COMPLEX $q$-ANALYSIS AND SCALAR FIELD THEORY ON A $q$-LATTICE

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Abstract

We develop the basic formalism of complex $q$-analysis to study the solutions of second order $q$-difference equations which reduce, in the $q \rightarrow 1$ limit, to the ordinary Laplace equation in Euclidean and Minkowski space. After defining an inner product on the function space we construct and study the properties of the solutions, and then apply this formalism to the Schrödinger equation and two-dimensional scalar field theory.

1 Introduction

One of the most recently researched aspects of quantum group applications to physics has been on the role they could play as symmetries in quantum field theory. For example, several developments on field theory with
compact quantum group internal symmetries can be found in the literature [1, 6]. Although quantum versions of Lorentz and Poincaré algebras have been given by several authors in refs. [2, 3], the search for an analytical formulation of field theory on quantum space-time is still an open problem [4]. A starting point taken by several authors has been the study of one-dimensional deformations of quantum mechanics based on $q$-deformations of the Heisenberg algebra. The analytical framework for these approaches is based on $q$-analysis [5] and the theory of basic hypergeometric functions. In addition, as discussed in references [6, 7], $q$-analysis provides a concrete realization of $q$-deformed quantum mechanics as standard quantum mechanics on a non-uniform lattice.

In this paper we extend the formalism given in Ref. [6] to two dimensions. In particular, we consider the simplest case, which is quantum mechanics and scalar field theory on a complex $q$-plane with commuting coordinates.

Thus, the formalism in this work gives a preliminary insight that could lead to more general formulations involving quantum plane (non-commuting) coordinates. At the beginning of Sec. 2 we introduce complex $q$-analysis by establishing a correspondence between the $q$-plane derivatives and difference operators, and then we proceed by defining an inner product consistent with the invariance of the coordinate algebra under the adjoint operation. In Sec. 3 we define a second-order difference operator whose eigenfunctions are orthonormal with respect to the inner product previously introduced. This result allows us to expand a general solution in terms of a $q$-analogue of the Fourier series. In Sec 4 we apply this formalism to the Schrödinger equation, and in Sec. 5 we construct the action which leads to the field equation reducing, in the $q \to 1$ limit, to a scalar field theory in two-dimensional Minkowski
space. In particular, the discreteness of the spectrum is exhibited for the case \( q \approx 0 \).

2 Complex \( q \)-analysis

We start by considering two commuting coordinates \( \hat{x}^i \) satisfying the following relations with differentials \( d\hat{x}^i \) and derivatives \( \hat{\partial}_i \)

\[
\begin{align*}
\hat{x}^i d\hat{x}^j &= p_i d\hat{x}^i \hat{x}^j \\
\hat{\partial}_i \hat{x}^j &= 1 + p_i \hat{x}^i \hat{\partial}_i \\
\hat{\partial}_i d\hat{x}^i &= p_i^{-1} d\hat{x}^i \hat{\partial}_i \\
d\hat{x}^i d\hat{x}^j &= -d\hat{x}^j d\hat{x}^i, \\
\end{align*}
\]

where the two parameters \( p_i \) are real numbers. One can introduce a \( q \)-complex number \( \hat{x} \equiv \hat{x}^1 \) and its adjoint \( \overline{\hat{x}} \equiv \hat{x}^2 \) if and only if \( p_1 = p_2^{-1} = q \).

One can check that the relations in Eq. (1) can be written involving two matrices \( B \) and \( C \) as follows

\[
\begin{align*}
\hat{x}^i \hat{x}^j &= B_{kl}^{ij} \hat{x}^k \hat{x}^l \\
\hat{\partial}_j \hat{x}^i &= \delta^i_j + C_{jk}^{il} \hat{x}^k \hat{\partial}_l \\
\hat{x}^i d\hat{x}^j &= C_{kl}^{ij} d\hat{x}^k \hat{x}^l \\
\hat{\partial}_j d\hat{x}^i &= (C^{-1})_{ik}^{jl} d\hat{x}^l \hat{\partial}_k, \\
\end{align*}
\]

where

\[
B = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - q & q & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\( \Box \)
and

\[ C = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \quad (7) \]

are a particular choice of the matrices discussed in ref. \[8\]. The well known consistency conditions given in ref. \[9\] are then fulfilled

\[ B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23} \quad (8) \]

\[ (B_{12} - 1)(C_{12} + 1) = 0 \quad (9) \]

\[ B_{12}C_{23}C_{12} = C_{23}C_{12}B_{23} \quad (10) \]

\[ C_{12}C_{23}C_{12} = C_{23}C_{12}C_{23}. \quad (11) \]

The invariance of the commutations relations in Equations (1) is restricted to the group of scale transformations

\[ t = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad (12) \]

with \([a, d] = 0\). Since the coordinate \(\hat{x}\) and its adjoint commute, we can now identify \(\hat{x}\) with the usual complex coordinate \(w\), such that to the adjoint \(\overline{\hat{x}}\) will correspond the complex conjugate \(x^*\). Thus, the \(q\)-derivatives \(\hat{\partial} \equiv \hat{\partial}_1\), \(\overline{\hat{\partial}} \equiv \hat{\partial}_2\) and differentials \(d\hat{x}, d\overline{\hat{x}}\) are realized on the usual complex plane in terms of \(q\)-difference operators according to the following relations

\[ \hat{\partial} : D_+ = w^{-1} \frac{1 - T}{1 - q} \quad (13) \]

\[ \overline{\hat{\partial}} : D^*_+ = w^{* -1} \frac{1 - T^{* -1}}{1 - q^{-1}} \quad (14) \]

\[ d\hat{x} : d_q w^{-1} T^{-1} \quad (15) \]

\[ d\overline{\hat{x}} : d_q w^{*} T^{*} \quad (16) \]

\[ 4 \]
where \( T = q^{w\partial_w} \) is the scaling operator and hereafter \( 0 < q < 1 \). Equations (1) written in terms of the variables \( w \) and \( w^* \) are consistent if we can define an inner product such that the adjoint of the \( q \)-difference operators and \( q \)-differentials satisfy

\[
D_+^* = -q^{-1}D_-^*
\]  
(17)

\[
\overline{d_q w} \propto d_q w^*.
\]  
(18)

In fact, Equation (17) is satisfied if we define our inner product in terms of a double \( q \)-integral

\[
< \phi, \psi > \equiv \int d_q w d_q w^* \phi^* \psi,
\]  
(19)

provided that the space functions satisfy the appropriate boundary conditions such that the boundary term

\[
\int d_q w d_q w^* D_+^* \left( \phi^* T^s - 1 \psi \right) = 0.
\]  
(20)

This inner product corresponds to the simplest two-dimensional generalization of the one defined in [10]. By introducing an independent complex variable \( z \) on a linear lattice and considering the coordinate \( w \) as a function \( w(z) = q^z \), Eq. (19) can be interpreted as an equation taking values on a non-uniform lattice (\( q \)-lattice) of step size \( \Delta w(z) \equiv w(z) - w(z+1) = (1-q)w(z) \). Explicitly, Eq. (19) reads

\[
< \phi, \psi > = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta w_n \Delta w_m^* \phi^*(w_n, w_m^*) \psi(w_n, w_m^*),
\]  
(21)

where \( w_n \equiv q^n w \) is the point after either \( n \) scalings in the \( q \)-lattice or \( n \) unit translations in the linear lattice. In fact, for \( q = e^{-\lambda} \) very close to one we can approximate at first order in \( \lambda \) as \( q \approx 1 - \lambda \) and the \( q \)-difference operators become proportional to difference operators on a linear lattice.
3 Orthogonal functions

Starting with the $q$-difference operators discussed in the previous section, there is no unique second order $q$-difference operator one could write which would lead to the ordinary laplacian as $q \to 1$. Some aspects of this lack of uniqueness have been discussed in [10] wherein a hermitian free hamiltonian was found to be given by the operator

$$h_0 = -x^{-1}[x\partial]x^{-1}[x\partial_x],$$

(22)

with $[x\partial] \equiv T^{1/2} - T^{-1/2}/q^{1/2} - q^{-1/2}$. Eigenfunctions of this operator are the basic exponentials $E(\sqrt{q}; x)$ defined by the formula [11]

$$E(\sqrt{q}; x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}.$$  

(23)

But in contrast to the (continuous) $q = 1$ case and the one (linear lattice) with exponentials with integer momentum, the basic exponentials are not orthogonal with respect to an inner product of the type given in Eq.(19). In fact, the basic sine $S(x)$ and cosine $C(x)$ functions defined by

$$E(\sqrt{q}, ix) = C(x) + iS(x),$$

(24)

satisfy in the interval $[-x,t, xt]$ the following orthogonality relations

$$\int_{-x}^{xt} d_q x S(\kappa x) S(\kappa' x) = M(\kappa, x) \delta_{\kappa, \kappa'}$$

(25)

$$\int_{-x}^{xt} d_q x S(\kappa x) C(\kappa' x) = 0$$

(26)

$$\int_{-x}^{xt} d_q x C(\sqrt{q}\kappa x) C(\sqrt{q}\kappa' x) = N(\sqrt{q}\kappa, x) \delta_{\kappa, \kappa'},$$

(27)

where $M$ and $N$ denote normalizations and the allowed values of $\kappa$ are those which satisfy $S(\kappa xt) = 0$. The generalization of the operator in Eq. (22)
to the two-dimensional euclidean case should lead in the $q \to 1$ limit to the laplacian $\Delta = \partial_w \partial_{w^*}$. Although this generalization is in principle non unique, we see that according to Eqs.(25),(26) and (27) its eigenfunctions should correspond to the $q$-analogue of either $\sin(\kappa^* w + \kappa w^*)$ or $\cos(\kappa^* w + \kappa w^*)$. In order to obtain a $q$-analogue of these functions we define the symbol

$$(a, b)^{(n)} = \sum_{m=0}^{n} \frac{[n]!}{[m]![n-m]!} a^m b^{n-m} = (b, a)^{(n)},$$

which can be checked to satisfy the following difference equation

$$a^{-1}[a \partial_a] (a, b)^{(n)} = [n](a, b)^{(n-1)}. \tag{29}$$

This symbol vanishes for $b = -aq^{\pm 1/2}$ with $n$ even, and $b = -a$ with $n$ odd. In particular

$$(a, b)^{(1)} = a + b, \tag{30}$$

and

$$(a, b)^{(2)} = a^2 + [2]ab + b^2 = (aq^{-1/2}, b)^{(1)}(aq^{1/2}, b)^{(1)} \tag{31}$$

is a $q$-analogue of the binomial theorem. Therefore, the function

$$S(\kappa w^*, \kappa^* w) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n (\kappa w^*, \kappa^* w)^{(2n+1)}}{[2n+1]!}, \tag{32}$$

is an eigenfunction of the operator

$$\nabla^2_q \equiv w^{-1}[w \partial_w] w^{-1} [w^* \partial_{w^*}], \tag{33}$$

satisfying the difference equation

$$- \nabla^2_q S(\kappa w^*, \kappa^* w) = \kappa \kappa^* S(\kappa w^*, \kappa^* w). \tag{34}$$
The orthogonality of these functions follow from the simple identity

\[ S(a, b) = S(a)C(b) + C(a)S(b), \]  
(35)

and the orthogonality relations in Eqs. (25), (26) and (27). This function can be generalized to include more variables. For example, introducing a new symbol

\[ ((a, b), (c, d))^{(m)} = \sum_{m=0}^{n} \frac{[n]!}{[m]![n-m]!} (a, b)^{(m)} (c, d)^{(n-m)}, \]  
(36)

one can define the function

\[ S((a, b), (c, d)) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n ((a, b), (c, d))^{(2n+1)}}{[2n+1]!}, \]  
(37)

such that

\[ S((a, b), (c, d)) = S(a, b)C(c, d) + C(a, b)S(c, d). \]  
(38)

In the two-dimensional case, restricting the values of \( \kappa \) to those which satisfy \( S(\kappa w^*) = 0 \) we write

\[ \int_{-w^*}^{w^*} \int_{-w^*}^{w^*} D_q w^* D_q w S(\kappa w^*, \kappa^* w) S(\kappa' w^*, \kappa'^* w) = 2M(\kappa, w^*)N(\kappa^*, w^*)\delta_{\kappa, \kappa'}, \]  
(39)

where the values of \( M \) and \( N \) have to be computed numerically for a particular choice of \( q \). Since the zeros of the function \( S(x) \) are real, \( \kappa w^* = \kappa^* w^* \), these normalizations satisfy \( M(\kappa, w^*)N(\kappa^*, w^*) = M(\kappa^*, w^*)N(\kappa, w^*) \). On the other hand, the identity

\[ C(a, b) = C(a)C(b) - S(a)S(b), \]  
(40)

shows that if \( \kappa w^* \) is a zero of either \( S(x) \) or \( C(x) \), the functions \( C(\kappa w^*, \kappa^* w) \) are not orthogonal. Thus, the only \( q \)-orthonormal eigenfunctions of the
operator in Eq. (33) are given by the set
\[ f_\kappa(w, w^*) = \frac{1}{\sqrt{2M(\kappa, w^*)N(\kappa^*, w^*')}} S(\kappa w^*, \kappa^* w^*). \] (41)

From these properties, we can formally expand an arbitrary function
\[ F(w, w^*) \] as a q-analogue of a Fourier series
\[ F(w, w^*) = \sum_{\kappa > 0} A(\kappa) f_\kappa(w, w^*), \] (42)
where the coefficients \( A(\kappa) \) are given by the formula
\[ A(\kappa) = \int_{-w^*}^{w^*} \int_{-w^*}^{w^*} dq dw q w^* F(w, w^*) f_\kappa(w, w^*). \] (43)

4 Quantum Mechanics

In this section, based on the discussion in Section (3), we solve the Schrödinger equation and show the closure property of its solutions. Introducing a standard time variable, the time dependent Schrödinger equation becomes the following difference equation
\[ -\nabla_q^2 \Psi(w, w^*, t) = i\partial_t \Psi(w, w^*, t). \] (44)

The corresponding stationary state solutions
\[ f_\kappa(w, w^*) e^{-iE_\kappa t} \] (45)
are written in terms of the q-orthonormal functions given in Equation (11).

As already discussed in the previous section, the set of \( \kappa \) values are those which make the functions \( f_\kappa \) to vanish at the boundary points. Then, a general solution of the time dependent equation can be expanded as
\[ \Psi(w, w^*, t) = \sum_{\kappa > 0} c_\kappa f_\kappa(w, w^*) e^{-iE_\kappa t}, \] (46)
with eigenvalues $E_{\kappa} = \kappa \kappa^*$. According to Equation (43) the coefficients are given by the inverse transformation

$$c_{\kappa} = 2 \sum_{n,m=0}^{\infty} \Delta w_n \Delta w_m^* \Psi(w_n, w_m^*, 0) f_{\kappa}(w_n, w_m^*) + 2 \sum_{n,m=0}^{\infty} \Delta w_n \Delta w_m^* \Psi(w_n, -w_m^*, 0) f_{\kappa}(w_n, -w_m^*), \quad (47)$$

such that replacing back into Equation (46), a simple algebraic manipulation leads to the closure relation for the functions $f_{\kappa}$

$$\sum_{\kappa>0} f_{\kappa}(w_{n}, w_{m}^*) f_{\kappa}(w_{r}, w_{s}^*) = \frac{\delta_{n,r} \delta_{m,s}}{2 \Delta w_n \Delta w_m^*}. \quad (48)$$

The kernel $K(w_{n}, w_{m}^*; w_{r}, w_{s}^*; t)$ is therefore given by

$$K(w_{n}, w_{m}^*; w_{r}, w_{s}^*; t) = \sum_{\kappa>0} f_{\kappa}(w_{n}, w_{m}^*) f_{\kappa}(w_{r}, w_{s}^*) e^{iE_{\kappa}t}. \quad (49)$$

and relates solutions of the free equation according to

$$\Psi(w_{r}, w_{s}^*; t) = \sum_{n,m=0}^{\infty} \Delta w_n \Delta w_m^* K(w_{n}, w_{m}^*; w_{r}, w_{s}^*; t) \Psi(w_n, w_m^*; 0). \quad (50)$$

5 Scalar Field Theory

In this section we apply the formalism discussed in sections 2 and 3 to study some basic aspects of scalar field theory on a $q$-lattice. In particular, we are interested in the case which corresponds in the $q \rightarrow 1$ limit to a field theory in two-dimensional Minkowski space. In order to resolve the problem of uniqueness, it is natural to require that the field equation be derived from an action, in a similar way as it is done for the $q = 1$ case. Therefore, based on the discussion in Section 2, we define a exterior derivative $d$ in terms
of the right derivatives $D_+$ and $D^+_*$ as follows

$$d = d_q w D_+ + d_q w^* D^+_*. \quad (51)$$

The lagrangian density $\mathcal{L}$ for a real scalar field can therefore be defined in terms of the usual Hodge operation $*d_q w = id_q w^*$ such that

$$d_q w d_q^* w^* \mathcal{L} \equiv -iq^{1/2} d\phi * d\phi, \quad (52)$$

and the action becomes

$$S = \frac{q^{1/2}}{2} \int \int d_q w d_q^* w^* \left[ [D_+ \phi(w, w^*)]^2 + [D^+_* \phi(w, w^*)]^2 \right]$$

$$= \frac{q^{1/2}}{2} \sum_{n,m=0}^\infty \Delta w_n \Delta w^*_m \left[ \left( \frac{\Delta \phi(w_n, w^*_m)}{\Delta w_n} \right)^2 + \left( \frac{\Delta^* \phi(w_n, w^*_m)}{\Delta w^*_m} \right)^2 \right], \quad (53)$$

where it is understood that $S$ is being valued at the boundaries. Variation of the action with respect to $\phi$ results in the field equation

$$- \left[ \left( w^{-1}[w \partial_w] \right)^2 + \text{c.c.} \right] \phi = 0, \quad (54)$$

corresponding in the $q \to 1$ limit to the flat space equation $(-\partial_x^2 + \partial_y^2) \phi = 0$.

A solution of the field equation (54) can be written as a sum in terms of the orthonormal sets

$$g_\kappa(w, w^*) = \frac{1}{\sqrt{M}} S(\kappa w, \kappa^* w^*), \quad (55)$$

and

$$h_\kappa(w, w^*) = \frac{1}{\sqrt{N}} \tilde{C}(\kappa w, \kappa^* w^*), \quad (56)$$

where the normalizations are written in terms of the values $M$ and $N$, as displayed in Eqs. (25) and (27). The function $\tilde{C}(a,b)$ is obtained by conveniently modifying the function $C(a,b)$ in Eq. (40) such that

$$\tilde{C}(a,b) = C(\sqrt{qa})C(\sqrt{qb}) - S(a)S(b). \quad (57)$$
Therefore, we define the real field $\phi$

$$\phi(w, w^*) = \sum_{\kappa \geq 0} [A_\kappa g_\kappa(w, w^*) + B_\kappa h_\kappa(w, w^*)],$$

(58)

where the values of $\kappa$ satisfy $S(\kappa w^t) = 0$. The real coefficients $A_\kappa$ and $B_\kappa$ can be written in terms of the inverse transformations

$$A_\kappa = \int_{-w^t}^{w^t} \int_{-w^*}^{w^*} d_q w d_q w^* g_\kappa(w, w^*) g_\kappa^*(w, w^*),$$

(59)

and

$$B_\kappa = \int_{-w^t}^{w^t} \int_{-w^*}^{w^*} d_q w d_q w^* h_\kappa(w, w^*) h_\kappa^*(w, w^*).$$

(60)

Without loss of generality we can define

$$\kappa = \frac{k_0 - i k_1}{\sqrt{2}} \equiv \frac{\alpha}{w^t},$$

(61)

where $\alpha \in \mathbb{R}$ are zeros of $S(x)$. The condition $\kappa^2 + \kappa^*2 = 0$ relates the boundary points such that $w^t = iw^t$. As a concrete example, we consider the case $q \approx 0$. In this case, a very good approximation to the zeros of $S(x)$ and $C(x)$ is given [11] respectively by the set of values

$$\alpha_m = \frac{q^{-m+1/4}}{1-q},$$

(62)

and

$$\beta_m = \frac{q^{-m+3/4}}{1-q},$$

(63)

where $m = 1, 2, 3, \ldots$. From the definition of the function $\tilde{C}$ in Eq. (57) and the relation between the zeros $\beta_m = \sqrt{q} \alpha_m$, we see that the expansion for the field at the lattice point $(w_n^t, w_m^*)$, $n \leq m$, reduces to the finite sum

$$\phi(w_n^t, w_m^*) = \sum_{l=1}^{m} \left[ A_l S \left( \frac{q^{-l+1/4}(q^m, q^n)}{1-q} \right) + B_l \tilde{C} \left( \frac{q^{-l+1/4}(q^m, q^n)}{1-q} \right) \right],$$

(64)
and therefore $\phi(w, w^*) = 0$. The corresponding discrete spectrum $k_0$ is given by

$$k_0 = \pm \frac{q^{-l+1/4}}{\sqrt{2|w^*|(1 - q)}}.$$  

(65)

6 Discussion

In this letter, we have generalized the formalism of real $q$-analysis to the complex case. We constructed the solutions of second order $q$-difference operators which in the $q \rightarrow 1$ limit reduce to the laplacians in Euclidean and Minkowski space. In order to understand the meaning of the adjoint operation and therefore to study the basic properties of these solutions we defined, generalizing a previous work [10], an inner product as a double $q$-integral. This inner product introduces a $q$-lattice structure and it specifies the type of solutions that satisfy the orthogonality and closure properties. We have seen that given the solutions for any of the two $q$-difference operators here considered, most of these solutions will fail to form an orthogonal set. This case contrasts with the case of a linear lattice, $q \approx 1$, wherein the sine, cosine and exponential functions are all invariant under lattice translations, therefore sharing the orthogonality and closure properties. We illustrated our results by considering the Schrödinger equation for the Euclidean case and scalar field theory for the Minkowski case. From the point of view of noncommutative geometry, this work provides a preliminary basis to search for an analytical approach to quantum mechanics and field theory involving quantum plane coordinates. On the other hand, by taking several copies of the coordinate pair $(w, w^*)$, one could generalize this work to a higher dimensional $q$-lattice with the goal that it could be used as a tool to regularize standard field theory. It would be of much interest to check if some of
the problems that arise when one defines a Dirac action on a linear lattice still remain when a more general geometry such as a $q$-lattice is considered.

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