I. INTRODUCTION

Squeezed states [1] have been recognized as a non-classical resource that can be exploited to improve the performance of several tasks beyond their classical limit [2–4], e.g., enhanced sensitivity of the LIGO detector [5–7], and real-time phase-tracking [8, 9]. Squeezing is usually associated with the variance of a single mode (reduced beyond the classical limit), however recently, a measure to quantify the squeezing of a multi-mode state has been proposed, i.e., the squeezing of formation (SoF) [10], which quantifies the minimum amount of squeezing needed to create a Gaussian state (spread over multiple modes).

Another fundamental resource in quantum information is entanglement [11]. Entanglement is a property of quantum systems that is manifested as correlations of quantum observables that cannot be classically reproduced [12]. This can lead to the violation of Bell inequalities [13], steering [14], or protocols such as quantum teleportation [15], all of which are impossible classically. There is no unique quantifier of entanglement; two examples are entanglement of formation (EoF) [16] and logarithmic negativity (LogNeg) [17]. Given that different entanglement measures are, in general, inequivalent to each other [18], the choice of a quantifier depends on the problem that is considered. A typical motivation for using LogNeg is that, unlike EoF for example, it is easy to compute. Entanglement and squeezing are closely related to each other. In continuous-variable (CV) quantum information, squeezing is considered a fundamental resource for entanglement. While a state with squeezed quadratures is not necessarily entangled, any squeezed state can be transformed into an entangled state by passive operations (assuming access to ancillary vacuum modes) [19]. Indeed, the maximum entanglement generated using passive operations, as quantified by LogNeg, has been previous investigated – and was shown to be analytically computable for any two-mode Gaussian state [19].

In this work, we introduce the entanglement potential of a given two-mode Gaussian state as the maximum attainable entanglement of formation through passive linear optics. For certain special cases, we derive closed form solutions for this quantity. In the more general case, we show that the entanglement potential can be upper-bounded by the minimum amount of squeezing needed to synthesize the state. This provides an analytical relation between non-classicality of a quantum state, and non-classicality of correlations in the Gaussian setting.

This manuscript proceeds as follows. We begin in Sec. II with a brief review regarding Gaussian states, Gaussian operations, and the quantification of squeezing and entanglement. In Sec. III we define the notion of the entanglement potential between two optical modes and demonstrate how it may be bounded from above by the squeezing of formation. In Sec. IV we then derive analytical expressions for the entanglement potential, while Sec. V concludes with avenues of further research.

II. PRELIMINARIES

A. Gaussian States

Quantized bosonic modes, that are the main focus in this work, are described by CV states [20–22]. Those modes are associated with the quadrature field operators \( \hat{x}_j := \hat{a}_j + \hat{a}^\dagger_j \) and \( \hat{p}_j := i(\hat{a}^\dagger_j - \hat{a}_j) \), where \( \hat{a}_j \) and \( \hat{a}^\dagger_j \) are the annihilation and creation operators, respectively, with \( [\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij} \), the kronecker delta. Gaussian states are a subclass of CV states that can be fully described by the first two statistical moments (mean value and variance) of the quadratures field operators. For the purposes of this work we can ignore the mean value (fix it for simplicity to zero as it does not contribute to the squeezing or the entan-
lement of the state), and fully describe the $n$-mode Gaussian states through a real, symmetric and positive-definite matrix called the covariance matrix, whose elements are given by $\sigma_{ij} := \frac{1}{2} \langle \{ \hat{q}_i, \hat{q}_j \} \rangle$, where $\hat{q} := (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)^T$ is the vectorial operator.

Every Gaussian state $\sigma$ can be non-uniquely decomposed as [23]

$$\sigma = \pi + \varphi,$$

(1)

where $\pi$ is a pure Gaussian state, i.e., $\det(\pi) = 1$, and $\varphi \geq 0$ is a positive-semidefinite matrix representing random correlated displacements in the quadrature fields.

**B. Gaussian Operations**

The action of a Gaussian operation (any operation that transforms a Gaussian state into another Gaussian state) on the covariance matrix $\sigma$ can be described by the following symplectic transformation $\Sigma$

$$\sigma \mapsto \Sigma \sigma \Sigma^T,$$

(2)

where $\Sigma \Omega \Sigma^T = \Omega$, with $\Omega := \bigoplus_{i=1}^n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, known as the symplectic form.

Quantum operations can be distinguished as passive or active, depending on whether the total average energy of the system is conserved or not, respectively. From an operational point of view, passive operations are considered “free”, since they can be straightforwardly implemented in the laboratory. On the other hand, active operations can alter the energy of the system, and they are typically more demanding.

Any symplectic operation $\Sigma$ can be decomposed uniquely (up to a permutation of the symplectic spectrum) through the Bloch-Messiah decomposition [24, 25], into a sequence of passive and active operations (see Fig. 1) as follows

$$\Sigma = K \bigoplus_{i=1}^n S(r_i) L,$$

(3)

where $K$ and $L$ are symplectic passive operations, e.g., phase shifts or beam-splitters [26], while $S(r_i)$ is a set of single-mode squeezing active operations, defined as

$$S(r_i) := \begin{bmatrix} e^{r_i} & 0 \\ 0 & e^{-r_i} \end{bmatrix},$$

(4)

with $r_i \in \mathbb{R}$. The Bloch-Messiah decomposition will be used in the next section as an intuitive way for quantifying squeezing in pure Gaussian states.

**C. Squeezing of Formation**

Experimentally, squeezing is a sophisticated operation that involves non-linear processes, and thus practical limitations emerge in our ability to construct certain quantum states. For a single mode, the quantification of squeezing is straightforward, since we only have to check how much the variance of $\hat{x}$ or $\hat{p}$ beats the shot noise (the variance of any quadrature of a vacuum state), which can also be considered as a non-classicality measure [27]. For multi-mode quantum states, though, where multiple interfering modes can be squeezed, the situation is more complicated, and defining a measure that reflects the “total” amount of squeezing in a state is a non-trivial task.

Given an $n$-mode pure Gaussian state $\sigma$, we can write $\sigma = \Sigma \Sigma^T$ where $\Sigma$ has a Bloch-Messiah decomposition with $L = 1$. Thus, applying the appropriate passive $n$-mode operation $K$, we end up with a covariance matrix in the following form

$$\sigma \mapsto K \pi K^T = \bigoplus_{i=1}^n \begin{bmatrix} e^{2r_i} & 0 \\ 0 & e^{-2r_i} \end{bmatrix}.$$

(5)

This means that the pure state $\pi$ can be decomposed into $n$ uncorrelated pure squeezed states through the passive operation $K$. The $2n$ eigenvalues $\lambda_i$ of this covariance matrix come in reciprocal pairs, so if we arrange them in an increasing order, i.e., $\lambda_1 : = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2n}$, we can fully characterize the eigenspectrum of $\pi$ through the first half of them. The squeezing of this multi-mode pure state $\pi$ can be characterized through the following function [10]

$$F(\pi) := -\frac{1}{2} \sum_{i=1}^n \ln |\lambda_i(\pi)| = \sum_{i=1}^n |r_i|,$$

(6)

which gives the sum of the absolute squeezing parameters of each mode for a pure state in the Bloch-Messiah decomposition.

Squeezing of formation (SoF) for an arbitrary state $\sigma$ is defined [10] as the convex-roof extension of Eq. (6), i.e.,

$$S(\sigma) := \inf_{\pi} \{ F(\pi) \mid \sigma = \pi + \varphi \},$$

(7)

and it quantifies the least amount of squeezing of the pure state $\pi$ on which we can apply random correlated displacements.
\( \varphi \) to create the state \( \sigma \). The merit of this measure is that it rigorously characterizes the amount of squeezing in a state, since the function \( S(\sigma) \) satisfies properties such as convexity and continuity [10].

**Remark 1.** From the the Bloch-Messiah decomposition it follows that two states with the same SoF cannot in general be transformed under passive operations into one another. Consider for example two pure states with the same SoF. The symplectic operation (decomposed through the unique Bloch-Messiah decomposition) applied onto the vacuum modes (for each state) has, in general, different local squeezing parameters \( r_i \), and thus subsequent passive operations do not have an impact on them. For example, with current technology the highest amount of single mode squeezing has a parameter of \( r \sim 1.7 \) (or \( \sim 15 \text{ dB} \)) [28]. So, a given pure two-mode state of \( S(\sigma) \sim 2 \) can only be constructed if the squeezing is equally distributed in each mode, but not if it is concentrated in a single mode.

### III. ENTANGLEMENT POTENTIAL

In this section, we introduce the concept of entanglement potential using EoF as the entanglement measure, and we present the main result of this work in Proposition 1, where we upper-bound the entanglement potential for any two-mode Gaussian state.

#### D. Entanglement of Formation

**Entanglement of formation** (EoF) is defined as the convex-roof extension of the entropy of entanglement [16], and quantifies the entanglement in terms of the entropy of entanglement of the least entangled state needed to prepare it (under local operations and classical communication).

For two-mode Gaussian states EoF is given by [29–33]

\[
E(\sigma) := \inf_{\pi} \{ E(\pi) \mid \sigma = \pi + \varphi \} ,
\]

(8)

where \( E(\pi) \) is the entropy of entanglement of the bipartite pure state \( \pi \). \( E(\pi) \) is defined as the von Neumann entropy of the reduced state, and given by [34, 35]

\[
E(\pi) := \begin{cases} h \left[ \nu_- (\pi^T) \right] & \text{for } \nu_- (\pi^T) < 1 \\ 0 & \text{for } \nu_- (\pi^T) \geq 1 \end{cases} ,
\]

(9)

where \( \nu_- (\pi^T) \) is the lowest (of the two) symplectic eigenvalue of the partially transposed pure state \( \pi \), and \( h(x) \) is the auxiliary function defined as

\[
h(x) := \frac{(1+x)^2}{4x} \log_2 \left[ \frac{(1+x)^2}{4x} \right] - \frac{(1-x)^2}{4x} \log_2 \left[ \frac{(1-x)^2}{4x} \right] .
\]

(10)

**Remark 2.** Eq. (8) technically defines the Gaussian-EoF, which for multi-mode Gaussian states, in general, is an upper bound to the EoF, but for two-mode Gaussian states the two measures coincide [36, 37]. For the rest of the work we focus only on two-mode Gaussian states.

Consider a two-mode Gaussian state with covariance matrix \( \sigma \), and access to \( n \) ancillary vacuum modes and \((n+2)\)-mode passive operations \( K_j \). **Entanglement of formation potential** (EoF-potential) quantifies the maximum attainable EoF of a state \( \sigma \) as follows

\[
\mathcal{P}(\sigma) := \sup_j \{ E(\sigma_j) \mid \sigma_j = \text{tr}_n \left[ K_j (\sigma \oplus \mathbb{I}_n) K_j^T \right] \} ,
\]

(11)

where \( \text{tr}_n \) denotes the partial trace over the last \( n \) modes. The introduction of the ancillary vacuum modes in the definition of the EoF-potential might seem redundant at first sight, but a simple example where those extra modes can indeed be useful is the following: consider two uncorrelated modes, a squeezed vacuum and a thermal mode. Then, in order to increase the entanglement of the final state over passive operations, it is always beneficial to exploit a vacuum mode instead of the thermal one.

Also, note that the definition given in Eq. (11) for the EoF-potential can be modified for any other entanglement measure or monotone. For instance, based on this context the proposition 2 of Ref. [19] corresponds to the LogNeg-potential.

**B. Bounds of EoF-potential**

By construction, the EoF of any state \( \sigma \) is a lower bound to the EoF-potential, i.e.,

\[
E(\sigma) \leq \mathcal{P}(\sigma) .
\]

(12)

For two-mode Gaussian states we can always numerically estimate the value of \( E(\sigma) \), and for special classes we can also achieve analytical expressions [29–34]. A more interesting bound to the EoF-potential would be an upper bound, that we derive below.

**Proposition 1** For an arbitrary two-mode Gaussian state \( \sigma \), the EoF-potential is upper-bounded as follows

\[
\mathcal{P}(\sigma) \leq h \left[ e^{-S(\sigma)} \right] ,
\]

(13)

where \( S(\sigma) \) is the SoF of \( \sigma \).
**Proof.** The EoF-potential in Eq. (11) can be written as
\[ \mathcal{P}(\sigma) = \sup_j \mathcal{E}(\sigma_j) = \sup_j \inf_\pi \mathcal{E}(\pi_j) = \sup_j \mathcal{E}(\pi_{e,j}), \]
(14)
where \( \pi_{e,j} \) is the state that achieves the \( \inf_\pi \mathcal{E}(\pi_j) \), and
\[ \sigma_j = \text{tr}_a \left[ K_j (\pi + \varphi) K_j^T \right] = \pi_j + \varphi. \]
(15)
Based on Ref. [19], for any pure state \( \pi \) we have
\[ \nu_- (\pi^T) \geq \sqrt{\lambda_1^2(\pi) \cdot \lambda_2^2(\pi)} \Rightarrow h \left[ \nu_- (\pi^T) \right] \leq h \left[ e^{-S(\pi)} \right], \]
(16)
and for pure states we can always find a passive operation (without introducing ancillary vacuum modes) that saturates the above inequality, so for a generic pure state \( \pi \) we have
\[ \sup_j \mathcal{E}(\pi_j) = h \left[ e^{-S(\pi_{e,j})} \right], \]
(17)
where the right-hand side does not depend on the operation \( K_j \). Thus, the EoF-potential in Eq. (14) takes the following form
\[ \mathcal{P}(\sigma) = h \left[ e^{-S(\pi_{e,j})} \right]. \]
(18)
A passive operation cannot affect the SoF of a state, so
\[ S(\sigma_j) = \inf_\pi S(\pi_j) = S(\pi_{s,j}) = S(\sigma), \]
(19)
where \( \pi_{s,j} \) is the state that achieves the \( \inf_\pi S(\pi_j) \). For any passive operation \( K_j \) we have
\[ E(\pi_{e,j}) \leq E(\pi_{s,j}) \Rightarrow \sup_j E(\pi_{e,j}) \leq \sup_j E(\pi_{s,j}), \]
(20)
and, so, we finally get
\[ \mathcal{P}(\sigma) \leq h \left[ e^{-S(\sigma)} \right], \]
(21)
which completes the proof. \( \blacksquare \)

Remark 3. Regardless of how each mode is individually squeezed, EoF-potential is related to the overall squeezing, quantified by SoF. Take for example a pure state \( \pi_1 \) that consists of a position-quadrature squeezed vacuum \( (r_1 = 1) \) and a non-squeezed vacuum \( (r_2 = 0) \), and another state \( \pi_2 \) that consists of two squeezed vacuum modes, one in the position quadrature \( (r_1 = 1) \) and the other in the momentum quadrature \( (r_2 = -1) \). Those two states have the exact same amount of SoF, since based on Eq. (6) we have \( S(\pi_1) = |2| + |0| = 2 \) and \( S(\pi_2) = |1| + |-1| = 2 \). The EoF for the two states is also equal, which can be seen as follows. We apply to both states a balanced (50:50) beam-splitter which (as a passive operation) keeps the SoF constant. Then, through local unitary operations (both passive and active), which keep the EoF constant, we can end up with the exact same covariance matrix (called the standard form of the covariance matrix [38, 39]), so \( \mathcal{E}(\pi_1) = \mathcal{E}(\pi_2) \).

**IV. EXACT VALUE FOR EOF-POTENTIAL IN SPECIAL CASES**

In this section, we derive analytical expressions for the SoF of specific classes of two-mode Gaussian states, which in conjunction with the corresponding expressions for the EoF, give a closed formula for the EoF-potential.

Let us reorder the vectorial operator into \( \hat{q}' := (\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2)^T \), so the covariance matrix of a two-mode Gaussian state takes the following form
\[ \sigma = \begin{bmatrix} \langle \hat{x}_1^2 \rangle & \langle \hat{x}_1 \hat{x}_2 \rangle & \frac{1}{2} \langle \hat{x}_1 \hat{p}_1 \rangle & \frac{1}{2} \langle \hat{x}_1 \hat{p}_2 \rangle \\ \langle \hat{x}_2 \hat{x}_1 \rangle & \langle \hat{x}_2^2 \rangle & \frac{1}{2} \langle \hat{x}_2 \hat{p}_1 \rangle & \frac{1}{2} \langle \hat{x}_2 \hat{p}_2 \rangle \\ \frac{1}{2} \langle \hat{p}_1 \hat{x}_1 \rangle & \frac{1}{2} \langle \hat{p}_1 \hat{x}_2 \rangle & \langle \hat{p}_1^2 \rangle & \langle \hat{p}_1 \hat{p}_2 \rangle \\ \frac{1}{2} \langle \hat{p}_2 \hat{x}_1 \rangle & \frac{1}{2} \langle \hat{p}_2 \hat{x}_2 \rangle & \langle \hat{p}_2 \hat{p}_1 \rangle & \langle \hat{p}_2^2 \rangle \end{bmatrix}. \]
(22)

Consider a state with no cross quadrature correlations\(^1\), i.e., \( \langle \hat{x}_i \hat{p}_j \rangle = \langle \hat{p}_i \hat{x}_j \rangle = 0, \forall \{i, j\} \), so Eq. (22) becomes
\[ \sigma = \begin{bmatrix} a_1 & c_1 \\ c_1 & b_1 \end{bmatrix} \oplus \begin{bmatrix} a_2 & c_2 \\ c_2 & b_2 \end{bmatrix} = c_x \oplus c_p, \]
(23)
where we require that \( a_i b_i - c_i \geq 1 \) for the state to be physical. In order to simplify the calculation of SoF for states in the form of Eq. (23) we show below that instead of minimizing over all pure states we can restrict the optimization over only a subset of them.

It has been shown in Ref. [29, 41] that a two-mode Gaussian state is pure if and only if its covariance matrix has the following form
\[ \pi(z, y) = \begin{bmatrix} z & zy \\ yz & yzy + z^{-1} \end{bmatrix}, \]
(24)
where \( z = z^T > 0 \) and \( y = y^T \) are real and symmetric 2×2 sub-matrices.

**Proposition 2** Among all pure two-mode Gaussian states \( \pi(z, y) \), squeezing of formation is minimized by pure states of the form \( \pi(z, 0) \), i.e.,
\[ S[\pi(z, y)] \geq S[\pi(z, 0)]. \]
(25)
**Proof.** Based on Eq. (6), SoF for a state \( \pi(z, y) \) is equal or larger than that of a state \( \pi(z, 0) \) if we have
\[ \lambda_i^\dagger[\pi(z, y)] \leq \lambda_i^\dagger[\pi(z, 0)], \text{ for } i = \{1, 2\}, \]
(26)
where \( \lambda_i^\dagger[\pi(z, y)] \) are the eigenvalues of each state in an increasing order. Then, for a given orthogonal projection \( \Pi = (1, 1, 0, 0)^T \) on \( \pi(z, y) \), i.e.,
\[ z = \Pi^T \pi(z, y) \Pi, \]
(27)

---

\(^1\) This is not the most general class of states, since using only passive operations we cannot always end up to this type of covariance matrix, but it is an experimentally relevant class (see Ref. [40] for a method to verify whether a state has no cross quadrature correlations has been constructed by some of us).
the Cauchy interlacing theorem (Theorem 4.3.17 in Ref. [42]) 

\[ \lambda_1^z[\pi(z, y)] \leq \lambda_1^z(z) = \lambda_1^z[\pi(z, 0)], \]  

(28)

which completes the proof. 

Based on the above proposition, for states in the form of Eq. (23) the SoF needs to be minimized over only pure states of the form \( \pi(z, 0) \), and due to Eq. (1) the constraint \( \sigma \geq \pi \) becomes \( c_p^{-1} \leq z \leq c_x \), i.e.,

\[ S[\sigma = c_x \oplus c_p] := \inf _z \{ F[\pi(z, 0)] \mid c_p^{-1} \leq z \leq c_x \}. \]  

(29)

Below, we present the analytical expressions for two specific classes of two-mode Gaussian states.

A. Symmetric States

Symmetric states have quadratures with equal variance, but (in general) different correlations in \( \hat{x} \) and \( \hat{p} \). The covariance matrix of symmetric states is given by

\[ \sigma_{\text{sym}} = \begin{bmatrix} a & c_1 \\ c_1 & a \end{bmatrix} \oplus \begin{bmatrix} a & c_2 \\ c_2 & a \end{bmatrix}. \]  

(30)

Interfering the two modes on a balanced beam-splitter, we end up with the following covariance matrix

\[ \begin{bmatrix} a + c_1 & 0 \\ 0 & a - c_1 \end{bmatrix} \oplus \begin{bmatrix} a + c_2 & 0 \\ 0 & a - c_2 \end{bmatrix}, \]  

(31)

which represents two uncorrelated modes, and the constraint \( c_p^{-1} \leq z \leq c_x \) takes the form

\[ \begin{bmatrix} \frac{1}{a + c_2} & 0 \\ 0 & \frac{1}{a - c_2} \end{bmatrix} \leq z \leq \begin{bmatrix} a + c_1 & 0 \\ 0 & a - c_1 \end{bmatrix}. \]  

(32)

Squeezing of formation, then, is equal to

\[ S(\sigma_{\text{sym}}) = - \ln(\mu_+ \mu_-), \]  

(33)

where \( \mu_{\pm} = \min\{1, \sqrt{a \pm c_1}, \sqrt{a \pm c_2}\} \). The EoF of symmetric states is given by [43]

\[ \mathcal{E}(\sigma_{\text{sym}}) = h(\mu_+ \mu_-) = h\left[ e^{-S(\sigma_{\text{sym}})} \right], \]  

(34)

and since the lower bound in Eq. (12) coincides with the upper bound in Eq. (13), the EoF-potential of symmetric states is given by

\[ \mathcal{P}(\sigma_{\text{sym}}) = h(\mu_+ \mu_-). \]  

(35)

B. Balanced Correlated States

Balanced correlated states have an equal amount of correlations in \( \hat{x} \) and \( \hat{p} \), i.e., \( c_1 = -c_2 = c > 0 \), as well as quadratures with the same variance in \( \hat{x} \) and \( \hat{p} \), i.e., \( a_1 = a_2 = a \), and \( b_1 = b_2 = b \). Their covariance matrix has the following form

\[ \sigma_{bc} = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \oplus \begin{bmatrix} a & -c \\ -c & b \end{bmatrix}. \]  

(36)

These states correspond to the output of a two-mode squeezed state passing through two independent phase-invariant channels, and they are typically encountered in quantum communication protocols, e.g., quantum teleportation [44], and quantum key distribution [45]. If \( 1 - a - b + ab \geq c^2 \) all the eigenvalues of \( \sigma_{bc} \) are greater or equal to one, and thus the state is classical and SoF vanishes.

Unlike symmetric states, these states cannot be brought into a diagonal form by a passive transformation. In App. A, we show that in this case the optimal \( z \) is given by

\[ z_{\text{opt}} = \frac{\lambda_+}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{\lambda_-}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \]  

(37)

where \( \lambda_- \) and \( \lambda_+ \) are the two eigenvalues of \( z_{\text{opt}} \), given by

\[ \lambda_+ = \frac{1}{\lambda_-} = \frac{a + b + 2c}{1 + ab - c^2 + \sqrt{(1 - ab + c^2)^2 - (a - b)^2}}, \]  

(38)

and the SoF is

\[ S(\sigma_{bc}) = \ln \lambda_+. \]  

(39)

The EoF for balanced correlated states has been calculated for the first time in Ref. [34] (where the name “GMEM” was given to the same type of states), but a simpler expression can be found in Refs. [32, 33]. As we show in App. B, the EoF is equal to

\[ \mathcal{E}(\sigma_{bc}) = h \left[ e^{-S(\sigma_{bc})} \right] = h(\lambda_-), \]  

(40)

and, since the lower bound in Eq. (12) coincides with the upper bound in Eq. (13), the EoF-potential for balanced correlated states is given by

\[ \mathcal{P}(\sigma_{bc}) = h(\lambda_-). \]  

(41)

V. DISCUSSION

In continuous-variable optics, the class of operations that is typically considered the most simple to realize is passive linear operations. In this work we ask, given two-mode Gaussian states, what is the maximum amount of entanglement that we can synthesize using such operations? We defined this quantity as the entanglement potential, and analytically computed it for certain special cases when entanglement is quantified by the entanglement of formation. For more general states, we demonstrated that entanglement potential can be bounded from above given the squeezing of formation of the same state – the amount of squeezing needed to synthesize these modes. This, thus, presents another interesting connection between non-classicality of localized quantum states and the non-classicality of correlations in the Gaussian regime.
There are a number of interesting future directions. One immediate question is the tightness of the bounds derived; another being to what extent such relations can be generalized to other measures of quantum correlations and non-classicality in light. This would become especially pertinent should we consider non-Gaussian states, whereby existing measures of non-classicality have already been defined from the perspectives of metrological advantage and non-equilibrium energy [27, 46, 47]. Such directions could well shed light to potential connections to entanglement distillation; where a no-go theorem prevents distillation through local Gaussian operations [48–50].

Note added: Upon completion of this work we learnt about a similar paper that recently appeared on arXiv [51], where the authors bound EoF using a non-classicality measure called monotone of total noise.

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Appendix A: SoF for Balanced Correlated States

In this appendix, we compute the SoF for a balanced correlated state [defined in Eq. (36)] through Eq. (29). Following the analysis of Ref. [29], we set

\[
\mathbf{c}_x := \begin{bmatrix} x_0 + x_1 & x_2 \\ x_2 & x_0 - x_1 \end{bmatrix}, \quad \mathbf{c}_p^{-1} := \begin{bmatrix} p_0 + p_1 & p_2 \\ p_2 & p_0 - p_1 \end{bmatrix},
\]

and

\[
\mathbf{z} := \begin{bmatrix} z_0 + z_1 & z_2 \\ z_2 & z_0 - z_1 \end{bmatrix},
\]

and we represent the matrices \( \mathbf{z} \), \( \mathbf{c}_x \), and \( \mathbf{c}_p^{-1} \), as points in three dimensions with coordinates, e.g. \( (x_1, x_2, x_0) \). With this representation, and given the constraint \( c_p^{-1} \leq z \leq c_x \), the coordinates of \( z \) must lie within the intersection of two 45-degrees vertical cones with vertices corresponding to \( c_x \) and \( c_p^{-1} \). For states where both \( c_x \) and \( c_p \) have one eigenvalue greater than one and one eigenvalue less than one (\( \sigma_{BC} \) falls in this category), the optimal \( z \) must also lie on the surface of both these cones. This surface is parameterized by \( \det(z-c_x) = 0 \) and \( \det(c_p^{-1}-z) = 0 \). With these two constraints, the problem of quantifying the SoF reduces to finding the optimal \( z \), restricted to the ellipse defined by the intersection of the two surfaces of the cones.

The coordinates of this ellipse can be parametrized by the angle \( \theta \) as follows

\[
\begin{align*}
    z_1(\theta) &= \frac{x_1 + p_1}{2} + r_1 \cos \gamma \sin \theta - r_2 \sin \gamma \cos \theta, \\
    z_2(\theta) &= \frac{x_2 + p_2}{2} + r_1 \sin \gamma \sin \theta + r_2 \cos \gamma \cos \theta, \\
    z_0(\theta) &= x_0 - \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2},
\end{align*}
\]

where \( \gamma := \arctan(c_2/c_1) \), \( c_i := (a_i - b_i)/2 \), \( r_1 := c_0 \), and \( r_2 := \sqrt{c_0^2 - (c_1^2 + c_2^2)} \). The SoF of \( z \) is \( S[\pi(z,0)] = \frac{1}{2} \log \frac{\lambda_+}{\lambda_-} \), where

\[
\lambda_{\pm} = z_0 \pm \sqrt{z_1^2 + z_2^2} = z_0 \left( 1 \pm \sqrt{\frac{z_1^2 + z_2^2}{z_0^2}} \right),
\]

are the two eigenvalues of \( z \). Minimizing \( S[\pi(z,0)] \) is equivalent to minimizing \( (z_1^2 + z_2^2)/z_0^2 \), so, we need to find the angle \( \theta \) that minimizes \( (z_1^2 + z_2^2)/z_0^2 \). For an arbitrary \( c_x \) and \( c_p^{-1} \), this involves solving a transcendental equation which does not have an analytical solution. However, for the balance correlated state, this minimization can be done analytically, and through long but straightforward computations, we find that \( (z_1^2 + z_2^2)/z_0^2 \) is minimized when

\[
z_0 = \frac{\lambda_+ + \lambda_-}{2}, \quad z_1 = \frac{\lambda_- - \lambda_+}{2}, \quad \text{and} \quad z_2 = 0,
\]

with

\[
\lambda_+ = \frac{1}{\lambda_-} = \frac{a + b + 2c}{1 + a b - c^2 + \sqrt{(1 - a b + c^2)^2 - (a - b)^2}},
\]

Appendix B: Proof of Eq. (40)

Based on Refs. [32, 33] the EoF for entangled balanced correlated states is given by

\[
\mathcal{E}(\sigma_{BC}) = \cosh^2 r_o \log_2 (\cosh^2 r_o) - \sinh^2 r_o \log_2 (\sinh^2 r_o),
\]

where

\[
r_o = \frac{1}{4} \ln \left[ \frac{a^2 (2b^2 - 1) + 2ab (1 - 2c^2) - b^2 + 2c^4 + 2 + 2 \sqrt{(ab - c^2 + 1)^2 [(a^2 - 1) (b^2 - 1) - 2c^2 (ab - 1) + c^4]}}{(a + b - 2c)^2} \right].
\]
Let us simplify the above expression by setting

\[ \kappa := ab - c^2 + 1, \quad (B3a) \]

\[ \tau_1 := 2c + a + b, \quad (B3b) \]

\[ \tau_2 := 2c - a - b, \quad (B3c) \]

so that

\[ r_o = \frac{1}{4} \log \left[ \frac{2\kappa^2 + \tau_1 \tau_2 - 2\sqrt{\kappa^2 (\kappa^2 + \tau_1 \tau_2)}}{\tau_1^2 \tau_2^2} \right]. \quad (B4) \]

The parameter \( \lambda_- \) in Eq. (38) can analogously be written as

\[ \lambda_- = \frac{\kappa + \sqrt{\kappa^2 + \tau_1 \tau_2}}{\tau_1}. \quad (B5) \]

In order to prove Eq. (40) we need to show that

\[ \cosh^2 r_o = \frac{(1 + e^{-2r_o})^2}{4e^{-2r_o}} = \frac{(1 + \lambda_-)^2}{4\lambda_-} \quad \Rightarrow \quad e^{-2r_o} = \lambda_- \quad \Rightarrow \quad e^{-4r_o} = \lambda_-^2. \quad (B6) \]

Consider

\[ \lambda_-^2 e^{4r_o} = \left[ \frac{2\kappa^2 + \tau_1 \tau_2 - 2\sqrt{\kappa^2 (\kappa^2 + \tau_1 \tau_2)} - 2\kappa^2 + \tau_1 \tau_2 + 2\kappa \sqrt{\kappa^2 + \tau_1 \tau_2}}{\tau_1^2 \tau_2^2} \right] \]

\[ = \frac{(2\kappa^2 + \tau_1 \tau_2)^2 - 4\kappa^2 (\kappa^2 + \tau_1 \tau_2)}{\tau_1^2 \tau_2^2} \quad (B7a) \]

\[ = 1. \quad (B7b) \]

Eq. (B7b) holds only when \( \kappa \geq 0 \), which is true for all quantum states, which completes the proof.
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