Description of Generalized Albanese Varieties by Curves

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Abstract
Let $X$ be a projective variety over an algebraically closed base field, possibly singular. The aim of this paper is to show that the generalized Albanese variety $\text{Alb}(X, X_{\text{sing}})$ of Esnault-Srinivas-Viehweg can be computed from one general curve $C$ in $X$, if the base field is of characteristic 0. We illustrate this by an example, which we also use to unravel some mysterious properties of $\text{Alb}(X, X_{\text{sing}})$.

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0 Introduction

In [Ru1] the author considered generalized Albanese varieties \( \text{Alb}_{\mathcal{F}}(X) \) associated with categories of rational maps from a variety \( X \) to algebraic groups. If such a generalized Albanese variety is generated by a curve \( C \), then the dimension of \( \text{Alb}_{\mathcal{F}}(X) \) is bounded by the dimension of the generalized Albanese of the curve \( C \), which is easy to compute. For example if \( \text{Alb}_{\mathcal{F}}(X) \) is the classical Albanese \( \text{Alb}(X) \) of a smooth proper variety \( X \) or the Albanese of Esnault-Srinivas-Viehweg \( \text{Alb}(X, X_{\text{sing}}) \) of a (singular) projective variety \( X \) (see [ESV], cf. [Ru1]), then \( \text{Alb}(C) \) resp. \( \text{Alb}(C, C_{\text{sing}}) \) is isomorphic to the Picard variety \( \text{Pic}^0(C) \). In this way the existence of the classical Albanese was shown in [Lg] and the existence of the Albanese of Esnault-Srinivas-Viehweg in [ESV].

The purpose of the present paper is to show that the functorial description of generalized Albanese varieties from [Ru1] is not only a purely theoretical one, but allows a concrete computation, using the interplay with curves. Here we restrict ourselves to the case that the base field \( k \) is algebraically closed of characteristic 0. The Albanese of Esnault-Srinivas-Viehweg \( \text{Alb}(X, X_{\text{sing}}) \) is an extension of the classical Albanese \( \text{Alb}(\tilde{X}) \), where \( \tilde{X} \to X \) is a resolution of singularities, by an affine group \( L_X \), whose Cartier dual \( \text{Div}^0_{\tilde{X}/X} \) is described in [Ru1, Prop.s 3.23, 3.24]. The main result of this work is a significant simplification of the presentation of \( \text{Div}^0_{\tilde{X}/X} \) (Theorem 3.8). Moreover, the functorial description allows to explain some pathological properties of the Albanese of Esnault-Srinivas-Viehweg. This is accomplished in an example (Section 4).

0.1 Leitfaden

Section 1. We recall some facts about the Picard variety of curves that allow us to compute \( \text{Pic}^0(C) \) of a singular curve \( C \). Here we decompose \( \text{Pic}^0(C) \) as an extension of the Picard variety \( \text{Pic}^0(\tilde{C}) \) of the normalization \( \tilde{C} \) of \( C \) by a linear group \( L \) that takes care of the singularities of \( C \).

Section 2. We consider a formal group \( \mathcal{F} \subset \text{Div}_X \) of relative Cartier divisors on \( X \) and a curve \( C \) in \( X \). We give a sufficient condition for the injectivity of the restriction map \( \mathcal{F} \to \text{Div}_C \) from \( \mathcal{F} \subset \text{Div}_X \) into the group sheaf of relative Cartier divisors on \( C \) (Lemma 2.2).

Section 3. The Albanese of Esnault-Srinivas-Viehweg \( \text{Alb}(X, X_{\text{sing}}) \) is the dual of the 1-motivic functor \( [\text{Div}^0_{\tilde{X}/X} \to \text{Pic}^0_{\tilde{X}}] \), where \( \tilde{X} \to X \) is a resolution of singularities, see [Ru1] Theorem 0.1. Here \( \text{Div}^0_{\tilde{X}/X} \) is the “kernel of the push-forward of relative Divisors from \( \tilde{X} \) to \( X \)”, if \( X \) is a curve (see [Ru1] Proposition 3.23). For higher dimensional \( X \) the definition of \( \text{Div}^0_{\tilde{X}/X} \) is derived from the one for curves by intersecting the formal groups \( \text{Div}^0_{\tilde{C}/C} \) associated with curves \( C \) in \( X \), where the intersection ranges over all Cartier curves \( C \) in \( X \) relative to the singular locus of \( X \) (see [Ru1] Proposition 3.24]). So a priori this object looks hard to grasp. We explain how \( \text{Div}^0_{\tilde{X}/X} \) can be computed from one single complete intersection curve in \( X \) (Corollary 3.7). Moreover, the curves with this property are dense in any space of sufficiently ample complete intersection curves (Theorem 3.8).

Sections 1, 2 and 3 provide the necessary tools in order to reduce the computation of \( \text{Alb}(X, X_{\text{sing}}) \) to the curve case. This is demonstrated in an example.
in the next section.

**Section 4**. The classical Albanese of smooth projective varieties \( X_i \) is compatible with products, i.e. \( \text{Alb}(\prod X_i) = \prod \text{Alb}(X_i) \). More generally, all universal objects for categories of rational maps to semi-abelian varieties have this property. However, due to additive subgroups it is possible for singular projective varieties \( X_i \) that \( \dim \text{Alb}(\prod X_i, (\prod X_i)_{\text{sing}}) > \sum \dim \text{Alb}(X_i, (X_i)_{\text{sing}}) \). Moreover, if \( X \) is a smooth projective variety of dimension \( d \) and \( L \) a very ample line bundle on \( X \), then for a complete intersection \( C \) of \( d-1 \) general divisors in the linear system \( |L| \) the Gysin map \( \text{Alb}(C) \to \text{Alb}(X) \) will be surjective. This is not true in general for the Albanese of Esnault-Srinivas-Viehweg of a singular \( X \), but a sufficiently high power \( L^\otimes N \) will again have this property.

We discuss an example that illustrates these pathological properties. This example was computed in the diploma of Alexander Schwarzhaupt [Sch] by means of the Hodge theoretic description given in [ESV]. The functorial description from [Ru1] yields a considerable simplification for the computation and gives an explanation for the strange behaviour of \( \text{Alb}(X, X_{\text{sing}}) \). In particular we obtain a formula for the dimension of \( \text{Alb}(\prod X_i, (\prod X_i)_{\text{sing}}) \) as a function of the dimensions of \( \text{Alb}(X_i, (X_i)_{\text{sing}}) \) (Proposition 4.2). Using a formula from [ESV], Schwarzhaupt computes in this example bounds \( N_0, N_1 \) such that the Gysin map \( \text{Alb}(C, C_{\text{sing}}) \to \text{Alb}(X, X_{\text{sing}}) \) is surjective for any complete intersection curve \( C \) of general divisors in \( |L^\otimes N| \) if \( N > N_1 \) and not surjective if \( N < N_0 \). With our method we can optimize the bound \( N_1 \), in fact we can show that the Gysin map is surjective if \( N \geq N_0 \), i.e. the condition we obtain is necessary and sufficient for the surjectivity of the Gysin map (Proposition 4.5).

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## 1 Picard Variety of Curves

Let \( C \) be a projective curve over a field \( k \). The fact that divisors equal 0-cycles on \( C \) yields an identification of the Picard scheme \( \text{Pic}^0 C \) with the relative Chow group \( \text{CH}_0(C, C_{\text{sing}}) \) from [LW].

We describe the algebraic group \( \text{Pic}^0 C \) as an extension of the abelian variety \( \text{Pic}^0 \tilde{C} \) by an affine algebraic group \( L \), using an intermediate curve \( C' \) (the semi-normalization) between \( C \) and its normalization \( \tilde{C} \). The methods of this Section are taken from [BLR] Section 9.2.

### 1.1 Normalization and Semi-Normalization of a Curve

Let \( C \) be a curve over a perfect field.

**Definition 1.1.** A point \( p \) of a curve \( C \) is called an *ordinary multiple point*, if it marks a transversal crossing of smooth formal local branches. More precisely, \( p \in C \) is an *ordinary \( m \)-ple point* if \( \mathcal{O}_{C,p} \cong k[[t_1, \ldots, t_m]]/\sum_{i \neq j} (t_i t_j) \).
Definition 1.2. The normalization $\nu : \tilde{C} \to C$ factors as $\tilde{C} \xrightarrow{\sigma} C' \xrightarrow{\nu} C$ for a unique curve $C'$ which is homeomorphic to $C$ and has only ordinary multiple points as singularities, see [BLR, Section 9.2, p. 247]. $C'$ is called the largest curve homeomorphic to $C$, or the semi-normalization of $C$.

Since $C$ is reduced, the smooth locus is dense in $C$, hence the singular locus $S$ is finite. The curve $C'$ is obtained from $C$ by identifying the points $\tilde{p}_i \in \tilde{C}$ lying over $p \in S$. For an explicit description of $C'$ see [BLR, Section 9.2, p. 247].

Notation 1.3. We write $\mathcal{O} := \mathcal{O}_C$, $\mathcal{O}' := \rho_*\mathcal{O}_{C'}$, $\tilde{\mathcal{O}} := \nu_*\mathcal{O}_{\tilde{C}}$.

1.2 Decomposition of the Picard Variety

Proposition 1.4. Let $C'$ be a connected projective curve having only ordinary multiple points as singularities. Let $\tilde{C}$ be the normalization of $C'$. Then $\Pic^{0}(C')$ is an extension of the abelian variety $\Pic^{0}(\tilde{C})$ by a torus $T$. If $k$ is algebraically closed, $T \cong (\Gamma_m)^t$ is a split torus of rank $t = \sum_{m \geq 1}(m-1)\#S_m - \#\text{Cp}(C') + 1$, where $\text{Cp}(C')$ is the set of irreducible components of $C'$ and $S_m$ is the set of $m$-ple points (see Definition 1.4).

Proof. (See also [BLR, Section 9.2, Proposition 10] for the first statement.) If $S$ denotes the singular locus of $C'$ and $\text{Cp}(\tilde{C})$ the set of components of $\tilde{C}$, we obtain from $1 \to \mathcal{O}^* \to \mathcal{O}^{*} \to \mathcal{Q}^* \to 1$ with $\mathcal{Q}^* = \prod_{p \in S} (\mathcal{O}_p)^*/(\mathcal{O}_p^*)^*$ the long exact cohomology sequence

$$1 \to k_\mathcal{Z} \to \prod_{Z \in \text{Cp}(\tilde{C})} k_{Z} \to \prod_{p \in S} T_p(k) \to \Pic(C') \to \prod_{Z \in \text{Cp}(\tilde{C})} \Pic(Z) \to 1$$

where $k_\mathcal{Z} = H^0(C', \mathcal{O}^*_{C'})$, $k_{Z} = H^0(Z, \mathcal{O}_Z^*)$, and

$$T_p(k) = (\mathcal{O}_p)^*/(\mathcal{O}_p^*)^* = \frac{\prod_{q \neq p} k(q)^*}{k(p)^*} \cong (k^*)^{m_p-1}$$

since each $p \in S$ is an ordinary multiple point (see Definition 1.4). Here $\prod_{p \in S} T_p(k)$ maps to the connected component of the identity of $\text{Pic}(C')$. Then the affine part $T$ of $\Pic^{0}(C')$ is the torus given by

$$T(k) = \text{coker} \left( \prod_{Z \in \text{Cp}(\tilde{C})} k_{Z} \to \prod_{p \in S} T_p(k) \right) = \frac{\prod_{p \in S} T_p(k)}{\prod_{Z \in \text{Cp}(\tilde{C})} k_{Z} / k_\mathcal{Z}} \cong (k^*)^t$$

with $t = \sum_{m \geq 1}(m-1)\#S_m - \#\text{Cp}(\tilde{C}) + 1$ and $\#\text{Cp}(\tilde{C}) = \#\text{Cp}(C')$.

Proposition 1.5. Let $C$ be a projective curve, let $C'$ be the largest homeomorphic curve between $C$ and its normalization $\tilde{C}$. Then $\Pic^{0}(C')$ is an extension of $\Pic^{0}(C')$ by a unipotent group $U$. If $k$ is algebraically closed, $U$ is characterized by $U(k) = \prod_{p \in S}(1 + m_{C',p})/(1 + m_{C,\tilde{C}})$, if moreover char($k$) = 0, the exponential map yields an isomorphism $U(k) \cong \prod_{p \in S} m_{C',p}/m_{C,\tilde{C}}$ where $S = C_{\text{sing}}$ is the singular locus of $C$.

Proof. (See also [BLR, Section 9.2, Proposition 9] for the first statement.) The exact sequence $1 \to \mathcal{O}^* \to \mathcal{O}^{*} \to \mathcal{Q}^* \to 1$ with $\mathcal{Q}^* = \prod_{p \in S} \mathcal{O}_p^*/\mathcal{O}_p^*$ yields the exact cohomology sequence $1 \to H^0(\mathcal{O}^*) \to H^1(\mathcal{O}^*) \to H^1(\mathcal{O}^*) \to 1$, since
\[ \rho : C' \to C \text{ is a homeomorphism, thus } H^0(O^*) \cong H^0(O^*) . \] Moreover it holds \[ H^0(Q^*) = \prod_{P \in S} (O^*_P) / \prod_{P \in S} (1 + \mathfrak{m}_{C, P}). \] As \( U(k) := H^0(Q^*) \) is a connected unipotent group, the image of \( U(k) \) is contained in the connected component of the identity of \( \text{Pic}(C) = H^1(O^*) \).  

**Proposition 1.6.** Let \( C \) be a projective curve and \( \tilde{C} \) its normalization. Then the Picard variety \( \text{Pic}^0 C \) is an extension of the abelian variety \( \text{Pic}^{0} \tilde{C} \) by an affine algebraic group \( L = T \times_k U \), which is the product of a torus \( T \) and a unipotent group \( U \). If \( k \) is algebraically closed, \( T \) and \( U \) are characterized as in Propositions 1.4 and 1.5.

**Proof.** Follows directly from Propositions 1.4 and 1.5, cf. [BLR, Section 9.2, Remark 1.7]. Let \( S \) be a subvariety of \( \text{Pic}^0 C \) and \( \delta \in \text{Lie}(\text{Pic}^0 C) \) be a deformation of the zero divisor in \( C \). Then \( \delta \) determines an effective divisor by the poles of its local sections. Hence for each generic point \( \eta \) of height 1 in \( C \), with associated discrete valuation \( v_{\eta} \), the expression \( v_{\eta}(\delta) \) is well defined and \( v_{\eta}(\delta) \leq 0 \). Thus we obtain a homomorphism \( \nu_{\eta} : \text{Lie}(\text{Pic}^0 C) \to \mathbb{Z} \).

**Definition 2.1.** For an ample line bundle \( \mathcal{L} \) on \( X \) and an integer \( c \) with \( 1 \leq c \leq \dim X \) write \( |\mathcal{L}|^c = \mathbb{P}(H^0(X, \mathcal{L}) \times \ldots \times \mathbb{P}(H^0(X, \mathcal{L})) \) (c copies). Let \( H_1, \ldots, H_c \in |\mathcal{L}| \) and \( V = \bigcap_{i=1}^{c} H_i \). By abuse of notation we write \( V \in |\mathcal{L}|^c \) instead of \( (H_1, \ldots, H_c) \in |\mathcal{L}|^c \).

**Lemma 2.2.** Let \( \mathcal{F} \) be a formal subgroup of \( \text{Div}^0_X \) s.t. \( \mathcal{F} \cong \mathbb{Z}^t \times_k (\mathbb{G}_{a})^v \) for \( t, v \in \mathbb{N} \), where \( \mathbb{G}_{a} \) denotes the completion of \( \mathbb{G}_{a} \) at 0. Let \( S \) be the set of generic points of \( \text{Supp}(\mathcal{F}) \) and \( S_{\text{inf}} \) the corresponding set for \( \text{Supp}(\mathcal{F}_{\text{inf}}) \). If \( \eta \) is a generic point of height 1 in \( X \), denote by \( E_{\eta} \) the associated prime divisor. For an ample line bundle \( \mathcal{L} \) on \( X \) there is an open dense \( U \subset |\mathcal{L}|^{c-1} \) such that \( (?, C) |_{\mathcal{F}} : \mathcal{F} \to \text{Div}^0_C \) is injective for \( C \in U \) if \( \#(C \cap E_{\eta}) \geq \dim_k (\text{Lie} \mathcal{F})_{\eta}^{-v} \)
for each $0 \leq k$.

Then the map $\Gamma (K_X/O_X) \ni \text{Lie} F \mapsto \Gamma (K_C/O_C)$ induces a map of characteristic $0$. Let $\text{Lie}(F)$ be a projective variety of dimension $d$. For an ample divisor $H$, consider the following diagram:

$$
\begin{array}{ccc}
\Gamma (K_X/O_X) \ni \text{Lie} F & \mapsto & \Gamma (K_C/O_C) \\
\downarrow i & & \downarrow i \\
\bigoplus_{\eta \in \Sigma} (K_X/O_X)_\eta & \ni \text{im} (\text{Lie} F) & \mapsto \bigoplus_{\eta \in \Sigma} (K_C/O_C)_\eta.
\end{array}
$$

For each $\eta \in S_{\text{inf}}$ choose a local parameter $t_\eta$ of $m_{X,\eta}$. Since $C$ intersects $E_\eta$ properly and if $C$ intersects $E_\eta$ transversally in general points, we may assume that each $q \in C \cap E_\eta$ is a regular closed point of $C$. Then the image $t_q \in \text{OC}_C$ of $t_\eta \in O_X$ is a local parameter of $m_{C,q}$ and $v_\eta (\text{Lie} F \cdot C) = v_\eta (\text{Lie} F)$. Set $-n_\eta = \min \{v_\eta (\text{Lie} F)\}$. We may consider $\text{Lie} F \cdot C$ as a $k$-linear subspace of the $k$-vector space

$$
\bigoplus_{\eta \in \Sigma} \bigoplus_{q \in C \cap E_\eta} t_q^{-n_\eta} \text{OC}_{C,q}/\text{OC}_{C,q}.
$$

Then the map

$$
\text{im} \left( \text{Lie} F \rightarrow (K_X/O_X)_\eta \right) \rightarrow \bigoplus_{q \in C \cap E_\eta} t_q^{-n_\eta} \text{OC}_{C,q}/\text{OC}_{C,q}
$$

$$
\sum_{q \in C \cap E_\eta} [f] t_q^{-\nu} \rightarrow \sum_{q \in C \cap E_\eta} [f] t_q^{-\nu}
$$

is injective if $\dim_k (\text{Lie} F)_\eta^{-\nu} \leq \#(C \cap E_\eta)$ for all $-\nu \in v_\eta (\text{Lie} F)$ and the intersection points of $C$ and $E_\eta$ are in general position. $(? \cdot C) |_{|L|} : \text{Lie} F \rightarrow \text{Lie} (\text{Div}_C)$ is injective if these maps are injective for all $\eta \in S_{\text{inf}}$. Since $F \cong \mathbb{Z}^l \times_k (\mathbb{G}_a)^w$ by assumption, $F$ is already determined by $\text{Lie} F$ and $\text{Lie} F$.

Proof. Let $C = \bigcap_{i=1}^{d-1} H_i$ be a complete intersection curve in $|L|^{d-1}$. As an ample divisor $H_i$ intersects each closed subscheme of codimension $1$ and $H_i$ restricted to $H_1 \cap \ldots \cap H_{i-1}$ is again ample for all $i = 2, \ldots, d - 1$, it follows by induction that $C \cap \text{Supp}(D) \neq \emptyset$ for all $0 \neq D \in F(R)$. Let $R$ be a finite dimensional $k$-algebra.

For the infinitesimal part of $F$ consider the following diagram:

$$
\begin{array}{ccc}
\Gamma (K_X/O_X) \ni \text{Lie} F & \mapsto & \Gamma (K_C/O_C) \\
\downarrow i & & \downarrow i \\
\bigoplus_{\eta \in \Sigma} (K_X/O_X)_\eta & \ni \text{im} (\text{Lie} F) & \mapsto \bigoplus_{\eta \in \Sigma} (K_C/O_C)_\eta.
\end{array}
$$

For each $\eta \in S_{\text{inf}}$ choose a local parameter $t_\eta$ of $m_{X,\eta}$. Since $C$ intersects $E_\eta$ properly and if $C$ intersects $E_\eta$ transversally in general points, we may assume that each $q \in C \cap E_\eta$ is a regular closed point of $C$. Then the image $t_q \in \text{OC}_C$ of $t_\eta \in O_X$ is a local parameter of $m_{C,q}$ and $v_\eta (\text{Lie} F \cdot C) = v_\eta (\text{Lie} F)$. Set $-n_\eta = \min \{v_\eta (\text{Lie} F)\}$. We may consider $\text{Lie} F \cdot C$ as a $k$-linear subspace of the $k$-vector space

$$
\bigoplus_{\eta \in \Sigma} \bigoplus_{q \in C \cap E_\eta} t_q^{-n_\eta} \text{OC}_{C,q}/\text{OC}_{C,q}.
$$

Then the map

$$
\text{im} \left( \text{Lie} F \rightarrow (K_X/O_X)_\eta \right) \rightarrow \bigoplus_{q \in C \cap E_\eta} t_q^{-n_\eta} \text{OC}_{C,q}/\text{OC}_{C,q}
$$

$$
\sum_{q \in C \cap E_\eta} [f] t_q^{-\nu} \rightarrow \sum_{q \in C \cap E_\eta} [f] t_q^{-\nu}
$$

is injective if $\dim_k (\text{Lie} F)_\eta^{-\nu} \leq \#(C \cap E_\eta)$ for all $-\nu \in v_\eta (\text{Lie} F)$ and the intersection points of $C$ and $E_\eta$ are in general position. $(? \cdot C) |_{|L|} : \text{Lie} F \rightarrow \text{Lie} (\text{Div}_C)$ is injective if these maps are injective for all $\eta \in S_{\text{inf}}$. Since $F \cong \mathbb{Z}^l \times_k (\mathbb{G}_a)^w$ by assumption, $F$ is already determined by $\text{Lie} F$ and $\text{Lie} F$.

3 Computation of the Kernel of the Push-forward of Divisors

Let $X$ be a projective variety of dimension $d$ over an algebraically closed field $k$ of characteristic $0$. Let $S$ denote the singular locus of $X$. For a line bundle $\mathcal{L}$ on $X$ we define $|\mathcal{L}|_S = \{H \in |\mathcal{L}| \mid H \text{ intersects } S \text{ properly}\}$. Let $\pi : \tilde{X} \rightarrow X$ be a projective resolution of singularities. For a curve $C$ we denote the normalization of $C$ by $\nu_C : \tilde{C} \rightarrow C$. The functor of formal divisors on $C$ is the formal group given by $\text{FDiv}_{\nu_C} = \bigoplus_{p \in (C(k))} \text{Hom}_{\text{Ab}/k} (\text{OC}_{C,p}, k^*)$ (see [Ha3, Def. 2.1]). A finite
morphism $\zeta : Z \to C$ induces an obvious push-forward of formal divisors $\zeta_* : \text{FDiv}_X \to \text{FDiv}_C$ (see [Ru3 Def. 2.4]). If $C$ is normal, there is a canonical homomorphism $\text{fml} : \text{Div}_C \to \text{FDiv}_C$ given by $D \mapsto \sum_{p \in C(\kappa)} (D, ?)_p$, where $(?, ?)_p$ is the local symbol at $p \in C$ (see [Ru3 Prop. 2.5]).

We use the definition of $\text{Div}^0_{\tilde{C}/X}$ given in [Ru3 Def. 2.6, 2.7], which coincides with the one given in [Ru1 Prop. 3.23, 3.24], as is easily verified (see [Ru3 Rmk. 2.8]):

**Definition 3.1.** If $C$ is a projective curve, then

$$\text{Div}^0_{\tilde{C}/C} = \ker \left( \text{Div}^0_C \xrightarrow{\text{fml}} \text{FDiv}_{\tilde{C}} \xrightarrow{\nu} \text{FDiv}_C \right)$$

where $\nu : \tilde{C} \to C$ is the normalization. For higher dimensional $X$

$$\text{Div}^0_{\tilde{X}/X} = \bigcap_C \left( ? \cdot \tilde{C} \right)^{-1} \text{Div}^0_{\tilde{C}/C}$$

where the intersection ranges over all Cartier curves in $X$ relative to the singular locus of $X$ (see [Ru1 Definition 3.1]), and $(? \cdot \tilde{C})$ is the pull-back of relative Cartier divisors on $\tilde{X}$ to those on $\tilde{C}$.

The functor $\text{Div}^0_{\tilde{X}/X}$ is a torsion-free dual-algebraic ([Ru2 Def. 1.20]) formal group (see [Ru3 Thm. 4.5] or [Ru1 Prop. 3.24]), this means that the Cartier dual of $\text{Div}^0_{\tilde{X}/X}$ is a connected algebraic affine group (by definition and [Ln (5.2)]). Equivalently, $\text{Div}^0_{\tilde{X}/X} \cong \mathbb{Z}^t \times (\hat{\mathbb{G}}_a)^v$ for some $t, v \in \mathbb{N}$ (cf. [Ln (4.2)]).

As $\text{char}(k) = 0$, a formal group $E$ is completely determined by its $k$-valued points $E(k)$ and its Lie-algebra $\text{Lie}(E)$ (see [Ru1 Cor. 1.7]). If $E$ is torsion-free and dual-algebraic, then $E(k)$ is a free abelian group of finite rank and $\text{Lie}(E)$ is a finite dimensional $k$-vector space. Then the dimension of the Cartier dual $E^\vee$ of $E$ is $\dim E^\vee = \text{rk} E(k) + \dim \text{Lie}(E)$ (cf. [Ln (5.2)])

**Proposition 3.2** (Bertini’s Theorem). Let $L$ be a line bundle on $X$. Then for almost all $C \subseteq |L|$ the inverse image $C_X = C \times_C \tilde{X} = \pi^{-1}C \subseteq |\pi^*L|$ is smooth.

**Proof.** [Ha II, Theorem 8.18] and induction. ■

**Proposition 3.3.** Let $B$ be a variety parametrizing Cartier curves in $X$, $C_0 \subseteq B$ and $D \in \text{Div}^0_X$ such that $(\nu_{C_0} \circ \text{fml})(D \cdot C_0) \neq 0$. Then there exists an open neighbourhood $U \ni C_0$ in $B$ such that $(\nu_{C} \circ \text{fml})(D \cdot \tilde{C}) \neq 0$ for all $C \in U$. In other words: The zero locus of the function $(\nu_{C} \circ \text{fml})(D \cdot ?) : B \to \bigoplus_{C \in B} \text{FDiv}_C$ is closed.

**Proof.** Let $\mathcal{C} \xrightarrow{\beta} B$ be the universal curve over $B$, let $Z := \mathcal{C} \times_X \tilde{X}$ be the relative curve over $B$ of preimages of $\mathcal{C}$ in $\tilde{X}$. We may assume that the fibre $Z_{\beta(C_0)}$ of $Z$ over the point $\beta(C_0) \in B$ corresponding to $C_0$ is normal, i.e. $C_0 = Z_{\beta(C_0)}$. Otherwise this can be achieved by blowing up $\tilde{X}$.

If $C$ is a curve, $S \subseteq C(k)$ and $F \in \text{FDiv}_C$, we denote by $F_S := \sum_{p \in S} F_p \in \text{Hom}_{\text{Ab} / k}(\hat{\mathcal{O}}_{C,S}, k^*)$ the sum of those components $F_p \in \text{Hom}_{\text{Ab} / k}(\hat{\mathcal{O}}_{C,p}, k^*)$ with $p \in S$. 

7
By definition, \((\nu_{\mathcal{C}_0, \ast} \circ \text{fml}) (D \cdot \mathcal{C}_0) \neq 0\) implies that there is \(p \in C_0(k)\) and \(f_0 \in \mathcal{O}_{C_0, p}^*\) such that \(\sum_{q \rightarrow p} (D \cdot \mathcal{C}_0, f_0)_q \neq 0\), where \(q \rightarrow p\) are the points \(q \in \mathcal{C}_0\) over \(p \in C_0\). Let \(S(D)\) denote the support of \(D\). Then necessarily \(q \in S(D)\). Let \(L_R := \mathbb{G}_m(\mathfrak{r} \otimes R)\) be the Weil restriction of \(\mathbb{G}_m, R\) from \(R\) to \(k\). We are going to construct a regular map \(\Phi : T \rightarrow L_R\) from a neighbourhood \(T\) of \(\beta(C_0)\) in \(B\) to \(L_R\), such that \(\Phi(b) = (\nu_{\mathcal{C}_0, \ast} \circ \text{fml})(D \cdot \mathcal{Z}_b)\). Let \(\beta(C_0)\) be a modulus for the rational map \(\Phi(\beta(C_0)) \neq 0\) implies that there is an open neighbourhood \(U \ni \beta(C_0)\) in \(B\) such that \(\Phi(u) \neq 0\) for all \(u \in U\), proving the assertion.

Let \(G(D) \in \text{Ext}_{\text{Ab}/k}(\text{Alb}(\mathcal{X}), L_R) \equiv \text{Pic}^0_{\text{Ab} \mathcal{X}}(R) \equiv \text{Pic}^0_{\mathcal{X}}(R)\) be the algebraic group corresponding to \(\mathcal{O}_{\mathcal{X}, \mathcal{D}}(D)\). The canonical 1-section of \(\mathcal{O}_{\mathcal{X}, \mathcal{D}}(D)\) induces a rational map \(\varphi^D : \mathcal{X} \rightarrow \mathcal{G}(D)\), which is regular away from \(S(D)\).

If \(f \in \mathcal{O}^*_{Z_b, q}\) for some \(b \in \mathcal{B}\) and \(q \in Z_b\), then according to [Rau, Lem. 3.16] \((\varphi^D|_{Z_b}, f)_q\) is contained in the fibre of \(G(D)\) over \(0 \in \text{Alb}(\mathcal{X})\), which is \(L_R\), and \((\varphi^D|_{Z_b}, f)_q = (D \cdot \mathcal{Z}_b, f)_q\).

Let \(M\) be a modulus for the rational map \(\varphi^D|_{\mathcal{C}_0}\). By the Approximation Lemma we find \(f_p \in \mathcal{O}_{\mathcal{C}_0, p}^*\) with \(f_0/f_p \equiv 1\) mod \(M\) at all points \(q \rightarrow p\) and \(f_p \equiv 1\) mod \(M\) at all points of \(S(D) \cap \mathcal{C}_0 \backslash \nu_{\mathcal{C}_0}(p)\). Then

\[
\sum_{q \rightarrow p} \left( \varphi^D|_{\mathcal{C}_0}, f_0 \right)_q = \sum_{q \in S(D) \cap \mathcal{C}_0} \left( \varphi^D|_{\mathcal{C}_0}, f_p \right)_q = -\sum_{c \in \mathcal{C}_0 \cap S(D)} v_c(f_p) \varphi^D(c)
\]

Let \(T \subset B\) be an affine neighbourhood of \(\beta(C_0)\), and let \(f \in \mathcal{O}^*_{\mathcal{X}, \mathcal{D}, q}\) be a lift of \(f_p \in \mathcal{O}_{\mathcal{C}_0, p}^*\). We consider \(f\) as an element of \(\mathcal{O}^*_{\mathcal{Z}_b, S(D), q}\). Let \(S(f)\) denote the support of \(\text{div}(f)\) in \(\mathbb{Z}_T\). Shrinking \(T\) if necessary, we may assume that \(S(f) \cap S(D) = \emptyset\), that \(S(f)\) intersects the fibres \(Z_b\) over \(T\) transversally and that \(S(f) \rightarrow T\) is étale. We choose \(f\) in such a way that \(\beta(C_0) \subset T\). Furthermore we may assume that all fibres \(Z_b\) over \(T\) are normal by Bertini’s Theorem (cf. Proposition 3.3). We obtain a regular map \(\lambda : S(f) \rightarrow G(D)\) defined by

\[
\lambda(s) = -v_s(\text{fml} \circ \varphi^D(s)) = -v_{E(s)}(f) \varphi^D(s)
\]

for \(s \in S(f)\), where \(E(s)\) is the unique irreducible component of \(S(f)\) containing \(s\). While \(S(f) \rightarrow T\) is finite étale, taking the trace of \(\lambda\) over \(T\) yields the map \(\text{Tr} \lambda : T \rightarrow L_R\), given by

\[
\text{Tr} \lambda(b) = \sum_{s \rightarrow b} \lambda(s) = -\sum_{c \in \mathbb{Z}_b \cap S(D)} v_c(f|_{Z_b}) \varphi^D(c) = -\sum_{c \in \mathbb{Z}_b \cap S(D)} (\varphi^D|_{Z_b}, f|_{Z_b})_c
\]

Then \(\text{Tr} \lambda(\beta(C_0)) = (\nu_{\mathcal{C}_0, \ast} \circ \text{fml})(D \cdot \mathcal{C}_0)\). As \(\lambda\) is regular on \(S(f)\), \(\text{Tr} \lambda\) is regular on \(T\), according to [Sag, III, No. 5, Prop. 8]. Thus \(\Phi := \text{Tr} \lambda\) gives the desired map.

**Corollary 3.4.** Let \(\mathcal{E} \cong \mathbb{Z}^t \times_k (\mathcal{G}_a)^n\) be a formal subgroup of \(\text{Div}^0_{\mathcal{X}}\) which contains \(\text{Div}^0_{\mathcal{X}, \mathcal{X}^t}\) and let \(B\) be a variety parametrizing Cartier curves in \(X\).

The function \(\dim \left( \mathcal{E} \times \text{Div}^0_{\mathcal{X}^t/\mathcal{X}} \right)^\vee : B \rightarrow \mathbb{N}\) is upper semi-continuous.
Proof. For $C \in B$ write $d(C) := \dim \left( \mathcal{E} \times \text{Div} \mathcal{D}_{/C} \right)^\vee$. We have to show that the sets $\{ C \in B \mid d(C) \leq n \}$ are open for all $n \in \mathbb{N}$. Let $C_0 \in B$ with $d(C_0) = n$. We show: there is an open neighbourhood $U \ni C_0$ in $B$ such that $d(C) \leq n$ for all $C \in U$.

For some $c_0, c_1 \in \mathbb{N}$ with $c_0 + c_1 = c := \dim \mathcal{E}^\vee - n$ one finds $\mathbb{Z}$-linearly independent elements $\delta^{(0)}_1, \ldots, \delta^{(0)}_{c_0} \in (\mathcal{E} \cdot \mathcal{C}_0)(k) \setminus \text{Div}^0_{\mathcal{C}_0/C_0}(k)$ and $k$-linearly independent elements $\delta^{(1)}_1, \ldots, \delta^{(1)}_{c_1} \in \text{Lie} (\mathcal{E} \cdot \mathcal{C}_0) \setminus \text{Lie} \left( \text{Div}^0_{\mathcal{C}_0/C_0} \right)$ that extend a basis of $\text{Div}^0_{\mathcal{C}_0/C_0}(k)$ resp. $\text{Lie} \left( \text{Div}^0_{\mathcal{C}_0/C_0} \right)$ to a basis of $\mathcal{E}(k)$ resp. $\text{Lie}(\mathcal{E})$. Let $\mathcal{D}^{(i)} \in \mathcal{E}$ with $\mathcal{D}^{(i)} \cdot \mathcal{C}_0 = \delta^{(i)}$ for $j = 1, \ldots, c_1$ and $i = 0, 1$. By Proposition 3.3 there exists an open neighbourhood $U \ni C_0$ in $T$ such that we have $(\nu_{C, s} \circ \text{fml}) (\mathcal{D}^{(i)} \cdot \mathcal{C}) \neq 0$ for $j = 1, \ldots, c_1$ and $i = 0, 1$ for all $C \in U$, and the locus in $B$ where $(\nu_{C, s} \circ \text{fml}) (\mathcal{D}^{(i)} \cdot \mathcal{C}), \ldots, (\nu_{C, s} \circ \text{fml}) (\mathcal{D}^{(i)} \cdot \mathcal{C})$ are linearly dependent mod $\overline{\text{Div}}^0_{\mathcal{C}/C}$ is a closed proper subset $V \subset U$. Then $d(C) \leq \dim \mathcal{E}^\vee - c = n$ for all $C \in U \setminus V$. ■

Proposition 3.5. Let $\mathcal{L}$ be an ample line bundle on $X$. Then there exists $N \in \mathbb{N}$ such that $\text{Div}^0_{\mathcal{L}/X}$ can be computed from curves in the parameter space $B := |\mathcal{L}^N|_{S}^{d-1}$: this means by definition

$$\text{Div}^0_{\mathcal{L}/X} = \bigcap_{C \in B} \left( \mathcal{C} \cdot \mathcal{C} \right)^{-1} \text{Div}^0_{\mathcal{C}/C}.$$

Proof. Cartier curves in $X$ are complete intersections locally in a neighbourhood of the singular locus $S$. For the computation of $\text{Div}^0_{\mathcal{L}/X}$ only the formal neighbourhood of $S$ is relevant, since $\text{Div}^0_{\mathcal{L}/X}$ is defined via the push-forward of formal divisors (see Definition 3.1 and its support is contained in $S_{\mathcal{L}} := S \times_X \mathcal{L}$). Thus we can replace the range of all Cartier curves by a set of complete intersection curves. As $\text{Div}^0_{\mathcal{L}/X}$ is dual-algebraic, there exists an effective divisor $E$ on $\mathcal{L}$ with support in $S_{\mathcal{L}}$ such that $\text{Div}^0_{\mathcal{L}/X} \subset \mathcal{F}_{\mathcal{L}/E}$ (see [Ru2, Prop. 3.21] or proof of [Ru1, Prop. 3.24]), where $\mathcal{F}_{\mathcal{L}/E}$ is the formal group associated with the modulus $E$ (see [Ru2, Def. 3.13]). If $m$ is the maximum multiplicity of $E$, then only the $(m-1)^{th}$-infinitesimal neighbourhood of $S$ is relevant. (This follows e.g. from [Ru1, Lem. 3.21].) Any Cartier curve $C \subset X$ can be approximated by a curve $C_N \in |\mathcal{L}^N|$ such that the $(m-1)^{th}$-infinitesimal neighbourhood of $S \cdot C$ coincides with the one of $S \cdot C_N$ for sufficiently large $N \in \mathbb{N}$. ■

Proposition 3.6. Let $\mathcal{L} \cong \mathbb{Z}^r \times_k (\mathcal{G}_a)^n$ be a formal subgroup of $\text{Div}^0_{\mathcal{L}/X}$ containing $\text{Div}^0_{\mathcal{L}/X}$, and let $B$ a parameter space of curves such that $\text{Div}^0_{\mathcal{L}/X}$ can be computed from $B$. Then there are finitely many curves $C_1, \ldots, C_r \in B$ such that

$$\text{Div}^0_{\mathcal{L}/X} = \bigcap_{i=1}^{r} \left( C_i \cdot C_i \right)^{-1} \text{Div}^0_{\mathcal{C}/C_i}.$$

Proof. For $C \in B$ set $\mathcal{F}_C := \left( C \cdot C \right)^{-1} \text{Div}^0_{\mathcal{C}/C}$. Then it holds $\text{Div}^0_{\mathcal{L}/X} = \bigcap_{C \in B} \mathcal{F}_C$. For every sequence $\{ C_i \}$ of curves in $B$ the sequence $\{ \mathcal{E}_i \}$ with
\( \mathcal{E}_0 := \mathcal{E}, \mathcal{E}_{i+1} := \mathcal{E}_i \cap F_{C_i} \) becomes stationary, due to the noetherian properties of the formal group \( \mathcal{E} \), cf. [Ru1] Remark 3.26.

**Corollary 3.7.** If \( C_i = H_1^{(i)} \cap \ldots \cap H_{d-1}^{(i)} \), then \( C_0 := \left( \sum \right. \left. H_1^{(i)} \right) \cap \ldots \cap \left( \sum \right. \left. H_{d-1}^{(i)} \right) \) satisfies:

\[
\text{Div}_X^{0} = \left( ? \cdot \mathcal{C}_0 \right)_{X/X}^{-1} \mathcal{C}_0/C_0
\]

**Theorem 3.8.** Let \( \mathcal{E} \cong \mathbb{Z}^t \times_k (\hat{G}_a)^v \) be a formal subgroup of \( \text{Div}_X^{0} \) such that \( \text{Div}_{X/X}^{0} \subset \mathcal{E} \). Let \( \mathcal{L} \) be an ample line bundle on \( X \). For \( C \in |\mathcal{L}^N|_S \) and sufficiently large \( N \) the property

\[
\text{Div}_{X/X}^{0} = \left( ? \cdot \mathcal{C}_0 \right)_{X/X}^{-1} \mathcal{C}_0/C_0
\]

is open and dense.

**Proof.** By Propositions 3.5, 3.6 and Corollary 3.7 there exist numbers \( r, N \in \mathbb{N} \) and a curve \( C_0 \in |\mathcal{L}^N|_S \) with \( \text{Div}_{X/X}^{0} = \left( ? \cdot \mathcal{C}_0 \right)_{X/X}^{-1} \mathcal{C}_0/C_0 \). Let \( C \) be a general curve in \( |\mathcal{L}^N|_S \). By definition of \( \text{Div}_{X/X}^{0} \) we have

\[
\left( ? \cdot \mathcal{C} \right)_{X/X}^{-1} \mathcal{C}_0/C_0 \supset \text{Div}_{X/X}^{0} = \left( ? \cdot \mathcal{C}_0 \right)_{X/X}^{-1} \mathcal{C}_0/C_0,
\]

i.e. \( \dim \left( \mathcal{E} \times \text{Div}_{C_0} \mathcal{C}_0/C_0 \right) \geq \dim \left( \mathcal{E} \times \text{Div}_{C_0} \mathcal{C}_0/C_0 \right) \). According to Proposition 3.4 the expression \( \dim \left( \mathcal{E} \times \text{Div}_{C_0} \mathcal{C}_0/C_0 \right) \), as a function in \( C \in B \), is upper semi-continuous. Thus a general curve \( C \in B \) satisfies

\[
\left( ? \cdot \mathcal{C} \right)_{X/X}^{-1} \mathcal{C}_0/C_0 = \left( ? \cdot \mathcal{C}_0 \right)_{X/X}^{-1} \mathcal{C}_0/C_0 = \text{Div}_{X/X}^{0}.
\]

\[\blacksquare\]

**4 Example: Product of two Cuspidal Curves**

We conclude this paper with the discussion of an example that was the subject of the diploma of Alexander Schwarzhaupt [Sch]. This example illustrates some pathological properties: the Albanese of Esnault-Srinivas-Viehweg is not in general compatible with products, in this example we obtain (writing \( \text{Alb}^{\text{ESV}}(X) := \text{Alb}(X, X_{\text{sing}}) \))

\[
\dim \left( \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta) \right) > \dim \left( \text{Alb}^{\text{ESV}}(\Gamma_\alpha) \times \text{Alb}^{\text{ESV}}(\Gamma_\beta) \right).
\]

Moreover, given a very ample line bundle \( \mathcal{L} \) on the surface \( X = \Gamma_\alpha \times \Gamma_\beta \) and a curve \( C_N \in |\mathcal{L}^N|_S \) in general position, we work out a necessary and sufficient condition on the integer \( N \) for the surjectivity of the Gysin map \( \text{Alb}^{\text{ESV}}(C_N) \longrightarrow \text{Alb}^{\text{ESV}}(X) \). The base field \( k \) is assumed to be algebraically closed and of characteristic 0.
4.1 Cuspidal Curve

Let $\Gamma_\alpha \subset \mathbb{P}_k^2$ be the projective curve defined by

$$\Gamma_\alpha : \quad X^{2\alpha + 1} - Y^2 Z^{2\alpha - 1} = 0$$

where $X : Y : Z$ are homogeneous coordinates of $\mathbb{P}_k^2$ and $\alpha \geq 1$ is an integer. The singularities of this curve are cusps at $0 := [0 : 0 : 1]$ and $\infty := [0 : 1 : 0]$. The normalization $\tilde{\Gamma}_\alpha$ of $\Gamma_\alpha$ is the projective line: $\tilde{\Gamma}_\alpha = \mathbb{P}_k^1$. Then $\text{Alb}(\tilde{\Gamma}_\alpha) = \text{Alb}(\mathbb{P}_k^1) = 0$. Since $\text{Alb}^{\text{ESV}}(\Gamma_\alpha)$ is an extension of $\text{Alb}(\tilde{\Gamma}_\alpha)$ by the linear group $L_{\Gamma_\alpha} = \left(\text{Div}^0_{\Gamma_\alpha/\Gamma_\alpha}\right)^\vee$, we obtain

$$\text{Alb}^{\text{ESV}}(\Gamma_\alpha) = L_{\Gamma_\alpha} = \left(\text{Div}^0_{\Gamma_\alpha/\Gamma_\alpha}\right)^\vee.$$ 

Moreover, $\Gamma_\alpha$ is homeomorphic to $\mathbb{P}_k^1$, i.e. the normalization $\tilde{\Gamma}_\alpha$ is given by the largest homeomorphic curve $\Gamma_\alpha$. This implies that $L_{\Gamma_\alpha}$ is a unipotent group (see Theorem [11]) and equivalently $\text{Div}^0_{\Gamma_\alpha/\Gamma_\alpha}$ is an infinitesimal formal group (see [Ru2] Proposition 1.17). We have $\text{Lie}\left(\text{Div}^0_{\Gamma_\alpha/\Gamma_\alpha}\right) = \text{Hom}_k\left(L_{\Gamma_\alpha}(k), k\right)$.

The $k$-valued points of $L_{\Gamma_\alpha}$ are given by (see Theorem [11])

$$L_{\Gamma_\alpha}(k) = \frac{1 + m_{\Gamma_\alpha,0}}{1 + m_{\Gamma_\alpha,0}} \times \frac{1 + m_{\Gamma_\alpha,\infty}}{1 + m_{\Gamma_\alpha,\infty}} \cong \frac{m_{\Gamma_\alpha,0}}{m_{\Gamma_\alpha,0}} \oplus \frac{m_{\Gamma_\alpha,\infty}}{m_{\Gamma_\alpha,\infty}}.$$

The dimensions are computed in [Sch] Proposition 1.5 as

$$\dim_k \left(\frac{m_{\Gamma_\alpha,0}}{m_{\Gamma_\alpha,\infty}}\right) = \alpha$$

$$\dim_k \left(\frac{m_{\Gamma_\alpha,\infty}}{m_{\Gamma_\alpha,0}}\right) = 2\alpha(\alpha - 1)$$

hence

**Proposition 4.1.**

$$\dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha) = \dim L_{\Gamma_\alpha} = \dim_k \text{Lie}\left(\text{Div}^0_{\Gamma_\alpha/\Gamma_\alpha}\right) = \alpha(2\alpha - 1).$$

As $\text{Lie}\left(\text{Div}^0_{\Gamma_\alpha/\Gamma_\alpha}\right) = \text{fm}^{-1} \bigoplus_{q = 0, \infty} \text{Hom}_k\left(\frac{m_{\Gamma_\alpha,q}}{m_{\Gamma_\alpha,\infty}}, k\right)$ it holds

$$\# v_q \left(\text{Lie}\left(\text{Div}^0_{\Gamma_\alpha/\Gamma_\alpha}\right)\right) = \# \left(v_q(O_{\tilde{\Gamma}_\alpha}) \cup v_q(O_{\Gamma_\alpha})\right) = \dim_k \left(\frac{m_{\Gamma_\alpha,q}}{m_{\Gamma_\alpha,\infty}}\right),$$

for $q \in \{0, \infty\} \subset \tilde{\Gamma}_\alpha$, cf. [Ru1] No. 3.3. A basis of $\text{Lie}\left(\text{Div}^0_{\Gamma_\alpha/\Gamma_\alpha}\right)$ is given by the following set of representatives

$$\Theta_{\Gamma_\alpha} = \left\{ t_q^{-\nu} \mid -\nu \in v_q \left(\text{Lie}\left(\text{Div}^0_{\Gamma_\alpha/\Gamma_\alpha}\right)\right), \quad q = 0, \infty \right\}$$

where $t_q$ is a local parameter of $m_{\Gamma_\alpha,q}$ at $q$ and $t_q^{-\nu} \in O_{\tilde{\Gamma}_\alpha,p}$ for all $p \neq q$.

4.2 Cuspidal Surface

Let $X$ be the product of the cuspidal curves $\Gamma_\alpha, \Gamma_\beta$ from Subsection 4.1

$$X = \Gamma_\alpha \times \Gamma_\beta.$$
where $\alpha, \beta \geq 1$ are integers. The singular locus of $X$ is
\[ X_{\text{sing}} = \{0\} \times \Gamma_\beta \cup (\{\infty\} \times \Gamma_\beta) \cup (\Gamma_\alpha \times \{0\}) \cup (\Gamma_\alpha \times \{\infty\}). \]

The normalization $\tilde{X}$ of $X$ is a resolution of singularities and given by $\tilde{X} = \tilde{\Gamma}_\alpha \times \tilde{\Gamma}_\beta = \mathbb{P}_k^1 \times \mathbb{P}_k^1$. Then $\text{Alb}(\tilde{X}) = \text{Alb}(\mathbb{P}_k^1) \times \text{Alb}(\mathbb{P}_k^1) = 0$. Thus the Albanese of Esnault-Srinivas-Viehweg of $X$ coincides with its affine part:
\[ \text{Alb}^{\text{ESV}}(X) = L_X = \left( \text{Div}^0_{\tilde{X}/X} \right)^\vee \]
and $\text{Div}^0_{\tilde{X}/X}$ is an infinitesimal formal group, since the normalization is a homeomorphism. The task is now to determine $\text{Div}^0_{\tilde{X}/X}$. The support of $\text{Div}^0_{\tilde{X}/X}$ is the preimage of $X_{\text{sing}}$:
\[ \text{Supp} \left( \text{Div}^0_{\tilde{X}/X} \right) = \{0\} \times \tilde{\Gamma}_\beta \cup (\{\infty\} \times \tilde{\Gamma}_\beta) \cup (\tilde{\Gamma}_\alpha \times \{0\}) \cup (\tilde{\Gamma}_\alpha \times \{\infty\}). \]

By Proposition 3.8 we can compute $\text{Div}^0_{\tilde{X}/X}$ by the preimage of $\text{Div}^0_{\tilde{C}/C}$ in $\text{Div}_{\tilde{X}}$ under pull-back to $\tilde{C}$, for a sufficiently ample curve $C \subset X$ in general position. We may take $C = \bigcup_{i=1}^k \{(p_i) \times \Gamma_\beta\} \cup \bigcup_{j=1}^l \{(\Gamma_\alpha \times \{q_j\}) \}$ for sufficiently many points $p_i \in \Gamma_\alpha \setminus \{0, \infty\}$ and $q_j \in \Gamma_\beta \setminus \{0, \infty\}$. Then $\text{Div}^0_{\tilde{C}/C} = \Pi_{i=1}^k \text{Div}^0_{\tilde{C}/\Gamma_\beta} \times \Pi_{j=1}^l \text{Div}^0_{\tilde{C}/\Gamma_\alpha}$. Now $\text{Lie} \left( \text{Div}^0_{\tilde{X}/X} \right)$ is the space of those $\delta \in \text{Lie} \left( \text{Div}^0_{\tilde{C}/C} \right)$ with support in the preimage of $X_{\text{sing}}$ such that $\delta \cdot \{(p) \times \tilde{\Gamma}_\beta\} \in \text{Lie} \left( \text{Div}^0_{\tilde{C}/\Gamma_\beta} \right)$ and $\delta \cdot (\tilde{\Gamma}_\alpha \times \{q\}) \in \text{Lie} \left( \text{Div}^0_{\tilde{C}/\Gamma_\alpha} \right)$ for all $p \in \tilde{\Gamma}_\alpha \setminus \{0, \infty\}$ and all $q \in \tilde{\Gamma}_\beta \setminus \{0, \infty\}$. From this we see: If $\Theta_{\Gamma_\beta}$ is a set of representatives of a basis of $\text{Lie} \left( \text{Div}^0_{\tilde{C}/\Gamma_\beta} \right)$ for $i = \alpha, \beta$, then
\[ \Theta_{\Gamma_\alpha \times \Gamma_\beta} = \left\{ \vartheta_{\Gamma_\alpha} \otimes \vartheta_{\Gamma_\beta}, | (\vartheta_{\Gamma_\alpha} \otimes \vartheta_{\Gamma_\beta}) \in \left( (\Theta_{\Gamma_\alpha} \cup \{1\}) \times (\Theta_{\Gamma_\beta} \cup \{1\}) \right) \setminus \{(1,1)\} \right\} \]
is a set of representatives of a basis of $\text{Lie} \left( \text{Div}^0_{\tilde{X}/X} \right)$.

Thus the dimension of $\text{Alb}^{\text{ESV}}(X)$ is given by
\[ \dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta) = \dim_k \text{Lie} \left( \text{Div}^0_{\Gamma_\alpha \times \Gamma_\beta} \right) \]
\[ = \left( \dim_k \text{Lie} \left( \text{Div}^0_{\Gamma_\alpha \times \Gamma_\beta} \right) + 1 \right) \cdot \left( \dim_k \text{Lie} \left( \text{Div}^0_{\Gamma_\alpha} \right) + 1 \right) = (\alpha(2\alpha - 1) + 1) \cdot (\beta(2\beta - 1) + 1) - 1. \]

With Proposition 4.11 this yields

\textbf{Proposition 4.2.}

$\dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta) = (\dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha) + 1) \cdot (\dim \text{Alb}^{\text{ESV}}(\Gamma_\beta) + 1) - 1$.

We obtain a basis of $\text{im} \left( \text{Lie} \left( \text{Div}^0_{\tilde{X}/X} \right) \to \left( \mathcal{K}_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}} \right)_{\Gamma_\alpha \times \{q\}} \right)$ for $q \in \{0, \infty\} \subset \tilde{\Gamma}_\beta$ from the following set of representatives
\[ \Theta_{\Gamma_\alpha \times \{q\}} = \left\{ \vartheta_{\Gamma_\alpha} \otimes t_q^{-\nu}, | \vartheta_{\Gamma_\alpha} \in \Theta_{\Gamma_\alpha} \cup \{1\}, -\nu \in v_q \left( \text{Lie} \left( \text{Div}^0_{\tilde{C}/\Gamma_\beta} \right) \right) \right\}. \]

Now $v_q \left( \text{Lie} \left( \text{Div}^0_{\tilde{C}/\Gamma_\beta} \right) \right) = v_{\Gamma_\alpha \times \{q\}} \left( \text{Lie} \left( \text{Div}^0_{\tilde{X}/X} \right) \right)$, therefore...
Proposition 4.3.

\[ \dim_k \left( \text{Lie} \left( \text{Div}_X^0(\Gamma_\alpha \times q) \right) \right) = \dim_k \text{Lie} \left( \text{Div}_X^0(\Gamma_\alpha) \right) + 1 \]

for \( q \in \{0, \infty\} \subset \Gamma_\beta \) and all \( -\nu \in \nu_{\Gamma_\alpha \times \{q\}} \left( \text{Lie} \left( \text{Div}_X^0(\Gamma_\alpha) \right) \right) \), where we use the notation from Lemma 3.2. Analogously for \( \{p\} \times \Gamma_\beta, p \in \{0, \infty\} \subset \Gamma_\alpha \).

4.3 Gysin Map

Consider the following divisor \( D^{k,l} \) on \( X \)

\[ D^{k,l} = \sum_{i=1}^{k} \{p_i \times \Gamma_\beta\} + \sum_{j=1}^{l} \{\Gamma_\alpha \times \{q_j\}\} \]

where \( p_i \in \Gamma_\alpha \setminus \{0, \infty\} \) for \( i = 1, \ldots, k \) and \( q_j \in \Gamma_\beta \setminus \{0, \infty\} \) for \( j = 1, \ldots, l \). The normalization of \( D^{k,l} \) is isomorphic to the disjoint union of \( k + l \) copies of \( \mathbb{P}^1 \). Therefore the Picard variety of the normalization \( \text{Pic}^0 \, D^{k,l} \) is trivial. Then by Theorem 1.6 using the explicit formulas of Propositions 1.4 and 1.5

\[ \text{Pic}^0 \, D^{k,l} = T \times V \]

where \( T \cong (\mathbb{G}_m)^t \) is a torus of rank

\[ t = \#S_2 - \#C_\beta(D^{k,l}) + 1 = k \cdot l - (k + l) + 1 = (k - 1) \cdot (l - 1), \]

and \( V \cong (\mathbb{G}_a)^v \) is a vectorial group of dimension

\[ v = k \cdot \dim L_{\Gamma_\alpha} + l \cdot \dim L_{\Gamma_\beta} = k \cdot \beta(2\beta - 1) + l \cdot \alpha(2\alpha - 1). \]

For general \( p_i \in \Gamma_\alpha \) and \( q_j \in \Gamma_\beta \) the divisor \( D^{2\alpha+1,2\beta+1} \) is very ample (see \cite{Sch} Lemma 3.2). Set \( \mathcal{L} = \mathcal{O}(D^{2\alpha+1,2\beta+1}) \) and choose \( C_N \in |\mathcal{L}^N| \) in general position. As \( \dim \text{Pic}^0 C_N = \text{const.} \) and \( \dim \text{Pic}^0(C_N \times_X \tilde{X}) = \text{const.} \), \( C_N \in |\mathcal{L}^N| \) and \( C_N \times_X \tilde{X} = C_N' \) is the semi-normalization, the dimension of the vectorial part \( V_{C_N} = \ker(\text{Pic}^0 C_N \to \text{Pic}^0 C_N') \) of \( \text{Pic}^0 C_N = \text{Alb}^{\text{ESV}}(C_N) \) is constant among \( C_N \in |\mathcal{L}^N| \) and hence

\[ \dim V_{C_N} = \dim V_{C_N} D^{2\alpha+1,2\beta+1} = N(\alpha(\alpha(2\alpha - 1)(2\beta + 1) + \beta(2\beta - 1)(2\alpha + 1)). \]

Since \( \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta) \) is a vectorial group, the map \( \text{Alb}^{\text{ESV}}(C_N) \to \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta) \) cannot be surjective if \( \dim V_{C_N} < \dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta) \).

Therefore a comparison of dimensions yields:

Proposition 4.4. The Gysin map \( \text{Alb}^{\text{ESV}}(C_N) \to \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta) \) is not surjective for

\[ N < \frac{(\alpha(2\alpha - 1) + 1) \cdot (\beta(2\beta - 1) + 1) - 1}{\alpha(2\alpha - 1)(2\beta + 1) + \beta(2\beta - 1)(2\alpha + 1)} \]

In the case \( \alpha = \beta \), this expression simplifies to

\[ N < \frac{\alpha(2\alpha - 1) + 2}{2(2\alpha + 1)}. \]
The homomorphism of vectorial groups $V_{C_0} \to V_X = \text{Alb}_{\text{ESV}}(X)$ is dual to the map between Lie algebras $\beta \cdot \tilde{C}_N : \text{Lie} \left( \text{Div}^0_{\tilde{\chi}/X} \right) \to \text{Lie} \left( \text{Div}^0_{\tilde{C}/C} \right)$, and the surjectivity of the first homomorphism is equivalent to the injectivity of the latter one. Here Definition 3.1 of $\text{Div}^0_{\tilde{\chi}/X}$ implies immediately that the image of $\text{Div}^0_{\tilde{\chi}/X}$ under pull-back $\beta \cdot \tilde{C}_N : \text{Div}^0_{\tilde{\chi}/X} \to \text{Div}^0_{\tilde{C}/C}$ is contained in $\text{Div}^0_{\tilde{C}/C}$.

The estimation of Proposition 4.4 yields a necessary condition for surjectivity of the Gysin map, i.e. a bound for $N$ from below.

The criterion of Lemma 2.2 gives a sufficient condition for the surjectivity of the Gysin map: $\# \left( \tilde{C}_N \cap (\tilde{\Gamma}_\alpha \times \{q\}) \right) \geq \dim_k \left( \text{Lie} \left( \text{Div}^0_{\tilde{\chi}/X} \right) / \tilde{\Gamma}_\alpha \times \{q\} \right)^{-\nu}$ for all $\nu \in v_{\tilde{\Gamma}_\alpha \times \{q\}}(\text{Div}^0_{\tilde{\chi}/X})$, all $q \in \{0, \infty\} \subset \tilde{\Gamma}_\beta$, and the same formula with $(\tilde{\Gamma}_\alpha \times \{q\})$ replaced by $(\{p\} \times \tilde{\Gamma}_\beta)$ for all $p \in \{0, \infty\} \subset \tilde{\Gamma}_\alpha$. Since $\dim_k \left( \text{Lie} \left( \text{Div}^0_{\tilde{\chi}/X} \right) / \tilde{\Gamma}_\alpha \times \{q\} \right) = \dim_k \text{Lie} \left( \text{Div}^0_{\tilde{\chi}/X} \right) / \tilde{\Gamma}_\alpha + 1$ (see Proposition 4.3) this is equivalent to $\# \left( \tilde{C}_N \cap (\tilde{\Gamma}_\alpha \times \{q\}) \right) \geq \dim_k \text{Lie} \left( \text{Div}^0_{\tilde{\chi}/X} \right) / \tilde{\Gamma}_\alpha + 1$.

Then since $\# \left( \tilde{C}_N \cap (\tilde{\Gamma}_\alpha \times \{q\}) \right) = N \deg (D^{2\alpha+1,2\beta+1})_X = N(2\alpha + 1 + 2\beta + 1) = N(2\alpha + \beta + 1)$, where $(D^{k,l})_X = \sum_{i=1}^k (\{p_i\} \times \tilde{\Gamma}_\beta) + \sum_{j=1}^l (\tilde{\Gamma}_\alpha \times \{q_j\})$, we obtain with Proposition 4.4

**Proposition 4.5.** The Gysin map $\text{Alb}_{\text{ESV}}(C_N) \to \text{Alb}_{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta)$ is surjective if

$$N \geq \frac{\alpha(2\alpha - 1) + 1}{2(\alpha + \beta + 1)} \quad \text{and} \quad N \geq \frac{\beta(2\beta - 1) + 1}{2(\alpha + \beta + 1)}.$$

For $\alpha = \beta$ this condition is necessary and sufficient.

**Proof.** The first statement follows from the discussion above. In the case $\alpha = \beta$ we need to show that the estimation above and the formula from Proposition 4.4 yield the same bound for $N \in \mathbb{N}$. As the difference is $\frac{1}{2(2\alpha + 1)} < 1$, it suffices to check that $2(2\alpha + 1)$ does not divide $\alpha(2\alpha - 1) + 1 = \alpha(2\alpha + 1) - (2\alpha - 1)$, which is obvious.

In [ESV] Variant 6.4 the following sufficient condition for surjectivity of the Gysin map is given:

$$\dim_k \text{im} \left( H^0(X, \mathcal{L}^N) \to H^0(Z, \mathcal{L}^N|_Z) \right) \geq 2 \dim L_{C_N} + \# \text{Cp}(X) + 2$$

for all $Z \in \text{Cp}(X)$, where $L_{C_N}$ is the largest connected affine subgroup of $\text{Pic}^0C_N$ for $C_N \in \mathcal{L}^N$ in general position. For $X = \Gamma_\alpha \times \Gamma_\beta$ it holds $\text{Cp}(X) = \{X\}$ and $\text{Pic}^0C_N = L_{C_N} = V_{C_N}$. Alexander Schwarzhaupt showed in his diploma [Sch] that in our example and for $\alpha = \beta$ this condition leads to the estimation

$$N > \frac{3(2\alpha - 1) + 1}{2(\alpha + 1)} + 1.$$

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