Exposé
ON A PAPER OF DRONOV AND KAPLITSKII

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In [1] Dronov and Kaplitzki showed that every complemented subspace of a nuclear Köthe space $E$ with a regular basis of type $(d_1)$ has a basis so, in particular, solving the long standing problem whether any complemented subspace of the space $(s)$ of rapidly decreasing sequences has a basis. We present a slightly modified version of their proof which shows that the range of every closed-range operator in $E$ has a basis.

Let $\lambda(A)$ be a nuclear Köthe space a regular basis of type $(d_1)$. The latter means that $E$ has property (DN). Without restriction of generality we may assume:

1. $a_{1,n} = 1$ for all $n$.
2. $a_{k,n}^2 \leq a_{k+1,n}$ for all $k, n$.
3. $\sum_{n=1}^{\infty} \frac{a_{k,n}}{a_{k+1,n}} \leq 1$ for all $k$.
4. $\frac{a_{k,n+1}}{a_{k+1,n+1}} \leq \frac{a_{k,n}}{a_{k+1,n}}$ for all $k, n$.

Due to nuclearity we may use the following two equivalent norm systems

- $|x|_k = \sup_n |x_n| a_{k,n}, k \in \mathbb{N}$.
- $\|x\|_k = \left(\sum_{n=1}^{\infty} |x_n|^2 a_{k,n}^2 \right)^{1/2}, k \in \mathbb{N}$.

Because of 3. above we obtain:

- $|x|_k \leq \|x\|_k \leq |x|_{k+1}, k \in \mathbb{N}$.

The respective local Banach spaces are

$G_k = c_0(a_k)$ and $H_k = \ell_2(a_k)$.

We consider an operator $T \in L(E)$ and we set $F = T(E) \subset E$, We want to study properties of $F$. $T$ is given in the form

$Tx = \left(\sum_{j=1}^{\infty} t_{i,j} x_j \right)_{i \in \mathbb{N}}.$
We define an operator $|T|$ by

$$|T|x = \left( \sum_{j=1}^{\infty} |t_{i,j}|x_j \right)_{i \in \mathbb{N}}.$$ 

To see that this defines an operator $|T| \in L(E)$ we recall the explicit description of the matrices of operators in $c_0$. $(t_{i,j})_{i,j \in \mathbb{N}}$ defines an operator in $c_0$ iff 1. $\sup_i \sum_{j=1}^{\infty} |t_{i,j}| < \infty$ and 2. $\lim_i t_{i,j} = 0$ for all $j$. This description depends on $|t_{i,j}|$ only.

Without restriction of generality we may assume

5. $\| |T|x \|_k \leq \frac{1}{2} |x|_{k+1}$, $k \in \mathbb{N}$.

Next we define three “dead-end spaces” that is continuously imbedded Banach spaces, given by two weights

$$a^2_{\infty,n} := \sum_{k=1}^{\infty} \delta_k^2 a^2_{k,n},$$

where the $\delta_k$ will be determined later, and

$$b_{\infty,n} := a_{n,n}.$$

We set $H_\infty = \ell_2(a_\infty)$ with the norm

$$\|x\|_\infty^2 := \|x\|_{H_\infty}^2 = \sum_{k=1}^{\infty} \delta_k^2 \|x\|_k^2.$$

We set $G_\infty = c_0(a_\infty)$ with the norm

$$|x|_\infty := |x|_{G_\infty} = \sup_n |x_n| a_{\infty,n}.$$

Moreover we set $G_{\infty,0} = c_0(b_\infty)$ with the norm

$$|x|_{\infty,0} := |x|_{G_{\infty,0}} = \sup_n |x_n| b_{\infty,n}.$$

We obtain

$$G_{\infty,0} \subset H_k \subset G_k, \quad k \in \mathbb{N}.$$ 

The second inclusion is obvious, the first one we get from

$$\sum_{n=k+1}^{\infty} a^2_{k,n} |x_n|^2 = \sum_{n=k+1}^{\infty} \frac{a^2_{k,n}}{a^2_{n,n}} a^2_{n,n} |x_n|^2 \leq \left( \sum_{n=k+1}^{\infty} \frac{a^2_{k,n}}{a^2_{n,n}} \right) |x|_{\infty,0}^2 \leq |x|_{\infty,0}^2.$$

With some constant $D_k$ we have

$$\|x\|_k \leq D_k |x|_{\infty,0}.$$ 

2
We may assume $D_k \leq D_{k+1}$ for all $k$. We obtain

$$\|Tx\|_\infty^2 = \sum_{k=1}^{\infty} \delta_k^2 \|Tx\|^2_k \leq \sum_{k=1}^{\infty} \delta_k^2 \|x\|^2_{k+1} \leq \left( \sum_{k=1}^{\infty} \delta_k^2 D_{k+1}^2 \right) \|x\|_{\infty,0}^2 \leq \|x\|_{\infty,0}^2,$$

where we have chosen, also under consideration of later application, $\delta_k \leq 1/(2^k D_{k+2})$. So we have shown

$$(1) \quad \|Tx\|_\infty \leq |x|_{\infty,0}$$

This shows that $L := \{Tx : x \in G_{\infty,0}\} \subset H_\infty$. We set $F_k$ the completion of $L$ with respect to $\|\cdot\|_k$ and $F_\infty$ the completion with respect to $\|\cdot\|_\infty$. The embedding $J : F_\infty \hookrightarrow F_1$ is clearly nuclear, hence we can expand it as

$$J(x) = \sum_{j=1}^{\infty} \langle x, f_j \rangle_{H_1} f_j$$

where $(f_j)_{j \in \mathbb{N}}$ is orthogonal in $H_\infty$, orthonormal in $H_1$. We set

$$T_n(x) = \sum_{j=1}^{n} \langle Tx, f_j \rangle_{H_1} f_j.$$

For every $x \in G_{\infty,0}$ we get $T_n(x) \to Tx$ in $H_\infty$. We want to show that the family of maps $\{T_n : n \in \mathbb{N}\}$ is equicontinuous in $L(E)$. This will imply

$$Tx = \sum_{j=1}^{\infty} \langle Tx, f_j \rangle_{H_1} f_j$$

for all $x \in E$, the series converging in $E$.

**Theorem:** If $T(E)$ is closed, then $F_\infty \subset T(E)$, hence all $f_j \in T(E)$. So the $f_j$ are a basis in $T(E)$.

Because of orthogonality we have

- $\|T_n x\|_1 \leq \|Tx\|_1$.
- $\|T_n x\|_\infty \leq \|Tx\|_\infty$.

For $x \in \varphi^+$ we have $\|Tx\|_1 \leq \|T|x\|_1$ and therefore we have

- $\|T_n x\|_1 \leq \|T|x\|_1$. 

3
To get an estimate between the $\| \cdot \|_\infty$ and the $| \cdot |_\infty$ norm, we fix some $r \in \mathbb{N}$ and obtain

$$
\| x \|_\infty = \| x \|_{\ell_2(\infty, n)} \leq \| x \|_{\ell_1(\infty, n)} = \sum_{n=1}^{\infty} |x_n| a_{\infty, n} \leq \sum_{n=1}^{\infty} \frac{a_{r, n}}{a_{r+1, n}} \left( \frac{a_{r+1, n}}{a_{r, n}} |x_n| a_{\infty, n} \right)
$$

(2)

$$
\leq \sum_{n=1}^{\infty} \frac{a_{r, n}}{a_{r+1, n}} \sup_{n \in \mathbb{N}} \left\{ \frac{a_{r+1, n}}{a_{r, n}} |x_n| a_{\infty, n} \right\}
$$

$$
\leq \left| \frac{a_{r+1}}{a_r} x \right|_\infty
$$

for all $x \in G_{\infty, 0}$.

To see that $\frac{a_{r+1}}{a_r} x \in G_{\infty}$ for all $x \in G_{\infty, 0}$ we use the estimate:

$$
\| \frac{a_{r+1}}{a_r} x \|_k^2 \leq \| a_k x \|_k^2 = \sum_{n=1}^{\infty} a_{k, n} |x_n|^2 \leq \sum_{n=1}^{\infty} a_{k+1, n} |x_n|^2 = \| x \|_{k+1}^2
$$

for all $k > r$. From that we obtain

$$
\| \frac{a_{r+1}}{a_r} x \|_k^2 = \sum_{k=1}^{\infty} \delta_k^2 \left( \frac{a_{r+1}}{a_r} x \right)_k^2
$$

$$
\leq \sum_{k=1}^{r} \delta_k^2 \left( \frac{a_{r+1}}{a_r} x \right)_k^2 + \sum_{k=r+1}^{\infty} \delta_k^2 \left( \frac{a_{r+1}}{a_r} x \right)_k^2
$$

$$
\leq \| x \|_{r+1}^2 \sum_{k=1}^{r} \delta_k^2 + \sum_{k=r+1}^{\infty} \delta_k^2 \| x \|_{k+1}^2
$$

$$
= \| x \|_{r+1}^2 \sum_{k=1}^{r} \delta_k^2 + \sum_{k=r+1}^{\infty} \delta_k^2 \| x \|_{k+1}^2
$$

$$
\leq (D_{r+1}^2 + \sum_{k=r+1}^{\infty} \delta_k^2 D_{k+1}^2) \| x \|_{\infty, 0}^2
$$

$$
\leq (D_{r+1}^2 + 1) \| x \|_{\infty, 0}^2
$$

Therefore $\frac{a_{r+1}}{a_r} x \in H_{\infty} \subset G_{\infty}$ for all $x \in G_{\infty, 0}$.

We apply the previous to $|T| x$ and obtain:

$$
\| \frac{a_{r+1}}{a_r} |T| x \|_\infty^2 \leq \| |T| x \|_{r+1}^2 \sum_{k=1}^{r} \delta_k^2 + \sum_{k=r+1}^{\infty} \delta_k^2 \| |T| x \|_{k+1}^2
$$

$$
\leq \| x \|_{r+2}^2 \sum_{k=1}^{r} \delta_k^2 + \sum_{k=r+1}^{\infty} \delta_k^2 \| x \|_{k+2}^2
$$

$$
\leq (D_{r+2}^2 + \sum_{k=r+1}^{\infty} \delta_k^2 D_{k+2}^2) \| x \|_{\infty, 0}^2
$$

$$
\leq (D_{r+2}^2 + 1) \| x \|_{\infty, 0}^2.
$$
We define $J_r x := \frac{a_{r+1}}{a_r} x$ and we have shown

$$\|J_r |T| x\|_\infty \leq M_r |x|_{\infty,0}$$

with $M_r^2 = D_r^2 + 1$.

We will use the canonical projection in the sequence space $E$:

$$(Q^N x)_n = \begin{cases} x_n, & n = 1, 2, \ldots, N \\ 0, & n > N. \end{cases}$$

We obtain for all $x \in G_1$

$$|J_r J'_r x|_\infty \leq |J_r |T| J'_r x|_\infty \leq |T x|_\infty \leq \| |T| x\|_\infty \leq |J_r |T| x|_\infty.$$

The last inequality comes from (2).

For $x \in \varphi^+$ we get the estimates:

$$|T_n J'_r x|_\infty \leq |J_r |T| J'_r x|_\infty = |A_r x|_\infty$$

$$|T_n J'_r x|_1 \leq |J_r |T| J'_r x|_1 \leq |x|_1$$

In the second line the first estimate comes from

$$\|x\|_1^2 = \sum_n |x_n|^2 = \sum_n \frac{a_r^2}{a_{r+1}^2} \frac{a_{r+1}^2}{a_r^2} |x_n|^2 \leq |J_r x|_1^2$$

which implies

$$|T_n x|_1 \leq \|T_n x\|_1 \leq \|T x\|_1 \leq \| |T| x\|_1 \leq |J_r |T| x|_1.$$

The second estimate is (4).

We define

$$S_{n,r} = T_n J'_r \quad \text{and} \quad S_{n,r}^{(N)} = S_{n,r} Q^{(N)}.$$
The first of the inequalities above is not applicable for interpolation, but it becomes applicable when we restrict it to a suitable cone.

We set
\[ Q_{r,N} = \{ x \in \omega^+ : x \geq A_r^{(N)}x \}. \]

Then \( Q_{r,N} \) is a cone and we have:
\[ |S_{n,r}^{(N)}x|_1 \leq |x|_1 \text{ for } x \in G_1^+, \]
\[ |S_{n,r}^{(N)}x|_{\infty} \leq |x|_{\infty} \text{ for } x \in Q_{r,N} \cap G_{\infty}^+. \]

By use of the interpolation theorem for cones [1, Theorem 1] we obtain that for every \( r \) there is a constant \( C(r) \) such that
\[ |S_{n,r}^{(N)}x|_r \leq C(r) \| x \|_r \text{ for } x \in Q_{r,N} \cap G_r^+. \]

Since \( A_r^{(N)} \in L(G_r) \) with \( \| A_r^{(N)} \| \leq 1/2 \) the operator \( B := I - A_r^{(N)} \) is invertible in \( L(G_r) \) and \( \| B^{-1} \| \leq 2 \). We can write for \( x \in G_r \)
\[ x = B^{-1}Bx = B^{-1}(Bx)_+ - B^{-1}(Bx)_. \]

Clearly \( B^{-1}(Bx)_+, B^{-1}(Bx)_+ \in Q_{r,N} \cap G_r^+ \). Positivity is seen by the Neumann series. Since \( \| B^{-1}B \| \leq 4 \), we have shown that every \( x \in G_r \) can be written as \( x = y - z \) here \( y, z \in Q_{r,N} \cap G_r^+ \) and \( \| y \|_r \leq 4 \| x \|_r \) and \( \| z \|_r \leq 4 \| x \|_r \).

It follows that
\[ |S_{n,r}^{(N)}x|_r \leq 8C(r) \| x \|_r \text{ for } x \in G_r. \]

With \( N \to \infty \) we conclude that
\[ |T_nJ_r'x|_r \leq 8C(r) \| x \|_r \text{ for } x \in G_r. \]

Finally we have
\[ |T_nx|_r \leq 8C(r) \| a_{r+2}x \|_r \leq 8C(r) \| x \|_{r+3}. \]

Since that can be done for all \( r \) the family \( (T_n)_{n \in \mathbb{N}} \) is equicontinuous in \( E \) which had to be shown.

We have to verify the conditions of the cone interpolation theorem [1, Theorem 1].

1. \( Q_{r,N} \) is a lower semi-lattice. This follows immediately from the fact that \( A_r^{(N)} \) is monotone on \( \omega^+ \).

2. \( Q_{r,N} \cap G_{\infty}^+ \) is total in \( G_{\infty} \). We have seen that \( I - A_r^{(N)} \) is invertible in \( G_r \), hence injective on \( G_{\infty} \). We show that \( A_r^{(N)} \in L(G_{\infty}) \). Then, as \( A_r^{(N)} \) is finite dimensional, \( I - A_r^{(N)} \) is bijective in \( G_{\infty} \). Arguing as above we obtain that \( Q_{r,N} \cap G_{\infty}^+ - Q_{r,N} \cap G_{\infty}^+ = G_{\infty} \).

We have the following chain of inequalities, where the second one is \( \| \) the last one finite dimensionality of \( Q^{(N)} \).
\[ |A_r^{(N)}x|_{\infty} = |J_r|P|J_r'Q^{(N)}x|_{\infty} \leq \| J_rP|J_r'Q^{(N)}x\|_{\infty} \leq |J_r'Q^{(N)}x|_{\infty,0} \leq |Q^{(N)}x|_{\infty,0} \leq C(N) \| x \|_{\infty}. \]
This completes the argument for 2.

3. Finally we have to show that $Q_{r,N} \cap G^+_\infty$ contains a strictly positive element. To show that we choose a strictly positive element $x_0 \in G_\infty$ and put $x = (I - A^{(N)}_r)^{-1}x_0$. $x$ can be calculated by means of the Neumann series in $G_r$. Since $A^{(N)}_r$ is positive this shows that $x \geq x_0$ which shows the claim.

References

[1] A. K. Dronov, V. M. Kaplitzkii, On the existence of a basis in a complemented subspace of a nuclear Köthe space of class $(d1)$. (Russian), Mat. Sb.209(2018), no.10, 50–70; translation in Sb. Math. 209 (2018), no.10, 1463–1481.

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