A PERFECT PAIRING FOR MONOIDAL ADJUNCTIONS

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Abstract. We give another proof of the fact that there is a dual equivalence between the ∞-category of monoidal ∞-categories with left adjoint oplax monoidal functors and that with right adjoint lax monoidal functors by constructing a perfect pairing between them.

1. Introduction

In category theory the right adjoint of an oplax monoidal functor between monoidal categories is lax monoidal. It is reasonable to expect the similar statement holds in higher category theory.

In [3, 4] Lurie has developed higher category theory using quasi-categories, which are models for (∞, 1)-categories. In particular, he has proved that the right adjoint of a strong monoidal functor is lax monoidal between monoidal ∞-categories in [4]. In [5] we showed that the right adjoint of an oplax monoidal functor is lax monoidal. Haugseng-Hebestreit-Linskens-Nuiten [1] proved that the ∞-category $\text{Mon}_{\mathcal{O}}^{\text{oplax},L}(\text{Cat}_\infty)$ of $\mathcal{O}$-monoidal ∞-categories with left adjoint oplax $\mathcal{O}$-monoidal functors is dual equivalent to the ∞-category $\text{Mon}_{\mathcal{O}}^{\text{lax},R}(\text{Cat}_\infty)$ of $\mathcal{O}$-monoidal ∞-category with right adjoint lax $\mathcal{O}$-monoidal functors for each ∞-operad $\mathcal{O}^\otimes$.

Theorem 1.1 ([1]). There is an equivalence

$$\text{Mon}_{\mathcal{O}}^{\text{oplax},L}(\text{Cat}_\infty)^{\text{op}} \xrightarrow{\sim} \text{Mon}_{\mathcal{O}}^{\text{lax},R}(\text{Cat}_\infty)$$

of ∞-categories, which is identity on objects, and which assigns to a left adjoint oplax $\mathcal{O}$-monoidal functor its right adjoint lax $\mathcal{O}$-monoidal functor.

The purpose of this note is to give another proof of Theorem 1.1. For this purpose we study functorialities of the monoidal Yoneda embeddings. After that, we will prove Theorem 1.1 by constructing a perfect pairing between $\text{Mon}_{\mathcal{O}}^{\text{oplax},L}(\text{Cat}_\infty)$ and $\text{Mon}_{\mathcal{O}}^{\text{lax},R}(\text{Cat}_\infty)$ (Theorem 4.4).

The author thinks that the method of the proof might be interesting in its own right.

In fact, Haugseng-Hebestreit-Linskens-Nuiten [1] proved a more general statement than Theorem 1.1. They showed that there is a bidual equivalence between the $(\infty, 2)$-category of $\mathcal{O}$-monoidal ∞-categories with left adjoint oplax $\mathcal{O}$-monoidal functors and that of $\mathcal{O}$-monoidal ∞-categories with right adjoint lax $\mathcal{O}$-monoidal functors. We will not give a proof of the $(\infty, 2)$-categorical statement since it needs a more sophisticated method to organize equivalences between mapping $(\infty, 1)$-categories of the two $(\infty, 2)$-categories. Instead, we will compare the equivalence in this note with the restriction of the equivalence of Haugseng-Hebestreit-Linskens-Nuiten in [6].

Now, we will describe an outline of our proof. A small ∞-category $\mathcal{C}$ can be embedded into the ∞-category $\mathcal{P}(\mathcal{C})$ of presheaves on $\mathcal{C}$ by the Yoneda lemma. If $\mathcal{C}$ is an $\mathcal{O}$-monoidal ∞-category, then we can construct an $\mathcal{O}$-monoidal ∞-category $\mathcal{P}_{\mathcal{O}}(\mathcal{C})$ such that $\mathcal{P}_{\mathcal{O}}(\mathcal{C})_X \simeq \mathcal{P}(\mathcal{C}_X)$ for each $X \in \mathcal{O}$ by Day convolution product [4]. We will construct a pairing between the ∞-category $\text{Mon}_{\mathcal{O}}^{\text{oplax}}(\text{Cat}_\infty)$ of $\mathcal{O}$-monoidal ∞-categories with oplax $\mathcal{O}$-monoidal functors and the ∞-category...
Mon_{\text{oplax}}(\mathcal{C}, \infty) of \mathcal{O}-monoidal \infty-categories with lax \mathcal{O}-monoidal functors. The pairing corresponds to a functor $\text{Mon}_{\text{oplax}}(\mathcal{C}, \infty) \times \text{Mon}_{\text{oplax}}(\mathcal{C}, \infty) \to \mathcal{S}$, which assigns to a pair $(\mathcal{C}, \mathcal{D})$ the mapping space $\text{Map}_{\text{Mon}_{\text{oplax}}}(\mathcal{P}_{\mathcal{O}}(\mathcal{C}), \mathcal{P}_{\mathcal{O}}(\mathcal{D}))$. By restricting the pairing to the full subcategory spanned by those vertices corresponding to right adjoint lax $\mathcal{O}$-monoidal functors $\mathcal{C} \to \mathcal{D}$, we obtain a perfect pairing between $\text{Mon}_{\text{oplax}}(\mathcal{C}, \infty)$ and $\text{Mon}_{\text{op}}(\mathcal{C}, \infty)$, which gives the desired dual equivalence.

The organization of this note is as follows: In \textsection 3 we study monoidal structures on $\infty$-categories of presheaves and Yoneda embeddings. In \textsection 4 we consider monoidal functorialities for the construction of $\infty$-categories of presheaves. In \textsection 5 we prove Theorem \textsection 3 by constructing the perfect pairing.

**Notation 1.2.** We fix Grothendieck universes $\mathcal{U} \in \mathcal{V} \in \mathcal{W} \in \mathcal{X}$ throughout this note. We say that an element of $\mathcal{U}$ is small, an element of $\mathcal{V}$ is large, an element of $\mathcal{W}$ is very large, and an element of $\mathcal{X}$ is super large.

We denote by $\text{Cat}_{\infty}$ the large $\infty$-category of small $\infty$-categories, and by $\hat{\text{Cat}}_{\infty}$ the very large $\infty$-category of large $\infty$-categories. We denote by $\text{Pr}^{1}_{\infty}$ the subcategory of $\hat{\text{Cat}}_{\infty}$ spanned by presentable $\infty$-categories and left adjoint functors. We write $\mathcal{S}$ for the large $\infty$-category of small spaces, $\hat{\mathcal{S}}$ for the very large $\infty$-category of small spaces, and $\hat{\mathcal{S}}$ for the super large $\infty$-category of very large spaces.

For a small $\infty$-category $\mathcal{C}$, we denote by $\mathcal{P}(\mathcal{C})$ the $\infty$-category $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$ of presheaves on $\mathcal{C}$ with values in $\mathcal{S}$. For an $\infty$-operad $\mathcal{O}^{\otimes} \to \text{Fin}_{\infty}$, we write $\mathcal{O}$ for $\mathcal{O}^{\otimes}_{(1)}$. For an $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, we denote by $\mathcal{C}$ the underlying $\infty$-category $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{O}$, and we say that $\mathcal{C}$ is an $\mathcal{O}$-monoidal $\infty$-category for simplicity.

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## 2. Monoidal structure on presheaves

Let $\mathcal{O}^{\otimes}$ be a small $\infty$-operad. When $\mathcal{C}$ is a small $\mathcal{O}$-monoidal $\infty$-category, we can construct an $\mathcal{O}$-monoidal $\infty$-category $\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ by using Day convolution product. In this section we study the $\mathcal{O}$-monoidal $\infty$-category $\mathcal{P}_{\mathcal{O}}(\mathcal{C})$.

First, we recall that there is an $\mathcal{O}$-monoidal $\infty$-category $\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ by Day convolution product if $\mathcal{C}$ is a small $\mathcal{O}$-monoidal $\infty$-category. We denote by $\mathcal{S}^{\infty}_{\mathcal{O}} \to \text{Fin}_{\infty}$, the symmetric monoidal $\infty$-category for the $\infty$-category $\mathcal{S}$ of spaces with Cartesian symmetric monoidal structure. Taking pullback along the map $\mathcal{O}^{\infty} \to \text{Fin}_{\infty}$, we obtain an $\mathcal{O}$-monoidal $\infty$-category $\mathcal{S}^{\infty}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$, where $\mathcal{S}^{\infty}_{\mathcal{O}} = \mathcal{S}^{\infty} \times_{\text{Fin}_{\infty}} \mathcal{O}^{\otimes}$.

For an $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, there is an opposite $\mathcal{O}$-monoidal $\infty$-category $(\mathcal{C}^{\infty})^{\otimes} \to \mathcal{O}^{\otimes}$ such that $(\mathcal{C}^{\infty})^{\otimes} \simeq c_{\mathcal{X}}^{\mathcal{O}}$ for each $X \in \mathcal{O}$. By \textsection 3 Construction 2.2.6.7, we can construct a fibration $\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ of $\infty$-operads, where $\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes} = \text{Fun}(\mathcal{C}^{\infty}, \mathcal{S}^{\infty}_{\mathcal{O}})$. In fact, it is a coCartesian fibration of $\infty$-operads by \textsection 3 Proposition 2.2.6.16.

**Lemma 2.1** (\textsection 3 Definition 2.2.6.1]). If $\mathcal{C}$ is a small $\mathcal{O}$-monoidal $\infty$-category, then there is an $\mathcal{O}$-monoidal $\infty$-category $\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ in $\text{Pr}^{1}_{\infty}$ by Day convolution product such that $\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes} \simeq \mathcal{P}(\mathcal{C})$ for each $X \in \mathcal{O}$.

**Notation 2.2.** For $\mathcal{O}$-monoidal $\infty$-categories $\mathcal{C}$ and $\mathcal{D}$, we denote by

$$\text{Fun}^{\text{lax}}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$$

the $\infty$-category $\text{Alg}_{\mathcal{O}}(\mathcal{D})$ of lax $\mathcal{O}$-monoidal functors between $\mathcal{C}$ and $\mathcal{D}$.

By the universal property of Day convolution product (\textsection 3 Definition 2.2.6.1]), we can characterize a lax $\mathcal{O}$-monoidal functor $\mathcal{C} \to \mathcal{P}_{\mathcal{O}}(\mathcal{D})$ by its associated functor $\mathcal{C} \times_{\mathcal{O}} \mathcal{D}^{\infty} \to \mathcal{S}_{\mathcal{O}}$.
Lemma 2.3 (II Definition 2.2.6.1]). Let $C$ be a small $O$-monoidal $\infty$-category and let $D$ be an $O$-monoidal $\infty$-category. There is a lax $O$-monoidal functor $P(O) \times_\infty C^\vee \to \mathcal{S}_O$ which induces an equivalence

$$
\text{Fun}^\text{lax}_O(D, P(O)) \cong \text{Fun}^\text{lax}_O(D \times_\infty C^\vee, \mathcal{S}_O)
$$

of $\infty$-categories.

When $C$ is a small $\infty$-category, we have the Yoneda embedding $j : C \to P(C)$ which is fully faithful, and the $\infty$-category $P(C)$ is freely generated by $j(C)$ under small colimits in the sense of [III Theorem 5.1.5.6]. We will show that $j$ is promoted to a strong $O$-monoidal functor if $C$ is a small $O$-monoidal $\infty$-category.

Lemma 2.4. Let $C$ be a small $O$-monoidal $\infty$-category. There is a strong $O$-monoidal functor $J : C \to \mathbb{P}_O(C)$ such that $J_X : C_X \to \mathbb{P}_O(C)_X \simeq P(C_X)$ is equivalent to the Yoneda embedding for each $X \in O$.

Proof. First, we shall show that there is a lax $O$-monoidal functor $J : C \to \mathbb{P}_O(C)$ such that $J_X$ is equivalent to the Yoneda embedding for each $X \in O$. By Lemma 2.3, it suffices to construct a lax $O$-monoidal functor $C^\vee \times_\infty C \to \mathcal{S}_O$ whose restriction $C_X^\vee \times C_X \to \mathcal{S}$ is equivalent to the mapping space functor for each $X \in O$.

By [II Example 5.2.2.23], we have a pairing of $O$-monoidal $\infty$-categories

$$\lambda^\otimes : \text{TwArr}(C)^\otimes \to C^\otimes \times_\infty (C^\vee)^\otimes.$$

Note that this is a strong $O$-monoidal functor. By taking opposite $O$-monoidal $\infty$-categories, we obtain a strong $O$-monoidal functor

$$((\lambda^\vee)^\otimes : (\text{TwArr}(C^\vee))^\otimes \to (C^\vee)^\otimes \times_\infty C^\otimes).$$

The restriction $(\lambda^\vee)^\otimes_X : (\text{TwArr}(C^\vee))^\otimes_X \to (C^\vee)^\otimes_X \times C_X^\otimes$ is a left fibration for any $X \in O^\otimes$. By using coCartesian pushforward, we can show that $(\lambda^\vee)^\otimes$ is a left fibration. Thus, we obtain a $(C^\vee)^\otimes \times_\infty C^\otimes$-monoid object of $S$. This means there is a lax $O$-monoidal functor $C^\vee \times_\infty C \to \mathcal{S}_O$ by [II Proposition 2.4.2.5]. Furthermore, we can see that the restriction over each $X \in O$ is equivalent to the mapping space functor by [II Proposition 5.1.5.6].

Next, we shall show that $J$ is strong $O$-monoidal. Let $\phi : X \to Y$ be an active morphism of $O^\otimes$ with $Y \in O$. Let $C \in C_X^\otimes$. If $X \simeq X_1 \oplus \cdots \oplus X_n$, then $C \simeq C_1 \oplus \cdots \oplus C_n$, where $C_i \in C_{X_i}$ for $1 \leq i \leq n$. There is a coCartesian morphism $C \to C'$ of $C^\otimes$ over $\phi$, where $C' \simeq \otimes_{\phi} C_i$. By [II Proof of Corollary 2.2.6.14], we have to show that

$$\alpha : \times \circ (\prod j(C_i)) \longrightarrow j(C') \circ \otimes_{\phi}$$

exhibits $j(C')$ as a left Kan extension of $\times \circ \prod j(C_i)$ along $\otimes_{\phi}$:

$$\begin{align*}
C_{X_1}^\text{op} \times \cdots \times C_{X_n}^\text{op}^\prod j(C_i) \quad \longrightarrow & \quad S \times \cdots \times S \quad \longrightarrow \quad S \\
\otimes_{\phi} \downarrow & \quad \downarrow j(C') \quad \quad \quad \quad \\
C_{Y}^\text{op} \quad & \quad \\
\end{align*}$$

This follows from the facts that a left Kan extension of a representable functor is also representable by Yoneda’s lemma and that $\times \circ \prod j(C_i)$ is represented by the object $(C_1, \ldots, C_n) \in C_{X_1} \times \cdots \times C_{X_n}$. □
Remark 2.5. Note that \( J^\otimes : \mathcal{C}^\otimes \to \mathbb{P}_\mathcal{O}(\mathcal{C})^\otimes \) is fully faithful. Hence, it induces a fully faithful functor
\[
J_* : \text{Fun}^{\text{lax}}(\mathcal{D}, \mathcal{C}) \to \text{Fun}^{\text{lax}}(\mathcal{D}, \mathbb{P}_\mathcal{O}(\mathcal{C}))
\]
for any small \( \mathcal{O} \)-monoidal \( \infty \)-category \( \mathcal{D} \). We also call \( J : \mathcal{C} \to \mathbb{P}_\mathcal{O}(\mathcal{C}) \) the Yoneda embedding.

Definition 2.6. Suppose that \( \mathcal{C} \) and \( \mathcal{D} \) are compatible with small colimits in the sense of [4, Variant 3.1.1.19]. We denote by
\[
\text{Fun}^{\text{lax,cp}}(\mathcal{C}, \mathcal{D})
\]
the full subcategory of \( \text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D}) \) spanned by those lax \( \mathcal{O} \)-monoidal functors \( f \) such that \( f_X : \mathcal{C}_X \to \mathcal{D}_X \) is colimit-preserving for each \( X \in \mathcal{O} \).

Proposition 2.7. Let \( \mathcal{C} \) be a small \( \mathcal{O} \)-monoidal \( \infty \)-category, and let \( \mathcal{D} \) be an \( \mathcal{O} \)-monoidal \( \infty \)-category which is compatible with small colimits. Then the Yoneda embedding \( J : \mathcal{C} \to \mathbb{P}_\mathcal{O}(\mathcal{C}) \) induces an equivalence
\[
J^* : \text{Fun}^{\text{lax,cp}}(\mathbb{P}_\mathcal{O}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D})
\]
of \( \infty \)-categories.

In order to prove Proposition 2.7 we need the following lemma.

Lemma 2.8. Let \( \mathcal{C} \) be a small \( \mathcal{O} \)-monoidal \( \infty \)-category. For any \( X \in \mathcal{O} \) and \( G \in \mathbb{P}(\mathcal{C}_X) \), the inclusion map \( \mathcal{C}_X/G \to \mathcal{C}_\mathcal{O}/G \) is cofinal.

Proof. For an object \( \overline{\mathcal{D}} = (D, J^\otimes(D) \to G) \) of \( (\mathcal{C}_\mathcal{O}/G) \), we set \( \mathcal{E}_{\overline{\mathcal{D}}} = \mathcal{C}_X/G \times (\mathcal{C}_\mathcal{O}/G)_{/G}((\mathcal{C}_\mathcal{O}/G)_{/G})_{/\overline{\mathcal{D}}}/\mathcal{E}_{\overline{\mathcal{D}}} \). By [4, Theorem 4.1.3.1], it suffices to show that \( \mathcal{E}_{\overline{\mathcal{D}}} \) has an initial object, which implies that it is weakly contractible. Let \( \phi \) be an active morphism in \( \mathcal{O}^\otimes \) over which \( J^\otimes(D) \to G \) lies. We take a coCartesian morphism \( D \to \phi_1 \) in \( \mathcal{C}^\otimes \) covering \( \phi \). Since \( J^\otimes : \mathcal{C}^\otimes \to \mathbb{P}_\mathcal{O}(\mathcal{C})^\otimes \) preserves coCartesian morphisms by Lemma 2.4, we can factor \( J^\otimes(D) \to G \) as \( J^\otimes(D) \to J^\otimes(\phi_1 \) \) \( D \to G \). Then we can see that \( (D \to \phi_1 \) \( D \to J^\otimes(\phi_1 \) \( D \to G \) is an initial object of \( \mathcal{E}_{\overline{\mathcal{D}}} \). □

Proof of Proposition 2.7 By [4, Proposition 3.1.1.20 and Corollary 3.1.3.4], the functor
\[
J^* : \text{Fun}^{\text{lax,cp}}(\mathbb{P}_\mathcal{O}(\mathcal{C}), \mathcal{D}) \to \text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D})
\]
admits a left adjoint \( J_* \). Lemma 2.8 implies that the restriction \( J_*(f)_X : \mathbb{P}_\mathcal{O}(\mathcal{C})_X \to \mathcal{D}_X \) can be identified with a left Kan extension of \( f_X \) along the Yoneda embedding \( j : \mathcal{C}_X \to \mathbb{P}(\mathcal{C}_X) \) for any \( X \in \mathcal{O} \). Hence \( J_*(f) \) lands in the full subcategory \( \text{Fun}^{\text{lax,cp}}(\mathbb{P}_\mathcal{O}(\mathcal{C}), \mathcal{D}) \) and \( J^*J_*(f) \simeq f \). In particular, we see that \( J_* \) is fully faithful and the essential image is \( \text{Fun}^{\text{lax,cp}}(\mathbb{P}_\mathcal{O}(\mathcal{C}), \mathcal{D}) \). This completes the proof. □

Definition 2.9. For \( \mathcal{O} \)-monoidal \( \infty \)-categories \( \mathcal{C} \) and \( \mathcal{D} \), we set
\[
\text{Fun}_\mathcal{O}^{\text{oplax}}(\mathcal{C}, \mathcal{D}) = \text{Fun}_\mathcal{O}^{\text{lax}}(\mathcal{C}^\vee, \mathcal{D}^\vee)^{\text{op}},
\]
and call it the \( \infty \)-category of oplax \( \mathcal{O} \)-monoidal functors. For \( f \in \text{Fun}_\mathcal{O}^{\text{oplax}}(\mathcal{C}, \mathcal{D}) \), we say that \( f \) is left adjoint if \( f_X : \mathcal{C}_X \to \mathcal{D}_X \) is left adjoint for all \( X \in \mathcal{O} \). We denote by
\[
\text{Fun}_\mathcal{O}^{\text{oplax},L}(\mathcal{C}, \mathcal{D})
\]
the full subcategory of \( \text{Fun}_\mathcal{O}^{\text{oplax}}(\mathcal{C}, \mathcal{D}) \) spanned by those oplax \( \mathcal{O} \)-monoidal functors which are left adjoint.

For \( f \in \text{Fun}_\mathcal{O}^{\text{lax}}(\mathcal{C}, \mathcal{D}) \), we also say that \( f \) is right adjoint if \( f_X : \mathcal{C}_X \to \mathcal{D}_X \) is right adjoint for all \( X \in \mathcal{O} \). We denote by
\[
\text{Fun}_\mathcal{O}^{\text{lax},R}(\mathcal{C}, \mathcal{D})
\]
the full subcategory of $\text{Fun}_O^{\text{lax}}(\mathcal{C}, \mathcal{D})$ spanned by those lax $O$-monoidal functors which are right adjoint.

Note that there is an equivalence
\[
\text{Fun}_O^{\text{oplax}}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}_O^{\text{lax}}(\mathcal{C}^\vee, \mathcal{D}^\vee)^{\text{op}}
\]
of $\infty$-categories by definition.

**Definition 2.10.** Let $\mathcal{C}$ and $\mathcal{D}$ be small $O$-monoidal $\infty$-categories. We define a functor
\[
P^*_O(\mathcal{C}, \mathcal{D}) : \text{Fun}_O^{\text{oplax}}(\mathcal{C}, \mathcal{D})^{\text{op}} \to \text{Fun}_O^{\text{lax}, cp}(P_O(D), P_O(C))
\]
by the following commutative diagram
\[
\begin{array}{ccc}
\text{Fun}_O^{\text{oplax}}(\mathcal{C}, \mathcal{D})^{\text{op}} & \to & \text{Fun}_O^{\text{lax}, cp}(P_O(D), P_O(C)) \\
\cong & & \cong \\
\text{Fun}_O^{\text{lax}}(\mathcal{C}^\vee, \mathcal{D}^\vee) & \underset{J^*}{\to} & \text{Fun}_O^{\text{lax}}(\mathcal{C}^\vee, P_O(D^\vee)) \\
\end{array}
\]
(2.1)
in $\hat{\text{Cat}}_\infty$, where the top right horizontal arrow is an equivalence by Proposition 2.7 and the right vertical and the bottom right horizontal arrows are equivalences by Lemma 2.3. Note that $P^*_O(\mathcal{C}, \mathcal{D})$ is fully faithful since $J^*$ is fully faithful by Remark 2.6. For an oplax $O$-monoidal functor $f : \mathcal{C} \to \mathcal{D}$, we simply write $f^*$ for $P^*_O(\mathcal{C}, \mathcal{D})(f)$.

**Proposition 2.11.** Let $f : \mathcal{C} \to \mathcal{D}$ be an oplax $O$-monoidal functor between small $O$-monoidal $\infty$-categories. Then $(f^*)_X$ is equivalent to the functor $(f_X)^* : P(D_X) \to P(C_X)$ for each $X \in O$.

**Proof.** This follows by restricting diagram (2.1) over $X \in O$. $\square$

**Corollary 2.12.** Let $f : \mathcal{C} \to \mathcal{D}$ be a left adjoint oplax $O$-monoidal functor between small $O$-monoidal $\infty$-categories. Then there exists a right adjoint lax $O$-monoidal functor $f^R : \mathcal{D} \to \mathcal{C}$ such that $(f^R)_X \simeq (f_X)^R$, where $(f_X)^R$ is a right adjoint to $f_X$. In this case we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{D} & \underset{f^R}{\to} & \mathcal{C} \\
\downarrow & & \downarrow \\
P_O(D) & \underset{f^*}{\to} & P_O(C)
\end{array}
\]
in $\text{Mon}^{\text{lax}}(\hat{\text{Cat}}_\infty)$.

**Proof.** By Lemma 2.4 and Definition 2.10 the composite $f^* \circ J : \mathcal{D} \to P_O(C)$ is lax $O$-monoidal. We can regard $\mathcal{C}$ as a full subcategory of $P_O(C)$ through $J$ by Remark 2.6. For each $X \in O$, we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{D}_X & \underset{(f_X)^R}{\to} & \mathcal{C}_X \\
\downarrow & & \downarrow \\
P(D_X) & \underset{(f_X)^*}{\to} & P(C_X),
\end{array}
\]
where the vertical arrows are the Yoneda embeddings. By Lemma 2.4 and Proposition 2.11 $f^* \circ J$ factors through $J : \mathcal{C} \to P_O(C)$, and we obtain the desired lax $O$-monoidal functor $f^R : \mathcal{D} \to \mathcal{C}$. $\square$
**Remark 2.13.** Let \( f : \mathcal{C} \to \mathcal{D} \) be a right adjoint lax \( \mathcal{O} \)-monoidal functor between small \( \mathcal{O} \)-monoidal \( \infty \)-categories. Then \( f^\vee : \mathcal{C}^\vee \to \mathcal{D}^\vee \) is a left adjoint oplax \( \mathcal{O} \)-monoidal functor. By Corollary 2.12 we have a right adjoint lax \( \mathcal{O} \)-monoidal functor \( (f^\vee)^R : \mathcal{D}^\vee \to \mathcal{C}^\vee \). We set \( f^L = ((f^\vee)^R)^\vee : \mathcal{D} \to \mathcal{C} \). Then \( f^L \) is a left adjoint oplax \( \mathcal{O} \)-monoidal functor such that \( (f^L)_X \simeq (f_X)^L \), where \( (f_X)^L : \mathcal{D}_X \to \mathcal{C}_X \) is a left adjoint to \( f_X : \mathcal{C}_X \to \mathcal{D}_X \).

**Definition 2.14.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be small \( \mathcal{O} \)-monoidal \( \infty \)-categories. We define a functor

\[
\mathcal{P}_{\mathcal{O}_!(\mathcal{C}, \mathcal{D})} : \text{Fun}_{\mathcal{O}}^\lax(\mathcal{C}, \mathcal{D}) \to \text{Fun}_{\mathcal{O}}^\lax,\text{cp}(\mathcal{P}_\mathcal{O}(\mathcal{C}), \mathcal{P}_\mathcal{O}(\mathcal{D}))
\]

by the following commutative diagram

\[
\begin{array}{ccc}
\text{Fun}_{\mathcal{O}}^\lax(\mathcal{C}, \mathcal{D}) & \xrightarrow{\mathcal{P}_{\mathcal{O}_!(\mathcal{C}, \mathcal{D})}} & \text{Fun}_{\mathcal{O}}^\lax,\text{cp}(\mathcal{P}_\mathcal{O}(\mathcal{C}), \mathcal{P}_\mathcal{O}(\mathcal{D})) \\
J_* \downarrow & & \downarrow J^* \\
\text{Fun}_{\mathcal{O}}^\lax(\mathcal{C}, \mathcal{P}_\mathcal{O}(\mathcal{D}))
\end{array}
\]

in \( \widehat{\text{Cat}}_{\infty} \), where the right vertical arrow is an equivalence by Proposition 2.7. Note that \( \mathcal{P}_{\mathcal{O}_!(\mathcal{C}, \mathcal{D})} \) is fully faithful since \( J_* \) is fully faithful by Remark 2.5. For a lax \( \mathcal{O} \)-monoidal functor \( f : \mathcal{C} \to \mathcal{D} \), we simply write \( f_! \) for \( \mathcal{P}_{\mathcal{O}_!(\mathcal{C}, \mathcal{D})}(f) \).

**Remark 2.15.** Let \( f : \mathcal{C} \to \mathcal{D} \) be a lax \( \mathcal{O} \)-monoidal functor between small \( \mathcal{O} \)-monoidal \( \infty \)-categories. By construction, we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{D} \\
\downarrow J & & \downarrow J \\
\mathcal{P}_\mathcal{O}(\mathcal{C}) & \xrightarrow{f_*} & \mathcal{P}_\mathcal{O}(\mathcal{D})
\end{array}
\]

in \( \text{Mon}^\lax(\widehat{\text{Cat}}_{\infty}) \).

**Lemma 2.16.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be small \( \mathcal{O} \)-monoidal \( \infty \)-categories. If \( f : \mathcal{D} \to \mathcal{C} \) is a left adjoint oplax \( \mathcal{O} \)-monoidal functor, then we have an equivalence \( f^* \simeq (f^R)_! \) in \( \text{Fun}_{\mathcal{O}}^\lax,\text{cp}(\mathcal{P}_\mathcal{O}(\mathcal{C}), \mathcal{P}_\mathcal{O}(\mathcal{D})) \). Dually, if \( g : \mathcal{C} \to \mathcal{D} \) is a right adjoint lax \( \mathcal{O} \)-monoidal functor, then we have \( (g^L)^* \simeq g_* \).

**Proof.** The Yoneda embedding induces an equivalence of \( \infty \)-categories \( \text{Fun}_{\mathcal{O}}^\lax,\text{cp}(\mathcal{P}_\mathcal{O}(\mathcal{C}), \mathcal{P}_\mathcal{O}(\mathcal{D})) \) \( \simeq \) \( \text{Fun}_{\mathcal{O}}^\lax(\mathcal{C}, \mathcal{P}_\mathcal{O}(\mathcal{D})) \) by Proposition 2.7. If \( f \) is a left adjoint oplax \( \mathcal{O} \)-monoidal functor, then Corollary 2.12 and Remark 2.15 imply that \( f^* \circ J \simeq J \circ f^R \simeq (f^R)_! \circ J \) in \( \text{Fun}_{\mathcal{O}}^\lax(\mathcal{C}, \mathcal{P}_\mathcal{O}(\mathcal{D})) \). Hence \( f^* \simeq (f^R)_! \). If \( g \) is a right adjoint lax \( \mathcal{O} \)-monoidal functor, then we have \( (g^L)^* \simeq ((g^L)^R)_! \simeq g_* \).

**Proposition 2.17.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be small \( \mathcal{O} \)-monoidal \( \infty \)-categories. There is a natural equivalence

\[
D_{(\mathcal{C}, \mathcal{D})} : \text{Fun}_{\mathcal{O}}^{\text{oplax},L}(\mathcal{C}, \mathcal{D})^{\text{op}} \xrightarrow{\simeq} \text{Fun}_{\mathcal{O}}^{\text{lax},R}(\mathcal{D}, \mathcal{C})
\]

of \( \infty \)-categories, which makes the following diagram commute

\[
\begin{array}{ccc}
\text{Fun}_{\mathcal{O}}^{\text{oplax},L}(\mathcal{C}, \mathcal{D})^{\text{op}} & \xrightarrow{D_{(\mathcal{C}, \mathcal{D})}} & \text{Fun}_{\mathcal{O}}^{\text{lax},R}(\mathcal{D}, \mathcal{C}) \\
\downarrow & & \downarrow \\
\text{Fun}_{\mathcal{O}}^{\text{lax},\text{cp}}(\mathcal{P}_\mathcal{O}(\mathcal{D}), \mathcal{P}_\mathcal{O}(\mathcal{C}))
\end{array}
\]

in \( \widehat{\text{Cat}}_{\infty} \). The functor \( D_{(\mathcal{C}, \mathcal{D})} \) associates to a left adjoint oplax \( \mathcal{O} \)-monoidal functor \( f \) its right adjoint lax \( \mathcal{O} \)-monoidal functor \( f^R \).
Proof. Since $\mathcal{P}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ and $\mathcal{P}_{\mathcal{O}_1}(\mathcal{C}, \mathcal{D})$ are fully faithful, it suffices to show that the essential images are the same for the existence of the equivalence $D_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$. This follows from Lemma 2.10. We also obtain the last part by Lemma 2.10.

3. Presheaf functors

In this section we study monoidal functorialities of the construction $\mathcal{P}_{\mathcal{O}}(\mathcal{C})$ from a small $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}$.

3.1. Contravariant functor $\mathcal{P}_{\mathcal{O}}^\ast$. There is a functor

$$\mathcal{P}^\ast : \mathcal{C}^\text{op}_\infty \to \mathcal{P}^L,$$

which associates to a small $\infty$-category $\mathcal{C}$ the $\infty$-category $\mathcal{P}(\mathcal{C})$ of presheaves on $\mathcal{C}$, and to a functor $f : \mathcal{C} \to \mathcal{D}$ the functor $f^\ast : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$ that is obtained by composing with $f^\text{op}$. In this subsection we will construct a functor

$$\mathcal{P}_{\mathcal{O}}^\ast : \mathcal{O}^{\text{oplax}}(\mathcal{C})^\text{op} \to \mathcal{O}^{lax}(\mathcal{P}^L),$$

which is a lifting of the functor $\mathcal{P}^\ast$.

First, we consider the composite of functors

$$\mathcal{F}^\ast_{\mathcal{O}} : \mathcal{O}^{\text{oplax}}(\mathcal{C})^\text{op} \xrightarrow{\sim} \mathcal{O}^{lax}(\mathcal{C})^\text{op} \to \mathcal{O}^{lax}(\mathcal{P}^L),$$

where the first arrow is an equivalence given by $\mathcal{C} \mapsto \mathcal{C}^\vee$ and the second arrow is given by $\mathcal{C} \mapsto \text{Fun}(\mathcal{C}, \mathcal{O}) = \mathcal{P}(\mathcal{C}^\vee)$. We note that $\mathcal{F}^\ast_{\mathcal{O}}$ assigns $\mathcal{P}_{\mathcal{O}}(\mathcal{C})$ to a small $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}$.

For any oplax $\mathcal{O}$-monoidal functor $f : \mathcal{C} \to \mathcal{D}$, the restriction of $\mathcal{F}^\ast_{\mathcal{O}}(f) : \mathcal{P}_{\mathcal{O}}(\mathcal{D}) \to \mathcal{P}_{\mathcal{O}}(\mathcal{C})$ over $X \in \mathcal{O}$ is equivalent to $(f_X)^\ast : \mathcal{P}(\mathcal{D}_X) \to \mathcal{P}(\mathcal{C}_X)$. Since $(f_X)^\ast$ admits a right adjoint $(f_X)^!$, the functor $\mathcal{F}^\ast_{\mathcal{O}}$ factors through $\mathcal{O}^{lax}(\mathcal{P}^L)$. Hence we obtain the desired functor $\mathcal{P}_{\mathcal{O}}^\ast$.

For small $\mathcal{O}$-monoidal $\infty$-categories $\mathcal{C}$ and $\mathcal{D}$, the functor $\mathcal{P}_{\mathcal{O}}^\ast$ induces a functor

$$\mathcal{M}_{\mathcal{O}}^{\text{oplax}}(\mathcal{C}, \mathcal{D}) \to \mathcal{M}_{\mathcal{O}}^{lax}(\mathcal{P}^L)(\mathcal{P}_{\mathcal{O}}(\mathcal{D}), \mathcal{P}_{\mathcal{O}}(\mathcal{C}))$$

of mapping spaces. We notice that it is equivalent to the functor obtained from $\mathcal{P}_{\mathcal{O},(\mathcal{C}, \mathcal{D})}^\ast$ in Definition 2.10 by taking core. Hence we obtain the following proposition.

**Proposition 3.1.** There is a functor

$$\mathcal{P}_{\mathcal{O}}^\ast : \mathcal{O}^{\text{oplax}}(\mathcal{C})^\text{op} \to \mathcal{O}^{lax}(\mathcal{P}^L),$$

which associates to a small $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}$ the $\mathcal{O}$-monoidal $\infty$-category $\mathcal{P}_{\mathcal{O}}(\mathcal{C})$, and to an oplax $\mathcal{O}$-monoidal functor $f : \mathcal{C} \to \mathcal{D}$ the lax $\mathcal{O}$-monoidal functor $f^\ast : \mathcal{P}_{\mathcal{O}}(\mathcal{D}) \to \mathcal{P}_{\mathcal{O}}(\mathcal{C})$.

3.2. Covariant functor $\mathcal{P}_{\mathcal{O}_1}$. There is a functor

$$P_1 : \mathcal{C}^\infty \to \mathcal{P}^L,$$

which associates to a small $\infty$-category $\mathcal{C}$ the $\infty$-category $\mathcal{P}(\mathcal{C})$ of presheaves on $\mathcal{C}$, and to a functor $f : \mathcal{C} \to \mathcal{D}$ the functor $f_1 : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ that is obtained by left Kan extension. In this subsection we will construct a functor

$$\mathcal{P}_{\mathcal{O}_1} : \mathcal{O}^{lax}(\mathcal{C}) \to \mathcal{O}^{lax}(\mathcal{P}^L)$$

which is a lifting of $P_1$.

**Remark 3.2.** The functor $P_1$ has two definitions: One is as a free cocompletion and the other is as a left adjoint to $P^\ast$. They are recently shown to be equivalent in [2].
First, we consider a functor
\[ F_{\Omega} : \text{Mon}_{\infty}^\text{lax}(\text{Cat}_\infty)^{\text{op}} \to \text{Fun}(\text{Mon}_{\infty}^\text{lax}(\text{Pr}^L), \hat{S}) \]
whose adjoint \( \text{Mon}_{\infty}^\text{lax}(\text{Cat}_\infty)^{\text{op}} \times \text{Mon}_{\infty}^\text{lax}(\text{Pr}^L) \to \hat{S} \) is given by \((C, D) \mapsto \text{Map}_{\text{Mon}_{\infty}^\text{lax}(\text{Cat}_\infty)}(C, D)\). By Proposition 2.7, the Yoneda embedding induces an equivalence \( J^* : \text{Map}_{\text{Mon}_{\infty}^\text{lax}(\text{Pr}^L)}(\mathbb{P}_\Omega(C), D) \xrightarrow{\cong} \text{Map}_{\text{Mon}_{\infty}^\text{lax}(\text{Cat}_\infty)}(C, D) \). This implies that the functor \( F_{\Omega} \) factors through the Yoneda embedding \( \text{Mon}_{\infty}^\text{lax}(\text{Pr}^L)^{\text{op}} \to \text{Fun}(\text{Mon}_{\infty}^\text{lax}(\text{Pr}^L), \hat{S}) \). Hence we obtain the desired functor \( \mathbb{P}_{\Omega} \)

For small \( \mathcal{O} \)-monoidal \( \infty \)-categories \( \mathcal{C} \) and \( \mathcal{D} \), the functor \( \mathbb{P}_{\Omega} \) induces a functor
\[ \text{Map}_{\text{Mon}_{\infty}^\text{lax}(\text{Cat}_\infty)}(\mathcal{C}, \mathcal{D}) \to \text{Map}_{\text{Mon}_{\infty}^\text{lax}(\text{Pr}^L)}(\mathbb{P}_\Omega(C), \mathbb{P}_\Omega(D)) \]
of mapping spaces. We notice that it is equivalent to the functor obtained from \( \mathbb{P}_{\Omega!(\mathcal{C}, \mathcal{D})} \) in Definition 2.14 by taking core. Hence we obtain the following proposition.

**Proposition 3.3.** There is a functor
\[ \mathbb{P}_{\Omega} : \text{Mon}_{\infty}^\text{lax}(\text{Cat}_\infty) \to \text{Mon}_{\infty}^\text{lax}(\text{Pr}^L), \]
which associates to a small \( \mathcal{O} \)-monoidal \( \infty \)-category \( \mathcal{C} \) the \( \mathcal{O} \)-monoidal \( \infty \)-category \( \mathbb{P}_\Omega(C) \), and to a lax \( \mathcal{O} \)-monoidal functor \( f : \mathcal{C} \to \mathcal{D} \) the lax \( \mathcal{O} \)-monoidal functor \( \mathbb{P}_\Omega(f) : \mathbb{P}_\Omega(C) \to \mathbb{P}_\Omega(D) \).

### 4. Perfect Pairing for Monoidal Adjunctions

In this section we will prove Theorem 1.1 by constructing a perfect pairing between the \( \infty \)-category of \( \mathcal{O} \)-monoidal \( \infty \)-categories with left adjoint oplax monoidal functors and that with right adjoint lax monoidal functors (Theorem 4.4).

We denote by
\[ \hat{\lambda} : \hat{\mathcal{M}}_{\mathcal{O}}^\text{lax} \to \text{Mon}_{\mathcal{O}}^\text{lax}(\text{Pr}^L) \times \text{Mon}_{\mathcal{O}}^\text{lax}(\text{Pr}^L)^{\text{op}} \]
the perfect pairing associated to the mapping space functor \( \text{Map}_{\text{Mon}_{\mathcal{O}}^\text{lax}(\text{Pr}^L)}(-, -) : \text{Mon}_{\mathcal{O}}^\text{lax}(\text{Pr}^L)^{\text{op}} \times \text{Mon}_{\mathcal{O}}^\text{lax}(\text{Pr}^L) \to \hat{S} \). We define
\[ \lambda : \mathcal{M}_{\mathcal{O}}^\text{lax} \to \text{Mon}_{\mathcal{O}}^\text{oplax,L}(\text{Cat}_\infty)^{\text{op}} \times \text{Mon}_{\mathcal{O}}^\text{lax,R}(\text{Cat}_\infty)^{\text{op}} \]
to be a right fibration obtained from \( \hat{\lambda} \) by pullback along the functor
\[ \mathbb{P}_\mathcal{O}^{\text{op}} \times \mathbb{P}_\mathcal{O}^{\text{op}} : \text{Mon}_{\mathcal{O}}^\text{oplax,L}(\text{Cat}_\infty)^{\text{op}} \times \text{Mon}_{\mathcal{O}}^\text{lax,R}(\text{Cat}_\infty)^{\text{op}} \to \text{Mon}_{\mathcal{O}}^\text{lax}(\text{Pr}^L) \times \text{Mon}_{\mathcal{O}}^\text{lax}(\text{Pr}^L)^{\text{op}}, \]
An object of \( \mathcal{M}_{\mathcal{O}}^\text{lax} \) corresponds to a triple \((\mathcal{C}, \mathcal{D}, f)\), where \( \mathcal{C} \) and \( \mathcal{D} \) are small \( \mathcal{O} \)-monoidal \( \infty \)-categories and \( f : \mathbb{P}_\Omega(C) \to \mathbb{P}_\Omega(D) \) is a morphism in \( \text{Mon}_{\mathcal{O}}^\text{lax}(\text{Pr}^L) \).

For small \( \mathcal{O} \)-monoidal \( \infty \)-categories \( \mathcal{C} \) and \( \mathcal{D} \), the functor \( \mathbb{P}_{\Omega} \) induces a fully faithful functor \( \text{Map}_{\text{Mon}_{\mathcal{O}}^\text{lax}(\text{Cat}_\infty)}(\mathcal{C}, \mathcal{D}) \to \text{Map}_{\text{Mon}_{\mathcal{O}}^\text{lax}(\text{Pr}^L)}(\mathbb{P}_\Omega(C), \mathbb{P}_\Omega(D)) \) of mapping spaces. We define
\[ \mathcal{M}_{\mathcal{O}}^{\text{lax,R}} \]
to be the full subcategory of \( \mathcal{M}_{\mathcal{O}}^\text{lax} \) spanned by those objects \( v \) corresponding to triples \((\mathcal{C}, \mathcal{D}, f)\) where \( f \simeq g \) for some right adjoint \( \mathcal{O} \)-monoidal functor \( g : \mathcal{C} \to \mathcal{D} \). We let
\[ \lambda^\text{R} : \mathcal{M}_{\mathcal{O}}^{\text{lax,R}} \to \text{Mon}_{\mathcal{O}}^\text{oplax,L}(\text{Cat}_\infty)^{\text{op}} \times \text{Mon}_{\mathcal{O}}^\text{lax,R}(\text{Cat}_\infty)^{\text{op}} \]
be the restriction of \( \lambda \) to \( \mathcal{M}_{\mathcal{O}}^{\text{lax,R}} \).

First, we show that \( \lambda^\text{R} \) is a paring of \( \infty \)-categories.

**Lemma 4.1.** The functor \( \lambda^\text{R} : \mathcal{M}_{\mathcal{O}}^{\text{lax,R}} \to \text{Mon}_{\mathcal{O}}^\text{oplax,L}(\text{Cat}_\infty)^{\text{op}} \times \text{Mon}_{\mathcal{O}}^\text{lax,R}(\text{Cat}_\infty)^{\text{op}} \) is a paring of \( \infty \)-categories.
Proof. We shall prove that $\lambda^R$ is a right fibration. For this purpose, since $\lambda$ is a right fibration, it suffices to show the following: For a morphism $m : v \to v'$ of $\mathcal{M}^{\text{lax}, R}_\mathcal{O}$, if $v' \in \mathcal{M}^{\text{lax}, R}_\mathcal{O}$, then $v \in \mathcal{M}^{\text{lax}, R}_\mathcal{O}$.

We denote by $k : \mathcal{P}_\mathcal{O}(\mathcal{C}) \to \mathcal{P}_\mathcal{O}(\mathcal{D})$ and $k' : \mathcal{P}_\mathcal{O}(\mathcal{C}') \to \mathcal{P}_\mathcal{O}(\mathcal{D}')$ morphisms in $\text{Mon}^{\text{lax}}_\mathcal{O}(\text{Pr}_L)$ corresponding to $v$ and $v'$, respectively. Since $v' \in \mathcal{M}^{\text{lax}, R}_\mathcal{O}$, we can write $k' \simeq f_1$, where $f : \mathcal{C}' \to \mathcal{D}'$ is a right adjoint lax $\mathcal{O}$-monoidal functor. Let $\lambda(m) \simeq (g, h)$. Then $k \simeq h_1 \circ k' \circ g^*$. Since $g$ is a left adjoint op-lax $\mathcal{O}$-monoidal functor, $g^* \simeq (g^R)$: by Lemma 2.16. Thus, $k \simeq (h \circ f \circ g^R)$. Since $h_1 \circ f \circ g^R$ is a right adjoint lax $\mathcal{O}$-monoidal functor, we see that $v \in \mathcal{M}^{\text{lax}, R}_\mathcal{O}$.

Next, we shall show that the pairing $\lambda^R$ is perfect. For this purpose, we need the following lemma.

**Lemma 4.2.** Let $v$ be an object of $\mathcal{M}^{\text{lax}, R}_\mathcal{O}$ which corresponds to a triple $(\mathcal{C}, \mathcal{D}, f)$. Then the following conditions are equivalent:

1. The object $v$ is left universal.
2. The object $v$ is right universal.
3. $f \simeq g_1$, where $g : \mathcal{C} \to \mathcal{D}$ is an equivalence in $\text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})$.

Proof. First, we shall prove the equivalence between (1) and (3). Let $\lambda^R_\mathcal{O}$ be the right fibration obtained from $\lambda^R$ by restriction to $\{\mathcal{C}\} \times \text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}}$. It is classifiable by the functor $\text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty}) \to \mathcal{S}$ given by $\mathcal{D}' \mapsto \text{Map}_{\text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})}(\mathcal{C}, \mathcal{D}')$. Hence $\lambda^R_\mathcal{O}$ is equivalent to the map $\text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}} \to \text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}}$ as right fibrations. Thus, we see that $v$ is left universal if and only if $f \simeq g_1$ for an equivalence $g$.

Next, we shall prove the equivalence between (2) and (3). We consider the right fibration $\lambda^R_\mathcal{O} : \mathcal{N} \to \text{Mon}^{\text{oplax}}_{\mathcal{O}}(\text{Cat}_{\infty})^{\text{op}}$ obtained from $\lambda^R$ by restriction to $\text{Mon}^{\text{oplax}}_{\mathcal{O}}(\text{Cat}_{\infty})^{\text{op}} \times \{\mathcal{D}\}$, where $\mathcal{N} = \mathcal{M}^{\text{lax}, R}_\mathcal{O} \times_{\text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}}} \{\mathcal{D}\}$. We have a commutative diagram of right fibrations

$$
\begin{array}{ccc}
\mathcal{N} / v & \xrightarrow{\sim} & \mathcal{N} \\
\downarrow & & \downarrow \\
\text{Mon}^{\text{oplax}, L}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}} / \mathcal{C} & \xrightarrow{\sim} & \text{Mon}^{\text{oplax}, L}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}}.
\end{array}
$$

We notice that the left vertical arrow is an equivalence since it is induced on overcategories by a right fibration. We would like to show that the map $\mathcal{N} / v \to \mathcal{N}$ is an equivalence if and only if $f$ is an equivalence.

For any $\mathcal{C}' \in \text{Mon}^{\text{oplax}, L}_\mathcal{O}(\text{Cat}_{\infty})$, we have a pullback diagram

$$
\begin{array}{ccc}
\mathcal{N} / v \times_{\text{Mon}^{\text{oplax}, L}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}}} \{\mathcal{C}'\} & \xrightarrow{\sim} & \text{Map}_{\text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})}(\mathcal{C}', \mathcal{D}) \\
\downarrow & & \downarrow \mathcal{P}_\mathcal{O} \\
\text{Map}_{\text{Mon}^{\text{oplax}, L}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}}}(\mathcal{C}', \mathcal{C}) & \xrightarrow{f \circ \mathcal{P}_\mathcal{O}} & \text{Map}_{\text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})}(\mathcal{P}_\mathcal{O}(\mathcal{C}'), \mathcal{P}_\mathcal{O}(\mathcal{D}))
\end{array}
$$

in spaces, where the left vertical arrow is an equivalence. Suppose that $f \simeq g_1$ with $g : \mathcal{C} \to \mathcal{D}$ a right adjoint lax $\mathcal{O}$-monoidal functor. By Lemma 2.16 and Proposition 2.17, we see that the map

$$
\mathcal{N} / v \times_{\text{Mon}^{\text{oplax}, L}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}}} \{\mathcal{C}'\} \to \mathcal{N} \times_{\text{Mon}^{\text{oplax}, L}_\mathcal{O}(\text{Cat}_{\infty})^{\text{op}}} \{\mathcal{C}'\}
$$

is equivalent to the composite

$$
\text{Map}_{\text{Mon}^{\text{oplax}, L}_\mathcal{O}(\text{Cat}_{\infty})}(\mathcal{C}, \mathcal{C}') \xrightarrow{\sim} \text{Map}_{\text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})}(\mathcal{C}', \mathcal{C}) \xrightarrow{g_0(-)} \text{Map}_{\text{Mon}^{\text{lax}, R}_\mathcal{O}(\text{Cat}_{\infty})}(\mathcal{C}', \mathcal{D})
$$
where the first arrow is an equivalence induced by Proposition 2.17 and the second arrow is the composition with $g$. Thus, each fiber of the map $\mathcal{N}/v \to \mathcal{N}$ is contractible if and only if $g$ is an equivalence. Therefore, (2) and (3) are equivalent.

\textbf{Proposition 4.3.} The paring $\lambda^R$ is perfect.

\textit{Proof.} The proposition follows from [3 Corollary 5.2.1.22] and Lemma [1.2].

We obtain the main theorem of this note.

\textbf{Theorem 4.4.} There is an equivalence

$$T : \text{Mon}_{\mathcal{O}}^{\text{lax},R}(\text{Cat}_{\infty}) \overset{\sim}{\to} \text{Mon}_{\mathcal{O}}^{\text{plax},L}(\text{Cat}_{\infty})^{\op}$$

of $\infty$-categories, which is identity on objects and assigns to right adjoint lax $\mathcal{O}$-monoidal functors their left adjoint oplax $\mathcal{O}$-monoidal functors. The equivalence $T$ fits into the following commutative diagram

$$\begin{array}{ccc}
\text{Mon}_{\mathcal{O}}^{\text{lax},R}(\text{Cat}_{\infty}) & \xrightarrow{T} & \text{Mon}_{\mathcal{O}}^{\text{plax},L}(\text{Cat}_{\infty})^{\op} \\
P_{\mathcal{O}} & & P_{\mathcal{O}}^{\op} \\
\downarrow & & \downarrow \\
\text{Mon}_{\mathcal{O}}^{\text{lax}}(\text{Pr}_{\mathcal{L}}^{L}). & & \\
\end{array}$$

\textit{Proof.} The first part follows from Proposition [1.3]. We have a left (and right) representable morphism $\lambda^R \to \hat{\lambda}$ of perfect pairings. The second part follows from [3 Proposition 5.2.1.17].

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