EXCEPTIONAL LOCI IN LEFSCHETZ THEORY

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Abstract. Let $\phi : X \to \mathbb{P}^n$ be a morphism of varieties. Given a hyperplane $H$ in $\mathbb{P}^n$, there is a Gysin map from the compactly supported cohomology of $\phi^{-1}(H)$ to that of $X$. We give conditions on the degree of the cohomology under which this map is an isomorphism for all but a low-dimensional set of hyperplanes, generalizing results due to Skorobogatov, Benoist, and Poonen-Slavov. Our argument is based on Beilinson’s theory of singular supports for étale sheaves.

1. Introduction and statement of results

In this note, we prove a generalization of the following theorem due to Benoist and written here in a form due to [7].

Theorem 1.1 ([2], Théorème 1.4). Let $X \subset \mathbb{P}^N$ be a geometrically irreducible quasiprojective variety over a field $k$. Define $M_{\text{bad}} \subset \mathbb{P}^N$ as the locus of hyperplanes $H$ such that $X_H := X \cap H$ is not geometrically irreducible. Then $\text{codim } M_{\text{bad}} \geq \dim X - 1$.

This result is geometric, but has a cohomological reformulation: the top degree compactly supported cohomology groups of $X$ and $X_H$ are isomorphic. In this note, we prove a similar result for all cohomology of sufficiently high degree on $X$.

Theorem 1.2. Let $X$ be a separated scheme of finite type over a separably closed field $k$, let $\phi : X \to \mathbb{P}^n$ be a morphism, and let $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ for $\ell$ a prime power not divisible by the characteristic of $k$. Set $r = \dim X \times_{\mathbb{P}^n} X$.

Then for each $c \geq 1$, there is a closed subscheme $Z_c \subset \mathbb{P}^n$ of dimension at most $n - c$ such that for $H \in \mathbb{P}^n \setminus Z_c$ and $q > c + r$ (resp. $q = c + r$), the Gysin map

$$H^{q-2}_{c}(\phi^{-1}(H), \Lambda(-1)) \to H^{q}_{c}(X, \Lambda)$$

is an isomorphism (resp. surjective).

In the setting of Theorem 1.1, taking $q = 2\dim X$, we see that $M_{\text{bad}} \subset Z_{\dim X - 1}$, which has dimension $\leq n - \dim X + 1$, recovering the assertion of loc. cit.

We remind the construction of Gysin maps in Section 2.

Remark. If $X$, $\phi$, and $H$ are defined over an arbitrary field $k'$ with separable closure $k$, then our construction shows that $Z_c$ is naturally defined over $k'$. Moreover, a Gysin map $H^{q-2}_{c}(\phi^{-1}(H) \times_{\text{Spec}(k')} \text{Spec}(k), \Lambda(-1)) \to H^{q}_{c}(X \times_{\text{Spec}(k')} \text{Spec}(k), \Lambda)$ is compatible.
with the action of $\text{Gal}(k/k')$, so if $H \not\in Z_c$ is defined over $k'$ and $q$ is as in Theorem 1.2, the Gysin map is an isomorphism or surjection of Galois representations.

The main new tool in the proof of this result is Beilinson and Saito’s works [1, 8] on the singular support of constructible sheaves in arbitrary characteristic. In Section 2, we will use their work to prove a result, Theorem 2.2, that is at the core of the argument; we will also collect a couple lemmas we will need. In Section 3, we will use these tools to prove Theorem 1.2.

1.1. Past results. Theorem 1.2 generalizes a number of results beyond Theorem 1.1. Poonen-Slavov [7] establish the $q = 2 \dim X$ case of Theorem 1.2 under the additional assumption that $\phi$ has equidimensional fibers. If $\phi$ is the immersion of a normal projective complex variety and $c = 1$, Theorem 1.2 is Corollary 7.4.1 of [5], and is proven as a special case of the paper’s Lefschetz hyperplane theorem for intersection homology.

And if $\phi$ is the closed immersion of a smooth projective variety, this result is known; by [9, Theorem 2.1], there exists an isomorphism $H^{q-2}_c(\phi^{-1}(H), \Lambda(-1)) \to H_q^c(X, \Lambda)$ so long as $q \geq n + s + 3$, where $s$ is the dimension of the singular locus of $\phi^{-1}(H)$. So the locus where there exists no isomorphism $H^{q-2}_c(\phi^{-1}(H), \Lambda(-1)) \to H_q^c(X, \Lambda)$ is contained in the locus of hyperplanes such that $\phi^{-1}(H)$ is singular in dimension at least $q - n - 2$. This locus in turn has codimension at least $q - n - 1$.

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2. The Gysin map for bounded complexes

2.1. Notation and the basic setup. Let $k$ be a separably closed field, let $\ell$ be a prime power not divisible by the characteristic of $k$, and set $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$. For the remainder of this paper, sheaves will be constructible étale sheaves of $\Lambda$-modules, and for any variety $V$ we will use $D(V)$ to denote the bounded derived category of constructible sheaves of $\Lambda$-modules on $V$. Given a $k$-point $H \in \mathbb{P}^n$, let $i : H \to \mathbb{P}^n$ denote the corresponding inclusion, and let $j : \mathbb{P}^n \setminus H \to \mathbb{P}^n$ be the inclusion of the complement. For the remainder of the paper, all functors will be derived; for instance, we will use $j_*$ to denote the derived pushforward associated to $j$. Finally, we will use the notation $\mathbb{H}^q(\mathcal{F})$ to denote the hypercohomology of $\mathcal{F} \in D(V)$ in degree $q$.

On the level of sheaves, the map in Theorem 1.2 is induced by applying $R\Gamma$ to a composition of two arrows in $D(\mathbb{P}^n)$. Given a bounded complex of sheaves $\mathcal{F}$ on $\mathbb{P}^n$, the two maps we will use are:

- The counit map $i_!i^! \mathcal{F} \to \mathcal{F}$, which appears in the localization triangle
  
  \[ i_! i^! \mathcal{F} \to \mathcal{F} \to j_*j^* \mathcal{F} \xrightarrow{+1}. \]

- The Gysin map $i^* \mathcal{F}(-1)[-2] \to i^! \mathcal{F}$, as discussed in [4].
Applying $R\Gamma$ to these two maps produces maps

$$H^q(H, i^! F) \to H^q(F)$$

and

$$H^q-2(H, i^* F(-1)) \to H^q(H, i^1 F)$$

respectively. Our main effort will be proving that these maps are isomorphisms or surjections assuming certain hypotheses on $F$ and $H$.

### 2.2. The counit map.

Of the two maps above, the counit map is the easier one to understand, as its cone is $j^* j^* F$.

**Lemma 2.1.** Let $F \in D(\mathbb{P}^n)$. Fix an integer $r$. Suppose that for all $p$, the sheaf $\mathcal{H}^p(F)$ has support of dimension at most $r - p$. Then the sheaf $\mathcal{H}^p(j_* j^* F)$ has support of dimension at most $r - p$ and we have $H^q(\mathbb{P}^n, j^* j^* F) = 0$ for $q > r$.

**Proof.** Let $F$ and $r$ satisfy the hypotheses of the lemma. For each $p$, the support of $\mathcal{H}^p(j^* F) \cong j^* \mathcal{H}^p(F)$ has dimension at most $r - p$. Since $\mathbb{P}^n \setminus H$ is affine, we then have that affine vanishing [6, Theorem VI.7.3] implies that $R^s j_*(\mathcal{H}^p(j^* F))$ is supported in dimension at most $r - p - s$ and $H^s(\mathcal{H}^p(j^* F)) = 0$ if $s + p > r$. The results follow by applying a spectral sequence to the filtered complex $\tau_{\geq p} F$. □

**Remark.** The hypothesis of the lemma is equivalent to asking that $F$ sit in perverse degrees $\leq r$. From this perspective, the claim follows from right exactness of affine pushforwards with respect to the perverse $t$-structure.

### 2.3. The Gysin map.

The following result is the most nonstandard ingredient of the proof of Theorem 1.2.

**Theorem 2.2.** Let $F$ be an object of $D(\mathbb{P}^n)$, let $\Phi \subset \mathbb{P}^n \times \mathbb{P}^n$ be the universal hyperplane, and let $\pi_1, \pi_2$ denote the projections. Then there is a closed subscheme $Z \subset \Phi$ of dimension $\leq n - 1$ such that for $H \in \mathbb{P}^n$, the cone of the Gysin map

$$i^* F(-1)[-2] \to i_! F,$$

associated to $i : H \to \mathbb{P}^n$ is supported on $\pi_1(Z \cap \pi_2^{-1}(H))$.

In the proof of Theorem 1.2 we will use the following corollary of this theorem.

**Corollary 2.3.** With $F$ as in Theorem 2.2, for any positive integer $c$ there is a closed subscheme $Z_c \subset \mathbb{P}^n$ of dimension at most $n - c$ such that for any $H \in \mathbb{P}^n \setminus Z_c$ the cone of the Gysin map associated to $i : H \to \mathbb{P}^n$ is supported on a subscheme of dimension at most $c - 2$.

**Proof.** Take $Z$ as in Theorem 2.2. Let $Z_c \subset \mathbb{P}^n$ be the closed subscheme where the fibers of the projection $Z \to \mathbb{P}^n$ have dimension $\geq c - 1$. As $\dim Z = n - 1$, we have $\dim Z_c \leq n - c$, and it satisfies the conclusion by definition of $Z$. □
The theory of singular support for étale sheaves was developed by Beilinson [1] and Saito [8]. We use it in the following form.

**Theorem 2.4** (Theorem 1.3, [1]). Let $X$ be a smooth variety of dimension $n$ and let $\mathcal{F}$ be a bounded constructible complex on it.

Then there exists a closed, conical subscheme $SS(\mathcal{F}) \subset T^*X$ of dimension $n$ such that for every pair $h : U \to X$ smooth and $f : U \to Y$ a morphism to a smooth variety $Y$ such that $df(T^*Y) \cap dh(SS(\mathcal{F}))$ is contained in the zero section $U \subset T^*U$, the map $f$ is locally acyclic with respect to $h^*(\mathcal{F})$ (in the sense of [3], 2.12).

**Remark.** The most serious part of the theorem is the calculation of the dimension of $SS(\mathcal{F})$.

**Lemma 2.5.** Suppose $X$ is smooth and let $i : D \to X$ be the embedding of a smooth divisor. Suppose $\mathcal{F} \in D(X)$ has the property that the intersection

$$SS(\mathcal{F})|_D \cap N^*_X/D \subset T^*_X \times D$$

is contained in the zero section.

Then the Gysin map

$$i^*\mathcal{F}(-1)[-2] \to i^!\mathcal{F}$$

is an isomorphism.

**Proof.** This result is a special case of [8] Proposition 7.13. We outline a direct proof below.

The assertion is local on $X$, so we may assume there is a map $f : X \to \mathbb{A}^1$ with $D = f^{-1}(0)$.

By assumption, the image of $\mathbb{P}(SS(\mathcal{F}) \cap df(X)) \to X$ avoids $D$, so we may assume this intersection is zero. Then taking $U = X$ and $Y = \mathbb{A}^1$ in the definition of singular support, we see that $\mathcal{F}$ is locally acyclic with respect to $f$ by hypothesis, so its vanishing cycles $\phi_f(\mathcal{F})$ are 0.

In general, recall that we have exact triangles

$$i^*(\mathcal{F}) \to \psi_f(\mathcal{F}) \xrightarrow{can} \phi_f(\mathcal{F}) \xrightarrow{+1}$$

$$\phi_f(\mathcal{F}) \xrightarrow{\varphi} \psi_f(\mathcal{F})(-1) \to i^!(\mathcal{F})[2] \xrightarrow{+1}$$

such that the composition

$$i^*(\mathcal{F})(-1)[-2] \to \psi_f(\mathcal{F})(-1)[-2] \to i^!(\mathcal{F})$$

is the Gysin map. As $\phi_f(\mathcal{F}) = 0$ by the above, we obtain our claim. □
Proof of Theorem 2.2. Recall that $\Phi$ is canonically isomorphic to $\mathbb{P} \mathbb{T}^* \mathbb{P}^n$. Let $Z = \mathbb{P} SS(\mathcal{F})$ as in Theorem 2.4. By the theorem, it has dimension $n - 1$.

Fix $H \in \mathbb{P}^n$, and let $B \subset H$ be the subscheme $\pi_1(Z \cap \pi_2^-(H))$. By construction, the closed embedding $H \setminus B \to \mathbb{P}^n \setminus B$ satisfies the hypotheses of Lemma 2.5 so the cone of the Gysin map is supported on $B$ as desired.

3. Proof of Theorem 1.2

In this section, we continue to use notation from the previous section. Let $\phi : X \to \mathbb{P}^n$ be a morphism of separated schemes of finite type over $k$. In this section we establish Theorem 1.2 for the map $\phi$. Let $\Lambda_X$ denote the constant sheaf on $X$ with fiber $\Lambda$.

We first note that $R^p \phi_* \Lambda_X$ is supported at points over which $\phi$ has fiber dimension at least $\frac{n}{2}$, because the stalk of $R^p \phi_* \Lambda_X$ at a point $x$ is just $H^p_\phi(\phi^{-1}(X), \Lambda)$ by [10, Lemma 0F7L]. As a consequence, if $X \times_{\mathbb{P}^n} X$ has dimension $r$, we have that $H^p(\phi_* \Lambda_X)$ has support in dimension at most $r - p$. So by Lemma 2.1 we have that for any inclusion of a hyperplane $i : H \to \mathbb{P}^n$, the hypercohomology groups $\mathbb{H}^q(\mathbb{P}^n, j_* j^* \phi_* \Lambda_X)$ vanish for $q > r$. So we have that the counit map $\mathbb{H}^q(H, i^* \phi_* \Lambda_X) \to H^q_\phi(X, \Lambda)$ is an isomorphism if $q \geq r + 2$ and is a surjection if $q = r + 1$.

We now show that the Gysin map induces an isomorphism or surjection on cohomology. Fixing $c \geq 1$, let $Z_c$ be the exceptional set in Corollary 2.3 relative to the complex of sheaves $\phi_* \Lambda_X$. Fix some $k$-point $H \in \mathbb{P}^n \setminus Z_c$, and let $Q \in D(H)$ denote the cone of the morphism

$$i^* \phi_* \Lambda_X(-1)[-2] \to i^! \phi_* \Lambda_X.$$

By Corollary 2.3 $Q$ is supported on a closed subscheme $B$ of dimension at most $c - 2$. Moreover, $\mathcal{H}^p(j_* j^* \phi_* \Lambda_X)$ has support of dimension at most $\min(r - p, n)$ by Lemma 2.1. Since we also have that $R^p \phi_* \Lambda_X$ is supported in dimension $\min(r - p, n)$, the distinguished triangle $i^! \phi_* \Lambda_X \to \phi_* \Lambda_X \to j_* j^* \phi_* \Lambda_X \to$ gives the bound on the dimension of the support of $\mathcal{H}^p(i^! \phi_* \Lambda_X)$,

$$\dim \text{supp}(\mathcal{H}^p(i^! \phi_* \Lambda_X)) \leq \min(r - p + 1, n).$$

Likewise, $\mathcal{H}^p(i^* \phi_* \Lambda_X(-1)[-2])$ is supported on a set of dimension at most $\min(r - p + 2, n - 1)$. From these two observations and the defining triangle for $Q$, we see that $\mathcal{H}^p(Q)$ is supported on a subscheme of dimension at most $\min(r - p + 1, c - 2)$. Therefore, $\mathbb{H}^{q-p}(H, \mathcal{H}^p(Q)) = 0$ for $q - p > 2 \min(r - p + 1, c - 2)$. Observe that if $q - p \leq 2 \min(r - p + 1, c - 2)$, then $2(q - p) \leq 2(r - p + 1) + 2(c - 2)$, so $q \leq r + c - 1$. Therefore, whenever $q > r + c - 1$, $\mathbb{H}^{q-p}(H, \mathcal{H}^p(Q)) = 0$. Applying the Grothendieck spectral sequence, we find that $\mathbb{H}^q(H, Q) = 0$ for $q > r + c - 1$. 

From this, we conclude that the Gysin map
\[
H^q_c(\varphi^{-1}(H), \Lambda(-1)) \cong \mathbb{H}^{q-2}(H, i^*\varphi!\Lambda_X(-1)) \to \mathbb{H}^q(H, i^!\varphi!\Lambda_X)
\]
is an isomorphism for \(q > r + c\) and a surjection if \(q = r + c\). Combining this with the counit map above, and noting \(c \geq 1\), we have that the map on cohomology
\[
H^q_c(\varphi^{-1}(H), \Lambda(-1)) \to H^q_c(X, \Lambda)
\]
is an isomorphism for \(q > r + c\) and a surjection if \(q = r + c\), proving Theorem 1.2 for \(X\).

**Remark.** In the above setting, note that \(\varphi!(\Lambda_X) \in D(P^n)\) is in perverse degrees \(\leq r\). The argument shows that, suitably understood, the conclusion of Theorem 1.2 holds for any \(F \in D(P^n)\) a bounded complex of constructible sheaves in perverse degrees \(\leq r\).

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