Super-symmetric informationally complete measurements

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Abstract: Symmetric informationally complete measurements (SICs in short) are highly symmetric structure in the Hilbert space. They possess many nice properties which render them an ideal candidate of fiducial measurements. The symmetry of SICs is intimately connected with the geometry of the quantum state space and also has profound implications for foundational studies. Here we study those SICs that are most symmetric according to a natural criterion and show that all of them are covariant with respect to the Heisenberg-Weyl groups, which are characterized by the discrete analog of the canonical commutation relation. Moreover, their symmetry groups are subgroups of the Clifford groups. In particular, we prove that the SIC in dimension 2, the Hesse SIC in dimension 3, and the set of Hoggar lines in dimension 8 are the only three SICs up to unitary equivalence whose symmetry groups act transitively on pairs of SIC projectors. Our work is of intrinsic interest to studying the geometry of quantum state space and foundational issues entangled with the geometry.

1. Introduction

Symmetry plays a fundamental role in all areas of natural science. Of special interest are those objects possessing highest symmetry, which are the targets of constant quest. In a $d$-dimensional Hilbert space, a symmetric informationally complete measurement (SIC in short) is usually composed of $d^2$ subnormalized projectors onto pure states $|\psi_j\rangle\langle\psi_j|/d$ with equal pairwise fidelity [21,26,31,33],

$$|\langle\psi_j|\psi_k\rangle|^2 = \frac{d\delta_{jk} + 1}{d + 1}. \quad (1)$$

In this paper, by a “SIC” we shall mean the set of SIC projectors $\Pi_j = |\psi_j\rangle\langle\psi_j|$, which sum up to $d$ times of the identity, $\sum_j \Pi_j = d$. SICs have many nice properties rooted in their high symmetry, which make them an ideal candidate of
fiducial measurements. They play a crucial role in studying quantum Bayesianism [14] and in understanding the geometry of quantum state space [2,8]. They also have intriguing connections with many other interesting subjects, such as equiangular lines, 2-designs, mutually unbiased bases (MUB), Lie algebras, and Galois theory; see Ref. [5] and references therein.

A SIC gives rise to a regular simplex in the operator space. Although this perspective is fruitful in understanding the properties of a SIC [5], it has an obvious limitation: most permutations among the SIC projectors cannot be realized by unitary or antiunitary transformations, those transformations that preserve the quantum state space. This conflict between permutation symmetry and unitary symmetry has profound implications for foundational studies and quantum information science [2,4,8,14]. It is closely connected with the fact that the state space is not a ball except for dimension 2. A better understanding of the symmetry of SICs is crucial to decoding the geometry of quantum state space as well as foundational and practical issues entangled with the geometry.

How symmetric is a SIC? This is a basic question we need to answer before we can conceive a clear picture about the quantum state space. Motivated by this question, here we determine those SICs that are most symmetric according to a natural criterion. In particular, we show that there are exactly three SICs up to unitary equivalence whose symmetry groups act transitively on pairs of SIC projectors.

The symmetry group $G$ of a SIC $\{\Pi_j\}$ is composed of all unitary operators $U$ that leave the set of SIC projectors invariant; that is, $U\Pi_jU^\dagger = \Pi_{\sigma(j)}$ for a suitable permutation $\sigma$. By convention, operators that differ only by overall phase factors are identified, as indicated by the overline notation in $\overline{G}$. The extended symmetry group is the larger group that also contains antiunitary operators. Every SIC known so far is group covariant in the sense that it can be generated from a single state—the fiducial state—by a group composed of unitary operators, that is, the SIC projectors form a single orbit under the action of the symmetry group. How much additional symmetry can we expect beyond group covariance?

A SIC is super-symmetric if every ordered pair of (distinct) SIC projectors can be mapped to every other such pair under its symmetry group, that is, the symmetry group acts 2-transitively on the set of SIC projectors in the language of permutation groups [10,13]. As a natural extension, a SIC is $k$-covariant if its symmetry group acts $k$-transitively, that is, every ordered $k$-tuple of SIC projectors can be mapped to every other such $k$-tuple. A $k$-covariant SIC is also referred to as doubly covariant when $k = 2$ and triply covariant when $k = 3$. Hence a super-SIC is doubly covariant and vice versa. To complete the picture, a SIC is $k$-homogeneous if every unordered $k$-tuple of SIC projectors can be mapped to every other such $k$-tuple under its symmetry group. A SIC in dimension $d$ is $k$-homogeneous if and only if it is $(d^2 - k)$-homogenous. In addition, any $k$-covariant SIC is $k$-homogeneous (the converse is also true when $k \leq d^2/2$ according to Lemma 1 below, but is not so obvious). When the symmetry group is replaced by the extended symmetry group, the terminologies are modified by adding the prefix “quasi-”. For example, a SIC is quasi-super-symmetric if every two ordered pairs of SIC projectors can be mapped to each other under its extended symmetry group.
2. Main results

As we shall see shortly, no triply covariant SICs can exist. Therefore, super-SICs are the most symmetric structures that can appear in the Hilbert space. Does there exist any super-SIC at all? An obvious candidate is the SIC in dimension 2, whose symmetry group can realize all even permutations among the SIC projectors (this SIC is also quasi-4-covariant, since the extended symmetry group can realize all permutations). After careful inspection of all SICs known in the literature [33], we found two other super-SICs. The first one is the Hesse SIC in dimension 3 [17,12,19,27,31,33] which is generated by the Heisenberg-Weyl (HW) group from the fiducial ket

\[ |\psi_3\rangle \equiv \frac{1}{\sqrt{2}}(0, 1, -1)^T. \]  

Its symmetry group is the whole Clifford group, which has order $9 \times 24$ in dimension 3 [32]. The second one is the set of Hoggar lines [18,31,33] generated by the three-qubit Pauli group from

\[ |\psi_8\rangle \equiv \frac{1}{\sqrt{6}}(1+i, 0, -1, 1, -i, -i, 0, 0)^T. \]  

It has an exceptionally large symmetry group, which has order $64 \times 6048$ [33]. It turns out that the three super-SICs we have identified exhaust all super-SICs.

Theorem 1. The SIC in dimension 2, the Hesse SIC in dimension 3, and the set of Hoggar lines in dimension 8 are the only three (quasi)-super-SICs up to unitary equivalence. They are also the only three (quasi)-2-homogeneous SICs up to unitary equivalence.

Here the second part of the theorem is an immediate consequence of the first part and the following lemma.

Lemma 1. A SIC in dimension $d$ is (quasi)-$k$-covariant with $k \leq d^2/2$ if and only if it is (quasi)-$k$-homogeneous. In particular, a SIC is (quasi)-super-symmetric if and only if it is (quasi)-2-homogeneous.

Proof. According to Theorem 1 of Kantor [21] (see also Theorem 9.4B in Ref. [13]), any $k$-homogenous permutation group of degree $n$ (with $n \geq 2k$) a perfect square is $k$-transitive, except possibly some 4-homogenous permutation group $G$ of degree 9. In addition, such a group $G$ must contain a subgroup isomorphic to $\text{PSL}(2,8)$, which is a nonabelian simple group. However, any group covariant SIC in dimension 3 is covariant with respect to the HW group, and its extended symmetry group is a subgroup of the extended Clifford group [32], which is solvable for dimension 3 and thus cannot contain any subgroup isomorphic to $\text{PSL}(2,8)$.

\[ ^1 \text{The stabilizer of each fiducial ket in the set of Hoggar lines is a nonabelian simple group of order 6048, which turns out to be isomorphic to } \text{PSU}(3,3). \]
2.1. Symmetry and triple products. Before proving our main result, we first show that no triply covariant SICs can exist and that for prime-power dimensions, (quasi)-super-SICs can only exist in dimensions 2, 3, and 8. The proof is based on simple observation on triple products among SIC projectors,

\[ T_{jkl} = \text{tr}(\Pi_j \Pi_k \Pi_l), \quad (4) \]

which have played an important role in studying the symmetry of and equivalence relations among SICs \[3, 32, 33\]. Note that \(|T_{jkl}|\) is equal to \((d + 1)^{-3/2}\) if the three indices are distinct, while it is equal to \(1/(d + 1)\) or 1 if two or three indices coincide. The normalized triple products \(\tilde{T}_{jkl} = T_{jkl}/|T_{jkl}|\) satisfy

\[ \tilde{T}_{jkl} = \tilde{T}_{klj} = \tilde{T}_{ljk} = \tilde{T}^{*}_{jkl}, \quad (5) \]

\[ \tilde{T}_{jkl} = \tilde{T}_{mjk}\tilde{T}_{mk}\tilde{T}_{mlj}. \quad (6) \]

**Lemma 2.** No SIC is triply covariant.

**Remark 1.** If a SIC is triply covariant, then all triple products among distinct SIC projectors are equal. Consequently, all permutations among SIC projectors can be realized by unitary transformations according to Theorem 3 in Ref. [3] (see also Chap. 10 in Ref. [33]). Such a SIC would be too symmetric to exist!

**Proof.** Suppose on the contrary that the SIC \(\{\Pi_j\}\) is triply covariant. Then each \(\tilde{T}_{jkl}\) for distinct \(j, k, l\) equals \(\pm 1\), where the sign is independent of \(j, k, l\), so

\[ \sum_j T_{jkl} = \pm \frac{d^2 - 2}{(d + 1)^{3/2}} + \frac{2}{d + 1}, \quad j \neq k. \quad (7) \]

Since \(\Pi_l\) sum up to \(d\), we also have

\[ \sum_l T_{jkl} = d\text{tr}(\Pi_j \Pi_k) = \frac{d}{d + 1}. \quad (8) \]

The two equations above cannot be satisfied simultaneously. This contradiction completes the proof.

**Lemma 3.** No SIC in dimension \(d \geq 3\) is quasi-triply covariant.

Since this lemma is not essential in proving our main result, the proof is relegated to the appendix.

**Lemma 4.** Suppose there exists a quasi-super-SIC in prime-power dimension \(d\). Then \(d\) equals 2, 3, or 8.

This lemma follows from Lemmas 5, 6, and 7 below.

**Lemma 5.** Suppose \(\{\Pi_j\}\) is a quasi-super-SIC with normalized triple products \(\tilde{T}_{jkl}\). Then all \(\tilde{T}_{jkl}\) are \(2d^2\)th roots of unity, that is, \(\tilde{T}_{jkl}^{2d^2} = 1\).
Proof. If \( \{\Pi_j\} \) is super-symmetric, then the multi set \( \{\tilde{T}_{mjk}, m = 1, 2, \ldots, d^2\} \) is identical with \( \{\tilde{T}_{mkj}, m = 1, 2, \ldots, d^2\} \). On the other hand, the two multisets are conjugates of each other. Therefore, both of them are conjugation invariant and \( \prod_m \tilde{T}_{mjk} = \pm 1 \), where the sign is independent of \( j \) and \( k \) as long as they are distinct. Now taking the product over \( m \) in Eq. (6) yields \( \tilde{T}_{d^2jkl} = \pm 1 \), which implies the lemma.

If \( \{\Pi_j\} \) is quasi-super-symmetric but not super-symmetric, then \( \{\Pi_j\} \) is quasi-2-homogenous but not 2-homogenous according to Lemma 1. It follows that every ordered pair of distinct SIC projectors can be mapped to the same pair with the reverse order under the symmetry group. So the multi set \( \{\tilde{T}_{mjk}, m = 1, 2, \ldots, d^2\} \) is still conjugation invariant and the lemma holds as before.

**Lemma 6.** Suppose there exists a quasi-super-SIC for \( d \geq 3 \), then \( (d - 2)\sqrt{d + 1} \in \mathbb{Z}[\zeta_{2d^2}] \) and \( \sqrt{d + 1} \in \mathbb{Q}[\zeta_{2d^2}] \), where \( \zeta_{2d^2} \) is a primitive \( 2d^2 \)th root of unity.

**Remark 2.** Here \( \mathbb{Z}[\zeta_{2d^2}] \) is the ring generated by integers and \( \zeta_{2d^2} \), and \( \mathbb{Q}[\zeta_{2d^2}] \) is the field generated by rational numbers and \( \zeta_{2d^2} \).

Proof. From the two equations

\[
\sum_k \text{tr}(\Pi_j \Pi_k \Pi_l) = \frac{2}{d+1} + \sum_{k \neq j,l} \frac{1}{(d+1)^{3/2}} \tilde{T}_{jkl},
\]

\[
\sum_k \text{tr}(\Pi_j \Pi_k \Pi_l) = d \text{tr}(\Pi_j \Pi_l) = \frac{d}{d+1}, \quad j \neq l,
\]

we deduce that

\[
\sum_{k \neq j,l} \tilde{T}_{jkl} = (d - 2)\sqrt{d + 1}.
\]

The lemma follows from the fact that \( \tilde{T}_{jkl} \) are \( 2d^2 \)th roots of unity according to Lemma 5.

Now Lemma 6 is a consequence of Lemma 4 and the following lemma, whose proof is relegated to the appendix.

**Lemma 7.** Suppose \( d \) is a prime power. Then \( \sqrt{d + 1} \in \mathbb{Q}[\zeta_{2d^2}] \) if and only if \( d = 3, 8 \).

2.2. Special role of Heisenberg-Weyl group. Another stepping stone besides Lemma 4 for establishing our main result is the following theorem.

**Theorem 2.** Every quasi-super-SIC is covariant with respect to a tensor HW group; its (extended) symmetry group is a subgroup of the (extended) Clifford group.

**Remark 3.** By tensor HW group we mean the tensor power of the HW group in a prime dimension, which is characterized by the discrete analog of the canonical commutation relation [30]. The Clifford group is its normalizer [11,15,33]. Remarkably, the symmetry requirement on a SIC naturally leads to the canonical commutation relation and the tensor HW group. When the dimension is a prime,
this conclusion is consistent with the earlier conclusion of the author [32] that
every group covariant SIC is covariant with respect to the HW group and that
its (extended) symmetry group is a subgroup of the (extended) Clifford group.
This theorem implies that super-SICs can only exist in prime-power dimensions.

To prove Theorem 2, we need to introduce several technical results concerning
permutation groups, especially Burnside’s theorem on 2-transitive permutation
groups; see Appendix C and Refs. [10, 13] for more details.

**Theorem 3 (Burnside).** Any 2-transitive permutation group \( G \) on a finite set
\( \Omega \) has a unique minimal normal subgroup, which is either elementary abelian
acting regularly on or nonabelian simple acting primitively on \( \Omega \).

**Remark 4.** A minimal normal subgroup \( N \) of a group \( G \) is a subgroup other than
the identity that contains no normal subgroup of \( G \) other than the identity and
itself. A group action is **regular** if it is transitive with trivial point stabilizer; it is
**primitive** if it is transitive and preserves no nontrivial partition. If \( N \) is regular
elementary abelian, then it can be identified as a vector space over some finite
Galois field, and \( G \) as a subgroup of the group of affine semilinear transformations
on the vector space and is called **affine**. If \( N \) is nonabelian simple, then \( G \) can
be identified as a subgroup of the automorphism group of \( N \) and is called **almost
simple** [10, 13].

The following two technical lemmas are proved in the appendix.

**Lemma 8.** Suppose \( H \) is a subgroup of index 2 of a 2-transitive permutation
group \( G \) with \( |G| > 2 \) on \( \Omega \). Then \( H \) has a unique minimal normal subgroup,
which coincides with the counterpart of \( G \).

**Lemma 9.** Suppose \( G \) is an almost simple 2-transitive permutation group whose
degree \( n \) is a perfect square. Let \( N \) be the minimal normal subgroup of \( G \). Then
one of the following three cases holds.

1. \( N \) is isomorphic to the alternating group \( A_n \) with \( n \geq 5 \) a perfect square.
2. \( N \) is isomorphic to \( \text{PSL}(k,q) \) with \((k,q) = (2,8), (4,7), \) or \((5,3) \) and \( n \) is
equal to \( 3^2, 20^2, \) or \( 11^2 \) accordingly.
3. \( N \) is isomorphic to \( \text{Sp}(6,2) \) and \( n \) is equal to \( 6^2 \).

**Lemma 10.** Suppose \( \{ \Pi_i \} \) is a quasi-super-SIC. Then its symmetry group and
extended symmetry group have a unique and identical minimal normal subgroup,
which is elementary abelian and regular.

**Proof.** Let \( \overline{G} \) and \( \overline{EG} \) be the symmetry group and extended symmetry group of
\( \{ \Pi_i \} \). Then \( \overline{G} \) is either identical with \( \overline{EG} \) or is a subgroup of index 2. So the
first two statements follow from Burnside’s theorem and Lemma 8. Let \( \overline{N} \) be
their common minimal normal subgroup. Then \( \overline{N} \) acts transitively on the set of
SIC projectors, so it is irreducible according to Theorem 2.34 of Zauner [31] or
Theorem 7.3 in the author’s thesis [33].

Suppose on the contrary that \( \overline{N} \) is not elementary abelian or regular. Then
\( \overline{N} \) is a nonabelian simple group, and \( \overline{EG} \) is an almost simple 2-transitive permuta-
tion group of degree \( d^2 \), which is a perfect square. According to Lemma 8
\( \overline{N} \) cannot be isomorphic to \( A_n \) for \( n \geq 5 \), because \( A_n \) is \((n - 2)\)-transitive [10,13].
In view of Lemma 9, \(N\) is a faithful irreducible projective representation (over the complex numbers) of \(\text{PSL}(k,q)\) with \((k,q) = (2,8), (4,7), (5,3),\) or \(\text{Sp}(6,2)\), and the degree of the representation (which equals the square root of the degree of the permutation group) is 3, 20, 11, or 6, respectively. According to Table II in Ref. [28], however, the minimal degree of such a representation is 7, 399, 120, or 7. This contradiction completes the proof.

Before proving Theorem 2, we need the following folklore lemma; see the appendix for a proof.

**Lemma 11.** Suppose \(H\) is an elementary abelian group of order \(p^{2n}\) with \(p\) a prime and \(n\) a positive integer. Then every faithful irreducible projective representation of \(H\) has degree \(p^n\) and the image is projectively equivalent to the tensor HW group in dimension \(p^n\).

**Proof (of Theorem 2).** Suppose \(\{\Pi_j\}\) is a quasi-super-SIC in dimension \(d\) with symmetry group \(G\) and extended symmetry group \(EG\). Then \(G\) and \(EG\) have a unique common minimal normal subgroup, say \(N\). In addition, \(N\) is elementary abelian and acts regularly on the set of SIC projectors according to Lemma 10. Therefore, \(N\) has order \(d^2\) and is irreducible according to Theorem 2.34 in Ref. [31] or Theorem 7.3 in Ref. [33]. In a word, \(N\) is a faithful irreducible projective representation of an elementary abelian group, so it is projectively equivalent to the tensor HW group according to Lemma 11. Given that \(N\) is normal in \(G\) and \(EG\), we conclude that \((EG)\overline{G}\) is a subgroup of the (extended) Clifford group.

### 2.3. Proof of the main results.

**Proof (of Theorem 1).** In view of Theorem 2, Lemma 1, and Lemma 4, to complete the proof of Theorem 1, it remains to show the uniqueness of quasi-super-SICs in dimensions 3 and 8.

In dimension 3, every group covariant SIC is covariant with respect to the HW group and its symmetry group is a subgroup of the Clifford group, which is isomorphic \(\text{SL}(2,3) \times (\mathbb{Z}_3)^2\) [32]. Therefore, the stabilizer of each SIC projector is isomorphic to a subgroup of \(\text{SL}(2,3)\). If the SIC is quasi-super-symmetric, then the order of the stabilizer (within the symmetry group) is divisible by 4. Observing that \(\text{SL}(2,3)\) contains a unique element of order 2, we conclude that the stabilizer contains an element of order 4. Since all order-4 Clifford transformations in dimension 3 are conjugated to each other [32], without loss of generality we may assume that one fiducial ket is stabilized by (is an eigenket of) the order-4 Clifford transformation

\[
\frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}
\]  

(11)

Calculation shows that this Clifford transformation has three nondegenerate eigenkets, one of which happens to be the fiducial ket that generates the Hesse SIC (see Eq. 2), while the other two are not fiducial kets. Therefore, in dimension 3 the Hesse SIC is the only quasi-super-SIC up to unitary equivalence.
In dimension 8, according to Theorem 2, every quasi-super-SIC is covariant with respect to the three-qubit Pauli group, and its symmetry group is a subgroup of the Clifford group, which has order $2^{15}\cdot 3^4\cdot 5\cdot 7$. Suppose $|\psi\rangle$ is a fiducial ket that generates a quasi-super-SIC. Then its stabilizer contains an order-7 Clifford transformation. Observe that all Sylow 7-subgroups of the Clifford group are cyclic of order 7 and are conjugated to each other. Without loss of generality, we may assume that $|\psi\rangle$ is stabilized by the order-7 Clifford transformation $U_7 = \frac{\omega^5}{\sqrt{2}}$.

$$
U_7 = \frac{\omega^5}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 1 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & i & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -i & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -i & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
-i & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -i & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & i
\end{pmatrix},
$$

(12)

where $\omega = e^{2\pi i/8}$. Calculation shows that $U_7$ has six nondegenerate eigenkets, none of which are fiducial kets. The two-dimensional eigenspace corresponding to the eigenvalue 1 contains two fiducial kets, which happen to be the only two normalized kets in the eigenspace that satisfy the following three equations,

$$
\langle \psi | \sigma_z \otimes 1 \otimes 1 | \psi \rangle = \pm \frac{1}{3}, \quad \langle \psi | 1 \otimes \sigma_z \otimes 1 | \psi \rangle = \pm \frac{1}{3}, \quad \langle \psi | 1 \otimes \sigma_x \otimes 1 | \psi \rangle = \pm \frac{1}{3}.
$$

(13)

The first fiducial ket happens to be the one that generates the set of Hoggar lines (see Eq. (3)). The second one

$$
\frac{1}{\sqrt{6}} (-i, -1, 0, 0, -1 + i, 0, 1, 1)^T
$$

(14)

generates a SIC which turns out to be equivalent to the set of Hoggar lines under the Clifford transformation

$$
\begin{pmatrix}
0 & U \\
V & 0
\end{pmatrix}, \quad U = \text{diag}(-i, -i, -1, 1), \quad V = \text{diag}(1, 1, i, -i),
$$

(15)

which can be computed using an algorithm described in Chap. 10 of the author’s thesis [32]. Therefore, the set of Hoggar lines is the unique quasi-super-SIC in dimension 8 up to unitary equivalence.

### 3. Summary

We have introduced super-SICs as the most symmetric structure that can appear in the Hilbert space. We proved that the SIC in dimension 2, the Hesse SIC in dimension 3, and the set of Hoggar lines in dimension 8 are the only three (quasi)-super-SICs up to unitary equivalence. They are also the only three (quasi)-2-homogeneous SICs up to unitary equivalence. Such general statements are of intrinsic interest but are quite rare in the literature due to the enormous difficulty in decoding SICs. In the course of our study, we revealed an intriguing connection between symmetry and canonical commutation relation, which
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deserves further study. Our study represents a significant step in understanding the elusive symmetry of SICs and the geometry of quantum state space. It may have profound implications for foundational studies, especially quantum Bayesianism [4]. In addition, our work establishes diverse links between SIC study and various other disciplines, such as number theory, representation theory, combinatorics, and permutation groups, which are of interest to researchers from respective fields. The techniques we introduced here may also have wide applications beyond the current focus.

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A. Proof of Lemma 3

Proof. Suppose on the contrary that the SIC \{Π_j\} is quasi-triply covariant. Recalling that the symmetry group of the SIC is a normal subgroup of the extended symmetry group of index at most 2, we conclude that the SIC is doubly covariant, that is, super-symmetric. In addition, \tilde{T}_{jkl} for distinct j, k, l can take at most two possible values, which are conjugate phase factors, say t and \bar{t}. If t = ±1, then the same proof of Lemma 2 applies. Similar reasoning also shows that t cannot equal ±i for d ≥ 3.

Now suppose that t^4 ≠ 1. Since the SIC \{Π_j\} is super-symmetric, the multiset \{\tilde{T}_{mjk}, m = 1, 2, \ldots, d^2\} is invariant under complex conjugation and is independent of j, k as long as j, k are distinct. It follows that the multiset contains two copies of 1 and (d^2 - 2)/2 copies of t and \bar{t}. This observation implies the lemma immediately when d is odd. In general, choose a triple j, k, l such that T_{jkl} = t. According to Eq. (6) and the assumption t^4 ≠ 1, if m is distinct from j, k, l, then two of the three numbers \tilde{T}_{mjk}, \tilde{T}_{mkl}, \tilde{T}_{mlj} are equal to t and one equal to \bar{t}. It follows that the three multisets \{\tilde{T}_{mjk}, m = 1, 2, \ldots, d^2\}, \{\tilde{T}_{mkl}, m = 1, 2, \ldots, d^2\}, and \{\tilde{T}_{mlj}, m = 1, 2, \ldots, d^2\} contain at least 2(d^2 - 3) copies of t in total. Therefore, 2(d^2 - 3) ≤ 3(d^2 - 2)/2, that is, d^2 ≤ 6, which can hold only when d = 2.

B. Proof of Lemma 7

Proof. Suppose \sqrt{d + 1} ∈ \mathbb{Q}[ζ_{2d^2}]. Then \sqrt{d + 1} belongs to \mathbb{Q} or a quadratic extension of \mathbb{Q}. In the former case d + 1 must be a perfect square, which implies that d = 3, 8 given that d is a prime power.

Observing that \mathbb{Q}[ζ_{2d^2}] is a Galois extension of \mathbb{Q}, we conclude that the quadratic extensions of \mathbb{Q} contained in \mathbb{Q}[ζ_{2d^2}] are in one-to-one correspondence
with the subgroups of the Galois group Gal(\(\mathbb{Q}[\zeta_d]\)/\(\mathbb{Q}\)) of index 2 \[6\]. The group Gal(\(\mathbb{Q}[\zeta_d]\)/\(\mathbb{Q}\)) is isomorphic to the automorphism group of \(\mathbb{Z}_{2d}\), which in turn is isomorphic to the group \(\mathbb{Z}_{2d}^*\) of invertible elements (or units) in \(\mathbb{Z}_{2d}\) \[22\].

When \(d\) is a power of 2, \(\mathbb{Z}_{2^k}\) is isomorphic to \(\mathbb{Z}_2 \times \mathbb{Z}_2\), which contains three subgroups of index 2. Consequently, \(\mathbb{Q}[\zeta_{2k}]\) contains three quadratic extensions over \(\mathbb{Q}\), which are identical with the three quadratic extensions over \(\mathbb{Q}\) contained in \(\mathbb{Q}[\zeta_2]\). It is easy to verify that the three quadratic extensions are \(\mathbb{Q}[\sqrt{2}], \mathbb{Q}[i],\) and \(\mathbb{Q}[\sqrt{-2}]\), none of which can contain \(\sqrt{d+1}\) except when \(d = 8\).

When \(d = p^n\) with \(p\) an odd prime and \(n\) a positive integer, \(\mathbb{Z}_{2^k}\) is cyclic of order \((p^n-1)(p-1)\) and contains a unique subgroup of index 2. Consequently, \(\mathbb{Q}[\zeta_{2k}]\) contains a unique quadratic extension over \(\mathbb{Q}\), which is identical with the quadratic extension over \(\mathbb{Q}\) contained in \(\mathbb{Q}[\zeta_p]\). According to the famous quadratic reciprocity (see Proposition 7-3-1 in Ref. \[29\] for example), this unique quadratic extension is \(\mathbb{Q}[\sqrt{(-1)^{(p-1)/2}d}]\), which cannot contain \(\sqrt{d+1}\) except when \(d = p = 3\).

C. Permutation groups

For the convenience of the reader, in this appendix we introduce several basic concepts and results about permutation groups that are relevant to our study in the main text; see Refs. \[10, 13, 22\] for more details.

Given a finite set \(\Omega = \{\alpha, \beta, \ldots\}\) with \(n\) elements, the group composed of all permutations on \(\Omega\) is called the symmetry group on \(\Omega\) and denoted by \(S_\Omega\). The group \(S_\Omega\) is isomorphic to the symmetry group of the set \(\{1, 2, \ldots, n\}\), which is usually denoted by \(S_n\). A group action of \(G\) on \(\Omega\) is a map from the Cartesian product \(G \times \Omega\) to \(\Omega\) that satisfies \(1\alpha = \alpha\) and \(g(h\alpha) = (gh)\alpha\), where \(1\) denotes the identity of \(G\) (also denotes the trivial group with only one element) and \(g\) denotes the image of the pair \((g, \alpha)\) under the map. An orbit is the set of images of a point \(\alpha \in \Omega\) (elements of \(\Omega\) are usually referred to as points) under the action of \(G\), that is, \(\{g\alpha : g \in G\}\). All orbits of the action form a partition of \(\Omega\). The stabilizer \(G_\alpha\) of a point \(\alpha\) is the group composed of all elements \(g\) that leave \(\alpha\) invariant, that is, \(g\alpha = \alpha\). The kernel of the action is the group composed of all elements \(g\) that act trivially on \(\Omega\), that is, \(g\alpha = \alpha\) for all \(\alpha \in \Omega\). By definition, the kernel is the intersection \(\cap_{\alpha \in \Omega} G_\alpha\) of all point stabilizers. Alternatively, a group action of \(G\) on \(\Omega\) is a homomorphism from \(G\) to \(S_\Omega\), and the kernel of the action coincides with the kernel of the homomorphism. The action is faithful if the kernel is trivial. A permutation group on \(\Omega\) is a group \(G\) that acts faithfully on \(\Omega\), which can be identified as a subgroup of \(S_\Omega\). The degree of the permutation group is the cardinality of \(\Omega\).

A permutation group \(G\) on \(\Omega\) is transitive if every element in \(\Omega\) can be mapped to every other under the action of \(G\). In that case, all point stabilizers are conjugated to each other, and the order of \(G\) is equal to the product of the order of each point stabilizer and the cardinality of \(\Omega\), that is, \(|G| = |G_\alpha||\Omega|\). A transitive group is regular if each point stabilizer is trivial, in which case \(|G| = |\Omega|\). The group \(G\) is \(k\)-transitive if every ordered \(k\)-tuple of distinct elements of \(\Omega\) can be mapped to every other such \(k\)-tuple. When \(k > 1\), a group is \(k\)-transitive if and only if each point stabilizer is \((k - 1)\)-transitive on the remaining points. A \(k\)-transitive group is sharply \(k\)-transitive if each \(k\)-point stabilizer is trivial. The group \(G\) is \(k\)-homogeneous if every unordered \(k\)-tuple of distinct elements
can be mapped to every other such \( k \)-tuple. By definition, a permutation group on \( \Omega \) with \( |\Omega| = n \) is \( k \)-homogeneous if and only if it is \((n-k)\)-homogeneous. In addition, every \( k \)-transitive permutation group is \( k \)-homogeneous.

A block \( \Delta \) (also called a set of imprimitivity) of the action of \( G \) on \( \Omega \) is a nonempty subset of \( \Omega \) such that \( g\Delta \) for any \( g \in G \) is either identical with \( \Delta \) or disjoint from \( \Delta \). The block is nontrivial if it is a proper subset of \( \Omega \) that contains more than one elements. A transitive permutation group is imprimitive if there exists a nontrivial block and primitive otherwise. Alternatively, the group is primitive if no nontrivial partition of \( \Omega \) is left invariant, where a trivial partition means the partition that is composed of one component or the one each component of which is composed of only one point. Primitive permutation groups play a crucial role in the study of permutation groups. Their basic properties are listed below for easy reference.

1. A permutation group \( G \) is primitive if and only if each point stabilizer is a maximal subgroup of \( G \).
2. Every normal subgroup \( N \neq 1 \) of a primitive permutation group \( G \) on \( \Omega \) acts transitively on \( \Omega \).
3. Every 2-transitive permutation group is primitive.

Remark 5. A maximal subgroup \( M \) of a group \( G \) is a proper subgroup that is not contained in any other proper subgroup.

To better understand the properties of primitive and 2-transitive permutation groups, we need to introduce several new concepts. A minimal normal subgroup \( N \) of a group \( G \) is a normal subgroup other than the identity that contains no normal subgroup of \( G \) other than the identity and itself. Any minimal normal subgroup is a direct product of isomorphic simple groups. Any two distinct minimal normal subgroups of a given group have a trivial intersection. The socle of a group \( G \) is the product of all minimal normal subgroups of \( G \) and is denoted by \( \text{soc}(G) \); it is a characteristic subgroup of \( G \). Recall that a characteristic subgroup of \( G \) is a subgroup that is invariant under all automorphisms of \( G \). The structure of minimal normal subgroups of a primitive permutation group is characterized by Theorem 4.3B in Ref. [13], as reproduced here.

Theorem 4. If \( G \) is a finite primitive permutation group on \( \Omega \), and \( N \) a minimal normal subgroup of \( G \), then exactly one of the following holds:

1. \( N \) is a regular elementary abelian group, and \( \text{soc}(G) = N = C_G(N) \).
2. \( N \) is a regular nonabelian group, \( C_G(N) \) is a minimal normal subgroup of \( G \) which is permutation isomorphic to \( N \), and \( \text{soc}(G) = N \times C_G(N) \).
3. \( N \) is nonabelian, \( C_G(N) = 1 \), and \( \text{soc}(G) = N \).

Remark 6. An elementary abelian group is the direct product of certain copies of a cyclic group of a prime order. Here \( C_G(N) \) denotes the centralizer of \( N \) in \( G \). Two permutation groups on \( \Omega \) are permutation isomorphic if they are conjugated to each other under \( S_\Omega \). This theorem implies that a primitive permutation group has at most two minimal normal subgroups. It has only one minimal normal subgroup if the socle is abelian. Note that the socle of a primitive permutation group is abelian if and only if one minimal normal subgroup is abelian.
D. Proof of Lemma 8

Proof. Note that any subgroup of index 2 is normal, that any 2-transitive permutation group is primitive, and that any nontrivial normal subgroup of a primitive permutation group is transitive \[10,13,22\]. It follows that \(H\) is a transitive normal subgroup of \(G\). In addition, the point stabilizer \(H_\alpha\) of any point \(\alpha \in \Omega\) has either one orbit or two orbits of equal length on the remaining points of \(\Omega\). In the former case, \(H\) is 2-transitive and is thus primitive; in the later case, it is also straightforward to show that \(H\) is primitive. According to Theorem 4, \(H\) has either one or two minimal normal subgroups, and in the later case the two minimal normal subgroups are nonabelian regular and are centralizers of each other.

Let \(N\) be the unique minimal normal subgroup of \(G\). Then \(N\) is contained in \(H\) and contains one of the minimal normal subgroups of \(H\), say \(M\). According to Burnside’s theorem on 2-transitive permutation groups \[10,13\], \(N\) is either elementary abelian regular or nonabelian simple, and the centralizer of \(N\) is either itself or trivial accordingly. Therefore, \(M\) must be identical with \(N\) since it is either a transitive subgroup of the regular group \(N\) or a normal subgroup of the simple group \(N\). Furthermore, the centralizer of \(M\) is also contained in \(N\) and thus identical with \(N\) if it is nontrivial by the same token. It follows that \(H\) has a unique minimal normal subgroup, which coincides with the unique minimal normal subgroup of \(G\).

E. Proof of Lemma 9

All 2-transitive permutation groups have been classified by Huppert \[20\] and Hering \[16,17\] (see also Refs. \[10,13\]), with the aid of the classification of finite simple groups (CFSG) \[11\]. Almost simple 2-transitive permutation groups are listed in Table 7.4 in Ref. \[10\]. According to this table, except for the alternating groups, most of them have degrees that are not perfect squares, which are easy to verify in most cases. Those less obvious cases are dealt with in the following four lemmas, according to which \(N\) is isomorphic to one of the four groups listed in Lemma 9, with degrees as specified.

Lemma 12. Suppose \(q\) is a prime power and \(m\) a positive integer. Then \(q + 1\) is a perfect square if and only if \(q = 3\) or \(q = 8\); \(q^m + 1\) is a perfect square if and only if \((m, q) = (1, 3), (1, 8)\) or \((3, 2)\).

Proof. Obviously \(q + 1\) is a perfect square when \(q = 3\) or \(q = 8\). Suppose \(q = p^j\) and \(q + 1 = b^2\), where \(p\) is a prime, and \(j, b\) are positive integers. Then \(p^j = (b - 1)(b + 1)\). If \(q > 3\) then \(b > 2\), so \(p\) divides both \(b - 1\) and \(b + 1\) and thus equals 2. Consequently, both \(b - 1\) and \(b + 1\) must be powers of 2. It follows that \(b = 3\) and \(q = 8\). The second part of the lemma is an immediate consequence of the first part given that \(q^m\) is also a prime power.

Lemma 13. Suppose \(q\) is a power of the prime \(p\) and \(j \geq k\) are nonnegative integers. Then \(q^j + q^k\) is a perfect square if and only if one of the following conditions is satisfied

\[\text{\textsuperscript{2} The author is grateful to Dragomir Ž. Doković for simplifying the proof of Lemma 8.}\]
1. \( j = k \) is odd and \( q \) is an odd power of 2.
2. \( q^k \) is a perfect square and \((j - k, q) = (1, 3), (1, 8), \) or \((3, 2)\).

Proof. If \( j = k \), then \( q^j + q^k = 2q^j \), which is a perfect square if and only if \( j \) is odd and \( q \) is an odd power of 2. If \( j > k \), then \( q^j + q^k = q^k(q^m + 1) \) with \( m = j - k \). If \( q^k \) is not a perfect square, then \( q^j + q^k \) is a perfect square if and only if \( p(q^m + 1) \) is a perfect square, which is impossible. Otherwise, \( q^m + 1 \) is a perfect square, so \((m, q) = (1, 3), (1, 8), \) or \((3, 2)\) according to Lemma [12].

Lemma 14. Suppose \( j > k \) are nonnegative integers. Then \( 2^j + 2^k \) is a perfect square if and only if \( k \) is even and \( j = k + 3; 2^j - 2^k \) is a perfect square if and only if \( k \) is even and \( j = k + 1 \).

Proof. The first statement follows from Lemma [13] If \( k \) is odd, then \( 2^j - 2^k \) is a perfect square if and only if \( 2(2^m - 1) \) with \( m = j - k \) is a perfect square, which is impossible. Otherwise, \( 2^j + 2^k \) is a perfect square if and only if \( (2^m - 1) \) is a perfect square, say \( 2^m - 1 = b^2 \) with \( b \) a positive integer. This is possible if and only if \( m = 1 \) given that \( b^2 + 1 \) is not divisible by 4.

Lemma 15. Suppose \( q \) is a prime power and \( k \geq 2 \) a positive integer. Then \((q^k - 1)/(q - 1)\) is a perfect square if and only if the pair \((k, q)\) takes on one of the four possible values \((2, 3), (5, 3), (4, 7), (2, 8)\).

Proof. The equation
\[
\frac{q^k - 1}{q - 1} = d^2
\]
with \( d \) a positive integer is a special instance of the Nagell-Ljunggren equation. If \( k \geq 3 \), then the only integer solutions are \((k, q, d) = (5, 3, 11)\) and \((k, q, d) = (4, 7, 20)\) [9, 23, 25, 14].

If \( k = 2 \), then \((q^k - 1)/(q - 1) = q + 1 \) is a perfect square if and only if \( q = 3 \) or \( q = 8 \) according to Lemma [12].

F. Proof of Lemma [11]

Proof. This lemma is closely related to Weyl’s theorem that the HW group is uniquely characterized by the discrete analog of the canonical commutation relation [30]. Suppose \( H \) has a \( d \)-dimensional faithful irreducible projective representation \( A \mapsto U_A \) for \( A \in H \). Let \( A_1 \) be an arbitrary element in \( H \) other than the identity. Given that the representation is faithful, \( U_{A_1} \) cannot be proportional to the identity and thus cannot commute with all elements in the representation according to Schur’s lemma. So there exists an element \( B_1 \in H \) such that
\[
U_{A_1}U_{B_1}U_{A_1}^\dagger U_{B_1}^\dagger = e^{i\phi}
\]
with \( e^{i\phi} \neq 1 \) a phase factor. Observing that \( U_{B_1}^\dagger \) is proportional to the identity, we conclude that \( e^{i\phi} \) is a \( p \)th root of unity. Replacing \( B_1 \) with a suitable power if necessary, we may assume that
\[
U_{A_1}U_{B_1}U_{A_1}^\dagger U_{B_1}^\dagger = \omega^{-1},
\]

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1 The author is grateful to Gergely Harcos for helping proving Lemma [13] by introducing the concept of Nagell-Ljunggren equation and Ref. [23] and to Daniel El-Baz for recommending Ref. [25] in response to a question posed by the author on MathOverflow.
where $\omega = e^{2\pi i/p}$ is a primitive $p$th root of unity. Note that $A_1$ and $B_1$ generate a group $H_1$ of order $p^2$. If $n = 1$, then $H = H_1$, the representation must have degree $p$, and the image is projectively equivalent to the HW group in dimension $p$ according to Weyl’s theorem [32].

If $n > 1$, let $A_2$ be an arbitrary element in $H$ not contained in $H_1$. Then $U_{A_2}$ commutes with $U_{A_1}$ and $U_{B_1}$ up to phase factors that are $p$th roots of unity. By multiplying $A_2$ with a suitable element in $H_1$ if necessary, we can ensure that $U_{A_2}$ commutes with $U_{A_1}, U_{B_1}$ and, consequently, the representations of all elements in $H_1$. By the same reasoning as in the previous paragraph, there exists an element $B_2 \in H$ such that

$$U_{A_2}U_{B_2}U_{A_2}^\dagger U_{B_2}^\dagger = \omega^{-1}.$$ (19)

Note that $B_2$ cannot belong to $H_1$. By multiplying $B_2$ with a suitable element in $H_1$ if necessary, we may assume that $U_{B_2}$ commutes with $U_{A_1}$ and $U_{B_1}$. Continuing this procedure, we can eventually find $n$ pairs of generators $A_1, B_1, \ldots, A_n, B_n$ of $H$ such that $U_{A_1}, U_{B_1}, \ldots, U_{A_n}, U_{B_n}$ satisfy the canonical commutation relations

$$U_{A_j}U_{B_k}U_{A_j}^\dagger U_{B_k}^\dagger = \omega^{-\delta_{jk}},$$

$$U_{A_j}U_{A_k}U_{A_j}^\dagger U_{A_k}^\dagger = 1,$$

$$U_{B_j}U_{B_k}U_{B_j}^\dagger U_{B_k}^\dagger = 1, \quad j, k = 1, 2, \ldots, n.$$ (20)

According to the multipartite analog of the Weyl’s theorem, the representation must have degree $p^n$ and the image must be projectively equivalent to the tensor HW group in dimension $p^n$.

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