On Supersymmetric Geometric Flows and $\mathcal{R}^2$ Inflation From Scale Invariant Supergravity

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Abstract

Models of geometric flows pertaining to $\mathcal{R}^2$ scale invariant (super) gravity theories coupled to conformally invariant matter fields are investigated. Related to this work are supersymmetric scalar manifolds that are isomorphic to the Kählerian spaces $\mathcal{M}_n = SU(1, 1 + k)/U(1) \times SU(1 + k)$ as generalizations of the non–supersymmetric analogs with $SO(1, 1 + k)/SO(1 + k)$ manifolds. For curved superspaces with geometric evolution of physical objects, a complete supersymmetric theory has to be elaborated on nonholonomic (super) manifolds and bundles determined by non–integrable superdistributions with additional constraints on (super) field dynamics and geometric evolution equations. We also consider generalizations of Perelman’s functionals using such nonholonomic variables which result in the decoupling of geometric flow equations and Ricci soliton equations with supergravity modifications of the $\mathcal{R}^2$ gravity theory. As such, it is possible to construct exact non–homogeneous and locally anisotropic cosmological solutions for various types of (super) gravity theories modelled as modified Ricci soliton configurations. Such solutions are defined by employing the general ansatz encompassing coefficients of generic off–diagonal metrics and generalized connections that depend generically on all spacetime coordinates. We consider nonholonomic constraints resulting in diagonal homogeneous configurations encoding contributions from possible nonlinear parametric geometric evolution scenarios, off–diagonal interactions and anisotropic polarization/mutation of physical constants. In particular, we analyze small parametric deformations when the underlying scale symmetry is preserved and the nontrivial anisotropic vacuum corresponds to generalized de Sitter spaces. Such configurations may mimic quantum effects whenever transitions to flat space are possible. Our approach allows us to generate solutions with scale violating terms induced by geometric flows, off–diagonal interactions and supersymmetric modifications of effective potentials. We study reconstructing the formalism for (super) geometric flows and modified gravity cosmological scenarios. We also analyze the conditions under which such modified mimetic type theories and solutions reproduce the Starobinsky inflationary models in the double scalar approach.

Keywords: supergeometric flows, super Ricci solitons, modified (super) gravity, Perelman’s superfunctionals, exact solutions in (super) gravity, anisotropic cosmological solutions and inflation.
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1 Introduction

Physical models with geometric evolution are well known in quantum field theory (QFT) since the late 70's, due to research on gauge and nonlinear sigma theories inspired by A. M. Polyakov and D. Friedan [1, 2, 3, 4]. Certain new directions in modern mathematics, with important achievements in geometric analysis and topology, were elaborated upon, beginning with the works of R. Hamilton [5, 6, 7]. It induced a great social interest in the field after the famous proofs of Thurston’s geometrization program and the Poincaré conjecture by G. Perelman [8, 9, 10] (for reviews of mathematical results see [11, 12, 13]). Such significant ideas and results intrigued many physicists to pursue research in the field, with the hope that the new geometric methods would offer surprising and far reaching perspectives in mathematical particle physics, QFT, string theory, (modified) gravity, geometric mechanics and thermodynamics etc. Here we call attention to certain important applications of the Ricci flow theory in the study of nonlinear sigma models [14, 15, 16, 17], research on geometric flow evolution of modified (non) holonomic commutative and noncommutative gravity theories [18, 19, 20, 21] and exact solutions for (modified) gravity and geometric flows, Ricci solitons [22, 23, 24, 25, 26, 27, 28].

In spite of the many constructions elaborated upon by physicists that have not yet passed the censorship of mathematicians (who request proofs of theorems on hundred pages using sophisticate geometric analysis methods), the field offers new physical ideas and results, constituting a source of inspiration for new interesting developments both in mathematics and physics. In this paper, we neither concentrate on QFT issues nor consider how certain geometric models of the space of probability measures over Riemannian metric measure spaces provide a natural connection between nonlinear sigma models and Ricci flow theory [17]. Our goals and methods are very different from the ones employed in "pure" mathematics elaborated in geometric analysis with relations to quantum theories. Instead, we follow a series of our recent results relating to exact solutions for nonlinear off–diagonal systems and gravity. Such geometric constructions and methods manifest that via geometric flows with Ricci soliton configurations we can model modified gravity theories, MGTs, and general relativity, GR, using new classes of physically important solutions in modern gravity and cosmology:

1. Nonholonomic constraints on geometric flows may result in the evolution of a (pseudo) Riemannian geometry, for instance, into a generalized Lagrange-Finsler type, or metric–affine geometry, almost Kähler configurations, noncommutative geometries, Hořava–Lifshits gravity theories, \( f(R) \) and \( R^2 \) modified gravity theories etc., see [19, 20, 18, 23, 24]. Complex manifolds and (in various different approaches) supermanifolds are defined by certain respective classes of nonholonomic distributions. It is expected that the method of nonholonomic deformations of fundamental functionals, actions and derived field equations and physical/geometric objects in supergravity and superstring theories will result in the elaborating of various models of supergeometric flows and (super) Ricci soliton spaces.

2. For self–similar fixed configurations, the (non) holonomic geometric flow equations transform into modified Ricci soliton equations which (for well defined conditions) are equivalent to gravitational field equations in MGTs, or GR. In a certain sense, the bulk of MGTs can be realized as some generalized Ricci soliton geometries arising in corresponding fixed points of parametric
geometric flow evolution of certain classes of fundamental geometric/physical geometric objects on (modified) spacetime manifolds.

3. Mathematically, it has not been elaborated upon as yet a well-defined geometric analysis formalism which allows to study rigorously the geometric evolution of metrics of pseudo-Euclidean signature and to find exact solutions of Ricci flow equations for generic off-diagonal metrics. There are a number of conceptual and fundamental issues which have not been addressed by mathematicians in order to develop methods of nonlinear functional analysis and the geometry of Lorentz manifolds for (modified) gravity theories. For physical models of relativistic spacetimes and gravity, the Ricci tensor in 4-d does not approximate in the limit of weak gravitational interactions to the Laplace (diffusion) operator but to the D’Alambert (wave) operator. The standard geometric evolution paradigm proposed initially in the Hamilton-Perelman program (with nonlinear evolution equations and associated entropy type functionals, all defined by Riemannian metrics) must be modified in order to study realistic geometric flow evolution models related to physical theories. For instance, further research on relativistic Ricci flows is possible, for instance, following methods on nonlinear relativistic hydrodynamics generalized in certain forms which allows to consider nonlinear geometric "heat" transfers, propagating relativistically entropy and temperature. Such issues go back to the fundamental works on relativistic definition of temperature and heat in the framework of relativistic kinetics and thermodynamics in special and general relativity theories (see details and references in [23]; on Einstein-Ott-Plank thermodynamics, Einstein-Vlasov kinetic theory, relativistic diffusion and hydrodynamics etc.).

4. In our research, we have found that it is possible to introduce such nonholonomic variables when the geometric flow evolution and Ricci soliton equations can be decoupled in certain general forms. This allows us to construct various classes of exact solutions with generic off-diagonal metrics and generalized connections. The coefficients of such objects depend on all space-time coordinates via generating and integration functions and various types of commutative and noncommutative parameters and integration/physical constants. Such geometric techniques of constructing exact solutions in geometric flow evolution and MGTs have been elaborated following the so-called anholonomic frame deformation method, AFDM. For details, examples of exact solutions, and various applications, we cite [29, 30, 31, 32] and references therein. Having generated certain classes of parametric solutions describing modifications by geometric flows on certain hypersurfaces, with 3+1 splitting, or with effective temperature and entropy, for instance, of certain black hole, or cosmological, solutions, we can analyze in explicit form possible physical consequences.

5. Another very important direction for research is connected to generalized geometric flow theories via corresponding modifications of Perelman’s functionals. In his original work [8], G. Perelman introduced two Lyapunov type functionals called F-entropy and W-entropy, which were crucial to the proof of the Poincaré conjecture. It allowed to elaborate an associated statistical thermodynamics model characterizing geometric flows. Such functionals can be formally generalized for various types of commutative and noncommutative geometric models also encoding interactions of matter fields [19, 20, 29, 21]. For relativistic geometric flows, the (modified) F- and W–functionals do not have a simple interpretation as entropy types and do not result in well-defined nonlinear diffusion type evolution equations. Nevertheless, redefining and adapting the geometric constructions to certain nonholonomic double 2+2 and 3+1 splitting, we can provide a thermodynamic characterization of MGTs and the corresponding classes of exact solutions. Such an approach is more general, being elaborated on different fundamental principles, than that considered for black holes using the Hawking-Bekenstein entropy (derived for 2-d hypersurface
gravity). In general, Perelman’s W-entropy can be computed, for instance, for general cosmological inhomogeneous and cosmological solutions even in supergravity theories (as we prove in this work). For such constructions, we can speculate on how Ricci flow conjugated initial data sets can be related to new concepts of entropy and produce cosmological averaging data [25].

The goal of this work is to elaborate on certain most important principles and methods for formulating supergeometric flow evolution models in order to encode modified (super) gravity theories which are compatible with the modern accelerating cosmology paradigm, see [34, 35, 21] and references therein. It should be noted that the first supersymmetric geometric flow constructions were considered for two dimensional Kähler–Ricci flows and solitons [36, 37]. In our approach, we develop a different approach orientated to supersymmetric geometric flow generalizations of MGTs and GR. We shall construct exact solutions with total metrics which in general are not Kähler, contain pseudo-Riemann bosonic parts of necessary finite dimension, and describe certain equivalent almost Kähler configurations. This can be important for research in classical and quantum gravity, string theory, deformation quantization, geometric flows with algebroid structure etc. [38, 39, 40, 41]. The present work can be considered as a supersymmetric / superflow partner of our recent works [24, 23, 32, 33] where the direction is developed for supergeometric flow models and exact solutions in superstring MGTs, in particular, related to $R^2$ theories and cosmology. We construct new classes of generalized inhomogeneous cosmological and geometric flow solutions for modified Ricci soliton equations (such solutions can be extended to higher dimensions as in [42, 43] using noncommutative and/or supersymmetric variables).

The last twenty years has witnessed the changing of GR cosmological paradigm to other ones related to MGTs and performed in order to include observational data (beginning late 90’s) on late-time acceleration of our Universe [41]. For reviews of results and some insightful studies, we cite [45, 46, 47, 48], when it is important to consult the works on early–time inflation [49, 50, 51], modified $F(R)$ gravity [52] with certain attempts to address the early-time and late-time acceleration within the same theoretical framework [45, 46, 47, 53]. The late-time acceleration is attributed to dark energy modelled, for instance, as a negative pressure perfect fluid which dominates our Universe’s energy density at the percentage $\Omega_{DE} \sim 68$. The rest of energy is controlled by cold dark matter, $\Omega_{DE} \sim 27$, and ordinary matter $\Omega_m \sim 5$. There is no up to date experimentally and/or observationally confirmed theory which predicts the nature of dark matter and dark energy. There have been developed a number of MGTs and cosmological scenarios, all approached by different constructions and methods, but addressing the same fundamental problems of modern cosmology.

In our recent papers, we elaborated on two main ideas related to MGTs and cosmology: The cosmological acceleration scenarios may involve 1) certain geometric spacetime flow evolution of 2) generic off–diagonal modified Ricci soliton configurations both resulting in modification of standard model with effective running and/or locally anisotropic polarization of physical constants. In a certain sense, such constructions can be modelled by generic off–diagonal solutions of MGTs and GR, which (for applications in cosmology) can be treated equivalently in the framework of (modified) mimetic gravity theories [24, 55, 56]. In order to study possible supergravity contributions to geometric flows and cosmology, the reconstruction method with $F(R)$ modifications [57, 58, 59] is more convenient. We shall develop such approaches in order to include geometric flows and off-diagonal configurations.

The article is organized as follows: In section 2 we briefly recall the geometry of nonholonomic manifolds enabled with nonlinear connection, N–connection, 2+2 splitting and introduce nonholonomic variables with deformation of the linear connection and frame structures. We define modified Perelman’s functionals in nonholonomic variables and show how geometric relativistic flow equations can be derived. For self-similar and fixed parameter configurations, such equations define Ricci soliton spacetimes modelling (for corresponding classes of nonholonomic constraints) MGTs and generic
off-diagonal Einstein spacetimes. Important examples of nonholonomic geometric evolution of and Ricci solitons corresponding to $R^2$ gravity with conformally coupled matter are also presented. Also considered are $SO(1, 1 + k)$ generalizations of geometric flow and pure $R^2$ gravity models.

Section 3 is devoted to supersymmetric extensions of Perelman’s functionals and nonholonomic Ricci solitons for $R^2$ gravity with $SU(1, 1 + k)$ modifications. This is possible due to using the constructions elaborated for multi-field and $SO(1, 1 + k)$ configurations. There is considered the proof of geometric flow equations in nonholonomic supersymmetric variables and analysis of possible configurations of the vacuum structure of MGTs determined by effective scalar potentials and associated cosmological constants. We speculate on actions which are associated with such supersymmetric modifications of Ricci solitons and related $R^2$ gravity models. Such effective modified potentials are important for constructing exact solutions and elaborating on reconstruction modified models in the following sections.

In section 4 we develop the AFDM for generating cosmological solutions of supersymmetric modifications of Ricci flows and $R^2$ gravity. Following such geometric methods, we prove that nonlinear systems of PDEs with explicit flow parameter and time coordinate dependencies can be decoupled in general forms for generic off–diagonal metrics, generalized connections with nonholonomically induced torsions and (effective) matter field sources encoding contributions from supergravity models. There are constructed in general forms the corresponding classes of inhomogeneous and locally anisotropic Ricci soliton metrics corresponding to gravitational field equations in $R^2$ and GR theories. We perform a generalization of such systems for geometric flows with factorized dependence of geometric evolution parameter. We show also that the AFDM allows to generate exact cosmological solutions of geometric flow equations encoding nontrivial vacuum configurations with general flows (non-factorized) on evolution parameter. We also show how to constrain such cosmological solutions for extracting torsionless configurations in GR. In order to be able to provide certain physical interpretation of new classes of geometric flow and Ricci soliton solutions, we formulate a procedure of small deformations on a $\varepsilon$ parameter which allows us to study generic off–diagonal deformations and geometric evolution of cosmologically important metrics in MGTs.

Finally, in section 5 we study how cosmological geometric flow and Ricci soliton effects can be connected to modified gravity scenarios and observable cosmological data. We follow a reconstruction method which allows us to construct effective mimetic potential and Lagrange multiplier determined by generalized supersymmetric and Starobinsky type potentials in modified double scalar models. Also are computed and analyzed possible contributions of running of physical constants and locally anisotropic interactions on observational indices. The approach is elaborated for generalized mimetic $F(A, R^2)$ theories with flexibility for constraints to $R^2$ and GR configurations. Discussions and conclusions are presented at the end of the article in section 6.

2 Nonholonomic Deformations of Perelman’s Functionals and the $R^2$ Gravity Theory

There are a few works on nonholonomic variables in supergravity and superstring theories. The geometric constructions for supersymmetric Finsler models on tangent superbundles [60] can be re-defined to nonholonomic Einstein configurations and generalized for fibered superspaces [42] and supersymmetric Kähler–Ricci flows and solitons [36, 37]. By the same token, there is extensive work on supergravity generalizations and modifications, see [34, 61, 62, 63, 64, 65], of the Starobinsky model [66, 67, 68]. Our approach provides a set up for topological and geometric flow evolution models and scale invariant theories in a de Sitter background, extending the $R^2$ plus conformally invariant matter theories, in
the framework of $N = 1$ supergravity and exact cosmological solutions for the accelerated expansion cosmology.

2.1 Geometric objects adapted to nonlinear connection splitting

In focusing on exact solutions for geometric flows, Ricci solitons and MGTs and GR, we must bypass three obstructions (two technical and one physical): the first one is to decouple such systems of nonlinear partial differential equations, PDEs; the second one is to integrate (find solutions) the equations in general form; the third one is to provide physical interpretations of such new classes of solutions. The first two mathematical tasks can be solved by using the so-called nonholonomic variables with deformations of the frame, metric and nonlinear and linear connections structures. We outline the key ideas and notations on nonholonomic variables in (super) gravity theories, elaborated and reviewed in our previous works 31, 32, 42, 60.

Let us consider a (pseudo) Riemannian manifold $V$ (we can consider any dimension $n + m$, with $n, m \geq 2$ like in 32, 33); for simplicity, we shall formulate the main results and discuss with some examples for $n = m = 2$.

We can introduce a conventional 2+2 (or $n + m$) splitting into horizontal (h) and vertical (v) components defined by a Whitney sum

$$N : TV = hV \oplus vV,$$

where $TV$ is the tangent bundle on a spacetime manifold $V$ (in 4-d, we can consider a Lorentzian manifold with local pseudo-Euclidean signature $(+++)$). Any $N$–connection structure (1) is determined locally by a corresponding set of coefficients $N^a_i$, when $N = N^a_i dx^i \otimes \partial_a$.

There are structures, elongated linearly on $N$–coefficients, respectively, of partial derivatives and differentials, $e_\nu = (e_i, e_a)$, and cobases, $e^\mu = (e^i, e^a)$, when

$$e_\nu = \begin{cases} (e_i = \partial/\partial x^i - N_i^a \partial/\partial y^a, \ e_a = \partial/\partial y^a), & \text{and} \\ e^\mu = (e^i = dx^i, e^a = dy^a + N_i^a dx^i). & \end{cases}$$

Such $N$–adapted bases (local frames) are nonholonomic because, in general, there are satisfied relations of the type

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W_{\alpha\beta}^\gamma e_\gamma,$$

with nontrivial anholonomy coefficients $W_{\alpha\beta}^\gamma = \partial_\alpha N_{\beta}^\gamma, W_{ij}^a = \Omega^a_{ij} = e_j (N_i^a) - e_i (N_j^a)$. We obtain holonomic (integrable) bases if and only if $W_{\alpha\beta}^\gamma = 0$. In a more general context, we can consider an arbitrary local basis $e^\alpha = (e^i, e^a)$ and the corresponding dual one, co-basis, $e_\beta = (e_\beta, e_\beta)$.

A local coordinate bases is denoted by $\partial_\nu = (\partial_{x^i}, \partial_{y^a})$ [for instance, $\partial_\nu = \partial/\partial x^\nu$, and the respective dual cobasis is written as $du^\nu = (dx^i, dy^a)$. The frame (vierbein) transforms can be written in the form $e_\beta = A_\beta^\nu (u) \partial_\nu$ and $e^\alpha = A^{\alpha}_\gamma (u) du^\gamma$ (in particular, we can parameterize the $A$–matrices via $N$–coefficients in (2) and (3)). We shall use boldface symbols in order to emphasize that a geometric space/object/construction is adapted to a $N$–connection structure. Similar symbols will be written in "non-boldface" form for any geometric object/equation if the adapting to a $N$–connection splitting is not considered. For convenience, we shall use primed, underlined indices etc., and the Einstein summation convention on repeated up–low indices will be assumed if not explicitly stated.

The local coordinates on $V$ are labelled $u^\nu = (x^i, y^a)$, (in brief, we set $u = (x, y))$, where indices respectively take values of type $i, j, ... = 1, 2, ..., n$ and $a, b, ... = n + 1, n + 2, ..., n + m$. The cumulative small Greek indices run with values $\alpha, \beta, ... = 1, 2, n + m$. In four dimensions, we consider $u^i = y^i = t$ as a time like coordinate.
A manifold \((V)\) endowed with a nontrivial \(N\)-connection structure \(N\), with nonzero \(W^\gamma_{\alpha\beta}\), is called nonholonomic. In a similar form, we can introduce the concept of nonholonomic supermanifold which depends on the type of nonholonomic distributions used for the definition of the supersymmetric/supergravity structures \([2, 60]\). We shall omit such considerations in this article and work only with the so-called bosonic part of certain modified supergravity/superstring theory.

We can elaborate any geometric construction, geometric physical theory with a \(N\)-adapted differential and integral calculus and a corresponding variational formalism for (modified) gravity theories using the \(N\)-elongated operators \([2, 3] \) and \(3\). Using frame transformations, such constructions can be re-defined with respect to arbitrary frame of reference. We say that the geometric constructions are performed with distinguished objects (in brief, \(d\)-objects) whenever the coefficients are determined using the \(N\)-elongated operators \((2)\) and \(3\). Using frame transformations, such constructions can be re-defined with respect to arbitrary frames/coordinate bases.

\[\alpha\beta\]

...such values are defined in standard form, when for \(du\) with respect to a dual local coordinate basis \(2\)-splitting, any \(d\)-metric can be written in two \(d\)-metric \(g\) in brief, \(d\)-connections, \(d\)-spinors etc. If we do not adapt the \(N\)-adapted form, the conventional \(h\)-\(v\)-splitting of formulas is not preserved under general frame/coordinate transforms.

In this article, we shall work with a special class of linear connections: A distinguished connection, \(d\)-connection, \(D = (hD, vD)\) is a linear connection, one preserving, under parallel transport, the \(N\)-connection splitting \([1]\). A general linear connection \(D\) is not adapted to a chosen \(h\)-\(v\)-decomposition, i.e. it is not a \(d\)-connection. A well known example is the Levi–Civita, LC, connection in GR which is not a \(d\)-connection even if it can be written with respect to \(N\)-adapted frames. Any \(d\)-connection \(D\) defines an operator of covariant derivative, \(D_X Y\), for a \(d\)-vector \(Y\) in the direction of a \(d\)-vector \(X\). With respect to \(N\)-adapted frames \([2, 3] \) and \(3\), we can compute \(N\)-adapted coefficients for \(D \{\Gamma_{\alpha\beta}^\gamma\} = \{L^j_{\alpha k}, L^b_{\alpha k}, C_{\alpha j}^j, C_{\alpha b}^b\}\) or of any linear connection \(D\) defined by certain geometric/physical principles, see details and explicit formulas in Refs. \([2, 3, 31, 32, 33]\). The \(N\)-adapted coefficients \(\Gamma_{\alpha\beta}^\gamma\) are defined and computed geometrically for the \(h\)-\(v\)-components of \(D_{\alpha\beta} e_\gamma := D_{\alpha\beta} e_\gamma\) using \(X = e_\alpha\) and \(Y = e_\beta\).

Any metric tensor \(g\) on a nonholonomic pseudo–Riemannian manifold \(V\) can be parameterized in off–diagonal form,

\[g = g_{\alpha\beta} du^\alpha \otimes du^\beta, \quad \text{where} \quad g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_i^a g_{ae} \\ N_j^b g_{ae} & g_{ab} \end{bmatrix}, \quad (5)\]

with respect to a dual local coordinate basis \(du^\alpha\). Equivalently, we can write a metric as a \(d\)-tensor (d–metric)

\[\tilde{g} = g_\alpha(u) e^\alpha \otimes e^\beta = g_\alpha(x) dx^i \otimes dx^j + g_\alpha(x, y) e^a \otimes e^a, \quad (6)\]

in brief, \(g = (h g, v g) = \tilde{g}\), with respect to a tensor product of \(N\)-adapted dual frame \([2]\). For a \(2 \times 2\) splitting, any \(d\)-metric can be written in two \(2 \times 2\) block diagonal form. We emphasize that a metric \(g\)

\[A\]
with N–coefficients $N_j^e$ is generic off–diagonal if the anholonomy coefficients $W_{\alpha\beta}$ are not zero. It is more convenient to work with d–metrics, canonical d–connections and N–adapted frames if we want to decouple and find general solutions for certain fundamental geometric/physical equations.

Following such geometric conditions, there are two very important linear connection structures determined by the same metric structure:

$$
\mathbf{g} \rightarrow \begin{cases} 
\nabla : & \nabla \mathbf{g} = 0; \quad T[\nabla] = 0, \quad \text{the Levi–Civita connection;} \\
\hat{D} : & \hat{D} \mathbf{g} = 0; \quad h\hat{T} = 0, \quad v\hat{T} = 0. \quad \text{the canonical d–connection.}
\end{cases}
$$

The LC–connection $\nabla$ can be introduced without any N–connection structure but can be always canonically distorted to a necessary type of d–connection. The canonical d–connection $\hat{D}$ depends generically on a prescribed nonholonomic h– and v–splitting. In formulas (7), $h\hat{T}$ and $v\hat{T}$ are respective torsion components which vanish on conventional h– and v–subspaces. There are also nonzero torsion components, $hv\hat{T}$, with nonzero mixed indices with respect to a N–adapted basis [2] and/or [3].

On any (pseudo) Riemannian manifold $\mathbf{V}$, all geometric constructions can be performed equivalently with $\nabla$ and/or $\hat{D}$ and related via a canonical distortion relation

$$
\hat{D}[\mathbf{g}, \mathbf{N}] = \nabla[\mathbf{g}, \mathbf{N}] + \hat{\mathbf{Z}}[\mathbf{g}, \mathbf{N}].
$$

In this formula both the linear connections and the distorting tensor $\hat{\mathbf{Z}}$ are uniquely determined by some data $(\mathbf{g}, \mathbf{N})$. The d–tensor $\hat{\mathbf{Z}}$ is an algebraic combination of coefficients $\hat{T}_{\alpha\beta}^\gamma$. The N–adapted coefficients for $\hat{D}$ and the corresponding torsion, $\hat{T}_{\alpha\beta}^\gamma$, Ricci d–tensor, $\hat{R}_{\gamma\beta\alpha}$, and Einstein d–tensor, $\hat{\mathbf{E}}_{\gamma\beta\alpha}$, can be computed in a simple way in N–adapted form, see details in [29, 30, 31, 32, 33]. The canonical distortion relation (3) can be used for computing respective distortion tensors of the Riemann, Ricci and Einstein tensors and corresponding curvature scalars [such values are also uniquely determined by data $(\mathbf{g}, \mathbf{N})$]. Thus, any (pseudo) Riemannian geometry and gravity theory based on such a (pseudo) geometry can be equivalently formulated using $(\mathbf{g}, \nabla)$ and/or $(\mathbf{g}, \hat{D})$.

Explicitly, the N–adapted coefficients of $\hat{D} = \{ \hat{\Gamma}_{\alpha\beta}^\gamma, \hat{L}_{jk}^i, \hat{\mathcal{C}}_{jc}^a, \hat{\mathcal{C}}_{bc}^a \}$ in (7) and (8) are

$$
\hat{L}_{jk}^i = \frac{1}{2} g^{ir} \left( e_k g_{jr} + e_j g_{kr} - e_r g_{jk} \right), \quad \hat{\mathcal{C}}_{jc}^a = \frac{1}{2} g^{ad} \left( e_c h_{bd} + e_d h_{cd} - e_d h_{bc} \right).
$$

The N–adapted coefficients of nonholonomically induced torsion $\hat{T} = \{ \hat{T}_{\alpha\beta}^\gamma \}$ are computed using (9) for the d–metric (5). Such coefficients satisfy the conditions $\hat{T}_{jk}^i = 0$ and $\hat{T}_{bc}^a = 0$, but with nontrivial h–v–coefficients

$$
\hat{T}_{jk}^i = \hat{L}_{jk}^i - \hat{L}_{kj}^i, \quad \hat{T}_{ji}^a = \hat{\mathcal{C}}_{ji}^a, \quad \hat{T}_{ji}^c = -\hat{\mathcal{C}}_{ji}^c, \quad \hat{\mathcal{C}}_{aj}^c = \hat{L}_{aj}^c - e_a(N_j^c), \quad \hat{T}_{bc}^a = \hat{\mathcal{C}}_{bc}^a - \hat{\mathcal{C}}_{bc}^a.
$$

We can consider N–splitting with zero nonholonomically induced d–torsion, when $\hat{T}_{\alpha\beta}^\gamma = 0$, i.e.

$$
\hat{\mathcal{C}}_{jb}^a = 0, \quad \mathcal{O}_{ji}^a = 0 \quad \text{and} \quad \hat{L}_{aj}^c = e_a(N_j^c).
$$

These conditions follow from formulas (9) and (11), see details in [31, 32]. If the Levi–Civita conditions, LC–conditions, (11) are satisfied, we obtain in N–adapted frames [2] and [3] $\hat{\mathbf{Z}}_{\alpha\beta}^\gamma = 0$ and $\hat{\Gamma}_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma$.

---

3The definition and the frame/coordinate transformation laws of a d–connection are different from that of a "usual" linear connection. In general, $\hat{D} \neq \nabla$ but we can impose additional conditions on coefficients $(\mathbf{g}_{\alpha\beta}, N_j^e)$ which allows us to generate LC–configurations.
Using nonholonomic variables \((\hat{\mathbf{g}}, \hat{\mathbf{D}})\), we can introduce the Lagrange density \(\hat{g}\mathcal{L} = \hat{\mathbf{R}}\), where the scalar curvature \(\hat{\mathbf{R}} := \hat{g}^{\alpha\beta} \hat{\mathbf{R}}_{\alpha\beta}\) is computed in standard form. For simplicity, the Lagrange density for matter, \(m\hat{\mathcal{L}}\), takes a simple form as it depends only on the coefficients of a metric field but not on their derivatives. The energy–momentum d–tensor can be computed via N–adapted variational calculus,

\[
m\hat{T}_{\alpha\beta} := -\frac{2}{\sqrt{|\hat{\mathbf{g}}|}} \frac{\delta(\sqrt{|\hat{\mathbf{g}}|} m\hat{\mathcal{L}})}{\delta \hat{g}^{\alpha\beta}} = m\hat{\mathcal{L}}\hat{g}^{\alpha\beta} + 2 \frac{\delta(m\hat{\mathcal{L}})}{\delta \hat{g}^{\alpha\beta}},
\]

for \(|\hat{\mathbf{g}}| = \text{det} |\hat{\mathbf{g}}_{\mu\nu}| = \text{det} |g_{\mu\nu}| = |\mathbf{g}|\). Following an N–adapted variational calculus with action

\[
g\mathbf{S} + m\mathcal{S} = \int d^d \mathbf{u} \sqrt{|\hat{\mathbf{g}}|}(g\hat{\mathcal{L}} + m\hat{\mathcal{L}}),
\]

we obtain the nonholonomically modified gravitational field equations

\[
\hat{\mathbf{R}}_{\mu\nu} = \hat{\mathbf{T}}_{\mu\nu},
\]

where \(\hat{\mathbf{T}}_{\mu\nu} = \varkappa( m\hat{T}_{\alpha\beta} - \frac{1}{2} \hat{\mathbf{g}}_{\alpha\beta} m\hat{T})\), for \(m\hat{T} := \hat{g}^{\alpha\beta} m\hat{T}_{\mu\nu}\) and \(\varkappa\) is the gravitational coupling constant.

The canonical d–connection \(\hat{\mathbf{D}}\) was used for elaborating the AFDM as a geometric method of constructing exact solutions in geometric flows and MGTs. Such a connection allows to decouple the gravitational and matter field equations with respect to N–adapted frames of reference. In Finsler geometry, similar decoupling properties also occur for the Cartan d–connection, see [19, 40]. The AFDM can not be applied completely if we work from the very beginning only with \(\nabla\). After constructing certain general classes of solutions for \(\hat{\mathbf{D}}\), or any d–connection uniquely distorted and determined by \((\mathbf{g}, \nabla)\) or \((\mathbf{g}, \hat{\mathbf{D}})\), we can impose at the end the condition \(\hat{T} = 0\) and extract LC–configurations \(\hat{\mathbf{D}}|_{\hat{T}=0} = \nabla\). In brief, the main principle of the AFDM is to deform the LC connection to an auxiliary one defined in such a way defined that it allows with ease the integration of certain important geometric/ physical equations. At the end of the procedure, we can consider imposing additional constraints if it is necessary to extract solutions with zero, or any prescribed value, of d–connection.

### 2.2 Modified Perelman’s functionals in nonholonomic variables

Let us consider a family of d–metrics \(g(\tau) = g(\tau, \mathbf{u})\) of signature \((+++–)\) and N–connections \(\mathbf{N}(\tau)\) parameterized by a positive parameter \(\tau, 0 \leq \tau \leq \tau_0\). Any nonholonomic manifold \(\mathbf{V} \subset \mathbf{V}(\tau)\) can be enabled with a double nonholonomic 2+2 and 3+1 splitting [23]. Respectively on the manifold are defined families of Lagrange densities \(g\mathcal{L}(\tau)\) and \(m\hat{\mathcal{L}}(\tau)\). We can consider local coordinates labeled as \(u^a = (x^1, y^a) = (x^1, u^2 = t)\) for \(i, j, k, \ldots = 1, 2; a, b, c, \ldots = 3, 4; \) and \(i, j, k = 1, 2, 3\). For 3+1 splitting, we can choose distributions such that any open region \(U \subset \mathbf{V}\) is covered by a family of 3-d spacelike hypersurfaces \(\hat{\Xi}_t\) parameterized by a time like parameter \(t\).

For arbitrary frame transformations on 4-d nonholonomic Lorentz manifolds with variables \((\mathbf{g}(\tau), \hat{\mathbf{D}}(\tau))\), we modify/ generalize the Perelman’s functionals in the form

\[
\hat{\mathcal{F}}(\tau) = \int_{t_1}^{t_2} \int_{\hat{\Xi}_t} e^{-\hat{\mathcal{F}}} \sqrt{|\hat{\mathbf{g}}|} d^4 u \left(\frac{1}{2} \hat{\mathbf{R}} + \frac{1}{2} \hat{\mathbf{m}} \mathcal{L} + |\hat{\mathbf{D}}\hat{\mathbf{f}}|^2\right),
\]

for 4+0 splitting

\[
\hat{\mathcal{W}}(\tau) = \int_{t_1}^{t_2} \int_{\hat{\Xi}_t} (4\pi)^{-3} e^{-\hat{\mathcal{W}}} \sqrt{|\hat{\mathbf{g}}|} d^4 u \left(\frac{1}{2} \hat{\mathbf{R}} + \frac{1}{2} \hat{\mathbf{m}} \mathcal{L} + |\hat{\mathbf{D}}\hat{\mathbf{f}}| + |\hat{\mathbf{D}}\hat{\mathbf{f}}|^2 + \hat{\mathbf{f}} - 8\right).
\]
where the normalizing function $\hat{\mathcal{F}}(\tau, u)$ satisfies $\int_{\Sigma_t} \int_{\mathbb{R}} (4\pi \tau)^{-3} e^{-\hat{\mathcal{F}}} \sqrt{|g|} d^4u = 1$. It should be noted that $\hat{\mathcal{W}}$ does not have a character of entropy for pseudo–Riemannian metrics as in the original Perelman’s work [8]. In this paper, we consider $\hat{R} + m \hat{\mathcal{L}}$ instead of $R$ used in the former mathematical works and in our previous articles. We go a step further and consider flows of the Lagrange density for matter fields, $\hat{m} \hat{\mathcal{L}}$, and other possibilities like modifying the term $\hat{R}$ into a quadratic one, or more general type of corrections and generalizations. In our approach, above F- and W–functionals may characterize relativistic nonlinear hydrodynamic flows of families of metrics and generalized connections [23]. We can compute entropy like values of type (14) and (15) for any 3+1 splitting with 3-d closed hypersurface fibrations $\hat{\Xi}_t$. In general, it is possible to work with any class of functions $\hat{f}(\tau, u)$ which can be fixed for certain constant values. In many cases, such a function is chosen in a non–explicit form which allows us to study non–normalized geometric flows and decouple certain systems of nonlinear PDEs.

If a family of nonholonomic Lorentz manifolds $\mathbf{V}(\tau)$ is determined by a 3+1 splitting with a family of d–metrics $\mathbf{g}(\tau) = [\mathbf{q}(\tau), \mathbf{N}(\tau)]$, for a 3-d hypersurface metrics $\mathbf{q}_{ij}(\tau)$ and lapse function $\mathbf{N}(\tau) = -g_{44}$, we can define and compute analogs of the above F– and W–functionals on 3-d space like hypersurfaces. Considering $\hat{\mathbf{D}} = \hat{\mathbf{D}}_{\hat{\Xi}_t}$ for the canonical d–connection $\hat{\mathbf{D}}$ defined on a 3-d hypersurface $\hat{\Xi}_t$, when all values depend on temperature like parameter $\tau$, we define also $\hat{\mathcal{R}} := \hat{R}_{\hat{\Xi}_t}$ for a corresponding N–adapted fixing of the local systems of references and necessary types of normalization functions. Using $(q_i) = (q_1, q_3)$, the Perelman’s functionals for 3-d spacelike hypersurfaces parameterized in N–adapted form are defined

$$\hat{\mathcal{F}}(\tau) = \int_{\hat{\Xi}_t} e^{-\hat{\mathcal{F}}} \sqrt{|q_{ij}|} d^3x (\frac{1}{2} \hat{\mathcal{R}} + \frac{1}{2} m \hat{\mathcal{L}} + |\hat{\mathbf{D}}f|^2),$$

and

$$\hat{\mathcal{W}}(\tau) = \int_{\hat{\Xi}_t} (4\pi \tau)^{-3} e^{-\hat{\mathcal{W}}} \sqrt{|q_{ij}|} d^3x [\tau (\frac{1}{2} \hat{\mathcal{R}} + \frac{1}{2} m \hat{\mathcal{L}} + |\hat{\mathcal{D}}f|^2 + |\hat{\mathcal{D}}\hat{f}|^2 + f - 6)],$$

where we have chosen a necessary type scaling function $f$ which satisfies $\int_{\hat{\Xi}_t} (4\pi \tau)^{-3} e^{-\hat{\mathcal{F}}} \sqrt{|q_{ij}|} d^3x = 1$. These functionals transform into standard Perelman functionals for 3-d Riemannian metrics on $\hat{\Xi}_t$ if $\hat{\mathbf{D}} \to \nabla$. To consider only such functionals is enough for 4-d stationary configurations. We can compute their evolution on a time interval by introducing additional integration along a timelike curve if certain coefficients of the metric and connections are modified to depend on the timelike coordinate.

Let us discuss possible physical implications of above F- and W-functional, normalizing (scaling) functions, and their relativistic generalizations and/or nonholonomic deformations. G. Perelman wrote (see section 1, paragraphs 1.3 and 1.4, in [8]) that such a "F-functional and its first variation formula can be found in the literature on the string theory, where it describes the low energy effective action; the function $f$ is called diaton field:...". He also remarked that exists a natural interpretation of $F$ in terms of Bochner-Lichnerowicz formulas (respectively, for differential forms and spinors). For 3-d Riemannian configurations, we can chose $f$ and respective normalization conditions in a form when variations of such functionals with the measure $e^{-\hat{\mathcal{F}}} \sqrt{|q_{ij}|} d^3x$ result in appropriate diffeomorphisms turning the evolution equations from weakly parabolic into strongly parabolic. This considerably simplify the proof of important theorems on short time existence and uniqueness of solutions. Such results can be used for various models of relativistic evolution with 3+1 splitting.

The choice and interpretation of $F$- and $W$–functionals depend also on the class of physical models where such values are introduced. Such values can be associated to some critical order parameters and respective thermodynamical values in nonlinear sigma models [11 12 13 14], for nonlinear sigma models [13 15 16 17], with generalizations for models of (non) holonomic commutative and noncommutative
gravity theories \cite{18,19,20,21}. Fixing certain values of \( f \) and/or relating such a normalization function to a scalar field \( \phi \) (see Ref. \cite{24} and developments in next sections of the present work), the relativistically generalized Perelman’s functionals \( (14) \) and \( (15) \) transforms into effective actions for \( R^2 \) gravity and/or equivalent Einstein equations with scalar fields. In a general sense, such functionals describe both relativistic evolution scenarios and, for respective self-similar fixed configurations, actions for various classes of non-Riemannian gravitational theories. Using variables involving the canonical d–connection \( \hat{D} \), we can prove that the gravitational field equations in such models can be integrated in general off-diagonal forms depending, in principle, on all spacetime coordinates \cite{29,30,31,32,42,33}.

Cosmological models with locally anisotropic and/or inhomogeneous generic off-diagonal metrics in MGTs (with possible mass terms, non-minimal couplings, noncommutative and/or supersymmetric parameters etc.) have been studied in our works \cite{84,85,86,87,88,89}. Those classes of solutions were found with effective and/or locally anisotropic polarization and variation of constants, deformed horizons, nontrivial vacuum structure etc. In this work, we study explicit examples when such cosmological models can be derived from generalized Perelman functionals for certain normalized self-similar configurations resulting in relativistic Ricci solitons being equivalent to \( R^2 \) gravity (reproducing modified Starobinsky cosmological scenarios). Here we also note that such constructions are motivated both by fundamental results in modern mathematics and by the fact that \( \hat{F} \)– and \( \hat{W} \)–functionals allows to elaborate an unified approach to geometric evolution theories, string theory, cosmology of MGTs and geometric thermodynamics. The last statement follows from the fact that \( \hat{W} \) \( (17) \) transforms into the standard Perelman’s entropy if \( \hat{D} \) is constrained to the Levi-Civita configurations. Various generalized Hamiltonians for geometric evolution and MGTs are generated by respective self-similar configurations determined by a corresponding choice, or fixing, of normalized \( f \)-function with possible interpretation as a dilaton/ effective scalar field.

### 2.3 Nonholonomic geometric evolution of \( R^2 \) gravity with conformally coupled matter

In the standard geometric analysis applied to Ricci flow theory, mathematicians are usually interested in studying the flow evolution of geometric objects determined by the metric structure (usually, of Riemannian signature or metrics related to Kähler configurations). For applications to the real world and generate realistic physical models and in modern cosmology, physicists have to create geometric relativistic models/ theories with Lorentz signature and relate to MGTs and GR and investigate possible geometric evolution scenarios (more complex, and less determined, than nonlinear diffusion processes) using exact solutions and nonholonomic deformations.

#### 2.3.1 Geometric evolution functionals and modified flow equations

Let us conformally rescale the metric in \( (14) \) and \( (15) \) in the following way, \( g_{\mu \nu} \rightarrow \tilde{g}_{\mu \nu} = g_{\mu \nu} e^{-\ln|1+2\tilde{t}|} \), for \( \sqrt{2/3} \phi = \ln|1+2\tilde{t}| \), and introduce a specific Lagrange density for matter

\[
\mathcal{L} = -\frac{1}{2} e_{\mu \phi} e^{\mu \phi} - V(\phi).
\]

In the above formula, we consider a nonlinear potential for scalar field \( \phi \)

\[
V(\phi) = \mu^2(1 - e^{-\sqrt{2/3}\phi})^2, \mu = \text{const},
\]

when \( V(\phi \gg 0) \rightarrow \mu^2, \ V(\phi = 0) = 0, \ V(\phi \ll 0) \sim \mu^2 e^{-2\sqrt{2/3}\phi}, \)
and employ the N–elongated partial derivative and differential operators, respectively, \( (2) \) and \( (3) \).

From \( (10) \) and \( (17) \), respectively, we get the functionals

\[
\hat{F}(\tau) = \int_{t_1}^{t_2} \int_{\mathcal{M}} e^{-\hat{f}} \sqrt{|g|} d^4u \left( \frac{1}{2} \hat{R} - \frac{1}{2} e_\mu \phi e^\mu \phi - V(\phi) + |\hat{D} \phi|^2 \right),
\]

and

\[
\hat{W}(\tau) = \int_{t_1}^{t_2} \int_{\mathcal{M}} (4\pi)^{-3} e^{-\hat{f}} \sqrt{|g|} d^4u \left( \frac{1}{2} \hat{R} - \frac{1}{2} e_\mu \phi e^\mu \phi - V(\phi) + |h \hat{D} f| + |v \hat{D} f|^2 + \hat{f} - 8 \right),
\]

These values present a relativistic generalization of Perelman’s functionals (describing geometric evolution of 3-d Riemannian metrics) to the case of normalized evolution of a class of MGTs with nonholonomic deformed connection \( \hat{D} \) and nonlinear potential for scalar field \( \phi \). Fixing explicit values of \( \hat{f} \), we can generate effective actions for nonholonomic Ricci soliton configurations which are equivalent to actions of respective gravitational-scalar models related to \( R^2 \) gravity. For applications of geometric flows methods in modern gravity and cosmology theories, it present a special interest to study, and find physical important solutions, of the systems of generalized Hamilton equations which can be derived from \( (20) \) and/or \( (21) \):

We can follow an N–adapted variational calculus with the above functionals. The procedure is similar to that presented in N–adapted form in \([19, 20]\). As a result, we derive a system of nonlinear PDEs that generalize the R. Hamilton equations to nonholonomic geometric evolution with canonical deformation of the linear connection structure adapted to \( h \)– and \( v \)–splitting of relativistic systems \( (g_{\mu\nu}, N^a_i, \hat{D}, \phi) \),

\[
\begin{align*}
\partial_\tau g_{ij} &= -2(\hat{R}_{ij} - \gamma_{ij}(\phi)), \\
\partial_\tau g_{ab} &= -2(\hat{R}_{ab} - \gamma_{ab}(\phi)), \\
\hat{R}_{ia} &= \hat{R}_{ai} = 0; \hat{R}_{ij} = \hat{R}_{ji}; \hat{R}_{ab} = \hat{R}_{ba}; \\
\partial_\tau \phi &= -2(\Box \phi + \frac{d}{d\phi} V(\phi)); \\
\partial_\tau f &= -\Box f + \left| \hat{D} f \right|^2 - h \hat{R} - v \hat{R} + \Box \phi + V(\phi).
\end{align*}
\]

The conditions \( \hat{R}_{ia} = 0 \) and \( \hat{R}_{ai} = 0 \) are necessary if we want to keep the total metric to be symmetric under Ricci flow evolution, see \([69]\) for theories of geometric evolution with nonsymmetric metrics. The general relativistic characteristic of 4-d geometric flow evolution is encoded in operators like \( \Box = \hat{D}^\alpha \hat{D}_\alpha \), d-tensor components \( \hat{R}_{ij} \) and \( \hat{R}_{ab} \), theirs scalars \( h \hat{R} = g^{ij} \hat{R}_{ij} \) and \( v \hat{R} = g^{ab} \hat{R}_{ab} \) with data \( (g_{ij}, g_{ab}, \hat{D}_\alpha) \).

The matter source d–tensor \( \gamma_{\alpha\beta}(\phi) = (\gamma_{ij}(\phi), \gamma_{ab}(\phi)) \) in the above formulae is computed as in \([13]\) for the energy momentum tensor \( (12) \) determined by \( (18) \), i.e.

\[
\phi T_{\mu\nu}(\phi) = \hat{D}_\mu \phi \hat{D}_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} \hat{D}_\alpha \phi \hat{D}^\alpha \phi + V(\phi) \right],
\]

when the potential \( V(\phi) \) is chosen in a form corresponding to evolution of a MGT with quadratic gravity and applications in modern cosmology.

### 2.3.2 Modified Ricci solitons and \( R^2 \) gravity

For self–similar fixed configurations with \( \tau = \tau_0 \), when \( \partial_\tau g_{\alpha\beta} = 0 \), \( \partial_\tau \phi = 0 \), the modified Hamilton equations \( (22) \) transform into canonically nonholonomically deformed gravitational and matter field

\footnote{In abstract geometric form, such proofs can be performed by changing \( \partial_\mu \rightarrow e_\mu \) and \( \nabla \rightarrow \hat{D} \) in the method presented in Perelman’s preprint \([8]\), or any detailed proof in \([11, 12, 13]\).}
equations for $R^2$ gravity supplemented with scalar field (studied for $\tilde{D} \to \nabla$ in [34] and references therein),

\begin{align}
\hat{R}_{\alpha\beta} &= \Upsilon_{\alpha\beta} \\
\Box \phi + \frac{d}{d\phi} V(\phi) &= 0.
\end{align}

(23)

into a relativistic nonholonomic variant of Ricci soliton equations (for simplicity, we omit the last formula working with non-normalized geometric flow evolution). These equations can be derived alternatively (equivalently) from the action

\begin{align}
S &= \int d^4u \sqrt{|g|} \left( \frac{1}{2} \hat{R} - 3 \frac{e^\mu e^\nu \tilde{\eta}}{(1 + 2\tilde{t})^2} - \mu^2 \frac{(2\tilde{\eta})^2}{(1 + 2\tilde{t})^2} \right) \\
&= \int d^4u \sqrt{|g|} \left( \frac{1}{2} \hat{R} - \frac{1}{2} e_\mu e^\mu \phi - V(\phi) \right),
\end{align}

(24)

which for any $\tau = \tau_0$ and corresponding parametrization of normalization function can be considered as the corresponding limits of functionals (20) and (21).

Let us prove that the equations (23) really define a relativistic version of Ricci soliton equations modeling $R^2$ gravity with the action

\begin{align}
S = \int d^4u \sqrt{|g|} \left( \frac{1}{2} \hat{R} + \frac{1}{16\mu^2} \hat{R}^2 \right).
\end{align}

(25)

Considering $\tilde{t}$ as a Lagrange multiplier, we can replace in the above equation $\hat{R}^2 \to 2\tilde{t} \hat{R} - 8\mu^2 \tilde{t}^2$ (working in nonholonomic variables with $\hat{R} \to \hat{R}$). Using the equation of motion for $\tilde{t}$ as an effective scalar field,

$$\tilde{t} = \frac{\hat{R}}{8\mu^2},$$

we can reproduce a theory with $\hat{R}^2$ term determined by a standard action for the $R^2$ gravity if $\tilde{t} \to \hat{R}$. Thus the action (25) can be written in an equivalent form (in the Jordan frame)

\begin{align}
S = \int d^4u \sqrt{|g|} \left[ \frac{1}{2} (1 + 2\tilde{t}) \hat{R} - 8\mu^2 \tilde{t}^2 \right].
\end{align}

(26)

This action is equivalent to actions (24) and (25).

It is well known that the pure $R^2$ gravity constitutes an example of a minimal version of a scale invariant theory without ghosts when instead of (26) we take

$$S = \int d^4u \sqrt{|g|} \frac{1}{16\mu^2} \hat{R}^2 \to S = \int d^4u \sqrt{|g|} \left[ \frac{1}{2} (2\tilde{t} \hat{R} - 8\mu^2 \tilde{t}^2) \right].$$

This theory is invariant (both for $\nabla$ and $\tilde{D}$) under global dilatation symmetry with a constant $\sigma$, $g_{\mu\nu} \to e^{-2\sigma} g_{\mu\nu}$, $\tilde{t} \to e^{2\sigma} \tilde{t}$. Passing from the Jordan to the Einstein frame with a redefinition $\phi = \sqrt{3/2} \ln |2\tilde{t}|$, we obtain

$$E S = \int d^4u \sqrt{|g|} \left( \frac{1}{2} \hat{R} - \frac{1}{2} e_\mu e^\mu \phi - 2\mu^2 \right),$$

14
where the scalar potential $\phi V(\phi)$ in [24] is transformed into the cosmological constant term $\mu^2$ which can be positive / negative / zero, respectively for de Sitter / anti de Sitter / flat space. The field equations derived from $E_\text{S}$ are

\begin{align}
\hat{R}_{\mu\nu} - e_\mu \phi e_\nu \phi - 2\mu^2 \hat{g}_{\mu\nu} &= 0, \\
\hat{D}^2 \phi &= 0.
\end{align}

Such equations constitute an example of relativistic nonholonomically deformed Ricci soliton equation considered in [23].

We conclude that the $R^2$ gravity for any linear connection determined in a metric compatible form by the metric structure can be modelled as a relativistic Ricci soliton configuration with effective scalar fields. In a more general context, we can preserve such an interpretation but keep in mind that additional nonholonomic deformations can be determined by a nonlinear scalar potential $\phi V(\phi)$, or other type modifications with contributions of matter fields and (we shall study in next sections) from supergravity theories. Homogeneous relativistic Ricci solitons constitute explicit examples of physically important cosmological spaces. For large positive values of $\tilde{t}$ and integrable configurations, the relativistic Ricci soliton is described by an approximate de Sitter space, which grows exponentially with Hubble parameter proportional to $\varsigma$,

$$3H^2 = \mu^2, \quad H = \frac{\dot{a}}{a}$$

for $a(t)$ denoting the scale factor of the metric $ds^2 = -dt^2 + a^2(dx^i)^2$. Starobinsky [66] proposed to describe a realistic inflationary cosmology using contributions from $R^2$ gravity. In order to create a successful density perturbation, $\delta \rho/\rho \sim 10^{-5}$, the mass scale must be as low as $\mu \sim 10^{-5} M \sim 10^{13}$GeV. Cosmological scenarios can be modified by generic off–diagonal interactions and geometric flows determining polarizations and running of physical constants (see, for instance, [70]).

\subsection*{2.4 The $SO(1, 1+k)$ generalizations of geometric flow and pure $R^2$ gravity models}

In order to realize realistic physical models, we generalize the pure $R^2$ geometric flow, Ricci solitons and gravity models by adding extra matter fields. For conformally coupled scalars, fermions and gauge bosons, one attributes corresponding scale weights in order to promote the global scale symmetry of the matter sector (at the classical level of theory) to a local conformal symmetry. It is considered that the interactions are of gauge or Yukawa type and that the scalar potential is quartic. For instance, the scalar potential from (18) is completed by a quartic one, $c V(\Phi_i)$, with corresponding modifications of $\phi V(\phi) \rightarrow \mu V(\phi, \Phi_i)$, when (respectively)

\begin{align*}
\text{cV}(\Phi_i) &= \lambda_{ijkl} \Phi_i \Phi_j \Phi_k \Phi_l, \quad \text{for certain constants } \lambda_{ijkl}, \\
\mu V(\phi, \Phi_i) &= \mu^2(1 + e^{-\sqrt{23/3} \phi / \Phi_i^2 / 6}), \\
&= \mu^2(1 - e^{-\sqrt{23/3} \phi} + e^{-\sqrt{23/3} \Phi_i^2 / 6}), \quad \text{for scale non–invarint models}.
\end{align*}

We define

$$\nabla(\phi, \Phi_i) := -e^{-2\sqrt{23/3} \phi} \, c V - \mu V,$$

\[ (29) \]

\footnote{We shall use a "caligraphic" symbol $\mathcal{R}$ in order to emphasize that such scalar curvature can be for any necessary type of linear connection.}
for some scalar fields $\Phi_\tau$ with values in $SO(1,1+k)$, for $k = 1,2,3,...$, where $\tau, i, j, \ldots = 2,3,\ldots, k+1$. Our notations differ from those in [34] (see this paper on additional speculations about classical and quantum $SO(1,1+k)$ theories and their generalizations for supergravity and superstring ones) and is motivated by constructions to be used for constructing exact solutions.

\[
\hat{F}(\tau) = \int_{t_1}^{t_2} \int_{\mathbb{R}^4} e^{-\hat{f}} \sqrt{|g|} d^4 u \frac{1}{2} (\hat{R} - \epsilon_{\mu\nu} \epsilon^{\mu\nu} - e^{-\sqrt{2}3\phi} \epsilon_{\mu\nu} \Phi_\tau e^{\mu} - \nabla(\phi, \Phi_\tau) + |D\hat{f}|^2) \tag{30}
\]

and

\[
\hat{W}(\tau) = \int_{t_1}^{t_2} \int_{\mathbb{R}^4} (4\pi)^{-3} e^{-\hat{f}} \sqrt{|g|} d^4 u \tau (\frac{1}{2} \hat{R} - \frac{1}{2} \epsilon_{\mu\nu} \epsilon^{\mu\nu} - e^{-\sqrt{2}3\phi} \epsilon_{\mu\nu} \Phi_\tau e^{\mu} \Phi_\tau - \nabla(\phi, \Phi_\tau) + |D\hat{f}| + |D^{-1}D\hat{f}|)^2 + \hat{f} - 8]. \tag{31}
\]

We can compensate contributions of terms like $\epsilon_{\mu\nu} \epsilon^{\mu\nu}$ and/or $e^{-\sqrt{2}3\phi} \epsilon_{\mu\nu} \Phi_\tau e^{\mu} \Phi_\tau$ by choosing a corresponding set of normalization functions $\hat{f}$ but the nonlinear scalar potential $\Phi V$ will contribute always to geometric flow evolution. These generalizations of Perelman’s functions are defined in such forms which include $SO(1,1+k)$ generalizations of $\mathcal{R}^2$ gravity models, which admit further extensions to supergravity theories (see next section) and new applications in elaborating cosmological evolution scenarios.

For evolution of symmetric $d$–metrics, the functionals result in modified Hamilton equations

\[
\begin{align*}
\partial_\tau g_{ij} &= -2(\hat{R}_{ij} - \mathcal{Y}_{ij}(\phi, \Phi_\tau)), \tag{32} \\
\partial_\tau g_{ab} &= -2(\hat{R}_{ab} - \mathcal{Y}_{ab}(\phi, \Phi_\tau)), \\
\hat{R}_{ia} &= \hat{R}_{ai} = 0; \hat{R}_{ij} = \hat{R}_{ji}; \hat{R}_{ab} = \hat{R}_{ba}; \\
\partial_\tau \phi &= -2(\Box \phi + \frac{\partial}{\partial \phi} \nabla(\phi, \Phi_\tau)); \\
\partial_\tau \Phi_\tau &= -2(\Box \Phi_\tau + \frac{\partial}{\partial \Phi_\tau} \nabla(\phi, \Phi_\tau)); \\
\partial_\tau f &= -\hat{\Box} f + |D\hat{f}|^2 - h \hat{R} - \nabla \phi + V(\phi).
\end{align*}
\]

The matter source $d$–tensor $\mathcal{Y}_{\alpha\beta}(\phi, \Phi_\tau) = (\mathcal{Y}_{ij}(\phi, \Phi_\tau), \mathcal{Y}_{ab}(\phi, \Phi_\tau))$ in the above formulae is computed as in [33] for the energy momentum tensor [12] determined by [29], with

\[
m\hat{L} \to \hat{L}(\phi, \Phi_\tau) = -\frac{1}{2} \epsilon_{\mu\nu} \epsilon^{\mu\nu} - \frac{1}{2} e^{-\sqrt{2}3\phi} \epsilon_{\mu\nu} \Phi_\tau e^{\mu} \Phi_\tau - \Phi V(\phi, \Phi_\tau). \tag{33}
\]

One gets

\[
\hat{T}_{\alpha\beta}(\phi, \Phi_\tau) := \frac{\delta}{\delta \phi_{\alpha\beta}} \hat{L}(\phi, \Phi_\tau) = \frac{\delta}{\delta \phi_{\alpha\beta}} \hat{L}(\phi, \Phi_\tau)g^{\alpha\beta} + 2(\hat{L}(\phi, \Phi_\tau)),
\]

and

\[
\mathcal{Y}_{\mu\nu}(\phi, \Phi_\tau) = \kappa(\hat{T}_{\alpha\beta}(\phi, \Phi_\tau) - \frac{1}{2} g_{\alpha\beta} \hat{T}(\phi, \Phi_\tau)),
\]

when the Lagrange density $\hat{L}(\phi, \Phi_\tau)$ is chosen in such a form which correspond to $SO(1, k+1)$ models of $\mathcal{R}^2$ gravity theory studied in [34].

For self-similar configurations and a fixed value of parameter $\tau_0$ (when terms with $\partial_\tau$ are fixed to zero), we obtain the equations of nonholonomic relativistic Ricci solitons with values in a scalar manifold

\[
\mathcal{M}(\phi, \Phi_\tau) = H^{k+1} \equiv SO(1,1+k)/SO(1+k)
\]
which is isomorphic to a maximally symmetric spaces, i.e. the hyperbolic space $H^{k+1}$ which is an Euclidean AdS space with non-negative cosmological constant $\mu^2$. The nonholonomic structure is given by the base spacetime $V$. Such equations can be obtained equivalently by an N–adapted variational calculus. The action in the Einstein frame is

$$S = \int d^4u \sqrt{|g|} \frac{1}{2} \left [ \hat{R} - e_{\mu} \Phi e^{\mu} - e^{-\sqrt{2/3} \Phi} e_{\mu} \Phi \Phi - \nabla(\phi, \Phi) \right ] .$$

Let us prove that the nonholonomic $SO(1,k+1)$ Ricci solitons model a $R^2$ theory of gravity. Write the corresponding action in the Jordan N–adapted frame with a Lagrange multiplier field $\tilde{t}$ of type

$$S = \int d^4u \sqrt{|g|} \frac{1}{2} \left ( (2\tilde{t} - \frac{1}{6} \Phi_\tau^2) \mathcal{R} - e_{\mu} \Phi \Phi - 2 \mathcal{V}(\Phi_\tau) - 8\mu^2 \phi^2 \right ) ,$$

where the linear term $\mathcal{R} \rightarrow \hat{R}$ is dressed by the fields $\Phi^2_i$ which has the conformal weight $w_\Phi = 1$. Performing a conformal rescaling of the metric

$$g_{\mu\nu} \rightarrow g_{\mu\nu} e^{-\ln \frac{\sqrt{2/3} \phi}{\tilde{t} - \frac{1}{6} \Phi_\tau^2}} ,$$

with $\sqrt{2/3} \phi = \ln |2\tilde{t} - \frac{1}{6} \Phi_\tau^2|$, we obtain a nonholonomic $\hat{R}^2$ theory with conformally coupled matter is given by action

$$S = \int d^4u \sqrt{|g|} \frac{1}{2} \left ( \frac{1}{8\mu^2} \hat{R}^2 - \frac{1}{6} \Phi_\tau^2 \hat{R} - e_{\mu} \Phi \Phi - 2 \mathcal{V}(\Phi_\tau) + ... \right ) .$$

In this formula, ellipses denote the fermionic and gauge boson matter parts which do not affect the vacuum structure of the model (minimal extension with fermionic and gauge boson parts of the action are invariant under conformal transformations). We note that the canonical Einstein term $R$ is absent since it is not permitted by the scale symmetry. Such a term in the action breaks the classical invariance since it shifts the field $2\tilde{t} \rightarrow 2\tilde{t} = 2\tilde{t} + 1$. Both last actions present explicit examples of two measure theories studied, for instance, in [71, 72, 73] and references therein.

The nonholonomic Ricci soliton configurations are of three types which correspond to the following three phases of the equivalent $SO(1,1+k)$ modified gravity of the type $R^2$ :

1. The scale invariant de Sitter era when the induced square mass of the canonically normalized fields $\Phi_\tau$ satisfy the condition $m^2_i = 2\mu^2 (1 - e^{-\sqrt{2/3} \phi}) > 0$ and $\mu V \rightarrow \mu^2$.

2. The flat space era when for any vacuum the total potential $tV = cV + \mu V = 0$. When there are flat directions in $cV$, there is a degeneracy vacuum in the flat direction of $\mu V$, which allows to compute the mean values $<\Phi^2_i> = e^{-\sqrt{2/3} \phi} - 1 >$, with $cV = \mu V = 0$.

3. The scale non–invariant era which is characterized by a more complex structure of the vacuum, see section 2.3.3 in [34].

We note that under geometric flow evolution the vacuum structure of the MGTs can be changed by relating different types of nonholonomic Ricci soliton configurations. Such solutions of generalized Hamilton equations (including self-similar configurations) will be presented in following sections.
3 Supersymmetric Extensions of Perelman’s Functionals and Ricci Solitons for $R^2$ Supergravity

Our main goal is to investigate possible connections with gauged supergravity of scale invariant geometric flows in a de Sitter background extending the $R^2$ gravity plus conformally invariant matter theories. The constructions will be performed in nonholonomic variables which will allow one to construct exact solutions in following sections. We shall work with the canonical d–connection which for $SU(1, 1 + k)$ extensions posses a quadratic scalar curvature $\hat{R}^2$ considering that via nonholonomic constraints $\hat{D}_{\hat{T}} = \nabla$ we can always extract standard $R^2$ and related GR configurations.

3.1 Nonholonomic variables and the minimal scale invariant $SU(1, 1 + k)$ supersymmetric extension of Perelman’s functionals

The minimal N–adapted supersymmetric extension of the $SO(1, 1 + k)$ $R^2$ - model to $SU(1, 1 + k)$ configurations is realized by introducing the supersymmetric scalar partners of $\bar{t}$ and $\Phi^2$ considered on a nonholonomic manifold. We shall use complex fields

$$\psi^I = \{T = \bar{t} + ib, z^I = |z^I|e^{i\theta^I}, \text{ for } |z^I| = \Phi^I/\sqrt{6}.$$  

Introducing the complex function $Y$ and it partial complex derivatives,

$$Y = T + \bar{T} - |z^I|^2, \quad Y_I = \frac{\partial Y}{\partial \psi^I}, \quad Y_{\bar{I}} = \frac{\partial Y}{\partial \bar{\psi}^I},$$

we can formulate scale invariant $SU(1, 1 + k)$ supergeometric flow and supergravity models. In such terms, the supersymmetric extension of generalized Perelm an’s functionals (30) and (31) can be performed in certain equivalent forms following respective formulae:

\begin{align}
\tilde{F}(\tau) &= \int_{t_1}^{t_2} \int_{\Sigma_t} e^{-\tilde{f}} \sqrt{|g|} d^4u \left[ \int \hat{R} - K_{\tau} e^\mu \psi^I e^\mu \psi^\tau - E V + |\hat{D} \hat{f}|^2 \right] \\
&= \int_{t_1}^{t_2} \int_{\Sigma_t} e^{-\tilde{f}} \sqrt{|g|} d^4u \left( \int \hat{R} - \frac{3}{4Y^2} (D_\mu D^\mu + J_\mu J^\mu) + \frac{3}{8} Y_{\tau} \bar{e}_\mu \psi^I e^\mu \psi^\tau - E V + |\hat{D} \hat{f}|^2 \right) \\
&= \int_{t_1}^{t_2} \int_{\Sigma_t} e^{-\tilde{f}} \sqrt{|g|} d^4u \left( \int \hat{R} - \frac{1}{2} e^\mu \phi e^\mu \phi - \frac{3}{4} e^{-2\sqrt{2/3}\phi} J_\mu J^\mu - 3e^{-2\sqrt{2/3}\phi} e_\mu z^I e^\mu \bar{z}^I - E V + |\hat{D} \hat{f}|^2 \right)
\end{align}

and

\begin{align}
\tilde{W}(\tau) &= \int_{t_1}^{t_2} \int_{\Sigma_t} (4\pi)^{-3} e^{-\tilde{f}} \sqrt{|g|} d^4u [\tau(\hat{R} - K_{\tau} e^\mu \psi^I e^\mu \psi^\tau - E V + |\hat{D} \hat{f}|^2 + \tilde{f} - 8)] \\
&= \int_{t_1}^{t_2} \int_{\Sigma_t} (4\pi)^{-3} e^{-\tilde{f}} \sqrt{|g|} d^4u [\tau(\hat{R} - \frac{3}{4Y^2} (D_\mu D^\mu + J_\mu J^\mu) + \frac{3}{8} Y_{\tau} \bar{e}_\mu \psi^I e^\mu \psi^\tau - E V + |\hat{D} \hat{f}|^2 + \tilde{f} - 8)] \\
&= \int_{t_1}^{t_2} \int_{\Sigma_t} (4\pi)^{-3} e^{-\tilde{f}} \sqrt{|g|} d^4u [\tau(\hat{R} - \frac{1}{2} e^\mu \phi e^\mu \phi - \frac{3}{4} e^{-2\sqrt{2/3}\phi} J_\mu J^\mu - 3e^{-2\sqrt{2/3}\phi} e_\mu z^I e^\mu \bar{z}^I - E V + |\hat{D} \hat{f}|^2 + \tilde{f} - 8)].
\end{align}

The nonholonomic and supersymmetric variables in the above formulae are defined in the following forms:
• The auxiliary vector field $A_\mu$ and the axial current $J_\mu$ (neglecting fermionic contributions, see details in [76] for holonomic configurations) are given in terms of the scalar fields in the following expression

$$A_\mu = \frac{3J_\mu}{2Y} = \frac{3i}{2Y}(Y_\tau e_\mu \tilde{\tau} - Y_\tau e_\mu \tilde{\tau}^I),$$

for

$$J_\mu = i(Y_\tau e_\mu \tilde{\tau} - Y_\tau e_\mu \tilde{\tau}^I), D_\mu = Y_\tau e_\mu \tilde{\tau} + Y_\tau e_\mu \tilde{\tau}^I = e_\mu Y.$$

These $\delta$-vector fields are necessary for the supersymmetric extension of the model and are a member of the $\mathcal{R}$ supermultiplet. They appear naturally together with the Einstein term $R$.

• In supergravity, the potential $^cV$ can be written in terms of the $Y$ function and superpotential $\tilde{W}(\psi^I)$ (see details in [34] on different supersymmetric generalizations of $\mathcal{R}^2$ theory with corresponding choices of $\tilde{W}$; in this work, we use "symbol" in order to avoid confusions with $W$ from anholonomy relations [1]),

$$^cV = Y^2 E_V,$$

$$E_V = e^K \{(\tilde{W}_I + K_I \tilde{W})K_I^T(\tilde{W}_I^T + K_I \tilde{W}) - 3|\tilde{W}|^2\} + D - \text{terms}.$$

• $K = -3\ln Y$ is the Kähler potential (a real function of scalars) which defines the symplectic metric

$$K_{IJ} = \frac{\partial^2 K}{\partial \psi^I \partial \psi^J} = \frac{3}{Y^2}(Y_I Y_I - Y Y_I).$$

• Like in the $SO(1,1+k)$ case, we use the no scale field $\phi$ determined by conditions $\sqrt{\frac{2}{3}} \phi = \ln Y$.

We use different supersymmetric $N$–adapted variables in order to be able to model for self–similar fixed configurations different models of modified supergravity in the so–called "old" and "new" formalisms corresponding to constructions in Refs. [74, 75, 76, 77, 78, 79, 80, 81, 82, 83]. The nonholonomic versions of Perelman’s like superfunctionals (34) and (35) are postulated in such forms which result (for self-similar Ricci supersoliton configurations) in mentioned type modified supergravity theories. The normalizing function and superfields, in respective nonholonomic variables, are parameterized in such a form which allows to describe alternatively the Ricci supersoliton configurations as certain $\mathcal{R}^2$ supergravity theory (described by an effective supersymmetric action, see below (41), and exact cosmological type solutions). These way, such supersymmetric generalizations of the Hamilton-Perelman theory (with respective evolution supersymmetric Hamilton like equations, to be derived in the next subsection) have explicit motivations from modern supergravity and superstring theory. The cosmological solutions for such effective $\mathcal{R}^2$ supergeometric flows and supergravity will be constructed in explicit form providing new types of supersymmetric accelerating cosmological scenarios.

3.2 Geometric flow equations with supersymmetric $SU(1, 1 + k)$ modifications

The explicit form of modified relativistic Hamilton equations depend on the type of nonholonomic variables, parameterizations and normalization functions we chose for functionals (34) and (35). For instance, the normalization function can be redefined $\tilde{f} \to f$ and constrained to satisfy the conditions

$$|\tilde{D} f|^2 = \frac{3}{4} e^{-2\sqrt{2/3} \phi} |J_\mu J^\mu + 3e^{-\sqrt{2/3} \phi} e_\mu z^\tau e^{\mu z^\tau}.$$

$$37$$
Applying these variables and an N-adapted variational formalism (for instance, to the first above functional) we obtain the following supersymmetrically modified Hamilton equations

\[
\begin{align*}
\partial_\tau g_{ij} &= -2(\tilde{R}_{ij} - E\bar{\Upsilon}_{ij}), \\
\partial_\tau g_{ab} &= -2(\tilde{R}_{ab} - E\bar{\Upsilon}_{ab}), \\
\tilde{R}_{ia} &= \tilde{R}_{ai} = 0; \quad \tilde{R}_{ij} = \tilde{R}_{ji}; \quad \tilde{R}_{ab} = \tilde{R}_{ba}; \\
\partial_\tau \phi &= -2(\tilde{\Box}\phi + \frac{\partial E\bar{V}}{\partial \phi}); \\
\partial_\tau f &= -\tilde{\Box} f + |\tilde{D} f|^2 - h\tilde{R} - v\tilde{R} + \tilde{\Box}\phi + E\bar{V}.
\end{align*}
\]

(38)

The matter source d–tensor \(E\bar{\Upsilon}_{\alpha\beta} = (E\bar{\Upsilon}_{ij}, E\bar{\Upsilon}_{ab})\) in the gravitational part of the above formulae is computed as in \((13)\) for the energy momentum tensor \((12)\) determined by \((36)\), with

\[
m\tilde{\mathcal{L}} \rightarrow E\tilde{\mathcal{L}} = -\frac{1}{2}e^\mu \phi e^\mu \phi - E\bar{V}.
\]

This allows to define

\[
E\tilde{\Upsilon}_{\alpha\beta} = \kappa(E\tilde{T}_{\alpha\beta} - \frac{1}{2}\tilde{g}_{\alpha\beta} E\tilde{T})
= \kappa[-e^\alpha \phi e^\beta \phi + \tilde{g}_{\alpha\beta} e^\mu \phi e^\mu \phi + \tilde{g}_{\alpha\beta} E\bar{V}]
\text{for}
\]

\[
E\tilde{T}_{\alpha\beta} = E\tilde{\mathcal{L}}\tilde{g}_{\alpha\beta} + 2\frac{\delta(E\tilde{\mathcal{L}})}{\delta\tilde{g}^{\alpha\beta}} = \left(-\frac{1}{2}e^\mu \phi e^\mu \phi - E\bar{V}\right)\tilde{g}_{\alpha\beta} - e^\alpha \phi e^\beta \phi,
\]

\[
E\tilde{T} = -3e^\mu \phi e^\mu \phi - 4E\bar{V},
\]

where the Lagrange density \(E\tilde{\mathcal{L}}\) is chosen in such a form which correspond to \(SU(1,1+k)\) models of \(\mathcal{R}^2\) gravity theory studied in \([34]\) and referenced therein.

The form of the system of nonlinear PDEs \((38)\) depend on the type of normalizing conditions we impose on geometric flows. The nonholonomic constraint \((37)\) encode the evolution dynamics of fields \(J_\mu\), and/or \(A_\mu\), and \(z^\tau\) into normalizations functions and geometric pseudo–Riemannian and Kähler structure. For other types of nonholonomic conditions, the evolution equations for such fields appear in the modified Hamilton equations. For MGTs, we can compute such contributions considering actions for gravitational and matter fields associated with functionals \((34)\) and \((35)\) for certain self–similar configurations.

### 3.3 Nonholonomic Ricci solitons with supersymmetric \(SU(1,1+k)\) modifications and the \(\mathcal{R}^2\) gravity theory

Fixing to zero the terms with \(\partial_\tau\) for fixed value of parameter \(\tau_0\), we obtain the equations nonholonomic Ricci solitons with supersymmetric modifications determined by \(E\bar{V}\). Depending on the type of nonholonomic variables, such equations can be alternatively derived from actions which present nonholonomic deformations in \(\mathcal{R}^2\) gravity. Writing such action with standard \(\mathcal{R} \rightarrow \tilde{\mathcal{R}}\) term in two sets
of variables, we obtain

\[ E S = \int d^4 u \sqrt{|g|} \left[ \frac{1}{2} Y \left( \hat{R} + \frac{2}{3} A_\mu A^\mu - J_\mu A^\mu + 3 Y T e_\mu \phi e^\mu \phi - c V \right) \right] \]

\[ = \int d^4 u \sqrt{|g|} \left[ \frac{1}{2} \hat{R} - K T e_\mu \phi e^\mu \phi - E V \right] \]

\[ = \int d^4 u \sqrt{|g|} \left[ \frac{1}{2} \hat{R} - \frac{3}{4 Y^2} (D_\mu D^\mu + J_\mu J^\mu) + \frac{3}{Y} Y T e_\mu \phi e^\mu \phi - E V \right] \]

\[ = \int d^4 u \sqrt{|g|} \left[ \frac{1}{2} \hat{R} - \frac{1}{2} e_\mu \phi e^\mu \phi - 3 \left( \frac{\sqrt{2/3} \phi}{4} \right)^2 - E V \right]. \quad (41) \]

Let us consider the main geometric and physical properties of such supersymmetric Ricci soliton configurations. The kinetic part of the SU(1,1 + k) geometric evolution and supergravity theory is manifestly scale invariant in a form which is very similar to the SO(1,1 + k) non-supersymmetric model. In the Einstein frame the scale symmetry acts as follows:

\[ T \to e^{2\sigma} T, \quad z^i \to e^{\sigma} z^i, \quad Y \to e^{2\sigma} Y, \quad e^{\alpha \phi} \to e^{2\sigma} e^{\alpha \phi}, \quad b \to e^{2\sigma} b. \]

The scale symmetry constraint implies that \( \tilde{W} \) has scaling weight 3: \( \tilde{W} \to e^{3\sigma} \tilde{W} \), see [34]. This requirement leads to the following three possibilities (or linear combinations of these):

1. The anti de Sitter (AdS) realization of the \( \mathcal{R}^2 \) theory, when \( \tilde{W} = e T^{3/2}, c = \text{const} \), give rise to a scale invariant model with negative cosmological term, i.e. the parameter \( \mu^2 \) is negative. The theory is conformally equivalent to a pure \( \mathcal{R}^2 \) supersymmetric theory. The potential turns out to be negative

\[ E V = e^K \left( \frac{|K T \tilde{W} + \tilde{W} T|^2}{K T T} - 3 |\tilde{W}|^2 \right) = \frac{3|c|^2}{8} \left( \frac{|T|^2 b^2}{l^3} - \frac{|T| (l^2 + b^2)}{l^3} \right) = \frac{3|c|^2}{8} \sqrt{1 + \frac{b^2}{l^2}}. \]

There is also a non-trivial gravitino mass term, \( m_{3/2}^2 = e^K |\tilde{W}|^2 = \frac{|c|^2}{8} \left( 1 + \frac{b^2}{l^2} \right)^{3/2} \). The stationary point occurs at \( b = 0 \). The classical vacuum corresponds to an AdS space with non-vanishing gravitino mass term \( m_{3/2}^2 = \frac{|c|^2}{8} \) and \( V = -\frac{3|c|^2}{8} \).

2. Flat space realizations and the connection to no-scale models are possible for \( \tilde{W} = e i z^i \tilde{T}^i \tilde{T}^k \), with nontrivial constants \( c_{ijk} \). For instance, under the so called \( U(1)_R \) gauging, \( U(1)_R : z^i \to e^{i w} z^i \) and \( W \to e^{3 i w} \tilde{W} \). The classical vacua of the model are characterized by \( \tilde{W}_i = 0 \) and \( D_\mu = 0 \) (F-flatness and D-flatness) with \( E V = 0 \). The no scale modulus \( \phi \) and the gravitino mass remain undetermined at the classical level. In this class of models the F-part of the potential can never generate a non-zero cosmological term. The only remaining way is via a contribution from non trivial Fayet-Iliopoulos D–terms (FI-term, with nontrivial antisymmetric field \( B_{\mu \nu} \)). We cite section 4 in [34] on emergence of inflationary potentials for certain anomaly free consistency conditions, which must be valid at the quantum level of the theory.

3. De Sitter realizations of the \( \mathcal{R}^2 \) theory are possible for \( \tilde{W} = e^{-K} z^i \tilde{T}^i T \), with nontrivial vacuum constants. The classical vacuum of the theory is de Sitter space if and only if there is non-trivial contribution from a non trivial \( U(1)_R \) symmetry D–term necessary for the stabilization of the \( z^i \).
fields. Under this particular case with $U(1)_R$, the superpotential transforms with a non-trivial phase giving rise to a non-zero FI term. It was shown that $N = 1$ superstring constructions provide metastable de Sitter vacua supported by FI $D_R$–terms associated to the several anomalous $U(1)_A$ gauge symmetries. The superstring resolution of these anomalies is achieved via local axion shifts promoting the several $U(1)_R$ symmetries to $U(1)_t$ or $U(1)_d$ symmetries.

Ricci soliton configurations with $SU(1,1 + k)$ modifications can be generated by different types of generating and integration functions and effective sources following the AFDM. The off–diagonal nonlinear interactions may preserve the realizations 1-3 above or change substantially the vacuum structure of certain classes of solutions.

4 Cosmological Solutions for Supersymmetric Modifications of Ricci Flows & $R^2$ Gravity

We develop the AFDM \cite{29 30 31 32 33} in order to construct in general forms certain classes of locally anisotropic and inhomogeneous cosmological solutions for geometric flows of Ricci soliton configurations modelling $R^2$ gravity. Metrics of such solutions are not stationary (like we considered for black ellipsoid solutions in $R^2$ gravity \cite{24}) but depend in general form on all spacetime coordinates and on evolution parameter $\tau$, i.e. $g_{\alpha\beta} = g_{\alpha\beta}(\tau, x^i, y^3, y^4 = t)$. Such geometric flows and Ricci soliton configurations are with nontrivial evolution of nonholonomically induced torsion. Having constructed certain general classes of solutions, we can always consider additional constraints for geometric flows with zero torsion and/or cosmological metrics of type $g_{\alpha\beta}(\tau, t)$ encoding variations of constants, off–diagonal deformations of standard cosmological solutions to nonholonomic cosmological Ricci solitons and other type cosmological solutions in MGTs.

4.1 PDEs for time like dependence of off–diagonal geometric flows and Ricci solitons

Using N–adapted 2+2 frame and coordinate transformations of the metric and source $E \gamma_{\alpha\beta}$,

$$g_{\alpha\beta}(\tau, x^i, t) = e^{\alpha'}_{\alpha}(\tau, x^i, y^a) e^{\beta'}_{\beta}(\tau, x^i, y^a) \tilde{g}_{\alpha'\beta'}(\tau, x^i, y^a) \quad \text{and}$$

$$E \gamma_{\alpha\beta}(\tau, x^i, t) = e^{\alpha'}_{\alpha}(\tau, x^i, y^a) e^{\beta'}_{\beta}(\tau, x^i, y^a) \tilde{\gamma}_{\alpha'\beta'}(\tau, x^i, y^a),$$

for a time like coordinate $y^4 = t$ ($i', i, k, k', \ldots = 1,2$ and $a, a', b, b', \ldots = 3,4$), we introduce another type of ansatz for metrics (comparing to those in \cite{24}, when $\partial_t$ was considered as a Killing vector for various classes of prime and target metrics). The new canonical parameterizations of $d$–metric and $N$–connection coefficients and (effective) sources are with generic dependence on time like coordinate $t$, applicable even for certain cases when there will be considered solutions with Killing like symmetry on $\partial_3$ where $y^3$ is a spacelike coordinate. This will allow us to decouple and solve physically important systems of PDEs in order to generate, in general, inhomogeneous and locally anisotropic solutions for relativistic geometric flows and MGTs. The generic off–diagonal metric ansatz is taken in the form

$$g = g_{\alpha'\beta'} e^{\alpha'} \otimes e^{\beta'} = g_i(\tau, x^k) dx^i \otimes dx^j + \omega^2(\tau, x^k, y^3, t) h_{ij}(\tau, x^k, t) e^a \otimes e^a \quad (42)$$

$$= q_i(\tau, x^k) dx^i \otimes dx^i + g_3(\tau, x^k, y^3, t) e^3 \otimes e^3 - N_2(\tau, x^k, y^3, t) e^i \otimes e^i, \quad (43)$$

$$e^3 = dy^3 + n_i(\tau, x^k, t) dx^i, \quad e^i = dt + w_i(\tau, x^k, t) dx^i.$$
The parametrization (43) is written as a 4–d metric with extension of a 3–d metric \( q_{ij} = \text{diag}(q_i) = (q_1, q_3) \) on a hypersurface \( \Sigma_t \) where

\[
q_3 = g_3 = \omega^2 h_3 \quad \text{and} \quad \tilde{N}^2(\tau, u) = -\omega^2 h_4 = -g_4,
\]

are related to the lapse function \( \tilde{N}(\tau, u) \).

The ansatz (42) are characterized by geometric evolution on parameter \( \tau \) of such \( N \)–connection and d-metric coefficients and are denoted by

\[
\begin{align*}
N_i^3(\tau) &= n_i(\tau, x^k, t); \quad N_i^4 = w_i(\tau, x^k, t) \quad \text{and} \\
g_{i'j'}(\tau) &= \text{diag}[g_i], \quad g_1 = g_2 = q_1 = q_2 = \exp(\psi, x^k) \\
g_{a'b'}(\tau) &= \text{diag}[\omega^2 h_a] = h_a = h_a(\tau, x^k, y^3), \quad q_3 = \omega^2 h_3, \quad \tilde{N}^2 = \tilde{N}^2(\tau, x^k, y^3, t).
\end{align*}
\]

To be able to construct exact solutions in explicit form we must parameterize sources with respect to \( N \)–adapted frames in certain diagonal form and with dependencies on \( \tau \) and \( (x^i, t) \),

\[
\bar{\Upsilon}_{\alpha\beta} = \text{diag}[\Upsilon_i(\tau); \Upsilon_a(\tau)], \quad \text{for} \quad \Upsilon_1(\tau) = \Upsilon_2(\tau) = \bar{\Upsilon}(\tau, x^k), \quad \Upsilon_3(\tau) = \Upsilon_4(\tau) = \Upsilon(\tau, x^k, t).
\]

We suppose also that the scalar and other matter fields can be described with respect to \( N \)–adapted frames when the exact solutions for \( \omega = 1 \) are with Killing symmetry on \( \partial_\tau \) which is a space like vector. Geometric methods of constructing exact solutions elaborated in [29, 30, 31, 32, 33] for "non–Killing" configurations allow us to construct very general classes of generic off–diagonal solutions depending on all spacetime variables.

In \( N \)–adapted frames, we consider certain special conditions for the effective scalar field \( \phi \) when \( e_a \phi = \phi_{a[0]} = \text{const} \). This results in \( D^2\phi = 0 \). We restrict our models to configurations of \( \phi \), which can be encoded into \( N \)–connection coefficients

\[
e_{i}\phi = \partial_i\phi - n_i\phi^* - w_i\phi^\circ = \phi_i^{[0]}, \quad \phi^* = \phi_3^{[0]}, \quad \phi^\circ = \phi_4^{[0]},
\]

for \( \phi_1^{[0]} = \phi_2^{[0]} \) and \( \phi_3^{[0]} = \phi_4^{[0]} \).

In this part of our work, there are considered brief denotations for partial derivatives \( a^* = \partial_1 a, \quad b^* = \partial_2 b, \quad h^\circ = \partial_3 h = \partial_4 h \). We used \( \phi^* = \partial_3 \phi \) in [24]. This way we encode the scalar field configurations into additional source \( \bar{\Upsilon}(\phi) = \tilde{\Lambda}_0(\phi) = \text{const} \) and \( \Upsilon(\phi) = \Lambda_0(\phi) = \text{const} \), where notations \( \tilde{\Lambda}_0(\phi) \) and \( \Lambda_0(\phi) \) emphasize that such constants are fixed for respective functionals \( \bar{\Upsilon}(\phi) \) and \( \Upsilon(\phi) \).

Self–consistent running of \( N \)–connection coefficients and scalar fields under geometric flows can be modeled for

\[
e_{a}\phi(\tau, x^k, y^a) = \phi_{a[0]} + \phi_{a[1]}(\tau)
\]

which modifies the effective h– and v–sources as

\[
\bar{\Upsilon}(\phi) = \tilde{\Lambda}_0(\phi) + \tilde{\Lambda}(\phi, \tau) \quad \text{and} \quad \Upsilon = \Lambda_0(\phi) + \hat{\Lambda}(\phi, \tau).
\]

We can use such effective sources with small values of \( \tilde{\Lambda}(\phi, \tau) \) and \( \hat{\Lambda}(\phi, \tau) \) and formula (40) in order to compute additional deformations of the evolution and modified gravitational field equations (see similar discussion in [24]).

In order to study important physical effects of geometric flows, we use the possibility to encode matter fields and various supersymmetric contributions into effective sources by imposing additional
conditions on the normalizing functions \( f \), or any nonholonomic modification to \( \hat{f} \). Thus for scalar fields we get,

\[
E \hat{\Upsilon}_{\alpha \beta} = \kappa \left( E \hat{T}_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} E \hat{T} \right) = \kappa \left[ -e_{\alpha} \phi \ e_{\beta} \phi + \hat{g}_{\alpha \beta} \left( e_{\mu} \phi \ e^{\mu} \phi + E V \right) \right].
\]

We write

\[
E \hat{\Upsilon}_{\alpha \beta} = \kappa \hat{g}_{\alpha \beta} E V \tag{49}
\]

for two classes of nonholonomic configurations: a) if \( e_{\beta} \phi \approx 0 \) is very small (this can be considered for gravitational field equations) and/or b) we choose such a normalization function in (34) and (35), i.e. for geometric flows, that \( |\hat{D} \hat{f}|^2 = \frac{1}{4} e_{\mu} \phi e^{\mu} \phi \). For such non-normalized geometric flows, the effective source are determined by \( E V \) encoding supergravity modifications of geometric flows and Ricci soliton configurations.

There are also other types of normalization conditions with N–elongated partial derivatives of the scalar field introduced in (33). Such conditions can be expressed in the following three equivalent forms

\[
K \hat{T} e_{\mu}^{\psi} e^{\mu} \psi \hat{T} + |\hat{D} \hat{f}|^2 = 0; \tag{50}
\]

\[
\frac{3}{4} \hat{Y} \left( D_{\mu} D^{\mu} + J_{\mu} J^{\mu} \right) + \frac{3}{2} \hat{Y} \left( e_{\mu} \psi e^{\mu} \psi \hat{T} + |\hat{D} \hat{f}|^2 \right) = 0;
\]

\[
\frac{1}{2} e_{\mu} \phi e^{\mu} \phi - 3 e^{-2 \sqrt{2/3} \phi} e_{\mu} \phi e^{\mu} \phi = 0. \tag{51}
\]

Supersymmetric contributions change the nonholonomic structure. The effect of this is that the normalizing function \( \hat{f} \) is different for different types of nonholonomic variables. Nevertheless, this is not a problem for finding exact solutions even in the non-normalized form. The main idea is to construct certain classes of physically important solutions for certain special normalisation when the decoupling of PDEs is possible. Fixing certain evolution points for \( \tau \), the metrics and connections will determine corresponding cosmological Ricci soliton configurations and/or MGTs.

We conclude that supersymmetric \( SU(1,1+k) \) models are characterized by modified geometric evolution equations (38) when all terms depending on \( \phi \) can be encoded into normalizing function \( f \) using the equation (39) keeping a nontrivial value of potential \( E V \). This results in \( m \hat{L} \rightarrow E \hat{L} \rightarrow - E V \) for a N–adapted parametrization \( E \hat{Y} \rightarrow \text{diag}[E \hat{Y}, E \hat{Y}] \). In such cases, sources of type (48) will be encoded into certain configurations \( [E \hat{Y}, E \hat{Y}] \), when \( E \hat{Y} \neq E \hat{Y} \). For any of normalizations (50), or source (49), we shall be able to integrate in explicit form the geometric flow equations for any \( E \hat{Y}(\tau, x^k) \) and \( E \hat{Y} = \kappa E V(\tau, x^k, t) \). In particular, we can consider stationary distributions of matter with \( E \hat{Y}(\tau, x^k) = E \hat{Y} = \kappa E V(\tau, x^k) \) which under geometric flows and via generic off–diagonal interactions result in inhomogeneous and locally anisotropic cosmological metrics \( \hat{g}_{\alpha \beta}(\tau, x^k, y^3, t) \).

4.1.1 Geometric flows and time evolution of d–metric coefficients

In this work, the solutions will be generated with respect to two generating functions \( \psi(\tau, x^i) \) and \( \Psi(\tau, x^i, t) \). The first one will be determined to generate solutions for certain 2-d Poisson type equations with (effective) source modified by geometric flows. To understand the properties of the second one, we consider a set of coefficients \( \alpha_\beta = (\alpha_i, \alpha_3 = 0, \alpha_4) \) determined by a generating function \( \Psi \) when

\[
\alpha_i = h_3^i \partial_i \Psi / \Psi, \quad \alpha_4 = h_3^4 \Psi ^\circ / \Psi, \quad \gamma = \left( \ln |h_3|^{3/2} / |h_4| \right)^{\circ}
\]

\[
\text{for } \Psi := h_3^0 / \sqrt{|h_3 h_4|}. \tag{52}
\]
A tedious computation (see details of such computation in [31, 32]) of the N–adapted components of the Ricci tensors for $\frac{\n}{\n}$, ansatz for $\frac{\n}{\n}$-metric (12) and supersymmetric sources (16) prove that we can transform the nonholonomic Ricci evolution equations (58) into the following system of nonlinear PDEs,

$$
\psi^{**} + \psi'' = 2\left( E\n\n - \frac{1}{2}\partial_s\psi \right),
$$

$$
\Psi^0 h^2_3 = 2h_3 h_4 (E\n\n - \partial_s \ln |\omega^2 h_3|) \Psi,
$$

$$
\partial_s \ln |\omega^2 h_3| = \partial_s \ln |\omega^2 h_4| = \partial_s \ln |N^2| ,
$$

$$
n^\gamma_i = 0,
$$

$$
\alpha^2 w_i - \alpha_i = 0,
$$

$$
e_k \omega = \partial_n^k \omega^0 + n_k \omega^0 = 0,
$$

The unknown functions are $\psi(\tau, x^i)$, $\omega(\tau, x^k, y^3, t)$, $h_0(\tau, x^k, t)$, $n_i(\tau, x^k, t)$ and $n_i(\tau, x^k, t)$. The first two equations contain possible additional sources determined by other effective polarized cosmological constants or matter fields written as $E\n\n(\tau, x^k)$ and $E\n\n(\tau, x^k, t)$. The system (53)–(58) has an important decoupling property when we can find $\psi$ from the first equation, $h_3$ and $h_4$ from the second and third equation and so on for any unknown function.

### 4.1.2 Inhomogeneous cosmological Ricci soliton equations

For such models, we consider fixed evolution parameter configurations with $\partial_s g_{\alpha\beta} = 0$ and $\tau = \tau_0$. The equations (53), (54) and (55) transform into self-similar Ricci soliton equations which for the off–diagonal ansatz can be written in the form

$$
\psi^{**}(\alpha) + \psi''(\alpha) = 2 E\n\n(\alpha) \Psi(\alpha) \Psi(\alpha),
$$

$$
\Psi^0(\alpha) h^2_3(\alpha) = 2 h_3(\alpha) h_4(\alpha) E\n\n(\alpha) \Psi(\alpha).
$$

In this paper, we write $\psi(\alpha, \tau, x^i)$ and $\Psi(\alpha, \tau, x^i, y^3 = t)$ instead of, respectively, $\psi(\alpha, x^i)$ and $\Psi(\alpha, x^i, y^3)$ used in [24] for generating stationary $R^2$ Ricci solitons (for a fixed evolution parameter those solutions depend generically only on space like coordinates $(x^i, y^3)$). For nonhomogeneous and locally anisotropic Ricci solitonic configurations, the time like evolution is possible being determined by nonholonomically deformed Einstein equations.

The equation (59) is just the 2-d Poisson equation which can be solved in general form for any prescribed source $E\n\n(\alpha, x^k)$. We note that such an equation is obtained with respect to $N$–adapted frames and that the right hand side encode "static" h–distributions of matter and possible supersymmetric deformations.

The system of nonlinear PDEs (52) and (61) can be integrated for any source $E\n\n(\alpha, x^k, t)$ encoding contributions from supergravity modified theories and matter fields. We point some key differences in constructing such solutions compared to various classes of solutions studied in [31, 32, 33, 84, 85, 86, 87, 88]. The generic off–diagonal cosmological solutions possess a nonlinear symmetry for re–definition of generating function, $\Psi \leftrightarrow \Psi$, and (effective source), $E\n\n(\alpha) \leftrightarrow \Lambda_0 \neq 0$, when

$$
\Lambda_0(\Psi^2(\alpha)) = |E\n\n(\Psi^2(\alpha))|, \text{ or } \Lambda_0 \Psi^2(\alpha) = |\Psi^2(\alpha)| E\n\n(\alpha) - \int dt \Psi^2(\alpha)| E\n\n(\alpha) |.
$$

This property can be used for re–definition of generation and source functions in order to simplify the method of generating exact solutions and in order to find new classes of solutions by nonholonomic deformations.
For generating off–diagonal locally anisotropic nonsingular cosmological solutions depending on \(y^i = t\), we have to consider generating functions involving \(\Psi^0(x)\). The system (51)–(57) with Killing symmetry on \(\partial_3\) leads to the following system of nonlinear PDEs

\[
\begin{align*}
\Psi^0(x) h_3^2(x) & = 2 h_3(x) h_4(x) E \Upsilon(x) \Psi(x), \\
\sqrt{h_3(x) h_4(x)} |\Psi(x)| & = h_3^2(x), \\
\Psi^0(x) \psi_i(x) - \partial_i \psi(x) & = 0, \\
n_i^i(x) + \left( \ln |h_3(x)|^{3/2} / h_4(x) \right) \circ n_i^i(x) & = 0.
\end{align*}
\]  

(61) and (62) can be expressed respectively as

\[
\Psi^0(x) h_3^2(x) = -2q^2 E \Upsilon(x) \Psi(x) \quad \text{and} \quad h_3^2(x) = q \Psi(x).
\]

(66)

Introducing \(h_3^2(x)\) from the second equation into the first one, we find

\[
q = -\frac{1}{2} \frac{\Psi^0(x)}{E \Upsilon(x)}.
\]

(67)

Substituting this value in the second equation of (66) and integrating on \(t\), we find

\[
h_3(x) = h_3(x) \circ - \frac{1}{4} \int dt \frac{(\Psi(x))}{E \Upsilon(x)},
\]

(68)

where \(h_3^0(x)\) is an integration function. The coefficient \(h_4\) follows from considering (67), (65) and formula (68),

\[
h_4(x) = \frac{1}{4h_3} \left[ \frac{\Psi^0(x)}{E \Upsilon(x)} \right]^2 = \frac{1}{4} \left\{ \frac{\Psi^0(x)}{E \Upsilon(x)} \right\}^2 \left( h_3^0 - \frac{1}{4} \int dt \frac{(\Psi^2(x))}{E \Upsilon(x)} \right)^{-1}.
\]

(69)

The N–connection coefficients \(n_i(x^k, t)\) are found by integrating two times on \(t\) in (61) using the value of coefficient \(\gamma\) found from (68) and (69). The first integration results in \(n_i^i(x) = 2n_i(x^k) / \circ h_4(x) / \circ h_3(x)^{3/2}\) with an integration functions \(2n_i(x^k)\). Integrating again on \(t\) and considering other set of integration functions \(1n_i(x^k)\), we find

\[
\begin{align*}
\gamma n_k(x, \tau_0) & = 1n_k + 2n_k \int dt \frac{h_4(x)}{|h_3(x)|^{3/2}} = 1n_k + 2\bar{n}_k \int dt \frac{|\Psi^0(x)|^2}{|h_3(x)|^{5/2} (E \Upsilon(x))^2} \\
& = 1n_k + 2n_k \int dt \frac{(\Psi^0(x))^2}{(E \Upsilon(x))^2} \left( h_3^0 - \frac{1}{4} \int dt \frac{(\Psi^2(x))}{E \Upsilon(x)} \right)^{-5/2},
\end{align*}
\]

(64)
with redefined \( 2n_i \rightarrow \tilde{2}\tilde{n}_k(x^i) \) including certain constant coefficients before the source and generating function. For any generating function \( \Psi(\varphi) \), we can solve the linear algebraic equations (63) and find the second sub-set of \( N \)–connection coefficients, \( w_i(\varphi) = \partial_i \Psi(\varphi)/\Psi^\circ(\varphi) \).

Summarizing in this subsection, we obtain the set of formulae for computing the coefficients of a d–metric and a \( N \)–connection which determine, in general form and dynamical in time, Ricci solitons with Killing symmetry on \( \partial_3 \) as solutions for the system (53)–(57),

\[
g_i(\varphi,\tau_0) = e^{\psi(\varphi,\tau_0,x^k)} \text{ as a solution of 2-d Poission equations (59)};
\]

\[
h_3(\varphi,\tau_0) = -\frac{1}{4} \left( \Psi^\circ(\varphi) \right)^0 \left( h^0_3 - \frac{1}{4} \int dt \left( \frac{\Psi^\circ(\varphi)}{E \Upsilon(\varphi)} \right)^{-1} \right); \tag{70}
\]

\[
h_4(\varphi,\tau_0) = h^0_4(\varphi,\tau_0) - \frac{1}{4} \int dt \left( \frac{\Psi^\circ(\varphi)}{E \Upsilon(\varphi)} \right)^0;
\]

\[
n_k(\varphi,\tau_0) = 1n_k + 2n_k \int dt \left( \frac{\Psi^\circ(\varphi)}{E \Upsilon(\varphi)} \right)^2 \left( h^0_3 - \frac{1}{4} \int dt \left( \frac{\Psi^\circ(\varphi)}{E \Upsilon(\varphi)} \right)^{-1/2} \right); \tag{71}
\]

The d–metric coefficients are determined functionally, respectively, by generating functions and sources, \( g_i[\psi(\varphi), \ E \Upsilon(\varphi)] \) and \( h_i[\Psi(\varphi), \ E \Upsilon(\varphi)] \). We can solve the equations (53) for a nontrivial \( \omega^2(\varphi) = \left| h_3(\varphi) \right|^{-1} \). The quadratic elements for such solutions with nonholonomically induced torsion are parameterized as

\[
ds^2 = g_{\alpha\beta} \partial_\alpha \partial_\beta = e^{\psi(\varphi)[(dx^1)^2 + (dx^2)^2] +}
\]

\[
\omega^2(\varphi) \ h_3[\Psi(\varphi), \ E \Upsilon(\varphi)] [dy^3 + (1n_k + 2\tilde{n}_k) \int dt \left( \frac{\Psi^\circ(\varphi)}{E \Upsilon(\varphi)} \right)^2] \left( h^0_3 - \frac{1}{4} \int dt \left( \frac{\Psi^\circ(\varphi)}{E \Upsilon(\varphi)} \right)^{-1/2} \right) dx^k \right|^2 - \frac{\omega^2(\varphi)}{4 h_3(\varphi)} \left[ \frac{\Psi^\circ(\varphi)}{E \Upsilon(\varphi)} \right]^2 [dt + \partial_i \Psi^\circ(\varphi) / \Psi^\circ(\varphi)] dx^i]^2. \tag{71}
\]

This class of metrics define also exact cosmological inhomogeneous and locally anisotropic solutions for the canonical d–connection \( \tilde{\mathbf{D}} \) in \( R^2 \) gravity with nonholonomically induced torsion and effective scalar field encoded into a nonholonomically polarized vacuum. We can impose additional constraints on generating functions and sources in order to extract Levi–Civita configurations (LC–configurations) as it is described in section 4.2. If in above formulas, we take instead of \( E \Upsilon(\varphi) = E V \), for instance, \( E \Upsilon(\varphi) = V(\phi, \Phi) \), we generate nonholonomic cosmological Ricci soliton solutions generalizing the constructions for \( SO(1, k + 1) \) models of \( R^2 \) gravity theory [34, 21].

4.1.3 Geometric evolution of cosmological Ricci solitons with factorized dependence on flow parameter

For simplicity, we construct solutions with Killing symmetry on \( \partial_3 \) (when \( \omega = 1 \)) and \( E \Upsilon(\varphi) = \Upsilon[0] = \text{const}. \) If \( E \Upsilon(\varphi) \) is not constant, it is a more difficult task to construct exact solutions in explicit form (we shall consider such examples in section 4.1.4). Considered here are generated functions, \( \psi(\tau, x^k) \) and \( \Psi(\tau, x^k, t) \), and effective sources, \( \hat{\Upsilon}(\tau, x^k) \) and \( \Upsilon(\tau, x^k, t) \), depending in factorized form on flow parameter \( \tau \),

\[
\psi(\tau, x^k) = \psi(\tau, x^k), \Psi = \Psi(\tau, x^k, t),
\]

\[
\text{for } h_3 = h_3(\tau) \ h_3(\tau, x^k, t), h_4 = h_4(\tau) \ h_4(\tau, x^k, t) \tag{72}
\]

\[
\text{and } \hat{\Upsilon}(\tau, x^k) = \hat{\Upsilon}(\tau) + E \Upsilon(\tau, x^k), \Upsilon(\tau, x^k, t) = \Upsilon(\tau) + E \Upsilon(\tau, x^k, t),
\]
The coefficients corresponding values \( h \) Ricci soliton but the d–metric coefficients are with factorized integration functions when the N–connection coefficients are completely determined by the data for a be integrated "step by step" as follows:

\[ e \, \psi_{\perp} = A_0 e^{2 \int d\tau \tilde{\Lambda}(\tau)}, A_0 = const, \]

where the integration constant can be set to \( A_0 = 1 \). This corresponds to possible variation of constants induced by effective scalar fields and effective cosmological constant and other possible matter sources.

To model the evolution of certain Ricci soliton configurations described by \((74)\) it is necessary to satisfy the conditions \(| h_{\perp 3} | = | h_{\perp 4} |\) and

\[ h_{\perp 4}[1 + \frac{1}{\Psi[0]} \left( \Psi_{\perp}(\tau) - \frac{\partial_\tau h_{\perp 4}}{h_{\perp 4}} \right)] = 1. \]

The solution of this equation is

\[ h_{\perp 3}(\tau) = 1 + \varepsilon_{\perp}(\tau), \text{ for } \varepsilon_{\perp}(\tau) = S_0 e^{\lambda_1 \tau} + S_1 e^{\lambda_3 \tau} \int d\tau e^{-\lambda_1 \tau} [\Psi_{\perp}(\tau)], \]

with integration constants \( S_0 \) and \( S_1 \) and \( \lambda_1 := (\Psi[0]). \) Such configurations have physical importance if there is an interval \( 0 \leq \tau \leq \tau_0 \) with \( 0 h_{\perp 4} \to 1 \) for increasing \( \tau_0 \). For certain deformations of stationary solutions in MGTs, the function \( \varepsilon_{\perp}(\tau), | \varepsilon_{\perp}(\tau) | \ll 1 \), has to be found from experimental data. We can express

\[ h_a = | h_{\perp a}(\tau)| h_a(\tau, x^k, t) \]

where \( h_a(\tau) \) are those from \((70)\) but with

\[ h_3(\tau) = h_3[0](x^k) - \frac{1}{4 \Psi[0]}(\Psi(\tau))^2 \quad \text{and} \quad h_4(\tau) = -\frac{1}{4 h_3(\tau)}(\frac{\Psi(\tau)}{\Psi[0]})^2. \]

Summarizing the above solutions, we obtain the following sets of d–metric coefficients,

\[ g_1(\tau, x^k) = g_2 = e \, \psi_{\perp} e^{\psi(\tau, x^k)} \text{ for } e \, \psi_{\perp} = A_0 e^{2 \int d\tau \tilde{\Lambda}(\tau)}, A_0 = const; \]

\[ h_3(\tau, x^k, t) = | h_{\perp 3}(\tau)| \left[ h_3[0](x^k) - \frac{1}{4 \Psi[0]}(\Psi(\tau))^2 \right]; \]

\[ h_4(\tau, x^k, t) = -| h_{\perp 3}(\tau)| \frac{1}{4 h_3(\tau)} \left( \frac{\Psi(\tau)}{\Psi[0]} \right)^2 \text{ for } h_{\perp 3}(\tau) \text{ taken as in (78)}; \]
and N–connection coefficients,

\[ n_k(\tau, x^i, t) = 1 n_k(\tau, x^i) + 2 n_k(\tau, x^i) \int dt \frac{h_4}{|h_3|^{3/2}} = 1 n_k(\tau, x^i) + 2 \tilde{n}_k(\tau, x^i) \int dt \frac{(\Psi(\tau))^2}{|h_4(\tau)|^{5/2}} \]

\[ w_i(\tau, k, t) = \partial_i \Psi(\tau)/ \Psi(\tau); \]

for certain re–defined integration and generation functions. In the above formulae, the generation functions and sources, the integration functions and the constants depend on the evolution parameter \( \tau \). This determines additional anisotropic polarizations of physical values and running of physical constants. The off–diagonal terms \( w_i(\tau, k, t) \) do not depend on the evolution parameter. If we take \( 2n_k = 0 \) and \( 1n_k = n_k(x^k) \), not all N–connection coefficients now depend on geometric evolution parameter being determined by a prescribed cosmological Ricci soliton configuration. Such configurations can be restricted to LC ones.

The generic off–diagonal quadratic elements with coefficients \( [80] \) are for solutions of relativistic geometric flows that induce anisotropic polarizations and running constants of cosmological Ricci solitons,

\[ ds^2 = g_{\alpha \beta}(\tau, x^k, t)du^\alpha du^\beta = e^2 \int dr \tilde{\Lambda}(\tau)e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + \]
\[ + \{1 + \varepsilon_\perp(\tau)\} \left[ dy^3 + (1 n_k(\tau, x^i) + 2 n_k(\tau, x^i) \int dt \frac{h_4(\tau)}{|h_3(\tau)|^{3/2}} dx^k \right]^2 \]
\[ - \frac{1}{4} \frac{h_4(\tau)}{|h_3(\tau)|^{3/2}} \left[ dt + \frac{\partial_1 \Psi(\tau)}{\Psi(\tau)} dx^1 \right]^2, \]

The nonholonomic geometric flow evolution described by the above cosmological d–metrics is for the canonical d–connection \( \tilde{D} \) in \( R^2 \) gravity with effective scalar field encoded into a nonholonomically polarized vacuum. For realistic cosmological models, we shall consider, for instance, modifications of the FLRW metric with explicit running/ polarized physical constants, see section [5]. Variations of constants should be taken from certain observational data and theoretical arguments (see, for instance, [70]).

### 4.1.4 Geometric cosmological flows of effective sources determined by evolution of d–metric and N–connection coefficients

Applying the AFDM, we can find exact solutions of the geometric flow equations when the d–metric and N–connection coefficients and generating functions depend in a general form on evolution parameter \( \tau \). For instance, we impose certain physically motivated constraints on the generating functions and then compute some well–defined horizontal and vertical effective sources.

We consider an effective source \( \Lambda_0(\tau) \) which constraints the source with supersymmetric corrections \( E\tilde{\Psi} \) and the generating function \( \partial_\tau \psi \) to satisfy the condition

\[ E\tilde{\Psi} - \frac{1}{2} \partial_\tau \psi = \Lambda_0(\tau). \]

Let us chose \( \psi(\tau, x^k) \) for \([53]\) as a solution of parametric on \( \tau \) 2-d Poisson equation,

\[ \psi^* + \psi'' = 2\Lambda_0(\tau). \]
In such an approach, the supersymmetric source (of any type 1-3 with corresponding scale symmetry on $\tilde{W}$ considered at the end of section [3]).

Another family of solutions can be generated if we use for the h–metric factorizations (72) with $\psi(\tau, x^k) = \psi_\perp(\tau) + \psi(\tau, x^k)$ and $\tilde{\Upsilon}(\tau, x^k) = \tilde{\Upsilon}_\perp(\tau) + E\tilde{\Upsilon}(\tau, x^k)$.

The next step is to integrate in certain general forms the equations for the vertical components of d–metric. We can generate a class of solutions of geometric flow equations (54)–(58) for arbitrary $h_3(\tau, x^k, t), h_3^3 \neq 0$ (it can be considered also as a generating function) if we introduce an effective cosmological constant $\Lambda_0$ and treat $E\tilde{\Upsilon}(\tau, x^k, t)$ as an effective source determined by the condition

$$E\tilde{\Upsilon} - \partial_\tau \ln |\omega^2 h_3| = \Lambda_0 \neq 0. \quad (82)$$

As a result, the system of equations (52) and (54) can be written as

$$\sqrt{|h_4|} = \frac{h_3^3}{\Psi \sqrt{|h_3|}} \text{ and } h_4 = \frac{\Psi^4 h_3}{2h_3} \Lambda_0,$$

for two unknown functions $h_3(\tau, x^k, t)$ and $\Psi(\tau, x^k, t)$. Using the square of the first equation with $h_3 = \epsilon_a |h_3|, \epsilon_a = \pm 1$, we compute

$$\Psi^2 = B(\tau, x^k) - \frac{4}{\Lambda_0} h_3 \text{ and } h_4 = -\frac{(h_3^3)^2}{h_3(\Lambda_0 B(\tau, x^k) - 4h_3)} \quad (83)$$

for an integration function $B(\tau, x^k)$.

A class of solutions of (55) can be generated for any $h_3 = h_4$ considering both such values determined by the same generating function $h_4(\tau, x^k, t)$. Using (83), we can generate solutions with $h_3 \neq h_4$ but it is difficult to solve in explicit form such equations for arbitrary $\omega$.

We find the first set of N–connection coefficients by integrating two times on $t$ in (56) using the condition $h_3 = h_4$,

$$n_k(\tau, x^i, t) = n_k(\tau, x^i) + 2\tilde{n}_k(\tau, x^i) \int dt \frac{1}{(\sqrt{|h_3|})^{-1}}. \quad (84)$$

The algebraic equation (57) for the second set of N–connection coefficients are solved in general form by considering the generating function from (83),

$$w_i(\tau, x^k, t) = \frac{\partial_\omega \Psi}{\Psi} = \frac{\partial_\omega \Psi^2}{\partial_\omega (\Psi^2)} = (h_3^3)^{-1}\partial_\omega [h_3 - \frac{\Lambda_0}{4} B(\tau, x^k)]. \quad (85)$$

It is possible to generalize the class of solutions by introducing a vertical conformal factor $\omega(\tau, x^k, t)$ as a solution of (58),

$$\partial_\omega \omega + n_k(\tau, x^i, t) \omega^* + w_k(\tau, x^i, t) \omega^0 = 0,$$

for N–connection coefficients determined by $h_3$ and respective integration functions, see (83) an (84). In particular, we can take $\omega = 1$ and generate solutions for geometric evolution of stationary configurations. As solutions of the equations (58), we can consider also distributions of a scalar field subject to the conditions (47) and (48) resulting in modifications with an effective cosmological constant.

Putting together the above formulae, we construct the following quadratic line element

$$ds^2 = g_{\alpha\beta}(\tau, x^k, t)du^\alpha du^\beta = e^{\psi(\tau, x^k)} [(dx^1)^2 + (dx^2)^2] + \omega^2(\tau, x^i, t)h_3(\tau, x^i, t) \{ [dy^3 + (1n_k(\tau, x^i) + \tilde{n}_k(\tau, x^i) \int dt (|h_3(\tau, x^i, t)|^{-1})dx^k]^2 + [dt + \partial_\omega (4h_3(\tau, x^i, t) - \Lambda_0 B(\tau, x^k)) \frac{4h_3^3(\tau, x^i, t)}{4h_3^3(\tau, x^i, t)} dx^i]^2 \}. \quad (86)$$
This class of inhomogeneous cosmological solutions and their geometric flows is general, with evolution of N–connection coefficients and respective nonholonomically induced torsion. Such geometric flows may transform one class of cosmological Ricci solitons into another one. In a more general context, we can consider evolution of a (supersymmetric) MGT into another MGT, or into certain classes of solutions in GR. We can always put at the end certain constraints for zero torsion. Mutual transformations of classes of (off-) diagonal solutions in GR can be described as some particular examples of such geometric flow evolution models. Using as sources effective potentials with supersymmetric corrections, \( E\Upsilon \rightarrow EV \), we can encode such contributions into a nontrivial effective vacuum structure of MGT, in particular, for \( R^2 \) or \( R \) gravity models.

4.2 Extracting geometric cosmological evolution of Levi–Civita configurations

We are required to impose additional constraints and parameterizations for the d–metric and N–connection coefficients in order to satisfy the zero torsion conditions (11) and extract Levi–Civita, LC, configurations. For the generic off–diagonal ansatz (42) with dependence on flow parameter \( \tau \), such conditions are equivalent to the system equations (see details in [31, 32, 33])

\[
\begin{align*}
    w_i^\circ(\tau,x^i,t) &= \left[ \partial_i - w_i(\tau,x^i,t)\partial_t \right] \ln \sqrt{|h_3(\tau,x^i,t)|},
    \left[ \partial_i - w_i(\tau,x^i,t)\partial_t \right] \ln \sqrt{|h_4(\tau,x^i,t)|} = 0, \\
    \partial_i w_j(\tau,x^i,t) &= \partial_j w_i(\tau,x^i,t),
    n_i^\circ(\tau,x^i,t) = 0, \partial_i n_j(\tau,x^i,t) = \partial_j n_i(\tau,x^i,t),
\end{align*}
\]

where we denote in brief \( \partial_4 h_4 = \partial_t h_4 = h_4^t \). By imposing such constraints, we can always make zero the d–torsion coefficients (10).

Following the same procedure as in [24], for simplicity, for classes of solutions with factorized parameterizations of d–metrics like (72), we prove that (87) can be satisfied if there are considered constraints such as follows:

1. We consider generating functions \( \Psi = \tilde{\Psi}(\tau,x^k,t) \) for which

\[
(\partial_i \tilde{\Psi}[\tau])^\circ = \partial_i (\tilde{\Psi}^\circ[\tau]),
\]

2. For sources, we take \( E\Upsilon = \text{const} \) or express any effective source as a functional \( E\Upsilon(x^i,t) = E\Upsilon[\tilde{\Psi}(x^i,t),\Lambda(\tau)] \) with parametric coordinate dependencies.

3. The conditions that \( \partial_i w_j = \partial_j w_i \) can be expressed in conventional form via any function \( \tilde{A} = \tilde{A}(\tau,x^k,t) \) for which

\[
w_i = \bar{w}_i = \partial_i \tilde{\Psi}/\tilde{\Psi}^\circ = \partial_i \tilde{A}.
\]

If \( \tilde{\Psi} \) is prescribed, we solve a system of first order PDEs which allows to find a function \( \tilde{A}[\tilde{\Psi}] \).

4. We choose \( n_j(\tau,x^k) = \partial_j n(\tau,x^k) \) for a function \( n(\tau,x^k) \). It is also possible to consider, running on a geometric flow parameter, like \( n(\tau,x^k) \) a more generalized class of integration functions.

Variations of values \( E\Upsilon(\tau), \tilde{\Lambda}(\tau), \Lambda(\tau), \mu(\tau) \) etc. have to be taken from observational data [10]. Dirac’s idea on variation of physical constants is considered in MGTs and geometric flows [28]. We conclude that geometric flow solutions can explain possible locally anisotropic polarizations and running of d–metric and N–connection coefficients and running of fundamental physical constants.
4.3 Small parametric deformations of off–diagonal cosmological solutions

We constructed and studied such off–diagonal deformations of cosmological solutions in various MTGs, see [56, 57, 58, 59, 60] and references therein. For geometric flows of stationary metrics in $R^2$, such constructions are also possible [24]. In general, it is not clear if various classes of inhomogeneous and locally anisotropic cosmological solutions may have physical importance. There were not constructed such off–diagonal cosmological solutions for geometric flows and Ricci solitons considered in other works. We can solve an number of theoretical, phenomenological and observational problems if we consider sub–sets of solutions generated from well known ones as deformations depending on a small parameter. Mathematically, using the AFDM various cosmological solutions can be constructed as exact ones for certain sets of prescribed parameters and generating functions. In this section, we show how the so-called $\varepsilon$–deformation procedure on a small positive parameter $\varepsilon$ of an off–diagonal "prime" metric, $\hat{g}(\tau,x^i,y^3,t)$ (for physically important known solutions such a metric can be diagonalizable under coordinate transforms) into a "target" metric, $g(\tau,x^i,y^3,t)$, works for cosmological geometric flows and Ricci solitons with generic dependence on $t$.

A "prime" pseudo–Riemannian cosmological metric $\hat{g} = [\hat{g}_i, \hat{h}_a, \hat{N}_i^2]$ can be parameterized in the form

$$ds^2 = \hat{g}_i(\tau,x^k,t)(dx^i)^2 + h_a(\tau,x^k,t)(\hat{e}^a)^2,$$

$$\hat{e}^3 = dy^3 + \hat{n}_i(\tau,x^k,t)dx^i, \hat{e}^4 = dt + \hat{w}_i(\tau,x^k,t)dx^i.$$

We suppose that such a metric is diagonalizable via a coordinate transform $u^\alpha' = u^\alpha(u^\alpha)$

$$ds^2 = \hat{g}'(\tau,x^k',t)(dx^i')^2 + \hat{h}_a'(\tau,x^k',t)(dy^\alpha')^2,$$

with $\hat{w}' = \hat{n}' = 0$. In arbitrary systems of coordinates, $\hat{W}_a^\alpha(u^\mu) = 0$, see (4). In order to avoid singular coordinate conditions it is important to work with "formal" off–diagonal parameterizations when the coefficients $\hat{n}_i(\tau,x^k,t)$ and/or $\hat{w}_i(\tau,x^k,t)$ are not zero but the anholonomy coefficients vanish for diagonalizable solutions. We suppose that some data $(\hat{g}_i, \hat{h}_a)$ may define a cosmological solution in MGT or in GR (for instance, a Friedman–Lamaître-Robertson-Walker, FLRW, metric). In general, $\hat{g}$ may not be a solution of any geometric evolution/ gravitational field equations, but it will be nonholonomically deformed into such solutions.

Let us prove that it is possible to realize a self–consistent procedure of $\varepsilon$–deforming prime cosmological metrics into certain target generic off–diagonal cosmological metrics,

$$ds^2 = \eta_i(\tau,x^k,t)\hat{g}_i(\tau,x^k,t)(dx^i)^2 + \eta_a(\tau,x^k,t)\hat{g}_a(\tau,x^k,t)(\hat{e}^a)^2,$$

$$\hat{e}^3 = dy^3 + \eta_i(\tau,x^k,t)\hat{n}_i(\tau,x^k,t)dx^i, \hat{e}^4 = dt + \eta_i(\tau,x^k,t)\hat{w}_i(\tau,x^k,t)dx^i.$$

In this formula, the coefficients $(g_a = \eta_a^\alpha g_\alpha, n_i^\mu n_i, w_i^\mu w_i)$ define, for instance, a cosmological Ricci soliton configuration determined by a class of solutions (70). We can express

$$\eta_i = \eta_i(\tau,x^k,t)[1 + \varepsilon \chi_i(\tau,x^k,t)], \eta_a = 1 + \varepsilon \chi_a(\tau,x^k,t)$$

for a small parameter $0 \leq \varepsilon \ll 1$, when $g = g(\varepsilon) = (g_i(\varepsilon), h_a(\varepsilon), N_i^j(\varepsilon))$ \(\rightarrow \hat{g}\) for $\varepsilon \rightarrow 0$. We emphasize that in general smooth limits are not possible for such nonholonomic deformations which can be satisfied for arbitrary parameterizations of sources, generation and integration functions, integration constants and general (effective) sources. Nevertheless, it is possible to construct exact solutions for
certain types of well-defined conditions on generating functions resulting in small deformations with clear physical interpretation of new classes of solutions.

The goal of the next subsection is to analyze such conditions when \( \varepsilon \)-deformations with nontrivial \( N \)-connection coefficients can be related to new classes of cosmological solutions of Ricci soliton and gravitational field equations (with possible supergravity corrections) and their factorized geometric flows.

### 4.3.1 \( \varepsilon \)-deformations for cosmological Ricci solitons

We provide a detailed proof for such deformations which can be applied for similar \( \varepsilon \)-formulae considered, for instance in [86, 87, 88, 89]. In this work, the formulae are provided in certain forms which can be applied for generating cosmological Ricci solitons.

Deformations of \( h \)-components of the cosmological \( \mathbf{d} \)-metrics with \( \tau = \tau_0 \) are characterized by

\[
g_i(\varepsilon, x, x_k) = \tilde{g}_i(x_k, t)\tilde{\eta}_i(x_k, t)[1 + \varepsilon \chi_i(x_k, t)] = e^{\psi(x_k)}
\]

being a solution of the Poisson equation (59). We do not consider summation on repeating indices in this formula. For \( \psi(\varepsilon) = 0, \psi(\varepsilon, x_k) + \varepsilon H(\psi(x_k), x_k) \) and \( E^0(\tilde{\psi}(x_k) + \varepsilon E^1(\tilde{\psi}(x_k)), x_k) \), we compute the deformation polarization functions

\[
\chi_i = e^{-\psi(x_k)} H(\psi(x_k), x_k) = E^0(\tilde{\psi}(x_k), x_k) .
\]  

(93)

In particular, we consider \( 1\psi(\varepsilon) = 0 \) and \( E^1(\tilde{\psi}(x_k), x_k) = 0 \).

Let us compute \( \varepsilon \)-deformations of \( v \)-components using formulas for a general source \( E^0(\tilde{\psi}(x_k), x_k), t) \),

\[
h_3(x, \varepsilon) = h_3^{[0]}(x_k) - \frac{1}{4} \int \frac{d\tilde{\psi}(x_k)}{E^0(x_k, t)} (1 + \varepsilon \chi_3) \tilde{g}_3 ;
\]

(94)

\[
h_4(x, \varepsilon) = \frac{1}{4} \left( \frac{\tilde{\psi}(x_k)^2}{E^0(x_k, t)} \right) \left( h_4^{[0]} - \frac{1}{4} \int \frac{d\tilde{\psi}(x_k)}{E^0(x_k, t)} \right)^{-1} = (1 + \varepsilon \chi_4) \tilde{g}_4.
\]

(95)

We parameterize the generation function in the form

\[
\Psi(\varepsilon) = \Psi(x, \varepsilon) = \tilde{\Psi}(x_k, t)[1 + \varepsilon \chi(x_k, t)]
\]

(96)

and introduce this value in (94). This allows us to compute

\[
\chi_3 = -\frac{1}{4\tilde{g}_3} \int \frac{d\tilde{\psi}(x_k)^2}{E^0(x_k, t)} \text{ and } \int \frac{d\tilde{\psi}(x_k)^2}{E^0(x_k, t)} = 4(h_3^{[0]} - \tilde{g}_3).
\]

(97)

As a result, we can find \( \chi_3 \) for any deformation \( \chi \) from a time-like oriented family of 2-hypersurfaces \( t = t(x_k) \) defined in non-explicit form from \( \tilde{\Psi} = \tilde{\Psi}(x_k, t) \) when the integration function \( h_3^{[0]}(x_k) \), the prime value \( \tilde{g}_3(x_k) \) and the fraction \( \tilde{\psi}(x_k)^2/E^0(x_k) \) satisfy the condition (97).

The formula for a hypersurface \( \tilde{\Psi}(x_k, t) \) can be written by choosing an explicit value of \( E^0(\tilde{\psi}(x_k), x_k) \). Introducing (96) into (94), we get

\[
\chi_4 = 2(\chi + \frac{\tilde{\psi}(x_k)^2}{E^0(x_k, t)}) - \chi_3 = 2(\chi + \frac{\tilde{\psi}(x_k)^2}{E^0(x_k, t)}) + \frac{1}{4\tilde{g}_3} \int \frac{d\tilde{\psi}(x_k)^2}{E^0(x_k, t)}.
\]

Thus, we can compute \( \chi_3 \) for any data \((\tilde{\psi}, \tilde{g}_3, \chi)\). Here we note that the formula for a compatible source is

\[
E^0(\tilde{\psi}(x_k), \tilde{\psi}(x_k), \chi) = \pm \tilde{\psi}/2\sqrt{|\tilde{g}_3 h_3^{[0]}|},
\]

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which transforms (97) into a time oriented family of 2-d hypersurface formulas \( t = t(x^k) \) defined in non-explicit form from

\[
\int dt \dot{\Psi} = \pm (h_3^{[0]} - \ddot{g}_3) / \sqrt{\vert \dddot{g}_4 h_3^{[0]} \vert}.
\]  

(98)

The \( \varepsilon \)-deformations of \( \overline{d} \)-metric and \( N \)-connection coefficients are not important if we restrict our research only to LC-configurations. Further approximations of type \( \tilde{\varepsilon} \)-coefficients (we omit such details which are not important if we restrict our research only to LC-configurations).

As a result of (93)-(98), we obtain the following coefficients for \( \varepsilon \)-deformations of a prime metric (90) into a target cosmological Ricci soliton metric:

\[
g_\varepsilon(\varepsilon, \tau, \tau_0) = [1 + \varepsilon \chi_i(\varepsilon, x^k)]_\varepsilon \tilde{g}_i = [1 + \varepsilon E \varphi(\varepsilon) \chi(x^k)] E [0 \varepsilon, \varphi(x^k)] \tilde{g}_i \text{ is a solution of (59)};
\]

\[
h_\varepsilon(\varepsilon, \tau, \tau_0) = [1 + \varepsilon \chi^3]_\varepsilon \tilde{g}_3 = \left[ 1 - \frac{1}{4g_3} \int dt \left( \frac{\dot{\tilde{g}}^2 \chi^\varepsilon}{\dot{\Psi} E \chi^\varepsilon} \right) \tilde{g}_3 \right];
\]

\[
n_\varepsilon(\varepsilon, \tau, \tau_0) = [1 + \varepsilon \chi^3]_\varepsilon \tilde{g}_3 = \left[ 1 + \varepsilon \left( 2(\chi + \frac{\dot{\tilde{g}}}{\dot{\tilde{g}}}) \int [\frac{1}{E \chi^\varepsilon} + \frac{5}{8g_3} \left( \frac{\dot{\tilde{g}}^2 \chi^\varepsilon}{\dot{\Psi} E \chi^\varepsilon} \right) \right] \tilde{g}_3 \right];
\]

\[
w_\varepsilon(\varepsilon, \tau, \tau_0) = [1 + \varepsilon \chi_i(\varepsilon, x^k)]_\varepsilon \tilde{g}_i = \left[ 1 + \varepsilon \left( \frac{\partial}{\partial \chi^\varepsilon} - \frac{(\chi^\varepsilon)^\varepsilon}{\dot{\Psi} E \chi^\varepsilon} \right) \right] \tilde{g}_i.
\]

The factor \( \tilde{g}_i(x^k) \) is a re-defined integration function including contributions from the prime metric.

The quadratic element for such inhomogeneous and locally anisotropic cosmological spaces is parameterized in the form

\[
ds^2 = g_{\alpha\beta}(\varepsilon, \varepsilon, x^k, t) dx^\alpha dx^\beta = g_\varepsilon(\varepsilon, x^k) \left[ (dx_1)^2 + (dx_2)^2 \right] +
\]

\[
+ h_{\varepsilon}(\varepsilon, x^k, t) \left[ dy^3 + n_i(\varepsilon, x^k, t) dx^i \right]^2 + h_{4\varepsilon}(\varepsilon, x^k, t) \left[ dt + w_k(\varepsilon, x^k, t) dx^k \right]^2,
\]

where the coefficients are taken from (99). We can subject additional constraints in order to extract LC-configurations as we considered in (88) and (89). Further approximations of type \( g_{\alpha\beta}(\varepsilon, \varepsilon, x^k, t) \simeq g_{\alpha\beta}(\varepsilon, \varepsilon, t) \) and/or for small off–diagonal contributions are possible in order to make such solutions to be compatible with experimental data. The source \( E \tilde{\chi}(\varepsilon, x^k, t) \) can be approximated to a potential \( E \varphi(x^k) \) for certain supergravity models as we considered at the end of section 3.

### 4.3.2 Geometric flow evolution of \( \varepsilon \)-deformed cosmological Ricci solitons

Employing (100) into (81), we construct quadratic elements with factorized geometric flow evolution of cosmological Ricci solitons

\[
ds^2 = g_{\perp\alpha\beta}(\varepsilon, \varepsilon, \tau, x^k, t) du^\alpha du^\beta = e^2 \int d^4 \tilde{\chi}(\tilde{\tau}) \quad g_\varepsilon(\varepsilon, x^k) \left[ (dx_1)^2 + (dx_2)^2 \right] + [1 + \varepsilon \perp(\tau)] (101)
\]

\[
\{ h_{3\varepsilon}(\varepsilon, x^k, t) \left[ dy^3 + n_i(\varepsilon, x^k, t) dx^i \right]^2 - \frac{1}{4 h_{3\varepsilon}(\varepsilon, x^k, t)} \left( \frac{\tilde{\psi}^\varepsilon(\varepsilon, x^k, t)}{E \tilde{\chi}(\varepsilon, x^k, t)} \right)^2 \left[ dt + w_k(\varepsilon, x^k, t) dx^k \right] \}. 
\]

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For simplicity, we do not linearize in $\varepsilon$ in $(\Psi^\circ(\varepsilon, \phi))^2/ h_3(\varepsilon, \phi)$, which is determined by any generating function $\chi(x^k, t)$ and respective integration functions. In explicit form, we can consider $\varepsilon$–deformations of a FLRW metric with dependence on $\tau$ locally anisotropic polarizations of constants. Such effects result in modified locally anisotropic cosmological scenarios. Having constructed an explicit class of solutions, we can study limits of type $g_{\alpha\beta}(\varepsilon, \phi, x^k, t) \simeq g_{\alpha\beta}(\varepsilon, \tau, t)$ for almost diagonal cosmological metrics.

5 Effective Mimetic $F(A, R^2)$ Theories for (Super) Geometric Evolution and Modified Ricci Solitons

In this section, we elaborate on effective mimetic theories associated with (super) geometric evolution and modified Ricci soliton models with sources $E \hat{\Upsilon}_{\alpha\beta} = \chi \check{\eta}_{\alpha\beta}$ encoding supersymmetric contributions. It is supposed that running and locally anisotropic polarizations of physical constants and geometric flows of nontrivial vacuum configurations (described by generic off–diagonal metrics and for generalized connections) can be modelled equivalently in the framework of effective mimetic $F(A, R^2)$ theories. Cosmological implications of generic off–diagonal solutions of type

$$
\check{g}_{\alpha\beta} = \begin{cases}
  g_{\alpha\beta}(\varepsilon, x^k, t) \simeq g_{\alpha\beta}(\varepsilon, \tau, t), & \text{see (71)}; \\
  g_{\perp\alpha\beta}(\varepsilon, \tau, x^k, t) \simeq g_{\perp\alpha\beta}(\varepsilon, \tau, t), & \text{see (81)}; \\
  g_{\alpha\beta}(\tau, x^k, t) \simeq g_{\alpha\beta}(\tau, t), & \text{see (86)}; \\
  g_{\alpha\beta}(\varepsilon, \tau, x^k, t) \simeq g_{\alpha\beta}(\varepsilon, \tau, t), & \text{see (100)}; \\
  g_{\perp\alpha\beta}(\varepsilon, \tau, x^k, t) \simeq g_{\perp\alpha\beta}(\varepsilon, \tau, t), & \text{see (101)},
\end{cases}
$$

will be studied in the Einstein and Jordan $N$–adapted and coordinate frames. We shall also exemplify our findings that are equivalent to $R^2$ and GR theory for well defined nonholonomic constraints extracting LC–configurations and diagonalizations (after the solutions are constructed in certain general forms) to metrics of type $g_{\alpha\beta}(\tau, t)$.

5.1 Einstein $N$–adapted frames, associated mimetic geometric flows and $F(A, R^2)$ gravity

We use specific normalizations for the nonholonomic geometric flows when the resulting cosmological Ricci soliton configurations are described by certain actions which define equivalently $F(A, R^2)$ gravity models. In addition to working with supersymmetric modifications of $R^2$ theories, we consider certain equivalent non-quadratic functional dependencies in order to be able to study observational effects (different, or equivalent for certain conditions) in modified gravity theories (see, for instance, [50] [90] [91] [92] [84] [85] [86] [87] [88] [89] and references therein) with generalized $F$–functional dependence on certain quantities denoted by $A$ and $R^2$ to be defined below.

5.1.1 Mimetic transforms of Perelman’s potentials and modified Hamilton’s equations

Let us consider a cosmological geometric flow, or Ricci soliton solution $\check{g}_{\alpha\beta}$. Redefining the normalization function $\check{f} \rightarrow f$ in modified Perelman’s functionals [50] and [51] for $\check{R}[\check{g}]$, $\check{D}[\check{g}]$ and $\check{e}_{\mu} \phi = \check{D} \phi$, we get

$$
-\frac{3}{4} e^{-\sqrt{2/3} \phi} \mathbf{J}_\mu J^\mu - 3 e^{-\sqrt{2/3} \phi} e_\mu \zeta e^\mu \zeta - E V + |\check{D}\check{f}|^2 = \\
-\ell(\phi) e^{-\sqrt{2/3} \phi} \mathbf{g}^{\mu \nu} (\check{e}_\mu \phi)(\check{e}_\nu \phi) - E \check{V} (\tau, \phi, \psi) + |\check{D}f|^2,
$$

(102)
where
\[ E\dot{V}(\tau, \phi, \varphi) = -U(\tau, \phi) - E\dot{V}(\tau, \varphi)e^{-2\sqrt{2/3}k\phi} + \ell(\tau, \varphi)e^{-2\sqrt{2/3}k\phi}. \] (103)

We introduce new nonholonomic variables with two effective scalar fields, \( \phi(u^\alpha) \) and \( \varphi(u^\alpha) \), with explicit gravitational constant \( k \) and Lagrange multiplier \( \ell(\varphi) \) which in the cosmological approximation is a function of type \( \ell(t) \). The potential \( U(\phi) \) is considered as a generalization of the Starobinski potential [68]. In general, the relation between \( E\dot{V}(\hat{W}) \) and \( E\dot{V}(\phi, \varphi) \) is not explicit and depend on the type of (anti) de Sitter / flat scenarios we consider to be determined by supergravity corrections as it is studied in [34], see points 1-3 after formula (41). We analyze models with effective running of \( \kappa \) gravitational coupling constant in order to define below a corresponding class of functionals \( E \) for a fixed value \( \tau_0 \), i.e. for cosmological soliton configurations with sources (49). In the above formulae, \( \hat{W} \) include possible distortions and off–diagonal terms of the Ricci d–tensor computed for \( \hat{\mathbf{g}}_{\alpha\beta} \) and \( \hat{\mathbf{N}} = \{ \hat{N}^0_i \} \) computed for off–diagonal cosmological solutions introduced in [38].

The system of modified Hamilton equations [38] corresponding to normalization (102) and potential (104) which is derived from modified Perelman’s potentials functionals (50) and (50) can be written as

\[ \partial_\tau \hat{g}_{ij} = -2\hat{R}_{ij}(\hat{g}_{kl}) - E\hat{Y}_{ij}, \] (105)

\[ \partial_\tau \hat{g}_{ab} = -2\hat{R}_{ab}(\hat{g}_{cd}) - E\hat{Y}_{ab}, \]

\[ \hat{R}_{ia} = \hat{R}_{ai} = 0; \hat{R}_{ij} = \hat{R}_{ji} = \hat{R}_{ab} = \hat{R}_{ba}; \]

\[ \partial_\tau \phi = -2(\hat{\square}\phi + \partial E\dot{V} / \partial \phi); \]

\[ \partial_\tau \varphi = -2(\hat{\square}\varphi + \partial E\dot{V} / \partial \varphi); \]

\[ \partial_\tau f = -\hat{\square}f + \left| \hat{\mathbf{D}}f \right|^2 - h\hat{\mathbf{R}} - v\hat{\mathbf{R}} + \hat{\square}\phi + \hat{\square}\varphi + E\dot{V}, \]

where the \( \hat{\mathbf{D}} \) 'Alambert operator \( \hat{\square} \) is defined by \( \hat{\mathbf{D}}[\hat{\mathbf{g}}_{\alpha\beta}] \). In the above notice the effective nonholonomically modification under geometric flows Ricci tensors and the two scalar fields involvement.

5.1.2 Cosmological Ricci solitons and mimetic \( F(A, R^2) \) gravity with supergravity modified potential

For self–similar configurations of the system (105), we obtain cosmological Ricci soliton spacetimes described by the action

\[ E\mathcal{S} = \int d^4u \sqrt{|\hat{\mathbf{g}}|} \left[ \frac{1}{2\kappa^2} \hat{\mathbf{R}}[\hat{\mathbf{g}}] - \frac{1}{2} \hat{\mathbf{g}}^{\mu\nu}(\hat{\mathbf{e}}_\mu\phi)(\hat{\mathbf{e}}_\nu\varphi) - \ell(\varphi)e^{-2\sqrt{2/3}k\phi}\hat{\mathbf{g}}^{\mu\nu}(\hat{\mathbf{e}}_\mu\phi)(\hat{\mathbf{e}}_\nu\varphi) - E\dot{V}(\tau, \phi, \varphi) \right], \] (106)

which is equivalent (up to nonholonomic deformations and re–definition of normalization functions) to the action (11). This implies that phenomenologically such models are similar to those from [93].
The action (II) describes a nonholonomic supersymmetric and imaginary deformation and generalization of the Starobinsky model [60] with potential
\[ U(\tau = \tau_0, \phi) = \phi V(\phi) = \frac{3}{4} \kappa^2 M^2 (1 - e^{-2\sqrt{2/3} \kappa \phi})^2, \quad \text{for} \quad \mu = \frac{3}{4} \kappa^2 M^2 = \text{const}, \]
see [19]. It is possible to calculate the observational induced inflation for such two scalar models and \( U(\phi) \) by N-adapting the techniques developed in Refs. [96, 97, 98].

For any cosmological configuration determined by geometric and/or off-diagonal flows, we can define an analogous MGT following such a procedure:

Let us consider a functional \( F(\tau, A) \) and express
\[ U(\tau, \phi) = \frac{A}{F^{\circ}(\tau, A)} - \frac{F(\tau, A)}{|F^{\circ}(\tau, A)|^2} \]
for an auxiliary scalar field \( A \) as in Ref. [59], but in our case we may take at the end \( A = \hat{R} \) defined by \( \mathbf{D} \), or (for certain additional assumptions) \( A = R \) defined by \( \nabla \). In this formula, \( F^{\circ} := dF/\partial A \). For \( \tau = \tau_0 \), we consider that such an auxiliary parametrization transforms (106) into the action
\[
E \hat{S} = \int d^4u \sqrt{|g|} \left\{ \frac{1}{2\kappa^2} \hat{R} \hat{g} - \frac{1}{2} \left[ F^{\circ}(\tau, A) \right]^2 \hat{g}^\mu_\nu (\hat{\mathbf{e}}_\mu A)(\hat{\mathbf{e}}_\nu A) - \frac{1}{2\kappa^2} \frac{A}{F^{\circ}(\tau, A)} - \frac{F(\tau, A)}{|F^{\circ}(\tau, A)|^2} \right\} - E \hat{V} (\tau, \phi)e^{-2\sqrt{2/3} \kappa \phi} + \ell(\tau, \phi)e^{-\sqrt{2/3} \kappa \phi} \hat{g}^\mu_\nu (\hat{\mathbf{e}}_\mu \phi)(\hat{\mathbf{e}}_\nu \phi) + \ell(\tau, \phi)e^{-2\sqrt{2/3} \kappa \phi}.
\]

In N-adapted frames \( \hat{\mathbf{e}}_\mu \), we imply the following transforms and identifications:
\[
g_{\mu\nu} = e^{-2\sqrt{2/3} \kappa \phi} \hat{g}_{\mu\nu}, \sqrt{|g|} = e^{-2\sqrt{2/3} \kappa \phi} \sqrt{|\hat{g}|}
\]
and \( \phi(\tau, u^\alpha) = -\sqrt{\frac{3}{2\kappa^2}} \ln F^{\circ}(\tau, A) \), i.e. \( \phi(\tau, t) = -\sqrt{\frac{3}{2\kappa^2}} \ln F^{\circ}(\tau, A(t, t)) \).

The metric \( g_{\mu\nu} \) is considered in the so-called Jordan metric if we work (it is preferred for observational computations) with local coordinates. The action \( E \hat{S} \) transforms into
\[
E,F \hat{S} = \int d^4u \sqrt{|\hat{g}|} \left\{ F^{\circ}(\tau, A) \left[ (\hat{\mathbf{R}} \hat{g}) - A \right] + F(A) - E \hat{V} (\tau, \phi) + \ell(\tau, \phi)[g^\mu_\nu (\partial_\mu \phi)(\partial_\nu \phi) + 1] \right\}.
\]

By varying this action with respect to \( A \), we obtain \( A = \hat{R} \), which means that this action is mathematically equivalent to the Jordan frame action for a mimetic \( F(\hat{R}) \) theory, in our case, with effective scalar potential \( E \hat{V} (\tau, \phi) \) (encoding contributions of geometric flows, off–diagonal terms and supersymmetric sources) and a Lagrange multiplier \( \ell(\tau, \phi) \). In an equivalent form, we consider that \( E \hat{V} (\tau, \phi) \) encode all distortions of connections and work with a \( F(\hat{R}) \) theory with LC scalar curvature \( \hat{R} \), when
\[
F \hat{S} = \int d^4u \sqrt{|\hat{g}|} \left\{ F[R(\hat{g}_{\mu\nu})] - E \hat{V} (\tau, \phi) + \ell(\tau, \phi)[g^\mu_\nu (\partial_\mu \phi)(\partial_\nu \phi) + 1] \right\}.
\]

For small \( \varepsilon \)–deformations studied in section 4.3, we assume that \( g_{\mu\nu} \) is a physical metric in the FLRW form with possible additional dependence on \( 7 \)-parameter (to be determined in next subsections)
\[
ds^2 = a^2(\tau, t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - dt^2,
\]
when \( R = 6(H^2 + 2H^2) \) is computed for the Hubble rate \( H(\tau, t) = a^\circ/a \). Mimetic gravity was introduced in [54, 55]. In this section, we derived the action (107) which is similar to the F(R) modification elaborated in [56] with the difference that in our case \( E \hat{V} (\tau, \phi) \) and \( \ell(\tau, \phi) \) encode possible contributions.
from (supersymmetric) geometric flows and off–diagonal cosmological Ricci solitons. Different (super- symmetric) geometric flow, Ricci soliton and MGT models are characterized by different $E\hat{V}(\tau,\varphi)$ and $\ell(\tau,\varphi)$; such values are different also for different classes of the solutions of the same theory.

In non-explicit form, the physical metric is a functional on generic off–diagonal auxiliary cosmological metric $\hat{g}_{\mu\nu}$ and auxiliary scalar field $\varphi(\tau, u^\alpha)$, i.e. $g_{\mu\nu} = g_{\mu\nu}[\hat{g}_{\alpha\beta}, \varphi]$, subject to conditions

$$g^{\mu\nu}(\hat{g}_{\alpha\beta}, \varphi)(\partial_\alpha \varphi)(\partial_\beta \varphi) = \delta^{\mu\nu}.$$  \hspace{1cm} (109)  

Upon variation of $FS$ with respect to $g_{\mu\nu}$, we obtain the effective gravitational field equations,

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F^\circ(R) + [\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box]F^\circ(R) +$$

$$\frac{1}{2}g_{\mu\nu}[- E\hat{V}(\tau, \varphi) + \ell(\tau, \varphi)(g^{\alpha\beta}(\partial_\alpha \varphi)(\partial_\beta \varphi) + 1)] - \ell(\tau, \varphi)(\partial_\mu \varphi)(\partial_\nu \varphi) = 0.$$ \hspace{1cm} (110)  

We note that exact cosmological solutions in various MGTs were studied in [84, 85, 86, 87, 88, 89] for $\nabla_\mu \rightarrow \hat{D}_\mu$. In this work, it is more convenient to work directly with LC-configurations of certain modified theories and consider physical metrics for effective F–modifications. The variation of action (107) in terms of $\nabla_\mu$ with respect to the effective scalar field $\varphi$ results in the equation

$$\nabla^\mu(\ell \partial_\mu \varphi) = -\frac{1}{2} \frac{\partial E\hat{V}}{\partial \varphi}. \hspace{1cm} (111)$$

Finally, we emphasize that the equations (109) can be obtained by varying the action (107) on the Lagrange multiplier $\ell$.

5.2 Reconstructing (super) geometric flow and MGTs cosmological scenarios

Resorting to observational cosmological data, we derive certain conditions relating to the compatibility of effective MGTs encoding (super) geometric non-holonomic flows with dependence on $\tau$–parameter.

5.2.1 Compatible effective supersymmetric potentials, MGTs and Lagrange multipliers

For a flat FLRW background (108) and approximation that the auxiliary metrics and scalar field depend only on flow parameter $\tau$ and cosmological time $t$, we write the equations (110), (111) and (109) (see similar details in [59]), respectively, in the following form

$$6(H^2 + H^2)F^\circ(R) - 6H[F^\circ(R)]^\circ - F(R) - \ell[(\varphi^\circ)^2 + 1] + E\hat{V}(\tau, \varphi) = 0,$$

$$2[F^\circ(R)]^\circ + 4H[F^\circ(R)]^\circ + F(R) - 2(H^2 + 3H^2) - \ell[(\varphi^\circ)^2 - 1] - E\hat{V}(\tau, \varphi) = 0,$$

$$2(\ell \varphi^\circ)^\circ + 6H\ell \varphi^\circ - \frac{\partial E\hat{V}}{\partial \varphi} = 0,$$

$$(\varphi^\circ)^2 - 1 = 0.$$  

The potential $E\hat{V}(\tau, \varphi)$ can be considered as an effective source of all possible geometric flows, non-holonomic deformations, running of constants depending on $\tau$ for certain admissible modifications of mimetic type gravity theories.
In [54], it was suggested the identification \( t = \varphi \) which is used also in F(R) modified gravity [59]. In our case, such identifications in the first and second equations of the above system allow to express the effective potential and Lagrange multiplier by corresponding formulas depending on \( F(R) \) and \( H \),

\[
E \tilde{V}(\tau, \varphi) = t = 2[F^\circ(R)]^\circ + 4H[F^\circ(R)]^\circ + F(R) - 2(H^\circ + 3H^2) \text{ and}
\]

\[
\ell(\tau, t) = -3H[F^\circ(R)]^\circ + 6(H^\circ + H^2) - \frac{1}{2} F(R).
\]

Prescribing any value \( E \tilde{V} \) and possible modification \( F(R) \), for instance, we can analyse if there is a viable \( \ell(\tau, t) \) for certain \( H \) taken from experimental data and compatible with the assumption that \( \tau = \tau_0 \). In different MGTs, one studies functionals of the type

\[
F(R) = p_3R + p_1R^{p_2} + c_1R^{c_2}
\]

with arbitrary parameters \( p_1, p_2, p_3, c_1, c_2 \). In particular, we can consider that \( F(R) = R^2 \).

The novelty of the reconstruction methods considered in [59, 84, 85, 86, 87, 88, 89] is that by choosing any possible generic off–diagonal cosmological solution for a modified (super) geometric flow and/or MGT we can produce any cosmological model (in general, nonhomogeneous and local anisotropic and with nontrivial torsion) and produce certain cosmological evolution scenarios which allows us to verify their concordance with experimental data.

5.2.2 Observational indices for double scalar models encoding (super) geometric flows and Ricci solitons

Our purpose is to find if a (super) symmetric mimetic effective scalar potential \( E \tilde{V}(\tau, \varphi = t) \) and corresponding Lagrange multiplier, both depending on geometric evolution parameter \( \tau \). Such a formalism was elaborated for the calculation of the Jordan frame observational indices [57, 58, 59] and developed for generic off–diagonal cosmological solutions in MGTs in [84, 85, 86, 87, 88, 89].

For observational indices and all physical quantities, we can use functions of the e–folding number \( N \) (we write a calligraphic symbol in order to avoid confusions with \( N \) and \( N \) used for the N–connection and its coefficients) instead of cosmic time \( t \). Let us denote

\[
P^\circ = \frac{\partial P}{\partial t} = H(N) \frac{\partial P}{\partial N} = H(N) P^a = \mathcal{H} P^a, \text{ for } P^a := \frac{\partial P}{\partial N}, \mathcal{H} := H(N);
\]

\[
P^{\ddagger a} = \frac{\partial^2 P}{\partial t^2} = H^2(N) P^{\ddagger a} + H(N) H^\circ P = \mathcal{H}^2 P^{\ddagger a} + \mathcal{H} P^a, \text{ for } P^{\ddagger a} := \frac{\partial^2 P}{\partial N^2}, \mathcal{H}^\circ := H^\circ(N).
\]

As a result of this, the slow-roll indices can be written in the following form (such relations are valid for very small values)

\[
e(\tau, N) = -\frac{\mathcal{H}}{4\mathcal{H}^\circ} \left( \frac{H^{\ddagger a}/\mathcal{H} + 6H^\circ/\mathcal{H} + (H^\circ/H)^2}{3 + H^\circ/\mathcal{H}} \right)^2 \ll 1,
\]

\[
\eta(\tau, N) = -\frac{18H^\circ/\mathcal{H} + 6H^{\ddagger a}/\mathcal{H} + (H^\circ/H)^2 - (H^{\ddagger a}/\mathcal{H}^\circ)^2 + 6H^{\ddagger a}/\mathcal{H}^\circ + 2H^{\ddagger a}/\mathcal{H}^\circ}{4(3 + H^\circ/\mathcal{H})} \ll 1.
\]

Considering the same formalism [57, 59] but with dependence on \( \tau \), we obtain respective formulae for the spectral index of primordial curvature perturbations, \( n_s \), and the scalar-to-tensor ratio, \( r \),

\[n_s \simeq 1 - 6e(\tau, N) + 2\eta(\tau, N) \text{ and } r = 16e(\tau, N).\]
Let us approximate
\[ H = [-q_0(\tau)e^{b_0(\tau)N} + q_1(\tau)]^{b_1(\tau)}, \]
where the values \( q_0(\tau), b_0(\tau), q_1(\tau) \) and \( b_1(\tau) \) may run with respect to geometric flow parameter and have to be chosen to obtain concordance with observational data. For instance, two sets of values
\[
q_0 = 0.5, \text{ or } 0.5; \\
b_0 = 0.024, \text{ or } 0.0222; \\
q_1 = 12, \text{ or } 10; \\
b_1 = 0.5, \text{ or } 1;
\]
are compatible with the recent Plank observational data [99]: \( n_s = 0.9644 \pm 0.0049 \) and \( r < 0.10 \). The second column results in \( n_s = 0.96567 \pm 0.0049 \) and \( r < 0.0640848 \) as it was computed in [59]. There is certain flexibility in the \( \tau \)–running of coefficients and type of MGTs. In that work, there were studied in details the issues of existence and stability of de Sitter solution in the framework of mimetic \( F(R) \) gravity with potential and Lagrange multiplier for various values of constants/parameters \( p_1, p_2, p_3, c_1, c_2 \) in functional (112). If any of such models is not compatible with observations for certain fixed data for \( \tau = \tau_0 \), we can obtain compatibility (also exit from inflation, stability of dS vacua etc.) under evolution of parameters and because of generic off–diagonal interactions which change the effective functional \( F(R) \).

5.3 Double scalar model calculations of observational indices for cosmological Ricci (super) solitons and mimetic \( F(A, R^2) \) gravity

We can chose nonholonomic geometric flow variables and constraints for cosmological Ricci solitons when the results of computation of physically important values are related to observational indices in cosmology. For well–defined conditions, such results can be related to and compared with those in mimetic \( F(A, R^2) \) gravity.

5.3.1 Effective configuration space

Let us provide an example of how the slow–roll indices for the two scalar model of (106) for the Starobinsky potential (19) work. For computations, we use the methods elaborated in [97, 59]. Consider a two–scalar formalism which is an example with potentials considered in 2.4 but for \( E\hat{V}(\tau, \phi, \phi I) = \hat{V}(\tau, \phi I) \), with labels \( I, J = 1, 2 \) and \( \phi^1 = \phi, \phi^2 = \phi \). Introducing a conventional scalar field configuration space endowed with metric
\[ G_{IJ}(\tau, \phi I) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\sqrt{\kappa} \phi^I} \ell(\tau, \phi^2) \end{pmatrix}, \]
the action for our model can be written as
\[
S = \int d^4u \sqrt{|\bar{g}|} \left[ \frac{1}{2\kappa^2} \bar{R}[\bar{g}] - \frac{1}{2} G_{IJ}(\phi^K) \bar{g}^{IJ}(\bar{e}_\mu \phi^I)(\bar{e}_\nu \phi^J) - E\hat{V}(\tau, \phi I) \right]. \tag{113}
\]
In the cosmological approximation with \( g_{\alpha\beta}(\tau, t) \) and \( \phi^I(\tau, t) \), the equations of motion corresponding to (106) in the form (113) are
\[
\begin{align*}
H^2 &= \frac{\kappa^2}{3} \left[ \frac{1}{2} \sigma^\phi + E\hat{V} \right], \tag{114} \\
H^\phi &= -\frac{\kappa^2}{2} (\sigma^\phi)^2; \\
\sigma^\phi + 3H\sigma^\phi + \sigma^I \partial(E\hat{V})/\partial\phi^I &= 0,
\end{align*}
\]

40
for \( \sigma^\circ := \sqrt{G_{IJ}(\phi^I)\circ(\phi^J)\circ} \) and \( \tilde{\sigma}^I := (\phi^I)/\sigma^\circ \). This system of equations is sensitive on the type of potential \( \tilde{E}\tilde{V} \). In configuration two-scalar space, we get a very simplified picture encoding certain values running on geometric evolution flow parameter \( \tau \).

## 5.3.2 Slow–roll parameters determined by supersymmetric potential with geometric flow parametric dependence

Using cosmological equations \((114)\) in 2-d configuration space, we compute in standard form the slow–roll parameters

\[
\epsilon = \frac{H^\circ}{H^2} = \frac{3(\sigma^\circ)^2}{(\sigma^\circ)^2 + \tilde{E}\tilde{V}} \quad \text{and} \quad \eta = \frac{M_{\sigma\sigma}}{\kappa^2 \tilde{E}\tilde{V}},
\]

for

\[
M_{\sigma\sigma} = \phi^K (\phi^I \partial^2 (\tilde{E}\tilde{V})/\partial \phi^K \partial \phi^I - \Gamma^K_{IJ}(\tilde{E}\tilde{V})/\partial \phi^I]/(\sigma^\circ)^2
\]

and \( \Gamma^K_{IJ} \) are the Christoffel symbols of the configuration metric \( G_{IJ} \). A tedious computation similar to that presented in Sec. V. A and Appendix B of \([59]\) in the approximation of small parameter \( \epsilon \), using our values \( \tilde{E}\tilde{V} \) and \( \ell \) result in the formulae

\[
\epsilon(\tau) = \frac{3(\phi^\circ)^2 + e^{-\sqrt{6}\kappa\phi}[\ell(\tau, \varphi)(\phi^\circ)^2]}{(\phi^\circ)^2 + e^{-\sqrt{6}\kappa\phi}[\ell(\tau, \varphi)(\phi^\circ)^2] - U(\tau, \varphi) - \tilde{E}\tilde{V}(\tau, \varphi)e^{-2\sqrt{2/3}\kappa\phi} + \ell(\tau, \varphi)[e^{-2\sqrt{2/3}\kappa\phi}]},
\]

\[
\eta(\tau) \simeq \epsilon - \sigma^\circ/\tilde{H}\sigma^\circ + \mathcal{O}(\epsilon^2),
\]

for \( \sigma^\circ = -3H\sqrt{1 + e^{-\sqrt{6}\kappa\phi}[\ell(\tau, \varphi)] - e^{-\sqrt{6}\kappa\phi}[\ell(\tau, \varphi)] \}} \)

\[
\frac{\{\phi^\circ[\sqrt{6}\kappa\tilde{E}\tilde{V}(\tau, \varphi) - \ell(\tau, \varphi)] + \partial U/\partial \phi]| + \phi^\circ \partial \ell/\partial \varphi[(\tau, \varphi)]\}}{\tilde{E}\tilde{V}(\tau, \varphi)} = \sigma^\circ = \sqrt{(\phi^\circ)^2 + e^{-\sqrt{6}\kappa\phi}[\ell(\tau, \varphi)(\phi^\circ)^2]}.
\]

(115)

As a result, the respective spectral index of primordial curvature perturbations, \( n_s \), and the scalar-to-tensor ratio, \( r \), are determined to be

\[
n_s = 1 - 6\epsilon(\tau) + 2\eta(\tau) \quad \text{and} \quad r = 16\epsilon(\tau).
\]

These potential slow-roll parameters are related to the Hubble slow-roll parameters which can be computed following formulas \([100]\), where it was proven that such parameters coincide at first order in the slow–roll approximation. The second Hubble slow-roll parameter is

\[
\eta_H(\tau) = -H^\circ/2H^2H.
\]

Using the cosmological equations \((114)\), we obtain the formula \( H^\circ = -\kappa\sigma^\circ\sigma^\circ \). For \( \eta_H \), one employs the formula

\[
\eta_H(\tau) = \frac{3H(\sigma^\circ)^2 + \phi^\circ \partial \tilde{E}\tilde{V}/\partial \phi + \psi^\circ \partial \tilde{E}\tilde{V}/\partial \psi}{(\sigma^\circ)^2[6(\sigma^\circ)^2 + 2 \tilde{E}\tilde{V}]},
\]

where values \((115)\) allow us to express \( \eta_H \) in terms of \( U, \ell \) and \( \tilde{E}\tilde{V} \). We omit the resulting cumbersome formulae in this work.
It should be noted that the Hubble and potential slow–roll parameters and spectral indices are related,
\[ n_s = 1 - 4\epsilon(\tau) + 2\eta_H(\tau) \quad \text{and} \quad n_H = -\epsilon(\tau) + \eta(\tau). \]

We reduced the (super) geometric evolution dynamics and cosmological Ricci soliton models to mimetic $F(R)$ gravity seen as a dynamical system with certain constants running with respect to $\tau$. It is possible to find the fixed points of such dynamical systems and study the stability of such points as in Ref. [59]. Such a study for special classes of solutions with running of cosmological constant will be undertaken in future studies correlated other specific reconstruction methods.

6 Discussions and Conclusions

In this article we presented a new theoretical framework that allows us to study applications of geometric flow theory in modified (super) gravity and cosmological theories in a consistent and relatively clear manner, making it easier, we hope, for physicists to follow. The approach is based on geometric techniques formulated for nonholonomic variables which allow to decouple in general ways physically important geometric evolution and modified gravitational field equations. This paper develops for modified gravity theories, MGTs, (in special for $R^2$ models and cosmology), our former results on black holes and geometric flows of anisotropic cosmological solutions in [34, 18] and can be considered as extension of the works [24, 23].

The Hamilton-Perelman theory of Ricci flows provides fundamental results on topology of three dimensional spaces which can be related to the geometric and topological structure of our Universe. The definition of Perelman’s functionals is of crucial importance in order to elaborate the proof of such results in modern mathematics and suggests a number of new ideas on analogous statistical thermodynamics for gravitational fields, on possible connections to string theory [2, 3, 4, 14, 15, 16, 17] and applications in geometric mechanics and modern commutative and noncommutative gravity [19, 20, 27, 28, 41].

The main problem for further applications of the geometric flow theory in modern gravity and quantum field theory is that existing rigorous mathematical methods are based on specific techniques elaborated in geometric analysis for flows of Riemannian metrics of Euclidean signature. We can consider various formal modifications of Perelman functionals with nonholonomic, commutative and noncommutative variables. Such values and derived geometric evolution equations do not have a standard nonlinear diffusion and entropy like interpretation for pseudo-Euclidean signatures and generalized connections. We can investigate possible physical implications of such generalized functionals and "relativistic evolution" equations if certain classes of exact solutions can be constructed in general forms. Positively, modified Perelman functionals and generalized Hamilton Ricci flow equations for non–Riemannian geometries do have physical importance in modern gravity and cosmology. This follows at least from the fact that we can always consider non-relativistic limits with corresponding 3+1 splitting of geometric models and gravitational theories. In order to construct exact solutions in explicit form, we work with nonholonomic 2+2 splitting but a second 3+1 fibration can be always considered if it is necessary for computing certain physical important values in non-relativistic limits.

In addition to the problem on elaborating models of relativistic geometric flows with applications in modern gravity and cosmology, we emphasize that it is of fundamental importance to study physically viable models of supergeometric flows and their connections to string theory. This is a very difficult task because one has not been elaborated generally accepted by mathematicians concepts of supermanifolds and relevant directions for supergeometric analysis, nonlinear functional analysis etc. Using certain recent results on viable effective scalar potentials in modified models of $R^2$ gravity
and cosmology, we can formulate supersymmetric extensions of Perelman functionals and study cosmological Ricci soliton configurations related to MGTs. This work can be considered as a first step to $SU(1, 1 + k)$ supersymmetric modifications of $R^2$ gravity, geometric flow theory and acceleration cosmology. We support our approach by constructing in explicit form various types of inhomogeneous and locally anisotropic cosmological solutions. Corresponding spacetime models encode nontrivial off–diagonal vacuum configurations, running constants and polarizations of physical constants depending on geometric flow parameter $\tau$. The main conclusion is that geometric evolution flow effects may result in running of physical constants at classical level and as such, in general, off–diagonal effects may be important in modern astroparticle physics relating to understanding dark energy and dark matter.

In order to compare our results on geometric flows and modified Ricci solitons to those developed in other MGTs, we consider effective mimetic F(R) gravity models accompanied by a potential with supersymmetric corrections $E^V$ and a Lagrange multiplier. It is not surprising that there is a strong analogy between nonholonomic geometric evolution models, Ricci solitons, and dynamical (generalized mechanical) models with Lagrange multipliers. Both methods complement each other, being important in constructing exact solutions and in order to study possible physical and observational implications. The presence of generalized sources determined by modified potentials, generating functions, nonholonomic constraints and Lagrange multiplies offer many possible ways and various classes of exact solutions for realization of geometric flow and cosmological evolution. Our particular interest is to understand if certain supersymmetric modifications determining modified geometric flows and new classes of cosmological Ricci solitons can produce cosmological spacetimes that are in concordance with observations. Given a generalized cosmological geometric flow/ Ricci soliton configuration, we can use a reconstruction procedure elaborated in the text to find mimetic potentials and Lagrange multiplier which can be modelled equivalently by a viable F(R) cosmology, in particular by supersymmetric modifications of $R^2$ theory. The reconstructing method elaborated in \cite{52, 56, 57, 59} and developed for nonholonomic configurations in \cite{84, 85, 86, 87, 88, 89} can be applied for arbitrary MGTs and for geometric flows of generic off–diagonal metrics and generalized connections. This way we can study physical viability of various flow evolution/ modified gravity theory via an unified description of late–time acceleration cosmology and existing observational data.

Furthermore, in the framework of nonholonomic geometric flow theory it is possible to study issues on stable and unstable (towards liner configurations) cosmological configurations. In principle, there is a strong dependence on MGT details, off–diagonal geometric evolution / interactions scenarios etc. For any classes of such generalized solution, we can consider at the end certain cosmological limits with running constant configurations and study further if there are stable de Sitter vacua, for instance, as final attractors of the trajectories of corresponding nonholonomic dynamical systems and speculate on inflation and acceleration scenarios. Interestingly enough, we can analyze if certain supergravity potentials $E^V$ via geometric flows and off–diagonal interactions lead to cosmological spacetimes with appealing cosmological properties. Our methods were applied in adapted form both for an effective Jordan frame $R^2$ theory, in order to calculate the spectral index of primordial perturbations, and in the Einstein frame for running of the spectral index. An effective two scalar field formalism was developed in the slow–roll limit.

Finally, we note that models of geometric flow evolution and mimetic MGTs offer many possibilities for realizing cosmological scenarios with various classes of effective sources and different types of generating functions. Computation of Perelman entropy may allow us to decide what model of geometric flows and cosmological Ricci solitons are physically viable. Such computations were provided in \cite{23, 24} for certain locally anisotropic cosmological models and black hole and solitonic configurations in $R^2$ gravity. We hope to address such issues for geometric flows related to superstring theory and classify respective cosmological Ricci solitons in our future publications.
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