ON THE UPPER SEMICONTINUITY OF
THE GLOBAL ATTRACTOR FOR A POROUS
MEDIUM TYPE PROBLEM WITH LARGE DIFFUSION

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Abstract. In this article, we are concerned with the asymptotic behavior
of a class of degenerate parabolic problems involving porous medium type
equations, in a bounded domain, when the diffusion coefficient becomes large.
We prove the upper semicontinuity of the associated global attractor as the
diffusion increases to infinity.

1. Introduction. In this paper, we study the solution and associated global at-
trator of the nonlinear reaction-diffusion equation
\[ \partial_t \beta(u) - D \Delta u + f(u) = g, \]
coupled with an initial data \( u_0 \in L^2(\Omega) \) and homogeneous Neumann boundary
condition, in a smooth bounded domain, as the diffusion coefficient \( D \) becomes
large. We consider \( \beta \) an increasing continuous function in \( \mathbb{R} \) with linear growth,
\( g \in L^\infty(\Omega) \) and the nonlinearity \( f \) satisfies certain conditions specified below.

These type of equations appear in the modelling of several biological and physical
problems, among them the filtration of a fluid in a partially saturated porous media
(see [1]) and the evolution of a biological population (see [10]).

The asymptotic behaviour of parabolic partial differential equations with large
diffusion has been extensively studied in the literature, particularly for systems of
the type
\[ \partial_t u = DLu + f(x, u, \nabla u), \]
where \( L \) is a linear operator of second-order (see [3], [4], [5] and [11]). As it is expected,
from a physical point of view, since diffusion enhances spatial homogeniza-
tion, the large diffusivity will cause the solutions to converge to a time dependent
spatially constant function. Hence, the asymptotic behaviour is determined by an

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ordinary differential equation. This result has been extended to include equations that depend on a nonlinear operator, as in the case of the $p$-Laplace and porous medium type equations (see [16], [12] and [13]). We remark that the conditions on $\beta$ and the nonlinearily $f$ studied presently differ from the conditions imposed in [12] and [13]. Moreover, our contribution is based mainly on the study of the upper semicontinuity of the associated global attractor.

There are several studies on the existence of a global attractor for equations of porous medium type. We refer to [9] for the study of the problem in an unbounded domain. In [2] the existence of a global attractor in $L^\infty(\Omega)$ is obtained, whereas in [8] the authors proved the existence of a global attractor in $L^1(\Omega)$.

One of the main issues when studying the long time behaviour of the equation as a dynamical system is the well posedness of the problem under relative generality of the nonlinearity $f$. It may provide difficulties when trying to apply the usual Galerkin method or the abstract nonlinear semigroup theory, especially to assure the uniqueness, the continuous dependence on the initial data or the necessary energy inequalities. Taking this into account, we study the problem with conditions as imposed in [6], where the existence is proved using an approximation method, studying first problem (1) with a more regular $\beta$ and then passing to the limit. This approximation will be used several times to study the uppersemicontinuity of the global attractor in $L^2(\Omega)$.

By the results in [6], we know that the Neumann boundary problem associated to equation (1) determines, for each $D \geq 1$, a semigroup $(S_D(t))_{t \geq 0}$ on $L^2(\Omega)$ by $S_D(t)u_0 = u_D(\cdot,t)$ where $u_D$ is the weak solution of the problem corresponding to (1). The semigroup associated with (1) admits a compact global attractor $\{A_D\}$ in $L^2(\Omega)$.

Following the line of ideas in [16], we prove that the family of attractors $\{A_D\}$ is upper semicontinuous at infinity, which means,

$$\sup_{a_D \in A_D} \text{dist}(a_D, A_\infty) \to 0 \quad \text{as} \quad D \to \infty,$$

where $A_\infty$ is the attractor of the limit problem defined below, for $u_0$ such that $u_0 \to u_0$ in $L^2(\Omega)$, as $D$ goes to infinity, and $\int v := \frac{1}{|\Omega|} \int_\Omega v$,

$$\begin{cases} 
(\beta(u))_t + f(u) = \int g, \quad t > 0 \\
\beta(u(0)) = \int \beta(u_0). \end{cases}$$

(2)

This paper is organized as follows. In Section 2, we present the definition of weak solution of the initial value problem associated to (1) and collect well known results on its well posedness and the existence of the attractor. We also recall apriori estimates uniform with respect to the diffusion parameter, which we use in Section 3, to obtain the convergence of the solutions as $D$ goes to infinity and the problem it satisfies at the limit. In Section 4, we study the global attractor associated to the ODE obtained at the limit and finally in Section 5, we prove the upper semicontinuity of the attractor.

2. Preliminaries: Existence of solution and associated attractor. We recall in this section known results on the global existence of solutions and associated attractor, as well as apriori estimates uniform with respect to the parameter $D$. |
which we will need later to study the upper semicontinuity of the global attractor.

We first give some notation that will be used throughout the paper.

Let \( \beta \) be a continuous increasing function with \( \beta(0) = 0 \). We define, for \( t \in \mathbb{R} \),
\[
\psi(t) = \int_0^t \beta(\tau)d\tau. \tag{3}
\]

We also define the sign function by
\[
\tau \rightarrow \text{sign}(\tau) = \begin{cases} 1 & \text{if } \tau > 0, \\ 0 & \text{if } \tau = 0, \\ -1 & \text{if } \tau < 0. \end{cases}
\]

We denote by \( W^{1,2}_{\omega}((0, T) \times \Omega) \), for all \( T > 0 \) and \( \Omega \), an open, bounded set of \( \mathbb{R}^N \), the set of all functions \( v \) such that
\[
\int_0^T \int_\Omega \left( |v|^q + |Dv|^q + |D^2v|^q + \left| \frac{\partial v}{\partial t} \right|^q \right) dxdt < \infty.
\]

We review the definition of subdifferential. Let \( X \) be a Hilbert space and let \( \Psi : X \rightarrow (-\infty, \infty] \) be a proper lowersemicontinuous convex function. The subdifferential \( \partial \Psi \) of \( \Psi \) at \( u \) in \( X \) is defined as follows:
\[
\partial \Psi(u) := \{ \xi \in X : \Psi(v) - \Psi(u) \geq (\xi, v - u), \forall v \in D(\Psi) \}.
\]

We consider the following initial boundary value problem
\[
\begin{align*}
\frac{\partial \beta}{\partial t} - D\Delta u + f(u) &= g(x) \quad \text{in } Q = \Omega \times (0, \infty) \\
\frac{\partial u}{\partial \eta} &= 0 \quad \text{on } S = \partial\Omega \times (0, \infty) \\
\beta(u(0)) &= \beta(u_0),
\end{align*} \tag{4}
\]
in a bounded smooth domain \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \), with given \( g(x) \in L^\infty(\Omega) \) and \( u_0 \in L^2(\Omega) \) for each \( 1 \leq D \in \mathbb{R} \). We assume the following conditions.

(H1) \( \beta \) is an increasing continuous function from \( \mathbb{R} \) into \( \mathbb{R} \), \( \beta(0) = 0 \), and there exist \( c_1 > 0 \) and \( c_2 > 0 \) such that \( |\beta(t)| \leq c_1|t| + c_2 \) for all \( t \in \mathbb{R} \).

(H2) The function \( f \) is continuous and we assume there exist \( c_3 > 0 \), \( c_4 > 0 \) and \( c_5 > 0 \) such that
\[
\text{sign}(s)f(s) \geq c_3|s|^{p-1} - c_4, \quad p > 2,
\]
and
\[
|f(s)| \leq c_5(|s|^{p-1} + 1).
\]

(H3) There exists \( c_6 > 0 \) such that \( f(s) + c_6\beta(s) \) is increasing almost everywhere.

Remark 1. An example is given by \( \beta(s) = |s|^{1/m} \text{sign}(s) \) (porous-medium type-like equation) and \( f(s) = \sum_{j=0}^{2l-1} b_j s^j \) with \( b_{2l-1} \geq \delta > 0 \).

By the results in [6], there exists a unique weak solution of problem (4) in the following sense.

Definition 2.1. By a weak solution of problem (4), we mean an element
\[
u \in L^p(0, T; L^p(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^\infty(\eta, T; L^\infty(\Omega)),
\]
for all $\eta > 0$, such that
\[
\frac{\partial \beta(u)}{\partial t} \in L^p(0,T; L^p(\Omega)) + L^2(0,T; H^{-1}(\Omega)),
\]
and for all $\phi \in L^\infty(0,T; L^\infty(\Omega)) \cap L^2(0,T; H^1(\Omega))$
\[
\int_0^T \left< \frac{\partial \beta(u)}{\partial t}, \phi \right>_{V',V} dt + D \int_0^T \int_\Omega \nabla u \cdot \nabla \phi dxdt + \int_0^T \int_\Omega f(u)\phi dxdt = \int_0^T \int_\Omega g\phi dxdt,
\]
where we have $V' = L^p'(\Omega) + H^{-1}(\Omega)$, $V = L^p(\Omega) \cap H^1(\Omega)$, and if $\partial_t \phi \in L^2(0,T; L^2(\Omega))$, $\phi(T) = 0$, then
\[
\int_0^T \left< \frac{\partial \beta(u)}{\partial t}, \phi \right>_{V',V} dt = \int_0^T \int_\Omega (\beta(u(t)) - \beta(u_{0_D})) \frac{\partial \phi}{\partial t} dxdt.
\]

**Remark 2.** It is also known by [6] that $\beta(u) \in C([0,T]; L^p(\Omega))$. Hence, the initial boundary condition makes sense for $u_0 \in L^2(\Omega)$.

To prove the existence of a unique solution of problem (4) in [6], the following approximation was used.

**Proposition 1.** Let $u_D$ be the solution of problem (4). Then, there exists a regular approximation $\beta_j \in C^\infty(\mathbb{R})$ to $\beta$ with $1/j \leq \beta'_j \leq M_j$, $|\beta_j(t)| \leq c_{11}|t| + c_{12}$ and $\beta_j \to \beta$ in $C_{loc}(\mathbb{R})$ as $j \to \infty$, such that for $u_{0_j} \in C_0^\infty(\Omega)$ with $u_{0_j} \to u_{0_D}$ in $L^2(\Omega)$, as $j \to \infty$, the problem
\[
\begin{cases}
\frac{\partial \beta_j(u)}{\partial t} - D\Delta u + f(u) = g(x) & \text{in } Q = \Omega \times (0,\infty) \\
\frac{\partial u}{\partial \eta} = 0 & \text{on } S = \partial \Omega \times (0,\infty) \\
\beta_j(u(0)) = \beta_j(u_{0_j}),
\end{cases}
\]
has a unique solution $u_j \in W^{1,2}_q((0,T) \times \Omega) \cap L^2(0,T; H^1(\Omega))$ satisfying, as $j \to \infty$,
(i) $u_j \to u_D$ a.e. in $(0,T) \times \Omega$
(ii) $\beta_j(u_{0_j}) \to \beta(u_{0_D})$ in $L^2(\Omega)$
(iii) $\beta_j(u_j) \to \beta(u_D)$ in $L^2(0,T; L^2(\Omega))$
(iv) $f(u_j) \to f(u_D)$ in $L^1(0,T; L^1(\Omega))$ and $L^r(\eta,T; L^r(\Omega))$ for any $r \geq 1$.

In the next lemma, we collect some apriori estimates which are independent of the parameter $D$ and will be used to prove the convergence of solutions of problem (4).

**Lemma 2.2.** The unique weak solution $u_D$ of problem (4) satisfies, for all $t \leq T$,
\[
D\|\nabla u_D\|_{L^2(0,t; L^2(\Omega))} + \|u_D\|_{L^p(\Omega)} \leq C(T,c_3,p,\|g\|_{L^\infty(\Omega)},\|u_{0_D}\|_{L^2(\Omega)},\|\Omega\|),
\]

\[
\|\beta(u_D(t))\|_{L^p(\Omega)} \leq \|\beta(u_{0_D})\|_{L^p(\Omega)} + C(\|g\|_{L^\infty(\Omega)})\|\beta(u_D)\|_{L^p(\Omega)} + 1,
\]
and
\[
\|f(u_D)\|_{L^P(0,t; L^p(\Omega))} \leq C(c_5,p)\|u_D\|_{L^p(0,T; L^p(\Omega))} + 1.
\]
Moreover, if $u_D$ is a solution of problem (4) associated to $g$ and $u_{0,D}$, while $\hat{u}_D$ is a solution of problem (4) associated to $\hat{g}$ and $u_{0,D}$, then we have the following inequality

$$\int_\Omega |\beta(u_D) - \beta(\hat{u}_D)| \leq \left( \int_\Omega |\beta(u_{0,D}) - \beta(\hat{u}_{0,D})| + T \int_\Omega |\hat{g} - g| \right) e^{\epsilon_0 t}.$$  

We move on to the estimates on the solutions in $L^\infty(\Omega)$ and $H^1(\Omega)$ that will allow us to conclude the existence of a global attractor, which we denote by $A_D$.

**Lemma 2.3.** If $u_D$ is a solution of problem (4), then there exist $c'_1, c'_2 > 0$ such that for all $t > 0$,

$$\|u_D(t)\|_{L^\infty(\Omega)} \leq \max (\beta^{-1}(c(t)), |\beta^{-1}(c(t))|),$$

where

$$c(t) = \left( \frac{c'_2 + \|g\|_{L^\infty(\Omega)}}{c'_1} \right)^{1/p-1} + \frac{1}{(c'_1(p-2)t)^{1/p-2}}.$$  

**Remark 3.** The previous lemma implies the existence of an absorbing set in $L^q(\Omega)$, for any $q \geq 1$.

**Lemma 2.4.** If $u_D$ is a solution of problem (4), then there exists a positive constant $c(\eta) > 0$, such that

$$\|u_D(t)\|_{H^1(\Omega)} \leq c(\eta),$$

for each $t \geq \eta + 1$ and $D \geq 1$, for any $\eta > 0$, where $c(\eta)$ depends only on $\|g\|_{L^\infty(\Omega)}$, $\|u_{0,D}\|_{L^2(\Omega)}$ and the bound on $\|u_D\|_{L^\infty(\Omega)}$. Thus, by the previous lemmas, we obtain the following result.

**Corollary 1.** The semigroup $S_D(t)$ associated to problem (4) possesses a global attractor $A_D$ which is uniformly bounded in $H^1(\Omega) \cap L^\infty(\Omega)$ and is compact and connected in $L^2(\Omega)$, for each $D \geq 1$.

3. Passage to the limit. In the present section, we obtain uniform bounds in space and time, which imply the strong convergence of $\beta(u_D)$ in $L^1((0,T];L^1(\Omega))$, as $D$ goes to infinity. This, together with apriori estimates obtained in Lemma 2.2, provide us with the appropriate tools to obtain the problem, satisfied at the limit, when $D$ goes to infinity.

Using the regular approximation in Proposition 1 and the arguments of Lemma 2.2 in [15] and Lemma 1.8 in [1], we have the following result.

**Lemma 3.1.** Let $u_D$ be the solution of problem (4), then it satisfies

$$\lim_{h \rightarrow 0} \int_0^{T-h} \|\beta(u_D(t + h)) - \beta(u_D(t))\|_{L^p(\Omega)} = 0.$$  

**Proof.** We work with $u_j$ the solutions of the approximation problem (5) and then pass to the limit. Our first aim is to prove that there exists $C > 0$ such that

$$\int_0^{T-h} \langle \beta_j(u_j(t + h)) - \beta_j(u_j(t)), u_j(t + h) - u_j(t) \rangle dt \leq hC. \quad (6)$$

Integrating for $t' \in (t, t + h)$ the equation in (5), we have

$$\int_t^{t+h} \frac{d}{dt} \beta_j(u_j) dt' = D \int_t^{t+h} \Delta u_j(t') dt' - \int_t^{t+h} f(u_j(t')) dt' - \int_t^{t+h} g dt'.$$
Using the change of variable \( t' = t + h' \), we obtain
\[
\beta_j(u_j(t + h)) - \beta_j(u_j(t)) = h(DH_1(t) + H_2(t) + H_3),
\]
where
\[
H_1(t) = \frac{1}{h} \int_0^h \Delta u_j(t + h')dh', \quad H_2(t) = \frac{1}{h} \int_0^h f(u_j(t + h'))dh',
\]
and
\[
H_3 = \frac{1}{h} \int_0^h gdh'.
\]
Now \( H_1(t) \) and \( H_2(t) \) are convolutions respectively of \( \Delta u_j \) and \( f(u_j) \) with the Dirac sequence \( h^{-1}[0,h] \), so that in particular,
\[
\int_0^T \|H_1(t)\|_{H^{-1}(\Omega)}^2 \leq \int_0^T \|\Delta u_j\|_{H^{-1}(\Omega)}^2 \leq \int_0^T \|\nabla u_j\|_{L^2(\Omega)}^2 \tag{7}
\]
and
\[
\int_0^T \|H_2(t)\|_{L^p(\Omega)}^p \leq \int_0^T \|f(u_j(t))\|_{L^{p'}(\Omega)}^{p'} \tag{8}
\]
Applying the equation to \( u_j(t + h) - u_j(t) \) and integrating with respect to \( t \in [0,T-h] \), we obtain
\[
\int_0^{T-h} \langle \beta_j(u_j(t + h)) - \beta_j(u_j(t)), u_j(t + h) - u_j(t) \rangle_{V',V} dt
= h \int_0^{T-h} \langle DH_1(t) - H_2(t) + H_3, u_j(t + h) - u_j(t) \rangle_{V',V} dt.
\]
Estimating
\[
\langle DH_1(t) - H_2(t) + H_3, u_j(t + h) - u_j(t) \rangle_{V',V}
\leq D\|H_1\|_{H^{-1}(\Omega)} \|u_j(t + h)\|_{H^1(\Omega)} + (\|H_2\|_{L^{p'}(\Omega)} + \|H_3\|_{L^{p'}(\Omega)})\|u_j(t + h)\|_{L^p} + D\|H_1(t)\|_{H^{-1}(\Omega)} \|u_j(t)\|_{H^1(\Omega)} + (\|H_2(t)\|_{L^{p'}(\Omega)} + \|H_3\|_{L^{p'}(\Omega)})\|u_j(t)\|_{L^p(\Omega)},
\]
and using Hölder’s inequality, we have
\[
\int_0^{T-h} \langle DH_1(t) - H_2(t) + H_3, u_j(t + h) - u_j(t) \rangle dt
\leq D \left( \int_0^{T-h} \|H_1(t)\|_{H^{-1}(\Omega)}^2 \right)^{1/2} \left( \int_0^{T-h} \|u_j(t + h)\|_{H^1(\Omega)}^2 \right)^{1/2}
+ \left[ \left( \int_0^{T-h} \|H_2(t)\|_{L^{p'}}^p \right)^{1/p'} + \left( \int_0^{T-h} \|H_3\|_{L^{p'}}^p \right)^{1/p'} \right] \left( \int_0^{T-h} \|u_j(t + h)\|_{L^p}^p \right)^{1/p}
+ D \left( \int_0^{T-h} \|H_1(t)\|_{H^{-1}(\Omega)}^2 \right)^{1/2} \left( \int_0^{T-h} \|u_j(t)\|_{H^1(\Omega)}^2 \right)^{1/2}
+ \left[ \left( \int_0^{T-h} \|H_2\|_{L^{p'}}^p \right)^{1/p'} + \left( \int_0^{T-h} \|H_3\|_{L^{p'}}^p \right)^{1/p'} \right] \left( \int_0^{T-h} \|u_j(t)\|_{L^p}^p \right)^{1/p}.
\]
Hence, using Young’s inequality,
\[
\int_0^{T-h} \langle \beta_j(u_j(t+h)) - \beta_j(u_j(t)), u_j(t+h) - u_j(t) \rangle dt \\
\leq h[D \int_0^T \|H_1(t)\|_{L^2}^2 + \frac{1}{2} \int_0^T \|u_j(t)\|_{H^1}^2] \\
+ \frac{1}{p'} \left( \int_0^T \|H_2\|_{L^{p'}}^p + \int_0^T \|H_3\|_{L^{p'}}^p \right) + \frac{2}{p} \int_0^T \|u_j(t)\|_{L^p}^p \\
\leq hC,
\]
where \( C \) is independent of \( j \) and \( D \) by inequalities (7), (8) and Lemma 2.2.

We then claim the existence of a continuous function \( w_M \) with \( w_M(0) = 0 \) such that if we have the bounds \( \|u\|_{H^1(\Omega)}, \|v\|_{L^p(\Omega)} \leq M, \|u\|_{L^p(\Omega)}, \|v\|_{L^p(\Omega)} \leq M \) and \( \langle \beta(u) - \beta(v), u - v \rangle \leq \delta \) then \( \|\beta(u) - \beta(v)\|_{L^p(\Omega)} \leq w_M(\delta) \). Indeed, assume \( u_\delta, v_\delta \in H^1(\Omega) \cap L^p(\Omega) \) satisfy the proposed estimates, but the inequality \( \|\beta(u_\delta - \beta(v_\delta))\|_{L^p} \geq k > 0 \) holds. We have by the boundedness of \( u_\delta \) that \( u_\delta \to u \) in \( L^p(\Omega) \) and recall \( \beta(u_\delta) \to \beta(u) \) in \( L^p(\Omega) \). Similarly, for \( v_\delta \) and \( \beta(v_\delta) \). By assumption
\[
\langle \beta(u_\delta) - \beta(v_\delta), u_\delta - v_\delta \rangle \leq \delta,
\]
and for \( \delta \to 0 \), we obtain \( \langle \beta(u) - \beta(v), u - v \rangle \leq 0 \). By monotonicity \( \langle \beta(u) - \beta(v), u - v \rangle = 0 \) and thus, \( \langle \beta(\cdot), u - v \rangle \) is constant along the line segment between \( u \) and \( v \), in particular, \( \langle \beta(u + \theta(v - u)), u - v \rangle = \langle \beta(u), u - v \rangle \), for any \( \theta \in [0,1] \). Now, we recall that \( \Psi'(w) = \beta(w) \) for any \( w \), where \( \Psi \) is defined in (3). By the convexity of \( \Psi \), we deduce
\[
\int_\Omega \Psi(u + w) - \Psi(v) = \int_\Omega \Psi(u + w) - \Psi(u) - \Psi'(u + \theta(v - u))(v - u) \\
\geq \int_\Omega \Psi'(u)(v + w - u) - \int_\Omega \Psi'(v)(v - u) \\
= \int_\Omega \Psi'(u)w = \int_\Omega \Psi'(v)w,
\]
for any \( w \), and therefore \( \beta(u) = \beta(v) \) a.e., which contradicts \( \|\beta(u) - \beta(v)\|_{L^p} \geq k > 0 \) and the claim follows.

Now consider
\[
E := \{ t \in [0, T-h] : \|u_j(t+h)\|_{H^1(\Omega) \cap L^p(\Omega)} \leq M, \|u_j(t)\|_{H^1 \cap L^p} \leq M, \\
(\beta_j(u_j(t+h)) - \beta_j(u_j(t)), u_j(t+h) - u_j(t))_{L^p(\Omega)} \leq hM \}.
\]

Let \( t \in E^c \), then by (6) and the fact that the solution \( u_j \) is uniformly bounded in \( L^p(0,T;L^p(\Omega)) \cap L^2(0,T;H^1(\Omega)) \),
we have
\[
M(T-h) \leq \int_0^{T-h} \frac{1}{h} \langle \beta_j(u_j(t+h)) - \beta_j(u_j(t)), u_j(t+h) - u_j(t) \rangle \\
+ \int_0^{T-h} \|u_j(t+h)\|_{H^1 \cap L^p} + \int_0^{T-h} \|u_j(t)\|_{H^1 \cap L^p} \leq C,
\]
and hence $|E^c| \leq T - h \leq C/M$. Split the integral of $\|\beta_j(u_j(t+h)) - \beta_j(u_j(t))\|_{L^{p'}}$ over $(0, T - h)$ into an integral over $E$ and $E^c$. By the claim,

$$\int_{E^c} \|\beta_j(u_j(t+h)) - \beta_j(u_j(t))\|_{L^{p'}}(\Omega) \leq w_M(hM)|E| \leq Tw_M(hM)$$

and

$$\int_{E^c} \int_\Omega |\beta_j(u_j(t+h)) - \beta_j(u_j(t))|^p' \leq \int_{E^c + h} \int_\Omega |\beta_j(u_j)|^p' + \int_{E^c} \int_\Omega |\beta_j(u_j)|^p' \leq 2C'|E| \leq 2C'(C/M),$$

where $C'$ is independent of $j$ and $D$, since $\beta_j(u_j)$ is uniformly bounded in $L^\infty(0, T; L^{p'}(\Omega))$ by Lemma 2.2. Hence

$$\int_0^{T-h} \|\beta_j(u_j(t+h)) - \beta_j(u_j(t))\|_{L^{p'}} \leq w_M(hM)T + 2C'C/M.$$

Choosing $M$ such that for any $\epsilon > 0$, $w_M(hM)T \leq \epsilon/2$ and $2C'C/M \leq \epsilon/2$ and using that $\beta_j(u_j) \to \beta(u_D)$ in $L^2(0, T; L^2(\Omega))$, the result follows.

**Theorem 3.2.** let $u_D$ be the solution of problem (4). Then, we have that $\beta(u_D)$ is relatively compact in $L^1([0,T]; L^1(\Omega))$.

**Proof.** Suppose that $u_j(x,t)$ is a solution of the approximation problem (5), then by the translation invariance of the Laplacian, we have that $u_j(x+y, t)$ also satisfies the equation in (5) in $\Omega' \times (0, T)$ for $\Omega' \subset \subset \Omega$ and $y \in \mathbb{R}^N$ small enough. Then, setting $v_j(x, t) = u_j(x+y, t) - u_j(x, t)$ and $w_j(x, t) := \beta_j(u_j(x+y, t)) - \beta_j(u_j(x, t))$, we have

$$\partial_t w_j(x, t) - \Delta v_j(x, t) + f(u_j(x+y, t)) - f(u_j(x, t)) = g(x+y) - g(x).$$

Hence

$$\partial_t w_j(x, t) - \Delta v_j(x, t) + (f(u_j(x+y, t)) + c_6 \beta_j(u_j(x+y, t))) - (f(u_j(x, t))c_6 \beta_j(u_j(x, t))) = c_6 v_j(x, t) + g(x+y) - g(x).$$

Then multiplying by $h_\delta(v_j(x, t))$, where $h_\delta$ is a smooth function such that $h_\delta' \geq 0$, $0 \leq h_\delta \leq 1$ and $h_\delta(r) \to \text{sign}(r)$ as $\delta \to 0$ and integrating in $\Omega'$, we obtain

$$\int_{\Omega'} \partial_t w_j(x, t)h_\delta(v_j(x, t)) + \int_{\Omega'} h_\delta'(v_j(x, t))|\nabla v_j(x, t)|^2$$

$$+ \int_{\Omega'} [(f(u_j(x+y)) + c_6 \beta_j(u_j(x+y))) - (f(u_j(x)) + c_6 \beta_j(u_j(x)))] h_\delta(v_j(x))$$

$$= c_6 \int_{\Omega'} w_j(x, t)h_\delta(v_j(x, t)) + \int_{\Omega'} (g(x+y) - g(x))h_\delta(v_j(x, t)).$$

We recall the conditions on $f$ and take $\delta \to 0$ to get

$$\frac{d}{dt} \int_{\Omega'} |w_j(x, t)| \leq c_6 \int_{\Omega'} |w_j(x, t)| + \int_{\Omega'} |g(x+y) - g(x)|.$$
Thus, integrating in \((\eta, t)\) for \(0 < \eta \leq t \leq T\)
\[
\int_{\Omega'} |w_j(x,t)| \leq \int_{\Omega'} |w_j(x,\eta)| + c_6 \int_{\eta}^{t} \int_{\Omega'} |w_j(x,t)| + \int_{\eta}^{t} \int_{\Omega'} |g(x+y) - g(x)|.
\]
We can then make \(j \to \infty\) and \(\eta \to 0\) to deduce
\[
\int_{\Omega'} |\beta(u_D(x+y,t)) - \beta(u_D(x,t))| \leq \int_{\Omega'} |\beta(u_{0_D}(x+y)) - \beta(u_{0_D}(x))| + c_6 \int_{0}^{t} \int_{\Omega'} |\beta(u_D(x+y,t)) - \beta(u_D(x,t))| + \int_{0}^{t} \int_{\Omega'} |g(x+y) - g(x)|.
\]
Finally by Gronwall's inequality and passing to the limit as \(|y|\) tends to 0, we conclude
\[
\lim_{|y| \to 0} \int_{\Omega'} |\beta(u_{D_D}(x+y)) - \beta(u_D(x))| = 0.
\]
Since we also have that \(u_D\) is uniformly bounded in \(L^\infty(\Omega)\) for \(t > 0\), we obtain the uniform boundedness in space. The uniform boundedness in time comes from Lemma 3.1.

Theorem 3.3. Let \(u_D\) be the unique weak solution of problem (4). Then, there exists a unique function \(u\) such that, as \(D \to \infty\),
\[
\beta(u_D) \to \beta(u) \quad \text{in} \quad L^1((0,T];L^1(\Omega)),
\]
where \(u\) is the unique solution of the ODE problem (2).

Proof. We deduce from Theorem 3.2 that there exists \(\chi\) such that
\[
\beta(u_D) \to \chi \quad \text{in} \quad L^1((0,T];L^1(\Omega)),
\]
as \(D \to \infty\). Now consider,
\[
J(u) := \begin{cases} 
\int_{0}^{T} \int_{\Omega} \Psi(v) dxd\tau & \text{if} \quad \Psi(v) \in L^1(0,T,L^1(\Omega)) \\
0 & \text{otherwise,}
\end{cases}
\]
where \(\Psi\) is as defined in (3). We know from the subdifferential theory in [7] that
\[
\partial J(u) = \{v \in L^2(0,T;L^2(\Omega)); v = \beta(u) \quad \text{a.e.}\}.
\]
Then, in particular, for any \(w \in L^\infty(0,T;L^\infty(\Omega))\), and any \(\eta > 0\),
\[
\int_{\eta}^{T} \int_{\Omega} \Psi(w) + \int_{\eta}^{T} \int_{\Omega} \beta(u_D)u_D \geq \int_{\eta}^{T} \int_{\Omega} \beta(u_D)w + \int_{\eta}^{T} \int_{\Omega} \Psi(u_D).
\]
Using the lower semicontinuity of \(u \mapsto \Psi(u)\), the strong convergence of \(\beta(u_D)\) and the weak star convergence of \(u_D\) in \(L^\infty(\eta,T;L^\infty(\Omega))\), we have
\[
\int_{\eta}^{T} \int_{\Omega} \Psi(w) + \int_{\eta}^{T} \int_{\Omega} \chi u \geq \int_{\eta}^{T} \int_{\Omega} \chi w + \int_{\eta}^{T} \int_{\Omega} \Psi(u)
\]
and thus \(\chi = \beta(u) \quad \text{a.e. in} \quad (0,T) \times \Omega\).
In summary, we have, for any $\eta > 0$,
\[
\beta(u_D) \to \beta(u) \text{ in } L^1([\eta, T]; L^1(\Omega)), \\
\|u_D\|_{L^2(0,T; H^1(\Omega))} \cap L^p(0,T; L^p(\Omega)), \\
\|u_D\| \to \|u\|_{L^\infty(\eta, T; L^\infty(\Omega))}
\]
and
\[
f(u_D) \to f(u) \text{ in } L^p(0,T; L^p(\Omega)).
\]
Now consider $\phi \in L^\infty(0,T)$ independent of space, then, passing to the limit,
\[
\int_0^T \langle \frac{\partial \beta(u)}{\partial t}, \phi \rangle_{V',V} \, dt + \int_0^T \int_\Omega f(u) \phi = \int_0^T \int_\Omega g \phi.
\]
Since, by Lemma 2.2,
\[
\int_0^T \|\nabla u\|^2 \leq \liminf_{D \to \infty} \int_0^T \int_\Omega |\nabla u_D|^2 = 0,
\]
we have that $u$ and hence also $\beta(u)$ and $f(u)$ are independent of space and we conclude, for any $\phi \in L^\infty(0,T)$
\[
\int_0^T \frac{d}{dt} \beta(u) \phi dt + \int_0^T f(u) \phi = \int_0^T \left( \frac{1}{|\Omega|} \int_\Omega g \right) \phi.
\]
and therefore $u$ is the solution of the equation in (2). Moreover, suppose that $\frac{\partial \phi}{\partial t} \in L^2(0,T)$ and $\phi(T) = 0$. Then, again passing to the limit,
\[
|\Omega| \int_0^T \frac{d}{dt} \beta(u) \phi = \int_0^T \int_\Omega (\beta(u) - \beta(u_0)) \frac{\partial \phi}{\partial t} = \int_0^T (\beta(u)|\Omega| - |\Omega| \int_\Omega \beta(u_0)) \frac{\partial \phi}{\partial t} = |\Omega| \int_0^T (\beta(u) - \int_\Omega \beta(u_0)) \frac{\partial \phi}{\partial t}.
\]
Hence, $\beta(u(0)) = \int_\Omega \beta(u_0)$ and the result follows.

4. **Global attractor of the limit problem.** In the previous section, we determined that, as $D$ goes to infinity, the solution of problem (4) converges to the solution of the following ODE

\[
\begin{cases}
(\beta(u))_t + f(u) = \int g, & t > 0 \\
\beta(u(0)) = \int \beta(u_0).
\end{cases}
\] (9)

We now prove the existence of a global attractor in $\mathbb{R}$, which we denote by $A_\infty$, associated to this limit problem. We recall that a semigroup $S(t)$ belongs to the class $K$ if, for each $t > 0$, the operator $S(t)$ is compact and it is $B$-dissipative if it has a bounded global $B$-attractor, as defined in [14].

**Theorem 4.1.** The problem (9) defines a semigroup of class $K$, which is $B$-dissipative and so there exists a global $B$-attractor $A_\infty$ associated with it. Moreover, the attractor $A_\infty$ is equal to the union of all the bounded complete trajectories in $\mathbb{R}$. 
Proof. We define $S(t) : \mathbb{R} \to \mathbb{R}$ by $S(t) \int u_0 = u(t)$ with $u$ the unique solution of problem (9). We will show that $S(t)$ is of class $K$ and also it is $B$-dissipative. To that end, we multiply the equation $(\beta(u))_t + f(u) = \int g$ by $	ext{sign} (\beta(u)) = \text{sign} (u)$ to obtain,

$$
\frac{d}{dt} |\beta(u)| + f(u) \text{sign} (u) = \int g \text{sign} (u).
$$

Then, using the conditions on $f$,

$$
\frac{d}{dt} |\beta(u)| + c'_1 |\beta(u)|^{p-1} - c_4 \leq |\int g|.
$$

Then, for all $t \geq \eta$, for any $\eta > 0$,

$$
|\beta(u)| \leq \left( \frac{c_4 + |\int g|}{c'_1} \right)^{1/(p-1)} + \frac{1}{(c'_1(p-2)t)^{1/(p-2)}} = c(t) \leq c(\eta)
$$

On the other hand, from (10), we also deduce integrating for $t \in [0, \eta]$

$$
|\beta(u(t))| \leq \left| \int \beta(u_0) \right| + c_4 + |\int g|.
$$

We conclude that for each $t > 0$, $S(t)$ maps bounded sets to bounded sets. As a result, $S(t)$ is compact in $\mathbb{R}$ and thus has a maximal compact invariant global $B$-attractor $A_{\infty}$, given as the union of all bounded complete trajectories in $\mathbb{R}$. 

5. Continuity of attractors. The following lemma and theorem follow as in [16]. For completeness, we give the details of the proof of Theorem 5.2. First, we have that if the elements of the global attractors $A_D$ converge, as $D$ goes to infinity, to a function, then such function must be equal to its spatial average.

**Lemma 5.1.** If for each $D \geq 1$, $v^D_0 \in A_D$ and $v_0 := \lim_{D \to \infty} v^D_0$ in $L^2(\Omega)$, then $v_0 = \int v_0$

We can, at last, obtain the upper semicontinuity at infinity.

**Theorem 5.2.** The family of global $B$-attractors $\{A_D : D \geq 1\}$ of the problem (4) is upper semicontinuous at infinity.

**Proof.** Let us consider a sequence $\{v^D_0\}$ with $v^D_0 \in A_D$, for each $D \geq 1$. Due to lemma 2.4, there exists a subsequence $\{v^{D_j}_0\}$ such that $v^{D_j}_0 \to v_0$ in $L^2(\Omega)$ as $D$ goes to infinity. Moreover, by lemma 5.1, $v_0 = \int v_0 \in \mathbb{R}$. To get the upper semi continuity of the family of global $B$-attractors it is sufficient to show that $v_0$ belongs to the global attractor $A_{\infty}$ of the limit problem. However, since by Theorem 4.1, $A_{\infty}$ is equal to union of all bounded complete trajectories in $\mathbb{R}$, it is actually sufficient to construct a bounded complete trajectory $\phi : \mathbb{R} \to \mathbb{R}$ through $v_0$.

To this aim, let $\{S_D(\tau)\}$ and $\{S(\tau)\}$ be the semigroups associated with (4) and (9) respectively. For each $t_j > j$, $t_1 < t_2 < \ldots < t_j < \ldots$, consider $x_j \in A_{D_j}$ such that $S^{D_j}(t_j)(x_j) = v^{D_j}_0$ and then
\[
S^{D_j}(t_j + t)(x_j) = S^{D_j}(t_j)v_0^{D_j} \to S(t) \int v_0 = S(t)v_0 \text{ strongly in } L^2(\Omega), \quad (11)
\]
for all \( t > 0 \). Now consider \( \{S^{D_j(t_j-1)}(x_j)\} \subset \cup_{D \geq 1} A_D \), then there exists \( z_1 = \int z_1 \) such that
\[
S^{D_j}(t_j - 1)(x_j) \to z_1 \text{ in } L^2(\Omega),
\]
and we obtain
\[
S^{D_j}(t_j - 1 + t)(x_j) = S^{D_j}(t_j)(S^{D_j}(t_j - 1)(x_j)) \to S(t) \int z_1 = S(t)z_1.
\]
Thus
\[
S^{D_j}(t_j + t)(x_j) = S^{D_j}(1)[S^{D_j}(t_j - 1 + t)(x_j)]
\]
\[
\to S(1) \int S(t)z_1 = S(1)S(t) \int z_1 = S(t)S(1)z_1,
\]
and therefore, by (11), \( v_0 = S(1)z_1 \).

Proceeding inductively, we find for each \( r = 0, 1, 2, \ldots \), a real number \( z_r \) such that \( S(1)z_{r+1} = z_r \). Given \( t \in \mathbb{R} \), we define \( \phi(t) \) to be the common value \( S(t + r)z_r \) for \( r \geq -t \), i.e., \( S(t + r)z_r = S(t)S(1)S(r - 1)z_r = S(t)S(1)z_1 = S(t)v_0 \). Then, we have \( \phi \) is a bounded complete trajectory through \( v_0 \).

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