Determination of Bending Angle of Light Deflection
Subject to Weak and Strong Quantum Gravity Effects

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Abstract
Explicit expressions for the bending angle of light deflection subject to weak and strong quantum gravity effects, respectively, are obtained, by a highly effective method. This method enables a full control of truncation errors and permits, in principle, acquisition of higher-order contributions of any desired levels.

Keywords: Bending angle, gravitational light deflection, quantum gravity, Schwarzschild line element, impact parameter.

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1 Introduction
Gravitational deflection of light is one of the three classical experimental tests of General Relativity proposed by Einstein himself. In modern theoretical physics, these tests and their computational realizations are of relevance and interest in the studies of various extended and modified theories as well, developed to enrich and improve Einstein’s theory. In these extended situations, it is often difficult to determine the deflection or bending angle with full precision and suitable approximations are inevitable. Among these, explicit calculations [1–15] may involve evaluating some complicated integrals and implicit calculations [16–26] amount to solving some sophisticated nonlinear equations. Recently, in [18], a study on the determination of the bending angle of light deflection subject to weak and strong quantum gravity effects is conducted. Specifically, in the weak quantum gravity effect situation, quantum deviation is measured in terms of a small parameter, $\kappa > 0$, which is present to deform the usual Schwarzschild black hole metric. As a consequence, it is shown [18] that the bending angle is now given by

$$\hat{\alpha} = 2 \left(2 + \frac{\kappa}{4}\right) \frac{GM}{\xi},$$  \hspace{1cm} (1.1)
where $G$ is Newton’s gravitational constant, $M$ the mass of a radially symmetric gravitational source, $\xi$ the impact parameter, and the speed of light in vacuum is taken to be unity. Furthermore, in the strong quantum gravity effect situation, quantum deviation of the gravitational metric is measured in terms of a positive integer, $n$, so that the bending angle is shown to be given by

$$\hat{\alpha} = 4(1 + 4n) \frac{GM}{\xi}.$$  \hspace{1cm} (1.2)

Both formulas are derived based on linear approximations and seen to overwhelm the classical Einstein angle, $\theta_E = \frac{4GM}{\xi}$. In order to derive these formulas, in the Friedmann-type differential equation. In order to overcome this difficult structure, the equation is further differentiated and a linearization then taken. Solving the linearized equation leads to a relation between the radial variable and the azimuthal angle. Using the leading-order approximation of this relation in the second-order differential equation obtained from differentiating the Friedmann-type equation and taking approximation again a non-linear functional equation is obtained. Finally solving the leading-order approximation of this last equation results in the bending angle. Thus, we have seen that, in order to find the bending angle, many steps of approximations are taken and the errors so accumulated are hard to keep track of. In fact, such an approach is well known and widely used. On the other hand, since in the context of gravitational scales, quantum effects are often small compared with the underlying classical ones, it will be useful and interesting to know detailed properties of the bending angle with regard to its higher-order terms, without and with quantum gravity effects. Notably, in an analytic calculation of the bending angle in general relativity is carried out to the second order in $\frac{GM}{\xi}$. In the Schwarzschild coordinates, their result reads

$$\hat{\alpha} = \frac{4GM}{\xi} + \left(\frac{15\pi}{4} - 4\right) \left(\frac{GM}{\xi}\right)^2,$$  \hspace{1cm} (1.3)

and, in based on a semiclassical calculation, it is found that the bending angle is given by

$$\hat{\alpha} = \frac{4GM}{\xi} + \frac{15\pi}{4} \left(\frac{GM}{\xi}\right)^2 + c_\xi \hbar \frac{G^2M}{\xi^3},$$  \hspace{1cm} (1.4)

where $\hbar$ is the Planck constant and $c_\xi$ a $\xi$-dependent quantity. This last formula is seen to take a clear quantum-effect departure from its classical limit. See also for some other studies on the fine structures of the bending angle. In view of these studies, it will be interesting to uncover the possibly hidden higher-order terms in the bending angle formulas and, so that both classical and quantum gravity effects, as well as their interplay, are clearly exhibited, through the bending angle. Indeed, in the current work, we set forth to extend the study in the bending angle containing all higher-order terms. In particular, we see that the linear-approximation results are actually underestimates of the bending angle in both situations, and hence, all higher-order additional terms serve to contribute to getting more accurate knowledge of the bending angle. Methodologically, comparing with that in our approach in the determination of the bending angle is more direct and effective in that we work directly on the integration of the Friedmann-type equation without taking further approximation.
The integral assumes a difficult form. However, we will show that its Taylor expansion is quite manageable to allow well-controlled calculations which provide precise information in the Taylor expansions and associated truncation errors. This method thus enables a determination of the bending angle of the problem within any desired accuracy threshold.

The content of the rest of the paper is as follows. In Section 2 we calculate the bending angle, $\hat{\alpha}$, under weak quantum gravity effect. We first review the Schwarzschild line element in the presence of weak quantum gravity effects following [18] and arrive at a nonlinear equation governing the radial variable. The complexity of this equation does not allow an explicit calculation of the bending angle and an approximation is necessary. The work of Section 2 is based on linear approximations of this equation. From these linear approximations we derive two closely related full-structure formulas for $\hat{\alpha}$. In leading orders, these formulas agree with (1.1) and (1.3). In Section 3 we calculate $\hat{\alpha}$ subject to strong quantum gravity effects. We show that this situation allows a complete determination of $\hat{\alpha}$ in the sense that all coefficients in the Taylor expansion of the integral that gives rise to $\hat{\alpha}$ may be computed explicitly. As an example, we present an expression for $\hat{\alpha}$ with a 4th-order truncation error in $\frac{GM}{\xi}$ whose leading term is as stated in (1.2). Interestingly but not surprisingly we see in this situation the appearance of the formula (1.3) again in leading orders. In Section 4 we reconsider the weak quantum gravity situation and solve the concerned nonlinear equation by a quadratic equation approximation. In this sharpened situation we see that the resulting solution for $\hat{\alpha}$ again agrees with (1.1) and (1.3) in leading orders and the differences between the higher-order terms of the formulas based on linear and quadratic approximations are quite insignificant. This indicates that linear approximations of the concerned nonlinear equation are rather robust. In the ending paragraph, we conclude the article with a summary. We note that, in order to facilitate our calculation, we have benefited from and resorted to the symbolic computational tools provided by MAPLE 10.

2 Light deflection subject to weak quantum gravity effects based on linear approximation

Following [18], within the framework of the Finsler geometry [33, 34], the line element of a phenomenological spacetime subject to weak quantum-gravity effect characterized by a small dimensionless parameter $\kappa$, beyond the Schwarzschild sphere $r = 2GM$, is given by

$$ds^2 = a dt^2 - \frac{1}{a} dr^2 - r^2 (\sin^2 \theta \, d\phi^2 + d\theta^2) + \kappa \sqrt{a} \frac{GM}{r} \left(1 - a \frac{J^2}{E^2 r^2} \right)^{\frac{3}{4}} dt \sqrt{dt dr}, \quad (2.1)$$

where $G$ is the Newton gravitational constant, $M$ the mass of a radially symmetric gravitational source, $a = 1 - \frac{2GM}{r}$ the Schwarzschild factor, $r$ the radial coordinate, $\theta$ the colatitude or polar angle coordinate, $\phi$ the longitude or azimuthal angle coordinate, $t$ time, and $E, J$ are some constants. When the motion of the particle is assumed to be confined in the equatorial plane $\theta = \frac{\pi}{2}$, the line element (2.1) becomes

$$ds^2 = a dt^2 - \frac{1}{a} dr^2 - r^2 \, d\phi^2 + \kappa \sqrt{a} \frac{GM}{r} \left(1 - a \frac{J^2}{E^2 r^2} \right)^{\frac{3}{4}} dt \sqrt{dt dr}. \quad (2.2)$$

To proceed further, we use $\tau$ to denote a generic trajectory coordinate variable and dot the corresponding derivative with respect to $\tau$. Then the null condition $ds^2 = 0$ for the light-like motion of
the particle leads to \[18\]:

\[
a t^2 - \frac{t^2}{a} - r^2 \phi^2 + \kappa \sqrt{a} \frac{G M}{r} \left( 1 - a \frac{J^2}{E^2 r^2} \right)^{\frac{3}{4}} \hat{t}^2 \hat{r}^2 = 0. \tag{2.3}
\]

On the other hand, integration of the autoparallel geodesic equations resulting from the line element \(2.1\) leads to the conservation laws \[18\]:

\[
a \dot{t} + \kappa \sqrt{a} \frac{3 GM}{4r} \left( 1 - a \frac{J^2}{E^2 r^2} \right)^{\frac{3}{4}} \hat{t} \dot{r} = E, \tag{2.4}
\]

\[
r^2 \phi = J. \tag{2.5}
\]

In view of \(2.4\) and \(2.5\), we see that the equation \(2.3\) becomes

\[
\dot{r}^2 = -\frac{1}{3} a^2 \dot{t}^2 + \frac{4}{3} E a \dot{t} - \frac{a J^2}{r^2}. \tag{2.6}
\]

On the other hand, \(2.4\) may be solved for \(\dot{t}\) to give us

\[
\sqrt{\hat{t}} = \frac{1}{2} \left[ \frac{9}{16a} \left( \frac{\kappa GM}{r} \right)^2 \left( 1 - a \frac{J^2}{E^2 r^2} \right)^{\frac{3}{4}} \hat{r} + \frac{4E}{a} - \frac{3\kappa GM}{8\sqrt{a}} \left( 1 - a \frac{J^2}{E^2 r^2} \right)^{\frac{3}{4}} \sqrt{\hat{r}}. \tag{2.7}
\]

Rewriting this relation as \(\dot{t} = f(\dot{r})\), we see that the equation \(2.6\) assumes the form

\[
\dot{r}^2 = -\frac{1}{3} a^2 f^2(\dot{r}) + \frac{4}{3} E a f(\dot{r}) - \frac{a J^2}{r^2}
\]

\[
\equiv g(\dot{r}), \tag{2.8}
\]

which is still too complicated to solve. Nevertheless, since \(\eta = \kappa \frac{GM}{r}\) is small, we may expand the expression \(\dot{r}^2 - g(\dot{r})\) around \(\eta = 0\) to obtain

\[
\dot{r}^2 - g(\dot{r}) = \dot{r}^2 - E^2 + \frac{a J^2}{r^2} + \frac{\kappa E^2}{2} \left( 1 - a \frac{J^2}{E^2 r^2} \right)^{\frac{3}{4}} \frac{G M}{r} \sqrt{\hat{r}} + O(\eta^3). \tag{2.9}
\]

Note that the quadratic term in such an expansion is absent such that the linear part already achieves a high-accuracy (second-order) approximation. Note also that, in the classical gravity limit \(\kappa = 0\), \(2.3\)–\(2.5\) lead to the solution

\[
\dot{r}_0 \equiv \dot{r} \mid_{\kappa = 0} = \sqrt{E^2 - \frac{a J^2}{r^2}}. \tag{2.10}
\]

Thus, expanding the right-hand side of \(2.9\) around the classical solution \(2.10\), we get

\[
\dot{r}^2 - g(\dot{r}) = \dot{r}_0^2 \frac{\kappa GM}{2r} + \dot{r}_0 \left( 2 + \frac{\kappa GM}{4r} \right) (\dot{r} - \dot{r}_0) + O(\dot{r}^3) + O(\eta^3). \tag{2.11}
\]

Neglecting the higher-order error terms in \(2.11\), we can solve the equation \(\dot{r}^2 - g(\dot{r}) = 0\) to obtain the solution

\[
\dot{r} = \dot{r}_0 \left( 1 - \frac{\kappa GM}{8r} \right) \left( 1 + \frac{\kappa GM}{8r} \right)^{-1}. \tag{2.12}
\]
Of course, one may take further approximations of (2.12) in order to facilitate the computation. First, since \( \eta = \kappa \frac{G M}{r} \) is small, we may use a linear truncation in (2.12) in terms of \( \eta \) which enables us to arrive at

\[
\dot{r} = \dot{r}_0 \left( 1 - \kappa \frac{G M}{4r} \right).
\]  

(2.13)

This equation is what studied in [18]. We will focus on (2.13) first as a computational illustration. Following [18], we insert (2.5) into (2.13) to arrive at

\[
\frac{1}{r^2} \frac{dr}{d\phi} = \sqrt{\frac{E^2}{J^2} - \frac{a}{r^2} \left( 1 - \kappa \frac{G M}{4r} \right)}.
\]  

(2.14)

Thus, with \( u = \frac{G M}{r} \), the above equation conveniently becomes [18]:

\[
\frac{du}{d\phi} = -\sqrt{\left( \frac{E G M}{J} \right)^2 - u^2 (1 - 2u) \left( 1 - \frac{\kappa u}{4} \right)}.
\]  

(2.15)

Since the light ray is assumed to pass around the gravitational source at the shortest distance \( r = \xi \) (the impact parameter) where \( \frac{dr}{d\phi} = 0 \) or \( \frac{du}{d\phi} = 0 \) and \( u_0 = \frac{G M}{\xi} \), we obtain from (2.15) the result

\[
\left( \frac{E G M}{J} \right)^2 = u_0^2 (1 - 2u_0),
\]  

(2.16)

which fixes the ratio of \( E \) and \( J \) as a by-product. Substituting (2.16) into (2.15), integrating (2.15), and noting the correspondence

\[
0 = u_0, \quad \phi = \pm \left( \frac{\pi}{2} + \alpha \right); \quad u = u_0, \quad \phi = 0,
\]  

(2.17)

between the variable \( u \) and the azimuthal angle \( \phi \) where \( \hat{\alpha} = 2\alpha \) is the angle of light ray deflection, we see that the branch \( 0 < \phi < \frac{\pi}{2} \) is given by the integral

\[
\phi(u) = -\int_{u_0}^{u} \frac{du'}{\sqrt{u_0^2 (1 - 2u_0) - u'^2 (1 - 2u') \left( 1 - \frac{\kappa u'}{4} \right)}}, \quad 0 < u < u_0.
\]  

(2.18)

To proceed further, we set

\[
p(u) = \left( u_0^2 (1 - 2u_0) - u^2 (1 - 2u) \right) \left( 1 - \frac{\kappa u}{4} \right)^2,
\]  

(2.19)

and \( u = u_0 v \). Then we have

\[
\frac{\pi}{2} + \alpha = \int_0^{u_0} \frac{du}{\sqrt{p(u)}} = \int_0^1 \frac{dv}{\sqrt{q(u_0, v)}} = Q(u_0),
\]  

(2.20)

where

\[
q(u_0, v) = \left( 1 - 2u_0 - v^2 (1 - 2u_0v) \right) \left( 1 - \frac{\kappa u_0 v}{4} \right)^2.
\]  

(2.21)
It remains to compute \( Q(u_0) \) effectively for \( u_0 > 0 \). Of course, we have
\[
Q(0) = \int_0^1 \frac{dv}{\sqrt{1-v^2}} = \frac{\pi}{2}.
\] (2.22)

Besides, we also have
\[
Q'(0) = \int_0^1 \left( \frac{1-v^3}{(1-v^2)^{3/2}} + \frac{\kappa v}{4\sqrt{1-v^2}} \right) dv = 2 + \frac{\kappa}{4},
\] (2.23)
\[
Q''(0) = \int_0^1 \left( \frac{3(1-v^3)^2}{(1-v^2)^{5/2}} + \frac{\kappa (1-v^3)v}{2(1-v^2)^{3/2}} + \frac{\kappa^2 v^2}{8 \sqrt{1-v^2}} \right) dv
= \left( \frac{15\pi}{4} - 4 \right) + \frac{1}{2} \left( \frac{3\pi}{4} - 1 \right) \kappa + \frac{\pi}{32} \kappa^2.
\] (2.24)

In principle, there is no difficulty in getting the values of derivatives of \( Q \) at \( u_0 = 0 \) of any orders such that the exact value of \( Q(u_0) \) may be estimated within arbitrary accuracy. It is interesting that all such values stay positive for any \( \kappa \geq 0 \) so that we always approximate the true value of \( Q(u_0) \) from below. In fact, as an illustration, we similarly obtain
\[
Q'''(0) = \left( \frac{122 - \frac{45\pi}{2}}{2} \right) + \frac{9}{4} (5 - \pi) \kappa + \left( 1 - \frac{3\pi}{16} \right) \kappa^2 + \frac{1}{16} \kappa^3,
\] (2.25)
which stays positive for all \( \kappa \geq 0 \). Since \( u_0 \) is small, we are ensured with \( Q'''(v) > 0 \) for any \( v \in (0, u_0) \). Thus, in view of (2.20), (2.22)–(2.25), we get the following formula for the deflection or bending angle:
\[
\hat{\alpha} = 2Q'(0)u_0 + Q''(0)u_0^2 + \frac{1}{3} Q'''(v)u_0^3 \quad \text{(some} \ v \in (0, u_0))
= 2 \left( 2 + \frac{\kappa}{4} \right) GM \xi + \left( \frac{15\pi}{4} - 4 \right) + \frac{1}{2} \left( \frac{3\pi}{4} - 1 \right) \kappa + \frac{\pi}{32} \kappa^2 \left( \frac{GM}{\xi} \right)^2 + O \left( \left[ \frac{GM}{\xi} \right]^3 \right),
\] (2.26)
in which the linear part, in \( \frac{GM}{\xi} \), is as stated in (1.1), obtained in [18], the quadratic correction is new, whose leading term is as given in (1.3), obtained in [29], and the cubic error is positive. Thus, even with the addition of a positive quadratic correction, the estimate for the deflection angle is still an underestimate.

We next consider the sharper equation (2.12) without the linear truncation. With the same reformulation, we see that (2.15) is now replaced with
\[
\frac{du}{d\phi} = -\sqrt{\left( \frac{EGM}{J} \right)^2 - u^2(1-2u) \left( 1 - \frac{\kappa u}{8} \right) \left( 1 + \frac{\kappa u}{8} \right)^{-1}}.
\] (2.27)
Thus, as before, we arrive similarly at the formula
\[
\frac{\pi}{2} + \alpha = \int_0^1 \frac{dv}{\sqrt{q_1(u_0, v)}} \equiv Q_1(u_0),
\] (2.28)
where now
\[
q_1(u_0, v) = (1 - 2u_0 - v^2(1-2u_0)) \left( 1 - \frac{\kappa u_0 v}{8} \right)^2 \left( 1 + \frac{\kappa u_0 v}{8} \right)^{-2}.
\] (2.29)
A direct computation gives us the results

\[ Q_1(0) = \frac{\pi}{2}, \quad Q'_1(0) = 2 + \frac{\kappa}{4}, \]
\[ Q''_1(0) = \left( \frac{15\pi}{4} - 4 \right) + \frac{1}{2} \left( \frac{3\pi}{4} - 1 \right) \kappa + \frac{\pi}{64} \kappa^2, \]
\[ Q'''_1(0) = \left( 122 - \frac{45\pi}{2} \right) + \frac{9}{4} (5 - \pi) \kappa + \frac{1}{2} \left( 1 - \frac{3\pi}{16} \right) \kappa^2 + \frac{1}{64} \kappa^3, \]
\[ Q^{(4)}_1(0) = \left( \frac{10395\pi}{16} - 1560 \right) + \left( \frac{945\pi}{16} - 147 \right) \kappa + \left( \frac{315\pi}{128} - 6 \right) \kappa^2 \]
\[ + \left( \frac{45\pi}{512} - \frac{3}{16} \right) \kappa^3 + \frac{9\pi}{4096} \kappa^4. \]  

(2.30)

All these quantities are again positive. Thus, as in (2.26), we have

\[ \hat{\alpha} = 2 \left( 2 + \frac{\kappa}{4} \right) \frac{GM}{\xi} \]
\[ + \left( \left[ \frac{15\pi}{4} - 4 \right] + \frac{1}{2} \left[ \frac{3\pi}{4} - 1 \right] \kappa + \frac{\pi}{64} \kappa^2 \right) \left( \frac{GM}{\xi} \right)^2 \]
\[ + \left( \left[ \frac{122}{3} - \frac{15\pi}{2} \right] + \frac{3}{4} [5 - \pi] \kappa + \frac{1}{6} \left[ 1 - \frac{3\pi}{16} \right] \kappa^2 + \frac{1}{192} \kappa^3 \right) \left( \frac{GM}{\xi} \right)^3 \]
\[ + O \left( \left[ \frac{GM}{\xi} \right]^4 \right), \]  

(2.31)

which is slightly different in the coefficient of its second-order term and again provides an effective underestimate for the deflection angle. Note that there is no difficulty in finding all \( Q^{(m)}_1(0) \) explicitly such that we may obtain all higher-order terms in \( \hat{\alpha} \) as illustrated in the above manner with well-described truncation errors.

### 3 Light deflection subject to strong quantum gravity effects

Following the phenomenological approach in [18], we consider in the equatorial plane the line element

\[ ds^2 = adt^2 - \left( \frac{4n + 1}{a} \right)^2 dr^2 - r^2 d\phi^2, \]  

(3.1)

where the integer \( n = 0, 1, 2, \ldots \) is a quantum deformation parameter and \( a \) the Schwarzschild factor defined in the previous section. Thus, as before, the null trajectory condition gives us the equation

\[ at^2 - \left( \frac{4n + 1}{a} \right)^2 r^2 = 0. \]  

(3.2)

On the other hand, it follows from integrating the autoparallel geodesic equations under the given line element that there are two additional conserved relations [18]:

\[ at = E, \]  

(3.3)

\[ r^2 \dot{\phi} = J, \]  

(3.4)
with $E, J$ two positive parameters. Thus, inserting (3.3) and (3.4) into (3.2), we arrive at the exact equation [18]:

$$(4n + 1)^2 r^2 = E^2 - a J^2. \tag{3.5}$$

Therefore, with the same change of variable, $u = \frac{GM}{r}$, the updated asymptotic correspondence

$$u = 0, \quad \phi = \pm \left( (4n + 1) \frac{\pi}{2} + \alpha \right); \quad u = u_0, \quad \phi = 0, \tag{3.6}$$

and the Friedmann-type differential equation

$$(4n + 1) \frac{du}{d\phi} = -\sqrt{\left( \frac{EGM}{J} \right)^2 - u^2 (1 - 2u)}, \tag{3.7}$$

we have as in (2.20) the conclusion

$$(4n + 1) \frac{\pi}{2} + \alpha = (4n + 1) \int_0^1 \frac{dv}{\sqrt{q_2(u_0,v)}} \equiv (4n + 1) Q_2(u_0), \tag{3.8}$$

where we have used the relation (2.16) to get

$$q_2(u_0,v) = 1 - 2u_0 - v^2 (1 - 2u_0v), \tag{3.9}$$

which is much simpler than those considered in the weak quantum-gravity-effect cases in the previous section. As a consequence, there is no difficulty to obtain all $Q^{(m)}(0)$ explicitly among which

$$Q_2(0) = \frac{\pi}{2}, \quad Q'_2(0) = 2, \quad Q''_2(0) = \frac{15\pi}{4} - 4,$$

$$Q'''_2(0) = 122 - \frac{45\pi}{2}, \quad Q^{(4)}_2(0) = \frac{10395\pi}{16} - 1560, \tag{3.10}$$

which are all positive. Hence the associated angle of deflection $\hat{\alpha} = 2\alpha$ is given up to the third order (say) of $\frac{GM}{\xi}$ by

$$\frac{\hat{\alpha}}{4n + 1} = 4 \left( \frac{GM}{\xi} \right) + \left( \frac{15\pi}{4} - 4 \right) \left( \frac{GM}{\xi} \right)^2 + \left( \frac{122}{3} - \frac{15\pi}{2} \right) \left( \frac{GM}{\xi} \right)^3 + O \left( \left[ \frac{GM}{\xi} \right]^4 \right). \tag{3.11}$$

The first term on the right-hand side of (3.11) is obtained earlier in [18] as stated in (1.2). It is interesting that the second-order term on the right-hand side of (3.11) is again as that given in (1.3), obtained in [29]. In fact, this term appears in all the bending angle formulas we present in our current study.

4 Bending angle subject to weak quantum gravity effects based on quadratic approximation

In Section 2, we have seen that the use of the fuller solution (2.12) of the first-order (linear) approximation of the equation (2.8) gives us the formula (2.31) which is slightly different from that based on the solution of its further approximated equation (2.13) (which is as used in [18]). Thus,
it may be interesting to know what happens when we replace (2.12) with a further, yet, improved approximation such as a quadratic one. In this section, we investigate this issue.

First we note that the quadratic approximation of the right-hand side of (2.9) in terms of $\dot{r} - \dot{r}_0$ is

$$\dot{r}^2 - g(\dot{r}) = \dot{r}_0^2 \frac{\kappa GM}{2r} + \dot{r}_0 \left( 2 + \frac{\kappa GM}{4r} \right) (\dot{r} - \dot{r}_0) + \left( 1 - \frac{\kappa GM}{16r} \right) (\dot{r} - \dot{r}_0)^2 + O(|\dot{r} - \dot{r}_0|^3) + O(\eta^3). \quad (4.1)$$

Neglecting the truncation errors and solving the quadratic equation, we obtain

$$\dot{r} = \dot{r}_0 \left( 1 - \frac{2}{16 - \eta} \left[ 8 - \sqrt{64 - 16\eta + 3\eta^2 + \eta} \right] \right), \quad (4.2)$$

where again $\eta = \frac{\kappa GM}{r}$. Thus, as in (2.20), we have

$$\frac{\pi}{2} + \alpha = \int_0^1 \frac{dv}{\sqrt{q_3(u_0, v)}} \equiv Q_3(u_0), \quad (4.3)$$

where

$$q_3(u_0, v) = (1 - 2u_0 - v^2(1 - 2u_0v)) \left( 1 - \frac{2}{16 - \eta} \left[ 8 - \sqrt{64 - 16\eta + 3\eta^2 + \eta} \right] \right)^2. \quad (4.4)$$

Thus we are led to arrive at the formula

$$\hat{\alpha} = 2 \left( 2 + \frac{\kappa}{4} \right) \frac{GM}{\xi}$$

$$+ \left( \frac{15\pi}{4} - 4 \right) \left[ \frac{3\pi}{4} - 1 \right] \frac{\kappa}{32} \kappa^2 \left( \frac{GM}{\xi} \right)^2$$

$$+ \left( \frac{122}{3} - \frac{15\pi}{2} \right) + \frac{3}{4}(5 - \pi)\kappa + \left( \frac{9}{16} \right) \kappa^2 + \frac{7}{128} \kappa^3 \left( \frac{GM}{\xi} \right)^3 + O \left( \left( \frac{GM}{\xi} \right)^4 \right), \quad (4.6)$$

for the bending angle, where the fourth-order error term is again positive.

It is interesting to note that, up to all second-order terms in $\frac{GM}{\xi}$, the formulas (4.6) and (2.26) are identical, and that (4.6) and (2.31) differ in one second-order term and in two third-order terms. As before, (4.6) contains (1.3) as its classical gravity limit.
In summary, we have presented a series of exact results which determine with high accuracy the bending angle of light deflection subject to weak and strong quantum gravity effects. Our method is direct and effective and provides precise information about the detailed properties of the bending angle and its truncation errors so that classical and quantum gravity effects including their interplay are clearly exhibited through these exact formulas for the bending angle.

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