Stanley-Wilf limits are typically exponential

Jacob Fox
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A permutation $\sigma = \sigma_1 \cdots \sigma_n$ contains another permutation $\pi = \pi_1 \cdots \pi_k$ if there exists indices $i_1 < \ldots < i_k$ such that $\sigma_{i_j} < \sigma_{i_\ell}$ if and only if $\pi_j < \pi_\ell$. Otherwise, $\sigma$ is said to avoid $\pi$.

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**Definition**

$S_n(\pi)$ is the number of $n$-permutations avoiding $\pi$. 

Theorem: (McMahon 1915, Knuth 1968) For each 3-permutation $\pi$, 

$$S_n(\pi) = \frac{1}{n+1} \left(2^n - \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^n\right).$$
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**Theorem: (McMahon 1915, Knuth 1968)**

For each 3-permutation $\pi$, 

$$S_n(\pi) = \frac{1}{n+1} \binom{2n}{n}.$$
Conjecture: (Stanley-Wilf 1980)
For each \( \pi \), there is \( L(\pi) \) such that \( \lim_{n \to \infty} S_n(\pi)^{1/n} = L(\pi) \).
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Theorem: (Marcus-Tardos 2004)
For each $k$-permutation $\pi$, $L(\pi)$ exists and satisfies
\[
L(\pi) \leq 15^{2k^4}\binom{k^2}{k}.
\]
Problem

How large can $L(\pi)$ be for a $k$-permutation $\pi$?
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A permutation is *layered* if it is a concatenation of decreasing sequences, the letters of each sequence being smaller than the letters in the following sequences.

Theorem: (Claesson-Jelínek-Steingrímsson 2012)

Every layered \( k \)-permutation \( \pi \) satisfies \( L(\pi) \leq 4k^2 \).
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**Theorem (F.)**

There is a $k$-permutation $\pi$ with

$$L(\pi) = 2^{\Omega(k^{1/4})}.$$
Extremal Problem for Matrices

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Matrix \( A \) \textit{contains} a \( k \times \ell \) matrix \( P = (p_{ij}) \) if there is a \( k \times \ell \) submatrix \( D = (d_{ij}) \) of \( A \) such that if \( p_{ij} = 1 \), then \( d_{ij} = 1 \). Otherwise, \( A \) \textit{avoids} \( P \).
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For every permutation $\pi$, $\text{ex}(n, \pi) = O(n)$. 
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Equivalent to $c(\pi) := \lim_{n \to \infty} \frac{\text{ex}(n, \pi)}{n}$ exists.
Klazar proved $L(\pi) \leq 15c(\pi)$.

**Theorem: (Marcus-Tardos 2004)**

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For every permutation $\pi$, $L(\pi) = O(c(\pi)^2)$ and $c(\pi) = O(L(\pi)^{4.5})$. 
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**Definition: Contraction**

The *contraction* of two consecutive rows of a matrix replaces the two rows by a single row, with a one in an entry of the new row if at least one of the two entries in the original two rows is a one. Contraction of columns is defined similarly.
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\[ \exists \; \ell^2 \text{-permutation } \pi \text{ whose matrix contains } J_{\ell} \text{ as an interval minor.} \]
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Lemma

$\exists \ell^2$-permutation $\pi$ whose matrix contains $J_\ell$ as an interval minor.

$\pi$ is given by $\pi(a\ell + b + 1) = b\ell + a + 1$ for $0 \leq a, b \leq \ell - 1$. 
Theorem: (F.)

Let \( r = \frac{1}{8} \ell \) and \( N = 2^r \). There is an \( N \times N \) matrix \( M \) with mass at least \( N^{3/2} \) which avoids \( J_\ell \) as an interval minor.

Proof: Let \( q = \frac{1}{8^r} \) and \( N' = 2N - 1 \).

Let \( B = (b_{IJ}) \) be the \( N' \times N' \) matrix with a row for each \( I \in V(T_R) \) and a column for each \( J \in V(T_C) \) and each entry is one with probability \( 1 - q \) independently of the other entries.

Let \( M = (m_{ij}) \) be the \( N \times N \) matrix with \( m_{ij} = 1 \) iff \( b_{IJ} = 1 \) for every ancestor \( I \) of \{i\} in \( T_R \) and every ancestor \( J \) of \{j\} in \( T_C \).

There is a choice of \( B \) that is \( J_\ell \)-free and \( M \) has mass at least \( N^{3/2} \).

Suppose for contradiction that, in \( M \), \( I_1, \ldots, I_\ell \) are intervals of rows and \( L_1, \ldots, L_\ell \) are intervals of columns which contract to make \( J_\ell \).

Assign each \( I_a \) a vertex \( v_a \) of \( T_R \) of largest height which contains \( I_a \).

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\( v_1, \ldots, v_\ell \) are distinct and \( u_1, \ldots, u_\ell \) are distinct.

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Let \( r = \frac{1}{8} \ell^{1/2} \) and \( N = 2^r \). There is an \( N \times N \) matrix \( M \) with mass at least \( N^{3/2} \) which avoids \( J_\ell \) as an interval minor.
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$v_1, \ldots, v_\ell$ are distinct and $u_1, \ldots, u_\ell$ are distinct.
Lower bound construction

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Let \( r = \frac{1}{8} \ell \frac{1}{2} \) and \( N = 2^r \). There is an \( N \times N \) matrix \( M \) with mass at least \( N^3/2 \) and avoids \( J_\ell \) as an interval minor.

Let \( k = \ell^2 \).

As there exists a \( k \)-permutation \( \pi \) whose matrix contains \( J_\ell \) as an interval minor, then \( M \) avoids \( \pi \).

Hence, \( \text{ex}(N, \pi) \geq N^3/2 \).

Since \( \text{ex}(n, \pi) \) is super-additive, \( c(\pi) \geq N^{1/2} \).

As \( L(\pi) \) and \( c(\pi) \) are polynomially related, \( L(\pi) = c(\pi) \Omega(1) = 2^{\Omega(k^{1/4})} \).
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Let \( r = \frac{1}{8} \ell^{1/2} \) and \( N = 2^{r} \). There is an \( N \times N \) matrix \( M \) with mass at least \( N^{3/2} \) and avoids \( J_{\ell} \) as an interval minor.

Let \( k = \ell^{2} \).

As there exists a \( k \)-permutation \( \pi \) whose matrix contains \( J_{\ell} \) as an interval minor, then \( M \) avoids \( \pi \).

Hence,

\[
\text{ex}(N, \pi) \geq N^{3/2}.
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Since \( \text{ex}(n, \pi) \) is super-additive, \( c(\pi) \geq N^{1/2} \).
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As \( L(\pi) \) and \( c(\pi) \) are polynomially related,

\[
L(\pi) = c(\pi)^\Omega(1) = 2^{\Omega(k^{1/4})}.
\]
Upper Bound

Let $T_n(\pi)$ be the number of $n \times n$ matrices which avoid $\pi$.

Lemma: (Klazar 2000) $T_n(\pi) = 2^{\Theta(ex(n,\pi))}$

This follows by induction from $T_{2n}(\pi) \leq T_n(\pi)^2$.

Theorem: (Cibulka) $L(\pi) = O(c(\pi)^2)$

New simple proof: For $N = t_n$, we have $S_N(\pi) \leq T_n(\pi) t^2 N$.

For $t = c(\pi)$, this is $S_N(\pi) \leq 2^{O(N)} c(\pi)^2 N$ and we are done.
Let $T_n(\pi)$ be the number of $n \times n$ matrices which avoid $\pi$. 

Lemma: (Klazar 2000) $T_n(\pi) = 2^{\Theta(ex(n, \pi))}$.

This follows by induction from $T_2(n(\pi)) \leq T_n(\pi) \cdot \frac{2}{ex(n, \pi)}$.

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New simple proof: For $N = t_n$, we have $S_N(\pi) \leq T_n(\pi) \cdot t^2 N$. For $t = c(\pi)$, this is $S_N(\pi) \leq 2^{O(N \cdot c(\pi)^2)}$ and we are done.
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**Lemma: (Klazar 2000)**

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**Lemma: (Klazar 2000)**

$$T_n(\pi) = 2^\Theta(\text{ex}(n, \pi))$$

This follows by induction from

$$T_{2n}(\pi) \leq T_n(\pi)15^{\text{ex}(n, \pi)}$$
Let $T_n(\pi)$ be the number of $n \times n$ matrices which avoid $\pi$.

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New simple proof: For $N = tn$, we have

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For $t = c(\pi)$, this is $S_N(\pi) \leq 2^{O(N)c(\pi)^{2N}}$ and we are done.
Marcus-Tardos theorem

Theorem: (Marcus-Tardos 2004)\\
$ex(n, \pi) \leq 2k^{4/k^2}n$.

Proof: This follows by induction from $ex(n, \pi) \leq (k-1)2ex(n/k^2, \pi) + 2k^3(k^2/k^2)n$.

Partition $n \times n$ matrix $A$ which avoids $\pi$ into $k^2 \times k^2$ blocks. Define a block to be wide (tall) if it contains $k$ different columns (rows).

Form $n^{k^2} \times n^{k^2}$ matrix $B$ from $A$ by contracting intervals of size $k^2$. Each column of $B$ has less than $k(k^2/k^2)$ ones from wide blocks. Hence, mass of $A$ in wide or tall blocks is at most $2n^{k^2}k^{4/k^2}k(k^2/k^2)$.

$B$ avoids $\pi$ and hence has mass at most $ex(n^{k^2}, \pi)$. The blocks which are neither wide nor tall each have at most $(k-1)2$ ones, and the desired inequality follows.
Theorem: (Marcus-Tardos 2004)

\[ \text{ex}(n, \pi) \leq 2k^4 \binom{k^2}{k} n. \]
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Proof: This follows by induction from

\[ \text{ex}(n, \pi) \leq (k - 1)^2 \text{ex} \left( \frac{n}{k^2}, \pi \right) + 2k^3 \binom{k^2}{k} n. \]
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Partition \( n \times n \) matrix \( A \) which avoids \( \pi \) into \( k^2 \times k^2 \) blocks. Define a block to be wide (tall) if it contains 1-entries in at least \( k \) different columns (rows).
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Hence, mass of \( A \) in wide or tall blocks is at most \( 2 \cdot \frac{n}{k^2} \cdot k^4 \cdot k \binom{k^2}{k} \).
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