NON-LOCAL INITIAL PROBLEM FOR SECOND ORDER
TIME-FRACTIONAL AND SPACE-SINGULAR EQUATION

ERKINJON KARIMOV, MURAT MAMCHUEV, AND MICHAEL RUZHANSKY

Abstract. In this work, we consider an initial problem for second order partial
differential equations with Caputo fractional derivatives in the time-variable and
Bessel operator in the space-variable. For non-local boundary conditions, we present
a solution of this problem in an explicit form representing it by the Fourier-Bessel
series. The obtained solution is written in terms of multinomial Mittag-Leffler
functions and first kind Bessel functions.

1. Introduction and formulation of a problem

It is well-known that partial differential equations are playing a key role in con-
structing mathematical models for many real-life processes. Especially, during the
last decades, many applications of various kinds of fractional differential equations
became target of intensive research due to both theoretical and practical reasons, see
e.g. [6] for an overview. Many kinds of boundary problems, including direct [11]
and inverse problems [4], were formulated for different type of PDEs of integer order
including several differential operators of fractional order.

We note works [3]-[9] devoted to studying partial differential equations with mul-
tiple Caputo derivatives. Precisely, in [9] the authors studied fractional differential
equations with Caputo fractional derivatives and using the operational method, solu-
tions of initial boundary problem for those equations were obtained in an explicit form
involving a multinomial Mittag-Leffler function. Certain properties of this function
were obtained by Li, Liu and Yamamoto [7] and applied to studying initial-boundary
problems for time-fractional diffusion equations with positive constant coefficients.
Later, Liu [8] established strong maximum principle for fractional diffusion equa-
tions with multiple Caputo derivatives and investigated a related inverse problem.
Daftardar-Gejji and Bhalikar [3], using the method of separation of variables, solved
some boundary-value problems for multi-term fractional diffusion-wave equation.

We also note works related to the Bessel operator. In [10], the initial inverse
problem for the heat equation with Bessel operator was investigated. Inverse initial
and inverse source problems for time-fractional diffusion equation with zero order
Bessel operator were recently studied in [2]. Direct and inverse problems for PDEs
containing two-term time fractional Caputo derivatives of orders up to 1, and Bessel
operator of order \( \nu \) were investigated in [1].

Date: February 15, 2018.

1991 Mathematics Subject Classification. 35R11, 33E12.
Key words and phrases. fractional derivatives, Cauchy problem, Bessel operator.

The last author was supported in parts by the EPSRC grant EP/K039407/1 and by the Lever-
hulme Grant RPG-2014-02. No new data was collected or generated during the course of research.
The consideration of non-local initial conditions is often justified by practical usage in certain real-life processes. For instance, when the initial temperature for the heat equation is not given instantly, but there is an information related with the temperature on a certain time interval that can be described by a non-local initial condition in a simple form. Boundary-value problems with non-local initial conditions were considered in works [12] for reaction-diffusion equations, in [16] for heat equation, in [13]-[14] for degenerate parabolic equations, and in [5] for a mixed parabolic equation.

For integer orders much more is known, and for a review of different questions of time decay of solutions for hyperbolic equations with integer order derivatives we can refer to [15].

In the present work we deal with the non-local initial boundary problem for multi-term time fractional PDE with Bessel operator of order \( \nu \). We use Fourier-Bessel series expansion in order to find the explicit solution for the considered problem, yielding also its existence. Because of the singularities in the Bessel operator such conditions appear naturally in space variables.

We note that most of the current literature deals with diffusion type equations considering time-derivatives of orders up to 1, see e.g. [7]. One of the novelties of the present paper is that we consider wave type equations allowing fractional derivatives of order up to 2, with additional fractional dissipation type terms. If there are multiple fractional time-derivative terms, a multinomial Mittag-Leffler function appears in the representation of solutions.

Let us now describe the problem in more detail. We consider the equation

\[
L(u) - B_\nu(u) = f(t, x)
\]  

in a rectangular domain \( D = \{(x, t) : 0 < x < 1, \ 0 < t < T\}, \ T > 0 \), where \( f(t, x) \) is a given function,

\[
L(u) = \partial_{0t}^\alpha u(t, x) - \sum_{i=1}^{n} \lambda_i \partial_{0t}^{\alpha_i} u(t, x)
\]

is the time component of the equation, with orders

\[ 0 < \alpha_i \leq 1, \ \alpha_i \leq \alpha \leq 2, \ n \in \mathbb{N}, \ \lambda_i \in \mathbb{R}, \]

and

\[
B_\nu(u) = u_{xx}(t, x) + \frac{1}{x} u_x(t, x) - \frac{\nu^2}{x^2} u(t, x)
\]

is the Bessel part of the equation with \( \nu > 0 \). Here

\[
\partial_{0t}^\alpha g(t) = \begin{cases} 
1 & \text{for } \alpha \in \mathbb{N}, \\
\frac{1}{\Gamma(k - \alpha)} \int_0^t \frac{g^{(k)}(z)}{(t - z)^{\alpha-k+1}} dz, & \alpha \notin \mathbb{N}_0, \\
\frac{d^k g(t)}{dt^k}, & \alpha = 0,
\end{cases}
\]

is a fractional differential operator of Caputo type, where \( k = [\alpha] + 1 \), and \( [\alpha] \) is the integer part of \( \alpha \). We can refer to [6] for further details on the Caputo fractional derivative operators.
The non-local initial boundary problem for equation (1.1)-(1.3) is formulated as follows:

**Problem:** Let \( M \in \mathbb{R} \). To find a solution \( u(t, x) \) of equation (1.1)-(1.3) in \( D \), which satisfies

(i) regularity conditions \( u \in W \) with

\[
W = \left\{ u(t, x) : u \in C(\overline{D}), \; u_{xx}, \partial_{\alpha}^n u \in C(D), \; \int_0^1 \sqrt{\int u(t, x)} \; dx < +\infty \right\}; \tag{1.4}
\]

(ii) boundary and non-local initial conditions

\[
\lim_{x \to 0} x u_x(t, x) = 0, \; u(t, 1) = 0, \tag{1.5}
\]

\[
u(0, x) + M u(T, x) = 0, \; 0 \leq x \leq 1, \; [\alpha] \cdot u_t(0, t) = 0, \; 0 < x < 1. \tag{1.6}
\]

2. **Main result**

The main result of this note is the following well-posedness theorem for the initial problem (1.1)-(1.6). The interesting part are the conditions on \( f \) allowing one to handle the singularities in the coefficients of the Bessel operator, and the non-resonance conditions (2.1) relating the parameter \( M \) with the length \( T \) of the time interval, coefficients and fractional orders of time-derivatives, through the multinomial Mittag-Leffler function.

**Theorem 2.1.** Assume that

- \( f(x, t) \) is differentiable four times with respect to \( x \);
- \( f(0, t) = f'(0, t) = f''(0, t) = f'''(0, t) = 0 \), \( f(1, t) = f'(1, t) = f''(1, t) = 0 \);
- \( \partial_x f(x, t) \) is bounded;
- \( f(x, t) \) is continuous and continuously differentiable with respect to \( t \),

and non-resonance conditions

\[
M \neq -\frac{1}{E_{(\alpha-a_1, \alpha-a_2, \ldots, \alpha)\cdot 1}(\lambda_1 T^{\alpha-a_1}, \ldots, \lambda_n T^{\alpha-a_n}, -\gamma_k^2 T^\alpha)} \tag{2.1}
\]

hold for all \( k = 1, 2, \ldots \). Then there exists a unique solution of the problem (1.1)-(1.6), and it can be written in the following form:

\[
u(t, x) = \sum_{k=1}^{\infty} \left[ \int_0^T z^{a-1} E_{(\alpha-a_1, \alpha-a_2, \ldots, \alpha-a_n, \alpha)}(\lambda_1 z^{a-a_1}, \ldots, \lambda_n z^{a-a_n}, -\gamma_k^2 z^\alpha) f_k(t - z) dz - \right.
\]

\[
\left. \frac{M}{1 + ME_{(\alpha-a_1, \alpha-a_2, \ldots, \alpha)\cdot 1}(\lambda_1 T^{\alpha-a_1}, \ldots, \lambda_n T^{\alpha-a_n}, -\gamma_k^2 T^\alpha) \times \right.
\]

\[
\times \int_0^T z^{a-1} E_{(\alpha-a_1, \alpha-a_2, \ldots, \alpha-a_n, \alpha)}(\lambda_1 z^{a-a_1}, \ldots, \lambda_n z^{a-a_n}, -\gamma_k^2 z^\alpha) f_k(T - z) dz +
\]

\[
+ E_{(\alpha-a_1, \alpha-a_2, \ldots, \alpha)\cdot 1}(\lambda_1 T^{\alpha-a_1}, \ldots, \lambda_n T^{\alpha-a_n}, -\gamma_k^2 T^\alpha) J_{\nu}(\gamma_k x). \tag{2.2}
\]
The numbers $\gamma_k$ and the functions appearing in the formula (2.2) are explained in Section 2.1. Briefly, here $E_{(\alpha_{-}\alpha_{1},\alpha_{-}\alpha_{2},\ldots,\alpha_{-}\alpha_{n},\alpha)}(\cdot)$ is the multinomial Mittag-Leffler function, $J_\nu$ are the first kind Bessel functions, $\gamma_k$ are their positive zeros, $\lambda_k$ are coefficients in the operator (1.2), and $f_k$ are Bessel expansions of $f$.

For the proof of Theorem 2.1 we start finding a formal solution in a series form and the convergence of the appearing series will be shown in Section 2.2.

2.1. Representation of a solution. We start by a formal discussion of the representation of solutions in (2.2). Let

$$J_\nu(z) = \sum_{i=0}^{\infty} \frac{(-1)^i (z/2)^{2i+\nu}}{i!(i+\nu)!}$$

be the Bessel function of the first kind (see e.g. [18]). It is known that for $\nu > 0$, the Bessel function $J_\nu(z)$ has countably many zeros, moreover, they are real and have pairwise opposite signs. Denote the $k^{th}$ positive root of the equation $J_\nu(z) = 0$ by $\gamma_k$, $k = 1, 2, \ldots$. For large $k$, we have (see [17])

$$\gamma_k \simeq k\pi + \frac{\nu\pi}{2} - \frac{\pi}{4}.$$

We now expand functions $u(t, x)$ and $f(t, x)$ in the Fourier-Bessel series (see e.g. [17]), writing them in the form

$$u(t, x) = \sum_{k=1}^{\infty} U_k(t) J_\nu(\gamma_k x),$$  
$$f(t, x) = \sum_{k=1}^{\infty} f_k(t) J_\nu(\gamma_k x),$$

where

$$U_k(t) = \frac{2}{J_{\nu+1}^2(\gamma_k)} \int_0^1 u(x, t) x J_\nu(\gamma_k x) \, dx,$$

$$f_k(t) = \frac{2}{J_{\nu+1}^2(\gamma_k)} \int_0^1 f(x, t) x J_\nu(\gamma_k x) \, dx.$$

It is known that if a function $g = g(x)$ is piecewise continuous on $[0, l]$ and satisfies

$$\int_0^l \sqrt{x}|g(x)| \, dx < +\infty,$$

then for $\nu > -1/2$, the Fourier-Bessel series converges at every point $x_0 \in (0, l)$, see e.g. [17]. Since we are looking for a function $u(t, x) \in W$, it satisfies these required conditions in order to be represented by a Fourier-Bessel series.

We substitute (2.3)-(2.4) into equation (1.1) and obtain the eigenvalue problem

$$L(U_k) + \gamma_k^2 U_k(t) = f_k(t).$$  

(2.7)
According to [9], the solution for equation (2.7) satisfying initial conditions
\[ U_k(0) = A, \quad [\alpha]U'_k(0) = 0, \] (2.8)
can be represented in the form
\[
U_k(t) = \int_0^t z^{\alpha-1} E_{(\alpha-\alpha_1, \alpha-\alpha_2, ..., \alpha-\alpha_n, \alpha)}(\lambda_1 z^{\alpha-\alpha_1}, ..., \lambda_n z^{\alpha-\alpha_n}, -\gamma_k z^\alpha) \times \\
\times f_k(t - z) dz + A U_0(t), \tag{2.9}
\]
where
\[
\overline{U}_0(t) = 1 + \sum_{i=1}^{n} \lambda_i t^{\alpha-\alpha_i} E_{(\alpha-\alpha_1, \alpha-\alpha_2, ..., \alpha-\alpha_n, \alpha)}(1+a_1 t^{\alpha-\alpha_1}, ..., a_n t^{\alpha-\alpha_n}, -\gamma_k t^\alpha) - \\
- \gamma_k t^\alpha E_{(\alpha-\alpha_1, \alpha-\alpha_2, ..., \alpha-\alpha_n, \alpha)}(1+a_1 t^{\alpha-\alpha_1}, ..., a_n t^{\alpha-\alpha_n}, -\gamma_k t^\alpha), \tag{2.10}
\]
and
\[
E_{(a_1, a_2, ..., a_n), b}(z_1, z_2, ..., z_n) = \sum_{k=0}^{\infty} \sum_{l_1 + l_2 + ... + l_n = k \atop l_1 \geq 0, ..., l_n \geq 0} \frac{k!}{l_1! l_2! ... l_n!} \prod_{i=1}^{n} z_i^{l_i} \Gamma(b + \sum_{i=1}^{n} a_i l_i) \tag{2.11}
\]
is the multinomial Mittag-Leffler function ([9]). From (2.9) we find that
\[
U_k(T) = \int_0^T z^{\alpha-1} E_{(\alpha-\alpha_1, \alpha-\alpha_2, ..., \alpha-\alpha_n, \alpha)}(\lambda_1 z^{\alpha-\alpha_1}, ..., \lambda_n z^{\alpha-\alpha_n}, -\gamma_k z^\alpha) \times \\
\times f_k(T - z) dz + A \overline{U}_0(T). \tag{2.12}
\]
Considering \(U_k(0) = A\), from the first condition in (1.5), we get the relation
\[ A + M U_k(T) = 0. \]
Using (2.12) we find \(A\) to be
\[
A = -\frac{M}{1 + M \overline{U}_0(T)} \int_0^T z^{\alpha-1} E_{(\alpha-\alpha_1, \alpha-\alpha_2, ..., \alpha-\alpha_n, \alpha)}(\lambda_1 z^{\alpha-\alpha_1}, ..., \lambda_n z^{\alpha-\alpha_n}, -\gamma_k z^\alpha) \times \\
\times f_k(T - z) dz.
\]
Substituting this value of $A$ into (2.9), we rewrite it as

$$U_k(t) = \int_0^t z^{a-1} E_{(a-1, a-2, \ldots, a-n, a)}(\lambda_1 z^{a-\alpha_1}, \ldots, \lambda_n z^{a-\alpha_n}, -\gamma_k^2 z^\alpha) f_k(t - z) dz - \frac{MU_0(t)}{1 + MU_0(T)} \int_0^T z^{a-1} E_{(a-1, a-2, \ldots, a-n, a)}(\lambda_1 z^{a-\alpha_1}, \ldots, \lambda_n z^{a-\alpha_n}, -\gamma_k^2 z^\alpha) \times f_k(T - z) dz.$$ (2.13)

If we use the formula (see [8])

$$1 + \sum_{j=1}^{n+1} \lambda_j t^{a-\alpha_j} E_{(a-1, a-2, \ldots, a-n+1)}(1 + a - \alpha_j, \lambda_1 t^{a-\alpha_1}, \ldots, \lambda_n t^{a-\alpha_n+1}) = E_{(a-1, a-2, \ldots, a-n+1)}(1, \lambda_1 t^{a-\alpha_1}, \ldots, \lambda_n t^{a-\alpha_n+1}),$$

representation (2.10) can be rewritten as

$$U_0(t) = E_{(a-1, a-2, \ldots, a-n+1)}(1, \lambda_1 t^{a-\alpha_1}, \ldots, \lambda_n t^{a-\alpha_n}, -\gamma_k^2 t^\alpha).$$

Denoting

$$F_k(t) = \int_0^t z^{a-1} E_{(a-1, a-2, \ldots, a-n, a)}(\lambda_1 z^{a-\alpha_1}, \ldots, \lambda_n z^{a-\alpha_n}, -\gamma_k^2 z^\alpha) f_k(t - z) dz,$$

$$U(t) = E_{(a-1, a-2, \ldots, a)}(1, \lambda_1 t^{a-\alpha_1}, \ldots, \lambda_n t^{a-\alpha_n}, -\gamma_k^2 t^\alpha),$$ (2.14)

we rewrite the function (2.13) as

$$U_k(t) = F_k(t) - \frac{M}{1 + MU_0(T)} F_k(T) U_0(t).$$ (2.15)

We note that the above expression is well-defined in view of the non-resonance conditions (2.11), that is

$$M \neq -\frac{1}{U_0(T)} = -\frac{1}{E_{(a-1, a-2, \ldots, a)}(1, \lambda_1 T^{a-\alpha_1}, \ldots, \lambda_n T^{a-\alpha_n}, -\gamma_k^2 T^\alpha)}$$

holds for all $k$. Finally, based on (2.15), we rewrite our formal solution as

$$u(t, x) = \sum_{k=1}^{\infty} \left[ F_k(t) - \frac{M}{1 + MU_0(T)} F_k(T) U_0(t) \right] J_{\nu}(\gamma_k x).$$ (2.16)

2.2. Justification of formal solution. In this section we prove convergence of the obtained infinite series corresponding to functions $u(t, x)$, $u_{xx}(t, x)$ and $\partial^2_{\nu} u(t, x)$.

In order to prove the convergence of these series, we use the estimate for the Mittag-Leffler function (2.11), obtained in [7] Lemma 3.2, of the form

$$|E_{(a-1, a-2, \ldots, a-n, \rho)}(z_1, z_2, \ldots, z_n)| \leq \frac{C}{1 + |z_1|}.$$ (2.17)
Let us first prove the convergence of series (2.16). For this, we collect several other known estimates. First, we use the following theorem on the estimate of the Fourier-Bessel coefficient:

**Theorem 2.2 ([17, p. 231])**. Let \( f(x) \) be a function defined on the interval \([0, 1]\) such that \( f(x) \) is differentiable \(2s\) times \((s \in \mathbb{N})\) and such that

1. \( f(0) = f'(0) = \ldots = f^{(2s-1)}(0) = 0; \)
2. \( f^{(2s)}(x) \) is bounded (this derivative may not exist at certain points);
3. \( f(1) = f'(1) = \ldots = f^{(2s-2)}(1) = 0. \)

Then the following inequality is satisfied by the Fourier-Bessel coefficients of \( f(x) \):

\[
|f_k| \leq c \gamma_k^{2s-1/2} k (c = \text{const}), \tag{2.18}
\]

where \( \gamma_k \) is the \(k\)th positive zero of the function \( J_\nu(x) \).

In our case we have

\[ f_k(t) = F_k(t) - \frac{M}{1 + MU_0(T)} F_k(T) U_0(t). \]

According to (2.14), using estimate (2.17) and imposing conditions (1)-(3) of Theorem 2.2 in the case \( s = 1 \) on the function \( f(t, x) \), we get

\[
\left| F_k(t) - \frac{M}{1 + MU_0(T)} F_k(T) U_0(t) \right| \leq \frac{C_1}{\gamma_k^{3/2}}. \tag{2.19}
\]

The series (2.15) then converges absolutely and uniformly on \([0, 1]\) in view of the following theorem:

**Theorem 2.3 ([17, p. 225])**. If \( \nu \geq 0, C > 0, \) and if the constants \( c_k \) satisfy

\[
|c_k| \leq \frac{C}{\gamma_k^{1+\varepsilon}},
\]

for some \( \varepsilon > 0 \), then the series

\[
c_1 J_\nu(\gamma_1 x) + c_2 J_\nu(\gamma_2 x) + c_3 J_\nu(\gamma_3 x) + \ldots
\]

converges absolutely and uniformly on \([0, 1]\).

According to [17, Theorem 2, p. 236], sufficient condition for differentiating the series (2.3) twice term by term, i.e. for the validity of the relation

\[
u_{xx}(t, x) = \sum_{k=1}^{\infty} U_k(t) \gamma_k^2 J''_\nu(\gamma_k x) \tag{2.20}
\]

will be

\[
|U_k(t)| \leq \frac{C_2}{\gamma_k^{5/2+\varepsilon}}, \tag{2.21}
\]

where \( U_k \) are as in (2.15), and \( C_2 \) and \( \varepsilon \) are positive constants. Based on this estimation and considering also (see [17, p. 233])

\[
|J_\nu(\gamma_k x)| \leq \frac{C_3}{\sqrt{\gamma_k x}} (C_3 = \text{const}),
\]

we deduce (see e.g. [17], p. 236)

\[ |U_k(t)\gamma_k^2 J''_\nu(\gamma_k x)| \leq \frac{C_2}{\gamma_k^{2+\epsilon}x^2} + \frac{C_3(|\nu| + \nu^2)}{\gamma_k^{3+\epsilon}x^2\sqrt{x}} + \frac{C_4}{\gamma_k^{1+\epsilon}\sqrt{x}}, \]

which provides the convergence of the series (2.20). We note that if we impose conditions on the given function \( f(x, t) \) of the form

- \( f(x, t) \) is differentiable four times with respect to \( x \);
- \( f(0, t) = f'(0, t) = f''(0, t) = f'''(0, t) = 0, \ f(1, t) = f'(1, t) = f''(1, t) = 0; \)
- \( \frac{\partial^4 f(x, t)}{\partial x^4} \) is bounded,

one can see that Theorem 2.3 implies estimation (2.21).

The convergence of series corresponding to \( \partial^4_\nu u(x, t), \ u_x(x, t) \) can be shown in similar ways, completing the proof of Theorem 2.1.

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Erkinjon Karimov:
Institute of Mathematics
National University of Uzbekistan
Tashkent, 100125
Uzbekistan
E-mail address erkinjon@gmail.com

Murat Mamchuev:
Department of Theoretical and Mathematical Physics
Institute of Applied Mathematics and Automation
Shortanova str. 89-A, Nalchik, 360000
Kabardino-Balkar Republic
Russia
E-mail address mamchuev@rambler.ru

Michael Ruzhansky:
Department of Mathematics
Imperial College London
180 Queen’s Gate, London, SW7 2AZ
United Kingdom
E-mail address m.ruzhansky@imperial.ac.uk