DETERMINISTIC SPECTRAL PROPERTIES OF ANDERSON-TYPE HAMILTONIANS

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ABSTRACT. This paper concerns the deterministic spectral properties of the self-adjoint operator \( H_\omega = H + V_\omega \), where \( V_\omega = \sum_n \omega_n (\cdot, \varphi_n) \varphi_n \), on a separable Hilbert space \( \mathcal{H} \) with a sequence \( \{ \varphi_n \} \subset \mathcal{H} \). Here \( \omega = (\omega_1, \omega_2, \ldots) \) is a random variable corresponding to a probability measure \( P \) on \( \mathbb{R}^\infty \) which is merely assumed to satisfy Kolmogorov’s 0-1 law.

The deterministic dynamical properties of \( H_\omega \) are the focus of this article.

The main result - Theorem 3.2 - states that under mild conditions the essential parts of \( H_\omega \) and \( H_\eta \) are almost surely with respect to the product measure \( P \times P \) unitary equivalent modulo a rank one perturbation. Its proof is based on an application of a method involving the Krein–Lifshits spectral shift function which was developed by A.G. Poltoratski (see [13]) in the case of purely singular spectral measures.

1. Introduction

In 1958 P.W. Anderson suggested that sufficiently large impurities in a semi-conductor could lead to spatial localization of electrons, see [2]. Since then the field has developed into a rich theory. A vast amount of progress has been made by both the mathematical and the physical community. Despite being one of the most studied problems in Mathematical Physics, the Anderson model, is far from being completely understood. Many problems with striking physical relevance remain unsolved, e.g. the so-called Anderson localization.

Here we describe the deterministic spectral structure of so-called Anderson-type Hamiltonians, a generalization of most commonly considered Anderson models.

1.1. Anderson-type Hamiltonian. Let \( H \) be a self-adjoint operator on a separable Hilbert space \( \mathcal{H} \). Let \( \{ \varphi_n \} \subset \mathcal{H} \) be a sequence of mutually linearly independent unit vectors in \( \mathcal{H} \), and let \( \omega = (\omega_1, \omega_2, \ldots) \) be a random variable corresponding to a probability measure \( P \) on \( \mathbb{R}^\infty \). Assume that \( P \) satisfies Kolmogorov’s 0-1 law (see subsection 2.5 below).

The Anderson-type Hamiltonian is formally given by

\[
H_\omega = H + V_\omega \quad \text{on} \quad \mathcal{H}, \quad V_\omega = \sum_n \omega_n (\cdot, \varphi_n) \varphi_n.
\]

Assume that \( H_\omega \) is almost surely essentially self-adjoint.

In general, the perturbation \( V_\omega \) is almost surely a non-compact operator. Therefore, many results from classical perturbation theory cannot immediately be applied here.

Remarks. (a) Literature provides many sufficient conditions which guarantee that the formal expression (1.1) yields an almost surely essentially self-adjoint operator. For example, if the vectors \( \varphi_n \) are mutually orthogonal, then it suffices to assume \( \sum |\omega_n| < \infty \) almost surely.

(b) Let \( \sum |\omega_n| < \infty \) almost surely. Then we have the almost sure self-adjointness, if the sequence \( \{ \varphi_n \} \) consists of linear combinations of vectors from an orthonormal basis \( \{ f_m \} \) where each \( f_m \) occurs only in finitely many \( \varphi_n \)'s. In this case, the vectors in the sequence \( \{ \varphi_n \} \) are not mutually orthogonal (we still assume that they are mutually independent).

(c) It is possible to consider so-called singular form bounded perturbations, where the vectors...
φ_n come from a larger Hilbert space which is usually denoted by H_{-1}(H) (see e.g. [10]). Such extensions are useful, as they allow us to include the continuous Schrödinger operator with random potential. (Note that the delta distributions do not belong to the underlying Hilbert space.)

(d) For more singular perturbations, e.g. the case where the vectors φ_n ∉ H_{-1}(H), the difficulty consists of defining the self-adjoint extension uniquely, see e.g. [9].

This definition generalizes the Anderson-type Hamiltonians that were considered in [4] and [5] to the case of non-orthogonal sequences of vectors \{φ_n\}. The results in [5] provide an interesting (but not complete) picture of the deterministic structure and properties of the spectrum of Anderson-type Hamiltonians.

The main result of the paper at hand (Theorem 3.2 below) basically states that, under mild conditions, the essential parts of two Anderson-type Hamiltonians are unitarily equivalent modulo a rank one perturbation almost surely (with respect to the product measure \( P \times P \)).

On the one hand, this greatly restricts the possible deterministic properties of Anderson-type Hamiltonians. On the other hand, it tells us how ‘difficult’ rank one perturbations can be.

Probably the most important special case of such Anderson-type Hamiltonians is the discrete Schrödinger operator with random potential on \( l^2(\mathbb{Z}^d) \) given by

\[
H f(x) = -\Delta f(x) = -\sum_{|n|=1} (f(x+n) - f(x)), \quad φ_n(x) = δ_n(x) = \begin{cases} 1 & x = n, \\ 0 & \text{else}. \end{cases}
\]

In fact, many Anderson models are special cases of the Anderson-type Hamiltonian given by (1.1).

1.2. Outline. We conclude this introductory section explaining some notation which is used throughout this article.

In section 2, we review selected results from perturbation theory, and remind the reader of the definition of and a few facts about the Krein–Lifshits spectral shift function for rank one perturbations. Further, we explain Kolmogorov’s 0-1 law, its meaning for Anderson-type Hamiltonians.

The results and their proofs can be found in section 3.

1.3. Notation. Briefly recall one of the standard ways to split up the spectrum of a normal, i.e. \( T^*T = TT^* \), operator \( T \). By the spectral theorem operator \( T \) is unitarily equivalent to \( M_z = \text{mult} z \) in a direct sum of real Hilbert spaces

\[
\mathcal{H} = \bigoplus \mathcal{H}(z) \, d\mu(z)
\]

with positive spectral measure \( d\mu \) on \( \mathbb{C} \).

Of course, in our case, i.e. for self-adjoint operators, the spectrum is real. In other words, the spectral measure \( d\mu \) lives on \( \mathbb{R} \).

By \( T_{ac} \) denote the restriction of \( T \) to its absolutely continuous part, i.e. \( T_{ac} \) is unitarily equivalent to \( M_z \big|_{\bigoplus \mathcal{H}(z) \, d\mu_{ac}(z)} \). Similarly, define the singular, singular continuous and the pure point parts of operator \( T \), denoted by \( T_s, T_{sc} \) and \( T_{pp} \), respectively.

Recall that an operator \( T \) is said to be cyclic, if there exists \( φ \) such that we have

\[
\mathcal{H} = \text{clos span}\{(T - \lambda I)^{-1} f : \lambda \in \mathbb{C} \setminus \mathbb{R}\};
\]

or equivalently, if \( L^2(\mu) = \bigoplus \mathcal{H}(z) \, d\mu(z) \), that is if there is only one fiber in this direct sum of Hilbert spaces for some real-valued Borel measure \( \mu \) on the real line. We say that \( T_{ac} \) is cyclic, if \( L^2(\mu_{ac}) = \bigoplus \mathcal{H}(z) \, d\mu_{ac}(z) \).

Let \( \sigma(T) \) denote the spectrum of a (closed) operator \( T \). We use

\[
\sigma_{ess}(T) = \sigma(T) \setminus \{\text{isolated point spectrum of finite multiplicity}\}
\]

to denote the essential spectrum of \( T \).
The essential support of the absolutely continuous part of a measure $\tau$ (on $\mathbb{R}$) is given by

$$\text{ess-supp} \tau_{ac} = \left\{ x \in \mathbb{R} : \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} d\tau > 0 \text{ and } < \infty \right\}. \quad (1.2)$$

At this point it is worth mentioning that $\text{ess-supp} \tau_{ac} \subsetneq \text{supp} \tau_{ac}$ may happen. For example, let $\tau_{ac}$ be given by the Lebesgue measure on intervals that have all rational points of $[0,3]$ as centers and with width $2^{-n}$. Outside those intervals, $\tau_{ac}$ is the zero measure. For the Lebesgue measures we have $|\text{ess-supp} \tau_{ac}| = 2 \neq 3 = |\text{supp} \tau_{ac}|$.

Further, we write $A \sim B$ for two operators $A$ and $B$, if the operators are unitary equivalent, i.e. $U A U^{-1} = B$ for some unitary operator $U$. The notation $A \sim B (\text{mod Class } X)$ is used, if there exists a unitary operator $U$ such that $U A U^{-1} - B$ is an element of Class $X$. At this Class $X$ can be any class of operators, e.g. compact, trace class, or finite rank operators.

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2. Preliminaries

2.1. Perturbation Theory. Perturbation theory is concerned with the general question: Given some information about the spectrum of an operator $A$, what can be said about the spectrum of the operator $A + B$ for $B$ in some operator class? Depending on which class of operators the perturbation $B$ is taken from, we obtain different results of spectral stability, i.e. preserving of parts of the spectrum under such perturbations.

For self-adjoint operators $A$ and $B$ let us recall the following well-known theorems that will be used in the proof of Theorem 3.1 below.

**Theorem 2.1** (Weyl–von Neumann, see e.g. [8]). The essential spectrum of two self-adjoint operators $A$ and $B$ satisfies $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$ if and only if $A \sim B (\text{mod compact operators})$.

**Theorem 2.2** (Kato–Rosenblum, see e.g. [8]). If for two self-adjoint operators we have $A \sim B (\text{mod trace class})$, then their absolutely continuous parts are equivalent, i.e. $A_{ac} \sim B_{ac}$.

Remarks. (a) In [8] Carey and Pincus found a complete characterization of when we have $A \sim B (\text{mod trace class})$ in terms of the operators’ spectrum.

(b) In the case of purely singular measures, Theorem 2.6 below resembles a characterization for $A \sim B (\text{mod rank one})$.

(c) Two operators satisfy $A_{ac} \sim B_{ac}$ if and only if the essential support of their spectral measures are equal up to a set of measure zero.

2.2. Cauchy transform and rank one perturbations. The deep connection between operator theory and the Cauchy transform

$$K \tau(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\tau(t)}{t - z}, \quad z \in \mathbb{C}_+, \quad \text{of an operator’s spectral measure } \tau \text{ poses - although well studied - still a wonderful source of information. This connection is frequently used to learn about the spectral properties of the operator under investigation.}$$

It is well-known that the density of the absolutely continuous part of the measure can be recovered via

$$d\tau_{ac}(x) = \lim_{y \downarrow 0} \Im K \tau(x + iy) \, dx, \quad x \in \mathbb{R}, \quad (2.1)$$
where \( \Im \) denotes the imaginary part.

In Aleksandrov–Clark Theory, the following Theorem plays an essential role.

**Theorem 2.3.** (Poltoratski [12], also see [6]) Let \( \tau \) and \( \tilde{\tau} \) be two measures such that \( \tilde{\tau} = f\tau + \tilde{\tau}_s \). Then

\[
\frac{K\tilde{\tau}}{K\tau}(x + i\varepsilon) \xrightarrow{\varepsilon \to 0} f(x) \quad \tau_s - \text{almost everywhere}.
\]

For many measures \( \tau \) of interest in this paper we have \( \int_{\mathbb{R}} \frac{d\tau(t)}{|t| + 1} = \infty \), but \( \int_{\mathbb{R}} \frac{t\,d\tau(t)}{|t|^2 + 1} < \infty \).

In order to avoid difficulties with convergence, it is standard to introduce an alternative definition of the Cauchy transform

\[
K_1\tau(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{t - z} - \frac{t}{t^2 + 1} \, d\tau(t), \quad z \in \mathbb{C}_+.
\]

It is worth mentioning that (for \( \tau \) such that \( K\tau \) is defined on \( \mathbb{C}_+ \)) the real part of \( K_1\tau \) differs from the conjugate Poisson integral by a finite additive constant. The advantage of introducing this alternative definition is that it is possible to define \( K_1\tau \) for more general measures \( \tau \) (because the kernel decays faster at infinity). Further notice that locally the \( K\tau \) and \( K_1\tau \) behave alike.

We use both \( K\tau \) and \( K_1\tau \) below.

The connection between operator theory and the Cauchy transform also plays a central role in the spectral theory of rank one perturbations. Due to space limitations, we merely recall the results that are applied later in this article. An accessible exposition of rank one perturbations can be found e.g. in [15].

Consider the formal expression

\[
A_\alpha = A + \alpha(\cdot, \varphi)\varphi, \quad \alpha \in \mathbb{R},
\]

which represents the rank one perturbation of a self-adjoint operator \( A \) with cyclic vector \( \varphi \). It is well-known that \( \varphi \) is then also a cyclic vector of the operator \( A_\alpha \) for all \( \alpha \in \mathbb{R} \). By \( \mu_\alpha \) denote the spectral measure of \( A_\alpha \) with respect to \( \varphi \), i.e. \((A_\alpha - zI)^{-1}\varphi, \varphi)_H = \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{t - z} \) for \( z \in \mathbb{C} \setminus \mathbb{R} \). We use the notation \( \mu = \mu_0 \).

With the resolvent formula, it is easy to see that the measures \( \mu \) and \( \mu_\alpha \) of the rank one perturbation [22] are related via the Aronszajn–Krein formula

\[
(2.3) \quad K\mu_\alpha = \frac{K\mu}{1 + \pi \alpha K\mu}.
\]

The Aronszajn–Donoghue theory gives a good picture of the spectrum of the perturbed operator for rank one perturbations. One of its results says that the singular part of rank one perturbations must move when we change the perturbation parameter \( \alpha \).

**Theorem 2.4 (Aronszajn–Donoghue).** For coupling constants \( \alpha \neq \beta \in \mathbb{R} \), the singular parts of the corresponding spectral measures \( \mu_\alpha \) and \( \mu_\beta \) are mutually singular, i.e. \((\mu_\alpha)_s \perp (\mu_\beta)_s\).

Another result within this theory gives a necessary condition for a point to be in the essential support of the singular spectrum of \( A_\alpha \). The theorem in this form can easily be extracted, e.g., from Theorem II.2 of [15].

**Theorem 2.5 (Aronszajn–Donoghue).** We have \((\mu_\alpha)_s(\{x : \lim_{y \to 0} K\mu(x + iy) \neq -\alpha^{-1}\}) = 0\).
2.3. Krein–Lifshits Spectral Shift for Rank One Perturbations. We present the Krein–Lifshits spectral shift function in the case of rank one perturbations. For more detailed explanations, examples and proofs we refer to [14] and the references within.

Consider the rank one perturbations \( A_\alpha \) given by (2.2) and their spectral measures \( \mu_\alpha \) corresponding to the cyclic vector \( \varphi \).

Since the spectral measure \( \mu \) is non-negative, the imaginary part of its Cauchy transform \( K\mu(z) \) is non-negative for \( z \in \mathbb{C}_+ \). Recall that the angular boundary values of the Cauchy transform exit almost everywhere with respect to the Lebesgue measure. For every \( \alpha \in \mathbb{R} \) it is hence possible to find an essentially bounded by \( -\pi < u(t) \leq \pi \), \( t \in \mathbb{R} \), function and a constant \( c \in \mathbb{R} \) such that

\[
1 + \pi \alpha K\mu = e^{K_1u+c}.
\]

(2.4)

Function \( u \) is called the Krein–Lifshits spectral shift of the rank one perturbation \( A_\alpha \).

Using the Aronszajn–Krein formula (2.3) we obtain a relation between the shift function and the measure \( \mu_\alpha \)

\[
1 - \pi \alpha K\mu_\alpha = e^{-K_1u-c}.
\]

(2.5)

We can label \( A \) and \( A_\alpha \) so that \( \alpha > 0 \).

For such \( \alpha \), it is possible to define \( u \) via the principal argument

\[
u = \arg(1 + \pi \alpha K\mu) \quad \in [0, \pi].
\]

Indeed, consider the logarithm of (2.4), take its imaginary part and recall the relation (2.1).

Now by breaking \( K\mu \) into real and imaginary part \( K\mu = iP\mu - Q\mu \) (where \( P \) denotes the Poisson integral and \( Q \) denotes the conjugate Poisson integral), one can see that \( u \) jumps from 0 to \( \pi \) at isolated points of \( \text{supp } \mu_\alpha \). Similarly, the analog

\[
u = -\arg(1 - \pi \alpha K\mu_\alpha)
\]

of (2.6) for \( \mu_\alpha \) implies that \( u \) drops from \( \pi \) to 0 at isolated points of \( \text{supp } (\mu_\alpha)_s \).

In the non-isolated case, a characterization of the point masses of \( \mu \) and \( \mu_\alpha \) was established in [11].

Further the set where \( u \in (0, \pi) \) is equal (up to a set of Lebesgue measure zero) to \( \text{ess-supp}(\mu)_{ac} = \text{ess-supp}(\mu_\alpha)_{ac} \).

Remark. These observations about the relationship between the spectrum of \( A \) and \( A_\alpha \), and the behavior of \( u \) give an alternative proof for the fact that the discrete spectrum of two purely singular operators in the same family of rank one perturbations must be interlacing (the Krein–Lifshits spectral shift essentially jumps from 0 to \( \pi \) and then back).

Vice versa, it is well-known that for fixed \( \alpha > 0 \) any measurable function \( u \) which is essentially bounded by \( 0 \leq u \leq \pi \) is the Krein–Lifshits spectral shift of the rank one perturbation \( M_\mu + \alpha(\cdot, 1)1 \) of the multiplication operator \( M_\mu \) by the independent variable on \( L^2(\mu) \). In fact, given such a function \( u \) and \( \alpha > 0 \) we obtain a unique pair of measures \( \mu \) and \( \nu = \mu_\alpha \), if we impose a normalization condition on the measures. For \( \alpha = 1 \), we say that the measures \( \mu \) and \( \nu \) correspond to \( u \).

2.4. Equivalence up to rank one perturbation. The next theorem can be seen as an inverse spectral problem in the following sense: It gives conditions on the spectrum of two purely singular operators which guarantee that the operators are unitary equivalent modulo a rank one perturbation.

Recall that two operators \( A \) and \( B \) are said to be completely non-equivalent, if there are no non-trivial closed invariant subspaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) of \( \mathcal{H} \) such that \( A|_{\mathcal{H}_1} \sim A|_{\mathcal{H}_2} \). Clearly, two
operators are completely non-equivalent, if and only if their spectral measures are mutually singular.

**Theorem 2.6** (Poltoratski [13]). Let $K \subset \mathbb{R}$ be closed. By $I_1 = (x_1; y_1), I_2 = (x_2; y_2), \ldots$ denote disjoint open intervals such that $K = \mathbb{R} \setminus \bigcup I_n$. Let $A$ and $B$ be two cyclic self-adjoint completely non-equivalent operators with purely singular spectrum. Suppose $\sigma(A) = \sigma(B) = K$ and assume that for the pure point spectra of $A$ and $B$ we have $\sigma_{pp}(A) \cap \{x_1, y_1, x_2, y_2, \ldots\} = \sigma_{pp}(B) \cap \{x_1, y_1, x_2, y_2, \ldots\} = \emptyset$. Then we have $A \sim B (\text{mod rank one})$.

The proof of our main result (Theorem 3.2, below) applies the latter Theorem as well as Lemma 3.3 below which allows us to introduce absolutely continuous spectrum while retaining precise control of the Radon–Nikodym derivatives of the singular measure after with respect to the measure before it was changed.

2.5. Kolmogorov’s 0-1 law and Anderson-type Hamiltonians. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and consider the sequence of independent random complex variables $X_n(w), w \in \Omega$. We explain a simple application of Kolmogorov’s 0-1 law to Anderson-type Hamiltonians using the standard probabilistic setup described in [7] (also see [1]), where the reader can find all the necessary definitions and basic properties.

As in the above setting of Anderson-type Hamiltonians we assume that $\Omega = \prod_{n=0}^{\infty} \Omega_n$, where $\Omega_n$ are different probability spaces, $w = (w_1, w_2, ...), w_n \in \Omega_n$ and the probability measure on $\Omega$ is introduced as the product measure of the corresponding measures on $\Omega_n$. Each of the independent random variables $X_n(w)$ depends only on the $n$-th coordinate, $w_n$, of $w$.

It is a standard observation that any sequence of independent random variables on an abstract probability space is similar to such a sequence $X_n$ defined, for instance, on an infinite dimensional torus: So without loss of generality we have $\Omega = \prod_{n=0}^{\infty} \Omega_n$, where each $\Omega_n$ is a copy of the unit circle with normalized Lebesgue measure, see e.g. [7] (where the unit interval was used instead of $\mathbb{T}$).

It is well-known that the properties we are interested in (cyclicity, etc.) are in fact an event, i.e. that the set $A$ of $w$, such that the function corresponding to the sequence $\{X_n(w)\}$ satisfies the desired property, is measurable: $A \in \mathcal{A}$. We will be mostly interested in the events $A$ that do not depend on the values of any finite number of variables $X_n$, i.e. the sets $A \in \mathcal{A}$ with the property that if $w \in A$ and $X_n(w) = X_n(w')$ for all but finitely many $n$ then $w' \in A$. By the zero-one law the probability of any such event is 0 or 1.

**Observation 2.7** (Kolmogorov’s 0-1 law applied to Anderson-type Hamiltonians). Consider the Anderson-type Hamiltonian $H_\omega$ given by (1.1). Assume that the probability distribution $\mathbb{P}$ satisfies the 0-1 law. Then those spectral properties that are invariant under finite rank perturbations are enjoyed by $H_\omega$ almost surely or almost never.

3. Deterministic spectral structure

3.1. Deterministic absolutely continuous part and essential spectrum. This subsection contains two simple results about the deterministic spectral structure of Anderson-type Hamiltonians.

**Theorem 3.1.** Let $H_\omega$ be given by (1.1) and assume the hypotheses of section 1.1. Then almost surely with respect to the product measure $\mathbb{P} \times \mathbb{P}$:

1) $(H_\omega)_{ac} \sim (H_\eta)_{ac}$ and
2) $H_\omega \sim H_\eta (\text{mod compact operator})$. 

Remark. Part 1 is essentially proven in [3], if the sequence \( \{\varphi_n\} \) forms an orthonormal basis. Note, however, that while spectral properties are invariant under conjugation by a unitary operator, they may change under a general (non-unitary) basis transformation.

Proof of Theorem 3.1. The words ‘almost surely’ in this proof refer to almost surely with respect to the product measure \( \mathbb{P} \times \mathbb{P} \), unless otherwise stated.

Let \( H_\omega \) denote finite rank perturbations of \( H \), i.e. \( \omega \neq 0 \) only in finitely many components.

In particular, \( H_\omega \) are compact and trace class perturbations of \( H \).

To show part 1, fix point \( x \in \mathbb{R} \). Without loss of generality, let \( \mu_\omega \) denote the fiber of the spectral measure of \( H_\omega \) for which \( \text{ess-sup} \mu_\omega \) is maximal (with respect to the inclusion of sets). Let \( \mu_\omega \) be the analog measure for \( H_\omega \). By the Kato–Rosenblum theorem, Theorem 2.2, for almost every \( x \in \mathbb{R} \) we have \( x \in \text{ess-sup}(\mu_0,0,0,...)_{ac} \) if and only if \( x \in \text{ess-sup}(\mu_\omega)_{ac} \).

In virtue of observation 2.7 we have that \( x \in \text{ess-sup}(\mu_\omega)_{ac} \) almost surely or almost never. The set (up to a set of measure zero) of points \( x \) for which the latter is almost surely true is hence deterministic and part 1 is proven.

Part 2 follows almost in analogy from the Weyl–von Neumann theorem 2.1 (instead of Theorem 2.2).

Remark. (a) In fact, we have proven the stronger (than part 1 of Theorem 3.1) statement that the essential support of the absolutely continuous spectrum is a deterministic set (up to a set of Lebesgue measure zero). Namely, for some measurable set \( A \subset \mathbb{R} \) we have that the symmetric difference \( A \triangle \text{ess-sup}(\mu_\omega)_{ac} \) has Lebesgue measure zero \( \mathbb{P} \) almost surely \( \omega \).

(b) Similarly for part 2 of Theorem 3.1 it follows that there exists a deterministic set \( K \) such that \( K = \sigma_{ess}(H_\omega) \) almost surely.

3.2. Almost sure unitary equivalence modulo rank one perturbation. The following theorem, the main result of this paper, states that the essential parts of two Anderson-type Hamiltonians are unitarily equivalent modulo a rank one perturbation.

On the one hand, this result greatly restricts the possible deterministic properties of Anderson-type Hamiltonians. On the other hand, it tells us how ‘difficult’ rank one perturbations can be.

Theorem 3.2. Assume the hypotheses of Theorem 3.1. Assume that \( (H_\omega)_{ess} \) is cyclic \( \omega \in \mathbb{P} \) almost surely and \( \mathbb{P} = \Pi_k \mu_k \) is a product measure of purely absolutely continuous measures \( \mu_k \) on \( \mathbb{R} \). Let \( \mu \) denote the spectral measure of the operator \( (H_\omega)_{ess} \) with respect to some cyclic vector. If \( |\partial \text{ess-sup}(\mu_\omega)_{ac}| = 0 \) almost surely, then \( (H_\omega)_{ess} \sim (H_\eta)_{ess} \) (mod rank one) almost surely with respect to the product measure \( \mathbb{P} \times \mathbb{P} \).

Remarks. (a) In the conclusion of Theorem 3.2, it is necessary to restrict to the essential parts of the operators. The statement \( H_\omega \sim H_\eta \) (mod rank one) is not true. Indeed, on the finite isolated point spectra of \( H_\omega \) and \( H_\eta \) might not interlace: One of the necessary conditions for two operators to be unitarily equivalent up to rank one perturbation. In fact, between two points in the discrete spectrum of \( H_\omega \) there may be any number of points from the discrete spectrum of \( H_\eta \) (almost surely).

(b) Theorem 3.2 cannot be concluded trivially, e.g., by using Theorem 2.6 part 1 of Theorem 3.1 and a separation of the singular from the absolutely continuous part, because embedded singular spectrum can possibly occur. In particular, the singular spectrum of one operator may be interlaced with the absolutely continuous spectrum of the other operator.

(c) If we assume that the vectors \( \varphi_n \) are orthonormal and a cyclic family for \( H_\omega \) almost surely, then the almost sure cyclicity of \( (H_\omega)_{ac} \) implies the almost sure cyclicity of \( (H_\omega)_{ess} \). Indeed, under the above assumptions, the restricted operator \( (H_\omega)_{\sigma} \) is cyclic almost surely.
by Theorem 1.2 of [5]. Also recall that the operators $\(H_\omega\)_{ac}$ and $\(H_\omega\)_{s}$ are completely non-equivalent, because the essential supports of their spectral measures are mutually singular. So the cyclicity of $\(H_\omega\)_{ac}$ and $\(H_\omega\)_{s}$ implies the cyclicity of $\(H_\omega\)_{ess}$.

As mentioned before, the proof of Theorem 3.2 applies Theorem 2.6 as well as the following Lemma which allows us to introduce absolutely continuous spectrum while retaining precise control of the Radon–Nikodym derivatives of the singular measure after with respect to the measure before it was changed.

**Lemma 3.3.** Let $u$ be a Krein–Lifshits spectral shift function with range in $\{0, \pi\}$. Let $\mu$ and $\nu$ be the corresponding spectral measures. Take an open set $O \subset \mathbb{R}$ such that $|O| < \infty$. For $c > 0$ define a new shift function by

$$
\tilde{u}(x) = \begin{cases} 
u(x) & \text{on } \mathbb{R} \setminus O \\
|u(x) - \min\{\text{dist}(\mathbb{R} \setminus O, x), \pi/2\}|, & \text{if } x \in O.
\end{cases}
$$

For the measures $\tilde{\mu}$ and $\tilde{\nu}$ that correspond to $\tilde{u}$, we have $\tilde{\mu}|_{\mathbb{R} \setminus O} \sim \mu|_{\mathbb{R} \setminus O}$ and $\tilde{\nu}|_{\mathbb{R} \setminus O} \sim \nu|_{\mathbb{R} \setminus O}$.

**Proof.** For $x \in \mathbb{R} \setminus O$ we have on $\mathbb{R} \setminus O$

$$
|K_1(u - \tilde{u})| = \int_{O} \frac{\text{dist}(\mathbb{R} \setminus O, x)}{|x - t|} dt \leq |O|,
$$

and with (2.4), it follows that

$$
0 < c < \frac{1 + \pi \tilde{K} \tilde{\mu}}{1 + \pi K \mu} < C < \infty \quad \mu|_{\mathbb{R} \setminus O} \text{ almost everywhere.}
$$

(Since $\tilde{\mu}$ and $\tilde{\nu}$ correspond to $\tilde{u}$, we have by convention $\alpha = 1$.)

By definition $\mu|_{\mathbb{R} \setminus O}$ and $\tilde{\mu}|_{\mathbb{R} \setminus O}$ are purely singular. Therefore, we have

$$
0 < \tilde{c} < \frac{K \tilde{\mu}}{K \mu} < \tilde{C} < \infty \quad \mu|_{\mathbb{R} \setminus O} \text{ almost everywhere.}
$$

If (on $\mathbb{R} \setminus O$) measure $\mu$ has a part that is singular with respect to $\tilde{\mu}$ (denote it by $\eta$), then the ratio of Cauchy integrals $\frac{\tilde{K}_\eta}{\tilde{K}_\mu}$ tends to zero with respect to $\eta$ almost everywhere. This contradicts the lower bound of the last estimate (3.1).

Hence we must have $\tilde{\mu}|_{\mathbb{R} \setminus O} \gg \mu|_{\mathbb{R} \setminus O}$.

The other direction $\tilde{\mu}|_{\mathbb{R} \setminus O} \ll \mu|_{\mathbb{R} \setminus O}$ follows in analogy and we have proven $\tilde{\mu}|_{\mathbb{R} \setminus O} \sim \mu|_{\mathbb{R} \setminus O}$.

The result for $\nu$ can be proven in analogy. $\square$

**Proof of Theorem 3.2.** Most of this proof is to be understood almost surely with respect to the product measure $\mathbb{P} \times \mathbb{P}$, although this might not be stated everywhere explicitly.

By $\mu$ denote the spectral measure of the operator $(H_\omega)_{ess}$ with respect to some cyclic vector, and similarly for $\nu$ and $(H_\eta)_{ess}$. At this $(\omega, \eta) \in \mathbb{P} \times \mathbb{P}$. It is worth mentioning that the spectral measures of an operator corresponding to any two cyclic vectors are equivalent.

Consider the measure $\tau$ on $\mathbb{R}$ given by $d\tau(t) = (t^2 + 1)^{-1} dt$.

The goal is to produce a spectral shift function with corresponding spectral measures that are equivalent to the spectral measures $\mu$ and $\nu$, respectively.

By part 1 of Theorem 3.1 the symmetric difference $\text{ess-supp} \mu_{ac} \setminus \text{ess-supp} \nu_{ac}$ is a set of measure zero (almost surely with respect to the product measure). Let us denote the intersection of these sets by $F = \text{ess-supp} \mu_{ac} \cap \text{ess-supp} \nu_{ac}$. Notice that by the hypothesis, without loss of generality, we can assume $|\partial \text{ess-supp} \mu_{ac}| = |\partial \text{ess-supp} \nu_{ac}| = 0$. Hence we have $|\partial F| = 0$. 


Further, by part 2 of Theorem 3.1 and the Weyl–von Neumann theorem, Theorem 2.1 their essential spectra satisfy \( \sigma_{\text{ess}}(H_\omega) = \text{supp } \mu = \text{supp } \nu \). Let us denote this set by \( E \).

First observe that, by definition of \( E = \sigma_{\text{ess}}(H_\omega) \), operators \( (H_\omega)_{\text{ess}} \) and \( (H_\eta)_{\text{ess}} \) have dense purely singular spectrum on the set \( E \setminus \text{clos}(F) \). By the definition of \( F \) and since \( |\partial F| = 0 \), it is possible to choose two purely singular measures \( \mu' \) and \( \nu' \) such that \( \mu' \) and \( \nu' \) are mutually singular \( (\mu' \perp \nu') \), \( \mu' \big| _{E \setminus (F \setminus \partial F)} = \nu' \big| _{E \setminus (F \setminus \partial F)} = 0 \) and so that \( \mu_1 = \mu + \mu' \) and \( \nu_1 = \nu + \nu' \) have dense (alternating) spectrum on \( E \).

In virtue of Lemma 3.4 (below) we have that \( \mu_s \perp \nu_s \) almost surely with respect to product measure.

By Theorem 2.6 the measures \( \mu_1 \) and \( \nu_1 \) possess a spectral shift function \( u_1 \), i.e. there exists a function \( u_1 \) which is essentially bounded by \( 0 \leq u_1 \leq \pi \) and such that

\[
\arg(1 + \pi K \mu_1) = - \arg(1 - \pi K \nu_1).
\]

(Note that the hypothesis that there are no point masses at the endpoints is satisfied almost surely. So we can assume this condition without loss of generality.)

In order to destroy the artificially created singular spectrum and introduce the appropriate absolutely continuous spectrum, we define

\[
u_2(x) = \begin{cases} u_1(x), & \text{if } x \in \mathbb{R} \setminus (F \setminus \partial F), \\ |u_1(x) - \min\{\text{dist}(\mathbb{R} \setminus (F \setminus \partial F), x), \pi/2\}|, & \text{if } x \in F \setminus \partial F, \end{cases}
\]

and let \( \mu_2 \) and \( \nu_2 \) be the measures corresponding to \( u_2 \).

It remains to prove that \( \mu_2 \sim \mu \) and \( \nu_2 \sim \nu \). We will explain the equivalence of \( \mu_2 \) and \( \mu \). The same fact for \( \nu \) follows in analogy.

Let us begin with the absolutely continuous parts. Recall the hypothesis \( |\partial F| = 0 \). So on the set \( F \) we have \( u_2 \in (0, \pi) \) Lebesgue almost everywhere. By equations (2.6), (2.7) and (2.11), it follows that \( \frac{d\mu_2}{dx}(x) > 0 \) and \( < \infty \) for Lebesgue almost all \( x \in F \). This means that \( (\mu_2)_{ac}|_F \sim (\mu)_{ac}|_F \).

The equivalence of the absolutely continuous part on \( \mathbb{R} \setminus F \) follows similarly from the fact that \( u_2 \) takes only the values 0 or \( \pi \) on \( \mathbb{R} \setminus F \).

We have shown that \( (\mu_2)_{ac} \sim (\mu)_{ac} \) and \( (\nu_2)_{ac} \sim (\nu)_{ac} \).

By the definition of \( \mu_1 \) we ensured the equality of measures

\[
\mu_1|_{\mathbb{R} \setminus (F \setminus \partial F)} = (\mu_1)_{s}|_{\mathbb{R} \setminus (F \setminus \partial F)} = (\mu)_{s}|_{\mathbb{R} \setminus (F \setminus \partial F)}
\]

and Lemma 3.3 implies that

\[
\mu_2|_{\mathbb{R} \setminus (F \setminus \partial F)} \sim (\mu_2)_{s}|_{\mathbb{R} \setminus (F \setminus \partial F)} \sim (\mu)_{s}|_{\mathbb{R} \setminus (F \setminus \partial F)}.
\]

It remains to check the singular parts on \( F \setminus \partial F \). Recall that in definition (1.2) the points where the limit-superior is infinite are excluded. So by the definition of \( F \) via the intersection of essential supports of the absolutely continuous measures we have that \( \mu_s|_{F \setminus \partial F} \equiv 0 \). By the definition of \( u_2 \) on \( F \setminus \partial F \), the same is true for \( (\mu_2)_{s} \). Indeed, for any closed set \( X \subset F \setminus \partial F \) there exists an \( \varepsilon > 0 \) such that \( u_2(x) \in (\varepsilon, \pi - \varepsilon) \) for all \( x \in X \). By equation (2.7), this means that \( \lim_{|y| \to 0} 3Ku_2(x + iy) \neq 0 \) for all \( x \in X \). In virtue of Theorem 2.5 (with \( \mu_\alpha = \mu_2 \) and \( \mu = \nu_2 \)) it follows that \( (\mu_2)_{s}(X) = 0 \).

If the \( \{\varphi_n\} \) form an orthonormal sequence, the following lemma is also mentioned as a corollary to the main theorem in [4]. However, since the proof provided there is rather lengthy, we decided to include the four-line argument (below).
Lemma 3.4. Assume the hypotheses of Theorem 3.1 and assume that $\mathbb{P}$ is a product of absolutely continuous measures. Then $(\mu_\omega)_s \perp (\nu_\eta)_s$ almost surely with respect to the product measure. In particular (with the notation of the proof of Theorem 3.2), we have $\mu_\omega \perp \nu_\eta$ almost surely with respect to the product measure.

Proof. Assume that the set $S = \{ (\omega, \eta) : (\mu_\omega)_s \not\perp (\nu_\eta)_s \}$ has positive product measure. Because $\mathbb{P}$ is assumed to be a product of absolutely continuous measures, there then exists a pair $(\omega, \eta) \in S$ such that $H_\omega$ is a rank one perturbation of $H_\eta$. But by Aronszajn–Donoghue theory, see Theorem 2.4, this is not possible. $\Box$

3.3. Intersection of the essential spectrum with open sets. Assume the setting of Theorem 3.1. Recall that $\sigma_{\text{ess}}(H_\omega)$ is a deterministic set, by part 2 of Theorem 3.1.

Theorem 3.5. Assume the hypotheses of Theorem 3.1 and assume that $\mathbb{P}$ is a product of absolutely continuous measures. Let $O$ be an open set and let $X = O \cap \sigma_{\text{ess}}(H_\omega)$. Then almost surely either $X = \emptyset$ or the Lebesgue measure $|X| > 0$.

Proof. Assume $|X| = 0$ and $X \neq \emptyset$. Take $x \in X$.

Since $O$ is open, there exists $\varepsilon > 0$ such that the interval $(x - \varepsilon, x + \varepsilon) \subset O$. Consider $X_\varepsilon = X \cap (x - \varepsilon, x + \varepsilon)$. Clearly we have $|X_\varepsilon| = 0$.

Recall part 1 of Theorem 3.1. This implies that almost surely

$$(\mu_\omega)_{\text{ac}}((x - \varepsilon, x + \varepsilon)) = (\mu_\omega)_{\text{ac}}(X_\varepsilon) = 0.$$

In virtue of Lemma 3.6 below $(\mu_\omega)_s(X_\varepsilon) = 0$ almost surely.

Therefore $x \notin \sigma_{\text{ess}}(H_\omega)$ almost surely, in contradiction to the fact that $x \in X$. Hence almost surely either $X = \emptyset$ or $|X| > 0$. $\Box$

Lemma 3.6. Assume the hypotheses of Theorem 3.1 and assume that $\mathbb{P}$ is a product of absolutely continuous measures $\mu_k$. If set $A \subset \mathbb{R}$ satisfies $|A| = 0$, then we have $(\mu_\omega)_s(A) = 0$ almost surely.

Proof. Recall that $\mathbb{P}$ is a product of absolutely continuous measures $\mu_k$.

Assume that $(\mu_\omega)_s(A) > 0$ with positive probability. Then (for all $k \in \mathbb{N}$) there exist $\omega_0$ and $\chi \subset \mathbb{R}$ such that $\mu_k(\chi) > 0$ and such that for all $\alpha \in \chi$ we have $(\mu_\omega)_s(A) > 0$ where $\omega_0 = \omega_0 + \alpha \delta_k$.

But this contradicts the Aronszajn–Donoghue Theorem 2.4 for rank one perturbations. Notice that $\chi$ contains at least two points, since all $\mu_k$ are absolutely continuous. $\Box$

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