Curves with many points over finite fields: the class field theory approach

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Abstract

The problem of constructing curves with many points over finite fields has received considerable attention in the recent years. Using the class field theory approach, we construct new examples of curves ameliorating some of the known bounds. More precisely, we improve the lower bounds on the maximal number of points \( N_q(g) \) for many values of the genus \( g \) and of the cardinality \( q \) of the finite field \( \mathbb{F}_q \), by looking at all unramified coverings of all genus three smooth projective curves over \( \mathbb{F}_q \), for \( q \) is an odd prime less than 19.

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1 Introduction to the problem

In this paper $C$ is a smooth projective curve over a finite field. An interesting question is how many points there can be on a curve of given genus over a given finite field. Let $\#C(\mathbb{F}_q)$ denote the number of points on $C$ over $\mathbb{F}_q$. We have the following classical result:

**Theorem 1 (Weil bound).** Let $g$ be the genus of the curve $C$, then

$$|q + 1 - \#C(\mathbb{F}_q)| \leq 2g\sqrt{q}.$$  

The problem of improving this bound is very difficult. Many well-known mathematicians, such as J.-P. Serre, V. Drinfeld and others, devoted considerable efforts to its study. A reason why this problem attracts much attention is that it has quite a few applications to the coding theory, cryptography, etc.

Let us denote by $N_q(g)$ the maximum of the number of points on a smooth projective curve $C$ of genus $g$ over $\mathbb{F}_q$. By using the Weil bound, one immediately sees that $N_q(g) \leq q + 1 + 2g\sqrt{q}$.

The goal of this paper is to find new lower bounds for the number $N_q(g)$ for small $g$ and $q$, that is for $g \leq 50$ and $q \leq 19$. Nowadays we do not know exactly the number $N_q(g)$ for many pairs $(q, g)$ from the above intervals, except for the cases $g = 1, 2, 3$ or $4$. For example, $N_2(1) = 5$, $N_2(2) = 6$, $N_2(4) = 8$, but $N_2(16)$ is either 17 or 18. Just a little improvement of the known bounds requires in many cases both modern mathematical tools and computer support. The up-to-date tables for $N_q(g)$ are available on the website manypoints.org.

2 Methods

2.1 Serre’s Example

Here we describe one of the ideas for constructing curves with many points. This idea belongs to Serre and its generalization is very important for our purposes. The following examples are taken from [5].

**Theorem 2 (Serre).** $N_2(4) \geq 8$ and $N_2(11) \geq 14$.

**Proof.** Consider the elliptic curve $C$ defined over $\mathbb{F}_2$ by the equation $y^2 + y = x^3 + x$. It is easy to see that $C$ has exactly five rational points: $P_0 = \infty$, $P_1 = (0, 0)$, $P_2 = (1, 1)$, $P_3 = (1, -1)$, $P_4 = (x, y)$.
\( P_1 = (0, 0), P_2 = (1, 0), P_3 = (1, 1) \) and \( P_4 = (0, 1) \) and the map \( \phi: P_i \mapsto i \), from \( C(\mathbb{F}_2) \) to \( \mathbb{Z}/5\mathbb{Z} \) is an isomorphism of abelian groups. Hence, there is a function \( f \) on \( C \) with the divisor \( D = [a_0; a_1; a_2; a_3; a_4] = \sum a_i P_i \) if and only if \( \sum a_i = 0 \) and \( \sum i * a_i = 0 \mod 5 \).

So, the divisor \( D_1 = [-3, -1, 2, 1, 1] \) is the divisor of some function \( f_1(x) \). One may take the Artin-Schreier extension \( C_1: w^2 + w = f_1(x) \). According to the Riemann-Hurwitz formula, it has genus 4. Moreover it is easy to see that it has exactly eight rational points. This shows that \( N_2(4) \geq 8 \). For the same reason, the divisor \( D_2 = [-1, -3, 1, 1, 2] \) is the divisor of some function \( f_2(x) \). The Artin-Schreier covering \( C_2: w^2 + w = f_2(x) \) also has genus 4 and eight rational points. On can take the fibre product of \( C_1 \) and \( C_2 \). It is a genus 11 curve with 14 rational points.

So, we have \( N_2(4) \geq 8 \) and \( N_2(11) \geq 14 \). Let us compare this result with the Weil-bound:

\[
N_2(4) \leq 2 + 1 + 8\sqrt{2} \approx 14.313
\]

and

\[
N_2(11) \leq 2 + 1 + 22\sqrt{2} \approx 34.11.
\]

It is possible to improve this bound. The point is that in both cases mentioned above \( g \) is “sufficiently greater” than \( q \). Let us illustrate this idea by the following theorem:

**Theorem 3** (Ihara bound). We have

\[
N_q(g) \leq q + 1 + 1/2(\sqrt{8q + 1}g^2 + 4g(q^2 - q) - g).
\]

In particular if \( g \geq \frac{\sqrt{9q(q-1)}}{2} \), then Ihara bound improves the Weil bound.

**Proof.** See [4] or [1].

Let us take here \( q = 2 \) and \( g = 4, 11 \). The Ihara bound gives us:

\[
N_2(4) \leq 1 + 2\sqrt{19} \approx 9.7177
\]

and

\[
N_2(11) \leq \frac{1}{2}(\sqrt{2145} - 5) \approx 20.6571.
\]

Actually \( N_2(4) = 8 \) and \( N_2(11) = 14 \), but it is quite a non-trivial result, which requires the so called Weil-explicit formulae and Oesterlé bound. See [6].
2.2 A generalization

Here we describe a generalization of the above method. This generalization has been used in many papers (e.g. [3]). For clarity we save the original notation.

First of all, we notice that the equation $w^2 + w = f_1(x)$ from the above theorem gives an abelian extension where the points $(x, y)$ such that $f_1(x) = 0$ split completely. In contrast to the above example, for the sake of clarity, we restrict ourselves to working with unramified abelian covers. The unramified abelian covers of $C$ are parameterized by the subgroups of the Picard group $Pic^0(C)$. Thus, we find ourselves working with the class field theory.

Let us fix the ground field $k = \mathbb{F}_q$ and let us consider a curve $C$ of genus $g$ over $k$. We denote by $F$ the function field $\mathbb{F}_q(C)$. By using class field theory we want to construct abelian extensions $F'$ of $F$ corresponding to new curves $C'$ with many points.

Let $S = \{P_i\}$ be the set of all rational points of $C$. Let us fix one rational point $O$ and consider the $S_O$-Hilbert class field which we denote by $F_O$. By definition, $F_O$ is the maximal unramified abelian extension of $F$ in which $O$ splits completely.

The class field theory gives us the isomorphism:

$$\phi: Pic^0(C) \rightarrow Gal(F_O/F).$$

The isomorphism $\phi$ maps the class $[P - \deg(P)O]$ to the Artin symbol of $P$ (the Frobenius map at point $P$). By using this isomorphism, one produces new curves. Namely, if $G$ is a subgroup of $Pic^0(C)$ of index $d = [Pic^0(C) : G]$, then its image $\phi(G)$ is a subgroup of $Gal(F_O/F)$. Hence, we have the subfield $F_{\phi(G)}$ of $F_O$ fixed by $\phi(G)$. By definition of $\phi$ a rational point $P$ splits completely if and only if $[P - \deg(P)O] \in G$. This gives us an unramified abelian extension, that corresponds to a new curve. Since the extension is unramified we can control its genus and its number of rational points.

Thus, for any pair $[G, O]$ we get the unramified extension $F_{\phi(G)}$ of degree $d$. It has genus $d(g - 1) + 1$ and $d \cdot |G \cap \{[P_i - O]\}|$ rational points.

By searching through all the subgroups of $Pic^0(C)$ and through all the points of $C$, one can improve the known lower bounds on $N_q(g)$. Note that for a fixed $G$ and different places $O$ we get different extensions. Despite the fact that such extensions have isomorphic Galois groups, the corresponding curves may have different numbers of rational points. We will see concrete examples in the next section.
3 Examples and Calculations

3.1 How to organize the computation

Previously, the above method was used mostly for hyper-elliptic curves of small genus. For example, for the case of genus two curves over finite fields \( \mathbb{F}_q \) with cardinality \( q \leq 16 \) this search was implemented in [3]. In our search, we do the same calculation for all genus three curves. We use Magma software to perform a computer search.

If we have a genus 3 curve \( C \) then it is either a hyper-elliptic curve, or a plane quartic.

A plane quartic is a geometrically smooth projective curve given by an equation of the form \( f(x, y, z) = 0 \), where \( f(x, y, z) \in \mathbb{F}_q[x, y, z] \) is a homogeneous polynomial degree 4. By using Invariant theory, it is possible to describe very explicitly the “moduli space” of plane quartics over finite fields. This was done by Lercier, Ritzenthaler, Rovetta, and Sijsling in [2].

Otherwise, if \( C \) is a hyper-elliptic curve, then its affine part has a smooth model \( y^2 = f(x) \), where \( f(x) \) is a polynomial of degree 7 or 8 without multiple roots over algebraic closure of the ground field. To obtain a projective model one has to take the normalization of the projective closure \( C \). A good thing here is that Magma allows one to work with the projective model \( C \) when only \( f \) is given. But unfortunately, we do not have an analogue of the database [2] in this case, so it takes much machine resources to provide calculation for all hyper-elliptic curves for a given genus.

Finally, we run the algorithm for the base field \( \mathbb{F}_p \), for the prime \( p \) equal to 3, 5, 7, 11, 13, 17 and 19. We refer the reader to the next section for details.

Let us show how it works for some concrete examples.

3.2 Concrete examples

Unfortunately, for \( p = 3 \) we could not find any new curves. Otherwise, we can take the base field to be \( \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}, \mathbb{F}_{13}, \mathbb{F}_{17} \) or \( \mathbb{F}_{19} \) and provide many examples improving previously known bounds.

Let us consider the case \( k = \mathbb{F}_7 \). By ranging over plane quartics, we have found the following examples.

Let \( C \) be the curve defined in \( \mathbb{P}^2 \) by the equation \( 6x^4 + y^3z + 6x^2z^2 + 4xz^3 + 6z^4 = 0 \). The curve \( C \) has exactly 14 rational points, namely: \( P_0 = (0 : 1 : 1) \), \( P_1 = (0 : 0 : 1) \), \( P_2 = (1 : 0 : 1) \), \( P_3 = (0 : 1 : 0) \), \( P_4 = (1 : 1 : 0) \), \( P_5 = (1 : 0 : 0) \), \( P_6 = (0 : 0 : 1) \), \( P_7 = (1 : 1 : 0) \), \( P_8 = (0 : 1 : 0) \), \( P_9 = (1 : 0 : 1) \), \( P_10 = (0 : 0 : 1) \), \( P_11 = (1 : 1 : 0) \), \( P_12 = (0 : 1 : 0) \), \( P_13 = (1 : 0 : 1) \), \( P_14 = (0 : 0 : 1) \). These points are all the rational points on the curve \( C \).
According to Magma, $\text{Pic}^0(C)$ is an abelian group isomorphic to $\mathbb{Z}/90\mathbb{Z}$. Let us take the index 7 subgroup $G$ spanned by 7, which is isomorphic to $\mathbb{Z}/129\mathbb{Z}$, and $O = P_0$. Then it gives us a curve of genus 15 with 56 rational points. This improves the previous lower bound for $g = 15$, which was 52.

Next, let us consider the curve $C$ over $\mathbb{F}_7$ defined by the equation $6x^4 + y^3z + 2x^2z^2 + z^4 = 0$. It has 12 rational points: $P_0 = (0 : 3 : 1)$, $P'_0 = (0 : 5 : 1)$, $P''_0 = (0 : 6 : 1)$, $P_2 = (2 : 0 : 1)$, $P_3 = (3 : 3 : 1)$, $P'_3 = (3 : 5 : 1)$, $P''_3 = (3 : 6 : 1)$, $P_4 = (4 : 3 : 1)$, $P'_4 = (4 : 5 : 1)$, $P''_4 = (4 : 6 : 1)$, $P_5 = (5 : 0 : 1)$, $\infty = (0 : 1 : 0)$.

Its class group is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/42\mathbb{Z}$. If one takes $O = P_2$ and the subgroup $G$ spanned by $v = (1, 2, 2)$, then we get a curve of genus 9 with 36 rational points. The previous bound for $N_{7}(9)$ was 32.

Let us change the base field and take $\mathbb{F}_{13}$. Here we have also found some interesting examples improving the previous bounds. For instance, consider the curve $C$ defined by $12x^4 + y^3z + z^4 = 0$ which has the class group $(\mathbb{Z}/3\mathbb{Z})^3 \oplus (\mathbb{Z}/12\mathbb{Z})^2$. One takes $O = (0 : 4 : 1)$ and the group $G$ defined as follows: let $m_1 = g(2)$, $m_2 = g(4)$, $m_3 = g(1) + g(5)$, where $g(k)$ denotes the generator of the $k$-th component of $\text{Pic}^0(C)$ in the above decomposition. Let $G$ be spanned by $m_1, m_2$ and $m_3$. Then, we get an extension of genus 19 with 108 rational points. The previous bound for $N_{13}(19)$ was 90.

We now summarize the results of our computations.

4 Main results

In this part we summarize all results we have found during this research. We provide this data in the following way: we give the genus and the number of rational points of a certain covering of the base curve. We also give the previously known bound to compare it with value we obtained.
Table 1: Results for the base fields $\mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}$

| The base Field | Genus | Improvements | Previous Result |
|----------------|-------|--------------|-----------------|
| $\mathbb{F}_5$ | 25    | 60           | 55–66           |
| $\mathbb{F}_7$ | 9     | 36           | 32–41           |
| $\mathbb{F}_7$ | 15    | 56           | 60–77           |
| $\mathbb{F}_7$ | 21    | 70           | 60–77           |
| $\mathbb{F}_7$ | 29    | 84           | unknown–98      |
| $\mathbb{F}_7$ | 31    | 90           | unknown–103     |
| $\mathbb{F}_7$ | 33    | 96           | unknown–109     |
| $\mathbb{F}_7$ | 35    | 102          | unknown–114     |
| $\mathbb{F}_7$ | 37    | 108          | unknown–119     |
| $\mathbb{F}_7$ | 39    | 95           | unknown–125     |
| $\mathbb{F}_7$ | 43    | 105          | unknown–135     |
| $\mathbb{F}_7$ | 45    | 110          | unknown–140     |
| $\mathbb{F}_7$ | 47    | 115          | unknown–145     |
| $\mathbb{F}_7$ | 49    | 120          | 114–150         |
| $\mathbb{F}_{11}$ | 9    | 52           | 48–59           |
| $\mathbb{F}_{11}$ | 13    | 66           | 60–77           |
| $\mathbb{F}_{11}$ | 21    | 90           | 80–110          |
| $\mathbb{F}_{11}$ | 23    | 99           | 88–119          |
| $\mathbb{F}_{11}$ | 25    | 108          | 96–127          |
| $\mathbb{F}_{11}$ | 27    | 104          | unknown–135     |
| $\mathbb{F}_{11}$ | 31    | 120          | unknown–139     |
| $\mathbb{F}_{11}$ | 35    | 136          | unknown–164     |
| $\mathbb{F}_{11}$ | 37    | 144          | unknown–171     |
| $\mathbb{F}_{11}$ | 43    | 168          | unknown–192     |
| $\mathbb{F}_{11}$ | 45    | 176          | unknown–199     |
| $\mathbb{F}_{11}$ | 47    | 161          | unknown–206     |
| $\mathbb{F}_{11}$ | 49    | 192          | unknown–213     |
Table 2: Results for the base fields $\mathbb{F}_{13}, \mathbb{F}_{17}$

| The base field | Genus | Improvements | Previous Result |
|----------------|-------|--------------|-----------------|
| $\mathbb{F}_{13}$ | 5     | 42           | 40–44           |
| $\mathbb{F}_{13}$ | 19    | 108          | 90–115          |
| $\mathbb{F}_{13}$ | 23    | 110          | unknown–133     |
| $\mathbb{F}_{13}$ | 25    | 120          | unknown–142     |
| $\mathbb{F}_{13}$ | 27    | 130          | unknown–152     |
| $\mathbb{F}_{13}$ | 31    | 135          | unknown–170     |
| $\mathbb{F}_{13}$ | 33    | 160          | 128–179         |
| $\mathbb{F}_{13}$ | 37    | 162          | 144–195         |
| $\mathbb{F}_{13}$ | 39    | 152          | unknown–203     |
| $\mathbb{F}_{13}$ | 41    | 180          | 160–211         |
| $\mathbb{F}_{13}$ | 47    | 184          | unknown–235     |
| $\mathbb{F}_{13}$ | 49    | 192          | unknown–243     |
| $\mathbb{F}_{17}$ | 5     | 48           | unknown–53      |
| $\mathbb{F}_{17}$ | 7     | 60           | unknown–70      |
| $\mathbb{F}_{17}$ | 9     | 72           | unknown–83      |
| $\mathbb{F}_{17}$ | 11    | 80           | unknown–96      |
| $\mathbb{F}_{17}$ | 13    | 90           | unknown–107     |
| $\mathbb{F}_{17}$ | 15    | 98           | unknown–118     |
| $\mathbb{F}_{17}$ | 17    | 112          | unknown–129     |
| $\mathbb{F}_{17}$ | 19    | 117          | unknown–140     |
| $\mathbb{F}_{17}$ | 21    | 130          | unknown–150     |
| $\mathbb{F}_{17}$ | 23    | 132          | unknown–161     |
| $\mathbb{F}_{17}$ | 25    | 144          | unknown–172     |
| $\mathbb{F}_{17}$ | 27    | 143          | unknown–183     |
| $\mathbb{F}_{17}$ | 29    | 154          | unknown–194     |
| $\mathbb{F}_{17}$ | 31    | 165          | unknown–205     |
| $\mathbb{F}_{17}$ | 33    | 192          | unknown–216     |
| $\mathbb{F}_{17}$ | 35    | 187          | unknown–226     |
| $\mathbb{F}_{17}$ | 37    | 216          | unknown–237     |
| $\mathbb{F}_{17}$ | 39    | 190          | unknown–248     |
| $\mathbb{F}_{17}$ | 41    | 240          | unknown–258     |
| $\mathbb{F}_{17}$ | 43    | 210          | unknown–269     |
| $\mathbb{F}_{17}$ | 45    | 220          | unknown–280     |
| $\mathbb{F}_{17}$ | 47    | 207          | unknown–291     |
| $\mathbb{F}_{17}$ | 49    | 240          | unknown–301     |
Table 3: Results for the base field $\mathbb{F}_{19}$

| The base field | Genus | Improvements | Previous Bound |
|---------------|-------|--------------|----------------|
| $\mathbb{F}_{19}$ | 5     | 54           | unknown–60     |
| $\mathbb{F}_{19}$ | 7     | 66           | unknown–76     |
| $\mathbb{F}_{19}$ | 9     | 80           | unknown–92     |
| $\mathbb{F}_{19}$ | 11    | 90           | unknown–104    |
| $\mathbb{F}_{19}$ | 13    | 96           | unknown–117    |
| $\mathbb{F}_{19}$ | 15    | 112          | unknown–128    |
| $\mathbb{F}_{19}$ | 17    | 112          | unknown–140    |
| $\mathbb{F}_{19}$ | 19    | 126          | unknown–153    |
| $\mathbb{F}_{19}$ | 21    | 140          | unknown–164    |
| $\mathbb{F}_{19}$ | 23    | 143          | unknown–175    |
| $\mathbb{F}_{19}$ | 25    | 168          | unknown–186    |
| $\mathbb{F}_{19}$ | 27    | 156          | unknown–198    |
| $\mathbb{F}_{19}$ | 29    | 196          | unknown–209    |
| $\mathbb{F}_{19}$ | 31    | 180          | unknown–221    |
| $\mathbb{F}_{19}$ | 33    | 192          | unknown–232    |
| $\mathbb{F}_{19}$ | 35    | 204          | unknown–244    |
| $\mathbb{F}_{19}$ | 39    | 209          | unknown–267    |
| $\mathbb{F}_{19}$ | 43    | 231          | unknown–289    |
| $\mathbb{F}_{19}$ | 45    | 242          | unknown–301    |
| $\mathbb{F}_{19}$ | 47    | 253          | unknown–312    |
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