Entrainment of a Van der Pol-Type Circadian Pacemaker to Daylight Cycle

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Abstract
A van der Pol self-sustained oscillator with higher-order nonlinearity exhibits a rich dynamics, with multiple periodic attractors, and still the model allows analytical approximations. Some of these properties can be conveniently exploited in the framework of circadian oscillations. When interpreted as a biological oscillator that determines the alternation sleep/awake, the dynamic variable exhibits some interesting features that can be related to biological behavior. We analyze in the paper the phenomenon of entrainment of van der Pol-type circadian pacemaker to daylight cycle. We determine the amplitude and frequency of the circadian model without natural forcing light, and find that the agreement between analytical and numerical results hardly depends on the stiffness coefficient, μ of circadian oscillations. It is shown that a practical and precise drive which imitates the effects of the conditions of natural light can be introduced in the system and analytically treated. Considering the effects of forcing light on the sleep-aware cycle model, we find that a strong drive destroys the circadian oscillations and thus we confirm the importance of darkness for regular circadian oscillations. Moreover, we observe the reverse situation when we take into account the phase in the forcing light. For instance, for \( \phi = \pi / 2 \) and the duration of daylight \( D_L = 12h \), the phenomenon of quenching of circadian oscillations disappears. The comparison between analytical treatment and numerical simulations of effects of light is discussed.

Keywords Van der Pol oscillator · Natural light · Circadian oscillations

1 Introduction
The van der Pol oscillator is a paramount system to describe self-sustained oscillations of artificial as well as biological systems. No wonder, it has been employed to model one of the fundamental oscillations of human, and even of primitive bacterial, life: the circadian alternate phases of sleep and awake [1]. To such purpose the van der Pol original version has been enriched with higher-order nonlinearity, as well as with the coupling of other degrees of freedom to describe the circadian oscillations of various body functions, along with the external drive of the light [2]. At the same time, the mathematical model of self-sustained oscillations has been investigated to include several effects of higher-order polynomial dissipation and to describe biorhythmicity [3, 4], or the response to external drives to describe forcing, noise [5], as well as delayed perturbation [6–8].

The aim of this paper is to interpret the mathematical developments in the context of the sleep/wake cycle which is a biological phenomenon under active investigation with quantitative methods which can lead to predicting the circadian phase [10]. The general objective is to seek analytical and numerical approximations which describe the modified van der Pol type oscillator in the presence of a light stimulus and which may be useful for biological research. To do so, we assume that the van der Pol oscillator simulates. To do so, we assume that the van der Pol oscillator simulates the stimulator whose biological name is supra-chiasmatic nucleus circadian stimulator responsible for the synchronization of the internal clock of the human organism located at the level of the brain [9] and considered that the circadian oscillator is subjected to an interaction with the environment.
[11] and trained to periodic light stimuli [12]. In this work, we propose to bring in a particular circadian light stimulus, which is extinguished for a certain time (the “night”) and in the remaining part is sinusoidal in shape, to mimic the behavior of light from the sun. We first investigate whether such behavior is able to lock the system and thus impose circadian rhythm, and then we highlight other interesting effects on the wake/sleep cycle. Thus, a purpose of the work is to modify the van der Pol oscillators to propose a specific realization of the nonlinear self-sustained oscillator that might fit the dynamic of the sleep/awake cycles with a peculiar form of the light stimulus. A second novelty consists in exploiting the mathematical treatability of the system to derive some analytical results.

The work is organized as follows: Sect. 2 presents the mathematical formulation of the basic oscillator, and the main analytical features of the undriven system and for which we propose an analytical treatment in Sect. IIIB. Section 3 proposes a model for the light influence that is shown to be analytically manageable in Sect. IIIB, and whose effects are described in Sect. 4. Section 5 collects the conclusions.

2 The Sleep/Awake Cycle Model and Dynamical Analysis

This section describes the main features of the mathematical model to be investigated, in the interpretation of the sleep/awake circadian oscillations.

2.1 The Undriven Oscillator to Model the Sleep/Awake Cycle

The model used in our analysis for the sleep-aware cycle model is the core self-sustained oscillator of the model for circadian oscillations determines [13–15], described by the following differential equation:

\[
\left(\frac{12}{\pi}\right)^2 \ddot{x} + \mu \left(-\frac{1}{3} - 4x^2 + \frac{256}{15} x^6\right) \left(\frac{12}{\pi}\right) \dot{x} + \left(\frac{24}{\tau_i}\right)^2 x = 0.
\] (1)

In the original model [14], \(x\) is the component of the human circadian pacemaker that closely reflects the endogenous core body temperature cycle, as measured during constant routine conditions [16]. \(\tau_i\) represents the intrinsic period of the circadian oscillator and \(\mu\) the stiffness of the oscillator that also regulates the nonlinear damping. At variance with the model of Eq. (1), the birhythmic model [3] is more general, as it contains also a term \(\propto x^4\). It is however important to underline that more refined models of the sleep/awake cycle are higher dimensional, inasmuch they contain also the processes that generate the circadian limit [17]. In this work, we employ the nonlinear oscillator Eq. (1) as the most important oscillator that regulates the sleep/awake cycle, and we interpret some of the features of the oscillations of Eq. (1) as indicative of the sleep/awake cycle.

2.2 Analysis of the Dynamics

We are going to determine analytically the amplitude and the frequency of the circadian oscillator model [13–15], for which we rewrite Eq. (1) in the form below:

\[
x + \left(\frac{2\pi}{\tau_i}\right)^2 x = -\mu \left(-\frac{1}{3} - 4x^2 + \frac{256}{15} x^6\right) \left(\frac{\pi}{12}\right) \dot{x}.
\] (2)

Equation (2) is the main equation of this paper, whose analysis, also in the presence of a peculiar drive, is the subject of our investigation. Lindstedt perturbation method [18, 19] can be used to find the amplitudes and frequencies. Let us set \(\tau = \omega t\), where \(\omega\) is the unknown frequency, and to assume that the periodic solution of the differential Eq. (2) has the following approximate form:

\[
x(\tau) = x_0(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) + \ldots
\] (3)

where \(x_n(\tau)(n = 0, 1, 2, \ldots)\) are functions of \(\tau\) of period \(2\pi\). It is possible to write the frequency \(\omega\) in the form:

\[
\omega = \omega_0 + \mu \omega_1 + \mu^2 \omega_2 + \ldots
\] (4)

where \(\omega_n(n = 0, 1, \ldots)\) are unknown constants. Using the perturbation method, the periodic solution of Eq. (2) can be approximated by the following expression:

\[
x(\tau) = A \cos \omega t + \mu \frac{\pi}{\omega_0} \left[ \left(\frac{73}{720} A^3 - \frac{3}{96} A^3\right) \sin \omega t 
- \left(\frac{1}{40} A^7 - \frac{1}{96} A^3\right) \sin 3\omega t - \frac{1}{216} A^7 \sin 5\omega t
- \frac{1}{2160} A^7 \sin 7\omega t\right] + O(\mu^2),
\] (5)

where the amplitude \(A\) obeys to the following equation:

\[
A \left(\frac{1}{6} + \frac{1}{2} A^2 - \frac{2}{3} A^6\right) = 0,
\] (6)

and the frequency \(\omega\) given by:

\[
\omega = \frac{2\pi}{\tau_i} - \frac{\pi^2}{\omega_0} \left[ \frac{31}{10800} A^{12} - \frac{67}{17280} A^8 - \frac{73}{51840} A^6
+ \frac{1}{1152} A^4 + \frac{1}{2304} A^2\right] \mu^2 + O(\mu^3).
\] (7)

It is possible to solve Eq. (6) to identify the conditions that give one root, and therefore one limit cycle, with
amplitude $A \approx 1.0$ and the frequency $\omega \approx 0.25745$ for the stiffness coefficient $\mu = 0.23$. Figure 1 shows the effects of stiffness coefficient $\mu$ on the behavior of the frequency (see Fig. 1a) and the period $T$ (see Fig. 1b) of the circadian cycle. The comparison between the analytical and the numerical results shows a nice agreement for $\mu \leq 0.35$, while for $\mu > 0.35$ the analytical and numerical results diverge. These results are in agreement with the one obtained in 1995 by Choe and Czeiser [20], who have estimated that the stiffness of human circadian pacemakers was between 0.005 and 0.34. Therefore, to use $\mu = 0.23$ for the van der Pol oscillator is reasonable. This oscillator is the basic element of the forced system, that is the subject of next Section.

3 A Model for the Natural Light Forcing of the Sleep-Awake Cycle

To properly insert a drive into van der Pol system with quintic dissipation to mimic the light/darkness effect, one should consider a function that is blank for some part of the period 24h (the night time), and displays a variable luminosity for the remaining part (the increase from the sunrise to noon, and then the decrease until dusk). A simple mathematical function that we propose is as follows:

$$I(t) = \theta(D_L - t_{24})I_0 \sin (\omega t_{24} + \phi),$$

$$\omega = \frac{\pi}{D_L},$$

$$t_{24} = \text{mod}(t, 24),$$

here $I(t)$ is the forcing, 24h is the cycle period, $D_L$ is the duration of the daylight, $\theta$ the Heaviside function. The model so far introduced employs arbitrary units. However, we find it convenient to have the time normalized to 1h, so that a day corresponds to $t = 24$, to facilitate an intuitive interpretation of the results. Thus, $D_L = 12$ denotes the equinox duration of the day (neglecting twilight effects). The corresponding angular velocity reads $\omega = \pi/12$, the phase $\phi$ mimics a shift of the daylight respect to the cycle. This shift might be relevant for the transient analysis, or to describe a sudden shift respect to the daylight variation, as can occur in intercontinental flights, where sudden time shift $\Delta t$ can appear that induces a change in the phase $\Delta \phi = \omega \Delta t = \pi \Delta t / D_L$.

Implicitly, we are assuming that on the scale of the oscillations of the variable “$x$” (some proxy of circadian biological oscillations such as the body temperature), the frequency of the irradiation is too fast to have an effect, and only its intensity is relevant.

Figure 2 shows the variation of the light Eq. (8) in three days with two different values of the amplitude $I_0$ of the light and for $D_L = 5h$ (a) and $D_L = 18h$ (b), consistent results are obtained for other values of the daylight duration. It is important to underline that the Heaviside function multiplies a sinusoidal function, thus altering the drive that is not anymore sinusoidal as in [5]. However, the analytic estimation for the purely sinusoidal drive can still be a guideline for the system, as will be shown below.

4 The Effects of the Light on the Sleep-Aware Cycle Model

In this section, we analyze the influence of natural light parameters on the dynamic behavior of the Jewett et al. model Eq. (2) [15]. At the core of the Jewett et al. model is the following modified driven van der Pol (mvdP) differential equation:

$$\left(\frac{12}{\pi}\right)^2 \ddot{x} + \mu \left(-\frac{1}{3} - 4x^2 + \frac{256}{15}x^6\right) \left(\frac{12}{\pi}\right) \dot{x} + \left(\frac{24}{\tau_x}\right)^2 x = I(t).$$

It will therefore be a question to check weather the duration of the daylight $D_L$ and of the intensity $I_0$ of the forcing influences the sleep-wake cycle. In other words, it is relevant to study the effects of $I_0$ and $D_L$ on the period and the amplitude of the drive of the nonlinear oscillator. We will proceed

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**Fig. 1** Variation of the period $T$ (a) and of the frequency $\omega$ (b) of circadian oscillations versus the stiffness coefficient $\mu$. These results are obtained using Eq. (7)
to an analytical investigation and we complete the analysis with numerical simulations.

4.1 Analytical Investigations

It is possible to gain an analytical insight of the system Eq. (9) under the influence of the peculiar shape of the light Eq. (8) using the Fourier series of the signal. As the signal is periodic, the Fourier expansion (to the order $N$) is promptly written in the form:

$$I_n(t) = \frac{a_0}{2} + \sum_{i=1}^{n} (a_i \cos(\omega_i t) + b_i \sin(\omega_i t)), \tag{10}$$

with the coefficients given by:

$$a_0 = \frac{\omega}{\pi} \int_{0}^{\pi} I(t) dt, \tag{11}$$

$$a_n = \frac{\omega}{\pi} \int_{0}^{\pi} I(t) \cos(n\omega t) dt, \tag{12}$$

$$b_n = \frac{\omega^2}{\pi} \int_{0}^{\pi} I(t) \sin(n \omega t) dt. \tag{13}$$

Inserting the light Eq. (8) it is possible to retrieve the coefficients:

$$a_0 = \frac{2I_0}{\omega} \cos \phi, \quad a_1 = \frac{I_0}{2} \cos \phi, \quad b_1 = \frac{I_0}{2} \sin \phi. \tag{14}$$

For $n = 1$, the first Fourier component reads:

$$I_1(t) = \frac{I_0}{\omega} \cos \phi + \frac{I_0}{2} \sin(\omega t + \phi) \tag{15}$$

Figure 12 compares the amplitude of the system oscillations as a function of the original light fluctuations $I(t)$ of Eq. (9) and of the first Fourier component $I_1(t)$ estimated through Eq. (15). The results show that the full model and the first component produce almost identical oscillations, and therefore it seems that just the first Fourier component captures most of the effect. Therefore, we can now proceed for analytical investigations of the model Eq. (9) by consider the mvdP model driven by the first component:

$$\ddot{x} + \alpha \dot{x} = -\mu \left( \frac{1}{3} - 4x^2 + \frac{256}{15} \omega^2 \right) \left( \frac{\pi}{12} \right) x + I_{1a} + I_{1b} \sin(\omega t + \phi), \tag{16}$$

with $I_{1a} = \left( \frac{\pi}{12} \right)^2 \frac{I_0}{2} \cos(\phi)$ and $I_{1b} = \left( \frac{\pi}{12} \right)^2 \frac{I_0}{2}.$

For this purpose, let us set the solution of Eq. (14) in the form:

$$x(t) = A_1 \cos \omega t + A_2 \sin \omega t + A_3 = A \cos(\omega t - \phi_1) + A_3, \tag{17}$$

where $A = \sqrt{A_1^2 + A_2^2}$ and $\phi_1 = \arctan(\frac{A_2}{A_1}).$

Inserting Eq. (17) in Eq. (16) and equating the cosine and sine terms separately, gives:

$$I_{1b} \cos \phi = (\Omega^2 - \omega_0^2) A_1 + \left( \frac{5}{64} c \Omega \Lambda^6 + \frac{15}{8} c \Omega \Lambda_3^2 A_4 \right) + \left( \frac{1}{4} \beta \Omega + \frac{15}{4} c \Omega \Lambda_3^2 A_2 \right) A_2, \tag{18}$$

$$I_{1b} \sin \phi = -\left( \frac{5}{64} c \Omega \Lambda^6 + \frac{15}{8} c \Omega \Lambda_3^2 A_4 + \left( \frac{1}{4} \beta \Omega + \frac{15}{4} c \Omega \Lambda_3^2 A_2 \right) A_1 + (\Omega^2 - \omega_0^2) A_2, \tag{19}$$

$$A_3 = \frac{I_{1b}}{\omega_0^2}, \tag{20}$$

with $a = \mu \frac{\pi}{12},$ $b = \mu \frac{r}{3},$ $c = -\mu \frac{256 \pi}{180}.$

The amplitude $A$ satisfies the following equation:
Having derived the above equation, we propose in the following Section the numerical approach to handle the model (17). The results thus obtained will be compared to numerical simulations of Eq. (9).

4.2 Numerical Simulations and Discussions

4.2.1 Case of Zero Initial Phase of the Light

Let us assume that the initial phase of the natural light reads $\phi = 0.0$. The duration of daylight $D_L$ is chosen between 10 and 20 hours, with an average duration, including dawn and dusk, of about 15 hours. Indeed, we can note for example that in the Northern hemisphere, the summer days are longer while the winter days are shorter, and the twilight lasts about one hour. Figures 3 and 4 show the effects of the amplitude $I_0$ and the duration of the daylight $D_L$, respectively, on the temporal evolution $x(t)$ of the mvdP model under the action of light. It appears that the duration of the daylight $D_L$ influences the behavior of the mvdP. To be concrete, the figure might represent the effects of the daylight from darkness for almost all day ($D_L \approx 0$, e.g., summer at the south polar circle) to longer light hours (e.g., as one moves North during summer) the case for all-day light ($D_L \approx 24$, e.g. summer at the north polar circle). The resulting behavior is quite peculiar. The oscillator displays a finite, relatively large amplitude only if the light duration is short enough,
and disappears as the light condition extends. This is indicative of possible sleep disturbances if the subject is exposed to darkness for a too short period. It is also noticeable that a too strong drive destroys the circadian oscillations, again confirming the importance of darkness for regular circadian oscillations. Thus the duration of the light and the light intensity play a role in the amplitude of sleep; in fact, the amplitude of sleep decreases as light intensity increases until it tends to be canceled. In other words, it is hardly possible to fall asleep under strong illumination for a long time. Figure 13 shows the partial adherence between the numerical and analytical results of the effect of the light intensity for several different values of the duration of daylight $D_L$. It appears that the period of oscillations changes automatically depending for the value of the amplitude $I_0$ (low $I_0$ or great $I_0$). These effects are shown on Fig. 7 where we plot the evolution of $x(t)$ and the phase shift for several different values of the amplitude $I_0$ with the daylight $D_L = 12h$.

In Fig. 8, we present the variation of the period $T$ of circadian oscillations versus the amplitude $I_0$ of the light for several different values of the duration of daylight $D_L$. It appears that when the amplitude $I_0$ increases from zero, the light does not induces in the mvdP model oscillations at 24h below a critical value $I_{0c}$ at which synchronization (that is, the period of the mvdP oscillator matches the period of the drive within numerical approximation) of the mvdP model and the light appears. Let us note that this critical value $I_{0c}$ depends on the system parameters, such as the duration of the daylight. Table 1 shows how $I_{0c}$ depends on the duration of the daylight $D_L$. It appears that the increase in the duration of light $D_L$ leads to the decrease in the intensity $I_{0c}$, moreover to have a large critical value, a value of the duration of very small light ($D_L \leq 0.001$). In other words, depending on the season, it emerges that a critical intensity is needed in the summer is about $I_{0c} = 0.064$ and in the winter $I_{0c} = 0.2$ to have a standby cycle / 24h sleep. Fig. 9 shows in the $(I_{0c}, D_L)$ plane, the boundary between the region where the light synchronizes the system to 24h (SP:

![Figure 7](image1.png)

![Figure 8](image2.png)

**Table 1** Boundary between synchronization states and unsynchronization states in the $(I_0, D_L)$ plane ($SP = $ Synchronized period / USP = Unsynchonized period). These results are obtained by numerical simulations of Eqs. (9, 13) and shown in Fig. 11.

| $D_L$ | 0.001 | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
|------|------|---|---|---|---|---|----|----|----|----|----|----|----|----|
| $I_{0c}$ | 15 | 0.632 | 0.316 | 0.162 | 0.114 | 0.090 | 0.076 | 0.068 | 0.064 | 0.066 | 0.066 | 0.066 | 0.076 | 0.09 |
$T = 24h$) and the region where the light does not synchronize
the system to 24h (USP: $T \neq 24h$).

Given that the period of the undriven oscillator is not
24 hours, a certain time, called transit time, is needed, dur-
ing which the forcing brings the oscillator to a period of 24
hours. Table 2 gives more information on the duration of
the transit time as a function of $I_0$ and $DL$.

Table 2 Transient time(days) for several different values of $I_0$ and $DL$. These results are obtained by numerical simulations of Eqs. (9, 13) and
shown on Fig. 12

| $DL$ | $I_0$ | 0.0025 | 0.005 | 0.5 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|------|-------|-------|-----|---|---|---|---|---|---|---|---|
| 10   |      | 11.46 | 11.04 | 1.33 | 2.42 | 9.33 | 21.37 | 4.33 | 11.33 | 57.29 | 184.17 | 366.67 |
| 12   |      | 11.46 | 11.04 | 1.37 | 2.50 | 21.25 | 12.5 | 7.08 | 54.17 | 258.33 | 504.17 | 600.6 |
| 16   |      | 11.46 | 11.04 | 1.46 | 6.67 | 21.87 | 8.50 | 95.83 | 345.83 | 1145.83 | 2862.50 | 5916.67 |

Figure 11 shows its variation as a function of the intensity
$I_0$ of the light. One notices that the phase remains constant,
independent of the season, and it is located around the value
$\psi = \frac{\pi}{2}$.

4.2.2 Case of Nonzero Initial Phase of the Light: $\phi \neq 0$

In this subsection, we consider that there is a nonzero
phase difference between the forcing and the oscillator.
In other words, it is a question of studying the wake/sleep
cycle in the case where the light acts with a delay or an
advance on the circadian stimulator ($\phi \neq 0$). Figures 14
and 15 show the effects of the phase $\phi$ on the variation of
the period of circadian oscillations versus the amplitude
of the light $I_0$ (see Fig. 14) and the duration of daylight $DL$. 

![Fig. 9](image_url)  
**Fig. 9** Effects of the intensity of the light $I_0$ on the temporal evolution
of $x(t)$ (a) and phase portrait (b) with $DL = 12h$. The other parameters
are defined in Fig. 3 and results are obtained using Eqs. (9, 13)

![Fig. 10](image_url)  
**Fig. 10** Effects of the duration of daylight $DL$ on the variation of the
period $T$ of circadian oscillations versus the amplitude of the light $I_0$.  
$DL = 10h$ (a) and $DL = 16h$ (b). The other parameters are defined in
Fig. 3 and results are obtained using Eqs. (9, 13)
(see Fig. 15). It shows that the phase shift influences the period of the circadian oscillations if the light intensity is low and leaves the period at 24 hours if the light intensity increases. Figure 16 shows the effects of the duration of daylight $D_L$ on the variation of the period of circadian oscillations versus phase $\phi$ for two values of the intensity $I_0$. It appears in this figure that for very small values of the amplitude of light, $I_0$, the period of circadian oscillations oscillates between 24.35 and 24.55, whereas when $I_0$ becomes large, the period of circadian oscillations remains constant around 24 hours. Figures 17 and 18 show the analytical and numerical amplitude-response versus the amplitude $I_0$ for $\phi = \pi/4$ and $\pi/2$ (see Fig. 17) and versus $\phi$ for $I_0 = 0.2$ and $I_0 = 4.0$ (see Fig. 18). Figure 19, we present the numerical effects of the daylight duration $D_L$ on the amplitude-response to the phase $\phi$ for two different values of $I_0$. Figure 19 shows that the phase shift plays a big role on the amplitude of the sleep/wake cycle. Indeed the amplitude increases when the phase shift approaches towards $\pi/2$ and decreases when it moves away.

On the other hand according to Figs. 20 and 21, it appears that the phase shift $\phi$ does not really play a big

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**Fig. 11** Boundary in the $(I_0, D_L)$ plane between the region of synchronized period (SP) and the region of unsynchronized period (USP). The other parameters are defined in Fig. 3 and results are obtained using Eqs. (9, 13)

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**Fig. 12** Effects of the duration of daylight $D_L$ on the transient time versus $I_0$. The other parameters are defined in Fig. 3 and results are obtained using Eqs. (9, 13)

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**Fig. 13** Variation of the phase of circadian oscillations versus the amplitude $I_0$ of the light for several different values of the daylight $D_L$. (a) $D_L = 10h$ and (b) $D_L = 16h$. The other parameters are defined in Fig. 4 and results are obtained using Eqs. (9, 13)

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**Fig. 14** Effects of the phase $\phi$ on the variation of the period $T$ of circadian oscillations versus the amplitude of light $I_0$ for $D_L = 12h$. The other parameters used are defined in Fig. 4 and results are obtained using Eqs. (9, 13)
The role at the level of the phase since it is practically constant \( (\psi \sim \frac{\pi}{2}) \). We can see in Fig. 22 that the phase shift does not have a great influence on the critical intensity (intensity around which the forcing imposes its frequency on the oscillator).

Figure 23 gives more information on the transit time (time at which the sleep-wake cycle becomes stable after the force is applied). We find that the transit time is relatively small when \( \phi \) approach \( \frac{\pi}{2} \), but increases otherwise. Table 3 gives us more information on this subject.

The main point is that when the daylight rises earlier or later than usual, it creates a dysfunction in the wake/sleep cycle. Indeed this can create a variation of the desire to sleep (amplitude of sleep Fig. 19) in particular by increasing or decreasing it according to the delay or the advance. However, it does not play a relevant role in the duration of the cycle which varies slightly when the intensity of the external light is very small while it remains constant for high intensity values (Fig. 16). The comparison between analytical and numerical results is acceptable. Thus, it emerges that the first Fourier component and the ansatz Eq. (17) well describe the main features of the nonlinear oscillator under the effect of day/night light oscillations. We can therefore conclude that the first component of the Fourier series is the dominant drive.
4.3 Discussions

Two more interesting effects occur in the mvdP nonlinear oscillator Eq. (9) that are related to the regularity of the sleep/awake cycle. First, the time necessary to adapt to light forcing of Fig. 23. This is relevant to investigate the effects on the sleep/awake cycle of strong perturbations to the light cycle, as can occur in intercontinental flights in the North or South directions, or in artificial light conditions. As already shown in Fig. 6, the nonlinear oscillator exhibits a 24 hour period only if the drive intensity is large enough the threshold depends upon the daylight duration. The effect is illustrated in more details in Figs. 7 and 8. In the latter figure it is shown the minimal amplitude to obtain synchronization. It is therefore interesting to see that the accommodation time increases with the duration of daylight; indeed this time could take weeks or even months depending on the light intensity.

Another interesting effect might be caused by a sudden change in the phase of the oscillator respect to the light cycle, as can occur in intercontinental flights toward East or West [21]. Both have a straightforward interpretation in the sleep/awake interpretation of the model. The former corresponds to the time necessary to adapt to different light conditions, e.g. in intercontinental flights. The latter is the minimal amplitude of the light oscillations that cause a regular, 24h period, sleep/awake cycle.

![Fig. 18](image1.png)

**Fig. 18** Analytical and numerical amplitude-response versus the phase $\phi$ for $I_0 = 0.2$ (a) and $I_0 = 4$ (b). The duration of daylight used is $D_L = 12h$ and the other parameters are defined in Fig. 3. (Analytical and numerical results are obtained from Eq. (17) and Eqs. (9, 13), respectively)

![Fig. 19](image2.png)

**Fig. 19** Effects of the duration of daylight $D_L$ on the variation of the amplitude-response versus the phase $\phi$ for (a) $I_0 = 0.02$, (b) $I_0 = 4$. The other parameters are defined in Fig. 3 and results are obtained using Eqs. (9, 13)

![Fig. 20](image3.png)

**Fig. 20** Variation of the phase of circadian oscillations versus the amplitude $I_0$ of the light for several different values of $\phi$ ($D_L = 12h$). The other parameters are defined in Fig. 3 and results are obtained using Eqs. (9, 13)
Conclusions

A nonlinear oscillator with higher-order polynomial dissipation is at the core of a model for the sleep/awake cycle [2]. This oscillator is mathematically very tractable [3–7, 22], and has offered a number of results, from response to sinusoidal drive to effects of noise [8, 23] (whose effects are not studied here). In the present work we have shown how the properties of the oscillator can be interpreted in the case of sleep/awake cycle. Moreover, we have compared the sinusoidal forcing to some more realistic forcing for the light cycle. The drive can be adapted to reproduce the seasonal changes in the duration of the daylight, as well as some particular disorders of the light stimulus, such as intercontinental flights that cause a sudden change of the phase of the drive.

The simulations show first that a function which is empty for a certain time (the “night”) to imitate the behavior of daylight is capable, within certain limits, of entraining the

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Table 3

| $D_L$ | $I_0$ | 0.0025 | 0.005 | 0.5  | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    |
|-------|-------|--------|-------|------|------|------|------|------|------|------|------|------|
| 12 : $\pi/3$ | 10.63 | 9.38  | 2.70  | 2.50 | 1.67 | 1.04 | 14.16| 35.41| 11.25| 5.21 | 18.75|
| 12 : $\pi/2$ | 10.41 | 9.37  | 1.25  | 1.67 | 1.67 | 1.67 | 1.67 | 1.67 | 1.67 | 1.67 | 1.67 | 1.67 |
| 16 : $\pi/3$ | 10.41 | 9.38  | 0.83  | 1.25 | 2.29 | 31.25| 100  | 8.75 | 5.83 | 29.16| 108.33|
| 16 : $\pi/2$ | 10.41 | 9.37  | 0.83  | 1.46 | 1.46 | 1.46 | 1.46 | 1.46 | 1.46 | 1.46 | 1.46 | 1.46|

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5 Conclusions

A nonlinear oscillator with higher-order polynomial dissipation is at the core of a model for the sleep/awake cycle [2]. This oscillator is mathematically very tractable [3–7, 22], and has offered a number of results, from response to sinusoidal drive to effects of noise [8, 23] (whose effects are not studied here). In the present work we have shown how the properties of the oscillator can be interpreted in the case of sleep/awake cycle. Moreover, we have compared the sinusoidal forcing to some more realistic forcing for the light cycle. The drive can be adapted to reproduce the seasonal changes in the duration of the daylight, as well as some particular disorders of the light stimulus, such as intercontinental flights that cause a sudden change of the phase of the drive.

The simulations show first that a function which is empty for a certain time (the “night”) to imitate the behavior of daylight is capable, within certain limits, of entraining the
oscillator which remains, within certain ranges of parameters, locked to the external drive period 24 hours, the adaptation time depends on the intensity of the stimulus and the duration of the daylight; the desire to sleep is under the control of each of the stimulus parameters. Finally, the presence of the time difference between the light and the wake-up/sleep cycle slightly modifies the period of the system for very small values of light intensity and locks the system to the period of the stimulus (24h) at or beyond certain \( I_c \) values. This shift plays a considerable role in the response, which can be interpreted as the desire to sleep. The agreement between the analytical results obtained with a treatment of the light features through Fourier analysis and numerical results is acceptable.

Few words of caution. The proposed oscillator is but a part of more complete mathematical models that describe the sleep/awake cycle. In addition, it is important to note that noise, which is unavoidable in biological systems, was not included in our study. We wish to underline that a random term should be included in future work with more complete models.

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