ON THE COHOMOLOGY OF POLYHEDRAL PRODUCTS WITH SPACE PAIRS \((D^1, S^0)\)

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Abstract. In this note we prove that \(H(\mathbb{R}Z_K)\), namely the integer cohomology ring of polyhedral products with space pairs \((D^1, S^0)\), can also be described explicitly by a multiplicative Hochster’s formula.

The objective of this note is to understand \(H(\mathbb{R}Z_K)\), in a way parallelly to the work related to the cohomology of moment-angle complexes \(Z_K\) (see e.g. [3], [2], [4]) by Buchstaber, Panov and Baskakov.

After some constructions, an explicit formula for \(H(\mathbb{R}Z_K)\) is obtained in Theorem 1.6, the whole proof of which is completed later after Proposition 2.1. Some applications are discussed in Remark 1.8.

1. ON THE ADDITIVE STRUCTURE

In this note let \(\mathbb{Z}\) be the ring of integers and let all coefficients be from \(\mathbb{Z}\).

For a non-negative integer \(m\), we write \([m]\) for the set \(\{1, \ldots, m\}\). A simplicial complex \(K\) on \([m]\) means the vertex set of \(K\) is identified with \([m]\), such that

(1) there is an unique emptyset which belongs to every simplex of \(K\);
(2) as a subset of \([m]\), \(\sigma\) is a simplex of \(K\) implies that any face (subset) of \(\sigma\) is a simplex.

For instance, if \(K\) contains \([m]\) as a simplex, then every subset \(\sigma = \{i_0, \ldots, i_p\}\) is a \(p\)-simplex of \(K\).

The geometric realization of \(K\) will be denoted by \(|K|\). By \(\mathbb{R}Z_K\) we denote a specific polyhedral product (we follow [3] on this notation) defined as the union

\[
\mathbb{R}Z_K = \bigcup_{\sigma \in K} \{(x_1, \ldots, x_m) \in (D^1)^m | x_i \in S^0, \text{ when } i \not\in \sigma\},
\]

thus \(\mathbb{R}Z_K\) can be embedded in the product \((D^1)^m\) in an obvious way.

Example 1.1. It is known that if \(K\) is a simplicial \((n - 1)\)-sphere (a simplicial complex homeomorphic to the \((n - 1)\)-sphere), then \(\mathbb{R}Z_K\) is an \(n\)-dimensional manifold (see [3, p. 98]). Let \(K\) be the pentagon on \([5]\), namely it is a 1-dimensional simplicial complex with
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edges \(\{i, i + 1\}\) for \(i = 1, 2, 3, 4\) and \(\{1, 5\}\). For the five 1-simplices (resp. five 0-simplices) of \(K\), let each 1-simplex (resp. 0-simplex) correspond to the \(2^3\) squares (resp. the \(2^4\) edges) \((D^1)^2 \times (S^0)^3\) (resp. \(D^1 \times (S^0)^4\)) and let \((S^0)^5\) be the \(2^5\) vertices as in definition (1). Then by counting the Euler number we can identify this \(\mathbb{R}Z_K\) as a surface with genus 5.

1.1. A Cell Decomposition. Now we give \((D^1)^m\) a CW structure by a simplicial decomposition to each component \(D^1_i\) \((i = 1, \ldots, m)\), which descends to \(\mathbb{R}Z_K\) as a subcomplex: let \(01_i\) be the 1-simplex and \(0, 1_i\) be two 0-simplices of the unit interval \([0, 1]\] \(\cong D^1_i\) respectively, then each oriented cell of \((D^1)^m\) can be written as

\[ e_1 \times \cdots \times e_m, \quad e_i = 0, 1_i \text{ or } 01_i. \]

With the decomposition above, we find

**Lemma 1.2.** A cell \(e = e_1 \times \cdots \times e_m\) of \((D^1)^m\) belongs to its subspace \(\mathbb{R}Z_K\) if and only if

\[ \sigma_e := \{i | e_i = 01_i\} \]

is a simplex of \(K\).

In what follows we will abuse the notation \(e_i\): it may mean a simplex or the corresponding simplicial chain, and we will declare what it means depending on different situations. For a topological space \(X\), we denote \(S(X)\) (resp. \(S^*(X)\)) to be its singular chains (resp. cochains); \(C_e(X)\) (resp. \(C^*_e(X)\)) will mean the cellular chains (resp. cochains) when \(X\) is a CW complex, and \(C(X)\) (resp. \(C^*(X)\)) will be denoted for the simplicial chains (resp. cochains) when \(X\) is a simplicial complex. The notation \(H(X)\) (resp. \(H^*(X)\)) means the singular homology (resp. cohomology) of \(X\).

1.2. A Brief Review. Let \(X = \prod_{i=1}^m |K_i|\) be a product space by topologized finite simplicial complexes \(K_i\), note that now each \(C^*(K_i)\) is finitely generated. Let \((\otimes_{i=1}^m C^*(K_i); d)\) be the differential graded abelian group with generators

\[ e_1^* \otimes \cdots \otimes e_m^* \]

where \(e_i^*\) is the dual of the simplicial chain generated by \(e_i\) in \(K_i\), and whose differential \(d\) satisfies

\[ d(e_1^* \otimes \cdots \otimes e_m^*) = \sum_{i=1}^m (-1)^{\deg(e_j)} e_1^* \otimes \cdots \otimes e_{i-1}^* \otimes de_i^* \otimes e_{i+1}^* \otimes \cdots \otimes e_m^* \]

on each generator (on the right hand side the differential \(d\) is the usual simplicial coboundary operator). For instance, assume that \(i = 1, 2, 3\), from \(d01_i^* = -01_i^*, d1_i^* = 01_i^*\) and \(d01_i^* = 0,\)
one has
\[ d(0_1^* \otimes 01^*_2 \otimes 1_3^*) = -01_1^* \otimes 01_2 \otimes 1_3^* - 01_1^* \otimes 01_2^* \otimes 01_3^*. \]

\( C_e^*(X) \) means the set of cellular cochains generated by dual cells
\[ (e_1 \times \cdots \times e_m)^*, \]
on which we define a morphism
\[
\begin{array}{ccc}
C^*(K_1) \otimes \cdots \otimes C^*(K_m) & \longrightarrow & C_e^*(X) \\
(\epsilon_1^* \otimes \cdots \otimes \epsilon_m^*) & \longrightarrow & (e_1 \times \cdots \times e_m)^*,
\end{array}
\]
which is a 1-1 correspondence by a comparison of basis, thus we can equip \( C_e^*(X) \) with a differential \( d \) such that \( f \) is a cochain map, and it will be denoted as \( (C_e^*(X); d) \).

**Proposition 1.3.** Consider \( X = \prod_{i=1}^m |K_i| \) as a CW complex with cellular cochains \( (C_e^*(X); d) \), and assume that \( A \) is a subcomplex of \( X \) subject to this cell decomposition, then there is an isomorphism

\[ H^*(A) \cong H(C_e^*(A); d) \]

preserving degrees. For instance, we have isomorphisms
\[
H(\bigoplus_{\sum_{i=1}^m p_i = p} C^{p_1}(K_1) \otimes \cdots \otimes C^{p_m}(K_m); d) \xrightarrow{f} H^p(C_e^*(X); d) \cong H^p(X)
\]
where \( p_i \) are non-negative integers.

A standard cellular (co)homology argument (see e.g. [6, Theorem 57.1, p. 339]) will provide a proof without involving cup products. We will give full version of this proposition later (i.e. Proposition 2.1), but here the simplicial cochain algebra of \( C^*(D_1) \) with simplicial cup products will be presented, because it motivates the algebraic construction later. Henceforth let us work with the following basis
\[
\{1_i = 1_i^* + 0_i^*, t_i = 1_i^*, u_i = 01_i^*\}
\]
of \( C^*(D_1^1) \) rather than the original \( \{0_i^*, 1_i^*, 01_i^*\} \). By definition (see e.g. [6, p. 292]), we have the following relations on simplicial cup products:
\[
1_i \cup t_i = t_i \cup 1_i, \quad 1_i \cup u_i = u_i \cup 1_i, \quad t_i \cup t_i = t_i, \quad u_i \cup u_i = 0, \quad t_i \cup u_i = 0, \quad u_i \cup t_i = u_i.
\]
1.3. An Algebraic Description. In what follows, \( x_\sigma \) will mean the monomial \( x_{i_0} \ldots x_{i_p} \) when \( \sigma = \{i_0, \ldots, i_p\} \) is a subset of \([m]\). By \( \mathbb{Z}[u_1, \ldots, u_m; t_1, \ldots, t_m] \) we denote the differential graded free \( \mathbb{Z} \)-algebra with \( 2m \) generators, such that

\[
\text{deg} u_i = 1, \quad \text{deg} t_i = 0; \quad du_i = 0, \quad dt_i = u_i,
\]

where the differential \( d \) follows the rule (2) on each monomial. Let \( R \) be the quotient algebra of \( \mathbb{Z}[u_1, \ldots, u_m; t_1, \ldots, t_m] \) subject to the following relations

\[
u_i t_i = u_i, \quad t_i u_i = 0, \quad u_i t_j = t_j u_i, \quad t_i t_j = t_j t_i, \quad u_i u_j = -u_j u_i, \quad t_i t_j = t_j t_i,
\]

for \( i, j = 1, \ldots, m \) such that \( i \neq j \), in which the Stanley-Reisner ideal \( I_K \) is defined to be generated by all square-free monomials \( u_\sigma \) such that \( \sigma \) is not a simplex of \( K \).

One can check that the relations (6) listed above are compatible with the differential \( d \), for instance,

\[
d(t_i t_i) = u_i t_i + t_i u_i = u_i t_i = u_i = dt_i.
\]

Theorem 1.4. There is a (degree-preserving) ring isomorphism

\[
H(R/I_K; d) \cong H(\mathbb{R}Z_K).
\]

Proof. Here we only prove the additive part of this theorem, and the whole proof involving products will be completed later after Proposition 2.1.

We start from a cochain map, which is generated by mapping each square-free monomial to a tensor product of simplicial cochains, namely

\[
\theta : R \longrightarrow C^*(D_1^1) \otimes \cdots \otimes C^*(D_m^1)
\]

\[
x_1 \ldots x_m \longrightarrow e_1^* \otimes \cdots \otimes e_m^*,
\]

in which (we are using the basis (4))

\[
e_i^* = \begin{cases} 
1_i, & \text{if } x_i = 1, \\
t_i, & \text{if } x_i = t_i, \\
u_i, & \text{if } x_i = u_i.
\end{cases}
\]

The additive generators of \( R \) are square-free monomials follows from the relations (6), and by definition \( \theta \) induces an isomorphism between abelian groups such that \( d \circ \theta = \theta \circ d \). Then from the following diagram about cochain maps (in which \( q \) is the quotient and \( i^* \) is induced
by the inclusion)
\[ R \xrightarrow{f \circ \theta} C_e^*(D_1^1 \times \cdots \times D_m^1) \]
\[ R/I_K \xrightarrow{\simeq} C_e^*(\mathbb{R}Z_K), \]
(8)
\[ R/\mathcal{I}_K \xrightarrow{\simeq} C_e^*(\mathbb{R}Z_K), \]
together with Lemma 1.2, we observe that the quotient part in the Stanley-Reisner ideal corresponds to the cells not contained in \( \mathbb{R}Z_K \), thus by Proposition 1.3 this part of proof is completed.

Note 1.4. A Combinatorial Description (Hochster’s Formula). With the notation \( x_{\sigma} \), we can write each square-free monomial of \( R/\mathcal{I}_K \) in the form
\[ u_{\sigma} t_{\tau}, \quad \sigma \cap \tau = \emptyset, \quad \sigma \in K \]
(for instance, \( u_2 t_3 u_4 t_5 t_6 = u_{2,4} t_{3,5,6} \)).

When \( \omega \) is a subset of \([m]\) (may be empty), we denote \( K_\omega \) to be the full subcomplex of \( K \) in \( \omega \), namely it is the subcomplex
\[ \{ \sigma \cap \omega | \sigma \in K \}, \]
and \( R/\mathcal{I}_K |_\omega \) will be denoted as a subgroup of \( R \) (as a group) generated by
\[ \{ u_{\sigma} t_{\tau} | \sigma \in K \text{ s.t. } \sigma \sqcup \tau = \omega, \ \sigma \cap \tau = \emptyset \}. \]

We observe that \( R/\mathcal{I}_K |_\omega \) is closed under the differential \( d \) and \( R/\mathcal{I}_K = \bigoplus_{\omega \subset [m]} R/\mathcal{I}_K |_\omega \), therefore
\[ H(R/\mathcal{I}_K; d) \cong \bigoplus_{\omega \subset [m]} H(R/\mathcal{I}_K |_\omega; d). \]

Each \( \omega \subset [m] \) yields a 1-1 cochain map (on the right hand side we use the usual simplicial coboundary operator)
\[ \lambda_\omega : R/\mathcal{I}_K |_\omega \xrightarrow{\cong} C^*(K_\omega) \]
\[ u_{\sigma} t_{\tau} \longrightarrow \sigma^*, \]
(9)
which induces an additive isomorphism of cohomology
\[ H^p(R/\mathcal{I}_K |_\omega) \xrightarrow{\lambda_\omega} \tilde{H}^{p-1}(K_\omega), \]
because
\[ d(t_1 \ldots t_m) = \sum_{i=1}^{m} t_1 \ldots t_{i-1} u_i t_{i+1} \ldots t_m \]
behaves as the augmentation and $\lambda_0^*$ maps $H^0(R)$ isomorphically onto $\tilde{H}^{-1}(\emptyset)$.

**Remark 1.5.** Indeed, $\lambda$ induces cochain maps since every time we jump over an index in $\sigma$, we will get a $-1$. Here the construction of $\lambda$ is motivated by Baskakov’s idea in [1].

Therefore we can equip $\bigoplus_{\omega \subset [m]} \tilde{H}^{-1}(K_\omega)$ with a product structure described by simplicial cochains, such that cochain map $\lambda$ behaves as the augmentation and $\lambda_0$ induces an isomorphism between cochain algebras. Together with Theorem [1.3] we have the following theorem, the full proof of which will be completed later.

**Theorem 1.6.** There are isomorphisms

$$\bigoplus_{\omega \subset [m]} \tilde{H}^{-1}(K_\omega) \xrightarrow{\cong} H^p(R/I_K) \xrightarrow{\cong} H^p(\mathbb{R}Z_K)$$

for all positive integers $p$. Moreover, after endowing $\bigoplus_{\omega \subset [m]} \tilde{H}^{-1}(K_\omega)$ the product structure induced by $\lambda$, they are isomorphisms between $\mathbb{Z}$-algebras.

Thus the cohomology ring of $\mathbb{R}Z_K$ will be clear once we write down all cochains representing the (reduced) simplicial cohomology for each $K_\omega$.

**Example 1.7.** Assume that $K$ is the pentagon described in Example [1.1] let us compute the cohomology of $\mathbb{R}Z_K$. First we can write down all the full subcomplexes of $K$ with non-trivial cohomology, namely

$$1 : \tilde{H}^{-1}(K_\emptyset);$$

$$\alpha_1 - \alpha_5 : \tilde{H}^0(K_{1,3})|u_{1t3}, \tilde{H}^0(K_{1,4})|u_{1t4}, \tilde{H}^0(K_{2,4})|u_{2t4}, \tilde{H}^0(K_{2,5})|u_{2t5}, \tilde{H}^0(K_{3,5})|u_{3t5};$$

$$\beta_1 - \beta_5 : \tilde{H}^0(K_{1,3,4})|u_{1t3,4}, \tilde{H}^0(K_{2,4,5})|u_{2t4,5}, \tilde{H}^0(K_{3,5,1})|u_{3t1,5}, \tilde{H}^0(K_{4,1,2})|u_{4t1,2}, \tilde{H}^0(K_{5,2,3})|u_{5t2,3};$$

$$\gamma : \tilde{H}^1(K_{1,2,3,4,5})|u_{12t3,4,5};$$

in which we write a generator for each group, via $\lambda$. Thus (by Theorem [1.6]) $H(\mathbb{R}Z_K)$ has ten generators in degree 1 (i.e. $\alpha_i, \beta_i, i = 1, \ldots, 5$), and one generator in degree 0 (i.e. 1 the unit) and degree 2 (i.e. $\gamma$) respectively. Note that these generators may not be unique: for instance,

$$d(u_{1t23,4,5}) = -u_{12t3,4,5} - u_{15t2,3,4} + \text{terms equivalent to 0},$$

and in the same way $-u_{1t3,4}$ is equivalent to $u_{3t1,4} + u_{4t1,3}$. A little computation shows, up to equivalences in cohomology, all non-trivial cup products are listed as follows (where we omit the notation $\cup$)

$$\gamma = \alpha_1 \beta_2 = \alpha_2 \beta_5 = \alpha_3 \beta_3 = \alpha_4 \beta_1 = \alpha_5 \beta_4 = \beta_1 \beta_2 = \beta_2 \beta_3 = \beta_3 \beta_4 = \beta_4 \beta_5 = \beta_5 \beta_1.$$
We can use the construction (8) in the proof of Theorem 1.4 to identify all corresponding cellular cochains.

**Remark 1.8.** From Theorem 1.6 after a comparison with the result in [3], it follows that there is an additive isomorphism between \( H(\mathbb{Z}_K) \) and \( H(\mathbb{R}_K) \) if we forget about degrees. For instance, we can also find arbitrary torsions in \( H(\mathbb{R}_K) \) when \( K \) are special simplicial spheres (see [5, Theorem 11.11]). As algebras, the difference between \( H(\mathbb{Z}_K) \) and \( H(\mathbb{R}_K) \) comes from different sign rules and square rules.

## 2. On the Cup Product

In this section we will describe cup products in Proposition 1.3, and then finish the proofs of Theorem 1.4 and Theorem 1.6.

Assume that \( X \) and \( Y \) are two topological spaces, we define a map \( g: S^*(X) \otimes S^*(Y) \to \text{Hom}(S(X) \otimes S(Y), \mathbb{Z}) \) generated by

\[
(10) \quad g(c_X^* \otimes c_Y^*)(c_X^* \otimes c_Y^*) = c_X^*(c_X^*)c_Y^*(c_Y^*), \quad c_X^* \in S^*(X), \quad c_Y^* \in S^*(Y),
\]

(the evaluation is trivial when degrees do not match) for each singular triangles \( c_X^* \) and \( c_Y^* \).

By Eilenberg-Zilber theorem, there is a natural chain equivalence \( \mu \) between singular chain complexes (unique up to chain homotopy, see [8]; the differential on tensors follows the rule (2))

\[
S(X \times Y) \cong S(X) \otimes S(Y),
\]

and the cup product is defined via the following

\[
S^*(X) \otimes S^*(X) \xrightarrow{\mu_{X \times Y} \circ g} S^*(X \times X) \xrightarrow{d^*} S^*(X)
\]

in which \( d^* \) is induced by the diagonal map \( X \to X \times X \) (the *cross product* is defined by \( \mu_{X \times X} \circ g \)).

From now on let Top\(^m\) be the category with \( m \)-products of topological spaces as objects, whose morphisms are \( m \)-products of continuous maps, and let \( \mathcal{C} \) be the category of chain complexes equipped with chain maps.

Consider \( X = \prod_{i=1}^m |K_i| \) as an object of Top\(^m\): a product by finite topologized simplicial complexes.

Thus in the following diagram, all morphisms can be considered as natural transformations between functors from Top\(^m\) to \( \mathcal{C} \), where \( d_i \) is the diagonal chain map from \( S(|K_i|) \) to \( S(|K_i|) \otimes S(|K_i|) \), and \( \tilde{\mu} \) is defined by using \( \mu \) for \( m - 1 \) times, factoring out one component.
each time:

\[
S(\prod_{i=1}^{m} |K_i|) \otimes S(\prod_{i=1}^{m} |K_i|) \xrightarrow{\mu_{X \times X \circ d}} \otimes_{i=1}^{m} S(|K_i|) \otimes \otimes_{i=1}^{m} S(|K_i|) \\
\xrightarrow{\bar{\mu}_X} S(|K_1|) \otimes \cdots \otimes S(|K_m|) \xrightarrow{\otimes_{i=1}^{m} \mu_{|K_i| \times |K_i|} \circ d_i} \otimes_{i=1}^{m} S(|K_i|) \otimes S(|K_i|),
\]

in which \(T\) is defined as follows. For each tensor by singular triangles \(\otimes_i c_i \otimes c_i' \in \otimes_{i=1}^{m} S(|K_i|) \otimes S(|K_i|)\)

\[
T(c_1 \otimes c_1' \otimes \cdots \otimes c_m \otimes c_m') = (-1)^{\varepsilon(c, c')} c_1 \otimes \cdots \otimes c_m \otimes c_1' \otimes \cdots \otimes c_m'
\]

where the mod 2 invariant \(\varepsilon(c, c')\) is generated by this rule: each operation of changing the order of any adjacent two terms will produce a multiplication of their degrees, thus we obtain \(\varepsilon(c, c')\) by summing up these multiplications for an arbitrary sequence of operations, for instance one choice for \(\varepsilon(c, c')\) can be

\[
\varepsilon(c, c') = \sum_{i=1}^{m} \deg c_i' \sum_{j>i} \deg c_j.
\]

It is straightforward to check that \(T\) is a chain map. From the method of acyclic model with the model category as \(m\)-products of topologized simplices (see \([3]\), the case for 2-products can be found in \([9\ p. 252]\)), it follows that there is a natural chain homotopy between

\[
\bar{\mu}_X \otimes \bar{\mu}_X \circ \mu_{X \times X} \circ d \quad \text{and} \quad T \circ (\otimes_{i=1}^{m} \mu_{|K_i| \times |K_i|} \circ d_i) \circ \bar{\mu}_X.
\]

Therefore after taking Hom to diagram (11), we obtain the following (where \(g_1, g_2\) are defined from (10))

\[
\begin{align*}
\otimes_{i=1}^{m} C^*(K_i) \otimes \otimes_{i=1}^{m} C^*(K_i) & \xrightarrow{T^*:=g_2^{-1} \circ T^* \circ g_1} \otimes_{i=1}^{m} C^*(K_i) \otimes C^*(K_i) \\
\Hom(\otimes_{i=1}^{m} S(K_i) \otimes \otimes_{i=1}^{m} S(K_i), \mathbb{Z}) & \xrightarrow{T^*} \Hom(\otimes_{i=1}^{m} S(K_i) \otimes S(K_i), \mathbb{Z}),
\end{align*}
\]

in which \(T^*\) is well defined since \(\text{Im}(T^* \circ g_1) \subset \text{Im}(g_2)\), and here every \(C^*(K_i)\) is considered as a subgroup of \(S^*(|K_i|)\) \((i = 1, \ldots, m)\), in which only simplicial chains are non-trivially evaluated.
Proposition 2.1. Assume that $e^* = \bigotimes_{i=1}^{m} e_i^*$ and $e'^* = \bigotimes_{i=1}^{m} e_i'^*$ are tensors of dual simplices in $(\bigotimes_{i=1}^{m} C^*(K_i) \subset) \otimes_{i=1}^{m} S^*(|K_i|)$, then as elements in $S^*(\prod_{i=1}^{m} |K_i|)$ on both sides, we have

\begin{equation}
\tilde{\mu}_X \circ g(\otimes_{i=1}^{m} e_i^*) \cup \tilde{\mu}_X \circ g(\otimes_{i=1}^{m} e_i'^*) = (-1)^{\varepsilon(e^*, e'^*)}\tilde{\mu}_X \circ g(\otimes_{i=1}^{m} e_i^* \cup e_i'^*),
\end{equation}

up to a cochain homotopy. Moreover, consider $X = \prod_{i=1}^{m} |K_i|$ as a CW complex whose cells are $m$-products of (topologized) simplices with $i$-th component in $K_i$, and suppose that $A$ is a subcomplex of $X$ with $e^*$ and $e'^*$ in $C^*(A)$, then as elements in $S^*(A)$, (12) holds up to a cochain homotopy.

Proof. After a little chasing on the diagram (11), the first statement follows from

\[
\tilde{\mu}_X \circ g(\otimes_{i=1}^{m} e_i^*) \cup \tilde{\mu}_X \circ g(\otimes_{i=1}^{m} e_i'^*) = d^* \circ \mu^*_{X \times X} \circ \tilde{\mu}_X \circ g(\otimes_{i=1}^{m} e_i^* \otimes \otimes_{i=1}^{m} e_i'^*) \\
= \tilde{\mu}_X \circ (\otimes_{i=1}^{m} \mu_{|K_i| \times |K_i|} \circ d_i)^* \circ g(\otimes_{i=1}^{m} e_i^* \otimes \otimes_{i=1}^{m} e_i'^*) \\
= (-1)^{\varepsilon(e^*, e'^*)}\tilde{\mu}_X \circ (\otimes_{i=1}^{m} \mu_{|K_i| \times |K_i|} \circ d_i)^* \circ g(\otimes_{i=1}^{m} e_i^* \otimes e_i'^*) \\
= (-1)^{\varepsilon(e^*, e'^*)}\tilde{\mu}_X \circ g(\otimes_{i=1}^{m} e_i^* \cup e_i'^*),
\]

in which from the second line to the third we use the (co)chain homotopy and change $\tilde{T}^*$ for the sign $(-1)^{\varepsilon(e^*, e'^*)}$. Now we prove the part about the subcomplex $A$. Denote all cells of $A$ by $E_A$, we can consider $E_A$ as a subcategory of $\text{Top}^m$ with inclusions as morphisms, then there is a chain equivalence (see e.g. [9, Theorem 31.5], together with the fact that each $e$ in $E_A$ is a neighborhood deformation retract of $A = \bigcup_{e \in E_A} e$)

$$S(A) \cong \text{colim}_{e \in E_A} S(\bigotimes_{i=1}^{m} e_i)$$

where $e = \prod_{i=1}^{m} e_i$ with each simplex $e_i$ in $K_i$. Next, from the naturality of $\mu$ we have the diagram

$$
\begin{array}{ccc}
S(X) & \xrightarrow{\mu_X} & \bigotimes_{i=1}^{m} S(|K_i|) \\
\uparrow & & \uparrow \\
S(A) & \cong & \text{colim}_{e \in E_A} S(\bigotimes_{i=1}^{m} e_i) \xrightarrow{\mu_X \circ g} \text{colim}_{e \in E_A} \bigotimes_{i=1}^{m} S(e_i),
\end{array}
$$

to which after taking Hom, it is not difficult to observe

$$
C_e^*(X) \cong \bigotimes_{i=1}^{m} C^*(K_i) \xrightarrow{\mu_X \circ g} S^*(X) \\
\downarrow \\
C_e^*(A) \xrightarrow{\mu_X \circ g} S^*(A).
$$
where \( g_A : C^*_e(A) \to \text{Hom}(\text{colim}_{e \in E_A} \otimes_{i=1}^{m} S(e_i), \mathbb{Z}) \) is defined by evaluating each \( \tilde{e}^* \in C^*_e(A) \) non-trivially on its dual \( \tilde{e} \in \text{colim}_{e \in E_A} \otimes_{i=1}^{m} S(e_i) \) only. Then we can replace everything of diagram (11) by colimits, and repeat the process for \( X \).

It is not difficult to find the sign rule for commutativity defined by \( \varepsilon \) coincides with the one in \( R/I_K \), which is defined in (6). Therefore we have completed all proofs.

**References**

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