FUNCTORS OF MODULES ASSOCIATED WITH FLAT AND PROJECTIVE MODULES II

ADRIÁN GORDILLO-MERINO, JOSÉ NAVARRO, PEDRO SANCHO

Abstract. Let $R$ be an associative ring with unit. Given an $R$-module $M$, we can associate the following covariant functor from the category of $R$-algebras to the category of abelian groups: $S \mapsto M \otimes_R S$. With the corresponding notion of dual functor, we prove that the natural morphism of functors $M \to M^{\vee \vee}$ is an isomorphism. We prove several characterizations of the functors associated with flat modules, flat Mittag-Leffler modules and projective modules.

1. Introduction

Let $R$ be an associative ring with unit. Consider the functor from the category of $R$-algebras $R$-$\text{Alg}$ to the category of right $R$-modules $R$-$\text{Mod}$, $o: R$-$\text{Alg} \to R$-$\text{Mod}$, $o(S) := S$ for any $R$-algebra $S$, and the functor $r: R$-$\text{Mod} \to R$-$\text{Alg}$, $r(N) = R\langle N \rangle$, where $R\langle N \rangle$ is the $R$-algebra generated by $N$ (see [3.1]). It is well known the functorial isomorphism

$$\text{Hom}_R(N, o(S)) = \text{Hom}_{R-alg}(r(N), S)$$

for any right $R$-module $N$ and any $R$-algebra $S$. Hence, it is easy to obtain a functorial isomorphism

$$\text{Hom}_{grp}(G \circ o, F) = \text{Hom}_{grp}(G, F \circ r)$$

for any covariant functors of abelian groups $G: R$-$\text{Mod} \to Z$-$\text{Mod}$, and $F: R$-$\text{Alg} \to Z$-$\text{Mod}$.

Let $\mathcal{R}$ be the covariant functor from the category of $R$-algebras, to the category of $R$-algebras, defined by $\mathcal{R}(S) := S$, for any $R$-algebra $S$.

Definition 1.1. A functor of $\mathcal{R}$-modules is a covariant functor $M: R$-$\text{Alg} \to Z$-$\text{Mod}$ together with a morphism of functors of sets $\mathcal{R} \times M \to M$ that endows $M(S)$ with an $S$-module structure, for any $R$-algebra $S$.

A morphism of $\mathcal{R}$-modules $f: M \to M'$ is a morphism of functors such that the morphisms $f_S: M(S) \to M'(S)$ are morphisms of $S$-modules.

If $G: R$-$\text{Mod} \to Z$-$\text{Mod}$ is additive, then $G^o := G \circ o$ is naturally a functor of $\mathcal{R}$-modules. Let $in, h_x: R\langle N \rangle \to R\langle N \oplus x \cdot R \rangle$ be the morphisms of $R$-algebras induced by the morphisms of $R$-modules $N \to R\langle N \oplus x \cdot R \rangle$, $n \mapsto n, x \cdot n$. Given a

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functor of \( R \)-modules \( F \), let \( F^r: R\text{-Mod} \to Z\text{-Mod} \) be defined as follows: \( F^r(N) \) is the kernel of the morphism

\[
F(R(N)) \xrightarrow{F(h_x) - x \cdot F(in)} F(R(N \oplus x \cdot R))
\]

for any right \( R \)-module \( N \) and any \( n' \in F(R(N)) \). We prove \( 4.8 \) the following theorem.

**Theorem 1.2.** Let \( F: R\text{-Alg} \to Z\text{-Mod} \) be a covariant functor of \( R \)-modules and \( G: R\text{-Mod} \to Z\text{-Mod} \) an additive covariant functor of abelian groups. Then, we have a functorial isomorphism

\[
\text{Hom}_{grp}(G, F^r) \cong \text{Hom}_R(G^o, F)
\]

In Algebraic Geometry, functors of \( R \)-modules from the category of \( R \)-algebras to the category of abelian groups are featured more frequently than those from the category of right \( R \)-modules to the category of abelian groups. Nevertheless, the results about reflexivity of modules and characterizations of flat Mittag-Leffler modules are obtained more naturally with the latter functors (see \( 5 \)). The results below are a consequence of Theorem 1.2 and the results obtained in \( 5 \).

Any \( R \)-module \( M \) can be thought as a functor of \( R \)-modules: Consider the following covariant functor of \( R \)-modules \( M \), defined by

\[
M(S) := S \otimes_R M,
\]

for any \( R \)-algebra \( S \). We will say that \( M \) is the quasi-coherent \( R \)-module associated with \( M \). It is easy to prove that the category of \( R \)-modules is equivalent to the category of quasi-coherent \( R \)-modules.

Given a functor of \( R \)-modules \( M \), we will say that the functor of right \( R \)-modules \( M^\vee \) defined by

\[
M^\vee(S) = \text{Hom}_R(M, S),
\]

for any \( R \)-algebra \( S \), is the dual functor of \( M \). We will say that \( M^\vee \) is the \( R \)-module scheme associated with the \( R \)-module \( M \). We are now in a position to state the main results of this paper, grouped together herein according to the concepts involved:

**Theorem 1.3.** Let \( M \) be an \( R \)-module. The natural morphism of \( R \)-modules

\[
M \to M^{\vee \vee}
\]

is an isomorphism.

When \( R \) is a commutative ring, this theorem has been proved for finitely generated modules using the language of sheaves in the big Zariski topology, in \( 6 \), and it is implicit in \( 4 \). The reflexivity of these quasi-coherent \( R \)-modules \( M \) has been used for a variety of applications in theory of linear representations of affine group schemes \( 1, 2, 3 \). Likewise, we think that this new reflexivity theorem will be useful in the theory of comodules over non-commutative rings.

**Theorem 1.4.** Let \( M \) be an \( R \)-module such that \( M^r(N) = N \otimes_R M \), for any right \( R \)-module \( N \). Then,
(1) $M$ is a finitely generated projective module if and only if $M$ is a module scheme.

(2) $M$ is a flat module if and only if $M$ is a direct limit of module schemes.

(3) $M$ is a flat Mittag-Leffler module if and only if $M$ is a direct limit of submodule schemes.

(4) $M$ is a flat strict Mittag-Leffler module if and only if $M$ is a direct limit of submodule schemes, $N_i \subseteq M$, and the dual morphism $M^\vee \rightarrow N_i$ is an epimorphism, for any $i$.

(5) $M$ is a countably generated projective module if and only if there exists a chain of module subschemes of $M$,

$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n \subseteq \cdots,$$

such that $M = \cup_{n \in \mathbb{N}} N_n$.

(6) $M$ is projective if and only if there exists a chain of $R$-submodules of $M$,

$$W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n \subseteq \cdots,$$

such that $M = \cup_{n \in \mathbb{N}} W_n$, where $W_n$ is a direct sum of module schemes and the natural morphism $M^\vee \rightarrow W_n^\vee$ is an epimorphism, for any $n \in \mathbb{N}$.

Also, the theorem below establishes several statements which are equivalent to an $R$-module being a flat strict Mittag-Leffler module.

**Theorem 1.5.** Let $M$ be an $R$-module such that $M^\vee(N) = N \otimes_R M$, for any right $R$-module $N$. Then, the following, equivalent conditions are met:

(1) $M$ is a flat strict Mittag-Leffler $R$-module.

(2) Let $\{M_i\}_{i \in I}$ be the set of all finitely generated submodules of $M$, and $M^*_i := \text{Im}[M^* \rightarrow M^*_i]$. The natural morphism $M \rightarrow \lim_{\rightarrow} M_i^\vee$ is an isomorphism.

(3) There exists a monomorphism $M \hookrightarrow \prod_{J} R$.

(4) Every morphism of $R$-modules $f : M^\vee \rightarrow R$ factors through the quasi-coherent module associated with $\text{Im} f_R$.

2. **Preliminaries**

Let $R$ be an associative ring with unit, and let $\mathcal{R}$ be the covariant functor from the category of $R$-algebras to the category of $R$-algebras, defined by $\mathcal{R}(S) := S$, for any $R$-algebra $S$.

**Definition 2.1.** A functor of $\mathcal{R}$-modules is a covariant functor $\mathcal{M} : R\text{-Alg} \rightarrow \mathbb{Z}\text{-Mod}$ together with a morphism of functors of sets $\mathcal{R} \times \mathcal{M} \rightarrow \mathcal{M}$ that endows $\mathcal{M}(S)$ with an $S$-module structure, for any $R$-algebra $S$.

A morphism of $\mathcal{R}$-modules $f : \mathcal{M} \rightarrow \mathcal{M}'$ is a morphism of functors such that the morphisms $f_S : \mathcal{M}(S) \rightarrow \mathcal{M}'(S)$ are morphisms of $S$-modules.

**Definition 2.2.** If $\mathcal{M}$ is an $\mathcal{R}$-module, the dual $\mathcal{M}^\vee$ is the following functor $R\text{-Alg} \rightarrow \mathbb{Z}\text{-Mod}$

$$\mathcal{M}^\vee(S) := \text{Hom}_{\mathcal{R}}(\mathcal{M}, S),$$

which is a functor of right $\mathcal{R}$-modules.
If $S$ is an $R$-algebra, the restriction of an $R$-module $M$ to the category of $S$-algebras will be written

$$M_{|S}(S') := M(S'),$$

for any $S$-algebra $S'$.

**Definition 2.3.** The functor of homomorphisms $\mathbb{Hom}_R(M, M')$ is the covariant functor $R$-$\text{Alg} \to \mathbb{Z}$-$\text{Mod}$ defined by

$$\mathbb{Hom}_R(M, M')(S) := \text{Hom}_S(M_{|S}, M'_{|S}),$$

where $\text{Hom}_S(M_{|S}, M'_{|S})$ stands for the set of all morphisms of $S$-modules from $M_{|S}$ to $M'_{|S}$.

In the following, it will also be convenient to consider another notion of dual module: $M^*$ is the functor of right $R$-modules defined by

$$M^* := \mathbb{Hom}_R(M, R).$$

**Definition 2.4.** The quasi-coherent $R$-module associated with an $R$-module $M$ is the following covariant functor $\mathcal{M} : R$-$\text{Alg} \to \mathbb{Z}$-$\text{Mod}$,

$$\mathcal{M}(S) := S \otimes_R M.$$

Quasi-coherent modules are determined by its global sections. In particular, we will make use of the following statement, whose proof is immediate:

**Proposition 2.5.** Restriction to global sections $f \mapsto f_R$ defines a bijection:

$$\text{Hom}_R(\mathcal{M}, \mathcal{M}) = \text{Hom}_R(M, M(R)),$$

for any quasi-coherent $R$-module $\mathcal{M}$ and any $R$-module $M$.

As a consequence, both notions of dual module introduced above coincide on quasi-coherent modules; that is, $\mathcal{M}^* = \mathcal{M}^\vee$.

In fact, if $S$ is an $R$-algebra, then

$$\mathcal{M}^\vee(S) = \text{Hom}_R(\mathcal{M}, S) = \text{Hom}_R(M, S)$$

and, as $\mathcal{M}_{|S}$ is the quasi-coherent $S$-module associated with $S \otimes_R M$,

$$\mathcal{M}^*(S) = \text{Hom}_S(\mathcal{M}_{|S}, S) = \text{Hom}_S(S \otimes_R M, S) = \text{Hom}_R(M, S) = \mathcal{M}^\vee(S).$$

**Definition 2.6.** We will say that a functor from the category of right $R$-modules to the category of abelian groups is a functor of abelian groups.

**Definition 2.7.** Given a functor of abelian groups $\mathbb{G}$, let $\mathbb{G}^\circ$ be the functor from the category of $R$-algebras to the category of abelian groups defined by

$$\mathbb{G}^\circ(S) := \mathbb{G}(S),$$

for any $R$-algebra $S$. Given a morphism of $R$-algebras $w : S \to S'$ then $\mathbb{G}^\circ(w) := \mathbb{G}(w)$.

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1In this paper, we will only consider well-defined functors $\text{Hom}_R(M, M')$, that is to say, functors such that $\text{Hom}_S(M_{|S}, M'_{|S})$ is a set, for any $R$-algebra $S$. 
Remark 2.8. Observe that we can define $s \ast g := \mathbb{G}(s \cdot g)$, for any $g \in \mathbb{G}^o(S)$ and $s \in S$ (where $s \cdot S \to S$ is defined by $s \cdot (s') := s \cdot s'$). If $\mathbb{G}$ is additive, then $\mathbb{G}^o$ is a functor of $R$-modules.

Any morphism $\phi: \mathbb{G} \to \mathbb{G}'$ of functors of abelian groups defines the morphism $\phi^o: \mathbb{G}^o \to \mathbb{G}'^o$, $\phi^o_S := \phi_S$ for any $R$-algebra $S$. Obviously,

$$\phi^o_S(s \ast g) = \phi_S(\mathbb{G}(s \cdot g)) = \mathbb{G}'(s \cdot (\phi_S(g))) = s \ast \phi^o_S(g).$$

Finally, any definition or statement in the category of $R$-modules has a corresponding definition or statement in the category of right $R$-modules, that we will use without more explicit mention.

As examples, if $M$ is an $R$-module, then $M^* := \text{Hom}_R(M, R)$ is a right $R$-module. If $N$ is a right $R$-module, then the dual module defined by

$$N^* := \text{Hom}_R(N, R)$$

is an $R$-module, etc.

3. Extension of a functor on the category of algebras to a functor on the category of modules

Notation 3.1. If $M$ is an $R$-module, observe that $M \otimes_Z R$ is an $R$-bimodule and we can consider the tensorial $R$-algebra

$$R(M) := T_R(M \otimes_Z R) = (T_Z^R M) \otimes_Z R.$$ 

Remark 3.2. If $N$ is a right $R$-module, then:

$$R(N) := T_R^R(R \otimes_Z N).$$

Lemma 3.3. The following functorial map is bijective:

$$\text{Hom}_{R-\text{alg}}(R(M), S) \to \text{Hom}_R(M, S), \quad f \mapsto f',$$

where $f'(m) := f(m \otimes 1)$ for any $m \in M$.

Proof. $\text{Hom}_{R-\text{alg}}(T_R(M \otimes_Z R), S) = \text{Hom}_{R \otimes Z R}(M \otimes_Z R, S) = \text{Hom}_R(M, S).$ \hfill $\square$

Any $R$-linear morphism $\phi: M \to M'$ uniquely extends to a morphism of $R$-algebras $\phi: R(M) \to R(M')$, $m \otimes 1 \mapsto \phi(m) \otimes 1$.

If we use the notation

$$M \cdot^n R := M \otimes_Z \cdots \otimes_Z M \otimes_Z R, \quad m_1 \cdots m_n \cdot r \mapsto m_1 \otimes \cdots \otimes m_n \otimes r,$$

then

$$R(M) = \bigoplus_{n=0}^{\infty} M \cdot^n R,$$

and the product in this algebra can be written as follows:

$$(m_1 \cdots m_n \cdot r) \cdot (m'_1 \cdots m'_n \cdot r') = m_1 \cdots m_n \cdot (r m'_1) \cdot m'_2 \cdots m'_n \cdot r'.$$

Notation 3.4. Let us use the following notation

$$M \oplus Rx := M \oplus R, \quad (m, r \cdot x) \mapsto (m, r).$$

Likewise, if $M$ is a right $R$-module $M \oplus xR := M \oplus R$, $(m, x \cdot r) \mapsto (m, r)$. 

Notation 3.5. Let $M$ be a right $R$-module. Consider the morphisms of $R$-algebras
\[ in, h_x : R(M) \to R(M \oplus xR) \]
induced by the morphism of $R$-modules $M \to R(M \oplus xR)$, $m \mapsto m$, $x \cdot m$ for any $m \in M$.

Definition 3.6. Given a functor of $R$-modules $F$, let $F^\prime$, be the functor of abelian groups, defined as follows: $F^\prime(M)$ is the kernel of the morphism
\[
\begin{align*}
F(R(M)) & \xrightarrow{F(h_x) - x \cdot F(in)} F(R(M \oplus xR)) \\
\end{align*}
\]
for any right $R$-module $M$ and any $f \in F(R(M))$.

If $w : M \to M'$ is a morphism of $R$-modules, it induces morphisms of $R$-algebras
\[ R(w) : R(M) \to R(M') \]
and $R(w \oplus 1) : R(M \oplus xR) \to R(M' \oplus xR)$, $R(w \oplus 1)(m) = w(m)$, $R(w \oplus 1)(x) = x$. Observe that $R(w \oplus 1) \circ h_x = h_x \circ R(w)$. Hence, we have the morphism
\[ F^\prime(w) : F^\prime(M) \to F^\prime(M') \]
for any $f \in F^\prime(M) \subset F(R(M))$.

Note 3.7. In a similar vein, we can define the extension of a functor $F$ of right $R$-modules, which is a functor $F^\tau$ from the category of $R$-modules to the category of abelian groups.

Notation 3.8. Let $N$ be a right $R$-module and let $M$ be an $R$-module. Consider the sequence of morphisms of groups
\[
N \otimes_R M \xrightarrow{i} N \otimes_R M \otimes_R R \xrightarrow{p_1 \otimes p_2} N \otimes_R M \otimes_R R \otimes_R R
\]
where $i(n \otimes m) := n \otimes m \otimes 1$, $p_1(n \otimes m \otimes r) := n \otimes m \otimes r \otimes 1$ and $p_2(n \otimes m \otimes r) := n \otimes m \otimes 1 \otimes r$, is exact.

Lemma 3.9. Let $M$ be an $R$-module and $N$ a right $R$-module. Then,
\[
\mathcal{N}^\tau(M) = \text{Ker}[N \otimes_R M \otimes_R R \xrightarrow{p_1 \otimes p_2} N \otimes_R M \otimes_R R \otimes_R R]
\]
Proof. It is easy to prove that the kernel of the morphism
\[ N \otimes_R R(M) \to N \otimes_R R(M)[x], \quad n \otimes p(m) \mapsto n \otimes (p(m)x - p(mx)) \]
is included in $N \otimes_R M \otimes_R R$. Observe that the morphism of $R$-algebras $R(M \oplus Rx) \to R(M)[x]$, $m \mapsto m$ and $x \mapsto x$, is an epimorphism.

Then,
\[
\mathcal{N}^\tau(M) \subseteq N \otimes_R M \otimes_R R
\]
and $\mathcal{N}^\tau(M) = \text{Ker}(p_1 - p_2)$.
Remark 3.10. Observe that
\[ N^r(M) = \text{Ker}[N \otimes R M \otimes Z R_p \otimes R Q N \otimes R M \otimes Z R] = M^r(N) . \]

Proposition 3.11. Let \( N \) be a right \( R \)-module and let \( M \) be an \( R \)-module. If \( M \) (or \( N \)) is an \( R \)-bimodule or a flat module, then
\[ N^r(M) = N \otimes R M . \]

Proof. By Lemma 3.9 we have to prove that \( \text{Ker}(p_1 - p_2) = N \otimes R M \).

Suppose that \( M \) is a bimodule. It is clear that \( \text{Im} i \subseteq \text{Ker}(p_1 - p_2) \). Let
\[ s: N \otimes R M \otimes Z R \to N \otimes R M, \quad s(n \otimes m \otimes r) = n \otimes m r \text{ and} \]
\[ s': N \otimes R M \otimes Z R \to N \otimes R M \otimes Z R, \quad s'(n \otimes m \otimes r \otimes r') = n \otimes m r \otimes r' . \]

Observe that \( s \circ i = \text{Id} \), so that \( i \) is injective. Also, \( s' \circ p_2 = \text{Id} \) and \( s' \circ p_1 = i \circ s \).

Thus, if \( x \in \text{Ker}(p_1 - p_2) \), then \( p_1(x) - p_2(x) = 0 \); hence, \( 0 = s'(p_1(x)) - s'(p_2(x)) = i(s(x)) - x \) and \( x \in \text{Im} i \). Then, \( \text{Ker}(p_1 - p_2) = N \otimes R M \).

In particular, taking the bimodule \( M = R \), the following sequence of morphisms of groups is exact:
\[ N \xrightarrow{i} N \otimes Z R \xrightarrow{p_1} N \otimes Z R \otimes R . \]

Thus, if \( M \) is flat, tensoring by \( M \) it also follows that \( \text{Ker}(p_1 - p_2) = N \otimes R M \). □

Proposition 3.12. If there exists a central subalgebra \( R' \subseteq R \) such that \( Q \to Q \otimes R' R \) is injective, for any \( R' \)-module \( Q \), then
\[ N^r(M) = N \otimes R M . \]

Proof. Let us write \( M' := M \otimes R' R \), which is a bimodule as follows:
\[ r_1 \cdot (m \otimes r) \cdot r_2 = r_1 m \otimes rr_2 . \]

The morphism of \( R \)-modules \( i: M \to M' \), \( i(m) := m \otimes 1 \) is universally injective: Given an \( R \)-module \( P \), put \( Q := P \otimes R M \). Then, the morphism \( P \otimes R M = Q \to Q \otimes R' R = P \otimes R M' \) is injective.

Put \( Q := M'/M \) and \( M'' := Q \otimes R' R \). Let \( p_1 \) be the composite morphism \( M' \to M'/M = Q \to Q \otimes R' R = M'' \). The sequence of morphisms of \( R \)-modules
\[ 0 \to M \xrightarrow{i} M' \xrightarrow{p_1} M'' \]
is universally exact. Consider the following commutative diagram
\[
\begin{array}{ccc}
0 & \to & N \otimes R M \\
\downarrow & & \downarrow \text{Id} \\
0 & \to & N \otimes R M \otimes Z R \\
\downarrow & & \downarrow \text{Id} \\
0 & \to & N \otimes R M \otimes Z R \otimes R \\
\downarrow & & \downarrow \text{Id} \\
0 & \to & N \otimes R M \otimes Z R \otimes Z R \\
\downarrow & & \downarrow \text{Id} \\
0 & \to & N \otimes R M \otimes Z R \otimes Z R \otimes R \\
\downarrow & & \downarrow \text{Id} \\
0 & \to & N \otimes R M \otimes Z R \otimes Z R \otimes Z R \\
\end{array}
\]

(where \( i' = \text{Id} \otimes i \otimes \text{Id} \otimes \text{Id} \) and \( p' = \text{Id} \otimes p \otimes \text{Id} \otimes \text{Id} \)) whose rows are exact, as well as both the second and third columns, by Proposition 3.11. Hence, the first column is exact too. □
Proposition 3.13. Let $F$ be a functor of $R$-modules. Then,
$$F^{v^R}(M) = \text{Hom}_R(F, M),$$
for any $R$-module $M$. Hence, $F^{v^R} = F$.

Proof. By Lemma 3.9 and Proposition 3.11, the sequence of morphisms
$$S \otimes_R M \longrightarrow S \otimes_R R\langle M \rangle \longrightarrow S \otimes_R R\langle M \oplus Rx \rangle$$
is exact for any $R$-algebra $S$, that is, if $M$, $R\langle M \rangle$ and $R\langle M \oplus Rx \rangle$ are the quasi-coherent modules associated with $M$, $R\langle M \rangle$ and $R\langle M \oplus Rx \rangle$, respectively, then the sequence of morphisms
$$M \longrightarrow R\langle M \rangle \longrightarrow R\langle M \oplus Rx \rangle$$
is exact. Hence, $F^{v^R}(M) = \text{Hom}_R(F, M)$.

4. ADJOINT FUNCTOR THEOREM

Given an $R$-algebra $S$, let $\pi_S: R(S) \rightarrow S$ be the morphism of $R$-algebras $s \mapsto s$, for any $s \in S$.

Definition 4.1. Let $F$ be an $R$-module. We have a natural morphism $\pi_F: F^{v^R} \rightarrow F$ defined as follows
$$\pi_{F,S}(m) = F(\pi_S)(m), \text{ for any } m \in F^{v^R}(S) = F^r(S) \subseteq F(R(S)).$$

Proposition 4.2. For any $s \in S$ and $m \in F^{v^R}(S) = F^r(S) \subseteq F(R(S))$,
$$\pi_{F,S}(s \cdot m) = s \cdot F(\pi_S)(m).$$
(Recall Remark 2.5).

Proof. Given $m \in F^r(S)$, we know that $F(h_x)(m) - x \cdot F(in)(m) = 0$, by Definition 3.6. Let $h_s: R\langle S \rangle \rightarrow R\langle S \rangle$ be defined by $h_s(s') = s \cdot s' \in S \cdot S \subseteq R(S)$. Consider the morphism of $R$-algebras $R\langle S \oplus Rx \rangle \xrightarrow{x=s} R\langle S \rangle, s' \mapsto s'$ and $x \mapsto s$. We have the commutative diagrams

$$
\begin{array}{ccc}
R\langle S \rangle & \xrightarrow{h_x} & R\langle S \oplus xR \rangle \\
\downarrow h_s & & \downarrow \text{id} \\
R\langle S \rangle & & R\langle S \rangle \\
\end{array}
\begin{array}{ccc}
R\langle S \rangle & \xrightarrow{\text{id}} & R\langle S \oplus xR \rangle \\
\downarrow x=s & & \downarrow x=s \\
R\langle S \rangle & & R\langle S \rangle \\
\end{array}
$$

Then,
$$0 = F(x = s)(F(h_x)(m) - x \cdot F(in)(m)) = F(h_x)(m) - s \cdot m.$$}

Observe that $\pi_S \circ h_s = \pi_S \circ R\langle s \rangle$. Then,
$$0 = F(\pi_S)(F(h_x)(m) - s \cdot m) = F(\pi_S)(F(R\langle s \rangle))(m) - s \cdot m$$
$$= F(\pi_S)(s \cdot m - s \cdot m) = F(\pi_S)(s \cdot m) - s \cdot F(\pi_S)(m)$$
$$= \pi_{F,S}(s \cdot m) - s \cdot \pi_{F,S}(m).$$

□
Proposition 4.3. Let $\phi : F \to F'$ be a morphism of $R$-modules. The diagram

$$
\begin{array}{ccc}
F^{ro} & \xrightarrow{\phi^{ro}} & F'^{ro} \\
\downarrow{\pi_F} & & \downarrow{\pi_{F'}} \\
F & \xrightarrow{\phi} & F'
\end{array}
$$

is commutative.

Proof. The diagram

$$
\begin{array}{ccc}
F^{ro}(S) & \xrightarrow{\phi^{ro}(S)} & F'^{ro}(S) \\
\downarrow{\pi_{F}(S)} & & \downarrow{\pi_{F'}(S)} \\
F(S) & \xrightarrow{\phi_S} & F'(S)
\end{array}
$$

is commutative. \qed

Given a right $R$-module $N$, let $i_N : N \to R\langle N \rangle$ be the morphism of $R$-modules $n \mapsto n$, for any $n \in N$.

Definition 4.4. Let $G$ be a functor of abelian groups. We have a natural morphism $i_G : G \to G^{or}$ defined as follows:

$$
i_G,N(g) := G(i_N)(g), \text{ for any } g \in G(N)
$$

Let us check that $G(i_N)(g) \in G^{or}(N) \subset G^{or}(R(N)) = G(R(N))$: The composite morphism

$$
N \xrightarrow{i_N} R\langle N \rangle \xrightarrow{h_x - x \cdot in} R\langle N \oplus xR \rangle
$$

is zero. Hence,

$$
0 = G(h_x - x \cdot in)(G(i_N)(g)) = (G(h_x) - G(x \cdot in))(G(i_N)(g))
= (G^o(h_x) - x \cdot G^o(in))(G(i_N)(g))
$$

and $G(i_N)(g) \in G^{or}(N)$.

Proposition 4.5. Let $\phi : G \to G'$ be a morphism of functors of groups. The diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\phi} & G' \\
\downarrow{i_G} & & \downarrow{i_{G'}} \\
G^{or} & \xrightarrow{\phi^{or}} & G'^{or}
\end{array}
$$

is commutative.
Proof. The diagram
\[ \begin{array}{ccc}
G(N) & \xrightarrow{\phi_N} & G'(N) \\
\downarrow i_{G,N} & & \downarrow i'_{G,N} \\
G^{or}(N) & \xrightarrow{\phi_N} & G^{or}(N) \\
\downarrow & & \downarrow \\
G^r(R(N)) & \xrightarrow{\phi_{R(N)}} & G'^r(R(N)) \\
\downarrow & & \downarrow \\
G(R(N)) & \xrightarrow{\phi_{R(N)}} & G'(R(N))
\end{array} \]
is commutative. □

Lemma 4.6. Let $G$ be a functor of abelian groups. The composite morphism
\[ G^a \circ i_{G^a} \circ \pi_G \circ G \rightarrow G^a \]
is the identity morphism.

Proof. The diagram
\[ \begin{array}{ccc}
G^a(S) & \xrightarrow{i^a_{G,S}} & G^{or}(S) \\
\downarrow & & \downarrow \\
G(S) & \xrightarrow{i_{G,s}} & G^{or}(S) \\
\downarrow & & \downarrow \\
G(R(S)) & \xrightarrow{G^{or}(s)} & G^a(R(S)) \\
\downarrow & & \downarrow \\
G_R(N) & \xrightarrow{\phi_R(N)} & G^r(R(N))
\end{array} \]
is commutative. Hence, $\pi_{G^a,S} \circ i^a_{G,S} = G(\pi_S) \circ G(i_S) = G(\pi_S \circ i_S) = Id$. □

Lemma 4.7. Let $F$ be an $R$-module. The composite morphism
\[ F^r \circ i_{F^r} \circ F^{or} \circ \pi_F \circ F \rightarrow F^r \]
is the identity morphism.

Proof. The diagram
\[ \begin{array}{ccc}
F^r(N) & \xrightarrow{i_{F^r,N}} & F(R(N)) \\
\downarrow & & \downarrow \pi_{F,R(N)} \\
F^{or}(N) & \xrightarrow{F^{or}(i_N)} & F^r(R(N)) \\
\downarrow & & \downarrow \pi_{F,R(N)} \\
F^r(N) & \xrightarrow{i_{F^r,N}} & F(R(N)) \\
\downarrow & & \downarrow \\
F(R(N)) & \xrightarrow{F(R(i_N))} & F(R(N))
\end{array} \]
is commutative. Then, $\pi_{F^r,N} \circ i_{F^r,N} = Id$ since
\[ F(\pi_{R(N)}) \circ F(i_{R(N)}) = F(\pi_{R(N)} \circ R(i_N)) = F(Id) = Id. \]
Hence, $\pi_F^a \circ i_{F^r} = Id$. □
Theorem 4.8. Let $F$ be an $R$-module and $G$ an additive functor of abelian groups. Then, we have the functorial isomorphism

$$\text{Hom}_{\text{grp}}(G, F^\circ) \cong \text{Hom}_R(G^\circ, F)$$

\begin{equation}
\phi \mapsto \pi_F \circ \phi^o \\
\varphi \circ i_G \mapsto \pi_F \circ i_G \circ \varphi \\
\varphi \circ i_G \mapsto \pi_F \circ i_G \circ \varphi 
\end{equation}

Proof. It is a check:

\begin{align*}
\phi &\mapsto \pi_F \circ \phi^o \mapsto \pi_F \circ \phi^o \circ i_G = \pi_F \circ i_F \circ \phi \\
\varphi &\mapsto \varphi \circ i_G \mapsto \pi_F \circ \varphi^o \circ i_G = \pi_F \circ i_F \circ \varphi
\end{align*}

\hfill \Box

Corollary 4.9. Let $F$ be an $R$-module such that $\pi_F : F^\circ \to F$ is an isomorphism. If $F'$ is another $R$-module, the natural morphism

$$\text{Hom}_R(F, F') \to \text{Hom}_{\text{grp}}(F^\circ, F'), \ f \mapsto f^\circ.$$

Proof. $\text{Hom}_R(F, F') = \text{Hom}_R(F^\circ, F') \cong \text{Hom}_{\text{grp}}(F^\circ, F')$. \hfill \Box

5. Reflexivity theorem

Let $M$ be an $R$-module. The functor $\mathcal{M}^{\vee r}$ is precisely the functor of (co)points of $M$ in the category of $R$-modules: if $Q$ is another $R$-module, in virtue of Proposition 3.13:

$$\mathcal{M}^{\vee r}(Q) = \text{Hom}_R(M, Q) = \text{Hom}_R(M, Q).$$

Lemma 5.1. Let $M$ be a right $R$-module and $G$ an additive functor of abelian groups. Then,

$$\text{Hom}_{\text{grp}}(\mathcal{M}^{\vee r}, G) = G(M).$$

Proof. It is Yoneda’s lemma. \hfill \Box

Theorem 5.2. Let $M$ be an $R$-module and $N$ be a right $R$-module. Then,

$$\text{Hom}_R(\mathcal{M}^{\vee}, N) = N^r(M).$$

Proof. $\mathcal{M}^{\vee}$ satisfies the hypothesis of Theorem 5.1 (see Proposition 3.13), so that

$$\text{Hom}_R(\mathcal{M}^{\vee}, N) \cong \text{Hom}_{\text{grp}}(\mathcal{M}^{\vee r}, N^r) \cong N^r(M).$$

\hfill \Box

Theorem 5.3. Let $M$ be an $R$-module. The natural morphism of $R$-modules

$$\mathcal{M} \to \mathcal{M}^{\vee \vee}$$

is an isomorphism.

Proof. It is a consequence of Theorem 5.2 and Proposition 3.11

$$\mathcal{M}^{\vee \vee}(S) = \text{Hom}_R(\mathcal{M}^{\vee}, S) = \mathcal{M}^{\vee}(S) = S \otimes_R M = \mathcal{M}(S).$$

\hfill \Box
Theorem 5.4. Let $M$ be an $R$-module. The natural morphism of $R$-modules
\[ M \rightarrow M^{**} \]
is an isomorphism.

Proof. Let $S$ be an $R$-algebra. $M |_{S}$ is the $S$-quasi-coherent module associated
with $S \otimes_{R} M$ and $M |_{S}^{*} = M |_{S}^{\vee}$. Then,
\[ M^{**}(S) = \text{Hom}_{S}(M |_{S}^{*}, S) = \text{Hom}_{S}(M |_{S}^{\vee}, S) \]
\[ \cong S \otimes_{S} (S \otimes_{R} M) = S \otimes_{R} M = \mathcal{M}(S). \]
\[ \square \]

6. Quasi-coherent modules associated with flat modules

Given an $R$-module $M$, let $M_{r}$ be the functor of abelian groups defined by
\[ M_{r}(N) := N \otimes_{R} M. \]
for any right $R$-module $N$. Observe that there exists a natural morphism $M_{r} \rightarrow M^{r}$ and $M_{r}^{o} = M^{ro} = M$, by Proposition 3.11.

In [5], it is given several characterizations of $M_{r}$, when $M$ is a flat or projective module. The adjoint functor theorem will give us the corresponding characterizations of $M$, when $M$ is a flat or projective module. We have only to add the following hypothesis.

Hypothesis 6.1. From now on we will assume that $M_{r} = M^{r}$ (if $M$ is a flat $R$-module, then $M_{r} = M^{r}$, by Proposition 3.11.)

Definition 6.2. We will say that $N^{r}$ is the module scheme associated with $N$.

Theorem 6.3. $M$ is a finitely generated projective module iff $M$ is a module scheme.

Proof. $\Rightarrow$ By [5 3.1], there exists an isomorphism $M_{r} \simeq N^{vr}$. Then,
\[ M = M_{r}^{o} \simeq N^{vr} = N^{r}. \]
\[ \Leftarrow\] $M \simeq N^{r}$. Then, $M_{r} = M^{r} \simeq N^{vr}$. By [5 3.1], $M$ is a finitely generated projective module.

$\square$

Theorem 6.4. $M$ is a flat module iff $M$ is a direct limit of module schemes.

Proof. $\Rightarrow$ By [4 3.2], there exists an isomorphism $M_{r} \simeq \lim_{\rightarrow i \in I} N_{i}^{vr}$. Then,
\[ M = M_{r}^{o} \simeq (\lim_{\rightarrow i \in I} N_{i}^{vr})^{o} = \lim_{\rightarrow i \in I} N_{i}^{vr} = \lim_{\rightarrow i \in I} N_{i}^{r}. \]
\[ \Leftarrow\] $M \simeq \lim_{\rightarrow i \in I} N^{r}$. Then, $M_{r} = M^{r} \simeq \lim_{\rightarrow i \in I} N^{vr}$. By [4 3.2], $M$ is flat.

$\square$

Lemma 6.5. Let $N_{1}, N_{2}$ be right $R$-modules. Then,
1. $\text{Hom}_{R}(N_{1}^{r}, N_{2}^{r}) = \text{Hom}_{\text{grp}}(N_{1}^{vr}, N_{2}^{vr}).$
2. $\text{Hom}_{R}(N_{1}, N_{2}) = \text{Hom}_{\text{grp}}(N_{1}^{r}, N_{2}^{r}).$
3. $\text{Hom}_{R}(N_{1}^{r}, M) = \text{Hom}_{\text{grp}}(N_{1}^{vr}, M^{r}).$
(4) \( \text{Hom}_R(\mathcal{M}', N_1) = \text{Hom}_{\text{grp}}(\mathcal{M}'^r, N_{1r}). \)

Proof. 1. It is Corollary 6.5.
  2. \( \text{Hom}_R(N_1, N_2) = \text{Hom}_{\text{grp}}(N_{1r}, N_{2r}). \)
  3. \( \text{Hom}_R(N_1', \mathcal{M}) = \text{Hom}_{\text{grp}}(N_1'^r, \mathcal{M}^r) = \text{Hom}_{\text{grp}}(N_1^r, \mathcal{M}_r). \)
  4. \( \text{Hom}_R(\mathcal{M}', N_1) = \text{Hom}_{\text{grp}}(\mathcal{M}'^r, N_{1r}). \)

\( \square \)

**Lemma 6.6.** Let \( f : \mathbb{F}_1 \to \mathbb{F}_2 \) be a morphism of \( \mathcal{R} \)-modules. Then, \( f \) is a monomorphism iff the morphism of functors of groups \( f^r : \mathbb{F}_1^r \to \mathbb{F}_2^r \) is a monomorphism.

Proof. If \( f \) is a monomorphism, then \( f^r \) is a monomorphism, since \( f^r_S = f_{R(N)} \) on \( \mathbb{F}_1^r(N) \subseteq \mathbb{F}_1(R(N)). \) If \( f^r \) is a monomorphism, then \( f = f^{ro} \) is a monomorphism since \( f^{ro}_S = f^S. \)

\( \square \)

**Theorem 6.7.** Let \( M \) be an \( \mathcal{R} \)-module. The following statements are equivalent.

1. \( M \) is a flat Mittag-Leffler module.
2. Every morphism of \( \mathcal{R} \)-modules \( \mathcal{N}' \to \mathcal{M} \) factors through an \( \mathcal{R} \)-submodule scheme of \( \mathcal{M} \), for any right \( \mathcal{R} \)-module \( N \).
3. \( \mathcal{M} \) is equal to a direct limit of \( \mathcal{R} \)-submodule schemes.

Proof. 1. \( \iff \) 2. It is an immediate consequence of [5, 4.5], Lemma 6.6, and Lemma 6.6.

1. \( \iff \) 3. It is an immediate consequence of [5, 4.5] and Lemma 6.6 since \( \mathcal{M} \cong \lim_{\to \mathcal{N}^r_i} \mathcal{M}_r \cong \lim_{\to \mathcal{N}^r_i} \mathcal{M}_r. \)

\( \square \)

**Lemma 6.8.** A morphism of \( \mathcal{R} \)-modules \( f : \mathcal{M}' \to \prod_{i \in I} \mathcal{N}_i \) is an epimorphism iff the corresponding morphism of functors of groups \( \mathcal{M}'^r \to \prod_{i \in I} \mathcal{N}_{ir} \) (see 6.7 (4)) is an epimorphism.

Proof. \( \Rightarrow \) \( f = (\sum_j n_n \otimes m_{ij})_{i \in I} \) through the equality

\[
\text{Hom}_R(\mathcal{M}', \prod_{i \in I} \mathcal{N}_i) = \text{Hom}_R(\mathcal{M}'^r, \prod_{i \in I} \mathcal{N}_{ir}) \cong \prod_{i \in I} \mathcal{N}_{ir}(M) = \prod_{i \in I} (N_i \otimes \mathcal{M}_r(M)).
\]

We have to prove that the morphism

\[
\text{Hom}_R(M, N) \xrightarrow{1} \mathcal{M}'^r(N) \xrightarrow{r} \prod_{i \in I} \mathcal{N}_{ir}(N) \xrightarrow{\prod_{i \in I} (N_{ir} \otimes N)}
\]

\[
h \xrightarrow{(\sum_j n_n \otimes h(m_{ij}))_{i \in I}} (\sum_j n_n \otimes h(m_{ij}))_{i \in I}
\]

is an epimorphism, for any \( \mathcal{R} \)-module \( N \). If \( N \) is an \( \mathcal{R} \)-algebra, then it is an epimorphism, since \( f \) is an epimorphism. We can suppose that \( N \) is a free \( \mathcal{R} \)-module, since the functor \( \prod_{i \in I} \mathcal{N}_{ir} \) preserves epimorphisms. In this case \( N \) is naturally a bimodule. Let \( \pi : R(N) \to N \) be the composition of the obvious morphisms of \( \mathcal{R} \)-modules \( R(N) \to N \cdot \mathcal{R} \to N \). Obviously, \( \pi \) is an epimorphism. Then, we can suppose that \( N \) is an \( \mathcal{R} \)-algebra. We conclude. 

\( \square \)
Theorem 6.9. Let $M$ be an $R$-module. The following statements are equivalent:

1. $M$ is a flat strict Mittag-Leffler module.
2. Any morphism $f : M^\vee \to N$ factors through the quasi-coherent module associated with $\text{Im} f_R$, for any right $R$-module $N$.
3. Any morphism $f : M^\vee \to R$ factors through the quasi-coherent module associated with $\text{Im} f_R$.
4. Let $\{M_i\}_{i \in I}$ be the set of all finitely generated $R$-submodules of $M$, and $M_i' := \text{Im}[M^* \to M_i^*]$. The natural morphism $M \to \lim_{\rightarrow} M_i^\vee$ is an isomorphism.
5. $M$ is a direct limit of submodule schemes, $N_i^\vee \subseteq M$ and the dual morphism $M^\vee \to N_i$ is an epimorphism, for any $i$.
6. There exists a monomorphism $M \hookrightarrow \prod R$.

Proof. 1. $\iff$ 2. It is a consequence of [5, 4.9 (2)] and Lemma 6.5 4.
   1. $\iff$ 3. It is a consequence of [5, 4.10] and Lemma 6.5 4.
   1. $\iff$ 4. It is a consequence of [5, 4.9 (3)].
   1. $\iff$ 5. It is a consequence of [5, 4.9 (4)] and Lemma 6.8.
   1. $\iff$ 6. It is a consequence of [5, 4.7 (3)] and Lemma 6.9. \qed

Proposition 6.10. An $R$-module $M$ is a projective $R$-module of countable type if and only if there exists a chain of submodule schemes of $M$

$$N_1^\vee \subseteq N_2^\vee \subseteq \cdots \subseteq N_n^\vee \subseteq \cdots$$

such that $M = \bigcup_{n \in \mathbb{N}} N_n^\vee$.

Proof. It is a consequence of [5, 4.11,13] and Lemma 6.0. \qed

Theorem 6.11. An $R$-module $M$ is projective if and only if there exists a chain of $R$-submodules of $M$

$$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_n \subseteq \cdots$$

such that $M = \bigcup_{n \in \mathbb{N}} W_n$, where $W_n$ is a direct sum of module schemes and the natural morphism $M^\vee \to W_n^\vee$ is an epimorphism, for any $n \in \mathbb{N}$.

Proof. It is a consequence of [5, 4.14] and Lemma 6.8. \qed

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FUNCTORS OF MODULES ASSOCIATED WITH FLAT AND PROJECTIVE MODULES II

Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, s/n, 06006 Badajoz (SPAIN)
E-mail address: adgorner@unex.es, navarroarmendia@unex.es, sancho@unex.es