IFOHAM—an iterative algorithm based on the first-order equation of HAM: exploratory preliminary results

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Abstract

In this work we present and study an iterative algorithm used to asymptotically solve nonlinear differential equations. This algorithm (Iterative First Order HAM or IFOHAM) is based on the first order equation of the Homotopy Analysis Method, HAM. We show that IFOHAM generalizes Picard-Lindeloff ’s iteration algorithm. Moreover, IFOHAM shares with HAM the possibility of ensuring convergence by adequately choosing \( c_0 \), a convergence control parameter. Preliminary results show that IFOHAM exhibits a very good performance both in aspects related to the speed of convergence and in aspects related to the CPU calculation time. It should also be noted that the IFOHAM is a very low complexity algorithm easily programmable in a symbolic computing environment.

Keywords:
IFOHAM; HAM; Picard-Lindeloff ’s iteration algorithm; Successive approximation method

1. Introduction

The Homotopy Analysis Method (HAM) was introduced in 1992 by Shijun Liao in its PhD thesis [1] and subsequently developed and applied by this author [2–4] and by a growing community of researchers.

An extensive and complete state of the art concerning the HAM technique can be found in [4].

This technique inserts or relates to the so-called asymptotic methods [5] and analytical approximation methods [6, 7].
Basically, the HAM technique transforms the original nonlinear problem (nonlinear differential equation, nonlinear differential equation system or even nonlinear partial differential equation system, for instance) into a set of linear differential equations (to be solved recursively) whose analytic solutions constitute the terms of a series of functions representing the solution of the original problem. This transformation is based on the concept of homotopy (under which an initial guess of the solution is continuously deformed to the solution of the original equation) and is built from the so-called zeroth-order deformation equation.

Consider the Initial Value Problem (IVP) described by

\[ N \left[ x(t) \right] = 0, \quad (1) \]

where \( N \) represents a nonlinear operator. So, depending on the order of the problem, the solution \( x = x(t) \) must satisfy initial conditions, such as

\[
\begin{align*}
  x(t_0) &= x_0^{(0)} \\
  x^{(1)}(t_0) &= x_0^{(1)} \\
  &\vdots \\
  x^{(r-1)}(t_0) &= x_0^{(r-1)}
\end{align*}
\]

if we assume that (1) is defined by an ordinary differential equation of order \( r \).

This work will be centered on the basic formulation of HAM developed in [8] which is supported by the corresponding zeroth-order deformation equation (3). Based on the previously mentioned equation an iterative algorithm (iterative first-order HAM: IFOHAM) to solve (1) will be proposed and its main features will be presented and discussed.

2. Basic HAM

2.1. Zeroth-order deformation equation

Following [8], a zeroth-order deformation equation (3) is defined, where \( \mathcal{L} \) represents an appropriate linear operator, \( c_0 \neq 0 \) stands for the convergence control parameter of HAM (to be described later), \( q \in [0, 1] \) and \( N \) represents the non-linear operator describing the problem (1) to be solved:

\[
(1 - q) \mathcal{L} [\phi(t; q) - u_0(t)] = c_0 q \left\{ N \left[ \phi(t; q) \right] \right\}. \quad (3)
\]
In expression (3), \( \phi = \phi (t; q) \) represents the so-called homotopy MacLaurin series which is a power series of the embedding parameter \( q \):

\[
\phi (t; q) = u_0 (t) + \sum_{n=1}^{+\infty} u_n (t) q^n, \quad q \in [0, 1].
\] (4)

Observe that in (3), \( u_0 (t) \) represents an initial guess (to be postulated), satisfying the initial conditions of the solution, \( u = u (t) \), of our original problem (1). Note that \( u_0 (t) \) is also the zeroth-order term of the homotopy Maclaurin series (4), that is,

\[
\phi (t; 0) = u_0 (t).
\] (5)

Setting \( q = 0 \), in the zeroth-order deformation equation (3), we obtain

\[
\mathcal{L} [\phi (t; 0) - u_0 (t)] = \mathcal{L} [0] = 0.
\]

Setting \( q = 1 \) we obtain \( N [\phi (t; 1)] = 0 \). This fact shows that, converging

\[
\phi (t; 1) = u_0 (t) + \sum_{n=1}^{+\infty} u_n (t),
\] (6)

is solution of (1). So, the coefficients of the homotopy Maclaurin series (4) are precisely the terms \( u_n (t), \ n \in \mathbb{N}_0 \), of the series of functions representing the searched solution

\[
u (t) = u_0 (t) + \sum_{n=1}^{+\infty} u_n (t)
\] (7)

of our problem (1).

Typically the zeroth-order deformation equation (3) is indexed in the parameter \( q \) (embedding parameter) and constitutes an homotopic family of differential equations with homotopic solutions under the embedding parameter \( q \), each one described by \( \phi = \phi (t; q) \). If \( q = 0 \), then \( \phi (t; 0) = u_0 (t) \), will be the trivial solution of

\[
\mathcal{L} [u (t)] = \mathcal{L} [u_0 (t)].
\] (8)

If \( q = 1 \), then \( N [\phi (t; 1)] = 0 \), that is, \( \phi (t; 1) \) will be our searched solution.
It should be noted that the zeroth-order deformation equation (3) is the starting point of HAM. Generalized formulations of the zeroth-order deformation equation can be built to be applied more efficiently as well as in broader contexts. [3, 4].

HAM users are interested in \( \phi(t; 1) \), that is, in the solution of problem (1). Let’s see how (6) can be obtained using this method.

2.2. High order deformation equations

Define the operator

\[
\mathcal{D}_k = \left. \frac{1}{k!} \frac{\partial^k}{\partial q^k} \right|_{q=0}
\]

and let’s apply it to the zeroth-order deformation (3). One obtain (see [3]):

\[
\mathcal{L}[u_1(t)] = c_0 [N[u_0(t)]]
\]

and

\[
\mathcal{L}[u_n(t) - u_{n-1}(t)] = c_0 \mathcal{D}_{n-1} [N[\phi(t; q)]], \quad n \in \mathbb{N} \text{ and } n \geq 2.
\]

Equations (10) and (11) constitutes the so-called high order deformation equations. These equations are linear and can be recursively solved to obtain each term \( u_n(t) \), \( n \in \mathbb{N}_0 \) of (6). Typically, using a symbolic computer environment, such as Mathematica, Maple or Matlab, for instance, one can automatically solve (10) and (11) and obtain an approximate solution

\[
u^m(t) = u_0(t) + \sum_{n=1}^{m} u_n(t)
\]

of order \( m \) of the problem (1). This approximate solution can be called \( m \)-th-order solution.

Note additionally that (12) must satisfy the initial conditions (2) of our problem and \( u_0(t) \) already does. Therefore \( u_n(t) \) and their derivatives up to order \( r-1 \) must satisfy null initial conditions for \( n = 1, \ldots, m \). In summary:

\[
\begin{align*}
\begin{cases}
u_0(t_0) = x_0^{(0)} \\
u_0^{(1)}(t_0) = x_0^{(1)} \\
\vdots \\
u_0^{(r-1)}(t_0) = x_0^{(r-1)}
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
u_n(t_0) = 0 \\
u_n^{(1)}(t_0) = 0 \\
\vdots \\
u_n^{(r-1)}(t_0) = 0
\end{cases}, \quad \forall n = 1, \ldots, m
\end{align*}
\]

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2.3. A trivial example of application of HAM

For the sake of clarity in exposition let’s apply the HAM technique to a nonlinear initial value problem with the known the solution \( x = \tan t \):

\[
\begin{cases}
x' = 1 + x^2 \\
x(0) = 0
\end{cases}
\] (14)

This IVP will be also used later as a simple test case.

Let’s consider

\[
N[x] = x' - x^2 - 1,
\] (15)

define

\[
\mathcal{L} [\cdot] = \frac{d}{dt} [\cdot],
\]

consider the convergence control parameter \( c_0 = -1 \), define the homotopy Maclaurin series

\[
\phi (t; q) = u_0 (t) + \sum_{n=1}^{+\infty} u_n (t) q^n, \quad q \in [0, 1],
\]

and choose the following initial guess (satisfying the initial conditions)

\[
u_0 (t) = t.
\] (16)

Hence, the zeroth-order deformation equation is

\[
(1 - q) \frac{d}{dt} [\phi (t; q) - t] = -q \left\{ \frac{\partial \phi (t; q)}{\partial t} - [\phi (t; q)]^2 - 1 \right\}, \quad q \in [0, 1].
\] (17)

and the corresponding high-order homotopy equations are

\[
\frac{d}{dt} [u_m (t) - \chi_m u_{m-1} (t)] = -\mathcal{D}_{m-1} \left[ \frac{\partial \phi (t; q)}{\partial t} - [\phi (t; q)]^2 - 1 \right],
\] (18)

\[
m \geq 1 \text{ and } \chi_m = \begin{cases} 0 & \text{if } m = 1 \\ 1 & \text{if } m > 1 \end{cases}.
\]

Applying (9) one deduce from (18) the following high-order deformation equations:

\[
\frac{d}{dt} [u_m (t)] = (\chi_m - 1) \left( \frac{d}{dt} [u_{m-1} (t)] - 1 \right) + \sum_{k=0}^{m-1} u_k (t) u_{m-1-k} (t),
\] (19)

\[
m \geq 1 \text{ and } \chi_m = \begin{cases} 0 & \text{if } m = 1 \\ 1 & \text{if } m > 1 \end{cases}.
\]
From (16) and (19) let’s present the first four linear ordinary differential equations as well as the corresponding solutions recursively solved:

| m | Linear equation | Solution       |
|---|----------------|----------------|
| 1 | $\frac{du_1}{dt} = t^2$ | $u_1 = \frac{t^3}{3}$ |
| 2 | $\frac{du_2}{dt} = 2tu_1$ | $u_2 = \frac{2t^5}{15}$ |
| 3 | $\frac{du_3}{dt} = u_1^2 + 2tu_2$ | $u_3 = \frac{17t^7}{315}$ |
| 4 | $\frac{du_4}{dt} = 2u_1u_2 + 2tu_3$ | $u_4 = \frac{62t^9}{2835}$ |

Based on (20) one can write the fourth order solution of the IVP (14):

$$u^4(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9.$$  \hspace{1cm} (21)

Observe and compare (21) with the Maclaurin series of $\tan t$:

$$\tan t = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{1382}{155 925}t^{11} + \frac{21 844}{6081 075}t^{13} + \frac{929 569}{638 512 875}t^{15} + O(t^{17}).$$

It can be stated that HAM “surgically” determines the terms of the Maclaurin series of the solution of our problem.

2.4. Main features of HAM

All the information needed to find the terms of (6) are contained in the zeroth-order equation (3). One important parameter in this equation, see [1–4], is precisely $c_0$ which controls the convergence/divergence of the series solution of (1). This parameter is called convergence control parameter and need to be carefully chosen. In [1–3] are presented some practical approaches to choose $c_0$ in order to ensure the convergence as well as the speed of convergence of the series solution built in the frame of HAM. Besides, the user of HAM has a great freedom in choosing the linear operator $L$ as well as the initial guess, $u_0(t)$, of the solution. All these facts underlies some remarkable advantages of HAM, namely:

1. Guarantee of convergence by adequately choosing $c_0$, the convergence control parameter;
2. Flexibility on the choice of base functions and decide about the solution expression by adequately choosing $L$ and the initial guess $u_0(t)$;
3. Ability to find the main parameters, such as amplitude and frequency, of periodic solutions of nonlinear evolution problems;
4. Great generality of application ranging from solving weakly to strong nonlinear differential equations or even fractional differential equations.

3. IFOHAM-Iterative first order HAM

3.1. Motivation
Consider the original IVP problem (1) and (2). Suppose that (6) converges and consider the first-order deformation equation (10) 
\[ \mathcal{L}[u_1(t)] = c_0 [N[u_0(t)]] , \]
from which we can obtain \( u_1 \). It will be reasonable to conjecture that \( u_0(t) + u_1(t) \) will be a best initial guess than the (postulated) original one \( u_0(t) \). This argument suggest the following iterative procedure to improve the initial guess \( u_0 \) for the solution of (1):
\[ \mathcal{L}[u_{m+1}(t)] = c_0 \left[ N \left( \sum_{k=0}^{m} u_k(t) \right) \right], \quad m \geq 0. \] (22)

As was the case in applying HAM, in accordance with (13), one must assure that \( u_k(t) \) and their derivatives up to order \( r - 1 \) must satisfy null initial conditions for \( k = 1, \ldots, m + 1 \). For instance, if \( N[\cdot] \) is defined by a first-order nonlinear differential equation, then
\[
\begin{aligned}
  u_0(t_0) &= x_0^{(0)} \\
  u_1(t_0) &= 0 \\
  &\vdots \\
  u_m(t_0) &= 0 \\
  &\vdots 
\end{aligned}
\] (23)

Algorithm (22) is entirely based on the first-order deformation equation (10) of HAM. So, let’s call it iterative first-order HAM: IFOHAM. Define
\[ x_m(t) = \sum_{k=0}^{m} u_k(t) . \] (24)
and call (24) an \( m \)th-order solution of the problem (1) and (2).

Some interesting issues arise immediately:
1. Does (24) converge for the solution of (1)? In what circumstances?
2. How does compare or relate (22) with other iterative algorithms?
3. How does the performance of (22) relates to the performance of HAM?
4. What features (22) share with HAM? In what features is (22) better effective than HAM?

In the following we will respond these issues and we will present some exploratory preliminary results.

3.2. IFOHAM and Picard-Lindelöf 's iteration algorithm

Consider the IVP described in the following first-order ordinary differential equation and the corresponding initial condition:

\[
\begin{align*}
\frac{dx}{dt} &= f(t, x) \\
x(t_0) &= x^{(0)}
\end{align*}
\]  

(25)

Note that in this case the nonlinear operator \( N[\cdot] \) can be identified with an ordinary differential equation in the canonical form, that is

\[
N[x] \equiv \frac{dx}{dt} - f(t, x).
\]  

(26)

Due to (26), IFOHAM (22) reduces to

\[
L[u_{m+1}(t)] = c_0 \left[ \sum_{k=0}^{m} u'_k(t) - f \left( t, \sum_{k=0}^{m} u_k(t) \right) \right], \quad m \geq 0.
\]  

(27)

Let

\[
u_0(t) = x^{(0)}_0,\]

(28)

be our initial guess, and assume

\[
u_k(t_0) = 0, \quad \forall k \in \mathbb{N}.
\]  

(29)

Consider

\[
c_0 = -1,
\]  

(30)

and define

\[
L [h(t)] = \frac{dh}{dt}(t) \quad \text{and} \quad L^{-1} [h(t)] = \int_{t_0}^{t} h(\xi) d\xi.
\]  

(31)
From (27) using (23), (28), (30) and (31) one deduce
\[ \sum_{k=0}^{m+1} u_k(t) = x_0^{(0)} + \int_{t_0}^{t} f\left(\xi, \sum_{k=0}^{m} u_k(\xi)\right) d\xi, \quad (32) \]
That is,
\[ \begin{align*}
x_0(t) &= x_0^{(0)} \\
x_{m+1}(t) &= x_0^{(0)} + \int_{t_0}^{t} f(\xi, x_m(\xi)) d\xi, \quad m \geq 0,
\end{align*} \quad (33) \]
where
\[ x_m(t) = \sum_{k=0}^{m} u_k(t). \quad (34) \]
Clearly, (33) represents Picard-Lindelöf’s iterative algorithm. So, in this case and under the described restrictions IFOHAM (27) and Picard-Lindelöf’s iteration algorithm (33) generate the same sequence of functions. The following result can be stated:

**Proposition 1.** Consider the IVP
\[ \begin{align*}
\frac{dx}{dt} &= f(t, x) \\
x(t_0) &= x_0^{(0)},
\end{align*} \]
where \( f \) is a continuous real function on an open set \( D \subseteq \mathbb{R}^2 \) and suppose that \( (t_0, x_0^{(0)}) \in D \). Consider additionally the corresponding nonlinear operator
\[ N[x] = \frac{dx}{dt} - f(t, x), \]
the IFOHAM algorithm
\[ \begin{align*}
u_0(t) &= x_0^{(0)} \\
\mathcal{L}[u_{m+1}(t)] &= c_0 \left[ N \left[ \sum_{k=0}^{m} u_k(t) \right] \right], \quad m \geq 0,
\end{align*} \quad (35) \]
where \( c_0 = -1 \), \( \mathcal{L}[h(t)] = \frac{dh}{dt}(t) \) and \( \mathcal{L}^{-1}[h(t)] = \int_{t_0}^{t} h(\xi) d\xi \) and consider further the Picard-Lindelöf’s iteration algorithm
\[ \begin{align*}
x_0(t) &= x_0^{(0)} \\
x_{m+1}(t) &= x_0^{(0)} + \int_{t_0}^{t} f(\xi, x_m(\xi)) d\xi, \quad m \geq 0.
\end{align*} \quad (36) \]
Then,
\[ x_m(t) = \sum_{k=0}^{m} u_k(t) , \forall m \in \mathbb{N}_0 \] (37)
whenever \((t, x_k(t)) \in D\) for \(k = 1, \ldots, m - 1\).

**Proof.** This statement is the instance \(c_0 = -1\) of Proposition □

This means that under the described restrictions the convergence of IFO-HAM is ensured if (25) satisfies the classical Picard-Lindelöf’s conditions for the existence and uniqueness of a solution. In short:

**Proposition 2.** Let \(D\) be an open set in \(\mathbb{R}^2\). Let \((t_0, x_0^{(0)}) \in D\) and let \(a\) and \(b\) be positive constants such that the set
\[ R = \left\{ (t, x) : |t - t_0| \leq a \text{ and } |x - x_0^{(0)}| \leq b \right\} \]
is contained in \(D\). Suppose function \(f\) is continuous and defined on \(D\) and satisfies a Lipschitz condition with respect to \(x\) in \(\mathbb{R}\). Let \(M = \max_{(t,x) \in R} |f(t,x)|\) and \(A = \min \left\{a, \frac{b}{M} \right\}\). Then,

**a)** the IVP
\[ \begin{cases}
\frac{dx}{dt} = f(t, x) \\
x(t_0) = x_0^{(0)}
\end{cases} \]
has a unique solution \(x = x(t)\) on the open interval \(I = ]t_0 - A, t_0 + A[\).

**b)** the sequence \(\{x_n(t)\}\), where
\[ \begin{aligned}
x_0(t) &= x_0^{(0)} \\
x_{n+1}(t) &= x_0^{(0)} + \int_{t_0}^{t} f(\xi, x_n(\xi)) d\xi , \ n \geq 0
\end{aligned} \]
converges uniformly on \(I\) to \(x(t)\);

**c)** the sequence \(\{\sum_{k=0}^{n} u_k(t)\}\) where
\[ \begin{aligned}
&u_0(t) = x_0^{0} \\
&\mathcal{L}[u_{n+1}(t)] = c_0 \cdot N[\sum_{k=0}^{n} u_k(t)] , \ n \geq 0 \\
&\text{with } u_k(t_0) = 0, \forall k \in \mathbb{N}
\end{aligned} \]
with \(N[x] = \frac{dx}{dt} - f(t,x), c_0 = -1, \mathcal{L}[h(t)] = \frac{dh}{dt}(t)\) and \(\mathcal{L}^{-1}[h(t)] = \int_{t_0}^{t} h(\xi) d\xi\) converges uniformly on \(I\) to \(x(t)\).
**Proof.** The proof of parts (a) and (b) are classic and can be found for instance in [9]. Part (c) is an immediate consequence of Proposition 4. □

Let us now study the role of the convergence control parameter $c_0$ in the behavior of IFOHAM.

### 3.3. IFOHAM and the convergence control parameter $c_0$

Consider again the IVP described in (25) and let the corresponding nonlinear operator $N[\cdot]$ be

$$N [x] \equiv \frac{dx}{dt} - f (t, x). \quad (38)$$

So, IFOHAM (22) reduces to

$$\mathcal{L} [u_{m+1} (t)] = c_0 \left[ \sum_{k=0}^{m} u_k (t) - f \left( t, \sum_{k=0}^{m} u_k (t) \right) \right], \quad m \geq 0. \quad (39)$$

Let our initial guess be

$$u_0 (t) = x_{0}^{(0)}, \quad (40)$$

and define

$$\mathcal{L} [h (t)] = \frac{dh}{dt} (t) \quad \text{and} \quad \mathcal{L}^{-1} [h (t)] = \int_{t_0}^{t} h (\xi) \, d\xi. \quad (41)$$

From (39) one deduce using (23), (40) and (41),

$$\sum_{k=0}^{m+1} u_k (t) = (1 + c_0) \sum_{k=0}^{m} u_k (t) - c_0 \left[ x_{0}^{(0)} + \int_{t_0}^{t} f \left( \xi, \sum_{k=0}^{m} u_k (\xi) \right) \, d\xi \right], \quad (42)$$

or, equivalently using (34)

$$x_{m+1} (t) = (1 + c_0) x_m - c_0 \left[ x_{0}^{(0)} + \int_{t_0}^{t} f \left( \xi, x_m (\xi) \right) \, d\xi \right], \quad (43)$$

Note that interestingly (43) can be interpreted as a weighted average between $x_m$, the previous iteration, and

$$x_{0}^{(0)} + \int_{t_0}^{t} f \left( \xi, x_m (\xi) \right) \, d\xi.$$
the iterate $m + 1$ computed using Picard-Lindelöf’s iterative algorithm (33). This fact suggests the decrease of the convergence speed of the algorithm for increasing values of $c_0$ in the interval $[-1, 0]$. In reality this conjecture will be corroborated by expression (52) from Proposition 5.

From (43) one can deduce the equivalent useful expression:

$$x_{m+1} = x_0^{(0)} + (1 + c_0) \int_{t_0}^{t} \frac{dx_m}{dt} (\xi) d\xi - c_0 \int_{t_0}^{t} f (\xi, x_m (\xi)) d\xi.$$  \hspace{1cm} (44)

Let’s summarize these results:

**Proposition 3.** Consider the IVP

$$\begin{cases}
\frac{dx}{dt} = f(t, x) \\
x(t_0) = x_0^{(0)}
\end{cases},$$

where $f$ is a continuous real function on an open set $D \in \mathbb{R}^2$ and suppose that $(t_0, x_0^{(0)}) \in D$. Let $c_0 \in \mathbb{R}$ and consider algorithm

$$\begin{cases}
u_0 (t) = x_0^{(0)} \\
L[u_{m+1} (t)] = c_0 \left[ \sum_{k=0}^{m} u_k' (t) - f (t, \sum_{k=0}^{m} u_k (t)) \right], m \geq 0
\end{cases}$$

with $u_k (t_0) = 0, \forall k \in \mathbb{N}$ \hspace{1cm} (45)

with $L[h(t)] = \frac{dh}{dt} (t)$ and $L^{-1} [h(t)] = \int_{t_0}^{t} h(\xi) d\xi$ and algorithm

$$\begin{cases}
x_0 (t) = x_0^{(0)} \\
x_{m+1} = x_0^{(0)} + (1 + c_0) \int_{t_0}^{t} \frac{dx_m}{dt} (\xi) d\xi - c_0 \int_{t_0}^{t} f (\xi, x_m (\xi)) d\xi.
\end{cases}$$ \hspace{1cm} (46)

Then,

$$x_m (t) = \sum_{k=0}^{m} u_k (t), \forall m \in \mathbb{N}_0$$ \hspace{1cm} (47)

whenever $(t, x_k (t)) \in D$ for $k = 1, \ldots, m - 1$.

**Proof.** Let’s argue by mathematical induction. For $m = 0$ (47) is trivially true from definition. Consider now the inductive hypothesis. Suppose that (47) is true for some $p \in \mathbb{N}$, that is, $x_p (t) = \sum_{k=0}^{p} u_k (t)$ and $(t, x_k (t)) \in D$ for $k = 1, \ldots, p - 1$. Let’s prove that

$$u_{p+1} (t) = x_{p+1} (t) - x_p (t),$$
that is, \( x_{p+1}(t) = \sum_{k=0}^{p+1} u_k(t) \). From (43) and using the inductive hypothesis we successively deduce

\[
\begin{align*}
    u'_{p+1}(t) &= c_0 \left[ \sum_{k=0}^{p} u'_k(t) - f(t, \sum_{k=0}^{p} u_k(t)) \right] \\
    u'_{p+1}(t) &= c_0 \left[ x'_p(t) - f(t, x_p(t)) \right],
\end{align*}
\]

and from (46)

\[
    x'_{p+1}(t) = (1 + c_0) x'_p(t) - c_0 f(t, x_p(t)) \Rightarrow
\]

\[
    (x_{p+1}(t) - x_p(t))' = c_0 \left[ x'_p(t) - f(t, x_p(t)) \right].
\]

Furthermore, \( u_{p+1}(t_0) = 0 \) from (45) and \( x_{p+1}(t_0) = x_p(t_0) = x^{(0)} \) from (46), hence

\[
    u_{p+1}(t_0) = x_{p+1}(t_0) - x_p(t_0) = 0.
\]

So, \( u_{p+1}(t) = x_{p+1}(t) - x_p(t) \) for all \( t \) such that \( (t, x_p(t)) \in D \). This completes the inductive step. \( \square \)

We are interested in knowing for what values of \( c_0 \) can we guarantee the convergence of the IFOHAM algorithm (39) in the context of choices (40) and (41). In this way, we will establish some sufficient conditions for convergence of this algorithm.

Let us first present a trivial lemma that we will need.

**Lemma 4.** Let \( \alpha \) and \( \beta \) real constants and \( h(x) = (1 + x) \alpha - x \beta \) with \( |\alpha| \leq A \) and \( |\beta| \leq A \). If \( x \in [-1, 0] \) then \( |h(x)| \leq A \).

**Proof.** Let \( \alpha - \beta = \delta \). Then, \( h(x) = \alpha + x\delta \) and \( h(x) = (1 + x)\delta + \beta \). If \( \delta = 0 \) then \( h(x) = \alpha = \beta \forall x \in \mathbb{R} \). Hence, \( |h(x)| = |\alpha| \leq A \). If \( \delta > 0 \) then \( h(x) = \alpha + x\delta \leq \alpha \forall x \in [-1, 0] \). Hence, \( -A \leq \beta \leq h(x) \leq \alpha \leq A \). Then, \( |h(x)| \leq A \). If \( \delta < 0 \) then \( h(x) = \alpha + x\delta \geq \alpha \) and \( h(x) = (1 + x)\delta + \beta \leq \beta \forall x \in [-1, 0] \). Hence, \( -A \leq \alpha \leq h(x) \leq \beta \leq A \). Then, \( |h(x)| \leq A \). So, If \( x \in [-1, 0] \) then \( |h(x)| \leq A \). \( \square \)
Proposition 5. Let $D$ be an open set in $\mathbb{R}^2$. Let $\left( t_0, x_0^{(0)} \right) \in D$ and let $a$ and $b$ be positive constants such that the set 
\[ R = \left\{ (t, x) : |t - t_0| \leq a \text{ and } |x - x_0^{(0)}| \leq b \right\} \]

is contained in $D$. Suppose function $f$ is continuous and defined on $D$ and satisfies a Lipschitz condition with respect to $x$ in $R$ with Lipschitz constant $L$. Let $M = \max_{(t,x) \in R} |f(t,x)|$ and $A = \min \left\{ a, \frac{b}{M} \right\}$. Consider the IVP
\[ \begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0^{(0)} \end{cases} \tag{48} \]

and its unique solution $x = x(t)$ on the open interval $I = ]t_0 - A, t_0 + A[. \quad \text{Consider also the IFOHAM algorithm}

\[ \begin{cases} u_0(t) = x_0^{(0)} \\ \mathcal{L}[u_{m+1}(t)] = c_0 \left[ \sum_{k=0}^{m} u_k(t) - f(t, \sum_{k=0}^{m} u_k(t)) \right], m \geq 0 \\ \text{with } u_k(t_0) = 0, \forall k \in \mathbb{N} \end{cases} \tag{49} \]

with $\mathcal{L}[h(t)] = \frac{dh}{dt}(t)$ and $\mathcal{L}^{-1}[h(t)] = \int_{t_0}^{t} h(\xi) d\xi$ and algorithm

\[ \begin{cases} x_0(t) = x_0^{(0)} \\ x_{m+1}(t) = x_0^{(0)} + (1 + c_0) \int_{t_0}^{t} \frac{dx_m}{dt}(\xi) d\xi - c_0 \int_{t_0}^{t} f(\xi, x_m(\xi)) d\xi \end{cases} \tag{50} \]

and its associated operator

\[ F(x(t)) = (1 + c_0) x(t) - c_0 \left( x_0^{(0)} + \int_{t_0}^{t} f(\xi, x(\xi)) d\xi \right). \tag{51} \]

1. If $c_0 \in [-1, 0]$ then \( \{ \sum_{k=0}^{n} u_k(t) \} \) converges uniformly on $I$ to $x(t)$.
2. Define

\[ S = \left\{ x(t) \in C(J) : |x(t) - x_0^{(0)}| \leq b, |t - t_0| \leq A \right\}, \]

let $\tilde{L}$ be any constant $\tilde{L} > L$, $J = [t_0 - A, t_0 + A]$ and consider the norm defined as follows:

\[ \|x\|_e = \max_{t \in J} \left| x(t) e^{-\tilde{L}|t-t_0|} \right|. \]
If \( x(t) \) and \( y(t) \) belongs to \( S \) and \( c_0 \in [-1,0[ \) then
\[
\| F(x(t)) - F(y(t)) \|_e \leq k\| x(t) - y(t) \|_e
\]
with
\[
0 < k = 1 + \left( 1 - \frac{L}{L} \right) \left( 1 - e^{-LA} \right) c_0 < 1. \tag{52}
\]

**Proof.** We begin by demonstrating part 1. The demonstration of part 2 will follow from the latter. From Proposition 3 one knows that
\[
x_n(t) = \sum_{k=0}^{n} u_k(t).
\]
Hence, it is sufficient to show that if \( c_0 \in [-1,0[ \) then \( \{x_n(t)\} \) converges uniformly on \( I \) to \( x(t) \). Therefore consider algorithm (50) and its associated operator (51).

Let's show that \( \{x_n(t)\} \) converges uniformly on \( I \) to some \( y(t) \) using Banach's fixed point theorem. The missing details of this elementary approach can be found in [10] and [11], for instance.

Let \( J = [t_0 - A, t_0 + A] \) and define the (non-empty, closed) subset
\[
S = \left\{ x \in C(J) : \| x(t) - x_0 \|_0 \leq b, \ |t - t_0| \leq A \right\},
\]
of the Banach space \( C(J) \) with the norm \( \| \cdot \|_\infty \). Note that
\[
\left\| F(x(t)) - x_0 \right\|_\infty = \left\| (1 + c_0) \left( x(t) - x_0 \right) - c_0 \int_{t_0}^{t} f(\xi, x(\xi)) d\xi \right\|_\infty.
\]
If \( x(t) \in S \), then \( \| x - x_0 \|_\infty \leq b \) and
\[
\left\| \int_{t_0}^{t} f(\xi, x(\xi)) d\xi \right\|_\infty = \max_{t \in J} \left( \int_{t_0}^{t} f(\xi, x(\xi)) d\xi \right) \leq AM \leq b.
\]
Note also that \( c_0 \in [-1,0[ \), so we can conclude from Lemma 4 that
\[
\| F(x(t)) - x_0 \|_\infty \leq b.
\]
Hence,
\[
x(t) \in S \Rightarrow F(x(t)) \in S.
\]
Let $\tilde{L}$ be any constant $\tilde{L} > L$ and consider the norm
\[
\|x\|_e = \max_{t \in J} \left| x(t) e^{-\tilde{L}|t-t_0|} \right|.
\]
Observe that norms $\|\cdot\|_e$ and $\|\cdot\|_\infty$ are equivalent. Suppose $x(t)$ and $y(t)$ are in $S$ and consider now the expression
\[
|F(x(t)) - F(y(t))| =
= \left| (1 + c_0) (x(t) - y(t)) - c_0 \left( \int_{t_0}^t (f(\xi, x(\xi)) - f(\xi, y(\xi))) \, d\xi \right) \right|,
\]
obtained from (51). Clearly
\[
|F(x(t)) - F(y(t))| e^{-\tilde{L}|t-t_0|} \leq
\leq |1 + c_0| |x(t) - y(t)| e^{-\tilde{L}|t-t_0|} + |c_0| L \left| \int_{t_0}^t |x(\xi) - y(\xi)| \, d\xi \right| e^{-\tilde{L}|t-t_0|}
\]
and
\[
\|F(x(t)) - F(y(t))\|_e \leq
\leq |1 + c_0| \|x(t) - y(t)\|_e + |c_0| L \left\| \int_{t_0}^t |x(\xi) - y(\xi)| \, d\xi \right\|_e.
\]
One can deduce that
\[
\left\| \int_{t_0}^t |x(\xi) - y(\xi)| \, d\xi \right\|_e \leq \frac{\|x(t) - y(t)\|_e}{\tilde{L}} \left( 1 - e^{-\tilde{L}A} \right),
\]
therefore
\[
\|F(x(t)) - F(y(t))\|_e \leq
\leq \left\{ |1 + c_0| + |c_0| \frac{L}{\tilde{L}} \left( 1 - e^{-\tilde{L}A} \right) \right\} \|x(t) - y(t)\|_e.
\]
If $c_0 \in [-1, 0[$, observe that
\[
0 < k = |1 + c_0| + |c_0| \frac{L}{\tilde{L}} \left( 1 - e^{-\tilde{L}A} \right) = (1 + c_0) - c_0 \frac{L}{\tilde{L}} \left( 1 - e^{-\tilde{L}A} \right) < 1,
\]
that is
\[ 0 < 1 + \left( 1 - \frac{L}{L} (1 - e^{-L}) \right) c_0 < 1. \]

So,
\[ \| F(x(t)) - F(y(t))\|_e \leq k \| x(t) - y(t)\|_e, \]

is a contraction. Therefore, from Banach’s fixed point theorem one conclude that \( \{x_n(t)\} \) converges uniformly on \( J \) to some fixed point \( y(t) \) of (51). Clearly, if \( y(t) \) is the fixed point of (51) then, one deduce also that
\[ y(t) = x_0^{(0)} + \int_{t_0}^{t} f(\xi, y(\xi)) \, d\xi, \]

that is, \( y(t) \) is the solution the IVP (48) on the interior of \( J \). From the uniqueness of the solution we will conclude that \( y(t) = x(t) \) on \( I \). This completes the proof of both parts. □

We would like to stress that Proposition 5 establishes sufficient conditions for the convergence of IFOHAM under the corresponding context. The convergence also depends on the structure of \( f \). So, it will not come as a surprise if convergence is also verified in a wider range \([c, 0] \) with \( c < -1 \).

Moreover, expression (52) suggest that the minimum on \([-1, 0]\) of the contraction constant \( k \) is attained at \( c_0 = -1 \). This means that in this frame and in the absence of information about the convergence of IFOHAM for \( c_0 \) less than \(-1\) the best choice for this parameter will be \( c_0 = -1 \), that is, the best choice will be Picard-Lindelöf’s iteration algorithm. So, the knowledge of the structure of \( f \) in (38) is of primordial importance for the useful use of the IFOHAM algorithm in the studied context.

4. Results and discussion

In order to preliminary compare the relative performance of HAM and IFOHAM we will address again the IVP (14).

In Figures 1 and 2 we display for different values of the convergence control parameter \( c_0 \) the squared residuals \( E_m \) corresponding to different \( m \)-th order solutions obtained using HAM and IFOHAM. The squared residuals were computed using expression
\[ E_m = \int_{\Omega} N \left( \sum_{i=0}^{m} u_i \right) \, dt \]
where $N$ represents operator (15) and $\Omega = [-1, 1]$. In the bottom sub-figures we display a more detailed zoom to improve the determination of the location of the value of the parameter $c_0$ that minimizes $E_m$.

With respect to Figure 1 and concerning the HAM, data suggest that:

- HAM converges for $c_0 \in [-1, 0]$ and diverges for $c_0 > 0$;
- Performance of the HAM algorithm for this test case improves in the neighborhood of $c_0 = -1$.

With respect to Figure 2 and concerning IFOHAM, data suggest that:

- IFOHAM converges for $c_0 \in [-1.3, 0]$ and diverges for $c_0 \geq 0$;
Performance of the IFOHAM algorithm is the best in the neighborhood \( c_0 = -1.2 \).

Note that the convergence of IFOHAM is assured if \( c_0 \in [-1, 0] \) in agreement with Proposition 5. However, depending on the structure of \( f \) in (38), convergence of IFOHAM, as noted in this case, can occur over a wider range \([c, 0]\) with \( c < -1 \).

One observe also that, the performance of IFOHAM, for \( c_0 \in [-1, 0] \), is best at the left end of this range. This fact is in agreement with expression (52) since the minimum value of the contraction constant \( k \) on \([-1, 0]\) interval is attained at \( c_0 = -1 \). As previously mentioned at the end of the last section, this means that in the absence of information about the convergence of IFOHAM for \( c_0 \) less than \(-1\), the best choice for this parameter will
Table 1: HAM effectiveness, $c_0 = -1$ in computing the $m$th-order approximate solution

| $k/m$ | $u_k$ | $E_m$ | CPU time [s] |
|-------|-------|-------|--------------|
| 0     | $t$   | $t$   | $4.00e -01$  |
| 1     | $t^3$ | $t + t^3$ | $1.28e -01$  |
| 2     | $t^5$ | $t + t^3 + 2t^5$ | $3.38e -02$  |
| 3     | $t^7$ | $t + t^3 + 2t^5 + 17t^7$ | $7.88e -03$  |
| 4     | $t^9$ | $t + t^3 + 2t^5 + 17t^7 + 62t^9$ | $1.70e -03$  |

be $c_0 = -1$, that is, the best choice will be Picard-Lindelöff's iteration algorithm. So, the knowledge of the structure of $f$ in (38) is essential for an effective use of the IFOHAM algorithm in the studied context.

In Tables 1, 2 and 3 we display the computed squared residuals $E_m$ as well as the computational CPU time consumed to obtain the corresponding $m$th-order approximate solutions for cases $c_0 = -1$ (HAM), $c_0 = -1$ (IFOHAM) and $c_0 = -1.2$ (IFOHAM). The above cases have been chosen especially because:

- HAM is better effective in the neighborhood of $c_0 = -1$ as was suggested from the analysis of Figure 1;

- IFOHAM with $c_0 = -1$ (that is, Picard-Lindelöff’s iteration algorithm) is the best blind implementation of IFOHAM in the absence of information regarding the structure of $f$;

- IFOHAM in the neighborhood of $c_0 = -1.2$ is the best informed implementation of IFOHAM as was suggested from the analysis of Figure 2.

Considering the extension of some expressions of the $m$th-order terms and $m$th-order approximate solutions these expressions were only partially reproduced in the Tables 2 and 3. However, the missing terms replaced by suspension points can be easily obtained by applying the IFOHAM technique on a symbolic computer environment.

The tabulated data suggest that in addressing our test case, the IVP (14), Picard-Lindelöff’s iteration algorithm (IFOHAM with $c_0 = -1$) is better effective than the best implementation of HAM (HAM with $c_0 = -1$) and the implementation of IFOHAM with $c_0 = -1.2$ is the best of all the illustrated implementations.
Table 2: IFOHAM effectiveness, $c_0 = -1$ in computing the $m$th-order approximate solution

| $k$ | $u_k$ |
|-----|-------|
| 0   | $t$   |
| 1   | $\frac{t^3}{3}$ |
| 2   | $\frac{t^5}{5} + \frac{2t^3}{3} + \frac{t}{3} + t$ |
| 3   | $\frac{t^7}{7} + \frac{2t^5}{5} + \frac{t^3}{3} + t + \frac{t}{3} + t$ |
| 4   | $\frac{t^9}{9} + \frac{2t^7}{7} + \frac{t^5}{5} + \frac{t^3}{3} + t + t$ |

| $m$ | $\sum_{k=0}^{m} u_k$ | $E_m$ | CPU time [s] |
|-----|----------------|------|--------------|
| 0   | $t$ | $4.00e-01$ | 0.000 |
| 1   | $\frac{t^3}{3}$ | $1.28e-01$ | 0.609 |
| 2   | $\frac{t^5}{5} + \frac{2t^3}{3} + \frac{t}{3} + t$ | $2.42e-02$ | 1.031 |
| 3   | $\frac{t^7}{7} + \frac{2t^5}{5} + \frac{t^3}{3} + t + \frac{t}{3} + t$ | $2.69e-03$ | 1.406 |
| 4   | $\frac{t^9}{9} + \frac{2t^7}{7} + \frac{t^5}{5} + \frac{t^3}{3} + t + t$ | $1.87e-04$ | 1.938 |

Table 3: IFOHAM effectiveness, $c_0 = -1.2$ in computing the $m$th-order approximate solution

| $k$ | $u_k$ |
|-----|-------|
| 0   | $t$   |
| 1   | $\frac{2t^3}{3} + t$ |
| 2   | $\frac{2t^3}{3} + t + \frac{2t^3}{3} + t + \frac{t}{3} + t$ |
| 3   | $\frac{2t^3}{3} + t + \frac{2t^3}{3} + t + \frac{t}{3} + t$ |
| 4   | $\frac{2t^3}{3} + t + \frac{2t^3}{3} + t + \frac{t}{3} + t$ |

| $M$ | $\sum_{k=0}^{M} u_k$ | $E_M$ | CPU time [s] |
|-----|----------------|------|--------------|
| 0   | $t$ | $4.00e-01$ | 0.000 |
| 1   | $\frac{2t^3}{3} + t$ | $1.03e-01$ | 0.609 |
| 2   | $\frac{2t^3}{3} + t + \frac{2t^3}{3} + t + \frac{t}{3} + t$ | $5.73e-03$ | 1.063 |
| 3   | $\frac{2t^3}{3} + t + \frac{2t^3}{3} + t + \frac{t}{3} + t$ | $3.54e-05$ | 1.438 |
| 4   | $\frac{2t^3}{3} + t + \frac{2t^3}{3} + t + \frac{t}{3} + t$ | $5.45e-06$ | 2.078 |
Note that sequences of approximate solutions generated by HAM or IFOH- 
HAM converge to the MacLaurin series of \( x = \tan t \) (the exact known solu-
tion of our problem). Despite this fact, it should be noted that the terms 
of each approximate solution already calculated in one iteration using IFO-
HAM may be modified in the next iteration contrary to what happens using 
HAM. As was noted before, HAM can “surgically” determines the terms of 
the Maclaurin series of the solution of our problem.

Moreover, in a few iterations the IFOHAM algorithm has to handle par-
ticularly long expressions. This may constitute a drawback of this algorithm.

However, these preliminary tests suggest that IFOHAM exhibits an inter-
esting performance both in aspects related to the speed of convergence and 
in aspects related to the CPU calculation time.

5. Conclusion and future work

In addressing the classic IVP problem

\[
\begin{align*}
\frac{dx}{dt} &= f(t, x) \\
x(t_0) &= x_0^{(0)},
\end{align*}
\]

we found that, conveniently defining \( \mathcal{L}[h(t)] = \frac{dh}{dt}(t) \), IFOHAM

\[
\mathcal{L}[u_{m+1}(t)] = c_0 \left[ N \sum_{k=0}^{m} u_k(t) \right], \quad m \geq 0,
\]

with \( c_0 = -1 \) coincides exactly with Picard-Lindelöf’s iteration algorithm. 
We concluded also that IFOHAM converges if \( c_0 \in ]-1, 0[ \) and depending 
on the structure of \( f \) IFOHAM can still converge with a better convergence 
speed to the searched solution if \( c_0 < -1 \). Clearly, the knowledge of the 
structure of \( f \) is of primordial importance for the future useful use of the 
IFOHAM algorithm in the studied context. Given these facts one can state 
that IFOHAM generalizes Picard-Lindelöf’s iteration algorithm.

Preliminary tests showed that IFOHAM exhibited a very good perfor-
mance both in aspects related to the speed of convergence and in aspects 
related to the CPU calculation time.

A very favorable aspect of IFOHAM lies in the ease of its implementa-
tion which is simple and without complexities. However, in a few iterations the 
IFOHAM algorithm has to handle particularly long expressions. This may 
constitute a drawback of this algorithm.
With regard to future work we would like to mention some possible interesting directions we are presently dealing with:

- To study the convergence of IFOHAM with respect the structure of $f$ in (38) or more generally regarding the structure of the operator $N$ in (1);
- To study the existence of flexibility of IFOHAM on the choice of base functions and decide about the solution expression by adequately choosing $L$ and the initial guess $u_0(t)$ as in the use of HAM;
- To study the ability of IFOHAM to find the main parameters, such as amplitude and frequency, of periodic solutions of nonlinear evolution problems;
- Study of the applicability of IFOHAM in addressing other classes of evolution non-linear problems.

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