On maximal $S$-free sets and the Helly number for the family of $S$-convex sets

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Abstract

Let $S$ be a subset of $\mathbb{R}^d$. A subset $K$ of $\mathbb{R}^d$ is said to be $S$-free if $K$ is closed, convex and the interior of $K$ is disjoint with $S$. An $S$-free set $K$ is said to be maximal if $K$ is not properly contained in another $S$-free set. We present a condition on $S$ which guarantees that every maximal $S$-free set is a polyhedron with at most $f$ facets, where the bound $f$ depends only on $S$. This condition on $S$ is formulated in terms of the Helly number for the family of $S$-convex sets. The presented result yields corollaries related to the cutting-plane theory from integer and mixed-integer optimization.

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1 Introduction

Let $d \in \mathbb{N}$ and $S \subseteq \mathbb{R}^d$. A subset $K$ of $\mathbb{R}^d$ is said to be $S$-free if $K$ is closed, convex and the interior of $K$ is disjoint with $S$. An $S$-free set $K$ in $\mathbb{R}^d$ is said to be maximal if there exists no $S$-free set properly containing $K$. Dey and Morán [7] have recently studied maximal $S$-free sets in the case that $S$ is the intersection of $\mathbb{Z}^d$ and a convex set. In particular, in [7] the following result was obtained.

**Theorem 1.** (Dey & Morán, [7, Theorem 3.4]). Let $S = \mathbb{Z}^d \cap C$, where $C \subseteq \mathbb{R}^d$ is convex. Then every $d$-dimensional maximal $S$-free set is a polyhedron with at most $2^d$ facets.

In the case $S = \mathbb{Z}^d$ Theorem [1] was formulated by Lovász [11, Theorem 3.4] (a proof can be found in [4, §2.2]). Various special cases of Theorem [1] were considered and used in [4, 5, 8, 10]. This note presents a theorem (Theorem [4]), which implies Theorem [1] and also yields the following mixed-integer analog of Theorem [1].

**Theorem 2.** Let $d, n \in \mathbb{N}$ and $S = (\mathbb{Z}^d \times \mathbb{R}^n) \cap C$, where $C \subseteq \mathbb{R}^d \times \mathbb{R}^n$ is convex. Then every $d$-dimensional maximal $S$-free set is a polyhedron with at most $2^d$ facets.
We remark that maximal $S$-free sets of dimension less than $d$ can be characterized in rather simple terms: a subset $K$ of $\mathbb{R}^d$ is maximal $S$-free and of dimension less than $d$ if and only if $K$ is a hyperplane and in both open subspaces determined by $K$ one can find points of $S$ lying arbitrarily close to $K$. Hence in what follows our considerations are restricted to the case of $d$-dimensional maximal $S$-free sets.

A natural question in the context of the cutting-plane theory (from integer and mixed-integer programming) is whether for a given $S$ one can find $f \geq 0$ such that every maximal $S$-free set is a polyhedron with at most $f$ facets. The motivation provided by the cutting-plane theory is based on the fact that maximal $S$-free sets for $S := \mathbb{Z}^d \cap C$ (where $C$ is convex) can be used as ‘cutting objects’ for generation of intersection cuts (see, for example, [1, 3, 10]). It is thus desirable to have an upper bound on the combinatorial complexity of such cutting objects.

Our question on the existence of an upper bound $f$ can be expressed in terms of a parameter $f(S)$, which we introduce as follows. If, for a given $S$, there exist maximal $S$-free sets which are not polyhedra or if there exist $d$-dimensional maximal $S$-free polyhedra with arbitrarily large number of facets we let $f(S) := +\infty$. If there exist no $d$-dimensional maximal $S$-free sets (e.g., for $S = \mathbb{R}^d$) we let $f(S) := -\infty$. In the remaining cases the set of $d$-dimensional maximal $S$-free sets is nonempty and consists of polyhedra whose number of facets is bounded in terms of $S$; in such cases we denote by $f(S)$ the largest possible number of facets in a $d$-dimensional maximal $S$-free polyhedron. Thus, we ask for conditions on $S$ which ensure $f(S) < +\infty$. The main message of this note is that there exists a strong relation between $f(S)$ and the Helly number associated to the family of $S$-convex sets.

**Definition 3.** Let $\mathcal{F}$ be a nonempty family of sets with $\mathcal{F} \neq \{\emptyset\}$. Then the Helly number $h(\mathcal{F})$ of $\mathcal{F}$ is defined to be the minimal $h \in \mathbb{N}$ such that the following implication holds: If $X$ is an arbitrary finite subfamily of $\mathcal{F}$ such that $X$ contains at least $h$ sets and every $h$-element subfamily of $X$ has nonempty intersection, then also $X$ has nonempty intersection. If no $h \in \mathbb{N}$ as above exists, we let $h(\mathcal{F}) := +\infty$. We also define the Helly number of $\{\emptyset\}$ by $h(\{\emptyset\}) := 0$.

A subset $A$ of $\mathbb{R}^d$ is called $S$-convex if $A = S \cap C$ for some convex subset $C$ of $\mathbb{R}^d$. The notion of $S$-convexity is reduced to the standard notion of convexity for $S = \mathbb{R}^d$ and to the notion of lattice-convexity for $S = \mathbb{Z}^d$. Let $h(S)$ denote the Helly number for the family of all $S$-convex sets. That is

$$h(S) := h\left(\left\{S \cap C : C \text{ is a convex subset of } \mathbb{R}^d\right\}\right).$$

The equality

$$h(\mathbb{R}^d) = d + 1$$

represents the classical theorem of Helly (see, for example, [13, Theorem 1.1.6]). Doignon [9, (4.2)] proved the equality

$$h(\mathbb{Z}^d) = 2^d,$$

which is the analog of Helly’s theorem for $\mathbb{Z}^d$-convex sets. See also [14, §16.5] for interpretation of (2) in terms of integer optimization. The result of Doignon and its special cases have often been rediscovered (see [6, 12, 15]). Based on [11] and [2] the authors of [2] showed

$$h(\mathbb{Z}^d \times \mathbb{R}^n) = (n + 1)2^d$$

(3)
for all \(d, n \in \mathbb{N}\). Equality (3) is the mixed-integer analog of Helly’s theorem.

Now we are ready to formulate our main result.

**Theorem 4.** Let \(S \subseteq \mathbb{R}^d\). Then \(f(S) \leq h(S)\).

As a consequence of Theorem 4 and (2) we obtain

**Theorem 5.** Let \(d, n \in \mathbb{N}\). Let \(A \subseteq \mathbb{R}^d\) and \(B \subseteq \mathbb{R}^d \times \mathbb{R}^n\) be convex. Then

\[
\begin{align*}
f(A \cap \mathbb{Z}^d) &\leq 2^d, \\
f(\mathbb{Z}^d) &\leq 2^d, \\
f(B \cap (\mathbb{Z}^d \times \mathbb{R}^n)) &\leq 2^d, \\
f(\mathbb{Z}^d \times \mathbb{R}^n) &\leq 2^d.
\end{align*}
\]

Comparing (5), (7) with (2), (3) we see that, for different choices of \(S\), in Theorem 4 one can have the equality \(f(S) = h(S)\) as well as the strict inequality \(f(S) < h(S)\).

Inequalities (4) and (6) represent the assertions of Theorems 1 and 2, respectively. The authors of [7] indicate that their proof of Theorem 1 is quite technical (see [7, p. 382, remark after Proposition 3.3]). In contrast to this, our arguments lead to a shorter and less technical proof of Theorem 1.

2 Proofs

We use standard terminology from the theory of polyhedra (see, for example, [14, Part III]). For \(n \in \mathbb{N}\) let \([n] := \{1, \ldots, n\}\). The standard scalar product of \(\mathbb{R}^d\) is denoted by \(\langle \cdot, \cdot \rangle\). By \(\text{int}\) we denote the interior with respect to the Euclidean topology of \(\mathbb{R}^d\).

**Lemma 6.** Let \(S \subseteq \mathbb{R}^d\) and \(f \in \mathbb{N}\). Assume that every \(d\)-dimensional \(S\)-free rational polyhedron \(P\) is contained in an \(S\)-free polyhedron \(Q\) with at most \(f\) facets. Then every \(d\)-dimensional maximal \(S\)-free set is a polyhedron with at most \(f\) facets.

**Proof.** Let \(K\) be an arbitrary \(d\)-dimensional maximal \(S\)-free set. It suffices to show that \(K\) is contained in an \(S\)-free polyhedron with at most \(h\) facets. We consider a sequence \((P_n)_{n=1}^{+\infty}\) of \(d\)-dimensional rational polytopes such that

\[
P_n \subseteq P_{n+1} \quad \forall n \in \mathbb{N}
\]

and

\[
\text{int}(K) = \bigcup_{n=1}^{+\infty} P_n.
\]

Such polytopes \(P_n\) can be constructed as follows. Let \((z_n)_{n=1}^{+\infty}\) be a sequence of all rational points of \(\text{int}(K)\) such that the first \(d+1\) points \(z_1, \ldots, z_{d+1}\) are affinely independent. Then, for every \(n \in \mathbb{N}\), we define \(P_n\) to be the convex hull of \(\{z_1, \ldots, z_{n+d}\}\).

By the assumption, each \(P_n\) is contained in an \(S\)-free polyhedron \(Q_n\) having at most \(f\) facets. Every \(Q_n\) can be represented by

\[
Q_n = \{x \in \mathbb{R}^d : \langle u_{1,n}, x \rangle \leq \beta_{1,n}, \ldots, \langle u_{f,n}, x \rangle \leq \beta_{f,n}\}
\]
where \( u_{1,n}, \ldots, u_{f,n} \in \mathbb{R}^d \) are vectors of unit (Euclidean) length and \( \beta_{1,n}, \ldots, \beta_{f,n} \in \mathbb{R} \). There exists an infinite subset \( \mathbb{N}_\infty \) of \( \mathbb{N} \) such that, for every \( i \in [f] \), the vector \( u_{i,n} \) converges to some unit vector \( u_i \) and \( \beta_i \) converges to some \( \beta_i \in (-\infty, +\infty) \), as \( n \) goes to infinity over points of \( \mathbb{N}_\infty \). We define the polyhedron

\[
Q := \left\{ x \in \mathbb{R}^d : \langle u_1, x \rangle \leq \beta_1, \ldots, \langle u_f, x \rangle \leq \beta_f \right\}.
\]

By construction, \( P_1 \subseteq P_n \subseteq Q_n \) for every \( n \in \mathbb{N} \). Hence \( P_1 \subseteq Q \), which shows that \( Q \) is \( d \)-dimensional. Let us show that \( Q \) is \( S \)-free. We assume the contrary. Then there exists \( x \in S \) belonging to \( \text{int}(Q) = \{ x \in \mathbb{R}^d : \langle u_1, x \rangle < \beta_1, \ldots, \langle u_f, x \rangle < \beta_f \} \). The latter implies \( \langle u_{i,n}, x \rangle < \beta_{i,n} \) for all \( i \in [f] \) if \( n \in \mathbb{N}_\infty \) is sufficiently large. This implies \( x \in S \cap \text{int}(Q_n) \) for all sufficiently large \( n \in \mathbb{N}_\infty \), contradicting the fact that \( Q_n \) is \( S \)-free. We also show \( \text{int}(K) \subseteq Q \) arguing by contradiction. If \( x \) is a point belonging to \( \text{int}(K) \) but not to \( Q \), then one can fix \( i \in [f] \) such that \( \langle u_i, x \rangle > \beta_i \). Consequently, \( \langle u_{i,n}, x \rangle > \beta_{i,n} \) for all sufficiently large \( n \in \mathbb{N}_\infty \). The inequality \( \langle u_{i,n}, x \rangle > \beta_{i,n} \) implies \( x \notin Q_n \) and, by this, \( x \notin P_n \). Thus, \( x \notin P_n \) for all sufficiently large \( n \in \mathbb{N}_\infty \). Since the sequence of \( P_n \)'s is monotone (as described by (3)), we get \( x \notin P_n \) for every \( n \in \mathbb{N} \). Consequently, \( x \notin \bigcup_{n=1}^{\infty} P_n \). In view of (9), we obtain \( x \notin \text{int}(K) \), which is a contradiction. We have verified the inclusion \( \text{int}(K) \subseteq Q \). Taking the closure of the left and the right hand side we arrive at \( K \subseteq Q \). This finishes the proof. \( \square \)

**Proof of Theorem 4** Let us first consider degenerate cases. If \( S = \emptyset \), we have \( h(S) = h(\{\emptyset\}) = 0 \). On the other hand, for \( S = \emptyset \), the whole space \( \mathbb{R}^d \) is the only maximal \( S \)-free set, and thus \( f(S) = 0 \). If \( S \) is nonempty we have \( h(S) \in \mathbb{N} \) or \( h(S) = +\infty \). In the case \( h(S) = +\infty \) the assertion is trivial. Now assume \( h(S) \in \mathbb{N} \).

Let us verify the assumption of Lemma 6 for \( f := h(S) \). Let \( P \) be an arbitrary \( d \)-dimensional \( S \)-free rational polyhedron in \( \mathbb{R}^d \). We represent \( P \) by \( P = H_1 \cap \cdots \cap H_n \), where \( n \in \mathbb{N} \) and \( H_1, \ldots, H_n \) are closed rational halfspaces. Then \( \text{int}(H_1) \cap \cdots \cap \text{int}(H_n) \cap S \) are \( S \)-convex sets whose intersection is empty. By the definition of the Helly number \( h(S) \), there exist indices \( i_1, \ldots, i_f \in [n] \) such that \( (\text{int}(H_{i_1}) \cap \cdots \cap \text{int}(H_{i_f})) \cap S = \emptyset \). It follows that \( P \subseteq Q := H_{i_1} \cap \cdots \cap H_{i_f} \), where \( Q \) is an \( S \)-free polyhedron with at most \( f \) facets. Thus, the assumption of Lemma 6 is fulfilled. Lemma 6 yields the assertion. \( \square \)

**Proof of Theorem 5** Directly from the definition of the Helly number it follows that for every \( S \subseteq \mathbb{R}^d \) and every convex set \( A \subseteq \mathbb{R}^d \) one has

\[
h(S \cap A) \leq h(S).
\]

(10)

Using Theorem 3 (10) and Doignon’s theorem (represented by (2)) we obtain \( f(A \cap \mathbb{Z}^d) \leq h(A \cap \mathbb{Z}^d) \leq h(\mathbb{Z}^d) = 2^d \), which shows (11).

For the verification of (5) it suffices to establish the existence of maximal \( \mathbb{Z}^d \)-free polyhedra with \( 2^f \) facets. Such polyhedra can easily be constructed. Let \( \| \cdot \|_1 \) be the \( l_1 \)-norm in \( \mathbb{R}^d \) and let \( c \) be the vector in \( \mathbb{R}^d \) whose components are all equal to \( 1/2 \). Let \( P := \{ x \in \mathbb{R}^d : \| x - c \|_1 \leq d/2 \} \). The polytope \( P \) is \( \mathbb{Z}^d \)-free since \( \| z - c \|_1 \geq d/2 \) for all \( z \in \mathbb{Z}^d \) and is maximal \( \mathbb{Z}^d \)-free since each of the \( 2^f \) facets of \( P \) is ‘blocked’ by a point from \( \{0, 1\}^d \). This shows \( f(\mathbb{Z}^d) \geq 2^f \) and yields (5).

For every \( S \subseteq \mathbb{R}^d \) the trivial equality \( f(S \times \mathbb{R}^n) = f(S) \) holds. The latter equality and Doignon’s theorem yield \( f(\mathbb{Z}^d \times \mathbb{R}^n) = f(\mathbb{Z}^d) = 2^d \), which verifies (7). Inequality (11) is a straightforward consequence of (7) and (10). \( \square \)
Remark 7. As can be seen from the proof of Theorem 4, the inequality \( f(S) \leq h(S) \) can be improved to
\[
f(S) \leq h \left( \{ \text{int}(P) \cap S : P \text{ is a rational polyhedron in } \mathbb{R}^d \} \right).
\]
Thus, in the proof of Theorem 5 it is sufficient to apply the following ‘rational’ version of Helly’s theorem:
\[
h \left( \{ \text{int}(P) \cap S : P \text{ is a rational polyhedron in } \mathbb{R}^d \} \right) = 2^d.
\]
(11)

Note that, for (11), replacing int\((P)\) by \(P\) does not change the family on the left hand side. A very short proof of (11) was given by Bell [6].

We also remark that (11) can be used to show \( h(\mathbb{Z}^d) = 2^d \) (the full version of Doignon’s theorem). This is done as follows. Let \( n \in \mathbb{N}, n \geq 2^d \) and let \( A_1, \ldots, A_n \) be convex sets in \( \mathbb{R}^d \) such that for all \( 1 \leq i_1 < \cdots < i_{2^d} \leq n \) one has \( A_{i_1} \cap \cdots \cap A_{i_{2^d}} \cap \mathbb{Z}^d \neq \emptyset \). Using (11) we now show \( A_1 \cap \cdots \cap A_n \cap \mathbb{Z}^d \neq \emptyset \). Consider the box \( B := [-N, N]^d \) with \( N > 0 \) large enough to guarantee that for all \( 1 \leq i_1 < \cdots < i_{2^d} \leq n \) one has \( A_{i_1} \cap \cdots \cap A_{i_{2^d}} \cap B \cap \mathbb{Z}^d \neq \emptyset \). Since, for every \( i \in [n] \), the convex hull of \( A_i \cap B \cap \mathbb{Z}^d \) \((i \in [n])\) is an integral polytope, one can determine rational polytopes \( P_1, \ldots, P_n \) such that, for every \( i \in [n] \), one has \( A_i \cap B \cap \mathbb{Z}^d = \text{int}(P_i) \cap \mathbb{Z}^d \). Applying (11) to the sets \( \text{int}(P_1) \cap \mathbb{Z}^d, \ldots, \text{int}(P_n) \cap \mathbb{Z}^d \) we deduce \( \text{int}(P_1) \cap \cdots \cap \text{int}(P_n) \cap \mathbb{Z}^d \neq \emptyset \). Hence \( A_1 \cap \cdots \cap A_n \cap \mathbb{Z}^d \neq \emptyset \). This implies \( h(\mathbb{Z}^d) \leq 2^d \). The inequality \( h(\mathbb{Z}^d) \geq 2^d \) follows by considering the family \( \mathcal{F} \) of 2^d sets \( \{0,1\}^d \setminus \{z\} \) with \( z \in \{0,1\}^d \). The elements of \( \mathcal{F} \) are \( \mathbb{Z}^d \)-convex, the intersection of \( \mathcal{F} \) is empty, and the intersection of every nonempty proper subfamily of \( \mathcal{F} \) is nonempty.

Remark 8. We indicate that the authors of [7] work under somewhat more general assumptions than the assumptions of Theorem 4. They consider the following sets:

- an affine subspace \( W \) of \( \mathbb{R}^n \) (where \( n \in \mathbb{N} \));
- a subset \( S \) of \( W \subseteq \mathbb{Z}^n \) such that \( S = \mathbb{Z}^d \cap C \) for some convex set \( C \subseteq W \);
- a closed, convex set \( K \) such that \( K \subseteq W \), the relative interior of \( K \) does not contain points of \( S \) and such that \( K \) is inclusion-maximal with respect to the above properties.

Also in this more general situation the polyhedrality of \( K \) and a bound on the number of facets of \( K \) can be determined using Theorem 4 and Doignon’s theorem. Let \( d \) be the dimension of \( W \). It suffices to consider the case that \( K \) is \( d \)-dimensional and \( S \neq \emptyset \). Let us fix a nonsingular affine transformation \( T : W \to \mathbb{R}^d \) which maps some point of \( W \cap \mathbb{Z}^d \) to the origin. Then \( \Lambda := T(W \cap \mathbb{Z}^d) \) is a lattice in \( \mathbb{R}^d \) and \( T(K) \) is a maximal \( T(S) \)-free set. Doignon’s theorem implies \( h(\Lambda) = 2^r \), where \( r \) is the rank of \( \Lambda \). Furthermore, \( h(T(S) \cap \Lambda) \leq h(\Lambda) \). Taking into account Theorem 4 we deduce that \( h(T(K)) \) (and, by this, also \( K \)) is a polyhedron with at most \( 2^r \) facets.

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