Uncertainty and complementarity relations based on generalized skew information

Huaijing Huang\textsuperscript{1}, Zhaoqi Wu\textsuperscript{1} (a) and Shao-Ming Fei\textsuperscript{2,3}

\textsuperscript{1} Department of Mathematics, Nanchang University, Nanchang 330031, P R China
\textsuperscript{2} School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
\textsuperscript{3} Max Planck Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

PACS 03.67.-a – Quantum information
PACS 03.65.Ta – Foundations of quantum mechanics; measurement theory

Abstract – Uncertainty relations and complementarity relations are core issues in quantum mechanics and quantum information theory. By use of the generalized Wigner-Yanase-Dyson (GWYD) skew information, we derive several uncertainty and complementarity relations with respect to mutually unbiased measurements (MUMs), and general symmetric informationally complete positive operator valued measurements (SIC-POVMs), respectively. Our results include some existing ones as particular cases. We also exemplify our results by providing a detailed example.

Introduction . – As embodiment of the Heisenberg uncertainty principle, uncertainty relations form a central part of our understanding on quantum mechanics, providing fundamental constraints on how well the outcomes of various incompatible measurements can be predicted. Heisenberg first noted the uncertainty in the measurements of position and momentum \[1\]. Robertson further generalized it to two arbitrary observables and presented a lower bound on the total variance of two observables \[2\]. Uncertainty relations are generally referred to as the lower bounds on the quantifiers. Various quantitative characterizations including entropy \[3-6\], Wigner-Yanase skew information \[7-12\], variance \[13\] and statistical distance \[14\] have been extensively studied.

Quantum measurement plays fundamental roles in quantum mechanics. Different kinds of measurements including von Neumann measurements \[15\], Lüders measurements \[16\], the dissipative adiabatic measurements (DAMs) \[17\], symmetric informationally complete positive operator valued measures (SIC-POVMs) \[18\], general SIC-POVMs \[19\] have been introduced and investigated. Since quantum coherence is basis-dependent, it is natural to study uncertainty relations of coherence with respect to a given measurement basis.

As the most basic feature in quantum mechanics, quantum coherence is extremely significant physical resource. The problem of
properly quantifying coherence at the quantum level has attracted considerable attention, there are many different ways to measure coherence \[20,23\]. In [24], the author revealed the close relationship between coherence and the quantum part of uncertainties. The concept of quantum uncertainty relations of quantum coherence has been introduced in [25,26]. By deriving the upper bounds on the sum of the corresponding measures, the complementarity relations of quantum coherence in different bases have been studied [27,28]. In addition, by using the Wigner-Yanase (WY) skew information, Luo et al. [29] not only studied a quantitative complementarity relation in the ubiquitous state-channel interaction, but also extended the coherence of a mixed state \(\rho\) with respect to an orthonormal basis to the one with respect to a quantum channel \(\Phi\). Furthermore, Wu et al. [30] discussed the complementarity relations and coherence measures by using the modified GWY skew information. Uncertainty relations for quantum coherence with respect to mutually unbiased bases (MUBs) has also been investigated [31].

In terms of the coherence measure based on the WY skew information, several uncertainty relations for coherence with respect to von Neumann measurements, MUBs and general SIC-POVMs have been established, respectively [32]. The average coherence of a state with respect to any complete set of mutually unbiased measurements (MUMs) and general SIC-POVMs has been also evaluated, respectively [33]. A natural question arises: can we consider the uncertainty and complementarity relations for measures based on GWY skew information with respect to MUMs and general SIC-POVMs? We study these problems in this paper.

GWYD skew information and quantum uncertainty. – Let \(\mathcal{H}\) be a \(d\)-dimensional Hilbert space, and \(S(\mathcal{H})\) and \(D(\mathcal{H})\) the set of Hermitian operators and density operators on \(\mathcal{H}\), respectively. For a density operator \(\rho\) \(\in D(\mathcal{H})\) and an observable \(A \in S(\mathcal{H})\), the WY skew information \([34]\) is defined by

\[
I_\rho(A) = -\frac{1}{2} \text{Tr}(\rho A^2),
\]

where \([X,Y] := XY - YX\) is the commutator of \(X\) and \(Y\). A more general quantity was proposed by Dyson,

\[
I_\rho^\alpha(A) = -\frac{1}{2} \text{Tr}(\rho^\alpha A [\rho^{1-\alpha}, A]), \quad 0 \leq \alpha \leq 1,
\]

which is now called the Wigner-Yanase-Dyson (WYD) skew information. The quantity in Eq. (2) was further generalized to \([34]\)

\[
I_\rho^{\alpha,\beta}(A) = -\frac{1}{2} \text{Tr}(\rho^\alpha A [\rho^{\beta}, A] \rho^{1-\alpha-\beta}),
\]

with \(\alpha, \beta \geq 0, \alpha + \beta \leq 1\), which is termed as GWY skew information. It is easy to see that when \(\alpha + \beta = 1\), \(I_\rho^{\alpha,\beta}(A)\) reduces to \(I_\rho^\alpha(A)\), and \(I_\rho^{\alpha,\beta}(A)\) reduces to \(I_\rho(A)\) when \(\alpha = \frac{1}{2}\). \(I_\rho^{\alpha,\beta}(A)\) can be equivalently expressed as

\[
I_\rho^{\alpha,\beta}(A) = \frac{1}{2}(\text{Tr}(\rho A^2) + \text{Tr}(\rho^{\alpha+\beta} A \rho^{1-\alpha-\beta}))
- \text{Tr}(\rho^\alpha A \rho^{1-\alpha} A) - \text{Tr}(\rho^\beta A \rho^{1-\beta} A)),
\]

where \(\alpha, \beta \geq 0\) and \(\alpha + \beta \leq 1\).

The set of all observables on \(\mathcal{H}\) constitutes a real \(d^2\)-dimensional Hilbert space \(M\) with inner product \(\langle A, B \rangle = \text{Tr}AB\). Let \(\{K_i\}_{i=1}^d\) be any complete orthonormal base of \(M\). In Ref. [35], the quantum uncertainty of a mixed state \(\rho\) is defined as

\[
Q(\rho) = \sum_{i=1}^{d^2} I_\rho(K_i).
\]

Denote \(\{\lambda_i\}_{i=1}^d\) the spectrum of \(\rho\). One has

\[
Q(\rho) = \sum_{i<j} (\sqrt{\lambda_i} - \sqrt{\lambda_j})^2 = d - (\text{Tr}\sqrt{\rho})^2.
\]

With respect to the WY skew information, Li et al. [37] proposed the quantum uncertainty of a mixed state \(\rho\) as

\[
Q_\alpha(\rho) = \sum_{i=1}^{d^2} I_\rho^{\alpha}(K_i), \quad 0 \leq \alpha \leq 1,
\]

where
which can be further expressed as
\[
Q_\alpha(\rho) = \sum_{i,j} (\lambda_i^a - \lambda_j^a)(\lambda_i^{1-a} - \lambda_j^{1-a})
\]
\[
= d - \text{Tr} \rho^a \text{Tr} \rho^{1-a}.
\]

It can be proved that \( Q_\alpha(\rho) \leq Q(\rho) \) for \( 0 \leq \alpha \leq 1 \). Similarly, by using the GWYD skew information defined in Eq. (3), we define the following generalized quantum uncertainty,
\[
Q^{\alpha,\beta}(\rho) = \sum_{i=1}^{d^2} I^{\alpha,\beta}(K_i), \quad \alpha, \beta \geq 0, \quad \alpha + \beta \leq 1,
\]
which has the form in terms of the spectrum of \( \rho \),
\[
Q^{\alpha,\beta}(\rho) = \frac{1}{2} \sum_{i<j} ((\lambda_i^a - \lambda_j^a) \times (\lambda_i^{1-a} - \lambda_j^{1-a})(\lambda_i^{1-a} + \lambda_j^{1-a})).
\]

It can be seen that \( Q^{\alpha,\beta}(\rho) \) reduces to \( Q_\alpha(\rho) \) when \( \alpha + \beta = 1 \), and \( Q_\alpha(\rho) \) reduces to \( Q(\rho) \) when \( \alpha = 1/2 \).

**Remark** It should be noted that the quantity defined in Eq. (11) is different from the one defined in Ref. [35], in which the quantum extensions of the Fisher information have been investigated and the quantity \( Q_{\alpha,\beta}(\rho) = \sum_{i=1}^{d^2} I_{\alpha,\beta}(\rho, K_i) \) has been defined as a measure of the quantum uncertainty, where \( I_{\alpha,\beta}(\rho, K_i) = \frac{1}{2\alpha(1-\alpha)}(\text{Tr}(\rho K_i^2) + \text{Tr}(\rho K_i^{1-a}K_i^{1-a}K_i) - \text{Tr}(\rho^a K_i^{1-a} K_i) - \text{Tr}(\rho^a K_i^{1-a} K_i)) \) for \( \alpha, \beta \geq 0, \quad \alpha + \beta \leq 1 \). Further calculations show that \( Q_{\alpha,\beta}(\rho) = \frac{1}{2\alpha} \sum_{i,j=1}^{d^2} [(\lambda_i^a - \lambda_j^a)(\lambda_i^{1-a} - \lambda_j^{1-a})(\lambda_i^{1-a} + \lambda_j^{1-a})] \).

By using the inequality \( \lambda_i^a \lambda_j^{1-a} \geq \lambda_i^{1-a} \lambda_j^a \geq \lambda_i^{1-a} + \lambda_j^{1-a} \), \( \alpha, \beta \in [0, 1] \) with \( \alpha + 2\beta \leq 1 \) and \( 2\alpha + \beta \leq 1 \) and Eq. (13), we can prove the following lemma.

**Lemma 1** \( Q^{\alpha,\beta}(\rho) \) satisfies the following inequality
\[
Q^{\alpha,\beta}(\rho) \leq \frac{1}{2}(d - \text{Tr} \rho^a \text{Tr} \rho^{1-a}),
\]
for \( \alpha, \beta \in [0, 1] \) with \( \alpha + 2\beta \leq 1 \) and \( 2\alpha + \beta \leq 1 \).
With respect to a set of MUBs $\mathcal{P}_{\text{MUB}} = \{P^{(b)}\}_{b=1}^{d+1}$, the following quantity has been defined \cite{32}: $C(\rho, \mathcal{P}_{\text{MUB}}) = \frac{1}{d+1} Q(\rho, \mathcal{P}_{\text{MUB}}) = \frac{1}{d+1} \sum_{b=1}^{d+1} Q(\rho, P^{(b)})$, where $Q(\rho, P^{(b)}) = \sum_{k=1}^{d} I_{\rho}(P_{k}^{(b)})$. Based on the GWYD skew information, we define the following generalized quantity: $C^{\alpha,\beta}(\rho, \mathcal{P}_{\text{MUB}}) = \frac{1}{d+1} Q^{\alpha,\beta}(\rho, \mathcal{P}_{\text{MUB}}) = \frac{1}{d+1} \sum_{k=1}^{d} Q^{\alpha,\beta}(\rho, P_{k}^{(b)})$, where $Q^{\alpha,\beta}(\rho, P^{(b)}) = \sum_{k=1}^{d} I_{\rho}^{\alpha,\beta}(F_{k}^{(b)})$. It is obvious that $C^{\frac{1}{2},\frac{1}{2}}(\rho, \mathcal{P}_{\text{MUB}}) = C(\rho, \mathcal{P}_{\text{MUB}})$.

**Theorem 1** With respect to MUBs $\mathcal{P}_{\text{MUB}} = \{P^{(b)}\}_{b=1}^{d+1}$, $C^{\alpha,\beta}(\rho, \mathcal{P}_{\text{MUB}})$ satisfies the following quantum uncertainty relations,

$$C^{\alpha,\beta}(\rho, \mathcal{P}_{\text{MUB}}) = \frac{\kappa d - 1}{(d^2 - 1)} Q^{\alpha,\beta}(\rho),$$

(15)

where $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$.

**Proof.** Note that $\sum_{b=1}^{d+1} \sum_{k=1}^{d} \text{Tr}[F_{k}^{(b)}]^{2} \rho] = (1 + \sqrt{d})^2 (d^2 - 1)$ \cite{32}. Taking into account the relations $\sum_{b=1}^{d+1} \sum_{k=1}^{d} \text{Tr}[\rho^{\alpha,\beta} F_{k}^{(b)}] = (d + \sqrt{d})^2 \sum_{b=1}^{d+1} \sum_{k=1}^{d} \text{Tr}(\rho^{\alpha,\beta} F_{k}^{(b)}), \sum_{b=1}^{d+1} \sum_{k=1}^{d} (F_{k}^{(b)})^2 = (d - \frac{1}{2}) I$ \cite{33}, we have

$$Q^{\alpha,\beta}(\rho, \mathcal{P}_{\text{MUB}}) = \frac{1}{2} \sum_{b=1}^{d+1} \sum_{k=1}^{d} \text{Tr}[F_{k}^{(b)}]^{2} \rho]
+ \text{Tr}(\rho^{\alpha,\beta} F_{k}^{(b)} \rho^{\frac{1}{2} - \alpha,\beta} F_{k}^{(b)})
- \text{Tr}(\rho^{\alpha,\beta} F_{k}^{(b)} \rho^{1 - \alpha,\beta} F_{k}^{(b)}),
= \frac{d^2}{2} (1 + \sqrt{d})^2 (d^2 - 1)
+ (d + \sqrt{d})^2 \sum_{k=1}^{d} \sum_{b=1}^{d+1} \text{Tr}(\rho^{\alpha,\beta} F_{k}^{(b) \rho^{\frac{1}{2} - \alpha,\beta} F_{k}^{(b)}})
- \text{Tr}(\rho^{\alpha,\beta} F_{k}^{(b) \rho^{1 - \alpha,\beta} F_{k}^{(b)}})\text{Tr}(F_{k}^{(b)})
= \frac{1}{2} (1 + \sqrt{d})^2 (d^2 - 1)
+ (d + \sqrt{d})^2 \sum_{k=1}^{d} \sum_{b=1}^{d+1} (2 I_{\rho}^{\alpha,\beta}(F_{k}^{(b)})\text{Tr}(F_{k}^{(b)})^2)
= \frac{\kappa d - 1}{d} Q^{\alpha,\beta}(\rho).$$

The theorem holds from the definition of $C^{\alpha,\beta}(\rho, \mathcal{P}_{\text{MUB}})$. □

In particular, taking $\kappa = 1$ in Theorem 1, we obtain the following corollary.

**Corollary 1** The quantum uncertainty relations based on the generalized skew information with respect to MUB are given by

$$C^{\alpha,\beta}(\rho, \mathcal{P}_{\text{MUB}}) = \frac{1}{(d+1)} Q^{\alpha,\beta}(\rho),$$

(16)

where $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$.

Corollary 1 can be viewed as a generalization of the corresponding result in \cite{10}. Taking $\alpha = \beta = \frac{1}{2}$ in Theorem 1, we obtain the following corollary corresponding to the results given in Ref. \cite{33}.

**Corollary 2** The average coherence of a state $\rho$ with respect to the $\mathcal{P}_{\text{MUB}} = \{P^{(b)}\}_{b=1}^{d+1}$ with parameter $\kappa$ is given by

$$C(\rho, \mathcal{P}_{\text{MUB}}) = \frac{\kappa d - 1}{(d^2 - 1)} \left[ d - \text{Tr} (\sqrt{\rho})^2 \right].$$

(17)

By using Lemma 1, we can further prove the following theorem.

**Theorem 2** The quantum complementary relations based on generalized skew information with respect to MUBs are given by

$$C^{\alpha,\beta}(\rho, \mathcal{P}_{\text{MUB}}) \leq \frac{\kappa d - 1}{2(d^2 - 1)} (d - \text{Tr} \rho^{\alpha} \text{Tr} \rho^{1 - \alpha})$$

(18)

for $\alpha, \beta \in [0, 1]$ with $\alpha + 2\beta \leq 1$ and $2\alpha + \beta \leq 1$.

Taking $\kappa = 1$ in Theorem 2, we obtain the following corollary.

**Corollary 3** The complementarity relations based on generalized skew information with respect to MUBs are given by

$$C^{\alpha,\beta}(\rho, \mathcal{P}_{\text{MUB}}) \leq \frac{1}{2(d+1)} (d - \text{Tr} \rho^{\alpha} \text{Tr} \rho^{1 - \alpha})$$

(19)

for $\alpha, \beta \in [0, 1]$ with $\alpha + 2\beta \leq 1$ and $2\alpha + \beta \leq 1$.

Uncertainty and complementarity relations based on GWYD skew information with respect to general SIC-POVMs

---

In this section, we study quantum uncertainty and complementarity relations based on GWYD skew information with respect to general SIC-POVMs. A set of $d^2$ positive-semidefinite operators $\{P_{i}\}_{i=1}^{d^2}$ is called a general SIC-POVM if
where \( \frac{1}{d} < a \leq \frac{1}{d^2} \). \( a = \frac{1}{d^2} \) if and only if all \( P_i \) are rank one, that is, the general SIC-POVM becomes the SIC-POVM. Any general SIC-POVM can be constructed as follows [44]. Let \( \{F_i\}_{i=1}^{d^2-1} \) be a set of traceless Hermitian operators on \( \mathcal{H} \), satisfying \( \text{Tr}(F_i F_k) = \delta_{i,k} \). Set \( F = \sum_{i=1}^{d^2-1} F_i \). For any \( t \) such that \( P_i \geq 0 \) and

\[
a = \frac{1}{d^2} + t^2(d - 1)(d + 1)^3,
\]

one has

\[
P_i = \begin{cases} \frac{1}{d^2} I + t[F - d(d + 1)F_i], & i = 1, \ldots, d^2 - 1; \\ \frac{1}{d^2} I + t(d + 1)F, & i = d^2. \end{cases}
\]

In Ref. [33], the coherence of a state with respect to a general SIC-POVM \( \{P_i\}_{i=1}^{d^2} \) with the parameter \( a \) is defined as

\[
\lambda(\rho, \mathcal{P}_{\text{GSM}}) = \sum_{i=1}^{d^2} \lambda_i(\rho_i).
\]

Now we define a generalized quantity with respect to the GWYD skew information, \( C_{\alpha,\beta}(\rho, \mathcal{P}_{\text{GSM}}) = \sum_{i=1}^{d^2} C_{\alpha,\beta}(\rho_i), \) \( \alpha, \beta \geq 0, \alpha + \beta \leq 1 \). It is straightforward to verify that

\[
C_{\alpha,\beta}(\rho, \mathcal{P}_{\text{GSM}}) = C(\rho, \mathcal{P}_{\text{GSM}}).
\]

**Theorem 3** The quantum uncertainty relations based on generalized skew information with respect to a general SIC-POVM are given by

\[
C_{\alpha,\beta}(\rho, \mathcal{P}_{\text{GSM}}) = \frac{(ad^2 - 1)}{d(d^2 - 1)} Q_{\alpha,\beta}(\rho),
\]

where \( \alpha, \beta \geq 0 \) and \( \alpha + \beta \leq 1 \).

**Proof.** Taking into account the relations

\[
\sum_{i=1}^{d^2} \text{Tr}(P_i) = d [\rho_{P_{\text{GSM}}}] \quad \text{and} \quad \sum_{i=1}^{d^2} \text{Tr}(\rho P_i) \rho^{1-\alpha} P_i = \frac{1}{d^2} + t^2(d + 1)^2 \sum_{i=1}^{d^2-1} \text{Tr}(\rho^\alpha F_i \rho^{1-\alpha}),
\]

we have

\[
C_{\alpha,\beta}(\rho, \mathcal{P}_{\text{GSM}}) = \frac{d^2}{2} \sum_{i=1}^{d^2} \left[ \text{Tr}(P_i)^2 \rho^\alpha + \text{Tr}(\rho^{\alpha+\beta} P_i \rho^{1-\alpha-\beta} P_i) \right] - \text{Tr}(\rho^\alpha P_i \rho^{1-\alpha} P_i) - \text{Tr}(\rho^\beta P_i \rho^{1-\beta} P_i)
\]

\[
= \frac{1}{2} \left[ (d - 1)^2 \sum_{i=1}^{d^2-1} \text{Tr}(\rho^\alpha F_i \rho^{1-\alpha-\beta} F_i) \right] - \left( \frac{1}{d^2} + t^2(d + 1)^2 \sum_{i=1}^{d^2-1} \text{Tr}(\rho^\alpha F_i \rho^{1-\alpha} F_i) \right) + \frac{1}{2} \left( \text{Tr}(\rho^\alpha F_i \rho^{1-\beta} F_i) - \text{Tr}(\rho^\beta F_i \rho^{1-\alpha} F_i) \right) = \frac{(ad^2 - 1)}{d(d^2 - 1)} Q_{\alpha,\beta}(\rho).
\]

Setting \( a = \frac{1}{d^2} \) in Theorem 3, we obtain the following corollary.

**Corollary 4** The quantum uncertainty relations based on generalized skew information with respect to a SIC-POVM are of the form,

\[
C_{\alpha,\beta}(\rho, \mathcal{P}_{\text{SIC}}) = \frac{Q_{\alpha,\beta}(\rho)}{d(d + 1)}
\]

for \( \alpha, \beta \geq 0 \) and \( \alpha + \beta \leq 1 \).

In particular, taking \( \alpha = \beta = \frac{1}{2} \), we obtain the following corollary corresponding to result in Ref. [33].

**Corollary 5** The coherence with respect to a general SIC-POVM is given by

\[
C(\rho, \mathcal{P}_{\text{GSM}}) = \frac{(ad^3 - 1)(d - (\text{Tr}\sqrt{\rho}))^2}{d(d^2 - 1)}.
\]

By using Lemma 1, we can prove the following theorem.

**Theorem 4** The quantum complementarity relations based on generalized skew information with respect to a general SIC-POVM are
given by
\[ C^{\alpha,\beta}(\rho, \mathcal{P}_{GSM}) \leq \frac{(ad^2 - 1)}{2d(d^2 - 1)}(d - \text{Tr}\rho^\alpha\text{Tr}\rho^{1-\alpha}) \]
for \(\alpha, \beta \in [0, 1]\) with \(\alpha + 2\beta \leq 1\) and \(2\alpha + \beta \leq 1\).

Taking \(a = \frac{1}{d^2}\) in Theorem 4, we obtain the following corollary.

**Corollary 6** The complementarity relations based on generalized skew information with respect to a SIC-POVM are given by
\[ C^{\alpha,\beta}(\rho, \mathcal{P}_{SIC}) \leq \frac{1}{2d(d+1)}(d - \text{Tr}\rho^\alpha\text{Tr}\rho^{1-\alpha}) \]
for \(\alpha, \beta \in [0, 1]\) with \(\alpha + 2\beta \leq 1\) and \(2\alpha + \beta \leq 1\).

**Example 1** Consider the Werner state,
\[ \rho_w = \frac{1}{d} \begin{pmatrix} \frac{1}{2}(3 - 2p) & 0 & 0 & 0 \\ 0 & \frac{1}{2}(4p - 3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(4p - 3) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(3 - 2p) \end{pmatrix}, \]
where \(p \in [0, 1]\). \(\rho_w\) is separable when \(p \in [0, \frac{1}{3}]\). Take \(\kappa = 1\) and \(a = \frac{1}{d^2}\). Figures 1 and 2 illustrate the complementarity relations of Eqs. (19) and (25) with different values of \(\alpha\) and \(\beta\), respectively.

![Fig. 1: The C-axis shows the complementarity and its upper bounds. Red solid (dotted) line represents the value of the right hand side of Eq. (19) with \(\alpha = \frac{5}{12} (\alpha = \frac{1}{3})\) and \(\beta = \frac{1}{6} (\beta = \frac{1}{4})\) for \(\rho_w\); blue dashed (dotdashed) line represents the value of the left hand side of Eq. (19) with \(\alpha = \frac{5}{12} (\alpha = \frac{1}{3})\) and \(\beta = \frac{1}{6} (\beta = \frac{1}{4})\) for \(\rho_w\).](image1)

![Fig. 2: The C-axis shows the complementarity and its upper bounds. Red solid (dotted) line represents the value of the right hand side of Eq. (25) with \(\alpha = \frac{5}{12} (\alpha = \frac{1}{3})\) and \(\beta = \frac{1}{6} (\beta = \frac{1}{4})\) for \(\rho_w\); blue dashed (dotdashed) line represents the value of the left hand side of Eq. (25) with \(\alpha = \frac{5}{12} (\alpha = \frac{1}{3})\) and \(\beta = \frac{1}{6} (\beta = \frac{1}{4})\) for \(\rho_w\).](image2)

**Conclusions.** Based on GWYD skew information, we have derived the uncertainty and complementarity relations with respect to MUMs and general SIC-POVMs, which include some uncertainty relations and complementarity relations in [32] and [33] as special cases. It is worth noting that the uncertainty and complementarity relations we obtained are all state-dependent. Our approaches and results may shed some new light on further investigations on quantum coherence and complementary measurements.

***

This work was supported by National Natural Science Foundation of China (Grant Nos. 11701259, 11461045, 11771198, 11675113); Jiangxi Provincial Natural Science Foundation (Grant No. 20202BAB201001); Beijing Municipal Commission of Education (KZ201810028042); Beijing Natural Science Foundation (Grant No. Z190005); Academy for Multidisciplinary Studies, Capital Normal University; Shenzhen Institute for Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen 518055, China (No. SIQSE202001).
REFERENCES

[1] Heisenberg W., Z. Phys., 43 (1927) 172.
[2] Robertson H. P., Phys. Rev., 34 (1929) 163.
[3] Deutsch D., Phys. Rev. Lett., 50 (1983) 631.
[4] Maassen H. and Uffink J. B. M., Phys. Rev. Lett., 60 (1988) 1103.
[5] Rastegin A. E., Eur. Phys. J. D, 67 (2013) 269.
[6] Zhang F. and Li Y., Sci. China-Phys. Mech. Astron., 61 (2018) 080312.
[7] Ma Z. H., Chen Z. H. and Fei S.-M., Sci. China-Phys. Mech. Astron., 60 (2017) 010321.
[8] Li J. and Fei S.-M., Entropy, 20 (2018) 132.
[9] Chen B., Fei S.-M. and Long G. L., Quantum Inf. Process., 15 (2016) 2639.
[10] Luo S. and Sun Y., Phys. Lett. A, 383 (2019) 2869.
[11] Fu S., Sun Y. and Luo S., Quantum Inf. Process., 18 (2019) 258.
[12] Wu Z., Zhang L., Wang J., Li-Jost X.-Q. and Fei S.-M., Int. J. Theor. Phys., 59 (2020) 704.
[13] Chen B., Cao N.-P., Fei S.-M. and Long G.-L., Quantum Inf. Process., 15 (2016) 3909.
[14] Qin H.-H., Zhang T.-G., Jost L., Sun C.-P., Li-Jost X.-Q. and Fei S.-M., Phys. Rev. A, 99 (2019) 032107.
[15] von Neumann J., Mathematical Foundations of Quantum Mechanics (Princeton University Press, Princeton) 1955.
[16] Lüders G., Annalen der Physik, 443 (1950) 322.
[17] Zhang D.-J. and Gong J. B., Phys. Rev. Research, 2 (2020) 023418.
[18] Renes J. M., Blume-Kohut R., Scott A. J. and Caves C. M., J. Math. Phys., 45 (2004) 2171.
[19] Appleby D. M., Opt. Spectrosc., 103 (2007) 416.
[20] Zhang D.-J., Liu C. L., Yu X.-D. and Tong D. M., Phys. Rev. Lett., 120 (2018) 170501.
[21] Baumgratz T., Cramer M. and Plenio M. B., Phys. Rev. Lett., 113 (2014) 140401.
[22] Zhu X. N., Jin Z. X. and Fei S.-M., Quantum Inf. Process., 18 (2019) 179.
[23] Xu J. W., Shao L. H. and Fei S.-M., Phys. Rev. A, 102 (2020) 012411.
[24] Luo S. and Sun Y., Phys. Rev. A, 96 (2017) 022130.
[25] Singh U., Pati A. and Bera M., Mathematics, 4 (2016) 47.
[26] Yuan X., Bai G., Peng T. and Ma X., Phys. Rev. A, 96 (2017) 032313.
[27] Cheng S. and Hall M. J. W., Phys. Rev. A, 92 (2015) 042101.
[28] Pan F., Qiu L. and Liu Z., Sci. Rep., 7 (2017) 43919.
[29] Luo S. and Sun Y., Phys. Rev. A, 98 (2018) 012113.
[30] Wu Z., Zhang L. and Fei S.-M., Quantum Inf. Process., 19 (2020) 154.
[31] Rastegin A. E., Front. Phys., 13 (2018) 130304.
[32] Luo S. and Sun Y., Commun. Theor. Phys., 71 (2019) 1443.
[33] Chen B. and Fei S.-M., Average coherence with respect to complementary measurements. arXiv:1908.03863.
[34] Wigner E. P. and Yanase M. M., Proc. Natl. Acad. Sci. USA, 49 (1963) 910-918.
[35] Chen P. and Luo S., Front. Math. China, 2 (2007) 359.
[36] Luo S., Phys. Rev. A, 73 (2006) 022324.
[37] Li X., Li D., Huang H., Li, X. and Kwek L. C., Eur. Phys. J. D, 64 (2011) 147.
[38] Wooters W. K., Found. Phys., 16 (1986) 391.
[39] Wootters W. K. and Fields B. D., Ann. Phys., 191 (1986) 363.
[40] DurT T., Englert B.-G., Bengtsson I. and ˙Zyczkowski K., Int. J. Quantum Inf., 8 (2010) 535.
[41] Kalev A. and Gour G., New J. Phys., 16 (2014) 053038.
[42] Chen B. and Fei S.-M., Commun. Theor. Phys., 72 (2020) 065106.
[43] Luo S., Theor. Math. Phys., 151 (2007) 693.
[44] Gour G. and Kalev A., J. Phys. A: Math. Theor., 47 (2014) 335302.