Differential Calculus and Discrete Structures

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Abstract

There is a deformation of the ordinary differential calculus which leads from the continuum to a lattice (and induces a corresponding deformation of physical theories). We recall some of its features and relate it to a general framework of differential calculus on discrete sets. This framework generalizes the usual (lattice) discretization.

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1 Introduction

In the context of ‘noncommutative geometry’ \[1, 2\] the following structure – which generalizes the notion of differential forms (on a manifold) – plays a crucial role. A differential calculus for an associative algebra \(A\) (over \(\mathbb{R}\) or \(\mathbb{C}\)) is a \(\mathbb{Z}\)-graded associative algebra \(\Lambda(A) = \bigoplus_{r=0}^{\infty} \Lambda^r(A)\) (where \(\Lambda^r(A)\) are \(A\)-bimodules and \(\Lambda^0(A) = A\)) together with a linear operator \(d: \Lambda^r(A) \to \Lambda^{r+1}(A)\) satisfying \(d^2 = 0\) and \(d(\omega \omega') = (d\omega) \omega' + (-1)^r \omega d\omega'\) where \(\omega \in \Lambda^r(A)\). We will assume that \(\Lambda(A)\) has a unit \(1_I\) such that \(d1_I = 0\). By now there is a vast literature dealing with differential calculi on various types of mostly non-commutative algebras, in particular quantum groups. But even commutative algebras exhibited in this context rather unexpected features. In physics, models of elementary particle physics were built with a space-time of the form \(M \times \mathbb{Z}_2\) where \(M\) is a four-dimensional differentiable manifold \[3\]. Using differential calculus on (the algebra of functions on) the two-point space \(\mathbb{Z}_2\), it was possible to extend the Yang-Mills action to \(M \times \mathbb{Z}_2\). Its \(\mathbb{Z}_2\)-part turned out to be the usual Higgs potential. Later, it was demonstrated that a certain deformation of the ordinary calculus of differential forms (on \(\mathbb{R}^n\)) leads to lattice calculus \[4, 5\]. In particular, the Wilson action of lattice gauge theory was rederived as a corresponding deformation of the usual continuum Yang-Mills action. Also a relation with \(q\)-calculus \[6\] has been established \[4\]. Some of these aspects will be briefly recapitulated in section 2. In section 3 we discuss differential calculus on an arbitrary discrete set (see also \[7\]). In particular, the lattice calculus of section 2 is then recovered as a ‘reduction’ of the universal differential calculus (the ‘universal differential envelope’ of \(A\) \[1, 2\]). Relations with ‘posets’ (partially ordered sets) point towards some interesting applications in physics (cf \[8, 9\]).

2 Differential calculus and lattices

Let \(x\) be the identity function on \(\mathbb{R}\) and \(A\) the algebra of \(\mathbb{C}\)-valued functions of \(x\). The following commutation relation is then consistent with the structure of a differential algebra,

\[
f(x) dx = dx f(x + a) \quad (\forall f \in A)
\]

(2.1)

where \(a \in \mathbb{R}, a > 0\). The (formal) differential \(dx\) is then a basis of \(\Lambda^1(A)\) as a right (or left) \(A\)-module. This allows us to introduce right and left partial derivatives, \(\overrightarrow{\partial}_x\) and \(\overleftarrow{\partial}_x\) respectively, with respect to \(x\) via

\[
df = dx \overrightarrow{\partial}_x f = (\overrightarrow{\partial}_x f) dx.
\]

(2.2)

Then

\[
dx \overrightarrow{\partial}_x f = \frac{1}{a} [x, dx] \overrightarrow{\partial}_x f = \frac{1}{a} [x, df] = \frac{1}{a} (f(x) dx - dx f(x))
\]

\[
= dx \frac{1}{a} (f(x + a) - f(x))
\]

(2.3)

where we have used (2.1) and the Leibniz rule for \(d\) shows that \(\overrightarrow{\partial}_x\) is the discrete (right) derivative. An integral associated with the differential calculus should satisfy

\[
\int df = f + ‘\text{constant’}.
\]

(2.4)
In the case under consideration, a function $h$ should be called ‘constant’ if $dh = 0$ which is the case iff $h$ is a periodic function with period $a$. One finds that the condition (2.4) is sufficient to enable us to calculate $\int dx f(x)$ for any $f \in A$. Since the integral of a function is only determined up to a periodic function, a definite integral is only well-defined over an interval the length of which is an integer multiple of $a$. In this case,

$$\int_{x_o - ma}^{x_o + na} dx f(x) = a \sum_{k=-m}^{n-1} f(x_o + k a).$$

Hence, the integral (for $a > 0$) is the usual approximation of the Riemann integral.

Let us consider a coordinate transformation $y = q^{x/a}$ where $q \in \mathbb{C}$ is not a root of unity. For a function $f(y)$ we then have

$$dx \frac{\partial}{\partial x} f = df = dy \frac{\partial}{\partial y} f = dx \frac{q-1}{a} y \frac{\partial}{\partial y} f$$

which leads to the $q$-derivative

$$\frac{\partial}{\partial y} f = \frac{f(q y) - f(y)}{(q-1)y}.$$

A similar expression is obtained for the left partial derivative $\frac{\partial}{\partial y}$. The integral introduced above yields in particular

$$\int_0^\infty dy f(y) = (q-1) \sum_{k=-\infty}^{\infty} f(q^k) q^k.$$

The rhs is defined in the mathematical literature as the $q$-integral (see [6], for example). It also makes sense when $q^N = 1$, $N \in \mathbb{Z}$, which corresponds to the case of a closed (i.e., periodic) lattice in $x$-space. Furthermore, we obtain representations of the $q$-plane and $q$-deformed canonical commutation relations, namely

$$\frac{\partial}{\partial y} - q \frac{\partial}{\partial y} \frac{\partial}{\partial y} = 0, \quad \frac{\partial}{\partial y} y - q y \frac{\partial}{\partial y} = 1$$

where $y$ has to be regarded as a multiplication operator. These relations are known to be invariant under the coaction of the quantum group $SL_q(2)$.

There is an obvious generalization of the differential calculus to higher dimensions [5]. It provides us with a ‘universal’ framework for the deformation of continuum to lattice theories. We refer to [4, 5] for further details, results and references.

### 3 Differential calculus on a discrete set

Let us consider a discrete set $M$. $A$ denotes the algebra of $\mathbb{C}$-valued functions on $M$ (with pointwise multiplication). There is a distinguished set of functions $e_i$ on $M$, defined by $e_i(j) = \delta_{ij}$ where $i, j \in M$. They satisfy the relations

$$e_i e_j = \delta_{ij} e_i, \quad \sum_i e_i = 1$$

Here by ‘discrete’ we mean finite or denumerable.
where \( I \) denotes the constant function \( I(i) = 1 \) which will be identified with the unit in \( \Lambda(A) \). Each \( f \in A \) can then be written as
\[
f = \sum_i f(i) e_i .
\] (3.2)

In the following we consider the universal differential calculus on \( A \). From the properties of the set of functions \( e_i \) we obtain
\[
e_i de_j = -(de_i) e_j + \delta_{ij} de_i , \quad \sum_i de_i = 0
\] (3.3)
(assuming that \( d \) commutes with the sum) which shows that the \( de_i \) are linearly dependent.

Let us introduce
\[
e_{ij} := de_i e_j \quad (i \neq j) , \quad e_{ii} := 0
\] (3.4)
(note that \( de_i e_i \neq 0 \)) and
\[
e_{i_1 \cdots i_r} := e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{r-1} i_r} \quad (r > 1) .
\] (3.5)

Then
\[
e_k e_{i_1 \cdots i_r} = \delta_{ki_1} e_{i_1 \cdots i_r} , \quad e_{i_1 \cdots i_r} e_k = e_{i_1 \cdots i_r} \delta_{i_k k} , \quad e_{i_1 \cdots i_r} e_{k\ell} = \delta_{i_k k} e_{i_1 \cdots i_r \ell}
\] (3.6)
and it can be shown that \( e_{i_1 \cdots i_r} \) with \( i_k \neq i_{k+1} \) for \( k = 1, \ldots, r - 1 \) form a basis over \( \mathbb{C} \) of \( \Lambda^{r-1}(A) \), the space of \( (r-1) \)-forms \( (r > 1) \). In particular, one finds
\[
de_i = \sum_j (e_{ij} - e_{ji}) .
\] (3.7)

From the above relations it is obvious, that we can set some of the \( e_{ij} \) to zero without generating further relations or running into conflict with the rules of differential calculus. This leads to (consistent) reductions of the universal differential calculus. They are conveniently represented by graphs in the following way. We regard the elements of \( M \) as vertices and associate with \( e_{ij} \neq 0 \) an arrow from \( j \) to \( i \). The universal differential calculus then corresponds to the graph where all the vertices (corresponding to the elements of \( M \)) are connected pairwise by arrows in both directions. Deleting arrows leads to graphs which represent reductions of the universal calculus. Some interesting examples of differential calculi arising in this way are considered in the following.

(1) We choose \( M = \mathbb{Z}^n \) and impose the condition
\[
e_{k\ell} \neq 0 \iff k = \ell + \hat{\mu} \quad \text{for some} \ \mu
\] (3.8)
where \( \hat{\mu} := (0, \ldots, 0, 1, 0, \ldots, 0) \) with a 1 in the \( \mu \)-th entry. The associated graph is a lattice with an orientation. Using
\[
x^\mu := a \sum_k k^\mu e_k \quad (a \in \mathbb{R}, a > 0)
\] (3.9)
we can express any function \( f \in A \) as \( f(x) \). The (reduced) differential calculus then implies
\[
f(x) \, dx^\mu = dx^\mu \, f(x + \mu a) \quad (\mu = 1, \ldots, n)
\]
(3.10)
which corresponds to the calculus considered in the previous section and underlies the usual lattice theories (where \( a \) is the lattice constant).

(2) We consider the differential calculus on \( \mathbb{Z}^n \) associated with the ‘symmetric lattice’, i.e.,
\[
e_{k\ell} \neq 0 \quad \Leftrightarrow \quad k = \ell + \mu \text{ or } k = \ell - \mu \text{ for some } \mu .
\]
(3.11)
Let us define
\[
e^{\pm \mu} := \sum_k e_{k\pm\mu,k} , \quad \tau^\mu := a^2 (e^{+\mu} + e^{-\mu}) .
\]
(3.12)
Using (3.7), we obtain
\[
dx^\mu = a (e^{+\mu} - e^{-\mu}) \text{ with } x^\mu \text{ defined as in (3.9) and}
\]
\[
[f(x), dx^\mu] = \frac{a^2}{2} dx^\mu \Delta_\mu f(x) + \tau^\mu \bar{\partial}_\mu f(x)
\]
(3.13)
\[
[f(x), \tau^\mu] = a^2 dx^\mu \bar{\partial}_\mu f(x) + \frac{a^2}{2} \tau^\mu \Delta_\mu f(x)
\]
(3.14)
where
\[
\partial_{\pm \mu} f := \pm \frac{1}{a} (f(x \pm a \hat{\mu}) - f(x))
\]
(3.15)
\[
\bar{\partial}_\mu f := \frac{1}{2} (\partial_{+\mu} f + \partial_{-\mu} f)
\]
(3.16)
\[
\Delta_\mu f := \partial_{+\mu} \partial_{-\mu} f = \frac{1}{a} (\partial_{+\mu} f - \partial_{-\mu} f) .
\]
(3.17)
Furthermore,
\[
df = \sum_\mu (dx^\mu \bar{\partial}_\mu f + \frac{1}{2} \tau^\mu \Delta_\mu f) .
\]
(3.18)
This differential calculus looks very much like a lattice version of a (generalization of the) calculus which arises in the classical limit of bicovariant differential calculus on the quantum group \( SL_q(2, \mathbb{R}) \) [10].

(3) Let \( S \) be a topological space (one may think of spacetime, for example). In general, the collection of all open sets will be infinite. A finite subtopology \( \mathcal{T} \) may then be regarded as an approximation of \( S \) (related, e.g., to inaccurate position and time measurements). Naturally associated with such a finite collection of open sets covering \( S \) is a graph in the following way. The vertices are the elements of \( \mathcal{T} \) (the open sets). If \( U \subset V \) for \( U, V \in \mathcal{T} \) we draw an arrow from the vertex representing \( U \) to the one representing \( V \). From the discussion above we know that there is a differential calculus associated with such a graph. These graphs (or ‘posets’) generalize the ordinary lattice and allow us to include nontrivial topology in discretized field theories (see also [9]).
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