Charged gravitational instantons in five-dimensional
Einstein-Gauss-Bonnet-Maxwell theory

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We study a solution of the Einstein-Gauss-Bonnet theory in 5 dimensions coupled to a Maxwell field, whose euclidean continuation gives rise to an instanton describing black hole pair production. We also discuss the dual theory with a 3-form field coupled to gravity.
1. Introduction.

Vacuum solutions of four-dimensional Einstein gravity with positive cosmological constant are characterized by the presence of a cosmological horizon. In case of vanishing mass the solutions are given by the maximally symmetric de Sitter spacetime, but for nonzero mass one can obtain spherically symmetric black hole solutions which contain an event horizon besides the cosmological one. In the limiting case in which the two horizons coincide these solutions reduce to a non-trivial spacetime metric, known as Nariai metric, which has topology $H^2 \times S^2$. Its euclidean continuation describes a gravitational instanton which can be used to compute the creation rate for black hole pairs in a cosmological background [1]. When a Maxwell field is coupled, more general black hole solutions can also be obtained carrying electric or magnetic charge. In this case one can have up to three horizons, and a few special solutions giving rise to instantons by euclidean continuation can be found [2-3].

In higher dimensions, the Einstein-Hilbert action can be generalized by the addition of the so called Gauss-Bonnet terms [4-5]. The generalized action gives rise to field equations which are still second order and no new degrees of freedom are introduced in the theory. In four dimensions or less the Gauss-Bonnet terms are total derivatives and do not contribute to the field equations. It is well known that the actions so generalized admit asymptotically de Sitter solutions even in absence of a cosmological constant. In particular, the most general spherically symmetric solutions of the Einstein-Gauss-Bonnet theory coupled to a Maxwell field have been obtained in arbitrary dimensions [6]. They include solutions containing both cosmological and event horizons. One may thus wonder if solutions with properties similar to the Nariai metric or its charged generalizations are available also in this case for special values of the parameters.

In this paper we consider the simplest non-trivial example, namely the five-dimensional theory. This case is quite peculiar, since contrary to the higher-dimensional ones, the metric function has only one root, leading in general to the appearance of naked singularities, and so it turns out that the only physically interesting case is that in which the root is double, corresponding to two coinciding horizons. The properties of this metric are therefore similar to those of the Nariai solution. On the other hand, five dimensions is the only case
that can be treated analytically, because in higher dimensions one cannot in general find a closed algebraic expression for the location of the horizons.

2. The Lorentzian solution.

The five dimensional Einstein-Gauss-Bonnet-Maxwell action is

\[ I = \frac{1}{16\pi} \int d^5x \sqrt{-g} \left( R + \bar{\alpha}S - \frac{1}{4} F^2 \right), \]  

(1)

where \( S = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \) is the Gauss-Bonnet term and \( \bar{\alpha} \) is a coupling constant. The field equations are\(^{\dagger}\)

\[ G_{ab} + \bar{\alpha}S_{ab} = T^F_{ab}, \]  

(2)

where

\[ S_{ab} = 2R_{acde}R^c_{eb} - 4R_{abcd}R^{cd} - 4R_{ac}R^c_b + 2RR_{ab}, \]  

(3)

\[ T^F_{ab} = \frac{1}{2} F_{ac}F^c_b - \frac{1}{8} F^2 g_{ab}. \]  

(4)

The theory admits two maximally symmetric solutions with vanishing charge, namely flat space and 5-dimensional de Sitter or anti-de Sitter space (depending on the sign of \( \bar{\alpha} \)), with curvature \( R = -10/\bar{\alpha} \). These can be considered as the ground states of the theory. It has been argued \([5]\) that the de Sitter background is unstable under small perturbations, so that the actual ground state is flat Minkowski space. However, in this paper we are mainly interested in the de Sitter sector of the theory. Hence, we shall consider the case \( \bar{\alpha} < 0 \), which admits the existence of a cosmological horizon, and ignore the asymptotically flat solution. Charged spherically symmetric solutions are given by \([6]\)

\[ F_{01} = \frac{Q}{r^3}, \]  

(5)

\[ ds^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2d\Omega_3^2, \]  

(6)

with

\[ V(r) = 1 - \frac{r^2}{2\alpha} \left[ 1 \pm \sqrt{1 - 4\alpha \left( \frac{2m}{r^4} - \frac{q^2}{r^6} \right)} \right], \]  

(7)

\(^{\dagger}\) Throughout this paper, we adopt orthonormal indices.
where \( q^2 = Q^2/12 \) and \( \alpha = -2\bar{\alpha} > 0 \). The branch with the minus sign is asymptotically flat, while the branch with the plus sign, which we shall consider in the following, is asymptotically de Sitter. In the last case, a horizon can be present if \(-\alpha < 2m < 3\alpha\). More precisely, if \( 4q^2 > (\alpha + 2m)^2 \), \( V(r) \) is always negative. If instead \( 4q^2 \leq (\alpha + 2m)^2 \), \( V(r) \) has a unique zero at \( r_0^2 = \frac{\alpha + 2m - \sqrt{(\alpha + 2m)^2 - 4q^2}}{2} \). In this case, however, a branch singularity is present at a point \( r_s > r_0 \), where the square root in (7) becomes negative. The only possibility to avoid this singularity is in the extremal case, \( 4q^2 = (\alpha + 2m)^2 \), when \( r_0 \) is a double root. In this case apparently there is no physical region, since one has two coinciding horizons. However, this is not the case, as can be shown by adopting a limit procedure similar to that originally discussed in [1] for the Nariai solution.

In fact, for \( 4q^2 \to (\alpha + 2m)^2 \), one can expand \( V(r) \) in a neighborhood of \( r_0 = \sqrt{\frac{\alpha + 2m}{2}} \) as
\[
r = r_0 \left( 1 + \frac{\epsilon}{2} \cos \chi + o(\epsilon^2) \right),
\]
so that
\[
r^2 = \frac{\alpha + 2m}{2} \left( 1 + \epsilon \cos \chi + o(\epsilon^2) \right),
\]
where \( \chi \) is a new coordinate. One can also expand \( q^2 \) as \( q^2 = \frac{(\alpha + 2m)^2}{4} (1 - \epsilon^2) \) and define a rescaled time coordinate
\[
\tau = \frac{8\sqrt{\alpha + 2m}}{3\alpha - 2m} \epsilon t.
\]
Substituting in (6), one finally gets, at leading order in \( \epsilon \),
\[
ds^2 = \frac{1}{A} (-\sin^2 \chi dr^2 + d\chi^2) + \frac{1}{B} d\Omega_3^2,
\]
where
\[
A = \frac{8}{3\alpha - 2m} = \frac{4}{2\alpha - |q|}, \quad B = \frac{2}{\alpha + 2m} = \frac{1}{|q|},
\]
Moreover, in this limit, \( F_{01} = QB^{3/2} = 2\sqrt{12}/(\alpha + 2m) \). The constants \( A \) and \( B \) are both positive since, as discussed previously, \(-\alpha < 2m < 3\alpha\) (or equivalently \(|q| < 2\alpha\)). The metric (11) has the form of a product of a two-dimensional de Sitter space of size \( 1/\sqrt{A} \) with a three-sphere of radius \( 1/\sqrt{B} \) and is therefore perfectly regular. Its Penrose diagram is that of two-dimensional de Sitter spacetime. In particular, in analogy with the Nariai
metric [1], an observer sees two cosmological horizons, both in the positive and negative \( \chi \)-directions. The solution can therefore be interpreted as a pair of black holes at antipodal points on the spatial section of a de Sitter spacetime.

The previous solution can also be obtained directly from the field equations (2), which for a metric of the form (11) and electric field \( F_{01} = QB^{3/2} \) reduce to

\[
-3B = -\frac{Q^2}{4} B^3, \quad -A(1 - 2\alpha B) - B = \frac{Q^2}{4} B^3,
\]

(13)

and hence, recalling that \( Q^2 = 12q^2 \), \( B = \frac{1}{|q|} \), \( A = \frac{4}{2\alpha - |q|} \), in accordance with (12).

From (13) follows that in five dimensions a non-flat solution is available only if \( Q \neq 0 \).

The metric (11) can also be obtained by duality in the case of Einstein-Gauss-Bonnet gravity coupled to a 3-form field \( H_{abc} \), with action

\[
I = \frac{1}{16\pi} \int d^5x \sqrt{-g} \left( R + \bar{\alpha}S - \frac{1}{12} H^2 \right).
\]

(14)

The gravitational field equations are now

\[
G_{ab} + \bar{\alpha}S_{ab} = T^H_{ab},
\]

(15)

with

\[
T^H_{ab} = \frac{1}{4} H_{acd} H^c_{bd} - \frac{1}{24} H^2 g_{ab}.
\]

(16)

With the ansatz \( H_{abc} = QB^{3/2} \epsilon_{abc} \), one has \( T^H_{ab} = T^F_{ab} \), and hence the field equations reduce to those of the electromagnetic case.

3. The euclidean metric.

The euclidean continuation of the metric (11) is given by

\[
ds^2 = \frac{1}{A} (\sin^2 \chi d\psi^2 + d\chi^2) + \frac{1}{B} d\Omega^2_3,
\]

(17)

where \( \psi = i\tau \) is the euclidean time and \( 0 \leq \psi < 2\pi \), \( 0 \leq \chi < \pi \). The metric (17) describes the product of two round spheres, \( S^2 \times S^3 \). In analogy with the charged Nariai metric in four dimensions [1], it can be interpreted as a gravitational instanton mediating the creation of a pair of black holes in a background de Sitter spacetime.
As discussed in ref. [3], according to the no-boundary proposal, the pair creation rate can be estimated as

\[
\Gamma = \exp\left[-2(I_{S^2 \times S^3} - I_{S^5})\right],
\]

where \(I_{S^2 \times S^3}\) is the action of the half-instanton with boundary the spacelike surface \(\Sigma\) of topology \(S^2 \times S^2\), corresponding to the maximal spatial section of (11), and \(I_{S^5}\) is the action of the half euclidean de Sitter space of radius \(\sqrt{\alpha}\), with spatial boundary \(S^4\).

In the case of the Maxwell field, the euclidean action is given by

\[
I = -\frac{1}{16\pi} \int d^5x \sqrt{g} \left(R + \bar{\alpha}S - \frac{1}{4} F^2\right) + \text{boundary terms}.
\]

The gravitational boundary terms do not contribute, since in the case under study the second fundamental form vanishes on the boundaries, but an electric contribution should be added in order to keep the charge fixed at the boundary [3]. This has the form

\[
I_b = \frac{1}{16\pi} \int_{\Sigma} d^4x \sqrt{h} F^{ab} n_a A_b,
\]

where \(n^a\) is the outgoing normal to \(\Sigma\). Using the trace of the field equations, the five-dimensional integral in (19) reduces to

\[
I = \frac{1}{8\pi} \int d^5x \sqrt{g} R.
\]

For \(S^5\), eq. (21) gives

\[
I = \frac{5}{4\pi \alpha} Vol(S^5) = \frac{5\pi^2}{8} \alpha^{3/2},
\]

where \(Vol(\ )\) is the volume of the half-instanton. It is important to notice that, due to the contribution of the Gauss-Bonnet term, the sign of \(I\) is opposite to the one that would have been obtained in the case of Einstein theory with a cosmological constant. The high value of the action is another sign of the instability of the de Sitter space in the Einstein-Gauss-Bonnet model. For \(S^2 \times S^3\),

\[
I = \frac{A + 3B}{8\pi \alpha} Vol(S^2 \times S^3) = \pi^2 \frac{A + 3B}{AB^{3/2}},
\]

and moreover \(I_b\) gives a further contribution \(\frac{3\pi^2}{AB^{3/2}}\). The total action is therefore

\[
I_{S^2 \times S^3} = \pi^2 \frac{A + 6B}{AB^{3/2}} = \frac{\pi^2}{2} \sqrt{|q|}(6\alpha - |q|).
\]

The value of the action goes to zero for \(q \to 0\), since the volume of the instanton vanishes in this limit. Finally, substituting in (18),

\[
\Gamma = \exp \left[-\pi^2 \left(\sqrt{|q|}(6\alpha - |q|) - \frac{5}{4} \alpha^{3/2}\right)\right].
\]
In the interval of allowed values for $|q|, 0 < |q| < 2\alpha$, the exponent of $\Gamma$ can assume both positive and negative sign. In particular, it is positive for small $|q|$, corresponding to a high production rate. Five dimensional de Sitter space appears therefore to be unstable in our model, for creation of black hole pairs of charge $|q| \ll \alpha$.

The same calculation can be done for the 3-form field $H$. In this case, the euclidean action is

$$I = -\frac{1}{16\pi} \int d^5x \sqrt{g} \left( R + \tilde{\alpha}S - \frac{1}{12} H^2 \right) = \frac{1}{8\pi} \int d^5x \sqrt{g} \left( R + \frac{1}{12} H^2 \right),$$

(24)

where the trace of the field equations has been used. Also in this case the gravitational boundary terms vanish, while no boundary term is necessary for the 3-form field [3]. Substituting the values of the fields in (24), one gets the same result (22) obtained in the Maxwell case. It appears therefore that one can extend also to higher dimensions the validity of the conjectures of [3] on the invariance under duality of the pair production rate of black holes.

4. Conclusion.

We conclude with some considerations on the higher-dimensional generalizations of the results discussed here. Although in principle straightforward, the possibility of making explicit calculations is prevented because one can no longer obtain an explicit expression for the location of the horizon, since one should solve algebraic equations of higher degree. One can show however an important qualitative difference in higher dimensions, since in $D \geq 6$ it is possible to get two distinct horizons and hence a physical region even in the non-extremal case. Thus solutions with properties similar to the lukewarm black holes of ref. [2] are possible.

One can still, however, look directly for solutions of the $D$-dimensional theory of the form $\text{deS}^2 \times S^d$, with $d = D - 2$, which generalize the one we have discussed in this paper. In this case, the gravitational field equations for a electromagnetic field $F_{01} = Q$ or a $d$-form field $F_{abc\ldots} = Q\epsilon_{abc\ldots}$ give:

$$-\frac{1}{2} d(d-1)B[1 + (d-2)(d-3)\tilde{\alpha}B] = -\frac{Q^2}{4},$$

$$-A[1 + 2(d-1)(d-2)\tilde{\alpha}B] - \frac{1}{2}(d-1)(d-2)B[1 + (d-3)(d-4)\tilde{\alpha}B] = \frac{Q^2}{4}.$$  

(25)
For $d > 3$ (i.e. $D > 5$), one has solutions even if $Q = 0$, with $B = -[(d-2)(d-3)\tilde{\alpha}]^{-1}$, $A = -(d-1)[(d+1)(d-2)\tilde{\alpha}]^{-1}$, while, in general, a 1-parameter class of solutions is available for $Q \neq 0$. This also includes the special case $A = B$. It remains an open question if these solutions can still be interpreted as limiting cases of black hole solutions.

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