Nonperturbative effects on the ferromagnetic transition in repulsive Fermi gases

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Nonperturbative effects on the ferromagnetic transition in repulsive Fermi gases can be understood as a result of the competition between the repulsive interaction and the Pauli exclusion principle. Itinerant ferromagnetism in condensed-matter physics, which can be dated back to the basic model proposed by Stoner \cite{2}. While the problem of itinerant ferromagnetism in electronic systems is quite complicated and the phase transition theory is still qualitative, a dilute spin-\(\frac{1}{2}\) Fermi gas with repulsive interactions may serve as a clean system to simulate the Stoner model. It is generally thought that the repulsive Fermi gas could undergo a ferromagnetic phase transition (FMPT) to a spin-polarized state with increased interaction strength \cite{3}. Recently, the experimentalists realized a two-component “repulsive” Fermi gas of \(^{6}\text{Li}\) atoms by using a nonadiabatic field switch to the upper branch of a Feshbach resonance with a positive \(s\)-wave scattering length. Our theory predicts a second-order phase transition, which indicates that ferromagnetic transition in dilute Fermi gases is possibly a counterexample to the BKV argument. The predicted critical gas parameter is expected to be of order \(O(1)\), perturbative predictions may be unreliable. In this paper we study the non-perturbative effects on the ferromagnetic phase transition by summing the particle-particle ladder diagrams to all orders in the gas parameter. We consider a universal repulsive Fermi gas where the effective range effects can be neglected, which can be realized in a two-component Fermi gas of \(^{6}\text{Li}\) atoms by using a nonadiabatic field switch to the upper branch of a Feshbach resonance with a positive \(s\)-wave scattering length. Our theory predicts a second-order phase transition, which indicates that ferromagnetic transition in dilute Fermi gases is possibly a counterexample to the BKV argument. The predicted critical gas parameter \((k_Fa)\) is 0.858 in good agreement with the recent quantum Monte Carlo result \((k_Fa)\) = 0.86 for a nearly zero-range potential [S. Pilati, \textit{et al.}, \textit{Phys. Rev. Lett.} \textbf{105}, 030405 (2010)]. We also compare the spin susceptibility with the quantum Monte Carlo result and find good agreement.

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\textbf{I. INTRODUCTION}

Itinerant ferromagnetism is a fundamental problem in condensed-matter physics, which can be dated back to the basic model proposed by Stoner \cite{1}. While the problem of itinerant ferromagnetism in electronic systems is quite complicated and the phase transition theory is still qualitative, a dilute spin-\(\frac{1}{2}\) Fermi gas with repulsive interactions may serve as a clean system to simulate the Stoner model. It is generally thought that the repulsive Fermi gas could undergo a ferromagnetic phase transition (FMPT) to a spin-polarized state with increased interaction strength \cite{3}. Recently, the experimentalists realized a two-component “repulsive” Fermi gas of \(^{6}\text{Li}\) atoms in a harmonic trap by using a nonadiabatic field switch to the upper branch of a Feshbach resonance with a positive \(s\)-wave scattering length \cite{3,4}. Therefore, it is possible to investigate itinerant ferromagnetism in cold Fermi gases. The experimental progress in this direction has attracted intense theoretical interest \cite{5,6}

The physical picture of the ferromagnetism in repulsive Fermi gases can be understood as a result of the competition between the repulsive interaction and the Pauli exclusion principle. The former tends to induce polarization and reduce the interaction energy, while the latter prefers balanced spin populations and hence a reduced kinetic energy. With increasing repulsion, the reduced interaction energy for a polarized state will overcome the gain in kinetic energy, and a FMPT should occur when the minimum of the energy landscape shifts to nonzero polarization or magnetization. Quantitatively, to study the FMPT in dilute Fermi gases at zero temperature, we should calculate the energy density \(E\) as a function of the spin polarization or magnetization \(x\) = \(n_1 - n_2)/(n_1 + n_2)\) at given dimensionless gas parameter \(k_Fa\) which represents the interaction strength \cite{3}. Here, \(k_F\) is the Fermi momentum related to the total density \(n = n_1 + n_2\) by \(n = k_F^2/(3\pi^2)\) and \(a > 0\) is the \(s\)-wave scattering length. Generally, the energy density can be expressed as

\begin{equation}
E(x) = \frac{3}{5} nE_F f(x),
\end{equation}

where \(E_F = k_F^2/(2M)\) is the Fermi energy with \(M\) being the fermion mass. The dimensionless function \(f(x)\), which depends on the gas parameter \(k_Fa\), represents the energy landscape with respect to the magnetization \(x\).

For the order of the FMPT, Belitz, Kirkpatrick, and Vojta (BKV) \cite{37} have argued that the phase transition in clean itinerant ferromagnets is generically of first order at low temperatures, due to the correlation effects or the coupling of the order parameter to gapless modes that lead to a nonanalytic term in the free energy. The general form of the Ginzburg-Landau free energy for clean itinerant ferromagnets takes the form

\begin{equation}
f_{GL}(x) = ax^2 + \nu x^4 \ln |x| + \beta x^4 + O(x^6),
\end{equation}

where we can keep \(\beta > 0\). If the coefficient \(\nu\) is positive, the phase transition is always of first order. On the other hand, for negative \(\nu\), one always has a second-order phase transition. The BKV argument is based on the assumption \(\nu > 0\), motivated by perturbation theory. \cite{37}. This is true for many solid-state systems where the FMPT occurs at weak coupling. However, for dilute Fermi gases where the critical gas parame-

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ter is expected to be of order $O(1)$, the assumption of a positive $\nu$ is not reliable.

In this paper, we study the nonperturbative effects on the FMPT by summing a set of particle-particle ladder diagrams to all orders in the gas parameter, motivated by the large-dimension expansion proposed by Steele [38]. We consider a universal repulsive Fermi gas where the effective range effect can be neglected, corresponding to a two-component “upper branch” Fermi gas with a positive $s$-wave scattering length. The prediction may also apply to the hard-sphere gas since the effective range corrections are sub-leading-order contributions in the large-dimension expansion. Our main conclusions for the order and the critical gas parameter of the FMPT can be summarized as follows:

1. **Order of phase transition.** We predict a second-order phase transition, in contrast to the BKV argument. This suggests that the FMPT in dilute Fermi gas may correspond to the case of negative $\nu$.

2. **Critical gas parameter.** We predict a critical gas parameter $(k_F a)\sim 0.858$ where the spin susceptibility $\chi$ diverges. The critical gas parameter and the spin susceptibility we obtained are in good agreement with the quantum Monte Carlo results [5].

The paper is organized as follows. In Sec. II we briefly review the perturbative predictions for FMPT in dilute Fermi gases. In Sec. III we introduce the effective field theory approach to the two-body scattering problem and show how we can recover the scattering amplitude by ladder resummation. We study the nonperturbative effects on FMPT in the theory of ladder resummation in Sec. IV and investigate the role of hole-hole ladders in Sec. V. We summarize in Sec. VI.

## II. PERTURBATIVE PREDICTIONS

In the perturbation theory, the gas parameter $k_F a$ is treated as a small number. Up to the order $O((k_F a)^2)$, the expression for $f(x)$ is universal, that is, independent of the details of the short-range interaction. We have

$$f(x) = \frac{1}{2}(\eta_1^2 + \eta_1^4) + \frac{10}{9\pi}k_F a^2 \eta_1^2 + \frac{(k_F a)^2}{21\pi^2} \xi(\eta_1, \eta_1),$$

(3)

where $\eta_1 = (1 + x)^{1/3}$ and $\eta_1 = (1 - x)^{1/3}$. The zeroth-order term corresponds to the kinetic energy, and the first-order term coincides with the Hartree-Fock mean-field theory [2]. The coefficient $\xi(\eta_1, \eta_1)$ in the second-order term was first evaluated by Kanno [39]. Its explicit form is

$$\xi = 22\eta_1^2 \eta_1^2(\eta_1 + \eta_1) - 4\eta_1^2 \ln \eta_1 + \frac{\eta_1 + \eta_1}{\eta_1 - \eta_1} - 4\eta_1^2 \ln \eta_1 + \frac{\eta_1 + \eta_1}{\eta_1 - \eta_1}$$

$$+ \frac{1}{2}(\eta_1 - \eta_1)^2 \eta_1(\eta_1 + \eta_1) [15(\eta_1^2 + \eta_1^2) + 11\eta_1 \eta_1]$$

$$+ \frac{7}{4}(\eta_1 - \eta_1)^4(\eta_1 + \eta_1)(\eta_1^2 + \eta_1^2 + 3\eta_1 \eta_1) \ln \left| \frac{\eta_1 - \eta_1}{\eta_1 + \eta_1} \right|. \quad (4)$$

Setting $x = 0$, we recover the well-known equation of state for dilute Fermi gases,

$$E = \frac{3}{5}nE_F \left[ 1 + \frac{10}{9\pi}k_F a + \frac{4(11 - 2 \ln 2)}{21\pi^2}(k_F a)^2 \right],$$

(5)

which was first obtained by Huang, Yang, and Lee [40] and recovered by Hammer and Furnstahl [41] in recent years using effective field theory.

In the first-order perturbation, the FMPT is of second order and occurs at $(k_F a) = \pi/2$ [2]. However, taking into account the second-order corrections, one finds a first-order FMPT at $(k_F a) = 1.054$ [14, 15]. This can be understood by noticing the nonanalytical term $\propto x^4 \ln |x|$ with positive coefficient in the small-$x$ expansion of the coefficient $\xi(\eta_1, \eta_1)$. The small-$x$ expansion of the energy density (3) takes the form

$$f(x) = f(0) + ax^2 + \nu x^4 \ln |x| + \beta x^4 + O(x^6),$$

(6)

which is consistent with the BKV argument [37] that the correlation effects or the coupling of the order parameter to gapless modes generally leads to nonanalytical terms in the free energy. The coefficient $\nu$ can be evaluated as

$$\nu = \frac{40(k_F a)^2}{243\pi^2}. \quad (7)$$

Therefore, up to the order $O((k_F a)^2)$, the Fermi gas problem corresponds to the case $\nu > 0$ which is assumed in the BKV argument.

In general, we expect that the critical parameter is of order $O(1)$. Therefore, the perturbative predictions for the FMPT are probably unreliable. There naturally arises a serious problem: Does the dilute Fermi gas problem really correspond to the case $\nu > 0$ if the nonperturbative effects at $k_F a \sim O(1)$ are taken into account? For the two-body problem in the vacuum, it is well known that an infinite set of bubble diagrams with the leading-order contact interaction must be resummed in order to reproduce the correct scattering amplitude if the two-body scattering length is large [42]. Therefore it is natural to extend the idea of ladder resummation to finite density so that the predicted equation of state works well even at $k_F a \sim O(1)$. We can also compare the non-perturbative predictions with the results from recent quantum Monte Carlo (QMC) simulations [5, 6].

## III. LADDER RESUMMATION FOR TWO-BODY SCATTERING

Before we establish a nonperturbative description for the FMPT in dilute Fermi gases, it is instructive to start with the low-energy effective field theory in vacuum [41, 42] and to see how the two-body scattering amplitude is reproduced from the ladder resummation method.

For nonrelativistic two-body scattering in the $s$-wave channel associated with a short-range interaction, the scattering amplitude $A(k)$ is related to the $s$-wave scattering phase shift $\delta$ by

$$A(k) = \frac{4\pi}{M \cot \delta - ik}, \quad (8)$$
where \( k \equiv |k| \) is the scattering momentum in the center-of-mass frame. If there exist bound states for attractive interactions, the scattering amplitude should exhibit some imaginary parts, \( k \cot \delta \) can be expanded as a Taylor series in \( k^2/\Lambda^2 \). In quantum scattering theory, this is called the effective range expansion,

\[
k \cot \delta = -\frac{1}{a} + \frac{1}{2} \sum_{n=0}^{\infty} r_n \Lambda^2 \left( \frac{k^2}{\Lambda^2} \right)^{n+1}
= -\frac{1}{a} + \frac{1}{2} r_0 k^2 + \ldots,
\]

(9)

where \( a \) is the scattering length and \( r_0 \) is the effective range. For a natural system, \( |a| \sim 1/\Lambda \) and \( |r_0| \sim 1/\Lambda \). An example commonly studied is a hard-sphere gas with radius \( R \), in which case \( a = R \) and \( r_0 = 2R/3 \). For cold atomic gases the interatomic interaction can be tuned by means of the Feshbach resonance, and we can have \( |a| \gg |r_0| \sim 1/\Lambda \).

According to the effective range expansion, one can construct the low-energy effective field theory \([41, 42]\) describing scattering at momenta \( k \ll \Lambda \). Since we assume \( k \ll \Lambda \), all interactions in the effective Lagrangian are contact interactions. The low-energy effective Lagrangian contains infinite contact interaction terms and is given by \([41]\)

\[
L_{\text{eff}} = \psi^\dagger \left( i \partial_t + \frac{\sqrt{2}}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 + \frac{C_2}{16} (\psi^\dagger \psi)^3 (\sqrt{2} \nabla^2 \psi) + \text{H.c.} + \ldots,
\]

(10)

where \( C_0 \) and \( C_2 \) are dimensionful coupling constants, \( \nabla = \nabla_x - \nabla_y \) is a Galilei invariant derivative, and \( \ldots \) denotes interactions with more derivatives (\( -\nabla^{2n} \), \( n \geq 2 \)) which generally have coupling constants \( C_{2n} \). The coupling constants \( C_{2n} (n = 0, 1, 2, \ldots) \) should be determined by reproducing the scattering amplitude \( \mathcal{A}(k) \).

In practice, we can reproduce the scattering amplitude \( \mathcal{A}(k) \) order by order in a Taylor expansion in \( k/\Lambda \). For small scattering length \((|a| \sim 1/\Lambda \) and \(|ak| \ll 1)\), we can expand the scattering amplitude as

\[
\mathcal{A}(k) = \frac{4\pi a}{M} \left[ 1 - iak + \left( \frac{ar_0}{2} - a^2 \right) k^2 + \ldots \right].
\]

(11)

However, for large scattering length \((|a| \gg 1/\Lambda)\), Kaplan et al. showed that one needs to expand \( \mathcal{A}(k) \) in powers of \( k/\Lambda \) while retaining \( ak \) to all orders \([42]\):

\[
\mathcal{A}(k) = \frac{4\pi}{M} \frac{1}{1/\Lambda + i k} \left[ 1 + \frac{r_0/2}{1/\Lambda + i k} k^2 + \ldots \right].
\]

(12)

This means if the scattering length is large, the loop diagrams with the leading-order interaction \( C_0 \) have to be resummed.

According to the free fermion propagator \( \mathcal{G}_0(p_0, p) = 1/(p_0 - \omega_p + i\epsilon) \) with the free dispersion \( \omega_p = \sqrt{p^2}/(2M) \), the one-loop bubble diagram \( B_0(p_0, p) \) [Fig.11] is given by

\[
B_0(p_0, p) = i \int \frac{d^4q}{(2\pi)^4} \left( \frac{p_0 + q_0 - (p - q)^2/2M + i\epsilon}{p_0 - q_0 - (p - q)^2/2M + i\epsilon} \right) = \int \frac{d^4q}{(2\pi)^4} \left( \frac{p_0 - q_0 - p^2/4M + i\epsilon}{M^2 - q^2/4M + i\epsilon} \right).
\]

(13)

Here \( P_0 \) and \( P \) are the total energy and momentum of the pairs in the bubble diagram [see Fig.11(a)]. Let \( p_1 \) and \( p_2 \) be the momenta of the scattering fermions, and we have \( P = p_1 + p_2 \) and \( \mathbf{k} = (p_1 - p_2)/2 \). Further, if the on-shell condition \( P_0 = (p_1^2 + p_2^2)/(2M) = P^2/(4M) + k^2/M \) is imposed, we find that \( B_0 \) depends only on the relative momentum \( \mathbf{k} \), corresponding to the translational invariance.

The integral over \( q \) in Eq. (13) is linearly divergent and therefore needs to be regularized. A popular regularization scheme is to use a momentum cutoff equal to \( \Lambda \) \([43]\). In this paper, we employ the dimensional regularization scheme. To this end, we change the dimensional regularization scheme. To this end, we change the space-time dimension from 4 to \( D \) and multiply the integral by a factor \( (\mu/2)^D \). Here \( \mu \) is an arbitrary mass scale introduced to allow the couplings \( C_{2n} \) multiplying operators containing \( \nabla^{2n} \) to have same dimensions for any \( D \). In general, the integral \( B_0 \) in \( D \) dimension can be evaluated as

\[
B_0(P_0, P) = -\frac{\Gamma\left(\frac{D-3}{2}\right)}{2} \frac{(\mu/2)^{4-D}}{(4\pi)^{D-1/2}} \times M \left( -MP_0 + \frac{P^2}{4} \right)^{(-D-3)/2}.
\]

(14)

For small scattering length, it is convenient to use the minimal subtraction (MS) scheme which subtracts any \( 1/(D - 4) \) pole before taking the \( D \to 4 \) limit. However, for large scattering length, it is more convenient to use the power divergence subtraction (PDS) scheme. The PDS scheme involves subtracting from the dimensionally regularized loop integrals not only the \( 1/(D - 4) \) poles corresponding to log divergences, as in MS, but also poles in lower dimensions which correspond to power law divergences at \( D = 4 \). The integral \( B_0 \) has a pole
in $D = 3$ dimensions. It can be removed by adding a counterterm $\delta B_0 = M_{\mu}/[4\pi(3 - D)]$ to $B_0$ \cite{42}. Finally, the subtracted integral in $D = 4$ dimensions is

$$
B_0(P_0, P) = -\frac{M}{4\pi} \left( \mu - \sqrt{-MP_0 + \frac{P^2}{4} - i\epsilon} \right).
$$

Note that the MS scheme corresponds to the $\mu = 0$ case.

The dependence of $C_{2\sigma}(\mu)$ on $\mu$ is determined by the requirement that the scattering amplitude is independent of the arbitrary mass scale $\mu$. To this end, we impose the on-shell condition, $P_0 = P^2/(4M) + k^2/M$. Then the one-loop bubble diagram becomes $B_0(k) = -M(\mu + ik)/(4\pi)$. Summing the bubble diagrams with $C_0$ vertices, we obtain \cite{42}

$$
\mathcal{A}(k) = \frac{C_0(\mu)}{1 - C_0(\mu)B_0(k)} + \frac{C_2(\mu)k^2}{[1 - C_0(\mu)B_0(k)]^2} + \ldots
$$

Comparing this result with the expansion \cite{42}, we obtain

$$
C_0(\mu) = \frac{4\pi}{M} \frac{1}{-\mu + 1/a},
$$

$$
C_2(\mu) = \frac{4\pi}{M} (1 - \frac{1}{-\mu + 1/a})^2 \frac{r_0}{2}.
$$

It was shown that these results fulfill the renormalization group equations \cite{42}. We note that the mass scale $\mu$ is similar to the cutoff $\Lambda$. In the cutoff scheme, we have $C_0(\Lambda) = (4\pi/M)(-2\Lambda/\pi + 1/a)$ \cite{43}.

In the following, we mainly consider a short-range potential with a positive scattering length $a$ and negligible effective range $r_0 \ll a$. In this case, we are able to obtain a universal result for $f(x)$ which is independent of the details of the interaction. In this case, the pair propagator $S_0(P_0, P)$ in the vacuum is given by

$$
S_0(P_0, P) = \frac{C_0(\mu)}{1 - C_0(\mu)B_0(P_0, P)} = \frac{4\pi}{M} \frac{1}{1/a - \sqrt{-MP_0 + \frac{P^2}{4} - i\epsilon}}
$$

For positive scattering length, the pair propagator has a pole given by $P_0 = -1/(Ma^2) + P^2/(4M)$. This pole corresponds to a bound state with binding energy $E_0 = -1/(Ma^2)$ and effective mass $2M$. Therefore, if the effective range is negligible, the underlying potential must be attractive and the ground state is a bound molecule of size $a$. However, for two-body scattering state with positive center-of-mass energy $E = k^2/M > 0$, the effective force is repulsive. This is the so-called “upper branch,” which is well defined in the two-body picture. For the many-body problem, a metastable “repulsive” Fermi gas can be realized if all fermions are forced on the upper branch of a Feshbach resonance with a positive s-wave scattering length \cite{3, 44}.

**IV. LADDER RESUMMATION AT FINITE DENSITY**

We now turn to the many-body problem of the repulsive Fermi gases. The main purpose of this paper is to present a nonperturbative calculation for function $f(x)$ from which we can give a better prediction for the FMPT. In general, we expect that the new nonperturbative result for $f(x)$ satisfies the following two criteria: (i) The function $f(x)$ recovers the perturbative result, Eq. (3), at weak coupling $\alpha$; and (ii) we consider a short-range potential with $r_0 \ll a$, the physical result should be universal, i.e., $f(x)$ depends only on the gas parameter $k_F a$ and not depend on other parameters such as the renormalization scale $\mu$. The criterion (ii) is hard to fulfill since the loop corrections in quantum field theory generally bring the renormalization scale dependence and weaken the prediction power due to the arbitrariness in the choice of the renormalization scale. However, in the following we show that the result from the particle-particle ladder resummation, corresponding to the leading order of the large-dimension expansion, is independent of the arbitrary mass scale $\mu$.

**A. Nonperturbative energy density**

We first construct the nonperturbative version of the energy density $\mathcal{E}(x)$ using the vertex $C_0(\mu)$ determined in Sec. III and the free propagators for the two spin components at finite density \cite{45}

$$
\mathcal{G}_\sigma(p_0, p) = \frac{\Theta(p - k_F^\sigma)}{p_0 - \omega_p + i\epsilon} + \frac{\Theta(k_F^\sigma - |p|)}{p_0 - \omega_p - i\epsilon}.
$$

Here $\sigma = \uparrow, \downarrow$, $k_F^{\uparrow, \downarrow} = k_F \eta_{1,1}$ are the Fermi momenta of the two spin components and $\Theta(z)$ is the Heaviside step function. For each spin component, the propagator \cite{19} describes two types of excitations, particles with momentum $|p| > k_F^\sigma$ and holes with $|p| < k_F^\sigma$.

The dilute imperfect Fermi gases are best described by resumming the multiple interactions in terms of the scattering amplitude. The Galitskii integral equations \cite{45} for the effective two-particle interaction or scattering amplitude in the medium are given by the ladder resummation \cite{10}. On the other hand, for large gas parameter $k_F a$, one may look...
for other expansion parameters instead of \( k_\sigma \) itself. Steele [38] and Schäfer et al. [47] have suggested a new expansion method using 1/\( D \) as the expansion parameter, where \( D = 2^{D/2} \) with \( D \) being the space-time dimension. Most important, they have shown that the contribution of the particle-particle (pp) ladder resummation, \( E^{(0)}_{\text{int}} \), is the leading-order contribution of the 1/\( D \) expansion [38, 47], that is,

\[
E = E_{\text{kin}} + E^{(0)}_{\text{int}} + O(1/D).
\]

All other contributions like hole-hole (hh) ladder sum and effective range corrections are suppressed by a factor 1/\( D \). According to the above arguments, we expect that the most important nonperturbative contributions come from the leading order of the 1/\( D \) expansion. The interaction energy density \( E^{(0)}_{\text{int}} \) to this order is given by all the particle-particle scattering terms (i.e., the \( n_{\text{pp}} \) 1hh bubbles for all \( n = 0, 1, 2, \cdots \)).

\[
E_{\text{int}}^{(0)} = \sum_{n=0}^{\infty} E_n.
\]

To evaluate the interaction energy density, we first calculate the elementary in-medium particle-particle bubble \( B(P_0, P) \) shown in Fig. 1(a). The fermion lines in the bubble diagram correspond to the particle terms of the free propagator [19]. According to the finite-density Feynman rules [41], it is given by

\[
B(P_0, P) = i \int \frac{d^3q}{(2\pi)^3} \frac{\Theta(|P/2 + q| - k_F^\pm)}{\frac{k^2}{2} + q_0 - \frac{P^2 + m^2}{2M} + i\epsilon} \frac{\Theta(|P/2 - q| - k_F^\pm)}{\frac{k^2}{2} - q_0 - \frac{P^2 - m^2}{2M} + i\epsilon}.
\]

For vanishing densities, \( k_F^\pm = 0 \), the in-medium particle-particle bubble recovers the vacuum result \( B_0 \). If the on-shell condition is imposed, the in-medium particle-particle bubble \( B \) depends on not only the relative momentum \( p \) but also the total momentum \( P \). This is due to the loss of translational invariance in the presence of Fermi sea.

We can separate \( B \) into a vacuum part and a medium part using the identity

\[
\Theta(|P/2 + q| - k_F^\pm)\Theta(|P/2 - q| - k_F^\pm) = 1 - \Theta(k_F^\pm - |P/2 + q|) - \Theta(k_F^\pm - |P/2 - q|) + \Theta(k_F^\pm - |P/2 + q|)\Theta(k_F^\pm - |P/2 - q|).
\]

The vacuum part (corresponding to 1) is identical to \( B_0 \) defined in the last section and is linearly divergent. The medium part is convergent. For the vacuum part, it is natural to use the dimensional regularization with PDS scheme introduced in the last section.

Then the \( n_{\text{pp}} \) 1hh bubble [see Fig. 1(b) for a typical example] at given \( n \) reads

\[
E_n = -C_0^{n+1} \int \frac{d^3P}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} e^{iP_0P} \Theta(k_F^\pm - |P/2 + k|) \Theta(k_F^\pm - |P/2 - k|) \int \frac{dP_0}{2\pi i} \frac{[B(P_0, P)]^n}{P_0 - \frac{m^2}{2M} - k_F^\pm - i\epsilon}.
\]

where \( e^{iP_0P} \) with \( n \to 0^+ \) is a convergence factor [41]. The integration over \( P_0 \) picks up the pole or imposes the on-shell condition \( P_0 = P^2/(4M) + k^2/M \), that is,

\[
\int \frac{dP_0}{2\pi i} e^{iP_0P} \frac{[B(P_0, P)]^n}{P_0 - \frac{m^2}{2M} - k_F^\pm - i\epsilon} = [B(P, k)]^n,
\]

where the on-shell version of \( B \) is given by

\[
B(P, k) = M \int \frac{d^3q}{(2\pi)^3} \frac{\Theta(P/2 + q + k_F^\pm)\Theta(P/2 - q - k_F^\pm)}{k^2 - q^2 + i\epsilon}.
\]

The total interaction energy density \( E_{\text{int}}^{(0)} \) is given by

\[
E_{\text{int}}^{(0)} = \sum_{n=0}^{\infty} E_n.
\]

Completing the summation of this geometric series, we obtain the interaction energy density at the leading order of the 1/\( D \) expansion,

\[
E_{\text{int}}^{(0)} = C_0 \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} \Theta(k_F^\pm - |p_1|)\Theta(k_F^\pm - |p_2|) \int \frac{dp_1}{2\pi i} \frac{[B(P_0, P)]^n}{1 - C_0 B(P, k)}.
\]

where \( p_{1,2} = P/2 \pm k \) as defined in Sec. III. The imaginary part of \( B \) can be evaluated as

\[
\text{Im} B(P, k) = -\frac{M|k|}{4\pi} \Theta(|p_1| - k_F^\pm)\Theta(|p_2| - k_F^\pm).
\]

This quantity is nonzero only when the momenta \( p_1 \) and \( p_2 \) are both above the Fermi surfaces. However, the final integration over \( p_1 \) and \( p_2 \) in the interacting energy density \( E_{\text{int}}^{(0)} \) is associated with a phase-space factor \( \Theta(k_F^\pm - |p_1|)\Theta(k_F^\pm - |p_2|) \).
Therefore, the interaction energy density is real and physical, as we expected.

\[ E / |E_b| = k_F a \]

FIG. 2: The pole energy \( E \) (divided by the binding energy \( |E_b| = 1/(Ma^2) \) in the vacuum) at zero pair momentum \( P = 0 \) as a function of the gas parameter \( k_F a \). The pole energy turns out to be positive for \( k_F a > \pi /2 \).

\[ E = P^2/(4M) \]

FIG. 3: (Color online) The pole energy \( E \) (divided by \( 2E_b \)) as a function of the pair momentum \( P = |P| \) (divided by \( k_F \)) for various values of the gas parameter \( k_F a \). The dashed line corresponds to the dispersion \( E(P) = P^2/(4M) \).

\[ C_0(\mu) \]

B. In-medium two-body problem

Since we adopt a zero-range potential, a bound state with binding energy \( E_b = -1/(Ma^2) \) always exists for positive scattering length \( a > 0 \). A key problem here is that how we can describe a metastable repulsive Fermi gas where all fermions are forced on the scattering states. Actually, the so-called upper branch has clear meaning only in the two-body picture, and so far it is not clear to what extent this two-body picture of a “repulsive” Fermi gas will persist. A recent study of three attractive fermions shows that there are many non-trivial avoided crossings between the two branches close to the resonance \( (a \rightarrow \infty) \), making it difficult to unambiguously identify a repulsive Fermi system [17]. To realize a metastable repulsive Fermi gas we have to exclude the molecule bound states of two atoms with unlike spins and enforce all atoms to the scattering states [48]. One possible prescription is to subtract the contribution from the bound-state poles within the Nozières-Schmitt-Rink (NSR) theory [48]. However, as designed, NSR theory works well only at temperature higher than the critical temperature of superfluidity. The particle-particle resummation theory we present complements the NSR theory and can be regarded as the zero-temperature analog of the NSR theory.

The key point in this problem is to consider the medium effects on the bound-state properties. To this end, we first construct the pair propagator \( S(P_0, P) \) in the presence of Fermi seas. With the in-medium elementary particle-particle bubble \( B(P_0, P) \), the in-medium pair propagator \( S(P_0, P) \) is given by the ladder resummation,

\[ S(P_0, P) = \frac{C_0(\mu)}{1 - C_0(\mu)B(P_0, P)}. \]

With this pair propagator, the interaction energy density \( \mathcal{E}_{\text{int}}^{(0)} \) can be expressed as

\[ \mathcal{E}_{\text{int}}^{(0)} = \int \frac{d^3P}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \Theta(k_F^+ - |p_1|)\Theta(k_F^+ - |p_2|) \times \int \frac{dP_0}{2\pi i} \frac{S(P_0, P)}{P_0 - \frac{P^2}{4M} - k_F^+ - i\epsilon}. \]

In general, the in-medium pair propagator \( S(P_0, P) \) has a real pole \( P_0 = E(P) \) corresponding to the in-medium bound state. However, we now show that such pole does not contribute to the energy density for the regime of the gas parameter \( k_F a \) we are interested in. As shown in the last subsection, in the calculation of the interaction energy density \( \mathcal{E}_{\text{int}} \), the on-shell condition \( P_0 = P^2/(4M) + k_F^2/M \) is imposed and the integrations over the momenta \( P \) and \( k \) are performed according to the finite-density Feynmann rules. Therefore, if the energy dispersion of the pole \( E(P) \) satisfies the condition

\[ E(P) < \frac{P^2}{4M} \]

for arbitrary \( P \), its contribution to the energy density is naturally excluded.

Since the main purpose of this paper is to study the FMPT which corresponds to an instability toward a small polarization \( x \), we can set \( x = 0 \) here. For convenience, we define two dimensionless quantities \( s = |P|/(2k_F) \) and \( z = \sqrt{MP_0 - P^2/4 + i\epsilon/k_F} = \sqrt{P_0/(2E_F)} - s^2 + i\epsilon \). The in-medium pair propagator can be evaluated as

\[ S(P_0, P) = \frac{4\pi}{M} \frac{1}{1/a - (k_F/\pi)W(s, z)}, \]

where \( W(s, z) \) is given by
The same result was also obtained in a recent paper \[49\].

For zero pair momentum \( P = 0 \), the condition \( E(P) < P^2/(4M) \) implies \( E(0) < 0 \). We thus focus on the regime of the gas parameter where the pole \( E(0) \) is negative. In this case, \( E(0) \) is determined by a simple equation,

\[
\frac{\pi}{2k_Fa} = 1 + \frac{-E(0)}{2E_F} \arctan \sqrt{\frac{-E(0)}{2E_F}}. \tag{34}
\]

This equation has negative solution only for \( 0 < k_Fa < \pi/2 \), where the solution represents the binding energy of an in-medium bound state. The numerical result for \( E(0) \) is shown in Fig. 2. In the low-density limit \( k_F \to 0 \), \( E(0) \) recovers the vacuum result \( E_{\text{vac}}(0) = E_b = -1/(Ma^2) \). However, at finite \( k_F \), the medium shields the bound state and reduces the binding energy, that is, \( |E(0)| < |E_{\text{vac}}(0)| \). For \( k_Fa > \pi/2 \) and \( k_Fa < 0 \), Eq. (34) has a positive solution which corresponds to the positive energy pole of the in-medium pair propagator. Such a pole is associated with Cooper pairs, and its appearance represents the BCS instability. This positive energy pole does not lead to singularities in the energy density integration, as can be seen in the next subsection, and does not need special treatment.

Note that \( E(0) < 0 \) is not a sufficient condition for \( E(P) < P^2/(4M) \). We thus have to check the energy dispersion \( E(P) \) carefully. The numerical results for some values of the gas parameter \( k_Fa \) are shown in Fig. 3. For \( k_Fa < 1.34 \), the condition \( E(P) < P^2/(4M) \) is fulfilled for all values of \( P \). However, for \( 1.34 < k_Fa < \pi/2 \), there exists a regime \( P_1 < |P| < P_2 \) where \( E(P) > P^2/(4M) \).

In conclusion, the condition \( E(P) < P^2/(4M) \) is fulfilled for \( k_Fa < 1.34 \). Therefore, in the parameter regime \( k_Fa < 1 \) investigated in the following, the contribution from the bound state can be naturally excluded in the ladder resummation scheme.

### C. Evaluating the energy density

Now we evaluate the explicit form of the energy density \( \mathcal{E}(x) \) and the dimensionless function \( f(x) \). First, the elementary particle-particle bubble \( B(P,k) \) can be decomposed into four parts

\[
B(P,k) = B_0(P,k) + B_{1\uparrow}(P,k) + B_{1\downarrow}(P,k) + B_{1\uparrow\downarrow}(P,k), \tag{35}
\]

where \( B_0 \) is the vacuum part discussed in Sec. III and the other parts are given by

\[
B_{1\uparrow}(P,k) = -M \int \frac{d^3q}{(2\pi)^3} \frac{\Theta(k_F^\uparrow - |P/2 + q|)}{k_F^\uparrow - q^2 + i\epsilon}, \quad B_{1\downarrow}(P,k) = -M \int \frac{d^3q}{(2\pi)^3} \frac{\Theta(k_F^\downarrow - |P/2 - q|)}{k_F^\downarrow - q^2 + i\epsilon}, \quad B_{1\uparrow\downarrow}(P,k) = M \int \frac{d^3q}{(2\pi)^3} \frac{\Theta(k_F^\uparrow - |P/2 + q|)\Theta(k_F^\downarrow - |P/2 - q|)}{k_F^\uparrow - q^2 + i\epsilon}. \tag{36}
\]

For convenience, we define another dimensionless quantity \( t = |k|/k_F \) together with \( s \) as defined in the last subsection. Since the imaginary part of \( B \) does not contribute to the interaction energy, we need only to evaluate the real part of \( B \). We have

\[
\text{Re}B_0(s,t) = -\frac{M\mu}{4\pi}, \quad \text{Re}B_1(s,t) = \frac{Mk_F}{4\pi^2}R_{1\uparrow}(s,t), \quad \text{Re}B_{1\downarrow}(s,t) = \frac{Mk_F}{4\pi^2}R_{1\downarrow}(s,t), \quad \text{Re}B_{1\uparrow\downarrow}(s,t) = \frac{Mk_F}{4\pi^2}R_{1\uparrow\downarrow}(s,t). \tag{37}
\]

where \( R_{1\sigma}(s,t) (\sigma = \uparrow, \downarrow) \) reads

\[
R_{1\sigma}(s,t) = \frac{\eta_{\sigma\sigma}^2 - (s + t)^2}{4s} \ln \frac{\eta_{\sigma\sigma} + s + t}{\eta_{\sigma\sigma} - s - t} + \frac{\eta_{\sigma\sigma}^2 - (s - t)^2}{4s} \ln \frac{\eta_{\sigma\sigma} + s - t}{\eta_{\sigma\sigma} - s + t} \eta_{\sigma\sigma}, \tag{38}
\]

and the function \( R_{1\uparrow\downarrow}(s,t) \) is

\[
R_{1\uparrow\downarrow}(s,t) = \begin{cases} 
-\Theta(x)R_{1\uparrow}(s,t) - \Theta(-x)R_{1\downarrow}(s,t), & 0 < s < \frac{1}{2} |\eta_{\uparrow} - \eta_{\downarrow}| \\
K_{1\uparrow\downarrow}(s,t) + K_{1\downarrow\uparrow}(s,t), & \frac{1}{2} |\eta_{\uparrow} - \eta_{\downarrow}| < s < \frac{1}{2} |\eta_{\uparrow} + \eta_{\downarrow}| \\
0, & \text{elsewhere.}
\end{cases} \tag{39}
\]
Here $K_{\sigma}(s, t)$ is defined as
\[
K_{\sigma}(s, t) = \frac{n^2_s - s^2 - t^2}{4s} \ln \left| \frac{\eta_s - s - t}{\eta_s - s + t} \right| + \frac{t}{2} \ln \left| \frac{\eta_s - s + t}{\eta_s - s - t} \right| + \frac{s - \eta_s}{2},
\]
where $r^2 = (\eta_s^2 + \eta_l^2)/2$.

Finally, the elementary particle-particle bubble reads
\[
B(s, t) = -\frac{M\mu}{4\pi} + \frac{Mk_F}{4\pi^2} R_{pp}(s, t),
\]
where the function $R_{pp}(s, t)$ is defined as
\[
R_{pp}(s, t) = R_1(s, t) + R_1(s, t) + R_1(s, t),
\]
Substituting the result of $B(s, t)$ into the expression of $E^{\text{int}}_{\text{in}}$, we observe that the energy density is independent of the renormalization mass scale $\mu$. Converting the integration variables $p_1$ and $p_2$ to $P$ and $k$, we find that the function $f(x)$ can be expressed as
\[
f(x) = \frac{1}{2} (\eta_1^2 + \eta_2^2) + \frac{80}{\pi} \int_0^\infty s^2 ds \int_0^\infty t dt I(s, t) F(s, t),
\]
where $F(s, t)$ is given by
\[
F(s, t) = \frac{k_F a}{1 - \frac{1}{\pi k_F a} R_{pp}(s, t)}.
\]

The function $I(s, t)$ appears due to integration over the angle between $P$ and $k$. Its explicit form is
\[
I(s, t) = \left[ \frac{\eta_1^2 - (s + t)^2}{4s} \Theta(s + t - \eta_1) + (\eta_1 - \eta_1) + t \right] \times \Theta(r^2 - s^2 - t^2) \Theta(\eta_1 - |s - \eta_1|) \Theta(\eta_1 - |s - \eta_1|).
\]

As we mentioned in the beginning of this section, it is important to check whether the present result for $f(x)$ is consistent with the perturbative expression (3) for weak coupling $k_F a \ll 1$. To this end, we expand the function $F(s, \kappa)$ as
\[
F(s, t) = k_F a + \frac{1}{\pi} (k_F a)^2 R_{pp}(s, t) + O((k_F a)^3).
\]

Using the expressions for $I(s, t)$ and $R_{pp}(s, t)$, we can show that
\[
\frac{80}{\pi} \int_0^\infty s^2 ds \int_0^\infty t dt I(s, t) = \frac{10}{9\pi^2} \eta_1^3 \eta_1,
\]
and
\[
\frac{80}{\pi^2} \int_0^\infty s^2 ds \int_0^\infty t dt I(s, t) R_{pp}(s, t) = \frac{\xi (\eta_1, \eta_1)}{21\pi^2}.
\]

Therefore, our nonperturbative expression (27) exactly recovers the perturbative result (3) at weak coupling. This convinces us that the present theoretical approach is suitable to study the universal upper-branch Fermi gas with a positive scattering length. In addition, we can compare the results from our theory and the second-order perturbation on the same footing and study the nonperturbative effects on the FMPT.

### D. Results and discussion

(A) **Energy density and compressibility.** We first study the equation of state for the unpolarized case $x = 0$. The gas parameter dependence of the energy density $E_s$ in the regime $0 < k_F a < 1$ is shown in Fig. 4. We find that the result from the ladder resummation is consistent with the perturbative result (3) for small gas parameters $k_F a < 0.4$. However, significant deviations are found for $k_F a > 0.4$, consistent with recent quantum Monte Carlo simulations [5, 6]. For the quantum Monte Carlo simulations of the attractive interactions with a negligible effective range (corresponding to UB and UB2 in Fig. 4), the exclusion of molecular bound states is implemented by choosing a two-body Jastrow factor [5, 6] to be the scattering solution of the attractive potential corresponding to positive energy, which, by construction, is orthogonal to the bound molecules. Therefore, the accuracy of the quantum Monte Carlo data depends on the choice of the Jastrow factor. Actually, exact orthogonality of the many-body variational wave function to the superfluid ground state (molecular condensation) can not be achieved in the quantum Monte Carlo simulations [5, 6]. We note that our theoretical curve agrees better with the UB2 data than with the UB data. The reason could be that the UB2 data from [6] are obtained with a Jastrow factor which imposes a better orthogonality to the superfluid ground state.

An important issue is whether the system is mechanically stable. The mechanical stability of the system requires a positive compressibility $\kappa$, which is defined as
\[
\frac{1}{\kappa} = n^2 \frac{\partial^2 E_s}{\partial n^2}.
\]

For the present ladder resummation theory, the explicit form of $\kappa$ can be evaluated as
\[
\frac{\kappa_0}{\kappa} = 1 + \frac{144}{\pi} \int_0^\infty s^2 ds \int_0^\infty t dt I(s, t) G(s, t),
\]
where $\kappa_0 = 3/(2nE_F)$ is the compressibility for noninteracting Fermi gases, and the function $G(s, t)$ is given by
\[
G(s, t) = F(s, t) + \frac{5R_{pp}(s, t)}{9\pi} F^2(s, t) + \frac{R_{pp}(s, t)}{9\pi^2} F^3(s, t).
\]

The compressibility $\kappa$ as a function of the gas parameter $k_F a$ is shown in Fig. 5. Comparing to the result from the second-order perturbation theory,
\[
\frac{\kappa_0}{\kappa} = 1 + \frac{2}{\pi k_F a} + \frac{8(11 - 2 \ln 2)}{15\pi^2} (k_F a)^2,
\]
good agreement is found for small gas parameters, as we expected. In the regime $0 < k_F a < 1$ we are interested in, we
find that the compressibility $\kappa$ is positive, indicating that the system is mechanically stable.

(B) Spin susceptibility. Next we study the response of the energy density to an infinitesimal polarization $x$. This response is referred to as the spin (or magnetic) susceptibility. The spin susceptibility $\chi$ can be defined as

$$\frac{1}{\chi} = \frac{1}{n^2} \frac{\partial^2 E}{\partial x^2} \bigg|_{x=0} = \frac{3E_F}{5n} \frac{\partial^2 f(x)}{\partial x^2} \bigg|_{x=0}. \quad (53)$$

In the present ladder resummation theory, an explicit form of $\chi$ is hard to obtain. In practice, we expand the function $f(x)$ near $x = 0$ as $f(x) = f(0) + ax^2 + \cdots$. The coefficient $a$ is related to the spin susceptibility by

$$\frac{\chi_0}{\chi} = \frac{9}{5} a, \quad (54)$$

where $\chi_0 = 3n/(2E_F)$ is the spin susceptibility of noninteracting Fermi gases. Therefore, a diverging spin susceptibility generally indicates a FMPT, as long as the transition is of second order.

In the second-order perturbation theory, an analytical result for $\chi$ can be achieved,

$$\frac{\chi_0}{\chi} = 1 - \frac{2}{\pi} k_F a - \frac{16(2 + \ln 2)}{15\pi^2} (k_F a)^2, \quad (55)$$

which indicates a diverging spin susceptibility at $k_F a = 1.058$. However, this differs from the critical gas parameter $(k_F a)_c = 1.054$, because the phase transition is of first order in the second-order perturbation theory due to the appearance of the nonanalytical term $\nu x^3 \ln |x|$ with $\nu > 0$.

Our result for the spin susceptibility $\chi$ as a function of the gas parameter $k_F a$ is shown in Fig. 6 and compared with the perturbative result. We find that the spin susceptibility predicted by the ladder resummation deviates significantly from the second-order perturbative result for $k_F a > 0.4$. Further, the spin susceptibility diverges at $k_F a = 0.858$, in contrast to the value 1.058 from the second-order perturbation theory. The data from the quantum Monte Carlo simulations [5] are also shown in Fig. 6 as a comparison. Our theoretical result is in good agreement with the data for the upper branch of the square well potential where the effective range $r_0$ is tuned to be much smaller than the scattering length [5]. The gas parameter $k_F a = 0.86$ where $\chi$ diverges is very close to our prediction $k_F a = 0.858$. For the purely repulsive potential, that is, the hard-sphere potential, the effective range effect cannot be neglected a priori. However, we find that our result still has nice agreement with the data for the hard-sphere case. The gas parameter $k_F a = 0.82$ where $\chi$ diverges is also close to our prediction $k_F a = 0.858$. Actually, the difference between the upper branch and the hard sphere cases [i.e., $0.86 - 0.82 = 0.04$] is very small compared with the critical gas parameters. This indicates that the contribution from the effective range effect is relatively small even for $k_F a \sim O(1)$, if the quantum Monte Carlo results are reliable. This can be understood from the large-dimension expansion [38, 47] introduced in the beginning of this section: The particle-particle ladder sum is the leading-order contribution in the $1/D$ expansion, and all other contributions including the effective range corrections are suppressed by a factor $1/D$. 

FIG. 5: The compressibility $\kappa$ [divided by its value $\kappa_0 = 3/(2nE_F)$ for noninteracting Fermi gases] as a function of the gas parameter $k_F a$ ($0 < k_F a < 1$) for the unpolarized case $x = 0$. The solid line is the result calculated from our particle-particle ladder resummation theory. The dashed line is result of the second-order perturbation theory. 

FIG. 6: The compressibility $\kappa$ [divided by its value $\kappa_0 = 3/(2nE_F)$ for noninteracting Fermi gases] as a function of the gas parameter $k_F a$ ($0 < k_F a < 1$) for the unpolarized case $x = 0$. The solid line is the result calculated from our particle-particle ladder resummation theory. The dashed line is result of the second-order perturbation theory.
(C) Ferromagnetic transition. While a diverging spin susceptibility indicates a ferromagnetic phase transition, the order of the ferromagnetic phase transition and the critical gas parameter \( k_{qA} \), should be obtained by studying carefully the shape of the energy landscape, that is, the full \( x \) dependence of the function \( f(x) \). To very high numerical accuracy, we have not found any maximum at \( x \neq 0 \) in the energy landscape. Instead, we find a second-order phase transition at \( k_{qA} = 0.858 \), where the function \( f(x) \) starts to develop a minimum at \( x \neq 0 \), consistent with the gas parameter where the spin susceptibility diverges. This is in contrast to the second-order perturbation theory which predicts a first-order phase transition at \( k_{qA} = 1.054 \) \[^{[14]}\] , where the spin polarization \( x \) jumps from zero to \( x_c = 0.573 \). A second-order FMPT for a zero-range potential model was also obtained by Heiselberg \[^{[18]}\] recently using a completely different many-body method.

It seems that our result of a second-order phase transition is in contradiction to the BKV argument \[^{[37]}\]. However, the BKV argument is based on the assumption that \( \nu > 0 \). Actually, we have fitted the energy density of the form \( f(x) = f(0) + ax^2 + bx^3 \ln|x| + \beta x^4 \) for small \( x \). For small gas parameter \( k_{qA} < 0.3 \), the coefficient \( \nu \) agrees well with the perturbative result \( \nu = 40(k_{qA})^2/(243\pi^2) \). However, for larger \( k_{qA} \) (especially around the critical gas parameter), it turns out to be negative due to the nonperturbative effects. This indicates that the FMPT in the systems of dilute repulsive Fermi gases corresponds to the case \( \nu < 0 \) and is a counterexample to the BKV argument where the assumption \( \nu > 0 \) is adopted.

Since an analytical expression for the function \( f(x) \) as well as the coefficient \( \nu \) cannot be achieved in the present ladder resummation theory, we cannot understand analytically how the nonperturbative effects modify the order of the phase transition. In fact, analytical results cannot be obtained from the order \( O((k_{qA})^3) \) even for the unpolarized case \( x = 0 \) in the perturbation theory \[^{[41]}\]. However, some definite conclusions can be drawn from our numerical results: (1) Higher-order terms in the gas parameter can also generate nonanalytical terms of the form \( x^4 \ln|x| \) and may generate other important non-analytical terms which are not known due to the mathematical limitation. (2) The coefficients of the nonanalytical terms generated by the higher-order contributions are certainly not always positive, and they are generally proportional to \((k_{qA})^n\) for the \( n \)-th order contributions. Since the phase transition occurs at a gas parameter \( k_{qA} \sim O(1) \), the nonperturbative effects from the sum of the higher order contributions are very important. As we have shown numerically, their effects are not only reducing the critical value of the gas parameter but also changing the order of the phase transition.

FIG. 7: The elementary bubbles organized in the number of the MI. The solid line with a cut represents the MI part of the propagator, and the pure solid line corresponds to the vacuum part.

V. INCLUSION OF HOLE-HOLE LADDERs

In this section we check whether our conclusion that the FMPT is of second order is changed by other contributions. We consider the contributions from the hole-hole ladder diagrams by summing the combined particle-particle and hole-hole ladders to all orders in \( k_{qA} \) while keeping the criteria (i) and (ii) satisfied. Such a resummation scheme for the unpolarized case \( x = 0 \) has been performed by Kaiser \[^{[50]}\]. Following the treatment by Kaiser \[^{[50]}\], we rewrite the propagator \[^{[19]}\] in an alternative form

\[
G_{\sigma}(p_0, p) = G_0(p_0, p) + 2\pi \delta(p_0 - \omega_p)\Theta(k_{qA} - |p|), \tag{56}
\]

where the first term corresponds to the vacuum propagator \( G_0(p_0, p) = (p_0 - \omega_p + i\epsilon)^{-1} \) and the second term is a so-called medium insertion (MI) \[^{[50]}\]. The elementary bubbles in this treatment are shown in Fig. 7. The first diagram \( B_0 \) is identical to the vacuum part studied in Sec. III and it can be renormalized using the PDS scheme. For our purpose of resummation, we are interested in the following two quantities, \( B_0 + B_1 + B_2 \) and \( B_0 + B_1 \), which are mutually complex conjugate. We have

\[
B_0 + B_1 + B_2 = -\frac{M_\mu}{4\pi} + \frac{M_{k_q}}{4\pi^2} [R(s, t) - \imath \pi I(s, t)],
\]

\[
B_0 + B_1 = -\frac{M_\mu}{4\pi} + \frac{M_{k_q}}{4\pi^2} [R(s, t) + \imath \pi I(s, t)], \tag{57}
\]
where $R(s,t) = R^1(s,t) + R^2(s,t)$ and $I(s,t)$ is the function defined in (30).

To sum all ladder diagrams built from the elementary bubbles, we first notice that the nonvanishing contributions to the interaction energy come from diagrams with at least two adjacent MIs. Then a typical nth-order contribution would look like the ring diagram of Fig. 1(b) with $n$ vertices and at least two adjacent MIs. Naively, all these nth-order diagrams are summed to give $g^n[(B_0 + B_1 + B_2)^n - (B_0 + B_1)^n]$, where the subtraction gets rid of those diagrams which have no adjacent MI pairs. However, this expression is complex and therefore cannot be the correct one. The crucial observations are that as the naive expression does. These amendments lead to the final result for the energy density does not depend on $\mu$ and the function $f(s)$ in this resummation scheme also takes the form of (41), while the function $F(s,t)$ becomes

$$F(s,t) = \frac{\ln \left[ 1 - \frac{1}{2}k_F aR(s,t) + ik_F aI(s,t) \right] - \text{c.c.}}{2I(s,t)}. \tag{58}$$

For small gas parameter $k_F a \ll 1$, $F(s,t)$ can be expanded as

$$F(s,t) = k_F a + \frac{1}{\pi} (k_F a)^2 R(s,t) + O((k_F a)^3). \tag{59}$$

We can also check that

$$\frac{80}{\pi^2} \int_0^{\infty} s^2 ds \int_0^{\infty} tdI(s,t)R(s,t) = \frac{\xi(\eta_1, \eta_1)}{21\pi^2}, \tag{60}$$

which reflects the fact that the hole-hole ladders start to contribute at the order $O((k_F a)^3)$ [38, 41, 47]. Therefore, the criteria (i) and (ii) are also fulfilled in the present resummation theory. Numerically, we also find a second-order phase transition, which occurs at a smaller gas parameter $k_F a = 0.786$. We note that the inclusion of hole-hole ladders may not improve the quantitative result, since it only includes part of the beyond-leading-order contribution in the large-$D$ expansion.

VI. SUMMARY

In summary, we have studied the nonperturbative effects on the ferromagnetic phase transition in repulsive Fermi gases by summing the ladder diagrams to all orders in the gas parameter $k_F a$. The nonperturbative effects not only reduce the critical gas parameter but also change the order of the phase transition. The resummation of particle-particle ladders, which corresponds to the leading order of the large-dimension expansion, predicts a second-order phase transition occurring at $k_F a = 0.858$, in good agreement with the quantum Monte Carlo result [5]. The spin susceptibility calculated from our resummation theory is also in good agreement with the quantum Monte Carlo results. Therefore, the resummation of the ladder diagrams provides a more quantitative way to study the ferromagnetic transition in repulsive Fermi gases. In this paper, we have considered only a zero-range potential model. It will be interesting to study the nonuniversal shape-dependent contributions using the finite-density effective range expansion [47].

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Nonperturbative effects on the ferromagnetic transition in repulsive Fermi gases

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It is generally believed that a dilute spin-1/2 Fermi gas with repulsive interactions can undergo a ferromagnetic phase transition to a spin-polarized state at a critical gas parameter $(k_Fa)$. Previous theoretical predictions of the ferromagnetic phase transition are based on the perturbation theory which treats the gas parameter as a small number. On the other hand, Belitz, Kirkpatrick, and Voja (BKV) have argued that the phase transition in clean itinerant ferromagnets is generically of first order at low temperatures, due to the correlation effects that lead to a nonanalytic term in the free energy. The second-order perturbation theory predicts a first order phase transition at $(k_Fa)_c = 1.054$, consistent with the BKV argument. However, since the critical gas parameter is expected to be of order $O(1)$, perturbative predictions may be unreliable. In this paper we study the non-perturbative effects on the ferromagnetic phase transition by summing the particle-particle ladder diagrams to all orders in the gas parameter. We consider a universal repulsive Fermi gas where the effective range effects can be neglected, which can be realized in a two-component Fermi gas of $^6$Li atoms by using a nonadiabatic field switch to the upper branch of a Feshbach resonance with a positive s-wave scattering length. Our theory predicts a second order phase transition, which indicates that ferromagnetic transition in dilute Fermi gases is possibly a counter example of the BKV argument. The predicted critical gas parameter $(k_Fa)_c = 0.858$ is in good agreement with recent Quantum Monte Carlo result $(k_Fa)_c = 0.86$ for a nearly zero-range potential [S. Pilati, \textit{et al.}, Phys. Rev. Lett. \textbf{105}, 030405 (2010)]. We also compare the spin susceptibility with the Quantum Monte Carlo result and find good agreement.

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I. INTRODUCTION

Itinerant ferromagnetism is a fundamental problem in condensed matter physics, which can be dated back to the basic model proposed by Stoner [1]. While the problem of itinerant ferromagnetism in electronic systems is quite complicated and the phase transition theory is still qualitative, a dilute spin-1/2 Fermi gas with repulsive interactions may serve as a clean system to simulate the Stoner model. It is generally thought that the repulsive Fermi gas could undergo a ferromagnetic phase transition (FMPT) to a spin-polarized state with increased interaction strength [2]. Recently, the experimentalists realized a two-component “repulsive” Fermi gas of $^6$Li atoms in a harmonic trap by using a nonadiabatic field switch to the upper branch of a Feshbach resonance with a positive s-wave scattering length [3]. Therefore, it is possible to investigate itinerant ferromagnetism in cold Fermi gases. The experimental progress in this direction has attracted intense theoretical interest [3–56].

The physical picture of the ferromagnetism in repulsive Fermi gases can be understood as a result of the competition between the repulsive interaction and the Pauli exclusion principle. The former tends to induce polarization and reduce the interaction energy, while the latter prefers balanced spin populations and hence a reduced kinetic energy. With increasing repulsion, the reduced interaction energy for a polarized state will overcome the gain in kinetic energy, and a FMPT should occur when the minimum of the energy landscape shifts to nonzero polarization or magnetization.

Quantitatively, to study the FMPT in dilute Fermi gases at zero temperature, we should calculate the energy density $E$ as a function of the spin polarization or magnetization $x = (n_1 - n_2)/(n_1 + n_2)$ at given dimensionless gas parameter $k_Fa$ which represents the interaction strength $[2]$. Here, $k_F$ is the Fermi momentum related to the total density $n = n_1 + n_2$ by $n = k_F^2/(3\pi^2)$ and $a > 0$ is the s-wave scattering length. Generally, the energy density can be expressed as

$$E(x) = \frac{3}{5} n E_F f(x),$$

(1)

where $E_F = k_F^2/(2M)$ is the Fermi energy with $M$ being the fermion mass. The dimensionless function $f(x)$, which depends on the gas parameter $k_Fa$, represents the energy landscape with respect to the magnetization $x$.

For the order of the FMPT, Belitz, Kirkpatrick, and Voja (BKV) [37] have argued that the phase transition in clean itinerant ferromagnets is generically of first order at low temperatures, due to the correlation effects or the coupling of the order parameter to gapless modes that lead to a nonanalytic term in the free energy. The general form of the Ginzburg-Landau free energy for clean itinerant ferromagnets takes the form

$$f_{GL}(x) = ax^2 + \nu x^4 \ln |x| + \beta x^4 + O(x^6),$$

(2)

where we can keep $\beta > 0$. If the coefficient $\nu$ is positive, the phase transition is always of first order. On the other hand, for negative $\nu$, one always has a second order phase transition. The BKV argument is based on the assumption $\nu > 0$ motivated by perturbation theory [37]. This is true for many solid state systems where the FMPT occurs at weak coupling. However, for dilute Fermi gases where the critical gas parameter is
expected to be of order $O(1)$, the assumption of a positive $\nu$ is not reliable.

In this paper, we will study the non-perturbative effects on the FMPT by summing a set of particle-particle ladder diagrams to all orders in the gas parameter, motivated by the large-dimension expansion proposed by Steele [38]. We consider a universal repulsive Fermi gas where the effective range effect can be neglected, corresponding to a two-component “upper branch” Fermi gas with a positive s-wave scattering length. The prediction may also apply to the hard sphere gas where the effective range corrections are subleading order contributions in the large-dimension expansion. Our main conclusions for the order and the critical gas parameter of the FMPT can be summarized as follows:

(1) Order of phase transition. We predict a second order phase transition, in contrast to the BKV argument. This suggests that the FMPT in dilute Fermi gas may correspond to the case of negative $\nu$.

(2) Critical gas parameter. We predict a critical gas parameter $(k_Fa)_c = 0.858$ where the spin susceptibility $\chi$ diverges. The critical gas parameter and the spin susceptibility we obtained are in good agreement with the Quantum Monte Carlo results [5].

The paper is organized as follows. In Section II we briefly review the perturbative predictions for FMPT in dilute Fermi gases. In Section III we introduce the effective field theory approach to the two-body scattering problem and show how we can recover the scattering amplitude by ladder resummation. We study the non-perturbative effects on FMPT in the theory of ladder resummation in Section IV and investigate the role of hole-hole ladders in Section V. We summarize in Section VI.

II. PERTURBATIVE PREDICTIONS

In the perturbation theory, the gas parameter $k_Fa$ is treated as a small number. Up to the order $O((k_Fa)^3)$, the expression for $f(x)$ is universal, i.e., independent of the details of the short range interaction. We have

$$f(x) = \frac{1}{2}(\eta_1^5 + \eta_1^5) + \frac{10k_Fa}{9\pi} \eta_1^3 \eta_1^3 + \frac{(k_Fa)^2}{21\pi^2} \xi(\eta_1, \eta_1),$$

(3)

where $\eta_1 = (1 + x)^{1/3}$ and $\eta_1 = (1 - x)^{1/3}$. The 0th-order term corresponds to the kinetic energy, and the 1st-order term coincides with the Hartree-Fock mean-field theory [2]. The coefficient $\xi(\eta_1, \eta_1)$ in the 2nd-order term was first evaluated by Kanno [39]. Its explicit form is

$$\xi = 22\eta_1^3 \eta_1^3 (\eta_1 + \eta_1) - 4\eta_1^5 \ln \eta_1^5 \eta_1^5 \eta_1^5 \eta_1^5 - 4\eta_1^5 \ln \eta_1^5 \eta_1^5 \eta_1^5 \eta_1^5 + \frac{1}{2} (\eta_1 - \eta_1)^2 \eta_1(\eta_1 + \eta_1)[15(\eta_1^2 + \eta_1^2) + 11\eta_1^2] + \frac{1}{4} (\eta_1 - \eta_1)^4 (\eta_1 + \eta_1)(\eta_1^2 + \eta_1^2 + 3\eta_1^2) \ln \frac{\eta_1 - \eta_1}{\eta_1 + \eta_1}. \tag{4}$$

Setting $x = 0$, we recover the well-known equation of state for dilute Fermi gases,

$$E = \frac{3}{2} n E_F \left[ 1 + \frac{10}{9\pi} k_F a + \frac{4(11 - 2 \ln 2)}{21\pi^2} (k_F a)^2 \right], \tag{5}$$

which was first obtained by Huang, Yang, and Lee [40] and recovered by Hammer and Furnstahl [41] in recent years using effective field theory.

In the 1st-order PTh, the FMPT is of second order and occurs at $k_Fa = \pi/2$ [2]. However, taking into account the 2nd-order corrections, one finds a first order FMPT at $k_Fa = 1.054$ [14, 15]. This can be understood by noticing the non-analytical term $\propto x^4 \ln x$ with positive coefficient in the small-$x$ expansion of the coefficient $\xi(\eta_1, \eta_1)$. The small-$x$ expansion of the energy density (3) takes the form

$$f(x) = f(0) + ax^2 + \nu x^4 \ln |x| + \beta x^4 + O(x^5), \tag{6}$$

which is consistent with the BKV argument [37] that the correlation effects or the coupling of the order parameter to gapless modes generally leads to non-analytical terms in the free energy. The coefficient $\nu$ can be evaluated as

$$\nu = \frac{40(k_F a)^2}{243\pi^2}. \tag{7}$$

Therefore, up to the order $O((k_F a)^3)$, the Fermi gas problem corresponds to the case $\nu > 0$ which is assumed in the BKV argument.

In general, we expect that the critical parameter is of order $O(1)$. Therefore, the perturbative predictions for the FMPT are probably unreliable. There naturally arises a serious problem: Does the dilute Fermi gas problem really correspond to the case $\nu > 0$ if the non-perturbative effects at $k_Fa \sim O(1)$ are taken into account? For the two-body problem in the vacuum, it is well known that an infinite set of bubble diagrams with the leading-order contact interaction must be resummed in order to reproduce the correct scattering amplitude if the two-body scattering length is large [42]. Therefore it is natural to extend the idea of ladder resummation to finite density so that the predicted equation of state works well even at $k_Fa \sim O(1)$. We can also compare the non-perturbative predictions with the results from recent Quantum Monte Carlo (QMC) simulations [5, 6].

III. LADDER RESUMMATION FOR TWO-BODY SCATTERING

Before we establish a non-perturbative description for the FMPT in dilute Fermi gases, it is instructive to start with the low energy effective field theory in vacuum [41, 42] and to see how the two-body scattering amplitude is reproduced from the ladder resummation method.

For non-relativistic two-body scattering in the s-wave channel associated with a short range interaction, the scattering amplitude $A(k)$ is related to the s-wave scattering phase shift $\delta$ by

$$A(k) = \frac{4\pi}{M \cot \delta - ik}, \tag{8}$$

where $M$ is the mass of the particles, and $k$ is the modulus of the wave vector.
where \( k \equiv |k| \) is the scattering momentum in the center-of-mass frame. If there exist bound states for attractive interactions, the scattering amplitude should exhibit some imaginary poles, \( k = i \sqrt{-M E_b} \), on the complex \( k \) plane with \( E_b < 0 \) being the binding energy. In general, the short range interaction is characterized by a momentum scale \( \Lambda \). Therefore, for low energy scattering, i.e., \( k \ll \Lambda \), the quantity \( k \cot \delta \) can be expanded as a Taylor series in \( k^2/\Lambda^2 \). In quantum scattering theory this is called the effective range expansion,

\[
k \cot \delta = -\frac{1}{a} + \frac{1}{2} \sum_{n=0}^{\infty} r_n \Lambda^2 \left( \frac{k^2}{\Lambda^2} \right)^{n+1}
\]

where \( a \) is the scattering length, and \( r_n \) is the effective range. For a natural system, \cite{42}, we have \( |a| \sim 1/\Lambda \) and \( |r_n| \sim 1/\Lambda \). An example commonly studied is a hard-sphere gas with radius \( R \), in which case \( a = R \) and \( r_0 = 2R/3 \). For cold atomic gases the inter-atomic interaction can be tuned by means of the Feshbach resonance, we can have \( |a| \sim 1/\Lambda \) and \( |r_n| \sim 1/\Lambda \).

According to the effective range expansion, one can construct the low energy effective field theory \cite{41,42} describing scattering at momenta \( k \ll \Lambda \). Since we assume \( k \ll \Lambda \), all interactions in the effective Lagrangian are contact interactions. The low energy effective Lagrangian contains infinite contact interaction terms and is given by \cite{41}

\[
\mathcal{L}_{\text{eff}} = \psi \left( i \partial_0 + \frac{\sqrt{2}}{2M} \mathbf{\nabla} \right) \psi - \frac{C_0}{2} (\psi \psi)^2 \\
+ \frac{C_2}{16} (\psi \psi)^4 (\psi \psi^* \psi^* \psi + h.c.) + \ldots,
\]

The integral over \( q \) in Eq. (13) is linear divergent and therefore needs to be regularized. A natural regularization scheme is to use a momentum cutoff equal to \( \Lambda \) \cite{43}. In this paper, we employ the dimensional regularization scheme. To this end, we change the space-time dimension from 4 to \( D \) and multiply the integral by a factor \( (\mu/2)^D \). Here \( \mu \) is an arbitrary mass scale introduced to allow the couplings \( C_{2n} \) multiplying operators containing \( \nabla^{2n} \) to have same dimensions for any \( D \). In general, the integral \( B_0 \) in \( D \) dimension can be evaluated as

\[
B_0(P_0, \mathbf{P}) = i \int \frac{d^Dq}{(2\pi)^D} \frac{1}{p_0 - q_0 + (\mathbf{p}^2 - q^2)/2M + i\epsilon} \frac{1}{p_0 - q_0 - (\mathbf{p}^2 - q^2)/2M + i\epsilon} = \int \frac{d^Dq}{(2\pi)^D} \frac{1}{P_0 - \mathbf{P}^2/4M + i\epsilon}.
\]

For small scattering length, it is convenient to use the minimal subtraction (MS) scheme which subtracts any \( 1/(D - 4) \) pole before taking the \( D \rightarrow 4 \) limit. However, for large scattering length, it is more convenient to use the power divergence subtraction (PDS) scheme. The PDS scheme involves subtracting from the dimensionally regularized loop integrals not only the \( 1/(D - 4) \) poles corresponding to log divergences, as in MS, but also poles in lower dimensions which correspond to power law divergences at \( D = 4 \). The integral \( B_0 \) has a pole in \( D = 4 \) dimensions. It can be removed by adding a counter

where \( C_0 \) and \( C_2 \) are dimensionful coupling constants and \( \mathbf{\nabla} = \mathbf{\nabla} - \nabla \) is a Galilei invariant derivative, and \( \ldots \) denotes interactions with more derivatives (\( \sim \nabla^{2n}, n \geq 2 \)) which generally have coupling constants \( C_{2n} \). The coupling constants \( C_{2n} \) \((n = 0, 1, 2, \ldots) \) should be determined by reproducing the scattering amplitude \( \mathcal{A}(k) \).

In practice, we can reproduce the scattering amplitude \( \mathcal{A}(k) \) order by order in a Taylor expansion in \( k/\Lambda \). For small scattering length \((|a| \sim 1/\Lambda \) and \(|ak| \ll 1)\), we can expand the scattering amplitude as

\[
\mathcal{A}(k) = \frac{4\pi a}{M} \left[ 1 - iak + \left( \frac{a r_0}{2} - a^2 \right) k^2 + \ldots \right].
\]

However, for large scattering length \((|a| \gg 1/\Lambda)\), Kaplan et al. showed that one needs to expand \( \mathcal{A}(k) \) in powers of \( k/\Lambda \) while retaining \( ak \) to all orders \cite{42}:

\[
\mathcal{A}(k) = \frac{4\pi}{M} \frac{1}{1/ak + i k} \left[ 1 + \frac{r_0/2}{1/ak + i k} k^2 + \ldots \right].
\]

This means, if the scattering length is large, the loop diagrams with the leading order interaction \( C_0 \) have to be resummed.

According to the free fermion propagator \( G_0(p_0, \mathbf{p}) = 1/(p_0 - \omega_{\mathbf{p}} + i\epsilon) \) with the free dispersion \( \omega_{\mathbf{p}} = \mathbf{p}^2/(2M) \), the one-loop bubble diagram \( B_0(P_0, \mathbf{P}) \) (Fig.1b) is given by

\[
B_0(P_0, \mathbf{P}) = -i\left( \frac{3 - D}{2} \right) \left( \frac{\mu/2}{(4\pi)^{(D-1)/2}} \right) \times M \left( -M P_0 + \frac{\mathbf{P}^2}{4} - i\epsilon \right)^{(D-3)/2}.
\]

For small scattering length, it is convenient to use the minimal subtraction (MS) scheme which subtracts any \( 1/(D - 4) \) pole before taking the \( D \rightarrow 4 \) limit. However, for large scattering length, it is more convenient to use the power divergence subtraction (PDS) scheme. The PDS scheme involves subtracting from the dimensionally regularized loop integrals not only the \( 1/(D - 4) \) poles corresponding to log divergences, as in MS, but also poles in lower dimensions which correspond to power law divergences at \( D = 4 \). The integral \( B_0 \) has a pole in \( D = 3 \) dimensions. It can be removed by adding a counter

\[
\text{Here } P_0 \text{ and } \mathbf{P} \text{ are the total energy and momentum of the pairs in the bubble diagram (see Fig.1a). Let } p_1 \text{ and } p_2 \text{ be the momenta of the scattering fermions, we have } \mathbf{P} = p_1 + p_2 \text{ and } k = (p_1 - p_2)/2. \text{ Further, if the on-shell condition } P_0 = (p_1^2 + p_2^2)/(2M) = \mathbf{P}^2/(4M) + k^2/M \text{ is imposed, we find that } B_0 \text{ depends only on the relative momentum } \mathbf{k}, \text{ corresponding to the translational invariance.}
\]
term $\delta B_0 = M\mu/[4\pi(3-D)]$ to $B_0$. Finally, the subtracted integral in $D = 4$ dimensions is

$$ B_0(P_0, \mathbf{P}) = -\frac{M}{4\pi} \left( \mu - \sqrt{-MP_0 + \mathbf{P}^2/4 - i\epsilon} \right). \quad (15) $$

Note that the MS scheme corresponds to the $\mu = 0$ case.

The dependence of $C_{2\sigma}(\mu)$ on $\mu$ is determined by the requirement that the scattering amplitude is independent of the arbitrary mass scale $\mu$. To this end, we impose the on-shell condition, $P_0 = P^2/(4M) + k^2/M$. Then the one-loop bubble diagram becomes $b_0(k) = -M(\mu + ik)/(4\pi)$. Summing the bubble diagrams with $C_0$ vertices, we obtain

$$ \mathcal{R}(k) = \frac{C_0(\mu)}{1 - C_0(\mu)B_0(k)} + \frac{C_2(\mu)k^2}{[1 - C_0(\mu)B_0(k)]^2} + \ldots. \quad (16) $$

Comparing this result with the expansion (12), we obtain

$$ C_0(\mu) = \frac{4\pi}{M} \frac{1}{\mu + 1/a}, \quad C_2(\mu) = \frac{4\pi}{M} \left( \frac{1}{\mu + 1/a} \right)^2 \frac{r_0}{2^2}. \quad (17) $$

It was shown that these results fulfill the renormalization group equations [42]. We note that the mass scale $\mu$ is similar to the cutoff $\Lambda$. In the cutoff scheme, we have $C_0(\Lambda) = (4\pi/M)(-2\Lambda/\pi + 1/a)$.

In the following, we mainly consider a short range potential with a positive scattering length $a$ and negligible effective range $r_0 \ll a$. In this case, we are able to obtain a universal result for $f(x)$ which is independent of the details of the interaction. In this case, the pair propagator $S_0(P_0, \mathbf{P})$ in the vacuum is given by

$$ S_0(P_0, \mathbf{P}) = \frac{C_0(\mu)}{1 - C_0(\mu)B_0(P_0, \mathbf{P})} = \frac{4\pi}{M} \frac{1}{1/a - \sqrt{-MP_0 + \mathbf{P}^2/4} - i\epsilon}. \quad (18) $$

For positive scattering length, the pair propagator has a pole given by $P_0 = -1/(Ma^2) + P^2/(4M)$. This pole corresponds to a bound state with binding energy $E_0 = -1/(Ma^2)$ and effective mass $2M$. Therefore, if the effective range is negligible, the underlying potential must be attractive and the ground state is a bound molecule of size $a$. However, for two-body scattering state with positive center-of-mass energy $E = k^2/M > 0$, the effective force is repulsive. This is the so-called “upper branch” which is well defined in the two-body picture. For the many-body problem, a metastable “repulsive” Fermi gas can be realized if all fermions are forced on the upper branch of a Feshbach resonance with a positive s-wave scattering length [3, 44].

IV. LADDER RESUMMATION AT FINITE DENSITY

We now turn to the many-body problem of the repulsive Fermi gases. The main purpose of this paper is to present a non-perturbative calculation for function $f(x)$ from which we can give a better prediction for the FMPT. In general, we expect that the new non-perturbative result for $f(x)$ satisfies the following two criteria: (i) The function $f(x)$ recovers the perturbative result Eq. (5) at weak coupling $ka \to 0$; (ii) Since we consider a short range potential with $r_0 \ll a$, the physical result should be universal, i.e., $f(x)$ depends only on the gas parameter $ka$ and does not depend on other parameters such as the renormalization scale $\mu$. The criterion (ii) is hard to be fulfilled since the loop corrections in quantum field theory generally brings the renormalization scale dependence and weakens the prediction power due to the arbitrariness in the choice of the renormalization scale. However, in the following we will show that the result from the particle-particle ladder resummation, corresponding to the leading order of the large-dimension expansion, is independent of the arbitrary mass scale $\mu$.

A. Non-Perturbative Energy Density

We first construct the non-perturbative version of the energy density $E(x)$ using the vertex $C_0(\mu)$ determined in Section III and the free propagators for the two spin components at finite density [45]

$$ G_{\sigma}(p_0, \mathbf{p}) = \frac{\Theta(p_0 - k_F^\sigma)}{p_0 - \omega_\sigma + i\epsilon} + \frac{\Theta(k_F^\sigma - |\mathbf{p}|)}{p_0 - \omega_\sigma - i\epsilon}. \quad (19) $$

Here $\sigma = \uparrow, \downarrow, k_F^{\uparrow, \downarrow} = k_F\eta_{\uparrow, \downarrow}$ are the Fermi momenta of the two spin components, and $\Theta(z)$ is the Heaviside step function. For each spin component, the propagator describes two types of excitations, particles with momentum $|\mathbf{p}| > k_F^\sigma$ and holes with $|\mathbf{p}| < k_F^\sigma$.

The dilute imperfect Fermi gases are best described by resumming the multiple interactions in terms of the scattering amplitude. The Galitskii integral equations [45] for the effective two-particle interaction or scattering amplitude in the medium are given by the ladder resummation [46]. On the other hand, for large gas parameter $ka$, one may look...
for other expansion parameters instead of \( k_{q_{o}} \) itself. Steele and Schäfer et al. have suggested a new expansion method using \( 1/D \) as the expansion parameter, where \( D = 2^{D/2} \) with \( D \) being the space-time dimension. Most importantly, they have shown that the contribution of the particle-particle (pp) ladder resummation, \( E^{(0)}_{\text{pp}} \), is the leading-order contribution of the \( 1/D \) expansion \[^{38,47}\], i.e.,

\[
E = E_{\text{kin}} + E^{(0)}_{\text{pp}} + O(1/D).
\] (20)

All other contributions like hole-hole (hh) ladder sum and effective range corrections are suppressed by a factor \( 1/D \). According to the above arguments, we expect that the most important non-perturbative contributions come from the leading order of the \( 1/D \) expansion. The interaction energy density \( E^{(0)}_{\text{int}} \) to this order is given by all the particle-particle scattering terms (i.e., the np-p hh bubbles for all \( n = 0, 1, 2, \cdots \)).

To evaluate the interaction energy density, we first calculate the elementary in-medium particle-particle bubble \( B(P_0, P) \) shown in Fig. 1(a). The fermion lines in the bubble diagram correspond to the particle terms of the free propagator \[^{19}\]. According to the finite density Feynmann rules \[^{41}\], it is given by

\[
B(P_0, P) = i \int \frac{d^2 q}{(2\pi)^2} \frac{\Theta([P/2 + q] - k^+_F)}{2q_0 - \frac{q^2 - (P/2 + q)^2}{2M} + i\epsilon} \frac{\Theta([P/2 - q] - k^+_F)}{2q_0 - q^2 - \frac{(P/2 - q)^2}{2M} + i\epsilon}.
\] (21)

For vanishing densities, \( k^+_F = 0 \), the in-medium particle-particle bubble recovers the vacuum result \( B_0 \). If the on-shell condition is imposed, the in-medium particle-particle bubble \( B \) depends on not only the relative momentum \( p \) but also the total momentum \( P \). This is due to the loss of translational invariance in the presence of Fermi sea.

We can separate \( B \) into a vacuum part and a medium part using the identity

\[
\Theta([P/2 + q] - k^+_F)\Theta([P/2 - q] - k^+_F) = 1 - \Theta(k^+_F - [P/2 + q]) - \Theta(k^+_F - [P/2 - q]) + \Theta(k^+_F - [P/2 + q])\Theta(k^+_F - [P/2 - q]).
\] (22)

The vacuum part (corresponding to 1) is identical to \( B_0 \) defined in the last section and is linearly divergent. The medium part is convergent. For the vacuum part, it is natural to use the dimensional regularization with PDS scheme introduced in the last section.

Then the np-p hh bubble (see Fig. 1(b) for a typical example) at given \( n \) reads

\[
E_n = C_n^{0+n} \int \frac{d^3 P}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} \Theta(k^+_F - [P/2 + k]) \Theta(k^+_F - [P/2 - k]) \int \frac{d^3 p}{(2\pi)^3} \epsilon^{\text{np}_p} \frac{[B(P_0, P)]^n}{P_0 - \frac{p^2}{2M} - k^+_k - i\epsilon},
\] (23)

where \( \epsilon^{\text{np}_p} \) with \( \eta \rightarrow 0^+ \) is a convergence factor \[^{41}\]. The integration over \( P_0 \) picks up the pole or imposes the on-shell condition \( P_0 = P^2/(4M) + k^2/M \), i.e.,

\[
\int \frac{d^3 P}{2\pi^3} \epsilon^{\text{np}_p} \frac{[B(P_0, P)]^n}{P_0 - \frac{p^2}{2M} - k^+_k - i\epsilon} = [B(P, k)]^n,
\] (24)

where the on-shell version of \( B \) is given by

\[
B(P, k) = M \int \frac{d^3 q}{(2\pi)^3} \frac{\Theta([P/2 + q] - k^+_F)\Theta([P/2 - q] - k^+_F)}{k^2 - q^2 + i\epsilon}.
\] (25)

The total interaction energy density \( E^{(0)}_{\text{int}} \) is given by

\[
E^{(0)}_{\text{int}} = \sum_{n=0}^{\infty} E_n.
\] (26)

Completing the summation of this geometric series, we obtain the interaction energy density at the leading order of the \( 1/D \) expansion,

\[
E^{(0)}_{\text{int}} = C_0 \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} \frac{\Theta(k^+_F - |p_1|)\Theta(k^+_F - |p_2|)}{1 - C_0 B(P, k)},
\] (27)

where \( p_{1,2} = P/2 \pm k \) as defined in Section III. The imaginary part of \( B \) can be evaluated as

\[
\text{Im} B(P, k) = \frac{M|k|}{4\pi} \Theta(|p_1| - k^+_F)\Theta(|p_2| - k^+_F).
\] (28)

This quantity is nonzero only when the momenta \( p_1 \) and \( p_2 \) are both above the Fermi surfaces. However, the final integration over \( p_1 \) and \( p_2 \) in the interacting energy density \( E_{\text{int}} \) is associated with a phase-space factor \( \Theta(k^+_F - |p_1|)\Theta(k^+_F - |p_2|) \).
Therefore, the interaction energy density is real and physical, as we expected.

![FIG. 2: The pole energy $E$ (divided by the binding energy $|E_b| = 1/(Ma^2)$ in the vacuum) at zero pair momentum $P = 0$ as a function of the gas parameter $k_Fa$. The pole energy turns out to be positive for $k_Fa > \pi/2$.](image)

![FIG. 3: (Color-online) The pole energy $E$ (divided by $2E_b$) as a function of the pair momentum $P = |P|$ (divided by $k_F$) for various values of the gas parameter $k_Fa$. The dashed line corresponds to the dispersion $E(P) = P^2/(4M)$.](image)

**B. In-Medium Two-Body Problem**

Since we adopt a zero range potential, a bound state with binding energy $E_b = -1/(Ma^2)$ always exists for positive scattering length $a > 0$. A key problem here is that how we can describe a metastable repulsive Fermi gas where all fermions are forced on the scattering states. Actually, the so-called “upper branch” has clear meaning only in the two-body picture, and so far it is not clear to what extent this two-body picture of a “repulsive” Fermi gas will persist. Recent study on three attractive fermions shows that there are many non-trivial avoided crossings between the two branches close to the resonance ($a \to \infty$), making it difficult to unambiguously identify a repulsive Fermi system [17].

To realize a metastable repulsive Fermi gas we have to exclude the molecule bound states of two atoms with unlike spins and enforce all atoms to the scattering states [48]. One possible prescription is to subtract the contribution from the bound-state poles within the Nozieres-Schmitt-Rink (NSR) theory [48]. However, as designed, NSR theory works well only at temperature higher than the critical temperature of superfluidity. The particle-particle resummation theory we present complements the NSR theory and can be regarded as the zero-temperature analogue of the NSR theory.

The key point in this problem is to consider the medium effects on the bound state properties. To this end, we first construct the pair propagator $S(P_0, P)$ in the presence of Fermi seas. With the in-medium elementary particle-particle bubble $B(P_0, P)$, the in-medium pair propagator $S(P_0, P)$ is given by the ladder resummation,

$$S(P_0, P) = \frac{C_0(\mu)}{1 - C_0(\mu)B(P_0, P)}.$$  (29)

With this pair propagator, the interaction energy density $\varepsilon_{\text{int}}^{(0)}$ can be expressed as

$$\varepsilon_{\text{int}}^{(0)} = \int \frac{d^3P}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \Theta(k_F^+ - |p_1|)\Theta(k_F^- - |p_2|)$$
$$\times \int \frac{dP_0}{2\pi i} e^{iP_0P} \frac{S(P_0, P)}{P_0 - P^2/4M - k_F^2/4M - i\epsilon}.$$  (30)

In general, the in-medium pair propagator $S(P_0, P)$ has a real pole $P_0 = E(P)$ corresponding to the in-medium bound state. However, we now show that such pole does not contribute to the energy density for the regime of the gas parameter $k_Fa$ we are interested in. As shown in the last subsection, in the calculation of the interaction energy density $\varepsilon_{\text{int}}$, the on-shell condition $P_0 = P^2/(4M) + k_F^2/M$ is imposed and the integrations over the momenta $P$ and $k$ are performed according to the finite density Feynmann rules. Therefore, if the energy dispersion of the pole $E(P)$ satisfies the condition

$$E(P) < \frac{P^2}{4M}$$  (31)

for arbitrary $P$, its contribution to the energy density is naturally excluded.

Since the main purpose of this paper is to study the FMPT which corresponds to an instability toward a small polarization $\chi$, we can set $\chi = 0$ here. For convenience, we define two dimensionless quantities $s = |P|/(2k_F)$ and $z = \sqrt{MP_0 - P^2}/4 + i\epsilon/k_F = \sqrt{P_0/(2E_F)} - s^2 + i\epsilon$. The in-medium pair propagator can be evaluated as

$$S(P_0, P) = \frac{4\pi}{M} \frac{1}{1/a - (k_F/\pi)W(s, z)},$$  (32)

where $W(s, z)$ is given by
The vacuum result in Fig. 2. In the low density limit medium bound state. The numerical result for \( R \) where \( \mu \) as can be seen in next subsection, and does not need special treatment. Such a pole is associated with Cooper pairs and its appearance represents the binding energy of an in-medium bound state. The numerical result for \( E(0) \) is shown in Fig. 2. In the low density limit \( k_F \to 0 \), \( E(0) \) recovers the vacuum result \( E_{\text{vac}}(0) = E_b = -1/(M \sigma^2) \). However, at finite \( k_F \), the medium shields the bound state and reduces the binding energy, i.e., \( |E(0)| < |E_{\text{vac}}(0)| \). For \( k_F \sigma > \pi/2 \) and \( k_F \sigma < 0 \), Eq. (34) has a positive solution which corresponds to the positive energy pole of the in-medium pair propagator. Such a pole is associated with Cooper pairs and its appearance represents the BCS instability. This positive energy pole does not lead to singularities in the energy density integration, as can be seen in next subsection, and does not need special treatment.

Note that \( E(0) < 0 \) is not a sufficient condition for \( E(\mathbf{P}) < \mathbf{P}^2/(4M) \). We thus have to check the energy dispersion \( E(\mathbf{P}) \) carefully. The numerical results for some values of the gas parameter \( k_F \sigma \) are shown in Fig. 3. For \( k_F \sigma < 1.34 \), the condition \( E(\mathbf{P}) < \mathbf{P}^2/(4M) \) is fulfilled for all values of \( \mathbf{P} \). However, for \( 1.34 < k_F \sigma < \pi/2 \), there exists a regime \( P_1 < |\mathbf{P}| < P_2 \) where \( E(\mathbf{P}) > \mathbf{P}^2/(4M) \).

In conclusion, the condition \( E(\mathbf{P}) < \mathbf{P}^2/(4M) \) is fulfilled for \( k_F \sigma < 1.34 \). Therefore, in the parameter regime \( k_F \sigma < 1 \) investigated in the following, the contribution from the bound state can be naturally excluded in the ladder resummation scheme.

### C. Evaluating the Energy Density

Now we evaluate the explicit form of the energy density \( \mathcal{E}(x) \) and the dimensionless function \( f(x) \). First, the elementary particle-particle bubble \( B(\mathbf{P}, \mathbf{k}) \) can be decomposed into four parts

\[
B(\mathbf{P}, \mathbf{k}) = B_0(\mathbf{P}, \mathbf{k}) + B_1(\mathbf{P}, \mathbf{k}) + B_1(\mathbf{P}, \mathbf{k}) + B_{11}(\mathbf{P}, \mathbf{k}),
\]

where \( B_0 \) is the vacuum part discussed in Section III, and the other parts are given by

\[
B_1(\mathbf{P}, \mathbf{k}) = -M \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \Theta(k_F^2 - |\mathbf{P}/2 + \mathbf{q}|), \quad B_{11}(\mathbf{P}, \mathbf{k}) - M \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \Theta(k_F^2 - |\mathbf{P}/2 - \mathbf{q}|).
\]

For convenience, we define another dimensionless quantity \( t = |\mathbf{p}|/k_F \) together with \( s \) defined in the last subsection. Since the imaginary part of \( B \) does not contribute to the interaction energy, we need only to evaluate the real part of \( B \). We have

\[
\text{Re}B_0(s, t) = -\frac{M \mu}{4\pi}, \quad \text{Re}B_1(s, t) = \frac{M k_F}{4\pi^2} R_1(s, t), \quad \text{Re}B_{11}(s, t) = \frac{M k_F}{4\pi^2} R_{11}(s, t).
\]

where \( R_{\sigma}(s, t) \) \((\sigma = \uparrow, \downarrow)\) reads

\[
R_{\sigma}(s, t) = \frac{\eta_{\sigma}^2 - (s + t)^2}{4s} \ln \left| \frac{\eta_{\sigma} + s + t}{\eta_{\sigma} - s - t} \right| + \frac{\eta_{\sigma}^2 - (s - t)^2}{4s} \ln \left| \frac{\eta_{\sigma} + s - t}{\eta_{\sigma} - s + t} \right| + \eta_{\sigma},
\]

and the function \( R_{11}(s, t) \) is

\[
R_{11}(s, t) = \begin{cases} -\Theta(x) R_1(s, t) - \Theta(-x) R_1(s, t), & 0 < s < \frac{1}{2} |\eta_1 - \eta_1|, \\ K_1(s, t) + K_1(s, t), & \frac{1}{2} |\eta_1 - \eta_1| < s < \frac{1}{2} |\eta_1 + \eta_1|, \\ 0, & \text{elsewhere.} \end{cases}
\]
where $r^2 = (\eta_1^2 + \eta_2^2)/2$.

Finally, the elementary particle–particle bubble reads

$$B(s, t) = -\frac{M\mu}{4\pi} + \frac{Mk_F}{4\pi^2}R_{pp}(s, t),$$

where the function $R_{pp}(s, t)$ is defined as

$$R_{pp}(s, t) = R_1(s, t) + R_1(s, t) + R_1(s, t),$$

Substituting the result of $B(s, t)$ into the expression of $E_{\text{int}}^{(0)}$, we observe that the energy density is independent of the renormalization mass scale $\mu$. Converting the integration variables $p_1$ and $p_2$ to $P$ and $K$, we find that the function $f(x)$ can be expressed as

$$f(x) = \frac{1}{2}(\eta_1^2 + \eta_2^2) + \frac{80}{\pi} \int_0^\infty s^2 ds \int_0^\infty \eta_1 t I(s, t)F(s, t),$$

where $F(s, t)$ is given by

$$F(s, t) = \frac{k_F a}{1 - \frac{1}{\pi}k_F a R_{pp}(s, t)}.$$

The function $I(s, t)$ appears due to integration over the angle between $P$ and $K$. Its explicit form is

$$I(s, t) = \frac{\eta_1^2 - (s + t)^2}{4s} \Theta(s + t - \eta_1) + (\eta_1 - \eta_2) + t$$

$$\times \Theta(r^2 - s^2 - t^2) \Theta(\eta_1 - |s - t|) \Theta(\eta_2 - |s - t|).$$

As we mentioned in the beginning of this section, it is important to check whether the present result for $f(x)$ is consistent with the perturbative expression [5] for weak coupling $k_F a \ll 1$. To this end, we expand the function $F(s, k)$ as

$$F(s, t) = k_F a + \frac{1}{\pi}(k_F a)^2 R_{pp}(s, t) + O((k_F a)^3).$$

Using the expressions for $I(s, t)$ and $R_{pp}(s, t)$, we can show that

$$\frac{80}{\pi} \int_0^\infty s^2 ds \int_0^\infty t I(s, t) = \frac{10}{9\pi} \eta_1^3 \eta_1,$$

and

$$\frac{80}{\pi^2} \int_0^\infty s^2 ds \int_0^\infty t I(s, t)R_{pp}(s, t) = \frac{\xi(\eta_1, \eta_2)}{21\pi^2}.$$

Therefore, our non-perturbative expression (27) exactly recover the perturbative result (3) at weak coupling. This convinces us that the present theoretical approach is suitable to study the universal upper branch Fermi gas with a positive scattering length. In addition, we can compare the results from our theory and the second-order perturbation on the same footing and study the non-perturbative effects on the FMPT.

### D. Results and Discussion

(A) Energy density and compressibility. We first study the equation of state for the unpolarized case $x = 0$. The gas parameter dependence of the energy density $E$ in the regime $0 < k_F a < 1$ are shown in Fig. 4. We find that the result from the ladder resummation is consistent with the perturbative result (3) for small gas parameters $k_F a < 0.4$. However, significant deviations are found for $k_F a > 0.4$, consistent with recent Quantum Monte Carlo simulations [5, 6]. For the Quantum Monte Carlo simulations of the attractive interactions with a negligible effective range (corresponding to UB and UB2 in Fig. 4), the exclusion of molecular bound states is implemented by choosing a two-body Jastrow factor [5, 6] to be the scattering solution of the attractive potential corresponding to positive energy, which, by construction, is orthogonal to the bound molecules. Therefore, the accuracy of the Quantum Monte Carlo data depends on the choice of the Jastrow factor. Actually, exact orthogonality of the many-body variational wave function to the superfluid ground state (molecular condensation) can not be achieved in the Quantum Monte Carlo simulations [5, 6]. We note that our theoretical curve agrees better with the UB2 data than with the UB data. The reason could be that the UB2 data from [6] are obtained with a Jastrow factor which imposes a better orthogonality to the superfluid ground state.

An important issue is whether the system is mechanically stable. The mechanical stability of the system requires a positive compressibility $\kappa$, which is defined as

$$\frac{1}{\kappa} = n^2 \frac{\partial^2 E}{\partial n^2}.$$  

For the present ladder resummation theory, the explicit form of $\kappa$ can be evaluated as

$$\frac{k_0}{\kappa} = 1 + \frac{144}{\pi} \int_0^\infty s^2 ds \int_0^\infty t I(s, t)G(s, t),$$

where $k_0 = 3/(2n_F)$ is the compressibility for non-interacting Fermi gases, and the function $G(s, t)$ is given by

$$G(s, t) = F(s, t) + \frac{5R_{pp}(s, t)}{9\pi} F^2(s, t) + \frac{R_{pp}(s, t)}{9\pi^2} F^3(s, t).$$

The compressibility $\kappa$ as a function of the gas parameter $k_F a$ is shown in Fig. 5. Comparing to the result from the second-order perturbation theory,

$$\frac{k_0}{\kappa} = 1 + \frac{2}{\pi} k_F a + \frac{8(11 - 2\ln 2)}{15\pi^2} (k_F a)^2,$$

good agreement is found for small gas parameters, as we expected. In the regime $0 < k_F a < 1$ we are interested in, we
For UB and UB2 cases, the diamonds for the upper branch (UB2) of a square well potential \[5\], and the green diamonds for the hard sphere (HS) potential \[5\], the red circles for the order perturbation theory (PTh). The dash-dotted horizontal line is much smaller than the s-wave scattering length \(a\). The solid line is the result calculated from our particle-particle ladder resummation theory (RTh). The dashed line is result of the 2nd-order perturbation theory. The dashed line is the result of the 2nd-order perturbation theory (PTh). The dash-dotted horizontal line corresponds to the energy of the fully polarized state \((x = 1)\), i.e., \(f(1) = 2^{1/3}\). The blue squares are the Quantum Monte Carlo (QMC) data for the hard sphere (HS) potential \[5\], the red circles for the upper branch (UB) of a square well potential \[5\], and the green diamonds for the upper branch (UB2) of an attractive short range potential \[6\]. For UB and UB2 cases, the effective range \(r_0\) is much smaller than the s-wave scattering length \(a\) \[5, 7\].

In the present ladder resummation theory, an explicit form of the spin susceptibility \(\chi\) is hard to obtain. In practice, we expand the function \(f(x)\) near \(x = 0\) as \(f(x) = f(0) + ax^2 + \cdots\). The coefficient \(a\) is related to the spin susceptibility by

\[
\frac{\chi_0}{\chi} = \frac{9}{5}\alpha, \tag{54}
\]

where \(\chi_0 = 3n/(2E_F)\) is the spin susceptibility of non-interacting Fermi gases. Therefore, a diverging spin susceptibility generally indicates a FMPT, as long as the transition is of second order.

In the second-order perturbation theory, an analytical result for \(\chi\) can be achieved,

\[
\frac{\chi_0}{\chi} = 1 - \frac{2}{\pi}k_Fa - \frac{16(2 + \ln2)}{15\pi^2}(k_Fa)^2, \tag{55}
\]

which indicates a diverging spin susceptibility at \(k_Fa = 1.058\). However, this differs from the critical gas parameter \((k_Fa)_c = 1.054\), because the phase transition is of first order in the second-order perturbation theory due to the appearance of the non-analytical term \(\nu \lambda^4 \ln |x|\) with \(\nu > 0\).

Our result for the spin susceptibility \(\chi\) as a function of the gas parameter \(k_Fa\) is shown in Fig. 6 and compared with the perturbative result. We find that the spin susceptibility predicted by the ladder resummation deviates significantly from the second-order perturbative result for \(k_Fa > 0.4\). Further, the spin susceptibility diverges at \(k_Fa = 0.858\), in contrast to the value 1.058 from the second-order perturbation theory. The data from the Quantum Monte Carlo simulations \[5\] are also shown in Fig. 6 as a comparison. Our theoretical result is in good agreement with the data for the upper branch of the square well potential where the effective range \(r_0\) is tuned to be much smaller than the scattering length \[5\]. The gas parameter \(k_Fa = 0.86\) where \(\chi\) diverges is very close to our prediction \(k_Fa = 0.858\). For the purely repulsive potential, i.e., the hard sphere potential, the effective range effect can not be neglected in prior. However, we find that our result still has nice agreement with the data for the hard sphere case. The gas parameter \(k_Fa = 0.82\) where \(\chi\) diverges is also close to our prediction \(k_Fa = 0.858\). Actually, the difference between the upper branch and the hard sphere cases, i.e., \(0.86 - 0.82 = 0.04\), is very small compared with the critical gas parameters. This indicates that the contribution from the effective range effect is relatively small even for \(k_Fa \sim O(1)\), if the Quantum Monte Carlo results are reliable. This can be understood from the large-dimension expansion \[38, 47\] introduced in the beginning of this section: The particle-particle ladder sum is the leading-order contribution in the \(1/D\) expansion, and all other contributions including the effective range corrections are suppressed by a factor \(1/D\).
(C) Ferromagnetic transition. While a diverging spin susceptibility indicates a ferromagnetic phase transition, the order of the ferromagnetic phase transition and the critical gas parameter \(k_Fa\), should be obtained by studying carefully the shape of the energy landscape, i.e., the full \(x\)-dependence of the function \(f(x)\). To very high numerical accuracy, we haven’t found any maximum at \(x = 0\) in the energy landscape. Instead, we find a second order phase transition at \(k_Fa = 0.858\) where the function \(f(x)\) starts to develop a minimum at \(x = 0\), consistent with the gas parameter where the spin susceptibility diverges. This is in contrast to the second-order perturbation theory which predicts a first order phase transition at \(k_Fa = 1.054\) where the spin polarization \(x\) jumps from zero to \(x = 0.573\). A second order FMPT for a zero range potential model was also obtained by Heiselberg \cite{18} recently using a completely different many-body method.

It seems that our result of a second order phase transition is in contradiction to the BKV argument \cite{37}. However, the BKV argument is based on the assumption that \(\nu > 0\). Actually, we have fitted the energy density of the form \(f(x) = f(0) + ax^2 + ax^3\ln|x| + bx^4\) for small \(x\). For small gas parameter \(k_Fa < 0.3\), the coefficient \(\nu\) agrees well with the perturbative result \(\nu = 40(k_Fa)^2/(243\pi^2)\). However, for larger \(k_Fa\) (especially around the critical gas parameter), it turns out to be negative due to the non-perturbative effects. This indicates that the FMPT in the systems of dilute repulsive Fermi gases corresponds to the case \(\nu < 0\), and is a counterexample to the BKV argument where the assumption \(\nu > 0\) is adopted.

Since an analytical expression for the function \(f(x)\) as well as the coefficient \(\nu\) cannot be achieved in the present lattice resummation theory, we cannot understand analytically how the non-perturbative effects modify the order of the phase transition. In fact, analytical results cannot be obtained from the order \(O((k_Fa)^3)\) even for the unpolarized case \(x = 0\) in the perturbation theory \cite{41}. However, some definite conclusions can be drawn from our numerical results: (1) Higher-order terms in the gas parameter can also generate non-analytical terms of the form \(x^4\ln|x|\) and may generate other important non-analytical terms which are not known due to the mathematical limitation; (2) The coefficients of the non-analytical terms generated by the higher-order contributions are certainly not always positive, and they are generally proportional to \((k_Fa)^n\) for the \(n\)-th-order contributions. Since the phase transition occurs at a gas parameter \(k_Fa \sim O(1)\), the non-perturbative effects from the sum of the higher order contributions are very important. As we have shown numerically, their effects are not only reducing the critical value of the gas parameter but also changing the order of the phase transition.

V. INCLUSION OF HOLE-HOLE LADDERS

In this section we check whether our conclusion that the FMPT is of second order is changed by other contributions. We consider the contributions from the hole-hole ladder diagrams by summing the combined particle-particle and hole-hole ladders to all orders in \(k_Fa\) while keeping the criteria (i) and (ii) satisfied. Such a resummation scheme for the unpolarized case \(x = 0\) has been performed by Kaiser \cite{50}.

Following the treatment by Kaiser \cite{50}, we rewrite the propagator \cite{19} in an alternative form

\[
\mathcal{G}_0(p_0, p) = \mathcal{G}_0(p_0, p) + 2\pi i \delta(p_0 - \omega_F)\Theta(k_F^2 - |p|), \tag{56}
\]

where the first term corresponds to the vacuum propagator \(\mathcal{G}_0(p_0, p) = (p_0 - \omega_F + \imath\epsilon)^{-1}\) and the second term is a so-called “medium-insertion” (MI) \cite{50}. The elementary bubbles in this treatment are shown in Fig. 7. The first diagram \(B_0\) is identical to the vacuum part studied in Section III and it can be renormalized using the PDS scheme. For our purpose of resummation, we are interested in the following two quantities, \(B_0 + B_1 + B_2\) and \(B_0 + B_1\), which are mutually complex conjugate. We have

\[
B_0 + B_1 + B_2 = \frac{-M\mu}{4\pi} + \frac{Mk_F}{4\pi^2} [R(s, t) - \imath nI(s, t)], \tag{57}
\]

\[
B_0 + B_1 = \frac{-M\mu}{4\pi} + \frac{Mk_F}{4\pi^2} [R(s, t) + \imath nI(s, t)],
\]
where \( R(s, t) = R_1(s, t) + R_2(s, t) \) and \( I(s, t) \) is the function defined in (30).

To sum all ladder diagrams built from the elementary bubbles, we first notice that the non-vanishing contributions to the interaction energy come from diagrams with at least two adjacent MIs \([50]\). Then a typical \( n \)-th-order contribution would look like the ring diagram of Fig. 1(b) with \( n \) vertices and at least two adjacent MIs. Naively, all these \( n \)-th-order diagrams are summed to give \( g^\nu[(B_0 + B_1 + B_2) \cdot (B_0 + B_1)^\nu] \) where the subtraction gets rid of those diagrams which have no adjacent MI pairs. However, this expression is complex and therefore cannot be the correct one. The crucial observations are that: (1) Each \( n \)-th-order ring diagram has a rotational symmetry. Therefore, we should introduce an additional factor \( 1/n \); (2) An \( n \)-th-order ring diagram comes from closing two open MI-lines of an \( n \)-th-order ladder diagram, which introduces an integration over the allowed phase space \( |p|^2 < k_F^4 \) \( \text{and} \ |p|^2 < k_F^4 \), but does not contribute a factor \( B_2 \) to the energy as the naive expression does. These amendments lead to the correct \( n \)-th-order contribution to the interaction energy \([50]\): \( g^\nu[(B_0 + B_1 + B_2) \cdot (B_0 + B_1)^\nu]/(2\pi n) \). The summation over \( n \) leads to two complex-conjugated logarithms and the final result is real.

The final result for the energy density does not depend on the renormalization scale \( \mu \), and the function \( f(x) \) in this resummation scheme also takes the form \([43]\), while the function \( F(s, t) \) becomes

\[
F(s, t) = \ln \left[ 1 - \frac{1}{2} k_{\text{FD}} R(s, t) + ik_{\text{FD}} I(s, t) \right] - c.c. \tag{58}
\]

For small gas parameter \( k_{\text{FD}} \ll 1 \), \( F(s, t) \) can be expanded as

\[
F(s, t) = k_{\text{FD}} + \frac{1}{\pi} (k_{\text{FD}})^2 R(s, t) + O((k_{\text{FD}})^3). \tag{59}
\]

We can also check that

\[
\frac{80}{\pi^2} \int_0^\infty \hat{s}^2 ds \int_0^\infty t dt I(t, s) R(s, t) = \frac{\xi(\eta_1, \eta_2)}{21\pi^2}, \tag{60}
\]

where \( R(s, t) = R_1(s, t) + R_2(s, t) \) and \( I(s, t) \) is the function defined in (30).

VI. SUMMARY

In summary, we have studied the non-perturbative effects on the ferromagnetic phase transition in repulsive Fermi gases by summing the ladder diagrams to all orders in the gas parameter \( k_{\text{FD}} \). The non-perturbative effects not only reduce the critical gas parameter but also change the order of the phase transition. The resummation of particle-particle ladder diagrams, which corresponds to the leading order of the large-dimension expansion, predicts a second order phase transition occurring at \( k_{\text{FD}} = 0.858 \), in good agreement with the Quantum Monte Carlo result \([5]\). The spin susceptibility calculated from our resummation theory are also in good agreement with the Quantum Monte Carlo results. Therefore, the resummation of the ladder diagrams provides a more quantitative way to study the ferromagnetic transition in repulsive Fermi gases. In this paper, we have only considered a zero range potential model. It will be interesting to study the non-universal shape-dependent contributions using the finite-density effective range expansion \([47]\).

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