The Existence of Planar 4-Connected Essentially 6-Edge-Connected Graphs with No Claw-Decompositions

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Received: 28 June 2022 / Revised: 31 October 2022 / Accepted: 6 November 2022 / Published online: 5 December 2022
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Abstract
In 2006 Barát and Thomassen conjectured that every planar 4-edge-connected 4-regular simple graph of size divisible by three admits a claw-decomposition. Later, Lai (2007) disproved this conjecture by a family of planar graphs with edge-connectivity 4 which the smallest one contains 24 vertices. In this note, we first give a smaller counterexample having only 18 vertices and next construct a family of planar 4-connected essentially 6-edge-connected 4-regular simple graphs of size divisible by three with no claw-decompositions. This result provides the sharpness for two known results which say that every 5-edge-connected graph of size divisible by three admits a claw-decomposition if it is essentially 6-edge-connected or planar.

Keywords Modulo orientation · Claw-decomposition · Star-decomposition · Planar graph · Edge-connectivity

1 Introduction

In this article, graphs have no loops, but multiple edges are allowed, and a simple graph has neither loops nor multiple edges. Let $G$ be a graph. The vertex set, the edge set, and the maximum degree of vertices of $G$ are denoted by $V(G)$, $E(G)$, and $\Delta(G)$, respectively. We denote by $d_G(v)$, $d^-_G(v)$, and $d^+_G(v)$, the degree, the in-degree, and the out-degree of a vertex $v$ in the graph $G$. For a vertex set $A$, we denote by $e_G(A)$ the number of edges with both ends in $A$. An orientation of the graph $G$ is said to be $p$-orientation, if for each vertex $v$, $d^+_G(v) \equiv p(v)$, where $p : V(G) \to Z_k$ is a mapping and $Z_k$ is the cyclic group of order $k$. For the zero function $p$, the graph $G$ has a $p$-orientation if and only if it admits a $k$-star-decomposition. A graph is termed essentially $\lambda$-edge-connected, if the edges of any edge cut of size strictly less than $\lambda$ are incident with a common vertex.

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In 2006 Barát and Thomassen \[1\] conjectured that every planar 4-edge-connected 4-regular simple graph $G$ of size divisible by 3 admits a claw-decomposition. Lai \[9\] disproved this conjecture by a class of planar graphs with vertex-connectivity two. Lai and Li \[8\] proved the following stronger assertion for planar 5-edge-connected graphs, in terms of $Z_3$-connectivity, using the duality of planar graphs with graph colouring. A direct proof of Theorem 1.1 is found by Richter, Thomassen, and Younger \[12\] and this theorem is recently developed to projective planar graphs by de Jong and Richter \[5\].

**Theorem 1.1** ((\[8\])) Let $G$ be a planar graph and let $p : V(G) \to Z_3$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. If $G$ is 5-edge-connected, then it has a $p$-orientation.

**Corollary 1.2** (see Theorem 4.2 in \[12\]) Every 5-edge-connected planar graph of size divisible by 3 admits a claw-decomposition.

In 2012 Thomassen \[13\] developed Theorem 1.1 to 8-edge-connected graphs and succeeded to confirm another conjecture proposed by Barát and Thomassen \[1\] about the existence of claw-decomposition in graphs with high enough edge-connectivity. Next, Lovász, Thomassen, Wu, and Zhang (2013) improved Thomassen’s result to edge-connectivity 6. In particular, they proved a stronger version which contains the following result as a corollary on essentially edge-connected graphs.

**Theorem 1.3** ([10], see also Theorem 1.1 in [3]) Every 5-edge-connected essentially 6-edge-connected graph of size divisible by 3 admits a claw-decomposition.

In this note, we show that Barát and Thomassen’s conjecture \[1\] does not hold in planar 4-connected essentially 6-edge-connected simple graphs by giving a new family of 4-regular graphs with no claw-decompositions. This shows that (i) the needed edge-connectivity in Corollary 1.2 is best possible even for essentially 6-edge-connected graphs and (ii) the needed edge-connectivity in Theorem 1.3 is best possible even in planar graphs. In Sect. 3, we also give another family of graphs with high essential edge-connectivity but with no $k$-star-decompositions, and provide a useful criterion for the existence of $k$-star-decompositions in graphs with maximum degree at most $2k - 1$; in particular, the existence of claw-decompositions in 4-regular graphs.

**2 Claw-Decompositions in 4-Regular Graphs**

In 1992 Jaeger, Linial, Payan, and Tarsi \[4\] constructed a 4-edge-connected 4-regular simple graph of order 12 with no claw-decompositions (the third graph in Fig. 1). By a computer search, we observed that there are only four 4-regular connected simple graphs of order 12 with no claw-decompositions using a regular generator due to Meringer \[11\]. In addition, we observed that among 4-regular connected simple graphs of order 15 (resp. 18) there are only 146 (resp. 15932).
graphs without claw-decompositions which is less than 0.02% (resp. 0.002%) of them. Recently, Delcourt and Postle [3] proved that this ratio must tend to zero.

For planar graphs, Lai [9] introduced a family of planar 2-connected 4-edge-connected 4-regular simple graphs with no claw-decompositions of which the smallest one contains 24 vertices. By a computer search (using a planar graph generator due to Brinkmann and McKay [2]), we observed that there is a smaller such planar graph containing only 18 vertices, illustrated in Fig. 2.

Motivated by Theorem 1.3, one may ask whether there is such a planar graph with higher essential edge-connectivity or even vertex-connectivity. By searching among planar 3-connected 4-regular graphs of order 21, we discovered a number of such desired planar graphs. Some of them have vertex-connectivity 4 and some essential edge-connectivity 6; for example, see Fig. 3. According to Corollary 3.4, one can easily prove these graphs do not have a claw-decomposition using independent sets. For instance, the graph in Fig. 2 has independence number 5, the right graph in Fig. 3 has independence number 6, and the left graph in Fig. 3 has a unique independent set of size 7 (up to isomorphism) such that by removing it the resulting graph has a component with two cycles.

It remains to decide whether Theorem 1.1 holds for planar 4-connected essentially 6-edge-connected graphs, except for a finite number of graphs. We show that the answer is surprisingly false by the following graph construction.
Theorem 2.1 There are infinitely many planar 4-connected essentially 6-edge-connected 4-regular simple graphs of size divisible by 3 with no claw-decompositions.

Proof Consider $3n$ copies of the graph in Fig. 4 and for every $i \in \mathbb{Z}_{3n}$, add three edges $z_iz_{i+1}$, $x_ia_{i+1}$, and $y_ib_{i+1}$ to the new graph. Call the resulting graph $G_{48n}$ which has $48n$ vertices. Figure 5 illustrates the graph $G_{48}$. As observed in [1, 9], if a 4-regular graph $G$ has a claw-decomposition, then the non-center vertices form an independent set of size $|V(G)|/3$. If $G_{48n}$ has a claw-decomposition, then it must have an independent set $X$ of size $16n$. But $X$ includes at most 5 vertices from every block and hence it has at most $15n$ vertices which is a contradiction. The vertex connectivity and essentially edge-connectivity of $G_{48n}$ can easily be verified. The proof is left to the reader.

3 Graphs with High Essential Edge-Connectivity and Without $k$-Star-Decompositions

It is known that every $(2k-1)$-edge-connected essentially $(3k-3)$-edge-connected graph $G$ of size divisible by $k$ with $k \geq 3$ admits a $k$-star-decomposition, and there are $(2k-2)$-edge-connected $(2k-2)$-regular graphs of size divisible by $k$ with no $k$-star-decompositions, see [3, 7, 10]. Motivated by Theorem 2.1, we are going to show that there are regular graphs with the highest essential edge-connectivity but without $k$-star-decompositions.
Theorem 3.1 For any integer $k$ with $k \geq 3$, there are infinitely many $(2k - 2)$-connected essentially $(4k - 6)$-edge-connected $(2k - 2)$-regular simple graphs of size divisible by $k$ with no $k$-star-decompositions.

Proof We may assume that $k \geq 4$ as the assertion holds by Theorem 2.1 for the special case $k = 3$. Take $G$ to be the Cartesian product of the cycle of order $kn$ and the complete graph of order $2k - 3$, where $n$ is an arbitrary positive integer. It is not hard to check that $G$ is $(2k - 2)$-connected essentially $(4k - 6)$-edge-connected $(2k - 2)$-regular simple graph of size divisible by $k$. We claim that $G$ has no $k$-star-decompositions. Otherwise, the number of stars must be $(1 - 1/k)|V(G)|$, since $G$ contains $(k - 1)|V(G)|$ edges. Thus the number of non-center vertices must be $|V(G)|/k$ and these vertices form an independent set of $G$. On the other hand, according to the construction, the graph $G$ has independence number at most $|V(G)|/(2k - 3)$. Since $2k - 3 > k$, we arrive at a contradiction. $\square$

In the following, we are going to present a helpful criterion for the existence of $k$-star-decompositions in terms of independent sets. For this purpose, we need the following well-known theorem due to Hakimi [6].

Lemma 3.2 ([6]) Let $G$ be a graph and let $p$ be an integer-valued function on $V(G)$. Then $G$ has an orientation such that for all $v \in V(G)$, $d^{-}_G(v) \leq p(v)$, if and only if for all $A \subseteq V(G)$,

$$e_G(A) \leq \sum_{v \in A} p(v),$$

Now, we are ready to prove the next assertion.

Theorem 3.3 Let $G$ be a graph of size divisible by $k$ satisfying $\Delta(G) \leq 2k - 1$ which $k$ is an integer number with $k \geq 3$. The graph $G$ admits a $k$-star-decomposition if and only if it has an independent set $S$ of size $|V(G)| - \frac{1}{k}|E(G)|$ such that for every $A \subseteq V(G) \setminus S$, 

![Fig. 5 A planar 4-connected essentially 6-edge-connected 4-regular graph of order 48 with no claw-decompositions](image-url)
\[
e_G(A) \leq \sum_{v \in A} (d_G(v) - k).
\]

**Proof** First assume that there is an independent set \( S \) of size \(|V(G)| - |E(G)|/k\) satisfying the theorem. By Lemma 3.2, there is an orientation of \( G \setminus S \) such that every vertex of it has in-degree at most \( d_G(v) - k \). According to the assumption, we also have

\[
\sum_{v \in V(G) \setminus S} k = k(|V(G)| - |S|) = |E(G)| = \sum_{v \in V(G) \setminus S} d_G(v) - |E(G \setminus S)|.
\]

Therefore, \( G \setminus S \) has size \( \sum_{v \in V(G) \setminus S} (d_G(v) - k) \) and hence every vertex \( v \) of it has in-degree \( d_G(v) - k \). Let us orient the remaining edges from \( V(G) \setminus S \) to \( S \) to obtain an orientation for \( G \) so that every vertex in \( S \) has out-degree zero and every vertex in \( V(G) \setminus S \) has out-degree \( k \). Obviously, this orientation induces a \( k \)-star-decomposition for \( G \).

Now, assume that \( G \) has a \( k \)-star-decomposition. Obviously, the number of stars must be \(|E(G)|/k\). Since \( G \) has maximum degree at most \( 2k - 1 \), every vertex is the center of at most one star. Thus the number of center vertices must be \(|E(G)|/k\). If we set \( S \) to be the set of all non-center vertices, then this set must be an independent set and we must have \(|S| = |V(G)| - |E(G)|/k\). Let us orient the edges of \( G \) such that the edges of every star leave the center. This implies that every vertex \( v \in V(G) \setminus S \) has in-degree at most \( d_G(v) - k \) in \( G \) and so does in \( G \setminus S \). By Lemma 3.2, for every \( A \subseteq V(G) \setminus S \), \( e_G(A) \leq \sum_{v \in A} (d_G(v) - k) \). Hence the proof is completed. \( \square \)

The following corollary is a useful tool to show why the left graph in Fig. 3 does not have a claw-decomposition. More precisely, it has a unique independence set of size \(|V(G)|/3\) (up to isomorphism).

**Corollary 3.4** Let \( G \) be a 4-regular graph of size divisible by three. Then \( G \) admits a claw-decomposition if and only if it has an independent set \( S \) of size \(|V(G)|/3\) such that every component of \( V(G) \setminus S \) contains exactly one cycle.

**Proof** Apply Theorem 3.3 and use the fact that a graph \( H \) of size \(|V(H)|\) satisfies \( e_H(A) \leq |A| \) for every \( A \subseteq V(H) \), if and only if every component of it contains exactly one cycle. Note that if \( S \) is an independent set of size \(|V(G)|/3\), then the number of edges of \( G \setminus S \) must be \(|V(G)\setminus S|\). \( \square \)

**Funding** No funding was received for conducting this study.

**Data availability** Not applicable.

**Declarations**

**Conflict of interest** None.
Code availability  Not applicable.

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