A New Version of the Exponentiated Exponential Distribution: Copula, Properties and Application to Relief and Survival Times

Hanaa Elgohari

Department of applied statistics, Faculty of commerce, Mansoura University, Egypt

Abstract In this paper, we introduce a new generalization of the Exponentiated Exponential distribution. Various structural mathematical properties are derived. Numerical analysis for mean, variance, skewness and kurtosis and the dispersion index is performed. The new density can be right skewed and symmetric with “unimodal” and “bimodal” shapes. The new hazard function can be “constant”, “decreasing”, “increasing”, “increasing-constant”, “upsidedown-constant”, “decreasing-constant”. Many bivariate and multivariate type model have been also derived. We assess the performance of the maximum likelihood method graphically via the biases and mean squared errors. The usefulness and flexibility of the new distribution is illustrated by means of two real data sets.

Keywords Exponentiated Exponential; Morgenstern family; Clayton Copula; Real Data Modeling; Hazard Function.

AMS 2010 subject classifications 62N01; 62N02; 62E10

DOI: 10.19139/soic-2310-5070-1093

1. Introduction and motivation

A random variable (RV) \( W \) is said to have the Exponentiated Exponential (EE) distribution if its probability density function (PDF) is given by

\[
\pi_{a,b}(w) |_{(w>0, a>0 \text{ and } b>0)} = ab e^{-bw} \left( 1 - e^{-bw} \right)^{a-1}.
\]

The corresponding cumulative distribution function (CDF) can be written as

\[
\Psi_{a,b}(w) |_{(w>0, a>0 \text{ and } b>0)} = \left( 1 - e^{-bw} \right)^a.
\]

Clearly, for \( a = 1 \), the EE reduces to the standard exponential (E) model. If \( 1 > a \), the function \( \pi_{a,b}(w) \) monotonically decreases with \( w \). If \( a > 1 \), the function \( \pi_{a,b}(w) \) attains a mode at \( w = \frac{1}{b} \log(a) \). The statistical properties of the EE model have been studied by many authors. Many authors have derived and studied the EE model, see Zheng [66], Zheng and Park [67], Kundu and Pradhan [48], Aslam et al. [15], Aryal et al. [13], Khalil et al. [42], Abouelmagd et al. ([1],[2]), Ibrahim et al. [35] and Bhatti et al. [16] among others. Recently, Alizadeh et al. [6] defined a new family based on the exponential model called the generalized odd generalized exponential family of distributions. Analogously, Hamedani et al. ([33] and [34]) defined the the type I and type I general exponential class of distributions. Other works can be cited such as Korkmaz et al. [46] (exponential Lindley odd log-logistic family) and Yadav et al. [62] (Burr-Hatke exponential model). In the work, we introduce a new version of the EE model using the Odd-Burr generalized (OB-G) family called the OBEE (OBEE). On the other hand, some new bivariate type OBEE are derived. Due to Alizadeh et al. [6], the CDF of the the OB-G family is given by
where $\mathbb{W}(w) = 1 - \mathbb{W}(w)$ and $\Phi$ refers to the parameter vector of the base line model. The PDF corresponding to (3) is given by

$$f_{\alpha, \beta, \Phi}(w) = \frac{\frac{\alpha \beta \pi(\Phi)}{\text{df}(\Phi)} \mathbb{W}(w)^{\alpha - 1} \mathbb{W}(w)^{\beta - 1}}{[\mathbb{W}(w)^{\alpha} + \mathbb{W}(w)^{\beta}]^{1 + \beta}},$$

where $\pi(\Phi) = d\mathbb{W}(w)/dw$. For $\beta = 1$, we get the Odd G (O-G) family. For $\alpha = 1$, we have the proportional reversed hazard rate family (PRHR). The OBEE CDF is given by

$$F_{\Lambda}(w) = 1 - \left\{ \frac{1 - [1 - e^{-bw}]^a}{[1 - e^{-bw}]^\alpha + \{1 - [1 - e^{-bw}]^a\}^\alpha} \right\}^{1+\beta},$$

where $\Lambda$ refers to the parameter vector of the new OBEE model. For $\beta = 1$, the OBEE reduces to the OEE. For $\alpha = 1$, the OBEE reduces to the PRHREE. The PDF corresponding to (5) is given by

$$f_{\Lambda}(w) = \alpha \beta a b e^{-bw} \left[ 1 - e^{-bw} \right]^{\alpha - 1} \frac{\{1 - \{1 - (1 - u)^{\beta - 1}\}^{\alpha - 1}\}}{\left(1 - u\right)^{a^*} + \{1 - [1 - e^{-bw}]^a\}^\alpha}^{1+\beta},$$

where $a^* = a\alpha - 1$. The hazard function (HRF) can be derived from $f_{\Lambda}(w)/S_{\Lambda}(w)$. For simulation of this new model, we obtain the quantile function (QF) of $w$ (by inverting (5)), say $w_u = F^{-1}(u)$, as

$$w_u = -b^{-1} \ln \left( 1 - \left\{ \frac{1 - (1 - u)^{\beta - 1}^{\alpha - 1}}{(1 - u)^{a^*} + \{1 - (1 - u)^{\beta - 1}^{\alpha - 1}\}} \right\}^{\frac{1}{\alpha}} \right).$$

Equation (7) is used for simulating the OBEE model. Figure 1 gives some PDF plots for some selected parameters value. Figure 2 gives some HRF plots for some selected parameters value. Based on Figure 1 the OBEE density can be right skewed and symmetric with unimodal and bimodal PDFs. Based on Figure 2 the OBEE HRF can be "constant", "decreasing", "increasing", "increasing-constant", "upside-downconstant" and "decreasing-constant".

2. Mathematical properties

2.1. Useful representations

Due to Alizadeh et al. (2016), the PDF in (6) can be expressed as

$$f(w) = \sum_{\kappa=0}^{\infty} \nabla_{\kappa} \pi_{a^*, \beta}(w),$$

where $a^* = a(1 + \kappa)$ and

$$\nabla_{\kappa} = \frac{\alpha \beta}{1 + \kappa} \sum_{i_1, i_2=0}^{\infty} (-1)^{i_2+k+\kappa} \binom{-1 + \beta}{i_1} \times \binom{-\left[ a(1 + i_1) + 1 \right]}{i_2} \binom{\alpha + i_1 + i_2 + 1}{i_3} \nabla_{\kappa}.$$
and \( \pi_{a^*,b}(w) \) refers to the density of the exponentiated exponential (EE) model with power parameter \( a^* \). By integrating (8), the CDF of \( W \) becomes

\[
F(w) = \sum_{k=0}^{\infty} \nabla_k \Pi_{a^*,b}(w),
\]

where \( \Pi_{a^*,b}(w) \) refers to the EE distribution with power parameter \( a^* \).

2.2. Asymptotics

Let \( c = \inf \{ F(w) \mid w, \pi_{a^*,b}(w) > 0 \} \), then

\[
F_\Delta(w) \sim \beta \left(1 - e^{-bw}\right)^{a\alpha} \text{ as } w \to 0,
\]

\[
f_\Delta(w) \sim \alpha \beta ab \left(1 - e^{-bw}\right)^{a} e^{-bw} \text{ as } w \to 0,
\]

and

\[
h_\Delta(w) \sim \alpha \beta ab \left(1 - e^{-bw}\right)^{a} e^{-bw} \text{ as } w \to 0.
\]

The asymptotics of CDF, PDF and HRF as \( w \to \infty \) are given by

\[
1 - F_\Delta(w) \sim \left\{ \alpha \left[1 - (1 - e^{-bw})^{a}\right]\right\}^{\beta} \text{ as } w \to \infty,
\]

\[
f_\Delta(w) \sim \frac{\alpha \beta ab \left(1 - e^{-bw}\right)^{a-1} e^{-bw}}{\left[1 - (1 - e^{-bw})^{a}\right]^{1-\beta}} \text{ as } w \to \infty,
\]

and

\[
h_\Delta(w) \sim \frac{\alpha \beta ab \left(1 - e^{-bw}\right)^{a} e^{-bw}}{\left[1 - (1 - e^{-bw})^{a}\right]^{1-\beta}} \text{ as } w \to \infty.
\]
Figure 2. HDF plots for some selected parameters value.

\[
h_{\mathbf{A}}(w) \sim \beta ab \left(1 - e^{-bw}\right)^{a-1} e^{-bw} \frac{1}{1 - (1 - e^{-bw})^a} \quad \text{as} \ w \to \infty.
\]

2.3. Moments and incomplete moments

The \(\zeta^{th}\) ordinary moment of \(W\) is given by

\[
\mu'_\zeta = \mathbf{E}(W^\zeta) = \int_{-\infty}^{\infty} w^\zeta f(w) \, dw.
\]
then we obtain
\[ \mu_\zeta \big|_{(\zeta > -1)} = b^{-\zeta} \Gamma (1 + \zeta) \sum_{n,h=0}^{\infty} \nabla^{(a_\zeta, \zeta)}_{\kappa, h}, \]  
(10)

where
\[ \nabla^{(a_\zeta, \zeta)}_{\kappa, h} = \nabla^{\kappa}_{\zeta} \frac{a_\zeta - 1}{h + 1} \left( a_\zeta - 1 \right) \]

and
\[ \Gamma (1 + \zeta) \big|_{(\zeta \in \mathbb{R}^+)} = \zeta! = \zeta^{-1} \big|_{(\zeta = 0)} (\zeta - \zeta). \]

where \( \mathbb{E}(w) = \mu'_1 \) is the mean of \( w \). The \( \zeta^{th} \) incomplete moment, say \( \varphi_\zeta (t) \), of \( w \) can be expressed, from (9), as
\[ \varphi_\zeta (t) = \int_{-\infty}^{t} w^\zeta f(w) dw = \sum_{\kappa=0}^{\infty} \nabla^{\zeta}_{\kappa} \int_{-\infty}^{t} w^\zeta \pi_{a_\zeta, b}(w) dw \]

then
\[ \varphi_\zeta (t) \big|_{(\zeta > -\delta)} = b^{-\zeta} \gamma (\zeta + 1, bt) \sum_{\kappa,h=0}^{\infty} \nabla^{(a_\zeta, \zeta)}_{\kappa, h}, \]  
(11)

where \( \gamma (\zeta, \theta) \) is the incomplete gamma function.

\[ \gamma (\zeta, \theta) \big|_{(\zeta \neq 0, -1, -2, \ldots)} = \int_{0}^{\theta} \exp (-w) dw \]
\[ = \frac{1}{\zeta} \theta^\zeta \{ \text{1F1} [\zeta; \zeta + 1; -\theta] \} \]
\[ = \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa}}{\kappa! \zeta^{\zeta + \kappa}} \theta^{\zeta + \kappa}, \]

and \( \text{1F1} [\cdot, \cdot, \cdot] \) is a confluent hypergeometric function. The first incomplete moment given by (11) with \( \zeta = 1 \) as
\[ \varphi_1 (t) = b\gamma \left( 2, \frac{1}{t} \right) \sum_{\kappa,h=0}^{\infty} \nabla^{(1, a_\zeta)}_{\kappa, h}. \]

2.4. Moment generating function (MGF)
The MGF \( M_W (t) = \mathbb{E} (\exp (t W)) \) of \( W \) can be derived from equation (8) as
\[ M_W (t) = \sum_{\kappa=0}^{\infty} \nabla^{\zeta}_{\kappa} M_{a_\zeta, b} (T), \]

where \( M_{a_\zeta, b} (T) \) is the MGF of the EW model with power parameter \( a_\zeta \).

\[ M_W (t) \big|_{(\zeta > -1)} = \sum_{\zeta=0}^{\infty} \sum_{\kappa,h=0}^{\infty} \frac{t^\zeta}{\zeta!} b^{-\zeta} \Gamma (1 + \zeta) \nabla^{(a_\zeta, \zeta)}_{\kappa, h}. \]
2.5. Residual life and reversed residual life functions

The $\rho^{th}$ moment of the residual life

$$a_\rho(t) = \mathbb{E}[(Z - t)^\rho | w > t, \rho = 1, 2, ...].$$

The $\rho^{th}$ moment of the residual life of $W$ is given by

$$a_\rho(t) = \frac{1}{1 - F_\Lambda(t)} \int_t^\infty (w - t)^\rho dF(w).$$

Therefore,

$$a_\rho(t) = \frac{1}{1 - F_\Lambda(t)} \sum_{\kappa, h=0}^\infty c^{(a^*, \rho)}_{\kappa, h} \Gamma (\rho + 1, bt) | (\rho > -1),$$

where

$$c^{(a^*, \rho)}_{\kappa, h} = \nabla_\kappa \sum_{r=0}^\rho \left( \begin{array}{c} \rho \\ r \end{array} \right) (-t)^{\rho-r}.\gamma,$$

$$\Gamma (\zeta, \varsigma) | \varsigma > 0 = \int_\varsigma^\infty w^{\varsigma-1} \exp (-w) dw$$

and

$$\Gamma (\zeta, \varsigma) = \Gamma (\zeta) - \gamma (\zeta, \varsigma).$$

The $\rho^{th}$ moment of the reversed residual life, say

$$A_\rho(t) = \mathbb{E}[(t - Z)^\rho | w \leq t, t > 0 and \rho = 1, 2, ...]$$

uniquely determines $F_\Lambda (w)$. We obtain

$$A_\rho(t) = \frac{1}{F_\Lambda(t)} \int_0^t (t - w)^\rho dF(w).$$

Then, the $\rho^{th}$ moment of the reversed residual life of $W$ becomes

$$A_\rho(t) = \frac{1}{F_\Lambda(t)} \sum_{\kappa, h=0}^\infty C^{(a^*, \rho)}_{\kappa, h} \gamma (\rho + 1, bt) | (\varsigma > -1),$$

where

$$C^{(a^*, \rho)}_{\kappa, h} = \nabla_\kappa \sum_{r=0}^\rho (-1)^r \left( \begin{array}{c} \rho \\ r \end{array} \right) t^{\rho-r}.$$

2.6. Numerical analysis

Table 1 gives Numericals results for the variance ($V(Z)$), mean ($E(Z)$), kurtosis ($K(Z)$), skewness ($S(Z)$) and dispersion index (DisIx$(Z)$). Based on Table 1, we note that: 1-The skewness of the OBEE distribution can range in the interval $(-2.7792, 8.2978)$. 2-The spread for the OBEE kurtosis is much larger ranging from $-46.275$ to $35.526$. 3-DisIx$(Z)$ can be “between 0 and 1” or “equal 1” or more than 1.
Table 1: Mean, variance, skewness, kurtosis and dispersion index.

| α  | β  | a   | b   | \( \alpha \) | \( \beta \) | \( a \) | \( b \) | \( E(Z) \) | \( V(Z) \) | \( S(Z) \) | \( K(Z) \) | \( \text{DisIx}(Z) \) |
|----|----|-----|-----|------------|------|-----|------|--------|--------|--------|--------|----------------|----------------|
| 0.5| 2  | 1.5 | 1.5 | 0.502266   | 0.485247 | 2.478655 | 11.13074 | 0.9661068 |
| 1  | 0.4849408 | 0.1495599 | 1.598563 | 6.885345 | 0.3084085 |
| 5  | 0.5970219 | 0.0104331 | -0.0226027 | 3.506458 | 0.0174753 |
| 20 | 0.6444730 | 0.0007732 | -0.4386726 | 4.015127 | 0.0011998 |
| 50 | 0.6552973 | 0.0001281 | -0.5222116 | 4.205749 | 0.0001955 |
| 100| 0.6590055 | 3.297×10^{-5} | -0.5496712 | 4.250314 | 4.9161×10^{-5} |
| 200| 0.6608784 | 8.1462×10^{-6} | -0.5636244 | 4.324122 | 1.2326×10^{-5} |
| 5  | 0.5  | 0.25 | 0.25 | 0.586532 | 0.40893890 | 4.073737 | 32.78264 | 0.6972151 |
| 1  | 0.3147107 | 0.05833588 | 2.792337 | 20.03788 | 0.1853635 |
| 10 | 0.08191879 | 0.00203066 | 0.665321 | 3.385543 | 0.0247887 |
| 50 | 0.03443559 | 0.00039684 | 0.678116 | 3.479092 | 0.0115240 |
| 100| 0.02328808 | 0.00019240 | 0.515537 | 3.49165 | 0.0082615 |
| 200| 0.01554562 | 9.0857×10^{-5} | 1.290022 | 3.473409 | 0.0058446 |
| 2  | 5  | 0.5  | 5   | 0.019935 | 0.00259295 | 1.431029 | 6.028403 | 0.01300695 |
| 1  | 0.068523 | 0.001137761 | 0.6051527 | 3.431216 | 0.01660408 |
| 20 | 0.553863 | 0.004671701 | -0.1685215 | 3.095139 | 0.00843472 |
| 50 | 0.733010 | 0.004878618 | -0.1930617 | 3.120336 | 0.00665559 |
| 100| 0.870260 | 0.004949493 | -0.2013219 | 3.130186 | 0.00568737 |
| 200| 1.008199 | 0.004985267 | -0.2054172 | 3.134844 | 0.00494473 |
| 500| 1.191042 | 0.005006845 | -0.2078738 | 3.13767 | 0.00420375 |
| 1000| 1.329533 | 0.005017667 | -0.2091019 | 3.139091 | 0.00377123 |
| 2000| 1.468093 | 9.0857×10^{-5} | 2.447838 | 2.87494 | 0.0029472 |
| 1.5| 1.5 | 1.5  | 0.5  | 1.7901830 | 1.055172 | 1.228803 | 5.744852 | 0.5894217 |
| 1  | 0.8950913 | 0.263793 | 1.228803 | 5.744886 | 0.2947109 |
| 5  | 0.1790183 | 0.010552 | 1.228803 | 5.744886 | 0.0589422 |
| 10 | 0.0895091 | 0.002638 | 1.228803 | 5.744886 | 0.0294711 |
| 50 | 0.0179018 | 0.0001055 | 1.113265 | 6.978685 | 0.0058942 |
| 100| 0.0089509 | 2.638×10^{-5} | 2.447838 | -2.87494 | 0.0029472 |
| 150| 0.0059673 | 7.114×10^{-6} | 9.999243 | -7.45278 | 0.0011922 |
| 1  | 1  | 1    | 1    | 1.0000978 | 1.3090×10^{-5} | 22.320 | -168.8085 | 0.00129637 |
| 2  | 0.0020196 | 1.0182×10^{-5} | -1.1216 | 1.136435 | 0.00504159 |
| 1  | 0.0117561 | 6.5598×10^{-6} | 128.59 | -1726.776 | 0.00055799 |

3. Copula under the OBEE model

In this section, we derive some new bivariate type OBEE (BvOBEE) model using FGM-copula, Clayton copula, modified FGM-copula and Renyi's entropy. The Multivariate OBEE (MvOBEE) type is also presented. Recently, many authors used and applied many different copulas in distribution theory such as Mansour et al. (49), (50), (51), (52), (53), (54), Elgohari and Yousof (20), (21), Salah et al. (?), Al-Babtain [3], Yousof et al. [64], Ibrahim et al. [39], Ali et al. ([4], [5]) and Yousof et al. [60].
3.1. FGM copula

First, we start with the joint CDF of the Morgenstern family (Morgenstern (1956)) of two RVs \( (W_1, W_2) \) which has the following form \( C_\lambda(\varsigma, \omega) = (1 + \lambda \varsigma \omega) \varsigma \omega \) where \( \lambda \in I_{(−1, 1)} \) and \( \varsigma, \omega \in (0, 1) \). Setting \( \varsigma = 1 - \varsigma \) and \( \omega = 1 - \omega \), then,

\[
\varsigma|_{(\bar{\theta}(w_1) = 1 - e^{-a_2 w_1})} = \left\{ 1 - \left[ \frac{\bar{\theta}(w_1)}{a_1} \right] \right\}^{\alpha_1} \beta_1 \\
\omega|_{(\bar{\theta}(w_2) = 1 - e^{-b_2 w_2})} = \left\{ 1 - \left[ \frac{\bar{\theta}(w_2)}{a_2} \right] \right\}^{\alpha_2} \beta_2,
\]

where \( a_i = a_{i-1, 2}, b_i = b_{i-1, 2} \) and

\[
C_\lambda(w_1, w_2) = \left( 1 - \left\{ 1 - \left[ \frac{\bar{\theta}(w_1)}{a_1} \right] \right\}^{\alpha_1} \beta_1 \right) \\
\times \left( 1 - \left\{ 1 - \left[ \frac{\bar{\theta}(w_2)}{a_2} \right] \right\}^{\alpha_2} \beta_2 \right) \\
\times \left( 1 + \lambda \left\{ \frac{1 - \left[ \frac{\bar{\theta}(w_1)}{a_1} \right]^{\alpha_1} \beta_1}{\left[ \frac{\bar{\theta}(w_1)}{a_1} \right]^{\alpha_1} + \left[ 1 - \left[ \frac{\bar{\theta}(w_1)}{a_1} \right] \right]^{\alpha_1} \beta_1} \right\} \right).
\]

3.2. Modified FGM copula

Consider the following modified version of the bivariate FGM copula defined as (see Farlie [26], Gumbel [31], Gumbel [32] and Morgenstern [57])

\[
C_{\Delta}(\varsigma, \omega)|_{\Delta \in (−1, 1)} = \varsigma \omega \left[ 1 + \Delta \hat{\theta}(\varsigma) \tilde{\omega}(\omega) \right] = \varsigma \omega + \Delta \hat{\theta}(\varsigma) \tilde{\omega}(\omega),
\]

where \( \hat{\theta}(\varsigma) = \varsigma \hat{\theta}(\varsigma) \), and \( \tilde{\omega}(\omega) = \omega \tilde{\omega}(\omega) \). Where \( \hat{\theta}(\varsigma) \) and \( \tilde{\omega}(\omega) \) are on \( I_{(0, 1)} \) where \( \hat{\theta}(0) = \hat{\theta}(1) = \tilde{\omega}(0) = \tilde{\omega}(1) = 0 \). Let

\[
\alpha = \inf \left\{ \frac{\partial}{\partial \varsigma} \hat{\theta}(\varsigma) : d_1(\varsigma) \right\} < 0, \xi = \inf \left\{ \frac{\partial}{\partial \omega} \tilde{\omega}(\omega) : d_2(\omega) \right\} > 0,
\]

\[
\beta = \sup \left\{ \frac{\partial}{\partial \varsigma} \hat{\theta}(\varsigma) : d_1(\varsigma) \right\} < 0, \eta = \sup \left\{ \frac{\partial}{\partial \omega} \tilde{\omega}(\omega) : d_2(\omega) \right\} > 0,
\]

and \( \min (\alpha \beta, \xi \eta) \geq 1 \) where

\[
\frac{\partial}{\partial \varsigma} \hat{\theta}(\varsigma) = \hat{\theta}(\varsigma) + \varsigma \frac{\partial}{\partial \varsigma} \hat{\theta}(\varsigma),
\]

\[
d_1(\varsigma) = \left\{ \varsigma : \varsigma \in I_{(0, 1)} | \frac{\partial}{\partial \varsigma} \hat{\theta}(\varsigma) \text{ exists} \right\},
\]

and

\[
d_2(\omega) = \left\{ \omega : \omega \in I_{(0, 1)} | \frac{\partial}{\partial \omega} \tilde{\omega}(\omega) \text{ exists} \right\}.
\]
3.2.1. Bivariate OBEE-FGM (Type-I) model The bivariate OBEE-FGM (Type-I) model

\[
C_{\Delta}(\varsigma, \omega) = \Delta \left[ \dot{\theta}(\varsigma) \dot{\Omega}(\omega) \right] + \left\{ 1 - \left( \frac{1 - [\Omega(\varsigma)]^{\alpha_1}}{[\Omega(\varsigma)]^{\alpha_1} + 1 - [\Omega(\varsigma)]^{\alpha_1}} \right)^{\alpha_2 \beta_2} \right\}
\times \left[ 1 - \frac{\{1 - [\Omega(\omega)]^{\alpha_1}\}^{\alpha_2 \beta_2}}{\{1 - [\Omega(\omega)]^{\alpha_1}\}^{\alpha_1} + 1 - [\Omega(\omega)]^{\alpha_1}} \right]^{\alpha_1 \beta_1},
\]

where

\[
\dot{\theta}(\varsigma) = \varsigma \frac{1 - [\Omega(\varsigma)]^{\alpha_1}}{[\Omega(\varsigma)]^{\alpha_1} + 1 - [\Omega(\varsigma)]^{\alpha_1}}^{\alpha_2 \beta_2},
\]

and

\[
\dot{\Omega}(\omega) = \omega \frac{1 - [\Omega(\omega)]^{\alpha_1}}{[\Omega(\omega)]^{\alpha_1} + 1 - [\Omega(\omega)]^{\alpha_1}}^{\alpha_2 \beta_2}.
\]

3.2.2. Bivariate OBEE-FGM (Type-II) model Consider \( \theta(\varsigma) \) and \( \Omega(\omega) \) where

\[
\theta^*(\varsigma) \big|_{\Delta_1 > 0} = \varsigma^{\Delta_1} (1 - \varsigma)^{1 - \Delta_1} \quad \text{and} \quad \Omega^*(\omega) \big|_{\Delta_2 > 0} = \omega^{\Delta_2} (1 - \omega)^{1 - \Delta_2}.
\]

The bivariate OBEE-FGM (Type-II) copula can be derived from

\[
C_{\Delta,\Delta_1,\Delta_2}(\varsigma, \omega) = \varsigma \omega [1 + \Delta \theta^*(\varsigma) \Omega^*(\omega)].
\]

3.3. The bivariate OBEE via Renyi’s entropy

Following Pougaza and Djafari [58], the joint CDF of the bivariate OBEE via Renyi’s entropy can be written as

\[
C(\varsigma, \omega) = w_2 \varsigma + w_1 \omega - w_1 w_2,
\]

then, the associated bivariate OBEE will be

\[
C(w_1, w_2) = w_2 \left[ 1 - \frac{\{1 - [\Omega(w_1)]^{\alpha}\}^{\alpha_1 \beta_1}}{[\Omega(w_1)]^{\alpha_1} + \{1 - [\Omega(w_1)]^{\alpha}\}^{\alpha_1}} \right]^{\alpha_2 \beta_2}
+ w_1 \left[ 1 - \frac{\{1 - [\Omega(w_2)]^{\alpha}\}^{\alpha_2 \beta_2}}{[\Omega(w_2)]^{\alpha_2} + \{1 - [\Omega(w_2)]^{\alpha}\}^{\alpha_2}} \right]^{\alpha_1 \beta_1} - w_1 w_2,
\]

where \( a_1 = a_2 = a, b_1 = b_2 = b \). Then, we get the BOBEE type distribution via Renyi’s entropy.

3.3.1. The bivariate OBEE extension Via Clayton copula The bivariate extension via Clayton copula can be considered as a weighted version of the Clayton copula, which is of the form

\[
C(\varsigma, \omega) \big|_{\eta \geq 0} = \left[ \varsigma^{-\eta} + \omega^{-\eta} - 1 \right]^{-\frac{1}{\eta}}.
\]

Next, setting \( \varsigma = 1 - \xi = \varsigma(x) \in I_{(0,1)} \) and \( \omega = 1 - \xi = \omega(y) \in I_{(0,1)} \). Then, the associated CDF bivariate OBEE type distribution will be
3.3.2. The Multivariate OBEE extension

A straightforward $d$-dimensional extension from the above will be
\[
C(w, w_2, \ldots, w_d) = \left\{ \sum_{i=1}^{d} \left[ 1 - \frac{\left\{ 1 - \left[ \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right] \right\}^\alpha}{\left( \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right) \left( \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} + \left( 1 - \left[ \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right] \right) \right)^\eta} \right] \right\}^{-\frac{1}{\eta}}.
\]

4. Maximum likelihood method

For getting the maximum likelihood estimates (MLE) of the vector $\Lambda$, we have the log-likelihood ($\ell(\Lambda)$) function
\[
\ell(\Lambda) = n \log(\alpha \beta ab) - b_i \log \left( \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right) + \left( \alpha \beta - 1 \right) \log \left( 1 - \left[ \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right] \right) - (1 + \beta) \log \left( \left[ \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right] \right).
\]

The components of the score vector $\frac{\partial \ell}{\partial \Lambda} = U(\Lambda) = (\partial \ell/\partial \alpha, \partial \ell/\partial \beta, \partial \ell/\partial a, \partial \ell/\partial b)$ are available if needed. We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing some sub-models of the OBEE distribution.

5. Simulations

In statistics, simulation is usually used for assessing the performance of a method, typically when there is a lack of theoretical background. In this section, we assess the performance of the maximum likelihood (ML) method. The assessment can be performed numerically or graphically. Graphically, we can perform the simulation experiments to assess of the finite sample behavior of the ML estimators (MLEs) via the biases and mean squared errors (MSEs). The following algorithm is considered for the assessment:

1. Using the inversion method, we we generate $N=1000$ samples of size $n$ from the OBEE distribution using (7).
2. Compute the MLEs for the 1000 samples, say
\[
\left( \hat{\alpha}_h, \hat{\beta}_h, \hat{a}_h, \hat{b}_h \right) \mid (h=1,2,\ldots,2000),
\]
3. Compute the standard errors (StErs) of the MLEs for the 1000 samples, say
\[
\left( S_{\hat{\alpha}_h}, S_{\hat{\beta}_h}, S_{\hat{a}_h}, S_{\hat{b}_h} \right) \mid (h=1,2,\ldots,2000).
\]

The StErs were computed by inverting the observed information matrix.
4. Compute the biases and MSEs given for $\mathbf{V} = (\alpha, \beta, a, b)$.

5. Repeated these steps for $n = 50, 100, \ldots, 300$ with $\alpha = 1, 2, \ldots, 100, \beta = 1, 2, \ldots, 100, a = 1, 2, \ldots, 100$ and $b = 1, 2, \ldots, 100$, so computing biases$(\text{Bias}_{\mathbf{V}}(n))$, MSEs $(\text{MSE}_{\mathbf{V}}(n))$ for $\alpha, \beta, a, b \ \forall \ n = 50, 100, \ldots, 300$ where

$$\text{Bias}_{\mathbf{A}}(n)|_{(\mathbf{A} = \nu, \theta, c_2, c_1)} = \frac{1}{2000} \sum_{h=1}^{2000} (\Lambda_h - \mathbf{V})$$

and

$$\text{MSE}_{\mathbf{A}}(n)|_{(\mathbf{A} = \nu, \theta, c_2, c_1)} = \frac{1}{2000} \sum_{h=1}^{2000} (\Lambda_h - \mathbf{V})^2$$

Figure 3. Biases (left) and MSEs (right) for the parameter $\alpha$.

Figures 3, 4, 5 and 6 gives the biases (left) and MSEs (right) for the parameters $\alpha, \beta, a$ and $b$ respectively. These figures (lefts) shows how the four biases vary with respect to $n$ and also shows how the four MSEs vary with respect to $n$. From Figures 3, 4, 5 and 6, the biases for each parameter are generally negative and getting close to zero as $n \to \infty$, the MSEs for each parameter decrease to zero as $n \to \infty$.

6. Real data applications

We shall compare the fits of the OBEE distribution with those of other competitive models, namely: Exponential (E), Odd Lindley Exponential (OLE), Marshall-Olkin Exponential (MOE), Moment Exponential (ME), The Logarithmic Burr-Hatke Exponential (LBHE), Generalized Marshall-Olkin Exponential (GMOE), Beta Exponential (BE), Marshall-Olkin Kumaraswamy Exponential (MOKwE), Kumaraswamy Exponential (KwE), the Burr X Exponential (BrXE) and Kumaraswamy Marshall-Olkin Exponential (KwMOE). Some other competitive model are can be derived based on Aryal, G. and Yousof...
Figure 4. Biases (left) and MSEs (right) for the parameter $\beta$.

Figure 5. Biases (left) and MSEs (right) for the parameter $a$. 
Figure 6. Biases (left) and MSEs (right) for the parameter $b$.

[14], Ibrahim et al. [36], Alizadeh et al. [8], Merovci et al. ([55], [59]), Korkmaz et al. [44], Karamikabir et al. [43] and Al-Babtain et al. [3]. For comparing models, we consider the Cramér-Von Mises ($C^1$) and the Anderson-Darling ($A^1$) and the Kolmogorov-Smirnov (KS) statistic. Moreover and for more accuracy, we consider another five goodness-of-fit measures: the Akaike Information Criterion (AIC) ($C^1$), Bayesian IC ($C^2$), Consistent AIC ($C^3$), Hannan-Quinn IC ($C^4$).

6.1. Modeling failure (relief) times

The first data set $\{1.1, 0.7, 1.9, 3.0, 1.7, 1.0, 1.8, 1.5, 1.2, 1.8, 1.6, 2.7, 4.1, 1.4, 1.3, 1.7, 2.2, 1.4, 2.3, 1.6, 2\}$ called the failure time data: The data represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic (see Gross and Clark [30]). This data was recently analyzed by Ibrahim et al. [40] and Al-Babtain et al. [3]. Table 2 lists the MLEs, StErrs confidence intervals (CIs). Table 3 lists the $C^1$, $C^2$, $C^3$, $C^4$, $A^1$, $C^1$, K.S. and p-value. Figure 7 gives the E-PDF, E-CDF, E-HRF and P-P plot for relief times data. Figure 8 below gives Kaplan-Meier survival plot for relief times data.
Table 2: MLEs, StErs, CIs for the relief times data.

| Models | MLEs, StErs and CIs |
|--------|---------------------|
| $E_{(b)}$ | MLE: 0.526, StEr: (0.117), LCI, UCI: (0.29, 0.75) |
| $OLE_{(b)}$ | MLE: 0.6044, StEr: (0.0535), LCI, UCI: (0.5, 0.7) |
| $ME_{(b)}$ | MLE: 0.950, StEr: (0.150), LCI, UCI: (0.66, 1.24) |
| $LBHE_{(b)}$ | MLE: 0.5263, StEr: (0.118), LCI, UCI: (0.43, 0.63) |
| $MOE_{(\alpha, b)}$ | MLE: 54.474, 2.316, StEr: (35.582), (0.374), LCI, UCI: (0, 124.2), (1.58, 3.05) |
| $GMOE_{(\lambda, \alpha, b)}$ | MLE: 0.519, 89.462, 3.169, StEr: (0.26), (66.28), (0.77), LCI, UCI: (0.02, 1), (0, 219.4), (1.66, 4.7) |
| $KwE_{(\alpha, \beta, b)}$ | MLE: 8.868, 34.826, 0.299, 4.899, StEr: (9.146), (22.312), (0.239), (3.176), LCI, UCI: (0, 28.8), (0, 78.6), (0, 0.8), (0, 11) |
| $MOKwE_{(\alpha, \beta, \lambda, b)}$ | MLE: 1.1635, 0.3207, StEr: (0.332), (57.85), (0.7), (1.8), LCI, UCI: (0, 0.8), (0, 146.59), (0, 1.98), (0, 5.22) |
| $KwMOE_{(\alpha, \beta, \lambda, b)}$ | MLE: 3.74, 0.27, 4.183, 1.366, StEr: (6.89), (0.19), (18.3), (2.94), LCI, UCI: (0, 16.9), (0, 0.65), (0, 40), (0, 7.2) |
Table 3: \( C_1, C_2, C_3, C_4, A', C', K.S. \) and \( p \)-value for the relief times data.

| Models | \( C_1, C_2, C_3, C_4 \) | \( A' \) | \( C' \) | K.S. and \( p \)-value |
|--------|-----------------------------|--------|--------|---------------------|
| E      | 67.70, 68.70, 67.89, 68.90 | 4.60   | 0.96   | 0.44 ( < 0.01 )     |
| OLE    | 49.12, 50.14, 49.33, 49.34 | 1.3    | 0.22   | 0.85 ( < 0.001 )    |
| ME     | 54.32, 55.31, 54.54, 54.50 | 2.76   | 0.53   | 0.32 (0.1)          |
| LBHE   | 67.70, 68.70, 67.89, 67.90 | 0.62   | 0.105  | 0.44 ( < 0.001 )    |
| MOE    | 43.51, 45.51, 44.22, 43.90 | 0.8    | 0.14   | 0.18 (0.55)         |
| GMOE   | 42.75, 45.74, 44.25, 43.34 | 0.51   | 0.08   | 0.15 (0.78)         |
| KwE    | 41.78, 44.75, 43.28, 42.32 | 0.45   | 0.07   | 0.14 (0.86)         |
| BE     | 43.48, 46.45, 44.98, 44.02 | 0.70   | 0.12   | 0.16 (0.80)         |
| MOKE   | 41.58, 45.54, 44.25, 42.30 | 0.60   | 0.11   | 0.14 (0.87)         |
| KMOE   | 42.82, 46.84, 45.55, 43.60 | 1.08   | 0.19   | 0.15 (0.86)         |
| BrXE   | 48.13, 50.15, 48.83, 48.52 | 1.39   | 0.24   | 0.248 (0.171)       |
| OBEE   | 38.95, 42.93, 41.62, 39.73 | 0.155  | 0.0268 | 0.0902 (0.9969)     |

Based on Table 3, we conclude that the proposed lifetime OBEE model is much better than all other mentioned models with \( C_1 = 38.95, C_2 = 42.93, C_3 = 41.62, C_4 = 39.73 \), \( A' = 0.155 \), \( C' = 0.0268 \), \( K.S = 0.09016 \) and \( p \)-value=0.9969 so the new lifetime model is a good alternative to these models in modeling relief times data set. According to Figures 8 and 9, the OBEE distribution provides adequate fits to the empirical functions.

### 6.2. Modeling survival times

The second data set \{0.10, 0.92, 0.93, 0.96, 0.33, 0.44, 0.56, 0.72, 0.74, 0.77, 1.0, 1.0, 1.02, 1.05, 1.07, 0.07, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 2.22, 2.3, 2.31, 2.4, 0.59, 1.08, 1.08, 1.08, 1.2, 1.21, 1.6, 1.09, 1.12, 1.13, 1.22, 1.22, 1.24, 1.30, 1.34, 1.36, 1.39, 1.44, 1.83, 1.95, 1.96, 1.97, 2.02, 1.15, 1.16, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 2.13, 2.15, 2.16, 1.46, 1.53, 1.59, 3.61, 4.02, 4.32, 4.58, 5.55\} called...
Figure 7. The box plot, Q-Q plot and TTT plot for the relief times data.

Figure 8. E-PDF, E-CDF and E-HRF for relief times data.
the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [17]. Table 4 lists the MLEs, StErs confidence intervals (CIs). Table 5 lists the $C_1$, $C_2$, $C_3$, $C_4$, $A^\cdot$, $C^\cdot$, K.S. and p-value. For many other real-life data sets see [19], [22], [18], [47], [45], [63], [23], [24], [25], [65], [38], [61], [37], [7]. Figure 9 gives the estimated PDF (E-PDF), E-CDF, E-HRF and P-P plot survival times data. Figure 10 gives the Kaplan-Meier survival plot survival times data. Based on Table 5, we conclude that the OBEE model is much better than all other competitive models with $C_1=203.84$, $C_2=212.94$, $C_3=204.44$, $C_4=207.46$, $A^\cdot=0.297$, $C^\cdot=0.046$, K.S=0.0675 and p-value=0.8987 so the new lifetime model is a good alternative to these models in modeling relief times data set. According to Figures 11 and 12, the OBEE distribution provides adequate fits to the empirical functions.
### Table 4: MLEs, StErs, CIs for the survival times data.

| Models         | MLE   | StEr   | LCI, UCI          |
|----------------|-------|--------|-------------------|
| $E_{(b)}$      | 0.540 | (0.063)| (0.4, 0.7)        |
| $OLE_{(b)}$    | 0.3815| (0.021)| (0.3, 0.4)        |
| $ME_{(b)}$     | 0.925 | (0.077)| (0.6, 1.1)        |
| $LBHE_{(b)}$   | 0.54  | (0.064)| (0.4, 0.67)       |
| $MOE_{(\alpha, b)}$ | 8.78, 1.38| (3.56, (0.19)| (1.8, 15.7), (1.00, 1.8) |
| $GMOE_{(\lambda, \alpha, b)}$ | 0.18, 47.64, 4.47| (0.07), (44.9), (1.33)| (0.04, 0.3), (0, 136), (1.8, 7) |
| $KwE_{(a,\beta, b)}$ | 3.304, 1.100, 1.037| (1.13, 5.47), (0, 2.59), (0, 2.24) |
| $BE_{(a,\beta, b)}$ | 0.807, 3.461, 1.331| (0.696), (1.003), (0.860) |
| $MOKwE_{(\alpha,\beta,\lambda)}$ | 0.008, 2.716, 1.986, 0.099| (0.002), (1.316), (0.784), (0.05) |
| $KwMOE_{(\alpha,\beta,\lambda)}$ | 0.373, 3.478, 3.306, 0.299| (0.136), (0.861), (0.779), (1.112) |
| $BrXE_{(a, b)}$ | 0.48, 0.21| (0.060), (0.012) |
| $OBEE_{(\alpha,\beta,a,b)}$ | 3.5, 0.59, 0.56, 0.28| (0.3138, 0.306, 0.711, 0.557) |

**Note:** The above table provides the Maximum Likelihood Estimates (MLE), Standard Errors (StEr), and Confidence Intervals (CI) for different models applied to survival times data. The table entries are rounded for display purposes.
Table 5: $C_1$, $C_2$, $C_3$, $C_4$, $A'$, $C'$, K.S. and (p-value) for survival times data.

| Models | $C_1$, $C_2$, $C_3$, $C_4$ | $A$ | $C$ | K.S. and (p-value) |
|--------|----------------|-----|-----|-------------------|
| E      | 234.60, 236.90, 234.68, 235.55 | 6.53 | 1.25 | 0.3 (0.06) |
| OLE    | 229.13, 231.43, 229.21, 230.11 | 1.94 | 0.33 | 0.5 (< 0.001) |
| ME     | 210.40, 212.68, 210.45, 211.30 | 1.52 | 0.25 | 0.15 (0.13) |
| LBHE   | 234.63, 236.92, 234.71, 235.51 | 0.71 | 0.115 | 0.28 (< 0.001) |
| MOE    | 210.37, 214.93, 210.52, 212.17 | 1.18 | 0.17 | 0.10 (0.43) |
| GMOE   | 210.54, 217.38, 210.89, 213.24 | 1.02 | 0.16 | 0.09 (0.5) |
| KwE    | 209.42, 216.24, 209.77, 212.12 | 0.74 | 0.11 | 0.09 (0.5) |
| BE     | 207.37, 214.21, 207.73, 210.09, 0.98 | 0.15 | 0.11 | 0.10 (0.34) |
| MOKwE  | 209.44, 218.56, 210.04, 213.04, 0.79 | 0.12 | 0.10 | 0.10 (0.44) |
| KwMOE  | 207.82, 216.94, 208.42, 211.42 | 0.61 | 0.11 | 0.09 (0.5) |
| BrXE   | 235.31, 239.92, 235.53, 237.14, 2.9 | 0.52 | 0.22 | 0.22 (0.002) |
| OBEE   | 203.84, 212.94, 204.44, 207.46 | 0.297 | 0.0460 | 0.0675 (0.8987) |

7. conclusions

In this article, we introduced and studied a new flexible version of the exponentiated exponential model called the odd Burr exponentiated exponential (OBEE) model. The new density can be right skewed and symmetric with unimodal and bimodal shapes. The new HRF can be "constant", "decreasing", "increasing", "increasing-constant", "upside-down-constant", "decreasing-constant". Some of its mathematical properties including the ordinary moments, incomplete moment, moment generating function are derived. Numerical calculations for the expected value, skewness, variance, kurtosis and the index of dispersion is presented. The skewness of the OBEE distribution can range in the interval (−2.779, 8.2978). The spread for the OBEE kurtosis is much larger ranging from −46.275 to 35.526. The index of dispersion can be "between 0 and 1" or "equal
1” or more than 1. Some bivariate and multivariate OBEE type model have been also derived. Estimation of OBEE parameters is performed by maximum likelihood estimation method. We assessed the performance of the maximum likelihood method. The assessment can be performed graphically via the biases and mean squared errors. The usefulness and flexibility of the new distribution is illustrated by means of two real data sets. The new model is much better than many other competitive models in modeling relief times and survival times data sets according to the Akaike Information Criterion, the Consistent Akaike Information Criterion, the Hannan-Quinn Information Criterion, the Bayesian Information Criterion, the Cramér-Von Mises, the Anderson-Darling statistics. As a future work, we can apply the Bagdonavičius–Nikulin goodness-of-fit test,
modified Bagdonavičius–Nikulin goodness-of-fit test, Nikulin-Rao-Robson goodness-of-fit test and modified Nikulin-Rao-Robson goodness-of-fit test to our new model (see Goual et al. ([27], [28] and [29]) and Yadav et al. [62] for more details). Characterization results and regression modeling can be derived based on OBEE model (see [9], [10], [11] and [12] for more details).

REFERENCES

1. Abouelmagd, T. H. M., Hamed, M. S. and Yousof, H. M. (2019a). Poisson Burr X Weibull distribution. Journal of Nonlinear Sciences & Applications (JNSA), 12(3), 173-183.
2. Abouelmagd, T. H. M., Hamed, M. S., Almamy, J. A., Ali, M. M., Yousof, H. M. and Korkmaz, M. C. (2019b). Extended Weibull log-logistic distribution. Journal of Nonlinear Sciences and Applications, 12(8), 523-534.
3. Al-Babtain, A. A. Elbatal, I. and Yousof, H. M. (2020). A new flexible three-parameter model: properties, clayton copula, and modeling real data. Symmetry, 12, 1-17. doi:10.3390/sym12030440
4. Ali, M. M., Yousof, H. M. and Ibrahim, M. (2021a). A new version of the generalized Rayleigh distribution with copula, properties, applications and different methods of estimation. Optimal Decision Making in Operations Research & Statistics: Methodologies and Applications, VOL 1, 1-25.
5. Ali, M. M., Ibrahim, M. and Yousof, H. M. (2021b). Expanding the Burr X model: properties, copula, real data modeling and different methods of estimation. Optimal Decision Making in Operations Research & Statistics: Methodologies and Applications, VOL 1, 26-49.
6. Alizadeh, M., Ghosh, I., Yousof, H. M., Rasekhi, M. and Hamedani, G. G. (2017). The generalized odd generalized exponential family of distributions: properties, characterizations and applications. Journal of Data Science, 15(3), 443-465.
7. Alizadeh, M., Rasekhi, M., Yousof, H. M., Ramires, T. G. and Hamedani, G. G. (2018). Extended exponentiated Nadarajah-Haghighi model: Mathematical properties, characterizations and applications. Studia Scientiarum Mathematicarum Hungarica, 55(4), 496-522.
8. Alizadeh, M., Yousof, H. M., Jahnanshahi, S. M. A., Najibi, S. M. and Hamedani, G. G. (2020). The transmuted odd log-logistic-G family of distributions. Journal of Statistics and Management Systems, 1-27.
9. Altun, E., Yousof, H. M. and Hamedani G. G. (2018). A Flexible Extension of Generalized Half-Normal Distribution: Characterizations and Regression Models. International Journal of Applied Mathematics and Statistics, 57(3), 27-49.
10. Altun, E., Yousof, H. M. and Hamedani G. G. (2018). A new flexible extension of the generalized half-normal lifetime model with characterizations and regression modeling. Bulletin of Computational Applied Mathematics, 6(1), 83-115.
11. Altun, E., Yousof, H. M. and Hamedani, G. G. (2018). A new generalization of generalized half-normal distribution: properties and regression models. Journal of Statistical Distributions and Applications, 5(1), 7.
12. Altun, E., Yousof, H. M. and Hamedani, G. G. (2018). A new log-location regression model with influence diagnostics and residual analysis. Facta Universitatis, Series: Mathematics and Informatics, 33(3), 417-449.
44. Korkmaz, M. C., Altun, E., Yousof, H. M. and Hamedani, G. G. (2020). The Hjorth's IDB Generator of Distributions: Properties, Characterizations, Regression Modeling and Applications. Journal of Statistical Theory and Applications, 19(1), 59-74.

45. Korkmaz, M. C. and Yousof, H. M. (2017). The one-parameter odd Lindley exponential model: mathematical properties and applications. Stochastics and Quality Control, 32(1), 25-35.

46. Korkmaz, M. C., Yousof, H. M. and Hamedani, G. G. (2018). The exponential Lindley odd log-logistic-G family: properties, characterizations and applications. Journal of Statistical Theory and Applications, 17(3), 554-571.

47. Korkmaz, M. C., Yousof, H. M., Rasekhi, M. and Hamedani, G. G. (2018). The Odd Lindley Burr XII Model: Bayesian Analysis, Classical Inference and Characterizations. Journal of Data Science, 16(2), 327-353.

48. Kundu, D. and Pradhan, B. (2009). Bayesian inference and life testing plans for generalized exponential distribution. Sc. China Ser. A. Math., 52, 1373–1388.

49. Mansour, M. M., Ibrahim, M., Aidi, K., Shafique Butt, N., Ali, M. M., Yousof, H. M. and Hamed, M. S. (2020a). A New Log-Logistic Lifetime Model with Mathematical Properties, copula, Modified Goodness-of-Fit Test for Validation and Real Data Modeling. Mathematics, 8(9), 1508.

50. Mansour, M. M., Butt, N. S., Ansari, S. I., Yousof, H. M., Ali, M. M. and Ibrahim, M. (2020b). A new exponentiated Weibull distribution's extension: copula, mathematical properties and applications. Contributions to Mathematics, 1 (2020) 57–66. DOI: 10.47443/cm.2020.0018

51. Mansour, M., Korkmaz, M. C., Ali, M. M., Yousof, H. M., Ansari, S. I. and Ibrahim, M. (2020c). A generalization of the exponentiated Weibull model with properties, copula and application. Eurasian Bulletin of Mathematics, 3(2), 84-102.

52. Mansour, M., Rasekhi, M., Ibrahim, M., Aidi, K., Yousof, H. M. and Elrazik, E. A. (2020d). A New Parametric Life Distribution with Modified Bagdonavičius-Nikulin Goodness-of-Fit Test for Censored Validation, Properties, Applications, and Different Estimation Methods. Entropy, 22(5), 592.

53. Mansour, M., Yousof, H. M., Shehata, W. A. and Ibrahim, M. (2020e). A new two parameter Burr XII distribution: properties, copula, different estimation methods and modeling acute bone cancer data. Journal of Nonlinear Science and Applications, 13(5), 223-238.

54. Mansour, M. M., Butt, N. S. Yousof, H. M., Ansari, S. I. and Ibrahim, M. (2020f). A Generalization of Reciprocal Exponential Model: Clayton copula, Statistical Properties and Modeling Skewed and Symmetric Real Data Sets. Pakistan Journal of Statistics and Operation Research, 16(2), 373-386.

55. Merovci, F., Alizadeh, M., Yousof, H. M. and Hamedani, G. G. (2017). The exponentiated transmuted-G family of distributions: theory and applications. Communications in Statistics-Theory and Methods, 46(21), 10800-10822.

56. Merovci, F., Yousof, H. M. and Hamedani, G. G. (2020). The Poisson Topp Leone Generator of Distributions for Lifetime Data: Theory, Characterizations and Applications. Pakistan Journal of Statistics and Operation Research, 16(2), 343-355.

57. Morgenstern, D. (1956). Einfache beispiele zweidimensionaler verteilen. Mitteilungsblatt fur Mathematische Statistik, 8, 234-235.

58. Pougaza, D. B. and Dijafari, M. A. (2011). Maximum entropies copulas. Proceedings of the 30th international workshop on Bayesian inference and maximum Entropy methods in Science and Engineering, 329-336.

59. Salah, M. M., El-Morshed, M., Elwia, M. S. and Yousof, H. M. (2020). Expanded Fréchet Model: Mathematical Properties, copula, Different Estimation Methods, Applications and Validation Testing. Mathematics, 8(11), 1949.

60. Yousof, H. M., Ali, M. M., Goual, H. and Ibrahim, M. (2021). A new reciprocal Rayleigh extension: properties, copulas, different methods of estimation and modified right censored test for validation, STATISTICS IN TRANSITION new series, forthcoming.

61. Yadav, A. S., Altun, E. and Yousof, H. M. (2019). Burr–Hatke Exponential Distribution: A Decreasing Failure Rate Model, Statistical Inference and Applications. Annals of Data Science, 1-20.

62. Yadav, A. S., Goual, H., Alotaibi, R. M., Ali, M. M. and Yousof, H. M. (2020). Validation of the Topp-Leone-Lomax model via a modified Nikulin-Rao-Robson goodness-of-fit test with different methods of estimation. Symmetry, 12(1), 57.

63. Yousof, H. M., Butt, N. S., Alotaibi, R. M., Rezk, H., Alomani, G. A. and Ibrahim, M. (2019). A new compound Fréchet distribution for modeling breaking stress and strengths data. Pakistan Journal of Statistics and Operation Research, 15(4), 1017-1035.

64. Yousof, H. M., Hamedani, G. G. and Ibrahim, M. (2020). The Two-parameter Xgamma Fréchet Distribution: Characterizations, copulas, Mathematical Properties and Different Classical Estimation Methods. Contributions to Mathematics, 2 (2020), 32-41.

65. Yousof, H. M., Korkmaz, M. C. and Sen, S. (2019). A new two-parameter lifetime model. Annals of Data Science, 1-16.

66. Zheng, G. (2002). On the fisher information matrix in type II censored data from the exponentiated exponential family. Biom. J., 44, 353–357.

67. Zheng, G. and Park, S. (2004). A note on time savings in censored life testing. J. Stat. Plan. Inference 124, 289–300.

Stat., Optim. Inf. Comput. Vol. 9, June 2021