Motion of halo compact objects in the gravitational potential of a low-mass model of the Galaxy

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ABSTRACT

Recently, we derived a lower bound for the Galaxy mass in the approximation of a point mass potential, assuming a spherical symmetric ensemble of test bodies representing compact objects of the halo. This result was obtained for a representative of most general phase-space distribution functions consistent with the measured radial velocity dispersion, assuming no constraints on the form of the velocity dispersion anisotropy parameter. In this paper we make use of the representative phase function to set the initial conditions for a simulation of test bodies in a more realistic gravitational potential with the same total mass. The predicted radial velocity dispersion profile evolves to forms still consistent with the measured profile, proving structural stability of the point mass approximation and the reliability of its lower bound estimate for Galaxy mass of about \(2.1 \times 10^{11} \, M_\odot\) within 150 kpc. We derive also a relationship holding in spherical symmetry between the radial velocity dispersion profile and a directly measured kinematical observable.

Subject headings: techniques: radial velocities - Galaxy: halo - Galaxy: disk - Galaxy: kinematics and dynamics - Galaxy: fundamental parameters - methods: numerical

1. Introduction

The motion of a system of compact objects orbiting the Milky Way can be used to infer the gravitational field for large Galactocentric radii. The method of Jeans modeling applied to this system enables one to link the gravitational potential with the available observational quantities, under the important assumption that the compact ingredients of the galactic halo form a collision-less system of test bodies in a steady-state equilibrium (Jeans 1915). The primary notion of this framework is the Phase-space Distribution Function (PDF). The physical observables are understood as secondary quantities being PDF-dependent functionals defined on the phase space (such as the number density, the eigenvalues associated with the main directions of the dispersion ellipsoid, the mean transversal velocity, etc).

At large distances, the Galactic gravitational potential can be described, to a fair degree of approximation, as a point mass potential. The crucial role in this approximation is played by a single parameter – the total mass, which is decisive for the asymptotics of the gravitational field of any compact mass distribution. In this approximation Bahcall & Tremaine (1981) proposed for the neighboring galaxies a simple estimator of their total masses in the form \(C \langle v^2 R \rangle\) (in cylindrical coordinates), where the averaging is performed over distant velocity tracers and \(C\) is a constant. More recently, Watkins et al. (2010) considered a version of this estimator with an arbitrary power of the radial distance, \(C \langle v^2 r^\gamma \rangle\), and applicable in spherical symmetry. With estimators of this kind, many authors attempted to estimate the value of the total mass based on the Radial Velocity Dispersion (RVD) data. The values given in the literature differ between each other. Among them, one can find low values of \(5.4^{+0.2}_{-1.0} \times 10^{11} M_\odot\) for a mass within 50 kpc given by Wilkinson & Evans (1999), the values of \(5.5^{+0.3}_{-0.2} \times 10^{11} M_\odot\) by Sakamoto et al. (2003) and \(4.2 \times 10^{11} M_\odot\) obtained by Deason et al. (2012a), both within 50 kpc, or \(5.8 - 6.0 \times 10^{11} M_\odot\) enclosed within 100 kpc given by Klypin et al. (2002). On the other hand, one also can find values higher than \(10^{12} M_\odot\) inferred by Battaglia et al. (2005) based on tracers within 150 kpc.

Any of the estimates above is related to a PDF restricted
in a particular way by some indirect constraints appearing because of the additional assumptions made about the secondary quantities. The restrictions usually concern the flattening of the velocity dispersion ellipsoid \( \beta \) which in general can be a position-dependent function. This quantity is hardly determinable for peripheral velocity tracers. Most often, to overcome this difficulty, \( \beta \) is assumed to be a constant parameter, the same for all positions. We consider this assumption as too much constraining. Any constraint on \( \beta \) indirectly imposes restrictions on the function space admissible for PDF's consistent with that \( \beta \) form, by which the lower bound for the mass functional defined on the phase space may increase and lead to overestimating the mass for a given RVD. We discussed this possibility in (Bratek et al. 2014). There is a hope that the future observational techniques, making possible measurements of proper motions of peripheral Galaxy satellites, will improve the halo models significantly and enable better determination of \( \beta \) function.

Not less important than the improvement in the observational techniques is the advancement in the theoretical methods making use of the measurement data. Recently, there is a growing interest in methods of determining the general assumption-free PDF's from the kinematical data, even for models with simple potentials. Magorrian (2014) proposed a framework in which the gravitational potential can be inferred from a discrete realization of the unknown distribution function by using the snapshots of galaxy’s stellar kinematics.

Our previous article (Bratek et al. 2014), where we proposed a method of determining a most general PDF from a given spherically symmetric RVD profile without imposing any constraints on the secondary quantities, is placed within this field of interest. The method is based on an orthogonal decomposition of the PDF, with the expansion coefficients adjusted so as to faithfully reconstruct the shape of the variable RVD profile, including its low-size and variable features. We applied it in the point mass approximation, however, the method could be generalized to systems with integrable (or approximately integrable) orbits, such as for tracers orbiting galactic black holes. Using this technique, we showed in this approximation for various PDF's with accurate reconstruction of the RVD of distant velocity tracers that, as far as there is no information about the transversal motions, there is no upper bound for the central mass. More importantly, we showed that there is a sharp lower bound for the Galaxy mass, a little lower than \( 2.0 \times 10^{11} M_\odot \). Below this lower bound no PDF could be found with the resulting RVD overlapping the measured one within the acceptable limits. We also conjectured that the lower bound may increase in response to additional constraints imposed on the PDF. In this paper we will also be assuming that only the radial velocity component in the LSR frame can be determined simultaneously for all velocity tracers.

1.1. The aim of the present work

The lower bound referred to above is small compared to Galaxy masses given in the literature. Thus, a natural question arises whether such a low mass is reliable and may appear in other models? In answering this question, a more realistic gravitational potential than the point mass field should be considered. But for more complicated potentials the distribution integral on the phase space cannot be explicitly constructed on account of the first integrals characterizing the admissible orbits not being known. To overcome this difficulty, a numerical simulation can be performed. In this paper we present an example of such a simulation in which test bodies represent compact objects of the Galactic halo. We use the simulation to test whether the predictions for \( \langle v_z^2 \rangle \) and total mass in the new potential are comparable with those made in the point mass approximation. In other words, we are testing structural stability of these predictions.

The essential problem we are faced with when trying to perform such a simulation is the choice of the initial conditions. They should be close to a known stationary solution of Jeans problem in the point mass potential. We therefore make use of a PDF found with the help of the method by Bratek et al. (2014). As for the central mass we choose a reference value \( M_{\text{ref}} = 1.8 \times 10^{11} M_\odot \) for two important reasons. Firstly, it is very close to the lower bound we have found. Secondly, it is consistent with the asymptotic mass of the gravitational field of a Galaxy model consisting: a) of a finite thickness disk with the column mass density derived from the Galactic rotation curve by iterations (Jalocha et al. 2014; Sikora et al. 2012), and b) of a spherically symmetric distribution of hot gas encircling the Galaxy, with a mass function inferred from observations of the \( O_\alpha K_\alpha \) absorption lines in the quasar spectra (Miller & Bregman 2013). This improved model approximates the Galactic gravitational field better than the point mass potential. We use it as the background potential for our simulation.

The structure of this paper is the following. In Sect. 2 we repeat briefly the main ideas behind the Keplerian ensemble method of obtaining a PDF in a point mass approximation. Next, in Sect. 3, we use a representative of possible PDF’s to set the initial condition for our n-body simulation, and we test the stability of the resulting predictions for the secondary quantities. Then conclusions follow.

2. A first approximation of PDF in the Keplerian Ensemble Method

2.1. Mathematical preliminaries.

This subsection summarizes the mathematical basis of our method. For a more detailed discussion we refer to our previous work (Bratek et al. 2014).

An elliptic orbit of a test particle bound in the gravitational field of a point mass \( M \) is fully characterized by five integrals of motion being functions of five parameters: the three Euler angles \( (\Phi, \Theta, \Psi) \) which determine the orbit orientation, the eccentricity \( e \) describing the orbit flattening, and the dimensionless energy parameter \( \epsilon = -\frac{E}{GM} \) describing the size of the large semi-axis \( (R \) is an arbitrary unit of length while \( E \) is the energy per unit mass). To ensure spherical symmetry we consider a spherically...
symmetric collection of confocal orbits called the Keplerian ensemble. From the Jeans theorem (Jeans 1915) it follows that PDF of a system in a steady state equilibrium is a function of isolating integrals. Accordingly, we look for a PDF expressed in terms of two first integrals $e$ and $\epsilon$.

This is done by applying the following mapping from the ordinary spherical coordinates $r, \theta, \phi, v_r, v_\theta, v_\phi$ to new coordinates $u, \theta, \phi, e, \epsilon, \psi$ defined by:

$$
\begin{align*}
r &= Ru, \\
v_r^2 &= \frac{GM}{R} \left( 2 - \frac{1 - e^2}{2 \epsilon u^2} - 2 \epsilon \right), \\
(v_\theta, v_\phi) &= \frac{1}{u} \sqrt{\frac{GM}{R}} \left( \frac{1 - e^2}{2 \epsilon} \right)^{1/2} \sin \psi (\cos \psi, \cos \psi).
\end{align*}
$$

(1)

Here $u$ is a dimensionless radial variable.

It is assumed for physical reasons that all orbits are contained entirely between two boundary spheres of radius $u_a < r/R < u_b$, that is, $u \in (u_a, u_b)$. This way all test particles that either would approach the Galaxy center too closely (where the point mass approximation is violated) or escape too far away (e.g. Local Group members), are excluded. The space of parameters $(e, \epsilon)$ for which the condition $u \in (u_a, u_b)$ is satisfied is restricted to a domain $S$ defined by the inequalities $\frac{u_a - u}{u_b - u} < \epsilon < \frac{1 + u}{2u}$ and $0 \leq \epsilon < \frac{u_b - u_a}{u_b + u_a} < 1$. On integrating out the angles $\theta, \phi, \psi$, the phase space principal integral $\int f(\vec{r}, \vec{v}) d^3\vec{r} d^3\vec{v}$ reduces then (to within a constant factor) to a triple integral

$$
\int_{u_a}^{u_b} \nu_u[f] \, du,
$$

where

$$
\nu_u[f] = \int_{S(u)} \frac{e \, d\epsilon \, d\theta}{\sqrt{\epsilon \left( \frac{1 - e - \epsilon}{2u} \right) (1 + \frac{u_b - u}{u_b + u} - \epsilon)}}.
$$

(2)

The integration domain $S(u) \subset S$ is a $u$-dependent quadrilateral region each point of which corresponds to a spherically symmetric pencil of conical elliptical orbits intersecting a sphere of a given radius $R \, u$, and the energy parameter of such orbits must satisfy the inequality $\frac{1 - e}{2u} \leq \epsilon \leq \frac{1 + u}{2u}$.

The functional $\nu_u[f]$ has the interpretation of the probability density for the variable $r/R$ to fall within a spherical shell $u < r/R < u + du$. Accordingly, given a distribution function $f(e, \epsilon)$, the expectation value $\langle g \rangle_r$ for an observable $g = g(e, \epsilon, u)$ measured inside that shell equals:

$$
\langle g \rangle_r = \frac{\nu_u[fg]}{\nu_u[f]}.
$$

(3)

In particular, for the RVD profile $(r v_r^2)/G$ the appropriate $g$ can be found from Eq. 1 and reads

$$
g(e, \epsilon, u) \equiv \frac{r v_r^2}{GM} = 2 - \frac{1 - e^2}{2 \epsilon u} - 2 \epsilon u.
$$

Given $M$ and $f(e, \epsilon, u)$, the RVD profile is obtained with the above expression substituted for $g$ in Eq. 3.\footnote{For technical reasons we found it more convenient in calculations of integrals like $\nu_u$ with $u$-dependent integration regions to use another set of coordinates $(\alpha, \beta, u)$ in place of $(e, \epsilon, u)$, such that $e = \frac{\beta - \alpha}{2 \sqrt{\alpha \beta}}$, $\epsilon = \frac{\beta - 1}{\sqrt{\alpha \beta}}$. The motivation behind this mapping and its explicit geometrical construction was given in (Bratek et al. 2014).}

But there is the inverse problem we are mainly interested in: given the RVD profile matching the observations, we want to derive a distribution function $f$ which the RVD profile would follow from. In answer to this question in (Bratek et al. 2014) we developed the following procedure. First, we consider an auxiliary function $h(e, \epsilon)$ such that $f = h^2$, which ensures that $f \geq 0$ as required for a probability density. Then, we make a function series expansion in a sequence of polynomials orthogonal on $S$:

$$
h(e, \epsilon) \approx \sum_{k=1}^{D_h} h_k Q_k(e, \epsilon).
$$

(4)

The consecutive elements of the orthogonal basis $Q_k$ we construct with the help of the Gramm-Schmidt orthogonalization method applied to the triangular region $S$. Next, given a mass parameter $M$ we find an optimum sequence of expansion coefficients $h_k$ by minimizing a discrepancy measure between $f$ and the experimentally established RVD profile defined as $\overline{r v_r^2}/G$ - where the averaging on the r.h.s. is understood as taken for a given $r$ over all halo compact objects within the corresponding spherical shell of some width, and $b$) the model $(g)_r$ profile calculated from Eq. 3 with the help of the function $h$ corresponding to the optimum $h_k$’s. This way a distribution function $f(e, \epsilon)$ consistent with the RVD measurements can be reconstructed. With these $h_k$’s it is possible to reduce the discrepancy measure further by replacing $M$ with a better fit value, e.g., $M \to M_{bf} = \sum_r (g)_r - \bar{p}_r)^2$ if the sum over $(g)_r$ is used.

2.2. RVD profile from measurements

To determine the RVD profile, the radial motions of kinematic tracers, measured with respect to the LSR frame, must be transformed to the Galactocentric frame. This cannot be done unambiguously without the transverse velocity components. However, for a spherically symmetric distribution of tracers one can try to assume a $\beta(r)$ profile or find a selfconsistent one by iterations.\footnote{The anisotropy function $\beta$ is a measure of the flattening of the velocity dispersion ellipsoid. The customary definition of $\beta$ is motivated by the form of a term in the Jeans equations. Provided that $\vec{v} \circ \vec{n} = 0$ for all directions $\vec{n}$, the form reduces in spherical coordinates to $\beta = (\langle v_\alpha^2 \rangle + \langle v_\beta^2 \rangle - 2 \langle v_\phi^2 \rangle) / 1 - \frac{1}{2} \langle v_\phi^2 \rangle$. In spherical symmetry, $\beta$ is a function of the radial variable only and $(\langle v_\phi^2 \rangle) = (\langle v_\phi^2 \rangle)$, then we write $\beta(r)$ instead of $\beta$.}

The particular model of $\beta(r)$ only affects the RVD inside a spherical region occupying a diameter of few $r_\odot$ distances, while it is not important for larger radii. But we must bear in mind a twofold influence of the particular model of $\beta$ on the total mass determination: both $\beta(r)$ itself and the so obtained $\beta$-dependent RVD profile enter the Jeans equations.

2.2.1. A formula relating the LSR radial motion measurements to the Galactocentric RVD

We consider a spherically symmetric ensemble of test bodies described by some PDF and the resulting $\beta(r)$, where we assume $\langle v_\phi^2 \rangle(r) = \langle v_\phi^2 \rangle(r)$ by spherical symmetry. Here, for every $r$ the averaging is taken over a sphere
of radius $r$. For a test body with the velocity vector $\vec{v}$ in the Galacto-centric coordinate frame, the radial and tangential components of $\vec{v}$ are $v_r = \vec{v} \circ \vec{e}_r$, $v_\theta = \vec{v} \circ \vec{e}_\theta$, $v_\phi = \vec{v} \circ \vec{e}_\phi$, with $\vec{e}_r = \frac{\vec{r}}{|\vec{r}|}$, $\vec{e}_\theta$, $\vec{e}_\phi$ forming an orthonormal basis tangent to the lines of constant spherical coordinates $r, \theta, \phi$. Although for closer objects $\vec{v}$ can be determined, only the direction of $\vec{v}$ along the line of sight from the Sun position, determined by the unit vector $\vec{e}_r = \frac{\vec{r}}{|\vec{r}|}$ can be measured for all objects in the sample and it reads $\vec{v}_r = \vec{v} \circ \vec{e}_r$. This is the only kinematical information available at large distances, suitable for constraining the total Galactic mass through the behavior of the resulting radial dispersion function $\langle \upsilon_r^2(r) \rangle$. It is connected with the direct measurement of the LSR relative velocity $v_\odot$ along the direction $\vec{e}_r$ through the relation $\vec{v}_r = v_\odot + \vec{v}_\odot \circ \vec{e}_r$. This information is insufficient for unambiguous transformation of the measured $v_\odot$ and so $\vec{v}_r$ onto the actual $v_r$ and then obtaining the Galacto-centric $\langle \upsilon_r^2 \rangle$. However, for the isotropic dispersion, $\beta(r) = 0$, we can make the (incorrect) identification $\vec{v}_r = v_r$ (valid only approximately at large distances where $\vec{e}_r \approx \vec{e}_r$) without making any error in equating the resulting dispersions $\langle \upsilon_r^2 \rangle = \langle \upsilon_r^2 \rangle$. When $\beta(r)$ is not known, the identification is impossible. However, under spherical symmetry, assuming a $\beta(r)$ allows us to relate $\langle \upsilon_r^2(r) \rangle$ to $\langle \upsilon_r^2(r) \rangle$ through the identity true both for $r < r_\odot$ and $r > r_\odot$:

$$\langle \upsilon_r^2(r) \rangle = \langle \upsilon_r^2(r) \rangle \left(1 - \frac{\beta(r)}{4} H(r)\right)$$

$$H(r) = 1 - \frac{(r^2 - r_\odot^2)^3}{2 r_\odot^3 \ln \frac{r + r_\odot}{r - r_\odot}} + \frac{r_\odot^3}{r^2}$$  \hspace{1cm} (5)

A derivation of this result is given in the appendix A.\(^3\) As noted earlier, $\langle \upsilon_r^2(r) \rangle = \langle \upsilon_r^2(r) \rangle$ only for the globally isotropic case ($\beta = 0$), and for large distances $H \sim 1 - \beta(r) \frac{r_\odot^3}{r^2} \rightarrow 1$ if $\beta$ is asymptotically bound (which may not hold for nearly circular orbits).

When the $\langle \upsilon_r^2(r) \rangle$ observable is used to determine a PDF $f = f(\vec{r}, \vec{v})$ within a model, then a self-consistent $\beta(r)$ may be looked for by iterations. In the first step of the recursion process we make the isotropic assumption $\langle \upsilon_r^2(r) \rangle = \langle \upsilon_r^2(r) \rangle$ as if $\beta(r) = 0$, then we use $\langle \upsilon_r^2(r) \rangle$ to determine the first approximation to $f$. With this $f$, we calculate the prediction for $\beta(r)$ for the next iteration step, and use that $\beta(r)$ in Eq. 5 to obtain a new $\langle \upsilon_r^2(r) \rangle$ from the measurement $\langle \upsilon_r^2(r) \rangle$. And the process is repeated until a stable $\beta(r)$ is reached. However, as will be seen from the numerical simulation in Sect. 3.4, the distinction between $\langle \upsilon_r^2(r) \rangle$ and $\langle \upsilon_r^2(r) \rangle$ is practically unimportant, unless the lower radii region is considered (which is not the case in this paper). For this reason, in preparing the RVD profile below, based on the measurements in the LSR frame, we could neglect this distinction.

\(^3\)Battaglia et al. (2005) derive a similar result, however, they conclude counter to the geometric intuition, that $\langle \upsilon_r^2 \rangle$ (and $\langle \upsilon_r^2 \rangle$) coincide for purely radial anisotropic ellipsoid ($\beta = 1$), not for purely isotropic one ($\beta = 0$). To dispel doubts as to which conclusion is correct we present an independent derivation of the relation between $\langle \upsilon_r^2 \rangle$ and $\langle \upsilon_r^2 \rangle$.

2.2.2. Measurements data

In our previous work (Bratek et al. 2014) we determined a RVD profile which we now assume as the basis for generating the initial conditions for the simulation in Sect. 3.1. We obtained this profile with the use of several position-velocity data: the halo giant stars catalogs (Dohm-Palmer et al. 2001; Starkenburg et al. 2009) based on the Spaghetti Project Survey (Morrison et al. 2000); the blue horizontal branch stars database (Clewley et al. 2004) from the United Kingdom Schmidt Telescope observations and SDSS; the field horizontal branch and A-type stars database (Wilhelm et al. 1999) based on the Beers et al. (1992) survey: the globular clusters (Harris 1996) and the dwarf galaxies (Mateo 1998) catalogs. The data was recalculated to epoch J2000 when necessary. In addition, we included the ultra-faint dwarf galaxies such as Ursa Major I and II, Coma Berenices, Canes Venatici I and II, Hercules (Simon & Geha 2007), Bootes I, Willman 1 (Martin et al. 2007), Bootes II (Koch et al. 2009), Leo V (Belokurov et al. 2008), Segue I (Geha et al. 2009), and Segue II (Belokurov et al. 2009).

To eliminate a possible decrease in the RVD at lower radii due to circular orbits in the disk, we excluded tracers in an ellipsoidal neighborhood of the mid-plane. We also did not take into account: a) Leo T located at $r > 400$ kpc, away from the other tracers, thus unsuitable in preparing the RVD profile, b) Leo I rejected for reasons largely discussed in (Bratek et al. 2014), and c) a single star for which $r \upsilon_r^2/(2G) > 5.6 \times 10^{11} M_\odot$. We rejected 4 additional objects for which $r \upsilon_r^2/(2G) \geq 3.5 \times 10^{11} M_\odot$. Had we not excluded them, the total expected mass would be increased by only a factor of $\approx 1.16$, which we showed in (Bratek et al. 2014) with the help of a simple asymptotic estimator. This factor is comparable to, or even less than, relative errors for the Milky Way mass in the literature.

In Fig. 1 we present the resulting RVD profile. We will use it as a reference profile for comparison with the simulation results in Sect. 3.4.

2.3. Obtaining a representative PDF from a given RVD profile.

By applying the method of Sect. 2 to the RVD profile in Fig. 1 for a range of central masses $M$, each time obtaining a PDF, we found in (Bratek et al. 2014) that there is a limiting value $M_{cut} \approx 2.0 \times 10^{11} M_\odot$ above which one can find a PDF for which the RVD profile is perfectly reconstructed, while much below this limit no satisfactory fit can be obtained. For $M > M_{cut}$, increasing $D_r$ (the number of the basis polynomials $Q_a$ in Eq. 4) efficiently decreases the fit residuals, but for $D_r$ high enough the residuals appear to tend to some small nonzero limit. However, for $M < M_{cut}$ the fit residuals remain very large regardless of the number of basis polynomials taken and rapidly increase with decreasing $M$. This observation led us to the conclusion that $M_{cut}$ is the lower bound for the Galaxy mass (in the point mass approximation). Motivated by this, we choose an example PDF solution with $M_{ref} = 1.8 \times 10^{11} M_\odot$, which is
close to the lower bound $M_{\text{cut}}$ (if the model RVD in the new potential turned out inconsistent with the reference PDF, the total mass, and accordingly the initial data, could be suitably rescaled relative to $M_{\text{cut}}$). In Fig. 2 we show the PDF on the $(e, \epsilon)$ plane. We take it as a starting point for the further analysis.

3. Simulation of RVD for test bodies in a background field

3.1. Procedure for preparing the initial conditions

The first stage toward determining the initial conditions corresponding to the PDF $f(e, \epsilon)$ shown in Fig. 2, is to generate a random set of initial radii $I_0 = \{u_i\}_{i=1}^N$ in the range $u_a < u_i < u_b$, and with the number density described by the integral $\nu_a [f]$ in Eq. 2. Here, $N$ stands for the number of all test bodies. This task, as well as similar problems of generating random numbers non-uniformly distributed in a range with a tabulated probability density $\nu (u)$ whose analytic form is unknown – as in our case with $\nu (u)$ being identified with a functional $\nu_a [f]$ – can be solved as follows. Once we find an approximate function interpolating an array of $P$ pairs $\{(u_p, \nu_a [f])\}_{p=1}^P$ ordered w.r.t the first argument, we can compute, by numerical integration, an array consisting of triples $\{(x_p = \int_{u_p}^{u_b} \nu_a [f] \, du, u_p, \nu_a [f] = (x_p)^{-1} = (\nu_a [f])^{-1})\}_{p=1}^P$.

and finally, by a suitable procedure interpolating between the knot points $u_p$, we obtain a smooth function $u(x)$. With a version of quartic spline interpolation, we can obtain a $u(x)$ smooth with continuous derivatives up to fourth order.

Indeed, apart from the tabulated $u(x)$, we know also the tabulated derivative $u' (x) \equiv 1/x' (u)$, in addition to which we may impose the requirement that the second derivatives of the adjacent polynomials defined on both sides of knot points $u_p$ should be equal while the fourth derivative be 0. Going on further, provided that $\nu_a [f]$ has been normalized to unity, $\int_{u_a}^{u_b} \nu_a [f] \, du = 1$, the smooth function $u(x)$ maps random numbers $x$ uniformly distributed in the range $(0, 1)$ to random numbers $u$ with the required nonuniform distribution $\nu (u)$ in the range $(u_a, u_b)$. At this point it may be instructive to note that $u(x)$ is the inverse function of the cumulative probability function $\chi (u) \equiv \int_{u_a}^{u} \nu (\tilde{u}) \, d\tilde{u}$ corresponding to probability density $\nu (u): x = \chi (u)$.5

In the next stage of finding the initial data, we have to account for the spherical symmetry assumption of the initial state. Accordingly, we assign to $I_0$ a set $\{(\theta_i, \phi_i)\}_{i=1}^N$ of spherical coordinates of directions uniformly distributed on the unit sphere. From the general result in footnote 5 it follows that if $\theta = \arccos (1 - 2x)$ and $\phi = 2\nu y$, then $(\theta, \phi)$ will be uniformly distributed on the unit sphere once both $x$ and $y$ are uniformly distributed on independent unit intervals. This gives us the initial positions $I_1 = \{(u_i, \theta_i, \phi_i)\}_{i=1}^N$.

In the third stage, by suitably choosing random parameters $(e, \epsilon, \psi)$ consistently with the initial PDF, we can as-
cribe to each \((u_k, \theta_k, \phi_k) \in I\) an elliptic orbit which a particular test body starting from the initial position would follow in the point mass potential. This will indirectly give us the initial velocities. To this end, we introduce an auxiliary variable \(X\) and a number \(f_S = \max\{f(e, \epsilon); (e, \epsilon) \in S\}\), and consider triples of random numbers \((e, \epsilon, X)\) uniformly distributed in their respective range: \(e \in (0, 1), \epsilon \in (0, 1/(2u_0))\) and \(X \in (0, f_S)\). Now, for each \(u_k \in I_0\) we go on with generating random triples \((e, \epsilon, X)\) until we encounter one which we label with a subscript \(k\) and for which both \(X < f(e_k, \epsilon_k)\) and \((e_k, \epsilon_k) \in S(u_k)\). Had we not stopped this procedure for a given \(u_k\), or considered pairs \((e_k, \epsilon_k)\) corresponding to a large number of various \(u_k's\) from a small range \(|u_i - u_k| < \delta u_k\) obtained in the same manner as \(u_k\), we would have seen their number being proportional to \(f(e_k, \epsilon_k)\). The procedure thus gives a set of random pairs \(I_2 = \{(e_i, \epsilon_i)\}\) with a non-uniform number density distribution \(f(e, \epsilon)\) and with each pair belonging to a \(u_k\)-dependent region \(S(u_k)\). With every \(u_k's\) we also associate their respective random velocity angles \(\psi_k's\) uniformly distributed in the range \((0, 2\pi)\) that fix planes of the corresponding ellipses. In effect we obtain a set \(I_3 = \{(e_i, \epsilon_i, \psi_i)\}\) and having appended the corresponding elements of sets \(I_2\) and \(I_3\) to each other, we have our set of initial data points \(I = \{(u_i, \theta_i, \phi_i, e_i, \epsilon_i, \psi_i)\}\) ready.

Finally, by applying the transformation Eq. 1 to each element of \(I\) we obtain the required set of initial positions and velocities expressed in spherical coordinates, and which leads to an initial RVD overlapping in the region of interest with the RVD in Fig. 1. Initial data obtained this way should be close to a possible stationary state in the modified potential to which the solution would tend starting from the initial state.

### 3.2. Gravitational potential

As mentioned in Sect. 1, we use an improved approximation of the Galactic potential, better than the simple point mass potential, with the same asymptotic value of the mass function of \(M_{\text{ref}} = 1.8 \times 10^{11} M_\odot\). The potential consists of two parts: a contribution from the Galactic disk and from a hot gaseous halo: \(\Psi = \Psi_{\text{disk}} + \Psi_{\text{halo}}\). As for \(\Psi_{\text{disk}}\) we use the potential of the infinitesimally thin disk of the form:

\[
\Psi_{\text{disk}}(\rho, \zeta) = -4G \int_0^\infty d\tilde{\rho} \frac{\tilde{\rho} K(k) \sigma(\tilde{\rho})}{\sqrt{\rho + \tilde{\rho}^2 + \zeta^2}},
\]

where \(K\) is the elliptic integral of the first kind defined in (Gradshtein et al. 2007), \(k = \sqrt{\frac{1-\rho^2}{1-\tilde{\rho}^2}}\) and \(\sigma(\rho)\) is a column mass density of a finite-width disk found by recursions from the Galactic rotation curve in (Jalocha et al. 2014). Although \(\sigma(\rho)\) is defined for all \(\rho > 0\), most of the disk mass is located within the inner disk \(\rho < 20\) kpc of mass \(M_{20} = 1.49 \times 10^{11} M_\odot\). Since the integrated mass inside 30 kpc is \(M_{30} = 1.51 \times 10^{11} M_\odot\), the contribution to the integral Eq. 6 from the outer disk \(\rho > 30\) kpc is negligible and we can restrict the integration region in Eq. 6 to the interval \(\tilde{\rho} \in (0, 30\text{kpc})\). To obtain a smooth function \(\Psi_{\text{disk}}\) we tabulated the integral Eq. 6 on a mesh \((\rho_j, \zeta_k)\)

and interpolated it with the help of a series of the form

\[
\Psi_{\text{disk}}(\rho, \zeta) = \sum_{a,b,c} \omega_{abc} \rho^a \zeta^b \left(\rho^2 + \zeta^2\right)^{-c/2},
\]

with integers \(a, b, c\) and the coefficients \(\omega_{abc}\) found by the least squares method, minimizing the discrepancy between Eq. 6 and the above series evaluated at the mesh points. Within the desired accuracy, we found that this approximation procedure is numerically more efficient than that of a two-dimensional interpolation.

The second contribution to the gravitational field comes from a spherically symmetric potential \(\Psi_{\text{halo}}\). Based on the \(O_{\gamma\gamma} / K\) absorption-line strengths in the spectra of galactic nuclei and galactic sources, Gupta et al. (2012) found large amounts of baryonic mass distributed in the form of hot gas around the Galaxy. Assuming a homogeneous sphere model, they found the electron density \(n_e\) of \(2.0 \times 10^{-4}\) cm\(^{-3}\) and the path length \(L\) of 72 kpc. Among other parameters, the total mass of the gas in their model depends on the gas metallicity and the oxygen-to-helium abundance. For reasonable values of parameters they found the total mass of the gas of \(1.2 - 6.1 \times 10^{10} M_\odot\). We may assume \(M_{\text{gas}} = 3.0 \times 10^{10} M_\odot\) consistently with these values. More recently, by applying the same observational method, Miller & Bregman (2013) found the mass function \(M(r)\) of the circumgalactic hot gas using a modified density profile

\[
n(r) = \frac{n_0}{(1 + (r/r_c)^2)^{3\lambda/2}}.
\]

We use this profile as the source of the spherical component \(\Psi_{\text{gas}}(r)\), with the parameters \(n_0 = 0.46\) cm\(^{-3}\), \(r_c = 0.35\) kpc and \(\lambda = 0.58\) allowable by the best fit to measurements. Then the integrated mass of the gas is \(M_{\text{gas}} = 3 \times 10^{10} M_\odot\) at \(r = 100\) kpc. For \(\lambda \leq 1\), the mass function is divergent and the integration must be cutoff at some radius, which is to some extent arbitrary. The cut-off at 100 kpc falls within the limits 18 kpc and 200 kpc on the minimum and maximum mass of the halo considered (Miller & Bregman 2013).

### 3.3. Numerical solution of the equations of motion

We consider a test particle of mass \(m\) in cylindrical coordinates \((\rho, \varphi, \zeta)\), moving in an axi-symmetric gravitational field described by the potential \(\Psi(\rho, \zeta)\). In the symmetry, the angular variable \(\varphi\) is cyclic – its respective conjugate momentum \(J_\varphi = m \rho^2 d\varphi/\psi\) is conserved. On account of \(\varphi\) being a monotonic function of the time \(t\) for orbits with \(J_\varphi \neq 0\), we may regard \(\varphi\) as the orbit independent parameter. In this parametrization, the Hamilton’s equations of motion on the \((\rho, \zeta)\) plane reduce to

\[
d_\rho \rho = J_\varphi^2 v_\rho, \quad d_\varphi v_\rho = J = \rho^2 \rho_\varphi \Psi(\rho, \zeta),
\]

\[
d_\rho \varphi = -\rho^2 J_\varphi, \quad d_\varphi \zeta = -\rho^2 \partial_\zeta \Psi(\rho, \zeta),
\]

where \(J = J_\varphi/m\) is the angular momentum per unit mass and \(v_\rho\) and \(v_\varphi\) are the velocity variables in the orthonormal basis of coordinate lines \(\rho, \zeta\). A derivation of these equations from a nonstandard Hamiltonian is presented in Appendix B. We solve these equations numerically by using the 4-order Runge-Kutta method with
adaptive step size. The step size is controlled so as to keep the relative change $|\Delta E/E|$ in the energy per unit mass $E = \frac{1}{2} \left( v^2_\rho + v^2_\phi + J^2/\rho^2 \right) + \Psi(\rho, \zeta)$ below some small threshold value. As a result, during our simulation the relative change of energy along each trajectory is always smaller than $10^{-6}$, that is $|E(t)/E(0) - 1| < 10^{-6}$ for all $t$, and this precision suffices for the purpose of this work.

3.4. Discussion of the results

Releasing indirect constraints on the PDF, such as those imposed on the form of the $\beta(r)$ profile, allowed us in (Bratek et al. 2014) to find a PDF of test bodies leading to a variable RVD profile overlapping at larger distances with the measured RVD, with a relatively small lower bound for Galaxy mass. We achieved this in the approximation of a point-mass background potential. But the question arises whether the resulting PDF model and predictions based on it are structurally stable, that is, not significantly changed by a modification in the potential at lower radii. The numerical simulation presented below is aimed at answering this question by showing that the low mass estimate occurs also for other models with unconstrained PDF and with a more realistic potential. The simulation can be understood as a means of obtaining a sequence of model PDF’s possible in the modified potential and enumerated by the simulation time.

By using the numerical procedure of Sect. 3.3 we prepared 3665 numerical trajectories for test bodies bound in the potential $\Psi_{\text{disk}} + \Psi_{\text{halo}}$ defined in Sect. 3.2. The initial conditions for these trajectories were generated as described in Sect. 3.1. The conditions agree with the initial PDF Fig. 2 obtained in the point mass approximation as a stationary solution of Jeans’s problem, and are consistent with the RVD of Galactic compact objects Fig. 1. Using the information about the evolution of whole system encoded in these trajectories, we could determine the RVD evolution in the modified potential from the initial state and visualize it by a sequence of snapshots taken at various instants. These snapshots are presented in Fig. 3 with a step size of $\approx 1$GY.

Each frame in Fig. 3 corresponding to a given simulation instant can be regarded as an independent RVD model used to estimate Galaxy mass by comparing the evolved RVD with the initial RVD (the background RVD in all snapshots) for which the mass estimation was determined by the use of the Keplerian ensemble method and equal to $M_{\text{ref}}$. In this approach, the mass estimate which we ascribe to the modified potential $\Psi_{\text{disk}} + \Psi_{\text{halo}}$ is a function of the simulation time, while the extent of the simulation time has no physical meaning and should not be compared with the Universe age.

At this point, it is appropriate to bring some features of the initial PDF to attention that persist during the simulation as model effects seen in the snapshots. Namely, the evolved RVD values are reduced in the lower radii region relative to those in the background RVD. The first reason is that for the initial PDF identified with that of the point mass field, a fraction of objects in the modified potential, due to a more extended mass distribution have too high velocities and either quickly escape to more distant regions or are unbound and therefore rejected from the analysis (in preparing the evolved RVD profiles we took into account only bound trajectories). In both cases the higher velocity values do not contribute to the RVD in this region, in consequence of which the RVD values are diminished. The other model effect is due to a cutoff in the PDF domain introduced in the Keplerian ensemble method to automatically prevent test bodies from penetrating the interior of a central spherical region which the point mass approximation is not applicable to. As so, there was no explicit limit on the number of almost nearly circular orbits an external neighborhood of this region could accommodate, too much elongated orbits could not occur there for geometrical reasons at all, while the admissible elongated orbits could enter this region with their pericentric sides only (on which radial motions are almost vanishing). In consequence of this, the overall mean RVD in this neighborhood could be easily reduced and by construction could not reach the high observed values. Now, since the initial PDF has been identified with the PDF of the Keplerian ensemble, a qualitatively similar reduction mechanism in the evolved RVD comes about in the modified potential, reflecting in the $\beta(r)$ reduced toward more negative values in the lower radii region (however, more circular motions in this region could be interpreted as consistent with a contribution from a cold disk). With a better initial PDF this model effect could be eliminated but it seems of no importance for the accuracy of the total mass determination for which the region of greater radii is more important.

Now, let us come back to the main issue. As mentioned earlier, we want to verify the expectation that the evolved PDF should be in a sense close to the initial PDF independently of the simulation instant if the point mass approximation well describes the real situation at higher distances. If as a result of the evolution in the modified potential the RVD turned out to be collapsing to much smaller values or change its shape completely, then this would mean that the mass estimate based on the initial PDF was wrong and inconsistent with the new evolved PDF. Thus, the simulation results provide also a test of the structural stability of the mass estimator in the point mass approximation.

As we can see from the RVD snapshots, although the evolved RVD changes with the simulation time – it decreases a little in some region and then it rises again - it generally remains high for the larger radii region. Similarly, the characteristic maxima in the initial RVD are not destroyed but oscillate. Besides the evolved RVD profile (blue-line RVD’s) corresponding to the initial mass $M_{\text{ref}} = 1.8 \times 10^{11}M_\odot$ of the point mass approximation, in every snapshot there is also shown a corrected RVD profile (red-line RVD’s) obtained by multiplying $M_{\text{ref}}$ and the radial variable with suitable factors close to unity, respectively $\mu$ and $\alpha$, so as to make the corrected RVD’s coincide with the background RVD as good as possible in the sense of the least squares. During the simulation run, the length scale factor $\alpha$ varied in the range (0.85; 1.02) with the mean 0.92 $\pm$ 0.03, while the mass factor $\mu$ varied in the range...
Fig. 3.— A sequence A, B, . . . , S, T of RVD models from an N-body simulation discussed in the text. Each frame corresponds to a distinct simulation instant and can be considered as an independent RVD model. The black thick line, the same for all frames, is the RVD profile from measurements and the light gray region is the RVD uncertainty region defined by vertical bars in Fig. 1. We refer to it in the text as the background RVD. The blue line is the evolved RVD profile (to compare with, the violet line is the result of a transformation of the radial motion measurements in the LSR frame to the Galactocentric frame, assuming isotropic dispersion ellipsoid; it overlaps with the blue line for larger radii). The red line RVD profile is obtained by rescaling the blue line.
Fig. 4.— Mass estimator $\mu M_{\text{ref}}$ in function of simulation time, with factors $\mu$ obtained from best fit model RVD at various threshold radii $R_T$.

responds to total mass estimate of $(2.12 \pm 0.13) \times 10^{11} M_{\odot}$, oscillating in the range $(1.85; 2.47) \times 10^{11} M_{\odot}$. Thus, the lower bound for Galaxy mass in the modified potential suffices to retain during the simulation run the qualitative features of the initial RVD profile and its values.

4. Conclusions

The lower bound for Galaxy mass of $\approx 2.1 \times 10^{11} M_{\odot}$ obtained within the Keplerian ensemble framework suffices to retain, during a numerical simulation run in a modified potential, the qualitative features of the evolved RVD profile and its values, consistently with the RVD from halo measurements within 150 kpc. In this sense the evolved RVD is stable. But these results also substantiate structural stability of the point mass approximation, showing that the lower range for Galaxy mass estimates assuming more general unconstrained PDF’s is reliable. Taking into account a possible correction factor 1.16 from rejecting 4 halo objects in Sect. 2.2.2, the simulation result gives us a value of $(2.5 \pm 0.2) \times 10^{11}$ for the Galaxy mass.

APPENDIX

A. Derivation of a relation between the observable $\langle \bar{v}^2 \rangle$ and the radial dispersion $\langle v^2 \rangle$

We want to relate the mean values $\langle \bar{v}_r^2 \rangle \equiv \langle (\bar{v}_r \circ \bar{v})^2 \rangle$ and $\langle v_r^2 \rangle \equiv \langle (v_r \circ v)^2 \rangle$ considered as function of Galacto-centric distance $r$. The mean values are taken over the sphere of radius $r$, where $\bar{v}$ is the Galacto-centric velocity vector, $\bar{v}_r = \frac{\bar{v}}{|\bar{v}|}$ and $\bar{v}_r = \frac{\bar{v}_r}{|\bar{v}_r|}$. In the standard spherical coordinates: $\bar{v}_r = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]$, $\bar{v}_r = [0, 0, 1]$ and $\bar{v}_r = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]$. The velocity vector can be expressed in terms of its radial $V_r$ and transversal components $V_{\theta}, V_{\phi}$: $\bar{v} = [V_r \cos \phi + V_{\theta} \cos \phi - V_{\phi} \sin \phi, (V_r \sin \theta + V_{\theta} \cos \theta) \sin \phi + V_{\phi} \cos \phi, V_r \cos \theta - V_{\phi} \sin \theta]$. By definition, for a spherical system

$$\langle \bar{v}_r^2 (r) \rangle = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \sin \theta \, d\theta \, d\phi \langle (\bar{v}_r (r, \theta, \phi) \circ \bar{v})^2 \rangle_{\text{int}}.$$ 

Here, $\langle \cdot \rangle_{\text{int}}$ denotes the averaging over the velocities weighted by the spherically symmetric PDF of the form $f(r, \bar{v}(r))$, normalized so as $\nu(r) \langle \cdot \rangle_{\text{int}} = \int f(r, \bar{v})(r) \, d^3\bar{v}$, with $\nu$ denoting the number density. The scalar product squared $(\bar{v}_r (r, \theta, \phi) \circ \bar{v})^2$ is a homogeneous form of degree 2 in the velocities $V_r, V_{\theta}, V_{\phi}$ with coefficients being functions of $r, \theta, \phi$. By direct inspection one can notice that integration over $\theta, \phi$ of the coefficients standing before mixed products $V_r V_{\theta}, V_{\theta} V_{\phi}, V_{\phi} V_r$ gives zero (by spherical symmetry the velocity components are independent of $\theta, \phi$). Thus, upon integration over velocities, we can consider only terms involving dispersions $\langle V_r^2 (r) \rangle, \langle V_{\theta}^2 (r) \rangle, \langle V_{\phi}^2 (r) \rangle$. Furthermore, it also follows from spherical symmetry that $\langle V_{r}^2 (r) \rangle = \langle V_{\theta}^2 (r) \rangle$ and, trivially, that the ratio $\langle V_{r}^2 (r) \rangle/\langle V_{\theta}^2 (r) \rangle$ define the same function of $r$. In accordance with the common convention assumed in the theory of Jeans equations under spherical symmetry, we express this function in terms of the flattening of the dispersion ellipsoid, $\beta(r)$. Then, by making the substitution $\langle V_r^2 (r) \rangle = (1 - \beta(r))\langle V_{\theta}^2 (r) \rangle$ and $\langle V_{\theta}^2 (r) \rangle = (1 - \beta(r))\langle V_{\phi}^2 (r) \rangle$, we obtain that

$$\langle \bar{v}_r^2 (r) \rangle = \langle v_r^2 (r) \rangle \left( 1 - \frac{\beta(r)}{1 + r^2/1_\odot} \mathcal{I}(\alpha(r)) \right),$$

where $\alpha(r) = \frac{2r_{1_\odot}}{r + r_{1_\odot}} < 1$ for $r \neq r_\odot$ and

$$\mathcal{I}(\alpha) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \sin \theta \frac{\cos^2 \theta \cos^2 \phi + \sin^2 \phi}{1 - \alpha \sin \theta \cos \phi}.$$ 

Recall, that all integrals that are zero by symmetries have been already omitted in the expression for $\mathcal{I}(\alpha)$. On account that the requirements for the integration of a functional series term by term and its limit are met for $0 \leq \alpha < 1$, the integral $\mathcal{I}(\alpha)$ can be calculated by a Taylor series expansion in $\alpha$ (note that owing to vanishing of the integrals $\int_0^{2\pi} \cos^m \phi \, d\phi$ with odd $m$, only even powers of $\alpha$ are present in the series). On reducing the summands with the help of the Pythagorean trigonometric identity, the remaining nonzero coefficients in the power series in $\alpha$ arrange to products of elementary definite integrals

$$\mathcal{I}(\alpha) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \alpha^{2n} \left( C_n - S_n + C_{n+1} - S_{n+1} \right),$$

where $S_n = \int_0^{\pi} \sin^{2n+1} \theta \, d\theta = 2(2n+1)! \cos^{2n+1} \theta$, and $C_n = \int_0^{2\pi} \cos^{2n} \phi \, d\phi = \frac{2\pi 2^{2n+1}}{(2n+1)!}$. Now, $C_n S_n = \frac{4\pi}{2n+3}$ and $C_{n+1} S_{n+1} = \frac{4\pi}{2n+5}$. Hence,

$$\mathcal{I}(\alpha) = \frac{1}{2\alpha} \sum_{n=0}^{\infty} \frac{2\alpha^{2n+1}}{2n+1} \left[ \frac{1}{2\alpha^3} - \frac{2\alpha + \sum_{n=0}^{\infty} \frac{2\alpha^{2n+1}}{2n+1}}{2n+3} \right],$$

where we have subtracted the excess term $2\alpha$ in the second series after renaming $n \rightarrow n + 1$. Both of the infinite series are Taylor series expansions of $\ln(1 + \frac{x}{1 - x})$, hence

$$\mathcal{I}(\alpha) = \frac{1}{\alpha^2} - \frac{1 - \alpha^2}{2\alpha^3} \ln \left( \frac{1 + \alpha}{1 - \alpha} \right), \quad 0 \leq \alpha < 1.$$
Next, using the earlier expression for $\langle \dot{v}_r^2 \rangle$ and substituting the definition of $\alpha(r)$ in place of $\alpha$, we finally obtain Eq. 5 showing the connection between $\langle \dot{v}_r^2(r) \rangle$ and $\langle v_r^2(r) \rangle$.

**B. Derivation of the equations of motion from a nonstandard Hamiltonian**

In this section we use cylindrical coordinates $(\rho, \varphi, \zeta)$. By axial symmetry of the potential $\Psi(\rho, \zeta)$ the $J_\varphi$ component of the angular momentum is conserved and we may use $\varphi$ as the independent parameter for trajectories with $J_\varphi \neq 0$. Then the Lagrangian for a test body of mass $m$ attains the form

$$L = \frac{m}{2} \frac{d}{d\varphi} \left( \frac{\rho^2 + \rho^2 + \dot{\zeta}^2}{\rho} \right) - m \frac{\partial \Psi}{\partial \rho}(\rho, \zeta) \frac{d}{d\varphi} \zeta t.$$

The dynamical variables, which we now regard as functions of $\varphi$, are $\rho$, $\zeta$ and the time variable $t$. The Hamiltonian is found, as usual, by means of the Legendre transformation

$$L \to H \equiv \frac{d}{d\varphi} \rho \frac{\partial L}{\partial \dot{\rho}} + \frac{d}{d\varphi} \zeta \frac{\partial L}{\partial \dot{\zeta}} + \frac{1}{2} \frac{d}{d\varphi} t \frac{\partial L}{\partial \dot{t}} - L = \frac{-m\rho^2}{2\rho^2 + \dot{\zeta}^2},$$

where the velocities must be expressed in terms of positions and momenta. Solving the canonical definitions of momenta $p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\frac{\dot{\rho}}{\rho}$, $p_\zeta = \frac{\partial L}{\partial \dot{\zeta}} = m\frac{\dot{\zeta}}{\rho}$, and $p_t = \frac{\partial L}{\partial \dot{t}} = -m\frac{\dot{\rho}^2 + \rho^2 + \dot{\zeta}^2}{2\rho^2 + \dot{\zeta}^2}$ for velocities, we find that the Hamiltonian $H$ reads

$$H = \pm \rho \sqrt{-\left( \frac{\rho^2}{J_\rho} + 2m \frac{p_\rho}{J_\rho} + \frac{p_\zeta^2}{J_\zeta} + 2m^2 \frac{\partial \Psi}{\partial \rho}(\rho, \zeta) \right)}.$$

For solutions this Hamiltonian equals the third component of the angular momentum. Since $H$ is not dependent explicitly on $\varphi$, $H$ is constant for solutions. Let denote this constant by $-J_\varphi$. Because there are two solutions for $H$ with the opposite sign and $H$ is constant for solutions, it is more convenient to rewrite Hamiltonian equations into the reduced form $\frac{d}{d\varphi} \rho = -\frac{\partial \Psi}{\partial \rho}(\rho, \zeta)$, $\frac{d}{d\varphi} \rho = \frac{-1}{2\rho^2} \{ \rho, H^2 \}$, where $(\cdot, \cdot)$ is the Poisson bracket on the phase space $(\rho, p_\rho, \zeta, p_\zeta, \zeta, p_t)$. In explicit form these equations read:

$$\frac{d}{d\varphi} \rho = \frac{J_\rho}{\rho^2} \rho, \quad \frac{d}{d\varphi} \zeta = \frac{J_\zeta}{\rho^2} \zeta, \quad \frac{d}{d\varphi} t = \frac{J_t}{\rho^2} t, \quad \frac{\partial}{\partial \rho} \rho = \frac{J_\rho}{\rho} - \frac{m^2 \rho^2}{J_\rho} \frac{\partial \Psi}{\partial \rho}, \quad \frac{\partial}{\partial \zeta} \zeta = -\frac{m^2 \rho^2}{J_\zeta} \frac{\partial \Psi}{\partial \zeta}, \quad \frac{\partial}{\partial t} t = 0.$$

The last equation states that $p_t$ is conserved. We denote this constant value of $p_t$ by $E$. Then, using the expression for $H^2$ or the definition of $p_t$ along with the Hamiltonian equations, it follows that $E = \frac{1}{2m} \left( \frac{\rho^2}{J_\rho} + \frac{\zeta^2}{J_\zeta} + \frac{J_t^2}{J_t^2} \right) + m \frac{\partial \Psi}{\partial \rho}(\rho, \zeta)$. The solution for function $t(\varphi)$ is not important for our purposes – the time averages along a trajectory may be expressed as averages over the angle $\varphi$, namely $\frac{1}{T} \int_0^T F(t) dt = \frac{1}{T} \int_0^T F \rho^2 d\varphi$. Therefore, given constants $J_\varphi$ and $E$, and having specified the initial conditions consistently with $E$ and $J_\varphi$, only four independent equations are left to be solved. On expressing $p_\rho$ and $p_\zeta$ by the physical velocities $v_\rho \equiv \frac{\partial L}{\partial \dot{\rho}}$ and $v_\zeta \equiv \frac{\partial L}{\partial \dot{\zeta}}$, the remaining first order equations for four functions $v_\rho(\varphi)$, $v_\zeta(\varphi)$, $\rho(\varphi)$ and $\zeta(\varphi)$ reduce to:

$$\frac{d}{d\varphi} \rho = \frac{\rho^2}{J_\rho/m} v_\rho, \quad \frac{d}{d\varphi} v_\rho = \frac{J_\rho}{m} \frac{\rho}{\rho} - \frac{\rho^2}{J_\rho/m} \frac{\partial}{\partial \rho} \Psi(\rho, \zeta),$$

$$\frac{d}{d\varphi} \zeta = \frac{\rho^2}{J_\zeta/m} v_\zeta, \quad \frac{d}{d\varphi} v_\zeta = -\frac{\rho^2}{J_\zeta/m} \frac{\partial}{\partial \zeta} \Psi(\rho, \zeta).$$

and the energy integral attains the form

$$E/m = \frac{v_\rho^2}{2} + \frac{v_\zeta^2}{2} + \frac{(J_\rho/m)^2}{2 \rho^2} + \Psi(\rho, \zeta).$$

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