SINGULAR EXTENSIONS AND TRIANGULATED CATEGORIES

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Dedicated in Loving Memory of Beso Jgarkava

1. Introduction

In this paper we propose a new look on triangulated categories, which is based on singular extensions of additive categories.

Let us recall that if $R$ is a ring and $M$ is a square zero two-sided ideal of $R$, then $M$ can be considered as a bimodule over the quotient ring $S = R/M$. Moreover the exact sequence

$$0 \rightarrow M \rightarrow R \rightarrow S \rightarrow 0$$

is a singular extension of the ring $S$ by the bimodule $M$, which is characterized by an element $e(R) \in \text{HH}^2(S, M)$. Here $\text{HH}^*$ denotes the Hochschild cohomology if $S$ is free as an abelian group and the Shukla cohomology [18], [4] in the general situation. Knowing the triple $(S, M, e(R))$ determines the ring $R$ up to isomorphism. This classical fact admits a straightforward generalization to preadditive categories known at least from the work of Mitchell [13].

The above relates to triangulated categories as follows. Let $T$ be a triangulated category as it was introduced by Puppe [17]. Thus we do not assume the octahedron axiom of Verdier [19] to hold in $T$. We first consider the category $T[1]$ of arrows of $T$ (see Section 3.1). Then for each morphism $f: A \rightarrow B$ of $T$ we choose a distinguished triangle:

$$A \xrightarrow{f} B \xrightarrow{u_f} C \xrightarrow{v_f} A[1].$$

Next we consider the category $\text{Triangles}_0(T)$ which has the same objects as $T[1]$, while morphisms $f \rightarrow f'$ in $\text{Triangles}_0(T)$ are commutative diagrams

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow a & & \downarrow b \\
A' & \xrightarrow{f'} & B'
\end{array} \quad \begin{array}{ccc}
C & \xrightarrow{u_f} & C' \\
\downarrow c & & \downarrow c' \\
C_f & \xrightarrow{v_f} & C_f'
\end{array} \quad \begin{array}{ccc}
A[1] & \xrightarrow{a[1]} & A'[1].
\end{array}$$

Consider the functor

$$\pi: \text{Triangles}_0(T) \rightarrow T[1]$$

which is identity on objects and assigns $(a, b)$ to the triple $(a, b, c)$. Obviously the functor $\pi$ is identity on objects and surjective on morphisms. We prove that the kernel of the functor $\pi: \text{Triangles}_0(T) \rightarrow T[1]$ is a square zero ideal in $\text{Triangles}_0(T)$ (see Section 3.2). It follows that there exists a bifunctor $\Theta: (T[1])^{op} \times T[1] \rightarrow \text{Ab}$ and a singular extension

$$0 \rightarrow \Theta \rightarrow \text{Triangles}_0(T) \xrightarrow{\pi} T[1] \rightarrow 0.$$

Hence the category $\text{Triangles}_0(T)$ and therefore the triangulated category structure on the category $T$ is completely determined by a bifunctor $\Theta$ and the corresponding class $\vartheta \in \text{HH}^2(T[1], \Theta)$.

The computation of the bifunctor $\Theta$ and of the class $\vartheta \in \text{HH}^2(T[1], \Theta)$ is a hard problem. Of course the bifunctor $\Theta$ and the class $\vartheta$ are not arbitrary and it is an interesting task to characterize such pairs $(\Theta, \vartheta)$. In Section 4 we give a reasonable solution of this problem. Our first observation is that the categories involved in our extension possess auto-equivalences induced by the translation functor of $T$. Thus our extension is in fact a singular $\tau$-extension as it is defined below. Our next observation is that there exists an easily defined bifunctor $\Delta$ (called the Toda bifunctor below), which does not depend on the triangulated structure at all and is related to the bifunctor $\Theta$ via a
binatural transformation $\theta : \Delta \to \Theta$ which is an isomorphism provided one of the arguments is a split morphism of the category $\mathcal{T}$. Hence $\Delta$ should be considered as a first approximation of $\Theta$. It turns out that in many cases, but not always our extension is a pushforward along $\theta$. For example this is so if $\mathcal{T}$ is a derived category of a ring (in the classical or in the brave new algebra sense) and it is not so if $\mathcal{T}$ is the triangulated category constructed by Muro [14]. These facts lead to the definition of a pseudo-triangulated category in Section 6.1. We will extend the notion of homology and Massey triple product from triangulated categories to pseudo-triangulated categories. Finally in Section 6 we characterize triangulated categories among all pseudo-triangulated categories.

2. Preliminaries

2.1. Pre-additive categories. A category $\mathbb{A}$ together with an abelian group structure on each of the sets of morphisms $\text{Hom}_\mathbb{A}(X, Y)$ is called a preadditive category provided all the composition maps $\text{Hom}_\mathbb{A}(Y, Z) \times \text{Hom}_\mathbb{A}(X, Y) \to \text{Hom}_\mathbb{A}(X, Z)$ are bilinear maps of abelian groups. Suppose $\mathbb{A}$ and $\mathbb{B}$ are preadditive categories. A functor $F : \mathbb{A} \to \mathbb{B}$ is said to be an additive functor if the induced maps

$$\mathbb{A}(X, Y) \to \mathbb{B}(F(X), F(Y)), \quad f \mapsto F(f)$$

are homomorphisms of abelian groups for all objects $X, Y \in \mathbb{A}$.

An additive category is a preadditive category $\mathbb{A}$ with zero object 0 and such that for all objects $X, Y$ there is given an object $X \oplus Y$ and morphisms

$$i_1 : X \to X \oplus Y, \quad i_2 : Y \to X \oplus Y,$$

$$r_1 : X \oplus Y \to X, \quad r_2 : X \oplus Y \to Y$$

with $r_1i_1 = \text{id}_X$, $r_2i_2 = \text{id}_Y$, $r_1i_2 = 0$, $r_2i_1 = 0$ and $i_1r_1 + i_2r_2 = \text{id}_{X \oplus Y}$. The object $X \oplus Y$ is called direct sum of $X$ and $Y$ in $\mathbb{A}$. It follows that $X \oplus Y$ together with $i_1$ and $i_2$ is a coproduct of $X$ and $Y$ and $X \oplus Y$ together with $r_1$ and $r_2$ is a product of $X$ and $Y$. The following fact is well known.

Lemma 2.1.1. For additive categories $\mathbb{A}$ and $\mathbb{B}$, a functor $F : \mathbb{A} \to \mathbb{B}$ is additive iff for all objects $X_1, X_2$ of the category $\mathbb{A}$ the canonical map

$$(F(r_1), F(r_2)) : F(X_1 \oplus X_2) \to F(X_1) \oplus F(X_2)$$

is an isomorphism.

2.2. Split idempotent and split morphisms. Let $e : A \to A$ be an endomorphism. If $e^2 = e$ then $e$ is called idempotent. If $e$ is an idempotent in an additive category $\mathbb{A}$ then $\text{id}_A - e$ is also an idempotent. For any objects $X_1$ and $X_2$ of an additive category $\mathbb{A}$, the morphism $e = i_1r_1 : X_1 \oplus X_2 \to X_1 \oplus X_2$ is an idempotent. An idempotent $e : A \to A$ is called split if there are arrows (called splitting data) $a : A \to B$ and $b : B \to A$, such that $e = ba$ and $ab = \text{id}_B$. An additive category $\mathbb{A}$ is called Karoubian provided all idempotents split, which is the same to require as that all idempotents have kernels (or cokernels).

A morphism $p : X \to Y$ of an additive category is called a splittable epimorphism if there exists a morphism $j : Y \to X$ such that $pj = \text{id}_Y$. For example the canonical projection $r : A \oplus B \to B$ is splittable. Morphisms isomorphic to such projections are called split epimorphisms. If $\mathbb{A}$ is Karoubian then any splittable epimorphism is actually a split epimorphism.

Dually a morphism $i : X \to Y$ is called a splittable monomorphism if there exists a morphism $r : Y \to X$ such that $ri = \text{id}_X$. For example the canonical inclusion $i : A \to A \oplus B$ is splittable. Morphisms isomorphic to such inclusions are called split monomorphisms. If $\mathbb{A}$ is Karoubian then any splittable monomorphism is actually a split monomorphism.

More generally a morphism $f : X \to Y$ is called splittable if there exists a morphism $s : Y \to X$ such that $fsf = f$. Examples of splittable morphisms are splittable epimorphisms, splittable monomorphisms and idempotents. Morphisms of the form $(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}) : X \oplus X' \to X \oplus X''$ are split morphisms. Morphisms isomorphic to such a morphism are called split morphisms. If $\mathbb{A}$ is a
Karoubian category then any splittable morphism \( f \) is actually a split morphism, i.e., it can be represented as a composite \( ir \), where \( r \) is a split epimorphism and \( i \) is a split monomorphism.

### 2.3. Subfunctors of additive functors and the second cross-effect.

Let \( \mathbb{A} \) be an additive category. Let \( F : \mathbb{A} \to \text{Ab} \) be a functor with \( F(0) = 0 \). The second cross-effect functor of \( F \) is a bifunctor \( \text{cr}_2(F) : \mathbb{A} \times \mathbb{A} \to \text{Ab} \) defined by
\[
\text{cr}_2(F)(X_1, X_2) := \text{Ker}(F(p_1), F(p_2)) : F(X_1 \oplus X_2) \to F(X_1) \oplus F(X_2)).
\]

Thus a functor \( F \) is additive iff \( F(0) = 0 \) and \( \text{cr}_2(F) = 0 \).

The proof of the following fact is an easy exercise on diagram chase and is left to the reader.

**Lemma 2.3.1.** For any short exact sequence of functors
\[
0 \to F_1 \to F \to F_2 \to 0
\]
one has a short exact sequence of bifunctors:
\[
0 \to \text{cr}_2(F_1) \to \text{cr}_2(F) \to \text{cr}_2(F_2) \to 0
\]
In particular any subfunctor of an additive functor is also additive.

\[\square\]

### 2.4. Ideals and quotient categories.

An *ideal* \( \mathcal{I} \) of \( \mathbb{A} \) is a subbifunctor of the bifunctor
\[
\text{Hom}_\mathbb{A}(-, -) : \mathbb{A}^{\text{op}} \times \mathbb{A} \to \text{Ab}.
\]

It follows from Lemma 2.3.1 that \( \mathcal{I} \) is biadditive. If \( \mathbb{A} \) and \( \mathbb{B} \) are additive categories and \( F : \mathbb{A} \to \mathbb{B} \) is an additive functor, one denotes by \( \text{Ker}(F) \) the ideal of \( \mathbb{A} \) consisting of morphisms \( f : A \to B \) such that \( F(f) \) is a zero morphism in \( \mathbb{B} \).

If \( \mathcal{I} \) is an ideal of \( \mathbb{A} \), then one can form the quotient category \( \mathbb{A}/\mathcal{I} \), which has the same objects as \( \mathbb{A} \), while morphisms in \( \mathbb{A}/\mathcal{I} \) are given by
\[
\text{Hom}_{\mathbb{A}/\mathcal{I}}(A, B) := \text{Hom}_\mathbb{A}(A, B)/\mathcal{I}(A, B).
\]

One has the canonical additive functor \( Q : \mathbb{A} \to \mathbb{A}/\mathcal{I} \). It is clear that \( \text{Ker}(Q) = \mathcal{I} \). Any additive functor \( F : \mathbb{A} \to \mathbb{B} \) factors through the category \( \mathbb{A}/\text{Ker}(F) \).

### 2.5. Nilpotent and square zero ideals.

Let \( \mathcal{I} \) and \( \mathcal{J} \) be ideals of \( \mathbb{A} \). For all object \( A \) and \( B \) we let \( \mathcal{I}(A, B) \) be the set of all products \( fg \), where \( f \in \mathcal{I}(C, B) \) and \( g \in \mathcal{I}(A, C) \), for some \( C \). We claim that \( \mathcal{I}(A, B) \) is a subgroup of \( \mathcal{A}(A, B) \). Indeed, if \( f \in \mathcal{I}(C, B) \), \( g \in \mathcal{I}(A, C) \) and \( f' \in \mathcal{I}(C', B) \), \( g' \in \mathcal{I}(A, C') \), then \( fg + f'g' = f''g'' \), where \( f'' = (f, f') : C \oplus C' \to B \) and \( g'' = \left(g, g'\right) : A \to C \oplus C' \).

We have \( \mathcal{I}(A, C \oplus C') = \mathcal{I}(A, C) \oplus \mathcal{I}(A, C') \) by Lemma 2.3.1. Since \( g \in \mathcal{I}(A, C) \) and \( g' \in \mathcal{I}(A, C') \), it follows that \( g'' \in \mathcal{I}(A, C \oplus C') \). Similarly \( f'' \in \mathcal{I}(C \oplus C', B) \), hence the claim. It is clear that \( \mathcal{I} \) is a subbifunctor of \( \mathcal{J} \) and \( \mathcal{J} \). Hence it is an ideal.

Having defined product of ideals, one can talk about powers \( \mathcal{I}^n \) of an ideal \( \mathcal{I} \). An ideal \( \mathcal{I} \) is *nilpotent* if \( \mathcal{I}^n = 0 \) for some \( n \). Of special interest are ideals with \( \mathcal{I}^2 = 0 \). They are called *square zero ideals*. We have the following easy but useful fact.

**Lemma 2.5.1.** For a square zero ideal \( \mathcal{I} \) of \( \mathbb{A} \) the bifunctor \( \mathcal{I} : \mathbb{A}^{\text{op}} \times \mathbb{A} \to \text{Ab} \) factors through the quotient category \( \mathbb{A}/\mathcal{I} \) in an unique way.

\[\square\]

This result can be used to prove the following simple result.

**Lemma 2.5.2.** Let \( \mathcal{I} \) be a nilpotent ideal of an additive category \( \mathbb{A} \). Then the quotient functor \( Q : \mathbb{A} \to \mathbb{A}/\mathcal{I} \) reflects isomorphisms and yields an isomorphism of monoids of isomorphism classes \( \text{Iso}(\mathbb{A}) \cong \text{Iso}(\mathbb{A}/\mathcal{I}) \).
and especially with relations between the second Hochschild cohomology and singular extensions of rings might wonder whether there is a cohomology theory which in dimension two has a section, then \( A \cong B \times D \).

2.6. **Singular extensions of additive categories.** Let \( B \) be an additive category and let \( D : B^{\text{op}} \times B \to \text{Ab} \) be a bifunctor. A *singular extension*

\[
0 \to D \overset{i}{\to} \mathbb{A} \overset{F}{\to} B \to 0
\]

of \( B \) by the bifunctor \( D \) is the following data:

(i) An additive category \( \mathbb{A} \) and an additive functor \( F : \mathbb{A} \to B \), such that \( \text{Ker}(F) \) is a square zero ideal and the canonical functor \( \mathbb{A}/\text{Ker}(F) \to B \) is an isomorphism of categories;

(ii) an isomorphism of bifunctors \( i : D(F(\cdot), F(\cdot)) \to \text{Ker}(F) \).

2.7. **Semidirect product.** Let \( B \) be an additive category and let \( D : B^{\text{op}} \times B \to \text{Ab} \) be a bifunctor. The *semidirect product* (compare with [5]) of \( B \) by \( D \) is the category \( B \times D \) which has the same objects as \( B \). Morphisms \( A \to B \) in \( B \times D \) are pairs \((f, a)\), where \( f : A \to B \) is a morphism in \( B \) and \( a \in D(A, B) \). Composition is defined by

\[
(f, a) \circ (g, b) = (fg, f_*(b) + g^*(a))
\]

Let \( I \) be the class of all morphisms of the form \((0, a)\). Then \( I^2 = 0 \), \( \mathbb{A}/I \cong B \), where \( \mathbb{A} = B \times D \) and \( i : D \to I \) is an isomorphism of bifunctors, given by \( i(a) = (0, a) \). Conversely, if

\[
0 \to D \overset{i}{\to} \mathbb{A} \overset{F}{\to} B \to 0
\]

is a singular extension and \( F \) has a section, then \( \mathbb{A} \cong B \times D \).

2.8. **Cohomology and singular extensions.** The reader familiar with the Hochschild cohomology and especially with relations between the second Hochschild cohomology and singular extensions of rings might wonder whether there is a cohomology theory which in dimension two would classify singular extensions of a small additive category \( B \) by a bifunctor \( D : B^{\text{op}} \times B \to \text{Ab} \). In fact such cohomology does exist and it is an obvious extension of the Shukla cohomology of rings [18], [4] to small preadditive categories.

As a matter of fact, let us mention here that there exists also Baues-Wirsching cohomology [5] which is defined for all small (maybe non-preadditive) categories. For additive categories the second Shukla cohomology and the second Baues-Wirsching cohomology \( H^2(\mathbb{A}, D) \) are isomorphic. This follows from [4], together with Proposition 3.4 of [11], which shows that any linear extension [5] of an additive category by an additive bifunctor is again an additive category. It must be mentioned that even for additive categories Shukla and Baues-Wirsching cohomologies are not isomorphic in dimensions \( \geq 3 \).

2.9. **Puppe triangulated categories.** Let \( \mathcal{T} \) be an additive category with an autoequivalence \( A \mapsto A[1] \). A *candidate triangle* in \( \mathcal{T} \) is a diagram

\[
X \to Y \to Z \to X[1].
\]

A morphism from a candidate triangle \( X \to Y \to Z \to X[1] \) to a candidate triangle \( X' \to Y' \to Z' \to X'[1] \) is a commutative diagram in \( \mathcal{T} \):

\[
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
| & & | & & | & & |
\downarrow a & \downarrow b & \downarrow c & \downarrow a[1] \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}
\]
We let \( \mathsf{Cand} \) be the category of candidate triangles. A candidate triangle \( X \to Y \to Z \to X[1] \) is \textit{acyclic} provided the sequence of abelian groups

\[
\cdots \to \text{Hom}_\mathcal{T}(X[1], A) \to \text{Hom}_\mathcal{T}(Z, A) \to \text{Hom}_\mathcal{T}(Y, A) \to \text{Hom}_\mathcal{T}(X, A) \to \cdots
\]

is exact for any object \( A \in \mathcal{T} \).

A \textit{Puppe triangulated category} structure, or simply triangulated category structure on \( \mathcal{T} \) is given by a collection of diagrams, called \textit{distinguished triangles}, of the form

\[
X \to Y \to Z \to X[1]
\]

such that

\begin{enumerate}
  \item [TR1)] Any candidate triangle isomorphic to a distinguished triangle in \( \mathsf{Cand} \) is a distinguished triangle.
  \item [TR2)] Any diagram of the following form is a distinguished triangle:
    \[
    X \xrightarrow{id_X} X \to 0 \to X[1]
    \]
  \item [TR3)] If
    \[
    X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]
    \]
    is a distinguished triangle, then
    \[
    Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]}, Y[1]
    \]
    is also a distinguished triangle.
  \item [TR4)] For any morphism \( f : X \to Y \) there is a distinguished triangle of the form
    \[
    X \xrightarrow{f} Y \to Z \to X[1].
    \]
  \item [TR5)] Suppose we have a diagram
    \[
    \begin{array}{c}
    X \quad Y \quad Z \\
    \downarrow a \quad \downarrow \quad \downarrow a[1] \\
    X' \quad Y' \quad Z' \\
    \end{array}
    \]
    in which the rows are distinguished triangles and the left rectangle commutes. Then there exists a morphism \( Z \to Z' \) making the diagram
    \[
    \begin{array}{c}
    X \quad Y \quad Z \\
    \downarrow a \quad \downarrow \quad \downarrow a[1] \\
    X' \quad Y' \quad Z' \quad X'[1] \\
    \end{array}
    \]
    commute.
\end{enumerate}

A category equipped with a triangulated structure is called a \textit{triangulated category}. We let \( \text{Triangles}(\mathcal{T}) \) be the full subcategory of \( \mathsf{Cand} \) formed by distinguished triangles.

Let \( \mathcal{T} \) be a triangulated category. An additive functor \( h : \mathcal{T} \to \text{Ab} \) is called \textit{homology} if, whenever

\[
X \xrightarrow{f} Y \to Z \to X[1]
\]

is a distinguished triangle, the sequence

\[
h(X) \to h(Y) \to h(Z) \to h(X[1])
\]

is exact. Then the sequence

\[
\cdots \to h^n(X) \to h^n(Y) \to h^n(Z) \to h^{n+1}(X) \to \cdots
\]

is also exact, where \( h^n(X) = h(X[n]) \).

It is well known that the functors \( \text{Hom}_\mathcal{T}(X, -) \) and \( \text{Hom}_\mathcal{T}(-, X) \) are homologies. In particular if \( X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1] \) is a distinguished triangle and \( h : Y \to V \) is a morphism such that \( hf = 0 \), then \( h \) factors through \( g \).
3. SINGULAR EXTENSIONS AND TRIANGULATED CATEGORIES

3.1. Category of arrows. Let \([1]\) be the category associated to the ordered set \(0 < 1\). For any category \(\mathcal{C}\) we let \(\mathcal{C}^{[1]}\) be the category of functors \(\mathbb{[1]} \to \mathcal{C}\). Thus \(\mathcal{C}^{[1]}\) is the category of arrows of \(\mathcal{C}\). For a morphism \(f : A \to B\) of the category \(\mathcal{C}\) considered as an object of the category \(\mathcal{C}^{[1]}\) we use the notation \(\hat{f}\) and the word "arrow" to denote the same morphism considered as an object of the category \(\mathcal{C}^{[1]}\). Hence objects of \(\mathcal{C}^{[1]}\) are arrows \(\hat{f}\), where \(f : A \to B\) is a morphisms of \(\mathcal{C}\), while morphisms \(\hat{f} \to \hat{f}'\) are pairs of morphisms \((a : A \to A', b : B \to B')\) in \(\mathcal{C}\) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{a} & & \downarrow{b} \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

commutes.

For any object \(A\) of \(\mathbb{A}\) we write \(\text{id}_A\) for the identity morphism in \(\mathbb{A}\) and use \(\text{id}_A\) for the corresponding arrow considered as an object of \(\mathcal{C}^{[1]}\). Hence \(\text{id}_A = \hat{\text{id}}_A\). Assume now that \(\mathcal{C}\) has a zero object. In this case we use the following notations. For an object \(A\) in \(\mathcal{C}\) we denote by \(A!\) (resp. \(!A\)) the object of \(\mathbb{A}^{[1]}\) corresponding to the unique morphism \(0 \to A\) (resp. \(A \to 0\)) in \(\mathcal{C}\).

The functors

\[
\text{id}_?!, \, ?!, \, \tilde{?}! : \mathcal{C} \to \mathcal{C}^{[1]}
\]

are full embeddings.

3.2. The main observation. Let \(\mathcal{T}\) be a triangulated category. For each morphism \(f : A \to B\) of \(\mathcal{T}\) we choose a distinguished triangle

\[
A \xrightarrow{f} B \xrightarrow{u_f} C_f \xrightarrow{v_f} A^{[1]},
\]

where \(A \to A^{[1]}\) is the translation functor. One of the axioms of triangulated categories asserts that such choice is always possible. Now we consider the category \(\text{Triangles}_0(\mathcal{T})\), whose objects are morphisms of \(\mathcal{T}\), thus the same as of the category \(\mathcal{T}^{[1]}\). For a morphism \(f : A \to B\) we let \([f]\) be the corresponding object of the category \(\text{Triangles}_0(\mathcal{T})\). The morphisms \([f] \to [f']\) in the category \(\text{Triangles}_0(\mathcal{T})\) are triples of morphisms \((a : A \to A', b : B \to B', c : C_f \to C_{f'})\) of the category \(\mathcal{T}\) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{u_f} & C_f & \xrightarrow{v_f} & A^{[1]} \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} & & \downarrow{a^{[1]}} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{u_{f'}} & C_{f'} & \xrightarrow{v_{f'}} & A'^{[1]}
\end{array}
\]

is commutative. Thus we have full subcategories \(\text{Triangles}_0(\mathcal{T}) \subseteq \text{Triangles}(\mathcal{T}) \subseteq \text{Cand}\). It is clear that the first inclusion \(\text{Triangles}_0(\mathcal{T}) \subseteq \text{Triangles}(\mathcal{T})\) is an equivalence of categories. Moreover the category \(\text{Triangles}(\mathcal{T})\) can be reconstructed from \(\text{Triangles}_0(\mathcal{T})\) as follows: A candidate triangle belongs to \(\text{Triangles}(\mathcal{T})\) iff it is isomorphic (in \(\text{Cand}\)) to an object of \(\text{Triangles}_0(\mathcal{T})\).

We let

\[
\pi : \text{Triangles}_0(\mathcal{T}) \to \mathcal{T}^{[1]}
\]

be the functor which is the identity on objects (thus \(\pi([f]) = \hat{f}\)) and assigns \((a, b)\) to the triple \((a, b, c)\). Another axiom of triangulated categories asserts that the functor \(\pi\) is surjective on morphisms.

Lemma 3.2.1. For arbitrary object \(X\) in a triangulated category \(\mathcal{T}\) and arbitrary morphism \(f : A \to B\), there exist isomorphisms

\[
\text{Hom}_{\mathcal{T}}(C_f, X) \cong \text{Hom}_{\text{Triangles}_0(\mathcal{T})}([f], !X)
\]

and

\[
\text{Hom}_{\mathcal{T}}(X, C_f[-1]) \cong \text{Hom}_{\text{Triangles}_0(\mathcal{T})}(X, [f]).
\]
These isomorphisms are natural in $X \in \mathcal{F}$ and in $f \in \text{Triangles}_0(\mathcal{F})$.

**Proof.** We prove the first isomorphism, second being similar. A morphism

\[
\begin{array}{cccc}
A & \xrightarrow{f} & B & \xrightarrow{u_f} C_f & \xrightarrow{v_f} A[1] \\
\downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
0 & \xrightarrow{b} & X & \xrightarrow{c} & 0 \\
\downarrow 0 & & \downarrow \text{id} & & \downarrow \text{id} \\
0 & \xrightarrow{v_0} & X & \xrightarrow{v_0} & 0
\end{array}
\]

is uniquely determined by $c$, which might be arbitrary. This implies the result. \[\square\]

The following easy but extremely important fact is new.

**Lemma 3.2.2.** The kernel of the functor $\pi : \text{Triangles}_0(\mathcal{F}) \to \mathcal{F}[1]$ is a square zero ideal.

**Proof.** Consider the following commutative diagram in $\mathcal{F}$

\[
\begin{array}{cccc}
A & \xrightarrow{f} & B & \xrightarrow{u_f} C_f & \xrightarrow{v_f} A[1] \\
\downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
A' & \xrightarrow{f'} & B' & \xrightarrow{u_{f'}} C_{f'} & \xrightarrow{v_{f'}} A'[1] \\
\downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
A'' & \xrightarrow{f''} & B'' & \xrightarrow{u_{f''}} C_{f''} & \xrightarrow{v_{f''}} A''[1]
\end{array}
\]

where rows are distinguished triangles. We have to prove that $c'c = 0$. Since $c'u_{f'} = 0$, there exist a morphism $d' : A'[1] \to C_{f'}$ such that $c' = d'v_{f'}$. Hence

\[c'c = d'v_{f'}c = d'0v_f = 0.\]

\[\square\]

**Corollary 3.2.3.** There exists a well-defined bifunctor $\Theta_{\mathcal{F}}$

\begin{equation}
\Theta_{\mathcal{F}} : (\mathcal{F}[1])^{\text{op}} \times \mathcal{F}[1] \to \text{Ab}
\end{equation}

such that

\[\Theta_{\mathcal{F}}(\hat{f}, \hat{f}') = \{c : C_{f'} \to C_{f'} | cu_{f'} = 0, v_{f'}c = 0\}.\]

The category $\text{Triangles}_0(\mathcal{F})$ is a singular extension of the category $\mathcal{F}[1]$ by the bifunctor $\Theta_{\mathcal{F}}$,

\begin{equation}
0 \to \Theta \to \text{Triangles}_0(\mathcal{F}) \xrightarrow{\pi} \mathcal{F}[1] \to 0.
\end{equation}

The class $\vartheta$ of the singular extension (3.2.3) in $\text{HH}^2(\mathcal{F}, \Theta)$ is independent of the choices of distinguished triangles (3.2.1). Hence the triangulated category structure on the category $\mathcal{F}$ is completely determined by the bifunctor $\Theta$ and the class $\Theta_{\mathcal{F}}$.

### 3.3. Categories with translation

Let $\mathbb{A}$ be an additive category. A **translation** on a category $\mathbb{A}$ is an autoequivalence $\mathbb{A} \to \mathbb{A}$; if such a translation is fixed, then we say that $\mathbb{A}$ is a **category with translation** or **$\tau$-category**. An evaluation of the translation functor on an object $A$ is denoted by $A[1]$ and is called translation of $A$. Moreover, for any object $A$ we choose an object $A[-1]$ together with an isomorphism $(A[-1])[1] \cong A$. Then $A \mapsto A[-1]$ can be extended as a functor $\langle \cdot \rangle[-1] : \mathbb{A} \to \mathbb{A}$ in a unique way. If $n$ is an integer, then one has objects $A[n]$ defined by induction: $A[n + 1] = (A[n])[1]$ if $n \geq 1$, $A[0] = A$ and $A[n - 1] = (A[n])[-1]$ if $n \leq -1$. Sometimes we write $\tau(A)$ instead of $A[1]$. Of course in this case we write $\tau^n(A)$ instead of $A[n]$ as well.

Let $\mathbb{A}$ and $\mathbb{B}$ be categories with translation. A **translation preserving functor**, or **$\tau$-functor** is an additive functor $F : \mathbb{A} \to \mathbb{B}$ such that $F(A[1]) = (F(A))[1]$ for all $A$.

Let $\mathbb{I}$ be an ideal in a $\tau$-category $\mathbb{A}$. We will say $\mathbb{I}$ is a $\tau$-**ideal** if for all objects $A$ and $B$ the isomorphism $\mathbb{A}(A, B) \to \mathbb{A}(A[1], B[1])$, $f \mapsto f[1]$, restricts to an isomorphism $\mathbb{I}(A, B) \to \mathbb{I}(A[1], B[1])$. In this case the quotient category $\mathbb{A}/\mathbb{I}$ carries a $\tau$-category structure and the quotient functor $\mathbb{A} \to \mathbb{A}/\mathbb{I}$ is a $\tau$-functor. Conversely, if $F : \mathbb{A} \to \mathbb{B}$ is a $\tau$-functor, then $\text{Ker}(F)$ is a $\tau$-ideal.
3.4. Koszul translation. For a morphism \( f : X \to Y \) in a \( \tau \)-category \( \mathcal{A} \) one puts:

\[
\tau(\hat{f}) = (-f[1] : X[1] \to Y[1]).
\]

Moreover, if \( f' : X' \to Y' \) is another morphism of the category \( \mathcal{A} \) and \((x : X \to X', y : Y \to Y')\) is a morphism \( \hat{f} \to \hat{f}' \) in the category \( \mathcal{A}[1] \), then one puts

\[
\tau(x, y) = ((x[1], [y]) : \tau(\hat{f}) \to \tau(\hat{f}')).
\]

In this way one gets a translation \( \tau : \mathcal{A}[1] \to \mathcal{A}[1] \) called the Koszul translation.

Let \( \mathcal{T} \) be a triangulated category. Then \( \text{Triangles}_0(\mathcal{T}) \) also possesses a Koszul translation, which on objects is given by the same rule

\[
\tau(\hat{f}) = (-f[1] : X[1] \to Y[1]),
\]

while on morphisms it is given by \( \tau(x, y, z) = (x[1], y[1], z[1]) \). Here \((x[1], y[1], z[1])\) is the following morphism in \( \text{Triangles}_0(\mathcal{T})\):

\[
\begin{array}{cccccc}
X[1] & \overset{-f[1]}{\longrightarrow} & Y[1] & \overset{-u_f[1]}{\longrightarrow} & C_f[1] & \overset{-v_f[1]}{\longrightarrow} & X[2] \\
\downarrow x[1] & & \downarrow y[1] & & \downarrow z[1] & & \downarrow x[2] \\
X'[1] & \overset{-f'[1]}{\longrightarrow} & Y'[1] & \overset{-u'_f[1]}{\longrightarrow} & C_f'[1] & \overset{-v'_f[1]}{\longrightarrow} & X'[2]
\end{array}
\]

It is clear that \( \pi : \text{Triangles}_0(\mathcal{T}) \to \mathcal{T}[1] \) is a \( \tau \)-functor.

3.5. \( \tau \)-bifunctors and singular \( \tau \)-extensions. Let \( \mathcal{A} \) be a \( \tau \)-category. A \( \tau \)-bifunctor on \( \mathcal{A} \) is a bifunctor \( D : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \text{Ab} \) together with a system of isomorphisms

\[
t_{A,B} : D(A, B) \to D(A[1], B[1]), \quad A, B \in \mathcal{A},
\]

which are natural in \( A \) and \( B \). If \( D \) and \( D' \) are two \( \tau \)-bifunctors, then a natural transformation \( \xi : D \to D' \) of bifunctors is called a \( \tau \)-transformation provided the following diagram commutes:

\[
\begin{array}{cccccc}
D(A, B) & \overset{t_{A,B}}{\longrightarrow} & D(A[1], B[1]) \\
\downarrow \xi(A,B) & & \downarrow \xi(A[1],B[1]) \\
D'(A, B) & \overset{t'_{A,B}}{\longrightarrow} & D'(A[1], B[1])
\end{array}
\]

For example the bifunctor \( \text{Hom}_\mathcal{A}(\cdot, \cdot) \) is a \( \tau \)-bifunctor, where \( t_{A,B}(f) = f[1] \). Moreover, if \( I \) is a \( \tau \)-ideal, then it is a \( \tau \)-subbifunctor of \( \text{Hom}_\mathcal{A} \).

A singular extension

\[
0 \to D \overset{i}{\to} \mathbb{B} \overset{p}{\to} \mathcal{A} \to 0
\]

of a \( \tau \)-category \( \mathcal{A} \) by a \( \tau \)-bifunctor \( D \) is called a singular \( \tau \)-extension if \( p \) is a \( \tau \)-functor and \( i \) yields an isomorphism \( D \to \text{Ker}(p) \) of \( \tau \)-bifunctors over \( \mathcal{A} \).

One easily sees that the singular extension \( \mathbb{B} \) is in fact a singular \( \tau \)-extension, where \( \mathcal{T}[1] \) and \( \text{Triangles}_0(\mathcal{T}) \) are equipped with Koszul translations. Here a \( \tau \)-bifunctor structure on \( \Theta \), i.e. isomorphisms

\[
t_{f,f'} : \Theta(\hat{f}, \hat{f}') \to \Theta(-\hat{f}[1], -\hat{f}'[1])
\]

are induced by \( c \mapsto c[1] \), for any \( c : C_f \to C_{f'} \) with \( cu_f = 0 = v_{f'}c \).

3.6. Toda bifunctor. Let \( \mathcal{A} \) be a category with translation. For morphisms \( f : A \to B \) and \( f' : A' \to B' \) we consider the homomorphism of abelian groups

\[
\phi_{f,f'} : \text{Hom}_\mathcal{A}(A[1], A') \oplus \text{Hom}_\mathcal{A}(B[1], B') \to \text{Hom}_\mathcal{A}(A[1], B')
\]

given by

\[
\phi_{f,f'}(g, h) = f'_*(g) - (f[1])^*(h) = f' \circ g - h \circ (f[1]).
\]

Here \( g : A[1] \to A' \) and \( h : B[1] \to B' \) are morphisms of \( \mathcal{A} \).

The Toda bifunctor \( \Delta_\mathcal{A} \), or simply \( \Delta \) is a bifunctor

\[
\Delta : (\mathcal{A}[1])^{\text{op}} \times \mathcal{A}[1] \to \text{Ab}
\]
given by
\[
\Delta(f, f') := \text{Coker}(\delta_{f,f'}) = \frac{\text{Hom}_\mathcal{A}(A[1], B')}{\text{Hom}_\mathcal{A}(A[1], A') - f^*\text{Hom}_\mathcal{A}(B[1], B')},
\]
where \(f: A \to B\) and \(f': A' \to B'\) are morphisms in \(\mathcal{A}\).

The following lemma is straightforward.

**Lemma 3.6.1.** Let \(\mathcal{A}\) be a category with translation. For any object \(X \in \mathcal{C}\) and any morphism \(f: A \to B\) one has
\[
\Delta(\text{Id}_X, \delta) = 0,
\]
\[
\Delta(f, \text{Id}_X) = 0,
\]
\[
\Delta(I_X, f) = 0,
\]
\[
\Delta(f', \delta) = 0,
\]
\[
\Delta(X!, f) = \text{Coker}(\text{Hom}_\mathcal{A}(X[1], A) \xrightarrow{f} \text{Hom}_\mathcal{A}(X[1], B)),
\]
\[
\Delta(f', X!) = \text{Coker}(\text{Hom}_\mathcal{A}(B[1], X) \xrightarrow{f'} \text{Hom}_\mathcal{A}(A[1], X)).
\]

There is a \(\tau\)-bifunctor structure on \(\Delta\) which we will use throughout. The isomorphisms
\[
t_{f, f'} : \Delta(f, f') \to \Delta(-f[1], -f'[1])
\]
are induced by \(a \mapsto a[1]\).

### 3.7. Natural transformation \(\theta\).

Let \(\mathcal{T}\) be a triangulated category. Then we have two \(\tau\)-bifunctors
\[
\Theta_{\mathcal{T}}, \Delta_{\mathcal{T}} : (\mathcal{T}[1])^{\text{op}} \times \mathcal{T}[1] \to \text{Ab}.
\]

It must be noticed that the Toda bifunctor depends only on the translation structure, while the bifunctor \(\Theta\) depends on the choice of the class of distinguished triangles. We now define the \(\tau\)-transformation
\[
\theta_{\mathcal{T}} : \Delta_{\mathcal{T}} \to \Theta_{\mathcal{T}}
\]
as follows.

Let \(f: A \to B\) and \(f': A' \to B'\) be morphisms in \(\mathcal{T}\). For any morphism \(x: A[1] \to B'\) we have the following morphism of distinguished triangles:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
| & | & | \\
0 & \xrightarrow{0} & c_x \\
\downarrow & \Downarrow{u_f} & \downarrow c_x \\
A' & \xrightarrow{f'} & B' \\
| & | & | \\
0 & \xrightarrow{0} & 0 \\
\end{array}
\begin{array}{ccc}
C_f & \xrightarrow{v_f} & A[1] \\
| & | & | \\
0 & \xrightarrow{0} & 0 \\
\downarrow & \Downarrow{v_f} & \downarrow 0 \\
C_{f'} & \xrightarrow{v'_{f'}} & A'[1],
\end{array}
\]

where \(c_x = u_{f'} x v_f\). One easily sees that the assignment \(x \mapsto (0, 0, c_x)\) yields the homomorphism
\[
\theta(f, f') : \Delta(f, f') \to \Theta(f, f'),
\]
hence a natural transformation \(\theta : \Delta \to \Theta\).

**Proposition 3.7.1.** Let \(f: A \to B\) and \(f' : A' \to B'\) be morphisms of \(\mathcal{T}\). If \(f\) or \(f'\) is splittable, then \(\theta(f, f') : \Delta(f, f') \to \Theta(f, f')\) is an isomorphism.

**Proof.** It is well known that for any triangulated category \(\mathcal{T}\) the Karoubian completion \(\mathcal{T}^{\text{K}\mathcal{A}}\) has also a triangulated category structure and the inclusion functor \(i\) is a morphism of triangulated categories. Hence without loss of generality we may assume that all idempotents split in \(\mathcal{T}\). By duality it suffices to consider the case, when \(f\) is splittabl. Since \(\mathcal{T}[1]\) is an additive category, \(\theta\) is a transformation of additive bifunctors and any splittable morphism considered as an object of \(\mathcal{T}[1]\) is isomorphic to a direct sum of objects of the form \(\text{Id}_A, 1_B\) or \(C^1\), we have to consider three cases \(f = \text{Id}_X, f = !_X\) and \(f = X!\). In the first case we have \(C_f = 0\) and therefore both groups
Δ(\tilde{f}, \tilde{f}') and Θ(\tilde{f}, \tilde{f}') are trivial. If \tilde{f} = ![X], then we have already shown that Δ(\tilde{f}, \tilde{f}') = 0. On the other hands if

\[
\begin{array}{c}
0 \to X \to X \to 0 \\
0 \to 0 \to c \to 0 \\
A' \to B' \to C' \to A'[1]
\end{array}
\]

is a morphism in \text{Triangles}_0(\mathcal{T}) then c = 0. Hence Θ(\tilde{f}, \tilde{f}') = 0 as well. Now consider the case, when \tilde{f} = ![X]. Let \(c : X[1] \to C'_X\) be a morphism in \mathcal{T}, then

\[
\begin{array}{c}
X \to X[1] \to 0 \\
0 \to 0 \to c \to 0 \\
A' \to B' \to C' \to A'[1]
\end{array}
\]

is a morphism of distinguished triangles iff

\[c \in \text{Ker}(\text{Hom}_A(X[1], C'_X) \to \text{Hom}_A(X[1], A'[1])).\]

But the last group is isomorphic to \(\text{Coker}(\text{Hom}_A(X[1], A') \to \text{Hom}_A(X[1], B')) = \Delta(\tilde{f}, \tilde{f}')\) and we are done.

\[\square\]

4. Pseudo-triangulated categories

4.1. Definition and examples. Let \(\mathcal{P}\) be an additive category with translation \(A \mapsto A[1]\).

**Definition 4.1.1.** We will say that there is given a pseudo-triangulated category structure on \(\mathcal{P}\) if there is given a singular \(τ\)-extension

\[0 \to τ \xrightarrow{i} \text{Ptr} \xrightarrow{p} \mathcal{P}[1] \to 0\]

of \(\mathcal{P}[1]\) by a \(τ\)-bifunctor

\[τ : (\mathcal{P}[1])^{\text{op}} \times \mathcal{P}[1] \to \text{Ab}\]

together with a \(τ\)-transformation \(φ : Δ \to τ\) from the Toda bifunctor to \(τ\) such that \(φ(\tilde{f}, \tilde{f}') : Δ(\tilde{f}, \tilde{f}') \to τ(\tilde{f}, \tilde{f}')\) is an isomorphism provided \(f\) or \(f'\) is splittable. If additionally \(φ\) is isomorphic then we say that \(\mathcal{P}\) is equipped with a Toda pseudo-triangulated category structure.

Abusing notation we will say that \(\mathcal{P}\) is a pseudo-triangulated category provided such a structure is given.

We have already seen that if \(\mathcal{P} = \mathcal{T}\) is a triangulated category then the extension

\[0 \to θ \to \text{Triangles}_0(\mathcal{T}) \xrightarrow{π} \mathcal{T}[1] \to 0\]

together with the transformation \(θ : Δ \to θ\) gives rise to a pseudo-triangulated structure on \(\mathcal{T}\). We refer to this example as the pseudo-triangulated category associated to a triangulated category \(\mathcal{T}\).

Unlike the triangulated category structure, any \(τ\)-category can be equipped with the structure of a pseudo-triangulated category: one can take \(τ = Δ\), and define \(\text{Ptr}\) to be the semidirect product of \(\mathcal{P}[1]\) with \(Δ\), or one can take any other singular \(τ\)-extension of \(\mathcal{P}[1]\) by \(Δ\). Thus triangulated category structures on a given category might be really different.

Let \(\mathcal{P}\) be a pseudo-triangulated category. Objects of the category \(\text{Ptr}\) are the same as of \(\mathcal{P}[1]\), i.e., they are still arrows, but now called pseudo-triangles. If a morphism \(f\) is considered as a pseudo-triangle, we use the notation \([f]\) instead of \(f\). Assume \([f]\) and \([f']\) are pseudo-triangles. Then one has the exact sequence of abelian groups

\[(4.1.1) \quad 0 \to τ(\tilde{f}, \tilde{f}') \xrightarrow{i} \text{Hom}_{\text{Ptr}}([f], [f']) \to \text{Hom}_{\mathcal{P}[1]}(\tilde{f}, \tilde{f}') \to 0.\]

It follows from Lemma 3.6.1 that

\[(4.1.2) \quad \text{Hom}_{\text{Ptr}}([f], [f']) = \text{Hom}_{\mathcal{P}[1]}(\tilde{f}, \tilde{f}')\]
provided one of the following equations holds: \( f = \text{id}_X \), \( f = \!_X \), \( f' = \text{id}_X \), \( f' = \!_X \), for an object \( X \in \mathcal{P} \). In particular
\[
\text{Hom}_{\mathcal{P}}(\!_X, \!_X) = \text{Hom}_{\mathcal{P}[1]}(\!_X, \!_X) = \text{Hom}_{\mathcal{P}}(A, X).
\]

It follows that the full embedding \( \mathcal{P} \to \mathcal{P}[1] \) given by \( X \mapsto \!_X \) has a unique lifting to \( \text{Ptr} \).

**Proposition 4.1.2.** If \( \mathcal{P} \) is a pseudo-triangulated category, then for any object \( X \) and for any morphism \( f : A \to B \) in \( \mathcal{P} \) one has the following exact sequences
\[
\cdots \to \text{Hom}_{\mathcal{P}}(A[n+1], X) \to \text{Hom}_{\mathcal{P}}(\tau^n([f]), \!_X) \to \text{Hom}_{\mathcal{P}}(B[n], X) \xrightarrow{f^*} \text{Hom}_{\mathcal{P}}(A[n], X) \to \cdots
\]
and
\[
\cdots \to \text{Hom}_{\mathcal{P}}(X, A[n-1]) \xrightarrow{\tau^*} \text{Hom}_{\mathcal{P}}(X, B[n-1]) \to \text{Hom}_{\mathcal{P}}(X^n([f]), \!_X) \to \text{Hom}_{\mathcal{P}}(X, A[n]) \to \cdots
\]

**Proof.** We prove exactness only for the first sequence. The proof for the second sequence is similar and therefore we omit it. By the exact sequence \( 4.1.1 \) we have
\[
0 \to \Delta(f, \!_X) \to \text{Hom}_{\mathcal{P}}([f], \!_X) \to \text{Hom}_{\mathcal{P}[1]}(\hat{f}, \!_X) \to 0.
\]

It follows from the definition of the category \( \mathcal{P}[1] \) that for \( f : A \to B \) one has the exact sequence
\[
0 \to \text{Hom}_{\mathcal{P}[1]}(\hat{f}, \!_X) \to \text{Hom}_{\mathcal{P}}(B, X) \xrightarrow{f^*} (A, X).
\]

This and Lemma 3.6.1 imply exactness of the following sequence:
\[
\text{Hom}_{\mathcal{P}}(B[1], X) \xrightarrow{f^*} \text{Hom}_{\mathcal{P}}(A[1], X) \to \text{Hom}_{\mathcal{P}}([f], \!_X) \to \text{Hom}_{\mathcal{P}}(B, X) \xrightarrow{f^*} \text{Hom}_{\mathcal{P}}(A, X).
\]
Replacing \( f \) by the translations of \( f \) we get the result. \( \square \)

### 4.2. Homology

We would like to introduce the notion of the homology in the setup of pseudo-triangulated categories generalizing the classical notion for triangulated categories. As in algebraic topology, a homology must satisfy the exactness and excision axioms. To introduce these axioms we need some preparations.

**Lemma 4.2.1.** Let \( \mathcal{P} \) be a pseudo-triangulated category. For any objects \( A \) and \( B \) of \( \mathcal{P} \) one has a natural isomorphisms
\[
\text{Hom}_{\mathcal{P}}(\!_A, \!_B) \cong \text{Hom}_{\mathcal{P}}(A[1], B)
\]

**Proof.** Since \( \text{Hom}_{\mathcal{P}[1]}(\!_A, \!_B) = 0 \), the result follows from the exact sequence \( 4.1.1 \) and the fact that
\[
\tau(\!_A, \!_B) = \Delta(\!_A, \!_B) = \text{Hom}_{\mathcal{P}}(A[1], B).
\]

In particular for any object \( A \) there is a canonical morphism
\[
j_A : \!_A \to \!_{A[1]}
\]
corresponding to \( \text{id}_{A[1]} \). It follows from our construction that
\[
p(j_A) = 0.
\]

Since \( \Delta(-, \!_A) = 0 \), it follows that \( \text{Hom}_{\mathcal{P}}(-, \!_A) = \text{Hom}_{\mathcal{P}[1]}(-, \!_A) \). In particular for any arrow \( f : A \to B \) there is a canonical morphism
\[
k_f : [f] \to \!_A
\]
in \( \text{Ptr} \) corresponding to the morphism \( (\text{id}_A, 0) : \hat{f} \to \!_A \) in \( \mathcal{P}[1] \). By construction it is functorial in \( [f] \in \text{Ptr} \).
Let \( f : X \to Y \) be a morphism in a pseudo-triangulated category \( \mathcal{P} \). Then we have the following commutative diagram in \( \mathcal{P} \):

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow f \\
0 & \longrightarrow & B \\
\downarrow & & \downarrow \text{id} \\
A & \longrightarrow & B
\end{array}
\]

which gives rise to the diagram in \( \mathcal{P}^{[1]} \):

\[
!_A \xrightarrow{(0,f)} !_B \xrightarrow{(0,\text{id})} \hat{f}.
\]

Since \( T(!_X, -) = 0 \), it has the unique lift

\[
!_A \xrightarrow{!f} !_B \xrightarrow{ij} [f]
\]

to \( \text{Ptr} \). Let

\[
(4.2.1) \quad j_f : [f] \to !_A[1].
\]

be the composite \( j_f = j_A \circ k_f \).

Gluing these sequences and applying the translation functor we obtain the sequence

\[
\cdots \to !_A[n] \to !_B[n] \to \tau^n([f]) \to !_A[n+1] \to \cdots
\]

which is functorial in \([f]\)\(\text{Ptr}\). Similarly one gets the sequence of morphisms:

\[
\cdots \to A[n]! \to B[n]! \to \tau^{n+1}([f]) \to A[n+1]! \to \cdots
\]

**Definition 4.2.2.** A morphism \( x : [f] \to [g] \) in \( \text{Ptr} \) is called **excising** if the induced map

\[
\text{Hom}_{\mathcal{P}^{[1]}}(X!, [f]) \to \text{Hom}_{\mathcal{P}^{[1]}}(X!, [g])
\]

is an isomorphism for any \( X \in \mathcal{P} \).

**Lemma 4.2.3.** For any object \( A \) the natural map

\[
j_A : A! \to !_A[1]
\]

is **excising**.

**Proof.** Since \( \text{Hom}_{\mathcal{P}^{[1]}}(X!, Y!) = 0 = T(X!, Y!) \) and \( T(X!, Y!) = \text{Hom}_{\mathcal{P}}(X[1], Y) \) the result follows. \( \square \)

Now we are ready to give the following definition.

**Definition 4.2.4.** A **homology** on a pseudo-triangulated category \( \mathcal{P} \) with values in an abelian category \( A \) is a covariant functor \( h : \text{Ptr} \to A \) satisfying the following two axioms:

(Exactness) For any morphism \( f : A \to B \) of the category \( \mathcal{P} \) the sequence

\[
h(!_A) \xrightarrow{!f} h(!_B) \xrightarrow{ij} h([f]) \xrightarrow{j_f} h(!_A[1])
\]

is exact.

(Excision) If \( x : [f] \to [g] \) is excising then \( h(x) : h([f]) \to h([g]) \) is an isomorphism.

In presence of the Excision Axiom, the Exactness Axiom is equivalent to the assertion that for any \( f : A \to B \) the sequence

\[
h(B[-1]!) \to h([f]) \to h(A!) \to h(B!)
\]

is exact. This easily follows from Lemma 4.2.3.

For a homology \( h \) we put

\[
h^n(A) := h(!_A[n]), \quad h^n([f]) := h(\tau^n([f])).
\]
Then we have an exact sequence
\[
\cdots \to h^n(A) \to h^n(B) \to h^n([f]) \to h^{n+1}(A) \to \cdots
\]
natural in \([f] \in \text{Ptr} \).

**Proposition 4.2.5.** For any object \(X \in \mathcal{P} \) the functor
\[
\text{Hom}_{\text{Ptr}}(X!, -) : \text{Ptr} \to \text{Ab}
\]
is a homology theory.

**Proof.** First we have to prove exactness of the sequence
\[
\text{Hom}_{\text{Ptr}}(X!, Y) \to \text{Hom}_{\text{Ptr}}(X!, B) \to \text{Hom}_{\text{Ptr}}(X!, [f]) \to \text{Hom}_{\text{Ptr}}(X!, \text{Id}_A).
\]
Since \(\text{Hom}_{\mathcal{P}[1]}(X!, Y) = 0 \) for all \(Y \in \mathcal{P} \), we have
\[
\text{Hom}_{\text{Ptr}}(X!, Y) = \Delta(X!, Y) = \Delta(X!, \text{Id}_Y) = \text{Hom}_{\mathcal{P}}(X[1], Y),
\]
thanks to Lemma 3.6.1 and Exact Sequence (4.1.1). Now exactness follows from Proposition 4.1.2.
It remains to prove that the functor \(\text{Hom}_{\text{Ptr}}(X!, -)\) transforms excising morphisms to isomorphisms. But this is obvious. \(\square\)

**Lemma 4.2.6.** Let \(\mathcal{T}\) be a triangulated category and \(E : \mathcal{T} \to \text{Ab}\) be a homology in the classical sense. Then the functor \(h : \text{Triangles}_0(\mathcal{T}) \to \text{Ab}\) defined by
\[
h([f]) := E(C_f)
\]
is a homology on the pseudo-triangulated category associated to the triangulated category \(\mathcal{T}\). In this way one gets an equivalence between the category of homologies in classical and new sense.

**Proof.** By our definition of the category \(\text{Triangles}_0(\mathcal{T})\) the assignment \(f \mapsto C_f\) can be considered as a well-defined functor \(\text{Triangles}_0(\mathcal{T}) \to \mathcal{T}\). By Lemma 3.2.1 a morphism \([f] \to [g]\) is excisable iff the induced morphism \(C_f \to C_g\) is an isomorphism. From these facts, the first part of the statement follows.

Assume \(h\) is a homology in the new sense. For any morphism \(f \in \mathcal{T}\) the morphism \((0, u_f) : [f] \to !C_f\) is excising. Hence \(h([f]) = E(C_f)\), where \(E : \mathcal{T} \to \text{Ab}\) is given by \(E(A) := h(!A)\). It follows easily from Exactness Axiom that \(E\) is a homology in the classical sense, hence the result. \(\square\)

### 4.3. Massey triple product.

Let \(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W\) be a diagram in a pseudo-triangulated category \(\mathcal{P}\). Suppose \(hg = 0\) and \(gf = 0\). Then we have the following commutative diagram in \(\mathcal{P}\):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow g \\
Y & \xrightarrow{g} & Z \\
\downarrow & & \downarrow h \\
0 & \to & 0 \\
\end{array}
\]

which can be considered as the following diagram in \(\mathcal{P}[1]\):

\[
X! \xrightarrow{(f,0)} Y! \xrightarrow{(0,g)} Z! \xrightarrow{h} W!.
\]

Observe that the composite morphism is zero in \(\mathcal{P}[1]\). Since the functor \(p : \text{Ptr} \to \mathcal{P}[1]\) is identity on objects and surjective on morphisms the diagram can be lifted to \(\text{Ptr}\):

\[
[X!] \xrightarrow{[f]} [g] \xrightarrow{w} [W].
\]
where $x$ and $w$ are morphisms of pseudo-triangles such that $p(x) = (f, 0)$ and $p(w) = (0, h)$. Then $p(wx) = 0$, hence

$$wx \in T(X!, !W);$$

since $X \rightarrow 0$ and $0 \rightarrow W$ are split morphisms, the last groups can be replaced by $\Delta(X!, !W)$. Hence Lemma 3.6.1 implies that

$$wx \in \text{Hom}_\mathcal{P}(X[1], W).$$

Actually, this element depends on lifting. If one chooses $x_1$ and $w_1$ instead of $x$ and $w$, then we can write $x_1 = x + a$ and $w_1 = w + b$, where

$$a \in T(X!, \tilde{g}) = \Delta(X!, \tilde{g}) = \text{Coker} \text{Hom}_\mathcal{P}(X[1], Y) \xrightarrow{g_\ast} \text{Hom}_\mathcal{P}(X[1], Z)$$

and

$$b \in T(\tilde{g}, !W) = \Delta(\tilde{g}, !W) = \text{Coker} \text{Hom}_\mathcal{P}(Z[1], W) \xrightarrow{g_\ast} \text{Hom}_\mathcal{P}(Y[1], W).$$

It follows that $w_1 x_1 = wx + bx + wa$, therefore the class $\{h, g, f\}$ of $wx$ in the quotient

$$\frac{\text{Hom}_\mathcal{P}(X[1], W)}{h_\ast \text{Hom}_\mathcal{P}(X[1], Z) + f_\ast \text{Hom}_\mathcal{P}(Y[1], W)}$$

is invariant; we call it the Massey product. By definition we have $wx \in \{h, g, f\}$. In the case of triangulated categories it coincides with the classical Massey product as defined in [8].

The following fact is well-known [10, Theorem 13.2].

**Lemma 4.3.1.** Let

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1] .$$

be an acyclic triangle in a triangulated category. Then it is a distinguished triangle if and only if $\text{id}_A[1] \in \{c, b, a\}$.

4.4. **$K_0$ for pseudo-triangulated categories.** Let $\mathcal{P}$ be a small pseudo-triangulated category. We let $K_0(\mathcal{P})$ be the abelian group generated by the symbols $[X]$ where $X$ is an object of $\mathcal{P}$, modulo the relations K1-K3 below.

- **K1** $[0] = 0$,
- **K2** $[X] = [Y]$ provided there exists an isomorphism $f : X \rightarrow Y$ in $\mathcal{P}$,
- **K3** For any arrows $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ in $\mathcal{P}$ and excising morphism $x : [f] \rightarrow [f']$ in $\text{Ptr}$ one has $[X] + [Y'] = [X'] + [Y]$.

One easily sees that this notion generalizes the Grothendieck’s original definition for triangulated categories.

5. **The class $\vartheta$ as the first obstruction**

Recent work of Muro and his coauthors [14], [15] shows that not all triangulated categories have models. It turns out that the class $\vartheta$ is the first obstruction for a triangulated category to have a model. Namely we will prove that if $\vartheta$ is not lies in the image of the canonical homomorphism $\text{HH}^2(\mathcal{T}[1], \Delta) \xrightarrow{\vartheta} \text{HH}^2(\mathcal{T}[1], \Theta)$ then $\mathcal{T}$ has no models. In other words we prove that if $\mathcal{T}$ is a triangulated category associated to a stable model category or a Frobenious category then the extension [3.2.3] is a pushforward construction along the transformation $\Theta : \Delta \rightarrow \Theta$ as it is defined in Section 5.1 and hence the class $\vartheta \in \text{HH}^2(\mathcal{T}[1], \Theta)$ lies in the image of the canonical homomorphism $\text{HH}^2(\mathcal{T}[1], \Delta) \xrightarrow{\vartheta} \text{HH}^2(\mathcal{T}[1], \Theta)$. Actually all this is an easy consequence of the work of Baues [2, 3].

We also check that for the triangulated category constructed in [14], [15] the class $\vartheta$ does not lies in the image of the homomorphism $\text{HH}^2(\mathcal{T}[1], \Delta) \xrightarrow{\vartheta} \text{HH}^2(\mathcal{T}[1], \Theta)$. This give an alternative proof of the corresponding result of [14], [15].
5.1. **Push-forward construction and domination.** Let \( \mathcal{P} \) be a \( \tau \)-category equipped with a pseudo-triangulated category structure given by a singular \( \tau \)-extension

\[
0 \to \mathcal{Y} \xrightarrow{i} \text{Ptr} \xrightarrow{p} \mathcal{P}^{[1]} \to 0
\]

and a \( \tau \)-transformation \( \varphi : \Delta \to \mathcal{Y} \).

Assume a \( \tau \)-transformation \( \xi : \mathcal{Y} \to \mathcal{Y}_1 \) of \( \tau \)-bifunctors is given which is an isomorphism as soon as one of the arguments is a split morphism. Consider the following category \( \text{Ptr}_1 \). The objects of \( \text{Ptr}_1 \) are the same as of the categories \( \mathcal{P}^{[1]} \) and \( \text{Ptr} \), i.e., they are arrows of the category \( \mathcal{P} \). Moreover \( \text{Hom}_{\text{Ptr}_1}([f], [g]) \) is defined using the pushout diagram of abelian groups:

\[
\begin{array}{ccc}
\mathcal{Y}(f, g) & \xrightarrow{} & \text{Hom}_{\text{Ptr}_1}([f], [g]) \\
\downarrow & & \downarrow \\
\mathcal{Y}_1(f, g) & \xrightarrow{} & \text{Hom}_{\text{Ptr}_1}([f], [g]) \\
\end{array}
\]

It is easy to see that in this way one gets a singular \( \tau \)-extension structure on \( \text{Ptr}_1 \), such that the following diagram commutes:

\[
\begin{array}{ccc}
0 & \xrightarrow{} & \mathcal{Y} \xrightarrow{i} \text{Ptr} \xrightarrow{p} \mathcal{P}^{[1]} \xrightarrow{i} 0 \\
\downarrow_{\xi} & & \downarrow_{j} & & \downarrow_{\text{id}} \\
0 & \xrightarrow{} & \mathcal{Y}_1 \xrightarrow{i} \text{Ptr}_1 \xrightarrow{p} \mathcal{P}^{[1]} \xrightarrow{i} 0. \\
\end{array}
\]

Hence \( \text{Ptr}_1 \) together with the \( \tau \)-transformation \( \xi \circ \varphi : \Delta \to \mathcal{Y}_1 \) is a pseudo-triangulated category structure on \( \mathcal{P} \), called the **pushforward construction**. In this situation we also say that the pseudo-triangulated category \( \text{Ptr} \) dominates \( \text{Ptr}_1 \) and write \( \text{Ptr}_1 \subseteq \text{Ptr} \).

The proof of the following easy fact is left to the reader.

**Lemma 5.1.1.** **Massey triple product is invariant under dominations.**

5.2. **Toda triangulated categories.** A **Toda triangulated category** is a triangulated category \( \mathcal{T} \) such that the associated pseudo-triangulated category \( \text{Triangles}_0 \) is dominated by a Toda pseudo-triangulated category. The following is a straightforward.

**Lemma 5.2.1.** A triangulated category \( \mathcal{T} \) is a **Toda triangulated category** iff the corresponding class \( \vartheta \in \text{HH}^2(\mathcal{T}^{[1]}, \Theta) \) lies in the image of the homomorphism \( \text{HH}^2(\mathcal{T}^{[1]}, \Delta) \to \text{HH}^2(\mathcal{T}^{[1]}, \Theta) \).

5.3. **Track categories.** In the recent work [2] Baues managed to construct Verdier triangulated categories from the data which he called **triangulated track categories**. Recall that a **track category** \( \mathbb{B} \) is a 2-category all of whose 2-morphisms are invertible. Such categories appeared already in the classical work [2] Ch. V. Thus \( \mathbb{B} \) consists of objects \( X, Y, \) etc., with 1-morphisms \( \xi, \eta \) and with 2-morphisms \( H : \xi \Rightarrow \eta \). If \( \xi, \eta : X \to Y \) are 1-morphisms and there exists a 2-morphism \( H : \xi \Rightarrow \eta \) then we say that \( \xi \) and \( \eta \) are homotopic. The corresponding quotient category is denoted by \( \mathbb{B}_{\simeq} \), which comes with the quotient functor \( Q : \mathbb{B} \to \mathbb{B}_{\simeq} \). Following [2] we use additive notation for the composite of 2-morphisms. A triangulated track category is a track category with some extra data. We refer to the original paper of Baues [2] for the exact definition. Here we point out that any pointed simplicial closed model category which is “stable” (meaning that the suspension induces an auto-equivalence of the homotopy category) gives rise to a triangulated track category structure on the track category \( \mathbb{B} \), which consists of fibrant-cofibrant objects, 1-morphisms are usual morphisms, while 2-morphisms are homotopy classes of homotopies.

5.4. **Hardie category.** We need the following construction due to Hadrie [3] which we learned from [3]. Let \( \mathbb{B} \) be a track category. Let \( \mathbb{A} \) be the corresponding homotopy category \( \mathbb{A} = \mathbb{B}_{\simeq} \). For each morphism \( f \) of \( \mathbb{A} \) we choose its representative \( \tilde{f} \) in the homotopy class of \( f \). Hence \( Q(f) = f \).

Objects of the **Hardie category** \( \mathcal{H}(\mathbb{B}) \) associated to the track category \( \mathbb{B} \) are morphisms of \( \mathbb{A} \). An object of \( \mathcal{H}(\mathbb{B}) \) corresponding to a morphism \( f \) is denoted by \( \{ f \} \). A morphism \( \{ f \} \to \{ g \} \) in the category \( \mathcal{H}(\mathbb{B}) \) corresponding to \( f : A \to B \) and \( g : X \to Y \) is an equivalence class of triples \( \{ f \} \to \{ g \} \).
\((\xi, \eta, H)\), where \(\xi : A \to X\) and \(\eta : B \to Y\) are 1-morphisms of the track category \(\mathbb{B}\), while \(H\) is a 2-morphism \(H : \eta \circ \tilde{f} \Rightarrow \tilde{g} \circ \xi\). Two such triples \((\xi, \eta, H)\) and \((\xi', \eta', H')\) are equivalent if there are 2-morphisms \(G : \eta' \Rightarrow \eta\) and \(K : \xi \Rightarrow \xi'\) such that \(H' = \tilde{g}K + H + \tilde{f}G\).

Let \(\{\xi, \eta, H\}\) be the equivalence class of \((\xi, \eta, H)\). Composition in the Hardie category is given by \(\{\xi, \eta, H\} \circ \{\xi_1, \eta_1, H_1\} = \{\xi\xi_1, \eta\eta_1, \eta_1H_1 + \xiH\}\).

**Lemma 5.4.1.** Let \(\mathbb{B}\) be a triangulated track category. Then there is a well-defined functor \(p : \mathcal{H}(\mathbb{B}) \to \mathbb{A}^{[1]}\) which is identity on objects and on morphisms is given by

\[
p(\xi, \eta, H) = (Q(\xi), Q(\eta))
\]

Moreover, if \(\mathbb{B}\) is a triangulated track category, then \(p\) is a part of a singular \(\tau\)-extension

\[
0 \to \Delta \to \mathcal{H}(\mathbb{B}) \xrightarrow{p} \mathbb{A}^{[1]} \to 0.
\]

**Proof.** This fact modulo notation is due to Baues \cite{3}. The extension is the same as his linear extension \cite{3} Equation (2), page 266, which is defined for much more general track categories. The only thing to check is that for triangulated track categories \(D^f\) in the notation of \cite{3} is the Toda bifunctor. But this follows immediately from the definition of \(D^f\) given in \cite{3} Equation (2.2) and the fact that \(D(X, Y) = \text{Hom}_\mathbb{A}(X[1], Y)\) for triangulated track categories, see \cite{2} Equation (2.7)).

\[
\Box
\]

5.5. **Pushforward construction in action.** Let \(\mathbb{B}\) be a triangulated track category. By \cite{2} the homotopy category \(\mathbb{A} := \mathbb{B}_{\sim}\) possesses a structure of triangulated category and therefore we have a singular \(\tau\)-extension (see Extension (3.2.3)):

\[
0 \to \Theta \to \text{Triangles}_0(\mathbb{A}) \to \mathbb{A}^{[1]} \to 0.
\]

In this section we prove the following result.

**Proposition 5.5.1.** Let \(\mathbb{B}\) be a triangulated track category. Then there is a functor \(T : \mathcal{H}(\mathbb{B}) \to \text{Triangles}_0(\mathbb{A})\) which makes the diagram

\[
\begin{array}{cccc}
0 & \to & \Delta & \xrightarrow{0} \mathcal{H}(\mathbb{B}) & \xrightarrow{p} & \mathbb{A}^{[1]} & \to & 0 \\
\Theta & \downarrow & & \downarrow & \text{id} & \downarrow & & \\
0 & \to & \Theta & \to & \text{Triangles}_0(\mathbb{A}) & \xrightarrow{\text{id}} & \mathbb{A}^{[1]} & \to & 0
\end{array}
\]

commute. In particular for the class \(\vartheta\) defined via Extension (3.2.3) one has

\[
\vartheta = \theta_\mathbb{A}(\beta),
\]

where \(\beta \in \text{HH}^2(\mathbb{A}^{[1]}, \Delta)\) is the class of the extension constructed in Lemma 5.4.1

**Proof.** In the notations of \cite{2} Section 4 the functor \(T\) is defined by

\[
T(\{f\}) = (A \xrightarrow{f} B \xrightarrow{u} C \xrightarrow{v} A[1])
\]

where \(f : A \to B\) is a morphism of \(\mathbb{A}\) and \(u = Q(i\tilde{f})\) and \(v = Q(q\tilde{f})\).

\[
\Box
\]

5.6. **Alternative approach.** To obtain the previous result that Extension (3.2.3) for a derived category of a differential algebra or a ring spectrum is pushforward along \(\theta\) instead of triangulated track categories we could have used systems of triangulated diagram categories in the sense of Franke \cite{6}. In fact let \(\mathcal{K}\) be a such system. In particular the categories \(\mathcal{K}_C\) are given for any (finite) poset \(C\) satisfying some extra conditions. It follows from these axioms that each category \(\mathcal{K}_C\) has a canonical structure of a Verdier triangulated category. These categories should be considered as refinement of the triangulated category \(\mathbb{A} = \mathcal{K}_{\mathbb{A}}\), which is the base of the system. Here \(\mathbb{A}\) denotes the poset \(\{0 \leq \cdots \leq n\}\). Based on the spectral sequence (32) \cite{6} Proposition I.4.10 one can prove that there is a singular \(\tau\)-extension

\[
0 \to \Delta \to \mathcal{K}_0 \to \mathbb{A} \to 0.
\]
and there is a functor $T : \mathcal{H} \rightarrow \text{Triangles}_0(A)$ which makes the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \Delta & \rightarrow & \mathcal{H}_1 & \rightarrow & A[1] & \rightarrow & 0 \\
\downarrow & & \downarrow T & & \downarrow \text{id} & & \downarrow 0 & & \\
0 & \rightarrow & \Theta & \rightarrow & \text{Triangles}_0(A) & \rightarrow & A[1] & \rightarrow & 0
\end{array}
\]

commute. The construction of the functor $T$ is similar to one constructed in the proof of Proposition 5.5.1 and is based on the cones constructed in [6, Section 1.4.6].

It should be point out that if a triangulated category is associated to a stable simplicial model category then both refinements – triangulated track category as well as system of triangulated diagram categories are available. One can prove that in this case Hardie category $\mathcal{H}$ is equivalent to $\mathcal{H}_1$ and hence both approach gives the same singular $\tau$-extensions.

5.7. Muro’s example. For a small preadditive category $S$ we let $\mathcal{F}(S)$ be the additive completion of $S$. If $S$ has only one object (and hence $S$ is just a ring) then $\mathcal{F}(S)$ is the category of finitely generated free $S$-modules. Muro [14] showed that the category $\mathcal{F}(\mathbb{Z}/4\mathbb{Z})$ with the identity translation functor has the unique triangulated category structure such that the triangle

\[
\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z}
\]

is distinguished. In this section we show that for this triangulated category the extension $3.2.3$ is not a pushforward along $\theta$. In the light of Section 5.5 and Section 5.6 it follows that this triangulated category does not admits any refinement as a triangulated track category [2] or as a system of triangulated diagram categories [6]. This fact sharpers some results from [14], [15].

Consider the preadditive category $R$ which is generated by the following graph

modulo the following relations: all arrows are annihilated by 4 and furthermore

\[
2 \cdot \delta \xi = 0, \quad 2 \cdot \eta \eta = 0,
\]

\[
\eta \delta = 0, \quad \phi \xi = 0, \quad \xi \gamma = 0,
\]

\[
\xi \delta = 2 \cdot \text{id}_d, \quad \eta \eta = 2 \cdot \text{id}_c, \quad \phi \gamma = 2 \cdot \text{id}_i,
\]

\[
\gamma \phi = \delta \xi + \xi \gamma.
\]

Then $\text{Hom}_R(d, c) = \text{Hom}_R(c, i) = \text{Hom}_R(i, d) = \text{Hom}_R(d, c) = 0$. Moreover the abelian groups $\text{Hom}_R(d, i)$, $\text{Hom}_R(d, t)$, $\text{Hom}_R(c, d)$, $\text{Hom}_R(c, t)$, $\text{Hom}_R(i, c)$, $\text{Hom}_R(t, c)$ and $\text{Hom}_R(t, c)$ are isomorphic to $\mathbb{Z}/4\mathbb{Z}$. The rings $\text{Hom}_R(d, d)$, $\text{Hom}_R(c, c)$ and $\text{Hom}_R(i, t)$ are isomorphic to $\mathbb{Z}/4\mathbb{Z}$, while $\text{Hom}_R(t, t)$ as a ring is isomorphic to the ring

\[
(5.7.1) \quad \text{Hom}_R(t, t) \cong \{(a, b, c) \in (\mathbb{Z}/4\mathbb{Z})^3 \mid a \equiv b \equiv c \equiv 0 \pmod{2}\}
\]

This isomorphism is given by

\[
(2, 2, 0) \mapsto \gamma \phi, \quad (0, 2, 2) \mapsto \delta \xi.
\]

Let $R_1$ be the quotient of $R$ by the relations

\[
2\xi = 0, \quad 2\zeta = 0, \quad \xi \zeta = 0.
\]

Finally let $R_2$ be the quotient of $R_1$ by the relation

\[
\gamma \phi = 2 \cdot \text{id}_i.
\]
We let \( q : R \to R_2 \) and \( p : R_1 \to R_2 \) be the quotient homomorphisms. We claim that neither \( q \) nor \( p \) has a section. This is clear for \( p \) because even the homomorphism of abelian groups \( \mathbb{Z}/4\mathbb{Z} = \text{Hom}_{R}(t, s) \to \text{Hom}_{R_2}(t, s) = \mathbb{Z}/2\mathbb{Z} \) does not have a section. For the functor \( p \) one observes that \( p(x, y) : \text{Hom}_{R_1}(x, y) \to \text{Hom}_{R_2}(x, y) \) is an isomorphism for all possible \( x, y \) except the case when \( x = y = t \). Hence, if \( p \) has a section \( s \), then \( s \) would respects all arrows indicated in the graph. But this contradicts to the fact that the equality \( \gamma\phi = 2 : \text{id}_t \) holds in \( R_2 \) but not in \( R_1 \).

Define \( R_2\text{-}R_2 \)-bimodules \( \Delta, \Theta, \Theta \) as follows. The bifunctor \( \Theta \) is zero everywhere but \( \Theta_1(t, t) = \mathbb{Z}/2\mathbb{Z} \). The left and right action of the endomorphism ring of \( t \) on \( \Theta_1(t, t) \) is given by the multiplication on \( a \) (which is the same as the multiplication by \( b \) or \( c \)). Here we used the identification \( Z \). Moreover, we have

\[
\Delta(i, -) = 0 = \Delta(-, i), \quad \Delta(d, -) = 0 = \Delta(-, c),
\]

\[
\Delta(c, d) = \mathbb{Z}/4\mathbb{Z}, \quad \Delta(t, d) = \Delta(t, t) = \Delta(c, t) = \mathbb{Z}/2\mathbb{Z}.
\]

The arrows of \( R_2 \) acts on \( \Delta \) as follows. The homomorphisms \( \Delta(c, \delta), \Delta(t, \delta) \) are natural epimorphisms \( \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \), the morphisms \( \Delta(c, \xi), \Delta(\xi, d) \) are natural inclusions \( \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \), finally we have \( \Delta(t, \xi) = 0 = \Delta(c, t) \), while \( \Delta(t, \delta), \Delta(\eta, t) \) are isomorphisms. The bifunctor \( \Theta \) on objects has the same values as the bifunctor \( \Delta \) and even morphisms act on \( \Delta \) and \( \Theta \) in the same way provided the group \( \Delta(t, t) \) is not involved. The rest actions are given as follows. The morphisms \( \Theta(t, \delta), \Theta(t, \xi), \Theta(\xi, t) \) are isomorphisms, while \( \Theta(\eta, t) = 0 \). Then one has a binatural transformation \( \theta : \Delta \to \Theta \), such that \( \theta(x, y) \) is the identity morphism for all possible \( x \) and \( y \) except the case when \( x = t = y \) and in this exceptional case we have \( \theta(t, t) = 0 \). One observes that we have the following diagram with exact columns and rows

\[
\begin{array}{cccccc}
\Delta & & & & & \\
\theta & & & & & \\
0 & \Theta & R & q & R_2 & 0 \\
\downarrow & & & & & \\
0 & \Theta_1 & R_1 & p & R_2 & 0 \\
\downarrow & & & & & \\
0 & & & & & 0 \\
\end{array}
\]

We have already seen that the bottom singular extension does not split. Hence the middle singular extension is not a pushforward along \( \theta \).

All this related to Muro’s example as follows. By mapping

\[
d \mapsto (0 \to \mathbb{Z}/4\mathbb{Z}), \quad c \mapsto (\mathbb{Z}/4\mathbb{Z} \to 0), \quad i \mapsto (\mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/4\mathbb{Z}), \quad t \mapsto (\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z})
\]

\[
\phi \mapsto (2, 1), \eta \mapsto (1, 0), \delta \mapsto (0, 1), \gamma \mapsto (1, 2), \zeta \mapsto (2, 0), \xi \mapsto (0, 2)
\]

one gets an equivalence of categories:

\[
\mathcal{F}(\mathbb{Z}/4\mathbb{Z})^{[1]} \cong \mathcal{F}(R_2),
\]

while mapping

\[
d \mapsto (0 \to \mathbb{Z}/4\mathbb{Z}), \quad c \mapsto (\mathbb{Z}/4\mathbb{Z} \to 0), \quad i \mapsto (\mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/4\mathbb{Z}), \quad t \mapsto (\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z})
\]

\[
\phi \mapsto (2, 1, 0), \eta \mapsto (1, 0, 2), \delta \mapsto (0, 1, 2), \gamma \mapsto (1, 2, 0), \zeta \mapsto (2, 0, 1), \xi \mapsto (0, 2, 1)
\]

one gets an equivalence of categories:

\[
\text{Triangles}_0 \cong \mathcal{F}(R)
\]

and these equivalences are compatible with bifunctors \( \Delta, \Theta, \) etc. This proves that for the Muro’s triangulated category the extension \( \mathbb{Z}/2\mathbb{Z} \) is not a pushforward along \( \theta \).
6. PSEUDO-TRIANGULATED VERSUS TRIANGULATED CATEGORIES

6.1. Embedding under domination. Let us recall that we have a full embedding \(!_?: \mathcal{P} \to \mathcal{P}^{[1]}\). Since \(\mathcal{T}(!_X,-) = 0\) this embedding has a unique lifting \(!?: \mathcal{P} \to \text{Ptr}\) which is still an embedding. Because of uniqueness it is invariant under domination. In fact we have the following result.

**Lemma 6.1.1.** Let

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & \mathcal{P} \\
\xi & \searrow & \Downarrow j \\
0 & \xrightarrow{f} & \text{Ptr} & \xrightarrow{\mathcal{T}} & \mathcal{P}^{[1]} & \xrightarrow{0} \\
& \Downarrow i & \searrow & \Downarrow \text{id} \\
0 & \xrightarrow{f} & \text{Ptr}_1 & \xrightarrow{\mathcal{T}_1} & \mathcal{P}^{[1]} & \xrightarrow{0}
\end{array}
\]

be part of a pushforward construction of pseudo-triangulated categories. Then the diagram

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{!} & \text{Ptr} \\
\Downarrow i & \searrow & \Downarrow j \\
\Downarrow ! & \searrow & \Downarrow \text{Ptr}_1
\end{array}
\]

commutes. Moreover the functor \(!?: \mathcal{P} \to \text{Ptr}\) has a left adjoint iff the functor \(!?: \mathcal{P} \to \text{Ptr}'\) does.

**Proof.** The first part is a consequence of the uniqueness of lifting. To prove the second part, we recall some general facts related to the adjoint functors. Let \(\mathcal{C}\) be a full subcategory of a category \(\mathcal{C}_1\) and \(x \in \mathcal{C}_1\). In these circumstances one denotes by \(x/\mathcal{C}\) the category of arrows \(x \to c\), where \(c \in \mathcal{C}\). It is well known that the inclusion \(\mathcal{C} \subseteq \mathcal{C}_1\) has a left adjoint iff for all objects \(x \in \mathcal{C}_1\) the category \(x/\mathcal{C}\) has an initial object.

According to Sequence 4.1.1 we have the following commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & \text{Hom}_{\mathcal{P}^{[1]}}([f], !A) \\
\searrow & \Downarrow j & \Downarrow \text{id} \\
0 & \xrightarrow{f} & \text{Hom}_{\mathcal{P}^{[1]}}([f], !A) \\
\end{array}
\]

Since \(\mathcal{T}(\cdot, !A) \to \mathcal{T}'(\cdot, !A)\) is an isomorphism, the middle vertical map is also an isomorphism. It follows that for a fixed \(f\) the category of arrows \([f] \to !A\) in \(\text{Ptr}\) where \(A\) runs over \(\mathcal{P}\), and the category of arrows \([f] \to !A\) in \(\text{Ptr}'\), \(A \in \mathcal{P}\), are equivalent. From this, the result follows. \(\square\)

A similar fact is true for ?! as well.

6.2. The main result.

**Theorem 6.2.1.** Let \(\mathcal{P}\) be a \(\tau\)-category equipped with a pseudo-triangulated category structure given by a singular \(\tau\)-extension

\[
0 \to \mathcal{T} \xrightarrow{i} \text{Ptr} \xrightarrow{P} \mathcal{P}^{[1]} \to 0.
\]

Assume the functor \(!?: \mathcal{P} \to \text{Ptr}\) has a left adjoint functor \(L: \text{Ptr} \to \mathcal{P}\) with counit of the adjunction \(w_f: [f] \to !L([f])\), where \(f: A \to B\) is a morphism in \(\mathcal{P}\). Declare a triangle

\[
X \to Y \to Z \to X[1]
\]

to be distinguished provided there is a morphism \(f: A \to B\) and a commutative diagram in \(\mathcal{P}\)

\[
\begin{array}{cccccc}
A & \xrightarrow{f} & B & \xrightarrow{w_f} & L([f]) & \xrightarrow{w_f} & A[1] \\
\downarrow a & \downarrow b & \downarrow c & \downarrow a[1] \\
X & \xrightarrow{f} & Y & \xrightarrow{f} & Z & \xrightarrow{f} & X[1]
\end{array}
\]

(6.2.1)
where \( a, b, c \) are isomorphisms in \( \mathcal{P} \), \( !_{uf} = w_{if} \), while \( v_f : L([f]) \to A[1] \) is the unique morphism such that \( j_f = v_f \circ w_f \). Here \( j_f \) and \( i_f \) are maps in the sequence

\[
!_A \xrightarrow{i} !_B \xrightarrow{ij} \xrightarrow{f} [f] \xrightarrow{ij} !_{A[1]}
\]

class constructed in Section 4.2. With this class of distinguished triangles Axioms TR1, TR2, TR4, TR5 of triangulated categories hold.

Moreover TR3 holds (and hence \( \mathcal{P} \) is a triangulated category) iff the functor \( f \to L(f)[-1] \) is right adjoint to the functor \( \mathcal{P} \to \text{Ptr} \) given by \( X \mapsto X^! \) and additionally

\[
\{v_f, uf, f\} = \text{id}_{A[1]}
\]

holds for all \( f \). If this is so then Triangles\(_q(\mathcal{P}) \leq \text{Ptr} \).

Proof. The functor \( L \) has the following universal property: for any morphism \( x : [f] \to X \) in \( \text{Ptr} \) there exists a unique morphism \( g : L([f]) \to X \) in \( \mathcal{P} \) such that \( x = !_g \circ w_f \). Applying this to the sequence (referred as the pretriangle corresponding to \([f]\))

\[
!_A \xrightarrow{i} !_B \xrightarrow{ij} \xrightarrow{f} [f] \xrightarrow{ij} !_{A[1]}
\]

we see that \( v_f \) indeed exists and is unique. Since \( !_f \) is full and faithful we have \( L(!_A) = A \). By applying the functor \( L \) to the pretriangle we obtain the sequence

\[
A \xrightarrow{f} B \xrightarrow{uf} L([f]) \xrightarrow{v_f} A[1],
\]

in \( \mathcal{P} \). Now we are in a position to check Axioms. The axioms TR1 and TR4 are obvious. Now we verify the axiom TR5. Consider a commutative diagram

\[
A \xrightarrow{f} B \xrightarrow{uf} L([f]) \xrightarrow{v_f} A[1]
\]

Then \( (a, b) : f \to f' \) is a morphism in \( \mathcal{P}[1] \). Hence there exists a morphism \( x : [f] \to [g] \) in \( \mathcal{P} \) such that \( p(x) = (a, b) \). By functoriality \( x \) induces corresponding morphism on pretriangles

\[
!_A \xrightarrow{i} !_B \xrightarrow{ij} \xrightarrow{f} [f] \xrightarrow{ij} !_{A[1]}
\]

Applying \( L \) we finally get the commutative diagram

\[
A \xrightarrow{f} B \xrightarrow{uf} L([f]) \xrightarrow{v_f} A[1]
\]

where \( c = L(x) \).

To verify the axiom TR2, we take \( f = \text{id}_A \). By adjointness we have

\[
\text{Hom}_{\text{Ptr}}(\text{id}_A, !_B) = \text{Hom}_{\mathcal{P}}(L(\text{id}_A), B).
\]

Since \( \text{Hom}_{\mathcal{P}[1]}(\text{id}_A, !_B) = 0 \) and \( \Upsilon(\text{id}_A, -) = \Delta(\text{id}_A, -) = 0 \), it follows that \( \text{Hom}_{\text{Ptr}}(\text{id}_A, !_B) = 0 \) for all \( B \in \mathcal{P} \). Now the Yoneda lemma shows that \( L(\text{id}_A) = 0 \). It follows that

\[
A \xrightarrow{\text{id}} A \to 0 \to A[1]
\]

is a distinguished triangle.
If Axiom TR3 holds, then \( \{v_f, u_f, f\} = \text{id}_{A[1]} \) by Lemma 3.2.1. It follows from Lemma 3.2.1 and analogue of Lemma 6.1.1 for right adjoint functors that the functor \( [f] \to L(f)[-1] \) is indeed the right adjoint functor.

Conversely, assume \( \{v_f, u_f, f\} = \text{id}_{A[1]} \) holds for all \( f \) and the the functor \( [f] \to L(f)[-1] \) is the right adjoint to the functor \( i^! \).

We start with an observation that the functor \( L \) takes any excising morphism into isomorphism. In fact if \( x : [f] \to [g] \) is an excising, then all morphisms

\[
\text{Hom}_{\mathcal{P}}(X, L(f)[-1]) \to \text{Hom}_{\mathcal{P}_{\text{tr}}}(X^!, [f]) \to \text{Hom}_{\mathcal{P}_{\text{tr}}}(X^!, [g]) \to \text{Hom}_{\mathcal{P}}(X, L(g)[-1])
\]

are isomorphism for all object \( X \in \mathcal{T} \). It follows from the Yoneda lemma that \( L(f) \to L(g) \) is also an isomorphism.

Next, remark that for any \( f : A \to B \) there are morphisms \( x : A^! \to [u_f] \) and \( w : [u_f] \to !_{A[1]} \) in \( \mathcal{P}_{\text{tr}} \) such that \( p(x) = (f, 0) \), \( p(w) = (0, v_f) \) and \( wx \in \mathcal{Y}(A^!, !_{A[1]}) = \text{Hom}_{\mathcal{P}}(A[1], A[1]) \) represents the identity morphism \( \text{id}_{A[1]} \). Hence \( wx = j \alpha \) is an excising morphism. It follows that \( L(wx) \) is an isomorphism. Thus \( L(w) : L([u_f]) \to A[1] \) is a split epimorphism. By 5-Lemma applied to exact sequences induced by \( u_f \) and \( f \) it follows that \( L(w) \) is in fact an isomorphism. This fact implies TR3. Hence \( \mathcal{P} \) is a triangulated category. The proof also shows that the triangulated category structure is dominated by \( \mathcal{P}_{\text{tr}} \).

Now we are in a position to prove our main result.

**Corollary 6.2.2.** Let \( \mathcal{P} \) be a \( \tau \)-category equipped with a pseudo-triangulated category structure given by a singular \( \tau \)-extension

\[
0 \to \mathcal{Y} \xrightarrow{i} \mathcal{P}_{\text{tr}} \xrightarrow{L} \mathcal{P}[1] \to 0.
\]

Then the following conditions are equivalent

i) There is a triangulated category structure \( \text{Triangles}(\mathcal{P}) \) on \( \mathcal{P} \) and a domination

\[
\text{Triangles}_{\omega}(\mathcal{P}) \subseteq \mathcal{P}_{\text{tr}}.
\]

ii) There is a functor \( L : \mathcal{P}_{\text{tr}} \to \mathcal{P} \) which is left adjoint to the functor \( X \mapsto !_{X} \), while \( [-1] \circ L \) is a right adjoint to the functor \( X \mapsto X^! \) and \( \{v_f, u_f, f\} = \text{id}_{A[1]} \) for all \( f \).

**Proof.** The implication i) \( \Rightarrow \) ii) follows from Lemma 3.2.1 the functor together with Lemma 6.1.1 and Lemma 4.3.1 The implication ii) \( \Rightarrow \) i) follows from Theorem .

7. **Idempotent completion**

7.1. **Karoubization.** Any additive category \( \mathbb{A} \) has a Karoubian completion \( \mathbb{A}^{\mathbb{A}_{\text{Kar}}} \), which is a Karoubian category with a full embedding \( i : \mathbb{A} \to \mathbb{A}^{\mathbb{A}_{\text{Kar}}} \) satisfying the following property. If \( \mathbb{B} \) is a Karoubian category and \( j : \mathbb{A} \to \mathbb{B} \) is an additive functor, then there exists an essentially unique functor \( f : \mathbb{A}^{\mathbb{A}_{\text{Kar}}} \to \mathbb{B} \) with \( j = f i \). Objects of \( \mathbb{A}^{\mathbb{A}_{\text{Kar}}} \) are pairs \( (A, e) \), where \( A \) is an object of \( \mathbb{A} \) and \( e : A \to A \) is an idempotent. A morphism \( (A, e) \to (A', e') \) is a morphism \( f : A \to A \) in \( \mathbb{A} \) such that

\[
f e = e' f = f.
\]

Let us observe that the identity morphism of \( (A, e) \) is \( e \) and the functor \( i \) is given by \( i(A) = (A, \text{id}_{A}) \).

**Lemma 7.1.1.** An idempotent \( e \) of the category \( \mathbb{A} \) is split iff \( (A, e) \) as an object of \( \mathbb{A}^{\mathbb{A}_{\text{Kar}}} \) is isomorphic to an object of the image of the functor \( i : \mathbb{A} \to \mathbb{A}^{\mathbb{A}_{\text{Kar}}} \).

**Proof.** One easily checks that having mutually inverse morphisms

\[
a : (A, e) \to (B, \text{id}_{B}), \quad b : (B, \text{id}_{B}) \to (A, e)
\]

is exactly the same as to have a splitting data for \( e \).
7.2. Lifting of idempotents. It is well know that if \( R \to S \) is a surjective homomorphism of rings with nilpotent kernel then any idempotent of \( S \) is an image of an idempotent of \( R \). We can specialize this for the ring homomorphism \( \text{Hom}_A(A, A) \to \text{Hom}_{A/1}(A, A) \) to get the following fact.

**Lemma 7.2.1.** Let \( \mathbb{I} \) be a nilpotent ideal of an additive category \( \mathbb{A} \). For any idempotent \( f : A \to A \) in the quotient category \( \mathbb{A}/\mathbb{I} \) there is an idempotent \( e : A \to A \) in \( \mathbb{A} \) such that \( Q(e) = f \).

7.3. Singular extensions and idempotent completion. Let

\[
0 \to D \xrightarrow{i} \mathbb{A} \xrightarrow{F} \mathbb{B} \to 0
\]

be a singular extension of an additive category \( \mathbb{A} \) by a bifunctor \( D : \mathbb{A}^{\text{op}} \times \mathbb{A} \to \mathbb{Ab} \). In this section we compare the categories \( \mathbb{A}^{K\mathbb{A}} \) and \( \mathbb{B}^{K\mathbb{A}} \). Actually our results here are very particular case of much more general results obtained in [10].

To make notations simpler we write \( \bar{A}, \bar{B} \) instead of \( \mathbb{A}, \mathbb{B} \). Since all idempotents in \( \mathbb{Ab} \) are idempotents and hence their images are in fact direct summands. It follows that the functor \( \bar{B} \to \mathbb{B}^{K\mathbb{A}} \) defined on objects by \( (A, e) \mapsto (A, \bar{e}) \) is full and faithful and in fact an equivalence thanks to Lemma 7.2.1. Having this equivalence in mind the bifunctor \( D^{K\mathbb{A}} \) can be considered as a bifunctor on \( \bar{B} \). We can now summarize our discussion.

**Lemma 7.3.1.** If

\[
0 \to D \xrightarrow{i} \mathbb{A} \xrightarrow{F} \mathbb{B} \to 0
\]

is a singular extensions of additive categories, then we have a singular extension of additive categories

\[
0 \to D^{K\mathbb{A}} \xrightarrow{i} \mathbb{A}^{K\mathbb{A}} \xrightarrow{\bar{F}} \mathbb{B}^{K\mathbb{A}} \to 0
\]

and an equivalence of categories \( \mathbb{B} \to \mathbb{B}^{K\mathbb{A}} \). Moreover, we have also a singular extension of additive categories

\[
0 \to D^{K\mathbb{A}} \to \mathbb{A} \to \mathbb{B}^{K\mathbb{A}} \to 0
\]

and an equivalence of categories

\[
\mathbb{A} \to \mathbb{A}^{K\mathbb{A}}.
\]

**Proof.** Only the second part of the statements needs some comments. It follows from the first part by notice that an equivalence of categories yields an isomorphism in the Baue-Wirsching cohomology [5].

Lemma 7.3.1 says that up to equivalence of categories a singular extension gives rise to a singular extension by passing trough the idempotent completion. Based on this fact we now prove the following easy fact.

\[
\text{Lemma 7.3.1}
\]
Proposition 7.3.2. Let
\[ 0 \to D \xrightarrow{i} A \xrightarrow{F} \mathbb{B} \to 0 \]
be a singular extensions of additive categories and let \( e : A \to A \) be an idempotent in \( A \). Then \( e \) splits iff \( F(e) \) splits.

Proof. Of course any functor takes split idempotents to split ones. Assume now \( \bar{e} = F(e) \) is split. This means that there is an isomorphism \((A, \bar{e}) \to (B, \text{id}_B)\) in \( \mathbb{B} \) (see Lemma 7.1.1). But both \((A, \bar{e})\) and \((B, \text{id}_B)\) are in the image of the functor \( \mathbb{B} \to \mathbb{B}^{Ka} \), which is an equivalence of categories (see Lemma 7.3.1). Hence there is an isomorphism \( \bar{x} : (A, e) \to (B, \text{id}_B) \) in \( \mathbb{B}^{Ka} \). The functor \( \mathbb{F} : A^{Ka} \to \mathbb{B} \) is full and reflects isomorphisms thanks to Lemma 7.3.1 and Lemma 2.5.2. It follows that there is an isomorphism \( x : (A, e) \to (B, \text{id}_B) \) and the result follows. \( \square \)

7.4. Split idempotents in the category of arrows. Let \( f : A \to B \) be a morphism of an additive category \( A \). It is clear that a morphism \((a, b) : \ast \to \ast \) in the category \( A^{[1]} \) is an idempotent iff \( a : A \to A \) and \( b : B \to B \) are idempotents.

Lemma 7.4.1. An idempotent \((a, b) : \ast \to \ast \) of \( A^{[1]} \) splits iff \( a \) and \( b \) are split idempotents in the category \( A \).

Proof. Assume \( a \) and \( b \) are split idempotents of the category \( A \). Let \( A \xrightarrow{c} C \xrightarrow{d} A \) and \( B \xrightarrow{e} D \xrightarrow{f} B \) be splitting data for \( a \) and \( b \). We set \( g = sfd : C \to D \). One easily checks that \( \hat{f} \xrightarrow{(c,d)} \hat{g} \xrightarrow{(d,t)} \hat{f} \) is a splitting data of the idempotent \((a, b) : \ast \to \ast \). The converse statement is obvious. \( \square \)

7.5. Split idempotents in the category of pseudo-triangles. Let \( \mathcal{P} \) be a \( \tau \)-category equipped with a pseudo-triangulated category structure given by a singular \( \tau \)-extension
\[ 0 \to \mathcal{T} \xrightarrow{i} \text{Ptr} \xrightarrow{p} \mathcal{P}^{[1]} \to 0. \]
and \( \tau \)-transformation \( \varphi : \Delta \to \mathcal{T} \).

Lemma 7.5.1. Let \( x : [f] \to [f] \) be an idempotent in \( \text{Ptr} \) with \( p(x) = (a, b) : \ast \to \ast \). Then \( x \) is a split idempotent in \( \text{Ptr} \) iff \( a \) and \( b \) are split idempotents in \( \mathcal{P} \).

Proof. If part is clear. Assume \( a \) and \( b \) are split idempotents. Then \((a, b) : \ast \to \ast \) is also split idempotent thanks to Lemma 7.4.1. Hence the result follows from Proposition 7.3.2. \( \square \)

As an immediate consequence of the above abstract non-sense we get the following crucial lemma in [12].

Corollary 7.5.2. Let \( \mathcal{T} \) be a triangulated category and let
\[ A \xrightarrow{f} B \xrightarrow{u_f} C_f \xrightarrow{v_f} A[1] \]
be a morphism of distinguished triangles. Assume \( a, b \) and \( c \) are idempotents. If \( a \) and \( b \) are split idempotents, then \((a, b, c) : [f] \to [g] \) is a split idempotent of the category \( \text{Triangles}_0 \). In particular \( c \) is a split idempotent of \( \mathcal{T} \).
7.6. **The full embedding** \( \varrho : (\mathcal{A}^{Ka})^{[1]} \to (\mathcal{A}^{[1]})^{Ka} \). In this section we compare categories \((\mathcal{A}^{Ka})^{[1]}\) and \((\mathcal{A}^{[1]})^{Ka}\).

Let \( \mathcal{A} \) be an additive category. Objects of \((\mathcal{A}^{Ka})^{[1]}\) are arrows \( f : (A, e) \to (A', e') \) in \( \mathcal{A}^{Ka} \), where \( A \) and \( A' \) are objects of \( \mathcal{A} \), while \( e \) and \( e' \) are idempotents of the category \( \mathcal{A} \). We can also say that the objects of the category \((\mathcal{A}^{Ka})^{[1]}\) are diagrams in \( \mathcal{A} \):

\[
\begin{array}{ccc}
A & \xrightarrow{e} & A \\
\downarrow{f} & & \downarrow{f} \\
A' & \xleftarrow{e'} & A'
\end{array}
\]

such that \( e^2 = e, (e')^2 = e', fe = f = e'f \). Such an object is denoted by \( (A, e, A', e', f) \).

On the other hand the objects of \((\mathcal{A}^{[1]})^{Ka}\) are pairs \( (f, x) \), where \( f : A \to B \) is an arrow in \( \mathcal{A} \) and \( x = (e, e') : f \to f \) is an idempotent in \( \mathcal{A}^{[1]} \). We can also say that the objects of the category \((\mathcal{A}^{[1]})^{Ka}\) are diagrams in \( \mathcal{A} \):

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow{e} & & \downarrow{e'} \\
A & \xrightarrow{f} & A'
\end{array}
\]

with \( e^2 = e, (e')^2 = e', fe = e'f \). Such an object is denoted by \([A, f, A', e, e']\). We see that the map

\[
\varrho((A, e, A', e', f)) = [A, f, A', e, e']
\]

yields an embedding of the class of objects of \((\mathcal{A}^{Ka})^{[1]}\) into the class of objects of \((\mathcal{A}^{[1]})^{Ka}\). The next lemma shows that \( \varrho \) can be extended as a full and faithful functor

\[
\varrho : (\mathcal{A}^{Ka})^{[1]} \to (\mathcal{A}^{[1]})^{Ka}.
\]

**Lemma 7.6.1.** For any objects \((A, e, A', e', f)\) and \((B, d, B', d', g)\) of the category \((\mathcal{A}^{Ka})^{[1]}\) we have

\[
\text{Hom}_{(\mathcal{A}^{Ka})^{[1]}}((A, e, A', e', f), (B, d, B', d', g)) \cong \text{Hom}_{(\mathcal{A}^{[1]})^{Ka}}([A, f, A', e, e'], [B, g, B', d, d'])
\]

**Proof.** A direct inspection shows that in both cases morphisms are pairs \((h, h')\), where \( h : A \to B \) and \( h' : A' \to B' \) are morphisms in \( \mathcal{A} \) such that

\[
dh = h = hc, \quad d'h' = h' = h'e', \quad h'f = gh.
\]

\[\square\]

7.7. **Idempotent completion of pseudo-triangulated categories.** In this section we show that Karubization of a pseudo-triangulated category carries a natural pseudo-triangulated category structure (compare with [1]). This is based on the previous relationship between the categories \((\mathcal{A}^{Ka})^{[1]}\) and \((\mathcal{A}^{[1]})^{Ka}\).

Let \( \mathcal{S} \) be a \( \tau \)-category equipped with a pseudo-triangulated category structure given by a singular \( \tau \)-extension

\[
0 \to \mathcal{Y} \xrightarrow{i} \text{Ptr} \xrightarrow{p} \mathcal{S}^{[1]} \to 0.
\]

and \( \tau \)-transformation \( \varphi : \Delta \to \mathcal{Y} \). By passing to the idempotent completion we obtain another singular \( \tau \)-extension (see lemma 7.3.1):

\[
0 \to \mathcal{Y}^{Ka} \to \text{Ptr} \to (\mathcal{S}^{[1]})^{Ka} \to 0.
\]

Now we can pull-back it along the \( \varrho \) to get a singular \( \tau \)-extension

\[
0 \to \mathcal{Y}^{Ka} \to \text{Ptr} \to (\mathcal{S}^{Ka})^{[1]} \to 0
\]

which in fact is a pseudo-triangulated category structure on \( \mathcal{S}^{Ka} \). One easily sees that for triangulated categories this is exactly the construction in [1].
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