A PROOF OF LEE-LEE’S CONJECTURE ABOUT GEOMETRY OF RIGID MODULES

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Abstract. This paper proves Lee-Lee’s conjecture that establishes a coincidence between the set of associated roots of non-self-intersecting curves in a $n$-punctured disc and the set of real Schur roots of acyclic (valued) quivers with $n$ vertices.

1. Introduction

Given an algebraically closed field $F$ and an acyclic quiver $Q$ with $n$ vertices, an indecomposable module $M$ is called by rigid if $Ext^1_{FQ}(M, M) = 0$ over the path algebra $FQ$. It is well-known that from the quiver $Q$ one may construct the irreducible symmetric Cartan matrix companion $A = A(Q)$, the root system $\Phi = \Phi(A)$ and the Coxeter-Weyl groups $G_n = G_n(A)$ with ordered simple reflections $\{s_1, \ldots, s_n\}$. In the root systems, positive real Schur roots are dimension vectors of indecomposable rigid modules. Such rigid modules and their real Schur roots play very important roles in representation theory and cluster algebras, see [2], [6], [8], [9], [10], [16] just to name. Hence it leads to necessarily characterize them among all real roots of $\Phi$. Remark that the real Schur roots depend on the orientation of the quiver $Q$, which makes it quite difficult to characterize them. In [11], K. H. Lee and K. Lee suggested a conjecture about their geometric property.

Conjecture ([11]). The real roots assigned by simple curves are precisely real Schur roots. (defined in Section 2).

In the paper, they proved it for acyclic quivers of 3 vertices with multiple arrows between every pair of vertices. When the paper was published on arxiv, A. Felikson and P. Tumarkin [2] proved it for all acyclic quivers of finite vertices with multiple arrows between every pair of vertices (called by 2-complete acyclic quivers). In the paper, we show the solution of the conjecture for all acyclic (valued) quivers of finite vertices.

Before going to the proof, we importantly need to reform the conjecture in another version. Given a quiver $Q$, we have a bijection between the orientation of the quiver and an admissible-sink (or admissible-source) order of its Coxeter element $c := s_1s_2...s_n$. Hence without any risk, in the paper we can fix the quiver $Q$ such that its Coxeter element is $c = s_1s_2...s_n$. In [10], Igusa and Schiffler showed that reflections corresponding real Schur roots are precisely prefix reflections in the factorization of the Coxeter element $c$. In [1], Hubery and Krause gave a characterization of real Schur roots in terms of simple partition. Their reflections...
are reflection elements belonging to the poset of generalized simple partitions \( NC = NC(Q) := \{ w \in G_n \mid 1 \leq w \leq c \} \) where \( \leq \) denotes the absolute order on \( G_n \).

In more general cases for a valued quiver \( Q \), given \( A \) be an irreducible symmetrizable generalized Cartan matrix with its symmetrizer \( D \) and orientation \( O \), in [5] Geiß, Leclerc and Schröer define an Iwanaga-Gorenstein \( \mathbb{F} \)-algebra \( H := H_{\mathbb{F}}(A, D, O) \) for any field \( \mathbb{F} \) in terms of a quiver with relations with a hereditary algebra \( \tilde{H} \) of the corresponding type and a Noetherian \( \mathbb{F}[c] \)-algebra \( \hat{H} \). In particular, in [4], their main results showed that the indecomposable rigid locally free \( H \)-modules are parametrized via their rank vector, by the real Schur roots associated to \( (A, O) \). Moreover, the left finite bricks of \( H \) are parametrized via their dimension vector by the real Schur roots associated to \( (C^T, O) \). Also as the symmetric case of acyclic quivers \( Q \), real Schur roots were proven to be real roots in its associated root system such that their reflections are reflection elements of the simple partition \( NC \) in its associated Weyl group. From the observations, Lee-Lee’s conjecture may be reformed as follow:

**Conjecture** ([11]) Three statements are equivalent:

1. A positive real root \( \beta \) is Schur.
2. Its reflection \( r_\beta \) is a prefix of the Coxeter element \( c \) i.e. there exists \( n-1 \) reflections \( r_2, r_3, ..., r_n \) such that \( r_\beta r_2 ..., r_n = c \).
3. The real root \( \beta \) may be presented by a simple curve and its reflection may be presented by a simple closed curve.

where [10] and [4] proved (1) \( \Leftrightarrow \) (2), so we only need to show (2) \( \Rightarrow \) (3).

Section 2 is devoted to recall the reader the constructions used before giving the proof of Lee-Lee’s conjecture in section 3. In section 4, we present another proof on finite types that also implies an one-side proof for affine types. We also give some other results and open problems that may be completed in the future.

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## 2. Reminders

Given an irreducible symmetrizable Cartan matrix \( A = (a_{ij}) = M_{n \times n}(\mathbb{Z}) \), it is well-known that one may construct its associated root system \( \Phi = \Phi(A) \) as follows.

We fix its simple roots \( \{ \alpha_1, ..., \alpha_n \} \) and define its simple reflections \( \{ s_1, ..., s_n \} \) by

\[
\begin{align*}
  s_i(\alpha_j) &= \alpha_j - a_{ij}\alpha_i, \quad i, j = 1, ..., n.
\end{align*}
\]

These reflections generates a group \( G_n = G_n(A) = \langle s_1, ..., s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle \)

called by the Coxeter/Weyl group where \( m_{ij} \) are defined from the table:

\[
\begin{array}{c|cccccccc}
  a_{ij} & 0 & 1 & 2 & 3 & \geq 4 \\
  m_{ij} & 2 & 3 & 4 & 6 & \infty \\
\end{array}
\]

Let \( B_n \) be the braid group on \( n \) strands and abstractly presented

\[
B_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \rangle
\]

with the action on \( n \) copies of an arbitrary group \( H \) as follow:

\[
\begin{align*}
  \sigma_i (g_1, ..., g_i, g_{i+1}, ..., g_n) &= (g_1, ..., g_i g_{i+1} g_i^{-1}, g_i, ..., g_n), \\
  \sigma_i^{-1} (g_1, ..., g_i, g_{i+1}, ..., g_n) &= (g_1, ..., g_i, g_{i+1}^{-1} g_i g_{i+1}, ..., g_n)
\end{align*}
\]
for any sequence \((g_1, \ldots, g_n) \in H^n\). It is clear that the action fixes the product \(g_1g_2\ldots g_n\). This implies that the position of reflections appearing in the factorization of the Coxeter element of \(G_n\) may be ignored by the following lemma.

**Lemma 2.1.** Assume that a reflection \(r_\beta\) belonging a factorization of the Coxeter element \(c\), then it is a prefix of \(c\) i.e. there exists \(n - 1\) reflections \(r_2, r_3, \ldots, r_n\) such that \(r_\beta r_2, r_3, \ldots, r_n = c\).

**Proof.** The smallest number of factorization of \(c\) into reflections is \(n\) (see [10]). Assume that \(c = r_1 r_2 \ldots r_n\) and \(r_j = r_\beta\) for some \(2 \leq j \leq n\), then \(r_\beta\) is a prefix of \(\sigma_1^{-1} \ldots \sigma_j^{-1} (r_1, r_2, \ldots, r_n)\). The action of \(B_n\) keeps the product invariant, hence the proof is completed.

Now we will present real roots as curves and their reflections as closed curves of by the following construction of D. Bessis in [5]. For convenience we will draw the disc \(D \subset \mathbb{C}\) as an upper half-plane \(\{z \in \mathbb{C} \mid \text{Im} z > 0\}\) with the punctured point \(p_i\) placed from left to right on the horizontal line \(\text{Im} z = 1\) and the point \(\infty\) at infinity. Denote by \(\ell_i\) a vertical ray \(\ell_i = \{z \mid \text{Re} z = \text{Re} p_i, \text{Im} z > \text{Im} p_i\}\).

**Definition 2.2.** A curve \(\gamma\) is a continuous map \(\gamma : [0, 1] \to D\) such that \(\gamma(0) = O, \gamma(1) \in \{p_1, p_2, \ldots, p_n\}\) and \(\gamma(t) \notin \{O, p_1, p_2, \ldots, p_n\}\) for \(t \in (0, 1)\).

A simple curve is a non-self-intersecting curve.

A closed curve \(\hat{\lambda}\) is a continuous map \(\hat{\lambda} : [0, 1] \to D\) such that \(\gamma(0) = \gamma(1) = O\) and \(\gamma(t) \notin \{O, p_1, p_2, \ldots, p_n\}\) for \(t \in (0, 1)\).

A non-self-intersecting closed curve is called by a simple loop.

Denote \(\text{Ins}(\hat{\lambda})\) be the interior region bounded by the closed curve \(\hat{\lambda}\).

**Remark 2.3.** One may associate such a curve \(\gamma\) uniquely (up to isotopy) to a closed curve \(\hat{\gamma}\) as follows: starting from \(\gamma(0) = O\), follow \(\gamma\); arriving close to \(\gamma(1)\), make a possible turn around a small circle centred on \(\gamma(1)\); return to \(O\) following \(\gamma\) backwards. Thus a simple curve associates to a simple loop and so the set of simple curves may inject as a subset of the set of simple loops.

**Definition 2.4.** Given a closed curve \(\hat{\lambda}\), then the word \(w_\hat{\lambda}\) presents an element of \(G_n\) constructed as follows: following \(\hat{\lambda}\), write \(s_j\) each time it crosses some rays \(\ell_j\).

**Definition 2.5.** Given a curve \(\gamma\) with \(\gamma(1) = p_i\) and its closed curve \(\hat{\gamma}\), then the word \(w_\gamma\) presents a real root constructed as follows: following \(\gamma\), write \(s_j\) each time it crosses some rays \(\ell_j\) and add \(\alpha_i\) as last word. It is clear that the word \(w_\gamma\) presents its reflection, denoted by \(r_\gamma\).

From the definition of curves and closed curves, now we may identify them and their words that present real roots and elements in \(G_n\). Since all real roots and elements in \(G_n\) may be presented by words, the set of real roots is the set of curves (up to isotopy) and \(G_n\) is the set of closed curves (up to isotopy). Note that two curves may be isotopic together but the former presents a positive real root \(\alpha\), the latter presents its negative real root \(-\alpha\) as the following example.

**Example 2.6.** In the example of \(G_3\), the simple red loop \(\hat{\lambda}\) induces the word \(s_1 s_2 s_3\) of the Coxeter element \(c\). In particular, the loop is isotopic to the boundary \(\partial D = S^1\) of the disc \(D\). The simple dark curve \(\gamma\) presents a positive real root \(\alpha\) and its simple green loop \(\hat{\gamma}\) presents a reflection \(r_\gamma = s_2 s_3 s_2\). From the curve \(\gamma\) we may draw another simple curve \(\beta\) that is isotopic with \(\gamma\) by going
around $p_3$ one time before ending at $p_3$. Then $\beta$ presents a negative real root $\alpha_\beta = s_2s_3\alpha_3 = -\alpha_\gamma$.

A real root of $\Phi$ and an element of $G_n$ may be presented by many different words i.e. curves and closed curves up to isotopy. However, if $G_n = W_n := \{s_1, \ldots, s_n \mid s_1^2 = \ldots = s_n^2 = 1\}$, called by universal Coxeter groups, then real roots of $\Phi$ and elements of $G_n$ are determined uniquely by their words because Cayley graphs of $W_n$ are trees. Hence real roots and elements of $G_n$ are presented uniquely by curves and closed curves. Now we prove main results that lead to the Lee-Lee’s conjecture.

**Lemma 2.7.** Given two simple curves with distinct end points such that they do not intersect each other except for $O$. We order them in a clockwise order of their emanating from $O$, written by the sequence $(\gamma_1, \gamma_2)$. Then

1. The product $r_{\gamma_1}r_{\gamma_2}$ may be presented by a simple loop $\hat{\gamma}$ such that two simple loops $\hat{\gamma}_1, \hat{\gamma}_2 \subset \text{Ins}(\hat{\gamma})$.

2. Braid group $B_2$ acts on $(r_{\gamma_1}, r_{\gamma_2})$ to be $\sigma_1.(r_{\gamma_1}, r_{\gamma_2}) = (r_{\gamma_1}r_{\gamma_2}r_{\gamma_1}, r_{\gamma_1})$ (similarly for $\sigma_1^{-1}.(r_{\gamma_1}, r_{\gamma_2}) = (r_{\gamma_2}r_{\gamma_1}r_{\gamma_2}, r_{\gamma_1})$), then it is possible to construct a simple curve $\gamma$ presenting the real root $r_{\gamma_1}\alpha_{\gamma_2}$ (thus the simple loop $\hat{\gamma}$ presenting $r_{\gamma_1}r_{\gamma_2}r_{\gamma_1}$) such that $\gamma$ does not intersect with $\gamma_2$ (thus $\hat{\gamma}$ does not intersect with $\hat{\gamma}_2$) and $\gamma(1) = \gamma_2(1)$. Hence braid group action preserves non-intersecting property of simple curves and loops.

**Proof.** For (1), since two simple curves $(\gamma_1, \gamma_2)$ do not intersect each other, neither do their simple loops $(\hat{\gamma}_1, \hat{\gamma}_2)$. Therefore we can connect them in a small neighbor at $O$ to become the loop $\hat{\gamma}$ qualified as the follow example.
For (2), we present \((r_{\hat{\gamma}_1}, r_{\hat{\gamma}_2})\) from \((r_{\hat{\gamma}_1}, r_{\hat{\gamma}_2})\) as follows: in a small neighbor of \(O\) we connect the loop \(\hat{\gamma}_1\) with the curve \(\gamma_2\), then we get a new simple curve \(\gamma'\) presenting the real roots \(r_{\hat{\gamma}_1}, r_{\hat{\gamma}_2}\) as qualified and its corresponding simple loop \(\hat{\gamma}'\) presents \(r_{\hat{\gamma}_1}r_{\hat{\gamma}_2}r_{\hat{\gamma}_1}\). Finally, we reorder two curve \(\gamma_1\) and \(\gamma'\) in clock wise order of emanating from \(O\), then we have the sequence \((\gamma', \hat{\gamma}_1)\) presenting for \((r_{\hat{\gamma}_1}, r_{\hat{\gamma}_2}r_{\hat{\gamma}_1}, r_{\hat{\gamma}_1})\). The remaining part \((r_{\hat{\gamma}_2}, r_{\hat{\gamma}_2}r_{\hat{\gamma}_1}r_{\hat{\gamma}_2})\) is similar. ■

**Corollary 2.8.** Given \(n\) simple curves with distinct end points such that their pairs do not intersect each other except for \(O\). We order them in a clock wise order of emanating from \(O\), possibly written by the sequence \((\gamma_1, ..., \gamma_n)\). Then

1. All products \(r_{\hat{\gamma}_i}...r_{\hat{\gamma}_j}, 1 \leq i \leq j \leq n\) may be presented by loops \(\hat{\gamma}_{ij}\) such that \(j-i+1\) simple loops \(\hat{\gamma}_1, ..., \hat{\gamma}_j \subset \text{Ins}(\hat{\gamma}_{ij})\) and \(\text{Ins}(\hat{\gamma}_{ij}) \subset \text{Ins}(\hat{\gamma}_{im})\) for \(1 \leq i \leq j \leq m \leq n\).

   In particular, the loop \(\hat{\gamma}_{1n}\) presents the product \(r_{\hat{\gamma}_1}...r_{\hat{\gamma}_n}\), all simple loops \(\hat{\gamma}_1, ..., \hat{\gamma}_m \subset \text{Ins}(\hat{\gamma}_{1n})\) and \(\text{Ins}(\hat{\gamma}_{ij}) \subset \text{Ins}(\hat{\gamma}_{1n})\) for \(1 \leq i \leq j \leq n\).

2. Braid group \(B_n\) acting on \((r_{\hat{\gamma}_1}, ..., r_{\hat{\gamma}_n})\) preserves non-intersecting property of simple curves and loops.

**Proof.** For (1), the proof is similar with (1) of Lemma 2.7 by applying it for \((\gamma_i, \gamma_{i+1}), 1 \leq i \leq j-1 \leq n\). For (2), we only need to check actions of \(\sigma_i\), \(1 \leq i \leq n-1\) on \((r_{\hat{\gamma}_1}, ..., r_{\hat{\gamma}_n})\). But the actions only locally transform two loops in the sequence and fix the remaining \(n-2\) loops, so its proof is completed from (2) of Lemma 2.7. ■
Proposition 2.9. The simple loop $\hat{\gamma}_{1n}$ is isotopic with the simple loop presenting the Coxeter element $c = s_1...s_n$. Hence $r_{\hat{\gamma}_1}...r_{\hat{\gamma}_n} = c$.

Proof. The idea of the proof is the same as Lemma 2.3 and 2.4 in [5]. Corollary 2.8 implies that $\text{Ins}(\hat{\gamma}_{1n})$ contains all punctured points $\{p_1,...,p_n\}$ and presents their product $r_{\hat{\gamma}_1}...r_{\hat{\gamma}_n}$. In the other hand, $c$ may be presented by a loop $\hat{\gamma}_c$ with the exact word that contains $\{p_1,...,p_n\}$ and it is isotopic with $\partial D = S^1$. Thus it may be chosen in its isotopic class such that $\text{Ins}(\hat{\gamma}_{1n}) \subset \text{Ins}(c)$. Since the annulus between $\hat{\gamma}_c$ and $\hat{\gamma}_{1n}$ contains no punched point in $\{p_1,...,p_n\}$, they are isotopic and so $r_{\hat{\gamma}_1}...r_{\hat{\gamma}_n} = c$. 

3. Proof of Lee-Lee’s conjecture

Theorem 3.1. Three statements are equivalent:

(1) A real root $\beta$ is Schur.

(2) Its reflection $r_\beta$ is a prefix of the Coxeter element $c$ i.e. there exists $n-1$ reflections $r_2,r_3,...,r_n$ such that $r_\beta r_2,...,r_n = c$.

(3) The real root $\beta$ may be presented by a simple curve and its reflection may be presented by a simple closed curve.

where [10] and [4] proved (1) ⇔ (2), so we only need to prove (2) ⇔ (3).

Proof. A real Schur root $\alpha$ may be presented by a simple curve. Indeed, in [10] and [4], a real root $\alpha$ is Schur if and only if its reflection $r_\alpha$ is a prefix of the Coxeter element $c$. Therefore it may be induced in a reflection sequence $(r_1,...,r_n)$ where $r_\alpha = r_j$ with some $j$ such that $r_1...r_n = c$. Since the braid group $B_n$ acts transitively on the factorization of $c$ (see [10]), there exists $\sigma \in B_n$ such that $\sigma(s_1,...,s_n) = (r_1,...,r_n)$. It is clear that $(s_1,...,s_n)$ can be presented by $n$ simple loops without pairwise intersection each other. But Corollary 2.8 shows that the action of $B_n$ preserves non-intersecting property of $n$ simple loops presenting them, so the real Schur root $\alpha$ may be presented by a simple curve.

Conversely, a simple curve can be induced in $n$ simple curves with no pairwise intersection because of induction on $n$ by cutting the disc $D$ along the curve giving rise to an $n-1$-punctured disc. We order them in a clock wise order of emanating from $O$, then Proposition 2.9 shows that the product of their reflections is the Coxeter element $c$. Thus roots corresponding reflections of these simple curve are Schur roots because of [10] and [4].

Corollary 3.2. In the case $G_n = W_n := \langle s_1,...,s_n \mid s_1^2 = ... = s_n^2 \rangle$, we have a bijective correspondence between simple curves (up to isotopy) and positive real Schur roots.

Proof. In the case, real roots and reflections are uniquely presented by their reduced words, thus non-isotopic curves present distinct real roots.

This is the same result obtained in [2].

Now we let $\Phi^{NC}$ be the set of simple loops and define an equivalence $\sim$ as follows: $\hat{\gamma} \sim \hat{\lambda}$ if they present the same elements in $G_n$ and denote by $\hat{\Phi}^{NC} := \Phi^{NC} / \sim$.

Remark that any two isotopic loops present the same elements but conversely, it is not true because a reflection may be presented by many non-isotopic loops.

Definition 3.3. For $\forall \hat{\gamma}, \hat{\lambda} \in \hat{\Phi}^{NC}$ a partial order $\subseteq$ in $\hat{\Phi}^{NC}$ is defined by

$\hat{\gamma} \subseteq \hat{\lambda} \iff \hat{\gamma}, \hat{\lambda}$ may be chosen in their equivalent class such that $\text{Ins}(\hat{\gamma}) \subseteq \text{Ins}(\hat{\lambda})$.
and number of punctured points in $\text{Ins}(\hat{\gamma})$ is fewer than in $\text{Ins}(\hat{\lambda})$.

The absolute length $l(w)$ of $w \in G_n$ is the minimal $k \geq 0$ such that $w$ can be written as product $w = t_1t_2...t_r$ of reflections $t_i \in G_n$.

**Definition 3.4.** For $\forall w, u \in G_n$ an absolute order $\leq$ on $NC$ is defined by

$$w \leq u \iff l(w) + l(w^{-1}u) = l(u).$$

Recall the simple partition $NC := \{ w \in G_n | 1 \leq w \leq c \}$.

**Theorem 3.5.** We have an order-preserving isomorphism between $(\hat{\Phi}^{NC}, \leq)$ and $(NC, \leq)$.

**Proof.** Bijection between $\hat{\Phi}^{NC}$ and $NC$ is trivial from definition of $\hat{\Phi}^{NC}$. Given $\hat{\gamma}, \hat{\lambda} \in \hat{\Phi}^{NC}$ such that $\hat{\gamma} \leq \hat{\lambda}$ and $w_{\gamma}, w_{\lambda} \in G_n$ are elements that they present for, respectively. We may assume that all punctured points in $\text{Ins}(\hat{\lambda})$ are $\{p_1, \ldots, p_k, p_{k+1}, \ldots, p_m\}$ and all punctured points in $\text{Ins}(\hat{\gamma})$ are $\{p_1, \ldots, p_t\}$ for $1 \leq i_1 < i_k < i_m \leq n$. In the annulus between $\hat{\gamma}$ and $\hat{\lambda}$ we draw $m - k$ simple loops $\hat{\beta}_{i_j}$ such that each loop contains exactly one punctured point $p_{i_j}$ (thus each one presents a reflection $r_{i_j}$) with $k + 1 \leq j \leq m$, they are pairwise non-intersecting, and they do not intersect with $\hat{\lambda}$. We order $m - k + 1$ loops $\{\hat{\gamma}_1, \hat{\beta}_{i_{k+1}}, \ldots, \hat{\beta}_{i_{m}}\}$ in a clockwise order of their eliminating from $O$, then the product of their corresponding elements is $w_{\lambda}$. Hence $w_{\gamma} \leq w_{\lambda}$. Similarly, $w_{\lambda} \leq c$. Conversely, given $1 \leq u \leq w \leq c$, then non-crossing partition $NC$ implies that there are $n$ reflections $r_{i_1}, \ldots, r_{i_n}$ such that $r_{i_j}r_{i_{j+1}}\ldots r_{i_{m}} = u$, $r_{i_j}r_{i_{j+1}}\ldots r_{i_{m}}r_{i_{k+1}}\ldots r_{i_m} = w$ and $r_{i_1}r_{i_2}\ldots r_{i_n} = c$ for $1 \leq j \leq k \leq m \leq n$. From Corollary 2.8, two simple loops $\hat{\gamma}, \hat{\lambda} \in \hat{\Phi}^{NC}$ presenting $u, w$ may be chosen such that $\hat{\gamma} \leq \hat{\lambda}$. ■

Denote $s(c)$ and $t(c)$ be first and last simple reflections of $c$, respectively. Let $s(c)(Q)$ be a new quiver obtained by conversing arrows adjacent to the vertex corresponding to $s(c)$ (similarly for $t(c)(Q)$). The new quivers obtained by this approach correspond to mutation of quivers (see [2], [14]) at sink-source vertices corresponding to $s(c)$ and $t(c)$ in the theory of quiver representations. These mutations maintain root systems of the quivers but change their orientation, thus change their set of real Schur roots. However, the set of real Schur roots of the new quivers and their corresponding simple curves may be obtained by the following proposition.

**Proposition 3.6.** A real root $\beta$ is Schur in the quiver $Q$ if and only if the real root $s(c)\beta$ is Schur in the quiver $s(c)(Q)$ (similarly for $t(c)(Q)$).

The word $w_\beta$ is obtained from a simple curve $c_\beta$ in the setting of the quiver $Q$. Then a simple curve presenting the real Schur root $s(c)\beta$ in the setting of the quiver $s(c)(Q)$ may be constructed from the word $s(c)w_\beta$ (similarly for $t(c)(Q)$).

**Proof.** Since $\beta$ is a real Schur root, there exists $n - 1$ reflections $r_1, \ldots, r_{n-1}$ such that $r_\beta r_1 \ldots r_{n-1} = c$, thus $(s(c)r_\beta s(c))(s(c)r_1 s(c)) \ldots (s(c)r_{n-1} s(c)) = (s(c) c e(c))$. The right hand side is the Coxeter element corresponding of the new quiver $s(c)(Q)$ so the first part of the proposition is proved. For the latter part, we may consider it when the Weyl group of the quiver $Q$ is an universal Coxeter group. In the case, all reduced words obtained from simple curves are unique, thus simple curve presenting for $s(c)\beta$ has to be constructed from the word $s(c)\beta$. ■
Example 3.7. We give an example for a rank-3 quiver with its Coxeter element $c := s_1 s_2 s_3$, so $s(c) = s_1$ and $t(c) = s_3$. A real Schur root $\beta := s_2 s_3 \alpha_2$ is presented by an simple curve $\gamma_\beta$. Mutating the quiver at the vertex 1 corresponding to $s(c)$ we obtain the new quiver $s(c)(Q)$ with its Coxeter element $\tilde{c} := s_2 s_3 s_1 = s(c)c s(c)$.

4. Some Remarks on Finite, Affine and rank-2 Types

In the section, we give another proof for the Lee-Lee’s conjecture in the case of finite and affine types. It also yields an algorithm to construct simple curves for all real Schur roots in finite, affine and rank-2 types. Assume $G_n$ is a Weyl group of finite types with the Dynkin diagram corresponding $\Delta$.

Lemma 4.1. Given a simple curve $\gamma$ with its real root $\alpha_\gamma$ and its reflection $r_\gamma$, then the real roots $c\alpha_\gamma$ and $c^{-1}\alpha_\gamma$ may be presented by simple curves with the exact words. Hence the action of the Coxeter element $c$ preserves non-self-intersecting property of simple curves. This yields the equivalence between $c$-orbit of simple curves and $c$-orbit of their real roots.

Proof. The proof is straight-forward from spiraling the simple curves clockwise or counterclockwise as the figures:
Proof. From \[ \sigma_{n-1}\sigma_{n-2}\ldots\sigma_1 \] and the transitive property of the action of \( B_2 \) on \( (s_1, s_2) \), one knew that reflections of real Schur roots are precisely elements appear in the orbit of \( B_2 \) on \( (s_1, s_2) \). In particular, for \( B_2 =< \sigma_1 > \), we have
\[
\sigma_1^{2h}(s_1, s_2) = (c^h s_1 c^{-h}, c^h s_2 c^{-h}) \quad \text{for} \quad h \in \mathbb{Z},
\]
\[
\sigma_1^{2k+1}(s_1, s_2) = (c^k s_1 s_2 s_1 c^{-k}, c^k s_1 c^{-k}) \quad \text{for} \quad k \in \mathbb{Z}^+,
\]
\[
\sigma_1^{-(2k+1)}(s_1, s_2) = (c^{-k} s_2 c^k, c^{-k} s_2 s_1 s_2 c^k) \quad \text{for} \quad k \in \mathbb{Z}^+.
\]
Moreover, \( s_1 s_2 s_1 = c s_2 c^{-1} \) and \( s_2 s_1 s_2 = c^{-1} s_1 c \). Hence Lemma 4.1 completed the proof.

Corollary 4.4. Lee-Lee’s conjecture holds for finite-type root systems.

Proof. In finite-type cases, real roots are precisely real Schur roots, so the proof of simple curves presenting real Schur roots is trivial. Conversely, let \( \beta_k = s_1 s_2 \ldots s_{k-1} \alpha_k \), for \( 1 \leq k \leq n \), then it is clear that these roots may be presented by simple curves with the exact word. Moreover, finite-type root systems have exact \( n \) distinct \( c \)-orbits where \( \beta_k \) belongs to the each one (see Proposition 33, chapter VI in \cite{15}). Thus Lemma 4.1 implies Lee-Lee’s conjecture.
In the case of affine types, the action of \( c \) also gives an one-side proof of Lee-Lee’s conjecture and the remain ones is delivered from Section 3.

**Corollary 4.5.** Lee-Lee’s conjecture holds for affine-type root systems.

**Proof.** In \([13]\) and \([14]\), the authors show explicit \( \Phi^c \) set of real Schur roots as follows: the set has exact \( 2n \) infinite \( c \)-orbits and \( n - 2 \) finite \( c \)-orbits with \( n \geq 2 \). The transversal set of \( 2n \) infinite \( c \)-orbits includes \( \beta_k = s_1s_2\ldots s_{k-1}\alpha_k \) and \( \delta_k = s_n\alpha_{n-1}\ldots s_{k+1}\alpha_k \), for \( 1 \leq k \leq n \) that clearly might be presented by simple curves. Moreover, all real roots in \( n - 2 \) finite \( c \)-orbits are of finite types that might be presented by simple curves because of Corollary 4.2. Hence Lemma 4.1 implies that all real Schur roots of affine types may be presented by simple curves. ■

Let \( R \) be the set of reflections of \( G_n \) and recall the action of \( B_n \) on \( s = (s_1, \ldots, s_n) \in R^n \). We denote \( c = c(\Delta) \) and \( h = h(\Delta) \) respectively being the Coxeter element and the Coxeter number of Dynkin diagram \( \Delta \). Let \( B_{G_n} \) be the stabilizer subgroup of \( s \) and \( B_{G_n}(s) \) is the \( B_n \)-orbit at \( s \). Transitivity action of \( B_n \) on the completed exceptional sequence implies that the orbit is all possibly completed exceptional sequences.

**Proposition 4.6.** We have a bijection between the coset \( B_n/B_{G_n} \) and the factorization of the Coxeter element \( c \), hence the index \( [B_n : B_{G_n}] = \frac{n!}{|\sigma_n|} \).

**Proof.** Bijection is trivial from the orbit-stabilizer theorem and the specific formula of number of the factorization of the Coxeter group is shown in \([12]\). ■

The following table in \([12]\) exhibits the index formula for the connected Dynkin diagrams \( \Delta \):

| \( \Delta \) | \( B_n \), \( C_n \), \( D_n \) | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( F_4 \) | \( G_2 \) |
|---------------|------------------|--------|--------|--------|--------|--------|
| \( \frac{n!}{|\sigma_n|} \) | \((n + 1)^{n-1} \) | \( n^n \) | \( 2(n - 1)^n \) | \( 2^6.3^4 \) | \( 2.3.5^2 \) | \( 2^4.3^3 \) | \( 2.3 \) |

Note that \( \{\sigma_1, \sigma_1\sigma_2\ldots\sigma_{n-1}\} \) generates \( B_n \) and the group \( B_{G_n} \) is a finitely generated subgroup because it is a finite index subgroup of \( B_n \). Unfortunately, now we cannot yet find the generating set of \( B_{G_n} \) but introduce some elements of \( B_{G_n} \). Let \( \Delta(i - j) \) to be subdiagrams of the Dynkin diagram \( \Delta \) with vertices \( \{i, i + 1, \ldots, j\} \) and \( h_{ij} := h(\Delta(i - j)) \) to be their corresponding Coxeter number; particularly \( h_i := h_{ii+1} \).

**Lemma 4.7.** For \( 1 \leq i < j \leq n - 1 \),

\[
(\sigma_{j-1}\sigma_{j-2}\ldots\sigma_1)^{(j-i+1)h_{ij}}, (\sigma_{n-1}\ldots\sigma_1)^{nh} \in B_{G_n}.
\]

In particular, if \( (s_is_{i+1})^{m_{ij}} = 1 \), then \( \sigma_i^{m_{ij}} \in B_{G_n} \) with \( m_{ij} = 2, 3, 4, 6 \).

**Proof.** We have

\[
\sigma_i^m(s_1, \ldots, s_i, s_{i+1}, \ldots, s_n) = (s_1, \ldots, c_is_i c_i^{-1}, c_is_{i+1} c_i^{-1}, \ldots, s_n),
\]

\[
(\sigma_{j-1}\sigma_{j-2}\ldots\sigma_1)^{(j-i+1)}, (s_1, \ldots, s_i, \ldots, s_{n-1}, \ldots, s_n) = (s_1, \ldots, c_i sjc_i^{-1}, \ldots, c_is_{n-1} c_i^{-1}, \ldots, s_n),
\]

\[
(\sigma_{n-1}\ldots\sigma_1)^{nh}(s_1, \ldots, s_n) = (c_{n} s^{c_{n}^{-1}}, \ldots, c_{n} s^{c_{n}^{-1}}),
\]

where \( c_i = c(\Delta(i - i + 1)) = s_is_{i+1}, c_{ij} = c(\Delta(i - j)) = s_is_{i+1} \ldots s_j \). Since \( c_i^m_i = c_i^{h_{ij}} = c_i^{h_{ij}} = c_i^{h_{ij}} = c_i^{h_{ij}} \) the proof is completed. ■

Remark that \( (\sigma_{n-1}\ldots\sigma_1)^{nh} \) belongs to the center of \( B_n \), hence \( (\sigma_{n-1}\sigma_2\ldots\sigma_1)^{nh} \in N_n := \bigcap_{\sigma \in A_n} \sigma B_{G_n} \sigma^{-1} \) i.e. \( B_n \) acts unfaithfully on \( s \). Since \( B_{G_n} \) is a finite-index subgroup, so is \( N_n \).
Example 4.8. For $G_2 = k_2$ we have $[B_2 : B_{k_2}] = 3$ and $B_{k_2} = < \sigma_1^3 >$.
For $G_2 = G_2$ we have $[B_2 : B_{G_2}] = 6$ and $B_{G_2} = < \sigma_1^6 >$.

Corollary 4.9. Assume that there exists a proper normal subgroup $M$ that is strictly larger than $B_{G_n}$ for $n \geq 3$, then $n = 2 \pmod{3}$. Hence $B_n$ and $B_n / N_n$ is not solvable for $n \geq 3$.

Proof. We consider the natural projection $\pi : B_n \rightarrow B_n / M$. Since $\sigma_1^3 \in B_{A_n}$ and $M$ is proper, the order of $\pi(\sigma_1)$ is 3 in $B_n / M$ so 3 divides $[B_n : B_{k_n}] = (n + 1)^{n-1}$. Thus $n \equiv 2 \pmod{3}$. This implies that there is no proper normal subgroup that is strictly larger than $B_3$ in $B_3$. Moreover, $B_3$ is not a normal subgroup of $B_3$, hence $B_3$ is not solvable. Since $B_3$ may be embedded in $B_n$ for $n \geq 3$, $B_n$ is not solvable and so is $B_n / N_n$. $\blacksquare$

Finally, we finish the paper with several further questions which might be of interest.

Question 1: How can we find the finite generating set of $B_{G_n}$ and $N_n := \bigcap_{\sigma \in A_n} \sigma B_{G_n} \sigma^{-1}$ when $G_n$ is a Weyl group of finite types?

Question 2: What is the classification of the finite non-solvable group $B_n / N_n$ for $n \geq 3$?

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