Exponential stability of the exact solutions and \(\theta\)-EM approximations to neutral SDDEs with Markov switching

Guangqiang Lan\(^{a\ast}\) and Chenggui Yuan\(^{b}\)

\(^a\)School of Science, Beijing University of Chemical Technology, Beijing 100029, China
\(^b\)Department of Mathematics, Swansea University, Singleton Park, Swansea, SA2 8PP, UK

Abstract

Exponential stability of the exact solutions as well as \(\theta\)-EM (\(\frac{1}{2} < \theta \leq 1\)) approximations to neutral stochastic differential delay equations with Markov switching will be investigated in this paper. Sufficient conditions are obtained to ensure the \(p\)-th moment (\(p \geq 1\)) and almost sure exponential stability of the exact solutions as well as \(\theta\)-EM approximations (\(p = 2\)). An example will be presented to support our conclusions.

MSC 2010: 60H10, 65C30.

Key words: neutral stochastic differential delay equation with Markov switching, \(\theta\) Euler-Maruyama approximation, exponential stability.

1 Introduction

Many dynamical systems depend not only on present states but also on past states. In such cases, stochastic differential delay equations provide an important tool for describing such systems. More generally, neutral stochastic differential delay equations could be used to describe the cases in which the delay argument occurs in the derivative of the state variable. On the other hand, many practical systems may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The hybrid systems driven by continuous time Markov chains have been developed to cope with such situations. Motivated by hybrid systems, Kolmanovskii et al. \(^9\) introduced the neutral stochastic differential delay equations with Markov switching (NSDDEs\(\text{wMS for short)\).

Let \((\Omega, \mathcal{F}, P)\) be a probability space endowed with a complete filtration \((\mathcal{F}_t)_{t \geq 0}\). Let \(d, m \in \mathbb{N}\) be arbitrarily fixed. For a given \(\tau > 0\), let \(C([\tau, 0]; \mathbb{R}^d)\) be a family of continuous
functions $\varphi$ from $[-\tau, 0]$ to $\mathbb{R}^d$, equipped with the supremum norm $||\varphi|| = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|$. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d)$ the family of bounded, $\mathcal{F}_0$ measurable, $C([-\tau, 0]; \mathbb{R}^d)$-valued random variables. If $x(t)$ is a continuous $\mathbb{R}^d$-valued stochastic process on $t \geq 0$, let $x_t = \{x(t + s) : -\tau \leq s \leq 0\}$ for $t \geq 0$ which is regarded as a $C([-\tau, 0]; \mathbb{R}^d)$-valued process. Let $r(t)$, $t \geq 0$ be a right continuous Markov chain adapted to $(\mathcal{F}_t)_{t \geq 0}$ on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with initial value $x_0 = i_0 \in S$.

We consider the following NSDDEswMS

$$d[X(t) - D(X(t - \tau), r(t))] = f(X(t), X(t - \tau), t, r(t))dt + g(X(t), X(t - \tau), t, r(t))dB_t, \quad (1.1)$$

where the initial

$$x_0 = \xi = \{\xi(\theta), \theta \in [-\tau, 0]\} \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d), \quad r(0) = i_0 \in S,$$

$(B_t)_{t \geq 0}$ is an $m$-dimensional standard $\mathcal{F}_r$-Brownian motion independent of $r(t)$, $D : (x, i) \in \mathbb{R}^d \times S \mapsto D(x, i) \in \mathbb{R}^d$, $f : (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \mapsto f(x, y, t) \in \mathbb{R}^d$ and $g : (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \mapsto g(x, y, t) \in \mathbb{R}^d \otimes \mathbb{R}^m$ are both Borel measurable functions.

It is well known that the Markov chain $r(t)$ can be represented as a stochastic integral with respect to a Possion random measure:

$$dr(t) = \int_\mathbb{R} \bar{h}(r(t), t) \nu(dt, dl), \quad t \geq 0$$

with initial value $r(0) = i_0 \in S$, where $\nu(dt, dl)$ is a Possion random measure with intensity $dt \times m(dl)$ in which $m$ is the Lebesgue measure on $\mathbb{R}$ while the explicit definition of $\bar{h}$ can be found in [2, 4].

Kolmanovskii et al. [2] studied the existence and uniqueness as well as the asymptotic moment boundedness and moment stability of the solutions of the NSDDEswMS (1.1). In Mao et al. [17], authors discussed the almost sure asymptotic stability of the exact solutions of the NSDDEswMS (1.1).

To make sure that equation (1.1) has a unique solution, which is denoted by $X(t) \in \mathbb{R}^d$, $t \geq -\tau$, throughout this paper, we assume that the coefficients satisfy local Lipschitz condition, that is, for each $h > 0$, there is $L_h > 0$ such that

$$|f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)| + |g(x, y, t, i) - g(\bar{x}, \bar{y}, t, i)| \leq L_h(|x - \bar{x}| + |y - \bar{y}|) \quad (1.2)$$

for all $(t, i) \in \mathbb{R}_+ \times S$ and $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq h$.

We also assume in this paper that for all $(x, y, i) \in \mathbb{R}^d \times \mathbb{R}^d \times S$, there exists $0 < \beta < 1$ such that

$$|D(x, i) - D(y, i)| \leq \beta|x - y| \quad (1.3)$$

and for all $(t, i) \in \mathbb{R}_+ \times S$,

$$D(0, i) = 0, \quad f(0, 0, t, i) = 0, \quad g(0, 0, t, i) = 0,$$

which implies that $x \equiv 0$ is the trivial solution of equation (1.1).
To study the exponential stability of equation (1.1), we introduce the following operator $L$. If $V \in C^2(\mathbb{R}^d \times \mathbb{R}_+ \times S; \mathbb{R})$, define $LV$ from $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times S$ to $\mathbb{R}$ by

$$LV(x, y, t, i) := V_t(x - D(y, i), t, i) + V_x(x - D(y, i), t, i)f(x, y, t, i)$$

$$+ \frac{1}{2}\text{trace}[g^T(x, y, t, i)V_{xx}(x - D(y, i), t, j)g(x, y, t, i)]$$

$$+ \sum_{j=1}^N \gamma_{ij}V(x - D(y, i), t, j),$$

where

$$V_t(x, t, i) = \frac{\partial}{\partial t}V(x, t, i), \quad V_x(x, t, i) = \left(\frac{\partial}{\partial x_1}V(x, t, i), \ldots, \frac{\partial}{\partial x_d}V(x, t, i)\right)$$

and

$$V_{xx}(x, t, i) = \left(\frac{\partial^2}{\partial x_i \partial x_j}V(x, t, i)\right)_{d \times d}.$$ 

Notice that if $V(x, t, i) = |x|^2$, then

$$LV(x, y, t, i) = 2\langle x - D(y, i), f(x, y, t, i) \rangle + \|g(x, y, t, i)\|^2,$$

where $| \cdot |$ is the Euclid norm in $\mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$ and $\| \cdot \|$ stands for the Hilbert-Schmidt norm of a matrix.

When there are no neutral term and no delay, in [12], Mao proved that $p$-th moment exponential stability plus linear growth condition on coefficients $f$ and $g$ implies almost sure exponential stability of the true solution of the corresponding SDEs. In [9] Theorem 5.3, authors presented a sufficient condition for almost sure exponential stability where the linear growth condition is also needed. For other stability results of the exact solutions, one can see e.g. [10, 14] and reference there in. In this paper, we will first study the sufficient condition for the almost sure exponential stability under the assumption of $p$-th moment exponential stability, which is different with that of [9], where the linear condition for $g$ is needed (see the following Remark 4.2).

Since in general both the explicit solutions and the probability distributions of the solutions are not known, discrete approximate solutions become necessarily. A particularly important issue is to determine the conditions under which the exact solution and approximate solution share same stability properties. There are plenty of literatures dealing with different types of numerical approximations for SDEs, for example, [11, 15, 14, 23] consider the moment or almost sure exponential stability of EM or backward EM approximations, in [11] the authors studied mean square polynomial stability of EM and backward EM approximations, while [3, 5, 25] deal with the moment or almost sure exponential stability of more general $\theta$-EM scheme. Recently, in [8], the authors presented sufficient conditions for mean square and almost sure exponential and polynomial stability of the $\theta$-EM scheme. However, as far as we know, there are few results on the exponential stability of the numerical approximations of NSDDEwMS. So the other aim of this paper is to present sufficient conditions for the exponential stability of numerical approximations of NSDDEwMS.

Choose a step size $\Delta > 0$ such that $\Delta = \tau/m$ for some integer $m$. Define the corresponding discrete $\theta$ Euler-Maruyama ($\theta$-EM) approximation (or the so called stochastic theta method)
of the NSDDEswMS (1.1) as
\[ X_j = \xi(j\Delta), j = -m, -m + 1, \ldots, 0, \quad (1.5) \]
while for \( k \geq 0 \),
\[
X_{k+1} - D(X_{k+1-m}, r((k+1)\Delta)) = X_k - D(X_{k-m}, r(k\Delta)) + [(1 - \theta)f(X_k, X_{k-m}, k\Delta, r(k\Delta)) \\
+ \theta f(X_{k+1}, X_{k+1-m}, (k+1)\Delta, r((k+1)\Delta))]\Delta \\
+ g(X_k, X_{k-m}, k\Delta, r(k\Delta))\Delta B_k,
\]
(1.6)
where \( \theta \in [0, 1] \) is a fixed parameter, \( \Delta B_k := B((k+1)\Delta t) - B(k\Delta t) \) is the increment of Brownian motion. Note that \( \theta \)-EM includes the classical EM method (\( \theta = 0 \)), the backward EM method (\( \theta = 1 \)) and the so-called trapezoidal method (\( \theta = \frac{1}{2} \)).

Since the scheme (1.6) is semi-implicit when \( \theta > 0 \), to make sure that the approximation scheme is well defined, a natural assumption is that \( f \) satisfies one-sided Lipschitz condition: There exists \( L > 0 \) such that for any \( x_1, x_2, y, i \in \mathbb{R}^d, t \geq 0, i \in S \),
\[
\langle x_1 - x_2, f(x_1, y, t, i) - f(x_2, y, t, i) \rangle \leq L|x_1 - x_2|^2.
\]
(1.7)
Under this condition, if \( L\theta\Delta < 1 \), then the \( \theta \)-EM scheme is well defined (see e.g. [8, 11, 15, 23]).

We will investigate the sufficient conditions under which the \( \theta \)-EM approximation is exponentially stable.

The rest of the paper is organized as follows. In Section 2, two lemmas will be introduced to prove the following stability results. In Section 3, sufficient conditions will be given to guarantee that both \( p \)-th moment and almost sure exponential stability hold. In Section 4, sufficient conditions will be presented to guarantee the almost sure stability under the assumption that the exact solution is exponentially stable in the \( p \)-th moment sense. Finally, in Section 5, we investigate the exponential stability of the \( \theta \)-EM scheme. We show that when \( \frac{1}{2} < \theta \leq 1 \), both the mean square and almost sure exponential stability of the \( \theta \)-EM scheme hold under given conditions. An example will be presented to illustrate our theory.

## 2 Preliminary

Before we state our main results, let us present two useful lemmas which will be used in the following Sections.

**Lemma 2.1** Let \( p \geq 1 \). If (1.3) holds, then
\[
|x - D(y, i)|^p \leq (1 + \beta)^{p-1}(|x|^p + \beta|y|^p), \quad \forall (x, y, i) \in \mathbb{R}^d \times \mathbb{R}^d \times S.
\]
For the proof, please see [12].

The second lemma is an elementary inequality which will be used in the following Sections. One can find this inequality in [18].

**Lemma 2.2**
\[
(a + b)^p \leq (1 + c)^{p-1}(a^p + c^{1-p}b^p)
\]
holds for all \( a, b > 0, p \geq 1, c > 0 \).
3 Sufficient conditions for exponential stability

In this Section, we will study the $p$-th moment exponential stability of $X(t)$. To do this, let us consider the following conditions:

There exist functions $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, $U \in C(\mathbb{R}^d \times [-\tau, \infty); \mathbb{R}_+)$, $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and $\psi \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and positive constants $c_1, \alpha_1, \alpha_2, \lambda$ such that

$$\int_0^\infty e^{\lambda t} \gamma(t) dt < \infty,$$

and

$$c_1|x|^p \leq V(x, t, i)$$

(3.1)

and

$$LV(x, y, t, i) \leq \gamma(t) + (\psi(t) - \lambda)V(x - D(y, i), t, i) - \alpha_1U(x, t) + \alpha_2U(y, t - \tau)$$

(3.2)

holds for $\alpha_1 \geq \alpha_2 e^\lambda \beta e^\lambda < 1$.

In general, conditions (1.2) and (1.3) will only guarantee a unique maximal local solution to SDE (1.1) for any given initial value $\xi$ and $i_0$. However, the additional conditions (3.1) and (3.2) will guarantee that this local solution is in fact a unique global solution. That is, we have the following

**Theorem 3.1** Assume that conditions (1.2), (1.3), (3.1) and (3.2) hold. Then there exist a unique global solution of SDE (1.1).

The proof is similar to that of Theorem A.1 in [17], so we omit it here.

Now let us consider the exponential stability of equation (1.1). We have the following

**Theorem 3.2** Assume that $p \geq 1$, and conditions (1.2), (1.3), (3.1) and (3.2) hold. Then the exact solution of equation (1.1) is $p$-th moment exponentially stable with Lyapunov exponent no greater than $-\lambda$.

**Proof** By the generalized Itô’s formula (see Skorohod [21] or Mao [13]), we have

$$d(e^{\lambda t}V(X(t) - D(X(t - \tau), r(t)), t, r(t))) = e^{\lambda t}[LV(X(t) - D(X(t - \tau), r(t)), t, r(t)) + LV(X(t), X(t - \tau), r(t))] dt + dM_t,$$

where

$$M_t := \int_0^t e^{\lambda s}[V_x(X(s) - D(X(s - \tau), r(s)), s, r(s))g(X(s), X(s - \tau), s, r(s)) dB_s$$

$$+ \int_0^t \int_\mathbb{R} (V(X(s) - D(X(s - \tau), r(s)), s, r(s) + \tilde{h}(r(s), l))$$

$$- V(X(s) - D(X(s - \tau), r(s)), t, r(s)))) \mu(ds, dl)]$$

is a local martingale.
Then
\[ e^{\lambda t}V(X(t)) - D(X(t - \tau), r(t)), t, r(t) \]
\[ \leq V(X(0)) - D(X(-\tau), i_0), 0, i_0) + M_t \]
\[ + \int_0^t e^{\lambda s} [\gamma(s) + \psi(s)V(X(s) - D(X(s - \tau), r(s)), s, r(s))] ds \]
\[ \leq V(X(0)) - D(X(-\tau), i_0), 0, i_0) + \int_0^t e^{\lambda s} \gamma(s) ds + M_t \] (3.3)
\[ + \int_0^t e^{\lambda s} \psi(s)V(X(s) - D(X(s - \tau), r(s)), s, r(s)) ds \]
\[ - \alpha_1 \int_0^t e^{\lambda s} U(X(s), s) ds + \alpha_2 \int_0^t e^{\lambda s} U(X(s - \tau), s - \tau) ds. \]

Taking expectation on both sides (cutting by stopping time if necessary). By conditions (3.1) and (3.2), we have
\[ e^{\lambda t} \mathbb{E}(V(X(t)) - D(X(t - \tau), r(t)), t, r(t)) \]
\[ \leq \mathbb{E}(V(X(0)) - D(X(-\tau), i_0), 0, i_0)) + \int_0^t e^{\lambda s} \gamma(s) ds \]
\[ + \int_0^t \psi(s)e^{\lambda s} \mathbb{E}(V(X(s) - D(X(s - \tau), r(s)), s, r(s))) ds \]
\[ - \alpha_1 \int_0^t e^{\lambda s} \mathbb{E}(U(X(s), s)) ds + \alpha_2 e^{\lambda t} \int_{-\tau}^0 e^{\lambda s} \mathbb{E}(U(X(s), s)) ds \]
\[ + \alpha_2 e^{\lambda t} \int_0^{t-\tau} e^{\lambda s} \mathbb{E}(U(X(s), s)) ds \]
\[ \leq C + \int_0^t \psi(s)e^{\lambda s} \mathbb{E}(V(X(s) - D(X(s - \tau), r(s)), s, r(s))) ds, \]
where
\[ C = \mathbb{E}(V(X(0)) - D(X(-\tau), i_0), 0, i_0)) \]
\[ + \int_0^{\infty} e^{\lambda s} \gamma(s) ds + \alpha_2 e^{\lambda t} \int_{-\tau}^0 e^{\lambda s} \mathbb{E}(U(X(s), s)) ds < \infty. \]

We have used the fact that \( \alpha_1 \geq \alpha_2 e^{\lambda t} \) in the last inequality of (3.4).

Now by Gronwall’s lemma, we have
\[ e^{\lambda t} \mathbb{E}(V(X(t)) - D(X(t - \tau), r(t)), t, r(t)) \leq C e^{\lambda t} \psi(s) ds \leq C e^{\lambda t} \psi(s) ds =: C' < \infty. \] (3.5)

On the other hand, according to Lemma (2.2), we have
\[ e^{\lambda t} |X(t)|^p \leq e^{\lambda t} [(1 - \beta)^{1-p}|X(t) - D(X(t - \tau), r(t))|^p + \beta^{1-p}|D(X(t - \tau), r(t))|^p] \]
\[ \leq \left( \frac{1 - \beta}{c_1} \right)^{1-p} e^{\lambda t} V(X(t) - D(X(t - \tau), r(t)), t, r(t)) \]
\[ + \beta e^{\lambda t} |X(t - \tau)|^p. \]

Thus, for any \( T > 0 \), we have
\[ \sup_{t \in [0, T]} e^{\lambda t} \mathbb{E}|X(t)|^p \leq C' (1 - \beta)^{1-p} + \beta e^{\lambda t} \mathbb{E}|X(t)|^p + \sup_{t \in [0, T]} e^{\lambda t} \mathbb{E}|X(t)|^p. \]
Then
\[ \sup_{t \in [0,T]} e^{\lambda t} \mathbb{E}[|X(t)|^p] \leq (1 - \beta e^{\lambda \tau})^{-1} \frac{C'(1 - \beta)^{1-p}}{c_1} + \beta e^{\lambda \tau} \sup_{t \in [-\tau,0]} e^{\lambda t} \mathbb{E}[|X(t)|^p] < \infty, \]
which implies
\[ \limsup_{t \to \infty} \frac{\log \mathbb{E}[|X(t)|^p]}{t} \leq -\lambda. \]

4 Almost sure exponential stability under the assumption of \( p \)-th moment exponential stability

In this Section, we investigate the almost sure stability of exact solution under the assumption that \( p \)-th moment exponential stability holds for the exact solution.

Consider the following conditions:

There exist functions \( V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+ \times S; \mathbb{R}_+) \), \( \varphi \in L^1(\mathbb{R}_+, \mathbb{R}_+) \) and \( h : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) and positive constants \( c_1, c_2, \lambda, C_0, \alpha \) such that such that \( \int_0^\infty e^{\lambda t} h(t) dt < \infty \),
\[ c_1 |x|^p \leq V(x, t, i) \leq c_2 |x|^p, \quad (4.1) \]
\[ LV(x, y, t, i) \leq h(t) + (C_0 + \varphi(t))V(x - D(y, i), t, i) \quad (4.2) \]
and
\[ |g^T(x, y, t, i)V_x(x - D(y, i), t, i)|^2 + \int_{\mathbb{R}} |V(x - D(y, i), t, i + \bar{h}(i, l)) - V(x - D(y, i), t, i)|^2 m(dl) \leq \alpha V^2(x - D(y, i), t, i). \quad (4.3) \]

We have the following

**Theorem 4.1** Assume that the exact solution of equation (1.1) is \( p \)-th moment exponentially stable \( (p \geq 1) \) with Lyapunov exponent no greater than \( -\lambda \). If conditions (4.1), (4.2) and (4.3) hold, and \( \beta < e^{-\lambda \tau} \), then the exact solution of equation (1.1) is almost surely exponentially stable with Lyapunov exponent no greater than \( -\frac{\lambda}{p} \).

**Proof** By the generalized Itô’s formula, we have
\[
\begin{align*}
    dV(X(t) - D(X(t - \tau), r(t)), t, r(t)) &= LV(X(t), X(t - \tau), t, r(t)) dt \\
    &+ V_x(X(t) - D(X(t - \tau), r(t)), t, r(t)) g(X(t), X(t - \tau), t, r(t)) dB_t \\
    &+ \int_{\mathbb{R}} (V(X(t) - D(X(t - \tau), r(t)), t, r(t) + \bar{h}(i, l)) - V(X(t) - D(X(t - \tau), r(t)), t, r(t))) \mu(dt, dl).
\end{align*}
\]
Thus, for any \( t \in [(k - 1)\delta, k\delta) \), \( \delta > 0 \), we have
\[
\begin{align*}
    V(X(t) - D(X(t - \tau), r(t)), t, r(t)) &= V(X((k - 1)\delta) - D(X((k - 1)\delta - \tau), r((k - 1)\delta)), (k - 1)\delta, r((k - 1)\delta)) \\
    &+ \int_{(k - 1)\delta}^t \left( LV(X(s), X(s - \tau), s, r(s)) ds + M_{1,t} + M_{2,t} \right),
\end{align*}
\]
where
\[ M_{1,t} := \int_{(k-1)\delta}^{t} V_{k}(X(s) - D(X(s - \tau), r(s)), s, r(s))g(X(s), X(s - \tau), s, r(s))dB_s \]
and
\[ M_{2,t} := \int_{(k-1)\delta}^{t} \int_{\mathbb{R}} (V(X(s) - D(X(s - \tau), r(s)), s, r(s) + \bar{h}(r(s), l)) - V(X(s) - D(X(s - \tau), r(s)), s, r(s)))\mu(ds, dl), \]
\( \mu(ds, dl) = \nu(ds, dl) - m(dl)ds \) is a martingale measure.

It is clear that both \( M_{1,t} \) and \( M_{2,t} \) are martingales on the interval \([(k-1)\delta, k\delta)\). Denote \( M_{j,t}^* = \sup_{s \in [(k-1)\delta, t]} M_{j,s}, j = 1, 2 \). Then by BDG inequality, there exists \( C \) (independent of \( t, k \) and \( \delta \)) such that
\[
\mathbb{E}(M_{1,t}^*) \leq C\mathbb{E}(\langle M_{2,t}^* \rangle_t^{\frac{1}{2}})
\]
\[
= C\mathbb{E}\left[ \int_{(k-1)\delta}^{t} \left| g(X(s), X(s - \tau), s, r(s))V_{k}(X(s) - D(X(s - \tau), r(s)), s, r(s)) \right|^2 ds \right]^{\frac{1}{2}}
\leq C\alpha \sqrt{\delta} \mathbb{E} \sup_{s \in [(k-1)\delta, t]} V(X(s) - D(X(s - \tau), r(s)), s, r(s)).
\]

Similarly, we have
\[
\mathbb{E}(M_{2,t}^*) \leq C\alpha \sqrt{\delta} \mathbb{E} \sup_{s \in [(k-1)\delta, t]} V(X(s) - D(X(s - \tau), r(s)), s, r(s)).
\]

We have used the fact that
\[
\langle M_{2,\cdot} \rangle_t = \int_{(k-1)\delta}^{t} ds \int_{\mathbb{R}} (V(X(s) - D(X(s - \tau), r(s)), s, r(s) + \bar{h}(r(s), l)) - V(X(s) - D(X(s - \tau), r(s)), s, r(s)))^2 m(dl)
\]
in the above inequality (see Situ \cite{Situ} Lemma 62).

Then we have
\[
\mathbb{E} \sup_{s \in [(k-1)\delta, t]} V(X(s) - D(X(s - \tau), r(s)), s, r(s)) \leq \mathbb{E} V(X((k-1)\delta) - D((k-1)\delta - \tau), r((k-1)\delta)), (k-1)\delta, r((k-1)\delta))
\]
\[
+ \mathbb{E} \int_{(k-1)\delta}^{t} [h(s) + (C_0 + \varphi(s))V(X(s) - D(X(s - \tau), r(s)), s, r(s))]ds
\]
\[
+ 2C\alpha \sqrt{\delta} \mathbb{E} \sup_{s \in [(k-1)\delta, t]} V(X(s) - D(X(s - \tau), r(s)), s, r(s))
\]
\[
\leq \mathbb{E} V(X((k-1)\delta) - D((k-1)\delta - \tau), r((k-1)\delta)), (k-1)\delta, r((k-1)\delta))
\]
\[
+ (C_0\delta + \int_{(k-1)\delta}^{k\delta} \varphi(s)ds + 2C\alpha \sqrt{\delta}) \mathbb{E} \sup_{s \in [(k-1)\delta, t]} V(X(s) - D(X(s - \tau), r(s)), s, r(s))
\]
\[
+ \int_{(k-1)\delta}^{t} h(s)ds.
\]
We can choose \( \delta \) small enough such that \( \tilde{C} := C_0 \delta + \int_{(k-1)\delta}^{k\delta} \varphi(s)ds + 2C\alpha \sqrt{\delta} < 1 \).

Thus,
\[
E \sup_{s \in [(k-1)\delta, t]} V(X(s) - D(X(s - \tau), r(s)), s, r(s)) \\
\leq (1 - \tilde{C})^{-1} \left[ EV(X((k-1)\delta) - D((k-1)\delta - \tau), r((k-1)\delta)), (k-1)\delta, r((k-1)\delta) \right] \\
+ \int_{(k-1)\delta}^{t} h(s)ds \\
\leq (1 - \tilde{C})^{-1} [c_2 E|X((k-1)\delta) - D((k-1)\delta - \tau), r((k-1)\delta)|^p + \int_{(k-1)\delta}^{t} h(s)ds].
\]

It follows that
\[
E \sup_{s \in [(k-1)\delta, t]} |X(s) - D(X(s - \tau), r(s))|^p \\
\leq c_1^{-1}(1 - \tilde{C})^{-1} [c_2 E|X((k-1)\delta) - D((k-1)\delta - \tau), r((k-1)\delta)|^p + \int_{(k-1)\delta}^{t} h(s)ds].
\]

Since \( X(t) \) is \( p \)-th moment exponentially stable, then by Lemma 2.1 for any \( \varepsilon > 0 \), there exists \( k_0 \) large enough such that for any \( k \geq k_0 \), we have
\[
E|X((k-1)\delta) - D((k-1)\delta - \tau), r((k-1)\delta)|^p \\
\leq (1 + \beta)^{p-1}(E|X((k-1)\delta)|^p + \beta E|X((k-1)\delta - \tau)|^p) \\
\leq (1 + \beta)^{p-1}(e^{-\lambda(t-\tau)} + \beta e^{-\lambda(t-\tau)(k-1)\delta}) \\
\leq (1 + \beta)^{p-1}(1 + \beta e^{\lambda\tau})e^{-\lambda+k-1)(k-1)\delta}.
\]

On the other hand, choosing \( c = \frac{\beta}{1-\beta} \) in Lemma 2.2, we have
\[
|X(t)|^p \leq (1 - \beta)^{1-p}|X(t) - D(X(t - \tau), r(t))|^p + \beta^{1-p}|D(X(t - \tau), r(t))|^p
\]
and \( \int_{(k-1)\delta}^{t} h(s)ds \leq e^{-\lambda(k-1)\delta} \int_{0}^{\infty} e^{\lambda s} h(s)ds \), then
\[
E \sup_{s \in [(k-1)\delta, t]} |X(s)|^p \\
\leq (1 - \beta)^{1-p} c_1^{-1}(1 - \tilde{C})^{-1} [c_2 (1 + \beta)^{p-1}(1 + \beta e^{\lambda\tau})e^{-\lambda+k-1)(k-1)\delta} \\
+ \int_{(k-1)\delta}^{t} h(s)ds] + \beta E \sup_{s \in [(k-1)\delta, t]} |X(s - \tau)|^p \\
\leq C' e^{-\lambda+\varepsilon(k-1)\delta} + \beta E \sup_{s \in [(k-1)\delta - \tau, t-\tau]} |X(s)|^p,
\]
where \( C' = (1 - \beta)^{1-p} c_1^{-1}(1 - \tilde{C})^{-1} [c_2 (1 + \beta)^{p-1}(1 + \beta e^{\lambda\tau}) + \int_{0}^{\infty} e^{\lambda s} h(s)ds \] is a constant independent of \( k \) and \( t \).

Let \( \tau/\delta = l \) be an integer. Then we have
\[
a_{k,t} \leq b_k + \beta a_{k-1,t},
\]
where \( a_{k,t} := E \sup_{s \in [(k-1)\delta, t]} |X(s)|^p, b_k = C' e^{-\lambda+\varepsilon(k-1)\delta}. \)
For any \( t \), we can choose integers \( n \) and \( k \) such that \( \left[ (k-1-n\ell)\delta, t-n\ell\delta \right] \subset [-\tau, 0] \), which implies that \( n \geq \frac{(k-1)\delta}{\tau} \). Notice also that \( \beta e^{(\lambda-\varepsilon)t} < 1 \) for any \( \varepsilon > 0 \). So we have
\[
a_{k,t} \leq \sum_{i=0}^{n-1} \beta^i b_{k,i} + \beta^n a_{k-n, t} \leq C' e^{(\lambda-\varepsilon)(k-1)\delta} + C'' \sup |x_0|^p.
\]
Therefore, for \( k \) large enough,
\[
\mathbb{E} \sup_{s \in [(k-1)\delta, t]} |X(s)|^p \leq 2\left[ C' e^{(\lambda-\varepsilon)(k-1)\delta} + \sup |x_0|^p \right] \leq C'' e^{-(\lambda-\varepsilon)(k-1)\delta},
\]
where \( C'' := \frac{2C'}{1-\beta e^{(\lambda-\varepsilon)t}} \). Hence, Chebyshev inequality yields that
\[
P\left( \sup_{s \in [(k-1)\delta, t]} |X(s)|^p \geq e^{-(\lambda-2\varepsilon)(k-1)\delta} \right) \leq \frac{\mathbb{E} \sup_{s \in [(k-1)\delta, t]} |X(s)|^p}{e^{-(\lambda-2\varepsilon)(k-1)\delta}} \leq C'' e^{-(\lambda-\varepsilon)(k-1)\delta}.
\]
By Borel-Cantelli lemma, we have for almost all \( \omega \in \Omega \),
\[
\sup_{s \in [(k-1)\delta, k\delta]} |X(s)| \leq e^{-\frac{(\lambda-2\varepsilon)(k-1)\delta}{p}} \tag{4.4}
\]
holds for all but finitely many \( k \). Hence there exists a \( k_0(\omega) \), for all \( \omega \in \Omega \) excluding a \( P \)-null set, for which equation (4.4) holds whenever \( k \geq k_0 \). Consequently, for almost all \( \omega \in \Omega \),
\[
\frac{1}{t} \log |X(t)| \leq -(\lambda-2\varepsilon)(k-1)\delta \leq -\frac{(\lambda-2\varepsilon)(k-1)}{pk}
\]
if \( (k-1)\delta \leq t \leq k\delta \) and \( k \geq k_0 \).
This yields immediately that
\[
\limsup_{t \to \infty} \frac{1}{t} \log |X(t)| \leq \limsup_{k \to \infty} \frac{(\lambda-2\varepsilon)(k-1)}{pk} = \frac{(\lambda-2\varepsilon)}{p}, \quad a.s.
\]
Letting \( \varepsilon \to 0 \). We complete the proof. \( \square \)

**Remark 4.2** Notice that our conditions (4.2) and (4.3) contain some cases where \( f \) and \( g \) do not satisfy the linear growth condition. For example, suppose \( d = 2, m = 1, r > 0 \). Let \( D(y,i) \) satisfies condition (4.3), and
\[
g(x, y, t, i) = |x - D(y, i)|^r(-x_2 + D_2(y, i), x_1 - D_1(y, i))^T,
\]
\[
f(x, y, t, i) = -|x - D(y, i)|^r(x - D(y, i))^T - (x - D(y, i))^T,
\]
where \( D_j \) is the \( j \)-th coordinate of \( D \). It is clear that the local Lipschitz condition (4.2) holds for \( f \) and \( g \). Moreover, by letting \( V(x, t, i) = |x|^2 \), we have
\[
LV(x, y, t, i) = -2|x - D(y, i)|^2 - 2|x - D(y, i)|^2 \leq -2|x - D(y, i)|^2 \leq \frac{1}{1+c_0} |x|^2 + \frac{\beta^2}{c_0} |y|^2, \quad \forall c_0 > \frac{\beta^2}{1-\beta^2}.
\]
Thus, by [17] Theorem A.1, there exist a unique global solution for equation (1.1). On the other hand, according to [9] Theorem 5.2, equation (1.1) is exponentially stable in mean square sense. Since our conditions (4.1), (4.2) and (4.3) hold for $p = 2$, $V(x, t, i) = |x|^2$, then by Theorem 4.1, equation (1.1) is also almost surely exponentially stable. However, the linear growth condition for $g$ does not hold in our case.

Remark 4.3 By taking $U \equiv 0$, condition (4.2) becomes to

$$LV(x, y, t, i) \leq \gamma(t) + (\psi(t) - \lambda)V(x - D(y, i), t, i).$$

(4.5) Obviously, it is stronger than condition (4.2).

Then by Theorem 4.1 and Theorem 3.2, we have the following

Corollary 4.4 Suppose conditions (4.1), (4.3) and (4.5) hold. Then the exact solution of equation (1.1) is $p$-th moment and almost surely exponentially stable.

Remark 4.5 Notice that the convergence theorem of nonnegative semi-martingales method could not be used here to prove the almost sure stability in our case since we cannot prove directly that in (3.3) the integral term

$$\int_0^t e^{\lambda s} \psi(s)V(X(s) - D(X(s - \tau), r(s)), s, r(s))ds$$

is finite almost surely. However, by using Theorem 4.1, we can overcome the difficulty in proving the almost sure exponential stability in Corollary 4.4.

5 Exponential stability of $\theta$-EM solution (1.5) and (1.6)

Let us now consider the exponential stability of the corresponding $\theta$-EM approximation of the exact solution (1.1). Since one-sided Lipschitz condition (1.7) holds,

$$F(x, y, t, i) := x - D(y, i) - \theta \Delta f(x, y, t, i)$$

is well defined for all $0 \leq \theta \leq 1$. Then (1.6) becomes to

$$F(X_{k+1}, X_{k+1-m}, (k+1)\Delta, r((k+1)\Delta)) = F(X_k, X_{k-m}, k\Delta, r(k\Delta)) + f(X_k, X_{k-m}, k\Delta, r(k\Delta))\Delta + g(X_k, X_{k-m}, k\Delta, r(k\Delta))\Delta B_k.$$  

(5.1)

Denote $F_k := F(X_k, X_{k-m}, k\Delta, r(k\Delta))$ for simplicity, and $f_k, g_k$ in the following are defined in the same way.

Consider the following condition:

$$2\langle x - D(y, i), f(x, y, t, i) \rangle + |g(x, y, t, i)|^2 \leq -C_1|x|^2 + C_2|y|^2, C_1 > C_2 > 0.$$  

(5.2)

It is clear that under the Lipschitz conditions (1.2) and (5.2), there exists a unique global solution of equation (1.1).

For the stability of $\theta$-EM numerical approximation (1.5) and (1.6), we have the following
Theorem 5.1 Assume that $\beta^2 < \frac{1}{3+2\sqrt{2}}$, and conditions (1.7) and (5.2) hold for $C_1 > \frac{3+2\sqrt{2}}{1-(3+2\sqrt{2})\beta^2}C_2$. If $\frac{1}{2} < \theta \leq 1$, then the $\theta$-EM numerical approximation (1.12) and (1.4) is mean square and almost surely exponentially stable.

Proof By (5.1), we have

$$|F_{k+1}|^2 = |F_k|^2 + 2\langle X_k - D(X_{k-m}, r(k\Delta)), f_k \rangle + |g_k|^2 + (1 - 2\theta)|f_k|^2|\Delta| + M_k,$$

where

$$M_k := |g_k\Delta B_k|^2 - |g_k|^2\Delta + 2\langle f_k, g_k\Delta B_k \rangle + 2\langle F_k, g_k\Delta B_k \rangle.$$ (5.3)

By condition (5.2), for any $c_0 > 0$, if $0 < C < \frac{C_1}{1+c_0}$, then we claim that

$$2\langle X_k - D(X_{k-m}, r(k\Delta)), f_k \rangle + |g_k|^2 + (1 - 2\theta)|f_k|^2\Delta 
\leq -C|F_k|^2 + \left(\frac{C_1}{c_0}\beta^2 + C_2\right)|X_{k-m}|^2.$$ (5.4)

Indeed,

$$(2\theta - 1)|f_k|^2\Delta - C|F_k|^2 = [(2\theta - 1)\Delta - C\theta^2\Delta^2]|f_k|^2
\quad + 2C\theta \Delta \langle X_k - D(X_{k-m}, r(k\Delta)), f_k \rangle
\quad - C|X_k - D(X_{k-m}, r(k\Delta))|^2
\quad = a|f_k|^2 + b(X_k - D(X_{k-m}, r(k\Delta)))^2
\quad - (ab^2 + C)|X_k - D(X_{k-m}, r(k\Delta))|^2,$$

where $a := (2\theta - 1)\Delta - C\theta^2\Delta^2$, $b := \frac{C\theta\Delta}{a}$.

Since $\frac{1}{2} < \theta \leq 1$ and $0 < C < \frac{C_1}{1+c_0}$, we can choose $\Delta$ small enough such that $a \geq 0$ and $ab^2 + C \leq \frac{C_1}{1+c_0}$, and therefore

$$\langle X_k - D(X_{k-m}, r(k\Delta))\rangle^2
\geq -\frac{C_1}{1+c_0}((1+c_0)|X_k|^2 + \frac{1+c_0}{c_0}\beta^2|X_{k-m}|^2)
\geq -C_1|X_k|^2 - \frac{C_1}{c_0}\beta^2|X_{k-m}|^2
\geq 2\langle X_k - D(X_{k-m}, r(k\Delta)), f_k \rangle + |g_k|^2
\quad - (\frac{C_1}{c_0}\beta^2 + C_2)|X_{k-m}|^2.$$

The second inequality holds because of Lemma 2.2.

Then (5.4) holds for all $\frac{1}{2} < \theta \leq 1$.

So

$$|F_{k+1}|^2 \leq |F_k|^2 - C\Delta|F_k|^2 + \left(\frac{C_1}{c_0}\beta^2 + C_2\right)|X_{k-m}|^2 + M_k.$$
Then for any $A > 1$,

$$A^{(k+1)\Delta} |F_{k+1}|^2 - A^{k\Delta} |F_k|^2$$

$$\leq A^{(k+1)\Delta} \|F_k\|^2 (1 - C\Delta) + \left( \frac{C_1}{c_0} \beta^2 + C_2 \right) |X_{k-m}|^2 + M_k - A^{k\Delta} |F_k|^2$$

$$= A^{(k+1)\Delta} |F_k|^2 (1 - C\Delta - A^{-\Delta}) + \left( \frac{C_1}{c_0} \beta^2 + C_2 \right) \Delta A^{(k+1)\Delta} |X_{k-m}|^2 + A^{(k+1)\Delta} M_k.$$  

For simplicity, denote $R_1 = 1 - C\Delta - A^{-\Delta}$, $R_2 = \frac{C_1}{c_0} \beta^2 + C_2$. Then

$$A^{k\Delta} |F_k|^2 \leq |F_0|^2 + R_1 \sum_{i=0}^{k-1} A^{(i+1)\Delta} |F_i|^2$$

$$+ R_2 \Delta \sum_{i=0}^{k-1} A^{(i+1)\Delta} |X_{i-m}|^2 + \sum_{i=0}^{k-1} A^{(i+1)\Delta} M_i.$$  

Note that

$$\sum_{i=-m}^{k-1} A^{(i+1)\Delta} |X_{i-m}|^2 = A^\tau \sum_{i=-m}^{k-1} A^{(i+1)\Delta} |X_i|^2 + A^\tau \sum_{i=k-m}^{k-1} A^{(i+1)\Delta} |X_i|^2$$

$$\leq A^\tau \sum_{i=-m}^{k-1} A^{(i+1)\Delta} |X_i|^2 + A^\tau \sum_{i=0}^{k-1} A^{(i+1)\Delta} |X_i|^2.$$  

On the other hand, by the definition of $F_k$, condition (5.2) and Lemma 2.2, we know that

$$|F_k|^2 \geq |X_k - D(X_{k-m}, r(k\Delta))|^2 + C_1 \theta \Delta |X_k|^2 - C_2 \theta \Delta |X_{k-m}|^2$$

$$\geq \left( \frac{1}{1 + c_0^2} \right) |X_k|^2 - \frac{\beta^2}{c_0} |X_{k-m}|^2 + C_1 \theta \Delta |X_k|^2 - C_2 \theta \Delta |X_{k-m}|^2$$

$$= \left( \frac{1}{1 + c_0^2} + C_1 \theta \Delta \right) |X_k|^2 - \left( \frac{\beta^2}{c_0} + C_2 \theta \Delta \right) |X_{k-m}|^2.$$  

(5.5)

Note that $R_1 |_{\Delta=0} = 0$ and

$$R_1'(\Delta) = -C + A^{-\Delta} \log A \leq 0$$

for $1 < A \leq e^C$.

Thus, $R_1 \leq 0$ for $A$ close to 1.

Therefore,

$$A^{k\Delta} |F_k|^2 \leq |F_0|^2 + R_1 \sum_{i=0}^{k-1} A^{(i+1)\Delta} \left[ \left( \frac{1}{1 + c_0^2} + C_1 \theta \Delta \right) |X_i|^2 - \left( \frac{\beta^2}{c_0} + C_2 \theta \Delta \right) |X_{i-m}|^2 \right]$$

$$+ R_2 \Delta \sum_{i=0}^{k-1} A^{(i+1)\Delta} |X_{i-m}|^2 + \sum_{i=0}^{k-1} A^{(i+1)\Delta} M_i$$

$$= |F_0|^2 + K_1 \sum_{i=0}^{k-1} A^{(i+1)\Delta} |X_i|^2 + K_2 \sum_{i=0}^{k-1} A^{(i+1)\Delta} |X_{i-m}|^2 + \sum_{i=0}^{k-1} A^{(i+1)\Delta} M_i$$

$$\leq |F_0|^2 + K_2 A^\tau \sum_{i=-m}^{k-1} A^{(i+1)\Delta} |X_i|^2 + (K_1 + K_2 A^\tau) \sum_{i=0}^{k-1} A^{(i+1)\Delta} |X_i|^2 + \sum_{i=0}^{k-1} A^{(i+1)\Delta} M_i,$$
where

\[ K_1 = R_1 \left( \frac{1}{1+c_0} + C_1 \theta \Delta \right), \]

\[ K_2 = R_2 \Delta - R_1 \left( \frac{\beta^2}{c_0} + C_2 \theta \Delta \right). \]

Now

\[ K_1 + K_2 A^T = R_1 \left( \frac{1}{1+c_0} - A^T \frac{\beta^2}{c_0} \right) + (R_1 C_1 \theta + R_2 - R_1 C_2 \theta) \Delta \]

\[ = \Delta \left[ \frac{R_1}{\Delta} \left( \frac{1}{1+c_0} - A^T \frac{\beta^2}{c_0} \right) + (R_2 + R_1 (C_1 - C_2) \theta) \right]. \]

It is obvious that

\[ \lim_{\Delta \to 0} \frac{R_1}{\Delta} = \log A - C < 0. \]

So if

\[ h := (\log A - C) \left( \frac{1}{1+c_0} - A^T \frac{\beta^2}{c_0} \right) + R_2 \leq 0, \]

then we have \( K_1 + K_2 A^T \leq 0. \)

On the other hand, since \( \beta^2 < \frac{1}{3 + 2\sqrt{2}} \), then there exists \( c_0 > 0 \) such that

\[ \beta^2 < \frac{c_0}{c_0^3 + 3c_0 + 2} \leq \frac{1}{3 + 2\sqrt{2}}. \]

Choose \( c_0 = \sqrt{2} \), then

\[ c_0 - (1 + c_0)^2 \beta^2 - (1 + c_0) \beta^2 = \sqrt{2}(1 - (3 + 2\sqrt{2}) \beta^2) > 0. \]

Since

\[ C_1 > \frac{3 + 2\sqrt{2}}{1 - (3 + 2\sqrt{2}) \beta^2} C_2 = \frac{c_0 (1 + c_0)^2}{c_0 - (1 + c_0)^2 \beta^2 - (1 + c_0) \beta^2} C_2, \]

then there exists \( \varepsilon \) small enough such that

\[ \frac{c_0 (1 - \varepsilon) - (1 + c_0)^2 \beta^2 - (1 - \varepsilon)(1 + c_0) \beta^2}{c_0 (1 + c_0)^2} C_1 > C_2. \]

By taking \( C = \frac{(1-\varepsilon)C_1}{1+c_0} \), we have

\[ \frac{c_0 C - (1 + c_0) \beta^2 C_1 - (1 + c_0) \beta^2 C}{c_0 (1 + c_0)} > C_2. \]

That is,

\[ \frac{C_1 \beta^2}{c_0} + C_2 - C \left( \frac{1}{1+c_0} - \frac{\beta^2}{c_0} \right) < 0. \]

Thus, \( h \leq 0 \) holds for some \( A (> 1) \) close to 1.

Then

\[ A^{k \Delta} |F_k|^2 \leq |F_0|^2 + K_2 A^T \sum_{i=-m}^{1-1} A^{(i+1) \Delta} |X_i|^2 + \sum_{i=0}^{k-1} A^{(i+1) \Delta} M_i. \]
Take expectation on both sides, we have
\[ A^k \Delta \mathbb{E} |F_k|^2 \leq \mathbb{E}(|F_0|^2 + K_2 A^r \sum_{i=-m}^{k-1} A^{(i+1)\Delta} |X_i|^2) =: K < \infty. \]

By (5.5), we have
\[ A^k \Delta \mathbb{E} |X_k|^2 \leq \left( \frac{1}{1 + c_0} + C_1 \theta \Delta \right)^{-1} \left( A^k \Delta \mathbb{E} |F_k|^2 + \left( \frac{\beta^2}{c_0} + C_2 \theta \Delta \right) A^k \Delta \mathbb{E} |X_{k-m}|^2 \right) \]
\[ \leq K (1 + c_0) + (1 + c_0) (\frac{\beta^2}{c_0} + C_2 \theta \Delta) A^r A^{(k-m)\Delta} \mathbb{E} |X_{k-m}|^2. \]

Notice that since \( \beta^2 \frac{1+c_0}{c_0} < \frac{1}{(1+c_0)^2} < 1 \), we can choose \( \Delta \) small enough and \( A \) close to 1 such that \( q := (1 + c_0) (\frac{\beta^2}{c_0} + C_2 \theta \Delta) A^r \) is a constant.

Denote \( a_k := A^k \Delta \mathbb{E} |X_k|^2, b := K(1 + c_0) \).

Then
\[ a_k \leq b + qa_{k-m} \]
\[ \leq b \sum_{i=0}^{\lfloor k/m \rfloor} q^i + q^{\lfloor k/m \rfloor+1} a_{k-(\lfloor k/m \rfloor+1)m} \]
\[ \leq \frac{b}{1-q} + |\xi((k-(\lfloor k/m \rfloor+1)m)\Delta)|^2. \]

Then there exists \( K' \) (independent of \( k \)) such that
\[ A^k \Delta \mathbb{E} |X_k|^2 \leq K' < \infty. \] (5.6)

This immediately yields that
\[ \limsup_{k \to \infty} \frac{\log \mathbb{E} |X_k|^2}{k \Delta} \leq -\log A < 0. \]

Let us now consider the almost sure stability. By Chebyshev inequality, inequality (5.6) implies that
\[ P(|X_k|^2 > e^{-k\Delta (\log A - \epsilon)}) \leq K'e^{-k\Delta \epsilon}, \forall k \geq 1, \epsilon > 0. \]

Then by Borel-Cantelli lemma, we see that for almost all \( \omega \in \Omega \)
\[ |X_k|^2 \leq e^{-k\Delta (\log A - \epsilon)} \] (5.7)
holds for all but finitely many \( k \). Thus, there exists a \( k_0(\omega) \), for all \( \omega \in \Omega \) excluding a \( P \)-null set, for which (5.7) holds whenever \( k \geq k_0 \).

Therefore, for almost all \( \omega \in \Omega \),
\[ \frac{1}{k \Delta} \log |X_k| \leq -\frac{\log A - \epsilon}{2} \] (5.8)
whenever \( k \geq k_0 \). Letting \( k \to \infty \) and \( \epsilon \downarrow 0 \). Then
\[ \limsup_{k \to \infty} \frac{1}{k \Delta} \log |X_k| \leq -\frac{\log A}{2}. \] (5.9)

The proof is then complete. □
Remark 5.2 When there is no Markovian switching, in \cite{22}, the authors considered the mean square stability of the semi implicit Euler method for NSDDEs. The numerical method they considered is a special case of our $\theta$-EM scheme (1.6) ($\theta = 1$). The sufficient conditions are the global Lipschitz continuity of both $f$ and $g$ plus the monotonicity condition (3.4) on $f$. However, we only need the local Lipschitz condition on the coefficients $f$ and $g$, the one sided Lipschitz condition (5.2) on $f$ and the monotonicity condition (5.2). Moreover, they only got the mean square stability while we get the mean square exponential stability of $\theta$-EM scheme. We also remark that in \cite{22} condition (C1) implies the linear growth condition (C2) since $f(t, 0, 0) = g(t, 0, 0) \equiv 0$. That is, condition (C2) is not necessary.

Let us now give an example to illustrate the theory.

Example Let $r(t)$ be a Markov Chain on the state space $S = \{1, 2\}$ with generator $\Gamma$. Consider the following scalar neutral stochastic differential delay equation with Markov switching

$$
\frac{d}{dt} [X(t) - \frac{1}{6r(t)} \sin X(t - \tau)] = (-6X(t) - X^{5}(t) - \frac{1}{2} \sin X(t - \tau))r(t)dt + 2 \frac{X^3(t)X(t - \tau)}{(1 + X^2(t - \tau))} dB_t
$$

with the initial value $x_0(\theta) = \xi(\theta) \equiv 1, \theta \in [-1, 0]$. It is clear that the coefficients $f(x, y, i) = -(6x + x^5 + \frac{1}{2} \sin y)i$ and $g(x, y, i) = 2 \frac{x^3y}{(1+y^2)i}$ satisfy the local Lipschitz condition, and $D(x, i) = \frac{1}{6} \sin x$ satisfies condition \cite{13} for $\beta_0 = \frac{1}{6} (< \frac{\sqrt{3+2\sqrt{2}}}{2})$. Moreover, $f$ satisfies condition (1.7).

On the other hand, we have $|g(x, y, i)|^2 \leq |x|^i$, and

$$
2(2 - D(y, i))f(x, y, i) = -12ix^2 - 2ix^6 + (2 - i)x \sin y + \frac{1}{3} x^5 \sin y + \frac{1}{6} \sin^2 y
$$

$$
\leq -12ix^2 - 2ix^6 + (2 - i)(2x^2 + \frac{\sin^2 y}{8})
$$

$$
+ \left\{ \frac{5}{6}x^6 + \frac{1}{6} \sin^6 y \right\} + \frac{1}{6} \sin^2 y
$$

$$
\leq -(8 + 2i)x^2 - (2i - \frac{5}{18})x^6 + \left(\frac{2 - i}{8} + \frac{18}{16} + \frac{1}{6}\right) y^2.
$$

Then

$$
2(2 - D(y, i))f(x, y, i) + |g(x, y, i)|^2 \leq -10x^2 + \frac{25}{2} y^2.
$$

Thus (5.2) holds for $C_1 = 10$, $C_2 = \frac{25}{2}$. By \cite{14} Theorem A.1 (Take $\gamma \equiv 0, U(x, i) = \frac{25}{12} x^2$ there), there exists a unique global solution $X(t)$ on $t \geq -1$. By taking $\beta_0 = \frac{1}{6}$ in (1.3) and $V(x, t, i) = U(x, i) = \left|x^2\right|$, $\gamma = \psi = 0, \lambda = 1, \alpha_1 = 8, \alpha_2 = \frac{25}{12}$, we know that (5.2) holds. Then by Theorem 3.2, the exact solution is mean square exponentially stable. On the other hand, since $\frac{C_1}{C_2} = 28.8 > \frac{3+2\sqrt{2}}{1-(3+2\sqrt{2})\beta_0}$, then by Theorem 5.1 the $\theta$-EM approximation is also mean square and almost surely exponentially stable.

However, the coefficient of the diffusion term does not satisfy the linear growth condition $|g(x, y, i)| \leq k(|x| + |y|)$ in this case. That is, we can not get the mean stability from Theorem 3.1 in \cite{22}. 

16
References

[1] Baker, C. and Buckwar, E., Exponential stability in $p$-th mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations, J. Comput. Appl. Math., 2005, 184(2):404-427.

[2] Basak, G., Bisi, A. and Ghosh, M.K., Stability of a random diffusion with linear drift, J. Math. Anal. Appl. 1996, 202, 604-622.

[3] Chen, L. and Wu, F., Almost sure exponential stability of the $\theta$-method for stochastic differential equations, Statistics and Probability Letters, 2012, 82:1669-1676.

[4] Ghosh, M., Arapostathis, A. and Marcus, S.I., Optimal control of switching diffusions with applications to flexible manufacturing systems, SIAM J. Control Optim., 1993, 31, 1183-1204.

[5] Higham, D., Mean-square and asymptotic stability of the stochastic theta method. SIAM Journal on Numerical Analysis, 2001, 38(3):753-769.

[6] Higham, D., Mao, X. and Stuart, A.M., Exponential mean-square stability of numerical solutions to stochastic differential equations. LMS J. Comput. Math., 2003, 6:297-313.

[7] Higham, D., Mao, X. and Yuan, C., Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations, SIAM J. Number. Anal. 2007, 45(2):592-609.

[8] Hu, Y., Lan, G. and Zhang, C., Polynomial and exponential stability of $\theta$-EM approximations to a class of stochastic differential equations, arXiv:1407.1486.

[9] Kolmanovskii, V., Koroleva, N., Maienberg, T. and Mao, X., Neutral stochastic differential delay equations with Markovian switching, Stoch. Anal. Appl., 2003, 21(4), 839-867.

[10] Liu, K. and Chen, A., Moment decay rates of solutions of stochastic differential equations, Tohoku Math. J., 2001, 53:81-93.

[11] Liu, W., Foondun, M. and Mao, X., Mean Square Polynomial Stability of Numerical Solutions to a Class of Stochastic Differential Equations, arXiv:1404.6073v1.

[12] Mao, X., Stability of stochastic differential equations with Markovian switching, Stoch. Proc. Appl., 1999, 79, 45-67.

[13] Mao, X. and Yuan, C., Stochastic Differential Equations with Markovian Switching, Imperial College Press, 2006.

[14] Mao, X., Stochastic differential equations and applicatons, 2nd edition, Horwood, Chichester, 2007.

[15] Mao, X. and Szpruch, L., Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients, J. Comput. Appl. Math., 2013, 238:14-28.
[16] Mao, X. and Szpruch, L., Strong convergence rates for backward Euler-Maruyama method for nonlinear dissipative-type stochastic differential equations with super-linear diffusion coefficients. Stochastics, 2013, 85(1):144-171.

[17] Mao, X., Shen, Y. and Yuan, C., Almost surely asymptotic stability of neutral stochastic differential equations with Markovian switching. Stoch. Proc. Appl., 2008, 118, 1385-1406.

[18] Milošević, M., Almost sure exponential stability of solutions to highly nonlinear neutral stochastic differential equations with time-dependent delay and the Euler-Maruyama approximation, Mathematical and Computer Modelling, 2013, 57, 887-899.

[19] Rodkina, A. and Schurz, H., Almost sure asymptotic stability of drift-implicit \( \theta \)-methods for bilinear ordinary stochastic differential equations in \( \mathbb{R}^1 \), J. Comput. Appl. Math., 2005, 180:13-31.

[20] Situ, R., Theory of Stochastic Differential Equations with Jumps and Applications, Springer, 2005.

[21] Skorohod, A., Asymptotic Methods in the Theory of Stochastic Differential Equations, American Mathematical Society, Providence, 1989.

[22] Wang, W. and Chen, Y., Mean-square stability of semi-implicit Euler method for nonlinear neutral stochastic delay differential equations, Applied Numerical Mathematics 2011, 61, 696-701.

[23] Wu, F., Mao, X. and Szpruch, L., Almost sure exponential stability of numerical solutions for stochastic delay differential equations, Numer. Math., 2010, 115:681-697.

[24] Yuan, C. and Lygeros, J., Stabilization of a class of stochastic differential equations with Markovian switching, Syst. Contr. Letters, 2005, 54:819-833.

[25] Zong, X. and Wu, F., Choice of \( \theta \) and mean-square exponential stability in the stochastic theta method of stochastic differential equations, J. Comput. Appl. Math., 2014, 255:837-847.