Entangled states of two quantum dots mediated by Majorana fermions

To cite this article: Z C Shi et al 2016 New J. Phys. 18 023005

View the article online for updates and enhancements.
Entangled states of two quantum dots mediated by Majorana fermions

Z C Shi1,2, W Wang1 and X X Yi1
1 Center for Quantum Sciences and School of Physics, Northeast Normal University, Changchun 130024, People’s Republic of China
2 School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, People’s Republic of China
E-mail: yixx@nenu.edu.cn

Keywords: Majorana fermions, superconductor, quantum dots

Abstract

With the assistance of a pair of Majorana fermions, we propose schemes to entangle two quantum dots by Lyapunov control in the charge and spin degrees of freedom. Four different schemes are considered, i.e., the teleportation scheme, the crossed Andreev reflection scheme, the intradot spin flip scheme, and the scheme beyond the intradot spin flip. We demonstrate that the entanglement can be generated by modulating the chemical potential of quantum dots with square pulses, which is easily realized in practice. In contrast to Lyapunov control, the preparation of entangled states by adiabatic passage is also discussed. There are two advantages in the scheme by Lyapunov control, i.e., it is flexible to choose a control Hamiltonian, and the control time is much shorter with respect to the scheme by adiabatic passage. Furthermore, we find that the results are quite different by different adiabatic passages in the scheme beyond the intradot spin flip, which can be understood as an effect of quantum destructive interference.

1. Introduction

Majorana fermions (MFs), which are predicted to exist at the boundary or in the vortex core of a topological superconductor (TS), have widely been studied recently both in theories [1–4] and in experiments [5–9]. For example, MFs have been applied into topological quantum computation [10, 11] due to their robustness against perturbations.

From another aspect, two spatially separated MFs can form a Dirac fermion. This nonlocality feature can offer the MFs an opportunity as the medium to entangle quantum systems, such as quantum dots (QDs). The advantages to entangle QDs via MFs are twofold. First, conventional schemes to entangle QDs are limited by spatial distance due to direct proximity coupling with each other [12–14]. As a result, it is difficult to manipulate quantum states at such a short distance. The entanglement preparation mediated by MFs can effectively solve this drawback. Second, from the side of topological quantum computation, the MFs are difficult to couple together as the wave-function overlap of two MFs decays exponentially with spatial distance between them. Additionally, braiding operations solely are insufficient in the universal quantum computation for Majorana-based qubits [11]. Those difficulties inevitably can be overcome in hybrid systems [15–21] where the combination of the robustness of topological qubits (e.g., Majorana-based qubit) and the universality of conventional qubits (e.g., QD-based qubit, flux qubit, etc) can form a universal computation.

The adiabatic passage, which is widely applied to quantum information processing, is one way to prepare entanglement. The main idea is as follows. Given a quantum system where its ground state is separable and is easy to prepare, one can adiabatically manipulate some physical parameters such that the system evolves into the ground state of the new Hamiltonian, which is the target entangled state. Since the adiabatic dynamics is protected, it is then immune to some types of perturbations. However, the price one should pay is the long time needed to finish the evolution. To overcome this difficulty, a shortcut to adiabaticity [22, 23] is introduced. The key point is that the time-dependent Hamiltonian $H(t)$ is brought into the Schrödinger equation. Nevertheless,
it is always hard to implement such a Hamiltonian for most systems, and $H_{t1}$ may not exist in complicated systems. This gives rise to a question—are there any other control strategies for the quantum system better than adiabatic passage?

In this work, we will focus on this issue. We propose to entangle two QDs by Lyapunov control, which has been employed in manipulating quantum states [24–29]. The basic principle of Lyapunov control is to design control fields to drive a quantum system to approach the target state. Note that the target state must be a steady state such that the system cannot evolve when the control fields vanish. In order to design control fields, a Lyapunov function has to be defined. To be specific, one first defines a Lyapunov function $V$ for the given target state $|\psi_T\rangle$. Then, by restricting nonpositivity of the time derivative of the Lyapunov function $V(t)$ (i.e., $\dot{V} \leq 0$), one can obtain the control fields $f_i(t)$. Driven by these control fields, the system would evolve to target state $|\psi_T\rangle$ with time.

The system we considered is a hybrid quantum system primarily consisting of two QDs and a TS wire with MFs. The two QDs are well spatially separated so that they do not have direct interaction, but they are coupled to a common TS wire. The entanglement between the two QDs is then induced via the nonlocality property of the Dirac fermion defined by MFs. We adopt the Lyapunov control to explore entanglement generation in the following. For comparison, the entanglement generated by adiabatic passage also is presented and is discussed.

### 2. Model

The system under consideration is illustrated in figure 1 (a), which consists of a TS wire coupled to two QDs and another bulk superconductor via tunneling. The free Hamiltonian of such a system contains two parts. The first part of the free Hamiltonian is for the TS wire ($\hbar = 1$),

$$H_0^{TS} = H_{\text{TS}} + H_{\text{FS}},$$

where $H_{\text{FS}} = \sum_x \gamma_x^\dagger \gamma_x$. The energy spectrum is $\varepsilon_x$, and the Bogoliubov quasiparticle operators $\gamma_x$ are related to the electron operators $c(x)$ of the TS wire in the standard way, where $x$ denotes the electron coordinate in the TS wire [3]. Note that we have dropped the spin subscript in this section as the spin degeneracy in the TS wire is broken due to the Zeeman effect, which suggests that we might consider only one spin direction. Considering the situation that the TS wire is in the topological nontrivial phase, there is a pair of MFs $\gamma_1$ and $\gamma_1^\dagger$ at the ends of the TS wire [30, 31]. These two zero-energy MFs can form a nonlocal Dirac fermion, i.e., $f = \gamma_1^\dagger f_0$, where $f$ is the annihilation operator of the Dirac fermion. When the energy scale of the TS wire is smaller than the superconductor gap, there are no other quasiparticle excitations, except for the zero-energy MFs in the TS wire. As a result, we can safely ignore the Hamiltonian $H_{\text{FS}}$ in the model. Note that, when the TS wire is of mesoscopic size and is linked to a capacitor, Earth grounded, there exists an additional Hamiltonian $H_{\text{FS}}$ describing the

![Figure 1.](image-url)
finite charging energy. The corresponding Hamiltonian takes [32]
\[ H_{\text{TS}} = E_c (2N_f - n_g + n_j)^2, \]
where \( N_f \) is the number of Cooper pairs, \( n_g \) denotes the dimensionless gate charge determined by the gate voltage \( V_g \), and \( n_j = f^\dagger f \) stands for the number of Dirac fermions formed by the MFs. The single electron charging energy is \( E_c = \frac{e^2}{2C} \) with capacitance \( C \). It has been shown that the single electron charging energy \( E_c \) plays a key role in the long-range entanglement generation of two QDs [33].

The second part of the free Hamiltonian is for the QDs,
\[ H_0^{\text{QD}} = \sum_{n=1}^2 \epsilon_n d_n^\dagger d_n, \]
where \( d_n \) and \( d_n^\dagger \) are the annihilation and creation operations of electrons in the \( n \)th QD. \( \epsilon_n \) denotes the chemical potential that can be changed by the gate voltage \( V_g \) (\( n = 1, 2 \)). We should emphasize that we have assumed both QDs in the Coulomb blockade regime in Hamiltonian \( H_0^{\text{QD}} \), i.e., the electron can only occupy a single fermion level. This requires that \( U_n \) [see equation (16)] is large compared to the other relevant energy in the system.

Next, we turn to the interaction Hamiltonian of the system. The first term describes the Cooper pair exchange between the TS wire and the bulk superconductor,
\[ H_T^{\text{TS}} = E_J \cos \phi = \frac{E_J}{2} (e^{-i\phi} + e^{i\phi}). \]
where \( E_J \) is the Josephson coupling, and \( \phi \) is the phase difference between the two superconductors. Here, the phases of the bulk superconductor have been set to zero for simplicity. The operator \( e^{-i\phi} \) represents the creation (annihilation) of a Cooper pair, i.e., \( e^{-i\phi} = \sum_n |N_e + 1\rangle \langle N_e| \). The number operator of Cooper pairs and the phase of superconductor are canonically conjugate, i.e., \( \{N_e, e^{-i\phi}\} = e^{-i\phi} \).

The second term of interaction Hamiltonian describes the electron tunneling between the TS wire and the QDs. For later use, we write down the electron operator \( c(x) \) in terms of quasiparticle operator \( \gamma_k \) in the TS wire, i.e., \( c(x) = \sum_k \gamma_k^\dagger \gamma_k + \cdots \), where \( \gamma_k \) is the wave function of the spatial coordinate. As we have mentioned before, there are no other quasiparticles, except for the MFs in the TS wire. This allows us to consider the first two terms in the expression of \( c(x) \), i.e., \( c(x) = \sum_k \gamma_k^\dagger \gamma_k + \sum_{QD} \gamma_n^\dagger \gamma_n \). In addition, if the length of the TS wire is long enough, there are no overlaps between \( g_i \) and \( g_j \), i.e., no coupling between \( \gamma_n \) and \( d_k(d_k^\dagger) \). As a consequence, the effective Hamiltonian describing tunnel coupling between the electron in the QDs and the Dirac fermion formed by the MFs is given by substituting \( c(x) \) into the bare tunneling terms \( \sim \int dx \, t_{n} d_{n}^\dagger c(x) + \text{h.c.} \) [34–36],
\[ H_T^{\text{QD}} = \sum_{n=1}^2 \lambda_n [f^\dagger + (-1)^{n-1}e^{-i\phi}] d_n + \text{h.c.}, \]
where \( \lambda_n \) represents the coupling strength. In the derivation of Hamiltonian \( H_T^{\text{QD}} \), we have considered charge conservation since it cannot create or annihilate charge 2\( e \) without any energy cost in the TS wire with \( E_c = 0 \). In a later discussion, the terms \( f^\dagger d_n \) (or \( d_n^\dagger f \)) and \( f^\dagger f \) (or \( d_n^\dagger d_n^\dagger f \)) are referred to the normal and anomalous tunneling processes, respectively.

As we study the long-range entanglement generation in this paper, the length of the TS wire is considered so sufficiently long that there does not exist direct coupling between the two MFs. Therefore, the interaction Hamiltonian \( H_T^{\text{MF}} = \lambda \gamma_1^\dagger \gamma_1^\dagger \gamma_1 \gamma_1 \) has also been neglected throughout this work.

Collecting these terms, we can write down the Hamiltonian for the whole system,
\[ H_0 = H_0^{\text{TS}} + H_0^{\text{QD}} + H_T^{\text{TS}} + H_T^{\text{QD}}. \]
In order to prepare the maximally entangled state of the two QDs, e.g., Bell states, we choose \( \epsilon_1 = \epsilon_2 = \epsilon \) and \( \lambda_1 = \lambda_2 = \lambda \) and work in the large-\( E_c \) limit (compared to the parameters \( \epsilon \) and \( \lambda \)) that can always be satisfied by changing the size of the TS wire. Since the total fermion parity, which is quantized by the number of the electrons in the two QDs plus the Dirac fermion formed by the MFs, is conserved in this model, we will study the even-parity case in the following. The extension from the even-parity case to the odd-parity case is straightforward.

### 3. Teleportation scheme

The teleportation refers to the nonlocal transfer of fermions across the TS wire [32, 33]. As an example, we briefly present the physical process that electrons transfer from the first QD to the second QD via the TS wire. When an electron tunnels from the first QD to the TS wire, the charges in the TS wire increase one unit with energy cost \( E_c \). If \( E_c \) cannot match \( \epsilon \), the tunneling process is virtual. In order to keep the conservation of charge in the TS wire, the electron may transfer to the second QD. We first consider a case in which the Josephson coupling is \( E_J = 0 \) and the gate charge is \( n_g = 1 \). The ground state corresponds to the number of Cooper pairs \( N_c = 0 \), the Dirac
fermion \( n_f = 1 \), and the electrons \( n_1 + n_2 = 1 \) in the even-parity case (cf., equations (2) and (3)). Namely, the ground states are \(|0110\rangle\) and \(|1010\rangle\). We have employed the notation \([n_f,n_1,n_2,N_f]\) to describe the system state, where \(n_f(n_2)\) is the number of electrons in the first (second) QD, \(n_j\) is the number of Dirac fermions formed by the MFs, and \(N_f\) is the number of Cooper pairs in the TS wire. \(|0001\rangle\) and \(|1100\rangle\) are the low excited states coupled directly to the ground state. We ignore the higher-energy excited states due to large gaps in the ground state. Since the entanglement generation is based on the nonlocality of the Dirac fermion defined by the MFs (cf., figure 1(b)), we refer this proposal to the teleportation scheme.

According to the transitions shown in figure 1(b), the Hilbert space is spanned by \(|0001\rangle\), \(|0110\rangle\), \(|1010\rangle\), \(|1100\rangle\). In this space, the Hamiltonian \(H_0\) can be written as a \(4 \times 4\) matrix, i.e.,

\[
H_0 = \begin{pmatrix}
E_1 & -\lambda & \lambda & 0 \\
-\lambda & \epsilon & 0 & \lambda \\
\lambda & 0 & \epsilon & \lambda \\
0 & \lambda & \lambda & E_c + 2\epsilon
\end{pmatrix}.
\]

(7)

The system eigenstates, then, can be found analytically,

\[
|E_i\rangle = N_j(|0110\rangle - |0011\rangle - A_1|0001\rangle),
\]

\[
|E_2\rangle = N_j(|1010\rangle - |0110\rangle + A_1|0001\rangle),
\]

\[
|E_3\rangle = N_j\left(-\frac{A_1}{2}|1010\rangle - \frac{A_2}{2}|0110\rangle + |1100\rangle\right),
\]

\[
|E_4\rangle = N_j\left(\frac{1}{A_3}|1010\rangle + \frac{1}{A_3}|0110\rangle + |1100\rangle\right),
\]

(8)

and the corresponding eigenvalues are given by \(E_1 = \epsilon - \lambda A_1\), \(E_2 = E_1 + \lambda A_1\), \(E_3 = E_2 + 2\epsilon - \lambda A_3\), and \(E_4 = \epsilon + \lambda A_3\), where \(A_1 = \frac{\epsilon - E_c + i\sqrt{E_c - \epsilon^2 + 8\lambda^2}}{2\lambda}, A_2 = \frac{E_c - E_c + i\sqrt{E_c - \epsilon^2 + 8\lambda^2}}{2\lambda}, A_3 = \frac{E_c + \epsilon + i\sqrt{E_c + \epsilon^2 + 8\lambda^2}}{2\lambda}, N_j \) \((j = 1-4)\) is the normalization constant. In equation (8), one can find that \(|E_2\rangle \simeq |0001\rangle\) if \(\epsilon \gg E_c\), while, if \(\epsilon \ll E_c, |E_1\rangle \equiv \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)|10\rangle = |\psi_T\rangle\), which is the Bell state of the two QDs. This observation paves the way towards preparing Bell state \(|\psi_T\rangle\) by adiabatic passage. To be explicit, we first initialize the system in state \(|0001\rangle\). Then, we adiabatically decrease the chemical potential \(\epsilon\). The system would stay in eigenstate \(|E_3\rangle\) and would arrive at Bell state \(|\psi_T\rangle\), finally. Similarly, when the initial state is \(|1100\rangle\), the system would follow the eigenstate \(|E_4\rangle\) to arrive at the Bell state.

In order to compare the adiabatic passage with Lyapunov control, we reformulate the aforementioned results from the point of quantum control. That is, consider the chemical potentials of the two QDs being manipulated in the adiabatic passage, and the control Hamiltonians read

\[
H_1 = d_1^\dagger d_1, \quad H_2 = d_2^\dagger d_2.
\]

(9)

Hence, the system evolution is governed by the following Schrödinger equation:

\[
i\dot{|\psi}\rangle = [H_0 + \sum_{k=1}^2 f_k(t)H_k]|\psi\rangle,
\]

(10)

where \(f_k(t)\) denotes the chemical potential changing with time. We refer to \(f_k(t)\) as the control field, hereafter, and set two control fields the same to simplify experimental realizations, i.e., \(f_k(t) = f(t)\). In figure 2, we plot the time evolution when linearly decreasing the chemical potentials of the two QDs. It suggests that one can, indeed, achieve Bell state \(|\psi_T\rangle\) when the control fields \(f(t)\) change sufficiently slowly with time (for instance, in figure 2(a)), but it takes a long manipulation time. With increasing the change rate of the chemical potentials, shown in figure 2(b), the performance gets worse due to the breakdown of the adiabatic condition. Therefore, the fact that it is difficult to decrease the manipulation time becomes a bottleneck for the adiabatic passage.

Next, we employ Lyapunov control to speed up the entanglement generation. In order to determine the form of the control fields, one has to choose a Lyapunov function \(V\) first, e.g.,

\[
V = 1 - \langle |\psi_T\rangle|\psi\rangle^2,
\]

(11)

where \(|\psi_T\rangle\) is the target state (i.e., the Bell state). The first-order derivative of \(V\) yields

\[
\dot{V} = -2\sum_k f_k(t)\text{Im}[\langle \psi_T|\psi\rangle\langle \psi_T|H_k|\psi\rangle]
\]

\[
= -2\sum_k \langle \psi_T|\psi\rangle f_k(t)\text{Im}[e^{i\arg\langle \psi_T|\psi\rangle}|\psi\rangle|H_k|\psi\rangle]
\]

(12)

where \(\text{Im}(\cdots)\) stands for the imaginary part of \((\cdots)\) and \(\arg\langle \psi_T|\psi\rangle\) is the phase difference between \(|\psi\rangle\) and \(|\psi_T\rangle\). Thus, the condition \(\dot{V} \leq 0\) can be satisfied naturally if we choose the control fields,
where the constant is $B_0 > 0$. Figures 3(a), (b) demonstrate how the system arrives at Bell state $|\psi_T\rangle$ by control fields $f_k(t)$ when the initial state is $|0001\rangle$. In particular, we have designed the control fields for both QDs to be the same, i.e., $f_2(t) = f_1(t)$. We further observe that the total manipulation time is related to the constant $B_1$, e.g., figures 3(c), (d) are plotted with $B_1 = 300$. An inspection of figures 3(a)–(d) shows that the amplitude of the control fields is time dependent, which may make experimental realization difficult. Actually, by virtue of the fundamental principle of Lyapunov control, the dynamics performance is insensitive to the amplitude of the control fields. Instead, it depends sharply on the sign of the control fields. Due to this flexibility feature, the time-dependent amplitude of the control fields can be replaced by square pulses as follows:

$$
f_k(t) = -B_k \text{Im}[e^{i\pi\sigma^z}\langle \psi_T | H_0 | \psi_T \rangle], \quad k = 1, 2,
$$

where the constant is $B_k > 0$. Figures 3(e)–(h) show the performance to realize Bell state $|\psi_T\rangle$ by square pulses where the forms of the control fields are much simpler than that given by equation (13). Furthermore, the control time is also shortened by square pulses.

Now, we turn to the case of the existing Josephson coupling. The Hilbert space is now spanned by $\{|0000\rangle, |0001\rangle, |0110\rangle, |1010\rangle, |1100\rangle, |1101\rangle\}$ as sketched in figure 1(b). We have neglected the higher-energy excited states once more. At first, we decrease the chemical potentials of the two QDs adiabatically when
the initial state is \( |0001\rangle \). The dynamical behaviors are plotted in figure 4(a). It can be observed that the adiabatic passage is invalid for perfectly generating Bell state \( |\psi_T\rangle \) since the population cannot reach 1. The reason is as follows. The evolution process mainly contains two physical mechanisms: (i) the Rabi oscillation between \( |0000\rangle \) and \( |0001\rangle \) for the existence of the Josephson coupling; (ii) the population adiabatic transfer from \( |0001\rangle \) to Bell state \( |\psi_T\rangle \). The population adiabatic transfer dominates only in the middle stage and, throughout the existence of Rabi oscillation, leads to failure generation of Bell state \( |\psi_T\rangle \). But when employing Lyapunov control, we find, in figures 4(b)–(e), that whether the system exists, the Josephson coupling makes no difference in the entanglement generation since Bell state \( |\psi_T\rangle \) is still the system eigenstate. The only difference from the aforementioned case is the shape of control field \( f_1(t) \).

Alternatively, we can utilize the Cooper pair exchange between the TS wire and the bulk superconductor to be the control Hamiltonian in Lyapunov control, i.e.,

\[
H_3 = \cos \phi = \frac{1}{2}(e^{-i\phi} + e^{i\phi}).
\]  

As shown in figure 5, the control field \( f_3(t) \) designed by equation (13) or (14) can steer the system into Bell state \( |\psi_T\rangle \). Thus, we can generate Bell state \( |\psi_T\rangle \) by modulating the strength of the Cooper pair exchanges between the TS wire and the bulk superconductor.

---

**Figure 4.** Population of Bell state \( |\psi_T\rangle \) (green-solid line) as a function of time in the presence of the Josephson coupling \( E_J = 0.5 \). Panel (a) demonstrates the results by the adiabatic passage, while panels (b)–(e) demonstrate the system evolution by Lyapunov control. The red-dashed (cyan-dashed) line denotes the population of \( |0000\rangle \) \( |0001\rangle \). \( P \approx 1 \) (purple-solid line) shows it has two major distinct physical mechanisms during the evolution since the populations of the other states almost vanish. The other parameters are the same as in figure 3. (a) \( T = 20 \). The control field in panel (c) is given by equation (13) with \( B_3 = 300 \), while it is calculated by equation (14) with \( F = 5 \) in panel (e).

**Figure 5.** The population of Bell state \( |\psi_T\rangle \) as a function of time where the control Hamiltonian is the Cooper pair exchange. (b) \( B_1 = 100 \). (d) \( F = 2 \).
4. Crossed Andreev reflection scheme

Crossed Andreev reflection (CAR), also known as nonlocal Andreev reflection, occurs when two spatially separated electrodes in a normal state form two separate junctions with a superconductor. In our model, a CAR process refers to the situation that two electrons from different QDs tunnel into the TS wire to form a Cooper pair in the TS wire and the bulk superconductor. Hence, we refer this process as the CA reflection scheme. Although the eigenstates can be calculated analytically, the expressions are tedious. Hence, we adopt numerical solutions to discuss the occupation of the two lowest eigenstates of the Hamiltonian $H_n$.

Figure 6 shows that the eigenstate is nearly Bell state $|\psi_T^+\rangle = ((00) \pm |11\rangle) |00\rangle$ when $|\epsilon| < 20$, and there is no eigenstate that approximately equals one of the bases. This indicates that it is difficult to prepare the Bell state by adiabatic passage. To prepare Bell state $|\psi_T^+\rangle$ by Lyapunov control, we first explore a situation that the control Hamiltonians are the particle number of the two QDs (i.e., $H_n = d_n^+ d_n$, $n = 1, 2$). With the control fields given by equation (13), we plot the results in figure 7(a). We find that the final state can be approximately expressed as $|\psi_c\rangle \approx \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) |00\rangle$, which is actually the Bell state. For the situation that the control Hamiltonian is the Cooper pair exchange (i.e., $H_3 = \cos \phi \hat{\sigma}_z$), the results are plotted in figure 7(b). We find that the final state is not a Bell state, indicating it fails to prepare Bell state $|\psi_T^+\rangle$ in this situation. Further observations reveal that the populations on the bases $|00\rangle$, $|01\rangle$, and $|10\rangle$ nearly equal each other, which implies that Bell state $|\psi_T^+\rangle$ may be obtained by measuring the parity of the Dirac fermion. Indeed, further examination yields that, if $n_f = 0$, the final state collapses to $|\psi_T^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) |00\rangle$. If $n_f = 1$, the final state collapses to $|\psi_c\rangle \approx \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) |10\rangle$, leading to the failure for the generation of Bell state $|\psi_T^+\rangle$. 

![Figure 6](image_url)
5. Scheme with the intradot spin flip

In the last section, we proposed a scheme to entangle two QDs regardless of the electron spin states. We can also employ the spin up and spin down to encode quantum information due to their long coherence time \[12-14\]. In the following, we will investigate this issue.

As sketched in figure 8 (a), the system does not differ much from the setup in figure 1 (a). It does not have the bulk superconductor and the gate charge. In particular, in order to keep spin degeneracy in the QDs, we do not apply magnetic fields but employ a magnetic insulator contacting with the TS wire to induce the effective Zeeman coupling \[41, 42\]. The Hamiltonian reads \[43-46\]

\[H_0 = \sum_{n=1}^{2} \left( \sum_\nu \epsilon_{\nu,n} d_{\nu,n}^\dagger d_{\nu,n} + U_n d_{\uparrow,n}^\dagger d_{\downarrow,n}^\dagger d_{\uparrow,n} d_{\downarrow,n} + t d_{\uparrow,n}^\dagger d_{\downarrow,n} \right) + \lambda_1 (f^\dagger + f) d_{\uparrow,1} + \lambda_2 (f^\dagger - f) d_{\downarrow,2} + \text{h.c.} \quad (16)\]

The first term describes the energy of the two QDs with the chemical potential \(\epsilon\) for spin projection \(\nu = \uparrow, \downarrow\). The second term describes the two electrons occupying the same QD with Coulomb interaction \(U_n\). In the Coulomb block regime, the electron can only occupy the single fermion level in the two QDs, and we focus on this regime in the following. The third term describes the intradot spin flip with strength \(t\), which stems from spin–orbit interactions and has been studied in \[47\]. Note that the spin of the QD is no longer a good quantum number as the transitions are allowed between distinct spin states. Since the spin flip term plays an essential role in the entanglement generation as shown by the green line in figure 8 (b), we will call this process the intradot spin flip scheme. The fourth (fifth) term describes the tunnel coupling between the Dirac fermion and the spin down (up) electron in the first (second) QD with strength \(\lambda_1, \lambda_2\).

---

**Figure 7.** The population on each basis as a function of time. The eight bases are ordered as \{0000, 0001, 011−1, 0110, 1010, 101−1, 110−1, 1100\} on the horizontal axis. The initial state is \(0110\), and the target state is \(|e^{x}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)|00\rangle\). The control Hamiltonians are \(H_1 = d_{\uparrow,1}^\dagger d_{\uparrow,1},\ H_2 = d_{\downarrow,2}^\dagger d_{\downarrow,2},\ H_5 = \cos \phi\). All control fields are designed by equation (13).

**Figure 8.** The schematic setup consisting of a TS wire tunnel coupling to two QDs. Only the spin down (up) electron in the first (second) QD can be tunnel coupled to the TS wire since the left (right) MF \(\gamma_1 (\gamma_2)\) is spin down (up) as well. At the same time, the intradot spin flip process exists in the two QDs. (b) The transition between distinct states. The green line represents the intradot spin flip process.
We can also prepare the Bell states based on a scheme beyond the intradot spin flip. The system considered is composed of two TS wires coupling to two common QDs as illustrated in figure 10(a). The Hamiltonian describing such a system reads [43–45]

6. Scheme beyond the intradot spin flip

In general, the MFs always have a definite spin polarization at the two ends of the TS wire. Thus, the wire can only send or can only receive electrons with the same spin polarization. The spin polarizations of the MFs are determined by the boundary between topological and nontopological regions [18, 49], which are antiparallel at the two ends of the TS wire [30, 41] in an ideal case. We should emphasize that the spin polarization is not antiparallel in real cases, which would cause an error in the entanglement preparation [19]. Nevertheless, we can manipulate the chemical potentials around the ends of the TS wire to achieve nearly perfect antiparallel spin polarization. This is the reason why, in the fourth (fifth) term, we have assumed the electron state is spin down (up) on the left (right) MF $\gamma_i (\bar{\gamma}_i)$, and it can be only coupled to the spin down (up) electron in the first (second) QD via tunneling.

When we consider the even-parity case, the Hilbert space is spanned by $\{|000\}, |0 \downarrow 0\rangle, |0 \uparrow 1\rangle, |0 \uparrow 0\rangle, |0 \downarrow 1\rangle, |1 \downarrow 0\rangle, |1 \uparrow 0\rangle, |1 \downarrow 0\rangle, |1 \uparrow 0\rangle\}$. We have employed the label $|n_1 n_2 n_3\rangle$ to describe system state, where $n_1, n_2 = 0, 1$ denotes the electron state in the first (second) QD and $n_3$ denotes the number of Dirac fermions combined by the MFs. Note that the Hamiltonian $H_i$ can be represented by a $9 \times 9$ matrix in these bases. As the analytical expressions of the system eigenstates are involved, we choose to show the behaviors of the two lowest eigenstates by numerical calculations.

Figures 9(a), (b) describe the amplitude of each basis in two lowest eigenstates as a function of the chemical potential. It demonstrates that the eigenstate approximately equals state $|000\rangle$ when the chemical potential is $\epsilon \gg 0$, while the eigenstate nearly becomes Bell state $|\psi_{T\gamma}^{0} \rangle = (| \uparrow \uparrow \rangle - | \downarrow \downarrow \rangle) |0\rangle$ when the chemical potential is $\epsilon \ll 0$. In addition to this, in figure 8(b), we find that the transition path $|000\rangle \leftrightarrow | \uparrow \downarrow \rangle$ and the transition path $|000\rangle \leftrightarrow | \downarrow \downarrow \rangle$ are completely symmetrical so that it is possible to generate the Bell state by adiabatic passage. The main control procedures are as follows. The system is initialized in state $|000\rangle$ with a large chemical potential. Then, we adiabatically decrease the chemical potential. The system would evolve to Bell state $|\psi_{T\gamma}^{0} \rangle = (| \uparrow \uparrow \rangle - | \downarrow \downarrow \rangle) |0\rangle$, finally, which is plotted in figure 9(c). A universal drawback of the adiabatic passage is that it takes a long control time in order to meet the adiabatic condition. To reduce the control time, we turn to Lyapunov control where the control fields are designed by equation (13), instead of decreasing gradually (in adiabatic control). The dynamics evolution is demonstrated in figure 9(d). One can find the total control time of implementing Bell state $|\psi_{T\gamma}^{0} \rangle$ by Lyapunov control is much shorter than that by adiabatic passage.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{The amplitude of each basis in the two lowest eigenstates as a function of the chemical potential in panels (a), (b). The horizontal axis denotes the nine bases ordered as $\{|000\}, |0 \downarrow 0\rangle, |0 \uparrow 1\rangle, |0 \uparrow 0\rangle, |0 \downarrow 1\rangle, |1 \downarrow 0\rangle, |1 \uparrow 0\rangle, |1 \downarrow 0\rangle, |1 \uparrow 0\rangle\}$. One can find that the amplitude mainly locates at bases $| \uparrow \downarrow \rangle$ and $| \downarrow \downarrow \rangle$ when $\epsilon \ll 0$ in panel (b). The population of each basis as a function of time in panels (c), (d). $\epsilon = -10$. The control fields are $c f(t) = -2t + 20$ in the adiabatic passage, (d) designed by equation (13) with $B_1 = 1000$ in Lyapunov control. After completing controls, the amplitude of the final state that is mainly equally weighted locates at bases $| \uparrow \uparrow \rangle$ and $| \downarrow \uparrow \rangle$.}
\end{figure}
where the notation is the same as in equation (16) and we adopt the convention $n \mod 2$ if $n > 2$. We use $|n_1 n_2 n_{f_1} n_{f_2}\rangle$ to denote the system state, where $n_1 (n_2) = 0, \uparrow, \downarrow$ stands for an electron state in the first (second) QD and $n_{f_k} = f_k^+ f_k$ ($n_{f_k} = f_k^+ f_k$) denotes the number of Dirac fermions defined by the MFs in the left (right) TS wire.

Similar to the analysis presented in the last section, we first analyze the behaviors of the two lowest eigenstates as a function of the chemical potential with numerical simulations. When $\epsilon = 0$, the eigenstate is very close to basis $|0000\rangle$ (see figure 11(a)). When $\epsilon \ll 0$, the eigenstate is approximately a superposition of $|\uparrow\downarrow\downarrow\downarrow\rangle$ and $|\downarrow\uparrow\uparrow\uparrow\rangle$ with almost equal weight (see figure 11(b)). It is believed that the system can be steered into the Bell state by adiabatic passage when the initial state is $|0000\rangle$. Here, we explore this issue by two different adiabatic passages. The first is to simultaneously decrease the chemical potentials of the two QDs with the same rate as shown in figure 12(a). The system, finally, is driven into Bell state $|\psi_1\rangle = |\uparrow\downarrow\downarrow\downarrow\rangle + e^{-i\epsilon}\lambda_n f_{n+1} f_{n+1}\rangle$. Another method is to adiabatically decrease the chemical potential of the first QD while the chemical potential of the second QD remains unchanged. After completing this operation, the system state becomes $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle)$. We, then, adiabatically decrease the chemical potential of the second QD while the chemical potential of the first QD remains unchanged. The results are shown in figure 12(b). The final state of the system becomes $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\downarrow\rangle + |\uparrow\uparrow\downarrow\uparrow\rangle)|00\rangle - \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle)|11\rangle$, which is actually not the Bell state of the two QDs. However, with the help of the measurement results on the Dirac fermion parity (e.g., $n_{f_1}$), the system would collapse into Bell state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle)|00\rangle$ if $n_{f_1} = 0$, while the system would collapse into Bell state $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle)|11\rangle$ if $n_{f_1} = 1$. Remarkably, the control results are quite different for the two adiabatic passages. This originates from the fact that quantum destructive interference
exists in the first adiabatic passage. To clarify this point, we examine the transition paths between $|0000\rangle$ and $|\uparrow \downarrow 00\rangle$ in figure 10(b), i.e.,

$$|0000\rangle \xrightarrow{\lambda_0} |0 \downarrow 01\rangle \xrightarrow{\lambda_1} |\uparrow \downarrow 01\rangle, \quad |0000\rangle \xrightarrow{\lambda_0} |\uparrow 00\rangle \xrightarrow{\lambda_2} |\uparrow \downarrow 01\rangle. \quad (18)$$

Due to the presence of the minus sign in the coupling constant, these two transition paths destructively interfere in the first method (the mechanism is the same for the transition paths between $|0000\rangle$ and $|\downarrow \uparrow 00\rangle$). However, this effect cannot happen in the second method.

From the aspect of Lyapunov control, since both $|\psi_1\rangle$ and $|\psi_2\rangle$ are the eigenstates of the system when $\epsilon \ll 0$ ($|\psi_2\rangle$ is not shown in figure 11), it is then possible to generate Bell states by Lyapunov control where the control fields are designed by equation (13). The results are presented in figures 12(c), (d). Compared with the adiabatic control, a specific type of Bell state (e.g., $|\psi_1\rangle$ and $|\psi_2\rangle$) can be prepared without measurement.

7. Discussion and conclusion

Before concluding, we discuss the validity of the assumptions made in our model and the experimental feasibility of our proposal.

The first assumption in this work is that there are no other quasiparticles in the TS wire except for the MFs. This assumption is true if gap $\Delta$ in the superconductor is sufficiently large. Recent experiments [5, 6, 50–52] report the observation of MFs in the TS wire (e.g., a InAs or InSb nanowire), and the superconducting gap $\Delta$ is on the order of 0.1–1 meV, which is much larger than thermal fluctuations $~k_B T$ with temperatures of $\lesssim 100$ mK.

In the second assumption, we suppose that both QDs are in the Coulomb block regime. Experimentally, the two QDs can be constructed at the two ends of the same TS wire. The electrostatic gates underneath the TS wire create a confinement potential for the electron to form a QD. It has been demonstrated that the Coulomb interaction $U$ in such a QD device is on the order of $1–10$ meV [53, 54], which is much larger than the superconducting gap $\Delta$ and the thermal fluctuation at the operating temperature. Therefore, the QDs in the Coulomb block regime are realistic in our model.

Besides, the charging energy $E_c$ is an essential parameter for generating long-range entanglement, which is on the order of 1 meV [52]. To meet the condition $E_c < \Delta$, one can increase the length of the TS wire or capacitance to reduce the charging energy $E_c$. If the charging energy $E_c$ is larger than the superconducting gap $\Delta$, quasiparticles would appear in the TS wire, which participate in the entanglement preparation. When such quasiparticles are in the bulk of the TS wire, i.e., no remarkable overlap with the MFs, they do not have an effect on the Bell state preparation. Nevertheless, when the quasiparticles are located near the ends of the TS wire, it may be invalid to prepare the Bell states by adiabatic passage or Lyapunov control.
Since the chemical potential of the QDs can be modulated by electrostatic gates, we mainly discuss how to change the gate voltage to simulate the control fields. To prepare the Bell states by adiabatic passage, we just need to decrease the gate voltage with time linearly, which is easily realized in experiments. In Lyapunov control, it may not be easy to manipulate the gate voltage to simulate the time-dependent amplitude of the control fields (e.g., figure 3(b)), but it is quite easy to realize the square pulses required by the control fields (e.g., figure 3(f)). One may notice that the tunnel coupling $\lambda$ is fixed in our calculation. In fact, the tunnel coupling depends on both the chemical potential of the QDs and the tunneling barriers. As the tunnel coupling changes with the chemical potentials of the QDs, one can employ additional electrostatic gates on the tunnel barriers to keep the tunnel coupling fixed [55].

Finally, we discuss the effect of decoherence on the preparation of the Bell states in the charge qubit since the spin qubit has a coherence time (on the order of microseconds) longer than the charge qubit. Due to the electron–phonon interactions, an intrinsic decoherence mechanism in the QDs, the lifetime of the charge qubit is on the order of 16 ns [56, 57]. In realistic situations, the tunnel coupling $\lambda$ can be modulated by electrostatic gates and can reach the order of 1–10 $\mu$eV, so the total manipulation time is on the order of 6–60 ns by the adiabatic passage. Hence, the adiabatic passage would be invalid for the Bell state preparation if the tunnel coupling $\lambda$ is too weak. However, the manipulation time is on the order of 0.7–7 ns by Lyapunov control, which is much shorter than the lifetime of the charge qubit. Therefore, Lyapunov control is feasible for the entangled state preparation when the tunnel coupling is not very large.

In conclusion, by the teleportation scheme, the crossed AR scheme, the intradot spin flip, and the scheme beyond the intradot spin flip, we show how to entangle two QDs mediated by a pair of MFs. The Bell states can be prepared in both the charge degrees and the spin degrees of QDs by Lyapunov control. In contrast, we compare our results with those by the adiabatic passage. The Lyapunov control manifests advantages over the adiabatic passage at flexibility designing control fields and accelerating control time.

In the teleportation scheme, the system can be driven into Bell state $|\psi_T\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)|10\rangle$ by both adiabatic passage and Lyapunov control when the initial state is $|0001\rangle$. When the Josephson coupling is taken into account, it is not available to prepare Bell states perfectly by the adiabatic passage due to the existence of Rabi oscillation. However, the Lyapunov control can still work well by modulating the shape of the control fields. In addition, we find that the Cooper pair exchange can also be served as the control Hamiltonian to generate Bell states. In the crossed AR scheme since a low-energy eigenstate, whose amplitude mainly locates at one of the bases, does not exist, we directly turn to Lyapunov control. The results show that the system can reach Bell state $|\psi_T^0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|00\rangle$ when we regulate the chemical potentials of the two QDs. However, if we choose the Cooper pair exchange as the control Hamiltonian, whether it can generate the Bell state successfully or not depends on the parity measurement results of the Dirac fermion formed by the MFs.

As to the entanglement in the spin degree of freedom, we have studied the system in the presence of the intradot spin flip process. Through exploring the low-energy eigenstates of the system, it demonstrates that one can achieve Bell state $|\psi_{Tf}^s\rangle = (|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)|00\rangle$ by both adiabatic passage and Lyapunov control. For the scheme beyond the intradot spin flip, the system consists of two TS wires and two QDs. Interestingly, we find that the results are quite different in distinct adiabatic passages, i.e., decreasing the chemical potentials of the two QDs simultaneously or in turns. This diversity originates from the existing quantum destructive interference in adiabatic evolution. Finally, we have shown that the system can certainly be driven into distinct Bell states by Lyapunov control.

Acknowledgments

We thank H F Lü for helpful discussions. This work was supported by the National Natural Science Foundation of China (Grants No. 11534002 and No. 61475033).

References

[1] Kitaev A 2001 Phys. Usp. 44 131
[2] Qi X L and Zhang S C 2011 Rev. Mod. Phys. 83 1057
[3] Alicea J 2012 Rep. Prog. Phys. 75 076501
[4] Beenakker C W J 2013 Annu. Rev. Condens. Mat. Phys. 4 113
[5] Das A, Ronen Y, Most Y, Oreg Y, Heiblum M and Shtrikman H 2012 Nat. Phys. 8 887
[6] Mourik V, Zuo K, Frollov S M, Plassard S R, Bakkers E P A M and Kouwenhoven L P 2012 Science 336 1003
[7] Rokhinson L P, Liu X and Furdyna J K 2012 Nat. Phys. 8 795
[8] Lee E H, Jiang X, Houzet M, Aguado R, Lieber C M and Franceschi S D 2014 Nat. Nanotechnol. 9 79–84
[9] Perge S N, Drozdov I K, Li J, Chen H, León S, Seo J, MacDonald A H, Bernevig B A and Yazdani A 2014 Science 346 602
[10] Kitaev A Y 2003 Ann. Phys. 303 2
[11] Nayak C, Simon S H, Stern A, Freedman M and Sarma S D 2008 Rev. Mod. Phys. 80 1083
[12] Loss D and Vincenzo D P Di 1998 Phys. Rev. A 57 120
[13] Petta J R, Johnson A C, Taylor F M, Laird E A, Yakoby A, Lukin M D, Marcus C M, Hanson M P and Gossard A C 2005 Science 309 2180
[14] Koppens F H L, Folk J A, Elzerman J M, Hanson R, van Beveren L H W, Vink I T, Tranitz H P, Wegscheider W, Kouwenhoven L P and Vandersypen L M K 2005 Science 309 1346
[15] Hassler F, Akhemerov A R, Hou C Y and Beenakker C W J 2010 New J. Phys. 12 125002
[16] Sau J D, Tewari S and Sarma S D 2010 Phys. Rev. A 82 052322
[17] Jiang L, Kane C L and Preskill J 2011 Phys. Rev. Lett. 106 130505
[18] Bonderson P and Lutchyn R M 2011 Phys. Rev. Lett. 106 130505
[19] Leijnse M and Flensberg K 2012 Phys. Rev. B 86 104511
[20] Kovalev A A, de A and Shtengel K 2014 Phys. Rev. Lett. 112 106402
[21] Xue Z Y, Gong M, Liu J, Hu Y, Zhu S L and Wang Z D 2015 Sci. Rep. 5 12233
[22] Demirplak M and Rice S A 2003 J. Phys. Chem. A 107 9937
[23] Berry M V 2009 J. Phys. A: Math. Theor. 42 365303
[24] Beauchard K, Koron J M, Mirrahimi M and Rouchon Z 2007 Syst. Control Lett. 56 388
[25] Corong J M, Grigoriu A, Lefser C and Turinici G 2009 New J. Phys. 11 105043
[26] Wang X T and Schirmer S G 2009 Phys. Rev. A 80 042305
[27] Yi X X, Huang X L, Wu C F and Oh C H 2009 Phys. Rev. A 80 052316
[28] Wang X T, Bayat A, Bose S and Schirmer S G 2010 Phys. Rev. A 82 012330
[29] Shi Z C, Zhao X L and Yi X X 2015 Phys. Rev. A 91 032301
[30] Oreg Y, Refael G and von Oppen F 2010 Phys. Rev. Lett. 105 177002
[31] Lutchyn R M, Sau J D and Sarma S D 2010 Phys. Rev. Lett. 105 077001
[32] Fu L 2010 Phys. Rev. Lett. 104 056402
[33] Plugge S, Zazunov A, Sedano F and Egger R 2015 Phys. Rev. B 91 241507
[34] Zazunov A, Yeyati A L and Egger R 2011 Phys. Rev. B 84 165440
[35] Hüttzen R, Zazunov A, Braunecker B, Yeyati A L and Egger R 2012 Phys. Rev. Lett. 109 166403
[36] Didier N, Gibertini M, Moghaddam A G, König J and Fazio R 2013 Phys. Rev. B 88 024512
[37] Recher P, Sukhorukov E V and Loss D 2001 Phys. Rev. B 63 165314
[38] Lesovik G B, Martin T and Blatter G 2001 Eur. Phys. J. B 24 287
[39] Nilsson J, Akhemerov A R and Beenakker C W J 2008 Phys. Rev. Lett. 101 120403
[40] Leijnse M and Flensberg K 2013 Phys. Rev. Lett. 111 060501
[41] Fu L and Kane C L 2008 Phys. Rev. Lett. 100 096407
[42] Sau J D, Lutchyn R M, Tewari S and Sarma S D 2010 Phys. Rev. Lett. 104 040502
[43] Tewari S, Zhang C, Das Sarma S, Nayak C and Lee D H 2008 Phys. Rev. Lett. 100 027001
[44] Flensberg K 2011 Phys. Rev. Lett. 106 090503
[45] Leijnse M and Flensberg K 2011 Phys. Rev. Lett. 107 210502
[46] Ke S S, Lv H F, Yang H J, Guo Y and Zhang H W 2015 Phys. Lett. A 379 170
[47] Khatrikivi A V and Nazarov Y V 2000 Phys. Rev. B 61 12639
[48] Sticlet D, Beno C and Simon P 2012 Phys. Rev. Lett. 108 096802
[49] Kjaergaard M, Wolms K and Flensberg K 2012 Phys. Rev. B 85 020503
[50] Deng M T, Yu C L, Huang G Y, Larsson M, Caroff P and Xu H Q 2012 Nano Lett. 12 6414
[51] Finck A D R, Van Harlingen D J, Mohseni P K, Jung K and Li X 2013 Phys. Rev. Lett. 110 126406
[52] Higginbotham A P, Albrecht S M, Kirsanskas G, Chang W, Kuemmeth F, Krosguptr P, Jespersen T S, Nygard J, Flensberg K and Marcus C M 2015 Nat. Phys. 11 1017
[53] Fasch C, Fuhrer A, Samuelson L, Golovach V N and Loss D 2007 Phys. Rev. Lett. 98 266801
[54] Nilsson H A, Caroff P, Thelander C, Larsson M, Wagner J B, Wernersson L E, Samuelson L and Xu H Q 2009 Nano Lett. 9 3151
[55] Nady-Perge S, Frolov S M, Bakkers E P A M and Kouwenhoven L P 2010 Nature 468 1084
[56] Petta J R, Johnson A C, Marcus C M, Hanson M P and Gossard A C 2004 Phys. Rev. Lett. 93 186802
[57] Petersson K D, Petta J R, Hu H and Gossard A C 2010 Phys. Rev. Lett. 105 246804