The Electric Field of a Uniformly Charged Non-Conducting Cubic Surface

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Abstract

As an integrative and insightful example for undergraduates learning about electrostatics, we discuss how to use symmetry, Coulomb’s law, superposition, Gauss’s law, and visualization to understand the electric field $E(x, y, z)$ produced by a non-conducting cubic surface that is covered with a uniform surface charge density. We first discuss how to deduce qualitatively, using only elementary physics, the surprising fact that the electric field inside the cubic surface is nonzero and has a complex structure, pointing inwards towards the cube center from the midface of each cube and pointing outwards towards each edge and corner. We then discuss how to understand the quantitative features of the electric field by plotting an analytical expression for $E$ along symmetry lines and on symmetry surfaces. This example would be a good choice for group problem solving in a recitation or flipped classroom.
FIG. 1: A three-dimensional charge distribution that consists of a uniform positive surface charge density $\sigma$ on a non-conducting cubic surface. The center of the cube coincides with the origin $O$ of the $xyz$ Cartesian coordinate system used in this paper. The cube has sides of length $L = 1\, \text{m}$ so that the eight vertices lie at the coordinates $(x, y, z) = (\pm 1/2, \pm 1/2, \pm 1/2)$ in meters.

I. INTRODUCTION

An undergraduate learning about electrostatics has to learn many challenging concepts such as symmetry (of a charge distribution or of a field), a three-dimensional electric vector field $\mathbf{E}(x, y, z)$, a three-dimensional electric potential scalar field $V(x, y, z)$, and related ideas such as superposition, Coulomb’s law, and Gauss’s law. Introductory and upper-level textbooks explain these concepts and provide examples that illustrate how to solve problems related to these concepts, but many physics curricula, some of which cover material at the rapid rate of one textbook chapter per week, lack the time to give students sufficient practice to master these concepts. Especially lacking are integrative examples that use multiple concepts (say spanning several chapters of a textbook) and that use different problem solving strategies (say qualitative, analytical, numerical, and visual).

To help students gain a deeper understanding of electrostatics and to improve their qualitative and quantitative problem solving skills, we discuss an electrostatic problem that is conceptually and technically appropriate for students taking an introductory physics course or an upper-level course on electricity and magnetism. The problem concerns understanding the electric field produced by a symmetric three-dimensional charge distribution that consists of a non-conducting cubic surface that is covered with a uniform positive charge density $\sigma$ (see Fig. 1).

This example is insightful in several ways. First, it shows students how to use qualitative
reasoning based on symmetry, superposition, Coulomb’s law, and Gauss’s law to deduce the properties of an electric field that is substantially more complicated than what most undergraduate physics courses discuss. For example, we explain how to deduce qualitatively that the electric field inside the charged non-conducting cubic surface points inwards near the middle of cube faces and points outwards towards the edges and vertices. The fact that the electric field inside this surface is nonzero by itself will likely be surprising to students because this example seems similar to other introductory physics examples for which the electric field is zero inside a symmetric charge distribution. (One example is the electric field inside a spherical surface covered with a uniform surface charge density, while a second example is the field inside a charged conducting cubic surface, see Section II E below.) That many interesting insights can be deduced qualitatively using elementary physics is one of the main contributions of this paper.

It turns out that one can obtain an explicit, although long, analytic expression for the potential \( V(x, y, z) \) everywhere in space for a uniformly charged cubic surface (see Appendix A and Ref. [8]). Via the relation \( \mathbf{E} = -\nabla V \), the expression for \( V \) leads to an even longer explicit expression for the electric field. We use the analytical expression for \( \mathbf{E} \) to confirm and then to extend the insights obtained by qualitative reasoning. Because the expression for \( \mathbf{E} \) is too long to provide insight, we demonstrate the value of plotting the analytical expression for \( \mathbf{E} \) on certain lines and areas that have symmetries such that the electric field is everywhere parallel to these lines and areas. Continuity of the electric field then allows one to extend knowledge of the field on the symmetry lines and surfaces to the full three-dimensional interior, and so provides a rather complete understanding of the interior electric field.

We envision this paper being used in several ways by physics instructors. Simplest is to use parts of this paper as supplementary homework problems or to stimulate thinking and discussion in class by using parts of this paper for interactive questions, say via anonymous polling [9,10]. But we feel that this paper will be most useful as a group learning project in a recitation or flipped class, in which students in small groups collaborate to work through portions of this paper in guided steps.

The rest of this paper is organized as follows. In Section II we explain how to use qualitative reasoning to deduce key properties of the electric field inside the uniformly charged cubic surface. This section is divided into subsections that represent pedagogical milestones that a group of students could discuss and solve in succession. In Section III we use an
analytical expression for the electric field of this cubic surface to confirm the qualitative arguments of Section [II] and then discuss some new quantitative features of the electric field and of the potential \( V \). In Section [IV] we summarize our results and discuss some of their pedagogical implications. Appendix [A] discusses the analytical expression for the potential and electric field of the uniformly charged cubic surface, and uses the analytical expression for \( \mathbf{E} \) to show that it diverges at the edges of the cubic surface.

II. QUALITATIVE INSIGHTS

In this section, we use qualitative reasoning based on symmetry, superposition, continuity, Coulomb’s law, and Gauss’s law to deduce the properties of the electric field \( \mathbf{E} \) produced by a non-conducting cubic surface that is covered with a uniform positive surface charge density \( \sigma > 0 \). A surprising conclusion is that the interior electric field is not zero and has a complicated geometry. Section [III] then uses an analytical expression for \( \mathbf{E} \) to confirm and to extend these qualitative insights by providing quantitative details.

A. Only the interior electric field is interesting to consider

A first point is that only the electric field interior to the uniformly charged non-conducting cubic surface is interesting to explore. This is because the electric field exterior to the cubic surface is qualitatively similar to the familiar electric field of a positive point charge at the center of the cubic surface: \( \mathbf{E} \) points away from the center of the cubic surface, and it diminishes in magnitude with increasing distance from the center of the cube.

One way to see this is to observe that, for any point \( P \) outside the cubic surface, one can find a plane such that the point \( P \) is on one side of the plane and the entire cubic surface is on the other side. (For example, the line segment connecting \( P \) and the center \( O \) of the cubic surface must intersect a face of the cube. Any plane outside the cubic surface that is parallel to that face and that intersects the interior part of the line segment \( PO \) will suffice.) Since the infinitesimal point charges that make up the cubic surface then lie on one side of the plane, the total electric field \( \mathbf{E} \) at point \( P \) due to these point charges must point away from the cubic surface (although generally not radially, i.e., not parallel to the line segment \( PO \)).
FIG. 2: A charge distribution $\rho$ (here a sphere filled with a spherically symmetric charge density $\rho(r)$) has a mirror symmetry plane $\Pi$ if the plane divides the charge distribution into two distinct sets such that, for every point $P_1$ in one set, there is a corresponding point $P'_1$ in the other set such that $P_1$ and $P'_1$ are mirror images with respect to $\Pi$. The mirror symmetry of $\rho$ implies that electric field $\mathbf{E}$ created by $\rho$ has a mirror symmetry. This means that the electric field vector $\mathbf{E}(P_2)$ at some point $P_2$ and the electric field vector $\mathbf{E}(P'_2)$ at the mirror image point $P'_2$ are themselves mirror images of each other as shown. As the points $P_2$ and $P'_2$ approach the symmetry plane $\Pi$, the corresponding electric field vectors become identical and parallel to $\Pi$.

In contrast, the electric field interior to the cubic surface can be complicated precisely because, at some point $P$ inside the cubic surface, there are contributions to the electric field at $P$ from all possible directions, associated with the infinitesimal point charges making up the surface charge density. However, as we now discuss, these contributions generally do not cancel to give zero.

**B. Electric field vectors are parallel to mirror planes at points on such planes**

A next step in our qualitative analysis is to take advantage of the symmetries of the uniformly charged cubic surface. These symmetries provide a way to deduce quickly and without calculation some information about the direction of the electric field on certain planes and lines, which are then the locations to consider first when trying to understand the electric field. In Section III we will see that these symmetry planes and lines are also good places to plot quantitative information about the electric field.

A charge distribution $\rho(x, y, z)$ is said to have a mirror symmetry plane (or mirror plane for short) if there is a plane $\Pi$ that divides the charge distribution into two distinct parts.
that are mirror reflections of each other with respect to Π (see Fig. 2). This means that for every point \( P_1 \) of the charge distribution that lies on one side of the plane Π, there is a corresponding point \( P'_1 \) (the mirror image of \( P_1 \) with respect to plane Π) on the other side of the plane such that the plane is perpendicular to and bisects the line segment \( P_1P'_1 \). The points \( P_1 \) and \( P'_1 \) are related in the same way that one finds the image \( P'_1 \) of a point \( P_1 \) with respect to a planar mirror via geometric optics.

Most charge distributions discussed in introductory physics courses have mirror planes, for example a point charge, a uniformly charged line segment, a cylindrical surface filled with an azimuthally symmetric charge density, a sphere filled with a spherically symmetric charge density, and an infinite uniformly charged plane. As shown in Fig. 3 a uniformly charged cubic surface has nine distinct mirror planes. Three of these planes (Fig. 3(a)) have the property of passing through the center \( O \) of the cube and of passing through the midpoints of four parallel edges. The six other mirror planes (Fig. 3(b)) have the property of passing through the center of the cube and of passing through two pairs of diagonally opposite vertices.

A mirror plane \( \Pi \) of a charge distribution is useful because, at any point \( P \) on such a plane, the electric field vector \( \mathbf{E} \) at \( P \) is parallel to \( \Pi \). This follows from the experimentally observed uniqueness of motions governed by Newton’s second law \( ma = \mathbf{F} \) when applied to a point particle of mass \( m \) and charge \( q \) in an electric field so that the force is given by \( \mathbf{F} = q\mathbf{E} \). If a charged point particle is placed at some point \( P_2 \) in some electric field with zero initial velocity (see Fig. 2), the particle is always observed to move along a unique path, which implies that there can only be one force vector at \( P_2 \) and so only one electric field vector \( \mathbf{E} = \mathbf{F}/q \). It then follows that the electric field at any point on a mirror plane cannot have a component perpendicular to the plane (i.e., \( \mathbf{E} \) must be parallel to \( \Pi \)). Otherwise, the mirror symmetry of the charge distribution would imply that there must be two distinct electric field vectors at \( P \) that have equal and opposite components perpendicular to the plane, a contradiction.

Further, for any point \( P \) that lies on a line \( L \) that is the intersection of two non-parallel mirror planes of the same charge distribution, the electric field at \( P \) must be parallel to the line \( L \). This follows since the electric field vector at any such point has to be parallel to two non-parallel planes at the same time, which is possible only if the vector is parallel to their line of intersection. For example, in Fig. 3(c), the mirror plane \( y = 0 \) that contains the
FIG. 3: (a) The square IJKL lies in one of three mirror planes of the cube that passes through the cube’s center \( O \) and through the midpoints of four parallel edges. (b) The rectangle ACGE lies in one of six mirror planes that pass through the cube’s center \( O \) and through two pairs of diagonally opposite corners. (c) The line segment \( M_1OM_2 \) is one of three symmetry line segments that pass through the cube’s center and that connects the midpoints \( M_1 \) and \( M_2 \) of two opposing faces. (d) The line segment \( AOG \) is one of four symmetry line segments that passes through \( O \) and through two diagonally opposite corners, here \( A \) and \( G \).

green square intersects the mirror plane \( x = 0 \) that contains the purple square at the line corresponding to the \( z \) axis, and so the electric field anywhere on the \( z \) axis is parallel to \( \hat{z} \), and so has the form \( \mathbf{E}(0,0,z) = (0,0,E_z) \). Similarly, in Fig. 3(d), we see that the mirror plane containing the blue rectangle \( ACGE \) and the mirror plane containing the orange rectangle \( ADGF \) intersect at a line that contains the diagonal line segment \( AOG \), and so the electric field anywhere on the line passing through \( AOG \) is parallel to this line.

If a charge distribution has three mutually non-parallel mirror planes that intersect at some point \( P \), the electric field must be zero at \( P \). This follows since the electric field at \( P \) has to be parallel to three distinct mirror planes, which is only possible if the vector is the zero vector. For example, we see in Fig. 3(c) that the center of the cubic surface \( O \) lies
at the intersection of the mirror planes \( x = 0, \ y = 0, \) and \( z = 0 \) and so the electric field at \( O \) lies entirely within each of these planes simultaneously which requires all three of its components to be zero.

A last observation is that, because an electric field varies continuously except where the charge density changes discontinuously (for the charged cubic surface, \( \mathbf{E} \) is continuous along any path that does not cross the surface), the internal electric field of the charged cubic surface close to a mirror plane must be almost parallel to that plane. Similarly, the electric field close to the intersection of two mirror planes must be nearly parallel to their line of their intersection, and the electric field near a point that is common to three distinct mirror planes must be close to zero in magnitude. So symmetry and continuity substantially constrain the qualitative properties of the electric field near regions of symmetry associated with the cubic surface.

C. The interior electric field near the midpoints of faces points inwards, towards the cube’s center

Now that we understand that the electric field at a point on a mirror plane is parallel to the mirror plane, we begin to obtain a qualitative understanding of the electric field inside a uniformly charged cubic surface. We claim that the internal electric field close to any midpoint of a face must point inwards towards the center \( O \) of the cubic surface. When this insight is combined with Gauss’s law in the next subsection, we will see that there must be places inside the cubic surface where the electric field points outwards, from the cube’s center towards the cube’s edges and corners.

To simplify some estimates that we make in the following discussion, we assume that the cubic surface has unit length and unit surface charge density in SI units so that \( L = 1 \) m, and \( \sigma = 1 \) C/m\(^2\). With these values, the total charge on any face of the cubic surface is \( Q = \sigma L^2 = 1 \) C and the total charge of the cubic surface is \( Q_{\text{tot}} = 6Q = 6 \) C. We will also occasionally not indicate the physical units when referring to spatial coordinates, which should be understood to always be in units of meters.

To understand why the internal electric field must point inwards near the center of any face, we use our knowledge of the electric field of a point charge and of the electric field of an infinite uniformly charged plane. In Fig. 3(c), let us consider a point \( P \) that lies just above
the bottom midpoint \( M_2 = (0, 0, -1/2) \) on the line segment \( M_1 O M_2 \). If \( P \) is sufficiently close to \( M_2 \), the electric field at \( P \) due to the bottom face \( EFGH \) will approximately be equal to the electric field of an infinite plane with uniform charge density \( \sigma \). (See for example page 762 of Ref. 4, where this is shown analytically to be the case for a point sufficiently close to and above the center of a uniformly charged disk.) Since the charge density \( \sigma \) is positive, we conclude that the electric field \( \mathbf{E}(P)_{EFGH} \) at \( P \) due to the face \( EFGH \) is given approximately by

\[
\mathbf{E}(P)_{EFGH} \approx \frac{\sigma}{2\epsilon_0} \hat{z} \approx 2\pi K \hat{z},
\]  

(1)

where we used the fact that the vacuum permittivity \( \epsilon_0 \) and Coulomb’s constant \( K \) are related by \( 1/\epsilon_0 = 4\pi K \). To one significant digit in multiples of \( K \), the electric field at \( P \) due to the bottom face has magnitude \( E \approx 6K \) and points in the positive \( z \)-direction, away from the charged face and towards \( O \).

Now let us consider the contribution \( \mathbf{E}(P)_{ABCD} \) to the electric field at \( P \) from the top face \( ABCD \) in Fig. 3(c). Since \( P \) lies on the intersection of two distinct mirror planes of the face \( ABCD \), the electric field at \( P \) due to \( ABCD \) must be parallel to the \( z \) axis. Further, since \( ABCD \) is covered with a positive charge density, we conclude that \( \mathbf{E}(P)_{ABCD} \) must point in the \( -\hat{z} \) direction. Now if square \( ABCD \) were extended to be an infinite uniformly charged plane \( z = 1/2 \) with density \( \sigma \) that contains square \( ABCD \), then the electric field at \( P \) due to this plane would be exactly opposite in direction and equal in magnitude to the electric field due to the bottom face and the electric field at \( P \) would be approximately zero. But since \( ABCD \) is a finite portion of an infinite charged plane, the electric field at \( P \) due to \( ABCD \) must be smaller in magnitude than the electric field magnitude \( 2\pi K \) of an infinite plane with the same charge density. We conclude that the total electric field at a point \( P \) that is sufficiently close to \( M_2 \) and that is due to the top and bottom faces of the cube must point upwards, towards the cube’s center \( O \).

We can get a quick estimate of the approximate magnitude of \( \mathbf{E}(P)_{ABCD} \) by approximating the distributed charge on the face \( ABCD \) with a single point charge with total charge \( Q = 1 \text{ C} \) at the center \( M_1 \) of this face. Coulomb’s law then tells us that the magnitude of the electric field at \( P \) due to \( ABCD \) is

\[
E(P)_{ABCD} \approx \frac{KQ}{L^2} = \frac{K[L^2\sigma]}{L^2} = K\sigma = K,
\]  

(2)

since we have assumed \( \sigma = 1 \text{ C/m}^2 \).
This simple estimate is too big for two reasons. (And indeed, the quantitative calculations of Section III show that the value of the electric field at \( M_2 \) due to the top face is \( 0.8K \) to one digit, so that the estimate Eq. (2) is too large by about 20%.) First, we can think of approximating the face \( ABCD \) with a single point charge \( Q = 1 \) C at its center as being achieved by relocating each infinitesimal point charge \( dq \) on that face in turn to the center of that face. But changing the position of a point charge \( dq \) to the face’s center moves the charge closer to \( P \) because the center of that face is closer to \( P \) than all the other points of the face. Since smaller distances \( d \) in Coulomb’s law \( E \propto \frac{1}{d^2} \) imply bigger electric fields, relocating all the point charges on a face to its center makes the estimated total electric field at point \( P \) larger than the actual value. Relocating a point charge \( dq \) on \( ABCD \) to the face’s center also changes the orientation of the electric field vector at \( P \) due to \( dq \) to be more parallel to the \( z \)-axis, which increases the component along the \( z \) axis compared to its original component.

We now know that the total field at a point \( P \) near \( M_2 \) due to only the top and bottom faces gives an upwards vertical electric field of approximate magnitude \( 6K - 1K \approx 5K \). However, since a point \( P \) near \( M_2 \) lies below the symmetry plane \( z = 0 \) that divides the cubic surface horizontally in half, there are more infinitesimal charges on each vertical side that lie above \( P \) than below \( P \). This implies that the net electric field at \( P \) due to the four side faces must have a net downwards \( z \)-component that also needs to be considered when estimating the total electric field vector \( E(P) \) at \( P \).

We can again obtain a quick estimate by replacing each vertical side face with a single point particle of charge \( Q = 1 \) C at its center as shown in Fig. 3(a) and by adding up the four electric field vectors at \( P \) due to these four point charges. It is harder now to determine whether this single-point-charge approximation will cause an overestimate or underestimate of the exact value \( E(P)_{\text{side-faces}} \) since, upon relocating an infinitesimal charge \( dq \) on a side face to the center of that side face, the distance to \( P \) can become larger or smaller, and the electric field vector at \( P \) due to \( dq \) can become more or less parallel to the \( z \) axis.

In any case, by approximating the four side faces with point charges \( Q = 1 \) C at their centers, by defining \( \mathbf{X}_P = (0, 0, -1/2) \) to be the position vector of point \( P \) at the bottom midface and \( \mathbf{X}_{F_1} = (1/2, 0, 0) \) to be the position vector of the point charge at the front midface of the cubic surface, by defining \( d_{PF_1} = \|\mathbf{X}_P - \mathbf{X}_{F_1}\| = 1/\sqrt{2} \) to be the distance between the two vectors, by defining \( \mathbf{r}_{PF_1} = (\mathbf{X}_P - \mathbf{X}_{F_1})/d_{PF_1} = (-1/\sqrt{2}, 0, -1/\sqrt{2}) \) to be
FIG. 4: The electric field $\mathbf{E}$ at a point $P$ at the midpoint of a charged cubic surface (here the top) due to a side face (here the front face) can be estimated by replacing the distributed charge of the front face with one or more point charges. In (a), a single point charge with $Q = 1\, \text{C}$ at the center of the face is used. Panels (b) and (c) show two ways that the charge on the front panel could be approximated by two equal point charges with charge $Q = 1/2\, \text{C}$, by dividing the face into two equal smaller rectangles and replacing the charges on each rectangle by a point charge at its midpoint.

the unit vector that points from the front midface to $P$, and finally by observing that, by symmetry, the final $z$-component of the electric field at $P$ due to the four sides is four times the $z$ component from any one side, we find that

$$E(P)_{\text{side-faces}} \approx 4 \left( \hat{z} \cdot \frac{KQ}{d_{PF_i}^2} \hat{r}_{PF_i} \right) \approx -4\sqrt{2}K \hat{z} \approx -5.7K \hat{z}. \quad (3)$$

So by approximating the bottom face with an infinite plane and the other five side faces with point charges $Q = 1\, \text{C}$ at their centers, we get a total electric vector near $M_2$ in the $-z$ direction of length $2\pi K - 1.0K - 5.7K \approx -0.4K$ pointing downwards. This suggests that the electric field just inside the cubic surface and near the midpoint of a face points outwards, away from the origin $O$.

However, since the estimate $E_z \approx (2\pi - 1.0 + 5.7)K \approx -0.4K$ involves cancellations of estimates bigger than $K$ in magnitude to give a final answer smaller than $K$ in magnitude, we need to worry about whether using a point charge to approximate the electric field of a charged face is sufficiently accurate. Since the four side faces contribute the second largest amount to the total electric field at $P$, it would be useful to explore whether a more careful approximation of the electric field from each side face might affect the sign of $E_z$. A next step that is easy to compute would be to approximate the surface charge of each side face with two equal point charges with values $q = Q/2 = 1/2\, \text{C}$ as shown in panels (b) and (c)
of Fig. 4. For panel Fig. 4(b), the total electric field at $P$ due to the eight point charges of the sides is given by

$$E_z(P)_{8\text{-point-charges}} \approx 4\hat{z} \cdot \left( \frac{K(Q/2)}{d_{PF_1}^2} \hat{r}_{PF_1} + \frac{K(Q/2)}{d_{PF_2}^2} \hat{r}_{PF_2} \right) \approx -4.9K,$$

(4)

to two digits. Here we define $X_{F_1} = (1/2, 0, 1/4)$ and $X_{F_2} = (1/2, 0, -1/4)$ to be the position vectors of the two point charges with charge $Q/2 = (1/2)$ C on the front face in Fig. 4(b), and then, as before, we define $d_{PF_1} = \|X_P - X_{F_1}\|$, $\hat{r}_{PF_1} = (X_P - X_{F_1})/d_{PF_1}$, and similarly for $d_{PF_2}$ and $\hat{r}_{PF_2}$. A calculation similar to Eq. (4) for Fig. 4(c) gives a value of $E_z \approx -4.7K$ so both arrangements of two equal charges per side lead to the same conclusion, that the magnitude of $E_z(P)$ due to all six faces is about $(2\pi - 1.0 - 4.9)K \approx 0.4K$ pointing in the $+\hat{z}$ direction, which represents a reversal in direction of the electric field near $M_2$ compared to a one-charge-per-face approximation.

One would presume that this new estimate using two charges per face is more accurate than an estimate based on one charge per face since two points charges per face should do a better job of getting the magnitude and direction of the face’s electric field at $M_2$ correct. Other calculations using more point charges per face indeed confirm that the electric field near the midpoint of a face indeed points towards the center. It is possible to calculate the exact total electric field at a point $P$ quite close to $M_2$ (see Appendix A) and one finds that $E(P) \approx 1.9K\hat{z}$ to two digits. The exact contribution to the electric field at $M_2$ due to the side faces is therefore $3.6K$ in the $-\hat{z}$ direction, compared to the estimates of $4.9K$ or $4.7K$ for panels (b) and (c) of Fig. 4. So two point charges per side face get the sign right but produce an error in magnitude of the electric field at $P$ of about 40%.

We conclude that the internal electric field points towards the center of the cubic surface for points sufficiently close to the midpoint of any face. From this, we can deduce what is the qualitative form of the electric field $E$ along the entire line segment $M_1OM_2$. (By symmetry, this will also be the qualitative form of the electric field along the other two line segments connecting midpoints of opposite faces.) The symmetry arguments of Sec II B imply that $E$ at any point on $M_1OM_2$ is parallel to $M_1OM_2$ and so $E$ must have the form $(0, 0, E_z)$ on this line segment. The $z$-component $E_z(z)$ is positive near $M_2$ and, by symmetry, must be negative near $M_1$ and we further know that $E_z(0) = 0$ since the center of the cube lies at the intersection of at least three distinct mirror planes and so $E$ must vanish there. We thus expect $E_z(z)$ to be a smoothly varying odd function of $z$, $E_z(-z) = -E_z(z)$, that is
FIG. 5: (a) The cubic surface \( S = A'B'C'D'E'F'G'H' \) is a Gaussian surface that is concentric with the charged cubic surface \( ABCDEFGH \) and that lies just within the charged cubic surface. The total charge enclosed by \( S \) is zero which implies, by the symmetry of the cubic surface, that the flux through each face such as \( A'B'E'F' \) is zero.

positive for \( z < 0 \) and that decreases through zero to negative values for \( z > 0 \). Further, the magnitude of \( E_z \) along \( M_1OM_2 \) must always be less than the magnitude 2\( \pi K \approx 6K \) of the electric field produced by an infinite plane with surface charge density \( \sigma = 1 \text{ C/m}^2 \). The quantitative calculation Fig. III(a) of Section III over the range \(-1/2 \leq z \leq 1/2 \) shows that this qualitative thinking is correct.

D. Gauss’s law implies that there are places where the interior electric field points outwards, towards edges and towards vertices

We now combine the insight that the internal electric field points inwards near the center of faces with a qualitative application of Gauss’s law to deduce that there have to be locations inside the charged cubic surface where the electric field points away from the cube’s center \( O \). Consider a Gaussian cubic surface \( S = A'B'C'D'E'F'G'H' \) that is concentric with and lies just inside the charged cubic surface \( ABCDEFGH \) as shown in Fig. □. Because there is no charge inside the surface \( S \), Gauss’s law gives:

\[
\frac{Q_{\text{enclosed}}}{\epsilon_0} = 0 = \Phi_{\text{total}} = \int_S \mathbf{E} \cdot d\mathbf{A} = 6 \int_\square \mathbf{E} \cdot d\mathbf{A} = 6\Phi_{\square},
\]

so the flux \( \Phi_{\square} \) through any face \( \square \) of the cubic surface \( S \) is zero. Here we have used the symmetry of the cube to deduce that the flux \( \Phi_{\square} \) through any face must be the same so the flux integral \( \Phi_{\text{total}} \) over the surface \( S \) is six times the flux \( \Phi_{\square} \) through any one face.
If we consider a particular face of the surface $S$, say the front face $A'B'F'E'$ that lies at a coordinate $x_0$ that is just less than $x = (1/2) \text{m}$, the flux integral for that face becomes a two-dimensional integral of the $x$-component $E_x(x_0, y, z)$ of the electric field vector since the face $A'B'F'E'$ is perpendicular to the $x$ axis:

$$0 = \Phi = \Phi_{A'B'F'E'} = \int_{A'B'F'E'} \mathbf{E} \cdot d\mathbf{A} = \int_{A'B'F'E'} \mathbf{E} \cdot (dA\hat{x}) = \int_{A'B'F'E'} E_x dy \, dz. \quad (6)$$

The last integral $\int E_x \, dx \, dy$ can be thought of as the limit of a finite sum of values $E_x(x_0, y_i, z_i)\Delta A_i$ over some fine uniform grid of tiny identical square areas $\Delta A_i$ that all have the same area $dA_i = \Delta y \Delta z = \Delta A$. So the flux integral Eq. (6) can be thought of as approximately equal to $\Delta A \sum_i E_x(x_0, y_i, z_i)$, i.e., it is proportional to the sum of the $x$-components of the electric field values over the face $A'B'F'E'$. Since this sum is zero by Eq. (6), we conclude that the values of $E_x$ cannot be everywhere positive or everywhere negative, else the sum $\sum_i E_x(x_0, y_i, z_i)$ would be respectively positive and negative, a contradiction.

But we know from Sec. II C that, near the center of the face $ABFE$ of the charged cubic surface, the interior electric field points inwards towards the origin $O$. Since the face $ABFE$ is perpendicular to the $x$-axis and lies at the coordinate $x = 1/2 \text{m}$, this specifically implies that $E_x$ must be negative near the center of $ABFE$. But the electric field $\mathbf{E}$ varies continuously everywhere inside the cubic surface. (Only along a path that crosses the charged surface would $\mathbf{E}$ change discontinuously.) Provided that the face $A'B'F'E'$ is sufficiently close to $ABFE$, continuity of $E_x$ implies that $E_x$ must also be negative over some finite region near the center of the face $A'B'F'E'$. But then the only way that Eq. (6) can hold is for $E_x$ to be positive on $A'B'F'E'$ in regions away from the middle of the face so that the negative and positive values of $E_x$ over the entire face add to zero. We conclude that there must be points inside the charged cubic surface where the electric field points away from the center of the cube. This immediately implies that the interior electric field must have a complicated structure, pointing inwards in some locations (near the middle of each face) and outwards in other locations.

A simple way that $E_x$ could be negative near the middle of $A'B'F'E'$ and positive away from the middle consistent with the symmetry of a cube would be for $E_x$ to be negative in some square-like region near the face center and positive elsewhere on the face. The quantitative calculations of Sec. III show that this simplest case is what actually occurs, see
Fig. 9 below.

If we assume this simplest case, then we can understand qualitatively that the electric field inside the cubic surface must actually point towards edges or towards corners for points near edges or near corners. For example, for interior points close to the points $A'$, $B'$, $F'$, and $E'$ in Fig. 5, the components of $E$ point outwards on three planes parallel to respective $y = 0$, $x = 0$, and $z = 0$. This tells us that the electric vectors near these points must point outwards towards the corners. These qualitative insights are confirmed by our quantitative discussion in Section III.

The qualitative arguments in this section only hold for interior Gaussian cubic surfaces $A'B'C'D'E'F'G'H'$ whose faces are sufficiently close to the faces of the charged cubic surface. How the electric field varies more deeply in the cube's interior cannot be worked out qualitatively and one has to turn to quantitative calculations to understand the bigger picture. The quantitative calculations show that the arguments of this subsection hold generally for all interior cubic Gaussian surfaces: everywhere in the interior, the electric field points inwards along the faces of a Gaussian cube and outwards near the edges and corners of the Gaussian cube.

E. A qualitative comparison of the non-conducting cubic surface with three similar problems

Before discussing our quantitative results, we compare our qualitative conclusions about the nonzero electric field inside a uniformly charged non-conducting cubic surface with three related problems for which the electric field inside some interior region of a symmetric charge distribution is zero. This comparison helps to clarify why the electric field inside the non-conducting charged cubic surface is nonzero.

First we ask: why is the electric field nonzero inside the charge distribution $\sigma_{\text{cube}}$ consisting of a uniformly charged cubic surface while the electric field is zero everywhere inside the charge distribution $\sigma_{\text{sphere}}$ consisting of a uniformly charged spherical surface? At first glance, these seem to be similar problems since the charge distributions are both highly symmetric.

One answer is that a spherical surface has many more symmetries than a cubic surface. For example, instead of having nine mirror planes like $\sigma_{\text{cube}}$ (see Fig. 3), the distri-
bution $\sigma_{\text{sphere}}$ has infinitely many mirror planes since any plane passing through the center of the spherical surface is a mirror plane of $\sigma_{\text{sphere}}$. These infinitely many mirror planes allow one to show\textsuperscript{12} that the electric field $\mathbf{E}$ inside $\sigma_{\text{sphere}}$ is radial, $\mathbf{E} = E\hat{r}$, and further that the electric field magnitude $E$ depends only on radius, $E = E(r)$. These two facts then lead to the usual argument given in introductory physics textbooks\textsuperscript{4}, that the flux integral $\Phi = \int \mathbf{E} \cdot d\mathbf{A}$ in Gauss’s law, applied to a spherical Gaussian surface of radius $r$ concentric with and inside the uniformly charged spherical surface, becomes a simple product $\Phi = E(r)A$ and so the vanishing of the flux (since there is no charge inside the spherical Gaussian surface) implies the vanishing of the electric field. In contrast, the charge distribution $\sigma_{\text{cube}}$ does not have enough symmetry for one to conclude that the interior electric field is radial (which it isn’t), so the flux integral cannot be written as the product of some area times some constant electric field magnitude.

We next consider an empty cubic region that lies within a symmetric charge distribution that consists of three identical parallel pairs of uniformly charged infinite non-conducting planes, see Fig. 6. This figure can be obtained by extending each face of the cubic surface in Fig. 1 to an infinite plane that contains that face and that has the same surface charge density $\sigma$. Since the electric field due to an infinite charged plane is uniform on a given side of the plane and points away from the plane if $\sigma > 0$, we conclude that the electric field $\mathbf{E}$ must be zero everywhere in the cubic region of Fig. 6 since, at any point $P$ inside the cube, the electric field vectors from opposing planes are equal and opposite and so cancel exactly. The nonzero internal electric field of Fig. 1 therefore arises from the finite size of the faces of the cubic surface, which in turn implies that each face produces a nonuniform electric field, that decreases in magnitude with increasing distance from a face. Since a planar charge distribution in introductory physics textbooks is often represented visually by a finite rectangle contained in the plane, it is easy for students to conclude incorrectly that the electric field inside the cubic surface Fig. 1 must be the same as that of Fig. 6.

The third example that we consider is the electric field inside a charged conducting cubic surface that has the same size and same total charge as the charged non-conducting surface of Fig. 1. Introductory physics textbooks discuss the fact that the electric field everywhere inside a hollow conductor must be zero if that conductor, charged or not, is in electrostatic equilibrium\textsuperscript{1–5}. So we have another situation that is confusing to student who are learning
FIG. 6: A charge distribution consisting of three identical pairs of parallel planes, each with a uniform surface charge density $\sigma$. The front and rear pair of planes are not shown to make the diagram easier to understand. The electric field is zero everywhere inside the central cubic region since the electric fields of opposing planes cancel exactly at any interior point.

about electrostatics: how can one have two identical cubic surfaces with identical total charges, and yet one surface has a nonzero interior electric field while the other surface has a zero interior electric field?

The key insight here is that, on a conducting cubic surface, the charges are mobile and move about (because of mutual repulsion) until, in electrostatic equilibrium, the surface charge density $\sigma$ is nonuniform in just such a way that the surface and interior are all equipotential. If there are no other charges inside the surface, this implies a zero interior electric field and that the external electric field is everywhere locally perpendicular to the cubic surface (except at edges and corners). Introductory physics textbooks mention briefly and qualitatively that charged non-spherical three-dimensional conductors in equilibrium have nonuniform surface charge densities, and that these densities are larger where the radius of curvature of the surface is smaller. But these books and even the commonly used upper-level books on electricity and magnetism do not discuss quantitative examples of nonuniform charge densities.

There is a complimentary insight, which is that the surface of a uniformly charged non-conductor of non-spherical shape cannot be equipotential. This follows since a charged non-spherical conductor has a non-uniform surface charge density.

Figure 7 clarifies these two insights in the context of a cubic surface. Fig. 7(a) shows a numerical approximation to the nonuniform surface charge density $\sigma$ for a charged conducting equipotential cubic surface in electrostatic equilibrium. (Note that this $\sigma$ is what the
FIG. 7: (a) Color density plot of the nonuniform surface charge density $\sigma$ on the faces of a conducting equipotential cubic surface with potential $V = 1$ V, as calculated using a commercial computer code. The density is approximately constant in the blue regions with value $\sigma \approx 8 \times 10^{-11}$ C/m$^2$ and increases near the edges (red brown regions) by a factor of about five. (b) The non-conducting uniformly charged cubic surface is not equipotential, as shown by the surface plot of the potential $V(0.499, y, z)$ of Eq. (A3) on the $yz$-face of a uniformly charged non-conducting cubic surface.

uniform charge density of Fig. 1 would evolve into if the uniformly charged non-conducting cubic surface were to become conducting.) For this calculation with a spatial resolution of $120 \times 120$ grid points per face, about 10% of the area of each face contains about 70% of the total charge per face so most of the surface charge ends up near the edges. As the spatial resolution becomes finer, the surface charge density on the edges increases, corresponding to the fact that the surface charge density mathematically diverges to infinity where sharp edges occur on a charged conductor.

Fig. 7(b) shows the complimentary result of how the potential $V$ is nonuniform over one face of the uniformly charged non-conducting cube. (The potential $V$ is easily evaluated numerically using the analytical expressions Eqs. (A2) and (A3) of Appendix A) The potential $V$ varies modestly in magnitude by about 30 percent, from about 7.7 V at the corners to about 9.8 V at the face’s center. However, it is the gradient of $V$ that determines the electric field via $E = -\nabla V$ and it is not apparent in Fig. 7(b) that the local slope (magnitude of the gradient) is becoming vertical and so diverges at the corners of the non-conducting cubic surface. This divergence is not easy to understand qualitatively, and we demonstrate this fact with a short Mathematica calculation in Appendix A.

We note that the calculation of the surface charge density on a charged cubic conductor is a difficult calculation that is not discussed even in graduate textbooks on electricity and
magnetism like Jackson. The authors only know of numerical calculations using specialized computer codes that have been carried out mainly by electrical engineers who have been interested in the capacitance of a cube-shaped capacitor. Nevertheless, the fact that the surface charge density on a conducting cubic surface in electrostatic equilibrium is not uniform is easily understood by undergraduates and they should be familiar with at least one quantitative example like the one discussed in this paper.

III. QUANTITATIVE INSIGHTS

In this section, we use an analytical expression for the electric field $E(x, y, z)$ of a uniformly charged cubic surface (see Appendix A) to confirm and to extend the qualitative results of the previous section. The expression for $E$ is so long (several pages) that it provides little insight by itself. So a key contribution of this section is a discussion of how to visualize the three-dimensional internal electric field $E$ by plotting it on line segments and on surfaces that lie within mirror planes (see Sect. II B). We can then use the fact that the electric field varies continuously in space along paths that do not cross a charged surface to extend knowledge from the symmetry plots to other spatial regions. We also discuss briefly the properties of the electric potential $V$ as revealed by several line and surface plots.

We note that, for obtaining the results of this section, it was convenient but not essential that an explicit mathematical expression for $E(x, y, z)$ was available. In fact, for general charge distributions, even in one spatial dimension, one would not expect to be able to obtain explicit mathematical expressions for the potential or for the electric field since the integrals analogous to Eq. (A1) in Appendix A can generally not be evaluated in closed form in terms of elementary functions.

Instead, we could have easily obtained all the results of this section by using an elementary numerical method based on discretization and on superposition, in which an arbitrary charge distribution $\rho$ is approximated by some finite set of point charges. (See Sect. II C, where we approximated some faces of the cubic surface with one or two point charges, and also see the brief description of the general numerical method in Appendix A.) Such a numerical method has the advantage over analytical expressions of simplicity and generality since all the details can be understood at the freshman-physics level and the method applies to arbitrarily complicated charge distributions. But one drawback of a numerical method is that
FIG. 8: Three-dimensional vector plot of the electric field \( \mathbf{E} \) inside the uniformly charged non-conducting cubic surface, over the region \(|x|, |y|, |z| < 0.45\). (The plot was created using the Mathematica command \texttt{VectorPlot3D}.) The geometry of \( \mathbf{E} \) is difficult to determine from this plot since interior vectors are obscured by vectors closer to the viewer.

A student would have to know how to write a computer program (or use a previously written program) to carry out a summation over the point charges, and this can be challenging for charge distributions of irregular shape such as the cubic surface. A second drawback is that, for each calculation that uses a numerical method, one has to explore how the accuracy of the result depends on the number and locations of the point charges used to approximate the charge distribution. In this paper, we avoided this last step by using the exact expression for \( \mathbf{E} \).

Whether an exact expression or a numerical method that approximates the exact field is used, there is still the challenge of how to visualize and to understand three-dimensional spatially varying vector fields. One might think that a first step would be to create a vector plot of \( \mathbf{E} \) on some regular 3D grid of points like Fig. 8, but this is not helpful since the vectors near the front of the plot partially hide the vectors that are further back, and the orientations of small vectors are difficult to determine visually. Instead, we have found it useful to plot \( \mathbf{E} \) on line segments and on rectangular regions that lie within mirror planes of the cubic surface since, as discussed in Sect. II.B, the electric field is parallel to such regions so such one- and two-dimensional plots provide complete knowledge of the electric field.
FIG. 9: Surface plot of the $x$-component $E_x(x_0, y, z)$ of the electric field vector on the front face $x_0 = 0.4L$ of a Gaussian cubic surface centered on the origin. This plot is proportional to the local flux $d\Phi = E_x \Delta A$. The orange plane indicates where $E_x$ has the value zero, so the surface above the plane in the middle is where the flux is negative (the electric field points into the Gaussian surface), and the surface below the plane is where the flux is positive (electric field points out of the Gaussian surface). The total flux through this front face is zero.

A. Confirmation of the qualitative insights of Sect. II

We begin our quantitative discussion by using the exact expression for $\mathbf{E}(x, y, z)$ in Appendix A to show in Fig. 9 how the local electric flux $d\Phi(x_0, y, z) = \mathbf{E}(x_0, y, z) \cdot d\mathbf{A}(x_0, y, z) = E_x(x_0, y, z) \Delta y \Delta z$ varies over the front face $A'B'F'E'$ of an interior cubic Gaussian surface (see Fig. 5) whose sides have length 0.8 m. Figure 9 shows a surface plot of $E_x(x_0, y, z)$ over the region $|y|, |z| \leq 0.4$ m. The middle bulge above the orange plane where $E_x = 0$ denotes where the electric field points inwards (where $E_x$ is negative).

This figure confirms the qualitative conclusions of Sects. II C and II D that the interior electric field near the midpoints of the faces points inwards. But now we understand quantitatively how the flux is zero over this face: the component $E_x$ is negative in a larger central area but with smaller magnitude $|E_x|$, while the component $E_x$ is positive in a smaller area but with a larger magnitude outside the central region. The larger number of smaller negative values balance the smaller number of larger positive values, giving a net flux of zero.

Panels (a) and (b) of Figure 10 next show how the electric field varies quantitatively...
along two symmetry lines of the charged cubic surface. Fig. 10(a) shows how $E = E_x \hat{x}$ varies along the line that passes through the two opposing midpoints $(x, y, z) = (1/2, 0, 0)$ and $(-1/2, 0, 0)$. This plot confirms the earlier qualitative conclusion that the electric field points inwards near the midpoints of faces and vanishes at the center $O$, and shows further that the electric field has a small magnitude over much of the interior of cubic surface. (Note the flat approximately zero behavior of $E_x$ for $|x| \lesssim 0.3$.) The magnitude of the interior electric field on this symmetry line is everywhere smaller than the electric field magnitude $2\pi K \sigma \approx 6.2K$ of an infinite plane with the same charge density (the two horizontal tick marks on the $x = 0$ vertical axis denote this magnitude). As one proceeds along this symmetry line from just inside to just outside the cubic surface (so $|x|$ increases from just less than 1/2 to just greater than 1/2), the electric field magnitude $E$ changes discontinuously to a finite value that is larger than the electric field magnitude of an infinite plane of the same charge density. That the electric field magnitude is larger just outside the surface was explained in Sect. IIA.

Figure 10(b) shows a similar plot except along a symmetry line $x(u) = (u/2, u/2, u/2)$ that connects the diagonally opposite corners $(-1/2, -1/2, -1/2)$ and $(1/2, 1/2, 1/2)$. From the sign of the electric field component $E_{\text{diag}} = E \cdot \hat{n}$ along this line, we see that, inside the cubic surface, the electric field everywhere except at the origin points outwards towards the corners, in agreement with our qualitative discussion in Sect. IIB. We further see that the electric field magnitude diverges to infinity at the corners, which we explain briefly mathematically in Sect. A2 of the appendix. In contrast, the electric field is finite in magnitude at the midpoints of the cubic surface (Fig. 10(a)).

In panels (a) and (b) of Fig. 10 we also show by thin black curves the electric field corresponding to a point particle at $O$ with the same total charge $Q = 6$ C as the cubic surface. We expect the external electric field of the cubic surface to converge to that of the point charge for distances sufficiently far from the origin, but the quantitative calculation shows the surprising result that the external field already acts accurately like that of a point charge for distances that are as close as one cube side $L$ from the center.

These quantitative observations are reinforced by figures 11, 12, and 13 which show how the electric field vectors vary over two-dimensional regions that lie within mirror planes of the charged cubic surface. Fig. 11 plots the electric vector field over a square $MNP0$ that lies within the mirror plane $x = 0$. In agreement with Sect. IID and with Fig. 9 the vector
FIG. 10: (a) Plot of $E_x(x, 0, 0)/K$ versus $x$ along the $x$-axis where $K$ is Coulomb’s constant. The two thin black curves are the electric field component $E_x/K = \pm 6/x^2$ at location $(x, 0, 0)$ for a point charge $Q = 6 \text{ C}$ at the origin. The two horizontal tick marks at $E/K = \pm 2\pi \approx \pm 6.3$ on the $x = 0$ vertical line denote the magnitude of the electric field for an infinite plane with the same charge density $\sigma = 1$. (b) Plot of the electric field component $E_{\text{diag}}/K = [\mathbf{E}(u, u, u) \cdot \hat{n}]/K$ parallel to the cube diagonal $\mathbf{x}(s) = s\hat{n}$ where $\hat{n} = (1, 1, 1)/\sqrt{3}$. (c) Potential $V(x, 0, 0)$ plotted along the $x$-axis. The thin black lines correspond to the potential $V = 6/x$ associated with a point charge $Q = 6 \text{ C}$ placed at the center of the cubic surface. (d) Potential $V(u, u, u)$ plotted along the same diagonal as in panel (b).

field plot Fig. 11(a) shows that the electric field points inwards near midpoints $M$ and $P$, and that the electric field points outwards towards the edge $N$.

Because the interior electric field magnitude increases as one approaches the cubic surface and diverges at the edges, the vectors near the center of the cubic surface are not visible when the vectors near the surface are displayed with moderate lengths. This difficulty can be avoided by using a so-called streamline plot, which consists of displaying unit electric field vectors $\hat{\mathbf{E}}$ on some fine regular mesh of spatial points, and then by drawing continuous
FIG. 11: (a) Schematic showing the relation of the plotting region, square $MNOP$ in the mirror plane $x = 0$, to the charged cubic surface. (b) Vector plot of the electric field $\mathbf{E}(0, y, z) = (0, E_y, E_z)$ over the square region $MNOP$ defined by $x = 0$ and by $0 \leq y < 1/2$, $0 \leq z < 1/2$. The electric field points inwards near the centers $M$ and $P$ of the top and side faces, and point outwards towards the edge that passes perpendicular to the point $N$. (c) A streamline plot of $\mathbf{E}$ over the same region reveals the geometry of $\mathbf{E}$ everywhere in the interior. This plot was created using the Mathematica command StreamPlot.

curves (the streamlines) that are locally tangent to these unit vectors, see Fig. 11(b). The streamline plot shows everywhere in the interior how the electric field points inwards near midpoints of faces and then gradually changes orientation to point outwards towards the edge at point $N$.

Finally, Fig. 13 shows a vector plot of the electric field over a square region $TUVO$ in the mirror plane $x = 0$ that includes the field external to the charged cubic surface. As was already shown in Fig. 11(a), the magnitude of the electric field in the plane $x = 0$ is substantially larger just outside the cubic surface than anywhere inside so, on the scale of this plot such that the vectors just outside the surface are of modest size, the interior electric field vectors are barely visible. The vector plot confirms the discussion of Sect. 11A in that
FIG. 12: Plots similar to Fig. 11 but now over the rectangle QRSO that lies in a mirror plane that contains the diagonal line \( \mathbf{x}(u) = (u, u, u) \). The plots were generated by plotting the quantity \( \mathbf{E}(u, u, z) \) over the ranges \( 0 \leq u \leq 0.7 \) and \( 0 \leq z \leq 0.46 \). The vector field \( \mathbf{E} \) now points outward from \( O \) to the edge at \( S \) and outward to the corner at \( R \).

The external electric field is qualitatively similar to that of a point charge at the center \( O \) of the cubic surface, vectors point roughly away from the center (but this is not a radial field) and decreases in magnitude as one moves further away from the cubic surface.

**B. The electric potential \( V \) associated with the charged non-conducting cubic surface.**

In this subsection, we briefly discuss some properties of the electric potential \( V \) associated with the uniformly charged cubic surface, using the analytical expression for \( V \) given in Appendix A. We conclude that for the electric field inside the charged cubic surface, plotting the vector electric field is more insightful than plotting equipotential surfaces of the scalar potential \( V \). This seems to contradict the discussions of introductory physics textbooks\(^{134} \), but most examples considered in those books involve just one or two point charges, or one...
FIG. 13: (a) Relation of the plotting region, square TUV0, to the charged cubic surface. The gray area is the set of points \( x = 0, 0.25 \leq y < 0.75, \) and \( 0.25 \leq z < 0.75. \) (b) Vector plot of the electric field \( \mathbf{E}(0, y, z) \) over the square TUV0. We see that, in the plane \( x = 0, \) the external electric field is substantially stronger than the internal field, that the electric field is particularly large near an edge, and that the external electric field is approximately radial.

or two conductors of simple geometry such that the electric field can be mainly understood by looking at equipotential contours in a single plane.

As was the case for the electric field (see subsection II A), only the potential \( V \) inside the charged cubic surface requires understanding since the potential outside the surface is qualitatively similar to that of a point charge at the center \( O \) of the surface, as shown in Fig. 14.

Panels (c) and (d) of Figure 10, we plot the potential \( V \) along the same two symmetry lines as in panels (a) and (b) of the same figure. Because the electric field is parallel to these symmetry lines for points on these lines (see subsection II B), the negative of the local slope of \( V \) in these plots directly gives the component of \( \mathbf{E} \) parallel to the symmetry line. In both cases, the potential \( V \) asymptotes to the potential \( KQ/d \) of a point charge at \( O \) (thin black curves in panels (c) and (d)) at distances that are close to the surface of the cube.

From Fig. 10(c), we see that the potential inside the cubic surface is nonuniform and varies over the modest range \( 9.6 < V < 9.8, \) but this modest variation in value is misleading since it is the slope of this curve that determines the magnitude of the electric field. The flat central portion of the curve near \( x = 0 \) implies a zero slope and so small electric field
FIG. 14: Equipotential contours of values $V = 9, 8, 7, 6, 5, 4 \cdots 3$ of the potential $V(0, y, z)$ in the plane $x = 0$ external to the charged cubic surface of Fig. 1. The contours change smoothly from squarish contours just outside the charged cubic surface ($V = 9$) to the circular contours of a point charge located at the center $O$ of the cubic surface ($V = 3$).

component, which is consistent with the central region of Fig. 10(c). The discontinuous changes in $E_x$ from negative to positive finite values at $x = \pm L/2$ in Fig. 10(a) at $x = 1/2$ are difficult to see from Fig. 10(a) so plotting the electric field component provides more insight here than plotting $V$.

Fig. 10(d) shows how the potential varies along the symmetry line $x(u) = (u, u, u)$ that passes through the diagonally opposite vertices $\pm(1/2, 1/2, 1/2)$. The range of $V$ inside the cubic surface along this line is somewhat broader than in panel (a), $7.5 \leq V \leq 9.2$. The potential in the middle region has again an approximately zero slope, consistent with the small values of $E$ near the center of the cubic surface. Barely visible at the coordinate values $u = \pm 1/2$ is the fact that the slope of $V$ becomes vertical, corresponding to the logarithmic divergence to infinity of the electric field magnitude at a vertex of the cubic surface (see Appendix A2).

Panels 15(a) and 15(b) are surface plots of $V$ for the same symmetry regions as respectively Fig. 11(a) and Fig. 12(a). The vector and streamline plots of $E$ clearly provide more insight about the magnitude and direction of the interior electric field than what is provided from the corresponding surface plots of $V$. We observe that, since the electric field is parallel to these symmetry rectangles at points on these rectangles, the two-dimensional gradient $(-\partial_y V, -\partial_z V)$ for panel (a) or $(-\partial_y V, -\partial_z V)$ for panel (b) give the full gradient of $V$. This
FIG. 15: (a) Plot of the potential $V(0, y, z)$ over the square $MNOP$ of Fig. 11(a). The negative local gradient $-\nabla V$ of this surface, which is difficult to determine visually from this plot, corresponds to the directions of the electric field in Fig. 11. (b) Plot of the potential $V(u, u, z)$ over the rectangle $QRSO$ defined by Fig. 12(a).

means that the direction and magnitude of the electric field is determined from just these surface plots (which would not be true for a surface plot on a rectangle that does not lie within a mirror plane).

For example, in Fig. 15(a) and in Fig. 15(b), the surface is approximately flat near the origin, which implies that the gradient and so the electric field $-\nabla V$ has a small magnitude near the center of the cubic surface. The subtle change in concavity of $V$ in Fig. 15(a) corresponds to the electric field pointing inwards along the face center $P$ and outwards towards an edge at $N$. For example, the surface $V(0, y, z)$ has a negative slope from $y = 0$ to $y = 1/2$ along the axis $z = 1/2$, corresponding to the electric field pointing towards the edge at $N$, while the same surface $V(0, y, z)$ has a positive slope from $z = 0$ to $z = 1/2$ along $y = 0$, corresponding to the electric field pointing inwards from the face’s midpoint $M$. In contrast, in Fig. 15(b), $V(u, u, z)$ decreases as $u$ increases along any constant $z$, corresponding to the electric field pointing outwards towards points $R$ and $S$ as shown more clearly in the electric field vector plot Fig. 12. These surface plots of the potential simply do not provide as much insight as the plots of the electric field vectors.

Finally, Fig. 16 shows two three-dimensional equipotential surfaces for the potential values $V = 9.6$ and $V = 9.4$ inside the charged cubic surface. (These were calculated using the Mathematica command ContourPlot3D.) For $V = 9.6$ in Fig. 16(a), the equipotential surface consists of six disconnected roughly spherical caps near the midpoint of each face. Since the electric field is perpendicular to an equipotential surface at any point on that surface, Fig. 16(a) tells us that, near the center of each face, the internal electric field points inward.
FIG. 16: Panels (a) and (b) are equipotential surfaces for respectively $V = 9.6$ and $V = 9.4$ inside the cube $|x|, |y|, |z| < 0.45$ as computed via the Mathematica function ContourPlot3D using the exact expression Eq. (A3) for the potential $V(x, y, z)$. At each point on these equipotential surfaces, the electric field $E = -\nabla V$ is perpendicular to the surface and points inwards in (a) and outwards towards edges and vertices in (b).

but over a spread of angles, in agreement with Fig. 11(c) near the point $M$. For $V = 9.4$ in Fig. 16(a), the equipotential surface is geometrically interesting and complex, with the local normals to this surface being consistent with, but less easy to understand physically, than the vector plots of Fig. 11(c) and Fig. 12(c). However, Fig. 16(b) does help one to appreciate visually the complexity of the electric field inside the charged cubic surface.

IV. CONCLUSIONS

In this paper, we have discussed an electrostatics problem concerning what is the electric field inside and outside of a three-dimensional symmetric charge distribution consisting of a cubic non-conducting surface with constant charge density $\sigma > 0$ (see Fig. 1). Although this problem at first glance seems similar to other problems that students learn about in an introductory physics course such as the electric field inside a spherical shell or inside the cavity of a cubic conductor (see Sect. II E), the problem has rather different and interesting features that make it a good choice for improving a student’s conceptual understanding of electrostatics and for strengthening a student’s qualitative and quantitative problem solving skills. Despite the seeming high symmetry of this charge distribution, the electric field is
not zero inside the cubic surface and in fact has a rather intricate geometry.

The main purpose of this paper was not so much to describe a new electrostatics problem but to use this problem as an opportunity to encourage undergraduate students to use qualitative physical reasoning at a deeper level than what is traditionally used, even in upper-level courses in electrodynamics. The authors feel that this kind of qualitative reasoning, which is used routinely by physics researchers, is under-emphasized in current physics textbooks. We hope that this paper will encourage more physics instructors to incorporate this kind of qualitative reasoning at several points throughout the semester, by discussing examples that integrate knowledge over multiple chapters and that apply qualitative, quantitative, analytical, and numerical approaches to the same problem.

A second contribution of this paper was to emphasize the value of plotting three-dimensional electric fields over lines or planes that have certain symmetries with respect to the charge distribution that produced the electric field (assuming that such symmetries are present). Nearly all examples of electric fields that are discussed analytically in undergraduate courses have a simple geometry, e.g., they are radial or azimuthal or uniform. The uniformly charged cubic surface is one of the simplest three-dimensional continuous charge distributions for which the electric field is intricate and yet can be mainly understood by qualitative reasoning.

If a course does not have time for students to work through some of the details of this paper, it would still be worthwhile for the instructor to mention some of the pedagogical insights that one can learn from the example of a uniformly charged nonconducting cubic surface. Some of these insights are the following:

1. The electric field inside the empty cavity of some static charge distribution can be nonzero provided that the charge distribution does not have spherical symmetry. Only a conducting surface can insure that the electric field is zero everywhere inside the interior, no matter what is the symmetry of the conductor.

2. Symmetry is not a binary property that either exists or not exists for some object in that one object can be more symmetric than another and the differing amounts of symmetry have physical consequences. For example, the surfaces of a sphere and of a cube are symmetric but the spherical surface is much more symmetric than the cubic surface because a sphere has an infinity of distinct mirror planes while a cube
has finitely many. The greater amount of symmetry of a spherical surface is enough to force the interior electric field to be zero everywhere.

3. The electric field magnitude can diverge to infinity for a charge distribution that has a sharp bend like the edges of the uniformly charged cubic surface. This can happen even though the charge density does not itself diverge in magnitude near an edge or vertex, as is the case for a charged conducting surface with a sharp bend.

4. For electric fields that vary in magnitude and direction in three spatial dimensions, it can be more insightful to visualize the electric field rather than equipotential surfaces, contrary to what is discussed in many introductory physics textbooks\(^\text{1,3,4}\).

5. Fig. [7] makes an important point that should help students learning about electrostatics: for non-spherical conductors in electrostatic equilibrium, the surface and interior are equipotential but the surface charge density is nonuniform. Conversely, for a non-spherical surface covered with a uniform charge density, the surface and the interior are generally not equipotential.

6. Three-dimensional vector fields like the electric field produced by this charged cubic surface are generally difficult to visualize and it is rarely insightful to plot the vector field directly (see Fig. [8]). Instead, several visualization techniques can be used to obtain insight that include plotting the field on lines and on rectangles that lie in mirror planes, and using streamline plots in addition to vector plots to determine the field geometry.

7. Just because an analytical expression is available for describing some problem does not automatically imply that the expression is scientifically insightful. Although students are exposed to this point even during their first few weeks of an introductory physics course (it is difficult to understand almost any symbolic mathematical expression the first time, say even something like the kinematic relation \(x = x_0 + v_0t + (1/2)at^2\) for the position of a one-dimensional particle undergoing constant acceleration), nearly all mathematical expressions that undergraduate physics majors work with tend to be less than one page in length, so students get the false impression that all physics expressions are this length or shorter. As shown in Appendix [A], the potential \(V\) and electric field \(E\) are known explicitly in terms of elementary mathematical functions.
everywhere in space for the charged cubic surface, but the expressions are so long so as to be almost useless for insight. So even if analytical solutions are available, it is important to find ways to understand their properties qualitatively as we did in Sect. II and to find ways to visualize their mathematical properties.

Finally, we mention that this problem has many components that make it well suited for students in a discussion section or flipped class to work on in groups. Students could work through the subsections of Sect. II in succession, first learning about how to use symmetry to constrain the electric field on certain spatial regions, then using Gauss’s law in a qualitative way. They could then either be given the figures of Sec. III to compare with their qualitative results or be given a computer code that evaluates the electric field and potential anywhere in space and be guided with how to plot these quantities in insightful ways. Students could also be challenged to explore topics that extend this paper, for example:

1. Most of the conclusions obtained by using mirror planes in Section II B can also be obtained using the discrete rotational symmetry of various symmetry lines, which would be interesting for students to explore.

2. Students could explore a two-dimensional version of this paper: consider a square frame that consists of four equal non-conducting line segments, each of which that has the same constant linear charge density $\lambda > 0$. Qualitatively and then quantitatively, investigate what is the electric field inside and outside the square frame?

3. Students could plot the analytical expressions for $E$ and $V$ given in Appendix A 1 for a uniformly charged planar rectangle. Using these plots, they could confirm that $E$ looks like that of an infinite plane for points close to any point that is closer to the rectangle than to an edge, that the electric field is not perpendicular to the plane of the rectangle as a point of interest approaches an edge, and that the electric field asymptotes to that a point charge field far from the rectangle. Students could also confirm that electric field magnitude $E$ diverges near edges, which clarifies that the divergence of $E$ near edges of the charged cubic surface arises from the properties of each face, not from the cubic geometry.

4. Instead of the charged non-conducting cubic surface, students could consider a non-conducting cube filled with a constant volume charge density $\rho$. They could then
investigate qualitatively and then quantitatively (say by approximating the density by a finite regular grid of point charges) the electric field and potential. (An analytical solution for \( V \) is known in this case\(^{20} \), and a qualitative discussion for a uniform cubic mass has been given by Sanny and Smith\(^{21} \)). How do the potential and electrical field differ from those for a sphere of constant charge density \( \rho \)?

5. Students could investigate qualitatively and then quantitatively the internal and external electric field for the surface of a tetrahedron that is covered with a uniform surface charge density.

**Appendix A: Analytical expressions for the electric field and potential**

1. **Derivation of the potential and electric field**

In this Appendix, we use superposition and Coulomb’s law to obtain an exact mathematical expression for the potential \( V_{\text{rect}}(x, y, z) \) of a uniformly charged rectangle\(^{22} \). Knowing \( V_{\text{rect}} \), it is then straightforward to use superposition to obtain an explicit expression for the potential \( V_C \) anywhere in space of a uniformly charged cubic surface since this surface consists of six identical uniformly charged squares, see Eq. (A3) below. From \( V_C \), one can then obtain an analytical expression for the electric field anywhere in space via the relation \( \mathbf{E} = -\nabla V_C \). The authors first learned about some of these results from an interesting blog of Michael Trott, who showed how to use Mathematica to calculate and plot the three-dimensional potential of some charge distributions that have sharp edges\(^8 \).

Rather than explain the following details to students, we recommend that instructors give the students a black-box computer program that returns as its output the electric field \( \mathbf{E}(x, y, z) = (E_x, E_y, E_z) \) and the potential \( V(x, y, z) \) at any point \((x, y, z)\). This output can then be plotted or studied numerically.

Consider a rectangle of dimensions \( a \times b \) that has a constant surface charge density \( \sigma \). We assume that the rectangle lies in the \( xy \)-plane of an \( xyz \)-Cartesian coordinate system such that the rectangle’s vertices lie at the four points \((x, y, z) = (\pm a/2, \pm b/2, 0)\). The potential \( V_{\text{rect}}(x, y, z) \) at some point \((x, y, z)\) is then given by the following two-dimensional
This integral is a statement of superposition, and is the limit of a discrete sum over the infinitesimal potentials $dV = K dq/d$ at $(x, y, z)$ created by infinitesimal squares of area $dA' = dx' \times dy'$ and of infinitesimal charges $dq = \sigma dA'$ that are centered on the point $(x', y', 0)$. By direct evaluation or by using a symbolic integrator such as those available in Mathematica or Maple, one finds that this integral has the following value:

\[
V_{\text{rect}}(x, y, z) = \frac{K\sigma}{2} \left( (b - 2y) \log \left( \sqrt{(a - 2x)^2 + (b - 2y)^2 + 4z^2} + a - 2x \right) \\
+ (a - 2x) \log \left( \sqrt{(a - 2x)^2 + (b - 2y)^2 + 4z^2} + b - 2y \right) \\
- (b - 2y) \log \left( \sqrt{(a - 2x)^2 + (b - 2y)^2 + 4z^2} - a - 2x \right) \\
+ (a + 2x) \log \left( \sqrt{(a + 2x)^2 + (b - 2y)^2 + 4z^2} + b - 2y \right) \\
- (a - 2x) \log \left( \sqrt{(a - 2x)^2 + (b + 2y)^2 + 4z^2} - b - 2y \right) \\
- (a + 2x) \log \left( \sqrt{(a + 2x)^2 + (b + 2y)^2 + 4z^2} - b - 2y \right) \\
+ (b + 2y) \left( \log \left( \sqrt{(a - 2x)^2 + (b + 2y)^2 + 4z^2} + a - 2x \right) \\
- \log \left( \sqrt{(a + 2x)^2 + (b + 2y)^2 + 4z^2} - a - 2x \right) \right) \\
- 2z \left[ \tan^{-1} \left( \frac{(a - 2x)(b - 2y)}{2z \sqrt{(a - 2x)^2 + (b - 2y)^2 + 4z^2}} \right) \\
+ \tan^{-1} \left( \frac{(a + 2x)(b - 2y)}{2z \sqrt{(a + 2x)^2 + (b - 2y)^2 + 4z^2}} \right) \\
+ \tan^{-1} \left( \frac{(a - 2x)(b + 2y)}{2z \sqrt{(a - 2x)^2 + (b + 2y)^2 + 4z^2}} \right) \\
+ \tan^{-1} \left( \frac{(a + 2x)(b + 2y)}{2z \sqrt{(a + 2x)^2 + (b + 2y)^2 + 4z^2}} \right) \right]. \tag{A2}
\]

The potential $V_{\text{cube}}(x, y, z)$ at any point $(x, y, z)$ for a uniformly charged cubic surface centered at the origin with side length $L$ is then obtained by first setting the rectangular
lengths $a = b = L$ and then by adding shifted versions of Eq. (A2) like this:

$$V_C(x, y, z) = V_{\text{rect}}(x, y, z + 1/2) + V_{\text{rect}}(x, y, z - 1/2) + V_{\text{rect}}(z, x, y + 1/2) + V_{\text{rect}}(z, x, y - 1/2) + V_{\text{rect}}(z, y, x + 1/2) + V_{\text{rect}}(z, y, x - 1/2).$$  \hspace{1cm} (A3)

Using a symbolic manipulation program like Mathematica\textsuperscript{15-23}, one can then obtain an explicit symbolic expression for the electric field $E_{\text{cube}} = -\nabla V_{\text{cube}}$ by symbolic partial differentiation of Eq. (A3) with respect to the variables $x$, $y$, and $z$. For example, the following brief Mathematica code defines a function $ECubicSurface$ that returns a symbolic expression for the electric field vector $E(x, y, z)$ at a given point $(x, y, z)$:

```mathematica
ECubicSurface[ x_, y_, z_ ] := Module[
{ x0, y0, z0 } ,
- Grad[ VCubicSurface[ x0, y0, z0 ] , { x0, y0, z0 } ]
/ . { x0 -> x, y0 -> y, z0 -> z }
]
```

The line $Grad[ VCubicSurface[ x0, y0, z0 ] , { x0, y0, z0 } ]$ takes the symbolic gradient (partial derivatives) of the expression $V_{\text{cube}}(x_0, y_0, z_0)$ with respect to the vector $(x_0, y_0, z_0)$. The following line $/ . { x0 -> x, y0 -> y, z0 -> z }$ performs a symbolic substitution, replacing all symbols $x_0, y_0, z_0$ in the expression for the electric field with respectively the values $x, y, z$. The resulting expression for $E$ is several pages long but is readily evaluated as needed.

2. Logarithmic divergence of the electric field near a corner

Using these exact expressions and Mathematica, we can show that the magnitude of the electric field diverges logarithmically as one approaches any edge or vertex of the charged cubic surface. This means that, if $P$ is some point and $E(P)$ is the magnitude of the electric field at $P$, then $E \propto |\log(d)|$ in the limit that the distance $d$ of $P$ to an edge or vertex becomes small ($d \ll L$). For example, the Mathematica code

```mathematica
Series[
ECubicSurface[ 1/2 - x, 1/2 - x, 1/2 - x ] ,
```
\{ x, 0, 2 \}, 
\text{Assumptions} \rightarrow ( x > 0 )
\]
evaluates the Taylor series of \(E(1/2 - x, 1/2 - x, 1/2 - x)\) to second order in the small quantity \(x\) about \(x = 0\), which corresponds to the vertex \((1/2, 1/2, 1/2)\). Evaluating this code gives the answer

\[E = (-2 \log(x) - 2.5 - 0.59x + 3.4x^2) \hat{x} + \hat{y} + \hat{z}. \quad (A4)\]

Eq. (A4) says that, as one approaches the vertex \((1/2, 1/2, 1/2)\) along the line \((1/2 - x, 1/2 - x, 1/2 - x)\), with \(x\) becoming small, the electric field diverges as \(-2 \log(x)\).

3. Validation of the symbolic expression using a simple numerical code based on discretization and superposition

Knowing the exact result Eq. (A2) and its gradient does not automatically imply that one can evaluate it correctly with a computer program since there are multiple ways that an error can enter during the process of writing and executing a Mathematica program. To make sure that our results were correct, we developed an independent numerical method and then used that method to confirm the correctness of all the figures in this paper.

We did this by using a simple algorithm whose technical details can be easily understood by freshman physics students, although it can be challenging to program the algorithm for general charge densities \(\rho(x, y, z)\). The key idea is illustrated in Fig. 17 and the key steps are summarized here:

1. the continuous charge distribution on each face of the cubic surface is approximated with a square mesh of \(N \times N\) identical point charges, each of charge \(\Delta Q = Q/N^2\);

2. at some point \(P = (x, y, z)\) of interest, sum the contributions of each of the \(6N^2\) point charges to the electric field and potential at \(P\), using the elementary expressions \(\Delta E_i = K \Delta Q_i \hat{r_i}/d_i^2\) and \(\Delta V_i = K \Delta Q_i / d_i\). Here \(i\) is some integer label that goes over all the point charges \(\Delta Q_i\), \(d_i\) is the distance between \(P\) and the \(i\)th charge, and \(\hat{r_i}\) is the unit vector pointing from the \(i\)th point charge to \(P\).
FIG. 17: The electric field or potential at a given point $P$ in space is obtained by approximating each face of the charged cubic surface with a regular square grid of identical point charges and then by adding up the electric field or potential due to each point charge at $P$.

We found that a value of $N \geq 10$ (at least 100 point charges per face) gave identical results at the level of the figures when compared to the exact answer. Some small differences between the exact and numerical results were found near edges (where the electric field magnitude diverges) and when some point of interest was close to the discrete grid of point charges (less than a few multiples of the spacing between the grid points).

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While the assumptions $L = 1 \text{ m}$ and $\sigma = 1 \text{ C/m}^2$ are convenient to make mathematically, they correspond to a huge electrical charge on the cubic surface that would be difficult to work with in a laboratory. In particular, the electric field magnitude near such a charged cubic surface would be of order $KQ_{\text{tot}}/L^2 \approx 10^{11} \text{ V/m}$ where $K \approx 10^{10} \text{ N} \cdot \text{m}^2/\text{C}^2$ is Coulomb’s constant. This magnitude greatly exceeds the order-of-magnitude of the dielectric breakdown of air, $E \approx 10^6 \text{ V/m}$, so that this charged cubic surface would have to be kept in a high vacuum to prevent spontaneous discharge via sparks.

The argument briefly goes like this. If at some point $P$ inside the uniformly charged spherical shell distribution $\rho_S$ the electric field $\mathbf{E}(P)$ is not radial, consider the plane $\Pi$ defined by two intersecting lines, the line through the origin that is parallel to the radial position vector $\mathbf{r}_P$ to $P$, and the line through the origin that is parallel to $\mathbf{E}(P)$. Then the mirror plane that is parallel to $\mathbf{r}_P$ and that is perpendicular to $\Pi$ allows one to deduce that there are two distinct electric field vectors at $P$, a contradiction, so $\mathbf{E}(P)$ must be radial at $P$ and so radial everywhere.

To show that $\mathbf{E}(P)$ can depend only on the radial distance $r$ of $P$ to the center of the shell, consider some other point $P'$ that lies on the spherical surface of radius $r$. Then the mirror plane $\Pi$ that passes through the center of the shell and that bisects the line segment $PP'$ can be used to prove that the electric field vector $\mathbf{E}(P')$ at $P'$ is the mirror image of $\mathbf{E}(P)$ and so has the same length, i.e., its length is the same everywhere on the sphere of radius $r$ and so can depend only on $r$.

Figure 7(a) represents an approximate numerical solution obtained with the commercial finite-element code COMSOL. In the code, a cube of side-length $L = 0.1 \text{ m}$ was specified to be an equipotential surface with $V = 1 \text{ V}$, and then a mesh of $120 \times 120$ points was specified for each face of the cube. The cube was then placed in a large enclosing equipotential sphere of radius $R = 1 \text{ m}$ and potential $V = 0 \text{ V}$, which was a practical way to approximate an
equipotential cubic surface in an infinite spatial domain. The COMSOL code then automatically introduced spatial discretizations of the volume between the cube and sphere and inside the cube, and then solved Poisson’s equation for the potential $V$ at all points in space. The surface charge density at some point on the given surface was then obtained from the electric field $\mathbf{E} = -\nabla V$ by applying Gauss’s law to a small area on the surface. This numerical problem was not the same as a equipotential cubic conductor with a specified total charge but is similar enough to illustrate the key points semi-quantitatively.

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