Instanton-Sphaleron Transition in the $d = 2$ Abelian–Higgs Model on a Circle

D.K. Park, a,b H.J.W. Müller–Kirsten, c J. Q. Liang a,c and A.V. Shurgaia a,d a) Department of Physics, University of Kaiserslautern, 67653 Kaiserslautern, Germany
b) Department of Physics, Kyungnam University, Masan, 631–701, Korea
c) Department of Physics and Institute for Theoretical Physics, Shanxi University, Taiyuan, Shanxi 030006, P.R. China
d) Mathematical Institute, Georgian Academy of Sciences, Ruchadze Str., 380093 Tbilisi, Georgia

Abstract
The transition from the instanton–dominated quantum regime to the sphaleron–dominated classical regime is studied in the $d = 2$ Abelian–Higgs model when the spatial coordinate is compactified to $S^1$. Contrary to the noncompactified case, this model allows both sharp first–order and smooth second–order transitions depending on the size of the circle. This finding may make the model a useful toy model for the analysis of baryon number violating processes. Since the model can to a large extent be treated analytically, it can also serve as a transparent prototype for the application of our method to more complicated cases, such as those in higher dimensions.

1 Introduction

After the sphaleron solution in the Weinberg–Salam model had been found [1, 2], the temperature dependence of baryon number violating processes (BNVP) was studied extensively. To understand the overall features of BNVP over the entire range of temperature, the computation of periodic instantons [3] and their corresponding classical actions is required. However, the calculation of these in the Weinberg–Salam model is a highly non–trivial problem, even if numerical techniques are employed. Hence in many cases simple toy models were used to explore the temperature dependence of BNVP.
An immediate candidate as a simple toy model is the $d = 2$ Mottola–Wipf (MW) model\cite{4}, which shares many common features with $d = 4$ electroweak theory. The scale invariance of the nonlinear $O(3)$ model is broken in the MW model by adding an explicit mass term. This has a close analogy to the fact that the conformal invariance of the electroweak theory is broken in the Higgs sector. Also, neither model supports a vacuum instanton which gives a dominant contribution to the winding number transition at low temperature. The transition between thermally assisted quantum tunneling dominated by periodic instantons and the classical crossover dominated by the sphaleron in the MW model has been analyzed in Refs.\cite{3, 4} using the method of \cite{7}, and it has been shown that the instanton–sphaleron transition is of the sharp first–order type in the full range of parameter space.

Recently, however, a numerical study\cite{8, 9} of the $d = 4$ SU(2)–Higgs model – which is a bosonic sector of the electroweak theory – has shown that a smooth second–order transition occurs when $6.665 < M_H/M_W < 12.03$ although the first–order transition occurs when $M_H/M_W < 6.665$\cite{10}. This implies that the MW model does not exhibit a proper transition of BNVP when heavy Higgs’s are involved.

Another candidate as a toy model is the $d = 2$ Abelian-Higgs model which supports vortex solutions\cite{11}, in particular the vacuum instanton and the sphaleron\cite{12} simultaneously. The simultaneous existence of instanton and sphaleron causes the model to yield phase diagrams for the instanton–sphaleron transition which are completely different from those of electroweak theory, as shown in Ref.\cite{13}. Furthermore, numerical\cite{14} and analytical\cite{15} approaches have shown that the instanton–sphaleron transition in this model is always of the second–order type, regardless of the ratio $M_H/M_W$. Hence, contrary to the MW model, the ordinary Abelian–Higgs model does not describe the instanton–sphaleron transition of the electroweak theory properly when the Higgs mass is small.

In the following we study the instanton–sphaleron transition in the $d = 2$ Abelian-Higgs model when the spatial coordinate is compactified to $S^1$. Quite apart from the question of the physical relevance of the investigation below, it is a natural theoretical curiosity to inquire what the order of thermal transitions would be in this case, and we present the answer here. Physically, of course, the transitions we investigate are not those with respect to an order parameter as in the Weinberg–Salam theory, but with respect to temperature or inverse period of the periodic instantons in the potential barrier. These transitions have physically the meaning of transitions between classical and quantum behavior.
 Nonetheless, as stated we consider the model as an analogy, which enables us to investigate corresponding behavior. Since, to our knowledge, the effect of the compactification of the spatial coordinate of this model has not yet been investigated, this is also of interest on its own. Furthermore, we show that this model exhibits both first–order and second–order transitions depending on the size of the circumference of the spatial coordinate domain, i.e. the first order transition disappears in the limit of the circumference becoming infinitely large, in fact, even beyond a finite critical value. This means that the Abelian–Higgs model defined on a circle can be a better toy model than the MW model or the uncompactified Abelian-Higgs model for an analysis which can be compared with that of BNVP. One may wonder how, if at all, this situation compares with finite size scaling effects, i.e. of lattices with periodic boundary conditions, in lattice gauge theory contexts. In the latter, see e.g.\cite{17}, lattice sizes of $4^4$ to $16^4$ are used and the possible dependence of the order of thermal transitions on these is investigated. Nonetheless the lattice sizes used are presumably still too small to permit definite conclusions about the scaling regime. We do not think that our case is really comparable to that. Rather we view the present model as a testing ground for various aspects related to phase transitions, since the study of the latter, as can be seen from the computations needed in the following, are highly nontrivial (and we therefore have to present some technical details), so that any model that can be handled to a large extent analytically, is worth studying. Thus before higher dimensional cases can be attacked convincingly, it is essential to have a thorough understanding of a lower dimensional one like the one we study here. This is therefore another main objective of the following.

2 The Sphaleron Configuration

We begin with the Euclidean action

$$S_E^{(0)} = \int d\tau dx \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu\phi)^* D_\mu\phi + \lambda (|\phi|^2 - \frac{v^2}{2})^2 \right]$$

and its field equations

$$\partial_\mu F_{\mu\nu} = ig [\phi^* (D_\nu \phi) - (D_\nu \phi)^* \phi],$$

$$D_\mu D_\mu \phi = 2\lambda \phi (|\phi|^2 - \frac{v^2}{2}),$$
where \( D_\mu = \partial_\mu - igA_\mu \). We define as mass–dimensional parameters

\[
M_H \equiv \sqrt{2\lambda v}, \\
M_W \equiv gv,
\]

which correspond to Higgs mass and gauge particle mass in electroweak theory respectively. It is easy to show that the static sphaleron solution in the \( A_0 = 0 \) gauge is given by

\[
A_1 = A = \text{const}, \\
\phi_{sph} = \frac{kb(k)}{\sqrt{\lambda}} e^{igAx} sn[b(k)x],
\]

where \( sn[z] \) is a Jacobian elliptic function, \( k \) is the modulus of the elliptic function, and

\[
b(k) = \sqrt{\frac{\lambda}{2v} \left( \frac{2}{1 + k^2} \right)^{1/2}}.
\]

Since \( sn[z] \) has period \( 4K(k) \), where \( K(k) \) is the complete elliptic integral of the first kind, the circumference \( L \) of \( S^1 \) is defined by

\[
L_n = \frac{4nK(k)}{b(k)}, \quad n = 1, 2, 3 \cdots.
\]

Since the transition rate is negligible for large \( n \) \([18]\), we restrict ourselves to the \( L = L_1 \) case here. In view of \( K(1) = \infty \) we see that \( k = 1 \) gives the uncompactified limit we investigated previously \([15]\). Thus, since this case does not lead to a first order transition, we can expect one, if at all, only in the domain of small values of the elliptic modulus \( k \), and in fact, we shall see that this is the case.

In order to examine the type of instanton–sphaleron transition we have to introduce the fluctuation fields around the sphaleron and expand the field equations (2) up to the third order in these fields. If, however, one expands Eq. (2) naively, one will realize that the fluctuation operators are not diagonalized and, hence, the computation of the spectra of these operators becomes a very non–trivial problem. To avoid this difficulty, we choose the \( R_\xi \) gauge \([19]\) by adding to the original action \([1]\) the gauge fixing term

\[
S_{gf} = \frac{1}{2\xi} \int d\tau dx \left[ \partial_\mu A_\mu + \frac{ig}{2} \xi (\phi^2 - \phi^*2) \right]^2.
\]
Then the field equations for the total Euclidean action $S_E = S^{(0)}_E + S_{gf}$ become

$$\partial_\mu F_{\mu\nu} + \frac{1}{\xi} [\partial_\mu \partial_\nu A_\mu + ig \xi (\phi \partial_\nu \phi - \phi^* \partial_\nu \phi^*)] = ig [\phi^*(D_\nu \phi) - (D_\nu \phi)^* \phi],$$

$$D_\mu D_\nu \phi + ig \phi^* \left[ \partial_\mu A_\mu + \frac{ig \xi}{2} (\phi^2 - \phi^{*2}) \right] = 2\lambda \phi (|\phi|^2 - \frac{v^2}{2}).$$

One can show that the sphaleron in this gauge is the same as that of Eq.(4) if $A = 0$:

$$A_1 = 0,$$

$$\phi_{sph} = \frac{kb(k)}{\sqrt{\lambda}} sn[b(k)x].$$

We have therefore determined the sphaleron configuration in the most optimal way to permit continuation with the following difficult computations.

### 3 Fluctuations about the Sphaleron

We now introduce the fluctuation fields around the sphaleron configuration by setting

$$A_0(\tau, x) = a_0(\tau, x),$$

$$A_1(\tau, x) = a_1(\tau, x),$$

$$\phi(\tau, x) = \phi_{sph}(x) + \frac{1}{\sqrt{2}} \left( \eta_1(\tau, x) + i\eta_2(\tau, x) \right),$$

where $a_0, a_1, \eta_1, \eta_2$ are real fields. Inserting (10) into Eq.(1) and Eq.(7) one can express $S_E$ for $\xi = 1$ as

$$S_E = \frac{E_{sph}}{T} + S_2 + S_3 + S_4$$

where $1/T$ is the period of the sphaleron \cite{20, 21} and

$$E_{sph} = \sqrt{2\lambda v^3} \left[ \left( \frac{2}{1 + k^2} \right)^{-\frac{1}{2}} + \frac{1 + 2k^2}{3} \left( \frac{2}{1 + k^2} \right)^{\frac{3}{2}} - 2 \left( \frac{2}{1 + k^2} \right)^{\frac{1}{2}} \right] K(k)$$

\cite{22, 23}. 

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\[ E(k) = \left[ \frac{2}{1 + k^2} \right]^{\frac{1}{2}} - \frac{1 + k^2}{3} \left( \frac{2}{1 + k^2} \right)^{\frac{1}{2}} \]

\[
S_2 = \int d\tau dx \left[ \frac{1}{2} a_0 [-\partial_\mu \partial_\mu + 2g^2 \phi_{sph}^2] a_0 + \frac{1}{2} a_1 [-\partial_\mu \partial_\mu + 2g^2 \phi_{sph}^2] a_1 \\
+ \frac{1}{2} \eta_1 \left[ -\partial_\mu \partial_\mu + 2\lambda (3\phi_{sph}^2 - \frac{v^2}{2}) \right] \eta_1 + \frac{1}{2} \eta_2 \left[ -\partial_\mu \partial_\mu + 2(\lambda + g^2) \phi_{sph}^2 - \lambda v^2 \right] \eta_2 \\
+ 2\sqrt{2} g \phi'_{sph} a_1 \eta_2 \right],
\]

\[
S_3 = \int d\tau dx \left[ 2g (a_0 \eta_1 \eta_2 + a_1 \eta_1' \eta_2) + \sqrt{2} g \phi_{sph} (a_0^2 + a_1^2) \eta_1 \\
+ \sqrt{2} \lambda \phi_{sph} \eta_1^3 + \sqrt{2} (\lambda + g^2) \phi_{sph} \eta_1 \eta_2^2 \right],
\]

\[
S_4 = \int d\tau dx \left[ \frac{g^2}{2} (a_0^2 + a_1^2) (\eta_1^2 + \eta_2^2) + \frac{\lambda}{4} (\eta_1^2 + \eta_2^2)^2 + \frac{g^2}{2} \eta_1^2 \eta_2^2 \right].
\]

In these equations \( E(k) \) is the complete elliptic integral of the second kind, and the dot and the prime denote differentiation with respect to \( \tau \) and \( x \) respectively. Owing to the final term in \( S_2 \) the fluctuation operators for \( a_1 \) and \( \eta_2 \) are not diagonalized although the \( R_{\xi=1} \) gauge has been chosen. To obtain the diagonalization we introduce the fluctuation fields \( \rho_\pm \) defined as

\[
\rho_+ = v_1 a_1 + v_2 \eta_2, \quad \rho_- = -v_2 a_1 + v_1 \eta_2,
\]

where

\[
v_1 = \sqrt{\frac{1 - (\phi_{sph}^2 - \frac{v^2}{2}) f_1^{-\frac{1}{2}}}{2}}, \quad v_2 = \sqrt{\frac{1 + (\phi_{sph}^2 - \frac{v^2}{2}) f_1^{-\frac{1}{2}}}{2}},
\]

and

\[
f_1 = (\phi_{sph}^2 - \frac{v^2}{2})^2 \cosh^2 \alpha - \frac{v^4}{4} \left( \frac{1 - k^2}{1 + k^2} \right) \sinh^2 \alpha.
\]

Here \( \alpha = \sinh^{-1} 2\theta \) and \( \theta \) is the dimensionless parameter

\[
\theta \equiv \frac{2M_W}{M_H} = \sqrt{\frac{2g^2}{\lambda}}.
\]
Using the field redefinition (13) and the first–order differential equation for $\phi_{sph}$,

$$\phi'_{sph} + \sqrt{\lambda} \left[ \frac{v^4}{4} \left( \frac{2k}{1 + k^2} \right)^2 - v^2 \phi_{sph}^2 + \phi_{sph}^4 \right]^{\frac{1}{2}} = 0,$$

it is straightforward to show that $S_2$ becomes

$$S_2 = \frac{1}{2} \int d\tau dx [a_0 \rho_{a_0} + \eta_1 D_1 \eta_1 + \rho_+ D_+ \rho_+ + \rho_- D_- \rho_-],$$

where

$$D_0 = -\partial_\mu \partial_\mu + 2g^2 \phi_{sph}^2,$$

$$D_1 = -\partial_\mu \partial_\mu + 2\lambda (3\phi_{sph}^2 - \frac{v^2}{2}),$$

$$D_\pm = -\partial_\mu \partial_\mu + 2g^2 \phi_{sph}^2 + \lambda (\phi_{sph}^2 - \frac{v^2}{2}) \mp \lambda \sqrt{f_1}.$$ 

After inserting the field redefinition (13) into $S_3$ and $S_4$, one can derive the field equations for the fluctuation fields by varying the total action $S_E$, i.e. (the method of Ref. [7] to determine the order of thermal transitions requires all the terms written out explicitly here)

$$\hat{l} \left( \begin{array}{c} a_0 \\ \rho_+ \\ \rho_- \\ \eta_1 \end{array} \right) = \hat{h} \left( \begin{array}{c} a_0 \\ \rho_+ \\ \rho_- \\ \eta_1 \end{array} \right) + \left( \begin{array}{cccc} G_{a_0}^{a_0} \\ G_{a_0}^{\rho_+} \\ G_{a_0}^{\rho_-} \\ G_{a_0}^{\eta_1} \end{array} \right) + \left( \begin{array}{cccc} G_{3}^{a_0} \\ G_{3}^{\rho_+} \\ G_{3}^{\rho_-} \\ G_{3}^{\eta_1} \end{array} \right) + \cdots$$

where

$$\hat{l} = \left( \begin{array}{cccc} \frac{\partial^2}{\partial z_0^2} & 0 & 0 & 0 \\ 0 & \frac{\partial^2}{\partial z_0^2} & 0 & 0 \\ 0 & 0 & \frac{\partial^2}{\partial z_0^2} & 0 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial z_0^2} \end{array} \right),$$

$$\hat{h} = \left( \begin{array}{cccc} \hat{h}_{a_0} & 0 & 0 & 0 \\ 0 & \hat{h}_{\rho_+} & 0 & 0 \\ 0 & 0 & \hat{h}_{\rho_-} & 0 \\ 0 & 0 & 0 & \hat{h}_{\eta_1} \end{array} \right),$$

and

$$G_{2}^{a_0} = \frac{2g}{b(k)} (v_2 \rho_+ + v_1 \rho_-) \eta_1 + \frac{2\sqrt{2}g^2}{b^2(k)} \phi_{sph} a_0 \eta_1,$$
\[ G_{30}^0 = \frac{g^2}{b^2(k)} a_0 \left[ \eta_1^2 + (v_2\rho_+ + v_1\rho_-)^2 \right], \]
\[ G_{2}^{0+} = \frac{2g}{b(k)} [v_2 a_0 \eta_1 + (v_1^2 - v_2^2) \rho_- \eta_1' + 2v_1 v_2 \rho_+ \eta_1'], \]
\[ + \frac{2\sqrt{2} \lambda}{b^2(k)} \phi_{sph} [v_2^2 \rho_+ \eta_1 + v_1 v_2 \rho_- \eta_1] + \frac{2\sqrt{2} g^2}{b^2(k)} \phi_{sph} \rho_+ \eta_1, \]
\[ G_{3}^{0+} = \frac{g^2}{b^2(k)} \left[ \rho_+ \eta_1^2 + v_2^2 a_0 \rho_+ + v_1 v_2 a_0 \rho_- + 2v_1^2 v_2^2 \rho_+^3 + 3v_1 v_2 (v_1^2 - v_2^2) \rho_+^2 \rho_- \right. \]
\[ + (v_1^4 - 4v_1^2 v_2^2 + v_2^4) \rho_+^2 \rho_- - v_1 v_2 (v_1^2 - v_2^2) \rho_+^2 \rho_- \]
\[ + \frac{\lambda}{b^2(k)} \left[ v_1^2 \rho_- \eta_1^2 + v_1 v_2 \rho_+ \eta_1^2 + v_2^2 \rho_+^3 + 3v_1 v_2 \rho_+ \rho_-^2 + 3v_1^2 v_2 \rho_+ \rho_-^2 + 3v_2^2 \rho_+ \rho_-^2 \right], \]
\[ G_{3}^{0-} = \frac{g^2}{b^2(k)} \left[ \rho_- \eta_1^2 + v_1^2 a_0 \rho_- + v_1 v_2 a_0 \rho_+ + 2v_1^2 v_2^2 \rho_-^3 + v_1 v_2 (v_1^2 - v_2^2) \rho_-^2 \right. \]
\[ + (v_1^4 - 4v_1^2 v_2^2 + v_2^4) \rho_-^2 \rho_+ - 3v_1 v_2 (v_1^2 - v_2^2) \rho_-^2 \rho_+ \]
\[ + \frac{\lambda}{b^2(k)} \left[ v_1^2 \rho_- \eta_1^2 + v_1 v_2 \rho_+ \eta_1^2 + v_2^2 \rho_+^3 + 3v_1 v_2 \rho_+ \rho_-^2 + 3v_1^2 v_2 \rho_+ \rho_-^2 + 3v_2^2 \rho_+ \rho_-^2 \right], \]
\[ G_{2}^{m} = -\frac{2g}{b(k)} \left[ v_2 (a_0 \rho_+ + a_0 \rho_-) + v_1 (a_0 \rho_+ - a_0 \rho_-) + 2(v_1 v'_2 - v_2 v'_1) \rho_+ \rho_- \right. \]
\[ + (v_1^2 - v_2^2) (\rho'_+ \rho_- + \rho_+ \rho'_-) + v'_1 v_2 (\rho_+^2 - \rho_-^2) + v'_1 v_2 (\rho_+^2 - \rho_-^2) + 2v_1 v_2 (\rho_+ \rho'_+ - \rho_- \rho'_-) \]
\[ + \frac{\sqrt{2} \lambda}{b^2(k)} \phi_{sph} (3\eta_1^2 + v_2^2 \rho_-^2 + v_1^2 \rho_-^2 + 2v_1 v_2 \rho_+ \rho_-) + \frac{2\sqrt{2} g^2}{b^2(k)} \phi_{sph} (a_0^2 + \rho_-^2 + \rho_+^2), \]
\[ G_{3}^{m} = \frac{g^2}{b^2(k)} (a_0^2 + \rho_-^2 + \rho_+^2) \eta_1 + \frac{\lambda}{b^2(k)} [\eta_1^3 + (v_2 \rho_+ + v_1 \rho_-)^2 \eta_1]. \]

Here \( z_0 \equiv b(k) \tau \), \( z_1 \equiv b(k) x \), and the dot and the prime denote differentiation with respect to \( z_0 \) and \( z_1 \) respectively. Also, the fluctuation operators \( h_{a0}, h_{\rho+} \),
\[ \hat{h}_{\rho_-} \text{ and } \hat{h}_{\eta_1} \text{ are} \]

\[
\begin{align*}
\hat{h}_{a_0} &= -\frac{\partial^2}{\partial z_1^2} + \frac{2g^2}{b^2(k)} \phi_{\text{sph}}^2, \\
\hat{h}_{\rho_+} &= -\frac{\partial^2}{\partial z_1^2} + \frac{1}{b^2(k)} \left[ 2g^2 \phi_{\text{sph}}^2 + \lambda(\phi_{\text{sph}}^2 - \frac{v^2}{2}) + \lambda \sqrt{f_1} \right], \\
\hat{h}_{\rho_-} &= -\frac{\partial^2}{\partial z_1^2} + \frac{1}{b^2(k)} \left[ 2g^2 \phi_{\text{sph}}^2 + \lambda(\phi_{\text{sph}}^2 - \frac{v^2}{2}) - \lambda \sqrt{f_1} \right], \\
\hat{h}_{\eta_1} &= -\frac{\partial^2}{\partial z_1^2} + \frac{2\lambda}{b^2(k)} \left[ 3\phi_{\text{sph}}^2 - \frac{v^2}{2} \right].
\end{align*}
\]
method of Appendix A, one can show that the eigenstates of $\hat{h}_{\rho+}$ also consist of only positive modes which we do not need. What we need (as pointed out earlier), is only the negative mode of $\hat{h}_{\rho-}$. If one performs the numerical calculation, one finds that $\hat{h}_{\rho-}$ has two negative modes, one of which is $2K$–periodic and the other $2K$–antiperiodic. Fig. 1 shows the $k$–dependence of the negative eigenvalues for $\theta = 1$. Since the $2K$–antiperiodic boundary condition is required for the proper continuum limit, we have to use the solid line in Fig. 1 as a negative eigenvalue. One should note that this negative eigenvalue approaches zero in the small $k$ region. We show in the following that this effect guarantees that the instanton–sphaleron transition in the small $k$–region is different from that in the large $k$–region. Fig. 2 shows normalized $2K$–antiperiodic eigenfunctions for the negative mode of $\hat{h}_{\rho-}$ at $(\theta = 1, k = 0.6)$ and $(\theta = 1, k = 0.99)$. Their Gaussian shape is indicative of their ground–state nature (below the zero–eigenvalue of the translational mode).

We let $\psi^{(\rho-)}_{-1}$ and $\epsilon^{(\rho-)}_{-1}$ be respectively the $2K$–antiperiodic eigenfunction and corresponding eigenvalue for the negative mode. To obtain the criterion for the sharp first–order instanton–sphaleron transition we have to compute the nonlinear correction to the frequency of the periodic instanton around the sphaleron. This can be carried out by solving Eq.(20) perturbatively. The perturbation procedure is briefly summarized in Appendix B. The criterion for the first-order transition is expressed as an inequality\cite{6,7}

$$\Omega - \Omega_{sph} > 0,$$  

where $\Omega$ is the frequency involving the nonlinear correction and $\Omega_{sph} \equiv \sqrt{-\epsilon^{(\rho-)}_{-1}}$.

In Appendix B it is shown that the inequality (24) can be expressed as

$$< \psi^{(\rho-)}_{-1} \mid D_1(z_1) > < 0$$  

where

$$D_1(z_1) = D_1^{(1)}(z_1) + D_1^{(2)}(z_1) + D_1^{(3)}(z_1).$$  

Here

$$D_1^{(1)}(z_1) = \frac{2 \sqrt{2(1+k^2)}}{v} \psi^{(\rho-)}_{-1}(z_1) \left[ k \left( v_1^2 + \frac{s_1(s_1+1)}{2} \right) sn[z_1]g_{0,1}(z_1) - \sqrt{s_1(s_1+1)} v_1v_2g'_{0,1}(z_1) \right],$$
\[ D_1^{(2)}(z_1) = \frac{\sqrt{2(1 + k^2)}}{v} \psi_\perp(-1)(z_1) \left[ k \left( v_1^2 + \frac{s_1(s_1 + 1)}{2} \right) sn[z_1]g_{n,2}(z_1) ight. \\
- \sqrt{s_1(s_1 + 1)}v_1v_2g'_{n,2}(z_1) \left. \right] , \]
\[ D_1^{(3)}(z_1) = \frac{3(1 + k^2)}{4v^2} [v_1^4 + s_1(s_1 + 1)v_1^2v_2^2] \psi_\perp(-3)(z_1) , \tag{27} \]

where \( s_1 \equiv \sqrt{\theta^2 + \frac{1}{4}} - \frac{1}{2} \) and

\[ g_{n,1}(z_1) = \hat{h}_{n}^{-1} | q(z_1) > , \tag{28} \]
\[ g_{n,2}(z_1) = (\hat{h}_{n} + 4\Omega_{sph}^2)^{-1} | q(z_1) > , \]
\[ | q(z_1) > = -\frac{1}{v} \sqrt{\frac{1 + k^2}{2}} \left[ \theta \left( (v_1v_2)'\psi_\perp(-2) + 2v_1v_2\psi_\perp(-2) \right) \\
+ k(v_1^2 + \frac{\theta^2}{2})sn[z_1]\psi_\perp(-2) \right] . \]

It is now necessary to evaluate \( g_{n,1} \) and \( g_{n,2} \) explicitly. Although one can calculate \( g_{n,1} \) exactly by following the procedure given in the Appendix of Ref.[23], this is not necessary here. We already know the type of instanton–sphaleron transition at \( k = 1 \)[14, 15] so that our interest concerns only the domain of small \( k \). We can therefore adopt the following approximate procedure which has been shown to be valid in the small \( k \) region[23]. Using the completeness relation one can express \( g_{n,1} \) as

\[ g_{n,1} = \sum_{n=0}^{\infty} \frac{\langle \psi_{n}(q) | q \rangle}{\epsilon_{n}(m)} | \psi_{n}(m) > . \tag{29} \]

Since \( | q \rangle \) is an odd function, the zero mode of \( \hat{h}_{n} \) does not contribute to the r.h.s. of Eq.(29). Hence the first approximation of \( g_{n,1} \) is

\[ g_{n,1} \approx \frac{\langle \psi_{1}(m) | q \rangle}{\epsilon_{1}(m)} | \psi_{1}(m) > . \tag{30} \]

which can be evaluated numerically. In fact, this approximation is valid when \( | \psi_{1}(m) > \) is an isolated discrete mode and the density of higher states is very
dilute. Ref. [23] shows these conditions are fulfilled in the small $k$–region if $\hat{h}_{\eta_1}$ is a Lamé operator as is the case here. In the same way $g_{\eta,2}$ is approximately

$$g_{\eta,2} \approx \frac{\langle \psi_{1}^{(m)} | q | \psi_{1}^{(m)} \rangle}{3k^2 + 4\Omega^2_{sph}}.$$  (31)

The plots of Fig. 3 show the $k$-dependence of

$$J_i \equiv \langle \psi_{1}^{(\rho -)} | D_{1}^{(i)} \rangle$$

and of the sum $J_1 + J_2 + J_3$ for $\theta = 1$. One can see that this sum becomes negative at approximately $k = 0.2$ and therefore satisfies the inequality (25) and so (24) for the existence of a first order transition. Thus Fig. 3 demonstrates that the sharp first–order instanton–sphaleron transition occurs at $k < k_c \approx 0.2$ for $\theta = 1$. Although the result is not included in this paper, we have checked also the $\theta = 3$ case and have found a similar behavior: a sharp transition occurs in the small $k$ region.

5 Conclusions

The study of phase transitions is of considerable significance in many areas of physics, but – as is also evident from the above – this requires highly nontrivial efforts, both analytically and numerically. In the above we studied a model which permits a considerable fraction of analytical investigation, but finally requires also highly nontrivial computational work. The results we presented above answer the naturally asked question as to what behavior the Abelian–Higgs model would reveal if the spacial coordinate is compactified on a circle. We have found that indeed a change occurs as compared to the uncompactified case, i.e. in the region of small elliptic modulus $k$ of the periodic instantons that we used, which corresponds to circle–circumferences below a critical size (a specific critical value was given for appropriate values of other parameters). Hence, depending on $k$, this model allows both smooth second–order transitions in the large $k$ region and sharp first–order transitions in the small $k$ region. These findings are similar to those of $d = 4$ SU(2)–Higgs theory in which the type of transition depends on the ratio of $M_H$ and $M_W$. Thus our findings can be seen as an analogy. Of course, our model lacks direct physical application, but this was also not anticipated. Rather we explored the model also for the other reasons stated, i.e.
as a matter of curiosity as to what type of thermal behavior will be found once the spatial coordinate is compactified, and as a further testing ground for methods of investigation of phase transitions, here in the sense of transitions from quantum to classical behavior. Such investigations are usually very complicated and there are few models that permit also transparent analytical investigation, at least in part.

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| Eigenvalue of $h_{m}$ | Eigenfunction of $h_{m}$ |
|------------------------|---------------------------|
| $\lambda_{0}^{(m)} = 0$ | $\psi_{0}^{(m)}(z_{1}) = N_{0}cn[z_{1}]dn[z_{1}]$ |
| $\lambda_{1}^{(m)} = 3k^{2}$ | $\psi_{1}^{(m)}(z_{1}) = N_{1}sn[z_{1}]dn[z_{1}]$ |

**Table**
Appendix A

Here we explain how the spectrum of $\hat{h}_{\rho_-}$ is obtained. The spectrum of $\hat{h}_{\rho_+}$ can be obtained similarly. The eigenvalue equation of $\hat{h}_{\rho_-}$ is

$$
\left[-\frac{\partial^2}{\partial z_1^2} + f(k, \theta, z_1)\right] \psi_n^{(\rho_-)} = \zeta^{} \psi_n^{(\rho_-)} \quad (32)
$$

where

$$
f(k, \theta, z_1) = (1 + \theta^2) k^2 \text{sn}^2[z_1] - \sqrt{(1 + 4\theta^2) [k^2 \text{sn}^2[z_1] - \frac{1 + k^2}{2}]}^2 - \theta^2 (1 - k^2)^2,
$$

$$
\zeta = \epsilon^{(\rho_-)} + \frac{1 + k^2}{2}. \quad (33)
$$

We first choose the $4K$–periodic boundary condition. In this case we can use the Fourier expansions

$$
f(k, \theta, z_1) = \sum_{n=-\infty}^{\infty} a_n e^{i \frac{n\pi}{l} z_1}, \quad \psi_n^{(\rho_-)} = \sum_{n=-\infty}^{\infty} b_n e^{i \frac{n\pi}{l} z_1}, \quad (34)
$$

where $l = 2K$ and the coefficients $a_n$ are given by

$$
a_n = \frac{1}{2l} \int_{-l}^{l} f(k, \theta, z_1) e^{-i \frac{n\pi}{l} z_1}. \quad (35)
$$

Inserting (34) into (32) and using the property of linear independence of the exponential functions one obtains

$$
\sum_m \left[ \left(\frac{n\pi}{l}\right)^2 \delta_{mn} + a_{n-m} \right] b_m = \zeta^{} b_n. \quad (36)
$$

Solving this matrix equation numerically, one can evaluate the eigenvalue $\epsilon_n^{(\rho_-)}$ and eigenfunction $\psi_n^{(\rho_-)}$. After that we choose only $2K$–antiperiodic eigenfunctions and determine the corresponding eigenvalues for the proper $k = 1$ limit.

Appendix B

In this appendix we show briefly how the inequality (25) is derived for the criterion of the sharp first–order transition by solving Eq.(20) perturbatively. First we choose an ansatz

$$
\begin{pmatrix}
a_0 \\
r_+ \\
r_- \\
n_1
\end{pmatrix}
= \Delta \begin{pmatrix}
a_0,0(z_1) \\
r_+,0(z_1) \\
r_-,0(z_1) \\
n_1,0(z_1)
\end{pmatrix} \cos \Omega_{\text{sph}} z_0 \quad (37)
$$

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where $\Delta$ is a small oscillation amplitude around the sphaleron. After inserting (37) into Eq.(20) and neglecting higher order terms, one obtains
\begin{equation}
\begin{align*}
\Omega_{sph} &= \sqrt{-\epsilon_{-1}}, \\
a_{0,0} &= 0, \\
\rho_{-0} &= \psi_{-1}^{(\rho-)}, \\
\rho_{+0} &= 0,
\end{align*}
\end{equation}

(38)

For the next order perturbation we set
\begin{equation}
\begin{pmatrix}
a_0 \\
rho_- \\
\eta_1
\end{pmatrix} = \begin{pmatrix}
\Delta^2 a_{0,1}(z_0, z_1) \\
\Delta^2 \rho_{+,1}(z_0, z_1) \\
\rho_{-0}(z_1) \cos \Omega z_0 + \Delta^2 \rho_{-,1}(z_0, z_1)
\end{pmatrix}.
\end{equation}

(39)

Inserting Eq.(39) into Eq.(20) and considering only terms up to quadratic order, one can show there is no frequency shift to this order. It is also straightforward to show that $a_{0,1} = 0$, $\rho_{+,1} = 0$, $\rho_{-,1} = 0$, and
\begin{equation}
\eta_{1,1} = g_{\eta,1}(z_1) + g_{\eta,2}(z_1) \cos 2\Omega_{sph} z_0
\end{equation}

(40)

where $g_{\eta,1}$ and $g_{\eta,2}$ are given by Eq.(28). For the next order perturbation we set
\begin{equation}
\begin{pmatrix}
a_0 \\
rho_+ \\
rho_- \\
\eta_1
\end{pmatrix} = \begin{pmatrix}
\Delta^3 a_{0,2}(z_0, z_1) \\
\Delta^3 \rho_{+,2}(z_0, z_1) \\
\rho_{-0}(z_1) \cos \Omega z_0 + \Delta^3 \rho_{-,2}(z_0, z_1) \\
\Delta^2 \eta_{1,1}(z_0, z_1) + \Delta^3 \eta_{1,2}(z_0, z_1)
\end{pmatrix}.
\end{equation}

(41)

Inserting this into Eq.(20) and considering contributions up to cubic order, one can show that there is a frequency change in this order given by
\begin{equation}
\Omega_{sph}^2 - \Omega_0^2 = \Delta^2 < \rho_{-,0} \mid D_1 >
\end{equation}

(42)

which proves Eq.(25).
Figure Captions

Fig.1
$k$–dependence of the negative eigenvalues $\epsilon^{(\rho-)}$ for $\hat{h}_{\rho-}$ for $\theta = 1$. The dotted line and the solid line represent the negative eigenvalues for the $2K$–periodic and $2K$–antiperiodic eigenfunctions respectively. For the correct $k = 1$ limit we have to choose the solid line as the negative eigenvalue.

Fig.2
The normalized $2K$–antiperiodic eigenfunctions for the negative mode of $\hat{h}_{\rho-}$ for (a) $\theta = 1$, $k = 0.6$, and (b) $\theta = 1$, $k = 0.99$.

Fig.3
$k$–dependence of $J_1$, $J_2$, $J_3$, and $J_1 + J_2 + J_3$ for $\theta = 1$. This shows that the sharp first–order instanton–sphaleron transition occurs for $k < k_c \approx 0.2$. 
Fig. 1
$\theta = 1, k = 0.6$

Fig. (2a)
Fig. (2b)

\[ \theta = 1, \ k = 0.99 \]

\[ \psi_{-1}^{(\rho)} \]
Fig. 3