Left-Right Symmetric Model from Geometric Formulation of Gauge Theory in $M_4 \times Z_2 \times Z_2$

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The left-right symmetric model (LRSM) with gauge group $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ is reconstructed from the geometric formulation of gauge theory in $M_4 \times Z_2 \times Z_2$ where $M_4$ is the four-dimensional Minkowski space and $Z_2 \times Z_2$ the discrete space with four points. The geometrical structure of this model becomes clearer compared with other works based on noncommutative geometry. As a result, the Yukawa coupling terms and the Higgs potential are derived in more restricted forms than in the standard LRSM.

§1. Introduction

Recently gauge theories on the space $M_4 \times Z_N$, where $M_4$ is the four-dimensional Minkowski space and $Z_N$ the discrete space with $N$ points, have been formulated in terms of noncommutative geometry (NCG) by Connes1) and in various alternative versions of this theory2) to apply to spontaneously broken gauge theories based on the Higgs mechanism. In these theories the Higgs fields are regarded as gauge fields along the discrete space $Z_N$.

All existing approaches based on NCG, however, appear to be too algebraic, rather than geometric, and hence their geometrical meanings have not been clarified in depth. Recently an alternative formulation of the gauge theory in $M_4 \times Z_N$ has been proposed from purely geometrical point of view, without employing the entire context of NCG.3) In this approach the Higgs fields are introduced as mapping functions between any pair of vector fields, each of which is defined independently on a sheet in the $N$-sheeted space-time. This theory has been applied to several physical models and their geometrical structures have been clarified.4)

In the present paper, we reconstruct the left-right symmetric model $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ (LRSM)5) in terms of the geometric formulation of the gauge theory in $M_4 \times Z_2 \times Z_2$. Our approach leads to the LRSM quite successfully and the geometrical structure of the model becomes clearer compared to other works based on NCG.6)

It has long been suspected that the standard model is not a final theory. Rather it will be replaced by some more unified theory at higher energies. The recent

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discovery of neutrino flavor oscillation stimulated our imagination to find such a theory. The LRSM is one of such unified theories that generates the light neutrino mass by the seesaw mechanism. Let us summarize the LRSM in its simplest form in the following. The quark sectors are not considered here and only the lepton sector of the first generation will be taken into account. The standard Lagrangian of the LRSM is given by

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_Y + \mathcal{L}_B,$$ (1.1)

where

$$\mathcal{L}_F = i\bar{l}_L \gamma^\mu \left( \partial_\mu + i \frac{1}{2} g_1 B_\mu - i g_2 W^L_\mu \right) l_L$$

$$+ i\bar{l}_R \gamma^\mu \left( \partial_\mu + i \frac{1}{2} g_1 B_\mu - i g_2 W^R_\mu \right) l_R$$ (1.2)

$$\mathcal{L}_Y = -\bar{l}_L (f \phi + \tilde{f} \tilde{\phi}) l_R + \text{h.c.}$$

$$-i\bar{l}_L^T C \sigma_2 h \Delta_L l_L + \text{h.c.}$$

$$-i\bar{l}_R^T C \sigma_2 h \Delta_R l_R + \text{h.c.},$$ (1.3)

$$\mathcal{L}_B = \text{tr} |D_\mu \Delta_L|^2 + \text{tr} |D_\mu \Delta_R|^2 + \text{tr} |D_\mu \phi|^2$$

$$+ \text{Yang-Mills terms of } B_\mu, W^L_\mu, W^R_\mu$$

$$- V(\text{Higgs potential of } \phi, \Delta_L, \Delta_R),$$ (1.4)

and

$$l_L = \left( \begin{array}{c} \nu_e \\ e \end{array} \right), \quad l_R = \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_R,$$ (1.5)

$$\tilde{\phi} \equiv \tau_2 \phi^* \tau_2.$$ (1.6)

Here $l_L$ is the left-handed lepton which is the $SU(2)_L$ doublet with the $U(1)$ charge $-1$, while $l_R$ is the right-handed lepton which is the $SU(2)_R$ doublet with the $U(1)$ charge $-1$. Three Higgs fields $\phi, \Delta_L$ and $\Delta_R$ carry the following $SU(2)_L, SU(2)_R$ and $U(1)_{B-L}$ quantum numbers, respectively;

$$\phi(1/2, 1/2^*, 0), \quad \Delta_L(1, 0, 2), \quad \Delta_R(0, 1, 2).$$ (1.7)

The representations of the fields are conveniently given by the $2 \times 2$ matrices as

$$\phi = \begin{pmatrix} \phi_1^0 \\ \phi_2^0 \end{pmatrix}, \quad \phi^* = \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix},$$ (1.8)

$$\Delta_L = \begin{pmatrix} \delta_L^+ / \sqrt{2} \\ \delta_L^0 \end{pmatrix}, \quad \Delta_R = \begin{pmatrix} \delta_R^+ / \sqrt{2} \\ \delta_R^0 \end{pmatrix},$$ (1.9)

The gauge fields associated with $SU(2)_L, SU(2)_R$ and $U(1)_{B-L}$ are $W^L_\mu, W^R_\mu$ and $B_\mu$, respectively. This Lagrangian is invariant under the left-right symmetry defined by

$$l_L \leftrightarrow l_R, \quad \Delta_L \leftrightarrow \Delta_R, \quad \phi \leftrightarrow \phi^\dagger.$$ (1.10)
The most general Higgs potential which is invariant under gauge transformations and the operation \((1\cdot10)\) is given in Ref.[9]. Inspite of the left-right symmetry of the Higgs potential, we can choose a vacuum such that the vacuum expectation values (VEV) of the Higgs fields break the left-right symmetry. Let us parametrize the VEV of each Higgs field as

\[
\langle \phi \rangle_0 = \left( \frac{\kappa_1}{\sqrt{2}} \ 0 \right), \quad \langle \Delta_{L,R} \rangle_0 = \left( \begin{array}{c} 0 \\ v_{L,R} \end{array} \right),
\]

where we assume

\[
|v_L| \ll |\kappa_1|, |\kappa_2| \ll |v_R|
\]

from the phenomenological reason. Then the charged-lepton mass is given by

\[
m_{l^+} \simeq \frac{1}{\sqrt{2}} |f\kappa_2 + \tilde{f}\kappa_1|,
\]

whereas the neutrino masses are

\[
m_{\nu_R} \simeq \sqrt{2}|hv_R|,
\]

\[
m_{\nu_L} \simeq \sqrt{2}\left|h v_L - \frac{f\kappa_1 + \tilde{f}\kappa_2}{4h v_R}\right|.
\]

The gauge boson masses are

\[
m^2_{W_L} \simeq \frac{1}{4} f^2 \left(|\kappa_1|^2 + |\kappa_2|^2\right) \sim m^2_Z,
\]

\[
m^2_{W_R} \simeq \frac{1}{2} g^2_2 |v_R|^2 \sim M^2_X.
\]

It is found from (1.12) that \(m_{\nu_R}, m_{W_R}\) and \(m_X\) are too heavy to observe.

\[\text{§2. LRSM in } M_4 \times Z_2 \times Z_2 \text{ space}\]

2.1. The structure of \(Z_2 \times Z_2\) group

Let the discrete points of the first \(Z_2\) be \(p_1 = (1,-1)\) and those of the second \(Z_2\) be \(p_2 = (1,-1)\). Then four points of \(Z_2 \times Z_2\) are give by (see Fig.1)

\[
g_p = (p_1, p_2), \quad p = 0, 1, 2, 3
\]

that is,

\[
g_0 = (1,1), g_1 = (-1,1), g_2 = (1,-1), g_3 = (-1,-1).
\]

They are subject to the algebraic relations

\[
g_0 g_p = g_p, \quad g_p g_0 = g_0, \quad (p = 0, 1, 2, 3)
\]

\[
g_1 g_2 = g_3, \quad \text{and permutations of } (1, 2, 3).
\]

We attach the \(SU(2) \times U(1)\) internal vector space to each point of \((x, g_p)\), where \(x \in M_4\). The four internal spaces are independent of each other. Fermionic fields \(\psi(x, g_p) \equiv \psi_p\) are assigned on \((x, g_p) \equiv p\), and they are extended spinors \(\times SU(2)\) spinors in each manifold.
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\[ g_0 = (1, 1) \quad g_1 = (-1, 1) \]

\[ g_2 = (-1, 1) \quad g_3 = (-1, -1) \]

Fig. 1. Four discrete points of \( Z_2 \times Z_2 \).

2.2. Fundamental requirements

We require that our model has the following invariances:

(i) Local invariance under the \( SU(2) \times U(1) \) gauge transformation at each point \( p \).

(ii) The extended \( G \)-conjugation invariance, which will be explained in \( \S 2.5 \).

(iii) Invariance under the \( Z_2 \times Z_2 \) discrete transformation.

According to (i), the \( SU(2) \) and \( U(1) \) gauge fields in \( M_4 \) are introduced through the covariant derivatives of \( \psi_p \) at \( p \) as

\[ D_\mu \psi_p = (\partial_\mu - iA_{p\mu}) \psi_p, \quad (2.3) \]

where

\[ A_{p\mu} = B_{p\mu} + W_{p\mu}, \quad (2.4) \]

\( B_{p\mu} \) being the \( U(1) \) gauge field while \( W_{p\mu} \) the \( SU(2) \) gauge field. Connections in \( Z_2 \times Z_2 \) are introduced as mapping functions of \( \psi_q \) from \( q \) to \( p \) as

\[ \psi_q \rightarrow \Phi_{pq} \psi_q \quad (2.5a) \]

with the property

\[ \Phi_{pq} = \Phi_{qp}^\dagger, \quad (2.5b) \]

where the mapping function \( \Phi_{pq} \) “parallel-transport” \( \psi_q \) to the fiber on \( p \) and will be identified with the Higgs fields. Here we add the fourth requirement for our model construction:

(iv) The mapping functions \( \Phi_{pq} \) are given only along each constituent \( Z_2 \), namely, along each side of the square depicted in Fig.1: \( 0 \leftrightarrow 1, 2 \leftrightarrow 3, 0 \leftrightarrow 2 \) and \( 1 \leftrightarrow 3 \).

(There are no direct connections along the diagonals: \( 0 \leftrightarrow 3 \) and \( 1 \leftrightarrow 2 \).)

Under gauge transformations \( U_p \) of \( \psi_p \), the mapping function \( \Phi_{pq} \) should obey the following rule,

\[ \Phi_{pq} \rightarrow \Phi'_{pq} = U_p \Phi_{pq} U_q^{-1}, \quad (2.6) \]

just like in lattice gauge theories. According to this transformation rule, we can regard \( \Phi_{pq} \) as gauge fields in \( Z_2 \times Z_2 \).
2.3. Fermionic Lagrangian

The fermionic Lagrangian in $M_4 \times Z_2 \times Z_2$ without gauge fields is assumed to be

$$\mathcal{L}_F^0 = i \sum_p \bar{\psi}_p \gamma^\mu \partial_\mu \psi_p - \sum_{p,q}^\prime \kappa_{p+q} \bar{\psi}_p \gamma^{p+q} \psi_q,$$

(2.7)

where $\kappa_{p+q}$ are real mass parameters and $\sum^\prime$ implies the summation over $p$ and $q$ with the constraint $p + q = h = 1, 2$ is taken. The last condition follows since we take into account only of interactions between nearest-neighbor points in Fig.1. The Dirac matrices $\gamma^h$ ($h = 1, 2$) are associated with the $Z_2 \times Z_2$ space and the whole set of the $\gamma$ matrices $\{\gamma^\mu, \gamma^h\}$ satisfies the six-dimensional Clifford algebra. The extra Dirac matrices are chosen as

$$\gamma^h = i \gamma_5 \sigma^h \ (h = p + q = 1, 2)$$

(2.8)

where $\sigma^h$ are the Pauli matrices. We have discarded here the term with $p = q$, which would arise if one naively replaces the derivative in the continuum part by the difference. This amounts to proposing a new type of interaction between fermion fields on a discrete space. As will be seen below, this choice of interaction is crucial in our theory.

If we require that the Lagrangian (2.7) is locally invariant under the $SU(2) \times U(1)$ gauge transformations in $M_4 \times Z_2 \times Z_2$, we should introduce gauge fields (2.4) and $\Phi_{pq} = \Phi^\dagger_{qp}$ into $\mathcal{L}_F^0$ as

$$\mathcal{L}_F = i \sum_p \bar{\psi}_p \gamma^\mu D_\mu \psi_p - \sum_{p,q}^\prime \kappa_{h} \bar{\psi}_p \gamma^h \Phi_{pq} \psi_q, \ (h = p + q = 1, 2).$$

(2.9)

The relevant mapping functions are

$$\Phi_{01}, \Phi_{23} \text{ for } h = 1,$$

(2.10)

and

$$\Phi_{20}, \Phi_{31} \text{ for } h = 2.$$

(2.11)

The interaction terms in Eq.(2.9) are composed of $A_{p\mu}$ and $\Phi_{pq}$ as

$$\mathcal{L}_I = \mathcal{L}_1 + \mathcal{L}_2,$$

where

$$\mathcal{L}_1 = \sum_p \bar{\psi}_p \gamma^\mu A_\mu \psi_p,$$

$$\mathcal{L}_2 = -i \sum_{p,q}^\prime \kappa_h \bar{\psi}_p \gamma_5 \sigma^h \Phi_{pq} \psi_q, \ (h = p + q = 1, 2).$$

(2.12)

2.4. Superselection rules

For the fermionic interaction Lagrangian $\mathcal{L}_I$ we find two kinds of superselection rules:

(I) Let us define $\Gamma_5$ in the (4+2)-dimensional space by

$$\Gamma_5 = \gamma_5 \sigma_3.$$
Then $\Gamma_5 = 1$ and $\Gamma_5 = -1$ components of $\psi_p$ are decoupled with each other.

(II) Let us define $\nu$ for the group element $g_p = (p_1, p_2)$ by

$$\nu = p_1 p_2 = \pm 1.$$  \hspace{1cm} (2.14)

One can see that the interaction $\mathcal{L}_1$ does not change signs of $\nu$ and $\sigma_3$, whereas the interaction $\mathcal{L}_2$ changes $\nu \rightarrow -\nu$ and $\sigma_3 \rightarrow -\sigma_3$. Hence, the product $\nu \sigma_3$ does not change the sign for both $\mathcal{L}_1$ and $\mathcal{L}_2$. Therefore the components of $\psi_p$ with $\nu \sigma_3 = 1$ and $\nu \sigma_3 = -1$ decouple with each other.

From (I) and (II) we can divide the fermionic field into four sectors. Let us consider one of them,

$$\Gamma_5 = -1 \text{ and } \nu \sigma_3 = 1,$$ \hspace{1cm} (2.15)

for example. Since $\Gamma_5 = \gamma_5 \sigma_3 = -1$ for this choice, the fermionic field $\psi_p$ should be of the form

$$\psi_p = \begin{pmatrix} \psi_{pL} \\ \psi_{pR} \end{pmatrix} \cdots \sigma_3 = +1, \cdots \sigma_3 = -1,$$ \hspace{1cm} (2.16)

where $\psi_L (\psi_R)$ is the left(right)-handed component of $\psi$, satisfying $\gamma_5 \psi_L = -\psi_L (\gamma_5 \psi_R = \psi_R)$. Then one obtains from $\nu \sigma_3 = 1$

$$\sigma_3 = \nu = +1 : \psi_0 = \begin{pmatrix} \psi_{0L} \\ 0 \end{pmatrix}, \psi_3 = \begin{pmatrix} \psi_{3L} \\ 0 \end{pmatrix},$$ \hspace{1cm} (2.17)

$$\sigma_3 = \nu = -1 : \psi_1 = \begin{pmatrix} 0 \\ \psi_{1R} \end{pmatrix}, \psi_2 = \begin{pmatrix} 0 \\ \psi_{2R} \end{pmatrix}. \hspace{1cm} (2.18)$$

In terms of these assignments of $\psi_p$, the interaction Lagrangians can be written as

$$\mathcal{L}_1 = \bar{\psi}_{0L} \gamma \cdot A_0 \psi_{0L} + \bar{\psi}_{1R} \gamma \cdot A_1 \psi_{1R} + \bar{\psi}_{2R} \gamma \cdot A_2 \psi_{2R} + \bar{\psi}_{3L} \gamma \cdot A_3 \psi_{3L},$$ \hspace{1cm} (2.19)

$$\mathcal{L}_2 = -i \kappa_1 \left( \bar{\psi}_{0L} \gamma^5 \sigma^1 \Phi_{01} \psi_0 + \bar{\psi}_{2R} \gamma^5 \sigma^1 \Phi_{23} \psi_3 \right) + h.c. = \bar{\psi}_{0L} \gamma^5 \Phi_{01} \psi_0 \psi_1 - \bar{\psi}_{2R} \gamma^5 \Phi_{23} \psi_3 \psi_1 + h.c.$$ \hspace{1cm} (2.20)

2.5. **Extended $G$-conjugation**

Let us return to the section in which we have not selected one of the sectors. The extended $G$-conjugation of $\psi_p$ is defined by

$$\psi_p \rightarrow U_G \psi_p U_G^{-1} = \lambda_p (-i \sigma_2)(-i \tau_2) \psi_p^c \equiv \lambda_p \psi_p^G, \hspace{1cm} (2.21)$$

where $\psi_p^G$ is the charge conjugate field of $\psi_p$ and $\lambda_p$ is the $G$-parity of $\psi_p$. We set $\lambda_p = 1$ in the following, for simplicity, and hence

$$\psi_p^{GG} = \psi_p.$$ \hspace{1cm} (2.22)
For each of $\sigma_3$-components of $\psi_p$, we have
\[
\begin{pmatrix}
\psi_{p+} \\
\psi_{p-}
\end{pmatrix}^G = \begin{pmatrix}
-\psi_{p-}^g \\
\psi_{p+}^g
\end{pmatrix},
\] (2.23)
where $\psi_{p\pm}^g$ is the conventional (i.e., without $-i\sigma_2$) $G$-conjugation of $\psi_{p\pm}$ and
\[
\psi_{p\pm}^{gg} = -\psi_{p\pm}.
\] (2.24)

Note the following formulae
\[
\bar{\psi}_p(1, \gamma_5)\psi_q = \psi_{Gp}^G(1, \gamma_5)\psi_{Gq}^G,
\] (2.25)
\[
\bar{\psi}_p(\gamma_\mu, \vec{\tau}, \sigma_h)\psi_q = -\psi_{Gq}^G(\gamma_\mu, \vec{\tau}, \sigma_h)\psi_{Gp}^G
\] (2.26)
and
\[
\Omega \rightarrow U_G\Omega U_G^{-1} = \lambda_\Omega (-i\tau_2)\Omega^* (-i\tau_2)^{-1} = \lambda_2 \tilde{\Omega} \equiv \lambda_\Omega \Omega^G,
\] (2.27)
where $\Omega$ stands for the gauge field $A_{\mu\nu}$ or $\Phi_{pq}$. The $G$-parity of each gauge field should be $\lambda_\Omega = -1$, according to Eqs. (2·24) $\sim$ (2·26), for the interaction Lagrangian $L_I$ be $G$-invariant.

2.6. Kinetic terms of Higgs fields

Let us calculate the curvature associated with paths in Fig. 2. The fermionic field $\psi_p(x)$ is mapped from $(x, g_p)$ to $(x + \delta x, g_q)$ through paths $C_1$ and $C_2$. The two images are
\[
\psi(C_1) = \Phi_{qp}(x + \delta x) [1 + iA_{\mu\nu}(x)\delta x^\mu] \psi_p(x),
\] (2.28)
\[
\psi(C_2) = [1 + iA_{\mu\nu}(x)\delta x^\mu] \Phi_{qp}(x)\psi_p(x).
\] (2.29)
The difference between them yields
\[
\psi(C_1) - \psi(C_2) = D_\mu \Phi_{qp}(x)\delta x^\mu \psi_p(x),
\] (2.30)
where
\[
D_\mu \Phi_{qp} = \partial_\mu \Phi_{qp} - i [A_{\mu\nu}, \Phi_{qp} - \Phi_{qp}A_{\mu\nu}] .
\] (2.31)
The covariant derivative of $\Phi_{qp}$ is, therefore, just the curvature associated with Fig 2.
2.7. Identification of fields

We first note that our system is $G$-invariant and also has a symmetry under the translation $p \rightarrow p + h$ ($h = 1, 2$). The latter translation symmetry is obvious for $i \sum_p \bar{\psi}_p \gamma^\mu D_\mu \psi_p$ and $\sum_{p,q} |D_\mu \Phi_{pq}|^2$. However, it is not so clear that $L_2$ has this invariance. Let us write $L_2$ explicitly again to prove this;

$$\sum_{p,q} i \kappa_h \bar{\psi}_p \gamma^h \Phi_{qp} \psi_p = \kappa_1 \left( \bar{\psi}_0 \gamma^1 \Phi_{01} \psi_1 + \bar{\psi}_2 \gamma^1 \Phi_{23} \psi_3 \right) + \kappa_2 \left( \bar{\psi}_2 \gamma^2 \Phi_{20} \psi_0 + \bar{\psi}_3 \gamma^2 \Phi_{31} \psi_1 \right) + \text{h.c.} \quad (2.32)$$

Equation (2.32) is invariant under the interchange $0 \leftrightarrow 1, 2 \leftrightarrow 3$, with suffices of $\kappa$ and $\gamma$ kept fixed, due to the Hermiticity $\Phi_{pq} = \Phi_{qp}^\dagger$. This is the translational symmetry of the type $p \rightarrow p + 1$. It is also invariant under $0 \leftrightarrow 2, 1 \leftrightarrow 3$, namely, the translational symmetry of the type $p \rightarrow p + 2$.

Consequently our system is invariant under the $G$-conjugation combined with the translation $p \rightarrow p + h$ ($h = 1, 2$), i.e.,

$$\psi_p \rightarrow \psi_{p+h}^G, \quad A_p \rightarrow -A_{p+h}^G, \quad \Phi_{pq} \rightarrow -\Phi_{p+q,h}^G. \quad (2.33)$$

From Eqs.(2.16 ~ 18) and (2.22), the first one $\psi_p \rightarrow \psi_{p+h}^G$ is reduced to

$$\psi_{pL} \rightarrow -\psi_{p+h,R}^g (p = 0, 3), \quad \psi_{pR} \rightarrow \psi_{p+h,L}^g (p = 1, 2). \quad (2.34)$$

Now we have too many independent fields to reconstruct the LRSM from our geometric formulation of gauge theory in $M_4 \times Z_2 \times Z_2$. Some of the fields must be removed. In view of the invariance (2.33) and (2.34), it is natural to assume

$$\psi_p = \psi_{p+2}^G, \quad A_p = -A_{p+2}^G, \quad \Phi_{pq} = -\Phi_{p+2,q+2}^G. \quad (2.35)$$

Another possible choice $\psi_p = \psi_{p+1}^G$, etc. will lead to the same LRSM. Let the independent fields be $\psi_{0L}, \psi_{1R}, A_0, A_1, \Phi_{01}, \Phi_{20}, \Phi_{31}$. Then we set them in familiar notations introduced in §1,

$$\begin{align*}
\psi_{0L} &= l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad \psi_{1R} = l_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}, \\
A_0 &= B_0 + W_L, \quad A_1 = B_1 + W_R, \\
\Phi_{01} &= \phi, \\
\Phi_{20} &= \Delta_L, \quad \Phi_{31} = \Delta_R.
\end{align*} \quad (2.36)$$

(see Fig.3). From Eq.(2.35) other fields are given by

$$\begin{align*}
\psi_{2R} &= \psi_{0L}^g = l_L^g = -i \tau_2 C^{-1} l_L^T, \\
\psi_{3L} &= -\psi_{1R}^g = -l_R^g = i \tau_2 C^{-1} l_R^T, \\
A_2 &= -A_0^g = -\bar{A} = -(B_0 + W_L) = -B_0 + W_L, \\
A_3 &= -A_1^g = -\bar{A}_1 = -B_1 + W_R, \\
\Phi_{23} &= -\phi_{01}^g = -\phi^g = -\phi.
\end{align*} \quad (2.37)$$
Here we have still two kinds of $U(1)$ gauge fields $B_0$ and $B_1$, which are defined independently on $M_4 \times \{g_0\}$ and $M_4 \times \{g_1\}$. If we identify $B_0$ with $B_1$, i.e., $B_0 = B_1 = B$, we have the standard LRSM, except for the Higgs potential. This will be studied in the next subsection.

In order to normalize the kinetic term, we rescale the fermions as $\psi_p \rightarrow \frac{1}{\sqrt{2}} \psi_p$. The $\gamma_5$ factor in $L_2$ can be removed by redefining $\psi_p$ as

$$e^{i\pi \gamma_5/4} \psi_p \rightarrow \psi_p,$$

where the use has been made of the identity $i\gamma_5 = e^{i\pi \gamma_5/2}$. The interaction Lagrangians $L_1$ and $L_2$ in Eqs.(2.19) and (2.20) are reduced to

$$L_1 = \frac{1}{4} \bar{l}_L \gamma \cdot A_0 l_L + \frac{1}{2} \bar{l}_R \gamma \cdot A_1 l_R + \frac{1}{4} \bar{l}_L \gamma \cdot A_2 l_L^g + \frac{1}{2} \bar{l}_R \gamma \cdot A_3 l_R^g,
L_2 = -\kappa_1 (\bar{l}_L \phi_R + \bar{l}_R \phi_L)
- \frac{1}{2} i\kappa_2 \left( \bar{l}_L \Delta_L l_L + \bar{l}_R \Delta_R l_R - \bar{l}_L \Delta_L \bar{l}_L^g l_L^g - \bar{l}_R \Delta_R \bar{l}_R^g l_R^g \right).$$

Covariant derivatives of Higgs fields (2.30) are

$$D\phi = \partial\phi - i(W_L\phi - \phi W_R),
D\Delta_L = \partial\Delta_L + 2iB\Delta_L - i(W_L \Delta_L - \Delta_L W_L),
D\Delta_R = \partial\Delta_R + 2iB\Delta_R - i(W_R \Delta_R - \Delta_R W_R).$$

2.8. Higgs potential

Let us calculate the curvature associated with the paths in Fig.4, where mapping functions $\Phi_{p0} = \Phi_{p0}^\dagger$ are still independent of each other. Identification of the fields will be introduced later. The fermion field $\psi_p$ on $(x, g_p)$ is mapped from $(x, g_p)$ to $(x, g_{p+3})$ through two paths $p \rightarrow p + 1 \rightarrow p + 3$ and $p \rightarrow p + 2 \rightarrow p + 3$. The difference between the two mappings, namely the holonomy associated with the paths, yields the curvature

$$G_{p+3,p} = \Phi_{p+3,p+1} \Phi_{p+1,p} - \Phi_{p+3,p+2} \Phi_{p+2,p} = G_{p,p+3}^L.$$

Fig. 3. Assignments of fields on $Z_2 \times Z_2$. 

\[
\begin{array}{c}
\begin{array}{ccc}
  & l^0_L & -\phi \\
 g_2 & & g_3
\end{array} \\
\hline
\Delta_L & & \Delta_R \\
\hline
\begin{array}{ccc}
  l_L & \phi & l_R \\
g_0 & & g_1
\end{array}
\end{array}
\]

Here we have still two kinds of $U(1)$ gauge fields $B_0$ and $B_1$, which are defined independently on $M_4 \times \{g_0\}$ and $M_4 \times \{g_1\}$. If we identify $B_0$ with $B_1$, i.e., $B_0 = B_1 = B$, we have the standard LRSM, except for the Higgs potential. This will be studied in the next subsection.

In order to normalize the kinetic term, we rescale the fermions as $\psi_p \rightarrow \frac{1}{\sqrt{2}} \psi_p$. The $\gamma_5$ factor in $L_2$ can be removed by redefining $\psi_p$ as

$$e^{i\pi \gamma_5/4} \psi_p \rightarrow \psi_p,$$

where the use has been made of the identity $i\gamma_5 = e^{i\pi \gamma_5/2}$. The interaction Lagrangians $L_1$ and $L_2$ in Eqs.(2.19) and (2.20) are reduced to

$$L_1 = \frac{1}{4} \bar{l}_L \gamma \cdot A_0 l_L + \frac{1}{2} \bar{l}_R \gamma \cdot A_1 l_R + \frac{1}{4} \bar{l}_L \gamma \cdot A_2 l_L^g + \frac{1}{2} \bar{l}_R \gamma \cdot A_3 l_R^g,
= \bar{l}_L \gamma \cdot (B + W_L) l_L + \bar{l}_R \gamma \cdot (B + W_R) l_R,$$

$$L_2 = -\kappa_1 (\bar{l}_L \phi_R + \bar{l}_R \phi_L)
- \frac{1}{2} i\kappa_2 \left( \bar{l}_L \Delta_L l_L + \bar{l}_R \Delta_R l_R - \bar{l}_L \Delta_L \bar{l}_L^g l_L^g - \bar{l}_R \Delta_R \bar{l}_R^g l_R^g \right).$$

Covariant derivatives of Higgs fields (2.30) are

$$D\phi = \partial\phi - i(W_L\phi - \phi W_R),
D\Delta_L = \partial\Delta_L + 2iB\Delta_L - i(W_L \Delta_L - \Delta_L W_L),
D\Delta_R = \partial\Delta_R + 2iB\Delta_R - i(W_R \Delta_R - \Delta_R W_R).$$

2.8. Higgs potential

Let us calculate the curvature associated with the paths in Fig.4, where mapping functions $\Phi_{p0} = \Phi_{p0}^\dagger$ are still independent of each other. Identification of the fields will be introduced later. The fermion field $\psi_p$ on $(x, g_p)$ is mapped from $(x, g_p)$ to $(x, g_{p+3})$ through two paths $p \rightarrow p + 1 \rightarrow p + 3$ and $p \rightarrow p + 2 \rightarrow p + 3$. The difference between the two mappings, namely the holonomy associated with the paths, yields the curvature

$$G_{p+3,p} = \Phi_{p+3,p+1} \Phi_{p+1,p} - \Phi_{p+3,p+2} \Phi_{p+2,p} = G_{p,p+3}^L.$$
Fig. 4. The mapping functions between points of $Z_2 \times Z_2$. The holonomy associated with two paths defines the curvature $G_{p+3,p}$.

Fig. 5. The path defining the curvature $F_{p,p+h,p}$.

There is also a different type of curvature arising from Fig. 5. This is the same type as in the Weinberg-Salam electroweak model. That is, $\psi_p$ is compared with its image $\psi_p(C_3 \cdot C_4)$, which is obtained by mapping $\psi_p$ through the path $C_3 \cdot C_4$, i.e.,

$$
\psi_p(C_3 \cdot C_4) - \psi_p = (\Phi_{p,p+h}\Phi_{p+h,p} - 1) \psi_p.
$$

(2.45)

This defines the curvature

$$
F_{p,p+h,p} \equiv \Phi_{p,p+h}\Phi_{p+h,p} - 1, \quad (h = 1, 2).
$$

(2.46)

These curvatures take finite values in a discrete space although they vanish in a continuous space.

Since Higgs fields are gauge fields, the Higgs potential must be of the Yang-Mills type and gauge-invariant. Gauge invariant combinations of curvatures are as follows:

$$
A_{p:h,k} \equiv \frac{1}{2} \text{tr} (F_{p,p+h}F_{p+p+k,p}) \quad (p = 0, 1, 2, 3; \ h, k = 1, 2)
$$

(2.47)

$$
B_{p,q:h,k} \equiv \frac{1}{2} \text{tr} (F_{p,p+h,p}) \frac{1}{2} \text{tr} (F_{q,q+k,q}) \quad (p, q = 0, 1, 2, 3; \ h, k = 1, 2)
$$

(2.48)

$$
\Gamma_p \equiv \frac{1}{2} \text{tr} (G_{p,p+3}G_{p+3,p}) \quad (p = 0, 1)
$$

(2.49)

Then the Higgs potential is written as a linear combinations of them,

$$
V = \alpha_{11} \sum_{p=0}^{1} A_{p:1,1} + \alpha_{12} \sum_{p=0}^{1} A_{p:2,2} + \alpha_{2} \sum_{p=0}^{3} A_{p:1,2}
$$

$$
+ \beta_{11} \sum_{p=0}^{1} B_{p,p:1,1} + \beta_{12} \sum_{p=0}^{1} B_{p,p:2,2} + \beta_{2} \sum_{p=0}^{3} B_{p,p:1,2}
$$
where \( \alpha, \beta \) and \( \gamma \) are arbitrary real parameters.

In the following, we use following formulae for the \( G \)-conjugation \( \tilde{\mathcal{A}} = \tau_2 A^* \tau_2 \):

\[
\tilde{\mathcal{A}} = A,
\tilde{\tau}_i = -\tau_i, \text{ for } i = 1, 2, 3,
\tilde{\mathcal{A}} \mathcal{B} = \tilde{\mathcal{B}} \mathcal{A},
\text{tr}(\tilde{\mathcal{A}}) = \text{tr}(A^*).
\]

After identifications of fields (2.35) and (2.36), the Higgs fields \( \Delta_{L,R} \) have a form

\[
\Delta = \sum_{i=1}^{3} \tau_i \Delta_i,
\]
and hence

\[
\tilde{\Delta}_{L,R} = -\Delta^\dagger_{L,R}.
\]

From these formulas one obtains the identity

\[
\Gamma_0 = \Gamma_1,
\]

since

\[
2\Gamma_1 = \text{tr}(G_{12}G_{21})
= \text{tr}\{ (\phi_{13}\phi_{32} - \phi_{10}\phi_{02}) (\phi_{20}\phi_{01} - \phi_{23}\phi_{31}) \}
= \text{tr}\{ (-\Delta_L^{\dagger}\phi^\dagger - \phi^\dagger \Delta_L) (\Delta_L^\dagger \phi + \phi \Delta_L) \}
= \text{tr}\{ (-\Delta_R^{\dagger}\phi^\dagger + \phi^\dagger \Delta_R) (-\Delta_R^\dagger \phi - \phi^\dagger \Delta_R) \}
= \text{tr}\{ (-\Delta_L^{\dagger}\phi^\dagger - \phi^\dagger \Delta_L) (-\Delta_R^{\dagger}\phi^\dagger + \phi^\dagger \Delta_R) \}
= \text{tr}\{ (-\phi^\dagger \Delta_L^\dagger - \Delta_R^{\dagger} \phi^\dagger) (\phi \Delta_R^{\dagger} + \Delta_L^\dagger \phi) \}
\]

and

\[
2\Gamma_0 = \text{tr}(G_{03}G_{30})
= \text{tr}\{ (\phi_{02}\phi_{23} - \phi_{01}\phi_{13}) (\phi_{31}\phi_{10} - \phi_{32}\phi_{20}) \}
= \text{tr}\{ (-\Delta_L^{\dagger}\phi^\dagger - \phi^\dagger \Delta_L) (-\Delta_R^{\dagger}\phi^\dagger + \phi^\dagger \Delta_R) \}
= \text{tr}\{ (-\phi^\dagger \Delta_L^\dagger - \Delta_R^{\dagger} \phi^\dagger) (\phi \Delta_R^{\dagger} + \Delta_L^\dagger \phi) \}
= 2\Gamma_1.
\]

Similarly it can be shown that

\[
\text{tr} (\hat{\phi}^\dagger \Delta_L \Delta_L^\dagger) = \text{tr} (\phi^\dagger \Delta_L \Delta_L^\dagger) \sim = \text{tr} (\phi^\dagger \Delta_L \Delta_L^\dagger) = \text{tr} (\phi^\dagger \Delta_L \Delta_L^\dagger),
\]

\[
\text{tr} (\hat{\phi}^\dagger \Delta_R \Delta_R^\dagger) = \text{tr} (\phi^\dagger \Delta_R \Delta_R^\dagger) = \text{tr} (\phi^\dagger \Delta_R \Delta_R^\dagger),
\]

from which one can prove the equality

\[
\text{tr} (F_{010}F_{020}) = \text{tr} (F_{232}F_{202}) = \text{tr}\{ (\frac{1}{2} \phi \phi^\dagger - 1) (\Delta_L \Delta_L - 1) \}.
\]
It can be also shown that
\[
\text{tr} (F_{101}F_{131}) = \text{tr} (F_{323}F_{313}) = \text{tr} \left\{ \left( \frac{1}{2} \phi^\dagger \phi - 1 \right) \left( \Delta_R^\dagger \Delta_R - 1 \right) \right\}. 
\]
\[
(2.57)
\]
Taking account of these identities the Higgs potential turns out to be of the form
\[
V = \alpha_{11} \text{tr} \left( F_\phi^2 \right) + \frac{1}{2} \alpha_{12} \left[ \text{tr} \left( F_\phi^2 \right) + \text{tr} \left( F_R^2 \right) \right] + \alpha_2 \left[ \text{tr} \left( F_\phi F_L \right) + \text{tr} \left( F_\phi' F_R \right) \right] \\
+ \frac{1}{2} \left( \beta_{11} + \frac{1}{2} \beta_{31} \right) (\text{tr} F_\phi)^2 + \frac{1}{4} \beta_{12} \left[ (\text{tr} F_L)^2 + (\text{tr} F_R)^2 \right] \\
+ \frac{1}{2} \beta_2 (\text{tr} F_\phi) [(\text{tr} F_L) + (\text{tr} F_R)] + \frac{1}{4} \beta_{32} (\text{tr} F_L) (\text{tr} F_R) \\
+ \gamma \text{tr} (G_{03}G_{30}), \quad (2.58)
\]
where
\[
F_\phi = F_{010} = \frac{1}{2} \phi^\dagger \phi - 1, \quad F_\phi' = F_{101} = \frac{1}{2} \phi^\dagger \phi - 1,
F_L = F_{020} = \Delta_L^\dagger \Delta - 1, \quad F_R = F_{131} = \Delta_R^\dagger \Delta_R - 1,
G_{30} = G_{03} = \Delta_R^\dagger \phi^\dagger + \check{\phi}^\dagger \Delta_L.
\]
\[
(2.59)
\]
The factor $1/2$ in $F_\phi$ is necessary when we identify fields $\Phi_{01}$ and $\Phi_{23}$ with $\phi$ and $-\check{\phi}$, respectively. In this case, two Higgs kinetic terms, $\text{tr} |D\Phi_{01}|^2$ and $\text{tr} |D\Phi_{23}|^2$, agree with each other, so that we should redefine the field $\phi$ as $\phi \rightarrow \phi/\sqrt{2}$. As for other fields $W_L, W_R$ and $B$ the same thing happens, but one can rescale their coupling constans by the factor $1/\sqrt{2}$.

The Higgs potential thus obtained is left-right symmetric under the operation $(1 \cdot 10)$. This property comes from our fundamental requirement (iii) in §2.2. Since our Higgs fields $\phi, \Delta_L$ and $\Delta_R$ are dimensionless, they should be redefined so as to be [Higgs fields] $= L^{-1}$. In the redefinition of the fields we need three new parameters. Consequently our Higgs potential (2.58) contains eleven free parameters.

If the symmetry between 1 and 2, namely the exchange of two $Z_2$'s, is required, one obtains
\[
\alpha_{11} = \alpha_{12} \equiv \alpha_1, \\
\beta_{11} = \beta_{12} \equiv \beta_1, \\
\beta_{31} = \beta_{32} \equiv \beta_3.
\]
\[
(2.60)
\]

§3. Conclusion

The LRSM with gauge group $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ has been reconstructed from the geometric point of view of gauge theory in $M_4 \times Z_2 \times Z_2$. We have started with the fundamental requirements (i) $\sim$ (iv) stated in §2.2. Our new results are then summarized as follows:

1. Three Higgs fields $\phi, \Delta_L$ and $\Delta_R$ are gauge fields in $Z_2 \times Z_2$. 

2. The Higgs potential, therefore, should be of Yang-Mills type, i.e., it is given by (2.58) as a sum of
\[ \text{tr} | \text{curvatures of Higgs fields} |^2, \]
which contains eleven free parameters. This should be compared with the general Higgs potential given by Deshpande et al. which contains eighteen parameters.

3. As a result we have obtained the property that our Higgs potential is left-right-symmetric under the operation (1.3). This property comes from our fundamental requirement (iii) in §2.2, that is, our model should be invariant under the discrete $Z_2 \times Z_2$ transformation.

4. The Yukawa coupling Lagrangian $L_2$ given by (2.40) does not contain the term $\bar{l}_L \phi_R + h.c.$, which appears in $L_Y$ of (1.3) in the standard LRSM.

5. In §2.7, we have identified the $U(1)$ gauge field $B_0$ with the other one $B_1$, i.e., $B_0 = B_1 = B$. However, if we do not identify them, we have another type of LRSM with two kinds of $U(1)$ gauge fields, $B_0 = B_L$ and $B_1 = B_R$.

Actual phenomenological analyses of the latter LRSM with two $U(1)$ gauge fields and of obtaining the left-right asymmetric vacuum based on our Higgs potential will be left for future study.

In conclusion, our approach led to the LRSM quite successfully and the geometrical structure of this model has been better understood compared to other works based on NCG. Finally we would like to thank M. Kubo for discussions.

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