Can spacetime geometry gives us spinors?

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Abstract. We live in a space, where the line element in a $n$-dimensional space-time is given by $(dx_0^2 - \sum_{i=1}^{n-1} dx_i^2)^{\frac{1}{2}}$. Based on this geometry of space-time, the general theory of relativity provides a complete geometric theory of gravity. However, it does not explain the other three forces of nature, i.e. electromagnetism, weak and strong interactions. We require the quantum field theory (QFT) to explain those forces. In quantum field theory we consider two kinds of particles, the Fermions, which are the spin half particles and the Bosons which are the integer spin particles and act as force carrier in QFT. No Fermion has been discovered with any other fractional spin except spin $\frac{1}{2}$ or $-\frac{1}{2}$. In this article, we define a parametric coordinate system in the tangent space of a Minkowski space, and show that these parametric coordinates can be assigned a tensor weight $\frac{1}{2}$. In fact, in a space where the line element is given by $(dx_0^2 - \sum_{i=1}^{n-1} dx_i^2)^{\frac{1}{2}}$, we can define similar parametric coordinate system in the tangent space that can be assigned a tensor weight $\frac{1}{p}$. We define 8 sets of such parametric coordinate systems, show how the parametric coordinates transform under different gauge transformations and how can these coordinates be used to represent different Fermions.
1 Introduction

“Philosophy is to be studied, not for the sake of any definite answers to its questions, since no definite answers can, as a rule, be known to be true, but rather for the sake of the questions themselves; because these questions enlarge our conception of what is possible, enrich our intellectual imagination and diminish the dogmatic assurance which loses the mind against speculation”- Russell (1912)

The general theory of relativity (GR) is the best known theory of gravity. It is based on two principle assumptions. Firstly, the acceleration of a particle can be understood by the curvature of the space-time. Secondly, in a small enough region of spacetime it is impossible to distinguish between the gravitational field and the acceleration. In other words, the outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime. The second assumption have been challenged by researchers over time. There may be multiple reasons, such as if the gravitational constant varies over space and time, then the above statement can not be true. These possibilities have been explored by Bekenstein (2009); Brans and Dicke (1961); Brownstein and Moffat (2006); Moffat (2006) etc. Also here it is assumed that the initial mass and the passive gravitation mass of a particle are same. Even if, $G$ is constant, and the ratio between the inertial and the passive gravitation masses vary over spacetime, then the second assumption does not hold true. In fact, the inertial mass of a particle is defined based on
the inertial properties of the particles. On the other hand the gravitational mass is defined either based on its ability of influence other particles gravitationally (active gravitational mass) or based on its susceptibility or response in some gravitational field (passive gravitational mass). A detailed discussion of the inertial and the gravitational mass can be found in Das (2012a,c); Jammer (2009). Therefore, there are no obvious reason to believe that the masses are same.

In Das (2012a,b,c), we consider the possibility that the ratio of the inertial mass and passive gravitational mass varies over the space-time. We introduced a 5-dimensional line element that takes the form $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - \left(\frac{\ell}{2}\right)^2 d\zeta^2$. We proposed a new theory of gravity and had shown that using this 5 dimensional line element we can explain the galactic velocity profile, galaxy cluster mass profile and cosmic expansion history etc. However, the geometric approach is limited to the gravity. Other three forces of nature are not explained by this simple geometric formalism. Instead, for explaining these forces we need the quantum field theory. In particle physics, we break the particles in two main categories. The elementary Bosons consists of photons, gluons and W and Z bosons as the force carrier and the recently discovered scalar particle Higgs Boson. On the other hand, the elementary Fermions consists of quarks and laptons. There exists, three are generations of these particles which are carbon copy of one another, except their mass. In each generation, there are two laptons, such as $e$ and $\nu_e$ for the first generation. There are also two types of quarks for each generations, e.g. $u$ and $d$ quark for the first generations. The quarks can carry 3 types of color charges, namely red green and blue. So independently for the first generation, there are total 6 quarks, 3 $u$s or $u_t$, $u_s$ and $u_b$ and 3 $d$s, namely, $d_t$, $d_b$, $d_s$. Combining with laptons there are total $3 + 3 + 2 = 8$ types of Fermions. Each of these Fermions comes in two different form based of their helicity – right handed and left handed (considering that the right handed neutrinos exist). Each of these particles, also have their anti-particles, giving total $2 \times 2 \times 8 = 32$ forms of particles for each generation of Fermions. In the QFT the Fermions don’t arise naturally. Instead we need to separately add the Fermions into the equations.

In general relativity a space-time is a Lorentzian manifold $(\mathcal{M},g)$, i.e. a smooth (infinitely differentiable) manifold $\mathcal{M}$ with a globally defined tensor field $g : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$. $T\mathcal{M}$ is the tangent space on $\mathcal{M}$. Suppose $X \in T\mathcal{M}$, then according to the general theory of relativity, the quantity $g_{ab}X^aX^b, \forall a,b = 0,..,3$ remains constant under the change of the coordinate system. This is a quadratic relation and this is how the line element in our universe behaves. A quadratic curve consists of two sets of straight lines. In this article, we parameterize the set of straight lines on a quadratic curve using 2 complex numbers, which we call the parametric space coordinate system. We show that we can assign a tensor weight $\frac{1}{2}$ to these coordinates and they behave as massless spinors. One interesting aspect is to note that there is no apparent mathematical reason why we can’t mathematically consider an $L^p$ space and a tensor field $g : (T\mathcal{M})^p \to \mathbb{R}$, such that $g_{a_1a_2...a_p}X^{a_1}X^{a_2}...X^{a_p}$ remain unchanged under any type of coordinate transformation. In this article, we show that a similar complex parametric space coordinate system defined on the tangent space of an $L^p$ space, can be assigned a tensor weight of $\frac{1}{2}$.

Our parametric coordinate system may be used to represent Fermions. There are total 3 color charges in quantum chromodynamics (QCD) and there are also 3 spatial dimensions in our coordinate system. Therefore, one can speculate if these color charges can somehow be related to the spatial dimensions. In this article, we introduce an internal complex $S^3$ space on the tangent-space of a null manifold. This allows us to define total 8 independent sets of such parametric-space coordinate systems. Among these 8 set of parametric space coordinates, we get two triplets which can rotate within itself under SU(3) transformation. 7 sets of these parametric space coordinate system can couple with an $U(1)$ field with different coupling strength – one of them couple with a unit strength, 3 others from one triplet can couple with $\frac{1}{2}$ of the strength and rest 3 from the other triplet couple with the same $U(1)$ field with a strength of $\frac{1}{2}$.

This article is organized as follows. In Sec. 2, we discuss about the system of linear equations. In the third section we define the new parametric space coordinate system over the tangent space of a 4-dimensional null hyper-surface, based on the work of Fock and Veblen Fock (1929); Goenner (2004, 2014); Veblen (1933a,b). In the next two sections, we describe the co-variant derivative, parallel transport, additional degrees of freedom etc. In Sec. 6.1, we define 8 different sets of parametric space
coordinate systems and discuss their inter transformations through gauge rotation. Finally we have discussion and conclusion. In the appendix we show how a similar analysis can be done on an $L^p$ space that can lead to a spin $\frac{1}{p}$ system. We also provide appendices for helping the reader better understand the rotation in the complex internal space and visualise SU(3) rotation.

2 The system of linear equations

Any quadratic can be expressed as system of linear equations. Most basic example is a circle,

$$X^2 + Y^2 = K^2.$$  \hfill (2.1)

$K \in \mathbb{R}$ is the radius of the circle. As $(K+X)(K-X) = Y^2$, this can be written as a pair of straight lines as

$$X = K + \lambda Y, \quad X = K - \frac{1}{\lambda} Y.$$  \hfill (2.2)

Here $\lambda$ is a parameter. Given any $\lambda \in \mathbb{R} - (0, \infty, -\infty)$, we get one line from each of the set, which meet at a point, $P(\lambda)$, the locus of which gives the circle.

Provided $K$ is a variable, lets say $K = T$, the circle becomes a cone, i.e. $X^2 + Y^2 = T^2$. In this case, each of the system of linear equations represent a plane through the origin. These planes cut each other at a straight line and these systems of straight lines together create the cone. Therefore, any point $\lambda$ represents a unique straight line on the cone.

Lets consider a hyperboloid by adding some constant term, $K$, with the cone

$$X^2 + Y^2 - T^2 = K^2.$$  \hfill (2.3)

Here the lines are difficult to visualize at the first place, as usually, the intersection of a plane with a quadratic yields a curve. However, that is not the case here and the hyperboloid indeed contain straight lines. For instance, the line $(K, 0, 0) + \lambda(0, 1, 1)$ is on the hyperboloid and it can be tested by substituting the point $(K, \lambda, \lambda)$ on the hyperboloid. In Fig. 1, we show the system of straight lines

![Figure 1](image-url)

Figure 1. Figure shows that a hyperboloid can be represented by the set of two straight lines. On the right, there are two systems of straight lines, one is given in blue and the other is shown in red. None of two lines of the same system meet each other and any two lines from opposite system meets each other at one and only one point. Therefore, these system of straight lines can be used as a coordinate system on the hyperboloid.
on the hyperboloid, which consists of two sets of lines. To check it mathematically we can rewrite the equation as
\[(X + T)(X - T) = (K + Y)(K - Y).\] (2.4)
For a given number, say \(\mu\), the planes \(X + T = \mu(K + Y)\) and \(\mu(X - T) = (K - Y)\) cut each other on the hyperboliod, giving one sets of lines. To get the other set of lines we can consider another parameter \(\lambda\). The planes \((X + T) = \lambda(K - Y)\) and \(\lambda(X - T) = (K + Y)\) produces another sets of lines on the hyperboliod. No two lines of same group intersect with each other and any two lines of the opposite groups intersect in one and only one point. Therefore, any set \((\mu, \lambda)\) represents a particular point \(P_2(\lambda, \mu)\), on the hyperboliod.

If we allow to vary \(K\), lets say \(K = Z\), the set of straight lines, becomes set of planes. and the point \(P_2(\lambda, \mu)\) now becomes a straight line. All these lines meet each other only at the origin and they span the entire space. For any line we have a unique \((\lambda, \mu)\), and the location on that straight line can be given by the value of \(Z\).

Here we should note that, if we take an hyperboliod of the form \(T^2 = X^2 + Y^2 + Z^2\), then the system of planes become imaginary, however the lines still remain real, as \((T, X, Y, Z)\) are real. The entire space is spanned by the series of straight lines, each of which can be parameterised by two parameters \((\lambda, \mu)\). Note that this \(\lambda\) and \(\mu\) may be real or complex numbers.

### 3 Mathematical setup for tangent space and parametric space coordinates

#### 3.1 Tangent space coordinate system

Consider a 4-dimension null spacetime manifold, \(\mathcal{M}\), represented by the coordinate system \((x^0, x^1, x^2, x^3)\) and a line element is given by \(g_{\mu\nu}dx^\mu dx^\nu = 0\) on the manifold, where \(\mu, \nu \in (0, \ldots, 3)\). At each point, \(\mathcal{P}\), of this manifold there is a tangent space \(T\mathcal{M}\), whose coordinates are \(\frac{dx^\mu}{dx}\). In tangent space, this coordinates system follows the quadratic \(g_{\mu\nu}\frac{dx^\mu}{dx} \frac{dx^\nu}{dx} = 0\). \(\lambda\) is an affine parameter on the null hyperspace. The tangent space is a flat space. Hence, we can define a Minkowski coordinate system on the tangent space, given by \((X^0, X^1, X^2, X^3)\). Therefore, at any point \(\mathcal{P}\), on the manifold if we transform the coordinate system \(\frac{dx^\mu}{dx} \rightarrow X\), then these new coordinates should follow the quadratic (conic section)

\[
(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 = 0,
\] (3.1)
which can also be written as \(\eta_{ab}X^aX^b = 0\), where \(\eta_{ab}\) is the Minkowskian metric, and \(a, b \in (0, \ldots, 3)\) are the indices to represent the coordinate system on the tangent space.

The transformation from the \(\frac{dx^\mu}{dx}\) coordinate system to \(X^a\) coordinate can be written as

\[
X^a = \Lambda^a_\mu \frac{dx^\mu}{dx}\quad \text{and hence} \quad \eta_{ab}\Lambda^a_\mu \Lambda^b_\nu = g_{\mu\nu}.
\] (3.2)
For the shake of mathematical benefit we can define

\[
\Lambda^{\mu a} = \Lambda^a_\mu = g^{\mu\nu}\Lambda^a_\nu \quad \text{and} \quad \Lambda_{a\mu} = \eta_{ab}\Lambda^b_\mu.
\] (3.3)
This also gives

\[
\Lambda^{\mu a} \Lambda_{\nu a} = g^{\mu\lambda}\Lambda^a_\lambda \eta_{ab}\Lambda^b_\nu = g^{\mu\lambda}g_{\lambda\nu} = \delta_\mu^\nu
\] (3.4)
and

\[
\Lambda^a_\nu = \Lambda^a_0 \delta^\nu_0 \quad \text{and} \quad \Lambda^a_\mu = \Lambda^a_0 \delta^\mu_0.
\] (3.5)
Here we are free to assume any coordinate transformation in the \(x^\mu\) reference frame. It does not change the tangent space. For any transformation in \(x^\mu\), each \(a\) component of \(\Lambda^a_\mu\), i.e. \(\Lambda^a_0, \Lambda^a_1, \Lambda^a_2, \Lambda^a_3\) transforms as contravariant vectors and each \(a\) component of \(\Lambda^a_\mu\) transforms as a co-variant vector. On the other hand the tangent space coordinates can be subjected to any Lorentz transformation

\[
X^a = L^a_b X^b \quad \text{and hence} \quad \tilde\Lambda^a_\mu = L^a_b \Lambda^b_\mu
\] (3.6)
3.2 Parameterizing the tangent space

For parameterizing the tangent space coordinates we follow the discussion from Sec. 2. We start with Eq. 3.1, which can be written in the determinant form as

\[
\begin{vmatrix}
X^0 + X^3 & X^1 + iX^2 \\
X^1 - iX^2 & X^0 - X^3
\end{vmatrix} = 0. \tag{3.7}
\]

The determinant of a 2 \times 2 matrix is zero, if and only if its two rows are equal up to some scaling factor. Therefore, if the determinant has to vanish then we must have variables like \(\psi^1, \psi^2, \psi^3, \psi^4\), such that

\[
X^0 + X^3 = \psi^1 \psi^3, \quad X^1 + iX^2 = \psi^1 \psi^4, \\
X^1 - iX^2 = \psi^2 \psi^3, \quad X^0 - X^3 = \psi^2 \psi^4. \tag{3.8}
\]

\(\psi^1, \psi^2, \psi^3, \psi^4\) are complex variables.

In Sec. 2 we discuss about two systems of planes, describe by two sets of linear equations, which are parameterize by two parameters, \(\lambda\) and \(\mu\). For our parameterization in Eq. 3.8, the first sets of planes can be obtained by the intersection of the linear equations \((X^1 + iX^2) = (\psi^1/\psi^3)(X^0 + X^3)\) and \((X^0 - X^3) = (\psi^4/\psi^3)(X^1 - iX^2)\). If \(\psi^3\) and \(\psi^4\) are multiplied by the same factor, this plane remains unaltered. There is one and only one such plane for each value of this ratio \(\psi^3/\psi^4\) and if the ratio changes the plane changes. Similarly, the other system of linear equations on the cone, i.e. \((X^1 + iX^2) = (\psi^1/\psi^2)(X^0 - X^3)\) and \((X^0 + X^3) = (\psi^2/\psi^1)(X^1 + iX^2)\) is represented by the value of \(\psi^1/\psi^2\). Each plane from system cut each plane of the other at a straight line. These system of straight-lines span the entire space given by Eq. 3.7. Therefore, \(\psi^1, \psi^2\) and \(\psi^3, \psi^4\) give a parametric representation of the tangent space light cone.

As \(\psi^1, \psi^2, \psi^3, \) and \(\psi^4\) are complex quantities, we have total 8 degrees of freedom, i.e. 5 extra degrees of freedom from the original equation, which has 4 variables and one constraints. Imposing the reality condition, i.e. \((X^0 + X^3)\) and \((X^0 - X^3)\) are real, we should get

\[
\psi^*^3 = k\psi^1 \quad \text{and} \quad \psi^*^4 = k\psi^2, \tag{3.9}
\]

where we are using the * to indicate complex conjugates and \(k\) is a real number. The effect of changing the value of \(k\) is shifting the point \(X^a\) along the same line on the light cone. Hence, no generality is lost by assuming \(k = 1\). The points \(X^a\)'s on the light cone are given by

\[
\begin{align*}
X^1 &= \frac{\psi^1 \psi^*^2 + \psi^2 \psi^*^1}{2}, \\
X^3 &= \frac{\psi^1 \psi^*^1 - \psi^2 \psi^*^2}{2}, \\
X^2 &= \frac{\psi^1 \psi^*^2 - \psi^2 \psi^*^1}{2i}, \\
X^0 &= \frac{\psi^1 \psi^*^1 + \psi^2 \psi^*^2}{2}.
\end{align*} \tag{3.10}
\]

This reduces the degrees of freedom to 4. We still have one extra degrees of freedom, which we will discuss in detail in Sec. 5 of this article. Each point on the tangent space \(TM\), can be parameterized by the quantity \(\psi^a\), where \(A \in (1, 2)\). In this article we refer \(\psi^A\) as parametric space coordinate. The indices are referred as spin index and we use the capital roman letters, \(A, B, \ldots\) to represents the spin indices of \(\psi^A\).

The transformation relation between \(\psi^A\) and \(X^a\) coordinate system can be written as

\[
X^a = G^a_{AB} \psi^A \psi^B, \tag{3.11}
\]

where \(G^a_{AB}\) are the matrices

\[
G^0_{AB} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G^1_{AB} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G^2_{AB} = \frac{1}{2i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad G^3_{AB} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.12}
\]
We can also write a reverse transformation as

\[ \psi^A \psi^B = G_a^{AB} X^a. \] (3.13)

Here \( G_a^{AB} \) are the matrix inverse of the matrices in Eq. 3.12. This leads us to the relations

\[ G_a^{AB} G_b^{CD} = \delta_a^b \quad g^{a} A^{B} = \delta^{A} C^{B} \] (3.14)

We must note that \( G_a^{AB} \) are not symmetric. The first index is used for the non-starred coordinates and the second index is used for the starred coordinates.

### 3.3 Transformation laws for parametric space coordinates

If we subject \( \psi^1 \) and \( \psi^2 \) to a linear transformation then we get

\[ \bar{\psi}^A = T_A^B \psi^B. \] (3.15)

It can be easily verified that \( \bar{\psi}^A \) also satisfy the quadratic relation given by Eq. (3.7). However, the equation gets multiplied by the square of the determinant

\[ T = |T_A^B|. \] (3.16)

We may consider the Eq.(3.15) as a transformation of the parametric coordinates \( \psi^A \) into new coordinates \( \bar{\psi}^C \), i.e. as a change of reference system. Under such linear transformation, \( X^a \)'s also undergo a linear transformation which can be verified by simple algebraic manipulation.

For providing \( \psi^A \)s with a tensor like structure, we define their covariant counterparts as \( \psi_A \). Subjected to the transformation of Eq. 3.15, these covariant components also go through a covariant transformation as

\[ \bar{\psi}_A = t_B^A \psi_B \quad (A, B = 1, 2) \quad \text{where,} \quad t_A^B t_C^D = \delta_A^C. \] (3.17)

Let us define a quantity \( \epsilon_{AB} \), that can lower the index of \( \psi^A \). The inverse of this matrix \( \epsilon^{AB} \) can be used to raise the indices. As we use \( \epsilon_{AB} \) to lower the indices, in the barred and unbarred reference frame it should satisfy

\[ \epsilon_{AB} \psi^A \psi^B = \bar{\epsilon}_{CD} \bar{\psi}^C \bar{\psi}^D = \epsilon_{CD} T_A^C T_B^D \psi^A \psi^B, \quad \text{giving,} \quad \epsilon_{AB} = \bar{\epsilon}_{CD} T_A^C T_B^D. \] (3.18)

Provided we take \( \epsilon^{AB} \), \( \epsilon_{AB} \) to be the Levi-Civita symbols, i.e.

\[ \epsilon^{11} = \epsilon^{22} = 0, \quad \epsilon^{12} = -\epsilon^{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon_{12} = -\epsilon_{21} = 1, \] (3.19)

and we demand that \( \epsilon_{AB} \) are invariant under such linear transform, then the left hand side of Eq. 3.18 get multiplied with \( T \), i.e. we get \( \epsilon_{AB} = \epsilon_{CD} T^{11} T^{22} \). If we take \( t = |T_A^B| \) then we can write the equations in terms of \( t \) as

\[ \epsilon_{CD} = \frac{1}{t} \epsilon_{AB} t_A^C t_B^D, \quad \epsilon^{CD} = t \epsilon^{AB} T_A^C T_B^D. \] (3.20)
Therefore, $\epsilon_{AB}$ and $\epsilon^{AB}$ behave as a covariant tensor density of weight $-1$ and contravariant tensor density of weight 1 respectively. Hence, raising the indices by $\epsilon^{AB}$ increases the weight by +1 and lowering indices by $\epsilon_{AB}$ decreases it by $-1$. If we assign $\psi^A$ to be of weight $+\frac{1}{2}$ then $\psi_A$ must be of weight $-\frac{1}{2}$. Assuming that $\psi^A$ are of weight $+\frac{1}{2}$, linear transformation of the $\psi^A$-coordinate can be re-written as

$$\tilde{\psi}^A = t^2 T^A_B \psi^B.$$ (3.21)

This transformation has an unit determinant and does not multiply the light-cone (given by Eq. 3.1) by $T^2$. In this article, we refer such transformation as the spin-space coordinate transformation or simply the spin transformation.

Raising and lowering indices works as follows

$$\epsilon_{BA} \psi^B = \psi_A, \quad \epsilon^{AB} \psi_B = \psi^A$$ (3.22)

which implies

$$\psi_1 = -\psi^2, \quad \psi_2 = \psi^1.$$ (3.23)

Here one should note that $\epsilon$ is not a symmetric metric. We use the first index to raise the indices and the second index to lower the indices. This raising and lowering the indices and the choice of $\epsilon_{AB}$ are purely mathematical construct for simplifying calculations.

We can also define such parametric space coordinates of spin $\frac{1}{p}$ in the tangent space of an $L^p$ space. Check Appendix A for detail discussion.

### 3.4 An illustrative example

Suppose at any point $P$ on a null manifold $\mathcal{M}$, the line element is given by $g_{\mu\nu} dx^\mu dx^\nu = 0$ and $\lambda$ is an affine parameter on the null manifold. The 4-velocity of a particle on the manifold is given by $\frac{dx^\mu}{d\lambda}$.

If the tangent space coordinates at $P$ are given by $X^a$, then these coordinates actually represent the four-velocity of the particle at $P$ and not the real space-time coordinates. We relate $X^a$'s with the 4-velocities as $X^a = \Lambda^a_\mu \frac{dx^\mu}{d\lambda}$, such that $X^a$'s satisfy Eq. 3.1 and the curvature of the spacetime gets absorbed in $\Lambda^a_\mu$. Therefore, these tangent space coordinates are free from any gravitation effect. We define the parametric space coordinate system, i.e. $\psi^A$, on the tangent space and hence they are also free from any spacetime curvature at that point.

As $\psi^A$ are complex numbers, we can write them as $\psi^1 = |\psi^1|e^{i\theta^1}$ and $\psi^2 = |\psi^2|e^{i\theta^2}$, where $|\psi^A|$ are the amplitudes and $\theta^A$ are the arguments of the complex quantities. This gives

$$X^1 = |\psi^1||\psi^2| \cos(\theta^1 - \theta^2), \quad X^2 = |\psi^1||\psi^2| \sin(\theta^1 - \theta^2),$$

$$X^3 = \frac{1}{2} (|\psi^1|^2 - |\psi^2|^2), \quad X^0 = \frac{1}{2} (|\psi^1|^2 + |\psi^2|^2).$$ (3.24)

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1 A tensor density or relative tensor is a generalization of tensor. A tensor density transforms as a tensor when coordinate system transformation except that it is additionally multiplied by a power $w$ of the Jacobian determinant of the coordinate transition function.

Considering an arbitrary transformation from a general coordinate system to another, a relative tensor of weight $w$ is defined by the following tensor transformation:

$$\hat{\Lambda}^{ab...c}_{lm...n} = \frac{\partial x^a}{\partial \bar{x}^l} \frac{\partial x^b}{\partial \bar{x}^m} \frac{\partial x^c}{\partial \bar{x}^n} \frac{\partial \bar{x}^l}{\partial x^m} \frac{\partial \bar{x}^m}{\partial x^n} \Lambda^{ab...c}_{lm...n}.$$}

This is useful in calculating area or volume. We can get an example from the cross product

$$\bar{u} \times \bar{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_2 - u_2 v_1.$$ This is an invariant, because it represents the area of a triangle, which should not change due to the change in coordinate system. Now let under the change of coordinate system, $\tilde{u} = A^{-1} u$ and $\tilde{v} = A^{-1} v$. Therefore, under such coordinate transformation $(A^{-1})^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} A^{-1}$ the original expression but multiplied by $\det A^{-1}$. This could be thought of as a two index tensor transformation, but instead, it is computationally easier to think of the tensor densities.
Given $X^a$ we can uniquely determine $|\psi^1|$, $|\psi^2|$ and $(\theta^1 - \theta^2)$. However, we have one degrees of freedom unfixed. Unless we fix one of $\theta^1$ or $\theta^2$, $\psi^A$ can not be determined uniquely. This additional degrees of freedom plays an important role in gauge choice and we discuss about this additional degrees of freedom in Sec. 5.

4 Covariant differentiation of the $\psi$ system

In accordance with our calculation in the previous section, $\psi^A$ behaves as a tensor density of weight $\frac{1}{2}$ and $\psi_A$ as a weight $-\frac{1}{2}$. Hence, co-variant derivative of $\psi^A$ can be written as

$$\psi^A_{;\alpha} = \frac{\partial \psi^A}{\partial x^\alpha} + \Upsilon^A_{\beta\alpha} \psi^B - \frac{1}{2} \Upsilon^A_{\beta\alpha} \psi^A$$

(4.2)

where $\Upsilon^A_{\beta\alpha}$ is a connection parameter. The $\frac{1}{2}$ comes in the second term because of the weight of $\psi^A$.

Under coordinate transformation, $\psi^A_{;\alpha}$ must transform as vector with respect to gauge and the spin transform ( by gauge transform we mean the change of the extra free parameter of the system, which is discussed in Sec. 5).

$$\left( \frac{\partial \bar{\psi}^A}{\partial \bar{x}^\alpha} + \bar{\Upsilon}^A_{\beta\alpha} \bar{\psi}^C - \frac{1}{2} \bar{\Upsilon}^A_{\beta\alpha} \bar{\psi}^A \right) = \frac{1}{2} \Upsilon^B_{\beta\alpha} \psi^D + \frac{1}{2} \Upsilon^D_{\beta\alpha} \psi^A (4.3)$$

Let us consider $\Gamma^A_{\beta\alpha} = \Upsilon^A_{\beta\alpha} - \frac{1}{2} \Upsilon^A_{\beta\alpha} \delta^A_B$. Therefore, in terms of this new variable the covariant derivative of $\Psi^A$ can be written as

$$\psi^A_{;\alpha} = \frac{\partial \psi^A}{\partial x^\alpha} + \Gamma^A_{\beta\alpha} \psi^B$$

(4.4)

If we take the complex conjugate of $\psi^A$, we should get similar equations for the complex conjugates. Therefore, we can write

$$\psi^A_{;\alpha} = \psi^A \left( \frac{\partial \psi^B}{\partial x^\alpha} + \Gamma^B_{\alpha\beta} \psi^C \right) + \left( \frac{\partial \psi^A}{\partial x^\alpha} + \Gamma^A_{\alpha\beta} \psi^C \right) \psi^B$$

(4.5)

We can multiply this equation with $G^a_{AB}$ and use Eq.3.13 and Eq. 3.11 to get

$$X^a_{;\alpha} = \frac{\partial X^a}{\partial x^\alpha} + G^a_{AB} \Gamma^B_{\alpha\beta} \psi^A \psi^C + G^a_{BC} \Gamma^B_{\alpha\beta} \psi^A \psi^C = \frac{\partial X^a}{\partial x^\alpha} + \Gamma^a_{\beta\alpha} \psi^A \psi^B$$

(4.6)

Here $X^a$ are the tangent space coordinates. $\Gamma^a_{\beta\alpha}$ are connection parameters for the covariant derivative of $X^a$ coordinates. Therefore, we can relate the connection parameter of the parametric space with that of the tangent space coordinate system as

$$\Gamma^a_{\beta\alpha} = G^a_{AB} \Gamma^B_{\alpha\beta} \psi^AC + G^a_{BC} \Gamma^B_{\alpha\beta} \psi^AC$$

(4.7)

We know that the tangent space coordinates $X^a$ are related to the coordinate of the space-time manifold, i.e. $dx^\mu$ as $\frac{dx^\mu}{dx^a} = \Lambda^a_\mu X^a$. Therefore, we can relate the Christoffel symbols on the manifold with the connections, $\Gamma^a_{\beta\alpha}$ as

$$2$$

\[ A^1_q = \partial_q A^1 + \Gamma^1_{aq} A^a - w A^1 \Gamma^1_{aq} , \]

(4.1)

i.e. we just need to add the negative term with normal covariant derivative if it was a tensor. Here $\Gamma^a_{aq}$ are the connection parameters, which are also known as the Christoffel symbol for Romanian space.
\[ \Gamma_b^a = \left( \Gamma^a_{\alpha \lambda} - \frac{\partial \Lambda^a}{\partial x^\alpha} \right) \Lambda^b. \]  

(4.8)

Using simple algebraic manipulations, this relation can also be inverted, which is given by

\[ \Gamma^\mu_{\nu a} = \left( \Gamma^a_{\nu \beta} + \frac{\partial \Lambda^a}{\partial x^\beta} \right) \Lambda^\mu. \]  

(4.9)

Inverting the relation given in Eq. 4.7 is tricky. For inverting it we can multiply both the sides with \( g^{PQ}_a \).

\[ \Gamma^a_{\nu P} = g^{PQ}_a g_{AB}^{\nu Q} \Gamma^B_{a C} g_c^{AC} + g^{PQ}_a g_{BC}^{\nu Q} \Gamma^B_{Ab} g_c^{AC} \]

\[ = \delta^P_B \Gamma^R_{a c} g_c^{AC} + \delta^P_B \delta^R_C \Gamma^B_{Ab} g_c^{AC} = \Gamma^R_{a c} g_c^{PC} + \Gamma^P_B g_c^{AQ}. \]

(4.10)

Multiplying it with another \( g^{PQ}_R \) we get

\[ \Gamma^P_R g^{PQ}_a \Gamma^P_{RQ} = \Gamma^R_{c a} g_c^{PC} g^{RS} + \Gamma^P_{Ac} g_c^{AQ} g^{RS} \]

\[ = \delta^P_R \delta^Q_C + \Gamma^P_{Ac} g_c^{AQ} g^{RS} = \Gamma^R_{c a} \delta^P_C + \Gamma^P_{Ac} \delta^Q_C. \]

(4.11)

Considering \( Q = S \) we get

\[ \frac{1}{2} g^{PQ}_R \Gamma^P_{RQ} = \Gamma^R_{c a} \delta^P_C + \Gamma^P_{Ac} \delta^Q_C. \]

(4.12)

Note that, here we just replace \( S \) by \( Q \), and there is not summation over \( Q \), in the \( \Gamma \). We put a bracket around \( Q \) to point that out. We can assume \( \Gamma^{(Q)}_{(Q) a} = C_a \), where \( C_a \) is a complex vector field. This gives \( \Gamma^{P}_{Ra} = \frac{1}{2} g^{PQ}_R \Gamma^P_{RQ} - C_a \delta^P_{R}. \) However, we also need to satisfy the relation Eq. 4.11, which requires \( C_{a*} = -C_a \). Therefore, the complex vector field has to be imaginary, giving us \( C_a = -i A_a \), where \( A_a \) is a real vector field. Therefore, we can write

\[ \Gamma^{P}_{Ra} = \frac{1}{2} g^{PQ}_R \Gamma^P_{RQ} + i A_a \delta^P_{R}. \]

(4.13)

If we know the Christoffel symbols on a manifold, we can find out \( \Gamma^a_{Ra} \) and calculate \( \Gamma^{P}_{Ra} \). However, one interesting thing here is that the covariant derivative of \( \psi^A \) is connected with an additional vector field, that does not come from the spacetime curvature.

### 4.1 Rule of transformations for the connection parameters

For calculating the rule of transformation for the connection parameter \( \Gamma^{P}_{Ra} \) and \( \Sigma^{P}_{Ra} \), we first differentiate both sides of the first relation of Eq. 3.20 with respect to \( x^a \)

\[ 0 = \epsilon_{AB} \frac{\partial}{\partial x^a} \left( \frac{1}{t} c^A c^B t^C t^D \right) + \epsilon_{AB} \frac{1}{t} \frac{\partial}{\partial x^a} \left( t^A c^B t^C t^D \right) + \epsilon_{AB} \frac{1}{t} t^A \frac{\partial}{\partial x^a} \left( t^B c^C t^D \right). \]

(4.14)

Multiplying this with the second relation of the same equation set, we get

\[ 0 = \epsilon_{AB} \frac{\partial}{\partial x^a} \left( \frac{1}{t} t^A t^B t^C t^D \right) \]

\[ + \epsilon_{AB} \frac{1}{t} t^A \frac{\partial}{\partial x^a} \left( t^B t^C t^D \right) \]

\[ + \epsilon_{AB} \frac{1}{t} t^A \frac{\partial}{\partial x^a} \left( t^B t^C t^D \right) \]

\[ + 2 t \frac{\partial}{\partial x^a} \left( \frac{1}{t} \right) + 4 T^A \frac{\partial}{\partial x^a} \left( t^A \right). \]

(4.15)

As \( \psi^B \) transform as \( t^A T^B_{AB} \psi^B \), we get
\[ \frac{\partial \bar{\psi}^B}{\partial x^\alpha} = \frac{\partial \left(t^\frac{1}{2} T_A^B \psi^A\right)}{\partial x^\alpha} = t^\frac{1}{2} T_A^B \frac{\partial \psi^A}{\partial x^\alpha} + t^\frac{1}{2} \psi^A \frac{\partial T_A^B}{\partial x^\alpha} + \frac{1}{2} T_A^B \psi^A t^\frac{1}{2} \frac{\partial \ln t}{\partial x^\alpha}. \] (4.16)

However, \( \bar{\psi}^A \) is the covariant derivative. Therefore, under the spin transformation, it should transform as \( \bar{\psi}^B = t^\frac{1}{2} T_A^B \psi^A \). Using some simple algebraic manipulations, we can show that under coordinate and spin transformation the connection parameters, \( \Gamma^A_{B\beta} \) transforms as

\[ \bar{\Gamma}^A_{B\beta} = \left( \Gamma^C_{D\alpha} \frac{\partial t^D_B}{\partial x^\alpha} \right) T^A_C - \frac{1}{2} \frac{\partial \ln t}{\partial x^\alpha} \delta^A_B \]. (4.17)

As \( \Gamma^A_{B\alpha} = \Upsilon^A_{B\alpha} - \frac{1}{2} \Upsilon^C_{D\alpha} \delta^A_B \), we can show that to satisfy Eq. 4.17, the variable \( \Upsilon^A_{B\alpha} \) must transform as

\[ \bar{\Upsilon}^C_{D\beta} = \left( \Upsilon^A_{B\alpha} t^D_B + \frac{\partial t^A_B}{\partial x^\alpha} \right) T^A_C \frac{\partial x^\alpha}{\partial x^\beta}. \] (4.18)

### 4.2 Calculating the parallel transport equations

In the above section, we discuss the covariant derivative of the \( \psi^A \) coordinate system. We can use these covariant derivatives to calculate the parallel transport equation for the \( \psi^A \) system, i.e. if we move the particle freely from one point on the spacetime to another, how will the parametric coordinate system change.

If we assume \( V^\mu = \frac{dx^\mu}{d\lambda} \), then the parallel transport equation for the vector \( V^\mu \) is given by

\[ \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho} V^\rho \frac{dx^\nu}{d\lambda} = 0 \] (4.19)

According to Eq. 3.2, the quantities \( X^a \) and \( V^\mu \) are related as \( V^\mu = \Lambda^\mu_a X^a \). Thus, the parallel transport equations for \( X^a \)'s look like

\[ \frac{d(\Lambda^\mu_a X^a)}{d\lambda} + \Gamma^\mu_{\nu\rho} \Lambda^\nu_a X^a \frac{dx^\rho}{d\lambda} = 0 \] (4.20)

\[ \Rightarrow \Lambda^\mu_a \frac{dX^a}{d\lambda} + X^a \frac{d\Lambda^\mu_a}{d\lambda} + \Gamma^\mu_{\nu\rho} \Lambda^\nu_a X^a \frac{dx^\rho}{d\lambda} = 0 \] (4.21)

Using Eq. 4.8, we get the equation for the parallel transport of \( X^a \) as

\[ \frac{dX^a}{d\lambda} + \Gamma^a_{\nu b} X^b \frac{dx^\nu}{d\lambda} = 0. \] (4.22)

For calculating the parallel transport equations for \( \Psi^A \)'s, we can use the relation Eq. 3.11. Considering \( G^a_{AB} \) as constants we get

\[ \frac{d(\psi^A \psi^* B)}{d\lambda} + G^a_{AB} \Gamma^a_{\nu C} \psi^* C \psi^* D \frac{dx^\nu}{d\lambda} = 0 \] (4.23)

Using Eq. 4.11 we get

\[ \psi^A \frac{d\psi^A}{d\lambda} + \psi^* B \frac{d\psi^* B}{d\lambda} + (\Gamma^B_{\nu C} \psi^* A + \Gamma^A_{C\nu} \psi^* D) \psi^* D \frac{dx^\nu}{d\lambda} = 0 \] (4.24)

\[ \Rightarrow \psi^A \left( \frac{d\psi^A}{d\lambda} + \Gamma^B_{\nu C} \psi^* B \frac{dx^\nu}{d\lambda} \right) + \psi^* B \left( \frac{d\psi^* B}{d\lambda} + \Gamma^A_{C\nu} \psi^* D \frac{dx^\nu}{d\lambda} \right) = 0 \] (4.25)

Here we have two similar equations which are just the complex conjugate to each other and the sum is zero. To separate these we again need to add and subtract some quantity like \( i B_\mu \psi^A \psi^* B \) as it was done in Eq. 4.13, giving
This gives the equation for parallel transport for the $\Psi^A$. $B_\nu$ is an arbitrary vector field. We can take $B_\mu = 0$ as it can be absorbed in the definition of $\Gamma^A_{A\nu}$, which already contains an arbitrary vector field as given in Eq. 4.13.

5 Exploring the additional degrees of freedom

As $\psi^1$ and $\psi^2$ are complex numbers, there are total 4 degrees of freedom. However, as we are trying to parameterize tangent space of a 4 dimensional manifold and there is one constrain, we have total 3 degrees of freedom. Therefore, we have introduced one additional degrees of freedom in our parameterization of the tangent space. This can be seen as follows. In Eq. 3.11, if $\psi^1$ and $\psi^2$ are multiplied with a number $c$ and $\psi^{*1}$ and $\psi^{*2}$ are multiplied with a number $\frac{1}{c}$ then $X^a$ remains unchanged. This implies that if we take $cc^* = 1$, or

$$c = \exp(i\theta), \quad \text{where} \quad \theta \in \mathbb{R}$$

then $X^a$s remain unchanged under transformation $\psi^A \rightarrow e^{i\theta}\psi^A$. Here we must note that no arbitrary choice of $c$ can alter the tangent space coordinate system. Therefore, $c$ does not have to be constant. It can vary over space-time, i.e. over $x^\mu$ for $\mu \in (0,3)$. Let us consider a new coordinate transformation given by

$$\theta' \rightarrow \theta + \rho \left(x^0, x^1, x^2, x^3\right).$$

$\psi^A$ transform as $\psi'^A \rightarrow e^{i\theta + i\rho \left(x^0, x^1, x^2, x^3\right)} \psi^A$. $\psi'^A$ still gives the same unique point on the light-cone as that of $\psi^A$. In this article, we refer these types of transformations in $\psi^A$ as gauge transformation, and name the coordinate $\theta$ as internal coordinate.\footnote{Note that in this article, we use primed coordinate system for the gauge transformation and the barred coordinate system for the spin transformation.}

We can further generalize the transformation by writing $\rho \left(x^0, x^1, x^2, x^3\right)$ as a path integration from point $P_0 = \{x^\nu|\nu \in (0,\ldots,3)\}$ to $P = \{x^\nu|\nu \in (0,\ldots,3)\}$ as

$$\theta' \rightarrow \theta + \int_{P_0}^P \lambda_\mu dx^\mu,$$

where $\lambda_\mu$ is an arbitrary vector field. This gives

$$\psi'^A \rightarrow \psi^A \exp \left(i\theta + i\int_{P_0}^P \lambda_\mu dx^\mu\right).$$

Here we introduce a parameterization of the light-cone that depends on the path of the integration, without changing the point on the light-cone. Suppose we take two points $P_0$ and $P$ on a manifold $\mathcal{M}$. The tangent space at these two points of the manifold are fixed. Suppose a particle moves from $P_0$ to $P$ through two different paths, and the initial and the final points on the tangent space are same. That does not ensure that the parametric coordinate, $\psi^A$, of the particle remains same because that path through which the the particle reach from the first to the second point are not same. At every point on the path there is a vector field $\lambda_\mu$, and the phase of the coordinate $\psi^A$ depends on the path integral through which it moved from one point to the another. In other words, the information of the vector field $\lambda_\mu$ on the path remains encoded on the $\psi^A$ coordinate system.
5.1 Holomorphicity of the $\psi^A$ coordinates

We have shown that $\psi^A$ are the complex functions of the tangent-space coordinates $X^a$ and the internal coordinate $\theta$, i.e. $\psi^A = \psi^A(X^a, \theta)$. Given values of $\psi^A$ we can uniquely determine $(X^a, \theta)$ and the vice versa. There is an interesting property of $\psi^A$. If we club the internal coordinate $\theta$ with any of the tangent space coordinates and create a complex coordinate $Z^a = X^a \exp(i2\theta)$ then $\psi^A$ is a holomorphic function of $Z^a$.

Proof: Let us consider $\psi^1 = |\psi^1| \exp(2i\phi_1)$ and $\psi^2 = |\psi^2| \exp(2i\phi_2)$. Let $\theta = \phi_1 + \phi_2$ and $\phi = \phi_1 - \phi_2$, where in accordance with Eq. 3.10, $\phi = \tan^{-1}(X^2/X^1)$. So using $\theta$ and $\phi$ we can write $\psi^A$ coordinates as $\psi^1 = |\psi^1| \exp(i\theta + i\phi)$ and $\psi^2 = |\psi^2| \exp(i\theta - i\phi)$.

As $X^a$ are on the Minkowski space

$$G_{AB}^a \psi^A \psi^B = \frac{X^a}{X^b} = \frac{\partial X^a}{\partial X^b} = \frac{\partial (G_{AB}^a \psi^A \psi^B)}{\partial X^b}. \quad (5.5)$$

Calculating Eq. 5.5 for $a = 0$ and $a = 3$ and summing them up we get

$$\frac{\psi^1 \psi^*-1}{X^b} = \frac{\partial (\psi^1 \psi^*)}{\partial X^b} = \frac{\psi^1 \partial \psi^1}{\partial X^b} + \frac{\psi^* \partial \psi^1}{\partial X^b} = 2 \frac{\psi^1 \partial \psi^1}{\partial X^b} \quad (5.6)$$

$$\Rightarrow \frac{\psi^1}{X^b} = 2 \frac{\partial \psi^1}{\partial X^b}. \quad (5.7)$$

This same relation will also be valid for $\psi^2$. Putting $\psi^1 = |\psi^1| \exp(i\theta + i\phi)$ in the above equation we get

$$\frac{|\psi^1| \cos(\theta + \phi) + i \sin(\theta + \phi)}{X^b} = 2 \left( \frac{\partial |\psi^1|}{\partial X^b} \right) (\cos(\theta + \phi) + i \sin(\theta + \phi))$$

$$+ 2 |\psi^1| \left( \frac{\partial (\theta + \phi)}{\partial X^b} \right) [\sin(\theta + \phi) + i \cos(\theta + \phi)]. \quad (5.8)$$

Separating out the real and the imaginary parts we get

$$\frac{|\psi^1| \cos(\theta + \phi)}{X^b} = 2 \frac{\partial |\psi^1|}{\partial X^b} \cos(\theta + \phi) - 2 |\psi^1| \frac{\partial (\theta + \phi)}{\partial X^b} \sin(\theta + \phi), \quad (5.9)$$

$$\frac{|\psi^1| \sin(\theta + \phi)}{X^b} = 2 \frac{\partial |\psi^1|}{\partial X^b} \sin(\theta + \phi) + 2 |\psi^1| \frac{\partial (\theta + \phi)}{\partial X^b} \cos(\theta + \phi). \quad (5.10)$$

Now, $\phi$ is a function of $X^a$ but $\theta$ is a completely free parameter. Let us consider a complex variable $Z^a = X^a \exp(2i\theta)$. We can write the following derivatives as

$$\frac{\partial (|\psi^1| \cos(\theta + \phi))}{\partial X^a} = \frac{\partial |\psi^1|}{\partial X^a} \cos(\theta + \phi) - |\psi^1| \frac{\partial (\theta + \phi)}{\partial X^a} \sin(\theta + \phi), \quad (5.11)$$

$$\frac{\partial (|\psi^1| \sin(\theta + \phi))}{\partial X^a} = \frac{|\psi^1| \cos(\theta + \phi)}{X^a \partial \theta}, \quad (5.12)$$

$$\frac{\partial (|\psi^1| \sin(\theta + \phi))}{\partial X^a} = \frac{\partial |\psi^1|}{\partial X^a} \sin(\theta + \phi) + |\psi^1| \frac{\partial (\theta + \phi)}{\partial X^a} \cos(\theta + \phi), \quad (5.13)$$

$$- \frac{\partial (|\psi^1| \cos(\theta + \phi))}{\partial X^a} = \frac{|\psi^1| \sin(\theta + \phi)}{X^a \partial \theta}. \quad (5.14)$$

Using Eq. 5.9 - Eq. 5.14, we can see that $\psi^1$ follows the Cauchy–Riemann equation, i.e.

$$\frac{\partial (|\psi^1| \cos(\theta + \phi))}{\partial X^a} = \frac{\partial (|\psi^1| \sin(\theta + \phi))}{\partial X^a}, \quad \frac{\partial (|\psi^1| \sin(\theta + \phi))}{\partial X^a} = - \frac{\partial (|\psi^1| \cos(\theta + \phi))}{\partial X^a} \quad (5.15)$$
Hence, $\psi^1$ is a holomorphic function of $Z^a = X^a \exp(i2\theta)$. Similar analysis with $\psi^2$ (subtracting $a = 3$ component from $a = 0$ in Eq. 5.5 and redoing the above calculations), we can show that $\psi^2$ is also holomorphic function of $Z^a$. This concludes the proof.

According to our previous discussion, $\theta$ is an additional degrees of freedom, that the $\psi^A$ coordinate system preserve, along with the $X^A$ coordinates. Mathematically, there is no restriction that the $\theta$ has to be single number. Instead of considering a single $\theta$, if we consider four independent angles $\theta^0, \theta^1, \theta^2, \theta^3$ such that $\theta = \theta^0 + \theta^1 + \theta^2 + \theta^3$ and define $Z^a = X^a \exp(2i\theta^a)$ where $a \in \{0, \ldots, 3\}$, then $\psi^A$ becomes holomorphic functions of $Z^a$.

Therefore, under this setup the tangent space coordinate axis are complex coordinates, and our parameter space coordinate $\psi^A$ are the functions of the complex coordinate systems. $\psi^A$ preserve the value of the total phase $\theta$. However, it does not independently save the phases associated with each coordinate axis. The complex phases with the coordinate axis allow us to store an independent quantity along each of the coordinate axis. As an explainer, we can take example of an electromagnetic field. While analyzing EM wave, we can write the electric field as a complex number which simplifies different equations. The real part of the complex number gives us the actual electric field. While analyzing EM wave, we can write the electric field as a complex number which can be associated with some real coordinate and the argument of the coordinate can be associated with some additional quantity when $\theta^a \neq 0$.

Here we should also note that the tangent space of the 4 dimensional null hyperspace is a 3 dimensional space. Therefore, instead of 4 independent values of $\theta^a$ we may consider that there are only 3 independent $\theta$s and the fourth value is somehow related to the other three values.

6 Exploring internal coordinate system

6.1 Defining different differential forms

The parametric coordinate $\psi^A$, parameterise the tangent space coordinate system $X^a$ and the sum of the angles $\theta^a$. These angles create a 3-sphere ($S^3$) internal subspace at every point on the tangent space (assuming that there are only three independent $\theta^a$s). The parametric coordinate $\psi^A$ can not individually parameterise $\theta^a$s. However, they poses some interesting properties. As all the $\theta^a$ angles appear in the exponent, the function remains the same if we integrate or differentiate any of these parametric space coordinate, $\psi^A$, with respect to any $\theta^a$. Therefore, we can construct differential forms $\psi^A e^p, \psi^A_{pq} e^p \wedge e^q$ and $\psi^A_{pqr} e^p \wedge e^q \wedge e^r$, which are analytically same as $\psi^A$ with added phase. Here $e_p = \delta_p^\theta$ are the basis vector along the $\theta^p$ coordinates and the basis in the covector space is $e^p$ where $p \in \{1, 2, 3\}$. Here wedge represent the wedge product of the vector spaces, $e^p \wedge e^q = 0$ and $e^p \wedge e^r = -e^q \wedge e^r$ for $p \neq q$. We use the indices $p, q, r, \ldots$ to represent the $\theta^p$ coordinates in the internal $S^3$ space. These indices can vary from 1 to 3, while the tangent space indices which are represented by $a, b, \ldots$ can vary from 0 to 3. We also refer these indices $p, q, r, \ldots$ as gauge index. These differential forms can also be used as parameters for parameterizing the light-cone. However, these new parameters have the directional dependence in the internal $S^3$ space.

Here we ignore the $a = 0$ component because the null hyperspace is a 3D surface. Among the 4 coordinates in the tangent space, one is not independent. So, without the loss of generality we consider 3 spacial coordinates as independent coordinates and we associate a complex phase with each of the space coordinate axis. The time coordinate behaves differently from the spatial coordinates as it comes with a negative signature. So, we consider it as a dependent coordinate. There should be some complex phase associated with the time coordinate too. However, we consider that phase in the time axis will somehow be related to the other three phases. We can write the light cone as $(X^0)^2 - \sum_{p=1}^{3} (Z^p Z^*^p) = 0$. Here, we use $Z^p = X^p \exp(2i\theta^p)$ as we define in the previous section.

Let us take the differential 0-form, 1-form, 2-form, 3-form on $S^3$ as
\[ \Omega_0 : \psi^A \] (6.1)  
\[ \Omega_1 : \psi_p^A e^p \quad \cdots \quad \forall p \in (1,2,3) \] (6.2)  
\[ \Omega_2 : \psi_{pq}^A e^p \wedge e^q \quad \cdots \quad \forall p, q \in (1,2,3) \quad \& \quad p, q \text{ in circular order} \] (6.3)  
\[ \Omega_3 : \psi_{pqr}^A e^p \wedge e^q \wedge e^r \quad \cdots \quad \forall p, q, r \in (1,2,3) \quad \& \quad p < q < r \] (6.4)

Under this construct the 0th form is the standard \( \psi^A \), used in the previous section and the tangent space coordinates from this is given by Eq. 3.11. The 1–forms have 3 directions in the internal \( S^3 \) space. We can take the dot product of them in the internal space. \( \psi^A \) are complex numbers.

However, the total strength of \( \psi_p^A \), i.e. \( \sum_{p=1}^3 \psi_{p}^A \psi_p^A \) for \( \forall A \in (1,2) \) component remains constant under any type of rotation of \( S^3 \). For any particular \( A \in (1,2) \), the magnitude of each \( p \) component i.e. \( \sqrt{\psi_p^A \psi_p^A} \) for \( p \in (1,2,3) \), rotates as SO(3) and rotation of complex \( \psi_p^A \) follows the rotation group SU(3) (check Appendix B and Appendix C for further discussion). For different \( p \) components we can get the contribution to the tangent space coordinate system as \( X^a = G_{AB}^a \psi_p^A \psi_p^B \) for each \( p \in (1,2,3) \).

The total contribution from the all three components is \( X^a = \sum_{p=1}^3 G_{AB}^a \psi_p^A \psi_p^B \). Under any SU(3) transformation of the \( \psi_p^A \) coordinates, the total contribution to the tangent space coordinate does not change.

The 2–forms being dual to the 1-forms, show similar properties under any rotation in the internal \( S^3 \) space and the dot products i.e. \( \psi_{12}^A \psi_{12}^B + \psi_{23}^A \psi_{23}^B + \psi_{31}^A \psi_{31}^B \) remains constant under any SU(3) transformation. The 3–form is dual to the 0th form and is a volume form. Therefore, it does not transform under rotation between the basis vectors \( e^p \). The tangent space coordinate comes from the dot product as \( X^a = G_{AB}^a \psi_{123}^A \psi_{123}^B \).

### 6.2 Rotation of \( \Omega_1 \) and \( \Omega_2 \) in complex coordinates

If we rotate the coordinate system in the internal space, then the basis vectors \( e^p \) change to lets say \( e'^q \). These can be related as \( e'^q = R_q^p e^p \), where \( R \) is a complex rotation matrix and \( R_q^p \) is the element in the \( p \)th row and \( q \)th column of the matrix. In \( S^3 \), it is a \( 3 \times 3 \) matrix whose elements can be complex numbers. So, the rotation along the complex axis in a single direction, that is prohibited in the a real rotation like SO(3) are allowed in the complex rotation matrix (check appendix C for proper visualization). For SU(3) rotation the matrix must satisfy \( R^T R = 1 \) and \( \text{det}(R) = 1 \). For small rotation we can write the SU(3) rotation matrix as \( \exp(i \sum_{i=1}^8 \omega_i \tau_i) \), where \( \tau_i \) for \( i = 1, \cdots, 8 \) are the generators of the SU(3) group, and the rotation along these generators are given by \( \omega_i \)s respectively.

Under SU(3) transformation the differential 1-form transforms as \( \psi'^A_p = R_p^q \psi_q^A \), which in matrix format we can write as

\[
[\psi'^A_p] = \left[ \exp(i \sum_{i=1}^8 \omega_i \tau_i) \right] [\psi^A_p]. \tag{6.5}
\]

Under covariant differentiation, \( \psi^A_p \) have two connection parameters, one corresponding to the spin index \( A \) and other corresponds to the gauge index \( p \). Thus the covariant differentiation can be written as

\[
\psi^A_{p;\mu} = \frac{\partial \psi^A_p}{\partial x^\mu} - \Gamma^q_{p\mu} \psi^A_q + \Gamma^A_{B\mu} \psi^B_p. \tag{6.6}
\]

The transformation rules for the \( \Gamma^A_{B\mu} \) are given by Eq. 4.17. \( \Gamma^p_{q\mu} \) is the connection parameter for the gauge variable. If we rotate the basis vectors of internal coordinate system by a rotation matrix \( R_q^p \), we can write \( \psi'^A_p = R_p^q \psi_q^A \) which gives
\[ \psi^A_{\mu \nu} = \frac{\partial}{\partial x^\mu} (R^s_p \psi^A_q) - \Gamma^s_{\mu \nu} \left( R^s_p \psi^A_q \right) = R^s_p \frac{\partial \psi^A_q}{\partial x^\mu} - \left( \Gamma^s_{\mu \nu} R^s_p \right) \psi^A_q R^s_p R^s_w \delta^w \delta^q v \\
= R^q_p \left[ \frac{\partial \psi^A_q}{\partial x^\mu} - \left( \Gamma^s_{\mu \nu} R^s_p \right) \psi^A_q R^s_p R^s_w \delta^w \delta^q v \right] = R^q_p \left[ \frac{\partial \psi^A_q}{\partial x^\mu} - \Gamma^s_{q \mu} \psi^A_s \right] = R^q_p \psi^A_{q \mu} . \] (6.7)

\( R \) being rotation matrix \( R R^T = 1 \), giving \( R^p_q R^q_r \delta^r \delta^p = \delta^p_p \), where \( \delta^p_q = \delta^q_p = 1 \) when \( p = q \), and 0 otherwise. This simplifies the transformation rules for the gauge connection parameter and simple algebraic manipulation shows that \( \Gamma^t_{\nu \mu} \) under internal basis rotation transforms as

\[ \Gamma^t_{q \mu} = \left( \Gamma^t_{\nu \mu} R^s_{\nu \mu} + \frac{\partial R^s_{\nu \mu}}{\partial x^\mu} \right) R^s_v \delta^v \delta^q . \] (6.8)

Under the space-time coordinate \( x^\mu \) transformation, each \((s, q)\) component of \( \Gamma^s_{q \mu} \) transforms as normal vector as there is only one space time index. For SU(3) transformation, we can write the connection parameter \( \Gamma^s_{q \mu} \) as a sum of 8 vectors as \( \Gamma^s_{q \mu} = \sum_{i=1}^{8} A^s_i [\tau_i]_q \). Here \( \tau_i \)s are the generators of the SU(3) matrix and the \([\tau_i]_q \) is the \((s, q)\) element of the matrix \( \tau_i \).

In the internal space, the 2-forms are dual to the 1-forms and they also transform as SU(3). For the two form \( \psi^A_{pq} = 0 \) if \( p = q \) and \( \psi^A_{pq} = -\psi^A_{qp} \). For simplifying, if we define \( \varphi^{Ap} = \psi^A_{pq} e^{pqr} \) and \( \varphi^{Ar} = \psi^A_{pq} \varepsilon^{pqr} \), then under the internal rotation \( \varphi^{Ap} \) transforms as \( \varphi^{Ap} = R^p_q \varphi^{Aq} \). Here, \( \varepsilon^{pqr} \) are Levi-Civita symbol, \( \varepsilon^{123} = \varepsilon^{213} = \varepsilon^{312} = 1 \), \( \varepsilon^{231} = \varepsilon^{132} = \varepsilon^{123} = -1 \) otherwise \( \varepsilon^{rr} = 0 \). In matrix format we can write this transformation as

\[ \begin{bmatrix} \varphi^{Ar} \\ \end{bmatrix} = \begin{bmatrix} \exp(i \sum_{i=1}^{8} \omega_i \tau_i) \end{bmatrix} \begin{bmatrix} \varphi^{Ap} \\ \end{bmatrix} \] (6.9)

Here also the covariant derivative of \( \varphi^{Ap} \) is given by the same connection parameter as

\[ \varphi^{Ap}_{\mu} = \frac{\partial \varphi^{Ap}}{\partial x^\mu} + \Gamma^p_{q \mu} \varphi^{Aq} + \Gamma^A_{p \mu} \varphi^{Bp} . \] (6.10)

The 0-form does not change under any coordinate transformation. The 3-form is the volume form in this case and does not also change under the rotation of the coordinates.

### 6.3 Unitary transformation of the coordinates

Apart from the rotational transformation SU(3), we can also have the unitary U(1) transformation in the complex space. As \( \psi^A \) are the complex numbers, each and every coordinate axis has its own submanifold, which gives an U(1) transformation. Under U(1) gauge translation, the basis transforms as \( e^p = \exp(i \alpha) e^p \), where \( \alpha \) is a small rotation angle. In matrix format we can write this as

\[ \begin{bmatrix} e^1 \\ e^2 \\ e^3 \end{bmatrix} = \begin{bmatrix} \exp(i \alpha) & 0 & 0 \\ 0 & \exp(i \alpha) & 0 \\ 0 & 0 & \exp(i \alpha) \end{bmatrix} \begin{bmatrix} e^1 \\ e^2 \\ e^3 \end{bmatrix} . \] (6.11)

Under this transformation, different forms shown in Eq. 6.4 transforms as

\[ \begin{align*}
\Omega'_0 & : \psi^{tA} \\
\Omega'_1 & : \psi^{tA} \exp(i \alpha) e^p & \forall p \in (1, 2, 3) \\
\Omega'_2 & : \psi^{tA} \exp(2i \alpha) e^p & \forall p, q \in (1, 2, 3) \quad & p, q \text{ in circular order} \\
\Omega'_3 & : \psi^{tA} \exp(3i \alpha) e^p & \forall p, q, r \in (1, 2, 3) \quad & p < q < r
\end{align*} \] (6.12-6.15)
Therefore, the 0-form remains unchanged and \( \psi'^A = \psi^A \). As in the internal \( S^3 \) space \( \psi^A \) is a vector, it should not change under any coordinate transformation and only its components along different basis vector changes, giving \( \psi'^A e^p = \psi^A e^p \). Hence, the 1-forms transform as

\[
\psi'^A_p = \exp(-i\alpha)\psi^A_p .
\]

(6.16)

Here one should note that \( \exp(i\alpha) \) is \( \frac{1}{3} \) of the tress of the \( 3 \times 3 \) matrix given in Eq. 6.11. The connection parameter for the covariant derivative for this transformation is also similar to the connection parameter for the SU(3) rotation, except it only affects the \( \Gamma^p_{\mu\nu} \) parameters. If we don’t take any changes in the spin coordinate then the covariant derivative of \( \psi'^A_p \) and \( \psi^A_p \) can be related as

\[
\psi'^A_p = \exp(-i\alpha) \frac{\partial \psi^A_p}{\partial p} - i \frac{\partial \alpha}{\partial x^\mu} \exp(-i\alpha) \psi^A_p - \Gamma^p_{\mu\nu} \exp(-i\alpha) \psi^A_p
\]

\[
= \exp(-i\alpha) \left[ \frac{\partial \psi^A_p}{\partial x^\mu} - \left( \Gamma^p_{\mu\nu} + i \frac{\partial \alpha}{\partial x^\mu} \right) \psi^A_p \right] = \exp(-i\alpha) \left[ \frac{\partial \psi^A_p}{\partial x^\mu} - \Gamma^p_{\mu\nu} \psi^A_p \right] .
\]

(6.17)

Hence, under U(1) transformation the connection parameter, i.e. \( \Gamma^p_{\mu\nu} \), transforms as \( \Gamma^p_{\mu\nu} = \Gamma^p_{\mu\nu} - i \frac{\partial \alpha}{\partial x^\mu} \). Note that, there is no summation over \( p \). If we assume no SU(3) gauge transformation and only U(1) gauge transformation, then for such transformation we can consider the connection parameters \( \Gamma^1_{\mu\nu} = \Gamma^2_{\mu\nu} = \Gamma^3_{\mu\nu} = i A_\mu \), where \( A_\mu \) is an arbitrary vector field and all the other \( \Gamma^p_{\mu\nu} \) are 0.

Similarly, under U(1) transformation, the two forms show satisfy \( \psi'^A_p e^p \land e^q = \psi^A_p e^p \land e^q \), giving \( \psi'^A_{pq} = \exp(-2i\alpha)\psi^A_{pq} \) for all \( p, q \in \{1, 2, 3\} \) and \( p, q \) in circular order as shown in Eq. 6.14. The 3-form transforms as \( \psi'^A_{123} = \exp(-3i\alpha)\psi^A_{123} \). The covariant derivative for the 2-form and 3-form can be written as

\[
\psi'^A_{pq,\mu} = \frac{\partial \psi^A_{pq}}{\partial x^\mu} - \Gamma^p_{\mu\nu} \psi^A_{pq} - \Gamma^q_{\nu\mu} \psi^A_{pq} = \frac{\partial \psi^A_{pq}}{\partial x^\mu} - 2i A_\mu \psi^A_{pq}
\]

(6.18)

and

\[
\psi'^A_{pqr,\mu} = \frac{\partial \psi^A_{pqr}}{\partial x^\mu} - \Gamma^p_{\mu\nu} \psi^A_{pqr} - \Gamma^q_{\nu\mu} \psi^A_{pqr} - \Gamma^r_{\mu\nu} \psi^A_{pqr} = \frac{\partial \psi^A_{pqr}}{\partial x^\mu} - 3i A_\mu \psi^A_{pqr}
\]

(6.19)

Here one can see that coupling strength of the coupling parameter for the 2-form is 2 times and the 3-form is 3 times that of the 1-form.

### 6.4 Renaming the coordinate systems

In the previous section, we define total 8 sets of parametric space coordinate systems. Each of these coordinate systems can represent an unique point in the tangent space. In these coordinate systems there are two triplets, the 1-forms and the 2-forms. These are sensitive to the rotation in the internal coordinate system. Therefore, if a particle, represented by these forms, moves freely from one point to another on space-time, then the phase of the parametric coordinate system saves the information about all the rotations in each of the direction in the internal \( S^3 \) space through its path.

We can associate these 8 coordinates to represent different Fermions. The 0-form neither couple with any U(1) field nor with any SU(3) field. Therefore, we can use it can represent a neutrino or an anti-neutrino. The 3-form does not couple with any SU(3) field, however couples with an U(1) field. Hence, it can be used to represent an electron or a position. The 1-forms couple with an U(1) field with \( \frac{1}{3} \) strength of that of the 3-forms. Therefore, it can be used to represent a down quark or anti-down quark. Each 1-form has three components which can be rotated as SU(3) group. They represent 3 color charges of the d-quark triplet (\( \Psi^A_{d_u}, \Psi^A_{d_d}, \Psi^A_{d_s} \)). The SU(3) connection parameters represent 8 gluons. The 2-forms can be used to represent the up or the anti-up quarks as they couple with an U(1) field with a \( \frac{2}{3} \) strength of that of the 3-forms. Also they couple with the SU(3) field. Therefore, we can rename the parametric space coordinate systems as
The spin indices represent the spin or the right or left handedness of the particles (for massless particles helicity and chirality are same). In this article we use the 4 dimensional spacetime to generate parametric space coordinates to represent the massless particles. As the spinors in the article are massless, its not possible to distinguish between the particle and the antiparticles. However, according to our previous articles the spacetime should be 5 dimensional. It can be shown that in higher dimension we can introduce the similar parametric space coordinate system with mass terms and get separate coordinates for particles and antiparticles. However, this discussion is beyond the scope of this article and will be addressed in future article. We also need to come up with theories of weak interaction, i.e. the SU(2) transformation between the electrons and the neutrinos. We want the doublets \((\Psi^A\nu, \Psi^A\bar{\nu})\), \((\Psi^A\mu, \Psi^A\bar{\mu})\) to transform as a SU(2) system. This issue will also be addressed in future articles.

7 Discussion and Conclusion

In this paper we explore the motion of a massless particle moving freely on a null hyper-surface of a 4 dimensional spacetime manifold. We define a tangent space on every point of the manifold. For a massless particle the tangent space coordinates actually represent the space of 4-velocity of the particle. We define a complex parametric space coordinate system on the tangent space. The parametric space coordinate system behaves like spinors. We show that the parametric space coordinates have an additional degrees of freedom that can be used to store the path integral of a vector field through the path of the particle. We explore different properties of the parametric space coordinates. We define a 3-sphere \((S^3)\) space on the tangent space and define 8 differential forms from the parametric space coordinate system. These coordinate systems can couple with different SU(3) and U(1) fields. Therefore, they can be used to represent different elementary Fermions. In this paper the exercise has been carried out on a 4-dimensional manifold which gives only massless spinors. However, in higher dimensions it is possible to accommodate massive spinors which we are planning to address in our subsequent research.

A Consequence in \(L^p\) space

In an \(L^2\) space the length between any two points is given by the square root of sum of square of the distances between those two points in different directions, i.e. \(d = \sqrt{\sum_{i=1}^{n-1}(X^i)^2}\) in an \(n\) dimensional Minkowski space-time. (In a Minkowski space-time the \(X^0\), i.e. the time component comes up with positive signature and rest spatial dimensions appear with negative signature.) In this paper we take a null manifold in a 4 dimensional spacetime and define a parametric coordinate system on its tangent space. We show that these parametric coordinates can be assigned a tensorial weight of \(\frac{1}{2}\), which we call spin weight. We also lower the spin indexes of our parametric coordinate system to get a system of spin \(-\frac{1}{2}\) variables. Thereby we obtain spin \(\frac{1}{2}\) and \(-\frac{1}{2}\) variables. Even though our space is an \(L^2\) space, mathematically we are free to consider an \(L^p\) space where the distances in different dimensions add up in power of \(p\). In such a case, for a \(n\) dimensional space, the distance element is given by \(d = \sqrt{\sum_{i=0}^{n-1}(X^i)^p}\). Therefore, if we consider a null manifold on a 4 dimensional \(L^p\) space, then the tangent space is given by

\[
(X^0)^p - (X^1)^p - (X^2)^p - (X^3)^p = 0.
\]

(A.1)

Here we consider the spatial coordinates to have a negative signature as a general convention. However, it does not affect any calculation. If we consider \(\omega_p = -\exp(2i\pi/p)\) as the \(p^{th}\) root of \(-1\), then we
can factorize the above equation as
\[(X^0 - X^1)(X^0 + \omega X^1) \ldots (X^0 + \omega^{p-1} X^1) = (X^2 + X^3)(X^2 - \omega X^3) \ldots (X^2 - \omega^{p-1} X^3),\]  
(A.2)
and follow the logic of Sec. 3, to parameterize the tangent space. For simplicity let us consider an \(L^3\) space and break it as
\[(X^0 - X^1)(X^0 + \omega X^1)(X^0 + \omega^2 X^1) = (X^2 + X^3)(X^2 - \omega X^3)(X^2 - \omega^2 X^3),\]  
(A.3)
where \(\omega\) is the imaginary cube-root of \(-1\). We can write it as a system of three linear equations
\[
\begin{align*}
X^1 + X^2 &= \lambda_1(X^0 - X^3), \\
X^1 - \omega X^2 &= \lambda_2(X^0 + \omega X^3), \\
X^1 - \omega^2 X^2 &= \frac{1}{\lambda_1 \lambda_2}(X^0 + \omega^2 X^3),
\end{align*}
\]
(A.4) \hspace{1cm} (A.5) \hspace{1cm} (A.6)
Here \(\lambda_1\) and \(\lambda_2\) are two arbitrary numbers. \(\lambda_1\) has to be real as the coordinates are real number. However, \(\lambda_2\) can be a real or complex number. We can permute the combinations in the left and right hand side of the equations to get different systems of linear equations. The parameterization of the tangent space can be done as
\[
\begin{align*}
X^1 + X^2 &= \Psi^1 \Psi^{-1}, \\
X^0 - X^3 &= \Psi^2 \Psi^2, \\
X^1 - \omega X^2 &= \Psi^3 \Psi^2, \\
X^0 + \omega X^3 &= \Psi^1 \Psi^1, \\
X^1 - \omega^2 X^2 &= \Psi^3 \Psi^2, \\
X^0 + \omega^2 X^3 &= \Psi^3 \Psi^1.
\end{align*}
\]
(A.7)
As we can see, we need 3 complex numbers, \(\Psi^A\) for \(A \in (1, 2, 3)\) to parameterize this space. In fact, similar calculations can show that we need total \(p\) complex numbers to parametrize tangent space of an \(L^p\) space. As there are only 3 degrees of freedom in the tangent space, the parametric space have \(2p - 3\) additional degrees of freedom, which can be used as gauge freedom. As before we refer \(A \in (1, 2, 3)\) as the spin index.

Under linear transformation, \(\Psi^A\), transforms as \(\bar{\Psi}^B = T^{AB}_B \Psi^B\), which we call the spin space transformation. This linear transformation also satisfy the cubic equation Eq. A.3, though the equation gets multiplied with \(T^2\) where \(T = |T^A_B|\). For giving a tensorial structure to these parametric space coordinates, let us define \(\bar{\Psi}^A\) as the covariant counterpart of \(\Psi^A\). Suppose we use some quantity \(\epsilon_{AB}\) to lower the index of \(\Psi^A\). This gives
\[
\begin{align*}
\epsilon_{AB} \Psi^A \Psi^B &= \tilde{\epsilon}_{CD} \bar{\Psi}^C \bar{\Psi}^D = \tilde{\epsilon}_{CD} T^C_B T^D_B \Psi^A \Psi^B, \\
&= \epsilon_{AB} \tilde{\epsilon}_{CD} T^C_B T^D_B.
\end{align*}
\]
(A.8) \hspace{1cm} (A.9)
This may be a good choice for raising or lowering the indices, but if we demand the \(\epsilon_{AB}\) to remain unchanged under coordinate transformation, then mathematically it is impossible to make \(\epsilon_{AB}\) and \(\tilde{\epsilon}_{CD}\) equal. As the parametric space is a 3 dimensional space, \(\epsilon_{AB}\) is a two form and it has some direction in the parametric space. When we make a linear transformation of the parametric coordinates, it is changing the direction of \(\epsilon_{AB}\) in the parametric space. If we want a quantity that does not change under such linear transform we need to have a 3-form or a volume form, e.g. \(\epsilon_{ABC}\). Consequently, in \(L^p\) space where we have \(p\) parametric space coordinates, we need to have a \(p\)-form which can remain unaltered under coordinate transformation.

Therefore, we get
\[
\begin{align*}
\epsilon_{ABC} \Psi^A \Psi^B \Psi^C &= \tilde{\epsilon}_{DEF} \bar{\Psi}^D \bar{\Psi}^E \bar{\Psi}^F = \tilde{\epsilon}_{DEF} T^D_A T^E_B T^F_C \Psi^A \Psi^B \Psi^C, \\
&= \epsilon_{ABC} \tilde{\epsilon}_{DEF} T^D_A T^E_B T^F_C.
\end{align*}
\]
(A.10) \hspace{1cm} (A.11)
Provided we choose \(\epsilon_{ABC}\) to be the Lavi-Civita symbol in three dimension, then based of the properties of Lavi-Civita symbol, this transformation multiplies the left-hand side of Eq. A.11 by the
determinant of $T_B^A$. To keep the $\epsilon_{ABC}$ same in both the frames, we need to multiply the right side by $T^{-1}$, where $T = [T_B^A]$. Thus the transformation looks like

$$
\epsilon_{ABC} = T^{-1} \epsilon_{DEF} T_A^D T_B^E T_C^F .
$$

(A.12)

Lowering or raising indices using $\epsilon_{ABC}$ and $\epsilon^{ABC}$ takes the form

$$
\Psi_{BC} = \epsilon_{ABC} \Psi^A , \quad \Psi_{BC} = \epsilon_{ABC} \Psi^A , \quad \Psi^{AB} = \epsilon^{ABC} \Psi_C , \quad \Psi^A = \epsilon^{ABC} \Psi_{BC} .
$$

(A.13)

It is important to note that the Levi-Civita symbol is not a symmetric quantity. The indices at the right are used to lower an index and indices at the left are used to raise an index. Also, in this setup when we raise or lower an index we get two indices at the top or bottom respectively.

Here $\epsilon^{ABC}$ and $\epsilon_{ABC}$ behaves as a contravariant and covariant tensor density with weight +1 and -1 respectively. Raising indices with $\epsilon^{ABC}$ and lowering indices with $\epsilon_{ABC}$ increases and decreases the weight of a quantity by +1 and -1. Therefore, if we assign a spin weight of $\frac{1}{3}$ to each of the $\Psi^A$ coordinates, then $\Psi_{AB}$ should have a spin weight $\frac{2}{3}$. $\Psi_A$ and $\Psi^{AB}$ get spin weight $\frac{1}{3}$ and $\frac{2}{3}$ respectively. In such case, under spin space transformation the $\Psi$-coordinates transform as

$$
\bar{\Psi}^A = T^{-\frac{1}{3}} T_B^A \Psi_B .
$$

(A.14)

This transformation has a determinant 1. Therefore, if we introduce it back to the cubic polynomial, given in Eq. A.3, it does not multiple the equation by $T^2$.

This concept can be generalized for any $L^p$ spaces and we introduce a spin $\frac{1}{p}$ parametric space coordinate system in $L^p$ spaces. We need $p$ components of $\Psi^A$ to span the space and need a $p$ dimensional Lavi-Civita symbol for raising and lowering the spin indices. However, these parametric space coordinate systems are not unique specially when $p$ is not a prime number. For instance if $p = 6$, then we can introduce a spin $\frac{1}{2}$ or spin $\frac{1}{3}$ coordinate system instead of a spin $\frac{1}{6}$ coordinate system and these coordinate systems can span the space equally well. However, by doing so we loose some gauge freedom which were available in the system.

B Understanding the rotation in the internal coordinate system

For understanding the rotations in the internal coordinate system we consider a 3 dimensional spatial manifold. The coordinate system on this manifold is given by $x^\mu$ for $\mu \in (1, 3)$. Therefore, the line element is given by $ds^2 = g_{\mu \nu} dx^\mu dx^\nu$. Let us take a car that is moving through a geodesic path on this manifold. The velocity of the car is given by $\dot{\mu} = \frac{dx^\mu}{ds}$. Suppose there is a coordinate system, $X^a$, that is attached to the car. These $X^a$’s represent the tangent space coordinate system. As this coordinate system is fixed on the car, there is no curvature and it is a flat coordinate system. Provided we relate $X^a = \Lambda_\mu^a \frac{dx^\mu}{ds}$, then $X^a$ actually gives the velocity of the car.

Let us consider there is a setup installed on the car and an arrow is attached with the setup. The arrow can now rotate in 3 dimension. There are three anemometers in three directions those measure the wind velocity in the path of the car and rotate the arrow accordingly. This setup introduces 3 extra degrees of freedom, or three new coordinates. We can consider the basis vectors for this coordinate system as $e^p$, and the rotation of the arrow is measured with respect to these basis vectors. The arrow can only rotate with respect to these basis vectors and does not have any translational or any other kind of motion with respect to $e^p$. This rotational space represents the internal space of our analysis. Note that, for the visualization purpose we are working on a real space and hence we can always align the basis $e^a$ to coincide the tangent space coordinate axis, $X^a$. However, in reality that is not possible as these basis vectors and the tangent space denotes two distinct hyperspaces.

If we assume that $\vec{V}$ is representing the direction of this arrow on this setup, then the vector $\vec{V}$ should be measured with respect to the local basis vector $e^p$. We can write its covariant derivative with respect to the space-time coordinate as

$$
V_{;\nu}^p = \frac{\partial V^p}{\partial x^\nu} + \Gamma^p_{\nu q} V^q .
$$

(B.1)
Here the connection parameters, i.e. $\Gamma^p_{\nu\mu}$, are the connection parameters for representing the rotation of the arrow with respect to the basis vectors $e^p$. $\Gamma^p_{\nu\mu}$ has only one spatial index. Therefore, essentially it is sum of some vector fields.

Suppose the basis vectors $e^p$ are not fixed through the path of the car and they can rotate. If the basis vectors rotate then the component of $V^p$ along different basis vectors also change. If we make any transformation in the basis vectors from $e^p \rightarrow e'^p$, then the $V^p$ must transform as

$$V'^q = V^p \frac{\partial e'^q}{\partial e^p} \Rightarrow \frac{\partial V'^q}{\partial x^\nu} = \frac{\partial V^p}{\partial x^\nu} \frac{\partial e'^q}{\partial e^p} + V^p \frac{\partial^2 e'^q}{\partial x^\nu \partial e^p}.$$  \hspace{1cm} (B.2)

Consider that the coordinate system $e^p$ and $e'^q$ are related by small rotation $\theta_3$ along $e^3$. Then the transformation of the basis vectors gives us

$$\begin{bmatrix} e'^1 \\ e'^2 \\ e'^3 \end{bmatrix} = \begin{bmatrix} \cos(\theta_3) & \sin(\theta_3) & 0 \\ -\sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^1 \\ e^2 \\ e^3 \end{bmatrix} = \begin{bmatrix} (1 - \theta^2_3 + \frac{\theta^4_3}{1} - \cdots) \\ (\theta_3 - \frac{\theta^3_3}{2} + \frac{\theta^5_3}{3!} - \cdots) \\ 0 \end{bmatrix} \begin{bmatrix} e^1 \\ e^2 \\ e^3 \end{bmatrix} \hspace{1cm} (B.3)$$

As we are working in a 3D real space, any general the rotation of the basis is given by SO(3) transformation, which has three generators

$$\tau_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  \hspace{1cm}

Therefore, in general, for small rotation $\theta_1$, $\theta_2$ and $\theta_3$ of the coordinate system along the basis vectors $e^1$, $e^2$ and $e^3$, the new basis vectors are given by

$$\begin{bmatrix} e'^1 \\ e'^2 \\ e'^3 \end{bmatrix} = \exp \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \theta_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \theta_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^1 \\ e^2 \\ e^3 \end{bmatrix}.$$  \hspace{1cm} (B.4)

Therefore, the second term of the Eq. B.2, can be written in the matrix format as

$$\frac{\partial^2 V^p}{\partial x^\nu \partial e^q} = \left( \sum_{i=1}^3 \frac{\partial \theta_i}{\partial x^\nu} \tau_i \right) \exp \left( \sum_{i=1}^3 \theta_i \tau_i \right).$$  \hspace{1cm} (B.5)

In Eq. B.1, if the covariant derivatives come only from the rotation of the arrow then $V'^p_{\nu} = \Gamma^p_{\nu\mu} V^q$. If the basis vectors of the rotational axis keeps of changing over the space then with respect to those basis vectors $V^p$ also changes as $V'^p = \left[ \exp(-\sum_{i=1}^3 \theta_i \tau_i) \right]_q^p V^q$, where $[\ldots]_q^p$ is the $p,q$th term of the matrix. Therefore, the conformal derivative of $V^i$ is given by

$$V'^i_{\nu} = \left( \Gamma^p_{\nu\mu} - \sum_{i=1}^3 \frac{\partial \theta_i}{\partial x^\nu} \tau_i \right) \left[ \exp(-\sum_{i=1}^3 \theta_i \tau_i) \right]_q^p V'^q.$$  \hspace{1cm} (B.6)

As discussed before, $\Gamma^p_{\nu\mu}$ only has a single spatial index. Therefore it behaves as a linear combination of vectors. If we consider $\Gamma^p_{\nu\mu} = \sum_{i=1}^3 W^p_{\nu\mu} \tau_i$ then it simplifies the equations, where $W^p_{\nu\mu}$ are three vector fields. In this case $W^p_{\nu\mu}$ transforms as $W'^p_{\nu\mu} = W^p_{\nu\mu} - \frac{\partial \theta_i}{\partial x^\nu} \tau_i$, which is equivalent to the expression shown in Eq. 6.8. However, this vector format is a much familiar format for gauge transformation.
Suppose, some wind blows through the path, it rotates the anemometers and changes the direction of the arrow. Suppose the the wind is interacting with the 3 anemometer blades to give torques $W_1, W_2$, and $W_3$, respectively. So, as car moves from point $P$ to point $Q$, through different paths, the anemometers interact with the wind on the path differently and it changes the direction of the arrow based on the path. Therefore, the final direction of the arrow will be different for two different path.

For checking the dependence on the path of the car, multiply Eq. B.1 with $\frac{dx^\nu}{ds}$ to get

$$
\frac{dV^p}{ds} = V^p_{\nu} \frac{dx^\nu}{ds} = \partial V^p_{\nu} \frac{dx^\nu}{ds} + \left[ \sum_{i=1}^{3} W_{i\nu} \tau_i \right]^{p}_{q} V^q x^\nu \frac{ds}{ds}.
$$

(B.7)

Here $s$ is line element through the path, which is an affine parameter. When the car moves from point $P$ to $Q$, the total change in the direction of $V^q$ from the wind is $\int_{P}^{Q} \sum_{i=1}^{3} W_{i\nu} \tau_i ^{p}_{q} V^q x^\nu \frac{ds}{ds}$. Therefore, the final $V^p$ stores the information of the external vector field through the path.

In this example the rotation of the arrow vector with respect to the basis vectors is given by a SO(3) rotation because the arrow vector is a real vector. However, for complex vectors in the internal-space as discussed in Sec. 6.1, this rotation is given by a SU(3) rotation group, which has total 8 independent $\theta$s and the connection parameter is linked to 8 independent vector fields. We can call these vector fields gluon fields. When some particle, denoted by parametric space coordinate $\nu$, moves through these vector fields, it can rotate in the internal coordinate space. The final direction of the $\psi_p^A$ in the internal coordinate system stores the information of the path integral with the 8 external fields. The component of $\psi_p^A$ along different basis vectors in the internal coordinate system gives the component of $\psi_p^A$ along different color charges.

C Visualizing rotation on a complex manifold

C.1 A setup for visualizing complex rotations

In this section, we discuss the SU(3) transformations for the complex coordinates (Coddens, 2018). Even though the generators of these rotation groups can be determining mathematically, its complicated to visualize the complex rotations. Therefore, here we try to provide an intuitive understanding of the complex rotations which may help the readers to comprehend the rotations discussed in the paper, especially in Sec. 6.2.

For visualizing the complex rotations, we take an example of an electromagnetic field. It has an electric and a magnetic component, given by $\vec{E} = (E_x, E_y, E_z)$ and $\vec{B} = (B_x, B_y, B_z)$ respectively. The total energy density of the field can be written as $E^2 = \epsilon_0 \left( E_x^2 + E_y^2 + E_z^2 + B_x^2 + B_y^2 + B_z^2 \right)$, where $\epsilon_0$ is the permittivity of vacuum and we consider $c = 1$. We define the field-strength along different direction as $F_x = \sqrt{E_x^2 + B_y^2}$, $F_y = \sqrt{E_y^2 + B_x^2}$, $F_z = \sqrt{E_z^2 + B_z^2}$. Here, $E^2$ and $\vec{F}$ are real quantities.

Therefore, under coordinate transform, we want $E^2$ to remain unaltered and $\vec{F}$ to transform as a real vector, i.e. $\vec{F}$ must follow the rotation group SO(3). Lets assume

$$
\mathcal{F}_x = E_x + i B_x, \quad \mathcal{F}_y = E_y + i B_y, \quad \mathcal{F}_z = E_z + i B_z,
$$

(C.1)
i.e. $F_1^2 = \mathcal{F}_i \mathcal{F}_i, \forall i \in (x, y, z)$. We can treat the directions of the electric field and the magnetic field as two independent directions. As in this setup the $\mathcal{F}$ is a complex number and $\vec{E}$ and $\vec{B}$ are the real and the imaginary components of a complex number, we have total 6 independent coordinates which are forming the basis of the electromagnetic field.

First of all, lets keep the $E_z$ and $B_z$ components constant and consider the rotation between the X-Y plane only. A 2D complex space has total 4 real coordinates with some addition structure. If there is a 4D real space then we can rotate the coordinate system using SO(4), which is easy to visualize. However, all the SO(4) rotations can not be an independent SU(2) rotation. To give an example, in X-Y plane the 4 basis vectors, can be rotated in total 6 ways in accordance with SO(4)
rotation. However, the constrain is that under any type of rotation in X-Y plane, the $\vec{F}$ must rotate as SO(2) in X-Y plane. If we make any rotation between $E_x$ and $B_y$, which is a permitted under SO(4), then $\vec{F}$ does not transform in proper way unless we rotate some other axes in such a way that they restore the rotation of $\vec{F}$. Hence, the transformations under SU(2) are constrained.

C.2 Understanding the SU(3) rotation

A rotation matrix $R \in U(3)$ if it satisfies the condition $RR^T = 1$, where 1 is the unity matrix. To belong to SU(3) it must also satisfies the condition $\det R = 1$. There can be three forms of rotation in SU(3) or in any SU($n$).

For understanding the first kind of rotation, lets rotate the complex field in the in the $x$ direction in the complex plane. Here we are not rotating the vector in the $X, Y$ and $Z$ axes, instead, we are rotating it in the complex $X$ plane and the $Y$ and $Y$ planes are kept fixed. We consider the initial fields as

$$E_x = F_x \cos(\chi^z_0), \quad B_x = F_x \sin(\chi^z_0). \quad (C.2)$$

After rotating the fields by $\delta_\chi$ we have $(E'_x + iB'_x) = (E_x + iB_x) \exp(i\delta_\chi)$. Thus the rotation matrix for such transformation can be written as

$$\begin{bmatrix}
E'_x + iB'_y \\
E'_y + iB'_z \\
E'_z + iB'_x
\end{bmatrix} = \begin{bmatrix}
\exp(i\delta_\chi^z) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
E_x + iB_x \\
E_y + iB_y \\
E_z + iB_z
\end{bmatrix}. \quad (C.3)$$

Such rotation neither change $F_x$ nor the $E$ and hence its a valid rotation in the complex plane. Unfortunately, the determinant of the matrix is not unity and hence it is a U(3) transformation but not SU(3). In fact its a combination of U(1) and SU(3). However, if we combine this with another rotation in the $Y$ direction, i.e.

$$\begin{bmatrix}
\exp(i\delta_\chi^z) & 0 & 0 \\
0 & \exp(-i\delta_\chi^x) & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad (C.4)$$

then such transformation do comply with the unit determinant condition. We can use this technique with the other two axis pairs. The rotation matrices for those cases are

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & \exp(i\delta_\chi^z) & 0 \\
0 & 0 & \exp(-i\delta_\chi^x)
\end{bmatrix}, \quad \begin{bmatrix}
\exp(-i\delta_\chi^y) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \exp(i\delta_\chi^y)
\end{bmatrix}. \quad (C.5)$$

However, if we apply all the three rotations together then the complete rotation matrix will be

$$\begin{bmatrix}
\exp(i\delta_\chi^z - i\delta_\chi^y) & 0 & 0 \\
0 & \exp(i\delta_\chi^x - i\delta_\chi^z) & 0 \\
0 & 0 & \exp(i\delta_\chi^y - i\delta_\chi^x)
\end{bmatrix}. \quad (C.6)$$

The interesting thing here is to note that the sum of all the exponents are zero. What it mean is that if we apply rotation $\delta_\chi^x$ and $\delta_\chi^y$ such that $\delta_\chi^x - \delta_\chi^y = 0$, then that gives the third rotation. In other words, there are only two degrees of freedom. The third rotation can be performed by a combination of the first two rotations. We can redefine the variables as

$$\delta_\chi^y - \delta_\chi^x = \delta_\sigma_1 \quad (C.7)$$
$$\delta_\chi^z - \delta_\chi^x = \delta_\chi^z - \delta_\chi^y + \delta_\chi^y - \delta_\chi^x = \delta_\sigma_2 + \delta_\sigma_1 \quad (C.8)$$
$$\delta_\chi^y - \delta_\chi^z = -\delta_\sigma_2 \quad (C.9)$$
For small rotations Eq. C.6 gives us the two generators of SU(3) as

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} + i \delta \sigma_2
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix} + i \delta \sigma_1
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

(C.10)

There are other kinds of rotations, i.e., the rotation between any two real coordinates. We want \( \vec{F} \) to rotate as SO(3). If \( F_x \) and \( F_y \) rotate by an angle \( \theta \) along \( Z \) axis then the transformation can be written as

$$\begin{bmatrix}
F'_x \\
F'_y \\
F'_z
\end{bmatrix} =
\begin{bmatrix}
\cos(\theta) & \sin(\theta) & 0 \\
-\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
F_x \\
F_y \\
F_z
\end{bmatrix}.$$  

(C.11)

Here \( F_i, \forall i \in (x, y, z) \) are the absolute values of the complex electromagnetic field. This transformation can be done in 2 ways. Simplest possibility is that both the \( E \) and \( B \) fields rotate in the same way, which gives us,

$$\begin{bmatrix}
E'_x + iB'_x \\
E'_y + iB'_y \\
E'_z + iB'_z
\end{bmatrix} =
\begin{bmatrix}
\cos(\theta) & \sin(\theta) & 0 \\
-\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
E_x + iB_x \\
E_y + iB_y \\
E_z + iB_z
\end{bmatrix}.$$  

(C.12)

Of course, it satisfies all the conditions required. For small values of \( \theta \), we get

$$\delta \begin{bmatrix}
E_x + iB_x \\
E_y + iB_y \\
E_z + iB_z
\end{bmatrix} = i \theta \begin{bmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

(C.13)

This transformation rotates both the \( E \) and the \( B \) field independently in the \( X \) – \( Y \) plane. If we rotate along 3 different axes then we get 3 SU(3) generators as

$$\begin{bmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & i
\end{bmatrix},\begin{bmatrix}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{bmatrix},\begin{bmatrix}
0 & 0 & i \\
0 & 0 & 0 \\
i & 0 & 0
\end{bmatrix}.$$  

(C.14)

The third possibility is that while rotating \( \vec{F} \) by an angle \( \phi \) along \( Z \) axis, instead of rotating \( E_x - E_y \) and \( B_x - B_y \), lets rotate between the coordinate \( E_x - B_y \) and \( E_y - B_x \). This type of transformation can also lead us to Eq. C.11. This can be done by making the rotation matrix as

$$\begin{bmatrix}
E'_x + iB'_x \\
E'_y + iB'_y \\
E'_z + iB'_z
\end{bmatrix} =
\begin{bmatrix}
\cos(\phi) & i \sin(\phi) & 0 \\
i \sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
E_x + iB_x \\
E_y + iB_y \\
E_z + iB_z
\end{bmatrix}.$$  

(C.15)

Its easy to see that the total field strength \( \vec{F} \) under such transformation rotate as expected. For small rotation we can write the change as

$$\delta \begin{bmatrix}
E_x + iB_x \\
E_y + iB_y \\
E_z + iB_z
\end{bmatrix} = i \phi \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

(C.16)

Therefore, rotating along 3 different axes give us the final 3 generators of SU(3), which are

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix},\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix},\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.$$  

(C.17)
This particular kind of transformation can be considered as the boost in classical electromagnetism. Through this analysis we try to explain the rotations in a complex field using Electromagnetic field as an example to depict a visual picture of SU(3) rotation. This can help the readers to visualize the rotation of the 1-forms and the 2-forms in the internal coordinate space.

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