Construction of new solutions of Korteweg-de Vries Caudrey-Dodd-Gibbon equation using two efficient integration methods

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Abstract

Korteweg-de Vries Caudrey-Dodd-Gibbon (KdV-CDG) equation describes many physical phenomena in plasma physics, optical fibers, dynamics of the ocean, quantum mechanics, acoustic waves and laser optical applications. In this paper, the KdV-CDG equation is analyzed via two reliable and efficient integrating approaches. The suggested techniques; the extended $G^0G^2$-expansion method and exponential ($\psi(\xi)$)-expansion method successfully extract hyperbolic function solutions, trigonometric function solutions and rational function solutions. The existence criteria for all the obtained solutions are also discussed in this paper. At the end, various 3D and 2D contour plots have been constructed for better understanding of constructed solutions.

Introduction

Nonlinear partial differential equations (NLPDEs) are used to investigate and model many physical problems occurring in the real-world. NLPDEs have immense applications in all sectors of life and have attracted the attention of researchers and scientists. It is observed that many of the NLPDEs are reported to have the exact solutions in the form of traveling wave functions, often depicting solitary waves or solitons.

Solitons were first discovered by Scott Russell in 1834. Since then, many scientists have contributed toward understanding solitons and their impact on real world applications. Solitons act like solitary waves which hold the law of conservation. Solitary waves have both particle-like and wave-like natures. Upon collision with other waves, soliton waves conserve their shape. Solitons occur due to the balance between two effects known as dispersive and nonlinear effects. These properties have motivated the researchers to find exact solutions to NLPDEs. Solitons have numerous applications in telecommunication, acoustics, optical fiber, fluid mechanics and in various sectors of physics. Much research has been done to find soliton solutions of many nonlinear partial differential equations as they link mathematics and physics together. In this regard major work has been done which includes the following studies. The construction of solitary waves, breather waves and hybrid waves for $(3+1)$-dimensional NLEE is studied in [1]. The higher-order nonlinear Schrödinger-Maxwell-Bloch equations are
investigated in [2]. Stability analysis, solitary wave and explicit power series solutions of a 
(2 + 1)-dimensional nonlinear Schrödinger equation is discussed in [3]. The Riemann-Hilbert 
problem is developed to study the nonlinear Schrödinger equation in [4]. Moreover, the higher 
order NLSE is studied using Darboux-dressing transformation with the Lax pair and asymptotic 
expansion method in [5]. The nonlinear wave transitions of (2 + 1)-dimensional Sawada-Kotera is investigated in [6]. Many techniques are used to extract soliton solutions such as the 
generalized projective Riccati equation method [7], improved tanh(x) method [8], (G'/G, 1/G)-expansion approach [9] and ψ-dressing method [10] and [11].

In this paper, the Korteweg-de Vries Caudrey-Dodd-Gibbon equation is studied using two 
powerful and efficient analytical techniques such as extended G'/G-expansion method and exponential (ψ(ξ))-expansion method. These methods have been found useful to solve many complex problems with impressive results. It has been observed that the extended G'/G-expansion method has been efficiently used for investigating problems such as time-fractional Burgers equation, fractional biological population model, space-time fractional Whitham-Broer-Kaup equations [12] and Triki-Biswas equation [13]. The exponential (ψ(ξ))-expansion method is applied on Nizhnik-Novikov-Veselov model [14] and on strain wave equation [15] to extract the soliton solutions. This research deals with the extraction of exact solutions for KdV-CDG equation using the proposed techniques.

This study is organized as follows: In Section 2, the governing equation is explained. Section 
3 contains detailed description of the proposed methods. Section 4 contains all newly con-
structed solutions extracted as a result of employing the suggested techniques. Section 5 con-
tains the graphical representation of few of the specified solutions and in the last section, 
concluding remarks are presented.

Korteweg-de Vries Caudrey-Dodd-Gibbon (KdV-CDG) equation

This study is based on investigating the combined form of Korteweg-de Vries and Caudrey-
Dodd-Gibbon equations; termed as Korteweg-de Vries Caudrey-Dodd-Gibbon (KdV-CDG) equation. Both KdV and CDG equations have numerous applications in water waves. They play a vital role in nonlinear studies such as: plasma physics, dynamics of ocean, quantum 
mechanics, acoustic wave and in laser optics [16]. The KdV-CDG equation is given, as

\[ v_t + l v_{xx} + m \left( \frac{1}{15} \beta v^3 + \beta v v_{xx} + v_{xxxx} \right) = 0. \]  

If \( m = 0 \) in Eq (1), then it reduces to Korteweg-de Vries (KdV) equation. If \( l = 0 \) in Eq (1), then it reduces to Caudrey-Dodd-Gibbon (CDG). In 1877, Boussinesq introduced the KdV equation for the first time and then later major work on it was done by Korteweg and de Vries. In 1895, they developed this linear model to describe solitary waves. Small amplitude shallow 
water waves, surface waves of long wavelength and internal waves in a shallow density-strati-
fied fluid are described by KdV equation [17, 18]. CDG is integrable model and provides vari-
ous solutions with limited number of conserved quantities [19]. The unique properties and 
numerous applications of the KdV-CDG equation has opened new horizons for researchers.

Recently, KdV-CDG model has been examined using different approaches including F-expansion method, extended hyperbolic function method and exponential function method [20–22].

Description of methods

This section contains the description of the proposed methods.
The general nonlinear evolution equation is considered, as
\[ E(v, D_t v, D_x v, D_{tt} v, D_{tx} v, D_{xx} v, \ldots) = 0, \]  
where \( v = v(x, t) \) is considered as an unknown function, \( E \) represents a polynomial function in \( v \). The following traveling wave transformation is used to convert Eq (2) into an ODE, as
\[ v(x, t) = s(\xi), \quad \xi = x - \rho t, \]
where \( \rho \) represents velocity of the wave profile. The transformed ODE has the form
\[ F(s, s', s'', s''', \ldots) = 0, \]
where \( ' \) indicates the derivative with respect to \( \xi \).

**Method I: The exponential \((\psi(\xi))\)-expansion method**

According to **Method I**, the solution of ODE Eq (3) has the form,
\[ s(\xi) = \sum_{i=0}^{n} w_i (\exp(-\psi(\xi)))^i, \]
where \( w_i \) represents the unknown constants that are determined later. \( \psi(\xi) \) obeys the relation
\[ \psi'(\xi) = \exp(-\psi(\xi)) + \mu \exp(\psi(\xi)) + \lambda. \]

Eq (5) possesses the following three types of solutions.

**Case 1**: \((\lambda^2 - 4\mu > 0 \text{ and } \mu \neq 0)\)

Hyperbolic solution is obtained, as
\[ \psi_1(\xi) = \ln \left( -\sqrt{\lambda^2 - 4\mu} \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + c) - \frac{\lambda}{2\mu} \right) \right). \]

**Case 2**: \((\lambda^2 - 4\mu < 0 \text{ and } \mu \neq 0)\)

Trigonometric solution is gained in Case 2, as
\[ \psi_2(\xi) = \ln \left( \sqrt{4\mu - \lambda^2} \tan \left( \frac{\sqrt{4\mu - \lambda^2}}{2} (\xi + c) - \frac{\lambda}{2\mu} \right) \right). \]

**Case 3**: \((\lambda^2 - 4\mu > 0 \text{ and } \mu = 0 \text{ and } \lambda \neq 0)\)

In this case, hyperbolic function solution is obtained, as
\[ \psi_3(\xi) = -\ln \left( \frac{\lambda}{\cosh(\lambda(\xi + c)) + \sinh(\lambda(\xi + c)) - 1} \right). \]

**Case 4**: \((\lambda^2 - 4\mu = 0 \text{ and } \mu \neq 0 \text{ and } \lambda \neq 0)\)
Rational solution is extracted in this case, as
\[ \psi_4(\xi) = \ln \left( -\frac{2(\lambda(\xi + c)) + 2}{\lambda^2(\xi + c)} \right). \]  
(9)

**Case 5:** \((\lambda^2 - 4\mu = 0 \text{ and } \mu = 0 \text{ and } \lambda = 0)\)

Solutions will be of the form
\[ \psi_5(\xi) = \ln(\xi + c), \]  
(10)
where \(c\) is a constant of integration.

Through homogenous balancing, the value of \(n\) is determined. Inserting Eq (4) into Eq (3) and using Eq (5), a system of equations in \(w_i\) is retrieved by computing the coefficients of every power of \(\exp(-\psi(\xi))\) to 0. Solving the obtained system gives the values of the unknown parameters.

**Method II: The extended \((\frac{G'}{G})\)-expansion method**

According to **Method II**, the assumed solution of Eq (3) has the form
\[ s(\xi) = c_0 + \sum_{i=1}^{n} c_i \left( \frac{G'}{G^2} \right)^i + d_i \left( \frac{G'}{G^2} \right)^{-i} \]  
(11)
where \(\frac{G'}{G^2}\) satisfies the ODE,
\[ \left( \frac{G'}{G^2} \right)^{\prime} = \lambda + \mu \left( \frac{G'}{G^2} \right)^2. \]  
(12)
\(\lambda \neq 1\) and \(\mu \neq 0\) are integers while \(c_0, c_i\) and \(d_i\) \((i = 1, 2, 3, \ldots, n)\) are arbitrary parameters which have to be determined. The value of \(n\) is obtained by homogenous balancing.

**Step 2**
Putting Eqs (11) and (12) into Eq (3), an algebraic system of equations is obtained by equating the coefficients of different powers of \(\left( \frac{G'}{G^2} \right)^j\), \((j = 0, \pm 1, 2, \ldots)\) to zero. Solving the obtained system gives the values of arbitrary parameters.

**Step 3**
Eq (12) has three different forms of solutions:

**Case 1**
Trigonometric form of solutions are obtained if \(\lambda \mu > 0\):
\[ \frac{G'}{G^2} = \sqrt{\frac{\lambda}{\mu}} \left[ H_1 \cos(\sqrt{\lambda \mu} \xi) + H_2 \sin(\sqrt{\lambda \mu} \xi) \right]. \]  
(13)

**Case 2**
Hyperbolic form of solutions are obtained if \(\lambda \mu < 0\):
\[ \frac{G'}{G^2} = -\sqrt{\frac{\lambda \mu}{\mu}} \left[ H_1 \sinh(2\sqrt{|\mu \lambda|} \xi) + H_2 \cosh(2\sqrt{|\mu \lambda|} \xi) + H_3 \right]. \]  
(14)

**Case 3**
When \( \lambda = 0 \) and \( \mu \neq 0 \), then rational solutions are obtained, as

\[
\frac{G'}{G^2} = -\frac{H_1}{\mu(H_1^2 + H_2)}.
\]  

(15)

\( H_1 \) and \( H_2 \) are considered as arbitrary parameters.

**Step 4**

By substituting \( c_0, c_i, d_i \) and \( \frac{\partial}{\partial \eta} \) in Eq (11), the solutions of Eq (3) are obtained.

**Mathematical analysis of KdV-CDG equation**

This part of the paper is dedicated to applying the afore mentioned techniques on the KdV-CDG equation to draw out new soliton solutions. The obtained results may be found helpful in understanding the complex nonlinear phenomena arising in plasma physics, optical fibers, dynamics of the ocean, quantum mechanics and acoustic waves. For applying both methods, the following traveling wave transformation is employed, as

\[
v(x, t) = s(\xi), \quad \xi = x - \rho t.
\]

This transformation, converts Eq (1) into the following ODE.

\[
-\rho s' + l \left( s' + \frac{1}{5} \beta s^2 \right)' + m \left( \frac{1}{15} \beta s^3 + \beta s' s + s'' \right)' = 0.
\]

(16)

After integrating Eq (16) and putting constant of integration to zero, the following equation is obtained.

\[
-\rho s + l \left( s' + \frac{1}{5} \beta s^2 \right) + m \left( \frac{1}{15} \beta s^3 + \beta s' s + s'' \right) = 0.
\]

(17)

**Construction of soliton solutions via technique I**

In this subsection, the technique I has been applied on the proposed model to obtain new soliton solutions. Application of homogenous balancing on Eq (17) yields \( n = 2 \). Inserting \( n = 2 \) in Eq (4), gives

\[
s(\xi) = w_0 + w_1 \exp \psi(\xi) + w_2 (\exp \psi(\xi))^2.
\]

(18)

Utilizing exponential \((\psi(\xi))\)-expansion method, an algebraic system of equations is obtained with the following solution sets.

**1\(^{st}\) Solution set:**

\[
w_0 = \frac{\sqrt{3} \sqrt{3l^2 - 10lm(\lambda^2 - 4\mu) + 35m^2(\lambda^2 - 4\mu)^2 + 3l + 15m(\lambda^2 + 4\mu)}}{4m},
\]

\[
w_1 = -30\lambda, \quad w_2 = -30, \quad \beta = 1.
\]

In **1\(^{st}\) Solution set** \((\lambda^2 - \mu)\) is taken as arbitrary constant so all the cases have been considered which are as follows:

**Case 1:** \( \lambda^2 - 4\mu > 0, \mu \neq 0 \)
Hyperbolic solution is obtained, as

\[
v_1(x, t) = -\sqrt{3} \sqrt{3l - 10lm(\lambda^2 - 4\mu)} + 35m^2(\lambda^2 - 4\mu)^2 + 3l + 15m(\lambda^2 + 4\mu) \frac{4m}{4m} \\
+ \frac{60\lambda\mu}{\sqrt{\lambda^2 - 4\mu} \tanh \left( \frac{1}{2}(c + \xi) \sqrt{\lambda^2 - 4\mu} \right) + \lambda} \\
- \frac{120\mu^2}{\left( \sqrt{\lambda^2 - 4\mu} \tanh \left( \frac{1}{2}(c + \xi) \sqrt{\lambda^2 - 4\mu} \right) + \lambda \right)^2}.
\] (19)

The graphical representation of the above obtained solution is represented in Fig 1.

Case 2: \( \lambda^2 - 4\mu < 0, \mu \neq 0 \)

Case 2 extracts trigonometric function solution, as

\[
v_2(x, t) = -\sqrt{3} \sqrt{3l - 10lm(\lambda^2 - 4\mu)} + 35m^2(\lambda^2 - 4\mu)^2 + 3l + 15m(\lambda^2 + 4\mu) \frac{4m}{4m} \\
+ \frac{60\lambda\mu}{\lambda - \sqrt{4\mu - \lambda^2} \tan \left( \frac{1}{2}(c + \xi) \sqrt{4\mu - \lambda^2} \right)} \\
- \frac{120\mu^2}{\left( \lambda - \sqrt{4\mu - \lambda^2} \tan \left( \frac{1}{2}(c + \xi) \sqrt{4\mu - \lambda^2} \right) \right)^2}.
\] (20)

The graphical representation of Eq (20) is displayed in Fig 2.

Case 3: \( \lambda^2 - 4\mu > 0, \mu = 0 \) and \( \lambda \neq 0 \)

Case 3 gives hyperbolic solution, as

\[
v_3(x, t) = -\sqrt{3} \sqrt{3l - 10lm(\lambda^2 - 4\mu)} + 35m^2(\lambda^2 - 4\mu)^2 + 3l + 15m(\lambda^2 + 4\mu) \frac{4m}{4m} \\
+ \frac{1}{8} (-15) (\lambda^2 \cosh(\lambda(c + \xi)) + 3\lambda^2) \text{csch}^2 \left( \frac{1}{2} \lambda(c + \xi) \right).
\] (21)

Case 4: \( \lambda^2 - 4\mu = 0, \mu \neq 0 \) and \( \lambda \neq 0 \)

Case 4 give rational solution, as

\[
v_4(x, t) = \frac{3\sqrt{F} + 3l + 15m(\lambda^2 + 4\mu)}{4m} + \frac{15\lambda^3(c + \xi)}{c\lambda + \lambda\xi - 1} - \frac{30\lambda^4(c + \xi)^2}{(2 - 2\lambda(c + \xi))^2}.
\] (22)

Case 5: \( \lambda^2 - 4\mu = 0, \mu = 0 \) and \( \lambda = 0 \)

Applying the condition of Case 5, the following solution is obtained, as

\[
v_5(x, t) = \left\{ -\frac{30}{(c + \xi)^2} - \frac{3\sqrt{F} + 3l}{4m} \right\}.
\] (23)
Fig 1. Bright soliton: $v_1(x, t)$ for $l = 1.9, m = 1.5, \lambda = 3.7, c = -2.4, v = 1.5, \beta = 1, \mu = 2.4.$

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Fig 2. Singular soliton: $|v_s(x, t)|$ for $l = 0.67, m = 2.1, \lambda = 1.5, c = -1.4, v = 0.5, \mu = 2.5$.

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2\textsuperscript{nd} Solution set:

\[ w_0 = -30\mu, \quad w_1 = -30\lambda, \quad w_2 = -30, \quad \beta = 1. \]

By inserting the values of 2\textsuperscript{nd} Solution set in Eq (18), the following solutions are obtained.

\textbf{Case 1: } \lambda^2 - 4\mu > 0, \mu \neq 0

In this case hyperbolic solution is obtained, as

\[ v_6(x, t) = -30\mu - \frac{60\lambda \mu}{-\sqrt{\lambda^2 - 4\mu} \tanh \left( \frac{1}{2} (c + \xi) \sqrt{\lambda^2 - 4\mu} \right) - \lambda} - \frac{60\mu}{-\sqrt{\lambda^2 - 4\mu} \tanh \left( \frac{1}{2} (c + \xi) \sqrt{\lambda^2 - 4\mu} \right) - \lambda}. \] (24)

The graphical representation of \( v_6(x, t) \) is presented in Fig 3.

\textbf{Case 2: } \lambda^2 - 4\mu < 0, \mu \neq 0

Trigonometric function solution has been obtained in this case, as

\[ v_7(x, t) = -30\mu - \frac{120\mu^2}{\left( \sqrt{4\mu - \lambda^2} \tan \left( \frac{1}{2} (c + \xi) \sqrt{4\mu - \lambda^2} \right) - \lambda \right)^2} - \frac{60\lambda \mu}{\sqrt{4\mu - \lambda^2} \tan \left( \frac{1}{2} (c + \xi) \sqrt{4\mu - \lambda^2} \right) - \lambda}. \] (25)

\textbf{Case 3: } \lambda^2 - 4\mu > 0, \mu = 0 \text{ and } \lambda \neq 0

Hyperbolic function solution is obtained, as

\[ v_8(x, t) = \frac{w_2 (l + 5\lambda^2 m)}{20m} + \frac{\lambda^2 w_2}{\sinh(\lambda(c + \xi)) + \cosh(\lambda(c + \xi)) - 1} \] (26)

\[ + \frac{\lambda^2 w_2}{\sinh(\lambda(c + \xi)) + \cosh(\lambda(c + \xi)) - 1}. \]

\textbf{Case 4: } \lambda^2 - 4\mu = 0, \mu \neq 0 \text{ and } \lambda \neq 0

Rational solution has been obtained, as

\[ v_9(x, t) = -30\mu - \frac{30\lambda^2 (c + \xi)^2}{(2 - 2\lambda(c + \xi))^2} - \frac{30\lambda^2 (c + \xi)}{2 - 2\lambda(c + \xi)}. \] (27)

\textbf{Case 5: } \lambda^2 - 4\mu = 0, \mu = 0 \text{ and } \lambda = 0

Following solution has been obtained, as

\[ v_{10}(x, t) = -\frac{30}{(c + \xi)^2}. \] (28)

3\textsuperscript{rd} Solution set:

\[ w_0 = \frac{w_2 (l + 5\lambda^2 m)}{20m}, \quad w_1 = \lambda w_2, \quad \mu = \frac{l + 5\lambda^2 m}{20m}, \quad \beta = -\frac{1800}{w_2 (w_2 + 90)}, \quad \rho = -\frac{4F}{25m}. \]
Fig 3. Singular soliton: $|\psi(x, t)|$ for $l = 3.1, m = 4.2, \lambda = 1.8, c = 3.8, v = 3, \mu = 3.6$.

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By putting the above values of 3rd Solution set different solutions has been obtained as follows:

**Case 1:** $\lambda^2 - 4\mu > 0, \mu \neq 0$

In this case hyperbolic solution is extracted, as

$$v_{11}(x, t) = \frac{lw_2(l + 5\lambda^2 m)}{4m \left( \sqrt{5} \sqrt{l} \sin \left( \frac{\sqrt{l}(c + \xi)}{2\sqrt{5} \sqrt{m}} \right) + 5\lambda \sqrt{m} \cosh \left( \frac{\sqrt{l}(c + \xi)}{2\sqrt{5} \sqrt{m}} \right) \right)^2}.$$  \hspace{1cm} (29)

provided that $lm < 0$.

**Case 2:** $\lambda^2 - 4\mu < 0, \mu \neq 0$

Trigonometric function solution is obtained, as

$$v_{12}(x, t) = \frac{lw_2(l + 5\lambda^2 m)}{4m \left( \sqrt{5} \sqrt{l} \sin \left( \frac{\sqrt{l}(c + \xi)}{2\sqrt{5} \sqrt{m}} \right) - 5\lambda \sqrt{m} \cos \left( \frac{\sqrt{l}(c + \xi)}{2\sqrt{5} \sqrt{m}} \right) \right)^2}.$$  \hspace{1cm} (30)

provided that $lm > 0$.

**Case 3:** $\lambda^2 - 4\mu > 0, \mu = 0$ and $\lambda \neq 0$

Hyperbolic solution is presented in this case.

$$v_{13}(x, t) = \frac{w_2(l + 5\lambda^2 m)}{20m} \frac{\lambda^2 w_2}{\sinh(\lambda(c + \xi)) + \cosh(\lambda(c + \xi)) - 1}$$

$$+ \frac{\lambda^2 w_2}{(\sinh(\lambda(c + \xi)) + \cosh(\lambda(c + \xi)) - 1)^2}.$$  \hspace{1cm} (31)

The graphical representation of the above constructed solution is represented in Fig 4.

**Case 4:** $\lambda^2 - 4\mu = 0, \mu \neq 0$ and $\lambda \neq 0$

Rational form of solution is obtained, as

$$v_{14}(x, t) = \frac{1}{20} w_2 \left( \frac{5\lambda^2}{(\lambda(c + \xi) - 1)^2} + \frac{l}{m} \right).$$  \hspace{1cm} (31)

**Case 5:** $\lambda^2 - 4\mu = 0, \mu = 0$ and $\lambda = 0$

Following solution for Case 5 has been obtained, as

$$v_{15}(x, t) = \frac{1}{20} w_2 \left( \frac{5(\lambda(c + \xi) + 2)^2}{(c + \xi)^2} + \frac{l}{m} \right),$$  \hspace{1cm} (32)

where $c$ is constant of integration.

**Construction of soliton solutions via technique II**

In this subsection, the technique II has been applied on the proposed model to obtain new soliton solutions. Applying homogenous balancing on Eq (17) yields $n = 2$. For $n = 2$, Eq (11)
Fig 4. Singular soliton: $|v_{13}(x, t)|$ for $l = -0.9, m = -1.8, \lambda = -2.3, c = 2.7, v = 3.6, \mu = 2.3$. 

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becomes
\[
\xi = c_0 + c_1 \left( \frac{G}{G'} \right) + c_2 \left( \frac{G'}{G} \right)^2 + d_1 \left( \frac{G}{G'} \right)^{-1} + d_2 \left( \frac{G'}{G} \right)^{-2}.
\]
(33)

Utilizing technique II, following solution sets have been derived.

**Solution set 1:**

- \( c_0 = 4(4\sqrt{30} - 15)\lambda \mu, \quad c_1 = 0, \quad d_1 = 0, \quad c_2 = -30\lambda^2, \quad d_2 = -30\mu^2, \)
- \( \rho = \frac{16}{5} (2 + \sqrt{30})\lambda \mu, \quad m = \frac{1}{80\lambda \mu}, \quad \beta = 1. \)

(34)

Upon inserting these values in Eq (33), the following solutions are obtained. \( \lambda \) and \( \mu \) are considered as arbitrary constants.

**Case 1: \( \lambda \mu > 0 \)**

In this particular case trigonometric function solution is obtained, as

\[
v_{16}(x, t) = 4(4\sqrt{30} - 15)\lambda \mu - \frac{30\lambda \mu (H_2 \cos (\sqrt{\lambda \mu}x) - H_1 \sin (\sqrt{\lambda \mu}x))^2}{(H_1 \cos (\sqrt{\lambda \mu}x) + H_2 \sin (\sqrt{\lambda \mu}x))^2} - \frac{30\lambda \mu (H_2 \cos (\sqrt{\lambda \mu}x) + H_1 \sin (\sqrt{\lambda \mu}x))^2}{(H_1 \cos (\sqrt{\lambda \mu}x) - H_2 \sin (\sqrt{\lambda \mu}x))^2}.
\]
(35)

**Case 2: \( \lambda \mu < 0 \)**

In this case hyperbolic function solution is obtained, as

\[
v_{17}(x, t) = 4(4\sqrt{30} - 15)\lambda \mu - \frac{30\lambda \mu^2 (H_2 \sinh (2\sqrt{\lambda \mu}x) + H_1 \cosh (H_2 - 2\sqrt{\lambda \mu}x))^2}{|\lambda \mu| (H_1 \sinh (2\sqrt{\lambda \mu}x) + H_2 \cosh (2\sqrt{\lambda \mu}x + H_2))^2} - \frac{30\lambda \mu (H_1 \sinh (2\sqrt{\lambda \mu}x) - H_2 \cosh (2\sqrt{\lambda \mu}x))^2}{(H_1 \sinh (2\sqrt{\lambda \mu}x) + H_2 \cosh (H_2 - 2\sqrt{\lambda \mu}x))^2}.
\]
(36)

**Case 3: \( \lambda = 0 \) and \( \mu \neq 0 \)**

In this case rational form of solution is obtained, as

\[
v_{19}(x, t) = \frac{30\mu^4 (H_1 \xi + H_2)^2}{H_1^2}.
\]
(37)

**Solution set 2:**

- \( c_0 = -\sqrt{3} \sqrt{30F} + 40\lambda \mu m + 560\lambda^2 \mu^2 m^2 + 3l + 60\lambda \mu m \)
- \( c_1 = 0, \quad d_1 = 0, \quad c_2 = -30\lambda^2, \quad d_2 = 0, \quad \beta = 1. \)

Taking values of **Solution set 2** in Eq (33) solutions are of the following form:

**Case 1: \( \lambda \mu > 0 \)**
Through this case trigonometric function solution is extracted, as

\[ v_{19}(x, t) = -\frac{3\sqrt{3}\sqrt{3l^2 + 40\lambda \mu \lambda m + 560\lambda m^2 + 3l + 60\lambda \mu m}}{4m} \]

\[ -\frac{30\lambda \mu (H_x \sin (\xi \sqrt{\lambda \mu}) + H_1 \cos (\xi \sqrt{\lambda \mu}))^2}{(H_1 \cos (\xi \sqrt{\lambda \mu}) - H_1 \sin (\xi \sqrt{\lambda \mu}))^2}. \]

Case 2: \( \lambda \mu < 0 \)

In this case hyperbolic function solution is obtained, as

\[ v_{20}(x, t) = -\frac{3\sqrt{3}\sqrt{3l^2 + 40\lambda \mu \lambda m + 560\lambda m^2 + 3l + 60\lambda \mu m}}{4m} \]

\[ -\frac{30\lambda \mu (H_x \sin (2\xi \sqrt{\lambda \mu}) + H_1 \cosh (2\xi \sqrt{\lambda \mu} + H_x))^2}{(H_x \sin (2\xi \sqrt{\lambda \mu}) + H_1 \cosh (2\xi \sqrt{\lambda \mu} - H_x))^2}. \]

The graphical representation of the above hyperbolic solution is plotted in Fig 5.

Case 3: \( \lambda = 0 \) and \( \mu \neq 0 \)

In this case rational form of solution is obtained, as

\[ -\frac{3\sqrt{3l^2 + 3l}}{4m}. \]

Solution set 3:

\[ c_0 = -\frac{30\mu(\sqrt{\beta(9\beta - 8)\lambda^4 + 3\beta\lambda^2})}{\beta\lambda}, \quad c_1 = 0, \quad d_1 = 0, \quad c_2 = -\frac{15(\sqrt{\beta(9\beta - 8)\lambda^4 + 3\beta\lambda^2})}{\beta}, \]

\[ d_2 = -\frac{15\mu^2(\sqrt{\beta(9\beta - 8)\lambda^4 + 3\beta\lambda^2})}{\beta^2}, \quad \rho = \frac{1}{5} (-64)\lambda \mu, \quad m = \frac{1}{80\lambda \mu}. \]

Inserting parameters of Solution set 3 in Eq (33), the solutions obtained are as follows:

Case 1: \( \lambda \mu > 0 \)

Here trigonometric function solution has been obtained, as

\[ v_{22}(x, t) = -\frac{15\mu(\sqrt{\beta(9\beta - 8)\lambda^4 + 3\beta\lambda^2})(H_x \sin (\xi \sqrt{\lambda \mu}) + H_1 \cos (\xi \sqrt{\lambda \mu}))^2}{\beta\lambda (H_x \cos (\xi \sqrt{\lambda \mu}) - H_1 \sin (\xi \sqrt{\lambda \mu}))^2} \]

\[ -\frac{15\mu(\sqrt{\beta(9\beta - 8)\lambda^4 + 3\beta\lambda^2})(H_x \cos (\xi \sqrt{\lambda \mu}) - H_1 \sin (\xi \sqrt{\lambda \mu}))^2}{\beta\lambda (H_x \sin (\xi \sqrt{\lambda \mu}) + H_1 \cos (\xi \sqrt{\lambda \mu}))^2} \]

\[ -\frac{30\mu(\sqrt{\beta(9\beta - 8)\lambda^4 + 3\beta\lambda^2})}{\beta\lambda}. \]

Case 2: \( \lambda \mu < 0 \)
Fig 5. Kink soliton: $|\psi_2(x, t)|$ for $\beta = 2.5$, $\nu = 7$, $\lambda = 3.7$, $\mu = -3.2$, $H_1 = 1.5$, $H_1 = 1$, $l = 1.3$, $m = 2.6$.  

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Hyperbolic function solution has been obtained, as

\[ v_{23}(x, t) = \]

\[-\frac{15\mu^2(\sqrt{\beta(9\beta - 8)\lambda^4} + 3\beta\lambda^2)(H_1 \sinh (2\xi \sqrt{\lambda\mu}) + H_1 \cosh (2\xi \sqrt{\lambda\mu} - H_2))^2}{\beta \lambda |H_1 \sinh (2\xi \sqrt{\lambda\mu}) + H_1 \cosh (2\xi \sqrt{\lambda\mu} + H_2)|^2} - \frac{15(\sqrt{\beta(9\beta - 8)\lambda^4} + 3\beta\lambda^2)\mu(H_1 \sinh (2\xi \sqrt{\lambda\mu}) + H_1 \cosh (2\xi \sqrt{\lambda\mu} + H_2))^2}{\beta \lambda (H_1 \sinh (2\xi \sqrt{\lambda\mu}) + H_1 \cosh (2\xi \sqrt{\lambda\mu} - H_2))^2} \]

\[-\frac{30\mu(\sqrt{\beta(9\beta - 8)\lambda^4} + 3\beta\lambda^2)}{\beta \lambda} , \]

where \( H_1 \) and \( H_2 \) are considered as arbitrary constants.

**Graphical overview of selected solution sets**

Graphical presentations of a few of the retrieved exact soliton solutions of the KdV-CDG equation are discussed in this section. Using a computer simulated program Maple, different 3D and contour plots have been plotted. The most appropriate values of arbitrary parameters have been chosen to construct 3D surface plots and 2D graphs.

**Conclusion**

In this article, two unique and reliable techniques, the extended \( \psi \)-expansion method and exponential \( (\psi(\xi)) \)-expansion method have been employed for constructing the exact solutions of the Korteweg-de Vries Caudrey-Dodd-Gibbon (KdV-CDG) equation. These technique efficiently extracted hyperbolic, rational and trigonometric solutions. These solutions can be found useful in investigating the governing model in different fields of science and engineering. 3D surface plots and 2D graphs have also been represented in this article to describe the dynamics of the obtained solutions.

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