AMENABILITY PROPERTIES OF THE CENTRAL FOURIER ALGEBRA OF A COMPACT GROUP

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Abstract. We let the central Fourier algebra, $Z_A(G)$, be the subalgebra of functions $u$ in the Fourier algebra $A(G)$ of a compact group, for which $u(xy^{-1}) = u(y)$ for all $x, y$ in $G$. We show that this algebra admits bounded point derivations whenever $G$ contains a non-abelian closed connected subgroup. Conversely when $G$ is virtually abelian, then $Z_A(G)$ is amenable. Furthermore, for virtually abelian $G$, we establish which closed ideals admit bounded approximate identities. We also show that if $Z_A(G)$ is weakly amenable, even hyper-Tauberian, exactly when $G$ admits no non-abelian connected subgroup. We also study the amenability constant of $Z_A(G)$ for finite $G$ and exhibit totally disconnected groups $G$ for which $Z_A(G)$ is non-amenable.

Let $G$ be a compact group with Fourier algebra $A(G)$ and $B$ be a group of continuous automorphisms on $G$. We let

$$Z_B A(G) = \bigcap_{\beta \in B} \{ u \in A(G) : u \circ \beta = u \}$$

and call this the $B$-centre of $A(G)$. In particular we let $\text{Inn}(G)$ be the group of inner automorphisms and let

$$Z_A(G) = Z_{\text{Inn}(G)} A(G)$$

and call this the $G$-centre of $A(G)$, or the central Fourier algebra on $G$. Of course, since $A(G)$ is a commutative algebra, this bears no relation to the centre of $A(G)$ as an algebra.

We are motivated by the results of [3], where amenability properties of the centre of the group algebra, $ZL^1(G)$, are studied. We note that this algebra is densely spanned by idempotents — i.e. normalized characters of irreducible representations, $d_\pi \chi_\pi$ — and hence is automatically weakly amenable, even hyper-Tauberian (see [43, Theo. 14]). Hence those properties were not discussed for compact $G$. However, it was shown for $G$ either non-abelian and connected, or an infinite product of non-abelian finite groups, that $ZL^1(G)$ is non-amenable. It is conjectured in that paper that $ZL^1(G)$ is

\footnotesize

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amenable if and only if if $G$ has an open abelian subgroup. Neither direction of that conjecture is resolved.

In the present article, we conduct a parallel investigation for $\text{ZA}(G)$. Our techniques are different and some of our conclusions sharper. In the case that the connected component of the identity, $G_e$, is non-abelian, we show that $\text{ZA}(G)$ admits a point derivation; see Section 2. In Section 3 we show that when $G$ is virtually abelian then $\text{ZA}(G)$ is amenable. We are able to use the structure of the central Fourier algebra for virtually abelian compact groups, to further establish that $\text{ZA}(G)$ is weakly amenable if and only if $G_e$ is abelian. We invest the extra effort to establish that this is exactly the case in which $G$ is hyper-Tauberian, a condition identified and studied by E. Samei ([43]). One of our major tools in this section is the relationship between these properties and certain conditions related to sets of spectral synthesis. In Section 4 we investigate the amenability constant $\text{AM}(\text{ZA}(G))$, and show that an infinite product of non-abelian finite groups, $P$ gives a non-amenable algebra $\text{ZA}(P)$.

1. Preliminaries and notation.

1.1. Amenability and weak amenability. We briefly note some fundamental definitions which go back to [26].

Let $\mathcal{A}$ be a commutative Banach algebra. A Banach $\mathcal{A}$-bimodule is any Banach space $X$ which admits a pair of contractive homomorphisms $a \mapsto (x \mapsto ax)$ and $a \mapsto (x \mapsto xa)$, each from $\mathcal{A}$ into bounded operators $\mathcal{B}(X)$, such that the ranges of these maps commute: $a(xy) = (axy)b$. The homomorphisms given by pointwise adjoints of these maps make the dual, $X^*$, into a dual Banach $\mathcal{A}$-module. We say $\mathcal{A}$ is amenable is every bounded derivation from $\mathcal{A}$ into any dual module, i.e. $D : \mathcal{A} \to X^*$ with $D(ab) = a(Db) + (Da)b$, is inner, i.e. $Da = af - fa$ for some $f$ in $X^*$.

Following [4], we say $\mathcal{A}$ is weakly amenable if there are no non-zero bounded derivations for $\mathcal{A}$ into any symmetric Banach $\mathcal{A}$-module, i.e. module $X$ satisfying $ax = xa$. In particular, $\mathcal{A}$ is a symmetric Banach $\mathcal{A}$-bimodule as is its dual $\mathcal{A}^*$. It is sufficient to see that there are no non-zero bounded derivations form $\mathcal{A}$ into $\mathcal{A}^*$ to show that $\mathcal{A}$ is weakly amenable. Given a multiplicative functional $\chi$ on $\mathcal{A}$, a point derivation at $\chi$ is any functional $D$ on $\mathcal{A}$ which satisfies $D(ab) = \chi(a)D(b) + D(a)\chi(b)$. The map $a \mapsto D(a)\chi : \mathcal{A} \to \mathcal{A}^*$ is then a derivation. Hence a weakly amenable algebra admits no bounded point derivations.

1.2. The central Fourier algebra. Throughout this article $G$ will denote a compact group. We let $\hat{G}$ denote the set of (equivalence classes of ) continuous irreducible representations. For $\pi$ in $\hat{G}$ we let $\mathcal{H}_\pi$ denote the space on which it acts and let $d_\pi = \dim \mathcal{H}_\pi$. We denote normalized Haar integration by $\int_G \ldots ds$. For integrable $u : G \to \mathbb{C}$ we let $\hat{u}(\pi) = \int_G u(s)\bar{\pi}(s)\,ds$, and
(34.4) provides the following description of the Fourier algebra:

\[ u \in A(G) \iff \|u\|_A = \sum_{\pi \in \hat{G}} d_\pi \|\hat{u}(\pi)\|_1 < \infty \]

where \(\|\cdot\|_1\) denotes the trace norm. This is a special case of the general definition of the Fourier algebra, for locally compact groups, given in [10].

We let \(VN(G) = \ell^\infty \bigoplus_{\pi \in \hat{G}} B(H_\pi)\) and we have dual identification \(A(G)^* \cong VN(G)\) via

\[ \langle u, T \rangle = \sum_{\pi \in \hat{G}} d_\pi \text{Tr}(\hat{u}(\pi)T_\pi). \]

Let \(Z_G : A(G) \to ZA(G)\) be given by \(Z_G u(x) = \int_G u(sxs^{-1}) \, ds\). This is easily verified to be a contractive linear projection. A well-known consequence of the Schur orthogonality relations is that for \(\pi \in \hat{G}\) and \(\xi, \eta \in H_\pi\) we have \(Z_G(\pi(\cdot)\xi|\eta) = d_\pi \chi_\pi \hat{\xi} \eta\), where \(\chi_\pi\) is the character of \(\pi\). Hence \(ZA(G) = \text{span}\{\chi_\pi : \pi \in \hat{G}\}\). Thus, since \(\hat{\chi}_\pi(\pi) = \frac{1}{d_\pi} I_{d_\pi}\) and \(\hat{\chi}_\pi(\pi') = 0\) for \(\pi' \neq \pi\), we have \(\|\hat{\chi}_\pi(\pi)\|_1 = 1\), and the description of the norm above gives that

\[
(1.1) \quad u \in ZA(G) \iff u = \sum_{\pi \in \hat{G}} \alpha_\pi \chi_\pi \text{ with } \|u\|_A = \sum_{\pi \in \hat{G}} d_\pi |\alpha_\pi| < \infty.
\]

Hence we have that \(ZA(G)^* \cong ZVN(G) = \ell^\infty \bigoplus_{\pi \in \hat{G}} CI_{d_\pi} \cong \ell^\infty(\hat{G})\).

In particular, we see that \(ZA(G)\) is the predual of a commutative von Neumann algebra. Thus, generally, we will have no need to discuss the completely bounded theory of this space. However, we shall require, some knowledge of the operator space structure on \(A(G)\) in Section 4 which we shall simply reference therein.

Let \(\sim_G\) denote the equivalence relation on \(G\) by conjugacy, i.e. \(x \sim_G x'\) if and only if \(x' = yxy^{-1}\) for some \(y \in G\). It is shown in [22] (34.37) that each element of the Gelfand spectrum of \(ZA(G)\) is given by evaluation functionals from \(G\). Since elements of \(ZA(G)\) are constant on conjugacy classes, we find that there is a continuous map \(\theta\) from Conj\((G) = G/\sim_G\) into this spectrum. Since \(A(G)\) is regular, given two distinct conjugacy classes \(C\) and \(C'\) of \(G\), there is \(u\) such that \(u|_C = 0\) and \(u|_{C'} = 1\). This property also holds so for \(Z_G u\), hence we find that \(\theta\), above, is injective. It is evident that \(Z_G\) has a certain expectation property: \(Z_G(uv) = uZ_G v\) for \(u\) in \(ZA(G)\) and \(v\) in \(A\).

We observe that \(ZA(G)\) is actually the hypergroup algebra \(\ell^1(\hat{G}, d^2)\). Indeed if we let \(\delta_\pi = \frac{1}{d_\pi} \chi_\pi\) then we obtain the hypergroup convolution \(\delta_\pi \delta_\pi' = \sum_{\sigma \subset \pi \otimes \pi'} \frac{d_\sigma}{d_\pi d_\pi'} \delta_\sigma\), where the sum is given all over irreducible subrepresentations of \(\pi \otimes \pi'\), allowing repetitions for multiplicity. Here the Haar measure is given by \(d^2(\{\pi\}) = d^2_\pi\). Notice that \(ZL^1(G)\), the algebra studied in [3], is the hypergroup algebra associated with the compact hypergroup Conj\((G)\). Moreover, \(\hat{G}\) and Conj\((G)\) are mutually dual hypergroups. See [25] to see that \(\hat{G}\) is the dual of Conj\((G)\). That Conj\((G)\) is the dual of \(\hat{G}\) was stated
in [3], but without reference nor proof. A proof of this fact will be given by the first-named author in a forthcoming article. It combines the facts that characters on $ZA(G) \cong \ell^1(\hat{G},\chi^2)$ are in bijective correspondence with elements of the spectrum of $ZA(G)$.

Let us end this section with a simple observation on quotient groups.

**Proposition 1.1.** Let $N$ be a closed normal subgroup of $G$. Then $P_Nu(x) = \int_H u(xn) \, dn$ (normalized Haar integration on $N$) defines a surjective quotient map from $ZA(G)$ to $ZA(G/N)$.

**Proof.** Since translation is an isometric action on $A(G)$, continuous in $G$, it is clear that $P_N : ZA(G) \to A(G/N)$ is a contraction. Let $q : G \to G/N$ be the quotient map. Then there is an obvious embedding $\pi \mapsto \pi \circ q : \hat{G/N} \to \hat{G}$. An easy calculation shows that for $\pi' \in \hat{G}$ we have

\[
P_{\pi'} = \begin{cases} 
\chi_{\pi} \circ q & \text{if } \pi' = \pi \circ q \\
0 & \text{otherwise.}
\end{cases}
\]

See, for example, the proof of [35, Prop. 4.14]. Further, an application of (1.1) shows that the map

\[
\sum_{\pi \in \hat{G/N}} \alpha_{\pi} \chi_{\pi} \mapsto \sum_{\pi \in \hat{G/N}} \alpha_{\pi} \chi_{\pi} \circ q : ZA(G/N) \to ZA(G)
\]

is an isometry whose range is the image of $P_N$. \qed

2. **Groups with non-abelian connected component**

In this section, we visit results which go back to [7] and [40]. However our method of proof seems to lead us more easily to Theorem 2.2. We comment more on this in Remark 2.3 (i), below.

We begin with the case of $G = SU(2)$, the group of $2 \times 2$ unitary matrices of determinant one. We recall that the conjugacy classes in $SU(2)$ are determined by the eigenvalues; i.e. for each $x$ in $SU(2)$ is $y$ in $SU(2)$ for which

\[
x = y \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} y^{-1} \text{ where } \zeta \text{ in } \mathbb{T} \text{ satisfies } \text{Im}\zeta \geq 0.
\]

Hence we may label the conjugacy class by the specified eigenvalue: $C_x = C_{\zeta}$, so $C_{\zeta} = C_{\zeta^{-1}}$. It is well known that

\[
\hat{SU}(2) = \{\pi_l : l = 0, 1, 2, \ldots\}
\]

where $d_{\pi_l} = l + 1$ and we have the associated character, evaluated at $x$ as in (2.1), given by

\[
\chi_{\pi_l}(x) = \chi_{\pi_l}(C_x) = \chi_{\pi_l}(C_{\zeta}) = \sum_{k=0}^l \zeta^{-2k} = \begin{cases} 
\frac{\zeta^{l+1} - \zeta^{-1}}{\zeta - \zeta^{-1}} & \text{if } \zeta \in \mathbb{T} \setminus \{-1, 1\} \\
\zeta^l(l+1) & \text{if } \zeta \in \{-1, 1\}.
\end{cases}
\]
We now recall that the $3 \times 3$ special orthogonal group $SO(3)$ is isomorphic to $SU(2)/\{-I, I\}$ and admits spectrum $\hat{SO}(3) = \{\pi_l : l = 0, 2, 4, \ldots\} \subset \hat{SU}(2)$.

**Proposition 2.1.** Let $S$ denote either $SU(2)$ or $SO(3)$. Then $ZA(S)$ admits a non-zero point derivation $D_z$ at each class $C_z$ of $S$ for which $z$ is a transcendental element of $T$.

**Proof.** Let $Z = \text{span}\{\chi_l : l = 0, 1, 2, \ldots\}$ which is dense in $ZA(SU(2))$. For $u$ in $Z$ and $z$ in $T$ with $\text{Im} z > 0$ let $D_z u = z \frac{d}{d\zeta} u(C_\zeta)\bigg|_{\zeta = z}$. For such $z$ we compute

$$D_z \chi_l = \frac{l(z^{l+2} - z^{-l-2}) - (l + 2)(z^l - z^{-l})}{(z - z^{-1})^2}$$

This implies that

$$|D_z \chi_l| \leq \frac{4l + 4}{|z - z^{-1}|^2}.$$

Moreover, if $z$ is transcendental, then $D_z \chi_l \neq 0$ for any $l > 0$. Now if $u = \sum_{j=1}^n \alpha_j \chi_{l_j}$ for $l_1 < \cdots < l_n$, we can use the formula for the norm (1.1) to see that

$$|D_z u| \leq \sum_{j=1}^n |\alpha_j| \frac{4l_j + 4}{|z - z^{-1}|^2} = \frac{4}{|z - z^{-1}|^2} \sum_{j=1}^n |\alpha_j|(l_j + 1) = \frac{4}{|z - z^{-1}|^2} \|u\|_A.$$

Hence $D_z$ extends to a continuous derivation on $ZA(SU(2))$.

By virtue of Proposition [1.1] and the identification $\hat{SO}(3) \subset \hat{SU}(2)$, above, we have

$$ZA(SO(3)) = \overline{\text{span}\{\chi_l : 0, 2, 4, \ldots\}} \subset ZA(SU(2)).$$

Hence each derivation $D_z$, with transcendental $z$, also defines a non-zero point derivation on $ZA(SO(3))$. \qed

It is tempting to think that all algebraic elements of $T$ are roots of unity. However, for any Pythagorean triple, $a^2 + b^2 = c^2$, the elements $\frac{a+ib}{c}$ contradict this notion.

**Theorem 2.2.** Let $G$ have non-abelian connected component $G_e$. Then $ZA(G)$ admits a non-zero point derivation.

**Proof.** According to the proof of [15 Theo. 2.1], $G_e$ admits a closed subgroup $S$ which is isomorphic to $SU(2)$ or $SO(3)$. [This uses a structure theorem for connected compact groups — see [38] — and the fact that any compact non-abelian Lie algebra admits a copy of $su(2)$.] Since $ZA(G)|_S \subset A(G)|_S = A(S)$ (see [20] or [22 (34.27)], for example), and since for $u$ in $ZA(G)$, $u(yxy^{-1}) = u(x)$ for $x, y$ in $G$, a fortiori in $S$, we have $ZA(G)|_S \subset ZA(S)$. Further, since $S$ is connected and $S \supset \{e\}$, $S$ is contained in more than one
conjugacy class of \( G \). Hence \( \text{ZA}(G)|_S \not\subset \mathbb{C}1 \), and there is \( \pi \) in \( \hat{G} \) for which 
\[
\chi_{\pi}|_S \not\in \mathbb{C}1.
\]
Thus we have 
\[
\pi|_S = \bigoplus_{j=1}^{n} m_j \pi_{l_j}
\]
where each \( m_j \) is a non-zero multiplicity and \( l_1 < \cdots < l_n \) with either \( n > 1 \) or \( l_1 > 0 \). It follows that 
\[
\chi_{\pi}|_S = \sum_{j=1}^{n} m_j \chi_{l_j}
\]
so 
\[
\chi_{\pi}(C_{\zeta}) = \sum_{j=1}^{n} \sum_{k_j=0}^{l_j} \zeta^{l_j-2k_j}
\]
where \( C_{\zeta} \) is the conjugacy class of elements with eigenvalue \( \zeta \) in \( S \). Thus for \( z \) in \( \mathbb{T} \) with \( \text{Im}z > 0 \) we have for the derivation \( D_z \), defined in the proposition above, that 
\[
D_z(\chi_{\pi}|_S) = \sum_{j=1}^{n} \sum_{k_j=0}^{l_j} m_j (l_j - 2k_j) z^{l_j-2k_j}.
\]
Notice that the above expression is a non-zero polynomial in \( z \) of degree \( l_n \). If \( z \) is transcendental over rationals, then 
\[
D_z(\chi_{\pi}|_S) \neq 0.
\]
Thus for such \( z \), 
\[
D = D_z \circ R_S : \text{ZA}(G) \to \mathbb{C},
\]
where \( R_S \) is the restriction map, is a non-zero point derivation. \( \square \)

**Remark 2.3.** (i) The existence of a bounded point derivations on \( \text{ZA}(\text{SU}(2)) \) is already known \([7]\). Moreover, for semisimple compact Lie \( G \), a simplification of a result in \([40]\) gives bounded point derivations at all regular points of \( G \). That result uses the Weyl character formula. Our proof of Theorem 2.2 though relying on some basic Lie theory, is more elementary than the Weyl character formula.

(ii) As mentioned in Section 1, \( \text{ZA}(G) \cong \ell^1(\hat{G}, d^2) \) is a discrete hypergroup algebra. There are other discrete hypergroups, namely certain polynomial hypergroups, whose hypergroup algebras are known to admit point derivations. See \([32]\).

(iii) There are other known examples of \( B \)-central Fourier algebras which admit point derivations. For example, if \( n \geq 3 \), then the algebra of radial elements of \( A(\mathbb{R}^n) \), \( Z_{\text{SO}(n)} A(\mathbb{R}^n) \), admits a point derivation at each infinite orbit \((39, 2.6.10)\). We remark that if we let \( H_n = \mathbb{R}^n \rtimes \text{SO}(n)_d \), \( \mathbb{R}^n \) acted upon by the discretized special orthogonal group, then for odd \( n \) we have \( \text{ZA}(H_n) = Z_{\text{SO}(n)} A(\mathbb{R}^n) \), while for even \( n \), \( \text{ZA}(H_n) = Z_{\text{SO}(n)/\{\pm I\}} A(\mathbb{R}^n \rtimes \{\pm I\}) \).

A general analysis of algebras \( \text{ZA}(H) \) for locally compact \( H \) is beyond the scope of our present investigation. For groups with pre-compact conjugacy classes (a class which does not include examples \( H_n \), above), there are some results for \( ZL^1(H) \) in \([3]\).
3. Virtually abelian groups and groups with abelian connected components

3.1. On sets of synthesis in certain fixed-point subalgebras. The purpose of this section is to gather some abstract results which will be useful for understanding $ZA(G)$ for a virtually abelian compact group $G$, in the next section.

For this section we let $\mathcal{A}$ denote a commutative, unital, semisimple Banach algebra which is regular in its Gelfand spectrum $X$. Given a group of automorphisms $B$ on $\mathcal{A}$ we let $Z_B \mathcal{A} = \bigcap_{\beta \in B} \{ u \in \mathcal{A} : \beta(u) = u \}$ denote the fixed point algebra. If $B$ is compact and acts continuously, we let $Z_B u = \int_B \beta(u) d\beta$ (normalized Haar measure), which may be understood as a Bochner integral. It is a surjective quotient map which satisfies the expectation property $Z_B(\mathcal{A}) = uZ_B \mathcal{A}$ for $u$ in $Z_B \mathcal{A}$ and $v$ in $\mathcal{A}$.

**Proposition 3.1.** Let $B$ be a compact group of continuous automorphisms on $\mathcal{A}$. The Gelfand spectrum of $Z_B \mathcal{A}$ is the orbit space $X/B$.

**Proof.** Since $X$ is the spectrum of $\mathcal{A}$, each $\beta$ in $B$ defines an automorphism $\beta^*|X$ of $X$. It is clear that the orbit space $X/B = \{ B^*x : x \in X \}$, with quotient topology comprises a closed subset of the spectrum of $Z_B \mathcal{A}$. The regularity of $\mathcal{A}$ passes immediately to the regularity of $Z_B \mathcal{A}$ on $X/B$. Indeed, if $B^*x \neq B^*x'$, there is $u$ in $\mathcal{A}$ for which $u|_{B^*x} = 1$ and $u|_{B^*x'} = 0$.

Let $\chi$ be any multiplicative functional on $Z_B \mathcal{A}$. Suppose for some $u_1, \ldots, u_n$ in $\ker \chi$, $\bigcap_{k=1}^n u^{-1}\{0\} \cap X/B = \emptyset$. Then for any $x$ in $X$, i.e. $B^*x$ in $X/B$, there are scalars $z_{x_1}, \ldots, z_{x_m}$ for which

$$\sum_{k=1}^n z_{x_k} u_k(x') > 0$$

for all $x'$ in a neighbourhood in $X/B$, $U_x$ of $x$.

Let $U_{x_1}, \ldots, U_{x_m}$ be a finite subcover of $X/B$. By regularity — see, for example, the proof of [22 (39.21)] — there is a partition of unity $w_1, \ldots, w_m$ in $Z_B \mathcal{A}$ subordinate to the cover $U_{x_1}, \ldots, U_{x_m}$. Let $v_k = \sum_{j=1}^m z_{x_j} k w_j$ and we have that

$$u = \sum_{j=1}^m w_j \sum_{k=1}^n z_{x_j} k u_k = \sum_{k=1}^n u_k v_k \in \sum_{k=1}^n u_k Z_B \mathcal{A} \subset \ker \chi$$

is pointwise positive, hence non-vanishing. Thus $u$ is non-vanishing on $X$ and hence admits an inverse $u'$ in $\mathcal{A}$. But then $Z_B u'$ is an inverse of $u$ in $Z_B \mathcal{A}$. We thus conclude that for all finite families $F \subset Z_B \mathcal{A}$, $\bigcap_{u \in F} u^{-1}\{0\} \cap X/B \neq \emptyset$, and a compactness argument yields that $\bigcap_{u \in Z_B \mathcal{A}} u^{-1}\{0\} \cap X/B \neq \emptyset$, whence $\chi \in X/B$. \qed
Remark 3.2. (i) If we assume that $A$ is conjugate-closed, qua functions on $X$, then for a family of elements $u_1, \ldots, u_n$ for which $\bigcap_{k=1}^n u_k^{-1}\{0\} \cap X/B = \emptyset$, we have that $u = \sum_{k=1}^n |u_k|^2$ forms the invertible element in the proof above. This would allow us to circumvent the partition of unity argument. See [22, (3.43)].

(ii) We can recover the result of [12] that for a compact subgroup $K$ of a locally compact group $H$, the algebra $A(H : K) = \{ u \in A(H) : u(xk) = u(x) \text{ for } u \in H \text{ and } k \in K \}$ has spectrum the coset space $G/K$. Indeed, consider the unitization $A(H) \oplus \mathbb{C}1$ and let $K$ act as automorphisms on this algebra by right translation; we obtain $A(H : K)$ as a subalgebra of $Z_K(A(H) \oplus \mathbb{C}1)$ of functions vanishing at infinity.

Now let $E$ be a closed subset of $X$. We let

$$I_A(E) = \{ u \in A : u|_E = 0 \}, \quad \text{and} \quad I^0_A(E) = \{ u \in A : \operatorname{supp} u \cap E = \emptyset \}.$$ 

We say that $E$ is

- spectral for $A$ if $\overline{I_A(E)} = I_A(E)$;
- Ditkin for $A$ provided each $u$ in $I_A(E)$ satisfies that $u \in \overline{I^0_A(E)}$;
- hyper-Ditkin if $I^0_A(E)$ possesses a bounded approximate identity for $I_A(E)$; and
- approximable if $I_A(E)$ possesses a bounded approximate identity.

Following [51], we will say that $E$ is strongly Ditkin for $A$ if $\overline{I^0_A(E)} = I_A(E)$ possesses a multiplier bounded approximate identity $(u_\alpha)$ or $I_A(E)$, i.e. so $\sup_{\alpha} \| u_\alpha u \| \leq C \| u \|$ for all $u$ in $I_A(E)$. Note that a sequential approximate identity is automatically multiplier bounded, thanks to the uniform boundedness principle. We shall not require this last notion but mention it only for comparative purposes. Notice that hyper-Ditkinness implies strong Ditkinness which, in turn, implies either of approximability or Ditkinness, and Ditkinness implies spectrality. We are unaware of any examples of approximable sets which are not spectral. Approximability and spectrality together imply hyper-Ditkinness.

Remark 3.3. A longstanding open question is whether finite unions of spectral sets are spectral. It is known, due to [49] (see also [29, 5.2.1]) that a finite union of Ditkin sets is Ditkin. An easy variant of an argument of [51] tells us that the same is true for hyper-Ditkin or approximable sets. Indeed, if $E$ and $F$ are approximable (respectively, hyper-Ditkin), let $(u_\alpha)$ be a bounded approximate identity for $I_A(E)$ and $(v_\beta)$ one for $I_A(F)$ (each contained in $I^0_A(E)$, respectively $I^0_A(F)$). Then $(u_\alpha v_\beta)$ (product directed set) is a bounded approximate identity for $I_A(E \cup F) = I_A(E) \cap I_A(F)$ (contained in $I^0_A(E \cup F) = I^0_A(E) \cap I^0_A(F)$), as is easily checked.

We now observe that finite groups of automorphisms preserve certain properties of sets which are stable under finite unions.
Theorem 3.4. Let $B$ be a finite group of automorphisms on $A$ and $E$ be a closed subset of $X$ which is Ditkin, hyper-Ditkin or approximable for $A$. Then the subset $B^*E$ of $X/B$ enjoys the same property for $Z_BA$.

Proof. We observe that for each automorphism $\beta$, we have $\beta(I_A(E)) = I_A(\beta^*(E))$ and $\beta(I_A^0(E)) = I_A^0(\beta^*(E))$. Hence Remark 3.3 shows that $B^*E = \bigcup_{\beta \in B} \beta^*E$ is Ditkin, hyper-Ditkin or approximable for $A$, based on the respective assumption for $E$. Furthermore, we observe that for any $u$ in $A$ we have $Z_Bu = \frac{1}{|B|} \sum_{\beta \in B} \beta(u)$, and hence

$$Z_BI_A^0(B^*E) = I_{Z_BA}^0(B^*E) \subseteq I_A^0(B^*E)$$

and the same sequence of inclusions holds for $I_A$. Suppose $u \in I_{Z_BA}(B^*E)$ and $(u_\alpha)$ is a net from $I_A^0(E)$ for which $\|uu_\alpha - u\| \to 0$. Then $uZ_Bu_\alpha - u = Z_B(uu_\alpha - u) \to 0$. We immediately see that Ditkenness or hyper-Ditkenness is preserved. By merely picking $(u_\alpha)$ from within $I_A(E)$, we see that approximability is preserved. \qed

Proposition 3.5. If the projective tensor product $A \hat{\otimes} A$ is semisimple, and $B$ is a compact group of automorphisms on $A$, then $Z_BA \hat{\otimes} Z_BA = Z_{B \times B}A \hat{\otimes} A$.

Proof. Since $Z_B$ is a quotient map, $Z_BA \hat{\otimes} Z_BA$ is isometrically a subspace of $A \hat{\otimes} A$. Moreover, $Z_B \otimes Z_B = Z_{B \times B}$. \qed

Suppose $A \hat{\otimes} A$ is semisimple. With our assumptions $A \hat{\otimes} A$ is regular on its spectrum $X \times X$ (48). Then, following [43 Theo. 6], we call $A$ hyper-Tauberian if the diagonal

$$X_D = \{(x, x) : x \in X\}$$

is spectral for $A \hat{\otimes} A$. It is a well-known interpretation of the splitting result of [19] (see also [8]) that approximability of $X_D$ for $A \hat{\otimes} A$ is equivalent to amenability of $A$. We further note that for a compact group of automorphisms on $A$ that

$$(X/B)_D = \{(B^*x, B^*x) : x \in X\} = (B \times B)^*X_D.$$

These comments combine with the last two results to give us the following.

Corollary 3.6. Suppose $A \hat{\otimes} A$ is semisimple and let $B$ be a finite group of continuous automorphisms on $A$.

(i) If $X_D$ is Ditkin for $A \hat{\otimes} A$, then $Z_BA$ is hyper-Tauberian.

(ii) If $A$ is amenable, then $Z_BA$ is amenable.

We observe that (ii), above, follows from a more general result of [30].

We shall say that a closed subset $E$ of $X$ is weakly spectral for $A$ if there is a fixed $n > 0$ for which $I_A(E)^n = \{u^n : u \in I_A(E)\} \subseteq I_A^0(E)$. We let the characteristic of $E$ with respect to $A$, $\xi_A(E)$, denote the minimal such $n$, so $\xi(E) = 1$ if $E$ is spectral. These concepts were introduced in [50].
**Proposition 3.7.** Suppose $B$ is a compact group of automorphisms on $A$ and $E = B^*E$ be weakly spectral for $A$. Then $E$ is weakly spectral for $Z_B A$ (with $\xi_{Z_B A}(E) \leq \xi_A(E)$).

**Proof.** It is evident that $Z_B I_A(E) = I_{Z_B A}(E)$. It is also true that $Z_B I^0_A(E) = I^0_{Z_B A}(E)$. Indeed, if $u \in I^0_A(E)$, then supp $u \cap E = \emptyset$. Then there there is open $U = B^*U \supset E$ such that supp $u \cap U = \emptyset$. If not, then

$$\text{suppu} \cap E = \text{suppu} \cap \bigcap \{U : U = B^*U \text{ open}, U \supset E\} \neq \emptyset$$

violating our initial assumption. Thus it follows that supp $(Z_G u) \cap E = \emptyset$.

We note that $I_{Z_B A}(E) \subset I_A(E)$. Hence if $E$ is weakly spectral for $A$, then for $u \in I_{Z_B A}(E)$, $u^{\xi_A(E)} \in I^0_A(E)$. But

$$u^{\xi_A(E)} = Z_B u^{\xi_A(E)} \in Z_B I^0_A(E) \subseteq Z_B I^0_A(E) = I^0_{Z_B A}(E).$$

Hence $\xi_{Z_B A}(E) \leq \xi_A(E)$.

\[ \square \]

**3.2. Virtually abelian groups.** We say a locally compact group is virtually abelian if it admits an abelian subgroup of finite index. In the case of a compact $G$, this is equivalent to having an open abelian subgroup.

**Theorem 3.8.** Let $G$ be virtually abelian. Then there exists a normal open abelian subgroup $T$. We have $T$-centre, i.e. when $T$ acts on $G$ by inner automorphisms, given by an isomorphic identification

$$Z_T A(G) = \bigoplus_{aT \in G/T} A(aT : R_a)$$

where $R_a = R_{aT}$ is a closed subgroup of $T$ and $A(aT : R_a) = \{u \in 1_{aT} A(G) : u(atr) = u(at) \text{ for } t \in T \text{ and } r \in R_a\}$. The algebra $Z_T A(G)$ admits spectrum $X = \bigsqcup_{aT \in G/T} aT/R_a$. We have

$$Z A(G) = Z_{G/T} Z_T A(G)$$

where the action of $G$, i.e. of $G/T$, on an element of $X$ is given by $bT \cdot aT R_a = bab^{-1}b^{-1}r R_{bab^{-1}}$. For each $a$ in $G$ and $t$ in $T$, we have conjugacy class

$$C_{at} = \{bab^{-1}b^{-1}r : b \in G \text{ and } r \in R_{bab^{-1}T}\}.$$

**Proof.** Let $S$ be an open abelian subgroup and $L$ a left transversal for $S$ in $G$. Then

$$T = \bigcap_{b \in L} bSb^{-1}$$

is an open normal subgroup. Indeed, $L$ is finite and $T$ is an intersection of finitely may open subgroups. Furthermore, the definition of $T$ is independent of choice of transversal and for any $a$ in $G$, $aL$ is another transversal, hence $aT a^{-1} = \bigcap_{b \in aL} bSb^{-1} = T$.

Now if $a$ in $G$ and $t$ in $T$ are fixed, then for $s$ in $T$ we have

$$sats^{-1} = a(a^{-1}sa)ts^{-1} = ats^{-1}(a^{-1}sa).$$
Since $T$ is abelian and compact we have that

$$R_a = \{ s^{-1}a^{-1}sa : s \in T \}$$

is a closed subgroup of $T$. Observe that for $a$ and $a'$ in $G$, if $Ta = T'a'$, then $R_a = R_{a'}$. Hence we may write $R_{Ta} = R_{aT} = R_a$. We recall that $A(G) = \bigoplus_{aT \in G/T} A(aT)$ where $A(aT) = 1_{aT}A(G) \cong A(T)$ and we obtain the desired form for $ZA(G)$ and its spectrum $X = G/\sim_T$.

It is evident that $ZA(G) \subseteq Z_{\ell^1}(G)$, and the action of $G$ by inner automorphisms on $G/\sim_T$ is really an action by $G/T$. In fact of we let $ZTu(x) = \int_T u(sxs^{-1})ds$, then the Weyl integral formula tells us that $Z_G = Z_G \circ Z_T = Z_{G/T} \circ Z_T$. Hence we gain the desired realization of $ZA(G)$.

To see the action of $G$ on $X$, and hence the structure of the conjugacy class $C_{at}$, we fix $a$ and $t$ as above, and for $b \in G$ and $s \in T$ we have

$$bab^{-1}(btb^{-1})[bs^{-1}b^{-1}(ba^{-1}b^{-1}bsb^{-1}bab^{-1})] = bab^{-1}(ba^{-1}b^{-1}bsb^{-1}bab^{-1})(btb^{-1})bs^{-1}b^{-1} = bs(at)(bs)^{-1}.$$

Since each $bsb^{-1}$ is a generic element of $T$, we get the desired result. \qed

**Theorem 3.9.** If $G$ is virtually abelian, then $ZA(G)$ is hyper-Tauberian and amenable.

**Proof.** We consider the algebra $ZA(G)$ and its spectrum $X$. Each $A(aT : R_a) \cong A(T/R_a)$ (which is the abelian group algebra $L^1(T/R_a)$). The diagonal $(T/R_a)_D^2$ in $A(T/R_a) \otimes A(T/R_a)$ is a subgroup and hence, thanks to [22], hyper-Ditkin. Thus Remark 3.3 shows us that $X_D \cong \bigcup_{aT \in G/T} (T/R_a)_D$ is hyper-Ditkin. Letting $B = G/T$, we appeal to Corollary 3.6. \qed

It is surprisingly easy, from this point, to give a characterization of those subsets $E$ of $\text{Conj}(G)$ which are approximable (as defined in the last section) for $ZA(G)$. For abelian $G$ this was achieved in [33] and for amenable $G$ this was achieved in [13]. A coset is any subset $K$ of $G$ which is closed under the ternary operation: $x, y, z \in K$ implies $xy^{-1}z \in K$. It is an exercise to see that this agrees with the “standard” notion of coset of some subgroup. We let $\Omega(G)$ denote the Boolean algebra generated by all cosets of $G$, and $\Omega_c(G)$ denote those elements of $\Omega(G)$ which are closed.

**Proposition 3.10.** Let $G$ be compact, and $E$ be a closed subset of $\text{Conj}(G)$. Then $E$ is approximable for $ZA(G)$ if and only if $\hat{E} = \bigcup_{C \in E} C \in \Omega_c(G)$.

**Proof.** If $(u_a)$ is a bounded approximate identity for $I_{ZA}(E)$. Then $(u_a)$ is a bounded net in $A(G) \subseteq B(G_d)$, where the latter space is the Fourier-Stieltjes algebra of the discretized group $G_d$. The embedding is an isometry thanks to [10]. Since $ZA(G)$ is regular, $(u_a)$ converges pointwise to the indicator function $1_{\text{Conj}(G) \setminus E}$ on $\text{Conj}(G)$, hence to $1_{G \setminus \hat{E}}$ in the weak* topology of $B(G_d)$. Hence by [23], $\hat{E} \in \Omega(G)$. Since $\hat{E}$, being the pre-image of $E$ in $G$ under the conjugation equivalence, is closed, we gain the desired conclusion.
If \( \tilde{E} \in \Omega_c(G) \), then by Remark 3.13, \( \tilde{E} \) is approximable for \( A(G) \). If \( (v_\alpha) \) is a bounded approximate identity for \( I_\Lambda(E) \), then \( (Z_G v_\alpha) \) is such for \( I_{ZA(E)} \).

We remark that the last proposition reduces the general question of amenability of \( ZA(G) \) into a question of whether \( \text{Conj}(G)_D \) is in \( \Omega(G \times G) \). (Indeed it follows form Lemma 4.1 that \( ZA(G) \otimes ZA(G) \cong ZA(G \times G) \). We do not know how to determine this for a general, even totally disconnected, group, unless it is a product of finite groups; see Theorem 4.3 for example.

Assessing which conjugacy-closed subsets are in \( \Omega_c(G) \) for virtually abelian \( G \) is surprisingly simple.

**Proposition 3.11.** If \( G \) is virtually abelian and \( E \in \Omega_c(G) \), then \( G^* E = \bigcup_{x \in E} C_x \in \Omega_c(G) \).

**Proof.** A result of [11] (after [13, 41]) shows that there are a finite number of closed subgroups \( H_1, \ldots, H_n \), elements \( a_1, \ldots, a_n \) of \( G \), and for each \( k \), open subgroups \( K_{k1}, \ldots, K_{km_k} \) of \( H_k \) and elements \( b_{k1}, \ldots, b_{km_k} \) of \( H_k \) such that \( E = \bigcup_{k=1}^n \sum_{j=1}^{m_k} a_{kj} \left( H_k \setminus \bigcup_{j=1}^{m_k} b_{kj} K_{kj} \right) \). Since each \( H_k \) is compact, each \( K_{kj} \) is of finite index. We can hence rearrange this result to show that \( E \) is simply a union of finitely many closed cosets of subgroups of \( G \). Let such a coset be given by \( aH \) where \( H \) is a closed subgroup. By taking intersection, we may suppose \( H \) is a subgroup of an open normal abelian subgroup \( T \), hence \( aH \subset aT \).

However, the calculations from the proof of Theorem 3.8 show that the orbit of \( aH \) under conjugation by \( G \) is \( \bigcup_{b \in G} bab^{-1} bHb^{-1} R_{bab^{-1}} \), which is clearly an element of \( \Omega_c(G) \).

**Remark 3.12.** Let \( G \) and \( T \) be as in Theorem 3.8. Using reasoning above, we see that \( 1_T ZA(G) = ZA(G) / T \), is amenable. In particular for \( G = T \times \{\text{id, t}\} \), where \( \iota(t) = t^{-1} \), we have that \( Z_{\{\text{id, t}\}} A(T) \cong Z_{\{\text{id, t}\}} \ell^1(T) \cong \ell^1(Z/\{\text{id, t}\}) \). Here, \( Z/\{\text{id, t}\} \cong \mathbb{N}_0 \) is the polynomial hypergroup with multiplication \( \delta_n * \delta_m = \frac{1}{2} (\delta_{n-m} + \delta_{n+m}) \). This hypergroup algebra is is also proved to be amenable in [31].

In fact, we may define a class of hypergroups by letting \( F \) be any finite subgroup of \( GL_n(\mathbb{Z}) \) and considering the orbit space \( \mathbb{Z}^n / F \). We let \( \ell^1(\mathbb{Z}^n / F) \) denote the closed subalgebra of \( \ell^1(\mathbb{Z}^n) \) generated by elements

\[
\delta_{F(v)} = \frac{1}{|F|} \sum_{\alpha \in F} \delta_{\alpha(v)}, \quad v \in \mathbb{Z}^n.
\]

We have that \( \ell^1(\mathbb{Z}^n / F) \cong Z_F \ell^1(\mathbb{Z}^n) \) is amenable.

**Remark 3.13.** Similarly as in Theorem 3.8 by working on the dense sub-space of continuous functions, we may obtain a decomposition

\[
ZL^1(G) \cong ZG/T ZT L^1(G) \cong ZG/T \left( \bigoplus_{aT \in G/T} \delta_a * L^1(T/RaT) \right).
\]
Hence if we could prove that $Z_T L^1(G)$ is amenable, we will gain the amenability of $ZL^1(G)$ thanks to the main result of [30].

### 3.3. Hyper-Tauberian property and weak amenability

We can give a complete characterization of both hyper-Tauberianess and weak amenability for $ZA(G)$.

**Theorem 3.14.** For any compact group $G$ the following are equivalent:

1. the connected component of the identity, $G_e$, is abelian;
2. $ZA(G)$ is hyper-Tauberian;
3. all singleton sets of $\text{Conj}(G)$ are spectral for $ZA(G)$;
4. $ZA(G)$ is weakly amenable; and
5. $ZA(G)$ admits no bounded point derivations.

**Proof.** That (ii) implies (iii) and (iv) are both from [43, Theo. 5]. A well-known observation from [45] is that a commutative Banach algebra admits a bounded point derivation at a multiplicative functional $\chi$ if and only if $(\ker \chi)^2$ is dense in $\ker \chi$. Hence (iii) implies (v). That (iv) implies (v) follows from a well-known fact mentioned in Section 1. Theorem 2.2 provides that (v) implies (i).

Hence it remains to see that (i) implies (ii). In the case that $G$ is virtually abelian, this is from Theorem 3.9.

Now let us consider the general case where $G_e$ is abelian. We follow the proof of [14, Theo. 3.3]. We let $\mathcal{N}$ be a net of closed normal subgroups, ordered by reverse inclusion, such that for any neighbourhood $U$ of $e$, we eventually have $N \subseteq U$ for some $N$ in $\mathcal{N}$, and for which $G/N$ is Lie for each $N$ in $\mathcal{N}$. For example, we may let $N = N_F = \bigcap_{\pi \in F} \ker \pi$ for the increasing net of all finite subsets of $\hat{G}$. Then each $G/N$ is virtually abelian, which follows from [21, (7.12)], for example. Hence by Proposition 1.1, $Z_N^{A}(G/N) \cong P_N(ZA(G))$, and hence is hyper-Tauberian. Also if $N \supset N'$ then $P_N(ZA(G)) \subseteq P_{N'}(ZA(G))$. It then is easy to check that for $u$ in $ZA(G)$, the difference $u - P_N u$ tends to 0 as $N$ tends to $e$. Hence $\bigcup_{N \in \mathcal{N}} P_N ZA(G)$ is dense in $ZA(G)$. We appeal to [43, Cor. 13] to see that $\ell^1 \bigoplus_{N \in \mathcal{N}} P_N ZA(G)$ is hyper-Tauberian, and hence the completion $Z_{\mathcal{N}}(G)$ of $\bigcup_{N \in \mathcal{N}} P_N ZA(G)$, with respect to the norm

$$
\|u\|_{\mathcal{N}} = \inf \left\{ \sum_{N \in \mathcal{N}} \|u_N\|_A : u = \sum_{N \in \mathcal{N}} u_N, u_N \in P_N ZA(G) \right\} \geq \|u\|_A
$$

is hyper-Tauberian, thanks to [43, Theo. 12]. Notice for $N \supset N'$ that $P_N ZA(G) P_{N'} ZA(G) \subseteq P_{N'} ZA(G)$, so $\|\cdot\|_{\mathcal{N}}$ is indeed an algebra norm. But the continuous inclusion with dense range, $Z_{\mathcal{N}}(G) \hookrightarrow ZA(G)$, shows again by [43, Theo. 12] that the latter algebra is hyper-Tauberian.

**Remark 3.15.** (i) We note that using the density of $\bigcup_{N \in \mathcal{N}} P_N ZA(G)$ in $ZA(G)$, above, we may have more easily shown that (i) implies (iv) directly. We found it more satisfying to obtain the stronger hyper-Tauberian property.
We note that [43, Rem. 24(ii)] shows that hyper-Tauberianness is stronger than weak amenability.

(ii) A technique employed in the proof of [16, Lem. 3.6] and [3, Theo. 2.4] can be used to allow us to bypass the algebra $Z_{A\lambda}(G)$, employed above. Our present technique allows us to avoid introducing local maps, which, admittedly, are used in the original definition of hyper-Tauberianness in [43].

The following partially recovers a non-spectral result of [37], but uses different methods. We defined weak spectrality at the end of the last section.

**Proposition 3.16.** Let $G$ be a non-abelian compact connected Lie group.

(i) There exists a $C$ in $\text{Conj}(G)$ for which $C$ is not spectral for $A(G)$.

(ii) Any finite $F \subseteq \text{Conj}(G)$ is weakly spectral for $Z_A(G)$ with $\xi_{Z_A}(F) \leq |F| + \sum_{C \in F} \dim C/2$ where $\dim C$ is the dimension of the manifold $C$.

**Proof.** The failure of spectrality for some $\{C\}$ follows from (iii), above, and Proposition 3.7. On the other hand, [36, Cor. 4.9] shows that a single conjugacy class $C$ is always weakly spectral for $A(G)$ with $\xi_A(C) \leq 1 + \dim C/2$. Hence, a result in [50] shows that

$$\xi_A\left(\bigcup_{C \in F} C\right) \leq \sum_{C \in F} \xi_A(C) = |F| + \sum_{C \in F} \dim C/2.$$ 

Again we appeal to Proposition 3.7. $\square$

### 4. Finite groups and their direct products

We recall from [27], that amenability of a Banach algebra $A$ is equivalent to having a bounded approximate diagonal (b.a.d.): a bounded net $(d_i) \subset A \hat{\otimes} A\!$ which for each $a$ in $A$ satisfies

$$(a \otimes 1)d_i - d_i(1 \otimes a) \xrightarrow{i} 0 \text{ and } m(d_i)a \xrightarrow{i} a$$

where $m : A \hat{\otimes} A \to A$ is the multiplication map. We let the amenability constant be given by

$$\text{AM}(A) = \inf\{M > 0 : \text{there is a b.a.d. } (d_i) \text{ for } A \text{ with } \|d_i\|_{A \hat{\otimes} A} \leq M\}$$

where we adopt the convention that $\inf\emptyset = \infty$. It will be useful for us to understand the tensor product $Z_{\text{A}(G)} \hat{\otimes} Z_{\text{A}(G')}$, where $G'$ is another compact group.

**Lemma 4.1.** We have an isometric isomorphism

$$Z_{\text{A}(G)} \hat{\otimes} Z_{\text{A}(G')} \cong Z_{\text{A}(G \times G')}.$$ 

**Proof.** Let us give two proofs.

For the first, we recall the theorem of [39] that

$$A(G) \hat{\otimes}^{op} A(G') \cong A(G \times G')$$

(4.1)
where \( \hat{\otimes}^{op} \) denotes the operator projective tensor product. The map \( Z_G : A(G) \to ZA(G) \) is easily verified to be a complete quotient map, so we have a completely isometric inclusion

\[
ZA(G) \hat{\otimes}^{op} ZA(G') = Z_G \otimes Z_{G'}(A(G) \hat{\otimes}^{op} A(G')) \subset A(G) \hat{\otimes}^{op} A(G').
\]

But in the identification (4.1), we have that \( Z_G \otimes Z_{G'} \cong Z_{G \times G'} \), so \( Z_G \otimes Z_{G'}(A(G) \hat{\otimes}^{op} A(G')) \cong ZA(G \times G') \). Since \( ZA(G)^* \cong ZVN(G') \) is a commutative von Neuman algebra, we obtain an isometric identification

\[
ZA(G) \hat{\otimes}^{op} ZA(G') \cong ZA(G) \hat{\otimes} ZA(G').
\]

For the second proof, we use the fact that \( ZA(G) \cong \ell^1(\hat{G}, d^2) \) as noted in Section 1. Using that \( \hat{G} \times G' \cong \hat{G} \times \hat{G}' \) (irreducible representations of products are exactly the Kronecker products of irreducible representations) we see that \( ZA(G \times G') \cong \ell^1(\hat{G} \times \hat{G}', d^2 \times d^2) \), where \( d^2 \times d^2(\pi, \pi') = d^2_{\pi} d^2_{\pi'} \).

Hence the usual tensor product formula shows that

\[
ZA(G) \hat{\otimes} ZA(G') \cong \ell^1(\hat{G}, d^2) \hat{\otimes} \ell^1(\hat{G}', d^2) \cong \ell^1(\hat{G} \times \hat{G}', d^2 \times d^2) \cong ZA(G \times G')
\]

with isometric identifications. \( \square \)

The following computation on a finite group mirrors [3, Theo. 1.8].

**Proposition 4.2.** Let \( G \) be a finite group. Then

\[
AM(ZA(G)) = \frac{1}{|G|^2} \sum_{\pi, \pi' \in \hat{G} \times \hat{G}} d_\pi d_{\pi'} \left| \sum_{C \in \text{Conj}(G)} |C|^2 \chi_\pi(C) \overline{\chi_{\pi'}(C)} \right|.
\]

In particular we see that \( 1 \leq AM(ZA(G)) \), with the bound achieved exactly when \( G \) is abelian.

**Proof.** Any bounded approximate diagonal admits a cluster point, which is a diagonal; i.e. \( d \in ZA(G) \hat{\otimes} ZA(G) \) for which \( m(d) = 1 \) and \( (u \otimes 1)d = d(1 \otimes u) \).

It was observed in [17] that for a finite dimensional amenable commutative algebra that the diagonal is unique. In fact this diagonal must be the indicator function of the diagonal of the spectrum of \( ZA(G) \), \( 1_{\text{Conj}(G)} = \sum_{C \in \text{Conj}(G)} 1_{C \times C} \). The Schur orthogonality relations provide Fourier series

\[
1_{C \times C} = \sum_{\pi, \pi' \in \hat{G} \times \hat{G}} \langle 1_{C \times C} | \chi_\pi \otimes \chi_{\pi'} \rangle \chi_\pi \otimes \chi_{\pi'}
\]

\[
= \sum_{\pi, \pi' \in \hat{G} \times \hat{G}} \left( \frac{1}{|G|^2} \sum_{x, y \in \hat{G} \times \hat{G}} 1_{C \times C}(x, y) \overline{\chi_\pi(x) \chi_{\pi'}(y)} \right) \chi_\pi \otimes \chi_{\pi'}
\]

\[
= \frac{1}{|G|^2} \sum_{\pi, \pi' \in \hat{G} \times \hat{G}} |C|^2 \chi_\pi(C) \overline{\chi_{\pi'}(C)} \chi_\pi \otimes \chi_{\pi'}
\]
and hence

\[ 1_{\text{Conj}(G)_D} = \frac{1}{|G|^2} \sum_{\pi, \pi' \in \hat{G} \times \hat{G}} \left( \sum_{C \in \text{Conj}(G)} |C|^2 \overline{\chi_{\pi}(C)} \chi_{\pi'}(C) \right) \chi_{\pi} \otimes \chi_{\pi'} \]

where we have exchanged \( \bar{\pi} \) for \( \pi \) to give our formula its “positive-definite” flavour. We then appeal to Lemma 4.1 and obtain

\[ \text{AM}(Z_A(G)) = \left\| 1_{\text{Conj}(G)_D} \right\|_{Z_A(G \times G)} \]

which, by (1.1) gives us the desired result.

Let us examine the lower bound. We restrict the outer sum to the diagonal to obtain

\[ \text{AM}(Z_A(G)) \geq \frac{1}{|G|^2} \sum_{\pi \in \hat{G}} d_{\pi}^2 \sum_{C \in \text{Conj}(G)} |C|^2 \overline{\chi_{\pi}(C)} \chi_{\pi}(C) \]

\[ \geq \frac{1}{|G|} \sum_{\pi \in \hat{G}} d_{\pi}^2 \sum_{C \in \text{Conj}(G)} \frac{|C|}{|G|} \overline{\chi_{\pi}(C)} \chi_{\pi}(C) \]

\[ = \frac{1}{|G|} \sum_{\pi \in \hat{G}} d_{\pi}^2 \chi_{\pi} \chi_{\pi} = \frac{1}{|G|} \sum_{\pi \in \hat{G}} d_{\pi}^2 = 1 \]

Notice that if \( G \) is non-abelian, then at least one conjugacy class satisfies \( |C|^2 > |C| \), and the second inequality, above, is strict. For an abelian group, \( Z_A(G) = A(G) \cong L^1(\hat{G}) \). The well-known diagonal \( \frac{1}{|G|} \sum_{\chi \in \hat{G}} \delta_{\chi} \otimes \delta_{\chi} \) shows that \( \text{AM}(L^1(\hat{G})) = 1 \).

For a finite non-abelian group, the lower bound of \( \text{AM}(ZL^1(G)) \geq 1 + \frac{1}{300} \) was derived based on a result in [41]. Some improvements were made in the investigation [2], however no better general lower bound, valid for all non-abelian finite groups, is known to the authors. Hence the following result stands as a pleasant contrast.

**Corollary 4.3.** If \( G \) is a non-abelian finite group, then \( \text{AM}(Z_A(G)) \geq \frac{2}{\sqrt{3}} \).

**Proof.** Because \( G \) is compact, we have that \( A(G) \) is its own multiplier algebra, even its own completely bounded multiplier algebra. As such each \( u \) in \( A(G) \) induces a Schur multiplier on \( G \times G \) matrices, \( [a_{st}] \mapsto [u(s^{-1}t)a_{st}] \), with norm the same as \( \|u\|_A \). See [5, 28] for details of this.

The reasoning above also applies to \( A(G \times G) \). The diagonal \( w = 1_{\text{Conj}(G)_D} \) is an element of \( Z_A(G \times G) \subset A(G \times G) \) is an idempotent, i.e. \( w^2 = w \), and \( \|w\|_A > 1 \). Hence by [17, Theo. 3.3] (using estimates which go back to [31]), we have that \( \text{AM}(Z_A(G)) = \|w\|_A \geq \frac{2}{\sqrt{3}} \).

**Lemma 4.4.** (i) If \( G_1, \ldots, G_n \) are finite groups and \( P = \prod_{i=1}^n G_i \), then

\[ \text{AM}(Z_A(P)) = \prod_{i=1}^n \text{AM}(Z_A(G_i)) \].
(ii) If $G = H \times F$ where $H$ is compact and $F$ is finite, then

$$\text{AM}(\text{ZA}(G)) \geq \text{AM}(\text{ZA}(F)).$$

Proof. To see (i), we use Lemma 4.1 and the isomorphism $P \times P \cong \prod_{i=1}^{n} G_i \times G_i$ to see that

$$\text{ZA}(G \times G) \cong \text{ZA}(G_1 \times G_1) \hat{\otimes} \ldots \hat{\otimes} \text{ZA}(G_1 \times G_1).$$

Hence the unique diagonal satisfies

$$1_{\text{Conj}(G)} \sim 1_{\text{Conj}(G_1)} \hat{\otimes} \ldots \hat{\otimes} 1_{\text{Conj}(G_n)}.$$ We appeal to the fact that $\hat{\otimes}$ gives a cross-norm.

To see (ii) we have that the map $u \otimes v \mapsto u(e)v : \text{ZA}(H) \hat{\otimes} \text{ZA}(F) \to \text{ZA}(F)$ extends to a contractive surjective homomorphism, and hence, again using Lemma 4.1 induces a contractive surjective homomorphism from $\text{ZA}(G)$ onto $\text{ZA}(F)$. It is standard and straightforward to check that if $\text{AM}(\text{ZA}(G)) < \infty$, then any bounded approximate diagonal for $\text{ZA}(G)$ is carried to such for $\text{ZA}(F)$, hence the diagonal for $\text{ZA}(F)$ has norm bounded above by $\text{AM}(\text{ZA}(G))$. □

We lend the following evidence to our conjecture that $\text{ZA}(G)$ is amenable if and only if $G$ is virtually abelian.

**Theorem 4.5.** Let $\{G_i\}_{i \in I}$ be an collection of finite groups and $P$ be the compact group product group $\prod_{i \in I} G_i$. Then $\text{ZA}(P)$ is amenable if and only if all but finitely many groups $G_i$ are abelian.

Proof. Suppose there is an infinite sequence of indices $i_1, i_2, \ldots$ for which each $G_{i_k}$ is non-abelian. Let $P_n = \prod_{k=1}^{n} G_{i_k}$ and $H_n = \prod_{i \in I \setminus \{i_1, \ldots, i_n\}} G_i$. We successively use parts (ii) and (i) of the lemma above, then Corollary 4.3 to see for each $n$ that

$$\text{AM}(\text{ZA}(P)) \geq \text{AM}(\text{ZA}(P_n)) = \prod_{k=1}^{n} \text{AM}(\text{ZA}(G_{i_k})) \geq (2/\sqrt{3})^n.$$ Thus we see that $\text{AM}(\text{ZA}(P)) = \infty$. □

Using techniques from the theory of hypergroups, the first-named author ([1]) has proved that if $G$ is tall, i.e. $\lim_{\pi \to \infty} d_{\pi} = \infty$, then $\text{ZA}(G)$ is non-amenable. There are examples of totally disconnected tall groups in [24]. Coupled with the last theorem, this gives two classes of totally disconnected and non-virtually abelian $G$ for which $\text{ZA}(G)$ is non-amenable.

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