Fast Min-Sum Algorithms for Decoding of LDPC over $GF(q)$

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Abstract—In this paper, we present a fast min-sum algorithm for decoding LDPC codes over $GF(q)$. Our algorithm is different from the one presented by David Declercq and Marc Fossorier in [1] only at the way of speeding up the horizontal scan in the min-sum algorithm. The Declercq and Fossorier’s algorithm speeds up the computation by reducing the number of configurations, while our algorithm uses the dynamic programming instead. Compared with the configuration reduction algorithm, the dynamic programming one is simpler at the design stage because it has less parameters to tune. Furthermore, it does not have the performance degradation problem caused by the configuration reduction because it searches the whole configuration space efficiently through dynamic programming. Both algorithms have the same level of complexity and use simple operations which are suitable for hardware implementations.

I. INTRODUCTION

LDPC (low density parity check) codes are the state of art technology [2], [3] for their near Shannon limit performance for channel error correction [4]. China has considered it for broadcasting digital video for terrestrial televisions. Europe has accepted for its next generation broadcasting digital video using satellites (DVB-S2). LDPC codes have also been considered by many industry standards such as IEEE 802.16 and IEEE 802.11n. The LDPC codes defined over Galois field $GF(q)$ of order $q > 2$ have shown significant improved performance over binary LDPC codes.

David Declercq and Marc Fossorier presented in [1] a fast min-sum algorithm for decoding LDPC codes over $GF(q)$. It is a generalization of the normalized/offset min-sum algorithm from the Galois field $GF(2)$ [2], [3] to any Galois field, $GF(q)$ for any $q > 2$. The Declercq and Fossorier’s algorithm has much less complexity than another generalization of the min-sum algorithm given in [5]. Their algorithm speeds up the computation by reducing the number of configurations evaluated at the horizontal scan of the min-sum algorithm.

In this paper, we present another min-sum algorithm which improves the Declercq and Fossorier’s one only at the horizontal scan. We use the dynamic programming technique to speed up the horizontal scan instead of reducing the number of configurations. Both techniques have the same level of complexity. The latter finds approximate solutions at the horizontal scan which may introduce some performance degradation, while the former finds exact solutions and does not cause any performance degradation. The former is also simpler to design because it does not need to tune the balance between the configuration reduction and performance.

II. GENERALIZED MIN-SUM ALGORITHMS FOR DECODING LDPC CODES OVER $GF(q)$

A. The Problem Statement

LDPC codes belong to a special class of linear block codes whose parity check matrix $H$ has a low density of ones. For a LDPC code over $GF(q)$, its parity check matrix $H$ has elements $h_{mn}$ defined over $GF(q)$, $h_{mn} \in GF(q)$. Let the code word length be $N$ (the number of symbols), then $H$ is a $M \times N$ matrix, where $M$ is the number of rows. Each row of $H$ introduces one parity check constraint on input data $x = (x_1, x_2, \ldots, x_N)$, i.e.,

$$\sum_{n=1}^{N} h_{mn} x_n = 0, \text{ for } m = 1, 2, \ldots, M.$$ 

Putting the $m$ constraints together, we have $Hx^T = 0$.

Let function $f_n(x_n)$ be defined as

$$f_n(x_n) = -\ln p(x_n/y_n),$$

where $p(x_n/y_n)$ is the conditional distribution of input data symbol $n$ at value $x_n$ given the output data symbol $y_n$. $f_n(0) - f_n(x_n)$, which is equal to $\ln (p(x_n/y_n)/p(0/y_n))$, is the log-likelihood ratio (LLR) of input data symbol $n$ at value $x_n$ versus value 0.

In these notations, the maximum likelihood decoding can be formulated as a constrained optimization problem,

$$\min_{x_1, x_2, \ldots, x_N} \sum_{n=1}^{N} f_n(x_n) \quad \text{s.t. (subject to) } Hx^T = 0. \quad (1)$$

The above function is called the objective function for decoding a LDPC code. The decoding problem is, thus, transferred as finding the global minimum of a multi-variate objective function.

Let $X$ be the set of all variables. Given the $m$th constraint be $H_{mn} x_n = 0$, let $X_m$ be the subset of variables corresponding to the non-zero elements in $H_m$, i.e.,

$$X_m \equiv \{x_n | h_{mn} \neq 0\}.$$
Let \( f_{X_m}(X_m) \) be a function defined over \( X_m \) as

\[
    f_{X_m}(X_m) = \begin{cases} 
        0, & \text{if } H_m x^T = 0; \\
        \infty, & \text{otherwise.}
    \end{cases}
\]

\( f_{X_m}(X_m) \) is called the constraint function representing the \( m \)th constraint. Using the constraint functions, the decoding problem can be reformulated as an unconstrained combinatorial optimization problem of the following objective function,

\[
    E(x) = \sum_{m=1}^{M} f_{X_m}(X_m) + \sum_{n=1}^{N} f_n(x_n). \tag{2}
\]

### B. Generalized Min-Sum Algorithm for LDPC over \( GF(q) \)

Dr. Wiberg [6] developed the min-sum algorithm as a generalization of the Viterbi algorithm. The min-sum algorithm is also proposed in [7] as an approximation to the belief propagation (BP) algorithm [8], [9]. It is also referred to as the BP-based algorithm. The min-sum algorithm is a soft-decision, iterative algorithm for decoding binary-LDPC codes.

Conventionally, a LDPC code is represented as a Tanner graph, a graphical model useful at understanding code structures and decoding algorithms. A Tanner graph is a bipartite graph with variable nodes on one side and constraint nodes on the other side. Edges in the graph connect constraint nodes to variable nodes. A constraint node connects to those variable nodes that are contained in the constraint. A variable node connects to those constraint nodes that use the variable in the constraints. Constraint nodes are also referred to as check nodes. During each iteration of the min-sum algorithm, messages are flowed from variables nodes to the check nodes first, then back to variable nodes from check nodes.

Let \( N(m) \) be the set of variable nodes that are connected to the check node \( m \). Let \( M(n) \) be the set of check nodes that are connected to the variable node \( n \). Let symbol \( \setminus \) denote the set minus. \( N(m) \setminus n \) denotes the set of variable nodes excluding node \( n \) that are connected to the check node \( m \). \( M(n) \setminus m \) stands for the set of check nodes excluding the check node \( m \) which are connected to the variable node \( n \).

The generalization of the min-sum algorithm for decoding LDPC codes over \( GF(q) \) is straightforward. At iteration \( k \), let \( Z_{mn}(x_n) \) denote the message sent from variable node \( n \) to check node \( m \). \( Z_{mn}(0) - Z_{mn}(x_n) \) is the log-likelihood ratio (LLR) of the \( n \)-th input symbol having the value \( x_n \) versus 0, given the information obtained via the check nodes other than check node \( m \). Let \( L_{mn}(x_n) \) denote the message sent from check node \( m \) to variable node \( n \). \( L_{mn}(x_n) \) is the log-likelihood ratio that the check node \( m \) is satisfied when input symbol \( n \) is fixed to value 0 versus value \( x_n \) and the other symbols are independent with log-likelihood ratios,

\[
    Z_{mn}(0) - Z_{mn}(x_n), \quad n \in N(m) \setminus n.
\]

The pseudo-code of the generalized min-sum algorithm for decoding LDPC over \( GF(q) \) is given as follows.

**Initialization**

For \( n = 1, 2, \ldots, N \), and \( m = 1, 2, \ldots, M \),

\[
    Z_{mn}^{(0)}(x_n) = f_n(x_n).
\]

**Iteration \((k = 1, 2, 3, \ldots)\)**

1. **Horizontal scan**
   - Compute \( L_{mn}^{(k)}(x_n) \), for each \( x_n \in GF(q) \),
   \[
   L_{mn}^{(k)}(x_n) = \min_{x_{n'} \in N(m) \setminus n} Z_{mn}^{(k-1)}(x_{n'}) \tag{3}
   \]
   s.t. \( \sum_{n' \in N(m)} h_{mn'} x_{n'} = 0 \).
   - Normalize \( L_{mn}^{(k)}(x_n) \).
     - For each \( m \), and each \( n \in N(m) \), offsetting \( L_{mn}^{(k)}(x_n) \) by \( L_{mn}^{(k)}(0) \),
     \[
     L_{mn}^{(k)}(x_n) = L_{mn}^{(k)}(x_n) - L_{mn}^{(k)}(0).
     \]
2. **Vertical scan**
   - For \( n = 1, 2, \ldots, N \),
   \[
   Z_{mn}^{(k)}(x_n) = f_n(x_n) + \sum_{m' \in M(n) \setminus m} L_{m'n}^{(k)}(x_n). \tag{4}
   \]
3. **Decoding**
   - For each symbol, compute its posteriori log-likelihood ratio (LLR)
   \[
   Z_{n}^{(k)}(x_n) = f_n(x_n) + \sum_{m \in M(n)} L_{mn}^{(k)}(x_n). \tag{5}
   \]
   - Then estimate the original codeword \( \hat{x}^{(k)} \),
   \[
   \hat{x}^{(k)}_n = \arg \min_{x_n} Z_{n}^{(k)}(x_n), \quad \text{for } n = 1, 2, \ldots, N.
   \]
   - If \( H (\hat{x}^{(k)})^T = 0 \) or the iteration number exceeds some cap, stop the iteration and output \( \hat{x}^{(k)} \) as the decoded codeword.

In the above algorithm, \( Z_{n}^{(k)}(0) - Z_{n}^{(k)}(x_n) \) is the posteriori LLR for variable \( x_n \) at iteration \( k \).

One way to possibly improve the performance of the generalized min-sum algorithm is to modify the Eq. (4) and Eq. (5) as

\[
    Z_{mn}^{(k)}(x_n) = f_n(x_n) + \alpha_k \sum_{m' \in M(n) \setminus m} L_{mn}^{(k)}(x_n),
\]
\[
    Z_{n}^{(k)}(x_n) = f_n(x_n) + \alpha_k \sum_{m \in M(n)} L_{mn}^{(k)}(x_n),
\]

where \( \alpha_k \) is a scaling constant at iteration \( k \) satisfying \( 0 < \alpha_k < 1 \). With these modifications, the decoding algorithm is called the normalized min-sum algorithm.

Another way to possibly improve the performance is to modify the Eq. (4) and Eq. (5) as

\[
    Z_{mn}^{(k)}(x_n) = f_n(x_n) + \sum_{m' \in M(n) \setminus m} \max(L_{mn}^{(k)}(x_n) - \beta_k, 0),
\]
\[
    Z_{n}^{(k)}(x_n) = f_n(x_n) + \sum_{m \in M(n)} \max(L_{mn}^{(k)}(x_n) - \beta_k, 0),
\]
where $\beta_k$ is an offset constant at iteration $k$ satisfying $\beta_k > 0$. With these modifications, the decoding algorithm is called the offset min-sum algorithm.

To possibly maximize the decoding power, the scaling factor $\alpha_k$ or the offset constant $\beta_k$ can be determined through experiments or the density evolution method [10].

C. Horizontal Scan via Dynamic Programming

Our algorithm for the horizontal scan is based on dynamic programming [11], which is, in principle, similar to the Viterbi algorithm [12] for decoding convolutional codes. It is a linear complexity algorithm for the minimization problem defined in (3) as long as all $x_n$s are in finite domains. For decoding LDPC codes over $GF(q)$, all variables are defined in $GF(q)$, a finite domain. The algorithm is applicable for this special case.

With loss of generality, we explain the dynamic programming algorithm with the assumption of $h_{mn} \neq 0$ for all $n,s$ to simplify notations. When some of elements $h_{mn} s$ are zero, we can apply the same algorithm simply on those variables with non-zero coefficients $h_{mn}$ in the parity check constraint $\sum_n h_{mn} x_n = 0$.

To simplify notations further, we define $g_n(x_n)$ as

$$g_n(x_n) = Z_{mn}^{(k-1)}(x_n).$$

The minimization problem defined in (5) can be rewritten in a more succinct form

$$L_{mn}^{(k)}(x_n) = \min_{X \setminus x_n} \sum_{n' = 1}^N g_{n'}(x_{n'}) \text{ s.t. } \sum_{n' = 1}^N h_{mn'} x_{n'} = 0. \quad (6)$$

We need to solve the minimization problem (6) $N$ times, one for each $x_n$, $n = 1, 2, \ldots, N$. This task can be done through two scans, each scan defines a dynamic programming process. One scan is started from variable $x_1$ and ended at variable $x_N$, so called the left scan. The other has the reverse order, from $x_N$ to $x_1$, so called the right scan.

Each scan has $N - 1$ steps, step $n = 1, 2, \ldots, N - 1$. We take the left scan as the case study. The right scan can be derived simply by reversing the order of variables.

For the left scan, at step $n$, we use variables $s_n, s_n \in GF(q)$, to represent the result of the following summation,

$$s_n = \sum_{n' = 1}^n h_{mn'} x_{n'}.$$

Also, we assign a real value $r_n^{(L)}(s_n)$ for each state $s_n$, which stores the result of the following constrained optimization problem,

$$r_n^{(L)}(s_n) = \min_{x_1, \ldots, x_n} \sum_{n' = 1}^n g_{n'}(x_{n'}) \text{ s.t. } \sum_{n' = 1}^n h_{mn'} x_{n'} = s_n,$$

where the superscript “L” stands for the left scan.

When $n = 1$, $r_1^{(L)}(s_1)$ is initialized as

$$r_1^{(L)}(s_1) = g_1(h_{m1}^{-1} s_1).$$

At each step $n, n = 2, 3, \ldots, N - 1$, the dynamic programming computes $r_n^{(L)}(s_n)$ for each state $s_n, s_n \in GF(q)$, as follows

$$r_n^{(L)}(s_n) = \min_{x_n, s_{n-1}} g_n(x_n) + r_{n-1}^{(L)}(s_{n-1}) \quad \text{ s.t. } \quad s_{n-1} + h_{mn} x_n = s_n. \quad (7)$$

Similarly, for the right scan, when $n = N$, $r_n^{(R)}(s_1)$ is initialized as

$$r_N^{(R)}(s_N) = g_N(h_{mN}^{-1} s_N).$$

We compute $r_n^{(R)}(s_n)$, for $n = N - 1, N - 2, \ldots, 2$, as follows

$$r_n^{(R)}(s_n) = \min_{x_n, s_{n+1}} g_n(x_n) + r_{n+1}^{(R)}(s_{n+1}) \quad \text{ s.t. } \quad s_{n+1} + h_{mn} x_n = s_n. \quad (8)$$

We can obtain the minimization result for (6) from $r_n^{(L)}(s_n)$ and $r_n^{(R)}(s_n)$ directly. For $1 < n < N$, we have

$$L_{mn}^{(k)}(x_n) = \min_{s_{n-1}, s_{n+1}} r_{n-1}^{(L)}(s_{n-1}) + r_{n+1}^{(R)}(s_{n+1}) \quad \text{ s.t. } \quad s_{n-1} + h_{mn} x_n + s_{n+1} = 0. \quad (9)$$

We can rewrite Eq. (2) to have a form clearer for computing

$$L_{mn}^{(k)}(x_n) = \min_{s_{n-1}} r_{n-1}^{(L)}(s_{n-1}) + r_{n+1}^{(R)}(-s_{n-1} + h_{mn} x_n). \quad (10)$$

For $GF(q), q = 2^m$, Eq. (10) can be simplified further to

$$L_{mn}^{(k)}(x_n) = \min_{s_{n+1}} r_{n-1}^{(L)}(s_{n-1}) + r_{n+1}^{(R)}(s_{n-1} + h_{mn} x_n). \quad (11)$$

When $n = N$, the result is

$$L_{mN}^{(k)}(x_N) = r_{N-1}^{(k)}(-h_{mN} x_N). \quad (12)$$

When $n = 1$, the result is

$$L_{m1}^{(k)}(x_1) = r_2^{(R)}(-h_{m1} x_1). \quad (13)$$

D. Computational Complexity

At each iteration, the vertical scan of the generalized (normalized/offset) min-sum algorithm has the computational complexity of $O(N d_v q)$, where $d_v$ is the average variable degrees.

For each constraint (check node), the dynamic programming horizontal scan carries $O(d_c (m) q^2)$ minimization operations and the same number of addition operations. $d_c (m)$ here is the degree of the $m$th check node. In total, the complexity of the horizontal scan is $O(M d_c q^2)$, where $d_c$ is the average check node degrees.

If we reduce the number of candidate symbols for each variable from $q$ to $n_m$ best candidate symbols, the complexity of the horizontal scan is $O(M d_c n_m q)$. The complexity of the horizontal scan of the algorithm proposed in [5] is $O(M d_c q^2)$. If $n_m$ is small compared to $q$, the complexity of our algorithm can be remarkably reduced. For example, for codes over $GF(256)$, we reduced $q = 256$ to $n_m = 16$ without noticing much degradation in performance in our experiments. In this case, the complexity of the dynamic programming horizontal scan is reduced by factor 16. Nevertheless, such a speedup can cause degradation in performance of the decoding algorithm if $n_m$ is too small compared with $q$. Furthermore, the degree of the degradation could vary from one code structure to another code structure.
Fig. 1. The bit error rates (BER) of decoding a $GF(4)$ LDPC code and a $GF(8)$ LDPC code using the standard sum-product algorithm (SPA) and our algorithm (COD).

III. EXPERIMENTAL RESULTS

We have used the two LDPC codes offered by Davey and Mackay in [13] to evaluate the performance of the generalized min-sum algorithm with the dynamic programming horizontal scan. The first code is defined over $GF(4)$ of a code length 9,000 and the second is over $GF(8)$ of a code length 6,000. The code rates of the both codes are $1/3$.

In our simulation, we use BPSK modulation and AWGN (additive White Gaussian Noise) channel. Figure 1 shows the performances of the normalized min-sum algorithm (COD) with the dynamic programming horizontal scan and the sum-product algorithm (SPA) at decoding both the codes. The factor $\alpha$ used by the normalized min-sum algorithm is 0.865 for the code over $GF(4)$ and 0.820 for the code over $GF(8)$. The normalized min-sum algorithm is a special case of a newly discovered optimization method called the cooperative optimization (see [14], [15]). The maximum numbers of iterations for both the algorithms are all set to 300.

From the figure we can see that the performances of the normalized min-sum algorithm are very close to those of the sum-product algorithm. The former is only around 0.05 $dB$ away from the latter at decoding the $GF(4)$ LDPC code. The degradation increases to 0.1 $dB$ for the $GF(8)$ LDPC code which is still negligible. The (normalized/offset) min-sum algorithm uses only additions and minimizations in its computation. The SPA in its computation uses addition operations, which are expensive for parallel hardware implementations due to the table looking up operations, which are expensive for parallel hardware implementations. Furthermore, the min-sum algorithm does not depend on the channel estimate while the sum-product algorithm needs to estimate the variance of the channel noise. The inaccuracy in the channel estimate can lead to noticeable performance degradations of the sum-product algorithm.

IV. CONCLUSION

We have presented in this paper a general (normalized/offset) min-sum algorithm for decoding LDPC codes over any Galois field $GF(q)$, $q \geq 2$. To speed up the horizontal scan of the algorithm, the dynamic programming technique has been applied. At each iteration, the computational complexity of the vertical scan of the algorithm is $O(N_d \alpha q)$ and the computational complexity of the horizontal scan is $O(M_d n_m q)$, $n_m < q$. In our experiments, compared with the belief propagation algorithm, the generalized min-sum algorithm with the dynamic programming horizontal scan has only around 0.1 $dB$ degradation in performance at water fall regions. It is suitable for hardware implementations because it is simple in computation and uses only minimization and addition operations.

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