# Key Issues Review

## Key issues review: numerical studies of turbulence in stars

**W David Arnett and Casey Meakin**

Steward Observatory, University of Arizona, 933 N. Cherry Avenue, Tucson, AZ 85721, USA

E-mail: darnett@as.arizona.edu

Received 21 February 2015, revised 4 July 2016
Accepted for publication 14 July 2016
Published 22 September 2016

**Abstract**

Three major problems of single-star astrophysics are convection, magnetic fields and rotation. Numerical simulations of convection in stars now have sufficient resolution to be truly turbulent, with effective Reynolds numbers of $Re > 10^4$, and some turbulent boundary layers have been resolved. Implications of these developments are discussed for stellar structure, evolution and explosion as supernovae. Methods for three-dimensional (3D) simulations of stars are compared and discussed for 3D atmospheres, solar rotation, core-collapse and stellar boundary layers. Reynolds-averaged Navier–Stokes (RANS) analysis of the numerical simulations has been shown to provide a novel and quantitative estimate of resolution errors. Present treatments of stellar boundaries require revision, even for early burning stages (e.g. for mixing regions during He-burning). As stellar core-collapse is approached, asymmetry and fluctuations grow, rendering spherically symmetric models of progenitors more unrealistic. Numerical resolution of several different types of three-dimensional (3D) stellar simulations are compared; it is suggested that core-collapse simulations may be under-resolved. The Rayleigh–Taylor instability in explosions has a deep connection to convection, for which the abundance structure in supernova remnants may provide evidence.

Keywords: turbulence, stellar evolution, supernovae, three-dimensional fluid simulations

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Stars are gravitationally bound thermonuclear reactors, made of plasma. Because of their large size, shearing motion implies enormous Reynolds numbers, and turbulence. Turbulent plasma generates magnetic fields, which in turn resist shearing motions, and so establish a deep connection between turbulence, rotation and magnetic fields in stars.

Consider a thought experiment in which the effects of plasma charges are removed so that there is no current to provide a magnetic field. Then we have a neutral fluid flow, turbulent, with rotation, similar in some respects to an atmosphere or an ocean. Now imagine there to be no rotation relative to an inertial frame. There remains a turbulent, self-gravitating fluid, heated in the center and cooled at the surface. We begin with this simpler state as a foundation to build up to the full problem.

Most of the history of the theory of stellar evolution is based on the approximation that stars have spherical symmetry, and are quasi-static objects. This flies in the face of observations of the surface of the Sun, as time-lapse videos of sunspots, eruptions and granulation show, so that the approximation must be stated more carefully as spherically symmetric on average. Spherical symmetry on average arises from the cancellation of effects, from the combined action of many chaotic fluctuations such as those which the solar videos capture. The

---

1 The Reynolds number is the ratio of inertial forces to viscous forces, and is expressed as $Re = \frac{ul}{v}$, where $l$ is a characteristic length (which is large for stars), $u$ a characteristic velocity, and $v$ the kinematic viscosity (Landau and Lifshitz 1959).

2 e.g. https://svs.gsfc.nasa.gov/cgi-bin/details.cgi?aid=3412
fortunate approximations of spherical symmetry and quasi-static behavior allow the reduction of a three-dimensional (3D) time-dependent problem to a one-dimensional (1D) quasi-static one, which is amenable to numerical simulation by computer as 50 years or so of progress have shown.

Numerical simulation of turbulent convection in stars is now feasible, and permits a theoretical study of the solutions to the 3D, time-dependent equations of fluid flow, and consequently a critical re-examination of the spherical, quasi-static approximation for stellar evolution. As a prelude to a solution to the full problem, involving rotation, magnetic fields and turbulence together, and ranging from solar physics to gamma-ray bursts (GRBs), gravitational wave generation and supernovae, we focus on stellar turbulence as one of the necessary and key components to an understanding of the complete evolution of stars.

The discussion will involve several fields (astrophysics, solar physics, numerical methods, supercomputing, fluid flow, turbulence, etc), each of which has developed its own jargon to facilitate concise discussion; the jargon is well known to insiders but difficult for readers new to that field. As a compromise for a broader readership, we attempt to introduce each new instance of ‘jargon’ with a physical definition and a reference, before using the shortened terminology.

2. Methods

Table 1 summarizes the numerical 3D simulations (using the PROMPI and MUSIC codes, (Meakin and Arnett 2007c, Viallet et al 2013, Arnett et al 2015)), upon which this discussion is based. Details may be found in those references. In 3D the turbulent cascade moves from large to small scales, so the implicit large-eddy simulation approach (ILES) of Boris (2007) maximizes effective use of computational resources. To attain higher resolution a ‘box-in-star’ grid configuration is used, so that only a part of the star is simulated on the grid. This gives a maximal numerical Reynolds number (minimal numerical viscosity), and allows truly turbulent behavior using presently available computers. There is a drawback: the limited size of the box, being smaller than the star, does not support the lowest order modes of motion. A resolution study (a range of 8 in linear resolution, or $8^3 = 4096$ in computational resources) suggests that the turbulent cascade and the convective boundaries may be adequately captured (Arnett et al 2015). These simulations use realistic microphysics for the equation of state (EOS), opacity, nuclear reaction network (25 nuclei), and neutrino emission. The initial model was mapped from a well-resolved 1D stellar evolutionary model onto the 3D computational grid.

A novel test is introduced, in which the residuals from conservation laws are monitored to check both the validity of the ILES approximation, and the zoning in boundary layers (Viallet et al 2013, Arnett et al 2015). This Reynolds-averaged Navier–Stokes (RANS) analysis points to regions of inadequate numerical resolution, and quantitatively measures those errors (Viallet et al 2013, Mocák et al 2014, Arnett et al 2015, Cristini et al 2015).

A number of other efforts have contributed to our improved understanding of 3D stellar convection. Herwig et al (2011), (2014) have conducted 3D simulations of similar resolution and character (on proton ingestion during He flash convection) using a ‘star-in-box’ approach (the entire star is enclosed by the computational grid, allowing even the lowest order modes to be represented). See earlier work in Porter et al (1999), Porter and Woodward (1994), (2000). Campbell et al (2016a) show ‘box-in-star’ simulations of the upper boundary of an $8 M_\odot$ star, which is a related ingestion problem.

The earliest successes of 3D radiative hydrodynamics were the pioneering simulations of the solar atmosphere by Nordlund and Stein, which reproduced the observed shapes of spectral lines without adjustable parameters. See Magic et al (2013), Nordlund (1985), Nordlund and Stein (2000), Nordlund et al (2009), Stein and Nordlund (1989), (1998). These simulations were in the ‘box-in-star’ mode, and treated magnetic fields on the grid in the MHD approximation, but contained no composition gradients or nuclear-burning. See Freytag et al (1996), (2012), Ludwig et al (1999) and Ludwig and Kucinskas (2012) for further work on 3D stellar atmospheres.

Juri Toomre and collaborators had a different focus, on rotation and magnetic fields in the ‘star-in-box’ mode, developing the MHD anelastic code ASH (Hurlburt et al 1984, 1986, Massaguer et al 1984, Cattaneo et al 1991, Brummell et al 2002, Brun et al 2004, Miesch 2005, Miesch et al 2008, Brun and Palacios 2009, Brun et al 2011). See also the discussion concerning energy conservation in Brown et al (2012) and Vasil et al (2013). The ASH simulations used a simplified 3

### Table 1. Some turbulence simulations with RANS analysis.

| Reference            | Type  | Code | Grid size | Name       |
|----------------------|-------|------|-----------|------------|
| Meakin and Arnett    | OB    | PROMPI | 400 × 100 | lo-res     |
| Viallet et al (2013) | OB    | PROMPI | 786 × 512 | hi-res     |
|                     | OB    | PROMPI | 384 × 256 | med-res    |
|                     | OB    | PROMPI | 192 × 128 | lo-res     |
|                     | RG    | MUSIC | 432 × 256 | m-med-res  |
|                     | RG    | MUSIC | 216 × 128 | m-lo-res   |
| Campbell et al (2015)| OB    | PROMPI | 1536 × 1024 | ultra-hi-res |

a Oxygen-burning shell in pre-collapse star of $23 M_\odot$.
b Red giant envelope convection, Viallet et al (2011).c Paper in preparation.
atmospheric boundary, and contained no composition gradients or nuclear-burning, but considered rotation and magnetic fields in detail.

Adaptive mesh refinement (AMR) is an attempt to attain better resolution with limited computational resources; this has been implemented in several stellar astrophysics codes: the compressible hydrodynamics code FLASH (Fryxell et al 2010), the low-Mach-number hydrodynamics code MAESTRO (Nonaka et al 2010), and the compressible hydrodynamics code CASTRO (Almgren et al 2010). The basic idea is to use fine zoning in regions where it is needed and coarser resolution elsewhere. For this to be effective there must be regions on the grid in which little is happening, so the degree of effectiveness of AMR may depend upon the problem being tackled. For example, turbulence expands to fill the volume allowed, and be a broadly distributed, while in contrast, nuclear-burning is highly sensitive to temperature, and may be localized. By omitting the AMR computations, PROMPI gains some computational efficiency for the turbulence problem, but FLASH (which originally was akin to PROMPI without AMR) should have an advantage for problems in which the action is more localized.

It would be desirable to have studies similar to Viallet et al (2013) to quantify how well the AMR codes reproduce the RANS variables determined from a fixed grid code like PROMPI, otherwise using the same solution algorithms. FLASH and CASTRO are more nearly comparable in nature; a RANS analysis of a MAESTRO simulation of turbulence is needed. If the energy error in ASH (Brown et al 2012, Vasil et al 2013) were fixed, it should be possible to do a RANS analysis as well. This would help find unresolved boundary behavior, which is even more an issue for ‘star-in-box’ simulations.

Extensive theoretical, numerical and experimental information from the geosciences has been analysed by Canuto 2012a, 2012b, 2012c, 2012d, 2012e, for application to the problem of stellar convection and mixing.

Given the variety in emphasis, resources and interests, these 3D simulations are strikingly consistent, which encourages this attempt to explore their implications.

3. Results

Qualitatively new and previously overlooked behavior appears in the simulations indicated in table 1 and referred to in section 2 above.

3.1. Dynamics

Stellar turbulent convection is a strikingly dynamic process (Meakin and Arnett 2007c). This is indicated in figure 1, which shows the dynamic onset of convection from a carefully zoned 1D initial model mapped to a 3D grid. Because of the inadequacy of the present algorithms for 1D turbulent convection, there is always an initial transient in which dynamically self-consistent convective flow is established. This is followed by bursts of turbulent kinetic energy, recurring on a time interval similar to the time needed to transit the convective layer. These bursts encounter stable layers, and drive wave motions (predominantly gravity modes) from both boundaries, most obviously from the top. The boundaries are eroded by entrainment. Both the rate of entrainment, and the amplitude of the waves, are affected by the stiffness of the stable layer (the value of the Brunt-Väisälä frequency), and are not some universal constant. The behavior is also modified by radiative effects, as shown by comparison of the oxygen-burning shell (OB) and the lower boundary of the surface convection zone of a red giant (RG) (Viallet et al 2013), which have significantly different values of the Péclet number (Viallet et al 2015).

3.2. Reynolds-averaged Navier–Stokes equations

At present the highest resolution grids have $\sim 10^9$ grid points, which for a 25-nucleus network implies a storage requirement of $\sim 3 \times 10^9$ individual numbers per timestep. The challenge is to abstract the essential features from this flood of data. The Reynolds-averaged Navier–Stokes (RANS) equations provide a systematic way to condense the data and to provide insight for the development of approximate equations to describe it (Mocák et al 2014). Two powerful features of this formulation are that it allows a quantitative evaluation of the sub-grid dissipation, and it explicitly allows the identification of errors due to inadequate resolution (Viallet et al 2013). The latter is particularly useful in assessing the numerical quality of simulations of boundaries between turbulent and non-turbulent regions.

3.3. Kolmogorov cascade

There is a balance, on average, between buoyancy and turbulent dissipation (Arnett et al 2009, Arnett and Meakin 2011b). Buoyancy occurs as a radial acceleration at the largest scales. At the large Reynolds numbers typical of stars, the motion is unstable and breaks-up into smaller scale motion,
becoming isotropic and losing its 1D character. This continues until the ‘Kolmogorov scale’ is reached, that is, the point dissipative processes can damp the motion. For these ILES simulations this damping occurs at the sub-grid scale. With sufficient resolution (e.g. see table 1, Herwig et al (2014) and Nonaka et al (2012)), the turbulent cascade is reasonably well approximated.

3.4. Lorenz convective roll

The largest scale (the integral scale) behaves like the convective roll of Lorenz (1963), which is a classic example of a dynamic system with a strange attractor and chaotic behavior (Manneville 2010). A combination of the Lorenz model and the Kolmogorov cascade give a simple but qualitatively consistent representation of the 3D simulation (Arnett et al 2009, Arnett and Meakin 2011b). The steady-state limit of this combination gives the steady vortex model, from which the mixing-length theory (MLT) cubic equation of Böhm-Vitense (1958) can be derived (Smith and Arnett 2014), a process which illustrates the strength and limitations of MLT.

3.5. Fluctuations and Rayleigh–Taylor instability

If a steady-state constraint is not imposed, the balance between buoyant driving and turbulent damping is not satisfied instantaneously, but only on average. Because driving and damping occur at widely different length scales, they are not synchronized in time, causing the intermittent pulse behavior seen in figure 1. If the driving is sufficiently strong, an explosion may develop instead of steady-state convection, with the rising plumes becoming Rayleigh–Taylor (RT) unstable and, depending upon the structure, turbulent (Swisher et al 2015). Expansion causes a ‘freeze-out’ of the motions (Abarzhi 2010) and of the compositional structure, tending toward a spherical explosion (a Hubble flow).

3.6. Turbulent kinetic energy flux

Convective regions possess a basic up–down asymmetry, which increases with stratification. In a hydrostatic stratification, rising material expands because it encounters decreasing pressure, and vice versa. To conserve mass, rising plumes will tend to be broader and slower than descending plumes. This difference requires a net downward flux of turbulent kinetic energy (TKE) (Meakin and Arnett 2010). This is pronounced in convective atmospheres, which are strongly stratified, containing many pressure scale heights. Net flux of turbulent kinetic energy is assumed to be zero in MLT, but is nonzero and a prominent feature of 3D simulations of stellar atmospheres (Nordlund and Stein 1995, Nordlund et al 2009).

3.7. Boundary layers

A key feature of sufficiently resolved 3D simulations (Herwig et al 2014, Arnett et al 2015) is the development of thin layers which separate regions of potential flow (non-turbulent) from those of solenoidal flow (turbulent). Because MLT is a local theory, it has no ability to describe such boundaries. By convention this gap is addressed by linear stability analysis, resulting in use of the criteria of Schwarzschild (no composition gradient) and Ledoux to define boundaries of mixing. In the geosciences a more sophisticated analysis leads to various versions of the Richardson criterion (see Turner (1973), section 10.2.3), which includes both the degree of convective vigor and the stiffness of the stable region. The Richardson criterion is a ratio of the potential energy required to mix a stably stratified layer, divided by the turbulent kinetic energy available to do so. This preference of a Richardson criterion for mixing is supported by the simulations (section 3.5, Arnett et al (2015)). The Ledoux criterion only includes the effects of a composition gradient, but not convective vigor, and so underestimates mixing. This may be why the Schwarzschild criterion is preferred over the Ledoux. The Schwarzschild criterion ignores the inhibiting composition gradient term, is simpler, and better agrees with astronomical observations. Accurate estimates of both the composition gradient and the convective speed are needed to define the true mixing boundary.

3.8. Composition currents

The mixing of composition is not imposed, the balance between buoyant driving and turbulent damping is not satisfied instantaneously, but only on average. Because driving and damping occur at widely different length scales, they are not synchronized in time, causing the intermittent pulse behavior seen in figure 1. If the driving is sufficiently strong, an explosion may develop instead of steady-state convection, with the rising plumes becoming Rayleigh–Taylor (RT) unstable and, depending upon the structure, turbulent (Swisher et al 2015). Expansion causes a ‘freeze-out’ of the motions (Abarzhi 2010) and of the compositional structure, tending toward a spherical explosion (a Hubble flow).
As collapse is approached, temperature rises. Neutrino cooling increases, balanced by increased heating from nuclear-burning, which gives increased turbulent velocity. The increased vigor of convection causes increased fluctuation amplitudes for density as well as tangential velocity. The core is the least stratified part of the star, so that the convective cells are larger and fewer, giving a less smooth cancellation of chaotic behavior. As collapse is approached, the equation of state (EOS) is softer (the adiabatic exponent nears the critical value $\Gamma \to 4/3$), so that radial restoring forces decrease, giving larger amplitude fluctuations.

This tendency for dynamic behavior was already indicated in 2D simulations, (Bazán and Arnett 1994, 1997a, 1997b, 1998, Asida and Arnett 2000, Meakin and Arnett 2006, Arnett and Meakin 2011a). At present, pre-collapse progenitor models have spherical symmetry, and no fluctuations or tangential velocities (e.g. Woosley and Weaver (1995) and Woosley et al (2002)). The simulations, both 2D and 3D, show that these progenitor models represent an extreme and non-physical limiting case.

The problem of inertial confinement fusion (ICF) is similar to core-collapse in that a compression by a large factor ($\sim 100$) is involved (Remington et al 1999, 2000). Experiments show the extreme importance of slight asymmetries in the initial state prior to compression; these can modify even the qualitative nature of the flow (Lindl 1998, Drake 2005), a feature shared with core-collapse (Couch and Ott 2013).

4.2. Precollapse dynamics

In addition to changes in shape, 3D simulations also show significant additional changes in time, which also can affect the evolution up to and into collapse. Waves are generated at the convective boundaries (figure 1), which give rise to entrainment and mass loss (Meakin and Arnett 2007c). For the pre-collapse stages, these waves may not have time to travel to the stellar surface, and they may dissipate along the way. The mass loss may be extensive (Quataert and Shiode 2013, Shiode and Quataert 2014), and the dynamics may even lead to eruptions prior to collapse (Arnett and Meakin 2011a), as suggested by observations of SNIIn (Smith and Arnett 2014). In order to get ‘Type I’ light curves for SNIbc supernovae, significant mass loss must occur prior to collapse (Arnett 1982); wave-driven mass-loss and eruptions may be alternatives to binary stripping (Arnett 1980).

The 3D morphology of silicon-burning (Couch et al 2015) changes from that of oxygen-burning, because the time scale for nuclear-burning approaches the turnover time scale for convective flow. The further evolution of Si-burning to nuclear statistical equilibrium (NSE) in a convection zone corresponds to a system having both mechanical and thermodynamic degrees of freedom (see Landau and Lifshitz (1969), section 123), an unexplored theoretical aspect of the approach to collapse indicated by 3D simulations.

4.3. Implications for core-collapse simulations

Core-collapse of a massive star leads to the formation of a neutron star or a black hole, but the details of this process have presented continuing difficulty. Janka et al (2016) have recently reviewed the physics of core-collapse in 3D. core-collapse simulations add the calculation of neutrino spectrum transport to the computational challenges already in 3D simulations of Arnett et al (2015) and Herwig et al (2014), and so cannot afford equal resolution. In addition, core-collapse simulations require a star-in-box approach (see section 2); a full 4π geometry is necessary. The Oak Ridge group used the CHIMERA code with a $540 \times 180^2$ grid, and the Garching group used the PROMETHEUS-VERTEX code with a $600 \times 90 \times 180$ grid. If these were box-in-star simulations, they would both correspond to the low-resolution examples in table 1. However, as star-in-box simulations, they correspond to a resolution still coarser by roughly a factor $\sim 4 \times 8 = 32$, as only 1/32 of the stellar volume needed to be simulated in the box-in-star case, with a comparable number of zones. The star-in-box simulation of Herwig et al (2014) used a grid of 1536$^3$ points and a refined numerical algorithm to improve resolution (see their appendix).

For comparison, Stein and Nordlund (1998) used grids of 125$^2 \times 82$ and 253$^2 \times 163$ for satisfactory resolution of the solar atmosphere, but in the box-in-star mode; they computed a $6^2 \times 3\text{Mm}^3$ box ($1.88 \times 10^{26}$ cm$^3$), or $7.5 \times 10^{-5}$ of the solar volume. Clearly a star-in-box simulation, which is necessary to represent core-collapse, would require additional zones to maintain resolution. This extreme requirement is softened, but not removed, by the fact that the solar opacity is more temperature dependent that the neutrino opacities (giving stronger gradients), and the solar density gradient is steeper (requiring more convective cells) for the solar case.

A comparison of zoning in the ASH code is more complex. Miesch et al (2008) quote a value $257 \times 1024 \times 2048$ for a star-in-box simulation, and state that ‘previous simulations did not have sufficient resolution to capture such dynamics’, which is consistent with the general trend discussed above.

With their lower demand on computational resources, 2D simulations might seem tempting, but they are inadequate for turbulent simulations due to the different cascade direction relative to 3D. The flows are simply different; see figure 4 in Meakin and Arnett (2007c). Simulations in 2D is more useful for debugging code (and ideas) than describing stars.

The 3D simulations in table 1 suggest that bulk features of turbulence are reproduced even in the low-resolution cases, but that more resolution is needed for boundaries. Are the problem areas discussed by Janka et al (2016) more akin to bulk features or to boundaries? The gain-region is sensitive to differences in tangential flow (Couch and Ott 2013), a feature itself affected by resolution. The standing accretion shock

\footnote{Previously the mechanical degrees of freedom were ignored during the approach to equilibrium; see Clayton (1983).}

\footnote{10 See the tortuous development in Arnett 1967, 1977a, 1977b, 1980, 1996 for example.}

\footnote{11 Except perhaps in cases in which there is a physical rather than computational motivation for 2D. A planetary atmosphere of shallow height, a strong magnetic field, or a strong rotation might give reason to use 2D, but history is littered with cases in which 3D turned out to be different in important ways from 2D.}
instability (SASI, Blondin et al. (2003)) seems more global, and might be captured with even the coarse resolution now used. The lepton-number emission asymmetry (LESA, Tamborra et al. (2014)) may be similar to convection currents seen in O-burning and proton ingestion simulations at higher resolution (see section 3). Comparison of figure 4 in Janka et al. (2016) to figure 2 in Herwig et al. (2014) and figure 1 in Miesch et al. (2008), all star-in-box simulations, suggests that higher resolution may be needed for the collapse simulations.

The 3D simulations of pre-collapse stages also have implications for the collapse process. They suggest that the boundaries may become steeper during the neutrino-cooled stages, as collapse is approached. This would affect core size and density stratification in the progenitor. It may decrease the rate of accretion on the newly formed neutron star; this would reduce the photodisintegration losses which kill the original bounce shock and oppose shock rebirth. A reduced accretion rate might also result from pre-collapse eruption or extensive mass loss.

It should be stressed that the integral scale turbulence carries the most momentum and kinetic energy, does the most transport, and is the least isotropic. Consequently the isotropy of collapse, or lack thereof, is an important feature.

4.4. Implications for thermonuclear supernovae

Carbon-burning under conditions of high electron degeneracy begins with a stage of convective instability (‘simmering’) which temporarily delays the thermonuclear runaway (Arnett 1968, 1969b). This ensures that the runaway begins in a fully 3D and chaotic manner. The time-dependent convection algorithm originally used was similar to that subsequently derived from 3D simulations (Arnett et al. 2015). Under-resolved simulations would tend toward laminar flow (a dipole velocity field), but finer zoning results in chaotic, turbulent flow.

The first simulations of the delayed detonation model of thermonuclear supernovae, with detailed nucleosynthesis and relatively high 3D resolution, were performed by Seitenzahl et al. (2013), using a 5123 grid and the LEAFS code. They did not model the ignition phase. They treated the ‘unresolved sub-grid acceleration of flame due to turbulence’ with a sub-grid model, and did nucleosynthesis by tracer particles. Fink et al. (2014) did pure deflagration models with the same code and grid.

The simmering phase was investigated by Nonaka et al. (2012) and Zingale et al. (2011) using the low-Mach code MAESTRO, with grids of 3843 and 5763; MAESTRO allowed time steps which were up to a factor of 18 larger than a compressible code. After the burning developed, this simulation was then continued with the compressible hydrodynamics code CASTRO, Malone et al. (2014); using AMR the effective resolution was quoted as 11523.

Extension of simulations to investigation of the boundary between core-collapse and thermonuclear explosion of ONe cores (Jones et al. 2016), and to the white dwarf merger channel for SNIa and merger in general and common envelope evolution (Ohlmann et al. 2016) is beginning to be made. The resolution issue seems particularly acute for mergers because of the steep gradient at the surface of each star (a boundary problem).

4.5. Young SN remnants

There is a deep connection between convection and the Rayleigh–Tayl instability (RTI). The analysis of the RTI leads to differential equations (Abarzhi 2010, Swisher et al. 2015) which are similar to those resulting from analysis of the 3D simulations of turbulent convection (Arnett et al. 2015). Roughly speaking, convection starts from a plume which breaks up into the turbulent cascade, and being contained in a given volume, repeats the process (figure 1). The RTI begins in a similar way, but is uncontained, so that expansion converts internal energy into kinetic energy by $\frac{\partial P}{\partial V}$ work. In this sense, the RTI is a process of convective freeze-out. The RTI converts the initial compositional structure of a supernova progenitor into a supernova remnant, so that the composition, distribution, and velocity of the filaments contain information about both the progenitor and the explosion. This is why young supernova remnants can be poorly mixed (Hughes et al. 2000).

This became clear with Cas A (Kirshner and Chevalier 1977, Chevalier and Kirshner 1978), which showed the previously predicted compositional signature of incomplete oxygen-burning—overabundances of Si, S, Ca as well as O; Truran and Arnett (1970)—and the velocity signature of plumes from incomplete overturn. This picture was extended with x-ray data (Becker et al. 1979), and Chandra observations (Hughes et al. 2000), which showed plumes, and an overabundance of Fe—which was predicted to be formed as radioactive 56Ni from explosive Si-burning (Truran et al. 1967, Woosley et al. 1973). Analysis by Dopita (1987) indicated abundances of Ar—explosive O-burning, and Ne—predicted from explosive C-burning; Arnett (1969a), Woosley et al. (1978). The radioactive nucleus 44Ti is a key link in the chain of reactions leading to the iron peak in an explosion (Truran et al. 1967), and was predicted to have an abundance that might be detected, as observations by NuStar (Harrison et al. 2013, Boggs et al. 2015) have now confirmed. Hubble space telescope (HST) observations (Fesen and Milisavljevic 2016) have refined and extended our understanding of the velocity-composition structure. Reliably connecting 3D simulations of the explosion to this observed structure is an ongoing challenge (Young et al. 2008), and one which requires self-consistent 3D models of both the progenitor (see sections 4.1 and 4.2) and the collapse-explosion process (sections 4.3 and 4.4).

4.6. Mixing boundaries and core helium-burning

The CoRoT and Kepler satellites have produced asteroseismological data which strain the present generation of stellar models. See Aerts et al. (2010) for an overview, and Schindler et al. (2015) and Constantino et al. (2015), (2016) for some examples. This is particularly acute for the stage

13While useful for this problem, this speed-up is unfortunately far too small to be of value for most stellar evolutionary stages.
of core helium-burning. Asteroseismology indicates that the real stellar cores during He-burning are significantly larger than those predicted, and that adjustment of parameters which were satisfactorily set during hydrogen-burning, is no longer adequate. Large adjustments of these parameters (‘overshoot’) within the accepted algorithmic framework produces implausible results, but still does not resolve the discrepancy with asteroseismology. It appears that a better understanding of the boundary physics during He-burning is required.

The 3D simulations indicate that the Schwarzschild criterion as normally used will underestimate the size of the mixing region, and this error may be exacerbated by the opacity behavior of the He-burning ashes. H-burning produces He, which reduces the opacity; He-burning produces C and O, which increase the opacity. An increased opacity tends to increase convective mixing and core size, so that there may be positive feedback on mixing during core He-burning.

5. Conclusion

5.1. Weather and climate

While 3D simulations of the evolution of supernova progenitors is becoming feasible, this is not true for earlier stages. Core convection during main sequence hydrogen-burning involves many (≥10⁶) turnover times to complete the exhaustion of fuel. The long-term evolution of a non-linear quasi-steady-state system is a difficult problem; it involves the challenge of extrapolating the weather to determine the climate. The steady-state approximation (Arnett and Meakin 2011b) discussed in section 3 is plausible but of unproven validity for stars.

5.2. Effective Reynolds number

Finite grid resolution requires a loss of information below the grid scale, so that numerical simulations have an effective Reynolds number which is determined by this ‘numerical viscosity’. The real Reynolds number in stars (Re ~ 10⁶ or more) is much larger than the effective Reynolds number in the present-day best-resolved simulations (Re ≲ 2 × 10⁴), so that we are forced to assume that it is valid to extrapolate the computed behavior to the stellar case (Arnett et al. 2014).

However, terrestrial experiments at the largest Reynolds numbers may not support this extrapolation (Dimotakis 2001). At present we must extrapolate, but should also test the results with due caution.

The Reynolds number is Re ~ u(ℓ)ν, where u is a characteristic flow velocity, ℓ is a characteristic length, and ν is the kinematic viscosity. If we define a numerical, effective viscosity by νnum ~ ΔuΔr, where the new length Δr is a typical zone size and Δu is the velocity fluctuation across Δr assuming a Kolmogorov cascade, we may define the corresponding effective Reynolds number as Re eff ~ [ℓ/(Δr)]⁴/³, where ℓ/(Δr) is a measure of the inverse of the linear resolution. Laboratory fluids exhibit a transition from laminar to turbulent flow at a critical value of Re; it varies with problem, but is Re ~ 10³ or so.¹⁴ The numerical resolution study mimics this behavior (table 1), with increasing resolution allowing more of the turbulent cascade to be manifest (Viallet et al. 2015). Simulations of low resolution may be badly misleading because of their high numerical viscosity, which tends toward laminar flow, suppressing turbulence.

5.3. Mach numbers

The ratio of fluid speed to sound speed, the Mach number Mₐ, varies enormously in stars. At the bottom of the solar convection zone the flow velocity is ~ 10⁻⁴ cm s⁻¹ so that Mₐ ~ 10⁻⁴, while at the surface the Mach number rises to ~ 0.2 or so. In pulsating stars and explosions, shocks form, so Mₐ > 1, and even >> 1. The numerical simulation of the flow at velocity v, on a space-time grid of length Δr and time step Δt, is limited by Δt < Δr/(ν + s), the causality condition, where s is the sound speed. If v ≪ s, the time step is limited by the sound speed, not the flow velocity. Various ‘sound-proofing’ algorithms (low Mach-number solvers) replace this condition with Δt < Δr/ν, where the sound speed is eliminated (as are sound waves), allowing larger time steps. The increase in time step size in practice is ~ 10 to 100 (see references above to ASH, MAESTRO, and MUSIC, for example). For flows with Mₐ ≪ 0.01, there are problems with the algorithms used in ‘explicit Riemann solvers’, which affect the PROMPI and FLASH codes; these methods over-damp small velocities, killing very weak convection. For H-, He- and C-burning, the variables (rates of heating and cooling) must be scaled to avoid this (Meakin and Arnett 2007c). Ne-burning may be a borderline case, and for later stages no such scaling is necessary; the convective Mach numbers are Mₐ ≥ 0.01, and such explicit solvers (with no sound-proofing) can scale well on the multi-core architectures of current supercomputers.

Scaling to larger numbers of processors is a crucial measure of usage of computer power: does the code take half the time on twice the processors (perfect scaling)? A small part of the work which cannot be shared can ruin scaling, leaving many of the multi-processors idle. MUSIC scales moderately well (see table 1, but PROMPI was designed for multi-processors and the multi-processors idle. Explicit solvers (with no sound-proofing) can scale well on the multi-core architectures of current supercomputers. While 3D simulations of the evolution of supernova progenitors is becoming feasible, this is not true for earlier stages. Core convection during main sequence hydrogen-burning involves many (≥10⁶) turnover times to complete the exhaustion of fuel. The long-term evolution of a non-linear quasi-steady-state system is a difficult problem; it involves the challenge of extrapolating the weather to determine the climate. The steady-state approximation (Arnett and Meakin 2011b) discussed in section 3 is plausible but of unproven validity for stars.

5.4. Carbon-, neon- and silicon-burning

Stellar evolutionary models of H-burning and He-burning may be tested by conventional astronomical observations;
for example, by their positions in the Herzspring–Russell (HR) diagram, by observations of eclipsing binaries, and by asteroseismology (g-modes and mixed modes; section 4.6). However the stages of C-, Ne-, O- and Si-burning are short-lived because of neutrino-cooling, and must be tested in other ways. Because of the different Péclet numbers\(^{15}\), calibrations from earlier photon-cooled stages will be invalid. Fortunately direct numerical simulations of these stages are feasible. We have results from O-burning (Meakin and Arnett 2007c, Viallet et al 2013, Arnett et al 2015), and are beginning to simulate C-burning (Cristini et al 2015), Ne-burning and Si-burning (Couch et al 2015). Because each stage is different, each needs to be studied with realistic turbulent flow. Preliminary indications are that C-, Ne- and O-burning have a family resemblance, but Si-burning and the ensuing neutronization by electron capture are clearly somewhat different; see section 4.3 and Couch et al (2015).

5.5. Entropy structure of atmospheres

In a recent summary of the character of 3D simulations of stellar atmospheres, Magic (2016) has shown that they may be characterized by two parameters, the entropy at depth, and the gradient of entropy just below the surface. This important result may be connected to the conventional parameterization of stellar atmospheres by gravity \(g\) and effective temperature \(T_{\text{eff}}\). We note that \(L/4\pi r^2 = F = \sigma T^4_{\text{eff}}\) and \(g = -1/\rho \frac{dP}{dr}\). The flux at the photosphere is a measure of the radiation entropy, \(F = 1.6 \times 10^{32} \frac{g_{\text{rad}}}{\text{erg/s}}\), where the term in square brackets is slowly varying (Arnett 1996). The gravitational acceleration \(g\) is a measure of the gradient of radiation entropy through the pressure gradient in hydrostatic equilibrium\(^{16}\). While not yet exact, these approximations do catch the trend of the simulations (Magic 2016), and would offer a significant improvement over fits to the flawed MLT, if the missing turbulent kinetic energy flux and realistic boundary physics are included.

5.6. Summary

Numerical simulations in 3D, now appearing for many different astrophysical problems, have become sufficiently mature to show new aspects of turbulent flow in stable and explosive stages of stellar evolution, as well as some errors in the conventional 1D theory. The issues of resolution, and behavior over long time scales, are basic challenges for a number of the problems discussed. With increasing computational power and better algorithms, the simulations are becoming truly turbulent, a significant step closer to reality. Some theoretical analysis of the simulations has been successful, leading to insight beyond the numbers.

\(^{15}\)The Péclet number is the ratio of the thermal energy convected to the thermal energy conducted. It becomes small in stellar atmospheres and large in neutrino-cooled stages.

\(^{16}\)After this paper was submitted, a preprint by Tanner et al (2016) has presented similar arguments.

Acknowledgments

This work was supported in part by the Theoretical Program in Steward Observatory. This work used the Extreme Science and Engineering Discovery Environment (XSEDE), which is supported by National Science Foundation grant number OCI-1053575, and made use of ORNL/Kraken and TACC/Stampede. This work was supported in part by resources provided by the Pawsley Supercomputing Centre with funding from the Australian Government and the Government of Western Australia, and through the National Computational Infrastructure under the National Computational Merit Allocation Scheme.

References

Abarizh S I 2010 Phil. Trans. R. Soc. A 368 1809
Aerts C, Christensen-Dalsgaard J and Kurtz D W 2010
Asteroseismology (Berlin: Springer)
Almgren A, Beckner V E, Bell J B, Day M S, Howell L H, Joggerst C C, Lijewski M J, Nonaka A, Singer M and Zingale M 2010 Astrophys. J. 715 1221
Arnett D and Bazán G 1994 Astrophys. J. 433 L41
Arnett D 1996 Supernovae and Nucleosynthesis (Princeton, NJ: Princeton University Press)
Arnett W D 1967 Can. J. Phys. 45 1621
Arnett W D 1968 Nature 219 1344
Arnett W D 1969a Astrophys. J. 157 339
Arnett W D 1969b Astrophys. Space Sci. 5 180
Arnett W D 1977a New York Acad. Sci. Ann. 302 90 (Proc. Texas Symp. on Relativistic Astrophysics (New York, 1977))
Arnett W D 1977b Astrophys. J. 218 815
Arnett W D 1980 New York Academy of Science Annals 336 366 (Proc. Texas Symp. on Relativistic Astrophysics (Munich, 1979))
Arnett W D 1982 Astrophys. J. 253 785
Arnett W D and Meakin C 2010 IAU Symp. 265 106
Arnett W D, Meakin C and Young P A 2009 Astrophys. J. 690 1715
Arnett W D, Meakin C and Young P A 2010 Astrophys. J. 710 1619
Arnett W D and Meakin C 2011a Astrophys. J. 733 78
Arnett W D and Meakin C 2011b Astrophys. J. 741 33
Arnett W D, Meakin C and Viallet M 2014 AIP Adv. 4 1010
Arnett W D, Meakin C A, Viallet M, Campbell S W, Lattanzio J C and Mocák M 2015 Astrophys. J. 800 30
Asida S M and Arnett D 2000 Astrophys. J. 545 435
Balbus S A 2009 Mon. Not. R. Astron. Soc. 395 2056
Bazán G and Arnett D 1997a Science 276 1359
Bazán G and Arnett D 1997b Nucl. Phys. A 621 607
Bazán G and Arnett D 1998 Astrophys. J. 496 316
Becker R H, Smith B W, White N E, Holt S S, Boldt E A, Mushotsky R F and Serlemitsos P J 1979 Astrophys. J. 234 773
Blondin J, Mezzakappa A and DeMarino D 2003 Astrophys. J. 584 971
Boggs S E et al 2015 Science 348 670
Böhm-Vitense E 1958 Z. Astrophys. 46 108
Boris J 2007 Implicit Large Eddy Simulations ed F F Grinstein et al (Cambridge: Cambridge University Press) p 9
Brown B P, Browning M K, Brun A S, Miesch M S and Toomre J 2010 Astrophys. J. 711 424
Brown B P, Vasil G M and Zweibel E G 2012 Astrophys. J. 756 109
Browning M K 2008 Astrophys. J. 676 1262
Brummell N H, Clune T L and Toomre J 2002 Astrophys. J. 570 825
Brun A S and Palacios A 2009 Astrophys. J. 702 1078
Brun A S, Miesch M S and Toomre J 2004 Astrophys. J. 614 1073
Van Dyke M 1982 *An Album of Fluid Motion* (Stanford, CA: The Parabolic Press)
Vasil G M, Lecoanet D, Brown B P, Wood T S and Zweibel E G 2013 *Astrophys. J.* 773 169
Viallet M, Baraffe I and Walder R 2011 *Astron. Astrophys.* 531 86
Viallet M, Meakin C, Arnett D and Mocák M 2013 *Astrophys. J.* 769 1
Viallet M, Meakin C, Prat V and Arnett D 2015 *Astron. Astrophys.* 580 61
Wood P and Arnett D 2011 *Astron. Soc. Pac. Conf.* 445 183
Wongwathanarat A, Janka H-T and Müller E 2010 *Astrophys. J.* 725 106
Woodward P R 2007 *Implicit Large Eddy Simulations* ed F F Grinstein et al (Cambridge: Cambridge University Press) p 130
Woodward P R, Herwig F and Lin P-H 2016 *Astrophys. J.* 798 49
Woodward P R, Porter D H and Jacobs M 2000 *3D Stellar Evolution* (ASP Conf. Series vol 293) ed S Turcotte et al (San Francisco, CA: Astronomical Society of the Pacific) p 45
Woodward P R, Porter D, Anderson S and Fuchs T 2006 *Numerical Modeling of Space Plasma Flows* ed N V Pogorelov and G P Zank *Astron. Soc. Pac. Conf. Ser.* 359 97
Woosley S E, Arnett W D and Clayton D D 1973 *Astrophys. J. Suppl. Ser.* 26 231
Weaver T A, Zimmerman G B and Woosley S E 1978 *Astrophys. J.* 225 1021
Woosley S E and Weaver T A 1995 *Astrophys. J. Suppl. Ser.* 101 181
Woosley S E, Heger A and Weaver T A 2002 *Rev. Mod. Phys.* 74 1015
Young P A, Ellinger D, Arnett D, Fryer C and Rockefeller G 2008 *Proc. 10th Symp. on Nuclei in the Cosmos* Zingale M, Nonaka A, Almgren A S, Bell J B, Malone C M and Woosley S E 2011 *Astrophys. J.* 740 8