Interaction Maxima in Distributed Systems

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Abstract
In this paper we study the maximum degree of interaction which may emerge in distributed systems. It is assumed that a distributed system is represented by a graph of nodes interacting over edges. Each node has some amount of data. The intensity of interaction over an edge is proportional to the product of the amounts of data in each node at either end of the edge. The maximum sum of interactions over the edges is searched for. This model can be extended to other interacting entities. For bipartite graphs and odd-length cycles we prove that the greatest degree of interaction emerge when the whole data is concentrated in an arbitrary pair of neighbors. Equal partitioning of the load is shown to be optimum for complete graphs. Finally, we show that in general graphs for maximum interaction the data should be distributed equally between the nodes of the largest clique in the graph. We also present in this context a result of Motzkin and Straus from 1965 for the maximal interaction objective.

Keywords: distributed systems, interaction maxima, quadratic programming, maximum clique.

1 Introduction
As technology advances, the amount of data humanity generates, stores and processes is increasing at rapid rates. One projection is for a global data sphere of 175 zettabytes (175 x 10^21 bytes) by 2025 [1]. Global mobile data traffic according to another projection [2] is expected to be 77 exabytes per month in 2022. Hence, means are needed for understanding the operation and interaction of communicating entities and data storage in networked systems in this context. Mathematical performance evaluation offers a method to accomplish this and had much success in the past and current environment going back to Erlang’s sizing of telephone systems using queueing theory. Some applications appear natural for the mathematical model discussed in this paper.

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In the first application consider networked systems where a network is represented mathematically by a graph and the \(i\)th node contains a memory of size \(m_i\). It is useful to assess the total amount of communication in order to size the communication infrastructure for this traffic. We make an assumption that two adjacent nodes interact and have an interaction activity proportional to the product of the sizes of their memories, \(m_i m_j\) for the \(i\)th and \(j\)th node. More specifically, the two-way traffic on the link connecting the two adjacent memories is \(cm_i m_j\) where \(m_i\) have units of [bits] and \(c\) is a proportionality constant with units of [1/bits×sec]]. This assumption makes some sense if one thinks that the larger a node’s memory the more likely is its node to query other nodes and be queried by other nodes. There is an implicit assumption here that nodal memories are fully utilized (a very large memory with no data would not show much activity). Overall, in a sense we are considering an intuitive model of a large database system emphasizing its interacting component memories.

The second application involves telecommunications between groups of communicating entities (people and/or machines) modeled as nodes (see a similar idea in [7]). The communication could be implemented physically in terms of a wired Internet or by cellular wireless communications. Using the same mathematical model as above, the network is represented by a graph where the \(i\)th node contains a population of communicating entities of size \(m_i\). Since members of the population both produce and receive information, we assume that two adjacent nodes’ populations interact and have a communication intensity proportional to the product of the size of their populations \(m_i m_j\) for the \(i\)th and \(j\)th nodes. Again, the two-way traffic on the link connecting the two adjacent populations is \(cm_i m_j\) where \(c\) is a proportionality constant with units of [bits/(entities \(^2 \times \) sec)] and the \(m_i\) have units of [entities]. It is practical to assess total communication flow created by the interacting entities to estimate their stress on the communication system and verify its suitability for such kind of interaction. The second model also makes some sense if one believes that the larger a node’s population of communicating entities, the more likely the node is to have communication with entities in other nodes. There is an implicit assumption here that each nodal population has the same level (percentage) of activity as that of other nodes. These are reasonable assumptions for a first study of this problem.

These two examples do not preclude other application areas. Such product models arise in physical systems involving charge or gravity, e.g., in N-body simulations. Intensity of interaction in scatter/gather and gossiping communications in computer networks can be modeled similarly. In the social sciences and economics one could see the nodal quantities, being influence or money, mathematically along the same lines as above. In this context there are interesting optimization problems which are presented below.
2 Problem formulation

For conciseness of presentation we will be referring to memory/population size in some node as the node load. We will refer to flow as the amount of interaction. The problem considered in this paper is formulated as follows:

**Data:** A connected graph $G = (V, E)$ with $n$ vertices (nodes) is given. Set $E$ of edges is defined by connectivity matrix: $l_{ij} = 1$ if nodes $i$ and $j$ are connected, and 0 otherwise, for $i, j = 1, \ldots, n$. Vertex loads (memory/population) sizes are $m_j \geq 0$, $j = 1, \ldots, n$. Loads of all vertices can be represented as a vector $\vec{m} = [m_1, \ldots, m_n]^T$. Let $D$ denote the total load volume in $G$, and let edge flows be defined as: $f_{ij} = c m_i m_j$ for $i, j = 1, \ldots, n$ where $c$ is the flow factor used for a conversion to bits per second. For brevity of exposition, we will assume that $c = 1$.

**Problem:**

$$\max_{\vec{m}=[m_1,\ldots,m_n]} \mathcal{F}(\vec{m}) = \sum_{i=1}^n \sum_{j=1}^n f_{ij} = \sum_{i=1}^n \sum_{j=1}^n c l_{ij} m_i m_j \quad (1)$$

s.t.:

$$\sum_{i=1}^n m_i = D \quad (2)$$

$$m_i \geq 0 \quad i = 1, \ldots, n \quad (3)$$

The goal here is to maximize the sum of the edge flows. This value provides useful information on sizing bandwidth of communication links over which the interaction is conducted, as maximum of $\mathcal{F}(\vec{m})$ is the worst case (upper bound) for the flow. The problem is non-trivial because of the constraints (2) and (3). More specifically, the problem we discuss is a quadratic optimization problem with inequality constraints. There are numerous applications of such problems to areas such as portfolio optimization, signal and image processing, least squares approximation and estimation and control theory. There is much work on the numerical solution of quadratic optimization problems. Numerical techniques that can be applied to this problem include active set and interior point methods. The website at [5] lists approximately thirty different solution codes. However, in this paper we are interested more in what can be found analytically. A quadratic programming formulation with inequalities is an NP-hard problem in general.

3 Single-Level Tree

Let 1 be the index of the root node, and 2, \ldots, $n$ the indices of the leaf nodes. By using (2) problem (1)-(3) can be re-stated as follows:

$$\max_{m_1,\ldots,m_n} \mathcal{F}(\vec{m}) = \sum_{j=2}^n f_{1j} \quad (4)$$
Thus, for a single-level tree $\mathcal{F}(\vec{m})$ is actually $\mathcal{F}(m_1)$, i.e. a function of one variable $m_1$. By (5) $\mathcal{F}(m_1)$ has maximum of $D^2/4$ at $m_1 = D/2$, $\sum^n_{i=2} m_i = D/2$. It is immaterial how the load is split between leaf nodes as long as together they have load $D/2$ in total. For maximum of $\mathcal{F}(\vec{m})$ the load equally well can be split in halves between the root and one leaf node.

4 Bipartite Graphs

In this section we show that any load distribution in a bipartite graph can be reduced to the single-level tree distribution without decreasing the flow objective (1). Actually, a bipartite graph can be reduced to a special case of a single-level tree: to just one edge. In this sense we say that bipartite graphs are equivalent to single-level trees. Note that bipartite graphs include trees, even-length cycles, and more generally, all graphs without odd-length cycles. Assume a bipartite graph $G(X,Y,E)$ is given, where $X,Y$ are the two parts of vertex set and $E$ is the set of edges.

**Theorem 1** Any load distribution in a bipartite graph can be reduced to a single-level tree distribution without decreasing the flow objective (1).

**Proof.** Consider a bipartite graph $G(X,Y,E)$. Let $\{x,y\} \in E$ denote an edge. Without loss of generality $x \in X, y \in Y$. The flow in $G$ is

$$\mathcal{F} = \sum_{\{x,y\} \in E} cm_x m_y \leq \sum_{x \in X} cm_x \left( \sum_{y \in Y} m_y \right) = c \left( \sum_{x \in X} m_x \right) \left( \sum_{y \in Y} m_y \right)$$

(6)

Thus, by inequality (6) load of all vertices in set $X$ can be moved to a single super-vertex $x'$ with load $\sum_{x \in X} m_x$ and of all vertices in set $Y$ to a single super-vertex $y'$ with load $\sum_{y \in Y} m_y$ connected by one edge without decreasing objective $\mathcal{F}$. As the super-vertices $x', y'$ any vertices can be chosen such that $x' \in X, y' \in Y, \{x', y'\} \in E$. \qed

Note that proof of Theorem 1 defines a load distribution transformation from any distribution in a bipartite graph to a load distribution on a single edge. In this load distribution transformation flow $\mathcal{F}$ is nondecreasing. But this transformation does not immediately construct optimal load partitioning. For optimality of the partitioning the load size in super-vertex $x'$ must be equal to the load size in super-vertex $y'$ as shown in Section 3.

4
5 Odd-Length Cycles

We will show that any load distribution in an odd-length cycle with \( n \geq 5 \) nodes can be reduced to a load distribution in a single-level tree, or even, a distribution over an edge.

**Theorem 2** Any load distribution in an odd-length cycle of length at least 5 can be reduced to a single-level tree distribution without decreasing the flow objective \( \mathcal{F} \).

**Proof.** Suppose that the load distribution is equal, i.e. \( m_1 = \ldots = m_n = D/n \). The total flow for equal load distribution is \( \mathcal{F}_{eq} = n \times (D/n)^2 = D^2/n \). However, equal distribution of the load between two neighboring nodes results in flow \( \mathcal{F}' = (D/2) \times (D/2) = D^2/4 \). Flow value \( \mathcal{F}' \) is bigger than \( \mathcal{F}_{eq} \) for all \( n \geq 5 \). Thus, equal load partitioning in odd-size cycles can be transformed to equal load distribution over a single edge without decreasing the value of the flow objective.

Suppose that the load distribution is not equal, i.e. \( m_i \) are not all simultaneously equal to the same value. Consider some node \( i \) and its neighbors with indices \( i^- = (i - 1) \mod n \) and \( i^+ = (i + 1) \mod n \). Let \( \phi(i) \) denote the sum of loads in neighbors of node \( i \). Node \( i \) together with its neighbors contribute to the flow value \( m_i (m_i^- + m_i^+) = m_i \phi(i) \). Since load distribution is not equal, nodes in the cycle can be ordered according to the increasing values of \( \phi \). Let us assume that node \( i \) has the smallest \( \phi(i) \) and node \( j \) has the largest \( \phi(j) \) in the cycle. If the set of \( i \) and \( j \) neighbors are disjoint (Fig.1a), i.e. \( \{i^-, i, i^+\} \cap \{j^-, j, j^+\} = \emptyset \) then by shifting the whole load of \( m_i \) to node \( j \) we lose flow value \( m_i \phi(i) = m_i (m_i^- + m_i^+) \), but we gain \( m_j (m_j^- + m_j^+) = m_j \phi(j) \). Overall we gain in this load shift because load distribution is unequal and \( \phi(i) \) is the smallest in the cycle.

Suppose \( |\{i^-, i, i^+\} \cap \{j^-, j, j^+\}| = 1 \), that is the set of \( i \) and \( j \) neighbors overlap in one node. But since \( \phi(i) \leq \phi(j) \), we do not lose on the value of flow by shifting whole load \( m_i \) to \( j \). Finally, suppose \( |\{i^-, i, i^+\} \cap \{j^-, j, j^+\}| = 2 \) (Fig.1b) which means that \( i \) is a direct neighbor of \( j \) and vice versa. Without loss of generality let us assume that \( j^+ \) is not a neighbor of \( i \) and...
$i^-$ is not a neighbor of $j$. Let $j^{++}$ be the neighbor of $j^+$ which is not $j$. Node $j^{++}$ exists because $n \geq 5$. By shifting whole load $m_i$ to $j^+$ we lose flow value $m_i \phi(i) = m_i(j^- + j^+) = m_i(j^+ - m_j)$, but we gain flow $m_i(m_j + m_{j^{++}}) = m_i \phi(j^{++})$. Overall, we gain as a result of this load shift because $\phi(i) \leq \phi(j^{++})$ (because $\phi(i)$ is the smallest in the cycle). The three cases of shifting load from $i$ with the smallest $\phi(i)$ result in load distribution with new $m_i = 0$. Hence, the load distribution is as in a chain, which is a bipartite graph. The load distribution can be further transformed to a single-level tree distribution as in Theorem 1 without decreasing value of flow $F$.

6 Clique

We will show by induction that equal partitioning is the optimum distribution of the load in a clique (a complete graph).

**Theorem 3** In a clique of size $n$ the flow is maximum if $m_i = D/n$, for $i = 1, \ldots, n$ and the value of the flow is $F = (D/n)^2 \times \binom{n}{2}$.

**Proof.** Note that this theorem holds for $n = 2$ (one edge) because $\binom{2}{2} = 1$ and by the result in Section 3. Let $OPT(x, y)$ denote the optimum value of the flow in a clique with $x$ nodes and volume of load $y$. Suppose that the theorem holds for $n \geq 2$ then it also holds for $n + 1$. The flow is

$$F = m_{n+1} \sum_{i=1}^{n} m_i + OPT(n, D - m_{n+1}) = \left(m_{n+1}(D - m_{n+1}) + OPT(n, D - m_{n+1})\right)$$

The first component in (8) corresponds to the flow between node $n + 1$ and the remaining part of the clique. The second component is the flow for the optimum distribution of the load of the remaining size $D - m_{n+1}$ in a clique with $n$ nodes. By the assumption of the inductive proof

$$OPT(n, D - m_{n+1}) = \frac{(D - m_{n+1})^2}{n^2} \times \binom{n}{2}$$

Let

$$\lambda = \frac{n}{n^2} = \frac{n!}{2!(n-2)!n^2} = \frac{(n-1)n}{2!n^2} = \frac{n-1}{2n}$$

Equation (8) can be rewritten as:

$$F = m_{n+1}(D - m_{n+1}) + (D - m_{n+1})^{2}\lambda = -(1 - \lambda)m_{n+1}^2 + D(1 - 2\lambda)m_{n+1} + D^2\lambda.$$
This is a quadratic function of \( m_{n+1} \) with a maximum at

\[
m^*_n = \frac{1 - 2\lambda}{2 - 2\lambda} D = \frac{1 - \frac{2(n-1)}{2n}}{2 - \frac{2(n-1)}{2n}} D = \frac{n - (n - 1)}{2n - (n - 1)} D = \frac{D}{n + 1}
\]

(13)

The load assignments of the other nodes, by the assumption of the inductive proof are

\[
m^*_i = \frac{D - D/(n + 1)}{n} = \frac{(n + 1)D - D}{n(n + 1)} = \frac{D}{n + 1}
\]

(14)

By (8), (9), (10) the optimum value of the objective function is:

\[
F = \frac{D}{n + 1} \frac{nD}{n + 1} + \frac{D^2 n^2}{(n + 1)^2} \frac{(n - 1)}{2n} = \frac{D^2}{(n + 1)^2} \left[ n + \frac{n(n - 1)}{2} \right] = \frac{D^2}{(n + 1)^2} \left( \frac{n + 1}{2} \right)
\]

(16)

Thus, from the assumption that the theorem holds for \( n \geq 2 \) it follows that it holds also for any \( n \).

7 General graphs

In Sections 4, 5 we have shown a set of load distribution transformations reducing the size (i.e., order) of the subgraph with non-zero node loads without decreasing value of the flow \( F \). In this section we generalize this approach and analyze the smallest subgraph which allows the value of flow \( F \) that is not smaller (but may be larger) than the initial one. Thus, if the initial load distribution was optimum with respect to \( F \), then also after the transformations we propose, the value of the flow remains optimum. Consequently, characterizing such a smallest subgraph also determines the optimum solutions. We will show that the smallest subgraph remaining after a sequence of flow-preserving transformations is the largest clique in the graph. This will be achieved by generalizing the transformation used in the proof of Theorem 2.

**Theorem 4** Flow \( F \) is maximum if the load is distributed equally between nodes of the largest clique in the graph.

**Proof.** Consider two nodes \( i, j \) of graph \( G \) and their direct neighborhoods \( N(i), N(j) \), respectively. Direct neighborhood of some nodes \( a, b \) means that there is edge \( \{a, b\} \in E \). Let \( \phi(a) \) denote the sum of the loads in neighbors
of some node \(a\). The contributions of nodes \(i, j\) to the value of flow are \(m_i \phi(i), m_j \phi(j)\), respectively. Suppose \(i \notin N(j), j \notin N(i)\). Without loss of generality we can assume that \(\phi(i) \leq \phi(j)\). Then, it is possible to transfer the whole load \(m_i\) to node \(j\) without decreasing the value of flow \(F\) (the value of flow may increase). As an additional effect of this transformation the size of the subgraph with nodes \(i\) holding \(m_i > 0\) decreases. This procedure may be continued as long as there are nodes \(i, j\) satisfying \(i \notin N(j), j \notin N(i)\). The procedure stops when all pairs of nodes are direct neighbors. Consequently, the load will be collected in one or more disconnected cliques in \(G\). By the lack of connectivity we mean here that there is no path connecting the cliques over nodes \(l\) with \(m_l > 0\). Suppose there are two cliques \(K_a\) and \(K_b\) in \(G\) remaining after the above sequence of load shifts and they hold loads \(x, y \geq 0\), respectively. \(K_a\) has size \(r\), \(K_b\) has size \(s\), and \(r \geq s\). According to Theorem 3 the maximum flows in \(K_a\) is \(x^2/r^2 \times \binom{r}{2} = x^2(1 - \frac{1}{r})/2\). Analogously, the maximum flows in \(K_b\) is \(y^2(1 - \frac{1}{s})/2\). Collecting loads \(x\) and \(y\) in the largest clique \(K_s\) results in flow \((x + y)^2(1 - \frac{1}{s})/2 \geq x^2(1 - \frac{1}{r})/2, (x + y)^2(1 - \frac{1}{s})/2 \geq y^2(1 - \frac{1}{s})/2\) and \((x + y)^2(1 - \frac{1}{s})/2 \geq (x + y)^2(1 - \frac{1}{s})/2\). Hence, concentrating the load in the largest clique in \(G\) results in the optimum flow \(F\). 

Let us observe that in the above proof we exploited property \(\phi(i) \leq \phi(j) \Rightarrow m_i \phi(i) + m_j \phi(j) \leq (m_i + m_j) \phi(j)\) of the flow function. Thus, the above proof can be extended to other forms of the flow objective which are additive functions of nodal flows \(f\) with the property: \(\phi(i) \leq \phi(j) \Rightarrow f(m_i, \phi(i)) + f(m_j, \phi(j)) \leq f(m_i + m_j, \phi(j))\).

8 Motzkin and Straus result

In 1965 article \[8\] T. S. Motzkin and E. G. Straus solved a problem suggested in 1963 by J.E. MacDonald \[6\]. Their result, expressed in terms of our paper, states that given weights \(m_1, \ldots, m_n\) on \(n\) vertices of graph \(G\), function \(F\) in equation \[4\] has maximum \(\frac{D^2}{2}(1 - \frac{1}{\omega(G)})\) when node loads are \(m_1 = \ldots = m_n = D/\omega(G)\), where \(\omega(G)\) is the size of the largest clique in \(G\). Their proof used the idea of maximization of flow \(F\) over an \(n\)-dimensional simplex on nodes of \(G\). Our proof of Theorem 4 provides an analogous result albeit on a different premise. Note that both Motzkin and Straus result and our Theorem 4 hinge on finding largest clique in a graph which is \(\mathbf{NP}\)-hard \[3\], but still, has many applications \[2\].

9 Conclusions

It is both timely and very interesting to see what is and isn’t possible with the use of \(m_i m_j\) modeling of interaction in this problem. We have related
such modeling to:

- a large number of application areas that are likely to be important both in the future as well as in the present,
- demonstrating simple solutions for maximal flow in single-level tree, bipartite graphs and clique interconnection networks,
- our Theorem 4 and the Motzkin-Straus result provide an elegant graph theoretic solution for this problem.

Furthermore, if the gain in forming a social group can be modeled as in our multiplicative model, then social implications, e.g. in emerging of communities, can be considered. The load, data, power, money can be concentrated in the largest clique $K$, a tightly connected subgraph of $G$, for maximum flow $F$. The remaining subgraph $G \setminus K$ is immaterial. Still, there may be practical reasons for using a non-optimal flow, such as resiliency. Thus, the multiplicative interaction model is an intriguing one which could be of good utility in future and current applications.

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