Sine-Gordon soliton as a model for Hawking radiation of moving black holes and quantum soliton evaporation

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The intriguing connection between black holes’ evaporation and physics of solitons is opening novel roads to finding observable phenomena. It is known from the inverse scattering transform that velocity is a fundamental parameter in solitons theory. Taking this into account, the study of Hawking radiation by a moving soliton gets a growing relevance. However, a theoretical context for the description of this phenomenon is still lacking. Here, we adopt a soliton geometrization technique to study the quantum emission of a moving soliton in a one-dimensional model. Representing a black hole by the one soliton solution of the sine-Gordon equation, we consider Hawking emission spectra of a quantized massless scalar field on the soliton-induced metric. We study the relation between the soliton velocity and the black hole temperature. Our results address a new scenario in the detection of new physics in the quantum gravity panorama.

I. INTRODUCTION

During the last ten years, analogue gravity systems have attracted major interest in the scientific community [1, 2]. These models aim at providing valuable scenarios to test inaccessible features of quantum gravity, as the Hawking radiation emission by black holes (BHs) [2]. Furthermore, the recent observation of gravitational waves (GWs) emitted by colliding BHs [2] opened new unexplored roads towards the search for quantum effects in gravity [3], as the Hawking’s BH evaporation [4]. Indeed, quantum BH emission might be observed by the concomitant monitoring of the BH collisions by gravitational and electromagnetic antennas. However, the collision process changes the original Hawking’s framework.

Originally, Hawking considered quantum fields in a stationary BH background, the Schwarzschild metric, and discovered that BHs emit thermal radiation and evaporate. His paper appeared exactly one year after a trailblazing article by Ablowitz, Kaup, Newell and Segur (AKNS), that cast new light on nonlinear waves by establishing the general method to solve classes of nonlinear field equations [5, 6]. Surprisingly, AKNS classes generate a metric and define an event horizon (EH). Indeed, it is known in the field of the nonlinear waves that the integrability condition of the GH equation determines a Riemannian surface with constant negative curvature [7, 8].

Recently, Hawking radiation analogues from solitons were considered in a huge variety of physical contexts, including light [9–12], ultracold gases [16–19], water and sound waves [20, 21]. Here, we study the geometrization of soliton equation by considering a canonical field quantization in the classical background of the Sine-Gordon (SG) soliton metric. Indeed, the 1 + 1 dimensional Sine-Gordon (SG) equation

$$\phi_{tt} - \phi_{xx} + m^2 \sin(\phi) = 0$$

is a nonlinear model that exhibits a Riemannian surface with constant negative curvature.

In this frame, the SG equation can be considered the AKNS counterpart of a two dimensional gravitational theory. Two dimensional theories of gravity are useful models to understand the quantum properties of higher-dimensional gravity. These theories capture essential features of higher-dimensional counterparts, and in particular have black hole solutions and Hawking radiation [22–25]. The link between the 1+1 dimensional gravity and the SG model introduces further simplifications since the quantum properties of this equation have been largely studied [24, 27]. As we shall recall in the next section, the integrability condition of the SG equation determines a metric, with a coordinate singularity, which defines an EH. In particular 1+1 dimensional BHs can be realized as solitons of the SG equation [28] and it has been shown with a one loop perturbative computation that this BH emits thermal radiation [29, 30].

In this paper, we show that SG soliton emits thermal particles with a specific Hawking temperature, finding the way the temperature changes with the velocity of the SG-BH. Afterward, we perform two different kinds of quantization, one for a massless scalar field and another for the soliton itself, and obtain their Hawking emission spectra. In both cases, we discover that an observer on the soliton tail detects a thermal radiation with a temperature directly proportional to the soliton speed. Furthermore, we analyze the temperature detected by an observer at rest by adding a Doppler effect.

Our paper is organized as follows: in sec. II we review the geometrization of the SG model; we show the connection between a soliton solution of an AKNS system

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and a metric on a two dimensional surface. In sec. III we study the BH metric induced by the SG equation and introduce suitable coordinate systems for the field quantization. In section IV we quantize massless scalar fields on the soliton background. In section V we quantize the SG soliton following the Faddeev semiclassical quantization [20], and show that the sine-gordon BH evaporates. Conclusions are drawn in section VI. A short appendix furnishes a minimal mathematical background to forms and curvature.

II. SINE-GORDON GEOMETRIZATION

We start reviewing the way integrable nonlinear equations generates surfaces with constant negative curvature [3]. By considering the SG equation defined in Eq. (1), we perform the coordinate transformation

\[
\chi = \frac{m}{2} (x + t), \quad \theta = \frac{m}{2} (x - t)
\]

and get

\[
\phi_{,\theta} = \sin \phi.
\]

As originally stated by Ablowitz, Kaup, Newel and Segur [8], for Eq. (3) the following system defines the scattering problem

\[
\begin{pmatrix}
V_\chi \\
V_\theta
\end{pmatrix} = \hat{L} \begin{pmatrix}
V_\chi \\
V_\theta
\end{pmatrix},
\]

where \(\hat{L}\) and \(\hat{M}\) are 2 \times 2 matrices, defining the Lax pair for Eq. (3). \(V\) is a vector. This system corresponds to the integrable Pfaffian system [31] (see appendix for an introduction to forms and surfaces)

\[
dV = \hat{\Omega} V, \quad V = \begin{pmatrix}
V_1 \\
V_2
\end{pmatrix},
\]

where \(\hat{\Omega}\) is a traceless matrix

\[
\hat{\Omega} = \hat{L} \chi + \hat{M} \theta = \begin{pmatrix}
\omega_1 & \omega_2 \\
\omega_3 & -\omega_1
\end{pmatrix},
\]

with the matrix elements \(\omega_i\) given by [9]

\[
\begin{align*}
\omega_1 &= \frac{1}{2} \lambda \chi - \frac{1}{2} \lambda \cos(\phi) \theta, \\
\omega_2 &= \frac{1}{2} \phi \chi - \frac{1}{2} \lambda \sin(\phi) \theta, \\
\omega_3 &= \frac{1}{2} \phi \chi - \frac{1}{2} \lambda \sin(\phi) \theta,
\end{align*}
\]

where \(\lambda\) is the spectral parameter of the SG scattering problem [8]. Following [16], the arclength of the induced Riemannian surface is written in terms of the matrix elements \(\omega_i\) as follows [8, 31]

\[
ds^2 = (\omega_2 + \omega_3)^2 + (2\omega_1)^2 = \\
= \lambda^2 \chi^2 + 2 \cos(\phi) \chi \theta + \frac{1}{\lambda^2} \theta^2.
\]

Eq. (8) defines the constant negative curvature metric induced by the ISM associated to the SG equation [3]. By changing the coordinates set as in the following, we write the first fundamental form \(ds^2\) as

\[
ds^2 = \sin^2 \left( \frac{\phi}{2} \right) d\tau^2 + \cos^2 \left( \frac{\phi}{2} \right) d\xi^2,
\]

which results to be associated with a SG equation of the form

\[
\phi_{\xi\xi} - \phi_{\tau\tau} = \sin(\phi),
\]

where

\[
\begin{aligned}
\xi &= \lambda \chi + \lambda^{-1} \theta, \\
\tau &= \lambda \chi - \lambda^{-1} \theta.
\end{aligned}
\]

Thus the metric tensor is

\[
\alpha = \begin{pmatrix}
\sin^2 \frac{\phi}{2} & 0 \\
0 & \cos^2 \frac{\phi}{2}
\end{pmatrix}.
\]

III. THE SINE-GORDON SOLITON BLACK HOLE

We show that the one-soliton solution of the ESG equation determines a BH metric.

The well known forward-propagating one-soliton solution of the Eq. (10) is

\[
\phi(\xi, \tau) = \arctan \left( \tan \frac{\phi_0}{2} \frac{\gamma^{-1}}{2} \frac{1}{\tau - \tau_0} \right).
\]

However, \(ds^2\) in Eq. (9) is not Lorentz invariant and it does not lead to a Schwarzschild-like metric. Following [28], in order to obtain a Minkowskii-like metric, we perform a Wick rotation \(\tau \rightarrow i\tau\) and obtain the elliptic SG (ESG) equation:

\[
\phi_{\xi\xi} + \phi_{\tau\tau} = \sin(\phi),
\]

whose corresponding metric is

\[
ds^2 = -\sin^2 \left( \frac{\phi}{2} \right) d\tau^2 + \cos^2 \left( \frac{\phi}{2} \right) d\xi^2.
\]

We adopt Eq. (16) hereafter. Substituting Eq. (15) in Eq. (14), we have

\[
ds^2 = ds^2_{\text{sol}} = -\operatorname{sech}^2(\rho) d\tau^2 + \tanh^2(\rho) d\xi^2,
\]
with \( \rho = \gamma (\xi - \beta_s \tau) \). Following [28], we adopt various coordinate transformations: first from \((r, \xi)\) to \((\mathcal{T}, \rho)\), with \( \rho \) as defined above and

\[
\mathcal{T} = \tau - \frac{1}{\beta_s} \left( \tanh^{-1}[\gamma^{-1} \tanh(\rho)] - \gamma^{-1} \rho \right). \tag{18}
\]

Next, we transform \((\mathcal{T}, \rho)\) to \((\mathcal{T}, r)\) by

\[
r = \frac{1}{\gamma} \sech(\rho). \tag{19}
\]

The result of the transformation is the line element

\[
ds^2 = (\beta_s^2 - r^2) d\mathcal{T}^2 - (\beta_s^2 - r^2)^{-1} dr^2. \tag{20}
\]

Eq. (20) is the metric of a 1+1 dimensional BH with EH at \( r_g := \beta_s \). Figures [1] and [2] show the EH positions \( \rho_g = \arcsch(\gamma r_g) \) on the soliton profiles and energy densities \( \mathcal{E} \), respectively for different velocities \( \beta_s \). The energy density, at fixed \( t \), is defined as follows [26]:

\[
\mathcal{E} = \frac{1}{2} (\partial_\xi \phi_s)^2 + [1 - \cos(\phi_s)].
\]

![FIG. 1. (Color online) The sine-Gordon soliton at fixed time \( \tau = 1 \), varying the velocity \( \beta_s \). The positions of the EHs \( \rho_g = m\gamma (\xi_\phi - \beta_s \tau) \) are in dashed lines.](image)

![FIG. 2. (Color online) The soliton energy density at various velocities \( \beta_s \), at fixed time \( \tau = 1 \). The positions of the EHs \( \rho_g = m\gamma (\xi_\phi - \beta_s \tau) \) are in dashed lines.](image)

It is now convenient to introduce two new sets of coordinates: the modified Regge-Wheeler coordinate, that we call the slug coordinate in analogy with the tortoise coordinate, as usually reported [2, 3], and the Kruskal-Szekeres coordinates.

We get the slug coordinate \( r^*(r) \) according to

\[
 dr^* = (\beta_s^2 - r^2)^{-1} dr,
\]

so that

\[
r^*(r) = \frac{1}{\beta_s} \tanh^{-1} \left( \frac{r}{\beta_s} \right) = \frac{1}{2\beta_s} \ln \left( \frac{\beta_s + r}{\beta_s - r} \right). \tag{22}
\]

Eq. (20) then becomes

\[
ds^2 = [\beta_s^2 - r^2 (r^*)] d\mathcal{T}^2 - (dr^*)^2. \tag{23}
\]

The slug coordinate is singular at \( r = \beta_s \) and it is defined on the exterior of the BH when \( r \to \pm \infty \) and \( r \to 0 \). In fact, as \( r \) approaches \( \beta_s \), \( r^* \) goes to \( +\infty \), while far away from the BH \( r^* \to 0 \) as \( r \to 0 \).

Introducing the slug lightcone coordinates

\[
\tilde{u} = \mathcal{T} - r^*, \quad \tilde{v} = \mathcal{T} + r^*, \tag{24}
\]

we write Eq. (20) as

\[
ds^2 = [\beta_s^2 - r^2 (\tilde{u}, \tilde{v})] d\tilde{u} d\tilde{v}. \tag{25}
\]

The slug lightcone coordinates are singular and they span only the exterior of the black hole. To describe the entire spacetime, we need another coordinate system. In order to be consistent with literature, we refer to them as the Kruskal-Szekeres (KS) coordinates. From Eq. (22) and Eq. (24) it follows that

\[
\beta_s^2 - r^2 = (\beta_s + r)^2 \exp[\beta_s (\tilde{u} - \tilde{v})]. \tag{26}
\]

The BH metric thus becomes

\[
ds^2 = [\beta_s + r (\tilde{u}, \tilde{v})]^2 e^{\beta_s (\tilde{u} - \tilde{v})} d\tilde{u} d\tilde{v}. \tag{27}
\]

In the KS lightcone coordinates, defined as

\[
u = \frac{e^{\beta_s \tilde{u}}}{\beta_s}, \quad v = \frac{e^{-\beta_s \tilde{v}}}{\beta_s}, \tag{28}
\]

Eq. (27) takes the form

\[
ds^2 = [\beta_s + r (\tilde{u}, \tilde{v})]^2 du dv, \tag{29}
\]

and it is regular at \( r = \beta_s \). The singularity occurring in the ESG-soliton metric is, as the Schwarzschild one, a coordinate singularity, which can be removed by a coordinate transformation. The KS coordinates, indeed, span the entire spacetime.

IV. MASSLESS SCALAR FIELD QUANTIZATION

We consider a field quantization on the classical soliton background metric. We first analyze a massless scalar field with the action

\[
S[\phi] = \frac{1}{2} \int g^\mu\nu \partial_\mu \phi \partial_\nu \phi \sqrt{-g} d^2 x. \tag{30}
\]
where $g^{\mu \nu}$ represents the inverse of a general metric tensor $g_{\mu \nu}$, $g$ is the determinant of $g_{\mu \nu}$ and $x = (x^0, x^1)$. The action in Eq. (30) is conformally invariant, and in terms of lightcone slug coordinates and lightcone KS coordinates it reads

$$S[\phi] = \int \partial_\nu \phi \partial_\nu \phi \, d\tilde{u} d\tilde{v},$$

$$S[\phi] = \int \partial_\nu \phi \partial_\nu \phi \, du dv.$$  

We write the solution of the scalar field equation in terms of lightcone slug coordinates

$$\phi = A(u) + B(v),$$

and in the lightcone KS coordinate as

$$\phi = A(u) + B(v),$$

where $A$, $A$ and $B$, $B$ are arbitrary smooth functions. In correspondence of the tail of the soliton, i.e., far away from the EH, the mode expansion of the field is

$$\hat{\phi} = \int_0^\infty \frac{d\Omega}{2\sqrt{\pi} \Omega} \left[ e^{-\Omega \tilde{u}} \hat{b}_{\Omega}^\dagger + e^{\Omega \tilde{u}} \hat{b}_{\Omega}^\dagger \right] + \text{left moving}. \quad (34)$$

In Eq. (34) the left moving part is given by the terms weighted by $e^{\pm \Omega \tilde{u}}$ in the mode expansion. The vacuum state $|0_B\rangle$, defined by $\hat{b}_{\Omega}^\dagger |0_B\rangle = 0$, is the Boulware vacuum (BV) and does not contain particles for an observer located far from the EH. However, as the slug coordinate is singular at horizon, the BV is also singular at the EH.

To obtain a vacuum state defined over the entire spacetime, we expand the field operator in terms of the KS lightcone coordinates

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{2\sqrt{\pi} \omega} \left[ e^{-i\omega \tilde{u}} \hat{a}_{\omega}^\dagger + e^{i\omega \tilde{u}} \hat{a}_{\omega}^\dagger \right] + \text{left moving}. \quad (35)$$

The creation and annihilation operators $\hat{a}_{\omega}^\dagger$ determine the Kruskal vacuum (KV) state $\hat{a}_{\omega}^\dagger |0_K\rangle = 0$. The KV is regular on the horizon and corresponds to true physical vacuum in the presence of the BH. For a remote observer the KV contains particles. To determine their number density, we follow the original calculations of Hawking and Unruh with the only difference in the definition of the KS coordinates (see chapters 8 and 9 of [2] for details).

We find that the remote observer moving with the soliton tail sees particles with the thermal spectrum

$$\langle \hat{N}_\Omega \rangle = \langle 0_K | \hat{b}_{\Omega}^\dagger \hat{b}_{-\Omega} | 0_K \rangle = \left[ \exp \left( \frac{2\pi \Omega}{\beta_s} \right) - 1 \right]^{-1} \delta(0).$$

If we consider a finite volume quantization we can put $V = \delta(0) \Box$ and we obtain the number density

$$n_\Omega = \left[ \exp \left( \frac{2\pi \Omega}{\beta_s} \right) - 1 \right]^{-1}, \quad (37)$$

corresponding to the temperature

$$T_H = \frac{\beta_s}{2\pi}. \quad (38)$$

In Fig. 3 we show the radiance $\Omega(\Omega) = \Omega^2 n_\Omega$. We observe that for a static soliton ($\beta_s = 0$) we get $T_H = 0$. This result may appear in contradiction with the Hawking original work, where he considered the emission from a static BH. However the result in eq. (38) is coherent with the structure of the metric induced by the SG equation, where the singularity occurs for $r = r_g = \beta_s$ and no emission can be observable for $r_g = 0$. This dependence of the Hawking radiations on the translation velocity is peculiar of soliton dynamics [2] and it is related to the structure of the spectral parameter in the IST [3, 4].

### A. Hawking temperature in the laboratory frame

Unlike the Schwarzschild BH, the ESG soliton is not static, but translates with velocity $\beta_s$. The frequency $\Omega$ seen by an observer at rest with respect to the soliton contains a Doppler shift. Letting $\Omega_s$ be the frequency emitted by the soliton in (30), the frequency measured by an observer moving with velocity $-\beta_s$, with respect to the soliton, and located at an angle $\theta_s$ with respect to the soliton direction is

$$\Omega_s = \frac{1 - \beta_s \cos \theta_s}{\sqrt{1 - \beta_s^2}} \Omega_s. \quad (39)$$

In the collinear case $\theta_s = 0$, and we have

$$\frac{\Omega_s}{\Omega_s} = \frac{1 - \beta_s}{1 + \beta_s}. \quad (40)$$

The corresponding Hawking temperature is (for small $\beta_s$)

$$T_H = \frac{\beta_s}{2\pi} \sqrt{\frac{1 - \beta_s}{1 + \beta_s}} \approx \frac{\beta_s}{2\pi} (1 - \beta_s). \quad (41)$$

This calculation also applies to a massive bosonic field, as the number density spectrum depends only on the statistics [24]. In the case of a fermionic field the theory is similar, but the number density spectrum follows the Fermi-Dirac statistics [2].
V. SOLITON QUANTIZATION

Previously we studied the BH evaporation following the works of Hawking and Unruh in [2, 35]. Now, we analyze a quantum perturbation of the BH metric given by the classical soliton solution of the ESG equation, and we obtain a BH evaporation without the interaction with a massless scalar field. We start from

$$\phi \simeq \phi_s + \phi_1,$$

(42)

where $\phi_s$ is the classical solution in Eq. (15) and $\phi_1$ represents a weak field perturbation. We consider the conformally invariant action

$$S[\phi] = \int \left[ \frac{1}{2} g^{\mu \nu} \partial_\mu \phi + \cos(\phi) \right] \sqrt{-g} \, d^2x,$$

(43)

which leads to a field equation

$$g^{\mu \nu} \partial_\mu \partial_\nu \phi + \sin(\phi) = 0.$$

(44)

The solutions of Eqs. (10, 13) differ for a Wick rotation. In other words, one passes from the SG soliton to the ESG one by the transformation

$$\tau \rightarrow i\tau, \quad \beta_s \rightarrow -i\beta_s.$$

(45)

We perform the inverse Wick rotation, i.e., $\tau \rightarrow -i\tau$, $\beta_s \rightarrow i\beta_s$, passing from the ESG to the SG, and substitute Eq. (12) into Eq. (14), hence we obtain

$$g^{\mu \nu} \partial_\mu \partial_\nu \phi_1 + \cos(\phi_s) \phi_1 = 0,$$

(46)

where we neglect terms $O(\phi_s^2)$. This equation expresses the interaction between a massive particle and the gravitational field, because the weak quantum field $\phi_1$ obeys a generalized Klein-Gordon equation with squared mass $\cos(\phi_s)$ depending on the soliton, and thus on the metric. Recalling Eq. (15), we have

$$g^{\mu \nu} \partial_\mu \partial_\nu \phi_1 + \cos\{4 \arctan[\exp(\rho)]\} \phi_1 = 0.$$

(47)

For an observer located on the tail of the soliton ($\rho \to \infty$), the field equation reduces to

$$g^{\mu \nu} \partial_\mu \partial_\nu \phi_1 + \phi_1 = 0,$$

(48)

while for an observer on the horizon ($\rho \to \rho_g$), we have

$$g^{\mu \nu} \partial_\mu \partial_\nu \phi_1 + F(\rho) \phi_1 = 0,$$

(49)

with $F(\rho)$ given by

$$F(\rho)_{\rho \to \rho_g} \simeq 1 + \frac{5}{2} \gamma^2 \beta_s^4 - 5 \gamma^2 \beta_s^2 \sqrt{1 - \gamma^2 \beta_s^2} (\rho - \rho_g).$$

(50)

Eq. (50) truncated at the order zero in $\rho - \rho_g$, i.e., exactly on the horizon, leads to

$$F \simeq 1 + \frac{5}{2} \gamma^2 \beta_s^4.$$

(51)

Due to the inverse Wick rotation, even if the action is conformally invariant, the quantization is not straightforward. We need to adapt both the slug and the KS lightcone coordinates in Eqs. (24, 28) to the rotated system. We obtain

$$r^* = \int_0^r \frac{dr'}{\beta_s^2 + r'^2} = \frac{i}{2\beta_s} \ln \left( \frac{i\beta_s + r}{i\beta_s - r} \right),$$

$$\hat{u} = \mathcal{T} - iv^*, \quad \hat{v} = \mathcal{T} + iv^*,$$

$$u = \frac{e^{-\beta_s \hat{u}}}{\beta_s}, \quad v = \frac{e^{\beta_s \hat{v}}}{\beta_s}.$$

(52)

Since the action (43) is conformally invariant, we thus write the field equation as follows

$$\partial_\mu \partial_\nu \phi_1 + \phi_1 = 0 \quad \text{slug lightcone},$$

$$\partial_\mu \partial_\nu \phi_1 + F \phi_1 = 0 \quad \text{K-S lightcone},$$

(53)

Eqs. (43) have exponential solution

$$\phi_1 \propto e^{i(K - \Omega_K)\hat{u} - i(K + \Omega_K)\hat{v}},$$

$$\phi_1 \propto e^{i(k - \omega_k)\hat{u} - i(k + \omega_k)\hat{v}},$$

(54)

with the following dispersion relations,

$$\Omega_K = \sqrt{K^2 + 1},$$

$$\omega_k = \sqrt{k^2 + F^2}.$$

(55)

From now on, we omit the $K$ and $k$ indices. We write the quantum fields as follows

$$\hat{\phi}_0 = \frac{1}{2\pi} \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}} \left[ \hat{\phi}_{\Omega} e^{i(K - \Omega)\hat{u} - i(K + \Omega)\hat{v}} \right]$$

$$= \frac{1}{2\pi} \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}} \left[ \hat{\phi}_{\Omega} e^{i(k - \omega)\hat{u} - i(k + \omega)\hat{v}} + \hat{\phi}_{\Omega}^* e^{-i(k - \omega)\hat{u} + i(k + \omega)\hat{v}} \right],$$

(56)

where, as in the non interacting case, the annihilation operators $\hat{\phi}_{\Omega}$ and $\hat{\phi}_{\Omega}^*$ define the Boulware vacuum $|0_B\rangle$ and the Kruskal vacuum $|0_K\rangle$, respectively. The operators $\hat{a}_{\omega}^\pm$
and \( \hat{b}_{1}^{\dagger} \) are related by the Bogolyubov transformations

\[
\hat{b}_{-} = \int_{0}^{\infty} d\omega \left( \alpha_{\omega} \hat{a}_{-}^{\dagger} - \beta_{\omega} \hat{a}_{+}^{\dagger} \right).
\]  
(57)

hence we obtain

\[
\alpha_{\omega} = \frac{1}{2\pi V} \sqrt{\frac{\Omega}{\omega}} \int d\omega \hat{\Omega} e^{i\Omega(\tilde{u} + \tilde{v}) - \Omega(u + v) + (\hat{b}_{-} - \hat{b}_{+})}.
\]  
(59)

By substituting this in Eq. (56), we find

\[
\frac{1}{\sqrt{\omega}} \int_{-\infty}^{\infty} d\omega d\theta e^{i(\theta + \omega - K(\tilde{u} + \tilde{v}) - \Omega(u + v) + (\hat{b}_{-} - \hat{b}_{+}))} = \int_{0}^{\infty} \frac{d\Omega}{\sqrt{\Omega}} \alpha_{\Omega,\omega}[2\pi \delta(\Omega - \Omega')],
\]  
(58)

Seemingly for \( \beta_{\omega} \), we have

\[
\beta_{\omega} = -\frac{1}{2\pi V} \sqrt{\frac{\Omega}{\omega}} \int d\omega d\theta e^{-i\omega(k - \omega) + i\omega(k + \omega)} e^{-i\tilde{v}(K - \Omega) + i\tilde{v}(k + \Omega)}.
\]  
(60)

Using now the KS coordinate (52), after lengthy but straightforward calculations, we find

\[
\alpha_{\Omega,\omega} = \frac{1}{2\pi V} \sqrt{\frac{\Omega}{\omega}} e^{i\pi \beta_{\omega} e^{iF(\Omega, K, \omega, k)} \Gamma \left[i + \frac{\Omega + K}{\beta_{\omega}} \right] \Gamma \left[i - \frac{\Omega - k}{\beta_{\omega}} \right]},
\]  
\[
\beta_{\Omega,\omega} = \frac{1}{2\pi V} \sqrt{\frac{\Omega}{\omega}} e^{-i\pi \beta_{\omega} e^{iG(\Omega, K, \omega, k)} \Gamma \left[i - \frac{\Omega + K}{\beta_{\omega}} \right] \Gamma \left[i - \frac{\Omega - k}{\beta_{\omega}} \right]}.
\]  
(61)

It follows that \( \alpha_{\Omega,\omega} \) and \( \beta_{\Omega,\omega} \) obey the useful relation

\[
|\alpha_{\Omega,\omega}|^2 = e^{4\pi \Omega / \beta_{s}} |\beta_{\Omega,\omega}|^2.
\]  
(62)

Therefore we can compute the expectation value of the particle number operator \( \hat{N}_{\Omega} = \hat{b}_{1}^{\dagger} \hat{b}_{1}^{\dagger} \) in the Kruskal vacuum, and obtain the number density

\[
n_{\Omega} = \left[ \exp \left( \frac{2\Omega}{T_{H}} \right) - 1 \right]^{-1}.
\]  
(63)

This corresponds to an emitted radiation with twice the frequency with respect to the simple massless case, of which spectral radiance \( B(\Omega) = \Omega^{2} n_{\Omega} \) is reported in figure 4. We observe that the Hawking temperature is equal to Eq. (58) for the massless scalar field. This is expected since the surface gravity is the same. For a moving observer with respect to the soliton the Hawking temperature, for small \( \beta_{s} \), reads

\[
T_{H} = \frac{\beta_{s}}{2\pi} \sqrt{\frac{1 - \beta_{s}}{1 + \beta_{s}}} \simeq \frac{\beta_{s}}{2\pi} (1 - \beta_{s}).
\]  
(64)

Eq. (64) provides the Hawking temperature of soliton evaporation in this toy model.

VI. CONCLUSIONS

We adopted the geometrization of the ESG model and reported on the connection between the one-soliton solution of the 1+1-dimensional elliptic sine-Gordon equation and a metric with a Schwarzschild-like coordinate singularity. We determined the BH metric and, by suitable coordinate systems, we eliminated the singularity and obtained a regular metric on the EH. We quantized a massless scalar field and found the thermal radiation detected by an observer far away on the BH exterior. We obtained that the temperature is proportional to the soliton velocity. We analyzed the temperature detected by an observer in the laboratory frame, by a Doppler effect. We studied also the quantum soliton evaporation, and

FIG. 4. (Color online) Spectral radiance for massive fields varying the soliton velocity.
elements of a basis. $X$ differents $d$ of same size, or also operators. A 2-form is a combination of the adopted coordinates and are hence 1-form. $\sigma^1$ and $\sigma^2$ contain the differentials of the adopted coordinates and are hence 1-form. $\sigma^1 \wedge \sigma^2$ is the elemental area on the surface. When one moves of an amount $d\mathbf{P}$, $e_{1,2}$ changes of amounts $de_{1,2}$. One considers a surface such that $de_1 = \omega e_2$ and $de_2 = -\omega e_1$ where $\omega$ depends on the shape of the surface, contains the differentials of the coordinate systems, and is a 1-form named the connection form. One finds the following equation

$$d\omega = -K \sigma^1 \wedge \sigma^2$$  \hspace{1cm} (70)

where $K$ is the Gaussian curvature. $\omega$, $\sigma^1$ and $\sigma^2$ are one forms that fix all the properties of the surface. In the particular case $K = -1$, one has from (70)

$$d\omega = \sigma^1 \wedge \sigma^2.$$  \hspace{1cm} (71)

By using (71) and considering the matrix 1-form \cite{10},

$$\Omega = \begin{pmatrix} -\frac{1}{2} \sigma^2 & \frac{1}{2} (\omega + \sigma^1) \\ \frac{1}{2} (-\omega + \sigma^1) & \frac{1}{2} \sigma^2 \end{pmatrix},$$  \hspace{1cm} (72)

one finds the Pfaff system in Eq. (68). In other words, considering the integrability condition \cite{15}, and retaining the element of $\Omega$ as the forms of a two-dimensional surface, Eq. (68) implies that the surface has a constant negative curvature $K = -1$. Hence integrability produces pseudospherical surfaces, i.e., surfaces of constant negative curvature.

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