Real Inflection Points of Real Linear Series on an Elliptic Curve

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\section*{ABSTRACT}
Given a real elliptic curve \( E \) with non-empty real part and a real effective divisor \( D \) on \( E \) arising via pullback from \( \mathbb{P}^1 \) under the hyperelliptic structure map, we study the real inflection points of distinguished subseries of the complete real linear series \( |D| \) on \( E \). We define \textit{inflection polynomials} whose roots index the \( (x\text{-coordinates of}) \) inflection points of the linear series, away from the points where \( E \) ramifies over \( \mathbb{P}^1 \). These fit into a recursive hierarchy, in the same way that division polynomials index torsion points. Our study is motivated by, and complements, an analysis of how inflectionary loci vary in the degeneration of real hyperelliptic curves to a metrized complex of curves with elliptic curve components that we carried out in an earlier joint work with I. Biswas.

\section*{1. Introduction}
This note is a companion to the papers [Garay 15] and [Biswas et al. 19]. In [Garay 15], the second author described the topology of real inflectionary loci of complete linear series on a real elliptic curve \( E \) as a function of the topology of the real locus \( E(\mathbb{R}) \), and used this as the inductive basis for a construction of real canonical curves of genus four in \( \mathbb{P}^3 \) with many real Weierstrass points. Then in [Biswas et al. 19], joint with I. Biswas, we used a degeneration borrowed from non-Archimedean geometry to relate two distinct generalizations of real inflectionary loci of complete series on \( E \), namely

\begin{enumerate}
\item real inflectionary loci of complete series on real hyperelliptic curves \( X \) of genus \( g \geq 2 \); and
\item real inflectionary loci of incomplete series on real elliptic curves.
\end{enumerate}

Very roughly, generalization (1) may be degenerated to generalization (2); a bit more precisely, the degeneration outputs a limit linear series on a metrized complex of real curves, whose models are smooth elliptic real curves \( E_i \), \( i = 1, \ldots, g \).

Even more precisely, the complete real series \( |L_{\mathbb{R}}| \) studied in [Biswas et al. 19] are all multiples of the \( g_1^2 \) obtained from pulling back a distinguished point \( \infty \) on \( \mathbb{P}^1 \) via the hyperelliptic structure morphism \( \pi : X \rightarrow \mathbb{P}^1 \). With respect to a suitable choice of affine coordinates, we can realize \( X \) as (a projective completion of) \( y^2 = f(x) \), where \( \deg(f) = 2g + 1 \) and the map \( \pi \) is given by \( (x,y) \mapsto x \). Assume that \( L_{\mathbb{R}} \) is represented by the divisor \( kD \) on \( X \), where \( D \) is the pull-back of \( \infty \) on \( \mathbb{P}^1 \). The interesting cases are associated with choices for which \( k > g \geq 2 \), so hereafter we assume this. There is a distinguished real basis of holomorphic sections for \( |L_{\mathbb{R}}| = |L_{\mathbb{R}}(kD)| \) given by

\[ F_g = \{1, x, \ldots, x^k, y, xy, \ldots, xy^{k-g-1}\}. \]

The ramification locus \( R_\pi \) of \( \pi \) is comprised of real points that degenerate to the ramification points of the elliptic components \( E_i \), viewed as double covers of \( \mathbb{P}^1 \). Furthermore, the \( 2k-g+1 \) sections \( F_g \) degenerate to \( 2k-g+1 \) sections that we will also abusively denote by \( F_g \); these, in turn, determine an inflectionary basis in every point of the ramification locus of \( E_i \rightarrow \mathbb{P}^1 \). Hereafter, we assume \( i = 1 \). Abusively, we let \( \pi \) denote the double cover \( E = E_1 \rightarrow \mathbb{P}^1 \). In [Biswas et al. 19], we explicitly computed the contribution to (real) inflection made by each point of \( R_\pi \). Note that in general, the inflection arising from \( R_\pi \) is not simple, which contrasts with the behavior of inflectionary loci of complete series. The behavior of inflectionary loci (including local multiplicities) away from \( R_\pi \),
however, was unclear. Here we will address this mystery behavior through a more careful analysis of the Wronskians whose zero loci select for (the x-coordinates of) inflectionary loci.

One upshot of our analysis will be that inflection is, in fact, simple (i.e. of multiplicity one) away from $R_x$, at least when the topology of $E(\mathbb{R})$ is maximally real. One of the key features of the complete series case on which the analysis of [Garay 15] is predicated is that inflection points of a complete linear series on an elliptic curve correspond to torsion points. Torsion points, in turn, form a sub-lattice of the lattice $\Lambda$ for which $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ as a topological space. Their $x$-coordinates are computed by the division polynomials described in [Silverman 85]. By analogy, we will define inflection polynomials that compute the $x$-coordinates of inflection points of $F_g$ on $E$ away from $R_x$.

Our inflection polynomials are extracted from Wronskians, which also play an important rôle in Griffiths and Harris’ modern treatment [Griffiths and Harris 78] of Poncelet’s theorem on polygons inscribed (and circumscribed) in conics. However, we organize the information encoded by these determinants in an apparently novel way.

When $E(\mathbb{R})$ is maximally real, we can realize $E = E_2$ as an elliptic curve in Legendre form, where $\lambda$ is a real parameter. The inflectionary loci of curves in the associated flat family then determine a planar inflection curve $C = C_{k,g}$, whose singularities control in a rather precise way the (real) inflectionary loci of the curves $E_2$. Our graphical experiments with the real loci of inflection curves in the maximally real setting suggest a strong topological uniformity depending only on the parity of $g$. As a result, we obtain a complete conjectural description of real inflection for the linear series $F_g$ in the maximally real case that we hope will convince the reader that the further study of inflection polynomials and curves is interesting in its own right.

Roadmap. The paper following this introduction is structured as follows. In Section 2, we define the inflection polynomials that are our main objects of study. These are polynomials $P_{\mu,n}(x, \lambda)$, where $\mu := k - g$ and $n := k + 1$; in this article, we focus primarily on the case $\mu = 1$. Conjecture 2.1 predicts that for every fixed value of $\lambda$, the corresponding inflection polynomial $P_{1,n}(x, \lambda)$ has only simple roots over $\mathbb{R}$ whenever $E = E_2$ is maximally real. Conjecture 2.2, on the other hand, makes an explicit prediction about the degree of the $x$-discriminant of $P_{1,\lambda}(x, \lambda)$ (in which $\lambda$ is no longer fixed) as a function of the number of connected components of $E(\mathbb{R})$.

The main contribution of Section 3 is Conjecture 3.1, which predicts that when $E(\mathbb{R})$ is maximally real, $F_g$ has precisely either $2(k-g)$ or $4(k-g)$ real inflection points away from $R_x$, depending upon whether $g$ is even or odd. In Section 4, we assemble some evidence for Conjecture 3.1, which is manifested graphically in Figures 1 and 2. Along the way, we highlight some important structural features of inflection polynomials and curves.

Lemmas 4.1 and 4.2 show that the $n$-th inflection polynomial $P_{1,n}$ is of $(x, \lambda)$-bidegree $(2n, n)$, and that it has distinguished automorphisms which in particular force the projective closure $\bar{C}_n$ of its associated curve to have analytically isomorphic singularities in $[0 : 0 : 1], [0 : 1 : 0]$, and $[1 : 1 : 1]$. Conjecture 4.3 posits, moreover, that these are the only singularities of $\bar{C}_n$.

In Theorem 4.4, we prove Conjecture 3.1 when $\mu = 1$ and $n \in \{2, 3, 4, 5\}$. In doing so, we reduce to proving specific instances of Conjecture 4.5, which characterizes what we call the topological profile of the real locus of $\bar{C}_n$, and describes the organization of the real branches of $\bar{C}_n$ near each of its distinguished three singularities. Our method of proof in the $n = 5$ case, which uses only local Puiseux parameterization solutions $\lambda$ in terms of $x$, should apply equally well in verifying Conjecture 3.1 for arbitrary fixed values of $(\mu, n)$.

Finally, in Section 5, we prove Theorem 5.1, which shows how to leverage the result of Conjecture 3.1, assuming it holds, to produce real complete linear series on real hyperelliptic curves with many real inflection points, in the spirit of [Biswas et al. 19].

Throughout we have made liberal use of Mathematica to calculate inflection polynomials and their discriminants, and to graph the real loci of the inflection curves $C_n$.

2. Wronskians, revisited

Borrowing notationally from [Biswas et al. 19], assume that $E$ is defined by an affine equation $y^2 = f$, where $f \in \mathbb{R}[x]$ is a separable cubic. The restriction of the real inflectionary locus on $E$ to the affine chart $U_y$ where the coordinate $y$ is nonvanishing is the zero locus of (the restriction of) a regular function $x$ equal to a Wronskian determinant of partial derivatives of sections of $F$, namely

$$x|_{U_y} = \det(f_i^{(0)})_{0 \leq i, j \leq 2k-g}.$$  

Here $f_i$ refers to the $i$-th element of the distinguished basis $(1)$ of $F_g$, and the superscript denotes the order of differentiation with respect to $x$.  

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Now, for any given $n \geq 1$, define $P_n(x) \in \mathbb{R}[x]$ by
\[ y^{(n)} = f^{-n}yP_n(x). \] (2)

Note that $x|_{U^k} = y^{(n)}$ precisely when $n = k + 1 \geq 3$ and $g = k - 1$. It follows in this situation that the roots $x = \gamma$ of $P_n(x)$ for which $f(\gamma) > 0$ are the $x$-coordinates of the inflection points of the codimension-$(g-1)$ subseries of $|\mathcal{L}_R|$ spanned by $\mathcal{F}_k$ away from the ramification locus $R_k$. (Note that the positivity of $f$ is required in order for the root of an inflection polynomial to lift to an inflection point.)

Accordingly, here we will focus on the calculation of the inflection polynomials $P_n(x)$, which may be done recursively. Differentiating (2) yields
\[
y^{(n+1)} = \frac{dP_n}{dx} \cdot f^{-n}y + P_n\left( -nf^{-(n+1)}f'y + f^{-n}y' \right)
= \frac{dP_n}{dx} \cdot f^{-n}y + P_n\left( -nf^{-(n+1)}f'y + f^{-n} \cdot \frac{1}{2}f'f^{-1}y \right)
= f^{-(n+1)}y \cdot \left( \frac{dP_n}{dx} \cdot f + P_nf'(-n + 1/2) \right)
\]

Figure 1. Real loci of (a) $P_2 = 0$, (b) $P_4 = 0$, (c) $P_6 = 0$, and (d) $P_8 = 0$. 

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from which we deduce that

$$P_{n+1} = \frac{dP_n}{dx} f + (-n + 1/2)P_n \cdot \frac{df}{dx}. \quad (3)$$

The inflection polynomials $P_n$, $n \geq 1$, are determined by the recursion (3) together with the seed datum

$$P_1 = \frac{1}{2} \frac{df}{dx}. \quad (4)$$

Empirically we observe the following phenomenon.

**Conjecture 2.1.** When the polynomial $f$ has three distinct real roots, the $n$-th inflection polynomial $P_n$ is a separable polynomial in $x$, for all $n \geq 1$ and for every fixed real value of $\lambda$.

Indeed, the assumption that $f$ has three distinct real roots means that the topology of $E(\mathbb{R})$ is maximally real, and that $E$ is isomorphic as a real curve to one
of the Legendre form
\[ y^2 = x(x-1)(x-\lambda) \]  
(5)
where \( \lambda \) is a real parameter, \( \lambda \neq 0, 1 \). With the help of Mathematica, we have checked that Conjecture 2.1 holds whenever \( n \leq 9 \) by explicitly computing the \( x \)-discriminant of \( P_n \) for the elliptic curve (5) and checking that the only real roots are 0 and 1 (which appear with large multiplicities). Note that the discriminant of \( P_0 \) is a polynomial in \( \lambda \) of degree 218.

On the other hand, when \( f \) has two conjugate roots, \( P_n \) is no longer separable in general, whenever \( n \geq 4 \). For example, one checks that the discriminant of \( P_4 \) of the curve with normal form
\[ y^2 = x(x-1)(x-\lambda)(x-1-\lambda i) \]  
(6)
has a nontrivial real root when \( \lambda = \pm \frac{1}{\sqrt{3}} \). So Conjecture 2.1, assuming it holds, strongly depends on the maximal reality of \( E(\mathbb{R}) \).

Curiously, it also appears that the degree of the \( x \)-discriminant of \( P_n \) depends upon the topological type of \( E(\mathbb{R}) \).

**Conjecture 2.2.** Let \( f = f(x, \lambda) \) be a cubic polynomial associated with a one-parameter family of elliptic curves \( y^2 = f \) as in (5) or (6). Define the associated inflection polynomials \( P_n = P_n(x, \lambda) \) according to the recursive hierarchy (3) together with the seed (4). When \( f \) is of Legendre type (5) (respectively, of type (6)), the degree of the \( x \)-discriminant of \( P_n \) is equal to \( 3n^2 - 3n + 2 \) (respectively, \( 4n^2 - 2n \)).

Hereafter, we focus our attention on elliptic curves \( E = E_3 \) arising from the real Legendre family (5). We will be concerned with **inflection polynomials** whose roots encode the \( x \)-coordinates of inflection points of \( F_g \), away from the ramification locus \( R_\pi \) of \( \pi : E \to \mathbb{P}^1 \), or equivalently, inflection points whose \( x \)-coordinates are not roots of \( f \). By substituting \( y^{(m)} = f^{-m}yP_m \) in the \( n = (k+1) \)-th Wronskian determinant that defines \( x|_{U_j} \), we can write \( x|_{U_j} = (f^{-n}y)^\mu P_{\mu, n} \), where \( P_{\mu, n} \) is the inflection polynomial associated with \( F_g \) when \( k = n \mu \geq 1 \). This \( P_{\mu, n} \) is a polynomial of degree \( \mu \) in the inflection polynomials \( P_m = P_{1, m} \). We now make this explicit in the cases \( \mu = 2, 3 \).

**Case: \( \mu = 2 \).** Here
\[ x|_{U_j} = \det \begin{pmatrix} y^{(n)} \\ (xy)^{(n)} \\ (xy)^{(n+1)} \end{pmatrix} = (f^{-n}y)^2 P_{2, n}; \]
(7)
applying the facts that
\[ y^{(m)} = f^{-m}yP_m \quad \text{and} \quad (xy)^{(m)} = my^{(m-1)} + xy^{(m)} \]  
(8)
for all \( m \geq 1 \) in (7) yields
\[ x|_{U_j} = (n+1)(y^{(n)})^2 - ny^{(n+1)}y^{(n-1)} \]
and
\[ P_{2, n} = (n+1)P_n^2 -nP_{n-1}P_{n+1} \]  
(9)
for all \( n \geq 2 \).

**Case: \( \mu = 3 \).** This time, we have
\[ x|_{U_j} = \det \begin{pmatrix} y^{(n)} \\ (xy)^{(n)} \\ (xy)^{(n+1)} \\ (xy)^{(n+2)} \\ (x^2y)^{(n)} \\ (x^2y)^{(n+1)} \\ (x^2y)^{(n+2)} \end{pmatrix} = (f^{-n}y)^3 P_{3, n}; \]
(10)
Evaluating the Wronskian determinant (10) and applying the derivative identities (8) together with the fact that
\[ (x^2y)^{(m)} = m(m-1)y^{(m-2)} + 2mxy^{(m-1)} + x^2y^{(m)} \]  
(11)
whenever \( m \geq 2 \) yields
\[ P_{3, n} = (n+1)^2(n+2)P_n^2 - n(n+1)P_{n-1}(2n+2)P_{n-1} \]
\[ + n(P_{n+1} - (n-1)P_{n-2}P_{n+2}) \]
\[ + n(n^2 + n - 2)P_{n-2}^2 P_{n+1} + n(n+1)P_{n-1}^2 P_{n+2} \]  
(12)
for all \( n \geq 3 \).

### 3. Real roots of inflection polynomials

In his Ph.D. thesis [Garay 15, Prop. 3.2.5], the second author obtained a complete characterization of the real inflectionary loci of complete real linear series \( |L_\pi| \) of degree \( d \geq 2 \) on a real elliptic curve \( E \). He showed, in particular, that when the real locus \( E(\mathbb{R}) \) has two real connected components, the number of real inflection points of \( |L_\pi| \) is either \( d \) or \( 2d \), depending upon whether \( d \) is odd or even. In the present setting, \( d = 2k \) is always even, and therefore \( |L_\pi| \) has precisely \( 4k \) real inflection points.

On the other hand, the linear series spanned by \( F_g \) is complete precisely when \( n = \mu + 2 \), i.e. when \( k = \mu + 1 \). In that situation, there are \( 4\mu + 4 \) real inflection points, of which \( 4\mu \) lie away from \( R_\pi \). It follows that the corresponding inflection polynomial \( P_{\mu, \mu+2} = P_{\mu, \mu+2}(\lambda) \) associated with any choice of real parameter \( \lambda \) has precisely \( 2\mu \) real roots \( x = \gamma \) for which \( f(\gamma, \lambda) > 0 \).

Surprisingly, it seems that the inflection polynomial \( P_{\mu, \mu+2} \) determines the real inflectionary behavior of every inflection polynomial \( P_{\mu, n} = P_{\mu, n}(\lambda) \) whenever \( n \geq \mu + 2 \), and irrespective of the choice of \( \lambda \).
Namely, graphing the zero loci \( (P_{\mu,n}(x,\lambda) = 0) \) when \( \mu = 1,2,3 \) for low values of \( n \geq \mu + 2 \), we are led to speculate the following.

**Conjecture 3.1.** Let \( \mu \geq 1 \) and \( n \geq \mu + 2 \) be nonnegative integers, and let \( \lambda \neq 0,1 \) be a fixed value. If \( n-\mu \) is odd (respectively, even), then the corresponding generalized key polynomial \( P_{\mu,n}(\lambda) \) has precisely \( \mu \) (respectively, \( 2\mu \)) real roots \( x = \gamma \) such that \( f(\gamma,\lambda) > 0 \).

Equivalently, the conjecture predicts that the codimension-\((g-1)\) linear subseries of \( |\mathcal{L}_R| \) spanned by \( F_g \) has precisely either \( 2(k-g) \) or \( 4(k-g) \) real inflection points away from \( R_n \), respectively, depending upon whether \( g \geq 1 \) is even or odd.

### 4. Real loci of plane curves defined by inflection polynomials

#### 4.1. Degrees and symmetries of inflection polynomials

We begin by showing that \( P_n = P_n(x,\lambda) \) is of degree exactly \( 2n \) in \( x \) (a consequence of Plücker’s formula for \( n \geq 4 \)) and of degree exactly \( n \) in \( \lambda \).

**Lemma 4.1.** For every positive integer \( n \geq 1 \), we have

\[
\deg_x(P_n) = 2n \quad \text{and} \quad \deg_{\lambda}(P_n) = n.
\]

**Proof.** The claim is clear for \( n = 1,2 \); so we assume that \( n \geq 3 \). First recall that since \( k = g+1 \), we have \( n = k+1 = g+2 \). So the geometric object of interest is a \( g^{2g+3} \) on the elliptic curve \( E \), whose total inflectionary degree may be realized as

\[
(2g+2)(g+3) = I_{I_n} + 2 \deg_x(P_{g+2})
\]

where \( I_{I_n} \) is the inflection concentrated in the ramification points, which is \( 4 \left( \frac{g+1}{2} \right) + 2(g-1) \). From (13), we deduce that \( \deg_x(P_{g+2}) = 2g + 4 = 2(g+2) \), as desired.

To prove the second statement, we argue by induction. The claim is obvious when \( n = 1 \), so assume that it holds for all \( k \leq n \) for some \( n > 1 \). We then have

\[
P_n = a_n(x)\lambda^n + Q_n(x,\lambda)
\]

with \( a_n(x) \in \mathbb{R}[x] \) and \( \deg_{\lambda}(Q_n) < n \). Applying (3) to \( P_n \) written in this form, we see that the coefficient of \( \lambda^{n+1} \) in \( P_{n+1} \) is the polynomial

\[
a_{n+1}(x) = -x(x-1) \frac{da_n(x)}{dx} + (-n+1/2)(1-2x)a_n(x).
\]

If \( a_{n+1}(x) = 0 \), then \( a_n(x) \) is a solution of the differential equation

\[
y' = R(x) = \frac{(-n + 1/2)(1-2x)}{x(x-1)}
\]

but \( y = K e^{\int R(x)dx} = K(x^2-x)^{(2-n)/2} \) is not a polynomial, so \( a_{n+1}(x) \neq 0 \) and \( \deg_{\lambda}(P_{n+1}) = n + 1 \).

We next show that the curve defined by \( P_n = 0 \) has special symmetry.

**Lemma 4.2.** For every positive integer \( n \geq 1 \), we have

\[
P_n(x,\lambda) = P_n(x,z) \quad \text{and} \quad P_n(x+1,\lambda+1) = P_n(-x,-\lambda).
\]

**Proof.** We prove (14) by applying the basic recursion (3). Clearly, both properties (14) hold when \( n = 1 \). For the first property, note that \( f(x,\lambda) = f(x,z) \); the fact that the analogous property holds for \( P_n \) is immediate by induction using (3). We now focus on the second symmetry in (14). By induction, we may assume it holds for all \( k \leq n \), for some \( n \geq 1 \). Then because (the second symmetry in) (14) is preserved under taking products and multiplying by scalars, \( (-n+\frac{1}{2})P_n \cdot df \) also satisfies (14). It is slightly more delicate, but nevertheless true, that \( \frac{df}{dx} \cdot f \) satisfies (14). Indeed, one checks easily that \( f(x+1,\lambda+1) = -f(-x,-\lambda) \), so it suffices to check that the \( x \)-derivative of \( P_n \) satisfies the same antisymmetric property, but this follows easily from the chain rule.

An upshot of Lemma 4.2 is that the monodromy group associated with the projection from the point \( [0:0:1] \) of the projective closure \( \mathcal{C}_n \) of the curve \( C_n \) defined by \( P_n = 0 \) inside of \( \mathbb{P}^2_{x,\lambda,z} \) contains transpositions that freely permute the points \( p_1, p_2, \) and \( p_3 \) with coordinates \( [0:0:1], [0:1:0], \) and \( [1:1:1] \). In particular, the singularities of \( \mathcal{C}_n \) in these three points are analytically isomorphic.

**Conjecture 4.3.** For every positive integer \( n \geq 1 \), the plane curve \( \mathcal{C}_n \) is nonsingular along \( \mathbb{P}^2 \setminus \{p_1,p_2,p_3\} \).

Note that Conjecture 4.3 is easy to verify by computer (i.e. algorithmically, using Gröbner bases) whenever \( n \) is small. However, we have thus far been unable to show “by hand” that the system of equations \( P_n(x,\lambda) = \frac{dx}{dx} = \frac{dy}{dy} \) is exactly solved by the coordinate pairs \( (0,0) \) and \( (1,1) \) in general. The next four subsections are devoted to proving Conjecture 3.1 in the first nontrivial cases.
Theorem 4.4. Conjecture 3.1 holds when $\mu = 1$ and $n = 2, 3, 4, 5$.

In fact, the case $n = 2$ is non-geometric (since $k = 1$ in that case) and when $n = 3$ the associated linear series is complete (so in particular Conjecture 4.3 holds on the basis of [Garay 15, Thm 3.2.5]), so the first geometrically interesting case is $n = 4$. However, as the argument used in the proof when $n = 4$ is a more elaborate version of the argument for $n = 3$ (which is itself a more elaborate version of that used for $n = 2$) we find it instructive to include a discussion of the $n = 2$ and $n = 3$ cases as well. The key point here is that the recursion (3) implies that the $n$-th inflection polynomial $P_n = P_n(x, \lambda)$, which is of degree $2n$ in $x$, is only of degree $n$ in $\lambda$. Accordingly, we can obtain a global parameterization $\lambda = \lambda(x)$ for $\lambda$ in terms of $x$ whenever $n \leq 4$.

4.2. Case: $n = 2$

We begin by writing $P_2$ as a quadratic polynomial in $\lambda$ with coefficients in $\mathbb{R}[x]$, namely

$$P_2(x, \lambda) = -\frac{1}{4} \lambda^2 + \left(-x^3 + \frac{3}{2} x^2\right) \lambda + \left(-x^3 + \frac{3}{4} x^4\right).$$

Applying the quadratic formula, we deduce that

$$\lambda = \lambda(x) = \left(-2x^3 + 3x^2\right) \pm 2 \sqrt{x(x-1)}$$

solves $P_2 = 0$. Conjecture 3.1 in this particular case follows easily.

4.3. Case: $n = 3$

We have

$$P_3(x, \lambda) = \left(-\frac{3}{4} x + \frac{3}{8}\right) \lambda^3 + \left(\frac{15}{8} x^2 - \frac{3}{4} x\right) \lambda^2
+ \left(\frac{3}{4} x^3 - \frac{15}{8} x^4\right) \lambda + \left(-\frac{3}{8} x^2 + \frac{3}{4} x^3\right).$$

Whenever $x \neq \frac{1}{2}$, the leading coefficient $-\frac{3}{8} x + \frac{3}{8}$ of $P_3$ is nonzero. Assuming this to be the case, we may divide by the leading coefficient. Doing so produces a monic cubic $\tilde{P}_3$ in $\lambda$ whose coefficients are (formally) power series in $x$, namely

$$\tilde{P}_3(x, \lambda) = \lambda^3 + \left(-\frac{5}{2} + \frac{1}{2} x\right) \lambda^2 - (1 + 4x) x^4 \lambda
+ \left(\frac{1}{2} + \frac{3}{2} x\right) x^5$$

where $x = \frac{1}{1-x^2}$. Now set $a_i := [x^i] \tilde{P}_3$, $i = 0, 1, 2$. Changing variables according to $\lambda \to \lambda^*$ where

$$\lambda^* = \lambda + \frac{a_2}{3} = \lambda + \left(-\frac{5}{6} + \frac{1}{2} x\right)x$$

allows $\tilde{P}_3$ to be rewritten as a depressed cubic, namely

$$\tilde{P}_3 = \tilde{P}_3(x, \lambda^*) = (\lambda^*)^3 + p \lambda^* + q$$

where

$$p = a_1 - \frac{a_2^2}{3} = -(1 + 4x) x^4 + \left(-\frac{25}{12} \frac{5}{6} x - \frac{1}{12} x^2\right) x^2$$

and

$$q = a_0 - \frac{a_1 a_2}{3} + \frac{2 a_2^3}{27} = \frac{1}{3} (2x^2 - 5x - 1) x^2 + \frac{1}{108} (x - 5) x^3.$$

Cardano’s formula now establishes that

$$\lambda^* = u + \nu$$

solves $\tilde{P}_3 = 0$, where

$$u = \sqrt{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

and

$$v = 3 \sqrt{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

Note that when

$$\Delta = \Delta(x) := \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$$

is negative, the expression for $u$ in (15) selects for an arbitrary cubic root, and $\nu := u$. In the latter situation, the corresponding cubic $\tilde{P}_3 = \tilde{P}_3(x)$ has three real roots.

In our case, we have $\Delta = \frac{1}{432} x^8 x^4 R$, where

$$R = -256 (x-1)^6.$$

In particular, $\Delta$ is negative whenever $x \notin \{0, 1, \frac{1}{2}\}$. From (15), we deduce that

$$u = \frac{x(x-1)}{x - \frac{1}{2}},$$

where

$$-3 \sqrt{\frac{1}{18} \left(x - \frac{1}{2}\right)^2 + \frac{5}{216} + \frac{1}{3 \sqrt{3}} (x(x-1)) \left(x - \frac{1}{2}\right) \sqrt{-1}.}
$$

(16)

Further, by negativity of the discriminant $\Delta$, we may always distinguish three disjoint roots of $\tilde{P}_3(x)$ given by $u + \zeta u + \zeta^2 u$, and $\zeta^2 u + \zeta u$, where $\zeta$ is a primitive cube root of unity and $u$ is as in (16), assuming that an arbitrary cubic root of $-q/2 + \sqrt{\Delta}$ is fixed.
Now let
\[ \bar{v}(x) := \frac{1}{18} \left( x - \frac{1}{2} \right)^2 + \frac{5}{216} \]
\[ + \frac{1}{3 \sqrt{3}} (x(x-1)) \left( x - \frac{1}{2} \right) \sqrt{-1}. \]
We can rewrite \( \bar{v} \) as \( \cos(\gamma) + \sin(\gamma) \sqrt{-1} \), where
\[ \gamma = \gamma(x^*) := \arctan \left( \frac{6}{\sqrt{3}} \cdot \frac{x^* \left( (x^*)^2 - \frac{1}{4} \right)}{(x^*)^2 + \frac{5}{12}} \right) \]
and where \( x^* = x - \frac{1}{2} \). Putting everything together, we obtain three distinguished solutions \( \bar{\lambda}_i = \bar{\lambda}_i(x^*) \), \( i = 1, 2, 3 \) of \( P_3 = P_3(x^*) = 0 \), given by
\[ \bar{\lambda}_1 = \left( \frac{(x^*)^2 - \frac{1}{4}}{x^*} \right) \cdot 2 \cos \left( \arctan \left( \frac{6}{\sqrt{3}} \cdot \frac{x^* \left( (x^*)^2 - \frac{1}{4} \right)}{(x^*)^2 + \frac{5}{12}} \right) \right) / 3 \]
\[ + \frac{5}{6} x^* + \frac{1}{2} + \frac{1}{24} \cdot \frac{1}{x^*}, \]
\[ \bar{\lambda}_2 = \left( \frac{(x^*)^2 - \frac{1}{4}}{x^*} \right) \cdot 2 \cos \left( \arctan \left( \frac{6}{\sqrt{3}} \cdot \frac{x^* \left( (x^*)^2 - \frac{1}{4} \right)}{(x^*)^2 + \frac{5}{12}} \right) \right) / 3 \]
\[ + 2 \pi / 3 \]
\[ + \frac{5}{6} x^* + \frac{1}{2} + \frac{1}{24} \cdot \frac{1}{x^*}, \]
and
\[ \bar{\lambda}_3 = \left( \frac{(x^*)^2 - \frac{1}{4}}{x^*} \right) \cdot 2 \cos \left( \arctan \left( \frac{6}{\sqrt{3}} \cdot \frac{x^* \left( (x^*)^2 - \frac{1}{4} \right)}{(x^*)^2 + \frac{5}{12}} \right) \right) / 3 \]
\[ + 4 \pi / 3 \]
\[ + \frac{5}{6} x^* + \frac{1}{2} + \frac{1}{24} \cdot \frac{1}{x^*}. \]

**Conjecture 3.1** now follows in the present case from the following subsidiary claims about the real locus of the affine plane curve \( C_3 \).

1. When \( x < 0 \), the real locus is comprised of three infinite \( x \)-monotone (i.e., nondecreasing with respect to \( x \)) curves \( \lambda_i = \lambda_i(x) \), \( i = 1, 2, 3 \) with \( \lambda_1 > 0 > \lambda_2 > x > \lambda_3 \). These curves intersect in the limit point \( (0, 0) \) (whose associated value of \( \lambda \) describes a singular rational curve). Moreover, \( \lambda_3 \) has a vertical asymptote at \( x = \frac{1}{2} \) and
\[ -\infty = \lim_{x \to -\infty} \lambda_2 = \lim_{x \to -\infty} \lambda_3 = -\lim_{x \to -\infty} \lambda_1. \]

2. When \( 0 < x < 1 \), the real locus is a union of monotone curves \( \lambda_i, 4 \leq i \leq 7 \). Here \( \lambda_4 > x > \lambda_5 > \lambda_6 \); \( \lambda_4, \lambda_5 \), and \( \lambda_6 \) intersect in \( (0, 0) \); \( \lambda_4 \) and \( \lambda_5 \) connect \( (0, 0) \) with \( (1, 1) \), where they also intersect \( \lambda_7 \). Moreover, we have \( \lambda_7 > \lambda_4 \) and
\[ \lim_{x \to -\frac{1}{2}} \lambda_6 = -\lim_{x \to -\frac{1}{2}} \lambda_7 = -\infty. \]

3. When \( x > 1 \), the real locus is comprised of three infinite monotone curves \( \lambda_i, i = 8, 9, 10 \), where \( \lambda_8 > x > \lambda_9 > \lambda_{10} \). These intersect in their common limit point \( (1, 1) \), in which the associated curve \( E_\lambda \) is singular. Moreover, we have
\[ \infty = \lim_{x \to -\infty} \lambda_8 = \lim_{x \to -\infty} \lambda_9 = -\lim_{x \to -\infty} \lambda_10. \]

**Conditions (1, 2), and (3), which describe the organization of the branches of \( C_3(\mathbb{R}) \) near its singularities, together comprise the topological profile of \( C_3(\mathbb{R}) \).**

4.4. **Case: \( n = 4 \)**

Here our analysis closely follows that of the \( n = 3 \) case. To begin, we have
\[ P_4(x, \lambda) = \left( -3x^4 + 3x^2 - \frac{15}{16} \right) \lambda^4 + \left( \frac{21}{2} x^3 - \frac{39}{4} x^2 + 3x \right) \lambda^3 \]
\[ + \left( -\frac{105}{8} x^4 + \frac{21}{2} x^3 - 3x^2 \right) \lambda^2 \]
\[ + \left( -\frac{3}{4} x^5 + \frac{21}{4} x^4 \right) \lambda + \left( \frac{9}{16} x^6 - \frac{3}{2} x^5 \right). \]

Now set \( \beta = \frac{1}{x^3 - \frac{1}{x^3} + \frac{1}{x^3}} = \frac{x^{6} - 1}{(x - 1)^{2} + x^2} \). Note that \( \beta \) is non-zero for all real values of \( x \). Dividing \( P_4 \) by its leading coefficient yields
\[ \bar{P}_4(x, \lambda) = \lambda^4 + \left( -\frac{x}{2} - \frac{1}{4} + \left( -\frac{5}{32} x + \frac{5}{64} \right) \beta \right) \lambda^3 \]
\[ + \left( \frac{35}{8} x^2 + \frac{7}{8} x + \frac{65}{128} + \left( \frac{15}{64} x - \frac{325}{2048} \right) \beta \right) \lambda^2 \]
\[ + \left( \frac{1}{2} x^5 - \frac{5}{4} x^4 - \frac{45}{32} x^3 - \frac{65}{64} x^2 - \frac{295}{512} x - \frac{265}{1024} \right) \lambda \]
\[ + \left( \frac{645}{8192} x^3 + \frac{1325}{16384} \right) \beta. \]

Let \( b_i := |z_i|^2 \bar{P}_4(x, \lambda), 0 \leq i \leq 3 \). Changing variables according to \( \lambda \to \lambda' \) with \( \lambda = \lambda' - \frac{b_1}{4} \) enables us to convert \( \bar{P}_4(x, \lambda) \) to a depressed quartic \( \bar{P}_4'(x, \lambda') \) of the form \( (\lambda')^4 + p(\lambda')^2 + q\lambda' + r, \) where
\[ p = b_2 - \frac{3}{8} b_3, \quad q = \frac{1}{8} b_1^2 - \frac{1}{2} b_2 b_3 + b_1, \quad \text{and} \]
\[ r = -\frac{3}{256} b_4^4 + b_0 - \frac{1}{4} b_1 b_3 + \frac{1}{16} b_2 b_3. \]

According to Ferrari, the roots of \( \tilde{P}_4(x; \lambda') \) are given by

\[ \lambda' = \frac{\pm 1 \sqrt{2m \pm 2} \sqrt{-(2p + 2m \pm 1)^2/\sqrt{m}}}{2} \]  

where \( m \) denotes any root of the resolvent cubic

\[ K(z) = 8z^3 + 8pz^2 + (2p^2 - 8r)z - q^2 \]

associated with \( \tilde{P}_4 \). Here the monic depressed cubic associated with \( K \) is

\[ K_0 = z^3 - \left( \frac{1}{12} p^2 + r \right) z + \left( -\frac{1}{108} p^3 + \frac{1}{3} pr - \frac{1}{8} q^2 \right) = z^3 + \left( -b_0 - \frac{1}{12} b_2^2 + \frac{1}{4} b_1 b_3 \right) z + \left( -\frac{1}{8} b_1^2 + \frac{1}{3} b_0 b_2 - \frac{1}{108} b_1 b_3 + \frac{1}{24} b_1 b_3 - \frac{1}{8} b_0 b_2^2 \right). \]

Now let \( p_0 := |z| K_0 \) and \( q_0 := |z^0| K_0 \). From Cardano, it follows that roots of \( K_0 \) are given by \( m_0 = u + v \), where

\[ u = 3 \sqrt{\frac{q_0}{2} + \sqrt{\Delta_0}} \quad \text{and} \quad v = 3 \sqrt{\frac{-q_0}{2} - \sqrt{\Delta_0}}. \]

Here \( \Delta_0 := \left( \frac{q_0}{2} \right)^2 + \left( \frac{p_0}{3} \right)^3 \) is the discriminant of \( K_0 \). In this case

\[ q_0 = \frac{b_3}{2} x^6 (x - 1)^6 \left( \frac{x - \frac{1}{2}}{2} \right)^4 + \frac{5}{6} \left( \frac{x - \frac{1}{2}}{2} \right)^2 + 11 \right \} \text{and} \]
\[ \Delta_0 = \frac{b_3}{256} x^{15} (x - 1)^{15} \left( \frac{x - \frac{1}{2}}{2} \right)^2 + \frac{5}{108}. \]

It follows that \( u = \frac{b_3}{4} x^2 (x - 1)^2 \frac{3}{2} \sqrt{U_1 + \sqrt{U_2}} \) and \( v = \frac{b_3}{4} x^2 (x - 1)^2 \frac{3}{2} \sqrt{U_1 - \sqrt{U_2}} \) where

\[ U_1 := \left( \frac{x - \frac{1}{2}}{2} \right)^4 + \frac{5}{6} \left( \frac{x - \frac{1}{2}}{2} \right)^2 + \frac{11}{432} \text{ and} \]
\[ U_2 := x^2 (x - 1)^3 \left( \frac{x - \frac{1}{2}}{2} \right)^2 + \frac{5}{108}. \]

Here \( U_1 \) is positive for all real values of \( x \), while \( U_2 \) is positive (respectively, negative) whenever \( x < 0 \) or \( x > 1 \) (respectively, whenever \( 0 < x < 1 \)). Moreover, we have

\[ U_1^2 - U_2 = \frac{64}{27} \left( \frac{x - \frac{1}{2}}{2} + \frac{1}{12} \right)^3. \]  

which is clearly positive for all real-valued \( x \). Accordingly, whenever \( x < 0 \) or \( x > 1 \), we take \( \sqrt{U_1^2(x)} \) to mean the positive square root of \( U_1(x) \); it follows that \( U_1 - \sqrt{U_2} \) and \( U_1 + \sqrt{U_2} \) are positive. In this situation, we denote by \( u = u(x) \) and \( v = v(x) \) the real functions that select for \( \beta \) \( x^2 (x - 1)^2 \) times the unique real positive cubic roots of \( U_1 + \sqrt{U_2} \) and \( U_1 - \sqrt{U_2} \), respectively. Similarly, whenever \( 0 < x < 1 \), we take \( \sqrt{U_1(x)} \) to mean \( -1 \) times the positive square root of \( -U_2(x) \), and then \( U_1 + \sqrt{U_2} \) and \( U_1 - \sqrt{U_2} \) become complex conjugates of one another, with positive real parts. Finally, we obtain

\[ m = m_0 - \frac{1}{3} p \]
\[ = \frac{\beta}{4} x^2 (x - 1)^2 \sqrt{U_1 + \sqrt{U_2}} \]
\[ + \frac{\beta}{12} x^2 (x - 1)^2 \sqrt{U_1 - \sqrt{U_2}} - \frac{1}{3} b_2 + \frac{1}{8} b_3 \]
\[ = \frac{\beta}{4} x^2 (x - 1)^2 \left( \gamma + \frac{7 \beta}{24} \left( x - \frac{1}{2} \right)^2 + \frac{1}{28} \right). \]

where \( \gamma = \gamma(x) := \sqrt{U_1 + \sqrt{U_2}} + \sqrt{U_1 - \sqrt{U_2}}. \) Note here that \( \gamma \) satisfies the algebraic equation \( \gamma^3 - 2U_1 - 4 \left( x - \frac{1}{2} \right)^2 + \frac{1}{12} \gamma = 0 \).

It follows from the above discussion that our choice of \( m \) in (19) is always positive, and it remains to control the sign of the expression inside the second radical in (17). Here

\[ p + m = m_0 + \frac{2}{3} \beta = \frac{\beta}{4} x^2 (x - 1)^2 \left( \gamma - \frac{7 \beta}{12} \left( x - \frac{1}{2} \right)^2 + \frac{1}{28} \right) \]

while

\[ q = \frac{b_3}{2} \left( x - \frac{1}{2} \right) x^3 (x - 1)^3 \left( x - \frac{1}{2} \right)^4 + \frac{7}{32} \left( x - \frac{1}{2} \right)^2 + \frac{1}{128}. \]

We claim that

i. \( -\left( \sqrt{2m(p + m) - q} \right) \geq 0 \Leftrightarrow x \geq 1, \) and
ii. \( -\left( \sqrt{2m(p + m) + q} \right) \geq 0 \Leftrightarrow x \leq 0. \)

Indeed, on the basis of the properties of the inflection polynomials \( P_n \) and their derivatives for small values of \( n \) (namely, the validity of Conjecture 4.3) it suffices to check that the signs of \( -\left( \sqrt{2m(p + m) \pm q} \right) \) are as claimed locally near \( x = 0 \) and \( x = 1 \). This may be achieved by computing the values of \( \sqrt{2m(p + m) \pm q} \) and its first derivative in \( x = 0 \) and \( x = 1 \). We omit the explicit calculation.

Finally, the explicit “parameterization” (17) for \( C_4 \) may be used to check that the real locus \( C_4(\mathbb{R}) \) has
the following topological profile, and the validity of Conjecture 3.1 when \( n = 4 \) follows easily.

1. There are no points in the real locus for which \( 0 < x < 1 \).
2. When \( x < 0 \), there are two monotone curves \( \lambda_i = \lambda_i(x) \), \( i = 1, 2 \) for which \( \lambda_1 > 0 > \lambda_2 > x \). These curves intersect in their common limit of \((0, 0)\), and satisfy
   \[
   \lim_{x \to -\infty} \lambda_1 = \infty = - \lim_{x \to -\infty} \lambda_2.
   \]
3. When \( x > 1 \), there are two monotone curves \( \lambda_i = \lambda_i(x) \), \( i = 3, 4 \) for which \( x > \lambda_3(x) > \lambda_4(x) \). These curves intersect in their common limit of \((0, 0)\) and satisfy
   \[
   \lim_{x \to \infty} \lambda_3 = \infty = - \lim_{x \to \infty} \lambda_4.
   \]

4.5. Case: \( n = 5 \)

This case is more difficult than the preceding ones. Indeed, as noted in Section 4, the monodromy group \( G \) of \( P_5(x, \lambda) \) over \( \mathbb{C}(x) \) associated with the projection of \( \mathbb{C}_5 \) from \((0, 0, 1)\) contains an element of order 3. On the other hand, the solvable subgroups of \( S_5 \) are precisely the subgroups of the Frobenius group \( F_{20} \) of order 20. It follows that \( G \) is not solvable, and therefore no global solution \( \lambda = \lambda(x) \) of the plane curve with equation \( P_5(x, \lambda) = 0 \) is available. On the other hand, local parameterizations \( \lambda = \lambda(x) \) that solve \( P_n(x, \lambda) = 0 \) may always be computed via Puiseux’s algorithm (see, e.g., [Brieskorn and Knörrer, 86, Sec. 8.3]), and we will carry this out now when \( n = 5 \) near the singular point \( p_1 \); in view of (4.2), the local descriptions of \( \mathbb{C}_5 \) near the singular points \( p_1, p_2 \), and \( p_3 \) are identical.

Local parameterizations at \((0, 0)\). We have

\[
P_5(x, \lambda) = -\frac{45}{32}x^{10} + \frac{75}{16}x^9 - \frac{675}{32}x^8 + \frac{75}{16}x^7 - 15x^6\lambda^2
+ \frac{135}{2}x^5\lambda^3 - \frac{945}{8}x^4\lambda^4 + \frac{1575}{16}x^3\lambda^5 + \frac{45}{2}x^2\lambda^6
- \frac{195}{2}x\lambda^7
+ \frac{2565}{16}x^4\lambda^8 - \frac{945}{8}x^5\lambda^9 - \frac{225}{16}x^6\lambda^{10} + \frac{1935}{32}x^7\lambda^{11}
- \frac{195}{2}x^8\lambda^{12} + \frac{135}{2}x^4\lambda^{13} + \frac{105}{32}x^5\lambda^{14} - \frac{225}{16}x^6\lambda^{15}
+ \frac{45}{2}x^2\lambda^{16} - 15x^3\lambda^{17}.
\]

\[[20]\]

From (20), it follows that the lower hull of the Newton polygon consists of the line segments \( L_1, L_2 \) between \((0, 5)\) and \((2, 3)\) (of slope \(-1\)); and \((2, 3)\) and \((9, 0)\) (of slope \(-\frac{1}{3}\)), respectively. So there are at least two local (complex) branches near \((0, 0)\), whose respective multiplicities are equal to the absolute values of the slopes of the corresponding segments \( L_i \).

Branches associated with \( L_1 \). Begin by writing

\[
\lambda_1 = x(c_1 + \lambda_1^1)
\]

where \( c_1 \in \mathbb{C}^* \) and

\[
\lambda_1^1 = \lambda_1^1(x) = cx^2 + cx^2 + \lambda_1^4 + \ldots
\]

is a Puiseux series that remains to be determined. Now substitute the first approximation (21) into the implicit equation \( P_5 = 0 \) and divide by the smallest common power in \( x \); the result is a polynomial in \( x \), \( c_1 \), and \( \lambda_1^1 \) the sum of whose lowest \( x \)-order terms is

\[
\frac{15}{32}x^5\left(7c_1^2 - 30c_1 - 48c_1 - 32\right)
= \frac{15}{32}x^5\left(c_1 - 2\right)^3(7c_1^2 - 16c_1 + 16).
\]

As the lowest \( x \)-order terms of \( P_5 \) must sum to zero, we deduce that either \( c_1 = 2 \) or else \( c_1 \) is one of the two complex conjugate roots of \( 7c_1^2 - 16c_1 + 16 \). In the latter case, we obtain two non-real branches, exchanged under conjugation. So say that \( c_1 = 2 \). Then upon dividing by \( \frac{15}{32}x^5 \) we obtain

\[
P_{5}^1(x, \lambda_1^1) = 16x - 64x^2 + 104x^3 - 80x^4 + 17x^5 + 48\lambda_1
\]

\[
-192x\lambda_1 + 280x^2\lambda_1 - 136x^3\lambda_1 - 45x^4\lambda_1
+ 10x^5\lambda_1 + 96\lambda_1^2 - 408x^2\lambda_1^2 + 648x^3\lambda_1^2
- 406x^4\lambda_1^2 + 88\lambda_1^3 - 376x^2\lambda_1^3 + 598x^3\lambda_1^3
- 380x^4\lambda_1^3 + 40\lambda_1^4 - 171x^2\lambda_1^4 + 272x^3\lambda_1^4
- 176x^4\lambda_1^4 + 7\lambda_1^5 - 30x\lambda_1^5 + 48x^2\lambda_1^5 - 32x^3\lambda_1^5.
\]

\[[22]\]

The lower hull of the Newton polygon of \( P_{5}^1(x, \lambda_1^1) \) is a single line segment of slope \(-1\) linking \((0, 1)\) and \((1, 0)\). It follows immediately that \( \gamma_1 = 1 \) for all \( i \geq 2 \). Substituting \( \lambda_1^1 = \sum_{i=1}^{\infty} c_{i+1}x^i \) in (22) and setting the result equal to zero, we may compute all of the remaining power series coefficients \( c_{i} \), \( i \geq 2 \). Clearly, the branch in question is real.

Branches associated with \( L_2 \). Consider now those branches arising from the line segment \( L_2 \). Much as before, begin by writing
\[ \lambda_1 = x^3 (c_1 + \lambda_1^i) \]  

(23)

where \( \lambda_1^i = \lambda_1^i(x) = c_2 x^{\delta_2} + c_3 x^{\delta_3 + \gamma_3} + \ldots \) is a Puiseux series that remains to be determined. Substituting (23) into the implicit equation \( P_5 = 0 \) and dividing by the smallest common power in \( y \) yields a polynomial in \( x \), \( c_1 \), and \( \lambda_1^i \) whose sum of lowest \( x \)-order terms is

\[ \frac{15}{32} x^9 (10 - 32 c_1^3). \]

It follows that \( c_1 \approx \pm \sqrt[4]{3} \). Now say that \( c_1 = \sqrt[4]{3} \). Then dividing by \(-\frac{15}{32} x^9 \) we obtain a polynomial

\[ P_5^2(x, \lambda_1^i) = -43008x + 80640x^2 + 7680\sqrt{5x^2 - 672000x^3} + 14080\sqrt{5x^4 + 3000x^4 - 27360\sqrt{5x^4 - 129000x^5} + 20160\sqrt{5x^8 + 20800x^8 - 175\sqrt{5x^{10}}} - 14400x^2 + 750\sqrt{5x^2} - 12000\sqrt{5x^8 + 800\sqrt{5x^9}} + 16384\sqrt{5x_1} - 73728\sqrt{5x_1^2} + 129024\sqrt{5x_1^3} + 189440x_1^4 \]

\[ + 107520\sqrt{5x_1^5} - 328320x_1^6 + 9600\sqrt{5x_1^7} + 241920x_1^8 - 41280\sqrt{5x_1^9} - 35000x_1^{10} + 66560\sqrt{5x_1^{11}} + 15000x_1^{12} - 46080\sqrt{5x_1^{13}} - 240000x_1^{14} + 16000x_1^{15} + 32768x_1^{16} - 147456x_1^{17} + 258048x_1^{18}; \]

\[ - 36864\sqrt{5x_1^{19}} - 215040x_1^{20} + 159744\sqrt{5x_1^{21}} + 57600x_1^{22} - 262656\sqrt{5x_1^{23}} - 247680x_1^{24} + 193536\sqrt{5x_1^{25}} + 399360x_1^{26} - 56000\sqrt{5x_1^{27}} - 276480x_1^{28} + 24000\sqrt{5x_1^{29}} - 38400\sqrt{5x_1^{31}} + 25600\sqrt{5x_1^{32}} - 49152x_1^{33} + 212992x_1^{34} - 350208x_1^{35} + 30720x_1^{36} + 258048x_1^{37} - 132096\sqrt{5x_1^{38}} - 224000x_1^{39} + 212992\sqrt{5x_1^{40}} + 96000x_1^{41} - 147456\sqrt{5x_1^{42}} - 153600x_1^{43} + 102400x_1^{44} + 30720x_1^{45} - 132096x_1^{46} + 212992x_1^{47} - 8960\sqrt{5x_1^{48}} - 147456x_1^{49} + 38400\sqrt{5x_1^{50}} - 61440\sqrt{5x_1^{51}} + 40960\sqrt{5x_1^{52}} - 7168x_1^{53} + 30720x_1^{54} - 49152x_1^{55} + 32768x_1^{56} \]

whose Newton polygon’s lower hull is a single line segment of slope \(-1\) linking \((0, 1)\) and \((1, 0)\). It follows just as before that \( \gamma_i = 1 \) for all \( i \geq 2 \), and that all remaining power series coefficients \( c_i, i \geq 2 \) may be computed by substituting \( \lambda_1^i = \sum_{i=1}^{\infty} g_{i+1} x^i \) in \( P_5^2(x, \lambda_1^i) = 0 \). The corresponding branch is real.

Finally, when \( c_1 = -\sqrt[4]{3} \) we obtain a polynomial

\[ P_5^2(x, \lambda_1^i) = P_5^2(x, -\lambda_1^i). \]

The corresponding branch is clearly real as well.

To conclude, it now suffices to show that \( C_5(\mathbb{R}) \) has the same topological profile as in the \( n=3 \) case. More generally, we conjecture on the basis of our graphical experiments that when \( E(\mathbb{R}) \) is maximally real, the topological profile of \( C_n(\mathbb{R}) \) depends only on the parity of \( n \).

**Conjecture 4.5.** Let \( n \geq 2 \) be any positive integer. Then \( C_n(\mathbb{R}) \) and \( C_{n+2}(\mathbb{R}) \) have the same topological profile.

Clearly, Conjecture 4.5 implies Conjecture 3.1. For the sake of completeness, we also record the following conjecture, which is related to Conjectures 4.5 and 4.3.

**Conjecture 4.6.** For all \( n \geq 2 \), the set of real roots of the discriminant \( \Delta_n = \Delta_n(x) \) of \( P_n = P_n(x, \lambda) \) with respect to \( \lambda \) is \( \{0, 1\} \).

Figure 1 includes computer-generated sketches of the real loci of \( C_n \) for some even values of \( n \), in which the \( f \)-nonnegative locus is shaded; the graphs give compelling evidence for Conjectures 4.5 and 3.1.

The affirmation of Conjecture 4.6 is a consequence by the monotonicity property of Conjecture 4.5, together with the conjectural characterization 4.3 of the real singular points of \( C_n \). In particular, Conjecture 4.6 holds when \( n=5 \).

It remains to verify Conjecture 4.5 when \( n=5 \). To this end, first note that implicitly differentiating the equation \( P_n(x, \lambda(x)) = 0 \) with respect to \( x \) shows that

\[ \frac{d\lambda}{dx} = 0 \iff \frac{dP_n}{dx} = 0 \]

away from the singular points of \( C_n \). On the other hand, for small values of \( n \) (including \( n=5 \)), the equations \( P_n = \frac{dP_n}{dx} = 0 \) have no common affine solutions \((x, y) \) in \( \mathbb{R}^2 \setminus \{(0,0), (1,1)\} \). In light of the Puiseux parameterizations of the real branches near \((0, 0)\) and the symmetries (4.2), the desired \( x \)-monotonicity properties of the solution curves \( \lambda = \lambda(x) \) follow immediately. It suffices now to check that \( P_5(x, \lambda) = 0 \) has exactly three solutions \( \lambda \) for each fixed value of \( x \) in \( \mathbb{R} \setminus \{0, 1, \frac{1}{2}\} \), i.e. that \( C_5(\mathbb{R}) \) is connected over each of the critical intervals \((-\infty, 0), (0, 1), \) and \((1, \infty) \) in \( x \). But this, in turn, is an immediate consequence of the Implicit Function theorem.

Finally, it remains to verify the inequalities of the real solution curves \( \lambda = \lambda(x) \) described in the topological profile of \( C_5(\mathbb{R}) \) above relative to \( x \). These hold locally near the singular points \( (0, 0) \) and \((1) \) of \( C_5 \) in the affine plane, so it suffices simply to show that the
intersection of the line $y = x$ with $C_5(\mathbb{R})$ is precisely supported along $\{(0,0),(1,1)\}$. And indeed, we have

\[
P_5(x,x) = -\frac{105}{32}x^5 + \frac{525}{32}x^6 - \frac{525}{16}x^7 + \frac{525}{16}x^8 - \frac{525}{32}x^9 + \frac{105}{32}x^{10} = -\frac{105}{32}x^5(x-1)^5.
\]

Figure 2 includes sketches of the real loci of $C_n$ for small odd values of $n$, in which the $f$-nonnegative locus is shaded; just as in Figure 1, the validity of Conjectures 4.5 and 3.1 in these cases is graphically apparent.

5. From incomplete series on an elliptic curve to complete series on a hyperelliptic curve

Conjectures 4.5 and 3.1, assuming they hold, may be leveraged to construct complete real linear series on real hyperelliptic curves of genus $g \geq 1$ with many real inflection points.

**Theorem 5.1.** Assume that Conjecture 3.1 holds. There exists a maximally-real hyperelliptic curve $X$ with affine equation $y^2 = f$ admits a complete real linear series $|\mathcal{L}_R| = |\mathcal{L}_R(kD)|$ with the distinguished basis $\mathcal{F}_g$ as in (1) whose total real inflectionary degree is

\[
\omega_R(k,g) = 2g(g + 1) + 2(k-g)(g-1) + 2g(1 + g \mod 2)(k-g).
\]

Here $f = f(x)$ is a polynomial of degree $2g-1$ ramified over $\infty \in \mathbb{P}^1$, and $D$ is the divisor represented by the pullback of $\infty$ to $X$.

**Proof.** Follows immediately from Theorems 5.8 and 6.1 of [Biswas et al. 19].

\[\square\]

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**Declaration of interest**

No potential conflict of interest was reported by the authors.

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