OPERATOR FORMALISM FOR $b$–$c$ SYSTEMS WITH $\lambda = 1$
ON GENERAL ALGEBRAIC CURVES

F. Ferrari$^a$, J. Sobczyk$^b$

$^a$Dipartimento di Fisica, Università di Trento, 38050 Povo (TN), Italy and INFN, Gruppo Collegato di Trento, E-mail: francof@galileo.science.unitn.it

$^b$Institute for Theoretical Physics, Wrocław University, pl. Maxa Borna 9, 50205 Wrocław, Poland, E-mail: jsobczyk@proton.ift.uni.wroc.pl

ABSTRACT

In this letter we develop an operator formalism for the $b$–$c$ systems with conformal weight $\lambda = 1$ defined on a general closed and orientable Riemann surface. The advantage of our approach is that the Riemann surface is represented as an affine algebraic curve. In this way it is possible to perform explicit calculations in string theory at any perturbative order. Besides the obvious applications in string theories and conformal field theories, (the $b$–$c$ systems at $\lambda = 1$ are intimately related to the free scalar field theory), the operator formalism presented here sheds some light also on the quantization of field theories on Riemann surfaces. In fact, we are able to construct explicitly the vacuum state of the $b$–$c$ systems and to define creation and annihilation operators. All the amplitudes are rigorously computed using simple normal ordering prescriptions as in the flat case.

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1. INTRODUCTION

In a recent paper [1] we have introduced an operator formalism for the $b-c$ systems with integer conformal weight $\lambda \geq 2$ on general Riemann surfaces. In our construction the fields are expanded in generalized Laurent series. Upon quantization, the coefficients of the series become either creation and annihilation operators or correspond to zero modes of the theory. The advantage of the approach used in [1] is that the Riemann surfaces are represented as algebraic curves, i.e. as $n$–sheeted coverings of the complex plane. This allows very explicit calculations and a better understanding of free field theories defined on nontrivial two dimensional topologies. A remarkable result which we obtain is the proof that the Hilbert space of physical states is splitted into a finite number of ”independent” spaces. With all these ingredients we have been able to calculate in an explicit way the correlation functions of the $b-c$ systems.

As might be expected, the main difficulty of the operator formalism is to deal properly with the zero modes. For that reason, in [1] we have restricted ourselves to the case $\lambda \geq 2$, where only the zero modes of the $b$ fields are present. In this paper we are going to generalize our approach also to the interesting case in which $\lambda = 1$. With respect to ref. [1], the calculations at $\lambda = 1$ are complicated by the presence of the $c$ zero mode, which modifies the definition of the vacuum state and of the normal ordering of the fields.

We notice that our formalism is valid on very general algebraic curves given by any Weierstrass polynomial that is nondegenerate. We also have assumed that all the branch points lie in a finite region of the complex plane. These restrictions are adopted only to simplify the technical aspects of the construction. In fact, the crucial point of the reasoning is the possibility of expanding the Weierstrass kernel in terms of multivalued modes (elements of the generalized Laurent expansions) of the fields. This possibility has been proven for arbitrary Weierstrass polynomials in [1]. For instance, in [1] we provide some examples showing that the way in which we introduce the operator formalism is entirely general. Let us mention also that the problem of constructing operator formalism for $b-c$ systems on Riemann surfaces has been discussed by other authors but in a different and less explicit (from the point of view of physical applications) mathematical framework [2].
There are various applications of our results. First of all, the new formalism should be useful in perturbative string theory, generalizing to any Riemann surface previous computations performed in the special case of hyperelliptic curves \[3\]. Moreover, there has been in recent times some interest in the quantization of two dimensional free field theories immersed in curved space-times \[4\]. From this point of view, the example of the \( b-c \) systems, in which the vacuum state is explicitly constructed and the amplitudes can be computed using simple normal ordering prescriptions, is remarkable. As a matter of fact, the Riemann surfaces are not globally hyperbolic manifolds and, as a consequence, there is not an unique definition of time \[5\]. Nevertheless, we are able to show that it is still possible to define creation and annihilation operators, corresponding to the creation and destruction of certain multivalued modes. Other applications are related to the representation of the solutions of deformed Kniznik-Zamolodchikov equations \[6\]–\[8\], integrable models \[9\]–\[10\] and relation between CFT on the complex plane and \( b-c \) systems on Riemann surfaces \[11\]–\[15\].

2. GENERALIZED LAURENT EXPANSIONS ON ALGEBRAIC CURVES

We consider a theory of free fermionic \( b-c \) fields with spin \( \lambda = 1 \) defined by the following action
\[
S_{bc} = \int_{\Sigma_g} d^2\xi \left( b\partial c + c.c. \right).
\] (2.1)
\( \xi \) and \( \bar{\xi} \) are complex coordinates on a Riemann surface \( \Sigma_g \) described by means of an algebraic equation in \( \mathbb{C}P_2 \) \[16\]–\[17\]:
\[
F(z, y) = P_n(z)y^n + P_{n-1}(z)y^{n-1} + \ldots + P_1(z)y + P_0 = 0.
\] (2.2)
The \( P_s(z) = \sum_{m=0}^{n-s} \alpha_{s,m}z^m \) are polynomials in \( z \) of degree at most \( n-s \), \( s = 0, \ldots, n \) and the \( \alpha_{s,m} \) are complex parameters. \( y \) can be viewed either as a multivalued function on the complex sphere \( \mathbb{C}P_1 \) or as a meromorphic function on the Riemann surface defined by eq. (2.2). In both cases we can think of \( \Sigma_g \) as of an \( n \)-branched covering of \( \mathbb{C}P_1 \), with the function \( y \) taking (in general) \( n \) different values on the \( n \) branches forming the Riemann surface. The function \( y(z) \) has poles only at \( z = \infty \). In order to fix the ideas (see the
discussion in [1], we require the algebraic curve (2.2) to be nondegenerate [17]. Then the genus of Σ is 
\[ g = \frac{(n-1)(n-2)}{2} \]. All the other Riemann surfaces can be obtained by some limiting procedure in the parameters \( \alpha_{s,m} \). In [1] we argue that our operator formalism is kept untouched while performing such limits.

The solutions of the classical equations of motion descending from eq. (2.1) can be represented as
\[
b(z) = \sum_{k=0}^{n-1} \sum_{i=-\infty}^{\infty} b_{k,i} z^{-i} f_k(z) \tag{2.3}
\]
\[
c(z) = \sum_{k=0}^{n-1} \sum_{i=-\infty}^{\infty} c_{k,i} z^{-i} \phi_k(z) \tag{2.4}
\]
with \( f_k \) and \( \phi_l \) chosen as follows \((k, l = 0, ..., n-1)\):
\[
f_k(z) = \frac{y^{n-1-k}(z) dz}{F_y(z, y(z))} \tag{2.5}
\]
\[
\phi_l(w) = y^l(w) + y^{l-1}(w) P_{n-1}(w) + y^{l-2}(w) P_{n-2}(w) + ... + P_{n-l}(w) \tag{2.6}
\]
We notice that we have introduced two different expansions for the fields \( b \) and \( c \). This is only for the sake of convenience in the formulation of the operator formalism.

In order to check the absence of spurious singularities in the amplitudes, we shall also need the following divisors:
\[
[dz] = \sum_{p=1}^{n_{bp}} (\nu_p - 1) a_p - 2 \sum_{j=0}^{n-1} \infty_j \tag{2.7}
\]
\[
[y] = \sum_{r=1}^{n} q_r - \sum_{j=0}^{n-1} \infty_j \tag{2.8}
\]
\[
[F_y] = \sum_{p=1}^{n_{bp}} (\nu_p - 1) a_p - (n-1) \sum_{j=0}^{n-1} \infty_j \tag{2.9}
\]
In eq. (2.8) the \( a_p \) are the branch points of the curve \( \Sigma \) determined by the condition [17]:
\[
F(z, y) = F_y(z, y) = 0 \tag{2.10}
\]
We suppose that there are \( n_{bp} \) finite branch points of multiplicity \( \nu_s \), where \( \nu_s \) describes the number of branches of \( y \) connected at the branch point \( a_s \). The \( q_j \) denote the zeros
of $y$ which, when projected from the Riemann surface on the $z$ complex plane, coincide with the zeros of $P_0(z)$. Moreover, $\infty_j$ describes the projection of the point at infinity on the $j$-th sheet of the Riemann surface. Finally, in our conventions positive and negative integers denote the order of the zeros and of the poles respectively. Starting from eqs. (2.7)-(2.9), it is also possible to find the divisors of the differentials (2.5):

$$[z^i f_k] = (n - 1 - k) \sum_{s=1}^{n} q_s + i \sum_{l=0}^{n-1} 0_l + (k - 2 - i) \sum_{l=0}^{n-1} \infty_l$$  \hspace{1cm} (2.11)

where, using the same notation exploited for the point at infinity, $0_l$ denotes the projection of the point $z = 0$ on the $j$-th sheet.

Next we explicitly derive the form of the zero modes (holomorphic differentials) associated to the equations of motion. We try the following ansatz:

$$\Omega_{k,i} = f_k(z)z^{-i-1}$$  \hspace{1cm} (2.12)

From (2.11) it turns out that $\Omega_{k,i}$ has no singularities whenever

$$i \leq -1 \quad \text{and} \quad k - 1 + i \geq 0$$  \hspace{1cm} (2.13)

Skipping the simple cases corresponding to genus zero and one Riemann surfaces ($n = 2, 3$), we obtain the conditions determining the $g$ independent zero modes of the form (2.12):

$$\begin{cases} k = 2, 3, \ldots, n - 1 \\ -k + 1 \leq i \leq -1 \end{cases}$$  \hspace{1cm} (2.14)

while

$$N_{b_k} = k - 1 \quad \text{for} \quad k = 2, \ldots, n - 1$$  \hspace{1cm} (2.15)

Obviously, the overall number of $b$ zero modes is equal to $\frac{(n-1)(n-2)}{2} = g$. Looking at (2.6) we also find out that the $c$ zero mode corresponds to $l = 0$.

3. THE OPERATOR FORMALISM

In the previous section we have shown how to identify the classical degrees of freedom as coefficients $b_{i,k}$ and $c_{i,k}$. Now we perform the quantization transforming $b_{i,k}$ and $c_{i,k}$
into creation and annihilation operators. We divide the degrees of freedom of the $b - c$ systems into $n$ sectors, numbered by the index $k = 0, \ldots, n - 1$ and characterized by the tensors $f_k$ for the $b$ fields and by the tensors $\phi_k$ for the $c$ fields. We assume that in our operator formalism the modes $z^j f_k(z)$ labelled by different indices $k$ do not interact. This hypothesis, to be proven a posteriori, implies that the space on which the $b - c$ fields defined on a Riemann surface act can be decomposed into a set of $n$ independent Hilbert spaces if the Riemann surface is represented as an $n$--sheeted branch covering over $\mathbb{CP}_1$.

In order to set up the operator formalism, we postulate the following commutation relations:

$$\{b_{k,i}, c_{k',i'}\} = \delta_{kk'} \delta_{i+i',0}. \quad (3.1)$$

Moreover, we define $n$ vacua $|0\rangle_k$, $k = 0, \ldots, n - 1$, in such a way that the modes with negative powers of $z$ are annihilation operators

$$b_{k,i}^- |0\rangle_k \equiv b_{k,i} |0\rangle_k = 0 \quad \begin{cases} k = 0, \ldots, n - 1 \\ i \geq 0 \end{cases} \quad (3.2)$$

$$c_{k,i}^- |0\rangle_k \equiv c_{k,i} |0\rangle_k = 0 \quad \begin{cases} k = 0, \ldots, n - 1 \\ i \geq 1 \end{cases} \quad (3.3)$$

Naively, the corresponding right vacua are given by:

$$k \langle 0 | b_{k,i}^+ \equiv k \langle 0 | b_{k,i} = 0 \quad \begin{cases} k = 2, \ldots, n - 1 \\ i \leq -k \end{cases} \quad (3.4)$$

$$k \langle 0 | b_{k,i}^+ \equiv k \langle 0 | b_{k,i} = 0 \quad \begin{cases} k = 0, 1 \\ i \leq -1 \end{cases} \quad (3.5)$$

$$k \langle 0 | c_{k,i}^+ \equiv k \langle 0 | c_{k,i} = 0 \quad \begin{cases} k = 1, \ldots, n - 1 \\ i \leq 0 \end{cases} \quad (3.6)$$

$$0 \langle 0 | c_{0,i}^+ \equiv 0 \langle 0 | c_{0,i} = 0 \quad \begin{cases} k = 0 \\ i \leq -1 \end{cases} \quad (3.7)$$

where $N_{b_k} = k - 1$ from eq. (2.15). In order to treat properly the zero modes, however, we have to introduce modified vacua $k \langle 0'|$ defined by:

$$k \langle 0'| = k \langle 0 | b_{k,-k+1} \ldots b_{k,-1} \quad k = 2, \ldots, n - 1 \quad (3.8)$$
\[ 1 \langle 0' | = 1 \langle 0 | \]  
\[ 0 \langle 0' | = 0 \langle 0 | c_{0,0} \]  
(3.9)  
(3.10)

This is the generalization on Riemann surfaces of the recipe given in the flat case in order to treat the zero modes (see ref. \[18\]). The modified vacua are normalized as follows:

\[ k \langle 0' | \langle 0 | = \prod_{k=0}^{n-1} k \langle 0 | \]  
(3.11)

Only the amplitudes containing at least a number \(N_{b_k}\) of \(b\) fields in the sectors \(k = 2, \ldots, n - 1\) and one \(c\) field in the sector \(k = 0\) do not vanish. The total vacuum \(\langle 0 |\) is given by:

\[ |0\rangle = \prod_{k=0}^{n-1} |0\rangle_k \]  
(3.12)

\[ \langle 0 | = \prod_{k=0}^{n-1} k \langle 0 | \]  
(3.13)

The following notations will be useful in the future:

\[ b_k(z) = f_k(z) \sum_{i=-\infty}^{\infty} b_{k,i}z^{-i-1} \]  
(3.14)

\[ c_k(z) = \phi_k(z) \sum_{i=-\infty}^{\infty} c_{k,i}z^{-i} \]  
(3.15)

\[ b(z) = \sum_{k=0}^{n-1} b_k(z) \]  
\[ c(z) = \sum_{k=0}^{n-1} c_k(z) \]  
(3.16)

Eq. (3.16) is equivalent to the decompositions (2.3) and (2.4). The only difference is that now the \(b_{k,i}\) and \(c_{k,i}\) are operators.

After the identification of \(b_{i,k}\) and \(c_{i,k}\) as creation or annihilation operators (see (3.2)-(3.7)) it is possible to introduce a natural notion of normal ordering. The only modification comes from the fact that some \(b_{i,k}\) as well as \(c_{0,0}\) operators are neither annihilation nor creation ones. According to eqs. (3.8)-(3.10), we define the normal ordering by requiring that the operators corresponding to zero modes stand to the left of the annihilation operators. Exploiting the commutation relations (3.11) and the conventions established above, it is easy to see that the relation between normal ordered and not ordered products of \(b - c\) fields is given by:

\[ b_k(z)c_k(w) =: b_k(z)c_k(w) : + \frac{1}{z - w} f_k(z)\phi_k(w) \]  
(3.17)
The “time ordering” is implemented by the requirement that the fields \( b(z) \) and \( c(w) \) are radially ordered with respect to the variables \( z \) and \( w \).

We are now ready to compute the correlation functions of the \( b - c \) systems. Using eq. (3.11) the proof of the following identity is straightforward [15], [1]:

\[
\langle 0 | b_k(z_1) \ldots b_k(z_{N_{b_k}}) | 0 \rangle_k = \det | \Omega_{k,j}(z_i) | \\
\text{subject to } \begin{cases} k = 2, \ldots, n - 1 \\ i, j = 1 \ldots, N_{b_k} \end{cases} \quad (3.18)
\]

The zero modes \( \Omega_{k,j}(z) \) can be expressed in terms of \( z \) and \( y \) from eqs. (2.5) and (2.12):

\[
\Omega_{k,j}(z) = \frac{y^{n-k-1}z^{j-1}}{F_y(z, y)} dz \\
\text{subject to } \begin{cases} k = 2, \ldots, n - 1 \\ j = 1, \ldots, k - 1 \end{cases} \quad (3.19)
\]

With the help of (3.18) it is possible to compute the correlator:

\[
S(u; z_1, \ldots, z_{N_{b}}) = \langle 0 | c(u) b(z_1) \ldots b(z_{N_{b}}) | 0 \rangle = \det | \Omega_I(z_{J}) | \quad (3.20)
\]

Since \( S(u; z_1, \ldots, z_{N_{b}}) \) does not depend on the coordinate \( u \) of the \( c \) zero mode, the following notation is more convenient: \( S(u; z_1, \ldots, z_{N_{b}}) \equiv S(z_1, \ldots, z_{N_{b}}) \). In (3.20) \( I, J = 1, \ldots, N_{b} \) and \( N_{b} \) denotes the total number of zero modes. The \( \Omega_{I}(z) \) represent all the possible zero modes in the \( b \) fields:

\[
\Omega_I(z) \in \{ \Omega_{k,i}(z) | 1 \leq i \leq N_{b_k}, \quad 2 \leq k \leq n - 1 \}
\]

To demonstrate (3.20), we rewrite the correlator (3.20) as follows:

\[
S(z_1, \ldots, z_{N_{b}}) = \sum_{s=0}^{n-1} \sum_{r_0 + \cdots + r_{n-1} = N_{b}} \sum_{\sigma} \text{sign}(\sigma) \prod_{k=0}^{n-1} \prod_{l_k=\alpha(k)}^{b(k)} b_k(z_{\sigma(l_k)}) c_s(w) | 0 \rangle_k \quad (3.21)
\]

where:

\[
\alpha(0) = 1 \quad \beta(0) = r_0 \\
\alpha(k) = 1 + \sum_{m=0}^{k-1} r_m \quad k = 1, \ldots, n - 1 \\
\beta(k) = \alpha(k) + r_k - 1 \\
\sigma(\alpha(k)) \leq \ldots \leq \sigma(\beta(k)) \quad k = 0, \ldots, n - 1
\]

In eq. (3.21) the products of the fields \( b \) and \( c \) have been expanded in their components \( b_k(z) \) and \( c_k(z) \), \( k = 0, \ldots, n - 1 \). It is easy to see that any contraction of a \( b \) field with
c_s(w) leads to a vanishing amplitude due to (3.11). For that reason, the only nonvanishing amplitudes are those for which:

\[ r_0 = r_1 = 0, \quad r_k = N_{bk} \]

and \( s = 0 \). Exploiting also eq. (3.18), we arrive at the final result:

\[
S(z_1, \ldots, z_{N_b}) = \sum_{\sigma} \text{sign}(\sigma) \prod_{k=2}^{n-1} \det[A_{i_k, i_{\sigma(k)}}] \tag{3.22}
\]

where \( i_k = 1, \ldots, N_{bk} \) and \( l_k = 1 + \sum_{m=2}^{k-1} N_{bk}, \ldots, \sum_{m=2}^k N_{bk} \). The right hand side of (3.22) is exactly the determinant \( \det[A_I(z_J)] \). This way of expanding the determinants is explained in [19]. Thus eq. (3.20) has been demonstrated. More details of the calculation are contained in ref. [1].

The next amplitude to be computed is the propagator \( G(z, w)dz \).

\[
G(z, w)dz \equiv \frac{\langle 0|b(z)c(w)c(u) \prod_{I=1}^{N_b} b(z_I)|0\rangle}{\langle 0|c(u) \prod_{I=1}^{N_b} b(z_I)|0\rangle} \tag{3.23}
\]

From eq. (3.17) the normal ordering between any two fields \( b \) and \( c \) becomes:

\[
b(z)c(w) =: b(z)c(w) : + K(z, w)dz \tag{3.24}
\]

where

\[
K(z, w)dz = \sum_{k=0}^{n-1} f_k(z)\phi_k(w) \left[ \frac{1}{z-w} \right] \tag{3.25}
\]

Using the Wick theorem we find:

\[
G(z, w)dz = K(z, w)dz - K(z, u)dz + \\
\sum_{J=1}^{N_b} (-1)^J K(z_J, w)dz_J S(z_1, \ldots, z_{J-1}, z, z_{J+1}, \ldots, z_{N_b}) S(z_1, \ldots, z_{N_b}) \\
\sum_{J=1}^{N_b} (-1)^J K(z_J, u)dz_J S(z_1, \ldots, z_{J-1}, z, z_{J+1}, \ldots, z_{N_b}) S(z_1, \ldots, z_{N_b}) \tag{3.26}
\]
The correlators $S(z_1, \ldots, z_{J-1}, z, z_{J+1}, \ldots, z_N)$ can be evaluated by means of eq. (3.20).

Let us introduce now the third kind differential $\omega_{wu}(z)\,dz = K(z, w)\,dz - K(z, u)\,dz = \frac{F(w, y^{(l)}(z))}{(y^{(l)}(z) - y^{(r)}(w)) F_y(z, y^{(l)}(z))} \frac{dz}{z - w} - \frac{F(u, y^{(l)}(z))}{(y^{(l)}(z) - y^{(s)}(u)) F_y(z, y^{(l)}(z))} \frac{dz}{z - w}$ (3.27)

$\omega_{wu}(z)\,dz$ is a differential in $z$ with simple poles at points $z = w$ and $z = u$ (it should be understood that by point $u$ or $w$ we mean a point on a definite sheet of $y$, namely $r$ for $w$ and $s$ for $u$, while the point $z$ lies on the sheet $l$). With these conventions eq. (3.26) becomes:

$$G(z, w)\,dz = \frac{1}{\det \Omega_I(z_J)}$$

It can be verified that the propagator (3.28) does not contain spurious poles. Further application of the Wick theorem enables us to calculate more complicated correlation functions.

$$G_{NM}(z_1 \ldots z_N; w_1 \ldots w_M) = \frac{1}{\det \Omega_I(z_J)}$$

The way in which (3.29) is written suggests that the point $w_M$ is treated in a special way. It is however only apparent due to the formula (3.27). In fact (3.29) has all the necessary properties of a "good" correlation function [21].

4. FINAL REMARKS

We have shown that the new operatorial formalism on Riemann surfaces proposed in [1] can be applied also to more complicated cases in which both the $b$ and $c$ zero modes are
present. All information about the zero modes is absorbed in the suitable definition of the modified vacuum state of the theory. With the prescriptions given here it is possible to compute the amplitudes of the $b-c$ systems at $\lambda = 1$. Due to the lack of space some details concerning the Wick theorem and the application of the operator formalism to the scalar fields have been omitted. These topics will be treated in a forthcoming paper, where we will also derive in an explicit way all the formulas required in the perturbative formulation of the string theory [22].
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