Integrability of potentials of degree $k \neq \pm 2$. Second order variational equations between Kolchin solvability and Abelianity

Guillaume Duval and Andrzej J. Maciejewski

7th February 2014

Abstract

In our previous paper: Integrability of Homogeneous potentials of degree $k = \pm 2$. An application of higher order variational equations, we tried to extract some particular structures of the higher variational equations (the VE$_p$ for $p \geq 2$), along particular solutions of some Hamiltonian systems. Then, we use them to get new Galois obstructions to the integrability of natural Hamiltonian with potential of degree $k = \pm 2$. In the present work, we apply the results of the previous paper, to the complementary cases, when the degrees of the potentials are relative integers $k$, with $|k| \geq 3$. Since these cases are much more general and complicated, we reduce our study only to the second variational equation VE$_2$.

1 Presentation

This paper completes the previous one [4] by applying the same results and strategies to the complementary cases. That is, we still study the integrability of homogeneous potentials along Darboux points, but here we assume that the degree $k$ is an arbitrary relative integer with $|k| \geq 3$. As we shall see thing are technically much more complicated for the following reasons. At first, the assumption over the degree are more general, and secondly, here in contrast to the cases when $|k| = 2$, the Morales Ramis table (see Table 1) gives discreet obstructions at the level of the first order variational equation VE$_1$. This will force the study to encompass a judge number of distinct cases. As a consequence, our major results, Proposition 5, Theorems 4 and 5 below, just concern the Galois group of the second variational equation VE$_2$. They are not definitive results which guarantee that the Galois groups of the associated systems VE$_2^{\alpha}$ and EX$_2^{\gamma,\alpha,\beta}$ are virtually Abelian in such and such cases, but they constitute effective algorithms to test such possibilities in particular cases. These results convert the virtual Abelianity of the Galois group to testing some Ostrowski relations between first level integrals $\Phi, \Psi, \Sigma$, etc, when they are so.

The reader will see that in general, these integrals are very complicated. And the main, difficulty will be the following: from the form of a given integrals, it will be in general very easy to predict that it can be algebraic if it is so. But in contrast when it does not have such specific form, it is very very difficult to decide when it is transcendental. Our main ingredient for this will be Remark 2 below.

In addition to the first paper, this one contains one additional idea which is interesting from a theoretical point of view. This is what we call the cohomological argument, which allow to
test that a second level integral is indeed computable in closed form, that is to test if a Galois group is virtually Abelian, without computing effectively this second level integral in closed form explicitly. This procedure was discovered when dealing with the present second level integrals, but it can be applied in more general contexts.

Let us mention finally, that this approach is not isolated. Combat in [2], deal with the same problem for homogeneous potentials of degree $k = -1$, while in the same time Weil in [1], is studying exactly the same problem than us but from the complementary point of view of gauge-transformations, which has the advantage to convert, the original systems into new ones which are much more simple in the sens that the virtual Abelianity of the Galois group can be directly read in the form of the new system.

In order to helps the reader, into the reading of this paper, let’s briefly present how it is organised. In Section 2, we present the systems $\text{VE}_{2,\alpha}$ and $\text{EX}_{2,\alpha,\beta}$. Here the idea is that as for the study of $\text{VE}_1$, these new systems are much more easy to study after the so called Yoshida-transformation, than in their original time-parametrisation. In Section 3, we present the associated second level integrals involved in those systems and their intrinsic hierarchy. Section 4, contains the technical ingredients both theoretical and practical which are going to be useful for the proofs of Theorems 4 and 5 (in Section 5), and for some effective remarks about $\text{VE}_{2,\alpha}$ and $\text{EX}_{2,\alpha,\beta}$. These practical studies will be done in Sections 6 and 7 below. Moreover, we recommend the reader, to first take a look, to Section 6.4 where we present some experimental facts about the complexity of the law that govern the virtual Abelianity of $\text{VE}_{2,\alpha}$. We hope that this will help him to understand how we are using the present criteria and why we where obliged to deal with such tremendous casuistic.

\section{From $\text{VE}_1$ to $\text{VE}_2$ through the Yoshida transformations}

In all the previous works concerning the integrability of homogeneous potentials along Darboux points, the key result for the analysis of the first variational equation $\text{VE}_1$, was the conversion of this differential system in time parametrisation to an equivalent one in new variable $z$. This is the Yoshida transformation given by

$$t \mapsto z = \varphi^k(t).$$

This transformation convert the initial variational equations over a hyperelliptic curve into a Fuchsian equation over $\mathbb{P}^1$, with singularities at $z \in \{0; 1; \infty\}$.

In this Section we shall recall some results concerning this transformation, and we show how it applies to the study of the second variational equation $\text{VE}_2$.

\subsection{The subsystems of $\text{VE}_2$ to deal with}

Here, we assume that $V''(c)$ is diagonalisable. Hence, $\text{VE}_1$ splits into a direct sum of equations which have the following form

$$\ddot{x} = -\lambda_x \varphi^{k-2} x,$$

where $\lambda_x$ are the eigenvalues of $V''(c)$, and $\varphi = \varphi(t)$ is a particular solution defined by a Darboux point $d$. For each of these equations we denote by $G_1 = G_\alpha = G(k, \lambda_x)$ its differential Galois group over the field $K = \mathbb{C}(\varphi, \dot{\varphi})$.

From Proposition 2.5 of [1], we know that the differential Galois group of $\text{VE}_2$ is virtually Abelian iff the same property hold true for the differential Galois groups of the systems

$$\text{VE}_{2,\alpha} \quad \text{and} \quad \text{EX}_{2,\alpha,\beta} \quad \text{for all} \quad \alpha; \beta; \gamma \quad \text{with} \quad \alpha \neq \beta.$$
In time parametrisation, these systems have the following form (see equations (2.20) and (2.22) of [4])

\[ \begin{align*}
VE_{2,\alpha}^\gamma \quad & \begin{cases} \dot{x} = -\lambda_\alpha \varphi^{k-2} x \\ \dot{y} = -\lambda_\gamma \varphi^{k-2} y + \varphi^{k-3} x^2 \end{cases} \\
EX_{2,\alpha,\beta}^\gamma \quad & \begin{cases} \ddot{x} = -\lambda_\alpha \varphi^{k-2} x \\ \ddot{y} = -\lambda_\beta \varphi^{k-2} y \\ \ddot{u} = -\lambda_\gamma \varphi^{k-2} u + \varphi^{k-3} x y \end{cases}
\end{align*} \]

and

Here, we directly assumed that the coefficients appearing in the non-homogeneous terms in the right hand sides of the last equations of those systems are non zero. Otherwise, the corresponding second variational equation will be reduced to the first variational equation.

2.2 The Morales-Ramis table

The full list of all values of \( \lambda \in \mathbb{C} \) for which the differential Galois group of equation

\[ \ddot{x} = -\lambda \varphi^{k-2} x, \]  

is virtually Abelian is given in the following table, where \( p \) denotes an integer.

| \( G(k,\lambda) \) | \( k \) | \( \lambda \) | Line Number |
|-------------------|-----|----------|------------|
| \( G_\alpha \)   | \( k = \pm 2 \) | \( \lambda \) arbitrary complex number | 1 |
| \( |k| \geq 3 \) | 3   | \( \lambda(k;p) = p + \frac{1}{2} p(p-1) \) | 2 |
|                   | 1   | \( p + \frac{1}{2} p(p-1), p \neq -1; 0 \) | 3 |
|                   | -1  | \( p - \frac{1}{2} p(p-1), p \neq 1; 2 \) | 4 |
| \( \{ \text{Id} \} \) | 1   | 0        | 5 |
|                   | -1  | 1        | 6 |
| \( |k| \geq 3 \) | 3   | \( \lambda(k;p) = p + \frac{1}{2} p(p+1)k \) | 7 |
|                   | -3  | \( \frac{1}{2} (1 + 3p)^2, \frac{1}{2} (1 + 4p)^2 \) | 8.9 |
|                   | -4  | \( \frac{1}{2} (1 + 3p)^2, \frac{1}{2} (1 + 5p)^2 \) | 10.11 |
|                   | -5  | \( \frac{1}{2} (1 + 3p)^2, \frac{1}{2} (1 + 5p)^2 \) | 12.13 |
|                   | 4   | \( \frac{1}{2} (1 + 3p)^2 \) | 14.15 |
|                   | -4  | \( \frac{3}{2} - \frac{1}{2} (1 + 3p)^2 \) | 16 |
|                   | -5  | \( \frac{3}{2} - \frac{1}{2} (1 + 3p)^2, \frac{5}{2} - \frac{1}{2} (2 + 5p)^2 \) | 17 |
|                   | 5   | \( \frac{3}{2} - \frac{1}{2} (1 + 3p)^2, \frac{7}{2} - \frac{1}{2} (2 + 5p)^2 \) | 18.19 |
|                   | -5  | \( \frac{3}{2} - \frac{1}{2} (1 + 3p)^2, \frac{7}{2} - \frac{1}{2} (2 + 5p)^2 \) | 20.21 |

Table 1:

Since we will now play with potential of degree \( k \neq \pm 2 \), the connected component of \( G_1 \) will be either \( G_\alpha \) or identity. See [5] and [3].

2.3 The Yoshida transformation of \( VE_1 \)

Here we reproduce some of the computations which were more detailed in [3]. The Yoshida transformation consists of the change of the independent variable

\[ t \mapsto z = \varphi^k(t), \]  

3
in the considered equation. Thanks to the chain rule
\[ \frac{d^2x}{dt^2} = \left( \frac{dz}{dt} \right)^2 \frac{d^2x}{dz^2} + \frac{d^2x}{dz^2} \frac{dx}{dz}, \]
equation (1) reads
\[ \varphi_k - 2(\lambda(t)) \left[ 2kz(1-z) \frac{d^2x}{dz^2} + (2(k-1)(1-z) - kz) \frac{dx}{dz} \right] = \frac{d^2x}{dt^2} \]
\[ \varphi_k^2(t) \left[ 2kz(1-z) \frac{d^2x}{dz^2} + (2(k-1)(1-z) - kz) \frac{dx}{dz} \right] = -\varphi_k^2(t) \lambda x \]
\[ 2kz(1-z) \frac{d^2x}{dz^2} + (2(k-1)(1-z) - kz) \frac{dx}{dz} = -\lambda x \]
\[ \frac{d^2x}{dz^2} + \frac{2(k-1)(1-z) - kz}{2kz(1-z)} \frac{dx}{dz} = -\lambda x \]
\[ \frac{d^2x}{dz^2} + p(z) \frac{dx}{dz} = s(z) \lambda x, \quad (3) \]
where
\[ p(z) = \frac{2(k-1)(z-1) + k}{2kz(z-1)} \quad \text{and} \quad s(z) = \frac{1}{2kz(z-1)}. \]
Now, after the classical Tschirnhaus change of dependent variable,
\[ x = f(z)\zeta, \quad f(z) = \exp \left( \frac{1}{2} \int p(z)dz \right) = z \frac{\Gamma(k-1)}{\Gamma(z-1)}, \quad (4) \]
equation (2.3) has the reduced form
\[ \frac{d^2\zeta}{dz^2} = [r_0(z) + \lambda s(z)]\zeta, \quad (5) \]
where
\[ r(z) := r_\lambda(z) := r_0(z) + \lambda s(z) = \frac{\rho^2 - 1}{4z^2} + \frac{\sigma^2 - 1}{4(z-1)^2} - \frac{1}{4}(1 - \rho^2 - \sigma^2 + \tau^2) \left( \frac{1}{z} + \frac{1}{1 - z} \right), \]
and
\[ \rho = \frac{1}{k}, \quad \sigma = \frac{1}{2}, \quad \tau = \sqrt{\frac{(k-2)^2 + 8k\lambda}{2k}}. \quad (6) \]
Since the three above numbers are respectively the exponents differences at \( z = 0, \ z = 1, \) and \( z = \infty, \) of the reduced hypergeometric equation \( L_2 = x'' - r(z)x = 0; \) the solutions of \( L_2 = 0, \) belong to the Riemann scheme
\[ P\left[ \frac{0}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{\infty}{2}; \frac{\Gamma(z)}{2}; \frac{\Gamma(z)}{2}; \right], \quad (7) \]
In Table [1] the group \( G(k, \lambda) \) appearing into the first column is precisely the differential Galois group of the equation \( L_2 = x'' - r(z)x = 0, \) with respect to the ground field \( \mathbb{C}(z). \)
2.4 Computation of the solution space of (1) when $G(k, \lambda)^{\circ} = G_{a}$.

We observed that after Yoshida transformation the new equation in $z$ variable is $L_2 = x' - r(z)x = 0$, and that the solutions of $L_2 = 0$ belong to the Riemann scheme

$$P_1 := P\{ \frac{1}{2} - \frac{1}{k} - \frac{1}{4} - \frac{1}{4} \rho_{\infty} z \}, \quad \rho_{\infty} = \frac{-1}{2}, \quad (8)$$

where

$$\rho_{\infty} = \frac{-1 - \tau}{2}, \quad \rho'_{\infty} = \frac{-1 + \tau}{2}, \quad \tau = p - \frac{1}{2} + \frac{1}{k}. \quad (9)$$

In Table 1, the group $G(k, \lambda)$ appearing into the first column is precisely the differential Galois group of the equation $L_2 = x' - r(z)x = 0$, with respect to the ground field $\mathbb{C}(z)$. But its connected component coincide with the connected component of (1) over $\mathbb{C}(\varphi, \check{\varphi})$.

Our Lemma 3.4 from [3] can be reformulated into the following way.

**Lemma 1** When $G_1 \simeq G_{a}$ then,

1. Up to a complex multiplicative constant, the algebraic solution $x_1$ is of the form $x_1 = z^a(z - 1)^b J(z)$ where

$$a \in \left\{ \frac{k - 1}{2k}, \frac{k + 1}{2k} \right\}, \quad b \in \left\{ \frac{1}{4}, \frac{3}{4} \right\}$$

and $J(z) \in \mathbb{R}[z]$ does not vanish at $z \in \{0, 1\}$.

2. The function $I := I_{z} = \int \frac{1}{x_z^2} dx$ has exponent $(1 - 2b)$ at $z = 1$, and, up to an additive constant, the monodromy around this point is $M_1(I) = -I$.

3. Around $z = 0$, $I$ has exponent $(1 - 2a)$, and its monodromy can be written into the form $M_0(I) = \exp(-4\pi i a)I + c_0$.

**Proof** (1 and 2) where proved in Lemma 3.4 of [3], (3) follows in the same way. □

From (8) and (9) we get

$$\rho_{\infty} = \frac{-1 - \tau}{2} \left( p + \frac{1}{2} + \frac{1}{k} \right), \quad \rho'_{\infty} = \frac{-1 + \tau}{2} \left( -\frac{3}{2} + \frac{1}{k} \right).$$

Since $x_1 = z^a(z - 1)^b J(z)$, we get that

$$x_1 \in P_1 := P\{ \frac{1}{2} - \frac{1}{k} - \frac{1}{4} - \frac{1}{4} \rho_{\infty} z \} \iff J \in P_2 := P\{ \frac{1}{2} - \frac{1}{k} - \frac{1}{4} - \frac{1}{4} \rho'_{\infty} a \rho_{\infty} + a + b z \}.$$
They belong to the Riemann scheme

\[
P_{J^*} = \begin{cases} 
-1 & \infty & 1 \\
0 & -n & 0 \\
-\beta & \alpha + \beta + n + 1 & -\alpha
\end{cases}.
\]

But here the singularities that we meet are \{-1, 1, \infty\} instead of \{0, 1, \infty\}. Hence we pass from the classical Jacobi polynomials to ours by putting \(t = 2z - 1\), i.e., we set

\[
J(z) = J^*(2z - 1),
\]

\[
J(z) \in P_J \begin{cases} 
0 & \infty & 1 \\
0 & -n & 0 \\
-\beta & \alpha + \beta + n + 1 & -\alpha
\end{cases} \iff J^*(t) \in P_{J^*} \begin{cases} 
-1 & \infty & 1 \\
0 & -n & 0 \\
-\beta & \alpha + \beta + n + 1 & -\alpha
\end{cases}.
\]

As a consequence, we get the following classification of the Jacobi polynomials \(J \in P_2\)

| Case | \(a\) | \(b\) | \(\alpha\) | \(\beta\) | \(n\) | \(p\) |
|------|------|------|------|------|------|------|
| Case 1 | \(\frac{1}{2} + \frac{\beta}{2}\) | \(\frac{1}{2} - \frac{\alpha}{2}\) | \(\frac{1}{2} - \frac{\beta}{2}\) | \(\frac{1}{2} + \frac{\alpha}{2}\) | \(p \in 2N + 1\) |
| Case 2 | \(\frac{1}{2} + \frac{\beta}{2}\) | \(\frac{1}{2} - \frac{\alpha}{2}\) | \(\frac{1}{2} - \frac{\beta}{2}\) | \(\frac{1}{2} + \frac{\alpha}{2}\) | \(p \in 2N + 2\) |
| Case 3 | \(\frac{1}{2} - \frac{\beta}{2}\) | \(\frac{1}{2} + \frac{\alpha}{2}\) | \(\frac{1}{2} - \frac{\beta}{2}\) | \(\frac{1}{2} + \frac{\alpha}{2}\) | \(p \in -2N\) |
| Case 4 | \(\frac{1}{2} - \frac{\beta}{2}\) | \(\frac{1}{2} + \frac{\alpha}{2}\) | \(\frac{1}{2} - \frac{\beta}{2}\) | \(\frac{1}{2} + \frac{\alpha}{2}\) | \(p \in -2N - 1\) |

Table 2: ()

Here, \(n := \deg(J(z))\). To obtain this, we identified \(P_2\) with \(P_J\). This gave \(\alpha\) and \(\beta\). To compute the degree, we observed that \(\alpha + \beta \notin \mathbb{Z}\). Hence, \(-n\) is the one of the two numbers \(\rho + a + b\), or \(\rho' + a + b\) that belongs to \(\mathbb{Z}\).

Thanks to the formula for \(J^* = J_n(\alpha, \beta)\) we get that up to a constant multiple

\[
J(z) = J_n(\alpha, \beta)(z) = (z - 1)^{-\alpha}(z)^{-\beta} \frac{d^n}{dz^n}(z - 1)^{\alpha + n}(z)^{\beta + n}.
\]

Hence, according to the previous table and Lemma 1, the precise forms of the integrals \(I\) depends on the four cases and are given by the following

| Cases | 1 | 2 | 3 | 4 |
|------|---|---|---|---|
| \(I^*\) | \(\frac{1}{z^{1+\nu}(z-1)^{1+\nu} t^2 J^*}\) | \(\frac{1}{z^{1+\nu}(z-1)^{1+\nu} t^2 J^*}\) | \(\frac{1}{z^{1+\nu}(z-1)^{1+\nu} t^2 J^*}\) | \(\frac{1}{z^{1+\nu}(z-1)^{1+\nu} t^2 J^*}\) |

Table 3:

**Remark 1** Compatibility between the above Table and Lemma is certainly true for \(|k| \geq 3\) but there is a problem for \(k = \pm 1\). Indeed, thanks to ([6] p.95), the differential equation for \(J(z) = J_n(\alpha, \beta)(z)\) is

\[
z(1 - z) J'' + [\beta + 1 - (\alpha + \beta + 2)z] J' + n(\alpha + \beta + n + 1)J = 0.
\]

So by plugging \(z = 0, 1\) in the later we get

\[
\begin{cases} 
(\beta + 1)J'(0) + n(\alpha + \beta + n + 1)J(0) = 0 \\
-(\alpha + 1)J'(1) + n(\alpha + \beta + n + 1)J(1) = 0
\end{cases}
\]

From Table 2 \(\alpha = \pm 1/2\) and \(\beta = \pm 1/k\) so \(n(\alpha + \beta + n + 1) \neq 0\). As a consequence, \(J(1) \neq 0\). But when \(\beta = -1\) that is when \(k = \pm 1\), we get that \(J(0) = 0\). Which is not compatible with the Lemma.
This is why, in this paper, we are going to work with the assumption that $|k| \geq 3$, which is in fact complementary to what was done in [2].

2.5 Yoshida transformation of $\text{VE}_{2,\alpha}^\gamma$ and $\text{EX}_{2,\alpha,\beta}^\gamma$.

With the above notations we get the following

**Proposition 1** By setting $x = f(z)\zeta$, the differential equation in time $t$:

$$\ddot{x} = -\lambda \varphi^{k-2}x + \varphi^{k-3}b,$$

is transformed into the differential equation in $z$ variable

$$\zeta'' = r_\lambda(z)\zeta - \frac{s(z)b}{f(z)z^{1/k}}.$$

**Proof** According to Yoshida, by dividing the original equation in time by $\varphi^{k-2}$ and multiplying by $-s(z)$ we get

$$x'' + p(z)x' = \lambda s(z)x - \frac{s(z)b}{\varphi}.$$

Now, the relation $x = f(z)\zeta$, gives $x'' + p(z)x' = f(z)\zeta'' + (f'' + pf')\zeta$, hence by dividing the previous equation by $f$ and using [3] again, we get

$$\zeta'' - r_0(z)\zeta = \frac{\lambda s(z)f}{f(z)\varphi} f\zeta' - \frac{s(z)b}{f(z)\varphi} f\zeta = r_\lambda(z)\zeta - \frac{s(z)b}{f(z)z^{1/k}}.$$

□

Let us set

$$\omega = z^{-(\frac{1}{k} + \frac{1}{k})} (z - 1)^{\frac{1}{k}}.$$

The field $\mathbb{C}(z)[\omega]$ is a finite Abelian extension of $\mathbb{C}(z)$.

**Corollary 1** The original systems in time $\text{VE}_{2,\alpha}^\gamma$ and $\text{EX}_{2,\alpha,\beta}^\gamma$ are equivalent to the following systems in $z$ variable that we still denote by the same symbols:

$$\begin{align*}
\text{VE}_{2,\alpha}^\gamma & \quad \begin{cases}
x'' = r_\alpha x, \\
y'' = r_\gamma y + \omega x^2,
\end{cases} \\
\text{EX}_{2,\alpha,\beta}^\gamma & \quad \begin{cases}
x'' = r_\alpha x, \\
y'' = r_\beta y, \\
u'' = r_\gamma u + \omega xy,
\end{cases}
\end{align*}$$

The coefficients of these systems are elements of $K_0 := \mathbb{C}(z)[\omega]$. Here, in order to simplify notations we set:

$$r_\alpha = r_{\lambda_\alpha}(z) = r_0(z) + \lambda_\alpha s(z).$$
Proof In $\text{EX}^{\gamma}_{2,\alpha,\beta}$, let’s perform the Tschirnhaus transformation by setting

$$x(t) := f(z)X(z); \quad y(t) := f(z)Y(z); \quad u(t) := f(z)U(z).$$

According to the above proposition, the system in time is therefore equivalent to

$$\text{EX}^{\gamma}_{2,\alpha,\beta} \begin{cases} X'' = r_\alpha X \\ Y'' = r_\beta Y \\ U'' = r_\gamma U - \frac{s(z)}{f(z)z^{1/k}}xy. \end{cases}$$

But now,

$$\frac{s(z)}{f(z)z^{1/k}}xy = \frac{s(z)}{f(z)z^{1/k}}f^2XY = \frac{s(z)}{z^{1/k}}f(z)XY.$$

Moreover direct computation gives

$$\frac{s(z)}{z^{1/k}}f(z) = \frac{\omega}{2k}.$$

Hence, the last equation of $\text{EX}^{\gamma}_{2,\alpha,\beta}$ becomes

$$U'' = r_\gamma U - \frac{\omega}{2k}XY.$$

Coming back to minuscule letter by setting

$$X = -2kx; \quad Y = y; \quad U = u,$$

we get the desired expression of $\text{EX}^{\gamma}_{2,\alpha,\beta}$ into $z$ variable. Similar computations hold for $\text{VE}^{\gamma}_{2,\alpha}$. □

3 Second level integrals and hierarchy

3.1 The second level integral involved in $\text{VE}^{\gamma}_{2,\alpha}$ and $\text{EX}^{\gamma}_{2,\alpha,\beta}$

From now on, we will only work with differential equation in $z$ variable, with coefficient in $K_0 = \mathbb{C}(z)[\omega]$.

**Proposition 2** Let $F$ be a differential field containing the elements $r, \omega, b$, and a basis of solutions $\{u_1; u_2\}$ of $u'' = ru$. Then, the field $F$ contains all solutions of the differential equation

$$u'' = ru + \omega b,$$

iff it contains the two following integrals

$$\Phi_i := \int \omega u_i b, \quad i = 1, 2.$$

**Proof** Classical variation of constant. □

Let us denote by $F_1/K_0$ the Picard-Vessiot extension associated to the homogeneous part of one of the systems $S := \text{VE}^{\gamma}_{2,\alpha}$, or $S := \text{EX}^{\gamma}_{2,\alpha,\beta}$. Let also denote by $F_2/K_0$ the Picard-Vessiot of $S$. From the above
The second solution is given by equation always have an algebraic solution that we shall always denote by an index one, i.e., we shall assume up to the end of this paper that $G$
 which is generated by first level integral with respect to $X$.

Proof In these cases $F_1/K_0$ is algebraic. It is also the algebraic closure of $K$, $K^2(F_2/K_0)$ is a vector group.

From now, since the casistic is sufficiently tremendous, with those cases only, we shall assume up to the end of this paper that $G_\alpha \simeq G_\beta \simeq G_\gamma \simeq G_\alpha$.

3.2 Hierarchy of the integrals involved in $VE^\gamma_{2,\alpha}$ and $EX^\gamma_{2,\alpha,\beta}$.

Since in the considered cases the group $G_\lambda$ of $w'' = \gamma w$ has the connected component $G_\alpha$, the equation always have an algebraic solution that we shall always denote by an index one, i.e., $w_1$.

The second solution is given by $w_2 = w_1 I_\lambda$ with $I = I_\lambda = \int \frac{1}{w_1^2}$.

With this convention the second level integral involved in $VE^\gamma_{2,\alpha}$ can be classify in following diagram

$\Phi = \int \omega y_1 x_1^2$,
$\Phi_\gamma = \int \omega y_2 x_1^2 = \int \Phi' I_\gamma$,
$\Phi_\alpha = \int \omega y_1 x_1 x_2 = \int \Phi' I_\alpha$,
$\Phi_\beta = \int \omega y_2 x_2 = \int \Phi' I_\beta$,
$\Phi_\gamma,\alpha = \int \omega y_2 x_1 x_2 = \int \Phi' I_\gamma I_\alpha$,
$\Phi_\gamma,\beta = \int \omega y_2 x_1 x_2 = \int \Phi' I_\gamma I_\beta$,
$\Phi_\alpha,\beta = \int \omega y_2 x_1 x_2 = \int \Phi' I_\alpha I_\beta$.

Observe that $\Phi$ is a first level integral with respect to the algebraic extension $K$ of $C(z)$, with $K := C(z)[\omega; y_1; x_1] = K_0[y_1; x_1]$.

The remaining five integrals, are of second level with respect to $K$. Their complexity grow at each change of line in the diagram.

For $EX^\gamma_{2,\alpha,\beta}$, we proceed similarly. Here, $\Phi = \int \omega u_1 y_1 x_1$ is first level with respect to $K := C(z)[\omega; u_1; y_1; x_1] = K_0[u_1; y_1; x_1]$.

The diagram of complexity is now the following

$\Phi = \int \omega u_1 y_1 x_1$,
$\Phi_\gamma = \int \Phi' I_\gamma$,
$\Phi_\beta = \int \Phi' I_\beta$,
$\Phi_\alpha = \int \Phi' I_\alpha$,
$\Phi_\gamma,\beta = \int \Phi' I_\gamma I_\beta$,
$\Phi_\gamma,\alpha = \int \Phi' I_\gamma I_\alpha$,
$\Phi_\alpha,\beta = \int \Phi' I_\alpha I_\beta$.
4 Ingredients and tools

4.1 A cohomological argument to decide between solvability and Abelianity

Here we recall one important result stated and proved in the first part of this paper (Theorem 3.1 of [4]). And we add to it some refinements which are going to be useful for effective testing of the virtual Abelianity.

**Theorem 2** Let $F_1/K$ and $F_2/K$ be two Picard-Vessiot extensions with $F_1 \subset F_2$ and $F_2/F_1$ generated by integral of second level. Then $G_2$ is virtually Abelian iff $G_1$ is virtually Abelian, and any second level integrals $\Phi \in F_2$ can be expanded into the form

$$\Phi = R_1 + J,$$

where $R_1 \in T(F_1/K)$ and $J' \in \bar{K}$. Here $\bar{K}$ is the algebraic closure of $K$ in $F_2$. Moreover, for all $\sigma \in G_2^\circ$, $\sigma(\Phi) - \Phi \in T(F_1/K)$.

Here we recall that $T(F_1/K)$ stands for the Picard-Vessiot ring of the extension $F_1/K$.

Let $K \subset F_1 \subset F_2$ be a tower a Picard-Vessiot extensions satisfying the assumptions of Theorem 2, with $F_1/K$ virtually Abelian. Let $\Phi \in F_2$ be a second level integral. If $G_2$ is virtually Abelian, the mapping

$$\sigma \mapsto C(\sigma) := \sigma(\Phi) - \Phi,$$

is a cocycle from $G_2^\circ$ with values in the $G_2^\circ$-module $M = T(F_1/K)$. Indeed, $C$ is a cocycle since it satisfies the relation

$$C(s\sigma) = s \cdot C(\sigma) + C(s).$$

Now, let $R \in M = T(F_1/K)$, and $h$ be an arbitrary mapping from $G_2^\circ$ to the field of constats $C$, we say that the mapping

$$B: G_2^\circ \to M; \quad \sigma \mapsto \sigma(R) - R + h(\sigma),$$

is an extended coboundary from $G_2^\circ$ with values in $M$. When moreover, $h \in \text{Hom}(G_2^\circ, C)$, then we say that $B$ is a coboundary. Direct computation shows that an extended coboundary $B$ is a cocycle iff $B$ is a coboundary.

In this language, Theorem 2 says that if $G_2^\circ$ is Abelian, then any cocycle associated to a second level integral is a coboundary. We may wonder about a converse. Let $\Phi$ be a second level integral such that there exists $R \in T(F_1/K)$, and some function $h : G_2^\circ \to C$, (here we do not assume a priory that $h$ is additive), satisfying

$$\sigma(\Phi) - \Phi = \sigma(R) - R + h(\sigma), \quad \forall \sigma \in G_2^\circ.$$

This relation yield the following implications

$$\sigma(\Phi - R) = \Phi - R + h(\sigma)$$

$$\sigma(\Phi' - R') = \Phi' - R'$$

$$\Phi' - R' \in \bar{K}.$$

Hence, there exists $J$ such that $J'$ is algebraic over $K$, and $\Phi = R + J$. As a consequence, we proved that the cocycle associated to a second level integral $\Phi$ is a coboundary iff $\Phi$ can be computed in closed form.
Let us conclude this sub-section by explaining how we will use these observations. For any given tower of Picard-Vessiot extensions and an explicate second level integral $\Phi$, the fact that the associated cocycle $C(\sigma) = \sigma(\Phi) - \Phi$ belong to $T(F_1/K)$, will give us necessary conditions for the virtual Abelianity of $G_2$. Next, thanks to the previous arguments, we will find sufficient conditions, showing that the cocycles are coboundaries.

4.2 Ostrowski relations and necessary conditions for virtual Abelianity

The following lemma was also stated into the first part of this paper as a consequence of Theorem 2 (see Lemma 3.3 of [4]). But we state it again in a more appropriated version to the present context. Moreover, we prove it again since its proof is better understandable into the framework of the cohomological arguments.

**Lemma 2** Let $K \subset F_1 \subset F_2$ be a tower of Picard-Vessiot extensions of $K$, with the same field of constants $C$, and satisfying the following conditions

- $G_1' = G_a$, $T(F_1/K) = K[I]$, and $I' \in K$,
- $F_2$ contains some second level integrals of the form $\Phi_1 := \int \Phi' I$, with $\Phi' \in K$, and $\Phi \in F_2$.

If $F_2/K$ is virtually Abelian, then $\Phi$ and $I$ satisfies Ostrowski relation of the form

$$\Phi - dI \in K, \text{for some } C.$$

Conversely, if such a relation holds then the extension $K(I, \Phi_1)/K$ is virtually Abelian.

**Proof** There exists $c \in \text{Hom}(G_2', \mathbb{C})$ such that for all $\sigma \in G_2'$, we have $\sigma(I) = I + c(\sigma)$. Hence,

$$\sigma(\Phi_1') = \Phi_1' + c(\sigma)\Phi' \Rightarrow \sigma(\Phi_1) - \Phi_1 = c(\sigma)\Phi + d(\sigma),$$

for some mapping $d : G_2' \rightarrow \mathbb{C}$. Now according to Theorem 2 the virtual Abelianity of $F_2/K$ implies that the cocycles $\sigma(\Phi_1) - \Phi_1 \in T(F_1/K) = K[I]$. Hence

$$\sigma(\Phi_1) - \Phi_1 = c(\sigma)\Phi + d(\sigma) \in T(F_1/K) = K[I].$$

Let us choose $\sigma = \sigma_0$ such that $c(\sigma_0) = 1$. The last relation implies that $\Phi \in K[I]$. Hence, the two primitive integrals over $K$, $\Phi$ and $I$ must be dependant and we conclude thanks to Ostrowski.

For the converse, let us assume that $\Phi = dI + f$ for some $f \in K$. Integrating by part, we can compute $\Phi_1$ in closed form thanks to first level integrals. Indeed we get

$$\Phi_1 = dI^2/2 + fI - \int fI'.$$

and the claim follows since, $\int fI'$ is a first level integral w.r.t $K$.

Alternatively, let us show on this example how the previous cohomological arguments are working here. If we assume that $\Phi = dI + f$ for some $f \in K$, we get a closed expression for the cocycle

$$\sigma(\Phi_1) - \Phi_1 = \Phi c(\sigma) + d(\sigma) = dIc(\sigma) + fc(\sigma) + d(\sigma) \in T(F_1/K) = K[I].$$

Let us show that this is an extended coboundary. If we set $P(I) := dI^2/2 + fI$, we get

$$\sigma(P(I)) - P(I) = P(I + c(\sigma)) - P(I)$$

$$= \frac{d}{2}(I + c(\sigma))^2 + f(I + c(\sigma)) - \frac{d}{2}I^2 - fI$$

$$\sigma(P(I)) - P(I) = dIc(\sigma) + fc(\sigma) + \frac{dc^2(\sigma)}{2}.$$
Hence, by comparing the above two formulae, we get

\[ \sigma(\Phi_1) - \Phi_1 = \sigma(P(I)) - P(I) + h(\sigma), \]

where the function \( h : G_2^2 \rightarrow C \), can be computed thanks to the formula given by \( h(\sigma) = -dc^2(\sigma)/2 + d(\sigma) \). Since the cocycle: \( \sigma(\Phi_1) - \Phi_1 \) is an extended coboundary, and, in fact, is a coboundary, \( \Phi_1 \) can be computed in closed form, and the result follows. \( \square \)

This proof explains why we introduced, the a priori artificial notion of an extended coboundary. Indeed, when looking to the complicated formula for \( h \) above, it not obvious that it is a group morphism from \( G_2^2 \) to \( C \). Nevertheless, what is really important for our purpose is that \( h \) takes constant values. In the more complicated cases that we shall meet below, we will not explicitly compute \( h \), but we will only show its existence. Moreover, the advantage of this cohomological approach is that it shows that a second level integrals can be explicitly computed in closed form without having to make this computation explicitly. This will make things simpler in the more complicated cases below.

4.3 Testing Ostrowski relations thanks to characters

Let \( K/\mathbb{C}(z) \) be an algebraic extension. Let \( \Phi \) and \( I \) be two primitive integrals of elements belonging to \( K \). In order to test if they satisfy an Ostrowski relation of the form

\[ \Phi + dI = p \in K \quad (11) \]

for some \( d \in \mathbb{C} \), we shall use the following observation by taking advantage that \( \Phi \) and \( I \) are primitive of algebraic functions.

Let \( \sigma \) be a Galois morphism fixing \( \mathbb{C}(z) \), for example a monodromy operator, and assume further that \( \sigma \) acts on \( \Phi' \) and \( I' \) by characters according to the formulae

\[ \sigma(\Phi') = \chi(\Phi)\Phi', \quad \sigma(I') = \chi(I)I'. \]

By integrating we get relations of the form

\[ \sigma(\Phi) = \chi(\Phi)\Phi + c_\Phi, \quad \sigma(I) = \chi(I)I + c_I. \]

As a consequence, applying \( \sigma \) to (11), we get a system of two equations

\[
\begin{pmatrix}
1 & 1 \\
\chi(\Phi) & \chi(I)
\end{pmatrix}
\begin{pmatrix}
\Phi \\
dI
\end{pmatrix} =
\begin{pmatrix}
p \\
\sigma(p) - c_\Phi - dc_I
\end{pmatrix} \in K^2.
\]

If \( \chi(\Phi) \neq \chi(I) \) the matrix is invertible and (11) implies that \( \Phi \) is algebraic. We have therefore proved the following criteria.

**Lemma 3**

1. Assume that \( \Phi \) is not algebraic, and that there exists a monodromy operator \( \sigma \) acting by character on \( \Phi' \) and \( I' \). If \( \chi(\Phi) \neq \chi(I) \), then (11) does not hold.

2. Here is a generalisation: Let \( \Phi, I_1, \ldots, I_n \) be \( n + 1 \) integrals over \( K \). Let \( \sigma \) be a Galois morphism acting by characters on the derivatives

\[ \sigma(\Phi') = \chi_\Phi \Phi', \quad \sigma(I'_j) = \chi_j I'_j, \quad \text{for} \ j = 1, \ldots, n. \]
If $\chi \notin \{\chi_1, \ldots, \chi_n\}$, then an arbitrary Ostrowski relation of the form

$$\Phi + \sum_{i=1}^{n} d_j I_j \in K,$$

implies that $\Phi \in K$.

The proof of the second point is similar to previous particular case. Indeed, by grouping together the $I_j$ corresponding to the same character, we are reduced to the case where all the $\chi_j$ are distinct. Then, the action of the $\sigma^p$ for $0 \leq p \leq n$ lead to an invertible Vandermonde $n \times n$ system, which allows a similar conclusion as in point (1).

We are going to use this lemma in the proofs of Propositions 6, 7 and 8 below, by showing thanks to monodromies that some Ostrowski relations are impossible. This is one of the great advantage of Yoshida transformation in comparison to the time parametrisation, where the singularities of the corresponding complex functions are not well understood. Another advantage of Yoshida transformation is going to be shown right now.

4.4 To be or not to be an algebraic integral

Remark 2 Let $F/K_0$ be the Picard-Vessiot extension of either $\text{VE}_{2,\alpha}$, or $\text{EX}_{2,\alpha,\beta}$, over the field $K_0 = \mathbb{C}(z)[\omega]$. The two above systems are Fuchsian with singularities at $z \in \{0, 1, \infty\}$. Indeed, according to Corollary 1 and Proposition 2, each solution of any of these systems is holomorphic in any simply connected domain of $\mathbb{C} \setminus \{0; 1; \infty\}$, with at most exponential growth at the singularities.

As a consequence, the Schlesinger theorem implies that $\text{Gal}(F/\mathbb{C}(z))$ is topologically generated by the two monodromies $M_0$ and $M_1$.

This observation will have the following important consequence. Let $\Gamma$ be a holonomic element of $F$ fixed by $M_0$, and having a finite orbit under $< M_1 >$. More generally, let us assume that $\Gamma$ has a finite orbit under the monodromy group $\mathcal{M} \subset \text{Gal}(F/\mathbb{C}(z))$. Then necessarily, $\Gamma$ is algebraic over $\mathbb{C}(z)$. Indeed, since $\Gamma$ is holonomic, there exists a $\mathbb{C}$-finite dimensional vector space $V$ containing $\Gamma$ on which $\text{Gal}(F/\mathbb{C}(z))$ acts algebraically. As a consequence, the map

$$f : \text{Gal}(F/\mathbb{C}(z)) \to V; \quad \sigma \mapsto \sigma(\Gamma),$$

is a morphism of algebraic variety. Since the image $f(\mathcal{M})$ is finite, it is a Zariski closed subset of $V$, so $f^{-1}(f(\mathcal{M}))$ is a closed subset of $\text{Gal}(F/\mathbb{C}(z))$ containing $\mathcal{M}$. Since $\mathcal{M}$ is dense in $\text{Gal}(F/\mathbb{C}(z))$, we get that $f^{-1}(f(\mathcal{M})) = \text{Gal}(F/\mathbb{C}(z))$ and the orbit of $\Gamma$ under $\text{Gal}(F/\mathbb{C}(z))$ is finite as has to be shown.

The most general integrals $\Phi$ and $\Psi$ we shall meet below are Abelian integrals of the form

$$\Gamma := \int \frac{P\Omega}{J^2},$$

where

- $\Omega(z) = z^{e_0}(1-z)^{e_1}$, where the exponents $e_0$ and $e_1$ are rational numbers $> -1$, with $e_0 + e_1 \notin \mathbb{Z}$.
- $P(z) \in \mathbb{C}[z]$.
- $J(z)$ is a Jacobi polynomial having $n$ simples roots $0 < z_1 < \cdots < z_n < 1$, if $n = \deg(J) \geq 1$. 

13
Precisely, $\Gamma$ will be an integral of the type $\Phi$, for $n = \deg(J) = 0$, and of the type $\Psi$ otherwise.

According to Remark 2, if one of the two exponents $e_0$, or $e_1$ is an integer, then the corresponding integral $\Phi$, or $\Psi$ has finite orbit under the monodromy group, hence is algebraic. This is a very surprising fact especially for the integrals $\Psi$. Indeed, this shows that here, the $P$, $\Omega$ and $J$ must be so specific that $P \Omega / J^2$ does not has residues at none of the $z_i$.

Away from those cases, we now have to investigate the integrals $\Gamma$, for which the two exponents are not integers in order to be able to test their eventual algebraicity.

### 4.4.1 Reduction of the integrals

When trying to compute $\Gamma$ in closed form we get the following formula

$$\forall R \in \mathbb{C}(z), \quad \left( \frac{R \Omega}{J} \right)' = \frac{T(R) \Omega}{J^2},$$

with

$$\begin{cases}
U(z) &:= \Omega(z)(1 - z) \\
T(R) &:= z(1 - z)JR' + [(e_0 + 1 - (e_0 + e_1 + 2)z)J + z(z - 1)J']R
\end{cases}$$

Viewed as a linear mapping of $\mathbb{C}(z)$ to itself, $T$ is injective. Indeed,

$$(T(R) = 0) \implies \left( \frac{R \Omega}{J} \right)' = 0 \implies (R = 0).$$

Now, if we restrict $T$ to $\mathbb{C}[z]$, by computing the leading term of $T(z^r)$, we see that the condition $e_0 + e_1 \notin \mathbb{Z}$ implies that $T$ increment the degree by $n + 1$. That is $\deg(T(R)) = \deg(R) + n + 1$. By counting dimensions we therefore get that for all $N \geq 0$, we have the following direct sum decomposition

$$\mathbb{C}_{N+n+1}[z] = T(\mathbb{C}_N[z]) \oplus \mathbb{C}_n[z].$$

Since it holds for all $N \geq 0$, we get

$$\mathbb{C}[z] = T(\mathbb{C}[z]) \oplus \mathbb{C}_n[z].$$

As a consequence, we can reduce any integral $\Gamma$ by lowering the degree of the numerator in the following way : $\forall P \in \mathbb{C}[z] \exists! (R, \Lambda) \in \mathbb{C}[z] \times \mathbb{C}_n[z]$ such that

$$P = T(R) + \Lambda. \quad (12)$$

By multiplying this equality by $\Omega / J^2$, and integrating we get

$$\int \frac{P \Omega}{J^2} = \frac{R \Omega}{J} + \int \frac{\Lambda \Omega}{J^2},$$

$$\Gamma(P) = \frac{R \Omega}{J} + \Gamma(\Lambda). \quad (13)$$

Since $R \Omega / J$ is algebraic, $\Gamma(P)$ is algebraic, iff $\Gamma(\Lambda)$ is algebraic. For the study of this problem we get the following.

**Theorem 3** Let $P, \Omega, J$ be as above with $e_0$ and $e_1$ in $\mathbb{Q}\setminus\mathbb{Z}$.

1. The integral $\Gamma(P)$ is algebraic iff it belongs to the field $\mathbb{C}(z)[\Omega]$, that is iff there exists $R \in \mathbb{C}(z)$ such that $P = T(R)$.  

14
2. \( \forall P \in \mathbb{C}[z] \text{ and } R \in \mathbb{C}(z), \text{ the relation } P = T(R) \text{ implies that } R \in \mathbb{C}[z]. \)

3. For a polynomial \( P \in \mathbb{C}[z], \text{ let } R \text{ and } \Lambda \text{ satisfy equation (12)}. \text{ Then } \Gamma(P) \text{ is algebraic iff } \Lambda = 0. \text{ Two integrals } \Gamma(P_1) \text{ and } \Gamma(P_2) \text{ satisfy an Ostrowski relation: } \Gamma(P_1) + d\Gamma(P_2) \text{ is algebraic iff for the corresponding } \Lambda s, \text{ we have } \Lambda_1 + d\Lambda_2 = 0. \\

4. If \( P \) is a non zero polynomial with \( \deg(P) \leq n \), then \( \Gamma(P) \) is transcendental. Moreover, the \( n + 1 \) integrals \( \int \frac{P}{BV} \) with \( 0 \leq s \leq n \) are algebraically independent.

The ideas behind this result are very closed to what we did in [3].

**Proof** 1. Since the exponents are rational but not integers, there exist a minimal integral power \( d \geq 2 \) such that \( \Omega^d \in \mathbb{C}(z) \). As a consequence, the field extension \( \mathbb{C}(z)[\Omega] = \mathbb{C}(z)[\Omega]/\mathbb{C}(z) \) is a Kummer extension of degree \( d \). Now, \( \Gamma(P) \) is algebraic iff it belongs to \( \mathbb{C}(z)[\Omega] \). It can therefore be expanded into the form

\[
\Gamma(P) = \int \frac{PO}{J^2} = \sum_{s=1}^{d} \frac{R_s}{J} \Omega^s \quad \text{with } R_s \in \mathbb{C}(z).
\]

Taking derivative of the above equation, we obtain the following expression

\[
\frac{PO}{J^2} = \sum_{s=1}^{d} \left( \frac{R'_s}{J} - \frac{J'}{J} + \frac{s \Omega}{J} \right) \frac{R_s}{J} \Omega^s
\]

Since in the right hand side, each coefficient of \( \Omega^s \) is in \( \mathbb{C}(z) \), we must have

\[
\left( \frac{R_s}{J} \Omega^s \right)' = 0 \quad \text{for } s \geq 2,
\]

and

\[
\frac{PO}{J^2} = \left( \frac{R_1}{J} \Omega \right)' = T(R_1) \Omega
\]

\[
\iff P = T(R_1),
\]

and the claim follows.

2. If \( P = T(R) \) with \( R \in \mathbb{C}(z) \), then the function

\[
z \mapsto \Gamma(z) = \int \frac{PO}{J^2} = \frac{R_1}{J} \Omega,
\]

is holomorphic in an arbitrary simply connected domain of \( \mathbb{C} \setminus \{0, 1, z_1, \ldots, z_n \} \). So, if \( R \) has got a pole, it must belong to \( \{0, 1, z_1, \ldots, z_n \} \). But for all \( p \geq 1 \), the leading term of \( T(1/z^p) \) is given by

\[
T(1/z^p) = J(0)(e_0 + 1 - p)/z^p \neq 0,
\]

since \( 0 \not\in \mathbb{Z} \) and \( J(0) \neq 0 \). As a consequence, \( z = 0 \) cannot be a pole of \( R \). A similar argument hold at \( z = 1 \), since the expansion of \( T(1/(z - 1)^p) \) begins with \( J(1)(p - e_0 - 1)/(z - 1)^p \).

Around \( z = z_i \), \( \Gamma'(z) \) is of the form

\[
\Gamma'(z) = a/(z - z_i)^2 + 0/(z - z_i) + h_1(z),
\]

with \( h_1 \) holomorphic. Indeed, because \( \Gamma \) is algebraic \( \Gamma' \) does not have residue. So,

\[
\Gamma(z) = -a/(z - z_i) + h_2(z) = R3/J = Rh_3(z)/(z - z_i),
\]

15
with $h_2$ and $h_3$ holomorphic around $z_i$. Since $h_3(z_i) \neq 0$, $R$ cannot have a pole at $z_i$. Hence, $R$ is polynomial.

3. and 4. follow from the following observation. Let $\Lambda \in \mathbb{C}_n[z]$ be such that $\Gamma(\Lambda)$ is algebraic. Point 1 implies the existence of some $R \in \mathbb{C}(z)$ with $\Lambda = T(R)$. But according to point 2, $R$ is a polynomial. If $R \neq 0$ then $\deg(\Lambda) = \deg(R) + n + 1 > n$ which is contradictory. So $\Lambda = T(0) = 0$ and the integral is algebraic iff $\Lambda = 0$. □

4.4.2 Linear forms and the equation $\Lambda = \Lambda(P) = 0$

The above result show that the algebraicity of an integral $\Gamma(P)$ reduces to the vanishing of the polynomial $\Lambda$ appearing in equation (12). Although the correspondence $P \mapsto \Lambda$ is linear, the decomposition given by equation (12), is very hard to perform effectively. Here, we are going to show that the vanishing of $\Lambda$ can be controlled by the vanishing of some linear forms on $P$ which can be directly computed thanks to some definite integrals.

In the most simple case, that is for $J = 1$, i.e., when $J(z)$ is a constant, this is achieved thanks to the following.

**Proposition 3** Let $e_0$ and $e_1$ be two real numbers greater than $-1$, and belonging to $\mathbb{Q}\setminus\mathbb{Z}$ with $e_0 + e_1 \notin \mathbb{Z}$. Let us set $\Omega := z^{e_0}(1 - z)^{e_1}$. Then we have

1. For any polynomial $P$, the primitive integral $\Phi := \int P\Omega$ is algebraic iff

   $$\mu(P) := \int_0^1 P\Omega(z)dz = 0.$$

2. For all $n \in \mathbb{N}$,

   $$\mu(z^n) = \frac{(e_0 + 1)n}{(e_0 + e_1 + 2)n} B(e_0 + 1; e_1 + 1),$$

   where $B(p; q)$ is the usual Euler Beta function and $(x)_n$ is the Pochammer symbol.

3. If $P(z) = \sum p_n z^n$, then $\Phi$ is algebraic iff

   $$\sum p_n \frac{(e_0 + 1)n}{(e_0 + e_1 + 2)n} = 0.$$

The condition on the two exponents to be greater than $-1$, guaranties the convergence of the generalised integrals between 0 and 1. The first point shows that the set of polynomials for which $\Phi$ is algebraic is an hyperplane given by the kernel of the linear form $\mu$.

Points 2 and 3, give explicit criterion on the coefficients of $P$ to decide whether or not $\Phi$ is algebraic.

**Proof** 1. Since $J = 1$, in the decomposition given by equation (12): $P = T(R) + \Lambda$, we have that $\Lambda$ is a number. The corresponding relation (13), can be written

$$\Phi = \int P\Omega = R\Omega + \Lambda \int \Omega = R(z)z^{e_0+1}(1 - z)^{e_1+1} + \Lambda \int \Omega.$$

Now let us compute $\Lambda$ by evaluating the integrals between 0 and 1. Since $e_p + 1 > 0$ for $p \in \{0, 1\}$, and $R \in \mathbb{C}[z]$, we have

$$\mu(P) = \int_0^1 P\Omega(z)dz = \Lambda \int_0^1 \Omega(z)dz \implies \Lambda = \mu(P)/ \int_0^1 \Omega(z)dz.$$
2. Is a direct consequence of the relation
\[ \mu(z^n) = \int_0^1 z^{e_0 + n} (1 - z)^{e_1} \, dz = B(e_0 + n + 1; e_1 + 1) = \frac{\Gamma(e_0 + n + 1)\Gamma(e_1 + 1)}{\Gamma(e_0 + e_1 + n + 2)}. \]

3. Follows directly from the previous considerations. □

On the basis of the same ideas we now treat the case when \( \text{deg}(J) = n \geq 1 \).

Let \( 0 < z_1 \leq \cdots \leq z_n < 1 \) be the roots of \( J \). Let \( \gamma_0 \) be a half of the circle going counterclockwise from 0 to 1. Let \( \gamma_i \) for \( 1 \leq i \leq n \) be some small trigonometric circles each enclosing \( z_i \) and no other root \( z_j \). Let us consider the \( n + 1 \) linear forms on \( \mathbb{C}[z] \) given by

\[ L_i(P) := \int_{\gamma_i} \frac{P \Omega}{J^2}, \quad 0 \leq i \leq n. \]

Then we get the following

**Proposition 4** With the previous notations

1. If \( P = T(R) + \Lambda \) as in relation (12), then for all \( 0 \leq i \leq n \), \( L_i(P) = L_i(\Lambda). \)

2. The \( n + 1 \) linear forms \( L_i \) are free and \( \int \frac{P \Omega}{J^2} \) is algebraic iff for all \( 0 \leq i \leq n \), \( L_i(P) = 0 \).

Let us observe that this property is a generalisation of the previous one. Indeed, \( L_0 = \mu \), for \( \text{deg}(J) = 0 \), that is when \( J = 1 \). Here the problem is that we do not find comparable simple closed formulae for the linear forms \( L_i \). We mention this difficulty because the \( L_i(P) \) got the flavour of some periods on some Abelian variety. But we did not find this link precisely. This is probably the deep reason why things are so complicated in our context. Maybe we did not find the proper geometric space where the actual notions would get some more transparent meaning.

**Proof**

1. This is a direct consequence of relation (13).

2. For \( P_0 = J^2 \) the function \( P \Omega/J^2 = \Omega \) has no residue at none of the \( z_i \), hence

\[ L_0(J^2) = \mu(1) \neq 0 \quad \text{and} \quad L_i(J^2) = 0, \quad \text{for} \quad 1 \leq i \leq n. \]

Now let us set \( P_s := J^2/(z - z_s) \), for \( 1 \leq s \leq n \). The function \( P_s \Omega/J^2 = \Omega/(z - z_s) \) has a non zero residue at \( z_s \), and zero residue elsewhere. So, \( L_s(P_s) \neq 0 \) and \( L_i(P_s) = 0 \) for \( i \neq s \), and \( 1 \leq i \leq n \). From this it follows immediately that the \( n + 1 \) linear forms \( L_s \) are free on \( \mathbb{C}[z] \).

According to point 1, their restriction to \( \mathbb{C}_n[z] \) form a basis of the dual space \( \mathbb{C}_n[z]^* \). As a consequence, any \( \Lambda \in \mathbb{C}_n[z]^* \) is zero iff it belongs to the common kernel of the linear forms. And we can therefore conclude that \( \int \frac{P \Omega}{J^2} \) is algebraic iff for all \( 0 \leq i \leq n \), \( L_i(P) = 0 \) according to Theorem 3. □

5 Reducing the virtual Abelianity of \( \text{VE}^\gamma_{2,\alpha} \) and \( \text{EX}^\gamma_{2,\alpha,\beta} \) to Ostrowski relations

In this section we exhibit the integrals and the Ostrowski relations which are going to govern the virtual Abelianity of the systems \( \text{VE}^\gamma_{2,\alpha} \) and \( \text{EX}^\gamma_{2,\alpha,\beta} \). Next in the two sections that follow we will test effectively these results.
5.1 Getting obstruction thanks to the integrals $\Phi_{\nu}$ for $\nu \in \{\alpha, \beta, \gamma\}$

**Proposition 5** With the notation of Section 3.2, we get the following necessary conditions

1. If the differential Galois group of $\text{VE}_{\gamma}^{2,\alpha}$ is virtually Abelian, then, we get two Ostrowski relations $\Phi + d_{\gamma} I_{\gamma}$ and $\Phi + d_{\alpha} I_{\alpha}$ are algebraic over $\mathbb{C}(z)$ for some constants $d_{\gamma}$ and $d_{\alpha}$ in $\mathbb{C}$. Here, $\Phi = \int \omega y_1 x_1$.

2. If the differential Galois group of $\text{EX}_{\gamma}^{2,\alpha,\beta}$ is virtually Abelian, then we get three Ostrowski relations $\Phi + d_{\gamma} I_{\gamma}, \Phi + d_{\beta} I_{\beta}$, and $\Phi + d_{\alpha} I_{\alpha}$ are algebraic over $\mathbb{C}(z)$. Here, $\Phi = \int \omega u_1 y_1 x_1$.

**Proof** Let $F/K$ be the Picard-Vessiot extension $\text{VE}_{\gamma}^{2,\alpha}$. According to Section 3.2, we get the following inclusion

\[ K \subset F_{\gamma}^1 := K(I_{\gamma}) \subset F_{\gamma}^2 := F_{\gamma}^1(\Phi_{\gamma}) \subset F, \]

and a similar one by changing $\gamma$ to $\alpha$. The fact that $F/K$ is virtually Abelian implies the same property for $F_{\gamma}^2/K$, and $F_{\alpha}^2/K$. Then we can conclude thanks to Lemma 2. Similar arguments hold when dealing with $\text{EX}_{\gamma}^{2,\alpha,\beta}$. \qed

5.2 Strategy and Game

Two cases may a priori happen, when applying the above Proposition: $\Phi$ is either transcendental or algebraic over $K$ or $\mathbb{C}(z)$.

5.2.1 When $\Phi$ is transcendental

If this occurs, then the virtual Abelianity of the differential Galois group of $S = \text{VE}_{\gamma}^{2,\alpha}$, or $S = \text{EX}_{\gamma}^{2,\alpha,\beta}$ implies that the corresponding constants $d_{\gamma}, d_{\beta}$ and $d_{\alpha}$ are non zero complex numbers. As a first consequence we must also get Ostrowski relations between the integrals $I$.

For instance, $d_{\gamma} I_{\gamma} - d_{\alpha} I_{\alpha}$ is algebraic...

5.2.2 When $\Phi$ is algebraic

In this situation, the Ostrowski relations of Proposition hold with $d_{\gamma} = d_{\beta} = d_{\alpha} = 0$ and the proposition is helpless. Unfortunately, as we shall see in Section 5 and 6 below, $\Phi$ is algebraic very often. This is the reason why we have to find new necessary conditions for the virtual Abelianity of differential Galois groups of $\text{VE}_{\gamma}^{2,\alpha}$ and $\text{EX}_{\gamma}^{2,\alpha,\beta}$ in this case. This will be the purpose of the next subsection.

5.3 Getting obstruction when $\Phi$ is algebraic

In this subsection, we give the two criteria such that groups of $\text{VE}_{\gamma}^{2,\alpha}$ and $\text{EX}_{\gamma}^{2,\alpha,\beta}$ are virtually Abelian when $\Phi$ is algebraic.

5.3.1 Integration by part and new second level integrals $\Psi_m$ associated to the $\Phi_m$
\( \Phi_{i,j,l} \) which have, according to the given hierarchy, one, two or three indices. For each multi-index \( m \in \{ \nu; (i,j); (i,j,l) \} \), let us symbolically write \( \Phi_m = \int \Phi'I^m \). Integration by part gives

\[
\Phi_m = \Phi I^m - \Psi_m \quad \text{with} \quad \Psi_m := \int \Phi'(I^m)' - \Psi_m.
\]

Since \( \Phi \in K \), \( \Phi I^m \in T(F_1/K) = K[I] \), and \( F/F_1 = \text{PV}(S)/F_1 \) is generated by the corresponding \( \Psi_m \).

Our motivation to introduce these new integrals \( \Psi_m \) is the fact that the computation of their associated cocycles \( \sigma(\Psi_m) - \Psi_m \) is simpler than for the cocycles \( \sigma(\Phi_m) - \Phi_m \).

Now, we give the precise formulae for the given \( \Psi_m \) for one, two and three indices respectively.

For one index  Here we have \( \Psi_\nu := \int \Phi I'_\nu \), for \( \nu \in \{ \alpha, \beta, \gamma \} \).

Remark 3 Let us observe that the integrals

\[
\Psi := \int \Phi I',
\]

are not well defined objects. This is because, as a primitive integral, \( \Phi \) is only defined up to an additive constant. Therefore, two integrals \( \Psi_1 \) and \( \Psi_2 \) defined for the same “\( \Phi \)” modulo constant terms, and the same \( I \), are related by a relation of the form

\[
\Psi_2 = \Psi_1 + dI + e,
\]

where \( (d,e) \in \mathbb{C}^2 \). Hence, having an Ostrowski relation \( \Psi_1 + dI \in K \) is therefore equivalent to have a representative \( \Psi_2 = \Psi_1 + dI \) which is algebraic. We will therefore use both of the two expressions.

Let us observe also that point (3) of Theorem 5 below is coherent when the two representatives \( \Psi_\gamma \) and \( \Psi_\alpha \) are defined with respect to the same \( \Phi \).

Observe also that the \( \Psi_\nu \) are integrals of first level with respect to \( K \), since \( \Phi \), and the \( I'_\nu \) are in \( K \).

For two indices  For \( i \neq j \), we have

\[
\Psi_{i,j} = \int \Phi(I'_i I_j + I'_j I_i) = \int \Psi_i' I_j + \Psi_j' I_i,
\]

\[
\Psi_{i,j} = \Psi_i I_j + \Psi_j I_i - M_{i,j} \quad \text{with} \quad M_{i,j} := \int \Psi_i' I_j' + \Psi_j' I_i'.
\]

In the particular case where \( i = j = \alpha \), by simplicity we divide by two the original \( \Psi \) by setting

\[
\Psi_{2\alpha} = \int \Phi I'_\alpha I_\alpha = \int \Psi'_\alpha I_\alpha,
\]

\[
\Psi_{2\alpha} = \Psi_\alpha I_\alpha - X \quad \text{with} \quad X := \int \Psi'_\alpha I_\alpha.'
\]
For three indices, we get
\[
\begin{align*}
\Psi_{\gamma,2\alpha} &= \int \Phi(I_\gamma I_\alpha^2) = \int \Phi(I_\alpha I_2^2 + 2I_\alpha I_\gamma), \\
\Psi_{\gamma,2\alpha} &= \int \Psi_\gamma I_\alpha^2 + 2\Psi_\alpha I_\gamma', \\
\Psi_{\alpha,\beta,\gamma} &= \int \Phi(I_\alpha I_\beta I_\gamma) = \int \Phi(I_\alpha I_\beta I_\gamma + I_\alpha I_\beta I_\gamma + I_\alpha I_\beta I_\gamma'), \\
\Psi_{\alpha,\beta,\gamma} &= \int \Psi_\alpha I_\beta I_\gamma + \Psi_\beta I_\gamma I_\alpha + \Psi_\gamma I_\alpha I_\beta.
\end{align*}
\]

**General formulae for the cocycles** \(C_m(\sigma) := \sigma(\Psi_m) - \Psi_m\) when \(\sigma \in G^0\). Here \(G\) denotes the corresponding Galois group. In order to simplify notations, we may some time write \(c_\nu\) instead of \(c_{\nu}(\sigma)\) in the relations \(\sigma(I_\nu) = I_\nu + c_\nu(\sigma)\), for \(\sigma \in G^0\). We get the following formulae:
\[
\begin{align*}
C_{i,j}(\sigma) &= c_i \Psi_j + c_j \Psi_j + l_{i,j}(\sigma), \\
C_{2\alpha}(\sigma) &= 2c_\alpha \Psi_\alpha + l_{2\alpha}(\sigma), \\
C_{\gamma,2\alpha}(\sigma) &= 2c_\alpha \Psi_{\alpha,\gamma} + 2c_\gamma \Psi_{2\alpha} + c_\alpha^2 \Psi_\gamma + 2c_\alpha c_\gamma \Psi_\alpha + l_{\gamma,2\alpha}(\sigma), \\
C_{\alpha,\beta,\gamma}(\sigma) &= c_\alpha(\sigma) \Psi_{\beta,\gamma} + c_\beta(\sigma) \Psi_{\alpha,\gamma} + c_\gamma(\sigma) \Psi_{\alpha,\beta} + c_\alpha c_\beta \Psi_\gamma + c_\beta c_\alpha \Psi_\alpha + c_\alpha c_\gamma \Psi_\beta + l_{\alpha,\beta,\gamma}(\sigma),
\end{align*}
\]

where the respective \(l_m\) are function from \(G^0\) to \(\mathbb{C}\).

We make only the computation for \(C_{\gamma,2\alpha}(\sigma)\) since the others are similar.

Since \(\Psi_{\gamma,2\alpha} = \Phi(I_\gamma I_\alpha^2 + 2I_\alpha I_\gamma I_\alpha),\) we get
\[
\begin{align*}
\sigma(\Psi_{\gamma,2\alpha}) &= \Phi(I_\gamma I_\alpha^2 + 2c_\alpha I_\gamma I_\alpha + c_\gamma^2) + 2\Phi(I_\alpha I_\gamma I_\alpha + c_\alpha)(I_\gamma + c_\gamma), \\
\sigma(\Psi_{\gamma,2\alpha})' &= \Psi_{\gamma,2\alpha} + 2c_\alpha \Psi_{\alpha,\gamma} + 2c_\gamma \Psi_{2\alpha} + c_\alpha^2 \Psi_{\gamma}' + 2c_\alpha c_\gamma \Psi_{\alpha}' , \\
C_{\gamma,2\alpha}(\sigma) &= 2c_\alpha \Psi_{\alpha,\gamma} + 2c_\gamma \Psi_{2\alpha} + c_\alpha^2 \Psi_{\gamma} + 2c_\alpha c_\gamma \Psi_{\alpha} + l_{\gamma,2\alpha}(\sigma).
\end{align*}
\]

The last relation has been obtained by integrating the previous one. As a consequence, the function \(l_{\gamma,2\alpha} : G^0 \rightarrow \mathbb{C}\) has constant values, but it is far to being a group morphism. This can be simply seen if we translate for \(l\) the cocycle relation for \(C\).

**Cocycles of degree in \(I\) and symmetric matrices** When proving the two theorems below, we will get explicit polynomial expressions of the above cocycles, since a necessary condition for the virtual Abelianity of \(G\) is going to be
\[
\forall \sigma \in G^0, \quad C_m(\sigma) \in T(F_1/K) = K[I] := K[I_1, \ldots, I_n].
\]

In practice, these cocycles are going to be of degree one or two in \(I\). Precisely, let us assume that \(F_1/K = K(I_1, \ldots, I_n)\), is generated by \(n\) independent first level integrals. Let us denote by \(I^T := (I_1, \ldots, I_n)\) and \(C^T(\sigma) := (c_1(\sigma), \ldots, c_n(\sigma))\).

We say that a cocycle \(C(\sigma) = \sigma(\Psi) - \Psi\) is of degree one in \(I\), if it has the form
\[
C(\sigma) = C(\sigma)^T A I + C(\sigma)^T \tilde{A} C(\sigma) + F^T C(\sigma) + l(\sigma),
\]
where \(A \in M_n(K)\) and \(\tilde{A} \in M_n(K)\) are \(n \times n\) matrices, \(F \in K^n\) and \(l\) is a constant valued function.
In the particular case where $A$ and $\tilde{A}$ are constant matrices (i.e., belong to $M_n(\mathbb{C})$), the mapping $\sigma \mapsto C(\sigma)^T \tilde{A}C(\sigma)$ is a constant valued function. So, the general expression of the degree one cocycles with constant matrix can be more simply written

$$C(\sigma) = C(\sigma)^T AI + C(\sigma)^T F + l(\sigma).$$

With these notations, we get the following property which simplifies the proof of the theorems, enlightening a link between the abelianity of a group, and the general idea of symmetry which is realised here by symmetric matrices.

**Lemma 4** Let us assume that $F_1/K = K(I_1, \ldots, I_n)$, is generated by $n$ independent first level integrals over $K$.

1. Let

$$C(\sigma) = C(\sigma)^T AI + C(\sigma)^T \tilde{A}C(\sigma) + F^T C(\sigma) + l(\sigma)$$

be a general cocycle of degree one in $I$. Then, $C$ coincides with a coboundary iff $A$ is a symmetric matrix, and $A - 2\tilde{A} \in M_n(\mathbb{C})$.

2. Let $C(\sigma) = \sigma(\Psi) - \Psi = C(\sigma)^T AI + C(\sigma)^T F + l(\sigma)$ be a degree one cocycle with constant matrices. Then $C$ coincides with a coboundary iff $A$ is symmetric. If it is the case, then the corresponding second level integral $\Psi$ can be computed in a closed form thanks to a quadratic expression of the form

$$\Psi = \frac{1}{2} I^T AI + F^T I + J \text{with} J' \in K.$$ 

3. For $n = 1$ every degree one cocycle with constant matrix is a coboundary.

**Proof** (1) Since $C$ is of degree one in $I$, if it coincides with a coboundary of the form $\Delta P + h(\sigma)$, then $P(I)$ must be quadratic in $I$. It can therefore be written into the form

$$P(I) = I^T SI + B^T I,$$

for some symmetric matrix $S \in M_n(K)$, and $B \in K^n$. The relation $C(\sigma) = \Delta P + h(\sigma) = P(I + c) - P(I) + h(\sigma)$, is therefore equivalent to having

$$I^T AC(\sigma) + C(\sigma)^T \tilde{A}C(\sigma) + F^T C(\sigma) + l(\sigma) = 2I^T SC(\sigma) + C(\sigma)^T SC(\sigma) + B^T C(\sigma) + h(\sigma).$$

For a fixed value of $\sigma$, both side of this equation are affine linear forms in $I$ with coefficients in $K$. Hence, we must have

$$AC(\sigma) = 2SC(\sigma),$$

for all $\sigma \in G^\circ$. Since $C(\sigma)$ span all $\mathbb{C}^n$, when $\sigma \in G^\circ$, we have that $A = 2S$ is symmetric. Moreover, the previous equation is reduced to

$$C(\sigma)^T (\tilde{A} - S)C(\sigma) + (F - B)^T C(\sigma) = h(\sigma) - l(\sigma).$$

By derivating both sides of this equation, we obtain

$$C(\sigma)^T (\tilde{A'} - S')C(\sigma) + (F' - B')^T C(\sigma) = 0, \text{ for all } \sigma \in G^\circ.$$

This is therefore equivalent to $\tilde{A'} - S' = 0$ and $F' - B' = 0$. That is to having $A - 2\tilde{A} \in M_n(\mathbb{C})$. Conversely, if those two conditions are satisfied, we just have to choose $B = F$ to get the desired coboundary.

21
When $C$ is of degree one with constant matrix, it coincides with a coboundary iff $A$ is constant and symmetric, since in this case there is no condition on $\tilde{A}$. Conversely, if $A$ is symmetric, the computation above with $\tilde{A} = 0$ shows that $\Psi$ and $\tilde{\Psi} := \frac{1}{2} I^T A I + F^T I$, have equal cocycles up to a constant valued function, therefore the difference $\Psi - \tilde{\Psi}$ is a first level integral.

(3) is obvious since a $1 \times 1$ matrix is always symmetric. □

Let’s observe that the computations of the second level integral $\Phi_1$ appearing in Lemma 2 is a particular case of points (2) and (3) of the above lemma.

5.3.2 The criteria for the virtual Abelianity of $EX^2_{\gamma,\alpha,\beta}$ when $\Phi$ is algebraic

We decided to begin with $EX^2_{\gamma,\alpha,\beta}$ because here, the role played by the integrals $I_{\nu}$ is symmetric. This is not the case when dealing with $VE^2_{\gamma,\alpha}$. As a consequence, although they proceed with the same methods, the proof of Theorem 4 is more transparent than the proof of Theorem 5 below.

For the statement and proof of the following result, we use the notations introduced above. The first point of the theorem gives a necessary condition for the virtual Abelianity of $EX^2_{\gamma,\alpha,\beta}$.

For the statement and proof of the following result, we use the notations introduced above. The first point of the theorem gives a necessary condition for the virtual Abelianity of $EX^2_{\gamma,\alpha,\beta}$.

Theorem 4 With the notation of Section 3.2, let us assume that $\Phi = \int \omega u_1 y_1 x_1$ is algebraic, i.e., $\Phi \in K$.

1. Let us denote by $\Psi = (\Psi_\alpha, \Psi_\beta, \Psi_\gamma)^T$, and similar notations for the three component vector $I$. If $EX^2_{\gamma,\alpha,\beta}$ has virtually Abelian differential Galois group, then there exists an Ostrowski relation between $\Psi$ and $I$ of the form:

$$\Psi = DI + F \in K^3,$$

for some constant $3 \times 3$ matrix $D$ and $F \in K^3$.

2. If the integrals $I_\alpha, I_\beta, I_\gamma$ are independent over $K$, then $EX^2_{\gamma,\alpha,\beta}$ has a virtually Abelian differential Galois group iff the following conditions are satisfied

a) There exists a unique determination of $\Phi$ modulo constants such that each $\Psi_\mu$ for $\mu \in \{\gamma; \beta; \alpha\}$ is algebraic. This correspond to having $D = 0$ in point (1).

b) For this determination of $\Phi$ the integrals $M_{i,j}$ are of first level, and can be expanded into the form

$$M = EI + G,$$

for some constant $3 \times 3$ symmetric matrix $E$, where $G \subset K^3$, and $M := (M_{\beta,\gamma}, M_{\gamma,\alpha}, M_{\alpha,\beta})^T$.

3. If the integrals $I_\alpha, I_\beta, I_\gamma$ form a system of rank one over $K$, that is if we have Ostrowski relations of the forms $I_\beta - \theta_\beta I_\alpha \in K$ and $I_\gamma - \theta_\gamma I_\alpha \in K$, then $EX^2_{\gamma,\alpha,\beta}$ has a virtually Abelian differential Galois group iff condition (1) holds and, for the first level integrals $N_{i,j}$ defined by equation (14) below, we get an Ostrowski relation of the form

$$N_{\beta,\gamma} + \theta_\beta N_{\gamma,\alpha} + \theta_\gamma N_{\alpha,\beta} = eI_\alpha + g,$$

with $e \in \mathbb{C}$ and $g \in K$. 

22
4. If the integrals \( I_\alpha, I_\beta, I_\gamma \) form a system of rank two over \( K \), that is if we get one Ostrowski relation of the form
\[
I_\gamma - bI_\beta - aI_\alpha \in K,
\]
and \( I_\alpha, I_\beta \) are independent. Then, \( \text{EX}_2^{\sigma}\alpha,\beta \) has a virtually Abelian differential Galois group iff the following conditions are satisfied

a) There exists a choice of \( \Phi \) modulo constants such that (1) can be written
\[
\Psi_\alpha = xI_\beta + f_\alpha, \quad \Psi_\beta = yI_\alpha + f_\beta, \quad \Psi_\gamma + b\Psi_\beta + a\Psi_\alpha \in K.
\]
with \( x \) and \( y \) in \( \mathbb{C} \).

b) For the first level integrals \( N_{i,j} \) defined by equation (15) below, we have a relation of the form
\[
\begin{pmatrix}
N_{\beta,\gamma} + aN_{\alpha,\beta} \\
N_{\alpha,\gamma} + bN_{\alpha,\beta}
\end{pmatrix} = E \begin{pmatrix} I_\alpha \\ I_\beta \end{pmatrix} + G.
\]
where, \( E \) is a \( 2 \times 2 \) constant symmetric matrix and \( G \in K^2 \).

**Proof** (1) Let us set \( F/K = \text{PV}(\text{EX}_2^{\sigma}\alpha,\beta) / K \), and assume that \( G \) is virtually Abelian. According to Theorem 2, since each \( \Psi_{i,j} \in F \), the cocycles
\[
C_{i,j} = \sigma(\Psi_{i,j}) - \Psi_{i,j} = c_i(\sigma)\Psi_j + c_j(\sigma)\Psi_i + l_{i,j}(\sigma) \in T(F_1 / K) = K[I] := K[I_\alpha, I_\beta, I_\gamma].
\]
By considering all those possible relations with \( i \neq j \), we get three relations which can be written into matrix form
\[
\begin{pmatrix}
0 & c_\gamma(\sigma) & c_\beta(\sigma) \\
c_\gamma(\sigma) & 0 & c_\alpha(\sigma) \\
c_\beta(\sigma) & c_\alpha(\sigma) & 0
\end{pmatrix}
\begin{pmatrix}
\Psi_\alpha \\
\Psi_\beta \\
\Psi_\gamma
\end{pmatrix} = C\Psi \in T(F_1 / K)^3.
\]
But \( \det(C) = 2c_\alpha(\sigma)c_\beta(\sigma)c_\gamma(\sigma) \neq 0 \), for some \( \sigma \in G^\circ \). So, by inverting this system we get that all \( \Psi_\mu \in T(F_1 / K) \). So, \( \Psi_\mu, I_\alpha, I_\beta, I_\gamma \) are four dependant integrals of first level over \( K \), from which we can deduce the desired Ostrowski relations.

(2) By plugging each \( \Psi_\mu = \sum \mu_{i,j}I_\mu + f_\mu \) into the above \( C_{i,j}(\sigma) \), we get that each \( \sigma(\Psi_{i,j}) - \Psi_{i,j} \) is a cocycle of degree one with constant matrix. Indeed, let us do this for \( \Psi_{\alpha,\beta} \):
\[
c_\alpha \Psi_\beta + c_\beta \Psi_\alpha = (c_\alpha, c_\beta, c_\gamma) \begin{pmatrix} d_{\beta,\alpha} & d_{\beta,\beta} & d_{\beta,\gamma} \\
d_{\alpha,\alpha} & d_{\alpha,\beta} & d_{\alpha,\gamma} \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix} I_\alpha \\ I_\beta \\ I_\gamma \end{pmatrix} + (c_\alpha, c_\beta, c_\gamma) \begin{pmatrix} f_\beta \\ f_\alpha \\ 0
\end{pmatrix}
\]
\[
C_{\alpha,\beta}(\sigma) = C(\sigma)^T A_{\alpha,\beta} I + C(\sigma)^T F_{\alpha,\beta} + I_{\alpha,\beta}(\sigma).
\]
According to Theorem 2 the Abelianness of \( G^\circ \) implies that the cocycle must be a coboundary. From Lemma 3 the equivalent condition is that \( A_{\alpha,\beta} \) is a symmetric matrix. As a consequence, we get that
\[
d_{\beta,\gamma} = d_{\alpha,\gamma} = 0 \quad \text{and} \quad d_{\alpha,\alpha} = d_{\beta,\beta}.
\]
Since the same arguments hold by considering \( C_{\beta,\gamma} \) and \( C_{\gamma,\alpha} \), we deduce that \( D = dI_3 \), for some \( d \in \mathbb{C} \). As a consequence, we can write
\[
\Psi_\mu = \int \Phi I_\mu = dI_\mu + f_\mu \quad \implies \quad \int (\Phi - d) I_\mu' = f_\mu \in K,
\]
23
for $\mu \in \{\alpha, \beta, \gamma\}$. Therefore, if we substitute, $\Phi - d$ to $\Phi$ in the definition of the corresponding $\Psi_{\mu}$, then they all are algebraic. This proves point (a). Let us assume from now that we are in this situation. Then, each $M_{i,j} = \int \Psi_{i}I_{i} + \Psi_{j}I_{j}$ is a first level integral.

Since, $\Psi_{\alpha,\beta,\gamma} = \int \Psi_{\gamma}I_{\beta}I_{\alpha} + \Psi_{\beta}I_{\alpha}I_{\gamma} + \Psi_{\alpha}I_{\gamma}I_{\beta}$ is a second level integral that belongs to $F$ its associated cocycles $C_{\alpha,\beta,\gamma}(\sigma) \in T(F_{1}/K)$ for all $\sigma \in G^{o}$. But

$$C_{\alpha,\beta,\gamma}(\sigma) = c_{\alpha}(\sigma)\Psi_{\beta,\beta} + c_{\beta}(\sigma)\Psi_{\gamma,\alpha} + c_{\gamma}(\sigma)\Psi_{\epsilon,\beta} + c_{\epsilon}c_{\alpha}\Psi_{\beta} + c_{\epsilon}c_{\beta}\Psi_{\gamma} + c_{\epsilon}c_{\gamma}\Psi_{\alpha} + c_{\epsilon}c_{\alpha}\Psi_{\beta} + l_{\alpha,\beta,\gamma}(\sigma).$$

As a consequence, each $\Psi_{i,j} \in T(F_{1}/K)$ since the vectors $C(\sigma) = (c_{\alpha}(\sigma), c_{\beta}(\sigma), c_{\gamma}(\sigma))^{T}$ ranges $\mathbb{C}^{3}$, when $\sigma$ ranges $G^{o}$. Therefore, each $M_{i,j} = \Psi_{i}I_{j} + \Psi_{j}I_{i} - \Psi_{i,j}I_{i,j}$ is a first level integral which belongs to $T(F_{1}/K)$. Hence, we get a matrix type Ostrowski relation of the form $M = EI + G$, where $E$ is constant $3 \times 3$ matrix, and $G \in K^{3}$. The previous relation implies that

$$Z := \begin{pmatrix} \Psi_{\beta,\gamma} \\ \Psi_{\gamma,\alpha} \\ \Psi_{\alpha,\beta} \end{pmatrix} = YI - EI - G, \quad \text{with} \quad Y := \begin{pmatrix} 0 & \Psi_{\gamma} & \Psi_{\beta} \\ \Psi_{\gamma} & 0 & \Psi_{\alpha} \\ \Psi_{\beta} & \Psi_{\alpha} & 0 \end{pmatrix} \in M_{3}(K).$$

As a consequence, we get the following formulae for the cocycle

$$C_{\alpha,\beta,\gamma}(\sigma) = C(\sigma)^{T}Z + \frac{1}{2}C(\sigma)^{T}YC(\sigma) + l_{\alpha,\beta,\gamma}(\sigma)$$

$$= C(\sigma)^{T}(YI - EI - G) + \frac{1}{2}C(\sigma)^{T}YC(\sigma) + l_{\alpha,\beta,\gamma}(\sigma),$$

$$C_{\alpha,\beta,\gamma}(\sigma) = C(\sigma)^{T}(Y - E)I + \frac{1}{2}C(\sigma)^{T}YC(\sigma) - C(\sigma)^{T}G + l_{\alpha,\beta,\gamma}(\sigma).$$

Hence, $C_{\alpha,\beta,\gamma}$ is a general coboundary of degree one. We get the formula of the first point of Lemma 4 by setting

$$A := Y - E \text{and} \tilde{A} := Y/2.$$  

Since $Y$ is symmetric, $C_{\alpha,\beta,\gamma}$ is a coboundary iff $E$ is a symmetric matrix. This proves the claim.

(3) For simplicity, we set $I := I_{\alpha}$ and $c := c_{\alpha}$, then by assumption we get $\forall \nu \in \{\alpha, \beta, \gamma\}$, $c_{\nu} = \theta_{\nu}c$, with $\theta_{\alpha} = 1$. Let’s assume that $G$ is virtually Abelian. Here, the Ostrowski relations of point (1) can be written

$$\forall \nu \in \{\alpha, \beta, \gamma\}, \Psi_{\nu} = d_{\nu}I + f_{\nu} \text{with} d_{\nu} \in \mathbb{C}, f_{\nu} \in K.$$  

As a consequence, the cocycles $C_{i,j}$ are of degree one with constant $1 \times 1$ matrices. Indeed we get

$$C_{i,j}(\sigma) = \sigma(\Psi_{i,j}) - \Psi_{i,j} = c(\theta_{i}d_{j} + \theta_{j}d_{i})I + c(\theta_{i}f_{j} + \theta_{j}f_{i})I + h_{i,j}(\sigma).$$

Therefore, according to Lemma 4 those cocycles are coboundary and there exist some first level integrals $N_{i,j}$ such that

$$\Psi_{i,j} = \frac{1}{2}(\theta_{i}d_{j} + \theta_{j}d_{i})I^{2} + (\theta_{i}f_{j} + \theta_{j}f_{i})I + N_{i,j}.$$  \hspace{1cm} (14)

Now, if we denote by the symbol $\bigoplus$ the sum over the three cyclic permutations of the indices
(α, β, γ), we get
\[ C_{α,β,γ}(σ) = \bigoplus c_hΨ_{i,j} + \bigoplus c_iC_jΨ_h + l_{α,β,γ}(σ). \]
\[ C_{α,β,γ}(σ) = c \bigoplus \left( \frac{θ_i}{2}(θ_id_j + θjd_i)I^2 + θ_h(θ_if_j + θjf_i)I + θ_hN_{i,j} \right) \]
\[ + c^2 \bigoplus θ_θ_j(d_hI + f_h) + l_{α,β,γ}(σ), \]
\[ C_{α,β,γ}(σ) = d[cI^2 + c^2I] + 2cfI + cf + c^2I + cN + l_{α,β,γ}(σ), \]
where we have set \( d := \bigoplus θ_θ_jd_h \in \mathbb{C}, \) \( f := \bigoplus θ_θ_iθ_if_j \in \mathbb{K} \) and \( N := \bigoplus θ_θ_hN_{i,j} \) is a first level integral. Since \( C_{α,β,γ}(σ) \in T(F_1/K), \) for all \( σ \in G^α, N \) also belongs to \( T(F_1/K). \) As a consequence, we get an Ostrowski relation of the form
\[ N = N_{β,γ} + θ_βN_{γ,α} + θ_γN_{α,β} = eI_a + g = eI + g. \]
This is precisely our additional necessary condition. Conversely, let's assume that (1) hold and \( N = eI + g. \) We already saw that all the \( Ψ_{i,j} \) can be computed in closed form. We will conclude by showing that in fact \( C_{α,β,γ} \) is a coboundary. By plugging \( N = eI + g \) inside the last expression of \( C_{α,β,γ}, \) we see that the latter is of degree two in \( I. \) But we are going to decrease its degree thanks to the following trick
\[ Δ(I^3/3) = (I + e)^3/3 - I^3/3 = eI^2 + e^2I + e^3/3. \]
Therefore,
\[ C_{α,β,γ}(σ) = Δ(dI^3/3) = 2cfI + cf + e^2f + eN + l_{α,β,γ}(σ) - c^3/3 \]
\[ = Δ(fI^3) + ceI + cg + l_{α,β,γ}(σ) - c^3/3 \]
\[ C_{α,β,γ}(σ) = Δ(df^3/3 + (f + e/2)I^2 + gI] + l(σ). \]
So, \( C_{α,β,γ} \) is a coboundary and \( G \) is virtually Abelian.

(4) Here we have \( c_c = b_c + ac_a. \) If we set: \( I^T := (I_α, I_β) \) and \( C(σ)^T := (c_α, c_β) \) then the Ostrowski relation of point (1) can be written
\[ Ψ = DI + F = \begin{pmatrix} Ψ_α \\ Ψ_β \\ Ψ_γ \end{pmatrix} = \begin{pmatrix} d_{α,α} & d_{α,β} \\ d_{β,α} & d_{β,β} \\ d_{γ,α} & d_{γ,β} \end{pmatrix} \begin{pmatrix} I_α \\ I_β \end{pmatrix} + \begin{pmatrix} f_α \\ f_β \end{pmatrix}. \]
Again, the cocycles \( C_{i,j} \) are going to be of degree one with constant matrices of size \( 2 \times 2 \) this time. If we write \( C_{i,j}(σ) = C(σ)^TF_{i,j}I + C(σ)^TF_{i,j}I + I_{i,j}(σ) \), the same computations as in point (3) give
\[ A_{β,γ} = \begin{pmatrix} a_{β,γ} & ad_{β,α} \\ d_{γ,α} + bd_{β,α} & d_{γ,β} + bd_{β,β} \end{pmatrix}, \quad F_{β,γ} = \begin{pmatrix} a_{β,γ} & af_β \\ f_γ + b_{f_β} \end{pmatrix}, \]
\[ A_{α,γ} = \begin{pmatrix} a_{α,γ} & ad_{α,α} \\ b_{dα,α} + ad_{α,β} & bd_{α,β} \end{pmatrix}, \quad F_{α,γ} = \begin{pmatrix} a_{α,γ} & af_α \\ b_{f_α} \end{pmatrix}, \]
\[ A_{α,β} = \begin{pmatrix} a_{α,β} & d_{β,β} \\ d_{α,α} & d_{α,β} \end{pmatrix}, \quad F_{α,β} = \begin{pmatrix} a_{α,β} & f_β \\ f_α \end{pmatrix}. \]
The three cocycles \( C_{i,j} \) are coboundary if the matrices \( A_{i,j} \) are symmetric. This translates to having \( D \) of the form
\[ D = \begin{pmatrix} d_{α,α} & d_{α,β} \\ d_{β,α} & d_{β,β} \\ d_{γ,α} & d_{γ,β} \end{pmatrix} = \begin{pmatrix} d & x \\ y & d \\ ad - by & bd - ax \end{pmatrix} \in M_{3,2}(\mathbb{C}). \]
If we look at the first line of $D$ this gives

$$\Psi_\alpha = \int \Phi I'_\alpha = dI_\alpha + xI_\beta + f_\beta \implies \int (\Phi - d) I'_\alpha = 0I_\alpha + xI_\beta + f_\alpha.$$  

As a consequence, if we change $\Phi$ to $\Phi - d$ in the definition of the $\Psi_\nu$, the new corresponding matrix $D$ will be simplified into the form

$$D = \begin{pmatrix} 0 & x \\ y & 0 \\ -by & -ax \end{pmatrix} \Rightarrow \begin{pmatrix} \Psi_\alpha \\ \Psi_\beta \\ \Psi_\gamma \end{pmatrix} = \begin{pmatrix} xI_\beta \\ yI_\alpha \\ -byI_\alpha - axI_\beta \end{pmatrix} + \begin{pmatrix} f_\alpha \\ f_\beta \\ f_\gamma \end{pmatrix},$$  

and this relation is equivalent to condition (4.a).

When this condition is satisfied, the matrices $A_{i,j}$ are symmetric, hence by the second point of the lemma, we get explicit formulæ of the form

$$\Psi_{i,j} = \frac{1}{2} T A_{i,j} I + F I_{i,j} I + N_{i,j}, \quad (15)$$

where the $N_{i,j}$ are first level integrals. Precisely, by expressing the $A_{i,j}$ in terms of $x, y, \theta_\alpha, \theta_\beta$, we get the following expressions

$$A_{\beta,\gamma} = \begin{pmatrix} ay & 0 \\ 0 & -ax \end{pmatrix} \quad \implies \quad \Psi_{\beta,\gamma} = \frac{ay}{2} I_\alpha^2 - \frac{ax}{2} I_\beta^2 + (af_\beta)I_\alpha + (f_\gamma + bf_\beta)I_\beta + N_{\beta,\gamma},$$

$$A_{\alpha,\gamma} = \begin{pmatrix} -by & 0 \\ 0 & bx \end{pmatrix} \quad \implies \quad \Psi_{\alpha,\gamma} = -\frac{by}{2} I_\alpha^2 + \frac{bx}{2} I_\beta^2 + (f_\gamma + af_\alpha)I_\alpha + (bf_\alpha)I_\beta + N_{\alpha,\gamma},$$

$$A_{\alpha,\beta} = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} \quad \implies \quad \Psi_{\alpha,\beta} = \frac{y}{2} I_\alpha^2 + \frac{x}{2} I_\beta^2 + (f_\beta)I_\alpha + (f_\alpha)I_\beta + N_{\alpha,\beta}.$$  

Next, by plugging these expressions in closed form of $\Psi_{i,j}$ and $\Psi_\nu$ into $C_{\alpha,\beta,\gamma}$ we get a formula of degree two in $I$ where $A$ is the symmetric matrix given by $A := \begin{pmatrix} af_\beta & f/2 \\ f/2 & bf_\alpha \end{pmatrix}$ with $f := f_\gamma + bf_\beta + af_\alpha$:

$$C_{\alpha,\beta,\gamma}(\sigma) = a[y[c_\alpha I_\alpha^3 + c_\beta I_\beta^3] + bx[c_\beta I_\beta^3 + c_\gamma I_\gamma^3] + l_{\alpha,\beta,\gamma}(\sigma) + C(\sigma)^T 2AI + C(\sigma)^T AC(\sigma) + C(\sigma)^T \left( \begin{array}{cc} N_{\beta,\gamma} + aN_{\alpha,\beta} \\ N_{\alpha,\gamma} + bN_{\alpha,\beta} \end{array} \right).$$

In the first line above we recognise an expression of the form $\Delta(\frac{ay}{3} I_\alpha^3 + \frac{bx}{3} I_\beta^3) + l(\sigma)$. Moreover, since $C_{\alpha,\beta,\gamma}(\sigma) \in T(F_1/K) \forall \sigma \in G^2$, the first level integrals $N_{\beta,\gamma} + aN_{\alpha,\beta}$ and $N_{\alpha,\gamma} + bN_{\alpha,\beta}$ are in $T(F_1/K)$. So we get an Ostrowski relation of the form

$$\left( \begin{array}{cc} N_{\beta,\gamma} + aN_{\alpha,\beta} \\ N_{\alpha,\gamma} + bN_{\alpha,\beta} \end{array} \right) = EI + G with E \in M_2(\mathbb{C}), G \in K^2.$$

Hence,

$$C_{\alpha,\beta,\gamma}(\sigma) = \Delta(\frac{ay}{3} I_\alpha^3 + \frac{bx}{3} I_\beta^3) + C(\sigma)^T (2A + E)I + C(\sigma)^T AC(\sigma) + C(\sigma)^T G + l(\sigma),$$

is a coboundary, iff $E$ is a constant $2 \times 2$ symmetric matrix. This proves the claim.  \qed

26
5.3.3 The criteria for the virtual Abelianity of $\text{VE}_{2,\alpha}^\gamma$ when $\Phi$ is algebraic

Here, we also use the notations of Section 3.2. Again, the first three points of the theorem below, give necessary conditions for the virtual Abelianity of $\text{VE}_{2,\alpha}^\gamma$. Points 5 and 6, give sufficient conditions according to the dependence of the two integrals $I_\alpha$ and $I_\gamma$.

**Theorem 5** With the notation of Section 3.2, let us assume that $\Phi = \int \omega y_3 x_1^2$ is algebraic (i.e. $\Phi \in K$). If $\text{VE}_{2,\alpha}^\gamma$ is virtually Abelian, then we get the following

1. There exists an Ostrowski relation between $\Psi_\alpha$ and $I_\alpha$ : $\Psi_\alpha - d_\alpha I_\alpha \in K$.

2. There exists an Ostrowski relation between $\Psi_\gamma$, $I_\gamma$ and $I_\alpha$ : $\Psi_\gamma = d_\gamma I_\gamma - d I_\alpha \in K$.

3. If in the previous relations $d_\gamma \neq d_\alpha$ then there exists an Ostrowski relation between $I_\gamma$ and $I_\alpha$.

4. Conversely, if $\Phi \in K$ and conditions (1),(2) and (3) hold true, then independently of the virtual Abelianity of $\text{VE}_{2,\alpha}^\gamma$, the second level integrals $\Psi_\alpha, \Psi_\gamma, \Psi_{2\alpha}, \Psi_{\gamma,\alpha}$ $X$ and $M$ can be computed in closed form. Moreover, $\text{VE}_{2,\alpha}^\gamma$ is virtually Abelian if the two following conditions are satisfied:

   - The integrals $X$ and $M$ have polynomial expressions of the form
     \[ \begin{pmatrix} M \\ X \end{pmatrix} = \begin{pmatrix} d I_\alpha^2/2 \\ 0 \end{pmatrix} + E \begin{pmatrix} I_\alpha \\ I_\gamma \end{pmatrix} + G \text{with } E \in M_2(\mathbb{C}), G \in K^2, \]
     where $M$ is defined in equation (16) below.
   - Moreover, $E$ is symmetric, that is we have $aX = b_M$ in equations (27) and (21) below.

5. Assume that $\Phi \in K$ and conditions (1),(2) and (3) hold true with $I_\alpha$ and $I_\gamma$ independent. According to points (1) and (3), there is unique choice of $\Phi$ modulo constants such that $\Psi_\alpha \in K$ and $\Psi_\gamma = dI_\alpha + f$. Then $\text{VE}_{2,\alpha}^\gamma$ is virtually Abelian iff the two following conditions are satisfied:

   - The integrals $X$ and $M$ have polynomial expressions of the form
     \[ \begin{pmatrix} M \\ X \end{pmatrix} = \begin{pmatrix} d I_\alpha^2/2 \\ 0 \end{pmatrix} + E \begin{pmatrix} I_\alpha \\ I_\gamma \end{pmatrix} + G \text{with } E \in M_2(\mathbb{C}), G \in K^2, \]
     where $M$ is defined in equation (16) below.
   - Moreover, $E$ is symmetric, that is we have $aX = b_M$ in equations (27) and (21) below.

6. Assume that $\Phi \in K$ and conditions (1),(2) hold true with $I_\alpha$ and $I_\gamma$ dependent, that is, we have an Ostrowski relation $I_\gamma = \theta I_\alpha \in K$. Let’s write (2) into the form $\Psi_\gamma - cI_\alpha \in K$. Then, $\text{VE}_{2,\alpha}^\gamma$ is virtually Abelian iff $M + \theta X$ can be computed into a polynomial form

\[ M + \theta X = \frac{e}{2} I_\alpha^2 + aI + g \text{ with } a, \theta \in \mathbb{C}, g \in K. \]

**Proof** (1) Again, let us assume that $F/K = \text{PV(VE}_{2,\alpha}^\gamma)/K$ is virtually Abelian. $F$ contains the first level integrals $\Psi_\nu = \int \Phi I_\nu$ for $\nu \in \{\gamma, \alpha\}$. It also contains the second level integral $\Psi_{2\alpha} = \int \Psi_\alpha I_\alpha$ and we get an Ostrowski relation between $\Psi_\alpha$ and $I_\alpha$ thanks to Lemma [2].

(2) Since the second level integral $\Psi_{\gamma,\alpha} := \int \Psi_\alpha I_\alpha + \Psi_\alpha I_\gamma$ also belongs to $F$, and since $F/K$ is virtually Abelian, from Theorem [2] the cocycles
\[ C_{\alpha,\gamma}(\sigma) = \sigma(\Psi_{\alpha,\gamma}) - \Psi_{\alpha,\gamma} = c_\alpha(\sigma)\Psi_\gamma + c_\gamma(\sigma)\Psi_\alpha + l(\sigma) \in T(F_1/K) = K[I_\gamma, I_\alpha], \]
From point (1), we already know that $\Psi_\alpha \in T(F_1/K)$. Therefore, $\Psi_\gamma \in T(F_1/K)$ and we get an Ostrowski relation between the three integrals $\Psi_\gamma, I_\gamma$ and $I_\alpha$.

(3) According to point (1) and Remark [K], we can choose a fixed representative of $\Phi$ such that $\Psi_\alpha \in K$. In other words we can assume that $d_\alpha = 0$ in (1). Let’s compute $\Psi_{\gamma,\alpha}$ thanks to the two previous Ostrowski relations modulo integral of first level and elements of $T(F_1/K)$. We get

\[ \Psi_{\alpha,\gamma} := \Psi_\gamma I_\alpha + \Psi_\alpha I_\gamma - M \text{ with } M := M_{\alpha,\gamma} = \int \Psi_\gamma I_\alpha' + \Psi_\alpha I_\gamma' \text{ (equation : M)} \] (16)
Since $\Psi_\alpha \in K$ and $\Psi_\gamma$ can be written $\Psi_\gamma = d_\gamma I_\gamma + dI_\alpha + f$ with $f \in K$, the expression $\Psi_\gamma I_\alpha + \Psi_\alpha I_\gamma \in T(F_1/K)$ and $M$ is a second level integral belonging to $F$. Moreover,

$$M = \int d_\gamma I_\gamma' + dI_\alpha' + (fI_\alpha' + \Psi_\alpha I_\gamma')$$

$$M = d_\gamma \int I_\gamma' I_\alpha + \frac{d}{2} I_\alpha^2 + \int fI_\alpha' + \Psi_\alpha I_\gamma' \in F.$$ 

If we set $J_1 := \int fI_\alpha' + \Psi_\alpha I_\gamma'$; this a first level integral over $K$ and the last relation tells us that

$$d_\gamma \int I_\gamma' I_\alpha + J_1 \in F.$$ 

But $F/K$ virtually Abelian implies that $F(J_1)/K$ is also virtually Abelian. Hence applying Lemma 2 again, we get an Ostrowski relation between $I_\gamma$ and $I_\alpha$ if $d_\gamma \neq 0$. \hfill $\square$

Before proving point (4), let us assume that $\Phi$ is algebraic and conditions (1),(2) and (3) of the theorem hold true. We have seen that these conditions can be restated into the following simpler form:

$$\begin{align*}
\Phi & \in K, \Psi_\alpha \in K, \\
\Psi_\gamma & = d_\gamma I_\gamma + dI_\alpha + f, \\
d_\gamma & \neq 0 \Rightarrow \exists (\theta, \kappa) \in C^* \times K|I_\gamma = \theta I_\alpha + \kappa
\end{align*}$$

(17)

Let us set $X := \int \Psi_\alpha I_\alpha'$ and $M := \int \Psi_\gamma I_\alpha' + \Psi_\alpha I_\gamma'$ as in (16). $X$ is a first level integral. Here we are going to show in addition that the second level integrals $M$ and $\Psi_{\gamma,\alpha}$ can also be computed in closed form. Precisely we shall prove that any such integral coincides with a polynomial in $I_\alpha, I_\gamma$ with coefficients in $K$ plus a first level integral.

**Proof** of point (4). Since $\Psi_{2\alpha} = \Psi_\alpha I_\alpha - X$, and $X$ is a first level integral and $\Psi_{2\alpha}$ can be computed in closed form. Now, from point (3) we get two possibilities. If $d_\gamma = 0$, then $M = \frac{d}{2} I_\alpha^2 + J_1$. If $d_\gamma \neq 0$, then

$$M = \frac{d_\gamma \theta + d_\alpha}{2} I_\alpha^2 + J_1 = \frac{d_\gamma \theta + d_\alpha}{2} I_\alpha^2 + J_2 with J_2 := J_1 + d_\gamma \int \kappa I_\alpha'.$$

Hence, $M$ can be computed in closed form since $J_2$ is a first level integral. For $\Psi_{\gamma,\alpha}$ this follows from (16). As a consequence, according to Theorem 2 and Section 3.2, $\text{VE}_{2,\alpha}$ is virtually Abelian iff $\Phi_{\gamma,2\alpha}$ can be computed in closed form. \hfill $\square$

**Computation of the cocycle $C_{\gamma,2\alpha}(\sigma) = \sigma(\Psi_{\gamma,2\alpha}) - \Psi_{\gamma,2\alpha}$ when (5.3.3) hold true** By substituting the previous integrals by their expression in closed form into the formula $C_{\gamma,2\alpha}(\sigma) = 2c_\alpha \Psi_{\alpha,\gamma} + 2c_\gamma \Psi_{2\alpha} + c_\gamma^2 \Psi_\gamma + 2c_\alpha c_\gamma \Psi_\alpha + l(\sigma)$, we get

$$C_{\gamma,2\alpha}(\sigma) = 2c_\alpha [\Psi_\gamma I_\alpha + \Psi_\alpha I_\gamma - M] + 2c_\gamma [\Psi_\alpha I_\alpha - X] + c_\alpha^2 [\Psi_\gamma] + 2c_\alpha c_\gamma [\Psi_\alpha] + l(\sigma)$$

(18)

According to Theorem 2, $G$ is virtually Abelian implies that

$$\forall \sigma \in G^\circ, C(\sigma) \in T(F_1/K) = K[I_\alpha; I_\gamma].$$

Therefore, if we compute $C_{\gamma,2\alpha}(\sigma)$ modulo $T(F_1/K) = K[I_\alpha; I_\gamma]$, we find a new necessary condition for Abelianness:

$$Gv. \ Ab \Rightarrow \forall \sigma \in G^\circ, c_\alpha X + c_\alpha M \in K[I_\alpha; I_\gamma].$$

(19)
Since the independence of $I_\gamma$ and $I_\alpha$ imposes to having $d_\gamma = 0$ that is $\Psi_\gamma = dI_\alpha + f$, and
$M = \frac{d}{2}I_\alpha^2 + J_1$. Therefore, $M \in K[I_\gamma; I_\alpha]$ iff we get an Ostrowski relation of the form

$J_1 = aMI_\alpha + bMI_\gamma + gM \Leftrightarrow M = \frac{d}{2}I_\alpha^2 + aMI_\alpha + bMI_\gamma + gM$.  

Now, if we substitute the actual expressions in closed polynomial form of $\Psi_\gamma, X$ and $M$ into (18), and expand the result as a polynomial in $\{I_\alpha; I_\gamma\} \text{ and } \{c_\alpha; c_\gamma\}$ with coefficients in $K$, this gives the following formula for the cocycle:

$$C_{\gamma, 2\alpha}(\sigma) = d[c_\alpha I_\alpha^2 + c_\gamma I_\gamma] + l_{\gamma, 2\alpha}(\sigma) + [2c_\alpha(f - aM) + 2c_\gamma(\Psi_\alpha - aX)]I_\alpha + 2c_\alpha(\Psi_\alpha - bM) + 2c_\gamma(-bX)I_\gamma + c_\alpha f + 2c_\gamma c_\alpha(\Psi_\alpha - 2c_\alpha gM - 2c_\gamma gX).$$

Again, as in point (4) of Theorem 3 this cocycle is of degree two, and by setting

$$A := \begin{pmatrix} 2(f - aM) & 2(\Psi_\alpha - bM) \\ 2(\Psi_\alpha - aX) & -2bX \end{pmatrix}, \bar{A} := \begin{pmatrix} f & \Psi_\alpha \\ \Psi_\alpha & 0 \end{pmatrix}, G := \begin{pmatrix} gM \\ gX \end{pmatrix}, I := \begin{pmatrix} I_\alpha \\ I_\gamma \end{pmatrix},$$

we get a formula

$$C_{\gamma, 2\alpha}(\sigma) = \Delta(d\frac{d}{2}I_\alpha^2) + C(\sigma)^T AI + C(\sigma)^T \bar{A}C(\sigma) - 2C(\sigma)^T G + I(\sigma).$$

But $A - 2\bar{A}$ is a constant matrix, and $A$ is symmetric iff $aX = bM$. So, $C_{\gamma, 2\alpha}$ is a coboundary iff the latter condition is satisfied. This proves the criteria given in point (5).

**Second case**: Here, we assume that $I_\gamma$ and $I_\alpha$ are dependant that is $I_\gamma = \theta I_\alpha + \kappa$ and $c_\gamma = \theta c_\alpha$. For simplicity we will write $I_\alpha = I$ and $c_\alpha = c \Rightarrow c_\gamma = \theta c$. The Ostrowski relation for $\Psi_\gamma$ will be written $\psi_\gamma = eI + f$ with $e := \theta d_\kappa + \eta$. As a consequence the formula for $M$ is now $M = \frac{e}{2}I^2 + J_2$ (independently of the possible vanishing of $d_\gamma$ the important number with that respect is now $e$). Now (19) is equivalent to having

$$M + \theta X = \frac{e}{2}I^2 + J_2 + \theta X \in K[I_\alpha; I_\gamma] = K[I] \Leftrightarrow J_2 + \theta X \in K[I].$$

Since, $J_2 + \theta X$ is a first level integral this equivalent to having an expansion in closed polynomial form:

$$J_2 + \theta X = aI + g \Leftrightarrow M + \theta X = \frac{e}{2}I^2 + aI + g \text{ with } a, g \in K.$$

Again, if we substitute the actual expressions of $\Psi_\gamma$ and $M + \theta X$ in closed polynomial form in (18) with $I_\gamma = \theta I + \kappa$ and $c_\gamma = \theta c$ and expand the result as a polynomial in $I$ and $c$, this gives the following formula for the cocycle

$$C_{\gamma, 2\alpha}(\sigma) = e[eI^2 + c^2I] + l_{\gamma, 2\alpha}(\sigma) + 2c(f + 2\theta \Psi_\alpha - a)I + c^2(f + 2\theta \Psi_\alpha) + 2c(\kappa \Psi_\alpha - g).$$
There is no condition on the size, and we recognise a coboundary
\[ C_{\gamma,2\alpha}(\sigma) = \Delta \frac{1}{3} I^3 + (f + 2\theta \Psi_{\alpha} - a) I^2 + 2(\kappa \Psi_{\alpha} - g) I + l(\sigma). \]
and we do not get more constrain in this case, and point (6) follows.

6 Effective test for $VE_{2,\alpha}^7$

Here according to Section 5.2 $K = \mathbb{C}[z[y_1; x_1]]$. We are therefore led to apply Proposition 6 with
\[ \Phi = \int \omega y_1 x_1^2. \]

6.1 Testing the algebraicity of $\Phi$ when $|k| \geq 3$

According to Lemma 1 when $|k| \geq 3$ and $G_{\gamma} = G_{\alpha} = G_{a}$, we can write $y_1$ and $x_1$ into the form
\[ y_1 = z^{a_1} (z - 1)^{b_1} J_{J_{2\alpha}} (z) \text{ and } x_1 = z^{a_1} (z - 1)^{b_1} J_{a} (z). \]
Since $\omega = z^{-\left(\frac{1}{2} + \frac{1}{k}\right)} (z - 1)^{-\frac{2}{k}}$ and $\Phi' = \omega y_1 x_1^2$ we get,
\[ \Phi' = z^{E_0} (z - 1)^{E_1} J_{J_{2\alpha}^2}, \]
with $E_0 = a_1 + 2a_1 - 3/2 - 1/2k$, and $E_1 = b_1 + 2b_1 - 5/4$. These values can be explicitly computed thanks to Table 2 and lead us to the following new table giving the explicit expression of $z^{E_0} (z - 1)^{E_1}$ in the expression $\Phi' = z^{E_0} (z - 1)^{E_1} J_{J_{2\alpha}^2}$.

| $\gamma\backslash\alpha$ | 1 | 2 | 3 | 4 |
|------------------------|---|---|---|---|
| 1                      | $z^{1/k}(z - 1)^{-1/2}$ | $z^{1/k}(z - 1)^{1/2}$ | $z^{-1/k}(z - 1)^{-1/2}$ | $z^{-1/k}(z - 1)^{1/2}$ |
| 2                      | $z^{1/k}$ alg | $z^{1/k}(z - 1)$ alg | $z^{-1/k}$ alg | $z^{-1/k}(z - 1)$ alg |
| 3                      | $(z - 1)^{-1/2}$ alg | $(z - 1)^{1/2}$ alg | $z^{-2/k}(z - 1)^{-1/2}$ | $z^{-2/k}(z - 1)^{1/2}$ |
| 4                      | 1 alg | $(z - 1)$ alg | $z^{-1/k}$ alg | $z^{-1/k}(z - 1)$ alg |

Table 4:

In this table the notation $<\text{alg}>$, means that necessarily, $\Phi$ is algebraic, and we count 10 cases over the 16 possibilities where this happens! In fact, this happens when at least one of the exponents $E_0$ or $E_1$ belong to $\mathbb{Z}$. This can be seen by direct computation, either by a consequence of Remark 2.

Now, for each of the 6 entries of the previous table, where $\Phi$ can be transcendental, $E_0 \in \{\pm 1/k; -2/\}$ and $E_1 \in \{\pm 1/2\}$, and we may write
\[ \Phi' = \Omega P \text{ with } \Omega := z^{E_0} (1 - z)^{E_1} \text{ and } P := J_{J_{2\alpha}^2}. \]
As a consequence, $\Phi$ is an integral of the type described in the previous section, so according to Proposition 3 the algebraicity of $\Phi$ reduces to the vanishing of $\mu(P) = \mu(J_{J_{2\alpha}^2})$. There are two ways of testing the vanishing of this number. One with the coefficients of $P$ if they are explicitly known. And the other with the evaluation of the definite integral between 0 and 1.

About this second method it applies sometime efficiently in our context. Indeed, for any Jacobi polynomial $J(z) = J_{(\alpha,\beta)}$, we can associate a kernel $\Omega = \Omega_J = z^\beta (1 - z)^\alpha$, whose explicit
formulas is given thanks to Table 2. Moreover we know that $J$ is orthogonal for the scalar product $<P; Q> = \int_0^1 PQdJ$, to any polynomial whose degree is smaller than $\deg(J)$.

For examples: thanks to Tables 2 and 4 we have

In case $(\gamma, \alpha) = (1; 1)$ then, $\Omega = \Omega_\gamma = \Omega_\alpha$ therefore,

$$\Phi = \int J_\gamma J_\alpha^2 \Omega_\gamma \Rightarrow \mu = \mu(J_\gamma J_\alpha^2) = <J_\gamma; J_\alpha^2>.$$ 

Hence, $2 \deg(J_\alpha) < \deg(J_\gamma) \Rightarrow \mu(P) = 0$ and the corresponding $\Phi$ is algebraic. This of course give new cases when $\Phi$ is algebraic. Unfortunately, we have got no converse and $\Phi$ could be algebraic outside of these cases.

In case $(\gamma, \alpha) = (1; 2)$ then $\Omega = \Omega_\gamma(1 - z) = \Omega_\alpha$ therefore,

$$\Phi = \int J_\gamma J_\alpha^2(1 - z)\Omega_\gamma \Rightarrow \mu = \mu(J_\gamma J_\alpha^2(1 - z)) > \gamma.$$ 

Hence, $2 \deg(J_\alpha) + 1 < \deg(J_\gamma) \Rightarrow \mu(P) = 0$ and the corresponding $\Phi$ is algebraic.

For the remaining four cases we have unfortunately no comparison of the expression of $\mu$ with one of the two scalar products. So in general we are not able to give any condition for the algebraicity! Nevertheless, for applications in specific examples one is therefore reduced just to check the relation given by point 3 of Proposition 3.

6.2 Getting obstruction when $\Phi$ can be transcendental in the 6 possible cases of Table 4

Our main result is going to be the following

Proposition 6 When $|k| \geq 3$ and $\Phi$ is transcendental, then in the six cases of Table 4, $\text{VE}_{2,\alpha}^\gamma$ is not virtually Abelian excepted maybe for $|k| = 3$ when $(\lambda_\gamma; \lambda_\alpha) \in \text{Case 3} \times \{\text{Case 3}; \text{Case 4}\}$.

Proof Let’s assume that $\text{VE}_{2,\alpha}^\gamma$ is virtually Abelian and look to the restrictions imposed by this condition thanks to Proposition 5 and its consequences. According to Lemma 1 the action of the monodromy around $z = 0$, is given by characters thanks to the following formulae

$$\mathcal{M}_0(I'_\gamma) = \exp(i2\pi(-2a_\gamma))I'_\gamma, \mathcal{M}_0(\Phi') = \exp(i2\pi E_0)\Phi'.$$ (22)

Since $\Phi$ is transcendental, the two constants $d_\gamma$ and $d_\alpha$ in Proposition 5.1 both are non-zero. Therefore, according to the later, we get an Ostrowski between $I_\gamma$ and $I_\alpha$. As a consequence, thanks to Lemma 3 the Ostrowski relation between $I_\gamma$ and $I_\alpha$ implies that

$$\exp(i2\pi(-2a_\gamma)) = \exp(i2\pi(-2a_\alpha)) \Leftrightarrow 2a_\gamma - 2a_\alpha \in \mathbb{Z}.$$ 

But, by writing $a = \frac{1}{2} + \varepsilon$, with $\varepsilon = 1$ in Cases 1 or 2, $\varepsilon = -1$ in Cases 3 or 4, we get that $2a_\gamma - 2a_\alpha \in \mathbb{Z}$ iff, $\varepsilon_\gamma = \varepsilon_\alpha$. In other words we proved that an Ostrowski relation between $I_\gamma$ and $I_\alpha$ implies that

$$a_\gamma = a_\alpha \Leftrightarrow (\gamma; \alpha) \in \{\text{Cases 1 or 2}\}^2 \cup \{\text{Cases 3 or 4}\}^2.$$ (23)

This in fact eliminates two possibilities in Table 4.

Now, the Ostrowski relation between $\Phi$ and $I_\gamma$ implies similarly that $E_0 + 2a_\gamma \in \mathbb{Z}$. But $E_0 = a_\gamma + 2a_\alpha = 3/2 - 1/2k$ and $a_\gamma = a_\alpha$, therefore,

$$E_0 + 2a_\gamma = 5a_\gamma = \frac{3}{2} - \frac{1}{2k} = 1 + \frac{5\varepsilon_\gamma - 1}{2k}.$$
For $\varepsilon = 1$, (Cases 1 or 2), $\frac{5\varepsilon - 1}{2\varepsilon} = \frac{2}{5} \not\in \mathbb{Z}$, Hence $\text{VE}_{2,\alpha}^2$ is not virtually Abelian.

For $\varepsilon = -1$, (Cases 3 or 4), $\frac{5\varepsilon - 1}{2\varepsilon} = \frac{3}{2} \in \mathbb{Z} \Leftrightarrow |k| = 3$. This proves that in general $\text{VE}_{2,\alpha}^2$ is not virtually Abelian except maybe in the exceptional cases mentioned in the proposition. \(\square\)

### 6.3 Getting Obstruction when $\Phi$ is algebraic

Here, according to the second point of Theorem 5, we have to test an Ostrowski relation between $\Psi_\alpha$ and $I_\alpha$. Again, if $\Psi_\alpha$ is algebraic such a relation hold and we get nothing new. As a consequence our first task is going to be the study of this problem.

#### 6.3.1 Testing the algebraicity of $\Psi_\alpha$ and $\Psi_\gamma$

Here, according to Lemma 1, $\Psi'_\alpha = \Phi I'_\alpha = \frac{\Phi}{x_1^2} = \frac{\Phi}{z^{2b_{\alpha}} (z - 1)^{2b_{\alpha}} J^2_\alpha(z)}$, and similar formula with $\Psi'_\gamma = \Phi I'_\gamma = \Phi/y_1^2$.

According to Remark 3, we must compute the two $\Psi'$ with the same $\Phi$ up to an additive constant. The precise forms of those expressions are respectively given for $\Psi'_\alpha$ resp for $\Psi'_\gamma$ in the following tables

| $\gamma \backslash \alpha$ | 1     | 2     | 3     | 4     |
|--------------------------|-------|-------|-------|-------|
| 1                        | $\frac{R}{J^2}$ | $\frac{(z-1)R}{J^2}$ | $\frac{R}{z^{1/2}J}$ | $\frac{(z-1)R}{z^{1/2}J}$ |
| 2                        | $\frac{R}{J^2}$ | $\frac{(z-1)R}{J^2}$ | $\frac{R}{z^{1/2}J}$ | $\frac{(z-1)R}{z^{1/2}J}$ |
| 3                        | $\frac{R}{J^2}$ | $\frac{(z-1)R}{J^2}$ | $\frac{R}{z^{1/2}J}$ | $\frac{(z-1)R}{z^{1/2}J}$ |
| 4                        | $\frac{R}{J^2}$ | $\frac{(z-1)R}{J^2}$ | $\frac{R}{z^{1/2}J}$ | $\frac{(z-1)R}{z^{1/2}J}$ |

Table 5:

| $\gamma \backslash \alpha$ | 1     | 2     | 3     | 4     |
|--------------------------|-------|-------|-------|-------|
| 1                        | $\frac{R}{J^2}$ | $\frac{(z-1)R}{J^2}$ | $\frac{R}{z^{1/2}J}$ | $\frac{(z-1)R}{z^{1/2}J}$ |
| 2                        | $\frac{R}{J^2}$ | $\frac{(z-1)R}{J^2}$ | $\frac{R}{z^{1/2}J}$ | $\frac{(z-1)R}{z^{1/2}J}$ |
| 3                        | $\frac{R}{J^2}$ | $\frac{(z-1)R}{J^2}$ | $\frac{R}{z^{1/2}J}$ | $\frac{(z-1)R}{z^{1/2}J}$ |
| 4                        | $\frac{R}{J^2}$ | $\frac{(z-1)R}{J^2}$ | $\frac{R}{z^{1/2}J}$ | $\frac{(z-1)R}{z^{1/2}J}$ |

Table 6:

Let’s explain briefly how those tables were computed. When we are in the ten cases of Table 4 where $\Phi$ is always algebraic, then $\Phi' = z^{E_0}(z - 1)^{E_1} P$, with at least one integral exponent. Assume for instance that $E_1 \in \mathbb{N}$, then we choose a primitive of the form $\Phi = z^{E_0+1} Q$ with $Q \in \mathbb{C}[z]$.

Now, when we are in the six remaining cases when the two exponents are not integers,

$$\Phi = \int \Omega P = \int z^{E_0}(z - 1)^{E_1} P.$$
According to Theorem 3, points 1 and 2, \( \Phi \) is algebraic iff it can be computed in closed form
\[
\Phi = \mathfrak{u}^R z^{E_0+1}(z - 1)^{E_1+1}R \text{ with } R \in \mathbb{C}[z].
\]
Surprisingly, in those six cases we get the following implication
\[
(\Phi \text{ alg}) \Rightarrow (\Psi_\alpha \text{ and } \Psi_\gamma \text{ alg}).
\]
Indeed in those six cases, \( E_1 + 1 \in \{1/2; 3/2\} \) For \( \nu \in \{\alpha; \gamma\} \),
\[
\Psi'_\nu = \frac{z^{E_0+1}(z - 1)^{E_1+1}R}{z^{2\nu}(z - 1)^{2\nu}f^\prime(z)}.
\]
Since \( 2b_\nu \) also belongs to \( \{1/2; 3/2\} \), the exponent of \( \Psi'_\nu \) at \( z = 1 \) is an integer \( \neq -1 \). As a consequence, \( \Psi'_\nu \) is fixed by \( M_1 \) and we can conclude thanks to Remark 2.

As a consequence, when \( \Phi \) is algebraic, we see from the above tables that \( \Psi_\alpha \) and \( \Psi_\gamma \) are simultaneously algebraic very often. In fact in 10 cases over the 16. Since it is much more difficult to prove the transcendence than to show the algebraicity of an integral, we therefore cannot say something general in the six remaining cases, when for example \( \lambda_\alpha \in \text{Case 4} \).

### 6.3.2 Getting obstruction when \( \Psi_\alpha \) or \( \Psi_\gamma \) is transcendental

Here our main result is going to be the following

**Proposition 7** If \( \Phi \) is algebraic, and \( \Psi_\alpha \) is transcendental, then \( \text{VE}_{2,\alpha}^\gamma \) is not virtually Abelian except maybe for the two cases \( (\lambda_\gamma; \lambda_\alpha) \in \text{Case 4} \times \{\text{Case 1 or 2}\} \).

Similarly, if \( \Psi_\gamma \) is transcendental, then \( \text{VE}_{2,\alpha}^\gamma \) is not virtually Abelian excepted maybe for the two cases \( (\lambda_\gamma; \lambda_\alpha) \in \text{Case 4} \times \{\text{Case 1 or 2}\} \) or when \( (\lambda_\alpha; \lambda_\gamma) \in \text{Case 2} \times \{\text{Case 3 or 4}\} \) if \( |k| = 3 \).

**Proof** From now we assume that \( \text{VE}_{2,\alpha}^\gamma \) is virtually Abelian. According to Theorem 3, we must get two Ostrowski relations: one between \( \Psi_\alpha \) and \( I_\alpha \), and another one between \( \Psi_\gamma \), \( I_\gamma \) and \( I_\alpha \). Let’s assume that we get such a relation between \( \Psi_\gamma \), \( I_\gamma \) and \( I_\alpha \). The arguments for \( \Psi_\alpha \) being simpler. According to Lemma 3, if \( \Psi_\gamma \) is transcendental, we must get the following relation between the exponents at \( z = 0 \):
\[
e_0(\Psi'_\gamma) \in \{e_0(I'_\gamma); e_0(I'_\alpha)\} \mod \mathbb{Z}.
\]
Now a direct comparison, of Table 8 with Table 3 in the six cases where \( \Psi_\gamma \) can be transcendental, gives that \( e_0(\Psi'_\gamma) \in \{e_0(I'_\gamma); e_0(I'_\alpha)\} \mod \mathbb{Z} \), when \( (\lambda_\gamma; \lambda_\alpha) \in \text{Case 4} \times \{\text{Case 1 or 2}\} \) or when \( (\lambda_\gamma; \lambda_\alpha) \in \text{Case 2} \times \{\text{Case 3 or 4}\} \), if \( |k| = 3 \). Moreover, when \( (\lambda_\gamma; \lambda_\alpha) \in \text{Case 4} \times \{\text{Case 1 or 2}\} \), we get the same conclusion for the exponents at \( z = 1 \). This is the reason why, these two cases cannot be a-priory refined.

### 6.3.3 Some results when \( \Phi, \Psi_\alpha \) and \( \Psi_\gamma \) are algebraic

As we said before those cases happen very often in 10 cases over the 16. This the right moment to apply Theorem 3, with \( d_\alpha = d_\gamma = c = 0 \). Here, \( X = \int \Psi_\alpha I'_\alpha \) and \( M = \int \Psi_\alpha I'_\gamma + \Psi_\gamma I'_\alpha \) are first level integrals and we first have to see if they can be computed in polynomial form. That is, if they satisfied Ostrowski relations with \( I_\alpha \) and \( I_\gamma \).

Precisely, according to the theorem we will have two cases
• When $I_\alpha$ and $I_\gamma$ are independent then we must check the following simple form of the relation given in point 5 of the theorem

$$
\begin{pmatrix} M \\ X \end{pmatrix} = E \begin{pmatrix} I_\alpha \\ I_\gamma \end{pmatrix} + G,
$$

(24)

where $E$ is symmetric.

• When $I_\alpha$ and $I_\gamma$ are dependant, that is $I_\gamma - \theta I_\alpha \in K$, then we must check a relation of the form

$$
M + \theta X = aI + g.
$$

(25)

Now, according to equation (23), we have a necessary condition for the independence of the two integrals $I_\alpha$ and $I_\gamma$, which we resume in the following new table

| $\gamma$ \(\backslash\) $\alpha$ | Case 1 | Case 2 | Case 3 | Case 4 |
|---|---|---|---|---|
| Case 1 | ? | ? | Ind | Ind |
| Case 2 | ? | ? | Ind | Ind |
| Case 3 | Ind | Ind | ? | ? |
| Case 4 | Ind | Ind | ? | ? |

Table 7:

So we see again, that in half of the cases the integrals are independent, and for the remaining cases we obviously do not know.

We have made explicit computations of the integrals $X$ and $M$, they are first level integrals so they will obey the rule given by Remark 2. Nevertheless, they are integrals of the form

$$
\int \frac{P\Omega}{J_\alpha J_\gamma^2} \text{ or } \int \frac{P\Omega}{J_\gamma J_\alpha^2}.
$$

As a consequence, they do not enter into the context of Theorem 3.

6.4 Experimental considerations for $\text{VE}_{2,\alpha}$.

Here we are going to join the information given by Propositions 6 and 7, the previous tables and some experiments given by computers. This will give a picture of the behaviour of the Galois group of $\text{VE}_{2,\alpha}$. For simplicity, we will assume that $|k| \geq 5$.

Indeed, the Propositions above showed, that for $|k| = 3$ some exceptional behaviour occur. Experiments also show that for $|k| = 4$, we also get new exceptions. This is quite normal since these two cases correspond geometrically to the situation where the hyper-elliptic curve parametrised by $t \mapsto (\varphi(t), \dot{\varphi}(t))$ is in fact an elliptic curve.

Step 1: when $\Phi$ is transcendental According to Section 5.1 we found that $\Phi$ can be transcendental in 6 over the 16 cases, with half of the possibilities when $(\gamma, \alpha) \in \{(1, 1); (1, 2)\}$. It seems that experiments are showing that away from the predicted cases, $\Phi$ is transcendental. As a consequence, thanks to Proposition 6 the probability to get obstruction when $\Phi$ is transcendental is

$$
p_\Phi = 5/16.
$$
Step 2: when \( \Phi \) is algebraic

Here for convenience, we present a new table resuming the case when \( \Phi \) is algebraic.

| \( \gamma \) | \( \alpha \) | Case 1 | Case 2 | Case 3 | Case 4 |
|---|---|---|---|---|---|
| Case 1  | \( A \) | \( A' \) |   |   |   |
| Case 2  | \( B \) | \( B' \) | \( C \) | \( C' \) |   |
| Case 3  | \( D \) | \( D' \) |   |   |   |
| Case 4  | \( E \) | \( E' \) | \( F \) | \( F' \) |   |

Table 8:

The empty cases correspond to the four cases where \( \Phi \) is transcendental. Again, for each letter \( X \), we will denote by \( p_X \) the probability to get obstruction in the context given by the corresponding letter. Here, the first thing to check is the possible transcendence of either \( \Psi_{\alpha} \) or \( \Psi_{\gamma} \) in order to apply Proposition 7. If nothing subsequent occurs from this test, then we check one of the two relations (24) or (25) depending of the possible dependence of the integrals \( I_{\alpha}, I_{\gamma} \) in order to apply Theorem 5.

**In \( A \) and \( A' \):** According to Section 6.1, \( \Phi \) is algebraic when \( 2 \deg(J_{\alpha}) < \deg(J_{\gamma}) \) in \( A \) and, when \( 2 \deg(J_{\alpha}) + 1 < \deg(J_{\gamma}) \) in \( A' \). In both cases, according to Tables 5 and 6 \( \Psi_{\alpha} \) and \( \Psi_{\gamma} \) are algebraic. Moreover, relation (24) hold for some symmetric matrix \( E \). Observe that here, we do not need to check the dependence of the integrals \( I \), since if this happens then we would get (24) \( \Rightarrow \) (25). As a consequence, there is no obstruction and

\[
p_A = p_{A'} = 0.
\]

For the remaining letters, \( \Phi \) is always algebraic.

**In \( B \) and \( B' \):** According to Tables 5 and 6 \( \Psi_{\alpha} \) and \( \Psi_{\gamma} \) are algebraic. Moreover, experiments give that \( M \) and \( X \) are algebraic. As a consequence, relation (24) is trivially satisfied with \( E = 0 \). Hence,

\[
p_B = p_{B'} = 0.
\]

**In \( C \) and \( C' \):** According to Table 5 \( \Psi_{\alpha} \) is algebraic. Nevertheless, experience shows that \( \Psi_{\gamma} \) is transcendental, hence according to Proposition 7 \( \text{VE}_{2,\alpha}^\gamma \) is not virtually Abelian and,

\[
p_C = p_{C'} = 1/16.
\]

**In \( D \) and \( D' \):** According to Tables 5 and 6 \( \Psi_{\alpha} \) and \( \Psi_{\gamma} \) are algebraic. Here, thanks to Table 7 the two integrals \( I_{\alpha} \) and \( I_{\gamma} \) are independent. In both cases, experiments show that \( M \) is algebraic and \( X \) is transcendental. In \( D \), we get an Ostrowski relation of the form \( X = aI_{\alpha} + g \). Therefore, in (24) the matrix

\[
E = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \text{ with } a \neq 0,
\]

is not symmetric. Therefore, there is obstruction. In \( D' \) the situation is quite similar excepted that there is no Ostrowski relation between \( X, I_{\alpha} \) and \( I_{\gamma} \), so (24) is not satisfied. Hence,

\[
p_D = p_{D'} = 1/16.
\]
In $E$ and $E'$: Here, we cannot apply directly Proposition. And in fact experiments show that we get two Ostrowski relations, one between $\Psi_\alpha$ and $I_\alpha$ and the other between $\Psi_\gamma$ and $I_\gamma$. Moreover, $\Psi_\alpha$ and $\Psi_\gamma$ are transcendental. But experiments also show that there is no possible Ostrowski relation between $M$, $I_\alpha$ and $I_\gamma$. Therefore, none of the equations (24) or (25) can be satisfied. So $VE_{2,\alpha}$ is not virtually Abelian and,

$$p_E = p_{E'} = 1/16.$$ 

In $F$ and $F'$: Experiments show that: in $F$ both $\Psi_\alpha$ and $\Psi_\gamma$ are algebraic iff $2 \deg(J_\alpha) > \deg(J_\gamma)$. Similarly, in $F'$, both $\Psi_\alpha$ and $\Psi_\gamma$ are algebraic iff $2 \deg(J_\alpha) + 1 > \deg(J_\gamma)$. Moreover, if these conditions on the degrees of the Jacobi polynomials are satisfied, then $M$ and $X$ are algebraic. Hence, we get half obstruction in each case and,

$$p_F = p_{F'} = 1/32.$$ 

As a consequence, the probability to get obstruction when $\Phi$ is algebraic is

$$p_{\text{alg}} = p_A + \cdots + p_{F'} = 7/16.$$ 

Hence, the total probability to get obstruction is

$$p_T = p_{\Phi} + p_{\text{alg}} = 5/16 + 7/16 = 3/4.$$ 

To our point of view, there are two significant conclusions that can be derived, from this study: First, that there is still a lot a obstruction at the level of the second variational equation. Indeed, it seems that there is a quite big distance between solvable Galois groups and virtually Abelian ones. Secondly, although it is comparatively much more complicated to test, the most important obstruction to the virtual Abelianity of $VE_{2,\alpha}$ happens when $\Phi$ is algebraic.

7 Considerations about $EX_{2,\alpha,\beta}^\gamma$

Here we follow the same strategy. But now the main technical difficulty comes in distinguishing the 64 possibilities for the cases satisfied by $\gamma, \beta, \alpha$. We will therefore have to deal with spatial tables of 64 entries! As a consequence, we will not give definitive results. Nevertheless, Theorem 4 can be used to deal with the complete study of specific potentials.

7.1 Counting the cases when $\Phi = \int \omega u_1 y_1 x_1$ is algebraic

Since the most check able obstructions are going to happen when $\Phi$ is transcendental, at first glance, we count the possible numbers of such occurrences. Here, by using similar arguments as in Table 4 we are going to count 44, possibilities where $\Phi$ is algebraic. Therefore, we will be left with at most 20 cases where $\Phi$ can be transcendental!

Here,

$$u_1 = z^{\alpha}(z-1)^{b_\gamma} J_\gamma; y_1 = z^{\alpha}(z-1)^{b_\beta} J_\beta; x_1 = z^{\alpha}(z-1)^{b_\alpha} J_\alpha \Rightarrow \Phi' = z^{E_0}(z-1)^{E_1} P(z),$$

With $P(z) = J_\gamma J_\beta J_\alpha$ and,

$$E_0 = a_\gamma + a_\beta + a_\alpha - \frac{3}{2} - \frac{1}{2k}; E_1 = b_\gamma + b_\beta + b_\alpha - \frac{5}{4}.$$ 

Again, as in Section 4.4 $\Phi$ is going to be algebraic when at least one of the two exponents $E_0$ or $E_1$ is an integer distinct from -1.


7.1.1 Values of $E_0$

If we write $a = \frac{1}{2} + \frac{\varepsilon}{2k}$ with $\varepsilon = 1$ in Cases 1 or 2 and $\varepsilon = -1$ in Cases 3 or 4, we get

$$E_0 = \frac{\varepsilon_\gamma + \varepsilon_\beta + \varepsilon_\alpha - 1}{2k}.$$ 

Then $E_0 \in \{1/k; -1/k; -2/k\}$, when $\text{Card}(i|\varepsilon_i = 1) \in \{3; 2; 1; 0\}$. As a consequence,

$$E_0 \in \mathbb{Z} \iff E_0 = 0 \iff \text{Card}\{i \in \{\gamma, \beta, \alpha\}|\varepsilon_i = 1\} = 2.$$ 

This happens in the $3 \times 8 = 24$ possibilities listed in the following table

| $E_0 \in \mathbb{Z}$ | $\gamma$ | $\beta$ | $\alpha$ |
|----------------------|---------|---------|---------|
| cases $L_1$          | 1 or 2  | 1 or 2  | 3 or 4  |
| cases $L_2$          | 1 or 2  | 3 or 4  | 1 or 2  |
| cases $L_3$          | 3 or 4  | 1 or 2  | 1 or 2  |

Table 9:

7.1.2 Values of $E_1$

Since $b = 1/4$ in Cases 1 or 3 and $b = 3/4$ in Cases 2 or 4, we get

$$
\begin{align*}
\begin{cases}
3 \text{ cases } 1 \text{ or } 3 & \Rightarrow E_1 = -1/2 \\
2 \text{ cases } 1 \text{ or } 3 & \Rightarrow E_1 = 0 \quad \in \mathbb{Z} \quad 24 \text{ possibilities} \\
1 \text{ cases } 1 \text{ or } 3 & \Rightarrow E_1 = 1/2 \\
0 \text{ cases } 1 \text{ or } 3 & \Rightarrow E_1 = 1 \quad \in \mathbb{Z} \quad 8 \text{ possibilities}
\end{cases}
\end{align*}
$$

As a consequence $E_1 \in \mathbb{Z}$ in 32 possibilities.

7.1.3 Counting when $E_0$ and $E_1$ both are integers

This happens when either $(E_0; E_1) = (0; 1)$ either $(E_0; E_1) = (0; 0)$.

**When** $(E_0; E_1) = (0; 1)$ Then, $E_1 = 1$ implies that $(\gamma; \beta; \alpha) \in \{\text{Cases 2 or 4}\}^3$ and the intersection with the table for $E_0 = 0$, gives 3 possibilities which are the cyclic permutations of

$$(\gamma; \beta; \alpha) \in (2; 2; 4).$$

**When** $(E_0; E_1) = (0; 0)$ Then $E_1 = 0$ implies the following possibilities

| $E_1 = 0$ | $\gamma$ | $\beta$ | $\alpha$ |
|-----------|---------|---------|---------|
| cases $L_1'$ | 1 or 3  | 1 or 3  | 2 or 4  |
| cases $L_2'$ | 1 or 3  | 2 or 4  | 1 or 3  |
| cases $L_3'$ | 2 or 4  | 1 or 3  | 1 or 3  |

Table 10:

As a consequence, if we compute $L_i \cap L_j$ we get $9 = 3 \times 3$ distinct cases. For example, $L_1 \cap L_1'$ implies $(\gamma; \beta; \alpha) \in (1; 1; 4)$.

Therefore, the two exponents both are integers in $3 + 9 = 12$ cases.
7.1.4 Conclusion
Since we get 24 possibilities for \( E_0 \in \mathbb{Z} \), 32 possibilities for \( E_1 \in \mathbb{Z} \) and 12 for both integral exponents. \( \Phi \) is going to be algebraic in \( 44 = 24 + 32 - 12 \) cases. As a consequence, in the spatial table of 64 entries for \( (\gamma; \beta; \alpha) \), there are at most 20 possibilities where \( \Phi \) can be transcendental.

7.2 Counting when all the \( \Psi_\mu \) are algebraic in the 44 case where \( \Phi \) is algebraic
This is also made in order to find the cases where there is no obstruction. In fact thanks to the consideration above we count 16 cases over 64 where everybody is certainly algebraic.

7.3 Getting obstruction with the assumption that \( \Phi \) is transcendental
Here our main result is going to be the following one which is very similar in its statement and proof to Proposition 6.

**Proposition 8** For \(|k| \geq 3\), in the 20 possible cases when \( \Phi \) can be transcendental, \( EX_{2,\alpha,\beta}^\gamma \) is not virtually Abelian excepted maybe for \(|k| = 3\) and \((\gamma; \beta; \alpha) \in \{\text{Cases } 3 \text{ or } 4\}\).

**Proof** From now we assume that \( EX_{2,\alpha,\beta}^\gamma \) is virtually Abelian. According to Proposition 5, we get three Ostrowski relations \( \Phi + dI_\gamma, \Phi + dI_\beta \) and \( \Phi + dI_\alpha \) are algebraic over \( \mathbb{C}(z) \). Since \( \Phi \) is transcendental, the three constants \( d_i \) are non-zero. As a consequence we get three Ostrowski relations between any two \( I_i \) and \( I_j \), for \( i \neq j \). But we have seen in the proof of Proposition 6 that such a relation implies that \( a_i = a_j \) (see equation (23)). And we can therefore deduce that \((\gamma; \beta; \alpha) \in \{\text{Cases } 1 \text{ or } 2\} \cup \{\text{Cases } 3 \text{ or } 4\}\).

Now, the Ostrowski relation between \( \Phi \) and \( I_\gamma \) implies similarly that \( E_0 + 2a_\gamma \in \mathbb{Z} \). But \( E_0 = a_\gamma + a_\beta + a_\alpha - 3/2 - 1/2k \) and \( a_\gamma = a_\beta = a_\alpha \), therefore,

\[
E_0 + 2a_\gamma = 5a_\gamma - \frac{3}{2} - \frac{1}{2k} = 1 + \frac{5\epsilon_\gamma - 1}{2k}.
\]

When \( \epsilon = 1 \), (Cases 1 or 2), \( \frac{5\epsilon_\gamma - 1}{2k} = \frac{5}{2} \notin \mathbb{Z} \), Hence \( EX_{2,\alpha,\beta}^\gamma \) is not virtually Abelian.

When \( \epsilon = -1 \), (Cases 3 or 4), \( \frac{5\epsilon_\gamma - 1}{2k} = \frac{1}{2k} \in \mathbb{Z} \Leftrightarrow |k| = 3 \). This proves that in general \( EX_{2,\alpha,\beta}^\gamma \) is not virtually Abelian excepted maybe in the exceptional cases mentioned in the proposition. 

\[\Box\]

7.4 Getting obstruction in the 44 cases when \( \Phi \) is algebraic
In order to be able to exploit the Ostrowski relations given by Theorem 4 in these 44 cases, we first have to investigate the possible algebraicity of the \( \Psi_\mu \), when \( \mu \in \{\gamma, \beta, \alpha\} \).

7.4.1 About the algebraicity of the integrals \( \Psi_\mu \) for \( \mu \in \{\gamma, \beta, \alpha\} \)
Here \( \Psi_\mu = \Phi I_\mu' = z^{E_0(z-1)^E_1 Q(z)} I_\mu' \). Let’s denote \( N_0^\mu \) and \( N_1^\mu \) the respective exponents of \( \Psi_\mu \) at \( z = 0 \) and \( z = 1 \), respectively. Up to addition of a positive integer we get

\[
N_0^\mu = E_0 - 2a_\mu; N_1^\mu = E_1 - 2b_\mu.
\]
Direct computation gives
\[ N^\mu_0 \in \{-1, \pm 1/k - 1, -2/k - 1\}; N^\mu_1 \in \{-3/2, -1, -1/2, 0, 1/2\}. \]

### 7.4.2 Getting obstruction when at least one \( \Psi_\mu \) is transcendental

#### Proposition 9

When \( \Phi \) is algebraic in the 44 cases mentioned above we get:

1. Let us assume that \( |k| \geq 4 \). If at least one of the \( \Psi_\mu \) is transcendental, then \( \text{EX}_2^{\alpha, \beta} \) is not virtually Abelian except maybe in the 12 cases where \( E_0 \) and \( E_1 \) both are integers.

2. For \( |k| = 3 \), we get the same conclusion if we assume that at least two of the three integrals \( \Psi_\mu \) are transcendental.

**Proof**

Now we assume that \( \text{EX}_2^{\alpha, \beta} \) is virtually Abelian. By symmetry let’s assume that \( \Psi_{\gamma} \) is transcendental. According to Theorem 3.2, the character of \( \Psi_{\gamma} \) must be equal to the character of one of the \( I_\mu \). But, as in the proof of Proposition 7, this condition is equivalent to having
\[ N^\gamma_1 - \frac{1}{2} \in \mathbb{Z} \iff E_1 - 2b_\gamma - \frac{1}{2} \in \mathbb{Z} \iff E_1 \in \mathbb{Z}. \]

Now let’s do the same job with \( M_0 \). As in the proof of Proposition 3.1 the Ostrowski relation imposes that at least one of the three following numbers is an integer:
\[ N^\gamma_0 + 2a_\gamma = E_0; N^\beta_0 + 2a_\beta = E_0 - 2a_\gamma + 2a_\beta; N^\alpha_0 + 2a_\alpha = E_0 - 2a_\gamma + 2a_\alpha. \]

Let us set \( \Delta_{\beta, \alpha} := E_0 - 2a_\beta + 2a_\alpha \). With this notation, we just have seen that \( \Psi_{\gamma} \) transcendental implies that
\[ E_0 \in \mathbb{Z} \text{or } \Delta_{\gamma, \beta} \in \mathbb{Z} \text{or } \Delta_{\gamma, \alpha} \in \mathbb{Z}. \]

If \( E_0 \in \mathbb{Z} \), both exponent at \( z = 0 \) and \( z = 1 \) are integers, we are in the 12 cases over the 44 where \( \Phi \) is algebraic and these arguments do not give any obstruction to the virtual Abelianness of \( \text{EX}_2^{\alpha, \beta} \).

Now let’s assume that \( E_0 \notin \mathbb{Z} \). We are led to find obstructions from the conditions \( \Delta \in \mathbb{Z} \). But, by writing \( a = 1/2 + \varepsilon/2k \), we get
\[ \Delta_{\beta, \alpha} = \frac{\varepsilon_\gamma - \varepsilon_\beta + 3\varepsilon_\alpha - 1}{2k}. \]

Its values depends on the eight possibilities given by \( \varepsilon_\mu = \pm 1 \). They are listed in the following table

| \( \varepsilon_\mu \) | \( \Delta_{\beta, \alpha} \) |
|-------------------|-----------------|
| \pm 1             | \pm 1/2         |
| \pm 1 \pm 1       | \pm 1 \pm 1/2   |
| \pm 1 \pm 1 \pm 1 | \pm 1 \pm 1 \pm 1/2 |

Here, we did not give the value of \( \Delta_{\beta, \alpha} \) for the three lines \( L_1, L_2, L_3 \) of Table 9 because they correspond to \( E_0 \in \mathbb{Z} \).

From this table, we get that \( \Delta \notin \mathbb{Z} \) except when \( |k| = 3 \), in the case of line \( L_6 \). This prove the first point of the proposition.

For the second point, let’s assume again that \( E_0 \notin \mathbb{Z} \) and \( \Psi_{\gamma} \) is transcendental. We must have \( \Delta_{\gamma, \beta} \in \mathbb{Z} \) or \( \Delta_{\gamma, \alpha} \in \mathbb{Z} \). But according to line \( L_6 \) we have the implications
\[ \Delta_{\gamma, \beta} \in \mathbb{Z} \Rightarrow (\gamma, \beta, \alpha) = (+, -, -) \Rightarrow \Delta_{\gamma, \alpha} \notin \mathbb{Z}. \]

Therefore, if for example \( \Psi_{\beta} \) is transcendental, we would get, according to the table
\[ \Delta_{\beta, \alpha} = -1/k \notin \mathbb{Z}, \quad \Delta_{\beta, \gamma} = 1/k \notin \mathbb{Z}. \]

So, the Ostrowski relation will not be satisfied for \( \Psi_{\beta} \). This prove the claim. \( \square \)
Table 11:

| $L_0$ | $e_x$ | $e_y$ | $e_\alpha$ | $\Delta_{3,\alpha}$ |
|-------|-------|-------|-----------|-----------------|
| $L_1$ | +     | +     | +         | $1/k$           |
| $L_2$ | -     | -     | -         | $-2/k$          |
| $L_3$ | -     | -     | +         | $1/k$           |
| $L_4$ | -     | +     | -         | $-3/k$          |
| $L_5$ | +     | -     | -         | $-1/k$          |

References

[1] A. Aparicio Monforte and J.-A. Weil. A reduction method for higher order variational equations of Hamiltonian systems. In *Symmetries and related topics in differential and difference equations*, volume 549 of *Contemp. Math.*, pages 1–15. Amer. Math. Soc., Providence, RI, 2011.

[2] Thierry Combot. Non-integrability of the equal mass; n-body problem with non-zero angular momentum. *Celestial Mechanics and Dynamical Astronomy*, pages 1–22.

[3] Guillaume Duval and Andrzej J. Maciejewski. Jordan obstruction to the integrability of Hamiltonian systems with homogeneous potentials. *Annales de l’Institut Fourier*, 59(7):2839–2890, 2009.

[4] Guillaume Duval and Andrzej J. Maciejewski. Integrability of Homogeneous potential of degree $k = \pm 2$. An application of higher variational equations. *submitted*, 2012.

[5] Juan J. Morales-Ruiz and Jean Pierre Ramis. A note on the non-integrability of some Hamiltonian systems with a homogeneous potential. *Methods Appl. Anal.*, 8(1):113–120, 2001.

[6] E. G. C. Poole. *Introduction to the theory of linear differential equations*. Dover Publications Inc., New York, 1960.