WAVE PHENOMENA IN A COMPARTMENTAL EPIDEMIC MODEL WITH NONLOCAL DISPERAL AND RELAPSE

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ABSTRACT. This paper is concerned with the wave phenomena in a compartmental epidemic model with nonlocal dispersal and relapse. We first show the well-posedness of solutions for such a problem. Then, in terms of the basic reproduction number and the wave speed, we establish a threshold result which reveals the existence and non-existence of the strong traveling waves accounting for phase transitions between the disease-free equilibrium and the endemic steady state. Further, we clarify and characterize the minimal wave speed of traveling waves. Finally, numerical simulations and discussions are also given to illustrate the analytical results. Our result indicates that the relapse can encourage the spread of the disease.

1. Introduction. In order to describe the propagation of a human or animal disease, it is often convenient to subdivide the total population, depending on disease status, into a small, tractable group of epidemiological classes or compartments, which leads to a compartmental model. Among the most often considered compartments are the class of susceptible individuals (S), the class of infective individuals (I) and the class of recovered individuals (R). Upon contracting the disease, the susceptible individuals enter the infective class and then, after their infective period ends, enter the recovered class. This type of model has been widely studied since the pioneer work of Kermack and McKendrick [18]. For a long time, the researchers try to modify or improve the existing models by various methods and techniques, involving many kinds of ordinary differential systems (see, e.g., [24, 27, 23, 12, 14]).

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functional differential equation systems (see, e.g., [13, 32, 34, 11, 22]), reaction-diffusion systems (see, e.g., [1, 7, 9, 8, 39, 21, 40, 31, 22, 43, 29, 10, 41, 15]) and lattice differential systems (see, e.g., [4, 39]).

Relapse phenomenon, that is, recovered individuals may experience relapse and reenter the infective class, is a very prevalent feature in disease transmission mechanism. Blower et al. [3] and Wildy et al. [36] observed that an individual, once infected herpes virus, remains infected all his life, passing regularly through episodes of relapse of infectiousness. In addition, for tuberculosis, Cox et al. [6] showed that the relapse can be caused by incomplete treatment or latent infection, being observed that HIV-positive patients are significantly more likely to relapse than HIV-negative patients, although it is often difficult to differentiate relapse from reinfection.

Van den Driessche and Zou [27] used a suitable function $P(t)$ to denote the fraction of recovered individuals remaining in the recovered individuals class $t$ time units after recovery. Then the proportion of recovered individuals can be expressed by

$$R(t) = \int_0^t \gamma I(\zeta)e^{-\mu(t-\zeta)}P(t-\zeta)d\zeta,$$

where $\gamma > 0$ is the recovery rate constant assuming that the infective period is exponentially distributed, and $e^{-\mu(t-\zeta)}$ accounts for the death of infectives with $\mu > 0$ being the death rate. Whereupon, they proposed an integro-differential system to model a general relapse phenomenon in infectious diseases including herpes

$$\begin{align*}
S'(t) &= B - \mu S(t) - \beta S(t)I(t), \\
I'(t) &= \beta S(t)I(t) - (\mu + \gamma)I(t) - \int_0^t \gamma I(\zeta)e^{-\mu(t-\zeta)}P(t-\zeta)d\zeta, \\
R'(t) &= \gamma I(t) - \mu R(t) + \int_0^t \gamma I(\zeta)e^{-\mu(t-\zeta)}P(t-\zeta)d\zeta,
\end{align*}$$

in which $B > 0$ is the recruitment rate of the population and $\beta > 0$ stands for the transmission rate. Note that the integral is in the Riemann-Stieltjes sense to allow for possible jump discontinuities of $P(t)$. The authors identified the basic reproduction number $R_0$ and then established the threshold property in terms of $R_0$. Letting $P(t) = e^{-\delta t}$, i.e., a negative exponential relapse distribution with relapse rate constant $\delta > 0$, and considering the disease-related death with rate $\alpha > 0$, one can obtain the following SIRI model of ODE type

$$\begin{align*}
S'(t) &= B - \mu S(t) - \beta S(t)I(t), \\
I'(t) &= \beta S(t)I(t) - (\mu + \gamma + \alpha)I(t) + \delta R(t), \\
R'(t) &= \gamma I(t) - (\mu + \delta)R(t),
\end{align*}$$

which has been studied by Vargas-De-León [28], mainly focusing on the global stability of disease-free steady state and endemic steady state. We refer the readers to [26, 23, 24, 2] for further understanding about model (1) as well as its related versions.

As we all know, due to the heterogeneity of the environment the species living and the large mobility of individuals in an area or even worldwide, which leads to that spatial uniform models are not sufficient to give a realistic picture of disease’s transmission, we should and indeed must distinguish the spatial locations in the mathematical models. In recent years, the following convolution operator

$$J * u(x, t) - u(x, t) = \int_{-\infty}^{\infty} J(x - y)u(y, t)dy - u(x, t)$$
has been widely introduced to biological models (see, e.g., [16, 5, 42, 20, 38, 30]) and infectious disease problems (see, e.g., [13, 21, 22, 40, 29]). This type of diffusion operator can describe the free and large-range migration of species, one called as \textbf{nonlocal dispersal}, in which the transition probability from one location to another depends on the distance the organisms traveled. Consequently, we consider the spatial non-homogeneous version of system (1) by introducing the nonlocal dispersal

\[
\begin{aligned}
\frac{\partial S(x,t)}{\partial t} &= J * S(x,t) - S(x,t) + B - \mu S(x,t) - \beta S(x,t)I(x,t), \\
\frac{\partial I(x,t)}{\partial t} &= J * I(x,t) - I(x,t) + \beta S(x,t)I(x,t) - (\mu + \gamma + \alpha)I(x,t) + \delta R(x,t), \\
\frac{\partial R(x,t)}{\partial t} &= J * R(x,t) - R(x,t) + \gamma I(x,t) - (\mu + \delta)R(x,t),
\end{aligned}
\]

(2) where \(S(x,t), I(x,t), R(x,t)\) are the population sizes of susceptible, infective and recovered classes at location \(x \in \mathbb{R}\) and time \(t\), respectively. In this paper we assume that all individuals possess the same dispersal ability described by the kernel \(J(x)\) having the property that

\(J) J \in C^1(\mathbb{R}), J(-x) = J(x) \geq 0, \int_{-\infty}^{\infty} J(x)dx = 1\) and \(J\) is compactly supported.

Following (J) and [28], system (2) always has a disease-free equilibrium \(E_0(\frac{B}{\mu}, 0, 0)\). When

\[R_0 := \frac{\beta B(\mu + \delta)}{\mu[\delta(\mu + \alpha) + \mu(\mu + \gamma + \alpha)]} > 1\]

which has been characterized as the basic reproduction number of the ODE system (1), model (2) admits a unique endemic steady state \(E_*(S^*, I^*, R^*)\), where

\[S^* = \frac{B}{\mu R_0}, \quad I^* = \frac{\mu}{\beta}(R_0 - 1), \quad R^* = \frac{\mu \gamma}{\beta(\mu + \delta)}(R_0 - 1)\]

Traveling wave, as a special solution maintaining its shape and moving at a constant speed, is a very important dynamical issue in mathematical epidemiology which can reveal whether the disease transmission is successful or not. Further, traveling waves can account for phase transitions between different disease states of the epidemiological system. Based on this importance, epidemic waves have attracted much attention in recent years, see e.g., [7, 9, 22, 40, 21, 34, 32, 31, 29, 37, 41, 15], mainly involving various versions of SIR type. In particular, when \(\delta = 0\) (no relapse), Yang et al. [40] established the existence and non-existence of traveling waves of system (2) without regard to the external supplies and death \((B = \mu = \alpha = 0)\). Very recently, Zhu et al. [43] studied the traveling waves for a nonlocal dispersal Susceptible-Infective-Removal-Healing (SIRH) model with relapse in the \textbf{weak} sense that the convergence to the endemic steady state is not clear.

With these in mind, in this paper we plan to investigate the existence and nonexistence as well as the minimal wave speed of the \textbf{strong} traveling waves for the general nonlocal parabolic-type SIRI model (2). As a preliminary, we first study the well-posedness of solutions for system (2). Since this system is non-monotone and possesses a convolution operator, it follows that the monotonicity theories and shooting method are not suitable. More precisely, we appeal to the method of constructing an invariant cone of initial functions defined in a large but bounded domain, then applying Schauder’s fixed-point theorem on this cone, and further extending to the whole spatial domain by a limiting argument, to show
the existence of strong traveling waves describing the transition from the disease-free equilibrium to the endemic steady state (see Theorem 3.5). In particular, we consider an eigenvalue problem to construct the so-called upper-lower solutions as the boundary of the cone and use a suitable Lyapunov like functional to derive their convergence towards the endemic equilibrium at positive infinity. This is non-trivial and interesting for a high-dimensional non-monotone system with integral items. Then, we prove the non-existence of traveling waves by using the two-sided Laplace transform (see Theorem 4.1), and further clarify the minimal wave speed by showing the existence of critical wave through a limiting argument (see Theorem 4.2). Note that for the traveling wave with critical speed, the asymptotic boundary at \( x + ct = -\infty \) cannot be obtained by the squeezing technique similar to that in Theorem 3.5. Thus, we take a detailed analysis technique to show the convergence of the critical wave to the disease-free equilibrium at negative infinity. We should point out that the emergence of relapse leads to a truly coupled wave profile system involving three equations, which makes the mathematical analysis more difficult. As such, the obtained results are a little flawed since we need a technical condition which, however, makes sense (see Remark 2). We leave the problem on refining this result and considering different dispersal ability for a further consideration.

The rest of this paper is organized as follows. Section 2 is devoted to the well-posedness of the solution for model (2) and an associated eigenvalue problem involving the linearization of the wave profile system. In Section 3, we construct the vector-value upper-lower solutions and use the fixed-point theorem together with a Lyapunov like functional to construct the so-called upper-lower solutions as trivial and interesting for a high-dimensional non-monotone system with integral. Then, we prove the non-existence of traveling waves by using the two-sided Laplace transform (see Theorem 4.1), and further clarify the minimal wave speed by showing the existence of critical wave through a limiting argument (see Theorem 4.2). Note that for the traveling wave with critical speed, the asymptotic boundary at \( x + ct = -\infty \) cannot be obtained by the squeezing technique similar to that in Theorem 3.5. Thus, we take a detailed analysis technique to show the convergence of the critical wave to the disease-free equilibrium at negative infinity. We should point out that the emergence of relapse leads to a truly coupled wave profile system involving three equations, which makes the mathematical analysis more difficult. As such, the obtained results are a little flawed since we need a technical condition which, however, makes sense (see Remark 2). We leave the problem on refining this result and considering different dispersal ability for a further consideration.

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2. Preliminaries. In this section, we give some preliminaries including the well-posedness of the solution for system (2) and an eigenvalue problem associated to the wave profile system.

2.1. The well-posedness. Let \( \mathbb{X} := BUC(\mathbb{R}, \mathbb{R}) \) be the set of all bounded and uniformly continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \) with norm \( \|u\|_X = \sup_{x \in \mathbb{R}} |u(x)| \), and \( \mathbb{X}_+ := \{ u \in \mathbb{X} : u(x) \geq 0, \forall x \in \mathbb{R} \} \). Set \( \mathbb{Y} = \mathbb{X} \times \mathbb{X} \times \mathbb{X} \) and \( \mathbb{Y}_+ = \mathbb{X}_+ \times \mathbb{X}_+ \times \mathbb{X}_+ \) with the norm

\[
\|(u, v, w)\|_Y := \sup_{x \in \mathbb{R}} \sqrt{|u(x)|^2 + |v(x)|^2 + |w(x)|^2},\quad (u, v, w) \in \mathbb{Y}.
\]

Let us define the following linear operators on \( \mathbb{X} \):

\[
[A_S\omega](x) = \int_{\mathbb{R}} J(x - y)\omega(y)dy - \omega(x) - \mu\omega(x),\quad \omega \in \mathbb{X},
\]

\[
[A_I\omega](x) = \int_{\mathbb{R}} J(x - y)\omega(y)dy - \omega(x) - (\mu + \gamma + \alpha)\omega(x),\quad \omega \in \mathbb{X},
\]

\[
[A_R\omega](x) = \int_{\mathbb{R}} J(x - y)\omega(y)dy - \omega(x) - (\mu + \delta)\omega(x),\quad \omega \in \mathbb{X}.
\]

Following assumption (J), \( A_S, A_I \) and \( A_R \) are bounded linear operators. Further, by the standard theory of semigroups (see, e.g., [25, Theorem 1.2]), they are generators of uniformly continuous semigroups \( \{T_S(t)\}_{t \geq 0}, \{T_I(t)\}_{t \geq 0} \) and \( \{T_R(t)\}_{t \geq 0} \), respectively. Similar to the argument in [17, Section 2.1.1], it follows that \( \{T_S(t)\}_{t \geq 0}, \{T_I(t)\}_{t \geq 0} \) and \( \{T_R(t)\}_{t \geq 0} \) are positive semigroups.
Proposition 1. Suppose that \((S(\cdot,0), I(\cdot,0), R(\cdot,0)) = (S_0, I_0, R_0) \in \mathbb{Y}_+\) with \(S_0 + I_0 + R_0 \leq B/\mu\). Then system (2) admits a unique solution \((S(\cdot,t), I(\cdot,t), R(\cdot,t)) \in \mathbb{Y}_+\) for all \(t \geq 0\).

Proof. Since \(A_S, A_I\) and \(A_R\) are generators of uniformly continuous semigroups \(\{T_S(t)\}_{t \geq 0}, \{T_I(t)\}_{t \geq 0}\) and \(\{T_R(t)\}_{t \geq 0}\), respectively, it easily follows that the solutions of (2) can be written by

\[
U(t, x) = [T(t)U(., 0)](x) + \int_0^t [T(t-s)K(U(., s)](x)ds, \quad t \geq 0, x \in \mathbb{R},
\]

where

\[
U(t, x) := \begin{pmatrix} S(x, t) \\ I(x, t) \\ R(x, t) \end{pmatrix}, \quad T(t) := \begin{pmatrix} T_S(t) \\ T_I(t) \\ T_R(t) \end{pmatrix}
\]

\[
K(U(x, s)) := \begin{pmatrix} B - \beta S(x, t)I(x, t) \\ \beta S(x, t)I(x, t) + \delta R(x, t) \\ \gamma I(x, t) \end{pmatrix}.
\]

By the arguments similar to [19, Lemma 3.1], we can show that \(K\) is continuously Fréchet differentiable on \(\mathbb{Y}\). Thus, in view of [33, Proposition 4.6], we see that there exists a \(T_0\) such that system (2) has a unique solution \((S(\cdot, t), I(\cdot, t), R(\cdot, t))\) for all \(t \in [0, T_0]\), and either \(T_0 = +\infty\) or \(\lim sup_{t \to T_0^-} ||(S(\cdot, t), I(\cdot, t), R(\cdot, t))||_\mathbb{Y} = +\infty\).

In the following, we consider the positivity of the solution \((S(\cdot, t), I(\cdot, t), R(\cdot, t))\) for \(t \in [0, T_0]\). Indeed, using the variation of constants formula, we see from the first equation of (2) that

\[
S(x, t) = S_0(x)e^{-\int_0^t (1+\mu+\beta I(x, \tau))d\tau} + \int_0^t e^{-\int_s^t (1+\mu+\beta I(x, \tau))d\tau} \left[ \int_{\mathbb{R}} J(x-y)S(y, s)ds + B \right] ds
\]

for all \(t \in (0, T_0)\) and \(x \in \mathbb{R}\). This, together with the facts that \(S_0 \in X_+\) and \(J(x) \geq 0\) on \(\mathbb{R}\), implies that \(S(\cdot, t) \in X_+\) for all \(t \in [0, T_0]\). Similarly, we can prove that \(I(\cdot, t) \in X_+\) and \(R(\cdot, t) \in X_+\) for all \(t \in [0, T_0]\).

On the other hand, setting \(N(x, t) = S(x, t) + I(x, t) + R(x, t)\), it then follows from (2) that

\[
\frac{\partial N(x, t)}{\partial t} = d[J \ast N(x, t) - N(x, t)] + B - \mu N(x, t) - \alpha I(x, t)
\]

\[
\leq d[J \ast N(x, t) - N(x, t)] + B - \mu N(x, t), \quad x \in \mathbb{R}, \quad t > 0.
\]

Notice that \(N(x, 0) = S_0(x) + I_0(x) + R_0(x) \leq B/\mu\). Then the comparison principle implies that

\[
N(x, t) = S(x, t) + I(x, t) + R(x, t) \leq \frac{B}{\mu}, \quad \forall x \in \mathbb{R}, \quad t > 0.
\]

Therefore, we obtain that \(T_0 = +\infty\) and the proof is thus complete. \(\square\)

Let \(S(x, t) = \phi(x + ct), I(x, t) = \varphi(x + ct)\) and \(R(x, t) = \psi(x + ct)\), where \(c > 0\) is the wave speed. Denoting the wave coordinate \(x + ct\) by \(\xi\), we get the following
wave profile system
\[
\begin{aligned}
  &c\phi'(\xi) = \int_{\mathbb{R}} J(\xi - y) \phi(y) dy - \phi(\xi) + B - \mu \phi(\xi) - \beta \phi(\xi) \varphi(\xi), \\
  &c\psi'(\xi) = \int_{\mathbb{R}} J(\xi - y) \varphi(y) dy - \varphi(\xi) + \beta \phi(\xi) \varphi(\xi) - (\mu + \gamma + \alpha) \varphi(\xi) + \delta \psi(\xi), \\
  &c\psi'(\xi) = \int_{\mathbb{R}} J(\xi - y) \psi(y) dy - \psi(\xi) + \gamma \varphi(\xi) - (\mu + \delta) \psi(\xi),
\end{aligned}
\]  
subjected to the asymptotic boundary conditions
\[
(\phi(-\infty), \varphi(-\infty), \psi(-\infty)) = \left(\frac{B}{\mu}, 0, 0\right), 
(\phi(\infty), \varphi(\infty), \psi(\infty)) = (S^*, I^*, R^*). 
\]

Note that (3) yields that \( \varphi(\xi) + \psi(\xi) \leq \frac{B}{\mu}, \forall \xi \in \mathbb{R} \). We intend to find the nonnegative solutions of system (4) satisfying (5).

2.2. The eigenvalue problem. Linearizing the equations of \( \varphi \) and \( \psi \) in (4) at \( E_0(\frac{B}{\mu}, 0, 0) \), we obtain that
\[
\begin{aligned}
  &c\varphi'(\xi) = \int_{\mathbb{R}} J(\xi - y) \varphi(y) dy - \varphi(\xi) + \beta \frac{B}{\mu} \varphi(\xi) - (\mu + \gamma + \alpha) \varphi(\xi) + \delta \psi(\xi), \\
  &c\psi'(\xi) = \int_{\mathbb{R}} J(\xi - y) \psi(y) dy - \psi(\xi) + \gamma \varphi(\xi) - (\mu + \delta) \psi(\xi). 
\end{aligned}
\]

Plugging \( (\varphi(\xi), \psi(\xi)) = e^{\lambda \xi} (\varphi_0, \psi_0) \) with some positive vector \( (\varphi_0, \psi_0) \), we get the following eigenvalue problem
\[
\begin{pmatrix}
  \int_{\mathbb{R}} J(y) e^{-\lambda y} dy - 1 - c\lambda + \beta \frac{B}{\mu} - (\mu + \gamma + \alpha) \varphi_0 + \delta \psi_0 = 0, \\
  \int_{\mathbb{R}} J(y) e^{-\lambda y} dy - 1 - c\lambda - (\mu + \delta) \psi_0 + \gamma \varphi_0 = 0.
\end{pmatrix}
\]

Letting
\[
\begin{aligned}
  &h_1(\lambda, c) = \int_{\mathbb{R}} J(y) e^{-\lambda y} dy - 1 - c\lambda + \beta \frac{B}{\mu} - (\mu + \gamma + \alpha), \\
  &h_2(\lambda, c) = \int_{\mathbb{R}} J(y) e^{-\lambda y} dy - 1 - c\lambda - (\mu + \delta),
\end{aligned}
\]

it follows that
\[
\begin{pmatrix}
  -\delta \psi_0 = h_1(\lambda, c) \varphi_0, \\
  -\gamma \varphi_0 = h_2(\lambda, c) \psi_0,
\end{pmatrix}
\]

which is equivalent to
\[
\begin{pmatrix}
  \varphi_0 \\
  \psi_0
\end{pmatrix}
= M(\lambda, c)
\begin{pmatrix}
  \varphi_0 \\
  \psi_0
\end{pmatrix}
:=
\begin{pmatrix}
  0 & -\frac{\delta}{h_1(\lambda, c)} \\
  -\frac{\gamma}{h_2(\lambda, c)} & 0
\end{pmatrix}
\begin{pmatrix}
  \varphi_0 \\
  \psi_0
\end{pmatrix}
\]

provided that \( h_1(\lambda, c) h_2(\lambda, c) \neq 0 \). This shows that \( (\varphi(\xi), \psi(\xi)) = e^{\lambda \xi} (\varphi_0, \psi_0) \) solves (6) if and only if the principal eigenvalue, denoted by \( \Theta(\lambda, c) \), of \( M(\lambda, c) \) equals 1 with \( (\varphi_0, \psi_0) \) being the corresponding eigenvector. By a direct calculation, we have
\[
\Theta(\lambda, c) = \sqrt{\frac{\gamma \delta}{h_1(\lambda, c) h_2(\lambda, c)}}.
\]

Lemma 2.1. Assume that \( \beta \frac{B}{\mu} < \mu + \gamma + \alpha \). Then there exist two positive constants \( \lambda_1 \) and \( \lambda_2 \) such that for any fixed \( c > 0 \), there hold
\[
\begin{aligned}
  & (i) \quad h_1(\lambda_1, c) = h_2(\lambda_2, c) = 0, \\
  & (ii) \quad h_1(\lambda, c) < 0, \forall \lambda \in (0, \lambda_1); \quad h_2(\lambda, c) < 0, \forall \lambda \in (0, \lambda_2).
\end{aligned}
\]
Proof. By a direct calculation, we have
\[ h_1(0, c) = \frac{\beta B}{\mu} - (\mu + \gamma + \alpha) \leq 0, \]
\[ h_2(0, c) = - (\mu + \delta) < 0, \]
\[ \frac{\partial^2 h_1(\lambda, c)}{\partial \lambda^2} = \frac{\partial^2 h_2(\lambda, c)}{\partial \lambda^2} = \int_{\mathbb{R}} J(y) y^2 e^{-\lambda y} dy > 0. \]
Then the assertion follows easily from the basic properties of \( h_i(\lambda, c) \).

Let \( \lambda_0 = \min\{\lambda_1, \lambda_2\} \). Then we have the following result.

**Lemma 2.2.** Assume that \( R_0 > 1 \) and \( \beta \frac{B}{\mu} < \mu + \gamma + \alpha \). Then there exist some \( \lambda^* \in (0, \lambda_0) \) and \( c^* > 0 \) such that
\[ \Theta(\lambda^*, c^*) = 1 \quad \text{and} \quad \frac{\partial \Theta(\lambda^*, c^*)}{\partial \lambda} = 0. \]
Furthermore, the following statements are valid:

(i) For any \( c > c^* \), the equation \( \Theta(\lambda, c) = 1 \) admits two real roots \( \lambda_1 := \Lambda_1(c) \) and \( \lambda_2 := \Lambda_2(c) \) with \( 0 < \lambda_1 < \lambda^* < \lambda_2 < \lambda_0 \) such that
\[ \Theta(\lambda, c) < 1, \quad \forall \lambda \in (\Lambda_1, \Lambda_2) \quad \text{and} \quad \Theta(\lambda, c) > 1, \quad \forall \lambda \in (0, \Lambda_1) \cup (\Lambda_2, \lambda_0). \]

(ii) If \( 0 < c < c^* \), then \( \Theta(\lambda, c) > 1 \) for all \( \lambda \in (0, \lambda_0) \).

**Proof.** For convenience, letting \( \Delta(\lambda, c) = \Theta^2(\lambda, c) \), it follows that
\[ \Delta(\lambda, c) = \frac{\gamma \delta}{h_1(\lambda, c) h_2(\lambda, c)}. \]
Since \( R_0 := \frac{\beta B (\mu + \delta)}{\mu (\mu + \alpha + \mu + \gamma + \alpha)} = \frac{\beta B (\mu + \delta)}{\mu (\mu + \alpha + \mu + \gamma + \alpha)} > 1 \), by Lemma 2.1, it follows that
\[ \Delta(0, c) = \frac{\gamma \delta}{h_1(0, c) h_2(0, c)} = \frac{\gamma \delta}{h_1(0, c) h_2(0, c)} > 1, \quad \forall c > 0, \]
\[ \frac{\partial \Delta(0, c)}{\partial \lambda} = \frac{\gamma \delta [h_1(0, c) + h_2(0, c)]}{[h_1(0, c) h_2(0, c)]^2} < 0, \quad \forall c > 0, \]
\[ \frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2} = \frac{\gamma \delta [(h_1 h_2)^2 + (h_1 h_2)^2 + (h_1 h_2)^2 - h_1 h_2 (h_1, h_2, h_1, h_2)]}{(h_1 h_2)^2} > 0, \]
where \( h_i = h_i(\lambda, c) \), \( h_{i, \lambda} = \frac{\partial h_i}{\partial \lambda} \) and \( h_{i, \lambda \lambda} = \frac{\partial^2 h_i}{\partial \lambda^2} \) for \( i = 1, 2 \). On the other hand, for all \( \lambda \in (0, \lambda_0) \), we have
\[ \frac{\partial \Delta(\lambda, c)}{\partial c} = \frac{\lambda \gamma \delta [h_1(\lambda, c) + h_2(\lambda, c)]}{[h_1(\lambda, c) h_2(\lambda, c)]^2} < 0. \]

Based on the above properties of \( \Delta(\lambda, c) \), we can easily obtain the conclusion of this lemma.

**Remark 1.** Under the assumptions of Lemma 2.2, for each \( c > c^* \), it follows that \( \Theta(\Lambda_1, c) = 1 \) and \( \frac{\partial \Theta(\lambda, c)}{\partial \lambda} < 0 \). Further, we have \( \Theta(\Lambda_1 + \epsilon, c) < 1 \) for some sufficiently small \( \epsilon > 0 \). Since \( \Theta(\Lambda_1 + \epsilon, c) \) is the eigenvalue of \( M(\Lambda_1 + \epsilon, c) \), there exists \((\varphi_1, \psi_1)\) such that
\[ M(\Lambda_1 + \epsilon, c) \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} = \Theta(\Lambda_1 + \epsilon, c) \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} < \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix}. \]
In view of the definition of $M(\Lambda_1 + \epsilon, c)$, it follows that
\[
\begin{cases}
\hat{h}_1(\Lambda_1 + \epsilon, c)\varphi_1 + \delta \psi_1 < 0, \\
\hat{h}_2(\Lambda_1 + \epsilon, c)\psi_1 + \gamma \varphi_1 < 0.
\end{cases}
\]

**Remark 2.** Note that the basic reproduction number
\[
R_0 := \frac{\beta B(\mu + \delta)}{\mu [\delta (\mu + \alpha) + \mu (\mu + \gamma + \alpha)]} = \frac{\beta B}{\mu + \alpha + \gamma \frac{\mu}{\mu + \delta}}
\]
is increasing with respect to the relapse rate $\delta$, which indicates that the relapse can encourage the spread of the disease. The assumption of Lemma 2.2 restricts that
\[
\mu + \alpha + \gamma \frac{\mu}{\mu + \delta} < \beta \frac{B}{\mu} < \mu + \alpha + \gamma,
\]
which makes sense by virtue of $\delta > 0$. We should mention that the existence of traveling waves with speeds $c > c^*$ is obtained under the hypothesis (7). Limited by current technical methods, we cannot address the wave dynamics of (2) with more general parameters in this paper. The main difficulty is that the wave system (4) is a truly coupled system involving three equations and we leave this issue for a future investigation. Meanwhile, it easily follows from (7) that when the relapse rate $\delta$ is relatively large, our theoretical results can include a wider range of parameters.

3. **Existence of traveling waves.** In this section, we always assume (7) holds and fix $c > c^*$. Define some continuous functions as follows
\[
\phi(\xi) = \frac{B}{\mu} \varphi(\xi) = e^{\Lambda_1 \xi} \varphi_0, \quad \psi(\xi) = e^{\Lambda_1 \xi} \psi_0,
\]

where $M_1, M_2, \epsilon_1$ and $\epsilon_2$ are some positive constants to be determined in the following lemmas.

**Lemma 3.1.** There hold that
\[
\begin{align*}
&c\varphi(\xi) \geq \int_{\mathbb{R}} J(\xi - y)\varphi(y)dy - \varphi(\xi) + B - \mu \varphi(\xi) - \beta \varphi(\xi) \varphi(\xi), \quad \forall \xi \in \mathbb{R}, \quad (8) \\
&c\varphi'(\xi) \geq \int_{\mathbb{R}} J(\xi - y)\varphi(y)dy - \varphi(\xi) + \beta \varphi(\xi) \varphi(\xi) \\
&\quad - (\mu + \gamma + \alpha)\varphi(\xi) + \delta \psi(\xi), \quad \forall \xi \in \mathbb{R}, \quad (9) \\
&c\psi(\xi) \geq \int_{\mathbb{R}} J(\xi - y)\psi(y)dy - \psi(\xi) + \gamma \varphi(\xi) - (\mu + \delta)\psi(\xi), \quad \forall \xi \in \mathbb{R}. \quad (10)
\end{align*}
\]

**Proof.** Since $\beta \varphi(\xi) \varphi(\xi) \geq 0$ and $\varphi(\xi) = \frac{B}{\mu}$ is a constant, (8) holds naturally.

Now we consider (9) and (10). In view of Lemma 2.2, it follows that
\[
h_1(\Lambda_1, c)\varphi_0 + \delta \psi_0 = 0, \quad h_2(\Lambda_1, c)\psi_0 + \gamma \varphi_0 = 0.
\]

Further, we have
\[
e^{\Lambda_1 \xi} h_1(\Lambda_1, c)\varphi_0 + e^{\Lambda_1 \xi} \delta \psi_0 = 0, \quad e^{\Lambda_1 \xi} h_2(\Lambda_1, c)\psi_0 + e^{\Lambda_1 \xi} \gamma \varphi_0 = 0.
\]

This ends the proof of (9) and (10). \[\square\]
Lemma 3.2. There exist some sufficiently large $M_i > 1$ ($i = 1, 2$) and small $\epsilon_1 > \epsilon_2 > 0$ such that

$$c\phi'(\xi) \leq \int_{\mathbb{R}} J(\xi - y)\phi(y)dy - \phi(\xi) + B - \mu \phi(\xi) - \beta \phi(\xi) \varphi(\xi), \quad \forall \xi \neq \xi_1,$$

(11)

$$c\varphi'(\xi) \leq \int_{\mathbb{R}} J(\xi - y)\varphi(y)dy - \varphi(\xi) + \beta \phi(\xi) \varphi(\xi)$$

$$- (\mu + \gamma + \alpha) \varphi(\xi) + \delta \psi(\xi), \quad \forall \xi \neq \xi_2,$$

(12)

$$c\psi'(\xi) \leq \int_{\mathbb{R}} J(\xi - y)\psi(y)dy - \psi(\xi) + \gamma \varphi(\xi) - (\mu + \delta) \psi(\xi), \quad \forall \xi \neq \xi_3,$$

(13)

where $\xi_1 = \frac{1}{\epsilon_1} \ln \frac{1}{M_1}, \xi_2 = \frac{1}{\epsilon_2} \ln \frac{\epsilon_0}{M_2 \varphi_1}$ and $\xi_3 = \frac{1}{\epsilon_2} \ln \frac{\epsilon_0}{M_2 \psi_1}$.

Proof. Without loss of generality, we only consider the case of $\xi_2 \leq \xi_3 \leq \xi_1$ and other cases can be treated similarly. In view of the definition of $\phi$, $\varphi$ and $\psi$ as well as the property (J), we have

$$\int_{\mathbb{R}} J(\xi - y)\phi(y)dy$$

$$\geq \max \left\{ \frac{B}{\mu} - \frac{B}{\mu} M_1 e^{\epsilon_1 \xi} \int_{\mathbb{R}} J(y) e^{-\epsilon_1 y}dy, \quad 0 \right\},$$

$$\int_{\mathbb{R}} J(\xi - y)\varphi(y)dy$$

$$\geq \max \left\{ e^{\lambda_1 \xi} \varphi_0 \int_{\mathbb{R}} J(y) e^{-\lambda_1 y}dy - e^{(\lambda_1 + \epsilon_2) \xi} M_2 \varphi_1 \int_{\mathbb{R}} J(y) e^{-(\lambda_1 + \epsilon_2) y}dy, \quad 0 \right\},$$

$$\int_{\mathbb{R}} J(\xi - y)\psi(y)dy$$

$$\geq \max \left\{ e^{\lambda_1 \xi} \psi_0 \int_{\mathbb{R}} J(y) e^{-\lambda_1 y}dy - e^{(\lambda_1 + \epsilon_2) \xi} M_2 \psi_1 \int_{\mathbb{R}} J(y) e^{-(\lambda_1 + \epsilon_2) y}dy, \quad 0 \right\}.$$

In the case where $\xi > \xi_1$, $\phi(\xi) = 0$, then (11) is obvious. In the case where $\xi < \xi_1$, $\phi(\xi) = \frac{B}{\mu} (1 - M_1 e^{\epsilon_1 \xi})$ and $\varphi(\xi) = e^{\lambda_1 \xi} \varphi_0$. It follows that

$$\int_{\mathbb{R}} J(\xi - y)\phi(y)dy - \phi(\xi) + B - \mu \phi(\xi) - \beta \phi(\xi) \varphi(\xi)$$

$$\geq \frac{B}{\mu} e^{\epsilon_1 \xi} \left[ M_1 \int_{\mathbb{R}} J(y)(1 - e^{-\epsilon_1 y})dy + \epsilon_1 c_1 M_1 - \beta(1 - M_1 e^{\epsilon_1 \xi}) e^{(\lambda_1 - \epsilon_1) \xi} \varphi_0 \right]$$

$$\geq \frac{B}{\mu} e^{\epsilon_1 \xi} \left[ M_1 \int_{\mathbb{R}} J(y)(1 - e^{-\epsilon_1 y})dy + \epsilon_1 c_1 M_1 - \beta e^{(\lambda_1 - \epsilon_1) \xi} \varphi_0 \right]$$

$$= \frac{B}{\mu} e^{\epsilon_1 \xi} \left[ M_1 \int_{\mathbb{R}} J(y)(1 - e^{-\epsilon_1 y})dy + \epsilon_1 c_1 M_1 - \beta \varphi_0 M_1^{1 - \frac{\Delta_1}{2}} \right].$$

Letting $M_1 = \frac{1}{\epsilon_1}$ and following (J), we obtain that

$$M_1 \int_{\mathbb{R}} J(y)(1 - e^{-\epsilon_1 y})dy = \int_{\mathbb{R}} J(y) \frac{1 - e^{-\epsilon_1 y}}{\epsilon_1} dy$$

$$= \sum_{n=1}^{\infty} \frac{(-\epsilon_1)^{2n-1}}{(2n)!} \int_{\mathbb{R}} J(y)y^{2n}dy \to 0,$$
and $M_1^{1-\frac{\omega_1}{\beta}} \to 0$ as $\epsilon_1 \to 0$. Therefore, we have
\[
\int_{\mathbb{R}} J(\xi - y)\varphi(y) dy - \varphi(\xi) + B - \mu \varphi(\xi) - \beta \overline{\varphi}(\xi) = 0.
\]
This ends the proof of (11).

When $\xi > \xi_2$, $\varphi(\xi) = 0$, then (12) holds naturally. When $\xi < \xi_2$, then $\varphi(\xi) = \frac{B}{\mu}(1 - M_1 e^{\epsilon_1 \xi})$, $\varphi(\xi) = e^{\Lambda_1 \xi}(\varphi_0 - M_2 \varphi_1 e^{\epsilon_2 \xi})$, and $\psi(\xi) = e^{\Lambda_1 \xi}(\psi_0 - M_2 \psi_1 e^{\epsilon_2 \xi})$. By a direct calculation, we have
\[
\int_{\mathbb{R}} J(\xi - y)\varphi(y) dy - \varphi(\xi) + \beta \overline{\varphi}(\xi) - (\mu + \gamma + \alpha) \varphi(\xi) + \delta \overline{\varphi}(\xi) - c\varphi'(\xi) \\
\geq e^{\Lambda_1 \xi}\varphi_0 \int_{\mathbb{R}} J(y) e^{-\Lambda_1 y} dy - e^{(\Lambda_1 + \epsilon_2) \xi} M_2 \varphi_1 \int_{\mathbb{R}} J(y) e^{-(\Lambda_1 + \epsilon_2) y} dy \\
- e^{\Lambda_1 \xi}(\varphi_0 - M_2 \varphi_1 e^{\epsilon_2 \xi}) + \beta \frac{B}{\mu}(1 - M_1 e^{\epsilon_1 \xi}) e^{\Lambda_1 \xi}(\varphi_0 - M_2 \varphi_1 e^{\epsilon_2 \xi}) \\
- (\mu + \gamma + \alpha)e^{\Lambda_1 \xi}(\varphi_0 - M_2 \varphi_1 e^{\epsilon_2 \xi}) + \delta e^{\Lambda_1 \xi}(\psi_0 - M_2 \psi_1 e^{\epsilon_2 \xi}) - c\Lambda_1 e^{\Lambda_1 \xi}\varphi_0 \\
+ cM_2 \varphi_1 (1 + \epsilon_2) e^{(\Lambda_1 + \epsilon_2) \xi} \\
\geq e^{\Lambda_1 \xi}[h_1(\Lambda_1, c)\varphi_0 + \delta \psi_0] - M_2 e^{(\Lambda_1 + \epsilon_2) \xi}[h_1(\Lambda_1 + \epsilon_2, c)\varphi_1 + \delta \psi_1] \\
- \beta \frac{B}{\mu} M_1 e^{(\Lambda_1 + \epsilon_1) \xi}(\varphi_0 - M_2 \varphi_1 e^{\epsilon_2 \xi}).
\]
By Lemma 2.2 and Remark 1, it follows that
\[
h_1(\Lambda_1, c)\varphi_0 + \delta \psi_0 = 0, \quad h_1(\Lambda_1 + \epsilon_2, c)\varphi_1 + \delta \psi_1 < 0.
\]
In order to prove (12), it suffices to show
\[
-M_2 e^{(\epsilon_2 - \epsilon_1) \xi}[h_1(\Lambda_1 + \epsilon_2, c)\varphi_1 + \delta \psi_1] \geq \beta \frac{B}{\mu} M_1 \varphi_0,
\]
which is equivalent to
\[
M_2 \geq \frac{\beta B M_1 \varphi_0}{-\mu[h_1(\Lambda_1 + \epsilon_2, c)\varphi_1 + \delta \psi_1]} e^{(\epsilon_1 - \epsilon_2) \xi}, \quad \forall \xi < \xi_2 := \frac{1}{\epsilon_2} \ln \frac{\varphi_0}{M_2 \varphi_1}.
\]
Note that $\epsilon_1 > \epsilon_2 > 0$ and $\xi < \xi_2$, we see that
\[
e^{(\epsilon_1 - \epsilon_2) \xi} < e^{(\epsilon_1 - \epsilon_2) \xi_2} = M_2^{1-\frac{\epsilon_2}{\epsilon_1}} \left(\frac{\varphi_0}{\varphi_1}\right)^{\frac{\epsilon_2}{\epsilon_1}}.
\]
Therefore, letting
\[
M_2 \geq \max \left\{ \left(\frac{\beta B M_1 \varphi_0}{-\mu[h_1(\Lambda_1 + \epsilon_2, c)\varphi_1 + \delta \psi_1]}\right)^{\frac{\epsilon_2}{\epsilon_1}} \left(\frac{\varphi_0}{\varphi_1}\right)^{\frac{\epsilon_1 - \epsilon_2}{\epsilon_1}}, 1 \right\},
\]
we find that (14) holds, and hence (12) is true.

Finally, we prove (13). In the case where $\xi > \xi_3$, $\varphi(\xi) = \psi(\xi) = 0$, then (13) holds naturally. In the following, we divide the case of $\xi < \xi_3$ into two subcases. When $\xi < \xi_2$, it follows that $\varphi(\xi) = e^{\Lambda_1 \xi}(\varphi_0 - M_2 \varphi_1 e^{\epsilon_2 \xi})$ and $\psi(\xi) = e^{\Lambda_1 \xi}(\psi_0 - M_2 \psi_1 e^{\epsilon_2 \xi})$,
and hence,
\[
\int R J(\xi - y)\hat{\psi}(y)dy - \hat{\psi}(\xi) + \gamma \varphi(\xi) - (\mu + \delta)\underline{\psi}(\xi) - c\psi'(\xi)
\geq e^{\lambda_1 \xi} \psi_0 \int R J(y)e^{-\lambda_1 y}dy - e^{(\lambda_1 + \epsilon_2)\xi} M_2 \psi_1 \int R J(y)e^{-(\lambda_1 + \epsilon_2) y}dy
\geq e^{\lambda_1 \xi} \psi_0 \int R J(y)e^{-\lambda_1 y}dy - e^{(\lambda_1 + \epsilon_2)\xi} M_2 \psi_1 \int R J(y)e^{-(\lambda_1 + \epsilon_2) y}dy
\]

by virtue of Remark 1. This implies (13). When \( \xi_2 \leq \xi < \xi_3 \), it follows that
\[
\underline{\psi}(\xi) = e^{\lambda_1 \xi}(\psi_0 - M_2 \psi_1 e^{\epsilon_2 \xi})
\]
and \( \overline{\varphi}(\xi) = 0 \). Further, we have
\[
\int R J(\xi - y)\underline{\psi}(y)dy - \underline{\psi}(\xi) + \gamma \varphi(\xi) - (\mu + \delta)\underline{\psi}(\xi) - c\psi'(\xi)
\geq e^{\lambda_1 \xi} \psi_0 \int R J(y)e^{-\lambda_1 y}dy - e^{(\lambda_1 + \epsilon_2)\xi} M_2 \psi_1 \int R J(y)e^{-(\lambda_1 + \epsilon_2) y}dy
\geq e^{\lambda_1 \xi} \psi_0 \int R J(y)e^{-\lambda_1 y}dy - e^{(\lambda_1 + \epsilon_2)\xi} M_2 \psi_1 \int R J(y)e^{-(\lambda_1 + \epsilon_2) y}dy
\]

which leads to (13). Now we complete all the proof of this Lemma. \( \square \)

Let \( X > \max \left\{ \frac{1}{\epsilon_1} \ln M_1, \frac{1}{\epsilon_2} \ln \frac{M_2 \psi_3}{\psi_0} \right\} \) and define the following set
\[
\Gamma_X = \left\{ (\eta, \theta, \varphi) \in C([-X, X], \mathbb{R}^3) \mid \begin{array}{l}
(\eta, \theta, \varphi)(-X) = (\phi, \varphi, \psi)(-X), \\
\varphi(\xi) \leq \eta(\xi) \leq B/\mu, \\
\varphi(\xi) \leq \varphi(\xi) \leq \min\{ e^{\lambda_1 \xi} \varphi_0, B/\mu \}, \\
\psi(\xi) \leq \varphi(\xi) \leq \varphi(\xi),
\end{array} \right\}
\]

It is easy to see that \( \Gamma_X \) is a bounded, closed and convex subset of \( C([-X, X], \mathbb{R}^3) \). For any given \( (\eta, \theta, \varphi) \in \Gamma_X \), we define the extension \( (\tilde{\eta}, \tilde{\theta}, \tilde{\varphi}) \in C(\mathbb{R}, \mathbb{R}^3) \) by
\[
(\tilde{\eta}(\xi), \tilde{\theta}(\xi), \tilde{\varphi}(\xi)) = \begin{cases}
(\eta(X), \theta(X), \varphi(X)), & \xi > X, \\
(\eta(\xi), \theta(\xi), \varphi(\xi)), & \xi \leq X, \\
(\phi(\xi), \varphi(\xi), \psi(\xi)), & \xi < -X.
\end{cases}
\]

and consider the following initial value problem:
\[
\begin{cases}
c_0'(\xi) = \int R J(y)\tilde{\eta}(\xi - y)dy - \phi(\xi) + B - \mu \phi(\xi) - \beta \eta(\xi)\theta(\xi), \\
c_2'(\xi) = \int R J(y)\tilde{\theta}(\xi - y)dy - \varphi(\xi) + \beta \eta(\xi)\theta(\xi) - (\mu + \gamma + \alpha) \varphi(\xi) + \delta \theta(\xi), \\
c_0'(\xi) = \int R J(y)\tilde{\psi}(\xi - y)dy - \psi(\xi) + \gamma \theta(\xi) - (\mu + \delta) \psi(\xi), \\
(\phi, \varphi, \psi)(-X) = (\phi, \varphi, \psi)(-X).
\end{cases}
\]
(15)
Following the standard ODE theory, system (15) admits a unique solution
\[(\phi_X, \varphi_X, \psi_X) \in C^1([-X, X], \mathbb{R}^3).\]

Further, we define an operator \(T = (T_1, T_2, T_3) : \Gamma_X \rightarrow C([-X, X], \mathbb{R}^3)\) by
\[T_1[\eta, \theta, \vartheta](\xi) = \phi_X(\xi), \quad T_2[\eta, \theta, \vartheta](\xi) = \varphi_X(\xi), \quad T_3[\eta, \theta, \vartheta](\xi) = \psi_X(\xi), \quad \forall \xi \in [-X, X].\]

**Lemma 3.3.** The operator \(T : \Gamma_X \rightarrow \Gamma_X\) is completely continuous.

**Proof.** In view of Lemmas 2.1 and 2.2, we can easily obtain that \(T(\Gamma_X) \subseteq \Gamma_X\). On the other hand, we can use the argument similar to some previous work (see e.g. [21, Theorem 2.5] or [29, Lemma 2.5]) to show the continuity and compactness of \(T\). Here we omit the details. \(\square\)

According to the above discussions, the Schauders fixed-point theorem implies that there exists \((\phi_X, \varphi_X, \psi_X) \in \Gamma_X\) such that
\[(\phi_X(\xi), \varphi_X(\xi), \psi_X(\xi)) = T[\phi_X, \varphi_X, \psi_X](\xi)\]
for all \(\xi \in [-X, X]\). Before letting \(X\) tend to infinity to obtain the existence of traveling waves, we first give some uniform estimates for \(\phi_X, \varphi_X\) and \(\psi_X\) in the space \(C^{1,1}([-X, X])\) defined by
\[C^{1,1}([-X, X]) = \{g \in C^1([-X, X]) | g\text{ and } g'\text{ are Lipschitz continuous}\}\]
with the norm
\[\|g\|_{C^{1,1}([-X, X])} = \max_{\xi \in [-X, X]} |g(\xi)| + \max_{\xi \in [-X, X]} |g'(\xi)| + \sup_{\xi, \xi' \in [-X, X], \xi \neq \xi'} \frac{|g'(\xi) - g'(\xi')|}{|\xi - \xi'|}.\]

**Theorem 3.4.** For any \(X > \max\left\{\frac{1}{r_1} \ln M_1, \frac{1}{r_2} \ln \frac{M_{\tilde{\varphi}_1}}{\varphi_0}, \frac{1}{r_2} \ln \frac{M_{\tilde{\psi}_1}}{\psi_0}, r\right\}\), there exists a positive constant \(C_0\) such that
\[\|\phi_X\|_{C^{1,1}([-X, X])} \leq C_0, \quad \|\varphi_X\|_{C^{1,1}([-X, X])} \leq C_0, \quad \|\psi_X\|_{C^{1,1}([-X, X])} \leq C_0,\]
where \(r\) is the radius of \(\text{supp } J\).

**Proof.** Note that \(\phi_X + \varphi_X + \psi_X \leq \frac{B}{\mu}\) satisfies
\[
\begin{aligned}
c\phi'_X(\xi) &= \int_{\mathbb{R}} J(y)\tilde{\phi}_X(\xi - y)dy - \phi_X(\xi) + B - \mu \phi_X(\xi) - \beta \phi_X(\xi)\varphi_X(\xi), \\
c\varphi'_X(\xi) &= \int_{\mathbb{R}} J(y)\tilde{\varphi}_X(\xi - y)dy - \varphi_X(\xi) + \beta \phi_X(\xi)\varphi_X(\xi) \\
&- (\mu + \gamma + \alpha)\varphi_X(\xi) + \delta \psi_X(\xi), \\
c\psi'_X(\xi) &= \int_{\mathbb{R}} J(y)\tilde{\psi}_X(\xi - y)dy - \psi_X(\xi) + \gamma \varphi_X(\xi) - (\mu + \delta)\psi_X(\xi),
\end{aligned}
\]
in which
\[\langle \tilde{\phi}_X(\xi), \tilde{\varphi}_X(\xi), \tilde{\psi}_X(\xi) \rangle = \begin{cases} 
(\phi_X(\xi), \varphi_X(\xi), \psi_X(\xi)), & \xi > X, \\
(\tilde{\phi}_X(\xi), \tilde{\varphi}_X(\xi), \tilde{\psi}_X(\xi)), & |\xi| \leq X, \\
(\phi(\xi), \varphi(\xi), \psi(\xi)), & \xi < -X.
\end{cases}\]

It easily follows that
\[\begin{align*}
|\phi'_X(\xi)| &\leq \frac{1}{c}\left(\frac{2B}{\mu} + 2B + \frac{\beta B^2}{\mu^2}\right) =: L_1, \\
|\varphi'_X(\xi)| &\leq \frac{B}{c\mu}\left(2 + \frac{\beta B}{\mu} + \mu + \gamma + \alpha + \delta\right) =: L_2, \\
|\psi'_X(\xi)| &\leq \frac{B}{c\mu}\left(2 + \gamma + \mu + \delta\right) =: L_3.
\end{align*}\]
Further, for any $\xi, \zeta \in [-X, X]$, we have

$$|\phi_X(\xi) - \phi_X(\zeta)| \leq L_1|\xi - \zeta|, \quad |\varphi_X(\xi) - \varphi_X(\zeta)| \leq L_2|\xi - \zeta|, \quad |\psi_X(\xi) - \psi_X(\zeta)| \leq L_3|\xi - \zeta|.$$ 

Following the equation satisfied by $\varphi_X$, we can obtain that

$$|\varphi'_X(\xi) - \varphi'_X(\zeta)| \leq \frac{1}{c} \int_{-\infty}^{\infty} |J(\xi - y) - J(\zeta - y)| \varphi_X(y) \, dy + \frac{1}{c} \left( \frac{\beta B}{\mu} + 1 + \mu + \gamma + \alpha \right) L_2|\xi - \zeta|$$

$$+ \frac{\beta B}{c\mu} L_1|\xi - \zeta| + \frac{\delta}{c} L_3|\xi - \zeta|, \quad \forall \xi, \zeta \in [-X, X].$$

Note that

$$\left| \int_{-\infty}^{\infty} [J(\xi - y) - J(\zeta - y)] \varphi_X(y) \, dy \right| \leq \int_{-\infty}^{\infty} \frac{1}{c} \ln \frac{\varphi(y)}{\varphi(|y|)} \, dy L_1|\xi - \zeta| + \int_{-\infty}^{\infty} \frac{1}{c} \ln \frac{\varphi(y)}{\varphi(|y|)} \varphi_X(y) \, dy + \frac{B}{\mu} \int_{\xi - X}^{\xi - X} |J(y)| \, dy,$$

where $L_1$ is the Lipschitz constant of kernel function $J$. By a direct calculation, we have

$$\left| \int_{-X}^{X} [J(\xi - y) - J(\zeta - y)] \varphi_X(y) \, dy \right|$$

$$= \left| \int_{-X}^{\xi + X} J(z) \varphi_X(\xi - z) \, dz - \int_{-X}^{\xi + X} J(z) \varphi_X(\zeta - z) \, dz \right|$$

$$= \left| \int_{\xi + X}^{\xi + X} J(z) \varphi_X(\xi - z) \, dz + \int_{\xi + X}^{\xi + X} J(z) [\varphi_X(\xi - z) - \varphi_X(\zeta - z)] \, dz \right.$$ 

$$+ \int_{\xi - X}^{\xi - X} J(z) \varphi_X(\xi - z) \, dz$$

$$\leq \left( \frac{2B\|J\|_{\infty}}{\mu} + L_2 \right) |\xi - \zeta|. $$
Thus, we further have such that $\lim_{|\xi| \to \infty} \frac{1}{\xi} \phi(y) dy$ is integrable. Therefore, we can arrive at

$$\left| \int_{\mathbb{R}} [J(\xi - y) - J(\xi - y)] \phi(y) dy \right| \leq \left( L_4 \int_{-\infty}^{\infty} \frac{1}{\xi} \phi(y) dy + \frac{3B\|J\|_\infty}{\mu} + L_2 \right) |\xi - \zeta|,$$

and hence,

$$|\phi'_X(\xi) - \phi'_X(\zeta)| \leq \frac{1}{C} \left( L_4 \int_{-\infty}^{\infty} \frac{1}{\xi} \phi(y) dy + \frac{3B\|J\|_\infty}{\mu} + \left( \frac{\beta B}{\mu} + 2 + \gamma + \alpha \right) L_2 + \frac{\beta B}{\mu} L_1 + \delta L_3 \right) |\xi - \zeta| = : L_4 |\xi - \zeta|.$$

Thus, we further have

$$\|\phi_X\|_{C^{1,1}([-X, X])} \leq \frac{B}{\mu} + L_2 + L_4.$$

By a similar argument, we can obtain the estimates for $\|\phi_X\|_{C^{1,1}([-X, X])}$ and $\|\psi_X\|_{C^{1,1}([-X, X])}$. To avoid redundancy, we omit the details. Then the proof is complete.

Next, choose a sequence $\{X_n\}_{n=1}^{+\infty}$ with

$$X_n > \max \left\{ \frac{1}{\epsilon_1} \ln M_1, \frac{1}{\epsilon_2} \ln M_2 \phi_1, \frac{1}{\epsilon_2} \ln M_2 \psi_1, r \right\},$$

such that $\lim_{n \to \infty} X_n = \infty$. Obviously, for each $n$, $\phi_{X_n}$, $\phi_{X_n}$, $\psi_{X_n}$ satisfy the estimates in Theorem 3.4. Thus, $\{\phi_{X_n}\}$, $\{\phi_{X_n}\}$, $\{\psi_{X_n}\}$ are relatively compact sets of $C^{1,1}([-X, X], \mathbb{R})$. Hence, by the Arzela-Ascoli theorem, there exist some convergent subsequences $\{\phi_{X_{n_k}}\}$, $\{\phi_{X_{n_k}}\}$, $\{\psi_{X_{n_k}}\}$ such that $\lim_{k \to \infty} X_{n_k} = \infty$ and

$$\phi_{X_{n_k}} \to \phi, \quad \phi_{X_{n_k}} \to \phi, \quad \psi_{X_{n_k}} \to \psi \quad \text{in} \quad C^{1,1}_{loc}(\mathbb{R}) \quad \text{as} \quad k \to \infty.$$

Following (J), the Lebesgue dominated convergence theorem implies that

$$\lim_{k \to \infty} \int_{\mathbb{R}} J(\xi - y) \phi_{X_{n_k}}(y) dy = \int_{\mathbb{R}} J(\xi - y) \phi(y) dy,$$

$$\lim_{k \to \infty} \int_{\mathbb{R}} J(\xi - y) \phi_{X_{n_k}}(y) dy = \int_{\mathbb{R}} J(\xi - y) \phi(y) dy,$$

$$\lim_{k \to \infty} \int_{\mathbb{R}} J(\xi - y) \psi_{X_{n_k}}(y) dy = \int_{\mathbb{R}} J(\xi - y) \psi(y) dy.$$

Thus, it easily follows that the limit $(\phi, \phi, \psi)$ satisfies system (4).

Now we are in a position to prove the main result of this section.

**Theorem 3.5.** Assume that (7) holds. Then for any $c > c^*$, system (2) admits a nontrivial nonnegative bounded traveling wave solution $(\phi(x+ct), \phi(x+ct), \psi(x+ct))$ connecting the disease-free equilibrium $E_0$ and the endemic equilibrium $E_*$. Further, the traveling wave has the following decay rates at $\xi = -\infty$

$$\lim_{\xi \to -\infty} e^{-\Lambda_1 \xi} \phi(\xi) = \varphi_0 \quad \text{and} \quad \lim_{\xi \to -\infty} e^{-\Lambda_1 \xi} \psi(\xi) = \psi_0.$$

(16)
Proof. Combining the previous discussion, we just need to show the asymptotic boundary of \((\phi(\xi), \varphi(\xi), \psi(\xi))\). By the definition of the functions \((\bar{\phi}, \bar{\varphi}, \bar{\psi})\) and \((\bar{\phi}, \bar{\varphi}, \bar{\psi})\), it follows that

\[
\lim_{\xi \to -\infty} (\phi(\xi), \varphi(\xi), \psi(\xi)) = \lim_{\xi \to -\infty} (\bar{\phi}(\xi), \bar{\varphi}(\xi), \bar{\psi}(\xi)) = (B/\mu, 0, 0).
\]

Then the squeezing technique implies that \(\lim_{\xi \to -\infty} (\phi(\xi), \varphi(\xi), \psi(\xi)) = \left(\frac{B}{\mu}, 0, 0\right)\) and

\[
\varphi_0 - M_2 \psi_1 e^{\gamma \xi} \leq e^{-\Lambda_1 \xi} \varphi(\xi) \leq \varphi_0 \quad \text{and} \quad \psi_0 - M_2 \psi_1 e^{\gamma \xi} \leq e^{-\Lambda_1 \xi} \psi(\xi) \leq \psi_0.
\]

This implies that (16) holds.

It remains to show that \(\lim_{\xi \to -\infty} (\phi(\xi), \varphi(\xi), \psi(\xi)) = E_* := (S^*, J^*, R^*)\). Motivated by [8, 28, 11, 22], here we plan to construct a Lyapunov functional to address this problem. We first claim that \(\phi(\xi), \varphi(\xi), \psi(\xi) > 0\), \(\forall \xi \in \mathbb{R}\). Next, we illustrate this point through proving \(\varphi(\xi) > 0\), \(\forall \xi \in \mathbb{R}\), and the others can be treated similarly. Indeed, suppose that there exists \(\xi_0 \in \mathbb{R}\) such that \(\varphi(\xi_0) = 0\), then \(\varphi'(\xi_0) = 0\) and \((J^* \varphi - \varphi)(\xi_0) \geq 0\) with equality holding if and only if \(\varphi = 0\). Since \(\varphi(\xi) \geq \max \{e^{\gamma \xi}(\varphi_0 - M_2 \psi_1 e^{\gamma \xi}), 0\}\), it follows that \((J^* \varphi - \varphi)(\xi_0) > 0\), and hence,

\[
0 = c \varphi'(\xi_0) = (J^* \varphi - \varphi)(\xi_0) + \delta \psi(\xi_0) > 0,
\]

which leads to a contradiction.

Let \(h(y) = y - 1 - \ln y, \forall y > 0\), and define

\[
\alpha_1(y) = \int_y^\infty J(x)dx, \quad \alpha_2(y) = \int_{-\infty}^y J(x)dx.
\]

Since \(J\) is compactly supported, denoting the compactly supported set of \(J\) by \(supp J = [-r, r]\) with \(r > 0\), it follows that

\[
\alpha_1(y) \equiv 0 \quad \text{for} \quad y \geq r, \quad \alpha_2(y) \equiv 0 \quad \text{for} \quad y \leq -r. \quad (17)
\]

Now, we consider the following functional

\[
V(\phi, \varphi, \psi)(\xi) = cV_3(\phi, \varphi, \psi)(\xi) + S^* U_1(\phi)(\xi) + I^* U_2(\varphi)(\xi) + R^* U_3(\psi)(\xi), \quad (18)
\]

for all \(\xi \in \mathbb{R}\), where

\[
V_1(\phi, \varphi, \psi)(\xi) = S^* h \left(\frac{\phi(\xi)}{S^*}\right) + I^* h \left(\frac{\varphi(\xi)}{I^*}\right) + \frac{\delta R^*}{\gamma I^*} R^* h \left(\frac{\psi(\xi)}{R^*}\right),
\]

\[
U_1(\phi)(\xi) = \int_0^{+\infty} \alpha_1(y) h \left(\frac{\phi(\xi - y)}{S^*}\right) dy - \int_{-\infty}^0 \alpha_2(y) h \left(\frac{\phi(\xi - y)}{S^*}\right) dy,
\]

\[
U_2(\varphi)(\xi) = \int_0^{+\infty} \alpha_1(y) h \left(\frac{\varphi(\xi - y)}{I^*}\right) dy - \int_{-\infty}^0 \alpha_2(y) h \left(\frac{\varphi(\xi - y)}{I^*}\right) dy,
\]

\[
U_3(\psi)(\xi) = \int_0^{+\infty} \alpha_1(y) h \left(\frac{\psi(\xi - y)}{R^*}\right) dy - \int_{-\infty}^0 \alpha_2(y) h \left(\frac{\psi(\xi - y)}{R^*}\right) dy.
\]

Due to the properties of \(h\), it is easy to see that \(V_1(\phi, \varphi, \psi)(\xi)\) is bounded from below for all \(\phi, \varphi, \psi > 0\). Further, by the boundedness of \((\phi, \varphi, \psi)\) and (17), it follows that \(U_1(\phi)(\xi), U_2(\varphi)(\xi)\) and \(U_3(\psi)(\xi)\) are bounded. Thus, \(V(\phi, \varphi, \psi)(\xi)\) is well-defined and bounded from below.
By a direct computation, we have

\[
\frac{dV_1(\phi, \varphi, \psi)(\xi)}{d\xi} = \frac{\phi(\xi) - S^*}{\phi(\xi)} c\phi'(\xi) + \frac{\varphi(\xi) - I^*}{\varphi(\xi)} c\varphi'(\xi) + \frac{\delta R^* \psi(\xi) - R^*}{\gamma I^*} c\psi'(\xi)
\]

and

\[
\frac{dU_1(\phi)(\xi)}{d\xi} = \frac{d}{d\xi} \int_0^{+\infty} \alpha_1(y) h \left( \frac{\phi(\xi - y)}{S^*} \right) dy - \frac{d}{d\xi} \int_{-\infty}^{0} \alpha_2(y) h \left( \frac{\phi(\xi - y)}{S^*} \right) dy
\]

Similarly, we can get that

\[
\frac{dU_2(\varphi)(\xi)}{d\xi} = h \left( \frac{\varphi(\xi)}{I^*} \right) - \int_{-\infty}^{+\infty} J(y) h \left( \frac{\varphi(\xi - y)}{I^*} \right) dy,
\]

\[
\frac{dU_3(\psi)(\xi)}{d\xi} = h \left( \frac{\psi(\xi)}{R^*} \right) - \int_{-\infty}^{+\infty} J(y) h \left( \frac{\psi(\xi - y)}{R^*} \right) dy.
\]
Further, using the formula (18), we can easily obtain that

\[
\frac{dV(\phi, \varphi, \psi)(\xi)}{d\xi} = \frac{\phi(\xi)}{\phi(\xi)} [(J * \phi - \phi)(\xi) + B - \mu \phi(\xi) - \beta \phi(\xi) \varphi(\xi)] \\
+ \frac{\varphi(\xi)}{\varphi(\xi)} [(J * \varphi - \varphi)(\xi) + \beta \phi(\xi) \varphi(\xi) - (\mu + \gamma + \alpha) \varphi(\xi) + \delta \psi(\xi)] \\
+ \frac{\delta R^* \psi(\xi) - R^*}{\gamma I^*} [(J * \psi - \psi)(\xi) + \gamma \varphi(\xi) - (\mu + \delta) \psi(\xi)] \\
+ \frac{S^*}{\gamma I^*} \frac{dU_1(\phi)(\xi)}{d\xi} + I^* \frac{dU_2(\varphi)(\xi)}{d\xi} + \frac{\delta R^* R^*}{\gamma I^*} \frac{dU_3(\psi)(\xi)}{d\xi} := X_1 + X_2 + X_3 + X_4,
\]

where

\[
X_1 = \left(1 - \frac{S^*}{\phi(\xi)}\right) [B - \mu \phi(\xi) - \beta \phi(\xi) \varphi(\xi)] \\
+ \left(1 - \frac{I^*}{\varphi(\xi)}\right) [\beta \phi(\xi) \varphi(\xi) - (\mu + \gamma + \alpha) \varphi(\xi) + \delta \psi(\xi)] \\
+ \frac{\delta R^*}{\gamma I^*} \left(1 - \frac{R^*}{\psi(\xi)}\right) [\gamma \varphi(\xi) - (\mu + \delta) \psi(\xi)],
\]

\[
X_2 = \left(1 - \frac{S^*}{\phi(\xi)}\right) (J * \phi - \phi)(\xi) + S^* \frac{dU_1(\phi)(\xi)}{d\xi},
\]

\[
X_3 = \left(1 - \frac{I^*}{\varphi(\xi)}\right) (J * \varphi - \varphi)(\xi) + I^* \frac{dU_2(\varphi)(\xi)}{d\xi},
\]

\[
X_4 = \frac{\delta R^*}{\gamma I^*} \left[\left(1 - \frac{R^*}{\psi(\xi)}\right) (J * \psi - \psi)(\xi) + R^* \frac{dU_3(\psi)(\xi)}{d\xi}\right].
\]

Using the relation at the endemic equilibrium \(E_*(S^*, I^*, R^*)\), by further computation and simplification, we have

\[
X_1 = -(\mu + \beta I^*) \frac{(\phi(\xi) - S^*)^2}{\phi(\xi)} - \delta R^* \left[\sqrt{I^* \varphi(\xi) R^*} - \sqrt{\varphi(\xi) R^*} \right]^2 \leq 0,
\]

\[
X_2 = -S^* \int_{-\infty}^{+\infty} J(y) h \left(\frac{\phi(\xi - y)}{\phi(\xi)}\right) dy \leq 0,
\]

\[
X_3 = -I^* \int_{-\infty}^{+\infty} J(y) h \left(\frac{\varphi(\xi - y)}{\varphi(\xi)}\right) dy \leq 0,
\]

\[
X_4 = -\frac{\delta R^*}{\gamma I^*} R^* \int_{-\infty}^{+\infty} J(y) h \left(\frac{\psi(\xi - y)}{\psi(\xi)}\right) dy \leq 0,
\]

and hence, \(\frac{dV(\phi, \varphi, \psi)(\xi)}{d\xi} \leq 0\), \(\forall \xi \in \mathbb{R}\). This indicates that \(V(\phi, \varphi, \psi)\) is decreasing on \(\mathbb{R}\).

Now, choose an increasing sequence \(\{\xi_n\}_{n \in \mathbb{Z}}\) such that \(\xi_n \to +\infty\) as \(n \to +\infty\) and let

\[
\phi_n(\xi) = \phi(\xi + \xi_n), \quad \varphi_n(\xi) = \varphi(\xi + \xi_n), \quad \psi_n(\xi) = \psi(\xi + \xi_n), \quad \forall \xi \in \mathbb{R}.
\]
By the proof of Theorem 3.4, we can show that \( \phi_n, \varphi_n \) and \( \psi_n \) are uniformly bounded in \( C^{1,1}(\mathbb{R}) \). Up to a subsequence, we can assume that \( \phi_n, \varphi_n \) and \( \psi_n \) converge to some nonnegative functions \( \phi_\infty, \varphi_\infty \) and \( \psi_\infty \) satisfying (4) as \( n \to +\infty \). Furthermore, since \( V(\phi, \varphi, \psi)(\xi) \) is non-increasing in \( \xi \in \mathbb{R} \) and bounded below, for large \( n \), there exists a constant \( A > 0 \) such that

\[
A \leq V(\phi_n, \varphi_n, \psi_n)(\xi) = V(\phi, \varphi, \psi)(\xi + \xi_n) \leq V(\phi, \varphi, \psi)(\xi).
\]

Thus there exists some \( \nu \in \mathbb{R} \) such that \( \lim_{n \to +\infty} V(\phi_n, \varphi_n, \psi_n)(\xi) = \nu \) for any \( \xi \in \mathbb{R} \). According to (18) and the Lebesgue dominated convergence theorem, it follows that

\[
\lim_{n \to +\infty} V(\phi_n, \varphi_n, \psi_n)(\xi) = V(\phi_\infty, \varphi_\infty, \psi_\infty)(\xi), \quad \forall \xi \in \mathbb{R},
\]

which implies that \( V(\phi_\infty, \varphi_\infty, \psi_\infty)(\xi) = \nu \).

On the other hand, \( \frac{dV(\phi, \varphi, \psi)(\xi)}{d\xi} = 0 \) if and only if \( \phi(\xi) = S^*, \varphi(\xi) = I^*, \psi(\xi) = R^* \) and \( \phi'(\xi) = 0, \varphi'(\xi) = 0, \psi'(\xi) = 0 \). Thus,

\[
\phi_\infty = S^*, \varphi_\infty = I^*, \psi_\infty = R^*.
\]

This shows that \( \lim_{\xi \to +\infty} (\phi(\xi), \varphi(\xi), \psi(\xi)) = (S^*, I^*, R^*) \) and completes the proof.

4. Minimal wave speed of traveling waves. In this section, we first establish the non-existence of traveling waves for system (2) in the case where \( \mathcal{R}_0 > 1 \) and \( c \in (0, c^*) \).

**Theorem 4.1.** Assume that \( \mathcal{R}_0 > 1 \) and \( \beta \frac{B}{\mu} + \delta < \gamma + \alpha \) (and hence, (7) holds). Then for each \( 0 < c < c^* \), system (4) has no bounded non-negative traveling waves satisfying \( \lim_{\xi \to -\infty} (\phi(\xi), \varphi(\xi), \psi(\xi)) = (B/\mu, 0, 0) \).

**Proof.** On the contrary, for any fixed \( c \in (0, c^*) \), we assume that \( (\phi, \varphi, \psi) \) is a pair of nontrivial bounded positive solution of (4) satisfying \( \lim_{\xi \to -\infty} (\phi(\xi), \varphi(\xi), \psi(\xi)) = (B/\mu, 0, 0) \). In the following, we will arrive at a contradiction by the method of two-sided Laplace transform, which has been used in our previous work [29] (see other applications in [21, 40]). To avoid redundancy, we only give the following rough outline of the proof process.

By the arguments similar to those in some previous work (see, e.g., [21, 40, 29]), we can conclude that there exists some \( \mu_0 \in (0, \lambda_0) \) such that

\[
\begin{align*}
\sup_{\xi \in \mathbb{R}} \{ \phi(\xi)e^{-\mu_0 \xi} \} &< \infty, \\
\sup_{\xi \in \mathbb{R}} \left\{ \int_{\mathbb{R}} J(\xi - y)\varphi(y)dye^{-\mu_0 \xi} \right\} &< \infty, \\
\sup_{\xi \in \mathbb{R}} \{ \varphi'(\xi)e^{-\mu_0 \xi} \} &< \infty, \\
\sup_{\xi \in \mathbb{R}} \left\{ \int_{\mathbb{R}} J(\xi - y)\psi(y)dye^{-\mu_0 \xi} \right\} &< \infty, \\
\sup_{\xi \in \mathbb{R}} \{ \psi(\xi)e^{-\mu_0 \xi} \} &< \infty, \\
\sup_{\xi \in \mathbb{R}} \left\{ \int_{\mathbb{R}} J(\xi - y)\psi(y)dye^{-\mu_0 \xi} \right\} &< \infty.
\end{align*}
\]

Note that the second and third equations of (4) are equivalent to the following system:

\[
\begin{align*}
J * \varphi(\xi) - \varphi(\xi) - c\varphi'(\xi) + \beta \frac{B}{\mu} \varphi(\xi) - (\mu + \gamma + \alpha)\varphi(\xi) + \delta\psi(\xi) \\
&= \beta \left( \frac{B}{\mu} - \phi(\xi) \right) \varphi(\xi), \quad (19) \\
J * \psi(\xi) - \psi(\xi) - c\psi'(\xi) - (\mu + \delta)\psi(\xi) + \gamma\varphi(\xi) &= 0. \quad (20)
\end{align*}
\]
For any \( \lambda \in \mathbb{C} \) with \( 0 < \text{Re}\lambda < \mu_0 \), taking two-side Laplace transform on (19) and (20), respectively, it follows that

\[
\begin{align*}
&h_1(\lambda, c) \int_{-\infty}^{+\infty} e^{-\lambda \xi} \varphi(\xi) d\xi + \delta \int_{-\infty}^{+\infty} e^{-\lambda \xi} \psi(\xi) d\xi \\
&= \int_{-\infty}^{+\infty} e^{-\lambda \xi} \beta \left( \frac{B}{\mu} - \phi(\xi) \right) \varphi(\xi) d\xi,
\end{align*}
\]

(21)

Substituting (22) into (21), we obtain that

\[
\begin{align*}
&h_2(\lambda, c) \int_{-\infty}^{+\infty} e^{-\lambda \xi} \psi(\xi) d\xi + \gamma \int_{-\infty}^{+\infty} e^{-\lambda \xi} \varphi(\xi) d\xi = 0.
\end{align*}
\]

(22)

By Lemma 2.2 (ii), \( \Theta(\lambda, c) = \sqrt{\frac{\gamma \delta}{h_1(\lambda, c) h_2(\lambda, c)}} > 1 \) for all \( \lambda \in (0, \lambda_0) \), it then follows that

\[
h_1(\lambda, c) - \frac{\delta \gamma}{h_2(\lambda, c)} > 0, \quad \forall \lambda \in (0, \lambda_0).
\]

Let \( L(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda \xi} \varphi(\xi) d\xi \). Now using the property of Laplace transform (see [35] and applications in [40, Page 1989] or [29, Page 151]), we can further show that \( L(\lambda) \) is well defined with \( \text{Re}\lambda > 0 \). However, (23) can be re-written as

\[
\int_{-\infty}^{+\infty} e^{-\lambda \xi} \left[ h_1(\lambda, c) - \frac{\delta \gamma}{h_2(\lambda, c)} \right] \varphi(\xi) d\xi = 0.
\]

(24)

In view of \( \beta \frac{B}{\mu} < -\delta + \gamma + \alpha \), we see that \( \beta \frac{B}{\mu} - (\mu + \gamma + \alpha) < -(\mu + \delta) \), and hence, \( \lambda_0 = \min\{\lambda_1, \lambda_2\} = \lambda_2 \). Note that \( h_2(\lambda, c) \to 0^+ \) as \( \lambda \to \lambda_2 \). Thus, \( h_1(\lambda, c) - \frac{\delta \gamma}{h_2(\lambda, c)} \to +\infty \) as \( \lambda \to \lambda_2 \). This contradicts the equation (24) and the proof is now complete.

Then, we further clarify the minimal wave speed by showing the existence of the strong traveling wave solution with critical speed \( c^* \).

**Theorem 4.2.** Suppose that (7) holds. Then system (4) with \( c = c^* \) has a nontrivial traveling wave solution \( (\phi_0(x + c^* t), \varphi_0(x + c^* t), \psi_0(x + c^* t)) \) satisfying

\[
(\phi_0(-\infty), \varphi_0(-\infty), \psi_0(-\infty)) = \left( \frac{B}{\mu}, 0, 0 \right)
\]

and

\[
(\phi_0(+\infty), \varphi_0(+\infty), \psi_0(+\infty)) = (S^*, I^*, R^*).
\]

**Proof.** Let \( \{c_n\} \subset (c^*, c^*+1) \) be a decreasing sequence such that \( \lim_{n \to \infty} c_n = c^* \). In view of Theorem 3.5, for each \( c_n \), there exists a traveling wave solution \( (\phi_n, \varphi_n, \psi_n) \) of (4) satisfying (5). Since \( (\phi_n(\cdot + r), \varphi_n(\cdot + r), \psi_n(\cdot + r)) \) for \( r \in \mathbb{R} \), are also solutions of (4)-(5), we can assume that

\[
\varphi_n(0) = \sigma \quad \text{and} \quad \varphi_n(\xi) \leq \sigma, \quad \forall \xi < 0,
\]

(25)

where \( \sigma \in (0, I^*) \) is sufficiently small. Further, using Theorem 3.4, Arzela-Ascoli theorem and a diagonalization argument, we can find a subsequence of \( (\phi_n, \varphi_n, \psi_n) \), still denoted by \( (\phi_n, \varphi_n, \psi_n) \), such that \( (\phi_n, \varphi_n, \psi_n) \) and \( (\phi'_n, \varphi'_n, \psi'_n) \) converge uniformly on every bounded interval (and hence, pointwise on \( \mathbb{R} \)) to functions...
if there exists $\xi$ such that $\lim_{\nu \to \infty} \phi_n(\xi) = 0$.

Without loss of generality, we further assume that $\lim_{n \to \infty} \phi_n(\xi) = 0$. Thus, we have

$$0 = \int_{\mathbb{R}} J(y) \phi_n(\xi_1 - y)dy + B > 0,$$

which is a contradiction. Then if there exists $\xi_2 \in \mathbb{R}$ such that $\phi_0(\xi_2) = 0$, it easily follows from the second equation of (4) that

$$0 = \int_{\mathbb{R}} J(y) \phi_0(\xi_2 - y)dy + \delta \psi_0(\xi_2),$$

which implies that $\phi_0(\xi) = 0$, $\forall \xi \in \mathbb{R}$ and $\psi_0(\xi_2) = 0$. However, (25) yields that $\phi_0(0) = 0$, $\forall \xi \in \mathbb{R}$ and $\phi_0(\xi) \leq \sigma$. Indeed, (25) shows that

$$0 = \int_{\mathbb{R}} J(y) \psi_0(\xi_3 - y)dy + \gamma \phi(\xi_3) > 0.$$

This contradiction shows that $\psi_0(\xi) > 0$, $\forall \xi \in \mathbb{R}$.

According to the proof of Theorem 3.5, we know that the Lyapunov functional to obtain the convergence towards the endemic equilibrium at $\xi = +\infty$ is independent of $c$. In the same way, for $c = c^*$, we can still get $(\phi_0(+\infty), \varphi_0(+\infty), \psi_0(+\infty)) = (S^*, R^*, R^*)$.

Next, we move to the proof of $(\phi_0(-\infty), \varphi_0(-\infty), \psi_0(-\infty)) = \left(\frac{B}{\mu}, 0, 0\right)$. Let us start with showing that $\phi_0(-\infty)$ exists and equals to $\frac{B}{\mu}$. Since $0 < \phi_0(\xi) \leq \frac{B}{\mu}$, $\forall \xi \in \mathbb{R}$, we can suppose, on the contrary, that

$$0 < \liminf_{\xi \to -\infty} \phi_0(\xi) < \limsup_{\xi \to -\infty} \phi_0(\xi) \leq \frac{B}{\mu}.$$

Without loss of generality, we further assume that $\liminf_{\xi \to -\infty} \phi_0(\xi) = \frac{\nu B}{\mu}$ for some $0 < \nu < 1$. It then follows that there exists a sequence $\{\xi_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} \xi_n = -\infty$ such that $\liminf_{n \to \infty} \phi_0(\xi_n) = \frac{\nu B}{\mu}$. By the continuity of $\phi_0(\xi)$, we can choose a sufficiently large number $N \in \mathbb{N}_+$ such that for any $n > N$ and some fixed $\epsilon > 0$ small enough, there holds that

$$\frac{B}{\mu} \left(\nu - \frac{1 - \nu}{4}\right) \leq \phi_0(\xi) \leq \frac{B}{\mu} \left(\nu + \frac{1 - \nu}{4}\right), \quad \forall \xi \in [\xi_n - \epsilon, \xi_n + \epsilon].$$

In particular, we take $n = N + 1$ for convenience. On the other hand, for each $n$, we have $\phi_n(-\infty) = \frac{B}{\mu}$. Thus, we have (if necessary, one can increase $N$)

$$\phi_n(\xi) \geq \frac{B}{\mu} \left(\nu + \frac{1 - \nu}{2}\right), \quad \forall \xi \in [\xi_n - \epsilon, \xi_n + \epsilon].$$
Combining (27) and (28), it follows that
\[
\phi_n(\xi) - \phi_0(\xi) \geq \frac{(1 - \nu)B}{4\mu} > 0 \quad \forall \xi \in [\xi_n - \epsilon, \xi_n + \epsilon].
\]
Letting \( n \to \infty \), we arrive at a contradiction, and hence, \( \phi_0(\infty) = \frac{B}{n} \). Now we claim that \( \phi_0(\infty) \) exists and equals to 0. Indeed, if
\[
0 \leq \varphi := \liminf_{\xi \to -\infty} \phi_0(\xi) < \limsup_{\xi \to -\infty} \phi_0(\xi) =: \varpi \leq \frac{B}{\mu},
\]
then there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) with \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = -\infty \) such that \( \lim \phi_0(x_n) = \varphi \) and \( \lim \phi_0(y_n) = \varpi \). In view of \( \phi_0(\infty) = \frac{B}{\mu} \), it follows that \( \phi_0'(\infty) = 0 \). Meanwhile, for any sequence \( \{\eta_n\} \) with \( \eta_n \to -\infty \) as \( n \to \infty \), Fatou lemma implies that
\[
\phi_0(\infty) \leq \liminf_{n \to \infty} J * \phi_0(\eta_n) \leq \limsup_{n \to \infty} J * \phi_0(\eta_n) \leq \phi_0(\infty).
\]
Thus,
\[
\lim_{n \to \infty} [J * \phi_0(\eta_n) - \phi_0(\eta_n)] = 0.
\]
Taking \( \xi = x_n, \ y_n \) in the first equation of system (4) and then letting \( n \to \infty \), respectively, we obtain that
\[
\begin{aligned}
& B - \mu \phi_0(\infty) - \beta \phi_0(\infty) \varphi = 0, \\
& B - \mu \phi_0(\infty) - \beta \phi_0(\infty) \varpi = 0,
\end{aligned}
\]
(29)
which yields that \( \varpi = \varphi \), a contradiction, and hence \( \phi_0(\infty) \) exists. In addition, according to (26), we see that \( \phi_0(\infty) \leq \sigma \), where \( \sigma \in (0, I^*) \) can be arbitrary small. Using (29) again, it follows that \( \phi_0(\infty) = 0 \). Finally, returning to system (4) and by a similar argument to the above, we can easily show that \( \psi_0(\infty) = 0 \). The proof is thereby complete. \( \square \)

5. Numerical simulations and discussion. In view of Theorems 3.5, 4.1 and 4.2, we show the existence and non-existence as well as the minimal wave speed of the strong traveling waves for SIRI model (2). In this section we present some numerical results and discussions. Choose the following particular kernel function for \( J \):
\[
J(x) = \begin{cases} 
0.1 \frac{1}{2(1-e^{-x})} e^{-\frac{|x|}{10}}, & -10 \leq x \leq 10, \\
0, & \text{elsewhere}.
\end{cases}
\]
Note that \( (\lambda^*, c^*) \) can be determined by the following equations:
\[
\begin{aligned}
& \int_{R} J(y) e^{-\lambda^* y} dy + c^* = 0, \\
& \left[ \int_{R} J(y) e^{-\lambda^* y} dy - 1 - c^* \lambda^* + \beta \frac{B}{\mu} - (\mu + \gamma + \alpha) \right] \\
& \times \left[ \int_{R} J(y) e^{-\lambda^* y} dy - 1 - c^* \lambda^* - (\mu + \delta) \right] = \gamma \delta.
\end{aligned}
\]
Choose \( \beta = 0.6, \ B = 0.7, \mu = 0.3 \) and \( \delta = \gamma = \alpha = 0.6 \). Then \( R_0 = \frac{14}{11} \) and (7) is satisfied. Moreover, \( E_0(B, 0, 0) = \left( \frac{7}{2}, 0, 0 \right) \) and \( E_*(S^*, I^*, R^*) = \left( \frac{11}{2}, \frac{3}{2}, \frac{1}{2} \right) \). With the help of Matlab, we can obtain \( (\lambda^*, c^*) \approx (0.1200, 3.4704) \). Choose \( c = 5 > c^* \). According to Theorem 3.5, system (2) admits a nontrivial nonnegative bounded traveling wave solution \( (S(x + ct), I(x + ct), R(x + ct)) \) connecting the disease-free equilibrium \( E_0 \) and the endemic equilibrium \( E_* \). See Figures 1-4 for details.
Furthermore, pick $c = 3.4704 = c^*$. According to Theorem 4.2, system (2) admits a nontrivial nonnegative bounded traveling wave solution $(S(x + c^*t), I(x + c^*t), R(x + c^*t))$ connecting the disease-free equilibrium $E_0$ and the endemic equilibrium $E_*$. See Figures 5-8 for details.

Now choose $\beta = 0.1$, $B = 0.5$, $\mu = 0.1$, $\delta = 0.1$, $\gamma = 0.5$ and $\alpha = 0.13$. Then $R_0 = \frac{25}{22} > 1$ and $\beta \frac{B}{\mu} + \delta < \gamma + \alpha$ is satisfied. Moreover, $E_0(\frac{B}{\mu}, 0, 0) = (5, 0, 0)$ and $(\lambda^*, c^*) \approx (0.0266, 0.6815)$. Choose $c = 0.1$. Then $c < c^*$. According to Theorem 4.1, system (4) has no bounded non-negative traveling waves satisfying $(S(-\infty), I(-\infty), R(-\infty)) = (\frac{B}{\mu}, 0, 0)$. See Figures 9-12 for details. In particular, Figures 11 and 12 indicate that the wave profiles of $I$ and $R$ oscillate around zero.

From Figures 1-8, we can see that the wave profiles share the same qualitative properties as $c > c^*$ and $c = c^*$. Further, we find that all these wave profiles involving the components $S$, $I$ and $R$ are monotone. Notice that the existence results of traveling wave solutions with speed $c \geq c^*$ are established under the
**Figure 5.** The profile of \((S,I,R)\) connecting \((\frac{7}{3}, 0, 0)\) and \((\frac{11}{9}, \frac{3}{22}, \frac{1}{17})\) for \(c = c^*\).

**Figure 6.** The profile of \(S\) connecting \(\frac{7}{3}\) and \(\frac{11}{9}\) for \(c = c^*\) at time steps \(t = 5, 12, 20\).

**Figure 7.** The profile of \(I\) connecting 0 and \(\frac{3}{22}\) for \(c = c^*\) at time steps \(t = 5, 12, 20\).

**Figure 8.** The profile of \(R\) connecting 0 and \(\frac{1}{17}\) for \(c = c^*\) at time steps \(t = 5, 12, 20\).

**Figure 9.** The profile of \((S,I,R)\) connecting \((5, 0, 0)\) and \((\frac{24}{5}, \frac{1}{22}, \frac{5}{35})\) for \(c < c^*\).

**Figure 10.** The profile of \(S\) connecting 5 and \(\frac{24}{5}\) for \(c < c^*\) at time steps \(t = 5, 12, 20\).
parameter setting (7) which is stronger than the condition $R_0 > 1$. We suspect that both monotone and non-monotone traveling waves exist for such considered epidemic model under more general parameter setting. It is a very interesting but challenging problem to rigorously show this conclusion and we leave it as a further investigation.

One purpose of the paper is to show the effect of relapse on disease propagation. From Figure 13, it follows that the minimal wave speed $c^*$ is monotone with respect to the relapse rate constant $\delta$ when other parameters are fixed. Hence, increase $\delta$ can encourage the disease spreading.

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REFERENCES

[1] Z. Bai and S. Zhang, Traveling waves of a diffusive SIR epidemic model with a class of nonlinear incidence rates and distributed delay, *Commun. Nonlinear Sci. Numer. Simul.*, 22 (2015), 1370–1381.

[2] S. Blower, Modelling the genital herpes epidemic, *Herpes 11*, 3 (2004), 138A–146A.

[3] S. M. Blower, T. C. Porco and G. Darby, Predicting and preventing the emergence of antiviral drug resistance in HSV-2, *Nat. Med.*, 4 (1998), 673–678.

[4] X. Chen and J. S. Guo, Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics, *Math. Ann.*, 326 (2003), 123–146.

[5] J. Coville and L. Dupaigne, On a non-local equation arising in population dynamics, *Proc. Roy. Soc. Edinburgh Sect. A.*, 137 (2007), 727–755.

[6] H. Cox, et al., Tuberculosis recurrence and mortality after successful treatment: Impact of drug resistance, *PLoS Med.*, 3 (2006), 1836–1843.

[7] O. Diekmann, Thresholds and travelling waves for the geographical spread of infection, *J. Math. Biol.*, 69 (1978), 109–130.

[8] A. Ducrot and P. Magal, Traveling wave solutions for an infection-age structured epidemic model with external supplies, *Nonlinearity*, 24 (2011), 2891–2911.

[9] A. Ducrot, P. Magal and S. Ruan, Travelling wave solutions in multigroup age-structured epidemic models, *Arch. Rational Mech. Anal.*, 195 (2010), 311–331.

[10] S. C. Fu, Traveling waves for a diffusive SIR model with delay, *J. Math. Anal. Appl.*, 435 (2016), 20–37.

[11] P. Guo, X. S. Yang and Z. C. Yang, Dynamical behaviors of an SIRI epidemic model with nonlinear incidence and latent period, *Adv. Difference Equ.*, 2014 (2014), 164–181.

[12] P. Georgescu and H. Zhang, A Lyapunov functional for a SIRI model with nonlinear incidence of infection and relapse, *Appl. Math. Comput.*, 219 (2013), 8496–8507.

[13] S. A. Gourley and J. Wu, Delayed non-local diffusive systems in biological invasion and disease spread, *Fields Inst. Commun.*, 48 (2006), 137–200.

[14] G. Huang, Y. Takeuchi, W. Ma and D. Wei, Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate, *Bull. Math. Biol.*, 72 (2010), 1192–1207.

[15] W. Huang and C. Wu, Non-monotone waves of a stage-structured SLIRM epidemic model with latent period, *Proc. Roy. Soc. Edinburgh Sect. A.*, 2013 (2013), 551–596.

[16] V. Hutson, S. Martinez, K. Mischaikow and G. T. Vickers, The evolution of dispersal, *J. Math. Biol.*, 47 (2003), 483–517.

[17] C. Y. Kao, Y. Lou and W. Shen, Random dispersal vs nonlocal dispersal, *Discrete Contin. Dyn. Syst.*, 26 (2010), 551–596.

[18] M. Kermack and A. McKendrick, Contributions to the mathematical theory of epidemics, *Proc. Roy. Soc. A.*, 115 (1927), 700–721.

[19] T. Kuniya and J. Wang, Lyapunov functions and global stability for a spatially diffusive SIR epidemic model, *Appl. Anal.*, 96 (2017), 1935–1960.

[20] W. T. Li, J. B. Wang and X.-Q. Zhao, Spatial dynamics of a nonlocal dispersal population model in a shifting environment, *J. Nonlinear Sci.*, 28 (2018), 1189–1219.

[21] W. T. Li and F. Y. Yang, Traveling waves for a nonlocal dispersal SIR model with standard incidence, *J. Integral Equations Appl.*, 26 (2014), 243–273.

[22] Y. Li, W. T. Li and F. Y. Yang, Traveling waves for nonlocal dispersal SIR model with delay and external supplies, *Appl. Math. Comput.*, 247 (2014), 723–740.

[23] J. Martins, A. Pinto and N. Stollenwerk, A scaling analysis in the SIRI epidemiological model, *J. Biol. Dyn.*, 3 (2009), 479–496.

[24] H. N. Moreira and Y. Wang, Global stability in an $S \to I \to R \to I$ model, *SIAM Rev.*, 39 (1997), 496–502.

[25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[26] D. Tudor, A deterministic model for herpes infections in human and animal populations, *SIAM Rev.*, 32 (1990), 136–139.

[27] P. van den Driessche and X. Zou, Modeling relapse in infectious diseases, *Math. Biosci.*, 207 (2007), 89–103.

[28] C. Vargas-De-León, On the global stability of infectious diseases models with relapse, *Abstr. Appl. Anal.*, 9 (2013), 50–61.
[29] J. B. Wang, W. T. Li and F. Y. Yang, Traveling waves in a nonlocal dispersal SIR model with nonlocal delayed transmission, *Commun. Nonlinear Sci. Numer. Simulat.*, 27 (2015), 136–152.

[30] J. B. Wang and C. Wu, Forced waves and gap formations for a Lotka-Volterra competition model with nonlocal dispersal and shifting habitats, *Nonlinear Anal. Real World Appl.*, 58 (2021), 103208.

[31] X. Wang, H. Wang and J. Wu, Travelling waves of diffusive predator-prey systems: Disease outbreak propagation, *Discrete Contin. Dyn. Syst.*, 32 (2012), 3303–3324.

[32] Z. C. Wang, J. Wu, Travelling waves of a diffusive Kermack-Mckendrick epidemic model with non-local delayed transmission, *Proc. R. Soc. Lond. Ser. A*, 466 (2010), 237–261.

[33] G. F. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics*, CRC Press, 1985.

[34] P. Weng and X. Q. Zhao, Spreading speed and traveling waves for a multi-type SIS epidemic model, *J. Differential Equations*, 229 (2006), 270–296.

[35] D. V. Widder, *Laplace Transform*, Princeton University Press, Princeton, NJ, 1941.

[36] P. Wildy, H. J. Field and A. A. Nash, Classical herpes latency revisited, *Virus Persistence Symposium*, 33 (1982), 133–167.

[37] C. Wu, Y. Yang, Q. Zhao, Y. Tian and Z. Xu, Epidemic waves of a spatial SIR model in combination with random dispersal and non-local dispersal, *Appl. Math. Comput.*, 313 (2017), 122–143.

[38] C. Wu, Y. Wang and X. Zou, Spatial-temporal dynamics of a Lotka-Volterra competition model with nonlocal dispersal under shifting environment, *J. Differential Equations*, 267 (2019), 4890–4921.

[39] C. C. Wu, Existence of traveling waves with the critical speed for a discrete diffusive epidemic model, *J. Differential Equations*, 262 (2017), 272–282.

[40] F. Y. Yang, Y. Li, W. T. Li and Z. C. Wang, Traveling waves in a nonlocal dispersal Kermack-Mckendrick epidemic model, *Discrete Contin. Dyn. Syst. Ser. B.*, 18 (2013), 1969–1993.

[41] F. Y. Yang and W. T. Li, Traveling waves in a nonlocal dispersal SIR model with critical wave speed, *J. Math. Anal. Appl.*, 458 (2018), 1131–1146.

[42] G. B. Zhang, W. T. Li and G. Lin, Traveling waves in delayed predator-prey systems with nonlocal diffusion and stage structure, *Math. Comput. Model.*, 49 (2009), 1021–1029.

[43] C. C. Zhu, W. T. Li and F. Y. Yang, Traveling waves in a nonlocal dispersal SIRH model with relapse, *Comput. Math. Appl.*, 73 (2017), 1707–1723.

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