Graceful Labellings of Various Cyclic Snakes

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Abstract

In this paper, we present a new sufficiency condition to obtain a graceful labelling for every $kC_{4n}$ snake and use this condition to label every such snake for $n = 1, 2, \ldots, 6$. Then, we extend this result to cyclic snakes where the cycles lengths vary. Also, we obtain new results on the (near) graceful labelling of cyclic snakes based on cycles of lengths $n = 6, 10, 14$, completely solving the case $n = 6$.

1 Introduction

Let $G = (V, E)$ be a graph with $m$ edges. Let $f$ be a labelling defined from $V(G)$ to $\{0, 1, 2, \ldots, m\}$ and let $g$ be the induced edge labelling defined from $E(G)$ to $\{1, 2, \ldots, m\}$ given by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling $f$ is graceful if $f$ is an injective mapping and $g$ is a bijection. If a graph $G$ has a graceful labelling, then it is graceful.

Alternatively, let $f$ be defined from $V(G)$ to $\{0, 1, 2, \ldots, m + 1\}$ and let $g$ be the induced edge labelling defined from $E(G)$ to $A$, where $A$ is $\{1, 2, \ldots, m - 1, m\}$ or $\{1, 2, \ldots, m - 1, m + 1\}$ given by $g(uv) = |f(u) - f(v)|$,

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for all $uv \in E$. Then $f$ is near graceful if $f$ is an injective mapping and $g$ is a bijection. If a graph $G$ has a near graceful labelling, then it is near graceful. In this paper, every near graceful labelling we find will omit the vertex label $m$ and the edge label $m$; that is the codomain of $f$ will be $\{1, 2, \ldots, m-1, m+1\}$ and the codomain of $g$ will be $\{1, 2, \ldots, m-1, m+1\}$.

A cyclic snake is a connected graph whose block-cutpoint graph is a path and each of the blocks is isomorphic to a fixed cycle. We define $kC_n$ to be a cyclic snake with $k$ blocks each of which is $C_n$. The string of a $kC_n$ is a sequence of integers $(d_1, d_2, d_3, \ldots, d_{k-2})$ where $d_i$ is the distance between the $i$th and $(i+1)$th cut vertex, counting cut vertices from one end of the snake to the other. Certainly, for fixed $n$ and $k$, the string is uniquely determined by the snake and vice versa. Note that if $k = 1$ or $k = 2$ the snake $kC_n$ has no string. A $kC_n$ is linear if all entries in its string are $\left\lceil \frac{n}{2} \right\rceil$, and it is even if all entries in its string are even numbers.

In [5], Rosa showed that all cycles $C_n$ with $n \equiv 0$ or 3 (mod 4) are graceful. Further, he introduced a necessary condition for an Eulerian graph to be graceful, namely if $G$ is a graceful Eulerian graph with $n$ edges, then $n \equiv 0$ or 3 (mod 4). A $kC_n$ is an Eulerian graph, and hence graceful only if $kn \equiv 0$ or 3 (mod 4).

Moulton, in [4], proved that a graceful labelling exists for every $kC_3$. Barrientos, in [1], proved that the cycle $C_n$ has a near graceful labelling if and only if $n \equiv 1$ or 2 (mod 4). This paper also showed that a graceful labelling exists for every $kC_4$, and for particular cases of snakes for $C_6$, $C_8$, and $C_{12}$. A complete survey of graph labellings is presented in A Dynamic Survey of Graph Labelling [2].

We define a variable snake, $n_1C_{m_1}n_2C_{m_2} \ldots n_iC_{m_i}$, to be a combination of different $n_jC_{m_j}$, where $n_jC_{m_j}$ is connected with $n_{j+1}C_{m_{j+1}}$ by identifying a vertex in the last cycle of the $n_jC_{m_j}$ with a vertex in the first cycle of the $n_{j+1}C_{m_{j+1}}$ (other than the cut vertex). The string for a variable snake is similar to the string for a $kC_n$.

We will represent all the cycle labellings in this paper as $n$-tuples, with the overline elements indicating the cut vertices, when necessary.

In Figure 1, we have a near gracefully labelled $5C_6$ with string $(3, 1, 2)$. We can represent the labelling of the $5C_6$ in Figure 1 in five 6-tuples as follows: $(20, 16, 17, 15, 18, 13), (13, 21, 11, 22, 10, 19), (22, 6, 25, 8, 23, 9), (6, 26, 4, 27, 3, 24), (3, 29, 2, 31, 0, 28).
In [5], Rosa also introduced an $\alpha$-labelling. An $\alpha$-labelling of a graph $G$ is a graceful labelling with an extra condition which is there exists an integer $w$ such that for any edge $uv \in G$, either $f(u) \leq w < f(v)$ or $f(v) \leq w < f(u)$. Any graph with an $\alpha$-labelling is necessarily bipartite.

We would also like to introduce an analogue of near graceful labellings. An $\hat{\alpha}$-labelling of a graph $G$ is a near graceful labelling with an extra condition which is there exists an integer $w$ such that for any edge $uv \in G$, either $f(u) \leq w < f(v)$ or $f(v) \leq w < f(u)$. Thus the snake in Figure 1 has an $\hat{\alpha}$-labelling with $w = 16$. In fact, by the nature of our constructions, the main results of this paper could be given in terms of $\alpha$- or $\hat{\alpha}$-labellings. While noting this to be true, we choose instead to state our results in the more familiar language of graceful and near graceful labellings.

In Lemmas 1.1 to 1.3, we give some results about (not necessarily graceful) labellings that will be useful later. In this paper, we adopt the convention that 0 is a natural number. Then, when we write $[x, y]$ with $x, y \in \mathbb{N}$ and $x < y$, we are indicating the set $\{z \in \mathbb{N}|x \leq z \leq y\}$.

If $G$ is any graph and $f$ is any labelling $G$ then we can relabel $G$ by adding a constant $c$, $h(v) = f(v) + c$. This technique preserves edge labels.

**Lemma 1.1.** If $c$ is an arbitrary integer and $f$ is a labelling of a graph $G = (V, E)$, defined by $f : V(G) \to [0, m]$ then $h : V \to [c, c + m]$ defined by $h(v) = f(v) + c$, is a labelling that preserves edge labels.
Proof. If \( v_1v_2 \in E \) and \( f \) is a labelling then \( f(v_1v_2) = |f(v_1) - f(v_2)| \). By definition, \( h(v_1v_2) = |h(v_1) - h(v_2)| = |(f(v_1) + c) - (f(v_2) + c)| \). But then \( h(v_1v_2) = |f(v_1) - f(v_2)| = f(v_1v_2) \). Therefore, \( h \) preserves edge labels. \( \square \)

Lemma 1.2. If \( c \) is an arbitrary integer and \( f \) is a labelling of a graph \( G = (V,E) \), defined by \( f : V(G) \rightarrow [0,m] \) then \( h : V \rightarrow [c - m,c] \) defined by \( h(v) = c - f(v) \), is a labelling that preserves edge labels. Further if \( f \) is graceful and \( c = m \), then \( h(v) = m - f(v) \) is graceful.

Proof. This proof follows the same argument as Lemma 1.1. If \( f \) is graceful and \( c = m \), then \( h(v) = m - f(v) \) is graceful. Let \( v_1,v_2 \in V \) such that \( h(v_1) = h(v_2) \). Then \( m - f(v_1) = m - f(v_2) \) which implies \( f(v_1) = f(v_2) \); since \( f \) is injective, then \( v_1 = v_2 \). Therefore, \( h \) is injective. Since \( h \) is also edge-preserving, it is graceful. \( \square \)

In Lemma 1.3 we present a similar result for near graceful labelling; we omit 1 from the range of \( f \) and the domain of \( h \), because if \( f(v) = 1 \) then \( h(v) = m \), and this would contradict the definition of a near graceful labelling.

Lemma 1.3. If \( f \) is a near graceful labelling of a graph \( G = (V,E) \), \( f : V(G) \rightarrow \{0,2,3,\ldots,m-1,m+1\} \) then \( h : V(G) \rightarrow \{0,2,3,\ldots,m-1,m+1\} \) defined by \( h(v) = (m+1) - f(v) \) is a near graceful labelling.

Proof. This proof follows the same argument as Lemma 1.2. \( \square \)

We describe the technique in Lemmas 1.2 and 1.3 as taking the complement of a (near) graceful labelling.

In [5] Rosa introduced a graceful labelling for \( C_{4n} \) with \( n \geq 1 \).

Lemma 1.4. [5] Let \( C_{4n} \) be a cycle with \( 4n \) edges and vertices \( v_i \), for \( 1 \leq i \leq 4n \). Then the following labelling \( f \) shows that \( C_{4n} \) is graceful:

\[
f(v_i) = \begin{cases} 
\frac{i-1}{2} & \text{if } i \text{ is odd}, \\
4n + 1 - \frac{i}{2} & \text{if } i \text{ is even, } i \leq \frac{4n}{2}, \\
4n - \frac{i}{2} & \text{if } i \text{ is even, } i > \frac{4n}{2}.
\end{cases}
\] (1)

Barrientos in [1] obtained the following results.
Theorem 1.5. [1] The $kC_4$ has a graceful labelling for any string.

Theorem 1.6. [1] The linear $kC_6$ is near graceful if $k$ is odd and graceful if $k$ is even.

Theorem 1.7. [1] The even $kC_8$ and $kC_{12}$ are graceful graphs.

Theorem 1.8. [1] The even $kC_{4n}$ with string $(d_1, d_2, \ldots, d_{k-2})$, where $d_i \in \{2, 4\}$, has a graceful labelling.

Theorem 1.9. [1] The even $kC_{4n}$, $4 \leq n \leq 5$, with string $(d_1, d_2, \ldots, d_{k-2})$, where $d_i \in \{2, 4, 2n\}$ has a graceful labelling.

In this paper, we introduce a new sufficiency condition to get a graceful labelling for every $kC_{4n}$. Then, we extend this result to $n_1C_{m_1}n_2C_{m_2} \ldots n_iC_{m_i}$. Further, we extend the results in Theorem 1.6 to 1.9 on (near) gracefully labelled $kC_n$ where $n = 6, 8, 12, 16, 20, 24$ for all possible strings. Also, we present new results on the (near) graceful labelling of $kC_n$ where $n = 10, 14$ and $k > 1$.

2 Graceful Labelling of $kC_m$ for $m \equiv 0 \pmod{4}$

Since the size of a $kC_{4n}$ is $4kn \equiv 0 \pmod{4}$, we have the potential to find a graceful labelling for any $kC_{4n}$. In Theorem 2.1, we give a new sufficient condition which, when satisfied, shows there is a graceful labelling of a $kC_{4n}$ for any string.

For fixed even $t$ and an arbitrary fixed positive integer $s$, an $s, t$-useful cycle with even distance $d$ is a $t$-cycle with vertices labelled from $[0, \frac{t}{2} - 1] \cup [st - \frac{t}{2}, st]$ and edge labels $[st - t + 1, st]$, with a vertex labelled 0 and a vertex labelled $\frac{t}{2} - 1$ at distance $d$. Similarly, an $s, t$-odd useful cycle with odd distance $d$ is a $t$-cycle with vertices labelled from $[0, \frac{t}{2}] \cup [st - \frac{t}{2} + 1, st]$ and edge labels $[st - t + 1, st]$, with a vertex labelled 0 and vertex labelled $st - \frac{t}{2} + 1$ at distance $d$.

A complete $s, t$-useful cycle set is a set of $s, t$-useful cycles of even and odd distances, the union of whose distances is $\{1, 2, \ldots, \frac{t}{2}\}$. Let $C_t^d$ be an element of a complete $s, t$-useful cycle set such that the distance between the vertices labelled 0 and $\frac{t}{2} - 1$ is $d$ if $d$ is even, and the distance between the vertices labelled 0 and $st - \frac{t}{2} + 1$ is $d$ if $d$ is odd.
Theorem 2.1. If there is a complete $s,4n$-useful cycle set with $s \geq 1$, then there exists a graceful labelling of any $kC_{4n}$.

Proof. To prove this result, we will in fact prove a slightly more complex result: namely, that given a complete $s,4n$-useful cycle set with $s \geq 1$ and for any $k \geq 1$, then there exists a graceful labelling of any $kC_{4n}$ with 0 in the last cycle in any position except the cut vertex.

If 0 is in the last cycle of a $kC_{4n}$, then up to symmetry its position is uniquely determined by the distance from the last cut vertex. These distances, $d$, can only be $1,2,\ldots,2n$.

We proceed by induction on $k$. For $k = 1$ use the graceful labelling for $C_t$ in Lemma 1.4, letting $t = 4n$. The vertex labels for this graceful $1C_{4n}$ are a subset of $[0,4n]$ and the edge labels are exactly $[1,4n]$.

For $k = 2$, we label $2C_{4n}$ while obtaining a vertex with label 0 at even distance $d$ from the unique cut vertex as follows. We label one cycle with the labelling used for $1C_{4n}$, with $2n-1$ added to each vertex, so that the vertex formerly labelled 0, and now labelled $2n-1$, is the cut vertex of $2C_{4n}$. The vertices have been labelled from the set $[2n-1,6n-1]$ and, by Lemma 1.1, the edge labels are $[1,4n]$. Apply the labelling $C_{4n}^d$ to the second cycle where $d$ is even, with the cut vertex receiving the label $2n-1$. Then this labelling of $2C_{4n}$ has all vertex labels from $[0,8n]$, and the edge labels are exactly $[1,8n]$, with no repeated vertex or edge label. That is, it is a graceful labelling of $2C_{4n}$ with the vertex labelled 0 at even distance $d$ from the cut vertex.

In the same way we label $2C_{4n}$ while obtaining a vertex with label 0 at odd distance $d$ from the unique cut vertex. We label one cycle with the labelling used for $1C_{4n}$, but replace each vertex label $x$ by $6n+1-x$. The vertices have been labelled from the set $[2n+1,6n+1]$ and, by Lemma 1.1, the edge labels are $[1,4n]$. Apply the labelling $C_{4n}^d$ to the second cycle where $d$ is odd, with the cut vertex receiving the label $6n+1$. Then this labelling of $2C_{4n}$ has all vertex labels from $[0,8n]$, and the edge labels are exactly $[1,8n]$, with no repeated vertex or edge label. That is, it is a graceful labelling of $2C_{4n}$ with the vertex labelled 0 at odd distance $d$ from the cut vertex.

Consider an arbitrary $kC_{4n}$ with $k \geq 3$, with the last entry in its string $d_{k-2}$. Let $G$ be the $(k-1)C_{4n}$ obtained by deleting a last cycle from this $kC_{4n}$. By the induction hypothesis, there is a graceful labelling of $G$ with a 0 on the vertex distance $d_{k-2}$ from the previous cut vertex. This labelling has vertex labels that are a subset of $[0,4nk-4n]$ and the edge labels are
exactly $[1, 4kn - 4n]$.

We label $kC_{4n}$ obtaining a vertex with label 0 at even distance $d$ from the unique cut vertex as follows. For the first $k - 1$ cycles, use the labelling of $G$ and add $2n - 1$ to each vertex label, so that the final cut vertex receives label $2n - 1$. Thus, the vertices have been labelled from the set $[2n - 1, 4kn - 2n - 1]$ and by Lemma 1.1 the edge labels are $[1, 4kn - 4n]$. Apply the labelling $C_{4n}^d$ to the final cycle, with the cut vertex receiving label $2n - 1$. Then this labelling of $kC_{4n}$ has all vertices labelled from $[2n - 1, 4kn - 4n]$ and the edge labels are exactly $[1, 4kn]$, with no repeated vertex or edge label. Thus, there is a graceful labelling of $kC_{4n}$ with the vertex labelled 0 at even distance $d$ from the cut vertex.

We label $kC_{4n}$ obtaining a vertex with label 0 at odd distance $d$ from the unique cut vertex as follows. For the first $k - 1$ cycles, use the labelling of $G$ with a 0 on the vertex distance $d_{k-2}$ from the previous cut vertex. Then subtract each vertex label from $4kn - 2n + 1$. Thus, the vertices have been labelled from the set $[2n + 1, 4kn - 2n + 1]$ and by Lemma 1.1 the edge labels are $[1, 4kn - 4n]$ . Apply the labelling $C_{4n}^d$ to the final cycle, with the cut vertex receiving label $4kn - 2n + 1$. Then this labelling of $kC_{4n}$ has all vertices labelled from $[0, 4kn]$, and the edge labels are exactly $[1, 4kn]$, with no repeated vertex or edge label. Thus, there is a graceful labelling of $kC_{4n}$ with the vertex labelled 0 at odd distance $d$ from the cut vertex.

□

In Table 1 we give labellings for $C_{4n}^{2j}$ where $1 \leq j \leq n$ and $n \leq 6$. For each $C_{4n}^{2j}$, we can use Lemma 1.2 with $c = t$ to obtain $C_{4n}^{2j-1}$. Then, $\{C_{4n}^m | 1 \leq m \leq 2n\}$ is a complete $s, 4n$-useful cycle set. Combining these sets with Theorem 2.1, we obtain the following corollary.

**Corollary 2.2.** If $1 \leq n \leq 6$ and $k \geq 1$ then every $kC_{4n}$ is graceful.

By adding $t - 4n$ to all even vertices in Equation (1) in Lemma 1.4 we obtain another labelling.

$$g(v_i) = \begin{cases} \frac{i - 1}{2} & \text{if } i \text{ is odd}, \\ t + \frac{i - 1}{2} & \text{if } i \text{ is even, } i \leq 2n, \\ t - \frac{i}{2} & \text{if } i \text{ is even, } i > 2n. \end{cases}$$ (2)

From Equation (2) we obtain $C_{4n}^2$ and $n \geq 1$, since the distance between the labels 0 and $2n - 1$ in $C_{4n}$ is 2. The same labelling also works as $C_{4n}^3$, since.
the distance between the labels 0 and \( t - (2n - 1) \) is 3. Further, applying Lemma 1.2 with \( c = t \) to the labelling \( C_{4n}^2 \) gives a new labelling with distance 4 between the labels 0 and \( 2n - 1 \), \( C_{4n}^4 \). Then, \( S = \{C_{4n}^m | 2 \leq m \leq 4 \} \) is a \( s, 4n \)-useful cycle set (though not complete). Combining the set \( S \) with Theorem 2.1, we obtain the following theorem.

**Theorem 2.3.** The snake \( kC_{4n} \), with string \( (d_1, d_2, \ldots, d_{k-2}) \) has a graceful labelling if \( d_i \in \{2, 3, 4\} \) for all \( i \).

From the previous discussion we can gracefully label a variable snake made from any \( kC_{4n} \) with \( n \leq 6 \). As an example of a variable snake, consider \( 3C_8 \), gracefully labelled via Theorem 2.1, with the vertices labelled from the set \( [0, 24] \) and the edge labels \( [1, 24] \). Then form a new labelling via

| \( C_{4n}^2 \) | \( (0, t, \overline{t}, t - 2) \) |
| \( C_{4n}^3 \) | \( (0, t, 3, t - 4, 2, t - 3, 1, t - 1) \) |
| \( C_{4n}^4 \) | \( (0, t, 1, t - 1, 7, t - 4, 2, t - 3) \) |
| \( C_{4n}^5 \) | \( (0, t, \overline{t}, t - 6, 4, t - 5, 3, t - 4, 2, t - 2, 1, t - 1) \) |
| \( C_{4n}^6 \) | \( (0, t, 4, t - 5, 5, t - 6, 2, t - 4, 3, t - 2, 1, t - 1) \) |
| \( C_{4n}^7 \) | \( (0, t, 1, t - 1, 2, t - 2, 6, 5, t - 5, 4, t - 2, 3, t - 4) \) |
| \( C_{4n}^8 \) | \( (0, t, \overline{t}, t - 8, 6, t - 7, 5, t - 6, 4, t - 5, 3, t - 2, 1, t - 1) \) |
| \( C_{4n}^9 \) | \( (0, t, t - 1, 7, t - 8, 6, t - 7, 5, t - 6, 4, t - 5, 2, t - 2, 3, t - 3) \) |
| \( C_{4n}^{10} \) | \( (0, t, 1, t - 1, 2, t - 2, 7, t - 8, 6, t - 7, 5, t - 6, 4, t - 3, 3, t - 5) \) |

| \( C_{4n}^{11} \) | \( (0, t, t - 1, 7, t - 12, t - 11, 9, t - 10, 8, t - 9, 7, t - 8, 6, t - 7, 5, t - 6, 4, t - 3, 3, t - 2, 1, t - 1) \) |
| \( C_{4n}^{12} \) | \( (0, t, 1, t - 1, 7, t - 12, 10, t - 11, 9, t - 10, 8, t - 5, 4, t - 4, 3, t - 3, 2, t - 2, 1, t - 1) \) |
| \( C_{4n}^{13} \) | \( (0, t, 1, t - 1, 2, t - 12, \overline{t}, t - 11, 10, t - 10, 9, t - 9, 8, t - 8, 7, t - 7, 6, t - 6, 5, t - 5, 4, t - 4, 3, t - 3, 2, t - 2, 1, t - 1) \) |
| \( C_{4n}^{14} \) | \( (0, t, 1, t - 1, 2, t - 12, \overline{t}, t - 11, 10, t - 10, 9, t - 9, 8, t - 8, 7, t - 7, 6, t - 6, 5, t - 5, 4, t - 4, 3, t - 3, 2, t - 2, 1, t - 1) \) |
| \( C_{4n}^{15} \) | \( (0, t, 1, t - 1, 2, t - 12, \overline{t}, t - 11, 10, t - 10, 9, t - 9, 8, t - 8, 7, t - 7, 6, t - 6, 5, t - 5, 4, t - 4, 3, t - 3, 2, t - 2, 1, t - 1) \) |
| \( C_{4n}^{16} \) | \( (0, t, 1, t - 1, 2, t - 12, \overline{t}, t - 11, 10, t - 10, 9, t - 9, 8, t - 8, 7, t - 7, 6, t - 6, 5, t - 5, 4, t - 4, 3, t - 3, 2, t - 2, 1, t - 1) \) |

| Labelling | \( [1, 3] \) |
| \( C_{4n}^{10} \) | \( (0, t, 1, t - 1, 2, t - 12, \overline{t}, t - 11, 10, t - 10, 9, t - 9, 8, t - 8, 7, t - 7, 6, t - 6, 5, t - 5, 4, t - 4, 3, t - 3, 2, t - 2, 1, t - 1) \) |
| \( C_{4n}^{11} \) | \( (0, t, t - 1, 7, t - 12, t - 11, 9, t - 10, 8, t - 9, 7, t - 8, 6, t - 7, 5, t - 6, 4, t - 5, 3, t - 2, 1, t - 1) \) |
| \( C_{4n}^{12} \) | \( (0, t, 1, t - 1, 2, t - 12, \overline{t}, t - 11, 10, t - 10, 9, t - 9, 8, t - 8, 7, t - 7, 6, t - 6, 5, t - 5, 4, t - 4, 3, t - 3, 2, t - 2, 1, t - 1) \) |

**Table 1:** Useful labellings \( C_{4n}^{2j} \), where \( t = 4kn \).
Lemma 1.1 with \( c = 5 \), so that a vertex in the last cycle obtains the label 5. Then add any \( C_{12}^j \) (from Table 1) to \( 3C_8 \). We obtain a gracefully labelled \( 3C_81C_{12} \). More generally, if we have a complete \( k, 4i \)-useful cycle set for all \( 1 \leq i \leq n \), then, by combining these sets with Theorem 2.1, we obtain the following corollary.

**Corollary 2.4.** If there is a complete \( s, 4i \)-useful cycle set with \( s \geq 1 \) for all \( 1 \leq i \leq n \), \( j \geq 1 \), and \( 1 \leq m_1, m_2, \ldots, m_j \leq n \), with \( n_1, n_2, \ldots, n_j \) positive integers then every \( n_1C_{4m_1}n_2C_{4m_2} \ldots n_jC_{4m_j} \) is graceful.

**Proof.** We follow the same method as in the proof of Theorem 2.1. That is, we will proceed by induction on the number of cycles \( k = n_1 + n_2 + \cdots + n_j \), and will prove the slightly more complex result that if there is a complete \( s, 4i \)-useful cycle set with \( s \geq 1 \) for all \( 1 \leq i \leq n \), \( j \geq 1 \), and \( 1 \leq m_1, m_2, \ldots, m_j \leq n \), with positive integers \( n_1, n_2, \ldots, n_j \) then every \( n_1C_{4m_1}n_2C_{4m_2} \ldots n_jC_{4m_j} \) is graceful with the label 0 in any position in the last cycle except the cut vertex. For \( k = 1 \) we have a \( k, 4i \)-useful cycle set from Theorem 2.1, so there exists a graceful labelling of any \( 1C_{4m} \) with 0 in any position in the last cycle.

Consider an arbitrary \( n_1C_{4m_1}n_2C_{4m_2} \ldots n_jC_{4m_j} \) with \( k > 1 \) and \( m = \sum_{l=1}^{j} n_l \left( 4m_l \right) \), the total number of edges. Let \( G \) be the graph obtained by deleting a last \( 4m_j \) cycle from \( n_1C_{4m_1}n_2C_{4m_2} \ldots n_jC_{4m_j} \). By the induction hypothesis, there is a graceful labelling of \( G \) with a 0 on the vertex distance \( d_{k-2} \) from the previous cut vertex. This labelling has vertex labels that are a subset of \( [0, m - 4m_j] \) and the edge labels are exactly \( [1, m - 4m_j] \).

We label \( n_1C_{4m_1}n_2C_{4m_2} \ldots n_jC_{4m_j} \) obtaining a vertex with label 0 at even distance \( d \) from the last cut vertex as follows. For the first \( k - 1 \) cycles, use the labelling of \( G \) and add \( 2m_j - 1 \) to each vertex label, so that the final cut vertex receives label \( 2m_j - 1 \). Thus, the vertices have been labelled from the set \( [2m_j - 1, m + 2m_j - 1] \) and by Lemma 1.1, the edge labels are \( [1, m - 4m_j] \).

Now the labelling of \( G \) has the label \( 2m_j - 1 \) at the last cycle. So apply the labelling of \( C_{4m_j}^d \) to the last cycle of \( G \), with the cut vertex receiving label \( 2m_j - 1 \) and 0 at position \( d \) (even distance) from the cut vertex. Then this labelling has all vertices labelled from \( [0, m] \), and edge labels exactly \( [1, m] \), with no repeated vertex or edge label. Thus, there is a graceful labelling of \( n_1C_{4m_1}n_2C_{4m_2} \ldots n_jC_{4m_j} \) with the vertex labelled 0 at even distance \( d \) from the cut vertex.
We label \( n_1 C_{4m_1}, n_2 C_{4m_2}, \ldots, n_j C_{4m_j} \) to obtain a vertex with label 0 at odd distance \( d \) from the last cut vertex as follows. For the first \( k - 1 \) cycles, use the labelling of \( G \) and subtract each vertex label from \( m - 2m_j + 1 \), so that the final cut vertex receives label \( m - 2m_j + 1 \). Thus, the vertices have been labelled from the set \( [2m_j - 1, m - 2m_j + 1] \) and by Lemma 1.1, the edge labels are \( [1, m - 2m_j + 1] \).

Now the labelling of \( G \) has the label \( m - 2m_j + 1 \) at the last cycle. So apply the labelling of \( C_{4m_j}^d \) to the last cycle of \( G \), with the cut vertex receiving label \( m - 2m_j + 1 \) and 0 at position \( d \) (odd distance) from the cut vertex. Then this labelling has all vertices labelled from \( [0, m] \), and edge labels exactly \( [1, m] \), with no repeated vertex or edge label. Thus, there is a graceful labelling of \( n_1 C_{4m_1}, n_2 C_{4m_2}, \ldots, n_i C_{4m_j} \) with the vertex labelled 0 at odd distance \( d \) from the cut vertex. \( \square \)

3 Graceful Labelling of \( kC_m \) for \( m \equiv 2 \mod 4 \)

In Section 2, we proved that if there is a complete \( s, 4n \)-useful cycle set with \( s \geq 1 \), then there exists a graceful labelling of any \( kC_{4n} \) for \( k \geq 1 \). The next natural question is can we prove the same results for \( kC_{4n+2} \)?

In \( kC_{4n} \) we have the nice property that \( kC_{4n} \) is always graceful regardless of the parity of \( k \). For \( kC_{4n+2} \) we will obtain graceful or near graceful labelings depending on the value of \( k \), because the size of a \( kC_{4n+2} \) is \( 4kn + 2k \) which is congruent to 0 modulus 4 for \( k \) even and \( 4kn + 2k \) which is congruent to 2 modulus 4 for \( k \) odd. Thus, we would essentially need to find two complete useful cycle sets for \( 4n + 2 \) because we are trying to change a graceful labelling to a near graceful one, or the reverse.

For \( kC_{4n} \) we obtained the complete cycle set by taking the complement as in Lemmas 1.2 to 1.3. Here for \( kC_{4n+2} \) we need to omit 1 from the labelling of the the near graceful cycles, because if we use 1 and take the complement we will obtain a labelling that is not near graceful. For example, if we take \( (0, 7, 3, 1, 2, 5) \) as a labelling of \( 1C_6 \), the complement would have 6 in the resulting vertex labelling and hence would not be near graceful. In this section we prove that there exists a (near) graceful labelling of any \( kC_6 \) (Theorem 3.1), because we found an analogue of a complete \( k, 6 \)-useful cycle set in Table 2 and did not use 1 for the near graceful useful cycles. An exhaustive analysis shows that no labelling of \( C_{10} \) exists that uses the
labels $[0, 5]$ and $[t - 5, t + 1] \setminus \{t\}$ that omits the label $1$. Thus, despite their effectiveness in the $4n$-cycle case, complete cycle sets cannot help us label even $kC_{10}$.

In [1], Barrientos proved that the $kC_4$ has a graceful labelling with any string, as summarized in Theorem 1.5. Recall the result of Barrientos from Theorem 1.6 that the linear $kC_6$ is near graceful if $k$ is odd and graceful if $k$ is even. In Theorem 3.1 we prove (near) graceful labellings exist for any $kC_6$.

In Table 2 we see four labellings of $C_6$. The labellings $C^a_6$ and $C^b_6$ use edge labels $[6k - 5, 6k + 1] \setminus \{6k\}$. The labellings $C^c_6$ and $C^d_6$ use edge labels $[6k - 6, 6k] \setminus \{6k - 5\}$.

| Labelling | $C^a_6$ | $C^b_6$ | $C^c_6$ | $C^d_6$ |
|-----------|---------|---------|---------|---------|
|           | $(0, 6k + 1, 2, 6k - 1, 3, 6k - 2)$ | $(0, 6k + 1, 3, 6k - 2, 2, 6k - 1)$ | $(0, 6k, 3, 6k - 3, 1, 6k - 1)$ | $(0, 6k, 1, 6k - 1, 3, 6k - 3)$ |

Table 2: Some useful labellings of $C_6$.

**Theorem 3.1.** If $k \geq 1$ then there exists a (near) graceful labelling of any $kC_6$.

**Proof.** As in proof of Theorem 2.1 we prove a slightly more complex result. Namely, we prove that if $k \geq 1$, then there exists a (near) graceful labelling of any $kC_6$ with $0$ in the last cycle in any position except the cut vertex.

If $0$ is in the last cycle of a $kC_6$, then up to symmetry its position is uniquely determined by the distance from the last cut vertex. These distances, $d$, can only be $1$, $2$ or $3$.

We proceed by induction on $k$. For $k = 1$, use $C^a_6$ or $C^b_6$ in Table 2 with $k = 1$ which will make it near graceful. For $k = 2$, use $(4, 7, 2, 5, 3, 6)$ and $(5, 0, 12, 1, 11, 3)$ to obtain a labelling with $d = 1$; $(8, 5, 10, 3, 7, 6)$ and $(3, 12, 0, 11, 1, 9)$ to obtain a labelling with $d = 2$; and $(4, 7, 2, 5, 6)$ and $(9, 1, 11, 0, 12, 3)$ to obtain a labelling with $d = 3$.

**Case 1:** Consider an arbitrary $kC_6$ with $k \geq 4$ and $k$ even, with the last entry in the string $d_{k-2}$. Let $G$ be the $(k - 1)C_6$ obtained by deleting
a last cycle from this $kC_6$. By the induction hypothesis, there is a near graceful labelling of $G$ with a 0 on the vertex distance $d_{k-2}$ from the previous cut vertex. This labelling has vertex labels that are a subset of $[0, 6k - 5] \setminus \{6k - 6\}$ and the edge labels are exactly $[1, 6k - 5] \setminus \{6k - 6\}$.

We label $kC_6$ obtaining a vertex with label 0 at distance $d$ from the cut vertex as follows. For the first $k - 1$ cycles, use the labelling of $G$ and add 3 to each vertex, so that the final cut vertex receives label 3. Thus, the vertices have been labelled from the set $[3, 6k - 2] \setminus \{6k - 3\}$ and by Lemma 1.1 the edge labels are $[1, 6k - 5] \setminus \{6k - 6\}$.

Apply the labelling $C^c_6$ or $C^d_6$ to the final cycle, with the cut vertex receiving label 3. Then this labelling of $kC_6$ has all vertices labelled from $[1, 6k]$, and the edge labels are exactly $[1, 6k]$ , with no repeated vertex or edge label. Then there is a graceful labelling of $kC_6$ with 0 in the $d = 2$ position.

We obtain a labelling with 0 in the $d = 1$ position by using the previously discussed labelling, ending with the $C^c_6$-labelling in the last cycle, then applying Lemma 1.2. We obtain $d = 3$ by using the $C^d_6$-labelling in the last cycle, and applying Lemma 1.2.

Case 2: Consider an arbitrary $kC_6$ with $k \geq 3$ and $k$ odd, with the last entry in the string $d_{k-2}$. We proceed in the same fashion as in Case 1, labelling all vertices except those in the final cycle, with vertices labelled from the set $[3, 6k - 3] \setminus \{6k - 6\}$.

Apply the labelling $C^a_6$ or $C^b_6$ to the final cycle, with the cut vertex receiving label 3. Then this labelling of $kC_6$ has all vertices labelled from $[1, 6k + 1] \setminus \{6k\}$, and the edge labels are exactly $[1, 6k + 1] \setminus \{6k\}$, with no repeated vertex or edge label. Then there is a graceful labelling of $kC_6$ with 0 in the $d = 2$ position.

We obtain a labelling with 0 in the $d = 1$ position by using the previously discussed labelling, ending with the $C^b_6$-labelling in the last cycle, then applying Lemma 1.3. We obtain $d = 3$ by using the $C^a_6$-labelling in the last cycle, and applying Lemma 1.3.

In Theorem 3.2 we prove (near) graceful labellings exist for some $kC_{10}$. In Table 3 we see 7 labellings of $C_{10}$. The labellings of $C^a_{10}, C^c_{10}, C^d_{10}$, and $C^b_{10}$ use edge labels $[10k - 9, 10k + 1] \setminus \{10k\}$. The labellings of $C^b_{10}, C^c_{10}, C^d_{10}$, and $C^f_{10}$ use edge labels $[10k - 10, 10k] \setminus \{10k - 9\}$. Since we cannot rely on the uniformity of a complete cycle set, this theorem uses a variety of different techniques to achieve similar effects.
Theorem 3.2. The $kC_{10}$ ($k \geq 1$) with string $(d_1, d_2, \ldots, d_{k-2})$ is graceful if $k$ is even and near graceful if $k$ is odd and one of the following is true:

1. $d_i \in \{4, 5\}$ if $i$ is odd and $d_i = 5$ if $i$ is even, where $1 \leq i \leq k - 2$,
2. $d_i \in \{3, 4\}$ if $i$ is odd and $d_i = 4$ if $i$ is even, where $1 \leq i \leq k - 2$,
3. $d_i \in \{2, 3\}$ if $i$ is odd and $d_i = 3$ if $i$ is even, where $1 \leq i \leq k - 2$, or
4. $d_i \in \{1, 2\}$ if $i$ is odd and $d_i = 2$ if $i$ is even, where $1 \leq i \leq k - 2$.

Proof. The proof is similar to the proof of Theorem 3.1 with some changes to the relabelling technique we use on vertex and edge labels.

Case 1: $d_i \in \{4, 5\}$ if $i$ is odd and $d_i = 5$ if $i$ even.

As in the proof of Theorem 3.1 we prove that the $kC_{10}$ ($k \geq 1$) with string $(d_1, d_2, \ldots, d_{k-2})$ satisfying the condition of part one with 0 in the $d = 4$ or $d = 5$ position in the last cycle and near graceful if $k$ is odd with $10k + 1$ in the $d = 5$ position in the last cycle.

We proceed by induction on $k$. For $k = 1$, use the labelling of $C_{10}^a$ in Table 3 with $k = 1$ which will make it near graceful. For $k = 2$, use $(11, 8, 13, 9, 16, 5, 14, 6, 12, 10)$ and $(0, 20, 2, 18, 1, 16, 3, 17, 7, 19)$ to get $d = 5$, and take this labelling with Lemma 1.2 to get $d = 4$.

Case 1a: The proof is similar to the proof of Case 1 in the proof of Theorem 3.1. Consider $kC_{10}$ to be an arbitrary snake with a string as indicated in the condition of Case 1 with $k \geq 4$ and $k$ even. The labelling of $(k - 1)C_{10}$ by the induction hypothesis has vertex labels that

| $C_{10}^a$ | Labelling |
|------------|-----------|
| $(0, 10k + 1, 4, 10k - 2, 3, 10k - 4, 5, 10k - 3, 1, 10k - 1)$ |
| $(0, 10k, 1, 10k - 7, 3, 10k - 3, 5, 10k - 1, 2, 10k - 2)$ |
| $(0, 10k, 2, 10k - 4, 7, 10k - 6, 1, 10k - 2, 3, 10k - 1)$ |
| $(0, 10k + 1, 3, 10k - 2, 1, 10k - 3, 4, 10k - 4, 5, 10k - 1)$ |
| $(0, 10k, 2, 10k - 2, 1, 10k - 7, 3, 10k - 3, 7, 10k - 1)$ |
| $(0, 10k + 1, 5, 10k - 4, 4, 10k - 3, 3, 10k - 2, 1, 10k - 1)$ |
| $(0, 10k, 5, 10k - 6, 2, 10k - 4, 3, 10k - 2, 1, 10k - 1)$ |

Table 3: Some useful labellings of $C_{10}$. 
are a subset of $[0, 10k - 9] \setminus \{2, 10k - 10\}$ and the edge labels are exactly $[1, 10k - 9] \setminus \{10k - 10\}$ with $10k - 9$ in the vertex distance $d_{k-2}$ from the previous cut vertex.

We label $kC_{10}$ as follows. For the first $k - 1$ cycles, use the labelling of $(k - 1)C_{10}$ obtained by induction and then subtract each vertex label from $10k - 5$. Thus, the vertices have been labelled from the set $[4, 10k - 5] \setminus \{5, 10k - 7\}$ and by Lemma 1.2, the edge labels are $[1, 10k - 9] \setminus \{10k - 10\}$.

Apply the labelling $C_{10}^b$ to the final cycle, with the cut vertex receiving label 4. Then this labelling of $kC_{10}$ has all vertices labelled from $[0, 10k]$, and the edge labels are exactly $[1, 10k + 1] \setminus \{10k\}$, with no repeated vertex or edge label.

By induction, a graceful labelling of $kC_{10}$ exists, with 0 in the $d = 4$ position.

(Note that a possible conflict occurs as $(10k - 5) - 2 = 10k - 7$, however, in the labelling of the $(k - 1)C_{10}$ no vertex is labelled 2, therefore we can use $C_{10}^b$ without any restriction).

We obtain a labelling with 0 in the $d = 5$ position by using the previously discussed labelling, ending with the $C_{10}^b$-labelling in the last cycle, then applying Lemma 1.2.

**Case 1b:** Consider $kC_{10}$ to be an arbitrary snake with a string as indicated in the condition of Case 1 with $k \geq 3$ and $k$ odd. The proof is similar to the proof of Case 1a, but instead of subtracting each vertex label from $10k - 5$, add 5 to each vertex and applying the labelling $C_{10}^a$ to the final cycle, with the cut vertex receiving label 5. Then this labelling of $kC_{10}$ has all vertices labelled from $[0, 10k]$, and the edge labels are exactly $[1, 10k + 1] \setminus \{10k\}$, with no repeated vertex or edge label. Then we obtain a near graceful labelling of $kC_{10}$ with $10k + 1$ in the $d = 5$ position from the cut vertex.

**Case 2:** $d_i \in \{3, 4\}$ if $i$ is odd and $d_i = 4$ if $i$ is even.

As in the proof of Case 1, we prove that $kC_{10}$ ($k \geq 1$) with string $(d_1, d_2, \ldots, d_{k-2})$ satisfying the condition of part two with 0 in the $d = 3$ or $d = 4$ position in the last cycle and near graceful if $k$ is odd with 0 in the $d = 4$ position in the last cycle.

We proceed by induction on $k$. For $k = 1$, use the labelling of $C_{10}$ in Table 3 with $k = 1$ which will make it near graceful. For $k = 2$, use $(10, 9, 11, 5, 13, 4, 15, 8, 12, 7)$ and $(0, 20, 2, 16, 4, 14, 1, 18, 3, 19)$ to get $d = 4$, and take this labelling with Lemma 1.2 to get $d = 3$.

**Case 2a:** Consider $kC_{10}$ to be an arbitrary snake with a string as indicated in the condition of Case 2 with $k \geq 4$ and $k$ even. The proof is similar to the proof of Case 1a.
We label $kC_{10}$ as follows. For the first $k - 1$ cycles, use the labelling of $(k - 1)C_{10}$ obtained by induction and then add 4 to each vertex label. Thus, the vertices have been labelled from the set $[4, 10k - 5]\setminus\{6, 10k - 6\}$ and by Lemma 1.1, the edge labels are $[1, 10k - 9]\setminus\{10k - 10\}$.

Apply the labelling $C_{c_{10}}$ to the final cycle, with the cut vertex receiving label 4. Then there is a graceful labelling of $kC_{10}$ and 0 in the $d = 4$ position, from the cut vertex. (Note that a possible conflict occurs as $(10k - 6) + 4 = 10k - 2$, however, in the labelling of the $(k - 1)C_{10}$ no vertex is labelled $10k - 2$, therefore we can use $C_{c_{10}}$ without any restriction).

We obtain a labelling with 0 in the $d = 3$ position by using the previously discussed labelling, ending with the $C_{c_{10}}$-labelling in the last cycle, then applying Lemma 1.2.

**Case 2b:** Consider $kC_{10}$ to be an arbitrary snake with a string as indicated in the condition of Case 2 with $k \geq 3$ and $k$ odd. The proof is similar to the proof of Case 1b, using the labelling of $(k - 1)C_{10}$ and adding 5 to each vertex. Finally, apply the labelling $C_{a_{10}}$ to the final cycle, with the cut vertex receiving label 5. Then we obtain a near graceful labelling of $kC_{10}$ and 0 in the $d = 4$ position, relative to cut vertex.

**Case 3:** $d_i \in \{2, 3\}$ if $i$ is odd and $d_i = 3$ if $i$ is even.

We prove that the $kC_{10}$ ($k \geq 1$) with string $(d_1, d_2, \ldots, d_{k-2})$ satisfying the condition of part three with 0 in the $d = 2$ or $d = 3$ position in the last cycle and near graceful if $k$ is odd with $10k + 1$ in the $d = 3$ position in the last cycle.

For $k = 1$, use the labelling of $C_{f_{10}}$ in Table 3 with $k = 1$ which will make it near graceful. For $k = 2$, use $(8, 14, 7, 12, 16, 5, 14, 10, 11, 9, 15, 6, 10, 9, 11)$ to get $d = 3$, and take this labelling with Lemma 1.2 to get $d = 2$. The rest of the proof is similar to the proof of Case 1, but uses $C_{d_{10}}$ instead of $C_{a_{10}}$, and $C_{e_{10}}$ instead of $C_{b_{10}}$.

**Case 4:** $d_i \in \{1, 2\}$ if $i$ is odd and $d_i = 2$ if $i$ is even.

We prove that $kC_{10}$ ($k \geq 1$) with string $(d_1, d_2, \ldots, d_{k-2})$ satisfying the condition of part four with 0 in the $d = 1$ or $d = 2$ position in the last cycle and near graceful if $k$ is odd with $10k + 1$ in the $d = 1$ position in the last cycle.

For $k = 1$, use the labelling of $C_{f_{10}}$ in Table 3 with $k = 1$ which will make it near graceful. For $k = 2$, use $(8, 14, 7, 12, 16, 5, 14, 10, 9, 11)$ to get $d = 2$, and take this labelling with Lemma 1.2 to get $d = 1$. The rest of the proof is similar to the proof of Case 2, but uses $C_{d_{10}}$ instead of $C_{a_{10}}$, and $C_{b_{10}}$ instead of $C_{c_{10}}$. □
In Theorem 3.3 we prove (near) graceful labellings exist for a $kC_{14}$ for particular strings. In Table 4 we see two labellings of $C_{14}$. The labelling $C^a_{14}$ uses edge labels $[14k - 13, 14k + 1] \setminus \{14k\}$. The labelling $C^b_{14}$ uses edge labels $[14k - 14, 14k] \setminus \{14k - 13\}$.

| Labelling |
|-----------|
| $C^a_{14}$ | $(0, 14k + 1, 4, 14k - 2, 5, 14k - 4, 6, 14k - 5, 7, 14k - 6, 2, 14k - 3, 1, 14k - 1)$ |
| $C^b_{14}$ | $(0, 14k, 4, 14k - 10, 2, 14k - 3, 3, 14k - 5, 6, 14k - 4, 5, 14k - 2, 1, 14k - 1)$ |

Table 4: Some useful labellings of $C_{14}$.

**Theorem 3.3.** If $k \geq 1$ and $d_i \in \{6, 7\}$ if $i$ is odd and $d_i = 7$ if $i$ is even, for $1 \leq i \leq k - 2$, then $kC_{14}$ with string $(d_1, d_2, \ldots, d_{k-2})$ is graceful if $k$ is even and near graceful if $k$ is odd.

**Proof.** The proof is similar to the proof of Theorem 3.1.

As in the proof of Theorem 3.1, we prove that if $k \geq 1$ and $d_i \in \{6, 7\}$ if $i$ is odd and $d_i = 7$ if $i$ is even, for $1 \leq i \leq k - 2$, then the $kC_{14}$ with string $(d_1, d_2, \ldots, d_{k-2})$ is graceful if $k$ is even with 0 in the $d = 6$ or $d = 7$ position in the last cycle and near graceful if $k$ is odd, with $14k + 1$ in the $d = 7$ position in the last cycle.

We proceed by induction on $k$. For $k = 1$, use the labelling of $C^a_{14}$ in Table 4 which will make it near graceful. For $k = 2$ use (21, 8, 20, 10, 19, 13, 14, 12, 15, 11, 16, 9, 17, 6) and (0, 28, 4, 18, 2, 25, 3, 23, 6, 24, 5, 26, 1, 27) to get $d = 6$, and take this labelling with Lemma 1.2 to get $d = 7$.

**Case 1:** Consider $kC_{14}$ to be an arbitrary snake with 0 in the $d = 6$ or $d = 7$ position in the last cycle with $k \geq 4$ and $k$ even. The labelling of $(k - 1)C_{14}$ by the induction hypothesis has vertex labels that are a subset of $[0, 14k - 15] \setminus \{3, 14k - 14\}$ and the edge labels are exactly $[1, 14k + 1] \setminus \{14k\}$ with $14k - 13$ in the vertex distance $d_{k-2}$ from the previous cut vertex.

We label $kC_{14}$ as follows. Subtract each vertex label from $14k - 7$ for the first $k - 1$ cycles of the $(k - 1)C_{14}$, so that the last cut vertex receives label 6. Then, apply the labelling $C^b_{14}$ to the final cycle, with the cut vertex receiving label 6. Then this labelling of $kC_{14}$ has all vertices labelled from $[0, 14k]$ and the edge labels are exactly $[1, 14k]$, with no repeated vertex or edge labels. Hence, we obtain a graceful labelling of $kC_{14}$ and 0 in the $d = 6$ position.
By using the previously discussed labelling, ending with the \( C_{14}^b \)-labelling in the last cycle, then applying Lemma 1.2 we obtain a labelling with 0 in the \( d = 7 \) position.

**Case 2**: Consider \( kC_{14} \) to be an arbitrary snake with 0 in the \( d = 6 \) position in the last cycle with \( k \geq 3 \) and \( k \) odd. The proof is similar to the proof of Case 1. Add 7 to each vertex label instead of subtracting each vertex label from \( 14k - 7 \) and apply the labelling \( C_{14}^a \) to the final cycle instead of \( C_{14}^b \), with the cut vertex receiving label 7 instead of 6. Then this labelling of \( kC_{14} \) has all vertices labelled from \([0, 14k + 1] \setminus \{3, 14k\}\), and the edge labels are exactly \([1, 14k + 1] \setminus \{14k\}\), with no repeated vertex or edge label. Thus, we obtain a near graceful labelling of \( kC_{14} \) and \( 14k + 1 \) in the \( d = 7 \) position. □

Recall that a \( kC_t \) is linear if all entries in its string are equal to \( \lfloor \frac{t}{2} \rfloor \). So, based on the results of Theorems 3.2 and 3.3 we now state a corollary for linear \( kC_{10} \) and \( kC_{14} \), following the style of Theorem 1.6.

**Corollary 3.4.** If \( k \geq 1 \) then the linear \( kC_{10} \) and linear \( kC_{14} \) are graceful if \( k \) is even and nearly graceful if \( k \) is odd.

## 4 Discussion

In this paper we (near) gracefully labelled several type of snakes. In section 2, we presented a new sufficient condition which when satisfied shows there is a graceful labelling of a \( kC_{4n} \) for any string. By using a complete \( s, t \)-useful cycle set we proved that if there is a complete \( s, 4n \)-useful cycle set with \( s \geq 1 \), then there exists a graceful labelling of any \( kC_{4n} \). We used the results in 2 with our results for \( kC_{4n} \) and proved that a graceful labelling exists for particular \( kC_{4n} \) with string \((d_1, d_2, \ldots, d_{k-2})\), where \( d_i \in \{2, 3, 4\} \). Expanding these results for any \( n \) and \( d \) is possible but hard to apply for large \( n \). We extended our main result to the case of cyclic snakes with cycles of varying sizes. Further, we extended the results in Theorems 1.6 to 1.9 on (near) gracefully labelled \( kC_n \) where \( n = 6, 8, 12, 16, 20, 24 \) for all possible snakes.

As we discussed in Section 3, new approaches must be found to gracefully label \( kC_{4n+2} \) snakes, even for fixed \( n \). Our collections of ad hoc methods work to give classes for fixed \( n \), but do not seem to generalize, even for “nice” subfamilies, such as linear snakes. Thus we pose the following open question.
**Question:** Can we (near) gracefully label every $kC_m$ with $k \geq 1$ and $m \equiv 2 \pmod{4}$?

In fact the technique we used is more general than indicated in our theorems. Suppose we have a gracefully labelled bipartite graph $G = K_{3,4}$ as in Figure 2. If we add 3 to each vertex label and use the cycle $H = C^2_8$ from Table 1 then we obtain a new gracefully labelled graph as in Figure 3. Thus we can in several cases gracefully label new graphs.

**Theorem 4.1.** If $G$ is graceful and $H$ is a $kC_{4n}$ with $1 \leq n \leq 6$, then the graph $GH^*$ obtained by identifying any vertex in $G$ that can be labelled 0 in some graceful labelling with any vertex in the first cycle of $H$ is graceful.

![Figure 2: Gracefully labelled $K_{3,4}$.](image1)

![Figure 3: Gracefully labelled $GH^*$.](image2)

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