The Dirac equation as a quantum walk: higher dimensions, observational convergence

Pablo Arrighi\textsuperscript{1,2}, Vincent Nesme\textsuperscript{1} and Marcelo Forets\textsuperscript{1}

\textsuperscript{1}LIG, Université Joseph Fourier, Grenoble, France
\textsuperscript{2}Université de Lyon, LIP, 46 allée d’Italie, F-69008 Lyon, France

E-mail: pablo.arrighi@imag.fr and vincent.nesme@imag.fr

Received 4 June 2014
Accepted for publication 16 September 2014
Published 4 November 2014

Abstract
The Dirac equation can be modelled as a quantum walk (QW), whose main features are being: discrete in time and space (i.e. a unitary evolution of the wave-function of a particle on a lattice); homogeneous (i.e. translation-invariant and time-independent) and causal (i.e. information propagates at a bounded speed, in a strict sense). This link, which was proposed already by Succi and Benzi, Bialynicki-Birula and Meyer, is shown to hold for Bargmann-Wigner equations and symmetric hyperbolic systems in general. We then analytically prove the convergence of the solution of the QW to the solution of the Cauchy problem for the Dirac equation. We do so by adapting a powerful method from standard numerical analysis, which is of general interest to the field of quantum simulation. At the practical level, this result provides precise error bounds and convergence rates, thereby validating the QW as a quantum simulation scheme. At the theoretical level, it reinforces the status of this QW as a simple, discrete toy model of relativistic particles.

Keywords: quantum computation, quantum cellular automata, quantum simulation
PACS numbers: 03.67.Ac, 02.60.Cb, 04.20.Gz

1. Introduction

The Dirac equation. This PDE is the main equation for describing the behaviour of relativistic quantum particles [1, 2]. For a free fermion of mass $m$, it takes the form (in Planck units $\hbar = c = 1$):
\[ \sum \partial \psi = D \psi, \quad \text{with} \quad D = m \alpha^0 - i \sum_j \alpha^j \partial_j, \]

where the Latin index \( j \) spans the spatial dimensions \( 1 \ldots n \) whereas Greek indices \( \mu, \nu \) will span the space–time dimensions \( 0 \ldots n \). In equation (1), \( \psi \) is a space–time wave-function from \( \mathbb{R}^{d+1} \) to \( \mathbb{C}^d \), with \( d \) a number that depends on \( n \), whereas \( \phi \) will denote a space-like wave-function from \( \mathbb{R}^n \) to \( \mathbb{C}^d \), e.g. we may write \( \phi = \psi (x_0 = 0) \) for the initial state. Finally, recall that the \( (\alpha^\nu) \) are \( d \times d \) hermitian matrices which must verify \( \{ \alpha^\nu, \alpha^\mu \} = 2 \delta_{\nu\mu} \text{Id} \), i.e. they square to the identity and pairwise anticommute [2].

Discretization. For the purpose of quantum simulation (on a quantum device) as envisioned by Feynman [3], or for the purpose of exploring the power and limits of discrete models of physics, we may wish to discretize the Dirac equation. There are (at least) two obvious directions one could follow. First, through finite-difference methods one gets (where \( \tau_{\mu, \epsilon} \) denotes translation by \( \epsilon \) along the \( \mu \)-axis):

\[
\psi (x_0 + \epsilon) = (\text{Id} - i \epsilon D_j) \psi (x_0),
\]

with \( D_j = m \alpha^0 - i \sum_j \alpha^j \tau_{j, \epsilon} - \text{Id} / \epsilon \),

\( (\tau_{j, \epsilon} \psi)(x_\mu) = \psi (x_\mu + \epsilon) \).

The problem with this crude approach is that \( (\text{Id} - i \epsilon D_j) \) does not conserve the \( \| \psi \|_2 \)-norm, in general. From the point of view of numerical simulation, this means one has to check the model’s convergence and stability. From the point of view of quantum simulation this simply bars the model as not implementable on a simulating quantum device. From the point of view of discrete toy models of physics, this means that the model lacks one of the fundamental, guiding symmetries: unitarity.

The second approach would be integrating exactly the original Dirac equation, and expressing \( \psi (x_0 + \epsilon) \) as a function of \( \psi (x_0) \). The transformation would be unitary, but it is unclear how to discretize space.

The Dirac Quantum Walk (QW). In [4–6], the Dirac equation is modelled as a QW. QWs [7] are dynamics having the following features: (i) the underlying spacetime is a discrete grid; (ii) the evolution is unitary; (iii) it is homogeneous, i.e. translation-invariant and time-independent; (iv) it is causal, i.e. information propagates strictly at a bounded speed.

In numerical analysis, in order to evaluate the quality of a numerical scheme model, two main criteria are used. The first criterion is consistency, a.k.a. accuracy. Intuitively it demands that, after an \( \epsilon \) of time, the discrete model approximates the solution to a given order of \( \epsilon \). Consistency of the \((1 + 1)\)-dimensional Dirac QW has been argued in [6], and for the \((1 + 1)\)-dimensional massless case in [8]. It has been observed numerically in \((1 + 1)\)-dimensions in [9] and in \((3 + 1)\)-dimensions in [10–12]. It has been proved in \((1 + 1)\)-dimensions in [13–15].

The second criterion is convergence. Intuitively it demands that, after an arbitrary time \( x_0 \) and if \( \epsilon \) was chosen small enough, the discrete model approximates the solution to a given order of \( \epsilon \). This criterion is stronger\(^3\). Convergence has been observed numerically in \((3 + 1)\)-dimensions in [10–12]. It has been proved in \((1 + 1)\)-dimensions in [13, 14, 16]. The difficulty to analyse the \((3 + 1)\)-dimensional Dirac QW is mentioned in [14, 17–19].

\(^3\) Of course convergence implies consistency, but the converse does not always hold. Indeed, consistency means that making \( \epsilon \) small will increase the precision of the simulation of an \( \epsilon \) of time step. But it will also increase the number of time steps \( k = x_0 / \epsilon \) which are required in order to simulate an \( x_0 \) of time evolution. Depending upon whether the two effects compensate, convergence may or may not be reached.
In this paper, our main contribution is to provide a formal, analytic derivation of both consistency and convergence. The novelty of the approach is to import a powerful technique of numerical analysis in order to provide a formal proof of convergence of solutions between the QW and the associated Dirac Cauchy problem in the appropriate function spaces where it is known to be well-posed. This method avoids the trouble of previous works (e.g. [14]) which rely on solving the QW in order to compare its solution against that of the Dirac equation. We believe that having adapted this method is a contribution by itself: indeed, for quantum simulation schemes without known solutions, the procedure will still apply. We also address the question of the discretization of the input wavefunction $\phi$. Altogether we prove that for any time $x_0$ and a sufficiently regular initial condition $\phi$, the probability of observing a discrepancy between the iterated walk $W_{\text{reconstruct}}(\text{discretize}(_0))$ and the solution of the Dirac equation $\psi(x_0) = T(x_0)\phi$, goes to zero, quadratically, as the discretization step $\epsilon$ goes to zero. Contrary to previous works, we do not limit ourselves to the massless case, nor to the $(1 + 1)$-dimensional case.

Other related works. The non-relativistic Dirac to Shrödinger limit of the Dirac QW is studied in [4, 13, 17, 20]. Decoherence, entanglement and Zitterbewegung are studied in [9, 14]. The relationship with the Klein–Gordon equation is studied in [8, 21]. The latter also studies the general continuous limit of $1D$ space and time-dependent QWs. Similarly, [22–24] provide variations aimed at accounting for the Maxwell–Dirac equations or the time-dependent Dirac equation, as well as faster convergence in numerical simulations. Algorithmic applications of the Dirac QW are studied in [25]. The issues of physical interpretation of the QW one-particle states are tackled in [26]. First principles derivations in $(1 + 1)$ and $(3 + 1)$-dimensions are provided in [15, 27]. The ideas behind the $(1 + 1)$-dimensional Dirac QW can be traced back to Feynman’s relativistic checkerboard [28], although early models were not unitary [29] and sometimes continuous-time Ising-like [30]. In $(2 + 1)$-dimensions, continuous-time models over the honeycomb lattice have been conceived in order to model electron transport in graphene [31].

Plan of the paper. We start with informal derivations in $(2 + 1)$ and $(3 + 1)$-dimensions (section 2). We recall well-posedness results for the Dirac equation (section 3.1), and continue with the formal analysis of the model, proving: consistency, stability and convergence (sections 3.2, 3.3 and 3.4). Finally, we discuss space discretization and other considerations such as generalizations and observational equivalence (sections 4 and 5). We summarize our results in section 6.

2. Informal derivations

2.1. In $(2+1)$-dimensions

A standard representation of the $(2 + 1)$-dimensional Dirac equation is:

$$id\psi = D\psi \quad \text{with} \quad D = m\sigma^z - i\sigma^1\partial_1 - i\sigma^2\partial_2$$

and $(\sigma^\mu)$ the Pauli matrices (with $\sigma^0$ the identity). Now, intuitively

$$\tau_{\mu,\epsilon}\psi = \left(\text{Id} + \epsilon\partial_\mu\right)\psi + O(\epsilon^2).$$

But this statement and its hypotheses will only be made formal and quantified in later sections. Meanwhile, substituting equation (2) into equation (3) for $\mu = 0$ yields:
\[ \tau_{0,\varepsilon} = (\text{Id} - i\varepsilon D) + O(\varepsilon^2) \]
\[ = (\text{Id} - i\varepsilon \sigma^2)(\text{Id} - i\varepsilon^3 \partial_j)(\text{Id} - i\varepsilon \sigma^3 \partial_j) + O(\varepsilon^2) \]
\[ = \exp(-i\varepsilon \sigma^2)H \left( \text{Id} - i\varepsilon^3 \partial_j \right)H \left( \text{Id} - i\varepsilon \sigma^3 \partial_j \right) + O(\varepsilon^2), \]

since \( \sigma^3 = H \sigma^3 H \) with \( H \) the Hadamard gate. Using the definition of \( \sigma^3 \), equation (3), and taking the convention that \( \mathbb{C}^2 \) is spanned by the orthonormal basis \( \{|l\rangle/l \in \{-1, 1\} \} \), we get:

\[ \tau_{0,\varepsilon} = C_\varepsilon HT_{1,\varepsilon}HT_{2,\varepsilon} + O(\varepsilon^2) \]

with 
\[ C_\varepsilon = \exp(-i\varepsilon \sigma^2) \]

and 
\[ T_{j,\varepsilon} = \sum_{l \in \{-1, 1\}} |l\rangle \langle l| \tau_{j,\varepsilon}. \]

Overall, we have:

\[ \psi(x_0 + \varepsilon) = \psi(x_0) + O(\varepsilon^2) \]

with 
\[ \psi = C_\varepsilon HT_{1,\varepsilon}HT_{2,\varepsilon}, \]

where the \( T \) matrices are partial shifts. This Dirac QW [4–6] models the \((2 + 1)\)-dimensional Dirac equation. It has a product form. Such ‘alternate QWs’ have the advantage of using a two-dimensional coin-space instead of a four-dimensional coin-space: fewer resources are needed for their implementation [32]. It is still just one QW, i.e. a translation-invariant causal unitary operator.

### 2.2. In \((3+1)\)-dimensions

From \((2+1)\) to \((3+1)\)-dimensions the Dirac equation changes form, the spin degree of freedom goes to degree four. The equation is:

\[ i\partial \psi = D\psi \quad \text{with} \quad D = m \left( \sigma^2 \otimes \sigma^0 \right) + i \sum_j \left( \sigma^3 \otimes \sigma^j \right) \partial_j. \]

Indeed, one can check that the matrices \( \sigma^2 \otimes \sigma^0 \) and \( (-\sigma^3 \otimes \sigma^j) \) are hermitian, that they square to the identity, and that they anticommute. Using the definition of \( \sigma^3 \), equation (3), and taking the convention that \( \mathbb{C}^4 \) is spanned by the orthonormal basis \( \{|l, r\rangle/ l, r \in \{-1, 1\}\} \):

\[ \left( \text{Id} + \varepsilon \left( \sigma^3 \otimes \sigma^3 \right) \partial_2 \right) \psi = T_{3,\varepsilon} \psi + O(\varepsilon^2) \]

with 
\[ T_{j,\varepsilon} = \sum_{r, l \in \{-1, 1\}} |r, l\rangle \langle r, l| \tau_{j,\varepsilon}. \]

Similarly

\[ \left( \text{Id} + \varepsilon \left( \sigma^3 \otimes \sigma^3 \right) \partial_2 \right) \psi = (\text{Id} \otimes F) T_{2,\varepsilon} \left( \text{Id} \otimes F^\dagger \right) \psi + O(\varepsilon^2) \]

as 
\[ \sigma^2 = Fe^2F^\dagger \]

with 
\[ F = R_2H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}. \]
Likewise

\[ (\text{Id} + \epsilon \left( \sigma^3 \otimes \sigma^1 \right) \partial_i ) \psi = (\text{Id} \otimes H) T_{1,\epsilon} (\text{Id} \otimes H) \psi + O(\epsilon^2) \]

as \( \sigma^1 = H \sigma^3 H. \)

Finally, let \( C_{\epsilon} = \exp \left( -i m (\sigma^2 \otimes \sigma^0) \right). \) We have:

\[ \psi(x_0 + \epsilon) = W_{\epsilon} \psi(x_0) + O(\epsilon^2) \]

with \( W_{\epsilon} = C_{\epsilon} (\text{Id} \otimes H) T_{1,\epsilon} (\text{Id} \otimes HF) T_{2,\epsilon} \left( \text{Id} \otimes F^\dagger \right) T_{3,\epsilon}, \)

where the \( T \) matrices are partial shifts. This is the \( (3+1) \)-dimensional Dirac QW. We now move on to the formal analysis of the model.

### 3. Formal analysis

#### 3.1. Well-posedness

Numerical analysis is mostly about finding discrete models to approximate the continuous solutions of a well-posed Cauchy problem.

Here, the Cauchy problem is to find the solution \( \psi \) given \( \psi(0) \) and \( i \partial_0 \psi = D \psi. \) Cauchy problems are well-posed if and only if the solution exists, is unique, and depends continuously upon \( \psi(0). \) Since the Dirac equation is a symmetric hyperbolic system, the problem is known \([33]\) to be well-posed for the Sobolev space \( H^s_m(\mathbb{R}^d). \) with \( s \geq 0 \) of the functions for which the \( \| \psi \|_{H^s_m} \)-norm is finite. This Sobolev norm

\[ \| \phi \|_{H^s_m} = \left( \int_{\mathbb{R}^d} \left( 1 + m^2 + \| k \|^2 \right)^s \| \hat{\phi}(k) \|^2 \, dk \right)^{\frac{1}{2}}, \]

and the well-posedness result are discussed in appendix \( B. \) Notice that \( H^s_m(\mathbb{R}^d) \) is the usual \( L^2(\mathbb{R}^d). \) Notice also that the Sobolev norm involves an integral in Fourier space. For this reason, and because the Dirac operator is just a pointwise multiplication in Fourier space, most of our derivations will use it. Conventions and basic facts about Fourier space are given in appendix \( A. \)

#### 3.2. Consistency

In numerical analysis, in order to evaluate the quality of a numerical scheme model, the first criterion is consistency, a.k.a. accuracy. Intuitively it demands that, after an \( \epsilon \) of time, the discrete model approximates the solution to a given order of \( \epsilon. \)

Formally, say a Cauchy problem is well-posed on \( X, \) with \( Y \) a dense subspace of \( X. \) The discrete model \( W_{\epsilon} \) is consistent of order \( r \) on \( Y \) if and only if there exists \( C \) such that for any solution \( \psi \) with \( \psi(x_0 = 0) \in Y, \) for all \( \epsilon \in \mathbb{R}^+, \) we have

\[ \| W_{\epsilon} \psi(0) - \psi(\epsilon) \|_X = \epsilon^{r+1} C \| \psi(0) \|_Y. \]

This is what we will now prove: that for \( s \geq 0, r = 1, X = H^s_m(\mathbb{R}^d) \) and \( Y = H^1_m, \) there exists \( C \) such that for all \( \phi, \epsilon: \)

\[ \| W_{\epsilon} \phi - T(\epsilon) \phi \|_{H^s_m} \leq \epsilon^2 C \| \phi \|_{H^{s+2}_m}, \]

with \( \phi = \psi(0), T(\epsilon) \phi = \psi(\epsilon), \) i.e. \( T(\epsilon) = T_{0,\epsilon} \) is the continuous solution’s time evolution operator.
We work on Fourier space and see \( \hat{W}_k(\epsilon) \) with fixed \( k \) as a function of the real-value \( \epsilon \). First, observe that the QW operator can generally be written as (we sometimes omit the \( k \) dependence in the notations of this section):

\[
\hat{W}_k = \prod_{\mu} e^{-i\hat{A}_\mu k}. \tag{4}
\]

With \( \hat{A}_\mu \) hermitian, \( \|\|\|\hat{A}_0\|\|_2 = m, \|\|\|\hat{A}_1\|\|_2 = k_j, \hat{A}_\mu \) hermitian and \( \sum_{\mu} \hat{A}_\mu = \hat{D} \) (see appendix A for further details). For instance, in \( (2 + 1) \)-dimensions, \( \hat{A}_0 \) is equal to \( m\sigma^\uparrow \), \( \hat{A}_1 \) is equal to \( k_1\sigma^\downarrow \) and \( \hat{A}_2 \) is equal to \( k_2\sigma^3 \) (see appendix A for further details).

As \( \hat{W}_k(\epsilon) \) is a matrix whose elements are products of trigonometric functions and exponentials, its entries are \( \infty \) functions (on the variable \( \epsilon \)). We will denote \( \partial_\epsilon \) the derivative with respect to variable \( \epsilon \) in each entry. Observe that \( \hat{W}_0 = \text{Id} \).

Now we will calculate the first and second order derivatives making use of equation (4). For the first order derivative we have

\[
(\partial_\epsilon \hat{W}_k)_{\epsilon=0} = -i\hat{D}.
\]

For the second order derivative, we have:

\[
(\partial_\epsilon^2 \hat{W}_k)_{\epsilon=0} = -\sum_{\mu} \left( \prod_{\kappa < \mu} e^{-i\hat{A}_\mu k} \hat{A}_\mu^2 \left( \prod_{\kappa > \mu} e^{-i\hat{A}_\mu k} \right) - 2 \sum_{\kappa < \mu} \left( \prod_{\kappa < \nu} e^{-i\hat{A}_\mu k} \hat{A}_\nu \left( \prod_{\kappa < \nu < \mu} e^{-i\hat{A}_\mu k} \right) \hat{A}_\mu \left( \prod_{\kappa > \mu} e^{-i\hat{A}_\mu k} \right) \right) \right)
\]

where we get to the preceding line using that for real numbers, \( (x_0 + \cdots + x_n)^2 \leq (n + 1)(x_0^2 + \cdots + x_n^2) \) and to the last line using \( \gamma^2 = m^2 + \|d\|_2^2 \). By application of Taylor’s formula with the integral form for the remainder [34] to each entry of the matrix \( \hat{W}_k \), we get

\[
\hat{W}_k = \text{Id} + \epsilon (\partial_\epsilon \hat{W}_k)_{\epsilon=0} + \int_0^\epsilon (\epsilon - \eta)(\partial_\epsilon^2 \hat{W}_k)_{\epsilon=\eta} \, d\eta
\]
and
\[ \hat{T}(\epsilon) = e^{-i\epsilon \hat{D}} = \text{Id} - i\epsilon \hat{D} \]
\[ + \int_0^\epsilon (\epsilon - \eta) \left(-\hat{D}^2 e^{-i\eta \hat{D}}\right) d\eta. \]

Let us define
\[ \hat{R}_\epsilon = \hat{W}_\epsilon - \hat{T}(\epsilon), \]
whose operator norm can be bounded after substitution of the previous expressions and application of the triangular inequality, thus obtaining
\[ \|\| \hat{R}_\epsilon \|\|_2 \leqslant \int_0^\epsilon |\epsilon - \eta| \left| \left| \partial_\eta^2 \hat{W}_\eta \right| \right|_2 \ d\eta 
+ \int_0^\epsilon |\epsilon - \eta| \left| \left| \hat{D}^2 e^{-i\eta \hat{D}} \right| \right|_2 \ d\eta 
\leqslant \int_0^\epsilon (\epsilon - \eta)(n + 1)\gamma^2 \ d\eta + \int_0^\epsilon (\epsilon - \eta)\gamma^2 \ d\eta 
\leqslant \epsilon^2 \gamma^2 \left(1 + \frac{n}{2}\right), \]
where we used that the eigenvalues of \( \hat{D} \) are \( \pm \gamma \) with \( \gamma^2 = m^2 + \|k\|_2^2 \), see appendix A.

Substituting this result into the Sobolev norm, i.e.
\[ \|W \varphi - T(\epsilon) \varphi\|_{H_s^m} \]
\[ = \sqrt{\int_{R^N} \left(1 + m^2 + \|k\|^2\right) \left| \hat{R}_\epsilon \hat{\varphi}(k) \right|^2 \ dk} 
= \sqrt{\int_{R^N} \left(1 + m^2 + \|k\|^2\right) \left| \hat{R}_\epsilon(k) \hat{\varphi}(k) \right|^2 \ dk} 
\leqslant \epsilon^2 C \sqrt{\int_{R^N} \left(1 + m^2 + \|k\|^2\right)^{s+2} \left| \hat{\varphi}(k) \right|^2 \ dk} 
\leqslant \epsilon^2 C \|\varphi\|_{H_s^{m+2}}, \]
which is what we wanted to prove, \( C \) being \( 1 + \frac{n}{2} \).

3.3. Stability

In numerical analysis, in order to evaluate the quality of a numerical scheme model, an intermediate criterion is stability. It demands the discrete model be a bounded linear operator. Thus, let us prove that for all \( \varphi \), for all \( s \geqslant 0 \), we have \( \|W \varphi\|_{H_s^m} = \|\varphi\|_{H_s^m} \). We proceed by applying the definition of Sobolev norm, which yields
\[ \left|\left| W \varphi \right|\right|_{H_s^m}^2 = \int_{R^N} \left(1 + m^2 + \|k\|^2\right) \left| \mathcal{T}(W \varphi)(k) \right|^2 \ dk 
= \int_{R^N} \left(1 + m^2 + \|k\|^2\right) \left| \hat{W}(\hat{\varphi})(k) \right|^2 \ dk 
= \int_{R^N} \left(1 + m^2 + \|k\|^2\right) \left| \hat{W}_\epsilon(k) \hat{\varphi}(k) \right|^2 \ dk, \]
where in the second to third lines we used the fact that as \( W \) is a translation-invariant unitary operator it is represented in Fourier space as a left multiplication by a unitary matrix \( \hat{W}_\epsilon(k) \).
which depends on \( k \). See appendix A for this particular case, and for instance [35] for the general case. We then have

\[
\left\| W_\epsilon \phi \right\|_{\mathcal{H}_0^s}^2 = \int_{\mathbb{R}^n} \left( 1 + m^2 + \| \phi \|_2^2 \right) \left\| \phi(k) \right\|_{\mathcal{H}_0^1}^2 \, dk
\]

Thus if \( \| \cdot \|_{\mathcal{H}_0^s} \) denotes the operator norm with respect to the norm \( \mathcal{H}_m^s \), we have \( \| W_\epsilon \|_{\mathcal{H}_0^s} \) equal to one as requested.

### 3.4. Convergence

In numerical analysis, in order to evaluate the quality of a numerical scheme model, the most important criterion for quality is convergence. Intuitively it demands that, after an arbitrary time \( x_0 \), and if \( \epsilon \) was chosen small enough, the discrete model approximates the solution to a given order of \( \epsilon \). Fortunately, the Lax theorem [36, 37] states that stability and consistency implies convergence. Unfortunately, as regards the quantified version of this result, the literature available comes in many variants, with various degrees of formalization, each requesting different sets of hypotheses. Thus, for clarity, we inline the proof here.

Formally, say a Cauchy problem is well-posed on \( X \) and \( Y \), with \( Y \) a dense subspace of \( X \). The discrete model \( W_\epsilon \) is convergent of order \( r \) on \( Y \) if and only if there exists \( C \) such that for any solution \( \psi \) with \( \psi(x_0) = 0 \in Y \), for all \( x_0 \in \mathbb{R}^* \), \( l \in \mathbb{N} \), we have:

\[
\left\| W_\epsilon^l \psi(0) - \psi(x_0) \right\|_X = \epsilon^l x_0 C \| \psi(0) \|_Y
\]

with \( \epsilon_l = x_0/l \). This is exactly what we will now prove: that for \( s \geq 0 \), \( r = 1 \), \( X = H_m^s(\mathbb{R}^n)^d \) and \( Y = H_m^{s+2}(\mathbb{R}^n)^d \), there exists \( C \) such that for all \( \phi \), \( \epsilon \):

\[
\left\| W_\epsilon^l \phi - T(\epsilon l) \phi \right\|_{\mathcal{H}_m^s} \leq \epsilon x_0 C \| \phi \|_{\mathcal{H}_m^{s+2}}.
\]

Take \( x_0 \in \mathbb{R}^* \). Consider the sequence \( \{ \epsilon_l \} \) such that \( \epsilon_l = x_0/l \). Because \( T(\epsilon l) = T(\epsilon_l) \), and because

\[
\sum_{j=0}^{l-1} W_\epsilon^l j T(\epsilon_l)^j - W_\epsilon^l j T(\epsilon_l)^j = 0
\]

\[
\sum_{j=0}^{l-1} W_\epsilon^l j T(\epsilon_l)^j - W_\epsilon^l j-1 T(\epsilon_l)^j+1 = W_\epsilon^l - T(\epsilon_l).
\]

We have:

\[
W_\epsilon^l \phi - T(\epsilon l) \phi = \sum_{j=0}^{l-1} W_\epsilon^l j-1 \left( W_\epsilon - T(\epsilon_l) \right) T(\epsilon_l)^j \phi.
\]

From consistency there exists \( C \) such that for all \( \phi \)

\[
\left\| W_\epsilon T(j \epsilon_l) \phi - T(\epsilon l) T(j \epsilon_l) \phi \right\|_{\mathcal{H}_m^s} \leq \epsilon_x^2 C \| \phi \|_{\mathcal{H}_m^{s+2}}.
\]
Hence
\[
\left\| W^l \phi - T(\epsilon_l)\phi \right\|_{H^s} \leq \sum_{j=0}^{l-1} \left\| W^{l-1-j} \right\|_{H^s} \epsilon_j^2 C \left\| \phi \right\|_{H^{s+2}} \\
\leq \epsilon^2 C \left\| \phi \right\|_{H^{s+2}} \leq \epsilon \lambda_0 C \left\| \phi \right\|_{H^{s+2}}
\]
as requested.

### 4. Space discretization

This paper aims at giving a QW model \( W_l: \ell_3(\mathbb{Z}^n)^d \rightarrow \ell_3(\mathbb{Z}^n)^d \) of the Dirac equation. So far we explained how we can discretize time the Dirac equation, but in order to get a QW, we need to discretize space as well. In a sense, this is already done since the walk operators \( W_l \) that we defined, although they take as input functions in \( H^s(\mathbb{R}^d) \), can equally well be defined on \( \ell_3(\mathbb{Z}^n)^d \), for the only shift operators involved in their definitions are multiples of the \( T_{i,e} \)-s. The question remains, however, of what initial state we can feed our QWs, and how we are to interpret their output. Answering this question is the aim of this section. One of the difficulties, in particular, is to construct, given \( \phi \in L^2(\mathbb{R}^d) \), a discretize \( \phi \in \ell_3(\mathbb{Z}^n)^d \). That the discretized version of \( \phi \) be normalized is essential so that the quantum simulation can be implemented on a quantum simulator, just like the unitarity of \( W_l \) was essential. This section relies heavily on notations introduced in appendix A.

**Discretization procedure.** We discretize by
\[
\text{discretize}(\phi) = \text{renormalize}\left( FS\left( FT\left( \phi \right) \right) \right).
\]
Notice that
\[
\phi_{\text{LP}} = FT^{-1}\left( \chi_{[-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]} FT(\phi) \right),
\]
where \( \chi_A \) denotes the indicator function of \( A \), applies an ideal low-pass filter, and that
\[
FS\left( FT\left( \phi_{\text{LP}} \right) \right)_{[-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]} = \epsilon^2 \lambda_2 \phi_{\text{LP}}|_{\mathbb{Z}^d}
\]
is, up to a constant, the sampling of \( \phi_{\text{LP}} \), see appendix A. \( \text{discretize}(\phi) \) is hence proportional to the function obtained by sampling \( \phi \) after it has been low-pass filtered. Since \( FS \) and \( FT \) are unitary, the renormalization is by a factor of \( \| \phi_{\text{LP}} \|^{-1}_{L^2} \). For it to be well-defined, we must check that \( \phi_{\text{LP}} \) does have a non-zero norm.

**Low-pass filtering.** For every \( s \geq 0 \), we have
\[
\| \phi - \phi_{\text{LP}} \|_{H^s} \\
= \int_{\mathbb{R}^d \setminus [-\pi, \pi]^d} \left( 1 + m^2 + \| k \|^2 \right) \| \hat{\phi}(k) \|^2 \, dk \\
= \int_{\mathbb{R}^d \setminus [-\pi, \pi]^d} \left( 1 + m^2 + \| k \|^2 \right)^{-2} \| \hat{\phi}(k) \|^2 \, dk \\
\leq \epsilon^2 C \| \phi \|_{H^{s+2}} \quad \text{with} \quad C' = \pi^{-2}.
\]

This tells us two things. First, if \( \varepsilon^2 < \frac{1}{C^2} \| \phi_{LP} \|_{H^s} \), then \( \phi_{LP} \neq 0 \), so it can be renormalized. Second, the loss induced by low-pass filtering is small, as needed below in order to bound the overall error.

**Reconstruction procedure.** We reconstruct by

\[
\text{reconstruct}(\tilde{\phi}) = FT^{-1}\left( FS^{-1}\left( \text{renormalize}^{-1}(\tilde{\phi}) \right) \right)
\]

with the convention that \( FS^{-1}(\tilde{\phi}) \in L^2([-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})^d \) is extended to \( L^2(\mathbb{R}^d) \) by the null function on \( \mathbb{R}^d \setminus [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})^d \), and the inverse renormalization is by a factor of \( \| \phi_{LP} \|_2 \). Notice that

\[
\phi_{LP} = \text{reconstruct}\left( \text{discretize}(\phi_{LP}) \right)
\]

and that this reconstruction is equivalent to the Whittaker–Kotelnikov–Shannon formula (see [38, 39] for the multidimensional case).

**Overall scheme.** Given a wave function \( \phi \), we approximate \( T(\varepsilon l)\phi \), the continuous evolution of \( \phi \), by \( \text{reconstruct}(W^l_i(\text{discretize}(\phi))) \), the reconstruction of the walk iterated on the discretization of \( \phi \). Let us bound the overall error. For all \( \phi \) we have (renormalizations cancel out by linearity of \( W^l_i \)):

\[
\text{reconstruct}(W^l_i(\text{discretize}(\phi))) = FT^{-1}\left( FS^{-1}\left( W^l_i\left( FS\left( FT\left( \phi \right) \right) \right) \right) \right)
\]

where the preceding step comes from the last line of appendix A. Now, since \( W^l_i \) is unitary, we have

\[
\| W^l_i(\phi_{LP}) - W^l_i(\phi) \|_{H^s_{\varepsilon}} = \| \phi_{LP} - \phi \|_{H^s_{\varepsilon}} \leq \varepsilon^2 \| \phi \|_{H^s_{\varepsilon}^{k+2}}.
\]

On the other hand in section 3.4 we had:

\[
\| W^l_i(\phi) - T(\varepsilon l)\phi \|_{H^s_{\varepsilon}} \leq \varepsilon^2 \| lC \| \phi \|_{H^s_{\varepsilon}^{k+2}}.
\]

And thus the bound on the overall error is:

\[
\| \text{reconstruct}\left( W^l_i(\text{discretize}(\phi)) \right) - T(\varepsilon l)\phi \|_{H^s_{\varepsilon}} \leq \varepsilon^2 \| l(C + C') \| \phi \|_{H^s_{\varepsilon}^{k+2}}.
\]

Where in the last inequality we should recall that \( \varepsilon \) is the discretization parameter and \( k \) the number of iterations, thus \( s_0 = \varepsilon l \) is for how long the evolution is simulated.
5. Further considerations

Generalizations. The method would work equally well for any symmetric hyperbolic systems with rational eigenvalues, i.e. equations of the form

$$i\partial_t \psi = D\psi \quad \text{with} \quad D = \beta^0 - i\sum_j \beta^j \partial_j,$$

where the $(\beta^j)$ are $d \times d$ hermitian having rational eigenvalues, and $\beta^0$ is hermitian. We can write $\beta^j = \frac{1}{q}(U_j)\Delta^j (U_j)^\dagger$, with $q \in \mathbb{N}^*$, $U_j$ unitary, and $\Delta^j$ diagonal with integer coefficients $\lambda_1^j$, ..., $\lambda_d^j$. The same procedure yields the QW:

$$W = C \prod_j U_j T_{j,\epsilon} U_j^\dagger \quad \text{with} \quad C_{\epsilon} = \exp\left(-i\epsilon q \beta^0\right) \quad \text{and} \quad T_{j,\epsilon} = \sum_{r,l} |r\rangle \langle l| \tau_{j,-\lambda_j^j\epsilon}.$$ 

More generally even, the method would work for equations of the form

$$i\partial_t \psi = D\psi \quad \text{with} \quad D = \sum_j D_j,$$

such that each $\exp\left(-iD_j\right)$ is a QW. Indeed, the same procedure yields the QW

$$W = \prod_j \exp\left(-iD_j\right).$$

Ultimately, it is the fact the Dirac Hamiltonian is a sum of logarithms of QWs, which enables us to model it as the product of these QWs.

Observational equivalence. Consider the case when $s = 0$. We then have $L^2(\mathbb{R}^d) = H^1_0(\mathbb{R}^d)^d$, as $\| \cdot \|_2 = \| \cdot \|_{H^1_0}^2$: the Sobolev norm then coincides with that of quantum theory, and we can interpret convergence in an operational manner. Convergence gives us the existence of $C$ such that if $\psi(x_0 = 0) \in H^2_0(\mathbb{R}^d)^d$, then for all $\epsilon = x_0/l$ we have

$$\left\| W_{\epsilon,l} \psi(0) - \psi(x_0) \right\|_2 \leq C \epsilon x_0 \psi(0) \|_{H^2_0}^2.$$ 

According to quantum theory the probability of observing through a measurement a discrepancy between the iterated walk $W_{\epsilon,l} \phi$ and the solution of the Dirac equation $\psi(x_0)$ is given by $\sin^2(\theta)$, with $\theta$ the angle between both vectors. Simple trigonometric reasoning shows that this is bounded above by $\epsilon^2 x_0^2 C^2 \| \psi(0) \|_{H^2_0}^2$, i.e. it diminishes quadratically as $\epsilon$ goes to zero.

6. Summary

Nowadays, simulation of physical processes is realized over classical computers, and the quality of the result is validated by numerical analysis. With the future development of quantum computers, the quantum simulation of physical processes will also need to be validated. One of our main contributions is thus to provide the field of quantum simulation with powerful methods of numerical analysis. Based on simple arguments, the approach allows to obtain convergence of the solutions from stability and consistency, without ever having to obtain the solutions themselves. This is a key point: this method will apply equally
well to more complicated QWs, e.g. Bargmann–Wigner equations and symmetric hyperbolic
systems in general.

In this article we have rigorously proven that the QW
\[ W_\varepsilon = C_\varepsilon (\text{Id} \otimes H)T_{1,\varepsilon} (\text{Id} \otimes HF)T_{2,\varepsilon} (\text{Id} \otimes F^\dagger)T_{3,\varepsilon} \]
models the Dirac equation. More precisely, for any time \( t \), we proved that the model converges to the continuous solution of the Dirac equation at time \( t \), i.e. the probability of observing a discrepancy between the model and the solution is upper bounded by a \( O(\varepsilon^2) \), with \( \varepsilon \) the discretization step. Indeed, consistency is ensured to first order and stability is given by unitarity, hence the model is convergent to first order. The result can be specialized elegantly to lower dimensions. It can also be generalized to other first-order PDEs, as well as to PDEs whose Hamiltonians can be expressed as a sum of logarithms of QWs. The model is suitable for quantum simulation, or as a discrete toy model. The QW is parametrized on \( \varepsilon \), the discretization step. It is of course tempting to set \( \varepsilon \) to in Planck units, and grant
\[ W = C (\text{Id} \otimes H)T_1 (\text{Id} \otimes HF)T_2 (\text{Id} \otimes F^\dagger)T_3, \]
a more fundamental status. One could even wonder whether some relativistic particles might behave according to this QW, rather than the Dirac equation. To our reader, we ask: could experimentalists really tell the difference? A decohered version of the QW model could be studied using the general techniques of [40]. We plan to study to which extent such discrete models retain some Poincaré-invariance.

Acknowledgments

The authors are indebted to Olivier Bournez and David Meyer for insightful discussions at the early stages of this work, to Stéphane Labbé and Stefano Facchini for some advice, and Alain Joye for his help. This work has been funded by the ANR-10-JCJC-0208 CausaQ grant.

Appendix A. Facts in Fourier space

Fourier transform. We recall that the Fourier transform of the wave-function \( \phi \in L^2(\mathbb{R}^n) \) is defined as the function \( (\mathcal{FT}\phi) = \hat{\phi} \): \( \mathbb{R}^n \rightarrow \mathbb{C}^n \) such that
\[ \hat{\phi}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \phi(x) e^{-i k \cdot x} \, dx, \]
where by \( k \cdot x \) we mean the scalar product in Euclidean space \( \mathbb{R}^n \), \( x = (x_j) \), and \( k = (k_j) \). The function \( \mathcal{FT} \) is unitary, its inverse is
\[ \phi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{\phi}(k) e^{i k \cdot x} \, dk. \]
From the above definition it is easily seen that for the spatial derivatives: \( \mathcal{FT}(\partial_j \phi)(k) = ik_j \hat{\phi} \). Is is also useful to recall that for translations:
\[ \mathcal{FT}(\phi(x \pm \varepsilon))(k) = e^{i k \cdot \varepsilon} \hat{\phi}(k). \]
In Fourier space the $(2+1)$-dimensional Dirac operator, equation (2), becomes:
\[
\hat{D}(k) = m\sigma^2 + k_1\sigma^1 + k_2\sigma^2
\]
\[
= \begin{pmatrix}
k_2 & k_1 - im \\
 k_1 + im & -k_2
\end{pmatrix}
\]
with eigenvalues $\pm |\gamma|$, being $\gamma^2 = m^2 + |k|^2$. The same formula for the eigenvalues holds true in three dimensions (i.e. there is a twofold degeneracy).

In Fourier space the $(2+1)$-dimensional Dirac QW operator $\hat{W}$, decomposes as a product of exponential matrices, using identities such as:
\[
H\hat{T}_{1,\epsilon}(k)H = H\begin{pmatrix}
e^{-ik_1\epsilon} & 0 \\
 0 & e^{ik_1\epsilon}
\end{pmatrix}H
\]
\[
= He^{-ik_1\epsilon}H = e^{-ik_1\epsilon}
\]
and likewise for the other directions. Eventually in $(n+1)$-dimensions it takes the form
\[
\hat{W}_n = \prod_{\mu} e^{-ik_{\mu}\lambda_{\mu}}
\]
with some known $\hat{\lambda}_{\mu}$.

**Fourier series.** We recall that the Fourier series of the wave-function $\phi \in L^2([-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]^d)$, $\epsilon \in \mathbb{R}^+$, is defined as the function $(FS\phi) = \hat{\phi} : \epsilon\mathbb{Z}^n \rightarrow \mathbb{C}^d$ such that
\[
\hat{\phi}(k) = \left(\frac{\epsilon}{2\pi}\right)^{n/2} \int_{[-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]^d} \phi(x) e^{ik\cdot x} dx.
\]
The function $FS$ is unitary, its inverse is
\[
\phi(x) = \left(\frac{\epsilon}{2\pi}\right)^{n/2} \sum_{k \in \mathbb{Z}^n} \hat{\phi}(k) e^{-ik\cdot x}.
\]
The sign conventions of the exponentials are non-standard; they have been chosen to that, whenever $\hat{\phi} = FT(\phi)$ has support in $[-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]^d$, then (with $|X|$ denoting restriction to $X$):

- $FS(\hat{\phi}|_{[-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]^d}) = e^{n/2}FT^{-1}(\hat{\phi})|_{\epsilon\mathbb{Z}^n} = e^{n/2}\phi|_{\epsilon\mathbb{Z}^n}$;
- $FS^{-1}(e^{n/2}\phi|_{\epsilon\mathbb{Z}^n}) = \hat{\phi}|_{[-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]^d}$.

Indeed, the first point follows from the definition, and the second is the reciprocal.

**Appendix B. Sobolev spaces and well-posedness**

**Sobolev spaces.** The usual wave-function space for quantum theory is the subspace $L^2(\mathbb{R}^n)^d$ of the functions $\mathbb{R}^n \rightarrow \mathbb{C}^d$ for which the $\|\|_2$-norm is finite. Recall that
\[
\|\phi\|_2 = \sqrt{\int_{\mathbb{R}^n} \|\phi(x)\|^2 dx}
\]
with $\|\|$ the usual two-norm in $\mathbb{C}^d$, $x = (x_j)$. For our approximations to hold, we need to restrict to the subspace $H^r_m(\mathbb{R}^n)^d$ of the functions $L^2(\mathbb{R}^n)^d$ for which the $\|\|_{H^r_m}$-norm is finite. Recall that
\[ \| \phi \|_{H^s} = \sqrt{\int_{\mathbb{R}^d} \left( 1 + |k|^2 \right) \left| \hat{\phi}(k) \right|^2 \, dk} \]

with \( \hat{\phi} \) the Fourier transform of \( \phi \), and again \( |k|^2 = \sum_j |k_j|^2 \).

Several remarks are in order. First, notice that \( \| \phi \|_{H^0} = \| \hat{\phi} \|_2 = \| \phi \|_2 \), thus \( \mathcal{H}_1(H^s) \). Second, notice that for continuous differentiable functions, \( \| \phi \|_{H^s} = \sum_j \| \partial_j \phi \|_s \), thus \( H^s_m(\mathbb{R}^d) \) is just the subset of \( L^2(\mathbb{R}^d) \) having first-order derivatives in \( H^s_m(\mathbb{R}^d) \). The same holds for \( H^s_m(\mathbb{R}^d) \) with respect to \( H^s_m(\mathbb{R}^d) \). Third, notice that \( \mathcal{H}_1(H^s) \) is dense in \( H^s_m(\mathbb{R}^d) \), as can be seen from mollification techniques [41]. Finally, notice that, on the one hand, the choice of having the \( H^s_m(\mathbb{R}^d) \)-norm to depend on \( m \) is slightly non-standard: usually this constant is set to zero. On the other hand, three elements argue in favour of this non-standard choice: (1) this fits nicely with the mathematics of this paper; (2) our main use of the \( H^s_m(\mathbb{R}^d) \)-norm is to impose a sufficiently regular initial condition on the particle’s wave-function, that this regularity condition may depend on the particle’s mass \( m \) does not seem problematic; (3) the above defined \( H^s_m(\mathbb{R}^d) \)-norm is equivalent to the usual \( H^s(\mathbb{R}^d) \)-norm:

\[ \| \phi \|_{H^s} = \sqrt{\int_{\mathbb{R}^d} \left( 1 + |k|^2 \right) \left| \hat{\phi}(k) \right|^2 \, dk} \]

in the sense of norm equivalence, because \( 1 + |k|^2 \leq 1 + m^2 + |k|^2 \leq (m^2 + 1)(1 + |k|^2) \). This last point is why the well-posedness of the Dirac equation with respect to the usual \( H^s(\mathbb{R}^d) \)-norm carries through with respect to the \( H^s_m(\mathbb{R}^d) \)-norm, see next.

\textbf{Well-posedness.} A Cauchy problem \( \partial_t \psi = D\psi \) is well-posed in a Banach space \( X \) if:

\begin{itemize}
  \item \( D \) is a densely defined operator of \( X \);
  \item There exists a dense subset \( Y \) of \( X \) such that for every initial condition in \( Y \), the Cauchy problem has a solution;
  \item There exists a non-decreasing function \( C : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that for every solution \( \psi \) (not necessarily from an initial condition in \( Y \)) and every \( x_0 \in \mathbb{R}^+ \), \( \|\psi(x_0)\|_X \leq C(x_0)\|\psi(0)\|_X \).
\end{itemize}

A hyperbolic symmetric system is a Cauchy problem of the form

\[ \partial_t \psi = D\psi \quad \text{with} \quad D = -i \beta^0 - \sum_j \beta^j \partial_j, \]

where the \( (\beta^a) \) are hermitian.

For symmetric hyperbolic systems, the Cauchy problem is known to be well-posed in \( H^s(\mathbb{R}^d) \) for any \( s \geq 0 \). \( D \) is defined on the subspace of \( \psi \in H^s(\mathbb{R}^d) \) such that \( D\psi \in H^s(\mathbb{R}^d) \), which is dense indeed, and every initial condition in this space yields a solution. The \( H^s \)-norm is constant for solutions of the problem, so that \( C(t) = 1 \) fulfills the requirement. For references, see [33] (1.6.21) or [42].

\textbf{References}

[1] Bjorken J D and Drell S D 1964 \textit{Relativistic Quantum Mechanics} vol 2 (New York: McGraw-Hill)
[2] Thaller B 1992 \textit{The Dirac Equation} (Berlin: Springer)
[3] Feynman R P 1982 \textit{Int. J. Theor. Phys.} 21 467
[4] Succi S and Benzi R 1993 \textit{Phys. D: Nonlinear Phenom.} 69 327
[5] Bialynicki-Birula I 1994 \textit{Phys. Rev.} D 49 6920–7
[6] Meyer D A 1996 \textit{J. Stat. Phys.} 85 551
[7] Kempe J 2003 Contemp. Phys. 44 307
[8] Chandrashekar C, Banerjee S and Srikanth R 2010 Phys. Rev. A 81 62340
[9] Love P and Boghosian B 2005 Quantum Inf. Process. 4 335
[10] Palpacelli S 2009 PhD Thesis Universit Degli Studi Roma Tre Facolt di Scienze Matematiche Fisiche e Naturali, Dottorato in Matematica XXI ciclo
[11] Lipatski D and Dellar P J 2011 Phil. Trans. R. Soc. A 369 2155
[12] Dellar P J, Lipatski D, Palpacelli S and Succi S 2011 Phys. Rev. E 83 046706
[13] Strauch F W 2006 Phys. Rev. A 73 054302
[14] Strauch F 2007 J. Math. Phys. 48 082102
[15] Bisio A, D’Ariano G M and Tosini A 2012 arXiv:1212.2839
[16] Shikano Y 2013 J. Comput. Theor. Nanoscience 10 1558
[17] Strauch F W 2006 Phys. Rev. A 74 030301
[18] Boghosian B M and Taylor W 1998 Physica D 120 30
[19] Cha M 2011 Master’s Thesis University of California, San Diego
[20] Boghosian B M and Taylor W 1998 Phys. Rev. E 57 54
[21] di Molfetta G and Debbasch F 2012 J. Math. Phys. 53 123302
[22] Lorin E and Bandrauk A 2011 Nonlinear Anal.: Real World Appl. 12 190
[23] Huang Z, Jin S, Markowich P A, Sparber C and Zheng C 2005 J. Comput. Phys. 208 761–89
[24] Fillion-Gourdeau F, Lorin E and Bandrauk A D 2012 Comput. Phys. Commun. 183 1403–15
[25] Childs A M and Goldstone J 2004 Phys. Rev. A 70 042312
[26] Bracken A, Ellinas D and Smyrnakis I 2007 Phys. Rev. A 75 022322
[27] D’Ariano G M and Perinotti P 2013 arXiv:1306.1934
[28] Bateson R and 2012 J. Phys.: Conf. Ser. 361 012009
[29] Kauffman L H and Noyes H P 1996 Phys. Lett. A 218 139
[30] Gersch H A 1981 Int. J. Theor. Phys. 20 491
[31] Kishigi K, Takeda R and Hasegawa Y 2008 J. Phys.: Conf. Ser. 132 012005
[32] Di Franco C, mc Gettrick M, Machida T and Busch Th 2011 Phys. Rev. A 84 042337
[33] Fattorini H O 1983 The Cauchy Problem (Encyclopedia of Mathematics and its Applications no 18) (Cambridge: Cambridge University Press)
[34] 2003 http://www.math.binghamton.edu/loya/papers/kl_taylor.pdf
[35] Davies E B 2007 Linear Operators and Their Spectra vol 106 (Cambridge: Cambridge University Press)
[36] Lax P D and Richtmyer R D 1956 Commun. Pure Appl. Math. 9 267
[37] 2011–2012 Operator Semigroups for Numerical Analysis Workshop of the 15th Internet Seminar on Evolutions Equations (https://isem-mathematik.uibk.ac.at/isemwiki/index.php/Lecture_4)
[38] Petersen D P and Middleton D 1962 Inf. Control 5 279
[39] Jingfan L and Gensun F 2004 Anal. Theory Appl. 20 52
[40] Kliesch M, Barthel T, Gogolin C, Kastoryano M and Eisert J 2011 Phys. Rev. Lett. 107 120501
[41] Sobolev S 1938 Rec. Math. [Mat. Sbornik] N.S. 4 471
[42] Benzoni-Gavage S and Serre D 2007 Multi-Dimensional Hyperbolic Partial Differential Equations (Oxford: Oxford University Press)