Relating chronology protection and unitarity through holography

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We give a simple nonsupersymmetric example in which chronology protection follows from unitarity and the AdS/CFT correspondence. We consider a ball of homogeneous, rotating dust in global AdS$_3$ whose backreaction produces a region of Gödel space inside the ball. We solve the Israel matching conditions to find the geometry outside of the dust ball and compute its quantum numbers in the dual CFT. When the radius of the dust ball exceeds a certain critical value, the spacetime will contain closed timelike curves. Our main observation is that precisely when this critical radius is exceeded, a unitarity bound in the dual CFT is violated, leading to a holographic argument for chronology protection.

Kurt Gödel was the first to emphasize that Einstein’s equation, in the presence of seemingly innocuous matter sources, can lead to causality violating geometries containing closed timelike curves (CTCs) [1]. Since then, classical solutions with CTCs have popped up ubiquitously, including supersymmetric versions of Gödel space in supergravity theories, both in 3+1 dimensions as well as in their higher-dimensional parent theories [2]. Such spacetimes lead to a variety of pathologies, both within classical general relativity as well as for interacting quantum fields propagating on them (see [3] for a review and further references). This led Hawking to propose the Chronology Protection Conjecture, stating that regions containing CTCs cannot be formed in any physical process [4]. It is expected [5] that a fully consistent treatment of such a dynamical argument behind chronology protection requires the issue to be addressed in a quantum theory where both matter and gravity itself are quantized.

The AdS/CFT correspondence [6] proposes that combined quantum gravity and matter systems on anti-de-Sitter (AdS) spaces have a holographic dual description in terms of a unitary conformal field theory (CFT) in one lower dimension. It is therefore ideally suited to study the issue of chronology protection in asymptotically AdS spaces. Indeed, several examples are known [7] where the appearance of CTCs in a BPS sector of the bulk theory is quantum mechanically forbidden as it would correspond to the violation of a unitarity bound in the dual CFT. In this Letter, we show that a similar conclusion holds for 2+1 dimensional Gödel space and give a simple argument that creating a patch of Gödel space large enough to contain CTCs would require violating unitarity in the dual CFT. An important novel feature of our example is that it doesn’t rely on supersymmetry but only on the general properties of gravity theories on AdS$_3$ that were established in [8, 9]. In particular, our argument only relies on the fact that, in a unitary CFT, all states have nonnegative conformal weights.

Let us now outline our argument. Gödel’s original 3+1 dimensional solution is the product of a nontrivial 2+1 dimensional space and a line, and we will here consider only the 2+1 dimensional part, henceforth referred to as Gödel space. Gödel space is a solution of 2+1 dimensional anti-de-Sitter gravity with a source of homogeneous rotating dust. The dust needs to rotate in AdS$_3$ in order to be stationary, since otherwise it would collapse to form either a conical defect or a BTZ black hole depending on its total mass [10, 11]. We will consider a 2 dimensional ball (i.e. a disc) of such rotating dust placed in global AdS$_3$ and give a detailed analysis of the resulting geometry. The metric inside the ball is that of Gödel space, and the one outside a generalized BTZ metric [12] describing an object with mass and angular momentum which we determine by solving a matching problem (a similar 3+1 dimensional problem was considered in [13]). The AdS/CFT dictionary then tells us the quantum numbers of the dust ball in the dual CFT. When we vary the radius of the dust ball while keeping the energy density of the dust constant, CTCs appear when the radius exceeds a critical value. The main question we want to address is what this critical radius corresponds to in the dual CFT. We will see that it corresponds precisely to the unitarity bound stating that conformal weights in the CFT have to be nonnegative, implying that the formation of a dust ball containing CTCs is forbidden by unitarity.

THE STATIONARY DUST BALL SOLUTION

We will consider a combined gravity and matter system in AdS$_3$, where we will not specify the microscopic
matter content in detail. We assume that the matter sector can effectively produce a source of pressureless dust. Hence we will consider Einstein’s equation with negative cosmological constant $\Lambda = -1/l^2$:
\[ R_{ab} - \frac{1}{2} R g_{ab} - \frac{1}{l^2} g_{ab} = 8\pi G T_{ab}, \tag{1} \]
where $G$ is the 2+1 dimensional Newton constant, $l$ is the AdS$_3$ radius and we take
\[ T_{ab} = \frac{\rho}{2\pi l^2} u_a u_b, \tag{2} \]
with $u$ a unit timelike vector and $\rho$ a dimensionless number parameterizing the energy density.

We will now solve (1) for a homogeneous ball of rotating dust, where we take the energy density $\rho$ to be nonzero and constant inside the ball and zero outside. Inside the ball, the metric will be that of Gödel space, while outside we expect a metric of generalized BTZ type characterized by a mass $M$ and angular momentum $J$.

This leads to the following ansatz for our matching problem. On the inside, we have the Gödel space metric
\[ ds_+^2 = l^2 \left[ -(du + M dt + J d\phi) + u d\phi^2 + \frac{1}{(1 - r^2)^2} dr^2 \right], \tag{3} \]
where $r$ runs between 0 and $r_0 \leq 1$, the radius where the dust region ends. The angular coordinate $\phi$ is identified with period $2\pi$. The Einstein equations (1),(2) determine $\mu$ in terms of the density of the dust as
\[ \mu = \frac{1}{1 - \rho}. \tag{4} \]
The physical values are $\rho \geq 0$, for positive energy, and $\rho < 1$, for a Minkowski signature of the resulting metric. Note that for $\mu = 1$, ($\rho = 0$), the metric describes global AdS$_3$. When $r_0$ exceeds the critical value $1/\sqrt{\mu}$, CTCs appear since $\partial_\phi$ becomes a timelike vector in the region $r > 1/\sqrt{\mu}$.

Outside of the dust ball, we take take a metric ansatz which is a vacuum solution to (1) and which generalizes the BTZ metric:
\[ ds_-^2 = l^2 \left[ -(dt + \phi) + u d\phi^2 + \frac{1}{(1 - r^2)^2} dr^2 \right], \tag{5} \]
where $f(u) = u^2 - Mu + \frac{J^2}{4}$.

The angle $\tilde{\phi}$ is identified with period $2\pi$ and the real parameters $M, J$ are the ADM mass and angular momentum (in convenient units) respectively. The function $f$ is related to the determinant of the induced metric $h_{i\bar{j}}$ on a surface of constant $u$ by $\det h_{i\bar{j}} = -f$. Let us review the properties of this class of metrics in the various regions of $(J,M)$ parameter space.

In the region $M^2 \geq J^2, M \geq 0$, which we will call region I, the metrics (5) describe BTZ black holes [12]. The function $f$ has two positive real zeroes, which correspond to the inner and outer horizons. Unlike their 3+1 dimensional cousins, BTZ black holes have no curvature singularities. Instead, in the region $u < 0$ the space contains CTCs. This ‘singularity in the causal structure’ is hidden behind a horizon. In the region $u \geq 0$, the standard radial BTZ coordinate is related to $u$ as $u = \bar{r}_{\text{BTZ}}^2$.

For $M^2 \geq J^2, M < 0$, henceforth referred to as region II, the metric describes a spinning conical defect. The function $f$ has two negative real zeroes, between which the signature of the metric becomes Euclidean. One can verify that at the largest zero $u_-$, the metric has a conical singularity arising from a pointlike source. The range of the $u$-coordinate is $u \geq u_+$ and, as before, there are CTCs in the region where $u_+ \leq u < 0$. Both the CTC region and the defect singularity are ‘naked’ and not hidden behind a horizon. There is one exceptional point, namely $M = -1$, $J = 0$, for which the geometry becomes smooth global AdS$_3$. This special point corresponds to the conformally invariant vacuum state in the dual CFT.

And finally, for $M^2 < J^2$, denoted by region III, the metric describes an overspinning object. The function $f$ has no real zeroes and the metric is free of curvature singularities. The range of $u$ is the real line and the space contains a ‘naked’ CTC region for negative values of $u$.

Let us now briefly discuss part of the AdS/CFT dictionary. The Virasoro quantum numbers of the spaces (5) can be extracted following the standard procedure of computing the renormalized boundary stress tensor and extracting its Fourier coefficients [16]. One finds that these spaces correspond to states with conformal weights
\[ L_0 = \frac{c}{24}(M + J + 1), \tag{6} \]
\[ \tilde{L}_0 = \frac{c}{24}(M - J + 1). \tag{7} \]

The central charge of the CFT is given by [8]
\[ c = \frac{3l^2}{2G}. \tag{8} \]

Unitarity implies that conformal weights in the CFT are positive, leading to the bound $L_0 \geq 0, \tilde{L}_0 \geq 0$. In terms of $M$ and $J$, this is equivalent to
\[ M + 1 \geq |J|. \tag{9} \]

States violating this bound are forbidden by unitarity and, according to the AdS/CFT conjecture, cannot be part of the spectrum in a consistent quantum gravity theory on AdS$_3$.

We will now match the inside metric for $r \leq r_0$ to the outside metric for $u \geq u_0$ for arbitrary values of the parameters $\rho$ and $r_0$. Since both metrics have a single $2\pi$ identification on the coordinates $\phi, \tilde{\phi}$ respectively, the coordinates $t, \phi$ and $\tilde{t}, \tilde{\phi}$ have to be related as follows
\[ t = c_1 \tilde{t}, \tag{10} \]
\[ \phi = c_2 \tilde{\phi} + \bar{\phi}, \tag{11} \]
with $c_1, c_2$ two constants.

The Israel matching conditions [17] require that the metric and the extrinsic curvature are continuous across the edge of the dust ball [19]:

$$h_{ij}^- = h_{ij}^+, \quad (12)$$

$$K_{ij}^- = K_{ij}^+, \quad (13)$$

Here $h_{ij}^-(h_{ij}^+)$ is the induced metric on the $r = r_0$ ($u = u_0$) boundary surface and $K_{ij}^-$ are the corresponding extrinsic curvatures. The conditions (12), (13) give 6 equations for the five undetermined parameters $M, J, u_0, c_1, c_2$ in terms of the two physical parameters $\rho, r_0$ of the ball of dust. After some algebra one can check that there are 2 solutions:

$$J = \pm \frac{2\rho r_0^4}{(1-r_0^2)(1-\rho)^2}, \quad (14)$$

$$M = -\frac{(1-\rho^2) - 2r_0^2}{(1-r_0^2)^2(1-\rho)^2}, \quad (15)$$

$$u_0 = \frac{r_0^2(1-\rho-r_0)}{(1-r_0^2)^2(1-\rho)^2}, \quad (16)$$

$$c_1 = \pm \left(1 - \frac{2r_0^2\rho}{(1-r_0^2)(1-\rho)}\right), \quad (17)$$

$$c_2 = \pm 1. \quad (18)$$

The two solutions have opposite angular momentum and are related by a change of sign for the angular coordinates $\phi, \dot{\phi}$. We will fix this freedom in what follows by taking $J$ to be positive, choosing the positive sign in these equations.

**DISCUSSION**

Let us now discuss the physical properties of the matched solution (14)-(18). First we examine how the $(r_0, \rho)$ parameter space is mapped into the $(J, M)$ plane. As mentioned before $(r_0, \rho)$ take values in $[0, 1) \times [0, 1)$. Their relation to $(J, M)$ is clearly not onto and also not injective, as the Jacobian of the map vanishes at $\rho = \frac{1-r_0^2}{1+r_0^2}$. It turns out that there are two branches in the $(J, M)$ plane, depending on whether the dust density $\rho$ is greater or smaller than $\frac{1-r_0^2}{1+r_0^2}$. More precisely the two branches are

- **Branch A** ($\rho \leq \frac{1-r_0^2}{1+r_0^2}$): 

  $$J - 1 \leq M \leq -1 + 2\sqrt{2J - J} \quad \text{when} \quad J \leq \frac{1}{2}, \quad (19)$$

  $$J - 1 \leq M \leq J \quad \text{when} \quad J \geq \frac{1}{2}.$$

- **Branch B** ($\rho \geq \frac{1-r_0^2}{1+r_0^2}$):

  $$-1 + 2\sqrt{2J - J} \leq M \leq J. \quad (20)$$

These branches are shown in figure 1 and we will now discuss them in more detail.

Within branch A, for values of $\rho \leq \left(\frac{1-r_0^2}{1+r_0^2}\right)^2$ (i.e. in the region $A_1$ in figure 1), the outside metric is of type II (conical defect). Note that there are two limits where the outside becomes the AdS$_3$ vacuum. One is simply setting $r_0 = 0$. There is only an outside space in this case, as $u_0 = 0$ and $M = -1, J = 0$. This outside space covers all of global AdS$_3$. The second limit is taking the energy density of the dust to be zero, $\rho = 0$. Now both the inside and outside metric become a patch of global AdS$_3$, albeit in different coordinate systems. The gluing conditions (11), (16) simply reduce to the appropriate coordinate transformation relating the two coordinate systems. For values of $\rho$ and $r_0$ lying in the region $A_2$ in figure 1, the outside metric is of type III (overspinning object). Within branch B, the outside metric is always also of type III. On the line $\rho = \frac{1-r_0^2}{1+r_0^2}$ where branch A meets branch B, $M$ becomes equal to $J$ and the outside metric is formally of type I, describing an extremal BTZ black hole. The edge of the dust ball $u_0$ coincides precisely with the black hole horizon. Since on this line $c_1$ becomes zero (17), the redshift between the AdS time and the G"odel time becomes infinite and the gluing singular. Already in [11] it was observed that glueings to a dust region cannot give rise to outside metrics of type I.

Now we address the issue of closed timelike curves in our matched solutions. Remember that a priori we have two regions in which closed timelike curves can appear in the solution. The inside part of the metric (G"odel space) has CTCs when $r_0^2 > \frac{1}{\rho} = 1 - \rho$, the outside
metric when \( u_0 < 0 \). But observe that by (16) these two conditions are equivalent, hence either no closed timelike curves appear at all, or they appear both in the inside and outside parts of the metric. In parameter space these closed timelike curves can only appear in part of branch B, which is denoted as region \( B_2 \) in figure 1. Hence on branch B, CTCs can be made to appear by smoothly varying the parameters \( (\rho, r_0) \), while the solutions remain seemingly well behaved in all other respects.

Now let us discuss the unitarity bound (9), which gives an extra constraint on which outside metrics are physically acceptable. On branch A, all the outside metrics satisfy the unitarity bound, while branch B is divided into a region where the bound is satisfied and one where it is violated. In fact, the bound for the absence of CTCs \( r_0^2 \leq 1 - \rho \) precisely coincides with the unitarity bound \( M + 1 \geq |J| \). This can be seen directly as by (14), (15)

\[
M + 1 - |J| = \frac{4\rho r_0^2(1 - \rho - r_0^2)}{(1 - r_0^2)^2(1 - \rho)}.
\]

Hence the condition of unitarity is equivalent to that of the absence of closed timelike curves.

**OUTLOOK**

In this Letter, we have discussed an example where the appearance of CTCs in a Gödel region within AdS\(_3\) was shown to precisely coincide with the violation of a unitarity bound in the dual CFT. Based on our result and other examples in the literature [7], it would be natural to propose an AdS version of the Chronology Protection Conjecture, stating that regions with CTCs in AdS spaces cannot be formed as a result of any unitary process. The AdS/CFT correspondence could in principle be used to address whether this proposal is true in general. If so, it would be very interesting to gain insight into the deeper dynamical mechanism that prevents the formation of regions with CTCs, see e.g. [18] for some proposals in the context of string theory.

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[26] The singularity in higher curvature corrections due to the sharp edge of our dust ball can be resolved by smoothing the density over a length scale \( L \). If \( L \) is large compared to \( G \) but small compared to \( l \), both curvature corrections (suppressed by \( G/L \)) and deviations from our solution (suppressed by \( L/l \)) remain small.