Phase Conjugation of a Quantum-Degenerate Atomic Fermi Beam

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We discuss the possibility of phase-conjugation of an atomic Fermi field via nonlinear wave mixing in an ultracold gas. It is shown that for a beam of fermions incident on an atomic phase-conjugate mirror, a time reversed backward propagating fermionic beam is generated similar to the case in nonlinear optics. By adopting an operational definition of the phase, we show that it is possible to infer the presence of the phase-conjugate field by the loss of the interference pattern in an atomic interferometer.

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In contrast with the situation for classical electromagnetic fields, there are considerable difficulties associated with the definition of a quantum mechanical phase operator  for bosonic fields [1, 2, 3]. This problem has been discussed at length in quantum optics [2], and has recently been extended in the context of Bose-Einstein condensation. With none of the numerous attempts at formally introducing a phase operator being fully satisfactory, Noh, Fougeres and Mandel adopted instead an operational approach based on an analysis of what is actually measured in an experiment, namely the relative phase between interfering fields. It can be represented by a combination of photon counting operators that depends on the particular experimental scheme [2]. A similarly operational approach has been adopted to discuss the interference of Bose-Einstein condensates [2, 3].

One fundamental difference between optical and matter-wave fields is that the latter ones can consist of either bosons or fermions. For fermions the question of phase of the field is even more difficult than it is for bosons. Yet, it is gaining considerably in relevance in view of the spectacular progress recently achieved toward the experimental realization of quantum-degenerate fermionic atomic gases, with temperatures reaching as low as \( T < 0.2 T_F \) where \( T_F \) is the Fermi temperature [4, 5, 10, 11, 12]. While one of the primary goals of this work has been to observe the BCS transition to a superfluid Fermi gas [13, 14, 15], future experiments will likely use these gases for atom optics experiments for which it will be necessary to understand what role, if any, the phase of the Fermi field plays.

Unfortunately, one cannot define a phase operator or phase eigenstates for an eigenmode of the Fermi field because of the Pauli exclusion principle: since number and phase are canonically conjugate variables, a well-defined phase requires a large uncertainty in particle number. Indeed, a straightforward generalization to fermions of the various phase operators and phase states discussed for bosonic fields [1, 2, 3] leads to mathematically ill-defined results. Therefore, it would appear that if phase is to have any meaning in Fermi systems, it must be associated with a multimode cooperative effect such as the order parameter, \( \Delta \), for a superfluid Fermi gas.

The goal of this Letter is to show that despite the apparent difficulties associated with the concept of phase for fermionic fields, it is possible to introduce it in an operational way. As a previous indication that this might be possible, we recall that it is possible in principle to operate atom interferometers with quantum-degenerate fermionic beams, thereby measuring the relative phase of the partial beams [16, 17]. Here, we go one step further and show that it is possible to phase conjugate a fermionic beam, so that its evolution is “time reversed”. This is clear evidence that from an operational point-of-view, the phase shift of a fermionic beam is a perfectly appropriate concept. To avoid any possible confusion, we emphasize at the outset that the phase under consideration is not the phase associated with the order parameter \( \Delta \) of a Fermi system undergoing a BCS superfluid transition but rather the phase associated with each eigenmode of the Fermi field.

Optical phase conjugation has been an active area of research in nonlinear optics for several decades [18]. In optical phase conjugation, an incident signal field interacts with a pump field inside a nonlinear medium to generate an idler field that is the time-reversed state of the signal field. This process can occur via three-wave mixing in a medium or by four-wave mixing in a medium [19]. In the context of classical optics, phase conjugation can be used to correct the phase aberrations incurred by the signal field while in quantum optics, phase conjugation via four-wave mixing can lead to the generation of squeezed states [20].

Four-wave mixing in normal Fermi gases has been demonstrated theoretically in Refs. [21, 22, 23, 24, 25]. They show that four-wave mixing could be interpreted in term of Bragg scattering off of density modulations. However, phase-conjugation via four-wave mixing requires a different configuration that necessitates the use of a superfluid gas. Specifically, we consider two counter-propagating beams of atomic fermions interacting with an atomic phase-conjugate mirror (PCM) see Fig. 1. This “mirror” is formed by a degenerate Fermi gas of alkali atoms at zero temperature confined in the region.
0 ≤ z ≤ L, with equal numbers of atoms, \(N_F\), occupying two hyperfine states that we refer to as spin up (↑) and spin down (↓). The confinement in the \(z\)-direction is provided by a trapping potential \(U_0 < 0\) that is constant in the region \(0 ≤ z ≤ L\) except for a small interval \(δz ≪ L\) around \(z = 0\) and \(L\) where it goes smoothly to 0. The extent of the gas in the \(x\) and \(y\) directions is taken to be infinite so that we can treat the gas as being spatially homogeneous with density \(n_F = N_F/V\) and volume \(V\).

The spin up and down atoms interact via an attractive two-body interaction characterized by the \(s\)-wave scattering length \(a < 0\). The existence of an attractive interaction leads to an instability in the normal state of the Fermi gas that results in the BCS transition to a superfluid state characterized by the nonzero order parameter \(\Delta\) [20].

\[
Δ = -g \sum_κ \langle \hat{c}_−κ|\hat{c}_κ\rangle
\]

where \(g = 4π\hbar|a|/V m\) and \(\hat{c}_κσ\) is the annihilation operator for a fermion of momentum \(\hbar k\) and spin \(σ\). The phase-conjugate mirror is therefore described by the linearized Hamiltonian in the region \(0 < z < L\),

\[
H = \sum_κ \left[ \hbar ω_κ (\hat{c}_κ↑\hat{c}_κ↑ + \hat{c}_−κ↓\hat{c}_−κ↓) + \hbar Δ + \hat{c}_κ↑\hat{c}_κ↓ + \hat{c}_−κ↓\hat{c}_κ↑ \right]
\]

where \(ω_κ = \hbar k^2/2m - \hbar k_F^2/2m\) and \(k_F = (6π^2n_F)^{1/3} ≫ L^{-1}\) is the Fermi wave number of the gas. For convenience we take \(Δ\) to be real.

Before proceeding we note that the PCM may also be produced by a three-wave mixing process that couples the fermions to a molecular condensate as in [14], in which case \(Δ\) in Eq. (2) is replaced by the expectation value of the molecular field. However, unlike four-wave mixing, phase-matching is no longer automatically satisfied in this case [15].

The Hamiltonian \(H\) may be diagonalized by the Bogoliubov transformation

\[
\hat{a}_κ↑ = \cos(θ_κ/2)\hat{c}_κ↑ - \sin(θ_κ/2)\hat{c}_−κ↓,
\]

\[
\hat{a}_−κ↓ = \cos(θ_κ/2)\hat{c}_−κ↓ + \sin(θ_κ/2)\hat{c}_κ↑,
\]

where \(α_κσ\) is an annihilation operator for a quasiparticle in the gas with energy \(\hbar ω_κ = \hbar \sqrt{ω_κ^2 + Δ^2}\) and \(\tan θ_κ = |Δ|/ω_κ\).

The cw beam impinging the PCM at \(z = 0\) consists of spin-up fermions with momenta \(\hbar k = \hbar k_ζ\), \(k > 0\). Similarly, at \(z = L\) spin-down fermions are incident on the gas with momenta \(\hbar k = −\hbar k_ζ\), \(k > 0\). The beams are considered to be sufficiently well collimated in the \(x\) and \(y\) directions that they can be treated as one-dimensional [17]. The number of atoms in these beams, \(N_B\), satisfy \(N_B ≪ N_F\) so that the superfluid gas forming the PCM can be treated as undepleted.

![FIG. 1: Schematic diagram of input-output relations for fields incident on superfluid Fermi gas.](image)

Now consider a fermion initially located at \(z < 0\) and described as a wave-packet with average momentum \(\hbar k_0\), \(ψ(z, t) \sim ψ(z − v_κ(t − t_0), 0)\) where \(v_κ = \hbar k_0/m\) is the group velocity. When the atom enters the superfluid gas it experiences the same confining potential as the trapped atoms, as well as an attractive Hartree potential, \(-g n_F\), [27] and propagates with the new group velocity \(v_κ = \hbar k_0/m\) where

\[
\bar{k}(k_0) = \sqrt{k_0^2 + 2m[(U_0 + g n_F)/\hbar^2]},
\]

reaching the other mirror surface after a time \(τ_{k_0} = L/v_κ\). Similarly, a wave-packet at \(z = L\) with mean momentum \(-\hbar k_0\) takes a time \(τ_{k_0}\) to reach \(z = 0\). The input fields are then related to the initial conditions, \(\hat{c}_κ↑|z = 0\rangle = \hat{c}_κ↑(t = 0)\) and \(\hat{c}_−κ↓|z = L\rangle = \hat{c}_−κ↓(t = 0)\).

To simplify the notation we define \(\hat{c}_κσ = \hat{c}_κσ(0)\) and \(\hat{c}_κ↑[0]\) and \(\hat{c}_−κ↓[0]\), are then obtained by integrating the equations of motion from \(t = 0\) to \(τ_κ\).

By using the Bogoliubov transformation [28] and the solution of the equations of motion for the quasiparticles, \(\hat{a}_κσ = \hat{a}_κσ(0)|\exp[−i\hat{c}_κσ t]|\), one readily obtains the output states in terms of the input states as

\[
\hat{c}_κ↑[L] = T_k(\hat{c}_κ↑[0] + R_k(\hat{c}_−κ↓[L])
\]

\[
\hat{c}_−κ↓[0] = T_k(\hat{c}_−κ↓[L]) + R_k(\hat{c}_κ↑[0])
\]

where

\[
T_k = \cos(ζ_k mL/\hbar k) − i \cos θ_k \sin(ζ_k mL/\hbar k),
\]

\[
R_k = i \sin θ_k \sin(ζ_k mL/\hbar k).
\]

Phase conjugation only occurs when \(R_k ≠ 0\), the output states being then a superposition of the transmitted input state plus its time-reversed state, \(\hat{c}_−κ↓[0] = T\hat{c}_κ↑[0] T^{-1}\) where \(T\) is the time-reversal operator. Note that \(\bar{k}_k = 0\) in the absence of a superfluid state, \(Δ = 0\), so that the existence of this state is essential for the operation of the PCM. Just as is normally the case in optics, the phase-conjugate mirror has a finite bandwidth, since \(\sin θ_k = |Δ|/ζ_k\) is only different from zero in the interval \(δk ≈ |Δ|/m\hbar k_F\) around \(k_F\). The phase-conjugate signal is therefore optimized by using an input state with
average momentum $\hbar k_0$ and bandwidth $\Delta k$ such that $\tilde{k}(k_0) = k_F$ and $|\tilde{k}(k_0 + \Delta k) - \tilde{k}(k_0 - \Delta k)| < \delta k$. In this case one can make $|R_{kF}| = 1$ for $L = (2j + 1)\pi \hbar k_F / 2m|\Delta|$ for $j = 0, 1, 2, \ldots$

An important difference from the optical case is that since $|T_k|^2 + |R_k|^2 = 1$, there is no amplification of the individual modes of the fermion beams. This is in stark contrast to the case of bosons where one has instead $|T_k|^2 - |R_k|^2 = 1$ in order to preserve the commutation relations and the transmitted field is always amplified since $|T_k| \geq 1$. The lack of amplification for a single mode of the fermion field is a necessary consequence of the Pauli exclusion principle. Note, however, that the total number of fermions in the output beams can be amplified. To see this, we take for definiteness the input state of the fermion beams to be

$$|\Psi\rangle = \prod_{|k - k_0| \leq \Delta k} \hat{c}^{\dagger}_{k\uparrow}[0]|0\rangle$$

with $k_0 > \Delta k > 0$. This corresponds to a beam of spin-up fermions with momenta centered around $\hbar k_0$ incident from $z < 0$ with no atoms incident from $z > L$. The occupation numbers for this state are $n_k = \langle \hat{c}^{\dagger}_{k\uparrow}|0\rangle \hat{c}_{k\uparrow}|0\rangle$. Defining the total number operators for the input and output fields as $N^{\text{in/out}} = \sum_{|k| > 0} \hat{c}^{\dagger}_{k\uparrow}|0/L\rangle \hat{c}_{k\uparrow}|0/L\rangle$ and $\hat{N}^{\text{in/out}} = \sum_{|k| > 0} \hat{c}^{\dagger}_{k\uparrow}|L/0\rangle \hat{c}_{k\uparrow}|L/0\rangle$, one finds that their expectation values are

$$\langle \hat{N}^{\text{out}}_\uparrow \rangle = \langle \hat{N}^{\text{in}}_\uparrow \rangle + \langle \hat{N}^{\text{out}}_\downarrow \rangle$$

$$\langle \hat{N}^{\text{out}}_\downarrow \rangle = \sum_k |R_{kF}|^2 (1 - n_k).$$

Eqs. 10 and 11 show that the number of atoms in both beams increase after having passed through the gas. However, Eq. 11 shows that the increase results from the scattering of atoms out of the superfluid gas and into those modes that are not occupied in the incident beam. Consequently, only incoherent amplification of the vacuum fluctuations occurs.

The identical increase in both beams reflects the underlying pair creation process given by the $\hat{c}^{\dagger}_{k\uparrow}\hat{c}^{\dagger}_{-k\downarrow}$ term in the Hamiltonian, as well as the fact that the number difference operator, $\hat{c}^{\dagger}_{k\downarrow}\hat{c}_{k\uparrow} - \hat{c}^{\dagger}_{-k\downarrow}\hat{c}_{-k\uparrow}$ commutes with the Hamiltonian. Defining the covariance matrix as $\text{Cov}[\hat{N}^{\text{out}}_1, \hat{N}^{\text{out}}_2] = \langle \hat{N}^{\text{out}}_1 \hat{N}^{\text{out}}_2 \rangle - \langle \hat{N}^{\text{out}}_1 \rangle \langle \hat{N}^{\text{out}}_2 \rangle$ we find that

$$\text{Cov}[\hat{N}^{\text{out}}_1, \hat{N}^{\text{out}}_2] = \sum_k |T_{kF}|^2 |R_{kF}|^2 (1 - n_k)$$

which shows that the intra-beam number fluctuations as well as the correlations of the beams are the same. Again, this reflects the fact that any atom created in one beam coincides with an atom created in the other beam so that the fluctuations in both beams must be the same.

Eqs. 11 and 12 show that for $|R_{kF}| = 1$ one can have amplification of the number of atoms in the output beam with no fluctuations. Thus by using an input state in which all the $k$ states are fully occupied except for a narrow window of width $\Delta k$ centered at $k_0$, one can have amplified output beams with negligible number fluctuations provided $\tilde{k}(k_0) = k_F$, $|\tilde{k}(k_0 + \Delta k) - \tilde{k}(k_0 - \Delta k)| < \delta k$, and $L = (2j + 1)\pi \hbar k_F / 2m|\Delta|$. However, since these are the same conditions required for a finite phase conjugate signal, one cannot simultaneously have a phase conjugate signal and an amplified output with reduced fluctuations.

These results show that despite the lack of a well-defined phase for fermionic fields, the phase-conjugation and time-reversal of these fields is readily possible in principle. In the present example, the phase-conjugate signal provides a direct signature of the BCS superfluid state since for a Fermi gas in the normal state, an incident beam of fermions would not generate a backward propagating reflected beam. Detailed studies of the reflected beam could then be used as a diagnostic tool to study the BCS state.

To further illustrate the analogy between fermionic and bosonic phase conjugation, we now show how phase-conjugate beams of fermionic atoms can compensate the relative phase accumulated between two paths of a matter-wave interferometer when one of the mirrors is replaced by a phase-conjugate mirror.

Fig. 2 shows a model Michelson interferometer. Here, BS labels a beam splitter with transmission and reflection amplitudes $t$ and $r$, respectively, with $|t|^2 + |r|^2 = 1$ and $rt^* + r^*t = 0$. We first let $M_1$ and $M_2$ be perfectly reflecting atom mirrors while $\ell_1$ and $\ell_2$ are the one way path lengths in the two arms of the interferometer, and $D$ represents an atom counter. The operators $\hat{a}_k$ and $\hat{b}_k$ denote annihilation operators for fermions with momentum $\hbar k$ that are incident on the two input ports of the interferometer. The input states only involve a single spin state so we drop the spin label for convenience. The outputs of the beam splitter are $\hat{a}_k = r\hat{a}_k + \hat{b}_k$ and $\hat{b}_k = t\hat{a}_k + \hat{b}_k$. The mirrors introduce a phase shift assumed to be the same for both mirrors, so that it can be ignored. After propagation through the arms and recombination at the beam splitter, the fermion annihilation operator at the atom counter is

$$\hat{d}_k = rt(e^{-i2k\ell_2} + e^{-i2k\ell_1})\hat{a}_k + (r^2 e^{-i2k\ell_2} + t^2 e^{-i2k\ell_1})\hat{b}_k.$$  

Using an input state of the same form as Eq. 11, $|\Psi\rangle = \prod_{|k - k_0| \leq \Delta k} \hat{a}^{\dagger}_k|0\rangle$, we find that the number of atoms at $D$ is

$$\langle \hat{N}_D \rangle = \sum_k \langle \hat{d}^{\dagger}_k \hat{d}_k \rangle$$

$$= 2|r|^2 |t|^2 N_B \left( 1 + \cos(\Delta \ell k_0) \frac{\sin(\Delta \ell \Delta k)}{\Delta \ell \Delta k} \right)$$

where $\Delta \ell = 2(\ell_1 - \ell_2)$ and $N_B$ is the number of incident atoms. There is a discernible interference pattern, although the broadband nature of the Fermi beam leads
to a loss of contrast due to dephasing for large path differences, $\Delta L \Delta k \gg 1$. It is worth emphasizing that even though the incident fermions are in Fock states which have completely random phases $\phi_k$ for each of the occupied $k$-states, this phase is the same in both arms of the interferometer so that the phase difference between the arms is independent of $\phi_k$.

We now replace the mirror $M_2$ by a phase-conjugate mirror. The reflected output of that mirror is

$$f'_k = R^*_{k(k)}f_k e^{i k \ell_2} + T_{k(k)}c_k$$

where $c_k$ is an annihilation operator for the fermions incident on the other side of the mirror in the opposite spin state, see Eqs. [16]. The annihilation operator for fermions at the atom counter is now

$$a_k = t e^{-i 2 k \ell_1} (r a_k + b_k) + R^*_{k(k)} r(t^* a_k + r^* b_k)$$

$$+ r T_{k(k)} e^{-i k \ell_2} c_k,$$

(15)

which, again using $|\Psi\rangle$, gives

$$\langle \tilde{N}_D \rangle =$$

$$\sum_k \left( |r|^2 |T_{k(k)}|^2 \Theta(\Delta k - |k - k_0|) + |r|^2 |R_{k(k)}|^2 \right)$$

where $\Theta$ is the unit step function. In this case, there is no detectable interference pattern: The phase in arm 1 is $2k \ell_1 + \phi_k$ while the accumulated phase after round trip propagation in arm 2 is $-\phi_k$ due to the phase-conjugate mirror. The phase difference between the two arms is then $2k \ell_1 + 2\phi_k$ but since $\phi_k$ is a random variable, there is no interference pattern. This effect has also been predicted to occur in optics with chaotic fields [20].

In conclusion we have shown that phase-conjugation of Fermi fields can be achieved using four-wave mixing in a superfluid Fermi gas, and that it is justified to discuss the relative phase of fermionic fields operationally. Surprisingly we have shown that the phase of the Fermi field can have observable effects even though we cannot define a Hermitian operator for the phase. In addition, our proposed experimental setup provides an unambiguous signature of superfluidity in the phase-conjugate mirror.

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