Abstract In view of the recently observed discrepancy of theory and experiment for muonic hydrogen [R. Pohl et al., Nature vol. 466, p. 213 (2010)], we reexamine the theory on which the quantum electrodynamic (QED) predictions are based. In particular, we update the theory of the $2P–2S$ Lamb shift, by calculating the self-energy of the bound muon in the full Coulomb+vacuum polarization (Uehling) potential. We also investigate the relativistic two-body corrections to the vacuum polarization shift, and we analyze the influence of the shape of the nuclear charge distribution on the proton radius determination. The uncertainty associated with the third Zemach moment $\langle r^3 \rangle_2$ in the determination of the proton radius from the measurement is estimated. An updated theoretical prediction for the $2S–2P$ transition is given.

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1 Introduction

The high accuracy of quantum electrodynamic (QED) predictions and the precise spectroscopy of simple atomic systems allow for the determination of fundamental constants like the Rydberg constant $R_\infty$ from the hydrogen spectrum, $\alpha$ from the helium fine structure, and the electron mass $m_e$ from the $g$ factor of hydrogenlike ions. In all these cases, nuclear structure effects are small or can be eliminated. On the other hand, from a comparison of the theoretical isotope shift of optical transitions with experimental values for such atoms as He, Li, and Be$^+$, the nuclear charge radii of the heavy, unstable isotopes, in comparison to the stable ones, have been determined with high accuracy [1,2]. The most elaborate measurement [3] of the hydrogen-deuterium (H–D) isotope shift of 1S–2S transition gives a deuterium-proton mean square charge radius difference accurate to 2 parts in $10^5$. In some cases, the excitation of nuclei by the orbiting electron, the so called nuclear polarizability correction, is a significant effect and has to be included in the theoretical predictions. For example, the 1S–2S transition in D is affected by about 20 kHz due to the deuteron polarizability [4], while the current experimental precision for the H-D isotope shift is below 100 Hz (see Ref. [3]).

Bound muons penetrate the electron charge cloud and undergo transitions close to the atomic nucleus, partially screened by the remaining electrons. The determination of absolute charge radii from the muonic transitions is an established technique for the determination of nuclear radii [5,6]. Also, the CODATA values [7] for the proton and deuteron nuclear radii are mainly determined by an analysis of the 23 most accurately measured transitions in these systems [8]. Both atomic hydrogen as well as bound muonic systems are essentially two-body bound systems, with a comparatively light orbiting particle (electron or muon) and a heavier nucleus. Still, the combined evaluation of relativistic, QED and two-body effects remains difficult, and the presently achieved accuracy is the result of decade-long work of a number of physicists. Since the original calculations of Lamb shift by Bethe, later by Feynman and others, theory has been worked out to a very high precision.

For hydrogen, present limitations come from the inaccuracy in the two-loop electron self-energy and amount to about 1 kHz for the 1S Lamb shift. This constitutes a relatively small uncertainty in comparison to the 1 MHz shift due to the proton finite size $r_p$, and allows for a determination of $r_p$ with an accuracy of 8 parts per thousand, namely

$$r_p = 0.8768(69) \text{ fm} \quad \text{(Refs. [7,8])}.$$  \hspace{1cm} (1.1)

This value is in excellent agreement with the recently obtained proton radius from electron scattering [9], which yields a value of $r_p = 0.879(8) \text{ fm}$. Here, we have added the statistical and systematic uncertainties given in Ref. [9] quadratically. The muonic hydrogen value

$$r_p = 0.84184(67) \text{ fm} \quad \text{(Refs. [10])}$$ \hspace{1cm} (1.2)

is in significant (5.0$\sigma$) disagreement with the CODATA and (4.6$\sigma$) with the electron scattering value. This proton radius significantly disagrees with a number of investigations, including determinations of the proton radius based on previous scattering experiments (“world scattering data”) reported in Refs. [11–13].

For atomic hydrogen and more generally for electronic bound systems, the energy shift due to the finite nuclear size is proportional to the mean square charge radius $\langle r^2 \rangle$, and the contribution from higher moments such as the convoluted, third-order Zemach moment $\langle r^3 \rangle_2$ is negligibly small [14], only about 50 Hz, and similarly, proton polarizability effects amount to only about 100 Hz (see Refs. [15,17]). Both of the latter effects thus have only a negligible effect on the proton radius determination from electronic hydrogen spectroscopy (we sometimes refer to atomic hydrogen as “electronic hydrogen” in the current work in order to uniquely distinguish the system from muonic hydrogen).

Indeed, the situation is different in muonic hydrogen. The muon Bohr radius is about 200 times smaller, and the energy levels are significantly affected by the proton finite size. The contribution from higher order moments of the nuclear charge distribution and from the proton polarizability, although still small, have to be accurately estimated in order to obtain a reliable proton charge radius. If one refrains from determining a new proton radius from the recent measurement [10] of the $2S (F=1) \leftrightarrow 2P_{3/2}(F=2)$ transition in D, one would obtain a new proton radius of $r_p = 0.87649(69) \text{ fm}$. This value is in excellent agreement with the recent measurements [18,19].
transition and uses the CODATA proton radius in order to predict the transition energy, then one finds that the experimental result reported in Ref. [10] deviates from QED theory by about 0.31 meV.

A reinvestigation of the theory of the $2S-2P$ Lamb shift transition in muonic hydrogen is thus indicated, especially because the theory of muonic atoms is not free of surprises [18]. We proceed as follows. In Sec. 2 we first discuss the size of the observed discrepancy in relation to the relativistic and QED effects which describe the energy levels of muonic hydrogen. We then continue to verify the current status of theory on the basis of the two-body Breit Hamiltonian and evaluate relativistic effects and quantum electrodynamic corrections (vacuum polarization and self energy, and combined effects) which are relevant on the level of the current disagreement. We then continue to calculate a few hitherto neglected corrections to theory (Sec. 3). Special attention is devoted to the calculation of the Bethe logarithm in the full Coulomb+Uehling potential, which amounts to a nonperturbative treatment of the vacuum polarization correction to the self energy. A final update of theory and a reevaluation of the proton radius is presented in Sec. 4. Conclusions are reserved for Sec. 5.

### 2 Verification of Theory

#### 2.1 Size of the Discrepancy

In view the recently observed discrepancy of theory and experiment in muonic hydrogen [10], we attempt to verify the theory [19–27] used for the theoretical evaluation of the recent measurement [10]. We base our considerations on the extensive list of corrections presented in Tables 1 and 2 of the supplementary information included with the recent paper [10], whose entries in turn are based on previous theoretical work [19–27].

The authors of Ref. [10] report on a measurement of the energy interval

$$\Delta E = E \left( 2P_{3/2}(F=2) \right) - E \left( 2S_{1/2}(F=1) \right)$$

(2.1)

in muonic hydrogen. Summing up all known effects, the authors of Ref. [10] use the following theoretical prediction

$$E_{\text{th}} = \left( 209.9779(49) - 5.2262 \frac{r^2_p}{\text{fm}^2} + 0.0347 \frac{r^3_p}{\text{fm}^3} \right) \text{meV.}$$

(2.2)

The experimental result is

$$E_{\text{exp}} = 206.2949(32) \text{meV.}$$

(2.3)

One thus infers the root-mean-square proton radius given in Eq. (1.2).

On the other hand, one can also interpret the result of the measurement [10] as a disagreement of theory and experiment in an important quantum electrodynamic measurement. Namely, if one uses the CODATA recommended proton radius given in Eq. (1.1) and evaluates theory according to Eq. (2.2), then one obtains a theoretical prediction of $E_{\text{th}} = 205.984(63) \text{meV}$ which deviates from the experimental result given in Eq. (2.3) by 5.0 $\sigma$. As already mentioned, an additional, hypothetical effect that would shift theory by (roughly) +0.31 meV might thus explain the discrepancy of the proton radius given in Eq. (1.2) and the 2006 CODATA value of $r_p = 0.8768(69) \text{fm}$.

The $2S_{1/2}(F=1) \leftrightarrow 2P_{3/2}(F=2)$ transition frequency is the sum of three contributions:

- (i) the $2S-2P_{1/2}$ Lamb shift,
- (ii) the $2P_{1/2}-2P_{3/2}$ fine-structure splitting, and
- (iii) the $2S$ and $2P_{3/2}$ hyperfine structure.

On the level of the theoretical-experimental discrepancy (0.31 meV), these contributions can be broken down into a set of well-established, conceptually simple relativistic and QED corrections, as detailed below.
2.2 Reduced Mass Dependence

Our goal is to verify the theory of the $2S (F=1) \leftrightarrow 2P_{1/2} (F=2)$ transition energy in muonic hydrogen on the level of the reported discrepancy [10], i.e., 0.31 meV in energy units. The unperturbed nonrelativistic Schrödinger Hamiltonian of the bound muon-proton system is

$$H_0 = \frac{\vec{p}^2}{2m_R} - \frac{\alpha}{r}, \quad (2.4)$$

and the corresponding unperturbed binding energies of bound states in muonic hydrogen are given by

$$E_0 = -\frac{\alpha^2 m_R}{2n^2}, \quad (2.5)$$

where $m_R = m_\mu m_N / (m_\mu + m_N)$ is the reduced mass of the system (we use units with $\hbar = c = \epsilon_0 = 1$). Here, $n$ is the principal quantum number. The proportionality to the reduced mass (not to the muon mass) follows from an elementary separation of the two-body Hamiltonian describing the proton and muon into relative and center-of-mass coordinates [28]. Relativistic corrections enter at relative order $\alpha^2$, i.e., at order $\alpha^4 m_R$. Their reduced mass dependence is captured in the two-body Breit Hamiltonian [29].

For muonic hydrogen, the mass ratio of orbiting particle and nucleus is

$$\frac{m_\mu}{m_p} = \frac{1}{8.880243 \ldots} \approx \frac{1}{9}, \quad (2.6)$$

and therefore not necessarily small compared to unity. The one-particle Dirac equation, which is valid only in the limit $m_\mu/m_p \to 0$, therefore cannot be used as a good approximation for the calculations of the relativistic effects.

Vacuum polarization effects shift the muonic hydrogen spectrum at relative order $\alpha$, i.e., at order $\alpha^3 m_R$. Although the vacuum polarization corrections are thus larger than the relativistic corrections, we start our discussion with the latter effects as they constitute the most natural extension of nonrelativistic atomic theory without field quantization.

2.3 Relativistic Effects and Fine Structure

The two-body Breit-Pauli Hamiltonian for the muon-proton system, including the reduced mass corrections but without the anomalous magnetic moment of the muon, reads as follows,

$$H_{BP} = -\frac{p_i^4}{8m_\mu^3} - \frac{p_i^4}{8m_p^3} - \frac{\alpha}{2m_\mu m_p} p_i^j \left( \frac{\delta^{ij}}{r} + \frac{r^i r^j}{r^2} \right) p^j$$

$$+ \frac{2\alpha}{3} \left( \langle r_p^2 \rangle + \frac{3}{4m_p^2} + \frac{3}{4m_\mu^2} \right) \pi \delta^3(r) + \frac{\alpha g_p}{3m_\mu m_p} \vec{\sigma}_\mu \cdot \vec{\sigma}_p \pi \delta^3(r)$$

$$+ \frac{\alpha g_p}{8m_\mu m_p} \left( \frac{\sigma_\mu^i \sigma_p^j}{r^3} \left( 3 r^i r^j - \delta^{ij} \right) \right) + \frac{\alpha g_p}{4r^3} \left( \frac{\vec{\sigma}_\mu}{m_\mu^2} + \frac{2\vec{\sigma}_\mu + g_p \vec{\sigma}_p}{m_\mu m_p} - \frac{(g_p - 1) \vec{\sigma}_p}{m_p^2} \right) \cdot \vec{r} \times \vec{p}. \quad (2.7)$$

The proton $g$ factor is $g_p = 5.585694712(46)$ (see Ref. [7]). The anomalous magnetic moment of the proton comes from its internal structure. In the leading QED approximation, one assumes a point proton, minimally coupled to the electromagnetic field with its physical value of the magnetic moment which determines the hyperfine splitting. Strictly speaking, this phenomenologically inserted intrinsic proton magnetic moment leads to a QED theory which is not renormalizable. The situation is different for the anomalous magnetic moment of the muon, which is described by vertex corrections within QED theory. Lacking a convenient way to describe ab initio the inner structure of the proton with its anomalous magnetic moment, the phenomenological insertion of $g_p$ into the proton-spin dependent terms in the Breit–Pauli Hamiltonian is the most obvious treatment in theoretical calculations adopted so far in the literature.
As already mentioned above, the $2S_{1/2}(F = 1) \leftrightarrow 2P_{3/2}(F = 2)$ transition frequency is the sum of three contributions: (i) the $2S_{1/2} - 2P_{1/2}$ Lamb shift, (ii) the $2P_{1/2} - 2F_{3/2}$ fine-structure splitting, and (iii) the $2S_{1/2}$ and $2P_{3/2}$ hyperfine structure. The $2S_{1/2}$ and $2P_{3/2}$ hyperfine structure splittings, as well as the $2P_{1/2} - 2P_{3/2}$ fine-structure splitting can directly be determined from the Breit–Pauli Hamiltonian given in Eq. (2.7). On the level of the theoretical-experimental discrepancy (0.31 meV), we find that the neglected terms beyond the Breit–Pauli approximation are already too small in magnitude to explain the discrepancy on the basis of either the fine-structure or hyperfine-structure intervals.

We start with the $2S$ hyperfine structure, which is exclusively given by the terms proportional to $\vec{\sigma} \cdot \vec{\sigma}$ in the Breit–Pauli Hamiltonian. It is sufficient to evaluate matrix elements of this operator, with nonrelativistic spinor wave functions that are solutions of the Schrödinger Hamiltonian of the system (with reduced mass $m_R$). The result is

$$E_{\text{hfs}}^{\text{BP}}(2S) = \langle 2S(F=1)|H_{\text{BP}}|2S(F=1)\rangle - \langle 2S(F=0)|H_{\text{BP}}|2S(F=0)\rangle \approx \frac{g_p}{6} \alpha^4 \frac{m_R^3}{m_\mu m_p} = 22.805 \text{ meV}. \quad (2.8)$$

After a consideration of the proton structure (Zemach), and QED radiative corrections, one obtains (see Ref. [26]) a theoretical prediction of

$$E_{\text{hfs}}^{\text{QED}}(2S) = 22.8148(78) \text{ meV}. \quad (2.9)$$

For the $2S(F=1)$ sublevel, the difference

$$\frac{1}{4} \left( E_{\text{hfs}}^{\text{QED}}(2S) - E_{\text{hfs}}^{\text{BP}}(2S) \right) = 0.0023 \text{ meV} \quad (2.10)$$

already is two orders of magnitude smaller than the experimental-theoretical discrepancy of 0.31 meV.

The $2P_{1/2}$ hyperfine splitting (energy difference of the $F = 1$ and $F = 0$ states) is obtained from the analysis of the experiment of Ref. [10]. Nevertheless, it is interesting to note that from Eq. (2.7), we have

$$E_{\text{hfs}}^{\text{BP}}(2P_{1/2}) = \langle 2P_{1/2}(F=1)|H_{\text{BP}}|2P_{1/2}(F=1)\rangle - \langle 2P_{1/2}(F=0)|H_{\text{BP}}|2P_{1/2}(F=0)\rangle = \frac{g_p \alpha^4 m_R^3}{18 m_\mu m_p} \left( 1 + \frac{g_p - 1}{2 g_p} \frac{m_\mu}{m_p} \right) = 7.953 \text{ meV}. \quad (2.11)$$

Including the anomalous magnetic moment of the muon, one obtains a result of [19]

$$E_{\text{hfs}}^{\text{QED}}(2P_{1/2}) = 7.963 \text{ meV}, \quad (2.12)$$

which differs from $E_{\text{hfs}}^{\text{BP}}(2P_{1/2})$ only on the level of 0.010 meV.

The $2P_{3/2}$ hyperfine splitting (energy difference of the $F = 2$ and $F = 1$ states) is obtained from the Breit–Pauli Hamiltonian as

$$E_{\text{hfs}}^{\text{BP}}(2P_{3/2}) = \langle 2P_{3/2}(F=2)|H_{\text{BP}}|2P_{3/2}(F=2)\rangle - \langle 2P_{3/2}(F=1)|H_{\text{BP}}|2P_{3/2}(F=1)\rangle = \frac{g_p \alpha^4 m_R^3}{45 m_\mu m_p} \left( 1 + \frac{5}{4} \frac{g_p - 1}{g_p} \frac{m_\mu}{m_p} \right) = 3.392 \text{ meV}. \quad (2.13)$$

The full QED result is minimally displaced by less than 0.001 meV (see Ref. [27]),

$$E_{\text{hfs}}^{\text{QED}}(2P_{3/2}) = 3.3926 \text{ meV}, \quad (2.14)$$

The Breit–Pauli Hamiltonian also leads to an off-diagonal coupling of the $2P_{1/2}(F=1)$ and $2P_{3/2}(F=1)$ sublevels [19], which evaluates to

$$V^{\text{BP}}(2P) = \langle 2P_{1/2}(F=1)|H_{\text{BP}}|2P_{3/2}(F=1)\rangle = -\frac{g_p \alpha^4 m_R^3}{144 \sqrt{2} m_\mu m_p} \left( 1 + 2 \frac{g_p - 1}{g_p} \frac{m_\mu}{m_p} \right) = -0.796 \text{ meV}, \quad (2.15)$$
in excellent agreement with the value given in Eq. (85) of Ref. [19].

Finally, the energy difference of the hyperfine centroids (the fine-structure splitting) of the 2P states is readily evaluated as

\[
E_{\text{BP}}^{\text{BP}}(2P) = \langle 2P_{3/2} | H_{\text{BP}} | 2P_{3/2} \rangle - \langle 2P_{1/2} | H_{\text{BP}} | 2P_{1/2} \rangle = \frac{\alpha^4 m_e^3}{32 m_\mu m_p} \left( 1 + \frac{2 m_\mu}{m_p} \right) = 8.329 \text{meV}.
\]

This result differs by only 0.023 meV from the full QED result of [19] [22].

If we define the states

\[
|1\rangle = |2P_{1/2}(F=0)\rangle, \quad |2\rangle = |2P_{1/2}(F=1)\rangle, \quad |3\rangle = |2P_{3/2}(F=1)\rangle, \quad |4\rangle = |2P_{3/2}(F=2)\rangle,
\]

and the matrix elements

\[
\beta_{1/2} = E_{\text{hfs}}(2P_{1/2}), \quad v = V(2P), \quad \beta_{3/2} = E_{\text{hfs}}(2P_{3/2}), \quad f = E_{\text{hfs}}(2P), \quad (2.17)
\]

and the zero point of the energy scale to be the hyperfine centroid of the 2P_{1/2} levels, then the Breit–Pauli Hamiltonian in the 2P state manifold assumes the following matrix form \( M_{\text{BP}} \):

\[
M = \begin{pmatrix}
-\frac{3}{4} \beta_{1/2} & 0 & 0 & 0 \\
0 & \frac{1}{4} \beta_{1/2} + v & 0 & 0 \\
0 & v & -\frac{3}{8} \beta_{3/2} + f & 0 \\
0 & 0 & 0 & \frac{5}{8} \beta_{3/2} + f
\end{pmatrix}.
\]

The off-diagonal elements \( v \) lead to admixtures to the \( |2P_{1/2}(F=1)\rangle \) levels from the \( |2P_{3/2}(F=1)\rangle \) levels and vice versa, and to a repulsive interaction as for any coupled two-level system. In agreement with this general consideration, a diagonalization of \( M_{\text{BP}} \) immediately leads to the conclusion, that the \( |2P_{1/2}(F=1)\rangle \) is lowered in energy by

\[
\Delta = 0.145 \text{meV},
\]

whereas the \( |2P_{3/2}(F=1)\rangle \) energy is increased by \( \Delta \). This is in full agreement with Ref. [19].

In the current derivation, we have not distinguished the magnetic projections of the hyperfine sublevels. If these are included, a 12-dimensional matrix is obtained from the singlet \( |1\rangle \), the three magnetic sublevels of states \( |2\rangle \) and \( |3\rangle \), and the five magnetic sublevels of state \( |4\rangle \). However, we have checked by an explicit calculation of the Hamiltonian matrix, using angular momentum algebra [30, 31], that the energies obtained by diagonalizing the full Hamiltonian matrix are the same as those obtained from \( (2.18) \), and that, in particular, the state \( |2P_{3/2}(F=2)\rangle \), which is so important for the experiment [10], remains uncoupled from the other hyperfine and fine-structure levels.

### 2.4 QED and Lamb Shift

The theory of the 2P_{1/2}−2S_{1/2} Lamb shift in muonic hydrogen is surprisingly simple. One here speaks of the 2P−2S Lamb shift because the 2P level is energetically higher than 2S, in contrast to electronic hydrogen where the situation is opposite, and the 2S−2P Lamb shift is observed. The one-loop vacuum-polarization (Uehling) correction due to virtual electron-positron pairs gives the main contribution, and already the first-order effect (in perturbation theory) accounts for 99.5 % of the Lamb shift, or 205.0074 meV. The second-order perturbation theory effect contributes 0.1509 meV, and the two-loop vacuum polarization gives a shift of 1.5081 meV. We note that in the 1970s, an error in the evaluation of the two-loop vacuum polarization [32] has led to a discrepancy of theory and experiment for heavy
muonic ions [33]. The discrepancy was reduced after the error was discovered [34–36]. This observation implies that a careful verification of the vacuum polarization effects appears worthwhile in the current situation.

Finally, the one-loop self-energy of the bound muon decreases the 2P–2S Lamb shift and amounts to $-0.6677$ meV (the sign is natural as the self-energy effect is the dominant effect in electronic hydrogen, where the sign of the entire Lamb shift is reversed compared to muonic hydrogen, due to this term). The four mentioned contributions to the Lamb shift, which are recalculated and verified below, already account for 205.9987 meV. The total QED result without the nuclear finite-size effect, according to Ref. [10], is 206.0573 meV. The difference, 0.0586 meV, is already much smaller than the observed discrepancy of 0.31 meV, and there are no cancellations among conceivably large, further QED effects not accounted for in the above list of dominant vacuum-polarization and self-energy effects. This illustrates that the theory of the Lamb shift in muonic hydrogen rests, to a large extent, on a very well established subset of QED effects.

### 2.5 One–Loop Vacuum Polarization

The one-loop vacuum polarization effect is described by the Uehling Potential $V_{vp}(r)$ which was evaluated already in 1935 (Ref. [37]). Muonic hydrogen is different from electronic hydrogen because the classical orbit of the bound muon is much closer to the proton than for the electron. It is thus useful to analyze the asymptotic behavior of the Uehling potential for large and small distances from the nucleus. We define the scaled coordinate $\rho = \alpha m_r R r$, where $\alpha$ is the fine-structure constant, and $m_R$ is the reduced mass of the bound system. The Coulomb potential scales as

$$V(r) = -\frac{\alpha}{r} = -\frac{\alpha^2 m_r}{\rho}.$$ (2.20)

The one-loop Uehling potential is

$$V_{vp}(r) = -\frac{\alpha^3 m_r}{\pi \rho} \int_1^\infty du e^{-2u \rho x} \frac{\sqrt{u^2 - 1} (2u^2 + 1)}{3u^4},$$ (2.21)

where $x = \frac{m_e}{\alpha m_r} = 0.73738368\ldots$ for muonic hydrogen. In atomic hydrogen, $x \approx \alpha^{-1} \gg 1$, and the Uehling potential can be approximated by a Dirac $\delta$, but this is not the case for muonic atoms. For small distances ($\rho \to 0$), one finds, in agreement with Ref. [34],

$$V_{vp}(r) \sim \frac{\alpha^3 m_r}{\pi \rho} \left[ \frac{2}{3} \left( \ln(\rho x) + \gamma_E \right) - \frac{\pi}{2} \rho x + \frac{5}{9} \right] + O(\rho).$$ (2.22)

As compared to the Coulomb potential, only a logarithmic divergence is added. For large distances, $\rho \to \infty$, by contrast, the Uehling potential decreases exponentially,

$$V_{vp}(r) \sim -\frac{\alpha^3 m_r}{\sqrt{\pi}} e^{-2\rho x} \left[ \frac{1}{4\rho^{5/2} x^{3/2}} - \frac{29}{64 \rho^{7/2} x^{5/2}} + \frac{2225}{2048 \rho^{9/2} x^{7/2}} + O\left(\frac{1}{\rho^{11/2}}\right) \right],$$ (2.23)

which leads to rapidly convergent radial integrals. On the Bohr radius scale of muonic atoms as measured by the scaled coordinate $\rho$, the Uehling potential goes as $\ln(\rho)/\rho$ for $\rho \to 0$. This is more singular than the Coulomb potential, but only by a logarithm. The smooth behavior excludes conceivable nonperturbative effects which could be expected for a singular behavior near the origin.

The question then is whether one should start the evaluation of the one-loop vacuum polarization correction from the nonrelativistic Schrödinger wave function or from relativistic Dirac theory. The latter approach is indicated for highly charged ions, where relativistic effects dominate over the reduced-mass corrections. Muonic hydrogen, on the other hand, is a light system, and the parameter characterizing the reduced-mass corrections ($m_\mu/m_p$) is much larger than the parameter characterizing the expansion into Coulomb vertices ($Z\alpha = \alpha$, where $Z = 1$ is the charge number of the proton). We therefore start from
the Schrödinger Hamiltonian and add reduced-mass effects later on the basis of a radiatively corrected Breit–Pauli Hamiltonian.

Up to now (Refs. [19,22]), the vacuum-polarization corrections have been evaluated using an analytic approach, with analytic representations of the reduced Green functions (see Ref. [19] for relevant formulas). Here, in order to check for any conceivable calculational errors in the numerically dominant vacuum polarization corrections, we choose a different approach. Namely, because the singularity of the Uehling potential at the origin is only logarithmic, the combined Coulomb+Uehling potential is eligible for a numerical solution of the Schrödinger equation on a lattice [38]. In this approach, one may choose to evaluate perturbative terms using unperturbed Schrödinger eigenstates (Green functions are represented by the pseudospectrum of states obtained from the finite lattice), or one may choose to perform a nonperturbative solution of the Schrödinger equation, for the combined Coulomb+Uehling potential.

Performing a matrix element calculation on the lattice composed of an exponential grid (see Ref. [39]) and observing the apparent convergence on lattices with more than $N = 300$ grid points, and increasing $N$ in steps of $\Delta N = 50$, we confirm the results,

$$\delta E^{(1)} = \langle 2P \mid V_{vp} \mid 2P \rangle - \langle 2S \mid V_{vp} \mid 2S \rangle = 205.0074 \text{ meV}$$

in first order and

$$\delta E^{(2)} = \left( \langle 2P \mid V_{vp} \frac{1}{(E_0 - H_0)} V_{vp} \mid 2P \rangle - \langle 2S \mid V_{vp} \frac{1}{(E_0 - H_0)} V_{vp} \mid 2S \rangle \right) = 0.1509 \text{ meV}$$

in second order. Here $H_0$ and $E_0$ are the Schrödinger Hamiltonian of the system and the binding energy given in Eqs. [22] and [23], and $[1/(E_0 - H_0)]$ denotes the reduced Green function.

The Hamiltonian of muonic hydrogen, including vacuum polarization, is

$$H_{0vp} = \frac{\overrightarrow{p}^2}{2m_R} - \frac{\alpha}{r} + V_{vp}(r).$$

For the numerical calculation, we add the Uehling potential to the Schrödinger Hamiltonian and diagonalize the Hamiltonian $H_{0vp}$ on a pseudospectrum of states that is generated by a finite, exponential grid [38,39]. We use the exact reduced mass of the system, which cannot be done consistently when using the Dirac equation, and a variable coupling parameter $\chi \equiv \alpha/\pi$, for various values of $\chi$, in order to control the convergence of the calculation. We find a smooth dependence on $\chi$, which can be fitted to excellent accuracy by a power series in $\chi$. The first two terms are consistent with the results derived in Eqs. [22] and [23]. For $\chi = \alpha/\pi$, which is the physically relevant coupling parameter for muonic hydrogen, our result for the total $2P-2S$ electronic vacuum polarization contribution to the Lamb shift of 205.1584(1) meV is in excellent agreement with the sum of the one-loop term (205.0074 meV), the iterated one-loop term (second order, 0.1509 meV) and the third-order perturbation theory effect of 0.00007 meV as discussed in Refs. [40,41].

### 2.6 Two–Loop Vacuum Polarization

The two-loop vacuum polarization correction corresponds to the diagrams in Fig. [1]. The expression for the diagrams was first derived by Kallen and Sabry in 1955 (Ref. [42]), and the corrections therefore carry their name. The two-loop calculation leading to their derivation is non-trivial, and it is thus imperative to clarify the status of the two-loop potential in the literature. A clear exposition of the derivation is not only given in volume III of the monograph [43], but the two-loop effect was also recalculate and later used as input for the three-loop corrections to the electron $g$ factor; explicit results are indicated in Ref. [44]. Finally, the result has later been generalized to non-Abelian gauge theories, see Refs. [45,46].

An evaluation of the two-loop potential with nonrelativistic Schrödinger wave functions, using the two-loop potential $V_{vp}^{(2)}$ given in Refs. [44,48], then leads to the result

$$\delta E_{KS} = \langle 2P \mid V_{vp}^{(2)} \mid 2P \rangle - \langle 2S \mid V_{vp}^{(2)} \mid 2S \rangle = 1.5081 \text{ meV}.$$  

(2.27)
Section 2: Verification of Theory

Figure 1: Two-loop vacuum-polarization diagrams.

For reference, it is quite instructive to divide the calculation into an evaluation of the loop-after-loop correction in the leftmost diagram of Fig. 1 (which gives a contribution of 0.25 meV to the $2S-2P$ Lamb shift), and the “true” two-loop effects diagrammatically represented in three rightmost diagrams in Fig. 1 which contribute 1.25 meV. The total value of 1.50 meV for the two-loop effect [19, 22] is thus confirmed.

2.7 Muon Self–Energy

The self-energy shifts of the $2S$ and $2P_{1/2}$ levels are given by

$$E_{SE}(2S_{1/2}) = \frac{\alpha^5 m_R^3}{8 \pi m^2_{\mu}} \left\{ \frac{4}{3} \ln \left( \frac{m_{\mu}}{\alpha^2 m_R} \right) + \frac{10}{9} - \frac{4}{3} \ln k_0(2S) + 4\pi\alpha \left( \frac{139}{128} - \frac{1}{2} \ln(2) \right) \right\} .$$ (2.28)

This expression has been derived originally in the early days of quantum electrodynamics [49], and the effect of relative order $\alpha$ has been derived in Refs. [50, 51]. The coefficients agree with very precise numerical investigations of the one-loop self-energy [52], in the domain of low nuclear charge numbers, and an extensive list of Bethe logarithms $\ln k_0(nP)$ has been given in Ref. [53].

There is a further effect due to virtual muon-antimuon pairs that modify the vacuum polarization potential,

$$E_{VP}(2S_{1/2}) = \frac{\alpha^5 m_R^3}{8 \pi m^2_{\mu}} \left\{ \frac{4}{15} + \pi \frac{5}{48} \right\} .$$ (2.29)

The reduced-mass dependence of the corrections listed in Eqs. (2.28) and (2.29) has been analyzed in Ref. [54].

The self-energy effect on the $2P$ state is given by

$$E_{SE}(2P_{1/2}) = \frac{\alpha^5 m_R^3}{8 \pi m^2_{\mu}} \left\{ -\frac{1}{6} \frac{m_{\mu}}{m_R} - \frac{4}{3} \ln k_0(2S) \right\} .$$ (2.30)

The first term in curly brackets is due to the anomalous magnetic moment of the muon. Its reduced-mass dependence, which is different from that of all other terms, follows from the Breit–Pauli Hamiltonian if one corrects the muon-spin dependent terms by the muon anomalous magnetic moment [54]. Otherwise, the coefficients in Eq. (2.30) have been verified against very precise numerical calculations [55] of the one-loop self-energy. There is no additional vacuum-polarization effect for the $2P_{1/2}$ state to the order under investigation ($\alpha^6 m_R$).

The final result for the self-energy and muonic-vacuum polarization shift is

$$\delta E_{SE} = E_{SE}(2P_{1/2}) - E_{VP}(2S_{1/2}) - E_{SE}(2S_{1/2}) = -0.6677 \text{ meV} .$$ (2.31)
Finally, the contributions given in Eqs. (2.24), (2.25), (2.27) and (2.31) add up to
\[ \delta E = \delta E^{(1)} + \delta E^{(2)} + \delta E_{\text{KS}} + \delta E_{\text{SE}} = 205.9987 \text{ meV}, \]  
(2.32)
which gives a total QED result of 206.0573 meV. Both individually as well as collectively, the remaining
effects are thus too small to explain the observed discrepancy of 0.31 meV.

3 Advancing Theory

3.1 Relativistic Effects and Vacuum Polarization

We have been unable to resolve the discrepancy of theory and experiment observed in the recent
measurement \[10\]. Therefore, we now turn our attention to the evaluation of a few numerically tiny, but still
important corrections to the Lamb shift, which have not yet been addressed in the literature.

The authors of Ref. \[10, 22\] use a value of 205.0282 meV for the first-order Uehling correction \( \delta E^{(1)} \)
to the \( 2P-2S \) muonic hydrogen Lamb shift, taken with the exact one-body Dirac wave function for both
the \( 2P_{1/2} \) as well as the \( 2S_{1/2} \) states. The relativistic correction thus is
\[ \delta E^{(1b)}_{\text{rel}} = (205.0282 - 205.0074) \text{ meV} = 0.0208 \text{ meV} \]  
(3.1)
for the one-body Dirac theory. However, the one-body Dirac theory cannot account for the two-body
reduced-mass corrections. As shown in Ref. \[56\], the full two-body treatment has to be based on a
generalization of Eq. (2.7) to a massive photon, integrated over the spectral function describing the
virtual electron-positron pairs in the vacuum polarization loop \[29\]. It is imperative to use the full two-
body treatment for muonic hydrogen because the parameter governing the relativistic corrections captured
in the one-body Dirac equation (the fine-structure constant \( \alpha \)) is much smaller for muonic hydrogen than
the parameter characterizing the reduced-mass corrections \( (m_{\mu}/m_p) \) which are captured in the Breit–
Pauli Hamiltonian. According to Eq. (25) of Ref. \[56\], the relativistic correction to the one-loop electronic
vacuum polarization thus reads
\[ \delta E^{(2b)}_{\text{rel}} = 0.0169 \text{ meV}, \]  
(3.2)
and we prefer to use this value for our final theoretical evaluations. The additional two-body relativistic
correction beyond the one-body treatment used in Refs. \[10, 22\] thus is
\[ \delta E_{\alpha} = \delta E^{(2b)}_{\text{rel}} - \delta E^{(1b)}_{\text{rel}} = -0.0039 \text{ meV}. \]  
(3.3)

3.2 Self-Energy Vacuum Polarization Corrections

Because the dominant effect to the Lamb shift is due to the one-loop vacuum polarization potential given
in Eq. (2.21), it would be highly desirable to carry out the calculation of the one-loop self-energy effect
cutoff $\epsilon$ can be conveniently expressed as part for solutions of the full Schrödinger–Uehling Hamiltonian on an exponential grid and find that it $V$ on the vacuum polarization potential $\delta E$. It follows from second-order perturbation theory with time-independent field operators and an upper, $\delta E = M$ element to the replacement $V \to V + V_{vp}$ in the Schrödinger Hamiltonian. The effect of high-energy virtual photons in the self-energy loops given in Fig. 2 can then be expressed in terms of the Dirac $\alpha$ form factor acting on the vacuum polarization potential $V_{vp}$. When rewritten in terms of the noncovariant photon energy cutoff $\epsilon$, which is a convenient overlapping parameter in Lamb shift calculations \[57\], we have

$$E_{H}^{(vp)} = \frac{\alpha}{3\pi m_{\mu}^{2}} \left< \psi \left| \nabla^{2}(V + V_{vp}) \right| \psi \right> \left( \frac{m_{\mu}}{2\epsilon} + \frac{10}{9} \right),$$  \hspace{1cm} (3.4)

where $\left| \psi \right>$ is the wave function of the bound state in the full binding potential $V + V_{vp}$. Denoting by $\phi$ the Schrödinger–Coulomb eigenstate, we have in leading order

$$\left< \phi \left| \nabla^{2}V \right| \phi \right> = \frac{4\alpha^{4} m_{R}^{3}}{n^{3}} \delta_{l0},$$  \hspace{1cm} (3.5)

and thus

$$E_{H}^{(0)} = \frac{4\alpha^{5} m_{R}^{3}}{3\pi m_{\mu}^{2} n^{3}} \delta_{l0} \left( \ln \left( \frac{m_{\mu}}{2\epsilon} \right) + \frac{10}{9} \right).$$  \hspace{1cm} (3.6)

Numerically, we find that the correction $\delta E_{H}$ to the high-energy part due to the vacuum polarization can be conveniently expressed in terms of a parameter $V_{61}$,

$$\delta E_{H} = E_{H}^{(vp)} - E_{H}^{(0)} = \frac{\alpha^{6} m_{R}^{3}}{\pi^{2} m_{\mu}^{2} n^{3}} V_{61} \left\{ \ln \left( \frac{m_{\mu}}{2\epsilon} \right) + \frac{10}{9} \right\}.$$  \hspace{1cm} (3.7)

By a numerical diagonalization of the Schrödinger–Coulomb Hamiltonian on an exponential grid \[39\], we find for the $V_{61}$ coefficient (the first subscript gives the power of $\alpha$, the second indicates the power of the logarithm),

$$V_{61}(2S_{1/2}) = 3.09, \quad V_{61}(2P_{3/2}) = -0.023.$$  \hspace{1cm} (3.8)

The anomalous magnetic moment of the electron gives rise to a further combined self-energy vacuum polarization correction,

$$\delta E_{M} = \frac{\alpha}{4\pi m_{\mu}^{2}} \left< \psi \left| \frac{1}{r} \frac{\partial}{\partial r} (V + V_{vp}) \left( \hat{\sigma} \cdot \hat{L} \right) \right| \psi \right> - \frac{\alpha}{4\pi m_{\mu}^{2}} \left< \phi \left| \frac{1}{r} \frac{\partial}{\partial r} \left( \hat{\sigma} \cdot \hat{L} \right) \right| \phi \right> = \frac{\alpha^{6} m_{R}^{3}}{\pi^{2} m_{\mu}^{2} n^{3}} M_{60},$$  \hspace{1cm} (3.9)

where $\psi$ again is the state in the Uehling+Coulomb potential, and $\phi$ is the unperturbed state. The matrix element $M_{60}$ is nonvanishing for $P$ states,

$$M_{60}(2S_{1/2}) = 0, \quad M_{60}(2P_{1/2}) = -0.022.$$  \hspace{1cm} (3.10)

The low-energy part of the muon self-energy, in the Coulomb potential, is

$$E_{L}^{(0)} = \frac{4\alpha^{5} m_{R}^{3}}{3\pi m_{\mu}^{2} n^{3}} \left[ \delta_{0} \ln \left( \frac{\epsilon}{\alpha m_{R}} \right) - \ln k_{0} \right].$$  \hspace{1cm} (3.11)

It follows from second-order perturbation theory with time-independent field operators and an upper, noncovariant cutoff for the photon energy \[57\] \[59\]. We have numerically calculated the low-energy $\delta E_{L}^{(vp)}$ part for solutions of the full Schrödinger–Uehling Hamiltonian on an exponential grid and find that it can be conveniently expressed as

$$\delta E_{L} = \delta E_{L}^{(vp)} - \delta E_{L}^{(0)} = \frac{\alpha^{6} m_{R}^{3}}{\pi^{2} m_{\mu}^{2} n^{3}} V_{61} \ln \left( \frac{\epsilon}{\alpha m_{R}} \right) - \frac{4\alpha^{6} m_{R}^{3}}{3\pi^{2} m_{\mu}^{2} n^{3}} L_{60},$$  \hspace{1cm} (3.12)
Figure 3: Light-by-light scattering diagrams.

where \( V_{61} \) parameterizes the modification of the logarithmic coefficient and equals the corresponding correction from the high-energy part, leading to a cancellation of the \( \epsilon \) parameter. The coefficient \( L_{60} \) gives the modification of the Bethe logarithm due to the Uehling potential. Numerically, we find

\[
L_{60}(2S_{1/2}) = 11.28, \quad L_{60}(2P_{1/2}) = -0.0146.
\]

In the total correction \( \delta E_H + \delta E_L \), the auxiliary overlapping parameter \( \epsilon \) cancels, and we have for the combined self-energy vacuum polarization correction,

\[
\delta E_{svp} = \frac{\alpha^6 m_{\mu}^3}{\pi^2 m_{\mu}^2 \hbar^3} \left\{ \ln \left( \frac{m_{\mu}}{2\alpha^2 m_R} \right) + \frac{10}{9} \right\} \Delta V_{61} + \Delta M_{60} - \frac{4}{3} \Delta L_{60}
\]

\[
= -0.0025 \text{ meV},
\]

when evaluated for the difference of the \( 2P \) and \( 2S \) states in muonic hydrogen. We have used the definitions,

\[
\Delta V_{61} = V_{61}(2P_{1/2}) - V_{61}(2S_{1/2}),
\]

\[
\Delta M_{60} = M_{60}(2P_{1/2}) - M_{60}(2S_{1/2}),
\]

\[
\Delta L_{60} = L_{60}(2P_{1/2}) - L_{60}(2S_{1/2}).
\]

Electronic vacuum polarization insertions into the bound muon line have previously been evaluated in Eqs. (40) and (45) of Ref. \[19\] in the leading logarithmic accuracy as

\[
\delta E_{svp}^{\log} = -0.006 \text{ meV}.
\]

The modification due to our complete treatment thus is

\[
\delta E_b = \delta E_{svp} - \delta E_{svp}^{\log} = +0.0035 \text{ meV}.
\]

### 3.3 Virtual Light–by–Light Scattering

Finally, we also include the results for the light-by-light scattering graphs shown in Fig. 3 as reported in Ref. \[60\], which add up to

\[
\delta E_{LL} = -0.00089(2) \text{ meV}
\]

for the \( 2P - 2S \) Lamb shift. These entries replace the results for the virtual Delbrück scattering and the Wichmann–Kroll term in \[10\] (entries \# 9 and \# 10 in Table I of the supplementary material included with Ref. \[10\]), which otherwise add up to

\[
\delta E_{WK+VD} = +0.00032(135) \text{ meV}
\]

The shift is

\[
\delta E_c = \delta E_{LL} - \delta E_{WK+VD} = -0.0012 \text{ meV},
\]

and the corresponding uncertainty of \( \pm 0.00135 \text{ meV} \) is eliminated from the theoretical prediction.
3.4 Zemach Moment Correction

For $S$ states bound to an infinitely heavy nucleus, the finite nuclear-size (NS) effect is given by

$$\delta E_{NS} = \frac{2\alpha^4 m_p^4}{3\hbar^3} \langle r^2 \rangle \left( 1 - \frac{\alpha}{2m_p} \frac{m_p \langle r^3 \rangle}{\langle r^2 \rangle} \right).$$

(3.21)

where $\langle r^2 \rangle$ is the mean-square charge radius of the proton, and the $\alpha^5$ correction to the nuclear size effect involves the so-called third Zemach moment $\langle r^3 \rangle$ (see Ref. [61]), which results from the elastic part of the two-photon exchange diagrams of bound particle and nucleus, when the both Coulomb photon interactions are corrected for the finite-size effect.

The coordinate-space representation of the third Zemach moment reads

$$\langle r^3 \rangle = \int d^3r \int d^3r' r^3 f_E(\vec{r} - \vec{r}') f_E(\vec{r}') ,$$

(3.22)

where $f_E$ is the normalized electric charge distribution of the proton, with $\int d^3r f_E^2(\vec{r}) = 1$.

The third Zemach moment can be obtained, in a model-independent way, from experimentally obtained scattering data for electron-proton scattering, in terms of the measured $G_E$ Sachs form factor as

$$\langle r^3 \rangle = \frac{48}{\pi} \int_0^\infty dq \frac{G_E^2(q^2) - 1 + \frac{1}{3} q^2 \langle r^2 \rangle}{q^4} ,$$

(3.23)

where the two subtraction terms ensure the convergence of the integral for small $q$. From a model-independent, careful analysis of the world scattering data performed in Ref. [14], Friar and Sick obtain the value

$$\langle r^3 \rangle = 2.71(13) \text{fm}^3.$$  

(3.24)

We note, in passing, that this analysis excludes the value recently assumed in Ref. [62], which reads

$$\langle r^3 \rangle = 36.6(7.3) \text{fm}^3.$$  

(3.25)

The magnitude of the estimate put forward in Ref. [62] has independently been called into question [63]. For muonic hydrogen, the mass ratio of the orbiting to the nuclear particle roughly is $\frac{1}{7}$, which is still small compared to unity. The relative uncertainty of $\langle r^3 \rangle$ therefore gives a realistic indication of the uncertainty that should be ascribed to the $r_p^3$ term in the formula for the Lamb shift in muonic hydrogen.

However, as stressed in Ref. [19], the third Zemach moment should not be used directly for an evaluation of the nuclear size effect in muonic hydrogen. In Eq. (58) of Ref. [19], it is shown how to express the two-photon exchange diagram, for a finite ratio of $m_p/m_p$, in terms of the proton form factors. Furthermore, in Eq. (59) of Ref. [19], it is shown that in leading order of $m_p/m_p$, the third Zemach moment correction is recovered. An evaluation of the complete two-photon exchange diagram is then performed for a simple dipole model of the nuclear charge distribution, with $G_E(-p^2) = \Lambda^4/\left(\langle p^2 \rangle + \Lambda^2 \right)^2$ and $G_M(-p^2) = \frac{1}{2} g_p G_E(-p^2)$. If one assumes the dipole model, then one may relate the third Zemach moment correction to the leading finite nuclear-size effect and express $\langle r^3 \rangle$ as being proportional to $r_p^3$, where $r_p$ is the root-mean-square radius of the proton. However, the relation of $\langle r^3 \rangle$ and $r_p$ is model-dependent, as emphasized in a particularly clear manner in Ref. [64] and illustrated on the Gaussian, uniform and exponential models for the nuclear charge distribution.

In order to obtain a realistic estimate for the (elastic part of the) two-photon exchange diagram, which captures the model dependence, we thus proceed as follows. We first observe that for the dipole model assumed in Ref. [19], the full two-photon exchange diagram yields a correction of +0.018 meV to the $2P$–$2S$ Lamb shift in muonic hydrogen. With a third Zemach moment based on the dipole model employed in Ref. [19], which roughly amounts to $\langle r^3 \rangle = 2.30 \text{fm}^3$ one obtains +0.021 meV in first order in $m_p/m_p$ [see Eq. (64) of Ref. [19] and the text following this equation], The actual third Zemach moment is somewhat larger [see Eq. (3.24)], but as shown in Ref. [19], the reduced-mass effect slightly reduces the nuclear-size correction of order $\alpha^5$. The ratio of the two results (+0.018 meV versus +0.021 meV), which is
equal to $6/7$, then gives a realistic estimate for the reduced mass correction to the third Zemach moment correction in muonic hydrogen, beyond the factor $m_\mu/m_p$ explicitly indicated in Eq. (3.21). However, the model-independent value of the third Zemach moment should be used for the final evaluation, as given in Eq. (3.24), and the correction factor $6/7$ should be applied in order to approximately account for the reduced-mass effects. This gives a result of

$$\delta E = +0.0212(12) \text{ meV}$$

for the $\alpha^5$ correction to the $2P-2S$ Lamb shift in muonic hydrogen, which replaces the model-dependent term

$$\delta E = +0.0347 \frac{r_p^3}{\text{fm}^3} \text{ meV}$$

used in Refs. [10,19].

4 Reevaluation of the Proton Radius

For convenience, we here recall that in Ref. [11], the theoretical expression already given above in Eq. (2.2),

$$E_{\text{th}} = \left( 209.9779(49) - 5.2262 \frac{r_p^2}{\text{fm}^2} + 0.0347 \frac{r_p^3}{\text{fm}^3} \right) \text{ meV},$$

has been used in order to analyze the measured energy interval $\Delta E$ defined in Eq. (2.1).

We here add the two-body treatment of the relativistic correction to the electronic vacuum polarization ($\delta E_a = -0.0039 \text{ meV}$), the complete result for the vacuum polarization correction to the muon self energy ($\delta E_b = +0.0026 \text{ meV}$), and the shift due to the light-by-light scattering graphs ($\delta E_c = -0.0012 \text{ meV}$), to obtain a total theoretical shift of

$$\delta E = \delta E_a + \delta E_b + \delta E_c = -0.0025 \text{ meV}$$

for the proton radius independent term. As the uncertainty due to the light-by-light scattering graphs is eliminated, the theoretical uncertainty of the proton-radius independent term shrinks from $\pm 0.0049 \text{ meV}$ to $\pm 0.0046 \text{ meV}$, to which we should add the uncertainty of $\pm 0.0012 \text{ meV}$ from Eq. (3.26). The model-dependent $r_p^3$ term given in Eq. (3.27), which is valid only for a dipole model of the nuclear charge distribution, is replaced by the model-independent term given in Eq. (3.26). Thus, our theoretical prediction is slightly shifted from Eq. (2.2) and reads

$$E_{\text{th}} = \left( 209.9974(48) - 5.2262 \frac{r_p^2}{\text{fm}^2} \right) \text{ meV}.$$  

A comparison with the experimental result given in Eq. (2.3) then gives a proton radius of

$$r_p = 0.84169(66) \text{ fm}.$$  

This value is only marginally shifted from the value given in Ref. [11], which reads $r_p = 0.84184(67) \text{ fm}$. The difference of the new radius given in Eq. (4.4) and the CODATA recommended value given above in Eq. (1.1) is 5.0 standard deviations and thus statistically highly significant.

5 Conclusions

The purpose of this paper has been twofold. First, in Sec. 2, we have rederived all relativistic and quantum electrodynamic contributions to the $2S(F=1) \leftrightarrow 2P_{3/2}(F=2)$ transition energy which are relevant on the order of the reported discrepancy of theory and experiment [11], which is 0.31 meV. We have emphasized......
that the relativistic and quantum electrodynamic corrections relevant on the level of the discrepancy represent theoretically well-established corrections. The QED theory of atomic bound states is generally recognized as a rather highly developed theory, and the theoretical predictions for muonic hydrogen have been obtained as a collective effort of QED theorists [19–27]. The main effects obtained in the cited literature references are independently being verified here. The numerically smaller QED corrections listed in Table II of Ref. [25] beyond those treated in Sec. 2 are said to be of “higher order” because they are parametrically suppressed (either by higher powers in the mass ratio $m_\mu/m_p$ or by higher powers in the fine-structure constant $\alpha$). A correction of a conceivable calculational insufficiency in one of the higher-order effects could thus only explain the discrepancy if it leads to a surprising enhancement of the corresponding correction that compensates its parametric suppression.

In Sec. 3 we calculate a number of hitherto neglected effects which contribute to the $2P-2S$ Lamb shift in muonic hydrogen. The most important of these probably is a fully nonperturbative treatment of the Uehling correction to the bound-muon self-energy, which is performed using a pseudospectrum of states obtained from an exponential grid on a lattice [38, 39]. We also refer to Ref. [60] for the results on the light-by-light scattering graphs, and we present an updated estimate for the $\alpha^5$ correction to the nuclear size effect, which uses a model-independent value for the third Zemach moment [14].

We have already stressed that the theory of muonic hydrogen is given, on the level of the discrepancy, by only few, simple relativistic and QED corrections not exceeding the two-loop level. In some sense, this makes the task easier for theoreticians as compared to the muon anomalous magnetic moment, where on the level of the observed $3.4\sigma$ discrepancy of theory and experiment, a multitude of physical effects (three-loop and beyond) contribute (see Refs. [65–68]). The observed discrepancy in muonic hydrogen should thus be interpreted as a large discrepancy, both in terms of standard deviations as well as in terms of its absolute magnitude in energy units.

The discrepancy is all the more surprising because the spectroscopy of muonic bound states is an established tool for the determination of nuclear radii [3, 5]. Nuclear radii determined from electron scattering and from muonic transitions have been observed to agree on the percent level already in a number of experiments, one of the first was reported in 1974 (Ref. [69]). Furthermore, the vacuum polarization contributions to the Lamb shift (including two-loop effects) have been verified in other muonic transitions [70]. Possibilities for a clarification of the discrepancy from a theoretical side seem to be somewhat limited and will be explored in a following investigation [71], to the extent possible.

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