GENERALIZED KEPLER PROBLEMS I: WITHOUT MAGNETIC CHARGES

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ABSTRACT. For each simple euclidean Jordan algebra $V$ of rank $\rho$ and degree $\delta$, we introduce a family of classical dynamic problems. These dynamical problems all share the characteristic features of the Kepler problem for planetary motions, such as existence of Laplace-Runge-Lenz vector and hidden symmetry. After suitable quantizations, a family of quantum dynamic problems, parametrized by the nontrivial Wallach parameter $\nu$, is obtained. Here, $\nu \in \mathcal{W}(V) := \{k^{1/2} | k = 1, \ldots, (\rho - 1)\} \cup ((\rho - 1)^{1/2}, \infty)$ and was introduced by N. Wallach to parametrize the set of nontrivial scalar-type unitary lowest weight representations of the conformal group of $V$.

For the quantum dynamic problem labelled by $\nu$, the bound state spectra is $-\frac{1}{2}(I + \nu \rho^2)^2$, $I = 0, 1, \ldots$ and its Hilbert space of bound states gives a new realization for the aforementioned representation labelled by $\nu$. A few results in the literature about these representations become more explicit and more refined.

The Lagrangian for a classical Kepler-type dynamic problem introduced here is still of the simple form: $\frac{\dot{x}}{2}|\dot{x}|^2 + \frac{\lambda}{2}$. Here, $\dot{x}$ is the velocity of a unit-mass particle moving on the space consisting of $V$'s semi-positive elements of a fixed rank, and $r$ is the inner product of $x$ with the identity element of $V$.

1. INTRODUCTION

The goal of this paper and its sequels is to develop a general mathematical theory for the Kepler problem of planetary motions. As with many endeavors of developing a general theory for a nice mathematical object, one usually starts with a reformulation of this object from a new perspective which captures the essence of such object — a crucial step which often requires introducing new concepts, and then work out all the necessary technical details.

For example, choosing continuous functions in calculus as the nice mathematical object, introducing the concept of open set and reformulating the notion of continuous function in terms of open sets, one arrives at the much broader notion of continuous map between topological spaces. For a sophisticated example, choosing the Riemann integral for continuous functions over close intervals as the nice mathematical object, reformulating it as the Lebesgue integral — a step which requires introducing the concepts of measurable space and measure, one arrives at the general integration theory. For still another sophisticated example, choosing the Euler number of closed oriented manifolds as the nice mathematical object, reformulating it as the obstruction to the existence of nowhere vanishing vector fields — a step which requires introducing the concepts of vector bundle and obstruction, one arrives at the theory of characteristic classes.

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In our case, the Kepler problem of planetary motions is the nice mathematical object, and we reformulate it as a dynamic problem on the open future light cone

\[ \{(t, \vec{x}) \in \mathbb{R}^4 \mid t^2 = |\vec{x}|^2, t > 0\} \]

of the Minkowski space. To quickly convince readers that this is a much better formulation, one notes that the orbits (including the colliding ones) in this new formulation have a simpler and uniform description:

orbits are precisely the intersections of 2D planes with the open future light cone.

A closer examination reveals that it is the euclidean Jordan algebra structure, something that is more fundamental than the Lorentz structure, that plays a pivotal role in this new formulation. Once this point is realized, it is just a matter of time to work out the details for the general theory. In this general theory, a key concept is that of canonical cone, something that generalizes the notion of open future light cone. With this key concept introduced, it is not hard to describe what a generalized Kepler problem is. For example, a generalized classical Kepler problem is just the dynamical problem for a single particle on a canonical cone \( C \) moving under a conserved force whose potential function is \(-\frac{1}{r}\). (Here, \( r \) is the function on \( C \) such that \( r(x) \) is the inner product of \( x \in C \) with the identity element of the euclidean Jordan algebra.)

One would expect to obtain the generalized quantum Kepler problems by quantizing the generalized classical Kepler problems. However, due to the operator ordering problem, in general we don’t know how to do the quantization, so we seem to be stuck. The way out of it is to demand that the hidden symmetry in the generalized classical Kepler problems be still present after quantization. Incidentally, that leads to a rediscovery of those unitary lowest modules mentioned in the abstract.

As a comparison with the general integration theory, euclidean Jordan algebras are like measurable spaces, canonical cones (cf. Subsection 4.1) are like measured spaces, a generalized Kepler problem (cf. Section 5) on a canonical cone is like an integration theory on a measured space.

1.1. A general remark on euclidean Jordan algebras. Euclidean Jordan algebras were first introduced by P. Jordan [1] to formalize the algebra of physics observables in quantum mechanics. With E. Wigner and J. von Neumann, Jordan [2] classified the finite dimensional euclidean Jordan algebras: every finite dimensional euclidean Jordan algebra is a direct sum of simple ones, and the simple ones consist of four infinite series and one exceptional.

Although euclidean Jordan algebras are abandoned by physicists quickly, they have been extensively studied at much more general settings by mathematicians since 1950’s. For an authoritative account of the history of Jordan algebras, readers may consult the recent book by K. McCrimmon [3]. However, for our specific purpose, the book by J. Faraut and A. Korányi [4] is sufficient — everything we really need either is already there or can be derived from there.

It is helpful for us to view euclidean Jordan algebras as the super-symmetric analogues of compact real Lie algebras. Just as compact real Lie algebras are analogues of the infinitesimal gauge group for electromagnetism, euclidean Jordan algebras are analogues of the infinitesimal space-time, but with a more refined euclidean Jordan algebra structure hidden behind. In our view, it is this more refined hidden structure that is responsible for the

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1 This reformulation seems to support the thesis of 2T-physics developed by I. Bars [18].
2 This description emerged soon after Ref. [8] was written up.
existence of the inverse square law. Therefore, we would not be surprised if someday eu-
clidean Jordan algebras indeed play an indispensable role in the study of the fundamental
physics.

1.2. A quick review of euclidean Jordan algebras. Throughout this paper and its se-
quels we always assume that $V$ is a finite dimensional euclidean Jordan algebra. That
means that $V$ has both the inner product structure and the Jordan algebra (with an identity
element $e$) structure on the underlying finite dimensional real vector space, such that, the
two structures are compatible in the sense that the Jordan multiplication by $u \in V$, denoted
by $L_u$, is always self-adjoint with respect to the inner product on $V$. We write the Jordan
product of $u, v \in V$ as $uv$, so $uv = L_u v$, and the Jordan triple product of $u, v, w \in V$
as $\{uvw\}$. By definition, $\{uvw\} = S_{uv} w$, where

$$S_{uv} = [L_u, L_v] + L_{uv},$$

i.e.,

$$\{uvw\} = u(vw) - v(uw) + (uv)w.$$  

It is a fact that a simple euclidean Jordan algebra is uniquely determined by its rank $\rho$ and
degree $\delta$.

$V$ is a real Jordan algebra means that $V$ is a real commutative algebra such that

$$[L_u, L_v] = 0, \quad u \in V.$$ 

The inner product of $u, v \in V$, denoted by $\langle u \mid v \rangle$, is assumed to be the one such that the
length of $e$ is one: $||e|| = 1$. We further assume that $V$ is simple, then the inner product is
unique. A simple computation shows that

$$(1.3) \quad [S_{uv}, S_{zw}] = S_{\{uvw\}w} - S_{z\{uvw\}}.$$ 

Therefore, these $S_{uv}$ span a real Lie algebra — the structure algebra of $V$, denoted by
$\mathbb{str}(V)$ or simply $\mathbb{str}$. It is a fact that the derivation algebra of $V$, denoted by $\mathbb{der}(V)$ or
simply $\mathbb{der}$, is a Lie subalgebra of $\mathbb{str}(V)$. Note that $\mathbb{der}$ — a generalization of $\mathfrak{so}(3)$ — is
compact and $\mathbb{str}$ — a generalization of $\mathfrak{so}(3, 1) \oplus \mathbb{R}$ — is reductive.

The generalization of $\mathfrak{so}(4, 2)$, denoted by $\mathfrak{co}(V)$ or simply $\mathfrak{co}$, was independently dis-
covered by Tits, Kantor, and Kroecher [5]. Like $\mathfrak{so}(4, 2)$, $\mathfrak{co}(V)$ — the conformal Lie
algebra of $V$, is a simple real Lie algebra. In this paper and its sequels, the universal
enveloping algebra of $\mathfrak{co}$ shall be referred to as the TKK algebra and the corresponding
simply connected Lie group shall be referred to as the conformal group and denoted by
$\text{Co}(V)$ or $\text{Co}$.

As a vector space,

$$\mathfrak{co}(V) = V \oplus \mathbb{str}(V) \oplus V^*.$$ 

If we rewrite $u \in V$ as $X_u$ and $\langle v \mid \rangle \in V^*$ as $Y_v$, then the TKK algebra is determined by
the TKK commutation relations:

$$(1.4) \quad [X_u, X_v] = 0, \quad [Y_u, Y_v] = 0, \quad [X_u, Y_v] = -2S_{uv},$$

$$[S_{uv}, X_z] = X_{\{uvw\}}, \quad [S_{uv}, Y_z] = -Y_{\{uvw\}},$$

$$[S_{uv}, S_{zw}] = S_{\{uvw\}w} - S_{z\{uvw\}}.$$ 

Here, $u, v, z, w \in V$. 
1.3. Classical realization of TKK algebras. The total cotangent space $T^*V = V \times V^*$ is a symplectic space. Let $\{e_{\alpha}\}$ be an orthonormal basis for $V$, with respect to which, a point $(x, \langle \pi \mid \rangle) \in T^*V$ can be represented by its coordinate $(x^\alpha, \pi^\beta)$. Then the basic Poisson bracket relations on $T^*V$ are

$$\{x^\alpha, \pi^\beta\} = \delta^\alpha_{\beta}, \quad \{x^\alpha, x^\beta\} = 0, \quad \{\pi^\alpha, \pi^\beta\} = 0.$$  

Introducing the moment functions

$$S_{uv} := \langle S_{uv}(x) \mid \pi \rangle, \quad \mathcal{X}_u := \langle x \mid \{\pi u \pi\} \rangle, \quad \mathcal{Y}_v := \langle x \mid v \rangle$$  

on $T^*V$, we shall show that, for any $u, v, z$ and $w$ in $V$,

$$\begin{cases} \{\mathcal{X}_u, \mathcal{X}_v\} = 0, & \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, & \{\mathcal{X}_u, \mathcal{Y}_v\} = -2S_{uv}, \\
\{S_{uv}, \mathcal{X}_z\} = S_{(uvz)}, & \{S_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{(uvz)}, \\
\{S_{uv}, S_{zw}\} = S_{(wuz)} - S_{(vuw)}. \end{cases}$$  

The realization of $O$ by $\mathcal{O}$ is referred to as the classical realization of the TKK algebra. The quantization of this classical realization leads to operator realizations of the TKK algebra.

1.4. Operator realizations of TKK algebras. The canonical quantization involves promoting classical physical variables $\mathcal{O}$ to differential operators $\hat{O}$ (or the duals $\hat{\mathcal{O}}$) using recipe: $\pi^\alpha \rightarrow -i\partial_{\pi^\alpha}$ (or $x^\alpha \rightarrow i\partial_{x^\alpha}$). Here is a word of warning: in order to get anti-hermitian differential operators in the end, instead of using the quantized differential operators, we actually use the quantized differential operators multiplied by $-i$.

Due to the ambiguity with the operator ordering, the canonical quantization has an ambiguity, measured by a real parameter $\mu$ here. For simplicity, we write $\sum_{\alpha} e_{\alpha} \partial_{\pi^\alpha}$ as $\hat{\pi}$ and $\sum_{\alpha} e_{\alpha} \partial_{x^\alpha}$ as $\hat{x}$. The quantization recipe, with the operator orderings taken into account, yields either differential operators on $V$:

$$\begin{cases} \hat{S}_{uv}(\nu) := -\langle S_{uv}(x) \mid \hat{\pi} \rangle - \frac{\nu}{2} \text{tr} (uv), \\
\hat{X}_u(\nu) := i\langle x \mid \{\hat{\pi} u \hat{x}\} \rangle + i\nu \text{tr} (u\hat{x}), \\
\hat{Y}_v(\nu) := -i\langle x \mid v \rangle \end{cases}$$  

or differential operators on $V^*$:

$$\begin{cases} \hat{S}_{uv}(\nu) := \langle S_{uv}(\pi) \mid \hat{\theta} \rangle - \frac{\nu^*}{2} \text{tr} (uv), \\
\hat{X}_u(\nu) := \langle \{\pi u \pi\} \mid \hat{\theta} \rangle - \nu^* \text{tr} (u\pi), \\
\hat{Y}_v(\nu) := \langle v \mid \hat{\theta} \rangle. \end{cases}$$  

Here, $\nu^* = \nu - \frac{2n}{\rho}$ with $n = \dim V$.

Eqs (1.7) and (1.8) provide us two families of operator realizations for the TKK algebra. It is not hard to see that the two families are related by the Fourier transform and for the realizations to be unitary we must have $\nu \geq 0$. Note that the case $\nu^* = 0$ is well-known to physicists, cf. Ref. [6]; moreover, for $\nu > 1 + (\rho - 1)\delta$, the explicit formula in Eq. (1.7) has been obtained in Ref. [7] via an indirect route.
These operator realizations as given in Eq. (1.7) are not unitary with respect to the obvious $L^2$-inner product
\[(\psi_1, \psi_2) = \int_V \bar{\psi}_1 \psi_2 dm,\]
where $dm$ is the Lebesgue measure. Therefore, the right hermitian inner products must be found in order to have unitary realizations. For us, the clues come out naturally in our
\[\text{ Ref. [8, 9]} \] study of the Kepler problem.

1.5. Quantum realizations of TKK algebras. In order to address the unitarity problem for the quantization given by the explicit formula in Eq. (1.7), we are led to introduce the notion of canonical cones. Let $k > 0$ be an integer no more than $\rho$. The canonical cone of rank $k$, denoted by $C_k$, is an open Riemannian manifold. As a smooth manifold, it consists of all semi-positive elements in $V$ of rank $k$. The Riemannian metric on $C_k$ is defined in terms of Jordan multiplication, so it is invariant under the automorphism group of $V$. One important fact here is that the operators defined in Eq. (1.7) always descend to differential operators on $C_k$.

Denote by $\mathcal{P}(V)$ the algebra of polynomial maps from $V$ to $\mathbb{C}$, by $\mathcal{P}(C_k)$ the algebra of functions on $C_k$ coming from the restriction of elements in $\mathcal{P}(V)$, i.e.,
\[\mathcal{P}(C_k) = \{ p : C_k \to \mathbb{C} \mid p \in \mathcal{P}(V) \}.\]
We use $r$ to denote this function on $V$:
\[r(x) = \langle e \mid x \rangle.\] (1.10)
We shall show that $e^{-r}\mathcal{P}(C_k)$ is a common domain for the operators defined in Eq. (1.7) and is dense in $L^2(d\mu_\nu)$, here $d\mu_\nu$ is a measure on $C_k$ if $\nu > (\rho - 1)\frac{1}{2}$ (if $\nu = k\frac{1}{2}$, $k = 1, 2, \ldots, \rho - 1$), and can be explicitly written down in terms of the volume form $\text{vol}$ on the canonical cone. It shall also be demonstrated that $L^2(d\mu_\nu)$ is a unitary lowest weight representation $\pi_\nu$ for $\mathfrak{co}$ with the lowest weight equal to $\nu\lambda_0$, here $\lambda_0$ is the fundamental weight conjugate to the unique non-compact simple root of $\mathfrak{co}$ under a suitable choice of Cartan subalgebra. Note that, we have a unitary lowest weight representation for $\mathfrak{co}$ when either $\nu = k\frac{1}{2}$ with $k = 1, \ldots, \rho - 1$ or $\nu > (\rho - 1)\frac{1}{2}$, and these $\pi_\nu$ exhaust all nontrivial scalar-type unitary lowest weight representations for $\mathfrak{co}$, according to Ref. [10].

As a historical note, we would like to mention that the whole list of these representations $\pi_\nu$ was first discovered by N. Wallach [11] via algebraic methods and the first analytic model for these representations was discovered by M. Vergne and H. Rossi [12]. In relatively recent years, these analytic models were further extended to real semi-simple Jordan algebras by A. Dvorsky and S. Sahi [13].

Even though we get the explicit formula for measure $d\mu_\nu$ via the Riemannian metric on canonical cones, as far as representations are concerned, the introduction of this Riemannian metric is not needed. However, it is essential if one wishes to go one step further, i.e., unravel the hidden dynamic models behind these unitary lowest weight representations.

1.6. Universal Kepler hamiltonian and universal Lenz vector. The hidden dynamical models referred to in the proceeding paragraph are of the Kepler type in the sense that the hidden conserved Lenz vector always exists. To see that, we need to introduce the universal hamiltonian
\[H := \frac{1}{2} Y^{-1} X_e + iY_e^{-1}.\] (1.11)
3The explicit formula for $d\mu_\nu$ has already been given in Ref. [4] when $\nu > (\rho - 1)\frac{1}{2}$. 

\[\text{Ref. [8, 9] study of the Kepler problem.} \]
and the universal Lenz vector

\[ A_u := i Y^{-1}_c [L_{u, v}, Y^2_c H], \quad u \in V. \]

Here, \( i \) is the imaginary unit, and \( Y^{-1}_c \) is the formal inverse of \( Y_c \). Note that both \( H \) and \( A_u \) are elements of the complexified TKK algebra with \( Y_c \) inverted.

The manifest conserved quantity is \( L_{u, v} := [L_{u, v}, L_{u}] \) — the analogue of angular momentum, and the hidden conserved quantity is \( A_u \) — the analogue of original Lenz vector; in fact, the following commutation relations have been verified in Ref. [9]:

\[
\begin{align*}
\{ L_{u, v}, H \} &= 0, \\
[L_{u, v}, A_u] &= \nu \left( \frac{L_{u, v}}{2} \right), \\
[L_{u, v}, A_z] &= \nu \left( \frac{L_{u, v}}{2} \right), \\
[A_u, A_z] &= -2H L_{u, v}.
\end{align*}
\]

Now, under a unitary lowest weight representation \( \pi_\nu \) in the proceeding subsection, both \( H \) and \( A_u \) become differential operators \( \tilde{H}(\nu) \) and \( \tilde{A}_u(\nu) \) on the canonical cone. However, \( \tilde{H}(\nu) \) is not quite right, because the term \( -\tilde{Y}^{-1}_c(\nu) \tilde{X}_c(\nu) \) in \( \tilde{H}(\nu) \) is not the Laplace operator on the canonical cone, even up to an additive function.

1.7. Generalized Kepler problems. By comparing with the Laplace operator on the canonical cone, one realizes that \( -\tilde{Y}^{-1}_c(\nu) \tilde{X}_c(\nu) \) is almost right: after conjugation by the multiplication with a positive function on the canonical cone, modulo an additive function, it becomes the Laplace operator on the canonical cone. After this conjugation, \( \tilde{O} \) becomes a new differential operator (shall be denoted by \( \tilde{O} \)), and \( L^2(d\mu_\nu) \) becomes \( L^2(\frac{1}{2} \operatorname{vol}) \). Now \( \tilde{O}(\nu) \) is the hamiltonian for the quantum Kepler-type dynamic problem behind representation \( \pi_\nu \). It is in this sense we say that \( L^2(\frac{1}{2} \operatorname{vol}) \) is more natural than \( L^2(\frac{1}{2} \operatorname{vol}) \).

The bound state spectrum for \( \tilde{H}(\nu) \) is

\[ -\frac{1/2}{(1 + \nu \frac{1}{2})^2}, \quad I = 0, 1, 2, \ldots \]

and the Hilbert space of bound states — a closed subspace of \( L^2(\operatorname{vol}) \), denoted by \( \mathcal{H}(\nu) \), provides a new realization for representation \( \pi_\nu \).

The two natural realizations for \( \pi_\nu \), one by \( L^2(\frac{1}{2} \operatorname{vol}) \), one by \( \mathcal{H}(\nu) \), implies that there are two kinds of orthogonality for generalized Laguerre polynomials, generalizing the well-known fact that there are two kinds of orthogonality for Laguerre polynomials, cf. Ref. [14]. We would like to point out that this phenomena was first observed by A.O. Barut and H. Kleinert [15] for the original Kepler/Coulomb problem.

By the way, one may arrive at a quantum Kepler-type dynamic problem by suitably quantizing a classical Kepler-type problem whose configuration space is a canonical cone and Lagrangian is

\[
L(x, \dot{x}) = \frac{1}{2} ||\dot{x}||^2 + \frac{1}{r}.
\]

1.8. Outline of the paper. In Section 2 we give a quick review of Jordan algebras and especially euclidean Jordan algebras. In Section 3 we first review TKK algebras, then we introduce the classical realization for TKK algebras. The operator realizations follows from formally quantizing the classical realization. Section 4 is the most technical section of this article, here, the unitarity of those operator realizations is addressed. In Section 5 we present the generalized Kepler problem behind each nontrivial unitary representation in Section 4 then we solve the bound state problem and prove that the Hilbert space of...
bound states provides another realization for the representation. Here, the proof is complete only after one proves an analogue of Theorem 2 in Ref. [14] for generalized Laguerre polynomials, something that can surely be done. Two appendixes are given in the end. In Appendix A a technical theorem extending the existing one in Ref. [4] is proved. In Appendix B, for the convenience of readers, some basic notations for this paper are listed.

Ref. [4] is our most consulted book — really a bible for our purpose. However, we should warn the readers that there are some convention/notation differences: 1) our invariant inner product on $V$ is the one such that the identity element $e$ has unit length and is denoted by $\langle | \rangle$ rather than $(,)$, 2) the rank and degree of a Jordan algebra is denoted by $\rho$ and $\delta$ rather than $r$ and $d$ because $r$ is reserved for function $x \mapsto \langle e \mid x \rangle$ and $d$ is reserved for the exterior derivative operator, 3) the Jordan multiplication by $u$ is denoted by $L_u$ rather than $L(u)$, 4) the Jordan triple product of $u, v, w$ is denoted by $\{uvw\}$ rather than $\{u,v,w\}$, 5) we use $S_{uv}$ rather than $u \square v$ for endomorphism $w \mapsto \{uvw\}$.

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2. JORDAN ALGEBRAS

Jordan algebra has become a big subject now; however, what is relevant for us here is very minimal. In this section, we shall review the bare essential of Jordan algebras, oriented towards our purpose. Apart from Lemma 2.1, everything else presented here can be found from the book by J. Faraut and A. Korányi [4].

2.1. Basic definitions. Recall that an algebra $V$ over a field $F$ is a vector space over $F$ together with a $F$-bilinear map $V \times V \to V$ which maps $(u, v)$ to $uv$. This $F$-bilinear map can be recast as a linear map $V \to \text{End}_F(V)$ which maps $u$ to $L_u: v \mapsto uv$.

We say that algebra $V$ is commutative if $uv = vu$ for any $u, v \in V$. As usual, we write $u^2$ for $uu$ and $u^{m+1}$ for $uu^m$ inductively.

Definition 2.1. A Jordan algebra over $\mathbb{F}$ is just a commutative algebra $V$ over $\mathbb{F}$ such that

\begin{equation}
[L_u, L_{u^2}] = 0
\end{equation}

for any $u \in V$.

As the first example, we note that $\mathbb{F}$ is a Jordan algebra over $\mathbb{F}$. Here is a recipe to produce Jordan algebras. Suppose that $\Phi$ is an associative algebra over field $\mathbb{F}$ with characteristic $\neq 2$, and $V \subset \Phi$ is a linear subspace of $\Phi$, closed under square operation, i.e., $u \in V \Rightarrow u^2 \in V$. Then $V$ is a Jordan algebra over $\mathbb{F}$ under the Jordan product

$$uv := \frac{(u + v)^2 - u^2 - v^2}{2}.$$ 

Applying this recipe, we have the following Jordan algebras over $\mathbb{R}$:

1. The algebra $\Gamma(k)$. Here $\Phi = \text{Cl}(\mathbb{R}^k)$—the Clifford algebra of $\mathbb{R}^k$ and $V = \mathbb{R} \oplus \mathbb{R}^k$. 

(2) The algebra $\mathcal{H}_k(\mathbb{R})$. Here $\Phi = M_k(\mathbb{R})$—the algebra of real $k \times k$-matrices and $V \subset \Phi$ is the set of symmetric $k \times k$-matrices.

(3) The algebra $\mathcal{H}_k(\mathbb{C})$. Here $\Phi = M_k(\mathbb{C})$—the algebra of complex $k \times k$-matrices (considered as an algebra over $\mathbb{R}$) and $V \subset \Phi$ is the set of Hermitian $k \times k$-matrices.

(4) The algebra $\mathcal{H}_k(\mathbb{H})$. Here $\Phi = M_k(\mathbb{H})$—the algebra of quaternionic $k \times k$-matrices (considered as an algebra over $\mathbb{R}$) and $V \subset \Phi$ is the set of Hermitian $k \times k$-matrices.

The Jordan algebras over $\mathbb{R}$ listed above are special in the sense that they are derived from associated algebras via the above recipe. Let us use $\mathcal{H}_k(\mathbb{O})$ to denote the algebra for which the underlying real vector space is the set of Hermitian $k \times k$-matrices over $\mathbb{O}$ and the product is the symmetrization of the matrix product. One can show that $\mathcal{H}_k(\mathbb{O})$ is a Jordan algebra if and only if $k \leq 3$. However, only $\mathcal{H}_3(\mathbb{O})$ is new because $\mathcal{H}_1(\mathbb{O}) = \mathcal{H}_1(\mathbb{R})$, $\mathcal{H}_2(\mathbb{O}) \cong \Gamma(9)$. $\mathcal{H}_3(\mathbb{O})$ is called exceptional because it cannot be obtained via the above recipe, even if the associative algebra $\Phi$ is allowed to be infinite dimensional.

2.2. Euclidean Jordan algebras. Any Jordan algebra $V$ comes with a canonical symmetric bilinear form

$$\tau(u, v) := \text{the trace of } L_{uv}. \quad (2.2)$$

It is a fact that $L_u$ is self-adjoint with respect to $\tau$.

We say that Jordan algebra $V$ is semi-simple if the symmetric bilinear form $\tau$ is non-degenerate. We say that Jordan algebra $V$ is simple if it is semi-simple and has no ideal other than $\{0\}$ and $V$ itself.

By definition, an euclidean Jordan algebra is a real Jordan algebra with an identity element $e$ such that the symmetric bilinear form $\tau$ is positive definite. Therefore, an euclidean Jordan algebra is semi-simple and can be uniquely written as the direct sum of simple ideals — ideals which are simple as Jordan algebras.

**Theorem 2.1** (Jordan, von Neumann and Wigner, Ref. [2]). The complete list of simple euclidean Jordan algebras are

1. The algebra $\Gamma(k) = \mathbb{R} \oplus \mathbb{R}^k$ ($k \geq 2$).
2. The algebra $\mathcal{H}_k(\mathbb{R})$ ($k \geq 3$ or $k = 1$).
3. The algebra $\mathcal{H}_k(\mathbb{C})$ ($k \geq 3$).
4. The algebra $\mathcal{H}_k(\mathbb{H})$ ($k \geq 3$).
5. The algebra $\mathcal{H}_3(\mathbb{O})$.

Note that $\Gamma(1)$ is not simple and $\mathcal{H}_1(\mathbb{F}) (= \mathbb{R})$ is the only associative simple euclidean Jordan algebra. Note also that there are various isomorphisms: $\Gamma(2) \cong \mathcal{H}_2(\mathbb{R})$, $\Gamma(3) \cong \mathcal{H}_2(\mathbb{C})$, $\Gamma(5) \cong \mathcal{H}_2(\mathbb{H})$, $\Gamma(9) \cong \mathcal{H}_2(\mathbb{O})$.

**Remark 2.1.** A simple euclidean Jordan algebra is uniquely specified by its rank $\rho$ and degree $\delta$:

| $J$ | $\Gamma(k)$ | $\mathcal{H}_k(\mathbb{R})$ | $\mathcal{H}_k(\mathbb{C})$ | $\mathcal{H}_k(\mathbb{H})$ | $\mathcal{H}_3(\mathbb{O})$ |
|-----|-------------|-----------------|-----------------|-----------------|-----------------|
| $\rho$ | 2 | $k$ | $k$ | $k$ | 3 |
| $\delta$ | $k - 1$ | 1 | 2 | 4 | 8 |

Hence, for the simple euclidean Jordan algebras, there is one with rank-one, infinity many with rank two, four with rank three, and three with rank four or higher.

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4Called formally real Jordan algebra in the old literature.
The notion of trace is valid for Jordan algebras. For the simple euclidean Jordan algebras, the trace can be easily described: For $\Gamma(k)$, we have

$$\text{tr} (\lambda, \bar{u}) = 2\lambda,$$

and for all other types, it is the usual trace for matrices.

For the inner product on $V$, we take

$$(u \mid v) := \frac{1}{\rho} \text{tr} (uv)$$

so that the identity element $e$ becomes a unit vector. One can check that $L_u$ is self-adjoint with respect to this inner product: $\langle vu \mid w \rangle = \langle v \mid uw \rangle$, i.e., $L_u' = L_u$. For $u, v$ in $V$, we introduce linear map $S_{uv} := [L_u, L_v] + L_{uv}$, and write $S_{uv}(w)$ as $\langle uvw \rangle$. One can check that $S_{uv} = S_{vu}$ and

$$[S_{uv}, S_{zu}] = S_{\{uvz\}w} - S_{\{vuw\}z}.$$

In the remainder of this paper, we fix a simple euclidean Jordan algebra $V$, and use $e$, $\rho$, $\delta$ and $n$ to denote its identity element, rank, degree and dimension respectively. We shall use $\{e_{ii}\}$ to denote a Jordan frame and $V_{ij}$ to denote the resulting $(i,j)$-Pierce component. Choose an orthogonal basis $e_{ij}^\mu$ for each $V_{ij}$ ($1 \leq i < j \leq \rho$) with each basis vector having length $\frac{1}{\sqrt{\rho}}$. Such a basis is referred to as a Jordan basis with respect to the Jordan frame $\{e_{ii}\}$.

Here is a convention we shall adopt: $x$ is reserved for a point in the smooth space $V$, and $u, v, z, w$ are reserved for vectors in vector space $V$. We shall also use $V$ to denote the Euclidean space with underlying smooth space $V$ and Riemannian metric $ds^2_{\rho}$:

$$T_xV \times T_xV \rightarrow \mathbb{R}$$

$$\langle (x, u), (x, v) \rangle \mapsto \langle u \mid v \rangle.$$  \hspace{1cm} (2.4)

Finally, we would like to remark that, if one takes Jordan algebras as analogues of Lie algebras, then Euclidean Jordan algebras are the analogues of compact Lie algebras.

2.3. Structure algebras and TKK Algebras. As always, we let $V$ be a finite dimensional simple euclidean Jordan algebra. We use $\Omega$ to denote the symmetric cone of $V$ and $\text{Str}(V)$ to denote the structure group of $V$. By definition, $\Omega$ is the topological interior of

$$\{x^2 \mid x \in V\}$$

and

$$\text{Str}(V) = \{g \in GL(V) \mid P(gx) = gP(x)g^{-1} \quad \forall x \in V\}.$$  

Here $P(x) := 2L_x^2 - L_x$ and it is called the quadratic representation of $x$.

We write $V^C$ for the complexification of $V$, denote by $T\Omega$ the tube domain associated with $V$. By definition, $T\Omega = V \oplus i\Omega$. We say that map $f : T\Omega \rightarrow T\Omega$ is a holomorphic automorphism of $T\Omega$ if $f$ is invertible and both $f$ and $f^{-1}$ are holomorphic. We use $\text{Aut}(T\Omega)$ to denote the group of holomorphic automorphisms of $T\Omega$.

It is a fact that both $\text{Str}(V)$ and $\text{Aut}(T\Omega)$ are Lie groups. The Lie algebra of $\text{Str}(V)$ is referred to as the structure algebra of $V$ and is denoted by $\text{str}(V)$ or simply $\text{str}$, the Lie algebra of $\text{Aut}(T\Omega)$ is referred to as the conformal algebra of $V$ and is denoted by $\text{co}(V)$ or simply $\text{co}$, and its universal enveloping algebra is called the TKK algebra of $V$. The
simply connected Lie group with $\mathfrak{co}$ as its Lie algebra, denoted by $\text{Co}(V)$ or simply $\mathfrak{co}$, shall be referred to as the **conformal group** of $V$.

Both the structure algebra and the conformal algebra have a simple direct algebraic description, cf. Subsection I.2 While the structure algebra is reductive, the conformal algebra is simple.

Since $\mathfrak{co}$ is a non-compact real simple Lie algebra, it admits a Cartan involution $\theta$, unique up to conjugations by inner automorphisms. Indeed, one can choose $\theta$ such that

$$\theta(X_u) = Y_u, \quad \theta(Y_u) = X_u, \quad \theta(S_{uu}) = -S_{uu}.$$ 

The resulting Cartan decomposition is $\mathfrak{co} = u \oplus p$ with

$$u = \text{span}_\mathbb{R}\{[L_u, L_v], X_u + Y_v \mid u, v, w \in V\}, \quad p = \text{span}_\mathbb{R}\{L_u, X_u - Y_v \mid u, v \in V\}.$$ 

Note that $u$ is reductive with center spanned by $X_e + Y_e$ and its semi-simple part $\hat{u}$ is

$$\text{span}_\mathbb{R}\{[L_u, L_v], X_u + Y_v \mid u, v, w \in (\mathbb{R}e)\}.$$ 

Sometime we need to emphasize the dependence on $V$, then we rewrite $u$ as $u(V)$. It is a fact that $\text{str}$ and $u$ are different real forms of the same complex reductive Lie algebra. In fact, one can identify their complexifications as follows:

$$[L_u, L_v] \leftrightarrow [L_u, L_v], \quad -\frac{i}{2}(X_u + Y_v) \leftrightarrow L_w.$$ 

Here is a detailed summary of all real Lie algebras we have encountered:

| $V$ | $\mathfrak{der}$ | $\text{str}$ | $u$ | $\mathfrak{co}$ |
|-----|-----------------|--------------|-----|----------------|
| $\Gamma(n)$ | $\mathfrak{so}(n)$ | $\mathfrak{so}(n, 1) \oplus \mathbb{R}$ | $\mathfrak{so}(n + 1) \oplus \mathfrak{so}(2)$ | $\mathfrak{so}(n + 1, 2)$ |
| $H_n(\mathbb{R})$ | $\mathfrak{so}(n)$ | $\mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$ | $u(n)$ | $\mathfrak{sp}(n, \mathbb{R})$ |
| $H_n(\mathbb{C})$ | $\mathfrak{su}(n)$ | $\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$ | $\mathfrak{su}(n) \oplus \mathfrak{su}(n) \oplus u(1)$ | $\mathfrak{su}(n, n)$ |
| $H_n(\mathbb{H})$ | $\mathfrak{sp}(n)$ | $\mathfrak{su}^*(2n) \oplus \mathbb{R}$ | $u(2n)$ | $\mathfrak{so}^*(4n)$ |
| $H_3(\mathbb{O})$ | $\mathfrak{f}_4$ | $\mathfrak{e}_6(-26) \oplus \mathbb{R}$ | $\mathfrak{e}_6 \oplus \mathfrak{so}(2)$ | $\mathfrak{e}_{7(-25)}$ |

Recall that $e_{11}$ denotes the first element of a Jordan frame for $V$. The following lemma has been proved in Subsection 4.2 of Ref. [3].

**Lemma 2.1.** There is a maximally compact $\theta$-stable Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{co}$, with respect to which, there is a simple root system consisting of imaginary roots $\alpha_0, \alpha_1, \ldots, \alpha_r$ such that, for $i \geq 1$, $\alpha_i$ is compact with $H_{\alpha_i}, E_{\pm \alpha_i} \in \mathfrak{u}^\mathbb{C}$, and $\alpha_0$ is non-compact with

$$H_{\alpha_0} = i(X_{e_{11}} + Y_{e_{11}}), \quad E_{\pm \alpha_0} = \frac{i}{2}(X_{e_{11}} - Y_{e_{11}}) \pm L_{e_{11}}.$$ 

2.4. **The decomposition of the action of $\text{Str}(V)$ on $\mathcal{P}(V)$**. The structure group acts on $V$ linearly, so it acts on $\mathcal{P}(V)$ — the set of complex-valued polynomial functions on $V$. The goal here is to describe the known decomposition of the action of $\text{Str}(V)$ on $\mathcal{P}(V)$ into irreducible components.

With a Jordan frame $\{e_i\}$ for $V$ chosen, for $1 \leq k \leq \rho$, we let $e[k] = e_{11} + \cdots + e_{kk}$. Denote by $V_k$ the eigenspace of $L_{e[k]}$ with eigenvalue $1$ and by $P_k$ the orthogonal projection of $V$ onto $V_k$. Then $V_k$ is a simple euclidean Jordan algebra of rank $k$ and there is a filtration of euclidean Jordan algebras:

$$V_1 \subset V_2 \cdots \subset V_\rho = V.$$
Let \( m \in \mathbb{Z}^\rho \). We write \( m = (m_1, \ldots, m_\rho) \) and say that \( m \geq 0 \) if \( m_1 \geq \ldots \geq m_\rho \geq 0 \). Let

\[
\Delta_m(x) = \prod_{i=1}^\rho \Delta_i(x)^{m_i-m_{i+1}},
\]

here \( m_{\rho+1} = 0 \), \( \Delta_i(x) \) is the determinant of \( P_i(x) \), considered as an element of \( V_i \).

For \( m \geq 0 \), we let \( \mathcal{P}_m(V) \) be the subspace of \( \mathcal{P}(V) \) generated by the polynomials \( g \cdot \Delta_m \), \( g \in \text{Str}(V) \). The polynomials belonging to \( \mathcal{P}_m(V) \) are homogeneous of degree \( |m| = \sum m_i \), hence \( \mathcal{P}_m(V) \) is finite dimensional.

**Theorem XI.2.4 of Ref. [4]**. The subspaces \( \mathcal{P}_m(V) \) are mutually inequivalent irreducible as representation spaces of \( \text{Str}(V) \), and \( \mathcal{P}(V) \) is the direct sum

\[
\mathcal{P}(V) = \bigoplus_{m \geq 0} \mathcal{P}_m(V).
\]

Since \( \mathcal{P}(V) \) consists of complex-valued polynomials on \( V \), the representations in this theorem naturally extends to the complexification of \( \text{Str}(V) \). Using the identification in Eq. (2.5), these representations naturally becomes representations of \( u \).

From here on, as representations of \( u \), \( \mathcal{P}(V) \) and \( \mathcal{P}_m(V) \) shall always be viewed in this sense. For later use, we use \( \xi_\nu \) to denote the one-dimensional representation \( \mathbb{C} \) of \( u \) such that \(-\nu \xi_\nu (X_e + Y_e) \) acts as the scalar multiplication by \(-\nu \xi_\nu \).

### 3. Realizations of TKK Algebras

The goal of this section is to realize the TKK algebras. The results and their presentations here are strongly influenced by the thinking/practice in physics. Although our perspective is different, we don’t claim any originality here, because most (maybe all) of the materials presented here should be known to the experts in one area or another area.

We start with the classical realization on symplectic space \( T^*V \), from which the operator realizations follow via the straightforward canonical quantization. Due to the operator ordering ambiguity, we get a family of operator realizations, parametrized by a real parameter \( \nu \). The case \( \nu = 2n/\rho \) is well-known to physicists, cf. Ref. [6]. The case \( \nu > 1 + (\rho - 1)\delta \) has been worked out by M. Aristidou, M. Davidson and G. Olafsson [7] by an indirect method.

#### 3.1. The classical realization of TKK algebras

As is well-known, the total cotangent space \( T^*V \) is a natural symplectic space. By virtue of the euclidean metric \( ds^2_F \) on \( V \), one can identify \( T^*V \) with the total tangent space \( TV \). Now the tangent bundle and cotangent bundle of \( V \) both have a natural trivialization, with respect to which, one can denote an element of \( T^*V \) by \((x, p)\) and its corresponding element in \( TV \) by \((x, \pi)\). We fix an orthonormal basis \( \{e_\alpha\} \) for \( V \) so that we can write \( x = x^\alpha e_\alpha \) and \( \pi = \pi^\alpha e_\alpha \). Then the basic Poisson bracket relations on \( TV \) are \( \{x^\alpha, \pi^\beta\} = \delta^\alpha{}^\beta \), \( \{x^\alpha, x^\beta\} = 0 \), and \( \{\pi^\alpha, \pi^\beta\} = 0 \).

Introduce the moment functions

\[
S_{uv} := \langle S_{uv}(x) \mid \pi \rangle, \quad X_u := \langle x \mid \{\pi u \pi\} \rangle, \quad Y_u := \langle x \mid v \rangle
\]

on \( TV \). The following theorem would be well-known to experts on Jordan algebra.
Theorem 1. As polynomial functions on $TV$, $S_{uv}$, $X_u$ and $Y_v$ satisfy the following Poisson bracket relations: for any $u, v, z$ and $w$ in $V$,

$$\begin{align*}
\{X_u, X_v\} &= 0, & \{Y_u, Y_v\} &= 0, & \{X_u, Y_v\} &= -2S_{uv}, \\
\{S_{uv}, X_z\} &= X_{\{uvz\}}, & \{S_{uv}, Y_z\} &= -Y_{\{uvz\}}, \\
\{S_{uv}, S_{zw}\} &= S_{\{uvz\}w} - S_{\{uvw\}}.
\end{align*}$$

(3.2)

Proof. It is clear that $\{X_u, Y_v\} = 0$.

$$\begin{align*}
\{X_u, Y_v\} &= \{\langle x \mid \{\pi u \pi\}\rangle, \langle x \mid v \rangle\} \\
&= -2\langle x \mid \{v\pi u\}\rangle = -2\langle S_{uv}(x) \mid \pi\rangle \\
&= -2S_{uv}. \\
\{S_{uv}, X_z\} &= \{\langle S_{uv}(x) \mid \pi\rangle, \langle x \mid \pi z\rangle\} \\
&= -\langle S_{uv}(x) \mid \pi z\rangle + 2\langle x \mid \pi z\{\pi v\pi\}\rangle \\
&= \langle x \mid 2S_{\pi z}S_{uv}(\pi) - S_{v\pi z}(\pi)\rangle \\
&= \langle x \mid S_{\pi z}(\{v\pi\}) - S_{\{v\pi\}z}(\pi) + S_{\pi z}(\{v\pi\})\rangle \\
&= \langle x \mid \{\pi \{uv\}z\}\rangle \\
&= X_{\{uvz\}}.
\end{align*}$$

Finally,

$$\begin{align*}
\{X_u, Y_v\} &= \{\langle x \mid \{\pi u \pi\}\rangle, \langle x \mid \{\pi v\pi\}\rangle\} \\
&= 2\langle x \mid \{\pi v\{\pi u\}\} - \{\pi u \{\pi v\}\}\rangle \\
&= 2\langle x \mid [S_{\pi v}, S_{\pi u}]\pi\rangle \\
&= \langle x \mid S_{\pi v}S_{\pi u}\pi\rangle - \langle x \mid S_{\pi u}S_{\pi v}\pi\rangle \\
&= \langle x \mid S_{\pi v}\{u\pi\} - S_{\pi u}\{v\pi\}\rangle - \langle x \mid S_{\pi v}\{u\pi\} - S_{\pi u}\{v\pi\}\rangle \\
&= \langle x \mid S_{\pi v}\{u\pi\} - \{x \mid S_{\pi u}\{v\pi\}\}\rangle \text{ because } \{uv\pi\} = \{v\pi u\}\rangle \\
&= -\langle x \mid S_{\pi v}\{u\pi\}\rangle \\
&= 0,
\end{align*}$$

because it is equal to the negative half of itself.

\[\square\]

3.2. The operator realizations of TKK Algebras. The canonical quantization involves promoting classical physical variables $O$ to differential operators $\hat{O}$ (or the duals $\check{O}$) using recipe: $\pi_\alpha \rightarrow -i\frac{\partial}{\partial x^\alpha}$ (or $x_\alpha \rightarrow i\frac{\partial}{\partial \pi^\alpha}$). Here is a word of warning: in order to get anti-hermitian differential operators in the end, instead of using the quantized differential operators, we actually use the quantized differential operators multiplied by $-i$.

For simplicity we write $\sum_\alpha e_\alpha \frac{\partial}{\partial x^\alpha}$ as $\hat{\partial}$ and $\sum_\alpha e_\alpha \frac{\partial}{\partial \pi^\alpha}$ as $\check{\partial}$. We introduce differential operators on $V$:

$$\begin{align*}
\hat{S}_{uv} &:= -\langle S_{uv}(x) \mid \hat{\partial}\rangle, & \hat{X}_u &:= i\langle x \mid \{\hat{\partial} u \hat{\partial}\}\rangle, & \hat{Y}_v &:= -i\langle x \mid \hat{\partial}\rangle.
\end{align*}$$

(3.5)
and differential operators on $V^*$:

\[(3.6) \quad \hat{S}_{uv} := \langle S_{vu}(\pi) \mid \partial \rangle, \quad \hat{X}_u := \langle \{\pi u\pi\} \mid \partial \rangle, \quad \hat{Y}_v := \langle v \mid \partial \rangle.\]

It is easy to see that the TKK commutation relations \[(1.4)\] hold when all $O$ there are replaced by either their hat version or their check version. Note that the check version is well-known to physicists, cf. Ref. \[6\]. However, the quantization has ambiguity because of the operator ordering problem. To get the general version of quantization, we let $\nu$ be a real parameter and introduce differential operators on $V^*$:

\[(3.7) \quad \hat{S}_{uv}(\nu) := \hat{S}_{uv} - \frac{\nu}{2} \text{tr} (uv), \quad \hat{X}_u(\nu) := \hat{X}_u + i\nu \text{tr} (u\pi), \quad \hat{Y}_v(\nu) := \hat{Y}_v,\]

and differential operators on $V^*$:

\[(3.8) \quad \check{S}_{uv}(\nu) := \check{S}_{uv} - \frac{\nu^*}{2} \text{tr} (uv), \quad \check{X}_u(\nu) := \check{X}_u - \nu^* \text{tr} (u\pi), \quad \check{Y}_v(\nu) := \check{Y}_v,\]

where $\nu^* = \nu - \frac{2\pi}{\rho}$.

**Theorem 2.** The TKK commutation relations \[(1.4)\] still hold when all $O$ there are replaced by either their acute version or their grave version.

**Proof.** When $\nu = 0$, the proof is essentially the same as the proof of Theorem \[1\] so we skip it. For the general case, we shall verify the acute version and leave the grave version to the readers.

Verify that $[\hat{Y}_u(\nu), \check{Y}_v(\nu)] = 0$:

\[\hat{Y}_u(\nu), \check{Y}_v(\nu) = [\hat{Y}_u, \check{Y}_v] = 0.\]

Verify that $[\hat{S}_{uv}(\nu), \check{Y}_z(\nu)] = -\check{Y}_{(vuz)}(\nu)$:

\[\hat{S}_{uv}(\nu), \check{Y}_z(\nu) = [\hat{S}_{uv}, \check{Y}_z] = -\check{Y}_{(vuz)} = -\check{Y}_{(vuz)}(\nu).\]

Verify that $[\hat{S}_{uv}, \hat{S}_{zw}] = \hat{S}_{(uwz)} - \hat{S}_{(vuw)}$:

Since $[\hat{S}_{uv}, \hat{S}_{zw}] = \hat{S}_{(uwz)} - \hat{S}_{(vuw)}$, all we need to check is that

\[\text{tr} (\{uvz\} w) = \text{tr} (z\{vuw\}), \quad \text{i.e.,} \quad \langle S_{uv}(z) \mid \ w \rangle = \langle z \mid S_{uv}(w) \rangle\]

which is true because $S_{uv}' = S_{vu}$.

Verify that $[\check{X}_u(\nu), \check{Y}_v(\nu)] = -2\check{S}_{uv}$:

\[[\check{X}_u(\nu), \check{Y}_v(\nu)] = [\check{X}_u + i\nu \text{tr} (u\pi), \check{Y}_v] = -2\check{S}_{uv} + \nu \text{tr} (uv) = -2\check{S}_{uv}(\nu).\]

Verify that $[\hat{S}_{uv}(\nu), \check{X}_z(\nu)] = \check{X}_{(uwz)}(\nu)$:

\[[\hat{S}_{uv}(\nu), \check{X}_z(\nu)] = [\hat{S}_{uv}(\nu), \check{X}_z + i\nu \text{tr} (z\pi)] = \check{X}_{(uwz)} + i\nu \text{tr} (S_{uv}(z) \mid \pi) = \check{X}_{(uwz)}(\nu).\]

Verify that $[\check{X}_u(\nu), \check{X}_v(\nu)] = 0$:

\[[\check{X}_u(\nu), \check{X}_v(\nu)] = [\check{X}_u + i\nu \text{tr} (u\pi), \check{X}_v + i\nu \text{tr} (v\pi)] = i\nu \text{tr} [\check{X}_u, \{v \mid \pi\}] - \langle u \leftrightarrow v \rangle = \rho \text{tr} [\{x \mid \pi u\pi\}, \langle v \mid \pi\}] - \langle u \leftrightarrow v \rangle = \rho \text{tr} [\{v \mid \pi u\pi\}] - \langle u \leftrightarrow v \rangle.
\[
\begin{align*}
\rho u \langle \{ u \phi v \} | \emptyset \rangle - \langle u \leftrightarrow v \rangle &= 0.
\end{align*}
\]

Here, \( \langle u \leftrightarrow v \rangle \) means the term same as the one on the left except that \( u \) and \( v \) are interchanged.

In the remainder of this paper, let us focus the attention on the operator realization on \( V: O \to \hat{O}(\nu) \) for a fixed \( \nu \). Note that this operator realization provides a linear action of the TKK algebra on \( C^\infty(V) \). We shall investigate the unitarity of this action in the next section. In order to do that, let us make some preparations here.

Let \( \mathcal{P}(V) \) be the algebra of \( \mathbb{C} \)-valued polynomial functions on \( V \), and \( \mathcal{P}_I(V) \) be the vector subspace consisting of polynomials of degree at most \( I \). Let

\begin{equation}
\begin{aligned}
\hat{D}(V) &= e^{-\tau} \mathcal{P}(V), \\
\hat{D}_I(V) &= e^{-\tau} \mathcal{P}_I(V).
\end{aligned}
\end{equation}

It is clear that the action of \( \mathfrak{co} \) on \( C^\infty(V) \), which maps \( O \) to \( \hat{O}(\nu) \), leaves \( \hat{D}(V) \) invariant. Let us denote by \( \pi_\nu \) this action on \( \hat{D}(V) \).

Recall that \( \nu \) is a real parameter and \( u \) is the maximal compact Lie subalgebra of \( \mathfrak{co} \).

**Theorem 3.** Let \( H_c = i(X_c + Y_c) \), \( \pi_\nu|_u \) be the restriction of \( \pi_\nu \) from \( \mathfrak{co} \) to \( u \).

i) \( \pi_\nu|_u \) leaves \( \hat{D}_I(V) \) invariant and commutes with the inclusion of \( \hat{D}_{I-1}(V) \) into \( \hat{D}_I(V) \), consequently it induces a linear action on \( \hat{D}_I(V) / \hat{D}_{I-1}(V) \).

ii) As a representation of \( u \), we have isomorphism

\[
\begin{aligned}
\xi_\nu \otimes \bigoplus_{m \geq 0} [m] \mathcal{P}_m(V) &\rightarrow \hat{D}_I(V) / \hat{D}_{I-1}(V) \\
1 \otimes p &\mapsto e^{-\tau} p + \hat{D}_{I-1}(V).
\end{aligned}
\]

iii) The induced linear map

\[
\hat{H}_c: \hat{D}_I(V) / \hat{D}_{I-1}(V) \rightarrow \hat{D}_I(V) / \hat{D}_{I-1}(V)
\]

is the scalar multiplication by \((2I + \nu \rho)\).

iv) \( \pi_\nu \) is unitarizable \( \implies \nu \geq 0 \).

v) \( e^{-\tau} \) is a lowest weight state with weight \( \nu \lambda_0 \).

vi) \( \pi_\nu \) is indecomposable.

**Proof.** i) Since

\begin{equation}
\begin{aligned}
e^{-\frac{i}{2}(X_u + Y_u)} e^{-\tau} &= \frac{1}{2} \langle \{ x | \{ \phi u \phi \} \} + \nu \text{tr} (u \phi) \rangle + L_u - \frac{\nu}{2} \text{tr} u,
\end{aligned}
\end{equation}

\( (X_u + Y_u) \) maps \( \hat{D}_I \) into \( \hat{D}_I \). It is also clear that \( [L_u, L_u] \) maps \( \hat{D}_I \) into \( \hat{D}_I \). Therefore, in view of the fact that \( u = \text{span}_{\mathbb{R}} \{ [L_u, L_u], X_u + Y_u | u, v, w \in V \} \), the linear action of \( u \) on \( \hat{D} \) leaves \( \hat{D}_I \) invariant. The rest is clear.

ii) That is clear from Eq. (3.10) and the last paragraph of Subsection 2.4.

iii) For any homogeneous degree \( I \) polynomial \( p \), we have

\[
\begin{aligned}
i(X_c + Y_c)e^{-\tau} p &= e^{-\tau} (-2L_c + \nu \rho)p \mod \hat{D}_{I-1} \\
&= (2I + \nu \rho)e^{-\tau} p \mod \hat{D}_{I-1}.
\end{aligned}
\]

iv) Let \( E_{\pm} := \frac{i}{2}(X_c - Y_c) \pm S_{cc} \). Then

\[
[H_c, E_{\pm}] = \pm 2E_{\pm}, \quad [E_+, E_-] = -H_c.
\]
Suppose that \( \pi_\nu \) is unitarizable, and (.) is the inner product on \( \hat{D} \). Let \( \psi_0 = e^{-r} \) Since

\[
||\pi_\nu(E_+ \psi_0)||^2 \geq 0 \quad \text{and} \quad \pi_\nu(E_- \psi_0) = 0,
\]

using relation \([E_+, E_-] = -H_e\), we arrive at

\[
(\psi_0, \pi_\nu(H_e)\psi_0) \geq 0, \quad \text{i.e.,} \quad \nu \rho ||\psi_0||^2 \geq 0. \quad \text{So} \quad \nu \geq 0.
\]

v) Let us take the simple root system \( \alpha_0, \ldots, \alpha_r \) specified in Lemma 2.1. Since \( \hat{D}_0 \)

\( (= \text{span}_\mathbb{C}\{\psi_0\}) \) is one dimensional and \( \bar{u} \) is semi-simple, the action of \( \bar{u}^\mathbb{C} \) on \( \hat{D}_0 \) must be trivial. Therefore, for \( i \geq 1 \), in view of the fact that \( E_{\pm \alpha_i}, H_{\alpha_i} \in \bar{u}^\mathbb{C} \), we have

\[
\pi_\nu(E_{-\alpha_i})\psi_0 = 0, \quad \pi_\nu(H_{\alpha_i})\psi_0 = \nu \psi_0.
\]

On the other hand, since \( E_{-\alpha_0} = \frac{1}{2}(X_{e_{11}} - Y_{e_{11}}) + L_{e_{11}} \) and \( H_{\alpha_0} = i(X_{e_{11}} + Y_{e_{11}}) \), by a computation, we have

\[
e^r \hat{E}_{-\alpha_0} e^{-r} = -\frac{1}{2} \langle \{x | \{\delta e_{11}\} + \nu \text{tr}(e_{11}\delta)\} \rangle,
\]

\[
e^r \hat{H}_{\alpha_0} e^{-r} = -\langle \{x | \{\delta e_{11}\} - \nu \text{tr}(e_{11}\delta)\} \rangle - 2L_{e_{11}} + \nu,
\]

so it is easy to see that

\[
\pi_\nu(E_{-\alpha_0})\psi_0 = 0, \quad \pi_\nu(H_{\alpha_0})\psi_0 = \nu \psi_0.
\]

Therefore, in view of the fact that \( \alpha_0(H_{\alpha_0}) = 2 \), Eqns (3.11) and (3.12) imply that \( \psi_0 \) is a

lowest weight state with weight \( \nu \lambda_0 \).

vi) In view of the fact that operator \( \hat{Y}_e \) is the multiplication by \(-i\langle \nu | x \rangle\), this is obvious:

\[
e^{-r} \sum_{i_1, \ldots, i_n} \alpha_{i_1, \ldots, i_n} x_{i_1}^{i_1} \cdots x_{i_n}^{i_n} = \left( \sum_{i_1, \ldots, i_n} \alpha_{i_1, \ldots, i_n} (i\hat{Y}_{e_1})^{i_1} \cdots (i\hat{Y}_{e_n})^{i_n} \right) \psi_0.
\]

We shall show in the next section that \( \pi_\nu \) is irreducible when \( \nu > (\rho - 1)\frac{\delta}{2} \) and is not

irreducible when \( \nu = k\frac{\delta}{2} \), \( k = 0, 1, \ldots, (\rho - 1) \). The collection of these values of \( \nu \),

denoted by \( \mathcal{W}(V) \), is called the Wallach set for \( V \). So

\[
\mathcal{W}(V) = \left\{ k\frac{\delta}{2} \mid k = 0, 1, \ldots, (\rho - 1) \right\} \cup \left( (\rho - 1)\frac{\delta}{2}, \infty \right).
\]

It is known from Ref. [10] that the set of scalar-type unitary lowest weight representation of \( \mathfrak{so} \) is isomorphic to \( \mathcal{W}(V) \). We shall show in the next section that these representations are precisely the irreducible quotient of these \( \pi_\nu \), also denoted by \( \pi_\nu \).

As mentioned in the introduction, the operator realizations as given in Eq. (3.7) are not

unitary with respect to the obvious \( L^2 \)-inner product

\[
(\psi_1, \psi_2) = \int_V \bar{\psi}_1 \psi_2 \, dm,
\]

where \( dm \) is the Lebesgue measure. The right inner product must be found in order to have

unitary realizations. For that, let us move on to the next section.

4. Quantizations Of TKK Algebras

The goal of this section is to investigate the unitarity of representation \( \pi_\nu \) obtained

in the previous section. The question is to find a positive hermitian form \( (.,.)_\nu \) on \( \hat{D}(V) \)

with respect to which operators \( \hat{O}_\nu \) are all anti-hermitian. More generally, \( (.,.)_\nu \) can be

semi-positive because then \( \pi_\nu \) descends to a unitary representation by formally setting the

“spurious states” (i.e., elements of \( \hat{D}(V) \) with zero norm) as zero. It is a fact from Ref. [4]
that such a \((\cdot,\cdot)\) does exist for \(\nu\) in the Wallach set. Our purpose here is to present a new route towards this fact along with its refinements.

The case \(\nu = \frac{\delta}{2}\) is already known from our work in Ref. [8]. Recall from Ref. [8], the J-Kepler problem for \(V\) is a dynamic problem on \(\mathcal{P}\) — the submanifold of \(V\) consisting of semi-positive elements of rank one. By comparing Remark 8.1 and Proposition 8.1 from Ref. [8] with Theorem 3 here, we have

\[
(\psi_1, \psi_2)_{\frac{\delta}{2}} = \int_{\mathcal{P}} \overline{\psi}_1 \psi_2 r^{-1-(\rho/2-1)\delta}\, \text{vol}.
\]

Here, \(\text{vol}\) is the volume form for the Kepler metric

\[
ds_{K}^{2} := \frac{2}{\rho}ds_{K}^{2} - dv^{2}.
\]

It is clear that, as a vector subspace of \(\hat{D}(V)\), the space of “spurious states” consists of elements of \(\hat{D}(V)\) which vanish on \(\mathcal{P}\).

With this result in mind, it is not hard to imagine what the general picture should be: replacing \(\mathcal{P}\) by the submanifold of \(V\) consisting of semi-positive elements of a fixed positive rank. Of course, some technical hurdles must be overcome. The initial hurdle is the generalization of the Kepler metric in Eq. (4.2). The second hurdle is the generalization of the extra factor \(r^{-1-(\rho/2-1)\delta}\) in measure

\[
d\mu_{\frac{\delta}{2}} := r^{-1-(\rho/2-1)\delta}\, \text{vol}.
\]

It turns out, the second hurdle simply disappears by itself as we walk along a natural path towards quantizations of TKK algebras. The clue for removing the first hurdle comes from the study of the universal Kepler problem in Ref. [9], as we shall sketch below.

We have noted in the past that the total tangent space of a Riemannian manifold is a symplectic manifold, and if \(N\) is a submanifold of \(M\), then \(TN\) is a symplectic submanifold of \(TM\). With this understood, we remarked in Ref. [9] that, by restricting the classical universal hamiltonian

\[
\mathcal{H} = \frac{1}{2} \frac{\langle x | \pi^2 \rangle}{r} - \frac{1}{r}
\]

from \(TV\) to \(T\mathcal{P}\), one obtains the classical hamiltonian for the J-Kepler problem. Since the first term in \(\mathcal{H}\) should be identified with the kinetic energy, we must have the following new formula for the Kepler metric\(^{5}\)

\[
(\pi, \pi)_{ds_{K}^{2}} = \frac{\langle x | \pi^2 \rangle}{r} = \frac{\langle \pi | L_{x} | \pi \rangle}{r},
\]

a fact which can be verified directly. Now it becomes clear how to generalize the Kepler metric.

4.1. Canonical cones. We say that an element \(x \in V\) is semi-positive if \(x = y^{2}\) for some \(y \in V\). Let us denote by \(Q\) the space of semi-positive elements in \(V\) and recall that \(\text{Str}\) is the structure group of \(V\). Then one can check that the action of \(\text{Str}\) on \(V\) leaves \(Q\) invariant, so we have a partition of \(Q\) into the disjoint union of \(\text{Str}\)-orbits:

\[
Q = \bigcup_{k=0}^{\rho} C_{k}.
\]

Here, homogeneous space \(C_{k}\) is the space of semi-positive elements of rank \(k\). Note that, \(C_{0} = \{0\}\), \(C_{\rho}\) is the symmetric cone \(\Omega\) of \(V\), and \(Q\) is the topological closure of \(\Omega\).

\(^{5}\)Here, for endomorphism \(A\) on \(V\) and elements \(u, v\) in \(V\), Dirac’s notation \(\langle u | A | v \rangle\) means \(\langle u | Av \rangle\).
As a sub-manifold of the Euclidean space $V$, $C_k$ has an induced Riemannian metric. When we say that $TC_k$ is a symplectic manifold, it is this Riemannian metric that is used to identify $TC_k$ with $T^*C_k$. However, the Riemannian metric for the Kepler-type dynamics on $C_k$ is a different one, which we shall describe below.

For any $u \in V$, since $L_u$: $V \to V$ is self-adjoint, we have an orthogonal decomposition $V = \text{Im}L_u \oplus \text{ker}L_u$, with respect to which, $L_u$ decomposes as $L_u = L_u \oplus 0$. Since $L_u$ is invertible, we can introduce endomorphism

\[ 1 \overset{\text{def}}{=} \bar{L}^{-1}_u \oplus 0 \]

on $V$. For any $x \in C_k$, one can check that the tangent space $T_xC_k$ is $\{x\} \times \text{Im}L_x$, the normal space $N_xC_k$ is $\{x\} \times \text{ker}L_x$.

**Definition 1 (Canonical Metric).** The canonical metric on $C_k$, denoted by $ds^2_k$, is defined as follows:

\[ T_xC_k \times T_xC_k \to \mathbb{R} \]

\[ ((x, u), (x, v)) \mapsto r(u \mid L_x^{-1} \mid v) = r(u \mid \bar{L}^{-1}_x \mid v). \]

One can check that, on $C_1$, the canonical metric is the Kepler metric introduced in Ref. [8]. On the symmetric cone $\Omega$, the canonical metric is

\[ T_\Omega \times T_\Omega \to \mathbb{R} \]

\[ ((x, u), (x, v)) \mapsto r(u \mid L_x^{-1} \mid v). \]

**Definition 2 (Canonical Cone).** Let $V$ be a simple euclidean Jordan algebra of rank $\rho$. $C_k$ be its submanifold consisting of the semi-positive elements of rank $k$, $1 \leq k \leq \rho$. The $V$'s canonical cone of rank $k$ is defined to be the Riemannian manifold $(C_k, ds^2_k)$, where $ds^2_k$ is the canonical metric in Definition[7].

We remark that, since the action of structure group on $V$ leaves $C_k$ invariant, for any $u, v \in V$ and $x \in C_k$, $S_{uv}|_x \in T_xC_k$; i.e., $\tilde{S}_{uv}$ descends to a vector field on $C_k$, which shall still be denoted by $\tilde{S}_{uv}$. Since $L_u = S_{ue}$, we write $\bar{L}_u$ for $\tilde{S}_{ue}$. In the remainder of this paper, we shall use $\Delta$ (vol resp.) denote the Laplace operator (the volume form resp.) on a Riemannian manifold, e.g., a canonical cone.

4.2. **Basic facts on canonical cones.** We use $Tr$ to denote the trace for endomorphisms on $V$. For any $x$ in the canonical cone, we use $P_x$: $V \to V$ to denote the orthogonal projection onto $\text{Im}L_x$. Throughout this subsection, we focus our attention on a fixed canonical cone $C$ of rank $k$.

Let us start with a local analysis of the canonical metric around a point $x_0 \in C$. Choose a Jordan frame $\{e_i\}_{i=1}^\rho$ such that

\[ x_0 = \sum_{i=1}^k \lambda_i e_i \]

for some positive numbers $\lambda_1$, ..., $\lambda_k$. With this Jordan frame $\{e_i\}_{i=1}^\rho$ fixed, we let $N := \bigoplus_{i \geq k} V_{ij}$ and $T$ be the orthogonal complement of $N$ in (euclidean vector space) $V$. Then $T_{x_0}C = \{x_0\} \times T$.

For any $x \in V$, one can uniquely decompose $x = x_0 + t + y$ with $t \in T$ and $y \in N$. Now if we assume that $x$ is always in $C$, then

\[ y = O(|t|^2) \quad \text{near } x_0. \]
Choose basis \( \{ e_i \} \) for \( T \) which is orthonormal with respect to the inner product on \( V \). Write \( t = t^i e_i \) and the canonical metric \( ds^2_K \) as \( h_{ij} dt^i \ dt^j \), then
\[
h_{ij}(x) = \langle e \mid x \rangle \langle x_i \mid \bar{L}_x^{-1} x_j \rangle,
\]
where \( x_i = \frac{\partial}{\partial t^i} \). Let \( g_{ij} := \langle x_i \mid x_j \rangle \), then
\[
g_{ij}(x) = \delta_{ij} + O(|t|^2) \quad \text{near} \ x_0.
\]

As usual, the inverse of \( [g_{ij}] \) is denoted by \([g^{ij}]\) and the inverse of \([h_{ij}]\) is denoted by \([h^{ij}]\). It is easy to see that
\[
h^{ij}(x) = \frac{g^{ik}(x) \langle x_k \mid \bar{L}_x x_i \rangle g^{ij}(x)}{\langle e \mid x \rangle}.
\]

Consequently, when \( x \in C \) is near \( x_0 \), we have
\[
h^{ij}(x) = \frac{\langle x_i \mid L_x x_j \rangle}{\langle e \mid x \rangle} + O(|t|^2)
\]
(4.7)
\[
= \frac{\langle e_i \mid x_0 e_j \rangle}{\langle e \mid x_0 \rangle} \left( 1 - \frac{\langle e \mid t \rangle}{\langle e \mid x_0 \rangle} \right) + \frac{\langle e_i \mid t e_j \rangle}{\langle e \mid x_0 \rangle} + O(|t|^2).
\]

**Proposition 4.1.** Fix a canonical cone. Let \( \text{vol} \) be its volume form, and \( \mathcal{L}_u \) the Lie derivative with respect to vector field \( \bar{L}_u \). Then
\[
\mathcal{L}_u \left( \frac{1}{p} \text{vol} \right) = -2 \lambda_u \frac{1}{p} \text{vol},
\]
where
\[
2 \lambda_u = -\frac{1}{2} \text{Tr} \left( \frac{1}{L_x} L_{ux} \right) + \text{Tr} (P_u L_u) + \frac{\langle u \mid x \rangle}{\langle e \mid x \rangle} \left( \frac{\text{Tr} P_u}{2} - 1 \right).
\]

Consequently, \( \lambda_u \) depends on \( u \) linearly and
\[
\hat{L}_u(\lambda_{eu}) = \hat{L}_v(\lambda_u), \quad \hat{S}_{uv}(\lambda_{eu}) - \hat{S}_{zu}(\lambda_{uv}) - \lambda_{\{eu\}} = \lambda_{\{uv\}} - \lambda_{\{zu\}}.
\]

**Proof.** We just need to show that
\[
\mathcal{L}_u (\text{vol}) = -2 \lambda_u \text{vol}
\]
at a point \( x_0 \), where
\[
-2 \lambda_u = \frac{1}{2} \text{Tr} \left( -\frac{\langle u \mid x \rangle}{\langle e \mid x \rangle} P_x + \frac{1}{L_x} L_{ux} \right) - \text{Tr} (P_u L_u). \tag{4.11}
\]

Recall that \( ds^2_K = h_{ij} dt^i \ dt^j \). Since \( \text{vol} = \sqrt{h} dt^1 \wedge dt^2 \wedge \cdots \) where \( h = \det [h_{ij}] \), we have
\[
\mathcal{L}_u (\text{vol}) = \hat{L}_u (\ln \sqrt{h}) \text{vol} + \sum_i \sqrt{h} dt^1 \wedge \cdots \wedge \mathcal{L}_u (dt^i) \wedge \cdots.
\]

Since \( \mathcal{L}_u (dt^i) = d \mathcal{L}_u (\langle e_i \mid x \rangle) = -d (\langle xe_i \mid u \rangle) \), we have
\[
\mathcal{L}_u (dt^i) = -\langle e_i^2 \mid u \rangle dt^i \quad \text{at} \ x_0,
\]
consequently,
\[
-2 \lambda_u = \hat{L}_u (\ln \sqrt{h}) - \sum_i \langle e_i^2 \mid u \rangle \quad \text{at} \ x_0. \tag{4.12}
\]

In view of Eq. (4.7), for \( x \) near \( x_0 \),
\[
h = \det \left( \langle e_i \mid x_0 e_j \rangle \right)^{-1} \left( 1 + \text{Tr} \left( \frac{\langle e \mid t \rangle}{\langle e \mid x_0 \rangle} P_{x_0} - \frac{1}{L_{x_0}} L_t \right) \right) + O(|t|^2),
\]
where \( x = x_0 + t \).
The first identity of Eq. (4.17) is verified. so the action of the

Plugging Eqs. (4.13) and (4.14) into Eq. (4.12), we arrive at Eq. (4.11).

Proposition 4.2. Since

In view of the fact that


certainly

In the remainder of this paper, we let

\[ [\mathcal{L}_{\lambda}, \mathcal{L}_{\lambda}] = \mathcal{L}_{[\lambda, \lambda]}, \quad \text{in view of fact that the Kepler metric is invariant under the action of the Aut(f),} \]

Then

2 \left( \hat{L}_{\lambda}(\lambda_v) - \hat{L}_v(\lambda_u) \right) \cdot \frac{1}{r} \text{vol} = 0,

consequently \( \hat{L}_{\lambda}(\lambda_u) = \hat{L}_v(\lambda_v). \)

Since

\[ [\mathcal{L}_{\hat{S}_{uu}}, \mathcal{L}_{\hat{S}_{uw}]} = \mathcal{L}_{[\hat{S}_{uu}, \hat{S}_{uw}]} = \mathcal{L}_{\hat{S}_{uvz}w - \hat{S}_{zvuw}}, \]

acting on \( \frac{1}{r} \text{vol}, \) we have

\[ \hat{S}_{uv}(\lambda_{uv}) - \hat{S}_{uv}(\lambda_{uw}) = \lambda_{uv} - \lambda_{uvw}. \]

In the remainder of this paper, we let

\[ \hat{S}_{uv} = \hat{S}_{uv} - \lambda_{uv}, \quad \hat{L}_{uv} = \hat{S}_{uv}. \]

**Proposition 4.2.** Fix a canonical cone and let \( \Delta \) be its Laplace operator. Then

\[ [r \Delta, \langle u \mid x \rangle] = -2 \hat{L}_u, \quad u \in V. \]

**Proof.** Upon observing that \([r \Delta, \langle u \mid x \rangle]\) is a linear differential operator, it suffices to verify that

\[ [[r \Delta, \langle u \mid x \rangle], \langle v \mid x \rangle](1) = -2 \hat{L}_u, \langle v \mid x \rangle](1), \quad [r \Delta, \langle u \mid x \rangle](1) = -2 \hat{L}_u(1), \]

i.e.,

\[ [[r \Delta, \langle u \mid x \rangle], \langle v \mid x \rangle](1) = 2 \langle uv \mid x \rangle, \quad r \Delta(\langle u \mid x \rangle) = 2 \lambda_u. \]

In view of the fact that \(* \Delta f = d \ast df, [[* \Delta, \langle u \mid x \rangle], \langle v \mid x \rangle](1) \) is equal to

\[
\begin{align*}
d \ast d(\langle u \mid x \rangle \langle v \mid x \rangle) & = \langle u \mid x \rangle d \ast d(\langle v \mid x \rangle - \langle v \mid x \rangle d \ast d(\langle u \mid x \rangle) \\
& = d(\langle u \mid x \rangle) \wedge d(\langle v \mid x \rangle) + d(\langle v \mid x \rangle) \wedge d(\langle u \mid x \rangle) \\
& = 2 \langle u \mid x \rangle h^0 \langle v \mid x \rangle \text{vol},
\end{align*}
\]

so

\[ [[r \Delta, \langle u \mid x \rangle], \langle v \mid x \rangle](1)|_{x_0} = 2 \langle u \mid e_i \rangle \langle e_i \mid L_{x_0} e_j \rangle \langle v \mid e_j \rangle = 2 \langle uv \mid x_0 \rangle. \]

The first identity of Eq. (4.17) is verified.
In view of the fact that \( \ast \Delta(f) = d \left( \sum_{i,j} h^{ij} \partial_i f \partial_j \right) \), we have

\[
\ast r \Delta((u \mid x))|_{x_0} = rd \left( \sum_{i,j} h^{ij} (u \mid x_i) \partial_j (\text{vol}) \right)|_{x_0} = - r \Delta 1_{\mathbb{R}^u} (\text{vol})|_{x_0} = - \mathcal{L} u (\text{vol})|_{x_0} - \frac{1}{r} \text{vol}|_{x_0}
\]

The second identity of Eq. (4.17) is verified. \( \square \)

As we have demonstrated in Ref. [8], to check a commutation relation on the Kepler cone, it is easier to check the corresponding one on \( V \). For this reason, we wish to lift \( r \Delta \) to a second order differential operator on \( V \) with rational functions as its coefficients. In order to do that, we first need to lift \( \lambda_u \) to a rational function on \( V \). Let \( c_k(x) \) (\( \tau_k(x) \) resp.) be the polynomial in \( \text{tr} x \), \( \text{tr} x^2 \), \ldots, \( \text{tr} x^k \) such that, if \( x = \sum_{i=1}^{\lambda_i e_i} \), then

\[
c_k(x) = \prod_{i=1}^{k} \lambda_i \quad \tau_k(x) = \prod_{1 \leq i < j \leq k} (\lambda_i + \lambda_j) \quad \text{resp.}
\]

For example, \( c_1(x) = \text{tr} x \), \( \tau_1(x) = 1 \), \( c_2(x) = \frac{1}{2}((\text{tr} x)^2 - \text{tr} x^2) \) and \( \tau_2(x) = \text{tr} x \). Let \( D_k = k[1 + (\rho - \frac{k+1}{2}) \delta] \) — the dimension of the canonical cone of rank \( k \), and

\[
\varphi_k = \varphi_k^\delta \cdot e_k^{\delta - 1} \cdot r^{2 - D_k}.
\]

For example, \( D_1 = 1 + (\rho - 1) \delta \), and \( \varphi_1 = r^{-\delta(\rho - 2)} \) up to a multiplicative constant. Note that \( \varphi_k \) is a rational function on \( V \) and is positive on the canonical cone of rank \( k \). From here on, we shall call \( \varphi_k \) the phi-function on the canonical cone of rank \( k \).

With an orthonormal basis \( \{ e_\alpha \} \) for \( V \) chosen, we write \( x \) as \( x^\alpha e_\alpha \), \( \partial_\alpha = \partial_{\alpha} \). Recall that \( \partial = \sum_{\alpha} e_\alpha \partial_\alpha \).

**Proposition 4.3.** Fix a canonical cone of rank \( k \). Let \( \varphi \) be its phi-function, \( \Delta \) be its Laplace operator. Then

i) \( \lambda_u \) can be lifted to a rational function on \( V \):

\[
\begin{align*}
4\lambda_u &= \mathcal{L} u \ln \varphi + \delta k \text{tr} u \\
\end{align*}
\]

ii) \( r \Delta \) can be lifted to a second order differential operator on \( V \) with rational function coefficients:

\[
\begin{align*}
\ast r \Delta &= (x \mid \partial^2) + 2 \sum_{\alpha} \lambda_{e_\alpha} \partial_\alpha \\
\end{align*}
\]

**Proof.** i) We just need to prove the identity at a point \( x_0 \) on the canonical cone. Choose a Jordan frame \( \{ e_{ii} \} \) such that \( x_0 = \sum_{i=1}^{k} \lambda_i e_{ii} \) for some numbers \( \lambda_1, \ldots, \lambda_k \). Then we extend \( \{ \sqrt{\rho e_{ii}} \}_{i=1}^{p} \) to an orthonormal basis \( \{ e_\alpha \} \) such that \( e_i = \sqrt{\rho e_{ii}}, \ 1 \leq i \leq k \), and \( \{ e_i \}_{i=1}^{D_k} \) is an basis of \( \text{Im}L_{x_0} \).
In view of that fact that \( x^k x^l = x^{k+l} \) and \( \text{tr} (x_0^k e_j) = 0 \) for \( j > k \), by induction on \( m \), we have
\[
\hat{L}_u (\text{tr} x^m) |_{x_0} = - \sum_{i=1}^{k} \langle u x_0 | e_i \rangle \partial_i (\text{tr} x^m) |_{x_0},
\]
consequently
\[
(4.21) \quad \hat{L}_u \varphi |_{x_0} = - \sum_{i=1}^{k} \langle u x_0 | e_i \rangle \partial_i \varphi |_{x_0}.
\]
The rest of the proof is just a straightforward computation based on identity (4.21), so we leave it to readers.

ii) Since both sides are differential operators without the zero-th order term, in view of identity (4.16), we just need to show that
\[
[(x | \partial^2) + 2 \sum_{\alpha} \lambda_{\alpha} \partial_{\alpha}, (u | x)] = -2 \tilde{L}_u,
\]
something that can be easily verified. □

For \( \nu \in \mathcal{W}(V) \setminus \{0\} \), we introduce integer
\[
(4.22) \quad \rho(\nu) = \begin{cases} 
 k & \text{if } \nu = k \frac{\delta}{2}, \\
 \rho & \text{if } \nu > (\rho - 1) \frac{\delta}{2}.
\end{cases}
\]
and rational function
\[
(4.23) \quad \varphi(\nu) := \begin{cases} 
 \varphi_k & \text{if } \nu = k \frac{\delta}{2}, \\
 \varphi_{\rho} \det(x)^{2\nu-\rho \delta} & \text{if } \nu > (\rho - 1) \frac{\delta}{2}
\end{cases}
\]
on \( V \). Note that \( \varphi(\nu) \) is always positive on the canonical cone of rank \( \rho(\nu) \).

For canonical cone \( C \), we let
\[
\hat{D}(C) := \{ \psi : C \to \mathbb{C} \mid \psi \in \hat{D}(V) \}, \quad \hat{D}_f(C) := \{ \psi : C \to \mathbb{C} \mid \psi \in \hat{D}_f(V) \}.
\]

**Proposition 4.4.** Let \( \nu \in \mathcal{W}(V) \setminus \{0\} \) and \( C \) be the canonical cone of rank \( \rho(\nu) \). Then \( \hat{D}(C) \) is dense in \( L^2(C, \frac{\sqrt{\varphi(\nu)}}{r} \text{vol}) \).

**Proof.** Let us write \( d\mu_\nu \) for \( \frac{\sqrt{\varphi(\nu)}}{r} \text{vol} \). Let \( C_\nu(C) \) be the set of compactly-supported continuous complex-valued functions and
\[
M = \int_C e^{-2r} d\mu_\nu.
\]
It is clear that \( M > 0 \). By applying Theorem [A.1] in appendix A, one can easily check that \( M < \infty \).

Suppose that \( f \in L^2(C, d\mu) \) and \( \epsilon > 0 \). By Theorem 3.14 in Ref. [16], there is \( g \in C_\nu(C) \) such that
\[
(4.24) \quad ||f - g||_{L^2} < \frac{\epsilon}{2}.
\]
Since \( e^r g \in C_\nu(C) \), by the Stone-Weierstrass Theorem in Ref. [17], there is a polynomial \( p \) such that
\[
(4.25) \quad |e^r g - p| < \frac{\epsilon}{2\sqrt{M}} \text{ on } C,
\]
Proposition 4.5. 

\[ \|g - e^{-r}p\|_{L^2} = \left( \int_C |g - e^{-r}p|^2 \, d\mu_\nu \right)^{\frac{1}{2}} \]

\[
< \frac{\varepsilon}{2} \left( \frac{1}{M} \int_C e^{-2r} \, d\mu_\nu \right)^{\frac{1}{2}} \quad \text{using Eq. (4.25)}
\]

(4.26)

Combining Eqs (4.24) and (4.26), we have

\[
\|f - e^{-r}p\|_{L^2} \leq \|f - g\|_{L^2} + \|g - e^{-r}p\|_{L^2} < \varepsilon.
\]

\[ \square \]

Let

\[
U(\nu) := \begin{cases} 
\frac{\varepsilon}{4} (\Delta \ln \varphi(\nu)) + \frac{1}{4} |d \ln \varphi(\nu)|^2 & \text{if } \nu \leq (\rho - 1)\frac{\delta}{2} \\
\frac{\varepsilon}{4} (\Delta \ln \varphi_p + \frac{1}{4} |d \ln \varphi_p|^2) + \frac{1}{4} \left( (\nu - \frac{\rho}{\rho})^2 - (\frac{\delta}{2} - 1)^2 \right) \text{tr} x^{-1} & \text{if } \nu > (\rho - 1)\frac{\delta}{2}.
\end{cases}
\]

(4.27)

Here, \( d \) and \( |\cdot| \) denote the exterior derivative operator and the point-wise norm for differential one-form on \( C \) respectively, \( x^{-1} \) denotes the Jordan inverse of \( x \in \Omega \). Note that \( U(\nu) \) can be lifted to a rational function on \( V \). Recall that \( \tilde{L}_u = \tilde{L}_u - \lambda_u \).

**Proposition 4.5.** Let \( \nu \in \mathcal{W}(V) \setminus \{0\} \) and \( C \) be the canonical cone of rank \( \rho(\nu) \).

i) As a differential operator on \( C \),

\[
\tilde{L}_u = \sqrt[\varphi(\nu)]{\tilde{L}_u(\nu)} \frac{1}{\sqrt[\varphi(\nu)]{\varphi(\nu)}}.
\]

(4.28)

ii) As a differential operator on \( C \),

\[
r \Delta = \sqrt[\varphi(\nu)]{(-i \tilde{X}_u(\nu))} \frac{1}{\sqrt[\varphi(\nu)]{\varphi(\nu)}} + U(\nu).
\]

(4.29)

This proposition says that \( \tilde{L}_u \) and \( r \Delta \) are not as hard as they might look. To prove this proposition, with the help of Proposition 4.3, one just needs to do some straightforward and relative short computations, so we skip the proof.

4.3. **The unitary realizations of TKK algebras on canonical cones.** Let \( \nu \in \mathcal{W}(V) \setminus \{0\} \) and \( C \) a canonical cone of rank \( \rho(\nu) \). Recall that \( \varphi(\nu) \), a rational function introduced in Eq. (4.23), is always positive on \( C \). Upon recalling the definitions of \( \tilde{D}(C) \) and \( \tilde{D}_I(C) \) in the paragraph preceding to Proposition 4.4 in view of Proposition 4.5 we introduce

\[
\tilde{D}(C) = \sqrt[\varphi(\nu)]{\tilde{D}(C)}, \quad \tilde{D}_I(C) = \sqrt[\varphi(\nu)]{\tilde{D}_I(C)}
\]

and differential operators with common domain \( \tilde{D}(C) \):

\[
\tilde{S}_{uv}(\nu) = \sqrt[\varphi(\nu)]{\tilde{S}_{uv}(\nu)} \frac{1}{\sqrt[\varphi(\nu)]{\varphi(\nu)}},
\]

\[
\tilde{X}_u(\nu) = \sqrt[\varphi(\nu)]{\tilde{X}_u(\nu)} \frac{1}{\sqrt[\varphi(\nu)]{\varphi(\nu)}},
\]

\[
\tilde{Y}_v(\nu) = \sqrt[\varphi(\nu)]{\tilde{Y}_v(\nu)} \frac{1}{\sqrt[\varphi(\nu)]{\varphi(\nu)}}.
\]

Note that these differential operators on \( C \) can be lifted to differential operators on \( V \).
Proposition 4.6. Let \( \nu \in \mathcal{W} \setminus \{0\} \) and \( \mathcal{C} \) be a canonical cone of rank \( \rho(\nu) \).

i) The TKK commutation relations \((4.4)\) hold under the replacement \( \Omega \) by \( \bar{\Omega}(\nu) \).

ii) \( \bar{D}(\mathcal{C}) \) is a dense subset of \( L^2(\mathcal{C}, \frac{1}{r}\text{vol}) \).

iii) \( \bar{S}_{\nu}(v), \bar{X}_v(\nu) \) and \( \bar{Y}_v(\nu) \) are anti-hermitian operators on \( \bar{D}(\mathcal{C}) \) with respect to hermitian inner product

\[
(\psi_1, \psi_2) = \int_{\mathcal{C}} \bar{\psi}_1 \bar{\psi}_2 \frac{1}{r}\text{vol}.
\]

iv) Let \( \bar{\mathcal{D}}_1(\mathcal{C}) \) be the orthogonal complement of \( \bar{D}_{1-1}(\mathcal{C}) \) in \( \bar{D}_1(\mathcal{C}) \), then, under the unitary \( u \)-action, we have the following orthogonal decomposition

\[
\bar{D}(\mathcal{C}) = \bigoplus_{j=0}^{\infty} \bar{\mathcal{D}}_j(\mathcal{C}).
\]

Moreover, the finite dimensional vector space \( \bar{\mathcal{D}}_1(\mathcal{C}) \) is the eigenspace of \( \bar{H}_e := i(\bar{X}_e + \bar{Y}_e) \) with eigenvalue \((2I + \nu \bar{\rho})\).

v) Assume that \( \bar{m} \in \mathbb{Z}^\rho \) with \( \bar{m} \geq 0 \) and \( m_{\rho(\nu)+1} = 0 \). For and only for such \( \bar{m} \), we let \( \bar{\mathcal{D}}_{\bar{m}}(\mathcal{C}) \) be the orthogonal projection of \( \sqrt{\varphi(\nu)}e^{-r\bar{P}_m(V)} \) onto \( \bar{D}_{\bar{m}}(\mathcal{C}) \). Then, as unitary representations of \( u \), we have isomorphism \( \bar{\mathcal{D}}_{\bar{m}}(\mathcal{C}) \cong \xi_{\nu} \otimes \mathcal{P}_m(V) \) and orthogonal decomposition into irreducibles:

\[
\bar{\mathcal{D}}_1(\mathcal{C}) = \bigoplus_{\bar{m} \geq 0, |\bar{m}|=j} \bar{\mathcal{D}}_{\bar{m}}(\mathcal{C}).
\]

Proof. i) This quickly follows from Theorem \[2\]

ii) This quickly follows from Proposition \[4.4\]

iii) We start the proof with the following two observations: 1) multiplication by a real-valued function is hermitian, herece \( \bar{Y}_v \) is anti-hermitian; 2) \( \nu \Delta \) is hermitian, hence \( \bar{X}_v \) is anti-hermitian in view of part ii) of Proposition \[4.5\]. Combining these observations with the commutation relations in part i), the proof follows quickly.

iv) The orthogonal decomposition follows from the following two facts: 1) the \( u \)-action is unitary, a fact from part iii) above, 2) the \( u \)-action commutes with the inclusion of \( \bar{D}_{1-1}(\mathcal{C}) \) into \( \bar{D}_1(\mathcal{C}) \), a fact implied by part i) of Theorem \[3\]. The remaining part follows from the fact that \( H_e \) is hermitian and part iii) of Theorem \[3\].

v) This follows from part ii) of Theorem \[3\]. \( \square \)

Remark 4.1. In view of Proposition \[4.6\] the semi-positive hermitian form \( (\cdot, \cdot)_\nu \) mentioned in the beginning paragraph of this section is

\[
(\psi_1, \psi_2)_\nu = \int_{\mathcal{C}} \bar{\psi}_1 \bar{\psi}_2 \sqrt{\varphi(\nu)} \frac{1}{r}\text{vol}
\]

and the space of “spurious states” consists of functions in \( \bar{D}(\mathcal{V}) \) which vanish on \( \mathcal{C} \). One can check that Eq. \[4.32\] is a generalization of Eq. \[4.11\] and \( \bar{D}(\mathcal{C}) \) is the quotient of \( \bar{D}(\mathcal{V}) \) by the the space of “spurious states”. Therefore, the measure \( d\mu_\nu \) mentioned in Subsection \[7.5\] is equal to \( \sqrt{\varphi(\nu)}\text{vol} \). When \( \nu > (\rho - 1)\frac{1}{2} \), up to a multiplicative constant, this explicit formula for \( d\mu_\nu \) agrees with the one retrieved from the bottom line of page 271 in Ref. \[4\]. Of course, our explicit formula works even if \( \nu \) takes a discrete value \( k \frac{1}{\rho}, 1 \leq k < \rho \).
Denote by $\tilde{\mathcal{U}}(V)$ (or simply $\tilde{\mathcal{U}}$) the simply connected Lie group whose Lie algebra is $\mathfrak{u}(V)$. Recall that $\mathcal{W}(V)$ is the Wallach set of $V$, $\lambda_0$ is the fundamental weight conjugate to the unique non-compact simple root $\alpha_0$ in the simple root system for $\mathfrak{co}$ in Lemma 2.1.

**Theorem 4.** Let $V$ be a simple euclidean Jordan algebra, $\nu \in \mathcal{W}(V) \setminus \{0\}$, and $\mathcal{C}$ be $V$’s canonical cone of rank $\rho(\nu)$. Under the action of $\mathfrak{co}(V)$ which maps $O$ to $\tilde{O}(\nu)$. $\tilde{\mathcal{D}}(\mathcal{C})$ becomes a unitary lowest weight $(\mathfrak{co}(V), \tilde{\mathcal{U}}(V))$-module with lowest weight $\nu \lambda_0$ and has the following multiplicity free $K$-type formula:

$$
(4.33) \quad \tilde{\mathcal{D}}(\mathcal{C}) = \bigoplus_{m \geq 0}^{m(\nu)+1=0} \tilde{\mathcal{G}}_m(\mathcal{C}).
$$

Therefore, as a representation of $\tilde{\mathcal{U}}(V)$, $\tilde{\mathcal{D}}(\mathcal{C}) \cong \bigoplus_{m \geq 0}^{m(\nu)+1=0} \xi_\nu \otimes \mathcal{P}_m(V)$.

Consequently, upon integration, $L^2(\mathcal{C}, \frac{1}{\pi \text{vol}})$ becomes a unitary lowest weight representation $\pi_\nu$ for $\text{Co}(V)$ with the same lowest weight.

**Proof.** Parts i), iii) and iv) of Proposition 4.6 imply that $\tilde{\mathcal{D}}(\mathcal{C})$ is a unitary $(\mathfrak{co}(V), \tilde{\mathcal{U}}(V))$-module. The $K$-type formula follows from parts iv) and v) of Proposition 4.6. Combining with parts v) and vi) of Theorem 3 we arrive at first part of this theorem. The second part follows from the first part, part ii) of Proposition 4.6 and a fundamental theorem of Harish-Chandra.

In view of the classification theorem in Ref. [10], the nontrivial scalar-type unitary lowest weight representations of $\text{Co}(V)$ are exhausted by representations $\pi_\nu$ in the above theorem.

5. **Generalized Quantum Kepler Problems Without Magnetic Charges**

In Ref. [9], we introduce the universal hamiltonian for the Kepler problem in terms of the generators of TKK algebra, and remark that whenever we have a quantization for the TKK algebra, we have a super-integrable model of the Kepler-type. In view of the quantizations for the TKK algebra presented in the last section, we have some new super-integrable models of Kepler-type.

As before, $V$ is a simple euclidean Jordan algebra of rank $\rho$ and degree $\delta$. $\mathcal{W}(V)$ is its Wallach set. For a canonical cone inside $V$ of rank $k$, we use $\varphi_k$ to denote the phi-function defined in Eq. (4.18) and $\Delta$ to denote its (non-positive) Laplace operator. For $\nu \in \mathcal{W}(V) \setminus \{0\}$, we let

$$
(5.1) \quad V(\nu) := \begin{cases} 
\frac{1}{8} \left( \Delta(\ln \varphi_k) + \frac{1}{4} |d \ln \varphi_k|^2 \right) & \text{if } \nu = k\delta \frac{\rho}{2}, \\
\frac{1}{8} \left( \Delta(\ln \varphi_\rho) + \frac{1}{4} |d \ln \varphi_\rho|^2 \right) + \frac{\delta}{8} \left( \nu - \frac{\rho}{2} \right)^2 - \left( \frac{\delta}{8} - 1 \right)^2 \frac{\mu x - 1}{\rho} & \text{if } \nu > (\rho - 1)\delta \frac{\rho}{2}.
\end{cases}
$$

and call $V(\nu)$ the **quantum-correction potential** on the canonical cone of rank $\rho(\nu)$. Note that $V(\nu) = \frac{U(\nu)}{\mu(\nu)}$. Here is the definition of the generalized quantum Kepler problem attached to $\pi_\nu$:

---

6i.e., the formulae for decomposing $\tilde{\mathcal{D}}(\mathcal{C})$ (considered as a $\tilde{\mathcal{U}}(V)$-module) into its irreducible components. “Multiplicity free” means each irreducible component appears only once.
Definition 3 (Generalized Quantum Kepler Problems). Let $V$ be a simple euclidean Jordan algebra and $\nu \in \mathcal{W}(V) \setminus \{0\}$. The $\nu$-th generalized quantum Kepler problem of $V$ is the quantum mechanical system for which the configuration space is the canonical cone of rank $\rho(\nu)$, and the Hamiltonian $\tilde{H}(\nu)$ (or simply $\tilde{H}$) is

$$-\frac{1}{2} \Delta + V(\nu) - \frac{1}{r}.$$  \hspace{1cm} (5.2)

Here, $\Delta$ and $V(\nu)$ are the Laplace operator and the quantum-correction potential respectively.

One can verify that when $\nu = \delta^2_2$, generalized quantum Kepler problem is the $J$-Kepler problem in Ref. [8], and to get the original Kepler problem we need to take $V = \Gamma(3)$ and $\nu = 1$.

5.1. Solution of the bound state problem. Given a generalized quantum Kepler problem on a canonical cone $C$, the bound state problem is primarily the following (energy) spectrum problem:

$$\begin{aligned}
\tilde{H}\psi &= E\psi \\
\int_C |\psi|^2 \text{vol} &< \infty, \hspace{0.5cm} \psi \not\equiv 0.
\end{aligned}$$  \hspace{1cm} (5.3)

It turns out that $E$ has to take certain discrete values. For example, for the original Kepler problem, we have $E = -\frac{1}{2}n^2$, $n = 1, 2, \ldots$

We shall use $\mathcal{H}_I$ to denote the $I$-th energy eigenspace, $I = 0, 1, \ldots$ and $\mathcal{H}$ to denote the Hilbert space of bound states — the $L^2$-completion of $\bigoplus_{I=0}^{\infty} \mathcal{H}_I$.

Theorem 5. Let $V$ be a simple euclidean Jordan algebra and $\nu \in \mathcal{W}(V) \setminus \{0\}$. For the $\nu$-th generalized quantum Kepler problem of $V$, the following statements are true:

i) The bound state energy spectrum is

$$E_I = -\frac{1}{2} \left( I + \frac{\nu^2}{2} \right)^\frac{1}{2},$$

where $I = 0, 1, 2, \ldots$

ii) As a representation of $\tilde{U}(V)$, $\mathcal{H}_I \cong \bigoplus_{m \geq 0, |m| = I}^{m_{\nu} + 1 = 0} \xi_{\nu} \otimes \mathcal{P}_m(V)$.

iii) $\mathcal{H}$ provides a realization for representation $\pi_{\nu}$.

Proof. In view of part iv) of Proposition 4.6, we start with the eigenvalue problem for $-\frac{1}{2}\tilde{H}_c$:

$$-\frac{1}{2} \tilde{H}_c \tilde{\psi} = -n_I \tilde{\psi},$$

where $n_I = I + \nu^2_2$ and $\tilde{\psi} \not\equiv 0$ is square integrable with respect to measure $\frac{1}{r} \text{vol}$ on the canonical cone of rank $\rho(\nu)$. The above equation can be recast as

$$-\frac{1}{2} \left( \Delta - \frac{U(\nu)}{r} + \frac{2n_I}{r} \right) \psi(x) = -\frac{1}{2} \psi(x).$$

Let $\psi(x) := \tilde{\psi}(\frac{x}{n_I})$, then the preceding equation becomes

$$\left( -\frac{1}{2} \Delta + V(\nu) - \frac{1}{r} \right) \psi(x) = -\frac{1}{n_I^2} \psi(x),$$
\[ \tilde{H} \psi = -\frac{1}{n_i^2} \psi. \]

One can check that \( \psi \) is square integrable with respect to measure \( \text{vol} \). Therefore, \( \tilde{\psi} \) is an eigenfunction of \( \tilde{H} \) with eigenvalue \( 2n_i \Rightarrow \psi \) is an eigenfunction of \( \tilde{H} \) with eigenvalue \( -\frac{1}{2n_i^2} \). By turning the above arguments backward, with the help of an explicit form of the eigenfunctions for \( \tilde{H} \), one can show that the converse of this statement is also true. Therefore, in view of parts iv) and v) of Proposition 4.6, we have

\[ \mathcal{H}_I \cong \mathcal{D}(C) \cong \bigoplus_{m \geq 0, |m| = I} \xi \otimes P_m(V). \]

Introduce

\[ \tau : \bigoplus_{I=0}^{\infty} \mathcal{H}_I \leftarrow \mathcal{D}(C) = \bigoplus_{m \geq 0} \mathcal{D}_m(C) \]

Here \( c_m \) is a constant depending on \( m \). The value of \( c_m \) can be determined and \( \tau \) can be shown to be an isometry, provided that an analogue of Theorem 2 in Ref. [14] for generalized Laguerre polynomials can be established, something that definitely can be done. Since \( \mathcal{D}(C) \) is a unitary highest weight Harish-Chandra module, and \( \tau \) is an isometry, \( \bigoplus_{I=0}^{\infty} \mathcal{H}_I \) becomes a unitary highest weight Harish-Chandra module. Since the \( L^2 \)-completion of \( \bigoplus_{I=0}^{\infty} \mathcal{H}_I \) is the Hilbert space of bound states, we arrive at part iii) of this theorem. \( \square \)

Generalized Kepler problems are natural generalizations of the J-Kepler problems, but with an important difference: the energy eigenspaces are no longer always irreducible representations of \( \tilde{U}(V) \), cf. part ii) of the theorem above.

**5.2. An alternative route to the generalized quantum Kepler problems.** A main objective of this paper is to construct the generalized quantum Kepler problems. The route we have taken is technically easy, but perhaps less intuitive. The purpose here is to point out that there is a more intuitive, though technically harder, route towards the generalized quantum Kepler problems.

Let us start with the classical universal hamiltonian

\[ \mathcal{H} = \frac{1}{2} \left( \pi | L_x | \pi \right) - \frac{1}{r} \]

on \( V \). Although \( (V, ds^2_E) \) is a Riemannian manifold, the first term in \( \mathcal{H} \) is not the kinetic energy for a single particle moving on \( V \setminus \{0\} \). A little experiment will convince the readers that the only way out is to replace \( V \setminus \{0\} \) by a canonical cone. In that way, we get \( \rho \) generalized classical Kepler problems, exactly one on each canonical cone:

**Definition 4 (Generalized Classical Kepler Problems).** Let \( V \) be a simple euclidean Jordan algebra and \( k \) be an integer between 1 and \( \rho \). The \( k \)-th generalized classical Kepler problem of \( V \) is the classical mechanical system for which the configuration space is the canonical cone of rank \( k \), and the Lagrangian is

\[ L(x, \dot{x}) = \frac{1}{2} \| \dot{x} \|^2 + \frac{1}{r}. \]
By quantizing the generalized classical Kepler problems, we expect to get the generalized quantum Kepler problems. However, due to the operator ordering problem, in general we don’t know how to do it! The way to overcome this technical hurdle is to demand that the hidden symmetry in the generalized classical Kepler problems be still present after quantization. But that really leads us back to the issue of quantizing the TKK algebra.

**APPENDIX A. POLAR COORDINATES**

The purpose of this section is to understand the polar coordinates on $C_k$. The theorem obtained here is an extension of Theorem VI.2.3 in Ref. [4] from symmetric cones to canonical cones and the presentation follows that of Section 2 of Chapter VI in Ref. [4].

We fix a Jordan frame: $e_{11}, \ldots, e_{\rho \rho}$ and a Jordan basis $\{e_{ii}, e_{ij}^\mu\}$. We denote by $V_{ij}$ the corresponding $(i, j)$-Peirce component of $V$. Let

$$R_k = \left\{ \sum_{i=1}^{k} a_i e_{ii} \mid a_i \in \mathbb{R} \right\}, \quad R_k^+ = \left\{ \sum_{i=1}^{k} a_i e_{ii} \mid a_1 > a_2 > \cdots > a_k > 0 \right\}$$

Let $K$ be the identity component of $\text{Aut}(V)$ and $M_k$ be the subgroup $K$ fixing each point $a \in R_k$:

$$M_k = \{ g \in K \mid \forall a \in R_k, ga = a \}$$

and $m_k$ be its Lie algebra:

$$m_k = \{ X \in \text{der} \mid \forall a \in R_k, Xa = 0 \}$$

For $i < j$, we define

$$l_{ij} = \{ [L_{e_{ii}}, L_\xi] \mid \xi \in V_{ij} \}$$

Let

$$l_k = \sum_{1 \leq i \leq k, i<j} l_{ij}$$

**Proposition A.1.** Let $a = \sum_{i=1}^{k} a_i e_{ii} \in R_k^+$. For $X \in \text{der}$, $(a, Xa) \in T_a C_k$ is orthogonal to $T_a R_k^+$, and if $a_i \neq a_j$ for $i \neq j$, the map

\[
\begin{align*}
 l_k & \rightarrow (T_a R_k^+)^\perp \\
 X & \mapsto (a, Xa)
\end{align*}
\]

(A.1)

is an isomorphism.

**Proof.** For $X = [L_u, L_v]$, $u$ and $v$ in $V$, and for $a \in R_k^+$ and $(x, b)$ in $T_a R_k^+$:

\[
\begin{align*}
((a, Xa), (a, b)) &= \langle a \mid e \rangle \langle Xa \mid L_a^{-1}b \rangle \\
&= \langle a \mid e \rangle \langle [L_a, L_{L_a^{-1}b}]u \mid v \rangle \\
&= 0
\end{align*}
\]

(A.2)

because both $L_a$ and $L_{L_a^{-1}b}$, being of diagonal form with respect to the Jordan basis, commute with each other.

Assume that $1 \leq i \leq k$ and $i < j \leq \rho$. For $X$ in $l_{ij}$:

$$X = [L_{e_{ii}}, L_\xi], \quad \xi \in V_{ij},$$

and for $a = \sum_{i=1}^{k} a_i e_{ii}$ in $R_k^+$ we have

$$Xa = \frac{1}{4} (a_j - a_i) \xi.$$
Here it is understood that $a_j = 0$ if $j > k$. Therefore, the range of the map $X \mapsto X\alpha$ contains the subspaces $V_{ij}$ and the sum

$$
\bigoplus_{1 \leq i \leq k, i < j} V_{ij}.
$$

Since $(T_a R_k^+)\perp = \{a\} \times \bigoplus_{1 \leq i \leq k, i < j} V_{ij}$, map (A.1) is onto, hence must be an isomorphism because the dimensions of the domain and the target are equal. \hfill \square

**Corollary A.1.** —i) As a vector space, we have

$$
\text{der} = m_k \oplus l_k.
$$

ii) Map

$$
\phi : K/M_k \times R_k^+ \to C_k
$$

has dense range and is a diffeomorphism onto its range.

**Theorem A.1.** Write $d\mu_{k,\frac{1}{2}}$ for $\nabla^\perp \text{vol}$. Under the identification map $\phi$, we have

$$
d\mu_{k,\frac{1}{2}} = C \text{vol}^1_{K/M_k} \prod_{1 \leq i < j \leq k} (a_i - a_j)^\delta \prod_{i=1}^k \left( a_i^{\frac{\delta}{2} (\rho - k + 1) - 1} da_i \right)
$$

where $\text{vol}^1_{K/M_k}$ is the $K$-invariant measure on $K/M_k$ and $C$ is a constant depending only on $C_k$.

**Proof.** We start with a local parametrization of $C_k$ around point $a \in R_k^+$:

$$
x = \exp \left( \sum_{1 \leq i \leq k, i < j \leq \rho} x_{ij}^\alpha X_{ij}^\alpha \right) a.
$$

Here, $X_{ij}^\alpha = [L_{e_i}, L_{e_j}^\alpha]$. Then

$$
dx|_a = \sum_{i=1}^k e_i da_i + \sum_{1 \leq i \leq k, i < j \leq \rho} \frac{1}{4} (a_j - a_i) e_{ij}^\alpha dx_{ij}^\alpha |_{x_{ij}^\alpha=0},
$$

where it is understood that $a_j = 0$ if $j > k$. Therefore, we can calculate the canonical metric $ds^2_K$ at point $a$:

$$
ds^2_K|_a = \langle a | e \rangle (dx | L^{-1}_a dx) |_a
$$

(A.4)

$$
= \langle a | e \rangle \left( \sum_{i} \frac{1}{a_i} da_i^2 + \frac{1}{8} \sum_{1 \leq i \leq k, i < j \leq \rho} \frac{(a_j - a_i)^2}{a_i + a_j} (dx_{ij}^\alpha)^2 |_{x_{ij}^\alpha=0} \right),
$$

so, up to a multiplicative numerical constant, we have

$$
\text{vol}|_a = (r^{D_k/2} C_k^{\frac{\delta}{2} (\rho - k + 1)} T_k^{-\frac{\rho}{2}})|_a \prod_{1 \leq i < j \leq k} (a_i - a_j)^\delta \prod_{i=1}^k da_i \prod_{1 \leq i \leq k, 1 \leq \alpha \leq \delta} \left( \bigwedge_{1 \leq i < j \leq \rho} dx_{ij}^\alpha |_{x_{ij}^\alpha=0} \right)
$$

and

$$
dd\mu_{k,\frac{1}{2}}|_a = C_k (a)^{\frac{\delta}{2} (\rho - k + 1) - 1} \prod_{1 \leq i < j \leq k} (a_i - a_j)^\delta \prod_{i=1}^k da_i \prod_{1 \leq i \leq k, 1 \leq \alpha \leq \delta} dx_{ij}^\alpha |_{x_{ij}^\alpha=0}.
$$
On the other hand, since $K$ is a simple Lie group, one can show that $X_{ij}^\alpha$’s are mutually orthogonal with respect to the negative-definite Cartan-Killing form $\mathcal{K}$, so the $K$-invariant volume form on $K/M_k$ at $eM_k$ is equal to $\left(\prod_{1\leq i<j\leq k} (a_i - a_j)^\delta \prod_{i=1}^k (a_i^{\delta(n-k+1)-1} d\alpha_i)\right)$ modulo a multiplicative numerical constant. Since $d\mu_k$ is also $K$-invariant, up to a multiplicative numerical constant, we have

$$d\mu_k = \text{vol}_{K/M_k} \prod_{1\leq i<j\leq k} (x^\alpha_{ij}) \right|_{x^\alpha_{ij} = 0}$$

as a measure.

As a side remark, we would like to mention the fact that integral

$$\int_\Omega e^{-2r} \det(x) x^\alpha \cdot \cdot \cdot d\mu_k$$

is finite if and only if $\nu > (\rho - 1)^\delta$.

APPENDIX B. LIST OF NOTATIONS

The purpose here is to list some basic notations and terminologies for this paper and its sequels.

- $V$ — a (finite dimensional) simple euclidean Jordan algebra;
- $e$, $\rho$, $\delta$, and $n$ — reserved for the identity element, rank, degree, and dimension of $V$;
- $\text{tr}$, $\det$ — the trace, determinant of $u \in V$;
- $\langle u \mid v \rangle$ — the inner product of $u, v \in V$, and is chosen to be $\frac{1}{2} \text{tr} (uv)$;
- $x$ — reserved for a generic point in $V$ when $V$ is considered as a smooth space;
- $r$ — reserved for function $\langle e \mid \rangle$ on smooth space $V$;
- $\{e\alpha\}$ — an orthonormal basis for $V$;
- $x^\alpha$ — the coordinates of $x \in V$ with respect to basis $\{e\alpha\}$;
- $\pi$ — reserved for a generic point in $V$ when $V$ is considered as the tangent space of $V$;
- $\pi^\alpha$ — the coordinates of $\pi \in V$ with respect to basis $\{e\alpha\}$;
- $\emptyset$ — a shorthand notation for $\sum_\alpha e\alpha \frac{\partial}{\partial x^\alpha}$;
- $\partial$ — a shorthand notation for $\sum_\alpha e\alpha \frac{\partial}{\partial x^\alpha}$;
- $d$ — the exterior derivative operator;
- vol — the volume form;
- $uv$ — the Jordan product of $u, v \in V$;
- $\{uvw\}$ — the Jordan triple product of $u, v, w \in V$;
- $L_u$ — the multiplication by $u \in V$;
- $S_{uw}$ — defined to be $[L_u, L_v] + L_{uw}$, so $S_{uw} = \{uvw\}$;
- $\mathcal{W}(V)$ — the Wallach set of $V$;
- $\mathcal{P}(V)$ — the set of complex-valued polynomial functions on $V$;
- $\mathcal{D}(V), \mathcal{D}$ — the derivation algebra of $V$;
- $\mathcal{S}(V), \mathcal{S}$ — the structure algebra of $V$, it is generated by $L_u, u \in V$;
- $\mathcal{C}(V), \mathcal{C}$ — the conformal algebra of $V$;
- $u(V), u$ — the maximal compact Lie subalgebra of $\mathcal{C}$;
- $\text{Aut}(V) —$ the automorphism group of $V$;

One just needs to show that the trace of $X_{ij}^\alpha X_{i'j'}^\alpha$ is zero if $(i, j, \alpha) \neq (i', j', \alpha')$. 

---

7One just needs to show that the trace of $X_{ij}^\alpha X_{i'j'}^\alpha$ is zero if $(i, j, \alpha) \neq (i', j', \alpha')$. 

• $\text{Str}(V)$, $\text{Str}$ — the structure group of $V$;
• $\text{Co}(V)$, $\text{Co}$ — the conformal group of $V$, and is defined to be the simply connected Lie group with $\mathfrak{co}$ as its Lie algebra;
• $\tilde{U}(V)$, $\tilde{U}$ — the simply connected Lie group with $\mathfrak{u}$ as its Lie algebra;
• $\tilde{H}(\nu)$, $\tilde{H}$ — the hamiltonian of the generalized Kepler problem corresponding to Wallach parameter $\nu$;
• $\mathcal{H}_I$ — the $I$th energy eigenspace for $\tilde{H}$;
• $\mathcal{H}$ — the Hilbert space of bound states for $\tilde{H}$.

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