Evolution from BCS Superconductivity to Bose Condensation: Analytic Results for the crossover in three dimensions

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We provide an analytic solution for the mean-field equations and for the relevant physical quantities at the Gaussian level, in terms of the complete elliptic integrals of the first and second kinds, for the crossover problem from BCS superconductivity to Bose-Einstein condensation of a three-dimensional system of free fermions interacting via an attractive contact potential at zero temperature. This analytic solution enables us to follow the evolution between the two limits, especially shorter than in conventional superconductors. In Section III our analysis is extended to quantities such as the phase coherence length \(\xi_{\text{phase}}\) and the sound velocity \(s\), by considering the Gaussian fluctuations about the mean field. To keep the presentation compact, the properties of the elliptic integrals used in our treatment are summarized in Appendix A. In Appendix B we report for the sake of comparison the solution for the two-dimensional case given previously by Ref. [11].

II. ANALYTIC RESULTS AT THE MEAN-FIELD LEVEL

We consider the Hamiltonian (\(\hbar = 1\))

\[
H = \sum_\sigma \int \text{d}r \psi_\sigma^\dagger (r) \left( -\frac{\nabla^2}{2m} - \mu \right) \psi_\sigma (r) + g \int \text{d}r \psi_\uparrow^\dagger (r) \psi_\downarrow^\dagger (r) \psi_\downarrow (r) \psi_\uparrow (r)
\]

where \(\psi_\sigma (r)\) is the fermionic field operator with spin projection \(\sigma\), \(m\) is the fermionic (effective) mass, and \(g = V\Omega\) is the strength of the short-range (contact) potential between fermions with \(V < 0\) (\(\Omega\) being the volume occupied by the system).

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At the zero temperature, the mean-field equations for the gap parameter $\Delta$ and the chemical potential $\mu$ are obtained by a suitable decoupling of the Hamiltonian (2.1) and are given by

\[ \frac{1}{V} = \sum_k \frac{1}{2E_k} \tag{2.2} \]

\[ n = \frac{N}{\Omega} = \frac{2}{\Omega} \sum_k k^2 \]  \tag{2.3}

where $k$ is the wave vector, $N$ the total number of fermions, and

\[ \xi_k = \frac{k^2}{2m} - \mu, \quad E_k = \sqrt{\xi_k^2 + \Delta^2}, \quad v_k = \frac{1}{2} \left( 1 - \frac{\xi_k}{E_k} \right). \]  \tag{2.4}

Owing to our choice of a contact potential, Eq. (2.2) diverges in the ultraviolet (both in two and three dimensions) and requires a suitable regularization. In three dimensions it is common practice to introduce the scattering amplitude $a_s$ defined via the equation

\[ \frac{m}{4\pi a_s} = \frac{1}{\Omega V} + \frac{1}{\Omega} \sum_k \frac{m}{k^2}, \]  \tag{2.5}

where the divergent sum on the right-hand side of Eq. (2.5) results in a finite value of $a_s$ by letting $V \to 0$ in a suitable way. Subtracting Eq. (2.5) from Eq. (2.2) we obtain

\[ \frac{m}{4\pi a_s} = \frac{1}{\Omega} \sum_k \left( \frac{1}{2E_k} - \frac{m}{k^2} \right) \]  \tag{2.6}

which has to be solved simultaneously with Eq. (2.5) to determine $\Delta$ and $\mu$ as functions of $a_s$.

In three dimensions it is convenient to introduce the following dimensionless quantities

\[ \begin{cases} x^2 = \frac{k^2}{2m} \frac{1}{\Delta}, & x_o = \frac{\mu}{\Delta}, \\ \xi_x = \frac{\xi_k}{\Delta} = x^2 - x_o, & E_x = \frac{E_k}{\Delta} = \sqrt{\xi_x^2 + 1} \end{cases} \]  \tag{2.7}

with $k = |k|$, and express Eqs. (2.3) and (2.6) in the form

\[ \frac{1}{a_s} = 2\pi (2m\Delta)^{1/2} I_1(x_o) \]  \tag{2.8a}

\[ n = \frac{1}{2\pi^2} (2m\Delta)^{3/2} I_2(x_o) \]  \tag{2.8b}

where

\[ I_1(x_o) = \int_0^{\infty} dx x^2 \left( \frac{1}{E_x} - \frac{1}{x^2} \right) \]  \tag{2.9}

\[ I_2(x_o) = \int_0^{\infty} dx x^2 \left( 1 - \frac{\xi_x}{E_x} \right) \]  \tag{2.10}

The integrals (2.9) and (2.10) were originally considered by Eagles, who evaluated them numerically as functions of the crossover parameter $x_o$. We shall soon show that the integrals (2.9) and (2.10) can actually be evaluated in a closed form for all values of $x_o$. Note that $I_2(x_o) \geq 0$ while $I_1(x_o)$ can take both signs.

Before proceeding further, it is convenient to render Eqs. (2.8) dimensionless by introducing the Fermi energy $\epsilon_F = k_F^2/2m = (3\pi^2 n)^{2/3}/2m$ where $k_F$ is the Fermi momentum. In this way Eq. (2.8b) becomes

\[ \frac{\Delta}{\epsilon_F} = \left[ \frac{2}{3I_2(x_o)} \right]^{2/3} \]  \tag{2.11}

and Eq. (2.8a) reduces to

\[ \frac{1}{k_F a_s} = -\frac{2}{\pi} \left[ \frac{2}{3I_2(x_o)} \right]^{1/3} I_1(x_o) \]  \tag{2.12}

Note that the right-hand sides of Eqs. (2.11) and (2.12) depend on $x_o$ only. Equation (2.12) can thus be inverted to obtain $x_o$ as a function of $k_F a_s$; from Eq. (2.11) and from $\mu/\epsilon_F = x_o \Delta/\epsilon_F$ one can then obtain the two parameters $\Delta/\epsilon_F$ and $\mu/\epsilon_F$ as functions of $k_F a_s$.

Alternatively, one can use $k_F \xi_{pair}$ as the independent variable in the place of $k_F a_s$, where $\xi_{pair}$ is the characteristic length for pair correlation given by the following expression in three dimensions at the mean-field level:

\[ \xi_{pair}^2 = \frac{1}{m^2} \int_0^{\infty} dk (k^4 \xi_k^2/E_k^6) \]  \tag{2.13}

where

\[ I_3(x_o) = \int_0^{\infty} dx x^4 \frac{\xi_x^2}{E_x^6} \]  \tag{2.14}

\[ I_4(x_o) = \int_0^{\infty} dx \frac{x^2}{E_x^4} \]  \tag{2.15}

are two additional integrals expressed in terms of the quantities (2.7). Contrary to the integrals $I_1$ and $I_2$, the integrals $I_3$ and $I_4$ are elementary and can be evaluated via the residues technique. One obtains:

\[ I_3(x_o) = \frac{\pi}{16} \frac{x_1 (1 + x_1^4)}{(1 + x_1^2)^{3/2}} \]  \tag{2.16}

\[ I_4(x_o) = \frac{\pi}{2} x_1 \]  \tag{2.17}

with the notation
\[
x^2 = \frac{1 + x_0^2}{2} x_0.
\]  
(2.18)

Making use of Eq. (2.14) we obtain eventually
\[
(k_F \xi_{\text{pair}})^2 = \frac{\epsilon F}{2} (1 + \xi_4^2) \left( \frac{1 + x_1^4}{1 + x_0^2} \right)^{1/2} \left( \frac{3 I_2(x_o)}{2} \right)^{2/3} .
\]
(2.19)

In this way Eq. (2.13) can be dropped in favor of Eq. (2.19), which can be inverted to obtain \( x_o \) as a function of \( k_F \xi_{\text{pair}} \).

There remains to evaluate the integrals \( I_1(x_o) \) entering Eq. (2.12) and \( I_2(x_o) \) entering Eqs. (2.11) and (2.19).

To this end, we introduce the auxiliary integrals
\[
\begin{align*}
I_5(x_o) &= \int_0^\infty \frac{x^2}{E_x} dx, \\
I_6(x_o) &= \int_0^\infty \frac{x^2 \xi}{E_x} dx,
\end{align*}
\]
(2.20)
(2.21)

such that
\[
\begin{align*}
I_1(x_o) &= 2 \left( x_o I_6(x_o) - I_5(x_o) \right) \\
I_2(x_o) &= \frac{2}{3} \left( x_o I_5(x_o) + I_6(x_o) \right)
\end{align*}
\]
(2.22)
(2.23)

after integration by parts. The auxiliary integrals \( I_5(x_o) \) and \( I_6(x_o) \) can, in turn, be expressed as linear combinations of the complete elliptic integrals of the first \( \{F(\frac{\pi}{2}, \kappa)\} \) and second \( \{E(\frac{\pi}{2}, \kappa)\} \) kinds. Reduction of a generic integral of the elliptic kind to the normal Legendre's form has been treated at length in the literature.\textsuperscript{4}

Here we proceed as follows. Integration by parts gives for \( I_6(x_o) \) [cf. Eq. (2.21)]:
\[
\begin{align*}
I_6(x_o) &= -\frac{1}{2} \int_0^\infty dx \frac{d}{dx} \frac{1}{E_x} = \frac{1}{2} \int_0^\infty dx \frac{1}{E_x} \\
&= \frac{1}{2} \int_0^\infty dx \frac{1}{(x^4 - 2x_o x^2 + x_o^2 + 1)^{1/2}} \\
&= \frac{1}{2(1 + x_o^2)^{1/4}} F(\frac{\pi}{2}, \kappa)
\end{align*}
\]
(2.24)

where in the last line use has been made of the results of Appendix A and where \( 0 \leq \kappa^2 < 1 \)
\[
\kappa^2 = \frac{x_1^2}{(1 + x_0^2)^{1/2}}
\]
(2.25)

with \( x_1 \) given by Eq. (2.18). For \( I_5(x_o) \) we obtain instead [cf. Eq. (2.20)]:
\[
\begin{align*}
I_5(x_o) &= \int_0^\infty dx \frac{x^2}{(x^4 - 2x_o x^2 + x_o^2 + 1)^{3/2}} \\
&= (1 + x_0^2)^{1/4} E(\frac{\pi}{2}, \kappa) - \frac{1}{4x_1^4(1 + x_o^2)^{1/4}} F(\frac{\pi}{2}, \kappa)
\end{align*}
\]
(2.26)

where again use has been made of the results of Appendix A.

In conclusion, we obtain for the quantities of interest [cf. Eqs. (2.11), (2.12), and (2.19)]:
\[
\begin{align*}
\frac{\Delta}{\epsilon F} &= \frac{1}{(x_o I_5(x_o) + I_6(x_o))^{2/3}} \\
\frac{\mu}{\epsilon F} &= \frac{\Delta}{\epsilon F} = \frac{x_o}{(x_o I_5(x_o) + I_6(x_o))^{2/3}} \\
\frac{1}{k_F \sigma} &= \frac{4}{\pi} \frac{(x_o I_5(x_o) - I_5(x_o))}{(x_o I_5(x_o) + I_6(x_o))^{1/3}} \\
k_F \xi_{\text{pair}} &= (1 + x_1^4)^{1/2} \frac{(x_o I_5(x_o) + I_6(x_o))^{1/3}}{(1 + x_o^2)^{1/4}}
\end{align*}
\]
(2.27)
(2.28)
(2.29)
(2.30)

with \( I_5(x_o) \) and \( I_6(x_o) \) given by Eqs. (2.26) and (2.24), respectively. It is sometimes convenient to normalize the chemical potential, when negative, with respect to the bound-state energy \( \epsilon_o \) of the associated two-fermion problem. In the three-dimensional case, \( \epsilon_o \) can be expressed in terms of \( a_s \) whenever \( a_s \geq 0 \). One finds
\[
\epsilon_o = \frac{1}{m a_s^2},
\]
(2.31)

so that [cf. Eq. (2.23)]
\[
\frac{\epsilon_o}{\epsilon F} = \frac{2}{(k_F \sigma)^2} = \frac{32}{\pi^2} \frac{x_o I_6(x_o) - I_5(x_o)^2}{(x_o I_5(x_o) + I_6(x_o))^{2/3}}.
\]
(2.32)

![FIG. 1. \( \Delta/\epsilon F \) vs \( k_F \xi_{\text{pair}} \), obtained from Eqs. (2.27) and (2.30). Full curve: exact solution; dashed curve: BCS approximation, obtained by including only the first two terms in Eqs. (A.7) and (A.8); dotted curve: BE approximation, obtained by including only the first two terms in Eqs. (A.5) and (A.6).](image-url)
Numerical values of the complete elliptic integrals $F$ and $E$ have been extensively tabulated. Otherwise, one may generate them with the required accuracy via Eqs. (A.5)-(A.8). In this way, the desired values of $I_5(x_o)$ and $I_6(x_o)$ can be obtained for given $x_o$ (or for given $k_F\xi_{pair}$ by inverting Eq. (2.30)). In Figs. 2-3 we report, respectively, the values of $\Delta/\epsilon_F$, $\mu/\epsilon_F$ for $\mu > 0$ and $\mu/(\epsilon_o/2)$ for $\mu < 0$, and $\epsilon_o/\epsilon_F$ vs $k_F\xi_{pair}$ obtained by this procedure. Within numerical accuracy, all values coincide with those calculated by solving numerically the gap equation and the normalization condition for the limiting case of a contact potential.

Besides the exact result (full curve), Figs. 2-3 show for comparison two additional curves obtained by approximating, respectively, the elliptic integrals $F$ and $E$ by the first two terms of Eqs. (2.25) and (2.30) (dotted curve) and by the first two terms of Eqs. (A.5) and (A.8) (dashed curve). In principle, these approximate results are expected to be reliable in the BE and BCS limits, in the order. Note, however, that the BE approximate result is surprisingly accurate on the BCS side of the crossover.

In the BCS and BE limits the values of $I_5(x_o)$ and $I_6(x_o)$ can be obtained by retaining only a few significant terms in the expansions (2.24)-(2.27). In particular, in the BCS limit $x_o \gg 1$ so that $\kappa^2 \approx 1 - 1/(4x_o^2)$ and

$$\begin{align*}
I_5(x_o) &\approx \sqrt{x_o} \\
I_6(x_o) &\approx \ln x_o/(2\sqrt{x_o}) .
\end{align*}$$

Then:

$$\begin{align*}
\Delta/\epsilon_F &\approx 1/x_o \\
\mu/\epsilon_F &\approx 1 \\
1/(k_Fa_s) &\approx -(2/\pi) \ln x_o \\
k_F\xi_{pair} &\approx x_o/\sqrt{2} .
\end{align*}$$

In the BE limit, on the other hand, $x_o < 0$ and $|x_o| \gg 1$ so that $\kappa^2 = 1/(4x_o^2)$ and

$$\begin{align*}
I_5(x_o) &\approx \pi/(16|x_o|^{3/2}) \\
I_6(x_o) &\approx \pi/(4|x_o|^{1/2}) .
\end{align*}$$

Then:

$$\begin{align*}
\Delta/\epsilon_F &\approx [16/(3\pi)]^{2/3}|x_o|^{1/3} \\
\mu/\epsilon_F &\approx -[16/(3\pi)]^{2/3}|x_o|^{4/3} \\
1/(k_Fa_s) &\approx [16/(3\pi)]^{1/3}|x_o|^{3/3} \\
\epsilon_o/\epsilon_F &\approx 2[16/(3\pi)]^{2/3}|x_o|^{4/3} \\
k_F\xi_{pair} &\approx 1/[16/(3\pi)]^{-1/3}|x_o|^{-2/3} .
\end{align*}$$

The limiting BCS (2.24) and BE (2.36) values coincide with those calculated previously by different methods.

We mention, finally, that another quantity which can be evaluated analytically at the mean-field level for all values of $x_o$ is the single-particle density of states.

All the above results hold for the three-dimensional system. Analogous results for the two-dimensional system are straightforwardly expressed in terms of elementary integrals and are reported for comparison in Appendix B.

III. ANALYTIC RESULTS AT THE GAUSSIAN LEVEL

Besides the quantities of Section II defined at the mean-field level, additional quantities whose definition...
requires the introduction of Gaussian fluctuations can also be expressed analytically in three dimensions at zero temperature for the Hamiltonian \( H \), in terms of the complete elliptic integrals \( \eta \) and \( \xi \), and the sound velocity \( s \) (associated with the Goldstone mode of the broken symmetry).

The matrix of the Gaussian fluctuations has elements

\[
\mathbf{\Gamma}(\mathbf{q}, \omega) = \begin{pmatrix} A(\mathbf{q}, \omega) & B(\mathbf{q}, \omega) \\ B(\mathbf{q}, \omega)^\star & A(-\mathbf{q}, -\omega) \end{pmatrix}
\]

where \( \omega \) is the frequency and (for a real order parameter)

\[
b_0 = a_0 \\
b_2 = \frac{1}{\Omega} \sum_k \frac{1}{32m} \left\{ -3\xi_k \frac{\Delta^2}{E_k^3} + \frac{k^2}{2} \frac{\Delta^2 (2\xi_k^2 - 3\Delta^2)}{E_k^3} \right\}
\]

\[
b_3 = \frac{1}{\Omega} \sum_k \frac{\Delta^2}{16E_k^6}
\]

(\( d \) being the dimensionality). In particular, to determine \( \xi_{\text{phase}} \) one has to consider the quantity \( A(\mathbf{q}, \omega = 0) + B(\mathbf{q}, \omega = 0) = 2a_0 + (a_2 + b_2)\mathbf{q}^2 + \ldots \)

(3.10)

from which

\[
\xi_{\text{phase}}^2 = \frac{a_2 + b_2}{2a_0}.
\]

(3.11)

To determine the sound velocity \( s \) one has instead to consider the full determinant

\[
A(\mathbf{q}, \omega) A(\mathbf{q}, -\omega) - B(\mathbf{q}, \omega)^2 = 2a_0(a_2 - b_2)\mathbf{q}^2 + [2a_0(a_3 - b_3) - a_1^2] \omega^2 + \ldots
\]

(3.12)

which vanishes for \( \omega = \omega(\mathbf{q}) = |\mathbf{q}| s \), with

\[
s^2 = \frac{2a_0(a_2 - b_2)}{2a_0(a_3 - b_3) + a_1^2}.
\]

(3.13)

We are left with evaluating the integrals entering Eqs. (3.11) and (3.13). In three dimensions we obtain:

\[
a_0 = \frac{m}{(2\pi)^2} (2m\Delta)^{1/2} I_5(x_o),
\]

(3.14)
Equation (3.11) thus reduces to

\[ a_1 = -\frac{m}{(2\pi)^2} \left( \frac{2m}{\Delta} \right)^{1/2} I_6(x_o), \]  

(3.15)

\[ a_2 + b_2 = \frac{1}{(2\pi)^2} \frac{1}{2} \left( \frac{2m}{\Delta} \right)^{1/2} \left[ 2 \int_0^\infty \frac{x^2}{E_x} dx - 2 \int_0^\infty \frac{x^2}{E_x^3} dx + \frac{10}{3} \int_0^\infty \frac{x^4}{E_x^5} dx \right], \]  

(3.16)

\[ a_2 - b_2 = \frac{1}{(2\pi)^2} \frac{1}{2} \left( \frac{2m}{\Delta} \right)^{1/2} \left[ I_6(x_o) + 2 \int_0^\infty \frac{dxx^2}{E_x^5} \right], \]  

(3.17)

\[ b_3 - a_3 = \frac{1}{(2\pi)^2} \frac{1}{2} \left( \frac{2m}{\Delta} \right)^{3/2} I_5(x_o), \]  

(3.18)

where use has been made of the notation (2.7) and of the integrals (2.20) and (2.21). The four new integrals appearing in Eqs. (3.16) and (3.17) can also be expressed as linear combinations of \( I_5(x_o) \) and \( I_6(x_o) \). Integrations by parts and simple manipulations lead to:

\[ \int_0^\infty \frac{x^2}{E_x} dx = \frac{1}{6} \int_0^\infty \frac{1}{E_x^3} dx = \frac{1}{6(1 + x_o^2)} \left[ \int_0^\infty \frac{x^4}{E_x} dx \right. \]

\[ \left. - 2 \int_0^\infty \frac{x^2}{E_x} dx + \int_0^\infty \frac{1}{E_x} dx \right] = x_o I_5(x_o) + I_6(x_o) \]

(3.19)

\[ \int_0^\infty \frac{x^2}{E_x} dx = I_6(x_o) - \int_0^\infty \frac{dxx^2}{E_x^3}; \]  

(3.20)

\[ \int_0^\infty \frac{x^4}{E_x} dx = \int_0^\infty \frac{x^2}{E_x} dx + x_o \int_0^\infty \frac{x^2}{E_x} dx \]

\[ = (1 + x_o^2) \left( \frac{1}{E_x} + \frac{x_o}{2} I_5(x_o) \right); \]  

(3.21)

\[ \int_0^\infty \frac{x^4}{E_x^2} dx = \frac{3}{10} \int_0^\infty \frac{x^2}{E_x} dx + \frac{1}{5} \int_0^\infty \frac{x^4}{E_x^5} dx. \]  

(3.22)

In conclusion we obtain:

\[ a_2 + b_2 = \frac{1}{(2\pi)^2} \frac{1}{2} \left( \frac{2m}{\Delta} \right)^{1/2} \]

\[ \times \left\{ 2 I_6(x_o) + \frac{(1 + 4x_o^2^2)}{3(1 + x_o^2)} [I_6(x_o) + x_o I_5(x_o)] \right\} \]  

(3.23)

and

\[ a_2 - b_2 = \frac{1}{(2\pi)^2} \frac{1}{2} \left( \frac{2m}{\Delta} \right)^{1/2} \left\{ I_6(x_o) + x_o I_5(x_o) \right\}. \]  

(3.24)

Equation (3.11) thus reduces to

\[ (k_F \xi_{\text{phase}})^2 = \frac{\epsilon_F}{\Delta} \frac{1}{12 I_5(x_o)} \]

\[ \times \left\{ 2 I_6(x_o) + \frac{(1 + 4x_o^2^2)}{3(1 + x_o^2)} [I_6(x_o) + x_o I_5(x_o)] \right\} \]  

(3.25)

with \( \epsilon_F / \Delta \) given by Eq. (2.27), while Eq. (3.13) becomes

\[ \left( \frac{s}{v_F} \right)^2 = \frac{1}{3} \frac{\Delta}{\epsilon_F} \frac{I_5(x_o)^2 + I_6(x_o)^2}{I_5(x_o)} \]  

(3.26)

where \( v_F = k_F / m. \)

The expressions (3.23) and (3.26) provide the desired analytic expressions of \( k_F \xi_{\text{phase}} \) and \( s/v_F \) for all values of \( x_o \), and again, that the BE approximation (dotted curve) is surprisingly accurate even on the BCS side.

To within numerical accuracy, we reproduce in this way the numerical results of Ref. [10] for \( k_F \xi_{\text{phase}} \). Note, again, that the BE approximation (dotted curve) is surprisingly accurate even on the BCS side.

In the BCS and BE limits one can use the approximate values (2.34) and (2.35), in the order, for the integrals \( I_5(x_o) \) and \( I_6(x_o) \). This gives:

\[ \begin{align*}
 k_F \xi_{\text{phase}} & \simeq x_o / 3 \\
 s/v_F & \simeq 1 / \sqrt{3}
\end{align*} \]  

(3.27)
in the BCS limit, and
\[
\begin{aligned}
k_F \xi_{\text{phase}} &\simeq (3\pi|x_o|/16)^{1/3} \\
s/v_F &\simeq (12\pi|x_o|)^{-1/3}
\end{aligned}
\tag{3.28}
\]
in the BE limit. Note that in the BE limit the product
\[
s \xi_{\text{phase}} = \frac{1}{m} \left(\frac{s}{v_F}\right) (k_F \xi_{\text{phase}}) \simeq \frac{1}{4m}
\tag{3.29}
\]
is a constant and coincides with the Bogoliubov result \((2m_B)^{-1}\) for composite bosons with mass \(m_B = 2m\), thus confirming the general results established in Ref. 10 for the mapping onto a bosonic system in the strong-coupling limit.

**FIG. 5.** \(s/v_F\) vs \(k_F \xi_{\text{pair}}\), obtained from Eqs. (2.31) and (3.26). Conventions are as in Fig. 4.

An additional quantity which can be evaluated analytically at the Gaussian level is the coefficient \(\gamma\) of the quartic term in the dispersion relation
\[
\omega(q)^2 = s^2q^2 + \gamma \left(\frac{q^2}{4m}\right)^2,
\tag{3.30}
\]
which is negative (and large), so that the dispersion relation
\[
\omega(q)^2 \simeq \frac{v_F^2q^2}{3} \left[1 - \frac{1}{3}q^2 \xi_{\text{pair}}^2\right]
\tag{3.32}
\]
holds for \(|q|\) smaller than a critical value \(q_c \propto 1/\xi_{\text{pair}}\). This implies that in the BCS limit the wavelength of the collective mode associated with the symmetry breaking cannot be smaller than the size of a Cooper pair.

The above results hold in three dimensions. In two dimensions both \(\xi_{\text{phase}}\) and \(s\) can be expressed in terms of elementary integrals, as shown in Appendix B.

**IV. CONCLUDING REMARKS**

In this paper we have provided the analytic solution of the crossover problem from BCS to BE in the three-dimensional case for a system of fermions interacting via an attractive contact interaction in free space, at the mean-field level and with the inclusion of Gaussian fluctuations. Although the assumptions required to obtain our analytic solution might be oversimplified in applications to realistic systems, it could be interesting yet to compare our analytic solution with numerical calculations describing more realistic cases. In particular, besides adopting a more sensible momentum-dependent form of the interaction potential, the free one-particle dispersion relation ought to be replaced by the actual band structure of the medium.

In addition, a detailed description of the crossover problem from BCS to BE would require one to introduce already at the mean-field level the coupling with the charge degrees of freedom, whose effects are expected to be especially important in the crossover region of interest, intermediate between these two limits.

In spite of these limitations, and considering the fact that analytic results for the crossover problem from BCS to BE were thus far limited to the two-dimensional case or to the two limits, our analytic solution is useful as it enables one to describe the crossover region in a compact way with very limited effort.

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**APPENDIX A: RELEVANT PROPERTIES OF ELLIPTIC INTEGRALS**

In this Appendix we briefly review the properties of elliptic integrals that are relevant to our treatment.

The elliptic integrals of the first and second kinds with modulus \(\kappa\) are defined by 14-17
\[
\begin{aligned}
F(\alpha, \kappa) &= \int_0^\alpha d\varphi \frac{1}{\sqrt{1 - \kappa^2 \sin^2 \varphi}} \tag{A1} \\
E(\alpha, \kappa) &= \int_0^\alpha d\varphi \sqrt{1 - \kappa^2 \sin^2 \varphi} \tag{A2}
\end{aligned}
\]
with $\kappa^2 < 1$. They satisfy the properties
\begin{align}
F(n\pi, \kappa) &= 2nF\left(\frac{\pi}{2}, \kappa\right) \quad \text{(A3)}
E(n\pi, \kappa) &= 2nE\left(\frac{\pi}{2}, \kappa\right) \quad \text{(A4)}
\end{align}

where $\kappa' = \sqrt{1 - \kappa^2}$ is known as the complementary modulus. The representations (A.5) and (A.6) are to be preferred when $\kappa^2 \ll 1$; when $\kappa^2 \simeq 1$ and $\kappa^2 \ll 1$ the representations (A.7) and (A.8) are to be preferred instead.

Equations (2.24) and (2.26) of the text are obtained by adapting tabulated results and using the properties (A.3) and (A.4). We obtain
\begin{align}
F\left(\frac{\pi}{2}, \kappa\right) &= \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 \kappa^2 + \left(\frac{1}{2} \cdot \frac{3}{4}\right)^2 \kappa^4 + \ldots + \left(\frac{2(2n-1)!!}{2^{2n}n!}\right)^2 \kappa^{2n} + \ldots \right\} \quad \text{(A.5)}
E\left(\frac{\pi}{2}, \kappa\right) &= \frac{\pi}{2} \left\{ 1 - \left(\frac{1}{2}\right)^2 \kappa^2 - \left(\frac{1}{2} \cdot \frac{3}{4}\right)^2 \kappa^4 - \ldots - \left(\frac{2(2n-1)!!}{2^{2n}n!}\right)^2 \kappa^{2n} + \ldots \right\} \quad \text{(A.6)}
F\left(\frac{\pi}{2}, \kappa\right) &= \ln \left(\frac{4}{\kappa}\right) + \left(\frac{1}{2}\right)^2 \left(\frac{2}{\kappa^2 - 1} - \frac{3}{4}\right) \kappa^2 + \left(\frac{3}{2} \cdot \frac{4}{4}\right) \left(\frac{2}{\kappa^2 - 1} - \frac{2}{3 - 4}\right) \kappa^4 + \ldots \quad \text{(A.7)}
E\left(\frac{\pi}{2}, \kappa\right) &= 1 + \frac{1}{2} \left(\ln \frac{4}{\kappa'} - \frac{1}{1 - 2}\right) \kappa^2 + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{4}{4} \left(\ln \frac{4}{\kappa'} - \frac{2}{1 - 2} - \frac{1}{3 - 4}\right) \kappa^4 + \ldots \quad \text{(A.8)}
\end{align}
in this paper, since it can be expressed in terms of elementary integrals. For the sake of comparison, we report in this Appendix the two-dimensional solution on equal footing of the three-dimensional solution discussed in the text.

Quite generally, in two dimensions the bound-state energy $\epsilon_o$ exists for any value of the interaction strength $\alpha$. For the contact potential we are considering, however, the bound-state equation
\begin{align}
-\frac{1}{g} &= \frac{1}{\Omega} \sum_k \frac{1}{k^2/2 + \epsilon_o} = \frac{m}{2\pi} \int_0^{\infty} dy \frac{1}{2y + \epsilon_o/\Delta} \quad \text{(B.1)}
\end{align}
(with $y = k^2/(2m\Delta)$) needs to be suitably regularized by introducing an ultraviolet cutoff. This cutoff can, in turn, be removed from further consideration by combining Eq. (B.1) with the gap equation (2.2), namely,
\begin{align}
-\frac{1}{2g} &= \frac{1}{2\Omega} \sum_k \frac{1}{\epsilon_o/\Delta + k^2} = \frac{m}{4\pi} \int_{-x_o}^{\infty} dz \frac{1}{(1 + z^2)^{1/2}} \quad \text{(B.2)}
\end{align}
(with $z = y - x_o$ and $x_o$ given by Eq. (2.7)) which requires an analogous regularization. Performing the elementary integrations in Eqs. (B.1) and (B.2) one obtains
\begin{align}
\frac{\epsilon_o}{\Delta} &= \sqrt{1 + x_o^2} - x_o \quad \text{(B.3)}
\end{align}
The normalization condition (2.3) gives further
\begin{align}
\epsilon_o &= \frac{m\Delta}{2\pi} \int_{-x_o}^{\infty} dz \left(1 - \frac{z}{\sqrt{1 + z^2}}\right) = \frac{m\Delta}{2\pi} \left(x_o + \sqrt{1 + x_o^2}\right) \quad \text{(B.4)}
\end{align}
Since in the normal state $n = k_F^2/(2\pi) = m\epsilon_F/\pi$, Eq. (B.4) reads
\begin{align}
\frac{\Delta}{\epsilon_F} &= \frac{2}{x_o + \sqrt{1 + x_o^2}} \quad \text{(B.5)}
\end{align}
Multiplying at this point both sides of Eqs. (B.3) and (B.5) yields

\[
\frac{\epsilon_o}{\epsilon_F} = 2 \sqrt{1 + x_o^2 - x_o} \quad .
\]  

(B.6)

Finally, the pair-correlation length can be obtained from its definition [cf. Eq. (2.13) for the three-dimensional case]:

\[
\xi^2_{\text{pair}} = \frac{1}{m^2} \frac{\int_0^\infty dk(k^2 \xi^2_k/E_k^0)}{\int_0^\infty dk(E_k)} = 2 \frac{m\Delta}{\int_0^\infty dy(y^2/1/E_y^2)}
\]  

(B.7)

with the dimensionless quantities

\[
\left\{ \begin{array}{c}
y = k^2/(2m\Delta) \quad , \quad x_o = \mu/\Delta \quad , \\
\xi_y = y - x_o \quad , \quad E_y = \sqrt{\xi_y^2 + 1}
\end{array} \right.
\]  

(B.8)

[in the place of (2.7) for the three-dimensional case]. The integrals in Eq. (B.7) are again elementary and give:

\[
(k_F \xi_{\text{pair}})^2 = \frac{1}{2} \frac{\epsilon_F}{\Delta} \left( x_o + \left( \frac{2 + x_o}{1 + x_o} \right) \left( \frac{\pi}{2} + \arctan x_o \right)^{-1} \right).
\]  

(B.9)

It is then clear that \(\Delta/\epsilon_F, \mu/\epsilon_F = x_o \Delta/\epsilon_F, \epsilon_o/\epsilon_F, \) and \(k_F \xi_{\text{pair}}\) can be expressed as functions of \(x_o\). [Note that no reference to the scattering amplitude \(a_o\) has been given in two dimensions].

Alternatively, Eq. (B.9) with \(\epsilon_F/\Delta\) given by Eq. (B.5) can be inverted to express \(x_o\) (as well as all other quantities) as a function of \(k_F \xi_{\text{pair}}\).

At the Gaussian level, all integrals entering the definitions (B.8) and (B.9) of the coefficients of the expansions (3.8) and (3.9) are elementary in two dimensions. We obtain:

\[
a_0 = \frac{m}{8\pi} \sqrt{1 + x_o^2 + x_o},
\]  

(B.10)

\[
a_1 = \frac{m}{8\pi} \frac{1}{\Delta \sqrt{1 + x_o}}
\]  

(B.11)

\[
a_2 + b_2 = \frac{1}{48\pi} \Delta \left( x_o + \frac{x_o^4 + 3x_o^2 + 1}{(1 + x_o^2)^{3/2}} \right)
\]  

(B.12)

\[
a_2 - b_2 = \frac{1}{16\pi} \Delta \left( x_o + x_o^2 + x_o \right)
\]  

(B.13)

\[
b_3 = \frac{m}{16\pi} \Delta^2 \left( x_o + x_o^2 + x_o \right).
\]  

(B.14)

These results give

\[
(k_F \xi_{\text{phase}})^2 = \frac{1}{6} \frac{\epsilon_F}{\Delta} \left( x_o + \frac{x_o^4 + 3x_o^2 + 1}{(1 + x_o^2)^{3/2}} \right)
\]  

(B.15)

and

\[
\frac{(s/v_F)}{2} = \frac{1}{4} \frac{\Delta}{\epsilon_F} \left( \sqrt{1 + x_o^2} \right)
\]  

(B.16)

Note that, owing to Eq. (B.5), \((s/v_F)^2 = 1/2\) is independent from \(x_o\). On the other hand, for \(k_F \xi_{\text{phase}}\) we obtain in the two limits:

\[
k_F \xi_{\text{phase}} \approx \left\{ \begin{array}{c} x_o \sqrt{\frac{6}{\pi}} \quad \text{BCS limit} \\
1/\sqrt{3} \quad \text{BE limit}
\end{array} \right.
\]  

(B.17)

thus confirming that the BE limit depends markedly on dimensionality and shows a peculiar behavior in two dimensions. [Note, however, that in the BE limit the product \(s \xi_{\text{phase}}\) still coincides with the Bogoliubov result \((4m)^{-1}\).

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