Mohamed Elouafi

On formulae for the determinant of symmetric pentadiagonal Toeplitz matrices

Abstract We show that the characteristic polynomial of a symmetric pentadiagonal Toeplitz matrix is the product of two polynomials given explicitly in terms of the Chebyshev polynomials.

Mathematics Subject Classification 15B05 · 65F40 · 33C45

1 Introduction

We consider here the problem of finding the determinant of the $m \times m$ symmetric pentadiagonal Toeplitz matrix

$$P_m = P_m(a, b, c) = \begin{pmatrix} a & b & c & 0 & \cdots & 0 \\ b & a & b & \cdots & \vdots \\ c & b & a & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & b & c \\ \vdots & \cdots & c & b & a & b \\ 0 & \cdots & 0 & c & b & a \end{pmatrix}.$$ 

This class of matrices arises naturally in many applications, such as signal processing, trigonometric moment problems, integral equations and elliptic partial differential equations with boundary conditions [9]. Computing the determinant of the matrix $P_m$ have intrigued the researchers for decades. If $c = 0$, then $P_m$ is reduced to a tridiagonal matrix and there exists a closed form of $\det(P_m)$ from which the eigenvalues of the matrix are explicitly given. It is becoming a challenge to find similar formulae for the general case and so far, little is known about the eigenvalues of $P_m$ [1, 2, 5, 8]. In [5, 7], $\det(P_m)$ is explicitly computed using the kernel of the Chebyshev polynomials $\{T_n\}$, $\{U_n\}$, $\{V_n\}$ and $\{W_n\}$ [11] and, as a consequence, the eigenvalues

M. Elouafi (✉)
Classes Préparatoires aux Grandes Ecoles d’Ingénieurs, Lycée Moulay Alhassan, Tangier, Morocco
E-mail: med3elouafi@gmail.com
of the matrix \( P_m \) are localized by means of explicitly given rational functions. The formulae are simplified to give \( \det(P_m) \) as polynomials of the parameters \( a, b, c \) [6].

In the new formula presented here, \( \det(P_m) \) is given as the product of two polynomials given in a standard form. Here is our main result:

**Theorem 1.1** We have

\[
\det(P_{2n+1}) = 2 \left( \sum_{k=0}^{n+1} \gamma_{n,k} c^{n+1-k} b^k T_k \left( \frac{a + 2c}{2b} \right) \right) \left( \sum_{k=0}^{n+1} \gamma_{n,k} c^{n+1-k} b^{k-1} U_{k-1} \left( \frac{a + 2c}{2b} \right) \right),
\]

and

\[
\det(P_{2n}) = \left( \sum_{k=0}^{n} \mu_{n,k} c^{n-k} b^k V_k \left( \frac{a + 2c}{2b} \right) \right) \left( \sum_{k=0}^{n} \mu_{n,k} c^{n-k} b^k W_k \left( \frac{a + 2c}{2b} \right) \right),
\]

where

\[
\gamma_{n,k} = (-1)^k \left( \frac{n + 1 + k}{n + 1 - k} \right), \quad \mu_{n,k} = (-1)^k \left( \frac{n + 1 + k}{n - k} \right).
\]

**2 Proof of the main result**

Since \( \det(P_m(a, b, c)) = \det(P_m \left( \frac{a}{c}, \frac{b}{c}, 1 \right)) \), then we can assume for simplicity that \( c = 1 \). We denote by \( \zeta_j, \frac{1}{\zeta_j}, j = 1, 2, \) the roots of the polynomial \( g(x) = x^3 + bx^3 + ax^2 + bx + 1 \) assumed pairwise distinct and different of \( \pm 1 \).

Recall that the Chebyshev polynomials \( \{T_n\}, \{U_n\}, \{V_n\} \) and \( \{W_n\} \) are orthogonal polynomials over \((-1, 1)\) with respect to the weight \( \frac{1}{\sqrt{1-x^2}}, \sqrt{1-x^2}, \frac{1}{\sqrt{1+x^2}}, \) and \( \frac{1}{\sqrt{1+x^2}} \), respectively, and we have for \( \zeta \in \mathbb{C}^* \)

\[
\begin{align*}
T_n \left( \frac{1}{2} \left( \zeta + \frac{1}{\zeta} \right) \right) &= \frac{1}{2} \left( \zeta^n + \zeta^{-n} \right), \quad U_n \left( \frac{1}{2} \left( \zeta + \frac{1}{\zeta} \right) \right) = \frac{\zeta^{n+1} - \zeta^{-n-1}}{\zeta - \zeta^{-1}}, \\
V_n \left( \frac{1}{2} \left( \zeta + \frac{1}{\zeta} \right) \right) &= \frac{\zeta^{n+1/2} + \zeta^{-n-1/2}}{\zeta^{1/2} + \zeta^{-1/2}}, \quad W_n \left( \frac{1}{2} \left( \zeta + \frac{1}{\zeta} \right) \right) = \frac{\zeta^{n+1/2} - \zeta^{-n-1/2}}{\zeta^{1/2} - \zeta^{-1/2}}.
\end{align*}
\]

We shall use the following formula for \( \det(P_m) \):

**Lemma 2.1** For \( J \subset \{1, 2\} \), let \( I_J(k) = \begin{cases} 1 & \text{if } k \in J \\ -1 & \text{if } k \notin J \end{cases} \) and

\[
\omega_J = \prod_{k=1}^{2} \gamma_{kJ(k)}, \quad \gamma_J = \prod_{1 \leq j < k \leq 2} \left( \zeta_j I_j(k) - \zeta_k I_k(j) \right).
\]

We have

\[
\det(P_m) = \frac{1}{d^2 \prod_{k=1}^{2} (\zeta_k - \zeta_k^{-1})} \left( \sum_{J} (-1)^{|J|} \gamma_J \omega_{J} \right)^{\frac{m+1}{2}},
\]

where \( d = \left( \zeta_2 + \frac{1}{\zeta_2} - \zeta_1 - \frac{1}{\zeta_1} \right) \).

**Proof** See [4].

Let us put \( \alpha = \zeta_1 \zeta_2, \beta = \zeta_1 \zeta_2^{-1} \) and \( u = \frac{1}{2} (\alpha + \alpha^{-1}), v = \frac{1}{2} (\beta + \beta^{-1}) \). We have by the Vieta’ formulae:

\[
u + v = \frac{1}{2} (\alpha + \alpha^{-1} + \beta + \beta^{-1}) = \frac{a}{2} - 1.
\]
and

\[ uv = \frac{1}{4}(\xi_1^2 + \xi_1^{-2} + \xi_2^2 + \xi_2^{-2}) \]

\[ = \frac{1}{4}((\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1})^2 - 2(2 + \alpha + \alpha^{-1} + \beta + \beta^{-1})) \]

\[ = \frac{b^2}{4} - \frac{a}{2}. \]

This implies that \((u + 1)(v + 1) = \left(\frac{b}{2}\right)^2\).

**Lemma 2.2** We have

\[
\text{det}(P_m) = \frac{U_{m+1}^2 \left(\sqrt{\frac{1+u}{2}}\right) - U_{m+1}^2 \left(\sqrt{\frac{1+v}{2}}\right)}{2(u-v)}. \tag{F1}
\]

**Proof** Using the notations from Lemma 2.1, we obtain

\[
\sum_J \gamma_I \omega_j^{m+1} = (\xi_2 - \xi_1)\alpha \frac{m+1}{2} + (\xi_2^{-1} - \xi_1^{-1})\beta \frac{m+1}{2} + (\xi_2^{-1} - \xi_1^{-1})\alpha^{-\frac{m+1}{2}} + (\xi_2 - \xi_1^{-1})\beta^{-\frac{m+1}{2}}
\]

\[= (\xi_2 - \xi_1) \left(\alpha \frac{m+1}{2} - \alpha^{-\frac{m+1}{2}}\right) + (\xi_2^{-1} - \xi_1^{-1}) \left(\beta \frac{m+1}{2} - \beta^{-\frac{m+1}{2}}\right). \]

Remark that

\[
\frac{(\xi_2 - \xi_1)}{(\xi_2^{-1} - \xi_1)} = \frac{\xi_2 \xi_1^{-1} - 1}{\xi_1 \xi_2^{-1} - 1}
\]

\[= \frac{\beta^{-1} - 1}{\alpha^{-1} - 1}
\]

\[= \frac{\alpha}{\alpha - 1} \times \frac{\beta - 1}{\beta}.
\]

and hence

\[
\sum_J \gamma_I \omega_j^{m+1} = (\xi_2^{-1} - \xi_1) \left[\left(\frac{\xi_2 - \xi_1}{\xi_2^{-1} - \xi_1}\right) \left(\alpha \frac{m+1}{2} - \alpha^{-\frac{m+1}{2}}\right) + \left(\beta \frac{m+1}{2} - \beta^{-\frac{m+1}{2}}\right)\right]
\]

\[= (\xi_2^{-1} - \xi_1) \left[\frac{\alpha}{\alpha - 1} \times \frac{\beta - 1}{\beta} \left(\alpha \frac{m+1}{2} - \alpha^{-\frac{m+1}{2}}\right) + \left(\beta \frac{m+1}{2} - \beta^{-\frac{m+1}{2}}\right)\right]
\]

\[= \left(\frac{\xi_2^{-1} - \xi_1}{\beta - 1}\right) \left(\frac{\alpha \frac{m+1}{2} - \alpha^{-\frac{m+1}{2}}}{\alpha - 1} + \frac{\beta \frac{m+1}{2} - \beta^{-\frac{m+1}{2}}}{\beta - 1}\right).
\]

On the other hand

\[
\frac{\alpha \frac{m+1}{2} - \alpha^{-\frac{m+1}{2}}}{\alpha - 1} = \frac{\alpha^{1/2} - \alpha^{-1/2}}{\alpha^{1/2} - \alpha^{-1/2}}
\]

\[= U_{m+1} \left(\frac{1}{2} \left(\alpha^{1/2} + \alpha^{-1/2}\right)\right)
\]

\[= U_{m+1} \left(\sqrt{\frac{1+u}{2}}\right).
\]

Similarly, we obtain that

\[
\frac{\beta \frac{m+1}{2} - \beta^{-\frac{m+1}{2}}}{\beta - 1} = U_{m+1} \left(\sqrt{\frac{1+u}{2}}\right).
\]
Consequently
\[ \sum_j \gamma_j \omega_j^{m+1} = \frac{(\xi_2^{-1} - \xi_1) (\beta - 1)}{\beta} \left( U_{m+1}\left(\sqrt{\frac{1+u}{2}}\right) + U_{m+1}\left(\sqrt{\frac{1+v}{2}}\right) \right). \]

By the same method, we get
\[ \sum_j (-1)^{|j|} \gamma_j \omega_j^{m+1} = \frac{(\xi_2^{-1} - \xi_1) (\beta - 1)}{\beta} \left( U_{m+1}\left(\sqrt{\frac{1+u}{2}}\right) - U_{m+1}\left(\sqrt{\frac{1+v}{2}}\right) \right). \]

Finally
\[
\det (P_m (a, b, 1)) = \frac{1}{d^2} \prod_{k=1}^{2} \left( \xi_k - \xi_k^{-1} \right) \left( \sum_j (-1)^{|j|} \gamma_j \omega_j^{m+1} \right) \times \left( \sum_j \gamma_j \omega_j^{m+1} \right) \\
= C \left( U_{m+1}^2\left(\sqrt{\frac{1+u}{2}}\right) - U_{m+1}^2\left(\sqrt{\frac{1+v}{2}}\right) \right),
\]
where
\[
C = \frac{(\xi_2^{-1} - \xi_1)^2 (\beta - 1)^2}{\beta^2 d^2 \prod_{k=1}^{2} \left( \xi_k - \xi_k^{-1} \right)}.
\]

A straightforward computation (using the Maple software for example) shows that
\[
C = \frac{1}{2 (u - v)},
\]
and this completes the proof of the Lemma.

\[ \square \]

Remark 2.3 We have \( u + v + 2 = \frac{a}{2} + 1 \) and \( (u + 1) \ (v + 1) = \left( \frac{b}{2} \right)^2 \). Then, \( u + 1 \) and \( v + 1 \) are the zeros of the second-order equation \( x^2 - \left( \frac{a}{2} + 1 \right) x + \left( \frac{b}{2} \right)^2 = 0 \). This gives for example
\[
u + 1 = \frac{1}{2} \left( \frac{a}{2} + 1 - \sqrt{\left( \frac{a}{2} + 1 \right)^2 - b^2} \right),
\]
and
\[
u + 1 = \frac{1}{2} \left( \frac{a}{2} + 1 + \sqrt{\left( \frac{a}{2} + 1 \right)^2 - b^2} \right).
\]

The term \( \frac{U_{m+1}^2\left(\sqrt{\frac{1+u}{2}}\right) - U_{m+1}^2\left(\sqrt{\frac{1+v}{2}}\right)}{2(u-v)} \) is a symmetric polynomial of \( u + 1 \) and \( v + 1 \) and, consequently, it can be expressed in terms of the elementary symmetric polynomials \( u + 1 + v + 1 = \frac{a}{2} + 1 \) and \( (u + 1) \ (v + 1) = \left( \frac{b}{2} \right)^2 \). For this, we distinguish two cases:

Case 1: \( m = 2n + 1 \). Using the following expression of \( U_{2n+2} (x) \)[3]:
\[
U_{2n+2} (x) = \sum_{k=0}^{n+1} (-1)^k \binom{2n + 2 - k}{k} (2x)^{2n+2-2k} \\
= (-1)^{n+1} \sum_{k=0}^{n+1} \gamma_{n,k} (2x)^{2k}, \quad \gamma_{n,k} = (-1)^k \binom{n + 1 + k}{n + 1 - k},
\]

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we obtain
\[ U_{2n+2} \left( \sqrt{\frac{1 + u}{2}} \right) + U_{2n+2} \left( \sqrt{\frac{1 + v}{2}} \right) = (-1)^{n+1} \sum_{k=0}^{n+1} \gamma_{n,k} 2^k \left( (1 + u)^k + (1 + v)^k \right), \]
\[ U_{2n+2} \left( \sqrt{\frac{1 + u}{2}} \right) - U_{2n+2} \left( \sqrt{\frac{1 + v}{2}} \right) = (-1)^{n+1} \sum_{k=1}^{n+1} \gamma_{n,k} 2^k \left( (1 + u)^k - (1 + v)^k \right). \]

On the other hand, we have for \( x, y \):
\[ x^{2k} + y^{2k} = (xy)^k \left( \left( \frac{x}{y} \right)^k + \left( \frac{x}{y} \right)^{-k} \right) = 2 (xy)^k T_k \left( \frac{x}{2y} + \frac{y}{2x} \right), \]
and for \( k \geq 1 \)
\[ x^{2k} - y^{2k} = (xy)^k \left( \left( \frac{x}{y} \right)^k - \left( \frac{x}{y} \right)^{-k} \right) = (x^2 - y^2)(xy)^{k-1} U_{k-1} \left( \frac{x}{2y} + \frac{y}{2x} \right). \]

Applying those formulae for \( x = \sqrt{1 + u} \) and \( y = \sqrt{1 + v} \) where
\[ xy = \frac{b}{2}, \quad x^2 - y^2 = u - v, \]
and
\[ \frac{x}{2y} + \frac{y}{2x} = \frac{x^2 + y^2}{2xy} = \frac{2 + u + v}{b} = \frac{a + 2}{2b}, \]
gives
\[ (1 + u)^k + (1 + v)^k = 2 \left( \frac{b}{2} \right)^k T_k \left( \frac{a + 2}{2b} \right), \]
and for \( k \geq 1 \):
\[ (1 + u)^k - (1 + v)^k = (u - v) \left( \frac{b}{2} \right)^{k-1} U_{k-1} \left( \frac{a + 2}{2b} \right). \]

Case 2: \( m = 2n \). We have [3]:
\[ U_{2n+1} (x) = \sum_{k=0}^{n} (-1)^k \binom{2n + 1 - k}{k} (2x)^{2n+1-2k} \]
\[ = (-1)^n \sum_{k=0}^{n} \mu_{n,k} (2x)^{2k+1}, \quad \mu_{n,k} = (-1)^k \binom{n + 1 + k}{n - k}, \]
and thus
\[
U_{2n+1} \left( \frac{\sqrt{1+u}}{2} \right) + U_{2n+1} \left( \frac{\sqrt{1+v}}{2} \right) = (-1)^n \sum_{k=0}^{n} \mu_{n,k} 2^{k+1/2} \left((1+u)^{k+1/2} + (1+v)^{k+1/2}\right),
\]

\[
U_{2n+1} \left( \frac{\sqrt{1+u}}{2} \right) - U_{2n+1} \left( \frac{\sqrt{1+v}}{2} \right) = (-1)^n \sum_{k=1}^{n} \mu_{n,k} 2^{k+1/2} \left((1+u)^{k+1/2} - (1+v)^{k+1/2}\right).
\]

As for the odd case, we have for \(x, y\):

\[
x^{2k+1} + y^{2k+1} = (xy)^{k+1/2} \left( \left( \frac{x}{y} \right)^{k+1/2} + \left( \frac{x}{y} \right)^{-k-1/2} \right)
\]

\[
= (xy)^{k+1/2} \left( \left( \frac{x}{y} \right)^{1/2} + \left( \frac{x}{y} \right)^{-1/2} \right) V_k \left( \frac{x}{2y} + \frac{y}{2x} \right),
\]

and

\[
x^{2k+1} - y^{2k+1} = (xy)^{k+1/2} \left( \left( \frac{x}{y} \right)^{k+1/2} - \left( \frac{x}{y} \right)^{-k-1/2} \right)
\]

\[
= (xy)^{k+1/2} \left( \left( \frac{x}{y} \right)^{1/2} - \left( \frac{x}{y} \right)^{-1/2} \right) W_k \left( \frac{x}{2y} + \frac{y}{2x} \right).
\]

This implies

\[
(1+u)^{k+1/2} + (1+v)^{k+1/2} = \left( \sqrt{1+u} + \sqrt{1+v} \right) \left( \frac{b}{2} \right)^k V_k \left( \frac{a+2}{2b} \right).
\]

and

\[
(1+u)^{k+1/2} - (1+v)^{k+1/2} = \left( \sqrt{1+u} - \sqrt{1+v} \right) \left( \frac{b}{2} \right)^k W_k \left( \frac{a+2}{2b} \right).
\]

which completes the proof of Theorem 1.1.

3 Numerical computation of \(\text{det} (P_m)\)

In this section, we shall derive from the formulae (1) and (2) an efficient algorithm for computing \(\text{det} (P_m)\). We are lead to evaluate sums of the form

\[
S_N = \sum_{k=0}^{N} \alpha_k P_k (x)
\]

where \(x = \frac{a+2c}{2b}\), and \(\{P_r\}\) are polynomials that satisfy the three-term recurrence

\[
P_r (x) - 2x P_{r-1} (x) + P_{r-2} (x) = 0.
\]

Such sums can be computed efficiently through the following method described in [11]:

Equation (4) may be written in matrix notation as \(M p = q\), where \(M\) is the \((N+1) \times (N+1)\) matrix
\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
-2x & 1 & \cdots & 0 \\
1 & -2x & 1 & \cdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & 1 & -2x & 1 \\
0 & 0 & 1 & -2x & 1
\end{pmatrix},
\]
\[
p = \begin{pmatrix}
P_0(x) \\
P_1(x) \\
\vdots \\
P_N(x)
\end{pmatrix}
\quad \text{and} \quad q = \begin{pmatrix}
P_0(x) \\
-2x P_0(x) + P_1(x) \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Let
\[
y^T = (y_0, y_1, \ldots, y_N)
\]
be the row vector such that
\[
y^T M = u^T = (\alpha_0, \alpha_1, \ldots, \alpha_N).
\]
Thus, \( y_k \) are computed by putting \( y_{N+1} = y_{N+2} = 0 \) and performing the three-term recurrence
\[
y_k = 2xy_{k+1} - y_{k+2} + \alpha_k, \quad \text{for } k = N, \ldots, 0.
\]

It follows that
\[
S_N = u^T p = y^T M p = y^T q = y_0 P_0(x) + (P_1(x) - 2x P_0(x)) y_1.
\]
For \( P_k = T_k \) and \( P_k = \frac{1}{b} U_{k-1} \), with \( U_{-1} = 0 \), respectively, we obtain
\[
\sum_{k=0}^{n+1} y_{n,k} c^{n+1-k} b^k T_k (x) = y_0 - xy_1
\]
and
\[
\sum_{k=1}^{n+1} y_{n,k} c^{n+1-k} b^{k-1} U_{k-1} (x) = \frac{1}{b} y_1,
\]
where \( y_{n+2} = y_{n+3} = 0 \) and
\[
y_k = 2xy_{k+1} - y_{k+2} + y_{n,k} c^{n+1-k} b^k, \quad \text{for } k = n + 1, \ldots, 0.
\]
For \( P_k = V_k \) and \( P_k = W_k \), respectively, we obtain
\[
\sum_{k=0}^{n} \mu_{n,k} c^{n-k} b^k V_k (x) = y_0 - y_1
\]
and
\[
\sum_{k=0}^{n} \mu_{n,k} c^{n-k} b^k W_k (x) = y_0 + y_1,
\]
where \( y_{n+1} = y_{n+2} = 0 \) and
\[
y_k = 2xy_{k+1} - y_{k+2} + \mu_{n,k} c^{n-k} b^k, \quad \text{for } k = n, \ldots, 0.
\]
Here is the implementation of the algorithm in Maple (To accelerate the algorithm, the terms $\gamma_{n,k}a^{n+1-k}b^k$ and $\mu_{n,k}c^{n-k}b^k$ are computed recursively at the same time as $y_k$. Implementation details are omitted):

```maple
## Computing det(P_{2n+1})
##
detP1:=proc(n,a,b,c)
local i,j,r,s,x,k,t,z;
i := 0;
j := 0;
r := (-1)^n*b^n;
x:=(a+2*c)/b;
t:=2*n;
z:=-c/b;
for k from 0 to n+1 do
s:=i;
i:=r+x*i-j;  # i:=simplify(r+x*i-j); if the purpose
    # is to compute the characteristic
    # polynomial with variable a
j:=s;
r:=r*z*((t+2)*(t+1))/((t+k+2)*(k+1));
t:=t-2;
od;
return 2*j*(i-(j*x/2)/b;
    # return simplify(2*j*(i-(j*x/2)/b);
    # if the purpose is to compute the
    # characteristic polynomial with
    # variable a
end;

## Computing det(P_{2n})
##
detP2:=proc(n,a,b,c)
local i,j,r,s,x,k;
i := 0;
j := 0;
r := (-1)^n*b^n;
x:=(a+2*c)/b;
t:=2*n;
z:=-c/b;
for k from 0 to n do
s:=i;
i:=r+x*i-j;  # i:=simplify(r+x*i-j); if the purpose
    # is to compute the characteristic
    # polynomial with variable a
j:=s;
r:=r*z*((t+1)*(t+1))/((t+k+1)*(k+1));
t:=t-2;
od;
return i^2-j^2;  # return simplify(i^2-j^2);
    # if the purpose is to compute
    # the characteristic polynomial
    # with variable a
end;
```

One can easily check that the complexity of the algorithm is about $7N$, where $N$ is the size of the matrix. Thus, the algorithm is the fastest among many other recently proposed (we exclude those based on the roots of certain polynomials which are approximative) [10]. Moreover, subject to minor modifications as explained in Algorithm 1, the algorithm is suitable for computing the characteristic polynomial of a symmetric
pentadiagonal Toeplitz matrix using computer algebra systems such as MAPLE, MATHEMATICA, MATLAB and MACSYMA.

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References

1. Barrera, M.; Grudsky, S.M.: Asymptotics of eigenvalues for pentadiagonal symmetric Toeplitz matrices. Oper. Theory Adv. Appl. 259, 179–212 (2017)
2. Chu, M.T.; Diele, F.; Ragnion, S.: On the inverse problem of constructing symmetric pentadiagonal Toeplitz matrices from three largest eigenvalues. Inverse Probl. 21, 1879–1894 (2005)
3. Doman, B.G.S.: The Classical Orthogonal Polynomials. World Scientific Publishing Company, Singapore (2015)
4. Elouafi, M.: A widom like formula for some Toeplitz plus Hankel determinants. J. Math. Anal. Appl. 422(1), 240–249 (2015)
5. Elouafi, M.: An eigenvalue localization theorem for pentadiagonal symmetric Toeplitz matrices. Linear Algebra Appl. 435, 2986–2998 (2011)
6. Elouafi, M.: A note for an explicit formula for the determinant of pentadiagonal and heptadiagonal symmetric Toeplitz matrices. Appl. Math. Comput. 219(9), 4789–4791 (2013)
7. Elouafi, M.: On a relationship between Chebyshev polynomials and Toeplitz determinants. Appl. Math. Comput. 229(25), 27–33 (2014)
8. Fasino, D.: Spectral and structural properties of some pentadiagonal symmetric matrices. Calcolo 25, 301–310 (1988)
9. Grenander, U.; Szegö, G.: Toeplitz Forms and their Applications. Chelsea, New York (1984)
10. Jia, J.T.; Yang, B.T.; Li, S.M.: On a homogeneous recurrence relation for the determinants of general pentadiagonal Toeplitz matrices. Comput. Math. Appl. 71, 1036–1044 (2016)
11. Mason, J.C.; Handscomb, D.: Chebyshev Polynomials. Chapman & Hall, New York (2003)

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