THE CR ALMOST SCHUR LEMMA AND THE POSITIVITY CONDITIONS

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ABSTRACT. We establish a new version of the CR almost Schur Lemma which gives an estimation of the pseudohermitian scalar curvature on a compact strictly pseudoconvex pseudohermitian manifold to be a constant in terms of the norm of the traceless Webster Ricci tensor and the pseudohermitian torsion under a certain positivity condition. In the torsion-free case, i.e. for a compact Sasakian manifold, our positivity condition coincides with the known one and we obtain a better estimate.

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1. INTRODUCTION

For a compact Riemannian manifold \((M^n, g)\) of dimension \(n \geq 3\) the famous Schur lemma states that if \((M^n, g)\) is Einstein, then it has constant scalar curvature, \(S = \text{Const.}\). The metric \(g\) is said to be Einstein, if the Ricci tensor is proportional to the metric, \(\text{Ric} = \frac{S}{n} g\). A generalization of the Schur lemma is the result of De Lellis and Topping [9] that states as follows.

**Theorem 1.1.** [9, Almost Schur Lemma] Let \((M^n, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\) with non-negative Ricci tensor, \(\text{Ric} \geq 0\). Then the following inequality holds

\[
\int_M (S - \bar{S})^2 \text{vol}_g \leq \frac{4n(n-1)}{(n-2)^2} \int_M \left| \text{Ric} - \frac{S}{n} g \right|^2_g \text{vol}_g,
\]

where \(\bar{S}\) means the average value of the scalar curvature \(S\) of \(g\). The equality holds if and only if the manifold is Einstein.
It is also shown in [9] that the positivity condition $Ric \geq 0$ assumed on the Ricci tensor is essential and can not be dropped.

In the CR case there are known two positivity conditions written in terms of the Webster Ricci curvature and the pseudohermitian torsion; one is used for obtaining a lower bound of the first eigenvalue of the sub-Laplacian (see e.g. [12]), while the other one appears in the CR Cordes type estimate (see [6]).

A CR version of the almost Schur lemma was first established in [7] in terms of the Tanaka-Webster connection, its Ricci curvature and the Webster torsion. The key assumption is the positivity condition (1.1) below. We present this result in real notations.

**Theorem 1.2.** [7, Theorem 1.2] Let $n \geq 2$ and $(M, J, \theta)$ be a $(2n+1)$-dimensional compact strictly pseudoconvex pseudohermitian manifold satisfying the condition

$$
(1.1) \quad Ric(X, X) + 4A(JX, X) = Rc(X, X) + 2(n+1)A(JX, X) \geq 0, \quad X \in H.
$$

Then the following inequality holds

$$
(1.2) \quad \int_M (S - \bar{S})^2 Vol_\theta \leq \frac{4n(n+1)}{(n-1)(n+2)} \int_M |Rc_0|^2 Vol_\theta - 8n \int_M \frac{\sum_{a,b=1}^{2n} A(e_a, Je_b)(\nabla^2 \varphi)_{[-1]}(e_a, e_b)}{Vol_\theta}.
$$

If the equality holds, then

$$
(1.3) \quad \int_M (S - \bar{S})^2 Vol_\theta = \frac{4n(n+1)}{(n-1)(n+2)} \int_M |Rc_0|^2 Vol_\theta
$$

and the manifold is CR equivalent to a pseudo-Einstein space.

In the inequalities (1.1) and (1.2), $Ric$ is the Ricci tensor of the Tanaka-Webster connection, $Rc$ is the Webster Ricci tensor, $A$ is the pseudohermitian torsion, $S$ and $\bar{S} = \int_M S Vol_\theta$ are the pseudohermitian scalar curvature and its average value, respectively, and $Rc_0 = Rc - \frac{\bar{S}}{n}g$, $|Rc_0|^2 = \sum_{a,b=1}^{2n} Rc_0(e_a, e_b)Rc_0(e_a, e_b)$ are the trace-free part of the Webster Ricci tensor and its horizontal norm, respectively. Finally, $\varphi$ is the unique solution of the sub-elliptic equation

$$
\Delta \varphi = S - \bar{S} \quad \text{with} \quad \int_M \varphi Vol_\theta = 0.
$$

The aim of this note is to present another version of the CR almost Schur lemma with a different positivity condition. We assume the positivity condition (1.4) below instead of (1.1). In the torsion-free case, i.e. for Sasakian manifolds, both positivity conditions coincide and we obtain a better estimate then that following from Theorem 1.2. Our main result is

**Theorem 1.3.** Let $n \geq 2$ and $(M, J, \theta)$ be a $(2n+1)$-dimensional compact strictly pseudoconvex pseudohermitian manifold satisfying the condition

$$
(1.4) \quad Ric(X, X) + 6A(JX, X) = Rc(X, X) + 2(n+2)A(JX, X) \geq 0, \quad X \in H.
$$

Then the following inequality holds

$$
(1.5) \quad \int_M (S - \bar{S})^2 Vol_\theta \leq \frac{2n(2n+3)}{(n-1)(n+3)} \int_M |Rc_0|^2 Vol_\theta - 8n \int_M \frac{\sum_{a,b=1}^{2n} A(e_a, Je_b)(\nabla^2 \varphi)_{[-1]}(e_a, e_b)}{Vol_\theta}.
$$

If the equality holds, then

$$
(1.6) \quad \int_M (S - \bar{S})^2 Vol_\theta = \frac{2n(2n+3)}{(n-1)(n+3)} \int_M |Rc_0|^2 Vol_\theta
$$

and the manifold is CR equivalent to a pseudo-Einstein space.

**Remark 1.4.** Note that the expression in the left-hand side of (1.1) is precisely the CR-Lichnerowicz condition used to find a lower bound of the first eigenvalue of the sub-Laplacian (see [12, 18, 8, 4, 5, 14]), while the expression in the left-hand side of (1.4) appears in the CR Cordes type a priori inequality between the (horizontal) Hessian and the sub-Laplacian of a function, derived in [6, Theorem 1], see Theorem 3.1 below.
The second Bianchi identity (4.4) below shows that a compact pseudo-Einstein space has constant pseudohermitian scalar curvature if and only if the next condition holds

\[(\nabla_{e_a} \nabla_{e_a} A)(e_a, J e_b) = 0,\]

and it seems natural to assume the condition (1.7) in order to have constant pseudohermitian scalar curvature.

**Corollary 1.5.** If, in addition to the conditions of Theorem 1.3, we suppose the equality (1.7) holds, then

\[
\int_M (S - \bar{S})^2 Vol_\theta \leq \frac{2n(2n + 3)}{(n - 1)(n + 3)} \int_M |Rc_0|^2 Vol_\theta.
\]

If we have equality in (1.8) then the compact pseudohermitian manifold is pseudo-Einstein with constant pseudohermitian scalar curvature.

The next result slightly improves [7, Corollary 1.3].

**Corollary 1.6.** If, in addition to the conditions of Theorem 1.2, we suppose the equality (1.7) holds, then

\[
\int_M (S - \bar{S})^2 Vol_\theta \leq \frac{4n(n + 1)}{(n - 1)(n + 2)} \int_M |Rc_0|^2 Vol_\theta.
\]

If we have equality in (1.9) then the compact pseudohermitian manifold is pseudo-Einstein with constant pseudohermitian scalar curvature.

In the torsion-free case we get from Corollary 1.5

**Corollary 1.7.** Let \(n \geq 2\) and \((M, J, \theta)\) be a \((2n + 1)\)-dimensional compact torsion-free strictly pseudoconvex pseudohermitian manifold, i.e. a Sasakian manifold, with non-negative Webster Ricci tensor, \(Rc \geq 0\). Then the following inequality holds

\[
\int_M (S - \bar{S})^2 Vol_\theta \leq \frac{2n(2n + 3)}{(n - 1)(n + 3)} \int_M |Rc_0|^2 Vol_\theta.
\]

If the equality in (1.10) holds, then the compact Sasakian space is pseudo-Einstein with constant pseudohermitian scalar curvature and therefore it is a Riemannian Sasaki \(\eta\)-Einstein space D-homothetic to a Riemannian Sasaki-Einstein space.

**Remark 1.8.** In the torsion-free case, \(A = 0\), the positivity assumptions (1.1) and (1.4) coincide and because the number \(\frac{2n(2n + 3)}{(n - 1)(n + 3)}\) is smaller than the number \(\frac{4n(n + 1)}{(n - 1)(n + 2)}\), we get a better estimate for Sasakian manifolds than the one following from [7, Theorem 1.2], i.e. Theorem 1.2 above.

In the proof of Corollary 1.6 we also present a proof of Theorem 1.2, [7, Theorem 1.2].

In the Appendix we record for self-sufficiency some of the results of [11, 12, 16] in real variables (see also [14, Appendix]) including the Greenleaf’s CR Bochner formula [12], the CR Paneitz operator and its non-negativity for \(n > 1\) [11].

**Convention 1.9.**

a) We shall use \(X, Y, Z, U\) to denote horizontal vector fields, i.e. \(X, Y, Z, U \in H\).

b) \(\{e_1, \ldots, e_{2n}\}\) denotes a local orthonormal basis of the horizontal space \(H\).

c) The summation convention over repeated vectors from the basis \(\{e_1, \ldots, e_{2n}\}\) will be used. For example, for a \((0,4)\)-tensor \(P\), the formula \(k = P(e_b, e_a, e_a, e_b)\) means \(k = \sum_{a,b=1}^{2n} P(e_b, e_a, e_a, e_b)\).

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2. PSEUDOHERMITIAN MANIFOLDS AND THE TANAKA-WEBSTER CONNECTION

In this section we will briefly review the basic notions of the pseudohermitian geometry of a CR manifold. Also, we recall some results (in their real form) from [16, 20, 22, 23], see also [10, 14, 15], which we will use in this paper.

A CR manifold is a smooth manifold $M$ of real dimension $2n+1$, with a fixed $n$-dimensional complex sub-bundle $\mathcal{H}$ of the complexified tangent bundle $\mathbb{C}TM$ satisfying $\mathcal{H} \cap \overline{\mathcal{H}} = 0$ and $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. If we let $H = \text{Re} \mathcal{H} \oplus \overline{\mathcal{H}}$, the real sub-bundle $H$ is equipped with a formally integrable almost complex structure $J$. We assume that $M$ is oriented and there exists a globally defined compatible contact form $\theta$ such that $H = \text{Ker} \, \theta$. In other words, the hermitian bilinear form
\[ 2g(X, Y) = -d\theta(JX, Y) \]
is non-degenerate. The CR structure is called strictly pseudoconvex if $g$ is a positive definite tensor on $H$. The vector field $\xi$ dual to $\theta$ with respect to $g$ satisfying $\xi \cdot d\theta = 0$ is called the Reeb vector field. The almost complex structure $J$ is formally integrable in the sense that
\[ ([JX, JY] + [X, JY]) \in H \]
and the Nijenhuis tensor
\[ N^J(X, Y) = [JX, JY] - [X, JY] - J[JX, Y] - J[X, JY] = 0. \]

A CR manifold $(M, \theta, g)$ with a fixed compatible contact form $\theta$ is called a pseudohermitian manifold. In this case the 2-form
\[ d\theta|_H := 2\omega \]
is called the fundamental form. Note that the contact form is determined up to a conformal factor, i.e. $\theta = \nu \theta$ for a positive smooth function $\nu$ defines another pseudohermitian structure called pseudo-conformal to the original one.

2.1. Invariant decompositions. As usual any endomorphism $\Psi$ of $H$ can be decomposed with respect to the complex structure $J$ uniquely into its $U(n)$-invariant $(2,0) + (0,2)$ and $(1,1)$ parts. In short we will denote these components correspondingly by $\Psi_{[-1]}$ and $\Psi_{[1]}$. Furthermore, we shall use the same notation for the corresponding 2-tensor, $\Psi(X, Y) = g(\Psi X, Y)$. Explicitly, $\Psi = \Psi_{[1]} + \Psi_{[-1]}$, where
\[ \Psi_{[1]}(X, Y) = \frac{1}{2} \left[ \Psi(X, Y) + \Psi(JX, JY) \right], \quad \Psi_{[-1]}(X, Y) = \frac{1}{2} \left[ \Psi(X, Y) - \Psi(JX, JY) \right]. \]
The above notation is justified by the fact that the $(2,0) + (0,2)$ and $(1,1)$ components are the projections on the eigenspaces of the operator
\[ \Upsilon = J \otimes J, \quad (\Upsilon \Psi)(X, Y) \overset{def}{=} \Psi(JX, JY), \]
corresponding, respectively, to the eigenvalues $-1$ and $1$. Note that both the metric $g$ and the 2-form $\omega$ belong to the $[1]$-component, since $g(X, Y) = g(JX, JY)$ and $\omega(X, Y) = \omega(JX, JY)$. Furthermore, the two components are orthogonal to each other with respect to $g$.

2.2. The Tanaka-Webster connection. The Tanaka-Webster connection [20, 22, 23] is the unique linear connection $\nabla$ with torsion $T$ preserving a given pseudohermitian structure, i.e., it has the properties
\[ \nabla \xi = \nabla J = \nabla \theta = \nabla g = 0, \]
\[ T(X, Y) = d\theta(X, Y)\xi = 2\omega(X, Y)\xi, \quad T(\xi, X) \in H, \]
\[ g(T(\xi, X), Y) = g(T(\xi, Y), X) = -g(T(\xi, JX), JY). \]
It is well known that the endomorphism $T(\xi, \cdot)$ is the obstruction for a pseudohermitian manifold to be Sasakian. The symmetric endomorphism $T_\xi : H \rightarrow H$ is denoted by $A$, $A(X, Y) := T(\xi, X, Y)$, and it is called the torsion of the pseudohermitian manifold or pseudohermitian torsion. The pseudohermitian torsion $A$ is completely trace-free satisfying
\[ A(e_a, e_a) = A(e_a, J e_a) = 0, \quad A(X, Y) = A(Y, X) = -A(JX, JY). \]
Let $R$ be the curvature of the Tanaka-Webster connection. The pseudohermitian Ricci tensor $Rc$, the pseudohermitian scalar curvature $S$ and the pseudohermitian Ricci 2-form $\rho$ are defined by

$$Rc(A, B) = R(e_a, A, B, e_a), \quad S = Ric(e_a, e_a), \quad \rho(A, B) = \frac{1}{2} R(A, B, e_a, Je_a).$$

We summarize below the well known properties of the curvature $R$ of the Tanaka-Webster connection [22, 23, 16] using real expression, see also [10, 15, 14, 13].

\[
\begin{align*}
(2.5) & \quad Rc(X, Y) = Rc(Y, X), \quad Rc(X, Y) - Rc(JX, JY) = 4(n - 1)A(X, JY), \\
(2.6) & \quad 2\rho(X, JY) = -Ric(X, Y) - Rc(JX, JY) = R(e_a, Je_a, X, JY), \\
(2.7) & \quad 2(\nabla_{e_a}Rc)(e_a, X) = dS(X).
\end{align*}
\]

The equalities (2.5) and (2.6) imply

\[
(2.8) \quad Rc(X, Y) = \rho(JX, Y) + 2(n - 1)A(X, JY),
\]

i.e. $\rho$ is the $(1, 1)$-part of the pseudohermitian Ricci tensor, while the $(2, 0) + (0, 2)$-part is given by the pseudohermitian torsion $A$.

The Webster Ricci tensor $Rc$ is defined by

\[
(2.9) \quad Rc(X, Y) = \rho(JX, Y) = Rc(JX, JY) = Rc(Y, X).
\]

The Webster Ricci tensor $Rc$ is symmetric, of type $(1, 1)$ with respect to $J$ and shares the same trace with the Ricci tensor $Rc$, $S = Ric(e_a, e_a) = \rho(Je_a, e_a) = Rc(e_a, e_a)$ due to (2.8).

The trace-free part $Rc_0$ of the Webster Ricci tensor is given by

\[
(2.10) \quad Rc_0(X, Y) = Rc(X, Y) - \frac{S}{2n} g(X, Y).
\]

We recall that a pseudohermitian manifold is called pseudo-Einstein if the trace-free part of the Webster Ricci tensor vanishes.

2.3. The Ricci identities for the Tanaka-Webster connection. Let $f$ be a smooth function on a pseudohermitian manifold $M$ with $\nabla f$ its horizontal gradient, $g(\nabla f, X) = df(X)$. The sub-Laplacian (or horizontal Laplacian) $\Delta f$ and the norm of the horizontal gradient $\nabla f = df(e_a)e_a$ of a smooth function $f$ on $M$ are defined respectively by

\[
\begin{align*}
\Delta f &= -tr_H(\nabla^2 f) = -\nabla^2 f(e_a, e_a), \quad |\nabla f|^2 = df(e_a) df(e_a).
\end{align*}
\]

We have the next Ricci identities for the Tanaka-Webster connection following from the general Ricci identities for a connection with torsion applying the properties of the pseudohermitian torsion listed in (2.3):

\[
\begin{align*}
\nabla^2 f(X, Y) - \nabla^2 f(Y, X) &= -2\omega(X, Y)df(\xi) \\
\nabla^2 f(X, \xi) - \nabla^2 f(\xi, X) &= A(X, \nabla f) \\
\nabla^3 f(X, Y, Z) - \nabla^3 f(Y, X, Z) &= -R(X, Y, Z, \nabla f) - 2\omega(X, Y)\nabla^2 f(\xi, Z) \\
\nabla^3 f(X, Y, Z) - \nabla^3 f(Z, Y, X) &= -R(X, Y, Z, \nabla f) - R(Y, Z, X, \nabla f) - 2\omega(X, Y)\nabla^2 f(\xi, Z) \\
&\quad - 2\omega(Y, Z)\nabla^2 f(\xi, X) + 2\omega(Z, X)\nabla^2 f(\xi, Y) + 2\omega(Z, X) A(Y, \nabla f).
\end{align*}
\]

An important consequence of the first Ricci identity is the following fundamental formula

\[
(2.13) \quad g(\nabla^2 f, \omega) = \nabla^2 f(e_a, Je_a) = -2n df(\xi).
\]

On the other hand, it follows from (2.11) that the trace with respect to the metric is the sub-Laplacian:

\[
\begin{align*}
g(\nabla^2 f, g) &= \nabla^2 f(e_a, e_a) = -\Delta f.
\end{align*}
\]

We also recall the horizontal divergence theorem [20]. Let $(M, g, \theta)$ be a pseudohermitian manifold of dimension $2n + 1$. For a fixed local 1-form $\theta$ the form $Vol_\theta = \theta \wedge \omega^n$ is a globally defined volume form since $Vol_\theta$ is independent on the local 1-form $\theta$. The (horizontal) divergence of a horizontal vector field/one-form
σ ∈ Λ^1 (H) is defined by ∇^* σ = -tr|_H ∇σ = -(∇_{e_a} σ)e_a. If the manifold is compact it is well known [20] that

\[ \int_M (\nabla^* \sigma) Vol_\theta = 0. \]

3. The positivity conditions, CR Cordes type inequality

In this section we discuss the importance of the positivity conditions (1.1) and (1.4), recover the CR Cordes type inequality established in [6] giving more information in the equality case.

We recall from [16], (see also [2] and [1]), that if \( n \geq 2 \), a function \( f \in \mathbb{C}^4 (M) \) on a compact \((2n+1)\)-dimensional strictly pseudoconvex pseudohermitian manifold is CR-pluriharmonic, i.e., locally it is the real part of a CR holomorphic function, if and only if the non-negative CR Paneitz operator of \( f \) vanishes, i.e. the \((1,1)\) trace-free part of the horizontal Hessian of \( f \) is zero, \((\nabla^2 f)_{[1][0]} = 0 \). In fact, as shown in [11], only one fourth-order equation \( Cf = 0 \) suffices for \((\nabla^2 f)_{[1][0]} = 0 \) to hold.

Now, we recover the CR Cordes estimate proved in [6, Theorem 1] and give a bit more information in the equality case for dimensions bigger than three.

**Theorem 3.1.** [6] On a compact strictly pseudoconvex pseudohermitian manifold of dimension \( 2n+1 \), for any real-valued function \( f \) and \( n > 1 \) we have the inequality

\[ \frac{n+2}{n} \int_M (\Delta f)^2 Vol_\theta \geq \int_M \left[ Rc(\nabla f, \nabla f) + 2(n+2) A(J\nabla f, \nabla f) + (\nabla^2 f)_{[1]}^2 + (\nabla^2 f)_{[-1]}^2 \right] Vol_\theta. \]

The equality is achieved only for CR-pluriharmonic functions.

If, moreover, the positivity condition (1.4) holds then

\[ \frac{n+2}{n} \int_M (\Delta f)^2 Vol_\theta \geq \int_M \left[ (\nabla^2 f)_{[1]}^2 + (\nabla^2 f)_{[-1]}^2 \right] Vol_\theta. \]

In the case \( n = 1 \), if we assume in addition that the CR-Paneitz operator is non-negative, then the inequality (3.1) holds also for \( n = 1 \).

**Proof.** Insert (6.16) into (6.18) and apply (2.8) and (2.9) to obtain

\[ \frac{n+2}{n} \int_M (\Delta f)^2 Vol_\theta = \int_M \left[ Rc(\nabla f, \nabla f) + 2(n+2) A(J\nabla f, \nabla f) \right] Vol_\theta \]

\[ + \int_M \left[ (\nabla^2 f)_{[1]}^2 + (\nabla^2 f)_{[-1]}^2 - \frac{2}{n} P(\nabla f) \right] Vol_\theta. \]

Due to Lemma 6.3 the last term in (3.2) is non-negative, \(-P(\nabla f) \geq 0\), and the inequality (3.1) follows from (3.2). Moreover, the equality in (3.1) can be achieved only when \( P(\nabla f) = 0 = (\nabla f)_{[1][0]} \) due again to Lemma 6.3. The assertion follows from the discussion above the formulation of the theorem. \( \square \)

**Remark 3.2.** It was pointed out in [5] that in the original Greenleaf’s CR-Bochner formula from [12] the coefficient in front of \( A \) is not correct. In the proof of [6, Theorem 1] it was used that incorrect CR-Bochner formula and therefore the coefficient in front of the pseudohermitian torsion \( A \) in (3.1) differs a bit from the coefficient in front of \( A \) in the formula (22) of [6, Theorem 1].

Applying (6.17) to the identity (3.2), taking into account (6.9), we also get

\[ \frac{n+1}{n} \int_M (\Delta f)^2 Vol_\theta = \int_M \left[ Rc(\nabla f, \nabla f) + 2(n+1) A(J\nabla f, \nabla f) \right] Vol_\theta \]

\[ + \int_M \left[ (\nabla^2 f)_{[1]}^2 + (\nabla^2 f)_{[-1]}^2 - \frac{3}{2n} P(\nabla f) \right] Vol_\theta. \]

**Remark 3.3.** Note that the expression in the right-hand side of the first line in (3.3) is precisely the CR-Lichnerowicz positivity condition used to find a lower bound of the first eigenvalue of the sub-Laplacian (see [12, 4, 5, 14]). Indeed, if one assumes the positivity condition (1.1) to be in the form

\[ Rc(X, X) + 2(n+1) A(JX, X) \geq k_0 \theta(X, X) \]
and $\Delta f = \lambda f$, one easily gets $\lambda \geq \frac{n}{n+1}k_0$ which is the Greenleaf’s estimation of the first eigenvalue of the sub-Laplacian.

4. Proof of Theorem 1.3

We follow the approach of [9] in the Riemannian case and [7] in the CR case. Denote by $\bar{S}$ the average value of the scalar curvature,

$$\bar{S} = \int_M S \text{Vol}_\theta.$$ 

Let $\varphi$ be the unique solution of the following PDE:

$$(4.1) \quad \Delta \varphi = S - \bar{S}, \quad \int_M \varphi \text{Vol}_\theta = 0.$$ 

We obtain

$$(4.2) \quad \int_M (S - \bar{S})^2 \text{Vol}_\theta = \int_M (S - \bar{S}) \Delta \varphi \text{Vol}_\theta = \int_M dS(\nabla \varphi) \text{Vol}_\theta,$$

where we used an integration by parts to achieve the last equality.

We write, using (2.10), the equality (2.8) in the form

$$(4.3) \quad \text{Ric}(X,Y) = R_{c_0}(X,Y) + \frac{S}{2n}g(X,Y) + 2(n-1)A(JX,Y).$$

In view of (4.3), the second Bianchi identity (2.7) takes the form

$$2(\nabla_{e_a} R_{c_0})(e_a, X) + \frac{1}{n}dS(X) + 4(n-1)(\nabla_{e_a} A)(e_a, JX) = dS(X),$$

which yields

$$(4.4) \quad dS(X) = \frac{2n}{n-1}(\nabla_{e_a} R_{c_0})(e_a, X) + 4n(\nabla_{e_a} A)(e_a, JX).$$

Substitute (4.4) into (4.2) to get after an integration by parts, applying (2.4) and (2.9), the following relations

$$(4.5) \quad \int_M (S - \bar{S})^2 \text{Vol}_\theta = \int_M dS(\nabla \varphi) \text{Vol}_\theta$$

$$= \frac{2n}{n-1} \int_M (\nabla_{e_a} R_{c_0})(e_a, \nabla \varphi) \text{Vol}_\theta + 4n \int_M (\nabla_{e_a} A)(e_a, J\nabla \varphi) \text{Vol}_\theta$$

$$= -\frac{2n}{n-1} \int_M R_{c_0}(e_a, e_b) \nabla^2 \varphi(e_a, e_b) \text{Vol}_\theta - 4n \int_M A(e_a, J\text{e}_b) \nabla^2 \varphi(e_a, e_b) \text{Vol}_\theta$$

$$= -\frac{2n}{n-1} \int_M R_{c_0}(e_a, e_b) (\nabla^2 \varphi)_{[1]}(e_a, e_b) \text{Vol}_\theta - 4n \int_M A(e_a, J\text{e}_b) (\nabla^2 \varphi)_{[-1]}(e_a, e_b) \text{Vol}_\theta.$$

Applying the Young’s inequality $2ab \leq \alpha a^2 + \frac{1}{\alpha}b^2$, we get from (4.5) that

$$(4.6) \quad \int_M (S - \bar{S})^2 \text{Vol}_\theta$$

$$\leq \frac{n}{n-1} \int_M \left[ \alpha |R_{c_0}|^2 + \frac{1}{\alpha} (|\nabla^2 \varphi|_{[1]}^2) \right] \text{Vol}_\theta - 4n \int_M A(e_a, J\text{e}_b) (\nabla^2 \varphi)_{[-1]}(e_a, e_b) \text{Vol}_\theta.$$ 

At this point we need to evaluate the norm of the $(1,1)$ trace-free part of the horizontal Hessian. To do this, we involve the positivity condition (1.4). We have

Proposition 4.1. On a compact strictly pseudoconvex pseudohermitian manifold of dimension $2n + 1$ for $n \geq 2$ and for any smooth function $f$ we have

$$(4.7) \quad \int_M (\Delta f)^2 \text{Vol}_\theta = \frac{2n}{2n+3} \int_M \left[ Rc(\nabla f, \nabla f) + 2(n+2)A(J\nabla f, \nabla f) \right] \text{Vol}_\theta$$

$$+ \int_M \left[ \frac{2n(n+3)}{(n-1)(2n+3)} \left( |\nabla^2 f|_{[1]}^2 \right)^2 + \frac{2n}{2n+3} \left| \nabla^2 f \right|_{[-1]}^2 + \frac{4n^2}{2n+3} (df(\xi))^2 \right] \text{Vol}_\theta.$$
Proof. It follows directly from (3.2), Lemma 6.3 and (6.9).

The positivity condition (1.4) and (4.7) yield the inequality
\[
\int_{\Omega} |(\nabla^2 f)_{1[0]}|^2 Vol_\theta \leq \frac{(n-1)(2n+3)}{2n(n+3)} \int_{\Omega} (\nabla f)^2 Vol_\theta,
\]
which taken with respect to the function \( \varphi \), then substituted into (4.6) and applying (4.1) lead to the inequality
\[
(4.8) \quad \left( 1 - \frac{2n+3}{2n} \right) \int_{\Omega} (S - \bar{S})^2 Vol_\theta \leq \frac{\alpha n}{n-1} \int_{\Omega} |Rc_0|^2 Vol_\theta - 4n \int_{\Omega} A(e_a, Je_b)(\nabla^2 \varphi)_{[-1]}(e_a, e_b) Vol_\theta.
\]
Take \( \alpha = \frac{2n+3}{2n} \) to obtain from (4.8) the inequality (1.5), which completes the proof of the first part of Theorem 1.3.

Suppose we have equality in (1.5). Take \( \alpha = \frac{2n+3}{2n} \) into (4.6) and then use the expression for \( |(\nabla^2 f)_{1[0]}|^2 \) from (4.7) substituted into (4.6) to get, taking into account (4.1), that
\[
(4.9) \quad 0 \leq -\frac{n}{2n+3} \int_{\Omega} Rc(\nabla \varphi, \nabla \varphi) + 2(n+2)A(J \nabla \varphi, \nabla \varphi) + |(\nabla^2 \varphi)_{[-1]}|^2 + 2n(d\varphi(\xi))^2 Vol_\theta.
\]
The positivity condition (1.4) and (4.9) imply
\[
(4.10) \quad Rc(\nabla \varphi, \nabla \varphi) + 2(n+2)A(J \nabla \varphi, \nabla \varphi) = 0; \quad (\nabla^2 \varphi)_{[-1]} = 0; \quad (d\varphi(\xi))^2 = 0.
\]
The equality in (1.5) and the second equality in (4.10) yield (1.6).

Substitute (4.10) into (4.7) and use (1.6) to get
\[
(4.11) \quad \int_{\Omega} |(\nabla^2 \varphi)_{1[0]}|^2 Vol_\theta = \frac{(n-1)(2n+3)}{2n(n+3)} \int_{\Omega} (\nabla \varphi)^2 Vol_\theta = \frac{(2n+3)^2}{(n+3)^2} \int_{\Omega} |Rc_0|^2 Vol_\theta.
\]
Applying (1.6) and (4.10) to (4.5) yields
\[
(4.12) \quad \int_{\Omega} (S - \bar{S})^2 Vol_\theta = \frac{2n(2n+3)}{(n-1)(n+3)} \int_{\Omega} |Rc_0|^2 Vol_\theta
\]
which together with (4.11) implies that we have equality in the Young's inequality. This shows that
\[
(4.13) \quad Rc_0(e_a, e_b) = -\frac{n+3}{2n+3} (\nabla^2 \varphi)_{1[0]}(e_a, e_b),
\]
which implies that the contact form \( \tilde{\theta} = exp(-\frac{2n+3}{2n+3}) \theta \) will be pseudo-Einstein by [10, Proposition 5.9]. This completes the proof of Theorem 1.3.

4.1. Proof of Corollary 1.5. Two integration by parts yield
\[
\int_{\Omega} A(e_a, Je_b)(\nabla^2 \varphi)_{[-1]}(e_a, e_b) Vol_\theta = \int_{\Omega} A(e_a, Je_b)\nabla^2 \varphi(e_a, e_b) Vol_\theta = -\int_{\Omega} (\nabla e_a A)(e_a, Je_b) Vol_\theta
\]
due to (1.7), which proves the first part.

To show the second part, suppose we have an equality in (1.8). Then (4.10) and (4.11) hold true. We obtain the following chain of equalities by integration by parts, using the second equality in (4.10):
\[
(4.14) \quad \int_{\Omega} A(J \nabla \varphi, \nabla \varphi) Vol_\theta = -\int_{\Omega} \varphi(\nabla e_a A)(e_a, Je_b) Vol_\theta - \int_{\Omega} \varphi A(e_a, Je_b)\nabla^2 \varphi(e_a, e_b) Vol_\theta
\]
\[
= -\int_{\Omega} \varphi(\nabla e_a A)(e_a, J \nabla \varphi) Vol_\theta - \int_{\Omega} \varphi A(e_a, Je_b)(\nabla^2 \varphi)_{[-1]}(e_a, e_b) Vol_\theta
\]
\[ = - \int_M \varphi(\nabla_{e_a} A)(e_a, J\nabla \varphi)\text{Vol}_\theta = \frac{1}{2} \int_M \varphi^2(\nabla_{e_a} \nabla_{e_a} A)(e_a, Je_b)\text{Vol}_\theta = 0. \]

Applying (4.14), Lemma 6.3 and the third equality in (4.10) to (6.17), we obtain using (4.11) that

\begin{equation}
0 = 2 \int_M A(\nabla \varphi, J\nabla \varphi)\text{Vol}_\theta = \int_M \left[ \frac{1}{2n}(\Delta \varphi)^2 - \frac{1}{n-1} \left| (\nabla^2 \varphi)_{[1]} \right|^2 \right]\text{Vol}_\theta
\end{equation}

\[ = \int_M \frac{1}{2n}(\Delta \varphi)^2 - \frac{2n+3}{2n(n+3)}(\Delta \varphi)^2 \text{Vol}_\theta = -\frac{1}{2(n+3)} \int_M (\Delta \varphi)^2 \text{Vol}_\theta. \]

Now, (4.15) shows that \( \Delta \varphi = 0 \) and the manifold is pseudo-Einstein with constant pseudohermitian scalar curvature, which completes the proof of Corollary 1.5.

The equality

\[ |Rc - \frac{S}{2n}g|^2 = |Rc - \frac{S}{2n}g|^2 + \frac{1}{2n}(S - \bar{S})^2 \]

together with Corollary 1.5 imply

Corollary 4.2. In the conditions of Corollary 1.5 one has

\[ \int_M |Rc - \frac{S}{2n}g|^2 \text{Vol}_\theta \leq \frac{n(n+4)}{(n-1)(n+3)} \int_M |Rc - \frac{S}{2n}g|^2 \text{Vol}_\theta. \]

4.2. Proof of Corollary 1.7. It remains to show only the last part of Corollary 1.7. In the torsion-free case, i.e. in Sasakian case, it is well known that the Ricci tensor \( Ric^h \) of the Riemannian metric \( h = g + \eta \otimes \eta \) and the Webster Ricci tensor \( Rc \) are connected by (see e.g. [10])

\begin{equation}
Ric^h(X, X) = Rc(X, X) - 2g(X, X), \quad Ric^h(\xi, \xi) = 2n.
\end{equation}

If we have pseudo-Einstein Sasakian structure, we obtain from (4.16)

\[ Ric^h(X, X) = \frac{S - 4n}{2n}g(X, X), \quad Ric^h = \frac{S - 4n}{2n}h + \left( 2n - \frac{S - 4n}{2n} \right) \eta \otimes \eta \]

showing that the Riemannian Sasaki structure is \( \eta \)-Einstein [19] and the Tanno’s D-homothetic deformation \( \bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a} \xi, \quad \bar{h} = ah + a(a-1)\eta \otimes \eta \) makes it a Sasaki-Einstein space for a suitable constant \( a \) [21] c.f. also [3].

5. Proof of Corollary 1.6

For completeness and a better understanding of the proof of Corollary 1.6, we give a proof of [7, Theorem 1.2], Theorem 1.2.

The next result involves the positivity condition (1.1). We have

Proposition 5.1. On a compact strictly pseudoconvex pseudohermitian manifold of dimension \( 2n + 1 \) for \( n \geq 2 \) and for any smooth function \( f \) we have

\begin{equation}
\int_M (\Delta f)^2 \text{Vol}_\theta = \frac{n}{n+1} \int_M \left[ Rc(\nabla f, \nabla f) + 2(n+1)A(J\nabla f, \nabla f) \right] \text{Vol}_\theta + \int_M \frac{n(n+2)}{n^2-1} \left| (\nabla^2 f)_{[1]} \right|^2 \text{Vol}_\theta.
\end{equation}

\begin{equation}
\int_M \left| (\nabla^2 f)_{[1]} \right|^2 \text{Vol}_\theta \leq \frac{n^2-1}{n(n+2)} \int_M (\Delta f)^2 \text{Vol}_\theta,
\end{equation}

which taken with respect to the function \( \varphi \), then substituted into (4.6) and applying (4.1) lead to the inequality

\begin{equation}
(1 - \frac{n+1}{\alpha(n+2)}) \int_M (S - \bar{S})^2 \text{Vol}_\theta \leq \frac{\alpha n}{n-1} \int_M |Rc_0|^2 \text{Vol}_\theta - 4n \int_M A(e_a, Je_b)(\nabla^2 \varphi)_{[-1]}(e_a, e_b) \text{Vol}_\theta.
\end{equation}

Proof. It follows directly from (3.3) and Lemma 6.3. \( \square \)
Taking $\alpha = \frac{2(n+1)}{n+2}$, we obtain from (5.2) the inequality (1.2), which completes the proof of the first part of Theorem 1.2 from [7].

The second part follows similarly to the proof of the second part of Theorem 1.3. Indeed, suppose we have equality in (1.2). Take $\alpha = \frac{2(n+1)}{n+2}$ into (4.6), then use the expression for $|\langle (\nabla^2 f)_{[1][0]} \rangle|^2$ from (5.1) substituted into (4.6) to get, taking into account (4.1), that

$$0 \leq -\frac{n}{2(n+1)} \int_M \left[ Rc(\nabla \phi, \nabla \phi) + 2(n+1)A(\nabla \phi, \nabla \phi) + |(\nabla^2 \phi)_{[-1]}|^2 \right] Vol_\theta.$$

The positivity condition (1.1) and (5.3) imply

$$\text{Rc}(\nabla \phi, \nabla \phi) + 2(n+1)A(\nabla \phi, \nabla \phi) = 0; \quad (\nabla^2 \phi)_{[-1]} = 0.$$

The equality in (1.2) and the second equality in (5.4) yield (1.3).

Substitute (5.4) into (5.1) and use (1.3) to get

$$\int_M |(\nabla^2 \varphi)_{[1][0]}|^2 Vol_\theta = \frac{n^2 - 1}{n(n + 2)} \int_M (\triangle \varphi)^2 Vol_\theta = \frac{(2n + 2)^2}{(n+2)^2} \int_M |RC_0|^2 Vol_\theta.$$

Applying (1.3) and (5.4) to (4.5) yields

$$\int_M (S - \tilde{S})^2 Vol_\theta = \frac{4n(n+1)}{(n-1)(n+2)} \int_M |RC_0|^2 Vol_\theta = -\frac{2n}{n-1} \int_M RC_0(e_a, e_b) (\nabla^2 \varphi)_{[1][0]} (e_a, e_b) Vol_\theta,$$

which, together with (5.5), implies that we have equality in the Young’s inequality. This shows that

$$RC_0(e_a, e_b) = -\frac{n+2}{2n+2} (\nabla^2 \varphi)_{[1][0]} (e_a, e_b),$$

yielding that the contact form $\bar{\theta} = \exp(-\frac{1}{n+1} \varphi) \bar{\theta}$ will be pseudo-Einstein by [10, Proposition 5.9] which completes the proof of Theorem 1.2 of [7].

To finish the proof of Corollary 1.6, it remains to show only that if we have equality in (1.9) then the manifold is pseudo-Einstein with constant pseudohermitian scalar curvature.

Indeed, in the equality case we have that (5.4) and (5.5) are valid. The second identity in (5.4) implies that (4.14) holds true. Apply (4.14), Lemma 6.3 and the second equality in (5.4) to (6.17) to get, using (5.5), that

$$0 = 2 \int_M A(\nabla \phi, J\nabla \phi) Vol_\theta = \int_M \left[ -\frac{1}{2n} g(\nabla^2 \phi, \omega)^2 + \frac{1}{2n} (\triangle \phi)^2 - \frac{1}{n-1} |(\nabla^2 \phi)_{[1][0]}|^2 \right] Vol_\theta$$

$$= \int_M \left[ -\frac{1}{2n} g(\nabla^2 \phi, \omega)^2 + \frac{1}{2n} (\triangle \phi)^2 - \frac{n+1}{n(n+2)} (\triangle \phi)^2 \right] Vol_\theta$$

$$= -\int_M \left[ \frac{1}{2(n+2)} (\triangle \phi)^2 + \frac{1}{2n} g(\nabla^2 \phi, \omega)^2 \right] Vol_\theta.$$

Now, (5.6) shows that $\triangle \phi = 0$ and the manifold is pseudo-Einstein with constant pseudohermitian scalar curvature which completes the proof.

6. Appendix

The purpose of this section is to record for self-sufficiency proofs of some of the results of [12, 16, 11] in real variables (see also [14, Appendix]) including the Greenleaf’s CR Bochner formula [12], the CR Paneitz operator and its non-negativity for $n > 1$ [11], etc.

6.1. The Greenleaf’s CR-Bochner formula.

**Theorem 6.1.** [12] On a strictly pseudoconvex pseudohermitian manifold of dimension $2n + 1$, $n \geq 1$, the following Bochner-type identity holds

$$-\frac{1}{2} \triangle |\nabla f|^2 = -g(\nabla(\triangle f), \nabla f) + Ric(\nabla f, \nabla f) + 2A(\nabla f, \nabla f) + |\nabla^2 f|^2 + 4\nabla df(\xi, J\nabla f).$$
Proof. By definition we have

\[
(6.2) \quad -\frac{1}{2} \triangle |\nabla f|^2 = \nabla^3 f(e_a, e_a, e_b)df(e_b) + \nabla^2 f(e_a, e_b)\nabla^2 f(e_a, e_b) = \nabla^3 f(e_a, e_a, e_b)df(e_b) + |\nabla^2 f|^2.
\]

To evaluate the first term in the right-hand side of (6.2), we use the Ricci identities (2.12). Taking into account \((\nabla_X T)(Y, Z) = 0\) and applying successively the Ricci identities (2.12), we obtain

\[
(6.3) \quad \nabla^3 f(e_a, e_a, e_b)df(e_b) = -g(\nabla(\triangle f), \nabla f) + \text{Ric}(\nabla f, \nabla f) + 2A(\nabla f, \nabla f) + 4\nabla^2 f(\xi, J\nabla f).
\]

A substitution of (6.3) into (6.2) completes the proof of (6.1).

The next integral formula was originally proved in [12]; we take it in real notations from [14].

Lemma 6.2. [12] On a compact strictly pseudoconvex pseudohermitian manifold of dimension \(2n + 1, n \geq 1\), we have the identity

\[
(6.4) \quad \int_M \nabla^2 f(\xi, J\nabla f) \text{Vol}_\theta = -\int_M \left[2n(df(\xi))^2 + A(\nabla f, \nabla f)\right] \text{Vol}_\theta.
\]

Proof. We consider the horizontal 1-form defined by \(D(X) = df(JX)\), whose divergence is, taking into account the second formula of (2.12),

\[
(6.5) \quad (\nabla_{e_a} D)(e_a) = \nabla^2 f(e_a, Je_a)df(\xi) - \nabla^2 f(\xi, J\nabla f) - A(\nabla f, \nabla f).
\]

Integrating (6.5) over \(M\) and using (2.13) implies (6.4), which completes the proof of the lemma.

6.2. The CR-Paneitz operator. The famous CR-Paneitz operator is defined as follows [17, 11].

Given a function \(f\) we define the one form,

\[
(6.6) \quad P(X) = Pf(X) = \nabla^3 f(X, e_b, e_b) + \nabla^3 f(JX, e_b, Je_b) + 4nA(X, J\nabla f),
\]

and also a fourth order differential operator (the so called CR-Paneitz operator in [8]),

\[
(6.7) \quad Cf = (\nabla_{e_a} P)(e_a) = \nabla^4 f(e_a, e_a, e_b, e_b) + \nabla^4 f(e_a, Je_a, e_b, Je_b) - 4n\nabla^* A(\nabla f, \nabla f) - 4n g(\nabla^2 f, JA).
\]

According to (2.1), the horizontal Hessian \(\nabla^2 f\) splits into two parts, \(\nabla^2 f = (\nabla^2 f)_{[1]} + (\nabla^2 f)_{[-1]}\), where

\[
(6.8) \quad (\nabla^2 f)_{[1]}(X, Y) = \frac{1}{2} \left[(\nabla^2 f)(X, Y) + (\nabla^2 f)(JX, JY)\right],
\]

\[
(\nabla^2 f)_{[-1]}(X, Y) = \frac{1}{2} \left[(\nabla^2 f)(X, Y) - (\nabla^2 f)(JX, JY)\right].
\]

In view of (2.13), the trace-free part \((\nabla^2 f)_{[1]}_{[0]}\) of the \((1,1)\) component of the horizontal Hessian is given by

\[
(6.9) \quad (\nabla^2 f)_{[1]}_{[0]}(X, Y) = (\nabla^2 f)_{[1]} + \frac{\triangle f}{2n} g(X, Y) + df(\xi) \omega(X, Y);
\]

\[
|((\nabla^2 f)_{[1]}_{[0]})|^2 = |(\nabla^2 f)_{[1]}|^2 - \frac{(\triangle f)^2}{2n} - 2n(df(\xi))^2.
\]

One of the basic results relating the above defined operators is the next lemma proved in [11], which we present in real notations, see e.g. [14].

Lemma 6.3. [11] On a compact strictly pseudoconvex pseudohermitian manifold of dimension \(2n + 1, n \geq 1\), the following identities hold true

\[
(6.10) \quad \nabla_{e_a} (\nabla^2 f)_{[1]}_{[0]}(e_a, X) = \frac{n-1}{2n} Pf(X),
\]

\[
(6.11) \quad \int_M |(\nabla^2 f)_{[1]}_{[0]}|^2 \text{Vol}_\theta = -\frac{n-1}{2n} \int_M Pf(\nabla f) \text{Vol}_\theta = \frac{n-1}{2n} \int_M f(Cf) \text{Vol}_\theta.
\]

In particular, if \(n > 1\) the CR-Paneitz operator is non-negative, \(\int_M f(Cf) \text{Vol}_\theta \geq 0\).
The sum of the above equalities gives

\begin{equation}
\begin{aligned}
\nabla^3 f(e_a, e_a, X) &= \nabla^3 f(X, e_a, e_a) + \text{Ric}(X, \nabla f) + 4\nabla^2 f(\xi, JX) + 2A(JX, \nabla f), \\
\nabla^3 f(e_a, Je_a, JX) &= \frac{1}{2} \left( \nabla^3 f(e_a, e_a, JX) - \nabla^3 f(Je_a, e_a, JX) \right) = -\rho(JX, \nabla f) - 2n\nabla^2 f(\xi, JX).
\end{aligned}
\end{equation}

The equality \((6.16)\) additionally in the compact case, the following integral identity holds true:

\begin{equation}
\begin{aligned}
\nabla^3 f(\xi, JX) &= \nabla^2 f(JX, \xi) - A(JX, \nabla f) = -\frac{1}{2n} \nabla^3 f(JX, e_a, Je_a) - A(JX, \nabla f).
\end{aligned}
\end{equation}

Therefore, using \((2.8)\) and \((6.13)\) we get from \((6.12)\) that

\begin{equation}
\begin{aligned}
2\nabla_{e_a}(\nabla^2 f)_{[1]}(e_a, X) &= \nabla^3 f(X, e_a, e_a) + \frac{n-2}{n} \nabla^3 f(JX, e_a, Je_a) + 4(n-1)A(X, J\nabla f).
\end{aligned}
\end{equation}

The divergence of the trace part of \((\nabla^2 f)_{[1]}\) is computed as follows

\begin{equation}
\begin{aligned}
\nabla_{e_a} \left( -\frac{1}{2n} \triangle f \cdot g - d\omega(\xi) \right) (e_a, X) &= \frac{1}{2n} \nabla^3 f(X, e_a, e_a) + \nabla^2 f(JX, \xi) \\
&= \frac{1}{2n} \nabla^3 f(X, e_a, e_a) - \frac{1}{2n} \nabla^3 f(JX, e_a, Je_a).
\end{aligned}
\end{equation}

Now, the identities \((6.14)\) and \((6.15)\) imply \((6.10)\). The second identity \((6.11)\) follows by an integration by parts from \((6.10)\).

The next result, essentially proved in [4], involves the CR-Paneitz operator. We present it in real notations from [14, Lemma 8.7]:

**Lemma 6.4.** On a strictly pseudoconvex pseudohermitian manifold of dimension \(2n + 1\), \(n \geq 1\), we have the identity

\begin{equation}
\nabla^2 f(\xi, Z) = \frac{1}{2} \nabla^3 f(Z, Je_a, e_a) - A(Z, \nabla f).
\end{equation}

Additionally, in the compact case, the following integral identity holds true:

\begin{equation}
\begin{aligned}
\int_M \nabla^2 f(\xi, J\nabla f) Vol_\theta &= \int_M \left[ -\frac{1}{2n} (\triangle f)^2 + A(J\nabla f, \nabla f) - \frac{1}{2n} P_f(\nabla f) \right] Vol_\theta.
\end{aligned}
\end{equation}

**Proof.** We compute using the first two Ricci identities in \((2.12)\)

\[2\nabla^3 f(Z, Je_a, e_a) = \nabla^3 f(Z, Je_a, e_a) - \nabla^3 f(Z, e_a, Je_a) = -2\omega(Je_a, e_a)\nabla^2 f(Z, \xi),\]

which proves the first formula. The second identity \((6.16)\) follows from the first formula, the definition \((6.6)\) of \(P_f\) and an integration by parts.

Combining \((6.16)\) with \((6.4)\), one gets

\begin{equation}
\begin{aligned}
2\int_M A(J\nabla f, \nabla f) Vol_\theta &= \int_M \left[ -\frac{1}{2n} g(\nabla^2 f, \omega)^2 + \frac{1}{2n} (\triangle f)^2 + \frac{1}{2n} P_f(\nabla f) \right] Vol_\theta.
\end{aligned}
\end{equation}

Integrating the Bochner type formula \((6.1)\) on a compact \(M\) gives

\begin{equation}
\begin{aligned}
0 &= \int_M \left[ - (\triangle f)^2 + |(\nabla^2 f)_{[1]}|^2 + |(\nabla^2 f)_{[-1]}|^2 \\
&\quad + \text{Ric}(\nabla f, \nabla f) + 2A(J\nabla f, \nabla f) + 4\nabla^2 f(\xi, J\nabla f) \right] Vol_\theta.
\end{aligned}
\end{equation}
We use Lemma 6.4 to represent the last term, which turns the identity (6.18) into the following

\[(6.19) \quad 0 = \int_M \left[ - (\triangle f)^2 + \left| (\nabla^2 f)_{[1]} \right|^2 + \left| (\nabla^2 f)_{[-1]} \right|^2 + \text{Ric}(\nabla f, \nabla f) + 6A(\nabla f, \nabla f) \right. \\
\left. - \frac{2}{n} (\triangle f)^2 - \frac{2}{n} P_f(\nabla f) \right] \text{Vol}_\theta.
\]

After a substitution of (6.17) in (6.19), taking into account (6.9), we get

\[(6.20) \quad 0 = \int_M \left[ \text{Ric}(\nabla f, \nabla f) + 4A(\nabla f, \nabla f) - \frac{n+1}{n} (\triangle f)^2 \right] \text{Vol}_\theta \\
+ \int_M \left[ \left| (\nabla^2 f)_{[1]} \right|_{[0]}^2 + \left| (\nabla^2 f)_{[-1]} \right|_{[-1]}^2 - \frac{3}{2n} P_f(\nabla f) \right] \text{Vol}_\theta.
\]

The above identities (6.19) and (6.20) are the key identities relating the CR-Paneitz operator and the Greenleaf’s CR-Bochner formula (6.18) on a compact manifold, and were essentially proved in [4]. We presented these identities in real notations from [14, (8.13), (8.15)].

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