A Note on the Real Fermionic and Bosonic quadratic forms: Their Diagonalization and Topological Interpretation

Abstract: We explain in this note how real fermionic and bosonic quadratic forms can be effectively diagonalized. Nothing like that exists for the general complex hermitian forms. Looks like this observation was missed in the Quantum Field theoretical literature. The present author observed it for the case of fermions in 1986 making some topological work dedicated to the problem: how to construct Morse-type inequalities for the generic real vector fields? This idea also is presented in the note.

Let us consider a Fock space $F_n^\pm$ generated by the finite number of creation operators $a_j$ from the vacuum vector $\Phi$ such that $a_j^+ \Phi = 0, j = 1, \ldots, n$. Here we have by definition following canonical fermionic (bosonic) commutation relations between the creation and annihilation operators $a_i, a_i^+$:

$$a_i^+ a_j + a_j a_i^+ = \delta_{ij},$$
$$a_i a_j + a_j a_i = 0, a_i^+ a_j^+ + a_j^+ a_i^+ = 0$$

for the space $F_n^-$ (fermions), and

$$[a_i, a_j] = [a_i^+, a_j^+] = 0$$
$$[a_i^+, a_j] = \delta_{ij}$$

for the Fock space $F_n^+$ (bosons). Here $[a, b] = ab - ba$ in the associative algebra generated by these symbols.

We introduce an euclidean inner product $<\eta, \zeta>$ in the Fock spaces such that $<\Phi, \Phi> = 1$, and the operators $a_i$ are adjoint to $a_i^+$:

$$<a_i \eta, \zeta> = <\eta, a_i^+ \zeta>$$

According to the standard lemma from the elementary Quantum Field Theory, all basic vectors in the Fock spaces $F_n^\pm$ have following form

$$\eta_m = a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n} \Phi$$

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where \( m = (m_1, m_2, \ldots, m_n) \). We have following vectors:

- \( m_j = 0, 1 \) for the case of fermions \( F_n^- \)
- \( m_j = 0, 1, 2, \ldots, \) i.e. \( m_j \in \mathbb{Z}_+ \) for bosons \( F_n^+ \).

Their inner products have a form

\[
\langle \eta_m, \eta_{m'} \rangle = m! \delta_{m, m'}
\]

where \( m! = m_1! m_2! \ldots m_n! \)

Therefore all these basic vectors have unit length for the case of fermions.

From the commutation relations (above) we can see that these Fock spaces are canonically isomorphic to the spaces of the total symmetric and external powers for the Euclidean space \( \mathbb{R}^n \):

\[
F_n^+ = \sum_{k \geq 0} S^k \mathbb{R}^n = S^* \mathbb{R}^n
\]

\[
F_n^- = \sum_{k=0}^{k=n} \Lambda^k \mathbb{R}^n = \Lambda^* \mathbb{R}^n
\]

The external power \( \Lambda^* \mathbb{R}^n \) is finite dimensional. As it was shown by E.Witten developing a new approach to the Morse Theory (see [1]), this presentation is very convenient for the study operators acting on the spaces of differential forms in topology. The present author used this idea investigating the following problem:

**What is a natural analog of the Morse inequalities for the singular points of the generic vector fields on manifolds?**

(see Appendices to the works [2,3]; algebraic part was repeated also in the Appendix to the work [4]; there is a more detailed exposition of the topological part in [5]). This problem led to the study real fermionic quadratic forms and their diagonalization in [3]. It turns out that this diagonalization never appeared in the Quantum Field theoretical literature before. In this note we develop similar approach for the diagonalization of the real bosonic quadratic forms. This problem also admits a simple and beautiful solution.

**Definition 1:** Let us call real bosonic (fermionic) quadratic form any self-adjoint operator \( H \) acting on the Fock spaces \( F_n^\pm \), of the following form

\[
H = U_{ij} a_i^+ a_j^+ + V_{ij} (a_i a_j^+ + a_j a_i^+) \pm U_{ij} a_i a_j + C
\]

where \( U_{ij} = \pm U_{ji} \) and \( V_{ij} = V_{ji} \), \( C = \text{const} \), the signs are (+) for bosons and (−) for fermions, all coefficients are real.
This formula represents all selfadjoint operators $H$ quadratic in the creation-
annihilation operators, with real coefficients.

The diagonalization of these quadratic forms we are doing by the following
series of steps:

**Step 1.** Any real bosonic (fermionic) quadratic form can be written in
the following **Standard Real form**:

$$H = T_{ij}(a_i + a_i^+)(a_j + a_j^+) + R_{ij}(a_i - a_i^+)(a_j - a_j^+) + \text{const}$$

where $T_{ij} + R_{ij} = U_{ij}, T_{ij} - R_{ij} = V_{ij}, T_{ij} = T_{ji}, R_{ij} = R_{ji}$ for bosons

$$H = C_{ij}(a_i + a_i^+)(a_j - a_j^+) + \text{const}$$

where $C_{ij} = U_{ij} + V_{ij}$ for fermions.

The proof of this is obvious.

**Remark.** Nothing like that exists for the general complex hermitian
quadratic forms.

**Definition 2:** By the real Bogolyubov Transformation $a_i, a_i^+ \rightarrow b_j, b_j^+$
(or canonical transformation) we call change of basis in the linear space $R^{2n}$
generated by the creation and annihilation operators (summation along the
repeated indices is assumed here)

$$a_i = P_{ij}b_j + Q_{ij}b_j^+, a_i^+ = Q_{ij}b_j + P_{ij}b_j^+$$

The new operators $b_j, b_j^+$ should satisfy to the same canonical bosonic (fermionic)
commutation relations; they are adjoint to each other.

**Step 2.** We prove that after the real Bogolyubov transformation $a = P^b + Q^b, a^+ = Q^b + P^b$ matrix coefficients $R, T, C$ of the Standard Real
forms are changing $R \rightarrow R', T \rightarrow T', C \rightarrow C'$ according to the following
rules:

$$T' = STS^t, R' = (S^{-1})^tR(S^{-1})$$

for bosons, where $S = P + Q$. The transformation is canonical if and only if
the following set of relations is satisfied:

$$(P + Q)(P^t - Q^t) = 1$$

This transformation is isospectral if matrices $P \pm Q$ preserve the orientation
of the space $R^n$. We call such Bogolyubov transformations positive.

For fermions we have

$$C' = O_+CO_-$$
where \( O_{\pm} = Q \pm P \). The transformation is canonical if and only if both matrices \( O_{\pm} \) are orthogonal. It is isospectral if Bogolyubov transformation is positive, i.e. \( O_{\pm} \in SO_n \).

The proof of these properties can be easily obtained by the elementary algebraic calculation.

**Step 3.** As a result of the step 2 we are coming to the following **Conclusion.**

I. **Bosons:** The symmetric matrices \( T, R \) transform together as a pair of real quadratic forms on the spaces \( R^n \) and \( R^n^* \); beginning from now we write \( T \) as a tensor with two upper indices and \( R \) as a tensor with two lower indices. We have to diagonalize them simultaneously by the same linear transformation \( S \) where \( \det S > 0 \). It is certainly possible if one of these forms is strictly positive (or negative).

**Theorem 1.** The diagonalization of the bosonic operator \( H \) by the real Bogolyubov transformation is possible if and only if the matrix \( (RT)^i_k = R_{ij}T^{jk} \) can be diagonalized over the field \( R \). It means that all eigenvalues of \( RT \) should be real, and all Jordan cells should be trivial. Finding the eigenbasis for the matrix \( RT = (U + V)(U - V) \), we diagonalize \( H \).

Finally, we represent the operator \( H \) in the form

\[
H = \sum_i H_i = \sum_i t_i (b_i + b_i^+)^2 + r_i (b_i - b_i^+)^2, i = 1, \ldots, n
\]

where \( b_i, b_i^+ \) are the new bosonic operators after the canonical transformation.

In the standard model we represent \( b_i, b_i^+ \) by the operators

\[
b = \frac{\partial + x}{\sqrt{2}}, b^+ = \frac{-\partial + x}{\sqrt{2}}
\]

acting in the Hilbert space \( L_2(R) \) of the variable \( x \). Here we have \( \Phi = (const) \exp\{-x^2/2\}, b^+ \Phi = 0 \). Therefore the operator \( H_i \) is represented by the oscillator:

\[
H_i = 2t_i x^2 + 2r_i \partial^2 = (-2r_i)(-\partial^2 - t_i/r_i x^2)
\]

assuming that \( r_i \neq 0 \). We are coming to the discrete spectrum if and only if \( t_i/r_i < 0 \) corresponding to the oscillator \(-\partial^2 + \omega^2 x^2 \) where \( \omega^2 = -t_i/r_i \). Our partial eigenvalues associated with the mode number \( i \) are equal to the numbers

\[
\lambda_{i,m} = -2r_i \frac{r_i}{|r_i|} \sqrt{-r_i(t_i(m + 1/2), m = 0, 1, 2, \ldots}
\]
The total eigenvalues are equal to their sum
\[ \lambda_{m_1,...,m_n} = \sum_i \lambda_{i,m_i} \]
for all possible choices of the integers \( m_i \geq 0 \).

In all other cases the spectrum is continuous.

\[ \text{II. Fermions:}\] The matrix \( C = U + V \) transforms as \( C \rightarrow C' = O_+CO_- \) where \( O_\pm \) is a pair of orthogonal transformations. We certainly can diagonalize \( C \) by these transformations. It follows from the following standard theorems of the linear algebra:

1. Any real matrix can be presented as a product of symmetric and orthogonal matrices \( C = C'O_1 \).
2. Any real symmetric matrix can be diagonalized by the orthogonal transformation \( C' = O_2C'O_2^t \).
3. The determinant \( \det C = \lambda_1 \ldots \lambda_n \) and eigenvalues \( \lambda_j^2 \geq 0 \) of the matrices \( CC^t \) and \( C'C \) (the so-called “s-numbers”) are exactly the full set of invariants of the positive Bogolyubov transformations.

Theorem 2. The diagonalization of the fermionic operator \( H \) by positive real Bogolyubov transformation is always possible following the procedure described above (i.e. the diagonalization of the matrix \( C = U + V \) by the transformations \( C \rightarrow O_1CO_2, O_s \in SO(n,R) \)). The eigenvalues of fermionic quadratic form \( H \) on the Fock space \( F_n^- = \Lambda^*Q^n = \Lambda^{even} + \Lambda^{odd} \) have the form
\[ 2 \sum_{l=k}^{i} \lambda_{i_l} - TrC = \lambda_{i_1} + \ldots + \lambda_{i_k} - \lambda_{j_1} - \ldots - \lambda_{j_{n-k}} = \sum_{p=1}^{w_k} w_k |\lambda_p|, w_k = \pm \]
where \( i_1 < i_2 < \ldots < i_k, j_1 < \ldots < j_{n-k} \) and \( i_p \neq j_r \). The corresponding set of eigenvalues is invariant under the change of sign for any number of \( \lambda_i \). We permit to change sign only for even number of them, so eigenvector belongs to the subspace \( \Lambda^{even} \) if and only if the number of \((-\) signs \( w_k \) is even; it belongs to \( \Lambda^{odd} \) otherwise.

Let us consider now the topological problem:

What kind of Morse-type inequalities might exist for the generic real vector fields on the closed manifolds?

This problem already has been discussed by the present author in 1986. This discussion led to the study diagonalization of the real fermionic quadratic forms. Let us remind it here.
We consider a closed \( C^\infty \)-manifold \( M_n \) with the generic vector field \( X \) (i.e. its singular points \( x^* \) where \( X = 0 \) are nondegenerate. In the local coordinates \( y^1, \ldots, y^n \) near the point \( x_j, y^k(x_j) = 0 \), we have \( X = (X^1(y), \ldots, X^n(y)) \) and

\[
X^k = C^k_i y^i + O(|y|^2), \quad C^k_i = \partial X^k / \partial y^i(x_j)
\]

In the generic case we have \( \det C^k_i \neq 0 \) for all singular points \( x^* \). The only local topological invariant of the vector field \( X \) in the point \( x^* \) is the sign of this determinant \( s(x^*) = \frac{\det C}{|\det C|} \). We have

\[
\sum_{x^*} s(x^*) = \chi(M^n)
\]

according to the Poincare-Hopf theorem for the Euler characteristics. Let \( m_{\pm} \) are the numbers of singular points \( x^* \) with the signs \( \pm \) correspondingly, so we have \( m_+ - m_- = \chi(M^n) \).

**Are there any separate inequalities for the numbers \( m_+ \) and \( m_- \)?** (like the separate lower estimates).

We use for that an arbitrary Riemannian metric \( g_{ij} \) on the manifold \( M^n \). It determines a 1-form \( \omega \) such that \( \omega_i = X^j g_{ij} \) in the local coordinates. We introduce a family of the operators \( d_t \omega = d + t \omega, t \in R \), acting on the space of all differential forms \( \Lambda^*(M^n) \):

\[
d_t \omega \Lambda^* = d\Lambda^* + t \omega \Lambda^*
\]

Using the metric, we construct the adjoint operators \( d^* \) and \( \omega^* \) and the family of second order operators

\[
H_t = (d_t \omega + d^*_t)^2 = -\Delta + t^2 (\omega^* + \omega^* \omega) + tQ
\]

\[
Q = \Omega + \Omega^* + d \omega^* + \omega^* d + d^* \omega + \omega^* d, \quad \Omega = d(\omega)
\]

**Lemma 1.** The \( t^2 \)-coefficient in the operator \( H_t \) is exactly equal to the multiplication operator by the function \( \langle \omega, \omega \rangle \).

The proof see below.

Therefore the zero modes \( \psi_t \) of these operators \( H_t \psi_t = 0 \) concentrate asymptoticaly \( (t \to \infty) \) near the critical points \( \omega = 0 \) equal to the singular points \( X = 0 \) by definition. The Morse-type inequalities for the critical numbers \( m_{\pm} \) of the generic vector field \( X \) are based on the fact that the
number of semiclassical zero modes is always not less than the number of the "genuine" zero modes.

Let us calculate the number of semiclassical zero modes. Following [1,2], we represent the space of differential forms through the Fock spaces of fermions. Our vacuum vector $\Phi$ corresponds to the constant function on the manifold $M^n$. We choose orthonormal basis in the tangent space of any given point, i.e. $g_{ij} = \delta_{ij}$ for the given point $x$. Let $a^i$ be a corresponding orthonormal basis of covectors (1-forms) in the same point. We have $\omega = \omega_i a^i$ for every 1-form. The space of real external forms in the point $x$ is identified with the real fermionic Fock space $F^{-}_n = \Lambda^* R^n$.

Proof of Lemma 1: We have

$$\omega = \omega_i a^i, \omega^* = \omega_j a_j^+$$

Therefore the $t^2$-coefficient operator $\omega \omega^* + \omega^* \omega$ has a form

$$\omega_i \omega_j (a_i a_j^+ + a_j^+ a_i) = \omega_i \omega_j \delta_{ij} = \langle \omega, \omega \rangle$$

Lemma 1 is proved.

For the study differential parts of these operators we choose now the special local coordinates near the point $x \in M^n$ such that

$$g_{ij}(x) = \delta_{ij}, \frac{\partial g_{ij}}{\partial y^k}(x) = 0$$

**Lemma 2.** In the special coordinates associated with the point $x$ the $t$-coefficient $Q$ of the operator $H_t$ is equal to the following expression in the point $x$:

$$Q = U_{ij} a_i a_j + V_{ij} (a_i a_j^+ + a_j^+ a_i) - U_{ij} a_i^+ a_j^+ + \text{const} = C_{ij} (a_i + a_i^+) (a_j - a_j^+)$$

where

$$C_{ij} = \frac{\partial \omega_i}{\partial y^j}(x)$$

In particular, the operator $Q$ is purely algebraic.

Proof of Lemma 2: We have $d = a_i \partial_i$ in the special coordinates in the point $x$ where $\partial_j (a_i) = 0$ in the point $x$. As a corollary we have $Q = A + A^t + B + B^t$ where

$$A = d \omega^* + \omega^* d = a_i \partial_i \omega_j a_j^+ + \omega_j a_j^+ a_i \partial_i$$
and $B$ is a multiplication operator by the 2-form $d(\omega)$:

$$B = d(\omega) = U_{ij} a_i a_j, U_{ij} = \omega_{ij} - \omega_{ji}$$

So in the point $x$ we have

$$A = \omega_{ij}(a_i a_j^+) + (a_j^+ a_i + a_i^+ a_j)\omega_j \partial_i = \omega_{ij} a_i a_j^+ + \omega_i \partial_i$$

$$\omega_{ij} = \frac{\partial \omega_i}{\partial y^j}$$

Lemma 2 follows from that immediately. Let us point out that this line of arguments essentially borrowed from [1], but the terms changing the number of particles did not appeared in [1] because the form $\omega = df$ was exact in this work. So nontrivial Bogolyubov transformations did not appeared as well as in the later works of Pajitnov where the closed (i.e locally exact) forms were analyzed.

As a result, we are coming to the following

Theorem 3. For every singular point $x^*$ of the vector field $X$ the operator $H_t$ for $t \to \infty$ has exactly one semiclassical zero mode. It belongs to the space $\Lambda^{even}(M^n)$ if this point is positive $s(x^*) = +$ and to the space $\Lambda^{odd}(M^n)$ if this point is negative $s(x^*) = -$.

Corollary. For any closed orientable manifold $M^n$, any Riemannian metric $g_{ij}$ and generic vector field $X$ the numbers of the ”genuine” zero modes for the operators $H_t$, $t$ is large enough, belonging to the spaces $\Lambda^{even}[\Lambda^{odd}]$, is no more than $m_+ [m_-]$ separately.

Proof: In the quadratic approximation made in the local special coordinates $y^k$ near the singular point $x^*$, we are coming to the operators

$$H_t = -\sum_i \partial_i^2 + t^2 C_{ij} C_{ik} y^j y^k + t C_{ij} (a_i + a_i^+) (a_j - a_j^+) + O(|y|^3)$$

Therefore we have a potential $(C^t C)_{jk} y^j y^k$ and a fermionic quadratic form who are diagonalizable simultaneously by the positive Bogolyubov transformation:

$$C^t C \to O^t C^t O = diag(\lambda_1^2, \ldots, \lambda_n^2)$$

by the rotation of the coordinates $y \to z$, and

$$C \to O_+ C O_- = diag(\lambda_1, \ldots, \lambda_n)$$
where \( \det O_+ = 1 \).

We are coming finally to the same operators as Witten in [1]:

\[
\sum_{i} \{-\partial_i^2 + \lambda_i^2 (z^i)^2 + 2\lambda_i a_i a_i^+\} - \sum_{i} \lambda_i
\]

The determinant \( \det C = \lambda_1 \ldots \lambda_n \) and all \( \lambda_i^2 \) remain unchanged under the positive Bogolyubov transformation. Therefore only even number of the quantities \( \lambda_i \) may change sign. The zero mode appears here exactly like in [1] as a vector in the exterior power corresponding to the set of all indices where our \( \lambda_i \) are negative. This number is well-defined only modulo 2.

This argument finishes the proof.

References.

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