A Simple City Equilibrium Model with an Application to Teleworking

Yves Achdou¹ · Guillaume Carlier²,³ · Quentin Petit⁴ · Daniela Tonon⁵

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Abstract
We propose a simple semi-discrete spatial model where rents, wages and the density of population in a city can be deduced from free-mobility and equilibrium conditions on the labour and residential housing markets. We prove existence and (under stronger assumptions) uniqueness of the equilibrium. We extend our model to the case where teleworking is introduced. We present numerical simulations which shed light on the effect of teleworking on the structure of the city at equilibrium.

Keywords Spatial equilibria · Teleworking · Non-atomic game · Optimal transport

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Yves Achdou
achdou@ljll-univ-paris-diderot.fr

Guillaume Carlier
carlier@ceremade.dauphine.fr

Daniela Tonon
tonon@math.unipd.it

1 Université de Paris Cité and Sorbonne Université, CNRS, Laboratoire Jacques-Louis Lions, (LJLL), 75006 Paris, France
2 CEREMADE, Université Paris Dauphine, PSL, Pl. de Lattre de Tassigny, 75775 Paris Cedex 16, France
3 INRIA-Paris, Mokaplan, Paris, France
4 CEREMADE, Université Paris Dauphine, PSL, and EDF R&D, Paris, France
5 Dipartimento di Matematica “Tullio Levi-Civita”, Università degli Studi di Padova, Via Trieste 63, 35121 Padua, Italy
1 Introduction

Whether the internal structure of cities can be determined endogenously by equilibrium (e.g. on the labour and residential markets) conditions is a central and generally complex issue in urban economics, see in particular Fujita and Ogawa [8], Lucas and Rossi-Hansberg [11] and the references therein. Due to the fact that equilibria are in general non-unique,\(^1\) difficult to compute explicitly or numerically\(^2\) and may depend quite crucially on some modelling assumption on the city itself (linear, circular, finite etc...), it is in general difficult to anticipate how changes in some parameters influence the city structure and how rents and spatial distributions of agents evolve under such changes. With the framework proposed in [3] and relying on optimal transportation, existence of equilibria for arbitrary city shapes can be proved, but the resulting model is far too complex to yield meaningful predictions on how spatial equilibria evolve under shocks on the technology used by the firms or on any other fundamentals in the city economy.

The tremendous development of telecommuting and the possibility of working from home (anticipated and analyzed in the seminal paper of Gaspar and Glaeser [10] in 1998), exacerbated since the beginning of the COVID-19 pandemic in 2020, has seriously challenged the traditional paradigm where the commuting cost to business districts is a key ingredient in workers’ housing choice which shapes the rents and residential distribution patterns. This has of course stimulated an active stream of recent research. Delventhal et al. [7] (also see [6]) developed a model to analyze the impact of the increase of teleworking in the Los Angeles metropolitan area; among their main findings: residents move to the periphery and average real estate prices fall, with declines in core locations and increases in the periphery. In the recent equilibrium model of Behrens et al. [1], existence and uniqueness of equilibria is established as well as the (mixed) effect of working from home on efficiency.

In the present paper, our goal is to present a simple spatial equilibrium model which is tractable enough to compute numerically equilibria in one or two-dimensional domains and analyze the effect of the increase of working from home. We assume that firms are located at finitely many fixed locations and that workers are identical and distributed continuously throughout the city. This semi-discrete setting has the advantage that firms and workers do not compete for land use, which simplifies a lot the analysis of equilibria with respect to the models considered by [11] or [3]. Another simplification of our model is that we do not consider production externalities between firms, their productivity is not affected by nonlocal effects such as the total number of workers in nearby firms as in [8, 11] or [3]. The structure of the model is as follows. Firstly, following Lucas and Rossi-Hansberg [11], one deduces (by profit maximization) the labour demand of firms as a function of wages and the spatial distribution of workers as a function of revenues (i.e. wage net of commuting cost) by the condition that, at equilibrium, the workers’ utility (which depends on their

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\(^1\) See in particular Fujita and Ogawa [8] for evidence of multiple equilibria. On the contrary, by variational and suitable convexity arguments, uniqueness results were derived by Blanchet, Mossay and Santambrogio [2] in a potential game setting which takes into account the housing market by congestion effects but not the labour market.

\(^2\) Except under additional assumptions, such as radial symmetry as in Lucas and Rossi-Hansberg [11].
space consumption) has to be constant within the city. Secondly, for a given spatial
distribution of workers, wages determine labour supply for the different firms through
some simple Gibbs distributions in the spirit of discrete choice models and optimal
transport methods for spatial equilibria, see Crippa, Jimenez and Pratelli [5] or Carlier
and Mallozzi [4], and the book [9] by Galichon for an overview of these methods
in economics. Even though our model is partly inspired by the variational optimal
transport viewpoint of [4, 5], in these references, the spatial distribution \( \mu \) of residents
is fixed and wages \( w \) are obtained by a suitable optimization problem. On the contrary,
in our setting, \( \mu \) depends on the equilibrium wages \( w \) which are obtained by clearing
the labour market; our model therefore determines wages, rents and the spatial distribution
of (possibly working from home) workers. In other words, in our model, finding
equilibria is not equivalent to finding critical points of a certain potential function (as
in [2, 5]) but requires an extra fixed-point argument.

On the theoretical side, we provide a simple existence argument and identify con-
ditions which guarantee uniqueness. As we will show, our model is well-suited to
capture the introduction of teleworking. On the applied side, we can numerically com-
pute equilibria, perform some comparative statics as working from home increases;
our findings are in line with those of [7]. Note however that our model differs from
those of [1, 7] in two ways: on the one hand, it is less general because we consider
identical workers (that may work from home or on site) and do not distinguish between
skilled (allowed to work from home) and unskilled (having to commute) workers; but,
on the other hand, it allows for general commuting costs, city shapes and a continuum
of housing locations.

The paper is organized as follows. The model is introduced and equilibria are defined
in Sect. 2. In Sect. 3, we establish existence of equilibria and also prove a uniqueness
result under additional assumptions. Section 4 is devoted to a modification of the model
which takes teleworking into account. Numerical simulations are presented in Sect. 5.
Some proofs are gathered in the appendix.

2 The Model

2.1 Setting and Assumptions

The city is modeled as a compact subset \( X \) of \( \mathbb{R}^d \) with positive \( d \)-dimensional Lebesgue
measure. In this city, there are firms and workers. Our aim is to describe a tractable
spatial equilibrium model for the labour and housing markets. Our model is of semi-
discrete type: the locations of firms are fixed and discrete whereas workers occupy
the whole city according to a certain (absolutely continuous) distribution \( \mu \) which is
determined by equilibrium conditions. Workers have to commute and will choose to
work at locations for which their revenue (i.e. wage net of commuting cost) is maximal.
We also assume that the total size of the population is fixed which (up to normalization)
amounts to saying that \( \mu \) belongs to \( \mathcal{P}(X) \), the set of Borel probability measures on
\( X \). The fact that the spatial distribution of firms is discrete and that the distribution
of workers is absolutely continuous implies that firms do not consume land whereas
workers do. Therefore, in this semi-discrete setting, there is no competition for land
use between firms and workers; this simplifies a lot the equilibrium condition (e.g. compared to the models of [11] or [3]) on the land market which is reduced to the residential housing market. We now detail the specifications of the model.

**Firms.** There are finitely many firms which are indexed by $i = 1, \ldots, N$, firm $i$ is located at a point of $X$ denoted $y_i$. Firms locations $\{y_1, \ldots, y_N\}$ are fixed. Firm $i$ hiring a quantity of labour $l_i$ produces a quantity $f_i(l_i)$. We shall always assume that the production functions $f_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are increasing, strictly concave, continuous on $\mathbb{R}_+$ and $C^1$ on $(0, +\infty)$ with

$$f_i(0) = 0, \quad \lim_{l \rightarrow +\infty} f_i(l) = +\infty, \quad \lim_{l \rightarrow 0^+} f_i'(l) = +\infty, \quad \lim_{l \rightarrow +\infty} f_i'(l) = 0. \quad (2.1)$$

If the wage paid by firm $i$ is denoted by $w \in \mathbb{R}_+$ and the selling price of its production is fixed (and normalized to 1), its employment level is obtained by maximizing profit:

$$\pi_i(w) := \sup_{l \geq 0} \{f_i(l) - wl\}. \quad (2.2)$$

Obviously, $\pi_i(0) = +\infty$, $\pi_i$ is strictly convex, nonnegative and nonincreasing and thanks to (2.1) and the envelope theorem, it is also $C^1$ on $(0, +\infty)$, and for every $w > 0$, firm $i$’s labour demand is given by

$$L_i(w) := \arg\max_{l \geq 0} \{f_i(l) - wl\} = -\pi_i'(w). \quad (2.3)$$

Moreover, one readily deduces from (2.1) that

$$\pi_i(0) = \lim_{w \rightarrow 0^+} \pi_i(w) = +\infty, \quad \lim_{w \rightarrow +\infty} \pi_i(w) = 0, \quad (2.4)$$

and

$$\lim_{w \rightarrow 0^+} L_i(w) = +\infty, \quad \lim_{w \rightarrow +\infty} L_i(w) = 0. \quad (2.5)$$

**Workers preferences, deducing density from revenues.** Workers spend their revenue in consumption $C$ on the one hand and renting their house on the other hand; the surface they rent is denoted $S$. Workers are assumed to all have the same Cobb-Douglas utility:

$$(C, S) \in \mathbb{R}_+^2 \mapsto C^\theta S^{1-\theta} \quad (2.6)$$

for some $\theta \in [0, 1)$. Normalizing the price of the consumption$^3$ good to 1, workers with revenue $R > 0$, facing a rent $Q > 0$, solve

$$\nu_\theta(R, Q) = \sup \left\{ C^\theta S^{1-\theta}: C + QS \leq R, C \geq 0, S \geq 0 \right\}, \quad (2.7)$$

$^3$ Note that we are not assuming that the goods produced by firms and this consumption good are the same, and we are not looking for equilibrium in the markets for these goods (which can be exported or imported from outside the city) that is why we can normalize these prices to 1.
the solution being
\[ C = \theta R, \quad \text{and} \quad S = (1 - \theta) \frac{R}{Q}. \] (2.8)

One gets the closed-form expression for the indirect utility \( U_\theta \)
\[ U_\theta(R, Q) = \theta (1 - \theta) \frac{R}{Q^{1-\theta}} \quad \text{if } \theta \in (0, 1) \] and \( U_0(R, Q) = \frac{R}{Q} \). (2.9)

Revenues, surface consumption and rents are functions of the workers’ residential location \( x \in X \), which we now denote \( R(x), S(x) \) and \( Q(x) \). If we denote by \( \mu(x) \) the density of the workers’ residential distribution and if we normalize the supply of land to be uniform,\(^4\) \( \mu \) and \( S \) are simply related by
\[ \mu(x) S(x) = 1, \quad \forall x \in X. \] (2.10)

In particular the support of \( \mu \) is the whole city \( X \). Since workers are identical and free to choose where to live, at equilibrium, utility should be constant which, in view of (2.9), yields
\[ Q \propto R^{1-\theta}, \]
where we write \( a \propto b \) to indicate the functions \( a \) and \( b \) are proportional. With (2.8) and (2.10), we deduce
\[ \mu \propto \frac{Q}{R} \propto R^{\theta}. \]

Since the total mass of \( \mu \) has been fixed to 1, we deduce an explicit dependence between density and revenue:
\[ \mu(x) = \frac{R(x)^{\theta}}{\int_X R(y)^{\theta} \, dy}, \quad \forall x \in X. \] (2.11)

**Free-mobility of labour, commuting costs, deducing revenues from wages.** Let us denote by \( w := (w_1, \ldots, w_N) \in (0, +\infty)^N \) the wages paid to workers hired by firm \( i = 1, \ldots, N \). If agents living at \( x \) choose to work for firm \( i \) located at \( y_i \), they incur a commuting cost \( c_i(x) \) so that their revenue is \( w_i - c_i(x) \). We assume that the commuting (or accessibility) costs are continuous functions:
\[ c_i \in C(X), \quad \forall i \in \{1, \ldots, N\}. \]

\(^4\) Of course, it is also possible to have a location dependent supply for land and to replace (2.10) by \( \mu(x)S(x) = \alpha(x) \) for a given nonnegative function \( \alpha \).
We finally assume that agents have the possibility to stay at home \((c_0 = 0)\) and receive a given exogeneous wage \(w_0 > 0\). In the absence of random shocks on revenues, the revenue of agents located at \(x\), is given by

\[
R_0(x, w) := \max_{i=0, \ldots, N} (w_i - c_i(x))
\]  

(2.12)

which is a convex function of wages but may have kinks. For mathematical simplicity, we will start by replacing it by its softmax (smooth) version:

\[
R_\sigma(x, w) := \sigma \log \left( \sum_{i=0}^{N} e^{w_i - c_i(x) / \sigma} \right)
\]  

(2.13)

where \(\sigma > 0\) is a regularization parameter, and we postpone the analysis of the limiting case \(\sigma = 0\) to paragraph 3.3. Note that \(R_\sigma(x, \cdot)\) is smooth and strictly convex \(^5\) and that it is an approximation of \(R_0\) when \(\sigma\) is small because of the obvious inequality:

\[
R_0(x, w) \leq R_\sigma(x, w) \leq \sigma \log(N + 1) + R_0(x, w), \forall (x, w) \in X \times \mathbb{R}^N. \tag{2.14}
\]

The regularized form (2.13) can also be justified by considering independent random shocks to the revenue, indeed (see Appendix A for details), one has

\[
R_\sigma(x, w) := \mathbb{E} \left( \max_{i=0, \ldots, N} (w_i - c_i(x) + \sigma \varepsilon_i) \right)
\]  

(2.15)

where \(\varepsilon_0, \ldots, \varepsilon_N\) are i.i.d. centered Gumbel random variables. So replacing \(R_0\) by its regularized form \(R_\sigma\) corresponds to a logit discrete choice model with a Gumbel distribution. More precisely, given the wages \(w\), the probability that an agent living at \(x\) works for firm \(i\), is given by

\[
P(x \text{ works at } i) = \mathbb{P} \left( w_i - c_i(x) + \sigma \varepsilon_i = \max_{j=0, \ldots, N} (w_j - c_j(x) + \sigma \varepsilon_j) \right);
\]

this probability is given by the Gibbs distribution (see Appendix A):

\[
P(x \text{ works at } i) = \frac{\partial R_\sigma(x, w)}{\partial w_i} = \frac{e^{w_i - c_i(x) / \sigma}}{\sum_{j=0}^{N} e^{w_j - c_j(x) / \sigma}}.
\]  

(2.16)

Hence the total labour supply for firm \(i\) induced by the vector of wages \(w\) is

\[
\int_X \frac{\partial R_\sigma(x, w)}{\partial w_i} \mu(x) dx = \int_X \frac{e^{w_i - c_i(x) / \sigma}}{\sum_{j=0}^{N} e^{w_j - c_j(x) / \sigma}} \mu(x) dx.
\]  

(2.17)

\(^5\) Note that the function \(R_\sigma(x, \cdot)\) defined in (2.13) is not strictly convex, seen as a function of all wages \((w_0, w_1, \ldots, w_N)\) (because it behaves in a linear way when adding a common constant to all wages including \(w_0\)), but it is strictly convex with respect to \(w = (w_1, \ldots, w_N)\) for fixed \(w_0\), which is the situation we consider here.
2.2 Equilibria

A spatial equilibrium is a configuration that clears both the labour and residential housing markets. It consists of a vector of wages \( w = (w_1, \ldots, w_N) \) and a probability density \( \mu \), such that for each \( i \), labour demand given by (2.3) matches labour supply given by (2.17):

\[
p_i'(w_i) + \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) \mu(x) dx = 0, \quad \forall i \in \{1, \ldots, N\}. \tag{2.18}
\]

But at the same time, by free-mobility of labour, revenues of workers can be deduced from wages by formula (2.13), and equilibrium on the housing market requires utility to be constant so that \( \mu \) can be deduced from \( R(\cdot) = R_\sigma(\cdot, w) \) by formula (2.11) i.e.

\[
\mu(x) = \mu_w(x) := \frac{R_\sigma(x, w) \frac{\partial}{\partial \theta} \Big|_{\theta = 0}}{\int_X R_\sigma(y, w) \frac{\partial}{\partial \theta} dy}, \quad \forall x \in X. \tag{2.19}
\]

This leads to the following definition:

**Definition 2.1** An equilibrium is a vector of wages \( w = (w_1, \ldots, w_N) \in (0, +\infty)^N \) such that the system of \( N \) equations (2.18) is satisfied for the probability density \( \mu = \mu_w \) given by (2.19).

3 Existence and Uniqueness

3.1 Existence of Equilibria

To prove existence of equilibria, it will be convenient to observe that for a given \( \mu \in \mathcal{P}(X) \), the system (2.18) is the first-order optimality condition equation for the convex minimization problem:

\[
\inf_{w \in \mathbb{R}^N_+} J_\mu(w) \text{ where } J_\mu(w) := \sum_{i=1}^N \pi_i(w_i) + \int_X R_\sigma(x, w) d\mu(x). \tag{3.1}
\]

Before going further, we would like to remark that the dual formulation of (3.1) is naturally related to entropic optimal transport. Indeed, the Fenchel-Rockafellar dual problem of (3.1) reads as the maximization problem over labour variables:

\[
\sup_{(l_1, \ldots, l_N) \in \mathbb{R}^N_+} \left\{ \sum_{i=1}^N f_i(l_i) - C_\sigma(l) \right\}
\]
where
\[ C_\sigma (l) := \sup_{w \in \mathbb{R}_+^N} \left\{ \sum_{i=1}^N l_i w_i - \int_X R_\sigma (x, w) d\mu(x) \right\} \quad (3.2) \]
it is obvious that \( C_\sigma (l) = +\infty \) unless \( l_i \geq 0 \) and \( \sum_{i=1}^N l_i \leq 1 \). For such \((l_1, \ldots, l_N)\) setting \( l_0 := 1 - \sum_{i=1}^N l_i \) we may view \( l = (l_0, l_1, \ldots, l_N) \) as a probability vector with \( l_0 \) being the probability of not working (for wage \( w_0 \) and zero commuting cost) and \( l_i, i \geq 1 \), being the probability of working for firm \( i \); in this case, the optimality conditions for the optimal \( w \) in (3.2) are
\[ l_i = \int_X e^{\frac{w_i - c_i(x)}{\sigma}} e^{-\frac{w_j - c_j(x)}{\sigma}} d\mu(x), \quad i = 0, \ldots, N \]
which is easily seen to imply that the probabilities
\[ \gamma(i, x) := \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sum_{j=0}^N e^{\frac{w_j - c_j(x)}{\sigma}}} \]
solve the entropic optimal transport problem
\[ \inf_{\gamma} \left\{ \sum_{i=0}^N c_i(x)\gamma(i, x)d\mu(x) + \sigma \sum_{i=0}^N \int_X \gamma(i, x) \log(\gamma(i, x))d\mu(x) \right\} \]
subject to the mass conservation constraints:
\[ \sum_{i=0}^N \gamma(i, x) = 1, \quad \forall x, \quad l_i = \int_X \gamma(i, x)d\mu(x), \quad i = 0, \ldots, N. \]
In other words, the dual of (3.1) consists in maximizing the total production net of (entropically regularized) transport. For more on connections between semi-discrete spatial equilibria and optimal transport, we refer to Crippa, Jimenez and Pratelli [5] and Carlier and Mallozzi [4].

**Lemma 3.1** For every \( \mu \in \mathcal{P}(X) \), (3.1) admits a unique minimizer \( w^*(\mu) \) and there exist constants \( w \) and \( \bar{w} \) that do not depend on \( \mu \) such that \( 0 < w < \bar{w} \) and \( w^*(\mu) \in [w, \bar{w}]^N \). Moreover \( w^*(\mu) \) is the only solution of (2.18) and the map \( \mu \in \mathcal{P}(X) \mapsto w^*(\mu) \in [w, \bar{w}]^N \) is weakly * continuous.

**Proof** Let \( M := \max_i \|c_i\|_{\infty} \); due to the form of \( R_\sigma \) in (2.13), one has
\[ \sum_{i=1}^N \pi_i (w_i) + \sigma \log(N + 1) + \max_{i=0,\ldots,N} w_i + M \]
\[ \geq J_\mu(w) \geq \sum_{i=1}^{N} \pi_i(w_i) + \max_{i=0, \ldots, N} w_i - M. \]

Let \( w \in \mathbb{R}_+^N \) be such that \( J_\mu(w) \leq J_\mu(w_0, \ldots, w_0) \), then, one has:

\[ \max_{i=1, \ldots, N} w_i + \sum_{i=1}^{N} \pi_i(w_i) \leq M + J_\mu(w_0, \ldots, w_0) \]

\[ \leq \overline{w} := 2M + \sum_{i=1}^{N} \pi_i(w_0) + \sigma \log(N + 1) + w_0 \]

since, for every \( i \), both \( w_i \) and \( \pi_i(w_i) \) are nonnegative, this yields

\[ \max_{i=1, \ldots, N} w_i \leq \overline{w}, \quad \text{and} \quad \max_{i=1, \ldots, N} \pi_i(w_i) \leq \sum_{i=1}^{N} \pi_i(w_i) \leq \overline{w} \]

and thanks to (2.4), there exists \( w \in (0, \overline{w}) \) such that \( \pi_i^{-1}((0, \overline{w})] \subset [w, +\infty) \). Hence the infimum in (3.1) coincides with \( \inf_{w \in [w, \overline{w}]^N} J_\mu(w) \) which is achieved by continuity of \( J_\mu \) on the compact set \( [w, \overline{w}]^N \). Uniqueness follows from the strict convexity of \( J_\mu \), \( w^*(\mu) \), lies in interior of \( \mathbb{R}_+^N \), it is the unique critical point of \( J_\mu \) hence the only solution of (2.18). Finally, let us assume that \( (\mu_n) \) is a sequence in \( \mathcal{P}(X) \) weakly * converging to \( \mu \), then \( w_n := w^*(\mu_n) \) taking values in the compact set \( [w, \overline{w}]^N \), \( (w_n) \) admits a (not relabeled) subsequence which converges to some \( w^* \in [w, \overline{w}]^N \). Since \( R_\sigma(\cdot, w_n) \) converges uniformly to \( R_\sigma(\cdot, w^*) \), we have

\[ \lim_{n} J_{\mu_n}(w_n) = J_\mu(w^*) \]

and since for every \( w \in (0, +\infty)^N \), \( J_{\mu_n}(w_n) \leq J_{\mu_n}(w) \), passing to the limit yields

\[ J_\mu(w^*) \leq J_\mu(w) \]

so that \( w^* = w^*(\mu) \) and the whole sequence \( (w_n) \) converges to \( w^*(\mu) \).

\[ \square \]

**Theorem 3.2** Under the general assumptions of paragraph 2.1, there exists an equilibrium (in the sense of definition 2.1) which corresponds to a vector of wages \( w \in [\underline{w}, \overline{w}]^N \) where \( 0 < \underline{w} < \overline{w} \) are the bounds from Lemma 3.1.

**Proof** Let us define for every \( w \in [\underline{w}, \overline{w}]^N \), \( \Phi(w) := w^*(\mu_w) \) where \( w^* \) is the continuous map defined in Lemma 3.1 and \( \mu_w \) is defined by (2.19). It follows from Lemma 3.1 that \( \Phi \) is a continuous self-map of \( [\underline{w}, \overline{w}]^N \), it therefore admits a fixed-point thanks to Brouwer’s fixed-point theorem, such a fixed-point being an equilibrium by construction, this ends the proof.

\[ \square \]
3.2 A Regime of Uniqueness

We are now going to combine the structure of the equilibrium conditions, the implicit function theorem and a continuation argument, to establish uniqueness of the equilibrium if the exponent \( \theta \) is small enough (see (2.6) for the Cobb-Douglas specification of agents preferences). Since we will need to differentiate the equilibrium conditions, we shall need an extra degree of smoothness of the production functions:

\[
\forall i \in \{1, \ldots, N\}, \quad f_i \in C^2((0, +\infty)) \quad \text{and} \quad f_i''(l) < 0, \quad \forall l \in (0, +\infty). \tag{3.3}
\]

Since \(-\pi'_i\) is the inverse of \(f'_i\), this implies

\[
\forall i \in \{1, \ldots, N\}, \quad \pi_i \in C^2((0, +\infty)) \quad \text{and} \quad \pi'_i(w) > 0, \quad \forall w \in (0, +\infty), \tag{3.4}
\]

so that, for every \(\mu \in \mathcal{P}(X)\), the function \(J_\mu\) is \(C^2\) and strongly convex on \([w, \overline{w}]^N\). The basic idea behind the proof is easy to grasp: setting \(\alpha := \frac{\theta}{1-\theta}\), recalling (2.19) and defining

\[
\tilde{\mu}(x, w, \alpha) := \frac{R_\sigma(x, w)^\alpha}{\int_X R_\sigma(y, w)^\alpha dy}, \quad \forall (x, w, \alpha) \in X \times \mathbb{R}_+^N \times \mathbb{R}_+ \tag{3.5}
\]

we see that finding an equilibrium amounts, for fixed \(\alpha\), to solving for \(w \in \mathbb{R}_+^N\) the system of \(N\) nonlinear equations

\[
G(w, \alpha) = 0, \tag{3.6}
\]

where \(G = (G_1, \ldots, G_N)\) is given by

\[
G_i(w, \alpha) = \pi'_i(w_i) + \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w)\tilde{\mu}(x, w, \alpha)dx. \tag{3.7}
\]

We already know that for every \(\alpha \geq 0\), (3.6) admits at least one solution and that all such solutions belong to the compact subset of \((0, +\infty)^N, [w, \overline{w}]^N\) (see Lemma 3.1). For \(\alpha = 0\), \(\tilde{\mu}_0 := \tilde{\mu}(., w, 0)\) does not depend on \(w\) and is the density of the uniform probability measure on \(X\), hence for \(\alpha = 0\) there exists a unique equilibrium which is the unique minimizer of the strictly convex function \(J_{\tilde{\mu}_0}\). Now observing that \(G\) is of class \(C^1\) on \((0, +\infty) \times \mathbb{R}_+\) and that the Jacobian of \(G\) can be written as a perturbation of order \(\alpha\) (see details in the proof below) of a symmetric definite positive matrix, (at least local) uniqueness for small \(\alpha\) follows from the implicit function theorem. More precisely, defining

\[
\alpha_0 := \frac{w_0}{N} \min_i \pi''_i(w), \quad i = \{1, \ldots, N\}, \quad w \in [\underline{w}, \overline{w}], \tag{3.8}
\]

we have:

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Theorem 3.3  In addition to the general assumptions of paragraph 2.1, suppose further that (3.3) holds. Defining $\alpha_0$ as in (3.8), and

$$\theta_0 := \frac{\alpha_0}{1 + \alpha_0},$$

then, for every $\theta \in [0, \theta_0]$, there exists a unique equilibrium (in the sense of definition 2.1).

Proof  Let us denote by $\cdot$ the usual scalar product on $\mathbb{R}^N$ and by $|.|$ the corresponding euclidean norm on $\mathbb{R}^N$. Take $(w, \alpha) \in [\underline{w}, \overline{w}]^N$, $\alpha \in [0, 1]$. 

Step 1: invertibility of the Jacobian of $G$. Let us denote by $A_{ij} := \frac{\partial G_i}{\partial w_j}(w, \alpha)$ the entries of the Jacobian (with respect to $w$) matrix of $G$ at $(w, \alpha)$:

$$A_{ij} = \pi''_i(w_i) \delta_{ij} + \int_X \frac{\partial^2 R_\sigma}{\partial w_j \partial w_i} \tilde{\mu} + \int_X \frac{\partial R_\sigma}{\partial w_i} \frac{\partial \tilde{\mu}}{\partial w_j}. \quad (3.9)$$

In matrix form, this reads as

$$A := \text{diag}(\pi''_1(w_1), \ldots, \pi''_N(w_N)) + B + C$$

where $B$ is symmetric positive definite and $C$ is a mixture of rank one matrices

$$B := \int_X D^2_{ww} R_\sigma(x, w) \tilde{\mu}(x, w, \alpha) dx, \; C := \int_X \nabla_w R_\sigma(x, w) \nabla_w \tilde{\mu}^T(x, w, \alpha) dx.$$ 

Hence, by Cauchy-Schwarz inequality, for $\xi \in \mathbb{R}^N$, we have

$$A\xi \cdot \xi \geq \left( v - \int_X |\nabla_w R_\sigma(x, w)| |\nabla_w \tilde{\mu}(x, w, \alpha)| dx \right) |\xi|^2 + B\xi \cdot \xi \quad (3.10)$$

for

$$v := \min_{i=1, \ldots, N} \pi''_i(w_i) \geq \min\{\pi''_i(w), \; i = 1, \ldots, N\}, \; w \in [\underline{w}, \overline{w}]. \quad (3.11)$$

Since, by (2.16)

$$0 \leq \frac{\partial R_\sigma}{\partial w_i}(x, w) \leq 1 \quad (3.12)$$

we have

$$|\nabla_w R_\sigma(x, w)| \leq \sqrt{N}. \quad (3.13)$$
Let us now estimate the partial derivatives of $\tilde{\mu}$ with respect to the $w_j$'s:

$$\frac{\partial \tilde{\mu}}{\partial w_j}(x, w, \alpha) = \frac{\alpha R_{\sigma}^{\alpha-1}(x, w) \partial R_{\sigma}(x, w)}{\int_X R_{\sigma}^\alpha} - \alpha \frac{R_{\sigma}^{\alpha}(x, w) \int_X R_{\sigma}^\alpha}{(\int_X R_{\sigma}^\alpha)^2} \int_X R_{\sigma}^{\alpha-1} \frac{\partial R_{\sigma}}{\partial w_j}$$

together with (3.12) and $R_{\sigma} \geq w_0$ so that $R_{\sigma}^{\alpha-1} \leq \frac{R_{\sigma}^{\alpha}}{w_0}$, this yields

$$0 \leq \frac{R_{\sigma}^{\alpha-1}(x, w) \partial R_{\sigma}(x, w)}{\int_X R_{\sigma}^\alpha} \leq \frac{R_{\sigma}^{\alpha}(x, w)}{w_0 \int_X R_{\sigma}^\alpha}$$

and

$$0 \leq \frac{R_{\sigma}^{\alpha}(x, w)}{(\int_X R_{\sigma}^\alpha)^2} \int_X R_{\sigma}^{\alpha-1} \frac{\partial R_{\sigma}}{\partial w_j} \leq \frac{R_{\sigma}^{\alpha}(x, w)}{w_0 \int_X R_{\sigma}^\alpha}$$

so that

$$\left| \frac{\partial \tilde{\mu}}{\partial w_j}(x, w, \alpha) \right| \leq \frac{\alpha}{w_0} \frac{R_{\sigma}^{\alpha}(x, w)}{\int_X R_{\sigma}^\alpha}$$

hence

$$\int_X |\nabla_w \tilde{\mu}(x, w, \alpha)| dx \leq \frac{\alpha \sqrt{N}}{w_0}. \quad (3.14)$$

Using (3.13)–(3.14) in (3.10), we thus get

$$A \xi \cdot \xi \geq \left( \nu - \frac{\alpha N}{w_0} \right) |\xi|^2 + B \xi \cdot \xi$$

which, since $B$ is symmetric positive definite, enables us to conclude with (3.11) that $A$ is invertible whenever $\alpha \in [0, \alpha_0]$.

**Step 2: uniqueness.** Let us define

$$I := \{ \alpha \in [0, \alpha_0] : \text{there exists a unique } w \in [w, \bar{w}]^N \text{ such that } G(w, \alpha) = 0 \}.$$ 

We have already observed that $0 \in I$ so that $I$ is nonempty, so proving that $I$ is closed and open in $[0, \alpha_0]$ will yield the desired uniqueness result. We can deduce from the previous step and the implicit function theorem that for every $\alpha \in [0, \alpha_0]$ and every $w \in \mathbb{R}_+^N$ such that $G(w, \alpha) = 0$ there is some neighbourhood of $(\alpha, w)$ say $((\alpha - \varepsilon, \alpha + \varepsilon) \cap [0, \alpha_0]) \times B(w, r)$ for some $\varepsilon > 0$, $r > 0$ and a $C^1$ curve $\gamma_{\alpha,w} : (\alpha - \varepsilon, \alpha + \varepsilon) \cap [0, \alpha_0] \to \mathbb{R}_+^N$ such that $G^{-1}(\{0\}) \cap (B(w, r) \times ((\alpha - \varepsilon, \alpha + \varepsilon) \cap [0, \alpha_0])) = \{ (\gamma_{\alpha,w}(\alpha'), \alpha') , \alpha' \in (\alpha - \varepsilon, \alpha + \varepsilon) \cap [0, \alpha_0] \}$. Let $(\alpha_n)_n \in I^\mathbb{N}$ converge to some $\alpha$; if $\alpha$ was not in $I$, we could find $w \neq \hat{w}$ with $G(w, \alpha) = G(\hat{w}, \alpha) = 0$, but
then for $n$, large enough we would have $\gamma_{\alpha, w}(\alpha_n) \neq \gamma_{\alpha, \hat{w}}(\alpha_n)$ contradicting the fact that $\alpha_n \in I$. Hence $I$ is closed. Now let $\alpha \in I$ and $w$ be the only root of $G(w, \alpha) = 0$. If $I$ was not a neighbourhood of $\alpha$ in $[0, \alpha_0]$, we could find a sequence $(\alpha_n)$ in $[0, \alpha_0] \setminus I$ converging to $\alpha$. For each $n$, we could pick $w_n \neq \hat{w}_n$ with $G(\hat{w}_n, \alpha_n) = G(w_n, \alpha_n)$, since both sequences $(w_n)$ and $(\hat{w}_n)$ take values in the compact set $[w, \hat{w}]^N$, both $(w_n)$ and $(\hat{w}_n)$ should converge to $w$ so that, for large enough $n$, we should have $w_n = \hat{w}_n = \gamma_{\alpha, w}(\alpha_n)$ which yields the desired contradiction. We thus have shown that $I$ is open in $[0, \alpha_0]$.

\[\Box\]

### 3.3 The Un-regularized Case (Zero Noise Limit)

We now consider the case where $\sigma = 0$, for which the revenue of workers is given by (2.12). In this case, given $w \in \mathbb{R}_+^N$, the set of workers locations for which working for firm $i$ (or staying home for wage $w_0$ if $i = 0$) is optimal is

$$V_i(w) := \{x \in X : R_0(x, w) = w_i - c_i(x)\}, \ i = 0, \ldots, N. \quad (3.15)$$

In case there are ties, i.e. several optimal choices, we also define the set of workers locations for which $i$ is strictly prefered to the other options:

$$V^s_i(w) := V_i(w) \cup \bigcup_{j \neq i} V_j(w). \quad (3.16)$$

Equilibrium on the residential housing market gives the population density as a function of $R_0$:

$$\mu_w(x) := \frac{R_0(x, w)^{\theta}}{\int_X R_0(y, w)^{\theta} \, dy}, \ \forall x \in X. \quad (3.17)$$

The labour demand of firm $i$ is determined by (2.3) exactly as in paragraph 2.1. As for the total labour supply for location $i = 1, \ldots, N$, because of possible ties\(^6\), it has to lie in the interval $[\mu_w(V^s_i(w)), \mu_w(V_i(w))]$. Equilibrium on the labour market then reads

$$-\pi'_i(w_i) \in [\mu_w(V^s_i(w)), \mu_w(V_i(w))], \ \forall i = 1, \ldots, N \quad (3.18)$$

supplemented with an additional consistency condition: workers who are not hired by any firm are those for which $w_0$ is optimal:

$$1 + \sum_{i=1}^N \pi'_i(w_i) \in [\mu_w(V^s_0(w)), \mu_w(V_0(w))]. \quad (3.19)$$

---

\(^6\) An easy way to rule out ties is the following. Since $\mu_w$ is absolutely continuous, one way to ensure that $V_i(w) \setminus V^s_i(w)$ is negligible is to assume that for every $i, j$ with $i \neq j$ and any $\lambda \in \mathbb{R}$, the level set $\{x \in X : c_i(x) - c_j(x) = \lambda\}$ is Lebesgue-negligible.
In this context, a spatial equilibrium is a collection of positive wages \( w \) for which (3.18)–(3.19) are satisfied with \( \mu_w \) given by (3.17). Existence is ensured by:

**Proposition 3.4** There exists \( w \in (0, +\infty)^N \) such that (3.18)–(3.19) are satisfied with \( \mu_w \) given by (3.17).

**Proof** Even though a fixed-point proof is possible, we prefer to give a short proof by passing to the (zero-noise) limit in the regularized equilibria whose existence is ensured by theorem 3.2. Indeed, we have seen (lemma 3.1 and theorem 3.2) that for every \( \sigma \in (0, 1) \), there exists \( w^\sigma \) belonging to the (independent of \( \sigma \in (0, 1) \)) compact subset \([w, \overline{w}]^N \) of \((0, +\infty)^N \) such that

\[
- \pi'(w^\sigma_i) = \int_X \frac{e^{w^\sigma_i - c_i(x)}}{\sum_{j=0}^N e^{w^\sigma_j - c_j(x)}} \mu_\sigma(x) dx, \quad \forall i \in \{1, \ldots, N\} \tag{3.20}
\]

where

\[
\mu_\sigma(x) = \frac{R_\sigma(x, w^\sigma)^{\theta \sigma}}{\int_X R_\sigma(y, w^\sigma)^{\theta \sigma} dy}, \quad \forall x \in X. \tag{3.21}
\]

Up to a subsequence, we may assume that \( w^\sigma \) converges to some \( w \in [w, \overline{w}]^N \) as \( \sigma \to 0^+ \). This implies that \( \pi'_i(w^\sigma_i) \) converges to \( \pi'_i(w_i) \) and \((R_\sigma(\cdot, w^\sigma), \mu_\sigma)\) converges uniformly to \((R_0(\cdot, w), \mu_w)\). Moreover, for \( i = 0, \ldots, N, \)

\[
x \in V^i_s(w) \Rightarrow \lim_{\sigma \to 0^+} \frac{e^{w^\sigma_i - c_i(x)}}{\sum_{j=0}^N e^{w^\sigma_j - c_j(x)}} = 1
\]

and

\[
x \in X \setminus V_i(w) \Rightarrow \lim_{\sigma \to 0^+} \frac{e^{w^\sigma_i - c_i(x)}}{\sum_{j=0}^N e^{w^\sigma_j - c_j(x)}} = 0.
\]

So letting \( \sigma \to 0^+ \) in (3.20), we easily get that \( w \) satisfies (3.18). As for (3.19), we remark that (3.20) implies

\[
1 + \sum_{i=1}^N \pi'_i(w^\sigma_i) = \int_X \frac{e^{w_0}}{\sum_{j=0}^N e^{w^\sigma_j - c_j(x)}} \mu_\sigma(x) dx
\]

by letting \( \sigma \to 0^+ \), we obtain (3.19). \( \square \)
4 A Teleworking Model

We now consider a variant of the previous model where teleworking is introduced. All workers are identical, they all have the same (Cobb-Douglas) utility and they all have the same productivity (conditional on the technology of the firm they work for, as in the previous sections, but also on whether they work on site or telework). Remote (teleworking) workers have no commuting costs and the production function of each firm \( i \) depends on both the numbers of on-site workers which we denote \( l_1^i \) and the numbers of teleworkers \( l_2^i \). We denote by \( \tilde{f}_i : \mathbb{R}_+^2 \to \mathbb{R}_+ \) this production function and assume that for \( i = 1, \ldots, N \), \( \tilde{f}_i \) is increasing in both arguments, strictly concave and continuous on \( \mathbb{R}_+^2 \) and \( C^1 \) on \( (0, +\infty)^2 \) as well as

\[
\tilde{f}_i(0, 0) = 0, \quad \lim_{l \to +\infty} \frac{\tilde{f}_i(l, s)}{l + s} = 0, \quad \forall s > 0.
\] (4.1)

and

\[
\lim_{(l, s) \in \mathbb{R}_+^2, l + s \to +\infty} \frac{\tilde{f}_i(l, s)}{l + s} = 0, \quad \forall s > 0.
\] (4.3)

which is the case as soon as \( \tilde{f}_i \) is homonogeneous of degree strictly less than 1. Typical examples of production functions which fulfill the previous conditions are Cobb-Douglas functions

\[
\tilde{f}_i(l, s) = A_i l_\alpha^i s_\beta^i, \quad \alpha_i > 0, \quad \beta_i > 0, \quad \alpha_i + \beta_i < 1
\]

or constant elasticity of substitution functions

\[
\tilde{f}_i(l, s) := A_i (l_\alpha^i + B_i s_\alpha^i)^{\frac{\beta_i}{\alpha_i}}, \quad A_i > 0, \quad B_i > 0, \quad (\alpha_i, \beta_i) \in (0, 1)^2.
\]

Let us define, for every \( \tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in \mathbb{R}_+^2 \), the profit of firm \( i \) when the vector of wages for on site/remote work is \( \tilde{w} \):

\[
\tilde{\pi}_i(\tilde{w}) = \sup_{l_1^i \geq 0, l_2^i \geq 0} \{ \tilde{f}_i(l_1^i, l_2^i) - w_1^i l_1^i - w_2^i l_2^i \}. \quad (4.4)
\]

It follows from (4.1), (4.2), (4.3) that \( \tilde{\pi}_i \) is strictly convex and nonincreasing in both arguments on \( \mathbb{R}_+^2 \), with

\[
\tilde{\pi}_i(0, w) = \tilde{\pi}_i(w, 0) = \lim_{\varepsilon \to 0^+} \tilde{\pi}_i(\varepsilon, w) = \lim_{\varepsilon \to 0^+} \tilde{\pi}_i(w, \varepsilon) = +\infty, \quad \forall w \geq 0 \quad (4.5)
\]

and \( \tilde{\pi}_i \) is \( C^1 \) on \( (0, +\infty) \) and for every \( (w^1, w^2) \in (0, +\infty) \) and for \( k = 1, 2 \):

\[
\frac{\partial \tilde{\pi}_i}{\partial w^k}(w^1, w^2) = -L_k^i(w^1, w^2) \quad (4.6)
\]
where

\[
(L_1^i(w^1, w^2), L_2^i(w^1, w^2)) = \text{argmax}_{l^1 \geq 0, l^2 \geq 0} \{ \tilde{f}_i(l^1, l^2) - w^1l^1 - w^2l^2 \} \tag{4.7}
\]

so that \(L_1^i\) represents the demand of on-site labour and \(L_2^i\) the demand for remote labour of firm \(i\).

Given a collection of wages \(\tilde{\omega} := (\tilde{\omega}_1, \ldots, \tilde{\omega}_N) \in \mathbb{R}^{2N}_+\), where \(\tilde{\omega}_i := (w_{1i}^1, w_{2i}^2)\) (the superscript \(k = 1\) corresponding to on site work and \(k = 2\) to remote work), the revenue of workers living at \(x\) takes the form

\[
\tilde{R}_\sigma (x, \tilde{\omega}) = \sigma \log \left( e^{w_{0i}^1} + \sum_{i=1}^{N} e^{w_{1i}^1 - c_i(x)} + \sum_{i=1}^{N} e^{w_{2i}^2} \right). \tag{4.8}
\]

Arguing as in Sect. 2.1, assuming the Cobb-Douglas form \((2.6)\) for workers’ preferences, equilibrium on the residential market implies that the density of workers can be expressed as:

\[
\tilde{\mu}_{\tilde{\omega}}(x) := \frac{\tilde{R}_\sigma (x, \tilde{\omega})}{\int_X \tilde{R}_\sigma (y, \tilde{\omega}) \, dy} \forall x \in X. \tag{4.9}
\]

Again arguing as in Sect. 2.1, the supply of on-site labour for firm \(i\) is given by

\[
\int_X \frac{\partial \tilde{R}_\sigma}{\partial w_{ik}}(x, w)\tilde{\mu}_{\tilde{\omega}}(x)dx = \int_X \frac{e^{w_{1i}^1-c_i(x)}}{e^{\frac{w_{0i}^1}{\sigma}} + \sum_{j=1}^{N} e^{w_{1j}^1-c_j(x)} + \sum_{j=1}^{N} e^{w_{2j}^2}}\tilde{\mu}_{\tilde{\omega}}(x)dx.
\tag{4.10}
\]

and the supply of remote labour is given

\[
\int_X \frac{\partial \tilde{R}_\sigma}{\partial w_{ik}}(x, w)\tilde{\mu}_{\tilde{\omega}}(x)dx = \int_X \frac{e^{w_{2i}^2}}{e^{\frac{w_{0i}^1}{\sigma}} + \sum_{j=1}^{N} e^{w_{1j}^1-c_j(x)} + \sum_{j=1}^{N} e^{w_{2j}^2}}\tilde{\mu}_{\tilde{\omega}}(x)dx.
\tag{4.11}
\]

In this setting, an equilibrium is a collection of wages \(\tilde{\omega} := (\tilde{\omega}_1, \ldots, \tilde{\omega}_N) = (w_1^1, w_1^2, \ldots, w_N^1, w_N^2) \in \mathbb{R}^{2N}_+\) for which supply and demand for on-site and remote labour coincide i.e.

\[
0 = \frac{\partial \tilde{\pi}_i}{\partial w^k_i}(w^1_i, w^2_i) + \int_X \frac{\partial \tilde{R}_\sigma}{\partial w^k_i}(x, w)\tilde{\mu}_{\tilde{\omega}}(x)dx, \ i = 1, \ldots, N, \ k = 1, 2. \tag{4.12}
\]
Defining for \( \mu \in \mathcal{P}(X) \), the strictly convex function,

\[
\begin{align*}
\tilde{J}_\mu(\tilde{w}) & := \sum_{i=1}^{N} \tilde{\pi}_i(w_i^1, w_i^2) + \int_X \tilde{R}_\sigma(x, \tilde{w})d\mu(x), \\
\forall \tilde{w} & = (w_1^1, w_1^2, \ldots, w_N^1, w_N^2) \in \mathbb{R}_{+}^{2N}
\end{align*}
\]

the system of \( 2N \) equations for equilibrium wages (4.12) reads

\[
\tilde{w} = \tilde{\Phi}(\tilde{w}) := \arg\min \tilde{J}_\mu(\tilde{w}).
\] (4.14)

Under the assumptions of this paragraph, the existence and uniqueness results from Sect. 3 can be extended to the case of teleworking (details can be found in Appendix B):

- there exist equilibria i.e. solutions to (4.12),
- if in addition, the production functions \( \tilde{f}_i \) are of class \( C^2 \) on \((0, +\infty)^2\) and

\[
D^2 \tilde{f}_i(l^1, l^2) \text{ is negative definite, for every } (l^1, l^2) \in (0, +\infty)^2,
\] (4.15)

then (4.12) admits a unique solution provided \( \theta \) is small enough (an explicit bound can be found in Appendix B).

5 Numerical Simulations

We approximate an equilibrium by solving equation (2.18) with the distribution \( \mu = \mu_w \) given by (2.19), i.e. we are solving

\[
\pi'_i(w_i) + \int_X \frac{\partial R_{\sigma}(x, w)}{\partial w_i} \mu_w(x)dx = 0, \forall i \in \{1, \ldots, N\}. \tag{5.1}
\]

This amounts to computing the zero of a function for which we apply a modification of Powell’s hybrid method, as detailed below.

5.1 The Algorithm

For convenience, we focus on the case when \( X = [0, 1] \) but the scheme can directly be extended to the case when \( X \) is a bounded domain of \( \mathbb{R}^d \), with \( d > 1 \). Let \( X_h \) be a uniform grid on \( X \) with step \( h = 1/K_h, K_h \in \mathbb{N} \). Let \( x_k \) denote a generic point in \( X_h \); the values of \( R_{\sigma}(\cdot, w), \frac{\partial R_{\sigma}}{\partial w_i}(\cdot, w) \) and \( \mu_w \) at \( x_k \) will be respectively denoted by \( R_k, DR_i R_k \) and \( \mu_k \) and computed by the explicit form of \( R_{\sigma}(\cdot, w), DR_i R_{\sigma}(\cdot, w) \) and \( \mu_w \). We use the trapezoidal rule to estimate the value of the integrals:

\[
\int_X \frac{\partial R_{\sigma}(x, w)}{\partial w_i} \mu_w(x)dx \simeq h \left( DR_0 \mu_0 + DR_{K_h} \mu_{K_h} + \sum_{k=1}^{K_h-1} DR_k \mu_k \right).
\]
Solving (5.1) numerically boils down to

\[ F(w) = 0, \]  

(5.2)

where for every \( i \in \{1, \ldots, N\} \),

\[ F_i(w) = \pi'_i(w_i) + h \left( \frac{D R_0 \mu_0 + D R K \mu}{2} + \sum_{k=1}^{K_h-1} D R_k \mu_k \right). \]

Note that when the dimension of \( X \) is greater than one, the scheme can be adapted by considering another approximation of the integrals.

We use Powell’s hybrid method [13, 14] to approximate a solution to (5.2). It consists in finding a minimizer of the least squares problem

\[ \min_{w \in \mathbb{R}^N} G(w) = \frac{1}{2} \sum_{i=1}^{N} F_i(w)^2. \]

We observe that for every \( w \in (\mathbb{R}^*_+)^d \),

\[ \nabla G(w) = DF(w) F(w), \]

where \( DF(w) \) is the Jacobian of \( F \) at \( w \). The algorithm combines the Gauss–Newton algorithm with the steepest descent algorithm, and uses an explicit trust region (see Algorithm 1 below).

**Algorithm 1** Calculate \( \hat{w} \in \arg\min_{w \in \mathbb{R}^N} G(w) \)

**Inputs** \( \Delta > 0, \) tol > 0, \( w^{(0)} \in \mathbb{R}^N. \)

\( w \leftarrow w^{(0)} \)

**while** \( G(w) > \) tol **do**

\( B \leftarrow DF(w) \)

\( \delta_{gn} \leftarrow -(B^T B)^{-1} B^T F(w) \)

\( \delta_{sd} \leftarrow -B^T F(w) \)

\( t \leftarrow \frac{||\delta_{sd}||^2}{||B\delta_{sd}||^2} \)

**if** \( ||\delta_{gn}|| \leq \Delta \) **then**

\( \delta \leftarrow \delta_{gn} \)

**else if** \( t||\delta_{sd}|| > \Delta \) **then**

\( \delta \leftarrow \Delta \delta_{sd}/||\delta_{sd}|| \)

**else**

find \( s \in [0, 1] \) such that \( ||(1-s)\delta_{sd} + s\delta_{gn}|| = \Delta \)

\( \delta \leftarrow (1-s)\delta_{sd} + s\delta_{gn} \)

**end if**

\( w \leftarrow w + \delta \)

**end while**
| Parameter | Value |
|-----------|-------|
| $\beta$   | 0.7   |
| $A$       | $10^{1.2}$ |
| $w_0$     | 12    |
| $\sigma$ | 0.1   |

An efficient implementation of Algorithm 1 can be found in the technical report MINPACK-1 [12], together with a suitable approximation $\tilde{DF}(w)$ of the Jacobian $DF(w)$.

### 5.2 Numerical Results: Some Comparative Statics

We present two simulations whose goal is to analyze the influence of some parameters on the equilibrium. They are carried out on a one-dimensional domain while a third simulation on a two-dimensional domain will be presented in paragraph 5.2.3. The setting is as follows: we set $X = [-10, 10]$, and we assume that there are three workplaces ($N = 3$) located at three different points in $X$. Let $y_i \in X$ be the location of the $i$th workplace. We assume that $y_1 = -7$, $y_2 = 0$, $y_3 = 3$, and that each workplace corresponds to a firm that seeks to maximize its profits; with a slight abuse of language, let $y_i$ be the name of the firm located at $y_i$. We assume that the transport cost to reach $y_i$ from $x \in X$ is given by

$$c_i(x) = \frac{|x - y_i|}{2}, \quad \forall i \in \{1, 2, 3\}.$$ 

Note that we could have used any other continuous function on $X$ to model the transport costs without changing the scheme.

#### 5.2.1 Comparative Statics as the Preference Parameter $\theta$ Varies

**Definition of the model and the parameters** We assume that the production of the firm $y_i$ is given by

$$f_i(l) = Al^\beta, \quad \forall l \in [0, +\infty).$$

The parameter $A$ can naturally be interpreted as the firm’s productivity (which may depend on its capital for instance). The parameters used in Test 1 are listed in Table 1 below.

**Numerical results** In the following three figures, we compare the results obtained for different values of $\theta$. In Fig. 1, we display the residential distribution of the people.
Fig. 1 Residential distributions of the people working at the different workplaces and of the independent workers, with $\theta = 0$ (top-left), $\theta = 0.2$ (top-right), $\theta = 0.4$ (middle-left), $\theta = 0.6$ (middle-right), $\theta = 0.8$ (bottom-left), $\theta = 0.99$ (bottom-right)
working at the different workplaces. Recall that these distributions are given by

\[ X \ni x \mapsto \frac{\partial R_\sigma}{\partial w_i}(x, w) \mu(x), \quad \forall i \in \{0, 1, 2, 3\}. \]

The lines (—), (—) and (—) are associated to the residences of the agents working at \(y_1, y_2\) and \(y_3\) respectively. The curve (—) corresponds to the residential distribution of independent workers (i.e. those who stay home for the wage \(w_0\)).

In Fig. 2, we display the wages and the number of workers in each workplace (we use the same color code as in Fig. 1). Finally, the rental price as a function of the spatial variable \(x\) is plotted in Fig. 3.

Let us focus on the case when \(\theta = 0\). The agents have only one source of utility, the surface that they rent. Therefore, as it clearly appears on Fig. 1, they distribute themselves uniformly on \(X\) (the supply of space is constant). This gives to \(y_3\) a positional advantage; indeed the basin of attraction of \(y_3\) is larger than those of \(y_1\) and \(y_2\). Therefore, the supply of labour at \(y_3\) is larger than at \(y_1\) or \(y_2\). We see that different locations lead to differences in labour supply.

Due to the advantage that \(y_3\) has, it attracts more workers and may pay them less than \(y_1\) or \(y_2\). Similarly, \(y_1\) has a positional advantage with respect to \(y_2\). This explains why \(w_3 < w_1 < w_2\), even though \(y_3\) attracts more workers than \(y_1\), which attracts more workers than \(y_2\), see Fig. 2.

As \(\theta\) increases, the relative importance of the surface of the house in the utility function of the agents decreases. As a consequence, the demand for surface decreases and so does the rental price, see Fig. 3. On the other hand, the relative importance of consumption in the utility function of the agents increases. Therefore, the tendency is to have a concentration of housing close to the workplaces, because the agents choose to reduce their transport costs in order to increase their consumption, see Fig. 1. In Fig. 2, we observe that, when \(\theta\) varies from 0 to 0.8, wages at \(y_1\) and \(y_2\) decrease, while wages at \(y_3\) increase. We may give two reasons for that. First, the concentration of houses near the workplaces tends to reduce the competition on the labour market between \(y_1\) and \(y_2\). Second, the number of workers living in the interval \([5, 10]\) decreases, so that the positional advantage of \(y_3\) decreases.

For \(\theta > 0.8\), the concentration phenomenon progressively isolates \(y_1\) from \(y_2\) and \(y_3\). Therefore, \(y_1\) enjoys a positional advantage similar to the one that \(y_3\) had when \(\theta\) was small. This allows \(y_1\) to decrease the level of wage. On the other hand, this extra competition pushes \(y_3\) to increase its wage.
Finally, when $\theta$ is close to 1, the size of the basins of attraction of the different workplaces becomes small, so that they are isolated of each other. When $\theta = 0.99$, the wages and the number of agents in each workplace is almost the same.

### 5.2.2 Comparative Statics as the Productivity of Teleworkers Varies

**Description of the model and the parameters** We use the teleworking model described in Sect. 4. Therefore, we approximate a solution of (4.12), the analogue equation of (5.1), by using the same method introduced in the beginning of this section. We assume that the production of $y_i$ is given by

$$\tilde{f}_i(l, s) = A(l^\alpha + B s^\alpha)^{\beta}, \quad \forall (l, s) \in [0, +\infty)^2.$$  

The parameter $B$ is related to the productivity of the teleworkers. We are going to let $B$ vary from 0 to 1.

The parameters used in Test 2 are listed in Table 2 below.

**Numerical results**

In the three figures below, we compare the results obtained for different values of $B$. In Fig. 4, we display the residential distributions for the workers of the different workplaces. The lines (---), (-----) and (----) are associated to the residences of the commuters working at $y_1$, $y_2$ and $y_3$ respectively. The curves (---), (-- -) and (----)
Table 2 The parameters used in Test 2

| Parameter | Value |
|-----------|-------|
| $\beta$   | 0.7   |
| $\alpha$  | 0.9   |
| $A$       | $10^{1.2}$ |
| $w_0$     | 12    |
| $\sigma$ | 0.1   |
| $\theta$ | 0.7   |

are respectively associated to the residences of the teleworkers working for $y_1$, $y_2$ and $y_3$. The curve (---) corresponds to the residences distribution of independent workers. In Fig. 5, we plot the wages and the number of workers in each workplace versus $B$ (we use the same color code). In Fig. 6, we plot the rental price versus $x$.

On Fig. 5, we observe a phenomenon similar to what happened in the first simulation related to the sensitivity with respect to $\theta$. Indeed, for $B = 0$, the wages depend on the workplace. Then, as the parameter $B$ increases, people choose to telecommute when their transport costs are high, see Fig. 4. As a result, $y_3$ loses some of its positional advantage, since $y_1$ and $y_2$ may hire teleworkers to the right of $y_3$ because the latter do not incur transportation costs. Progressively, as in Test 1, $y_3$ loses its positional advantage whereas $y_1$ becomes more attractive because its basin of attraction becomes isolated from those of the other two workplaces. As in Test 1, commuters’ residential distributions tend to concentrate in smaller and smaller areas, so that when $B$ is large, no firm has a geographical advantage on the others. Therefore, when $B = 1$, the wages and the number of workers in each workplace are the same, see Fig. 5. Note that for small values of $\sigma$, the wages of the teleworkers are the same in each workplace, because the firms compete for hiring the teleworkers and because the latter do not incur transportation costs. This is what we observe here for $\sigma = 0.1$ where the differences when $B \geq 0.2$ are of the order of $10^{-4}$. The differences observed for $B < 0.2$ are due to numerical approximations.

We also observe on Fig. 5 that commuters are more paid than teleworkers, even if $B = 1$. This comes from the fact that commuters have transport costs. Besides, commuters live in areas where the rental price is higher. Moreover, we observe that when $B$ increases, the wages of both commuters and teleworkers increase. There are two reasons which explain this phenomenon. First, the demand for teleworkers increases with their productivity, and so does their wage. Second, for $B > 0$, we observe that the partial derivative of the production function $\tilde{f}$ with respect to $\ell$ ($\ell = l, s$) explodes towards $+\infty$ when $\ell$ converges to zero. This implies that the production when hiring both types of workers is bigger than the one produced when hiring just one type.

On Fig. 6, we can see the effect of increasing the productivity of teleworkers on rental price. First note that it has little impact on the local maxima of the rent which are located at the firms locations. It results, on the contrary, in a significant increase of rents far from the production centers i.e. in zones with higher transport costs. Finally, as is particularly clear in the bottom right figure, increasing $B$ creates zones far from
Fig. 4 Residential distributions of the people working for the different workplaces and of the independent workers (left column), and a zoom on the residential distributions of teleworkers (right column), for $B = 0$ (top), $B = 0.6$ (middle-top), $B = 0.8$ (middle-bottom), $B = 1$ (bottom)

Fig. 5 Wages versus $B$ (on the left) and the number of workers versus $B$ (on the right)
Fig. 6 Rental price versus $x$, for $B = 0$ (top-left), $B = 0.2$ (top-right), $B = 0.4$ (middle-left), $B = 0.6$ (middle-right), $B = 0.8$ (bottom-left), $B = 1$ (bottom-right)

Table 3 The parameters used in Test 3

| Parameter | Value |
|-----------|-------|
| $\beta$   | 0.7   |
| $\alpha$  | 0.9   |
| $A$       | $10^{1.2}$ |
| $w_0$     | 12    |
| $\sigma$  | 0.1   |
| $\theta$  | 0.7   |

production centers (and mostly populated by teleworkers) where the rents become flat (because the revenues of teleworking residents are quite insensitive to the distance to firms there).

In the following paragraph, we present simulations on a two-dimensional domain using the same numerical method.

5.2.3 A Teleworking Model in a Two-Dimensional Domain

We aim at extending the teleworking simulation of the previous paragraph to a two-dimensional domain. We obtain similar results with a computation time which increases significantly due to the computational cost of the integrals in (4.12) and (4.9) (by a trapezoidal rule adapted to the two-dimensional case).
Fig. 7 Distributions of the houses of commuters working for the different workplaces, of teleworkers and of the independent workers, with $B = 0$ (top-left), $B = 0.33$ (top-right), $B = 0.5$ (middle-left), $B = 0.66$ (middle-right), $B = 1$ (bottom-left)

**Description of the model and the parameters** We assume that $X = [-10, 10]^2$, that the workplaces are located in

$$y_1 = (-7, 7), \quad y_2 = (0, 0), \quad y_3 = (3, -3)$$

and that the production of $y_i$ is given by

$$\bar{f}_i(l, s) = A(l^\alpha + B s^\alpha)^{\beta}, \quad \forall (l, s) \in [0, +\infty)^2.$$
In this setting, the transport cost to reach the $i$th workplace, for commuters living at $x \in X$, is given by

$$c_i(x) = \frac{\|x - y_i\|_2}{2},$$

while it is 0 for teleworkers.

As before, we are going to let $B$ vary from 0 to 1. The parameters used in Test 3 are the same as in Test 2 and are listed in Table 3 below.

**Numerical results**

In the figures below, we display the residential distributions of commuters, working at the different workplaces, for different values of $B$. The color (—), (—) and (—) are associated to the residences of the commuters working at $y_1$, $y_2$ and $y_3$ respectively. The color (—) is associated to the teleworkers’ residences, and (—) corresponds to the housing distribution of independent workers.

The analysis is the same as in the latter simulation. When $B$ increases, telecommuting develops in areas with high transport costs, see Fig. 7. The distributions displayed on the bottom-left of Fig. 7 highlight the same phenomenon of concentration identified in Test 1 and 2, namely the houses of commuters are located in smaller and smaller areas.

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**Declaration**

**Conflict of interest** We disclose any interest directly or indirectly related to the work submitted for publication.

**Appendix**

**A. Softmax as the Expectation of a Max**

Let us first recall that a random variable $\varepsilon$ has a centered standard Gumbel distribution if its cdf has the following double exponential form:

$$
\mathbb{P}(\varepsilon \leq t) = \exp(-\exp(-t - \gamma)), \quad \forall t \in \mathbb{R}, \quad \text{with } \gamma := -\int_0^{+\infty} \log(s) \exp(-s)ds.
$$
Consider now \( \varepsilon_0, \ldots, \varepsilon_N, N + 1 \) i.i.d. distributed with a centered standard Gumbel and for \( \beta = \beta_0, \ldots, \beta_N \in \mathbb{R}^{N+1} \) set

\[
V(\beta) := \mathbb{E}( \max_{i=0,\ldots,N} (\beta_i + \varepsilon_i))
\]

since

\[
\mathbb{P}(\max_{i=0,\ldots,N} (\beta_i + \varepsilon_i) \leq t) = \exp \left( -\Lambda(\beta) \exp(-t - \gamma) \right), \quad \text{with } \Lambda(\beta) = \sum_{i=0}^{N} e^{\beta_i}
\]

we have

\[
V(\beta) = \int_{\mathbb{R}} t \exp \left( -\exp(-t + \log \Lambda(\beta) - \gamma) \right) \exp(-t + \log \Lambda(\beta) - \gamma) dt
\]

\[
= \int_{\mathbb{R}} (s + \log(\Lambda(\beta))) \exp \left( -\exp(-s - \gamma) \right) \exp(-s - \gamma) ds
\]

\[
= \mathbb{E}(\varepsilon + \log(\Lambda(\beta))) = \log(\Lambda(\beta)) = \log \left( \sum_{i=0}^{N} e^{\beta_i} \right).
\]

Recalling the expression (2.13) for \( R_\sigma \), we thus have

\[
\mathbb{E}(\max_{i=0,\ldots,N} (w_i - c_i(x) + \sigma \varepsilon_i)) = \sigma \log \left( \sum_{i=0}^{N} e^{\frac{w_i - c_i(x)}{\sigma}} \right) = R_\sigma(x, w)
\]

which shows (2.15). Moreover, it follows from Lebesgue’s dominated convergence theorem that \( V \) is differentiable with

\[
\frac{\partial V}{\partial \beta_i}(\beta) = \mathbb{P}(\beta_i + \varepsilon_i \geq \beta_j + \varepsilon_j, \forall j = 0, \ldots, N) = \frac{e^{\beta_i}}{\sum_{j=0}^{N} e^{\beta_j}}
\]

which shows formula (2.16).

**B. Existence and Uniqueness of Equilibria in the Teleworking Model**

**Existence**

Under the assumptions of Sect. 4, we claim that there exists equilibrium wages i.e. a vector \( \tilde{w} \in \mathbb{R}^{2N}_+ \) solving the fixed-point equation (4.14). To see this, we first argue as in Lemma 3.1 and find that for every \( \mu \in \mathcal{P}(X) \), the functional \( \tilde{J}_\mu \) defined in (4.13) admits a unique minimizer \( \tilde{w} = (w^k_i)_{i=1,\ldots,N,k=1,2} \) which satisfies

\[
\max_{i,k} w^k_i + \sum_{i=1}^{N} \tilde{\pi}_i(w^1_i, w^2_i) \leq \bar{w} := 2M + \sum_{i=1}^{N} \tilde{\pi}_i(w_0, w_0) + \sigma \log(2N + 1) + w_0
\]
where \( M := \max_i \| c_i \|_\infty \). Thanks to the nonnegativity of \( \tilde{\pi}_i \) and (4.5), we find that the minimizer of \( \tilde{J}_\mu \) belongs to \( [w, \tilde{w}]^{2N} \) where \( 0 < w \leq \tilde{w} \) are bounds that do not depend on \( \mu \). The conclusion of Lemma 3.1 still holds for \( \tilde{J}_\mu \) so that the existence of an equilibrium follows from Brouwer’s theorem exactly as in the proof of theorem 3.2.

**Uniqueness**

Let us now further assume that the production functions \( \tilde{f}_i \) are of class \( C^2 \) on \((0, +\infty)^2\) and that (4.15) holds. Since \( -\nabla \tilde{f}_i \) is the inverse of \( \nabla \tilde{\pi}_i \), this implies that for every \((w^1, w^2) \in [w, \tilde{w}]^2\), the \( 2 \times 2 \) matrix \( D^2 \tilde{\pi}_i (w^1, w^2) \) is positive definite so that its smallest eigenvalue \( \lambda_{\text{min}}(D^2 \tilde{\pi}_i (w^1, w^2)) \) is positive (and depends on \((w^1, w^2) \in [w, \tilde{w}]^2\) in a continuous way). Setting \( \alpha := \theta - (\theta - \theta) \), and

\[
\tilde{\mu}(x, \tilde{w}, \alpha) := \frac{\tilde{R}_\sigma(x, \tilde{w})^\alpha}{\int_X \tilde{R}_\sigma(y, \tilde{w})^\alpha \, dy}, \quad \forall (x, \tilde{w}, \alpha) \in X \times \mathbb{R}^{2N}_+ \times \mathbb{R}_+
\]

we write the system of equilibrium conditions as the system of \( 2N \) equations in the \( 2N \) unknowns \( \tilde{w} := (w^1_1, w^2_1, \ldots, w^1_N, w^2_N) \)

\[
\tilde{G}(\tilde{w}, \alpha) = 0, \quad \text{where } \tilde{G}_i(\tilde{w}, \alpha) = \frac{\partial \tilde{\pi}_i}{\partial w_i}(w^1_i, w^2_i) + \int_X \frac{\partial \tilde{R}_\sigma}{\partial w_i} \tilde{\mu}(x, \tilde{w}, \alpha) \, dx
\]

so that the Jacobian matrix (with respect to \( \tilde{w} \)) \( \tilde{A} \) of \( \tilde{G} \) reads

\[
\tilde{A} = \begin{pmatrix}
D^2 \pi_1 (w^1_1, w^2_1) & O & O & \cdots & O \\
O & D^2 \pi_2 (w^1_2, w^2_2) & O & \cdots & O \\
& O & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
O & O & \cdots & \cdots & \cdots \\
O & O & \cdots & \cdots & \cdots \\
\end{pmatrix}
+ \int_X D^2_{\tilde{w} \tilde{w}} \tilde{R}_\sigma \tilde{\mu}
+ \int_X \nabla_{\tilde{w}} \tilde{R}_\sigma \nabla_{\tilde{w}} \tilde{\mu}^\top
\]

where the matrix in the first line is written in \( 2 \times 2 \) block-diagonal form (and \( O \) denotes the zero \( 2 \times 2 \) matrix) and the matrix on the second line is positive definite. Since all wages \( w^k_i \) belong to \([w, \tilde{w}] \subset (0, +\infty)\), setting

\[
v := \min_{i=1, \ldots, N} \min_{(w^1, w^2) \in [w, \tilde{w}]^2} \lambda_{\text{min}}(D^2 \pi_i (w^1, w^2)) > 0,
\]

we have, for every \( \tilde{\xi} \in \mathbb{R}^{2N} \setminus \{0\} \)

\[
\tilde{A} \tilde{\xi} \cdot \tilde{\xi} > (v - \int_X |\nabla_{\tilde{w}} \tilde{R}_\sigma| |\nabla_{\tilde{w}} \tilde{\mu}|) \| \tilde{\xi} \|^2.
\]
Obviously $|\nabla \tilde{w} \tilde{R}_\sigma| \leq \sqrt{2N}$ and assuming that $\alpha \leq 1$, arguing as in the proof of theorem 3.3, we get

$$\int_X |\nabla \tilde{w} \tilde{\mu}| \leq \frac{\alpha}{w_0} \sqrt{2N}$$

hence, we deduce that $\tilde{A}$ is invertible as soon as

$$\alpha \leq \frac{w_0 \nu}{2N}$$

For such a choice of $\alpha$, uniqueness of the equilibrium can be shown exactly as in the proof of theorem 3.3.

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