Why the Entropy of a Black Hole is $A/4$?

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Abstract

A black hole considered as a part of a thermodynamical system possesses the Bekenstein-Hawking entropy $S_H = A_H/(4l_P^2)$, where $A_H$ is the area of a black hole surface and $l_P$ is the Planck length. Recent attempts to connect this entropy with dynamical degrees of freedom of a black hole generically did not provide the universal mechanism which allows one to obtain this exact value. We discuss the relation between the 'dynamical' contribution to the entropy and $S_H$, and show that the universality of $S_H$ is restored if one takes into account that the parameters of the internal dynamical degrees of freedom as well as their number depends on the black hole temperature.

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According to the thermodynamical analogy in black hole physics, the entropy of a black hole in the Einstein theory of gravity equals $S_H = A_H/(4l_P^2)$, where $A_H$ is the area of a black hole surface and $l_P = (\hbar G/c^3)^{1/2}$ is the Planck length [1,2]. The calculations in the framework of the Euclidean approach initiated by Gibbons and Hawking [3,4] relate this quantity with the tree-level contribution of the gravitational action, namely the action of the Euclidean black hole instanton. In this approach the entropy of a black hole has pure topological origin, and it remains unclear whether there exist any real dynamical degrees of freedom which are responsible for it. The problem of the dynamical origin of the black hole entropy was intensively discussed recently (see e.g., [5–10, ?]). The basic idea which was proposed is to relate the dynamical degrees of freedom of a black hole with its quantum excitations. This idea has different realizations. In particular, it was proposed to identify the dynamical degrees of freedom of a black hole with the states of all fields (including the gravitational one) which are located inside the black hole [3,4]. By averaging over states located outside the black hole one generates the density matrix of a black hole and can calculate the corresponding entropy $S_1$. The main contribution to the entropy is given by inside modes of fields located in the very close vicinity of the horizon. It appears that the so defined $S_1$ is divergent. It was argued that quantum fluctuations of the horizon may provide natural cut-off and make $S_1$ finite. Simple estimations [3] of the cut-off parameter show that $S_1 \approx S_H$. The generic feature of this as well as other ‘dynamic’ approaches is that the entropy of a black hole arises at the one-loop level. The relation between the ‘topological’ (tree-level) calculations and one-loop calculations based on the counting dynamical degrees of freedom of a black hole remains unclear. In particular, $S_1$ depends on the number and characteristics of the fields, while $S_H$ does not. What is the general mechanism which provides the universal relation of the dynamically defined entropy with the universal thermodynamical value $S_H$? The aim of this paper is to give a simple explanation of this puzzle.

A black hole considered as a part of a thermodynamical system possesses a remarkable
property: its properties (size, gravitational field and so on) are determined only by one parameter (mass $M$), which in its turn in a state of a thermal equilibrium is directly connected with the temperature of the system. By varying the temperature one at the time inevitably changes all the internal parameters of the system. We show that namely this property together with scaling properties of the free energy results in the universality of the expression for $S_H$.

We illustrate the basic idea by considering a simple thermodynamical system described by the Hamiltonian

\[ \hat{H} = E_0 + \hat{H}^{(N)}, \quad \hat{H}^{(N)} = \sum_{i=1}^{N} \hat{H}_i. \]  

(1)

We suppose that $\hat{H}_i$ is a Hamiltonian of an oscillator of frequency $\omega_i$. The free energy of such a system is

\[ F = E_0 + T \sum_{i=1}^{N} \ln[1 - \exp(-\varepsilon_i/T)], \]  

(2)

where $\varepsilon_i = \hbar \omega_i$, while the entropy reads

\[ S = -\partial_T F = \sum_{i=1}^{N} s(\varepsilon_i/T), \quad s(x) = \frac{x}{\exp x - 1} - \ln[1 - \exp(-x)]. \]  

(3)

For a system of $N$ identical oscillators $S = N s(\bar{\varepsilon}/T)$. The variation of the free energy reads

\[ dF = -SdT + \left( \frac{\partial H}{\partial \lambda} \right) d\lambda. \]  

(4)

where $\lambda$ denotes additional parameters on which the Hamiltonian depends. In the case when the internal parameters $\lambda$ are uniquely defined by the temperature $T$ (as it happens for black holes) one has

\[ dF = -\tilde{S}dT \]  

(5)

\[ \tilde{S} = S - \frac{dE_0}{dT} - \frac{dN}{dT} \bar{\varepsilon} - N \frac{d\bar{\varepsilon}}{dT}, \]  

(6)

where $\bar{\varepsilon}$ is the average energy of the oscillator of frequency $\omega(T)$ at the temperature $T$. It is evident that for such a system the entropy $S$ determined by its dynamical degrees of freedom
differs from $\tilde{S}$. For convenience in order to distinguish between $S$ and $\tilde{S}$ we call the former ‘dynamical’ and the latter ‘thermodynamical’ entropy.

The above consideration being applied to the case of a black hole indicates that generally one cannot expect that the black-hole’s ‘thermodynamical’ entropy $S_H$ coincides with the ‘dynamical’ entropy $S$, obtained by summing the contributions to the entropy of all internal degrees of freedom of the black hole. In order to obtain $S_H$ one must add to $S$ the term defined by the tree-level free energy $F_0$ as well as the terms connected with the dependence of the energy and number of states on the temperature. The remarkable fact is that the additional terms which arise due to the dependence of the geometry of a black hole and the number of states of quantum fields inside it on the temperature, exactly compensate the main ’dynamical’ contribution $S$ of the quantum field to the entropy. As the result of this compensation the leading term $S$ simply does not contribute to $S_H$, and $S_H$ is determined by the dependence of the mass (and hence the energy) of a black hole on the temperature (i.e., by the tree-level free energy $F_0$). Because different fields contribute to $S$ additively it is sufficient to prove this for one of the fields.

Consider a static black hole located inside a spherical cavity of radius $r_B$. We suppose that the boundary surface has temperature $T_B \equiv \hbar \beta_B^{-1}$. The black hole will be in equilibrium with the thermal radiation inside the cavity if it has mass $M$ such that $\beta_B = 4\pi r_+ (1 - r_+/r_B)^{1/2}$, where $r_+ = 2M$ is the gravitational radius [14]. This relation can be used to express $r_+$ as the function of $\beta_B$ and $r_B$ : $r_+ = r_+(\beta_B, r_B)$. The equilibrium is stable if $r_B > 3r_+/2$. The tree-level contribution of the black hole to the free energy of the system is [12] [13]

$$F_0 = \hbar F_0 = \hbar r_B \left(1 - \sqrt{1 - r_+/r_B}\right) - \pi r_+^2 \beta_B^{-1}.$$  

(7)

The tree-level contribution $\tilde{S}_0$ to the entropy $S_H$ of a black hole defined as $\tilde{S}_0 = -\partial_{T_B} F_0 \equiv \hbar^{-1} \beta_B^2 \partial_{\beta_B} F_0$ is $S_0 = \pi r_+^2 (\equiv A/4l_p^2 \equiv S_H)$. Besides this tree-level contribution there are also one-loop contributions directly connected with dynamical degrees of freedom of the black hole, describing its quantum excitations. We consider them now in more details.
For simplicity we consider a contribution $F_1$ of massless conformal invariant fields to the free energy of a black hole. It is possible to show \[11\] that this contribution is directly connected with the renormalized value $\Gamma_{\text{ren}}$ of the Euclidean effective action $\Gamma = \frac{1}{2} \text{Tr} \ln D$ for the corresponding field $\varphi$ obeying the field equation $D\varphi = 0$ on a manifold with the metric

$$ds^2 = B d\tau^2 + B^{-1} dr^2 + r^2 d\Omega^2.$$  

Here $B = 1 - r_+/r$ and $d\Omega^2$ is a line element on the unit sphere. Namely one has $F_1 = \beta^{-1} \Gamma_{\text{ren}}$. The effective action is to be calculated on the manifold periodic in Euclidean time $\tau$ with the period $\beta$.

The renormalized free energies $F_1$ for two conformally related static spaces $\bar{g}_{\mu\nu} = e^{-2\omega} g_{\mu\nu}$ are connected as follows \[15\]

$$F_1[g] = F_1[\bar{g}] + \Delta F_1[\omega, g].$$ (9)

Here $\Delta F_1[\omega, g] = aA[\omega, g] + bB[\omega, g] + cC[\omega, g]$,

$$A[\omega, g] = \int d^3x g^{\frac{1}{2}} \left\{ \omega C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} + \frac{2}{3} \left[ R + 3(\Box \omega - \omega^\sigma \omega_\sigma) \right] (\Box \omega - \omega^\sigma \omega_\sigma) \right\},$$ (10)

$$B[\omega, g] = \int d^3x g^{\frac{1}{2}} \left\{ \omega * R_{\omega\beta\gamma\delta} * R^{\alpha\beta\gamma\delta} + 4 R_{\mu\nu} \omega^\mu \omega^\nu - 2 R \omega^\sigma \omega_\sigma + 2 (\omega^\sigma \omega_\sigma)^2 - 4 \omega^\sigma \omega_\sigma \Box \omega \right\},$$ (11)

$$C[\omega, g] = \int d^3x g^{\frac{1}{2}} \left\{ \left[ R + 3(\Box \omega - \omega^\sigma \omega_\sigma) \right] (\Box \omega - \omega^\sigma \omega_\sigma) \right\},$$ (12)

and the coefficients $a$, $b$, and $c$ are the coefficients of conformal anomalies

$$a = \frac{\hbar}{23040\pi^2} \left[ 12h(o) + 18h(1/2) + 72h(1) \right],$$ (13)

$$b = \frac{\hbar}{23040\pi^2} \left[ -4h(0) - 11h(1/2) - 124h(1) \right],$$ (14)

$$c = \frac{\hbar}{23040\pi^2} \left[ -120h(1) \right].$$ (15)

Here $h(s)$ is the number of polarizations of spin $s$. The term $\Delta F_1[\omega, g]$ evidently does not depend on the temperature $\beta$.

The useful expression can be obtained for $F_1[g]$ by applying the formula (9) to the particular case when $\omega = \frac{1}{2} \ln B$ and $\bar{g}_{\mu\nu} = e^{-2\omega} g_{\mu\nu}$ is the ultrastatic metric. In the high
temperature limit the leading terms of the renormalized free energy \( F_1[\bar{g}] \) in the ultrastatic space \( \bar{g} \) reads

\[
F_1[\bar{g}] = -\hbar \frac{\alpha}{\beta^4} \int d^3x \bar{g}^{1/2} + \ldots,
\]

where \( \alpha \equiv \frac{\pi^2}{90} \left[ h(0) + \frac{7}{8} h(1/2) + h(1) \right] (= 32\pi^4(a + b - c/2)) \). For the conformal massless scalar field \( D = -\Box + \frac{1}{6} R \) the renormalized free energy \( F_1[\bar{g}] \) in the ultrastatic space \( \bar{g} \) allows the following representation [15]

\[
\bar{h} F_1[\bar{g}] = -\frac{\pi^2}{90} \bar{c}_0 - \frac{1}{24} \bar{c}_1/\beta^2 - \bar{c}_3 \zeta_3(0, \infty) \frac{1}{2\beta} - \frac{\bar{c}_2}{16\pi^2} \left[ \gamma + \ln \left( \frac{\beta}{4\pi} \right) \right] \\
- \frac{\pi^{3/2}}{16} \sum_{l=3}^{\infty} \bar{c}_l \Gamma(l - 3/2) \zeta_l (2l - 3) \pi^{-2l} \left( \frac{\beta^2}{4} \right)^{l-2}.
\]

Here \( \bar{c}_l = \int d^3x \bar{g}^{1/2} \bar{a}_l(x, x) \), and the functions \( \bar{a}_l \) are the Minakshisundaram-DeWitt coefficients for the the ultrastatic metric \( \bar{g} \). The analogous representation is also valid for non-zero spins. For the conformal massless field \( \bar{a}_0 = 1 \) and \( \bar{a}_1 = 0 \).

The variation of \( F_1[g] \) with respect to the mertic \( g \) gives the renormalized stress-energy tensor \( T^\nu_\mu \). For \( \beta = \beta_H \) (i.e., for a quantum field is in the Hartle-Hawking state) this tensor remains finite at the horizon (at \( r = r_+ \)). For such a state the divergency of thermal contribution to \( T^\nu_\mu \) (obtained by variation of \( F_1[\bar{g}] \)) is exactly compensated by vacuum polarization (‘zero-point’) contribution (obtained by variation of \( \Delta F_1[\omega, g] \)). For thermal equilibrium the change of the temperature \( \beta \) implies the change of the black hole size \( r_+ \). In such a process the ‘zero-point’ contribution (depending on \( r_+ \)) is changed in such a way that it all the time exactly compensates the divergencies at the horizon of the termal contribution (depending on temperature).

Denote by \( F_1 \) the on-shell value of \( F_1[g] \), i.e. the value of this functional calculated on the manifold with the Euclidean metric \( 8 \) with the period \( \beta \) in the Euclidean time. \( (\beta = \hbar/T, \) where \( T \) is the temperature of the system measured at infinity). In general case the free energy \( F_1 \) contains volume divergences. The leading divergent near the horizon \( r = r_+ \) term is of the form

\[
F_1 \approx -\hbar \frac{\alpha \cdot 1}{4\pi^3 r_+ \varepsilon} f \left( \frac{\beta_H}{\beta} \right),
\]

(18)
where $\varepsilon = (l/r_+)^2$ and $l$ is the proper distance from the horizon, and $\beta_H = 2\pi r_+ = \hbar/T_H$, ($T_H$ is the black hole temperature). The function $f(x)$ in Eq.(18) for large $x$ is $\sim x^4$ and for $x = 1$ it vanishes $f(1) = 0$. This divergence (18) for $\beta \neq \beta_H$ reflects the fact that the number of modes that contribute to the free energy and entropy is infinitely growing as one consider regions closer and closed to the horizon [8]. In order to emphasize the dependence of $F_1$ on the dimensionless cut-off parameter $\varepsilon$ we shall write $F_1 = F_1(\beta, r_+, \varepsilon)$. For a black hole inside the cavity the spatial integration must be restricted by the cavity volume [16]. It means that $F_1$ depends also on $r_B$. We do not indicate this dependence explicitly because it is not important for our purpose. For $\beta = \beta_H$ the not only the leading but all possible volume divergencies disappear. The value $F_1(\beta_H, r_+, 0)$ in this limit is finite because the point $r = r_+$ is regular pint of the regular Euclidean manifold and the renormalized effective action calculated for a part $r \leq r_B$ of this manifold is finite.

The free energy has the same dimension as $\hbar r_+^{-1}$ and hence it can be presented in the form

$$F_1(\beta, r_+, \varepsilon) \equiv r_+^{-1} \hbar F(r_+^{-1} \beta, \varepsilon), \quad (19)$$

where $F$ is dimensionless function of two dimensionless variables. This property also directly follows from the following scaling property of the renormalized free energy [14]

$$F_1(\beta, r_+, \varepsilon) = \alpha F_1(\alpha \beta, \alpha r_+, \varepsilon). \quad (20)$$

The finiteness of $F_1(\beta_H, r_+, 0)$ and the scaling (20) are the properties which are not connected with the particular choice of the field and remain valid for an arbitrary quantum field.

The one-loop contribution of a quantum field $\varphi$ to the entropy is

$$S_1 = \frac{1}{\hbar} \left[ \beta^2 \frac{\partial F_1}{\partial \beta} \right]_{\beta = \beta_H}. \quad (21)$$

It should be stressed that one must put $\beta = \beta_H$ only after the differentiation. The leading near the horizon term of $S_1$ is

$$S_1 \approx \frac{2\alpha}{\pi^2 \varepsilon}. \quad (22)$$
If the proper-distance cut-off parameter $l$ is of the order of Planck length $l_p$ then the contribution of the field to the entropy $S_1$ is of order $S_1 \sim A/l_p^2$, where $A$ is the surface area of the black hole. In other words for the ‘natural’ choice of the cut-off parameter $l \sim l_p$ the ‘dynamical’ entropy $S_1$ of a black hole is of the same order of magnitude as the ‘thermodynamical’ entropy $S_H$.

We return now to our main problem: the relation between $S_1$ and $S_H$. One can write

$$dF_1 \equiv -\tilde{S}_1dT_H = -(S_1 + \Delta S)dT_H. \quad (23)$$

The first relation defines the one-loop contribution $\tilde{S}_1$ to the ‘thermodynamical’ entropy of the black hole. The additional term

$$\Delta S = \frac{\beta^2_H}{4\hbar\pi} \left[ \frac{\partial F_1}{\partial r_+} + \frac{\partial F_1}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial r_+} \right]_{\beta = \beta_H}. \quad (24)$$

is connected with the change of energy of states of the field when the parameters of the system ($r_+$ in our case) are changed (the first term in the square brackets of Eq.(24)), and with the change of the number of states in these process (the second term in Eq.(24)). This terms are similar to the analoguous terms arizing in Eq.(6). It is evident that $\tilde{S}_1$ can also be obtained by direct substitution of $\beta = \beta_H \equiv 4\pi r_+$ into $F_1$ before its differentiation with respect to $\beta_H$. By using Eq.(19) one gets

$$\tilde{S}_1 = 4\pi \beta_H \frac{\partial}{\partial \beta_H} \left[ \frac{F_1}{\beta^2_H} \right] = 4\pi \left[ -F_1(4\pi, \varepsilon) + \frac{\partial F_1(4\pi, \varepsilon)}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \ln \beta_H} \right]. \quad (25)$$

Because the free energy $F_1$ does not contain volume divergencies for $\beta = \beta_H$, the quantity in the square brackets of Eq.(23) is a finite number. It means that the additional contribution $\Delta S$ which arises due to the change the parameters of the system (of the black hole) exactly compensates the divergent terms of $S_1$ (‘dynamical’ entropy). That is why the contribution $\tilde{S}_1$ of the quantum field $\varphi$ to the ‘thermodynamical’ entropy of a black hole $S_H$ is of order of $O(\varepsilon^0)$. It is much smaller than $A/l_p^2$ and can be neglected. As the result of this compensation mechanism the dynamical degrees of freedom of the black hole practically do not contribute to its ‘thermodynamical’ entropy $S_H$, and the letter is defined by the tree-level quantity $\tilde{S}_0$. It is evident that this conclusion remains valid for contribution of any other field.
To make the basic idea more clear we restricted ourselves in the above discussion by considering a non-rotating black hole. The analysis is easily applied to the case of a charged rotating black hole as well as to their n-dimensional generalization. It is interesting that for black holes in the generalized gravitational theories the 'thermodynamical' $S_H$ and 'dynamical' $S_1$ entropy may have different dependence on the mass $M$ of a black hole. In particular for two dimensional dilaton black hole $S_H = 4\pi M/\sqrt{\lambda}$ \cite{17}, while $S_1 \sim \ln \varepsilon$.

To summarize we show that there is no contradiction between thermodynamical definition of the entropy and its statistical-mechanical calculation based on the existence of the internal degrees of freedom of the black hole. The dynamical degrees of freedom of the black hole are related with possibility for a black hole to have different internal structure for the same external parameters. A black hole as a thermodynamical system is singled out by the property that the parameters of its internal degrees of freedom depend on the temperature of the system in the universal way. This results in the universal cancellation of all those contributions to the thermodynamical entropy which depend on the particular properties and number of fields. That is why the 'thermodynamical' entropy of black holes in the Einstein theory is always $S_H$.

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REFERENCES

[1] J.D.Bekenstein, Nuov.Cim.Lett. 4, 737 (1972).

[2] J.D.Bekenstein, Phys.Rev. D7, 2333 (1973).

[3] G.W.Gibbons and S.W.Hawking, Phys.Rev. D15, 2752 (1976).

[4] S.W.Hawking, In: General Relativity: An Einstein Centenary Survey. (eds. S.W.Hawking and W.Israel), Cambridge Univ.Press, Cambridge, 1979.

[5] G.’t Hooft, Nucl.Phys. B256, 727 (1985).

[6] V.Frolov and I.Novikov, Phys.Rev. D48 4545 (1993).

[7] S.Carlip and C.Teitelboim, The Off-Shell Black Hole. IASSNS-93/84; UCD-93-34; gr-qc/9312002 (1993).

[8] L.Susskind and J.Uglum, Black Hole Entropy in Canonical Quantum Gravity and Superstring Theory. Preprint SU-ITP-94-1, hep-th/9401076 (1994).

[9] M.Maggiore, Black holes as quantum membranes, preprint IFUP-TH 6/94, gr-qc/9401027 (1994).

[10] D.Garfinkle, S.B.Giddings, and A.Strominger, Phys.Rev. D49, 958 (1994).

[11] A.I.Barvinsky, V.P.Frolov, and A.I.Zelnikov, Wavefunction of a Black Hole and the Dynamical Origin of Entropy. Preprint Alberta Thy 13-94, gr-qc/9404036 (1994) (submitted to Phys.Rev.D).

[12] J.M.York, Phys.Rev. D33, 2092 (1986).

[13] H.W.Braden, J.D.Brown, B.F.Whiting, and J.W.Jork, Phys.Rev. D42, 3376 (1990).

[14] We use units in which $G = c = 1$.

[15] J.S.Dowker and G.Kennedy, J.Phys. A 11, 895 (1978).
For a black hole located inside a cavity the field $\varphi$ must obey some boundary conditions at the surface of the cavity. For this case boundary terms must be also added to the expression (17). This modification is not important for the entropy problem. That is why we do not discuss these boundary terms.

[17] V.P.Frolov, Phys.Rev. D46, 5383 (1992).