DYNAMICS OF A SELF-GRAVITATING MAGNETIZED SOURCE.

A. Ulacia Rey*
Facultad de Tecnología de la Salud, Dr. Salvador Allende, Cerro
Ciudad de la Habana, cp-10400, Cuba
alain@icmf.inf.cu

A. Pérez Martínez
Instituto de Cibernética Matemática y Física (ICIMAF), Calle E No-309 Vedado
Ciudad de la Habana, cp-10400, Cuba
aurora@icmf.inf.cu

Roberto. A. Sussman
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México (UNAM)
Distrito Federal 04510, cp-70543, México
sussman@nuclecu.unam.mx

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We consider a magnetized degenerate gas of fermions as the matter source of a homogeneous but anisotropic Bianchi I spacetime with a Kasner-like metric. We examine the dynamics of this system by means of a qualitative and numerical study of Einstein-Maxwell field equations which reduce to a non-linear autonomous system. For all initial conditions and combinations of free parameters the gas evolves from an initial anisotropic singularity into an asymptotic state that is completely determined by a stable attractor. Depending on the initial conditions the anisotropic singularity is of the “cigar” or “plate” types.

Keywords: Magnetic Field; Kasner Metric; Anisotropy.

1. Introduction

Intense magnetic fields may play an important role in astrophysical objects, such as compact object like neutron or white dwarf stars. Weaker magnetic fields may also be important in connection with the magnetic plasma effects associated with local anisotropy in self-gravitating and collision-less systems (e.g., galactic halos of fermionic dark matter) and net radiation flow entering (or leaving) the gas clouds. Strong magnetic fields can also arise as plausible explanation behind the origin of anisotropy in highly dense systems, like solid core, exotic phase transitions.

*Present address: ICIMAF, Calle E No-309 Vedado, cp-10400, Ciudad Habana. Cuba
pion condensation or anisotropic compact object by ultra-strong self-magnetization \((B > 10^{14})\) Gauss. The latter phenomenon motivates the present work, as a theoretical explanation behind the anisotropy of stresses in these systems. This is connected with previous work \(^{4,5,7,6}\) in which we studied the thermodynamical properties of gases of electrons and vectorial bosons using the Electroweak model. We showed in that work that such a system was characterized by an anisotropic energy–momentum tensor in which the pressure transverse to the magnetic field \((p_{\perp})\) was (in general) different from that parallel to the field \((p_{\parallel})\). Under a Newtonian framework, we found that the vanishing of \(p_{\perp}\) would drive the system to gravitational collapse. We also discussed if critical values of the magnetic field exist, allowing for the vanishing of this pressure and its subsequent collapse. By assuming that the model could describe a Newtonian compact object, we concluded that the final state of such a star would never be a magnetar of the type of Duncan and Tompson\(^{8}\). Instead, such a state could be a strange star or a “black cigar–like” structure.

However, the proper study of highly magnetized sources of possible compact objects requires that we incorporate the non–linear effects of Einstein’s General Relativity theory, allowing us to examine the effects of the gas magnetization in the collapse of a magnetized self-gravitating Fermi gas. Ideally, we should consider minimally realistic rotating configurations, but this would require elaborate hydrodynamical numerical techniques. Instead and as a first step, we consider simple and mathematically more tractable, even if extremely idealized, configurations that could reveal (at least) some of the properties of the dynamical evolution of magnetized gases in General Relativity. We undertake such a study in the present article by considering what could be the simplest non–static geometry compatible with the anisotropies of a magnetized gas: the homogeneous but anisotropic Bianchi I spacetime in a Kasner–like coordinate basis.

2. Magnetized Fermi gas for a Kasner metric.

The Bianchi I spacetime can be described by the Kasner–like metric element \(^{9,10}\)

\[
ds^2 = -c^2 dt^2 + A^2(t) dx^2 + B^2(t) dy^2 + C^2(t) dz^2, \tag{1}
\]

corresponding to an homogeneous, but anisotropic, spacetime with zero spacial curvature and having all relevant quantities depending only on time. We consider as a source of \(^{11}\) a degenerated magnetized Fermi gas whose energy–momentum tensor is that obtained by A. Pérez Martínez and H.Pérez Rojas\(^{3}\).

The energy–momentum tensor for this gas in a comoving frame with coordinates \(x^0 = ct, x, y, z\) and 4–velocity \(u^a = \delta_0^a\) is given by

\[
T^a_b = (U + P) u^a u_b + P \delta^a_b + \Pi^a_b, \quad P = p - \frac{2}{3} B M, \tag{2}
\]

where \(U\) is the total matter–energy density, \(B\) the magnetic field and \(M\) the magnetization of the gas, \(P\) is the isotropic pressure and \(\Pi^a_b\) is the traceless anisotropic pressure tensor, which for \(^{11}\) takes the form \(\Pi^a_b = \text{diag}[\Pi, \Pi, -2\Pi, 0]\) with \(\Pi = -\frac{1}{3} B M.\)
Thus, $p = T^z_z$ is the stress tensor component (pressure) in the direction of the magnetic field.

The full equation of state for this gas takes the following form:

$$p = \lambda \beta \Gamma_p(\beta, \mu), \quad \mathcal{BM} = \lambda \beta \Gamma_M(\beta, \mu), \quad U = \lambda \beta \Gamma_U(\beta, \mu).$$

(3)

where the $\Gamma$ functions are given further ahead, $\beta$, the magnetic field normalized with a suitable critical value $B_c$, and $\lambda$ are given by:

$$\beta = \frac{B}{B_c}, \quad \lambda = \frac{m c}{4\pi^2 \lambda^3}.$$  

(4)

If we consider our gas to be made of by electrons, then $m$ is the electron mass, $m_e$, while $\lambda_c = \hbar/mc$ is the Compton wavelength and $B_c = m^2 c^3/e\hbar$ is the so-called critical magnetic field. In this limit of strong magnetic field, the functions $\Gamma(\beta, \mu)$ take the form:

$$\Gamma_p = \sum_{n=0}^{s} \alpha_n(a_n - b_n - c_n), \quad \Gamma_M = \Gamma_p - \sum_{n=0}^{s} \alpha_n c_n, \quad \Gamma_U = -\Gamma_p + 2 \sum_{n=0}^{s} \alpha_n a_n$$

(5)

where $a_n = \mu \sqrt{\mu^2 - 1 - 2n\beta}$, $b_n = \ln[(\mu + a_n/\mu)/\sqrt{1 + 2n\beta}]$, $c_n = 2n\beta b_n$, $\alpha_n = 2 - \delta_{0n}$, and $\mu = \mu_e/(m_e c^2)$ chemical potential of the fermions, $s = I[\frac{1}{2}(B_c/B)(\mu^2 - 1)]$ corresponds to the maximum Landau level for a given Fermi energy and magnetic field strength, while $I[X]$ denotes the integer part of its argument $X$.

Although (1) is obviously inappropriate to examine the magnetized gas as a source of a compact object, we will consider in our analysis the conditions prevailing inside a very high density compact object in which we can ignore the gas temperature.

### 3. Einstein–Maxwell field equations.

The comoving 4-velocity $u^a = \delta^a_0$ in (1) yields the expansion scalar $\Theta = u^a \partial_a$ and the traceless shear tensor $\sigma_{ab} = u_{(a;b)} - (1/3)\Theta h_{ab}$. In order to obtain a self-consistent system of ordinary differential equations suitable for numerical integration, it is convenient to eliminate second order derivatives of the metric functions by expressing the coupled Einstein-Maxwell field equations $G^a_b = \kappa T^a_b$, $F^{ab} \; ;_b = 0$ and $F_{[abc]} = 0$, as well as the balance equations $T^{ab} \; ;_b = 0$, in terms of $\Theta$ and $\sigma^a_b$. Following this approach and after tedious algebraic manipulations, we can reduce the field and balance equations to the following self-consistent system of 4 independent ordinary

*(e.g. for the electron $B^e_c = 4.41 \times 10^{13}$ Gauss)*
differential equations:
\[
\begin{align*}
\dot{U} &= -(U + p - \frac{2}{3}BM) \Theta - BM \Sigma \\
\dot{\Sigma} &= \frac{2}{3} \kappa BM - \Theta \Sigma \\
\dot{\Theta} &= \kappa (BM + \frac{3}{2}(U - p)) - \Theta^2 \\
\dot{\beta} &= \frac{2}{3} \beta (3\Sigma - 2\Theta)
\end{align*}
\] (6a)
(6b)
(6c)
(6d)

where \(\Sigma \equiv \sigma_z^z\) and a dot denotes derivative with respect to \(ct\). From (5), the state variables \(U, p\) and \(BM\) depend on the normalized magnetic field and chemical potential, \(\beta\) and \(\mu\).

It is convenient to assume the zero order approximation \((n = 0)\), so that we would be considering all electrons in the basic Landau level. Under this approximation the expansions of the \(\Gamma(\beta, \mu) \equiv \Gamma(\mu)\) functions in (5) simplify considerably: \(\Gamma_p = \Gamma_M = a_0 - b_0\) and \(\Gamma_U = a_0 + b_0\), where \(a_0, b_0\) are given by setting \(n = 0\) in (5).

Introducing the following new variables and dimensionless functions:
\[
\begin{align*}
H &= \frac{\Theta}{3}, \\
\frac{d}{d\tau} &= \frac{1}{H} \frac{d}{ct}, \\
S &= \frac{\Sigma}{H}, \\
\Omega &= \frac{\kappa \lambda \beta}{3H^2}, \\
\mathcal{H} &= \frac{H}{H_0},
\end{align*}
\] (7)

where \(H_0\) is a constant with inverse length units. Einstein–Maxwell equations become the following dimensionless system
\[
\begin{align*}
\Omega' &= 2 \left\{ 1 + S - \left[ \mu \sqrt{\mu^2 - 1} + 2 \ln \left( \mu + \sqrt{\mu^2 - 1} \right) \right] \right\} \Omega, \\
S' &= \left[ (2 - S) \mu \sqrt{\mu^2 - 1} - 2(1 + S) \ln \left( \mu + \sqrt{\mu^2 - 1} \right) \right] \Omega, \\
\mu' &= \left[ (2 - S) \ln \left( \mu + \sqrt{\mu^2 - 1} \right) - 3S \mu \sqrt{\mu^2 - 1} \right] \frac{1}{2\mu^2} \sqrt{\mu^2 - 1},
\end{align*}
\] (8a)
(8b)
(8c)

supplemented by
\[
\mathcal{H}' = \left\{ -3 + \left[ \mu \sqrt{\mu^2 - 1} + 2 \ln \left( \mu + \sqrt{\mu^2 - 1} \right) \right] \right\} \mathcal{H}.
\] (9)

where a prime denotes derivative with respect to the dimensionless time \(\tau\) and we have chose a length scale characterized by setting \((3H_0^2/\kappa \lambda) = 1\), so that (from (7)), we obtain
\[
\beta = \Omega \mathcal{H}^2.
\] (10)

Since we have \(\kappa \lambda = 0.749 \times 10^{-24} \text{cm}^{-2}\) for electrons, this choice leads to \(1/H_0 = 2 \times 10^{12} \text{cm}\), which is much smaller than the cosmological Hubble radius.\(^b\)

\(^b\)It is of the order of the distance from the Earth to the Sun.
4. Qualitative and numerical analysis.

The system (8) defines a three dimensional phase space \((\Omega, S, \mu)\) with \(\Omega \geq 0\) and \(\mu \geq 1\). The planes \(\Omega = 0\) and \(\mu = 1\) are invariant subsets corresponding to a Kasner vacuum. Besides these subsets, we also have:

\[
I = \{\mu = 1, S = -1, \Omega\}
\]

\[
II = \left\{\mu, S(\mu) = \frac{2 b_0 \left(-1 + 2 \mu^2 + 2 a_0\right)}{(-3 + 6 \mu^2 + 2 b_0) a_0 + (-1 + 2 \mu^2) b_0 - 6 \mu^2(1 - \mu^2)}, \Omega\right\},
\]

\[
III = \{\mu = 1.42, S = 0.34, \Omega = 0.42\}
\]

where \(a_0, b_0\) are given by setting \(n = 0\) in (5). The curves I and II are saddles, while III is a stable attractor. As shown in figure 1, some trajectories start in a singular state characterized by \(\Omega \to \infty\) while others start in a singular Kasner vacuum \(\Omega = 0\), though all terminate in the attractor III. Depending on initial conditions, some trajectories approach eras of Kasner vacuum I, II. The asymptotic state is given by III, which (from (10)) corresponds to \(\beta \to 0.42 H^2\). Numerical examination of (8) and (9) reveals that \(\beta\) has an asymptotic power law decay in terms of the physical time \(t\). Trajectories that initiate in a singular Kasner vacuum denote an unphysical evolution passing from \(\Omega = 0\) to the value \(\Omega = 0.42\) given by the attractor III. The most physically motivated trajectories correspond to those in which \(\Omega\) decreases monotonically to the asymptotic value in III, hence the normalized magnetic field \(\beta\) also exhibits monotonous decay approximately following a power law scaling law (in terms of the physical time \(t\)).

Since the physically interesting range of magnetic field intensities in a magnetized electron gas is \(B \geq B_c\) (or \(\beta \geq 1\)), it is important to examine the behavior of the trajectories near the initial singular states. The nature of the initial singularity can be appreciated by looking at the behavior of the scale factors \(A, B, C\) in (1) in the limit \(\tau \to -\infty\). The initial singularity (whether a Kasner vaccum or not) is generically anisotropic and can be of the “cigar” or “plate” types. In the former case we have \(C \to \infty\) with diverging \(\Omega\), while both \(A, B\) tend to zero, forming a singular line along the \(z\) direction parallel to the magnetic field.

5. Conclusion

We have examined the evolution of magnetized degenerate gas of fermions, considering the case in which the fermions are electrons in their basic Landau level \((n = 0\) in (5)). Einstein–Maxwell equations for a Bianchi I spacetime yield the dynamical system (5). Initial conditions exist for an evolution starting in a line singularity with \(\Omega\) (and \(\beta\)) decaying monotonously to the asymptotic value in the attractor III. This type of evolution is similar to that emerging in previous Newtonian studies. However, we have also the possibility of a non–vacuum initial singular state that is not unphysical (diverging \(\Omega\)) and is characterized by a “plate” type of singularity,
Fig. 1. Phase space $(\Omega, S, \mu)$. Notice how all trajectories evolve towards the attractor III. Some curves approach states of Kasner vacuum (the saddles $\Omega = 0$, I and II). Some curves have an initial singular vacuum state (III) and some initiate at $\Omega \to \infty$. Singular state are characterized by anisotropic singularities.

which is a counter–intuitive type of anisotropy that lacks a Newtonian equivalent. A more detailed and extensive study of the singular states in this gas, as well as a less restrictive form of the equation of state and the case of a gas of neutrons (instead of electrons), is currently under investigation.

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