Infinite Systems of Non-Colliding Generalized Meanders
and Riemann-Liouville Differintegrals

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Abstract. Yor’s generalized meander is a temporally inhomogeneous modification of the 2(ν+1)-dimensional
Bessel process with ν > −1, in which the inhomogeneity is indexed by κ ∈ [0, 2(ν + 1)). We introduce the
non-colliding particle systems of the generalized meanders and prove that they are Pfaffian processes, in
the sense that any multitime correlation function is given by a Pfaffian. In the infinite particle limit, we show
that the elements of matrix kernels of the obtained infinite Pfaffian processes are generally expressed by
the Riemann-Liouville differintegrals of functions comprising the Bessel functions Jν used in the fractional
calculus, where orders of differintegration are determined by ν − κ. As special cases of the two parameters
(ν, κ), the present infinite systems include the quaternion determinantal processes studied by Forrester,
Nagao and Honner and by Nagao, which exhibit the temporal transitions between the universality classes of
random matrix theory.

running head: Non-colliding generalized meanders

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1 Introduction

The random matrix (RM) theory was introduced originally as an approximation theory of statistics of nuclear
energy levels [30]. It should be noted that at the same time as the standard theory was established for three
ensembles called the Gaussian unitary, orthogonal, and symplectic ensembles (GUE, GOE, GSE) [11], Dyson
proposed to study such stochastic processes of interacting particles that the eigenvalue statistics of RMs are
realized in distribution of particle positions on R [10]. Dyson’s Brownian motion model is a one-parameter
family of N-particle systems, \( Z^{(β)}(t) = (Z_1^{(β)}(t), Z_2^{(β)}(t), \ldots, Z_N^{(β)}(t)) \), described by the stochastic differential equations

\[
\frac{dZ_i^{(β)}(t)}{dt} = dB_i(t) + \frac{β}{2} \sum_{1 \leq j \leq N, j \neq i} \frac{1}{Z_i^{(β)}(t) - Z_j^{(β)}(t)} dt, \quad t \in [0, \infty), 1 \leq i \leq N,
\]

where \( B_i(t), i = 1, 2, \ldots, N \) are independent standard Brownian motions and the parameter β equals 2, 1
and 4 for GUE, GOE and GSE, respectively. Due to the strong repulsive forces, which are long-ranged
and act between any pair of particles, intersections of particle trajectories are prohibited for $\beta \geq 1$ 143 (see also 17). In this one-parameter family, the $\beta = 2$ case (i.e. the GUE case) is the simplest and the most-understood, since its equivalence with the $N$ particle systems of Brownian motions conditioned never to collide with each other can be proved 10.

The standard (Wigner-Dyson) theory has been extended by adding three chiral versions of RM ensembles in the particle physics of QCD 52 51 19 10, and by introducing the four additional ensembles so-called the Bogoliubov-de Gennes classes in the mesoscopic physics 11 2. Here we note that the chiral ensembles have a parameter $\nu \in \{0, 1, 2, \cdots \}$ in addition to $\beta$. In these totally ten ensembles 11 54 4 2, chiral GUE (chGUE), class C and class D can be regarded as natural extensions of the GUE, in the sense that these eigenvalue statistics are also realized in appropriate non-colliding systems of stochastic particle systems: König and O’Connell showed that the chGUE with the parameter $\nu \in \{0, 1, 2, \cdots \}$ corresponds to the non-colliding systems of $2(\nu + 1)$-dimensional squared Bessel processes 28. The present authors clarified that the eigenvalue statistics in the classes C and D are realized by the non-colliding systems of the Brownian motions with an absorbing wall at the origin and of the Brownian motions reflecting at the origin 27 20. Since the absorbing and reflecting Brownian motions are directly related with the three-dimensional and one-dimensional Bessel processes, respectively (see, for example, 11), the stochastic differential equations of these non-colliding particle systems are generally given by

$$d\tilde{Z}^{(\nu)}_i(t) = dB_i(t) + \frac{2\nu + 1}{2} \frac{1}{\tilde{Z}^{(\nu)}_i(t)} + \sum_{1 \leq j \leq N, j \neq i} \left\{ \frac{1}{\tilde{Z}^{(\nu)}_i(t) - \tilde{Z}^{(\nu)}_j(t)} + \frac{1}{\tilde{Z}^{(\nu)}_i(t) + \tilde{Z}^{(\nu)}_j(t)} \right\} dt,$$

$$t \in [0, \infty), 1 \leq i \leq N,$$

with reflecting barrier condition at the origin in case $\nu = -1/2$. Therefore, the difference of (non-standard) RM ensembles can be attributed to the difference of dimensionality of the Bessel processes, whose non-colliding sets realize the statistics of the RM ensembles 29. Here we remind that the $d$-dimensional Bessel process is defined as the process of the radial coordinate (the modulus) of a Brownian motion in $\mathbb{R}^d$. To realize other $10 - 4 = 6$ RM ensembles by conditioned stochastic processes may be much more difficult (see 49), but we demonstrated that, if we consider appropriate non-colliding systems of temporally inhomogeneous processes defined only in a finite time-interval $[0, T]$, we can observe the transitions of distributions into the 6 distributions as the time $t$ approaches the final time $T$ 25 28. The interesting fact is that the processes that can be used instead of the Bessel processes 11 4 should have one more parameter $\kappa$ in addition to $\nu$. This two-parameter family of temporally inhomogeneous processes indexed by $(\nu, \kappa), \nu > -1, \kappa \in [0, 2(\nu + 1))$ is equivalent with the family of processes already studied by Yor. He called them the generalized meanders 53.

From the viewpoint of random matrix theory, studying time-development of stochastic systems by calculating, for example, the multitime correlation functions corresponds to considering multi-matrix models. In particular, the temporally inhomogeneous processes will be identified with such matrix models that matrices with different symmetries are coupled in a chain 23 24 33 22. Determination of all multitime correlation functions of systems, which allows us to determine scaling limits associated with the infinity limit of matrix sizes (i.e. the infinite-particle limit) is one of the main topics of the modern theory of RM 30. The finite and infinite particle systems showing the orthogonal-unitary and symplectic-unitary transitions, and transitions between class C to class CI were studied and multitime correlation functions were determined by Forrester, Nagao and Honner (FNH) 16, and by Nagao 32, respectively. The system in the Laguerre ensemble with $\beta = 1$ initial condition reported in the former paper can be regarded as the $\nu = \kappa \in \{0, 1, 2, \cdots \}$ case of the non-colliding system of the generalized meanders and the system reported in the latter paper as the $(\nu, \kappa) = (1/2, 1)$ case.

If we think about the system of generalized meanders apart from the RM theory, however, we can consider the parameters $\nu$ and $\kappa$ as real numbers, and not necessarily integers nor half-integers. In the present paper, we calculate the multitime correlation functions of non-colliding systems of (squared) generalized meanders for arbitrary values of parameters, provided they satisfy the condition $\nu > -1, \kappa \in [0, 2(\nu + 1))$ so that the systems are not collapsed. We first define the $N$ particle systems in a finite time-interval $[0, T]$ and take the $N = T \to \infty$ limit to construct the two-parameter family of infinite particle systems. We prove that the
multitime characteristic functions is given by a Fredholm Pfaffian [40] and thus any multitime correlation function is given by a Pfaffian. Similarly to the results by FNH [15] and Nagao [32], and their temporally-homogeneous version (the determinantal process with the extended Bessel kernel [50]), the elements of the matrix kernels of Pfaffians are expressed using the Bessel functions, but we clarify the fact that they are generally given by the Riemann-Liouville differintegrals of the functions comprising the Bessel functions, which are used in fractional calculus (see, for example, [36, 44, 38]). This structure will explain the origin of the multiple integral expressions for the elements of the matrix kernels reported by FNH [15] and Nagao [32].

The paper is organized as follows. In Section 2, the definitions of the generalized meanders of Yor and their non-colliding systems are given and the Riemann-Liouville differintegrals of the Bessel functions with appropriate factors are introduced. The main theorem for the infinite particle limit (Theorem 2.1) is then given. It is demonstrated that, if we take a further limit in the system of Theorem 2.1, we will obtain the temporally homogeneous system of infinite number of particles, which is a determinantal process with the extended Bessel kernel studied in [50] (see also [37]). Using the properties of the Riemann-Liouville differintegrals, we show that Theorem 2.1 includes the results by FNH [15] and Nagao [32] as special cases. Section 3 is devoted to prove that for any finite number of particles N, the present system is a Pfaffian process (Theorem 3.1), in the sense that any multitime correlation function is given by a Pfaffian. These Pfaffian processes may be regarded as the continuous space-time version of the Pfaffian point processes and Pfaffian ensembles in [6, 47, 48]. See also [39, 20, 45, 13, 17] in the context of study of nonequilibrium phenomena in the polynuclear growth models, and [35, 14] in that of shape fluctuations of crystal facets. The processes studied in [15, 32] are also Pfaffian processes, since the ‘quaternion determinantal expressions’ of correlation functions, introduced and developed by Dyson, Mehta, Forrester, and Nagao [12, 20, 50, 33, 31], are readily transformed to Pfaffian expressions. The method of skew-orthogonal functions associated with the Laguerre polynomials are used in Section 4 in order to perform matrix inversion and give explicit expressions for the elements of matrix kernels of Pfaffians. Asymptotics in T = N → ∞ are studied in Section 5. Appendices are given to show proofs of formulae and lemmas used in the text.

At the end of this introduction, we would like to refer to the papers [13, 8], which reported the further extensions of RM theory in physics and the representation theory. We hope that the present paper will demonstrate the fruitfulness of developing the probability theory of interacting infinite particle systems in connection with the extensive study of (multi-)matrix models in the RM theory.

2 Definition of Processes and Results

2.1 Non-colliding systems of generalized meanders

Let Z and R be the sets of integers and real numbers, respectively, and set N = {1, 2, . . .}, N0 = N ∪ {0}, Z− = Z \ N0, and R+ = {x ∈ R : x ≥ 0}. Let Γ(c), c ∈ R \ (Z− ∪ {0}), be the Gamma function: Γ(c) = ∫ 0 ∞ dy e−yyc−1 for c > 0, and Γ(c) = Γ(c + [−c]+1)/[c(c+1) · · · (c+[−c])} for c ∈ (−∞, 0) \ Z−, where [c] is the largest integer that is less than or equal to the real number c. For t > 0, x, y ∈ R+ and ν > −1 we denote by Gtν(t; y|x) the transition probability density of a 2(ν + 1)-dimensional Bessel process [41, 3].

\[
G^{(ν)}(t; y|x) = \frac{y^{ν+1}}{x^{ν}} e^{-(x^{2}+y^{2})/2t} I_{ν} \left(\frac{xy}{t}\right), \quad x > 0, y ∈ R_{+},
\]

\[
G^{(ν)}(t; y|0) = \frac{y^{2ν+1}}{2^{ν}Γ(ν+1)} e^{−y^{2}/2t}, \quad y ∈ R_{+},
\]

where \( I_{ν}(z) \) is the modified Bessel function: \( I_{ν}(z) = ∑_{n=0}^{∞} (z/2)^{2n+ν}/\{Γ(n+1)Γ(ν+n+1)\} \). For T > 0, κ ∈ [0, 2(ν + 1)], we put

\[
i_{T}^{(ν,κ)}(t, x) = ∫_{0}^{∞} dy G^{(ν)}(T−t; y|x)y^{−κ}, \quad x ∈ R_{+}, t ∈ [0, T],
\]
and
\[
G_T^{(\nu,\kappa)}(s, x; t, y) = \frac{1}{h_T^{(\nu,\kappa)}(s, x)} G^{(\nu)}(t - s; x|y) h_T^{(\nu,\kappa)}(t, y), \quad x > 0, y \in \mathbb{R}^+, \quad (2.1)
\]
\[
G_T^{(\nu,\kappa)}(0, 0; t, y) = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1 - \kappa/2)} (2T)^{\kappa/2} G^{(\nu)}(t; y|0) h_T^{(\nu,\kappa)}(t, y), \quad y \in \mathbb{R}^+, \quad (2.2)
\]
for \(0 \leq s \leq t \leq T\). This transition probability density \(G_T^{(\nu,\kappa)}(s, x; t, y)\) defines the temporally inhomogeneous process in a finite time-interval \([0, T]\), which is called a generalized meander. In particular, when \(\nu = 1/2\) and \(\kappa = 1\), it is identified with the process called a Brownian meander (see Chapter 3 in Yor [53]).

Now we consider the \(N\)-particle system of generalized meanders conditioned that they never collide in a finite time-interval \([0, T]\). Let
\[
\mathbb{R}_+^N = \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}_+^N : 0 \leq x_1 < x_2 < \cdots < x_N \right\}.
\]
According to the determinantal formula of Karlin and McGregor [21], the transition probability density is given as
\[
g_{N,T}^{(\nu,\kappa)}(s, x; t, y) = \frac{f_{N,T}^{(\nu,\kappa)}(s, x; t, y) \mathcal{N}_{N,T}^{(\nu,\kappa)}(T - t, y)}{\mathcal{N}_{N,T}^{(\nu,\kappa)}(T - s, x)}, \quad 0 \leq s \leq t \leq T, \quad x, y \in \mathbb{R}_+^N, \quad (2.3)
\]
where
\[
f_{N,T}^{(\nu,\kappa)}(s, x; t, y) = \det_{1 \leq j, k \leq N} \left[ G_T^{(\nu,\kappa)}(s, x_j, t, y_k) \right], \quad \mathcal{N}_{N,T}^{(\nu,\kappa)}(t, x) = \int_{\mathbb{R}_+^N} dy f_{N,T}^{(\nu,\kappa)}(T - t, x; T, y).
\]
Since \(h_T^{(\nu,\kappa)}(t, x) = 1\), \(G_T^{(\nu,\kappa)}(s, x; t, y) = G^{(\nu)}(t - s; y|x)\) and thus \(f_{N,T}^{(\nu,\kappa)}\) is temporally homogeneous and independent of \(T\), we will write \(f_N^{(\nu)}(t - s; y|x)\) for \(f_{N,T}^{(\nu,\kappa)}(s, x; t, y)\). Moreover, note that
\[
f_{N,T}^{(\nu,\kappa)}(s, x; t, y) = \frac{1}{h_T^{(\nu,\kappa)}(s, x)} f^{(\nu)}(t - s; x|y) h_T^{(\nu,\kappa)}(t, y),
\]
where \(h_T^{(\nu,\kappa)}(t, x) = \prod_{j=1}^N h_T^{(\nu,\kappa)}(t, x_j)\) and \(h_T^{(\nu,\kappa)}(T, x) = \prod_{j=1}^N x_j^{-\kappa}\). Then (2.3) can be written as
\[
g_{N,T}^{(\nu,\kappa)}(s, x; t, y) = \frac{1}{\mathcal{N}_N^{(\nu,\kappa)}(T - s, x)} f_N^{(\nu)}(t - s; y|x) \mathcal{N}_N^{(\nu,\kappa)}(T - t, y), \quad (2.4)
\]
where
\[
\mathcal{N}_N^{(\nu,\kappa)}(t, x) = \int_{\mathbb{R}_+^N} dy f_N^{(\nu)}(t; y|x) \prod_{j=1}^N y_j^{-\kappa}.
\]
In our previous paper [20] it was shown that, taking the limit \(\mathbf{x} \to 0 \equiv (0, 0, \ldots, 0)\) at the initial time \(s = 0\), (2.4) becomes
\[
g_{N,T}^{(\nu,\kappa)}(0, 0; t, y) = C_{N,T}^{(\nu,\kappa)}(t) \prod_{j=1}^N G^{(\nu)}(t, y_j|0) \prod_{1 \leq j < k \leq N} (y_k^2 - y_j^2) \mathcal{N}_N^{(\nu,\kappa)}(T - t, y), \quad (2.6)
\]
for \(\nu > -1\) and \(\kappa \in [0, 2(\nu + 1))\), where
\[
C_{N,T}^{(\nu,\kappa)}(t) = \frac{T^{(N+\kappa-1)/2}(N-1)^N}{2^{N(\nu-1)/2}} \prod_{j=1}^N \frac{\Gamma(\nu + 1) \Gamma(1/2)}{\Gamma(j/2) \Gamma((j + 1 + 2\nu - \kappa)/2)}.
\]
The $N$-particle system of \textit{non-colliding generalized meanders all starting from the origin $0$ at time $0$} is defined by the transition probability density $g_{N,T}^{(\nu,\kappa)}$ given above and it will be denoted by $X(t) \in \mathbb{R}_+^N$, $t \in [0, T]$ in the present paper. It makes a two-parameter family of temporally inhomogeneous processes parameterized by $\nu > -1$ and $\kappa \in [0, 2(\nu + 1)]$.

We denote by $\mathcal{X}$ the space of countable subsets $\xi$ of $\mathbb{R}$ satisfying $\sharp(\xi \cap K) < \infty$ for any compact subset $K$. For $x = (x_1, x_2, \ldots, x_n) \in \bigcup_{n=1}^{\infty} \mathbb{R}^n$, we denote $\{ x_i \}_{i=1}^{n} \subset \mathcal{X}$ simply by $\{ x \}$. Then $\Xi_{N}(t) = \{ X(t) \}, t \in [0, T]$, is the diffusion process on the set $\mathcal{X}$ with transition density function $g_{N,T}^{(\nu,\kappa)}(s, x; t, y)$, $0 \leq s \leq t \leq T$:

$$g_{N,T}^{(\nu,\kappa)}(s, x; t, y) = \begin{cases} g_{N,T}^{(\nu,\kappa)}(s, x; t, y), & \text{if } s > 0, \; \sharp(\xi) = \sharp(\eta) = N, \\ g_{N,T}^{(\nu,\kappa)}(0, 0; t, y), & \text{if } s = 0, \; \xi = \{ 0 \}, \; \sharp(\eta) = N, \\ 0, & \text{otherwise}, \end{cases}$$

where $x$ and $y$ are the elements of $\mathbb{R}_+^{N}$ with $\xi = \{ x \}, \eta = \{ y \}$.

For the given time interval $[0, T]$, we consider the $M$ intermediate times $0 < t_1 < t_2 < \cdots < t_{M+1} < T$. For convenience, we set $t_0 = 0$, $t_{M+1} = T$. For $x^{(m)} \in \mathbb{R}^N$, $1 \leq m \leq M+1$, and $N' = 1, 2, \ldots, N$, we put $x_{N'}^{(m)} = (x_{1}^{(m)}, x_{2}^{(m)}, \ldots, x_{N'}^{(m)})$ and $\xi_{N'} = \{ x_{N'}^{(m)} \}$. Then the multitime transition density function of the process $\Xi_{N}(t)$ is given by

$$g_{N,T}^{(\nu,\kappa)}(0, \{ 0 \}; t_1, \xi_{N_1}^{N_1}; \ldots; t_{M+1}, \xi_{N_{M+1}}^{N_{M+1}}) = \prod_{m=0}^{M} g_{N,T}^{(\nu,\kappa)}(t_m, \xi_{m}^{N}, t_{m+1}, \xi_{m+1}^{N}), \quad (2.7)$$

where we assume $\xi_{0}^{N} = \{ 0 \}$. For a sequence $\{ N_m \}_{m=1}^{M+1}$ of positive integers less than or equal to $N$, we define the $(N_1, N_2, \ldots, N_{M+1})$-multitime correlation function by

$$\rho_{N,T}^{X}(t_1, \xi_{1}^{N}; t_2, \xi_{2}^{N}; \ldots; t_{M+1}, \xi_{M+1}^{N+1}) = \int_{\prod_{m=1}^{M+1} \mathbb{R}_-^{N-N_m}} \frac{1}{(N-N_m)!} \prod_{j=N_{m}+1}^{N} dx_{j}^{(m)} g_{N,T}^{(\nu,\kappa)}(0, \{ 0 \}; t_1, \xi_{1}^{N}; \ldots; t_{M+1}, \xi_{M+1}^{N}). \quad (2.8)$$

Associated with the generalized meander $[2.4]$, $[2.5]$, we consider a temporally inhomogeneous diffusion process with transition probability density

$$p_{T}^{(\nu,\kappa)}(0, 0; t, y) = G_{T}^{(\nu,\kappa)}(0, 0; t, \sqrt{y}) \times \frac{1}{2} y^{-1/2}, \quad y \in \mathbb{R}_+, \quad$$

$$p_{T}^{(\nu,\kappa)}(s, x; t, y) = G_{T}^{(\nu,\kappa)}(s, \sqrt{x}; t, \sqrt{y}) \times \frac{1}{2} y^{-1/2}, \quad x > 0, y \in \mathbb{R}_+, \quad t \in [0, T],$$

call it a \textit{squared generalized meander}. The $N$-particle system of \textit{non-colliding squared generalized meanders} $Y(t), t \in [0, T]$, is then defined by

$$Y(t) = \left( X_1(t)^2, X_2(t)^2, \ldots, X_N(t)^2 \right), \quad t \in [0, T].$$

The correlation function $\rho_{N,T}^{Y}(t) = \{ Y(t) \}$ is obtained from $[2.8]$ through the relation

$$\rho_{N,T}^{Y}(t_1, \xi_{1}^{N}; t_2, \xi_{2}^{N}; \ldots; t_{M+1}, \xi_{M+1}^{N+1}) = \rho_{N,T}^{X}(t_1, \xi_{1}^{N}; t_2, \xi_{2}^{N}; \ldots; t_{M+1}, \xi_{M+1}^{N+1}) \prod_{m=1}^{M+1} \prod_{j=N_m+1}^{N_m} \frac{1}{2x_j},$$

where $\xi_{N_m} = \{ x_{N_m}^{(m)} \}, \eta_{N_m} = \{ y_{N_m}^{(m)} \}$ with $x_j^{(m)} = \sqrt{y_j^{(m)}}$, $1 \leq j \leq N_m, 1 \leq m \leq M+1$. 

5
2.2 Riemann-Liouville differintegrals of Bessel functions

We consider the following left and right Riemann-Liouville differintegrals for integrable functions \( f \) on \( \mathbb{R}_+ \),

\[
0 D^c_x f(x) = \frac{1}{\Gamma(n-c)} \left( \frac{d}{dx} \right)^n \int_0^x (x-y)^{n-c-1} f(y) dy, \tag{2.10}
\]

\[
x D^c_\infty f(x) = \frac{1}{\Gamma(n-c)} \left( -\frac{d}{dx} \right)^n \int_x^\infty (y-x)^{n-c-1} f(y) dy, \tag{2.11}
\]

where \( c \in \mathbb{R} \) and \( n = [c+1]_+ \) with the notation \( x_+ = \max\{x,0\} \). It is easy to confirm that, if \( c \in \mathbb{N}_0 \), both of them are reduced to the ordinary multiple derivative,

\[
0 D^c_x f(x) = (-1)^c x D^c_\infty f(x) = \left( \frac{d}{dx} \right)^c f(x),
\]

and, if \( c \in \mathbb{Z}_- \), they are equal to the multiple integrals,

\[
0 D^c_x f(x) = \int_0^x dy_{|c|-1} \int_0^{y_{|c|-1}} dy_{|c|-2} \cdots \int_0^{y_2} dy_1 \int_0^{y_1} dy_0 f(y_0),
\]

\[
x D^c_\infty f(x) = \int_x^\infty dy_{|c|-1} \int_{y_{|c|-1}}^{\infty} dy_{|c|-2} \cdots \int_{y_2}^{\infty} dy_1 \int_{y_1}^{\infty} dy_0 f(y_0).
\]

For \( c \in (-\infty,0) \setminus \mathbb{Z}_- \) \text{[2.10]} and \( 2.11 \) define fractional integrals, and for \( c \in \mathbb{R}_+ \setminus \mathbb{N}_0 \) fractional differentials. The Riemann-Liouville differintegrals are most often used in the fractional calculus (see, for example, \[36, 44, 38\]).

Let \( J_\nu(z) \) be the Bessel functions: \( J_\nu(z) = \sum_{\ell=0}^{\infty} (-1)^\ell (z/2)^{2\ell+\nu}/\{\Gamma(\nu+\ell+1)\ell!\} \). We define functions \( \tilde{J}_\nu \) and \( \hat{J}_\nu \) as

\[
\tilde{J}_\nu(\theta, \eta, x, s) = (\theta \eta x)^{\nu/2} J_\nu(2\sqrt{\theta \eta x}) e^{2s\theta \eta} = e^{2s\theta \eta} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (\theta \eta x)^{\ell+\nu}}{\Gamma(\nu+\ell+1)\ell!}, \tag{2.12}
\]

\[
\hat{J}_\nu(\theta, \eta, x, s) = (\theta \eta x)^{-\nu/2} J_\nu(2\sqrt{\theta \eta x}) e^{2s\theta \eta} = e^{2s\theta \eta} \sum_{\ell=0}^{\infty} \frac{(-\theta \eta x)^{\ell}}{\Gamma(\nu+\ell+1)\ell!}. \tag{2.13}
\]

We will use the following abbreviations for the Riemann-Liouville differintegrals of order \( c \in \mathbb{R} \) of \( \tilde{J}_\nu \) and \( \hat{J}_\nu \),

\[
\tilde{J}_\nu^{(c)}(\theta, \eta, x, s) = 0 D^c_x \tilde{J}_\nu(\theta, \eta, x, s), \quad \theta, \eta > 0, s \in \mathbb{R}, \tag{2.14}
\]

\[
\hat{J}_\nu^{(c)}(\theta, \eta, x, s) = x D^c_\infty \hat{J}_\nu(\theta, \eta, x, s), \quad \theta, \eta > 0, s < 0. \tag{2.15}
\]

We note that, if \( c \in \mathbb{R} \setminus \mathbb{N}_0 \), \( \tilde{J}_\nu^{(c)} \) can be expanded as

\[
\tilde{J}_\nu^{(c)}(\theta, \eta, x, s) = \frac{1}{\Gamma(-c)} \sum_{n=0}^{\infty} \frac{(-1)^n \eta^{n-c}}{n!(n-c)} \tilde{J}_\nu^{(n)}(\theta, \eta, x, s), \quad \theta, \eta > 0, s \in \mathbb{R}. \tag{2.16}
\]

It is also noted that, since \( \tilde{J}_\nu(\theta, \eta, x, s) \to 0 \) exponentially fast as \( \eta \to \infty \), if \( s\theta < 0 \),

\[
\tilde{J}_\nu^{(c)}(\theta, \eta, x, s) = \frac{1}{\Gamma(n-c)} \int_\eta^\infty d\xi (\xi - \eta)^{n-c-1} \tilde{J}_\nu^{(n)}(\theta, \xi, x, s), \quad \theta, \eta > 0, s < 0, \tag{2.17}
\]

where \( n = [c+1]_+ \).
2.3 Results

We put

$$a = a(\nu, \kappa) = \nu - \frac{\kappa}{2}, \quad b = b(\nu, \kappa) = \nu - \kappa,$$

and introduce functions $D(s, x; t, y)$, $\tilde{I}(s, x; t, y)$, $S(s, x; t, y)$ and $\tilde{S}(s, x; t, y)$, $x, y \in \mathbb{R}_+, s, t < 0$,

$$D(s, x; t, y) = \frac{1}{2}(xy)^{\kappa/2} \int_0^{\infty} d\theta \theta^{1-\kappa} \left[ \tilde{J}_\nu^{(-b-1)}(\theta, 1, x, -s) \tilde{J}_\nu^{(-b)}(\theta, 1, y, -t) - \tilde{J}_\nu^{(-b)}(\theta, 1, x, -s) \tilde{J}_\nu^{(-b-1)}(\theta, 1, y, -t) \right],$$

$$\tilde{I}(s, x; t, y) = -(xy)^{\kappa/2} \int_0^{\infty} d\theta \theta^{\kappa-1} \left[ \int_1^\infty d\xi \xi^a \tilde{J}_\nu^{(b+1)}(\theta, \xi, x, s) \tilde{J}_\nu^{(b+1)}(\theta, 1, y, t) - \tilde{J}_\nu^{(b+1)}(\theta, 1, x, s) \int_1^\infty d\xi \xi^a \tilde{J}_\nu^{(b+1)}(\theta, \xi, y, t) \right],$$

$$S(s, x; t, y) = \frac{1}{2} (xy)^{\kappa/2} \int_0^{\infty} d\theta \left[ \tilde{J}_\nu^{(b+1)}(\theta, 1, x, s) \tilde{J}_\nu^{(-b-1)}(\theta, 1, y, -t) - \{a \tilde{J}_\nu^{(-b-1)}(\theta, 1, y, -t) - \tilde{J}_\nu^{(-b)}(\theta, 1, y, -t) \} \right] \int_1^\infty d\xi \xi^a \tilde{J}_\nu^{(b+1)}(\theta, \xi, s),$$

and

$$\tilde{S}(s, x; t, y) = S(s, x; t, y) - \mathbf{1}_{(s < t)} \left( \frac{y}{x} \right)^{b/2} \mathbf{G}(s, x; t, y),$$

where $\mathbf{1}_{(\omega)}$ is the indicator function: $\mathbf{1}_{(\omega)} = 1$ if $\omega$ is satisfied and $\mathbf{1}_{(\omega)} = 0$ otherwise, and

$$\mathbf{G}(s, x; t, y) = \int_0^{\infty} d\theta J_\nu(2\sqrt{\theta x}) J_\nu(2\sqrt{\theta y}) e^{2(s-t)\theta}. $$

For an integer $N$ and a skew-symmetric $2N \times 2N$ matrix $A = (a_{ij})$, the Pfaffian is defined as

$$\text{Pf}(A) = \text{Pf}_{1 \leq i < j \leq 2N} (a_{ij}) = \frac{1}{N!} \sum_{\sigma} \text{sgn}(\sigma) a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} \cdots a_{\sigma(2N-1)} a_{\sigma(2N)},$$

where the summation is extended over all permutations $\sigma$ of $(1, 2, \ldots, 2N)$ with restriction $\sigma(2k-1) < \sigma(2k)$, $k = 1, 2, \ldots, N$. We put

$$\tilde{\Xi}_N^Y(s) = \{ Y_1(T_N + s), Y_2(T_N + s), \ldots, Y_N(T_N + s) \}, \quad s \in [-T_N, 0),$$

and $\tilde{\Xi}_N^Y(s) = \{ 0 \}$, $s \in (-\infty, -T_N)$. Then we can state the main theorem in the present paper.

**Theorem 2.1** Let $T_N = N$. Then the process $\tilde{\Xi}_N^Y(s), s \in (-\infty, 0)$ converges to the process $\tilde{\Xi}_N^Y(s), s \in (-\infty, 0)$, as $N \to \infty$, in the sense of finite dimensional distributions, whose correlation functions $\rho^Y$ are given by

$$\rho^Y \left( s_1; \{ Y_N^{(1)} \}; s_2; \{ Y_N^{(2)} \}; \ldots; s_M; \{ Y_N^{(M)} \} \right) = \text{Pf} \left[ A \left( y_N^{(1)}, y_N^{(2)}, \ldots, y_N^{(M)} \right) \right],$$

for any $M \geq 1$, any sequence $\{ N_m \}_{m=1}^M$ of positive integers, and any strictly increasing sequence $\{ s_m \}_{m=1}^{M+1}$ of nonpositive numbers with $s_{M+1} = 0$, where $A \left( y_N^{(1)}, y_N^{(2)}, \ldots, y_N^{(M)} \right)$ is the $2 \sum_{m=1}^M N_m \times 2 \sum_{m=1}^M N_m$ skew-symmetric matrix defined by

$$A \left( y_N^{(1)}, y_N^{(2)}, \ldots, y_N^{(M)} \right) = \left( A_{i,j}^{(m, n)} \right)_{1 \leq i \leq N_m, 1 \leq j \leq N_n, 1 \leq m, n \leq M}.$$
with $2 \times 2$ matrices $\mathcal{A}^{m,n}(x, y)$:

$$\mathcal{A}^{m,n}(x, y) = \begin{pmatrix} D(s_m, x; s_n, y) & \tilde{S}(s_n, y; s_m, x) \\ -\tilde{S}(s_m, x; s_n, y) & -I(s_m, x; s_n, y) \end{pmatrix}. $$

In the infinite-particle system defined by Theorem 2.1 we can take the further limit:

$$s_m \to -\infty \quad \text{with the time differences } s_n - s_m \text{ fixed, } 1 \leq m, n \leq M.$$ 

In this limit, $D(s_m, x; s_n, y)\tilde{I}(s_m, x; s_n, y) \to 0$, $1 \leq m, n \leq M$, as we show in Appendix E. Therefore, we can replace $D$ and $\tilde{I}$ by zeros in the matrices. Then the Pfaffian is reduced to an ordinary determinant of the $\sum_{m=1}^{M+1} N_m \times \sum_{m=1}^{M+1} N_m$ matrix, $\tilde{A}^\nu \left( Y_{N_1}^{(1)}, Y_{N_2}^{(2)}; \ldots ; Y_{N_M}^{(M)} \right) = \left( a^{m,n}(y_i^{(m)}, y_j^{(n)}) \right)_{1 \leq i \leq N_m, 1 \leq j \leq N_m, 1 \leq m, n \leq M}$ with the elements

$$a^{m,n}(y_i^{(m)}, y_j^{(n)}) = \tilde{S}(s_m, y_i^{(m)}; s_n, y_j^{(n)}),$$

where

$$\tilde{S}(s, x; t, y) = \begin{cases} \int_0^1 d\theta J_\nu(2\sqrt{\theta}x)J_\nu(2\sqrt{\theta}y)e^{2(s-t)\theta}, & \text{if } s > t, \\
\frac{J_\nu(2\sqrt{\theta}x)\sqrt{\theta}J_\nu'(2\sqrt{\theta}y) - J_\nu(2\sqrt{\theta}y)\sqrt{\theta}J_\nu'(2\sqrt{\theta}x)}{x - y}, & \text{if } s = t, \\
-\int_{-\infty}^1 d\theta J_\nu(2\sqrt{\theta}x)J_\nu(2\sqrt{\theta}y)e^{2(s-t)\theta}, & \text{if } s < t, \end{cases}$$

with $J'_\nu = dJ_\nu(z)/dz$. Hence, in this limit we obtain a temporally homogeneous system of infinite number of particles, whose correlation functions are given by

$$\tilde{\tilde{Y}}^\nu \left( s_1, \{ y_{N_1}^{(1)} \}; s_2, \{ y_{N_2}^{(2)} \}; \ldots ; s_M, \{ y_{N_M}^{(M)} \} \right) = \det \tilde{A}^\nu \left( y_{N_1}^{(1)}, y_{N_2}^{(2)}; \ldots ; y_{N_M}^{(M)} \right). \quad (2.23)$$

**Remark 1.** Forrester, Nagao, and Honner [15] studied the orthogonal-unitary and symplectic-unitary universality transitions in random matrix theory by giving the quaternion determinantal expressions of (two-time) correlation functions for parametric RM models. One of their results for the ‘Laguerre ensemble with $\beta = 1$ initial condition’, which shows the orthogonal-unitary transition, can be reproduced from Theorem 2.1 by setting

(i) \quad $\kappa = \nu \quad \iff \quad a = \frac{\nu}{2}, \quad b = 0,$ \quad where \quad $\nu \in \mathbb{N}.0.$

This fact may be readily seen, if we notice that by definition

$$J^-\nu^{-1}(\theta, 1, x, s) = \int_0^1 d\eta (\theta, \eta x)^{\nu/2} J_\nu(2\sqrt{\theta, \eta x})e^{2s\theta \eta} = \theta^{-1}x^{\nu/2} \int_0^\theta du u^{\nu/2} J_\nu(2\sqrt{u}x)e^{2su}. \quad \text{(2.24)}$$

**Remark 2.** Nagao’s result on the multitime correlation functions for vicious random walk with a wall [32] can be regarded as the special case of Theorem 2.1 in which

(ii) \quad $\nu = \frac{1}{2}, \quad \kappa = 1 \quad \iff \quad a = 0, \quad b = -\frac{1}{2}.$
This fact can be confirmed by noting that, by definition (2.14) with \( \tilde{J}_{1/2}(\theta, 0; x, s) = 0 \),

\[
\tilde{J}_{1/2}^{(-1/2)}(\theta, 1, x, s) = \frac{(\theta x)^{1/4}}{\sqrt{\pi}} \int_0^1 d\eta (1 - \eta)^{-1/2} \eta^{1/4} J_{1/2}(2\sqrt{\theta \eta} x) e^{2\theta \eta},
\]

\[
\tilde{J}_{1/2}^{(1/2)}(\theta, 1, x, s) = \frac{(\theta x)^{1/4}}{\sqrt{\pi}} \int_0^1 d\eta (1 - \eta)^{-1/2} \frac{d}{d\eta} \left\{ \eta^{1/4} J_{1/2}(2\sqrt{\theta \eta} x) e^{2\theta \eta} \right\},
\]

by (2.17),

\[
\tilde{J}_{1/2}^{(1/2)}(\theta, \eta, x, s) = -\frac{(\theta x)^{-1/4}}{\sqrt{\pi}} \int_\eta^\infty d\xi (\xi - \eta)^{-1/2} \frac{d}{d\xi} \left\{ \xi^{-1/4} J_{1/2}(2\sqrt{\theta \xi} x) e^{2\theta \xi} \right\}, \quad s < 0,
\]

and, by definition (2.15),

\[
\int_1^\infty d\xi \tilde{J}_{1/2}^{(1/2)}(\theta, \xi, x, s) = \frac{(\theta x)^{-1/4}}{\sqrt{\pi}} \int_1^\infty d\eta (\eta - 1)^{-1/2} \eta^{-1/4} J_{1/2}(2\sqrt{\theta \eta} x) e^{2\theta \eta}, \quad s < 0.
\]

In this case, the system shows the transition between the class \( C \) and class \( CI \) of the Bogoliubov-de Gennes universality classes of nonstandard RM theory [32, 27, 26].

**Remark 3.** From the results for finite non-colliding processes [26], we expect that, when

(iii) \( \kappa = \nu + 1 \quad \iff \quad a = \frac{\nu - 1}{2}, \quad b = -1, \quad \text{where} \quad \nu \in \mathbb{N}_0, \)

the present infinite particle system will show the transition from the chiral GUE to the chiral GOE of the universality classes and when

(iv) \( \nu = -\frac{1}{2}, \quad \kappa = 0 \quad \iff \quad a = b = -\frac{1}{2}, \)

that from the class \( D \) to the ‘real-component version’ of class \( D \) of the Bogoliubov-de Gennes universality classes [26].

**Remark 4.** Following the argument given in [39, 20], tightness in time can be proved and transition phenomena observed in the limit \( s_M \to 0 \) may be generally discussed, which will be reported elsewhere.

**Remark 5.** The homogeneous system (2.28) was studied in [50, 37].
where (2.4) and (2.6) with (2.5) are used.

Through the relation (2.9), the multitime transition density for the process \( \{Y(t)\}, t \in [0, T] \), denoted by \( p_{N,T}^{(\nu,\kappa)} \) is then written as

\[
p_{N,T}^{(\nu,\kappa)} \left( 0, \{0\}; t_1, \{y^{(1)}\}; \ldots; t_{M+1}, \{Y^{(M+1)}\} \right) = C_{N,T}^{\nu,\kappa}(t_1) h_N^{(\nu)}(y^{(1)}) \prod_{k=1}^{N} \tilde{G}^{(\nu,\kappa)}(t_k, y^{(1)}_k) \prod_{m=1}^{M} \det_{1 \leq i, k \leq N} \left[ \tilde{p}^{(\nu,\kappa)}(t_{m+1}-t_m, y^{(m+1)}_j y^{(m)}_k) \right],
\]

where

\[
h_N(y) = \prod_{1 \leq i < j \leq N} (y_j - y_i), \quad y \in \mathbb{R}^N,
\]

\[
\tilde{p}^{(\nu,\kappa)}(t, y|0) = \tilde{G}^{(\nu,\kappa)}(t, \sqrt{y}|0) \times \frac{1}{2} y^{-1/2}
\]

\[
= \frac{y^\alpha}{2^{\nu+1} \Gamma(\nu+1) t^{\nu+1}} e^{-y/2t}, \quad y \in \mathbb{R}_+,
\]

\[
\tilde{p}^{(\nu,\kappa)}(t-s, y|x) = \tilde{G}^{(\nu,\kappa)}(t-s, \sqrt{y}|x) \times \frac{1}{2} y^{-1/2}
\]

\[
= \frac{e^{-y/(2(t-s))} \sqrt{y}}{2(t-s)^b} I_\nu, \quad x > 0, \quad y \in \mathbb{R}_+.
\]

Expectations related to the process \( \{Y(t_1)\}, \{Y(t_2)\}, \ldots, \{Y(t_{M+1})\} \) are denoted by \( \mathbb{E}^{Y_{N,T}} \):

\[
\mathbb{E}^{Y_{N,T}} \left[ f(\{Y(t_1)\}, \{Y(t_2)\}, \ldots, \{Y(t_{M+1})\}) \right] = \left( \frac{1}{N!} \right)^{M+1} \int_{\mathbb{R}^{N(M+1)}} \prod_{m=1}^{M+1} dy^{(m)} \times f(\{y^{(1)}\}, \{y^{(2)}\}, \ldots, \{y^{(M+1)}\}) p_{N,T}^{(\nu,\kappa)} \left( 0, \{0\}; t_1, \{y^{(1)}\}; \ldots; t_{M+1}, \{Y^{(M+1)}\} \right).
\]

### 3.2 Fredholm Pfaffian representation of characteristic function and Pfaffian process

For simplicity of expressions, we assume from now on that the number of particles \( N \) is even. The references [33][32] will be useful to give necessary modifications to the following expressions in the case that \( N \) is odd. Let \( C_0(\mathbb{R}) \) be the set of all continuous real functions with compact supports. For \( f = (f_1, f_2, \cdots, f_{M+1}) \in C_0(\mathbb{R})^{M+1} \), \( \theta = (\theta_1, \theta_2, \cdots, \theta_{M+1}) \in \mathbb{R}^{M+1} \), the multitime characteristic function is defined for the process \( \{Y(t)\}, t \in [0, T] \) as

\[
\Psi^{Y_{N,T}}(f; \theta) = \mathbb{E}^{Y_{N,T}} \left[ \exp \left\{ \sqrt{-1} \sum_{m=1}^{M+1} \theta_m \sum_{i=1}^{N} f_m(Y_{i m}(t_m)) \right\} \right].
\]

Let \( \chi_m(x) = e^{\sqrt{-1} \theta_m f_m(x)} - 1, \; 1 \leq m \leq M + 1 \). Then by the definition of multitime correlation function (2.9) with (2.8), we have

\[
\Psi^{Y_{N,T}}(f; \theta) = \sum_{N_1=0}^{N} \cdots \sum_{N_{M+1}=0}^{N} \frac{1}{N^{M+1}} \prod_{m=1}^{M+1} \int_{\mathbb{R}_+^{N_1}} dy^{(1)}_N \int_{\mathbb{R}_+^{N_2}} dy^{(2)}_N \cdots \int_{\mathbb{R}_+^{N_{M+1}}} dy^{(M+1)}_{N_{M+1}} \times \prod_{m=1}^{M+1} \prod_{i=1}^{N} \chi_m(y^{(m)}_i) \rho^{\Psi^{Y_{N,T}}}(t_1; \{Y^{(1)}_{N_1}\}, t_2; \{Y^{(2)}_{N_2}\}, \ldots; t_{M+1}; \{Y^{(M+1)}_{N_{M+1}}\}),
\]
that is, the multitime characteristic function is a generating function of multitime correlation functions \( \rho^Y_{N,T} \).

We consider a vector space \( \mathcal{V} \) with the orthonormal basis \( \{ |m,x\rangle \}_{1 \leq m \leq M+1, x \in \mathbb{R}_+} \), which satisfies

\[
\langle m,x|n,y \rangle = \delta_{mn}\delta(x-y), \quad m,n = 1,2,\ldots,M+1, x,y \in \mathbb{R}_+, \tag{3.6}
\]

where \( \delta_{mn} \) and \( \delta(x-y) \) denote Kronecker’s delta and Dirac’s \( \delta \)-measure, respectively. We introduce the operators \( \hat{J}, \hat{p}, \hat{p}_+, \hat{p}_- \) and \( \hat{\chi} \) acting on \( \mathcal{V} \) as follows

\[
\begin{align*}
\langle m,x|\hat{J}|n,y \rangle &= 1_{(m=n=M+1)}\text{sgn}(y-x), \\
\langle m,x|\hat{p}|n,y \rangle &= 1_{(m<n)}\tilde{p}^{(v,\kappa)}(t_n - t_m, y|x) + 1_{(m>n)}\tilde{p}^{(v,\kappa)}(t_m - t_n, x|y) \\
&\quad + 1_{(m=n)}\delta(x-y), \\
\langle m,x|\hat{p}_+|n,y \rangle &= 1_{(m<n)}\tilde{p}^{(v,\kappa)}(t_n - t_m, y) = \langle n,y|\hat{p}_-|m,x \rangle, \\
\langle m,x|\hat{\chi}|n,y \rangle &= \chi_m(x)\delta_{mn}\delta(x-y), \tag{3.7-3.10}
\end{align*}
\]

and we will use the convention

\[
\langle m,x|\hat{A}|n,y \rangle \langle n,y|\hat{B}|\ell,z \rangle = \sum_{n=1}^{M+1} \int_{\mathbb{R}_+} dy A(m,x;n,y)B(n,y;\ell,z) = \langle m,x|\hat{A}\hat{B}|\ell,z \rangle
\]

for operators \( \hat{A} \) and \( \hat{B} \) with \( \langle m,x|\hat{A}|n,y \rangle = A(m,x;n,y) \) and \( \langle m,x|\hat{B}|n,y \rangle = B(m,x;n,y) \).

Let \( M_i(x) \) be an arbitrary polynomial of \( x \) with degree \( i \) in the form \( M_i(x) = b_i x^i + \cdots \) with a constant \( b_i \neq 0 \) for \( i \in \mathbb{N}_0 \). Since the product of differences \( \tilde{h}_N(x) \) is equal to the Vandermonde determinant, we have

\[
h_N(x) = \left\{ \prod_{k=1}^{N} b_{k-1} \right\}^{-1} \det_{1 \leq i,j \leq N} \left[ M_{i-1}(x_j) \right]. \tag{3.11}
\]

Then we consider the set of linearly independent vectors \( \{ |i\rangle : i \in \mathbb{N} \} \) in \( \mathcal{V} \) defined by

\[
|i\rangle = |m,x\rangle|m,x|i\rangle,
\]

where

\[
\langle m,x|i \rangle = \langle i|m,x \rangle = \int_{\mathbb{R}_+} dy M_{i-1}(y)\tilde{p}^{(v,\kappa)}(t_1, y|0)\tilde{p}^{(v,\kappa)}(t_m - t_1, x|y), \tag{3.12}
\]

\( i \in \mathbb{N}, m = 1,2,\ldots,M+1, x \in \mathbb{R}_+ \). We will use the convention

\[
\langle i|\hat{A}|j \rangle \langle j|\hat{B}|m,x \rangle = \sum_{j=1}^{\infty} A_{ij} B_j^{(m)}(x) = \langle i|\hat{A} \circ \hat{B}|m,x \rangle,
\]

for \( A_{ij} = \langle i|\hat{A}|j \rangle \) and \( B_j^{(m)}(x) = \langle j|\hat{B}|m,x \rangle \). It should be noted that the vectors \( \{ |i\rangle : i \in \mathbb{N} \} \) are not assumed to be mutually orthogonal. By these vectors, however, any operator \( \hat{A} \) on \( \mathcal{V} \) may have a semi-infinite matrix representation \( A = \left( \langle i|\hat{A}|j \rangle \right)_{i,j \in \mathbb{N}} \). If the matrix \( A \) representing an operator \( \hat{A} \) is invertible, we define the operator \( \hat{A}^\Delta \) so that its matrix representation is the inverse of \( A \):

\[
\left( \langle i|\hat{A}^\Delta|j \rangle \right)_{i,j \in \mathbb{N}} = A^{-1}, \tag{3.13}
\]

that is, \( \langle i|\hat{A}|j \rangle \langle j|\hat{A}^\Delta|k \rangle = \langle i|\hat{A} \circ \hat{A}^\Delta|k \rangle = \delta_{ik}, i,k \in \mathbb{N} \).
Let $\mathcal{P}_N$ be a linear operator projecting $\text{Span}\{i; i \in \mathbb{N}\}$ to its $N$-dimensional subspace $\text{Span}\{i; i = 1, 2, \ldots, N\}$ such that

$$\langle i|\mathcal{P}_N|m, x \rangle = \langle m, x|\mathcal{P}_N|i \rangle = \begin{cases} \langle i|m, x \rangle, & \text{if } 1 \leq i \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

We will use the abbreviation $\hat{A}_N = \mathcal{P}_N \hat{A} \mathcal{P}_N$ for an operator $\hat{A}$. If the $N \times N$ matrix defined by $A_N = ((\langle i|\hat{A}_N|j\rangle))_{1 \leq i, j \leq N}$ is invertible, then $(\hat{A}_N) \Delta$ is defined so that $\left( \langle i|(\hat{A}_N) \Delta|j\rangle \right)_{1 \leq i, j \leq N} = (A_N)^{-1}$, and $\langle i|(\hat{A}_N) \Delta|j\rangle = 0$, if $i \geq N + 1$ or $j \geq N + 1$.

As shown in [40], we can prove that

$$\left\{ \Psi_{N,T}(f, \theta) \right\}^2 = \text{Det} \left( I_2 \delta_{mn} \delta(x - y) + \left( \begin{array}{c} \tilde{S}^{m,n}(x, y) \\ \tilde{D}^{m,n}(x, y) \end{array} \right) \lambda_n(y) \right), \quad (3.14)$$

where $\text{Det}$ denotes the Fredholm determinant. Here $I_2$ is the unit matrix with size 2,

$$D^{m,n}(x, y) = -\langle m, x|\hat{J}_N|n, y \rangle,$$

$$S^{m,n}(x, y) = \langle m, x|\hat{\theta}_1 \hat{J}_N|n, y \rangle,$$

and

$$\tilde{S}^{m,n}(x, y) = S^{m,n}(x, y) - \langle m, x|\hat{\theta}_1 \hat{J}_N|n, y \rangle,$$

$$\tilde{I}^{m,n}(x, y) = I^{m,n}(x, y) + \langle m, x|\hat{\theta}_1 \hat{J}_N|n, y \rangle. \quad (3.15)$$

It implies that the multitime characteristic function is given by the Fredholm Pfaffian [40],

$$\Psi_{N,T}(f, \theta) = \text{PF} \left( J_2 \delta_{x,y} + \sqrt{\gamma(x)A^{m,n}(x, y)}\sqrt{\gamma(y)} \right), \quad (3.17)$$

where $J_2 = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ and

$$A^{m,n}(x, y) = J_2 \left( \begin{array}{cc} \tilde{S}^{m,n}(x, y) & \tilde{I}^{m,n}(x, y) \\ \tilde{D}^{m,n}(x, y) & \tilde{S}^{n,m}(y, x) \end{array} \right) = \left( \begin{array}{cc} D^{m,n}(x, y) & \tilde{S}^{m,n}(x, y) \\ -\tilde{S}^{n,m}(y, x) & \tilde{I}^{m,n}(x, y) \end{array} \right). \quad (3.18)$$

It is defined by

$$\text{PF} \left( J_2 \delta_{x,y} + \sqrt{\gamma(x)A^{m,n}(x, y)}\sqrt{\gamma(y)} \right) = \sum_{N_1 = 0}^{N} \sum_{N_2 = 0}^{N} \cdots \sum_{N_{M+1} = 0}^{N} \prod_{m=1}^{M+1} \frac{1}{N_m!} \int_{\mathbb{R}^{N_1}} dy_{N_1}^{(1)} \int_{\mathbb{R}^{N_2}} dy_{N_2}^{(2)} \cdots \int_{\mathbb{R}^{N_{M+1}}} dy_{N_{M+1}}^{(M+1)} \prod_{m=1}^{M+1} \prod_{i=1}^{N_m} \chi_m(y_{i(m)}^{(m)}) \text{Pf} \left( A^{(1)}(y_{N_1}, y_{N_2}, \ldots, y_{N_{M+1}}) \right), \quad (3.19)$$

where $A^{(1)}(y_{N_1}, y_{N_2}, \ldots, y_{N_{M+1}})$ denotes the $2 \sum_{m=1}^{M+1} N_m \times 2 \sum_{m=1}^{M+1} N_m$ skew-symmetric matrices constructed from (3.18) as

$$A^{(1)}(y_{N_1}, y_{N_2}, \ldots, y_{N_{M+1}}) = \left( \begin{array}{cc} A^{m,n}(y_{i(m)}^{(m)}, y_{j(m)}^{(m)}) \end{array}\right)_{1 \leq i, j \leq N_m, 1 \leq n \leq N_m}. \quad (3.18)$$
for $N_m = 1, 2, \ldots, N, 1 \leq m \leq M + 1$. Comparison of (3.12) and (3.14) with (3.19) immediately gives the following statement.

**Theorem 3.1** The $N$-particle non-colliding system of squared generalized meanders $Y(t), t \in [0, T]$ is a Pfaffian process, in the sense that any multitime correlation function is given by a Pfaffian

$$\rho_{N,T}(t_1, \{Y_{N_1}^{(1)}\}; t_2, \{Y_{N_2}^{(2)}\}; \ldots; t_{M+1}, \{Y_{N_{M+1}}^{(M+1)}\}) = \text{Pf} \left( A \left( Y_{N_1}^{(1)}, Y_{N_2}^{(2)}, \ldots, Y_{N_{M+1}}^{(M+1)} \right) \right).$$

### 4 Skew-Orthogonal Functions and Matrix Inversion

#### 4.1 Skew-symmetric inner products

Consider the $N \times N$ skew-symmetric matrix $A_0 = ((A_0)_{ij})_{1 \leq i, j \leq N}$ with

$$(A_0)_{ij} = \langle i|\hat{J}N|j \rangle = \langle i|m, x\rangle \langle m, x|\hat{J}n, y\rangle \langle n, y|j \rangle, \quad i, j = 1, 2, \ldots, N. \quad (4.1)$$

In order to clarify the fact that each element $(A_0)_{ij}$ is a functional of the polynomials $M_{i-1}(x)$ and $M_{j-1}(x)$ through (4.12), we introduce the skew-symmetric inner product

$$\langle f, g \rangle \equiv \int_0^\infty dx \int_0^\infty dy \, F(x, y)p^{(\nu, \kappa)}(t_1, x|0)p^{(\nu, \kappa)}(t_1, y|0)f(x)g(y), \quad (4.2)$$

where

$$F(x, y) = \int_0^\infty dw \int_0^w dz \left| \frac{\tilde{p}^{(\nu, \kappa)}(T - t_1, z|x)}{\tilde{p}^{(\nu, \kappa)}(T - t_1, z|y)} \frac{\tilde{p}^{(\nu, \kappa)}(T - t_1, w|x)}{\tilde{p}^{(\nu, \kappa)}(T - t_1, w|y)} \right|, \quad x, y \in \mathbb{R}_+. \quad (4.3)$$

Then we have the expression

$$(A_0)_{ij} = \langle M_{i-1}, M_{j-1} \rangle, \quad i, j = 1, 2, \ldots, N. \quad (4.4)$$

We now rewrite the skew-symmetric inner product (4.2) by using the simpler one

$$\langle f, g \rangle_+ = -\langle g, f \rangle \equiv \int_0^\infty dw \, e^{-w/2}w^a \int_0^w dz \, e^{-z/2}z^a \left\{ f(z)g(w) - f(w)g(z) \right\}, \quad (4.5)$$

which we call the **elementary skew-symmetric inner product**. Remind that $\tilde{p}^{(\nu, \kappa)}$ is given by (4.2) using the modified Bessel function. We will expand it in terms of the Laguerre polynomials, $L_j^{\nu}(x) = (x^{-\nu}e^x/j!)(d/dx)^j(e^{-x}x^{\nu+j})$, $\alpha \in \mathbb{R}, j \in \mathbb{N}_0$, using the formula

$$\sum_{j=0}^{\infty} \frac{\Gamma(j+1)L_j^{\nu}(x)L_j^{\nu}(y)r^j}{\Gamma(j+1+\nu)} = \frac{1}{1-r} e^{-\frac{(\xi+y)r}{2}}(xyr)^{-\nu/2}I_\nu \left( \frac{2\sqrt{xyr}}{1-r} \right), \quad |r| < 1, \nu > -1. \quad (4.6)$$

(See the corresponding calculation for the non-colliding Brownian particles in [22], where the heat kernel was expanded in terms of the Hermite polynomials.) For this purpose, it is useful to introduce the variables

$$c_n = \frac{t_n(2T - t_n)}{T}, \quad \chi_n = \frac{2T - t_n}{t_n}, \quad n = 1, 2, \ldots, M + 1,$$

since we can see that

$$\tilde{p}^{(\nu, \kappa)}(t_n - t_m, c_n \eta|c_m \xi) = \frac{1}{2(t_n - t_m)} I_\nu \left( \frac{2\sqrt{\xi\eta\chi_n/\chi_m}}{1 - \chi_n/\chi_m} \right) \left( \frac{c_n \eta}{c_m \xi} \right)^{b/2}$$

$$\times \exp \left[ -\left( \frac{1}{1 - \chi_n/\chi_m} - 1 + \frac{t_m}{2T} \right) \xi - \left( \frac{1}{1 - \chi_n/\chi_m} - \frac{t_n}{2T} \right) \eta \right].$$
and, if we apply the formula (4.6) with \( r = \chi_n/\chi_m, x = \xi \) and \( y = \eta \), it is written as

\[
\tilde{p}^{(\nu, \kappa)}(t_n - t_m, c_n\eta|c_m\xi) = \left(\frac{t_m}{t_n}\right)^{\nu+1} e^{-\frac{\nu}{2} - \frac{\nu}{\kappa} + \frac{\nu+1}{\kappa}} (c_n\eta)^{\frac{\nu}{2}} \exp\left[-\frac{t_m}{2T} \xi - (1 - \frac{t_n}{2T})\eta\right] \\
\times \sum_{j=0}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} \left(\frac{\chi_n}{\chi_m}\right)^j L_j^\nu(\xi)L_j^\nu(\eta).
\] (4.7)

That is, \( c_n \) and \( \chi_n \) give the spatial scale of spread of \( N \) particles and the proper temporal factor at time \( t_n \), respectively. (See equation (17) and explanation below it in [34], where the variable \( c_n \) was determined by showing that the one-particle density obeys Wigner’s semicircle law scaled by \( c_n \) for the non-colliding Brownian particles.) In particular, for \( n = M + 1 \) we have

\[
\tilde{p}^{(\nu, \kappa)}(T - t_m, T\eta|c_m\xi) = \left(\frac{t_m}{T}\right)^{\nu+1} e^{-\frac{\nu}{2} - \frac{\nu}{\kappa} + \frac{\nu+1}{\kappa}} (c_n\eta)^{\frac{\nu}{2}} \exp\left[-\frac{t_m}{2T} \xi - (1 - \frac{t_n}{2T})\eta\right] \\
\times e^{-\xi/2} e^{-\eta/2} \sum_{j=0}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} \chi_m^j L_j^\nu(\xi) L_j^\nu(\eta),
\] (4.8)

since \( c_{M+1} = T \) and \( \chi_{M+1} = 1 \). Then we obtain the relation

\[
\left\langle f \left(\frac{\cdot}{c_1}\right), g \left(\frac{\cdot}{c_1}\right) \right\rangle = \frac{2^{-2\nu-2T-\kappa}}{\Gamma(\nu+1)^2} \int_0^\infty dx \int_0^\infty dy \exp[-x^\nu y\nu] f(x)g(y) \\
\times \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \chi_1^{-j-k} L_j^\nu(x)L_k^\nu(y) \left\langle \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} \frac{\Gamma(k+1)}{\Gamma(k+1+\nu)} L_j^\nu \cdot L_k^\nu \right\rangle.
\] (4.9)

### 4.2 Skew-orthogonal polynomials

For \( \alpha \in \mathbb{R} \) and \( n \in \mathbb{Z} \) we define

\[
\binom{n+\alpha}{n} = \begin{cases} 
\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} & \text{if } n \in \mathbb{N}, \alpha \notin \mathbb{Z}_-, \\
\frac{\Gamma(n+\alpha)(-\alpha)}{(-1)^n\Gamma(-\alpha)} & \text{if } n \in \mathbb{N}, n+\alpha \in \mathbb{Z}_-, \\
0 & \text{if } n \in \mathbb{N}, \alpha \in \mathbb{Z}_-, n+\alpha \in \mathbb{N}_0, \\
1 & \text{if } n = 0, \\
0 & \text{if } n \in \mathbb{Z}_-.
\end{cases}
\] (4.10)

Note that for \( n \in \mathbb{N}, \alpha \in \mathbb{Z}_- \) with \( n+\alpha \leq -1 \), \( \binom{n+\alpha}{n} = (-1)^n \binom{-\alpha-1}{n} \). By this definition, the equality

\[
\frac{1}{n!} \left(\frac{d}{dx}\right)^n x^{n+\alpha} \bigg|_{x=1} = \binom{n+\alpha}{n}
\] (4.11)

holds for \( n \in \mathbb{N}_0, \alpha \in \mathbb{R} \). Then Laguerre polynomials can be expressed as

\[
L_j^\alpha(x) = \sum_{\ell=0}^{j} \frac{(-1)^\ell}{\ell!} \binom{j+\alpha}{j-\ell} x^\ell
\] (4.12)

for any \( \alpha \in \mathbb{R} \). Remark that applying 4.111 to the equation

\[
\frac{1}{n!} \left(\frac{d}{dx}\right)^n x^{n+\alpha} = \frac{1}{(n-1)!} \left(\frac{d}{dx}\right)^{n-1} x^{(n-1)+\alpha} + \frac{1}{n!} \left(\frac{d}{dx}\right)^n x^{n+(\alpha-1)},
\]
with \( x = 1 \) and putting \( \beta = \alpha + n \), we have the identity
\[
\binom{\beta}{n} = \binom{\beta - 1}{n} + \binom{\beta - 1}{n - 1}, \quad n \in \mathbb{Z}, \quad \beta \in \mathbb{R}.
\] (4.13)

We introduce the polynomials
\[
F_j(x) = -\frac{d}{dx}L_{j+1}^{2a}(x), \quad j \in \mathbb{N}_0,
\] (4.14)
\[
G_j(x) = \frac{d}{dx}\left\{L_{j+1}^{2a}(x) - \frac{j + 2a}{j}L_{j-1}^{2a}(x)\right\}, \quad j \in \mathbb{N}.
\] (4.15)

For \( k \in \mathbb{N}_0, j = 0, 1, 2, \ldots, k \), let
\[
\alpha_{k,j} = \binom{k - j + b}{k-j}, \quad \text{if } k \text{ is even},
\]
\[
\alpha_{k,j} = \frac{k + 2a}{k} \binom{k - 2 - j + b}{k - 2 - j} - \binom{k - j + b}{k-j}, \quad \text{if } k \text{ is odd},
\] (4.16)

In Appendix B we will give the proof of the following lemmas.

**Lemma 4.1** For \( \ell \in \mathbb{N}_0 \)
\[
F_{2\ell}(x) = \sum_{j=0}^{2\ell} \alpha_{2\ell,j}L_j^\nu(x),
\] (4.17)
\[
G_{2\ell+1}(x) = \sum_{j=0}^{2\ell+1} \alpha_{2\ell+1,j}L_j^\nu(x).
\] (4.18)

**Lemma 4.2** For \( q, \ell \in \mathbb{N}_0 \)
\[
\langle F_{2q}, G_{2\ell+1} \rangle = -\langle G_{2\ell+1}, F_{2q} \rangle = r_q^*\delta_q\ell, \quad (4.19)
\]
\[
\langle F_{2q}, F_{2\ell} \rangle = 0, \quad (4.20)
\]
\[
\langle G_{2q+1}, G_{2\ell+1} \rangle = 0, \quad (4.21)
\]
with
\[
r_q^* = \frac{4\Gamma(2q+2a+2)}{(2q+1)!} = 4\Gamma(2q+1)\binom{2q+2a+1}{2q+1}.
\] (4.22)

Then if we define the monic polynomials in \( x \) of degree \( k \) for \( k \in \mathbb{N}_0 \) as
\[
R_k(x) = k! \left(\frac{x}{c_1}\right)^k \sum_{j=0}^{k} \alpha_{k,j}L_j^\nu \left(\frac{x}{c_1}\right) \chi_1^j,
\] (4.23)

Lemma 4.2 gives the following through the relation (4.9) and the orthogonality of the Laguerre polynomials (B.1).

**Lemma 4.3** For \( q, \ell \in \mathbb{N}_0 \)
\[
\langle R_{2q}, R_{2\ell+1} \rangle = -\langle R_{2\ell+1}, R_{2q} \rangle = r_q\delta_q\ell, \quad (4.24)
\]
\[
\langle R_{2q}, R_{2\ell} \rangle = 0, \quad \langle R_{2q+1}, R_{2\ell+1} \rangle = 0,
\]
where
\[
r_q = 2^{-2\nu - \kappa} \left(\frac{t^2}{1}\right)^{4q+1} \frac{(2q)!\Gamma(2q+2+2a)}{\Gamma(\nu+1)^2}.
\] (4.24)
4.3 Matrix inversion

Let \( b_{2k} = b_{2k+1} = r_k^{-1/2}, k \in \mathbb{N}_0 \), and determine the polynomials \( \{M_i(x)\}_{0 \leq i \leq N-1} \) in (3.12) as

\[
M_i(x) = b_i R_i(x), \quad i = 0, 1, \ldots, N-1.
\]

Then by (4.1), (4.4) and Lemma 4.3, we have the equality

\[
\langle i|J_N|j \rangle = (J_N)_{ij}, \quad i, j = 1, 2, \ldots, N,
\]

(4.25)

where \( J_N = I_{N/2} \otimes J_2 \). It is interesting to compare this result with (3.7). Since \( J_N^2 = -I_N \), we can immediately obtain the inversion matrix appearing in (3.14) as

\[
\langle i|(J_N^\Delta)|j \rangle = -(J_N)_{ij}, \quad i, j = 1, 2, \ldots, N.
\]

(4.26)

If we consider a semi-infinite matrix

\[
J = \lim_{N \to \infty} J_N = \left( \langle i|J|j \rangle \right)_{i,j \in \mathbb{N}},
\]

its inverse matrix may be given by

\[
J^{-1} = \left( \langle i|J^\Delta|j \rangle \right)_{i,j \in \mathbb{N}} = -J.
\]

Using expansions (5.6) and (5.8) with Lemmas 4.1, 4.2 and C.1, we can show

\[
\langle m, x|p\hat{J}|n, y \rangle = \langle m, x|\hat{p}\hat{J}|i \rangle \langle i|J_N^\Delta|j \rangle \langle j|n, y \rangle
\]

and so

\[
\langle m, x|\hat{p}\hat{J}|n, y \rangle = \langle m, x|\hat{p}\hat{J}|i \rangle \langle i|J_N^\Delta|j \rangle \langle j|\hat{p}|n, y \rangle.
\]

Then the equations (3.10) are written as

\[
\hat{S}^{m,n}(x, y) = \begin{cases} 
\langle m, x|\hat{p}\hat{J}|i \rangle \langle i|(J_N^\Delta)|j \rangle \langle j|n, y \rangle, & \text{if } m \geq n, \\
-\langle m, x|\hat{p}\hat{J}|i \rangle \langle i|(J_N^\Delta - (J_N^\Delta)^\Delta)|j \rangle \langle j|n, y \rangle, & \text{if } m < n,
\end{cases}
\]

(4.27)

\[
\hat{I}^{m,n}(x, y) = \langle m, x|\hat{p}\hat{J}|i \rangle \langle i|(J_N^\Delta - (J_N^\Delta)^\Delta)|j \rangle \langle j|\hat{p}|n, y \rangle.
\]

Now we introduce the notations, just following the previous papers for multi-matrix models \[33 \ 15\ 31\], as

\[
R_i^{(m)}(x) = \frac{1}{b_i} \langle m, x|i + 1 \rangle
\]

(4.28)

\[
= \int_0^\infty dy R_i(y) \hat{p}^{(\nu, \kappa)}(t_1, y|0) \hat{p}^{(\nu, \kappa)}(t_m - t_1, x|y),
\]

\[
\Phi_i^{(m)}(x) = -\frac{1}{b_i} \langle m, x|\hat{p}\hat{J}|i + 1 \rangle
\]

(4.29)

\[
= \int_0^\infty dy R_i^{(m)}(y) F^{(m)}(y, x),
\]

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for \( i = 0, 1, \cdots, N - 1, m = 1, 2, \cdots, M + 1 \), where

\[
F^{(m)}(x, y) = \int_{0}^{\infty} dw \int_{0}^{w} dz \left| \frac{\tilde{p}^{(\nu,c)}(T - t_m, z|x)}{\tilde{p}^{(\nu,c)}(T - t_m, z|y)} - \frac{\tilde{p}^{(\nu,c)}(T - t_m, w|x)}{\tilde{p}^{(\nu,c)}(T - t_m, w|y)} \right|. \tag{4.30}
\]

It should be noted that \( R^{(1)}_i(x) = R_i(x)\tilde{p}^{(\nu,c)}(t_1, x|0), 0 \leq i \leq N - 1 \), and \( F^{(1)}(x, y) = F(x, y) \), where \( R_i(x) \) and \( F(x, y) \) were defined by (4.23) and (4.3), respectively. Then we arrive at the following explicit expressions for the elements of matrix kernel (3.18) of our Pfaffian processes,

\[
D^{m,n}(x, y) = D^{m,n}_N(x, y) = \sum_{\ell=0}^{(N/2)-1} \frac{1}{r_{\ell}} \left[ R^{(m)}_{2\ell}(x) R^{(n)}_{2\ell+1}(y) - R^{(m)}_{2\ell+1}(x) R^{(n)}_{2\ell}(y) \right],
\]

\[
\tilde{I}^{m,n}(x, y) = \tilde{I}^{m,n}_N(x, y) = -\sum_{\ell=\lfloor N/2 \rfloor + 1}^{\infty} \frac{1}{r_{\ell}} \left[ \Phi^{(m)}_{2\ell}(x) \Phi^{(n)}_{2\ell+1}(y) - \Phi^{(m)}_{2\ell+1}(x) \Phi^{(n)}_{2\ell}(y) \right],
\]

\[
S^{m,n}(x, y) = S^{m,n}_N(x, y) = \sum_{\ell=0}^{(N/2)-1} \frac{1}{r_{\ell}} \left[ \Phi^{(m)}_{2\ell}(x) R^{(n)}_{2\ell+1}(y) - \Phi^{(m)}_{2\ell+1}(x) R^{(n)}_{2\ell}(y) \right], \tag{4.31}
\]

and

\[
\tilde{S}^{m,n}(x, y) = \tilde{S}^{m,n}_N(x, y) = S^{m,n}(x, y) - \tilde{p}^{(\nu,c)}(t_m - t, y|x) \mathbf{1}_{(m < n)}. \tag{4.32}
\]

## 5 Asymptotic Behavior of Correlation Functions

In this section, we prove the lemma of our main theorem (Theorem 2.1), by estimating the \( N \to \infty \) asymptotic of matrix kernel (3.18) of Theorem 3.1. Elementary calculation needed for the estimation are summarized in Appendix D. Here \( a_N \sim b_N, N \to \infty \) means \( a_N/b_N \to 1, N \to \infty \). We assume that \( T = N, t_m = T + s_m, 1 \leq m \leq M + 1 \) with \( s_1 < s_2 < \cdots < s_M < s_{M+1} = 0 \). We put

\[
L^\nu_j (x - s_m) = L^\nu_j (x) \chi^j_m, \quad \text{and} \quad \tilde{L}^\nu_j (x, s_m) = \frac{\Gamma(j + 1)}{\Gamma(j + \nu + 1)} L^\nu_j (x) \chi^{j-1}_m. \tag{5.1}
\]

### 5.1 Asymptotics of \( R_k(x) \) and \( \tilde{R}^{(m)}_k(x) \)

Let

\[
\tilde{R}^\nu_k(x) = \frac{1}{k!} \left( \frac{t^2}{T} \right)^{-k} R_k(x) = \sum_{j=0}^{k} \alpha_{k,j} L^\nu_j (x, c_1, s_1). \]

Since \( c_1 \sim N = T \),

\[
\tilde{R}_{2\ell}(x) \sim I(2\ell, b),
\]

\[
\tilde{R}_{2\ell+1}(x) \sim \frac{2a}{2\ell+1} \left[ I(2\ell - 1, b) - I(2\ell + 1, b - 1) - I(2\ell, b - 1) \right] \sim \frac{a}{\ell} I(2\ell, b) - 2I(2\ell, b - 1), \quad N \to \infty, \tag{5.2}
\]

where

\[
I(q, c) = \sum_{j=0}^{q} \binom{q - j + c}{q - j} L^\nu_j \left( \frac{x}{N}, -s_1 \right) = \sum_{j=0}^{q} \binom{j + c}{j} L^\nu_{q-j} \left( \frac{x}{N}, -s_1 \right)
\]

for \( q \in \mathbb{N} \) and \( c \in \mathbb{R} \). We set

\[
2\ell = N\theta,
\]

\[
t_1 = N + 1, \quad t_2 = N + 2, \quad \ldots, \quad t_{M+1} = 2N - 1.
\]
and examine the asymptotic behavior of $I(2\ell, c)$ as $N \to \infty$ with some $\theta \in (0, \infty)$. When $c \in \mathbb{Z}_-$, 
\[
(j + c) \choose j = (-1)^j \binom{-c - 1}{j}.
\]
Then from (D.10) in Lemma D.2 with $j = 2\ell$ (i.e. $\eta = 1$ in (D.5)), we can easily see that
\[
I(2\ell, c) = \sum_{j=0}^{\ell-c} \binom{j + c}{j} L_{2\ell - j}^\nu \left( x/N, -s_1 \right) \sim \frac{(N\theta)^{c+\nu+1}}{(\theta x)^\nu} J_{\nu-\ell}^{(-c-1)}(\theta, 1, x, -s_1), \quad N \to \infty.
\]
This result is generalized to the following lemma.

**Lemma 5.1** For any $c \in \mathbb{R}$, $\theta \in (0, \infty)$, we have
\[
I(2\ell, c) \sim \frac{(N\theta)^{c+\nu+1}}{(\theta x)^\nu} J_{\nu-\ell}^{(-c-1)}(\theta, 1, x, -s_1), \quad N \to \infty. \tag{5.3}
\]

**Proof.**
\[
I(2\ell, c) = \sum_{p=0}^{2\ell} \binom{p + c}{p} L_{2\ell - p}^\nu \left( x/N, -s_1 \right)
\]
\[
= \sum_{p=0}^{2\ell} \binom{p + c}{p} \left\{ L_{2\ell - p}^\nu \left( x/N, -s_1 \right) - L_{2\ell}^\nu \left( x/N, -s_1 \right) \right\} + \sum_{p=0}^{2\ell} \binom{p + c}{p} L_{2\ell}^\nu \left( x/N, -s_1 \right)
\]
\[
= \sum_{p=0}^{2\ell} \binom{p + c}{p} \sum_{k=0}^{p-1} \left\{ L_{2\ell - k - 1}^\nu \left( x/N, -s_1 \right) - L_{2\ell - k}^\nu \left( x/N, -s_1 \right) \right\} + \sum_{p=0}^{2\ell} \binom{p + c}{p} L_{2\ell}^\nu \left( x/N, -s_1 \right)
\]
\[
= \sum_{p=0}^{2\ell} \binom{p + c}{p} \sum_{k=0}^{p-1} \sum_{q=0}^{1} (-1)^q \binom{1}{q} L_{2\ell - k - q}^\nu \left( x/N, -s_1 \right) + \sum_{p=0}^{2\ell} \binom{p + c}{p} L_{2\ell}^\nu \left( x/N, -s_1 \right).
\]
Repeating this procedure, we have
\[
I(2\ell, c) = \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} (-1)^k a_k(2\ell, c) \sum_{p=0}^{2\ell} \binom{p + c}{p} \binom{k}{q} L_{2\ell - q}^\nu \left( x/N, -s_1 \right) \tag{5.4}
\]
with
\[
a_k(2\ell, c) = \sum_{p=0}^{2\ell} \binom{p + c}{p} \sum_{j_1=0}^{p-1} \sum_{j_2=0}^{j_1-1} \cdots \sum_{j_k=0}^{j_{k-1}-1} 1 = \sum_{p=0}^{2\ell} \binom{p + c}{p} \binom{p}{k}.
\]
Using (4.13), we can rewrite $a_k(2\ell, c)$ as
\[
a_k(2\ell, c) = \sum_{p=0}^{2\ell} \left\{ \binom{p + c + 1}{p} - \binom{p + c}{p - 1} \right\} \binom{p}{k}
\]
\[
= \sum_{p=0}^{2\ell} \left[ \binom{p + c + 1}{p} \binom{p}{k} - \binom{p + c}{p - 1} \binom{p - 1}{k} + \binom{p + c}{p - 1} \left\{ \binom{p - 1}{k} - \binom{p}{k} \right\} \right]
\]
\[
= \binom{2\ell + c + 1}{2\ell} \binom{2\ell}{k} - a_{k-1}(2\ell - 1, c + 1).
\]
Using this equation recursively, we obtain
\[
a_k(2\ell, c) = \sum_{r=0}^{k} (-1)^r \binom{2\ell + c + 1}{2\ell - r} \binom{2\ell - r}{k - r} = \binom{2\ell + c + 1}{2\ell - k} \binom{c + k}{k}. \tag{5.5}
\]
Suppose that
\[ I(2\ell, c) = \sum_{k=0}^{\infty} (-1)^k \binom{2\ell + c + 1}{2\ell - k} \left( \frac{c + k}{k} \right) \sum_{q=0}^{k} (-1)^q \binom{k}{q} \int_{-s_1}^{x} L_{2\ell - q}^{\nu} \left( \frac{x}{N} \right). \]

By simple calculation with the estimate (D.2) and (D.10) of Lemma D.2, we obtain (5.3) through the expression (5.4).

Thus (5.4) with (5.5) gives
\[ \hat{R}_{2\ell}(x) \sim \frac{(N\theta)^{2a+1}}{(\theta x)^{\nu}} J_{\nu}^{(-b-1)}(\theta, 1, x, -s_1), \quad N \to \infty. \]

From above asymptotic of \( I(2\ell, c) \) with equations (5.2) we have the following proposition.

**Proposition 5.2**

1. Suppose that \( \ell \in \mathbb{N} \) and \( 2\ell \sim N\theta, N \to \infty \) for some \( \theta \in (0, \infty) \). Then
   \[ \hat{R}_{2\ell+1}(x) \sim \frac{2(N\theta)^{b+\nu}}{(\theta x)^{\nu}} \left[ a J_{\nu}^{(-b-1)}(\theta, 1, x, -s_1) - \tilde{J}_{\nu}^{(-b)}(\theta, 1, x, -s_1) \right], \quad N \to \infty. \]

We next examine asymptotic of \( R_{k}^{(m)}(x) \). From the definition (4.28) and the expression (4.7)
\[ R_{k}^{(m)}(x) = c_1 \int_{0}^{\infty} d\eta R_k(c_1\eta) \theta^{(\nu,\kappa)}(t_1, c_1\eta) p^{(\nu,\kappa)}(t_m - t_1, x) c_1 \]
\[ = \frac{k!}{2^{\nu+1} \Gamma(\nu + 1)} \left( \frac{c_1}{\chi_1} \right)^k \left( \frac{1}{t_m} \right)^{\nu+1} x^a \exp \left[ \left( -1 + \frac{t_m}{2T} \right) \frac{x}{c_m} \right] \sum_{j=0}^{k} \alpha_{k,j} J_{\nu}^{\nu} \left( \frac{x}{c_m}, -s_m \right). \]

We put
\[ \hat{R}_{k}^{(m)}(x) = \frac{2^{\nu+1} \Gamma(\nu + 1)}{\Gamma(\nu + 1 + 2a)} \left( \frac{\chi_1}{c_1} \right)^k \hat{R}_{k}^{(m)}(x). \]

If we set \( k \sim N\theta \) as \( N \to \infty \), (D.1) in Appendix D gives
\[ \frac{k! \Gamma^{\nu}}{\Gamma(\nu + 1 + 2a)} \left( \frac{1}{t_m} \right)^{\nu+1} \exp \left[ \left( -1 + \frac{t_m}{2T} \right) \frac{x}{c_m} \right] \sim N^{-(2a+1)} \theta^{-2a}, \quad N \to \infty, \]
then we obtain the following from Proposition 5.2.

**Proposition 5.3**

1. Suppose that \( \ell \in \mathbb{N} \) and \( 2\ell \sim N\theta, N \to \infty \) for some \( \theta \in (0, \infty) \). Then
   \[ \hat{R}_{2\ell}^{(m)}(x) \sim \theta^{1-\nu} x^{-\kappa/2} \tilde{J}_{\nu}^{(-b-1)}(\theta, 1, x, -s_m), \quad N \to \infty. \]

2. Suppose that \( \ell \in \mathbb{N} \) and \( 2\ell + 1 \sim N\theta, N \to \infty \) for some \( \theta \in (0, \infty) \). Then
   \[ \hat{R}_{2\ell+1}^{(m)}(x) \sim \frac{\theta^{1-\nu} x^{-\kappa/2}}{N} \left[ a \tilde{J}_{\nu}^{(-b-1)}(\theta, 1, x, -s_m) - \tilde{J}_{\nu}^{(-b)}(\theta, 1, x, -s_m) \right], \quad N \to \infty. \]
5.2 Asymptotics of $\Phi_k^{(m)}(x)$

Using (4.8), (4.20) is rewritten as

$$F^{(m)}(y, x) = \left(\frac{1}{T}\right)^{\kappa} \left(\frac{t_m}{c_m}\right)^{2(\nu+1)} (xy)^{\kappa/2} \exp\left\{ -\frac{t_m}{T} \frac{x+y}{2c_m} \right\} \times \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} \langle \tilde{L}_p^\nu, L_j^\nu \rangle_{L_p} \left( \frac{y}{c_m}, s_m \right) \tilde{L}_j^\nu \left( \frac{x}{c_m}, s_m \right).$$

Hence from (4.29) and (5.6), we have

$$\Phi_k^{(m)}(x) = c_m \int_0^\infty d\eta R_k^{(m)}(c_m \eta) F^{(m)}(c_m \eta, x)$$

$$= \frac{k!}{2^{\nu+1} \Gamma(\nu+1)} \left( \frac{c_1}{c_m} \right)^k \left( \frac{1}{c_m} \right)^{\nu+1} \frac{t_m+1}{T^{\nu+1}} x^{\kappa/2} \exp\left\{ -\frac{t_m x}{2Tc_m} \right\} \times \sum_{j=0}^{\infty} \sum_{p=0}^{k} \alpha_{k,p} L_p^\nu, L_j^\nu \tilde{L}_j^\nu \left( \frac{x}{c_m}, s_m \right) \right), \tag{5.8}$$

where we have used the orthogonal relation (3.1) of Laguerre polynomials. Put

$$\tilde{\Phi}_k^{(m)}(x) = \frac{2^{\nu} T^{-\nu} \Gamma(\nu+1)}{k!} \left( \frac{\chi_1}{c_1} \right)^k \Phi_k^{(m)}(x). \tag{5.9}$$

Then we have the following proposition.

**Proposition 5.4**

1. Suppose that $\ell \in \mathbb{N}$ and $2\ell \sim N\theta$, $N \to \infty$ for some $\theta \in (0, \infty)$. Then

$$\tilde{\Phi}^{(m)}_{2\ell}(x) \sim -\theta^{\nu} x^{\kappa/2} \int_1^\infty d\xi \xi^{\kappa} \tilde{J}_\nu^{(b+1)}(\theta, \xi, x, s_m), \quad N \to \infty. \tag{5.10}$$

2. Suppose that $\ell \in \mathbb{N}_0$ and $2\ell+1 \sim N\theta$, $N \to \infty$ for some $\theta \in (0, \infty)$. Then

$$\tilde{\Phi}^{(m)}_{2\ell+1}(x) \sim -\frac{2\theta^{-1+\nu} x^{\kappa/2}}{N} \tilde{J}_\nu^{(b+1)}(\theta, 1, x, s_m), \quad N \to \infty. \tag{5.11}$$

**Proof.**

$$\tilde{\Phi}_k^{(m)}(x) \sim \frac{T^{-\nu} x^{\kappa/2}}{2} \sum_{j=0}^{\infty} \sum_{p=0}^{k} \langle \alpha_{k,p} L_p^\nu, L_j^\nu \rangle_{L_p} \tilde{L}_j^\nu \left( \frac{x}{N}, s_m \right), \quad N \to \infty,$$

$$= \frac{T^{-\nu} x^{\kappa/2}}{2} \sum_{j=0}^{\infty} \sum_{q=0}^{j} \beta_j Q_q \tilde{L}_j^\nu \left( \frac{x}{N}, s_m \right).$$

Here we have used the notation (6.1) and introduced $\beta = (\beta_{j,k})$, which is the inverse of the matrix $\alpha = (\alpha_{k,j})$ given by Lemma C.1 in Appendix C. By the skew orthogonality of $\{Q_k\}$ given by Lemma 4.2, we have

$$\tilde{\Phi}_{2\ell}^{(m)}(x) \sim \frac{T^{-\nu} x^{\kappa/2} \nu^\nu}{2} \sum_{j=0}^{\infty} \beta_{2\ell+1} \tilde{L}_j^\nu \left( \frac{x}{N}, s_m \right), \quad N \to \infty, \tag{5.12}$$

$$\tilde{\Phi}_{2\ell+1}^{(m)}(x) \sim -\frac{T^{-\nu} x^{\kappa/2} \nu^\nu}{2} \sum_{j=0}^{\infty} \beta_{2\ell} \tilde{L}_j^\nu \left( \frac{x}{N}, s_m \right), \quad N \to \infty. \tag{5.13}$$

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By (4.22) and (C.4), (5.13) gives
\[ \hat{H}^{(m)}_{2\ell+1}(x) \sim -2\Gamma(2a+1)T^{-a}x^{\kappa/2}(2\ell + 1 + 2a) \times \sum_{j=2\ell}^{\infty} \binom{j - 2\ell - b - 2}{j - 2\ell} \hat{L}_j^\nu \left( \frac{x}{N}, s_m \right), \quad N \to \infty. \] (5.14)

From (D.8) we have
\[ \sum_{j=2\ell}^{\infty} \binom{j - 2\ell - b - 2}{j - 2\ell} \hat{L}_j^\nu \left( \frac{x}{N}, s_m \right) = \sum_{r=0}^{\infty} \binom{r - b - 2 + \alpha}{r} \hat{L}_{2\ell + r}^\nu \left( \frac{x}{N}, s_m \right) \sim (2\ell)^{-\alpha} \sum_{r=0}^{\infty} \binom{r - b - 2 + \alpha}{r} J^{(\alpha)}(\theta, \eta, x, s_m), \quad N \to \infty, \] (5.15)

where (D.11) of Lemma D.2 was applied. Setting \( j = 2\ell \eta = N\theta \eta \) and using (D.2) in Appendix D we conclude that
\[ \hat{H}^{(m)}_{2\ell+1}(x) \sim -2\Gamma(2a+1)T^{-a}x^{\kappa/2} \sum_{j=2\ell}^{\infty} \binom{j - 2\ell + 1}{j - 2\ell} \hat{L}_j^\nu \left( \frac{x}{N}, s_m \right) \]
\[ \sim -\frac{2\theta^{-1+a}x^{\kappa/2}}{\Gamma(-b-1+\alpha)} \int_1^{\infty} d\eta \frac{J^{(\alpha)}(\theta, \eta, x, s_m)}{(\eta - 1)^{b+2-\alpha}}, \quad N \to \infty. \] (5.16)

Through the expression (2.17) we obtain (5.14).

By (C.5) of Lemma C.1 with (C.2)
\[ \sum_{j=2\ell+1}^{\infty} \beta_{j,2\ell+1} \hat{L}_j^\nu \left( \frac{x}{N}, s_m \right) \]
\[ = -\frac{1}{b(1,2\ell+1)} \sum_{j=2\ell+1}^{\infty} \binom{r - b - 2 + \alpha}{r} \hat{L}_j^\nu \left( \frac{x}{N}, s_m \right) \binom{j - 2r - b - 1}{j - 2r + 1} \]
\[ = -\frac{1}{b(1,2\ell+1)} S(\ell), \]

where
\[ S(\ell) = \sum_{r=\ell+1}^{\infty} b(1,2r-1) \sum_{j=2r-1}^{\infty} \hat{L}_j^\nu \left( \frac{x}{N}, s_m \right) \binom{j - 2r - b - 1}{j - 2r + 1}. \]

By this equation with the estimate (D.3) for \( b(1,2r-1) \) and (4.22), (5.12) becomes
\[ \hat{H}_{2\ell}(x) \sim -2\Gamma(2\ell + 2)^{-a} \Gamma(2a + 1) \left( \frac{2\ell + 1 + 2a}{2\ell + 1} \right) x^{\kappa/2} S(\ell) \]
\[ \sim -2\Gamma(2\ell + 2)^{-a} x^{\kappa/2} S(\ell), \quad N \to \infty. \] (5.17)

From (5.15) with (D.3)
\[ S(\ell) \sim \sum_{r=\ell+1}^{\infty} (2r)^{a} \binom{1}{2\ell} \sum_{j=2r-1}^{\infty} \binom{j - 2r + 1}{j - 2r} \hat{L}_j^\nu \left( \frac{x}{N}, s_m \right) \]
\[ \sim \frac{(2\ell)^{\kappa/2}}{2\Gamma(-b-1+\alpha)} \int_1^{\infty} d\xi \int_1^{\infty} \frac{d\eta}{(\eta - \xi)^{b+2-\alpha}}, \quad N \to \infty. \]

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Thus
\[
\hat{\Phi}_2^{(m)}(x) \sim -\frac{\theta^\nu x^{\kappa/2}}{\Gamma(-b-1+\alpha)} \int_1^\infty d\xi \frac{\xi^a}{\Gamma(a+b-\alpha)} \int_1^\infty d\eta \frac{\hat{J}_\nu^{(1)}(\theta, \eta, x, s_m)}{(\eta-\xi)^{b+2-\alpha}}
\]
\[
= -\theta^\nu x^{\kappa/2} \int_1^\infty d\xi \frac{\xi^a \hat{J}_\nu^{(1)}(\theta, \xi, x, s_m)}{(\eta-\xi)^{b+2-\alpha}}, \quad N \to \infty,
\]
where we used the expression \((2.17)\). This completes the proof of Proposition 5.4.

5.3 Asymptotics of \(D^{m,n}(x, y)\), \(\tilde{I}^{m,n}(x, y)\), \(S^{m,n}(x, y)\) and \(\tilde{S}^{m,n}(x, y)\)

From Propositions 5.3 and 5.4 we obtain the following asymptotics:

\[
D^{m,n}_N(x, y) \sim \sum_{\ell=0}^{(N/2)-1} \left( \frac{2\ell}{N} \right)^{2a} \left[ \hat{R}_2^{(m)}(x) \hat{R}_{2\ell+1}^{(n)}(y) - \hat{R}_2^{(m)}(x) \hat{R}_{2\ell+1}^{(n)}(y) \right],
\]
\[
\tilde{I}^{m,n}_N(x, y) \sim -\sum_{\ell=0}^{(N/2)-1} \left( \frac{2\ell}{N} \right)^{-2a} \left[ \hat{\Phi}_2^{(m)}(x) \hat{\Phi}_{2\ell+1}^{(n)}(y) - \hat{\Phi}_2^{(m)}(x) \hat{\Phi}_{2\ell+1}^{(n)}(y) \right],
\]
\[
S^{m,n}_N(x, y) \sim \sum_{\ell=0}^{(N/2)-1} \left[ \hat{\Phi}_2^{(m)}(x) \hat{R}_{2\ell+1}^{(n)}(y) - \hat{\Phi}_2^{(m)}(x) \hat{R}_{2\ell+1}^{(n)}(y) \right], \quad N \to \infty.
\]

From Propositions 5.3 and 5.4 we obtain the following asymptotics:

\[
D^{m,n}_N(x, y) \sim D(s_m, x; s_n, y),
\]
\[
\tilde{I}^{m,n}_N(x, y) \sim \tilde{I}(s_m, x; s_n, y),
\]
\[
S^{m,n}_N(x, y) \sim S(s_m, x; s_n, y), \quad N \to \infty,
\]
where \(D, \tilde{I}, S\) are defined by \((2.19)\).

Next we study the asymptotic behavior of \(\overline{p}^{(\nu, \kappa)}(t_n - t_m, y|x)\). From \((1.4)\) we have
\[
\overline{p}^{(\nu, \kappa)}(t_n - t_m, y|x) = \left( \frac{t_m}{t_n} \right)^{\nu+1} e_m^{a-1} \left( \frac{x}{e_m} \right)^{\kappa/2} \exp \left[ \frac{t_m x}{2T_e_m} \right] \exp \left[ \left( -2 + \frac{t_n}{T} \right) \frac{y}{2e_n} \right]
\]
\[
\times \sum_{j=0}^\infty \bar{L}_j \left( \frac{x}{e_m} s_m \right) L_j \left( \frac{y}{e_n} s_n \right).
\]

Then by simple calculation with Lemma \((1.22)\) with \(\alpha = 0\), we have
\[
\overline{p}^{(\nu, \kappa)}(t_n - t_m, y|x) \sim \left( \frac{y}{x} \right)^{b/2} \sum_{j=0}^1 \frac{1}{N} \exp \left[ 2(s_m - s_n)\theta \eta \right] J_{\nu}(2\sqrt{\theta \eta} x) J_{\nu}(2\sqrt{\theta \eta} y),
\]
\[
\sim \left( \frac{y}{x} \right)^{b/2} \mathcal{G}(s_m, x; s_n, y), \quad N \to \infty,
\]
where \(\mathcal{G}\) is defined by \((2.21)\), and then \(\tilde{S}^{m,n}_N(x, y) \sim \tilde{S}(s_m, x; s_n, y), \quad N \to \infty\). Then, the proof of Theorem \((2.1)\) is completed.
Appendices

A Proof of (3.14)

We assume that the number of particles \( N \) is even. Consider the multiple integral

\[
Z_{N,T}[\chi] = \left( \frac{1}{N!} \right)^{M+1} \int_{\mathbb{R}^{N(M+1)}} \prod_{m=1}^{M+1} dx^{(m)} \det_{1 \leq i,j \leq N} \left[ M_{i-1}(x^{(1)}_j (t_1, x^{(1)}_j | 0) (1 + \chi(x^{(1)}_j))) \right] \\
\times \prod_{m=1}^{M} \det_{1 \leq i,j \leq N} \left[ \hat{p}^{(\nu, \kappa)}(t_{m+1} - t_m, x^{(m)}(t_m)|x^{(m)}_i) (1 + \chi_{m+1}(x^{(m+1)})) \right] \text{sgn} (h_N(x^{(M+1)})).
\]

By the definition (3.4) with (3.3) and (3.1), and by the equality (3.11), we have

\[
\Psi_{N,T}(\mathbf{f}, \mathbf{\theta}) = \frac{Z_{N,T}[\chi]}{Z_{N,T}[0]},
\]

where \( Z_{N,T}[0] \) is obtained from \( Z_{N,T}[\chi] \) by setting \( \chi_m(x) \equiv 0 \) for all \( m = 1, 2, \cdots, M + 1 \).

By repeated applications of the Heine identity

\[
\int_{\mathbb{R}^N} dx \det_{1 \leq i,j \leq N} \phi_i(x_j) \phi_i(x_j) = \det_{1 \leq i,j \leq N} \left[ \int_{\mathbb{R}^N} dx \phi_i(x) \phi_j(x) \right]
\]

for square integrable continuous functions \( \phi_i, \tilde{\phi}_i, 1 \leq i \leq N, \) we have

\[
Z_{N,T}[\chi] = \int_{\mathbb{R}^N} dy \det_{1 \leq i,j \leq N} \left[ \int_{\mathbb{R}^{M+1}} \prod_{m=1}^{M+1} dx^{(m)} \left( M_{i-1}(x^{(1)}_j (t_1, x^{(1)}_j | 0) (1 + \chi(x^{(1)}_j))) \right) \right] \\
\times \prod_{m=1}^{M} \left( \hat{p}^{(\nu, \kappa)}(t_{m+1} - t_m, x^{(m)}(t_m)|x^{(m)}_i) (1 + \chi_{m+1}(x^{(m+1)})) \right) \delta(y_j - x^{(M+1)}).
\]

Using the notations in Section 3.2, it is expressed as

\[
Z_{N,T}[\chi] = \int_{\mathbb{R}^N} dy \det_{1 \leq i,j \leq N} \left[ \langle i | \left( 1 + \frac{1}{1 - \chi^{(1)}_i \hat{p}} \right) \hat{\chi} \right]_{N} | M + 1, y_j \rangle \right] \\
= \int_{\mathbb{R}^N} dy \det_{1 \leq i,j \leq N} \left[ \langle M + 1, y_i | \left( 1 + \hat{\chi} \frac{1}{1 - \hat{p}^{-1}} \right) \right]_{N} | j \rangle \right],
\]

since \( \langle m, x | (\hat{p}^+)^k | n, x \rangle = \langle m, y | (\hat{p}^-)^k | n, x \rangle \equiv 0 \) for \( k > n - m \geq 0 \). Here we have used the Chapman-Kolmogorov equation, \( \int_{\mathbb{R}_+} dy \hat{p}^{(\nu, \kappa)}(t-s,y|x) \hat{p}^{(\nu, \kappa)}(u-t,z|x) = \hat{p}^{(\nu, \kappa)}(u-s,z|x), 0 \leq s \leq t \leq u \leq T, x, y \in \mathbb{R}_+ \). Next we use the formula of de Brujin \[3].

\[
\int_{\mathbb{R}_+^N} dy \det_{1 \leq i,j \leq N} \phi_i(y_j) = \text{Pf}_{1 \leq i,j \leq N} \left[ \int_{\mathbb{R}_+} dy \int_{\mathbb{R}_+} dy \text{sgn}(\hat{y} - y) \phi_i(y) \phi_j(y) \right]
\]

for integrable continuous functions \( \phi_i, 1 \leq i \leq N, \) in which the Pfaffian is defined by \[3]. Since \( (\text{Pf}(A))^2 = \det A \) for any even-dimensional skew-symmetric matrix \( A \), we have

\[
\left( Z_{N,T}[\chi] \right)^2 = \det_{1 \leq i,j \leq N} \left[ \langle i | \left( 1 + \frac{1}{1 - \chi^{(1)}_i \hat{p}} \right) \hat{\chi} \right]_{N} | j \rangle \right]_2 \\
= \det_{1 \leq i,j \leq N} \left[ (A_0)_{ij} + (A_1)_{ij} + (A_2)_{ij} + (A_3)_{ij} \right]
\]

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with

\[
(A_0)_{ij} = \langle i|J_N|j \rangle, \\
(A_1)_{ij} = \langle i| \left( \frac{1}{1 - \chi p_+} \hat{\chi} \hat{p} \right)_N |j \rangle = \langle i| \left( \hat{\chi} \frac{1}{1 - \hat{p} - \hat{\chi}} \right)_N |j \rangle, \\
(A_2)_{ij} = \langle i| \left( \hat{p} \hat{\chi} \hat{J} \frac{1}{1 - \hat{p} - \hat{\chi}} \right)_N |j \rangle, \\
(A_3)_{ij} = \langle i| \left( \hat{\chi} \frac{1}{1 - \hat{p} - \hat{\chi}} \right)_N |j \rangle.
\]

Then (3.14) is derived.

By our notation (3.13), (A.3) gives

\[
\left\{ \Psi_{N,T}(f; \theta) \right\}^2 = \det_{1 \leq i,j \leq N} \left[ (A_0)_{ij} \right],
\]

(A.1) gives

\[
\left\{ \Psi_{N,T}(f; \theta) \right\}^2 = \det_{1 \leq i,j \leq N} \left[ \delta_{ij} + (A_0^{-1} A_1)_{ij} + (A_0^{-1} A_2)_{ij} + (A_0^{-1} A_3)_{ij} \right].
\]

(A.2)

By our notation (3.13), (A_0^{-1})_{ij} = \langle i|(J_N) \hat{\chi} \rangle|j \rangle, and it is easy to confirm that (A.2) is written in the form

\[
\left\{ \Psi_{N,T}^Y (f; \theta) \right\}^2 = \det_{1 \leq i,j \leq N} \left[ \delta_{ij} + \langle i|B|m, x \rangle \langle m, x|C|j \rangle \right].
\]

(A.3)

where we have introduced B and C as the following two-dimensional row and column vector-valued operators,

\[
B = \begin{pmatrix}
(\hat{J}_N) \hat{\chi} \frac{1}{1 - \hat{p} + \hat{\chi}} & -(\hat{J}_N) \hat{\chi} \left( 1 + \frac{1}{1 - \hat{p} + \hat{\chi}} \right) \hat{J} \hat{\chi}
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
\hat{p} \hat{J} \\
\frac{-1}{1 - \hat{p} + \hat{\chi}}
\end{pmatrix}.
\]

The determinant (A.3) is equivalent with the Fredholm determinant,

\[
\text{Det}(m, x|I_2 + C \circ B|n, y).
\]

Introducing matrix-valued operators,

\[
K_+ = \begin{pmatrix}
1 - \hat{p}_+ \hat{\chi} & 0 \\
0 & 1
\end{pmatrix}, \quad K_- = \begin{pmatrix}
1 & 0 \\
0 & 1 - \hat{p}_- \hat{\chi}
\end{pmatrix}, \quad \hat{K} = \begin{pmatrix}
1 & -\hat{p}_+ \hat{\chi} \\
0 & 1
\end{pmatrix},
\]

we have

\[
I_2 + C \circ B = K_-^{-1} \left[ K_- K_+ + \left( \hat{p} \hat{J} \hat{J}_N^\Delta - \hat{p}_+ \hat{J} \hat{J}_N \hat{J}_N^\Delta \hat{J}_N \hat{p} \right) \hat{K} \right] K_+^{-1} \hat{K} = K_-^{-1} \left[ I_2 + \left( \hat{p} \hat{J} \hat{J}_N^\Delta - \hat{p}_+ + \hat{p} \hat{J} - \hat{p}_+ \hat{J} \hat{J}_N \hat{J}_N^\Delta \hat{J}_N \hat{p} \right) \hat{K} \right] K_+^{-1} \hat{K},
\]

where \( \hat{J}_N = \circ (\hat{J}_N) \hat{\chi} \hat{\chi} \). From the orthogonality (3.10) and the definitions (3.9) of the operators \( \hat{p}_+ \) and \( \hat{p}_- \), we have the fact that

\[
\text{Det}(m, x|K_+|n, y) = \text{Det}(m, x|K_-|n, y) = \text{Det}(m, x|\hat{K}|n, y) = 1.
\]

Then (3.14) is derived.
B Proofs of Lemmas 4.1 and 4.2

We use the following orthogonal relations and formulae on Laguerre polynomials, which hold for \( \alpha, \beta > -1; \)

\[
\int_0^\infty L_j^\alpha(x)L_k^\beta(x)x^\alpha e^{-x}dx = \frac{\Gamma(\alpha + j + 1)}{\Gamma(j + 1)}\delta_{jk}, \quad j, k \in \mathbb{N}_0, \tag{B.1}
\]

\[
x \frac{d}{dx} L_j^\alpha(x) = j L_j^\alpha(x) - (j + \alpha)L_{j-1}^\alpha(x), \quad j \in \mathbb{N}, \tag{B.2}
\]

\[
L_j^\alpha(x) = -\frac{d}{dx} L_{j+1}^\alpha(x) + \frac{d}{dx} L_j^\alpha(x), \quad j \in \mathbb{N}_0, \tag{B.3}
\]

\[
L_j^\beta(x) = \sum_{k=0}^j \binom{j - k + \beta - 1}{j - k} L_k^\alpha(x), \quad j \in \mathbb{N}_0. \tag{B.4}
\]

**Remark 6.** The identities (B.2) and (B.3) are given as Eqs. (6.2.6) and (6.2.7) in [3]. The relation (B.4)
is proved in [1] as (6.2.37) only when \( \beta \geq \alpha > -1. \) The identity (see (4.10) and [22])

\[
\sum_{\ell=0}^k \binom{\ell - \alpha - 1}{\ell} \binom{k - \ell + \alpha - 1}{k - \ell} = \delta_{k0},
\]

can be used to invert the relation (B.4) to the form

\[
L_j^\alpha(x) = \sum_{\ell=0}^j \binom{\ell - j + \alpha - 1}{\ell - j} L_j^\beta(x), \quad \ell \in \mathbb{N}_0.
\]

Therefore, the validity of (B.4) for \( \beta \geq \alpha > -1 \) implies that for \( \alpha > \beta > -1. \)

**B.1 Proof of Lemma 4.1**

In this subsection we prove Lemma 4.1, which gives the expansion formulae of \( F_k(x) \) and \( G_k(x) \) in terms of \( \{L_j^\nu(x)\}. \) Taking the summation of the equalities (B.3) from 0 to \( k \in \mathbb{N}_0 \) gives

\[
\sum_{n=0}^k L_n^{2a}(x) = -\frac{d}{dx} L_{k+1}^{2a}(x) + \frac{d}{dx} L_0^{2a}(x) = F_k(x). \tag{B.5}
\]

From (B.4), (B.5) and (4.13), we have

\[
F_k(x) = L_k^{2a}(x) + F_{k-1}(x)
\]

\[
= \sum_{j=0}^k \binom{k - j + b - 1}{k - j} L_j^\nu(x) + F_{k-1}(x)
\]

\[
= \sum_{j=0}^k \binom{k - j + b}{k - j} L_j^\nu(x) - \sum_{j=0}^{k-1} \binom{k - j + b - 1}{k - j - 1} L_j^\nu(x) + F_{k-1}(x).
\]

Since \( L_0^\nu(x) = 1 \) and \( F_0(x) = L_0^{2a}(x) = 1, \)

\[
F_k(x) - \sum_{j=0}^k \binom{k - j + b}{k - j} L_j^\nu(x) = F_{k-1}(x) - \sum_{j=0}^{k-1} \binom{k - 1 - j + b}{k - 1 - j} L_j^\nu(x)
\]

\[
= F_0(x) - \binom{b}{0} L_0^\nu(x) = 0.
\]
Then we have (4.17). From (4.14) and (4.15) we have

\[ G_k(x) = -F_k(x) + \frac{k + 2a}{k} F_{k-2}(x), \quad k \in \mathbb{N}. \]  

(B.6)

Then (4.17) gives (4.18). This completes the proof.

B.2 Proof of Lemma 4.2

We introduce a symmetric inner product

\[ (f, g) \equiv \int_{0}^{\infty} dx e^{-x^2} f(x) g(x). \]

It is easy to see that it is related with the elementary skew-symmetric inner product (4.5) as

\[ \langle f, g \rangle_* = 2(\mathcal{I} f, g) - \int_{0}^{\infty} dx e^{-x^2} f(x) \int_{0}^{\infty} dy e^{-y^2} g(y), \]  

(B.7)

where

\[ \mathcal{I} f(x) \equiv \int_{0}^{x} dz e^{-z/2} z^a f(z). \]

We consider the polynomials

\[ W_j(x) = L_j^{2a}(x) - \frac{j + 2a}{j} L_{j-1}^{2a}(x), \quad j \in \mathbb{N}. \]  

(B.8)

Lemma B.1 For \( j = N_0, \ k \in \mathbb{N} \)

\[ (W_k, F_j) = -\Gamma(j + 2a + 2) \delta_{k-1, j}. \]  

(B.9)

Proof. For \( 2a > -1 \) the above equation can be derived immediately from the definitions (1.3), (B.5), and the orthogonal relation (B.1). We can extend it to the case \( a > -1 \). For \( k = 2, 3, \ldots \) it is easy to see that

\[ (L_k^{2a}, x^j) = (-1)^k \Gamma(k + 2a + 1) \delta_{jk}, \quad j = 1, 2, \ldots, k. \]  

(B.10)

However, if \( 2a < -1 \), \( (L_k^{2a}, 1) = \infty \). Then we use the following equation:

\[ (L_k^{2a}, L_j^{2a} - L_j^{2a}(0)) = \frac{\Gamma(k + 2a + 1)}{k!} \delta_{jk}, \quad k \in \mathbb{N}_0, \]  

(B.11)

which is obtained from (B.10) for \( j = 1, 2, \ldots, k \), since \( L_k^{2a}(x) \) is a polynomial whose coefficient of the \( k \)-th order is \((-1)^k/k!\). By simple calculation we have

\[ (1, L_j^{2a} - L_j^{2a}(0)) = 0, \quad j \in \mathbb{N}_0, \]

and thus

\[ (L_k^{2a}, L_j^{2a} - L_j^{2a}(0)) = (L_k^{2a} - L_k^{2a}(0), L_j^{2a} - L_j^{2a}(0)) = 0, \quad j \geq k + 4. \]

From the definitions of \( W_k \) and \( F_0 \equiv 1 \) it is easily to see that

\[ (W_k, F_0) = -\Gamma(2a + 2) \delta_{k-1, 0}. \]  

(B.12)
When \(k, j \in \mathbb{N}\), from (B.12) and (B.11),
\[
(W_k, F_j) = \left( L_k^2 - \frac{k + 2a}{k} L_{k-1}^2, \sum_{p=0}^{j} L_p^2 \right)
\]
\[
= \sum_{p=1}^{j} \left( L_k^2 - \frac{k + 2a}{k} L_{k-1}^2, L_p^2 - L_p^2(0) \right)
\]
\[
= -\frac{\Gamma(k + 2a + 1)}{k!} \delta_{k-1 j}.
\]
This completes the proof. \(\blacksquare\)

Here we prove the following integral formula.

**Lemma B.2** For \(j \in \mathbb{N}, \ell \in \mathbb{N}_0\)
\[
\int_{0}^{z} dx \ e^{-x/2} x^a G_j(x) = 2e^{-z/2} z^a W_j(x), \tag{B.14}
\]
\[
\int_{0}^{z} dx \ e^{-x/2} x^a F_{2\ell}(x) = 2^{\alpha+1} \binom{\ell + a}{\ell} \gamma(a + 1, z/2)
\]
\[
-2e^{-z/2} z^a \binom{\ell + a}{\ell} \sum_{r=0}^{\ell-1} \binom{\ell - r + a}{\ell - r}^{-1} W_{2\ell-2r}(z), \tag{B.15}
\]
where \(\gamma(c, y), c > 0\) is the incomplete gamma function \(\gamma(c, y) = \int_{0}^{y} dx \ e^{-x} x^{c-1}\).

**Remark 7.** If we set \(a = 0\) in (B.14), we will have the simpler equation
\[
\int_{0}^{z} dx \ e^{-x/2} \frac{d}{dx} \left( L^0_{j+1}(x) - L^0_{j-1}(x) \right) = 2e^{-z/2} \left( L^0_j(z) - L^0_{j-1}(z) \right), \quad j \in \mathbb{N}.
\]

**Proof of Lemma B.2.** We first introduce the functions defined by
\[
\psi^2_j(z) = \int_{0}^{z} dx \ e^{-x/2} x^a \frac{d}{dx} L^2_j(x), \quad j \in \mathbb{N}_0,
\]
\[
\varphi^2_j(z) = \int_{0}^{z} dx \ e^{-x/2} x^a L^2_j(x), \quad j \in \mathbb{N}_0.
\]
Then (B.13) gives
\[
\varphi^2_j(z) = -\psi_{j+1}^2(z) + \psi_j^2(z). \tag{B.16}
\]
On the other hand, by (B.2),
\[
\psi_j^2(z) = j \int_{0}^{z} dx \ e^{-x/2} x^{a-1} W_j(x).
\]
Noting the assumption \(a > -1\) and the fact that \(W_j(x) = O(x)\), in \(x \to 0\),
\[
2a \psi_j^2(z) = 2j \left\{ e^{-x/2} z^a W_j(z) + \int_{0}^{z} dx \ x^a \left( \frac{1}{2} e^{-x/2} W_j(x) - e^{-x/2} \frac{d}{dx} W_j(x) \right) \right\}
\]
\[
= 2j e^{-z/2} z^a W_j(z) + j \varphi_j^2(z) - (j + 2a) \varphi_j^2(z) - 2j \psi_j^2(z) + 2(j + 2a) \psi_j^2(z).
\]

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Then we use (B.16) to eliminate $\varphi_j^{2\alpha}(z)$ and $\varphi_j^{2\alpha+1}(z)$, and the equality (B.13) is obtained from the relation

$$\int_{0}^{z} dx \ e^{-x/2} x^a G_j(x) = \psi_j^{2\alpha+1}(z) - \frac{j+2a}{j} \psi_j^{2\alpha}(z).$$

Note that (B.14) gives a recurrence relation for $\psi_j^{2\alpha}(z)$,

$$\psi_j^{2\alpha+1}(z) = \frac{j+2a}{j} \psi_j^{2\alpha}(z) + 2e^{-z/2} z^a W_j(z).$$

It is solved as

$$\psi_j^{2\alpha+1}(z) = \left( \ell + a \right) \psi_j^{2\alpha}(z) + 2e^{-z/2} z^a \left( \ell + a \right) \sum_{r=0}^{\ell-1} \left( \ell - r + a \right) \left( \ell - r \right)^{-1} W_{2\ell-2r}(z).$$

Since $L_1^2(x) = (1 + 2a) - x$,

$$\psi_1^{2\alpha}(z) = \int_{0}^{z} dx \ e^{-x/2} x^a \frac{d}{dz} L_1^2(z) = -\int_{0}^{z} dx \ e^{-x/2} x^a = -2^{a+1} \gamma(a+1, z/2).$$

Then we have (B.13). 

Then we can prove Lemma 4.2. From Lemma B.2 and two relations (B.16) and (B.17), we have

$$\langle F_{2q}, G_{2\ell+1} \rangle = -\langle G_{2\ell+1}, F_{2q} \rangle = -4(W_{2\ell+1}, F_{2q}),$$

$$\langle G_{2\ell+1}, G_{2\ell+1} \rangle = 4(W_{2\ell+1}, G_{2\ell+1})$$

$$= 4 \left\{ (W_{2\ell+1}, F_{2\ell+1}) + \frac{2\ell+1+2a}{2\ell+1} (W_{2\ell+1}, F_{2\ell-1}) \right\}.$$ 

Then (B.20) and (B.21) are derived from Lemma B.1. From Lemma B.3 and B.5, we have

$$\int_{0}^{\infty} dw \ e^{-w/2} w^a \gamma(a+1, w/2) F_{2q}(w)$$

$$= \left[ \Gamma(a+1) \right] \left[ \frac{\gamma(a+1, w/2)}{w} \right] \int_{0}^{\infty} dz \ e^{-z/2} z^a F_{2q}(z)$$

$$= \int_{0}^{\infty} dw \ 2^{-(1+a)} e^{-w/2} w^a \int_{0}^{\infty} dz \ e^{-z/2} z^a F_{2q}(z)$$

$$= 2^{a+1} \left( \ell + a \right) \left\{ \Gamma(a+1)^2 - \int_{0}^{\infty} dw \ 2^{-(1+a)} e^{-w/2} w^a \gamma(a+1, w/2) \right\}$$

$$= 2^{a+1} \left( \ell + a \right) \left\{ \Gamma(a+1)^2 - \frac{1}{2} \left( \gamma(a+1, w/2)^2 \right) \right\}$$

$$= 2^{a+1} \left( \ell + a \right) \Gamma(a+1)^2.$$ 

Then by using Lemma B.2 and B.5, we have (B.20), since

$$\langle F_{2q}, F_{2\ell} \rangle = 2 \int_{0}^{\infty} dw \ e^{-w/2} w^a F_{2q}(w) \int_{0}^{\infty} dz \ e^{-z/2} z^a F_{2\ell}(z)$$

$$- \int_{0}^{\infty} dw \ e^{-w/2} w^a F_{2q}(w) \int_{0}^{\infty} dz \ e^{-z/2} z^a F_{2\ell}(z)$$

$$= 2^{a+2} \left( q + a \right) \int_{0}^{\infty} dw \ e^{-w/2} w^a \gamma(a+1, w/2) F_{2q}(w)$$

$$- 2^{a+2} \left( q + a \right) \left( \ell + a \right) \Gamma(a+1)^2 = 0.$$ 

This completes the proof. 

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Inverse of \( \{\alpha_{k,j}\} \)

We put
\[
Q_{2\ell}(x) = F_{2\ell}(x) \quad \text{and} \quad Q_{2\ell+1}(x) = G_{2\ell+1}(x),
\]
for \( \ell \in \mathbb{N}_0 \), and \( Q_k \equiv 0 \) for \( k \in \mathbb{Z}_{-} \). Lemma \( [11] \) gives the expansion formula of \( Q_k(x) \) in terms of \( \{L^\nu_j(x)\} \). Here we give the formula to expand \( L^\nu_j(x) \) in terms of \( \{Q_k(x)\} \). In other words, we calculate the inverse of the matrix \( \alpha = (\alpha_{k,j}) \) given by (4.16), which is denoted by \( \beta = (\beta_{j,k}) \) and used in Section 5.2.

Let \( b(n) = (n + 2a)/n, n \in \mathbb{N}, \) and
\[
 b(m,n) = \begin{cases} 
 b(m)b(m + 2) \cdots b(n), & \text{if } m, n \text{ are odd and } m \leq n, \\
 1, & \text{if } m, n \text{ are odd and } m > n, \\
 0, & \text{otherwise}. 
\end{cases} 
\]
Then the following lemma holds.

**Lemma C.1** For \( j \in \mathbb{N}_0 \)
\[
L^\nu_j(x) = \sum_{k=0}^{j} \beta_{j,k}Q_k(x).
\]
where \( \beta_{j,k}, k, 1, \ldots, j, j \in \mathbb{N}_0 \) are defined by the following:

When \( k \) is even
\[
\beta_{j,k} = \begin{cases} 
0, & \text{if } j < k, \\
\binom{j - k - b - 2}{j - k}, & \text{if } j \geq k, 
\end{cases}
\]
and, when \( k \) is odd
\[
\beta_{j,k} = \begin{cases} 
0, & \text{if } j < k, \\
-\sum_{r=0}^{\lfloor (k+1)/2 \rfloor} b(k + 2, 2r - 1) \binom{j - 2r - b - 1}{j - 2r + 1}, & \text{if } j \geq k. 
\end{cases}
\]

**Proof.** From \( [11,6] \) and \( [11,6] \) we have
\[
Q_{2\ell}(x) = \sum_{j=0}^{2\ell} L^\nu_j(x), \quad Q_{2\ell+1}(x) = -\sum_{j=0}^{2\ell+1} L^\nu_j(x) + b(2\ell + 1) \sum_{j=0}^{2\ell-1} L^\nu_j(x).
\]
By simple calculations we see that
\[
L^\nu_j(x) = \sum_{k=0}^{j} \beta_{j,k}Q_k(x), \quad j \in \mathbb{N}_0
\]
where \( \beta_{j,k}, k = 0, \ldots, j, j \in \mathbb{N}_0, \) are defined by the following:

When \( k \) is even
\[
\beta_{j,k} = \begin{cases} 
1, & \text{if } j = k, \\
-1, & \text{if } j = k + 1, \\
0, & \text{otherwise}, 
\end{cases}
\]
and, when \( k \) is odd
\[
\beta_{j,k} = \begin{cases} 
b(k + 2, j - 1), & \text{if } j \text{ is even}, \\
-b(k + 2, j), & \text{if } j \text{ is odd}. 
\end{cases}
\]
Using the formula \( [11,4], [11,6] \) gives
\[
L^\nu_j(x) = \sum_{p=0}^{j} \binom{j - p - b - 1}{j - p} \sum_{k=0}^{p} \beta_{p,k}Q_k(x).
\]
If we define
\[ \beta_{j,k} = \sum_{p=k}^{j} \beta_{p,k} \left( \frac{j - p - b - 1}{j - p} \right) \]  
(C.7)
for \( j \geq k \) and \( \beta_{j,k} = 0 \) for \( j < k \), (C.3) is satisfied. The expressions (C.4) and (C.5) are derived from (C.7) by simple calculation.

D Elementary Calculation for Asymptotics Estimation

By Stirling’s formula \( \Gamma(x) \sim \sqrt{2\pi x^{x-1/2}e^{-x}}, \ x \to \infty \), we have
\[ \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)} \sim (n + 1)^{\alpha}, \ n \to \infty, \]  
(D.1)
\[ \left( \frac{n + \alpha}{n} \right) \sim \frac{(n + 1)^{\alpha}}{\Gamma(\alpha + 1)} \ n \to \infty, \]  
(D.2)
for any \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \), and
\[ b(1, 2\ell - 1) = \frac{\Gamma(2a + 1)}{2^a \Gamma(a + 1)} \left( \frac{2\ell + 2a}{2\ell} \right) / \left( \frac{\ell + a}{\ell} \right) \sim (2\ell)^a, \ \ell \to \infty \]  
(D.3)
\[ b(2\ell + 1, 2p - 1) = \frac{b(1, 2p - 1)}{b(1, 2\ell - 1)} \sim \left( \frac{p}{\ell} \right)^a, \ \ell \to \infty \]  
(D.4)
for \( \ell, p \in \mathbb{N} \) with \( \ell < p \), where \( b(m, n) \) is defined by (C.2).

From now on, we assume that \( T = N \), \( t_m = T + s_m \) with \( s_m < 0 \). We set
\[ 2\ell = N\theta \quad \text{and} \quad j = 2\ell \eta, \]  
(D.5)
and consider the limit \( N \to \infty \) with some \( \eta, \theta \in (0, \infty) \). Then we have
\[ \chi_j^m = \left( \frac{2T - t_m}{t_m} \right)^j = \left( 1 - \frac{2s_m}{t_m} \right)^{N\theta \eta} \sim \exp(-2s_m \theta \eta), \ N \to \infty, \]  
and
\[ \sum_{p=0}^{\alpha} (-1)^p \binom{\alpha}{p} \chi_j^{m-p} \sim (2\ell)^{-\alpha} \left( \frac{d}{d\eta} \right)^{\alpha} \exp(-2s_m \theta \eta), \ N \to \infty. \]  
(D.6)
We use the following identities (see [42] and pages 8, 201 and 202 in [42]).
(i) Let \( \alpha \in \mathbb{N}_0 \) and \( c \in \mathbb{R} \). Then
\[ \sum_{p=0}^{\alpha} (-1)^p \binom{\alpha}{p} \left( \frac{n - p + c}{n - p - j} \right) = \binom{n - \alpha + c}{n - j}. \]  
(D.7)
(ii) Let \( \alpha \in \mathbb{N}_0, \ c \in \mathbb{R} \) and \( a_k, k = 1, 2, \ldots \), be a sequence in \( \mathbb{R} \). Then
\[ \sum_{r=0}^{\infty} \binom{r + c}{r} a_r = \sum_{r=0}^{\infty} \binom{r + c + \alpha}{r} \sum_{p=0}^{\alpha} (-1)^p \binom{\alpha}{p} a_{r+p}. \]  
(D.8)
(iii) Let \( \alpha \in \mathbb{N}_0 \), and \( a_k, b_k, k = 1, 2, \ldots \), be sequences in \( \mathbb{R} \). Then
\[ \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha}{k} a_k b_k = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \sum_{p=0}^{\beta} (-1)^p \binom{\beta}{p} a_p \sum_{q=0}^{\alpha - \beta} (-1)^q \binom{\alpha - \beta}{q} b_q. \]  
(D.9)
Lemma D.1  For any $\alpha \in \mathbb{N}_0$ and $w \geq 0$ we have

$$\sum_{p=0}^{\alpha} (-1)^p \left(\frac{\alpha}{p}\right) L_{n-p}^{\nu} \left(\frac{w}{n}\right) \sim \left(\frac{w}{n}\right)^{\alpha-\nu} \left(\frac{d}{dw}\right)^{\alpha} \left\{ w^{\nu/2} J_\nu(2\sqrt{w}) \right\}, \quad n \to \infty.$$  

Proof. From the definition of the Laguerre polynomials (4.12) and (D.7), we have$$$$by (D.9), the asymptotic (D.10) is derived from (D.6) and Lemma D.1 with

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{n - \alpha + \nu}{n - j} \right) y^j.$$  

Hence, by (D.12),

$$\lim_{n \to \infty} \left(\frac{w}{n}\right)^{\nu-\alpha} \sum_{p=0}^{\alpha} (-1)^p \left(\frac{\alpha}{p}\right) L_{n-p}^{\nu} \left(\frac{w}{n}\right) = \lim_{n \to \infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{n - \alpha + \nu}{n - j} \right) \left(\frac{w}{n}\right)^{j+\nu-\alpha}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{w^{\nu+j-\alpha}}{\Gamma(j - \alpha + \nu + 1) j!}$$

$$= \left(\frac{d}{dw}\right)^{\alpha} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{w^{\nu+j}}{\Gamma(\nu + j + 1) j!}$$

Then we obtain the lemma.

Applying the above lemma, we obtain the following asymptotics, where $L_j^{\nu}$ and $\hat{L}_j^{\nu}$ are defined by (5.1).

Lemma D.2  For any $\alpha \in \mathbb{N}_0$, $\theta, \eta \in (0, \infty)$, and $x \in \mathbb{R}_+$ we have

$$\sum_{p=0}^{\alpha} (-1)^p \left(\frac{\alpha}{p}\right) L_j^{\nu} \left(\frac{x}{N}, -s_m\right) \sim \frac{(2\ell)^{\nu-\alpha}}{(\theta x)^{\nu}} J_\nu(\theta, \nu, x, -s_m), \quad N \to \infty, \quad (D.10)$$

$$\sum_{p=0}^{\alpha} (-1)^p \left(\frac{\alpha}{p}\right) \hat{L}_{j+p}^{\nu} \left(\frac{x}{N}, s_m\right) \sim (2\ell)^{-\alpha} J_\nu^{(\alpha)}(\theta, \eta, x, s_m), \quad N \to \infty. \quad (D.11)$$

Proof. Since

$$\sum_{p=0}^{\alpha} (-1)^p \left(\frac{\alpha}{p}\right) L_{j-p}^{\nu} \left(\frac{x}{N}\right) \chi_m^{j-p}$$

$$= \sum_{\beta=0}^{\alpha} \left(\frac{\alpha}{\beta}\right) \sum_{p=0}^{\beta} (-1)^p \left(\frac{\beta}{p}\right) L_{j-p}^{\nu} \left(\frac{x}{N}\right) \sum_{q=0}^{\alpha-\beta} (-1)^q \left(\frac{\alpha-\beta}{q}\right) \chi_m^{j-q},$$

by (D.9), the asymptotic (D.10) is derived from (D.6) and Lemma D.1 with $n = j = N\theta\eta, w = \theta\eta x$. From (D.29), we have

$$\sum_{k=0}^{\alpha} (-1)^k \left(\frac{\alpha}{k}\right) \hat{L}_{j+k} \left(\frac{x}{N}, s_m\right)$$

$$= \sum_{\beta=0}^{\alpha} \left(\frac{\alpha}{\beta}\right) \sum_{p=0}^{\beta} (-1)^p \left(\frac{\beta}{p}\right) \hat{L}_{j+p} \left(\frac{x}{N}\right) \chi_m^{-(j+p)} \sum_{q=0}^{\alpha-\beta} (-1)^q \left(\frac{\alpha-\beta}{q}\right) \frac{\Gamma(j + q + 1)}{\Gamma(j + q + 1 + \nu)}.$$  

(D.12)
By (D.1), we see
\[ \sum_{q=0}^{\alpha-\beta} (-1)^q \left( \frac{\alpha - \beta}{q} \right) \frac{\Gamma(j + q + 1)}{\Gamma(j + q + 1 + \nu)} = \nu \sum_{q=0}^{\alpha-\beta-1} (-1)^q \left( \frac{\alpha - \beta - 1}{q} \right) \frac{\Gamma(j + q + 1)}{\Gamma(j + q + 2 + \nu)} = \frac{\nu(\nu + 1) \cdots (\nu + \alpha - \beta - 1)\Gamma(j + \nu)}{\Gamma(j + 1 + \alpha - \beta + \nu)}, \]
\[ \sim (2\ell)^{-(\alpha-\beta+\nu)} \left( -\frac{d}{d\eta} \right)^{\alpha-\beta} \eta^{-\nu}, \quad N \to \infty. \]

On the other hand, (D.10) gives
\[ \sum_{p=0}^{\beta} (-1)^p \left( \frac{\beta}{p} \right) L_{j+p}^\nu \left( \frac{x}{N} \right) \sim (2\ell)^{\nu-\beta} \left( -\frac{d}{d\eta} \right)^{\beta} \left\{ \eta^\nu \mathcal{J}_\nu(\theta, \eta, s_m) \right\}. \]
Hence, the asymptotic (D.11) is derived from (D.12).

\[ \text{Lemma E.1} \]

**On Temporally Homogeneous Limit**

**Lemma E.1**  *For any \( c \in \mathbb{R} \) and \( \eta, \theta, x \geq 0 \), we have that as \( t \to \infty \)*
\[ \hat{J}_\nu^{(c)}(\theta, \eta, x, t) \sim (2t\theta)^c (\theta \eta)^{\nu/2} J_\nu(2\sqrt{\theta \eta} x) e^{2t\theta \eta}, \quad (E.1) \]
\[ \hat{J}_\nu^{(c)}(\theta, \eta, x, -t) \sim (2t\theta)^c (\theta \eta)^{-\nu/2} J_\nu(2\sqrt{\theta \eta} x) e^{-2t\theta \eta}, \quad (E.2) \]
\[ \int_1^\infty d\xi \, \xi^a \hat{J}_\nu^{(c+1)}(\theta, \xi, x, -t) \sim (2t\theta)^c (\theta x)^{-\nu/2} J_\nu(2\sqrt{\theta \eta} x) e^{-2t\theta}. \quad (E.3) \]

**Proof.** From the expression (2.10) with the definition (2.2), we have
\[ \hat{J}_\nu^{(c)}(\theta, \eta, x, t) = \frac{e^{2t\theta \eta}}{\Gamma(-c)} \sum_{k=0}^\infty \frac{(-1)^k \eta^k - c}{k!(k-c)} \sum_{j=0}^k \frac{\Gamma(j)}{\Gamma(j+1)} \hat{J}_\nu^{(j)}(\theta, \eta, 0)(2t\theta)^{k-j} \]
\[ \sim \frac{e^{2t\theta \eta}}{\Gamma(-c)} (2t\theta)^c \mathcal{J}_\nu(\theta, \eta, 0) \sum_{k=0}^\infty \frac{(-1)^k (2t\theta \eta)^{k-c}}{k!(k-c)}, \quad t \to \infty. \]

From the relation
\[ \frac{d}{dz} \sum_{k=0}^\infty \frac{(-1)^k z^k}{k!(k-c)} = z^{-c-1} e^{-z}, \]
and the equation
\[ \Gamma(-c) = \sum_{k=0}^\infty \frac{(-1)^k}{k!(k-c)} + \int_1^\infty dz \, z^{-c-1} e^{-z} \]
(see (1.1.19) in [3]), we have
\[ \Gamma(-c) = \lim_{t \to \infty} \sum_{k=0}^\infty \frac{(-1)^k (2t\theta \eta)^{k-c}}{k!(k-c)}. \]
Then we conclude
\[ \hat{J}_\nu^{(c)}(\theta, \eta, x, t) \sim e^{2t\theta \eta}(2t\theta)^c \mathcal{J}_\nu(\theta, \eta, x, t) = (2t\theta)^c \mathcal{J}_\nu(\theta, \eta, x, t), \quad t \to \infty. \]
Hence (E.1) is derived from (2.12).
Let \( n = [c + 1]_+ \) and \( \beta > 0 \) with \( c = n - \beta \). Since
\[
\left(-\frac{d}{d\eta}\right)^n \tilde{J}_\nu(\theta, \eta, x, -t) \sim (2t\theta)^n \tilde{J}_\nu(\theta, \eta, x, -t), \quad t \to \infty,
\]
(2.7) gives
\[
\tilde{J}^{(c)}(\theta, \eta, x, -t) \sim \frac{(2t\theta)^n}{\Gamma(\beta)} \int_0^\infty d\xi \xi^{\beta-1} \tilde{J}_\nu(\theta, \eta + \xi, x, -t)
\]
\[
= \frac{(2t\theta)^{n-\beta}}{\Gamma(\beta)} \int_0^\infty d\xi \xi^{\beta-1} \tilde{J}_\nu(\theta, \eta + \frac{\xi}{2t\theta}, x, -t)
\]
\[
\sim (2t\theta)^{n-\beta} \tilde{J}_\nu(\theta, \eta, x, -t), \quad t \to \infty.
\]
Then (2.12) is derived from (2.11).
From (2.10) we have
\[
\int_1^\infty d\xi \xi^a \tilde{J}^{(c+1)}(\theta, \xi, x, -t) \sim (2t\theta)^{c+1} \int_1^\infty d\xi \xi^a \tilde{J}_\nu(\theta, \xi, x, 0)e^{-2t\theta\xi}
\]
\[
\sim (2t\theta)^c \tilde{J}_\nu(\theta, 1, x, -t), \quad t \to \infty.
\]
This completes the proof. 

Applying the above lemma, we have as \( s_m, s_n \to -\infty \) with the difference \( s_n - s_m \) fixed
\[
D(s_m; x; s_n, y) \sim \frac{(xy)^{b/2}(s_n - s_m)}{2^{2b+1}(s_m s_n)^{b+1}} \int_1^\infty d\theta \theta^{-b} J_\nu(2\sqrt{\theta x})J_\nu(2\sqrt{\theta y})e^{-2(s_m + s_n)\theta}
\]
\[
\sim \frac{(xy)^{b/2}(s_n - s_m)}{2^{2b+1}(s_m + s_n)^{b+1}} J_\nu(2\sqrt{x})J_\nu(2\sqrt{y})e^{-2(s_m + s_n)},
\]
\[
\tilde{I}(s_m; x; s_n, y) \sim \frac{2^{2b+1}(s_m s_n)^b(s_n - s_m)}{(s_m + s_n)(xy)^{b/2}} \int_1^\infty d\theta \theta^b J_\nu(2\sqrt{\theta x})J_\nu(2\sqrt{\theta y})e^{2(s_m + s_n)\theta}
\]
\[
\sim \frac{2^{2b}(s_m s_n)^b(s_n - s_m)}{(s_m + s_n)(xy)^{b/2}} J_\nu(2\sqrt{x})J_\nu(2\sqrt{y})e^{2(s_m + s_n)},
\]
and
\[
S(s_m; x; s_n, y) \sim \left(\frac{y}{x}\right)^{b/2} \int_0^1 d\theta J_\nu(2\sqrt{\theta x})J_\nu(2\sqrt{\theta y})e^{2(s_m - s_n)\theta}.
\]
It is then clear that
\[
\lim_{{s_m, s_n \to -\infty}} D(s_m; x; s_n, y)\tilde{I}(s_m; x; s_n, y) = 0.
\]

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