A Sahlqvist-style Correspondence Theorem for Linear-time Temporal Logic

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Abstract
The language of modal logic is capable of expressing first-order conditions on Kripke frames. For instance, the modal formula ($\square q \rightarrow q$) is valid in exactly the reflexive frames, where reflexivity $\forall x R(x, x)$ is a first-order condition. The classic result by Henrik Sahlqvist identifies a significant class of modal formulas for which first-order conditions – or Sahlqvist correspondents – can be found in an effective, algorithmic way. Recent works have successfully extended this classic result to more complex modal languages. In this paper, we pursue a similar line and develop a Sahlqvist-style correspondence theorem for Linear-time Temporal Logic (LTL), which is one of the most widely used formal languages for temporal specification. LTL extends the syntax of basic modal logic with dedicated temporal operators Next $X$ and Until $U$. As a result, the complexity of the class of formulas that have first-order correspondents also increases accordingly. In this paper, we identify a significant class of LTL Sahlqvist formulas built by using modal operators $F$, $G$, $X$, and $U$. The main result of this paper is to prove the correspondence of LTL Sahlqvist formulas to frame conditions that are definable in first-order language.

Keywords: Linear Temporal Logic; Sahlqvist formula; Correspondence Theory; Kripke frame.

1 Introduction
One of the most well-known results in the model theory of modal logic is that modal languages are rich enough to express (first-order) conditions on Kripke frames. Results along this direction have been known as Correspondence Theory [15,3]. For instance, the modal formula ($\square q \rightarrow q$) is valid in exactly the

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reflexive frames, where reflexivity $\forall x R(x, x)$ is a first-order condition. Since the 1970's, much research in modal logic has been devoted to identifying classes of formulas for which such first-order correspondents exist, including algorithms for their automatic computation. The classic result by H. Sahlqvist [13] identifies a significant class of modal formulas for which first-order conditions – or Sahlqvist correspondents – can be found in an effective, algorithmic way. Since then, correspondence theory has been successfully extended to more complex and expressive modal languages [8,14,16].

**Contribution.** In this paper we develop a Sahlqvist-style correspondence theorem for Linear-time Temporal Logic (LTL), which is nowadays one of the most widely-used formal languages for temporal specification [1]. LTL extends the syntax of basic modal logic with dedicated temporal operators $\text{Next } X$ and $\text{Until } U$. Formulas in LTL are interpreted on infinite words – or paths – representing the execution of a reactive system. Interestingly, Kamp [12] proved that every temporal operator on a class of continuous, strict linear orderings that is definable in first-order logic is expressible in terms of $\text{Since } S$ and $\text{Until } U$. As a result, the complexity of the class of modal formulas that have first-order correspondents also increases accordingly. In this paper, we identify a significant class of LTL formulas built by using temporal operators $\text{Eventually } F$, $\text{Always } G$, $\text{Next } X$, and $\text{Until } U$. To accommodate the enhanced expressiveness, we extend the class of Sahlqvist formulas with some additional conditions. To facilitate the treatment, we introduce the “intermediate” logic LTL’, which is more expressive than LTL, but whose syntax is closer to that of normal modal logics. Our main result is to prove the correspondence of such Sahlqvist formulas in LTL to frame conditions that are definable in a first-order language.

**Related Work.** As we mentioned above, Sahlqvist correspondence theorem has been extended in a number of different directions, mainly considering more and more expressive modal languages. For instance, in [8] a correspondence theorem is proved for temporal modal logic, whereas in [16,2] similar results are proved for the $\mu$-calculus and modal fixed-point logic respectively. It has to be remarked that these works extend the proof given in [14], rather than Sahlqvist’s original result in [13]. More recently, correspondence results have been proved for hybrid logics [7], distributive modal logics [9], and polyadic modal logics [10]. Some efforts have also been applied to the problem of finding more general and efficient algorithms to compute first-order correspondents of modal formulas [5,6,17], including [14] mentioned earlier. Still, to the best of our knowledge, no comparable result has been proved for the kind of temporal logics used in the specification and verification of reactive and distributed systems [1]. We deem such a result of interest to theoreticians and practitioners in modal logics alike.

**Structure of the Paper.** In Sec. 2 we introduce the syntax and semantics of LTL as well as the auxiliary logic LTL’, and define correspondence between modal and first-order formulas. In Sec. 3 we define the class of Sahlqvist
formulas for LTL and LTL', and provide a few preliminary results. Finally, Sec. 4 is devoted to the main result of this paper, namely the proof of the correspondence theorem.

2 Preliminaries: Linear-time Temporal Logic

In this section we provide background information about Linear-time Temporal Logic (LTL) [1,11]. Specifically, in Sec. 2.1 we introduce its syntax, as well as the syntax of the auxiliary language LTL'. Then, in Sec. 2.2 we interpret both languages on infinite system executions. Finally, in Sec. 2.3 we define their standard translations [3].

2.1 LTL: Syntax

We fix a set Prop of atomic propositions (or atoms) and define the formulas \( \phi \) in Linear-time Temporal Logic in Backus-Naur form as follows:

\[ \phi = \text{Prop} \mid \bot \mid \top \mid \neg \phi \mid \phi \land \phi \mid \phi \lor \phi \mid G\phi \mid F\phi \mid X\phi \mid \phi U \phi \]

where \( G \) is read “always”, \( F \) “eventually”, \( X \) is the Next operator, and \( U \) is the Until operator [1]. The Boolean connectives \( \rightarrow \) and \( \leftrightarrow \) can be introduced as standard. Operators \( F \) and \( G \) can be defined in terms of \( U \), but for convenience we assume them as primitive.

In this paper, we consider also a variant of LTL, that we call LTL'. Let \( W \) be a set of possible worlds (which serves as the model of LTL and LTL'). Fix \( W \), then we define the syntax of LTL' w.r.t. this particular \( W \) as follows:

\[ \phi = \text{Prop} \mid \bot \mid \top \mid \neg \phi \mid \phi \land \phi \mid \phi \lor \phi \mid G_{w,w'}\phi \mid F_x\phi \mid \hat{G}_{w,w'}\phi \mid X\phi \]

where \( w \) and \( w' \) (\( w \neq w' \)) are states in \( W \), and \( x \) is a variable over states. We will also use the following convention: \( w < w' \). If it were the case that \( w' < w \), then it suffices to switch the place of \( w \) and \( w' \) and write \( \hat{G}_{w,w'} \) as \( \hat{G}_{w',w} \). Remark that \( \hat{G}_{w,w'} \) is not in the language of LTL'.

Remark 2.1 Although states are semantical notions, the symbols representing them can be treated syntactically. The difference between the use of symbols in \( F_x \) and \( \hat{G}_{w,w'} \) is that \( x \) in the former is a variable that ranges over possible states, of which \( w \) and \( w' \) in the latter are members. In this paper, \( x,y,z,\ldots \) would be used to denote variables, whereas \( w, w', s, s', \ldots \) would denote states that are fixed in the context. Also, \( u,u' \) and \( v,v' \) can be used interchangeably, whenever the context is clear.

2.2 LTL: Semantics

To provide a semantics to LTL, we consider transition systems \( T = (S,\rightarrow) \), where \( S \) is a set of states, and the transition relation \( \rightarrow \subseteq S \times S \) is a binary relation on \( S \). Normally, the relation \( \rightarrow \) is assumed to be serial: for all \( s \in S \), there exists \( s' \in S \) such that \( s \rightarrow s' \). Then, a path in a transition system is an infinite sequence \( s_1,s_2,s_3,\ldots \), where for all \( i \in \mathbb{N}, s_i \rightarrow s_{i+1} \).

We now define models for LTL. Let \( W \) be the set of all paths in \( T \); whereas \( \leq, <, \text{ and } S \) are all binary relations on \( W \), introduced as follows. Let \( w = \)
s_1, s_2, s_3, ... and v = s_1', s_2', s_3', ... be paths in T, then w ≤ v iff for some i ≥ 1, s_i = s_i' and for all j > 0, s_{i+j} = s'_{i+j}, that is, v is a subpath of w starting from some index i. Then, w < v iff w ≤ v and w ≠ v. Further, S means successor: v = S(w) iff for all i > 0, s_i' = s_{i+1}. When the context is clear, we sometimes simply write R(w, v) for w < v, w ≤ v or v = S(w).

A model for LTL is a tuple \( M = (T, h) \), where T is a transition system, and \( h : Prop \to 2^S \) is an assignment function from atoms to set of states in S. We lift the assignment h from states to paths so that \( w \in h(q) \) iff \( s_1 \in h(q) \).

**Definition 2.2** [Satisfaction] Given a model \( M, w, \) and formula \( \phi \) in LTL', the satisfaction relation \( \models \) is defined as follows:

\[
\begin{align*}
(M, w) \models q & \iff w \in h(q) \\
(M, w) \models \neg \phi & \iff (M, w) \not\models \phi \\
(M, w) \models \phi \land \varphi & \iff (M, w) \models \phi \text{ and } (M, w) \models \varphi \\
(M, w) \models \phi \lor \varphi & \iff (M, w) \models \phi \text{ or } (M, w) \models \varphi \\
(M, w) \models G\phi & \iff \text{for all } v \in W, w \leq v \text{ implies } (M, v) \models \phi \\
(M, w) \models F_x\phi & \iff \text{for some } x \in W, w \leq x \text{ and } (M, x) \models \phi \\
(M, w) \models G_{w,w'}\phi & \iff \text{for all } u \in W, w \leq u < w' \text{ implies } (M, u) \models \phi \\
(M, w) \models X\phi & \iff v = S(w) \text{ and } (M, v) \models \phi
\end{align*}
\]

Hereafter we use \( w \models \phi \) as an abbreviation for \( (M, w) \models \phi \). We write \( [\phi]_w = 1 \) iff \( (M, w) \models \phi \) for \( M = (T, h) \). For future references, we precisely define below assignments for arbitrary formulas.

**Definition 2.3** [Assignment] Let \( T = (S, \to) \) be a transition system, and \( h : Prop \to 2^W \) an assignment function as before. We extend the domain of h from the set of atoms Prop to the set Form of all formulas:

\[ h : Form \to 2^W \]

such that \( h(\phi) \) is defined as \( \{ w \in S \mid (T, h, w) \models \phi \} \).

To provide an interpretation for LTL, we replace the clause for \( G_{w,w'} \) with a clause for the Until operator \( U \), as follows:

\[
(M, w) \models \phi U \phi' \iff \text{for some } u \geq w, (M, u) \models \phi', \text{ and for all } v \in W, w \leq v < u \text{ implies } (M, v) \models \phi
\]

LTL also replaces \( F_x \) with the operator \( F \), where the variable path is no longer shown in the syntax. But its semantics remains the same.

Now it is possible to translate LTL into LTL'.

**Definition 2.4** Let Form_{LTL} be the class of all LTL formulas and Form_{LTL'} be the class of all LTL' formulas. Let

\[ \tau : Form_{LTL} \to Form_{LTL'} \]

be the translation from LTL to LTL' defined as follows:

\[ q \mapsto q \]
\[\neg \phi \mapsto \neg \tau(\phi)\]
\[\phi_1 \land \phi_2 \mapsto \tau(\phi_1) \land \tau(\phi_2)\]
\[\phi_1 \lor \phi_2 \mapsto \tau(\phi_1) \lor \tau(\phi_2)\]
\[G\phi \mapsto G\tau(\phi)\]
\[F\phi \mapsto F_x \tau(\phi)\]
\[X\phi \mapsto X\tau(\phi)\]
\[\phi_1 \U \phi_2 \mapsto F_x (\tau(\phi_2) \land \hat{G}_{w,x} \tau(\phi_1))\]

where \(x\) is a path variable, and \(w\) is the path at which we aim to evaluate the formula.

**Remark 2.5** [Variable Convention]

In the conjunctive and disjunctive clause, if a path variable \(x\) appears in both \(\tau(\phi_1)\) and \(\tau(\phi_2)\), then in \(\tau(\phi_1 \land \phi_2)\) we replace \(x\) in \(\tau(\phi_2)\) by another path variable \(x'\) that do occur in either \(\tau(\phi_1)\) or \(\tau(\phi_2)\).

If \(x\) in \(\tau(\phi_1 \U \phi_2)\) appears in \(\tau(\phi_1)\) or \(\tau(\phi_2)\), then we replace the occurrences of \(x\) in \(\tau(\phi_1)\) and \(\tau(\phi_2)\) by \(x_1\) and \(x_2\).

**Lemma 2.6** Let \(\tau\) be the translation from LTL to LTL’ in Def. 2.4. Then an LTL formula and its translation w.r.t. \(\tau\) are semantically equivalent.

**Proof.** The proof makes use of structural induction on the formula. We only consider the case for the LTL formula \(\phi = \phi_1 \U \phi_2\). Let \(w\) be any path, and \(\phi\) is evaluated at \(w\). The translation \(\tau(\phi)\) of \(\phi\) at \(w\) is

\[\exists x(w < x \land \tau(\phi_2) \land \forall u(w \leq u < x \rightarrow \tau(\phi_1)))\]

By induction hypothesis, \(x \vDash \tau(\phi_2) \iff x \vDash \phi_2\) and \(u \vDash \tau(\phi_1) \iff u \vDash \phi_1\). So \(w \vDash \tau(\phi) \iff w \vDash \phi\). Since \(w\) is arbitrary, \(\phi\) and \(\tau(\phi)\) are semantically equivalent. \(\Box\)

### 2.3 Standard Translation and Correspondence

The standard translation of formulas in LTL’ mirrors their semantics. For every atom \(q \in Prop\), we introduce a predicate symbol \(Q\). For an arbitrary formula \(\phi\) in LTL’, we denote the first-order standard translation of \(\phi\) at \(w\) as \(ST_w(\phi)\), and it is inductively defined as follows:

**Definition 2.7** [Standard Translation] The standard translation \(ST_w(\phi)\) of formula \(\phi\) at path \(w\) is inductively defined as case of \(\phi\):

\[q : Q(w)\]
\[\neg \phi : \neg ST_w(\phi)\]
\[\phi \land \varphi : ST_w(\phi) \land ST_w(\varphi)\]
\[\phi \lor \varphi : ST_w(\phi) \lor ST_w(\varphi)\]
\[G\phi : \forall v(w < v \rightarrow ST_v(\phi))\]
\[F_x \phi : \exists x(w < x \land ST_x(\phi))\]
\[\hat{G}_{s,s'} \phi : \forall v(s \leq v < s' \rightarrow ST_v(\phi))\]
\[X \phi : ST_{S(w)}(\phi)\]

To simplify the notation, instead of saying that \(ST_w(\phi)\) is the standard translation of \(\phi\) at \(w\), we say that it is the standard translation of \(\phi[w]\). Then,
the second-order standard translation of \( \phi[w] \) is obtained by prefixing universal quantification for every predicate \( Q_1, Q_2, \ldots, Q_k \) in \( ST_w(\phi) \). There is no abbreviated notation for this second-order standard translation. Whenever the context is clear, we will also call it the standard translation. For the most part, we work with the second-order standard translation.

Since the models for LTL' and for first-order logic are the same (they are both relational structures), we say that \( (M, w) \models \phi \), where \( \phi \) is a first order formula. However, when it comes to the second-order formulas, the models have to be modified. In second-order logic, quantification over predicates (sets) is allowed, and the domain of a predicate is determined by the assignment \( h \), i.e., \( \text{dom}(Q) = h(q) \). Therefore, assignments in transition systems are equivalent to (universal) quantification over predicates in second-order logic.

**Definition 2.8** [Correspondence] Let \( T = (S, \rightarrow) \) be a transition system, and \( w \in W \). An LTL' formula \( \phi(q_1, q_2, \ldots, q_k) \) is said to (locally) correspond to a formula \( \varphi \) in second order logic at \( w \) whenever \( \phi \) and \( \varphi \) are both evaluated to be true at \( w \) in \( T \).

The following lemma shows why local correspondence is defined the way it is. A proof can be obtain by a straightforward induction on the structure of formula \( \phi \).

**Lemma 2.9** An LTL' formula \( \phi(q_1, q_2, \ldots, q_k) \) (locally) corresponds to \( \forall Q_1 \forall Q_2 \ldots \forall Q_k \text{ST}_w(\phi) \) at \( w \), where \( \text{ST}_w(\phi) \) is the (first-order) standard translation of \( \phi[w] \).

**Remark 2.10** In light of this lemma, we will be using semantics and standard translation interchangeably in this paper.

The main result we prove in this paper can be stated informally as follows: there is a collection of LTL formulas \( \phi \), such that for all paths \( w \), the local correspondent \( \phi[w] \) of \( \phi \) at \( w \) can be expressed as a first-order formula. This is the basic content of Sahlqvist correspondence theorem, which will be stated later on in more precise terms. Note that, although the standard translation is only defined for LTL', the translation for LTL and its semantics can be defined in a similar manner, where the main difference is the following clause:

\[
\text{ST}_w(\phi U \phi') = \exists u, w \leq u \land \text{ST}_u(\psi) \land (\forall v, w \leq v < u \rightarrow \text{ST}_v(\phi'))
\]

### 3 Sahlqvist Formulas for LTL

In this section, we introduce two particular types of formulas that play key roles in the construction of Sahlqvist formulas: boxed formulas and negative formulas. We prove the monotonicity theorem in Sec. 3.2 and introduce Sahlqvist formulas for LTL in Sec. 3.3.

#### 3.1 Boxed Formulas

In standard modal logic, boxed formulas are defined as a sequence of boxes \( \Box \) followed by an atomic formula, i.e., they have the form \( \Box \ldots \Box q \) for a possible empty sequence of boxes. The sequence of boxes can be denoted as \( \Box^n \), for
We can prove the following auxiliary result concerning boxed formulas. Assume that for every formula \( \square q \), this lemma shows that the standard translation of every boxed formula.

**Remark 3.3** This lemma shows that the standard translation of every boxed formula \( \square q \) can be written in the form of \( \forall v, R(w, v) \rightarrow Q(v) \) using a unique

\[ n \in \mathbb{N}, \text{ whose semantics is similar to the one for a single box: } w \models \square q \text{ iff for all } v, R^a(w, v) \text{ implies } v \models q, \text{ where } R^a \text{ is not difficult to construct (see Lemma 3.2).} \]

Similarly for LTL’, the syntactic operators having universally quantified implication as semantics can be integrated into the LTL’ boxed formulas for the same reason. We denote an arbitrary boxed formula as \( \square^n q = \square \ldots \square q \), where each \( \square \) is a distinct element from \( \{G, \bar{G}_w, w', X\} \) (i.e., the set of boxed operators). Now we define the corresponding accessibility relation.

**Definition 3.1** [Accessibility Relation \( R_{\square^n} \)] We define the accessibility relation \( R_{\square^n} \) by induction on \( n \in \mathbb{N} \).

**Base case:** if \( n = 0 \), i.e. \( \square q = q \), then \( R^0(w, v) \text{ iff } w = v. \)

**Inductive cases:** let \( R_{\square^n} \) be defined, then

- If \( \square^{n+1} q = G \square^n q \), then \( R_{\square^n + 1}(w, v) \text{ iff for some } u \in W, w \leq u \) and \( R_{\square^n}(u, v). \)
- If \( \square^{n+1} q = X \square^n q \), then \( R_{\square^n + 1}(w, v) \text{ iff } R_{\square^n}(S(w), v). \)
- If \( \square^{n+1} q = \bar{G}_{s,s'} \square^n q \), then \( R_{\square^n + 1}(w, v) \text{ iff for some } u \in W, s \leq u < s' \) and \( R_{\square^n}(u, v). \)

Whenever the context is clear, we use \( R^n \) to denote \( R_{\square^n} \).

By Def. 3.1 we can prove the following auxiliary result concerning boxed formulas.

**Lemma 3.2** (Boxed Formulas Lemma) Let \( \square^n q \) be an LTL’ boxed formula with \( n \) boxed operators appearing in front of atom \( q \) (with \( n \) possibly equal to 0). Then \( w \models \square^n q \text{ iff for all } v \in W, R^n(w, v) \text{ implies } v \models q. \)

**Proof.** We prove this lemma by induction on \( n \). The base case for \( n = 0 \) is immediate: \( w \models q \text{ iff for all } v, w = v \text{ implies } v \models q, \) that is \( R_0(w, v). \) Now suppose that the lemma holds for an arbitrary \( n \), i.e., \( w \models \square^n q \text{ iff for all } v \in W, R^n(w, v) \text{ implies } v \models q. \) We have to show that \( w \models \square^{n+1} q \text{ iff for all } v \in W, R^{n+1}(w, v) \text{ implies } v \models q. \) We discuss by case the options for the first boxed operator \( \square_0 \) in \( \square^{n+1} q. \)

For \( \square_0 = \bar{G}_{s,s'}, w \models \square_0 \square^n q \text{ iff for every } v, s \leq v < s' \text{ implies } v \models \square^n q. \) By induction hypothesis, \( w \models \square_0 \square^n q \text{ iff for every } v, s \leq v < s' \text{ implies } v \models \square^n q. \) That is \( \square^n(v, u), u \models q. \) We want to show that this is equivalent to: for all \( u, R^{n+1}(w, u) \text{ implies } u \models q. \) Suppose \( w \models \square^n q \text{ is the case; fix } u_0. \) Let \( v_0 \) be a path such that \( s \leq v_0 < s' \) if \( R^n(v_0, u_0) \), then \( u_0 \models q \) by assumption. Since \( u_0 \) is arbitrary, we get \( R^{n+1}(w, u_0) \text{ for every } u_0. \) Conversely, assume that for every \( u, \text{ if for any } v \text{ such that } s \leq v < s' \text{ and } R^n(v, u), \) then \( u \models q. \) Fix \( v_0 \) such that \( s \leq v_0 < s' \). Then take an arbitrary \( u_0. \) If \( R^n(v_0, u_0) \), then by assumption, \( u_0 \models q. \) Since \( v_0 \) is arbitrary, for every \( v_0, s \leq v_0 < s' \text{ and } R^n(v_0, u_0) \text{ imply } u_0 \models q \text{ for arbitrary } u_0, \) as desired. This concludes the case for \( \square_0 = \bar{G}_{s,s'} \). The proofs for the cases \( \square_0 = G \text{ and } \square_0 = X \) are similar. \( \square \)
follows directly from the Lemma Proof. If the atomic variables appearing in $N$ let Corollary 3.6 \{\land\} from $q$ If for all $\varphi$ or $h$ the proof is by induction on the structure of $\varphi$. We show the proof is easy to see that $h$ is an LTL' positive formula; whereas $F_x \hat{G}_{w,w'} \varphi$ is negative. Let $N$ shall also call this formula negative. For example, if $\neg\exists y(s \leq y < s' \land ST_y(\neg N'))$ Remark that the part in the scope of the negation is in fact a positive fragment in the interpretation of LTL' formulas. By monotonicity lemma, if for all $q_j$ occurring in $N'$, $h_1(q_j) \subseteq h_2(q_j)$, then $h_1(\neg N') \subseteq h_2(\neg N')$. In other words,
if \((T, h_1, y) \models ST_y(\neg N')\), then \((T, h_2, y) \models ST_y(\neg N')\). Therefore, if there is a path \(y\) between \(s\) and \(s'\) such that \(y \models ST_y(\neg N')\) and \(y \in h_1(\neg N')\), then there is also such a path for \(h_2\). So if \((T, h_1, x) \models \exists y (s \leq y < s' \land ST_y(\neg N'))\), then \((T, h_2, x) \models \exists y (s \leq y < s' \land ST_y(\neg N'))\). It follows that \(h_2(N) \subseteq h_1(N)\), as desired.

3.3 Sahlqvist Formulas

The main goal of this paper is to find a significant class of Sahlqvist formulas for LTL, we therefore define them here. Then, we will show that this construction can be simplified by using the auxiliary language LTL’.

A formula \(A_{\text{LT}_L}\) is an LTL boxed formula if it is a sequence of boxes followed by an atom, where each element of the sequence belongs to \(\{X, G\}\). A formula is an LTL positive formula if it can be constructed from all logical symbols and modal operators of LTL except negation; a formula \(N_{\text{LT}_L}\) is an LTL negative formula if it is the negation of an LTL positive formula.

We now define LTL Sahlqvist formulas.

**Definition 3.7** [LTL Sahlqvist Formulas] Suppose \(\beta\) is an LTL boxed formula or negative formula. Then we define LTL untied formula as follows:

\[
\phi = A_{\text{LT}_L} \mid N_{\text{LT}_L} \mid \beta U \phi \mid \phi \land \phi
\]

The LTL Sahlqvist formulas are the conjunction of negations of LTL untied formulas.

**Remark 3.8** In the definition of LTL untied formula, \(F\phi\) can be retrieved using \(\top U \phi\).

As for LTL’, its Sahlqvist formulas are defined as follows:

**Definition 3.9** [LTL’ Sahlqvist Formulas] An LTL’ untied formula is constructed from LTL’ boxed formulas and LTL’ negative formulas using only \(F\), and conjunction:

\[
\phi = A_{\text{LT}_L'} \mid N_{\text{LT}_L'} \mid \phi \land \phi \mid F\phi
\]

As before, LTL’ Sahlqvist formulas are the conjunctions of negations of LTL’ untied formulas.

4 Correspondence Theorem

In this section we present the proof of the correspondence theorem for LTL. By embedding LTL Sahlqvist formulas into LTL’ Sahlqvist formulas, we only need to show that the theorem holds for the latter. We start by showing that the translation \(t\) from LTL to LTL’ in Sec. 2.2 preserves Sahlqvist formulas. Then we introduce the main lemma crucial to the theorem. Finally, a detailed proof of the theorem is provided.

4.1 Translation

We show that LTL Sahlqvist formulas can be translated into LTL’ Sahlqvist formulas.
Lemma 4.1 Let $\tau$ be the translation from LTL to LTL’ in Def. 2.4. Then the following claims are true:

1. The translation of an LTL untied formula w.r.t. $\tau$ is an LTL’ untied formula.
2. An LTL untied formula and its translation w.r.t. $\tau$ are semantically equivalent.

Proof.

1. The claim can be proved using structural induction on the formula. We only consider the case for the LTL untied formula $\phi = \beta U \psi$, where $\psi$ is also LTL untied. Let $w$ be any path, and $\phi$ is evaluated at $w$. By definition 2.4, $\tau(\phi) = F_x(\tau(\psi) \land \hat{G}_{w,x} \tau(\beta))$. If $\beta$ is an LTL boxed formula, then $\tau(\beta)$ is also an LTL’ boxed formula; so $\hat{G}_{w,x} \tau(\beta)$ is also an LTL’ boxed formula. If $\beta$ is an LTL negative formula, then $\tau(\beta)$ is an LTL’ negative formula; so $\hat{G}_{w,x} \tau(\beta)$ is also an LTL’ negative formula. Therefore, $\hat{G}_{w,x} \tau(\beta)$ is untied. By induction hypothesis, $\tau(\psi)$ is an LTL’ untied formula, hence $F_x(\tau(\psi) \land \hat{G}_{w,x} \tau(\beta))$ is LTL’ untied.

2. It follows immediately from Lemma 2.6.

Whenever two formulas are semantically equivalent, they have the same frame conditions. Therefore, having shown that for each LTL Sahlqvist formula, a semantically equivalent LTL’ formula exists and is also Sahlqvist, we can conclude the following lemma:

Lemma 4.2 If every LTL’ Sahlqvist formula locally corresponds to a first order formula, then every LTL Sahlqvist formula locally corresponds to a first order formula.

4.2 Main Lemma

In this section, we prove the main lemma, essential to the proof of the correspondence theorem for LTL’. The LTL’ untied formulas are solely built from boxed formula and negative formula, hence intuitively in order to find first-order correspondents for LTL’ Sahlqvist formula $\phi$, it suffices to find an assignment $h_0$ that satisfies the following for every boxed formula $A$ and every negative formula $N$ in $\phi$:

$$\exists Q, ST_w(A) \iff ST_w(A)[Q_0] \text{ and }$$
$$\exists Q, ST_w(N) \iff ST_w(N)[Q_0]$$

where $Q_0(x)$ is true iff $x \in h_0(q)$. $Q_0$ is called minimal predicate.

Definition 4.3 [Substitution] We first fix the notation on substitution in the minimal assignment. Let $\phi(q)$ be a formula and $h_0(q)$ be its minimal assignment for atom $q$ (to be defined subsequently). Let $Q_0$ be its corresponding minimal predicate. Suppose $t$ to be a symbol occurring in the expression of $Q_0$. Then we use $[t'/t]Q_0$ to denote the substitution of $t'$ for all occurrences of $t$ in $Q_0$. 
We can now introduce the notion of minimal assignment.

**Definition 4.4** [Minimal assignment] Let $\phi(q_1, \ldots, q_k)$ be an LTL' untied formula; let $w$ be a path. For every variable $q_j$ occurring in $\phi$, we define the minimal assignment $h_0(q_j)$ of $\phi$ at $w$ by induction on the structure of formula.

**Base cases:** Suppose that $\phi(q_j)$ is a boxed formula and its standard translation at $w$ is $\forall v(R_j(w, v) \rightarrow Q_j(v))$, then the minimal assignment for $q_j$ is $h_0(q_j) = \{u \in W \mid R_j(w, u)\}$.

Suppose $\phi$ is a negative formula, then $h_0(q_j) = \emptyset$ (and $Q_j(w) \equiv \bot$ for every $w$).

**Inductive cases:**

If the minimal assignment for $\phi_1(q_1, \ldots, q_k)$ and $\phi_2(q_1, \ldots, q_k)$ are respectively $h_0^1$ and $h_0^2$, then the minimal assignment for $\phi_1 \land \phi_2$ is $h_0^1 \cup h_0^2$.

If the minimal assignment for $\phi$ at $v$ is $h_0$, then the minimal assignment for $F_s \phi$ at $w = [x/v]h_0$.

If the minimal assignment for $\phi$ at $v$ is $h_0$, then the minimal assignment for $X\phi$ at $w = [S(w)/v]h_0$.

Suppose the minimal assignment for $\phi$ at $v$ is $h_0$. The minimal predicates for $q_j$ occurring in $\phi$ is defined as $Q_{j0}(z) \iff z \in h_0(q_j)$. Then the minimal assignment $h_0^0$ for $\bar{G}_{s, s'} \phi$ at $w$ is defined as $h_0^0(q_j) = \{y \in W \mid \exists s < x < s' \land [x/v]Q_{j0}(y)\}$ for every $q_j$.

**Remark 4.5** The minimal assignment for an LTL untied formula can be obtained by translating it into an LTL' untied formula.

Let $A$ be of the form $\exists^n q$ and $w \models A$ iff $\forall x(R^n(w, x) \rightarrow Q(x))$. Let $Q_0(x)$ be $R^n(w, x)$, we claim that $\exists Q(\forall x(R^n(w, x) \rightarrow Q(x))$ iff $\forall x(R^n(w, x) \rightarrow Q_0(x))$. The proof of this claim is immediate: the right hand side is always true; the right-to-left implication is also always true. It turns out that for every Sahlqvist formula, the recursive construction of the minimal assignment will always produce a first-order correspondent to its second-order translation. In particular, we need to show how the occurrences of negative formulas in a Sahlqvist formula can be given such first-order correspondents via minimal assignment.

**Lemma 4.6 (Main Lemma)** Let $E$ be an LTL’ untied formula, $w$ is a state, and $h_0$ is the minimal assignment of $E$ at $w$ (possibly empty). Let $h$ be an assignment. If there exists an assignment $g$ and a state $w$ such that $|E|_w^g = 1$, then the following are equivalent:

(a) For all $q_j \in \{q_1, \ldots, q_k\}$, $h_0(q_j) \subseteq h(q_j)$.

(b) $|B|_w^h = 1$.

where $B$ is defined is obtained from $E$ by replacing all occurrences of negative formulas $N_1, N_2, \ldots, N_m$ in $E$ by $\top$.

**Proof.** We proceed by induction on the structure of the formula.

For the base cases, we suppose that $E$ is either an LTL’ boxed formula $A$ or an LTL’ negative formula $N$. If $E$ is a negative formula $N$, then $h_0$ is empty,
therefore (a) must be true. As $B$ becomes $\top$, (b) is true, hence (a) and (b) are equivalent. If $E$ is a boxed formula $A$, then $B = A$. As only one atom appears in $A$, let it be $q$. Since $A(q)$ is true at $w$, $Q_0(x)$ is $R(w, x)$ where $R$ is obtained from the standard translation of $A(q)$. As $h(q) = \{x \in W \mid Q(x)\}$ and $h_0(q) = \{x \in W \mid Q_0(x)\} = \{x \in W \mid R(w, x)\}$, $h_0(q) \subseteq h(q)$ is therefore just saying that for all $x$, $(R(w, x) \rightarrow Q(x))$. But this is exactly what (b) says. Namely, $[E]_{w}^h = 1$ iff $w \models A$ iff $\forall x (R(w, x) \rightarrow Q(x))$. Therefore (a) and (b) are equivalent.

There are two cases for inductive steps: $E_1 \land E_2$, $F_x E$.

Case (i): Suppose that there is an assignment $g$ making $E = E_1 \land E_2$ true at $w$. Then $g$ also makes both $E_1$ and $E_2$ true at $w$. By induction hypothesis, $(a) \Leftrightarrow (b)$ holds for both $B_1$ and $B_2$. Let $h$ be an arbitrary assignment. Let $h_0^1, h_0^2, h_0$ denote the minimal assignments for $E_1, E_2, E$. We know $h_0^1, h_0^2 \subseteq h_0$. Also, for every atomic formula $q_j$ in $E$, it must be in either $E_1$ or $E_2$. Thus if for every $q_j$, $h_0(q_j) \subseteq h(q_j)$, then $[B_1]_{w}^h = [B_2]_{w}^h = 1$. So $(a) \Rightarrow (b)$ holds for $B$. Now assume $[B_1]_{w}^h = 1$. As before, $[B_1]_{w}^h = 1$ and $[B_2]_{w}^h = 1$. So for every atom $q_j$, if $q_j$ occurs in $B_1$, then $h_0^1(q_j) \subseteq h(q_j)$ ($i \in \{1, 2\}$). By definition of $h_0$, it is also the case that $h_0^1(q_j) \cup h_0^2(q_j) = h_0(q_j) \subseteq h(q_j)$ for every $q_j$. Hence $(b) \Rightarrow (a)$ holds for $B$.

Case (ii): Suppose that there is an assignment $g$ making $F_x E$ true at $w$. Then there is a state $x \geq w$ at which $g$ makes $E$ true. By induction hypothesis, (1) and (2) are equivalent:

(1) For every atomic variable $q_j$ in $E$, $h_0(q_j) \subseteq h(q_j)$.

(2) $[B]_{x}^h = 1$.

If $[B]_{x}^h = 1$, then $[F_x B]_{x}^h = 1$; so $(1) \Rightarrow [F_x B]_{x}^h = 1$. Also, if $(a) \Rightarrow [F_x B]_{x}^h = 1$, then there is a state $y$ at which $[B]_{y}^h = 1$. Since $x$ in (1) $\Leftrightarrow$ (2) is arbitrary, we get (1) back. Therefore, $(a) \Leftrightarrow (b)$ holds for $F_x B$. \hfill $\Box$

4.3 Correspondence Theorem

It finally only remains to show that all LTL’ Sahlqvist formulas $S$ have first-order correspondents.

Theorem 4.7 (LTL’ Correspondence Theorem) Let $S$ be an LTL’ Sahlqvist formula, then the local correspondent of $S[w]$ can be expressed in first-order terms, i.e., $\forall Q_1, \ldots, \forall Q_k, ST_w(S(q_1, \ldots, q_k))$ has a first-order correspondent.

Proof.

Let $S = \bigwedge_{i=1}^m \neg E_i$ where $E_i$ are LTL’ untied formulas. The second order standard translation of $S[x]$ is $\bigwedge_{i=1}^m \forall Q_1, \forall Q_2, \ldots, \forall Q_k, \neg ST_x(E_i)$. However, to simplify the task, we can work with each conjunctive clause $E_i$ individually. In addition, we are going to work with the first correspondence of its negation:

$\exists Q_1, \exists Q_2, \ldots, \exists Q_k, ST_x(E_i)$
We proceed by induction on the complexity of the formula.

**Base case:** Let’s write the formula for the base case as follows:

\[
\bigwedge_{j=1}^{m} \oplus a_j C_j
\]

where \( C_j \) is either an LTL’ boxed formula or an LTL’ negative formula, and \( a_j \) is the number of \( F_x \) appearing in front of each \( C_j \).

For each \( j \), if \( C_j \) is a boxed formula, then the standard translation of \( \oplus a_j C_j[x] \) can be written as

\[
\exists x_1, \ldots, \exists x_{a_j}( R_{j1}(x, x_{j1}) \land \ldots \land R_{ja_j}(x_{a_j-1}, x_{a_j}) \land (\forall y(R_j(x_{a_j}, y) \rightarrow Q_j(y))))
\]

However, if \( C_j \) is a negative formula, we do not need to write down the standard translation of \( \oplus a_j C_j[x] \). We can omit \( \oplus a_j \) because it can be part of the LTL’ negative formula. Therefore, the standard translation of \( E_i[x] \) can be written as

\[
\bigwedge_{j=1}^{t} (\exists x_1, \ldots, \exists x_{a_f} \bigwedge_{s=1}^{a_f} R_{fs}(x_{s-1}, x_s) \land (\forall y(R_f(x_{a_f}, y) \rightarrow Q_f(y)))) \land \bigwedge_{l=1}^{r} ST_x(N_l)
\]

where \( t + r = m \). Here, \( C_1, \ldots, C_m \) are \( A_1, \ldots, A_t, N_1, \ldots, N_r \).

For the first conjunct of this formula, the following two formulas are equivalent by definition of minimal assignments, where \( Q_{f0} \) is the minimal predicate of the atomic variable \( Q_f \):

\[
\exists Q_1, \ldots, Q_k \bigwedge_{j=1}^{t} (\exists x_1, \ldots, x_{a_f} \bigwedge_{s=1}^{a_f} R_{fs}(x_{s-1}, x_s) \land (\forall y(R_f(x_{a_f}, y) \rightarrow Q_f(y))))
\]

\[
\bigwedge_{j=1}^{t} (\exists x_1, \ldots, \exists x_{a_f} \bigwedge_{s=1}^{a_f} R_{fs}(x_{s-1}, x_s) \land (\forall y(R_f(x_{a_f}, y) \rightarrow Q_{f0}(y))))
\]

For the second conjunct of (1) \( \bigwedge_{l=1}^{r} ST_x(N_l) \), notice that \( E_i \) satisfies the condition of the main lemma: namely, there exists an assignment under which it is satisfied at \( x \). Also, let \( h \) be an arbitrary assignment, if \( [E_i]_h^0 = 1 \), then \( [B_i]_h^0 = 1 \), where \( B_i \) is obtained from \( E_i \) by substituting \( \top \) for every occurrence of negative formulas in \( E_i \). It follows that for all \( q_j \in \{q_1, \ldots, q_k\} \), \( h_0(q_j) \subseteq h(q_j) \). As \( N_l \) are negative formulas, by the monotonicity lemma for negative formulas (Corollary 3.6), \( h(N_l) \subseteq h_0(N_l) \). Therefore, for all \( h \), if \( [N_l]_h^0 = 1 \), then \( [N_l]_{h_0}^0 = 1 \). From this, we can easily prove that (4) and (5) below are equivalent.

\[
\exists Q_1, \exists Q_2, \ldots, \exists Q_k \bigwedge_{l=1}^{r} ST_x(N_l)
\]

\[
\bigwedge_{l=1}^{r} ST_x(N_l)[Q_{10}, Q_{20}, \ldots, Q_{h0}]
\]
From the equivalence (2) \land (4) \iff (3) \land (5), (1) obtains its first order correspondent by substituting minimal predicate \(Q_{10}, \ldots, Q_{k0}\) for \(Q_1, \ldots, Q_k\), hence the quantifiers over them can also be dropped. Therefore, \(\forall Q_1, \ldots, \forall Q_k, ST_x(S) \equiv \bigwedge_{i=1}^m \neg(\exists Q_1, \ldots, \exists Q_k, ST_x(E_i))\) also has first-order correspondent \(\bigwedge_{i=1}^m \neg ST_x(E_i)[Q_{10}/Q_1, \ldots, Q_{k0}/Q_k]\).

Now we proceed to the inductive steps. There are two cases:

**Case 1:** Suppose the untied formula \(E\) is of the form \(F_y C\), where \(C\) is an untied formula. If \(E\) is true at the state \(x\), then there is a state \(y\) such that \(x \leq y\) and \(C\) is true at \(y\). By induction hypothesis, we can find the minimal predicates for \(C\) which is \(Q^C_0 = Q_{10}, \ldots, Q_{k0}\) such that

\[
\exists Q ST_y(C) \iff [Q^C_0/Q]ST_y(C)
\]

Since the minimal predicate for \(E\) is \(Q^E_0 = [x/y]Q^C_0\), we get

\[
\exists Q ST_x(E) \iff [Q^E_0/Q]ST_x(E)
\]

**Case 2:** Suppose \(E = E_1 \land E_2\) where \(E_1\) and \(E_2\) are both untied. Let \(q_1, \ldots, q_k\) be the atoms appearing in both \(E_1\) and \(E_2\). Then by induction hypothesis, we have two sets of minimal predicates \(\{Q_{10}^1, \ldots, Q_{k0}^1\}\) and \(\{Q_{10}^2, \ldots, Q_{k0}^2\}\). The minimal predicates for \(E\) are defined as

\[Q_{j0} = Q^1_{j0} \lor Q^2_{j0}\]

By induction hypothesis, we know that

\[
\exists Q ST_x(E_1) \iff [Q^1_0/Q]ST_x(E_1)
\]

\[
\exists Q ST_y(E_2) \iff [Q^2_0/Q]ST_y(E_2)
\]

We want to show that

\[
\exists Q ST_x(E) \iff [Q_0/Q]ST_x(E)
\]

\((\Rightarrow)\): Assume \(\exists Q ST_x(E)\). Then, \(\exists Q ST_x(E_1)\) and \(\exists Q ST_x(E_2)\) hold. So both \([Q^1_0/Q]ST_x(E_1)\) and \([Q^2_0/Q]ST_x(E_2)\) are the case. Since \(h_{10} \subseteq h_0\), \(Q_0/Q]ST_x(E_1)\) is also true. Similarly, so is \([Q_0/Q]ST_x(E_2)\). Therefore, \([Q_0/Q]ST_x(E)\).

\((\Leftarrow)\): \(Q_0\) is an instance of \(Q\).

This concludes the proof of the Sahlqvist correspondence theorem for LTL'. First-order correspondents for LTL can be found by first translating the LTL Sahlqvist formulas into LTL'.

### 4.4 Example

The above proof of the correspondence theorem also yields an algorithm for translating the frame condition of an LTL Sahlqvist formula into a first-order formula. We do not elaborate the algorithm here. But the algorithm
for the Sahlqvist formula for standard modal logic applies with appropriate modification. Let’s see an example for LTL involving the Until operator. Let \( \varphi = \neg (\neg q U q) \), readers can easily verify that it is an LTL Sahlqvist formula. The standard translation of \( \varphi[w] \) is

\[
\neg \exists Q (\exists v, w \leq u \wedge (\forall u, v \leq u \rightarrow Q(u)) \wedge (\forall z, w \leq z < v \rightarrow \neg Q(z)))
\]

Taking the minimal assignment \( Q(x) \equiv v \leq x \), we reduce the \( ST_w(\varphi) \) to

\[
\neg (\exists v, w \leq v)
\]

Formula (6) identifies the empty class of structures, as there exists no class of frames over which formula (6) can be true at any state.

5 Conclusions

In this paper we introduced a notion of Sahlqvist formula for the Linear-time Temporal Logic LTL and proved a Sahlqvist correspondence theorem for this language. In some respects, they can be viewed as a generalization of the same result for standard modal logic, in the sense that we allow states to index temporal operators \( F_x \) and \( G_{x,x'} \). One should also remark that LTL’ Sahlqvist formulas are in fact very similar to the Sahlqvist formulas of standard modal logic to the extent that the proof for the completeness property \([3,8]\) for Sahlqvist formulas almost identically applies to the LTL’ Sahlqvist formulas.

Further research direction may consists in finding an even larger class of LTL Sahlqvist formulas. For standard modal logic, Chagrova \([4]\) has proved that it is undecidable if an arbitrary formula has a first-order correspondent. Therefore, the same problem is equally undecidable for LTL as the latter is strictly more expressive than the former.
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