CONFORMAL CURVATURE TENSORS IN A GENERALIZED RIEMANNIAN SPACE IN EISENHART SENSE

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Dedicated to Professor Svetislav Minčić on the occasion of his 90th birthday

In the present paper generalizations of conformal curvature tensor from Riemannian space are given for five independent curvature tensors in generalized Riemannian space \((GR_N)\), i.e. when the basic tensor is non-symmetric. In earlier works of S. Minčić and M. Zlatanović et al a special case has been investigated, that is the case when in the conformal transformation the torsion remains invariant (equitorison transformation). In the present paper this condition is not supposed and for that reason the results are more general and new.

1. INTRODUCTION

The use of non-symmetric basic tensor and non-symmetric connection starts to be actual specially in relation with the works of A. Einstein [1] – [7] devoted to the Unified Field Theory (UFT). M. Prvanović [15] and S. Minčić [13] gave geometric interpretation of the torsion, curvature tensors and curvature pseudotensors of non-symmetric connection.

In the sense of the definition of L. P. Eisenhart [4], a generalized Riemannian space \((GR_N)\) is a differentiable manifold endowed with non-symmetric basic tensor.
(g_{ij} \neq g_{ji}), where

\[ g_{ij} = g_{ij} + g_{ji}, \quad \text{det}(g_{ij}) = g \neq 0, \]

(1)

\[ g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}), \quad g_{ji} = \frac{1}{2}(g_{ij} - g_{ji}). \]

For lowering of indices in \( GR_N \) one uses \( g_{ij} \), and for the rising ones \( g^{ij} \), where

\[ g_{ij} g^{jk} = \delta^i_j, \quad \text{det}(g_{ij}) \neq 0. \]

(2)

Christoffel’s symbols in \( GR_N \) are given by:

\[ \Gamma^i_{jk} = \frac{1}{2}(g^{ij,k} - g^{jk,i} + g^{ik,j}), \]

(3)

\[ \Gamma^i_{jk} = g^{ij} \Gamma^i_{p,jk} = \frac{1}{2} g^{ij} (g^{pj,k} - g^{jk,p} + g^{pk,j}), \]

(4)

where, e.g., \( g_{ji,k} = \frac{\partial g_{ji}}{\partial x^k} \).

In the works of S. Minčić [10], [11], [14], 12 curvature tensors are obtained in \( GR_N \), and in S. Minčić [12] is proved that five among them are independent, because the others can be expressed by these 5 of 12 mentioned tensors and \( R^i_{jmn} \), formed by \( g_{ij} \).

The cited independent curvature tensors in \( GR_N \) are according to [12]:

\[ R^i_{1 jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm} \Gamma^i_{pn} - \Gamma^p_{jn} \Gamma^i_{pm}, \]

(5)

\[ R^i_{2 jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{mp}, \]

(6)

\[ R^i_{3 jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{pm} + \Gamma^p_{nm} (\Gamma^i_{pj} - \Gamma^i_{jp}), \]

(7)

\[ R^i_{4 jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{pm} + \Gamma^p_{mn} (\Gamma^i_{pj} - \Gamma^i_{jp}), \]

(8)

\[ R^i_{5 jmn} = \frac{1}{2} (\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} \]

\[ + \Gamma^p_{jm} \Gamma^i_{jn} + \Gamma^p_{mj} \Gamma^i_{np} - \Gamma^p_{jn} \Gamma^i_{mp} - \Gamma^p_{nj} \Gamma^i_{pm}). \]

(9)
In [12] the following notations are introduced (omitting the indices on the left sides):

\begin{align}
A &\equiv A_i^j m n = \Gamma_i^j m V_n, \quad B = \Gamma_p^j m \Gamma_i^p m n, \quad C = \Gamma_n^m \Gamma_p^l \nu
\end{align}

\begin{align}
A' &\equiv A_i^j n m = \Gamma_i^j n V_m, \quad B' = \Gamma_p^j n \Gamma_i^p n m,
\end{align}

where with \( n \) the covariant derivative on \( x^n \) with respect to symmetric connection \( \Gamma_i^j k \) is denoted. In \( \Gamma_i^j m n \) the covariant derivative \( ; n \) is with respect of \( \Gamma_i^j n \).

By virtue of (10), (11) in [12] the following relations are proved

\begin{align}
R_{1} &= R + A - A' + B - B' \\
R_{2} &= R - A + A' + B - B' \\
R_{3} &= R + A + A' - B + B' - 2C \\
R_{4} &= R + A + A' - B + B' + 2C \\
R_{5} &= R + B + B'.
\end{align}

Conformally flat spaces satisfying certain conditions were considered by M. Prvanovich, U. C. De, S. Bandhopadhyay at [16]. A. Velimirović at [19] has considered conformal equitorsion and concircular transformations.

Remark that at S. Mincic [12] \( R \) is denoted with \( \tilde{R} \).

Analogously to the definition for conformal transformation in \( R_N \), we give corresponding definition in \( GR_N \).

**Definition 1.1.** Conformal transformation in \( GR_N \) is transformation where basic tensor \( g_{ij} \) is changed by help of the next rule (K. Yano [20])

\begin{align}
\tilde{g}_{ij}(x) &= \rho^2(x) g_{ij}(x), \quad g_{ij} \neq g_{ii},
\end{align}

where \( \rho(x) = \rho(x^1, \ldots, x^N) \) is some differentiable function in \( GR_N \).
Geometric objects are observed in common coordinate system, that we see, e.g., from (17). In that case by overline the transformed object is denoted. From \( ds^2 = g_{ij}dx^i dx^j \) it follows that

\[
\bar{ds}^2 = \rho^2 ds^2, \quad \bar{ds} = \rho ds.
\]

Denoting

\[
(ln \rho)_i = \frac{\partial (ln \rho)}{\partial x^i} = \frac{1}{\rho} \rho'_i = \rho_i,
\]

based on (3), we get

\[
\bar{\Gamma}_{i,jk} = \rho^2 (\Gamma_{i,jk} + g_{ij} \rho_k - g_{jk} \rho_i + g_{ik} \rho_j),
\]

and \( \bar{\Gamma}_{i,jk} = \bar{g}^{ip} \bar{\Gamma}_{p,jk} \) is obtained by using the fact that it is

\[
\bar{g}^{ij} = \rho^{-2} g^{ij}.
\]

2. CONFORMAL TENSOR OF THE 1\textsuperscript{ST} KIND IN \( GR_N \)

From the equations (omitted indices):

\[
\bar{R}_1 = R + A - A' + B - B',
\]

\[
\bar{\bar{R}}_1 = \bar{R} + \bar{\bar{A}} - \bar{\bar{A}'} + \bar{B} - \bar{\bar{B}}',
\]

we obtain

\[
\bar{\bar{R}}_1 - \bar{\bar{A}} + \bar{\bar{A}'} - \bar{\bar{B}} + \bar{\bar{B}'} = R - A + A' - B + B' + \bar{R} - R.
\]

Because in conformal transformation in \( GR_N \) it is

\[
\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i \rho_{jk} g_{ip} (g_{jp} \rho_k - g_{jk} \rho_p + g_{pk} \rho_j),
\]

we have

\[a) \quad \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \xi^i_{jk},\]

\[b) \quad \xi^i_{jk} = g_{ip} (g_{jp} \rho_k - g_{jk} \rho_p + g_{pk} \rho_j) = -\xi^i_{kj} \]
\[ \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{jk}. \]

In the works of K. Yano [20], Part I, and N. Sinyukov [17] the conformal transformation of curvature tensor in Riemannian space \( R_N \) is considered and in Stanković et al [18], Zlatanović et al [21], A. Velimirović [19] "equitorsion" transformation is applied "because of non-possibility to find a generalization of conformal tensor in general case". In the following, we prove that it is possible. From (26) we have

\[ \bar{T}^i - T^i_{jk} = 2\xi^i_{jk}. \]

**Definition 2.2.** Equitorsion deformation in \( GR_N \) is such one where it is

\[ \bar{T}^i_{jk} = T^i_{jk} \iff \xi^i_{jk} = 0. \]

In the present work we do not use (28), that is we consider general conformal transformation in \( GR_N \).

In the Riemannian space \( R_N \) (Mikeš et al. [9], Eisenhart L. P. [8], Sinyukov, N.S. [17], Yano K. [20]) we have

\[ \bar{R}^i_{jmn} = R^i_{jmn} + \delta^i_m \rho_{jn} - \delta^i_n \rho_{jm} + \rho^i_m g_{jn} - \rho^i_n g_{jm}, \]

where [20]

\[ \rho_{jn} = \rho_{j;m} - \rho_{j} \rho_{m} + \frac{1}{2} \rho_{p} \rho_{m}^{p} g_{jm}, \]

and by (: :) the covariant derivative in relation with symmetric connection \( \Gamma^i_{jk} \) is denoted, while we have \( \rho^p = \theta^p_{\rho}. \)

Using the value, given in (5), from (24),(29) we obtain

\[ \bar{R}^i_{jmn} - \bar{\Gamma}^i_{v:jn} + \bar{\Gamma}^i_{v:mn} - \bar{\Gamma}^i_{v:jm} + \bar{\Gamma}^i_{v:pm} + \bar{\Gamma}^i_{v:vm} \]

\[ = R^i_{jmn} - \Gamma^i_{v:jn} + \Gamma^i_{v:mn} - \Gamma^i_{v:jm} + \Gamma^i_{v:pm} + \Gamma^i_{v:vm} \]

\[ + \delta^i_m \rho_{jn} - \delta^i_n \rho_{jm} + \rho^i_m g_{jn} - \rho^i_n g_{jm}. \]

Contracting with \( i = n \):

\[ \bar{R}_{j} - \bar{\Gamma}_{v:j}^{i} + \bar{\Gamma}_{v:jm}^{i} - \bar{\Gamma}_{v:jm}^{i} + \bar{\Gamma}_{v:pm}^{i} + \bar{\Gamma}_{v:vm}^{i} \]

\[ = R_{j} - \Gamma_{v:j}^{i} + \Gamma_{v:jm}^{i} - \Gamma_{v:jm}^{i} + \Gamma_{v:pm}^{i} + \Gamma_{v:vm}^{i} \]

\[ + \rho_{jn} - \rho_{jm} + \rho^i_{m} g_{jn} - \rho^i_{n} g_{jm}. \]
Because in $GR_N$

\[(33) \quad \Gamma^i_{vj} = \bar{\Gamma}^i_{vj} = 0,\]

it follows that

\[(34) \quad \bar{R}_{jm} - \Gamma^i_{vj;m} + \bar{\Gamma}^i_{vj} \bar{\Gamma}^i_{pm} = R_{jm} - \Gamma^i_{vj;m} + \bar{\Gamma}^i_{vj} \bar{\Gamma}^i_{pm} + (2 - N) \rho_{jm} - \rho^p_p g_{jm}.\]

Let us effect the composition of corresponding sides with

\[(35) \quad \rho^2 \bar{g}^{jm} = g^{jm}\]

and we obtain

\[(36) \quad \rho^2 \bar{R} - 0 + \rho^2 \bar{g}^{jm} \bar{\Gamma}^p_p \bar{\Gamma}^i_{pm} = R - 0 + g^{jm} \Gamma^p_p \Gamma^i_{pm} + (2 - N) \rho^p_p - N \rho^p_p,\]

wherefrom, changing the repeated indices: $j \rightarrow s$, $i \rightarrow r$, $m \rightarrow t$.

\[(37) \quad \rho^p_p = \rho^p_p = \frac{1}{2(N - 1)} \left[ R_{jm} - \rho^2 \bar{R} + g^{st} \bar{\Gamma}^p_p \bar{\Gamma}^r_r - \rho^2 \bar{g}^{st} \bar{\Gamma}^p_p \bar{\Gamma}^r_r \right].\]

This value we substitute in (34), from where

\[(38) \quad \rho_{jm} = \rho_{jm} = \frac{1}{N - 2} \left[ R_{jm} - \Gamma^r_{vj;r} + \Gamma^p_p \Gamma^r_r \Gamma^i_{pm} - \bar{R}_{jm} + \Gamma^r_{vj;r} - \bar{\Gamma}^p_p \bar{\Gamma}^r_r \right] + \frac{1}{2(N - 1)} \left[ \rho^2 \bar{R} - \rho^2 \bar{g}^{st} \bar{\Gamma}^p_p \bar{\Gamma}^r_r - \bar{R} + g^{st} \bar{\Gamma}^p_p \bar{\Gamma}^r_r \right] g_{jm}.\]

Introducing the notation $[m, n]$ for alternation on indices $m, n$, for example the equation (31) can be written in the form:

\[(39) \quad \bar{R}^i_{jm;n} - \Gamma^i_{vj;n} + \Gamma^p_p \bar{\Gamma}^i_{pm}[m,n] = R^i_{jm;n} - \Gamma^i_{vj;n} + \Gamma^p_p \Gamma^i_{pm}[m,n] + \left[ \delta^r_{jn} \rho_{jm} + \rho^d_{jm} g_{jn} \right][m,n].\]

With respect of (38), we put into (39):
Conformal curvature tensors in a generalized Riemannian space in Eisenhart sense

\[
\overline{R}^i_{jmn} - [\overline{\Gamma}^i_{jm;n} + \overline{\Gamma}^p_{jm} \overline{\Gamma}^i_{pn}]_{m,n} = R^i_{jmn} - [\Gamma^i_{jm;n} + \Gamma^p_{jm} \Gamma^i_{pn}]_{m,n} + \frac{1}{N-2} \{ \delta^i_m [R_{jn} - \Gamma^r_{jn;r} - \Gamma^r_{jm} \overline{\Gamma}^r_{pm}]_{m,n} \\
- \overline{R}_{jn} + \Gamma^r_{jm} \overline{\Gamma}^r_{pm} + \overline{\Gamma}^r_{pm} \Gamma^r_{jm} + \frac{1}{2(N-1)} (\rho^2 \overline{R}_1 - \rho^2 g^{at} \overline{\Gamma}^a_{t} \overline{\Gamma}^r_{pt} - R + g^{at} \Gamma^p_{rs} \Gamma^r_{pt} g_{jm}) [m,n] \\
+ \frac{1}{N-2} \{ g_{jn} [R^i_{im} - g^{is} \Gamma^r_{sm;r} - g^{is} \Gamma^r_{pm}]_{m,n} - \rho^2 g^{at} (\overline{R}_{sm} - \Gamma^r_{sm;r} - \Gamma^r_{pm} \Gamma^r_{jm}) \\
+ \frac{1}{2(N-1)} (\rho^2 \overline{R} - \rho^2 g^{at} \overline{\Gamma}^a_{t} \overline{\Gamma}^r_{pt} - R + g^{at} \Gamma^p_{rs} \Gamma^r_{pt} \delta^i_m) [m,n] \}. \]
\]

From (40) we conclude that the next theorem is valid:

Theorem 2.1. The tensor

\[
C^i_{jmn} = R^i_{jmn} - [\Gamma^i_{jm;n} + \Gamma^p_{jm} \Gamma^i_{pn}]_{m,n} + \frac{1}{N-2} \{ \delta^i_m [R_{jn} - \Gamma^r_{jn;r} - \Gamma^r_{jm} \overline{\Gamma}^r_{pm}]_{m,n} \\
+ g_{jn} [R^i_{im} - g^{is} \Gamma^r_{sm;r} - g^{is} \Gamma^r_{pm}]_{m,n} - \frac{1}{(N-2)(N-1)} (R - g^{at} \overline{\Gamma}^a_{t} \overline{\Gamma}^r_{pt}) (\delta^i_m g_{jm} - \delta^i_n g_{jm}) \}
\]

is an invariant of the conformal transformation in GRN, that is

(42) \( \overline{C}^i_{jmn} = C^i_{jmn} \).

Definition 2.3. The tensor \( C^i_{jmn} \) we call conformal tensor of the 1st kind in GRN.

3. CONFORMAL TENSOR OF THE 2ND KIND IN GRN

Analogously to the previous case, based on (13), now we have

(43) \( \overline{R} + \overline{A} - \overline{A}' - \overline{B} + \overline{B}' = R + A - A' - B + B' + \overline{R} - R. \)

From (43), (29) we obtain
\[
\mathcal{R}^i_{\ jmn} + \Gamma^i_{\ jm;n} - \Gamma^i_{\ jm;m} - \Gamma^p_{\ jm} \Gamma^i_{\ pm} + \Gamma^p_{\ jm} \Gamma^i_{\ pm} \\
= R^i_{\ jmn} + \Gamma^i_{\ jm;n} - \Gamma^i_{\ jm;m} - \Gamma^p_{\ jm} \Gamma^i_{\ pm} + \Gamma^p_{\ jm} \Gamma^i_{\ pm} \\
+ \delta^i_m \rho_{jn} - \delta^i_n \rho_{jm} + \rho^i_m g_{jn} - \rho^i_n g_{jm}.
\]

The contraction with \( i = n \) gives:

\[
\mathcal{R}^i_{\ jm} + \Gamma^i_{\ jm;i} + \Gamma^p_{\ jm} \Gamma^i_{\ pm} = R^i_{\ jm} + \Gamma^i_{\ jm;i} + \Gamma^p_{\ jm} \Gamma^i_{\ pm} \\
+ (2 - N) \rho_{jm} - \rho^p g_{jm}.
\]

From here, combining with (35), it follows that

\[
\rho^p_{\ 2} \equiv \rho^p_{\ 2} = \frac{1}{2(N - 1)} (R^2 - \rho^2 R^2 + g^{st} \Gamma^p_{\ \nu} \Gamma^r_{\ \nu} - \rho^2 g^{st} \Gamma^p_{\ \nu} \Gamma^r_{\ \nu}).
\]

This value we put into (45), and then one gets

\[
\rho^p_{\ 2} \equiv \rho^p_{\ jm} = \frac{1}{N - 2} \left( R^2 \right) \\
+ \frac{1}{2(N - 1)} (\rho^2 R^2 - \rho^2 g^{st} \Gamma^p_{\ \nu} \Gamma^r_{\ \nu} - R^2 + g^{st} \Gamma^p_{\ \nu} \Gamma^r_{\ \nu}) g_{jm}.
\]

By further procedure, as in the previous case, one proves that the next theorem is valid.

**Theorem 3.2.** The tensor

\[
C^i_{\ jm} = C^i_{\ jm} = R^i_{\ jm} + \Gamma^i_{\ jm;n} - \Gamma^i_{\ jm;m} - \Gamma^p_{\ jm} \Gamma^i_{\ pm} + \Gamma^p_{\ jm} \Gamma^i_{\ pm} \\
+ \delta^i_m \rho_{jn} - \delta^i_n \rho_{jm} + \rho^i_m g_{jn} - \rho^i_n g_{jm}.
\]

This value we put into (45), and then one gets

\[
\rho^p_{\ 2} \equiv \rho^p_{\ jm} = \frac{1}{N - 2} \left( R^2 \right) \\
+ \frac{1}{2(N - 1)} (\rho^2 R^2 - \rho^2 g^{st} \Gamma^p_{\ \nu} \Gamma^r_{\ \nu} - R^2 + g^{st} \Gamma^p_{\ \nu} \Gamma^r_{\ \nu}) g_{jm}.
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+ \delta^i_m \rho_{jn} - \delta^i_n \rho_{jm} + \rho^i_m g_{jn} - \rho^i_n g_{jm}.
\]

**Definition 3.4.** The tensor \( C^i_{\ jm} \) we call a conformal tensor of the 2\textsuperscript{nd} kind in \( GR_N \).
4. CONFORMAL TENSOR OF THE 3rd KIND IN $GR_N$

By virtue of (9), now we have

\[ \bar{R} - \bar{A} - \bar{A}' + \bar{B} - \bar{B}' + 2\bar{C} = R - A - A' + B - B' + 2C + \bar{R} - R. \]

and, using (29) it follows that

\[ \bar{R}^{i}_{jmn} - \bar{\Gamma}^{i}_{jm;n} + \bar{\Gamma}^{i}_{j;m,n} + \bar{\Gamma}^{p}_{jm} \bar{\Gamma}^{i}_{pm} - \bar{\Gamma}^{p}_{jn} \bar{\Gamma}^{i}_{pm} + 2\bar{\Gamma}^{p}_{m,n} \bar{\Gamma}^{i}_{pj} + \bar{R}^{i}_{jmn} = R^{i}_{jmn} - \Gamma^{i}_{jm;n} + \Gamma^{i}_{j;m,n} + \Gamma^{p}_{jm} \Gamma^{i}_{pm} - \Gamma^{p}_{jn} \Gamma^{i}_{pm} + 2\Gamma^{p}_{m,n} \Gamma^{i}_{pj} \]

and contraction with $i = n$, we obtain:

\[ \bar{R}^{i}_{jm} - \bar{\Gamma}^{i}_{jm;i} + \bar{\Gamma}^{p}_{m} \bar{\Gamma}^{i}_{pj} = R^{i}_{jm} - \Gamma^{i}_{jm;i} + \Gamma^{p}_{m} \Gamma^{i}_{pj} + (2 - N)\rho_{jm} - \rho^{p}_{p} g^{jm}. \]

From here, compounding by (35):

\[ \rho^{p}_{3} = \rho^{p}_{3} = \frac{1}{2(N - 1)} (R^{i}_{3} - \rho^{2} R^{i}_{3} + g^{2} \Gamma^{p}_{3} \Gamma^{i}_{3} + \rho^{2} g^{2} \bar{\Gamma}^{p}_{3} \bar{\Gamma}^{i}_{3}). \]

This value we put into (52), from where it is obtained

\[ \rho_{jm} = \frac{1}{N - 2} [R^{i}_{jm} - \Gamma^{i}_{jm;r} + \Gamma^{p}_{jm} \Gamma^{r}_{pm} - R^{i}_{jm} + \Gamma^{r}_{jm} - \Gamma^{p}_{jm} \Gamma^{r}_{pm} + \frac{1}{2(N - 1)} (\rho^{2} R^{i}_{3} - \rho^{2} g^{2} \bar{\Gamma}^{p}_{3} \bar{\Gamma}^{i}_{3} - R^{i}_{3} + g^{2} \Gamma^{p}_{3} \Gamma^{i}_{3}) g^{jm}]. \]

Further, the equation (51) we write in the form ($(m,n)$ means a symmetrization, and $[m,n]$ an alternation on $m$, $n$, without division):

\[ R^{i}_{jmn} - \bar{\Gamma}^{i}_{jmn;m,n} + \bar{\Gamma}^{p}_{jmn} \bar{\Gamma}^{i}_{mn[m,n]} + 2\bar{\Gamma}^{p}_{mn[m,n]} \bar{\Gamma}^{i}_{pj} = R^{i}_{jmn} - \Gamma^{i}_{jmn;m,n} + \Gamma^{p}_{jmn} \Gamma^{i}_{mn[m,n]} + 2\Gamma^{p}_{mn[m,n]} \Gamma^{i}_{pj} \]

and then exchange in this equation by virtue of (54).

In this manner, we have proved:
Theorem 4.3. The tensor

\[ C^i_{\; jmn} = R^i_{\; jmn} - \Gamma^i_{\; jm;n(m,n)} + \Gamma^p_{\; jm} \Gamma^i_{\; pn[m,n]} + 2\Gamma^p_{\; mn} \Gamma^i_{\; pj} \]

\[ + \frac{1}{N-2} \left[ \delta^i_m \left( R^r_{\; jm} - \Gamma^r_{\; jm;r} + \Gamma^r_{\; jm} \Gamma^r_{\; pn} \right) \right] + \frac{1}{(N-2)(N-1)} \left( R - g^{\frac{2}{3}} \Gamma^p_{\; r} \Gamma^r_{\; pt} \right) (\delta^i_m g_{jn} - \delta^i_n g_{jm}) \]

\[ \text{is an invariant of the conformal transformation in } GR_N, \text{ i.e. we have} \]

\[ \tilde{C}^i_{\; jmn} = C^i_{\; jmn}. \]

Definition 4.5. The tensor \( C^i_{\; jmn} \) we call a conformal tensor of the 3\(^{rd}\) kind in \( GR_N \).

5. CONFORMAL TENSOR OF THE 4\(^{TH}\) KIND IN \( GR_N \)

Using (10), by procedure as like in the previous case, the next theorem is proved.

Theorem 5.4. The tensor

\[ C^i_{\; jmn} = R^i_{\; jmn} - \Gamma^i_{\; jm;n(m,n)} + \Gamma^p_{\; jm} \Gamma^i_{\; pn[m,n]} - 2\Gamma^p_{\; mn} \Gamma^i_{\; pj} \]

\[ + \frac{1}{N-2} \left[ \delta^i_m \left( R^r_{\; jm} - \Gamma^r_{\; jm;r} + \Gamma^r_{\; jm} \Gamma^r_{\; pn} \right) \right] + \frac{1}{(N-2)(N-1)} \left( R - g^{\frac{2}{3}} \Gamma^p_{\; r} \Gamma^r_{\; pt} \right) (\delta^i_m g_{jn} - \delta^i_n g_{jm}) \]

\[ \text{is an invariant of the conformal transformation in } GR_N, \text{ i.e. we have} \]

\[ \tilde{C}^i_{\; jmn} = C^i_{\; jmn}. \]

Definition 5.6. The tensor \( C^i_{\; jmn} \) we call a conformal tensor of the 4\(^{th}\) kind in \( GR_N \).
6. CONFORMAL TENSOR OF THE 5TH KIND IN $GR_N$

By the procedure from previous cases, on the base of (11), (5), (6), we obtain

$$\bar{R}^i_{\ jmn} - \bar{\Gamma}^p_{\ jm} \bar{\Gamma}^r_{\ pn} - \bar{\Gamma}^p_{\ jm} \bar{\Gamma}^i_{\ pm}$$

$$= R^i_{\ jmn} - \Gamma^p_{\ jm} \Gamma^r_{\ pn} - \Gamma^p_{\ jm} \Gamma^i_{\ pm} + \delta^i_m \rho_j - \delta^i_n \rho_j + \rho^j_m g_{jn} - \rho^j_n g_{jm}.$$  

Contracting by $i = n$, we have:

$$\bar{R}^p_{\ jm} - \bar{\Gamma}^i_{\ jm} = R^p_{\ jm} - \Gamma^p_{\ jm} + \Gamma^i_{\ mn} + (2 - N) \rho_{jm} - \rho^p_n g_{jm}.$$  

Comporing (61) with (35), it follows that

$$\rho^p_5 \equiv \rho^p_5 = \frac{1}{2(N-1)} (R - \rho^2 \bar{R} - g^{st} \Gamma^p_{\ qr} \Gamma^r_{\ pq} + \rho^2 g^{st} \bar{\Gamma}^p_{\ qr} \bar{\Gamma}^r_{\ pq}).$$

Putting this value into (62), it is obtained

$$\rho_{jm} \equiv \rho_{jm} = \frac{1}{N - 2} R_{\ jm} - \Gamma^p_{\ jm} \Gamma^r_{\ pn} - \bar{R}^p_{\ jm} + \bar{\Gamma}^i_{\ jm} \bar{\Gamma}^i_{\ pm}$$

$$+ \frac{1}{2(N - 1)} (\rho^2 \bar{R} - \rho^2 g^{st} \bar{\Gamma}^p_{\ qr} \bar{\Gamma}^r_{\ pq} - R + g^{st} \bar{\Gamma}^p_{\ qr} \bar{\Gamma}^r_{\ pq} g_{jm}).$$

Analogously to previous cases, we prove that the next theorem is valid.

**Theorem 6.5.** The tensor

$$C^i_{\ jmn} = R^i_{\ jmn} - \Gamma^p_{\ jm} \Gamma^r_{\ pn} - \Gamma^p_{\ jm} \Gamma^i_{\ pm}$$

$$+ \frac{1}{N - 2} \big[ \delta^i_m (R_{\ jm} - \Gamma^p_{\ jm} + \Gamma^r_{\ pn}) + g_{jn} (R^i_{\ jm} - g^{st} \bar{\Gamma}^p_{\ qr} \bar{\Gamma}^r_{\ pq}) \big]_{[m, n]}$$

$$- \frac{1}{(N - 2)(N - 1)} (R - g^{st} \bar{\Gamma}^p_{\ qr} \bar{\Gamma}^r_{\ pq}) (\delta^i_m g_{jn} - \delta^i_n g_{jm})$$

is an invariant of the conformal transformation in $GR_N$, that is

$$\bar{C}^i_{\ jmn} = C^i_{\ jmn}.$$  

**Definition 6.7.** The tensor $C^i_{\ jmn}$ is called conformal tensor of the 5th kind in $GR_N$. 
**Conclusion.** All conformal tensors $C^{\theta}_{jmn}, (\theta = 1, \ldots, 5)$ in $GR_N$, that are obtained in the present work, in the Riemannian space $R_N$ (i.e. when $g_{ij} = g_{ji}$) reduce to known conformal tensor $C^{i}_{jmn}$. In $GR_N$ are obtained the tensors $C^{\theta}_{jmn}$ from other authors, Stanković et al [18], Zlatanović et al [21], in the case of equi-torsion transformation, i.e. if $\bar{T}^i_{jk} = T^i_{jk}$. That means that in the present work the general case is investigated.

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(Received 06.02.2020)  (Revised 30.10.2020)