Scattering off an SO(10) cosmic string

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Abstract

The scattering of fermions from the abelian string arising during the phase transition \( SO(10) \to SU(5) \times Z_2 \) induced by the Higgs in the 126 representation is studied. Elastic cross-sections and baryon number violating cross-sections due to the coupling to gauge fields in the core of the string are computed by both a first quantised method and a perturbative second quantised method. The elastic cross-sections are found to be Aharonov-Bohm type. However, there is a marked asymmetry between the scattering cross-sections for left and right handed fields. The catalysis cross-sections are small, depending on the grand unified scale. If cosmic strings were observed our results could help tie down the underlying gauge group.

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I. INTRODUCTION

Modern particle physics and the hot big-bang model suggest that the universe underwent a series of phase transitions at early times at which the underlying symmetry changed. At such phase transitions topological defects [1] could be formed. Such topological defects, in particular cosmic strings, would still be around today and provide a window into the physics of the early universe. In particular, cosmic strings arising from a grand unified phase transition are good candidates for the generation of density perturbations in the early universe which lead to the formation of large scale structure [2]. They could also give rise to the observed anisotropy in the microwave background radiation [3].

Cosmic strings also have interesting microphysical properties. Like monopoles [4] they can catalyse baryon violating processes [5,6]. This is because the full grand unified symmetry is restored in the core of the string, and hence grand unified, baryon violating processes are unsuppressed. In [6] it was shown that the cosmic string catalysis cross-section could be a strong interaction cross-section, independent of the grand unified scale, depending on the flux on the string. Unlike the case of monopoles, where there is a Dirac quantisation condition, the string cross-section is highly sensitive to the flux, and is a purely quantum phenomena. Defect catalysis is potentially important. It has already been used to bound the monopole flux [8], and could erase a primordial baryon asymmetry [10]. It is, thus, important to calculate the string catalysis cross-section in a realistic grand unified theory.

In [6] a toy model based on a $U(1)$ theory was used. In a grand unified theory the string flux is given by the gauge group, and cannot be tuned. A cosmic string is essentially a flux tube. Hence the elastic cross-section [11] is just an Aharonov-Bohm cross-section [12], depending on the string flux. This gives the dominant energy loss in a friction dominated universe [13]. Since the string flux is fixed for any given particle species it is important to check that the Aharonov-Bohm cross-section persists in a realistic grand unified theory.

In this paper we calculate the elastic and inelastic cross-sections for cosmic strings arising
from an $SO(10)$ grand unified theory [14]. Cosmic strings arise in the breaking scheme

$$SO(10) \rightarrow SU(5) \times Z_2$$

where the breaking is due to the 126 representation of the Higgs field, the self-dual anti-symmetric 5-index tensor of $SO(10)$. These stable strings survive the subsequent transitions to $SU(3) \times SU(2) \times U(1) \times Z_2$ [15]. They have been studied elsewhere [17].

Now the $SO(10)$ symmetry is restored inside the string core, and therefore there are baryon number violation processes mediated by the gauge fields $X, Y, X', Y'$ and $X_s$ of $SO(10)$. We therefore expect a non-zero inelastic cross-section which we will determine. This cross-section should be running from a small cross-section $O(\eta^{-1})$, where $\eta$ is the grand unified scale $\sim 10^{15}$ GeV to a much larger cross-section of the order of the strong interaction.

The plan of this paper is as follow: In section II we define an $SO(10)$ string model. We give 'top-hat' forms for the Higgs and gauge fields forming the string, since the 'top-hat' core model doesn’t affect the cross-sections of interest structure of the string core, we introduce the baryon number violating gauge fields of $SO(10)$ present in the core of the string.

In section III A we review the method used to calculate the scattering cross-sections. There are two different approaches. A fundamental quantum mechanical one and a perturbative second quantised method calculating the geometrical cross-section, i.e. the scattering cross-section for free fermionic fields. The catalysis cross-section is then enhanced by an amplification factor to the power of four.

In section III B we derive the equations of motion. In order to simplify the calculations and to get a fuller result, we also consider a 'top-hat' core model for the gauge fields mediating quark to lepton transitions.

In section III C and section III D we calculate the solutions to the equations of motion outside and inside the string core respectively, and in section III E we match our solutions at the string radius. In section III F we calculate the scattering amplitude for incoming plane waves of linear combinations of the quark and electron fields.

We use these results in section IV and section V in order to calculate the scattering cross-sections of incoming beams of pure single fermion fields. In section V we calculate
the elastic cross-sections. And in section VII we calculate the baryon number violation cross-sections.

In section VI we derive the catalysis cross-section using the second quantised method of ref. [5,6]. The second-quantised cross-sections are found to agree with the first quantised cross-section of section V.

There are 4 appendices. Appendix A gives a brief review on SO(10) theory, and gives an explicit notation used everywhere in this paper. Appendices B and C contain the technical details of the external and internal solutions calculations. Finally, Appendix D is a discussion of the matching conditions at the core radius.

II. AN SO(10) STRING

In the appendix A, we give a brief review of SO(10) theory. With that notation, the lagrangian is,

\[ L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi_{126})^\dagger (D^\mu \Phi_{126}) - V(\Phi) + L_F \]  

where \( F_{\mu\nu} = -i F_{\mu\nu}^a \tau_a \), \( \tau_a = 1,\ldots,45 \) are the 45 generators of SO(10). \( \Phi_{126} \) is the Higgs 126, the self-dual anti-symmetric 5-index tensor of SO(10). \( L_F \) is the fermionic part of the lagrangian. In the covariant derivative \( D_\mu = \partial_\mu + i e A_\mu \), \( A_\mu = A_\mu^a \tau_a \) where \( A_\mu^a \) \( a = 1,\ldots,45 \) are 45 gauge fields of SO(10).

We assume that the universe undergoes the following breaking scheme,

\[ SO(10) \rightarrow SU(5) \times Z_2 \rightarrow SU(3) \times SU(2) \times U(1) \times Z_2 \rightarrow SU(3) \times U(1)Q \times Z_2 \]

giving vacuum expectation values to the components of the 10 which correspond to the usual Higgs doublet. The decomposition of the 126 representation under \( SU(5) \times U(1) \) is given by,

\[ 126 = 1_{10} + \ldots \]
The first transition is achieved by giving vacuum expectation value to the component of the 126 in the $1_{10}$ direction. The first homotopy group $\pi_1(SO(10)/SU(5) \times Z_2)$ is $Z_2$, and therefore $Z_2$ strings are formed. In terms of $SU(5)$, the 45 generators of $SO(10)$ can be decomposed as follows,

$$45 = 24 + 1 + 10 + \bar{10}$$  \hspace{1cm} (3)

From the 45 generators of $SO(10)$, 24 belong to $SU(5)$, 1 generator corresponds to the $U(1)'$ symmetry in $SO(10)$ not embedded in $SU(5)$ and there are 20 remaining ones. Therefore the breaking of $SO(10)$ to $SU(5) \times Z_2$ induces the creation of two types of strings. An Abelian one, corresponding to the $U(1)'$ symmetry, and an non abelian one made with linear combinations of the 20 remaining generators. In this paper we are interested in the abelian strings since the non abelian version are Alice strings, and would result in global quantum number being ill-defined, and hence unobservable [7]. We note that there is a wide range of parameters where the non abelian strings have lower energy [17]. However, since the abelian string is topologically stable, there is a final probability that it could be formed by the Kibble mechanism [8].

If we call $\tau_{str}$ the generator of the abelian string, $\tau_{str}$ will be given by the diagonal generators of $SO(10)$ not lying in $SU(5)$ that is,

$$\tau_{str} = \frac{1}{5} (M_{12} + M_{34} + M_{56} + M_{78} + M_{910})$$  \hspace{1cm} (4)

where $M_{ij} : i, j = 1...10$ are the 45 $SO(10)$ generators defined in appendix A in terms of the generalised gamma matrices. Numerically, this gives,

$$\tau_{str} = diag(\frac{1}{2}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, -3, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, -3, \frac{1}{10}, -3, -3, -3).$$  \hspace{1cm} (5)

The results of Perkins et Al. [6] find that the greatest enhancement of the cross-section is for fermionic charges close to integer values. Thus, from equation (5), we expect no great enhancement; the most being due to the right-handed neutrino.

We are going to model our string as is usually done for an abelian $U(1)$ string. That is, we take the string along the $z$ axis, resulting in the Higgs $\Phi_{126}$ and the gauge fields $A_\mu$ of
the string to be independent of the z coordinate, depending only on the polar coordinates \((r, \theta)\). Here \(A_\mu\) is the gauge field of the string, obtained from the product \(A_\mu = A_{\mu, str} \tau_{str}\).

The solution for the abelian string can be written as,

\[
\Phi_{126} = f(r) e^{ir_{\tau_{str}} \theta} \Phi_0 = f(r) e^{i\theta} \Phi_0 \\
A_\theta = -\frac{g(r)}{er} \tau_{str} \\
A_r = A_z = 0
\] (6)

where \(\Phi_0\) is the vacuum expectation value of the Higgs 126 in the 1_{10} direction. The functions \(f(r)\) and \(g(r)\) describing the behaviour the Higgs and gauge fields forming the string are given by

\[
f(r) = \begin{cases} 
\eta & r \geq R \\
\eta \left(\frac{r}{R}\right) & r < R 
\end{cases}, \quad g(r) = \begin{cases} 
1 & r \geq R \\
\left(\frac{r}{R}\right)^2 & r < R 
\end{cases}
\] (8)

where \(R\) is the radius of the string. \(R \sim \eta^{-1}\), where \(\eta\) is the grand unified scale, assumed to be \(\eta \sim 10^{15} GeV\). In order to simplify the calculations and to get a fuller result we use the top-hat core model, since it has been shown not to affect the cross-sections of interest.

The top-hat core model assumes that the Higgs and gauge fields forming the string are zero inside the string core. Hence, \(f(r)\) and \(g(r)\) are now given by,

\[
f(r) = \begin{cases} 
\eta & r \geq R \\
0 & r < R 
\end{cases}, \quad g(r) = \begin{cases} 
1 & r \geq R \\
0 & r < R 
\end{cases}
\] (9)

The full \(SO(10)\) symmetry is restored in the core of the string. \(SO(10)\) contains 30 gauge bosons leading to baryon decay. These are the bosons \(X\) and \(Y\), and their conjugates, of SU(5) plus 18 other gauge bosons usual called \(X', Y'\) and \(X_s\), and their conjugates. Therefore inside the string core, there are quark to lepton transitions mediated by the gauge bosons \(X, X', Y, Y'\) and \(X_s\) and we expect the string to catalyse baryon number violating processes in the early universe.

The \(X, X', Y, Y'\) and \(X_s\) gauge bosons are associated with non diagonal generators of \(SO(10)\). For the electron family, the relevant part of the lagrangian is given by,
\[ L_x = \bar{\Psi}_{16} (ie^\mu (X_\mu \tau^X + X'_\mu \tau^{X'} + Y_\mu \tau^Y + Y'_\mu \tau^{Y'} + X_{s\mu} \tau^{X_s})) \Psi_{16} \]  

(10)

where \( \tau^X, \tau^{X'}, \tau^Y, \) and \( \tau^{Y'} \) are the non-diagonal generators of SO(10) associated with the \( X, X', Y, Y' \) and \( X_s \) gauge bosons respectively.

Expanding equation (10) gives [20],

\begin{align*}
L_x &= g \sqrt{2} X^\alpha \left[-\epsilon_{\alpha\beta\gamma} \bar{u}_L \gamma^\mu u^\beta_L + \bar{d}_L \gamma^\mu \nu_e^e + \bar{d}_R \gamma^\mu \nu_{eR}^{eR} \right] \\
&+ g \sqrt{2} Y^\alpha \left[-\epsilon_{\alpha\beta\gamma} \bar{u}_L \gamma^\mu d^\beta_L + \bar{d}_R \gamma^\mu \nu_{eR}^{eR} - \bar{u}_L \gamma^\mu \nu_{eL}^{eL} \right] \\
&+ g \sqrt{2} X'^\alpha \left[-\epsilon_{\alpha\beta\gamma} \bar{d}_L \gamma^\mu d^\beta_L - \bar{u}_R \gamma^\mu \nu_e^e - \bar{u}_L \gamma^\mu \nu_{eL}^{eL} \right] \\
&+ g \sqrt{2} Y'^\alpha \left[\epsilon_{\alpha\beta\gamma} \bar{d}_L \gamma^\mu u^\beta_L - \bar{u}_R \gamma^\mu \nu_{eR}^{eR} - \bar{d}_L \gamma^\mu \nu_{eL}^{eL} \right] \\
&+ g \sqrt{2} X_{s\mu} \left[\bar{d}_L \gamma^\mu \nu_{eL}^{eL} + \bar{d}_L \gamma^\mu \nu_{eR}^{eR} + \bar{u}_L \gamma^\mu \nu_{eL}^{eL} + \bar{u}_R \gamma^\mu \nu_{eR}^{eR} \right] \\
\end{align*}  

(11)

where \( \alpha, \beta \) and \( \gamma \) are colour indices. The \( X_s \) does not contribute to nucleon decay except by mixing with the \( X' \) because there is no vertex \( qqX_s \). We consider baryon violating processes mediated by the gauge fields \( X, X', Y \) and \( Y' \) of SO(10). In previous papers [3,11], baryon number violating processes resulting from the coupling to scalar condensates in the string core have been considered. In our SO(10) model we do not have such a coupling.

III. SCATTERING OF FERMIONS FROM THE ABELIAN STRING

A. The scattering cross-section

Here, we will briefly review the two methods used to calculate the scattering cross-section. The first is a quantum mechanical treatment. From the fermionic lagrangian \( L_F \), we derive the equations of motion inside and outside the string core. We then find solutions to the equations of motion inside and outside the string core and we match our solutions at the string core. Considering incoming plane waves of pure quarks, we then calculate the scattering amplitude. The matching conditions together with the scattering amplitude enable us to calculate the elastic and inelastic scattering cross-sections. The second method is a quantised one, where one calculates the geometrical cross-section \( \left( \frac{d\sigma}{d\Omega}\right)_{\text{geom}} \), i.e. using
free fermions spinors $\psi_{\text{free}}$. The catalysis cross-section is enhanced by a factor $A^4$ over the geometrical cross-section,

$$\sigma_{\text{inel}} = A^4 \left( \frac{d\sigma}{d\Omega} \right)_{\text{geom}}$$

(12)

where the amplification factor $A$ is defined by,

$$A = \frac{\psi(R)}{\psi_{\text{free}}(R)}.$$  

(13)

where R is the radius of the string, $R \sim \eta^{-1}$. This method has been applied in ref. [6] and [19].

B. The equations of motion

The fermionic part of the lagrangian $L_F$ is given in terms of 16 dimensional spinors as defined in Appendix A. We shall consider only one family in this work, and in particular the electron family. The fermionic lagrangian for only one family,

$$L_F = L_F^{(e)} = \bar{\Psi}_{16} \gamma^\mu D_\mu \Psi_{16} + L_M + L_x$$

(14)

where $L_M$ is the mass term and $L_x$ is the lagrangian describing quark to lepton transitions through the $X, X', Y, Y'$ and $X_s$ gauge bosons in SO(10) and given by (equation (11)). The covariant derivative is given by $D_\mu = \partial_\mu - ieA_{\mu,\text{str}} \tau_{\text{str}}$ where $A_{\mu,\text{str}}$ is the gauge field forming the string and $\tau_{\text{str}}$ is the string generator given by equation (5). Therefore, since $\tau_{\text{str}}$ is diagonal, there will be no mixing of fermions around the string. The lagrangian $L_F$ will split in a sum of eight terms, one for each fermion of the family. In terms of 4-spinors, this is

$$L_F = \sum_{i=1}^{8} L_f^i + L_x$$

(15)

where $L_f^i = i\bar{\psi}^i_L \gamma^\mu D_\mu \psi^i_L + i\bar{\psi}^{c,i}_L \gamma^\mu D_\mu \psi^{c,i}_L + L_m$, and i runs over all fermions of the given family. One can show that $i\bar{\psi}^{c,i}_L \gamma^\mu D_\mu \psi^{c,i}_L = i\bar{\psi}^i_R \gamma^\mu D_\mu \psi^i_R$ and $\tau_{\text{str}}^{L_f^i} = \tau_{\text{str}}^{R_f^i}$. Finally, $L_x$ is given by equation (11). It is easy to generalise to more families.
From equations (13) and (14) we derive the equations of motions for the fermionic fields. We take the fermions to be massless inside and outside the string core. This a relevant assumption since our methods apply for energies above the confinement scale. We consider the case of free quarks scattering from the string and coupling with electrons inside the string core. Outside the string core, the fermions feel the presence of the string only by the presence of the gauge field. We are interested in the elastic cross sections for all fermions and in the cross-section for these quark decaying into electron. The fermionic lagrangian given by equations (13) and (14) becomes,

\[
L_F(e, q) = i\bar{\psi}_L \gamma^\mu D_{\mu}^{(L)} e_L + i\bar{\psi}_R \gamma^\mu D_{\mu}^{(R)} e_R
\]

\[
+ i\bar{q}_L \gamma^\mu D_{\mu}^{(L)} q_L + i\bar{q}_R \gamma^\mu D_{\mu}^{(R)} q_R
\]

\[
- \frac{gG_{\mu}}{2\sqrt{2}} \bar{\psi}_L \gamma^\mu e_L - \frac{gG_{\mu}'}{2\sqrt{2}} \bar{q}_R \gamma^\mu e_R + H.C.
\]

(16)

giving the following equations of motion,

\[
i\gamma^\mu D_{\mu}^{(L)} e_L + \frac{gG_{\mu}'}{2\sqrt{2}} \gamma^\mu q_L = 0
\]

\[
i\gamma^\mu D_{\mu}^{(R)} e_R + \frac{gG_{\mu}'}{2\sqrt{2}} \gamma^\mu q_R = 0
\]

\[
i\gamma^\mu D_{\mu}^{(L)} q_L + \frac{gG_{\mu}'}{2\sqrt{2}} \gamma^\mu e_L = 0
\]

\[
i\gamma^\mu D_{\mu}^{(R)} q_R + \frac{gG_{\mu}'}{2\sqrt{2}} \gamma^\mu e_R = 0
\]

(17)

which are valid everywhere. The covariant derivatives \(D_{\mu}^{(L,R)} = \partial_{\mu} + ieA_{\mu,str}^{(L,R)}\) and \(D_{\mu}^{q,(L,R)} = \partial_{\mu} + ieA_{\mu,str}^{q,(L,R)}\). We have \(\tau_{str}^{R,u} = \tau_{str}^{L,u} = \tau_{str}^{L,e} = \tau_{str}^{L,d} = \frac{1}{10}\) and \(\tau_{str}^{R,e} = \tau_{str}^{R,d} = \frac{-3}{10}\) together with \(\tau_{str}^{L,e} = \tau_{str}^{R,i}\) and \(\tau_{str}^{L,e} = \tau_{str}^{R,i}\). \(G_{\mu}\) and \(G_{\mu}'\) stand for \(X_{\mu}\), \(X'_{\mu}\), \(Y_{\mu}\) or \(Y'_{\mu}\), depending on the chosen quark.

Since these equations involve quarks and lepton mixing, we do not find independent solution for the quark and lepton fields. However, we can solve these equations taking linear combinations of the the quark and lepton fields, \(q_L^c \pm e_L\) and \(q_R^c \pm e_R\). In this case, the effective gauge fields are

\[
e(A_{\mu,str} \tau_{str}^{f_L} \pm G_{\mu})
\]

(18)
and
\[ e (A_{\mu, str} \tau_{str}^{f_R} \pm G') \] (19)
respectively.

In order to make the calculations easier, we use a top-hat theta component for \( G \) and \( G' \) within the string core, since Perkins et al. \[6\] have shown that the physical results are insensitive to the core model used for the gauge fields mediating baryon violating processes.

C. The External Solution

Outside the string core, the gauge field of the string \( A_{\mu, str} \) has only, from equations 7 and 9, a non vanishing component \( A_{\theta} = \frac{1}{er} \tau_{str} \), and the effective gauge fields \( G \) and \( G' \) are set to zero. Therefore the equations of motion (17) for \( r > R \) become,
\[
\begin{align*}
    i \gamma^\mu D_{\mu}^{e,L} e_L &= 0 \\
    i \gamma^\mu D_{\mu}^{e,R} e_R &= 0 \\
    i \gamma^\mu D_{\mu}^{u,L} q^e_L &= 0 \\
    i \gamma^\mu D_{\mu}^{u,R} q^e_R &= 0
\end{align*}
\] (20)

where the covariant derivatives
\[
D_{\mu}^{e,(L,R)} = \partial_{\mu} + i e A_{\mu, str} \tau_{str}^{e,(L,R)}
\]
and
\[
D_{\mu}^{q^e,(L,R)} = \partial_{\mu} + i e A_{\mu, str} \tau_{str}^{q^e,(L,R)}.
\]

We take the usual Dirac representation \( e_L = (0, \xi_e) \), \( e_R = (\chi_e, 0) \), \( q^e_L = (0, \xi_{q^e}) \) and \( q^e_R = (\chi_{q^e}, 0) \) and the mode decomposition for the spinors \( \xi_{q^e} \), \( \xi_e \), \( \chi_{q^e} \) and \( \chi_e \),
\[
\begin{align*}
    \chi_{(e,q^e)}(r, \theta) &= \sum_{n=-\infty}^{n=+\infty} \left( \begin{array}{c}
    \chi^1_{(e,q^e)}(r) \\
    i \chi^2_{(e,q^e)}(r) e^{i\theta}
    \end{array} \right) e^{in\theta} \\
    \xi_{(e,q^e)}(r, \theta) &= \sum_{n=-\infty}^{n=+\infty} \left( \begin{array}{c}
    \xi^1_{(e,q^e)}(r) \\
    i \xi^2_{(e,q^e)}(r) e^{i\theta}
    \end{array} \right) e^{in\theta}.
\end{align*}
\] (21)

From appendix \[3\] we see that the fields \( \xi^1_{1,(e,q^e)} \), \( \xi^2_{2,(e,q^e)} \), \( \chi^1_{1,(e,q^e)} \) and \( \chi^2_{2,(e,q^e)} \) satisfy Bessel equations of order \( n - \tau_{str}^{R(e,q^e)} \), \( n + 1 - \tau_{str}^{R(e,q^e)} \), \( n - \tau_{str}^{L(e,q^e)} \) and \( n - \tau_{str}^{R(e,q^e)} \) respectively. The external solution becomes,
Therefore, outside the string core, we have got independent solutions for the quark and electron fields.

D. The Internal Solution

Inside the string core, the gauge field of the string, $A_\mu$, is set to zero whereas $G_\theta$ and $G'_\theta$ take the value $2\sqrt{2}A$ and $2\sqrt{2}A'$ respectively. Therefore, the equations of motion (17) become,

\begin{align*}
\imath\gamma^\mu \partial_\mu e_L + \frac{gG'_\mu}{2\sqrt{2}} \gamma^\mu q_L &= 0 \\
\imath\gamma^\mu \partial_\mu e_R + \frac{gG_\mu}{2\sqrt{2}} \gamma^\mu q_R &= 0 \\
\imath\gamma^\mu \partial_\mu q_L + \frac{gG'_\mu}{2\sqrt{2}} \gamma^\mu e_L &= 0 \\
\imath\gamma^\mu \partial_\mu q_R + \frac{gG_\mu}{2\sqrt{2}} \gamma^\mu e_R &= 0.
\end{align*}

(23)

Since these equations of motions involve quark-leptons mixings, there are no independent solutions for the quarks and electron fields. However, we get solutions for the fields $\rho^\pm$ and $\sigma^\pm$ which are linear combinations of the quarks and electron fields,

\begin{align*}
\rho^\pm &= \chi_{q^c} \pm \chi_e \\
\sigma^\pm &= \xi_{q^c} \pm \xi_e.
\end{align*}

(24) (25)

Using the mode decomposition (21) for the fields $\rho^\pm$ and $\sigma^\pm$, the internal solution becomes,
where $\rho_{n1}^\pm$ and $\rho_{n2}^\pm$ and $\sigma_{n1}^\pm$ and $\sigma_{n2}^\pm$ are the upper and lower components of the fields $\rho^\pm$ and $\sigma^\pm$ respectively. They are given in terms of hyper-geometric functions. From appendix C we get,

$$\rho_{n1}^\pm = (kr)^{|n|} e^{-ikr} \sum_{j=0}^{\infty} \alpha_j^\pm \frac{(2ikr)^j}{j!}$$

where $k^2 = w^2 - (eA)^2$, $e = \frac{g}{\sqrt{2}}$. $\alpha_{j+1}^\pm = (a_j^\pm + j) a_j^\pm$ with $a^\pm = \frac{1}{2} + |n| \pm \frac{eA(2n+1)}{2ik}$ and $b = 1 + 2|n|$. $\rho_{n2}^\pm$ can be obtained using the coupled equation (C1.2) of appendix C. We find,

$$\rho_{n2}^\pm = -\frac{1}{w} (kr)^{|n|} e^{-ikr} \sum_{j=0}^{\infty} \alpha_j^\pm \frac{(2ikr)^j}{j!} \left( \frac{|n| - n}{r} - ik + \frac{j}{r} \pm eA \right).$$

We get similar hyper-geometric functions for the fields $\sigma_{n1}^\pm$ and $\sigma_{n2}^\pm$.

**E. Matching at the String Core**

From now on, we will do calculations for the right-handed fields, the calculations for the left-handed ones being straight-forward. Once we have our internal and external solutions, we match them at the string core. We must take the same linear combinations of the quark and lepton fields outside and inside the core, and must have continuity of the solutions at $r = R$. The continuity of the solutions at $r = R$ implies,

$$(\chi_{1,q}^n \pm \chi_{1,e}^n)^{out} = \rho_{n1}^\pm_{in}$$

$$(\chi_{2,q}^n \pm \chi_{2,e}^n)^{out} = \rho_{n2}^\pm_{in}.$$
\[
\left( \frac{d}{dr} \pm eA \right) \rho_{n2}^{\pm in} = \left( \frac{d}{dr} - \frac{\tau^{R(e,q)}}{R} \right) \left( \chi_{2,q}^n \pm \chi_{2,e}^n \right)^{out} \tag{31}
\]
\[
\left( \frac{d}{dr} \pm eA \right) \rho_{n1}^{\pm in} = \left( \frac{d}{dr} + \frac{\tau^{R(e,q)}}{R} \right) \left( \chi_{1,q}^n \pm \chi_{1,e}^n \right)^{out}. \tag{32}
\]

These equations lead to a relation between the coefficients of the Bessel functions for the external solution, as derived in Appendix D,

\[
\frac{v_q^n \pm v_e^n}{v_n^q \pm v_n^e} = \frac{w l_n^\pm \left( J_{n+1-R}(wR) + J_{n-R}(wR) \right)}{w l_n^\pm \left( J_{-(n+1-R)}(wR) + J_{-(n-R)}(wR) \right)} \tag{33}
\]

where

\[
l_n^\pm = \frac{\sum_{j=0}^{n=+\infty} \alpha_j^{\pm (2ikr)^j}}{\sum_{j=0}^{n=+\infty} \alpha_j^{\pm (2ikr)^j} \left( \frac{|n-n-1|}{r} - ik + \frac{i}{r} \pm eA \right)} \tag{34}
\]

The relations (33) and (34) are the matching conditions at \( r = R. \)

**F. The Scattering Amplitude**

In order to calculate the scattering amplitude, we match our solutions to an incoming plane wave plus an outgoing scattered wave at infinity. However, since the internal solution, and therefore the matching conditions at \( r = R, \) are given in terms of linear combinations of quarks and leptons, we consider incoming waves of such linear combinations. Let \( f_n^\pm \) denote the scattering amplitude for the mode \( n, \) \( f_n^+ \) if we consider the scattering of (quarks + electrons) and \( f_n^- \) if we consider the scattering of (quarks - electrons). Then the matching conditions at infinity are,

\[
(-i)^n \left( J_n \frac{1}{iJ_{n+1}e^{i\theta}} \right) + f_n^\pm e^{ikr} \sqrt{r} \begin{pmatrix} 1 \\ i e^{i\theta} \end{pmatrix} = \begin{pmatrix} (v_q^n \pm v_e^n) J_{n-R} & (v_q^n \pm v_e^n) J_{-(n-R)} \\ i (v_q^n \pm v_e^n) J_{n+1-R} & (v_q^n \pm v_e^n) J_{-(n+1-R)} e^{i\theta} \end{pmatrix}. \tag{35}
\]

Using then the large \( r \) forms for the Bessel functions,

\[
J_\mu(\omega r) = \sqrt{\frac{2}{\pi \omega r}} \cos(\omega r - \frac{\mu \pi}{2} - \frac{\pi}{4}), \tag{36}
\]

13
and matching the coefficients of $e^{i\omega r}$ we find,

$$\sqrt{2\pi}\omega f_n^\pm e^{i\frac{\tau R}{2}} =
\begin{cases}
e^{-i\pi}(e^{i\tau R\pi} - 1) + (v_n^q \pm v_n^{e'})e^{i(n-\tau R)\frac{\pi}{2}}(1 - e^{-2i(n-\tau R)\pi}) \\
e^{i\pi}(e^{i(n-\tau R)\pi} - e^{-i\pi}) + (v_n^q \pm v_n^{e'})e^{-i(n-\tau R)\frac{\pi}{2}}(1 - e^{2i(n-\tau R)\pi})
\end{cases}.$$  \hspace{1cm} (37)

Matching the coefficients $e^{-i\omega r}$, we get relations between the Bessel functions coefficients,

$$(v_n^q \pm v_n^{e'}) = (1 - (v_n^q \pm v_n^{e'})e^{-i(n-\tau R)\frac{\pi}{2}}) e^{-i(n-\tau R)\frac{\pi}{2}}.$$  \hspace{1cm} (38)

The relations (37), (38), (33) and (34) determine the scattered wave.

**IV. THE ELASTIC CROSS-SECTION**

When there is no baryon number violating processes inside the string core, when the gauge fields mediating quark to lepton transitions are set to zero, we have elastic scattering. In this case, the scattering amplitude reduces to,

$$f_n^{\text{elast}} = \frac{1}{\sqrt{2\pi}\omega} e^{-i\frac{\tau R}{2}} e^{-i\pi n} \left\{ \begin{array}{ll} e^{-i\pi}(e^{i\tau R\pi} - 1) & n \geq 0 \\
e^{i\pi}(e^{-i\tau R\pi} - 1) & n \leq -1 \end{array} \right.$$  \hspace{1cm} (39)

The elastic cross-section per unit length is given by

$$\sigma_{\text{elast}} = \frac{1}{2\pi\omega} \left| \sum_{n=-\infty}^{+\infty} f_n^{\text{elast}} e^{i\theta} \right|^2.$$  \hspace{1cm} (40)

Using the relations $\sum_{n=a}^{+\infty} e^{inx} = \frac{e^{i\alpha}}{1-e^{i\alpha}}$ and $\sum_{n=-\infty}^{b} e^{inx} = \frac{e^{i\beta}}{1-e^{-i\alpha}}$, we find the elastic cross-section to be

$$\sigma_{\text{elast}} = \frac{1}{2\pi\omega} \frac{\sin^2(\tau R\pi)}{\cos^2(\frac{\theta}{2})}.$$  \hspace{1cm} (41)

This is an Aharonov-Bohm cross-section, and $\tau_R$ is the flux in the core of the string.

Now, remember that $\tau_{\text{str}}^{Lc,u} = \tau_{\text{str}}^{L,u} = \tau_{\text{str}}^{Lc,e} = \tau_{\text{str}}^{L,e} = \frac{1}{10}$ and $\tau_{\text{str}}^{Lc,d} = \tau_{\text{str}}^{Lc,i} = \tau_{\text{str}}^{R,i}$ and $\tau_{\text{str}}^{L,i} = \tau_{\text{str}}^{Rc,i}$. Hence,
\[ \sigma_{elast}^{eL} = \sigma_{elast}^{dR} > \sigma_{elast}^{eR} = \sigma_{elast}^{uR} = \sigma_{elast}^{dL} = \sigma_{elast}^{uL}. \tag{42} \]

We therefore have a marked asymmetry between fermions. We have got a marked asymmetry between left and right handed electrons, left and right handed down quarks or, since \( \sigma_{elast}^{iLc} = \sigma_{elast}^{iRc} \) and \( \sigma_{elast}^{iLc} = \sigma_{elast}^{iRc} \), between left handed particle and antiparticle, respectively right handed, for the electron and the down quark. But we have equal cross sections for right handed particles and left handed antiparticles for the electrons and the down quark, and equal cross sections for both left handed and right handed up quarks an anti-quarks. This is a marked feature of grand unified theories. If cosmic strings are found it may be possible to use this asymmetry to identify the underlying gauge symmetry.

V. THE INELASTIC CROSS-SECTION

The gauge fields \( X, X', Y \) and \( Y' \) are now 'switched on'. In this case we are calculating the baryon number violating cross-section. If we consider identical beams of incoming pure \( \rho^+ \) and \( \rho^- \), recalling that \( \rho^\pm = \chi_qc^\pm \chi_e \), this will ensure that we will have an incoming beam of pure quark. Therefore, the scattering amplitude for the quark field is given by half the difference of \( f_n^+ \) and \( f_n^- \), and the scattering amplitude for the electron field is given by half the sum of \( f_n^+ \) and \( f_n^- \). From equation (37) we get,

\[ \frac{1}{2} \sqrt{2\pi\omega} (f_n^+ - f_n^-) e^{i\frac{\pi}{4}} = v_n^e e^{-i(n-\tau_R)\frac{\pi}{2}} (1 - e^{-\tau_R^2\pi}). \tag{43} \]

The inelastic cross-section for the quark field is given by,

\[ \sigma_{inel} = | \sum_{n=-\infty}^{+\infty} (f_n^+ - f_n^-) e^{in\theta} |^2. \tag{44} \]

Hence, from equation (43),

\[ \sigma_{inel} \sim \frac{1}{\omega} \left| \sum_{n=-\infty}^{+\infty} v_n^e e^{-i(n-\tau_R)\frac{\pi}{2}} \right|^2. \tag{45} \]

Using equations (33), (34) and (38), we find,

\[ v_n^e = \frac{e^{i(n-\tau_R)\frac{\pi}{2}}}{2} \left( \frac{1}{\delta_n^+ + e^{i(n-\tau_R)\pi}} - \frac{1}{\delta_n^- + e^{i(n-\tau_R)\pi}} \right) \tag{46} \]
where

$$\delta_n^\pm = \frac{w l_n^\pm J_{n+1-\tau_R}(wR) + J_{n-\tau_R}(wR)}{w l_n^\pm J_{-(n+1-\tau_R)}(wR) + J_{-(n-\tau_R)}(wR)}$$  \hspace{1cm} (47)$$

and $\lambda^\pm$ are given by equations (34). Equations (45), (46) and (47) determine the inelastic cross-section. This is given in terms of a power series. However, using small argument expansions for Bessel functions, we conclude that this power series involves always one dominant term, the other terms being suppressed by a factor $(\omega R)^n$ where $n$ is an integer such that $n \geq 1$. Therefore the inelastic cross-section involves one dominant mode, the other modes being exponentially suppressed. If $d$ denotes the dominant mode we get $\sigma_{inel} \sim \frac{1}{\omega} |v_d^e|^2$. The value of the dominant mode depends on the sign of the the fractional flux $\tau_{str}$. Our results can be summarised as follow.

For $0 < \tau_R < 1$, the mode $n = 0$ is enhanced, and the other modes are exponentially suppressed. Hence,

$$\sigma_{inel} \sim \frac{1}{\omega} |v_0^e|^2.$$  \hspace{1cm} (48)$$

Using small argument expansions for Bessel functions, this yields

$$\sigma_{inel} \sim \frac{1}{\omega} (eAR)^2 (\omega R)^{4(1-\tau_R)}$$  \hspace{1cm} (49)$$

where $A$ is the value of the gauge field inside the string core, $e$ is the gauge coupling constant, and $R \sim \eta$, $\eta$ being the the grand unified scale $\sim 10^{15} GeV$. The greater amplification occurs for $eAR \sim 1$, giving $\sigma_{inel} \sim \frac{1}{\omega} (\omega R)^{4(1-\tau_R)}$.

For $-1 < \tau_R < 0$, the mode $n = -1$ is enhanced, and the other modes are exponentially suppressed. Hence,

$$\sigma_{inel} \sim \frac{1}{\omega} |v_{-1}^e|^2.$$  \hspace{1cm} (50)$$

Using small argument expansions for Bessel functions, this yields

$$\sigma_{inel} \sim \frac{1}{\omega} (eAR)^2 (\omega R)^{4(1+\tau_R)}.$$  \hspace{1cm} (51)$$
The greater amplification occurs for $eAR \sim 1$, giving $\sigma_{inel} \sim \frac{1}{\omega} (\omega R)^{4(1+\tau_R)}$. Thus, the baryon number violating cross-section is not a strong interaction cross-section but is suppressed by a factor depending on the grand unified scale $\eta \sim R^{-1} \sim 10^{15} GeV$. The baryon number violation cross-sections are very small. For $u_L$ and $d_L$ we obtain,

$$\sigma_{inel} \sim \frac{1}{\omega} (\omega R)^{3.6} .$$  \hspace{1cm} (52)

Whereas for $d_R$ we get,

$$\sigma_{inel} \sim \frac{1}{\omega} (\omega R)^{2.8} .$$  \hspace{1cm} (53)

Here again we have a marked asymmetry between left and right handed fields. We find an indeterminate solution for the left-conjugate up quark because its phase around the string ($\frac{1}{10}$) differs from the phase of the left-handed electron ($\frac{-3}{10}$) by a fractional value different from a half.

VI. THE SECOND QUANTISED CROSS-SECTION

We now derive the baryon number violating cross-sections using the perturbative method introduced in section III A.

Firstly, we calculate the geometrical cross-section. This is the cross-section for free fields $\psi_{free}$, where $\psi_{free}$ is a 2-spinor. In the case of gauge fields mediating catalysis it is given by,

$$\left( \frac{d\sigma}{d\Omega} \right)_{geom} = \frac{1}{\omega} (\omega R)^4 (eAR)^2$$  \hspace{1cm} (54)

where $\omega$ is the energy of the massless field $\psi_{free}$, $A$ is the value of the gauge field mediating quark to lepton transitions, $e$ is the gauge coupling constant and $R$ is the radius of the string with $R \sim \eta^{-1}$ with $\eta \sim 10^{15}$ GeV.

The second step is to calculate the amplification factor $A = \frac{\psi}{\psi_{free}}$, $\psi$ and $\psi_{free}$ being two 2-spinors. The catalysis cross-section is enhanced by a factor $A^4$ over the geometrical cross-section,
We now use the results of sections III C, III D and III E where we have solved the equations of motion for the fields $\psi$ and calculated the matching conditions. Using equation (52), we get the wave function $\psi$ at the string core, and for the mode $n$,

$$\psi^n = \begin{pmatrix}
(v_n^q \pm v_n^e) J_{n-\tau_{str}}(\omega R) + (v_n^q' \pm v_n^e') J_{-(n-\tau_{str})}(\omega R) e^{i\theta} \\
i ((v_n^q \pm v_n^e) J_{n+1-\tau_{str}}(\omega R) + (v_n^q' \pm v_n^e') J_{-(n+1-\tau_{str})}(\omega R)) e^{i(n+1)\theta}
\end{pmatrix}$$

Using equations (33) and (34) and using small argument expansions for Bessel functions, we conclude that for $n \geq 0$, $(v_n^q \pm v_n^e) \gg (v_n^q' \pm v_n^e')$, and for $n < 0$, $(v_n^q \pm v_n^e) \ll (v_n^q' \pm v_n^e')$. Now, from equation (38), we see that is one coefficient dominates that will be the $O(1)$. Hence, for $n \geq 0$, $(v_n^q \pm v_n^e) \sim 1$, and for $n < 0$, $(v_n^q' \pm v_n^e') \sim 1$. Therefore, using small argument expansions for Bessel functions we get for $n \geq 0$,

$$\psi^n \sim \begin{pmatrix}
(\omega R)^{n-\tau_{str}} \\
(\omega R)^{n+1-\tau_{str}}
\end{pmatrix}$$

which is to be compared with $\psi_{2}^{\text{free}} \sim 1$ for free spinors. The upper component of the spinor is amplified while the other one is suppressed by a factor $\sim (\omega R)$. For $n < 0$ we have,

$$\psi^n \sim \begin{pmatrix}
(\omega R)^{-(n-\tau_{str,R})} \\
(\omega R)^{-(n+1-\tau_{str,R})}
\end{pmatrix}$$

Hence we conclude that for $n < 0$ the lower component is amplified while the upper one is suppressed by a factor $\sim \omega R$.

Therefore, for $\tau_{str} = -\frac{3}{10}$, the amplification occurs for the lower component and for the mode $n = -1$. The amplification factor is

$$\mathcal{A} \sim (\omega R)^{\tau_{str}}$$

leading to the baryon number violating cross-section,

$$\sigma_{\text{inel}} \sim \frac{1}{\omega} (eAR)^2 (\omega R)^4(1+\tau_{str})$$

(60)
In the case $\tau_{str} = \frac{1}{10}$, the amplification occurs for the upper component and for the mode $n = 0$. The amplification factor is,

$$A \sim (\omega R)^{-\tau_{str}}$$

leading to the baryon number violating cross-section,

$$\sigma_{inel} \sim \frac{1}{\omega} (eAR)^2 (\omega R)^4 (1-\tau_{str}) .$$

This method shows explicitly which component of the spinor and which mode are enhanced. The results agree with scattering cross-sections derived using the first quantised method.

**VII. CONCLUSION**

We have investigated elastic and inelastic scattering off abelian cosmic strings arising during the phase transition $SO(10) \stackrel{<\phi_{126}}{\rightarrow} SU(5) \times Z_2$ induced by the Higgs in the 126 representation in the early universe. The cross-sections were calculated using both first quantised and second quantised methods. The results of the two methods are in good agreement.

During the phase transition $SO(10) \rightarrow SU(5) \times Z_2$, only the right-handed neutrino gets a mass. This together with the fact that we are interested in energies above the confinement scales allows us to consider massless particles.

The elastic cross-sections are found to be Aharonov-Bohm type cross-sections. This is as expected, since we are dealing with fractional fluxes. We found a marked asymmetry between left-handed and right-handed fields for the electron and the down quark fields. But there is no asymmetry for the up quark field. This is a general feature of grand unified theories. If cosmic strings were observed it might be possible to use Aharonov-Bohm scattering to determine the underlying gauge group.

The inelastic cross-sections result from quark to leptons transitions via gauge interactions in the core of the string. The catalysis cross-sections are found to be quite small, and
here again we have a marked asymmetry between left and right handed fields. They are 
suppressed from a factor $\sim \eta^{-3.6}$ for the left-handed up and down quark fields to a factor 
$\sim \eta^{-2.8}$ for the right-handed down quark field.

Previous calculations have used a toy model to calculate the catalysis cross-section. Here 
the string flux could be ‘tuned’ to give a strong interaction cross-section. In our case the flux 
is given by the gauge group, and is fixed for each particle species. Hence, we find a strong 
sensitivity to the grand unified scale. Our small cross-sections make it less likely that grand 
unified cosmic strings could erase a primordial baryon asymmetry, though they could help 
generate it [16]. If cosmic strings are observed our scattering results, with the distinctive 
features for the different particle species, could help tie down the underlying gauge group.

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APPENDIX A: BRIEF REVIEW OF SO(10)

The fundamental representation of SO(10) consists of 10 generalised gamma matrices. 
They can be written in an explicit notation, in terms of cross products,

\[ \Gamma_1 = \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 \]
\[ \Gamma_2 = \sigma_2 \times \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 \]
\[ \Gamma_3 = I \times \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 \]
\[ \Gamma_4 = I \times \sigma_2 \times \sigma_3 \times \sigma_3 \times \sigma_3 \]
\[ \Gamma_5 = I \times I \times \sigma_1 \times \sigma_3 \times \sigma_3 \]
\[ \Gamma_6 = I \times I \times \sigma_2 \times \sigma_3 \times \sigma_3 \]
\[ \Gamma_7 = I \times I \times I \times \sigma_1 \times \sigma_3 \]
$\Gamma_8 = I \times I \times I \times \sigma_2 \times \sigma_3$
$\Gamma_9 = I \times I \times I \times I \times \sigma_1$
$\Gamma_{10} = I \times I \times I \times I \times \sigma_2$  \hspace{1cm} (A1)

where the $\sigma_i$ are the Pauli matrices and I denotes the two dimensional identity matrix. They generate a Clifford algebra defined by the anticommutation rules

$$\{\Gamma_i, \Gamma_j\} = 2 \delta_{ij} \quad i = 1, ..., 10.$$

One can define the chirality operator $\chi$, which is the generalised $\gamma_5$ of the standard model by

$$\chi = (-i)^5 \prod_{i=1}^{10} \Gamma_i.$$  \hspace{1cm} (A3)

In terms of the cross-product notation, $\chi$ has the form,

$$\chi = \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3.$$  \hspace{1cm} (A4)

The 45 generators of SO(10) are also given in terms of the generalised gamma matrices

$$M_{ab} = \frac{1}{2i} [\Gamma_i, \Gamma_j] \quad i, j = 1...10.$$  \hspace{1cm} (A5)

They are antisymmetric, purely imaginary $32 \times 32$ matrices. One can write the diagonal $M$,

$$M_{12} = \frac{1}{2} \sigma_3 \times I \times I \times I \times I$$
$$M_{34} = \frac{1}{2} I \times \sigma_3 \times I \times I \times I$$
$$M_{56} = \frac{1}{2} I \times I \times \sigma_3 \times I \times I$$
$$M_{78} = \frac{1}{2} I \times I \times I \times \sigma_3 \times I$$
$$M_{910} = \frac{1}{2} I \times I \times I \times I \times \sigma_3.$$  \hspace{1cm} (A6)

In SO(N) gauge theories fermions are conventionally assigned to the spinor representation. For N even, the spinor representation is $2^\frac{N}{2}$ dimensional and decomposes into two equivalent
spinors of dimension $2^{N_2-1}$ by means of the projection operator $P = \frac{1}{2} (1 \pm \chi)$, where 1 is the $2^{N_2} \times 2^{N_2}$ identity matrix. Thus $SO(10)$ has got two irreducible representations,

$$\sigma^\pm = \frac{1 \pm \chi}{2}$$  \hspace{1cm} (A7)

of dimension 16. Therefore $SO(10)$ enables us to put all the fermions of a given family in the same spinor. Indeed, since each family contains eight fermions, we can put all left and right handed particles of a given family in the same 16 dimensional spinor. This is the smallest grand unified group which can do so. However, gauge interactions conserve chirality. Indeed,

$$\bar{\psi} \gamma^\mu A^\mu \psi = \bar{\psi}_L \gamma^\mu A^\mu \psi_L + \bar{\psi}_R \gamma^\mu A^\mu \psi_R.$$  \hspace{1cm} (A8)

Therefore $\psi_L$ and $\psi_R$ cannot be put in the same irreducible representation. Hence, instead of choosing $\psi_L$ and $\psi_R$, we chose $\psi^c_L$ and $\psi^c_L$. The fields $\psi_L$ and $\psi^c_L$ annihilate left-handed particles and antiparticles, respectively, or create right-handed antiparticles and particles. The fields $\psi_L$ and $\psi^c_L$ are related to the fields $\psi_R$ and $\bar{\psi}_R$ by the following relations,

$$\psi^c_L \equiv P_L \psi^c = P_L C \bar{\psi}^T = C (\bar{\psi} P_L)^T = C \bar{\psi}_R^T = C \gamma_0^T \psi^*_R \hspace{1cm} (A9)$$

$$\bar{\psi}^T_L \equiv \psi^T_L \gamma_0 = \psi^*_L \gamma_0^T C^T \gamma_0 = -\psi_R^T C^{-1} = \psi_R^T C \hspace{1cm} (A10)$$

where the projection operators $P_{L,R} = \frac{1}{2} (1 \pm \gamma_5)$ and $C$ is the usual charge conjugation matrix. For the electron family we get,

$$\Psi^{(e)}_L = (\nu^c_e, u^c_r, u^c_y, u^c_b, d^c_r, d^c_y, d^c_b, e^-, u_b, u_y, u_r, \nu_{(e)}, e^+, d^c_r, d^c_y, d^c_b)_L \hspace{1cm} (A11)$$

where the upper index $c$ means conjugate, and the sub-indices refer to quark colour. We find similar spinor $\Psi^{(\mu)}$ and $\Psi^{(\tau)}$ associated with the $\mu$ and the $\tau$ family respectively:

$$\Psi^{(\mu)} = (\nu^c_{(\mu)}, c^c_r, c^c_y, c^c_b, s_b, s_y, s_r, \mu^-, c_b, c_y, c_r, \nu_{(\mu)}, \mu^+, s^c_r, s^c_y, s^c_b)_L \hspace{1cm} (A12)$$

$$\Psi^{(\tau)} = (\nu^c_{(\tau)}, t^c_r, t^c_y, t^c_b, b_b, b_y, b_r, \tau^-, t_b, t_y, t_r, \nu_{(\tau)}, \tau^+, b^c_r, b^c_y, b^c_b)_L \hspace{1cm} (A13)$$
APPENDIX B: THE EXTERNAL SOLUTION

We want to solve equations (20). We set \( \partial_t = -i\omega \), where \( \omega \) is the energy of the electron and take the usual Dirac representation \( e_L = (0, \xi_e) \), \( e_R = (\chi_e, 0) \), \( q_L^e = (\xi_q, 0) \) and \( q_R^e = (\chi_q, 0) \). We use the usual mode decomposition for the spinors \( \xi_q, \xi_e, \chi_q \) and \( \chi_e \):

\[
\chi_{(e,q^e)}(r, \theta) = \sum_{n=-\infty}^{n=+\infty} \begin{pmatrix}
\chi_1^{n(e,q^e)}(r) \\
i \chi_2^{n(e,q^e)}(r) e^{i\theta}
\end{pmatrix} e^{in\theta}
\]

\[
\xi_{(e,q^e)}(r, \theta) = \sum_{n=-\infty}^{n=+\infty} \begin{pmatrix}
\xi_1^{n(e,q^e)}(r) \\
i \xi_2^{n(e,q^e)}(r) e^{i\theta}
\end{pmatrix} e^{in\theta}.
\]

Then, using the basis,

\[
\gamma^j = \begin{pmatrix}
0 & -i\sigma^j \\
i\sigma^j & 0
\end{pmatrix}
\]

the equations of motion (20) become,

\[
\begin{align*}
\omega \chi_1^{n(e,q^e)} & - (\frac{d}{dr} + \frac{n+1}{r} - \frac{\tau_{str}^{R(e,q^e)}}{r}) \chi_2^{n(e,q^e)} = 0 \\
\omega \chi_2^{n(e,q^e)} & + (\frac{d}{dr} - \frac{n}{r} + \frac{\tau_{str}^{R(e,q^e)}}{r}) \chi_1^{n(e,q^e)} = 0 \\
\omega \xi_1^{n(e,q^e)} & + (\frac{d}{dr} + \frac{n+1}{r} - \frac{\tau_{str}^{L(e,q^e)}}{r}) \xi_2^{n(e,q^e)} = 0 \\
\omega \xi_2^{n(e,q^e)} & - (\frac{d}{dr} - \frac{n}{r} + \frac{\tau_{str}^{L(e,q^e)}}{r}) \xi_1^{n(e,q^e)} = 0
\end{align*}
\]

It is easy to show that the fields \( \xi_1^{n(e,q^e)}, \xi_2^{n(e,q^e)}, \chi_1^{n(e,q^e)} \) and \( \chi_2^{n(e,q^e)} \) satisfy Bessel equations of order \( n - \tau_{str}^{R(e,q^e)} \), \( n + 1 - \tau_{str}^{R(e,q^e)} \), \( n - \tau_{str}^{L(e,q^e)} \) and \( n - \tau_{str}^{R(e,q^e)} \) respectively. Hence the external solution is,

\[
\begin{pmatrix}
\xi_{(e,q^e)}(r, \theta) \\
\chi_{(e,q^e)}(r, \theta)
\end{pmatrix} = \sum_{n=-\infty}^{n=+\infty} \begin{pmatrix}
(v^{n(e,q^e)}_1 Z^{1}_{n-\tau_{str}^{R(e,q^e)}}(\omega r) + v^{n(e,q^e)'}_1 Z^{2}_{n-\tau_{str}^{R(e,q^e)}}(\omega r)) e^{in\theta} \\
i (v^{n(e,q^e)}_2 Z^{1}_{n+1-\tau_{str}^{R(e,q^e)}}(\omega r) + v^{n(e,q^e)'}_2 Z^{2}_{n+1-\tau_{str}^{R(e,q^e)}}(\omega r)) e^{i(n+1)\theta}
\end{pmatrix}
\]

\[
\begin{pmatrix}
(w^{n(e,q^e)}_1 Z^{1}_{n-\tau_{str}^{L(e,q^e)}}(\omega r) + w^{n(e,q^e)'}_1 Z^{2}_{n-\tau_{str}^{L(e,q^e)}}(\omega r)) e^{in\theta} \\
i (w^{n(e,q^e)}_2 Z^{1}_{n+1-\tau_{str}^{L(e,q^e)}}(\omega r) + w^{n(e,q^e)'}_2 Z^{2}_{n+1-\tau_{str}^{L(e,q^e)}}(\omega r)) e^{i(n+1)\theta}
\end{pmatrix}
\]

The order of the Bessel functions will always be fractional. We therefore take \( Z^1_\nu = J_\nu \) and \( Z^2_\nu = J_{-\nu} \).
We get solutions for fields which are linear combinations of the quark and electron fields. Indeed, we get solutions for the fields \( \sigma^{\pm} = \xi_{q} \pm \xi_{e} \) and \( \rho^{\pm} = \chi_{q} \pm \chi_{e} \). Using the mode decomposition (21), the upper components of the fields \( \rho^{\pm} \) and \( \sigma^{\pm} \) are respectively \( \rho^{\pm}_{n1} = \chi^{n}_{1q} \pm \chi^{n}_{1e} \) and \( \rho^{\pm}_{n2} = \chi^{n}_{2q} \pm \chi^{n}_{2e} \) whilst the lower components are \( \sigma^{\pm}_{n1} = \xi^{n}_{1q} \pm \xi^{n}_{1e} \) and \( \sigma^{\pm}_{n2} = \xi^{n}_{2q} \pm \xi^{n}_{2e} \) respectively. The equations of motions (23) become

\[
\omega \rho^{\pm}_{n1} - \left( \frac{d}{dr} + \frac{n+1}{r} \mp eA' \right) \rho^{\pm}_{n2} = 0 \]
\[
\omega \rho^{\pm}_{n2} + \left( \frac{d}{dr} - \frac{n}{r} \pm eA' \right) \rho^{\pm}_{n1} = 0 \]  
(C1)

Combining the two first equations of (C1), one can see that \( \rho^{\pm}_{n1} \) satisfy an hyper-geometric equation giving,

\[
\rho^{\pm}_{n1} = (kr)^{|n|} e^{-ikr} \sum_{j=0}^{n=+\infty} \alpha_{j}^{\pm} \frac{(2ikr)^{j}}{j!} \]  
(C2)

where \( k^{2} = w^{2} - (eA)^{2} \), \( e = \frac{a}{2\sqrt{2}} \). \( \alpha_{j+1}^{\pm} = \frac{(a^{\mp} + j)}{(b+p)} \alpha_{j}^{\pm} \) with \( a^{\mp} = \frac{1}{2} + |n| \pm \frac{eA(2n+1)}{2ik} \) and \( b = 1 + 2|n| \). \( \rho^{\pm}_{n2} \) can be obtained using the coupled equation (C1.2). We find

\[
\rho^{\pm}_{n2} = -\frac{1}{w} (kr)^{|n|} e^{-ikr} \sum_{j=0}^{n=+\infty} \alpha_{j}^{\pm} \frac{(2ikr)^{j}}{j!} \left( \frac{|n| - n}{r} - ik + \frac{j}{r} \pm eA \right) . \]  
(C3)

\( \sigma^{\pm}_{n2} \) are also solutions of hyper-geometric equations, and using the coupled equation (C1.4) we get,

\[
\sigma^{\pm}_{n1} = (kr)^{|n|} e^{-ikr} \sum_{j=0}^{n=+\infty} \beta_{j}^{\pm} \frac{(2ikr)^{j}}{j!} \]  
(C4)

\[
\sigma^{\pm}_{n2} = -\frac{1}{w} (kr)^{|n|} e^{-ikr} \sum_{j=0}^{n=+\infty} \beta_{j}^{\pm} \frac{(2ikr)^{j}}{j!} \left( \frac{|n| - n}{r} - ik + \frac{j}{r} \pm eA' \right) \]  
(C5)

where \( k^{2} = w^{2} - (eA')^{2} \), \( \beta_{j+1}^{\pm} = \frac{(c^{\mp} + j)}{(b+p)} \beta_{j}^{\pm} \) with \( c^{\mp} = \frac{1}{2} + |n| \pm \frac{eA'(2n+1)}{2ik} \). And the internal solution is,
Therefore the internal solution is given by a linear combination of the quark and electron fields.

APPENDIX D: THE MATCHING CONDITIONS

The continuity of the solutions at \( r = R \) lead to,

\[
(kR)^{n_1} e^{-ikR} \sum_{j=0}^{\infty} \alpha_j^\pm \frac{(2ikR)^j}{j!} = (v_n^q \pm v_n^e) J_{n-(n-1)R}(\omega R) + (v_n'^q \pm v_n'^e) J_{-(n-1)R}(\omega R) \tag{D1}
\]

\[
-\frac{1}{w} (kR)^{n_1} e^{-ikR} \sum_{j=0}^{\infty} \alpha_j^\pm \frac{(2ikR)^j}{j!} \left( \frac{|n| - n}{R} - ik + \frac{j}{R} \pm eA \right) = (v_n^q \pm v_n^e) J_{n+1-(n-1)R}(\omega R) + (v_n'^q \pm v_n'^e) J_{-(n+1-(n-1)R)R}(\omega R). \tag{D2}
\]

Nevertheless, we will have discontinuity of the first derivatives. Indeed, inside we have

\[
\omega \rho_{n1}^\pm - \left( \frac{d}{dr} + \frac{n+1}{r} \mp eA' \right) \rho_{n2}^\pm = 0 \tag{D3}
\]

whereas outside we have

\[
\omega (\chi_{1,q^c}^n \pm \chi_{1,e}^n) - \left( \frac{d}{dr} + \frac{n+1}{r} - \frac{eA}{r} \right) R_{(e,q^c)} \left( \chi_{2,q^c}^n \pm \chi_{2,e}^n \right) = 0 \tag{D4}
\]

Now,

\[
(\chi_{1,q^c}^n \pm \chi_{1,e}^n)^{\text{out}} = \rho_{n1}^{\pm \text{in}} \tag{D5}
\]

\[
(\chi_{2,q^c}^n \pm \chi_{2,e}^n)^{\text{out}} = \rho_{n2}^{\pm \text{in}} \tag{D6}
\]

giving us the relations for the first derivatives,
\[ \left( \frac{d}{dr} \mp eA \right) \rho_{n2}^\pm = \left( \frac{d}{dr} - \frac{\tau_{R(e,q^c)}}{R} \right) \left( \chi_{2,q^c}^n \pm \chi_{2,e}^n \right)^{\text{out}} \]  
(D7)

\[ \left( \frac{d}{dr} \pm eA \right) \rho_{n1}^\pm = \left( \frac{d}{dr} + \frac{\tau_{R(e,q^c)}}{R} \right) \left( \chi_{1,q^c}^n \pm \chi_{1,e}^n \right)^{\text{out}} . \]  
(D8)

Dividing equation (D1) by equation (D2) or either replacing equation (D1) in equation (D7), we get the following relations

\[ \frac{v_{n}^d \pm v_{p}^{e'}}{v_{n}^q \pm v_{p}^{e}} = \frac{w l_{n}^\pm J_{n+1-\tau_R}(wR) + J_{n-\tau_R}(wR)}{w l_{n}^\pm J_{-(n+1-\tau_R)}(wR) + J_{-(n-\tau_R)}(wR)} \]  
(D9)

where

\[ l_{n}^\pm = \frac{\sum_{j=0}^{n=+\infty} \alpha_{j}^\pm (2ikr)^j j!}{\sum_{j=0}^{n=+\infty} \alpha_{j}^\pm (2ikr)^j j! \left( \frac{|n|-n}{r} - ik + \frac{i}{r} \pm eA \right)} . \]  
(D10)
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