Square-free discriminants of Frobenius rings

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Abstract

Let $E$ be an elliptic curve over $\mathbb{Q}$. It is well known that the ring of endomorphisms of $E_p$, the reduction of $E$ modulo a prime $p$ of ordinary reduction, is an order of the quadratic imaginary field $\mathbb{Q}(\pi_p)$ generated by the Frobenius element $\pi_p$. When the curve has complex multiplication (CM), this is always a fixed field as the prime varies. However, when the curve has no CM, very little is known, not only about the order, but about the fields that might appear as algebra of endomorphisms varying the prime. The ring of endomorphisms is obviously related with the arithmetic of $a_p^2 - 4p$, the discriminant of the characteristic polynomial of the Frobenius element. In this paper, we are interested in the function $\pi_{E,r,h}(x)$ counting the number of primes $p$ up to $x$ such that $a_p^2 - 4p$ is square-free and in the congruence class $r$ modulo $h$.

We give in this paper the precise asymptotic for $\pi_{E,r,h}(x)$ when averaging over elliptic curves defined over the rationals, and we discuss the relation of this result with the Lang-Trotter conjecture, and with some other problems related to the curve modulo $p$.

1 Introduction and statement of results

Let $E$ be an elliptic curve over $\mathbb{Q}$ with conductor $N_E$. For each prime $p$ of good reduction (i.e. $p \nmid N_E$), $E$ reduces to an elliptic curve over the finite field $\mathbb{F}_p$. The Frobenius endomorphism $(x, y) \mapsto (x^p, y^p)$ of $E/\mathbb{F}_p$ has characteristic polynomial

$$x^2 - a_p x + p = (x - \pi_p)(x - \overline{\pi}_p)$$

where $|a_p| \leq 2\sqrt{p}$ by the Hasse bound. Let $\text{End}(E/\mathbb{F}_p)$ be the ring of endomorphism of $E/\mathbb{F}_p$. If $p > 3$ is an ordinary prime for $E$ (or equivalently, $a_p \neq 0$), then

$$\text{End}(E/\mathbb{F}_p) \otimes \mathbb{Q} = \mathbb{Q}(\pi_p)$$

is completely determined by the Frobenius ring $\mathbb{Z}[\pi_p]$. The ring of endomorphism $\text{End}(E/\mathbb{F}_p)$ is more subtle, and the Frobenius ring $\mathbb{Z}[\pi_p]$ can be a proper subset of $\text{End}(E/\mathbb{F}_p)$. In fact, it follows from Deuring’s Theorem [De] that any order $R$ such that

$$\mathbb{Z}[\pi_p] \subseteq R \subseteq \mathbb{Q}(\pi_p)$$

can occur as the ring of endomorphism of some curve $E$ over $\mathbb{F}_p$ such that the Frobenius of $E$ has characteristic polynomial $x^2 - a_p x + p$. It is then a natural question to ask whether
\[ \mathbb{Z}[\pi_p] \simeq \text{End}(E/\mathbb{F}_p), \text{ or whether } \mathbb{Z}[\pi_p] \text{ is the maximal order of } \mathbb{Q}(\pi_p) \text{ (the second question is of course a refinement of the first one), and when this happens for a fixed curve } E \text{ when varying the prime } p. \]

For a fixed } E/\mathbb{F}_p, \text{ let } \Delta_p \text{ be the discriminant of the order } \text{End}(E/\mathbb{F}_p). \text{ Then, the “distance” between the endomorphism ring and the Frobenius ring is encoded by the unique positive integer } b_p \text{ defined by the relation } a_p^2 - 4p = b_p^2 \Delta_p. \text{ The } b_p \text{'s where studied by Duke and Toth in } [\text{DuTo}], \text{ and by Cojocaru and David } [\text{CoDa}] \text{ who showed that under the GRH}

\[ \# \{ p \leq x : p \nmid N_E \text{ and } b_p = 1 \} \sim \mathcal{C}_{\text{CD}}(E) \frac{x}{\log x}, \]

for an explicit non-zero constant } \mathcal{C}_{\text{CD}}(E). \text{ The condition } b_p = 1 \text{ is equivalent to the triviality of the Tate-Shafarevich group of } E \text{ over its function field } \mathbb{F}_p(E).

In general, properties of the Frobenius ring can be deduced from a study of the arithmetic of } a_p^2 - 4p \text{ and, in particular, in studying the square divisors of } a_p^2 - 4p. \text{ In this paper, we study the following question: given an elliptic curve } E \text{ over } \mathbb{Q}, \text{ for which primes } p \text{ is } a_p^2 - 4p \text{ square-free? For those primes, the Frobenius ring } \mathbb{Z}[\pi_p] \text{ is the maximal order in } \mathbb{Q}(\pi_p). \text{ This also provides a refinement of the question of Cojocaru and Duke. Since the result of Cojocaru and Duke is itself a refinement of Serre’s result about the cyclicity of the group } E(\mathbb{F}_p), \text{ we have the string of implications}

\[ a_p^2 - 4p \text{ squarefree} \implies b_p = 1 \implies E(\mathbb{F}_p) \text{ cyclic.} \]

Can we then show that there is a positive proportion of } a_p^2 - 4p \text{ which are square-free, refining Cojocaru and Duke’s and Serre’s result? Maybe surprisingly, we believe that this is a much more difficult question. If } E \text{ has complex multiplication (CM), } a_p^2 - 4p \text{ square-free is equivalent to } p \text{ lying in some quadratic progression. For example, let } E \text{ be the CM elliptic curve } y^2 = x^3 - x \text{ with complex multiplication by } \mathbb{Z}[i]. \text{ Let } p \text{ be an ordinary prime, which is a prime congruent to 1 modulo 4. Since } E \text{ has rational 2-torsion, } a_p \text{ is even, and then 4 divides } a_p^2 - 4p, \text{ and the natural question to consider is the square-freeness of } (a_p^2 - 4p)/4. \text{ Since } E \text{ has complex multiplication by } \mathbb{Z}[i], a_p^2 - 4p = -4f^2 \text{ for some } f \in \mathbb{Z}, \text{ and } (a_p^2 - 4p)/4 \text{ is square-free if and only if } f = 1 \text{ and only if } p = (a_p/2)^2 + 1.

Another reason to investigate square-freeness in the sequence } a_p^2 - 4p \text{ is that it might shed light on the following conjecture.}

**Conjecture 1 (Lang-Trotter conjecture [LaTy])** \text{ Let } K \text{ be an imaginary quadratic number field, and } E \text{ an elliptic curve over } \mathbb{Q} \text{ without complex multiplication. Let}

\[ \Pi_{E,K}(x) = \# \{ p \leq x : p \nmid N_E \text{ and } \mathbb{Q}(\pi_p) = K \}. \]

\text{Then as } x \to \infty

\[ \Pi_{E,K}(x) \sim \mathcal{C}_{\text{LT}}(E, K) \frac{\sqrt{x}}{\log x}, \]

\text{for some explicit constant } \mathcal{C}_{\text{LT}}(E, K) \text{ depending on } E \text{ and } K.

Upper bounds for } \Pi_{E,K}(x) \text{ were investigated by Serre, by Cojocaru, Fouvry and Murty } [\text{CoFoMu}], \text{ and by Cojocaru and David } [\text{CoDa}]. \text{ But there are no known lower bounds for } \Pi_{E,K}(x), \text{ in particular there are no known examples of (non-CM) elliptic curves such that}
\( \mathbb{Q}(\pi_p) = K \) for infinitely many primes \( p \). In fact, we know much less than that, as there are no known examples of \((\text{non-CM})\) elliptic curves such that \( D_p \), the discriminant of \( \mathbb{Q}(\pi_p) \), lies in a fixed congruence class for infinitely many primes \( p \). Here again, one could give a lower bound for the number of such primes by counting the number of primes \( p \) such that \( a_p^2 - 4p \) is square-free and in the prescribed congruence class, since in this case \( a_p^2 - 4p = D_p \).

Let \( h \) be a positive odd integer, and let \( r \) be any integer such that the greatest common divisor \((r, h)\) is square-free. Let

\[
\Pi^b_{E, r, h}(x) = \# \{ 2 < p \leq x, \text{ prime} : a_p^2 - 4p \in \Delta(r, h) \},
\]

where \( \Delta(r, h) \) is the set of square-free integers \( n \) such that \( n \equiv r \mod h \). We remark that if \((r, h)\) is not square-free, then \( \Pi^b_{E, r, h}(x) = 0 \). As for the restriction to \( h \) odd, it simplifies technical aspects of the proof in various places, but it is not an essential restriction.

Unfortunately, we cannot give an asymptotic (or a lower bound) for \( \Pi^b_{E, r, h}(x) \), but we can prove that the correct asymptotic holds on average over elliptic curves.

In all the following, \( E(a, b) \) denotes the elliptic curve given by the equation \( y^2 = x^3 + ax + b \), with \( 4a^3 + 27b^2 \neq 0 \).

**Theorem 2** Let \( h \) be a positive odd integer, and let \( r \) be any integer such that \((r, h)\) is square-free. Fix any \( \varepsilon > 0 \). Let \( A, B \) be such that \( AB > x \log^3 x \), \( A, B > x^{\varepsilon} \). Let \( C(A, B) \) be the set of all elliptic curves \( E(a, b) \) with integer coefficients \( a, b \) such that \( |a| \leq A \) and \( b \leq B \). Then, as \( x \to \infty \),

\[
\frac{1}{|C(A, B)|} \sum_{E(a, b) \in C(A, B)} \Pi^b_{E(a, b), r, h}(x) = \mathcal{C} \frac{x}{\log x} + O \left( \frac{x^{3}}{\log^2 x} \right),
\]

where \( \mathcal{C} \) is the positive constant

\[
\mathcal{C} = \frac{1}{3h} \prod_{\ell \mid h} \frac{\ell - 1}{\ell} \prod_{\ell \nmid r} \frac{\ell (\ell - 1 - (\ell \mid r))}{(\ell - 1) (\ell - (\ell \mid r))} \prod_{\ell \mid h} \frac{\ell^4 - 2\ell^2 - \ell + 1}{\ell^2 (\ell^2 - 1)},
\]

where all products are taken over odd primes \( \ell \) with the specified conditions.

The result above is similar in spirit, and by the techniques used to prove it, to other average results as [BaCoDa] for the Koblitz’s conjecture and [FoMu] and [DaPa] for the Lang-Trotter conjecture. All those results rely on the fact that the average over elliptic curves in \( C(A, B) \) can be rewritten as an average over elliptic curves over \( \mathbb{F}_p \) by interchanging the summations, and this average can be reduced to an average of class numbers by means of Deuring’s Theorem. Hence, Theorem 2 is equivalent to

**Theorem 3** Let \( h \) be a positive odd integer, and let \( r \) be any integer such that \((r, h)\) is square-free. Let

\[
\Pi^b(p) = \# \{ E \text{ over } \mathbb{F}_p : a_p^2 - 4p \in \Delta(r, h) \}.
\]

Then, as \( x \to \infty \),

\[
\sum_{p \leq x} \Pi^b(p) = \frac{\mathcal{C}}{3} \frac{x^3}{\log x} + O \left( \frac{x^3}{\log^2 x} \right),
\]

where \( \mathcal{C} \) is the constant of Theorem 2.
There is another sequence related to elliptic curves which was investigated for square-freeness, namely the sequence \( p + 1 - a_p = |E(F_p)| \). Again, there should be a positive proportion of square-free \( p + 1 - a_p \), but this is still an open question, and there are no known examples of non-CM elliptic curves over \( \mathbb{Q} \) with infinitely many square-free \( p + 1 - a_p \). The case of CM curves is very different. Let \( E \) be an elliptic curve with CM by an order \( \mathcal{O} \) in a quadratic imaginary field \( K \), and let \( p \) be a prime of ordinary reduction. Then, \( p + 1 - a_p \) is square-free if and only if \( \pi_p - 1 = \mathfrak{p}_1 \cdots \mathfrak{p}_s \) where the \( \mathfrak{p}_i \) are distinct split primes in \( \mathcal{O} \), and the number of such \( \pi \in \mathcal{O} \) can be counted. When \( E \) has CM by the maximal order of a quadratic imaginary field \( K \), the asymptotic for the number of primes \( p \) such that \( p + 1 - a_p \) is square-free was obtained by Cojocaru \[Co\]. The number of square-free \( p + 1 - a_p \) was also investigated in average over elliptic curves \( F_p \) as \( p \) tends to infinity by Gekeler \[Ge\]. The techniques that he uses are completely different from ours, and rely on the work of Howe \[Ho\] on the moduli spaces of elliptic curves over \( F_p \) with a given group structure. Gekeler then proved the analogue of our Theorem 3 for the sequence \( p + 1 - a_p \) (his results are stated in a different way, but the two forms are easily seen to be equivalent). As a corollary of his result \[Ge\] Proposition 4.2 (i), one obtains, by going from the average over \( F_p \) to the average over \( \mathbb{Q} \) as indicated in Section 2, that the number of primes \( p \leq x \) such that \( p + 1 - a_p \) is square-free follows the predicted asymptotic on average over all elliptic curves. It is unclear if the techniques used by Gekeler could be used to prove Theorem 2, as \( a_p^2 - 4p \) is not easily related to the group structure of \( E(F_p) \).

The structure of our paper is as follows. In section 2 we reduce the average over elliptic curves over \( \mathbb{Q} \) to an average over elliptic curves over \( F_p \). There are several ways to do that, and we use multiplicative characters as in \[BaSh\] and \[BaCoDa\] to obtain the short average of Theorem 2. Section 3 deals with some easy error terms of the asymptotic of Theorem 2 while Section 4 deals with the main error term by applying the Theorem of Barban, Davenport and Halberstam. Section 5 deals with the main term of the asymptotic, and it is basically a computation of the constant \( C \) of Theorem 2. Finally, we explain in Section 6 what is the conjectural asymptotic for \( \Pi_{E,r,h}(x) \). The conjectural constant is very similar to our average constant \( C \), and Theorem 2 then gives additional evidence for the conjecture.

2 Reduction to a short average over finite fields

As we have mentioned in the introduction, the proof of Theorem 2 relies in reducing the average of \( \Pi_{E,r,h}(x) \) to an average of class numbers by means of Deuring’s Theorem. This could be done in a straightforward manner by just writing

\[
\frac{1}{|\mathcal{C}(A,B)|} \sum_{E \in \mathcal{C}(A,B)} \Pi_{E,r,h}^p(x) = \sum_{p \leq x} \sum_{(s,t) \in S} \frac{1}{|\mathcal{C}(A,B)|} \sum_{|a| \leq A, \ |b| \leq B \atop a \equiv s \pmod{p} \atop b \equiv t \pmod{p}} 1, \tag{2}
\]

where

\[
S = S(p,r,h) = \{(s_1, s_2) \in F_p^2 : a_p(E(s_1, s_2))^2 - 4p \in \Delta(r,h)\}.
\]

Observe that the number of terms in the middle sum is \( \Pi^p(p) \). The innermost sum is
simply
\[
\left(\frac{2A}{p} + O(1)\right) \left(\frac{2B}{p} + O(1)\right) \sim \frac{4AB}{p^2}
\]
when \(A, B\) are big enough with respect to \(x\). Hence, we get the asymptotic
\[
\frac{1}{|C(A, B)|} \sum_{E \in C(A, B)} \Pi^b_{E,r,h}(x) \sim \sum_{p \leq x} \frac{\Pi^b(p)}{p^2}
\]
However, this approach results a poor average, as we need to take \(AB > x^{2+\epsilon}\) to get the asymptotic above.

Following [BaSh] and [BaCoDa], one can obtain a substantial improvement from a better use of the uniform distribution of equivalent elliptic curves. Indeed, since two elliptic curves \(E(s_1, s_2)\) and \(E(t_1, t_2)\) are isomorphic over \(\mathbb{F}_p\) if and only if \(t_1 = s_1 u^4\) and \(t_2 = s_2 u^6\) for some \(u \in \mathbb{F}_p^*\), there are \((p-1)/\# \text{Aut}(E(s_1, s_2))\) elliptic curves over \(\mathbb{F}_p\) which are \(\mathbb{F}_p\)-isomorphic to a given elliptic curve \(E(s_1, s_2)\). Hence, we can write
\[
\frac{1}{|C(A, B)|} \sum_{E \in C(A, B)} \Pi^b_{E,r,h}(x)
= \frac{1}{|C(A, B)|} \sum_{p \leq x} \sum_{(s,t) \in \mathbf{S}} \frac{\# \text{Aut}(E(s_1, s_2))}{p-1} \sum_{|a| \leq A, |b| \leq B, 31 \leq u < p; a \equiv s_1 u^4 \pmod{p}, b \equiv s_2 u^6 \pmod{p}} 1,
\]
where for each fixed \(s_1, s_2 \in \mathbb{F}_p^*\), the innermost sum is over all integers \(|a| \leq A, |b| \leq B\) such that there exists \(u \in \mathbb{F}_p^*, a \equiv s_1 u^4 \pmod{p}\) and \(b \equiv s_2 u^6 \pmod{p}\). We now want to approximate the innermost sum by its main term. This is the content of Lemma 4 of [BaCoDa], and we will use it to prove the short average of Theorem 2. We include it here for reading convenience.

**Lemma 4** [BaCoDa] *Lemma 4* For any positive integer \(k\), we have as \(x \to \infty\)
\[
\sum_{p \leq x} \left| \frac{1}{p} \sum_{(s,t) \in (\mathbb{F}_p^*)^2} \frac{1 - \frac{2AB}{p}}{\sum_{|u| \leq A, |v| \leq B, 31 \leq u < p; a \equiv s_1 u^4 \pmod{p}, b \equiv s_2 u^6 \pmod{p}} 1} \right| \ll ABx^{1 - \frac{k}{A} \log^2 x} + (A\sqrt{B} + B\sqrt{A})x^{1 - \frac{k}{A} \log^2 x} + \sqrt{AB}x^{3/2} \log^2 x.
\]
Now, since
\[
|C(A, B)| = 4AB + O(A + B),
\]
and \(\# \text{Aut}(E(s, t)) = 2\) except when \(p|a\) or \(p|b\), we can rewrite (3) as
\[
\frac{1}{AB} \sum_{p \leq x} \sum_{(s,t) \in \mathbf{S}} \frac{1}{2(p-1)} \sum_{|a| \leq A, |b| \leq B, 31 \leq u < p; a \equiv s_1 u^4 \pmod{p}, b \equiv s_2 u^6 \pmod{p}} 1 + O\left(\log \log x + \frac{A+B}{AB} \frac{x}{\log x}\right)
\]
(4)
using the trivial bound
\[ \frac{1}{|C(A, B)|} \sum_{p \leq x} \sum_{\substack{ab \equiv 0 \\ b \leq A \\ (b \mod p)}} 1 \ll \log \log x + \frac{A + B}{AB} \frac{x}{\log x}. \]

Then, using Lemma 4 to replace the first term in (4), and observing that the error term in (4) is smaller than the error term of Lemma 4, we have
\[ \frac{1}{|C(A, B)|} \sum_{E \in C(A, B)} \Pi^b_{E, r, h}(x) = \sum_{p \leq x} \frac{\Pi^b(p)}{p(p-1)} + O \left( x^{1 - \frac{1}{2} - \frac{1}{2}} \log^2 x \right). \]

By choosing \( A, B \) such that \( A, B > x^\varepsilon \) and \( AB > x \log^8 x \), and \( k \) large enough to have that \( \varepsilon k > 1 \), we have that \( \varepsilon k > 1 \), we have that
\[ \frac{1}{|C(A, B)|} \sum_{E \in C(A, B)} \Pi^b_{E, r, h}(x) = \sum_{p \leq x} \frac{\Pi^b(p)}{p(p-1)} + O \left( \frac{x}{\log^2 x} \right). \]

The heart of the proof of Theorem 2 then consists in an estimate for the main term in (6), and we will show that
\[ \sum_{p \leq x} \frac{\Pi^b(p)}{p(p-1)} = \frac{C}{\log x} x \log x + O \left( \frac{x}{\log^2 x} \right), \]
which will complete the proof of Theorem 2. With (7) and partial summation, we also get that
\[ \sum_{p \leq x} \Pi^b(p) = \frac{C}{3} \frac{x^3}{\log x} + O \left( \frac{x^3}{\log^2 x} \right), \]
and the proof of Theorem 3 also follows from (7). We first apply Deuring’s Theorem to rewrite the sum in (6) as a sum of class numbers.

**Theorem 5 (Deuring’s Theorem [De])**
Let \( p > 3 \) be a prime and let \( t \) be an integer such that \( t^2 - 4p < 0 \). Let \( \mathcal{E}_t(p) \) be the set of \( \mathbb{F}_p \)-isomorphism classes of elliptic curves defined over \( \mathbb{F}_p \) with \( \alpha_p = t \). Then,
\[ \sum_{E \in \mathcal{E}_t(p)} \frac{1}{\# \text{Aut}(E)} = H(t^2 - 4p), \]
where \( \text{Aut}(E) \) is the automorphism group of \( E \) and for any \( D < 0 \), \( H(D) \) is the Kronecker class number
\[ H(D) := \sum_{\substack{f \mid D \\ f^2 \equiv 0, 1 \mod 4}} \frac{h(D/f^2)}{w(D/f^2)} \]
Using the previous theorem, and noting that for square-free \(D\) we have

\[ H(D) = \frac{h(D)}{w(D)}, \]

we can write

\[
\sum_{p \leq x} \frac{\Pi^b(p)}{p(p-1)} = \sum_{p \leq x, \ t^2 - 4p \in \Delta(r,h)} \frac{h(t^2 - 4p)}{w(t^2 - 4p)p} = 2 \sum_{p \leq x} \sum_{\substack{1 \leq t \leq 2\sqrt{p} \atop t^2 - 4p \in \Delta(r,h)}} \frac{h(t^2 - 4p)}{w(t^2 - 4p)p} \]

since \(t^2 - 4p\) is not square-free when \(t\) is even.

3 Terms divisible by big squares

Let \(\beta > 0\) a parameter to be chosen later, and let

\[ K = \lfloor \log^{2\beta} x \rfloor, \quad Y = \frac{x}{K + 1}, \quad R = \log^\beta x. \]

We first split the sum over the primes up to \(x\) into \(K\) sums over the intervals \([kY, kY + Y]\) of length \(Y\), and write \((9)\) as

\[
\sum_{p \leq x} \frac{\Pi^b(p)}{p(p-1)} = 2 \sum_{1 \leq k \leq K} \sum_{kY < p \leq kY + Y} \sum_{\substack{1 \leq t \leq 2\sqrt{p} \atop t^2 - 4p \in \Delta(r,h)}} \frac{h(t^2 - 4p)}{w(t^2 - 4p)p} + O \left( \frac{x}{\log^{2\beta+1} x} \right), \]

where the error term is to account for the primes \(p \leq Y\). Using the M"obius function to detect squares, the main term of \((11)\) is

\[
2 \sum_{1 \leq k \leq K} \sum_{kY < p \leq kY + Y} \sum_{\substack{1 \leq t \leq 2\sqrt{p} \atop t^2 - 4p \equiv r \mod h}} \frac{h(t^2 - 4p)}{w(t^2 - 4p)p} \sum_{d | \mu(d) = T_1 + T_2 + T_3},
\]

where \(T_1, T_2\) and \(T_3\) will be sums over small, medium, and large divisors \(d\). In particular, for each \(p\) we define the intervals \(I_1 = [1, R], I_2 = (R, \sqrt{Y}], I_3 = (\sqrt{Y}, 2\sqrt{p}]\) and then

\[
T_i = 2 \sum_{1 \leq k \leq K} \sum_{kY < p \leq kY + Y} \sum_{d \in I_i} \mu(d) \sum_{\substack{1 \leq t \leq 2\sqrt{p} \atop t^2 - 4p \equiv r \mod h}} \frac{h(t^2 - 4p)}{w(t^2 - 4p)p},
\]

for \(i = 1, 2, 3\). We show in this section that the terms corresponding to big squares \(d\), namely \(T_2\) and \(T_3\), are small and will be part of the error term.
For $T_3$ we use the fact that there is a unique representation of the numbers $4p-t^2 = m^2u$, where $u$ is square-free, and that $d^2|4p-t^2$ if and only if $d|m$, to get the trivial upper bound

$$T_3 \ll \log x \sum_{1 \leq k \leq K} \sum_{kY < p \leq kY + Y} \frac{1}{\sqrt{p}} \sum_{\sqrt{Y} < m \leq 2\sqrt{p}} \tau(m)|S_m(p)|,$$

where $S_m(p) = \{1 \leq t \leq 2\sqrt{p} : 4p-t^2 = m^2u, u$ square-free}, and $\tau(m)$ denotes the usual divisor function. To get the above upper bound, and in other estimations to come, we use the well-known bound $h(-d) \ll \sqrt{d}\log d$ for the class number of a negative integer $-d$. Now, using the uniqueness of the representation, we can change the sum in $m$ to a sum over square-free $u \leq 4p/Y$ to obtain

$$T_3 \ll \frac{x^\varepsilon}{Y} \sum_{1 \leq k \leq K} \sum_{kY < p \leq kY + Y} \sqrt{p} \ll \frac{x^{3/2+\varepsilon}}{Y},$$

where we have used that, for each square-free $u$ fixed, the number of solutions to the equation $4p = t^2 + m^2u$ is bounded above by the number of ideal divisors of $4p$ in the ring of integers of $\mathbb{Q}(\sqrt{-u})$, together with the trivial bound $\tau(m) \ll m^\varepsilon$. For $T_2$ we switch the sums to get

$$T_2 \ll (\log x) \sum_{1 \leq k \leq K} \sum_{R < d \leq \sqrt{Y}} \frac{1}{\sqrt{Y}} \sum_{1 \leq t \leq 2\sqrt{Y} + Y} \sum_{d^2|4p-t^2} \frac{1}{d} \ll (\log x) \frac{\sqrt{\frac{k + 1}{k}}}{R},$$

$$\leq (\log x) Y \sum_{1 \leq k \leq K} \frac{1}{k} \sum_{R < d \leq \sqrt{Y}} \frac{1}{d^2} < (\log x) \frac{x}{R} \quad (13)$$

4 An average of class numbers

The main term of $T$ comes from $T_1$. On each interval $kY < p \leq kY + Y$, we approximate the sum over $\{1 \leq t \leq 2\sqrt{p} \}$ by $\{1 \leq t \leq 2\sqrt{kY} \}$ with an error term of $RY \log x$. Replacing (13) and (12) in (11), we have

$$\sum_{p \leq x} \frac{\Pi^b(p)}{p(p-1)} = 2 \sum_{1 \leq k \leq K} \sum_{1 \leq t \leq 2\sqrt{kY}} \sum_{d \leq R} \frac{\mu(d)}{d} \sum_{kY < \sqrt{d^2+4p} \leq \sqrt{t^2+4p} \leq \sqrt{t^2-4p}} \frac{h(t^2-4p)}{\omega(t^2-4p)p}$$

$$+ O(x/\log^{\beta-1} x)$$

$$= \frac{1}{\pi} \sum_{1 \leq k \leq K} \sum_{1 \leq t \leq 2\sqrt{kY}} \sum_{d \leq R} \frac{\mu(d)}{d} \sum_{kY < \sqrt{d^2+4p} \leq \sqrt{t^2+4p} \leq \sqrt{t^2-4p}} \frac{\sqrt{4p-t^2} L(1, \chi(t^2-4p))}{p}$$

$$+ O(x/\log^{\beta-1} x),$$

using the class number formula. Let

$$U = x^{1/2} R^2.$$

$$\quad$$

$$\quad$$
Since
\[ L(1, \chi^{t^2-4p}) = \sum_{n \geq 1} \frac{\chi^{t^2-4p}(n)}{n} = \sum_{n \geq 1} \frac{\chi^{t^2-4p}(n)}{n} + \frac{1}{2} \sum_{n \geq 1} \frac{\chi^{t^2-4p}(2n)}{n} \]
\[ = \left(1 + \frac{\chi^{t^2-4p}(2)}{2}\right) \sum_{n \geq 1} \frac{\chi^{t^2-4p}(n)}{n} + \frac{1}{4} \sum_{n \geq 1} \frac{\chi^{t^2-4p}(n)}{n}, \]
using Polya-Vinogradov inequality to bound the tail of the L-function, we obtain
\[ \sum_{p \leq x} \frac{\Pi^b(p)}{p(p-1)} = \frac{2}{3\pi} \sum_{1 \leq k \leq K} \sum_{1 \leq r \leq \sqrt{2\pi \alpha}} \frac{1}{n} \sum_{\alpha \equiv r \mod (n,h)} \left(\frac{\alpha}{n}\right) \sum_{\mu(d), 1 \leq d \leq R} \mu(d) \sum_{\nu(t,\alpha,r) \equiv 1, r \equiv 0 \mod (d^2,h)} \frac{\sqrt{4p - t^2}}{p} \chi^{t^2-4p}(n) \]
\[ + O(x/\log^{\beta-1} x), \quad (15) \]
since \( \chi^{t^2-4p}(2) = -1 \) for any \( t, p \) in the previous conditions. Using quadratic reciprocity, we can write \( \chi \) as a character mod \( n \), and we use this to rewrite the main term of (15) as
\[ M = \frac{2}{3\pi} \sum_{1 \leq k \leq K} \sum_{1 \leq \nu(t,\alpha,r) \equiv 1, r \equiv 0 \mod (d^2,h)} \frac{1}{n} \sum_{\alpha \equiv r \mod (n,h)} \left(\frac{\alpha}{n}\right) \sum_{\mu(d), 1 \leq d \leq R} \mu(d) S(k,Y,n,t,\alpha,d) \]
where
\[ S(k,Y,n,t,\alpha,d) = \sum_{p \equiv \nu(t,\alpha,r) \mod [nd^2,h]} \frac{\sqrt{4p - t^2}}{p}, \]
and \( \nu(t,\alpha,r) \) is the invertible residue modulo \( [nd^2,h] \) solving the congruences
\[ 4p \equiv t^2 - r \mod h, \quad \text{for} \quad (t^2 - r, h) = 1, \]
\[ 4p \equiv t^2 - \alpha \mod n, \quad \text{for} \quad (t^2 - \alpha, n) = 1, \]
\[ 4p \equiv t^2 \mod d^2, \quad \text{for} \quad (t, d) = 1, \]
under the conditions \( (n, d) = 1, r \equiv \alpha \mod (n, h) \) and \( r \equiv 0 \mod (d^2, h) \) insuring that the congruences are compatible.

In order to compute the asymptotic of \( M \), we first change the weights and consider
\[ M' = \frac{2}{3\pi} \sum_{1 \leq \nu(t,\alpha,r) \equiv 1, r \equiv 0 \mod (d^2,h)} \frac{1}{n} \sum_{\alpha \equiv r \mod (n,h)} \left(\frac{\alpha}{n}\right) \sum_{\mu(d), 1 \leq d \leq R} \mu(d) S'(k,Y,n,t,\alpha,d), \]
where
\[ S'(k,Y,n,t,\alpha,d) = \sum_{p \equiv \nu(t,\alpha,r) \mod [kd^2,h]} \frac{\sqrt{4kY - t^2 \log p}}{kY \log kY}. \]
Lemma 6 Let $M$ and $M'$ be the two sums defined above where $Y = x/(K + 1)$ and $K = \lfloor \log^{23} x \rfloor$ for some $\beta \geq 1$. Then as $x \to \infty$

$$M - M' \ll \frac{x}{\log^{23} x}.$$ 

Proof: It is enough to note that

$$S(k, Y; n, t, \alpha, d) - S'(k, Y; n, t, \alpha, d) = \sum_{kY \leq p \leq kY + Y \atop p \equiv (t, \alpha, r) \mod \left(\lfloor nd^2, h \rfloor\right)} \left(\frac{\sqrt{4p - t^2}}{p} - \frac{\sqrt{4kY - t^2}}{kY}\right)$$

$$+ \sum_{kY \leq p \leq kY + Y \atop p \equiv (t, \alpha, r) \mod \left(\lfloor nd^2, h \rfloor\right)} \left(\frac{4kY - t^2}{kY} - \frac{4kY - t^2 \log p}{kY \log kY}\right) \ll \frac{Y}{kn \log^{23} k}. $$

Summing over $k, n, t, \alpha, d$, with trivial bounds for the remaining terms, gives the result. □

We then have to evaluate the sum $M'$ above. We first define some notation. For any positive integers $a, q$ such that $(a, q) = 1$, let

$$\psi(X, Y; a, q) = \sum_{X < p < X+Y \atop p \equiv a \mod q} \log p$$

$$E(X, Y; a, q) = \psi(X, Y; a, q) - \frac{Y}{\varphi(q)},$$

where, as usual, $\varphi$ denotes the Euler function.

Subtracting and adding $Y/\varphi([nd^2, h])$ to the sum $M'$ above, we write it as $M' = S_1 + S_2$ where

$$S_1 = \frac{2Y}{3\pi} \sum_{1 \leq k \leq K} \sum_{n \leq U \atop (d^2 - r, h) = 1} \sum_{1 \leq t \leq \sqrt{kY} \atop (t^2 - r, h) = 1} \frac{\sqrt{4kY - t^2}}{kY \log kY} \sum_{\alpha \equiv r \mod \left(\lfloor nd^2, h \rfloor\right)} \left(\frac{\alpha}{n}\right) \sum_{\mu(d) \atop d \equiv 0 \mod (d^2, h)} \varphi([nd^2, h])$$

and

$$S_2 \ll \sum_{1 \leq k \leq K} \frac{1}{\sqrt{kY}} \sum_{1 \leq t \leq \sqrt{kY} \atop (t^2 - r, h) = 1} \sum_{d \leq R \atop (d, n) = 1} \sum_{\alpha \equiv r \mod \left(\lfloor nd^2, h \rfloor\right)} |E(kY, Y; \nu(t, \alpha, r), [nd^2, h])|$$

$$\ll \sum_{1 \leq k \leq K} \sum_{q \leq R^2Uh} \sum_{\alpha \equiv r \mod \left(\lfloor nd^2, h \rfloor\right)} |E(kY, Y; a, q)| \ll \sum_{1 \leq k \leq K} \sum_{q \leq R^2Uh} \sum_{\alpha \equiv r \mod \left(\lfloor nd^2, h \rfloor\right)} |E(kY, Y; a, q)|$$

since each $q \leq R^2Uh$ can be written as $q = [nd^2, h]$ in at most $R$ ways, and for each fixed integer $t$, and for any $\alpha \neq \alpha' \mod n, \nu(t, \alpha, r) \neq \nu(t, \alpha', r)$ modulo $q = [nd^2, h]$. We now use:
Theorem 7 (Barban-Davenport-Halberstam) Let \(X, Y\) be positive integers such that \(X + Y \leq x\). Then, for any \(N > 0\), there exist \(M > 0\) such that

\[
\sum_{q \leq Q} \sum_{a \mod q \atop (a, q) = 1} |E(X, Y; a, q)|^2 \ll \frac{x^2}{\log^N x}
\]

whenever \(Q \leq x \log^{-M} x\).

Using Cauchy-Schartz and Theorem 7 on (17), we get that for any \(A > 0\)

\[
S_2 \ll R^3 \sum_{1 \leq k \leq K} \left( \sum_{q \leq R^2 / h \atop a \mod q \atop (a, q) = 1} |E(kY, Y; a, q)|^2 \right)^{1/2} \left( \sum_{q \leq R^2 / h \atop a \mod q \atop (a, q) = 1} \frac{1}{q^2} \right)^{1/2} \ll x \log^{-A} x \quad \text{(18)}
\]

since

\[R^2 U h \leq \frac{x}{\log^M x}\]

for any \(M > 0\), and \(x\) sufficiently large.

We now evaluate \(S_1\). Consider the sum

\[
S(T) = \sum_{1 \leq t \leq T \atop (t^2 - r, h) = 1} \sum_{n \leq U} \frac{1}{n} \sum_{\alpha \mod n \atop (t^2 - r, n, \alpha) = 1} \left( \frac{\alpha}{n} \right) \sum_{d \leq N \atop (d, n, h) = 1} \frac{\mu(d)}{\varphi(nd^2, h)}. \quad \text{(19)}
\]

Theorem 8 Let \(R, U\) as before. Then, as \(T, x \to \infty\) we have for any \(\varepsilon > 0\)

\[
S(T) = \frac{3}{2} \mathcal{C} T + O \left( \frac{T \log R}{R} \right) + O \left( \frac{1}{T^{1-\varepsilon}} \right).
\]

where \(\mathcal{C}\) is the constant given in (1).

Let us assume for a moment the previous theorem, which will be proved in the next section, and let us show how the evaluation of \(S_1\) in (16) follows from the asymptotic of \(S(T)\). This will complete the proof of Theorem 2.

We define \(F(t)\) such that

\[
S(T) = \sum_{1 \leq t \leq T \atop (t^2 - r, h) = 1} F(t),
\]

where \(S(T)\) is the sum defined in (19), and consider

\[
S_1(X) = \sum_{1 \leq t \leq 2 \sqrt{X - t^2} \atop (t^2 - r, h) = 1} \frac{\sqrt{4X - t^2}}{X \log X} F(t).
\]
Using Theorem 8 and partial summation we get
\[
S_1(X) = \frac{3\zeta}{X \log X} \int_0^{2\sqrt{X}} t^2(4X - t^2)^{-1/2} dt = \frac{3\zeta \pi}{\log X} + O(1/R). \tag{20}
\]
Replacing (20) into (16), we get
\[
S_1 = \frac{2Y}{3\pi} \sum_{1 \leq k \leq K} S_1(kY) = \zeta Y \sum_{1 \leq k \leq K} \frac{1}{\log kY} + O(x/R) \tag{21}
\]
Using (21), and collecting the error terms from (18), Lemma 6 and (15) with \(\beta \geq 3\), this shows that
\[
\sum_{p \leq x} \Pi^b(p) = \zeta x \log x + O(x \log^2 x),
\]
and it completes the proof of Theorem 2.

5 Proof of Theorem 8

Let
\[
c_t(n) = \sum_{\alpha \equiv r \mod (n,h)} \left(\frac{a}{n}\right)
\]
be the sum over residues modulo \(n\) that appears in (19).

Lemma 9 Let \(r, h, t\) be integers such that \((t^2 - r, h) = 1\), and let \(p\) be an odd prime. Then, \(c_t(n)\) is a multiplicative function of \(n\), with value at prime powers given by

1. If \(p \mid h\), then \(c_t(p^\ell)p^{-\ell} = \frac{1}{(p^\ell, h)} \left(\frac{r}{p^\ell}\right)\).

2. If \(p \nmid h\), then
\[
c_t(p^\ell)p^{-\ell} = \begin{cases} 
0 & \text{if } p \mid t, \ell \text{ odd;} \\
-1/p & \text{if } p \nmid t, \ell \text{ odd;} \\
1 - 1/p & \text{if } p \mid t, \ell \text{ even;} \\
1 - 2/p & \text{if } p \nmid t, \ell \text{ even.}
\end{cases}
\]

Proof: This is a straightforward case by case computation, and we omit it.

First, for each fixed odd \(t\) in (19), such that \((t^2 - r, h) = 1\), we evaluate the sum
\[
\sum_{n \leq U} \frac{c_t(n)}{n} \sum_{\substack{d \mid n \\\text{if } (d, n) = 1 \\\text{if } (d, n) = 1}} \frac{\mu(d)}{\varphi(nd^2, h)}.
\]
For any \((d, n) = 1\), we have that

\[
\varphi([nd^2, h]) = \frac{\varphi(n)\varphi(d^2)\varphi(h)}{\varphi((n, h))\varphi((d^2, h))} > \varphi(n)\varphi(d^2).
\]

(22)

We also observe that

\[
\sum_{d > R \atop (d, nt) = 1} \frac{1}{d^2} \ll \sum_{d > R} \frac{\log \log d}{d^2} \ll \frac{1}{R^{1/2 - \varepsilon}},
\]

and

\[
\sum_{\text{odd } n > U} \frac{c_\ell(n)}{n\varphi(n)} \ll \sum_{\text{odd } n > U} \frac{\log \log n}{n} \ll \sum_{\text{odd } n > U^{1/2}} \frac{2^{\omega(n)} \log \log n}{n^2} < \frac{1}{U^{1/2 - \varepsilon}}
\]

for any \(\varepsilon > 0\). Then,

\[
\sum_{n \leq U} \frac{c_\ell(n)}{n} \sum_{d > R \atop (d, nt) = 1 \atop r \equiv 0 \mod (d^2, h)} \frac{\mu(d)}{\varphi([nd^2, h])} = C_\ell + O \left( \frac{\log R}{R} \right),
\]

(23)

where

\[
C_\ell = \sum_{n \geq 1} \frac{c_\ell(n)}{n} \sum_{d > R \atop (d, nt) = 1 \atop r \equiv 0 \mod (d^2, h)} \frac{\mu(d)}{\varphi([nd^2, h])} = \sum_{d > 1} \frac{\mu(d)}{\varphi((d^2, h))} \sum_{n > 1 \atop (n, d) = 1} \frac{c_\ell(n)}{n\varphi([nd^2, h])}.
\]

(24)

We now use the multiplicativity of the functions involved to write the constant \(C_\ell\) as an Euler product.

**Lemma 10** Let \(r, h, t\) be integers such that \((t^2 - r, h) = 1\), and \((h, r)\) is squarefree. Then,

\[
C_\ell = P(r, h) \prod_{p \mid h} \left( 1 + \frac{2p^2 + p - 1}{p^4 - p^3 - 2p^2 - p + 1} \right),
\]

where

\[
P(r, h) = \frac{1}{\varphi(h)} \prod_{p \mid h} p - \left( \frac{2}{p} \right) \prod_{p \mid r} \left( 1 - \frac{1}{p} \right) \prod_{p \mid h} \frac{p^4 - p^3 - 2p^2 - p + 1}{p(p - 1)(p^2 - 1)},
\]

and all the products run over odd prime numbers.

**Proof:** We first observe that

\[
\frac{p^4 - p^3 - 2p^2 - p + 1}{p(p - 1)(p^2 - 1)} = 1 - \frac{p^2 + 2p - 1}{p(p - 1)(p^2 - 1)}.
\]

(25)
so each of the products of $C_t$ is convergent. Although intricate, the proof consists of a combination of straightforward computations using Lemma 9. We always assume that every prime appearing in the products below is odd. We first note that

$$\sum_{\begin{array}{c} d \geq 1 \\ (d, nt) = 1 \\ r \equiv 0 \mod (d^2, h) \end{array}} d \mu(d) \frac{\varphi(d^2, h)}{\varphi(d^2)} = \prod_{p \nmid h \ n t} \left(1 - \frac{1}{p}\right) \prod_{p \mid h t} \left(1 - \frac{1}{p} \frac{1}{p(p - 1)}\right)^{-1} \prod_{p \mid n \ n t} \left(1 - \frac{1}{p} \frac{1}{p(p - 1)}\right)^{-1},$$

Hence, using (22), we get

$$C_t = F(h, t) \sum_{n \geq 1}^{\text{odd}} \frac{c_t(n) \varphi(n, h)}{n \varphi(n)} \prod_{p \mid (r, n)} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p \mid n \ n t} \left(1 - \frac{1}{p} \frac{1}{p(p - 1)}\right)^{-1},$$

for

$$F(h, t) = \frac{1}{\varphi(h)} \prod_{p \mid h t} \left(1 - \frac{1}{p} \frac{1}{p(p - 1)}\right) \prod_{p \mid n \ n t} \left(1 - \frac{1}{p}\right).$$

Again, writing the sum over $n$ as an Euler product, we get that

$$C_t = F(h, t) \prod_p \left(1 + \delta(p) \sum_{j \geq 1} \frac{c_t(p^j) \varphi((p^j, h))}{p^j \varphi(p^j)}\right), \quad (26)$$

where

$$\delta(p) = \begin{cases} \left(1 - \frac{1}{p} \frac{1}{p(p - 1)}\right)^{-1} & \text{when } p \nmid h t; \\ \left(1 - \frac{1}{p}\right)^{-1} & \text{when } p \mid h, p \mid r, p \nmid t; \\ 1 & \text{otherwise.} \end{cases}$$

Finally, we use Lemma 9 to note that when $p \mid h$, $c_t(p^j) \frac{\varphi((p^j, h))}{\varphi(p^j)} = \left(\frac{r}{p}\right)^j$, independently of $t$. For primes $p \nmid h$, we have that $\varphi((p^j, h)) = 1$. Hence, by splitting the inner sum in (26) into odd and even terms, we get

$$\sum_{j \geq 1} \frac{c_t(p^j) \varphi((p^j, h))}{p^j \varphi(p^j)} = \frac{p}{(p - 1)(p^2 - 1)} \left(\frac{c_t(p)}{p} + c_t(p^2)\right).$$

The result now follows by considering the different cases in Lemma 9. □
We now proceed to prove Theorem 8. In the following, all primes $p$ are odd and all products are restricted to odd primes. Let us call $G$ the multiplicative function with value at a prime $p$ given by $G(p) = \frac{2p^2 + p - 1}{p^3 - p^3 - 2p^2 - p + 1}$. Using (23) and Lemma 10 we have that

$$S(T) = P(r, h) \sum_{t \leq T} \sum_{d | t, (d, h) = 1} \mu^2(d)G(d) + O\left(\frac{T \log R}{R}\right).$$

To compute the inner sum, we write

$$1 = \sum_{t \leq T/d} \sum_{k | d^2t^2 - r, h} \mu(k) = \sum_{k | h} \mu(k) \sum_{t \leq T/d, (d^2t^2 - r, h) = 1} 1.$$  

(28)

Because of the Moebius function, we can suppose that $k$ is square-free. In that case, by the Chinese Remainder Theorem, the number of solutions modulo $2k$ to the congruences $t^2 \equiv rd^{-2} \mod k$ and $t \equiv 1 \mod 2$ is

$$\prod_{p | k} \left(1 + \left(\frac{r}{p}\right)\right).$$

Using that in (28), we have that

$$1 = \sum_{t \leq T/d} \sum_{k | d^2t^2 - r, h} \mu(k) = \frac{T}{2d} \left(\prod_{p | h} \left(1 + \left(\frac{r}{p}\right)\right) + O(\tau(h)2^{\omega(h)}).\right.$$  

(29)

Now we observe that

$$\sum_{d | T/d} \frac{\mu^2(d)G(d)}{d} = \prod_{p | h} \left(1 + \frac{G(p)}{p}\right) + O\left(\frac{1}{T^{2-\epsilon}}\right).$$

(30)

Using (29) and (30) in (27) completes the proof of Theorem 8 with

$$c = \frac{P(r, h)}{3} \prod_{p | h} \left(1 - \frac{1 + \left(\frac{r}{p}\right)}{p}\right) \prod_{p | h} \left(1 + \frac{G(p)}{p}\right)$$

$$= \frac{1}{2h} \prod_{p | h} \frac{p-1}{p} \prod_{p | h} \frac{p(p-1) - \left(\frac{r}{p}\right)}{(p-1)(p - \left(\frac{r}{p}\right))} \prod_{p | h} \frac{p^4 - 2p^2 - p + 1}{p^2(p^2 - 1)}.$$
6 Conjectures and Constants

We develop in this section a precise conjecture for the asymptotic behavior of

\[ \Pi_{E,r,h}^b(x) = \# \{ 2 < \ell \leq x, \text{ prime } : a_p^2 - 4p \in \Delta(r, h) \}, \]

following the standard heuristics, as in \[LaTr\] or \[Ko\] (for different but related questions). We want \( a_p^2 - 4p \) to be square-free, i.e. not divisible by \( \ell^2 \), for any prime \( \ell \). If \( \ell \mid h \), we also have to take into account the fact that \( a_p^2 - 4p \equiv r \mod h \).

Fix an odd prime \( \ell \), and suppose for now that \( \ell \nmid h \). To measure the probability that \( a_p^2 - 4p \) is not divisible by \( \ell^2 \), we use the Galois extension \( \mathbb{Q}(E[\ell^2])/\mathbb{Q} \), where \( \mathbb{Q}(E[\ell^2]) \) is the field obtained from \( \mathbb{Q} \) by adjoining the coordinates of all the \( \ell^2 \)-torsion points of \( E \). Since \( E[\ell^2] \), the group of \( \ell^2 \)-torsion points of \( E \), is isomorphic to \( \mathbb{Z}/\ell^2 \mathbb{Z} \times \mathbb{Z}/\ell^2 \mathbb{Z} \) as an abstract group, fixing a basis of \( E[\ell^2] \), we obtain an injective group homomorphism

\[ \rho_{\ell^2} : \text{Gal}(\mathbb{Q}(E[\ell^2])/\mathbb{Q}) \hookrightarrow \text{GL}_2(\mathbb{Z}/\ell^2 \mathbb{Z}). \]

Let \( G(\ell^2) \) be the image of \( \rho_{\ell^2} \) in \( \text{GL}_2(\mathbb{Z}/\ell^2 \mathbb{Z}) \). For any prime \( p \) unramified in \( \mathbb{Q}(E[\ell^2])/\mathbb{Q} \) (i.e. \( p \nmid \ell N_E \)), the Frobenius automorphism \( \sigma_p \) is sent to a conjugacy class of matrices in \( \text{GL}_2(\mathbb{Z}/\ell^2 \mathbb{Z}) \) such that for any \( g \) in the conjugacy class

\[ \text{tr}(g) \equiv a_p \mod \ell^2 \]
\[ \text{det}(g) \equiv p \mod \ell^2. \]

Let

\[ \Omega(\ell^2) = \{ g \in G(\ell^2) : \text{tr}(g)^2 - 4 \text{det}(g) \not\equiv 0 \mod \ell^2 \}. \]

We can then measure the probability that \( \ell^2 \nmid a_p^2 - 4p \) by the ratio

\[ P_1(\ell) = \frac{|\Omega(\ell^2)|}{|G(\ell^2)|}. \]

By Serre’s Theorem \[Se\], \( G(\ell^2) = \text{GL}_2(\mathbb{Z}/\ell^2 \mathbb{Z}) \) for all but finitely many primes \( \ell \). Furthermore, there is an integer \( M_E \), depending on the elliptic curve \( E \), such that the Galois group of the extension obtained by adjoining the coordinates of all torsion points of \( E \) to \( \mathbb{Q} \) is the full inverse image in \( \text{GL}_2(\mathbb{Z}) \) of \( \text{Gal}(\mathbb{Q}(E[M_E])/\mathbb{Q}) \subseteq \text{GL}_2(\mathbb{Z}/M_E \mathbb{Z}) \), where \( \mathbb{Z} \) is the ring of adeles of \( \mathbb{Z} \). In particular, \( \ell \nmid M_E \) implies that \( G(\ell^2) = \text{GL}_2(\mathbb{Z}/\ell^2 \mathbb{Z}) \). The integer \( M_E \) is not uniquely defined, and if some integer \( m \) satisfies the property above, so does any multiple of \( m \). We remark that \( M_E \) is always even.

Then, for \( \ell \nmid M_E \), we compute that

\[ P_1(\ell) = \frac{|\{ g \in \text{GL}_2(\mathbb{Z}/\ell^2 \mathbb{Z}) : \text{tr}(g)^2 - 4 \text{det}(g) \not\equiv 0 \mod \ell^2 \}|}{|\text{GL}_2(\mathbb{Z}/\ell^2 \mathbb{Z})|} = \frac{\ell^4 - 2\ell^2 - \ell + 1}{\ell^2(\ell^2 - 1)}. \]

Suppose now that \( \ell \mid M_E \), but \( \ell \mid h \). We write \( h = \ell^\alpha h' \) where \( \alpha \geq 1 \) and \( (h', \ell) = 1 \). There are two congruence conditions modulo powers of \( \ell \), namely \( a_p^2 - 4p \not\equiv 0 \mod \ell^2 \) and
\[ a_p^2 - 4p \equiv r \mod \ell^\alpha. \] Let \( \beta = \max(\alpha, 2) \). The probability that \( a_p^2 - 4p \) satisfies both congruences is then the ratio
\[
P_2(\ell) = \frac{|\{g \in G(\ell^\beta) : \text{tr}(g)^2 - 4 \det(g) \not\equiv 0 \mod \ell^2 \text{ and } \text{tr}(g)^2 - 4 \det(g) \equiv r \mod \ell^\alpha\}|}{|G(\ell^\beta)|}.
\]

Again, Serre’s theorem guarantees that \( G(\ell^\beta) = GL_2(\mathbb{Z}/\ell^\beta\mathbb{Z}) \) since \( \ell \nmid M_E \). For such an odd prime \( \ell \), one computes that the ratio above is
\[
P_2(\ell) = \begin{cases} \frac{\ell(\ell - 1 - (\frac{r}{\ell}))}{(\ell - 1)(\ell - (\frac{r}{\ell}))\ell^\alpha} & \text{when } \ell \nmid r; \\ \frac{\ell - 1}{\ell^{\alpha+1}} & \text{when } \ell \mid r \text{ and } \alpha = 1; \\ \frac{1}{\ell^\alpha} & \text{when } \ell \parallel r \text{ and } \alpha \geq 2; \\ 0 & \text{when } \ell^2 \mid r \text{ and } \alpha \geq 2. \end{cases}
\]

The last case \( (\ell^2 \mid r \text{ and } \alpha \geq 2) \) happens only when \( \ell^2 \mid (r,h) \), which is impossible when \( (r,h) \) is square-free. If \( (r,h) \) is not square-free, it is clear that \( \Pi_{E,r,h}^0(x) \) is empty.

We now put all the local probabilities together. When \( \ell_1, \ell_2 \nmid M_E \), the probabilities modulo \( \ell_1, \ell_2 \) are independent by Serre’s Theorem. Using the probability \( P_1(\ell) \) when \( \ell \nmid hM_E \) and \( P_2(\ell) \) when \( \ell \nmid M_E, \ell \mid h \), the conjectural constant is
\[
C_{SF}(E,r,h) = \frac{1}{h} P(M_E) \prod_{\ell \mid M_E} \prod_{\ell \mid h} \frac{\ell - 1}{\ell} \prod_{\ell \nmid h} \frac{\ell(\ell - 1 - (\frac{r}{\ell}))}{(\ell - 1)(\ell - (\frac{r}{\ell}))\ell^\alpha} \prod_{\ell \nmid h} \frac{\ell^4 - 2\ell^2 - \ell + 1}{\ell^2(\ell^2 - 1)}, \quad (31)
\]

where \( P(M_E) \) is the probability modulo \( M_E \). In order to give an explicit expression for \( P(M_E) \), we need some notation. Let
\[
M_E = \prod_{\ell \mid M_E} \ell^{\gamma(\ell)} \quad \text{and} \quad h = \prod_{\ell \mid h} \ell^{\alpha(\ell)}.
\]

If \( \ell \mid M_E \), but \( \ell \nmid h \), we define \( \alpha(\ell) \) to be 0. For each prime \( \ell \mid M_E \), let
\[
\beta(\ell) = \max\{2, \gamma(\ell), \alpha(\ell)\}.
\]

Finally, let
\[
m = \prod_{\ell \mid M_E} \ell^{\beta(\ell)},
\]

and let \( G(m) \) be the image of the Galois representation
\[
\rho_m : \text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q}) \hookrightarrow GL_2(\mathbb{Z}/m\mathbb{Z}).
\]

Then,
\[
P(M_E) = \frac{|\{g \in G(m) : \text{tr}(g)^2 - 4 \det(g) \not\equiv 0 \mod \ell^2 \text{ and } \text{tr}(g)^2 - 4 \det(g) \equiv r \mod \ell^{\alpha(\ell)} \forall \ell \mid M_E\}|}{|G(m)|}.
\]
Conjecture 11 Let $h$ be a positive odd integer, and $r$ an integer such that $(h, r)$ is square-free. Then as $x \to \infty$

$$
\Pi_{E,(a,b),r,h}^b(x) \sim C_{SF}(E, r, h) \frac{x}{\log x},
$$

where $C_{SF}(E, r, h)$ is the constant given by (31).

Then, Theorem 2 gives evidence for the asymptotic of Conjecture 11 and also for the constant $C_{SF}(E, r, h)$ appearing on the asymptotic.

A similar heuristic leads to the following conjecture for the number of primes $p$ smaller than $x$ such that $p + 1 - a_p(E)$ is square-free.

Conjecture 12 Let $\Pi_E(x)$ be the number of primes of good reduction such that $p + 1 - a_p(E)$ is square-free. Then as $x \to \infty$

$$
\Pi_E(x) \sim C_{SF}'(E) \frac{x}{\log x},
$$

where $C_{SF}'(E)$ is the constant

$$
C_{SF}'(E) = \frac{\Omega(M_E)}{G(M_E)} \prod_{\ell \mid M_E} 1 - \frac{\ell^3 - \ell - 1}{(\ell^2 - 1)(\ell - 1)}. \]

The work of Gekeler mentioned in the introduction then gives evidence for Conjecture 12 and in particular for the constant $C_{SF}'(E)$ appearing in the conjecture. We stress again that the techniques used by Gekeler and our techniques are completely different, but they both uncover the conjectural constants computed from Galois representations in that section (and of course, neither the work of Gekeler or the work of this present paper relies on Galois representations, and counting elements of Galois groups).

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