The symmetric Post Correspondence Problem, and errata for the freeness problem for matrix semigroups

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Abstract

We define the symmetric Post Correspondence Problem (PCP) and prove that it is undecidable. As an application we show that the original proof of undecidability of the freeness problem for 3×3 integer matrix semigroups works for the symmetric PCP, but not for the PCP in general.

1 Introduction

The Post Correspondence Problem (PCP) was introduced, and proved to be undecidable, by Emil Post in 1946 [8]. Let A be a finite alphabet of size at least 2, and let $A^*$ be the set of all finite strings, including the empty string $\varepsilon$. The statement of the PCP over the alphabet A is as follows.

INPUT: A non-empty finite set $\{(u_i, v_i) : 0 \leq i < k\}$ of ordered pairs of strings in $A^*$, for some $k > 0$.

QUESTION: Does there exist a non-empty finite sequence $(j_1, \ldots, j_n)$ of numbers in $\{0, \ldots, k-1\}$ such that $u_{j_1} \ldots u_{j_n} = v_{j_1} \ldots v_{j_n}$? Equivalently, does the subsemigroup $\langle \{(u_i, v_i) : 0 \leq i < k\}\rangle$ of $A^* \times A^*$ intersect $\{(x, x) : x \in A^*\}$?

The PCP is called bounded if and only if some upper-bound on k has been fixed beforehand.

Notation: $A^* \times A^*$ is the direct product of the free monoid $A^*$ with itself. For any set $S \subseteq A^* \times A^*$, $\langle S \rangle$ is the subsemigroup of $A^* \times A^*$ generated by S. Two sets X and Y are said to intersect if and only if $X \cap Y \neq \emptyset$. For a string $w \in A^*$, $|w|$ denotes the length of w; and for a finite set X, $|X|$ denotes the cardinality. We denote concatenation of strings $u, v \in A^*$ by $uv$. For $u, w \in A^*$ we call u a prefix of w if and only if there exists $v \in A^*$ such that $w = uv$. We say that $u, v \in A^*$ are prefix-comparable if and only if u is a prefix of v or v is a prefix of u.

Non-triviality assumption: In order to avoid trivial solutions of the PCP (consisting of a single input pair), we assume that the input satisfies $u_i \neq v_i$ for all i.

The symmetric Post Correspondence Problem (symPCP) has the same problem statement as the PCP, but with the additional restriction that the input relation should be symmetric; i.e., for every $(u, v) \in \{(u_i, v_i) : 0 \leq i < k\}$ we also have $(v, u) \in \{(u_i, v_i) : 0 \leq i < k\}$.

In other words, the PCP is symmetric if and only if for every $i \in \{0, \ldots, k-1\}$ there exists $i' \in \{0, \ldots, k-1\}$ such that $(u_i, v_i) = (v_{i'}, u_{i'})$. 

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In Section 2 we prove that the symmetric PCP is undecidable. In Section 3 we address some issues about the original proof of undecidability of the freeness problem for semigroups of $3 \times 3$ integer matrices in [6]. In Subsection 3.2 the most significant of these issues is resolved by reduction from the symmetric PCP instead of the general PCP; this was our initial motivation for looking at the symmetric PCP.

2 Undecidability of the symmetric Post Correspondence Problem

Proposition 2.1 For any alphabet of size at least 2, the symmetric bounded Post Correspondence Problem is undecidable, for some bound.

Proof. Our proof is based on the proof of undecidability of the PCP by Robert Floyd [4], with small modifications; other references for this proof are [2], [3], [5]. Floyd’s proof reduces the word problem of any semi-Thue system to a PCP. However, since the word problem is already undecidable for certain finitely presented semigroups (which are special semi-Thue systems, namely symmetric Thue systems that do not use the empty string), we immediately obtain a PCP that is almost symmetric.

More precisely, let $(B \mid R)$ be a finite presentation of a semigroup with undecidable word problem; here, $B$ is a finite alphabet and $R \subseteq B^+ \times B^+$ is a finite symmetric relation. The existence of such semigroups was proved by A.A. Markov and E. Post independently in 1947 [7, 9]. Note that for a semigroup presentation, only non-empty strings are used (i.e., the set $A$ is almost symmetric).

For the details, we follow Floyd’s proof in the formulation of [2, 3]. For the PCP we use the alphabet $A = \{\downarrow, \uparrow, \circ, \circ\} \cup B \cup \overline{B}$, where $\overline{B} = \{\overline{b} : b \in B\}$. This alphabet has size $|A| = 4 + 2|B|$, but we will later encode $A$ over $\{0, 1\}$. We will use the overline as an isomorphism from $(B \cup \{\circ\})^*$ onto $(\overline{B} \cup \{\overline{\circ}\})^*$, with $\ell_1\ell_2\ldots\ell_n = \overline{\ell_1}\overline{\ell_2}\ldots\overline{\ell_n}$ for all $\ell_1, \ell_2, \ldots, \ell_n \in B \cup \{\circ\}$.

An instance $x \overset{2}{\underset{(B \mid R)}{\rightarrow}} y$ of the word problem of the semigroup presentation $(B \mid R)$, with $x, y \in B^+$, is reduced to the PCP $P_{x,y}$ with the following input:

$\text{INPUT}(P_{x,y}) = \{(b, \overline{b}) : b \in B \cup \{\circ\}\} \cup \{\overline{b}, b : b \in B \cup \{\circ\}\}
\cup \{(u, \overline{\pi}) : (u, v) \in R\} \cup \{\overline{\pi}, v : (u, v) \in R\}
\cup \{(\downarrow, x_0, \downarrow), (\uparrow, \overline{x_0}, \uparrow)\}$.

Since $R$ can be assumed to be symmetric, we see that except for the two pairs $(\downarrow, x_0, \downarrow)$ and $(\uparrow, \overline{x_0}, \uparrow)$, this PCP is already symmetric. Floyd proves that this is indeed a many-one reduction, i.e., $x \overset{2}{\underset{(B \mid R)}{\rightarrow}} y$ is true if and only if the PCP $P_{x,y}$ has a solution. More precisely, there is a derivation $x = x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_{n+1} = y$ in $(B \mid R)$ if and only if the PCP $P_{x,y}$ has a solution

$$(\downarrow, x_1, \circ) (\overline{x}_2, x_1) (\circ, \circ) (x_3, x_2) (\circ, \overline{\circ}) \ldots (\overline{\circ}, \circ) (x_n, x_{n-1}) (\downarrow, \circ x_n, \downarrow)$$

$$= (\downarrow, x_1 \circ x_2 \circ x_3 \circ \ldots \circ x_n, \downarrow) \circ x_1 \circ x_2 \circ \overline{x}_3 \circ \ldots \circ \overline{x}_{n-1} \circ x_n, \downarrow) \circ x_1 \circ x_2 \circ \overline{x}_3 \circ \ldots \circ \overline{x}_{n-1} \circ x_n, \downarrow)$$

Here we may assume that $n$ is odd, because the pairs $(b, \overline{b})$ and $(\overline{b}, b)$ (for any $b \in B$) enable us to lengthen the solution of the PCP by one block in $(B^\circ \circ \times (B^\circ \circ \circ))$ or $(B^\circ \circ \circ \times (B^\circ \circ \circ))$.

Note that the sets $B$ and $R$ can be kept fixed, since the given semigroup $(B \mid R)$ has an undecidable word problem; only $x$ and $y$ are variable in the input. Hence the $P_{x,y}$ has a bounded number of input pairs.
Finally, we obtain a symmetric PCP by taking the PCP $sP_{x,y}$ with input

$$\text{INPUT}(sP_{x,y}) = \text{INPUT}(P_{x,y}) \cup \{ (\downarrow, \downarrow x \circ), (\uparrow y \downarrow, \downarrow) \}.$$

Claim: The symPCP $sP_{x,y}$ has a solution if and only if the PCP $P_{x,y}$ has a solution.

Proof of the Claim: Obviously, a solution for $P_{x,y}$ is also a solution for $sP_{x,y}$.

Conversely, suppose $sP_{x,y}$ has a solution. This solution starts either with the pair $(\downarrow x \circ, \downarrow)$ or the pair $(\downarrow, \downarrow x \circ)$, since those are the only pairs in $sP_{x,y}$ in which the two coordinates have a common prefix; in all other pairs, one coordinate starts with an overlined letter and the other coordinate starts with a non-overlined letter.

Case 1: The start pair is $(\downarrow x \circ, \downarrow)$. Now by the same reasoning as in [3, pp. 131-132] and [2], we can construct a derivation $x = (B | R) y$. In this construction, the 1st coordinate is always longer than the 2nd coordinate, until the derivation of $y$ is complete; then $(\uparrow y \downarrow, \downarrow)$ is the right-most pair of the solution of the PCP.

Case 2: The start pair is $(\downarrow, \downarrow x \circ)$. Then by just switching the roles of the 1st and 2nd coordinates, we can carry out the same reasoning as in Case 1; now $(\uparrow y \downarrow, \downarrow)$ is the right-most pair of the solution of the PCP. Again a derivation $x = (B | R) y$ is constructed.

[This proves the Claim.]

We still have to show that symPCP is undecidable for an alphabet of size 2, e.g., for $\{0, 1\}$. The symPCP $sP_{x,y}$ uses the alphabet $A = \{ \downarrow, \uparrow, \circ, \overline{\circ} \} \cup B \cup \overline{B}$, of size $|A| = 4 + 2|B|$. Let us choose any injective function $\text{code} : A \to \{0, 1\}^\ell$, where $\ell = \lceil \log_2 |A| \rceil$. Let $\text{code}(sP_{x,y}) = \{(\text{code}(q), \text{code}(r)) : (q, r) \in \text{INPUT}(sP_{x,y})\}$, and $\text{code}(P_{x,y}) = \{(\text{code}(q), \text{code}(r)) : (q, r) \in \text{INPUT}(P_{x,y})\}$. Obviously, $\text{code}(sP_{x,y})$ is a symmetric PCP.

Then the symPCP $\text{code}(sP_{x,y})$ has a solution if and only if the original symPCP $sP_{x,y}$ has a solution.

3 Clarifications and errata for the freeness problem

Let $\mathbb{N}^{3\times3}$ denote the monoid of 3-by-3 matrices over the natural numbers. For a subset $S$ of $\mathbb{N}^{3\times3}$, the subsemigroup generated by $S$ in $\mathbb{N}^{3\times3}$ is denoted by $\langle S \rangle$. Article [6] considers the following problem, called the freeness problem of subsemigroups of $\mathbb{N}^{3\times3}$.

INPUT: A finite set $S \subseteq \mathbb{N}^{3\times3}$.

QUESTION: Is $\langle S \rangle$ free over $S$? (Note that this is not equivalent to just asking whether $\langle S \rangle$ is free, i.e., is isomorphic to any free semigroup.)

The freeness problem is shown to be undecidable in [6], but the proof is incomplete (see Subsection 3.2 below, where the gap is filled). Many stronger forms of this result were proven later; e.g., the problem is undecidable for upper triangular matrices in $\mathbb{N}^{3\times3}$ [1] (whose proof is not based on [6]).

Subsection 3.1 corrects notational errors in [6], arising from mix-ups between reverse base 2 and ordinary base 4 notations.

In Subsection 3.2, the main claim of [6] is proved by using the symmetric PCP. In [6] it is claimed that certain finitely generated matrix semigroups have a relation if and only if the PCPs encoded by these matrices have solutions; this claim is true if the PCPs are symmetric. For non-symmetric PCPs there are counter-examples, found by the second author.

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3.1 Encoding a PCP by matrices

• Page 224, bottom paragraph of [6]: The word “reverse” should be removed.

Comment on this correction: The function \( \varphi : \{0,1,2,3\}^* \to \mathbb{N} \) performs base 4 conversion, i.e., \( \varphi(x_{n-1} \ldots x_1 x_0) = \sum_{i=0}^{n-1} x_i 4^i \); this is the ordinary base 4 representation, not reverse base 4. On the other hand, the function \( \beta : \{0,1\}^* \to \mathbb{N} \) performs reverse base 2 conversion, i.e., \( \beta(x_0 x_1 \ldots x_{n-1}) = \sum_{i=0}^{n-1} x_i 2^i \).

The reason for the difference is that \( \varphi \) is used in a lower-triangular \( 2 \times 2 \) matrix, whereas \( \beta \) is used in an upper-triangular \( 2 \times 2 \) matrix.

• The first paragraph on page 225 of [6] should be replaced by the following:

Next we want to encode an instance of the Post Correspondence Problem into \( 3 \times 3 \) matrices over \( \mathbb{N} \). We view the indices \( 0, 1, \ldots, k-1 \) of the Post Correspondence pairs as binary strings of uniform length \( h = \max\{\lceil \log_2 k \rceil, 1\} \). More precisely, we encode every \( i \in \{0, 1, \ldots, k-1\} \) as \( \bar{i} \in \{0, 1\}^h \); the choice is arbitrary, except that \( i \mapsto \bar{i} \) is injective. Since \( \{0, 1\} \subseteq \{0, 1, 2, 3\} \) we can decode the string \( \bar{i} \) into a natural number by \( \varphi \), in ordinary base 4 notation. The binary strings \( u_i \) and \( v_i \) will be decoded into natural numbers by \( \beta(u_i) \), respectively \( \beta(v_i) \), in reverse binary notation.

Then we represent the instance \( \{(u_i, v_i) : i = 0, 1, \ldots, k-1\} \) of the Post Correspondence Problem by \( 4k + 1 \) matrices as follows:

\[
L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 4 \end{bmatrix},
\]

and for \( i = 0, 1, \ldots, k-1 \):

\[
U_i = \begin{bmatrix} 2^{\|u_i\|} & \beta(u_i) & 0 \\ 0 & 1 & 0 \\ 0 & 2 + \varphi(\bar{i}) \cdot 4 & 4^{h+1} \end{bmatrix},
\]

\[
\bar{U}_i = \begin{bmatrix} 2^{\|u_i\|} & \beta(u_i) & 0 \\ 0 & 1 & 0 \\ 0 & 2 + \varphi(\bar{i}) \cdot 4 + 3 \cdot 4^{h+1} & 4^{h+2} \end{bmatrix},
\]

\[
V_i = \begin{bmatrix} 2^{\|v_i\|} & \beta(v_i) & 0 \\ 0 & 1 & 0 \\ 0 & \varphi(\bar{i}) + 2 \cdot 4^h & 4^{h+1} \end{bmatrix},
\]

\[
\bar{V}_i = \begin{bmatrix} 2^{\|v_i\|} & \beta(v_i) & 0 \\ 0 & 1 & 0 \\ 0 & \varphi(\bar{i}) + 3 \cdot 4^h & 4^{h+1} \end{bmatrix}.
\]

In summary, the correspondence between these matrices and pairs of strings is given by the following table (where \( i = 0, 1, \ldots, k-1 \):
\begin{array}{|c|c|c|c|c|}
\hline
L & U_i & \overline{U}_i & V_i & \overline{V}_i \\
(\varepsilon, 2) & (u_i, 2\bar{2}) & (u_i, 2\bar{3}) & (v_i, \bar{2}) & (v_i, \bar{3}) \\
\hline
\end{array}

Table 1: Correspondence between matrices and pairs of strings.

As we remarked earlier, when these matrices are multiplied, the pairs of strings that they encode are concatenated; for example, \(U_iU_j\overline{U}_r\) encodes \((u_iu_ju_r, 2\bar{2}j2\bar{3})\). Note again that all strings \(i\) are over \(\{0, 1\}\) and have the same length \(h = \max\{\lfloor \log_2 k \rfloor, 1\}\).

Comment about this correction: In [6], \(i\) appears in some places where \(\varphi(\hat{i})\) should have been used (although the explanations in the paper make it clear that \(\varphi(\hat{i})\) was intended). However, \(\varphi(\hat{i})\) is a different number than \(i\). Indeed, \(\hat{i}\) is an arbitrary binary string representing \(i\), whereas \(\varphi(\hat{i})\) is the natural integer represented by the binary string \(\hat{i}\) in base 4; note also that this is the ordinary base 4 decoding, not reverse base 4.

\bullet \textbf{Isomorphism between subsemigroups of } \mathbb{N}^{3 \times 3} \text{ and subsemigroups of } \{0, 1\}^* \times \{0, 1, 2, 3\}^*

The following was briefly mentioned in [6] (p. 224, bottom paragraph); here we give more details:

\textbf{Lemma 3.1} The subsemigroup of \(\mathbb{N}^{3 \times 3}\) generated by \(\{L\} \cup \{U_i, \overline{U}_i, V_i, \overline{V}_i : 0 \leq i < k\}\) can be embedded into \(\{0, 1\}^* \times \{0, 1, 2, 3\}^*\) (direct product of two free monoids). More precisely,

\(\langle \{L\} \cup \{U_i, \overline{U}_i, V_i, \overline{V}_i : 0 \leq i < k\} \rangle \) in \(\mathbb{N}^{3 \times 3}\)

is isomorphic to

\(\langle \{(\varepsilon, 2)\} \cup \{(u_i, 2\bar{2}), (u_i, 2\bar{3}), (v_i, \bar{2}), (v_i, \bar{3}) : 0 \leq i < k\} \rangle \) in \(\{0, 1\}^* \times \{0, 1, 2, 3\}^*\)

by the isomorphism given by the Table 1 above.

\textbf{Proof.} As we indicated above, the matrices represent pairs of strings, and we remarked that when these matrices are multiplied, the pairs of strings that they encode are concatenated.

For any matrix \(M\), let \(M_{i,j}\) denote the \((i, j)\)-entry (in row \(i\) and column \(j\)).

\textbf{Claim:} Let \(M\) be a product of a sequence of matrices in \(\{L\} \cup \{U_i, \overline{U}_i, V_i, \overline{V}_i : 0 \leq i < k\}\). Then the entries \(M_{1,1}, M_{1,2}, M_{3,3}, M_{3,2}\) determine a pair \((w, J) \in \{0, 1\}^* \times \{0, 1, 2, 3\}^*\), where \((w, J)\) is the concatenation of the corresponding pairs in \(\{(\varepsilon, 2)\} \cup \{(u_i, 2\bar{2}), (u_i, 2\bar{3}), (v_i, \bar{2}), (v_i, \bar{3}) : 0 \leq i < k\}\). Conversely, \((w, J)\) determines \(M\) by

\[(w, J) \mapsto M = \begin{bmatrix} 2^{|w|} \beta(w) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varphi(J) & 4^{|J|} \end{bmatrix} \]

Note that \(w \in \{0, 1\}^*\) could have trailing 0s, so in reverse base 2 representation, \(\beta(w) \in \mathbb{N}\) alone does not determine \(w\). But \(\beta(w)\) and \(|w|\) together determine \(w\). Similarly, \(J \in \{0, 1, 2, 3\}^*\) could have leading 0s, so in base 4 representation, \(\varphi(J) \in \mathbb{N}\) alone does not determine \(J\). But \(\varphi(J)\) and \(|J|\) together determine \(J\).

\textbf{Proof of the Claim:} We use induction on the number of matrices multiplied. For one matrix, the lemma holds by the Table 1 above. In general, suppose \(M\) determines \((w, J)\), as in the lemma, and consider the matrix \(MX\), where \(X \in \{L\} \cup \{U_i, \overline{U}_i, V_i, \overline{V}_i : 0 \leq i < k\}\).
If \( X = L \),
\[
ML = \begin{bmatrix} 2^{|w|} & \beta(w) & 0 \\ 0 & 1 & 0 \\ 0 & \varphi(J) + 2 \cdot 4^{|J|+1} & 4^{|J|+1} \end{bmatrix} = \begin{bmatrix} 2^{|w|} & \beta(w) & 0 \\ 0 & 1 & 0 \\ 0 & \varphi(J2) & 4^{|J2|} \end{bmatrix},
\]
which determines \((w, J2)\). Recall that the 1st coordinate of \((w, J)\) uses reverse base 2, and the 2nd coordinate uses the usual base 4 representation.

If \( X = U_i \),
\[
MU_i = \begin{bmatrix} 2^{|w|+|u_i|} & 2^{|w|} \beta(u_i) + \beta(w) & 0 \\ 0 & 1 & 0 \\ 0 & \varphi(J) + (2 + \varphi(i) \cdot 4) \cdot 4^{|J|} & 4^{|J|+h+1} \end{bmatrix} = \begin{bmatrix} 2^{|wu_i|} & \beta(wu_i) & 0 \\ 0 & 1 & 0 \\ 0 & \varphi(J2i) & 4^{|J2i|} \end{bmatrix},
\]
which determines \((wu_i, J2i)\).

If \( X = U_i \),
\[
MU_i = \begin{bmatrix} 2^{|w|+|u_i|} & 2^{|w|} \beta(u_i) + \beta(w) & 0 \\ 0 & 1 & 0 \\ 0 & \varphi(J) + (2 + \varphi(i) \cdot 4 + 3 \cdot 4^{h+1}) \cdot 4^{|J|} & 4^{|J|+h+2} \end{bmatrix} = \begin{bmatrix} 2^{|wu_i|} & \beta(wu_i) & 0 \\ 0 & 1 & 0 \\ 0 & \varphi(J2i3) & 4^{|J2i3|} \end{bmatrix},
\]
which determines \((wu_i, J2i3)\).

Similarly, \( MV_i \) and \( M\overline{V}_i \) determine \((w, J2)\), respectively \((w, Ji3)\).

[This proves the Claim.]

The function from matrices to pairs of strings is injective, since \((w, J)\) determines \(M\) by the formula given in the claim. \(\square\)

As a consequence of Lemma 3.1 we have:
\[
\langle\{L\} \cup \{U_i, \overline{U}_i, V_i, \overline{V}_i : 0 \leq i < k\}\rangle\]
is free over the given generators if and only if \(\langle\{(\varepsilon, 2)\} \cup \{(u_i, 2\varepsilon), (u_i, 2\varepsilon3), (v_i, 2\varepsilon), (v_i, 2\varepsilon3) : 0 \leq i < k\}\rangle\)
is free over the given generators.

### 3.2 An application of the symmetric PCP

The proof in [6] does not work for all PCPs. A counter-example was found by A. Talambutsa, who also observed that restricting the proof to the symmetric PCP would correct the mistake.

**Example:** Consider the PCP with input \(\{(00, 0)\}\), which obviously has no solution. (There are similar counter-examples, e.g., \(\{(00, 0), (u, 0)\}\), for any \(u \in \{0, 1\}^+\) with \(|u| \geq 2\).) The corresponding subsemigroup of \(\{0, 1\}^+ \times \{0, 1, 2, 3\}^*\), constructed in [6], is generated by \(\Gamma = \{(\varepsilon, 2), (00, 20), (00, 203), (02, 0), (00, 03)\}\); here the single input pair \((00, 0)\) is coded by the binary string \(0 = 0\) (as in Subsection 3.1). The subsemigroup \(\langle \Gamma \rangle\) satisfies the following relations (among others):
\[
(00, 20) (\varepsilon, 2) (\varepsilon, 2) (0, 03) = (\varepsilon, 2) (0, 02) (00, 203) ;
(00, 20) (\varepsilon, 2) (\varepsilon, 2) (0, 02) = (\varepsilon, 2) (0, 02) (00, 20) (\varepsilon, 2) ;
\]
The Lemma 3.2 (reduction). The symPCP with input \((u_i, v_i) : 0 \leq i < k\) over \(\{0, 1\}\) has a solution if and only if the subsemigroup of \(\{0, 1\}^* \times \{0, 1, 2, 3\}^*\) generated by

\[
\Gamma = \{ (\varepsilon, 2) \} \cup \{ (u_i, 2i2), (v_i, i23), (v_i, i3) : 0 \leq i < k \}
\]

is not free. The function \((u_i, v_i) : 0 \leq i < k \mapsto \Gamma\) is a one-one polynomial-time reduction from the symPCP to the non-freeness problem of the subsemigroup \(\langle \Gamma \rangle\) of \(\{0, 1\}^* \times \{0, 1, 2, 3\}^*\).

Proof. \(\Rightarrow\) If the PCP has a solution \(u_j \ldots u_j = v_j \ldots v_j\) then one verifies immediately that the following semigroup relation holds in \(\langle \Gamma \rangle\):

\[
(u_j, 2j) \ldots (u_j, 2j)(u_j, 2j3) = (\varepsilon, 2)(v_j, j2) \ldots (v_j, j2)(v_j, j3).
\]

So \(\langle \Gamma \rangle\) is not free.

Also, using symmetry we have \((u_j, v_j) = (v_j', u_j')\) for some \(j' \in \{0, \ldots, k-1\}\). So \((u_j, 2j2), (v_j, 2j2) \in \Gamma\) (in addition to \((u_j, 2j2), (v_j, j2) \in \Gamma\)). Note that \((u_i, v_i)\) has a code \(\hat{i} \in \{0, 1\}^*\) which is different from the code \(\hat{j}\) of \((v_i, u_i) = (v_i', v_i')\) (by the non-triviality assumption on INPUT(PCP)). Hence we also have the relation,

\[
(u_j, 2j) \ldots (u_j, 2j)(\varepsilon, 2)(u_j, 2j) = (\varepsilon, 2)(v_j, j2) \ldots (v_j, j2)(v_j, 2j)(\varepsilon, 2).
\]

The corresponding matrix relations are

\[
U_j \ldots U_j \overline{V}_j = LV_j \ldots V_j \overline{V}_j, \quad \text{and} \quad U_j \ldots U_j L^2 V^\prime_j = LV_j \ldots V_j U^\prime_j L.
\]

The first of these matrix relations was given in [6], the second is new and is based on the symmetry of the PCP input.

Remark (3.2R): The second, new, relation (based on the symPCP) shows that the letter 3, as well as the generators \((u_i, 2i23), (v_i, i23) : 0 \leq i < k\), are not needed for the construction of a subsemigroup of \(\{0, 1\}^* \times \{0, 1, 2\}^*\) with undecidable freeness problem. Similarly, the overlined matrices \(\{U^\prime_i, V^\prime_i : 0 \leq i < k\}\) are not needed in the input of the freeness problem for matrices.

We will nevertheless continue using 3 and the redundant generators and matrices, since we want to show that the proof in [6] is correct for symmetric PCPs. It is straightforward to rewrite the proof without the redundancies (by simply leaving out the redundancies); see Lemma 3.3 [End, Remark.]

\([\Leftarrow]\) Suppose the semigroup \(\langle \Gamma \rangle\) is not free, i.e., it has a non-trivial relation \(p_1 \ldots p_m = q_1 \ldots q_n\), with \(p_1, \ldots, p_m, q_1, \ldots, q_n \in \Gamma\).

Since the semigroup \(\{0, 1\}^* \times \{0, 1, 2, 3\}^*\) is cancellative, we can assume that \(p_1 \neq q_1 \) and \(p_m \neq q_n\). We abbreviate the generator sequence \((p_1, \ldots, p_m)\) by \(P\), and the generator sequence

\[(00, 20) (00, 203) (\varepsilon, 2) (02) (03) = (\varepsilon, 2) (02) (03) (00, 20) (00, 203).\]
(q_1, \ldots, q_n) by Q. The relation is abbreviated by \( \Pi P = \Pi Q \), where \( \Pi P \) is the product, in \( \{0,1\}^* \times \{0,1,2,3\}^* \), of the generators in the sequence \( P \); and similarly for \( \Pi Q \).

Claim 1:
(A) One of the generators \( p_1, q_1 \) is \((\varepsilon, 2)\), and the other belongs to \( \{(u_i, 2_i^2), (u_i, 2_i^3) : 0 \leq i < k \} \).

(B) If \( p_1 = (\varepsilon, 2) \), we have:
- \( p_1 = (u_j, 2_i^2) \) and \( q_1 = (\varepsilon, 2)(v_{j_i}, 2_i^2) = (v_j, 2_i^2) \), for some \( (u_j, v_{j_i}) \in \text{INPUT}(PCP) \);
- or \( p_1 = (u_j, 2_i^3) \) and \( q_1 = (\varepsilon, 2)(v_{j_i}, 2_i^3) = (v_j, 2_i^3) \), for some \( (u_j, v_{j_i}) \in \text{INPUT}(PCP) \).

If \( p_1 = (\varepsilon, 2) \) then, symmetrically, the conclusion is similar.

Proof of Claim 1(A): There are several cases. By cancellativity we already ruled out \( p_1 = q_1 \).

Case (1): \( p_1 = (\varepsilon, 2) \).
Then \( q_1 \in \{(u_i, 2_i^2), (u_i, 2_i^3) : 0 \leq i < k \} \), since the 2nd coordinate of \( q_1 \) must start with 2, and we ruled out \( p_1 = q_1 \).

Case (2): \( p_1 \in \{(u_i, 2_i^2), (u_i, 2_i^3) : 0 \leq i < k \} \).
Then \( q_1 \in \{(\varepsilon, 2) \cup \{(u_i, 2_i^2), (u_i, 2_i^3) : 0 \leq i < k \} \), as the 2nd coordinate of \( q_1 \) must start with 2.

If \( p_1 = (u_i, 2_i^2) \) and \( q_1 = (u_j, 2_j^2) \) then \( i \) and \( j \) are prefix-comparable, hence \( i = j \) since both have length \( \lceil \log_2 k \rceil \); this implies \( p_1 = q_1 \), which was ruled out. Similarly, if \( p_1 = (u_i, 2_i^3) \) and \( q_1 = (u_j, 2_j^3) \) then \( p_1 = q_1 \), which is ruled out.

If \( p_1 = (u_i, 2_i^2) \) and \( q_1 = (u_j, 2_j^3) \) then \( i \) and \( j \) are prefix-comparable, hence \( i = j \) since both have length \( \lceil \log_2 k \rceil \). Then there will be no possible choice for \( p_2 \) that could match the letter 3 in \( q_1 \); this contradicts the assumption that \( p_1 \ldots p_m = q_1 \ldots q_n \).

If \( p_1 = (u_j, 2_j^3) \) and \( q_1 = (u_i, 2_i^2) \) or \( q_1 = (u_j, 2_j^3) \), then we obtain the same contradictions as above, with the roles of \( p_1 \) and \( q_1 \) switched.

The only alternative left is \( q_1 = (\varepsilon, 2) \).

Case (3): \( p_1 \in \{(v_i, 2_i^2), (v_i, 2_i^3) : 0 \leq i < k \} \).
By symmetry of the input PCP, this is the same as Case (2).

Proof of Claim 1(B): By Claim 1(A) we can assume that \( q_1 = (\varepsilon, 2) \) (the case where \( p_1 = (\varepsilon, 2) \) is similar). Moreover, \( p_1 = (u_j, 2_j^1) \) or \( p_1 = (u_j, 2_j^13) \), for some \( j_1 \) (uniquely determined by \( \Pi P \) and \( \Pi Q \)). The fact that all \( \bar{j}_i \) have the same length \( \lceil \log_2 k \rceil \) implies: \( \Pi Q = q_1 q_2 \ldots \) for some \( q_2 \in \bar{\Gamma} \), and \( q_2 = (v_{j_1}, 2_j^1) \), or \( q_2 = (v_{j_1}, 2_j^13) \) with \( (u_j, v_{j_1}) \in \text{INPUT}(PCP) \).

[This proves Claim 1.]

Claim 2: The relation \( \Pi P = \Pi Q \) has one of two forms:
[2-2 block]: \( \Pi P = (u_{j_1}, \ldots, u_{j_r}, 2_{j_1} \ldots 2_{j_r}^2) \Pi R_1 = (v_{j_1}, \ldots, v_{j_r}, 2_{j_1} \ldots 2_{j_r}^2) \Pi R_2 = \Pi Q \),
[2-3 block]: \( \Pi P = (u_{j_1}, \ldots, u_{j_r}, 2_{j_1} \ldots 2_{j_r}^3) \Pi R_3 = (v_{j_1}, \ldots, v_{j_r}, 2_{j_1} \ldots 2_{j_r}^3) \Pi R_4 = \Pi Q \),
for some \( R_1, R_2, R_3, R_4 \in \{\bar{\Gamma}\} \).

In either case we have: \( (u_{j_1}, v_{j_1}), \ldots, (u_{j_r}, v_{j_r}) \in \text{INPUT}(PCP) \), and \( u_{j_1}, \ldots, u_{j_r} \), is prefix-comparable with \( v_{j_1}, \ldots, v_{j_r} \). (Note that \( u_{j_1}, \ldots, u_{j_r} \) and \( v_{j_1}, \ldots, v_{j_r} \) need not be equal.)

Proof of Claim 2: By Claim 1(B) we can have
\( \Pi P = p_1 \ldots = (u_{j_1}, 2_{j_1}^13) \ldots = (v_{j_1}, 2_{j_1}^13) \ldots = q_1 q_2 \ldots = \Pi Q \),
where \((u_{j_1}, v_{j_1}) \in \text{INPUT}(PCP)\), and \(u_{j_1}\) is prefix-comparable with \(v_{j_1}\). Then \(\Pi P = \Pi Q\) begins with a 2-3 block, so Claim 2 holds.

Or we have

\[ \Pi P = p_1 \ldots = (u_{j_1}, 2j_12) \ldots = (v_{j_1}, 2j_12) \ldots = q_1q_2 \ldots = \Pi Q, \]

where \((u_{j_1}, v_{j_1}) \in \text{INPUT}(PCP)\), and \(u_{j_1}\) is prefix-comparable with \(v_{j_1}\).

The presence of two letters 2 in the 2nd coordinate of \(q_1q_2\) implies that \(\Pi P = p_1p_2 \ldots\), for some \(p_2 \in \Gamma\) of the form \(p_2 = (\varepsilon, 2)\), or \(p_2 = (u_{j_2}, 2j_2)\), or \(p_2 = (u_{j_2}, 2j_23)\). Hence the relation \(\Pi P = \Pi Q\) takes one of the following forms:

Case 1: \((u_{j_1}, 2j_12) \ldots = (v_{j_1}, 2j_12) \ldots\), with \(p_2 = (\varepsilon, 2)\).

In this case the relation \(\Pi P = \Pi Q\) begins with a 2-2 block, so Claim 2 holds.

Case 2: \((u_{j_1} u_{j_2}, 2j_12j_23) \ldots = (v_{j_1}, 2j_12) \ldots\), where \(p_2 = (u_{j_2}, 2j_23)\).

Now \(\Pi Q = q_1q_2q_3 \ldots\), for some \(q_3\) of the form \(q_3 = (v_{j_2}, j_3) \in \Gamma\). Then the relation \(\Pi P = p_1p_2 \ldots = q_1q_2q_3 \ldots = \Pi Q\) takes the form

\[ (u_{j_1} u_{j_2}, 2j_12j_23) \ldots = (v_{j_1}, v_{j_2}, 2j_12j_23) \ldots, \]

where \((u_{j_1}, v_{j_1})\), \((u_{j_2}, v_{j_2}) \in \text{INPUT}(PCP)\), and \(u_{j_1}, u_{j_2}\) is prefix-comparable with \(v_{j_1}, v_{j_2}\). In this case the relation \(\Pi P = \Pi Q\) begins with a 2-3 block, so Claim 2 holds.

Case 3: \((u_{j_1} u_{j_2}, 2j_12j_23) \ldots = (v_{j_1}, 2j_12) \ldots\), where \(p_2 = (u_{j_2}, j_22)\).

Now \(\Pi Q = q_1q_2q_3 \ldots\), for some \(q_3 = (v_{j_2}, j_2) \in \Gamma\), with \(\nu \in \{2, 3\}\), and \((u_{j_2}, v_{j_2}) \in \text{INPUT}(PCP)\). Then the relation \(\Pi P = \Pi Q\) takes the form \((u_{j_1} u_{j_2}, 2j_12j_23) \ldots = (v_{j_1} v_{j_2}, 2j_12j_23) \ldots\). If we had \(\nu = 3\) then there would be no possible choice for the generator \(p_3\) that could match the letter 3 in the second coordinate of \(q_1q_2q_3\ldots\). Hence, we must have \(\nu = 2\). Now the relation \(\Pi P = \Pi Q\) takes the form

\[ (u_{j_1} u_{j_2}, 2j_12j_23) \ldots = (v_{j_1} v_{j_2}, 2j_12j_23) \ldots, \]

where \((u_{j_1}, v_{j_1})\), \((u_{j_2}, v_{j_2}) \in \text{INPUT}(PCP)\), and \(u_{j_1}, u_{j_2}\) is prefix-comparable with \(v_{j_1}, v_{j_2}\).

Since \(2j_12j_2 \neq j_12j_2, \Pi P\) must be of the form \(p_1p_2p_3 \ldots\), for some \(p_3 \in \Gamma\) of the form \(p_3 = (\varepsilon, 2)\), or \(p_3 = (u_{j_3}, 2j_3)\), or \(p_3 = (u_{j_3}, 2j_33)\) (just as at the beginning of the proof of Claim 2). For \(p_3 = (\varepsilon, 2)\) we go to case 1, and we obtain a 2-2 block. For \(p_3 = (u_{j_3}, 2j_33)\) we go to case 2, and we obtain a 2-3 block. For \(p_3 = (u_{j_3}, 2j_3)\), we go back to the beginning of case 3, and the relation \(\Pi P = \Pi Q\) takes the form

\[ (u_{j_1} u_{j_2} u_{j_3}, 2j_12j_22j_33) \ldots = (v_{j_1} v_{j_2} v_{j_3}, 2j_12j_22j_33) \ldots, \]

where \((u_{j_1}, v_{j_1})\), \((u_{j_2}, v_{j_2})\), \((u_{j_3}, v_{j_3}) \in \text{INPUT}(PCP)\), and \(u_{j_1} u_{j_2} u_{j_3}\) is prefix-comparable with \(v_{j_1} v_{j_2} v_{j_3}\). In this way case 3 could repeat itself a number of times, but since \(P\) and \(Q\) have finite length, case 3 must eventually lead to case 1 or case 2; i.e., \(P\) and \(Q\) both start with a 2-2 block, or both start with a 2-3 block.

[This proves Claim 2.]

Claim 3: The relation \(\Pi P = \Pi Q\) can be factored into blocks as \(\Pi P = B_1 \ldots B_b = C_1 \ldots C_b = \Pi Q\), with the following properties:

(3.1) Each block \(B_i\), \(C_i\) is a product of generators in \(\Gamma\) (for \(i = 1, \ldots, b\)).

(3.2) There is a matching between the generators in each pair of blocks \((B_i, C_i)\) (for \(i = 1, \ldots, b\)); i.e., \(B_i\) and \(C_i\) are of the form:

\[ B_i = (u_{h_1} u_{h_2} \ldots u_{h_s}, 2h_12h_2 \ldots 2h_s \nu) \]

and
\[ C_i = (v_{h_1}, v_{h_2}, \ldots, v_{h_\nu}, 2h_1 2h_2 \ldots 2h_\nu), \]
where \((u_{h_1}, v_{h_1}), (u_{h_2}, v_{h_2}), \ldots, (u_{h_\nu}, v_{h_\nu}) \in \text{INPUT(PCP)}, \text{ and } \nu \in \{2, 3\}.\]

Thus, \(B_i\) and \(C_i\) are either both 2-2 blocks, or both 2-3 blocks.

Notation for (3.3): For blocks \(B_i, C_i\), the 1st coordinates are \((B_i)_1 = u_{h_1} u_{h_2} \ldots u_{h_\nu}\), respectively \((C_i)_1 = v_{h_1} v_{h_2} \ldots v_{h_\nu}\). Similarly, the 2nd coordinates are \((B_i)_2 = C_i)_2 = 2h_1 2h_2 \ldots 2h_\nu\).

(3.3) For every \(j = 1, \ldots, b:\)
\[ (B_1)_1 (B_2)_1 \ldots (B_j)_1 \text{ is prefix-comparable with } (C_1)_1 (C_2)_1 \ldots (C_j)_1. \]
Moreover,
\[ (B_1)_1 (B_2)_1 \ldots (B_b)_1 = (C_1)_1 (C_2)_1 \ldots (C_b)_1; \]
and this common string is a solution of the PCP.

Remark: The numbers of blocks in \(P\) and \(Q\) are the same (that number is called \(b\) above); but the lengths (over \(\Gamma\)) of \(B_i\) and \(C_i\) (and hence, of \(P\) and \(Q\)) can be different; it depends on the number of blocks that start or end with \((\varepsilon, 2)\).

For example, the second relation in the \([\leq]\)-part of this proof has two blocks on either side.

Proof of Claim 3: By Claim 2, \(P\) and \(Q\) both start with a 2-2 block, or they both start with a 2-3 block.

It could happen that \(P\) consists of one such block, which implies that \(Q\) also has only one block. Indeed, the left-most block of \(Q\) has the same 2nd coordinate as the left-most block of \(P\), so the equality \(\Pi P = \Pi Q\) would be violated in the 2nd coordinate if \(Q\) had additional blocks. Similarly, if \(Q\) consists of one block, then \(P\) has only one block. Since \(\Pi P = \Pi Q\) holds, the 1st coordinate now yields \(u_{j_1} \ldots u_{j_n} = v_{j_1} \ldots v_{j_n}\) as a solution of the PCP.

If \(P\) and \(Q\) do not consist of just one block we look at the generators to the right of the first block in \(P\) and in \(Q\). The relation \(\Pi P = \Pi Q\) then has the form \(\Pi P = B_1 p_{s+1} \ldots = C_1 q_{t+1} \ldots = \Pi Q\), where \(s\) and \(t\) are the length of \(B_1\), respectively \(C_1\), over \(\Gamma\). Just as in Claim 1, the next generator after the left-most block is \((\varepsilon, 2)\) on one side of the relation, and an element of \([\{(u_i, 2\varepsilon) : 0 \leq i < k\}]\) on the other side of the relation. As in Claim 2, this will produce either two 2-2 blocks or two 2-3 blocks just to the right of \(B_1\) and \(C_1\); the process is the same as the construction of \(B_1\) and \(C_1\).

Eventually, \(P\) and \(Q\) are factored into a finite number of blocks. The equality \(\Pi P = \Pi Q\) implies equality in the 1st and 2nd coordinates. The 2nd coordinate gives the sequence of pairs chosen in \(\text{INPUT(PCP)}\). The 1st coordinate gives a solution of the PCP.

[This proves Claim 3.]

By Claim 3, any non-trivial relation in \((\Gamma)\) determines a solution of the PCP in the 1st coordinate. This concludes the proof of Lemma 3.2. \(\Box\)

Lemma 3.3 (reduction without letter 3). The symPCP with input \([(u_i, v_i) : 0 \leq i < k]\) over \(\{0, 1\}\) has a solution if and only if the subsemigroup of \(\{0, 1\}^* \times \{0, 1, 2\}^*\) generated by
\[ \{(\varepsilon, 2)\} \cup \{(u_i, 2\varepsilon), (v_i, 2\varepsilon) : 0 \leq i < k\}\]
is not free over the given generators. This holds if and only if the subsemigroup of \(\mathbb{N}^{3 \times 3}\) generated by
\[ \{L\} \cup \{U_i, V_i : 0 \leq i < k\}\]
is not free over the given generators.
Proof. This follows from the proof of Lemma 3.2 and Remark (3.2R). \qed

As a consequence of Lemma 3.2 we obtain the following undecidability results about the freeness problem (which was defined at the beginning of Section 3).

**Proposition 3.4**  The freeness problem of finitely generated subsemigroups is undecidable for the semigroups and groups below:

- the matrix monoid \( \mathbb{N}^{3 \times 3} \);
- the direct product of free monoids \( \{0,1\}^* \times \{0,1\}^* \);
- the direct product of free groups \( \text{FG}_2 \times \text{FG}_2 \).

The number of generators of the subsemigroup can be kept bounded without changing undecidability.

Proof. Lemma 3.2 yields this for \( \{0,1\}^* \times \{0,1,2,3\}^* \), and by Lemma 3.1 it then follows for \( \mathbb{N}^{3 \times 3} \). It follows for \( \{0,1\}^* \times \{0,1\}^* \), since \( \{0,1,2,3\}^* \) is embeddable into \( \{0,1\}^* \) (e.g., by coding \( \{0,1,2,3\} \) to \( \{00,01,10,11\} \)). It then follows for \( \text{FG}_2 \times \text{FG}_2 \) since \( \{0,1\}^* \) is a submonoid of the two-generator free group \( \text{FG}_2 \). Boundedness of the number of generators follows from Proposition 2.1. \qed

Boundedness of the number of generators was already observed in [1].

It remains an open problem whether the freeness problem for finitely generated subgroups of \( \text{FG}_2 \times \text{FG}_2 \) is undecidable.

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