STEREOLOGY WITH CYLINDER PROBES

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ABSTRACT

Intersection formulae of Crofton type for general geometric probes are well known in integral geometry. For the special case of cylinders with non necessarily convex directrix, however, no equivalent formula seems to exist in the literature. We derive this formula resorting to motion invariant probability elements associated with test systems, instead of using a traditional approach. Because cylinders are seldom used as probes in stereological practice, however, this note is mainly of a theoretical nature.

Keywords: Cylinders, integral geometry, motion invariant measures, ratio design, stereology, test systems.

INTRODUCTION

The fundamental equations of stereology - see for instance Miles (1972), Baddeley and Jensen (2005), (Section 2.2.3), or Cruz-Orive (2017) for their history, are based on intersections between a target set and a geometric probe. The latter is usually an \( r \)-plane, or a bounded portion of it. With rare exceptions (e.g. Horgan et al. (1993)) cylinder probes are seldom used in stereology - therefore, the present note is mainly theoretically oriented.

The classical stereological equations are usually ratios of motion invariant measures. For instance, the ratio \( B_A \) of the total planar curve length determined by a motion invariant test plane in the boundary of a compact three dimensional set, divided by the total planar section area determined in the set, is equal to \( (\pi/4)S_V \), where \( S_V \) is the surface to volume ratio of the set. The identity is the result of dividing side by side two integral identities which belong to the family of Crofton intersection formulae of integral geometry. Such ratio identities hold formally unchanged for probes other than \( r \)-planes, notably cylinders. The Crofton integrals in the numerator and the denominator of a ratio, however, do in general depend on probe shape.

To fix ideas, consider a cylindrical surface \( Z_2 \subset \mathbb{R}^3 \) whose generator \( L_{1[0]} \) is a straight line and its directrix, namely its cross section by a plane \( L_{2[0]} \) through the origin, perpendicular to the generator, is a piecewise smooth, simple closed curve \( Z_1 \subset L_{2[0]} \) of perimeter length \( b \), see Fig. 1. The object is a compact set \( Y \subset \mathbb{R}^3 \) of surface area \( S \) and volume \( V \). The motion invariant density of the cylinder is the kinematic density:

\[
dZ_2 = dx \, du \, d\tau, \quad x \in \mathbb{R}^2, \; u \in S^2, \; \tau \in S,
\]

where \( x \) is an associated point (AP) of the cross section \( Z_1 \), (namely a point rigidly attached to \( Z_1 \) according to a fixed rule), whereas \( u \) is a unit vector on the unit sphere \( S^2 \) giving the direction of the generator, and \( \tau \) is a rotation around the generator. The pertinent Crofton intersection formulae read as follows,

\[
\int B(dY \cap Z_2) \, dZ_2 = 2\pi^3 b S, \quad (2)
\]

\[
\int A(Y \cap Z_2) \, dZ_2 = 8\pi^2 b V, \quad (3)
\]

and side by side division yields the aforementioned stereological equation \( S_V = (4/\pi)B_A \). Analogous formulae arise for a solid cylinder \( Z_3 \subset \mathbb{R}^3 \) whose cross section \( Z_2 \subset L_{2[0]} \) is a domain of area \( a \) and perimeter length \( b \).

Fig. 1. Sketch of a cylinder \( Z_r \subset \mathbb{R}^3 \) with generator \( L_{1[0]} \) and directrix \( Z_{r-1} \subset L_{2[0]} \). For \( r = 1, 2, 3 \) the cylinder is a straight line, a cylindrical surface, or a solid cylinder, respectively. The remaining symbols are defined in the text.
While a Crofton intersection formula for general manifolds is well known, see for instance Santaló (1976), Eq. (15.20), we have not found an analogous formula for general cylinders in the literature. Schneider and Weil (2008) consider cylinders with convex directrix. Particular cases such as Eq. 2 and Eq. 3, formulaforgeneralcylindersintheliterature.Schneider (1976), Eq. (15.20), we have not found an analogous manifolds is well known, see for instance Santaló (1976), Eq. (15.20), we have not found an analogous.

Examples for $n = 3$ and $p = 1$.

- If $r = 1$, then the cylinder $Z^3$ is a straight line $L^3_1$ normal to a given plane $L^3_{2[0]}$.
- If $r = 2$, then $Z^2_2$ is a cylindrical surface whose cross section is a bounded curve $Z^2_2$ contained in a plane $L^3_{2[0]}$ normal to the straight line generator $L^3_{1[0]}$.
- If $r = 3$, then $Z^3_3$ is a solid cylinder whose cross section is a domain $Z^2_3$ of dimension 2 contained in $L^3_{2[0]}$.

An $n$–box $J^n_p \subset \mathbb{R}^n$ is defined as

$$J^n_p = [0, a_1) \times [0, a_2) \times \cdots \times [0, a_n),$$

where $a_1, a_2, \ldots, a_n$ are finite, positive real numbers.

A bounded cylinder $T^n_p \subset \mathbb{R}^n$ is a compact set defined as:

$$T^n_p = Z^n_{r-p} \times J^n_p.$$  

Example. If $p = 1$, then $T^n_1$ is a bounded right cylinder of base $Z^{n-1}_r$ and finite height $a_1 > 0$.

**MOTION INVARIANT DENSITIES**

The material in this section, given to make the note self contained, is well known - for general reference see Santaló (1976).

A non-oriented linear subspace $L^n_{p[0]}$ submitted to a rotation from the group $G_{p,n-p}$ of rotations about a fixed point in $\mathbb{R}^n$, called the Grassmann manifold, has a rotation invariant density denoted by

$$dL^n_{p[0]} = dL^n_{n-p[0]}.$$  

It can be shown that

$$\int_{G_{p,n-p}} dL^n_{p[0]} = \frac{O_{n-1} O_{n-2} \cdots O_{n-p}}{O_{p-1} O_{p-2} \cdots O_0},$$

where

$$O_k = \frac{2\pi^{(k+1)/2}}{\Gamma((k+1)/2)}, \quad k = 0, 1, \ldots, n,$$
denotes the surface area of the $k$–dimensional unit sphere $S_k$ ($O_0 = 2, O_1 = 2\pi, O_{k+2} = 2\pi O_k / (k + 1)$).

As given below, the motion invariant density for cylinders involves an oriented $p$–subspace $L^n_{p[0]}$, which we denote by $L^n_{\hat{p}[0]}$. Consequently, the measure given in Eq. 8 must be multiplied by $O_n = 2$.

For a compact set $T^n_r \subset \mathbb{R}^n$, not necessarily a bounded cylinder, the motion invariant density is the kinematic density, namely,

$$ dT^n_r = dx_n du_n, \quad x_n \in \mathbb{R}^n, \ u_n \in G_{n[0]}, \quad (10) $$

where $x_n$ is an AP fixed in $T^n_r$, whereas $u_n$ is an element of the special group of rotations $G_{n[0]}$, isomorphic to $SO(n)$, about a fixed point in $\mathbb{R}^n$. It can be shown that

$$ \int_{G_{n[0]}} du_n = O_{n-1}O_{n-2} \ldots O_1. \quad (11) $$

**Example.** For $n = 3$ we have $u_3 = (u_2, u_1)$, where $u_2 \equiv (\phi, \theta) \in S^2$ is a unit vector of spherical polar coordinates $(\phi, \theta)$ whereas $u_1 \equiv \tau \in S^1$ is a rotation about $u_2$. Thus,

$$ \int_{G_{3[0]}} du_3 = \int_{S^2} du_2 \int_{S^1} du_1 = 4\pi 2\pi = 8\pi^2. \quad (12) $$

For a cylinder $Z^n_r$ of cross section $Z_{r-p}^{n-p}$, where $1 \leq p \leq r \leq n$, the motion invariant density is

$$ dZ^n_r = dZ_{r-p}^{n-p} dL^n_{p[0]}, \quad (13) $$

(Santaló, 1976, Eq. (15.76)), where $dZ_{r-p}^{n-p}$ is the kinematic density in $L_{n-p}[0]$, namely,

$$ dZ_{r-p}^{n-p} = dx_{n-p} du_{n-p}. \quad (14) $$

Substitution into the right hand side of Eq. 13 yields,

$$ dZ^n_r = dx_{n-p} dL^n_{p[0]} du_{n-p}. \quad (15) $$

where $x_{n-p} \in L^n_{n-p}[0], \ L^n_{p[0]} \in G^*_{p,n-p}, \ u_{n-p} \in G_{n-p}[0]$ (this group is isomorphic to the group of rotations about $L^n_{p[0]}$). From Eq. 8 and Eq. 11 we get,

$$ \int_{G^*_{p,n-p}} dL^n_{p[0]} \int_{G_{n-p}[0]} du_{n-p} = O_{n-1} \ldots O_{p}. \quad (16) $$

**Example.** For a cylinder $Z^3_r, \ r = 2, 3, \ p = 1$, the motion invariant density given by Eq. 15 reduces to Eq. 1.

### TEST SYSTEMS OF CYLINDERS

Consider a test system $\Lambda_T \subset \mathbb{R}^n$ whose fundamental tile is an $n$–box $J^n_0$, whereas the fundamental probe is a bounded cylinder $T^n_r \subset J^n_0$ given by Eq. 6 with its AP at the origin $O$, see Fig. 2. For details pertaining to the construction of a test system see Santaló (1976) under the term "lattice of figures."
an infinite cylinder $Z_r^n$. Thus, $\Lambda_T$ coincides with a test system $\Lambda_Z \subset \mathbb{R}^n$ of cylinders congruent with $Z_r^n$, whose fundamental tile is $J_0^{n-p} \subset L_0^{n-p}[0]$, whereas the fundamental probe is the orthogonal projection $Z_r^{n-p}$ of $Z_r^n$ onto $L_0^{n-p}[0]$. The motion invariant probability element corresponding to $\Lambda_Z$ is the normalized density given by Eq. 15. Thus, for $x_{n-p} \in J_0^{n-p}$, $L^*_0[0] \in G_{p,n-p}, u_{n-p} \in G_{n-p}[0]$, 
\[
\mathbb{P}(d\Lambda_Z) = \mathbb{P}(dx_{n-p}, dL^*_0[0], du_{n-p}) = \frac{dx_{n-p}}{\nu_{n-p}(J_0^{n-p})} \frac{dL^*_0[0]}{d\nu_{n-p}}, \tag{18}
\]
where the integral involving the orientation variables is given by Eq. 16.

From the preceding considerations it follows that the test system $\Lambda_Z$ equipped with the probability element given by Eq. 18, has identical statistical properties as the test system $\Lambda_T$ equipped with the probability element given by Eq. 17. In particular, for a compact submanifold $Y_q \subset \mathbb{R}^n$ of dimension $q \in \{0, 1, \ldots, n\}$, with $q + r \geq n$, the following identity holds,
\[
\mathbb{E}\nu_{q+r-n}(Y_q \cap \Lambda_T) = \mathbb{E}\nu_{q+r-n}(Y_q \cap \Lambda_Z), \tag{19}
\]
the expectations being with respect to the corresponding motion invariant probability elements.

**CROFTON INTERSECTION FORMULA FOR BOUNDED PROBES**

For the compact submanifold $Y_q \subset \mathbb{R}^n$ just considered, hit by a compact probe $T_r^n$ equipped with the kinematic density $dT_r^n$ given by Eq. 10, the following identity holds,
\[
\int_{\mathbb{R}^n \times G_n[0]} \nu_{q+r-n}(Y_q \cap T_r^n) dT_r^n = c(q, r, n) \nu_q(Y_q) \nu_r(T_r^n), \tag{20}
\]
\[\text{Santalo} \ (1976), \ \text{Eq. (15.20)},\]
where
\[
c(q, r, n) = \frac{O_n O_{n-1} \cdots O_1 O_{q+r-n}}{O_q O_r}. \tag{21}
\]

**RESULTS**

**CROFTON INTERSECTION FORMULA FOR CYLINDERS**

The main purpose of this note is to prove the following identity.

**Proposition**

\[
\int_{\mathbb{R}^n \times G_{p,n-p} \times G_{n-p}[0]} \nu_{q+r-n}(Y_q \cap Z_r^n) \ dZ_r^n = c_Z(q, r, n) \nu_r(Z_r^{n-p}) \nu_q(Y_q) \tag{22}
\]
where
\[
c_Z(q, r, n) = \frac{O_n O_{n-1} \cdots O_p O_{q+r-n}}{O_q O_r}. \tag{23}
\]

**Proof**

Set $\nu \equiv \nu_{q+r-n}, Y \equiv Y_q, T \equiv T_r^n, Z \equiv Z_r^n, Z' \equiv Z_r^{n-p}$ and $c \equiv c(q, r, n)$, for short. In addition, the domains of integration $G_{n}[0]$ for $u_{n-p}, G_{p,n-p}$ for $L^*_0[0]$, and $G_{n-p}[0]$ for $u_{n-p}$, will be omitted in the sequel.

The proof is based on Eq. 19, whose left hand side becomes,
\[
\mathbb{E}\nu(Y \cap \Lambda_T) = \int_{x_n \in J_0^n} \nu(Y \cap \Lambda_T) \mathbb{P}(d\Lambda_T) = \frac{1}{\nu_n(J_0^n)} \int_{x_n \in J_0^n} \nu(Y \cap \Lambda_T) \ dx_n \ du_n \tag{24}
\]
where the last identity follows from Santaló’s identity for test systems, see Santaló (1976), chapter 8, or Cruz-Orive (2002). In combination with Eq. 20 we obtain,
\[
\mathbb{E}\nu(Y \cap \Lambda_Z) = \frac{c v_q(Y) v_r(T)}{\nu_n(J_0^n)} \int \ dx_n. \tag{25}
\]

Analogously,
\[
\mathbb{E}\nu(Y \cap \Lambda_Z) = \int_{x_{n-p} \in J_0^{n-p}} \nu(Y \cap \Lambda_Z) \mathbb{P}(d\Lambda_Z) = \int_{x_{n-p} \in \mathbb{E}_{n-p}} \nu(Y \cap Z) \ dZ = \frac{1}{\nu_{n-p}(J_0^{n-p})} \int dL^*_0[0] \ du_{n-p}. \tag{26}
\]

Finally, bearing Eq. 11 and Eq. 16 in mind, applying Eq. 19, and using the following identities,
\[
v_n(J_0^n) = v_{n-p}(J_0^{n-p}) v_p(J_0^p), \tag{27}
\]
\[
v_r(T) = v_{r-p}(Z') v_p(J_0^p), \tag{28}
\]
we obtain,
\[
\int_{x_{n-p} \in \mathbb{E}_{n-p}} \nu(Y \cap Z) \ dZ = \frac{O_n O_{n-1} \cdots O_p O_{q+r-n}}{O_q O_r} v_{r-p}(Z') v_q(Y). \tag{29}
\]
which is the identity given by Eq. 22, thus completing the proof of the proposition.

SPECIAL CASES FOR $n = 2, 3$ AND $p = 1$

A cylinder $Z^2 \subset \mathbb{R}^2$ is a solid stripe of thickness $t > 0$, say, in the plane. Its boundary $\partial Z^2 \equiv Z^2_1$ is the union of two parallel straight lines a distance $t$ apart.

Consider a compact set $Y_2 \subset \mathbb{R}^2$ of area $A > 0$ and piecewise smooth boundary $\partial Y_2 \equiv Y_1$ of length $B > 0$. Application of Eq. 22 with $r = 1$ yields:

$$\int v_0(Y_1 \cap Z^2_1) \, dZ^2_1 = \frac{O_2 O_1 O_0}{O_1 O_2} v_0(Z^2_0) \cdot B = 8B,$$

(30)
because the projection $Z^2_0$ of $Z^2_1$ onto an axis normal to the stripe is the union of two points a distance $t$ apart hence $v_0(Z^2_0) = 2$. As a cross-check, note that

$$\int v_0(Y_1 \cap Z^2_1) \, dZ^2_1 = 2 \int I(Y_1 \cap L^*_1) \, dL^*_1$$

$$= 4 \int I(Y_1 \cap L^*_1) \, dL^*_1 = 8B,$$

(31)
where $L^*_1$ is a straight line with motion invariant density $dL^*_1$, and $I(\cdot)$ denotes number of intersections - see Cruz-Orive (2017), Eq. (1), for references.

On the other hand

$$\int v_2(Y_2 \cap Z^2_2) \, dZ^2_2 = \frac{O_2 O_1 O_2}{O_2 O_1} v_1(Z^1_1) A = 2\pi A,$$

(32)
which is twice the value obtained in the classical manner (see Eq. (5.16) of Santaló (1976)), because we consider oriented stripes. Note that the projection $Z^1_1$ of the stripe $Z^2_1$ onto an axis normal to the stripe is a segment of length $t$ - hence $v_1(Z^1_1) = t$.

Further

$$\int v_1(Y_1 \cap Z^1_1) \, dZ^1_1 = \frac{O_2 O_1 O_1}{O_2 O_1} v_0(Z^1_0) A = 4\pi A,$$

(33)
which is equivalent to

$$\int v_1(Y_1 \cap Z^1_1) \, dZ^1_1 = 4 \int L(Y_2 \cap L^*_1) \, dL^*_1 = 4\pi A,$$

(34)
where $L(\cdot)$ denotes intercept length, and the first integral pertains to the two oriented straight lines constituting $Z^1_1$, yielding $2\pi A$ each.

For $n = 3$ and $r = q = 2$, Eq. 22 yields $c_Z(2, 2, 3, 1) = O_3 O_2 O_1 (O_2 O_2) = 2\pi^3$, and we obtain Eq. 2.

For $n = 3$, $q = 3$, and $r = 2$ Eq. 22 yields $c_Z(3, 2, 3, 1) = O_3 O_2 O_1 (O_2 O_2) = 8\pi^2$ and we obtain Eq. 3.

Finally, for $n = 3$, $r = 2$, and $q = 1$, namely for $Z^3 \subset \mathbb{R}^3$ hitting a curve $Y_1 \subset \mathbb{R}^3$ of length $L$, we have

$$c_Z(1, 2, 3, 1) = O_3 O_1 O_0 = 4\pi^2 \text{ and }$$

$$\int v_0(Y_1 \cap Z^3_2) \, dZ^3_2 = 4\pi^2 bL,$$

(35)
see Santaló (1976), p. 280.

STEREOLOGICAL EQUATIONS FOR TEST SYSTEMS OF CYLINDERS

Substitution of Eq. 29 into the right hand side of Eq. 26, yields the Hausdorff measure $v_q(Y_q)$ of a compact submanifold $Y_q \subset \mathbb{R}^n$ in terms of the measure of its intersection with a test system $\Lambda_{Z_r}$ of cylinders of dimension $r$, namely,

$$v_q(Y_q) = \frac{O_q O_r}{O_n O^{q+r-n} v_{r-p} (Z^{n-p}_r)} B \nu_{q+r-n} (Y_q \cap \Lambda_{Z_r}).$$

(36)
The numerical constant in the right hand side of the preceding identity is the same as that arising for $r-$probes in general, see for instance Voss and Cruz-Orive (2009), Eq. (A28), (A32).

In stereology, $v_q(Y_q)$ is often estimated via the ratio design, which is based on the identity,

$$v_q(Y_q) = v_n(X_n) \cdot R_{q,n}.$$

(37)
where $X_n \supset Y_q$ is a reference submanifold containing $Y_q$, whose volume is estimated separately (e.g. by the Cavalieri design). Ratio designs were studied in some detail by Cruz-Orive (1980) and Cruz-Orive and Weibel (1981), see also Baddeley and Jensen (2005). Thus, it only remains to estimate the ratio $R_{q,n} \equiv v_q(Y_q) / v_n(X_n)$ via the identity,

$$R_{q,n} = \frac{O_q O_r}{O_n O^{q+r-n} \nu_{q+r-n} (Y_q \cap \Lambda_{Z_r})} / \nu_r (X_n \cap \Lambda_{Z_r}).$$

(38)
Provided that the same test system $\Lambda_{Z_r}$ is used in the numerator and denominator, the right hand side of the preceding identity does not involve any properties of $\Lambda_{Z_r}$ itself - hence the relative popularity of ratios. Thus, with the usual conditions Eq. 38 holds for any $r-$dimensional test system; it was already obtained by Miles (1972), Eq. (2.16), and it encapsulates the classical stereological equations used in practice, see also Cruz-Orive (2002), Eq. (6.19).

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