Euclidean field theory on a sphere

Dirk Schlingemann
The Erwin Schrödinger International Institute for Mathematical Physics (ESI)
Vienna

March 28, 2022

Abstract

This paper is concerned with a structural analysis of euclidean field theories on the euclidean sphere. In the first section we give proposal for axioms for a euclidean field theory on a sphere in terms of C*-algebras.

Then, in the second section, we investigate the short-distance behavior of euclidean field theory models on the sphere by making use of the concept of scaling algebras, which has first been introduced by D. Buchholz, and R. Verch and which has also be applied to euclidean field theory models on flat euclidean space in a previous paper. We establish the expected statement that that scaling limit theories of euclidean field theories on a sphere are euclidean field theories on flat euclidean space.

Keeping in mind that the minkowskian analogue of the euclidean sphere is the de Sitter space, we develop a Osterwalder-Schrader type construction scheme which assigns to a given euclidean field theory on the sphere a quantum field theory on de Sitter space. We show that the constructed quantum field theoretical data fulfills the so called geodesic KMS condition in the sense of H. J. Borchers and D. Buchholz, i.e. for any geodesic observer the system looks like a system within a thermal equilibrium state.
1 Introduction

One basic motivation for studying structural aspects of euclidean field theory models on a sphere is that the finite volume of the sphere can be regarded as a natural infra-red regularizator. In fact, there are indications that non-trivial euclidean field theory models with an infra-red cutoff can be constructed. This is based on the work of J. Magnen, V. Rivasseau, and R. Sénéor [17] where it is claimed that the Yang-Mills$_4$ model exists within a finite euclidean volume.

**Euclidean field theory on the sphere.** In order to give an overview of the ideas and strategies we use, a brief description of the setup, we are going to use, is given here. Euclidean field theory on a sphere can be formulated in an analogous manner as euclidean field theory on $\mathbb{R}^d$ [19]. For a precise formulation of the axioms, we refer the reader to Section 2. The mathematical ingredients which model the concepts of euclidean field theory consist of two main objects, a C*-algebra and a particular class of states on it.

The C*-algebra $\mathfrak{B}$, the first ingredient, has the following structure: To each region $\mathcal{V}$, for our purpose $\mathcal{V}$ is a subset of the $d$-dimensional euclidean sphere, a C*-subalgebra $\mathfrak{B}(\mathcal{V}) \subset \mathfrak{B}$ is assigned. We require that this assignment is isotonous and local in the sense that, $\mathcal{V}_1 \subset \mathcal{V}_2$ implies $\mathfrak{B}(\mathcal{V}_1) \subset \mathfrak{B}(\mathcal{V}_2)$ and operators which are localized in disjoint regions commute, i.e. $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ implies $[\mathfrak{B}(\mathcal{V}_1), \mathfrak{B}(\mathcal{V}_2)] = \{0\}$. In addition to that, the assignment $\mathcal{V} \mapsto \mathfrak{B}(\mathcal{V})$ has a symmetry. The rotation group $O(d+1)$ acts covariantly on the algebra $\mathfrak{B}$ by automorphisms $\beta_h$, here $h$ is a rotation, such that the algebra $\mathfrak{B}(\mathcal{V})$ is mapped via $\beta_h$ onto the algebra of the corresponding rotated region $\mathfrak{B}(h \mathcal{V})$.

The states, which are of interest for our considerations, are rotation invariant reflexion positive regular states $\eta$. Rotation invariance just means that $\eta$ is invariant under the automorphisms $\beta_h$. The properties reflexion positivity and regularity are precisely formulated in Section 2. Roughly speaking, both reflexion positivity and regularity imply certain analytic properties within variables upon which particular correlation functions depend. This fact is an essential ingredient for constructing a quantum field theory from euclidean data.

**Scaling algebras and renormalization group.** A general approach for the analysis of the high energy properties of a given quantum field theory model has been developed by D. Buchholz and R. Verch [6, 8, 5, 4, 3, 2]. Such a short distance analysis can analogously be carried out for euclidean field theory models [21].

We show in Section 3 that the scaling limit theories of euclidean field theories within a finite volume, here on a sphere, are essentially independent of the volume cutoff, the radius of the sphere. As a result
euclidean field theories on flat euclidean space \( \mathbb{R}^d \) are the result of the scaling limit procedure. This is what one expects, and we mention at this point that the analogous situation has already been studied for the analogous situation in Minkowski space \( \mathbb{R}^d \). More precisely, the scaling limits of a quantum field theory in de Sitter spacetime are quantum field theories in flat Minkowski space.

For a point \( e \in rS^d \), the stabilizer subgroup of \( e \) in \( O(d+1) \) is isomorphic to \( O(d) \). If the scaling limit procedure is performed at \( e \), the invariance under the stabilizer subgroup should remain as a \( O(d) \) invariance within the scaling limit. The translation invariance should then enter from the fact that the state under consideration \( \eta \) is invariant under the full group \( O(d+1) \). The scaling limit procedure which we going to use can also be seen as blowing up the radius of the sphere and we always choose \( r = 1 \) for the unscaled theory.

Quantum field theory on de Sitter spacetime. Keeping in mind that the minkowskian analogue of the euclidean \( d \)-sphere \( S^d \subset \mathbb{R}^{d+1} \) is the de Sitter space, we show in Section \( \S \) by exploring the analytic structure of de Sitter space, that from a given euclidean field \((\mathfrak{B}, \beta, \eta)\) on the sphere \( S^d \) a quantum field theory \((\mathfrak{A}, \alpha, \omega)\) on de Sitter space can be constructed. The constructed state \( \omega \) satisfies the geodesic KMS condition which means that for any geodesic observer the state \( \omega \) looks like an equilibrium state. These type of states have been analyzed by H. J. Borchers and D. Buchholz \( \cite{Borchers} \). A constructive example which fits perfectly within our axiomatic framework has been given by R. Figari, R. Høegh-Krohn, and C. R. Nappi \( \cite{Figari} \).

2 Formulation of the axioms

The starting point in the framework of algebraic euclidean field theory is an isotonomous net

\[ \mathcal{V} \mapsto \mathfrak{B}(\mathcal{V}) \subset \mathfrak{B} \]

of \( C^* \)-subalgebras of \( \mathfrak{B} \), indexed by convex sets \( \mathcal{V} \subset S^d \). This net covers the kinematical aspects of a particular model. We require the following properties for the net:

O\((d+1)\)-covariance: There exists a group homomorphism \( \beta \) from the orthogonal group \( O(d+1) \) into the automorphism group of \( \mathfrak{B} \) such that for each convex set \( \mathcal{V} \subset S^d \) one has

\[ \beta_h \mathfrak{B}(\mathcal{V}) = \mathfrak{B}(h \mathcal{V}) \]

for each \( h \in O(d+1) \).
**Locality:** If \( \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset \), then \( [\mathcal{B}(\mathcal{V}_1), \mathcal{B}(\mathcal{V}_2)] = \{0\} \).

We consider a particular class of states on \( \mathcal{B} \) which contain the dynamical information of the particular model. In order to formulate the analogue property of reflexion positivity in the euclidean approach in flat space, we consider for each point \( e \in S^d \) on the sphere the half space

\[
H_{e}^{d+1} := \mathbb{R}_+ e + e^\perp \subset \mathbb{R}^{d+1}
\]

in the ambient space and we build the half sphere

\[
HS^d_e := H_{e}^{d+1} \cap S^d.
\]

We denote by \( \mathfrak{B}(e) \) the C*-algebra generated by operators which are localized in \( HS^d_e \). The reflexion \( \theta_e \in O(d + 1) \) at the hyperplane \( P_{e}^d = H_{e}^{d+1} \cap H_{e}^{d+1} \) maps \( HS^d_e \) onto \( HS^d_{-e} \) and it leaves the hypersphere \( S^d_e = HS^d_e \cap HS^d_{-e} \) stable.

We are now prepared to formulate sufficient properties, shared by those states on \( \mathcal{B} \), which enable us to construct quantum field theory model from the euclidean data.

**Rotation invariance:** A state \( \eta \) on \( \mathcal{B} \) is called euclidean invariant if for each \( h \in O(d + 1) \) the identity \( \eta \circ \beta_h = \eta \) holds true.

**Reflexion positivity:** A state \( \eta \) on \( \mathcal{B} \) is called reflexion positive if exists a point \( e \in S^d \) on the sphere such that the sesquilinear form

\[
\mathfrak{B}(e) \otimes \mathfrak{B}(e) \ni b_0 \otimes b_1 \mapsto \langle \eta, j_e(b_0) b_1 \rangle
\]

is positive semi definite. Here, \( j_e \) is the anti-linear involution which is given by \( j_e(b) = \beta_{b^*} \).

**Regularity:** A state \( \eta \) on \( \mathcal{B} \) is called regular if for each \( b_0, b_1, b_2 \in \mathcal{B} \) the map

\[
h \mapsto < \eta, b_0 \beta_h(b_1) b_2 >
\]

is continuous.

A triple \((\mathfrak{B}, \beta, \eta)\) consisting of a euclidean net of C*-algebras \((\mathfrak{B}, \beta)\) and a euclidean invariant reflexion positive regular state \( \eta \) is called a euclidean field on the sphere \( S^d \).
3 Short-distance analysis for EFTh on $S^d$

For $e \in S^d$ we consider the coordinate chart

$$\phi_e : HS^d_e \ni x \mapsto x - (ex)e \in \mathbb{R}^d$$

where $P^d_e$ is identified with $\mathbb{R}^d$ in a canonical manner. For an element $h \in O(d+1)$ we get

$$h \circ \phi_e = \phi_{he} \circ h .$$

In particular, $e$ is mapped to 0 via $\phi_e$. For $e_1 \in S^d$, $e_1 e = 0$, the hypersphere $S^{d-1}_{e_1}$ contains $e$. As a consequence $S^{d-1}_{e_1} \cap HS^d_e$ is mapped into the hyperplane $P^{d-1}_{\phi_e(e)} \subset \mathbb{R}^d$ and $HS^{d-1}_{e} \cap HS^d_e$ is mapped into the halfspace $\mathbb{R}^{d-1}_{\phi_e(e_1)} \subset \mathbb{R}^d$.

From a given euclidean field $(\mathcal{B}, \beta, \eta)$ on the sphere $S^d$ and a point $e \in S^d$, we obtain a net $\mathcal{B} \mapsto \mathcal{B}_e(\mathcal{B})$ of C*-algebras in a natural manner by setting

$$\mathcal{B}_e(\mathcal{B}) := \mathcal{B}(\phi_e^{-1}(\mathcal{B}))$$

for each bounded convex set $\mathcal{B} \subset \mathbb{R}^d$, where we use the convention $\mathcal{B}(\emptyset) := \mathbb{C}1$. Let $\beta_e$ be the restriction of $\beta$ to the stabilizer subgroup $O_e(d) \cong O(d)$ of $e$, then it is obvious that

$$\beta_{(e,h)} \mathcal{B}_e(\mathcal{B}) = \mathcal{B}_e(h\mathcal{B})$$

is valid for each $h \in O(d)$. Moreover, we have in general

$$\beta_h \mathcal{B}_e(\mathcal{B}) = \mathcal{B}(h\phi_e^{-1}(\mathcal{B})) = \mathcal{B}_e(\phi_e h\phi_e^{-1} \mathcal{B})$$

for each $h \in O(d+1)$ and for each bounded convex set $\mathcal{B} \subset \mathbb{R}^d$ with $h\phi_e^{-1}(\mathcal{B}) \subset HS^d_e$. The restriction $\eta_e := \eta|_{\mathcal{B}_e}$ of $\eta$ to the C*-subalgebra $\mathcal{B}_e$ is a reflexion positive $O(d)$-invariant regular state.

**Limit functionals.** A convenient method for labeling the different scaling limit theories makes use of limit functionals \[21\] (for the limit $\lambda \to 0$ in $\mathbb{R}_+$. These functionals are states $\zeta$ on the C*-algebra of $\mathcal{F}_b(\mathbb{R}_+)$ of all bounded functions on $\mathbb{R}_+$, which annihilate the closed ideal $\mathcal{F}_0(\mathbb{R}_+)$, which is generated by functions $f \in \mathcal{C}_0(\mathbb{R}_+)$ with $\lim_{\lambda \to 0} f(\lambda) = 0$. Indeed, for a function $f \in \mathcal{F}_b(\mathbb{R}_+)$ with $\lim_{\lambda \to 0} f(\lambda) = f_0$, we find $< \zeta, f > = f_0$ for each limit functional $\zeta$. Since each limit functional can be regarded as a measure on the spectrum of $\mathcal{F}_b(\mathbb{R}_+)$, we write

$$< \zeta, f > = \int d\zeta(\lambda) f(\lambda)$$

in a suggestive manner.
Taking scaling limits. We briefly review here, how scaling limit models can be constructed from the data $(\mathfrak{B}, \beta_e, \eta_e)$. First, we consider the C*-algebra of bounded $\mathfrak{B}$-valued functions on $\mathbb{R}^+_{+}$, $\mathcal{F}_b(\mathbb{R}^+_{+}, \mathfrak{B})$. We introduce for a bounded convex set $\mathcal{W} \subset \mathbb{R}^d$ by $\mathfrak{B}_e(\mathcal{W})$ the C*-subalgebra in $\mathcal{F}_b(\mathbb{R}^+_{+}, \mathfrak{B})$ which is generated by functions $\lambda \mapsto b(\lambda) = \int d\lambda f(\lambda) \beta_{\lambda} b_{\lambda}(\lambda)$ such that $b(\lambda) \in \mathfrak{B}_e(\lambda \mathcal{W})$ for each $\lambda$. Here $f \in \mathcal{C}^\infty(\mathcal{O}(d+1))$ is a smooth function on $\mathcal{O}(d+1)$ and $d\lambda$ is the Haar measure on $\mathcal{O}(d+1)$. The C*-algebra which is generated by all local algebras $\mathfrak{B}_e(\mathcal{W})$ is $\mathfrak{B}_e$.

For a limit functional $\zeta$, we introduce the ideal $J_\zeta$ in $\mathfrak{B}_e$ which consists of those functions $b$ for which the C*-seminorm $\|b\|_\zeta = \int d\zeta(\lambda) \|b(\lambda)\|$ vanishes. The scaling algebra $\mathfrak{B}(e, \zeta)$ is just given by the quotient $\mathfrak{B}_e / J_\zeta$ and $p_\zeta$ denotes in the subsequent the corresponding canonical projection onto the quotient. We formally interpret $\mathfrak{B}(e, \zeta)$ in terms of a direct integral decomposition with respect to the measure $\zeta$ and we write

$$p_\zeta[b] = \int d\zeta(\lambda) b(\lambda) .$$

The local algebras are given by $\mathfrak{B}(e, \zeta)(\mathcal{W}) := p_\zeta[\mathfrak{B}_e(\mathcal{W})]$. The group homomorphism $\beta_{(e, \zeta)}$ is given according to

$$\beta_{(e, \zeta)}(h)p_\zeta[b] = \int d\zeta(\lambda) \beta_{h}(\lambda) b(\lambda)$$

for each $h$ in the stabilizer subgroup $\mathcal{O}_e(d)$. According to [2], there exists a $\mathcal{O}(d)$-invariant reflexion positive state $\eta_{(e, \zeta)}$ on $\mathfrak{B}(e, \zeta)$ which is uniquely determined by

$$\langle \eta_{(e, \zeta)}, p_\zeta[b]\rangle = \int d\zeta(\lambda) \langle \eta_e, b(\lambda)\rangle$$

for each $b \in \mathfrak{B}_e$. We are now prepared to formulate the following statement:

**Theorem 3.1**: There exists a group homomorphism $\beta_{(e, \zeta)}$ from the euclidean group $\mathcal{E}(d)$ into the automorphism group of $\mathfrak{B}(e, \zeta)$ which acts covariantly on the net $\mathfrak{B}(e, \zeta)$ and which extends the homomorphism $\beta_{(e, \zeta)}^0$ such that the triple $(\mathfrak{B}(e, \zeta), \beta_{(e, \zeta)}, \eta_{(e, \zeta)})$ is a euclidean field on $\mathbb{R}^d$.  

6
**Sketch of the proof.** We postpone the complete proof of the theorem to Appendix [3]. We briefly sketch here the main idea of the proof which is quite simple. In order to construct an action of the translation group in $\mathbb{R}^d$ we make use of the rotations which do not leave the point $e$ stable. We choose an orthonormal basis $(e_0, \cdots, e_d)$ with $e_0 = e$. Let $L_{0\mu}$ be the generator of the rotations in the plane spanned by $e_0, e_\mu$ then for each $\mu = 1, \cdots, d$ an automorphism on $\mathfrak{B}(e, \zeta)$ is given by

$$
\beta(e, \zeta, s e_\mu) p_\zeta [b] = \int d\zeta(\lambda) \beta_{\exp(\lambda s L_{0\mu})} b(\lambda)
$$

for each $s \in \mathbb{R}_+$. Indeed, it turns out that these automorphisms generate an action of the translation group. In addition to that it can be shown that the automorphisms $\beta_{e, \zeta, h}$, where $h$ is in the stabilizer subgroup of $e$, together with the automorphisms $\beta(e, \zeta, s e_\mu), \mu = 1, \cdots, d$, generate an action of the full euclidean group $E(d)$ on $\mathfrak{B}(e, \zeta)$.

This is exactly what one expects by looking at the geometrical situation. Taking the scaling limit at the point $e$ can also be interpreted as blowing up the sphere $S^d$. Heuristically, the spheres $\lambda^{-1} S^d$ tend to $\mathbb{R}^d$ if the radius $\lambda^{-1}$ becomes infinite, i.e. $\lambda \to 0$, and the point $e$ is identified with the origin $x = 0$ in $\mathbb{R}^d$. During this limit process, the stabilizer subgroup of $e$ becomes the rotation group in $\mathbb{R}^d$ and the remaining rotations, generated by $L_{0\mu}, \mu = 1, \cdots, d$, can be identified with the translations in $\mathbb{R}^d$.

### 4 From EFTh on the sphere to QFTh on de Sitter space

This section is devoted to an analogous construction procedure as in [19] which relates a given euclidean field $(\mathfrak{B}, \alpha, \eta)$ on the sphere to a quantum field theory in de Sitter spacetime.

According to our axioms, the map

$$
\mathfrak{B}(e) \otimes \mathfrak{B}(e) \ni b_1 \otimes b_2 \mapsto \langle \eta, j_e(b_1)b_2 \rangle
$$

is a positive semidefinite sesquilinear form. By dividing the null-space and taking the closure we obtain a Hilbert space $\mathcal{H}$. The corresponding canonical projection onto the quotient is denoted by

$$
\Psi : \mathfrak{B}(e) \to \mathcal{H}
$$

and we write $\Omega := \Psi[1]$. The construction of the observables, which turn out to be bounded operators on $\mathcal{H}$, can be performed in several steps.
Construction of a representation of $\text{SO}(d, 1)$. The construction of a unitary strongly continuous representation $U$ of $\text{SO}(d, 1)$ can be performed by applying the theory of virtual group representations, as it has been worked out by J. Fröhlich, K. Osterwalder, and E. Seiler [13], to our situation. This leads to the result:

**Theorem 4.1**: There exists a unitary strongly continuous representation $U$ of the Lorentz group $\text{SO}(d, 1)$ on $\mathcal{H}$.

**Proof.** A strongly continuous unitary representation $W$ of the stabilizer subgroup $O_e(d) \subset O(d, 1)$ of $e$ can easily be constructed according to

$$W(h)\Psi[b] = \Psi[\beta_h b]$$

where $b$ is an operator in $\mathcal{B}(e)$. In order to construct the Lorentz boosts, we introduce the regions $\Gamma(e, \tau), \tau \in (0, \pi/2)$, which is the intersection $\Gamma(e, \tau) \cap S^d$, where $\Gamma(e, \tau)$ is the $O(d + 1)$ invariant cone in $e$ direction with opening angle $2\tau$. We choose an orthonormal basis $(e_0, \ldots, e_d)$ with $e = e_0$ and writing $L_\mu$ for the generator of the rotations in the plane spanned by $e_0, e_\mu$, on obtains a vector valued function

$$\Psi_{(h, \mu)}(is) = \Psi[\beta_{e_\mu(sL_\mu)}b]$$

which is defined for each $b \in \mathcal{B}(\Gamma(e, \tau))$ and for $|s| < \tau$. The function $\Psi_{(h, \mu)}$ has an holomorphic extension into the strip $\mathbb{R} + i(-\tau, \tau)$.

Assuming that the net $\mathcal{V} \mapsto \mathcal{B}(\mathcal{V})$ fulfills weak additivity in the sense that for each convex set $\mathcal{V} \subset S^d$ we have

$$\mathcal{B} = \bigcup_{h \in O(d+1)} \mathcal{B}(h\mathcal{V})^{\|\cdot\|},$$

then, as we show in the Appendix A, the space $D(e, \tau) := \Psi[\mathcal{B}(\Gamma(e, \tau))]$ is dense in $\mathcal{H}$ which allows to apply the results, shown in [13], directly to our case. In fact, as it has been carried out in [13], the analytic properties of one parameter groups of boosts can be exploit to get the result: A unitary strongly continuous representation $U$ of the Lorentz group $\text{SO}(d, 1)$ is uniquely determined by

$$U(h) = W(h)$$

$$U(\exp(tB_\mu))\Psi[b] = \Psi_{(b, \mu)}(t)$$

where $h$ is an element of $\text{SO}_e(d)$ and $B_\mu$ is the generator of the Lorentz boosts leaving the wedge $\{x \in \mathbb{R}^{d+1} | |x^0| \leq x^\mu \}$ invariant. □
Construction of the local net of observables. In order to keep our technical assumptions as simple as possible we assume that the euclidean net \((\mathcal{B}, \beta)\) fulfills the time zero condition (TZ). This condition states that the C*-algebra is generated by the \(j_e\)-invariant elements \(b \in \mathcal{B}(S^d_{e-1})\), which are contained in the intersection \(\mathcal{B}(e) \cap \mathcal{B}(-e)\), together with the transformed operators \(\beta_h b, h \in O(d+1)\).

The algebra \(\mathcal{B}(S^d_{e-1})\) is represented by bounded operators on \(\mathcal{H}\) where the representation \(\pi\) is given as follows:

\[
\pi(b)\Psi[b] = \Psi[bb].
\]

Analogously to the situation in Minkowski spacetime \(\mathbb{R}\) we assign to each bounded causally complete region \(\mathcal{O} \subset dS^d\) in \(d\)-dimensional de Sitter spacetime \(dS^d\) the von Neumann algebra \(\mathfrak{A}(\mathcal{O}) \subset B(\mathcal{H})\) which is generated by the bounded operators \(U(g)\pi(b)U(g)^*\) where \(b\) is localized in a convex set \(\mathcal{G}\) of the time slice \(S^d_{e-1}\) and \(g\mathcal{G} \subset \mathcal{O}\). The C*-algebra which is generated by all local algebras \(\mathfrak{A}(\mathcal{O})\) (\(\mathcal{O} \subset dS^d\) causally complete and bounded) is denoted by \(\mathfrak{A}\).

We also obtain a group homomorphism \(\alpha\) form the Lorentz group \(SO(d,1)\) into the automorphism group \(\text{Aut}\mathfrak{A}\) by setting \(\alpha_g := \text{Ad}U(g)\) for each Lorentz transformation \(g\). By construction \(\alpha\) acts covariantly on the net \(\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})\), i.e. for each causally complete region \(\mathcal{O}\) in de Sitter space, the automorphism \(\alpha_g\) maps the algebra \(\mathfrak{A}(\mathcal{O})\) onto \(\mathfrak{A}(g\mathcal{O})\).

The geodesic KMS condition. There is a canonical Lorentz invariant state \(\omega\) on \(\mathfrak{A}\) which is just given by \(\langle \omega, a \rangle := \langle \Omega, a\Omega \rangle\). For a Boost generator \(B_\mu\) we denote by \(\mathcal{W}_{B_\mu} := \{x \in dS^d| x^0 | \leq x^\mu\}\) the intersection of the de Sitter space \(dS^d\) with the wedge in ambient space associated to the boosts \(\exp(sB_\mu)\). We also consider for a boost generator \(B = B_\mu\) and the corresponding one-parameter group \(\alpha_B\) of automorphisms

\[
\alpha_{(B,t)} := \alpha_{\exp(tB)}
\]

which obviously maps the algebra \(\mathfrak{A}(\mathcal{W}_B)\) onto itself and we get a W*-dynamical system \((\mathfrak{A}(\mathcal{W}_B), \alpha_B)\). In the subsequent we prove that \(\omega\) fulfills the geodesic KMS condition which can be precisely formulated by the theorem which is proven in Appendix \(\mathbb{C}\):

**Theorem 4.2:** The restricted state \(\omega|_{\mathfrak{A}(\mathcal{W}_B)}\) is a KMS state with respect to the W*-dynamical system \((\mathfrak{A}(\mathcal{W}_B), \alpha_B)\) at inverse temperature \(2\pi\).

The modular conjugation associated with a wedge algebra. The geodesic KMS condition (Theorem \(\mathbb{C}\)) implies that the
vector Ω is cyclic and separating for the wedge algebra \( \mathcal{A}(\mathcal{W}_B) \) (compare [24]). We build the modular conjugation \( J_B \) as well as the modular operator \( \Delta_B \) with respect to the pair \((\mathcal{A}(\mathcal{W}_B), \Omega)\). We choose an orthonormal basis \((e_0, \cdots, e_d)\) with \( e = e_0 \).

For \( B = B_\mu \) the intersection \( HS^{d-1}_{e_0,e_\mu} := S^{d-1}_{e_0} \cap HS^{d}_{e_\mu} \) is the spatial base of the wedge \( \mathcal{W}_B \). The reflexion \( \theta_{e_\mu} \) at the hyperplane \( e_\mu \) is contained in the stabilizer group of \( e = e_0 \) and the prescription

\[
J_B \Psi[b] = \Psi[j_{e_\mu}(b)]
\]
defines a anti-unitary operator \( J_B \), a PCT operator, on \( \mathcal{H} \). Following the analysis, carried out in [24], one finds (see Appendix D: Theorem 4.3 : The modular conjugation \( J_B \) of the pair \((\mathcal{A}(\mathcal{W}_B), \Omega)\) coincides with the PCT operator \( J_B \):

\[
J_B = J_B.
\]

**Verification of the Haag-Kastler axioms.** The statement of Theorem 4.3 can be used to verify the Haag-Kastler axioms for the net \( \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \) in a very straightforward manner. We already know that there exists a group homomorphism \( \alpha \) from the Lorentz group \( SO(d,1) \) into the automorphism group of \( \mathcal{A} \) which is covariant with respect to the net structure, i.e. \( \alpha_g \mathcal{A}(\mathcal{O}) = \mathcal{A}(g \mathcal{O}) \) is valid for each causally complete set \( \mathcal{O} \) in de Sitter space and for each \( g \in SO(d,1) \). It remains to be proven that locality is satisfied which is formulated in the corollary:

**Corollary 4.4 :** The net \( \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \) fulfills locality, i.e. if \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are spacelike separated regions in de Sitter space, then the commutator \( [a_1, a_2] = 0 \) vanishes for each \( a_1 \in \mathcal{A}(\mathcal{O}_1) \) and for each \( a_2 \in \mathcal{A}(\mathcal{O}_2) \).

**Proof.** The geometric action of the PCT operator \( J_B \) implies that

\[
J_B \mathcal{A}(\mathcal{W}_B) J_B = \mathcal{A}(\mathcal{W}_B')
\]

where \( \mathcal{W}_B' \) is the causal complement of \( \mathcal{W}_B \) in de Sitter space. Since \( J_B \) coincides with the modular conjugation \( J_B \) by Theorem 4.3, we conclude

\[
\mathcal{A}(\mathcal{W}_B) = \mathcal{A}(\mathcal{W}_B').
\]

The net \( \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \) is \( SO(d,1) \) covariant and thus Equation [1] holds true for each wedge region \( \mathcal{W} \) in de Sitter space. Choosing \( \mathcal{W} \) in such a way that \( \mathcal{O}_1 \subset \mathcal{W} \) and \( \mathcal{O}_2 \subset \mathcal{W}' \), it follows that \( [a_1, a_2] = 0 \) for each \( a_1 \in \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{W}) \) and for each \( a_2 \in \mathcal{A}(\mathcal{O}_2) \subset \mathcal{A}(\mathcal{W}') = \mathcal{A}(\mathcal{W}_B') \). □
5 Conclusion and outlook

**Conclusion:** We have proposed to consider a finite volume euclidean field theory in $d$ dimensions as a field theory on the $d$-sphere $S^d \subset \mathbb{R}^{d+1}$. The corresponding euclidean net $\mathcal{B}$ carries a covariant action of the $d+1$-dimensional rotation group $O(d+1)$ and the functional $\eta$ is invariant under this action. For a given point $e \in S^d$ on the sphere and for a given limit functional $\zeta$, we have constructed the scaling limit theory $(\mathcal{B}_{(e,\zeta)}, \beta_{(e,\zeta)}, \eta_{(e,\zeta)})$ at $e$. The invariance under the stabilizer subgroup of $e$ remains as a $O(d)$ invariance of the state $\eta_{(e,\zeta)}$ whereas the translation invariance enter from the fact that the underlying state $\eta$ is invariant under the full group $O(d+1)$. This leads to the result, we expected, namely that the scaling limit theory of a euclidean field theory on the sphere is a euclidean field theory in an infinite volume.

Moreover, we have discussed how to construct a quantum field theory $(\mathfrak{A}, \alpha, \omega)$ in de Sitter space from a given euclidean field theory $(\mathcal{B}, \beta, \eta)$ on the sphere, by exploring the analytic structure of de Sitter space. In particular, we have proven that the reconstructed state $\omega$ fulfills the so called geodesic KMS condition, i.e. for any geodesic observer the state $\omega$ looks like an equilibrium state.

**Outlook:** Alternatively, one can consider a euclidean field theory within a compact region $\Lambda \subset \mathbb{R}^d$ with boundary $\partial \Lambda$. The corresponding euclidean net of C*-algebras $\mathcal{B} \to \mathcal{B}_\Lambda(\mathcal{W})$ is then indexed by convex regions $\mathcal{W} \subset \mathbb{R}^d$. By choosing the region $\Lambda$ rotationally invariant, it makes sense to consider an action of the rotation group $O(d)$ by automorphisms $\beta_{(\Lambda, h)}$ on the algebra $\mathcal{B}_\Lambda$. The axiom of $O(d)$-invariance and reflexion positivity can analogously be formulated for a state $\eta_{\Lambda}$.

For a given limit point $\zeta$, the corresponding scaling limit theory $(\mathcal{B}_{(\Lambda, \zeta)}, \beta_{(\Lambda, \zeta)}, \eta_{(\Lambda, \zeta)})$ at $x = 0 \in \Lambda$ can be constructed. There are two natural questions which one can ask within this context:

1. Is the scaling limit theory $(\mathcal{B}_{(\Lambda, \zeta)}, \beta_{(\Lambda, \zeta)}, \eta_{(\Lambda, \zeta)})$ a euclidean field theory on $\mathbb{R}^d$, i.e. within an infinite volume, where $\eta_{(\Lambda, \zeta)}$ is invariant under the full euclidean group $E(d)$?

2. Do the scaling limit theory $(\mathcal{B}_{(\Lambda, \zeta)}, \beta_{(\Lambda, \zeta)}, \eta_{(\Lambda, \zeta)})$ depend on the choice of boundary conditions at $\partial \Lambda$?

**Acknowledgment:**

I am grateful to Prof. Jakob Yngvason for supporting this investigation with hints and many ideas. I would also like to thank Prof. Detlev Buchholz for useful hints and remarks. This investigation is financially
supported by the Jubiläumsfonds der Oesterreichischen Nationalbank which is also gratefully acknowledged. Finally I would like to thank the Erwin Schrödinger International Institute for Mathematical Physics, Vienna (ESI) for its hospitality.
A On a Reeh-Schlieder-type theorem for local euclidean algebras.

Regularity, reflexion positivity, and the euclidean invariance of the functional $\eta$ imply certain analytic properties of correlation functions. This leads to a Reeh-Schlieder-type theorem for local euclidean algebras, by assuming that the net $\mathcal{V} \mapsto \mathfrak{B}(\mathcal{V})$ fulfills weak additivity in the sense that for each convex set $\mathcal{V} \subset S^d$:

$$\mathfrak{B} = \bigcup_{h \in O(d+1)} \mathfrak{B}(h\mathcal{V})$$

**Theorem A.1**: For each bounded open convex set $\mathcal{V} \subset H_{s}^{d}$ contained in half sphere $H_{s}^{d}$ the subspace $\mathcal{D}(\mathcal{V}) := \Psi[\mathfrak{B}(\mathcal{V})]$ is dense in $\mathcal{H}$.

**Proof.** The proof can be obtained by an application of a Reeh-Schlieder-type argument. Consider a unit vector $e_1$, perpendicular to $e$. Then we define linear operators $V_{e_1}(s) : \mathcal{D}(\mathcal{V}) \rightarrow \mathcal{H}$ according to

$$V_{e_1}(s)\Psi[b] = \Psi[\beta_{\exp(sL_{ee_1})}b]$$

for $s \in I(e_1, \mathcal{V})$, where the open interval $I(\mathcal{V})$ is given by

$$I(e_1, \mathcal{V}) := \{s|\exp(sL_{ee_1})\mathcal{V} \subset H_{s}^{d}\}$$

and $L_{ee_1}$ denotes the generator of rotations in the plane spanned by $e, e_1$. One easily checks that $V_{e_1}(s)$ is symmetric, i.e.

$$\langle \Psi_1, V_{e_1}(s)\Psi_2 \rangle = \langle V_{e_1}(s)\Psi_1, \Psi_2 \rangle$$

for each $\Psi_1, \Psi_2 \in \mathcal{D}(\mathcal{V})$ and we get

$$s - \lim_{s \rightarrow 0, s \in I(e_1, \mathcal{V})} V_{e_1}(s_1)\Psi = \Psi$$

for each $\Psi \in \mathcal{D}(\mathcal{V})$. Due to a theorem by J. Fröhlich [12] or by using the results of A. Klein and L. J. Landau [16], the operators $V_{e_1}(s)$ extends uniquely to self adjoint operators on $\mathcal{H}$. This implies that the vector valued function

$$is \mapsto V_{e_1}(s)\Psi$$
has an holomorphic extension into the open strip $\mathbb{R}+i\mathcal{I}(\mathcal{V})$ for each $\Psi \in \mathcal{D}(\mathcal{V})$. More general, for operators $b_1, \cdots, b_k \in \mathfrak{B}(\mathcal{V})$, the operator valued function

$$
\Psi_{[b_1, \cdots, b_k; e_1, \cdots, e_k]} : i(s_1, \cdots, s_k) \mapsto \Psi \left[ \prod_{j=1}^{k} \beta_{\exp(s_k L_{ee_k})} b_k \right]
$$

has an holomorphic extension into the tube $\mathbb{R}^k + iI(e_1, \cdots, e_k, \mathcal{V})$, where the region $I(e_1, \cdots, e_k, \mathcal{V})$ contains all points $(s_1, \cdots, s_k) \in \mathbb{R}^k$ such that

$$
\exp(s_1 L_{ee_1}) \mathcal{V}(s_2, e_2; \cdots; s_k, e_k) \subset \text{HS}_{\mathbb{R}}^d.
$$

The set $\mathcal{V}(s_2, e_2; \cdots; s_k, e_k)$ is recursively defined by

$$
\mathcal{V}(s_2, e_2; \cdots; s_k, e_k) := \exp(s_2 L_{ee_2}) \mathcal{V}(s_3, e_3; \cdots; s_k, e_k).
$$

In particular, by construction $I(e_1, \cdots, e_k, \mathcal{V})$ is an open connected set.

Let $\Psi'$ be a vector in the orthogonal complement of $\mathcal{D}(\hat{\mathcal{V}})$, where $\hat{\mathcal{V}}$ is a slightly larger region than $\mathcal{V}$. Then we conclude that there is an open connected subset $J \subset I(e_1, \cdots, e_k, \mathcal{V})$ such that

$$
\langle \Psi', \Psi_{[b_1, \cdots, b_k; e_1, \cdots, e_k]}(z) \rangle = 0
$$

for each $z \in \mathbb{R}^k + iJ$. Since $\Psi_{[b_1, \cdots, b_k; e_1, \cdots, e_k]}$ is holomorphic in the tube $\mathbb{R}^k + iI(e_1, \cdots, e_k, \mathcal{V})$ we conclude

$$
\langle \Psi', \Psi_{[b_1, \cdots, b_k; e_1, \cdots, e_k]}(z) \rangle = 0
$$

for all $z \in \mathbb{R}^k + iI(e_1, \cdots, e_k, \mathcal{V})$. By making use of the weak additivity of the net, the set of vectors

$$
\left\{ \Psi_{[b_1, \cdots, b_k; e_1, \cdots, e_k] \{is\} \middle| \ b_1, \cdots, b_k \in \mathfrak{B}(\mathcal{V}); s \in I(e_1, \cdots, e_k, \mathcal{V}), k \in \mathbb{N} \right\}
$$

span a dense subspace in $\mathcal{H}$ which implies $\Psi' = 0$ and the theorem follows. We mention here that a similar argument can also be found in [15].

\[\square\]

**B  Proof of Theorem 3.1**

We consider an orthonormal basis of $\mathbb{R}^{d+1}$, $(e_0, \cdots, e_d)$, $e = e_0$. Let $\beta_{(e_i, e_j)} : s \mapsto \beta_{(e_i, e_j, s)}$ be the one-parameter group of automorphisms, related to the rotations in the $e_i - e_j$-plane. The corresponding generators in the Lie algebra $\mathfrak{o}(d+1)$ are denoted by $L_{ij}$. For a scaling
parameter $\lambda \in \mathbb{R}_+$ and for a bounded convex region $\mathcal{U} \subset \mathbb{R}^d$ we consider the set $G(\lambda, \mathcal{U}) \subset \mathbb{R}^d$ which consists of all $s \in \mathbb{R}$ such that $\exp(\lambda s L_0) \phi_e^{-1}(\lambda \mathcal{U}) \in HS^d_d$. It is obvious that the definition is independent of the choice of the coordinate direction $\mu = 1, \ldots, d$ and that the inclusion $G(\lambda, \mathcal{U}) \subset G(\lambda', \mathcal{U})$ is valid for $\lambda' < \lambda$. For a function $b \in B(\mathcal{U})$ we put

$$\beta_{(e, \mathcal{U}, \mu)}^\circ \theta_{\mu} b(\lambda) := \begin{cases} \beta_{(e, \mathcal{U}, \mu)}^\circ \theta_{\mu} b(\lambda) ; & s \in G(\lambda, \mathcal{U}) \\ b(\lambda) ; & s \not\in G(\lambda, \mathcal{U}) \end{cases}$$

which defines a function in $\mathcal{F}_b(\mathbb{R}_+, \mathcal{B})$. In order to verify that $\beta_{(e, \mathcal{U}, \mu)}^\circ \theta_{\mu} b$ is contained in $B(\tau_{se_e \mu} \mathcal{U})$, we compute for $x \in \mathcal{U}$ and for each $s \in G(\lambda, \mathcal{U})$

$$\phi_e \exp(\lambda s L_0) \phi_e^{-1}(\lambda x)$$

(2)

We introduce the region

$$\tau_{se_e \mu} \mathcal{U} = \bigcup_{\lambda : s \in G(\lambda, \mathcal{U})} \lambda^{-1} \phi_e \exp(\lambda s L_0) \phi_e^{-1}(\lambda \mathcal{U})$$

which is compact since $\lambda^{-1} \sin(\lambda s)(1 - \lambda^2 x^2)^{1/2} = O(1)$. This implies that $\beta_{(e, \mathcal{U}, \mu)}^\circ \theta_{\mu} b$ is contained in $\mathcal{B}(\tau_{se_e \mu} \mathcal{U})$ and the prescription

$$\beta_{(e, \mathcal{U}, \mu)}^\circ \theta_{\mu} \beta_{\alpha(\mu)} \mathcal{P}_e b(\lambda) = \mathcal{P}_e \beta_{(e, \mathcal{U}, \mu)}^\circ \theta_{\mu} b$$

yields a well defined automorphism of $\mathcal{B}_{(e, \mathcal{U})}$. Analogously we define an automorphism $\beta_{(e, \mathcal{U}, s_1 e_{\mu_1} + \cdots + s_k e_{\mu_k})}$ by replacing $s L_0 \mu$ by $s_1 L_0 \mu_1 + \cdots + s_k L_0 \mu_k$ and we show that

$$\beta_{(e, \mathcal{U}, s_1 e_{\mu_1} + \cdots + s_k e_{\mu_k})} \alpha(e, s_2 e_{\mu_2}) = \beta_{(e, \mathcal{U}, s_1 e_{\mu_1} + s_2 e_{\mu_2})}$$

which implies that for $x = \sum_\mu x^\mu e_\mu$ the assignment

$$x \mapsto \beta_{(e, \mathcal{U}, x)}$$

is indeed a group homomorphism form the translation group $\mathbb{R}^d$ into the automorphism group of $\mathcal{B}_{(e, \mathcal{U})}$. Consider the function

$$b : \lambda \mapsto b(\lambda) = \int dh \, f(h) \alpha_b(\lambda)$$

15
then we get the estimate
\[
\| \beta(e,s_1e_\mu) b(\lambda) - \beta(e,s_2e_\nu) b(\lambda) \| \\
\leq \sup_{\lambda \in \mathbb{R}_+} \| b_0(\lambda) \| \\
\times \int dh \left| f(h) - f(e^{-\lambda s_1L_0} e^{-\lambda s_2L_0} e^{[s_1 L_0 + s_2 L_0] h}) \right| .
\]

Let \( V \) be a finite dimensional linear space on which \( O(d+1) \) is represented by unitary operators. We may assume that the function \( f \) is given by \( f(h) = f_V(hv) \) with \( v \in V \) and \( f_V \in C_0^\infty(V) \). A straightforward computation shows that there exists a linear operator \( M(\lambda) \in L(V) \) with \( \| M(\lambda) \| \leq \text{const.} \lambda^2 \) such that
\[
e^{-\lambda s_1L_0} e^{-\lambda s_2L_0} e^{[s_1 L_0 + s_2 L_0] h} = hv + M(\lambda) hv .
\]

Hence we conclude
\[
\lim_{\lambda \to 0} \sup_{h \in O(d+1)} \left| f(h) - f(e^{-\lambda s_1L_0} e^{-\lambda s_2L_0} e^{[s_1 L_0 + s_2 L_0] h}) \right| = 0
\]
which implies the desired result
\[
\mathbf{p}_\zeta(\beta(e,s_1e_\mu) b(\lambda) - \beta(e,s_2e_\nu) b(\lambda)) = 0 .
\]

For an element \( h \) of the stabilizer subgroup \( O_e(d) \) one easily checks the relation
\[
\beta(e,\zeta,se_\mu) B(e,\zeta) = \beta(e,\zeta,x) B(e,\zeta,se_\mu)
\]
and the existence of the homomorphism \( \beta(e,\zeta) \) follows. It remains to be proven that \( \beta(e,\zeta) \) acts covariantly on \( \mathcal{B}(e,\zeta) \). Let a bounded convex set \( \mathcal{U} \subset \mathbb{R}^d \) be given. According to Equation (2) we conclude that there exists \( r > 0 \) such that
\[
\lambda^{-1} \phi_e \exp(\lambda s L_0) \phi_e^{-1}(\lambda \mathcal{U}) \subset (\mathcal{U} + se_\mu) + B_d(r\lambda)
\]
for each \( \lambda \in \mathbb{R}_+ \). Here \( B_d(r\lambda) \) is the closed ball in \( \mathbb{R}^d \) with center \( x = 0 \) and radius \( r\lambda \). This implies
\[
\beta(e,\zeta,se_\mu) \mathcal{B}(e,\zeta)(\mathcal{U}) = \mathcal{B}(e,\zeta)(\mathcal{U} + se_\mu)
\]
which proves the covariance. Since the state \( \eta(e,\eta) \) is translationally invariant, which is due to the construction of \( \beta(e,\zeta) \), the theorem follows. \( \square \)
C Proof of Theorem 4.2

The main steps of the proof can be performed in complete analogy to the analysis of [22, 15]. We consider a family of operators \( b_1, \ldots, b_n \) which are contained in the time slice algebra, where \( b_j \in \mathfrak{B}(G_j) \) is localized in a convex subset \( G_j \subset S^{d-1}_e \cap W_{B_{\mu_j}} \). Let \( s \mapsto \beta_{(j,s)} \) be the one-parameter automorphism group, corresponding to the rotations in \( \mu_j - 0 \) direction. This implies that \( \beta_{(j,s)} b_j \in \mathfrak{B}(HS^d_e) \) for each \( s \in (0, \pi) \).

Let \( \mathcal{I}_j \) be the open subset in \( \mathbb{R}^2 \)
\[
\mathcal{I}_j \left( G_j \right) := \{ (\tau, s) \in \mathbb{R}^2 : \forall j: \exp(\tau L_j) \exp(sL) \subset HS^d_e \}
\]
which contains in particular the set \( \{0\} \times (0, \pi) \subset \mathcal{I}(G_j) \). By introducing the operators
\[
b_j(\tau_j) := V_{\mu_j}(\tau_j) \pi(b_j)V_{\mu_j}(-\tau_j)
\]
we obtain by an analogous computation as it has been carried out in [22]:
\[
\langle V(s_n)b_n(\tau_n) \cdots V(s_{k+1})b_{k+1}(\tau_{k+1})\Omega, V(s_k)b_k(\tau_k) \cdots \cdots V(s_1)b_1(\tau_1)\Omega \rangle
\]
\[
= \langle V(\pi - s_1 + \cdots + s_k)b_1(\tau_1)^*V(s_1)b_2(\tau_2)^* \cdots V(s_{k-1})b_k(\tau_k)^*\Omega, \times V(\pi - (s_{k+1} + \cdots + s_n))b_{k+1}(\tau_{k+1})^*V(s_{k+1})b_{k+2}(\tau_{k+2})^* \cdots \cdots V(s_{n-1})b_n(\tau_n)^*\Omega \rangle
\]
which expresses the KMS condition at inverse temperature \( 2\pi \) in the euclidean points. Finally, a straight forward application of the analysis of [15] proves the theorem. □

D Proof of Theorem 4.3

By following the analysis of [15], we choose a family of operators \( b_1, \ldots, b_n \) which are contained in the time slice algebra, where \( b_j \in \mathfrak{B}(G_j) \) is localized in a convex subset \( G_j \subset S^{d-1}_e \cap W_{B_{\mu_j}} \), and we choose directions \( e_0, \ldots, e_n \) which are perpendicular to \( e \). By using the same notations as for the proof of Theorem 4.2 we obtain by
putting $\mathcal{J} := \mathcal{J}_B$:

$$V(s_k)b_k(\tau_k) \cdots V(s_1)b_1(\tau_1)\Omega$$

$$= V(s_k)b_k(\tau_k)V(-s_k)V(s_k + s_{k-1})b_{k-1}(\tau_{k-1})V(-s_k - s_{k-1}) \cdots$$

$$\cdots V(s_1)b_1(\tau_1)\Omega$$

$$= V(s_k)b_k(\tau_k)V(-s_k) \cdots V(s_1 + \cdots + s_k)b_1(\tau_1)\Omega$$

$$= \Psi[b_k(\tau_k, s_k) \cdots b_1(\tau_1, s_1 + \cdots + s_k)] .$$

We compute for $s_1, \ldots, s_k \in \mathbb{R}^+$ and $(\tau_1, s_1 + \cdots + s_i) \in I(\mathcal{G}_i)$ for $k \leq i \leq 1$:

$$\mathcal{J}V(s_k)b_k(\tau_k) \cdots V(s_1)b_1(\tau_1)\Omega$$

$$= \Psi[j_{e_0}(b_k(\tau_k, s_k) \cdots b_1(\tau_1, s_1 + \cdots + s_k))]$$

$$= \Psi[b_1(\sigma_1\tau_1, \pi - (s_1 + \cdots + s_k))^* \cdots b_k(\sigma_k\tau_k, \pi - s_k)^*]$$

$$= V(\pi - s_1 + \cdots + s_k)b_1^*(\sigma_1\tau_1)V(s_1) \cdots b_k^*(\sigma_k\tau_k)^*V(s_k)\Omega$$

with $\sigma_j = 1$ if $e_j \perp e_0$ and $\sigma_j = -1$ if $e_j = e$. Performing an analytic continuation within the parameter $s_1, \ldots, s_k$ and $\tau_1, \ldots, \tau_k$ and taking boundary values at $s_j = \tau_j = 0$ yields the relation (compare [15] as well as [13] and [19])

$$\mathcal{J}_B \left[ \prod_{j=1}^{k} U(\exp(t_j B_{e_j}))b_j U(\exp(-t_j B_{e_j})) \right] \Omega$$

$$= V(\pi) \left[ \prod_{j=1}^{k} U(\exp(t_j B_{e_j}))b_j U(\exp(-t_j B_{e_j})) \right]^* \Omega$$

which implies that the Tomita operator is

$$J_B \Delta_B^{1/2} = \mathcal{J}_B V(\pi) .$$

Moreover, according to Theorem 4.2, the automorphism group

$$\alpha_B : t \mapsto \text{Ad}[U(\exp(tB))]$$

maps $\mathfrak{A}(\mathcal{W}_B)$ into itself and the state $\omega|_{\mathfrak{A}(\mathcal{W}_B)}$ is a KMS state at inverse temperature $2\pi$ and the theorem follows. □
References

[1] Borchers, H.-J. and Buchholz, D.:  
*Global properties of vacuum states in de Sitter space*  
e-Print Archive: gr-qc/9803036

[2] Buchholz, D.:  
*Short Distance Analysis in Algebraic Quantum Field Theory*  
Invited talk at 12th International Congress of Mathematical Physics (ICMP97), Brisbane, Australia, 13-19 Jul 1997. e-Print Archive: hep-th/9710092

[3] Buchholz, D. and Verch, R.:  
*Scaling algebras and renormalization group in algebraic quantum field theory. II. Instructive examples*  
hep-th/9708095

[4] Buchholz, D.:  
*Quarks, gluons, colour: Facts or fiction?*  
Nucl.Phys. B**469** (1996) 333-356

[5] Buchholz, D.:  
*Phase Space Properties of Local Observables and Structure of Scaling Limits*  
Annales Poincare Phys. Theor. **64** (1996) 433-460

[6] Buchholz, D.:  
*On the manifestations of particles*  
Published in "Mathematical Physics Towards the 21st Century", R. Sen, A. Gersten eds. Beer-Sheva, Ben Gurion University Press 1994

[7] Buchholz, D., Hennig, C.:  
Private communication.

[8] Buchholz, D. and Verch, R.:  
*Scaling algebras and renormalization group in algebraic quantum field theory*  
Rev. Math. Phys. **7**, 1195-1240, (1995)

[9] Driessler, W. and Fröhlich, J.:  
*The reconstruction of local algebras from the Euclidean Green's functions of relativistic quantum field theory.*  
Ann. Inst. Henri Poincaré **27**, 221-236, (1977)

[10] Figari, R., Höegh-Krohn, R. and Nappi, C. R.:  
*Interacting relativistic boson fields in de Sitter universe with two space-time dimensions*  
Commun. Math. Phys. **44**, 265-278, (1975)
[11] Fröhlich, J.: 
Some results and comments on quantized gauge fields. 
Cargese, Proceedings, Recent Developments In Gauge 
Theories, 53-82, (1979)

[12] Fröhlich, J.: 
Unbounded, symmetric semigroups on a separable Hilbert 
space are essentially selfadjoint 
Adv. Appl. Math. 1, 237-256, (1980)

[13] Fröhlich, J., Osterwalder, K. and Seiler E.: 
On virtual representations of symmetric spaces and their 
analytic continuation. 
Ann. Math. 118, 461-489, (1983)

[14] Glimm, J. and Jaffe, A.: 
Quantum physics, a functional integral point of view. 
Springer, New York, Berlin, Heidelberg (1987)

[15] Klein, A. and Landau L. J.: 
Stochastic processes associated with KMS states 
J. Func. Anal. 42, 368-429, (1981)

[16] Klein, A. and Landau L. J.: 
Construction of a unique self-adjoint generator for a local 
semigroup 
J. Func. Anal. 44, 121-137, (1981)

[17] Magnen, J., Rivasseau, V. and Sénéor, R.: 
Construction of YM-4 with an infrared cutoff. 
Commun. Math. Phys. 155, 325-384, (1993)

[18] Osterwalder, K. and Schrader, R.: 
Axioms for Euclidean Green's functions I. 
Commun. Math. Phys. 31, 83-112, (1973) 
Osterwalder, K. and Schrader, R.: 
Axioms for Euclidean Green's functions II. 
Commun. Math. Phys. 42, 281-305, (1975)

[19] Schlingemann, D.: 
From euclidean field theory to quantum field theory. 
To appear in Rev. Math. Phys. (1999)

[20] Schlingemann, D.: 
Constructive aspects of algebraic euclidean field theory. 
ESI preprint 622, math-ph/9902022

[21] Schlingemann, D.: 
Short-distance analysis for algebraic euclidean field theory. 
ESI preprint 737 (1999), hep-th/9907167
[22] Schlingemann, D.:  
Application of Tomita-Takesaki theory in algebraic euclidean field theories.  
preprint, [hep-th/9912219](https://arxiv.org/abs/hep-th/9912219)  

[23] Seiler, E.:  
Gauge theories as a problem of constructive quantum field theory and statistical mechanics.  
Berlin, Germany: Springer (1982) 192 P. (Lecture Notes in Physics, 159).  

[24] Streater, R.F. and Wightman, A.S.:  
PCT, spin and statistics and all that. Redwood City, USA: Addison-Wesley (1989) 207 p. (Advanced book classics).