Link Prediction: A Graphical Model Approach

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Abstract—We consider the problem of link prediction in networks whose edge structure may vary (sufficiently slowly) over time. This problem, with applications in many important areas including social networks, has two main variants: the first, known as positive link prediction or PLP consists in estimating the appearance of a link in the network. The second, known as negative link prediction or NLP consists in estimating the disappearance of a link in the network. We propose a data-driven approach to estimate the appearance/disappearance of edges. Our solution is based on a regularized optimization problem for which we prove existence and uniqueness of the optimal solution.

I. INTRODUCTION

The increased popularity of research in social-network analysis can be mainly attributed to explosion of social networks such as Facebook, Twitter, YouTube, etc. Despite the fact that one can collect plenty of information coming from social platforms, the topological and non-topological changes over time remain unknown due to the dynamical behavior exhibited by this kind of networks. Such underlying dynamical nature strongly motivates the interest in problems such as inferring unobserved edges in the current network or predicting edges that appear/disappear in the future, which are both referred to as Link Prediction problems [1], [2]. Link prediction techniques find important applications in e-mails networks [3] and in gene-expression networks in biology [4] just to mention a few. Generally speaking, link prediction problems can be basically distinguished in two categories: Positive Link Prediction (PLP) and Negative Link Prediction (NLP) problems. The most part of the research effort has been focused on the positive link prediction problem which aims to predict the formation of edges in the network. Although links’ disappearance models important behaviors in social networks such as the “unfollow” behavior in online social networks, less attention has been dedicated to the NLP problem, understood as the prediction of disappearing links in the network [4].

Many algorithms solving the PLP problem have been proposed in the literature. In general terms, most of these methods deal with PLP over static networks and concern the construction of the so-called score matrix, whose entry \((i, j)\) measures the probability of appearance of an edge between node \(i\) and node \(j\). Each element \((i, j)\) of the matrix is therefore determined by a proper choice of a similarity measure which, given the available prior information, estimates how likely the appearance (in the near future) of an edge between two specific nodes is. These similarity measures are based on some knowledge of the topology of the network in the past and some (usually qualitative) properties the network is expected to have. For instance, a real network is expected possesses the so-called “small world” property meaning that the network is highly clustered and most pairs of nodes are related through short chains. As a consequence, a couple of unconnected nodes whose path length is short should have a high similarity score. Among the most successful similarity measures that have been proposed, we recall common neighbors, Adamic/Adar, Katz and spreading activation measures, see [2] for more details. Few similarity measures have been proposed instead in the NLP framework. These are based on a procedure that essentially “reverses” the PLP process based on a certain similarity measure. The method proposed in [4], for instance, classifies as “likely to disappear” those edges that (after being removed) have a small similarity measure according to the selected PLP criterium; in other words, the edges that are estimated to disappear are the ones between the nodes that have the smallest similarity measure according to a PLP algorithm applied to the network deprived of those edges.

However, the aforementioned approaches have the following limitations:

- there are instances called “unfriendly prediction networks” for which most of the measures provide poor prediction performances, see for instance [5]; this evidence is due by the fact that the expected network properties do not coincide with the actual ones;
- in many applications both a prior for the network topology and a prior for the mathematical model are available; moreover, the available data allow to estimate the entire model instead of only its support and the latter contains far less information than the former; however, the similarity measures in the literature only exploit the topology of the graph.

The aim of this paper is to address the above limitations. More precisely, we consider the link prediction problem over static networks modeled by means of undirected graphical models for Gaussian random vectors. Then, we propose a data-driven link prediction approach based on a similarity measure accounting the knowledge on the past information (including the topology and the mathematical model) and some noisy piece of information. The latter is represented by noisy measurements of the network in which the new links have already appeared/disappeared. Therefore: 1) we
let emerge the network properties from the measurements rather than expected properties; 2) we use all the available information. It is worth remarking that it is standard in the literature to refer to a link prediction problem either if the focus is on a prediction problem or if the problem at hand is actually a detection problem in which the prediction comes from a combination of the past information (the current network) and some information (data) coming from the network where the new link has already appeared/disappeared. Our approach falls in the latter case. Moreover, it naturally settles into a covariance estimation problem with prior set-up [6]: we are looking for the graphical model that agrees with the data while being as close as possible to the current graphical model (i.e. the prior). The latter can also be understood as a static version of the covariance extension problems with prior for which a large body of literature is available [7], [8], [9], [10], [11], [12], [13], [14], [15].

Our main modeling tool will be graphical models for Gaussian random vectors which encode conditional dependence relations among a set of jointly normally distributed variables [16]. The reason why we consider conditional dependence among pairs of nodes is because it accounts for the information distributed on the whole network and has therefore a global nature which seems suitable for many of the applications mentioned before.

The topology of a Gaussian graphical model is reflected by the support of the inverse covariance matrix. Accordingly, the topology can be inferred from data by resorting to a regularized maximum likelihood problem where the regularizer induces sparsity on the inverse covariance matrix, see [17], [18]. Our link prediction approach has a similar spirit. The main differences are that: we consider the prior; we propose three different regularizers corresponding to PLP, NLP, and a mixed version of PLP and NLP, respectively. Moreover, our paradigm is strictly related to a generalized version of the Dempster’s problem [19], which has been extensively studied and generalized both in the static [20] and dynamic [21], [22], [23], [24], [25], [26], [27] case.

**Outline of the paper.** In Section II we review the basic properties of Gaussian graphical models that we need throughout the paper. In Section III we formulate the link prediction problems by means of Gaussian graphical models. In Section IV some experimental results are presented. Finally in Section V we draw the conclusions.

### A. Notation

Given a $p \times p$ real matrix $A \in \mathbb{R}^{p \times p}$, we will denote with $A^\top$ its transpose, by $A^{-1}$ its inverse, and by $\det A$ and $\text{tr} A$ its determinant and trace, respectively. The writings $A \geq 0$ and $A > 0$ denote the fact that $A$ is positive semi-definite and positive definite. The operator diag : $\mathbb{R}^{p \times p} \rightarrow \mathbb{R}^{p \times p}$ maps the matrix $A$ to the diagonal matrix $\text{diag}(A)$ obtained by setting to zero all the elements of $A$ outside the main diagonal. Given support a $\Omega \subseteq \{(i, j) : i, j = 1, \ldots, p\}$, we denote by $\Omega^c$ the complement of $\Omega$; the map $P_{\Omega} : \mathbb{R}^{p \times p} \rightarrow \mathbb{R}^{p \times p}$, defined by

$$[P_{\Omega}(A)]_{ij} = \begin{cases} A_{ij} & \text{if } (i, j) \in \Omega, \\ 0 & \text{if otherwise,} \end{cases}$$

is the orthogonal projection onto the subspace of matrices with support $\Omega$. The set of all symmetric $p \times p$ matrices will be denoted by $S_p$ while $S_p^+ \subseteq S_p$ will denote the cone of the positive definite matrices in $S_p$. $I_p$ denotes the identity matrix of order $p$. The symbol $\propto$ means “proportional to”.

### II. GAUSSIAN GRAPHICAL MODELS

Let $x$ be an $m$-dimensional Gaussian random vector and denote with $x_1, \ldots, x_m$ its components. Let $\mathcal{G} = (\mathcal{V}, E)$ be an undirected graph with vertexes $\mathcal{V} = \{1, \ldots, m\}$ and edges $E \subseteq \mathcal{V} \times \mathcal{V}$. We say that the random vector $x$ satisfies the (undirected) Gaussian graphical model $\mathcal{G}$, if $x$ admits probability density function $\mathcal{N}(0, \Sigma)$ with $\Sigma > 0$ and such that

$$(\Sigma^{-1})_{ij} = 0, \quad \text{for all } (i, j) \notin E. \quad (1)$$

Accordingly, a complete characterization of the graph $\mathcal{G}$ associated to the random vector $x$ is given in terms of the support of its concentration matrix $\Sigma^{-1}$. One can show [16] that the random variable $x_i$ is conditional independent from $x_j$ given the other components $\{x_k\}_{k \neq i,j}$, and we write $x_i \perp \perp x_j | \{x_k\}_{k \neq i,j}$, and if and only if $(\Sigma^{-1})_{ij} = 0$. Therefore,

$$x_i \perp \perp x_j | \{x_k\}_{k \neq i,j} \iff (i, j) \notin E. \quad (2)$$

The identification of Gaussian graphical models from data, in view of (2), reduces to the estimation of the covariance matrix $\Sigma$ whose inverse should be sparse. In particular, one can use the maximum likelihood principle. Given $N$ i.i.d. observations $x_1, \ldots, x_N$ of $x$, the negative log-likelihood function has the form

$$\ell(\Sigma; \hat{\Sigma}) \propto \frac{N}{2} \log \det \Sigma + \frac{N}{2} \text{tr}(\hat{\Sigma} \Sigma^{-1}) \quad (3)$$

where

$$\hat{\Sigma} = \frac{1}{N} \sum_{k=1}^{N} x_k x_k^\top \quad (4)$$

is the sample covariance matrix. Hence, the maximum likelihood estimator of $\Sigma$ boils down to the constrained optimization problem

$$\arg\min_{\Sigma \in \Theta} \log \det \Sigma + \text{tr}(\hat{\Sigma} \Sigma^{-1}) \quad (5)$$

where $\Theta \subseteq S_p^+$ is a suitable parametric family (e.g. the family of covariance matrices corresponding to a graphical model having a certain topology).

### III. LINK PREDICTION

This section is devoted to the introduction and the formalization of the link prediction problem in our setting. Let $\mathcal{G}_s = (\mathcal{V}, E_s)$ be the graphical model associated to an $m$-dimensional Gaussian random vector $x$ with covariance matrix $\Sigma$, modeling some kind of network at a certain time $s > 0$. By relation (1) the support of $\Sigma^{-1}$, $\Omega_s := \{(i, j) \in \mathcal{V} \times \mathcal{V} : (\Sigma^{-1})_{ij} \neq 0\}$,
coincides with the set $E_t$ and it is assumed to be known. Adopting the standard assumption that the number of nodes does not change over time, we model our system at time $t > s$ with the graphical model $G_t = (V, E_t)$ associated to the same random vector $x$. The (new) edges’ set is now related to the concentration matrix $T^{-1}$, through its support $\Omega_t$ which is considered unknown, and therefore it has to be estimated.

Given the application, it is reasonable to assume that $T$ will not be drastically different from $S$ because this would mean that the network has completely changed in a relatively short period of time, which is typically not the case. Notice that this is not a formal mathematical assumption but must be regarded simply as a justification of the fact that among all the models compatible with the data we select the one that is closer to the ‘prior’ given by $S$. We are now ready to state the following estimation problem.

**Problem 1:** Assume to have the “prior” covariance matrix $S$ of $x$ at time $s$. Let $\Omega_s$ be the support of $S^{-1}$. Given $N$ i.i.d. observations $x_1^t, \ldots, x_N^t$ of $x$ at time $t$, with $t > s$, compute an estimate of the covariance $T$, which is close to $S$ and fits the observation as much as possible.

Notice that Problem 1 includes the main variations of the link prediction problem. In fact, in link prediction we only need to estimate the support $\Omega_t$ (which is fully specified by $T$). In particular,

- $\Omega_t \subset \Omega_s$ corresponds to the PLP problem since conditional dependencies, and hence edges, are appearing between the variables;
- $\Omega_t \supset \Omega_s$ corresponds to the NLP problem since edges are disappearing between the variables.

Figure 1 explains the problems just presented for a simple network of four nodes.

![Diagram of link prediction problems for a four-nodes graph.](image)

**Fig. 1:** Link prediction problems for a four-nodes graph. The dashed red edges disappear at time $t$ while the blue ones appears. For simplicity $\Omega_s$ and $\Omega_t$ are identified with half of the support of $S^{-1}$ and $T^{-1}$, respectively.

We first make the simplistic assumption that we know the support $\Omega$. The solution to Problem 1 that we will derive under this assumption, will give us insights on how to deal with the actual problem where $\Omega$ is unknown. Let $\hat{T}$ be the sample covariance matrix (4) computed with the observations $x_1^t, \ldots, x_N^t$. In light of the previous observations, we propose the following mathematical formulation of Problem 1:

$$\begin{align*}
\arg\min_{T \in \mathcal{S}_t} & \ 2\mathcal{D}(T||S) \\
\text{subject to} & \quad P_{\Omega_t}(T - \hat{T}) = 0,
\end{align*}$$

where

$$\mathcal{D}(T||S) := \frac{1}{2} \left[ -\log \det(S^{-1}T) + \text{tr}(S^{-1}T) - m \right]$$

is the Kullback-Leibler divergence between the distributions $N(0, S)$ and $N(0, T)$. Some remarks concerning Problem (6) are in order. First of all, notice that if $S = I$ and $\{(i, i) : i = 1, \ldots, m\} \subset \Omega_s$, Problem (6) becomes the classical Dempster’s problem, widely studied in the literature [19]. Indeed, in this case, the linear term $\text{tr}(S^{-1}T) = \text{tr}T$ is in fact the constant $\text{tr}\hat{T}$, thanks to the constraint. The index that has to be optimized in (6) imposes invertibility of the solution $T$ so that even if the “true” covariance is singular we find a nonsingular approximation.

The constraint in (6) imposes that the entries of $T$ coincide with those of $\hat{T}$ on the support $\Omega$. Enforcing such a constraint may be confusing at first sight, as the edges of the predicted network have to do with the support of the concentration matrix rather than the support of the covariance matrix. However, conditional dependence relations among the variables are strictly related to their correlations and with that constraint we are precisely accounting only the most reliable correlations inferred from the data. As we will see at the end of the section, such constraint imposes the right structure of the solution $T_o^{-1}$ which will have as support the union $\Omega_o \cup \Omega_t$, that will allow to model the different link prediction scenarios.

The problem can be re-parametrized in term of $T^{-1}$ exploiting duality theory. To this end, we first eliminate the uninteresting constant term $\log \det(S) - m$ form the cost function in (6). Then we form the Lagrangian for Problem (6): it is

$$\mathcal{L}(T, \tilde{\Lambda}) = -\log \det T + \text{tr}(S^{-1}T) + \text{tr}\left[ P_{\Omega_t}(T - \hat{T}) \tilde{\Lambda} \right]$$

$$= -\log \det T + \text{tr}(S^{-1}T) + \text{tr}\left[ (T - \hat{T}) P_{\Omega_t}(\tilde{\Lambda}) \right]$$

where $\tilde{\Lambda} = \tilde{\Lambda}^\top \in \mathbb{R}^{m \times m}$ is the Lagrange multiplier and we have exploited the fact that $P_{\Omega_t}(\cdot)$, being an orthogonal projection, is a self-adjoint operator as it may be readily checked by applying the definition. Introducing the new multiplier $\Lambda := P_{\Omega_t}(\tilde{\Lambda})$, we can rewrite the Lagrangian as

$$\mathcal{L}(T, \Lambda) = -\log \det T + \text{tr}\left[ (S^{-1} + \Lambda)T \right] - \text{tr}\left( \tilde{T} \Lambda \right).$$

Since $\mathcal{L}$ is a strictly convex function of $T$, a sufficient condition for $T_o$ to be a minimum point for $\mathcal{L}$ is that the first variation of $\mathcal{L}$ in direction $\delta T$ is zero for every direction $\delta T \in \mathcal{S}_p$, namely

$$\mathcal{L}(T, \Lambda; \delta T) = \text{tr}(-T^{-1}\delta T + (S^{-1} + \Lambda)\delta T) = 0, \forall \delta T \in \mathcal{S}_p.$$ 

The form of the minimum is therefore $T_o = (S^{-1} + \Lambda)^{-1}$ provided that $\Lambda \in \mathcal{S}_p$ is chosen so that $S^{-1} + \Lambda > 0$. The
dual of Problem (6) is therefore
\[
\arg\max_{\Lambda \in Q_S} \mathcal{L}(T_o, \Lambda)
\]
subject to \( P_{\Omega} (\Lambda) = 0 \)
(8)
and \( Q_S := \{ \Lambda \in \mathbb{S}^p : S^{-1} + \Lambda > 0 \} \) is the domain of optimization. To remain in the convex setting, we change sign and minimize the opposite function; namely, we consider the following problem:

\[
\arg\min_{\Lambda \in Q_S} j_S(\Lambda)
\]
subject to \( P_{\Omega} (\Lambda) = 0 \),
(9)
where the dual functional (save for constant terms) is
\[
j_S(\Lambda) = -\mathcal{L}(T_o, \Lambda) + m = -\log \det (S^{-1} + \Lambda) + \text{tr} (T \Lambda).
\]

Remark 1: Performing the substitution \( T = (S^{-1} + \Lambda)^{-1} \) in (9), the dual functional \( j_S \) correspond (save for constant factors) to the negative log-likelihood \( \ell (T; \hat{T}) \) in (3). Hence, in view of (5), Problem (9) can be readily interpreted as a maximum-likelihood problem,

\[
\arg\min_{T \in \Theta_{S, A}} \ell (T; \hat{T}),
\]
over the parametric family \( \Theta_{S, A} := \{ T = (S^{-1} + \Lambda)^{-1} : \Lambda \in Q_S, P_{\Omega} (\Lambda) = 0 \} \).

The derivation of the dual problem shows that the support of the optimum \( T_o^{-1} \) is given by \( \Omega_o \cup \Omega_t \). If \( \Omega_t \) were known and if the relation \( \Omega_o \supset \Omega_t \) holds, then we could choose \( \Omega = \Omega_t \) so that the support of \( T_o^{-1} \) would be \( \Omega_o \); in this case, we have a PLP problem. Similarly, knowing \( \Omega_t \) and if the relation \( \Omega_o \supset \Omega_t \) holds, then we may choose \( \Omega = \Omega_o \) so that the support of \( T_o^{-1} \) would be \( \Omega_o \), provided that \( \Lambda \) is such that \( (\Lambda)_{ij} = -(S^{-1})_{ij}, \forall (i, j) \in \Omega_o \setminus \Omega_t \); this is the NLP problem. The requirement on \( \Lambda \) over \( \Omega_o \setminus \Omega_t \) produces zeros on \( T^{-1} \) so that its support becomes smaller than \( \Omega_o \), which is precisely what happens in NLP. In practice \( \Omega_t \) is unknown and the aforementioned constraints cannot be enforced. Equivalently, in view of (10), the parametric family \( \Theta_{S, A} \) is unknown. To overcome this issue, we estimate \( \Omega_t \) from the available data at time \( t \) by considering two different regularized versions of Problem (10) corresponding to the two link prediction problems:

- The PLP problem, for which \( \Omega_o \supset \Omega_t \), boils down to the regularized maximum-likelihood problem

\[
\arg\min_{T \in \Theta_{S, A}^P} \ell (T; \hat{T}) + \gamma_P h_P (T),
\]
(11)
where \( \Theta_{S, A}^P := \{ T = (S^{-1} + \Lambda)^{-1} : \Lambda \in \Omega_S \} \)
and
\[
h_P (T) = \sum_{(i, j) \in \mathcal{J}_P} |\Lambda_{ij}|
\]
with \( \mathcal{J}_P := \{(i, j) \in V \times V \setminus \Omega_o : i > j \} \). With this choice we are inducing sparsity among the \( \Lambda \)'s outside \( \Omega_o \) so that the support \( \Omega_o \) of \( T_o^{-1} = S^{-1} + \Lambda \) will contain \( \Omega_o \). It is worth noticing that from the definition of the index sets \( \mathcal{J}_P \) we are not penalizing the entries fixed by the prior support \( \Omega_t \). This will reduce the bias (i.e. the shrinking to zero of the entries) affecting the final estimate.

- The NLP problem, for which \( \Omega_o \subset \Omega_o \), can be formulated as a regularized maximum-likelihood problem as well,

\[
\arg\min_{T \in \Theta_{S, A}^N} \ell (T; \hat{T}) + \gamma_N h_N (T),
\]
(13)
where \( \Theta_{S, A}^N := \{ T = (S^{-1} + \Lambda)^{-1} : \Lambda \in \Omega_S, P_{\Omega} (\Lambda) = 0 \} \)
and
\[
h_N (T) = \sum_{(i, j) \in \mathcal{J}_N} |\Lambda_{ij} + (S^{-1})_{ij}|,
\]
with \( \mathcal{J}_N := \{(i, j) \in \Omega_o : i > j \} \). Here, we force \( \Lambda_{ij} = 0 \) on \( \Omega_o \) and induce \( \Lambda_{ij} \) to be equal to \( -S^{-1}_{ij} \) over \( \Omega_o \). Hence, in this case, \( \Omega_o \) will be contained in \( \Omega_o \).

Here \( \gamma_P, \gamma_N > 0 \) are the two regularization parameters, weighting the effect of the \( \ell_1 \)-penalization \( h_P \) and \( h_N \), respectively.

Remark 2: The approach outlined above, can be adapted in the case of a mixed link-prediction situation, in which one intends to predict both appearing and disappearing links. This may be achieved by considering the PLP regularized maximum-likelihood problem (11) and substituting the regularizer with a combination of (12) and (14),

\[
h_M (T) := \eta_P \sum_{(i, j) \in \mathcal{J}_P} |\Lambda_{ij}| + \eta_N \sum_{(i, j) \in \mathcal{J}_N} |\Lambda_{ij} + (S^{-1})_{ij}|
\]
where \( \eta_P > 0 \) and \( \eta_N > 0 \) are the two regularization parameters that contribute to enforce the sparsity of \( \Lambda \) over \( \Omega_o \) for the PLP and the fact that \( \Lambda_{ij} = -(S^{-1})_{ij} \) over \( \Omega_o \) for the NLP, respectively. The resulting regularized maximum-likelihood problem is

\[
\arg\min_{T \in \Theta_{S, A}^P} \ell (T; \hat{T}) + \gamma_P h_P (T) + \gamma_N h_N (T).
\]
(15)

Tuning \( \gamma_P \) and \( \gamma_N \) in (15) it is possible to emphasize the PLP or the NLP task, respectively.

It can be shown that Problems (11), (13) and (15) admit a unique solution. While uniqueness follows by showing that (in an equivalent formulation of the problem) the functional is strictly convex, the argument for existence is based on the fact that the functional goes to \( -\infty \) whenever its argument tends to the boundary of the feasible set or diverges.

A. Induced Similarity Measure

Let \( \hat{T}_o \) be the solution to Problem (11) (respectively (13), (15), depending on the link prediction problem we are considering). The latter not only characterizes the graphical model \( \hat{G}_t \) of \( x \) at time \( t \), but it also induces a similarity measure between the nodes of \( \hat{G}_t \). In fact, our estimation method naturally induces the score matrix which evaluates the inclination of pairs of nodes to be conditionally dependent:

\[
R_t := \text{diag}(\hat{T}_o)^{1/2} \hat{T}_o^{-1} \text{diag}(\hat{T}_o)^{1/2}.
\]
The measure of the similarity between node $i$ and node $j$ that comes out from our approach is therefore $r_{ij} = (R_t)_{ij}$. More precisely $-1 \leq r_{ij} \leq 1$ and the higher $|r_{ij}|$ is the more probable is that a link connecting node $i$ to node $j$ will appear (i.e. node $i$ and node $j$ will be conditionally dependent). It is worth noting that in order for our similarity measure to be a suitable topology-selection measure according to relations (1)-(2), a thresholding procedure is needed:

$$
(i, j) \in \Omega_t \iff |r_{ij}| > t_r, \quad (16)
$$

where $t_r > 0$ has to be set properly.

IV. Experimental Results

In this section we show how the proposed method works in predicting appearing/disappearing edges in a simple network of $m = 10$ nodes. The thresholding value $t_r$ is set to $10^{-4}$ throughout all the experiments. First we present the results for PLP. Figure 2 reports the support $\Omega_t$ of the prior $S^{-1}$ at time $s$ and the true support $\Omega_t$ of the concentration matrix $T^{-1}$ at time time $t > s$, that we want to estimate. Such a network is an unfriendly positive link prediction network. Indeed, consider the “common neighbors” similarity measure [2] at time $t$:

$$
(CN_0)_{ij} = \text{card}(N_i \cap N_j)
$$

where $N_i$ is the set of neighbors of node $i$, according to the topology of the “prior” network $\Omega_s(V, \Omega_s)$, and $\text{card}(N_i \cap N_j)$ denotes the cardinality of set $N_i \cap N_j$. Then, it is not difficult to see that: $(CN_0)_{ij} = 1$ for the pairs (3,6) and (8,9); $(CN_0)_{ij} = 0$ for the other $(i, j) \notin \Omega_s$. Therefore, this similarity measure mispredicted the appearance of 3 edges.

Starting from the left of Figure 3 we have three versions of the estimated support $\hat{\Omega}_t$ for $\gamma_P = 0.01$, 0.08, 0.5, respectively using our method. By comparison with Figure 2, $\gamma_P = 0.08$ is the best choice, in that the estimation procedure infers exactly the true $\Omega_t$. For $\gamma_P = 0.01$ the effect of regularization is too mild, i.e. sparsity is not properly enforced, while $\gamma_P = 0.5$ gives an exaggerated sparse estimate, as one should expect. The estimation has been performed by solving Problem (11) with the CVX package for Matlab [28], [29]. In particular, the sampling covariance matrix $\hat{\Sigma}$ has been computed as in (4) using $N = 1000$ i.i.d. observations drawn from the distribution $\mathcal{N}(0, T)$.

![Fig. 3: Estimates of the support $\Omega_t$ for different values of the regularization parameters $\gamma_P$.](image)

The NLP experiment has been performed along the same lines of the PLP one. As before, we consider a network of $m = 10$ nodes whose edges are prescribed by the support $\Omega_s$ of the prior concentration matrix $S^{-1}$. As depicted in Figure 4 (left). The true support $\Omega_t$ of the concentration matrix $T^{-1}$ has been obtained by setting to zero some of the elements in $\Omega_s$ (in blue) and it is represented in Figure 4 (right). This network appears to have the “unfriendly prediction” property as the method in [4, Sect. 4.2.2] applied by “reversely” implementing the PLP similarity measure (17) provides poor performances. More precisely, it correctly predicts only the disappearance of the edge (8,10), while it erroneously predicts the disappearance of edge (4,5) and does not predict the disappearance of the two edges (6,7) and (6,8) that indeed disappear. Thus, in total, 3 edges are mispredicted.

![Fig. 4: NLP case. Support $\Omega_s$ of the prior concentration matrix (left) and support $\hat{\Omega}_t$ of the concentration matrix $T^{-1}$ that has to be estimated (right).](image)
the previous examples. For the PLP case, the model with smallest $\hat{E}_r$ is the one with $\gamma_P = 0.08$, as expected. In regard to the NLP case, the model with smallest $\hat{E}_r$ is the one with $\gamma_N = 0.15$. It is worth noting that the latter is still better than the one obtained by the procedure in [4, Sect. 4.2.2]; indeed, our model with $\gamma_N = 0.15$ mispredicts only 1 edge.

$$\begin{array}{c|c|c|c|c}
\hat{E}_r(T_0) & \gamma_P & \gamma_N & \hat{E}_r(T_0) & \gamma_P \\
PLP & 0.026 & 0.01 & 0.15 & 0.0618 \\
& 0.0132 & 0.08 & 0.26 & 0.0877 \\
& 0.0432 & 0.5 & 2 & 0.3206 \\
NLP & & & & \\
\end{array}$$

Fig. 6: Relative estimation error $\hat{E}_r$ for the different regularization parameters $\gamma_P$'s and $\gamma_N$'s considered in the above PLP and NLP examples.

V. CONCLUSIONS

In this work an estimation method based on $\ell_1$-regularized maximum likelihood has been proposed and applied to link prediction problems. Both positive and negative link prediction problems have been formulated as convex optimization problems, whose solution is unique. The most significant contribution of the work is the introduction of a similarity measure exploiting not only the topology of the current network (prior), but also noisy information coming from the network at the current time rather than some properties the network is expected to have. The numerical examples have shown that our method can be a potential alternative to some of the existent link prediction approaches.

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