$\tau \rightarrow \rho \pi \pi \nu$ decays

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Abstract

Effective chiral theory of mesons is applied to study the four decay modes of $\tau \rightarrow \rho \pi \pi \nu$. Theoretical values of the branching ratios are in agreement with the data. The theory predicts that the $a_1$ resonance plays a dominant role in these decays. There is no new parameter in this study.
Effective chiral theory of pseudoscalar, vector, and axial-vector mesons proposed in Ref.[1] provides a unified description of meson physics at low energies. In Ref.[2] it shows that the chiral perturbation theory[3] is the low energy limit(< \textit{m}_\rho) of this effective theory and the 10 coefficients of the chiral perturbation theory have been predicted[2].

\(\tau\) mesonic decays have been systematically studied by this effective theory[4] and theoretical results agree with the data. In this paper the decay modes of \(\tau \to \rho\pi\pi\nu\) are investigated.

The branching ratios of \(\tau \to 4\pi\nu\) have been measured[5]. These decays are via CVC related to \(ee^+ \to 4\pi[6]\). In Ref.[7] a dynamical model has been developed to study \(4\pi\) modes of \(\tau\) decay. It is pointed out[7] that the a\(_1\) resonance is important in understanding the data of \(\tau \to 4\pi\nu\).

This paper is the continuation of the study of \(\tau\) mesonic decays presented in Ref.[4]. The decays \(\tau \to 4\pi\nu\) are caused by the vector current which is expressed as[4]

\[
\mathcal{L}^V = \frac{gW}{4}\cos\theta_C g\left\{-\frac{1}{2}(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu)(\partial^\mu \rho^{i\nu} - \partial^\nu \rho^{i\mu}) + A^i_\mu j^{i\mu}\right\},
\]

(1)

where \(A^i_\mu\) is the W boson field, \(g\) is the universal coupling constant defined in Ref.[1], and \(j^{i\mu}_\mu\) is the vector current derived by the substitution

\[
\rho^{i\mu}_\mu \to \frac{gW}{4}g\cos\theta_C A^i_\mu
\]

(2)
in the vertices involving \(\rho\) meson.
All the vertices contributing to these decay modes are derived from the Eq.(13) of Ref.[1a].

There are three kinds of vertices. The first kind of vertices consists of three meson fields:

\[
\mathcal{L}_{a_1\rho\pi} = \epsilon_{ijk} \left\{ A(q^2, p^2) a_i^\rho a_j^\mu a_k^\pi - B a_i^\rho \partial^\mu a_j^{\pi k} + D a_i^\rho \partial^\nu (a_j^\rho \partial^\mu a_k^\pi) \right\} 
\]

\[
A(q^2, p^2) = \frac{2}{f_\pi} f_a \left\{ \frac{F^2}{g^2} + p^2 \left[ \frac{2c}{g} + \frac{3}{4\pi^2 g^2} (1 - \frac{2c}{g}) \right] \right\} + q^2 \left[ \frac{1}{2\pi^2 g^2} - \frac{2c}{g} - \frac{3}{4\pi^2 g^2} (1 - \frac{2c}{g}) \right], 
\]

\[
c = \frac{f_\pi^2}{2gm_p^2}, 
\]

\[
F^2 = (1 - \frac{2c}{g})^{-2} f_\pi^2, 
\]

\[
f_a = \left( 1 - \frac{1}{2\pi^2 g^2} \right)^{-\frac{1}{2}}, 
\]

\[
B = -\frac{2}{f_\pi} f_a \left\{ \frac{1}{2\pi^2 g^2} (1 - \frac{2c}{g}) \right\}, 
\]

\[
D = -\frac{2}{f_\pi} f_a \left\{ \frac{2c}{g} + \frac{3}{2\pi^2 g^2} (1 - \frac{2c}{g}) \right\}, 
\]

\[
\mathcal{L}^{a_1\pi\pi} = \frac{2}{g} \epsilon_{ijk} \rho_i^\mu \pi^j \partial^\nu \pi^k - \frac{2}{\pi^2 f_\pi^2 g} \left\{ \left( 1 - \frac{2c}{g} \right)^2 - 4\pi^2 c^2 \right\} \epsilon_{ijk} \rho_i^\mu \partial^\nu \pi^j \partial^\rho \pi^k 
\]

\[-\frac{1}{\pi^2 f_\pi^2 g} \left\{ 3(1 - \frac{2c}{g})^2 + 1 - \frac{2c}{g} - 8\pi^2 c^2 \right\} \epsilon_{ijk} \rho_i^\mu \pi^j \partial^2 \partial^\mu \pi^k, 
\]

\[
\mathcal{L}^{a_1\pi\pi} = -\frac{2}{g} \epsilon_{ijk} \partial^\mu \pi^j \rho^{\mu \nu} \rho^\nu \pi^k, 
\]

\[
\mathcal{L}^{\omega\rho\pi} = -\frac{3}{\pi^2 g^2 f_\pi} \epsilon^{\mu\nu\alpha\beta} \partial_\mu \omega^{\nu \alpha} \rho^{\beta \pi^i}, 
\]

where q is the momentum of $a_1$ meson and p is the momentum of $\rho$ meson. The second kind of vertices is contact interaction between two meson fields

\[
\mathcal{L}^{\pi\pi} = \frac{1}{4\pi^2 f_\pi^2} \left( 1 - \frac{2c}{g} \right)^2 \partial^{\mu \nu} \pi^i \partial^{\rho \nu} \pi^i 
\]
\[ \mathcal{L}^{\pi\alpha} = \frac{f_{\pi}}{2\pi^2 g f_{\pi}} (1 - \frac{2c}{g}) \partial_{\mu\nu} \pi^i \partial^\mu \alpha^i \nu \],

(14)

\[ \mathcal{L}^{\alpha\alpha} = \frac{f_{\alpha}^2}{4\pi^2 g^2} (\partial_{\mu} \alpha^{i\mu})^2. \]

The third kind of vertices is the direct interaction between \( \pi\pi\rho \rho \)

\[ \mathcal{L}^{\pi\pi\rho\rho} = \frac{4}{f_{\pi}^2} \epsilon_{ijk} \epsilon_{ij'k'} \{ \frac{F^2}{2g^2} \rho_i^j \rho_i^{j'} \pi_k \pi_{k'} + \left[ -\frac{2c^2}{g^2} + \frac{3}{4\pi^2 g^2} (1 - \frac{2c}{g})^2 \right] \rho_i^j \rho_i^{j'} \partial_{\mu} \pi_k \partial_{\nu} \pi_{k'} - \frac{2c^2}{g^2} \rho_i^j \rho_i^{j'} \partial_{\nu} \pi_k \partial_{\mu} \pi_{k'} \right] 

+ \left[ -\frac{2c^2}{g^2} + \frac{1}{4\pi^2 g^2} (1 - \frac{2c}{g})^2 \right] \rho_i^j \rho_i^{j'} \partial_{\mu} \pi_k \partial_{\nu} \pi_{k'} - \frac{2c^2}{g^2} \rho_i^j \rho_i^{j'} \partial_{\mu} \pi_k \partial_{\nu} \pi_{k'} \}

+ \frac{3}{2\pi^2 g^2} (1 - \frac{2c}{g}) (\rho_i^j \pi_k \partial_{\nu} \pi_{j'} - \rho_i^j \pi_k \partial_{\mu} \pi_{j'}) \partial_{\nu}^\mu \rho_k \pi_{j'} - \frac{1}{2\pi^2 g^2} (1 - \frac{2c}{g}) \rho_i^j \pi_k \pi_{j'} \partial_{\nu} \pi_{k'}

+ \frac{1}{4\pi^2 g^2} \left[ \partial_{\nu} (\rho_i^j \pi_k) \partial_{\nu} (\rho_i^{j'} \pi_{k'}) + 2(1 - \frac{2c}{g}) \rho_i^j \pi_k \rho_i^{j'} \partial_{\nu} \pi_{k'} \right]. \]

(16)

From these vertices(3-12) the physical processes contributing to \( \tau \to \rho\pi\pi\nu \) are obtained and shown in Fig.1. Similar diagrams involving the vertices(13-15) can be drawn too. The decay \( \tau \to 4\pi\nu \) originates in \( \tau \to \rho\pi\pi\nu \).

In terms of Eq.(1) the vector current is derived from the vertices(3-16). In the chiral limit this vector current must be conserved. Derivative expansion is taken in the effective theory[1]. The vertices(3-16) are derived up to the fourth order in derivatives. Due to the anomaly the vector current conservation is guaranteed in Fig.(1e). It is proved that the vector current derived from Eqs.(3-11,13-15) is indeed conserved in the chiral limit. The proof of vector current conservation shows that the presence of the vertices(13-15) is necessary to
make the vector current conserved.

There are four decay channels \( \tau^- \rightarrow \rho^0 \pi^- \pi^0 \nu, \rho^- \pi^+ \pi^- \nu, \rho^- \pi^0 \pi^0 \nu, \rho^+ \pi^- \pi^- \nu \).

The matrix element of the vector current obtained from the diagram(fig.1(e)) is expressed as

\[
\langle \rho_l(p)\pi_m(p_1)\pi_n(p_2)|V^*_{\mu}|0 \rangle = \frac{1}{\sqrt{8E\omega_1\omega_2}} \frac{g}{\sqrt{2}} (\delta_{l1} - i\delta_{l2}) \delta_{jl} \epsilon_{\mu\nu}^\alpha \epsilon_{\nu}^\beta \epsilon_{\nu}^\gamma \epsilon_{\nu}^\delta \frac{m^2 - \rho^2 + i\sqrt{q^2\Gamma_\rho(q^2)}}{q^2 - m^2 - i\sqrt{q^2\Gamma_\rho(q^2)}} \{ \delta_{lm} \delta_{jn} \frac{p_{1\beta}p_{2\beta}}{k_1^2 - m_\omega^2} + \delta_{ln} \delta_{jm} \frac{p_{2\beta}p_{1\beta}}{k_2^2 - m_\omega^2} \}, \quad (17)
\]

where \( k_1 = q - p_1 \) and \( k_2 = q - p_2 \).

The matrix element of the vector current obtained from the diagrams(fig.1(a-d)) and the diagrams involving the vertices(13-15) is written as

\[
\begin{align*}
\langle \rho_l(p)\pi_m(p_1)\pi_n(p_2)|V^*_{\mu}|0 \rangle &= \frac{g}{\sqrt{8E\omega_1\omega_2}} \frac{1}{\sqrt{2}} (\delta_{l1} - i\delta_{l2}) \delta_{jl} \epsilon_{\mu\nu}^\alpha \epsilon_{\nu}^\beta \epsilon_{\nu}^\gamma \epsilon_{\nu}^\delta \frac{m^2 - \rho^2 + i\sqrt{q^2\Gamma_\rho(q^2)}}{q^2 - m^2 - i\sqrt{q^2\Gamma_\rho(q^2)}} \\
&\{ \delta_{km} \delta_{k'n'} [ (g_{\mu
u} - \frac{q_{\mu}q_{\nu}}{q^2}) f + (p_{1\mu} - \frac{p_1\cdot q}{q^2} q_{\mu}) (f_{11}p_{1\nu} + f_{12}p_{2\nu}) + (p_{2\mu} - \frac{p_2\cdot q}{q^2} q_{\mu}) (f_{12}p_{1\nu} + f_{22}p_{2\nu}) ] \\
&+ \delta_{kn} \delta_{k'm'} [ (g_{\mu
u} - \frac{q_{\mu}q_{\nu}}{q^2}) f' + (p_{2\mu} - \frac{p_2\cdot q}{q^2} q_{\mu}) (f'_{11}p_{2\nu} + f'_{12}p_{1\nu}) + (p_{1\mu} - \frac{p_1\cdot q}{q^2} q_{\mu}) (f'_{12}p_{2\nu} + f'_{22}p_{1\nu}) ] \}
\end{align*}
\]

where

\[ q = p + p_1 + p_2, \]

where \( f' \) and \( f'_{ij} \) are obtained from \( f \) and \( f_{ij} \) by exchanging \( p_1 \leftrightarrow p_2 \), \( \Gamma_\rho \) is the decay width of \( \rho \) meson of momentum \( q \)

\[
\Gamma_\rho(q^2) = \frac{q^2}{12\pi g^2 m_\rho} \{ 1 + \frac{q^2}{2\pi^2 f_\pi^2} [(1 - \frac{2c}{g})^2 - 4\pi^2 c^2] \} \{ 1 - \frac{4m_\omega^2}{q^2} \}^2. \quad (19)
\]
The contributions of $a_1$ meson (fig. 1b) to the functions $f$ and $f_{ij}$ are presented below. The contributions from other diagrams are shown in the appendix.

$$BW(k_1^2) = \frac{1}{k_1^2 - m_a^2 + i\sqrt{k_1^2\Gamma_a(k_1^2)}}$$

$$f = BW(k_1^2)A(q^2, k_1^2)A(m_\rho^2, k_1^2),$$

$$f_{11} = BW(k_1^2)A(m_\rho^2, k_1^2)B,$$

$$f_{12} = BW(k_1^2)\{[A(m_\rho^2, k_1^2) + A(q^2, k_1^2)]D + (k_1 \cdot p_2 - k_1 \cdot p_1)BD + p_1 \cdot p_2 B^2 - k_1^2 D^2\}$$

$$- BW(k_1^2)\frac{1}{m_a^2}\{-A(q^2, k_1^2) + k_1 \cdot p_1 B + k_1^2 D\}\{A(m_\rho^2, k_1^2) + k_1 \cdot p_2 B - k_1^2 D\},$$

$$f_{22} = BW(k_1^2)A(q^2, k_1^2)B,$$

(20)

The decay width of $a_1$ meson is derived as

$$\Gamma_a(k_1^2) = \frac{k_a}{12\pi} \frac{1}{\sqrt{k_1^2 m_a}} \{ (3 + \frac{k_1^2}{m_\rho^2})A^2(m_\rho^2, k_1^2) - 2A(m_\rho^2, k_1^2)B(k_1^2 + m_\rho^2)\frac{k_1^2}{m_\rho^2} + \frac{k_1^2 m_\rho^2}{m_\rho^2} k_1^2 B^2 \}. $$

$$k_a^2 = \frac{1}{k_1^2} (k_1^2 + m_\rho^2 - m_\rho^2) - m_\rho^2. $$

(21)

In the matrix elements (17,18) there are three parameters: $f_\pi$, $m_\rho$, and $g$. Obviously, the first two have been fixed, and the third one is determined from $\rho \rightarrow ee^+$ to be $g = 0.39$.

The mass of the $a_1$ meson has been derived as

$$ (1 - \frac{1}{2\pi^2 g^2}) m_a^2 = \frac{F^2}{g^2} + m_\rho^2. $$

Therefore, this study is parameter free.
The branching ratios of the four decay channels are calculated

\[ B(\tau \to \rho^0 \pi^0 \pi^- \nu) = B(\tau \to \rho^- \pi^+ \pi^- \nu) = 1.02\% \]

\[ B(\tau \to \rho^- \pi^0 \pi^0 \nu) = B(\tau \to \rho^+ \pi^- \pi^- \nu) = 1.27\%. \]  \hspace{1cm} (22)

The experimental data\[8\] are

\[ B(\tau \to h^- \rho \pi^0 \nu) = (1.33 \pm 0.20)\% \hspace{1cm} B(\tau \to h^- \rho^- h^+ \nu) = (1.15 \pm 0.23)\%, \]

\[ B(\tau \to h^- \rho^+ h^- \nu) = (4.4 \pm 2.2) \times 10^{-3} \hspace{1cm} B(\tau \to h^- (\rho \pi)^0 \nu) = (2.84 \pm 0.34)\%. \]

Except for the \( h^- \rho^+ h^- \) mode the theoretical results are in agreement with the data.

It is necessary to point out that in order to have the current conservation all the diagrams must be taken into account. However, the calculations show that the \( a_1 \) meson(fig1(b)) dominates the four decay channels. If only the diagram(fig.1(b)) is kept in the matrix elements(16) it is obtained that

\[ B(\tau \to \rho^- \pi^- \pi^+ \nu) = 1.25\%. \]

The dominance of the \( a_1 \) meson in other three channels is true too. There are two reasons for this dominance:

1. The mass of the \( a_1 \) meson is the energy range of the decay \( \tau \to \rho \pi \pi \nu \),
2. the strong coupling between \( a_1 \) and \( \rho \pi(3) \) which leads to the wide decay width of the \( a_1 \) meson\(^4\).

The distribution functions, \( \frac{d\Gamma}{dq^2} \), are calculated and shown in fig.2 and fig.3.

![Graph 1](image1)

![Graph 2](image2)
In conclusion, the branching ratios and distribution functions of the four decay modes of $\tau \to \rho \pi \pi \nu$ have been calculated in terms of the effective chiral theory of mesons[1]. The study is parameter free and theoretical results agree with the data. The theory predicts the $a_1$ dominance in these four decay modes.

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Appendix

Contributions of the diagrams to the matrix element(18):

1. diagrams involving the vertices(13-15)

$$f_{12} = -\frac{1}{\pi^2 g^2 f_\pi^2} \left\{ \frac{4c}{g} \left(1 - \frac{c}{g}\right) + \frac{1}{\pi^2 g^2} (1 - \frac{2c}{g})^2 + 2(1 - \frac{2c}{g}) \left[ 1 + \frac{m^2_\rho}{2\pi^2 f_\pi^2} (1 - \frac{2c}{g})^2 - 4\pi^2 c^2 \right] \right\}$$

$$\left\{ 4 \left(1 - \frac{c}{g}\right) + \frac{2q^2}{\pi^2 f_\pi^2} (1 - \frac{2c}{g})^2 + \frac{2k^2_1}{\pi^2 f_\pi^2} \left(1 - \frac{c}{g}\right) (1 - \frac{2c}{g}) \right\} - \frac{4k^2_1}{\pi^2 g^2 f_\pi^2} \left(1 - \frac{c}{g}\right)^2 \left(1 - \frac{2c}{g}\right)^2.$$  (23)

2. diagrams(fig.1(c))

$$f_{12} = \frac{1}{k^2_1 g^2} \left\{ 1 + \frac{q^2}{2\pi^2 f_\pi^2} [(1 - \frac{2c}{g})^2 - 4\pi^2 c^2] + \frac{k^2_1}{2\pi^2 f_\pi^2} (1 - \frac{c}{g})(1 - \frac{2c}{g}) \right\}$$

$$\left\{ 1 + \frac{m^2_\rho}{2\pi^2 f_\pi^2} [(1 - \frac{2c}{g})^2 - 4\pi^2 c^2] \right\} + \frac{8}{\pi^2 g^2 f_\pi^2} \left(1 - \frac{c}{g}\right) (1 - \frac{2c}{g}).$$  (24)

3. diagrams(fig.1(a))

$$f = \frac{4}{f_\pi^2} \left( \frac{F^2}{g^2} + (q^2 + m^2_\rho) \left[ \frac{2c^2}{g^2} - \frac{1}{4\pi^2 g^2} (1 - \frac{2c}{g})^2 + \frac{3}{4\pi^2 g^2} (1 - \frac{2c}{g}) \right] \right)$$
\[ k_1^2 \left\{ \frac{-2c^2}{g^2} + \frac{1}{4\pi^2g^2}(1 - \frac{2c}{g})^2 - \frac{3}{2\pi^2g^2}(1 - \frac{2c}{g}) + \frac{1}{2\pi^2g^2} \right\} + k_2^2 \left\{ \frac{-2c^2}{g^2} + \frac{1}{4\pi^2g^2}(1 - \frac{2c}{g})^2 \right\}, \]

\[ f_{11} = -\frac{2}{\pi^2g^2f_\pi^2}(1 - \frac{2c}{g}), \]

\[ f_{12} = -\frac{8}{f_\pi^2} \left\{ -\frac{2c^2}{g^2} + \frac{3}{4\pi^2g^2}(1 - \frac{2c}{g})^2 + \frac{3}{2\pi^2g^2}(1 - \frac{2c}{g}) \right\}, \]

\[ f_{21} = \frac{8}{f_\pi^2} \left\{ -\frac{4c^2}{g^2} + \frac{3}{4\pi^2g^2}(1 - \frac{2c}{g}) \right\}, \]

\[ f_{22} = f_{11}. \] (25)

4. diagrams(fig.1(d))

\[ BWR = \frac{1}{k_3^2 - m_\rho^2 + i \sqrt{k_3^2} \Gamma_\rho(k_3^2)} \]

\[ f = \frac{4}{g^2} BWR(k_1^2 - k_2^2) \left\{ 1 + \frac{p_1 \cdot p_2}{\pi^2f_\pi^2} \left[ (1 - \frac{2c}{g})^2 - 4\pi^2c^2 \right] \right\}; \]

\[ f_{12} = \frac{16}{g^2} BWR \left\{ 1 + \frac{p_1 \cdot p_2}{\pi^2f_\pi^2} \left[ (1 - \frac{2c}{g})^2 - 4\pi^2c^2 \right] \right\}, \]

\[ f_{21} = -f_{12}. \] (26)

where \( k_3 = q - p \).

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Figure Captions

Fig. 1 Diagrams of the decay $\tau \rightarrow \rho \pi \pi \nu$

Fig. 2 Distribution function of $\tau \rightarrow \rho^0 \pi^0 \pi^- (\rho^- \pi^- \pi^+) \nu$

Fig. 3 Distribution function of $\tau \rightarrow \rho^+ \pi^- \pi^- (\rho^- \pi^0 \pi^0) \nu$
