The energy equality for weak solutions to the equations of non-Newtonian fluids

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ABSTRACT

In this short paper, we extend the result of Shinbrot (1974) to an incompressible fluid with shear dependent viscosity. It is shown that a weak solution to the equations of non-Newtonian fluids lying in a space $L^q(0,T;L^p)$ satisfies an energy equality, where $\frac{2n}{r-1} \leq p \leq \frac{2n}{r-2}$ and $\frac{1}{p} + \frac{r}{q} \leq \frac{r-1}{r}$, if $r > 2$; $p \geq \frac{2n}{r-1}$ and $\frac{r-1}{p} + \frac{1}{q} = \frac{r-1}{2}$, if $\frac{2(n+1)}{n+2} < r \leq 2$. In particular, our result implies that the weak solution must satisfy the energy equality when $r \geq \frac{3n+2}{n+2}$, which is consistent with the known fact.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a domain with a sufficiently smooth boundary. We consider a non-Newtonian incompressible fluid which is governed by the following system

\[
\begin{aligned}
\begin{cases}
\begin{split}
\frac{\partial u}{\partial t} + u \cdot \nabla u - \text{div} \left(|D(u)|^{r-2}D(u)\right) + \nabla \pi = 0, & \text{in } \Omega \times (0,T), \\
\text{div } u = 0, & \text{in } \Omega \times (0,T), \\
u(0,x) = u_0(x), & \text{in } \Omega.
\end{split}
\end{cases}
\end{aligned}
\] (1)

where $u = (u_1,u_2,u_3)$ denotes the unknown velocity of the fluid and $\pi$ the pressure, and

\[D(u) = \frac{1}{2} \left(\nabla u + (\nabla u)^T\right).\]

We first give the definition of a weak solution to (1). To this end, we denote by $C_0^\infty(\Omega)$ the space of smooth functions with compact support. Let $C_0^\infty,\sigma(\Omega) = \{\varphi \in C_0^\infty(\Omega)|\nabla \cdot \varphi = 0\}$. $L^p_0(\mathbb{R}^3)$ denotes the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_p$. $\dot{H}^{1,p}_{0,\sigma}(\Omega)$ denotes the closure of $C_0^\infty,\sigma(\Omega)$ in the norm $\|\nabla \cdot \|_p$. We write

\[\|u\|_{p,q} = \left(\int_0^T \|u(t)\|^q_p dt\right)^{\frac{1}{q}}, \quad 1 \leq q < \infty,\]

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and
\[ \|u(t)\|_{p,\infty} = \text{ess sup}_{0 < t < T} \|u\|_p. \]

**Definition 1.** Let \( r \geq \frac{2n}{n+2} \), \( u_0 \in L^2_r(\Omega) \). A vector-valued function \( u \in L^\infty(0,T;L^2_r(\Omega)) \cap L^2(0,T;\dot{H}^1_{0,r}(\Omega)) \) is called a weak solution to (1) if the following identity
\[ (u_0, \varphi(0)) + \int_0^T \left[ (u, \partial_t \varphi) - (u \cdot \nabla u, \varphi) - \left( \|D(u)\|^{r-2}D(u), D(\varphi) \right) \right] dt = 0 \] holds for all \( \varphi \in C^\infty_0([0,T);C^\infty_0(\Omega)) \). Here, one restricts \( r \geq \frac{2n}{n+2} \) to make sure that the expression \( \int_0^T (u \cdot \nabla u, \varphi) dt \) makes sense.

The existence of weak solutions of (1) is shown in [1,2] with the periodic boundary condition, and in [3] for the whole space. In [4], J. Wolf showed the existence of weak solutions with Dirichlet boundary condition for \( r > \frac{2(n+1)}{n+2} \). Moreover, we know that a weak solution satisfies the global energy inequality
\[ \|u(t)\|^2_2 + 2 \int_0^t \|D(u)(\tau)\|^p d\tau \leq \|u_0\|^2_2, \quad t \geq 0. \]

A natural question is to consider the possible validity of the energy equality. For Newtonian fluids, i.e. \( r = 2 \), the pioneering results by Prodi [5] and Lions [6] concern the validity of energy equality for a weak solution such that
\[ u \in L^4(0,T, L^4(\Omega)). \]
Later, Shinbrot [7] enlarged the range of exponents proving that if a weak solution belongs to
\[ u \in L^q(0,T, L^p(\Omega)), \quad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad p \geq 4. \]

More results in this respect, the reader can see [8] and references therein. However, as far as I know, there is no result for the non-Newtonian fluids. In this paper, we will extend Shinbrot’s result to the non-Newtonian fluids. Our main result is stated as follows.

**Theorem 1.** Let \( r > \frac{2(n+1)}{n+2} \), \( r' \) is the Hölder conjugate of \( r \), \( u_0 \in L^2_r(\Omega) \), and let \( u \) be a weak solution of (1). If \( u \in L^q(0,T; L^p(\Omega)), \) where \( 2r' \leq p \leq \frac{2r}{r-2} \) and \( \frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} \), if \( r > 2; p \geq 2r' \) and \( \frac{r-1}{p} + \frac{1}{q} \leq \frac{r-1}{2} \), if \( r \leq 2 \). Then \( u \) satisfies the energy equality
\[ \|u(t)\|^2_2 + 2 \int_0^t \|D(u)(\tau)\|^p d\tau = \|u_0\|^2_2, \quad 0 \leq t < T. \]

**Remark 1.** Here, since the existence of weak solutions to (1) is still unsolved for \( r \leq \frac{2(n+1)}{n+2} \), see [4], we have to restrict \( r > \frac{2(n+1)}{n+2} \).

**Remark 2.** It is well known that the weak solution is strong and unique for \( r \geq \frac{3n+2}{n+2} \), see for example [2], thus the weak solution satisfies automatically the energy equality. It is remarkable that our result is consistent with this fact. Actually, by virtue of Gagliardo-Nirenberg inequalities and Korn’s inequality, one easily verifies
\[ L^\infty(0,T; L^2_r(\Omega)) \cap L^2(0,T; \dot{H}^1_{0,r}(\Omega)) \hookrightarrow L^{\frac{(n+2)r^2-2nr}{n}}(0,T; L^{2r'}(\Omega)). \]
When \( r \geq \frac{3n+2}{n+2} \), one can easily check that
\[ \frac{1}{2r'} + \frac{n}{(n+2)r^2-2nr} \leq \frac{1}{r'}, \]
which implies a weak solution of (1) must satisfy the energy equality for any \( r \geq \frac{3n+2}{n+2} \).
2. The proof of Theorem 1

Firstly, we have the following property of weak solutions.

Lemma 2. Let \( r > \frac{2(n+1)}{n+2} \), \( u_0 \in L^2_r(\Omega) \), and let \( u \) be a weak solution of (1). Then, after suitable redefinition of \( u \) on a set of values of \( t \) of one-dimensional measure zero, we have

\[
(u(t), \varphi(t)) = (u_0, \varphi(0)) + \int_0^t \left[ (u, \partial_t \varphi) - (u \cdot \nabla u, \varphi) - \left( |D(u)(\tau)|^{r-2} D(u)(\tau), D(\varphi)(\tau) \right) \right] d\tau = 0
\]

holds for all \( \varphi \in C_0^\infty (\Omega) \) and all \( 0 \leq t < T \).

This lemma is completely similar with that of Lemma 2.1 in [9], see also [5,10,11], we omit the details here.

Lemma 3. Let \( r > 1 \), \( \phi \in L^p(\Omega) \), \( \psi \in \dot{H}_0^1(\Omega) \), \( \chi \in L^2(\Omega) \cap L^p(\Omega) \), where

\[
2r' \leq p \leq \frac{2r}{r-2}, \quad \text{if } r > 2; \quad p \geq 2r', \quad \text{if } 1 < r \leq 2.
\]

Then

\[
|\langle \phi \cdot \nabla \psi, \chi \rangle| \leq \|\phi\|_p \|\nabla \psi\|_r \|\chi\|_2 \|\chi\|_p^{1-\theta},
\]

where

\[
\theta = \frac{1}{q} - \frac{1}{p} = \frac{2(p-q)}{2(p-2)},
\]

and \( q \) is defined by \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \).

Proof. If \( \chi \in L^q(\Omega) \), then

\[
|\langle \phi \cdot \nabla \psi, \chi \rangle| \leq \|\phi\|_p \|\nabla \psi\|_r \|\chi\|_q.
\]

From the assumption (5), one can easily check that \( 2 \leq q \leq p \). Hence we can use interpolation and write \( q \) as \( \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{q} \), which give us \( \theta := \frac{1}{2} - \frac{1}{p} - \frac{2(p-q)}{q(p-2)} \). Thus,

\[
|\langle \phi \cdot \nabla \psi, \chi \rangle| \leq \|\phi\|_p \|\nabla \psi\|_r \|\chi\|_2 \|\chi\|_p^{1-\theta}.
\]

Lemma 4. Let \( r > 1 \), and

\[
\phi \in L^q(0,T; L^p(\Omega)), \quad \psi \in L^r(0,T; \dot{H}_0^1(\Omega)), \quad \chi \in L^\infty(0,T; L^2(\Omega)) \cap L^\lambda(0,T; L^p(\Omega)),
\]

where \( p \) satisfies the assumption (5). Then

(i) If \( r > 2 \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \), one has

\[
\left| \int_0^T \langle \phi(t) \cdot \nabla \psi(t), \chi(t) \rangle \, dt \right| \leq T \left( \frac{1}{r} - \frac{1}{2} \right) \|\phi\|_{p,q-r} \|\nabla \psi\|_{r,r} \|\chi\|_{2,\infty}^2 \|\chi\|_{p,q-2,r}^{2-1},
\]

(ii) If \( 1 < r < 2 \) and \( \frac{r}{p-1} + \frac{1}{q} = \frac{r-1}{2} \), one has

\[
\left| \int_0^T \langle \phi(t) \cdot \nabla \psi(t), \chi(t) \rangle \, dt \right| \leq \|\phi\|_{p,q-r} \|\nabla \psi\|_{r,r} \|\chi\|_{2,\infty}^2 \|\chi\|_{p,q-2,r}^{2-1}.
\]
Proof. If $r > 2$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r'}$, one can easily check that
$$\frac{1}{q} + \frac{1}{r} + \frac{1}{q} + \left(\frac{1}{r'} - 1\right)$$
then by Hölder’s inequality, one can obtain (7) by Lemma 4. If $1 < r < 2$ and $\frac{1}{q} + \frac{r-1}{p} = \frac{r-1}{2}$, it is easy to know that
$$\frac{1}{q} + \frac{1}{r} + \frac{1}{q} = 1,$$
thus, one can get (8) by virtue of Lemma 4.

Now, using the above result, we can prove Theorem 1.

Proof of Theorem 1. The proof follows [7]. Let
$$(k_\varepsilon * \phi)(t) = \int_0^{t_0} k_\varepsilon(t - \tau)\phi(\tau)d\tau$$
be a mollifier, so that $k_\varepsilon$ is $C^\infty$, real-valued, nonnegative function, supported in $[-\varepsilon, \varepsilon]$, and integrating to unity. Let \( \{u_m\} \subset C_0^\infty([0, \infty); C_0^\infty(\Omega)) \) be a sequence converging to $u$ in $L^2(0, T; L^2(\Omega)) \cap L^r(0, T; \dot{H}^1(\Omega)) \cap L^p(0, T; L^p(\Omega))$. Set $t = t_0$ and $\varphi = k_\varepsilon \ast u_m$ in (4). One obtains
$$\begin{align*}
\int_0^{t_0} k_\varepsilon(t_0 - t)(u(t_0), u_m(t))dt \\
= \int_0^{t_0} k_\varepsilon(-t)(u_0, u_m(t)) + \int_0^{t_0} \int_0^{t_0} \partial_t k_\varepsilon(t - \tau)(u(t), u_m(\tau))d\tau dt \\
- \int_0^{t_0} \int_0^{t_0} k_\varepsilon(t - \tau) \left[(u(t) \cdot \nabla u(t), u_m(\tau)) + \left(|D(u(t)|r-2D(u(t), D(u_m)(\tau))\right)\right] d\tau dt.
\end{align*}$$
(9)

From Lemma 3, we have
$$\left|\int_0^{t_0} \int_0^{t_0} k_\varepsilon(t - \tau)(u(t) \cdot \nabla u(t), u_m(\tau) - u(\tau))d\tau dt\right|$$
$$\leq \int_0^{t_0} ||u(t)||_p ||\nabla u(t)||_r dt \int_0^{t_0} ||u_m(\tau) - u(\tau)||_2^\theta ||u_m(\tau) - u(\tau)||_p^1 - \theta d\tau.$$

Since $r' \leq q$, we have
$$\int_0^{t_0} ||u(t)||_p ||\nabla u(t)||_r dt \leq C(t_0)||u||_{p,q} ||\nabla u||_{r,r}.$$

On the other hand, since $\frac{1}{2} + \frac{1}{q} \leq 1$, we have
$$\int_0^{t_0} ||u_m(\tau) - u(\tau)||_2^\theta ||u_m(\tau) - u(\tau)||_p^1 - \theta d\tau \leq C(t_0)||u_m - u||_{2,2}^\theta ||u_m - u||_{p,q}^1 - \theta \to 0, \text{ as } m \to \infty.$$

Thus, let $m \to \infty$ in (9), we have
$$\begin{align*}
\int_0^{t_0} k_\varepsilon(t_0 - t)(u(t_0), u(t))dt \\
= \int_0^{t_0} k_\varepsilon(-t)(u_0, u(t)) + \int_0^{t_0} \int_0^{t_0} \partial_t k_\varepsilon(t - \tau)(u(t), u(\tau))d\tau dt \\
- \int_0^{t_0} \int_0^{t_0} k_\varepsilon(t - \tau) \left[(u(t) \cdot \nabla u(t), u(\tau)) + \left(|D(u(t)|r-2D(u(t), D(u(\tau))\right)\right] d\tau dt.
\end{align*}$$
(10)
The term here involving the derivative vanishes if \( k \) is chosen to be even. We send \( \varepsilon \) to zero in the remaining terms. Because of the usual properties of mollifiers,

\[
\int_0^{t_0} \int_0^{t_0} k_\varepsilon(t - \tau) \left( |D(u)(t)|^{-2} D(u)(t), D(u)(\tau) \right) d\tau dt \to \int_0^{t_0} \|D(u)(t)\|_p^p dt.
\]

In addition,

\[
\int_0^{t_0} k_\varepsilon(t_0 - t)(u(t_0), u(t)) dt = \int_0^{\varepsilon} k_\varepsilon(t) (u(t_0), u(t_0 - t)) dt,
\]

since as a function of \( t \), \( u \) is continuous in the weak topology of \( L^2_\sigma(\Omega) \) (see [4] Theorem 1.3 for example), we have

\[
\int_0^{t_0} k_\varepsilon(t_0 - t)(u(t_0), u(t)) dt = \int_0^{\varepsilon} k_\varepsilon(t) \|u(t_0)\|_2^2 + o(1) dt \to \frac{1}{2}\|u(t_0)\|_2^2.
\]

Similarly,

\[
\int_0^{t_0} k_\varepsilon(-t)(u_0, u(t)) dt \to \frac{1}{2}\|u_0\|_2^2.
\]

Finally, we consider the nonlinear term in (10). We have

\[
\int_0^{t_0} \int_0^{t_0} k_\varepsilon(t - \tau) (u(t) \cdot \nabla u(t), u(\tau)) d\tau dt - \int_0^{t_0} (u(t) \nabla u(t), u(t)) dt
\]

\[
= \int_0^{t_0} (u(t) \cdot \nabla u(t), (k_\varepsilon * u)(t) - u(t)) dt. \tag{11}
\]

By Lemma 4, this is bounded by

\[
C(t_0)\|u\|_{p,q}\|\nabla u\|_{r,r} \|k_\varepsilon * u - u\|_{2,\infty}^{\frac{p}{2}} \|k_\varepsilon * u - u\|_{p,q}^{\frac{2}{2} - 1}.
\]

This goes to zero because of usual properties of mollifiers. Thus (11) goes to zero. Now, we prove \( \int_0^{t_0} (u(t) \cdot \nabla u(t), u(t)) = 0 \). From Lemma 4, we have that the function \( F \) defined by

\[
F(\psi, \chi) = \int_0^{t_0} (u(t) \cdot \nabla \psi(t), \chi(t)) dt
\]

is continuous on \( L^2(0, t_0; \dot{H}^1_r(\Omega)) \times L^q(0, t_0; L^p(\Omega)) \). On the other hand, integration by parts shows that \( F(\psi, \psi) = 0 \) if \( \psi \) is smooth. Let \( \{u_m\} \) be a sequence from \( C_{0,\sigma}^\infty([0, T); C_{0,\sigma}^\infty(\Omega)) \) converging to \( u \) in the appropriate spaces. Then we find

\[
0 = F(u_m, u_m) \to F(u, u).
\]

All of this shows that

\[
\int_0^{t_0} \int_0^{t_0} k_\varepsilon(t - \tau) (u(t) \cdot \nabla u(t), u(\tau)) d\tau dt \to 0
\]

as \( \varepsilon \to 0 \). Now, let \( \varepsilon \to 0 \) in (9), we have

\[
\frac{1}{2}\|u(t_0)\|_2^2 + \int_0^t \|D(u)(t)\|_r^r d\tau = \|u_0\|_2^2,
\]

which is (3) for \( t = t_0 \). Since \( t_0 \) is arbitrary, we have finished the proof of Theorem 1.

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