + 2^{\frac{3}{4} - \frac{1}{4}} + \cdots + 2^{\frac{2k-3}{2} - \frac{1}{4}} \right) \right)
\]

\[
= 5 \cdot 2^{2k+1} + 2^{2k-\frac{3}{4}} \left( \frac{11}{2} \left( 1 + 2 + 2^2 + \cdots + 2^{k-2} \right) \\
+ 2^{\frac{11}{4}} \left( 1 + 2 + 2^2 + \cdots + 2^{k-3} \right) \right)
\]

\[
= 5 \cdot 2^{2k+1} + 2^{2k-\frac{3}{4}} \frac{11}{2} \left( 2^{k-1} - 1 + 2^{k-2} - 1 \right)
\]

\[
= 5 \cdot 2^{2k+1} + 2^{2k+2} (3 \cdot 2^{k-2} - 2) = 2^{2k+1}(1 + 3 \cdot 2^{k-1}).
\]
Let \( n = 2k \) for some positive integer \( k \geq 3 \). Then,

\[
F(2k) = 5 \cdot 2^{2k} + 2^{2k-\frac{7}{4}} \sum_{\ell=5}^{2k} 2^{\ell + \frac{(-1)^{\ell+1}}{4}}
\]

\[
= \frac{1}{2} \left( 5 \cdot 2^{2k+1} + 2^{2k-\frac{7}{4}} \sum_{\ell=5}^{2k+1} 2^{\ell + \frac{(-1)^{\ell+1}}{4}} \right)
\]

\[
= \frac{1}{2} F(2k + 1) - 2^{3k-1}
\]

\[
= 2^{2k}(1 + 3 \cdot 2^{-1}) - 2^{3k-1} = 2^{2k}(1 + 2^k).
\]

To prove \( F(n) = 2^{a(n)} + 2 F(n - 1) \) for \( n \geq 3 \), we proceed as follows: Let \( n = 2k + 1 \) for some \( k \geq 1 \). Then, \( F(n) = F(2k + 1) = 2 \cdot 2^{2k} + 3 \cdot 2^{3k} = 2^{2(2k+1)} + 2 F(2k) = 2^{a(n)} + 2 F(n - 1) \). Let \( n = 2k \) for some \( k \geq 2 \). Then, \( F(n) = F(2k) = 2^{2k} + 2^{3k} = 2^{3k-2} + 2^{k} + 2^{3k-2} = 2^{a(2k)} + 2 F(2k - 1) = 2^{a(n)} + 2 F(n - 1) \).

If \( n = 2 \), then \( F(2) = 2^2 + 2 F(1) \) directly follows from the definition.

For \( n \in \mathbb{N} \), we will denote the sequence of sets \( \alpha_{2a(n)}(v), \cup_{\omega \in I} S_\omega(\alpha_{2a(n-1)}(v)) \), \( \cup_{\omega \in I^2} S_\omega(\alpha_{2a(n-2)}(v)), \ldots, \cup_{\omega \in I^{n-4}} S_\omega(\alpha_{2a(4)}(v)) \), \( \cup_{\omega \in I^{n-3}} S_\omega(\alpha_{2a(3)}(v)) \), \( \cup_{\omega \in I^{n-2}} S_\omega(\alpha_{2a(2)}(v)) \), \( \cup_{\omega \in I^{n-1}} S_\omega(\alpha_{2a(1)}(v)) \), \( \{S_\omega(\omega) : \omega \in I^n\} \) by \( S(a(n)), S(a(n - 1)), S(a(n - 2)), \ldots, S(a(4)), S(a(3)), S(2), S(1), \) and \( S(0) \), respectively.

Also, for \( 1 \leq \ell \leq 2 \), write

\[
S(2)(\ell) := \cup_{\omega \in I^{n-\ell}} S_\omega(\alpha_{2^{\ell+1}}(v)) \quad \text{and} \quad S^{(2)}(2)(\ell) := \cup_{\omega \in I^{n-\ell}} S_\omega(\alpha_{2^{\ell+2}}(v))
\]

and for \( 3 \leq \ell \leq n \), write

\[
S(2)(a(\ell)) := \cup_{\omega \in I^{n-\ell}} S_\omega(\alpha_{2^{a(\ell)+1}}(v)) \quad \text{and} \quad S^{(2)}(2)(a(\ell)) := \cup_{\omega \in I^{n-\ell}} S_\omega(\alpha_{2^{a(\ell)+2}}(v))
\]

Further, we write \( S^{(2)}(0) := \{S_\omega(\alpha_2(P)) : \omega \in I^n\} = \{S_\omega(\frac{19}{90}), S_\omega(\frac{71}{90}) : \omega \in I^n\} \), and \( S^{(2)}(0) = \cup_{\omega \in I^n} S_\omega(\alpha_2(v)) \cup \{S_\omega(\frac{1}{2}) : \omega \in I^{n+1}\} \). Moreover, for any \( \ell \in \mathbb{N} \cup \{0\} \), if \( A := S(i) \), we identify \( S^{(2)}(i) \) and \( S^{(2)}(2)(i) \), respectively, by \( A^{(2)} \) and \( A^{(2)(2)} \), and if \( A := S(a(i)) \), we identify \( S^{(2)}(a(i)) \) and \( S^{(2)}(2)(a(i)) \), respectively, by \( A^{(2)} \) and \( A^{(2)(2)} \). Set

\[
\alpha_{F(n)} := \alpha_{F(n)}(P) = \begin{cases} 
S(1) \cup S(0) & \text{if } n = 1, \\
S(2) \cup S(1) \cup S(0) & \text{if } n = 2, \\
\bigcup_{\ell=3}^n S(a(\ell)) \cup S(2) \cup S(1) \cup S(0) & \text{if } n \geq 3.
\end{cases}
\]

For any real number \( x \), let \( \lfloor x \rfloor \) denote the greatest integer not exceeding \( x \). For \( n \in \mathbb{N} \), \( n \geq 1 \), set
\[ SF(n) := \{S(0), S(1), S(2), S(a(3)), S(a(4)), \ldots, S(a(n))\}, \]
\[ SF^{(1)}(n) := \{S(0), S(4), S(6), \ldots, S(a(2\lfloor \frac{n}{2} \rfloor))\}, \]
\[ SF^{(2)}(n) := SF(n) \setminus SF^{(1)}(n). \]

In addition, write
\[ SF^*(n) := \{S(0), S(1), S(2), S(a(3)), S(a(4)), \ldots, S(a(n)), S^{(2)}(0), S^{(2)}(a(4)), \]
\[ S^{(2)}(a(6)), \ldots, S^{(2)}(a(2\lfloor \frac{n}{2} \rfloor))\}. \tag{5} \]

For any element \( a \in A \subseteq SF^*(n) \), by the Voronoi region of \( a \) it is meant the Voronoi region of \( a \) with respect to the set \( \bigcup_{B \subseteq SF^*(n)} B \). Similarly, for any \( a \in A \subseteq SF(n) \), by the Voronoi region of \( a \) it is meant the Voronoi region of \( a \) with respect to the set \( \bigcup_{B \subseteq SF(n)} B \). Notice that if \( a, b \in A \), where \( A \in SF(n) \) or \( A \in SF^*(n) \), the error contributed by \( a \) in the Voronoi region of \( a \) is equal to the error contributed by \( b \) in the Voronoi region of \( b \). Let us now define an order \( \succ \) on the set \( SF^*(n) \) as follows: For \( A, B \in SF^*(n) \) by \( A \succ B \) it is meant that the error contributed by any element \( a \in A \) in the Voronoi region of \( a \) is larger than the error contributed by any element \( b \in B \) in the Voronoi region of \( b \). Similarly, we define the order relation \( \succ \) on the set \( SF(n) \).

**Lemma 3.2.3** For a given positive integer \( n \geq 4 \), let \( S(a(4)) \in SF^*(n) \). Then, \( S(a(4)) \succ B \) for any \( B \in SF^*(n) \setminus \{S(a(4))\} \).

**Proof** We first prove that \( S(a(4)) \succ S(a(\ell)) \) for \( 5 \leq \ell \leq n \). In order to that, for \( 5 \leq \ell \leq n \), we first prove the following inequality:
\[ \frac{18^{\alpha(\ell) - a(4)}}{75^{\ell - 4}} > 1. \tag{6} \]

First, assume that \( \ell = 2k + 1 \) for some \( k \geq 2 \). Then,
\[ \frac{18^{\alpha(\ell) - a(4)}}{75^{\ell - 4}} = \frac{18^{a(2k+1) - a(4)}}{75^{2k+1 - 4}} = \frac{18^{3k-3}}{75^{2k-4}} = \left( \frac{18^3}{75^2} \right)^{(k-1)} \frac{18}{75} > 1. \]
Next, assume that \( \ell = 2k \) for some \( k \geq 3 \). Then,
\[ \frac{18^{\alpha(\ell) - a(4)}}{75^{\ell - 4}} = \frac{18^{a(2k) - a(4)}}{75^{2k - 4}} = \frac{18^{3k-4}}{75^{2k-4}} = \left( \frac{18^3}{75^2} \right)^{(k-2)} > 1. \]

Thus, the inequality is proved.

The distortion error due to any element from the set \( S(a(\ell)) \) is given by \( \frac{1}{75^{\ell - 4}} \frac{1}{3} \frac{1}{18^{\alpha(\ell)}} W \). On the other hand, the distortion error due to any element from the set \( S(a(4)) \) is given by \( \frac{1}{75^{\ell - 4}} \frac{1}{3} \frac{1}{18^{\alpha(4)}} W \). Thus,
\[ \frac{1}{75^{\ell - 4}} \frac{1}{3} \frac{1}{18^{\alpha(4)}} W \succ \frac{1}{75^{\ell - 4}} \frac{1}{3} \frac{1}{18^{\alpha(\ell)}} W \text{ is true if } \frac{18^{\alpha(\ell) - a(4)}}{75^{\ell - 4}} > 1, \]
which is clearly true by (6). Next, take \( 2 \leq \ell \leq \lfloor \frac{n}{2} \rfloor \). Then, the distortion error due to the set \( S^{(2)}(a(\ell)) \) is given by
\[ \frac{1}{75^{\ell - 2}} \frac{1}{3} \frac{1}{18^{\alpha(\ell)+1}} W = \frac{1}{75^{\ell - 2}} \frac{1}{3} \frac{1}{18^{\alpha(4)}} W. \]
Thus,
\[ \frac{1}{75^{\ell - 4}} \frac{1}{3} \frac{1}{18^{\alpha(4)}} W \succ \frac{1}{75^{\ell - 4}} \frac{1}{3} \frac{1}{18^{\alpha(\ell)}} W \text{ is true if } \frac{18^{\alpha(\ell) - a(4)}}{75^{\ell - 4}} > 1, \]
which is clearly true as \( \ell \geq 2 \). Similarly, we can show that \( S(a(4)) \succ S^{(2)}(0) \). Now, take \( 1 \leq \ell \leq 3 \). Then, \( S(a(4)) \succ S(a(\ell)) \) is true since
\[ \frac{1}{75^{\ell - 4}} \frac{1}{3} \frac{1}{18^{\alpha(4)}} W \succ \frac{1}{75^{\ell - 4}} \frac{1}{3} \frac{1}{18^{\alpha(4)}} W. \]
Similarly, \( S(a(4)) \succ S(0) \). Thus, the assertion follows.
Lemma 3.2.4 Let $0 \leq k \leq n$. Then,

(i) $S(a(2k)) > S(a(2k + 2))$ for all $k \geq 2$.
(ii) $S(a(82)) > S(a(2k + 1)) > S(a(2k + 82)) > S(a(2k + 3))$ for all $k \geq 1$.
(iii) $S(a(83)) > S^{(2)}(a(2k)) > S(a(2k + 160)) > S(a(2k + 81)) > S^{(2)}(a(2k + 2))$
    for all $k \geq 2$.
(iv) $S(a(7)) > S(0) > S(a(88)) > S(a(9)), S(a(81)) > S(2) > S(a(162)) > S(a(83)), S(a(121)) > S^{(2)}(a(42)) > S(a(202)) > S^{(2)}(0) > S(a(123)) > S^{(2)}(a(44)), and$
    $S^{(2)}(a(80)) > S(a(240)) > S(1) > S(a(161)) > S^{(2)}(a(82))$.

Proof (i) For any $k \geq 2$, let $b \in S(a(2k)) = \bigcup_{\omega \in \mathcal{I}^N_a(T^N_2)} S_\omega(a_2(2k)(v))$. Then, $b = S_\omega(T^N_1(\frac{1}{2}))$ for some $\omega \in I^{n-2k}$ and $\tau \in I^a(2k)$. The error contributed by $b$ in its
    Voronoi region is given by
    \[
    \int_{S_\omega(T^N_1(\frac{1}{2}))} \left( x - S_\omega \left( T^N_1 \left( \frac{1}{2} \right) \right) \right)^2 dP = \frac{1}{75^{n-2k}} \frac{1}{3} \frac{1}{18 a(2k)} W.
    \]
    Similarly, the error contributed by any element in the set $S(a(2k + 2))$ is $\frac{1}{75^{n-2k}} \frac{1}{3} \frac{1}{18 a(2k + 2)} W$. Thus, $S(a(2k)) > S(a(2k + 2))$ will be true if $\frac{1}{75^{n-2k}} \frac{1}{3} \frac{1}{18 a(2k + 2)} W > \frac{1}{75^{n-2k}} \frac{1}{3} \frac{1}{18 a(2k)} W$, i.e., if $1 < \frac{18 a(2k + 2) - a(2k)}{75^2}$, which is clearly true.

(ii) $S(a(82)) > S(a(3))$ is true since $\frac{1}{75^{n-2}} \frac{1}{3} \frac{1}{18 a(83)} W > \frac{1}{75^{n-2}} \frac{1}{3} \frac{1}{18 a(83)} W$. For all $k \geq 1$, proceeding similarly as in (i), we have $S(a(2k + 1)) > S(a(2k + 82))$ and $S(a(2k + 82)) > S(a(2k + 3))$. Thus, (ii) is proved.

(iii) $S(a(83)) > S^{(2)}(a(4))$ is true since $\frac{1}{75^{n-2}} \frac{1}{3} \frac{1}{18 a(83)} W > \frac{1}{75^{n-2}} \frac{1}{3} \frac{1}{18 a(84)} W$. Proceeding similarly as in (i), we can show that $S^{(2)}(a(2k)) > S(a(2k + 160)), S(a(2k + 160)) > S(a(2k + 81))$, and $S(a(2k + 81)) > S^{(2)}(a(2k + 2))$ for all $k \geq 2$. Thus (iii) follows.

(iv) Since $\frac{1}{75^{n-2}} \frac{1}{3} \frac{1}{18 a(7)} W > \frac{1}{75^{n-2}} \frac{1}{3} \frac{1}{18 a(88)} W > \frac{1}{75^{n-2}} \frac{1}{3} \frac{1}{18 a(9)} W$, the inequality $S(a(7)) > S(0) > S(a(88)) > S(a(9))$ is true. Similarly we can show the other inequalities. \hfill $\Box$

Remark 3.2.5 The Lemma 3.2.4 defines the order among the elements of the sets $SF^*(n)$ for all $n \geq 1$. Since it is helpful in obtaining the optimal sets of $F(n)$-means, we exhibit the following cases for later use:

(i) $S(0) > S^{(2)}(0) > S(1)$ for $n = 1$;
(ii) $S(0) > S(2) > S^{(2)}(0) > S(1)$ for $n = 2$;
(iii) $S(a(3)) > S(0) > S(2) > S^{(2)}(0) > S(1)$ for $n = 3$;
(iv) $S(a(4)) > S(a(3)) > S(0) > S(2) > S^{(2)}(a(4)) > S^{(2)}(0) > S(1)$ for $n = 4$.

Lemma 3.2.6 Let $\alpha_{F(n)}, SF(n), SF^{(1)}(n)$, and $SF^{(2)}(n)$ be the sets as defined before. Then,

\[
\alpha_{F(n+1)} = \left( \bigcup_{A \in SF^{(1)}(n)} A^{(2)}(2) \right) \cup \left( \bigcup_{A \in SF^{(2)}(n)} A^{(2)}(2) \right).
\]
Proof Let $A \in SF^{(1)}(n)$. If $A = S(0)$, then $A^{(2)} = S^{(2)}(0) = \{S_\omega(\frac{1}{2}) : \omega \in I^{n+1}\} \cup (\cup_{n \in I} S_\omega(\alpha_2(v))) = \{S_\omega(\frac{1}{2}) : \omega \in I^{n+1}\} \cup (\cup_{n \in I} S_\omega(\alpha_2(v)))$. If $A = S(a(\ell))$ for some $\ell \in \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, then

$$A^{(2)} = \cup_{\omega \in I^{n-2\ell}} S_\omega(\alpha_{2a(2\ell)+2}(v)) = \cup_{\omega \in I^{n+1-(2\ell+1)}} S_\omega(\alpha_{2a(2\ell+1)}(v)).$$

Next, let $A \in SF^{(2)}(n)$. Then, $A = S(1)$, $S(2)$, or $S(a(2\ell + 1))$ for some $\ell \in \mathbb{N}$. If $A = S(1)$, then $A^{(2)} = \cup_{\omega \in I^{n-2\ell}} S_\omega(\alpha_{2a(2\ell+1)}(v)) = \cup_{\omega \in I^{n+1-(2\ell+1)}} S_\omega(\alpha_{2a(2\ell)}(v))$, and similarly, we have $S^{(2)}(2) = \cup_{\omega \in I^{n+1}} S_\omega(\alpha_{23}(v))$. If $A = S(a(2\ell + 1))$, then

$$A^{(2)} = \cup_{\omega \in I^{n-(2\ell+1)}} S_\omega(\alpha_{2a(2\ell+1)}(v)) = \cup_{\omega \in I^{n-(2\ell+1)}} S_\omega(\alpha_{2a(2\ell+1)}(v)),$$

yielding $A^{(2)} = \cup_{\omega \in I^{n+1-(2\ell+1)}} S_\omega(\alpha_{2a(2\ell+1)}(v))$. Thus, we see that

$$\alpha_{\Phi A^{(n+1)}} = \left(\bigcup_{A \in SF^{(1)}(n)} A^{(2)}\right) \cup \left(\bigcup_{A \in SF^{(2)}(n)} A^{(2)}\right),$$

which proves the assertion. \qed

Lemma 3.2.7 For $A, B \in SF(n)$ with $A > B$, the distortion error due to the set $(SF(n) \setminus A) \cup A^{(2)} \cup B$ is less than the distortion error due to the set $(SF(n) \setminus B) \cup B^{(2)} \cup A$.

Proof Let $V(\alpha_{\Phi A(n)})$ be the distortion error due to the set $\alpha_{\Phi A(n)}$ with respect to the condensation measure $P$. First take $A = S(a(k))$ and $B = S(a(k'))$ for some $3 \leq k' \leq n$. Then, the distortion error due to the set $(\alpha_{\Phi A(n)} \setminus A) \cup A^{(2)} \cup B$ is less than the distortion error due to the set $(\alpha_{\Phi A(n)} \setminus B) \cup B^{(2)} \cup A$ is

$${V(\alpha_{\Phi A(n)})} - {\frac{1}{\frac{1}{75n-k} - \frac{1}{3} \frac{1}{9a(k')}}} W + {\frac{1}{\frac{1}{75n-k} - \frac{1}{3} \frac{1}{9a(k'+1)}}} W + {\frac{1}{\frac{1}{75n-k} - \frac{1}{3} \frac{1}{9a(k')}}} W$$

yielding $${\frac{1}{\frac{1}{75n-k} - \frac{1}{3} \frac{1}{9a(k')}}} W > {\frac{1}{\frac{1}{75n-k} - \frac{1}{3} \frac{1}{9a(k')}}} W,$$

which is clearly true since by the hypothesis $A > B$. Similarly, we can prove the lemma in each of the following cases:

(i) $A = S(a(k))$ and $B = S(k')$, where $3 \leq k \leq n$ and $0 \leq k' \leq 2$;
(ii) $A = S(k)$ and $B = S(a(k'))$, where $0 \leq k \leq 2$ and $3 \leq k' \leq n$;
(iii) $A = S(k)$ and $B = S(a(k'))$, where $0 \leq k \leq 2$ and $0 \leq k' \leq 2$.

Thus, the proof of the lemma is complete. \qed

Lemma 3.2.8 For any two sets $A, B \in SF^*(n)$, let $A > B$. Then, the distortion error due to the set $(SF^*(n) \setminus A) \cup A^{(2)} \cup B$ is less than the distortion error due to the set $(SF^*(n) \setminus B) \cup B^{(2)} \cup A$. Using the similar technique as Lemma 3.2.7, the following lemma can be proved.
Remark 3.2.9 By Proposition 3.1.4, we know that $\alpha_{F(1)}$ is an optimal set of $F(1)$-means. Assume that $\alpha_{F(n)}$ is an optimal set of $F(n)$-means for some $n \geq 1$. Let $A \in SF(n)$ be such that $A \succ B$ for any other $B \in SF(n)$. By Lemma 3.2.7, we deduce that if $A = S(0) = \{S_\omega(\frac{1}{2}) : \omega \in I^n\}$, then the set $(\alpha_{F(n)} \setminus A) \cup \{S_\omega(\frac{10}{99}), S_\omega(\frac{71}{99}) : \omega \in I^n\}$ is an optimal set of $F(n) - 2^n + 2^{n+1}$-means. If $A = \cup_{\omega \in I^n - S_\omega(\alpha_{2\ell}(v))}$ for $1 \leq \ell \leq 3$, then the set $(\alpha_{F(n)} \setminus A) \cup (\cup_{\omega \in I^n - S_\omega(\alpha_{2\ell+1}(v)))}$ is an optimal set of $F(n) - 2^n + 2^{n+1}$-means. If $A = \cup_{\omega \in I^n - S_\omega(\alpha_{2\ell}(v))}$, then the set $(\alpha_{F(n)} \setminus A) \cup (\cup_{\omega \in I^n - S_\omega(\alpha_{2\ell+1}(v))})$ is an optimal set of $F(n) - 2^n - 2^{\ell} + 2^{n-\ell} - 2^{\ell+1}$-means.

Proposition 3.2.10 Let $\alpha(n)$ and $F(n)$ be the two sequences as defined by Definition 3.2.1. Then, for any $n \geq 1$, the set $\alpha_{F(n)}(P)$ is an optimal set of $F(n)$-means with quantization error given by $V_{F(n)}(P) = \begin{cases} \frac{1}{3} \int \omega + \frac{2}{75} \nu & \text{if } n = 1, \\ \frac{1}{3} \int \omega + \frac{1}{75} \nu + \left(\frac{2}{75}\right)^2 \nu & \text{if } n = 2, \\ \sum_{\ell=3}^{n} \left(\frac{2}{75}\right)^{n-\ell} \int \omega + \left(\frac{2}{75}\right)^{n-1} \frac{1}{3} \int \omega + \left(\frac{2}{75}\right)^{n} \nu & \text{if } n \geq 3. \end{cases}$

Proof By Proposition 3.1.4, for $n = 1$ the set $\alpha_{F(1)}$ is an optimal set of $F(1)$-means with quantization error $\frac{1}{3} \int \omega + \frac{2}{75} \nu$. We now show that $\alpha_{F(n)}$ is an optimal set of $F(n)$-means for any $n \geq 2$. Consider the following cases:

Case 1. $n = 2$.

Let $\alpha$ be an optimal set of $F(2)$-means. Recall that $\alpha$ does not contain any point from the open intervals $(\frac{1}{2}, \frac{2}{3})$ and $(\frac{3}{4}, \frac{4}{5})$. The distortion error due to the set $\alpha_{F(2)}(P) = \alpha_{2\ell}(v) \cup (\cup_{\omega \in I} S_\omega(\alpha_{2\ell}(v))) \cup \{S_\omega(\frac{1}{2}) : \omega \in I^2\}$ is given by

$$\int_{a \in \alpha_{F(2)}(P)} (x - a)^2 \, dP = \int_{a \in \alpha_{F(2)}(P)} (x - a)^2 \, dP \geq \int_{a \in \alpha_{F(2)}(P)} (x - a)^2 \, dP = \int_{a \in \alpha_{F(2)}(P)} (x - a)^2 \, dP = \frac{1}{3} \int \omega + \frac{2}{75} \nu + \left(\frac{2}{75}\right)^2 \nu = \frac{9283}{88695000}.$$ Since $V_{F(2)}(P)$ is the quantization error for $F(2)$-means, we have $V_{F(2)}(P) \leq \frac{9283}{88695000}$. Since $\alpha$ is an optimal set of $F(2)$-means, $F(2)$ is even, and $P$ is symmetric about the point $\frac{1}{2}$, $\alpha$ must contain equal number of points from each of the sets $S_1$ and $S_2$ implying that $\alpha$ contains even number of points from $L$. We now show that $\alpha$ contains $2^2$ elements from $L$. Suppose that $\alpha$ contains less than $2^2$ elements from $L$. Then, $\text{card}(\alpha \cap L) = 0$ or 2. But, $\alpha \cap L \neq \emptyset$, and so $\text{card}(\alpha \cap L) = 2$. Hence,

$$\int_{a \in \alpha_{F(2)}(P)} (x - a)^2 \, dP \geq \int_{a \in \alpha_{F(2)}(P)} (x - a)^2 \, dP = \frac{1}{3} \int \omega + \frac{1}{5400} > V_{F(2)}(P),$$

which is a contradiction. Next suppose that $\alpha$ contains more than $2^2$ elements from $L$. As $\alpha \cap S_i \neq \emptyset$ for $i = 1, 2$, we must have $\text{card}(\alpha \cap L) = 6, 8, \text{or 10}$. Suppose that $\text{card}(\alpha \cap L) = 6$. Then, for $1 \leq i \leq 2$, $\alpha \cap S_i$ is an optimal set of three-means with respect to the image measure $P \circ S_i^{-1}$, which by Lemma 2.1.8 yields that $S_i^{-1}(\alpha \cap S_i)$ is an optimal set of three-means for $P$, and so by Proposition 2.1.13, we have $S_i^{-1}(\alpha \cap S_i) = \{S_1(\frac{1}{2}), S_2(\frac{1}{2})\}$, i.e., $\alpha \cap J_1 = \{S_1(\frac{1}{2}), S_1(\frac{1}{2}), S_1(\frac{1}{2})\}$, and
\[ \alpha \cap J_2 = \{ S_{21}(\frac{1}{2}), S_2(\frac{1}{2}), S_{22}(\frac{1}{2}) \}. \] Then, the distortion error is \( V_{F(2)}(P) \geq 2 \left( \int_{J_{11}} (x - S_{11}(\frac{1}{2}))^2 dP + \int_{S_1(L)} (x - S_1(\frac{1}{2}))^2 dP + \int_{J_{12}} (x - S_{12}(\frac{1}{2}))^2 dP \right) = 2 \left( \frac{1}{75} V + \frac{1}{75} \frac{1}{3} W + \frac{1}{75} V \right) = \frac{203}{1642500} > V_{F(2)}(P), \] which is a contradiction. Similarly, we can show that if \( \text{card} (\alpha \cap L) = 8 \) or 10, contradiction will arise. Hence, \( \text{card} (\alpha \cap L) = 2^2 \), which implies that \( \alpha \) contains \( F(1) \) elements from each of the sets \( J_1 \) and \( J_2 \). Thus, we see that in this case \( \alpha = \alpha_{F(2)}(P) \) with quantization error \( V_{F(2)}(P) = \frac{1}{3} \frac{1}{9} W + \frac{2}{75} \frac{1}{3} \frac{1}{9} W + \left( \frac{2}{75} \right)^2 V. \)

Case 3. \( n = 3. \)

By Case 1, we know that \( \alpha_{F(2)} = S(0) \cup S(1) \cup S(2) \) is an optimal set of \( F(2) \)-means, where \( S(0), S(1), S(2) \in SF(2) \). By Remark 3.2.9, \( S(0) \supset S(2) \supset S(0)^{(2)} \supset S(1) \). Thus, by Remark 3.2.9, the set \( \{ \alpha_{F(2)} \setminus S(0) \} \cup S(0)^{(2)} = S(1) \cup S(2) \) is an optimal set of \( 2^2 + 2^2 + 2^3 \)-means. Similarly, the set \( S(1) \cup S(2) \cup S(0)^{(2)} \) is an optimal set of \( 2^2 + 2^3 + 2^3 \)-means, the set \( S(2) \cup S(2) \cup S(2)^{(2)} \) is an optimal set of \( 2^2 + 2^3 + 2^3 + 2^3 \)-means, the set \( S(2) \cup S(2) \cup S(2)^{(2)} \) is an optimal set of \( 2^3 + 2^3 + 2^3 + 2^3 \)-means. By Lemma 3.2.6, it is known that \( \alpha_{F(3)} = S(2)^{(2)} \cup S(2)^{(2)} \cup S(2)^{(2)}(0) \). Thus, \( \alpha_{F(3)} \) is an optimal set of \( F(3) \)-means with quantization error same as it is given in the hypothesis for \( n = 3 \).

Case 4. \( n \geq 4. \)

Let \( \alpha_{F(n)} \) be an optimal set of \( F(n) \)-means for some \( n \geq 3 \). We need to show that \( \alpha_{F(n+1)} \) is an optimal set of \( F(n+1) \)-means. We have \( \alpha_{F(n)} = \bigcup_{A \in SF(n)} A \). In the first step, let \( A(1) \in SF(n) \) be such that \( A(1) \supset B \) for any other \( B \in SF(n) \). Then, by Remark 3.2.9, the set \( \{ \alpha_{F(n)} \setminus A(1) \} \cup A(2) \) gives an optimal set of \( F(n) \) - card \( A(1) \) + card \( A(2) \) means. In the 2nd step, let \( A(2) \in (SF(n) \setminus \{ A(1) \}) \cup A(2) \) such that \( A(2) \supset B \) for any other set \( B \in SF(n) \setminus \{ A(1) \} \cup A(2) \). Then, using the similar technique as Lemma 3.2.7, we can show that the distortion error due to the following set:

\[ \left( \left( \{ \alpha_{F(n)} \setminus A(1) \} \cup A(2) \right) \setminus A(2) \right) \cup A(2) \]

with cardinality \( F(n) - \text{card}(A(1)) + \text{card}(A(2)) - \text{card}(A(2)) + \text{card}(A(2)) \) is smaller than the distortion error due to the set obtained by replacing \( A(2) \) in the set (7) by any other set \( A'(2) \) having the same cardinality as \( A(2) \). In other words, \( (\{ \alpha_{F(n)} \setminus A(1) \} \cup A(2)) \cup A(2) \) forms an optimal set of \( F(n) \) - card \( A(1) \) + card \( A(2) \) - card \( A(2) \) + card \( A(2) \) means. Proceeding inductively in this way, up to \( (n + 1 + 2) \left( \frac{9}{4} \right) \) steps, we can see that \( \alpha_{F(n+1)} = \left( \bigcup_{A \in SF(n+1)} A(2) \right) \cup \left( \bigcup_{A \in SF(n+1)} A(2) \right) \) forms an optimal set of \( F(n+1) \)-means.

The \( F(n) \)-th quantization error for all \( n \geq 3 \) is given by

\[
\int \min_{a \in \alpha_{F(n)}(P)} (x - a)^2 dP = \frac{1}{3} V_{2a(n)}(v) + \sum_{\omega \in I} \sum_{\tau \in I_{a(n-1)}} \int_{S_\omega T_\tau(L)} \left( x - S_\omega T_\tau \left( \frac{1}{2} \right) \right)^2 dP
\]
Lemma 3.2.11

For all $k \geq 2$, we have

$$V_{F(2k+1)} = \frac{1}{9a(2k+1)} \cdot \frac{79}{7224} \cdot \left(1 - \frac{324}{625}\right)^k - \frac{3571}{44347500} \cdot \left(\frac{2}{75}\right)^{2k-1}, \text{ and}$$

$$V_{F(2k+1)} = \frac{1}{9a(2k+1)} \cdot \frac{1}{9a(2k+1)} \cdot \frac{79}{7224} \cdot \left(1 - \frac{324}{625}\right)^k - \frac{3571}{44347500} \cdot \left(\frac{2}{75}\right)^{2k-1}. \text{ and}$$

Remark 3.2.12

By Proposition 3.2.10, we see that if $\alpha_{F(n)}$ is an optimal set of $F(n)$-means for some $n \geq 4$, then $\alpha_{F(n)}$ contains $F(n-1)$ elements from each of $J_1$ and $J_2$, and $9a(n)$ elements from $L$. Thus, by Proposition 3.1.8, we have $V_{F(n)}(P) = \frac{1}{3} \cdot \frac{1}{9a(n)} \cdot \frac{79}{7224} \cdot \left(1 - \frac{324}{625}\right)^k - \frac{3571}{44347500} \cdot \left(\frac{2}{75}\right)^{2k-1}$. So, $V_{F(n)}(P) = \frac{1}{3} \cdot \frac{1}{9a(n)} \cdot \frac{79}{7224} \cdot \left(1 - \frac{324}{625}\right)^k - \frac{3571}{44347500} \cdot \left(\frac{2}{75}\right)^{2k-1}$. Thus, the proof of the proposition is complete. \hfill \Box
3.3 Calculation of Optimal Sets of \(m\)-Means for all \(m \in \mathbb{N}\)

Now, we give the description of how to calculate the optimal sets of \(m\)-means for any \(m \in \mathbb{N}\).

For \(1 \leq m \leq 4\), the optimal sets of \(m\)-means and the \(m\)th quantization error are known by Lemma 2.1.4, Propositions 2.1.10, 2.1.13, and 3.1.4. If \(m = F(n)\) for some positive integer \(n\), then they are known by Proposition 3.2.10. Let \(m \in \mathbb{N}\) be such that \(F(n_m) < m < F(n_m + 1)\) for some positive integer \(n_m\). Notice that \(SF^\ast(n_m)\) contains \((n_m + 1 + \lfloor \frac{n_m}{2} \rfloor)\) elements. After rearranging the elements of \(SF^\ast(n_m)\) write

\[
SF^\ast(n_m) = \{C(0), C(1), C(2), \ldots, C(n_m + 1 + \lfloor \frac{n_m}{2} \rfloor)\},
\]

where \(C(0) \succ C(1) \succ C(2) \succ \ldots \succ C(n_m + 1 + \lfloor \frac{n_m}{2} \rfloor)\). Let \(\ell_m \in \mathbb{N}\) be such that

\[
F(n_m) + \text{card}(C(0)) + \ldots + \text{card}(C(\ell_m)) < m < F(n_m) + \text{card}(C(0)) + \ldots + \text{card}(C(\ell_m + 1)).
\]

If \(m = F(n_m) + \text{card}(C(0)) + \ldots + \text{card}(C(\ell_m))\), then the set

\[
\alpha_m(P) = \left(\alpha_{F(m)} \setminus (C(0) \cup C(1) \cup \ldots \cup C(\ell_m))\right) \cup \left((C(2)(0) \cup C(2)(1) \cup \ldots \cup C(2)(\ell_m)) \setminus (C(0) \cup C(1) \cup \ldots \cup C(\ell_m))\right)
\]

is a unique optimal set of \(m\)-means. Let \(F(n_m) + \text{card}(C(0)) + \ldots + \text{card}(C(\ell_m)) < m < F(n_m) + \text{card}(C(0)) + \ldots + \text{card}(C(\ell_m + 1))\). Let \(\beta_m \subset C(\ell_m + 1)\) with \(\text{card}(\beta_m) = m - (F(n_m) + \text{card}(C(0)) + \ldots + \text{card}(C(\ell_m))).\) The following three cases can arise:

Case 1. \(C(\ell_m + 1) = \bigcup_{\omega \in I^{n_k}} S_\omega(\alpha_{2m}(\nu))\).

Write \(\beta_{m,1} := \{\omega \in I^{n_k} : S_\omega T_\tau(1) \subseteq \beta_m\ \text{for some } \tau \in I^{a(k)}\}\) and \(\beta_{m,2} := \{\tau \in I^{a(k)} : S_\omega T_\tau(1) \subseteq \beta_m\ \text{for some } \omega \in I^{n_k}\}\). Set

\[
\beta^*_m := \{S_\omega T_\tau(T_1(\frac{1}{2})) : \omega \in \beta_{m,1} \text{ and } \tau \in \beta_{m,2}\}
\]

\[
\cup \{S_\omega T_\tau(T_2(\frac{1}{2})) : \omega \in \beta_{m,1} \text{ and } \tau \in \beta_{m,2}\}.
\]

Case 2. \(C(\ell_m + 1) = S_\omega(\frac{1}{2}) : \omega \in I^{n_m}\).

Write \(\beta_{m,1} := \{\omega \in I^{n_m} : S_\omega(\frac{1}{2}) \subseteq \beta\}\) and then \(\beta^*_m := \bigcup_{\omega \in \beta_{m,1}} \{S_\omega(\frac{19}{90}), S_\omega(\frac{71}{90})\}\).

Case 3. \(C(\ell_m + 1) = \bigcup_{\omega \in I^{n_m}} \{S_\omega(\frac{19}{90}), S_\omega(\frac{71}{90})\}\).
Write $\beta_{m,1} := \{\omega \in \mathcal{F}^m : S_{\omega}^{(19/90)} \in \beta_m \text{ or } S_{\omega}^{(71/90)} \in \beta_m\}$, and write
\[
\beta_m^* := \bigcup_{\{\omega \in \beta_{m,1} \& S_{\omega}^{(19/90)} \in \beta_m\}} \{S_{\omega}^{1/2}, S_{\omega}T_1^{1/2}\}
\bigcup_{\{\omega \in \beta_{m,1} \& S_{\omega}^{(71/90)} \in \beta_m\}} \{S_{\omega}T_2^{1/2}, S_{\omega}2^{1/2}\}.
\]

Let $\beta_m^*$ be the set that arises either in Case 1, Case 2, or in Case 3. Then, the set
\[
\alpha_m(P) = \left(\alpha_F(m) \setminus (C(0) \cup C(1) \cup \cdots \cup C(\ell_m))\right) \cup \left((C^{(2)}(0) \cup C^{(2)}(1)\right.
\]
\[
\left.\cup \cdots \cup C^{(2)}(\ell_m)) \setminus (C(0) \cup C(1) \cup \cdots \cup C(\ell_m))\right) \setminus \beta_m \cup \beta_m^*
\]
is an optimal set of $m$-means. The number of such sets in any of the above cases is given by $\text{card}(C(\ell_m + 1))C_{\text{card}(\beta_m)}$, where $^uC_v = \binom{u}{v}$ represent the binomial coefficient.

The following two examples illustrate the computations described above.

**Example 3.3.1** Let $m = 21$. Since $F(2) < m < F(3)$, we have $n_m = 2$. Since $S(0) > S(2) > S^{(2)}(0) > S(1)$, we have $SF^*(n_m) = SF^*(2) = (C(0), C(1), C(2), C(3))$, where $C(0) = S(0)$, $C(1) = S(2)$, $C(2) = S^{(2)}(0)$, and $C(3) = S(1)$. Again, $F(2) + \text{card}(C(0)) + \text{card}(C(1)) < m < F(2) + \text{card}(C(0)) + \text{card}(C(1)) + \text{card}(C(2))$ implying $\ell_m = 1$, and so $C(\ell_m + 1) = S^{(2)}(0)$. Take $\beta_m = \{S_{11}^{(19/90)}, S_{12}^{(19/90)}, S_{21}^{(71/90)}\}$, where $\beta_m \subset S^{(2)}(0)$ with $\text{card}(\beta_m) = m - (F(2) + \text{card}(C(0)) + \text{card}(C(1))) = 23 - 20 = 3$. Then,
\[
\beta_m^* = \{S_{11}^{(1/2)}, S_{11}T_1^{(1/2)}, S_{12}^{(1/2)}T_1^{(1/2)}, S_{21}^{(1/2)}T_2^{(1/2)}, S_{212}^{(1/2)}\}.
\]

Hence,
\[
\alpha_{23}(P) = \left((\alpha_{F(2)} \setminus (C(0)) \cup C(1)) \cup \left((C^{(2)}(0) \cup C^{(2)}(1)) \setminus (C(0) \cup C(1))\right) \setminus \beta_m \cup \beta_m^*
\right.
\]
\[
= \left(S(1) \cup S^{(2)}(0) \cup S^{(2)}(2) \setminus \beta_m \right) \cup \beta_m^*
\]
\[
= \bigcup_{\omega \in \mathcal{F}} S_{\omega}(\alpha_2(v)) \cup \alpha_{23}(v)
\]
\[
\bigcup \{S_{11}^{(71/90)}, S_{12}^{(71/90)}, S_{21}^{(19/90)}, S_{22}^{(19/90)}, S_{22}^{(71/90)}\}
\]
\[
\bigcup \{S_{11}^{(1/2)}, S_{11}T_1^{(1/2)}, S_{12}^{(1/2)}T_1^{(1/2)}, S_{21}^{(1/2)}T_2^{(1/2)}, S_{212}^{(1/2)}\}.
\]

The number of optimal sets of 23-means is given by $^8C_3 = 56$. 
Example 3.3.2 Let $m = 31$. As in the previous example, we have $n_m = 2$, $C(0) = S(0)$, $C(1) = S(2)$, $C(2) = S(2)(0)$, and $C(3) = S(1)$. Since $F(2) + \text{card}(C(0)) + \text{card}(C(1)) + \text{card}(C(2)) = 12 + 4 + 4 + 8 = 28 < m < 32 = F(2) + \text{card}(C(0)) + \text{card}(C(1)) + \text{card}(C(2)) + \text{card}(C(3))$, we have $\ell_m = 2$, and so $C(\ell_m + 1) = S(1)$. Take $\beta_m = \{S_1T_1(\frac{1}{2}), S_1T_2(\frac{1}{2}), S_2T_1(\frac{1}{2})\}$, where $\beta_m \subseteq S(1)$ with $\text{card}(\beta_m) = m - 28 = 3$. Then,

$$\beta_m^* = \{S_1T_1(\frac{1}{2}), S_1T_1T_2(\frac{1}{2}), S_1T_2T_1(\frac{1}{2}), S_1T_2T_2(\frac{1}{2}), S_2T_1T_1(\frac{1}{2}), S_2T_1T_2(\frac{1}{2})\}.$$ 

Hence,

$$\alpha_{23}(P) = \left( (\alpha_{F(2)} \setminus (C(0) \cup C(1) \cup C(2))) \cup \left( (C(2)(0) \cup C(2)(1) \cup C(2)(2)) \setminus (C(0) \cup C(1) \cup C(2)) \right) \right) \setminus \beta_m \cup \beta_m^*$$

$$= \left( (S(1) \cup S(2)(2) \cup S(2)(2)(0)) \setminus \beta_m \right) \cup \beta_m^*$$

$$= \alpha_{23}(v) \cup \bigcup_{\omega \in I^2} S_2(\omega) \cup \left\{ S_2(\frac{1}{2}) : \omega \in I^3 \right\} \cup \beta_m^*$$

is an optimal set of 31-means. The number of optimal sets of 31-means is given by $4C_3 = 4$.

3.4 Asymptotics for the $n$th Quantization Error $V_n(P)$

Having the optimal sets and the corresponding quantization errors are explicitly known, we now turn to the investigation of the quantization dimension and the quantization coefficients for the condensation measure $P$. If $D(v)$ is the quantization dimension of $v$, then $(\frac{1}{2}(\frac{1}{2}))^{\frac{D(v)}{D(v)}} + (\frac{1}{2}(\frac{1}{2}))^{\frac{D(v)}{D(v)}} = 1$ (see [13]), which yields $D(v) = \frac{\log 2}{\log 3} = \beta$, where $\beta$ is the Hausdorff dimension of the Cantor set generated by the similarity maps $T_1$ and $T_2$ (i.e., $\beta$ satisfies $(\frac{1}{3})^\beta + (\frac{1}{2})^\beta = 1$).

Theorem 3.4.1 Let $P$ be the condensation measure associated with the self-similar measure $v$. Then, $\lim_{n \to \infty} \frac{2 \log n}{\log V_n(P)} = D(v)$, i.e., the quantization dimension $D(P)$ of the measure $P$ exists and is equal to $D(v)$.

Proof For $n \in \mathbb{N}, n \geq 3$, let $\ell(n)$ be the least positive integer such that $F(2\ell(n) + 1) \leq n < F(2\ell(n) + 1) + 1$. Then, $V_{F(2\ell(n) + 1)} < V_n \leq V_{F(2\ell(n) + 1)}$. Thus, we have

$$\frac{2 \log (F(2\ell(n) + 1))}{-\log (V_{F(2\ell(n) + 1)})} < \frac{2 \log n}{-\log V_n} < \frac{2 \log (F(2\ell(n) + 1) + 1))}{-\log (V_{F(2\ell(n) + 1)})}$$

By Lemmas 3.2.2 and 3.2.11, we have $F(2\ell(n) + 1) = 2 \cdot 2^{\ell(n)} + 3 \cdot 2^{3\ell(n)}$, and $V_{F(2\ell(n) + 1)} = 9^{-3\ell(n)+1} \cdot 79 \cdot 1 - \left(\frac{324}{625}\right)^{\ell(n)+1} \cdot 7223 - \frac{3571}{44347500} \cdot \left(\frac{2}{75}\right)^{2\ell(n)+1}$. Notice that $\ell(n) \to \infty$ whenever $n \to \infty$. 

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Again,

\[
\lim_{\ell(n) \to \infty} \frac{\log (F(2\ell(n) + 1))}{-\log (V_{F(2\ell(n)+1)})} = \lim_{\ell(n) \to \infty} \frac{2 \cdot 2\ell(n) \log 2 + 9 \cdot 2^{3\ell(n)} \log 2}{2 \cdot 2\ell(n) + 3 \cdot 2^{3\ell(n)}}
\]

\[
\cdot \frac{9^{-3n(n) + 1}}{9^{3n(n) + 1}} \frac{\log 9}{\log 3} (9324 \frac{625}{7224} (1 - \left(\frac{9324}{625}\right)^{\ell(n)+1}) + 9324 \frac{625}{7224} (1 - \left(\frac{9324}{625}\right)^{\ell(n)+1}) - 3571 \frac{44347500}{625} \frac{2}{75}) \frac{2^{\ell(n)+1}}{2^{\ell(n)+1}}
\]

\[
\frac{\log 2}{2 \log 3}.
\]

and so

\[
\lim_{\ell(n) \to \infty} \frac{2 \log (F(2\ell(n) + 1))}{-\log (V_{F(2\ell(n)+1)})} = \frac{\log 2}{\log 3}. \quad \text{Similarly, } \lim_{\ell(n) \to \infty} \frac{2 \log (F(2\ell(n) + 1) + 1))}{-\log (V_{F(2\ell(n)+1)})} = \frac{\log 2}{\log 3}.
\]

Thus, \( \frac{\log 2}{\log 3} \leq \lim \inf_{\nu} \frac{2 \log n}{-\log V_{\nu}} \leq \lim \sup_{\nu} \frac{2 \log n}{-\log V_{\nu}} \leq \frac{\log 2}{\log 3} \) implying the fact that the quantization dimension of the measure \( P \) exists and is equal to \( D(\nu) = \beta \). \( \square \)

**Theorem 3.4.2** \( \beta \)-dimensional quantization coefficient for the condensation measure \( P \) does not exist; however, the \( \beta \)-dimensional lower and upper quantization coefficients for \( P \) are finite and positive.

**Proof** Since \( \beta = \frac{\log 2}{\log 3} \) for any \( k \geq 2 \), we have

\[
F(2k + 1)^{\frac{\beta}{3}} = (2 \cdot 2^{2k} + 3 \cdot 2^{3k})^{\frac{2}{3}} = 2^{6k} \left(\frac{2}{2k} + 3\right)^{\frac{2}{3}} = 9^{3k} \left(\frac{2}{2k} + 3\right)^{\frac{2}{3}}.
\]  

(8)

Similarly, \( F(2(k+1))^{\frac{\beta}{3}} = 9^{3k} \left(\frac{2}{2k} + 8\right)^{\frac{2}{3}}. \) Moreover, \( 9^{\alpha(2k+1)} = 9^{3k} \) and \( 9^{\alpha(2(k+1))} = 9^{3(k+1) - 2} = 9^{3k+1}, \) and \( 9^{3k} \left(\frac{2}{75}\right)^{2k-1} = (324 \frac{625}{7224})^{k} \left(\frac{75}{2}\right). \) Then, by Lemma 3.2.11, we have

\[
F(2k + 1)^{\frac{\beta}{3}} V_{F(2k+1)}(P)
\]

\[
= \left(\frac{2}{2k} + 3\right)^{\frac{2}{3}} \left(\frac{79}{7224} \left(1 - \left(\frac{324}{625}\right)^{k}\right) - 3571 \frac{44347500}{625} \left(\frac{324}{625}\right)^{k} \frac{75}{2}\right)
\]

yielding

\[
\lim_{k \to \infty} F(2k + 1)^{\frac{\beta}{3}} V_{F(2k+1)}(P) = 3^{\frac{2}{3}} \frac{79}{7224},
\]  

(9)
and

\[ F(2(k + 1))^2 \beta V_{F(2(k+1))}(P) = \left(\frac{4}{2^k} + 8\right)^2 \frac{2}{\beta} \left(\frac{1}{27} W + \frac{2}{75} \frac{79}{7224} \left(1 - \left(\frac{324}{625}\right)^k\right)\right) \]

\[ - \frac{3571}{44347500} \left(\frac{324}{625}\right)^k \]

yielding \( \lim_{k \to \infty} F(2(k + 1))^2 \beta V_{F(2(k+1))}(P) = 8^2 \left(\frac{1}{27} W + \frac{2}{75} \frac{79}{7224}\right) \). Since \( F(2k + 1)^2 \beta V_{F(2(k+1))}(P) \) and \( F(2(k + 1))^2 \beta V_{F(2(k+1))}(P) \) are two subsequences of \( (n^2 V_n(P))_{n \in \mathbb{N}} \) having two different limits, we can say that the sequence \( (n^2 V_n(P))_{n \in \mathbb{N}} \) does not converge, in other words, the \( \beta \)-dimensional quantization coefficient for \( P \) does not exist. For \( n \in \mathbb{N}, n \geq 3 \), let \( \ell(n) \) be the least positive integer such that \( F(2\ell(n) + 1) \leq n < F(2(\ell(n) + 1) + 1) \). Then, \( V_{F(2(\ell(n)+1)+1)} < V_n \leq V_{F(2(\ell(n)+1)} \) implying

\[ F(2(\ell(n) + 1))^2 \beta V_{F(2(\ell(n)+1)+1)} < n^2 \beta V_n < F(2(\ell(n) + 1) + 1)^2 \beta V_{F(2(\ell(n)+1)}. \]

As \( \ell(n) \to \infty \) whenever \( n \to \infty \), by Eq. (8), we have

\[ \lim_{n \to \infty} \frac{F(2(\ell(n) + 1))}{F(2(\ell(n) + 1) + 1)} = \lim_{\ell(n) \to \infty} \frac{g^{3\ell(n)}(\frac{2}{2^{\ell(n)+1}} + 3)^2 \beta}{g^{3\ell(n)+1}(\frac{2}{2^{\ell(n)+1}} + 3)^2 \beta} = \frac{1}{9}. \]

Hence, using (9), we have

\[ \lim_{n \to \infty} F(2(\ell(n) + 1) + 1)^2 \beta V_{F(2(\ell(n)+1)+1)} = \frac{1}{9} \lim_{\ell(n) \to \infty} F(2(\ell(n) + 1) + 1)^2 \beta V_{F(2(\ell(n)+1)+1)} = \frac{1}{9}(3^2 \frac{79}{7224}), \]

and similarly,

\[ \lim_{n \to \infty} F(2(\ell(n) + 1) + 1)^2 \beta V_{F(2(\ell(n)+1)+1)} = 9 \lim_{\ell(n) \to \infty} F(2(\ell(n) + 1) + 1)^2 \beta V_{F(2(\ell(n)+1)} = 9(3^2 \frac{79}{7224}), \]

yielding the fact that \( \frac{1}{9}(3^2 \frac{79}{7224}) \leq \liminf_{n \to \infty} n^2 \beta V_n \leq \limsup_{n \to \infty} \frac{n^2}{\beta} V_n \leq 9(3^2 \frac{79}{7224}) \), i.e., \( \beta \)-dimensional lower and upper quantization coefficients for \( P \) are finite and positive. \( \square \)
Recall that the critical value of the condensation system under consideration is the number \( \kappa \) satisfying \( \left( \frac{1}{3} \right)^3 \frac{2}{75} = \frac{1}{2} \). Hence, \( \kappa = \frac{2 \log 2}{\log 75 - \log 2} \); on the other hand, \( D(\nu) = \frac{\log 2}{\log 3} > \kappa \). Therefore, by Theorem 3.4.1, it follows that

**Proposition 3.4.3** \( D(P) = \max\{\kappa, D(\nu)\} \).

### 4 Condensation Measure \( P \) with Self-Similar Measure \( \nu \) Satisfying \( D(\nu) < \kappa \)

In this section, we study the optimal quantization for the condensation measure \( P \) generated by the condensation system \( \{S_1, S_2\}, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), \nu \)\), where the self-similar measure \( \nu \) is given by \( \nu = \frac{1}{2} \nu \circ T_1 + \frac{1}{2} \nu \circ T_2^{-1} \) with \( T_1(x) = \frac{1}{7} x + \frac{12}{35} \), and \( T_2(x) = \frac{1}{7} x + \frac{18}{35} \) for all \( x \in \mathbb{R} \) (i.e., the case \( s = \frac{1}{7} \)). From the general results obtained in Sect. 2, we have

- \( E(\nu) = \frac{1}{2} \), \( W = V(\nu) = \frac{3}{400} \); \( E(P) = \frac{1}{2} \), \( V = V(P) = \frac{131}{1168} \);
- \( \alpha_1 = \{ \frac{1}{2} \} \), with \( V_1 = V(P) \),
- \( \alpha_2 = \{ \frac{43}{210}, \frac{167}{210} \} \), \( V_2 = \frac{321877200}{321827} \), and
- \( \alpha_3 = \{ \frac{1}{7}, \frac{1}{7}, \frac{9}{10} \} \), \( V_3 = \frac{481}{87600} \).

Below, if the arguments of some statements are exactly the same as their counterparts in the previous section, we will omit their proofs to avoid repetition.

Notice that in this case, we have \( V_{2^n}(\nu) = \frac{1}{49^n} W \), and

\[
\int_{T_{\omega}(L)} (x - x_0)^2 d\nu(x) = \frac{1}{2^k} \left( \frac{1}{49^k} W + \left( T_{\omega} \left( \frac{1}{2} \right) - x_0 \right)^2 \right) .
\]

**Proposition 4.1** The set \( \{S_1(\frac{1}{2}), T_1(\frac{1}{2}), T_2(\frac{1}{2}), S_2(\frac{1}{2})\} \) is an optimal set of four-means with quantization error \( V_4 = \frac{13057}{4292400} \).

**Proposition 4.2** Let \( \alpha_n \) be an optimal set of \( n \)-means for all \( n \geq 3 \). Then, \( \alpha_n \cap J_1 \neq \emptyset \), \( \alpha_n \cap J_2 \neq \emptyset \), and \( \alpha_n \cap L \neq \emptyset \). Moreover, \( \alpha_n \) does not contain any point from the open intervals \( (\frac{1}{5}, \frac{2}{5}) \) and \( (\frac{3}{5}, \frac{4}{5}) \).

### 4.3 Canonical Sequence and Optimal Quantization

In this section, the two canonical sequences are \( \{a(n)\}_{n \geq 1} \) and \( \{F(n)\}_{n \geq 1} \) given by the following definition.

**Definition 4.3.1** The canonical sequences for this condensation system \( (S, p, \nu) \) is defined as

\[
a(n) = \begin{cases} 
  a(1) = 1 \\
  a(n) = n - 1 \text{ for all } n \geq 2, \text{ and }
\end{cases}
\]

\[
F(n) = (n + 3)2^{n-1} \text{ for all } n \geq 1.
\]
From these definitions it follows that, for any \( n \geq 1 \), \( 2^{a(n+1)} + 2F(n) = 2^n + (n + 3)2^n = (n + 4)2^n = F(n + 1) \); hence, we have

**Lemma 4.3.2** For the canonical sequences \( a(n) \) and \( F(n) \), \( F(n + 1) = 2^{a(n+1)} + 2F(n) \).

For \( 1 \leq \ell \leq n \), write \( S(\ell) := \bigcup_{\omega \in I^{n-\ell}S_\omega(\alpha_{2^\ell}(v))} \) and \( S^{(2)}(\ell) := \bigcup_{\omega \in I^{n-\ell}S_\omega(\alpha_{2^{\ell+1}}(v))}. \) Notice that if \( \ell = n \), then \( S(n) = \alpha_{2^n}(v) \). Moreover, write

\[
S(0) := \{ S_\omega \left( \frac{1}{2} \right) : \omega \in I^n \},
\]
\[
S^{(2)}(0) := \{ S_\omega(\alpha_2(P)) : \omega \in I^n \} = \left\{ S_\omega \left( \frac{43}{210} \right), S_\omega \left( \frac{167}{210} \right) : \omega \in I^n \right\}, \quad \text{and}
\]
\[
S^{(2)}(2) := \bigcup_{\omega \in I^n} S_\omega(\alpha_{2^{n+1}}(v)) \cup \left\{ S_\omega \left( \frac{1}{2} \right) : \omega \in I^{n+1} \right\}.
\]

For any \( \ell \in \mathbb{N} \cup \{0\} \), if \( A := S(\ell) \), we identify \( S^{(2)}(\ell) \) and \( S^{(2)}(2)(\ell) \) respectively by \( A^{(2)} \) and \( A^{(2)(2)} \). For \( n \in \mathbb{N} \), set

\[
\alpha_{F(n)} := S(n) \cup S(n-1) \cup S(n-2) \cup \cdots \cup S(1) \cup S(0), \quad \text{and}
\]
\[
SF(n) := \{ S(n), S(n-1), S(n-2), \cdots, S(1), S(0) \}. \quad (10)
\]

In addition, write

\[
SF^*(n) := \{ S(n), S(n-1), \cdots, S(0), S^{(2)}(0) \}.
\]

The order relation \( > \) on the elements of \( SF^*(n) \) and \( SF(n) \) are defined analogously as it is defined on the elements of \( SF^*(n) \) and \( SF(n) \) in Sect. 2.

**Remark 4.3.3** By Definition 4.3.1, \( \alpha_{F(n)} = S_1(\alpha_{F(n-1)}) \cup \alpha_{2^n}(v) \cup S_2(\alpha_{F(n-1)}) \).

**Lemma 4.3.4** Let \( > \) be the order relation on \( SF^*(n) \). Then,

\[
S(2) > S(0) > S(3) > S(4) > \cdots > S(11) > S^{(2)}(0) > S(12) > S(13) > \cdots > S(18) > S(1) > S(19) > S(20) > \cdots.
\]

**Proof** For any \( n \geq k \geq 1 \), the distortion error due to any element in the set \( S(k) \) is given by \( \frac{1}{75k-3} \), \( \frac{1}{2a(n)n} \), \( \frac{1}{4g(n)n} \) \( W \). On the other hand, the distortion error due to to the set \( S(0) \) and \( S^{(2)}(0) \) are, respectively, given by \( \frac{1}{75} V \) and \( \frac{1}{75} \frac{1}{2} V_2 \). Hence, \( S(2) > S(0) > S(3) \) will be true if \( \frac{1}{3} \frac{1}{98} W > V > \frac{1}{3} \frac{1}{98} W \) which is clearly true. Thus, \( S(2) > S(0) > S(3) \). For \( n \geq k \geq 2 \), the inequality \( S(k) > S(k + 1) \) is true if \( 1 > \frac{75}{98} \), which is obvious. Moreover, since \( \frac{1}{3} \frac{1}{98} W > \frac{1}{2} V_2 > \frac{1}{3} \frac{1}{98} W \), we have \( S(11) > S^{(2)}(0) > S(12) \). Again, \( \frac{75}{98} > \frac{75}{98} > \frac{75}{98} \) \( \frac{1}{2} W \), yields \( S(18) > S(1) > S(19) \). Combining all these inequalities, we see that the lemma follows. \( \square \)
Remark 4.3.5 Lemma 4.3.4 implies that if $n = 1$, then $S(0) > S^{(2)}(0) > S(1)$; if $n = 2$, then $S(2) > S(0) > S^{(2)}(0) > S(1)$; if $n = 3$, then $S(2) > S(0) > S(3) > S^{(2)}(0) > S(1)$; and so on.

From the definitions of $S(k)$ for all $0 \leq k \leq n$ and $S^{(2)}(0)$ it follows that

Lemma 4.3.6 Let $\alpha_{F(n)}$ and $SF(n)$ be the sets as defined before. Then,

$$\alpha_{F(n+1)} = \bigcup_{k=1}^{n} S^{(2)}(k) \cup S^{(2)}(0).$$

Lemma 4.3.7 For any two sets $A, B \in SF^*(n)$, let $A > B$. Then, the distortion error due to the set $(SF^*(n) \setminus A) \cup A^{(2)} \cup B$ is less than the distortion error due to the set $(SF^*(n) \setminus B) \cup B^{(2)} \cup A$.

Proof We have $SF^*(n) = \{S(n), S(n - 1), \ldots, S(1), S(0), S^{(2)}(0)\}$. Let $V_{SF^*(n)}$ be the distortion error due to the set $SF^*(n)$. First, take $A = S(k)$ and $B = S(k')$ for some $2 \leq k < k' \leq n$. Then, by Lemma 4.3.4, $A > B$. The distortion error due to the set $(SF^*(n) \setminus A) \cup A^{(2)} \cup B$ is given by

$$V_{SF^*(n)} = \frac{2}{75} n^{-k} \frac{1}{3} 49^{a(k)} + \frac{2}{75} n^{-k} \frac{1}{3} 49^{a(k) + 1} + \frac{2}{75} n^{-k'} \frac{1}{3} 49^{a(k')}$$

$$= V_{SF^*(n)} - \frac{2}{75} n^{-k} \frac{1}{3} 49^{k} + \frac{2}{75} n^{-k'} \frac{1}{3} 49^{k' - 1}. \quad (11)$$

Similarly, The distortion error due to the set $(SF^*(n) \setminus B) \cup B^{(2)} \cup A$ is

$$V_{SF^*(n)} = \frac{2}{75} n^{-k'} \frac{1}{3} 49^{k'} + \frac{2}{75} n^{-k} \frac{1}{3} 49^{k - 1}. \quad (12)$$

Thus, (11) will be less than (12) if $\left(\frac{98}{75}\right)^{k' - k} > 1$, which is clearly true since $k' > k$. Similarly, we can prove the lemma for any two elements $A, B \in SF^*(n)$. Thus, the proof of the lemma is complete. \qed

Proposition 4.3.8 For any $n \geq 1$ the set $\alpha_{F(n)}$ is an optimal set of $F(n)$-means for the condensation measure $P$ whose quantization error is given by

$$V_{F(n)} := V_{F(n)}(P) = \begin{cases} \frac{13057}{4292400} & \text{if } n = 1, \\ \frac{69071}{6170325} \left(\frac{2}{75}\right)^{n-1} - \frac{3}{368} \left(\frac{1}{4}\right)^{n-1} & \text{if } n \geq 2. \end{cases}$$

Proof By Proposition 4.1, the set $\alpha_{F(1)} = \{S_{1}(\frac{1}{2}), T_{1}(\frac{1}{2}), T_{2}(\frac{1}{2}), S_{2}(\frac{1}{2})\}$ is an optimal set of $F(1)$-means with quantization error $V_{4} = \frac{13057}{4292400}$. Proceeding in the same way as Proposition 3.2.10, we can show that for any $n \geq 2$, the set $\alpha_{F(n)}$ forms an optimal set of $F(n)$-means, and the quantization error $V_{F(n)}$ is given by...
\[ \int \min_{a \in \alpha_{F(n)}} (x - a)^2 \, dP \]
\[ = \sum_{k=0}^{n-1} \sum_{\omega \in I^k} \int_{J_\omega} \min_{a \in S_\omega(\alpha_{\omega(n-k)}(v))} (x - a)^2 \, dP + \sum_{\omega \in I^n} \int_{J_\omega} \left( x - S_\omega \left( \frac{1}{2} \right) \right)^2 \, dP \]
\[ = \sum_{k=0}^{n-1} \sum_{\omega \in I^k} \frac{1}{3} \frac{75^k}{49(n-k)} + \left( \frac{2}{75} \right)^n V = \sum_{k=0}^{n-1} \frac{1}{3} \left( \frac{2}{75} \right)^k \frac{W}{49(n-k)} + \left( \frac{2}{75} \right)^n V \]
\[ = \frac{1}{3} \frac{W}{49} \left( \frac{2}{75} \right)^{n-1} - \left( \frac{1}{49} \right)^{n-1} \frac{W}{49} + \left( \frac{2}{75} \right)^n V \]
\[ = \left( \frac{2}{75} \right)^{n-1} \left( \frac{1}{3} \frac{W}{49} \frac{3675}{23} + \frac{1}{3} \frac{W}{49} + \frac{2}{75} V \right) - \left( \frac{1}{49} \right)^{n-1} \frac{1}{3} \frac{W}{49} \frac{3675}{23} \]
yielding
\[ V_{F(n)} = \frac{69071}{6170325} \left( \frac{2}{75} \right)^{n-1} - \frac{3}{368} \left( \frac{1}{49} \right)^{n-1}. \]

Thus, the proof of the proposition is complete. \qed

4.4 Asymptotics for the \( n \)th Quantization Error \( V_n(P) \)

In this subsection, we turn to the investigation of the quantization dimension \( D(P) \) and the \( D(P) \)-dimensional quantization coefficient for the condensation measure \( P \). Notice that in this case \( D(\nu) = \beta \), where \( \beta = \frac{\log 2}{\log 7} \) which is the Hausdorff dimension of the limit set generated by the similarity maps \( T_1 \) and \( T_2 \) considered in this section.

**Theorem 4.4.1** \( D(P) = \lim_{n \to \infty} \frac{2 \log n}{\log V_n(P)} = \kappa \); hence, the quantization dimension of \( P \) exists and is equal to the critical value of the condensation system.

**Proof** Since \( \left( \frac{1}{3} \left( \frac{1}{7} \right)^2 \right)^{\frac{\kappa}{2}} = \frac{1}{3} \), \( \kappa = \frac{2 \log 2}{\log 7} \log 7 \). For \( n \in \mathbb{N} \), \( n \geq 4 \), let \( \ell(n) \) be the least positive integer such that \( F(\ell(n)) \leq n < F(\ell(n) + 1) \). Then, \( V_{F(\ell(n)+1)} < V_n \leq V_{F(\ell(n))} \). Thus, we have

\[ \frac{2 \log (F(\ell(n)))}{- \log (V_{F(\ell(n)+1)})} < \frac{2 \log n}{- \log V_n} < \frac{2 \log (F(\ell(n) + 1))}{- \log (V_{F(\ell(n))})}. \]

Notice that when \( n \to \infty \), then \( \ell(n) \to \infty \). Recall that \( F(\ell(n)) = (\ell(n) + 3)2^{\ell(n)-1} \), and so by Proposition 4.3.8, we have
Proof} For $n \in \mathbb{N}, n \geq 4$, let $\ell(n)$ be the least positive integer such that $F(\ell(n)) \leq n < F(\ell(n) + 1)$. Then, $V_F(\ell(n) + 1) < V_n \leq F_F(\ell(n))$ implying \( F(\ell(n)) \frac{2^n}{\kappa} V_F(\ell(n) + 1) < n^{2/\kappa} V_n < (F(\ell(n) + 1))^{2/\kappa} V_F(\ell(n)) \). As $\ell(n) \to \infty$ whenever $n \to \infty$, we have

\[
\lim_{n \to \infty} \frac{F(\ell(n))}{F(\ell(n) + 1)} = \lim_{n \to \infty} \frac{(\ell(n) + 3)2^{\ell(n) - 1}}{(\ell(n) + 4)2^{\ell(n)}} = \frac{1}{2}.
\]

Next, since $\kappa = \frac{2\log 2}{\log 75 - \log 2}$, we have

\[
\lim_{n \to \infty} (F(\ell(n)))^{2/\kappa} V_F(\ell(n) + 1) = \frac{1}{4^{1/\kappa}} \lim_{n \to \infty} (F(\ell(n) + 1))^{2/\kappa} V_F(\ell(n) + 1)
\]

\[
= \frac{1}{4^{1/\kappa}} \lim_{n \to \infty} (\ell(n) + 4)^{2/\kappa} 2^{\ell(n)/\kappa} \left( \frac{69071}{6170325} \left( \frac{2}{75} \right)^{\ell(n)} - \frac{3}{368} \left( \frac{1}{49} \right)^{\ell(n)} \right)
\]

\[
= \frac{1}{4^{1/\kappa}} \lim_{n \to \infty} (\ell(n) + 4)^{2/\kappa} \left( \frac{75}{2} \right)^{\ell(n)} \left( \frac{69071}{6170325} \left( \frac{2}{75} \right)^{\ell(n)} - \frac{3}{368} \left( \frac{1}{49} \right)^{\ell(n)} \right)
\]

\[
= \frac{1}{4^{1/\kappa}} \lim_{n \to \infty} (\ell(n) + 4)^{2/\kappa} \left( \frac{69071}{6170325} - \frac{3}{368} \left( \frac{75}{98} \right)^{\ell(n)} \right) = \infty,
\]

and similarly

\[
\lim_{n \to \infty} (F(\ell(n) + 1))^{2/\kappa} V_F(\ell(n)) = 4^{1/\kappa} \lim_{n \to \infty} (F(\ell(n)))^{2/\kappa} V_F(\ell(n)) = \infty
\]

yielding the fact that $\infty \leq \liminf_{n \to \infty} n^{2/\kappa} V_n(P) \leq \limsup_{n \to \infty} n^{2/\kappa} V_n(P) \leq \infty$, i.e., the $D(P)$-dimensional quantization coefficient for the condensation measure $P$ is infinity. \qed
Since $D(P) = \kappa = \frac{2 \log 2}{\log 75 - \log 2} > D(\nu) = \frac{\log 2}{\log 7}$, the following proposition is also true.

**Proposition 4.4.3** $D(P) = \max\{\kappa, D(\nu)\}$.

## 5 Condensation Measures $P$ with Self-Similar Measure $\nu$ Satisfying $D(\nu) > \kappa$ and $D(\nu) = \kappa$

In this section, we consider the quantization when $s = \frac{1}{5}$ and $s = \frac{\sqrt{5}}{15}$. Since the proofs are the same as in the previous two cases, we will summarize the results and make some concluding remarks. The condensation measures $P$ in these cases are generated by the systems $\{(S_1, S_2), (\frac{1}{5}, \frac{1}{5}, \frac{1}{5})\}$, where the measures $\nu$ are given by $\nu = \frac{1}{2} \nu \circ T_1^{-1} + \frac{1}{2} \nu \circ T_2^{-1}$ with

$T_1(x) = \frac{3}{5} x + \frac{1}{5}$ and $T_2(x) = \frac{3}{5} x + \frac{2}{5}$ in the case $s = \frac{1}{5}$, and

$T_1(x) = \frac{4\sqrt{5}}{15} x + \frac{3}{5} - \frac{2\sqrt{5}}{75}$ and $T_2(x) = \frac{4\sqrt{5}}{15} x + \frac{6}{5} - \frac{\sqrt{5}}{25}$ in the case $s = \frac{\sqrt{5}}{15}$.

Table 1 below outlines the information for the quantization of $P$ in each case. The results in table are consequence of the particular facts that:

(a) For $n \geq 3$, $\alpha_n \cap J_1 \neq \emptyset$, $\alpha_n \cap J_2 \neq \emptyset$, and $\alpha_n \cap L \neq \emptyset$, and $\alpha_n$ does not contain any point from the open intervals $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$.

(b) The sets $SF(n)$ and $SF^*(n)$, and the order relation $\succ$ among the elements of them are also defined the same way as in Sect. 3. Furthermore, $\kappa$ does not contain any point from the open intervals $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$.

| $s = \frac{1}{5}$ | $s = \frac{\sqrt{5}}{15}$ |
|-------------------|--------------------------|
| $E(\nu)$, $V(\nu)$ | $\frac{1}{2}$, 1350       | $\frac{1}{2}$, 77-10\sqrt{5} | 7500 |
| $E(P)$, $V(P)$   | $\frac{1}{2}$, 4938       | $\frac{1}{2}$, 2413-10\sqrt{5} | 21710 |
| $\alpha_1$, $V_1$ | $\{\frac{1}{2}\}$, 4938   | $\alpha_1$, same, 2413-10\sqrt{5} | 21710 |
| $\alpha_2$, $V_2$ | $\{\frac{31}{750}$, 119, 21211 | \{821250, $\frac{\sqrt{5}+90}{450}$, 30-\sqrt{5} | 450, 155430, $\sqrt{5}+4167521$ | 179853750 |
| $\alpha_3$, $V_3$ | $\{\frac{1}{10}$, 12, 9, 10, 3630 | $\alpha_3$, same, 10447-750\sqrt{5} | 1598700 |
| $\alpha_4$, $V_4$ | $\{S_1(\frac{1}{2}), T_1(\frac{1}{2}), T_2(\frac{1}{2}), S_2(\frac{1}{2})\}$, 841273750 | $\alpha_4$, same, 93298-740\sqrt{5} | 29975625 |
| $a(n)$           | 1 if $n = 1$, $n-1$ for $n > 1$ | Same |
| $F(n)$           | $(n+3)2^{n-1}$            | Same |
| $V_F(n)$         | $\frac{1}{2^{5n-1}} - \frac{164}{3525}(\frac{3}{75})^{n-1}$ if $n \geq 2$ | $\left(\frac{2}{75}\right)^{n-1}V(\nu) \frac{1}{2}$ $n+\frac{2}{3}V(\nu)$ for $n \geq 2$ | 
| $\kappa$         | $\frac{2 \log 2}{\log 75-\log 2}$ | $\frac{2 \log 2}{\log 75-\log 2}$ |
| $D(\nu)$         | $\log 2/\log 5$           | $\log 2/\log 5$ |
| $D(P)$           | $\log 2/\log 5$           | $\log 2/\log 5$ |
Table 2  Relationships among \(D(P), D(\nu)\) and \(\kappa\)

| Contracting factor | Quant. dimensions | Quant. coefficients | \(D(\nu)\) versus critical value |
|--------------------|-------------------|---------------------|----------------------------------|
| \(s = \frac{1}{3}\) | \(D(\nu) = D(P)\) | Finite, positive | \(D(\nu) > \kappa\) |
| \(s = \frac{1}{5}\) | \(D(\nu) = D(P)\) | Infinite | \(D(\nu) > \kappa\) |
| \(s = \frac{\sqrt{5}}{15}\) | \(D(\nu) = D(P)\) | Infinite | \(D(\nu) = \kappa\) |
| \(s = \frac{1}{7}\) | \(D(\nu) < D(P)\) | Infinite | \(D(\nu) < \kappa\) |

(i) \(\cdots > S(4) > S(3) > S(2) > S(0) > S^{(2)}(0) > S(1)\).
(ii) Although the canonical sequences \(\{a(n)\}_{n\geq 1}\) and \(\{F(n)\}_{n\geq 1}\) are identical, the order in the elements of \(SF^*(n)\) are different.
(iii) For \(A, B \in SF^*(n)\) with \(A > B\), the distortion error due to the set \((SF^*(n) \setminus A) \cup A^{(2)} \cup B\) is less than the distortion error due to the set \((SF^*(n) \setminus B) \cup B^{(2)} \cup A\).

Since, from \((\frac{1}{3}(\frac{1}{3})^2)^{\frac{x}{\nu}} = \frac{1}{2}\) we have \(\kappa = \frac{2\log 2}{\log 75 + \log 2}\) as the critical value, it follows that for these condensation systems \(D(P) = \max\{\kappa, D(\nu)\}\) as well.

4.1 Concluding Remarks

For the condensation system \((\{S_1, S_2\}, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}, \nu)\) generating the condensation measure \(P\), where \(S_1, S_2\) are the similarity mappings and \(\nu\) is a self-similar measure defined by \(\nu = \frac{1}{3} \nu \circ T_1^{-1} + \frac{1}{3} \nu \circ T_2^{-1}\) with \(T_1(x) = sx + (1 - s)^{\frac{2}{3}}\) and \(T_2(x) = sx + (1 - s)^{\frac{3}{3}}\), \(0 < s < \frac{1}{3}\), by Theorems 3.4.2, 4.4.2, and Table 2, we see that when \(s = \frac{1}{7}, \frac{\sqrt{5}}{15}, \frac{1}{3}\), the \(D(P)\) dimensional lower quantization coefficient for the condensation measure \(P\) is infinity; on the other hand, if \(s = \frac{1}{3}\), then the \(D(P)\)-dimensional lower and upper quantization coefficients are finite, positive and unequal. Notice that \(\frac{1}{7} < \frac{\sqrt{5}}{15} < \frac{1}{3} < \frac{1}{7}\). Thus, it is worthwhile to investigate the least upper bound of \(s\) for which the \(D(P)\) dimensional lower quantization coefficient for the condensation measure \(P\) is infinity. Such a problem still remains open.

Observe that, for \(s = \frac{1}{7}\) and \(s = \frac{\sqrt{5}}{15}\), although the quantization coefficients are infinity, whereas the relations between \(D(P)\) and \(D(\nu)\) are not the same. Again, it is worthwhile to investigate the least upper bound of \(s\) for which the \(D(P) = D(\nu)\) while \(D(P)\)-dimensional lower quantization coefficient for the condensation measure \(P\) is infinity.

Data Availability  Data sharing not applicable to this article as no datasets were generated or analyzed during the current study. However, detailed calculations for the items summarized in Table 1 are available from the corresponding author upon request.
References

1. Abaya, E.F., Wise, G.L.: Some remarks on the existence of optimal quantizers. Stat. Probab. Lett. 2(6), 349–351 (1984)
2. Abundo, C., Bodnar, T., Driscoll, J., Hatton, I., Wright, J.: City population dynamics and fractal transport networks. In: Proceedings of the Santa Fe Institute’s CSSS2013; Santa Fe Institute: Santa Fe, NM (2013)
3. Barnsley, M.F.: Fractals Everywhere. Academic Press, New York (1988)
4. Benhenni, K., Cambanis, S.: The effect of quantization on the performance of sampling designs. IEEE Trans. Inform. Theory 44, 1981–1992 (1998)
5. Bucklew, J.A., Wise, G.L.: Multidimensional asymptotic quantization theory with \( r \)th power distortion measures. IEEE Trans. Inf. Theory 28(2), 239–247 (1982)
6. Dettmann, C.P., Roychowdhury, M.K.: Quantization for uniform distributions on equilateral triangles. Real Anal. Exch. 42(1), 149–166 (2017)
7. Du, Q., Faber, V., Gunzburger, M.: Centroidal Voronoi tessellations: applications and algorithms. SIAM Rev. 41, 637–676 (1999)
8. Gruber, P.M.: Optimum quantization and its applications. Adv. Math. 186, 456–497 (2004)
9. Gersho, A., Gray, R.M.: Vector Quantization and Signal Compression. Kluwer Academy Publishers, Boston (1992)
10. Gray, R.M., Kieffer, J.C., Linde, Y.: Locally optimal block quantizer design. Inf. Control 45, 178–198 (1980)
11. György, A., Linder, T.: On the structure of optimal entropy-constrained scalar quantizers. IEEE Trans. Inf. Theory 48, 416–427 (2002)
12. Graf, S., Luschgy, H.: Foundations of Quantization for Probability Distributions. Lecture Notes in Mathematics, vol. 1730. Springer, Berlin (2000)
13. Graf, S., Luschgy, H.: The quantization of the Cantor distribution. Math. Nachr. 183, 113–133 (1997)
14. Graf, S., Luschgy, H.: Quantization for probability measures with respect to the geometric mean error. Math. Proc. Camb. Philos. Soc. 136, 687–717 (2004)
15. Graf, S., Luschgy, H.: The quantization dimension of self-similar probabilities. Math. Nachrichten 241, 103–109 (2002)
16. Graf, S., Luschgy, H., Pagés, G.: The local quantization behavior of absolutely continuous probabilities. Ann. Probab. 40(4), 1795–1828 (2012)
17. Gray, R., Neuhoff, D.: Quantization. IEEE Trans. Inform. Theory 44, 2325–2383 (1998)
18. Hao, Y., Chen, M., Hu, L., Song, J., Volk, M., Humar, I.: Wireless fractal ultra-dense cellular networks. Sensors 17, 841 (2017)
19. Kaza, K.R., Kshirsagar, K., Rajan, K.S.: A bi-objective algorithm for dynamic reconfiguration of mobile networks. In: Proceedings of the IEEE International Conference on Communications (ICC), Ottawa, ON, Canada, pp. 5741–5745 (2012)
20. Kesseböhmer, M., Zhu, S.: Some recent developments in quantization of fractal measures. In: Fractal Geometry and Stochastics V (Progress in Probability Book 70), pp. 105–120. Birkhäuser/Springer, Cham (2015)
21. Kesseböhmer, M., Zhu, S.: Stability of quantization dimension and quantization for homogeneous Cantor measures. Math. Nachr. 280(8), 866–881 (2007)
22. Lasota, A.: A variational principle for fractal dimensions. Nonlinear Anal. 64(3), 618–628 (2006)
23. Lindsay, L.J., Mauldin, R.D.: Quantization dimension for conformal iterated function systems. Nonlinearity (Institute of Physics Publishing) 15, 189–199 (2002)
24. Lu, Z., Zhang, H., Southworth, F., Crittenden, J.: Fractal dimensions of metropolitan area road networks and the impacts on the urban built environment. Ecol. Indic. 70, 285–296 (2016)
25. Okabe, A., Boots, B., Sugihara, K., Chiu, S.N.: Spatial Tessellations: Concepts and Applications of Voronoi Diagrams, 2nd edn. Wiley, Hoboken (2000)
26. Olsen, L., Snigireva, N.: Multifractal spectra of in-homogenous self-similar measures. Indiana U. Math. J. 57, 1789–1844 (2008)
27. Pagés, G.: A space quantization method for numerical integration. J. Comput. Appl. Math. 89, 1–38 (1997)
28. Pagès, G., Pham, H., Printemps, J.: Optimal quantization methods and applications to numerical problems in finance. In: Rachev, S. (ed.) Handbook of Computational and Numerical Methods in Finance, pp. 253–297. Birkhäuser Boston, Boston (2004)
29. Pötzelberger, K.: The quantization dimension of distributions. Math. Proc. Camb. Philos. Soc. 131, 507–519 (2001)
30. Roychowdhury, M.K.: Quantization and centroidal Voronoi tessellations for probability measures on dyadic Cantor sets. J. Fractal Geom. 4, 127–146 (2017)
31. Roychowdhury, M.K.: Optimal quantizers for some absolutely continuous probability measures. Real Anal. Exch. 43(1), 105–136 (2017)
32. Roychowdhury, M.K.: Least upper bound of the exact formula for optimal quantization of some uniform Cantor distributions. Discrete Contin. Dyn. Syst. Ser. A 38(9), 4555–4570 (2018)
33. Roychowdhury, M.K.: Quantization dimension function and ergodic measure with bounded distortion. Bull. Pol. Acad. Sci. Math. 57, 251–262 (2009)
34. Roychowdhury, M.K.: Quantization dimension for some Moran measures. Proc. Am. Math. Soc. 138, 4045–4057 (2010)
35. Roychowdhury, M.K.: Quantization dimension function and Gibbs measure associated with Moran set. J. Math. Anal. Appl. 373, 73–82 (2011)
36. Roychowdhury, M.K.: Quantization dimension estimate of inhomogeneous self-similar measures. Bull. Pol. Acad. Sci. Math. 61(1), 35–45 (2013)
37. Rosenblatt, J., Roychowdhury, M.K.: Optimal quantization for piecewise uniform distributions. Uniform Distrib. Theory 13(2), 23–55 (2018)
38. Song, Y.: Cost-Effective Algorithms for Deployment and Sensing in Mobile Sensor Networks. Ph.D. Thesis, University of Connecticut, Storrs, CT, USA (2014)
39. Zam, R.: Lattice Coding for Signals and Networks: A Structured Coding Approach to Quantization, Modulation, and Multiuser Information Theory. Cambridge University Press, Cambridge (2014)
40. Zador, P.L.: Asymptotic quantization error of continuous signals and the quantization dimensions. IEEE Trans. Inform. Theory 28, 139–148 (1982)

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