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LIMITING LAWS FOR LONG BROWNIAN BRIDGES PERTURBED BY THEIR ONE-SIDED MAXIMUM, III

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In homage to Professors E. Csaki and P. Revesz.

Abstract. Results of penalization of a one-dimensional Brownian motion \( (X_t) \), by its one-sided maximum \( (S_t = \sup_{0 \leq u \leq t} X_u) \), which were recently obtained by the authors are improved with the consideration-in the present paper- of the asymptotic behaviour of the likewise penalized Brownian bridges of length \( t \), as \( t \to \infty \), or penalizations by functions of \( (S_t, X_t) \), and also the study of the speed of convergence, as \( t \to \infty \), of the penalized distributions at time \( t \).

Key words and phrases : penalization, one-sided maximum, long Brownian bridges, local time, Pitman’s theorem

AMS 2000 subject classifications : 60B 10, 60G 17, 60G 40, 60G 44, 60J 25, 60J 35, 60J 55, 60J 60, 60J 65.

1 Introduction

1.1 Let \( (\Omega = C(\mathbb{R}^+, \mathbb{R}), (X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0}) \) be the canonical space with \( (X_t) \) the process of coordinates : \( X_t(\omega) = \omega(t); t \geq 0 \), \( (\mathcal{F}_t)_{t \geq 0} \) the canonical filtration associated with \( (X_t) \). We write \( \mathcal{F}_\infty \) for the \( \sigma \)-algebra generated by \( \bigcup_{t \geq 0} \mathcal{F}_t \). Let \( P_0 \) be the Wiener measure defined on the canonical space such that \( P_0(X_0 = 0) = 1 \).

In this paper, as well as in the previous ones \((\text{[13]}, \text{[14]}, \text{[15]})\), we consider perturbations of Brownian motion with certain processes \( (F_t)_{t \geq 0} \), which we call weight-processes; precisely, let \( (F_t)_{t \geq 0} \) be an
In this paper we only consider the case.

**Theorem 1.1** Let \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) satisfying (1.3) and \( \Phi(y) = \int_0^y \phi(x)dx, y \geq 0 \).

1. For every \( u \geq 0 \), and \( \Gamma_u \) in \( \mathcal{F}_u \), the quantity:

\[
Q_0^\phi(\Gamma_u) := \lim_{t \to \infty} \frac{E_0[1_{\Gamma_u} \phi(S_t)]}{E_0[\phi(S_t)]},
\]

exists; hence, \( Q_0^\phi \) may be extended as a p.m. on \( (\Omega, \mathcal{F}_\infty) \).
2. It is equal to \( E_0[1_{\Gamma_u} M_u^\phi] \), where \( (M_u^\phi)_{u \geq 0} \) is the martingale:

\[
M_u^\phi = \phi(S_u)(S_u - X_u) + 1 - \Phi(S_u); \quad u \geq 0.
\]

(The last \( (P_0, (\mathcal{F}_u)) \text{-martingales have been introduced in} \ [1].\)
3. The probability $Q^\phi_0$ may be disintegrated as follows:

(a) under $Q^\phi_0$, $S_\infty$ is finite a.s., and admits $\varphi$ as a probability density;

(b) $Q^\phi_0(S_\infty \in dy)$ a.e., conditionally on $S_\infty = y$, the law of $(X_t)$, under $Q^\phi_0$ is equal to $Q^{(y)}_0$,
where, for any $y > 0$, the p.m. $Q^{(y)}_0$ on the canonical space is defined as follows:

i. $(X_t ; t \leq T_y)$ is a Brownian motion started at 0, and considered up to $T_y$, its first hitting time of $y$,

ii. the process $(X_{T_y + t} ; t \geq 0)$ is a "three dimensional Bessel process below $y$", namely :

$(y - X_{T_y + t} ; t \geq 0)$ is a three dimensional Bessel process started at 0.

iii. the processes $(X_t ; t \leq T_y)$ and $(X_{T_y + t} ; t \geq 0)$ are independent.

(c) Consequently :

$$Q^\phi_0(\Gamma|S_\infty = y) := Q^{(y)}_0(\Gamma), \text{ for any } \Gamma \in \mathcal{F}_\infty,$$

$$Q^\phi_0(\cdot) = \int_0^\infty Q^{(y)}_0(\cdot) \varphi(y) dy.$$  

(1.7)

In the present paper, we develop a number of variants of this Theorem 1.1, by presenting either extensions or some new proofs of this theorem. Here are these variants, together with the organization of our paper.

In Section 2, we give, in particular, another proof of Theorem 1.1, which originates from the following considerations : the main step in [15] consisted in studying the asymptotics of $E[\varphi(S_t)|\mathcal{F}_s]$, for fixed $s$, as $t \to \infty$. In Section 2 here, we proceed in a dual manner by studying the asymptotics of

$$Q^{(y)}_0(\Gamma_u) := P(\Gamma_u|X_t = a),$$

(1.8)
as $t \to \infty$, where $u \geq 0$ and $\Gamma_u \in \mathcal{F}_u$ are fixed.

**Theorem 1.2** Let $y > 0$, $u \geq 0$ and $\Gamma_u \in \mathcal{F}_u$.

1. As $t \to \infty$, $Q^{(y)}_0(\Gamma_u)$ converges towards the probability $Q^{(y)}_0(\Gamma_u)$, where $Q^{(y)}_0$ is the probability introduced in Theorem 1.1, 3.

2. Moreover, $Q^{(y)}_0$ satisfies :

$$Q^{(y)}_0(\Gamma_u) = e^{-y^2/2u} \sqrt{\frac{2}{\pi u}} E_0[1_{\Gamma_u}(y - X_u)|S_u = y] + E_0[1_{\Gamma_u}1_{S_u < y}].$$

(1.9)

In Section 3, we strengthen the result obtained in Section 2 in that we consider the existence of the limits, as $t \to \infty$, of :

$$\frac{E_0[1_{\Gamma_u}\varphi(S_t)|X_t = a]}{E_0[\varphi(S_t)|X_t = a]}$$

(1.10)
and, in the spirit of the preceding Section 3 (or Theorem 1.2) :

$$Q^{a,y}_0(t)(\Gamma_u) := P_0(\Gamma_u|X_t = a, S_t = y),$$

(1.11)
where $u \geq 0, \Gamma_u \in \mathcal{F}_u, y \geq a_+$.

The title of the present paper originates from this central Section 3. The results are the following :

• concerning (1.11), we obtain :
Theorem 1.3 1. For any $u \geq 0$ and $\Gamma_u \in F_u,$

$$\lim_{t \to \infty} Q^{a,y}_0(\Gamma_u) := Q^{a,y}_0(\Gamma_u),$$

exists.

2. The p.m. $Q^{a,y}_0$ may be expressed as a convex combination of the laws $Q^{(z)}_0,$ $z \in \mathbb{R}_+ :$

$$(2y-a)Q^{a,y}_0(\cdot) = (y-a)Q^{(y)}_0(\cdot) + \int_0^y dzQ^{(z)}_0(\cdot).$$

(1.13)

Remark 1.4 1. Recall that Remark 1.4

$$\text{Theorem 1.3}$$

1. For any $u \geq 0$ and $\Gamma_u \in F_u,$

$$\lim_{t \to \infty} Q^{a,y}_0(\Gamma_u) := Q^{a,y}_0(\Gamma_u),$$

exists.

2. The p.m. $Q^{a,y}_0$ may be expressed as a convex combination of the laws $Q^{(z)}_0,$ $z \in \mathbb{R}_+ :$

$$(2y-a)Q^{a,y}_0(\cdot) = (y-a)Q^{(y)}_0(\cdot) + \int_0^y dzQ^{(z)}_0(\cdot).$$

(1.13)

3. Identity (1.13) implies that $(a,y) \mapsto Q^{a,y}_0$ is continuous.

4. We may recover $Q^{(y)}_0$ from $(Q^{a,y}_0 : y_+ \leq a)$ since $Q^{(y)}_0 = \frac{d}{dy}(yQ^{a,y}_0).$

5. Let $\mu^{a,y}$ the p.m. on $\mathbb{R}_+ : \mu^{a,y}(dz) = \frac{y-a}{2y-a} d\delta_y(dz) + \frac{1}{2y-a} \mathbb{1}_{[0,y]}(z) dz.$ The relation (1.13) admits the following probabilistic interpretation : first, $z$ is chosen at random following $\mu^{a,y};$ secondly, the dynamics of $(X_t)$ is given by $Q^{(z)}_0.$

6. From Lévy’s theorem, under $P_0,$ $((S_t - X_t, S_t ; t \geq 0)$ and $((X_t), L^0_t ; t \geq 0)$ have the same distribution. Let $Q^{(y)}_0$ be the unique p.m. on $(\Omega, \sigma([X_t], t \geq 0))$ satisfying :

$$Q^{(y)}_0(\Gamma_u) = e^{-y^2/2u} \sqrt{\frac{2}{\pi u}} E_0[1_{\Gamma_u}] E_0[|L^0_u = y|] + E_0[1_{\Gamma_u} 1_{L^0_u < y}],$$

(1.14)

for any $u \geq 0$ and $\Gamma_u \in \sigma([X_t], t \leq u).$

In a forthcoming paper it is proved that the analog of (1.12) and (1.13) is :

$$\lim_{t \to \infty} P_0(\Gamma_u \mid X_t = a, L^0_t = y) = \frac{a}{a+y} Q^{(y)}_0(\Gamma_u) + \frac{1}{a+y} \int_0^y Q^{(z)}_0(\Gamma_u) dz,$$

(1.15)

with $\Gamma_u$ any event in $\sigma([X_t], t \leq u),$ and an adequate extension of this result with $|X_s|$ being replaced by a Bessel process with dimension $d < 2$ is obtained.

- As for (1.10), we obtain :

Theorem 1.5 Let $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that :

$$\int_0^\infty (1 + x) \varphi(x) dx < \infty.$$ 

(1.16)

1. For any $u \geq 0,$ $\Gamma_u \in F_u$ and $a \in \mathbb{R},$ we have :

$$Q^{a,\varphi}_0(\Gamma_u) := \lim_{t \to \infty} \frac{E_0[1_{\Gamma_u} \varphi(S_t) \mid X_t = a]}{E_0[\varphi(S_t) \mid X_t = a]},$$

(1.17)

exists.
2. The p.m. $Q_0^{a,\varphi}$ may be expressed in terms of either of the two families $(Q_0^{a,y}, y > 0)$ and $(Q_0^{(y)}, y > 0)$:

$$Q_0^{a,\varphi}(\cdot) = \frac{1}{\int_{a+}^{\infty} (2y-a)\varphi(y)dy} \int_{a+}^{\infty} (2y-a)\varphi(y)Q_0^{a,y}(\cdot)dy$$

$$= \frac{1}{\int_{a+}^{\infty} (2y-a)\varphi(y)dy} \left[ \int_{a+}^{\infty} (y-a)\varphi(y)Q_0^{(y)}(\cdot)dy + \int_0^{\infty} (1 - \Phi(z \lor (a+)))Q_0^{(z)}(\cdot)dz \right].$$

We would like to generalize Theorem 1.5 by replacing the weight-process $(\varphi(S_t))$ with $(f(X_t, S_t))$, where $f : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is Borel.

**Theorem 1.6** To $f : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that:

$$\mathcal{F} := \int_{\mathbb{R}} da \int_{a+}^{\infty} (2y-a)f(a,y)dy < \infty$$

we associate $f^* = 1/\mathcal{F}$, and:

$$\varphi(y) = f^* \left[ \int_{\mathbb{R}} da \int_{y \lor a+}^{\infty} f(a, \eta)d\eta + \int_{-\infty}^{y} f(a,y)(y-a)da \right].$$

1. For every $u \geq 0$, and $\Gamma_u$ in $\mathcal{F}_u$,

$$\lim_{t \to \infty} E_0[1_{\Gamma_u}f(X_t, S_t)] = E_0[\mathcal{F}(\Gamma_u)],$$

where $Q_0^\varphi$ is the p.m. introduced in Theorem 1.1, associated with the $(P_0, (\mathcal{F}_t))$ martingale $(M_\varphi)$.

2. Moreover the following relations hold:

$$M_\varphi^\mathcal{F} = f^* \int_{\mathbb{R}} da \int_{a+}^{\infty} (2y-a)f(a+X_t, S_t \lor (y+X_t))dy,$$

$$Q_0^\varphi(\cdot) = f^* \int_{\mathbb{R}} da \int_{a+}^{\infty} (2y-a)f(a,y)Q_0^{a,y}(\cdot)dy,$$

where the p.m. $Q_0^{a,y}$ is defined in Theorem 1.3.

Theorem 1.6 led us to go further and to enquire what happens if $f : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ does not satisfy (1.20). Rather than trying to give a complete answer, we shall restrict ourselves to functions $f$ of exponential type:

$$f(a,y) = e^{\lambda y + \mu a}, \ y \geq a+; \lambda, \mu \in \mathbb{R}.$$  

It is easy to check (see Section 5) that, if $f$ is given by (1.25), then: $\mathcal{F} < \infty$ iff $\mu > 0$ and $\lambda + \mu < 0$. Then in this case Theorem 1.6 applies.

We claim that for any $\lambda, \mu \in \mathbb{R}$ a penalization principle holds and we are able to describe the limiting p.m. Before stating this result in Theorem 1.7 below, let us introduce the three disjoint sets:

$$R_1 = \{(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}; \lambda + \mu < 0, \mu \geq 0\},$$
See the figure below.

\[ R_2 = \{ (\lambda, \mu) \in \mathbb{R} \times \mathbb{R}; \lambda + 2\mu \geq 0, \lambda + \mu \geq 0 \} \quad \text{(1.27)} \]

\[ R_3 = \{ (\lambda, \mu) \in \mathbb{R} \times \mathbb{R}; \lambda + 2\mu < 0, \mu < 0 \} \quad \text{(1.28)} \]

See the figure below.

**Theorem 1.7** Let \( \lambda, \mu \in \mathbb{R} \).

1. For every \( u \geq 0 \), and \( \Gamma_u \) in \( \mathcal{F}_u \),

\[
\lim_{t \to 1} \frac{E_0[\Gamma_u e^{\mu X_t + \lambda S_t}]}{E_0[e^{\mu X_t + \lambda S_t}]} \quad \text{(1.29)}
\]

exists and is equal to \( E_0[\Gamma_u M_{\mu,\lambda}^u] \), with \( (M_{\mu,\lambda}^u) \) a positive \((\mathcal{F}_u,P_0)\) martingale, such that \( M_{\mu,\lambda}^u = 1 \), which is given by

\[
M_{\mu,\lambda}^u = \begin{cases} 
- (\lambda + \mu) e^{(\lambda+\mu)S_u} (S_u - X_u) + e^{(\lambda+\mu)S_u} & \text{if } (\lambda,\mu) \in R_1, \\
 e^{(\lambda+\mu)X_u - (\lambda+\mu)^2 u/2} & \text{if } (\lambda,\mu) \in R_2, \\
 e^{(\lambda+\mu)S_u - \mu^2 u/2} \left[ \cosh (\mu(S_u - X_u)) - \frac{\lambda + \mu}{\mu} \sinh (\mu(S_u - X_u)) \right] & \text{if } (\lambda,\mu) \in R_3.
\end{cases}
\]
2. Consequently, \( \Gamma_u(\in \mathcal{F}_u) \mapsto E_0[1_{\Gamma_u} M_{u,\lambda}^\varphi] \) induces a p.m. on \((\Omega, \mathcal{F}_\infty)\).

**Remark 1.8**

1. We have already observed that if \( f \) is defined by (1.23), then \( \overline{T} < \infty \) iff \( \mu > 0 \) and \( \lambda + \mu < 0 \). Thus, in this case, Theorem 1.4 implies that \( (M_t^\varphi) \) is a martingale of the type \( (M_t^\varphi) \) where \( \varphi \) is given by (1.28). An easy calculation yields: 
   \[
   \varphi(y) = -(\lambda + \mu)e^{(\lambda + \mu)t}, \quad y \geq 0,
   \]
   and:
   \[
   M_t^{\mu,\lambda} = M_t^\varphi = -(\lambda + \mu)e^{(\lambda + \mu)t} (S_t - X_t) + e^{(\lambda + \mu)t}, \quad t \geq 0.
   \]

2. In the third case (i.e. \( (\lambda, \mu) \in R_3 \)), the martingale belongs to the family of Kennedy martingales. These martingales were used in [1] and play a central role in [15]. Let us briefly recall the definition of these processes.

   To \( \psi : \mathbb{R} \to [0, \infty] \), a Borel function satisfying:
   \[
   \int_x^\infty \psi(z) e^{-\lambda z} dz < \infty, \quad \forall x \in \mathbb{R}.
   \]
   we associate the function \( \Phi : \mathbb{R} \mapsto \mathbb{R} : \)
   \[
   \Phi(y) = 1 - \frac{e^{\lambda y}}{\lambda} \int_y^\infty \psi(z) e^{-\lambda z} dz, \quad y \in \mathbb{R}.
   \]
   Let \( \varphi \) be the derivative of \( \Phi \); then, \( \varphi(y) := \Phi'(y) = \psi(y) - \lambda e^{\lambda y} \int_y^\infty \psi(z) e^{-\lambda z} dz \), and
   \[
   M_t^{\lambda,\varphi} := \left\{ \psi(S_t) \frac{\sinh(\lambda(S_t - X_t))}{\lambda} + e^{\lambda X_t} \int_{S_t}^\infty \psi(z) e^{-\lambda z} dz \right\} e^{-\lambda t/2},
   \]
   is a positive \((\mathcal{F}_t, P_0)\)-martingale.

3. Let \( Q_0^{\mu,\lambda} \) be the p.m. defined in point 2. of Theorem 1.3, and \( P_0^\delta \) be the law of Brownian motion with drift \( \delta \), starting at 0. Using Theorem 3.9 of [13], we may reformulate (1.30) as follows:

   \[
   Q_0^{\mu,\lambda} = \begin{cases} 
   Q_0^{\varphi(-\lambda t)} & \text{if } (\lambda, \mu) \in R_1, \\
   P_0^{\mu + \lambda} & \text{if } (\lambda, \mu) \in R_2, \\
   \frac{\lambda + 2\mu}{2\mu} e^{\lambda S_\infty} \cdot P_0^\mu & \text{if } (\lambda, \mu) \in R_3
   \end{cases}
   \]

   where \( \varphi_\delta(y) = e^{-\delta y} - e^{-\lambda y}, \quad \delta > 0, y \geq 0 \).

The proof of Theorem 1.7 is postponed to Section 3.

Let \( \varphi \) as in Theorem 1.3. We are now interested in the rate of convergence of \( Q_0^{\varphi}(\Gamma_u) := \frac{E_0[1_{\Gamma_u} \varphi(S_t)]}{E_0[\varphi(S_t)]} \) towards \( Q_0^\varphi(\Gamma_u) \), as \( t \to \infty \), for any \( \Gamma_u \in \mathcal{F}_u \). More generally, under additional assumptions, we are able to determine the asymptotic development of \( Q_0^{\varphi}(\Gamma_u) \) in powers of \( 1/t \), \( t \to \infty \).

**Theorem 1.9** Let \( \varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) satisfying (1.3) and the related function \( \Phi \) as in Theorem 1.4. We suppose that there exists an integer \( n \geq 1 \) such that:

   \[
   \int_0^\infty y^{2n+3} \varphi(y) dy < \infty.
   \]
1. There exists a family of functions \( (F_i^\omega)_{1 \leq i \leq n} \), \( F_i^\omega : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \), such that

(a) \( (F_i^\omega(X_t,S_t,t), t \geq 0) \) is a \(((\mathcal{F}_t), P_0)\)-martingale, for any \( 1 \leq i \leq n \),

(b) If \( i = 1 \), we have:

\[
F_1^\omega(X_t, S_t, t) = -\bar{F}_1^\omega(X_t, S_t) + \left(t + \int_0^\infty y^2 \varphi(y) dy\right) M_t^\omega,
\]

where

\[
\bar{F}_1^\omega(a, y) = \varphi(y) \frac{(y - a)^3}{3!} + \frac{1}{2} \int_y^\infty \varphi(v)(v - a)^3 dv, \quad t \geq 0, x \in \mathbb{R}.
\]

2. The following asymptotic development holds:

\[
\frac{E_0[1_{\Gamma_u \varphi(S_t)}]}{E_0[\varphi(S_t)]} = Q_0^\varphi(\Gamma_u) + \sum_{i=1}^n \frac{1}{\nu} E_0[1_{\Gamma_u F_i^\varphi(X_u, S_u, u)}] + \mathcal{O}\left(\frac{1}{t^{n+1}}\right), \quad t \rightarrow \infty.
\]

Theorem \([11]\) will be proved in Section \([3]\). We also give a complement of Theorem \([1.9]\) (Theorem \([1.3]\) in Section \([3]\)), taking as weight-process: \( \psi(S_t)e^{\lambda(S_t - X_t)} \), with \( \lambda > 0 \).

2 Proof of Theorem \([1.2]\), and of Theorem \([1.1]\), as a consequence

Our proof of Theorem \([1.2]\) is based on the following Lemma.

**Lemma 2.1** Let \( y > 0 \), \( u \geq 0 \), \( \Gamma_u \in \mathcal{F}_u \) and \( t > u \). Then:

\[
P_0(\Gamma_u | S_t = y) = \frac{p_{S_u}(y)}{p_{S_t}(y)} E_0 \left[ 1_{\Gamma_u \left( h(t - u, y - X_u) \mid S_u = y \right)} \right]
\]

\[
+ \frac{1}{p_{S_t}(y)} E_0 \left[ 1_{\Gamma_u \left( 1_{S_u < y} p_{S_t - u} (y - X_u) \right)} \right].
\]

where \( p_{S_r} \) denotes the density function of \( S_r \), for a fixed \( r > 0 \):

\[
p_{S_r}(z) = \sqrt{\frac{2}{\pi r}} e^{-z^2/2r} 1_{\{z > 0\}},
\]

and

\[
h(r, z) = P(S_r < z) = \int_0^z p_{S_r}(x) dx = \sqrt{\frac{2}{\pi r}} \int_0^z e^{-x^2/2r} dx, \quad r, z > 0.
\]

**Proof** of Lemma \([2.1]\) Let \( u \geq 0 \), \( \Gamma_u \in \mathcal{F}_u \) and \( t > u \). It is clear that:

\[
S_t = S_u \vee \left( X_u + \max_{0 \leq v \leq t - u} \{ X_{u+v} - X_u \} \right).
\]

Consequently if \( g : [0, +\infty] \rightarrow [0, +\infty] \) is Borel, applying the Markov property at time \( u \) leads to:

\[
E_0 \left[ 1_{\Gamma_u g(S_t)} \right] = E_0 \left[ 1_{\Gamma_u \tilde{g}(X_u, S_u)} \right],
\]

where

\[
\tilde{g}(x, y) = E_0 \left[ g(y \vee \{ x + S_{t-u} \}) \right], \quad x_+ \leq y.
\]

Then we easily obtain:

\[
\tilde{g}(x, y) = g(y) P_0(S_{t-u} \leq y - x) + E_0 \left[ g(x + S_{t-u}) 1_{\{ S_{t-u} > y - x \}} \right]
\]

\[
= g(y) h(t - u, y - x) + \int_0^\infty g(z) p_{S_{t-u}}(z - x) 1_{\{ z > y \}} dz.
\]

This proves \([2.1]\).
Proof of Theorem 1.2 The two estimates:
\[ p_{S_t}(y) \sim \sqrt{\frac{2}{\pi t}}, \quad h(t, y) \sim y \sqrt{\frac{2}{\pi t}}, \quad t \to \infty \quad (y > 0), \quad (2.5) \]
directly imply that \( Q_{0,t}^{(y)} \) converges weakly to \( \tilde{Q}_0^{(y)} \), as \( t \to \infty \), where:
\[ \tilde{Q}_0^{(y)}(\Gamma_u) = p_{S_u}(y)E_0[1_{\Gamma_u}(y - X_u)|S_u = y] + E_0[1_{\Gamma_u}1\{S_u < y\}], \quad \forall u \geq 0 \text{ and } \Gamma_u \in \mathcal{F}_u. \]
Thanks to (1.3), \( 1 - \Phi(y) = \int_y^\infty \varphi(z)dz, y \geq 0 \), then:
\[ \int_0^\infty \tilde{Q}_0^{(y)}(\Gamma_u)\varphi(y)dy = E_0\left[1_{\Gamma_u}((S_u - X_u)\varphi(S_u) + 1 - \Phi(S_u))\right] = E_0\left[1_{\Gamma_u}M^\varphi_t\right] = Q_0^\varphi(\Gamma_u). \]
Consequently (3.2) implies \( \tilde{Q}_0^{(y)} = Q_0^0, Q_0^\varphi(S_\infty) \in dy \) a.e.

Remark 2.2 It is interesting to point out that (1.3) permits to prove that \( y \mapsto Q_0^{(y)} \) is continuous, as the space of p.m.’s on the canonical space is endowed with the topology of weak convergence.

As indicated in Section 1, we now show how to prove Theorem 1.2, i.e. how to recover (1.4) from Theorem 1.2 and (1.7).
Indeed, let \( \varphi \) be as in Theorem 1.1. We have:
\[ \lim_{t \to \infty} \frac{E_0[1_{\Gamma_u}\varphi(S_t)]}{E_0[\varphi(S_t)]} = \int_0^\infty \frac{Q_0^{(y)}(\Gamma_u)\varphi(y)p_{S_t}(y)dy}{\int_0^\infty \varphi(y)p_{S_t}(y)dy}, \]
where \( u \geq 0, \Gamma_u \in \mathcal{F}_u \) and \( t > u \).
Using Theorem 1.2 (2.5) and the dominated convergence theorem, we get:
\[ \lim_{t \to \infty} \frac{E_0[1_{\Gamma_u}\varphi(S_t)]}{E_0[\varphi(S_t)]} = \frac{\int_0^\infty Q_0^{(y)}(\Gamma_u)\varphi(y)dy}{\int_0^\infty \varphi(y)dy} = \int_0^\infty Q_0^{(y)}(\Gamma_u)\varphi(y)dy. \]

3 Penlalization for long Brownian bridges perturbed by their one-sided maximum

We keep the notation given in Sections 1 and 2.
Let \( Q_{0,t}^\varphi(\Gamma) := E_0[\Gamma_t|X_t = x], \quad \Gamma_t \in \mathcal{F}_t \).
(note the difference with the p.m. \( Q_{0,t}^{(x)} \) defined in (1.8)).
Here, we make a simple remark concerning the weak limit of \( Q_{0,t}^\varphi \) as \( t \to \infty \).
Indeed, we observe that this limit is equal to the Wiener measure \( P_0 \) if \( u \geq 0 \) and \( \Gamma_u \in \mathcal{F}_u \) then:
\[ \lim_{t \to \infty} Q_{0,t}^\varphi(\Gamma_u) = P_0(\Gamma_u), \quad (3.2) \]
which follows from the fact that \( (X_s, 0 \leq s \leq u) \) under \( Q_{0,t}^\varphi \), may be represented as \( (B_s - \frac{s}{t}B_t + \frac{s}{t}x, 0 \leq s \leq u) \), where \( (B_s) \) is a Brownian motion started at 0.
The asymptotic study of long Brownian bridges penalized by their one-sided maximum is more involved; in fact, we determine the weak limit \( Q_{0,t}^{(y)} \) of \( Q_{0,t}^{(y)} \) as \( t \to \infty \), where \( Q_{0,t}^{(y)} \) is the p.m. defined in (1.11). The result is stated in Theorem 1.3.
We proceed as for the proof of Theorem 1.2. We need to generalize Lemma 1.1, taking conditional expectations with respect to \( (S_t, X_t) \).
Lemma 3.1 Let \( a \in \mathbb{R}, y > a_+, u \geq 0, \Gamma_u \in \mathcal{F}_u \) and \( t > u \). Then:
\[
P_0(\Gamma_u | X_t = a, S_t = y) = \frac{p_{s_t}(y)}{p_{X_t,S_t}(a,y)} E_0 \left[ \mathbf{1}_{\Gamma_u} \left( \int_{\mathbb{R}^+} p_{X_{t-u},S_{t-u}}(a - X_u, \xi) 1_{\{\xi < y - X_u\}} d\xi \right) | S_u = y \right] + \frac{1}{p_{X_t,S_t}(a,y)} E_0 \left[ \mathbf{1}_{\Gamma_u} 1_{\{S_u < y\}} p_{X_{t-u},S_{t-u}}(a - X_u, y - X_u) \right].
\tag{3.3}
\]
where \( p_{X_t,S_t} \) denotes the density function of \((X_v, S_v), v > 0:\)
\[
p_{X_t,S_t}(a,y) = \sqrt{\frac{2}{\pi v^3}} (2y - a) e^{-(2y-a)^2/2v} 1_{\{y > a_+\}},
\tag{3.4}
\]

**Proof.** We imitate the proof of Lemma 2.1.

Let \( g : [0, +\infty[ \times [0, +\infty[ \to [0, +\infty[ \) be a Borel function. Thanks to (2.4), we have :
\[
E_0 \left[ \mathbf{1}_{\Gamma_u} g(X_t, S_t) \right] = E_0 \left[ \mathbf{1}_{\Gamma_u} \tilde{g}(X_u, S_u) \right],
\]
where
\[
\tilde{g}(a,y) = E_0 \left[ g(\cdot + X_{t-u}, y \vee \{a + S_{t-u}\}) \right], \quad a_+ \leq y.
\]
It follows :
\[
\tilde{g}(a,y) = \tilde{g}_1(a,y) + \tilde{g}_2(a,y),
\]
with :
\[
\tilde{g}_1(a,y) = E_0 \left[ g(\cdot + X_{t-u}, y \vee \{a + S_{t-u}\}) 1_{\{S_{t-u} \leq y - a\}} \right],
\]
\[
\tilde{g}_2(a,y) = E_0 \left[ g(\cdot + X_{t-u}, a + S_{t-u}) 1_{\{S_{t-u} > y - a\}} \right].
\]
Since :
\[
\tilde{g}_1(a,y) = \int_{\mathbb{R} \times \mathbb{R}^+} g(b,y) p_{X_{t-u},S_{t-u}}(b-a,\xi) 1_{\{\xi < y - a\}} db d\xi,
\]
and
\[
\tilde{g}_2(a,y) = \int_{\mathbb{R} \times \mathbb{R}^+} g(b,z) p_{X_{t-u},S_{t-u}}(b-a,z-a) 1_{\{z > y\}} db dz,
\]
then (3.3) follows immediately.

**Proof of Theorem 1.3** Let \( u \geq 0 \) and \( \Gamma_u \in \mathcal{F}_u \).

1) Using (2.5) and (2.6), and
\[
p_{X_t,S_t}(a,y) \sim \sqrt{\frac{2}{\pi t^3}} (2y - a), \quad t \to \infty, \quad y \geq a_+.
\tag{3.5}
\]
we get :
\[
\lim_{t \to \infty} Q_{0,t}^{a,y}(\Gamma_u) = \frac{p_{s_t}(y)}{2y - a} E_0 \left[ \mathbf{1}_{\Gamma_u} \left( \int_{(a - X_u)_+} (2\xi - a + X_u) d\xi \right) | S_u = y \right] + \frac{1}{2y - a} E_0 \left[ \mathbf{1}_{\Gamma_u} 1_{\{S_u < y\}} (2y - a - X_u) \right].
\]
The first integral in the right-hand side of the previous identity may be computed, which yields :
\[
Q_{0,t}^{a,y}(\Gamma_u) = \frac{y - a}{2y - a} E_0 \left[ \mathbf{1}_{\Gamma_u} (y - X_u) | S_u = y \right] + \frac{1}{2y - a} E_0 \left[ \mathbf{1}_{\Gamma_u} 1_{\{S_u < y\}} (2y - a - X_u) \right].
\]
2) The relations (1.9) and (2.2) imply:

\[ (2y - a)Q_0^{y,a}(\Gamma_u) = (y - a)\{Q_0^{y}(\Gamma_u) - E_0[1_{\Gamma_u \cap \{S_u < y\}}]\} + E_0[1_{\Gamma_u \cap \{S_u < y\}}(2y - a - X_u)] \]

\[ = (y - a)Q_0^{y}(\Gamma_u) + E_0[1_{\Gamma_u \cap \{S_u < y\}}(y - X_u)]. \]

Applying (4.1) with \( \varphi_y = \frac{1}{y}1_{[0,y]} \), we get:

\[ \int_0^y Q_0^{y,z}(\Gamma_u)dz = yE_0[1_{\Gamma_u}M_u^{y,v}]. \]

But \( \Phi_y(z) := \int_0^z \varphi_y(r)dr = \frac{z ∧ y}{y} \), consequently:

\[ M^{y,v}_u = (S_u - X_u)\varphi_y(S_u) + 1 - \Phi_y(S_u) = \frac{y - X_u}{y}1_{\{S_u < y\}}. \]

This proves (1.13).

We now consider the Brownian bridge penalized by a function of its one-sided maximum (cf Theorem 1.3).

**Proof of Theorem 1.6**

Theorem 1.5 is a direct consequence of Theorem 1.3.

Let \( u \geq 0 \) and \( \Gamma_u \in \mathcal{F}_u \) and \( \varphi \) as in Theorem 1.3.

The relations (2.2) and (3.4) imply:

\[ P(S_t \in dy | X_t = a) = \frac{2}{t}e^{-\frac{2(y-a)}{t}}1_{\{y > a\}}dy. \]

Consequently:

\[ E_0[1_{\Gamma_u} \varphi(S_t) | X_t = a] = \frac{2}{t} \int_0^{a+} P_0(\Gamma_u | X_t = a, S_t = y)\varphi(y)(2y - a)e^{-\frac{2(y-a)}{t}}dy. \]

Hence:

\[ \frac{E_0[1_{\Gamma_u} \varphi(S_t) | X_t = a]}{E_0[\varphi(S_t) | X_t = a]} = \frac{\int_0^{a+} P_0(\Gamma_u | X_t = a, S_t = y)\varphi(y)(2y - a)e^{-\frac{2(y-a)}{t}}dy}{\int_0^{a+} \varphi(y)(2y - a)e^{-\frac{2(y-a)}{t}}dy}. \]

Applying Theorem 1.3 and the dominated convergence theorem we get:

\[ Q_0^{a,y}_u(\Gamma_u) = \frac{\int_0^{a+} (2y - a)\varphi(y)Q_0^{y,a}(\Gamma_u)dy}{\int_0^{a+} (2y - a)\varphi(y)dy}. \]

This proves (1.18). As for (1.19), it is a direct consequence of (1.13).

**4 Proof of Theorem 1.6**

1) Point 1. of Theorem 1.6 is a direct consequence of Lemma 3.1 and Theorem 1.3.

Taking the conditional expectation with respect to \((X_t, S_t)\), we obtain:

\[ \frac{E_0[1_{\Gamma_u}f(X_t, S_t)]}{E_0[f(X_t, S_t)]} = \frac{\int_R da \int_0^\infty Q_0^{a,y}(\Gamma_u)p_{X_t,S_t}(a,y)f(a,y)dy}{\int_R da \int_0^\infty p_{X_t,S_t}(a,y)f(a,y)dy}. \] (4.1)
where \( p_{X_t,S_t} \) denotes the density function of \((X_t,S_t)\), as given by (3.4).

Since \( f \) satisfies (1.20), we may apply the dominated convergence theorem; then taking the limit \( t \to \infty \), Theorems 1.3 and (3.5) imply:

\[
\lim_{t \to \infty} \frac{E_0[1_{\Gamma_u} f(X_t,S_t)]}{E_0[f(X_t,S_t)]} := \tilde{Q}_0(\Gamma_u),
\]

where:

\[
\tilde{Q}_0(\Gamma_u) = f^* \int_{\mathbb{R}} da \int_{a+}^{\infty} (2y - a) f(a,y) Q_0^{a,y}(\Gamma_u) dy.
\]

2) We need to identify \( \tilde{Q}_0(\cdot) \).

Let \( \varphi \) be the function defined by (1.21) and \( \Phi(y) = \int_0^y \varphi(z) dz, \ y \geq 0 \).

It is clear that \( \varphi \geq 0 \), then applying Fubini’s theorem, we easily obtain:

\[
\Phi(y) = f^* \left[ \int_{\mathbb{R} \times \mathbb{R}_+} f(a,\eta) 1_{\{\eta > y \lor a+\}} (\eta \land y + (\eta - a) 1_{\{\eta < y\}}) d\eta \right].
\] (4.2)

In particular, taking the limit \( y \to \infty \), we get: \( \lim_{y \to \infty} \Phi(y) = 1 \). This means that \( \varphi \) satisfies (1.3).

Moreover:

\[
1 - \Phi(y) = f^* \left[ \int_{\mathbb{R} \times \mathbb{R}_+} f(a,\eta) 1_{\{\eta > y \lor a+\}} (2\eta - a - y) d\eta \right].
\] (4.3)

Applying identity (1.13), we get:

\[
\tilde{Q}_0(\cdot) = f^* \int_{\mathbb{R}} da \left( \int_{a+}^{\infty} (y - a) Q_0^{a,y} (\cdot) dy + \int_0^y Q_0^{(\eta)} (\cdot) d\eta \right) f(a, y) dy
\]

\[
= f^* \int_{\mathbb{R} \times \mathbb{R}_+} \left\{ (\eta - a) f(a,\eta) 1_{\{\eta > a+\}} + \int_{\eta \lor a+}^{\infty} f(a, y) dy Q_0^{(\eta)} (\cdot) d\eta \right\}
\]

\[
= \int_0^{\infty} \varphi(\eta) Q_0^{(\eta)} (\cdot) d\eta.
\]

Property (1.7) implies: \( \tilde{Q}_0 = Q_0^\varphi \).

3) It remains to prove (1.23).

Let \( \tilde{M}_t \) be the process defined as the right-hand side of (1.23):

\[
\tilde{M}_t = f^* \int_{\mathbb{R}} db \int_{b+}^{\infty} (2y - b) f(b + X_t, S_t \lor (y + X_t)) dy.
\]

Setting: \( a = b + X_t \) and \( \eta = y + X_t \), we obtain:

\[
\tilde{M}_t = f^* \int_{\mathbb{R}} \tilde{\psi}_t(a) da,
\]

where:

\[
\tilde{\psi}_t(a) = \int_{\mathbb{R}} (2\eta - X_t - a) f(a, S_t \lor \eta) 1_{\{\eta > a, \eta > X_t\}} d\eta.
\]

We have:

\[
\tilde{\psi}_t(a) = 1_{\{S_t > a\}} f(a, S_t) \int_{a \lor X_t}^{S_t} (2\eta - X_t - a) d\eta + \int_{\mathbb{R}} (2\eta - X_t - a) f(a, \eta) 1_{\{\eta > S_t \lor a\}} d\eta
\]

\[
= 1_{\{S_t > a\}} f(a, S_t) \left( (S_t - a \lor X_t)(S_t - a \lor X_t - X_t - a) + \int_{\mathbb{R}} (2\eta - X_t - a) f(a, \eta) 1_{\{\eta > S_t \lor a\}} d\eta \right)
\]

\[
= 1_{\{S_t > a\}} f(a, S_t) (S_t - X_t) ((S_t - a) + \int_{\mathbb{R}} (2\eta - X_t - a) f(a, \eta) 1_{\{\eta > S_t \lor a\}} d\eta.
\]
Suppose that $\mu > P_2$. In our approach it is convenient to introduce
Consequently if this condition holds, then Theorem 1.6 applies.

Since $M_t^\varphi = (S_t - X_t)\varphi(S_t) + 1 - \Phi(S_t)$, using (1.21) and (4.3), we get:
\[
M_t^\varphi = f^* \int_R \psi_t(a) da,
\]
where:
\[
\psi_t(a) = (S_t - X_t) \left[ \int_R f(a, \eta) 1_{\eta > S_t \vee a+} d\eta + f(a, S_t)(S_t - a) 1_{a < S_t} \right] + \int_R f(a, \eta)(2\eta - a - S_t) 1_{\eta > S_t \vee a+} d\eta.
\]

It is now clear that $\psi_t(a) = \tilde{\psi}_t(a)$. Consequently $M_t^\varphi = \tilde{M}_t$.

**Remark 4.1**
1. Let $f$ be of the type: $f(a, y) = f_1(a) 1_{[0, A]}(y)$, where $A > 0$ and $f_1 : ]-\infty, A] \mapsto \mathbb{R}_+$ satisfies $\int_{-\infty}^A (1 + |a|) f_1(a) da < \infty$. Then it is easy to check that $\tilde{f} = A \int_{-\infty}^A (A - a) f_1(a) da < \infty$, and $\varphi(y) = \frac{1}{A} 1_{[0, A]}(y)$.

2. It is possible to recover the identity (1.13) from Theorem 1.6. Let $f$ as in Theorem 4.1. Using the first part of the proof of Theorem 1.6, (1.3) and (1.2A), we have:
\[
f^* \int_R da \int_{a+}^\infty (2y - a) f(a, y) Q_0^{a,y}(\cdot) dy = Q_0^\varphi(\cdot) = \int_0^\infty \varphi(y) Q_0^{(y)}(\cdot) dy,
\]
\[
\int_R da \int_{a+}^\infty (2y - a) f(a, y) Q_0^{a,y}(\cdot) dy = \int_0^\infty Q_0^{(y)}(\cdot) dy \left\{ \int_R da \int_{y \vee a+}^\infty f(a, \eta) d\eta + f(a, y)(y-a) 1_{a < y} \right\}.
\]

Using Fubini’s theorem, we easily obtain:
\[
\int_R da \int_{a+}^\infty (2y - a) f(a, y) Q_0^{a,y}(\cdot) dy = \int_R da \int_{a+}^\infty f(a, y) \left[ y-a + \int_y^\infty Q_0^{(y)}(\cdot) dy \right] dy,
\]
for any non-negative function $f$, satisfying (1.2A), but an easy application of Beppo-Levi theorem shows that (1.4) holds even without (1.2A) being satisfied.

## 5 Penalization with $e^{\lambda S_t + \mu X_t}$

1) In this section we focus on penalizations with weight-processes $f(X_t, S_t)$, where the function $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ belongs to the family $\{ f_{\lambda, \mu} : f_{\lambda, \mu}(a, y) = e^{\lambda y + \mu a}, \lambda, \mu \in \mathbb{R} \}$.

First, let us determine under which condition $f_{\lambda, \mu}$ satisfies (1.2A).

Using the Fubini theorem, we have:
\[
\mathcal{J}_{\lambda, \mu} = \int_0^\infty e^{\lambda y} dy \int_{-\infty}^y (2y - a) e^{\mu a} da.
\]

Consequently if $\mu \leq 0$ then $\mathcal{J}_{\lambda, \mu} = \infty$.

Suppose that $\mu > 0$. The integral with respect to $da$ may be computed, this yields to:
\[
\mathcal{J}_{\lambda, \mu} = \frac{1}{\mu^2} \int_0^\infty (1 + \mu y) e^{(\lambda + \mu)y} dy.
\]

As a result:
\[
\mathcal{J}_{\lambda, \mu} < \infty \iff \mu > 0 \text{ and } \lambda + \mu < 0.
\]

Consequently if this condition holds, then Theorem 1.4 applies.

2) In our approach it is convenient to introduce $P_\mu$, the law of Brownian motion with drift $\mu$, starting at 0, and $P_0^{(3)}$ the law of a three dimensional Bessel process started at 0.

Recall Pitman’s theorem (6, 8):
1. under $P_0$, the process $((2S_t - X_t, S_t), t \geq 0)$ is distributed as $((X_t, J_t), t \geq 0)$ under $P_0^{(3)}$, where $J_t = \inf_{u \geq t} X_u$.

2. let $(R_t)$ be the natural filtration associated with the process $(R_t = 2S_t - X_t, t \geq 0)$, then :

$$E_0[f(S_t)|R_t] = \frac{1}{R_t} \int_0^{R_t} f(u)du,$$

(5.2)
for any Borel function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

Pitman’s theorem has been extended to the case of Brownian motion with drift. From [3] (see also [3]), we know that $(2S_t - X_t, t \geq 0)$ is a diffusion with generator :

$$\frac{1}{2} \frac{d^2}{dx^2} + \mu \coth(\mu x) \frac{d}{dx},$$

(5.3)

**Proof of Theorem 1.7**

Let $u$ be a fixed positive real number, $\Gamma_u \in \mathcal{F}_u$, and define :

$$\Delta(\Gamma_u, t) := E_0[1_{\Gamma_u} e^{\lambda X_t + \mu X_u}].$$

1) First suppose that $(\lambda, \mu)$ belongs to $R_1$. We have already proved that if $\mu > 0$ then Theorem 1.7 is a direct consequence of Theorem 1.6. If $\mu = 0$ and $\lambda + \mu = \lambda < 0$, then Theorem 1.7 follows from Theorem 1.1.

2) We now investigate the last case : $(\lambda, \mu) \in R_3$.

If $\mu = 0$, then $\lambda < 0$ and Theorem 1.7 is a direct consequence of Theorem 1.1.

We suppose, in the sequel $\mu < 0$. We write $\Delta(\Gamma_u, t)$ as follows :

$$\Delta(\Gamma_u, t) = e^{\mu t/2} E_0^\mu[1_{\Gamma_u} e^{\lambda X_t}].$$

Applying the Markov property at time $u$, we get :

$$\Delta(\Gamma_u, t) = e^{\mu t/2} E_0^\mu[1_{\Gamma_u} h(X_u, S_u, t - u)],$$

(5.4)
where

$$h(a, y, r) = E_0^\mu[e^{(y+\lambda + a_S_t)}], \quad y \geq a_+, r \geq 0.$$

Since $\mu < 0$, it is well-known that, under $P_0^\mu$, $X_t \rightarrow -\infty$ as $t \rightarrow \infty$, $S_\infty < \infty$ and $P_0^\mu(S_\infty > x) = e^{2\mu x}, x \geq 0$.

Consequently :

$$\lim_{r \rightarrow \infty} h(a, y, r) = I := -2\mu \int_0^\infty e^{\lambda(y+\lambda + a_S_t)} e^{2\mu z} dz.$$

Obviously the above integral may be computed explicitly :

$$I = -2\mu \left[e^{\lambda y} \int_0^{y-a} e^{2\mu z} dz + e^{\lambda a} \int_{y-a}^\infty e^{(\lambda+2\mu)z} dz\right]$$

$$= e^{\lambda y} \left[1 - e^{2\mu(y-a)} + \frac{2\mu}{\lambda + 2\mu} e^{2\mu(y-a)}\right] = e^{\lambda y} \left[1 - \frac{\lambda}{\lambda + 2\mu} e^{2\mu(y-a)}\right].$$

Moreover, it is easy to check :

$$e^{(\lambda+\mu)y-\mu a} \left[\cosh (\mu(y - a)) - \frac{\lambda + \mu}{\mu} \sinh (\mu(y - a))\right] = \frac{\lambda + 2\mu}{2\mu} e^{\lambda y} \left[1 - \frac{\lambda}{\lambda + 2\mu} e^{2\mu(y-a)}\right].$$

Finally :

$$\lim_{r \rightarrow \infty} h(a, y, r) = \frac{2\mu}{\lambda + 2\mu} e^{(\lambda+\mu)y-\mu a} \left[\cosh (\mu(y - a)) - \frac{\lambda + \mu}{\mu} \sinh (\mu(y - a))\right].$$
Coming back to (5.4), we obtain:
\[
\lim_{t \to \infty} \frac{E_0[1_{\Gamma_u} e^{\mu X_t + \lambda S_t}]}{E_0[e^{\mu X_t + \lambda S_t}]} = \lim_{t \to \infty} \frac{E_0^\mu[1_{\Gamma_u} h(X_u, S_u, t - u)]}{E_0^\mu[h(0, 0, t)]} = E_0^\mu[1_{\Gamma_u} e^{(\lambda + \mu)S_u - \mu X_u} \{ \cosh (\mu(S_u - X_u)) - \frac{\lambda + \mu}{\mu} \sinh (\mu(S_u - X_u)) \}] = E_0[1_{\Gamma_u} e^{\mu X_u + \lambda S_u}].
\]

3) Let \((\lambda, \mu)\) be an element of \(R_2\).

a) Let us start with the additional assumption: \(\lambda + 2\mu > 0\). Since
\[
P_0^{\lambda+\mu} = e^{(\lambda+\mu)X_t - (\lambda+\mu)^2t/2}P_0 \quad \text{on} \quad \mathcal{F}_t,
\]
we have:
\[
\Delta(\Gamma_u, t) = e^{(\lambda+\mu)^2t/2}E_0^{\lambda+\mu}[1_{\Gamma_u} e^{\lambda(S_t - X_t)}].
\]

Recall Theorem 1.1 in [5]: under \(P_0^{\lambda+\mu}\), the process \((S_t - X_t; t \geq 0)\) is distributed as \(\{(Y_t), t \geq 0\}\), where \((Y_t)\) is the so-called bang-bang process with parameter \(\lambda + \mu\), i.e., the diffusion with infinitesimal generator:
\[
\frac{1}{2}d^2 - (\lambda + \mu)\text{sgn}(x)\frac{d}{dx}.
\]

Applying the Markov property at time \(u\) in (5.5), yields to:
\[
\Delta(\Gamma_u, t) = e^{(\lambda+\mu)^2t/2}E_0^{\lambda+\mu}[1_{\Gamma_u} e^{\lambda Y_{t-u}}],
\]

where \(P_x\) denotes a p.m. under which \((Y_t)\) is the diffusion process with generator (5.6) starting at \(x\). Under \(P_x\), \((Y_t)\) is a recurrent diffusion and \(\nu(dx) := (\lambda + \mu)e^{-2(\lambda+\mu)|x|}dx\) is its invariant p.m.

Consequently, for any \(x \in \mathbb{R}\),
\[
\lim_{r \to -\infty} E_x[\lambda Y_{t-u}] = (\lambda + \mu) \int_{-\infty}^0 e^{\lambda |y|}e^{-2(\lambda+\mu)|y|}dy = \frac{2(\lambda + \mu)}{\lambda + 2\mu}.
\]

Since \(\lambda + 2\mu > 0\) and \((\lambda, \mu) \in R_2\), then the integral in the right-hand side is finite and does not depend on \(x\). As a result:
\[
\lim_{t \to \infty} \frac{E_0[1_{\Gamma_u} e^{\mu X_t + \lambda S_t}]}{E_0[e^{\mu X_t + \lambda S_t}]} = \lim_{t \to \infty} \frac{E_0^{\lambda+\mu}[1_{\Gamma_u} E_{S_u-X_u} \{e^{\lambda|Y_{t-u}|}\}]}{E_0[e^{\lambda|Y_{t-u}|}]} = P_0^{\lambda+\mu}(\Gamma_u).
\]

Let us deal with the case \(\lambda + 2\mu = 0, \mu \neq 0\). Applying (5.5), we have:
\[
\Delta(\Gamma_u, t) = e^{\mu^2t/2}E_0^{-\mu}[1_{\Gamma_u} e^{-\mu(2S_t - X_t)}].
\]

The result follows from Pitman’s theorem (for Brownian motion with drift).

b) It remains to study the case: \(\lambda + 2\mu = 0\) and \(\lambda > 0\).

i) To begin with, we modify \(\Delta(\Gamma_u, t), \lambda\) and \(\mu\) being for now two real numbers, without restriction.

Applying the Markov property at time \(u\) leads to:
\[
\Delta(\Gamma_u, t) = E_0[1_{\Gamma_u} g(X_u, S_u, t - u)],
\]

where
\[
g(a, y, r) = E_0[\lambda Y_{y\vee(a+S_r)} + \mu(a+X_r)], \quad y \geq a, r \geq 0.
\]

Obviously, \(g(a, y, r)\) may be decomposed as follows:
\[
g(a, y, r) = e^{\lambda y + \mu a}g_1(a, y, r) + e^{(\lambda+\mu)a}g_2(a, y, r),
\]

(5.9)
We need to determine the asymptotic behaviour of $g$. Using (5.12) and (5.13) we have:
\[
\lambda ii) We suppose now that $b$
\]
Setting $\lambda$
\[
\mu \]
As a result:
\[
\mu \]
It turns out that if $\mu < 0$
\[
\mu \]
ii) We suppose now that $\lambda = -2\mu > 0$.
We need to determine the asymptotic behaviour of $g_2(a, y, r)$ as $r \to \infty$.
Using (5.12) and (5.13) we have:
\[
g_2(a, y, r) = E_0^0 \left[ e^{-\mu X_r} \mathbb{1}_{\{X_r \geq y-a\}} \right] = E_0^0 \left[ e^{-\mu X_r} \frac{X_r - y + a}{X_r} \right] = \sqrt{\frac{2}{\pi r}} \int_0^\infty z(z - y + a) e^{-\mu z - z^2/2r} dz
\]
\[
\mu \]
As a result:
\[
g_2(a, y, r) \sim 2\mu^2 r e^{a^2 r/2}, \quad r \to \infty.
\]
Due to (5.8), (5.11), (5.14), (5.15) and $\lambda = -2\mu$, we get:
\[
\Delta(\Gamma_u, t) \sim 2\mu^2 t e^{a^2 t/2} E_0 \left[ e^{-\mu X_u - \mu^2 u/2} \right], \quad t \to \infty.
\]
Proposition 5.2 \(\Delta(\Omega,t) = E_0\left[e^{-\mu X_t + \lambda S_t}\right] \sim 2\mu^2 t e^{\mu^2 t/2}, \ t \to \infty.\)

Finally:
\[
\lim_{t \to \infty} \frac{E_0\left[1_{\Gamma_u} e^{\mu X_t + \lambda S_t}\right]}{E_0\left[e^{\mu X_t + \lambda S_t}\right]} = E_0\left[1_{\Gamma_u} e^{-\mu X_u - \mu^2 u/2}\right].
\]

Remark 5.1 Here is another proof of Theorem \(1.7\) : keeping the notations introduced in point 3) b) i) of the proof above, recall that we have proved:
\[
E_0\left[1_{\Gamma_u} e^{\mu X_t + \lambda S_t}\right] = E_0\left[1_{\Gamma_u} \left\{ e^{\mu X_u + \lambda S_u} g_1(X_u, S_u, t - u) + e^{(\lambda + \mu) X_u} g_2(X_u, S_u, t - u) \right\}\right],
\]
where the functions \(g_1(a,y,r)\) and \(g_2(a,y,r)\) are given by (5.14) or (5.14) and (5.12). In our proof of Theorem \(1.7\) we only need the asymptotics of \(g_i(a,y,r)\) as \(r \to \infty, \ i = 1,2\) in the case \(\lambda + 2\mu = 0, \lambda > 0\). It is actually possible to determine the asymptotics of the previous quantities in any case. However tedious calculations are needed, this explains why we have given a short and direct proof of Theorem \(1.7\).

We now give a direct interpretation of Theorem \(1.7\) in terms of the three dimensional Bessel process and its post-minimum.

Proposition 5.2 Let \(\lambda, \mu \in \mathbb{R}\).

1. For every \(u \geq 0\), and \(\Gamma_u\) in \(\mathcal{F}_u\),
\[
\lim_{t \to \infty} \frac{E_0^{(3)}\left[1_{\Gamma_u} e^{\mu X_t + \lambda J_t}\right]}{E_0^{(3)}\left[e^{\mu X_t + \lambda J_t}\right]} := E_0^{(3)}\left[1_{\Gamma_u} \overline{M}_{u}^{\mu,\lambda}\right],
\]
where \((\overline{M}_u^{\mu,\lambda})\) is the positive \((\mathcal{F}_u, \mathcal{P}_0^{(3)})\) martingale:
\[
\overline{M}_u^{\mu,\lambda} = \begin{cases} 
1 & \text{if } \lambda + \mu < 0 \text{ and } \mu \leq 0, \\
\frac{e^{-(\lambda + \mu)u/2} \sinh \left( (\lambda + \mu) X_u \right)}{(\lambda + \mu) X_u} & \text{if } \lambda \geq 0 \text{ and } \lambda + \mu \geq 0, \\
\frac{e^{-\mu u/2} \sin \left( (\lambda + \mu) X_u \right)}{(\lambda + \mu) X_u} & \text{if } \lambda < 0 \text{ and } \mu > 0,
\end{cases}
\]
Note that \(\overline{M}_0^{\mu,\lambda} = 1\).

2. The map \(\Gamma_u(\in \mathcal{F}_u) \leftrightarrow E_0^{(3)}\left[1_{\Gamma_u} \overline{M}_{u}^{\mu,\lambda}\right]\) induces a p.m. on \((\Omega, \mathcal{F}_\infty)\).

Proof. Proposition 5.3 is a direct consequence of Theorem \(1.7\) and Pitman’s theorem. We have:
\[
\frac{E_0^{(3)}\left[1_{\Gamma_u} e^{\mu X_t + \lambda J_t}\right]}{E_0^{(3)}\left[e^{\mu X_t + \lambda J_t}\right]} = \frac{E_0\left[1_{\tilde{\Gamma}_u} e^{-\mu X_t + (\lambda + 2\mu) S_t}\right]}{E_0\left[e^{-\mu X_t + (\lambda + 2\mu) S_t}\right]},
\]
where \(\tilde{\Gamma}_u := \{ \omega \in \Omega; \tilde{\omega} \in \Gamma_u \}\), and \(\tilde{\omega}_t := \sup_{0 \leq u \leq t} \omega(u) - \omega(t)\).

Applying our Theorem \(1.7\), we obtain:
\[
\lim_{t \to \infty} \frac{E_0^{(3)}\left[1_{\Gamma_u} e^{\mu X_t + \lambda J_t}\right]}{E_0^{(3)}\left[e^{\mu X_t + \lambda J_t}\right]} = E_0\left[1_{\tilde{\Gamma}_u} \overline{M}_{u}^{-\mu,\lambda + 2\mu}\right].
\]
Since \( \hat{\Gamma}_u \in \mathcal{R}_u \) then :

\[
E_0[1_{\hat{\Gamma}_u} M_u^{-\mu, \lambda+2\mu}] = E_0[1_{\hat{\Gamma}_u} E_0[M_u^{-\mu, \lambda+2\mu} | \mathcal{R}_u]].
\]

We claim that :

\[
E_0[M_u^{-\mu, \lambda+2\mu} | \mathcal{R}_u] = \begin{cases} 
1 & \text{if } \lambda + \mu < 0 \text{ and } \mu \leq 0, \\
e^{-(\lambda+\mu)^2u/2} \sinh \left( \frac{(\lambda + \mu)(2S_u - X_u)}{(\lambda + \mu)(2S_u - X_u)} \right) & \text{if } \lambda \geq 0 \text{ and } \lambda + \mu \geq 0, \quad (5.18) \\
e^{-\mu^2u/2} \sinh \left( \frac{(2S_u - X_u)}{\mu(2S_u - X_u)} \right) & \text{if } \lambda < 0 \text{ and } \mu > 0,
\end{cases}
\]

Making again use of Pitman’s theorem, it is immediate to obtain (5.17).

As for (5.18), we only prove the third case. The two other cases may be proved similarly. Note that 
\((\lambda + 2\mu, -\mu) \in R_1\) (resp. \(R_2\)) iff \(\lambda + \mu < 0\) and \(\mu \leq 0\) (resp. \(\lambda \geq 0\) and \(\lambda + \mu \geq 0\)).

As for the third case, we have : \((\lambda + 2\mu, -\mu) \in R_3\) iff \(\lambda < 0\) and \(\mu < 0\).

Setting \(R_u := 2S_u - X_u\), then (3.30) and (5.2) imply :

\[
M_u^{-\mu, \lambda+2\mu} = e^{(\lambda+\mu)S-u-\mu^2u/2} \left[ \cosh (\mu(R_u - S_u)) - \frac{\lambda + \mu}{\mu} \sinh (\mu(R_u - S_u)) \right],
\]

\[
E_0[M_u^{-\mu, \lambda+2\mu} | \mathcal{R}_u] = \frac{e^{-\mu^2u/2}}{R_u} \int_0^{R_u} e^{(\lambda+\mu)y} \left[ \cosh (\mu(R_u - y)) - \frac{\lambda + \mu}{\mu} \sinh (\mu(R_u - y)) \right] dy
\]

\[
= \frac{e^{-\mu^2u/2}}{R_u} \left[ - \frac{1}{\mu} e^{(\lambda+\mu)y} \sinh (\mu(R_u - y)) \right]_{y=0}^{y=R_u}
\]

\[
= e^{-\mu^2u/2} \frac{\sinh (\mu R_u)}{\mu R_u}.
\]

This establishes the third case in (5.17), using again Pitman’s theorem.

\[ \square \]

**Remark 5.3** It seems natural to ask for :

\[
\lim_{t \to \infty} \frac{E_0^{(3)}[1_{\hat{\Gamma}_t} f(X_t, J_t)]}{E_0^{(3)}[f(X_t, J_t)]}. \tag{5.19}
\]

for some suitable Borel \( f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \).

Using Pitman’s theorem (see 1. in the proof of Proposition [5.4]), the above ratio is equal to :

\[
\frac{E_0[1_{\hat{\Gamma}_t} f(2S_t - X_t, S_t)]}{E_0[f(2S_t - X_t, S_t)]}.
\]

Consequently Theorem 1.6 applies as soon as :

\[
\tilde{f} := \int_{\mathbb{R}} da \int_{a_+}^{\infty} (2y - a)f(2y - a, y)dy = \int_{\mathbb{R}_+ \times \mathbb{R}_+} f(b, y)1_{\{b > y\}} dbdy < \infty. \tag{5.20}
\]

Suppose that this condition holds. Then

\[
\lim_{t \to \infty} \frac{E_0^{(3)}[1_{\hat{\Gamma}_t} f(X_t, J_t)]}{E_0^{(3)}[f(X_t, J_t)]} = E_0[1_{\hat{\Gamma}_u} M_u^u],
\]

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with
\[ \varphi(y) = f^1 \left[ \int_{\mathbb{R}_+ \times \mathbb{R}_+} f(b, \eta) 1_{\{b > \eta > y\}} dbd\eta + \int_y^\infty f(b, y) db \right], \]

and \( f^1 = 1/\tilde{f} \).

Proceeding as in the proof of Proposition 5.2 we may prove:

\[ E_0 [M_{\gamma} u | R u] = 1. \] (5.21)

Finally the limit in (5.19) equals \( P_0^{(3)}(\Gamma_u) \). In other words the penalization with \( f(X_t, J_t) \), \( f \) satisfying (5.20) does not generate a new p.m.

As an end to this section, we would like to discuss the relationship between Theorem 1.7 and the results obtained in [3]. Recall that these authors have proved that

\[ \lim_{t \to \infty} E_{\mu} [1_{\Gamma_u A_{t/2}} e^{\mu X_t} A_{t/2}] = E_0 [1_{\Gamma_u} e^{\mu X_t} A_{t/2}], \] (5.22)

exists where \( \mu, \lambda \in \mathbb{R}, P_{\mu}^{(0)} \) denotes the p.m. on canonical space which makes \((X_t)\) a Brownian motion with drift \( \mu \), started at 0, and:

\[ A_t = \int_0^t e^{2X_s} ds, \quad t \geq 0. \]

As for our Theorem 1.7, it is proved in [3], that a phase transition phenomenon occurs: there exists three disjoint regions in \( \mathbb{R} \times \mathbb{R} \) associated with three types of limit distributions in (5.22). It is striking to note that these regions coincide with the domains \( R_1, R_2 \) and \( R_3 \) introduced in (1.26)-(1.28).

We have actually no proof of this fact. Nevertheless if \( \lambda > 0 \), we have a heuristic argument:

\[ \frac{E_{\mu} [1_{\Gamma_u} A_{t/2}^\lambda]}{E_0 [A_{t/2}^\lambda]} = \frac{E_0 [1_{\Gamma_u} e^{\mu X_t} A_{t/2}^\lambda]}{E_0 [e^{\mu X_t} A_{t/2}^\lambda]}. \] (5.22)

Roughly speaking, the Laplace theorem tells us that \( A_t = \int_0^t e^{2X_s} ds \) has the same behaviour as \( e^{2S_t} \), see more precisely, the limit results in [2] (formulae (61) and (62) p 181) and [3]. Therefore replacing formally \( A_t \) by \( e^{2S_t} \), we get:

\[ \frac{E_0 [1_{\Gamma_u} e^{\mu X_t} A_{t/2}^\lambda]}{E_0 [e^{\mu X_t} A_{t/2}^\lambda]} \approx \frac{E_0 [1_{\Gamma_u} e^{\mu X_t + \lambda S_t}]}{E_0 [e^{\mu X_t + \lambda S_t}]}, \quad t \to \infty. \]

6 Asymptotic development

We first recall a penalization result obtained in ([13]), choosing as weight-process: \( \psi(S_t) e^{\lambda(S_t - X_t)} \), where \( \lambda > 0 \).

Let us start with some notations. Let \( \psi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) be a Borel function satisfying:

\[ \int_0^\infty \psi(z) e^{-\lambda z} dz = 1. \] (6.1)

To \( \psi \) we associate the functions \( \Phi \) and \( \varphi \):

\[ \Phi(y) := 1 - e^{\lambda y} \int_y^\infty \psi(z) e^{-\lambda z} dz, \quad y \geq 0, \] (6.2)

\[ \varphi(y) := \Phi'(y) = \psi(y) - \lambda e^{\lambda y} \int_y^\infty \psi(z) e^{-\lambda z} dz, \quad y \geq 0. \] (6.3)
Then :

\[ M_{t}^{\lambda, \varphi} := \left\{ \psi(S_t) \frac{\sinh(\lambda(S_t - X_t))}{\lambda} + e^{\lambda X_t} \int_{S_t}^{\infty} \psi(z)e^{-\lambda z}dz \right\} e^{-\lambda^2 t/2} \quad t \geq 0, \]

(6.4)
is a \((\mathcal{F}_t), P_0\) positive, and continuous martingale.

In this setting we have proved (see Theorem 3.9 in [15]).

**Proposition 6.1** Let \(\psi : \mathbb{R}_+ \mapsto \mathbb{R}\) be a Borel function satisfying (6.4). Then :

\[
\lim_{t \to \infty} \frac{E_0[1_{\Gamma_u}(S_t) e^{\lambda(S_t - X_t)}]}{E_0[\psi(S_t) e^{\lambda(S_t - X_t)}]} = E_0[1_{\Gamma_u} M_{u}^{\lambda, \varphi}],
\]

(6.5)

for any \(u \geq 0\) and \(\Gamma_u \in \mathcal{F}_u\).

**Remark 6.2**

1. In [13], we have determined the law of \(u\) for any \(t\) in \(\mathbb{R}\).

2. If we take \(\lambda = 0\) and \(\psi = \varphi\), then (6.3) holds, (6.4) corresponds to (1.3) and \((M_{t}^{\lambda, \varphi})\) coincides with the martingale \((M_{t}^{\varphi})\) defined by (1.3). With these conventions, Proposition 6.1 is an extension of points 1. and 2. of Theorem 1.4.

The aim of this section is to prove that, under suitable assumptions, we can obtain an asymptotic expansion of \(t \mapsto \frac{E_0[1_{\Gamma_u}(S_t) e^{\lambda(S_t - X_t)}]}{E_0[\psi(S_t) e^{\lambda(S_t - X_t)}]}\) as \(t \to \infty\). Note that Proposition 6.1 gives the first term.

**Theorem 6.3** Let \(\psi : \mathbb{R}_+ \mapsto \mathbb{R}\) satisfying (6.4). We suppose that there exists an integer \(n \geq 1\) such that :

\[
\int_{0}^{\infty} \psi(y)(1 + y^n)dy < \infty.
\]

(6.6)

1. There exists a family of functions \((F_{i}^{\lambda, \varphi})_{1 \leq i \leq n}\), \(F_{i}^{\lambda, \varphi} : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}\), such that

(a) \((F_{i}^{\lambda, \varphi}(X_t, S_t), t \geq 0)\) is a \((\mathcal{F}_t), P_0\)-martingale, for any \(1 \leq i \leq n\),

(b) If \(i = 1\), we have :

\[
F_{1}^{\lambda, \varphi}(X_t, S_t, t) = \frac{c(\lambda, \varphi)}{\lambda^3 \sqrt{2\pi}} (M_{t}^{\varphi} - M_{t}^{\lambda, \varphi}).
\]

(6.7)

where \(c(\lambda, \varphi) := \int_{0}^{\infty} \psi(x)(1 - \lambda x)dx\) and

\[
\varphi_1(y) := \frac{1}{c(\lambda, \varphi)} \left( \psi(y) - \lambda \int_{y}^{\infty} \psi(x)dx \right), \quad y \geq 0.
\]

(6.8)

(Note that \(\int_{0}^{\infty} \varphi_1(y)dy = 1\).)

2. The following asymptotic development as \(t \to \infty\), holds :

\[
\frac{E_0[1_{\Gamma_u}(S_t) e^{\lambda(S_t - X_t)}]}{E_0[\psi(S_t) e^{\lambda(S_t - X_t)}]} = \frac{1}{t} E_0[1_{\Gamma_u} M_{u}^{\lambda, \varphi}] + e^{-\lambda^2 t/2} \left( \sum_{i=1}^{n} \frac{1}{t^i} E_0[1_{\Gamma_u} F_{i}^{\lambda, \varphi}(X_u, S_u, u)] + O\left( \frac{1}{t^{n+1}} \right) \right).
\]

(6.9)
Note that the two asymptotic expansions (1.38) and (6.9) are drastically different, depending on whether \( \lambda > 0 \) or \( \lambda = 0 \). We have already observed in Remark 6.2, that taking formally \( \lambda = 0 \) in (6.3) gives (1.4). In other words the first term in (6.9) (with \( \lambda = 0 \)) coincides with the first term in (1.38). However the expansion is expressed in terms of powers \( t^{-(i+1/2)} \) instead of \( t^{-i/2} \).

**Proof of Theorems 1.9 and 6.3**

1) Let us start with some common features concerning the two cases \( \lambda > 0 \) and \( \lambda = 0 \), i.e. \( \lambda \geq 0 \). We adopt the convention that \( \psi = \varphi \) if \( \lambda = 0 \).

Let \( u \) be a fixed positive real number, \( \Gamma_u \in \mathcal{F}_u \), and

\[
\Delta(\lambda, \Gamma_u, t) := E_0[1_{\Gamma_u} \psi(S_t)e^{\lambda(S_t-X_t)}],
\]

where \( \lambda \geq 0 \).

Applying the Markov property at time \( u \), we get :

\[
\Delta(\lambda, \Gamma_u, t) = E_0[1_{\Gamma_u} g(\lambda, X_u, S_u, t-u)],
\]

where

\[
g(\lambda, a, y, r) := E_0[\psi(y \vee (a+S_r))e^{\lambda(y\vee(a+S_r))} - a - X_r]], \quad r \geq 0, y \geq a, a \in \mathbb{R}.
\]

We have :

\[
g(\lambda, a, y, r) = \psi(y)e^{\lambda(y-a)}E_0[e^{-\lambda(2J_r-x_r)}1_{\{J_r<y-a\}}] + E_0[\psi(a+S_r)e^{\lambda(S_r-x_r)}1_{\{S_r>y-a\}}].
\]

Using Pitman’s theorem, we get :

\[
g(\lambda, a, y, r) = \psi(y)e^{\lambda(y-a)}E_0^{(3)}[e^{-\lambda(2J_r-x_r)}1_{\{J_r<y-a\}}] + E_0^{(3)}[\psi(a+J_r)e^{\lambda(J_r-x_r)}1_{\{J_r>y-a\}}]
\]

\[
= E_0^{(3)}\left[\frac{e^{\lambda x_r}}{x_r} \int_0^{X_r} \left\{ \psi(y)e^{\lambda(y-a)}e^{-2\lambda z}1_{\{z<y-a\}} + \psi(a+z)e^{-\lambda z}1_{\{z>y-a\}} \right\} dz \right]
\]

\[
= \psi(y)e^{\lambda(y-a)}\int_{y-a}^\infty e^{-2\lambda z}h(\lambda, z, r)dz + \int_{y-a}^\infty e^{-\lambda z}h(\lambda, z, r)dz,
\]

where :

\[
h(\lambda, z, r) = E_0^{(3)}\left[\frac{e^{\lambda x_r}}{x_r}1_{\{x_r>z\}}\right].
\]

Applying (5.13), we get :

\[
h(\lambda, z, r) = \sqrt{\frac{2}{\pi r^3}}e^{\lambda^2r/2}\int_z^\infty be^{-(b-\lambda r)^2/2r}db = \sqrt{\frac{2}{\pi r}}e^{\lambda^2r/2}\int_{\frac{\lambda r}{\sqrt{\lambda}}}^\infty \frac{e^{-v^2/2}}{\sqrt{\lambda}}(\lambda\sqrt{r} + v)e^{-v^2/2}dv.
\]

2) Suppose that \( \lambda = 0 \). Therefore we replace in the sequel \( \psi \) by \( \varphi \).

a) Then :

\[
h(0, z, r) = \sqrt{\frac{2}{\pi r^3}}\int_z^\infty be^{-b^2/2r}db = \sqrt{\frac{2}{\pi r}}e^{-z^2/2r},
\]

\[
g(0, a, y, r) = \sqrt{\frac{2}{\pi r}}\tilde{g}(0, a, y, r),
\]

with :

\[
\tilde{g}(0, a, y, r) = \varphi(y)\int_0^{y-a} e^{-z^2/2r}dz + \int_{y}^{\infty} e^{-(v-a)^2/2r}\varphi(v)dv.
\]

Let us introduce :

\[
A_i(a, y) = \frac{\varphi(y)(y-a)^{2i+1}}{2i+1} + \int_y^{\infty} (v-a)^{2i}\varphi(v)dv,
\]

where \( i \geq 0 \).
To compute $F$ (note that $F$ is of class $C^\infty$ on $[0, 1/2]$ and):

$$g(0, a, y, t) = \sum_{i=0}^{n} \frac{(-1)^i}{(2i)!} A_i(a, y) + O(\frac{1}{t^{n+1}}).$$  \hspace{1cm} (6.13)

Moreover $\varepsilon \mapsto g(0, a, y, 1/\varepsilon)$ is of class $C^\infty$ on $[0, 1/2]$ and:

$$\left| \frac{\partial^i g(0, a, y, 1/\varepsilon)}{\partial \varepsilon^i} \right| \leq k_i \left( \varphi(y)(y - a)^{2i+1} + \int_0^\infty (v - a)^{2i} \varphi(v) dv \right).$$  \hspace{1cm} (6.14)

Suppose that $t \to \infty$, then:

$$\frac{g(0, a, y, t-u)}{g(0, 0, 0, t)} = \left(1 - \frac{u}{t}\right)^{-1/2} \sum_{i=0}^{n} \frac{(-1)^i}{2^i!} \frac{1}{t^i (1-u/t)} A_i(a, y) + O(\frac{1}{t^{n+1}})$$

$$= \sum_{i=0}^{n} \frac{1}{i!} F_i(a, y, u) + \frac{1}{i!} R(0, a, y, u, t),$$  \hspace{1cm} (6.15)

where $t \mapsto R(0, a, y, u, t)$ is bounded, and $F_i(a, y, u)$ may be written in the following form:

$$F_i(a, y, u) = \sum_{j=0}^{i} \alpha_{i,j}(u) A_j(a, y),$$  \hspace{1cm} (6.16)

$\alpha_{i,j}(u)$ being some polynomial function.

b) In particular:

$$F_0(a, y, u) = \frac{A_0(a, y)}{A_0(0, 0)} = A_0(a, y) = \varphi(y)(y - a) + \int_y^\infty \varphi(v) dv = \varphi(y)(y - a) + \Phi(y),$$  \hspace{1cm} (6.17)

(note that $F_0$ does not depend on $u$).

To compute $F_1(a, y, u)$ we need the first order term:

$$\frac{g(0, a, y, t-u)}{g(0, 0, 0, t)} = \frac{A_0(a, y) \left(1 - \frac{1}{2t} \frac{A_1(a, y)}{A_0(a, y)}\right) \left(1 - \frac{u}{t}\right)^{-1/2}}{A_0(0, 0) \left(1 - \frac{1}{2t} \frac{A_1(0, 0)}{A_0(0, 0)}\right)} + O(\frac{1}{t^2}).$$

Recall that $A_0(0, 0) = 1$, consequently:

$$F_1(a, y, u) = \frac{A_0(a, y)}{2} \left(A_1(0, 0) - A_1(a, y) + u\right)$$

$$= \frac{A_1(0, 0)}{2} A_0(a, y) - \frac{A_1(a, y)}{2}$$

$$= \int_0^\infty \frac{v^2 \varphi(v) dv}{2} A_0(a, y) - \varphi(y) \frac{(y - a)^3}{3!} + \frac{1}{2} \int_0^\infty (v - a)^2 \varphi(v) dv.$$  \hspace{1cm} (6.18)

c) We would like to obtain some estimates about the remainder term $R(0, a, y, u, t)$ in (6.15), as a function of $(a, y)$.
Taking $t \geq 2u+2$ and setting $\varepsilon = 1/t$ we have: $\varepsilon \leq 1/2$, $\varepsilon u \leq 1/2$, $1/t - u = \varepsilon u \leq 1 - u \varepsilon \in [0, 1]$. 

Let $\tilde{g}(0, a, y, \varepsilon) := \tilde{g}(0, a, y, (1 - u \varepsilon)/\varepsilon)$, $\varepsilon \in [0, 1/(2u+2)]$. Then property (6.14) implies:

$$\left| \frac{\partial^i \tilde{g}(0, a, y, \varepsilon)}{\partial \varepsilon^i} \right| \leq K_n \left( \sum_{j=1}^{n+1} \varphi(y) (y - a)^{2j+1} + \int_0^\infty (v - a)^{2j} \varphi(v) dv \right), \quad 0 \leq i \leq n + 1, \quad (6.18)$$

where, from now on, $K_n$ denotes a generic constant, which only depends on $u$.

Let us introduce $\hat{g}(\varepsilon) := \tilde{g}(0, 0, 0, 1/\varepsilon)$, $\varepsilon \in [0, 1/(2u+2)]$, then:

$$\left| \frac{\partial^i \hat{g}(\varepsilon)}{\partial \varepsilon^i} \right| \leq K_n \int_\varepsilon^\infty v^{2i+2} \varphi(v) dv, \quad 0 \leq i \leq n + 1. \quad (6.19)$$

Note that $\varepsilon \leq 1/2$, consequently:

$$\hat{g}(\varepsilon) \geq \int_0^\infty e^{-v^2/4} \varphi(v) dv > 0. \quad (6.20)$$

Finally, taking into account (6.18), (6.19), (6.21) and (1.35) we get:

$$\left| \frac{\partial^{n+1} \hat{g}(0, a, y, u, t)}{\partial \varepsilon^{n+1}} \right| \leq K_n \left( \sum_{j=1}^{n+1} \varphi(y) (y - a)^{2j+1} + \int_0^\infty (v - a)^{2j} \varphi(v) dv \right), \quad (6.21)$$

d) Using (6.10) and (6.15), we have:

$$E_0 \left[ \frac{1_{\Gamma_u} \varphi(S_t)}{\varphi(S_t)} \right] = E_0 \left[ 1_{\Gamma_u} \frac{g(0, X_u, S_t, t-u)}{g(0, 0, 0, t)} \right] = E_0 \left[ 1_{\Gamma_u} \left\{ \sum_{i=0}^n \frac{1}{i!} F_i(X_u, S_u, u) + \frac{1}{(n+1)!} R(0, X_u, S_u, u) \right\} \right].$$

Inequality (6.22) implies (1.33) and point 1. b) of Theorem 1.9.

e) It remains to prove that for any $i \in \{1, \cdots, n\}$, $(F_i^w(X_t, S_t, t), t \geq 0)$ is a $(\mathcal{F}_t, P_0)$ martingale.

From (6.16), we deduce that $E[|F_i^w(X_t, S_t)|] < \infty$.

Let $\Gamma_u \in \mathcal{F}_u$ and $u \leq v$. The asymptotic development (1.38) implies that:

$$\lim_{t \to \infty} \left\{ E_0 \left[ 1_{\Gamma_u} \frac{\varphi(S_t)}{\varphi(S_t)} \right] - Q_0^w(\Gamma_u) \right\} = E_0 \left[ 1_{\Gamma_u} F_i^w(X_u, S_u, t) \right].$$

Since $\Gamma_u \in \mathcal{F}_v$ then:

$$E_0 \left[ 1_{\Gamma_u} F_i^w(X_u, S_u, u) \right] = E_0 \left[ 1_{\Gamma_u} F_i^w(X_v, S_u, v) \right].$$

Consequently $(F_i^w(X_t, S_t, t), t \geq 0)$ is a $(\mathcal{F}_t, P_0)$ martingale.

Reasoning by induction, we easily prove that $(F_i^w(X_t, S_t, t), t \geq 0)$ is a $(\mathcal{F}_t, P_0)$ martingale, for any $1 \leq i \leq n$.

3) We now suppose $\lambda > 0$. We proceed as previously; the relation (6.12) implies:

$$h(\lambda, z, r) = \sqrt{2 \pi} e^{\lambda^2 r/2} \left[ \lambda \sqrt{2\pi} + \frac{1}{\sqrt{r}} e^{-\left( z - \lambda r \right)^2/2} - \lambda \Phi_0 \left( \frac{z - \lambda r}{\sqrt{r}} \right) \right],$$

where $\Phi_0$ denotes the function: $\Phi_0(x) := \int_{-\infty}^x e^{-u^2/2} du$. 

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Suppose $x < 0$. Integrating by parts we have:

$$
\Phi_0(x) = \int_{-\infty}^{x} \frac{1}{u} e^{-u^2/2} \, du = -\frac{1}{x} e^{-x^2/2} - \int_{-\infty}^{x} \frac{1}{u^2} e^{-u^2/2} \, du.
$$

Reasoning by induction we can easily prove:

$$
\Phi_0(x) = e^{-x^2/2} \sum_{i=0}^{n} (-1)^{i+1} \frac{a_i}{x^{2i+1}} + (-1)^{n+1} a_{n+1} \int_{-\infty}^{x} \frac{1}{u^{2n+2}} e^{-u^2/2} \, du,
$$

with $a_0 = 1$ and

$$
a_i = 1 \times 3 \times \cdots \times (2i-1) = \frac{(2i)!}{2^i i!} = E_0[X_i^2], \quad i \geq 1.
$$

This relation implies:

$$
\Phi_0(x) = e^{-x^2/2} \left[ \sum_{i=0}^{n} (-1)^{i+1} \frac{a_i}{x^{2i+1}} + O\left( \frac{1}{x^{2n+3}} \right) \right], \quad x \to -\infty.
$$

Consequently:

$$
h(\lambda, z, r) = \sqrt{\frac{2}{\pi}} e^{z^2/2} \left[ \lambda \sqrt{2\pi} + e^{-(z^2 - \lambda^2 x^2)/2} \right] h_1(\lambda, z, r)
$$

where:

$$
h_1(\lambda, z, r) := -\frac{z}{\lambda r^{3/2}(1 - \frac{z}{\lambda r})} + \sum_{i=1}^{n} (-1)^{i+1} \frac{a_i}{\lambda^{2i}} \left( \frac{1}{\sqrt{1 - \frac{z}{\lambda r}}} \right)^{2i+1} + O\left( \frac{1}{\lambda^{n+3/2}} \right), \quad r \to \infty.
$$

Setting $r = t - u$, where $u > 0$ is fixed and $t \to \infty$, we get:

$$
h_1(\lambda, z, t - u) = -\frac{1}{\lambda t^{3/2}(1 - u/t)^{3/2}(1 - \frac{z}{\lambda t})} + \sum_{i=1}^{n} \frac{(-1)^{i+1} a_i}{\lambda^{2i}} \left( \frac{1}{\sqrt{1 - u/t(1 - \frac{z}{\lambda t})}} \right)^{2i+1} + O\left( \frac{1}{\lambda^{n+3/2}} \right), \quad t \to \infty,
$$

This implies:

$$
h_1(\lambda, z, t - u) = \frac{1}{\sqrt{t}} \left[ \sum_{i=1}^{n} \alpha_i(\lambda, z, u), t \to \infty,\right]
$$

where, for any $i$, $(z, u) \mapsto \alpha_i(\lambda, z, u)$ is a polynomial function with degree less than $n$, with respect to $z$ or $u$. Moreover we have:

$$
\alpha_1(\lambda, z, u) = -\frac{z}{\lambda} + \frac{1}{\lambda^2}.
$$

If $r = t - u$ we have:

$$
e^{-(z^2 - \lambda^2 x^2)/2} = e^{\lambda z + \lambda^2 x^2/2} e^{-\lambda^2 t/2} e^{-\lambda^2 t^2/2(i-1)}.
$$

therefore:

$$
h(\lambda, z, t - u) = \sqrt{\frac{2}{\pi}} e^{\lambda^2(t-u)/2} \left[ \lambda \sqrt{2\pi} + e^{\lambda z + \lambda^2 x^2/2} e^{-\lambda^2 t/2} \frac{1}{\sqrt{t}} \left( \sum_{i=1}^{n} \beta_i(\lambda, z, u) \frac{1}{t^i} + O\left( \frac{1}{t^{i+1}} \right) \right) \right], \quad t \to \infty,
$$

where $(z, u) \mapsto \beta_i(\lambda, z, u)$ is a polynomial function with at most degree $n$ with respect to $z$ or $u$.

Note that $\beta_1(\lambda, z, u) = \alpha_1(\lambda, z, u) = -\frac{z}{\lambda} + \frac{1}{\lambda^2}$.
We are able to come back to relation (6.11):

\[ g(\lambda, a, y, t - u) = \sqrt{\frac{2}{\pi}} e^{\lambda^2(t-u)/2} \left[ \lambda \sqrt{2 \pi} \left( \psi(y) \frac{\sinh(\lambda(y-a))}{\lambda} \right) + e^{\lambda a} \int_y^\infty e^{-\lambda x} \psi(x) dx \right] \]

\[ + e^{\lambda^2 u/2} e^{-\lambda^2 t/2} \frac{1}{\sqrt{t}} \sum_{i=1}^n \gamma_i(\lambda, a, y, u) \frac{1}{t^i} + 0 \left( \frac{1}{t^{i+1}} \right), \quad t \to \infty, \]

where

\[ \gamma_i(\lambda, a, y, u) = \psi(y)e^{\lambda(y-a)} \int_0^{y-a} e^{-\lambda z} \beta_i(\lambda, z, u) dz + \int_y^\infty \psi(x) \beta_i(\lambda, x - a, u) dx. \]

Note that :

\[ |\gamma_i(\lambda, a, y, u)| \leq C(1 + u^n)(1 + |a|^n + \psi(y)e^{\lambda(y-a)}), \quad a \in \mathbb{R}, y \geq 0, u \geq 0. \]  

(6.22)

Introducing :

\[ F_0^{\lambda, \varphi}(a, y, u) := e^{-\lambda^2 u/2} \left( \psi(y) \frac{\sinh(\lambda(y-a))}{\lambda} \right) + e^{\lambda a} \int_y^\infty e^{-\lambda x} \psi(x) dx, \]  

(6.23)

We have :

\[ g(\lambda, a, y, t - u) = \sqrt{\frac{2}{\pi}} e^{\lambda^2 t/2} \left[ \lambda \sqrt{2 \pi} F_0^{\lambda, \varphi}(a, y, u) + e^{-\lambda^2 t/2} \frac{1}{\sqrt{t}} \sum_{i=1}^n \gamma_i(\lambda, a, y, u) \frac{1}{t^i} + 0 \left( \frac{1}{t^{i+1}} \right) \right], \quad t \to \infty. \]

Since \( F_0^{\lambda, \varphi}(0, 0, 0) = \int_0^\infty e^{-\lambda x} \psi(x) dx = 1, \) then

\[ \frac{g(\lambda, a, y, t - u)}{g(\lambda, 0, 0, t)} = \frac{\lambda \sqrt{2 \pi} F_0^{\lambda, \varphi}(a, y, u) + e^{-\lambda^2 t/2} \frac{1}{\sqrt{t}} \sum_{i=1}^n \gamma_i(\lambda, a, y, u) \frac{1}{t^i} + 0 \left( \frac{1}{t^{i+1}} \right)}{\lambda \sqrt{2 \pi} + e^{-\lambda^2 t/2} \frac{1}{\sqrt{t}} \sum_{i=1}^n \gamma_i(\lambda, 0, 0, 0) \frac{1}{t^i} + 0 \left( \frac{1}{t^{i+1}} \right)} = \frac{F_0^{\lambda, \varphi}(a, y, u) + e^{-\lambda^2 t/2} \frac{1}{\sqrt{t}} \sum_{i=1}^n F_i^{\lambda, \varphi}(a, y, u) \frac{1}{t^i} + 0 \left( \frac{1}{t^{i+1}} \right)}{1 + e^{-\lambda^2 t/2} \frac{1}{\sqrt{t}} \sum_{i=1}^n \gamma_i(\lambda, 0, 0, 0) \frac{1}{t^i} + 0 \left( \frac{1}{t^{i+1}} \right)}, \quad t \to \infty. \]

In particular :

\[ F_1^{\lambda, \varphi}(a, y, u) = \frac{1}{\lambda \sqrt{2 \pi}} \left( \gamma_1(\lambda, a, y, u) - \gamma_1(\lambda, 0, 0, 0) F_0^{\lambda, \varphi}(a, y, u) \right). \]

It is easy to compute \( \gamma_1(\lambda, a, y, u). \) We have :

\[ \gamma_1(\lambda, a, y, u) = \frac{1}{\lambda^2} \left( \psi(y)e^{\lambda(y-a)} \int_0^{y-a} e^{-\lambda z}(1 - \lambda z) dz + \int_y^\infty \psi(x)(1 - \lambda(x - y) - \lambda(y - a)) dx \right) \]

\[ = \frac{1}{\lambda^2} \left( c(\lambda, \varphi) \varphi_1(y)(y - a) + \int_y^\infty \psi(x)(1 - \lambda(x - y) dx \right), \]

with \( c(\lambda, \varphi) = \int_0^\infty \psi(x)(1 - \lambda x) dx \) and

\[ \varphi_1(y) = \frac{1}{c(\lambda, \varphi)} \left( \psi(y) - \lambda \int_y^\infty \psi(x) dx \right), \quad y \geq 0. \]
Setting:

\[ \Phi_1(y) := \int_0^y \varphi_1(z)dz, \]

we obtain:

\[ \Phi_1(+\infty) - \Phi_1(y) = \frac{1}{c(\lambda, \varphi)} \left( \int_y^\infty \psi(x)dx - \lambda \int_y^\infty \psi(x)(x-y)dx \right) \]
\[ = \frac{1}{c(\lambda, \varphi)} \int_y^\infty \psi(x)(1 - \lambda(x-y))dx. \]

In particular:

\[ \Phi_1(+\infty) - \Phi_1(0) = \int_0^\infty \varphi_1(z)dz = 1. \]

Moreover:

\[ F_{\lambda,\varphi}^1(a, y, u) = \frac{c(\lambda, \varphi)}{\lambda^3 \sqrt{2\pi}} \left( \varphi_1(y)(y-a) + \int_y^\infty \varphi_1(z)dz - F_{0,\varphi}^1(a, y, u) \right). \]

This proves point 1. (b) of Theorem 6.3.

7 Further discussions about Brownian penalizations

As a conclusion to this paper, we would like to mention that we are presently developing some further discussions about Brownian penalizations in three papers in preparation:

- in \([13]\), we study a number of extensions of Pitman’s theorem, which are closely related with the penalizations found in the present paper;
- in \([12]\), we extend most of the results found in the present paper when \((X_t)\) is replaced with \((R_t)\), a Bessel process with dimension \(d < 2\), the weight process being a function of the local time of \((R_t)\) at level 0;
- in \([11]\), we study penalization results for \(n\)-dimensional Brownian motion, when the weight process is \(\exp - \int_0^t 1_C(X_s)ds\), where \(C\) denotes a cone in \(\mathbb{R}^d\) with vertex 0.
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