Narasimhan-Simha pseudonorms, envelopes and submultiplicative norms on section rings

Siarhei Finski

Dedicated to Xiaonan Ma on the occasion of his 50th birthday

Abstract. We study the set of submultiplicative norms on section rings of ample line bundles over compact complex manifolds. As the main application, we establish that over canonically polarized manifolds, the convex hull of Narasimhan-Simha pseudonorms over pluricanonical sections is asymptotically equivalent to the sup-norm associated to the supercanonical metric of Tsuji, as the tensor power of the canonical line bundle tends to infinity. As another application, we deduce that in the same asymptotic regime the $L^p$-norms, $p \in [1, +\infty]$, on section rings of ample line bundles associated to continuous metrics are asymptotically equivalent to the $L^\infty$-norms associated to their plurisubharmonic envelopes. This refines previous results of Berman-Demailly and Berman, stating that similar relations hold on the weaker level of Fubini-Study convergence. An important step in our proof is to establish that injective and projective tensor norms on symmetric algebras of finitely dimensional normed complex vector spaces are asymptotically equivalent.

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1 Introduction

We study the set of submultiplicative norms on section rings of ample line bundles. From this study, we derive the asymptotics of Narasimhan-Simha pseudonorms over the vector space of pluricanonical sections, as the power of the canonical line bundle tends to infinity, and we show that the $L^p$-norms, $p \in [1, +\infty]$, on section rings of ample line bundles of continuous metrics are asymptotically equivalent to the $L^\infty$-norms of their plurisubharmonic envelopes.

More precisely, we fix a compact complex manifold $X$ of dimension $n$ and denote by $K_X \equiv \bigwedge^n T(1,0)^* X$ its canonical line bundle. Narasimhan-Simha in [20] defined pseudonorms $\| \cdot \|_{k}^{NS}$, $k \in \mathbb{N}^*$, over the vector space of $k$-th pluricanonical sections, $f \in H^0(X, K_X^k)$, as

$$\| f \|_{k}^{NS} := \left( \int_X \left( -\sqrt{-1} \right)^{k(n^2+2n)} \cdot (f \wedge \overline{f})^{\frac{n}{k}} \right)^k. \quad (1.1)$$

By a pseudonorm over a finitely dimensional vector space $V$, we mean a non-negative absolutely homogeneous continuous function over $V$, which is equal to 0 only at $0 \in V$. 

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Remark that the sequence of pseudonorms $\mathcal{NS}_k$, $k \in \mathbb{N}^*$, is defined without the use of any fixed metric on $K_X$. In particular, it depends only on the complex structure of $X$. Even more, it is a birational invariant, as birational equivalence between two complex manifolds $X$ and $Y$ induces the isometry with respect to $\mathcal{NS}_k$ between $H^0(X, K^0_X)$ and $H^0(Y, K^0_Y)$ for any $k \in \mathbb{N}^*$, cf. [20].

Assume now more generally that a pair $(X, \Delta)$ of $X$ with a $\mathbb{Q}$-divisor $\Delta$ is klt, see Section 3 for necessary definitions. We denote the log canonical $\mathbb{Q}$-line bundle by $K_X(\Delta) := K_X \otimes \mathcal{O}_X(\Delta)$ and let $r := r_\Delta \in \mathbb{N}^*$ be the minimal number such that $r_\Delta$ is a $\mathbb{Z}$-divisor. Any section $f \in H^0(X, K_X(\Delta)^{kr})$, $k \in \mathbb{N}^*$, can be then interpreted as a meromorphic section of $K^0_X$. The klt condition implies, see (3.3) and after, that the integrand in (1.1) is finite. We denote by $\mathcal{NS}_{kr}^\Delta := \| \cdot \|_{kr, \Delta}$ the pseudonorm on $H^0(X, K_X(\Delta)^{kr})$, given by this integral.

Now, in a different direction, assume that a holomorphic line bundle $L$ over $X$ is endowed with a continuous metric $h^L$. Then we have a natural sequence of norms $\text{Ban}^\infty_k(h^L) := \| \cdot \|_{k, \infty}^{h^L}$ over $H^0(X, L^k)$, defined for $f \in H^0(X, L^k)$ as follows

$$\|f\|_{k, \infty}^{h, L} = \sup_{x \in X} |f(x)|_{h^L}.$$  (1.2)

In this article, we answer the following basic question: is there a relation between the two constructions above? To describe precisely this question and our answer to it, we need to introduce a certain equivalence relation on graded pseudonorms over section rings.

For a holomorphic line bundle $L$ over $X$, we consider the section ring $R(X, L)$, defined as

$$R(X, L) := \oplus_{k=1}^\infty H^0(X, L^k).$$  (1.3)

The multiplication maps, see (4.8), endow $R(X, L)$ with the structure of a graded ring.

For a Hermitian metric $h^L$ on $L$, we define over $R(X, L)$ the induced graded norms

$$\text{Ban}^\infty_k(h^L) := \sum_{k=1}^\infty \text{Ban}^\infty_k(h^L).$$  (1.4)

Over log canonical ring, $R(X, K_X(\Delta)^r)$, we define the Narasimhan-Simha graded pseudonorm

$$\mathcal{NS}^\Delta := \sum_{k=1}^\infty \mathcal{NS}_{kr}^\Delta.$$  (1.5)

Recall that one can naturally associate the convex hull norm $\text{Conv}(N_V) := \| \cdot \|_{V}^{\text{conv}}$ for any pseudonorm $N_V := \| \cdot \|_V$ over a finitely dimensional vector space $V$ as follows

$$\|v\|_V^{\text{conv}} = \inf \left\{ \sum \|v_i\|_V : \sum v_i = v \right\}.$$  (1.6)

Geometrically, the unit ball of $\text{Conv}(N_V)$ is the convex hull of the unit ball of $N_V$. We extend this definition to graded pseudonorms by considering the convex hulls of graded pieces.

We say that two graded pseudonorms $N = \sum N_k$, $N' = \sum N'_k$ over $R(X, L)$ are equivalent ($N \sim N'$) if the multiplicative gap between the graded pieces, $N_k$ and $N'_k$, is subexponential. This means that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}^*$, such that for any $k \geq k_0$, we have

$$\exp(-\epsilon k) \cdot N_k \leq N'_k \leq \exp(\epsilon k) \cdot N_k.$$  (1.7)
Recall that a plurisubharmonic (or psh) metric $h^L$ on a holomorphic line bundle $L$ is a (singular) metric such that for any local holomorphic section $\sigma$ of $L$, $-\log |\sigma|_{h^L}$ is psh. A line bundle is called pseudoeffective (or psef) if it carries a psh metric.

The equivalence relation (1.7) is well-motivated since for ample $L$, it distinguishes graded norms $\text{Ban}^\infty(h^L)$ for different continuous psh metrics $h^L$ over $L$, cf. [14, Theorem 1.7].

Now, any psh metric $h^{K,\Delta}$ on $K_X(\Delta)^r$ induces a volume form (with singularities) on $X$, denoted by $dV_{h^{K,\Delta}}$, see (3.4) for details. If the pair $(X, \Delta)$ is klt, $dV_{h^{K,\Delta}}$ is of finite volume, see (3.3) and after. Recall that Tsuji in [26] defined the supercanonical metric $h_{\text{can}}^{K,\Delta}$ on $K_X(\Delta)^r$ over klt pairs $(X, \Delta)$ with psef $K_X(\Delta)^r$ through the following envelope construction: for $x \in X$, we let

$$h_{\text{can}}^{K,\Delta}(x) = \inf \left\{ h^{K,\Delta} : h^{K,\Delta} \text{ is a psh metric on } K_X(\Delta)^r, \text{ with } \int_X dV_{h^{K,\Delta}} \leq 1 \right\}. \quad (1.8)$$

Recall that a pair $(X, \Delta)$ is called log canonically polarized if $K_X(\Delta)^r$ is ample. Berman-Demailly in [8, Theorem 5.5] showed that $h_{\text{can}}^{K,\Delta}$ is continuous and psh for such pairs.

**Theorem 1.1.** For log canonically polarized klt pairs $(X, \Delta)$, the following equivalence of graded norms on the log canonical ring $R(X, K_X(\Delta)^r)$ holds

$$\text{Conv}(\mathcal{NS}^\Delta) \sim \text{Ban}^\infty(h_{\text{can}}^{K,\Delta}). \quad (1.9)$$

**Remark 1.2.** Taking convex hull is necessary. In fact, for any $f \in H^0(X, K_X(\Delta)^{kr}, k \in \mathbb{N}^*$, both the Narasimhan-Simha pseudonorms and sup-norms behave multiplicatively on the sequence $f^l$ for $l \in \mathbb{N}^*$. Hence, if the statement would hold without taking the convex hull, it would imply that Narasimhan-Simha pseudonorms coincide identically with the sup-norm, which is false.

The proof of Theorem 1.1 is based on the result of Berman-Demailly [8, Proposition 5.19], cf. Theorem 3.1 about the convergence of Fubini-Study metrics of Narasimhan-Simha pseudonorms to the supercanonical metric and on the classification of submultiplicative norms on section rings of ample line bundles, see Theorem 4.2 which refines our previous work [14].

An important ingredient in the proof of this classification theorem is to establish that injective and projective tensor norms on symmetric algebras of finitely dimensional normed complex vector spaces are asymptotically equivalent, see Theorem 5.1. This contrasts a lot with full tensor algebras, where the analogous norms are essentially never equivalent, see Remark 5.2. Surprisingly, the proof of the latter functional-analytic statement uses tools from complex geometry, as Ohsawa-Takegoshi extension theorem. In particular, the proof wouldn’t work for real vector spaces, which is hardly surprising, as for them the analogous statement fails, see Remark 5.2(b).

We will now describe another application of our classification theorem. Let $L$ be an ample line bundle on a compact complex manifold $X$ and $h^L$ be a continuous metric on $L$. Recall that the plurisubharmonic envelope $P(h^L)$ of $h^L$ is the metric on $h^L$, defined for any $x \in X$ as follows

$$P(h^L)(x) := \inf \left\{ h^L_0(x) : h^L_0 \text{ is a psh metric on } L, \text{ verifying } h^L \leq h^L_0 \text{ over } X \right\}. \quad (1.10)$$

It is possible to prove, cf. Lemma 5.1 that $P(h^L)$ is continuous and psh for continuous $h^L$.

We fix a positive volume form $d\mu$ over $X$. Then we define a sequence of norms $\text{Ban}^p_k(h^L, d\mu) := \| \cdot \|_{L^p_k(X, h^L, d\mu)}$ for $f \in H^0(X, L^k)$ as follows

$$\|f\|_{L^p_k(X, h^L, d\mu)} = \left( \int_{x \in X} |f(x)|_{h^L}^p d\mu(x) \right)^\frac{1}{p}. \quad (1.11)$$
Similarly to (1.4), we define over $R(X,L)$ the induced graded norms $\text{Ban}^p(h^L,d\mu)$. Clearly, the equivalence class of $\text{Ban}^p(h^L,d\mu)$ doesn’t depend on the choice of $d\mu$. Due to this, by an abuse of notations, we denote the above graded norms by $\text{Ban}^p(h^L)$.

**Theorem 1.3.** For any $p \in [1, +\infty]$, the following equivalence of graded norms on $R(X,L)$ holds
\[
\text{Ban}^p(h^L) \sim \text{Ban}^\infty(P(h^L)).
\] (1.12)

Theorem 1.3 refines a result of Berman [6 Theorem 4.11], cf. Lemma 8.3, stating that a similar relation holds on the weaker level of Fubini-Study convergence, see Remark 4.3b).

To conclude, we mention that historically, Narasimhan-Simha pseudonorms have been introduced in [20] to study moduli problems. In family setting, the study of positivity of related (pseudo)norms is linked to Iitaka conjecture and invariance of plurigenera problem, see Kawamata [16], Berndtsson-Păun [9], Păun-Takayama [21]. See also Amini-Nicolussi [1], [2] and Shivaprasad [25], [24] for other related result in singular family setting.

This paper is organized as follows. In Section 2, we recall the definition of the Fubini-Study operator for pseudonorms on the cohomology. In Section 3, we recall the klt condition and the result of Berman and Demailly on the convergence of Fubini-Study operators for Narasimhan-Simha pseudonorms. In Section 4, we state a classification theorem for submultiplicative graded norms, see Theorem 4.2 and then deduce Theorem 1.1 from it. In Section 5, we show that Theorem 4.2 follows from its specialization on the projective space, which can be reformulated purely in functional-analytic terms, see Theorem 5.1. In Section 6, we establish a special case of Theorem 5.1 for $l_1$-norms using subexponential bound in Bohnenblust–Hille inequality. In Section 7, we deduce the general case of Theorem 5.1 by relying on the established case for $l_1$-norms and on a version of Ohsawa-Takegoshi extension theorem. Finally, in Section 8, we prove Theorem 1.3.

**Notation.** A sequence of numbers (resp. positive numbers) $a_k, k \in \mathbb{N}$, is called subadditive (resp. submultiplicative) if $a_{k+l} \leq a_k + a_l$ (resp. $a_{k+l} \leq a_k a_l$) for any $k, l \in \mathbb{N}$. We extend those notions for sequences of functions and for metrics on powers of a line bundle. A sequence of positive real numbers $a_k$ is called subexponential if for any $\epsilon > 0, a_k \leq \exp(\epsilon k)$ for $k$ big enough.

Over $\mathbb{C}^l, l \in \mathbb{N}^*$, we denote by $l_1 = \| \cdot \|_1$ and $l_\infty = \| \cdot \|_\infty$ the norms, defined for $x = (x_1, \cdots, x_l)$ as follows $\|x\|_1 = \sum |x_i|, \|x\|_\infty = \max |x_i|$. By a multiplicative gap between the pseudonorms $N_1, N_2$ on a finitely dimensional vector space $V$, we mean the minimal constant $C > 0$, such that both inequalities $N_1 \leq CN_2$ and $N_2 \leq CN_1$ are satisfied.

By a singular metric on a line bundle $L$, we mean a metric $h^L$, which can be written as $e^{-\phi} \cdot h^L_0$, where $h^L_0$ is a smooth metric and $\phi \in L^1_{\text{loc}}$. We denote by $| \cdot |_{h^L}$ the induced pointwise norm on $L$. In the notations of (1.4), we sometimes denote the graded norm $\text{Ban}^\infty(h^L)$ by $\text{Ban}^\infty_X(h^L)$ to underline the dependence on the ambient manifold, $X$.

We also mention that some authors use the term Finsler metric for what we call a pseudonorm in this article. We decided to follow the notations from [20].

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## 2 Fubini-Study metrics of pseudonorms and their positivity

In this section we recall the definition of the Fubini-Study operator and its positivity properties.

We fix an ample line bundle $L$ over a compact complex manifold $X$. Recall that for $k$ so that
$L^k$ is very ample, Fubini-Study operator associates for any norm $N_k = \| \cdot \|_k$ on $H^0(X, L^k)$, a continuous metric $FS(N_k)$ on $L$, constructed in the following way. Consider the Kodaira embedding

$$Kod_k : X \hookrightarrow \mathbb{P}(H^0(X, L^k)^*).$$ (2.1)

The evaluation maps provide the isomorphism $L^{-k} \cong Kod_k^* \mathcal{O}(-1)$, where $\mathcal{O}(-1)$ is the tautological bundle over $\mathbb{P}(H^0(X, L^k)^*)$. We endow $H^0(X, L^k)^*$ with the dual norm $N_k^*$ and induce from it a metric $h_{FS}(N_k)$ on $\mathcal{O}(-1)$ over $\mathbb{P}(H^0(X, L^k)^*)$. We define the metric $FS(N_k)$ on $L^k$ as the only metric verifying under the dual of the above isomorphism the identity

$$FS(N_k) = Kod_k^*(h_{FS}(N_k)^*).$$ (2.2)

Sometimes, by an abuse of notations, we denote by $FS(N_k)$ the metric $h_{FS}(N_k)^*$ on $\mathcal{O}(1)$ over $\mathbb{P}(H^0(X, L^k)^*)$. A statement below can be seen as an alternative definition of $FS(N_k)$.

**Lemma 2.1.** For any $x \in X$, $l \in L^k_x$, the following identity takes place

$$\|l\|_{FS(N_k)} = \inf_{s \in H^0(X, L^k)} \|s\|_k.$$ (2.3)

**Proof.** An easy verification, cf. Ma-Marinescu [19, Theorem 5.1.3].

The above construction of the Fubini-Study metric works more generally for pseudonorms $N_k$. In this case, since the Fubini-Study operator uses the dual of the pseudonorm and double dual of a pseudonorm is equal to its convex hull, we clearly have

$$FS(N_k) = FS(Conv(N_k)).$$ (2.4)

When the norm $N_k$ comes from a Hermitian product on $H^0(X, L^k)$, the above construction is standard and explicit evaluation shows that in this case $c_1(\mathcal{O}(-1), h_{FS}(N_k))$ coincides up to a negative constant with the Kähler form of the Fubini-Study metric on $\mathbb{P}(H^0(X, L^k)^*)$ induced by $N_k$. In particular, $c_1(\mathcal{O}(-1), h_{FS}(N_k))$ is a negative $(1, 1)$-form.

Let us now discuss the positivity properties of the metric $FS(N_k)$ for general pseudonorms $N_k$. A pseudonorm $N_V := \| \cdot \|_V$ on a vector space $V$ defines a continuous function $F_V : V \to [0, +\infty]$, $v \mapsto \|v\|_V^2$. Following Kobayashi’s terminology on Finsler metrics, [17], we say that $N_V$ is pseudoconvex if we have $\sqrt{-1}\partial \bar{\partial} F_V \geq 0$ in the sense of currents. Clearly, if $N_V$ is a norm, the function $F_V$ is convex by triangle inequality and then $N_V$ is trivially pseudoconvex.

Pseudonorms $N_V$ on $V$ are in one-to-one correspondence with Hermitian metrics $h^{N_V}$ on the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}(V)$. According to [17, Lemma on p.160], pseudoconvexity of $N_V$ is equivalent to the negativity of the $(1, 1)$-current $c_1(\mathcal{O}(-1), h^{N_V})$. In particular, for any norm $N_V$ on $V$, we have $c_1(\mathcal{O}(-1), h^{N_V}) \leq 0$ in the sense of currents. Hence, from (2.2) and (2.4), the (singular) metric $FS(N_k)$ is psh for any pseudonorm $N_k$ on $H^0(X, L^k)$.

### 3 Kawamata log terminal divisors and the result of Berman-Demailly

The main goal of this section is to recall Kawamata log terminal (or klt) condition for $\mathbb{Q}$-divisors and a convergence result of Berman-Demailly.
We say that for a normal crossing divisor $\sum D_i$ on $X$, the $\mathbb{Q}$-divisor $\sum d_iD_i$, $d_i \in \mathbb{Q}$, is klt if for any index $i$, we have $d_i < 1$. More generally, a $\mathbb{Q}$-divisor $D$ is klt if for a resolution of singularities $\pi : \tilde{X} \to X$ of $|D|$ and the normal crossing $\mathbb{Q}$-divisor $\tilde{D}$, verifying
\[ K_{\tilde{X}} + \tilde{D} = \pi^*(K_X + D), \tag{3.1} \]
the pair $(\tilde{X}, \tilde{D})$ is klt. This definition doesn’t depend on the choice of the resolution $\pi$, cf. [18, Lemma 3.10].

The klt condition can be restated in the following more differential-geometric way. Let $r := r_D \in \mathbb{N}^*$ be the minimal number such that the element $rD$ is a $\mathbb{Z}$-divisor. We denote by $h^{rD}$ the canonical (singular) metric on the line bundle $\mathcal{O}_X(rD)$, defined as
\[ |s|_{h^{rD}}(x) = 1, \quad \text{for } x \notin |D|, \tag{3.2} \]
where $s$ is the canonical (meromorphic) section of $\mathcal{O}_X(rD)$. According to [18, Proposition 3.20], the klt condition is equivalent to the fact that
\[ \left( \frac{h^{rD}}{h^{sm}} \right)^{\frac{1}{r}} \text{ is integrable over } X, \tag{3.3} \]
where $h^{sm}$ is some (hence, any) smooth metric on $\mathcal{O}_X(rD)$.

Let us now recall that psh metrics on log canonical line bundles of klt pairs give rise to positive integrable volume forms. More precisely, assume first that $h_0^K$ is a smooth metric on $K_X$. We define the positive volume form $dV_{h^K}$ by requiring that for any $x \in X$, we have
\[ dV_{h^K}(x) = (-\sqrt{-1})^{n^2+2n}dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n, \tag{3.4} \]
where $|dz_1 \wedge \cdots \wedge dz_n|_{h^K}(x) = 1$. This construction can be extended to psh metrics $h^K$ on $K_X$ by writing $h^K = e^{-\phi} \cdot h_0^K$ for $\phi \in L^1_{\text{loc}}$ and defining $dV_{h^K} := e^\phi \cdot dV_{h^K_0}$. Clearly, the result doesn’t depend on the choice of $h_0^K$. The volume form $dV_{h^K}$ is bounded since any quasi-psh function $\phi$ is bounded. It might, nevertheless, vanish, as $\phi$ is allowed to take $-\infty$ values.

Now, more generally a psh metric $h^{K,D}$ on $K^*_X \otimes \mathcal{O}_X(rD)$ defines a singular metric $h^{K,D}_{\text{sing}} = \frac{h^{K,D}}{h^{rD}}$ on $K^*_X$. Then $h^{K,D}$ defines a (singular) volume form $dV_{h^{K,D}}$ as $dV_{h^{K,D}} = e^\phi \cdot dV_{h^{K}_{\text{can}}}$, where $\phi$ is so that $h^{K,D}_{\text{sing}} = e^{-r\phi} \cdot (h_0^K)^r$. For $(X, D)$ klt, $dV_{h^{K,D}}$ is integrable by (3.3).

This property was of utmost importance in the definition of supercanonical metric in (1.8). Let us now recall a related result about the convergence of Fubini-Study metrics associated to Narasimhan-Simha pseudonorms. Below we use the notations from Section[1]

**Theorem 3.1** (Berman-Demailly [8 Proposition 5.19 and Remark 5.23]). For a log canonically polarized klt pair $(X, \Delta)$, the sequence of Hermitian metrics $FS(\mathcal{N}^\Delta_S)^{\frac{1}{r}}$ on $K_X(\Delta)^r$ converges uniformly, as $k \to \infty$, to the supercanonical metric, $h^{K,\Delta}_{\text{can}}$.

**Remark 3.2.** In [8], authors assume that $\Delta$ is effective, but this is never used in the proof.

**Proof.** The result was not stated in [8] in exactly the same language, so we give more details.

Let us fix (smooth) Hermitian metrics $h^K$, $h^{\Delta}_{\text{sm}}$ over $K^*_X$ and $\mathcal{O}_X(r\Delta)$. For the canonical singular metric $h^{r\Delta}$ on $\mathcal{O}_X(r\Delta)$, we define the function $\gamma$ so that $h^{r\Delta} = e^{-r\gamma} \cdot h^{\Delta}_{\text{sm}}$. Following [8 Definition 5.3], for any $x \in X$, we define
\[ \phi^{\text{can}}(x) := \sup \left\{ \phi(x) : \frac{\sqrt{-1} \partial \overline{\partial} \phi}{2\pi} + c_1(K_X, h^K) + \frac{1}{r} c_1(\Theta_X(r\Delta), h^{r\Delta}_{\text{sm}}) \geq 0, \right. \]
\[ \left. \int_X e^{\phi - \gamma} dV_{h^K} \leq 1 \right\}, \tag{3.5} \]

the latter integral is finite because of the klt condition, see (3.3). Clearly, by (1.8), we have
\[ h^{K,\Delta}_{\text{can}} = e^{-r\phi^{\text{can}}} \cdot (h^K)^r \cdot h^{r\Delta}_{\text{sm}}. \tag{3.6} \]

Now, following [8, Remark 5.23], for any \( x \in X, k \in \mathbb{N}^* \), we define
\[ \phi^{\text{alg}}_k(x) := \frac{1}{kr} \sup \left\{ \log |\sigma(x)| : \sigma \in H^0(\Delta)^{kr}_X, \|\sigma\|^{NS,\Delta}_{kr} \leq 1 \right\}, \tag{3.7} \]

where the pointwise norm \( |\cdot| \) is induced on \( K_X(\Delta)^{kr}_X \) by \( h^K \) and \( h^{r\Delta}_{\text{sm}} \). Directly from Poincaré-Lelong equation, we see that \( \phi^{\text{alg}}_k \leq \phi^{\text{can}} \). From [8, Proposition 5.19 and Remark 5.23], there is \( C > 0 \), such that for any \( k \in \mathbb{N}^* \), we moreover have
\[ \phi^{\text{can}} \leq \phi^{\text{alg}}_k + \frac{C}{k}. \tag{3.8} \]

The two bounds above imply that \( \phi^{\text{alg}}_k \) converge uniformly to \( \phi^{\text{can}} \), as \( k \to \infty \).

Now, from (3.7), we see that the metric \( h_{K,k} := e^{-kr\phi^{\text{alg}}_k} \cdot (h^K)^{kr} \cdot (h^{r\Delta}_{\text{sm}})^k \) on \( K_X(\Delta)^{kr}_X \) can be alternatively described as follows. For \( x \in X, l \in K_X(\Delta)^r_X \), we have
\[ \|l\|_{h_{K,k}} = \inf_{\sigma \in H^0(X, K_X(\Delta)^{kr}_X)} \left\{ \frac{l^kr}{\sigma(x)} : \|\sigma\|^{NS,\Delta}_{kr} \leq 1 \right\}. \tag{3.9} \]

In particular, from Lemma 2.1 and (3.9), we see that
\[ FS(\mathcal{N}^{\Delta}_{kr})^{\frac{1}{k}} = h_{K,k}^{\frac{1}{k}} = e^{-r\phi^{\text{alg}}_k} \cdot (h^K)^r \cdot h^{r\Delta}_{\text{sm}}. \tag{3.10} \]

The result now follows from (3.6), (3.8) and (3.10). \( \square \)

### 4 Classification of submultiplicative norms on section rings

The main goal of this section is to provide a classification of submultiplicative norms on section rings in terms of sup-norms.

We say that a graded pseudonorm \( N = \sum N_k, N_k := \|\cdot\|_k \), over the section ring \( R(X, L) \) is submultiplicative if for any \( k, l \in \mathbb{N}^* \) and \( f \in H^0(X, L^k), g \in H^0(X, L^l) \), we have
\[ \|f \cdot g\|_{k+l} \leq \|f\|_k \cdot \|g\|_l. \tag{4.1} \]

**Lemma 4.1.** The sequence of Fubini-Study metrics \( FS(N^1_k) \), \( k \in \mathbb{N}^* \), is submultiplicative for any submultiplicative graded norm \( N = \sum N_k \). In particular, by Fekete’s lemma, the sequence of metrics \( FS(N^1_k) \) on \( L \) converges, as \( k \to \infty \), to a (possibly only bounded from above and even null) upper semi-continuous metric, which we denote by \( FS(N) \). If \( FS(N) \) is lower semi-continuous and everywhere non-null, then the convergence is uniform and \( FS(N) \) is psh.
Proof. The first part follows easily from Lemma 2.1. The second part is a consequence of the well-known subadditive analogue of Dini’s theorem and a statement asserting that a pointwise limit of subadditive sequence of continuous functions is upper semi-continuous, cf. [14, Appendix A].

The main goal of this section is to establish the following characterization of submultiplicative norms in terms of sup-norms on $R(X, L)$.

**Theorem 4.2.** Assume that a graded norm $N = \sum N_k$ over the section ring $R(X, L)$ of an ample line bundle $L$ is submultiplicative, and the metric $FS(N)$ on $L$ from Lemma 4.1 is continuous and non-null everywhere. Then $N \sim \text{Ban}^\infty(FS(N))$.

**Remark 4.3.** a) The continuity and non-vanishing of $FS(N)$ alone do not determine the equivalence class of the metric, see [14, Proposition 4.16] for an example.

b) A similar statement appeared in [14, Theorem 1.6]. The difference between the current statement and the one from our previous paper is that here our multiplicative type assumption is weaker and we work more generally with norms, which are not necessarily Hermitian. Also, instead of the lower multiplicative-type bound, which we weren’t able to obtain for Narasimhan-Simha pseudonorms directly, we assume here the continuity of $FS(N)$. This continuity, on the contrary, in the setting of [14] followed from the lower bound, see [14, Theorem 4.1].

c) Similarly to [14, Definition 1.3], one can lighten the submultiplicativity assumption by requiring that there is $p_0 \in \mathbb{N}$ and a function $f : \mathbb{N}_{\geq p_0} \to \mathbb{R}$, verifying $f(k) = o(k)$, as $k \to \infty$, such that for any $r \in \mathbb{N}^*, k; k_1, \ldots, k_r \geq p_0$, $k_1 + \cdots + k_r = k$, $f_i \in H^0(X, L^{k_i})$, $i = 1, \ldots, r$, we have

$$
\| f_1 \cdots f_r \|_k \leq \| f_1 \|_{k_1} \cdots \| f_r \|_{k_r} \cdot \exp \left( f(k_1) + \cdots + f(k_r) + f(k) \right).
$$

The proof in this case remains the same with only one modification: instead of the usual Fekete’s lemma for the proof of the convergence of $FS(N_k)^{\frac{1}{k}}$, one needs to rely on [14, Appendix A].

Before proceeding with the proof of this result, let us first deduce Theorem 1.1 from it. For this, we first need the following basic fact, the verification of which is left to the reader.

**Lemma 4.4.** Assume that a graded pseudonorm $N$ is submultiplicative. Then the graded norm $\text{Conv}(N)$ is submultiplicative as well.

**Proof of Theorem 1.1 assuming Theorem 4.2.** By Hölder’s inequality, $N_{S^\Delta}$ is submultiplicative. By the discussion before Theorem 1.1, $h_{\text{can}}^{k, \Delta}$ is continuous and non-null. Theorem 1.1 now follows from this, Theorems 3.1, 4.2, Lemma 4.4, and (2.4).

The proof of Theorem 4.2 is based on the asymptotic study of projective tensor norms, the definition of which we will now recall. Let $V_1, V_2$ be two finitely dimensional vector spaces endowed with norms $N_i = \| \cdot \|_{i}$, $i = 1, 2$. There is no single canonical constructions of a norm on the tensor product $V_1 \otimes V_2$. Instead, several definitions are widely used.

The *projective tensor norm* $N_1 \otimes_x N_2 = \| \cdot \|_{\otimes_x}$ on $V_1 \otimes V_2$ is defined for $f \in V_1 \otimes V_2$ as

$$
\| f \|_{\otimes_x} = \inf \left\{ \sum \| x_i \|_1 \cdot \| y_i \|_2 : f = \sum x_i \otimes y_i \right\},
$$

where the infimum is taken over different ways of partitioning $f$ into a sum of decomposable terms. The *injective tensor norm* $N_1 \otimes_{\psi} N_2 = \| \cdot \|_{\otimes_{\psi}}$ on $V_1 \otimes V_2$ is defined as

$$
\| f \|_{\otimes_{\psi}} = \sup \left\{ |(\phi \otimes \psi)(f)| : \phi \in V_1^*, \psi \in V_2^*, \| \phi \|^*_1 = \| \psi \|^*_2 = 1 \right\}
$$

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where \( \| \cdot \|_i^*, \ i = 1, 2 \), are the dual norms associated to \( \| \cdot \|_i \). Lemma below compares injective and projective tensor norms, see [23. Proposition 6.1], [3. Theorem 21] for a proof.

**Lemma 4.5.** The following inequality between the norms on \( V_1 \otimes V_2 \) holds

\[
N_1 \otimes \epsilon N_2 \leq N_1 \otimes \pi N_2 \leq N_1 \otimes \epsilon N_2 \cdot \min\{\dim N_1, \dim N_2\}.
\]  
(4.5)

If, moreover, the norms \( N_1 \) and \( N_2 \) are Hermitian, then

\[
N_1 \otimes \epsilon N_2 \leq N_1 \otimes N_2 \leq N_1 \otimes \pi N_2.
\]  
(4.6)

Recall also that a norm \( N_V = \| \cdot \|_V \) on a finitely dimensional vector space \( V \) naturally induces the norm \( \| \cdot \|_Q \) on any quotient \( Q, \pi : V \to Q \) of \( V \) through the following identity

\[
\|f\|_Q := \inf \left\{ \|g\|_V : g \in V, \pi(g) = f \right\}, \quad f \in Q.
\]  
(4.7)

By a slight abuse of notations, we sometimes denote the quotient norm by \( [N_V] \), i.e. without the reference to the quotient bundle \( Q \) or the projection \( \pi \).

Now, for any \( r \in \mathbb{N}^* \), \( k; k_1, \ldots, k_r \in \mathbb{N} \), \( k_1 + \cdots + k_r = k \), we define the multiplication map

\[
\text{Mult}_{k_1, \ldots, k_r} : H^0(X, L^{k_1}) \otimes \cdots \otimes H^0(X, L^{k_r}) \to H^0(X, L^k),
\]  
(4.8)

as follows \( f_1 \otimes \cdots \otimes f_r \mapsto f_1 \cdots f_r \). It is standard that there is \( p_0 \in \mathbb{N}^* \), such that for any \( k_1, \ldots, k_r \geq p_0 \), the map \( \text{Mult}_{k_1, \ldots, k_r} \) is surjective, cf. [14. Proposition 3.1].

Assume now that \( k, l \in \mathbb{N}^* \) are big enough so that \( \text{Mult}_{k,l} \) is surjective. A central idea of our approach to Theorem 4.2 is to interpret the submultiplicativity condition in terms of projective tensor norms. In fact, we see that in the notations of (4.1), the submultiplicativity condition can be reformulated in terms of inequalities between the norms on \( H^0(X, L^{k+l}) \) as follows

\[
N_{k+l} \leq [N_k \otimes \pi N_l].
\]  
(4.9)

Assume now for simplicity that \( L \) is very ample and the multiplication map \( \text{Mult}_{1,1} \) is surjective (it implies that all other multiplication maps are surjective as well). Let \( N_1 \) be a norm on \( H^0(X, L) \). Then by the surjectivity of the multiplication maps (4.8), one can endow \( H^0(X, L^k) \) with the norms \( N_k^\pi = [N_1 \otimes \pi \cdots \otimes \pi N_1] \) and \( N_k^\epsilon = [N_1 \otimes \epsilon \cdots \otimes \epsilon N_1] \), where the tensor powers are repeated \( k \) times. We denote by \( N^\pi = \sum N_k^\pi \) and \( N^\epsilon = \sum N_k^\epsilon \) the induced graded norms on \( R(X, L) \). The following result, established in Sections 5 and 6, lies in the core of our approach.

**Theorem 4.6.** The norms \( N^\pi, N^\epsilon \) and \( \text{Ban}^\infty(\text{FS}(N_1)) \) are equivalent.

**Remark 4.7.** In [14. Theorem 4.18], we established a similar statement, where we assumed that \( N_1 \) is Hermitian and projective/injective tensor norms were replaced by the Hermitian tensor norm. Since according to Lemma 4.5, the Hermitian tensor norm is pinched between the injective and projective tensor norms, Theorem 4.6 refines [14. Theorem 4.18]. The Hermitian assumption in [14] simplified substantially the proof, as it allowed us to do explicit calculations on the projective space, see [14. the first part of the proof of Theorem 4.18]. Circumventing those calculations is exactly the content of Sections 6 and 7 of this article.
Proof of Theorem 4.2.} Let us fix $\epsilon > 0$. By our assumption on the continuity of $FS(N)$ and Lemma 4.1, there is $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, we have
\[
\exp(-\epsilon/4) \cdot FS(N) \leq FS(N_k)^{\frac{1}{k}} \leq \exp(\epsilon/4) \cdot FS(N). \tag{4.10}
\]

Now, for a continuous psh metric $h^L$ on $L$, we denote by $\text{Hilb}(h^L)$ the $L^2$-norm on $H^0(X, L^k)$ induced by $h^L$ and the volume form $\frac{1}{k!} c_1(L, h^L)^n$ constructed using the Bedford-Taylor definition of the wedge power, see [14, §2.2] for details. Recall that in [14, Theorem 1.5], we proved that for any continuous psh metric $h^L$, the graded norm $\text{Hilb}(h^L) = \sum \text{Hilb}(h^L)$ is multiplicatively generated in the sense of [14, Definition 1.3]. This means, in particular, that there is $k_1 \in \mathbb{N}$, such that for any $k, l \geq k_1$, we have
\[
\exp(-\epsilon(k + l)/8) \cdot \text{Hilb}_{k+l}(h^L) \leq [\text{Hilb}_k(h^L) \otimes \text{Hilb}_l(h^L)] \leq \exp(\epsilon(k + l)/8) \cdot \text{Hilb}_{k+l}(h^L). \tag{4.11}
\]

Now, from the result of Berman-Boucksom-Witt Nyström [7, Theorem 1.14], we know that the graded norms $\text{Hilb}(h^L)$ and $\text{Ban}_\infty^\infty(h^L)$ are equivalent. Applying this for $h^L := FS(N)$ with the use of Lemma 4.5 we see that there is $k_2 \in \mathbb{N}$, such that for any $k, l \geq k_2$, we have
\[
\exp(-\epsilon(k + l)/4) \cdot \text{Ban}_{k+l}^\infty(FS(N)) \leq [\text{Ban}_k^\infty(FS(N)) \otimes_\pi \text{Ban}_l^\infty(FS(N))] \leq \exp(\epsilon(k + l)/4) \cdot \text{Ban}_{k+l}^\infty(FS(N)). \tag{4.12}
\]

We fix from now on $k' \geq \max\{k_0, k_1, k_2\}$.

Directly from Lemma 2.1, we see that for any $k \in \mathbb{N}^*$, we have
\[
N_k \geq \text{Ban}_k^\infty(FS(N_k)). \tag{4.13}
\]

In conjunction with (4.10), we see that for any $k \geq k'$, we have
\[
N_k \geq \exp(-\epsilon k/4) \cdot \text{Ban}_k^\infty(FS(N)). \tag{4.14}
\]

Now, through iteration of the submultiplicativity condition, (4.9), for any $l \in \mathbb{N}^*$, we have
\[
N_{k'l} \leq [N_{k'} \otimes_\pi \cdots \otimes_\pi N_{k'}], \tag{4.15}
\]
where the tensor product is repeated $l$ times. By the application of Theorem 4.6, (4.10) and (4.15), we see that there is $l_0 \in \mathbb{N}^*$, such that for any $l \geq l_0$, we have
\[
N_{k'l} \leq \exp(ekl/2) \cdot \text{Ban}_{k'l}^\infty(FS(N)). \tag{4.16}
\]

Remark that since the spaces $H^p(X, L^p)$, $p = k', \ldots, 2k' - 1$, are finitely dimensional, the norms $N_p$ and $\text{Ban}_p^\infty(FS(N))$ are comparable up to a uniform constant. From this and (4.12), we deduce that there is $l_1 \in \mathbb{N}$, such that for any $0 \leq r \leq k' - 1$, $l \geq l_1$, we have
\[
[\text{Ban}_{k'l}^\infty(FS(N)) \otimes_\pi N_{k'+r}] \leq \exp(ekl/4) \cdot \text{Ban}_{k'(l+1)+r}^\infty(FS(N)). \tag{4.17}
\]

A combination of (4.9), (4.16) and (4.17) yields for $k \geq 2k' \max\{l_0, l_1\}$ the following estimate
\[
N_k \leq \exp(\epsilon k) \cdot \text{Ban}_k^\infty(FS(N)). \tag{4.18}
\]

The result now follows directly from (4.14) and (4.18).\qed
5 Projective geometry and norms on symmetric algebras

In this section, we reduce the proof of Theorem 4.6 to a functional-analytic statement about the norms on the symmetric algebra of complex vector spaces. We also show that the latter statement can be seen as a special case of Theorem 4.6 applied for the projective space.

We fix a finitely dimensional complex vector space $V$ with a norm $N_V := \| \cdot \|_V$. Recall that for any $k \in \mathbb{N}^*$, we have the polarization map $\text{Pol} : \text{Sym}^k(V) \to V^k$ and the symmetrization map $\text{Sym} : V^k \to \text{Sym}^k(V)$. Consider two norms $\text{Sym}^k(N_V) := \| \cdot \|_{N_V,k}$ and $\text{Sym}^k(N_V) := \| \cdot \|_{N_V,k}$ on symmetric tensors $\text{Sym}^k(V)$, induced by the polarization map, and the norms $N_V \otimes \cdots \otimes N_V$, $N_V \otimes \pi \cdots \otimes \pi N_V$ on $V^k$. Define the norm $\text{Sym}^k(N_V) := \| \cdot \|_{N_V,k}$ on $\text{Sym}^k(V)$ as

$$\|P\|_{N_V,k}^v := \sup_{\|v\|_V \leq 1} |P(v)|, \quad P \in \text{Sym}^k(V). \quad (5.1)$$

As in [14], we construct from those norms the graded norms $\text{Sym}^k(N_V)$, $\text{Sym}(N_V)$ and $\text{Sym}_\pi(N_V)$ on the symmetric algebra $\text{Sym}(V)$. Similarly to [17], we define the equivalence relation on the set of graded norms over $\text{Sym}(V)$. The following result will be established in Sections 6 and 7.

**Theorem 5.1.** The norms $\text{Sym}_\pi(N_V)$, $\text{Sym}(N_V)$ and $\text{Sym}_\pi(N_V)$ are equivalent.

**Remark 5.2.** a) Restriction to symmetric tensors is absolutely necessary for this statement. In fact, by the recent result of Aubrun-Müller-Hermes [4, Theorem 1.1], in the full tensor algebra $T(V) := \sum_{k=1}^{\infty} V^k$, the gap between injective and projective tensor norms on the graded pieces is exponential for any normed vector space $(V, N_V)$ of dimension bigger than 1.

b) Surprisingly, the corresponding statement for real vector spaces is false. In fact, if we consider a polynomial $P(x, y) = xy(x^2 - y^2)$ and view it as a polynomial on $(\mathbb{R}^2, l_1)$, then an easy calculation shows that for any $k \in \mathbb{N}^*$, we have $\|P^k\|_{l_1,4k} = \sup_{-1 \leq x,y \leq 1} |P^k(x, y)| = \left(\frac{2\sqrt{3}}{9}\right)^k$, cf. [11] proof of Theorem 4.2. But from the proof of Theorem 5.1 we know that $\text{Sym}_\pi(l_1)$ corresponds to the sum of the absolute values of the coefficients. Hence, we have $\|P^k\|_{l_1,4k} = 2^k$.

We now explain that Theorem 5.1 is in fact a special case of Theorem 4.6. For this, we give geometric interpretations for some of the above norms. First of all, directly from (4.4), we have

$$\|P\|_{\text{Sym},e}^k := \sup_{v_1, \ldots, v_k \in V^* \atop \|v_i\|_V \leq 1} \left|\text{Pol}(P)(v_1, \ldots, v_k)\right|, \quad P \in \text{Sym}^k(V). \quad (5.2)$$

Hence, from Lemma 4.5 (5.1) and (5.2), the following chain of inequalities holds

$$\text{Sym}_e(N_V) \leq \text{Sym}_e(N_V) \leq \text{Sym}_\pi(N_V). \quad (5.3)$$

In particular, we see that for the proof of Theorem 5.1, it is enough to establish the equivalence of the norms $\text{Sym}_e(N_V)$ and $\text{Sym}_\pi(N_V)$.

**Remark 5.3.** By (5.2), the equivalence of $\text{Sym}_e(N_V)$ and $\text{Sym}_\pi(N_V)$ from Theorem 5.1 means exactly that the polarization constant, cf. [12] (4)) for a definition, for finitely dimensional normed complex vector spaces is equal to 1. This fact was recently established by Dimant-Galicer-Rodríguez [12, Theorem 1.1] for real and complex vector spaces through different methods.
Let us now give an interpretation of the norm $\text{Sym}_{ev}(N_V)$ through projective spaces. We view the symmetric algebra $\text{Sym}(V)$ as the section ring $R(\mathbb{P}(V^*), \mathcal{O}(1))$ through the identification

$$\text{Sym}^k(V) = H^0(\mathbb{P}(V^*), \mathcal{O}(k)). \quad (5.4)$$

Under this isomorphisms, we have the following identification of norms

$$\text{Sym}^k_{ev}(N_V) = \text{Ban}^{\infty}_{\mathbb{P}(V^*)}(FS(N_V)). \quad (5.5)$$

We now consider the norms $\text{Sym}^k_{\epsilon,0}(N_V)$ and $\text{Sym}^k_{\pi,0}(N_V)$ on $\text{Sym}^k(V)$, given by the quotients of $N_V \otimes_\epsilon \cdots \otimes_\epsilon N_V$, $N_V \otimes_\pi \cdots \otimes_\pi N_V$ on $V^\otimes k$ through the symmetrization map, $\text{Sym}$.

**Lemma 5.4.** The norms $\text{Sym}^k_{\epsilon,0}(N_V)$ (resp. $\text{Sym}^k_{\pi,0}(N_V)$) and $\text{Sym}^k(N_V)$ (resp. $\text{Sym}^k_{\epsilon}(N_V)$) over $\text{Sym}^k(V)$ coincide.

**Proof.** Clearly, permutations of coordinates are isometries for both norms $N_V \otimes_\epsilon \cdots \otimes_\epsilon N_V$, $N_V \otimes_\pi \cdots \otimes_\pi N_V$. Also for any $Q \in V^\otimes k$, $\text{Sym}(Q)$ can be represented as an average over all permutations. From this, we see that for any $P \in \text{Sym}^k(V)$, the element $Q = \text{Pol}(P)$ minimizes both norms $N_V \otimes_\epsilon \cdots \otimes_\epsilon N_V$, $N_V \otimes_\pi \cdots \otimes_\pi N_V$ among all elements, verifying $\text{Sym}(Q) = P$. \qed

Remark that symmetrization and multiplication maps (4.3) can be put under the isomorphisms (5.4) into the following commutative diagram

$$
\begin{array}{ccc}
H^0(\mathbb{P}(V^*), \mathcal{O}(1)) ^\otimes k & \overset{\text{Mult}}{\longrightarrow} & H^0(\mathbb{P}(V^*), \mathcal{O}(k)) \\
\left\|\right\| & & \left\|\right\|
\end{array}
\quad (5.6)

\begin{array}{ccc}
V^\otimes k & \overset{\text{Sym}}{\longrightarrow} & \text{Sym}^k(V)
\end{array}
$$

Lemma 5.4, 5.5 and 5.6 imply that Theorem 5.1 is a specialization of Theorem 4.6 to $X = \mathbb{P}(V^*)$, $L = \mathcal{O}(1)$ and $N_1 := N_V$. Remark, however, that our proof proceeds in another direction: we first establish Theorem 5.1 and then prove Theorem 5.6. To explain how this implication works, let us recall a version of Ohsawa-Takegoshi extension theorem.

Let $Y$ be a closed submanifold of a compact complex manifold $X$. Let $L$ be an ample line bundle over $X$ endowed with a continuous psh metric $h^L$. It is classical that there is $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, the map

$$\text{Res}_Y : H^0(X, L^k) \to H^0(Y, L|_Y^k), \quad (5.7)$$

is surjective. In particular, any norm on $H^0(X, L^k)$ induces a norm on $H^0(Y, L|_Y^k)$.

**Theorem 5.5.** The following equivalence of norms on $R(Y, L)$ holds

$$[\text{Ban}^{\infty}_X(h^L)] \sim \text{Ban}^{\infty}_Y(h^L). \quad (5.8)$$

**Remark 5.6.** For a refinement of this result for smooth positive $h^L$, see [13, Theorem 1.1]. See also [15, Theorems 1.1 and 1.3] for a more general statement about jet extensions.

**Proof.** This result was proved by Randriambololona [22] under laxer assumptions on the manifolds $X$ and $Y$ but with stronger assumption of strict positivity on the curvature of $(L, h^L)$. The proof of exactly this version of the theorem can be found in [14, Corollary 2.12]. \qed
Proof of Theorem 4.6 assuming Theorem 5.1} Let us first prove that the following inequalities hold
\[ \text{Ban}^\infty(FS(N_1)) \leq N^\epsilon \leq N^\pi. \] \hfill (5.9)

The first inequality is a direct consequence of (4.13) and [14, Lemma 4.3]. The second inequality follows directly from Lemma 4.5.

From (5.9), it is enough to establish that the norm \( N^\pi \) can be bounded from above by \( \text{Ban}^\infty(FS(N_1)) \), considered up to a subexponential factor. The proof of this result is essentially a word-to-word repetition of the proof of the second part of [14, Theorem 4.18]. We only need to replace the use of the first part of the proof of [14, Theorem 4.18] by Theorem 5.1. For the convenience of the reader, we reproduce the argument below.

Let us consider the Kodaira embedding \( \text{Kod}_1 \) from (2.1). We denote by \( \text{Res}_{\text{Kod}} : R(\mathbb{P}(H^0(X, L)^*), \mathcal{O}(1)) \to R(X, L) \) the associated restriction operator, and by \( \text{Res}_{\text{Kod},k}, k \in \mathbb{N}^* \), the restriction operators on the associated graded pieces. The multiplication operator \( \text{Mult}_{1,\ldots,1} \) from (4.8) factorizes under the identification (5.4) through symmetrization and restriction as
\[ H^0(X, L)^{\otimes k} \xrightarrow{\text{Sym}} \text{Sym}^k(H^0(X, L)) \xrightarrow{\text{Mult}_{1,\ldots,1}} H^0(\mathbb{P}(H^0(X, L)^*), \mathcal{O}(k)) \xrightarrow{\text{Res}_{\text{Kod},k}} H^0(X, L^k). \] \hfill (5.10)

Now, from (5.10), it is sufficient to show that by a subsequent quotient of the norm \( N^\pi \) through the symmetrization map \( \text{Sym} \) and the map \( \text{Res}_{\text{Kod}} \), we get the norm \( [N^\pi] \) on \( H^0(X, L^k) \). From Theorem 5.1 the quotient norm on \( \text{Sym}(H^0(X, L)) \) associated to \( N^\pi \) is equivalent to \( \text{Sym}_{\text{ev}}(N_V) \), which by (5.5) coincides with \( \text{Ban}^\infty_{\mathbb{P}(H^0(X, L)^*)}(FS(N_1)) \) under the identification (5.4). But by Theorem 5.5, the quotient of the norm \( \text{Ban}^\infty_{\mathbb{P}(H^0(X, L)^*)}(FS(N_1)) \) under the map \( \text{Res}_{\text{Kod}} \) is equivalent to \( \text{Ban}^\infty_X(FS(N_1)) \). This finishes the proof. \hfill \( \square \)

6 Norms on the space of polynomials and Bohnenblust-Hille inequality

The main goal of this section is to establish Theorem 5.1 in the special case \( V = \mathbb{C}^r, r \in \mathbb{N}^* \), and \( N_V := \| \cdot \|_V := l_1 \). To establish this, we rely on a recent result about the optimal estimate in Bohnenblust-Hille inequality, which we now recall. Consider a vector space \( V_{r,k} \) of homogeneous complex polynomials of degree \( k \) in \( r \) variables. We represent an element \( P \in V_{r,k} \) as
\[ P(x_1, \ldots, x_r) = \sum_{|\alpha| = k} a_\alpha x^\alpha. \] \hfill (6.1)

Since \( \dim V_{r,k} = \binom{r+k}{r} < +\infty \), any two norms on \( V_{r,k} \) are equivalent. In particular, for any \( \beta \geq 1 \), there is a constant \( B^\beta_{r,k} > 0 \), such that for any \( P \in V_{r,k} \) as in (6.1), we have
\[ \left( \sum_{|\alpha| = k} |a_\alpha|^{\beta} \right)^{\frac{1}{\beta}} \leq B^\beta_{r,k} \cdot \| P \|. \] \hfill (6.2)
where the sup-norm \( \| P \| \) is defined as follows
\[
\| P \| := \sup_{x_i \in \mathbb{C}, |x_i| \leq 1} |P(x_1, \cdots, x_r)|.
\]

(6.3)

We assume that the constants \( B_{r,k}^\beta \) for \( r, k \in \mathbb{N}^* \), \( \beta \geq 1 \), are the minimal constants verifying the inequality (6.2). The main result of this section goes as follows.

**Proposition 6.1.** For any fixed \( r \in \mathbb{N}^* \), the sequence \( B_1^{r,k}, k \in \mathbb{N} \), grows subexponentially in \( k \).

Recall that Bohnenblust–Hille in [10] showed that for \( \beta := \frac{2k}{k+1} \), the constant
\[
B_k := \sup_{r \in \mathbb{N}} B_{r,k}^\beta
\]

(6.4)
is finite. In other words, for this choice of \( \beta \), the bound like (6.2) can be made uniform in the number of variables. We need the following recent result about the asymptotics of \( B_k \).

**Theorem 6.2** (Bayart-Pellegrino-Seoane-Sepúlveda [5, Corollary 5.3]). The constants \( B_k \) grow subexponentially in \( k \).

**Proof of Proposition 6.1.** By the generalized mean inequality and (6.2), we have
\[
\sum_{|\alpha| = k} |a_\alpha| \leq B_k \cdot \left( \binom{r+k}{r} \right)^{1 - \frac{k+1}{k}} \cdot \| P \|,
\]
in the notations (6.1). In particular, since the binomial coefficients \( \binom{r+k}{r} \) are polynomials in \( k \) for fixed \( r \) (and, hence, subexponential in \( k \)), we deduce Proposition 6.1 from Theorem 6.2.

**Proof of Theorem 6.1 in the special case when \( V = \mathbb{C}^r \) and \( N_V := \| \cdot \|_V := l_1 \).** From (5.3), it is sufficient to show that \( \text{Sym}_\pi(N_V) \), considered up to a subexponential constant, is bounded from above by \( \text{Sym}_\pi(N_V) \).

Let us denote by \( x_1, \ldots, x_r \) the coordinate vectors in \( \mathbb{C}^r \). We use the notation (6.1) for \( P \in \text{Sym}^k(V), k \in \mathbb{N}^* \). Since the dual of the \( l_1 \)-norm is given by the \( l_\infty \)-norm on \( \mathbb{C}^r \), (5.1) gives us
\[
\| P \|_{l_1,k}^\infty = \| P \|.
\]

(6.6)

On another hand, since projective tensor norms behave multiplicatively on \( l_1 \)-spaces, i.e. \( (\mathbb{C}^n, l_1) \otimes_\pi (\mathbb{C}^n, l_1) = (\mathbb{C}^{nm}, l_1) \), cf. [23, Exercise 2.8], the norm \( \text{Sym}_\pi(l_1) \) corresponds to the sum of absolute values of the coefficients occurring in the representation (6.1), i.e. we have
\[
\| P \|_{l_1,k}^{\text{Sym}_\pi} = \sum_{|\alpha| = k} |a_\alpha|.
\]

(6.7)

We conclude by Proposition 6.1 and (6.6), (6.7) that \( \text{Sym}_\pi(N_V) \), considered up to a subexponential constant, can be bounded from above by \( \text{Ban}^\infty(FS(N_V)) \).
7 Projective tensor norms and holomorphic extension theorem

The main goal of this section is to prove Theorem 5.1 in its full generality. Surprisingly, our main technical tool in the proof of this purely functional-analytic statement comes from complex geometry, see Theorem 5.5. We also use the following classical result.

Lemma 7.1 (cf. [12, Lemma 2.2]). For any finitely dimensional normed complex vector space \((V, \| \cdot \|_V)\), and any \(\epsilon > 0\), there is \(l \in \mathbb{N}^*\) and a surjective map \(\pi : \mathbb{C}^l \to V\), such that \(\| \cdot \|_V\) is related to the quotient norm associated to the \(l_1\)-norm on \(\mathbb{C}^l\) as follows

\[
\exp(-\epsilon) \cdot [l_1] \leq \| \cdot \|_V \leq [l_1].
\] (7.1)

Proof of Theorem 5.1. Let us fix \(\epsilon > 0\) and consider a projection \(\pi : \mathbb{C}^l \to V\) as in Lemma 7.1.

For clarity of the presentation, we introduce the following notation

\[
U := \mathbb{C}^l, \quad N_U := \| \cdot \|_U := l_1.
\]

We consider the induced embedding

\[
\text{Im}_\pi : \mathbb{P}(V^*) \to \mathbb{P}(U^*).
\] (7.2)

An easy verification shows that under this embedding, the associated restriction operator, which we denote by \(\text{Res}_{\pi, k}\), and the projection map on the symmetric tensors induced by \(\pi\), which we denote by \(\text{Sym}^k \pi\), can be put with the identifications (5.4) into the following commutative diagram

\[
\begin{array}{ccc}
H^0(\mathbb{P}(U^*), \mathcal{O}(k)) & \xrightarrow{\text{Res}_{\pi, k}} & H^0(\mathbb{P}(V^*), \mathcal{O}(k)) \\
\| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
\text{Sym}^k(U) & \xrightarrow{\text{Sym}^k \pi} & \text{Sym}^k(V).
\end{array}
\] (7.3)

The condition (7.1) clearly implies that

\[
\exp(-\epsilon) \cdot FS([N_U]) \leq FS(N_V) \leq FS([N_U]).
\] (7.4)

From Theorem 5.5 (5.5) and (7.3), we conclude that there is \(k_0 \in \mathbb{N}^*\), such that for any \(k \geq k_0\), \(f \in \text{Sym}^k(V)\), there is \(g \in \text{Sym}^k(U)\), such that \(\text{Sym}^k \pi(g) = f\), and we have

\[
\| f \|_{[N_U], k}^\text{ev} \geq \exp(-\epsilon k) \cdot \| g \|_{N_U, k}^\text{ev}.
\] (7.5)

In conjunction with (5.5) and (7.4), the estimate (7.5) implies that for any \(k \geq k_0\), we have

\[
\| f \|_{N_V, k}^\text{ev} \geq \exp(-2\epsilon k) \cdot \| g \|_{N_U, k}^\text{ev}.
\] (7.6)

Now, from the version of Theorem 5.1 for \(l_1\)-norms, established in Section 6, we deduce that there is \(k_1 \in \mathbb{N}^*\), such that for any \(k \geq k_1, g \in \text{Sym}^k(U)\), we have

\[
\| g \|_{N_U, k}^\text{ev} \geq \exp(-\epsilon k) \cdot \| g \|_{N_U, k}^\text{ev}.
\] (7.7)

Remark now that the condition (7.1) implies that for any \(x \in U\), we have

\[
\| x \|_U \geq \| \pi(x) \|_V.
\] (7.8)
From this, the definition of the projective tensor norm and Lemma 5.4, we deduce that for any $k \in \mathbb{N}^*$, $f \in \text{Sym}^k(V)$ and $g \in \text{Sym}^k(U)$, verifying $\text{Sym}^k \pi_2(g) = f$, we have

$$\|g\|_{\text{Sym}, k} \geq \|f\|_{\text{Sym}, k}. \quad (7.9)$$

From (7.6), (7.7) and (7.9), we see that for any $k \geq \max\{k_0, k_1\}$, $f \in \text{Sym}^k(V)$, we have

$$\|f\|_{\text{ev}, k} \geq \exp(-3\epsilon k) \cdot \|f\|_{\text{Sym}, k}. \quad (7.10)$$

As $\epsilon > 0$ is arbitrary, from (5.3) and (7.10), we conclude that $\text{Sym}_{\epsilon}(N_V)$ and $\text{Sym}_\pi(N_V)$ are asymptotically equivalent. As described after (5.3), this finishes the proof. \[\square\]

8 Continuous plurisubharmonic envelopes and equivalence of $L^p$-norms

The main goal of this section is to prove that from the asymptotic point of view, the study of $L^p$-norms, $p \in [1, +\infty]$, on section rings of ample line bundles associated to continuous metrics reduces to the study of sup-norms for continuous psh metrics, i.e. we establish Theorem 1.3.

To prove Theorem 1.3, we need three lemmas, all of which are certainly well-known to experts. We conserve the notations from the statement of Theorem 1.3.

**Lemma 8.1.** For any continuous $h^L$, the envelope $P(h^L)$ is continuous and psh.

**Remark 8.2.** In particular, by this result and [13] Theorem 1.7, we see that $P(h^L)$ is the only continuous psh metric, verifying the statement of Theorem 1.3.

**Proof.** Recall that Berman in [6] Proposition 3.1 and Theorem 3.4 proved that for any smooth $h^L$, the envelope $P(h^L)$ is psh and $C^{1,1}$. Now, as any continuous metric can be uniformly approximated by smooth ones, Lemma 8.1 follows from the mentioned result of Berman and the trivial fact that under uniform convergence, the envelope construction is continuous. \[\square\]

**Lemma 8.3.** As $k \to \infty$, the metrics $FS(\text{Ban}^\infty_k(h^L))^{1/k}$ converge to $P(h^L)$ uniformly.

**Remark 8.4.** For smooth $h^L$, an equivalent result was established by Berman [6 Theorem 4.11] in a more general setting for line bundles which are not necessarily ample.

**Proof.** Directly from Lemma 2.1 we see that $FS(\text{Ban}^\infty_k(h^L))^{1/k} \geq h^L$. Moreover, by Section 2 we know that $FS(\text{Ban}^\infty_k(h^L))^{1/k}$ is psh. From this and (1.10), we see that $FS(\text{Ban}^\infty_k(h^L))^{1/k} \geq P(h^L)$. On another hand, we clearly have $FS(\text{Ban}^\infty_k(h^L)) \leq FS(\text{Ban}^\infty_k(P(h^L)))$. But since $P(h^L)$ is continuous and psh by Lemma 8.1 and Lemma 8.3, $FS(\text{Ban}^\infty_k(P(h^L)))^{1/k}$ converge to $P(h^L)$ uniformly, as $k \to \infty$, cf. [13] Theorem 2.15. Hence, $FS(\text{Ban}^\infty_k(h^L))^{1/k}$ is pinched between $P(h^L)$ and a sequence, converging uniformly to $P(h^L)$. From this, we infer the uniform convergence. \[\square\]

**Lemma 8.5.** The following lower bound holds $\lim\inf_{k \to \infty} FS(\text{Ban}^p_k(h^L))^{1/k} \geq h^L$.

**Proof.** From Lemma 2.1 we see that it is enough to establish that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, $f \in H^0(X, L^k)$, $x \in X$, we have $\|f\|_{L^p_k(X, h^L, \mu)} \geq \exp(-\epsilon k) \cdot |f(x)|_{h^L}$. But the latter bound is a consequence of continuity of $h^L$ and the mean-value inequality. \[\square\]
Proof of Theorem 1.3. Let us first remark that for $p = +\infty$, the result follows from Theorem 4.2, Lemmas 8.1, 8.3 and the trivial fact that $\operatorname{Ban}_{\infty}(h^L)$ is submultiplicative. Now, let us fix $p \in [1, +\infty[$. First of all, we trivially have $\operatorname{Ban}_p(h^L) \leq \operatorname{Ban}_{\infty}(h^L) \cdot \operatorname{vol}(d\mu)^{\frac{1}{p}}$, where $\operatorname{vol}(d\mu)$ is the volume of $d\mu$. Also, from Lemma 8.3 and (4.13), we conclude that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, we have $\operatorname{Ban}_p(h^L) \geq \exp(-ek) \cdot \operatorname{Ban}_k(h^L)$. Theorem 1.3 now follows from the above two bounds as well as from Theorem 1.3 for $p = +\infty$.

Remark 8.6. For any $\epsilon \in ]0, 1[$, the same proof would give us the following equivalence

\[
\operatorname{Conv}(\operatorname{Ban}_p(h^L)) \sim \operatorname{Ban}_{\infty}(P(h^L)),
\]

where the graded pseudonorm $\operatorname{Ban}_p(h^L)$ is defined by the same formula as (1.11).

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Siarhei Finski, CNRS-CMLS, École Polytechnique F-91128 Palaiseau Cedex, France.

E-mail: finski.siarhei@gmail.com.