REPRESENTATIONS OF RIGHT 3-NAKAYAMA ALGEBRAS

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Abstract. In this paper we study the category of finitely generated modules over a right 3-Nakayama artin algebra. First we give a characterization of right 3-Nakayama artin algebras and then we give a complete list of non-isomorphic finitely generated indecomposable modules over any right 3-Nakayama artin algebra. Also we compute all almost split sequences for the class of right 3-Nakayama artin algebras. Finally, we classify finite dimensional right 3-Nakayama algebras in terms of their quivers with relations.

1. INTRODUCTION

Let $R$ be a commutative artinian ring. An $R$-algebra $\Lambda$ is called an artin algebra if $\Lambda$ is finitely generated as an $R$-module. Let $\Lambda$ be an artin algebra. A right $\Lambda$-module $M$ is called uniserial (1-factor serial) if it has a unique composition series. An artin algebra $\Lambda$ is called Nakayama algebra if any indecomposable right $\Lambda$-module is uniserial. The class of Nakayama algebras is one of the important class of representation finite algebras whose representation theory completely understood [3]. According to [5, Definition 2.1], a non-uniserial right $\Lambda$-module $M$ of length $l$ is called $n$-factor serial ($l \geq n > 1$), if $\frac{M}{\text{rad}^{l-n}(M)}$ is uniserial and $\frac{M}{\text{rad}^{l-n+1}(M)}$ is not uniserial. An artin algebra $\Lambda$ is called right $n$-Nakayama if every indecomposable right $\Lambda$-module is $i$-factor serial for some $1 \leq i \leq n$ and there exists at least one indecomposable $n$-factor serial right $\Lambda$-module [5, Definition 2.2]. The authors in [3] showed that the class of right $n$-Nakayama algebras provide a nice partition of the class of representation finite artin algebras. More precisely, the authors proved that an artin algebra $\Lambda$ is representation finite if and only if $\Lambda$ is right $n$-Nakayama for some positive integer $n$ [5, Theorem 2.18]. The first part of this partition is the class of Nakayama algebras and the second part is the class of right 2-Nakayama algebras. Indecomposable modules and almost split sequences for the class of right 2-Nakayama algebras are classified in section 5 of [5]. In this paper we will study the class of right 3-Nakayama algebras. We first show that an artin algebra $\Lambda$ which is neither Nakayama nor right 2-Nakayama is right 3-Nakayama if and only if every indecomposable right $\Lambda$-module of length greater than 4 is uniserial and every indecomposable right $\Lambda$-module of length 4 is local. Then we classify all indecomposable modules and almost split sequences over a right 3-Nakayama artin algebra. We also show that finite dimensional right 3-Nakayama algebras are special biserial and we describe all finite dimensional right 3-Nakayama algebras by their quivers and relations. Riedtmann in [6] and [7], by using the covering theory, classified representation-finite self-injective algebras. By elementary

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proof, avoiding many technical things, we classify self-injective special biserial algebras of finite type which able us to classify self-injective right 3-Nakayama algebras.

The paper is organized as follows. In Section 2 we first study 3-factor serial right modules and after classification of right 3-Nakayama algebras we describe all indecomposable modules and almost split sequences over right 3-Nakayama artin algebras.

In Section 3 we show that any finite dimensional right 3-Nakayama algebra is special biserial and then we describe the structure of quivers and their relations of right 3-Nakayama algebras.

In the final section, we classify self-injective right 3-Nakayama algebras.

1.1. **notation.** Throughout this paper all modules are finitely generated right \( \Lambda \)-modules and all fields are algebraically closed fields unless otherwise stated. For a \( \Lambda \)-module \( M \), we denote by \( \text{soc}(M) \), \( \text{top}(M) \), \( \text{rad}(M) \), \( l(M) \), \( ll(M) \) and \( \text{dim}M \) its socle, top, radical, length, Loewy length and dimension vector, respectively. We also denote by \( \tau(M) \), the Auslander-Reiten translation of \( M \). Let \( Q = (Q_0,Q_1,s,t) \) be a quiver and \( \alpha : i \to j \) be an arrow in \( Q \). One introduces a formal inverse \( \alpha^{-1} \) with \( s(\alpha^{-1}) = j \) and \( t(\alpha^{-1}) = i \). An edge in \( Q \) is an arrow or the inverse of an arrow. To each vertex \( i \) in \( Q \), one associates a trivial path, also called trivial walk, \( \varepsilon_i \) with \( s(\varepsilon_i) = t(\varepsilon_i) = i \). A non-trivial walk \( w \) in \( Q \) is a sequence \( w = c_1c_2\cdots c_n \), where the \( c_i \) are edges such that \( t(c_i) = s(c_{i+1}) \) for all \( i \), whose inverse \( w^{-1} \) is defined to be the sequence \( w^{-1} = c_n^{-1}c_{n-1}^{-1}\cdots c_1^{-1} \). A walk \( w \) is called reduced if \( c_{i+1} \neq c_i^{-1} \) for each \( i \). For \( i \in Q_0 \), we denote by \( i^+ \) and \( i^- \) the set of arrows starting in \( i \) and the set of arrows ending in \( i \), respectively, and for any set \( X \), we denote by \( |X| \) the number of elements in \( X \).

2. **Right 3-Nakayama Algebras**

In this section we give a characterization of right 3-Nakayama artin algebras. We also classify all indecomposable modules and almost split sequences over right 3-Nakayama algebras.

**Definition 2.1.** [5] Definitions 2.1, 2.2 Let \( \Lambda \) be an artin algebra and \( M \) be a right \( \Lambda \)-module of length \( l \).

1. \( M \) is called 1-factor serial (uniserial) if \( M \) has a unique composition series.
2. Let \( l \geq n > 1 \). \( M \) is called \( n \)-factor serial if \( \text{rad}^{l-n}(M) \) is uniserial and \( \text{rad}^{l-n+1}(M) \) is not uniserial.
3. \( \Lambda \) is called right \( n \)-Nakayama if every indecomposable right \( \Lambda \)-module is \( i \)-factor serial for some \( 1 \leq i \leq n \) and there exists at least one indecomposable \( n \)-factor serial right \( \Lambda \)-module.

**Lemma 2.2.** Let \( \Lambda \) be an artin algebra and \( M \) be an indecomposable right \( \Lambda \)-module of length \( r \) and Loewy length \( t \). Then the following conditions are equivalent:

(a) \( M \) is a 3-factor serial right \( \Lambda \)-module.

(b) One of the following conditions hold:

(i) \( M \) is local and for every \( 1 \leq i \leq r - 4 \), \( \text{rad}^i(M) \) is local and \( \text{rad}^{r-3}(M) \) is not local that either

(1) \( r = t + 2 \), \( \text{soc}(M) = \text{rad}^{r-3}(M) \) and \( l(\text{soc}(M)) = 3 \) or
(2) \( r = t + 1 \) and \( \text{rad}^{r-2}(M) \) is simple.

(ii) \( M \) is not local, \( r = 3 \) and \( t = 2 \).

Proof. (a) \( \implies \) (b). Assume that \( M \) is a local right \( \Lambda \)-module, then by [5] Theorem 2.6, for every \( 0 \leq i \leq r - 4 \), \( \text{rad}^i(M) \) is local and \( \text{rad}^{r-3}(M) \) is not local. On the other hand by [5] Lemma 2.21, \( r \leq t + 2 \) and since \( M \) is not uniserial \( t < r \). If \( r = t + 2 \), by [5] Remark 2.7, \( \text{soc}(M) \subseteq \text{rad}^{r-3}(M) \) and since \( t = r - 2 \), \( \text{soc}(M) = \text{rad}^{r-3}(M) = S_1 \oplus S_2 \oplus S_3 \).

If \( r = t + 1 \), then \( \text{rad}^{r-2}(M) \neq 0 \). If \( l(\text{rad}^{r-2}(M)) = 2 \), then \( \frac{M}{\text{rad}^{r-2}(M)} \) is uniserial which gives a contradiction. Thus \( \text{rad}^{r-2}(M) \) is simple, which complete the proof of (i). If \( M \) is not local, then by [5] Corollary 2.8 \( r = 3, t = 2 \) and the result follows.

(b) \( \implies \) (a). If \( M \) is not local and \( r = 3 \), then by [5] Corollary 2.8, \( M \) is a 3-factor serial right \( \Lambda \)-module. Now assume that \( M \) satisfies the condition (i). Then \( \frac{M}{\text{rad}^{r-3}(M)} \) is uniserial. If \( M \) satisfies the condition (1), then \( \text{rad}^{r-2}(M) = 0 \) and so \( \frac{M}{\text{rad}^{r-2}(M)} \cong M \) is not uniserial. If \( M \) satisfies the condition (2), then \( \frac{M}{\text{rad}^{r-2}(M)} \) is non-uniserial and so \( M \) is a 3-factor serial right \( \Lambda \)-module.

Lemma 2.3. Let \( n > 1 \) be a positive integer, \( \Lambda \) be a right \( n \)-Nakayama artin algebra and \( M \) be an indecomposable \( n \)-factor serial right \( \Lambda \)-module. Then \( \text{top}(M) \) is simple if and only if \( M \) is projective.

Proof. Assume that \( \text{top}(M) \) is simple and \( M \) is not projective. Let \( P \to M \) be a projective cover of \( M \). Since \( \text{top}(M) \) is simple, \( P \) is indecomposable. Then by [5] Lemma 2.11, \( P \) is a \( t \)-factor serial right \( \Lambda \)-module for some \( t \geq n + 1 \) which gives a contradiction. Then \( M \) is projective. \( \square \)

Theorem 2.4. Let \( \Lambda \) be a right 3-Nakayama artin algebra and \( M \) be an indecomposable right \( \Lambda \)-module. Then \( M \) is either a factor of an indecomposable projective right \( \Lambda \)-module or a submodule of an indecomposable injective right \( \Lambda \)-module.

Proof. If \( M \) is uniserial or 2-factor serial, then by definition of uniserial modules and [5] Lemma 5.2, \( M \) is a factor of an indecomposable projective module. Now assume that \( M \) is 3-factor serial. If \( \text{top}(M) \) is simple, then by Lemma 2.3, \( M \) is projective. If \( \text{top}(M) \) is not simple, then by Lemma 2.7, \( l(M) = 3 \) and \( \text{soc}(M) \) is simple. This implies that \( M \) is a submodule of an indecomposable injective right \( \Lambda \)-module. \( \square \)

An artin algebra \( \Lambda \) is said to be of local-colocal type if every indecomposable right \( \Lambda \)-module is local or colocal (i.e. has a simple socle) [8].

Corollary 2.5. Let \( \Lambda \) be a right 3-Nakayama artin algebra. Then \( \Lambda \) is of local-colocal type.

An artin algebra \( \Lambda \) is said to be of right \( n \)-th local type if for every indecomposable right \( \Lambda \)-module \( M \), \( \text{top}^n(M) = \frac{M}{\text{rad}^i(M)} \) is indecomposable [2].

Proposition 2.6. Let \( \Lambda \) be a right 3-Nakayama artin algebra and \( M \) be an indecomposable right \( \Lambda \)-module. Then the following statements hold.

(a) If \( M \) is 2-factor serial, then \( l(M) = 3 \) and \( ll(M) = 2 \).

(b) If \( M \) is 3-factor serial and non-local, then \( l(M) = 3 \) and \( ll(M) = 2 \).
(c) If $M$ is 3-factor serial and local, then $l(M) = 4$ and $l(I(M)) = 3$.

Proof. Any indecomposable right $\Lambda$-module is either uniserial or 2-factor serial or 3-factor serial, then by definition of uniserial module, [5, Lemma 5.2] and Lemma 2.2, $\text{top}^2(M) = \frac{M}{\text{rad}(M)}$ is indecomposable and so $\Lambda$ is of right 2-nd local type. Let $M$ be a 2-factor serial right $\Lambda$-module, so by [2, Lemma 1.4], $\text{rad}(M)$ is not local and by [5, Corollary 2.8], $l(\text{rad}(M)) = 2$. This proves part (a).

The part (b) follows from Lemma 2.2

Let $M$ be a local 3-factor serial right $\Lambda$-module. By [2, Lemma 1.4], $\text{rad}(M)$ is not local and by [5, Theorem 2.6], $l(M) = 4$. By Lemma 2.2, $l(I(M)) = 3$ and the result follows. □

In the next theorem, we give a characterization of right 3-Nakayama artin algebras.

Theorem 2.7. Let $\Lambda$ be an artin algebra which is neither Nakayama nor right 2-Nakayama. Then $\Lambda$ is right 3-Nakayama if and only if every indecomposable right $\Lambda$-module $M$ with $l(M) > 4$ is uniserial and every indecomposable right $\Lambda$-module $M$ with $l(M) = 4$ is local.

Proof. Assume that $\Lambda$ is a right 3-Nakayama algebra. It follows from Proposition 2.6 that, every indecomposable right $\Lambda$-module $M$ with $l(M) > 4$ is uniserial. Assume that there exists an indecomposable right $\Lambda$-module $M$ with $l(M) = 4$ which is not local. Then by [5, Corollary 2.8], $M$ is 4-factor serial which is a contradiction. Conversely, assume that any indecomposable right $\Lambda$-module $M$ with $l(M) > 4$ is uniserial and every indecomposable right $\Lambda$-module $M$ with $l(M) = 4$ is local. Since $\Lambda$ is neither Nakayama nor right 2-Nakayama, there exists an indecomposable $t$-factor serial $\Lambda$-module $M$ for some $t \geq 3$. Also for any indecomposable right $\Lambda$-module $N$ of length 4, $N$ is local and by [5, Corollary 2.8], $N$ is $r$-factor serial right $\Lambda$-module for some $r \leq 3$. Therefor $M$ is 3-factor serial and $\Lambda$ is right 3-Nakayama. □

Corollary 2.8. Let $\Lambda$ be an artin algebra. Then the following statements hold.

(a) If every indecomposable right $\Lambda$-module of length greater than 4 is uniserial and every indecomposable right $\Lambda$-module of length 4 is local, then $\Lambda$ is either Nakayama or right 2-Nakayama or right 3-Nakayama.

(b) If every indecomposable right $\Lambda$-module of length greater than 4 is uniserial, every indecomposable right $\Lambda$-module of length 4 is local and there exists an indecomposable non-uniserial right $\Lambda$-module which is not projective, then $\Lambda$ is right 3-Nakayama.

Proof. (b) It is clear that $\Lambda$ is not Nakayama. If $\Lambda$ is a right 2-Nakayama, then by [5, Proposition 5.5] every indecomposable non-uniserial right $\Lambda$ is projective, which gives a contradiction. Thus by Theorem 2.7, $\Lambda$ is right 3-Nakayama. □

Lemma 2.9. Let $\Lambda$ be a right 3-Nakayama artin algebra and $M$ be an indecomposable 3-factor serial right $\Lambda$-module. Then $l(\text{soc}(M)) \leq 2$.

Proof. By Proposition 2.6, $l(M)$ is either 3 or 4. If $l(M) = 3$, then by Lemma 2.2, $\text{soc}(M)$ is simple. Now assume that $l(M) = 4$. Then by Proposition 2.6 and Lemma 2.3, $M$ is projective. If $M$ is injective, then $\text{soc}(M)$ is simple. Assume that $M$ is not injective. Assume on a contrary that $\text{soc}(M) = S_1 \oplus S_2 \oplus S_3$, for simple $\Lambda$-modules $S_1, S_2$ and $S_3$. Then $\text{rad}(M) = \text{Soc}(M)$ and we have a right minimal almost split morphism $S_1 \oplus S_2 \oplus S_3 \to M$. Thus $S_i$ is not injective for each $1 \leq i \leq 3$ and we have the following almost split sequences
Theorem 2.10. Let $\Lambda$ be a right 3-Nakayama artin algebra and $M$ be an indecomposable right $\Lambda$-module. Then the following statements hold.

(a) Assume that $M$ is 2-factor serial. Then submodules of $M$ are $S_1, S_2$ and $\text{rad}(M) = \text{soc}(M) = S_1 \oplus S_2$, where $S_i$ is a simple submodule of $M$ for each $i = 1, 2$.

(b) Assume that $M$ is local and non-colocal 3-factor serial. Then submodules of $M$ are $\text{rad}(M)$ which is indecomposable non-local 3-factor serial of length 3, two uniserial submodules $M_1$ and $M_2$ of length 2 and $S := \text{soc}(M)$ which is simple and $\ell(M) = 3$.

(c) Assume that $M$ is local and non-colocal 3-factor serial. Then submodules of $M$ are uniserial submodule $N$ of length 2, simple submodules $S$ and $S'$ that $\text{soc}(N) = S'$, $\text{rad}(M) = N \oplus S$ and $\text{soc}(M) = S' \oplus S$ and $\ell(M) = 3$.

(d) Assume that $M$ is colocal and non-local 3-factor serial. Then submodules of $M$ are two uniserial modules $M_1$ and $M_2$ of length 2 and $\text{rad}(M) = \text{soc}(M) = S$ which is simple and $\ell(M) = 2$.

Proof. (a) and (c) follow from Proposition 2.6 and Lemma 2.9.

(b). Since $M$ is local, $\text{rad}(M)$ is maximal submodule of $M$ and $\text{soc}(M) \subseteq \text{rad}(M)$ and since $\text{soc}(M)$ is simple, by [5, Theorem 2.6] and Proposition 2.6, $\text{rad}(M)$ is non-local indecomposable right $\Lambda$-module of length 3. Therefore $\text{rad}(M)$ has two uniserial maximal submodules $M_1$ and $M_2$ of length 2.

The proof of (d) is similar to the proof of (b). \qed

Proposition 2.11. Let $\Lambda$ be a right 3-Nakayama artin algebra and $M$ be an indecomposable right $\Lambda$-module. Then $M$ is not projective if and only if one of the following situations holds.

(a) $M$ is local and there is an indecomposable projective right $\Lambda$-module $P$ such that $M$ is a factor of $P$ where $P$ satisfy one of the following situations:

(i) $P$ is an uniserial projective right $\Lambda$-module. So $M \cong \frac{P}{\text{rad}(P)}$ for some $1 \leq i < \ell(P)$.

(ii) $P$ is a 2-factor serial projective right $\Lambda$-module that $\text{rad}(P) = \text{soc}(P) = S_1 \oplus S_2$ where $S_i$ is simple submodule for each $1 \leq i \leq 2$. So $M$ is isomorphic to either $\frac{P}{\text{rad}(P)}$ or $\frac{P}{S_i}$ for some $1 \leq i \leq 2$.

(iii) $P$ is a 3-factor serial projective-injective right $\Lambda$-module and submodules of $P$ are $\text{rad}(P)$ which is indecomposable non-local 3-factor serial right $\Lambda$-module of length 3, two uniserial modules $M_1$ and $M_2$ of length 2 and $S = \text{soc}(P)$ which is simple. Then $M$ is isomorphic to either $\frac{P}{\text{rad}(P)}$ or $\frac{P}{S_i}$ for some $1 \leq i \leq 2$ or $\frac{P}{S}$. 

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\[0 \rightarrow S_1 \rightarrow M \rightarrow \tau^{-1}(S_1) \rightarrow 0\]
\[0 \rightarrow S_2 \rightarrow M \rightarrow \tau^{-1}(S_2) \rightarrow 0\]
\[0 \rightarrow S_3 \rightarrow M \rightarrow \tau^{-1}(S_3) \rightarrow 0\]
\[0 \rightarrow M \rightarrow \tau^{-1}(S_1) \oplus \tau^{-1}(S_2) \oplus \tau^{-1}(S_3) \rightarrow \tau^{-1}(M) \rightarrow 0\]

Then $\tau^{-1}(M)$ is an indecomposable of length 5 and by Theorem 2.7, $\tau^{-1}(M)$ is uniserial. On the other hand for any $1 \leq i \leq 3$, the irreducible morphism $f_i: \tau^{-1}(S_i) \rightarrow \tau^{-1}(M)$ is a monomorphism and so $\tau^{-1}(S_i)$ is uniserial. Also by [5, Theorem 2.13], there exists $1 \leq i \leq 3$ such that $\tau^{-1}(S_i) \cong \frac{M}{S_i}$ is not uniserial, which gives a contradiction. Therefore $l(Soc(M)) \leq 2$ and the result follows. \qed
(iv) $P$ is a 3-factor serial projective non-injective right $\Lambda$-module. $\text{rad}(P) = N \oplus S$ where $N$ is an uniserial submodule of length 2 and $S$ is a simple submodule of $P$, $\text{soc}(P) = S' \oplus S$ where $S' = \text{soc}(N)$. So $M$ is isomorphic to either $\frac{P}{\text{rad}(P)}$ or $\frac{P}{N}$ or $\frac{P}{S}$ or $\frac{P}{S'}$.

(b) $M$ is non-local 3-factor serial of length 3 where submodules of $M$ are two uniserial modules $M_1$ and $M_2$ of length 2 and $\text{rad}(M) = \text{soc}(M) = S$ which is simple. $M$ is either injective or a submodule of 3-factor serial projective-injective indecomposable module.

Proof. It follows from Corollary 2.5, Theorem 2.10 and [5, Lemma 5.3]. \qed

Now we characterize almost split sequences of right 3-Nakayama algebras.

**Theorem 2.12.** Let $\Lambda$ be a right 3-Nakayama artin algebra and $M$ be an indecomposable non-projective right $\Lambda$-module. Then one of the following situations hold:

(A) Assume that $M \cong \frac{P}{\text{rad}_i(P)}$ where $P$ is uniserial projective for some $1 \leq i < l(P)$.

(i) If $M$ is a simple direct summand of $\text{top}(L)$, where $L$ is an indecomposable non-local, submodules of $L$ are including $M_1$ and $M_2$ that are uniserial modules of length 2 and $\text{soc}(L)$ which is a simple module. In this case $M \cong \frac{L}{M_j}$ for some $1 \leq j \leq 2$ and the following exact sequence

$$0 \longrightarrow M_j \overset{i_j}{\longrightarrow} L \overset{\pi_j}{\longrightarrow} \frac{L}{M_j} \longrightarrow 0$$

is almost split sequence for each $j = 1, 2$.

(ii) Otherwise, the sequence

$$0 \longrightarrow \frac{\text{rad}_i(P)}{\text{rad}_{i+1}(P)} \overset{\pi_1}{\longrightarrow} \frac{\text{rad}_i(P)}{\text{rad}_{i+1}(P)} \oplus \frac{P}{\text{rad}_{i+1}(P)} \overset{-\pi_3}{\longrightarrow} \frac{P}{\text{rad}_i(P)} \longrightarrow 0$$

is an almost split sequence.

(B) Assume that $M$ is a factor of 2-factor serial projective right $\Lambda$-module $P$, where $\text{rad}(P) = \text{soc}(P) = S_1 \oplus S_2$ that $S_1$ and $S_2$ are simple modules.

(i) If $M \cong \frac{P}{S_i}$ for some $1 \leq i \leq 2$, then the sequence

$$0 \longrightarrow S_i \overset{i_i}{\longrightarrow} P \overset{\pi_i}{\longrightarrow} \frac{P}{S_i} \longrightarrow 0$$

is an almost split sequence.

(ii) If $M \cong \frac{P}{\text{rad}(P)}$, then the sequence

$$0 \longrightarrow \frac{P}{S_1} \overset{\pi_5}{\longrightarrow} \frac{P}{S_1} \oplus \frac{P}{S_2} \overset{-\pi_7, \pi_8}{\longrightarrow} \frac{P}{\text{rad}(P)} \longrightarrow 0$$

is an almost split sequence.

(C) Assume that $M$ is a factor of 3-factor serial projective-injective right $\Lambda$-module $P$ and submodules of $P$ are $\text{rad}(P)$ which is indecomposable non-local 3-factor serial of length 3, two uniserial modules $M_1$ and $M_2$ of length 2 and $S = \text{soc}(P)$ that is simple.

(i) If $M \cong \frac{P}{\text{rad}(P)}$, then the sequence

$$0 \longrightarrow \frac{P}{S} \overset{\pi_9}{\longrightarrow} \frac{P}{M_1} \oplus \frac{P}{M_2} \overset{-\pi_{11}, \pi_{12}}{\longrightarrow} \frac{P}{\text{rad}(P)} \longrightarrow 0$$


is an almost split sequence.

(ii) If $M \cong \frac{P}{M_i}$ for some $1 \leq i \leq 2$, then the sequence

$$0 \longrightarrow M_i \xrightarrow{i_5} \frac{P}{S} \xrightarrow{\pi_{13}} \frac{P}{M_i} \longrightarrow 0$$

is an almost split sequence.

(iii) If $M \cong \frac{P}{S}$, then the sequence

$$0 \longrightarrow \text{rad}(P) \xrightarrow{i_6} \frac{P}{S} \oplus P \xrightarrow{[-i_7, \pi_{15}]} \frac{P}{S} \longrightarrow 0$$

is an almost split sequence.

(D) Assume that $M$ is a factor of 3-factor serial non-injective projective right $\Lambda$-module $P$. That $\text{rad}(P) = N \oplus S$ where $N$ is an uniserial submodule of length 2 and $S$ is a simple submodule of $P$ and $\text{soc}(P) = S' \oplus S$ where $S' = \text{soc}(N)$.

(i) If $M \cong \frac{P}{\text{rad}(P)}$, then the sequence

$$0 \longrightarrow \frac{P}{S} \xrightarrow{\pi_{16}} \frac{P}{N} \oplus \frac{P}{\text{soc}(P)} \xrightarrow{[-\pi_{18}, \pi_{19}]} \frac{P}{\text{rad}(P)} \longrightarrow 0$$

is an almost split sequence.

(ii) If $M \cong \frac{P}{N}$, then the sequence

$$0 \longrightarrow \frac{N}{S} \xrightarrow{i_8} \frac{P}{S} \xrightarrow{\pi_{20}} \frac{P}{N} \longrightarrow 0$$

is an almost split sequence.

(iii) If $M \cong \frac{P}{\text{soc}(P)}$, then the sequence

$$0 \longrightarrow P \xrightarrow{\pi_{21}} \frac{P}{S} \oplus \frac{P}{S} \xrightarrow{[-\pi_{23}, \pi_{24}]} \frac{P}{\text{soc}(P)} \longrightarrow 0$$

is an almost split sequence.

(iv) If $M \cong \frac{P}{S}$, then the sequence

$$0 \longrightarrow S \xrightarrow{i_9} P \xrightarrow{\pi_{25}} \frac{P}{S} \longrightarrow 0$$

is an almost split sequence.

(v) If $M \cong \frac{P}{S}$, then the sequence

$$0 \longrightarrow N \xrightarrow{i_{10}} \frac{N}{S} \oplus P \xrightarrow{[-i_{11}, \pi_{27}]} \frac{P}{S} \longrightarrow 0$$

(E) Assume that $M$ is a non-local 3-factor serial right $\Lambda$-module of length 3 and submodules of $M$ are two uniserial maximal submodules $M_1$ and $M_2$ of length 2 and $\text{rad}(M) = \text{soc}(M) = S$ which is simple.

Then the following exact sequence

$$0 \longrightarrow S \xrightarrow{[i_{12}, i_{13}]} M_1 \oplus M_2 \xrightarrow{[-i_{14}, i_{15}]} M \longrightarrow 0$$

is an almost split sequence.

Where $i_j$ is an inclusion for each $1 \leq j \leq 15$ and $\pi_j$ is a canonical epimorphism for each $1 \leq j \leq 27$. 
Proof. Put $g_1 = [-i_3, \pi_3], g_2 = [-\pi_7, \pi_8], g_3 = [-\pi_{11}, \pi_2], g_4 = [-i_7, \pi_{15}], g_5 = [-\pi_{18}, \pi_{19}], g_6 = [-\pi_{23}, \pi_{24}], g_7 = [-i_{11}, \pi_{27}]$ and $g_8 = [-i_{14}, i_{15}]$. It is easy to see that all given sequences are exact, non-split and have indecomposable end terms. It is enough to show that homomorphisms $\pi_1, g_1, \pi_4, g_2, \pi_3, g_3, \pi_13, g_4, \pi_5, g_5, \pi_20, g_6, \pi_25, g_7$ and $g_8$ are right almost split morphisms.

(A)(i) Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow M$ be a non-isomorphism. Since $M$ is a simple module, so $\nu$ is an epimorphism. If $j = 1$, then $V$ is isomorphic to either $M_2$ or $\frac{P}{\text{rad}(P)}$ for some $2 \leq i < l(P)$. Since $M_2$ is a submodule of $L$ and $\text{top}(\frac{P}{\text{rad}(P)})$ is a direct summand of $\text{top}(L)$ for each $2 \leq i < l(P)$, there exists a homomorphism $h : V \rightarrow L$ such that $\pi_1 h = \nu$. In case $j = 2$ the proof is similar.

(A)(ii) Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{P}{\text{rad}(P)}$ be a non-isomorphism. If $\nu$ is an epimorphism, then $V \cong \frac{P}{\text{rad}(P)}$ for some $s > i$. This implies that there is a homomorphism $h : V \rightarrow \frac{\text{rad}(P)}{\text{rad}(P)} \oplus \frac{P}{\text{rad}(P)^{t+1}}$ such that $\nu = g_1 h$. Now assume that $\nu$ is not an epimorphism, then $\text{Im}(\nu) = \frac{\text{rad}(P)}{\text{rad}(P)}$ for some $t < i$. Then there is a homomorphism $h : V \rightarrow \frac{\text{rad}(P)}{\text{rad}(P)} \oplus \frac{P}{\text{rad}(P)^{t+1}}$ such that $\nu = g_1 h$.

(B)(i) Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{P}{S_i}$ be a non-isomorphism. If $\nu$ is an epimorphism, then $V \cong P$. This implies that there is an isomorphism $h : V \rightarrow P$ such that $\nu = \pi_4 h$. Now assume that $\nu$ is not an epimorphism. Since $l(P) = 3, l(\frac{P}{S_i}) = 2$ and so $\text{Im}(\nu)$ is a simple submodule of $M$ which is isomorphic to the direct summand of $\text{soc}(P)$. Then there is a homomorphism $h : V \rightarrow P$ such that $\nu = \pi_4 h$.

(B)(ii) Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{P}{\text{rad}(P)}$ be a non-isomorphism. Since $\frac{P}{\text{rad}(P)}$ is simple, $\nu$ is an epimorphism and $V$ is isomorphic to either $\frac{P}{S_i}$ for some $1 \leq i \leq 2$ or $P$. So there is a homomorphism $h : V \rightarrow \frac{P}{S_i} \oplus \frac{P}{S_2}$ such that $g_2 h = \nu$.

(C)(i) The proof is similar to the proof of the B(ii).

(C)(ii) Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{P}{M_i}$ be a non-isomorphism. If $\nu$ is an epimorphism, then $V$ is isomorphic to either $\frac{P}{S}$ or $P$. So there is a homomorphism $h : V \rightarrow \frac{P}{S}$ such that $\nu = \pi_{13} h$. Now assume that $\nu$ is not an epimorphism. Then $\text{Im}(\nu)$ is simple and isomorphic to the direct summand of $\text{soc}(\frac{P}{S})$. This implies that there is a homomorphism $h : V \rightarrow \frac{P}{S}$ such that $\nu = \pi_{13} h$.

(C)(iii) Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{P}{S}$ be a non-isomorphism. If $\nu$ is an epimorphism, then $V \cong P$. This implies that there is a homomorphism $h : V \rightarrow \frac{\text{rad}(P)}{S} \oplus P$ such that $\nu = g_4 h$. Now assume that $\nu$ is not an epimorphism. Then $\text{Im}(\nu)$ is a submodule of $\frac{\text{rad}(P)}{S}$ and so there is a homomorphism $h : V \rightarrow \frac{\text{rad}(P)}{S} \oplus P$ such that $\nu = g_4 h$.

(D)(i) The proof is similar to the proof of the B(ii).

(D)(ii) Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{P}{N}$ be a non-isomorphism. If $\nu$ is an epimorphism, then $V$ is isomorphic to either $P$ or $\frac{P}{S}$. So there is a homomorphism $h : V \rightarrow \frac{P}{S}$ such that $\pi_{20} h = \nu$. Now assume that $\nu$ is not an epimorphism, so
\( \text{Im}(\nu) \cong S \) and \( S \) is a direct summand of \( \frac{\text{rad}(P)}{S} \). This implies that there is a homomorphism \( h : V \rightarrow \frac{P}{S} \) such that \( \pi_{25}h = \nu \).

(D)(iii) Let \( V \) be an indecomposable right \( \Lambda \)-module and \( \nu : V \rightarrow \frac{P}{S} \) be a non-isomorphism. If \( \nu \) is an epimorphism, then \( V \cong P \) such that \( \pi_{25}h = \nu \). If \( \nu \) is not an epimorphism, then \( \text{Im}(\nu) \) is a submodule of \( \frac{\text{rad}(P)}{S} \). This implies that there is a homomorphism \( h : V \rightarrow \frac{P}{S} \) such that \( \pi_{25}h = \nu \).

(D)(iv) Let \( V \) be an indecomposable right \( \Lambda \)-module and \( \nu : V \rightarrow \frac{P}{S} \) be a non-isomorphism. If \( \nu \) is an epimorphism, then \( V \cong P \) such that \( \pi_{25}h = \nu \). If \( \nu \) is not an epimorphism, then \( \text{Im}(\nu) \) is a submodule of \( \frac{\text{rad}(P)}{S} \) and \( \frac{\text{rad}(P)}{S} \cong N \). This implies that there is a homomorphism \( h : V \rightarrow P \) such that \( \pi_{25}h = \nu \).

(D)(v) Let \( V \) be an indecomposable right \( \Lambda \)-module and \( \nu : V \rightarrow \frac{P}{S} \) be a non-isomorphism. If \( \nu \) is an epimorphism, then \( V \cong P \) and there is a homomorphism \( h : V \rightarrow \frac{N}{S} \oplus P \) such that \( g\nu = \nu \). If \( \nu \) is not an epimorphism, then \( \text{Im}(\nu) \) is a submodule of \( \frac{P}{S} \). Therefore \( \frac{P}{S} \) is a submodule of \( \frac{\text{rad}(P)}{S} = \text{soc}(\frac{P}{S}) \cong S \oplus \frac{N}{S} \). This implies that there is a homomorphism \( h : V \rightarrow \frac{N}{S} \oplus P \) such that \( g\nu = \nu \).

(E) Let \( V \) be an indecomposable right \( \Lambda \)-module and \( \nu : V \rightarrow M \) be a non-isomorphism. Since \( \nu \) is not an isomorphism and \( M \) is not a local, by Theorem [2.10] \( \nu \) is not epimorphism. Therefore \( \text{Im}(\nu) \) is a submodule of \( M \) and there is a homomorphism \( h : V \rightarrow M_1 \oplus M_2 \) such that \( g\nu = \nu \).

\[ \square \]

3. QUIVERS OF RIGHT 3-NAKAYAMA ALGEBRAS

In this section we describe finite dimensional right 3-Nakayama algebras in terms of their quivers with relations.

A finite dimensional \( K \)-algebra \( \Lambda = \frac{KQ}{I} \) is called special biserial algebra provided \((Q, I)\) satisfying the following conditions:

1. For any vertex \( a \in Q_0 \), \(|a^+| \leq 2 \) and \(|a^-| \leq 2 \).
2. For any arrow \( \alpha \in Q_1 \), there is at most one arrow \( \beta \) and at most one arrow \( \gamma \) such that \( \alpha \beta \) and \( \gamma \alpha \) are not in \( I \).

Let \( \Lambda = \frac{KQ}{I} \) be a special biserial finite dimensional \( K \)-algebra. A walk \( w = c_1c_2\cdots c_n \) in \( Q \) is called string of length \( n \) if \( c_i \neq c_{i+1}^{-1} \) for each \( i \) and no subwalk of \( w \) nor its inverse is in \( I \). In addition, we have strings of length zero, for any \( a \in Q_0 \) we have two strings of length zero, denoted by \( 1_{(a,1)} \) and \( 1_{(a,-1)} \). We have \( s(1_{(a,1)}) = t(1_{(a,1)}) = s(1_{(a,-1)}) = t(1_{(a,-1)}) = a \) and \( 1_{(a,1)} = 1_{(a,-1)} \). A string \( w = c_1c_2\cdots c_n \) with \( s(w) = t(w) \) such that each power \( w^m \) is a string, but \( w \) itself is not a proper power of any strings is called band. We denote by \( S(\Lambda) \) and \( B(\Lambda) \) the set of all strings of \( \Lambda \) and the set of all bands of \( \Lambda \), respectively. Let \( \rho \) be the equivalence relation on \( S(\Lambda) \) which identifies every string \( w \) with its inverse \( w^{-1} \) and \( \sigma \) be the equivalence relation on \( B(\Lambda) \) which identifies every band \( w = c_1c_2\cdots c_n \) with the cyclically permuted bands \( w_{(i)} = c_ic_{i+1}\cdots c_nc_{i-1} \) and their inverses \( w_{(i)}^{-1} \) for each \( i \). Butler and Ringel in [4] for each string \( w \) defined a unique string module \( M(w) \) and for each band \( v \) defined a family of band modules \( M(v, m, \varphi) \) with \( m \geq 1 \).
and $\varphi \in Aut(K^m)$. Let $\widetilde{S}(\Lambda)$ be the complete set of representatives of strings relative to $\rho$ and $\widetilde{B}(\Lambda)$ be the complete set of representatives of bands relative to $\sigma$. Butler and Ringel in [4] proved that, the modules $M(w)$, $w \in \widetilde{S}(\Lambda)$ and the modules $M(v, m, \varphi)$ with $v \in \widetilde{B}(\Lambda)$, $m \geq 1$ and $\varphi \in Aut(K^m)$ provide complete list of pairwise non-isomorphic indecomposable $\Lambda$-modules. Indecomposable $\Lambda$-modules are either string modules or band modules or non-uniserial projective-injective modules (see [4] and [10]). If $\Lambda$ is a special biserial algebra of finite type, then any indecomposable $\Lambda$-module is either string module or non-uniserial projective-injective module.

**Remark 3.1.** Let $Q$ be a finite quiver, $I$ be an admissible ideal of $Q$, $Q'$ be a subquiver of $Q$ and $I'$ be an admissible ideal of $Q'$ which is restriction of $I$ to $Q'$. Then there exists a fully faithful embedding $F : rep_K(Q', I') \rightarrow rep_K(Q, I)$.

**Proposition 3.2.** Any basic connected finite dimensional right 3-Nakayama $K$-algebra is a special biserial algebra of finite type.

**Proof.** Let $\Lambda = KQ/I$ be a right 3-Nakayama algebra. By Theorem [5, Theorem 2.18], $\Lambda$ is of finite type. We show that for every $a \in Q_0$, $|a^+| \leq 2$. If there exists a vertex $a$ of $Q_0$ such that $|a^+| \geq 3$, then we have two cases.

- **Case 1:** The algebra $\Lambda_1 = KQ_1$ given by the quiver $Q_1$

  \[
  \begin{array}{c}
  1 \\
  \downarrow \alpha_1 \\
  4 \\
  \downarrow \alpha_2 \\
  2 \\
  \downarrow \alpha_3 \\
  3 \\
  \end{array}
  \]

  which is a subquiver of $Q$, is a subalgebra of $\Lambda$. There is an indecomposable representation $M$ of $Q_1$ such that $\text{dim}M = [1, 1, 1, 2]^t$. $M$ is not local and by [5, proposition 2.8], $M$ is a 5-factor serial right $\Lambda_1$-module. Therefore by using Remark 3.1 there is a 5-factor serial right $\Lambda$-module which is a contradiction.

- **Case 2:** The algebra $\Lambda_2 = \frac{KQ_2}{I_2}$ given by the quiver $Q_2$

  \[
  \begin{array}{c}
  2 \\
  \downarrow \gamma \\
  3 \\
  \downarrow \alpha \\
  1 \\
  \end{array}
  \]

  which is a subquiver of $Q$ and the ideal $I_2$ which is a restriction of $I$ to $Q_2$, is a subalgebra of $\Lambda$. There is an indecomposable representation $M$ of $(Q_2, I_2)$ such that $\text{dim}M = [1, 1, 3]^t$. $M$ is not local and by [5, proposition 2.8], $M$ is a 5-factor serial right $\Lambda_2$-module. Then, there is a 5-factor serial right $\Lambda$-module which is a contradiction.

Now we show that for every $a \in Q_0$, $|a^-| \leq 2$. Assume that there exists a vertex $a$ of $Q_0$ such that $|a^-| \geq 3$, then we have two cases.
• Case 1: The algebra $\Lambda_1 = KQ_1$ given by the quiver $Q_1$

which is a subquiver of $Q$, is a subalgebra of $\Lambda$. There is an indecomposable representation $M$ of $Q_1$ such that $\dim M = [2, 1, 1, 1]^t$. $M$ is not local and by [5, proposition 2.8], $M$ is a 5-factor serial right $\Lambda_1$-module. Then there is a 5-factor serial right $\Lambda$-module which is a contradiction.

• Case 2: The algebra $\Lambda_2 = \frac{KQ_2}{I_2}$ given by the quiver $Q_2$

which is a subquiver of $Q$ and $I_2$ is a restriction of $I$ to $Q_2$, is a subalgebra of $\Lambda$. There is an indecomposable representation $M$ of $(Q_2, I_2)$ such that $\dim M = [3, 1, 1]^t$. $M$ is not local and by [5, proposition 2.8], $M$ is a 5-factor serial right $\Lambda_2$-module. Therefore there is a 5-factor serial right $\Lambda$-module which is a contradiction.

Now we show that for any $\alpha \in Q_1$, there is at most one arrow $\beta$ and at most one arrow $\gamma$ such that $\alpha \beta$ and $\gamma \alpha$ are not in $I$. Now assume that there exist $\alpha, \beta_1, \beta_2 \in Q_1$ such that $\alpha \beta_1$ and $\alpha \beta_2$ are not in $I$. Then we have two cases.

• Case 1: The algebra $\Lambda_1 = KQ_1$ given by the quiver $Q_1$

which is a subquiver of $Q$, is a subalgebra of $\Lambda$. There is an indecomposable representation $M$ of $Q_1$ such that $\dim M = [1, 1, 2, 1]^t$ that $M$ is not local and by [5, proposition 2.8], $M$ is a 5-factor serial right $\Lambda_1$-module. Therefore there is a 5-factor serial right $\Lambda$-module which is a contradiction.

• Case 2: The algebra $\Lambda_2 = \frac{KQ_2}{I_2}$ given by the quiver $Q_2$

which is a subquiver of $Q$ and $I_2$ is a restriction of $I$ to $Q_2$, is a subalgebra of $\Lambda$. Since $R^n \subseteq I$ for some $n \geq 3$, $\alpha^n \in I$ and $\alpha^{n-1}\beta \in I$. Then there is an indecomposable representation $M$ of $(Q_2, I_2)$ such that $\dim M = [2, 4]^t$. $M$ is not
local and by [5, proposition 2.8], \( M \) is a 6-factor serial right \( \Lambda \)-module. Therefore there is a 6-factor serial right \( \Lambda \)-module which is a contradiction.

Now assume that there exist arrows \( \alpha, \gamma_1, \gamma_2 \in Q_1 \) such that \( \gamma_1 \alpha \) and \( \gamma_2 \alpha \) are not in \( I \). Then we have two cases.

- **Case 1:** The algebra \( \Lambda_3 = KQ_3 \) given by the quiver

\[
\begin{array}{c}
3 \\
\downarrow \gamma_1 \\
2 \\
\downarrow \alpha \\
1 \\
\uparrow \gamma_2 \\
4
\end{array}
\]

which is a subquiver of \( Q \), is a subalgebra of \( \Lambda \). There is an indecomposable representation \( M \) of \( Q_3 \) such that \( \dim M = [1, 2, 1, 1]^t \), \( M \) is not local and by [5, proposition 2.8], \( M \) is a 5-factor serial right \( \Lambda_3 \)-module. Therefore there is a 5-factor serial right \( \Lambda \)-module which is a contradiction.

- **Case 2:** The quiver \( Q_4 \) given by

\[
\begin{array}{c}
2 \\
\downarrow \beta \\
1 \\
\downarrow \alpha \\
\uparrow \beta
\end{array}
\]

is a subquiver of \( Q \). Let \( I_4 \) be the restriction of \( I \) to \( Q_4 \). Then \( \Lambda_4 = KQ_4/I_4 \) is a subalgebra of \( \Lambda \). Since \( R^n \subseteq I \) for some \( n \geq 3 \), \( \alpha^n \in I \) and \( \beta \alpha^{n-1} \in I \). There is an indecomposable representation \( M \) of \( (Q_4, I_4) \) such that \( \dim M = [4, 2]^t \), \( M \) is not local and by [5, proposition 2.8], \( M \) is a 6-factor serial right \( \Lambda_4 \)-module. Therefore, there is a 6-factor serial right \( \Lambda \)-module which is a contradiction.

\[\square\]

**Theorem 3.3.** Let \( \Lambda = \frac{KQ}{I} \) be a basic and connected finite dimensional \( K \)-algebra. Then \( \Lambda \) is a right 3-Nakayama algebra if and only if \( \Lambda \) is a special biserial algebra of finite type that \((Q, I)\) satisfying the following conditions:

1. If there exist a walk \( w \) and two different arrows \( w_1 \) and \( w_2 \) with the same target such that \( w_1^{i+1}w_2^{-1} \) is a subwalk of \( w \), then \( w = w_1^{i+1}w_2^{-1} \).
2. If there exist a walk \( w \) and two different arrows \( w_1 \) and \( w_2 \) with the same source such that \( w_1^{-1}w_2^{i+1} \) is a subwalk of \( w \), then \( \text{length}(w) \leq 3 \).
3. If there exist two paths \( p \) and \( q \) with the same target and the same source such that \( p - q \in I \), then \( \text{length}(p) = \text{length}(q) = 2 \).
4. At least one of the following conditions holds.
   1. There exists a vertex \( a \) of \( Q_0 \) such that, \( |a^+| = 2 \).
   2. There exist a walk \( w \) of length \( 3 \) and two different arrows \( w_1 \) and \( w_2 \) with the same source such that \( w_1^{-1}w_2^{i+1} \) is a subwalk of \( w \).
   3. There exist two paths \( p \) and \( q \) with the same target and the same source such that \( p - q \in I \) and \( \text{length}(p) = \text{length}(q) = 2 \).

**Proof.** Assume that \( \Lambda \) is a right 3-Nakayama algebra. By Proposition 3.2, \( \Lambda \) is a special biserial algebra of finite type. Assume that the condition (i) does not hold. Then there
exists a walk $w$ of length greater than or equal to 3, such that $w$ has a subwalk of the form $w_1^{-1}w_2^{-1}$. Since $\Lambda$ is an algebra of finite type, the walk $w_1^{-1}w_2^{-1}$ has one of the following forms:

- First case: The walk $w_1^{-1}w_2^{-1}$ is of the form

```
  1  \\
 w_1 ↗ ⌅ w_2 ↙
   2   3
```

In this case $w$ has a subwalk of one of the following forms:

(i)

```
  1  \\
 w_1 ↗ ⌅ w_2 ↙ ⌅ w_3 ↙ \\
   2   3   a
```

In this case the vertex $a$ can be either 2 or 3 or 4.

(ii)

```
  1  \\
 w_1 ↗ ⌅ w_2 ↙ ⌅ w_3 ↙ \\
   2   3
```

In this case the vertex $a$ can be either 1 or 2 or 3 or 4.

- Second case: The walk $w_1^{-1}w_2^{-1}$ is of the form

```
  1  \\
 w_1 ↗ ⌅ w_2 ↙
   1   2
```

In this case $w$ has a subwalk of one of the following forms:

(i)

```
  1  \\
 w_1 ↗ ⌅ w_2 ↙ ⌅ w_3 ↙ \\
   1   2   a
```

In this case the vertex $a$ can be either 2 or 3.

(ii)

```
  1  \\
 w_1 ↗ ⌅ w_2 ↙ ⌅ w_3 ↙ \\
   1   2
```

In this case the vertex $a$ can be either 1 or 2 or 3.
In this case the vertex $a$ can be either 2 or 3. In all the above cases, there is a non-local indecomposable right $\Lambda$-module of length 4 that by [5, proposition 2.8] is 4-factor serial, which gives a contradiction.

Now assume that the condition (ii) does not hold. Then there exists a walk $w$ of length greater than or equal to 4, such that $w$ has a subwalk of the form $w^{-1}_1 w_2^+1$. Since $\Lambda$ is an algebra of finite type, the walk $w_1^{-1} w_2^+1$ has one of the following forms:

- First case: The walk $w_1^{-1} w_2^+1$ is of the form

  $\begin{array}{c}
    2 \\
    \overrightarrow{w_1} \overrightarrow{w_2} \\
    \overleftarrow{1}
  \end{array}$

  In this case $w$ has a subwalk of one of the following forms:

  - (i) $\begin{array}{c}
      a \\
      \overrightarrow{w_3} \overrightarrow{w_1} \overrightarrow{w_2} \\
      \overleftarrow{1} \overleftarrow{1} \overleftarrow{1}
    \end{array}$

    In this case the vertices $a$ and $b$ can be either $a = 4$ and $b = 5$ or $a = 4$ and $b = 1$.

  - (ii) $\begin{array}{c}
      a \\
      \overrightarrow{w_1} \overrightarrow{w_2} \\
      \overleftarrow{1} \overleftarrow{1} \overleftarrow{1}
    \end{array}$

    In this case the vertices $a$ and $b$ can be either $a = 4$ and $b = 5$ or $a = 4$ and $b = 1$.

- Second case: The walk $w_1^{-1} w_2^+1$ is of the form

  $\begin{array}{c}
    1 \\
    \overrightarrow{w_1} \overrightarrow{w_2} \\
    \overleftarrow{2} \overleftarrow{2}
  \end{array}$
In this case \( w \) has a subwalk of one of the following forms:

(i)

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (0,-1) {3};
  \node (4) at (1,-1) {2};
  \node (5) at (0,-2) {w_3};
  \node (6) at (1,-2) {w_4};
  \draw (1) -- (2);
  \draw (2) -- (4);
  \draw (3) -- (5);
  \draw (4) -- (6);
\end{tikzpicture}
\end{array}
\]

(ii)

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (0,-1) {2};
  \node (4) at (1,-1) {2};
  \node (5) at (0,-2) {w_1};
  \node (6) at (1,-2) {w_3};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (5);
  \draw (5) -- (6);
  \draw (6) -- (1);
\end{tikzpicture}
\end{array}
\]

(iii)

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (1) at (0,0) {2};
  \node (2) at (1,0) {1};
  \node (3) at (0,-1) {2};
  \node (4) at (1,-1) {4};
  \node (5) at (0,-2) {w_2};
  \node (6) at (1,-2) {w_4};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (6);
  \draw (5) -- (1);
\end{tikzpicture}
\end{array}
\]

In all the above cases, there is a 4-factor serial indecomposable right \( \Lambda \)-module of length 5, which gives a contradiction.

Assume that the condition (iii) does not hold. Then there exist two paths \( p = p_1 \ldots p_l \) and \( q = q_1 \ldots q_r \) such that \( p_i, q_j \in Q_1 \), \( s(p_1) = s(q_1), t(p_1) = t(q_r), p - q \in I \) and \( l \geq 3 \).

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (p) at (0,0) {p_1};
  \node (q) at (0,-1) {q_1};
  \node (p1) at (2,0) {p_3};
  \node (q1) at (2,-1) {q_2};
  \node (p2) at (4,0) {p_{l-1}};
  \node (q2) at (4,-1) {q_{r-1}};
  \draw (p) -- (p1);
  \draw (q) -- (q1);
  \draw (p1) -- (p);
  \draw (q1) -- (q);
\end{tikzpicture}
\end{array}
\]

Then the string \( w = p_{l-1}^{-1} p_l^{-1} q_r^{-1} \in \tilde{S}(\Lambda) \). \( M(w) \) is a 4-factor serial right \( \Lambda \)-module which gives a contradiction. Now assume that the condition (iv) does not hold. Then by [5, Theorem 5.13] \( \Lambda \) is a right \( t \)-Nakayama algebra for some \( t \leq 2 \) which is a contradiction. Conversely, assume that \((Q, I)\) satisfies the conditions (i)-(v). By [4], every indecomposable right \( \Lambda \)-module is either string or band or non-uniserial projective-injective. Since \( \Lambda \) is representation finite, \( B(\Lambda) = \emptyset \). The conditions (i), (ii) and (iii) imply that for any \( w \in \tilde{S}(\Lambda) \), \( w \) is either \( w_1^{-1} \ldots w_n^{-1} \) or \( w_1^{-1} w_2^{-1} \) or \( w_1^{-1} w_2^{-1} \) or \( w_1^{-1} w_2^{-1} w_3^{-1} \). If \( w = w_1^{-1} \ldots w_n^{-1} \), then \( M(w) \) is uniserial. If \( w = w_1^{-1} w_2^{-1} \), then \( M(w) \) is 2-factor serial. If \( w = w_1^{-1} w_2^{-1} \) or \( w = w_1^{-1} w_2^{-1} w_3^{-1} \), then \( M(w) \) is 3-factor serial. By the condition (iii), if there exists a non-uniserial projective-injective right \( \Lambda \)-module \( M \), then \( M \) is 3-factor serial. The condition (v) implies that, there exists at least one string module \( M(w) \), where either \( w = w_1^{-1} w_2^{-1} w_3^{-1} \) or \( w_1^{-1} w_2^{-1} \). Thus there exists a 3-factor serial right \( \Lambda \)-module. Therefore \( \Lambda \) is right 3-Nakayama and the result follows.

\[ \square \]

**Remark 3.4.** If the condition (iii) of the Theorem 3.3 holds, then there exists a non-uniserial projective-injective 3-factor serial right \( \Lambda \)-module.
In this section, we first characterize self-injective finite dimensional special biserial algebras of finite type. Then we give a characterization of right 3-Nakayama self-injective algebras.

**Theorem 4.1.** Let $\Lambda = \frac{KQ}{I}$ be a basic and connected finite dimensional $K$-algebra. Then $\Lambda$ is non-Nakayama self-injective special biserial algebra of finite type if and only if $\Lambda$ is given by the quiver $Q = Q_{m,n,s}$ with $s \geq 1$ and $m, n \geq 2$, bounded by the following relations $R_{m,n,s}$:

(i) $\alpha_i^{[i]} \cdots \alpha_i^{[s]} = \beta_i^{[i]} \cdots \beta_i^{[s]}$ for all $i \in \{0, \ldots, s-1\}$;

(ii) $\beta_i^{[i]} \alpha_i^{[i+1]} = 0$, $\alpha_i^{[i]} \beta_i^{[i+1]} = 0$ for all $i \in \{0, \ldots, s-2\}$, $\beta_n^{[s-1]} \alpha_1^{[0]} = 0$ and $\alpha_m^{[s-1]} \beta_1^{[0]} = 0$;

(iii) (a) Paths of the form $\alpha_1^{[i]} \cdots \alpha_1^{[r]}$ of length $m + 1$ are equal to 0;

(b) Paths of the form $\beta_1^{[i]} \cdots \beta_1^{[r]}$ of length $n + 1$ are equal to 0.

**Proof.** It is easy to see that $\Lambda = \frac{KQ}{I}$, where $Q = Q_{m,n,s}$, $I$ is an ideal generated by the relations (i), (ii) and (iii), $m, n \geq 2$ and $s \geq 1$ is a non-Nakayama self-injective special biserial algebra of finite type. Let $\Lambda = \frac{KQ}{I}$ be a non-Nakayama self-injective special biserial algebra of finite type, we show that $Q = Q_{m,n,s}$ with $s \geq 1$ and $m, n \geq 2$ bounded by relations (i), (ii) and (iii).

Since $\Lambda$ is self-injective, $Q$ has no sources and no sinks. Since $\Lambda$ is special biserial self-injective and non-Nakayama, then there exists $b \in Q_0$ such that $|b^+| = 2$. We show that for every vertex $a$ of $Q$, $|a^+| = |a^-|$. Assume on contrary there exists a vertex $a$ such that
|a^-| = 2 and |a^+| = 1. The following quiver is a subquiver of Q.

$$\begin{array}{c}
\cdots \rightarrow a \leftarrow \cdots \\
|a^-| = 2 \quad \text{and} \quad |a^+| = 1.
\end{array}$$

Since \(\Lambda\) is special biserial then either \(\gamma\alpha \in I\) or \(\gamma\beta \in I\). If \(\gamma\alpha \in I\), then the indecomposable injective right \(\Lambda\)-module \(I(b)\) is not projective and if \(\gamma\beta \in I\), then the indecomposable injective right \(\Lambda\)-module \(I(c)\) is not projective which is a contradiction. The same argument shows that there is no vertex \(a \in Q_0\) such that \(|a^+| = 2\) and \(|a^-| = 1\). Consider a vertex \(a\) of quiver \(Q\) such that \(|a^+| = |a^-| = 2\). The following quiver is a subquiver of \(Q\).

$$\begin{array}{c}
\cdots \rightarrow b \leftarrow \cdots \\
|a^-| = 2 \quad \text{and} \quad |a^+| = 1.
\end{array}$$

Since \(\Lambda\) is of finite type, \(n \geq 2\) or \(m \geq 2\). Now we show that both \(m\) and \(n\) are grater than or equal to 2. Assume that \(Q\) has a subquiver of the form

$$\begin{array}{c}
\cdots \rightarrow a \leftarrow \cdots \\
\text{for some } n \geq 2. \quad \text{If for some } i, \beta_1 \cdots \beta_i \in I, \text{ then the indecomposable injective right } \Lambda\text{-module } I(b) \text{ is not projective, which gives a contradiction and if } \beta_1 \cdots \beta_n \notin I, \text{ then } \Lambda \text{ is representation infinite which gives a contradiction. Therefore } m \geq 2.\end{array}$$
For any subquiver $Q'$ of $Q$ of the form

, with $m, n \geq 2$, we show that $\alpha_1...\alpha_m - \beta_1...\beta_n \in I$. Assume on the contrary that $\alpha_1...\alpha_m - \beta_1...\beta_n \notin I$. If either $\alpha_1...\alpha_i \in I$ for some $2 \leq i \leq m$ or $(\beta_1...\beta_j \in I)$ for some $2 \leq j \leq n$, then the indecomposable injective right $\Lambda$-module $I(a)$ is not projective which gives a contradiction. If there is no relation in this subquiver, then $\Lambda$ is not of finite representation type which gives a contradiction.

Assume that $Q$ has a subquiver of the form

, with $m, n, r, l \geq 2$, bounded by relations $\alpha_1...\alpha_m - \beta_1...\beta_n \in I$, $\gamma_1...\gamma_r = \eta_1...\eta_l \in I$ and $\beta_n \gamma_1 = \alpha_m \eta_1 = 0$. We show that in this case $r = m$ and $n = l$. Assume on the contrary that $m > r$. In this case there are two vertices $a$ and $b$ such that the indecomposable projective right $\Lambda$-modules $P(a)$ and $P(b)$ have the same simple socle, which gives a contradiction.

If $m < r$, then there are two vertices $b$ and $c$ such that the indecomposable injective right $\Lambda$-modules $I(b)$ and $I(c)$ have the same simple top, which gives a contradiction.
The similar argument shows that \( n = l \). Finally we show that any paths of form \( \alpha_i^{[j]} \cdots \alpha_h^{[f]} \) of length \( m + 1 \) is zero. First we note that if there exist a positive integer \( t \) and a path \( w \) of the form \( w = \alpha_i^{[j]} \cdots \alpha_h^{[f]} \) of length \( t \) such that \( w = 0 \), then any path of the form \( \alpha_i^{[j]} \cdots \alpha_h^{[f]} \) of length \( t \) should be zero. Since otherwise we can find an indecomposable projective right \( \Lambda \)-module, which is not injective. Now since by the above arguments \( \alpha_1^{[1]} \cdots \alpha_m^{[1]} - \beta_1^{[1]} \cdots \beta_n^{[1]} = 0 \) and \( \beta_n^{[i]} \alpha_1^{[i+1]} = 0, \alpha_1^{[i]} \cdots \alpha_m^{[i]} \alpha_1^{[i+1]} = 0 \). Therefor any paths of form \( \alpha_i^{[j]} \cdots \alpha_h^{[f]} \) of length \( m + 1 \) is zero. The similar argument shows that any paths of the form \( \beta_i^{[j]} \cdots \beta_h^{[f]} \) of length \( n + 1 \) is zero. □

The following Proposition provide a large class of self-injective right \( m+n-1 \)-Nakayama algebras.

Proposition 4.2. Let \( \Lambda = \frac{KQ}{I} \) be a basic and connected finite dimensional \( K \)-algebra such that \( Q = Q_{m,n,s} \) and \( I = R_{m,n,s} \) with \( s \geq 1 \) and \( m, n \geq 2 \). Then \( \Lambda \) is a right \( (m+n-1) \)-Nakayama algebra.

Proof. There exists a projective-injective non-uniserial right \( \Lambda \)-module \( M \) of length \( m+n \), such that for every indecomposable right \( \Lambda \)-module \( N \), \( l(M) \geq l(N) \) and \( \text{rad}(M) \) is not local. Then by [5, Corollary 2.8], \( M \) is \( (m+n-1) \)-factor serial. Therefor \( \Lambda \) is right \( (m+n-1) \)-Nakayama.

Corollary 4.3. Let \( \Lambda = \frac{KQ}{I} \) be a basic, connected and finite dimensional \( K \)-algebra. Then \( \Lambda \) is right \( 3 \)-Nakayama self-injective if and only if \( Q = Q_{2,2,s} \) and \( I = R_{2,2,s} \).

Proof. It follows from Proposition 4.2 and Theorem 4.1 □

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