1. Introduction

This paper continues the study of non-general type subvarieties begun in [PSS99]. Consider for simplicity a submanifold \( A \) of a projective manifold \( X \) with ample normal bundle \( N_A \) (but we will also consider also subvarieties with weaker positivity assumptions on the normal bundle). In [PSS99] it was shown that if \( A \) is not of general type, then the Kodaira dimension \( \kappa(X) = -\infty \). Conjecturally any manifold of negative Kodaira dimension is uniruled, i.e. covered by rational curves. Here we establish that in the above situation. The new ingredient is the paper [BDPP04] where it is shown that a manifold is uniruled if its canonical bundle \( K_X \) is not pseudo-effective, i.e. its Chern class is not contained in the closure of the cone generated by effective classes. Alternatively, \( K_X \) is not pseudo-effective, if \( K_X \) does not admit a possibly singular metric with non-negative curvature (current).

Once \( X \) is uniruled, one can form the rational quotient \( f : X \twoheadrightarrow W \), see [Ca81],[Ko96]. This is just a rational map, but has some good properties, see (2.1). Now by a fundamental result of Graber-Harris-Starr [GHS03], \( W \) is no longer uniruled, hence again by [BDPP04] \( K_W \) is pseudo-effective (\( W \) can be taken smooth). Coming back to our submanifold \( A \), this can be used to show that \( f|A \) is onto \( W \); thus we obtain a bound for \( \dim W \) and often also for \( \kappa(W) \). It allows also to relate \( \pi_1(A) \) and \( \pi_1(X) \). Finally we give a cohomological criterion for a projective manifold to be rationally connected.

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so-called rational quotient (e.g. [Ca81], [CP99], [Ko96])

\[ f : X \rightarrow W \]

which has the remarkable property to be almost holomorphic, i.e. the locus of indeterminacy does not project onto W. By [GHS03], W does not carry any covering family of rational curves, i.e. W is not uniruled, and consequently the canonical bundle \( K_W \) is not pseudo-effective (if W is smooth, [BDPP04]).

If \( \dim W = 0 \), then X is rationally connected.

(2.2) In [Ca95], Campana introduced a refined Kodaira dimension, which potentially differs from the usual notion only in the case that X is uniruled. Namely, he considers subsheaves \( F \subset \Omega^p_X \) of an arbitrary rank \( r > 0 \) for any \( p \) and defines

\[ \kappa^+(X) \]

to be the maximum over all \( \kappa(\det F) \), where \( \det F = (\wedge^r F)^* \). Conjecturally \( \kappa^+(X) = \kappa(X) \) if X is not uniruled; and \( \kappa(X) = \kappa(W) \), when X is uniruled with rational quotient W - here we let \( \kappa(W) = -\infty \) if \( \dim W = 0 \). See [Ca95],[CP99] for details. In [CP04] it is proved that \( \kappa^+(X) = \dim X \) implies \( \kappa(X) = \dim X \).

(2.3) (1) We say that a vector bundle \( E \) is \( \mathbb{Q} \)-effective [PSS99], if some symmetric power \( S^k(E) \) is generically spanned by global sections.
(2) \( E \) is almost ample, if \( E \) is nef and \( E|C \) is ample on every curve not contained in some countable union of proper subvarieties.
(3) \( E \) is generically ample, if \( E \) is \( \mathbb{Q} \)-effective and if \( E|C \) is ample for all curves not contained in a countable union of proper subvarieties.
(4) \( E \) is big if for all torsion free quotients \( E \rightarrow S \rightarrow 0 \) of rank 1 the determinant \( \det S := S^{**} \) is big.

As an example, consider a birational map \( \pi : \hat{X} \rightarrow X \) from a projective manifold \( \hat{X} \). Consider a vector bundle \( E \) of the form \( E = \pi_* (\hat{E})^{**} \) with an ample vector bundle \( \hat{E} \) on \( \hat{X} \). Then \( E \) is \( \mathbb{Q} \)-effective, big and generically ample. Notice also that an almost ample vector bundle might not be big.

3. A CRITERION FOR UNIRULEDNESS

For the entire section we fix a projective manifold and a compact submanifold \( A \subset X \) whose normal bundle \( N_A = N_{A/X} \) is supposed to be \( \mathbb{Q} \)-effective. One could allow certain singularities (see remark 3.7(4)) but for sake of simplicity we only consider the smooth case.

3.1. Lemma. Let \( L \) be a pseudo-effective line bundle on X. Then the restriction \( L_A \) is again pseudo-effective.

Proof. Fix an ample line bundle \( H \) on X. Since \( L \) is pseudo-effective, we find for any \( m \gg 0 \) a positive number \( t \) such that

\[ H^0(X, t(mL + H)) \neq 0. \]

Then from [PSS99,2.1] we deduce the existence of some number \( k \) such that

\[ H^0(S^k N_A^* \otimes (mL_A + H_A)) \neq 0. \]

By assumption there exists \( l \) such that \( S^l N_A \) is generically spanned, hence \( S^l N_A^* \subset O_A^* \). Putting things together, we obtain a number \( t' \) such that

\[ H^0(t'(mL_A + H_A)) \neq 0. \]
Thus $L_A$ is pseudo-effective.

As application we prove

3.2. **Theorem.** If $N_A$ is $\mathbb{Q}$–effective and $\det N_A$ is big and $A$ not of general type, then $X$ is uniruled.

**Proof.** We are going to show that $K_X$ is not pseudo-effective; then the uniruledness follows from [BDPP04]. Assume to the contrary that $K_X$ is pseudo-effective. Then by the previous lemma $K_X|A$ is pseudo-effective. Thus by adjunction, $K_A$ is the sum of a pseudo-effective and a big divisor, hence big, contradicting our assumption. 

3.3. **Remark.** In [PSS99] it is shown that if $N_A$ is $\mathbb{Q}$–effective and $\kappa(\det N_A) > \kappa(A)$, then $\kappa(X) = -\infty$. So we would expect that in Theorem 3.2 that it is sufficient to assume $\kappa(\det N_A) > \kappa(A)$. From our discussion in the proof we see that things come down to the following. Suppose that $K_A = P + E$ with $P$ pseudo-effective and $E$ effective, then we should have $\kappa(A) \geq \kappa(E)$. This is discussed in [CP04] and proved in special cases. It is not unreasonable to expect that the assumption on the $\mathbb{Q}$–effectivity can be substituted by assuming that $N_A$ is nef.

3.4. **Corollary.** Let $X$ be a projective manifold, $A \subset X$ a submanifold with ample normal bundle. If $A$ is not of general type, then $X$ is uniruled.

3.5. **Setup.** Returning to submanifolds $A \subset X$ with $\mathbb{Q}$–effective normal bundle $N_A$ and big determinant $\det N_A$ and assuming again $A$ not of general type, we now know that $X$ is uniruled. So let $f : X \dasharrow W$ be “the” rational quotient; we choose $W$ to be smooth. Since $W$ is not uniruled, $K_W$ is pseudo-effective and if $p = \dim W$, there exists a locally free pseudo-effective sheaf $\mathcal{F} \subset \mathcal{O}_X^p$ of rank 1. To be more precise, consider a birational map $\pi : \hat{X} \to X$ from a projective manifold $\hat{X}$ such that the induced map $\hat{f} : \hat{X} \to W$ is holomorphic. Then we have an injective map $\hat{f}^*(\mathcal{O}_W^p) \to \mathcal{O}_{\hat{X}}^p$ and we set $\mathcal{F} = (\pi_*(\hat{f}^*(K_W)))^{**}$.

Then $\mathcal{F} \subset \mathcal{O}_X^p$ since this evidently holds outside a set of codimension at least 2 and $\mathcal{F}$ is clearly pseudo-effective. Moreover, if $A \subset X$ with $N_A$ $\mathbb{Q}$–effective is a subvariety then Lemma 3.1 shows that $\mathcal{F}_A$ is again pseudo-effective.

If $A$ does not meet the set of indeterminacies of $f$, then the almost ampleness of $N_A$ forces $f|A$ to be onto as soon as $\dim A \geq \dim W$. This actually also holds if $A$ does meet the indeterminacy locus:

3.6. **Lemma.** Let $X$ and $W$ be projective manifolds, $f : X \dasharrow W$ be almost holomorphic. Let $A \subset X$ be a submanifold or a local complete intersection with almost ample normal bundle and assume $\dim A \geq \dim W$. Then $A$ meets the general fiber of $f$, in particular $A$ is not contained in the indeterminacy locus of $f$, and $f : A \dasharrow W$ is dominant.
Proof. If \( N_A \) is ample, this is a special case of [FL82, Theorem 1 or Corollary 1, applied to a general fiber or a general hyperplane section of a general fiber. See also [Fu84, 12.2.4]. The more general case that \( N_A \) is almost ample can be done by an easy adaption of the proof of the ample case. \( \square \)

It is not clear whether the lemma is still true when \( N_A \) is only \( \mathbb{Q} \)-effective (or generically ample).

In the next theorem we make use of the notations of (3.5).

**3.7. Theorem.** Assume that \( A \) is not of general type with \( N_A \) big and \( \mathbb{Q} \)-effective.

Then:

1. \( \mathcal{F}_A \) is a (pseudo-effective) subsheaf of \( \Omega^p_X \).
2. \( \dim A \geq \dim W \), and if \( N_A \) is almost ample or if \( A \) is not in the indeterminacy locus of \( f \), then \( f|A \) is onto \( \mathbb{W} \).
3. If \( A \) is uniruled with rational quotient \( A \to B \), then even \( \dim B \geq \dim \mathbb{W} \).

Proof. Recall that indeed \( X \) is uniruled and consider the rational quotient \( f : X \to \mathbb{W} \).

First we show how to derive (2), (3) and (4) from (1).

(2) is obviously a consequence of (1); for the second part use Lemma 3.6 and observe that since \( \mathcal{F}_A \) is a subsheaf of \( \Omega^p_X \) via the canonical generically defined map \( f^*(\Omega^p_W) \to \Omega^p_A \), the restricted map \( f|A \) must be onto.

Concerning (3): if \( A \) is uniruled, then we cannot have \( \dim A = p \), because then \( K_A \) would contain a pseudo-effective line bundle, i.e. would be pseudo-effective. More generally, consider a general smooth fiber \( A_b \) of the almost holomorphic map \( g : A \to B \). Then \( \Omega^p_A|A_b \) has a filtration by terms

\[ \Omega^i_{A_b} \otimes \bigwedge^j N_{A_b}^* \]

Since \( A_b \) is rationally connected, \( \mathcal{F}_A \) cannot be in a term \( \Omega^i_{A_b} \otimes \bigwedge^j N_{A_b}^* \) with \( i > 0 \), hence we must have \( \mathcal{F}_A \subset g^*(\Omega^p_B) \), at least generically. But then \( \dim B \geq p \). If \( A \) is not contained in the indeterminacy locus of \( f \), then it is clear that the fibers \( A_b \) must be contracted by \( f \) (consider general rational curves in \( A_b \); their deformations cover all of \( X \)). Hence \( X \to \mathbb{W} \) factors over \( B \to \mathbb{W} \).

It remains to prove (1). By taking \( \bigwedge^p \) of the exact sequence

\[ 0 \to \bigwedge^p N^*_A \to \Omega^1_X|A \to \Omega^1_A \to 0 \]

we obtain a filtration of \( \Omega^p_X|A \) by the terms

\[ \Omega^i_A \otimes \bigwedge^j N^*_A \]

with \( i + j = p \). The map \( \mathcal{F} \to \Omega^p_X \) corresponds to a non-zero section \( s \in H^0(X, \Omega^p_X \otimes \mathcal{F}^*) \). If \( s|A = 0 \), choose \( k \) maximal such that

\[ s \in H^0(X, \Omega^p_X \otimes \mathcal{F}^* \otimes I_A^k) \]

Allowing \( k = 0 \) if \( s|A \neq 0 \), we always get a non-zero map

\[ S^k N_A \otimes \mathcal{F}_A \to \Omega^p_X|A. \]
This yields a non-zero map
\[ S^k N_A \otimes F_A \to \Omega^j_A \otimes S^i N_A^* \]
and therefore a non-zero map
\[ S^k N_A \otimes S^i N_A \otimes F_A \to \Omega^j_A. \]
Since $F_A$ is pseudo-effective (see 3.5) and since $N_A$ is assumed to be big, the determinant $L$ of the image of this map is big. If now $i > 0$ or $k > 0$, then by [CP04], we obtain $\kappa(A) = \dim A$, contrary to our assumption. Hence $F_A \subset \Omega^j_A$. □

3.8. **Corollary.** Assume that $A$ is not of general type and $N_A$ to be $\mathbb{Q}$-effective. Then the conclusions of Theorem 3.7 hold, if one of the following conditions is satisfied.

1.) $N_A$ is ample.

2.) $\det N_A$ big; $A$ is not contained in the indeterminacy locus of the rational quotient; $W$ is of general type and $f|A$ is generically finite.

3.) $A$ is uniruled, $N_A$ is generically ample.

4.) $A$ is uniruled, $\kappa(W) > 0$.

5.) For any map
\[ S^k N_A \otimes F_A \to \Omega^j_A, \]
the determinant of the image is big.

**Proof.** (1) is clear. For (2) we only need to remark that $F_A$ is big; then the proof of (3.7) still works. The bigness of $F_A$ follows from the fact that $f|A : A \dashrightarrow W$ is generically finite, so that it suffices to show that $K_W f(A)$ is big. This in turn follows from the bigness of $K_W$, the fact that the normal sheaf of $f(A) \subset W$ is $\mathbb{Q}$-effective and [PS99, 4.2].

For (3) and (4), consider a general rational curve $C$ in $A$ and first observe that the deformations of $C$ fill up $X$ so that $X$ is uniruled. Then consider the restricted morphism
\[ \kappa_C : S^k N_A|C \otimes F_C \to \Omega^j_A|C \otimes S^i N_A|C. \]
Since $F_A$ is pseduo-effective (3.5) and since $N_A|C$ is ample of $N_A$ is generically ample, $\kappa_C$ must vanish, the dual of $\Omega^j_A|C$ being nef. In case $N_A$ is just $\mathbb{Q}$-effective, we can only conclude that either $\kappa_C = 0$ for general $C$ or does not have any zeroes. Hence if the full map $\kappa \neq 0$, then it can vanish only in codimension 2. Then consider a general curve $B$ so that $\kappa_B$ does not have zeroes. This is only possibly when $F_B$ is not ample which implies $F \equiv 0$, hence $\kappa(W) = 0$.

(5) finally is a direct consequence from the proof of (3.7)(1). □

The critical point in the proof of Theorem 3.7 where we use the bigness of $N_A$ is the investigation of a line bundle $L \subset \bigwedge^r \Omega^j_A$. If say $N_A$ is generically ample, then this line bundle has the property that
\[ L \cdot C > 0 \]
for all movable curves $C$. Recall that a curve is movable, if the deformations of $C$ fill up the whole variety. In general, it is not true that a line bundle with this property is big. However the following should be true.
3.9. Conjecture. Let $A$ be a projective manifold, $\mathcal{L} \subset \bigwedge^r \Omega^1_A$ a line bundle such that $\mathcal{L} \cdot C > 0$ for all movable curves. Then $\mathcal{L}$ is big, in particular $A$ is of general type.

Even if $L = K_A$, this conjecture is unknown in dimension at least 4. However:

3.10. Proposition. Assume that $A$ admits a good minimal model $A'$, i.e., some multiple $mK_{A'}$ is spanned. Then Conjecture 3.9 holds in the weak form that $A$ is of general type.

Proof. It is easily checked that the line bundle $L$ induces a Weil divisor $L'$ on $A$ which is $\mathbb{Q}$-Cartier such that $L' \subset \bigwedge^r \Omega^1_{A'}$. This last sheaf is by definition the reflexive extension of the corresponding sheaf on the smooth part. Notice that the existence of $L$ forces $A$ to be non-uniruled.

Consider the exact sequence

$$0 \to L' \to \bigwedge^r \Omega^1_{A'} \to Q \to 0.$$ 

Then $\det Q$ is generically nef by [CP04,1.5]. Taking determinants, we obtain

$$NK_{A'} = L' + \det Q$$

for a suitable positive number $N$. Thus $K_{A'} \cdot C > 0$ for all movable curves. Since some multiple of $K_{A'}$ is spanned, $A$ must be of general type. □

3.11. Corollary. Assume $A \subset X$ not of general type and $\dim A \leq 3$. If $N_A$ is $\mathbb{Q}$-effective and generically ample with $\det N_A$ big, or, then Theorem 3.7 holds.

3.12. Remark. (1) The proof of (3.7) shows that $\kappa(W) \leq \kappa^+(A)$ (with the assumption that $A$ is not in the indeterminacy locus of $f$ in case $N_A$ is not almost ample).

(2) Via the conjectural equality $\kappa^+(X) = \kappa(W)$, one would arrive at

$$\kappa^+(X) \leq \min(\dim A, \kappa^+(A)).$$

(3) There should be a version of (3.7) in the case $\kappa(\det N_A) > \kappa(A)$.

(4) Everything in this section works also if $X$ and $A$ have canonical singularities and $\dim(A \cap \text{Sing}(X)) \geq 2$.

3.13. Example. Let $X$ be a projective manifold and $C \subset X$ an elliptic curve with ample normal bundle. Then $\dim W \leq 1$, i.e. $\kappa^+(X) \leq 0$. If $X$ is not rationally connected, i.e. $\dim W = 1$, then the rational quotient $X \to W$ is holomorphic and $C$ is multi-section. In particular $W$ is elliptic and $q(X) = 1$.

We will generalize that in the following section.

4. Computing invariants

In this section we compare the fundamental group and the spaces of $p$–forms for $A$ and $X$, when the normal bundle $N_A$ is almost ample and $\mathbb{Q}$–effective. If $N_A$ is ample, a theorem of Napier-Ramachandran [NR98] says that, given a submanifold $A \subset X$ with ample normal bundle, then the image of $\pi_1(A) \subset \pi_1(X)$ has finite index (instead of ampleness it actually suffices to assume finiteness of formal cohomology in degree 0 along $A$ of locally free sheaves; but the relation of this property to almost
ampleness is unclear). We find Napier-Ramachandran’s theorem again if \( A \) is not of general type, but we can also weaken the ampleness assumption. Furthermore we can deal with holomorphic forms of any degree.

4.1. **Theorem.** Let \( X \) be a projective manifold, \( A \subset X \) a submanifold not of general type. Suppose that \( N_A \) is big, \( \mathbb{Q} \)-effective and almost ample (or that \( N_A \) is \( \mathbb{Q} \)-effective, almost ample and one of the conditions of (3.8) or (3.11) holds). Then

1.) The image of the canonical map

\[
\pi_1(A) \to \pi_1(X)
\]

has finite index in \( \pi_1(X) \); the index is at most the number of connected components of the general fiber of \( A \) over the rational quotient \( W \).

2.) \( h^0(\Omega^q_X) \leq h^0(\Omega^q_A) \) for all \( q \geq 1 \).

**Proof.** (1) We are using the followig basic fact: if \( f : X \to Y \) is a surjective holomorphic map between normal compact complex spaces, then the image of \( f^* : \pi_1(X) \to \pi_1(Y) \) has finite index in \( \pi_1(Y) \) \cite{Ca91,1.3}, where the index is also computed. Since \( X \) is smooth, then the same still holds for meromorphic dominant \( f \) (blow up \( X \) to make \( f \) holomorphic; this does not change \( \pi_1 \)).

We apply this to the rational quotient \( f : X \to W \) to conclude via 3.6 and 3.7 that the image of

\[
\pi_1(A) \to \pi_1(W)
\]

has finite index. Now the claim follows from the isomorphism

\[
\pi_1(X) \simeq \pi_1(W)
\]

\cite{Ko93,5.2}.

(2) This follows immediately by considering a holomorphic model of \( f|A \) and by observing \( h^0(\Omega^q_X) = h^0(\Omega^q_W) \), the fibers of \( f \) being rationally connected. \( \square \)

We are now applying (4.1) to special cases.

4.2. **Corollary.** Let \( A \subset X \) be an abelian variety embedded in a projective manifold \( X \). Suppose that \( N_A \) is almost ample, \( \mathbb{Q} \)-effective and big.

1.) \( q(X) \leq \dim A \) and the Albanese map is surjective.

2.) The image of

\[
\pi_1(A) \to \pi_1(X)
\]

has finite index in \( \pi_1(X) \) and therefore \( \pi_1(X) \) is almost abelian.

**Proof.** (1) is a direct consequence of the almost ampleness of \( N_A \) while (2) is clear from (4.1). \( \square \)

4.3. **Corollary.** If \( A \subset X \) is a simply connected submanifold not of general type with almost ample, \( \mathbb{Q} \)-effective and big normal bundle, then \( \pi_1(X) \) is finite.

4.4. **Remark.** In (4.1) - (4.3) the assumption on almost ampleness is only used to make sure that \( A \) is not contained in the indeterminacy locus of the rational quotient. Thus we can omit “almost ampleness” by requiring that \( A \) is not contained in the indeterminacy locus.
If \( A \subset X \) is a submanifold whose normal bundle is almost ample without assumption on the Kodaira dimension or, more generally, generically ample (possible even with the generic spannedness assumption replaced by a generic nefness assumption), then one still would expect that the image of \( \pi_1(A) \to \pi_1(X) \) has finite index. Here is a partial result in terms of the Shafarevitch map [Ko93],[Ca95]

\[
\text{sh} : X \to \text{Sh}(X),
\]

which is almost holomorphic.

4.5. Proposition. Let \( X \) be a projective manifold. Let \( A \subset X \) be a submanifold or a local complete intersection. Suppose that

1. \( N_A \) is almost ample, or that
2. \( A \) contains a local complete intersection \( B \) such that \( N_{B/X} \) is ample.

If \( \dim A \geq \dim \text{Sh}(X) \), resp. \( \dim B \geq \dim \text{Sh}(X) \), then the image of \( \pi_1(A) \to \pi_1(X) \) has finite index.

Proof. We use the following fact which was communicated to me by F.Campana:

(*) If \( \text{sh}|A \) dominates \( \text{Sh}(X) \), then the image of \( \pi_1(A) \) in \( \pi_1(X) \) has finite index in \( \pi_1(X) \).

To prove (*), let \( h : \tilde{X} \to X \) be the universal cover and \( \tilde{A} = h^{-1}(A) \). It suffices to show that \( \tilde{A} \) has only finitely many irreducible components. Consider a general fiber \( F \) of \( \text{sh} \) and let \( \tilde{F} = h^{-1}(F) \). Then \( \tilde{F} \) is compact. Now every irreducible component \( \tilde{A}_i \) of \( \tilde{A} \) dominates \( A \), hence \( \tilde{A}_i \) meets \( F \). Since \( \tilde{A} \cap \tilde{F} \) has only finitely many components, we conclude (*).

Now an application of (3.7) yields (1).

(2) By (3.7) \( \text{sh}|B \) dominates \( \text{Sh}(X) \), hence \( \text{sh}|A \) dominates \( \text{Sh}(X) \), and we can apply (*).

\[ \square \]

Of course, this gives nothing when \( \dim X = \dim \text{Sh}(X) \), i.e. when there is no compact subvariety of positive dimension through the very general point of the universal cover \( X \). In that case \( X \) is of general type or \( \chi(O_X) = 0 \) [CP04].

5. A CRITERION FOR RATIONAL CONNECTIVITY

The following well-known conjecture is due to Mumford.

5.1. Conjecture. Let \( X \) be a projective manifold. Assume that

\[
H^0(X, (\Omega_X^1)^{\otimes m}) = 0
\]

for all \( m \in \mathbb{N} \). Then \( X \) is rationally connected.

This is slightly weaker than the conjecture that \( \kappa^+(X) = -\infty \) implies rational connectedness (see (2.2)). In fact, \( H^0(X, (\Omega_X^1)^{\otimes m}) = 0 \) implies that

\[
H^0(X, \Gamma(\Omega_X^1)) = 0
\]

for all tensor representations \( \Gamma \) and in particular \( \kappa^+(X) = -\infty \). In this direction we prove

5.2. Theorem. Let \( X \) be a projective manifold and \( L \) be a big line bundle on \( X \). If

\[
H^0(X, ((\Omega_X^1)^{\otimes m} \otimes L)^{\otimes N}) = 0
\]

for all \( m, N \gg 0 \), then \( X \) is rationally connected.
Proof. The condition (*) implies in particular that
\[ H^0(N(mK_X + L)) = 0 \]
for \( m, N \gg 0 \). Hence \( K_X \) is not pseudo-effective, because otherwise \( mK_X + L \) would be big. Therefore \( X \) is uniruled by [BDPP04]. Let \( f : X \to W \) be the rational quotient (\( W \) smooth) and suppose \( \dim W > 0 \). By eventually blowing up \( X \), we clearly may assume that \( f \) is holomorphic. Choose a big line bundle \( L' \) on \( W \) and choose \( k \) so large that \( kL - f^*(L') \) is still big. By substituting \( L' \) by a multiple and \( k \) by a multiple, we may assume that \( kL - f^*(L') \) has a section so that
\[ kL = f^*(L') + A \]
with \( A \) effective. By (*) we also have
\[ H^0(N(mK_W + kL')) = 0 \]
for large \( m, N \). Hence by induction \( W \) is rationally connected, contradiction. Actually it is sufficient to notice that \( H^0(N(mK_W + kL')) = 0 \) which shows that \( K_W \) is not pseudo-effective, hence \( W \) is uniruled, contradiction. □

Of course, if \( X \) is rationally connected, then
\[ H^0(X, (\Omega^1_X)^{\otimes m} \otimes kL)^{\otimes N}) = 0 \]
for \( m, N \gg 0 \) and \( N \in \mathbb{N} \). Just restrict to rational curves on which \( T_X \) is ample.

5.3. Remark. By the same method one shows easily the following. If for every \( p \) and every \( F \subset \Omega^p_X \) one has
\[ H^0(N(m \det F + L)) = 0 \]
for some fixed big line bundle and every \( m, N \), i.e. if \( \det F \) is not pseudo-effective, then \( X \) is rationally connected.

5.4. Corollary. Let \( X \) be a projective manifold, \( C \subset X \) a (possibly singular) curve. If \( T_X|C \) is ample, then \( X \) is rationally connected.

Proof. We are going to verify (*) in (5.2). Fix a big line bundle \( L \) and choose \( m_0 \) so large that
\[ (\Omega^1_X)^{\otimes m_0} \otimes L|C \]
is negative. Set
\[ E = (\Omega^1_X)^{\otimes m} \otimes L)^{\otimes N} \]
with \( m \geq m_0 \) and \( N \in \mathbb{N} \). Let \( \hat{C} \) be the formal completion of \( X \) along \( C \). Then
\[ H^0(E) \subset H^0(E|\hat{C}) \subset \bigoplus_{k \geq 0} H^0(E \otimes I^k/I^{k+1}). \]
Here \( I \) denotes the ideal sheaf of \( C \). Since \( \Omega^1_X|C \) is negative, the sheaves \((I^k/I^{k+1})/\text{torsion}\) are negative; in fact, the canonical map
\[ I^k/I^{k+1} \to S^k(\Omega^1_X|C) \]
is generically injective.
Hence any section in
\[ H^0(C, \mathcal{T}^k/\mathcal{T}^{k+1} \otimes \mathcal{E}) \]
is a torsion section, supported at most in the singular locus of \( C \). Thus if we take \( s \in H^0(\mathcal{E}) \), then the restriction to the \( k-\)th infinitesimal neighborhood is generically 0. Since this holds for all \( k \), we conclude that \( s = 0 \). This shows (*) \( \square \)

Similarly we prove:

5.5. **Proposition.** Let \( X \) be a projective manifold and suppose that \( T_X|_C \) is nef for some (possibly singular) curve \( C \).
(1) If \( -K_X \cdot C > 0 \), then \( X \) is uniruled;
(2) if \( -K_X \cdot C = 0 \), then \( \kappa(X) \leq \dim X - 1 \).

**Proof.** We adapt the proof of (5.4) and substitute \( \Omega^1_X \) by \( K_X \). We obtain for all \( m \) and all line bundles \( L \):
\[
h^0(mK_X - L) \leq h^0(mK_X - L|_{\hat{C}}) \leq \sum_k h^0(mK_X - L \otimes (\mathcal{T}^k/\mathcal{T}^{k+1})/\text{tor}).
\]
For \( L = \mathcal{O} \) we obtain (1), and by plugging in some ample \( L \), we deduce that \( h^0(mK_X - L) = 0 \) for all \( m \) so that \( K_X \) cannot be big. \( \square \)

5.6. **Remark.** Assume that
\[ H^0((\Omega^1_X)^{\otimes m}) = 0 \] for all positive \( m \). Suppose that \( K_X \) is pseudo-effective. *If \( K_X \) carries a metric with algebraic singularities* \([DPS01,2.14]\), then either \( \chi(\mathcal{O}_X) = 0 \) - which is excluded by (*) - or
\[ H^0(X, \Omega^q_X \otimes mK_X) \neq 0 \]
for some \( q \) and infinitely many \( m \). This also contradicts (*). Consequently if (*) holds, then \( K_X \) cannot have a metric with algebraic singularities of non-negative curvature. In general, without the assumption on the metric (but still assuming \( K_X \) to be pseudo-effective, of course), one can only conclude that \( \chi(X, \mathcal{O}((m+1)K_X \otimes \mathcal{I}(h_m))) = 0 \)
for all \( m \geq m_0 \) and all (singular) metrics \( h_m \) on \( mK_X \) with non-negative curvature (current). Actually all cohomology groups vanish:
\[ H^0(X, \mathcal{O}((m+1)K_X) \otimes \mathcal{I}(h_m)) = 0. \]

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