The Schonmann projection: how Gibbsian is it?
Dedicated to the memory of our dear friend Dima Ioffe

Aernout van Enter\textsuperscript{a} and Senya Shlosman\textsuperscript{b}

\textsuperscript{a}Bernoulli Institute of Mathematics, Computer Science and Artificial Intelligence
University of Groningen, Groningen, the Netherlands
avanenter@gmail.com, a.C.D.van.Enter@rug.nl

\textsuperscript{b}Skolkovo Institute of Science and Technology, Moscow, Russia;
\textsuperscript{c}Aix Marseille Univ, Universite de Toulon, CNRS, CPT, Marseille, France;
\textsuperscript{d}Inst. of the Information Transmission Problems, RAS, Moscow, Russia.
S.Shlosman@skoltech.ru, senya.shlosman@univ-amu.fr, shlor@iitp.ru

Abstract. We study the one-dimensional projection of the extremal Gibbs measures of the two-dimensional Ising model – the "Schonmann projection". These measures are known to be non-Gibbsian at low temperatures, since their conditional probabilities as a function of the (two-sided) boundary conditions are not continuous. We prove the conjecture that they are g-measures, which means that their conditional probabilities have a continuous dependence on one-sided boundary conditions.

Résumé. Nous étudions la projection unidimensionnelle des mesures extrémales de Gibbs du modèle d'Ising bidimensionnel - la "projection Schonmann". Ces mesures sont connues pour être non-Gibbsiennes à basses températures, puisque leurs probabilités conditionnelles en fonction des conditions aux limites (bilatérales) ne sont pas continues. Nous prouvons la conjecture que néanmoins celles-ci sont des g-mesures, ce qui signifie que leurs probabilités conditionnelles dépendent de façon continue des conditions aux limites unilatérales.

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1. Introduction

In 1989, Roberto Schonmann published the paper: 'Projections of Gibbs measures may be non-Gibbsian', [\textsuperscript{Sch}]. He was considering there the Ising model on $\mathbb{Z}^2$, and he showed that the projection of the $(-)$-phase on the $x$-axis $\mathbb{Z}^1 \subset \mathbb{Z}^2$ is 'too long-range' to be qualified as a Gibbs distribution for a well-behaved potential. In the paper [DS] the situation was mended – the definition of the Gibbs state was properly generalized there, and from this general point of view the Schonmann projection can still be viewed as a Gibbs measure. Still, this projection remains an interesting object to consider.

The peculiar property of this $2\text{D} \rightarrow 1\text{D}$ projection is the following. Consider a square box $V_M \subset \mathbb{Z}^2$ of size $2M$ with $(-)$ boundary condition, where additionally we put the $(+)$ boundary condition on two segments of the $x$-axis:

$I' = \left[(-N,0),(-n,0)\right]$, and $I'' = \left[(n,0),(N,0)\right],$

with $1 \ll n \ll N \ll M$. The corresponding ground state configuration $\bar{\sigma}$ (i.e. the one which minimizes the energy) is equal to $-1$ everywhere in $V_M$, except on the segments $I', I''$. So $\bar{\sigma}$ has two contours, $\gamma'$ and $\gamma''$, of unit thickness,
which surround the segments $I', I''$, and in particular $\sigma_{(0,0)} = -1$. However, at any positive temperature $T$ the typical configuration with these boundary conditions looks quite different! Namely, for any $n \gg 1$ and any low temperature $T$ one can find $N(n,T)$ such that for any $N > N(n,T)$ the typical configuration $\sigma$ looks as follows – it contains just one contour $\gamma$, surrounding both $I'$ and $I''$, and having (+)-phase inside. In particular, the probability of the event $\sigma_{(0,0)} = +1$ is quite large, and in fact is close to 1. This shows that the behavior of the projected Gibbs measure on the $x$-axis at the origin can be affected by changing the configuration on this axis quite far away from the origin.

In this note we want to study the following question: can one similarly influence the behavior of a typical 1D configuration $\sigma$ at the origin by conditioning the (−)-phase to take the value +1 on segments $I', I''$, ... which are allowed to be placed only to the left of the origin. The negative answer would be an indication that the Schonmann projection is a $g$-measure, i.e. that the corresponding distribution is continuous with respect to the one-sided conditioning. We remind the reader that a shift-invariant probability measure $\mu$ on $\{-, +\}^Z$ is called a regular $g$-measure, with $g$ being a continuous function on $\{-, +\}^Z$, if some version of the conditional expectation of the spin at the origin, given its complete past $\omega$ (that is all $\omega_i$ with $i < 0$), equals to $g$:

$\mu(+|\omega) - \mu(-|\omega) = g(\omega)$.

In this paper we get this negative answer and we also prove that the projection of the (−)-phase $\mu_-$ on $Z^1$ is a regular $g$-measure.

We will start by showing that one cannot influence the origin by one such segment of pluses. Namely, let the (+)-segment $I' \subset Z_1^1$ and let $\gamma'$ be the shortest possible contour surrounding $I'$ (i.e. ground state configuration contour). Let $T$ be some low temperature, and $\gamma$ denotes the exterior contour of a typical configuration $\sigma$, which surrounds the segment $I'$. It is well known, that the contour $\gamma$ fluctuates quite far away from the segment $I'$. (The phenomenon that a rigid wall repels an interface is called entropic repulsion. It has been studied in various contexts, and an early rigorous study can for example be found in [BMF]. For a recent one, see [IST].) However, not all fluctuations of $\gamma$ can happen! For example, the right tip of the contour $\gamma$ looks, typically, pretty much the same as the right tip of the ground state contour $\gamma'$. Moreover, the size of the fluctuations of the right tip of $\gamma$ away from the contour $\gamma'$ goes to zero with temperature. Therefore, the origin stays outside the contour $\sigma$, and in particular the probability of the event $\sigma_{(0,0)} = +1$ is very small; it is about the same as the probability of this event in the (−)-phase.

The main technical result of the present paper is the proof of the above picture: though the fluctuations of the contour $\gamma$ away from the $x$ axis can be quite large, they are, so to say, vertical; the abscissa of the rightmost tip of the contour $\gamma$ stays very close to that of $\gamma'$, notwithstanding the fact that the ordinate of the top of $\gamma$ can be arbitrarily high – it is of the order of $\sqrt{T}$. In other words, although the transversal fluctuations diverge, the longitudinal ones remain bounded uniformly in $n$ and $N$. Our proof uses cluster expansion and so is limited to low temperatures $T$, though we believe the result to be true for all $T < T_{cr}$.

In the next section we will describe our general strategy. We will then provide the details in the third section. Then we explain how this technical result implies $g$-measurability.

2. Main technical result

We are considering the 2-dimensional nearest-neighbour ferromagnetic Ising model, with formal Hamiltonian $-H = \sum_{i,j \in Z^2} \sigma_i \sigma_j$, at low temperature. It is known that there exist two extremal Gibbs measures for this model, $\mu^+$ (the maximal one) and $\mu^-$ (the minimal one), which are shift-invariant probability measures on $\{-, +\}^Z$. See e.g. [Geo] and/or [FV] for this and further background on Gibbs measure theory.

Our main result is about the one-dimensional marginal (the Schonmann projection) of the measures above. As mentioned before, these measures are known not to be Gibbs measures.

Our analysis will make extensive use of contours, which are curves made of bonds in the dual lattice of $Z^2$, separating pluses and minus spins. Contours have been a major tool in understanding the low-temperature Ising model since Peierls introduced them in the 1930’s. Studying measures on contour configurations is equivalent to studying measures on spin configurations. In particular, at low temperatures, long contours tend to be exponentially improbable, except in cases when they are unavoidable, which can happen due to a specific choice of boundary conditions.

We want to show that the spin at the origin depends in a continuous manner on the signs in its one-dimensional past, that is, on the spins on the $x$-axis to the left of the origin.

We concentrate on the low-$T$ 2D Ising model in a square box $V_M$ of size $2M$ with (−) boundary condition, where additionally we put the (+) boundary condition on a segment $I = [(-N,0),(0,0)], N < M$, thereby forcing a contour $\gamma$ around $I$. We are interested in the probability that the point $(n,0) \in \text{Int}(\gamma)$; we want to show that it is of the order $\exp \{ -\beta n \}$, uniformly in $N$ and $M$. 


We note that this result concerns the situation when the spins in the interval between $(0, 0)$ and $(n, 0)$ are not fixed. This then implies that for most fixed choices of spin configurations in this interval the dependence on what happens outside it decays exponentially, as was proven in [BC].

In section 4 we will explain that in fact the dependence decays for all fixed choices of spin configurations in this interval, but for some choices only algebraically. This decay property will prove the $g$-measure property of the Schonmann projection.

**Theorem 1.** The probability of the event that the point $(n, 0)$ happens to be inside the contour $\gamma$ is of the order $\exp \{-\beta n\}$, uniformly in $N$ and $M$, provided $\beta$ is large enough.

This claim is not at all immediate, due to the entropic repulsion of $\gamma$ from $I$. If the segment $I$ is very long, then the contour $\gamma$ would go away from $I$ by a distance $\sim \sqrt{|I|}$. Yet, this deviation of $\gamma$ from $I$ goes in vertical direction only, and the contour $\gamma$ passes quite close to the edges $(-N, 0)$, $(0, 0)$ of the segment, as we are going to show.

So let $\gamma$ be our (exterior) contour. Its distribution is given by the weight

$$w(\gamma) = \exp \left\{-\beta |\gamma| + \sum_{\Lambda: \Lambda \cap \gamma \neq \emptyset} \Phi(\Lambda) \right\},$$

where $|\gamma|$ is the length of the contour $\gamma$, the summation over $\Lambda$ goes over connected subsets $\Lambda \subset V_M \setminus I$, and $\Phi(\Lambda)$-s are exponentially small in $\text{diam}(\Lambda)$:

$$|\Phi(\Lambda)| \leq \exp \{-2\beta \text{diam}(\Lambda)\}.$$  \hspace{1cm} (2)

Such a representation is well-known; one can find it in [DKS], where the function $\Phi$ is also explicitly defined.

Let $L = \{(x, y) : x = -N\}$, $R = \{(x, y) : x = 0\}$ be two vertical lines, $S$ be the strip between them, and $H_\pm$ be the upper and lower half-planes. Define the set of four cut-points $u_1, u_2, v_1, v_2$ of $\gamma$ by the properties:

- $u_1, u_2 \in \gamma \cap L$, $v_1, v_2 \in \gamma \cap R$,
- the arcs $\gamma_1$ (piece of $\gamma$ from $u_1$ to $v_1$) and $\gamma_2$ (piece of $\gamma$ from $u_2$ to $v_2$) lie inside the strip $S$.

We denote by $\gamma_u, \gamma_v$ the remaining two arcs of $\gamma$.

Let us define

$$Z(u_1, u_2, v_1, v_2) = \sum_{\gamma: u_1, u_2, v_1, v_2 \in \gamma} w(\gamma),$$

where the summation goes over $\gamma$-s with cut-points $u_1, u_2, v_1, v_2$.

We first pretend that

$$w(\gamma) = w(\gamma_u) w(\gamma_1) w(\gamma_v) w(\gamma_2),$$

and moreover

$$Z(u_1, u_2, v_1, v_2) \approx \exp \{-\beta (u_1 - u_2 + v_1 - v_2)\} Z(u_1 \rightarrow v_1) Z(u_2 \rightarrow v_2),$$

where $Z(u_1 \rightarrow v_1) = \sum_{\gamma_1} w(\gamma_1)$, $Z(u_2 \rightarrow v_2) = \sum_{\gamma_2} w(\gamma_2)$, and the weights $w(\gamma_1), w(\gamma_2)$ are taken from (1), i.e.

$$w(\gamma_1) = \exp \left\{-\beta |\gamma_1| + \sum_{\Lambda: \Lambda \cap \gamma_1 \neq \emptyset} \Phi(\Lambda) \right\},$$

where now both $\gamma_1$ and all $\Lambda$-s stay in the upper semistrip $I \times \{1, 2, 3, \ldots\}$, and similarly for $w(\gamma_2)$. (Compare with (1).)

In what follows we keep $N$ fixed and we abuse notation by writing $u_i$ instead of $(-N, u_i)$, etc.

While considering the partition functions $Z(u_1 \rightarrow v_1), Z(u_2 \rightarrow v_2)$ we will separate between regimes when the points $u_i, v_i$ are of order $\sqrt{N}$ or smaller, or are of higher orders.

1. Suppose first that $u_1 > \sqrt{N}$, and $v_1 > u_1$. Then,

$$Z(u_1 \rightarrow v_1) < Z(u_1 \rightarrow u_1)$$
(meaning: \( Z(u_1 \to u_1) \equiv Z((-N, u_1) \to (0, u_1)) \)). Moreover, the function \( Z(u_1 \to v_1) \) is decreasing in \( v_1 \) in this regime. Because of the extra factor of \( \exp \{-\beta (v_1 - u_1)\} \), coming from the weight \( w(\gamma_v) \), we can disregard the contribution of the configurations with \( u_1 > \sqrt{N} \) and \( v_1 > u_1 \) to the partition function \( \sum w(u_1, u_2, v_1, v_2) \).

2. The same monotonicity in \( v_1 \) holds when \( u_1 \leq \sqrt{N} \) and \( v_1 > \sqrt{N} \), with the same conclusion.

3. Iterating 1, 2, we come to the remaining case, when all four variables \( u_1, u_2, v_1, v_2 \) are \( \leq \sqrt{N} \) in absolute values. In this regime we use the local limit theorem for the random point \( v \) in the ensemble \( Z(u_1 \to v) \) and claim the existence of a constant \( C \), which does not depend on \( \beta, u, v, \) and \( v \), such that

\[
\frac{Z(u_1 \to (v_1 + 1))}{Z(u_1 \to v_1)} < C. \tag{5}
\]

For \( \beta \) large, the factor \( \exp \{-\beta\} \) — which is the price, due to the weight \( w(\gamma_v) \), for the point \( v_1 \) to be one step higher — beats \( C \).

### 3. Splitting the partition function

We start by presenting the rigorous counterpart to the relation (3). We will use the standard SW convention to define the contours as self-avoiding loops. Such contours and self-avoiding lattice paths will be called legal. Let \( \gamma \) be an exterior contour surrounding the segment \([(-N, 0), (0, 0)]\). Then the cut-point \( u_1 \) is defined as the point of the last intersection of \( \gamma \) with the line \( L \) before getting to the line \( R \), provided \( \gamma \) is oriented clockwise. The other three points \( u_2, v_1, v_2 \) are defined similarly.

As a result, \( \gamma \) is a concatenation,

\[
\gamma = \gamma_1 \circ \gamma_v \circ \gamma_2 \circ \gamma_u,
\]

where

- \( \gamma_1 \) is any legal path in \( S \), joining \( u_1 \) and \( v_1 \) and lying above \([(-N, 0), (0, 0)]\),
- \( \gamma_2 \) is any legal path in \( S \), joining \( u_2 \) and \( v_2 \) and lying below \([(-N, 0), (0, 0)]\),
- \( \gamma_u \) (resp. \( \gamma_v \)) is any legal path joining \( u_1 \) and \( u_2 \) (joining \( v_1 \) and \( v_2 \)), not intersecting \([(-N, 0), (0, 0)]\), and such that the concatenation \( \gamma_1 \circ \gamma_v \circ \gamma_2 \circ \gamma_u \) is legal.

As this definition suggests, we will treat the arcs \( \gamma_1, \gamma_2 \) of \( \gamma \) as independent variables, which put restrictions on the allowed realizations of \( \gamma_u, \gamma_v \).

Having in mind the definition (1), we write

\[
Z(u_1, u_2, v_1, v_2) = \sum_{\substack{\gamma_1, \gamma_2, \gamma_u, \gamma_v \text{ legal}}} \left[ \exp \{ -\beta |\gamma_1| + \sum_{\Lambda: \Lambda \cap \gamma_1 \neq \emptyset, \Lambda \subset H_+} \Phi(\Lambda) \} \exp \{ -\beta |\gamma_2| + \sum_{\Lambda: \Lambda \cap \gamma_2 \neq \emptyset, \Lambda \subset H_-} \Phi(\Lambda) \} \right.
\]

\[
\times \left( \sum_{\gamma_v: \gamma_v \circ \gamma_1 \circ \gamma_2 \circ \gamma_u \text{ legal}} \exp \{ -\beta |\gamma_v| + \sum_{\Lambda: \Lambda \cap \gamma_v \neq \emptyset, \Lambda \subset (\gamma_1 \cup \gamma_2) = \emptyset} \Phi(\Lambda) \} \right) \exp \left\{ -\beta |\gamma_1| + \sum_{\Lambda: \Lambda \cap \gamma_1 \neq \emptyset, \Lambda \cap (\gamma_1 \cup \gamma_2 \cup \gamma_u) = \emptyset -} \Phi(\Lambda) \right\} \right]. \tag{6}
\]

#### 3.1. The vertical parts

Let us first consider the partition function

\[
Z(v_1, v_2) = \sum_{\gamma: \gamma \circ \gamma_1 \circ \gamma_2 \circ \gamma_u \text{ legal}} \exp \{ -\beta |\gamma| + \sum_{\Lambda: \Lambda \cap \gamma \neq \emptyset} \Phi(\Lambda) \}. \tag{7}
\]
Proof. Let $v$ and consider the event $\{v_1 > (1 + \varepsilon) \| v_1 - v_2 \| \}$ goes to zero as $\beta \to \infty$, uniformly in $v_1, v_2$. (For the benefit of the reader we will outline the proof of this statement at the end of this subsection.) Therefore, the probability $p_1(\beta)$ in the ensemble (7) of the event that the ‘first’ edge of $\gamma$ — i.e., the edge starting from $v_1$ — goes down in the direction $v_2$, thus connecting $v_1$ to $(v_1 - e_2)$, has the property that $p_1(\beta) \to 1$ as $\beta \to \infty$. Let $Z^\beta(v_1, v_2)$ be part of the partition function (7) restricted to such configurations. Clearly,

$$\frac{Z^\beta(v_1, v_2)}{Z(v_1 - e_2, v_2)} \sim e^{-\beta},$$

since the terms $\Phi(\Lambda)$ are of smaller order. Therefore,

$$\frac{Z(v_1, v_2)}{Z(v_1 - e_2, v_2)} = \frac{1}{p_1(\beta)} \frac{Z^\beta(v_1, v_2)}{Z(v_1 - e_2, v_2)} \leq e^{-c\beta},$$

for some $c \to 1$ as $\beta \to \infty$. The same argument, slightly modified, applies to the third and the fourth partition functions in (6).

**Lemma 2.** Let $v_1, v_2 \in \mathbb{Z}^2$ be two points on the lattice. Consider the ensemble $P$ of lattice paths $\gamma$, joining these two points, and defined by the weight $w(\gamma) = \exp \{-\beta |\gamma|\}$. Let $\|v_1 - v_2\|$ be the length of the shortest such path. Let $\varepsilon > 0$, and consider the event $\{\gamma : |\gamma| > (1 + \varepsilon) \| v_1 - v_2 \|\}$. Then $P \{\gamma : |\gamma| > (1 + \varepsilon) \| v_1 - v_2 \|\} \to 0$ as $\beta \to \infty$, uniformly in $v_1, v_2$.

**Proof.** Let $\gamma : v_1 \to v_2$ be one of the paths with $|\gamma| = \| v_1 - v_2 \|$. Then

$$P \{\gamma : |\gamma| > (1 + \varepsilon) \| v_1 - v_2 \|\} \leq \frac{\sum_{l > (1 + \varepsilon) \| v_1 - v_2 \|} \exp \{-\beta l\}}{\exp \{-\beta |\gamma|\}} = \sum_{k > 0} 3^{(1 + \varepsilon) \| v_1 - v_2 \| + k} \exp \{-\beta ([1 + \varepsilon] \| v_1 - v_2 \| + k)\} \to 0$$

as $\beta \to \infty$, since the factor $\exp \{-\beta [1 + \varepsilon] \| v_1 - v_2 \|\}$ beats $3^{(1 + \varepsilon) \| v_1 - v_2 \|}$ once $\beta$ is large enough (depending on $\varepsilon$), while $\exp \{-\beta k\}$ beats $3^k$.

3.2. The horizontal parts

Here we will treat the partition function

$$Z(u_1 \to v_1) = \sum_{\gamma_1 : u_1 \to v_1} \exp \{-\beta |\gamma_1| + \sum_{\Lambda : \Lambda \cap \gamma_1 \neq \emptyset, \Lambda \subset H_+} \Phi(\Lambda)\}.$$

The properties needed are obtained in [IOVW], which is based on the random walk approximation of the random line $\gamma_1$, worked out in [OV]. One can use instead the random walk description introduced in [DS], and used for a similar goal in [IST].

In this subsection we will drop the subscript 1 and will write $u, v, \gamma$ instead of $u_1, v_1, \gamma_1$.

The model we have to deal with is defined by assigning the weight

$$w(\gamma) = \exp \left\{-\beta |\gamma| + \sum_{\Lambda : \Lambda \cap \gamma \neq \emptyset, \Lambda \subset H_+} \Phi(\Lambda)\right\}.$$
to any path $\gamma \subset S$. As in [IST], we can pass to an enlarged ensemble, with more variables -- $(\gamma, \Lambda)$ -- consisting of a path $\gamma$ and a finite collection $\Lambda$ of connected sets, $\Lambda = \{\Lambda_i \subset \mathbb{Z}^2\}$, each intersecting $\gamma$, and defined by the weight

$$w(\gamma, \Lambda) = \exp \{-\beta |\gamma|\} \prod_{\Lambda_i \in \Lambda} \Psi(\Lambda_i).$$

The special case of $\Lambda = \emptyset$ is not excluded. Here the functional $\Psi$ satisfies the same estimate (2), but in addition is positive, which makes $w$ a legitimate statistical weight. The functional $\Psi$ and the new temperature $\beta'$ can be chosen in such a way that $\sum_{\Lambda} w(\gamma, \Lambda) = w(\gamma) = \text{so the partition functions for the weights } w \text{ and } w \text{ are the same -- while } |\beta - \beta'| \to 0 \text{ as } \beta \to \infty$. We will call a pair $(\gamma, \Lambda)$ a dressed path, or just a path. The idea of introducing the hidden variables $\Lambda$ to the ensemble $(\gamma, \Lambda)$ goes back to [DS], see [IST] for more details.

Let $x_0 \in \mathbb{Z}^1$. We call the point $x_0$ a splitting point of the dressed path $(\gamma, \Lambda)$, if the intersection of the line $l_{x_0} = \{ (x, y) : x = x_0 \}$ with the curve $\gamma$ is a single point, while all the intersections $l_{x} \cap \Lambda_i = \emptyset$, $\Lambda_i \in \Lambda$. Let $x_1, \ldots, x_k$ be all the splitting points of the path $(\gamma, \Lambda)$. Then $(\gamma, \Lambda)$ is split by the lines $l_{x_i}$ into $k + 1$ irreducible pieces $(\gamma_0, \Lambda_0), \ldots, (\gamma_k, \Lambda_k)$, and the dressed path $(\gamma, \Lambda)$ is their concatenation. We will call the irreducible pieces $(\gamma_i, \Lambda_i)$ the animals. Note that

$$w(\gamma, \Lambda) = \prod_{i=0}^{k} w(\gamma_i, \Lambda_i),$$

which paves the way to the definition of the random walk $S$ -- the effective random walk representation.

Let $u = (x, y), u' = (x', y')$ be two points in $\mathbb{Z}^2$, $x < x'$. We define the weight

$$s_{u, u'} = \sum_{\gamma; \gamma|u \to u'|} w(\gamma, \Lambda),$$

where the summation goes over dressed paths $(\gamma, \Lambda)$ such that

- the path $\gamma$ goes from $u$ to $u'$, and
- the dressed path $(\gamma, \Lambda)$ is its unique irreducible piece.

These weights define the distribution of the random vector $X = (\theta, \zeta)$, which defines the steps of the walk $S$. Its starting point will be $S_0 = (-N, u)$, while $S_i = (-N, u) + \sum_{j=1}^{i} X_j = u_i + (T_i, Z_i)$, where $T_i = \sum_{j} \theta_j$, $Z_i = \sum_{j} \zeta_j$, (we follow here the notations of [IOVW], definition (45)). The overall distribution of $S$ will be denoted by $P_{(-N,u)}$.

To study the ratio $\frac{Z(u \to (v+1))}{Z(u \to v)}$ in (5) we can pass to the study of the probabilities in the ensemble $P_{(-N,u)}$ of the event $\{ S : (-N, u) \to (0, v), S > 0 \}$ that the path $S$ stays positive and arrives to the point $(0, v)$, resp. $(0, v + 1)$. The very precise estimates of [IOVW], see the relations (47-49) there, tell us that

$$P_{(-N,u)} \{ S : (-N, u) \to (0, v); S > 0 \} \sim C \frac{h^+(u) h^-(v)}{N^{3/2}}$$

as $N \to \infty$, where

- the function $h^+(x) = x - E_x(Z_x)$, where the random walk $Z$ starts from the point $x \in \mathbb{Z}^1$, $x > 0$, and the stopping moment $\tau$ is defined by $\tau = \inf \{ n : Z_n \leq 0 \}$; 
- the function $h^-(x)$ is defined in the same way, but for the random walk $(-Z)$; 
- $C = C(\theta, \zeta) > 0$ is some constant; 
- the variables $u, v \in [1, N^{1/2-\delta}]$, with any small $\delta > 0$, which parameter will be fixed from now on (say, $\delta = \frac{1}{100}$).

For the region $u \in [1, N^{1/2-\delta}]$, $v \in [N^{1/2-\delta}, N^{1/2}]$ we have

$$P_{(-N,u)} \{ S : (-N, u) \to (0, v); S > 0 \} \sim C \frac{h^+(u) v \exp \{-v^2/2N\} \text{Var}(\zeta)}{N^{3/2}},$$

while in the region $u, v \in [N^{1/2-\delta}, N^{1/2}]$ we have

$$P_{(-N,u)} \{ S : (-N, u) \to (0, v); S > 0 \} \sim \frac{\psi(u/N^{1/2}, v/N^{1/2})}{N^{1/2}}.$$
for some positive continuous bounded function \( \psi \) on \([0, 1]^2\).

Since in our case the random variable \( \zeta \) is exponentially localized, i.e. \( \Pr \{ \zeta = k \} \sim \exp \{ -c(\beta) |k| \} \) with \( c(\beta) \to \infty \) as \( \beta \to \infty \), we have that

\[
\frac{\Pr_{(-N,u)} \{ S: (-N, u) \to (0, v + 1); S > 0 \}}{\Pr_{(-N,u)} \{ S: (-N, u) \to (0, v); S > 0 \}} < C
\]

uniformly for all \( N \) large enough and \( u, v \in [1, N^{1/2}] \). In words, though the probability \( \Pr_{(-N,u)} \) of the event \( \{ S: (-N, u) \to (0, v + 1); S > 0 \} \) can be higher than \( \Pr_{(-N,u)} \{ S: (-N, u) \to (0, v); S > 0 \} \) due to the entropic repulsion of the path \( S \) from the \( x \)-axis, their ratio is bounded by a constant. That proves (5).

Combining the horizontal and vertical parts implies that with large probability the cut points \( u_1, u_2, v_1 \) and \( v_2 \) all are at a not too large vertical distance of the segment, which distance does not grow with \( N \). Therefore the probability that a point at horizontal distance \( n \) to the right of the segment will be inside the contour containing the segment decays exponentially in \( n \), again uniformly in \( N \) and \( M \). For the projected system this implies the continuity of the spin expectation in the origin on our left configuration.

4. Uniform continuity

Here we prove the main result of our paper.

**Theorem 3.** The Schonmann projection, at sufficiently low temperature, is a regular \( g \)-measure

**Proof.** Let \( \omega \) be some semi-infinite string of spins \( \ldots \sigma_{-N-1}, \sigma_{-N}, \sigma_{-N+1}, \ldots, \sigma_{-1} \). Let us introduce the notation \( \omega_{N+} \) for the configuration \( \ldots, +, +, \sigma_{-N}, \sigma_{-N+1}, \ldots, \sigma_{-1} \), and denote by \( \omega_{N-} \) the configuration \( \ldots, -, -, \sigma_{-N}, \sigma_{-N+1}, \ldots, \sigma_{-1} \).

To prove the uniform continuity of our \( g \)-function we will look at the difference

\[
\mu_{M}(+0|\omega_{N+}) - \mu_{M}(+0|\omega_{N-}) \equiv g'(\omega, N) \geq 0.
\]

We will show that \( g'(\omega, N) \to 0 \) as \( N \to \infty \), uniformly in \( \omega \). Clearly, this implies the uniform continuity, because of the FKG property.

Depending on the context, we will consider below the quantities \( \mu_{M'}(+0|\omega_{M''}) \), where \( M' \geq M'' \), \( \mu_{M'} \) is the Gibbs distribution in the box \( V_{M'} \) with \((-\)\)-boundary condition, and \( \omega_{M''} \) is a configuration on the segment \([-M'', 1]\). To save on notation, we will sometime denote them by the same expression \( \mu_{M}(+0|\omega) \).

1. The simplest case to consider is the string \( \omega = \omega^+ \equiv +1 \). This case is basically the one considered above. We need to compute the difference

\[
\mu_{M}(+0|\omega_{M}^+) - \mu_{M}(+0|\omega_{M}^-) = \frac{\mu_{M}(+0, \omega_{M}^+)}{\mu_{M}(\omega_{M}^+)} - \frac{\mu_{M}(+0, \omega_{M}^-)}{\mu_{M}(\omega_{M}^-)},
\]

where the size of the box, \( M \), exceeds the length \( N \) of the segment. The ratio of two probabilities, \( \frac{\mu_{M}(+0, \omega_{M}^+)}{\mu_{M}(\omega_{M}^+)} \), is the ratio of the two partition functions: one is taken over all configurations containing a contour \( \gamma \) surrounding the segment \([-N, -1]\), while the other is restricted to those \( \gamma \)-s, which enclose an extra point \((0, 0)\). According to sections 3.1, 3.2, this ratio is equivalent to

\[
\exp \left\{ -2\beta \left( \frac{N^{3/2}}{(N + 1)^{3/2}} \right)^2 \right\},
\]

see (6.9 – 11). Indeed, the loop \( \gamma \) surrounding the segment \([-N, -1]\) roughly corresponds to a pair of paths (one above and one below the segment), and each of them has to pass right near the tip of the segment. (The square appears here because \( \gamma \) contains two horizontal strings.) The same analysis applies to the term \( \frac{\mu_{M}(+0, \omega_{M}^-)}{\mu_{M}(\omega_{M}^-)} \), with \( N \) replaced by \( M > N \). So their difference \( g'(\omega^+, N) \leq \frac{1}{M} \).

2. Now consider the string \( \omega^- \equiv -1 \). Here we need to know the behavior of the contour \( \gamma \), surrounding the \((+)\)-segment \([-M, -M + N]\), and we assume that \( M > N \). The contour \( \gamma \) must cross the \( x \)-axis to the right of \([-M, -M + N]\), and it has two options to do so: one is to cross at the location \(-N\); the other is to cross through a point at a positive semi-axis.


In the first case, the contour $\gamma$ is at distance $N$ from the origin, so the contribution to $g'(\omega^-, N)$ is exponentially small in $N$, as follows from the cluster expansion.

In the second case the (-)-segment $[-N, -1]$ gets inside the (+)-phase, which fills the interior $\text{Int}(\gamma)$ of the contour $\gamma$, so it is surrounded by an extra contour $\Gamma \subset \text{Int}(\gamma)$, which brings an extra cost of $\exp\{-2\beta |\Gamma|\}$. Note that under condition $\Gamma \subset \text{Int}(\gamma)$ the behavior of $\gamma$ can be quite different from the one we have seen above; in particular the event that $(0, 0) \in \text{Int}(\gamma)$ can be quite likely, in which case the magnetization at the origin is very different from $-m^+(\beta)$. However, the factor $\exp\{-4\beta N\}$ -- which is the probability of the appearance of $\Gamma$ -- beats all these complications, and we conclude that $g'(\omega^-, N) = o(\exp\{-2\beta N\})$.

3. Next, consider the key (and most delicate) case of the string $\omega^{-,k} = \ldots + + + - - - - - k$, with $k$ (-)-spins neighboring the origin, $k = 1, 2, \ldots$. Again, the contour $\gamma$, surrounding the (+)-string, can cross the $x$-axis either at the location $-k$ or at some location to the right of the point $(1,0)$. In the second case we again have the contour $\Gamma \subset \text{Int}(\gamma)$. But this time the small weight $\exp\{-2\beta |\Gamma|\}$ is small in $k$ only -- but not in $N$, so it gives us the estimate $g'(\omega^{-,k}, N) = o(\exp\{-\beta k\})$, which is not what we look for, since we need the estimate decaying in $N$.

3a. Anyway, the analysis of the first case, when the contour $\gamma$ crosses the $x$-axis at the location $-k$ is similar to the case 1, and we conclude that its contribution to the function $g'(\omega^{-,k}, N) \sim \frac{c(k)}{k}$ with $c(k) \to 0$ as $k \to \infty$, which is more than enough for our purposes.

3b. As for the second case, the presence of the contour $\Gamma$ can change the magnetization at the origin quite a bit. Our point now is that the change is almost the same for both cases -- when $\gamma$ surrounds the segment $[-N, -1]$ and when it surrounds the segment $[-M, -1]$ with $M > N$, their difference being $\sim \frac{1}{M}$.

To see it, let us fix some scale, growing with $N$ -- say, $\ln N$ -- and consider two cases. The first is when $|\Gamma| > \ln N$. Then the probability of appearance of such a $\Gamma$ decays as $N^{-\beta}$, so what happens with the spin at the origin due to the contour $\gamma$ under the condition $\Gamma \subset \text{Int}(\gamma)$ is immaterial. If, on the other hand, we consider the case $|\Gamma| \leq \ln N$, then for large $N$ we can use the relations (9 - 11) for the contour $\gamma$ over the landscape made by the union of the $x$-axis and the contour $\Gamma$ of height $\leq \ln N$. This is possible since $N^{1/2} \gg \ln N$. So, as in 1, we conclude that $g'(\omega^{-,k}, N) \sim \frac{1}{N}$.

4. The general case of several (-)-segments follows from the combinations of the three cases above.

5. Conclusions and further comments

We showed that the Schonmann projection of the extremal low-temperature Gibbs measures of the 2-dimensional zero-field Ising model changes weakly at the origin if conditioned on a long segment far from the origin, no matter how long the segment is. The property responsible for this is a lack of entropic repulsion from a long segment in the direction of the segment. We use it to show that this projection is a $g$-measure, i.e. it has a kind of one-sided Gibbsian property.

An earlier example of a non-Gibbsian $g$-measure was found in [FGM]. On the other hand, in [BEEL] a Gibbsian non-$g$-measure is displayed. The $g$-measure property thus cannot be seen as either weaker or stronger than the property of being a Gibbs measure.

Presumably our result remains true for all subcritical temperatures, by applying a coarse-graining argument as has been developed by Ioffe, Velenik and their collaborators on Ornstein-Zernike behavior, see e.g. [IOVW, OV].

The $g$-function, although continuous, cannot be too regular, as the Schonmann projection is known to be non-Gibbsian. It therefore cannot have the property of "summable variations" (otherwise known as "Dini continuity"), as was remarked before in [BC], and as also follows from [BFV]. One can wonder about "how continuous" or "how regular" the $g$-function might be. Our proof suggests the following answer. If we consider the magnetization at the origin $(0, 0)$ conditioned on a segment $[(-n, 0), (-1, 0)]$ to be all plus, the upper part of the enforced contour looks like a (Brownian) bridge, of length $n$, constricted to be positive. Our relations (9 - 11) allow us to conclude that the $n$-variation $\var_n(g)$ of $g$ at the "all-plus" configuration is $O(\frac{1}{n})$, which is non-summable. An early mentioning of such an argument is given in [BF], section VII. Notice again that the "all-plus" configuration which is responsible for this behavior, is in some sense the worst one, and is atypical for the measure $\mu$ under consideration. Indeed, for $\mu$-most configurations $\omega$, there exists a positive density of minuses, and changing $\omega$ at a distance larger than $n$, left of the origin, will only have an exponentially small effect at site $(0,0)$. Incidentally, in [BC] it was shown that for typical configurations a square summability condition is satisfied.

Although higher-dimensional versions of the non-Gibbsianess of the Schonmann projected measures have been proved, [MMR], due to a similar entropic repulsion argument there seems to be no natural higher-dimensional extension of the $g$-measure property. One possible interpretation of a one-sided conditioning would be requiring the Global Markov property, another one would be requiring a continuous dependence on the lexicographic past. However, in both these situations, there are counterexamples, or nearest-neighbour Gibbs measure, thus having the Local Markov property, but lacking the Global Markov property, or having conditional expectations which are discontinuous as a function of the lexicographic past. For a discussion on some of those and related issues, see e.g. [ELP].
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