Functional analysis

On connected Lie groups and the approximation property

Sur les groupes de Lie connexes et la propriété d'approximation

Søren Knudby 1

Mathematical Institute, University of Münster, Einsteinstraße 62, 48149 Münster, Germany

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ABSTRACT

Recently, a complete characterization of connected Lie groups with the Approximation Property was given. The proof used the newly introduced property ($T^*$). We present here a short proof of the same result avoiding the use of property ($T^*$). Using property ($T^*$), however, the characterization is extended to almost connected locally compact groups. We end with some remarks about the difficulty of going beyond the almost connected case.

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RÉSUMÉ

Une caractérisation complète des groupes de Lie connexes avec la propriété d’approximation a été obtenue récemment. La preuve utilisait la propriété ($T^*$), nouvellement introduite. Nous présentons ici une preuve courte du même résultat sans utiliser la propriété ($T^*$). En utilisant ($T^*$), cependant, la caractérisation est étendue aux groupes localement compacts presque connexes. Nous concluons avec quelques remarques sur la difficulté d’aller au-delà du cas presque connexe.

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The Fourier algebra $A(G)$ of a locally compact group $G$ was introduced by Eymard [5] and can be defined as the coefficient space of the left regular representation of $G$ on $L^2(G)$,

$$A(G) = \{ f \ast \tilde{g} \mid f, g \in L^2(G) \},$$

where $\tilde{g}(x) = g(x^{-1})$. One can identify $A(G)$ with the predual of the group von Neumann algebra associated with the left regular representation. A multiplier $\varphi$ of $A(G)$ is called completely bounded if its transposed operator $M_{\varphi}^*$ is a completely bounded bounded operator on the group von Neumann algebra. The completely bounded multiplier norm of $\varphi$ is defined as

$$\| \varphi \|_{M_{\varphi}^*} = \| M_{\varphi} \|_{cb}.$$
The space of completely bounded multipliers is denoted $M_0 A(G)$. One has an inclusion $A(G) \subseteq M_0 A(G)$. It was shown in [4, Proposition 1.10] that the space $M_0 A(G)$ has an isometric predual obtained as the completion of $L^1(G)$ in the norm

$$\|f\|_{M_0 A(G)_*} = \sup \left\{ \left| \int_G f(x) \varphi(x) \, dx \right| : \varphi \in M_0 A(G), \|\varphi\|_M \leq 1 \right\}.$$ 

Remark 1. In general, $M_0 A(G)$ does not have a unique predual nor a unique weak* topology. The predual always refers to the one constructed explicitly above, and the weak* topology on $M_0 A(G)$ is the weak* topology coming from this explicit predual.

Following Haagerup and Kraus [9], a locally compact group $G$ has the Approximation Property (short: AP) if there exists a net in $A(G)$ converging to $1$ in the weak* topology of $M_0 A(G)$. Haagerup and Kraus showed that many groups have this property, e.g., all weakly amenable groups. This includes solvable groups, compact groups, and simple Lie groups of real rank one [3,11]. They showed moreover that the AP is preserved by passing to closed subgroups and preserved under group extensions (as opposed to weak amenability). It is also well-known (and routine to verify) that if $K$ is a compact normal subgroup in $G$, then $G$ has the AP if and only if $G/K$ has the AP. We refer to [3,4,9] for details on completely bounded multipliers and the AP.

Haagerup and Kraus conjectured that the group $SL(3, \mathbb{R})$ does not have the AP, but the conjecture was settled only much later by Laflorgue and de la Salle [10]. Not long after, Haagerup and de Laat showed more generally that any simple Lie group of real rank at least two does not have the AP [6,7]. Finally, a complete characterization of connected Lie groups with the AP was given in [8] (see also Theorem 2 below). The proof in [8] involved among other things the property (T*), introduced in [8]. We give below a much shorter proof of the characterization of connected Lie groups with the AP that avoids the use of property (T*).

To state the characterization of connected Lie groups with the AP, we first recall the Levi decomposition of such a group (see [15, Section 3.18] for details). For a connected Lie group $G$, one can decompose its Lie algebra $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{r}$ is the solvable radical and $\mathfrak{s}$ is a semisimple subalgebra. One can further decompose $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ into simple summands $\mathfrak{s}_i$ ($1 \leq i \leq n$). Let $R$, $S$, and $S_1$ denote the corresponding connected Lie subgroups of $G$. One has $G = RS$ as a set. This is called a Levi decomposition of $G$.

The subgroup $R$ is solvable, normal, and closed. The subgroup $S$ is semisimple and locally isomorphic to the direct product $S_1 \times \cdots \times S_n$ of simple factors, but it need not be closed in $G$.

**Theorem 2 ([8]).** Let $G$ be a connected Lie group, let $G = RS$ denote a Levi decomposition, and suppose that $S$ is locally isomorphic to the product $S_1 \times \cdots \times S_n$ of connected simple factors. Then the following are equivalent.

1. The group $G$ has the AP.
2. The group $S$ has the AP.
3. The groups $S_i$, with $i = 1, \ldots, n$, have the AP.
4. The real rank of the groups $S_i$, with $i = 1, \ldots, n$, is at most 1.

**Proof.** The equivalence of (3) and (4) is given by [7, Theorem 5.1].

Suppose (4) holds. We show that (1) and (2) hold. The proof of this implication is the proof from [8]. For completeness, we include it: The group $R$ is solvable, so it has the AP. Since the AP is preserved under group extensions [9, Theorem 1.15], it is therefore enough to prove that $G/R$ has the AP. The Lie algebra of $G/R$ is the Lie algebra of $S$, so to prove (1) and (ii), it suffices to prove that any connected Lie group $S'$ locally isomorphic to $S_1 \times \cdots \times S_n$ has the AP. Using [9, Theorem 1.15] again, we may assume that the center of $S'$ is trivial. If $Z_i$ denotes the center of $S_i$, then $S' \simeq (S_1/Z_1) \times \cdots \times (S_n/Z_n)$. Each group $S_i/Z_i$ has real rank at most 1 and finite center and hence has the AP [3]. Hence $S'$ has the AP.

Suppose (4) does not hold. We show that (1) does not hold (which also proves that (2) does not hold). Fix some $i$. The closure $S_i'$ of $S_i$ in $G$ is of the form $S_iC$ for some connected, compact, central subgroup of $G$ (see [13, p. 614]). Let $G' = G/C$, and let $\pi: G \to G'$ be the quotient homomorphism. As $C$ is compact, $\pi$ is a closed map, and $\pi(S_i) = \pi(S_i')$ is a closed subgroup of $G'$. Since $\ker \pi \cap S_i$ is contained in the center of $S_i$ and therefore is discrete (in the topology of $S_i$), the group $\pi(S_i)$ is locally isomorphic to $S_i$. If now the real rank of $S_i$ is at least 2, then $\pi(S_i)$ does not have the AP [7, Theorem 5.1]. Hence $G'$ does not have the AP, and it follows that $G$ does not have the AP. \qed

A characterization of almost connected groups with the AP

Let $G$ be a locally compact group. Recall from [8] that there is a unique left invariant mean on $M_0 A(G)$, and we say that $G$ has property $(T^*)$ if this mean is weak* continuous. It is clear that non-compact groups with property $(T^*)$ do not have the AP. We establish the following converse for almost connected groups. Recall that $G$ is almost connected if the quotient group $G/G_0$ is compact, where $G_0$ denotes the connected component of the identity in $G$. 

Theorem 3. For an almost connected locally compact group $G$, the following are equivalent.

1. The group $G$ has the AP.
2. No closed non-compact subgroup of $G$ has property $(T^*)$.

Proof. $(1) \implies (2)$ is clear and holds also without the assumption of almost connectedness. We prove $(2) \implies (1)$. Suppose $G$ does not have AP. We prove the existence of a closed non-compact subgroup of $G$ with property $(T^*)$.

By [9, Theorem 1.15], we may assume that $G$ is connected. Then there is a compact normal subgroup $K$ of $G$ such that $G/K$ is a Lie group (see [12, Theorem 4.6]). Then $G/K$ is a connected Lie group without the AP. It follows from the remark after Theorem C in [8] that we can find a closed, non-compact subgroup $H_0 \leq G/K$ with property $(T^*)$. Let $H$ be the inverse image of $H_0$ in $G$. By [8, Proposition 5.13], $H$ has property $(T^*)$, and $H$ is clearly a closed non-compact subgroup of $G$. 

Remark 4. Theorem 3 is not true in general without the assumption of almost connectedness. In fact, there are discrete groups without the AP and without infinite subgroups with property $(T^*)$. This is not a new result, but simply a compilation of known results, the last ingredient being the recent result of Arzhantseva and Osajda (see [14, Theorem 2] and [1]) about the existence of discrete groups without property $A$ but with the Haagerup property. Let $G$ be such a group.

As property $(T^*)$ implies property $(T)$ (see [8, Proposition 5.3]), and property $(T)$ is an obstruction to the Haagerup property, it is clear that $G$ has no infinite subgroups with property $(T^*)$.

On the other hand, if the discrete group $G$ had the AP then its reduced group ${C}^*$-algebra would have the strong operator approximation property. This is known to imply exactness of the group ${C}^*$-algebra, which implies that $G$ is an exact group. Finally, exactness for groups is equivalent to property $A$. This shows that $G$ cannot have the AP. Proofs of all the claimed statements can be found in [2, Chapters 5 and 12].

References

[1] G. Arzhantseva, D. Osajda, Graphical small cancellation groups with the Haagerup property, preprint, arXiv:1404.6807, 2014.
[2] N. Brown, N. Ozawa, $C^*$-Algebras and Finite-Dimensional Approximations, Grad. Stud. Math., vol. 88, American Mathematical Society, Providence, RI, USA, 2008.
[3] M. Cowling, U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, Invent. Math. 96 (3) (1989) 507–540.
[4] J. de Cannière, U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups, Amer. J. Math. 107 (2) (1985) 455–500.
[5] P. Eymard, L’algèbre de Fourier d’un groupe localement compact, Bull. Soc. Math. Fr. 92 (1964) 181–236.
[6] U. Haagerup, T. de Laat, Simple Lie groups without the approximation property, Duke Math. J. 162 (5) (2013) 925–964.
[7] U. Haagerup, T. de Laat, Simple Lie groups without the approximation property, Trans. Amer. Math. Soc. 368 (6) (2016) 3777–3809.
[8] U. Haagerup, S. Knudby, T. de Laat, A complete characterization of connected Lie groups with the approximation property, Ann. Sci. Éc. Norm. Super. 49 (4) (2016), preprint, arXiv:1412.3033, 2014.
[9] U. Haagerup, J. Kraus, Approximation properties for group ${C}^*$-algebras and group von Neumann algebras, Trans. Amer. Math. Soc. 344 (2) (1994) 667–699.
[10] V. Lafforgue, M. de la Salle, Noncommutative $L^p$-spaces without the completely bounded approximation property, Duke Math. J. 160 (1) (2011) 71–116.
[11] M. Lemvig Hansen, Weak amenability of the universal covering group of $SU(1, n)$, Math. Ann. 288 (3) (1990) 445–472.
[12] D. Montgomery, L. Zippin, Topological Transformation Groups, Interscience Publishers, New York–London, 1955.
[13] C.D. Mostow, The extensibility of local Lie groups of transformations and groups on surfaces, Ann. of Math. (2) 52 (1950) 606–636.
[14] D. Osajda, Small cancellation labellings of some infinite graphs and applications, preprint, arXiv:1406.5015, 2014.
[15] V.S. Varadarajan, Lie Groups, Lie Algebras, and Their Representations, Grad. Texts Math., vol. 102, Springer-Verlag, New York, 1984, reprint of the 1974 edition.