Inferring the size of a collective from the motion of a few accessible units is a fundamental problem in network science and interdisciplinary physics. Here, we recognize stochasticity as the commodity traded in the units’ interactions. Drawing inspiration from the work of Einstein-Perrin-Smoluchowski on the discontinuous structure of matter, we use the random motion of one unit to identify the footprint of every other unit. Just as the Avogadro’s number can be determined from the Brownian motion of a suspended particle in a liquid, the size of the collective can be inferred from the random motion of any unit. For self-propelled Vicsek particles, we demonstrate an inverse proportionality between the diffusion coefficient of the heading of any particle and the size of the collective. We provide a rigorous method to infer the size of a collective from measurements of a few units, strengthening the link between physics and collective behavior.
Collective dynamics are ubiquitous in nature\textsuperscript{1,2}. From neural circuits to animal groups, there are countless instances where the interactions among large numbers of elementary units bestow surprisingly complex patterns of tantalizing beauty for the collective. Deciphering the interactions within a collective from observations of its units is the object of several successful methodological endeavors\textsuperscript{3,4}. Common to the vast majority of the existing approaches is the knowledge of the size of the collective, a hypothesis that is often defeated in experiments. For example, understanding bird migration requires tagging individuals to measure their motion\textsuperscript{5}, but there is no guarantee that these tagged individuals compose the totality of the flock.

Only a handful of recent studies have addressed the fundamental problem of estimating the size of a collective from measurements of some of its units\textsuperscript{6-9}. The approaches proposed by Haehne et al.\textsuperscript{5}, Porfiri\textsuperscript{7}, and Tang et al.\textsuperscript{8} are based on the computation of the rank of ancillary matrices that encode the transient response of some of the units, in the absence of forcing. In contrast, Tylo and Delabays\textsuperscript{9} tackle the inference problem from steady-state response to harmonic forcing. Although they may tolerate some level of added noise and measurement uncertainty, all of these methods are designed on the premise of completely deterministic dynamics.

Here, we propose a different paradigm to solve the problem of estimating the size of a collective, building upon seminal ideas of Einstein\textsuperscript{10}, Smoluchowski\textsuperscript{11}, and Perrin\textsuperscript{12} that led to the determination of the Avogadro’s number from kinetic theory. Using the words of Feynman regarding the experiments of Perrin\textsuperscript{13}, “one of the earliest determinations of the number of atoms was by the determination of how far a dirty little particle would move if we watched it patiently under a microscope for a certain length of time.” We demonstrate that the mean square dynamics of a single unit is sufficient to infer the size of a collective.

We focus on a system of self-propelled Vicsek particles\textsuperscript{14} as a universal model for collective dynamics, for which we show that the time rate of growth of the mean square heading of any particle is sufficient to predict the number of particles in the system. The concept is best explained by drawing an analogy between Vicsek models and the classical kinetic theory of gases\textsuperscript{15}, where the mean square displacement of any particle is given by $\lambda k T / 2 m$, with $\lambda$ being time and $D$ the diffusion coefficient. The latter can be estimated as $D \propto k T / V$, where $k_0$ is Boltzmann’s constant, $T$ the absolute temperature, $V$ the volume of the gas, $d$ the particle diameter, $m$ the particle mass, and $N$ the number of particles. Hence, one can estimate the number of particles in the gas from the rate of growth of the mean square displacement, upon some knowledge of the thermodynamic state of the gas and the properties of its particles.

Provided that the network of interactions underlying the collective dynamics of the Vicsek particles guarantees some average connectivity in time, the motion of one unit will contain a significant footprint of the size of the entire system, similar to existing approaches for the inference of the size of a collective\textsuperscript{6-9}. We numerically and mathematically demonstrate that the inference of the size of the system from the motion of a single particle only requires knowledge about the variance of the added noise, which, indeed, has been proposed as a thermodynamic analog of the temperature for collective dynamics\textsuperscript{16-19}. Different from the experiments of Perrin, our focal unit is not a macroscopic particle in a bath of microscopic particles, rather, it is one of the very particles comprising the collective whose trajectory is accessible by the experiment.

Results

Numerical results. We consider a system of $N$ self-propelled particles in a square domain of size $L \times L$ with periodic boundary conditions. At time $k \in \mathbb{Z}_{\geq 0}$, the planar position of particle $i$ is $r_i^k \in \mathbb{C}$ and its heading is $\theta_i^k \in [-\pi, \pi]$. Particles negotiate their heading with other particles within a unitary interaction radius under the effect of additive noise, moving at a constant speed,

$$\begin{align*}
\theta_{i+1}^k &= \arg \left( \sum_{j \in \mathcal{N}_i^k} \exp(i\theta_j^k) \right) + \psi_i^k, \\
\eta_{i+1}^k &= \eta_i^k + \exp(i\theta_i^k).
\end{align*}$$

Here $i$ is the imaginary unit; $\mathcal{N}_i^k = \{ j = 1, \ldots, N : \| \eta_i^k - \eta_j^k \| \leq 1 \}$ is the set of neighbors of particle $i$ at time $k$; $\psi_i^1, \ldots, \psi_i^N$ are independent uniformly distributed random variables in $[-\eta / 2, \eta / 2]$ with $\eta \in \mathbb{R}_{>0}$ being a control parameter that modulates the added noise so that the noise variance is $\sigma^2 = \eta^2 / 12$; and $s \in \mathbb{R}_{>0}$ is the common speed of all the particles, which corresponds to the number of interaction radii that are traveled in one time-step, so that neighbors are almost fixed for $s \ll 1$ and they randomly mix for $s \gg 1$. Parameters $N$ and $L$ control the density of the system, $\eta$ acts as a temperature, and $s$ regulates temporal correlations among particles.

We report simulation results for two values of $s$ (low and high), and several values of $L$ and $N$, spanning the simulation range considered by Vicsek et al.\textsuperscript{14}. For each parameter combination, we perform 1000 repetitions with the same initial configuration, randomly selected in a square of unit length at the center of the domain. The initial heading of each particle is randomly selected in $[-\pi, \pi]$, and $\eta$ is equal to 0.1, which corresponds to the “small noise” case originally considered by Vicsek\textsuperscript{14} to demonstrate ordered motion in the collective (with an average velocity, sometimes called polarization, close to one). In each repetition, we run model (1) for 150 time steps and track the heading of a single (randomly selected) focal particle, see Fig. 1. For each parameter combination and at every time-step $k$, we compute the sample variance $Y_k$ of the heading across the 1000 repetitions (for moderate noise, the sample variance is, in fact, equivalent to the sample circular variance). As one should expect from kinetic theory, $Y_k$ grows linearly with time (that is, $Y_k = Dk + Y_0$, with $Y_0$ being the initial variance) and the diffusion coefficient (estimated using linear regression on $Y_1, \ldots, Y_{100} \simeq \sum_{i=1}^{100} Y_i$) is inversely proportional to the number of particles $N$, see Fig. 1.

Irrespective of the particle speed, there is a wide region of the parameter space $(L^2, N)$ for which the diffusion coefficient per unit noise variance $(D / \sigma^2)$ is larger than the mean of the diffusion coefficients estimated from all the particles). This is equal to the inverse of the number of particles, see Fig. 2a, b. As a result, it is possible to exactly infer the size of the collective from knowledge of the heading of a focal particle and of the noise variance that captures the system temperature (for details on the precision of the estimate, see Supplementary Note 1). More specifically, for the low-speed case, the accuracy of the inference is not as precise as the changes in the domain size and the number of particles, with variations of at most 20% registered in the whole parameter space. For the high-speed case, we find a more marked dependence on the accuracy of the inference on $L$ and $N$. Although accurate inferences (within 20%) are obtained for $L^2 \leq 40$, further increasing the domain size challenges the inference. Such a detrimental effect is more evident for small values of $N$. The possibility of accurately performing the inference across widely different particle speeds is surprising, whereby the Vicsek model spans from the diluted XY-model ($s \ll 1$) to the mean-field behavior of a ferromagnet ($s \gg 1$) as a function of the speed\textsuperscript{14}. We offer some mathematical backing to this observation through the analytical treatment of these two limit cases, for which we establish closed-form expressions for the diffusion coefficient.
In the vast majority of the simulations in which the inference does not match $N$ (especially for large values of $L$ in the high-speed case), we document an underestimation of the size of the collective ($DN/s^2 > 1$). This can be explained by the formation of independent particle clusters\(^{14}\), which can be revealed from the study of the overall connectivity of the system. As a result, the diffusion coefficient will only be reflective of the size of the cluster to which the focal particle belongs, thereby leading to the observed underestimation. Numerical evidence supporting this explanation is presented in the bottom panels of Fig. 2c, d, where we plot the normalized Fiedler eigenvalue $\lambda_2/N$ of the Laplacian matrix associated with the average network of interactions during the simulation window, similar to the analysis by Baglietto et al.\(^{20}\).

The Fiedler eigenvalue, measuring the overall connectivity of the system, follows the trend similar to that of $DN/s^2$ in Fig. 2a, b. Until a threshold value for the normalized eigenvalue $-\log(\lambda_2/N)$, the method provides a nearly perfect inference of the collective size with a weak dependence on $\lambda_2/N$ that translates into a modest decrease in accuracy as $-\log(\lambda_2/N)$ increases. On the contrary, beyond the threshold, the accuracy of the inference significantly degrades, with a clear dependence on $\lambda_2/N$, as illustrated in Fig. 2e. The origin of this qualitative transition is presently unclear. Perhaps, it is associated with the emergence of ordered bands within the collective\(^{21}\), so that a few subgroups are able to synchronize along common headings in a bath of disordered motion for the rest of the collective. We do not favor this explanation, however, given the short length of the...
The theory of low particle speed. We begin with the study of low speed, for which the neighbor set of each particle can be approximated to be constant in time. We hypothesize that the noise is sufficiently small for the system to be in its ordered phase, where particles’ headings synchronize. Hence, the collective dynamics of the system is described by the classical noisy consensus problem with static network topology that is extensively studied in control theory.

Specifically, we consider the following linear dynamics:

$$\theta_k = A\theta_k + u_k,$$  

$$y_k = \theta_k^T e_f,$$  

where $\theta_k = [\theta_k^1, \ldots, \theta_k^N]^T$, $A$ is the state matrix, $u_k = [u_k^1, \ldots, u_k^N]^T$ is a white noise vector with variance $\sigma^2$ (that is, for all $k$ and $\tau \neq 0$: $E[u_k] = 0$, $E[u_k u_k^T] = \sigma^2 I_N$, and $E[u_k^T u_k] = 0$), with $E[\cdot]$ denoting expectation, $e_f$ is the vector with all zeros but one in the $f$-th entry, and $y_k$ is the heading of the focal particle. The state matrix $A$ corresponds to the adjacency matrix of an undirected, unweighted graph with self-loops for each node, such that $A_{ii} = I_N$ with $I_N$ being the vector of all ones. The reason why the adjacency matrix is row-stochastic is because the fully connected component the focal particle belongs to. Hence, the proposed inference approach would never overestimate the size of the system, see Fig. 2. Third, the approach is robust with respect to individual variations in the form of heterogeneous variances $\sigma_1^2$, $\ldots$, $\sigma_N^2$ across the particles. In this case, the same derivations would yield an equivalent linear dependence in time for $y_k$ with $D = \sum_{i=1}^{N} \sigma_i^2/N$, so that the inference is independent of the choice of the focal particle and the only required information is the average value of the variance. Likewise, the approach is robust with respect to weighted interactions, whereby the results would not change for any undirected primitive row-stochastic matrix $A$.

Finally, although the derivation of Eq. (2b) assumes the network to be constant in time and given once for all across the repetitions, its applicability is likely to extend to a more general setting that should be formalized in future studies. Specifically, one should consider slowly-varying topologies so that $A$ varies in time much slower than the rate of divergence of the variance of $y_k$, the same mathematical arguments might be applicable. Likewise, should the adjacency matrix differ across repetitions, we would not expect changes in Eq. (2b), provided each of the matrices is connected, since no topological feature of the adjacency matrix enters the estimation of the size of the collective. Simulation results in Fig. 2 and in Supplementary Note 2 offer compelling evidence for these claims, whereby the inference is accurate albeit the network topology changes in time due to additive noise, and initial conditions are varied across repetitions.

Theory: high particle speed. The analytical result in Eq. (5) is based on the assumption of a static topology. An equivalent claim can be established in the opposite scenario of random mixing.
where the speed of the particles is sufficiently large for the memory of past interactions to fade between two consecutive time steps. This limit case can be studied through the vectorial network model (VNM)\(^{24}\), which implements heading update by randomly assigning to any particle \(K\) neighbors drawn uniformly from the entire collective at each time step. In this high-speed limit, particles interact on average with \(nL^2(N-1)\) other particles within their unit circles so that \(L\) and \(K\) can be related by \(L^2 = n(N-1)/(K-1)\). For the case of small noise, the model reduces to\(^{25}\)

\[
\theta_{k+1} = A_0 \theta_k + u_k, \\
y_k = \theta_k^T \psi_f,
\]

where the matrices \(A_0, A_1, \ldots\) are independent identically distributed (i.i.d.) with common random variable \(A\), and the noise \(u_k\) and all the other variables are defined as in the case of static topology. Specifically, \(A\) is such that each of its rows is the sum of \(K \geq 2\) i.i.d. vectors, each one having all entries 0, except one (selected from a uniform distribution) that is equal to \(1/K\). Similar to the low-speed case in Eq. (2a) and consistent with the original Vicsek model (1), matrix \(A\) is row-stochastic so that \(A_1 = I_N\) by construction. This temporal patterning yields an all-to-all average network, \(E[A] = \frac{1}{N} I_N\). We acknowledge the possibility that the number of distinct neighbors is less than \(K\) within the VNM; however, the probability that two vectors will contribute a non-zero entry at the same location is very small.

Similar to Eq. (3), we write

\[
\Theta_{k+1} = E[A_k \theta_k | \theta_k^T A_k^T] + \sigma^2 I_N
\]

but, in this case, the adjacency matrix depends on time and cannot be moved out of the expectation. To establish a linear recursion for \(\Theta_k\), we resort to Kronecker algebra,

\[
\text{vec}(\Theta_{k+1}) = G \text{vec}(\Theta_k) + \sigma^2 \text{vec}(I_N).
\]

Here, “vec” identifies the process of vectorization and \(G = E[A \otimes A]\) can be explicitly computed as\(^{25}\)

\[
G = \frac{1}{KN} \text{vec}(I_N) \text{vec}(R)^T + \frac{1}{N^2} I_{N^2} I_N^T \otimes I_N I_N^T,
\]

with “\(\otimes\)” being the outer product and \(R = I_N - I_N I_N^T/N\) a projection matrix. None of the analytical developments by Porfiri\(^{25}\) can be utilized for the case at hand, where the dynamics considered therein was deprived of the mean to study the evolution of the synchronization error. In Supplementary Note 3, we study the spectral properties of \(G\) to determine that, for sufficiently large times, element \(ij\) of \(\Theta_k\) can be written as

\[
\Theta_{k+1}^{ij} \approx \frac{\sigma^2 y}{N^2} k,
\]

where \(y = (a-b)/(a-N+1)\), with \(a = N(N-1-2KN+2K+K^2N)\) and \(b = N(N-1-KN+K)\). Given that \(Y_{\hat{k}} = \Theta_{\hat{k}}^T\), we find the sought dependence on the size of the collective,

\[
Y_k \approx \frac{\sigma^2 y}{N} k.
\]

Different from the low-speed limit, the particle density enters the estimation of the diffusion coefficient through \(y\). As further detailed in Supplementary Note 3, for any choice of \(2 \leq K \leq N\) and \(N > 1\), \(y\) is between 1 and 2. The largest values of \(y\) are registered for low densities, that is, small systems in large domains. Increasing the density (that is, increasing \(K\)) lowers the value of \(y\), thereby leading to accurate inferences. This theoretical result is in qualitative agreement with numerical simulations in Fig. 2b, which indicate that in the high-speed case, the accuracy of the inference degrades with \(L\) and improves with \(N\). However, care should be placed when drawing conclusions for large values of \(L\) that lead to poorly connected average networks (as evidenced by the normalized Fiedler eigenvalue in Fig. 2d), in contrast with the all-to-all average network predicted by the vectorial network model.

In principle, the computation can be extended to weighted, random interactions, in which every node randomly weights the state of its \(K\) neighbors at each time step, such as blinking models\(^{26}\), activity-driven networks\(^{27}\), and averaging models\(^{28}\). However, this would likely lead to a different expression for \(y\) since the spectral properties of \(G\) would vary. The presence of heterogeneities in the added noise across the units will not produce a difference in the diffusion coefficient with respect to the static case, as shown in Supplementary Note 4.

Conclusions

There is a growing effort to build physical intuition and technical rigor in collective dynamics by recognizing analogies with thermodynamics\(^{-29-33}\). For example, Sinhuber and Oulette\(^{30}\) have introduced definitions of pressure and chemical potentials for insect swarms to study the coexistence of a condensed phase surrounded by a vapor phase observed in experiments. Likewise, Haeri et al.\(^{32}\) have recently elucidated thermodynamic variables of collectives governed by attractive and repulsive forces, presenting an equation of state similar to the ideal gas law.

Here, we unveil a further, microscopic analogy, upon which we establish a methodology to infer the size of a collective from the dynamics of a single unit. Different from the entire body of knowledge that relies on deterministic dynamics\(^{4-9}\), we treat stochastically as a commodity to solve the inference problem. Just as the motion of a Brownian particle in a liquid can be used to determine Avogadro’s number as shown in the classical experiments by Perrin\(^{12}\), so we demonstrate that the stochastic motion of a single unit is sufficient for the determination of the size of the entire collective.

We focus on the ordered (homogeneous) phase of the self-propelled particle model of Vicsek, for which we discover an inverse proportionality between the diffusion coefficient associated with the heading of any of the particles and the size of the collective. Given the sole knowledge of the temperature of the system, in the form of the variance of the added noise, one can successfully infer the size of the collective from the diffusion coefficient. We offer detailed analytical treatment of two limit cases of the Vicsek model, encompassing the XY-model and mean-field behavior of a ferromagnet. The mathematical study of these cases confirms numerical predictions that connectivity of the collective ensures that the diffusion coefficient is, within a first approximation, independent of the particle density—a counter-intuitive finding with respect to the case of an ideal gas where the geometry of the system modulates the diffusion coefficient\(^{15}\). While these limited cases may offer partial insight into the study of key nonlinear features of the Vicsek model, like the nature of its phase transition\(^{21}\), they accurately anticipate the observed evolution of the variance of the heading and its dependence on the system parameters at low temperatures.

This study is not free of limitations, which shall be addressed in future work toward the study of real collectives, from insect swarms to bird flocks, fish schools, and human crowds. First, the approach is demonstrated on a rather simplified model of collective behavior, which, albeit popular and general, does not capture all the intricacies and complexities of collective behavior. Future work should seek to apply the approach to more detailed models of collective behavior, including, for example, attraction and repulsion rules\(^2\). Second, the implementation of the approach requires knowledge of the noise variance, which may not be feasible all the times, especially when dealing with experimental observations. Should one have access to some video recordings of the collective and knowledge of an adequate mathematical model, it could be possible to
infer the value of the noise through dimensionality reduction techniques, without the need for tracking\(^9\).

Overall, this study provides a rigorous, mathematically backed method to infer the size of a realistic collective from measurements of some of its units, whose random motion contains the footprints of the entire system. The theoretical underpinnings of the method provide further evidence for the analogies identified by Einstein between interdisciplinary research in the collective behavior of animal groups and modern physics\(^13\).

**Data availability**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Code availability**

The code that supports the findings of this study is available from the corresponding author upon reasonable request.

Received: 15 November 2021; Accepted: 18 March 2022; Published online: 11 April 2022

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**Acknowledgements**

The authors wish to thank Agnieszka Truszkowska for setting up and performing numerical simulations for large collectives. P.D.L. was supported by the program “STAR 2018” of the University of Naples Federico II and Compagnia di San Paolo, Istituto Banco di Napoli - Fondazione, project ACROSS. M.P. was supported by the National Science Foundation under grant numbers CMMI 1561134 and CMMI 1932187.

**Author contributions**

P.D.L. and M.P. designed the study, performed the numerical analysis, developed the theoretical framework, analyzed and interpreted the results, and wrote the final draft.

**Competing interests**

The authors declare no competing interests.

**Additional information**

**Supplementary information** The online version contains supplementary material available at https://doi.org/10.1038/s42005-022-00864-9.

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**Peer review information** Communications Physics thanks Joseph Lizier, Robin Delabays, Melvyn Tyloo, and the other anonymous reviewer(s) for their contribution to the peer review of this work.

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