A weighted finite difference method for American and Barrier options in subdiffusive Black-Scholes Model

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Abstract

This paper is focused on American option pricing in the subdiffusive Black Scholes model. Two methods for valuing American options in the considered model are proposed. The weighted scheme of the finite difference (FD) method is derived and the main properties of the method are presented. The Longstaff-Schwartz method is applied for the discussed model and is compared to the previous method. In the article it is also shown how to valuate wide range of Barrier options using the FD approach. The proposed FD method has $2 - \alpha$ order of accuracy with respect to time, where $\alpha \in (0, 1)$ is the subdiffusion parameter, and 2 with respect to space. The paper is a continuation of [13], where the derivation of the governing fractional differential equation, similarly as the stability and convergence analysis can be found.

Keywords: Weighted finite difference method, subdiffusion, time fractional Black-Scholes model, American option, Barrier option, Longstaff-Schwartz method.

1. Introduction

Option pricing is the core content of modern finance and has fundamental meaning for global economy. By the recent announcement of Futures Industry Association trading activity in the global exchange-traded derivatives markets, in 2019 reached a record of 34.47 billion contracts, where 15.23 billion of them were options contracts [7]. The value of the global derivatives markets is estimated 700 trillion dollars to upwards of 1,5 quadrillion dollars (including so called shadow derivatives) [26].

American option is one of the most popular financial derivatives. It is widely accepted by investors for its flexibility of exercising time (see e.g. [15]). Barrier options are the simplest of all exotic options traded on financial markets [28]. This kind of security is a standard vanilla option which begins to be valid if the price of the underlying asset hits predetermined barrier (or barriers) before the maturity. They have become increasingly popular due to the lower costs and the ability to match speculating or hedging needs more closely than their vanilla equivalents. Moreover, barrier options play an important role in managing and modeling risks in finance as well as in refining insurance products such as variable annuities and equity-indexed annuities [3, 5].

Over the last two decades the Black-Scholes model has been increasingly attracting interest as effective tool of the options valuation. The model was of such great importance that the authors were awarded the Nobel Prize for Economics in 1997. The classical model was generalized in order to weaken its strict assumptions, allowing such features as stochastic interest [20], jumps model [21], stochastic volatility [9], and transactions costs [1, 4]. Analysis of empirical financial records indicates that the data can exhibit fat tails (see e.g. [2] and the references therein). This feature has been observed in many different markets (see e.g. [2] and the references therein). Such dynamics can be observed in emerging markets where the number of sellers and buyers is low. Also an interest rate often exhibits the feature of constant periods appearing - e.g. in US between 2002 and 2017 [14]. In response to the empirical evidence of fat tails, $\alpha$-stable distribution as an alternative to the Gaussian law was proposed [6, 19]. The stable...
distribution has found many important applications, for example in finance [24], physics [8,10,22] and electrical engineering [27]. With increasing interest of fractional calculus and non-local differential operators the family of fractional Black-Scholes equations has emerged in the recent literature (see e.g. [13] and references therein).

The subdiffusive B-S is the generalization of the classical B-S model to the cases, where the underlying assets display characteristic periods in which they stay motionless. The standard B-S model does not take this phenomena into account because it assumes the asset is described by continuous Gaussian random walk. As a result of an option pricing for such underlying asset, the fair price provided by the B-S model is misestimated. In order to describe this dynamics properly, the subdiffusive B-S model assumes that the underlying instrument is driven by \( \alpha \)-stable inverse subordinator [25]. Then the frequency of the constant periods appearing is dependent of subdiffusion parameter \( \alpha \in (0, 1) \). If \( \alpha \to 1 \), the subdiffusive B-S is reducing to the classical model. Due to its practicality and simplicity, the standard B-S model is one of the most widely used in option pricing. Although in contrast to the subdiffusive case it does not take into account the empirical property of the constant price periods in the underlying instrument dynamics. In Figure 1 we compare sample simulation of underlying asset in classical and subdiffusive market model. Even short stagnation of a market can not be simulated by standard B-S model. As a generalization of the classical B-S model, its subdiffusive equivalent can be used in wide range of markets - including all cases where B-S can be applied.

Since the subdiffusive Black-Scholes model was proposed [17] many open problems still have remained unsolved. One of them is the way of valuation American and Exotic options. In this paper we derive the Linear Complementarity Problem (LCP) system describing the fair price of an American option in subdiffusive B-S model. We apply the weighted scheme of the FD method and the Longstaff-Schwartz (LS) method to solve the system numerically. We compare both methods in terms of theoretical properties and practical applications. Moreover we show how to valuate wide range of barrier options in subdiffusive B-S model using the FD approach.

2. Subdiffusive Black-Scholes Model

The evolution of the market is taking place up to time horizon \( T \) and is contained in the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Here, \( \Omega \) is the sample space, \( \mathcal{F} \) is a filtration interpreted as the information about history of asset price and \( \mathbb{P} \) is the objective probability measure. The assumptions are the same as in the classical Black-Scholes model [12] with the exception that we do not have to assume the market liquidity and that the underlying instrument, instead of Geometric Brownian Motion (GBM), follows subdiffusive GBM [17]:

\[
\begin{align*}
Z_\alpha(t) &= \mathbb{Z}(S_\alpha(t)), \\
Z(0) &= Z_0,
\end{align*}
\]

where \( Z_\alpha(t) \) - the price of the underlying instrument, \( Z(t) = Z(0) \exp(\mu t + \sigma B_t) \), \( \mu \) - drift (constant), \( \sigma \) - volatility (constant), \( B_t \) - Brownian motion, \( S_\alpha(t) \) - the inverse \( \alpha \)-stable subordinator defined as \( S_\alpha(t) = \inf(\tau > 0 : U_\alpha(\tau) > t) \) [17], where \( U_\alpha(t) \) is the \( \alpha \)-stable subordinator [25], \( 0 < \alpha < 1 \). Here \( S_\alpha(t) \) is independent of \( B_t \) for each \( t \in [0, T] \). In Figure 1 we compare the samples trajectories of GBM and subdiffusive GBM for given \( \alpha \)-stable subordinator.

Let us introduce the probability measure

\[
\mathbb{Q}(A) = \int_A \exp\left(-\gamma B(\mathbb{S}_\alpha(T)) - \frac{\gamma^2}{2} \mathbb{S}_\alpha(T) \right) d\mathbb{P},
\]

(1)

where \( \gamma = (\mu + \sigma^2/2)/\sigma \), \( A \in \mathcal{F} \). As it is shown in [17], \( Z_\alpha \) is a \( \mathbb{Q} \)-martingale. The subdiffusive Black-Scholes model is arbitrage-free and incomplete [17]. Despite \( \mathbb{Q} \) defined in (1) is not unique, but it is the “best” martingale measure in the sense of criterion of minimal relative entropy. It means that the measure \( \mathbb{Q} \) minimizes the distance to the measure \( \mathbb{P} \) [18]. Between European put and call options the put-call parity holds [17].
Figure 1: The sample trajectory of GBM (upper panel) with its subdiffusive analogue (middle panel) and the corresponding inverse subordinator (lower panel). In the subdiffusive GBM the constant periods characteristic for emerging markets can be observed. The parameters are $Z_0 = \sigma = \mu = 1, \alpha = 0.9$.

3. Selected options

In Tables 1 and 2 we recall the payoff functions for options considered in this article. Recall that the payoff function $f(t)$ is the gain of the option holder at the time $t$.

| European | American |
|----------|----------|
| Plain    | $\max (Z_T - K, 0)$ | $\max (Z_T - K, 0)$ |
| Knock up-and-in | $\max (Z_T - K, 0) \mathbb{1} (M_T > H^+)$ | $\max (Z_T - K, 0) \mathbb{1} (M_T > H^+)$ |
| Knock up-and-out | $\max (Z_T - K, 0) \mathbb{1} (m_T < H^-)$ | $\max (Z_T - K, 0) \mathbb{1} (m_T < H^-)$ |
| Knock down-and-in | $\max (Z_T - K, 0) \mathbb{1} (m_T < H^-)$ | $\max (Z_T - K, 0) \mathbb{1} (m_T > H^+)$ |
| Knock down-and-out | $\max (Z_T - K, 0) \mathbb{1} (m_T > H^+)$ | $\max (Z_T - K, 0) \mathbb{1} (m_T > H^+)$ |
| Knock double-out | $\max (Z_T - K, 0) \mathbb{1} (H^+ > M_T, m_T > H^-)$ | $\max (Z_T - K, 0) \mathbb{1} (H^+ > M_T, m_T > H^-)$ |
| Knock double-in | $\max (Z_T - K, 0) - \max (Z_T - K, 0) \mathbb{1} (H^+ > M_T, m_T > H^-)$ | $\max (Z_T - K, 0) - \max (Z_T - K, 0) \mathbb{1} (H^+ > M_T, m_T > H^-)$ |

Table 1: Payoff functions for selected call options.

Here and in the rest of the paper $K$ - strike, $Z_t = Z_t(t)$ - value of underlying instrument at time $t$, $t \in [0, T]$. $M_t = \max_{\tau \in [0,t]} (Z_{\tau})$, $m_t = \min_{\tau \in [0,t]} (Z_{\tau})$, $H^+, H^-$ - barriers.

4. Valuation of American option as Free Boundary Problem

The next proposition explains why in context of American options we will proceed only with the put options.

**Proposition 4.1.** If the dividend rate $\delta = 0$, then the value of American call option is equal to its European analogue. Similarly it can be shown that if $r = 0$, then it is not worth to realize American put before $T$, so in this case value of American put is equal to his European equivalent.
Proof of this fact can be found for example in [12].

We proceed with the following main result of this section

**Theorem 4.1.** The fair price of an American put option in the subdiffusive B-S model is equal to $v(z, t)$, where $v(z, t)$ satisfies:

$$
\begin{align*}
    x &= \ln z, \\
    u(x, t) &= v(e^x, T - t).
\end{align*}
$$

and $u(x, t)$ is the solution of the system

$$
\begin{align*}
    u(x, 0) &= \max (K - \exp(x), 0), \\
    \max \left( \max (K - \exp(x), 0) - u(x, t), \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} - \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial u(x, t)}{\partial x} + ru(x, t) \right) &= 0, \\
    \lim_{x \to -\infty} u(x, t) &= 0, \\
    \lim_{x \to -\infty} u(x, t) &= K,
\end{align*}
$$

where $(x, t) \in (0, \infty) \times (0, T)$ and $\partial^\alpha g$ is Caputo fractional derivative defined as

$$
\partial^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d g(s)}{ds} (t-s)^{-\alpha} d s.
$$

**Proof.** We consider the subdiffusive Black-Scholes Equation [13].

$$
\partial^\alpha u(x, t) = \frac{1}{2} \sigma^2 \frac{\partial^2 u(x, t)}{\partial x^2} + \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial u(x, t)}{\partial x} - ru(x, t),
$$

with the initial condition determining put option

$$
\begin{align*}
    u(x, 0) &= \max (K - \exp(x), 0).
\end{align*}
$$

At the time $t \in [0, T]$ we can gain at least $\max (K - \exp(x), 0)$ (by exercising the option) and maybe even more. It leads us to the inequality:

$$
\begin{align*}
    u(x, t) &\geq \max (K - \exp(x), 0),
\end{align*}
$$

After the optimal exercise moment $v(x, t)$ can not describe the value of the option. So true is the following inequality:

$$
\begin{align*}
    \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} - \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial u(x, t)}{\partial x} + ru(x, t) &\leq 0.
\end{align*}
$$

At each moment we decide if it is worth to use the option, or keep it. Mathematically we can describe it as:

$$
\begin{align*}
    u(x, t) &= \max (K - \exp(x), 0),
\end{align*}
$$

| American | European |
|----------|----------|
| $\max (K - Z_t, 0)$ | $\max (K - Z_t, 0)$ |
| $\max (K - Z_t, 0)$ | $\max (K - Z_t, 0)$ |
| $\max (K - Z_t, 0)$ | $\max (K - Z_t, 0)$ |
| $\max (K - Z_t, 0)$ | $\max (K - Z_t, 0)$ |
| $\max (K - Z_t, 0)$ | $\max (K - Z_t, 0)$ |
| $\max (K - Z_t, 0)$ | $\max (K - Z_t, 0)$ |

Table 2: Payoffs for selected put options.
if we realize option or
\[ \partial_t u(x, t) - \frac{1}{2} \sigma^2 \partial^2_x u(x, t) - \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial u(x, t)}{\partial x} + ru(x, t) = 0, \]
if we keep it on. This can be written as follow:
\[ (u(x, t) - \max(K - \exp(x), 0)) \partial_t u(x, t) - \frac{1}{2} \sigma^2 \partial^2_x u(x, t) - \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial u(x, t)}{\partial x} + ru(x, t) = 0. \]  
Combining (5), (6) and (7) we get
\[ \max \left( \max(K - \exp(x), 0) - u(x, t) \right) \partial_t u(x, t) - \frac{1}{2} \sigma^2 \partial^2_x u(x, t) - \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial u(x, t)}{\partial x} + ru(x, t) = 0. \]
For the sufficiently high price of the underlying instrument we will not use the option. So:
\[ \lim_{x \to +\infty} u(x, t) = 0. \]
In analogy if the price of the underlying instrument will be low we will use the option, selling the underlying instrument for \( K \):
\[ \lim_{x \to -\infty} u(x, t) = K. \]

5. Numerical scheme for American put option

In this section we derive the numerical scheme for American put option. To do so, we will approximate limits by finite numbers and derivatives by finite differences. We will proceed for a \( \theta \)-convex combination of explicit (\( \theta = 1 \)) and implicit (\( \theta = 0 \)) discrete scheme, similarly as it was done for European options in [13]. We introduce parameter \( \theta \in [0, 1] \) by optimization purposes - similarly as for the case \( \alpha = 1, \theta = \frac{1}{2} \) has the best properties in terms of the error and unconditional stability/convergence [12]. Instead of the continuous space \(( -\infty, \infty ) \times (0, T) \), we take its discrete and finite equivalent \( \{ x_0 = x_{\min}, x_1, \ldots, x_N = x_{\max} \} \times \{ t_0 = 0, \ldots, t_N = T \} \), where \( x_{\min}, x_{\max} \) - lower and upper boundary of the grid. We consider a uniform grid, so \( t_j = j \Delta t \) and \( x_i = x_{\min} + i \Delta x \), where \( i = 0, \ldots, n, j = 0, \ldots, N, \Delta t = T/N, \Delta x = (x_{\max} - x_{\min})/n \). After obtaining the discrete analogue of (2) we will solve it recursively. As a result we will find its numerical solution \( \hat{u}^j, i = 0, \ldots, n, j = 0, \ldots, N \).
We begin discretizing initial condition for put option (4), we get
\[ \hat{u}^0_j = \max(K - \exp(x_{\min} + i \Delta x), 0), \]
for \( i = 0, \ldots, n \).
Similarly the discrete version of boundary conditions (8) and (9) has the form
\[ \begin{cases} \hat{u}^0_0 = 0, \\ \hat{u}^0_N = K, \end{cases} \]
for \( l = 0, \ldots, N \).
Discretizing (7) we get
\[ \begin{aligned} C \hat{u}^{l+1} &= \hat{u}^0 + (1 - \theta) G^{l+1} + \theta G^0 + \theta B \hat{u}^0, \\ \hat{u}^1 &= \max(\hat{u}^1, \hat{u}^0), \\ C \hat{u}^{k+1} &= \sum_{j=0}^{k-1} \left( b_j - b_{j+1} \right) \hat{u}^{k-j} + b_k \hat{u}^0 + (1 - \theta) G^{k+1} + \theta G^k + \theta B \hat{u}^k, \\ \hat{u}^{k+1} &= \max(\hat{u}^{k+1}, \hat{u}^0), \end{aligned} \]
(12)
for \( k \geq 1 \).

Here \( b_j = (j + 1)^{1 - \alpha} - j^{1 - \alpha} \), \( C = (\theta I + (1 - \theta) A) \), \( A = (a_{ij})_{(n-1) \times (n-1)} \), such that:

\[
a_{ij} = \begin{cases} 
1 + 2 \frac{ad}{\Delta x^2} + cd, & \text{for } j = i, i = 1, 2, ..., n - 1, \\
- \frac{ad}{\Delta x^2} + \frac{bd}{2\Delta x}, & \text{for } j = i - 1, i = 2, ..., n - 1, \\
- \frac{ad}{\Delta x^2} + \frac{bd}{2\Delta x}, & \text{for } j = i + 1, i = 2, ..., n - 2, \\
0, & \text{in other cases}, 
\end{cases}
\]

\[
B = (b_{ij})_{(n-1) \times (n-1)}, \text{ such that:}
\]

\[
b_{ij} = \begin{cases} 
- \frac{2ad}{\Delta x^2} + cd, & \text{for } j = i, i = 1, n - 1, \\
\frac{ad}{\Delta x^2} + \frac{bd}{2\Delta x}, & \text{for } j = i - 1, i = 1, 2, ..., n - 1, \\
\frac{ad}{\Delta x^2} - \frac{bd}{2\Delta x}, & \text{for } j = i - 1, i = 2, ..., n - 2, \\
0, & \text{in other cases}, 
\end{cases}
\]

\[
G^k = \left( \frac{ad}{\Delta x^2} - \frac{bd}{2\Delta x} \right) \hat{u}_0^k, 0, ..., 0, \left( \frac{ad}{\Delta x^2} + \frac{bd}{2\Delta x} \right) \hat{u}_{n-1}^k \right)^T,
\]

\[
u^k = (u_1^k, u_2^k, ..., u_{n-1}^k)^T.
\]

\[\begin{align*}
a = \frac{1}{2} \theta^2, b = \left( r - \frac{1}{2} \theta^2 \right), c = r, d = \Gamma (2 - \alpha) \Delta \nu, \Delta t = T/N, k = 1, 2, ..., N.
\end{align*}\]

Note that the analogical scheme for the European option [13] is

\[
\begin{align*}
C \hat{u}^1 &= \hat{u}^0 + (1 - \theta) G^1 + \theta G^0 + \theta B \hat{u}^0, \\
C \hat{u}^{k+1} &= \sum_{j=0}^{k-1} (b_j - b_{j+1}) \hat{u}^{k-j} + b_k \hat{u}^0 + (1 - \theta) G^{k+1} + \theta G^0 + \theta B \hat{u}^k,
\end{align*}
\]

with corresponding boundary conditions

\[
\begin{align*}
\hat{u}_l^0 &= \exp(x_{max}) - K \exp(-r(T - t_l)), \\
\hat{u}_0^0 &= 0,
\end{align*}
\]

and initial condition for a call option

\[
\hat{u}_l^0 = \max (\exp (x_{min} + i\Delta x) - K, 0),
\]

where \( l = 0, ..., N, i = 0, 1, ..., n \).
6. Numerical schemes for Barrier options

The systems (12) and (13) can be used to price different types of options, only if the initial-boundary conditions will be properly modified. Note that the initial condition defines type of an option (call or put), (12) and (13) determine style (American or European) and boundary conditions indicate it is barrier or plain option. Let us treat $x_{\min}$ and $x_{\max}$ not as approximations of infinite values, but as logarithm of lower and supreme barriers $H^-$ and $H^+$ defined in double barrier option. We take a logarithm because of change of variables $x = \ln z$ made in Theorem 4 and in [13]. The initial

$$\hat{u}_l^0 = \max(\exp(\ln H^+ + i\Delta x) - K, 0),$$

and boundary conditions

$$\begin{align*}
\hat{u}_n^l &= 0, \\
\hat{u}_0^l &= 0,
\end{align*}$$

where $l = 0, 1, \ldots, N$, $\Delta x = (\ln H^+ - \ln H^-)/n, i = 0, 1, \ldots, n,$

together with (13) is the scheme for the European double knock-out call option. The same boundary-initial conditions with (12) create the scheme for the American double knock-out call option. Analogously, prices of one-side barrier knock-out options can be obtained. Hence we have initial and boundary conditions for knock-out-and-down call option

$$\hat{u}_l^0 = \max(\exp(\ln H^+ + i\Delta x) - K, 0),$$

$$\begin{align*}
\hat{u}_n^l &= \exp(x_{\max}) - K \exp(-r(T - t_l)), \\
\hat{u}_0^l &= 0,
\end{align*}$$

$l = 0, \ldots, N$, $\Delta x = (x_{\max} - \ln H^-)/n, i = 0, 1, \ldots, n,$
and for knock-out-and-up call option

$$\hat{u}_i^0 = \max\left(\exp(x_{\min} + i\Delta x) - K, 0\right),$$  \hspace{1cm} (20)

$$\begin{cases} \hat{u}_i^l = 0, \\
\hat{u}_{\ell}^l = 0, \end{cases}$$  \hspace{1cm} (21)

$l = 0, \ldots, N$, $\Delta x = (\ln H^+ - x_{\min})/n$, $i = 0, 1 \ldots, n$.

Figure 3: The price of European down-and-out call option ($C_{d-o}$) in dependence of $Z_0$. The parameters are $n = 800$, $\sigma = 0.3$, $r = 0.08$, $N = 170$, $x_{\max} = 83$, $H^- = 3$, $T = 4$, $K = 4$.

If we want to price the knock-in options, it is helpful to use the fact that for fixed parameters there holds the so called in-out parity

$$\text{Knock}_{in} = \text{Van} - \text{Knock}_{out},$$  \hspace{1cm} (22)

where $\text{Van}$ - the price of Vanilla (plain) option, $\text{Knock}_{in}$, $\text{Knock}_{out}$ - option prices of knock-in and knock-out of the same type and style.

Please note, that the value of a double knock-out option for $Z_0$ outside of the interval $(\ln H^-, \ln H^+)$ (but being a positive number) is equal 0. Analogous remark applies for one-sided barrier options.

To summarize we present the way to price the considered options in Tables 3, 4 and 5.
Figure 4: The price of European double knock-out call option ($C_{K}^{2-o}$) in dependence of $Z_0$. The parameters are $n = 300$, $\sigma = 0.3$, $r = 0.08$, $N = 300$, $H^+ = 10$, $H^- = 1$, $T = 4$, $K = 2$. In this case, there is no clear relation between the prices of option for different $\alpha$ (like e.g. in vanilla equivalent for $T > 1$ where for higher $\alpha$ the price is higher for all $Z_0$). We can conclude that in this figure there is a critical point where all plots intersect. It is unknown under which conditions (if there is any) such point exists and what is its value.

| Style of option     | Numerical Scheme | Boundary conditions | Initial condition | Apply in-out parity? |
|---------------------|------------------|---------------------|-------------------|----------------------|
| Plain               | (13)             | (14)                | (15)              | No                   |
| Knock up-and-in     | (13)             | (21)                | (20)              | Yes                  |
| Knock up-and-out    | (13)             | (21)                | (20)              | No                   |
| Knock down-and-in   | (13)             | (19)                | (18)              | Yes                  |
| Knock down-and-out  | (13)             | (19)                | (18)              | No                   |
| Knock double-out    | (13)             | (17)                | (16)              | No                   |
| Knock double-in     | (13)             | (17)                | (16)              | Yes                  |

Table 3: European call options.

To price the European put options we can firstly compute their call equivalents and then apply the Put-Call parity. We can also use other initial conditions than in Table 3

| Style of option     | Numerical Scheme | Boundary conditions | Initial condition | Apply in-out parity? |
|---------------------|------------------|---------------------|-------------------|----------------------|
| Plain               | (13)             | (14)                | (10)              | No                   |
| Knock up-and-in     | (13)             | (21)                | (25)              | Yes                  |
| Knock up-and-out    | (13)             | (21)                | (25)              | No                   |
| Knock down-and-in   | (13)             | (19)                | (24)              | Yes                  |
| Knock down-and-out  | (13)             | (19)                | (24)              | No                   |
| Knock double-out    | (13)             | (17)                | (23)              | No                   |
| Knock double-in     | (13)             | (17)                | (23)              | Yes                  |

Table 4: European put options.

where

$$\hat{u}_0^i = \max(\exp(\ln H^+ + i\Delta x) - K, 0),$$

(23)
for $\Delta x = (\ln H^+ - \ln H^-)/n$ (double options),

$$\hat{u}_i^0 = \max (\exp (\ln H^+ + i\Delta x) - K, 0) , \quad (24)$$

for $\Delta x = (x_{\text{max}} - \ln H^-)/n$ (knock-down options),

$$\hat{u}_i^0 = \max (\exp (x_{\text{min}} + i\Delta x) - K, 0) , \quad (25)$$

for $\Delta x = (\ln H^+ - x_{\text{min}})/n$, $l \geq 0$ (knock-up options).

If there is no dividend (the case we consider in this paper), the American call is equal its European equivalent so Table 3 holds also for American call options. For American put options we have,

| Style of option        | Numerical Scheme | Boundary conditions | Initial condition | Apply in-out parity? |
|------------------------|------------------|---------------------|------------------|----------------------|
| Plain                  | (12)             | (14)                | (10)             | No                   |
| Knock up-and-in        | (12)             | (21)                | (25)             | Yes                  |
| Knock up-and-out       | (12)             | (21)                | (25)             | No                   |
| Knock down-and-in      | (12)             | (19)                | (24)             | Yes                  |
| Knock down-and-out     | (12)             | (19)                | (24)             | No                   |
| Knock double-in        | (12)             | (17)                | (23)             | Yes                  |
| Knock double-out       | (12)             | (17)                | (23)             | No                   |

Table 5: American put options.

Note, that for each type of barrier option, the definition of $\Delta x$ is different.

In Figures 2, 3, 4 we compare the fair prices given for different values of $\alpha$ for American put, European down-and-out call and European double knock-out call option respectively.

7. Finite difference method

In this section we show consistency and give the condition providing stability/convergence of the numerical schemes considered in this paper. It is important, because if the scheme is not stable/convergent the FD method cannot be used. We also present the optimal choice of parameter $\theta$ in terms of conservation of unconditional stability/convergence and minimization of potential numerical error. The unconditional stability/convergence is the property that numerical scheme is stable/convergent independently of $\Delta t$ and $\Delta x$ [13].

**Theorem 7.1.** For $\theta \in [0, 1]$ and $1 \leq i \leq n$, $1 \leq j \leq N$, the truncation error $R_i^j$ of the numerical scheme (12) satisfies

$$\left|R_i^j\right| \leq C_{\text{max}}^2 \Delta t^2 \left(\Delta x^2 + \Delta x^2 \right).$$

Moreover if

1. $$1 - \log_2 \left(2 - \frac{\theta}{1-\theta}\right) \leq \alpha,$$

or

2. $$1 - \log_2 \left(2 - \frac{\theta}{1-\theta}\right) > \alpha \text{ or } \theta = 1 \text{, and the inequality}$$

$$d \left(\theta - (1 - \theta) (b_0 - b_1)\right) \left(\frac{4a}{\Delta x^2} + c\right)^2 + \left(\frac{b}{\Delta x}\right)^2 \leq 2c (b_0 - b_1),$$

holds, then the scheme (12) with boundary-initial conditions (11) is stable and convergent. We obtain the same result for all considered in this article knock-out options, since the initial-boundary conditions have no influence for consistence-stability-convergence analysis. Indeed, if the boundary conditions are known (fixed) values then the numerical error corresponding to these conditions is equal 0, similarly as in the case of European call option [13].
The proof is the same as for theorems 3.1, 3.2 and 3.3 of [13].

We recall the observation from [13] that the optimal choice of $\theta$ for given $\alpha$ is such that $\bar{\theta}_\alpha = \frac{2 - 2^{1 - \alpha}}{3 - 2^{1 - \alpha}}$. Then the lowest boundary for an error is achieved without losing the unconditional stability/convergence. For $\theta > \bar{\theta}_\alpha$ the stability/convergence is not provided. In Figure 5 the relation between the fair price of American put $P_A$ and $\theta$ is presented. The real price of the option is close to 0.36. The jump presented in the figure is the result of the increasing error. It is the consequence of lack of the stability.

\[ \begin{align*}
0 & \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \\
0.36 & \quad 0.38 \quad 0.4 \quad 0.42 \quad 0.44 \quad 0.46 \quad 0.48 \quad 0.5
\end{align*} \]

$P_A$ vs $\theta$

Figure 5: The price of American put ($P_A$) in dependence of $\theta$. The explosion of the numerical result is the consequence of the lack of the unconditional stability outside of the interval $[0, \bar{\theta}_\alpha]$. The parameters are $n=5000, \sigma=1, r = 0.04, N = 140, T = 4, Z_0 = 2, K = 1, \alpha = 0.7$.

7.1. Longstaff-Schwartz method

The Longstaff-Schwartz method is one of the most popular approaches for valuing American/Bermudan options and their Asian equivalents [11,16]. Moreover it has an important applications in solving dynamic investment portfolio problems and in American/Bermuda style swaptions valuation (see e.g. [15] and references therein). All these applications have an important meaning in finance. Only for notional amount of interest rate swaps outstanding at the end of 1999 the losses caused by wrong exercise strategies were estimated on billions of dollars [15].

The main idea of this method is the use of least squares to estimate the conditional expected payoff to the option holder from continuation. This strategy allows to find the value of an option with the optimal exercise time (i.e. the moment where exercising option is the most profitable, if there are more than one such moments we choose the lowest of them). The method was introduced for the classical B-S model but it can be extended for many other cases [16]. In this paper we focus on American option, but note that the same method can be used to price Bermudan and American-style Asian options. The LS algorithm can be found in Appendix.

Note that the inverse-$\alpha$ stable process is not Markovian so the expected value in $A.1$ could be taken not only under current but also previous states [16]. Such proceeding could increase precision but will cost significant gain of running time. We decided to simplify the algorithm considering the expected value in $A.1$ only by the current state. Interesting could be problem of optimal choice of the set of states - e.g. using statistical background.

The LS method has its limitations. As it is shown in [23], for continuous underlying process and small values of $T$ the method is unstable. The reason is the ill-condition of the underlying regression problem [23]. The analytical formulas indicating regime where the method is stable and where is not are still unknown. This fact limits the range of possible applications of the algorithm. Note that the error in LS method could be of different origin. The first is produced by discretization the continuous stochastic processes into $m$ nodes and assumption that the American
option can be exercise only at these points. The second is related to Monte Carlo method i.e. that we estimate the expected value by the mean of size $M$. The next possible origin of the error is coming from approximation of the conditional expected value $A.1$ by the average of $l$ basis functions. The last possible type of the error is produced for non-Markovian underlying processes. Since these processes have memory, the expected value $A.1$ should be conditioned not only under the current state but also under the whole history of the underlying asset. Since in the algorithm the stochastic process is considered only at discrete nodes, even conditioning by current and all previous states produce an error. In contrast to LS, the FD method produces an error coming only from discretization of the variables (and approximation of infinities by $x_{min}$ and $x_{max}$), therefore its stability/convergence is easier to analyze.

7.2. Numerical examples

We compare both methods presented in this paper in pricing American put option. Simulations are made for $\sigma = 1, r = 0.04, Z_0 = 5, K = 2, N = 150 x_{max} = 10, x_{min} = -20, n = 200, \sigma = 1, m = 100, M = 3000$ and different values of $T$ and $\alpha$.

![Diagram showing the price of American put option computed by FD and LS for different $T$ and $\alpha$ (upper panel) with corresponding running time of the algorithms (lower panel). For small values of $T$, LS does not match the real solution properly. FD is precise and fast method for all $T$ and $\alpha$. We take $\theta = \tilde{\theta}_{1.2}$.](image)

Figure 6: The price of American put option computed by FD and LS for different $T$ and $\alpha$ (upper panel) with corresponding running time of the algorithms (lower panel). For small values of $T$, LS does not match the real solution properly. FD is precise and fast method for all $T$ and $\alpha$. We take $\theta = \tilde{\theta}_{1.2}$. 
8. Summary

In this paper:

– We have derived the system of equations describing the fair price of American put option in subdiffusive B-S model.

– We have introduced the weighted numerical scheme for this system.

– We have given condition under which the discrete scheme is stable and convergent. We have given the order of convergence.

– We have given the formula for the optimal choice of discretization parameter $\theta$ in dependence of subdiffusion parameter $\alpha$. Such numerical scheme has the lowest numerical error without losing unconditionally stability/convergence.
– We have shown how to modify previous results for valuing wide range of barrier options in frame of the same model.

– We have applied the Longstaff-Schwartz method for the subdiffusive B-S model. This method is worse than the FD method in terms of speed and precision of computation. Moreover for small values of $T$ this method is unstable so cannot be used in many different cases. By the other hand with the fair price of an option, the LS method finds also the optimal exercise strategy, what is not provided for FD method.

– We have presented some numerical examples to illustrate introduced theory.

Acknowledgements

This research was partially supported by NCN Sonata Bis 9 grant nr 2019/34/E/ST1/00360.

Appendix A. Longstaff-Schwartz Algorithm

The method uses a dynamic programming to find the optimal stopping time and Monte Carlo to approximate the fair price of an option. We start assuming the exercise time $\tau$ is equal $T$. Going backwards to 0, we replace $\tau$ by each moment we find where is better to exercise. Let us denote $V(Z(t), t)$ as the fair price of an American option with the payoff function $f(Z(t))$ and underlying asset $Z(t)$. It is easy to conclude that

$$V(Z_0, 0) = E(e^{-rt} f(Z(\tau))).$$

Let us divide the interval $[0, T]$ into $m$ subintervals (of the same length) using the grid $[t_0 = 0, t_1, \ldots, t_m = T]$, moreover we introduce

$$H(Z(t_i)) = E\left(e^{-r(T-t_i)} f(Z(\tau_i))|Z(t_i)\right),$$

where $\tau_i$ is the optimal exercise moment in $[t_{i+1}, \ldots, t_{i-1}, t_i]$, $i = 0, 1, \ldots, m - 2$. The interpretation of the function $H(Z(t_i))$ is the expected profit from keeping the option up to time $t_i$. For each trajectory we will proceed using the following algorithm

Algorithm 1

1. $\tau = t_m$, $V = f(Z(\tau))$
2. for $t$ from $t_{m-1}$ to $t_1$ do
3. \hspace{1em} $V \leftarrow e^{-r\Delta t}V$
4. \hspace{1em} if $H(Z(t)) < f(Z(t))$ then
5. \hspace{2em} $\tau \leftarrow t$
6. \hspace{2em} $V \leftarrow f(Z(\tau))$
7. \hspace{1em} end if
8. end for
9. $V \leftarrow e^{-r\Delta t}V$
10. Return $V$.

In other words at each grid point from $[t_1, \ldots, t_{m-1}]$ we compare profit from keeping and exercise the option and then decide is it more profitable to exercise the option or keep it on. In the algorithm above $V = V(Z(0), 0)$. The key question is how to estimate values $H(Z(t_i))$ for $i = 1, \ldots, m$. To do so, Longstaff and Schwartz proposed to use least squares regression. This can be done since the conditional expectation is an element of $L^2$ space, so it can be represented using its infinite countable orthonormal basis. To proceed with the computations, the finite set of $I$ such basis elements should be chosen. For the simulations we choose 3 first elements of Laguerre polynomials $L_0(x) = 1$, $L_1(x) = 1 - x$, $L_2(x) = 1/2 \left(2 - 4x - x^2\right)$. Note that the early exercise at $t$ can be profitable only if $f(Z(t)) > 0$, i.e. if option is in the money. The whole LS algorithm for the subdiffusive case will look as follow:
Algorithm 2

1: Generate $Z_{j(t)}$ for $j = 1, \ldots, M$, $i = 1, \ldots, m$
2: $\tau = [t_m, t_{m-1}, \ldots, t_0]$, $V = f(Z(\tau))$
3: for $t$ from $t_{m-1}$ to $t_0$ do
4:     Find in the money trajectories i.e. $w = \{j_1, \ldots, j_K\}$ s.t. $f(Z_k(t)) > 0$ for $k \in w$
5:     Put $Z_w \leftarrow [Z_{j_1}, \ldots, Z_{j_K}]$, $V_w \leftarrow [V_{j_1}, \ldots, V_{j_K}]$
6:     Find regression coefficients $\beta_0, \ldots, \beta_l$ such that $\sum_{i=0}^{l} \beta_i L_i(Z_w) = e^{-r\Delta t} V_w$
7:     For $k \in w$
8:         if $\sum_{i=0}^{l} \beta_i L_i(Z_k) < f(Z_k(t))$ then
9:             $\tau_k \leftarrow t$
10:            $V_k \leftarrow f(Z(\tau_k))$
11:        end if
12:    for $i \in [1, \ldots, M] \setminus w$ do
13:        $V_i \leftarrow e^{-r\Delta t} V_i$
14:    end for
15:    end for
16: $Price \leftarrow \sum_{i=1}^{M} e^{-r\Delta t} V_i / M$
17: Return $Price$.

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