De Alfaro, Fubini and Furlan from multi matrix systems

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ABSTRACT: We consider the quantum mechanics of an even number of space indexed hermitian matrices. Upon complexification, we show that a closed subsector naturally parametrized by a matrix valued radial coordinate has a description in terms of non interacting \textit{s}-state “radial fermions” with an emergent De Alfaro, Fubini and Furlan type potential, present only for two or more complex matrices. The concomitant \textit{AdS}_2 symmetry is identified. The large \textit{N} description in terms of the density of radial eigenvalues is also described.

KEYWORDS: Matrix Models, AdS-CFT Correspondence, 1/N Expansion, Gauge-gravity correspondence

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The study of multi-matrix systems, and particularly their large $N$ [1] limit is of great interest. They are of interest in their own right as, for instance, in providing an equivalent reduced description of QCD in terms of a finite number of matrices with quenched momenta [2–4], or, as is well known, and through the large $N$ limit of their description of D0 branes [5], in providing a possible definition of $M$ theory [6].

In the context of the AdS/CFT correspondence [7–9], they find many important applications. Due to supersymmetry and conformal invariance, correlators of supergravity and 1/2 BPS states reduce to calculation of free matrix model overlaps [10, 11] or consideration of related matrix hamiltonians [12]. For stringy states, in the context of the BMN limit [13] and $\mathcal{N} = 4$ SYM, similar considerations apply [14–16]. There is a precise phase space identification [17, 18] between the collective density description of the dynamics of a single matrix [19, 20], and the droplet description of the LLM [21] metric. Compactification of $\mathcal{N} = 4$ SYM on $R \times S^3$ yields a plane-wave matrix theory [22, 23], related to the $\mathcal{N} = 4$ SYM dilatation operator [24]. Multi-matrix, multi-trace operators with diagonal free two point functions have been identified [25–32], allowing for the calculation of correlators beyond the planar limit.

The AdS/CFT correspondence has in general provided many new insights into the properties of several different gravitational objects from their corresponding description in terms of the large $N$ limit of matrix valued (super) Yang-Mills theories. But it is also of great interest to understand how or if gravitational degrees of freedom emerge from the large $N$ limit of matrix theories, including in settings without supersymmetry or even conformal invariance.

1By multimatrix systems we have in mind the integral or the quantum mechanics of a finite number of matrices.
It is this problem of “emergent geometry” that we investigate in this communication, in the context of the quantum mechanics of an even number \( d = 2m \) of space valued hermitian matrices. These can be thought of as D0 coordinates, although we will not consider a Yang-Mills potential, in effect restricting ourselves mostly to the free case.

Geometries arise often with a high degree of symmetry, typically with spherical symmetry. It is an interesting question to ask if there is a sense in which a matrix valued “radial coordinate” can be defined. For an even number of hermitian matrices (or an arbitrary number of complex matrices) a closed sub sector\(^2\) parametrized by a single matrix with the properties expected of such radial matrix has indeed been identified [33, 34].

The key result of this communication is to show that in this radial subsector a De Alfar, Fubini and Furlan (dAFF) [35] \( 1/r^2 \) potential emerges, with strength fixed and proportional to \( N^2 \). This is present only when the number of complex matrices \( m \) is two or more.

Single particle conformal quantum mechanics with a \( 1/x^2 \) potential of arbitrary strength, the very context considered in the original dAFF work, has formed the basis of several studies of the \( AdS_2/CFT_1 \) correspondence [36–45], which has not yet been fully developed.

We will establish in this article a “fermionic” description of the large \( N \) ground state Hamiltonian in terms of \( N \) non-interacting higher dimensional \( s \)-state radial fermions in the presence of a dAFF \( 1/r^2 \) potential of well defined strength. We identify the generators of the SL(2, \( R \)) conformal group, isomorphic to the group of isometries of \( AdS_2 \).

It is hoped that the emergence of this conformal group in the radial sector of a higher dimensional system may shed further light into the \( AdS_2/CFT_1 \) correspondence, which often arises in the product space of near horizon geometries of higher dimensional black-hole geometries, although this is not the purpose of this communication.

The paper is organized as follows: in section 2, the matrix valued radial coordinate is identified, and the volume element for inner products of wavefunctions in the radial subsector is described. The fermionic description of this sector in terms of the sum of \( N \) noninteracting \( d + 1 \) dimensional \( s \)-state single particle Hamiltonians is obtained in section 3, and the emergence of a dAFF potential established. In section 4, the generators of the SL(2, \( R \)) conformal group are identified. Section 5 describes the large \( N \) eigenvalue density description of the underlying Calogero model, which is then explicitly obtained for the \( L_0 \) SO(2,1) generator. Section 6 is reserved for discussion and conclusions. The appendix derives a technical result needed in section 3.

2 Hamiltonian and radial sector

We consider the Hamiltonian of \( d = 2m \) hermitean matrices \( X_a \) labeled by a space index \( a = 1, \ldots, 2m \):

\[
\hat{H} = -\frac{1}{2} \text{Tr} \sum_{a=1}^{2m} \frac{\partial}{\partial M_a} \frac{\partial}{\partial M_a} + V \\
= -\frac{1}{2} V^2 + V.
\]  

\(^2\)In the sense of closure under Schwinger-Dyson equations.
We complexify by introducing complex matrices:

\[ Z_1 = X_1 + iX_2 \quad , \quad Z_2 = X_3 + iX_4 \quad , \quad \text{etc.} \]

Labeling these complex matrices \( Z_A \), \( A = 1, \ldots, m \), we consider the matrix

\[
\sum_{A=1}^{m} Z_A^\dagger Z_A . \tag{2.2}
\]

This matrix is hermitean and positive definite, and its eigenvalues

\[ \rho_i = r_i^2 , \quad i = 1, \ldots, N , \quad \rho_i \geq 0 , \tag{2.3} \]

have a natural interpretation as the eigenvalues of a matrix valued radial coordinate.

One can consider a parametrization of the complex matrices \( Z_A \), \( A = 1, \ldots, m \) in terms of the matrix valued radial coordinate and \( 2m - 1 \) unitary matrices.\(^3\) Although in this communication we will mostly discuss the laplacian operator, or the potential free case, considering potentials \( V \) that depend only the radial matrix eigenvalues allow us to consistently restrict ourselves to radial wavefunctions, i.e., wavefunctions depending only on the eigenvalues (2.3). This sector has an enhanced \( \text{U}(N)^{m+1} \) symmetry

\[ Z_A \to V_A Z_A V^\dagger , \quad A = 1, \ldots, m . \tag{2.4} \]

The measure in the inner product of two such wave function will then take the form

\[
\int \prod_A \prod_{ij} dZ_A^\dagger_{ij} dZ_{Aij} = \int \prod_i d\rho_i \mathcal{J}(\rho_i) d|\text{Angular}| ,
\]

with the “angular” degrees of freedom being integrated out.

\( \mathcal{J}(\rho_i) \) has recently been obtained in closed form [34]. This results from the remarkable fact that correlators in this sector, with the enhanced symmetry (2.4), close under Schwinger-Dyson equations. The result is:

\[
\mathcal{J}(\rho_i) = C_m \prod_i d\rho_i \rho_i^{m-1} \prod_{i>j} \rho_i^{m-1} \rho_j^{m-1} (\rho_i - \rho_j)^2
\]

\[
= D_m \prod_i dr_i r_i^{2m-1} \prod_{i>j} r_i^{2m-2} r_j^{2m-2} (r_i^2 - r_j^2)^2 \tag{2.5}
\]

\[
= C_m \prod_i d\rho_i \rho_i^{m-1} \Delta_{\text{RM}}^2 (\rho_i) = D_m \prod_i dr_i r_i^{2m-1} \Delta_{\text{RM}}^2 (r_i^2) ,
\]

where the antisymmetric product

\[
\Delta_{\text{RM}}(\rho_i) \equiv \prod_{i>j} \rho_i^{\frac{m+1}{2}} \rho_j^{\frac{m+1}{2}} (\rho_i - \rho_j)
\]

generalizes the well known Van der Monde determinant \( \Delta = \prod_{i>j} (\rho_i - \rho_j) \), and \( C_m \) and \( D_m \) are numerical constants.

\(^3\)For an explicit such parametrizations in the \( m = 1 \) case, see [33].
In the radial sector, the Laplacian then takes the form:

\[-\frac{1}{2} \nabla^2_{\text{Radial}} = -\frac{1}{2} \sum_i \frac{1}{\Delta^2_{\text{RM}}(r_i^2)} \frac{1}{r_i^{2m-1}} \frac{\partial}{\partial r_i} \Delta^2_{\text{RM}}(r_i^2) \frac{\partial}{\partial r_i} \]

\[= -2 \sum_i \frac{1}{\Delta^2_{\text{RM}}(\rho_i)} \frac{1}{\rho_i^{m-1}} \frac{\partial}{\partial \rho_i} \rho_i^m \Delta^2_{\text{RM}}(\rho_i) \frac{\partial}{\partial \rho_i}, \tag{2.6}\]

The Hamiltonian acts on symmetric wavefunctions \(\Phi(\rho_i)\) of the eigenvalues:

\[\hat{H}\Phi(\rho_i) = E\Phi(\rho_i)\]

\[\left( -\frac{1}{2} \nabla^2_{\text{Radial}} + V(\rho_i) \right) \Phi(\rho_i) = E\Phi(\rho_i), \tag{2.7} \]

3 Radial fermionic description and dAFF

It is a well known result that in the singlet sector of a single hermitean matrix, the first quantized hamiltonian can be mapped to a system of non-interacting single particle fermions \[46\]. This was shown to also be the case in the radial sector of the two (hermitean) matrix hamiltonian, which was mapped to a system of non interacting 2 + 1 dimensional s-state single particle radial fermions \[33\]. We show in this section that this result is true also in the general case of \(m > 1\) complex matrices but with the important difference that, in this case, an additional dAFF potential is induced.

Define the anti-symmetric wavefunction \(\Psi(\rho_i)\) as follows:

\[\Psi(\rho_i) = \Delta_{\text{RM}}(\rho_i) \Phi(\rho_i)\]

The Laplacian operator \(\nabla^2_{\text{Radial}}\) now acts on \(\Psi(\rho_i)\) as:

\[4 \sum_i \left( \frac{1}{\rho_i^{m-1}} \Delta'_{\text{RM}}(\rho_i) \frac{\partial}{\partial \rho_i} \Delta_{\text{RM}}(\rho_i) \right) \rho_i^m \left( \Delta_{\text{RM}}(\rho_i) \frac{\partial}{\partial \rho_i} \frac{1}{\Delta_{\text{RM}}(\rho_i)} \right) \Psi(\rho_i) \]

However, one has the identity:

\[4 \sum_i \left( \frac{1}{\rho_i^{m-1}} \Delta'_{\text{RM}}(\rho_i) \frac{\partial}{\partial \rho_i} \Delta_{\text{RM}}(\rho_i) \right) \rho_i^m \left( \Delta_{\text{RM}}(\rho_i) \frac{\partial}{\partial \rho_i} \frac{1}{\Delta_{\text{RM}}(\rho_i)} \right) \Psi(\rho_i) \]

\[= \left( \sum_i \frac{4}{\rho_i^{m-1}} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} - \frac{(N^2 - 1) (m - 1)^2}{\rho_i} \right) \Psi(\rho_i) \tag{3.1}\]

The proof of this identity is left to appendix A.

As a result, the Hamiltonian acting on \(\Psi(\rho_i)\) now takes the form:

\[-2 \sum_i \frac{1}{\rho_i^{m-1}} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} + \frac{(N^2 - 1) (m - 1)^2}{2 \rho_i} + V(\rho_i) \]

\[\left( -\frac{1}{2} r_i^{2m-1} \frac{\partial}{\partial r_i} \right)_i \frac{1}{r_i^{2m-1}} \frac{\partial}{\partial r_i} + \frac{(N^2 - 1) (m - 1)^2}{2 r_i^2} + V(r_i) \]

\[\Psi(\rho_i) = E\Psi(\rho_i), \quad \Psi(r_i) = E\Psi(r_i), \quad \Psi(\rho_i) = E\Psi(\rho_i), \quad \Psi(r_i) = E\Psi(r_i) \]

\[-4-\]
This is now the sum of single particle \( d + 1 = 2m + 1 \) dimensional s-state hamiltonians, but with an additional radial dAFF potential. The coefficient of this \( \frac{1}{r^2} \) potential is uniquely determined. The fact that it is proportional to \( N^2 \) ensures that the system has a smooth 't Hooft limit. It is absent in the case of a single complex matrix.

This first quantized hamiltonian acts on wavefunctions which are antisymmetric under the exchange of radial coordinates only, hence their referral to as radial fermions.

4 AdS\(_2\)

It is well known that the conformal quantum mechanical hamiltonian

\[
\hat{h} = \frac{1}{2} p^2 + \frac{\eta^2}{2x^2}
\]

has a SL(2, \( R \)) conformal symmetry generated by \( \hat{h} \) together with

\[
\hat{k} = \frac{x^2}{2}; \quad \hat{d} = \frac{1}{2} (xp + px).
\]

The commutator algebra is:

\[
[\hat{d}, \hat{h}] = 2i\hat{h} \quad [\hat{d}, \hat{k}] = -2i\hat{k} \quad [\hat{h}, \hat{k}] = -i\hat{d}
\] (4.1)

This is mapped to SO(2, 1) generators in the standard way:

\[
\hat{L}_0 = \frac{1}{2} (\hat{h} + \hat{k}) \quad \hat{L}_\pm = \frac{1}{2} (\hat{h} - \hat{k} \mp i\hat{d}),
\]

with algebra

\[
[\hat{L}_0, \hat{L}_\pm] = \pm \hat{L}_\pm; \quad [\hat{L}_{-1}, \hat{L}_1] = 2\hat{L}_0.
\]

In the fermionic picture, the single matrix hamiltonian with the same potential, and the other SL(2, \( R \)) generators, can be written in terms of second quantized fields as:

\[
\hat{H} = \int dx \Psi(x) \left( \frac{1}{2} p^2 + \frac{\eta^2}{2x^2} \right) \Psi(x)
\]

\[
\hat{K} = \int dx \Psi(x) \frac{x^2}{2} \Psi(x)
\]

\[
\hat{D} = \frac{1}{2} \int dx \Psi(x)(xp + px)\Psi(x)
\] (4.2)

with

\[
\{ \Psi(x), \Psi^\dagger(x') \} = \delta(x - x').
\] (4.3)

We now generalize to the higher dimensional case. At the 1st quantized level \((d = 2m)\) \( p_r = -i\partial_r \) one has:

\[
\hat{h}(p_r, r) = \frac{1}{2} r^{d-1} p_r r^{d-1} p_r + \frac{(N^2 - 1) (d - 2)^2}{8r^2}
\]

\[
= \frac{p_r^2}{2} - i(d-1)p_r + \frac{(N^2 - 1) (d - 2)^2}{8r^2}
\]

\[
\hat{d}(p_r, r) = rp_r - \frac{d}{2}; \quad \hat{k}(p_r, r) = \frac{r^2}{2}
\] (4.4)

\(^\text{4}\)We use the conventions in \([40]\), and consider our generators at \( t = 0 \).
One can verify that the algebra (4.1) is satisfied. In terms of second quantized operators satisfying
\[
\left\{ \Psi(r), \Psi^\dagger(r') \right\} = \frac{\delta(r - r')}{r^{d-1}},
\] (4.5)
one has as generators:
\[
\hat{H} = \int r^{d-1} dr \Psi^\dagger(r) \hat{h}(p_r, r) \Psi(r)
\]
\[
\hat{K} = \int r^{d-1} dr \Psi^\dagger(r) \hat{k}(p_r, r) \Psi(r)
\]
\[
\hat{D} = \int r^{d-1} dr \Psi^\dagger(r) \hat{d}(p_r, r) \Psi(r)
\]
(4.6)

The simplest way to verify that these generators close the appropriate algebra, is to note that (4.5) suggests that we redefine:
\[
\tilde{\Psi}(r) \equiv r^\frac{d-1}{2} \Psi(r), \quad \tilde{\Psi}^\dagger(r') \equiv r^\frac{d-1}{2} \Psi^\dagger(r')
\]
This is also the redefinition of the fields in terms of which \( p_r \) becomes explicitly hermitean. One finds:
\[
\hat{K} = \int dr \tilde{\Psi}^\dagger(r) \frac{r^2}{2} \tilde{\Psi}(r)
\]
\[
\hat{D} = \int dr \tilde{\Psi}^\dagger(r) \frac{1}{2}(r p_{r'} + p_r r) \tilde{\Psi}(r)
\]
\[
\hat{H} = \int dr \tilde{\Psi}^\dagger(r) \left( \frac{p_r^2}{2} + \frac{N^2(d-2)^2 - 1}{8r^2} \right) \tilde{\Psi}(r)
\] (4.7)

So the higher dimensional case has been mapped to a one-dimensional quantum mechanical conformal hamiltonian with
\[
\eta^2 = \frac{1}{4} \left( N^2(d-2)^2 - 1 \right),
\]
which has the required symmetry, as the quantum mechanical generators (4.2) close the conformal algebra for arbitrary \( \eta \).

This is the basis of the evidence of a possible emergence of an \( AdS_2 \) geometry, in a spirit similar to arguments presented in [40].

5 Density

In this section, we describe the large \( N \) description of the dynamics of eigenvalues in terms of their density. This provides a continuum description of the underlying Calogero model. We use well established collective field theory techniques [19, 20], which we apply to the radial sector of our Hamiltonian. Our results are new, and generalize previously obtained results for the single hermitean matrix, and also for two matrices. For a single hermitean matrix, this approach corresponds to the bosonization of the fermionic description of the system [47].
We recall the positive definite hermitian matrix defined from $m$ complex matrices $Z_A, A = 1, 2, 3, ..., m$:

$$\sum_A Z_A^\dagger Z_A.$$ 

Our collective field variables are defined as

$$\phi_k = \text{Tr} e^{ik \sum_A Z_A^\dagger Z_A} = \sum_i e^{ik \rho_i};$$ 

$$\phi(\rho) = \int dk e^{-ik\rho} \phi_k = \sum_i \delta(\rho - \rho_i) = \sum_i \delta(\rho - \rho_i),$$ 

The collective field construction is based on two operators (so called “joining” and “splitting” operators), which have been obtained previously $[33, 34]$:

$$\Omega(\rho, \rho'; [\phi]) = \int \frac{dk'}{2\pi} \int \frac{dk}{2\pi} e^{-ik\rho} e^{-ik'\rho'} \sum_A \frac{\partial \phi_k}{\partial Z_A^\dagger} \frac{\partial \phi_{k'}}{\partial Z_A}$$ 

$$= \partial_\rho \partial_{\rho'} [\rho \phi(\rho) \delta(\rho - \rho')] ,$$ 

$$\omega(\rho; [\phi]) = \int \frac{dk}{2\pi} e^{-ik\rho} \frac{\partial^2 \phi_k}{\partial Z_A^\dagger \partial Z_A}$$ 

$$= -\partial_\rho \left( \rho \phi(\rho) \left[ 2 \int \frac{dp' \phi(p')}{(\rho - p')} + \frac{N(m-1)}{\rho} \right] \right).$$ 

The leading (in $N$) form of the collective field hamiltonian $[19, 20]$ then takes the form:

$$H_{\text{coll}}[\rho; [\phi]] = 2 \int d\rho \int d\rho' \Pi(\rho) \Omega(\rho, \rho'; [\phi]) \Pi(\rho')$$

$$+ \frac{1}{2} \int d\rho \int d\rho' \omega(\rho; [\phi]) \Omega^{-1}(\rho, \rho'; [\phi]) \omega(\rho'; [\phi]) + V[\rho; [\phi]],$$

$$\equiv \hat{H}_K[\rho; \Pi, [\phi]] + \Delta V[\rho; [\phi]] + V[\rho; [\phi]],$$

where we have introduced the conjugate momentum $\Pi(\rho) = \partial/\partial \phi(\rho)$. Then

$$-\frac{1}{2} \nabla_{\text{Radial}}^2 \to \hat{H}_K + \Delta V$$

$$= 2 \int d\rho (\partial_\rho \Pi(\rho)) [\rho \phi(\rho)] (\partial_\rho \Pi(\rho))$$

$$+ \frac{1}{2} \int d\rho (\rho \phi(\rho)) \left[ 2 \int \frac{dp' \phi(p')}{(\rho - p')} + \frac{N(m-1)}{\rho} \right]^2$$

Some re-arrangement yields:

$$\Delta V = 2 \int_0^\infty d\rho \rho \phi(\rho) \left[ \int_0^\infty \frac{dp' \phi(p')}{(\rho - p')} \right]^2 + \frac{N^2(m-1)^2}{2} \int_0^\infty d\rho \frac{\phi(\rho)}{\rho}.$$

In terms of the underlying Calogero eigenvalue system, this is equivalently written as:

$$\Delta V = 2 \sum_i \rho_i \left[ \sum_{j \neq i} \frac{1}{(\rho_i - \rho_j)} \right]^2 + \frac{N^2(m-1)^2}{2} \sum_i \frac{1}{\rho_i}.$$
We again find in addition to the usual repulsive term amongst the eigenvalues, the emergent dAFF $1/\rho$ potential, which is present for $m > 1$.

When one extends to the whole line and define $\Phi(r) \equiv 2r\phi(r^2) = \Phi(-r)$, one can use the identity
\[
\int_{-\infty}^{\infty} dr \Phi(r) \left( \int_{-\infty}^{\infty} dr' \Phi'(r') \right)^2 = \frac{\pi^2}{3} \int_{-\infty}^{\infty} dr \Phi^3(r)
\]

Then
\[
-\frac{1}{2} \nabla_{\text{Radial}}^2 \rightarrow \hat{H}_K + \Delta V
\]
\[
= \frac{1}{2} \int_{0}^{\infty} dr \partial_r \Pi(r) \Phi(r) \partial_r \Pi(r)
\]
\[
+ \frac{\pi^2}{6} \int_{0}^{\infty} dr \Phi^3(r) + \frac{N^2(m-1)^2}{2} \int_{0}^{\infty} dr \left[ \frac{\Phi(r)}{r^2} \right]
\]

with
\[
[\Phi(r), \Pi(r')] = i\delta(r-r')
\]

It is worth while pointing out what (5.1) achieves: a continuous $r$ coordinate emerges, which combines naturally with the original time of quantum mechanics to yield a local theory. In this sense, a constructive holographic description has been achieved, which ultimately results from a change of variables in the gauge theory side, similarly to, and very much in the spirit of, the holographic construction of higher spin theories achieved in [48] from their $O(N)$ invariant duals.

5.1 2$\hat{L}_0$: large $N$ background

We now show how to obtain the large $N$ background associated with the $\hat{L}_0$ operator.\(^5\)

\[
2\hat{L}_0 = \hat{H} + \hat{K}
\]

One has
\[
H_{\text{coll}} = \frac{1}{2} \int_{0}^{\infty} dr \partial_r \Pi(r) \Phi(r) \partial_r \Pi(r)
\]
\[
+ \frac{\pi^2}{6} \int_{0}^{\infty} dr \Phi^3(r) + \frac{N^2(d-2)^2}{8} \int_{0}^{\infty} dr \left[ \frac{\Phi(r)}{r^2} \right]
\]
\[
+ \frac{\omega^2}{2} \int_{0}^{\infty} dr \, r^2 \Phi(r) - \mu \left( N - \int_{0}^{\infty} dr \Phi(r) \right)
\]

where the lagrange multiplier has been added to enforce the contraint $\int_{0}^{\infty} dr \Phi(r) = N$.To make powers of $N$ explicit, we rescale
\[
r \rightarrow \sqrt{N} r \quad \Phi(r) \rightarrow \sqrt{N} \Phi(r) \quad \mu \rightarrow N\mu; \quad \Pi(r) \rightarrow \Pi(r)/N
\]

We find
\[
V_{\text{eff}} \equiv \Delta V + V \rightarrow N^2 V_{\text{eff}} \quad H_K \rightarrow H_K/N^2
\]

\(^5\)For dimensional reasons we introduce a mass parameter $w$. 

- 8 -
The large $N$ is then the minimum of $V_{\text{eff}}$, and is easily shown to be:

$$\Phi_0(r) = \frac{1}{\pi} \left( \frac{\omega}{2} (d-1) - \omega^2 r^2 - \frac{(d-2)^2}{4} \frac{1}{r^2} \right)^{1/2} \quad r_- \leq r \leq r_+$$

$$r^2_{\pm} = \frac{1}{4\omega} \left( \frac{(d-1)^2}{16\omega^2} - \frac{(d-2)^2}{4\omega^2} \right)$$

We observe that a Wigner distribution is present only for the single complex matrix ($m = 1$), but that for two or more complex matrices the background has support within “hyper-annuli”, or two “event horizons”.

In [40], it has been argued that the difference between considering $\hat{H}$ versus $\hat{L}_0$ should be associated with different choices of time (Poincaré versus global time) on the gravity side. A discussion of the large $N^\text{H}$ background in the context of such scenario, and possible identification of metric structures, is beyond the scope of this communication.

It is interesting to observe that the presence of a dAFF potential plays an important role in the light front approach to the matter sector spectrum of QCD of [49]. Indeed the difference between the spectrum of $\hat{H}$ and that of $\hat{L}_0$ is related to a generated confining potential associated with $\hat{L}_0$.

### 6 Summary and conclusions

We considered in this communication the quantum mechanics of an even number of hermitean matrices in a restricted radial sector of the theory, and showed that a $1/r^2$ potential emerges with strength proportional to $N^2$, for sufficiently large number of matrices. This sector of the theory has a (radial) fermionic description in terms of higher dimensional single particle $s$-state hamiltonians in the presence of a dAFF potential. This generalizes previously known large $N$ fermionic descriptions of single hermitean matrices. SL(2, $R$) conformal generators and generators of SO(2, 1) were identified, indicative if an $AdS_2$ geometry. The large $N$ eigenvalue density description of the underlying Calogero model was obtained, and explicitly obtained for the $L_0$ SO(2, 1) generator.

Several issues of interest and for future investigation arise naturally from the results established in this communication. For instance, it is remarkable that the single particle hamiltonians and corresponding wave functions satisfy AdS wave equations, despite the fact that one would not a priori associate the radial coordinate, as defined in this communication, with the holographic $z$ of an AdS Poincaré patch. Also, there is a very natural metric associated with the large $N$ collective field theory description of the single hermitian matrix [50]. It is of great interest to investigate if the analogue metric in the context of the results of this communication provides an explicit construction of an emergent AdS metric. It is likely that this will require further understanding of the angular degrees of freedom (parametrizing $S_{d-1}$). A related issue is whether the original conformal quantum mechanical correlators can be obtained from generating functionals of gravity field sources [8, 9]. These issues are currently being investigated.

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6Generalizations of the dAFF system to include angular degrees of freedom have been considered in [51], for instance.
A  An identity

We wish to prove the identity

\[ 4 \sum_i \left( \frac{1}{\rho_i^{m-1}} \Delta_{\text{RM}}(\rho_i) \frac{\partial}{\partial \rho_i} \Delta_{\text{RM}}(\rho_i) \right) \rho_i^m \left( \Delta_{\text{RM}}(\rho_i) \frac{\partial}{\partial \rho_i} \Delta_{\text{RM}}(\rho_i) \right) = \left( \frac{\partial}{\partial \rho_i} - \frac{a}{\rho_i} - \sum_{k \neq i} \frac{1}{\rho_i - \rho_k} \right) \frac{1}{\Delta_{\text{RM}}(\rho_i)} \]

One has

\[ \Delta_{\text{RM}}(\rho_i) \frac{\partial}{\partial \rho_i} \Delta_{\text{RM}}(\rho_i) = \left( \frac{\partial}{\partial \rho_i} - \frac{a}{\rho_i} - \sum_{k \neq i} \frac{1}{\rho_i - \rho_k} \right) \frac{1}{\Delta_{\text{RM}}(\rho_i)} \]

where we have defined \( a \equiv (N - 1)(m - 1)/2 \). Similarly,

\[ \left( \frac{1}{\Delta_{\text{RM}}(\rho_i)} \frac{\partial}{\partial \rho_i} \Delta_{\text{RM}}(\rho_i) \right) = \left( \frac{\partial}{\partial \rho_i} + \frac{a}{\rho_i} + \sum_{k \neq i} \frac{1}{\rho_i - \rho_k} \right) \frac{1}{\Delta_{\text{RM}}(\rho_i)} \]

Defining \( b = m - 1 \) for simplicity, one then has

\[ \sum_i \frac{4}{\rho_i^p} \left( \frac{\partial}{\partial \rho_i} + \frac{a}{\rho_i} + \sum_{k \neq i} \frac{1}{\rho_i - \rho_k} \right) \rho_i^m \left( \frac{\partial}{\partial \rho_i} - \frac{a}{\rho_i} - \sum_{j \neq i} \frac{1}{\rho_i - \rho_j} \right) = \]

\[ \sum_i \frac{4}{\rho_i^p} \left( \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} - a \frac{\partial}{\partial \rho_i} \rho_i^b \right) - \sum_{j \neq i} \frac{1}{\rho_i - \rho_j} \rho_i^m \frac{\partial}{\partial \rho_i} + \sum_{k \neq i} \frac{1}{\rho_i - \rho_k} \frac{\partial}{\partial \rho_i} \rho_i^m - \]

\[ \sum_i \frac{4}{\rho_i^b} \left( \sum_{k \neq i} \rho_i^m \frac{1}{\rho_i - \rho_k} \right) \cdot \]

Except for the term \( \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} \), one now acts with the derivatives. One finds that all terms linear in the derivative \( \frac{\partial}{\partial \rho_i} \) cancel out and one is left with:

\[ = 4 \sum_i \left( \frac{1}{\rho_i^p} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} - \frac{ab + a^2}{\rho_i} - (m + 2a) \sum_{j \neq i} \frac{1}{\rho_i - \rho_j} \right) \]

\[ + 4 \sum_i \left( \sum_{j \neq i} \frac{\rho_i^{m-b}}{(\rho_i - \rho_j)^2} - \sum_{k \neq i} \sum_{j \neq i} \frac{\rho_i^{m-b}}{\rho_i - \rho_k} \frac{1}{\rho_i - \rho_j} \right) \]

\[ = 4 \sum_i \left( \frac{1}{\rho_i^p} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} - \frac{ab + a^2}{\rho_i} + \sum_{j \neq i} \frac{\rho_i}{(\rho_i - \rho_j)^2} - \sum_{k \neq i} \sum_{j \neq i} \frac{\rho_i}{\rho_i - \rho_k} \frac{1}{\rho_i - \rho_j} \right) \]

\[ = 4 \sum_i \left( \frac{1}{\rho_i^p} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} - \frac{ab + a^2}{\rho_i} + \sum_{j \neq i} \frac{\rho_i}{(\rho_i - \rho_j)^2} - \sum_{k \neq i} \sum_{j \neq i} \frac{\rho_i}{\rho_i - \rho_k} \frac{1}{\rho_i - \rho_j} \right) \]
The last two terms have already been encountered in [33], where it was already shown that they vanish. Indeed, one notices that:

\[
\left( \sum_i \sum_{k \neq i} \sum_{j \neq i} \frac{1}{\rho_i - \rho_k - \rho_j} - \sum_i \sum_{j \neq i} (\rho_i - \rho_j)^2 \right) = \sum_{i \neq j \neq k \neq i} \frac{1}{\rho_i - \rho_k - \rho_j}.
\]  

(A.1)

This is easily seen to vanish by considering any three distinct eigenvalues.

Hence, we have established the result (3.1):

\[
4 \sum_i \left( \frac{1}{\Delta_{RM}(\rho_i)} \rho_i^{m-1} \frac{\partial}{\partial \rho_i} \Delta_{RM}(\rho_i) \right) \rho_i^m \left( \Delta_{RM}(\rho_i) \frac{\partial}{\partial \rho_i} \frac{1}{\Delta_{RM}(\rho_i)} \right)
\]

\[
= 4 \sum_i \left( \frac{1}{\rho_i^{m-1}} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} \frac{ab + a^2}{\rho_i} \right)
\]

\[
= \sum_i \left( \frac{4}{\rho_i^{m-1}} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} \frac{(N^2 - 1)(m-1)^2}{\rho_i} \right).
\]

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