$A_n^{(1)}$ Affine Quiver Matrix Model

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Abstract

We introduce $A_n^{(1)}$ ($n = 1, 2, \cdots$) affine quiver matrix model by simply adopting the extended Cartan matrices as incidence matrices and study its finite $N$ Schwinger-Dyson equations as well as their planar limit. In the case of $n = 1$, we extend our analysis to derive the cubic planar loop equation for one-parameter family of models labelled by $\alpha$: $\alpha = 1$ and $\alpha = 2$ correspond to the non-affine $A_2$ case and the affine $A_1^{(1)}$ case respectively. In the case of $n = 2$, we derive three sets of constraint equations for the resolvents which are quadratic, cubic and quartic respectively.
1 Introduction

Schwinger-Dyson equation for matrix models played an important role in the development of 2d gravity and its extensions in nineties and takes the form of infinite dimensional algebraic constraints \[1, 2\]. Among other things, $A_n$ quiver (or conformal) matrix model was constructed such that it satisfies the $W_n$ constraints automatically \[3, 4\]. The model in its $\beta$ deformation has contributed a great deal to the recent understanding of the connection \[5\] between 2d conformal field theory and the instanton sum \[6\] that derives the Seiberg-Witten curve \[7\]. The understanding consists of the proof in some special cases \[8\], isomorphism of the curves in both sides \[9\] and direct checks in the $q$-expansion \[10, 11\]. The case in which the incidence matrices take the generalized Cartan matrices of the affine Lie algebra $A_n^{(1)}$ appears to us an interesting and natural generalization and deserves study for its own sake. In this paper, we provide such model and study its finite $N$ Schwinger-Dyson equations and their planar limit.

This paper is organized as follows. In the next section, we introduce one-parameter family of matrix models labelled by a parameter $\alpha$ with two species of eigenvalues. The non-affine $A_2$ and affine $A_n^{(1)}$ quiver matrix models correspond to the $\alpha = 1$ and $\alpha = 2$ cases respectively. In section three, we consider S-D equations of this “$\alpha$” model. We consider the finite $N$ S-D equations as well as their planar limit. We derive a cubic planar loop equation and the cubic curve associated with it. A drastic simplification is observed in the case where $\alpha = 2$ and $W_0 = -W_1$ and the cubic symmetry of the curve is made manifest. In section four, we introduce an $A_n^{(1)}$ affine quiver matrix model. In section five, we derive the planar loop equations for the case of $n = 2$. They take the form of quadratic, cubic and quartic constraints for the resolvents. In the appendix A, we outline the derivation of the S-D equations in section three. The appendix B gives the detail of the derivation of the planar loop equation in section five.

While it is not unlikely that, with a proper engineering of the potential $W_i$ and the choice of the contour \[14, 11\], the partition function of the model may get identified with the Nekrasov function specified by a set of gauge theory data, we are unable to find one so far.

\[1\] For a more extensive list of references till now, see, for instance, \[12\] as well as \[13\].
2 $A_1^{(1)}$ model and the $\alpha$ deformation

Following the punchlines in the introduction, let us consider the $\beta$-deformed matrix model with the partition function

$$Z := \int d^{N(0)} \mu \int d^{N(1)} \nu \prod_{1 \leq I < J \leq N(0)} |\mu_I - \mu_J|^{2\beta} \prod_{1 \leq I < J \leq N(1)} |\nu_I - \nu_J|^{2\beta} \prod_{I=1}^{N(0)} \prod_{J=1}^{N(1)} \frac{1}{|\mu_I - \nu_J|^{\alpha \beta}} \times \exp \left( \frac{\sqrt{\beta}}{g_s} \sum_{I=1}^{N(0)} W_0(\mu_I) + \frac{\sqrt{\beta}}{g_s} \sum_{J=1}^{N(1)} W_1(\nu_J) \right).$$

Here, we have left the range of the integrations unspecified except that it is designed such that the integrand vanishes at the end points of the integrations. The second “deformation” parameter $\alpha$ interpolates between the $\beta$-deformed matrix model of $A_1^{(1)}$ type ($\alpha = 2$) and that of $A_2$ type ($\alpha = 1$).

For notational simplicity, let us introduce the “effective” action

$$e^{-S_{\text{eff}}} := \prod_{1 \leq I < J \leq N(0)} |\mu_I - \mu_J|^{2\beta} \prod_{1 \leq I < J \leq N(1)} |\nu_I - \nu_J|^{2\beta} \prod_{I=1}^{N(0)} \prod_{J=1}^{N(1)} \frac{1}{|\mu_I - \nu_J|^{\alpha \beta}} \times \exp \left( \frac{\sqrt{\beta}}{g_s} \sum_{I=1}^{N(0)} W_0(\mu_I) + \frac{\sqrt{\beta}}{g_s} \sum_{J=1}^{N(1)} W_1(\nu_J) \right).$$

3 S-D equation of the “$\alpha$ model” and the planar limit

Let us begin with the Virasoro constraints:

$$0 = \int d^{N(0)} \mu \int d^{N(1)} \nu \sum_{I=1}^{N(0)} \frac{\partial}{\partial \mu_I} \left( \frac{1}{z - \mu_I} e^{-S_{\text{eff}}} \right),$$

we have

$$\left\langle \sum_{I=1}^{N(0)} \frac{1}{(z - \mu_I)^2} \right\rangle + 2\beta \left\langle \sum_{I=1}^{N(0)} \sum_{J=1}^{N(0)} \frac{1}{z - \mu_I \mu_I - \mu_J} \right\rangle$$

$$- \alpha \beta \left\langle \sum_{I=1}^{N(0)} \sum_{J=1}^{N(1)} \frac{1}{z - \mu_I \mu_I - \nu_J} \right\rangle + \frac{\sqrt{\beta}}{g_s} \left\langle \sum_{I=1}^{N(0)} \frac{W_0'(\mu_I)}{z - \mu_I} \right\rangle = 0.$$

Similarly, from

$$0 = \int d^{N(0)} \mu \int d^{N(1)} \nu \sum_{J=1}^{N(1)} \frac{\partial}{\partial \nu_J} \left( \frac{1}{z - \nu_J} e^{-S_{\text{eff}}} \right),$$

we have

$$\left\langle \sum_{J=1}^{N(1)} \frac{1}{(z - \nu_J)^2} \right\rangle - 2\beta \left\langle \sum_{I=1}^{N(0)} \sum_{J=1}^{N(1)} \frac{1}{z - \mu_I \mu_I - \nu_J} \right\rangle + \alpha \beta \left\langle \sum_{I=1}^{N(0)} \sum_{J=1}^{N(1)} \frac{1}{z - \mu_I \mu_I - \nu_J} \right\rangle + \frac{\sqrt{\beta}}{g_s} \left\langle \sum_{I=1}^{N(0)} \frac{W_0'(\mu_I)}{z - \mu_I} \right\rangle = 0.$$
we have
\[
\left\langle \sum_{j=1}^{N^{(1)}} \frac{1}{(z - \nu_j)^2} \right\rangle + 2\beta \left\langle \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(1)}} \frac{1}{z - \nu_i \nu_j - \nu_j} \right\rangle - \alpha \beta \left\langle \sum_{i=1}^{N^{(0)}} \sum_{j=1}^{N^{(1)}} \frac{1}{z - \nu_j \nu_j - \mu_i} \right\rangle + \frac{\sqrt{\beta}}{g_s} \left\langle \sum_{j=1}^{N^{(1)}} W_1(\nu_j) \right\rangle = 0. \tag{3.4}
\]

Let
\[
\tilde{\omega}_0(z) := \sqrt{\beta} g_s \sum_{i=1}^{N^{(0)}} \frac{1}{z - \mu_i}, \quad \tilde{\omega}_1(z) := \sqrt{\beta} g_s \sum_{j=1}^{N^{(1)}} \frac{1}{z - \nu_j}. \tag{3.5}
\]

Adding \(g_s^2\cdot(3.2)\) and \(g_s^2\cdot(3.1)\), we have
\[
\left\langle (\tilde{\omega}_0(z))^2 \right\rangle + \left\langle (\tilde{\omega}_1(z))^2 \right\rangle - \alpha \left\langle \tilde{\omega}_0(z) \tilde{\omega}_1(z) \right\rangle + \epsilon \left\langle \tilde{\omega}_0(z) \right\rangle + \epsilon \left\langle \tilde{\omega}_1(z) \right\rangle + W_0'(z) \left\langle \tilde{\omega}_0(z) \right\rangle + W_1'(z) \left\langle \tilde{\omega}_1(z) \right\rangle - \left\langle f_0(z) \right\rangle - \left\langle f_1(z) \right\rangle = 0 \tag{3.6}
\]
where
\[
\epsilon := \left( \sqrt{\beta} - \frac{1}{\sqrt{\beta}} \right) g_s. \tag{3.7}
\]

\[
\hat{f}_0(z) := \sqrt{\beta} g_s \sum_{i=1}^{N^{(0)}} \frac{W_0'(z) - W_0'(\mu_i)}{z - \mu_i}, \tag{3.8}
\]

\[
\hat{f}_1(z) := \sqrt{\beta} g_s \sum_{j=1}^{N^{(1)}} \frac{W_1'(z) - W_1'(\nu_j)}{z - \nu_j}. \tag{3.9}
\]

To take the planar limit \(g_s \to 0\), let
\[
\omega_{0,1}(z) := \lim \left\langle \tilde{\omega}_{0,1}(z) \right\rangle, \quad f_{0,1}(z) := \lim \left\langle \hat{f}_{0,1}(z) \right\rangle, \tag{3.10}
\]
We obtain
\[
\omega_0(z)^2 + \omega_1(z)^2 - \alpha \omega_0(z) \omega_1(z) + W_0'(z) \omega_0(z) + W_1'(z) \omega_1(z) - f_0(z) - f_1(z) = 0. \tag{3.11}
\]

Let us turn to the higher order constraints. In particular, let us consider
\[
0 = \int d^{N^{(0)}} \mu \int d^{N^{(1)}} \nu \sum_{i=1}^{N^{(0)}} \frac{\partial}{\partial \mu_i} \left( \frac{1}{z - \mu_i} \sum_{K=1}^{N^{(1)}} \frac{1}{\nu_K - \mu_i} e^{-S_{\text{eff}}} \right), \tag{3.12}
\]
as well as
\[
0 = \int d^{N^{(0)}} \mu \int d^{N^{(1)}} \nu \sum_{K=1}^{N^{(1)}} \frac{\partial}{\partial \nu_K} \left( \frac{1}{z - \nu_K} \sum_{I=1}^{N^{(0)}} \frac{1}{\nu_K - \mu_I} e^{-S_{\text{eff}}} \right). \tag{3.13}
\]
Taking the difference of these two equations and carrying out some algebra which is outlined in the appendix A, we obtain

\[
\left\langle \left( \frac{2(1-\alpha)}{\alpha} A - \alpha \beta B - \frac{(1-\alpha \beta)}{2} C + D \right) \right\rangle = 0, \tag{3.14}
\]
where

\[
A := -\frac{1}{2} \left( \sum_{I \neq J} \frac{1}{z - \mu_I z - \mu_J} \right)' + (2\beta - 1) \sum_{I \neq J} \frac{1}{z - \mu_I (\mu_I - \mu_J)^2}
\]
\[
+ \frac{2\beta}{3} \sum_{I \neq J \neq K} \frac{1}{(z - \mu_I)(z - \mu_J)(z - \mu_K)}
\]
\[
+ \frac{\sqrt{\beta} W_0'(z)}{g_s} \left( \sum_{I \neq J} \frac{1}{z - \mu_I z - \mu_J} \right) - \frac{\sqrt{\beta}}{g_s} \left( \sum_{I \neq J} \frac{W_0'(z) - W_0'(\mu_I)}{z - \mu_I} \right)
\]
\[
- (\mu_I \leftrightarrow \nu_I, W_0 \leftrightarrow W_1),
\]
\[
B := \sum_{I,J,K} \frac{1}{(z - \mu_I)(z - \nu_J)(z - \nu_K)} - \sum_{I,J,K} \frac{1}{(z - \mu_I)(z - \mu_J)(z - \nu_K)},
\]
\[
C := \sum_{I,K} \frac{1}{(z - \mu_I)(z - \nu_K)^2} - \sum_{I,K} \frac{1}{(z - \mu_I)^2(z - \nu_K)}
\]
\[
+ \frac{1}{\alpha \beta} \left( \sum_{I} \frac{1}{(z - \mu_I)^2} + 2\beta \sum_{I \neq K} \frac{1}{z - \mu_I \mu_I - \mu_K} \right)
\]
\[
+ \frac{\sqrt{\beta}}{g_s} \sum_{I} W_0'(\mu_I)
\]
\[
- (\mu_I \leftrightarrow \nu_I, W_0 \leftrightarrow W_1)',
\]
\[
D := \frac{\sqrt{\beta} W_0'(z)}{g_s} \frac{1}{\alpha \beta} \left( \sum_{I} \frac{1}{(z - \mu_I)^2} + 2\beta \sum_{I \neq K} \frac{1}{z - \mu_I \mu_I - \mu_K} \right)
\]
\[
+ \frac{\sqrt{\beta}}{g_s} \sum_{I} W_0'(\mu_I)
\]
\[
- \frac{\sqrt{\beta}}{g_s} \left( \sum_{I,K} \frac{W_0'(z) - W_0'(\mu_I)}{z - \mu_I} \right)
\]
\[
- (\mu_I \leftrightarrow \nu_I, W_0 \leftrightarrow W_1).
\]

This is a complicated equation but let us multiply by \(g_s^3 \sqrt{\beta}\) and take the planar limit.

Let

\[
u(z) := \omega_0(z) - \omega_1(z), \quad v(z) := \omega_0(z) + \omega_1(z). \tag{3.16}\]
Also, let

\[ h_0(z) := \lim \left\langle \beta g_s^2 \sum_{l \neq k} \frac{W_0' (z) - W_0' (\mu_I)}{z - \mu_I} - \frac{1}{\mu_I - \mu_K} \right\rangle, \]

\[ h_1(z) := \lim \left\langle \beta g_s^2 \sum_{l \neq k} \frac{W_1' (z) - W_1' (\nu_I)}{z - \nu_I} - \frac{1}{\nu_I - \nu_K} \right\rangle, \]

\[ g_0(z) := \lim \left\langle \beta g_s^2 \sum_{l, K} \frac{W_0' (z) - W_0' (\mu_I)}{z - \mu_I} - \frac{1}{\mu_I - \mu_K} \right\rangle, \]

\[ g_1(z) := \lim \left\langle \beta g_s^2 \sum_{l, K} \frac{W_1' (z) - W_1' (\nu_K)}{z - \nu_K} - \frac{1}{\nu_K - \nu_I} \right\rangle. \]

(3.17)

In terms of \( u, v \), the planar SD equations can be rewritten as

\[ \left( \frac{2 + \alpha}{4} \right) u(z)^2 + \left( \frac{2 - \alpha}{4} \right) v(z)^2 \]

\[ + \frac{1}{2} \left( W_0''(z) - W_1''(z) \right) u(z) + \frac{1}{2} \left( W_0'(z) + W_1'(z) \right) v(z) - f_0(z) - f_1(z) = 0, \]

\[ \frac{(3 \alpha - 2)(\alpha + 2)}{12 \alpha} u(z)^3 - \frac{(2 - \alpha)}{4 \alpha} (W_0'(z) - W_1'(z)) u(z)^2 - \frac{1}{2 \alpha} \left\{ (W_0'(z))^2 + (W_1'(z))^2 \right\} u(z) \]

\[ - \frac{1}{\alpha} \left\{ (2 - \alpha) u(z) + (W_0'(z) - W_1'(z)) \right\} \left( \frac{2 - \alpha}{4} v(z)^2 + \frac{1}{2} (W_0'(z) + W_1'(z)) v(z) \right) \]

\[ + \frac{1}{\alpha} \left( W_0''(z) f_0(z) - W_1''(z) f_1(z) \right) - 2 \left( 1 - \frac{1}{\alpha} \right) (h_0(z) - h_1(z)) + g_0(z) - g_1(z) = 0. \]

(3.18)

Using the planar Virasoro constraint (3.18), we can convert (3.19) into an equation for \( u \):

\[ \frac{(\alpha + 2)}{3 \alpha} u(z)^2 + \frac{1}{\alpha} (W_0'(z) - W_1'(z)) u(z)^2 - \frac{1}{\alpha} \left\{ W_0'(z) W_1'(z) + (2 - \alpha) (f_0(z) + f_1(z)) \right\} u(z) \]

\[ + \frac{1}{\alpha} \left( W_1''(z) f_0(z) - W_0''(z) f_1(z) \right) - \frac{2(\alpha - 1)}{\alpha} (h_0(z) - h_1(z)) + g_0(z) - g_1(z) = 0. \]

(3.20)

For simplicity, we assume \( \alpha \neq -2 \). Let

\[ x(z) := u(z) + \frac{1}{\alpha + 2} (W_0'(z) - W_1'(z)) = \omega_0(z) - \omega_1(z) + \frac{1}{\alpha + 2} (W_0'(z) - W_1'(z)). \]

(3.21)

The cubic equation (3.20) becomes

\[ x(z)^3 - p(z)x(z) - q(z) = 0, \]

(3.22)

where

\[ p(z) = \frac{3}{(\alpha + 2)^2} \left\{ (W_0'(z))^2 + (W_1'(z))^2 + \alpha W_0'(z) W_1'(z) \right\} + \frac{3(2 - \alpha)}{\alpha + 2} (f_0(z) + f_1(z)), \]

(3.23)
\[ q(z) = -\frac{1}{(\alpha + 2)^3} (W_0'(z) - W_1'(z)) \left\{ 2(W_0'(z))^2 + 2(W_1'(z))^2 + (3\alpha + 2)W_0'(z)W_1'(z) \right\} \]
\[ \quad - \frac{3}{(\alpha + 2)^2} \left\{ (2 - \alpha)W_0'(z) + 2\alpha W_1'(z) \right\} f_0(z) + \frac{3}{(\alpha + 2)^2} \left\{ 2\alpha W_0'(z) + (2 - \alpha)W_1'(z) \right\} f_1(z) \]
\[ \quad + \frac{6(\alpha - 1)}{\alpha + 2} (h_0(z) - h_1(z)) - \frac{3\alpha}{\alpha + 2} (g_0(z) - g_1(z)). \] (3.24)

At \( \alpha = 2 \), we get the cubic equation for \( A_1^{(1)} \) model:

\[ x^3 - \frac{3}{16} (W_0' + W_1')^2 x + \frac{1}{32} (W_0' - W_1') \left\{ (W_0')^2 + (W_1')^2 + 4W_0'W_1' \right\} \]
\[ \quad + \frac{3}{4} (W_1'f_0 - W_0'f_1) + \frac{3}{2} (-h_0 + h_1 + g_0 - g_1) = 0. \] (3.25)

Here

\[ x = \omega_0 - \omega_1 + \frac{1}{4} (W_0' - W_1'). \] (3.26)

At \( \alpha = 1 \), it turns into the loop equation for \( A_2 \) model

\[ x^3 - \frac{1}{3} \left\{ (W_0')^2 + (W_1')^2 + W_0'W_1' + 3(f_0 + f_1) \right\} x \]
\[ \quad + \frac{1}{27} (W_0' - W_1') \left\{ 2(W_0')^2 + 2(W_1')^2 + 5W_0'W_1' \right\} \]
\[ \quad + \frac{1}{3} (W_0' + 2W_1')f_0 - \frac{1}{3} (2W_0' + W_1')f_1 + g_0 - g_1 = 0, \] (3.27)

Here

\[ x = \omega_0 - \omega_1 + \frac{1}{3} (W_0' - W_1'). \] (3.28)

This cubic equation (3.27) can be rewritten as follows:

\[ (x - t_1(z))(x - t_2(z))(x - t_3(z)) - f_1(z)(x - t_3(z)) - f_0(z)(x - t_1(z)) - g_1(z) + g_0(z) = 0, \] (3.29)

where

\[ t_1(z) = \frac{1}{3} (2W_1'(z) + W_0'(z)), \quad t_2(z) = -\frac{1}{3} (W_1'(z) - W_0'(z)), \quad t_3(z) = -\frac{1}{3} (W_1'(z) + 2W_0'(z)). \] (3.30)

This is the form which have been analysed before.

Finally, let us consider the special case where \( \alpha = 2 \) and \( W_0 = -W_1 \). In this case, eq. (3.25) reduces to

\[ x^3 - \frac{1}{8} (W_0')^3 - \frac{3}{4} W_0'(f_0 + f_1) - \frac{3}{2} (h_0 - h_1 - g_0 + g_1) = 0, \] (3.31)

possessing the symmetry of \( x \) rotation by cubic root of unity \( x \to e^{\pm \frac{2\pi i}{3}} x \). This drastic simplification is understood as the prescription \( \sqrt[3]{\beta} \to -\sqrt[3]{\beta} \) for the second species of eigenvalues.
\( \nu_J, \; (J = 1, 2, \ldots, N^{(1)}) \). Let us introduce notation

\[
\begin{align*}
\nu_I &= \begin{cases} 
\mu_I, & (I = 1, 2, \ldots, N^{(0)}), \\
\nu_{I-N^{(0)}}, & (I = N^{(0)} + 1, \ldots, N^{(0)} + N^{(1)}),
\end{cases} \\
\operatorname{sgn} I &= \begin{cases} 
1, & (I = 1, 2, \ldots, N^{(0)}), \\
-1, & (I = N^{(0)} + 1, \ldots, N^{(0)} + N^{(1)}).
\end{cases}
\end{align*}
\]

The partition function in this case can be written as that of the \( \beta \) deformation of one-matrix model with positive and negative “charges” in the Coulomb gas analogy:

\[
Z := \int d^{N(0)+N(1)} z \prod_{1 \leq I < J \leq N(0)+N(1)} |z_I - z_J|^{2\beta (\operatorname{sgn} I)(\operatorname{sgn} J)} \exp \left( \frac{\sqrt{\beta}}{g_s} \sum_{I=1}^{N(0)+N(1)} (\operatorname{sgn} I)W_0(z_I) \right).
\]

The entire S-D equations can be formulated in terms of a single resolvant \( \hat{\omega}(z) := \hat{w}_0(z) - \hat{w}_1(z) \) and two kinds of quantum deformations \( \hat{f}(z) := \hat{f}_0(z) + \hat{f}_1(z) \) and \( \hat{h}(z) := (\hat{h}_0(z) - \hat{h}_1(z)) - (\hat{g}_0(z) - \hat{g}_1(z)) \), all of which are written succinctly in this one-matrix notation.

4 \( A_n^{(1)} \) affine quiver matrix model

The partition function for the \( \beta \)-deformed \( A_n^{(1)} \) quiver matrix model is defined by

\[
Z := \int d\lambda e^{-S_{\text{eff}}},
\]

where

\[
d\lambda = \prod_{i=0}^{n} \prod_{I=1}^{N(i)} d\lambda^{(i)}_I,
\]

\[
e^{-S_{\text{eff}}} := \prod_{i=0}^{n} \prod_{1 \leq I < J \leq N(i)} |\lambda^{(i)}_I - \lambda^{(i)}_J|^{2\beta} \prod_{i=0}^{n} \prod_{I=1}^{N(i)} \prod_{J=1}^{N(i+1)} |\lambda^{(i)}_I - \lambda^{(i+1)}_J|^{-\beta}
\]

\[
\times \exp \left( \frac{\sqrt{\beta}}{g_s} \sum_{i=0}^{n} \sum_{I=1}^{N(i)} W_i(\lambda^{(i)}_I) \right),
\]

with the periodicity of the index \( i \): \( \lambda^{(n+1)}_I = \lambda^{(0)}_I \) and \( N^{(n+1)} = N^{(0)} \). In the following part, we assume this kind of periodicity for the index \( i \): \( i = k + n + 1 \equiv k \).
For later convenience, we define the following functions:

\[ \tilde{\omega}_i(z) := \sqrt{\beta} g_s \sum_{I=1}^{N^{(i)}} \frac{1}{z - \lambda_I^{(i)}}, \]

\[ \tilde{R}_{i,j_1,j_2\ldots,j_k}^{(i)}(z) := \sqrt{\beta} g_s \sum_{I=1}^{N^{(i)}} \frac{\xi_{j_1,j_2\ldots,j_k}^{(i)}(\lambda_I^{(i)})}{z - \lambda_I^{(i)}}, \]

\[ \tilde{U}_{i,j_1,j_2\ldots,j_k}^{(i)}(z) := \sqrt{\beta} g_s \frac{\partial \xi_{j_1,j_2\ldots,j_k}^{(i)}(\lambda_I^{(i)})}{\partial \lambda_I^{(i)}}. \]  

(4.4)

Here \( i, j_1, \ldots, j_k = 0, 1, \ldots, n \) and

\[ \xi_{j_1,j_2\ldots,j_k}^{(i)}(\lambda_I^{(i)}) := \begin{cases} \xi_{j_1}^{(i)}(\lambda_I^{(i)})\xi_{j_2}^{(i)}(\lambda_I^{(i)})\ldots\xi_{j_k}^{(i)}(\lambda_I^{(i)}), & (k \geq 1), \\ 1 & (k = 0), \end{cases} \]  

(4.5)

with

\[ \xi_{j}^{(i)}(\lambda_I^{(i)}) := \begin{cases} \sqrt{\beta} g_s \sum_{I=1 \atop (J \neq I)}^{N^{(i)}} \frac{1}{\lambda_I^{(i)} - \lambda_J^{(i)}}, & (j = i), \\ \sqrt{\beta} g_s \sum_{J=1 \atop (J \neq i)}^{N^{(i)}} \frac{1}{\lambda_I^{(i)} - \lambda_J^{(i)}} = \tilde{\omega}_j(\lambda_I^{(i)}), & (j \neq i). \]  

(4.6)

Notice that \( \tilde{R}_{j_1\ldots,j_k}^{(i)}(z) \) with \( k = 0 \) coincide with \( \tilde{\omega}_i(z) \):

\[ \tilde{R}^{(i)}(z) = \sqrt{\beta} g_s \sum_{I=1}^{N^{(i)}} \frac{1}{z - \lambda_I^{(i)}} = \tilde{\omega}_i(z), \quad (k = 0), \]  

(4.7)

Later, we use several identities which relate products of \( \tilde{\omega}_i(z) \) to sums of these functions. For \( \{j_1, j_2, \ldots, j_k\} \) all different, the identity

\[ \prod_{\ell=1}^{k} \frac{1}{z - \lambda_I^{(j_\ell)}} = \sum_{\ell=1}^{k} \frac{1}{z - \lambda_I^{(j_\ell)}} \prod_{m=1 \atop (m \neq \ell)}^{k} \frac{1}{\lambda_I^{(j_m)} - \lambda_I^{(j_m)}} \]  

(4.8)

leads to the following identity:

\[ \tilde{\omega}_{j_1}(z)\tilde{\omega}_{j_2}(z)\ldots\tilde{\omega}_{j_k}(z) = \sum_{\ell=1}^{k} \tilde{R}_{j_1\ldots,j_{\ell-1}j_{\ell+1}\ldots,j_k}^{(j_\ell)}(z), \quad (\{j_i\} \text{ all different}). \]  

(4.9)
If some of indices $j_\ell$ coincide, there are $O(g_s)$ corrections:

$$\hat{\varpi}_j(z)\hat{\varpi}_k(z) = \sum_{\ell=1}^k \hat{R}^{(j_\ell)}_{j_{\ell-1}j_\ell1\cdots j_k}(z) + O(g_s).$$  \hspace{1cm} (4.10)

Explicit forms of (4.10) for $k = 2, 3$ are given by

$$\hat{\varpi}_i(z)\hat{\varpi}_j(z) = \hat{R}^{(i)}_{j}(z) + \hat{R}^{(j)}_{i}(z) - \sqrt{\beta g_s}\hat{\varpi}_i(z)\delta_{ij},$$  \hspace{1cm} (4.11)

$$\hat{\varpi}_i(z)\hat{\varpi}_j(z)\hat{\varpi}_k(z) = \hat{R}^{(i)}_{jk}(z) + \hat{R}^{(j)}_{ki}(z) + \hat{R}^{(k)}_{ij}(z), \quad (i \neq j \neq k \neq i),$$  \hspace{1cm} (4.12)

$$\left(\hat{\varpi}_i(z)\right)^2 = 2\hat{R}^{(i)}_{ij}(z) + \hat{R}^{(j)}_{ii}(z) - \sqrt{\beta g_s}d\hat{R}^{(i)}_j(z) + \sqrt{\beta g_s}\hat{\varpi}^{(i)}_j(z), \quad (i \neq j),$$  \hspace{1cm} (4.13)

The identity (4.11) for $j = i$, we have

$$2\hat{R}^{(i)}_i(z) = (\hat{\varpi}_i(z))^2 + \sqrt{\beta g_s}\hat{\varpi}^{(i)}_i(z).$$  \hspace{1cm} (4.14)

Substituting this identity into (4.15), we have

$$\left\langle \epsilon\hat{\varpi}^{(i)}_i(z) + (\hat{\varpi}^{(i)}_i(z))^2 - \hat{R}^{(i)}_{i-1}(z) - \hat{R}^{(i)}_{i+1}(z) + W'_i(z)\hat{\varpi}_i(z) - \hat{F}^{(i)}(z) \right\rangle = 0,$$  \hspace{1cm} (4.16)

where

$$\epsilon := \left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right) g_s.$$

The identity (4.11) for $j = i + 1$ gives

$$\hat{\varpi}_i(z)\hat{\varpi}_{i+1}(z) = \hat{R}^{(i)}_{i+1}(z) + \hat{R}^{(i+1)}_{i}(z).$$  \hspace{1cm} (4.17)
Summing over $i$, we have
\[
\sum_{i=0}^{n} (\hat{R}_{i+1}(z) + \hat{R}_{i-1}(z)) = \sum_{i=0}^{n} \hat{\omega}_i(z) \hat{\omega}_{i+1}(z). \tag{4.20}
\]

Only this combination of $\hat{R}_{i+1}(z) + \hat{R}_{i-1}(z)$ allows an expression in terms of the resolvents $\hat{\omega}_j(z)$. Hence the sum of (4.17) over $i$ gives the “Virasoro constraint”:
\[
\left\langle \sum_{i=0}^{n} (\epsilon \hat{\omega}'_i(z) + (\hat{\omega}_i(z))^2 - \hat{\omega}_i(z) \hat{\omega}_{i+1}(z) + W'_i(z) \hat{\omega}_i(z) - \hat{F}^{(i)}(z)) \right\rangle = 0. \tag{4.21}
\]

5 Planar loop equations for $n = 2$

For simplicity, we consider the S-D equations for the $A_n^{(1)}$ model in the planar limit: $g_s \rightarrow 0$.

Let
\[
\begin{align*}
R_{j_1j_2 \ldots j_k}^{(i)}(z) &:= \lim \left\langle \hat{R}_{j_1j_2 \ldots j_k}^{(i)}(z) \right\rangle, \tag{5.1} \\
F_{j_1j_2 \ldots j_k}^{(i)}(z) &:= \lim \left\langle \hat{F}_{j_1j_2 \ldots j_k}^{(i)}(z) \right\rangle. \tag{5.2}
\end{align*}
\]

In the planar limit, the SD equations (4.14) are given by
\[
2R_{j_1j_2 \ldots j_k}^{(i)}(z) - R_{(i-1)j_1j_2 \ldots j_k}^{(i)}(z) - R_{(i+1)j_1j_2 \ldots j_k}^{(i)}(z) + W'_i(z) R_{j_1j_2 \ldots j_k}^{(i)}(z) - F_{j_1j_2 \ldots j_k}^{(i)}(z) = 0. \tag{5.3}
\]

We write explicit constraints for the resolvents (loop equations) in the $A_2^{(1)}$ model. The planar Virasoro constraint is given by
\[
\begin{align*}
\omega_0^2 + \omega_1^2 + \omega_2^2 - \omega_0 \omega_1 - \omega_0 \omega_2 - \omega_1 \omega_2 \\
+ W'_0 \omega_0 + W'_1 \omega_1 + W'_2 \omega_2 - F^{(0)} - F^{(1)} - F^{(2)} = 0. \tag{5.4}
\end{align*}
\]

The cubic loop equation takes the form
\[
\begin{align*}
\frac{8}{3}(\omega_0^3 + \omega_1^3 + \omega_2^3) - \omega_0(\omega_1^2 + \omega_2^2) - \omega_1(\omega_0^2 + \omega_2^2) - \omega_2(\omega_0^2 + \omega_1^2) - 2\omega_0 \omega_1 \omega_2 \\
+ W'_0(3\omega_0^2 + W'_0 \omega_0 - F^{(0)}) + W'_1(3\omega_1^2 + W'_1 \omega_1 - F^{(1)}) + W'_2(3\omega_2^2 + W'_2 \omega_2 - F^{(2)}) \\
- 4F'_0 - F'_0 - F'_2 - 4F'_1 - F'_2 - F'_0 - 4F'_2 - F'_0 - F'_2 = 0. \tag{5.5}
\end{align*}
\]
The quartic loop equation is given by
\[ \frac{13}{2} (\omega_0^4 + \omega_1^4 + \omega_2^4) - \omega_0 (\omega_1^3 + \omega_2^3) - \omega_1 (\omega_0^3 + \omega_2^3) - \omega_2 (\omega_0^3 + \omega_1^3) \]
\[ - \frac{3}{2} (\omega_0^2 \omega_1^2 + \omega_0^2 \omega_2^2 + \omega_1^2 \omega_2^2) - 3 \omega_0 \omega_1 \omega_2 (\omega_1 + \omega_2 + \omega_2) \]
\[ + W_0' \left[ 9 \omega_0^3 + W_0' \left( \frac{9}{2} \omega_0^2 + W_0' \omega_0 - F^{(0)} \right) - 7 F_0^{(0)} - F_1^{(0)} - F_2^{(0)} \right] \]
\[ + W_1' \left[ 9 \omega_1^3 + W_1' \left( \frac{9}{2} \omega_1^2 + W_1' \omega_1 - F^{(1)} \right) - 7 F_1^{(1)} - F_2^{(1)} - F_0^{(1)} \right] \]
\[ + W_2' \left[ 9 \omega_2^3 + W_2' \left( \frac{9}{2} \omega_2^2 + W_2' \omega_2 - F^{(2)} \right) - 7 F_2^{(2)} - F_0^{(2)} - F_1^{(2)} \right] \]
\[ - 13 F_{00}^{(0)} - 5 F_{01}^{(0)} - 5 F_{02}^{(0)} - F_{11}^{(0)} - 2 F_{12}^{(0)} - F_{22}^{(0)} \]
\[ - 13 F_{11}^{(1)} - 5 F_{12}^{(1)} - 5 F_{01}^{(1)} - F_{22}^{(1)} - 2 F_{02}^{(1)} - F_{00}^{(1)} \]
\[ - 13 F_{22}^{(2)} - 5 F_{02}^{(2)} - 5 F_{12}^{(2)} - F_{00}^{(2)} - 2 F_{01}^{(2)} - F_{11}^{(2)} = 0. \]

The derivation of these constraints is given in Appendix B

**Acknowledgements**

We thank Nobuhiro Yonezawa for interesting discussion. The research of H. I. and T. O. is supported in part by the Grant-in-Aid for Scientific Research (2054278, 23540316) from the Ministry of Education, Science and Culture, Japan.

**A Derivations of (3.14), (3.15)**

In this appendix, we outline the derivation of (3.14) and (3.15), starting from the second set of S-D equations eq. (3.12), eq. (3.13) which are constraints higher than Virasoro constraints. Eq. (3.12) reads
\[ \left\langle \sum_{I,K} \frac{1}{(z - \mu_I)^2} \frac{1}{\mu_I - \nu_K} \right\rangle - \left\langle \sum_{I,K} \frac{1}{(z - \mu_I)} \frac{1}{(\mu_I - \nu_K)^2} \right\rangle \]
\[ + 2 \beta \left\langle \sum_{I,K} \frac{1}{z - \mu_I} \frac{1}{\mu_I - \nu_K} \sum_{J \neq I} \frac{1}{\mu_I - \mu_J} \right\rangle - \alpha \beta \left\langle \sum_{I,K,J} \frac{1}{z - \mu_I} \frac{1}{\mu_I - \nu_K} \frac{1}{\mu_I - \nu_J} \right\rangle \]
\[ + \frac{\sqrt{\beta}}{g_s} \left\langle \sum_{I,K} W_0' (\mu_I) \frac{1}{z - \mu_I} \frac{1}{\mu_I - \nu_K} \right\rangle = 0. \]

The counterpart read from eq. (3.13) is given by replacing \( \mu_I \) by \( \nu_K \) in eq. (A.1) and we will not spell it out. Let us subtract this one from eq. (A.1), which we refer to as eq. (A.1), and
analyse this in what follows. We will make a frequent use of the partial fraction formula

\[
\sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{1}{z_i - z_j} = 0. \quad (A.2)
\]

for a set of \( n \) complex numbers \((z_1, \cdots, z_n)\). For the developments of this formula in the context of, see [15]. Using (A.2) for \((z, \mu, \nu_K, \nu_J)\) and for \((z, \nu_K, \mu, \mu_J)\), we convert the fourth term of eq. (A.1) as

\[
2\alpha\beta \cdot \left\langle \sum_{I, K, J \atop (K \neq J)} \frac{1}{z - \nu_K} \frac{1}{\nu_J} \frac{1}{\nu_K - \mu_I - \nu_K} - \sum_{I, K, J \atop (I \neq J)} \frac{1}{z - \mu_I} \frac{1}{\mu_J} \frac{1}{\mu_I - \nu_K} \right\rangle
\]

\[
- \alpha\beta \cdot \left\langle \sum_{I, K, J \atop (K \neq J)} \frac{1}{z - \mu_I} \frac{1}{z - \nu_K} \frac{1}{z - \nu_J} - \sum_{I, K, J \atop (I \neq J)} \frac{1}{z - \mu_I} \frac{1}{z - \mu_J} \frac{1}{z - \nu_K} \right\rangle \quad (A.3)
\]

\[
- \alpha\beta \cdot \left\langle \sum_{I, K} \frac{1}{z - \mu_I} \frac{1}{(\mu_I - \nu_K)^2} - \sum_{I, K} \frac{1}{z - \nu_K} \frac{1}{(\nu_K - \mu_I)^2} \right\rangle.
\]

The first line of eq. (A.3) combined with the third term in eq. (A.1) as gives

\[
2(1 - \alpha)\beta \cdot \left\langle \sum_{I, K, J \atop (I \neq J)} \frac{1}{z - \mu_I} \frac{1}{\mu_J} \frac{1}{\mu_I - \nu_K} - \sum_{I, K, J \atop (K \neq J)} \frac{1}{z - \nu_K} \frac{1}{\nu_J} \frac{1}{\nu_K - \mu_I} \right\rangle. \quad (A.4)
\]

The third line of eq. (A.3) combined with the first and the second terms in eq. (A.1) as gives

\[
- \frac{1 + \alpha\beta}{2} \cdot \left\langle \sum_{I, K} \frac{1}{z - \mu_I} \frac{1}{(z - \nu_K)^2} - \sum_{I, K} \frac{1}{z - \nu_K} \frac{1}{(z - \mu_I)^2} \right\rangle
\]

\[
+ \frac{1 - \alpha\beta}{2} \cdot \left\langle \sum_{I, K} \frac{1}{(z - \mu_I)^2} \frac{1}{\mu_I - \nu_K} - \sum_{I, K} \frac{1}{(z - \nu_K)^2} \frac{1}{\nu_K - \mu_I} \right\rangle. \quad (A.5)
\]
Here we have used $\nu_K$ derivative and $\mu_I$ derivative of eq. (A.2) for $(z, \mu_I, \nu_K)$. All in all, we obtain

$$2(1-\alpha)^{-\beta} \cdot \left\langle \sum_{I,K,J \atop (I \neq J)} \frac{1}{z-\mu_I} \frac{1}{\mu_I-\mu_J} \frac{1}{\nu_K-\nu_J} - \sum_{I,K,J \atop (K \neq J)} \frac{1}{z-\nu_K} \frac{1}{\nu_K-\nu_J} \frac{1}{\nu_J-\mu_I} \right\rangle$$

$$-\alpha^{-\beta} \cdot \left\langle \sum_{I,J,K} \frac{1}{z-\mu_I} \frac{1}{z-\nu_K} \frac{1}{\nu_J-\mu_I} + \frac{1}{\nu_I-\mu_J} \right\rangle$$

$$-\frac{1-\alpha^{-\beta}}{2} \cdot \left\langle \sum_{I,K} \frac{1}{z-\mu_I} \frac{1}{(z-\nu_K)^2} \frac{1}{\nu_K-\mu_I} \right\rangle$$

$$+ \frac{\sqrt{\beta}}{g_s} \left\langle \sum_{I,K} \frac{1}{z-\mu_I} \frac{1}{(z-\nu_K)^2} \right\rangle - \sqrt{\beta} \left\langle \sum_{I,K} \frac{1}{z-\nu_K} \frac{1}{\nu_K-\mu_I} \right\rangle$$

$$= 0.$$  

In this expression, all except the first line take forms which are expressible in terms of the two resolvents $\tilde{\omega}_{0,1}(z)$, their derivatives and polynomials in $z$ once we invoke the original Virasoro constraints eq. (3.2)

$$\alpha^{-\beta} \cdot \left\langle \sum_{I,J} \frac{1}{z-\mu_I} \frac{1}{\mu_I-\mu_J} \right\rangle =$$

$$\left\langle \sum_{I,J} \frac{1}{z-\mu_I} \right\rangle + 2\beta \left\langle \sum_{I,J \atop (J \neq I)} \frac{1}{z-\mu_I} \frac{1}{\mu_I-\mu_J} \right\rangle + \frac{\sqrt{\beta}}{g_s} \left\langle \sum_{I=1} W'_0(\mu_I) \right\rangle.$$  

(A.7)

and the one eq. (3.4) obtained by $\mu_I \leftrightarrow \nu_K$.

In order to handle the first line of eq. (A.6), let us consider another S-D equation:

$$0 = \int d^{N(0)} \mu \int d^{N(1)} \nu \sum_{I,J} \frac{\partial}{\partial \mu_I} \left( \frac{1}{z-\mu_I} \sum_{J \atop (J \neq I)} \frac{1}{\mu_I-\mu_J} e^{-S_{\text{eff}}} \right).$$  

(A.8)

Exploiting eq. (A.2) for $(z, \mu_I, \mu_J)$, $(z, \mu_I, \mu_J, \mu_K)$ as well as its $z$ derivative? in eq. (A.8), we
reexpress the first line of eq. (A.6), using
\[\alpha \beta \left\langle \sum_{I,K,J \neq J} \frac{1}{z - \mu_I} \frac{1}{\mu_I - \mu_J} \frac{1}{\mu_I - \nu_K} \right\rangle = -\frac{1}{2} \left\langle \left( \sum_{I \neq J} \frac{1}{z - \mu_I} \frac{1}{z - \mu_J} \right)^{\prime} \right\rangle \]
+ (2\beta - 1) \left\langle \left( \sum_{I \neq J} \frac{1}{z - \mu_I} \frac{1}{(\mu_I - \mu_J)^2} \right) \right\rangle + \frac{2\beta}{3} \left\langle \left( \sum_{I \neq J \neq K \neq I} \frac{1}{(z - \mu_I)(z - \mu_J)(z - \mu_K)} \right) \right\rangle \] \hspace{1cm} (A.9)
+ \frac{\sqrt{\beta} W_0' (z)}{2} \left\langle \left( \sum_{I \neq J} \frac{1}{z - \mu_I} \frac{1}{z - \mu_J} \right) \right\rangle - \frac{\sqrt{\beta}}{g_s} \left\langle \left( \sum_{I \neq J} \frac{W_0'(z) - W_0' (\mu_I)}{z - \mu_I} \frac{1}{\mu_I - \mu_J} \right) \right\rangle

Substituting this into eq. (A.6), we obtain the expression quoted in the text.

\section*{B Derivations of the constraints (5.4), (5.5), (5.6)}

\subsection*{B.1 The constraint (5.4)}

The planar Virasoro constraint (5.4) can be obtained by taking the planar limit of the Virasoro constraint (4.21). But we rederive it from the planar S-D equations (5.3) because we will need (B.4), (B.5) and (B.6) to obtain higher order loop equations (5.5) and (5.6).

For \(n = 2\) and \(k = 0\), \((5.3)\) are explicitly given by

\[2R_0^{(0)} - R_2^{(0)} - R_1^{(0)} + W_0' \omega_0 - F^{(0)} = 0,\]
\[2R_1^{(1)} - R_0^{(1)} - R_2^{(1)} + W_1' \omega_1 - F^{(1)} = 0,\]
\[2R_2^{(2)} - R_1^{(2)} - R_0^{(2)} + W_2' \omega_2 - F^{(2)} = 0.\] \hspace{1cm} (B.1)

Notice that we have planar identities:

\[\omega_i(z) \omega_j(z) = R_i^{(i)} (z) + R_i^{(j)} (z).\] \hspace{1cm} (B.2)

In particular,

\[R_0^{(0)} = \frac{1}{2} \omega_0^2, \quad R_1^{(1)} = \frac{1}{2} \omega_1^2, \quad R_2^{(2)} = \frac{1}{2} \omega_2^2.\] \hspace{1cm} (B.3)

Using these relations, we find

\[R_2^{(0)} + R_1^{(0)} = \omega_0^2 + W_0' \omega_0 - F^{(0)},\] \hspace{1cm} (B.4)
\[R_0^{(1)} + R_2^{(1)} = \omega_1^2 + W_1' \omega_1 - F^{(1)},\] \hspace{1cm} (B.5)
\[R_1^{(2)} + R_0^{(2)} = \omega_2^2 + W_2' \omega_2 - F^{(2)}.\] \hspace{1cm} (B.6)

By adding these three constraints, we have the planar Virasoro constraint (5.4).
B.2 The cubic loop equation (5.5)

The planar S-D equations (5.3) for $n = 2$ and $k = 1$ are given by

\[ 2R_{00}^{(0)} - R_{02}^{(0)} - R_{01}^{(0)} + W_0' R_0^{(0)} - F_0^{(0)} = 0, \]
\[ 2R_{01}^{(0)} - R_{12}^{(0)} - R_{11}^{(0)} + W_0' R_1^{(0)} - F_1^{(0)} = 0, \]
\[ 2R_{02}^{(0)} - R_{22}^{(0)} - R_{12}^{(0)} + W_0' R_2^{(0)} - F_2^{(0)} = 0, \]

and similar six equations obtained by cyclic permutations of the indices $0 \to 1 \to 2 \to 0$ or $0 \to 2 \to 1 \to 0$.

Using
\[ R_{00}^{(0)}(z) = \frac{1}{3} \omega_0(z)^3, \] (B.10)

(B.7) leads to
\[ R_{02}^{(0)} + R_{01}^{(0)} = \frac{2}{3} \omega_0^3 + \frac{1}{2} W_0' \omega_0^2 - F_0^{(0)}. \] (B.11)

The sum of (B.8) and (B.9) gives
\[ R_{11}^{(0)} + 2R_{12}^{(0)} + R_{22}^{(0)} = 2(R_{02}^{(0)} + R_{01}^{(0)}) + W_0'(2R_2^{(0)} + R_1^{(0)}) - F_1^{(0)} - F_2^{(0)} \]
\[ = \frac{4}{3} \omega_0^3 + W_0'(2\omega_0 + W_0' \omega_0 - F^{(0)}) - 2F_0^{(0)} - F_1^{(0)} - F_2^{(0)}. \] (B.12)

From \{2 $\times$ (B.11) + (B.12)\} + (cyclic equations), we obtain the cubic loop equation (5.5).

B.3 The quartic loop equation (5.6)

The explicit form of planar SD equations (5.3) for $n = 2$ and $k = 2$ are given by

\[ 2R_{000}^{(0)} - R_{002}^{(0)} - R_{001}^{(0)} + W_0' R_{00}^{(0)} - F_{00}^{(0)} = 0, \]
\[ 2R_{001}^{(0)} - R_{012}^{(0)} - R_{011}^{(0)} + W_0' R_{01}^{(0)} - F_{01}^{(0)} = 0, \]
\[ 2R_{002}^{(0)} - R_{022}^{(0)} - R_{012}^{(0)} + W_0' R_{02}^{(0)} - F_{02}^{(0)} = 0, \]
\[ 2R_{011}^{(0)} - R_{112}^{(0)} - R_{111}^{(0)} + W_0' R_{11}^{(0)} - F_{11}^{(0)} = 0, \]
\[ 2R_{012}^{(0)} - R_{122}^{(0)} - R_{112}^{(0)} + W_0' R_{12}^{(0)} - F_{12}^{(0)} = 0, \]
\[ 2R_{022}^{(0)} - R_{222}^{(0)} - R_{122}^{(0)} + W_0' R_{22}^{(0)} - F_{22}^{(0)} = 0, \]

and similar equations.

The constraint (B.13) can be rewritten as
\[ R_{001}^{(0)} + R_{002}^{(0)} = \frac{1}{2} \omega_0^4 + \frac{1}{3} W_0' \omega_0^3 - F_{00}^{(0)}. \] (B.19)
The sum of \((B.14)\) and \((B.15)\) leads to

\[
R^{(0)}_{011} + 2R^{(0)}_{012} + R^{(0)}_{022} = \omega_0^4 + W'_0 \left( \frac{4}{3} \omega_0^3 + \frac{1}{2} W'_0 \omega_0^2 - F^{(0)}_0 \right) - 2F^{(0)}_{00} - F^{(0)}_{01} - F^{(0)}_{02}.
\]  

(B.20)

By taking the combination \((B.16) + 2 \times (B.17) + (B.18)\), we find

\[
R^{(0)}_{111} + 3R^{(0)}_{112} + 3R^{(0)}_{122} + R^{(0)}_{222} = 2\omega_0^4 + W'_0 \left\{ 4\omega_0^3 + W'_0(3\omega_0^2 + W'_0 \omega_0 - F^{(0)}_0) - 4F^{(0)}_{00} - F^{(0)}_{01} - F^{(0)}_{02} \right\}
\]  

(B.21)

\[
- 4F^{(0)}_{00} - 2F^{(0)}_{01} - 2F^{(0)}_{02} - F^{(0)}_{11} - 2F^{(0)}_{12} - F^{(0)}_{22}.
\]

From \(\{3 \times (\ (B.19) + (B.20) ) + (B.21)\} + (\text{cyclic equations})\), we find the quartic loop equation (5.6).

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