Ground-state magnetization curve of
a generalized spin-1/2 ladder

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Employing a method of exact diagonalization for finite-size systems, we investigate the magnetization curve in the ground state of an antiferromagnetic spin-1/2 ladder with additional exchange interactions on diagonal bonds, which is equivalent to an antiferromagnetic spin-1/2 chain with bond-alternating nearest-neighbor and uniform next-nearest-neighbor interactions. It is found that a half-plateau appears in the magnetization curve in a certain range of the interaction constants. This result is discussed in connection with the necessary condition for the appearance of the plateau, recently given by Oshikawa et al.

Key Words: ground-state magnetization curve, generalized spin-1/2 ladder, spin-1/2 chain with competing interactions, half-plateau

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There has been a considerable current interest in the study of quantum spin systems with competing interactions, which exhibit a variety of fascinating phenomena originating from frustration and quantum fluctuation. In this paper we investigate the magnetization curve in the ground state of an antiferromagnetic spin-1/2 ladder with additional exchange interactions on diagonal bonds. We express the Hamiltonian describing this system in an external magnetic field as

\[ H = H_0 + H_Z, \]

\[ H_0 = (1 + \alpha) \frac{N}{2} \sum_{\ell=1}^{N/2} \vec{S}_{1,\ell} \cdot \vec{S}_{2,\ell} + J \sum_{j=1}^{2} \sum_{\ell=1}^{N/2} \vec{S}_{j,\ell} \cdot \vec{S}_{j,\ell+1}, \]

\[ + (1 - \alpha) \frac{N}{2} \sum_{\ell=1}^{N/2} \vec{S}_{1,\ell} \cdot \vec{S}_{2,\ell+1}, \]

\[ H_Z = -H \sum_{j=1}^{2} \sum_{\ell=1}^{N/2} S_{j,\ell}^z, \]

where \( \vec{S}_{j,\ell} \) is the spin-1/2 operator at the \( j \)th leg and the \( \ell \)th rung; \( 1 + \alpha \) and \( 1 - \alpha \) (\( 0.0 \leq \alpha \leq 1.0 \)) are, respectively, the interaction constants between neighboring spins along the rung and diagonal bonds; \( J \) (\( J \geq 0.0 \)) is the interaction constant between neighboring spins along the leg bond; \( H \) (\( H \geq 0.0 \)) is the magnitude, in an appropriate unit, of the magnetic field applied along the \( z \)-axis; \( N \) is the total number of spins in the system and is assumed to be a multiple of four. We impose periodic boundary conditions (\( \vec{S}_{j,N+1} \equiv \vec{S}_{j,1} \)). It should be noted that this system is equivalent to an antiferromagnetic spin-1/2 chain with both bond-alternating nearest-neighbor interactions, the interaction constants being \( 1 + \alpha \) and \( 1 - \alpha \), and uniform next-nearest-neighbor interactions, the interaction constant being \( J \), which compete with each other.
The ground-state [1] and thermodynamic [2-4] properties of the system in the case of $\alpha=0.0$ and $H=0.0$ have been studied extensively. In particular, it has been found that as the value of $J$ increases, the phase transition from the massless spin-fluid phase to the massive dimer phase occurs at $J=J_c \sim 0.2412$ in the ground state [4-6]. As is well known, the exact ground-state magnetization curve in the case of $\alpha=J=0.0$ has been obtained by Griffiths [7]. The ground-state magnetization curves in the case of $\alpha=0.0$ [8] and in the case of $J=0.0$ [9] have also been numerically calculated for various values of $J$ and $\alpha$, respectively.

Obtaining the ground-state magnetization curve of the present system, we employ Sakai and Takahashi’s method [10], by the use of which they have discussed the ground-state magnetization curve for a uniform spin-1 chain. The outline of this method, which is based on a method of exact diagonalization for finite-size systems combined with the conformal field theory, can be summarized as follows. Let $E_0(N, M)$ be the lowest-energy eigenvalue, within the subspace determined by the value $M = \sum_{j=1}^{2} \sum_{\ell=1}^{N/2} S_j^z,\ell(=0, 1, \cdots, N/2)$, of the Hamiltonian $H_0$ for a given $N$. Then, the conformal field theory predicts that, if the state with the energy $E_0(N, M)$ is massless, the asymptotic behavior of $E_0(N, M)$ in the thermodynamic ($N \to \infty$) limit has the form [11]

$$\frac{E_0(N, M)}{N} \sim \varepsilon(m) - C(m) \frac{1}{N^2} \quad (N \to \infty)$$

with $m \equiv M/N$, where $\varepsilon(m)$ is the lowest energy per spin for a given $m$ in the thermodynamic limit and $C(m)$ is a positive constant which is proportional to the product of the central charge and the sound velocity. Minimizing with respect to $m$ the total energy, $\varepsilon_{\text{tot}} = \varepsilon(m) - Hm$, per spin of the system described by the Hamiltonian $\mathcal{H}$, we can obtain the equation which relates
the ground-state magnetization $\langle m \rangle$ per spin in the thermodynamic limit with $H$ as

$$\varepsilon'(\langle m \rangle) = H.$$  \hfill (3)

From Eq. (2) we can derive

$$\Delta E_0(N; M, M-1) \sim \varepsilon'(m_0) - \frac{1}{2} \varepsilon''(m_0) \frac{1}{N} \quad (N \to \infty), \quad (4a)$$

$$\Delta E_0(N; M+1, M) \sim \varepsilon'(m_{+0}) + \frac{1}{2} \varepsilon''(m_{+0}) \frac{1}{N} \quad (N \to \infty), \quad (4b)$$

where $\Delta E_0(N; M, M-1) \equiv E_0(N, M) - E_0(N, M-1)$. The essential point of Sakai and Takahashi’s method [10] is that as far as the massless states are concerned, Eqs. (4a,b) hold when $N$ is sufficiently large, and thus the value $H = \varepsilon'(m)$ [see Eq. (3)] of the magnetic field for a given $m$ ($0 < m < 1/2$) can be estimated by making an extrapolation which uses these equations to be $H = \varepsilon'(m_0) = \varepsilon'(m_{+0})$. For the estimation of $H_{c0} \equiv \varepsilon'(0)$, which is the critical field at which $\langle m \rangle$ starts to increase from zero in the ground-state magnetization curve, we apply Shanks’ transformation [12] to the sequences $\{\Delta E_0(N; 1, 0)\}$, following Sakai and Takahashi’s procedure [13]. Furthermore, the saturation field $H_s \equiv \varepsilon'(1/2)$ can be obtained analytically, since it is straightforward to diagonalize the Hamiltonian $\mathcal{H}_0$ within the $M = (N/2) - 1$ subspace; it is given by $H_s = \text{Max}(2, 1+2J+\alpha)$ in the present case. Recently, we have successfully applied the method to the case of an antiferromagnetic spin-1 chain with bond-alternating nearest-neighbor interactions and uniaxial single-ion-type anisotropy [14].

We obtain the ground-state magnetization curve in the thermodynamic limit for $J = 0.1$, 0.2, 0.3, and 0.4, for each of which various values of $\alpha$ are chosen. In the calculation we numerically diagonalize the Hamiltonian $\mathcal{H}_0$, using the computer program package KOBEPACK/S [15], to calculate $E_0(N, M)$ for $N=8, 12, \cdots, 24$. Then, we can make the analysis for
Our calculation shows that both $\Delta E_0(N; M + 1, M)$ and $\Delta E_0(N; M, M - 1)$ are almost linear functions of $1/N$ at least for $m = 1/8$ and $3/8$ in accordance with the forms given by Eqs. (4a,b). The values of $\varepsilon'(m_-0)$ and $\varepsilon'(m_+0)$ for these $m$’s can thus be estimated. In a similar way we estimate $\varepsilon'(m_-0)$ and $\varepsilon'(m_+0)$ for $m = 1/12, 1/6, 1/3, and 5/12$, assuming Eqs. (4a,b), although only two data are available. All the obtained results show that $\varepsilon'(m_-0)$ and $\varepsilon'(m_+0)$ coincide with each other within the numerical error. For $m = 1/4$, on the other hand, Eqs. (4a, b) do or do not hold depending upon the values of $\alpha$ and $J$. The case where Eqs. (4a, b) do not hold is the case where the state with $m = 1/4$ is massive and therefore $\varepsilon'(1/4_-0)$ is smaller than $\varepsilon'(1/4_+0)$. This means that, in the magnetization curve, there appears the half($\langle m \rangle = 1/4$)-plateau with the critical field $H_{c1} \equiv \varepsilon'(1/4_-0)$ at which the plateau starts and that $H_{c2} \equiv \varepsilon'(1/4_+0)$ at which it ends. We estimate the former and latter critical fields by applying Shanks' transformation \cite{12} to the sequences \{$\Delta E_0(N; N/4, N/4 - 1)$\} and \{$\Delta E_0(N; N/4 + 1, N/4)$\}, respectively.

We find that the half-plateau appears in the magnetization curve when $0.5 \lesssim \alpha \lesssim 0.95$ for $J = 0.1$, when $0.2 \lesssim \alpha \lesssim 0.85$ for $J = 0.2$, when $0.0 \lesssim \alpha \lesssim 0.8$ for $J = 0.3$, and when $0.0 < \alpha \lesssim 0.75$ for $J = 0.4$. As an example, we depict in Fig. 1 the magnetization curve with the half-plateau, obtained for $J = 0.2$ and $\alpha = 0.5$. Plotting versus $\alpha$ the critical fields $H_{c0}$, $H_{c1}$, and $H_{c2}$ as well as the saturation field $H_s$, we can draw the ground-state phase diagram on the $H$ versus $\alpha$ plane; the result for $J = 0.2$ is shown in Fig. 2.

We also calculate numerically the eigenfunctions of the lowest- and second-lowest-energy states within the $M = N/4$ subspace for the finite-$N$ systems, and find that at least for a set of $J$ and $\alpha$ giving the plateau,
the lowest-energy state for $M = N/4$ in the thermodynamic limit is doubly degenerate, one of the eigenfunctions of which has the periodicity $n = 4$ (in units of the lattice constant) concerning the translational symmetry. This result is consistent with the necessary condition $n(S - \langle m \rangle) = \text{integer}$ for the appearance of the plateau with the magnetization $\langle m \rangle$ ($S$ is the magnitude of spins), which has recently been given by Oshikawa, Yamanaka, and Affleck [16]. Finally, it should be noted that very recently Totsuka [17] has clarified in an excellent way the mechanism for the appearance of the plateau in the present system, using a bosonization technique. According to this work, it becomes clear that the next-nearest-neighbor interaction plays a crucial role in the appearance of the plateau.

**Acknowledgements**

We would like to thank Drs. M. Hagiwara, K. Totsuka, and M. Yamanaka for invaluable discussions. We also thank the Supercomputer Center, Institute for Solid State Physics, University of Tokyo and the Computer Center, Tohoku University for computational facilities. The present work has been supported in part by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture, Japan.
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Figure Captions

Fig. 1. Ground-state magnetization curve in the thermodynamic limit obtained for $J=0.2$ and $\alpha=0.5$. The closed circles show plots of $\langle m \rangle$ versus $H=\varepsilon'(\langle m \rangle)$. The solid lines are guides to the eye.

Fig. 2. Ground-state phase diagram on the $H$ versus $\alpha$ plane in the thermodynamic limit for $J=0.2$. The closed circles show plots versus $\alpha$ of the critical fields $H_{c0}$, $H_{c1}$, and $H_{c2}$ and also of the saturation field $H_s$. The solid lines are guides to the eye. The magnetization $\langle m \rangle$ is given by $\langle m \rangle=0$, $1/4$, and $1/2$ in the regions A, B, and C, respectively. In the remaining region $\langle m \rangle$ increases continuously as $H$ increases.
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