RECIProCAL SCHRÖDINGER EQUATION: DURATIONS OF DELAY AND OF FINAL STATES FORMATION IN PROCESSES OF SCATTERING

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Abstract

The reciprocal Schrödinger equation \( \partial S(\omega, r)/i\partial \omega = \hat{\tau}(\omega, r) S(\omega, r) \) for \( S \)-matrix with temporal operator instead the Hamiltonian is established via the Legendre transformation of classical action function. Corresponding temporal functions are expressed via propagators of interacting fields. Their real parts \( \tau_1 \) are equivalent to the Wigner-Smith delay durations at process of scattering and imaginary parts \( \tau_2 \) express the duration of final states formation (dressing). As an apparent example, they can be clearly interpreted in the oscillator model via polarization \( \tau_1 \) and conductivity \( \tau_2 \) of medium. The \( \tau \)-functions are interconnected by the dispersion relations of Kramers-Kröning type. From them follows, in particular, that \( \tau_2 \) is twice bigger than the uncertainty value and thereby is measurable; it must be negative at some tunnel transitions and thus can explain the observed superluminal transfer of excitations at near field intervals (M.E.Perel’man. In: arXiv. physics, Gen.Phys/0309123). The covariant generalizations of reciprocal equation clarifies the adiabatic hypothesis of scattering theory as the requirement: \( \tau_2 \to 0 \) at infinity future and elucidate the physical sense of some renormalization procedures.
I. INTRODUCTION

In the classical physics are used different "spaces" or sets of variables, best suited to consideration of different problems. For these aims are formulated the methods of transition between corresponding forms of dynamical equations, performing by Legendre transformations of action functions, and corresponding transformations in thermodynamics. In the quantum theory for similar aims are using expansions over complete sets of suitable functions: exponents, i.e. the Fourier transformation from \((t, r)\) to \((E, p)\) variables and vice versa, the Legendre functions, i.e. transition to moment variables, etc.

However, as far as we know, in quantum theory the possibilities of direct Legendre transformations, which must lead to reciprocal equations, are not considered. It can be partly attributable to the distinction with classic fields, where all equations are directly expressed via action functions, but equations of quantum dynamics are expressed via exponents of these fundamental magnitudes, which causes some complications at deducing the reciprocal expressions.

Are needed such reciprocal expressions or not, can they lead to some new physical results, or they will represent only a methodological interest? It is a crucial question and we shall demonstrate that the reciprocal Schrödinger equation permits, at least, to understand and investigate the duration of interactions. Such approach does not exclude, of course, possibility of performing these investigation by more common methods, but shows some prospects of developing theory.

The consideration of several problems of durations of interaction have aroused a lot of research and their discussions are continuing. For introduction into these problems we give overview of the status of some duration presentations [1-17]. As will be shown, the joined consideration of two magnitudes, the duration of delay at scattering processes and the duration of final states formation, their "dressing", essentially simplify problems. These two magnitudes, usually examined separately, are unified by the reciprocal Schrödinger equation and just this circumstance clarifies the necessity of proposed Legendre-type transformation
consideration.

The notions of delay (waiting time) in the process of signal transfer and of duration, needed for the final form restoration, were naturally perceived in classical physics, e.g. for oscillating systems; thus they were experimentally and theoretically investigated in the electrical engineering and in the acoustics, e.g. [1]. But in quantum theory during long time there was actually implied that all problems, connected with particles and states formation and with coupling of particles during scattering processes, should be restricted only and only by the frameworks of uncertainty principles.

The first (semi-qualitative) consideration of time delay in processes of tunneling had been performed, as far as we know, by Bohm [2]. The more constructive and physically more transparent notion of time delay under the elastic scattering was introduced by Wigner [3] through the derivative of partial phase shifts: \( \tau_l(\omega) = d\delta_l/d\omega \). This expression was generalized by Smith [4] via \( S \)-matrix of scattering:

\[
\tau = \text{Re} \frac{\partial}{\partial \omega} \ln S. \quad (1.1)
\]

On the base of (1.1) Goldberger and Watson derived a "coarse-grain" Schrödinger equation [5], it shown the generality of this magnitude. But at their approach the magnitude (1.1) had been introduced artificially, by decomposition of the logarithm of Fourier transformed response function \( R(t) \) of linear relation

\[
O(t) = R(t) \otimes I(t) \equiv \int dt R(t-t') I(t') \quad (1.2)
\]

near the selected frequency, without discussion of its imaginary part, higher terms and dependence on space variables.

In the case of photonic processes it is intuitively evident, that the delay time is the duration between absorption of single photon and its reemission or vice verse by a bound or free electron (this time duration coincidences with (1.1) and will be denoted as \( \tau_1 \)). It should be underlined that this time duration could be deduced in the course of QED calculations. So after summarizing of the complete sequence of the main \( S \)-matrix terms for some multiphoton processes the evaluated infinite series may be reformulated via the parameter

\[
j/j_0 = j\sigma_\text{tot}\tau_1 \text{ with } \tau_1,
\]

the flux density \( j \) and the total cross-section \( \sigma_\text{tot} \) of single
γ-e scattering, and this naturally appearing parameter determines the thresholds of some multiphoton processes saturation and of the new channels opening [6, 7]. It demonstrates that the duration time expressions are implicitly contained in the QED and therefore the corresponding magnitudes should be recognized in the common theory.

Another approach, which seems at first glance distinctive from the Wigner-Smith one, was suggested by Baz' [8] for the consideration of nonrelativistic tunneling processes. This approach consists in attributing to the scattering particle some moment, e.g. magnetic, and in analyzing its turns at the scattering process (the method of ”Larmor clocks”), some its variants are reviewed, in particular, in [9] and in several articles in [10]. This method will be very shortly discussed later with demonstration of its principal identity with the Wigner-Smith approach.

But even earlier Frank had been forced to introduce in the theory of Čerenkov radiation the notion of path length (or duration), necessary for the extended formation of real photon by ”superluminal electron in media” [11]. Without such concept was completely incomprehensible the discrete character of this emission, and Frank had been forced to consider the interference picture of continuously emitted (virtual) waves, which leads to the real emission at resonance conditions.

Then, independently, Ter-Mikaelyan [12] and Landau and Pomeranchuk [13] had considered the duration of photon formation in the theory of bremsstrahlung: it is the time duration needed for a virtual coat formation around particle, its dressing (cf. the reviews [14]). But as must be underlined, the concept of particle or state formation has more general sense: so Moshinsky had calculated, through the non-stationary Schrödinger equation, the duration necessary for establishment of the certain state of electron after its transition onto the upper level [15], and it is also the duration of final state formation.

Notice that the time duration of final state formation can be, in principle, naturally measurable in the processes of multiphoton ionization [7, 16]: it must be such extended period of time, during which the photoelectron, already absorbing enough energy for liberation, is yet in the virtual state, retains its association with the atom/ion till gain for the
moment corresponding to the absorbed energy. In such virtual state electron can absorb additional, above threshold photons. Thus the multiphoton processes, except some cases of high harmonics generations, require the correlation of two independent, generally speaking, time durations: the duration of energy absorption depending on the photon flux density and so on, and the duration of corresponding moment accumulation, depending on interaction with surrounded particles and fields.

Therefore multiphoton processes seem exceptionally interesting for all quantum theory: in these processes are examined the concept of virtual coats of particles and the dynamics of their formation.

All results cited above, which were established in various and, we think, artificial methods, can be calculated at the unified and simplified way by the expression

$$\tau_2 = \text{Im} \frac{\partial}{\partial \omega} \ln S$$

through the response function or matrix element of transition [15].

This expression may be formally considered as the analytically complementary to the Smith’s formulae (1.1). Since (1.3) can be rewritten as $$\tau_2 = \frac{\partial \ln |S|}{\partial \omega}$$, it can be considered as the measure of incompleteness of the final state or of the outgoing dressed particle formation. As far as we know, the similar expression for $$\tau_2$$ was introduced, for the first time, by Pollak and Miller [17] and was interpreted as the duration of tunneling process.

The main purpose of the paper consists in some simultaneous refinement of both notions, of delay and of formation of final state, and in the revealing of their place in the common scattering and general field theories. Thereby the investigation of several approaches to revealing a latent, as though, existence of temporal expressions or their equivalents in common theories is needed.

The most natural way in this direction can begin, as represented, with the more general formulation of considered magnitudes. So both definitions (1.1) and (1.3) can be formally combined as $$\tau(\omega, r) \equiv \tau_1 + i \tau_2 = \frac{\partial}{\partial \omega} \ln S(\omega, r)$$. Then it must be attempted to consider this expression as the consequence of some equation,
\[
\frac{\partial}{\partial \omega} S(\omega, \mathbf{r}) = \tau(\omega, \mathbf{r}) S(\omega, \mathbf{r}). \quad (1.4)
\]

Formally this relation seems analogical to the Schrödinger equation for $S$-matrix, but rewritten via some transformation of $x_\mu \longleftrightarrow p_\mu$ type and with a some ”temporal” operator \(\tau(\omega, \mathbf{r})\) instead the Hamiltonian.

The proposed transition to new variables can be performed by the Legendre transformation in classical theory, at least. On the other hand it can be performed by the Fourier transformation of response function in (1.2), and in the Section 2 both approaches are examined: they lead to approximately identical results.

The main properties of unified temporal functions \(\tau(\omega, \mathbf{r})\), their simplest interpretation and their interrelations with the uncertainty magnitudes are considered in the Section 3. As these functions are causal, for them can be established the certain dispersion relations and corresponding sum rules (Section 4), that demonstrate some principal properties of temporal functions. The received results are discussed in the Section 5 on the example of the simplest oscillator model of medium; it descriptively reveals the physical sense of both temporal functions.

As the temporal parameters can be considered as the results of interference of waves, coming from different points, it seems that the suitting functions for their comparative investigation should be the Wigner functions (Section 6). Their consideration shows that the expressions of temporal functions are close to propagators (resolvents or Green functions), and it will be proven in the Section 7 in the scope of formal theory of scattering.

In the Section 8 temporal functions and their covariant forms will be considered by the methods of quantum field theory, and it will be proven that the equations of (1.4)-type can be generalized till completely covariant analogues of the Tomonaga - Schwinger equations. Therefore it will be shown that the developed theory can be considered as the justification for the adiabatic hypothesis of quantum theory of interactions and as its generalization; it permits to understand the physical sense of such formal, as usually seems, mathematical procedure.
The Section 9 is devoted to some problems of QED. Their considerations are continued in the Section 10 by interpretation of renormalization procedures, the Pauli-Villars and the subtraction methods, and, more generally, the renormalization group equations via the temporal functions.

In the Conclusions are summed up the main results and are contemplated some perspectives of further investigations.

2. LEGENDRE TRANSFORMATIONS AND FOURIER TRANSFORMATION

The basic equation of quantum dynamics for the evolution operator (S-matrix),

\[ i \frac{\partial}{\partial t} S = H S, \quad (2.1) \]

can be formally deduced from the Hamilton-Jacobi equation for classical action function and Hamiltonian,

\[ \left( \frac{\partial}{\partial t} \right) S_{cl}(q_i; \partial S_{cl}/\partial q_i; t) = H_{cl}(q_i; \partial S_{cl}/\partial q_i; t). \quad (2.2) \]

For such transition is used the Schrödinger-type heuristic substitution:

\[ S_{cl} \rightarrow i \hbar \ln \left\{ \frac{S(t, r)}{\hbar} \right\} \quad (2.3) \]

with the determination: \( d \ln S \equiv (dS) S^{-1} \) (below \( c = \hbar = 1 \)) and classical variables \( x, p \) are replaced by corresponding operators.

The transition to new variables in the classical action function is realized by the Legendre transformation (e.g. [18]):

\[ S_{cl}(q; p; t) = S_{cl}(q; p; t) - \sum (qtq + pt + ttk). \quad (2.4) \]

Thus the canonical transformation from the time variable \( t \) to the variable of energy, \( t \rightarrow t' = H \rightarrow E \), in the equation (2.2) results in

\[ S_{cl}(t; ... - \hbar t = S_{cl}^H(E; ...), \quad (2.5) \]

and the canonical equation (2.2) is transformed into the temporal Hamilton-Jacobi equation:

\[ \left( \frac{\partial}{\partial E} \right) S_{cl}^H(E; ...) = -T_{cl}(E; ...), \quad (2.6) \]

in which the role of Hamiltonian plays (classical) function of duration of considered
process. It leads to classical temporal Hamilton equations and so on.

The employment of the analog of Schrödinger-type substitution (2.3),

$S^L_c(E; ... ) \rightarrow i \ln S^L(E, \mathbf{r})$, \hspace{1cm} (2.7)

leads to the quantum equation (we use more familiar for such notations symbol $\omega$ instead $E$):

$$\frac{\partial}{\partial \omega} S^L(\omega, \mathbf{r}) = \tau^L(\omega, \mathbf{r}) S^L(\omega, \mathbf{r}),$$ \hspace{1cm} (2.6')

from which follows the determination of temporal function in accordance with the Legendre transformation:

$$\tau^L(\omega, \mathbf{r}) = \frac{\partial}{\partial \omega} \ln S^L(\omega, \mathbf{r}).$$ \hspace{1cm} (2.8)

The Legendre transformation can be performed at nonzero values of the Hessian, i.e. the determinant consisted from second derivatives:

$$J(\tau \rightarrow \omega) = \left( \frac{\partial \tau \partial \tau \ln S^L}{\partial r \partial r \ln S^L} \right) - \left( \frac{\partial r \partial \tau \ln S^L}{\partial \tau \partial \tau \ln S^L} \right)^2 \neq 0.$$

(2.9)

It can be rewritten as

$$J(\tau \rightarrow \omega) = \partial_t H \partial_r \mathbf{P} - (\nabla H)^2 \neq 0$$ \hspace{1cm} (2.9')

and evidently determines conditions needed for possibility of introduction of the temporal functions $\tau^L(\omega, \mathbf{r})$. Note that as the Legendre transformation $L$ is performed by the involution operator, $L^2 = 1$, this transformation does not change the magnitudes of observables and Poisson brackets (commutation relations). Notice that the variation of function $\tau^L(\omega, \mathbf{r})$ immediately leads to the Fermat principle.

Further Legendre transformation $\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{k}$ of the function $S^L(\omega, \mathbf{r})$ leads to the equation:

$$\rho^L(\omega, \mathbf{k}) = \frac{\partial}{\partial \mathbf{k}} \ln S^L(\omega, \mathbf{k}),$$ \hspace{1cm} (2.10)

which must correspond to an extent of interaction region in dependence on energy-momentum.

If the Legendre transformation (2.5) is considered as the sum of infinitesimal canonical transformations, the expression (2.5) transfers, evidently, into equation

$$d S^L_c(t; ...) - d \sum_k \Delta_k(\mathbf{H}t) = d S^L(\omega; ...).$$ \hspace{1cm} (2.5'')

After passage to the limit, substitution of (2.7) and integration, this expression leads to
the representation:

\[ S^L(\omega, r) = S_0(\omega_0, r) \exp[-i \int_{\omega_0}^\omega \tau^L(\omega, r) \, d\omega], \quad (2.11) \]

which can be considered as the general solution of the equation (2.6'). Integration in it goes from some \( \omega_0 \), at which \( \tau^L(\omega_0, r) \) is known, till the examined value \( \omega \). Corresponding general solution of (2.10) will be expressed via density of the 4-volume of interaction.

Notice that the performed manipulations can be formulated as the prescription: the Legendre transformation to new variables must be executed in the exponents of quantum expressions.

Operators \( (\tau^L, \rho^L) = (\partial / i \partial \omega, i \partial / \partial k) \) form the 4-vector \( \hat{x}_\mu \), corresponding to the equation:

\[ [(\tau^L)^2 - (\rho^L)^2 - s^2] \, S^L(\omega, r) = -[\Box_{\omega k} + s^2] \, S^L(\omega, r) = 0 \quad (2.12) \]

with a 4-interval \( s \). It can be considered as the reciprocal one to the Klein-Gordon equation and as the differential analogue, at \( s^2 \geq 0 \), of the relativistic generalization of Kramers-Krönig dispersion relations [19, 20].

Let’s consider now the function \( S^F(\omega, r) \), the Fourier transform of response function of the linear relation for signal passed through a uniform passive linear medium:

\[ O(t, r) = \int dt' \, dr' \, R(t - t'; r - r') \, I(t', r'). \quad (2.13) \]

The law of energy conservation in classical theory or the unitarity principle in quantum theory are responsible for the existence of partial or complete Fourier transformations of this relation:

\[ O(\omega, r) = R(\omega, r) \, I(\omega, r). \quad (2.13') \]

The logarithm of response function can be expanded near characteristic frequency \( \omega_0 \) and this series can be restricted in some cases by the first terms [4]:

\[ \ln R(\omega, r) = \ln R(\omega_0, r) + i(\omega - \omega_0) \, \tau^F(\omega, r) + ... \quad (2.14) \]

with the designation:

\[ \tau^F(\omega, r) = -i \partial_\omega \ln R(\omega, r). \quad (2.15) \]

The subsequent terms of this series are expressed via derivatives of (2.15): the first of them shows a temporal spread of signal and is considered in the next Section.
Hence the response function can be expressed (approximately!) as
\[ R(\omega, r) \simeq R(\omega_0, r) \exp\{i(\omega - \omega_0) \tau_F(\omega, r)\}. \] (2.16)

The comparison of (2.11) to (2.16) shows that the Fourier form of temporal functions correspond to an averaged Legendre-type temporal functions and, generally speaking, there are necessity for estimations of omitted terms of (2.14) in comparison to (2.11).

In the classical theory of (2.13) temporal quantities can be evidently deduced by direct expansion of response function in (2.13’) near the separated frequency. But more interesting seems the revealing of temporal magnitudes in another way. Let us consider the duration of rotation in the classical mechanics, which is determined as
\[ T = 2 \int_a^b dx/v(x) = 2m \int_a^b dx \left[ 2m (E - V(x)) \right]^{-\frac{i}{2}}, \] (2.17)
where \( v \) is the velocity of rotating particle, \( E \) and \( V \) are the complete and potential energies. Via the action function \( A = 2 \int_a^b p(x)dx \) the duration of process is determined as
\[ T = 4\partial A/\partial E. \] (2.18)

The duration, for example, of the packet spreading over a system of equidistant levels was determined in the ”old” quantum mechanics as [21]
\[ \Delta T \sim 1/(\partial \Delta E/\partial A) \approx \partial^2 A/\partial E^2. \] (2.18’)

At transition from classical mechanics to quantum one in accordance (2.7) \( A \to -i\hbar \ln(S/\hbar) \), and just this substitution leads to the definition (2.3).

3. TEMPORAL FUNCTIONS

The general solution of (2.8) or (2.11) can be presented as (superscripts are omitted)
\[ S(\omega, r) = S_1(\omega_0, r) \exp\{i \int^\omega \tau_1(\eta, r) \, d\eta - \int^\omega \tau_2(\eta, r) \, d\eta\}, \] (3.1)
where low limits of integrals do not depend on \( \omega \);
\[ \tau_1(\omega, r) = \partial_\omega \arg S(\omega, r) \] (3.2)
and
\[ \tau_2(\omega, r) = \partial_\omega \ln |S(\omega, r)| \] (3.3)
are, correspondingly, the Wigner-Smith formula of time delay at the process of elastic
scattering and the expression of extended duration of physical state formation.

The unitarity of \( S(\omega, k) \) permits to conclude, with the consideration of Cauchy-Schwartz inequality

\[
|S(\omega, k)|^2 \equiv 1 = \left| \int S(\omega, r) e^{-ikr} dr \right|^2 \leq \left| \int S(\omega, r) dr \right|^2 + \int \exp\{-2 \int \tau_2(\omega, r) d\omega\} dr, \tag{3.4}
\]

that \( \tau_2(\omega, r) \) cannot retain the constant sign over all frequencies interval. Its alternating may show an incompleteness of response function in the given space point. It can be assumed that just \( \tau_2 \) must describe for the details of processes leading to the terminating of reaction, to processes that are usually named as particles (states) dressing.

At the simplified consideration it can be concluded that as (2.16) at \( \tau_2 \geq 0 \) leads to the equality

\[
|R(\omega)| = |R(\omega_0)| \exp[-(\omega - \omega_0) \tau_2], \quad (3.4')
\]

then the opportunity of Fourier-transformation of \( R(\omega, r) \), i.e. the existence of the response function \( R(t, r) \), dictate for the considered theory the inequality:

\[
(\omega - \omega_0)\tau_2(\omega_0, r) \geq 0. \quad (3.5)
\]

It shows that at \( \omega < \omega_0 \) the duration of formation \( \tau_2 \leq 0 \), i.e. in the certain frequencies range the advanced emission or even superluminal phenomena are not excluded. Just such situation has place at superluminal transfer of excitation and corresponds to a lot of experimental data [22].

Addition of the following term of decomposition of \( \ln R(\omega, r) \) to (2.4),

\[
\sigma(\omega, r) \equiv - (\partial_\omega)^2 \ln R(\omega, r) = -i\tau t(\omega, r), \quad (3.6)
\]

at the inverse Fourier transformation with \( \tau_2 \geq 0 \), i.e. for \( \omega \geq \omega_0 \), leads to the "normal" response function:

\[
R^{(+)}(t, r) = R(\omega_0)(8\pi\sigma)^{-1/2} \exp\{-i\omega_0 t - (t - \tau)^2 / 2\sigma\} \left[1 - \text{erf}((t - \tau) / \sqrt{2\sigma})\right]. \tag{3.7}
\]

This term shows the broadening of signals on their path.

With \( \tau_2 \leq 0 \), i.e. at \( \omega \leq \omega_0 \), such transformation results in the "anomalous" response function \( R^{(-)}(t, r) \), which will be distinguished by the sign before the errors function.

Thus, the complete response function is represented as the sum

\[
R(t) = \theta(\tau_2) \ R^{(+)}(t) + \theta(-\tau_2) \ R^{(-)}(t), \quad (3.8)
\]
which can be examined as an analogue of decomposition of the causal propagator 
\[ \Delta_c(x) = \theta(t)\Delta^{(-)} + \theta(-t)\Delta^{(+)}, \]
where \( \Delta^{(\pm)} \) propagators correspond to positive and negative frequencies parts.

Let’s consider some peculiarities of temporal functions connected with the uncertainty principle.

It seems that the most general formal deduction of these relations was given by Schrödinger in [23]. The decomposition of the operators product on the Hermitian and anti-Hermitian parts as 
\[ AB = \frac{1}{2}(AB + BA) + \frac{1}{2}(AB - BA), \]
subsequent quadrating of this expression, its averaging over complete system of \( \psi \)-functions and replacement for operators on difference of operators and their averaged values 
\( A \rightarrow A - \langle A \rangle \)
bring to such expression:
\[ (\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle AB - BA \rangle|^2 + \frac{1}{4}[(\langle AB + BA \rangle - 2\langle A \rangle\langle B \rangle)]^2, \quad (3.9) \]
which differs from the more usual form by the last term. The Heisenberg limit of this expression shows a minimal value of uncertainties, which can be achieved in the determined conditions.

The deduction of uncertainty principle with operator \( \hat{\tau} = \partial / i \partial \omega \) was shown by Wigner in [24]. In considered case the operators must be taken as 
\( A \rightarrow E - \langle E \rangle \) and 
\( B \rightarrow t - \langle t \rangle \)
and there is needed the averaging (instead of \( \psi \)-functions) by the complete system of \( S(E) \) function, non-unitary in general, as 
\[ \langle A \rangle = \int_{-\infty}^{\infty} dE S^* A S / \int dE |S|^2. \]
The evident calculations give such result:
\[ (\Delta E)^2(\Delta t)^2 \geq \frac{1}{4} \hbar^2 + \frac{1}{4} [\langle E \tau_1 \rangle - 2\langle E \rangle\langle \tau_1 \rangle]^2, \quad (3.9') \]
i.e. the general form of uncertainty does not depend on the formation duration \( \tau_2 \) as it is the internal property of forming particle, but can be enlarged by enlarging the duration of scattering process.

Note that this condition shows the possibility for enlarging the near field extent. It seems that just this possibility is used in the near field optics, where close to source are introducing additional macroscopic scatterers and energy is varied by an external light flux [18].

As must be noted, the time domain processes are usually estimated via the time-energy
uncertainty principle $\Delta E \Delta t \geq \frac{1}{2} \hbar$. But in the cited article Wigner specially underlined that these uncertainties depend on coordinates points, so if the process is progressing in the $z$ direction:

$$(\Delta t(z))^2 = \int dx dy dt (t - t_0)^2 |\psi(x, y, z, t)|^2 / \int dy dt |\psi(x, y, z, t)|^2,$$

$$(\Delta E(z))^2 = \int dx dy dE (E - E_0)^2 |\psi(x, y, z, E)|^2 / \int dy dE |\psi(x, y, z, E)|^2$$

(3.10)

and can be different, in general case, for different $z$. This peculiarity can be the starting point at investigation of phenomena of FTIR.

4. DISPERSION RELATIONS AND SUM RULES

Response functions in the $(\omega, r)$-representation obey the temporal equation and simultaneously they are subject to the causality principle, i.e. they are governed by the Kramers-Kröning dispersion relations:

$$S_c(\omega) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\eta}{\eta - \omega} S_c(\eta) \quad (4.1)$$

(we write them in the simplest form with $S_c(\omega) \to 0$ at $\omega \to 0$). This duality permits to obtain the principal results.

By differentiation of (4.1) or by its substitution into (1.4) dispersion relations can be represented in two forms:

$$\tau(\omega) S_c(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\eta}{(\eta - \omega)^2} S_c(\eta), \quad (4.2)$$

$$\tau(\omega) S_c(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\eta}{\eta - \omega} \tau(\eta) S_c(\eta). \quad (4.2')$$

Equating of their right sides leads to the sum rule:

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega} S_c(\omega) \left[ \tau(\omega) - \frac{i}{\omega} \right] = 0. \quad (4.3)$$

This expression can be satisfied, in particular, with the equalities

$$\tau_1(\omega) = 0, \quad \tau_2(\omega) = 1/\omega, \quad (4.4)$$

which show, and it is the principal conclusion, that even at the absence of delay there is needed the certain time duration (twice bigger than the uncertainties value) for formation of the out state (wave or particle, etc.).

Notice that from the temporal equation (1.4) at its Fourier transformation follows also
such expression:
\[ t \ S(t) = \int_0^\infty dt' \ S(t') \ \tau(t-t'), \quad t \geq 0. \quad (4.5) \]

Whether \( \lim_{t \to 0} t \ S(t) = 0 \) at \( t \to 0 \), there must be rewarding the sum rule:
\[ \int_0^\infty dt \ S(t) \ \tau(-t) = 0. \quad (4.5') \]

Further derivatives of the equation (1.4) lead to more complicate sum rules, by checking of which can be determined the singularities of \( S(t) \) at \( t \to 0 \).

As temporal functions \( \tau(t) \) must be causal, there exist the independent dispersion relations:
\[ \tau(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\eta}{\eta - \omega} \ \tau(\eta), \quad (4.6) \]

which evidently connect \( \tau_1 \) and \( \tau_2 \). They are consistent, in particular, with the conditions (4.4) and with the representations of these functions via propagators in the Section 6.

The analicity of causal response functions \( S(\omega) \) permits to write them in the form of Bläschke product:
\[ S(\omega) = \text{const} \ \omega^{-p} \prod_n \frac{\omega - \omega_n - i\gamma_n/2}{\omega - \omega_n + i\gamma_n/2}. \quad (4.7) \]

With taking into account the relations (4.6) the sum rule (4.3) can be rewritten via interaction operator \( T(\omega) = i(S(\omega) - 1) \) as:
\[ \int_{-\infty}^{\infty} d\omega \ \omega^{-1}T(\omega) \ [\tau(\omega) - i/\omega] = 0. \quad (4.3') \]

Since \( \omega^p S(\omega) \) is the meromorph function, the substituting of \( T(\omega) \) into this equality and closing the integration contour in the upper half-plane produces the representation:
\[ \tau(\omega) = \sum_n 1/[\omega - \omega_n + i\gamma_n/2] \pm \ i p/\omega, \quad p > 0. \quad (4.8) \]

Temporal functions have physical sense for positive frequencies, for negative frequencies they are determined by the analytical continuation \( \tau(-\omega) = \tau^*(\omega) \), which follows from the analyticity of \( S(\omega) \). It permits the determination of Fourier transforms:
\[ \tau(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \ e^{i\omega t} \ \frac{\partial}{\partial \omega} \ln S(\omega) = \sum \text{res} \ e^{i\omega t}, \quad (4.9) \]

the last equality follows from the meromorphy of (4.7) at integer \( p \). It leads to the representations:
\[ \tau_1(t) = -\sum_n \cos(\omega_n t) \ \exp(-\gamma_n |t|); \quad \tau_2(t) = -i \ sgn(t) \ \tau_1(t), \quad (4.10) \]
i.e. the temporal functions are represented by the set of damped oscillators with self-
frequencies modulated by the widths of transmission bands, where index \( n \) numerates self-frequencies.

It must be underlined the oscillator character ("time diffraction") in the expressions (4.10) and all oscillators are modulated by the self-band widths (cf. [16]). Note, that at the standard approach the duration of processes are usually taken as \( 1/\gamma \), without taking into account their oscillation character.

The analyticity of \( S(\omega + i\eta) \) in the upper half-plane permits to write such integral over the closed contour:

\[
\oint \tau(\omega) \, d\omega = \oint \tau_1(\omega) \, d\omega = 2\pi(N - P),
\]

(4.11)

where \( N \) and \( P \) are zeros and poles of temporal function into the closed contour. Poles of \( \tau_1(\omega) \) signify impossibility of signal transferring on these frequencies through the system (frequencies locking) or particles capture at the scattering processes. Zeros show that corresponding signals are passed via system without delays, etc. Really (4.11) represents a variant of the Levinson theorem of quantum scattering theory, e.g. [5].

The maximum-modulus principle for \( |S(\omega)| \) shows, that as \( \tau_2(\omega) \) is determined via its derivative, it can not be equal to zero at any frequency: the formation of outgoing signal (wave, particle, state) always requires of some extended time duration.

It represents the main physical result of this Section.

5. HARMONIC OSCILLATOR

Let’s illustrate some of obtained results via consideration of the simplest model, the oscillator with damping of (4.8)-type:

\[
\ddot{x} - \gamma \dot{x} + \omega_0^2 x = f(t).
\]

(5.1)

The complete causal solution of (5.1) can be written via the Green functions:

\[
x(t) = \int_{-\infty}^{t} dt' G(t - t') \, f(t'); \quad G(t) = G_0(t) + G_1(t),
\]

(5.2)

and (5.2) can be considered as a model description of (2.13). The response part of complete Green function is the solution of non-homogeneous equation, Fourier image of
which is

\[ G_1(\omega) = -1/2\pi(\omega - \omega_1 + i\gamma/2)(\omega + \omega_1 + i\gamma/2) \]  

(5.3)

with \(\omega_1^2 = \omega_0^2 - \gamma^2/4\).

The corresponding causal temporal functions are:

\[ \tau_1(\omega) = \gamma/2[(\omega - \omega_1)^2 + \gamma^2/4] + \{\omega_1 \to -\omega_1\}, \]  

(5.4)

\[ \tau_2(\omega) = (\omega - \omega_1)/[(\omega - \omega_1)^2 + \gamma^2/4] + \{\omega_1 \to -\omega_1\}. \]  

(5.5)

The last expression shows the possibility of advanced or superluminal propagation at \(\omega < \omega_1 - \gamma^2/8\omega_1\) (cf. [21] and the superluminal transferring in macroscopic oscillator systems [25]).

Apart of some exotic cases \(\gamma << \omega_0\) and at \(|\omega - \omega_0| > \gamma\) and then at \(\gamma \to 0\) it can be taken that

\[ \tau_1(\omega) \simeq \gamma/2[(\omega - \omega_0)^2 + \gamma^2/4] \to \pi\delta(\omega - \omega_0), \]  

(5.4’)

\[ \tau_2(\omega) \simeq (\omega - \omega_1)/[(\omega - \omega_1)^2 + \gamma^2/4] \to 1/(\omega - \omega_1), \]  

(5.5’)

which shows the proximity of last expression to the uncertainty values, but it must be specially underlined its twice bigger numerical value. It means possibility of measurements of these values and therefore the observability of connected phenomena.

It seems that the most evident and close to the intuitive physical representation of temporal functions may give their description in the Lorentz model of dispersing and absorbing media (e.g. [26]), where media are described as the set of oscillators with damping. Each oscillator is describing by the Green function (5.3) with corresponding factor depending on density of scatterers, etc.

The real part of dielectric susceptibility and conductance are expressed in this model, respectively, as

\[ \varepsilon_1(\omega) - 1 \simeq \omega_p^2 (\omega_0 - \omega)/2\omega[(\omega_0 - \omega)^2 + \gamma^2/4]; \]  

(5.6)

\[ \sigma_{el}(\omega) \simeq \omega_p^2/8\pi\gamma [(\omega_0 - \omega)^2 + \gamma^2/4], \]  

(5.7)

\(\omega_p\) is the plasma frequency.

The comparison of (5.6-7) to (5.4-5) suggest, excluding the nearest vicinity of resonance, the possibilities of approximations:
\[ \varepsilon_1(\omega) - 1 \simeq \left( \frac{\omega_p^2}{2\omega} \right) \tau_2(\omega), \quad (5.6') \]
\[ \sigma_{el}(\omega) \simeq \left( \frac{\omega_p^4}{4\pi^2} \right) \frac{\varepsilon_1}{\varepsilon_2}, \quad (5.7') \]

These relations give the evident interpretation of both temporal functions. So the polar-    
ization of media is reasonably determined by durations of waves formation. And, as it is also    
intuitively evident, the electrical conductivity, as (every) transfer process, is determined via    
the durations of EM waves delay, which can be induced by virtual moment transfers between    
charged particles, i.e. by their retarded movements in the EM flux direction.

The more general connection of temporal functions with characteristics of media can    
be established in such fashion. The principle of entropy grows requires of execution of the    
strong inequality for almost transparent passive dispersing media: \[ \frac{\partial(\omega\varepsilon)}{\partial\omega} \geq 0 \] [27]. With    
the substitution \[ R \rightarrow \varepsilon(\omega) - \varepsilon(\infty) = \varepsilon_1 + \varepsilon_2 \], i.e. by the equation \[ \frac{\partial\varepsilon}{\partial\omega} = i \tau \varepsilon \], the real    
part of this general inequality is rewritten as

\[ \tau_2 \leq \frac{1}{\omega} - \frac{\tau_1 \varepsilon_2}{\varepsilon_1}, \quad (5.8) \]

As for sufficiently low frequencies \[ \varepsilon_2 = \left( \frac{4\pi}{\omega} \right) \sigma_{el}(\omega) \], this inequality reduces to the    
simplest form:

\[ \tau_1 + \tau_2 \leq \frac{1}{\omega}, \quad (5.9) \]

which evidently show that \( \tau_2 \) can be negative in some frequencies regions. In particular it must be negative in the region of anomalous dispersion, where must be expected a discordance between maxima of \( \tau_1 \) and \( \tau_2 \) [21], but for their description are needed more complicated models.

6. TEMPORAL WIGNER FUNCTIONS

As the cited Frank theory explains the Čerenkov photons emission via interference of classical waves emitted at different points, this effect in the quantum theory would be interpreted via the Wigner functions that just describe the overlapping of space domains of states [28]:
\[ w(k;r;t) = \left(\frac{1}{2\pi}\right)^3 \int dq \ e^{i\mathbf{q}\cdot\mathbf{r}} \psi(\mathbf{r} - \mathbf{q}/2; t) \ \psi^*(\mathbf{r} + \mathbf{q}/2; t), \quad (6.1) \]

or via their covariant generalization [29]:

\[ w(k;x) = \left(\frac{1}{2\pi}\right)^4 \int dv \ e^{-iv\mathbf{k}\cdot\mathbf{x}} \psi(\mathbf{x} - \mathbf{v}/2; t) \ \psi^*(\mathbf{x} + \mathbf{v}/2), \quad (6.2) \]

with 4-vectors \( k, x, v \) that describe the time-space overlapping (interference) of the quantum self-states. The quantum field interpretation of (6.2) through the creation and destruction operators descriptively shows that the interference of oppositely shifted wave functions in it must sum the maps of their possible variation onto 4-intervals.

Let us consider the one-particle temporal Wigner functions as the special case of (6.2),

\[
w^{(+)}(\omega, t; \mathbf{r}) = \frac{1}{2\pi} \int_0^{\infty} d\tau \ e^{i\omega\tau} \psi(t - \tau/2; \mathbf{r}) \ \psi^*(t + \tau/2; \mathbf{r}); \quad (6.3)\\
w^{(-)}(\omega, t; \mathbf{r}) = w^{(+)}(-\omega, t; \mathbf{r}). \quad (6.3')
\]

These functions evidently describe the overlap of time-shifted wave functions at one space point and therefore just these functions should characterize the time delay at collision process and the duration of states formation (space arguments will be hereafter omitted).

By time shifts of wave functions with the Hamiltonian \( \mathbf{H} \),

\[
\psi(t - \tau/2) = \psi(t) \ \exp(i\mathbf{H}\tau/2);\\
\psi^*(t + t/2) = \exp(i\mathbf{H}\tau/2) \ \psi^*(t), \quad (6.4)
\]

the temporal Wigner function (6.3) is rewritten as

\[
w^{(+)}(\omega, t) = \psi(t) \ \delta_+(\omega - \mathbf{H}) \ \psi^*(t) \rightarrow \psi(t) \ W^{(+)}(\omega, t) \ \psi^*(t). \quad (6.5)
\]

These functions are the self-functions of the operator equation

\[
\frac{\partial}{\partial\omega} \ w^{(+)}(\omega, t) = i(\omega - E)^{-1}w^{(+)}(\omega, t) \quad (6.6)
\]

of the (1.4) type, where \( E \) is the (complex) energy of system, \( \mathbf{H}\psi = E\psi \). This equation can be considered as the reciprocal one to the Liouville equation in the Schrödinger representation. It shows that the durations of scattering processes and of states formation should be described as the self-values of corresponding Green operators.

It must be noted that in distinction from the space Wigner functions the temporal functions are non-symmetric relative to their parameters and therefore their self-values can be complex ones. It just corresponds to possibilities of retarded and advanced interactions.
Slightly another derivation of such equation can be examined on transition to the Heisenberg representation,
\[
\psi(t - \tau/2) = \exp(i\omega\hat{T}) \psi(t/2) \exp(-i\omega\hat{T}),
\]
\[
\psi^+(t + \tau/2) = \exp(i\omega\hat{T}) \psi^+(t/2) \exp(-i\omega\hat{T}),
\]
with a temporal operator \(\hat{T}\). The equation (6.6) can be rewritten as
\[
-i\partial_\omega w^{(+)}(\omega,t) = \exp(i\omega\hat{T}) \left[\hat{T},Q\right] \exp(-i\omega\hat{T}),
\]
with function
\[
Q(\omega) = \frac{1}{2\pi} \int_0^\infty dt \ e^{i\omega t} \psi(-\tau/2) \psi^+(\tau/2).
\]
This representation naturally leads to the Hamilton equations for temporal operators.

The function (6.3), just as the Wigner functions, can be rewritten via the conjugate variable, via the energy shifts,
\[
-i\partial_\omega w^{(+)}(\omega,t;\mathbf{r}) = \frac{1}{2\pi} \int_0^\infty d\eta \ e^{i\eta t} \psi(\omega - \eta/2;\mathbf{r}) \psi^+(\omega + \eta/2;\mathbf{r}).
\]

Therefore the state formation can be considered as a gradually process of variation of energy till their definite values for physical ("dressed") particles. This property can be evidently generalized on interactions of arbitrary number of particles. In an analagical way may be considered the gradual evolution of (establishment in) other particles characteristics in the processes of interaction.

It can be noted, in particular, that if it is possible to introduce the operator of complete moment \(\mathbf{K}\), the Wigner functions in the close analogy with all above can be symbolically written as
\[
w(\mathbf{k};\mathbf{r}) = \psi(\mathbf{r}) \delta(\mathbf{k} - \mathbf{K}) \psi^+(\mathbf{r});
\]
i.e. via the vector Green functions. (This possibility will not be considered here further.)

7. FORMAL THEORY OF SCATTERING

Let’s consider more scrupulously the representation of temporal functions via propagators for the process of elastic scattering:
The kinetics of interaction must be described by the operator $S = 1 - iT$, where $T$ is the operator of interaction, expressed via propagators $G(E) = (E - H)^{-1}$ and $g(E) = (E - H_0)^{-1}$, the complete Hamiltonian $H = H_0 - V$, self-values of the Hamiltonians are complex, $H\psi = (E + i\Gamma)\psi$ and $H_0\psi_0 = (E_0 + i\Gamma_0)\psi_0$, where $\Gamma_0$ and $\Gamma$ are the natural and complete widths of the upper level (for the sake of simplicity there is considered the simplest two-level system).

As it was shown in [7] the duration of scattering and duration of newly state formation are naturally expressed via propagators with account and without account of this interaction:

$$\Delta\hat{\tau} \equiv \hat{\tau} - \hat{\tau}_0 = i[G(E) - g(E)], \quad (7.2)$$

where $\hat{\tau}$ and $\hat{\tau}_0$ denote temporal characteristics of complete particle path with and without interaction.

The expression (7.2) follows from differentiation of the operator of interaction, $T = V/(1 - gV)$, with taking into account the Dyson equation $G = g + gV G$ and the definition of temporal operator (7.2) via equation

$$\frac{\partial T}{i\partial E} = \Delta\hat{\tau} \ T. \quad (7.3)$$

Under the transition to energy surface, $E = E(p)$, the matrix element of (7.2),

$$\langle p | \Delta\hat{\tau} | p \rangle = \pm i \sum \{ [E - E_n - i\Gamma_n/2]^{-1} - [E - E_n(0) - i\Gamma_n(0)/2]^{-1} \}, \quad (7.4)$$

clearly shows its properties. So $iG(E)$ can be interpreted as the time duration needed for particles flight and their elastic scattering and $ig(E)$ corresponds to the free transfer.

Transition in (7.2) into the coordinate representation,

$$G(r) - g(r) = -\frac{1}{(2\pi)^3} \int d\mathbf{p} \ \langle \mathbf{p} | \Delta\hat{\tau} | \mathbf{p} \rangle \ e^{i\mathbf{p}\mathbf{r}}, \quad (7.5)$$

demonstrates the similarity of our definition with the Smith derivation of time delay at scattering processes [4].

Notice that from (7.2) follows such expression for the temporal operator:

$$\Delta\hat{\tau} = ig \ V \ G, \quad (7.6)$$

which permits, in particular, the expansion of temporal functions into the series about the free Green functions and interaction vertices:

$$\Delta\hat{\tau} = igVg + igVgVg + \ldots, \quad (7.7)$$
natural for quantum theories and useful for interpretations of these processes via Feynman graphs, etc. These forms show that the measurement of time characteristics of process is equivalent to addition of specific vertex (vertices) to corresponding diagrams of process (we shall return to this interpretation below).

The third form of temporal operator, which follows from (7.2), can be expressed as

\[ \Delta \hat{\tau} = i g T g. \quad (7.6) \]

Its matrix element,

\[ i \langle p|T|p \rangle / [(E - E_0(p))^2 + \Gamma_0^2(p)/4], \quad (7.7) \]

by the substitution of the known expression for scattering amplitude on the angle zero, \( f(p,p) = 4\pi^2 m \langle p|T(p)|p \rangle \), and the transferring to energy surface \( E = E(p) \) leads to the expression:

\[ \langle p|\Delta \hat{\tau}|p \rangle = \frac{1}{2\pi^2m^2} f(p,p). \quad (7.8) \]

The real part of (7.8) can be expressed, with taking into account the optical theorem of scattering theory, via the total cross-section of scattering:

\[ \tau_1(p) = \frac{p}{(2\pi)^4 m^2} \sigma_{tot}(p). \quad (7.9) \]

Just this result clarifies the great delay with the beginning of investigation of temporal characteristics of scattering processes: the most part of this information is contained in the Green functions and cross-sections.

If we determine the volume of interaction as \( V = \sigma_{max} u \tau_{max} \), where \( u \) is the velocity of scattered particle, \( \tau_{max} = 2/\Gamma \) and \( \sigma_{max} \) is the resonance cross-section, the mean value of duration of interaction can be determined as the balance relation,

\[ \bar{\tau}_1(p) = \sigma_{tot}(p) \tau_{max}/\sigma_{max}. \quad (7.10) \]

For the most practically important optical region \( \Gamma \sim 10^8 \text{sec}^{-1} \), \( \sigma_{max} = 4\pi/k^2 \), \( \sigma_{tot} = (4\pi/k) r_0 \), \( r_0 = e^2/mc^2 \). Therefore for \( k = 6.3 \cdot (10^4 \div 10^3) \text{ cm}^{-1} \) the expression (7.10) leads for nonresonant frequencies to

\[ \bar{\tau}_1(p) \sim (k/\Gamma) r_0 = 1.6 \cdot (10^{-16} \div 10^{-15}) \text{ sec}, \quad (7.11) \]

which evidently corresponds to the observable data.

It can be shown that this value permits to estimate the mean value of index of refraction
in nonresonant region. As it had been shown in [30] the optical dispersion in the transparent, at least, region can be considered as the kinetic process of photons transfer through media. Such transfer must be described for the free path lengths \( \ell = 1/N\sigma_{\text{tot}} \) with the vacuum velocity \( c \), where \( N \) is the density of outer (optical) electrons, and the subsequent delays at each scattering for the mean time of order (7.11). So, the complete time, needed for photons transfer on distance \( L \), is equal to

\[
\bar{T} = (L/c) + (L/\ell) \tau_1. \tag{7.12}
\]

This estimation leads to the group velocity \( u = L/\bar{T} \) and, for nonresonant cases, to the group index of refraction:

\[
n_{\text{gr}} \equiv \frac{c}{u} = \frac{c}{\bar{T}} = 1 + cN\sigma_{\text{tot}}\tau_1 - 1 + N\frac{4\pi\Gamma r_0^2}{\ell} - 1 + 3 \times 10^{-22}N, \tag{7.13}
\]

which qualitatively corresponds to the observations (\( N \) is of order of the L"oschmidt number).

It must be underlined that the representation of temporal functions via propagators supports the results of the Section 6: their analytical properties and the existence of dispersion relations of Kramers-Kr"onig type (cf. the estimation of such relations with possible subtractions, connected with renormalization procedures [31]).

8. DURATION OF INTERACTION AND ADIABATIC HYPOTHESIS

Let us show that the magnitudes of duration of interaction are implicitly contained in the standard theory in the form of adiabatic hypothesis. This hypothesis asserts that for the correct quantum calculations of transition amplitude there is needed such artificial substitution for the Hamiltonian:

\[
V(t) \to V(t) \exp(-\lambda|t|) \tag{8.1}
\]

with passage to the limit \( \lambda \to 0 \) after all calculations (e.g. [5]).

Stueckelberg proposed more general approach to these problems via the causality condition [32]. Bogoliubov generalized this method by introduction of operations of ”the switching interaction on and off”, i.e. of some function \( q(x) \in [0,1] \), which characterizes the intensity
of interaction: in the space-time regions with \( q(x) = 0 \) interaction is completely absent and at \( q(x) = 1 \) is completely switched [33]. But the introduction of this switching function has not physical substantiation and can be justified a posteriori only.

In this theory \( S \)-matrix becomes a functional of function \( q(x) \) and the final state of system in the interaction representation is expressed as

\[
\Phi[q] = S[q] \Phi_0,
\]

(8.2)

where \( \Phi_0 \) is the initial state. For performing of this program the switching function is introduced into the (classical) action function, e.g.

\[
S_{cl} = \int dx \; q(x) \; \hat{L}(x),
\]

(8.3)

where \( \hat{L}(x) \) is the density of Lagrangian of interaction. In the quantum field theory, correspondingly, the operator of evolution will be represented as the functional:

\[
S[q] = T' \exp\{\frac{i}{\hbar} \int dx \; q(x) \; \hat{L}(x, q)\},
\]

(8.4)

\( T' \) is the chronologization operator and it is assumed that the relative value of Lagrangian depends on "intensity of interaction".

The variation of (8.2) over \( q(x) \) leads to the variational equation

\[
i\delta\Phi[q]/\delta q(x) = H(x; q) \; \Phi(q)
\]

(8.5)

with the Hamiltonian of interaction

\[
H(x; q) = i(\delta S[q]/\delta q(x)) \; S^*[q],
\]

(8.6)

which is the evident variational analog, at \( q = 1 \), of the Schrödinger equation for \( S \)-operator in the interaction representation. This form leads to the covariant Tomonaga-Schwinger equation.

The switching function \( q(x) \) describes the 4-region of interaction, and if we shall assume that the magnitude of this region depends on details of interaction, we rewrite (8.4) as

\[
S[\hat{L}] = T' \exp\{-i \int dx \; q(x, \hat{L}) \; \hat{L}(x)\} = T' \exp\{-i \int dk \; q(-k, \hat{L}) \; \hat{L}(k)\},
\]

(8.7)

in the last equality the existence of corresponding Fourier transforms is proposed. This transition from (8.4) to (8.7) can be considered as the Legendre-type transformation \( q \leftrightarrow \hat{L} \) of the classical action function (8.3), i.e. instead of consideration of switching of intensity of interaction there is considered a variable part of 4-volume of interaction (in particular, of
the duration of interaction). This assumption can be also justified only a posteriori.

Thus we vary (8.7) over $\hat{L}(k)$ and it leads to the equation

$$\delta S[\hat{L}]/i\delta \hat{L}(k) = q(-k, \hat{L}) S[\hat{L}], \quad (8.8)$$

or

$$q(-k, \hat{L}) = (\delta S[\hat{L}]/i\delta \hat{L}(k)) S^{-1}[\hat{L}], \quad (8.8')$$

i.e. to the evident variation-type analog of the equation for temporal operator. Notice

that in the complete accordance with the Bogoliubov method it can be considered the

singularity of $\hat{L}$ on a hypersurface $\sigma(\omega)$, which would lead to the equation

$$\delta S/i\delta \hat{L}(k, \sigma) = q(-k, \sigma) S(\sigma), \quad (8.9)$$

reciprocal to the Tomonaga-Schwinger equation.

From these equations naturally follows the equation (1.4), reciprocal to the Schrödinger
equation for $S$-matrix, with the formal temporal function

$$\tau(\omega) = \int dk \ q(-k, \hat{L}) (\delta \hat{L}(k)/\delta \omega). \quad (8.9)$$

The switching function $q(x)$ can be presented, in accordance with the adiabatic hypoth-
thesis (8.1), as

$$q(x) = \exp(-\gamma|t|/2) \quad \text{or} \quad q(-k) = \delta(k)/2\pi i(k_0 \pm i\gamma/2). \quad (8.10)$$

These expressions can be rewritten in the covariant form by introduction of any unit time-
like vector $n_\mu$ and replacement for $\tau \rightarrow n_\mu x_\mu$. The substitution of (8.10) into the expression

(8.9) with assuming of the $\delta$-type properties of $\delta \hat{L}(k)/\delta \omega$ and with the frequency’s shift

$\omega_0 \rightarrow \omega - \omega_0$, leads to the usual form of temporal function for the simplest two-level system,

$$\tau \equiv \tau_1 + i\tau_2 = 1/\pi(\gamma/2 \pm i(\omega - \omega_0)). \quad (8.11)$$

Thus it can be concluded that the adiabatic hypothesis presents a non-obvious introduc-
tion of the time duration concept in the theory.

9. QUANTUM ELECTRODYNAMICS

Let us begin the consideration of temporal functions of QED with examination of the

photon causal propagator of lowest order in vacuum (Feynman calibration, $\eta \rightarrow 0+$):
\[ D_c(\omega, \mathbf{k}) = \frac{4\pi}{(\omega^2 - \mathbf{k}^2) + i\eta}. \]  

(9.1)

In accordance with all above it conducts to such expressions for time delay and duration of formation:

\[
\tau_1 = -2\pi \delta(\omega^2 - \mathbf{k}^2), \quad (9.2)
\]

\[
\tau_2 = \frac{2\omega}{(\omega^2 - \mathbf{k}^2)} \cdot \frac{1}{\omega - |\mathbf{k}|}. \quad (9.3)
\]

The function \(\tau_1\) simply shows that the photon can be absorbed or emitted only completely. The function \(\tau_2\) qualitatively corresponds to the uncertainty principle, but is twice bigger, i.e. this time can be measurable; it shows the possibility of retarded, at \(\omega > |\mathbf{k}|\), or advanced, at \(\omega < |\mathbf{k}|\), emission of photon.

It should be noted that the consideration of complete propagators through replacements \(\mathbf{k}^2 \to \mathbf{k}^2 + P(\mathbf{k})\) in (9.1) with the polarization operator \(P(\mathbf{k})\) of QED or even transition to propagators of massive (scalar, for simplicity) particles does not change these general results.

The estimations of temporal values for elementary processes in the nearest orders can be achieved by such simple procedure: in accordance with (3.3) it can be suggested the expression for \(\tau_2\) via cross-section of scattering:

\[
\tau_2 = -\frac{1}{2} (\partial/\partial \omega) \ln \sigma. \quad (9.4)
\]

So as for the Rutherford scattering \(\sigma \sim E^{-2}\), it gives, in accordance with (9.3), \(\tau_2 = 1/E\); for the nonrelativistic limit of Compton scattering \(\sigma \sim (1 - 2\omega/m)\) and therefore \(\tau_2 = 1/m (1 - 2\omega/m)\), etc. The values of \(\tau_1\) can be estimated now via dispersion relations (4.6) and so on.

The complete covariant generalization of the temporal operator \(\hat{\tau}\) can be achieved by the Legendre transformation of equations for the 4-moments of interaction:

\[
i\partial S/\partial x_{\mu} = k_{\mu} S \quad \longleftrightarrow \quad \partial S/i\partial k_{\mu} = x_{\mu} S, \quad (9.5)
\]

where \(x_{\mu} = (t, \mathbf{r})\) represents the 4-vector of ”duration-space extent of interaction” in (2.12). Notice that for investigation of some process the substitution \(S \to M\), i.e. the consideration of concrete matrix element of the process is implied.

The time operator is now generalized as the covariant operator \(\partial/\partial p_{\mu}\), canonically con-
jugated to the energy-momentum operator $i\partial/\partial x_\mu$. The determination of corresponding functions can be established via the Ward-Takahashi identity:

$$\partial G/\partial p_\mu = G(p) \Gamma_\mu(p,p;0) G(p), \quad (9.6)$$

$G(p)$ is the particle Green function, $\Gamma_\mu(p,q;p-q)$ is the vertex part. From it follows the expression for self-values of 4-operator:

$$\xi_\mu(p) \equiv \partial \ln G/\partial p_\mu = \frac{1}{2}\{G(p) \Gamma_\mu(p,p;0) + \Gamma_\mu(p,p;0) G(p)\}. \quad (9.6')$$

The 4-vector $\xi_\mu$ consists of the temporal and space components, $\xi_0(p) \equiv \tau(\omega,k)$ and $\xi(p) \equiv \rho(\omega,k)$, which determines the space extents of interaction. (Similar operators were introduced for localized states of spin zero massive particles [34], but they are a matter of discussions for photons [35].)

The known representation of vertex operator $\Gamma_\mu(p,p;0) = \gamma_\mu - (\partial/\partial p_\mu)\Sigma$ with the mass operator $\Sigma$ shows that the expression (9.6) is connected with the extended formation of physical particles parameters.

The difference between both parts of $\xi_\mu$ can be demonstrated by consideration of the simplest case, the complete causal propagator $D_c = \overline{D} + D_1$ in the scope of scalar electrodynamics. In accordance with (9.6') both parts of temporal function in the $p$-representation are equal to

$$\xi_\mu_1(p) \equiv \text{Re} \xi_\mu(p) = p_\mu D_1(p;m) = p_\mu(D^{(+)} - D^{(-)}), \quad (9.7)$$
$$\xi_\mu_2(p) \equiv \text{Im} \xi_\mu(p) = p_\mu \overline{D}(p;m) = p_\mu(D_{ret} - D_{adv}). \quad (9.7')$$

In the $x$-representation these expressions are even more descriptive:

$$\xi_\mu_1(x) = (\partial/\partial x_\mu)(D^{(+)}(x) - D^{(-)}(x)); \quad (9.8)$$
$$\xi_\mu_2(x) = (\partial/\partial x_\mu)(D_{ret}(x) - D_{adv}(x)), \quad (9.8')$$

i.e. the duration of interaction describes the decreasing of negative-frequency part and increasing of positive-frequency part, the extended duration of state formation determine the difference of retarded and advanced parts alteration.

These expressions evidently show also the difference between uncertainty values and durations or space extents of interactions. So from the expression (9.7') and as $\overline{D}(p) = -P_{\frac{1}{k}}$ follows that $\tau_2$ and $\rho_2$ are approximately twice bigger than the corresponding uncertainty.
values.

Notice that these representations can lead to several particular models. Let’s consider as example the space extent of particle formation averaged over frequencies not excided its rest mass:

\[ \langle \Delta(r, m) \rangle = \frac{1}{m} \int_{0}^{m} d\mu \, \Delta(r, \mu) = \sin(mr) / 4\pi mr^2, \]

its gradient describes, via (9.8’), the space extent of interaction, and it approaches, in accordance with the uncertainty principle, to \( \delta(r) \) with increasing of the mass of particle.

The temporal functions for electron must be determined via the electron Green functions and in the nearest order they are represented through (9.6’) as

\[ \langle \tau(p) \rangle = \frac{1}{2} \text{Tr} \{ \gamma_0 S_{\psi}(p) \} = p_0 \Delta_{\psi}(p), \quad (9.9) \]

which at the substitutions for \( \omega \to (p_0^2 - p^2)^{1/2} \) and the Fourier transformation over moments variables coincides with (9.2-3).

The physical sense of these functions can be established in such a way. The expression (9.8) shows that the temporal measurement, for which \( \mu = 0 \), is equivalent to addition of zero-frequency scalar photon line to the appropriate electron lines of the Feynman graphs. Therefore the durations can be interpreted via the probed additional Coulomb fields of zero intensity (cf. with the Baz’ method of zero-intensity probe magnetic field and the ”Larmor clock” in it [9, 10]).

This examination demonstrates, in particular, that the superluminal phenomena may be observed, in principle, into all scattering processes, and not only in processes of QED.

For the spinor quantum electrodynamics this 4-vector must be determined, correspondingly, as

\[ \xi_{\mu} = \text{Tr} \left( M^+ \frac{\partial}{\partial \epsilon_{\mu}} M \right) / \text{Tr} (M^+M), \quad (9.10) \]

It seems interesting to check by this expression the results, obtained in [11 - 14] for bremsstrahlung. By insertion of its known matrix element (i.e. [36]) into (9.10) it can be easy shown that Re \( \xi_{\mu} = 0 \) in the lowest order. It corresponds to the absence of any delay at bremsstrahlung, but the components of Im \( \xi_{\mu} \), which are connected to the formation processes, are nonzero. So if \( \epsilon, k \) and \( \epsilon', k' \) are the initial and final electron energies and
moments, $\omega$ is the photon energy, $\vartheta$ is the angle of electron departure, we have that the duration and corresponding path extent of electron dressing are determined as

$$
\tau_2 = \frac{1}{\omega}, \quad \rho_2 = \frac{k'}{c\omega} + \frac{1}{2} \frac{k'k - k}{\epsilon\epsilon'} + \frac{m^2}{k' - k}
$$

(9.11)

at $\epsilon, \epsilon' \geq m$ and

$$
\tau_2 \approx |\rho_2| - \frac{2\epsilon(\epsilon' + \omega)}{m^2}, \quad \rho_{\perp} \approx \frac{2\epsilon\vartheta}{m^2},
$$

(9.12)

when $\epsilon, \epsilon' \gg m$.

These results correspond to the previous calculations, but are obtained by a shorter and more general way. Notice that the region of photons formation can be considered as the near field of classical electrodynamics.

Let’s consider shortly, as an example, some more general problems.

So if we investigate the scattering of scalar particles via the one-particle exchange, the lowest order values of $\xi_{\mu}$ are determined as the logarithmic derivatives of intermediate particle propagator. In the standard notation with taking into account the Ward-Takahashi identity it leads to the expression,

$$
\xi_{\mu}(k) = (\partial/\partial k_\mu) \ln D_c' = [2k_{\mu}/i(t - m^2)^2] \Gamma(t, t, 0) D_c',
$$

(9.13)

where $D_c'$ is the complete Green function.

The factor $2k_{\mu}/(t - m^2)$, that formally is close to the uncertainty principle, corresponds to the duration of outgoing particles formation $\tau_2 = |\rho_2|^{-1/2} E$. Time delay is connected with the imaginary part of propagator and arises at $t \geq 4m^2$, i.e. with possibilities of new particles birth.

As well as under photons formation the length of their formation (the near field region) appears, it can be proposed that in processes of formation of particles with additional internal parameters would manifest itself another regions of their formation with their own peculiarities.

10. TO INTERPRETATION OF SOME RENORMALIZATION PROCEDURES
Let's begin with the Pauli-Villars method of regularization.

This method consists in such substitutions:

\[
\Delta(p, m) \to \Delta(p, m) - \Delta(p, M) - \frac{m^2 - M^2}{(p^2 - m^2 + i\eta)(p^2 - M^2 + i\eta)}
\]  

(10.1)

with further passage to the limit \( M \to \infty \).

What is its physical sense? Such substitutions implies a decreasing of duration of new state formation with \( p^2 < M^2 \):

\[
\xi_{\mu 2} = 2p_{\mu}\{(p^2 - m^2)^{-1} + (p^2 - M^2)^{-1}\}, \quad (10.2)
\]

i.e. it seems as a procedure of alteration of the interaction 4-volume, which was discussed in connection with the adiabatic hypothesis.

In the \( x \)-representation such substitution, \( \xi_{\mu}(x) \to (\partial / \partial x_{\mu})\{\Delta_c(x, m) + \Delta_c(x, M)\} \), leads to increasing of the role of more energetic and more deep-seated virtual excitations at the beginning of calculations. And it actually means a partial account of higher terms of \( S \)-matrix in the process of particle formation.

Let's pass on to the subtraction procedures of renormalization.

The regularized mass function of the electron propagator is determined as

\[
\Sigma^{\text{reg}}(p) = \Sigma(p) - \frac{m - (\gamma p - m) (\partial \Sigma(p) / \partial p) \mid_{\gamma p = m}}{(p^2 - m^2 + i\eta)(p^2 - M^2 + i\eta)}.
\]  

(10.3)

Then the equalities

\[
\Sigma^{\text{reg}}(p) \mid_{\gamma p = m} = 0, \quad (\partial \Sigma^{\text{reg}}(p) / \partial p) \mid_{\gamma p = m} = 0,
\]  

(10.4)

postulated for its renormalization, now can be interpreted as the physically justified conditions: the mass of particle has definite magnitude and the process of its accumulation should be finished at the finite time.

The regularized self-energetic part of the photon propagator \( (k \partial_k \equiv k_{\nu} \partial / \partial k_{\nu}) \)

\[
\Pi^{\text{reg}}_{\mu\nu}(k) = \Pi_{\mu\nu}(k) - \{1 - k \partial_k - \frac{1}{2}(k \partial_k)^2\} \Pi_{\mu\nu}(k) \mid_{k=0}.
\]  

(10.5)

Apart from the evident and gauge equalities,

\[
\Pi^{\text{reg}}_{\mu\nu}(0) = 0; \quad k^{-2}\Pi^{\text{reg}}_{\mu\nu}(k) \mid_{k=0} = 0,
\]  

(10.6)

the conditions, that are usually simply postulated:

\[
k \partial_k \Pi^{\text{reg}}_{\mu\nu}(k) \mid_{k=0} = 0, \quad (k \partial_k)^2 \Pi^{\text{reg}}_{\mu\nu}(k) \mid_{k=0} = 0,
\]  

(10.7)
must be physically interpreted as the conditions for the completeness of physical photon formation and for the impossibility of its self-acceleration.

Thus it is stated that the subtraction regularization corresponds to mathematical formulation of the common physical conditions, primary imposed on the system, and therefore it is far from an artificial, ad hoc method.

It must be specially underlined that the method of renormalization group [37, 31] can be immediately interpreted via the temporal functions. Really, as the corresponding Lie equations contain logarithmic derivatives of propagators over energy-moment, they are still proportional to the temporal magnitudes.

As the denominators of propagators leads only to the trivial terms, connected with the uncertainty principles or twice bigger them, let’s consider for checking of such proposition the nondimensional Green functions $\tilde{\mathcal{G}}(q)$ with all 4-moments, except one, fixed. Then in accordance with the renorm-group equation of Callan and Symanzik [38] it can be written that

$$
|\xi_{\mu}| - q^2 \frac{\partial}{\partial q^2} \ln \tilde{\mathcal{G}}(q) = (\gamma_m - 1) \frac{\partial}{\partial m^2} \ln \tilde{\mathcal{G}} + \beta \frac{\partial}{\partial e} \ln \tilde{\mathcal{G}} - \gamma_G(m^2, e),
$$

where $\gamma_m$, $\beta$ and $\gamma_G$ are the structure functions of the renorm-group.

In the lowest order of $\varphi^4$ theory $\gamma_m = 0$, $\beta = \frac{3}{2}e^2$ and $\gamma_G = -\frac{3}{2}e$. Therefore in (10.8) for the 4-tail graphs are retained only the terms connected with the charge formation and accumulation of the observed mass:

$$
e \frac{\partial}{\partial e} \ln \tilde{\mathcal{G}} = e(1 - \tilde{\mathcal{G}}^{-1});$$
$$-rac{\partial}{\partial m^2} \ln \tilde{\mathcal{G}} = -\frac{e^2}{m^2 \tilde{\mathcal{G}}} \sum \frac{1}{y_k} \text{Arth } y_k,
$$

where $y_k = (z_k^2 - z_k)^{1/2}$, $(z_1, z_2, z_3) = (s, u, t)/m^2$.

It shows that in the $\varphi^4$ theory, and correspondingly in the QED, the charge increasing must extend the duration of formation, but in such gauge theories, where $\beta < 0$, this process should decrease $\xi_{\mu}$.

Note that in the UV limit $\gamma_m = 1$ in (10.8) and

$$
\ln \tilde{\mathcal{G}}(q^2, e) \to \ln \tilde{\mathcal{G}}(1, e) - 2\gamma_G \ln q,
$$

Hence in the asymptotically free theories, where $e \to -e$, the expression (10.8) can be
reduced to such relation:
\[
\xi_{\mu 2}(k) = \frac{2\nu}{q}(1 - 6|\nu|), \quad (10.11)
\]
i.e. at \(|\nu| = 1/6\nu\) the duration of formation in this approximation is equal to zero.

This result can be of general interest: it seems very tempting to attempt this form with problem of existence of only restrict types of particles generation, but this invites further more detailed investigations. Note that the absence of terms, which describe the delays in the (10.8), can be connected with the calculation of matrix elements in the one-loop approximation.

**CONCLUSIONS**

The main results of performed researches can be formulated in such points.

1. There are established the reciprocal forms of the Schrödinger or the Bloch equations in \(p\)-space and their covariant generalizations. By these equations are substantiated some different, as has been seemed, methods of calculation of durations of scattering.

2. The reciprocal equations permit, in particular, the unified determination of the Wigner-Smith function of delay at scattering process and the Pollak-Smith function of duration of final state formation.

3. The magnitudes of duration of scattering process (time of delay and duration of formation of final state) are implicitly contained in the usual field theory. The propagators of interacting fields actually can describe them and thereby it becomes evident why many problems of kinetics could be considered without explicit introducing of temporal magnitudes. Just it may explain a delayed beginning of researches of temporal parameters in the quantum theory.

4. The adiabatic hypothesis of quantum theory may be considered as the implicit expression of existence of the certain duration of formation (dressing) of physical particles. Therefore it is far from the formal, pure mathematical procedure and shows that at the consideration of arising of any physical state the account of its formation duration is inevitable.

5. Both basic methods of delay duration measurement, by Wigner and Smith and by
"Larmor clock", can be described as an addition of zero-energy line to the Feynman graph of process.

6. The dispersion relations for temporal functions are established. They prove, in particular, that the duration of final state formation is, at least, twice bigger than the uncertainty values and therefore is measurable.

7. The consideration of the Lorentz (oscillator) model of simple dispersing medium leads to an intuitively evident interpretation of temporal functions. In this model the function of time delay is proportional to the polarization of medium and the function of final state formation is proportional to the electric conductivity.

8. The transition from the Schrödinger equation to the reciprocal temporal equation corresponds to the Legendre transformation of classical action function. The averaged values of these functions are deduced via the Fourier transformation of response functions. The covariant form of temporal equation is deduced by the temporal variant of Stueckelberg-Bogoliubov variational method.

9. The methods of subtraction regularization in field theory can be physically substantiated and explained as the imposing the requirements of finishing of their formation and impossibility of particles self-acceleration on propagators of particles.

10. The concept of duration of interactions imparts the evident physical sense to the equations of renormalization group and demonstrates that the formation of each of particles parameters required of the certain (may be, specific) duration. Hence it gives a possibility to think that the coordination of durations of these partial processes will allow more detailed understanding of peculiarities of those or other particles. Such program requires, however, further researches.

We do not discuss here a lot of delays determinations known in the current literature. It can be suggested that just the revealed analytical properties of the composed temporal functions demonstrate their general significance. It does not exclude, of course, the possible usefulness of some other determinations in special cases.

Note that apart from the general problems of interpretation, the represented approach
simplify the consideration of such phenomena as multiphoton processes, in which the termination of process depends on intervals between subsequent interactions [6, 15], the general theory of optical dispersion [29]; it permits to consider the superluminal phenomena as processes connected with the properties of near fields of radiation [22], etc. Some other applications of this approach will be considered elsewhere.

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