Data complexity of answering conjunctive queries over $SHIQ$ knowledge bases

Technical Report
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Abstract In [6] the authors give an algorithm for answering conjunctive queries over $ALCNR$ knowledge bases which is coNP in data complexity. Their technique is based on the tableau technique for checking satisfiability in $ALCNR$ presented in [2]. In their algorithm, the blocking conditions of [2] are weakened in such a way that the set of models their algorithm yields suffices to check query entailment. The algorithm we propose consists on applying a similar technique to the tableaux algorithm in [4], which decides the satisfiability of $SHIQ$ knowledge bases. As a result we have an algorithm for answering conjunctive queries over $SHIQ$ knowledge bases that is also coNP in terms of data complexity.

1 Introduction

The idea of using description logic (DL) knowledge bases to represent the conceptual view of data repositories is becoming popular nowadays. In the context of large data repositories with a fixed schema, query answering becomes a key issue and the size of the data is the main parameter for measuring complexity. While atomic queries (A-Box reasoning) have always been considered an essential reasoning task in description logics, conjunctive queries and other kind of queries have recently become a topic of interest. Data complexity of query answering over DL knowledge bases was already studied in [7]. Many of the existing results correspond to the fragment of DLs for which the problem remains polynomial and the LogSPACE boundary of such logics, that has been studied in detail in [3]. It is known that for rather simple DLs, even less expressive than $AŁC$, the problem is already coNP hard [7,3]. However, results concerning complexity upper bounds are scarce. In [6] the authors prove that answering conjunctive queries over $ALCNR$ knowledge bases is in coNP w.r.t. data complexity and they provide a worst case optimal algorithm for solving the problem. In this work, we address the same problem for more expressive DLs, namely ones that have inverse roles and role hierarchies. In [5], a data complexity coNP upper bound for ground atomic queries over $SHIQ$ knowledge bases is given, but their technique does not yield such an upper bound for conjunctive queries.

In this work we use a tableau algorithm. The algorithm proposed in [6] is based on the tableau technique for checking satisfiability in $ALCNR$ presented.
in [2]. The key issue is that the blocking conditions of [2] are weakened in such a
way that it can be ensured that the query is entailed by the knowledge base iff it
is entailed by the models obtained via this algorithm. The algorithm we propose
consists basically on applying the same technique to the tableaux algorithm
in [4], which decides the satisfiability of a \(SHIQ\) knowledge base. As a result
we have an algorithm for answering conjunctive queries over \(SHIQ\) knowledge
bases that is CoNP in data complexity.

2 Preliminaries

2.1 \(SHIQ\) Knowledge Bases

The syntax and semantics of \(SHIQ\) are defined in the standard way.

**Definition 1 (\(SHIQ\) knowledge base).** Let \(C\) be a set of concept names and
\(R\) a set of role names with a subset \(R_+ \subseteq R\) of transitive role names. The set
of roles is \(R \cup \{R^- \mid R \in R\}\). The function Inv and Trans are defined on roles.
Inv is defined as \(Inv(R) = R^-\) and \(Inv(R^-) = R\) for any role name \(R\). Trans is
a boolean function, \(Trans(R) = true\) iff \(R \in R_+\) or \(Inv(R) \in R_+\).

A role inclusion axiom is an expression of the form \(R \sqsubseteq S\) where \(R\) and \(S\) are
roles. A role hierarchy is a set of role inclusion axioms. The relation \(\sqsubseteq^*\) denotes
the transitive closure of \(\sqsubseteq\) over a role hierarchy \(R \cup \{Inv(R) \sqsubseteq S \mid R \subseteq S \in R\}\). We say that \(R\) is a sub-role of \(S\) when \(R \sqsubseteq^* S\), and a super-role of \(S\) when
\(S \sqsubseteq^* R\). We will assume that it is never the case that \(R\) is both a sub-role and
a super-role of \(S\). A role is simple if its neither transitive nor has transitive
sub-roles.

The set of \(SHIQ\) concepts is the smallest set such that:

- Every concept name is a concept,
- If \(C\) and \(D\) are concepts, \(R\) is a role, \(S\) is a simple role and \(n\) is a non-
negative integer, then \(C \sqcap D, C \sqcup D, \neg C, \forall R.C, \exists R.C, \geq n S.C, \leq n S.C\)
are concepts.

A concept inclusion axiom is an expression of the form \(C \sqsubseteq D\) for two
concepts \(C\) and \(D\). A terminology or T-Box is a set of concept inclusion axioms.

Let \(I\) be a set of individual names. An assertion is an expression that can have the form \(C(a), R(a,b)\) or \(a \not\approx b\) where \(C\) is a concept, \(R\) is a role and
\(a,b \in I\). An A-Box is a set of assertions.

A \(SHIQ\) knowledge base is a triple \(K = \langle A, R, T \rangle\), where \(A\) is an A-Box,
\(R\) is role hierarchy and \(T\) is a terminology.

The semantics of \(SHIQ\) knowledge bases is given by interpretations.

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1 This consideration is done for practical purposes, however it does not restrict the
expressiveness of the language. It is clear that if \(R\) is at the same time a sub-role
and a super-role of \(S\) both roles will have the same extension and one of them can
be eliminated by replacing it by the other.
Definition 2 (Interpretation). An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ is defined for a set of individual names $\mathcal{I}$, a set of concepts $C$ and a set of roles $R$. The set $\Delta^\mathcal{I}$ is called domain of $\mathcal{I}$. The valuation $\cdot^\mathcal{I}$ maps each individual name in $\mathcal{I}$ to an element in $\Delta^\mathcal{I}$, each concept in $C$ to a subset of $\Delta^\mathcal{I}$, and each role in $R$ to a subset of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$. Additionally, for any concepts $C, D$, any role $R$ and any non-negative integer $n$, the valuation $\cdot^\mathcal{I}$ must satisfy the following equations:

\[
\begin{align*}
R^\mathcal{I} &= (R^\mathcal{I})^+ \quad \text{for each role } R \in R_+ \\
(R^-)^\mathcal{I} &= \{ (y, x) \mid (x, y) \in R^\mathcal{I} \} \\
(C \cap D)^\mathcal{I} &= C^\mathcal{I} \cap C^\mathcal{I} \\
(C \cup D)^\mathcal{I} &= C^\mathcal{I} \cup C^\mathcal{I} \\
(\neg C)^\mathcal{I} &= \Delta^\mathcal{I} \setminus C^\mathcal{I} \\
(\forall R.C)^\mathcal{I} &= \{ x \mid \text{for all } y, (x, y) \in R^\mathcal{I} \text{ implies } y \in C^\mathcal{I} \} \\
(\exists R.C)^\mathcal{I} &= \{ x \mid \text{for some } y, (x, y) \in R^\mathcal{I} \text{ and } y \in C^\mathcal{I} \} \\
(\geq n R.C)^\mathcal{I} &= \{ x \mid \{ y \mid (x, y) \in R^\mathcal{I} \text{ and } y \in C^\mathcal{I} \} \geq n \} \\
(\leq n R.C)^\mathcal{I} &= \{ x \mid \{ y \mid (x, y) \in R^\mathcal{I} \text{ and } y \in C^\mathcal{I} \} \leq n \} 
\end{align*}
\]

Definition 3 (Model of a knowledge base). An interpretation $\mathcal{I}$ satisfies an assertion $A$ iff:

- $a \in C^\mathcal{I}$ if $A$ is of the form $C(a)$
- $(a, b) \in R^\mathcal{I}$ if $A$ is of the form $R(a, b)$
- $a^\mathcal{I} \neq b^\mathcal{I}$ if $A$ is of the form $a \neq b$

An interpretation $\mathcal{I}$ satisfies an A-Box $A$ if it satisfies every assertion in $A$. $\mathcal{I}$ satisfies a role hierarchy $\mathcal{R}$ if $R^\mathcal{I} \subseteq S^\mathcal{I}$ for every $R \subseteq S$ in $\mathcal{R}$. $\mathcal{I}$ satisfies a terminology $\mathcal{T}$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ for every $C \subseteq D$ in $\mathcal{T}$. $\mathcal{I}$ is a model of $K = (A, R, T)$ if it satisfies $A$, $R$ and $T$.

A SHIQ concept is said to be in negation normal form (NNF) if negation occurs only in front of concept names. Since concepts can be translated into NNF in linear time [4], we will assume that all concepts are in NNF. We denote by $NNF(\neg C)$ the NNF of the concept $\neg C$. The closure of a concept $\text{clos}(C)$ is the smallest set containing $C$ that is closed under subconcepts and negation (in NNF). For a knowledge base $K$, $\text{clos}(K) = \cup_{C(a) \in K} \text{clos}(C)$.

Global Constraint Concepts.

A knowledge base $K$ has an associated set of concepts that we will call the global constraint concepts of $K$. This set contains two kinds of concepts:

- For each concept inclusion axiom $C \subseteq D$ in the TBox, there is a global constraint concept of the form $\neg C \cup D$. This way, if we assure that all individuals in a model belong to the extension of global constraint concepts, the model will satisfy the T-Box of $K^2$.

\footnote{In [4] the authors consider an internalised T-Box. We do not make this assumption.}
– We will consider that $K$, additionally to the A-Box, T-Box, R-Box, might have a set of distinguished concepts names that we will denote $C_K$. In order not to make the notation too cumbersome, we will not denote it explicitly as a part of $K$. For all concept names $C$ in $C_K$ the concept $C \sqcup \neg C$ belongs to the global constraints of $K$. In the algorithm we present in the following sections, we will use partial representations of models of a knowledge base to verify whether some formula $Q$ is entailed in them. In these partial representations it may remain undecided whether some individuals belong to the extension of a concept or of its negation. However, for the concepts that appear in $Q$, we want to assure that the decision is taken. We will later see that in our framework, the set $C_K$ will be used to represent the concepts that may appear in the queries to be answered.

Definition 4 (Global Constraint Concepts). Given a knowledge base $K = \langle A, T, R \rangle$ and a set of distinguished concept names $C_K$, the set of global constraint concepts for $K$ and $C_K$ is defined as $\text{const}(K, C_K) = \{\neg C \sqcup D \mid C \sqsubseteq D \in T \} \cup \{C \sqcup \neg C \mid C \in C_K\}$.

If not stated otherwise, in the following $K$ will denote a SHIQ knowledge base $K = \langle A, R, T \rangle$, $R_K$ the roles occurring in $K$ together with their inverses, $\text{clos}(K)$ the closure of the concept names occurring in $A$, $C_K$ will denote a distinguished set of concept names, and $I_K$ the individual names occurring in $A$.

2.2 Answering Conjunctive Queries over Knowledge Bases

In the traditional database setting, free variables in a query are called distinguished variables. For a query $Q$ that has $X$ as distinguished variables, the query answering problem over $K$ consists on finding all the possible tuples of constants $T$ of the same arity as $X$ such that when $X$ is substituted by $T$ in $Q$, it holds that $K \models Q$. The set of such tuples $T$ is the answer of the query. Query answering has an associated recognition problem: given a tuple $T$, the problem is to verify whether $T$ belongs to the answer of $Q$\textsuperscript{4}. We say that query answering for a certain description logic is in a class $C$ w.r.t. data complexity when the corresponding recognition problem is in $C$. Since we will only focus on the recognition problem, we allow conjunctive queries to contain constants and we are assuming that all variables in the query are existentially quantified.

Definition 5 (conjunctive query). A conjunctive query over a knowledge base $K$ is a sentence of the form

$$(\exists Y_1 \ldots \exists Y_n).p_1(Y_1) \land \ldots \land p_n(Y_n)$$

\textsuperscript{3} If $C_K = \text{clos}(K)$, the algorithm can be used to check entailment w.r.t. any concept in the knowledge base, however this may be inconvenient from an implementation perspective.

\textsuperscript{4} This problem is usually known as the query output problem.
where \( p_1, \ldots, p_n \) are either roles in \( R_K \) or concepts in \( C_K \); \( \overline{Y}_1, \ldots, \overline{Y}_n \) are tuples of variables and constants. \( V_Q = \overline{Y}_1 \cup \ldots \cup \overline{Y}_n \) denotes the set of variables and constants in \( Q \). The set of literals in \( Q \) is \( L_Q = \{ p_1(\overline{Y}_1), \ldots, p_n(\overline{Y}_n) \} \), and the cardinality of \( L_Q \) will be denoted by \( n_Q \).

Conjunctive queries are interpreted in the standard way, i.e. \( I = (\Delta^I, \cdot^I) \) is a model of \( Q \) if there is a mapping \( \sigma \) from the variables and constants in \( Q \) to objects in \( \Delta^I \) such that \( \sigma \) is the identity on all constants and \( \sigma(\overline{Y}) \in p^I \) for all \( p(\overline{Y}) \in L_Q \). For a knowledge base \( K \) and a query \( Q \), we say that \( K \models Q \) iff for every interpretation \( I, I \models K \) implies \( I \models Q \). Analogously, for a completion forest \( F \) and a query \( Q \), we say that \( F \models Q \) iff for every interpretation \( I, I \models F \) implies \( I \models Q \).

**Definition 6 (Conjunctive Query Entailment).** Let \( K \) be a knowledge base and let \( Q \) be conjunctive query. The conjunctive query entailment problem is to decide whether \( K \models Q \).

We are interested in solving the conjunctive query entailment problem. It is important to notice that the conjunctive query entailment problem is not reducible to satisfiability of knowledge bases, since the negation of the query cannot not be expressed as a part of a knowledge base. For this reason, the known algorithms for reasoning over knowledge bases do not suffice. A knowledge base \( K \) has an infinite number possibly infinite models, and we have to verify whether the query \( Q \) is entailed by all of them. In general, we want to provide an entailment algorithm, i.e. an algorithm for checking whether a sentence \( Q \) with a particular syntax (namely, a conjunctive query) is entailed by a \( SHIQ \) knowledge base \( K \). Informally, our algorithm differs from the one proposed in [4] for reasoning with individuals in \( SHIQ \) in the fact that, since they only focus on problems that can be reduced to checking satisfiability, they only need to ensure that if the knowledge base has some model then their algorithm will obtain a model. In our case, however, this is not enough. We need to make sure that the algorithm obtains a set of models \( M \) such that \( Q \) is entailed by \( K \) iff it is entailed by every model in \( M \).

### 3 A \( SHIQ \) Entailment Algorithm

We will provide an algorithm for checking entailment of some sentence \( Q \) in a \( SHIQ \) knowledge base \( K \), i.e. to check if all models of \( K \) are models of \( Q \). Like the algorithm in [4], we will use completion forests. A completion forest is a relational structure that captures sets of models of a knowledge base. A completion forest is always finite, and it represents a set of possibly infinite models. When defining completion forests, we will use a parameter \( n \) that is not present in [4]. This parameter will be crucial in ensuring that the application of our algorithm will yield a set of models \( M \) such that \( Q \) is entailed by \( K \) iff it is entailed by every model in \( M \). We will see later that this parameter will take values that depend on \( Q \).
3.1 Completion Forests

A forest will be defined as a set of variable trees. A variable tree is a tree where the nodes are variables, and where the nodes and arcs of the tree are labeled. For any nodes \( n_1 \) and \( n_2 \), \( \mathcal{L}(n_1) \) will denote the label of \( n_1 \) and \( \mathcal{L}((n_1, n_2)) \) will denote the label of the arc that goes from \( n_1 \) to \( n_2 \).

**Definition 7** *(n-tree equivalence).* Given a variable tree \( V \) s.t. \( v \) is a node of \( V \), the \( n \)-tree of \( v \) is the subtree of \( V \) that has \( v \) as its root and contains the successors of \( v \) that are at most \( n \) direct successor arcs away. We denote by \( V_n(v) \) the set of nodes of \( V \) that appear in the \( n \)-tree of \( v \). Two variables \( v, w \) in \( V \) are \( n \)-tree equivalent in \( V \) if there is an isomorphism \( \psi \) between their \( n \)-trees, i.e. \( \psi : V_n(v) \rightarrow V_n(w) \) is a mapping such that:

- \( \psi(v) = w \)
- For every node \( n \) in \( V_n(v) \), \( \mathcal{L}(n) = \mathcal{L}(\psi(n)) \)
- For every arc connecting two nodes \( n_1 \) and \( n_2 \) in \( V_n(v) \), \( \mathcal{L}((n_1, n_2)) = \mathcal{L}((\psi(n_1), \psi(n_2))) \).

**Definition 8** *(n-Witness).* Let \( V \) be a variable tree where both \( v \) and \( w \) are nodes. We say that \( w \) is an \( n \)-witness of \( v \) in \( V \) iff \( w \) is an ancestor of \( v \) in \( V \), \( w \) is \( n \)-tree equivalent to \( v \) in \( V \) and \( v \) is not in the \( n \)-tree of \( w \). Let \( t \) denote the \( n \)-tree of which \( v \) is root, \( t^\prime \) the \( n \)-tree that has \( w \) as root, and let \( \psi \) denote an isomorphism between \( t \) and \( t^\prime \). In this case, we say that \( t^\prime \) tree-blocks \( t \). For all variables \( x \) in \( t \), we say that \( \psi(x) \) tree-blocks \( x \).

**Definition 9** *(n-Completion Forest).*

A completion forest for a knowledge base \( K \) is given by a forest of trees and an inequality relation \( \not\approx \) which is assumed to be symmetric. The forest is a set of variable trees whose roots are the individuals in \( I_A \). The roots can be connected by edges in an arbitrary way. \( \mathcal{L}(x) \subseteq \text{clos}(K) \) denotes the label of a node \( x \), and \( \mathcal{L}((x, y)) \subseteq \text{R}_K \) denotes the label of an edge \( (x, y) \).

If two nodes \( x, y \) are connected by an edge with \( R \in \mathcal{L}(x, y) \) and \( R \subseteq S \) then \( y \) is an \( S \)-successor of \( x \) and \( y \) is an \( S \)-predecessor of \( x \). If \( y \) is an \( S \)-successor of \( x \), then \( y \) is an \( S \)-descendant of \( x \). If \( z \) is an \( S \)-descendant of \( x \), \( y \) is an \( S \)-descendant of \( z \) and \( S \in \text{R}_+ \), then \( y \) is an \( S \)-descendant of \( x \). If \( x \) is an \( S \)-successor or an \( S \)-predecessor of \( y \), then \( x \) is an \( S \)-neighbor of \( y \). If \( x \) is an \( S \)-successor of \( y \) for some role \( S \), then \( x \) is a successor of \( y \) and \( y \) is a predecessor of \( x \). The transitive closure of predecessor is called ancestor.

A node is blocked iff it is not a root node and it is either directly or indirectly blocked. A node is indirectly blocked iff one of its ancestors is blocked or if it’s a successor of a node \( x \) and \( \mathcal{L}(x, y) = \emptyset \). A node is directly blocked iff none of its ancestors are blocked and it is a leaf of an \( n \)-tree that is tree-blocked.

**Definition 10** *(Clash free completion forest).* A node \( x \) in a completion forest \( F \) contains a clash iff for some concept name \( C \), \( C \in \mathcal{L}(x) \) and \( \lnot C \in \mathcal{L}(x) \) or if \( \leq n \text{R.C.} \in \mathcal{L}(x) \) and \( x \) has \( n+1 \) \( R \)-successors \( y_0, \ldots, y_n \) such that \( C \in \mathcal{L}(y_i) \) for all \( y_i \) and \( y_i \neq y_j \in \mathcal{F} \) for all \( 0 \leq i < j \leq n \). A completion forest \( \mathcal{F} \) is clash free if none of its nodes contains a clash in \( \mathcal{F} \).
Definition 11 (Complete completion forest). A completion $F$ is complete if none of the rules in Table 1 can be applied to it.

3.2 The Completion Forest Algorithm

Given a knowledge base $K = \langle A, R, T \rangle$ and a blocking parameter $n$, the algorithm does the following: An initial completion forest for $K$ is built and it is expanded using the rules in Table 1 until no more expansions can be obtained. The (possibly empty) set of complete and clash-free $n$-completion forests obtained by this expansion induce a set of models for $K$. As we will see in the coming sections, this set of models can be used to check entailment of a conjunctive query $Q$ if a suitable $n$ (depending on $Q$) is used.

Initializing the Completion Forest. An initial completion forest $F_K$ for a knowledge base $K$ is constructed as follows:

- For each individual $a_i \in I_K$ a node $a_i$ is introduced.
- An edge $\langle a_i, a_j \rangle$ is created if $R(a_i, a_j) \in A$ for some role $R$.
- The labels of these nodes and edges as well as the $\not= \not=$ relation are initialized as follows:

\[
\begin{align*}
\mathcal{L}(a_i) & := \{ C \mid C(a_i) \in A \} \cup \text{const}(K, \mathcal{C}_K) \\
\mathcal{L}(\langle a_i, a_j \rangle) & := \{ R \mid R(a_i, a_j) \in A \} \\
a_i \neq a_j & \quad \text{iff} \quad a_i \neq a_j \in A
\end{align*}
\]

Expanding the Completion Forests. From the initial completion forest, new completion forests for $K$ can be obtained by applying the rules in Table 1. Note that the application of the rules is non-deterministic. Different choices for $E$ in the $\sqcup$-rule and the choose-rule generate different forests. The $\exists$-rule and the $\geq$-rule are called generating rules since they add new nodes to the forest.

The set of $n$-completion forests for a knowledge base $K$ is denoted by $\mathcal{F}_n^K$ and it is the smallest set satisfying the following conditions:

1. The initial completion forest $F_K$ is a completion forest for $K$.
2. If $F$ is a legal $n$-completion forest for $K$ and $F'$ can be obtained from $F$ by applying one of the rules in Table 1 using $n$-blocking, then $F'$ is a $n$-completion forest for $K$.

Completion Forests as Semantic Objects. Semantically, we can interpret a completion forest in the way we interpret a knowledge base. For a knowledge base $K$ and a completion forest $F$ for $K$, note that all the individuals in $I_K$ are nodes in $F$, node labels in $F$ are concepts in $\text{clos}(K) \cup \mathcal{C}_K$ and edge labels in $F$ are roles in $\mathcal{R}_K$, hence interpretations for $K$ can be interpretations for $F$ and vice-versa. We will see completion forests as a representation of a set of models of the knowledge base. It is not a common practice to give a semantical interpretation to completion forests. However, this reading will make easier some of our results and proofs.
| Rule  | Description                                                                 |
|-------|-----------------------------------------------------------------------------|
| ¬-rule | if \( C_1 \cap C_2 \in \mathcal{L}(x) \), \( x \) is not indirectly blocked and \( \{C_1, C_2\} \not\subseteq \mathcal{L}(x) \) then \( \mathcal{L}(x) := \mathcal{L}(x) \cup \{C_1, C_2\} \) |
| ∪-rule | if \( C_1 \cup C_2 \in \mathcal{L}(x) \), \( x \) is not indirectly blocked and \( \{C_1, C_2\} \cap \mathcal{L}(x) = \emptyset \) then \( \mathcal{L}(x) := \mathcal{L}(x) \cup \{E\} \) for some \( E \in \{C_1, C_2\} \) |
| ∃-rule | if \( \exists S, C \in \mathcal{L}(x) \), \( x \) is not blocked and \( x \) has no \( S \)-neighbour \( y \) with \( C \in \mathcal{L}(y) \) then create new node \( y \) with \( \mathcal{L}((x, y)) := \{S\} \) and \( \mathcal{L}(x) := \{C\} \cup \text{const}(K, C_K) \) |
| ∀-rule | if \( \forall S, C \in \mathcal{L}(x) \), \( x \) is not indirectly blocked and there is an \( S \)-neighbour \( y \) of \( x \) with \( C \notin \mathcal{L}(y) \) then \( \mathcal{L}(y) := \mathcal{L}(y) \cup \{C\} \) |
| ∀₁-rule | if \( \forall S, C \in \mathcal{L}(x) \), \( x \) is not indirectly blocked, there is some \( R \) with \( \text{Trans}(R) \) and \( R \subseteq^* S \) and there is an \( S \)-neighbour \( y \) of \( x \) with \( \forall R \in \mathcal{L}(y) \) then \( \mathcal{L}(y) := \mathcal{L}(y) \cup \{\forall R\} \) |
| choose-rule | if \( \leq n \ S, C \subseteq \mathcal{L}(x) \) or \( \geq n \ S, C \subseteq \mathcal{L}(x) \), \( x \) is not indirectly blocked and there is an \( S \)-neighbour \( y \) of \( x \) with \( \{C, \text{NNF}(-C)\} \cap \mathcal{L}(y) = \emptyset \) then \( \mathcal{L}(y) := \mathcal{L}(y) \cup \{\forall R, C\} \) |
| ≥-rule | if \( \geq n \ S, C \subseteq \mathcal{L}(x) \), \( x \) is not blocked and there are not \( S \)-neighbours \( y_1, \ldots, y_n \) of \( x \) such that \( C \notin \mathcal{L}(y_i) \) and \( y_i \neq y_j \) for \( 1 \leq i < j \leq n \) then create new nodes \( y_1, \ldots, y_n \) with \( \mathcal{L}((x, y_i)) := \{S\} \), \( \mathcal{L}(y_i) := \{C\} \cup \text{const}(K, C_K) \) and \( y_i \neq y_j \) for \( 1 \leq i < j \leq n \) |
| ≤-rule | if \( \leq n \ S, C \subseteq \mathcal{L}(x) \), \( x \) is not indirectly blocked, \( |\{y \mid y \text{ is an} \ S \text{-neighbour of} \ x \text{ and} \ C \subseteq \mathcal{L}(y)\}| > n \) and there are \( S \)-neighbours \( y, z \) of \( x \) with \( y \neq z \), \( y \) is neither a root node nor an ancestor of \( z \) and \( C \subseteq \mathcal{L}(y) \cap \mathcal{L}(z) \) then \( \mathcal{L}(z) := \mathcal{L}(z) \cup \mathcal{L}(y) \), if \( z \) is an ancestor of \( x \), then \( \mathcal{L}((z, x)) := \mathcal{L}((z, x)) \cup \text{Inv}(\mathcal{L}((x, y))) \), else \( \mathcal{L}((x, z)) := \mathcal{L}((x, z)) \cup \mathcal{L}((x, y)) \), \( \mathcal{L}((x, y)) := \emptyset \), set \( u \neq z \) for all \( u \) with \( u \neq y \) |
| ≤₁-rule | if \( \leq n \ S, C \subseteq \mathcal{L}(x) \), \( |\{y \mid y \text{ is an} \ S \text{-neighbour of} \ x \text{ and} \ C \subseteq \mathcal{L}(y)\}| > n \) and there are \( S \)-neighbours \( y, z \) of \( x \) with \( y \neq z \), both \( y \) and \( z \) are root nodes and \( C \subseteq \mathcal{L}(y) \cap \mathcal{L}(z) \) then \( \mathcal{L}(z) := \mathcal{L}(z) \cup \mathcal{L}(y), \mathcal{L}(y) := \emptyset \) for all edges \( \langle y, w \rangle \) \( \mathcal{L}((z, w)) := \mathcal{L}((z, w)) \cup \mathcal{L}((y, w)), \mathcal{L}((y, w)) := \emptyset \) for all edges \( \langle w, y \rangle \) \( 6\mathcal{L}((w, z)) := \mathcal{L}((w, z)) \cup \mathcal{L}((w, y)), \mathcal{L}((w, y)) := \emptyset \) set \( y \approx z \) and \( u \neq z \) for all \( u \) with \( u \neq y \) |

| Table 1. Expansion Rules |
Definition 12 (Model of a completion forest). For an $n$-completion forest $\mathcal{F}$ for $K$, $\mathcal{F} \in \mathcal{F}_n^K$, an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, x^\mathcal{I})$ is a model of $\mathcal{F}$, represented $\mathcal{I} \models \mathcal{F}$ if $\mathcal{I} \models K$ and for all nodes $x, y \in \mathcal{F}$ the following hold:

- if $C \in \mathcal{L}(x)$, then $x^\mathcal{I} \in C^\mathcal{I}$
- if $R \in \mathcal{L}(\langle x, y \rangle)$ then $\langle x^\mathcal{I}, y^\mathcal{I} \rangle \in R^\mathcal{I}$
- if $x \not\equiv y \in \mathcal{F}$, then $x^\mathcal{I} \neq y^\mathcal{I}$

We want to emphasize that in order to be a model of a completion forest for $K$, an interpretation must be a model of $K$. The initial completion forest is just an alternative representation of the knowledge base, and it has exactly the same models. When we expand the forest, we will make choices and obtain new forests that capture a subset of the models of the knowledge base. Note that if an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, x^\mathcal{I})$ is a model of $\mathcal{F}$, then all nodes in $\mathcal{F}$ will be mapped to an object in $\Delta^\mathcal{I}$, however there might be objects in $\Delta^\mathcal{I}$ that are not the image of any node in $\mathcal{F}$.

Lemma 1. An interpretation $\mathcal{I}$ is a model of $\mathcal{F}_K$ iff $\mathcal{I}$ is a model of $K$.

Proof. The if direction follows from Definition 12. To prove the other direction, it suffices to consider an arbitrary model $\mathcal{I}$ of $K$ and verify that for all nodes $x, y \in \mathcal{F}_K$ the following hold:

(i) if $C \in \mathcal{L}(x)$, then $x^\mathcal{I} \in C^\mathcal{I}$
(ii) if $R \in \mathcal{L}(\langle x, y \rangle)$ then $\langle x^\mathcal{I}, y^\mathcal{I} \rangle \in R^\mathcal{I}$
(iii) if $x \not\equiv y \in \mathcal{F}$, then $x^\mathcal{I} \neq y^\mathcal{I}$

By definition, the nodes in $\mathcal{F}_K$ correspond exactly to the individuals in $\mathcal{I}_K$. For each of these individuals $a_i$, the label of $a_i$ in $\mathcal{F}_K$ is given as $\mathcal{L}(a_i) = \{C \mid C(a_i) \in \mathcal{A}\} \cup \text{const}(K, \mathcal{C}_K)$. Since $\mathcal{I}$ is a model of $\mathcal{A}$, if $C(a_i) \in \mathcal{A}$ then $a_i^\mathcal{I} \in C^\mathcal{I}$. For any concept $C \in \text{const}(K, \mathcal{C}_K)$, either $C$ is of the form $\neg D \sqcup E$ for some $D \subseteq E$ in $\mathcal{T}$ or $C$ is of the form $D \sqcup \neg D$ for an arbitrary concept $D$. In the first case, $a_i^\mathcal{I} \in (\neg D \sqcup E)^\mathcal{I}$ must hold because $\mathcal{I}$ is a model of $\mathcal{T}$. In the other case, $x^\mathcal{I} \in (D \sqcup \neg D)^\mathcal{I}$ holds for any individual $x$ in $\Delta^\mathcal{I}$ and any concept $D$ by the definition of interpretation. So we have that $a_i^\mathcal{I} \in C^\mathcal{I}$ for every $C \in \mathcal{L}(a_i)$ and item (i) holds. The label of a pair of nodes $a_i, a_j$ in $\mathcal{F}_K$ is given by $\mathcal{L}(\langle a_i, a_j \rangle) = \{R \mid R(a_i, a_j) \in \mathcal{A}\}$. Since $\mathcal{I}$ is a model of $\mathcal{A}$, $\langle a_i^\mathcal{I}, a_j^\mathcal{I} \rangle \in R^\mathcal{I}$ for every $R(a_i, a_j)$ in $\mathcal{A}$, hence item (ii) holds. Analogously, the $\not\equiv$ relation was initialized with $a_i \not\equiv a_j$ for every $a_i \not\equiv a_j$ in $\mathcal{A}$, so item (iii) will also hold for any $\mathcal{I}$ model of $\mathcal{A}$.

Finally, for a set of completion forests $\mathcal{F}$, we will denote by $\mathcal{ccf}(\mathcal{F})$ the set of forests in $\mathcal{F}$ that are complete and clash free. For a knowledge base $K$, the union of all the models of the forests in $\mathcal{ccf}(\mathcal{F}_K^\mathcal{F})$ captures all the models of $K$, as we prove in Proposition 1. This result is crucial, since it allows us to ensure that checking the forests in $\mathcal{ccf}(\mathcal{F}_K^\mathcal{F})$ suffices to check all models of $K$. In order to prove this result, we will first prove the following lemma. It states that when applying any of the rules in Table 1, no models are lost.
Lemma 2. Let \( F \) be a completion forests in \( \mathbb{F}_K^n \), let \( r \) be a rule in Table 1 and let \( F \) be the set of \( n \)-completion forests that can be obtained from \( F \) by applying \( r \). Then for every \( \mathcal{I} \) such that \( \mathcal{I} \models F \) there is some \( F' \in \mathcal{F} \) such that \( \mathcal{I} \models F' \).

Proof. We will do the proof for each rule \( r \) in Table 1.

First we will consider the deterministic, non-generating rules. There is only one \( F' \) in \( \mathcal{F} \) and the models of \( F \) are exactly the models of \( F' \). For the case of the \( \nabla \)-rule, there is some node \( x \in F \) such that \( C_1 \cap C_2 \in \mathcal{L}(x) \). Since \( F \) is a model of \( F' \), then \( x^2 \in (C_1 \cap C_2)^2 \), and since \( F \) is a model of \( F' \), then both \( x^2 \in C_1^2 \) and \( x^2 \in C_2^2 \) hold. The inequality relation and all labels in \( F' \) are exactly as in \( F \), the only change is that \( \{C_1, C_2\} \subset \mathcal{L}(x) \) in \( F' \), so \( \mathcal{I} \models F' \).

The cases of the the \( \forall \)-rule and the \( \forall,+ \)-rule, are similar to the \( \nabla \)-rule. All labels of \( F \) are preserved in \( F' \). Only the label of the node \( y \) to which the rule was applied is modified, having in \( F' \) \( C \in \mathcal{L}(y) \) or \( \forall S.C \in \mathcal{L}(x) \) respectively. Since \( F \) is a model of \( F' \), then \( x^2 \in \forall S.C)^2 \) and \( y \) and \( S \)-neighbour of \( x \) imply \( y^2 \in C^2 \), and \( x^2 \in \forall S.C)^2 \) and \( y \) and \( R \)-neighbour of \( x \) for some transitive sub-role of \( S \) imply \( y^2 \in \forall S.C)^2 \), then trivially \( \mathcal{I} \models F' \) in both cases.

Let us analyze the non-deterministic rules. For the case of the \( \sqcup \)-rule, there is some node \( x \in F \) such that \( C_1 \sqcup C_2 \in \mathcal{L}(x) \). After applying the \( \sqcup \)-rule, we will have two forests \( F_1, F_2 \) with \( \{C_1\} \subset \mathcal{L}(x) \) in \( F'_1 \) and \( \{C_2\} \subset \mathcal{L}(x) \) in \( F'_2 \) respectively. For every \( \mathcal{I} \) such that \( \mathcal{I} \models F \), we have \( x^2 \in (C_1 \sqcup C_2)^2 \), and since \( \mathcal{I} \) is a model of \( F \), then either \( x^2 \in C_1^2 \) or \( x^2 \in C_2^2 \) hold. If it is the case that \( x^2 \in C_1^2 \), then \( \mathcal{I} \models F'_1 \), and otherwise \( \mathcal{I} \models F'_2 \), so the claim holds.

The proof of the choose rule is trivial, since after its application we will have two forests \( F'_1, F'_2 \) with \( \{C\} \subset \mathcal{L}(x) \) in \( F'_1 \) and \( \{\lnot C\} \subset \mathcal{L}(x) \) in \( F'_2 \) respectively, but since trivially \( x^2 \in (C \sqcup \lnot C)^2 \) holds for any \( x \), any \( C \) and any model of \( F \), then for every \( \mathcal{I} \) either \( \mathcal{I} \models F'_1 \) or \( \mathcal{I} \models F'_2 \) holds.

When the \( \leq \)-rule or the \( \leq,+ \)-rule are applied to a variable \( x \) in \( F \), there are some variables \( y, z \) neighbours of \( x \) s.t. \( y \) is identified with \( z \) in \( F' \). This can only be done if we do not have that \( z^2 \neq y^2 \) in \( I \), hence it must be the case that \( z^2 = y^2 \). In \( F' \), we will add the pair \( (z, y) \) to the extension of \( \approx \). Due to \( z^2 = y^2 \) the extension of all labels of \( F \) will be preserved in \( F' \) and so \( \mathcal{I} \models F' \) holds.

Finally we consider the two generating rules. For the case of the \( \exists \)-rule, since the propagation rule was applied, there is some \( x \) in \( F \) such that \( \exists R.C \in \mathcal{L}(x) \), which implies the existence of some \( o \in \Delta^2 \) with \( (x^2, o) \in R^o \) and \( o \in C^2 \). \( F' \) was obtained by adding to \( F \) a new node which we denote \( y \). This node will make explicit in \( F \) the existence of \( o \), and we will have that \( y^2 = o \), so \( \mathcal{I} \models F' \).

The case of the \( \geq \)-rule is analogous to the \( \exists \)-rule, since in models of \( F' \) we have that \( y^2 = o_i \) for \( 1 \leq i \leq n \), where \( \{y_1, \ldots, y_n\} \) are the variables added to \( F \) and \( o_1, \ldots, o_n \) denote the elements in \( \Delta^2 \) s.t. \( (x^2, o_i) \in R^o \) and \( o_i \in C^2 \) for the variable \( x \) in \( F \) to which the rule was applied.

Finally, we can prove that the union of models of the forests in \( \text{ccf}(F^n_K) \) is exactly the set of all models of \( K \).

Proposition 1. For every \( \mathcal{I} \) such that \( \mathcal{I} \models K \), there is some \( F \in \text{ccf}(F^n_K) \) with \( n \geq 0 \) such that \( \mathcal{I} \models F \).
3.3 Tableaux and Canonical Models

We will define a tableau for a knowledge base. A tableau is only a representation of a model of a knowledge base, however, if may be infinite. Intuitively, a tableau is a model captured by a complete and clash free completion forest $F$ and it will provide a natural way of building a canonical interpretation of $F$. Note that if $F$ contains blocked nodes, then it is capturing a set of potentially infinite models. In this case, its tableau must be an infinite structure. The tableau $T$ of a forest $F$ will correspond to the unraveling of $F$. i.e. the structure obtained by considering each path to a node in $F$ as a node of $T$. Following [4], we will give a rather complex definition of a tableau. Defining a model of $K$ from a tableau will be straightforward with this definition, and the many conditions required for a tableau are met by complete and crash free completion forests.

**Definition 13 (Tableau).** $T = (S, L, E, I)$ is a tableau for a knowledge base $K = (A, R, T)$ iff

- $S$ is a non-empty set,
- $L : S \rightarrow 2^{\text{clos}(K)}$ maps each element in $S$ to a set of concepts,
- $E : R_K \rightarrow 2^{S \times S}$ maps each role to a set of pairs of elements in $S$, and
- $I : I_K \rightarrow S$ maps each individual occurring in $A$ to an element in $S$.

Furthermore, for all $s, t \in S$; $C_1, C_2 \in \text{clos}(K)$ and $R, S \in R_K$, $T$ satisfies:

(P1) if $C \in L(s)$, then $\neg C \notin L(s)$,
(P2) if $C_1 \cap C_2 \in L(s)$, then $C_1 \in L(s)$ and $C_2 \in L(s)$,
(P3) if $C_1 \cup C_2 \in L(s)$, then $C_1 \in L(s)$ or $C_2 \in L(s)$,
(P4) if $\forall S . C \in L(s)$ and $\langle s, t \rangle \in E(S)$, then $C \in L(t)$,
(P5) if $\exists S . C \in L(s)$, then there is some $t \in S$ such that $\langle s, t \rangle \in E(S)$ and $C \in L(t)$,
(P6) if $\forall S . C \in L(s)$ and $\langle s, t \rangle \in E(R)$ for some $R \subseteq^* S$ with $\text{Trans}(R) = \text{true}$ then $\forall S . C \in L(t)$,
(P7) $\langle s, t \rangle \in E(R)$ iff $\langle t, s \rangle \in E(\text{inv}(R))$,
(P8) if $\langle s, t \rangle \in E(R)$ and $R \subseteq^* S$ then $\langle s, t \rangle \in E(S)$,
(P9) if $\leq n . S . C \in L(s)$, then $|\{t \in S \mid \langle s, t \rangle \in E(S) \text{ and } C \in L(t)\}| \leq n$,
(P10) if $\geq n . S . C \in L(s)$, then $|\{t \in S \mid \langle s, t \rangle \in E(S) \text{ and } C \in L(t)\}| \geq n$,
(P11) if $\langle s, t \rangle \in E(R)$ and either $\leq n . S . C \in L(s)$ or $\geq n . S . C \in L(s)$, then $C \in L(t)$ or $\text{NNF}(\neg C) \in L(t)$,
(P12) if \( C(a) \in A \) then \( C \in \mathcal{L}(I(a)) \),
(P13) if \( R(a,b) \in A \) then \( (I(a), I(a)) \in \mathcal{E}(R) \),
(P14) if \( a \neq b \in A \) then \( I(a) \neq I(a) \),
(P15) if \( C \in \text{const}(K, C) \), then for all \( s \in S \) \( C \in \mathcal{L}(s) \).

Trivially, we can obtain a canonical model of a knowledge base from a tableau for it.

**Definition 14 (Canonical Model of a Tableau).** Let \( T \) be a tableau. The canonical model of \( T \), \( \mathcal{I}_T = (\Delta_T, A_T) \) is defined as follows:

\[
\Delta_T := S
\]

for all concept names \( A \) in \( \text{clos}(K) \),

\[
A_T := \{ s \mid A \in \mathcal{L}(s) \}
\]

for all individual names \( a \) in \( I_K \),

\[
a_T := a
\]

for all role names \( R \) in \( \mathcal{R} \),

\[
R_T := \mathcal{E}(R)^\oplus
\]

where \( \mathcal{E}(R)^\oplus \) the closure of the extension of \( R \) under \( \mathcal{R} \), which is defined as:

\[
\mathcal{E}(R)^\oplus := \begin{cases} (\mathcal{E}(R))^+ & \text{if Trans}(R) \\ \mathcal{E}(R) \cup \text{sub}(\mathcal{E}(R)^\oplus) & \text{otherwise} \end{cases}
\]

where \( (\mathcal{E}(R))^+ \) denotes the transitive closure of \( \mathcal{E}(R) \) and

\[
\text{sub}(\mathcal{E}(R)^\oplus) = \bigcup_{P \in \mathcal{R}, P \neq R} \mathcal{E}(P)^\oplus.
\]

**Lemma 3.** Let \( T \) be a tableau for \( K \). The canonical model of \( T \) is a model of \( K \).

**Proof.** That \( \mathcal{I}_T \) is a model of \( \mathcal{R} \) and \( A \) can be proved exactly as in the proof of Lemma 2 in [4]. Due to (P15), it can be easily verified that \( \mathcal{I}_T \) is also a model of \( T \).

**Canonical Interpretation of a Completion Forest.** A completion forest \( F \) induces a tableau \( T_F \), and this tableau gives us a canonical model for \( F \).

**Definition 15 (Tableau induced by a completion forest).** A path in a completion forest \( F \) is a sequence of nodes of the form \( p = [\frac{z_1}{x_1}, \ldots, \frac{z_n}{x_n}] \). In such a path, we define \( \text{tail}(p) = x_n \) and \( \text{tail}'(p) = x'_n \); and \( [p \mid \frac{z_{n+1}}{x_{n+1}'}] \) denotes the path \( [\frac{z_1}{x_1}, \ldots, \frac{z_n}{x_n}, \frac{z_{n+1}}{x_{n+1}'}] \). For any path \( p \) and variable \( z \), if \( z \) is not blocked and \( z \) is an \( R \)-successor of \( \text{tail}(p) \), then \( [p \mid \frac{z}{x}] \) is an \( R \)-step of \( p \). If \( z' \) is blocked by \( z \) and \( z' \)
is an $R$-successor of $\text{tail}(p)$, then $[p \rightarrow z]$ is an $R$-step of $p$. If $q$ is an $R$-step of $p$ for some role $R$, then $q$ is a step of $p$ and $p$ is a prefix of $q$. The transitive closure of prefix is called subpath.

Given a completion forest $\mathcal{F}$, the set $\text{paths}(\mathcal{F})$ is defined inductively as follows:

- If $x_0^i$ is a root in $\mathcal{F}$, $[x_0^i] \in \text{paths}(\mathcal{F})$.
- If $p \in \text{paths}(\mathcal{F})$ and $q$ is a step of $p$, then $q \in \text{paths}(\mathcal{F})$.

The tableau $T_{\mathcal{F}} = (S, \mathcal{L}, \mathcal{E}, \mathcal{I})$ induced by the completion forest $\mathcal{F}$ is defined as follows:

- $S = \text{paths}(\mathcal{F}) \setminus \{p \mid p \in \text{paths}(\mathcal{F}) \text{ and } p = [x] \text{ for some } x \text{ with } \mathcal{L}(x) = \emptyset\}$
- $\mathcal{L}(p) = \mathcal{L}(\text{tail}(p))$
- $\mathcal{E}(R) = \{\langle p, q \rangle \mid p \in S \times S \mid q \text{ is an } R\text{-step of } p\} \cup \{\langle p, q \rangle \mid p \in S \times S \mid p \text{ is an } \text{Inv}(R)\text{-step of } q\} \cup \{\langle x, y \rangle \mid x, y \text{ are root nodes and } x \text{ is an } R\text{-neighbour of } y\}$

Lemma 4. Every $\mathcal{F} \in \text{ccf}(F^n_K)$ for $n \geq 1$ induces a canonical model $I_{\mathcal{F}}$ for $K$.

Proof. First, it is proved as in [4] that every $\mathcal{F} \in \text{ccf}(F^n_K)$ for $n \geq 1$ induces a tableau $T_{\mathcal{F}}$ for $K$. For the last item of the proof of $(P9)$, note that since $n \geq 1$, pairwise blocking is subsumed and the existence of the $u$ predecessor can be ensured. $(P15)$ also holds due to the following facts:

- All nodes $x$ are initialized with $\text{const}(K, C_K) \subseteq \mathcal{L}(x)$.
- The concept names in $\text{const}(K, C_K)$ are never removed from the label of a node unless the label is set to $\emptyset$ by the $\leq r$-rule. In this case, the label of the node is never modified again.

Since $T_{\mathcal{F}}$ is a tableau for $K$, it has a canonical model $I_{\mathcal{F}}$ that is a model of $K$. The canonical model of $\mathcal{F}$ is $I_{\mathcal{F}}$.

4 Answering Conjunctive Queries

For a knowledge base $K$ and a query $Q$, we say that $K \models Q$ iff for every interpretation $\mathcal{I}$, $\mathcal{I} \models K$ implies $\mathcal{I} \models Q$. Analogously, for a completion forest $\mathcal{F}$ and a query $Q$, we say that $\mathcal{F} \models Q$ iff for every interpretation $\mathcal{I}$, $\mathcal{I} \models \mathcal{F}$ implies $\mathcal{I} \models Q$. We are interested in solving the conjunctive query entailment problem. However, a knowledge base $K$ has an infinite number of possibly infinite models. The problem is then how to verify that the query $Q$ is entailed by all of them. The key issue is that for a given $Q$ it is sufficient to consider the set of complete and clash free $N$-completion forests for $K$, where $N$ is a number that depends on $Q$. Then, we only have to verify a finite number of structures, all of them of finite size. In order to provide a sound and complete algorithm for answering conjunctive queries, we have to prove the following:
I. If $K \models Q$ then for every $F \in ccf(F_K^N)$ we can find a mapping from the variables in $Q$ to the variables in $F$ that witnesses the entailment of the query.

II. If $K$ does not entail $Q$, then there will be some $F \in ccf(F_K^N)$ into which $Q$ can not be mapped.

From I and II, we have an algorithm for checking conjunctive query entailment that works as follows: an initial completion forest for $K$ is built and expanded using a suitable $N$-blocking as termination condition. Then $Q$ is entailed by $K$ iff the query can be mapped into every complete and clash free completion forest obtained.

In the following, we will use $Q$ to denote a conjunctive query. We say that $Q$ can be mapped into a completion forest $F$, denoted $|Q|\models F$, if there is a mapping $\sigma: V_Q \rightarrow V_F$ that is the identity mapping for all constants in $V_Q$ and that satisfies the following:

1. For all $C(x) \in L_Q$, $C \in L(\sigma(x))$.
2. For all $R(x,y) \in L_Q$, $\sigma(y)$ is an $R$-descendant of $\sigma(x)$.

We have already proved that every model of $K$ is a model of some $F \in ccf(F_K^N)$. Hence, if $K \not\models Q$, then $F \not\models Q$ for some $F$. To prove II, we only need to prove that if this is the case, then there is no mapping $\sigma$. This is done in the next lemma, which stated that the existence of $\sigma$ suffices to ensure that $I \models Q$ for every $I$ model of $F$.

**Lemma 5.** If $|\models F$, then $F \not\models Q$.

**Proof.** Since $|\models F$, there is a mapping $\sigma: V_Q \rightarrow V_F$ satisfying conditions 1 and 2. Take any arbitrary model $I = (\Delta_I, \cdot_I)$ of $F$. By definition, it satisfies the following:

- if $C \in L(x)$, then $x^I \in C^I$.
- if $x$ is an $R$-descendant of $y$, then $\langle x^I, y^I \rangle \in R^I$.
- if $x \not\approx y \in F$, then $x^I \not= y^I$.

We can define a mapping $\phi$ from the variables in $V_Q$ to objects in $\Delta^I$ as $\phi(x) = \sigma(x)^I$, and this mapping satisfies $\phi(\overline{Y}) \in p^I$ for all $p(\overline{Y}) \in L_Q$.

The next step is to prove I. We know that if $K \models Q$, then $I \models Q$ for any model $I$ of any $F \in ccf(F_K^N)$. We only need to ensure that if this is the case, then the mapping $\sigma$ can be found in $F$, i.e. we want to consider a suitable $N$ such that the set of complete and clash-free $N$-completion forests can witness on their own the entailment of the query. It suffices to prove that if there is model of $F$ that is a model of $Q$, then $Q$ can be mapped into $F$. In particular, we will see that if the canonical model of a forest entails $Q$, then a mapping of $Q$ into $F$ exists.

In this proof, the value of $N$ (and hence the termination condition) will play a crucial role. As we mentioned, it depends on $Q$. More specifically, it depends
in what we call maximal $Q$-distance. If the canonical model of a forest $F$ entails $Q$, then there is a mapping of the variables in $Q$ onto the nodes of the tableau induced by $F$. Intuitively, the maximal $Q$-distance is the length of the longest path between two connected nodes of the graph defined by the image of the query when mapped on the tableau. For a maximal $Q$-distance of $d$ it will be possible to find a mapping in an $d$-completion forest that is isomorphic to the image of the query under $\sigma$, since this image does not contain any path of length greater than $d$. For this reason, we will use the maximal $Q$-distance as blocking condition when expanding the completion forest.

Formally, for a given forest $F$ in $\text{ccf}(\mathbb{F}_K^n)$ for some $n$, let $T_F = \langle S, L, E, I \rangle$ denote its tableau and $\mathcal{I}_F$ the canonical interpretation of $T_F$. If $\mathcal{I}_F \models Q$, then there is a mapping $\sigma : V_Q \rightarrow S$ such that for every $R(x, y) \in L_Q$, $\langle \sigma(x), \sigma(y) \rangle \in E(R)^\circ$. For each such $R(x, y) \in L_Q$, we use $d^R(\sigma(x), \sigma(y))$ to denote the length of the shortest path from $\sigma(x)$ to $\sigma(y)$ in the graph $\langle S, \bigcup_{P \subseteq R} E(P) \rangle$ and call it the $R$-distance between $\sigma(x)$ and $\sigma(y)$. For any $x, y$ in $V_Q$, let $d^Q(x, y)$ be the maximal $d^R(\sigma(x), \sigma(y))$ that is defined for all $R$ (and it is 0 if it is not defined for any $R$). Let $p$ be a path in the graph $G(Q) = \langle V_Q, \{\{x, y\} | R(x, y) \in L_Q, R \in R_K\} \rangle$, then $d^Q(p) = \sum_{(x, y) \in p} d^Q(x, y)$, and

$$\text{max}d^Q(x, y) = \max \{d^Q(p) | p \text{ is a path from } x \text{ to } y \text{ in } G(Q)\}$$

Finally, the maximal $Q$-distance, denoted $d_Q$, is the maximal $\text{max}d^Q(x, y)$ that is defined for all $x, y$ in $V_Q$, and it is zero if it is not defined for all $x, y$. The maximal $Q$-distance is bounded by the length of the longest path in $G(Q)$ (which is bounded by $n_Q$) times the maximal $d^Q(x, y)$ that is defined for all $x, y$ in $V_Q$.

Now we prove that for any complete and crash free $d_Q$-completion forest $F$, if $\mathcal{I}_F \models Q$, then there is a mapping $\sigma' : V_Q \rightarrow F$ that witnesses the entailment of $Q$.

**Proposition 2.** Consider any $F \in \text{ccf}(\mathbb{F}_K^n)$, and let $\mathcal{I}_F$ be the canonical model of the tableau induced by $F$. If $\mathcal{I}_F \models Q$ then $\models F Q$.

**Proof.** Since $\mathcal{I}_F \models Q$, then there is a mapping $\sigma : V_Q \rightarrow \Delta^{T_F}$ s.t.

- For all $C(x) \in L_Q$, $\sigma(x) \in C^{T_F}$.
- For all $R(x, y) \in L_Q$, $\langle \sigma(x), \sigma(y) \rangle \in R^{T_F}$.

Since $\Delta^{T_F} = \text{V}_{T_F}$, $\sigma(x)$ and $\sigma(y)$ are nodes in $T_F$ and correspond to paths in $F$. By the definition of $\mathcal{I}_F$, the mapping $\sigma$ satisfies that for all $C(x) \in L_Q$, $C \in L(\sigma(x))$ and for all $R(x, y) \in L_Q$, $\langle \sigma(x), \sigma(y) \rangle \in E(R)^\circ$.

We will define a new mapping $\sigma' : V_Q \rightarrow V_F$. In order to define $\sigma'$, we will first consider the pairs of variables that are mapped by $\sigma$ to nodes in the forest such that the path connecting them goes through a leaf of a blocked tree. The set of this pairs will be denoted $\text{throughLeaves}(V_Q)$. For each $R(x, y) \in L_Q$, if there is some $s \in S$ s.t. $\langle \sigma(x), s \rangle \in E(R)^\circ$, $\langle s, \sigma(y) \rangle \in E(R)^\circ$ and $\text{tail}(s) \neq \text{tail}(s)$, then $\langle x, y \rangle \in \text{throughLeaves}(V_Q)$. The set $\text{afterblocked}(V_Q)$ will contain the variables in $V_Q$ that occur in the second position of some pair in $\text{throughLeaves}(V_Q)$ or that
are mapped to a descendant of one such node. If \( (x, y) \in \text{throughLeaves}(V_Q) \) or if \( R(x, y) \in L_Q \) and \( x \in \text{afterblocked}(V_Q) \), then \( y \in \text{afterblocked}(V_Q) \). For all variables \( v \in V_Q \setminus \text{afterblocked}(V_Q) \), if \( \text{tail}'(\sigma(v)) \) is tree blocked let \( \psi(\text{tail}'(\sigma(v))) = \text{tail}(\sigma(v)) \) denote the variable that tree blocks it. Otherwise, let \( \psi \) be the identity function. The mapping \( \sigma' : V_Q \rightarrow V_X \) is defined as follows:

\[
\sigma'(x) = \begin{cases} 
\text{tail}'(\sigma(x)) & \text{if } x \in \text{afterblocked}(V_Q) \\
\psi(\text{tail}'(\sigma(x))) & \text{otherwise}
\end{cases}
\]

Now we will show that the mapping \( \sigma' \) has the following properties:

1. If \( C \in \mathcal{L}(\sigma(x)), \) then \( C(x) \in L_Q, \) \( C \in \mathcal{L}(\sigma'(x)). \)
2. If \( \langle \sigma(x), \sigma(y) \rangle \in \mathcal{E}(R)^\oplus, \) then \( \sigma'(y) \) is an \( R \)-descendant of \( \sigma'(x). \)

The proof of 1 is trivial, since \( \mathcal{L}(\sigma(x)) = \mathcal{L}(\text{tail}'(\sigma(x))) = \mathcal{L}(\psi(\text{tail}'(\sigma(x)))), \) so \( \mathcal{L}(\sigma(x)) = \mathcal{L}(\sigma'(x)). \) To prove 2, first we see that the following hold:

\((*)\) If both \( x \) and \( y \) are in \( \text{afterblocked}(V_Q) \) and \( \langle \sigma(x), \sigma(y) \rangle \in \mathcal{E}(R)^\oplus \) then \( \text{tail}(\sigma(y)) \) can not be a blocked leaf.

Since \( x \) is in \( \text{afterblocked}(V_Q) \), then by definition there must be some \( z \in V_Q \) such that there is a path from \( \sigma(z) \) to \( \sigma(x) \) in the image of the query that goes through a blocked leaf node, and since there is also a path from \( \sigma(x) \) to \( \sigma(y) \), if \( \text{tail}(\sigma(y)) \) was a blocked leaf then there would be a path from \( \sigma(z) \) to \( \sigma(y) \) that goes through a blocked leaf and finishes in another blocked leaf. Since we used \( d_Q \)-blocking, the minimal distance between two blocked leaves is \( d_Q + 1 \), and then the path from \( \sigma(z) \) to \( \sigma(y) \) would have a length strictly greater than \( d_Q \), which is a contradiction.

\((**)\) If both \( x \) and \( y \) are not in \( \text{afterblocked}(V_Q) \) and \( \langle \sigma(x), \sigma(y) \rangle \in \mathcal{E}(R)^\oplus \) then \( \text{tail}(\sigma(x)) \) can not be a blocked leaf.

If \( \text{tail}(\sigma(x)) \) is a blocked leaf and \( x \) is not in \( \text{afterblocked}(V_Q) \), then \( (x, y) \) is in \( \text{throughLeaves}(V_Q) \) by definition, and then \( y \) is in \( \text{afterblocked}(V_Q) \).

By the definition of \( \mathcal{E}(R)^\oplus \) and of \( R \)-step, \( \langle \sigma(x), \sigma(y) \rangle \in \mathcal{E}(R)^\oplus \) implies that \( \text{tail}'(\sigma(y)) \) is an \( R \)-descendant of \( \text{tail}'(\sigma(x)) \). We will now prove that if this is the case, then then \( \sigma'(x) \) is an \( R \)-descendant of \( \sigma'(y) \). Note that since \( \sigma(y) \) is an \( R \)-descendant of \( \sigma(x) \), it can not be the case that \( x \) is in \( \text{afterblocked}(V_Q) \) and \( y \) is not. We have the following cases:

(a) Both \( x \) and \( y \) are in \( \text{afterblocked}(V_Q) \).

In this case we have that \( \sigma'(x) = \text{tail}'(\sigma(x)) \) and \( \sigma'(y) = \text{tail}'(\sigma(y)) \). By \((*)\), \( \sigma(y) \) is not a blocked leaf, and then from \( \text{tail}(\sigma(y)) = \text{tail}'(\sigma(y)) \) we have that \( \text{tail}(\sigma(y)) \) is an \( R \)-descendant of \( \text{tail}(\sigma(x)), \) so \( \psi(\text{tail}(\sigma(y))) = \text{tail}(\sigma(y)) \) is an \( R \)-descendant of \( \psi(\text{tail}(\sigma(x))) = \text{tail}(\sigma(x)) \) and \( \sigma'(y) \) is an \( R \)-descendant of \( \sigma'(x) \) as desired.

(b) Neither \( x \) nor \( y \) are in \( \text{afterblocked}(V_Q) \).

By \((**)\), \( \sigma(x) \) is not a blocked leaf, so \( \text{tail}(\sigma(x)) = \text{tail}'(\sigma(x)) \) an then \( \text{tail}'(\sigma(y)) \) is an \( R \)-descendant of \( \text{tail}'(\sigma(x)) \), so \( \psi(\text{tail}'(\sigma(y))) \) is an \( R \)-descendant of \( \psi(\text{tail}'(\sigma(x))) \) as desired.
(c) $x$ is not in afterblocked($V_Q$), but $y$ is.

In this case we have that $\sigma(x)$ is a blocked leaf and tail($\sigma(x)$) = $\psi$(tail($\sigma(x)$)), so tail($\sigma(y)$) = $\sigma'(y)$ is an $R$-descendant of $\psi$(tail($\sigma(x)$)) = $\sigma'(x)$.

Since the mapping $\sigma'$ has properties 1 and 2, $\models F$.

In the absence of transitive roles, $d^R(\sigma(x), \sigma(y)) = 1$ for every pair of variables $x$, $y$ that appear in some $R(x,y)$ in $Q$, and then the maximal $Q$-distance is bounded by $n_Q$. Due to this fact, it is sufficient to consider $n_Q$-blocking as a termination condition when expanding the completion forest.

**Corollary 1.** Let $K$ be a knowledge base with $R_+ = \emptyset$. Consider any $F \in \text{ccf}(\mathbb{R}^n_K)$, and let $I_F$ be the canonical model of the tableau induced by $F$. If $I_F \models Q$ then $\models Q$.

In the presence of transitive roles, if does not suffice to consider $n_Q$-blocking as a termination condition. Since $d^R(\sigma(x), \sigma(y))$ may be arbitrarily big for each $R(x,y)$, then also the maximal $Q$-distance is unbounded and an isomorphic mapping may not exist on a structure of bounded depth. However, as we will now show, if a there is some mapping from the query variables into a tableau for $K$ satisfying $Q$, then there is a mapping that also satisfies $Q$ where the maximal $Q$-distance is bound by a number that depends on $K$. This will allow us to find an isomorphic mapping of the query variables into a completion forest of fixed size. We denote by $c$ the cardinality of $\text{clo}(K) \cup \mathcal{C}_K$ and by $r$ the cardinality of $R_K$. The bound will be given as $D = 2^{2c+r}$. We prove that any mapping where the maximal $d^R(\sigma(x), \sigma(y))$ that is defined for some $R$, $x$, $y$ exceeds $D$ can be modified into one that does not.

**Lemma 6.** Consider a tableau $T = \langle S, L, E, I \rangle$ for $K$. If there is a mapping $\sigma' : V_Q \to S$ that satisfies

1. For all $C(x) \in L_Q$, $C \in L(\sigma(x))$.
2. For all $R(x,y) \in L_Q$, $\langle \sigma(x), \sigma(y) \rangle \in E(R)$. 

then there is a mapping $\sigma' : V_Q \to S$ that also satisfies 1 and 2, and that additionally satisfies that for all $R(x,y) \in L_Q$, $d^R(\sigma'(x), \sigma'(y)) \leq D$.

**Proof.** If $\langle \sigma(x), \sigma(y) \rangle \in E(R)$, then there is a sequence of nodes $n_0, \ldots, n_m$ s.t. $n_0 = \sigma(x)$, $n_m = \sigma(y)$ and for all $0 \leq i \leq m$, $\langle n_i, n_{i+1} \rangle \in E(S)$ for some $S$ subrole of $R$, and $d^R(\sigma(x), \sigma(y)) = m$. We can prove that if $m > D$, then there is a mapping $\sigma_m$ with $d^R(\sigma_m(x), \sigma_m(y)) < m$. Since there are at most $2^c$ node labels and $2^r$ arc labels, there are at most $D = 2^{2c+r}$ possible different labellings for a pair of nodes and an edge. This implies that if $m > D$, there is some node $m'$ in $n_0, \ldots, n_m$ that had previously occurred with the same predecessor and the same incoming edge, and hence $n_0, \ldots, n_m$ contains a cycle. In this case we can consider the path $n_0, \ldots, m'$ and the new mapping is given as $\sigma_m(x) = \sigma_m(x)$, and $\sigma_m(y) = m'$. Inductively, we can prove that there is a mapping $\sigma'$ that satisfies $d^R(\sigma'(x), \sigma'(y)) \leq D$ for every $R(x,y) \in L_Q$. Since $\sigma'$ preserves all the labels in $\sigma$ and all $R$-descendant relations, $\sigma'$ also satisfies 1 and 2.
Now we know that in the presence of transitive roles, since $d^R(\sigma(x), \sigma(y))$ is bounded by $D$, the maximal $Q$-distance is bounded by $Dn_Q$, so we can use $Dn_Q$-blocking as a termination condition when expanding the completion forest.

**Corollary 2.** Consider any $F \in \text{ccf}(F_{K_{Q^n}}^n)$, and let $I_F$ be the canonical model of the tableau induced by $F$. If $I_F \models Q$ then $\models F Q$.

Summing up, to solve the conjunctive query entailment problem, it suffices to check for entailment the set of complete and crash free completion forests for $K$, no matter the $n$ that is used as a termination condition.

**Proposition 3.** $K \models Q$ iff $\models F Q$ for every $F \in \text{ccf}(F_{K_n})$ for any $n$.

**Proof.** The only if direction is trivial. Consider any $F \in \mathbb{F}_K^n$. Since any model $I$ of $F$ is a model of $K$ by definition, then $K \models Q$ implies $F \models Q$. The if direction can be done by contraposition. If $K \not\models Q$, then there is some model $I$ of $K$ such that $I \not\models Q$. By Proposition 1, $I \models F$ for some $F \in \text{ccf}(F_{K_n})$, and we have that $F \not\models Q$ for some $F \in \text{ccf}(F_{K_n})$.

However, if we choose a suitable $n$-blocking, checking for entailment in all the models of a completion forest can be reduced to finding a mapping of the query into the completion forest itself.

**Theorem 1.** $K \models Q$ iff $\models F Q$ for every $F \in \text{ccf}(F_{K_{Q^n}})$. \[d\]

**Proof.** First we prove that if $K \models Q$ then $\models F Q$. Take any arbitrary $F \in \text{ccf}(F_{K_{Q^n}})$. Since $K \models Q$, then $F \models Q$ (Proposition 3). In particular, we have that $I_F \models Q$, where $I_F$ is the canonical model of the tableau induced by $F$. Thus, by Proposition 2, $\models F Q$.

To prove the other direction, observe that from $\models F Q$ and Lemma 5, we have that $F \models Q$ for every $F \in \text{ccf}(F_{K_{Q^n}})$. Finally, by Proposition 3, $K \models Q$.

**Corollary 3.** If $R_+ = \emptyset$ in $K$, then $K \models Q$ iff $\models F Q$ for every $F \in \text{ccf}(F_{K_{Q^n}})$.

**Corollary 4.** $K \models Q$ iff $\models F Q$ for every $F \in \text{ccf}(F_{K_{Dn_Q}})$.

## 5 Complexity

In this section, for a knowledge base $K$, we will use $c$ to denote the cardinality of $\text{clos}(K) \cup C_K$, $r$ the cardinality of $R_K$ and $m_C$ the maximum $m$ occurring in a concept of the form $\leq m R.C$ or $\geq m R.C$ in $\text{clos}(K) \cup C_K$. $|A|$ denotes the number of assertions in $A$. By $|K|$ we will denote the total size of the (string encoding the) knowledge base. Note that $c$, $r$ and $m_C$ are linear on $|K \cup C_K|$ assuming unary coding of numbers in number restrictions and constant on $|A|$, while $|I_K|$ is linear on both.

**Lemma 7.** The maximal number $T_n$ of non-isomorphic $n$-trees in a completion forest for $K$ is given by $T_n = O((2^c(m_C)r^*)^{em_Cr^*})$. 

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Proof. Since $L(x) \subseteq \text{clo}(K) \cup C_K$, there are at most $2^x$ different node labels in a completion forest. Each successor of a node can be the root of a tree of depth $(n - 1)$. Considering a single role $R$, if a node $v$ has $x_R$ successors, then there is a maximum number of $(T_{n-1})^x$ trees of depth $(n - 1)$ rooted at $v$. A generating rule can be applied to each node at most $c$ times. Each time it is applied, it generates at most $m_C$ $R$-successors for each role $R$. This gives a bound of $cm_C$ $R$-successors for each role. The number of $R$-successors of a node might range from 0 to $cm_C$, and for each number of $R$-successors, we have at most $(T_{n-1})^{cm_C}$ trees of depth $(n - 1)$. So, each node can be the root of at most $(cm_C)(T_{n-1})^{cm_C}$ trees of depth $(n - 1)$ if we consider one single role. Since at most the same number of trees can be generated for every role in $R_K$, there is a bound of $((cm_C)(T_{n-1})^{cm_C})^r$ trees of depth $(n - 1)$ rooted at each node. The number of different roots of an $n$-tree is bounded by $2^c$. We now give an upper bound on the number of non-isomorphic $n$-trees as

$$T_n = O((2^c(cm_C)(T_{n-1})^{cm_C})^r)$$

To simplify the notation, let’s consider $x = 2^c(cm_C)^r$ and $a = cm_Cr$. Then we have

$$T_n = O(x(T_{n-1})^a) = O(x^{1+a+\ldots+a^{n-1}}(T_0)^a) = O((xT_0)^a)$$

The maximal number of trees of depth 0 is also bounded by $2^c$. Returning to the original notation we get

$$T_n = O((2^{cm_C}(cm_C)^r)^{cm_Cr})^n$$

**Corollary 5.** The maximal number $T_n$ of non-isomorphic $n$-trees in a completion forest for $K$ is:
- single exponential in $n$
- double exponential in $|K|$ if $n$ is constant on $|K|
- triple exponential in $|K|$ if $n$ is single exponential on $|K|$.

**Lemma 8.** The number of nodes in a completion forest $F \in \mathbb{F}_K^n$ is bounded by

$$O((|I_K|(cm_Cr)^n(2^c(cm_C)^r)^{cm_Cr})^n)$$

**Proof.** The claim follows from the following properties:

i) The outdegree of $F$ is bounded by $cm_Cr$.

Nodes are only added to the forest by applying a generating rule. Only concepts of the form $\exists R.S$ or $\geq n R.C$ trigger the application of a generating rule, and there are at most $c$ such concepts. Each such rule generates at most $m_C$ successors for each role, and there are $r$ roles. Note that if a node $v$ is identified with another by the $\forall$-rule or the $\forall_r$-rule, then the rule application which led to the generation of $v$ will never be repeated [4].

ii) The depth of $F$ is bounded by $d = (T_n + 1)n$.

This is due to the fact that there is a maximum of $T_n$ non-isomorphic $n$-trees. If there was a path of length greater than $(T_n + 1)n$ to a node $v$ in $F$,
this would imply that \( v \) occurred after a sequence of \( T_n + 1 \) non overlapping \( n \)-trees, and then one of them would have been blocked and \( v \) would not have been generated.

iii) The number of variables in a variable tree in \( \mathcal{F} \) is bounded by \( O((cmCr)^{d+1}) \).

iv) The number of variables in \( \mathcal{F} \) is bounded by \( O(|I_K|(cmCr)^{d+1}) \).

**Corollary 6.** If \( n \) is constant on \( |K| \), then the maximum number of nodes in a completion forest \( \mathcal{F} \in \mathbb{F}_n^c K \) is 4-exponential on \( (|K| + n) \), 3-exponential on \( |K| \), double exponential on \( n \) and linear in \( |A| \).

**Corollary 7.** If \( n \) is single exponential on \( |K| \), then the maximum number of nodes in a completion forest \( \mathcal{F} \in \mathbb{F}_n^c K \) is 5-exponential on \( (|K| + n) \), 4-exponential on \( |K| \), double exponential on \( n \) and linear in \( |A| \).

**Proposition 4.** The expansion of \( \mathcal{F}_K \) into some \( \mathcal{F} \in \mathbb{F}_n^c K \) terminates in time:
- nondeterministic 3-exponential on \( |K| \) if \( n \) is constant on \( |K| \),
- nondeterministic 4-exponential on \( (|K| + n) \) if \( n \) is constant on \( |K| \),
- nondeterministic 4-exponential on \( |K| \) if \( n \) is single exponential on \( |K| \),
- nondeterministic 5-exponential on \( (|K| + n) \) if \( n \) is single exponential on \( |K| \),
- nondeterministic double exponential on \( n \),
- nondeterministic polynomial (linear) in \( |A| \).

**Proof.** Let \( M = O(|I_K|(cmCr)^{(2^{2c(cmCr)^{r}})}[(cmCr)^{r})^n]) \) denote the maximal number of nodes in \( \mathcal{F} \). We will obtain an upper bound of the number of rules that are applied to expand \( \mathcal{F}_K \) into \( \mathcal{F} \).

i) For a single node \( v \), the \( \sqcap \)-rule, the \( \sqcup \)-rule and the choose-rule can be applied \( O(c) \) times, since they are applied at most once for each concept in \( \mathcal{L}(v) \).

ii) For the \( \exists \)-rule, \( \forall \)-rule, \( \forall^+ \)-rule, \( \geq \)-rule and \( \leq \)-rule, the bound on the number of times it can be applied to \( v \) is given by the maximal number of successors of \( v \), i.e. \( O(cmCr) \).

iii) Rules 1 to 8 can be applied at most \( O(McmCr) \) times to obtain \( \mathcal{F} \).

iv) The \( \leq_r \)-rule can be applied at most once to each root node in \( \mathcal{F}_K \), hence it is bounded by \( |I_K| \).

v) The total rule applications required to expand \( \mathcal{F}_K \) into \( \mathcal{F} \) is \( O(|I_K| + (McmCr)) \).

### 5.1 Complexity of answering Conjunctive Queries

**Lemma 9.** For an \( \mathcal{F} \in \text{ccf}(\mathbb{F}_n^c K) \), checking whether \( \models_{\mathcal{F}} Q \) can be done in polynomial time.

**Proof (Sketch).** \( \mathcal{R} \) and \( \mathcal{F} \) can be expressed as a relational database. The complexity of verifying whether \( \models_{\mathcal{F}} Q \) is the complexity of answering a conjunctive query over a relational database, which can be done in polynomial time [1].

**Theorem 2.** Let \( K \) be a knowledge base with \( R_+ = \emptyset \). The algorithm answers the conjunctive query entailment problem in 3coNEXPTIME w.r.t. the size of \( K \).
Proof. As Theorem 1 states, \( K \not \models Q \) iff there is some \( F \in \text{ccf}(F^n_K) \) such that \( \not \models F \models Q \). Since \( K \) does not contain transitive roles, \( n = n_Q \) is constant on \( |K| \), and by Proposition 4, this \( F \) can be obtained in time nondeterministic 3-exponential on \( |K| \). From this and Lemma 9, we have that non-entailment is in 3NEXPTIME and the claim follows.

**Theorem 3.** Let \( K \) be a knowledge base. The algorithm answers the conjunctive query entailment problem in 4coNEXPTIME w.r.t. the size of \( K \).

Proof. As Theorem 1 states, \( K \not \models Q \) iff there is some \( F \in \text{ccf}(F^n_K) \) such that \( \not \models F \models Q \). Since \( n = 2^{2^{cr \cdot n_Q}} \) is single exponential on \( |K| \), by Proposition 4 \( F \) can be obtained in time nondeterministic 4-exponential on \( |K| \). From this and Lemma 9, we have that non-entailment is in 4NEXPTIME and the claim follows.

### 5.2 Data Complexity

**Theorem 4.** The conjunctive query entailment problem over a knowledge base \( K \) in any DL from \( \text{ALE} \) to \( \text{SHIQ} \) is in coNP w.r.t. data complexity.

Proof. Once again, by Theorem 1 we have that \( K \not \models Q \) iff there is some \( F \in \text{ccf}(F^n_K) \) such that \( \not \models F \models Q \). Proposition 4 states that this \( F \) can be obtained in time nondeterministic linear in \( |A| \), and by Lemma 9 it can be checked in polynomial time, hence non-entailment is in NP in data complexity, and entailment is in coNP.

**Theorem 5.** The conjunctive query entailment problem over a knowledge base \( K \) in any DL from \( \text{ALE} \) to \( \text{SHIQ} \) is coNP-complete w.r.t. data complexity.

Proof. The first such hardness result was given in [7], where coNP-hardness was proved for \( \text{ALC} \). In [3] the same result is given for logics even less expressive than \( \text{ALE} \). Membership for \( \text{SHIQ} \) is proved in Theorem 4.

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