Generalized holographic dark energy model described at the Hubble length

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We generalize the holographic dark energy model described in Hubble length IR cutoff by assuming a slowly time varying function for holographic parameter $c^2$. We calculate the evolution of EoS parameter and the deceleration parameter as well as the evolution of dark energy density in this generalized model. We show that the phantom line is crossed from quintessence regime to phantom regime which is in agreement with observation. The evolution of deceleration parameter indicates the transition from decelerated to accelerated expansion. Eventually, we show that the GHDE with HIR cutoff can interpret the pressureless dark matter era at the early time and dark energy dominated phase later.

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Nowadays we are strongly believed that our universe experiences an accelerated expansion. The complementary astronomical data gathered from SNe Ia [1], WMAP [2], SDSS [3] and X-ray [4] experiments confirm this cosmic acceleration. Within the framework of general relativity (GR), a dark energy component with negative pressure is introduced to explain this acceleration. Dark energy scenario have got a lot of attention in modern cosmology. In recent years a plenty theoretical models have been investigated to interpret the dynamical properties of dark energy. One can see [5, 6] for a review of dark energy models. The holographic dark energy (HDE) model is one of these models to explain a dark energy scenario. This model is constructed based on the holographic principle in quantum gravity [7]. In quantum gravity, a short distance ultra-violet (UV) cut-off is related to the long distance infra-red (IR) cut-off, due to the limit set by the formation of a black hole [7].

Based on the holographic principle, the energy of a system with size \( L \) does not exceed the energy of black hole with the same size, i.e.,

\[
L^3 \rho_d \leq L m_p^2
\]

where \( m_p \) is the reduced plank mass. By saturating the inequality (1), the energy density of HDE model in cosmology is identified by [8]

\[
\rho_d = 3 c^2 m_p^2 L^{-2},
\]

where \( c^2 \) is a numerical constant of order unity and the factor 3 was introduced for convenience. An interesting feature of HDE is that it has a close connection with the space-time foam [9]. Another features of HDE model can be found in Section 3 of [10]. From the observational point of view, the HDE model has been constrained by various astronomical observation [13, 15, 16, 29]. The HDE model has been also investigated widely in the literature [14, 17]. For a recent review on different HDE models and their consistency check with observational data see [18]. From the theoretical point of view there are some motivations leading to the form of HDE model [19]. It should be noted that various HDE models have been investigated via assuming different IR cutoffs. The simple choice for IR cutoff is the Hubble radius, i.e., \( L = H^{-1} \). In this case, the accelerated expansion of the universe can not be achieved and we get a wrong equation of state for this model [7]. However, in the presence of interaction between dark matter and dark energy, the HDE model can derive the cosmic acceleration and also, in this case, the cosmic coincidence problem can be solved.
Event horizon is the another choice for IR cutoff. Although, in this case the accelerated expansion can be achieved, but the generalized second law (GSL) does not satisfy in a universe enveloped by event horizon IR cutoff. The other choice for IR cutoff is the particle horizon. In this case, the HDE model can not also obtain the late time accelerated expansion. Here same as, we assume Hubble horizon as an IR cutoff for HDE model. In this case the GSL is also satisfied in the interacting accelerating universe. Therefore the Hubble horizon is preferred from thermodynamical point of view.

It is worthwhile to mention that the parameter $c^2$ in HDE model has an essential role in characterizing the properties of HDE model. For example, the HDE model can behave as a phantom or quintessence dark energy models at the future for the values of $c^2$ bigger or smaller than 1, respectively.

In all above references the HDE model was assumed to have a constant value for holographic parameter $c^2$. However there are no strong evidences telling us that $c^2$ should be a constant parameter. In general the term $c^2$ can be assumed as a function of time. By slowly vary function with time, $(\dot{c^2})/c^2$ is upper bounded by the Hubble expansion rate, i.e.,

$$\frac{\dot{c^2}}{c^2} \leq H$$

In this case the time scale of the evolution of $c^2$ is shorter than $H^{-1}$ and one can be satisfied to consider the time dependency of $c^2$. It has been also shown that the parameter $c^2$ can not be constant for all times during the evolution of the universe.

As was mentioned above, in the presence of interaction between dark matter and dark energy the HDE model with the Hubble horizon IR cutoff can solve the the coincidence problem and late time accelerated expansion. However, another alternative approach instead of interaction between dark components is that the holographic parameter $c^2$ varies slowly with time to solve the coincidence problem and explain late time acceleration. It has been shown that the interacting model of dark energy in which the coincidence problem is alleviated can be recast as a noninteracting model in which the holographic parameter $c^2$ evolves slowly with time.

In the line of above studies, we consider the HDE model with time-varying holographic parameter $c^2(z)$, namely: generalized holographic dark energy (GHDE, hereafter). We also consider the Hubble horizon as an IR cutoff (HIR, hereafter). We investigate the EoS parameter of the model as well as the deceleration parameter and discuss the density
evolution of dark energy in this model.

Let us start with flat Friedmann-Robertson-Walker (FRW) universe. In this case the first Friedmann equation is given by

\[ H^2 = \frac{1}{3m_p^2} (\rho_m + \rho_d) \]  

(4)

where \( \rho_m \) and \( \rho_d \) are, respectively, the energy densities of pressureless dark matter and dark energy and \( m_p \) is the reduced planck mass. For Hubble radius IR cutoff, \( L = H^{-1} \), the energy density of GHDE model from (2) can be given by

\[ \rho_d = 3m_p^2 c^2(z) H^2 \]  

(5)

where the holographic parameter is considered as a function of redshift.

We now define the dimensionless energy density parameters as

\[ \Omega_m = \frac{\rho_m}{\rho_c} = \frac{\rho_m}{3M_p^2 H^2}, \quad \Omega_d = \frac{\rho_d}{\rho_c} = \frac{\rho_d}{3M_p^2 H^2} = c^2(z) \]  

(6)

According to these definitions, the first Friedmann equation in spatially flat universe can be written as follows

\[ \Omega_m + \Omega_\Lambda = 1. \]  

(7)

The conservation equations for pressureless dark matter and dark energy, respectively, are given by

\[ \dot{\rho}_m + 3H\rho_m = 0, \]  

(8)

\[ \dot{\rho}_d + 3H(1 + w_d)\rho_d = 0. \]  

(9)

Taking the time derivative of Friedmann equation (4) and using (7, 8, 9), one can obtain

\[ \frac{\dot{H}}{H^2} = -\frac{3}{2} [1 + w_\Lambda \Omega_d] \]  

(10)

Also it is obvious to see that differentiating Eq.(5) with respect to time yields

\[ \dot{\rho}_d = 2\rho_d \left( \frac{\dot{c}}{c} + \frac{\dot{H}}{H} \right) \]  

(11)

Inserting (11) and (5) in conservation equation for dark energy (9) and using (10), we find the equation of state, \( w_d \), for GHDE model with Hubble length as

\[ w_d = -\frac{2c'}{3c(1 - c^2)} \]  

(12)
where prime represents the derivative with respect to \( \ln a \). It is clear that the above relation reduce to \( w_d = 0 \) for constant holographic parameter \( c \). Hence, as expected, the HDE model in HIR gets to wrong equation of state for dark energy which can no describe the expanding universe. As was mentioned before, this problem for HDE model can be solved, if we consider the interaction between dark matter and dark energy (see \[22\] for more detail). In is worthwhile to mention that from (12) one can see that the GHDE model in which the holographic parameter \( c \) is considered as a function of redshift can get \( w_d < 0 \) in HIR without assuming the interaction parameter. For this aim we use the Wetterich parametrization in which the holographic parameter \( c \) is considered in terms of redshift as follows \[28\]

\[
c(z) = \frac{c_0}{1 + c_1 \ln (1 + z)}
\]  

(13)

Putting \( c_1 = 0 \), the above holographic parameter reduces to \( c = c_0 \) indicating the constant value for HDE model. At the present time: \( z \to 0, c(z) \to c_0 \), and at the early time: \( z \to \infty, c(z) \to 0 \). Hence the holographic parameter varies slowly from zero to \( c_0 \) during the history of the universe. Also to have positive energy density for dark energy, \( \rho_d \geq 0 \), we should take the condition: \( c_0 > 0 \) and \( c_1 \geq 0 \). In numerical procedure, we chose matter density parameter \( \Omega_m = 1 - c_0^2 \), and dark energy density parameter \( \Omega_d = c_0^2 \), indicating the spatially flat universe. In Fig.(1), by solving (12) and using (13), we plot the evolution of EoS parameter, \( w_d \), in terms of redshift \( z \) for different illustrative values of \( c_0 \) and \( c_1 \). Here we see that the EoS parameter, \( w_d \) of GHDE with HIR can transit from quintessence regime \( (w_d > -1) \) to phantom regime \( (w_d < -1) \). The observations favor dark energy models which cross the phantom line \( w = -1 \) from up \( (w_d < -1) \) to down \( (w_d < -1) \) in near past \[29\]. Therefore this model is compatible with observations. Contrary the GHDE model HIR, the EoS parameter for interacting HDE model with HIR is constant during the history of the universe (see Eq.(8) of \[22\]). However, Sheykhi showed that by applying some restrictions on the interaction parameter \( b \) and model parameter \( c \), the EoS parameter of the HDE model with HIR can behave as a quintessence or a phantom type dark energy. But, neither the quintessence nor the phantom alone can fulfill the transition from \( w_d > -1 \) to \( w_d < -1 \).

We now calculate the evolution of energy density of GHDE model described at HIR. From Eq. (6), the dark energy density of GHDE equals to square of varying holographic parameter, \( \Omega_d = c^2(z) \). Using (13), in Fig.(2) the evolution of dark energy density is plotted in terms of redshift for some illustrative values of model parameters \( c_0 \) and \( c_1 \). At the early time
(\(z \rightarrow \infty\)) the parameter \(\Omega_d \rightarrow 0\) which represents the dark matter dominated universe at the early time. Then the parameter \(\Omega_d\) increases to its present value \(c_0\) which indicates the dark energy dominated epoch. The important note is that in standard HDE model under HIR cutoff, since the model parameter \(c\) is constant therefore the energy density \(\Omega_d = c^2\) has no evolution during the history of the universe, i.e., \(\Omega_d\) is constant from early time to present time. Unlike the standard HDE model, in GHDE model with HIR the parameter \(\Omega_d\) increases from zero at the early time and tends to its present value at the present epoch. This behavior of GHDE model can interpret the decelerated expansion at the early time dark matter dominated universe and also the accelerated expansion at the dark energy dominated epoch.

For completeness, we calculate the deceleration parameter \(q\) for GHDE model with HIR. The positive value of deceleration parameter \((q > 0)\) indicates the decelerated phase of expansion and the negative value \((q < 0)\) represents the accelerated phase of expansion of the universe. The parameter \(q\) is defined as

\[
q = -1 - \frac{\dot{H}}{H^2} \tag{14}
\]

Inserting (10) in (14) and using (12) as well as \(\Omega_d = c^2\), the parameter \(q\) for GHDE with HIR can be obtained as

\[
q = \frac{1}{2} - \frac{c'c}{1 - c^2} \tag{15}
\]

In the limiting case of standard HDE model with constant value of \(c\), the parameter \(q\) reduces to \(q = 1/2\) which describes the decelerated phase and can not represents the accelerated phase. However, including the interaction parameter and some restrictions on the interaction parameter \(b\) and model parameter \(c\) in standard HDE model with HIR can result the negative value for deceleration parameter \(q\) \[22\], but since the parameter \(c\) is constant therefore the transition from decelerated to accelerated expansion can not be achieved.

In Fig.(3), by solving (15) and using (13), we plot the parameter \(q\) in GHDE with HIR model as a function of redshift parameter \(z\). We see that at the early times the parameter \(q\) is 1/2 indicating the decelerated phase at dark matter-dominated universe. Then the parameter \(q\) reaches to negative values representing the accelerated phase at dark energy-dominated background. This property of GHDE with HIR is consistent with this observational fact that the universe has entered to the accelerated phase at past times \[30\].
In summery, we considered the generalized holographic dark energy model in spatially flat universe described in Hubble length an an IR cutoff (GHDE with HIR cutoff). The holographic parameter \( c \) generally is not constant and can be assumed as a function of cosmic redshift. The standard HDE model described by HIR cutoff gets to wrong equation of state for dark energy [7]. The observations favor dark energy models which cross the phantom line \( w = -1 \) from quintessence regime \( (w_d < -1) \) to phantom regime \( (w_d < -1) \) in near past [29] and also the models in which the deceleration parameter transit from positive value to negative value [30]. Although, including the interaction between dark matter and dark energy in standard HDE model described by HIR cutoff can solve the coincidence problem and late time accelerated expansion [21, 22], but the EoS parameter of this model behaves as a quintessence or phantom model and can not transit from quintessence regime \( (w_d > -1) \) to phantom regime \( (w_d < -1) \) [22]. Also in the context of interacting HDE with HIR cutoff the deceleration parameter \( q \) is negative for all times in the history of the universe and therefore can not explain the transition from decelerated to accelerated expansion. However, in the case of GHDE with HIR cutoff, we obtained the EoS parameter as well as the deceleration parameter and evolution of dark energy density. Here we assumed the holographic parameter \( c^2 \) varies slowly with time instead of adding the interaction term. We showed that in this model the phantom line is crossed from up \( (w_d > -1) \) to down \( (w_d < -1) \) which is in agreement with observation [29]. Also it has been shown that in this model the evolution of deceleration parameter \( q \) indicates the decelerated phase at the early time \( (q > 0) \) and accelerated phase at later \( (q < 0) \). The evolution of energy density of dark energy represents the pressureless dark matter-dominated universe at the early time and dark energy-dominated phase at the present time.
FIG. 1: The evolution of EoS parameter of GHDE model with HIR cutoff versus redshift parameter $z$ for different values of model parameters $c_0$ and $c_1$. Here we take $\Omega_m^0 = 1 - c_0^2$ and $\Omega_d^0 = c_0^2$.

FIG. 2: The evolution of energy density of GHDE model with HIR cutoff versus redshift parameter $z$ for different values of model parameters $c_0$ and $c_1$. Here we take $\Omega_m^0 = 1 - c_0^2$ and $\Omega_d^0 = c_0^2$. 
FIG. 3: The evolution of deceleration parameter $q$ in GHDE with HIR model versus redshift parameter $z$ for different illustrative values of model parameters $c_0$ and $c_1$. Here we take $\Omega_m^0 = 1 - c_0^2$ and $\Omega_d^0 = c_0^2$.

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