Symmetric and generating functions for some generalized polynomials

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Abstract
In this paper, we define generalized Gaussian Padovan numbers, generalized Gaussian Padovan polynomials and generalized trivariate Fibonacci polynomials, we present the new generating functions of these generalizations, and as special cases, we investigate Gaussian Padovan numbers and polynomials, Gaussian Pell Padovan numbers and polynomials and trivariate Fibonacci and Lucas polynomials with their generating functions. Moreover, we give the new generating functions of some generalized Vieta polynomials.

Keywords
Symmetric functions, generating functions, generalized Gaussian Padovan numbers, generalized trivariate Fibonacci polynomials, generalized Vieta-Pell polynomials, generalized Vieta-Jacobsthal polynomials.

AMS Subject Classification
05E05, 11B39.

1. Introduction

Many numbers and polynomials sequences can be defined, characterized, evaluated, and classified by linear recurrence relations with certain orders. The third order recurrence sequence have been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation. There is a many generalizations of numbers and polynomials of third order in literature which are defined recursively. For example, Soykan and Tasdemir in [13], given the Gaussian generalized Tribonacci numbers

\[ \{ GV_n \}_{n \geq 0} = \{ GV_n (GV_0, GV_1, GV_2) \}_{n \geq 0} \]

by

\[
\begin{align*}
GV_n &= GV_{n-1} + GV_{n-2} + GV_{n-3}, \quad n \geq 3 \\
GV_0 &= c_0 + i(c_2 - c_1 - c_0), \quad GV_1 = c_1 + ic_0, \\
GV_2 &= c_2 + ic_1.
\end{align*}
\]

Let now we define some generalizations for some numbers and polynomials of third order linear recurrence sequences, we begin by the recurrence relation of generalized Gaussian Padovan numbers \( \{ GN_n \}_{n \geq 0} = \{ GN_n (a, b) \}_{n \geq 0} \) as:

\[
\begin{align*}
GN_n &= aGN_{n-2} + GN_{n-3}, \quad n \geq 3 \\
GN_0 &= 1 + bi, \quad GN_1 = GN_2 = 1 + i.
\end{align*}
\]

Special cases of \( \{ GN_n \}_{n \geq 0} \) are Gaussian Padovan numbers \( GN_n (1, 0) = GP_n \) and Gaussian Pell Padovan numbers \( GN_n (2, -1) = GR_n \). We formally define them as follows:

Gaussian Padovan numbers is defined by

\[ GP_n = GP_{n-2} + GP_{n-3}, \quad \text{for all } n \geq 3, \]

with initial conditions \( GP_0 = 1, \GP_1 = GP_2 = 1 + i \) and Gaussian Pell Padovan numbers is defined by

\[ GR_n = 2GR_{n-2} + GR_{n-3}, \quad \text{for all } n \geq 3, \]

with initial conditions \( GR_0 = 1 - i, \GR_1 = GR_2 = 1 + i, \) see the paper [5].
Also, we define the generalized trivariate Fibonacci polynomials \( \{W_n(x,y,t)\}_{n \geq 0} \) as follows

\[
\begin{align*}
W_0(x,y,t) &= xW_{n-1}(x,y,t) + yW_{n-2}(x,y,t) + tW_{n-3}(x,y,t), n \geq 3 \\
W_0(x,y,t) &= a, W_1(x,y,t) = b + cx, \\
W_2(x,y,t) &= dx + ey + f x^2.
\end{align*}
\]

(1.1)

If we take \( \{a = c = e = f = 0, b = d = 1\} \) and \( \{a = 3, b = d = 0, c = f = 1, e = 2\} \) in (1.1) we give the following definitions.

**Definition 1.1.** [6] For any integer \( n \geq 0 \), the trivariate Fibonacci polynomials, denoted by \( \{H_n(x,y,t)\}_{n \geq 0} \) is defined recursively by

\[
H_n(x,y,t) = xH_{n-1}(x,y,t) + yH_{n-2}(x,y,t) + tH_{n-3}(x,y,t),
\]

with the initials

\[
H_0(x,y,t) = 0, H_1(x,y,t) = 1 \text{ and } H_2(x,y,t) = x.
\]

**Definition 1.2.** [6] For any integer \( n \geq 0 \), the trivariate Lucas polynomials, denoted by \( \{K_n(x,y,t)\}_{n \geq 0} \) is defined recursively by

\[
K_n(x,y,t) = xK_{n-1}(x,y,t) + yK_{n-2}(x,y,t) + tK_{n-3}(x,y,t),
\]

with the initials

\[
K_0(x,y,t) = 3, K_1(x,y,t) = x \text{ and } K_2(x,y,t) = x^2 + 2y.
\]

Next, we define the generalized Gaussian Padovan polynomials \( \{GM_n(x)\}_{n \geq 0} \) by the following third order recurrence relation:

\[
\begin{align*}
GM_n(x) &= axGM_{n-2}(x) + GM_{n-3}(x), n \geq 3 \\
GM_0(x) &= 1 + bi, \quad GM_1(x) = GM_2(x) = 1 + i
\end{align*}
\]

(1.2)

If we take \( \{a = 1, b = 0\} \) and \( \{a = 2, b = -1\} \) in the relationship (1.2), then we get the recurrence relations of Gaussian Padovan and Gaussian Pell Polynomials \( \{GP_n(x)\}_{n \geq 0} \) and \( \{GR_n(x)\}_{n \geq 0} \) given in the Table 1.

| Gaussian polynomials | Linear recurrence sequences | Initial conditions |
|----------------------|----------------------------|-------------------|
| Gaussian Padovan polynomials | \(GP_n(x) = xGP_{n-2}(x) + GP_{n-3}(x), n \geq 3\) | \(GP_0(x) = 1, \quad GP_1(x) = 1 + i, \quad GP_2(x) = 1 + i\) |
| Gaussian Pell Padovan polynomials | \(GR_n(x) = 2xGR_{n-2}(x) + GR_{n-3}(x), n \geq 3\) | \(GR_0(x) = 1 - i, \quad GR_1(x) = 1 + i, \quad GR_2(x) = 1 + i\) |

In this part we define some generalized Vieta polynomials.

**Definition 1.3.** [11] For \( n \in \mathbb{N} \), the generalized Vieta-Jacobshthal polynomials, denoted by \( \{GI_{k,n}(x)\}_{n \in \mathbb{N}} \) is defined recursively by

\[
\begin{align*}
GI_{k,n}(x) &= GI_{k,n-1}(x) - 2^k xGI_{k,n-2}(x), n \geq 2 \\
GI_{k,0}(x) &= 0, \quad GI_{k,1}(x) = 1
\end{align*}
\]

**Definition 1.4.** [11] For \( n \in \mathbb{N} \), the generalized Vieta-Jacobshthal-Lucas polynomials, denoted by \( \{GL_{k,n}(x)\}_{n \in \mathbb{N}} \) is defined recursively by

\[
\begin{align*}
GL_{k,n}(x) &= GL_{k,n-1}(x) - 2^k xGL_{k,n-2}(x), n \geq 2 \\
GL_{k,0}(x) &= 2, \quad GL_{k,1}(x) = 1
\end{align*}
\]

**Definition 1.5.** [12] For \( n \in \mathbb{N} \), the generalized Vieta-Pell polynomials, denoted by \( \{P_{k,n}(x)\}_{n \in \mathbb{N}} \) is defined recursively by

\[
\begin{align*}
P_{k,n}(x) &= 2^k xP_{k,n-1}(x) - P_{k,n-2}(x), n \geq 2 \\
P_{k,0}(x) &= 0, \quad P_{k,1}(x) = 1
\end{align*}
\]

**Definition 1.6.** [12] For \( n \in \mathbb{N} \), the generalized Vieta-Pell-Lucas polynomials, denoted by \( \{Q_{k,n}(x)\}_{n \in \mathbb{N}} \) is defined recursively by

\[
\begin{align*}
Q_{k,n}(x) &= 2^k xQ_{k,n-1}(x) - Q_{k,n-2}(x), n \geq 2 \\
Q_{k,0}(x) &= 2, \quad Q_{k,1}(x) = 2^k x
\end{align*}
\]

**Definition 1.7.** [12] For \( n \in \mathbb{N} \), the generalized Vieta-modified Pell polynomials, denoted by \( \{q_{k,n}(x)\}_{n \in \mathbb{N}} \) is defined recursively by

\[
\begin{align*}
q_{k,n}(x) &= 2^k xq_{k,n-1}(x) - q_{k,n-2}(x), n \geq 2 \\
q_{k,0}(x) &= 1, \quad q_{k,1}(x) = 2^k - 1 x
\end{align*}
\]

The remainder of this paper is organized as follows:

- **In section 2**, we first give the notion of the symmetric function and then we present and prove our main result which relates the symmetric function defined in this section with the symmetrizing operator \( \delta_{p_1p_2} \).

- **In section 3**, we derive the new generating functions of generalized Gaussian Padovan numbers, generalized Gaussian Padovan polynomials and generalized trivariate Fibonacci polynomials. In particular, the generating functions of Gaussian Padovan numbers and polynomials, Gaussian Pell Padovan numbers and polynomials, trivariate Fibonacci polynomials and trivariate Lucas polynomials are obtained.

- **In section 4**, by making use of the symmetric function we obtain the new generating functions of generalized Vieta-Jacobshthal polynomials, generalized Vieta-Jacobshthal-Lucas-polynomials, generalized Vieta-Pell polynomials, generalized Vieta-Pell-Lucas polynomials and generalized Vieta-modified Pell polynomials.
2. Preliminaries and main results

In this section, we give definitions and properties of the symmetric functions (for more details, we can see [7]). Let us now start at the following definitions.

Definition 2.1. Let $k$ and $n$ be two positive integers and $\{p_1, p_2, \ldots, p_n\}$ be a set of given variables the $k$-th complete homogeneous symmetric function $h_k(p_1, p_2, \ldots, p_n)$ is defined by

$$ h_k(p_1, p_2, \ldots, p_n) = \sum_{i_1 + i_2 + \ldots + i_n = k} p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n} \quad (k \geq 0), $$

with $i_1, i_2, \ldots, i_n \geq 0$.

Remark 2.2. Set $h_0(p_1, p_2, \ldots, p_n) = 1$, by usual convention. For $k < 0$, we set $h_k(p_1, p_2, \ldots, p_n) = 0$.

Definition 2.3. [1] Let $A$ and $P$ be any two alphabets. We define $S_n(A-P)$ by the following form:

$$ \prod_{p \in P} (1 - pz) = \prod_{a \in A} S_n(A-P)z^n, \quad (2.1) $$

with the condition $S_n(A-P) = 0$ for $n < 0$.

Equation (2.1) can be rewritten in the following form

$$ \sum_{n=0}^{\infty} S_n(A-P)z^n = \left( \sum_{n=0}^{\infty} S_n(A)z^n \right) \left( \sum_{n=0}^{\infty} S_n(-P)z^n \right), \quad (2.2) $$

where

$$ S_n(A-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(A). $$

Remark 2.4. Taking $A = \{0\}$ in (2.1) gives

$$ \sum_{n=0}^{\infty} S_n(-P)z^n = \prod_{p \in P} (1 - pz). $$

Definition 2.5. [2] Given a function $g$ on $\mathbb{R}^n$, the divided difference operator is defined as follows

$$ \partial_{p_i, p_i+1}(g) = \frac{g(p_1, p_2, \ldots, p_i, p_i+1, \ldots, p_n) - g(p_1, p_2, \ldots, p_{i-1}, p_i+1, p_i, p_{i+2}, \ldots, p_n)}{p_i - p_{i+1}}. $$

Definition 2.6. Let $n$ be a positive integer and $P_2 = \{p_1, p_2\}$ be a set of given variables, then, the $n$-th symmetric function $S_n(p_1 + p_2)$ is defined by

$$ S_n(P_2) = S_n(p_1 + p_2) = \frac{p_1^{n+1} - p_2^{n+1}}{p_1 - p_2}, $$

with

$$ S_0(p_2) = S_0(p_1 + p_2) = 1, $$

$$ S_1(p_2) = S_1(p_1 + p_2) = p_1 + p_2, $$

$$ S_2(p_2) = S_2(p_1 + p_2) = p_1^2 + p_1 p_2 + p_2^2, $$

and so on.

Definition 2.7. [3] Given an alphabet $P_2 = \{p_1, p_2\}$, the symmetrizing operator $\delta^k_{p_1, p_2}$ is defined by

$$ \delta^k_{p_1, p_2}g(p_1) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2}, \text{ for all } k \in \mathbb{N}_0. \quad (2.3) $$

If $g(p_1) = p_1$, the operator (2.3) gives us

$$ \delta^k_{p_1, p_2}g(p_1) = \frac{p_1^{k+1} - p_2^{k+1}}{p_1 - p_2} = S_k(p_1 + p_2). $$

The following theorem is one of the key tools of the proof of our main results. It has been proved in [4]. For the completeness of the paper we state its proof here.

Theorem 2.8. Given two alphabets $P_2 = \{p_1, p_2\}$ and $A_3 = \{a_1, a_2, a_3\}$, we have

$$ \sum_{n=0}^{\infty} S_n(A_3) \partial_{p_1, p_2}(p_1^{n+1})z^n = \frac{\sum_{n=0}^{\infty} S_n(-A_3) \partial_{p_1, p_2}(p_1^{n+1})z^n}{\left( \sum_{n=0}^{\infty} S_n(-A_3)p_1^{n}z^n \right) \left( \sum_{n=0}^{\infty} S_n(-A_3)p_2^{n}z^n \right)} \quad (2.4) $$

with $S_0(-A_3) = 1, S_2(-A_3) = a_1 a_2 + a_1 a_3 + a_2 a_3, S_3(-A_3) = -a_1 a_2 a_3$.

Proof. Let $\sum_{n=0}^{\infty} S_n(A_3)z^n$ and $\sum_{n=0}^{\infty} S_n(-A_3)z^n$ be two sequences such that $\sum_{n=0}^{\infty} S_n(A_3)z^n = \frac{1}{\sum_{n=0}^{\infty} S_n(-A_3)z^n}$. On one hand, since

$$ g(p_1) = \sum_{n=0}^{\infty} S_n(A_3)p_1^n z^n \text{ and } g(p_2) = \sum_{n=0}^{\infty} S_n(A_3)p_2^n z^n, $$

we have

$$ \delta^k_{p_1, p_2}g(p_1) = \delta_{p_1, p_2}(\sum_{n=0}^{\infty} S_n(A_3)p_1^n z^n) \quad = \frac{p_1 \sum_{n=0}^{\infty} S_n(A_3)p_2^n z^n - p_2 \sum_{n=0}^{\infty} S_n(A_3)p_1^n z^n}{p_1 - p_2} \quad = \frac{\sum_{n=0}^{\infty} S_n(A_3) \left( p_1^{n+1} - p_2^{n+1} \right) z^n}{p_1 - p_2}. $$

On the other part, since $g(p_1) = \frac{1}{\sum_{n=0}^{\infty} S_n(-A_3)p_1^n z^n}$ and $g(p_2) =
Given an alphabet $A$, we have

\[ \sum_{n=0}^{\infty} S_n a^n r^2 \delta p^2 R(p_1) = \frac{\sum_{n=0}^{\infty} S_n a^n p_1^2 r^n}{\sum_{n=0}^{\infty} S_n a^n p_2^2 r^n} = \frac{p_1}{p_1 - p_2} \sum_{n=0}^{\infty} S_n (a^n p_1 - p_2) a^n.
\]

This completes the proof.

### 3. The generating functions for some well-known generalized numbers and polynomials of third-order linear recurrence sequences

In this part, we now derive the new generating functions of generalized Gaussian Padovan numbers, generalized trivariate Fibonacci polynomials and generalized Gaussian Padovan polynomials. The technique used is based on the theory of the so called symmetric functions.

- For the case $A_3 = \{a_1, a_2, a_3\}$ and $P_2 = \{1, 0\}$ in theorem (2.8) we deduce the following lemma.

**Lemma 3.1.** Given an alphabet $A_3 = \{a_1, a_2, a_3\}$, we have

\[ \sum_{n=0}^{\infty} S_n (A_3) z^n = \frac{1}{(1-a_1 z)(1-a_2 z)(1-a_3 z)}. \]  

Proof. By applying the operator $\delta_{a_2 a_3}$ to the identity

\[ \sum_{n=0}^{\infty} S_{n-1} (a_1 + a_2) z^n = \frac{z}{(1-a_1 z)(1-a_2 z)}, \]

we obtain

\[ \delta_{a_2 a_3} \sum_{n=0}^{\infty} S_{n-1} (a_1 + a_2) z^n = \delta_{a_2 a_3} \frac{z}{(1-a_1 z)(1-a_2 z)}, \]

\[ \sum_{n=0}^{\infty} S_{n-2} (a_1 + a_2 + a_3) z^n = \frac{z}{(1-a_1 z)(1-a_2 z)} \delta_{a_2 a_3} \frac{1}{1 - a_3 z}. \]

Therefore

\[ \sum_{n=0}^{\infty} S_{n-1} (A_3) z^n = \frac{z}{(1-a_1 z)(1-a_2 z)(1-a_3 z)}. \]

Hence, we obtain the desired result.

**Proposition 3.3.** Given an alphabet $A_3 = \{a_1, a_2, a_3\}$, we have

\[ \sum_{n=0}^{\infty} S_{n-2} (A_3) z^n = \frac{z^2}{(1-a_1 z)(1-a_2 z)(1-a_3 z)}. \]  

Proof. By applying the operator $\delta_{a_2 a_3}^0$ to the identity

\[ \sum_{n=0}^{\infty} S_{n-1} (a_1 + a_2) z^n = \frac{z}{(1-a_1 z)(1-a_2 z)}, \]

we have

\[ \delta_{a_2 a_3}^0 \sum_{n=0}^{\infty} S_{n-1} (a_1 + a_2) z^n = \delta_{a_2 a_3}^0 \frac{z}{(1-a_1 z)(1-a_2 z)} \]

\[ \sum_{n=0}^{\infty} S_{n-2} (a_1 + a_2 + a_3) z^n = \frac{z}{(1-a_1 z)} \delta_{a_2 a_3}^0 \frac{1}{1 - a_3 z}. \]

Therefore

\[ \sum_{n=0}^{\infty} S_{n-2} (A_3) z^n = \frac{z^2}{(1-a_1 z)(1-a_2 z)(1-a_3 z)}. \]

This completes the proof.
This part consists of three cases.

**Case 1.** The substitution of
\[
\begin{align*}
a_1 + a_2 + a_3 &= 0 \\
a_1a_2 + a_1a_3 + a_2a_3 &= -a \\
a_1a_2a_3 &= 1
\end{align*}
\]  

in the relationships (3.1), (3.2) and (3.3), we obtain
\[
\sum_{n=0}^{\infty} S_n(A_3)z^n = \frac{1}{1 - az^2 - z^3},
\]

(3.5)

\[
\sum_{n=0}^{\infty} S_{n-1}(A_3)z^n = \frac{z}{1 - az^2 - z^3},
\]

(3.6)

\[
\sum_{n=0}^{\infty} S_{n-2}(A_3)z^n = \frac{z^2}{1 - az^2 - z^3},
\]

(3.7)

respectively.

Multiplying the equation (3.5) by \((1 + bi)\) and adding it to the equation obtained by (3.6) multiplying by \((1 + i)\) and adding it to the equation obtained by (3.7) multiplying by \((1 - a + i(1 - ab))\), then we obtain
\[
\sum_{n=0}^{\infty} \left( (1 + bi) S_n(A_3) + (1 + i) S_{n-1}(A_3) + (1 - a + i(1 - ab)) S_{n-2}(A_3) \right) z^n
\]
\[
= \frac{1 + bi + (1 + i)z + (1 - a + i(1 - ab))z^2}{1 - az^2 - z^3},
\]

and we have the following theorem.

**Theorem 3.4.** For \(n \in \mathbb{N}\), the new generating function of generalized Gaussian Padovan numbers \(G_{N_n}\) is given by
\[
\sum_{n=0}^{\infty} G_{N_n}z^n = \frac{1 + bi + (1 + i)z + (1 - a + i(1 - ab))z^2}{1 - az^2 - z^3},
\]

(3.8)

with
\[
G_{N_n} = (1 + bi) S_n(A_3) + (1 + i) S_{n-1}(A_3) + (1 - a + i(1 - ab)) S_{n-2}(A_3).
\]

**Proof.** The generalized Gaussian Padovan numbers can be considered as the coefficients of the formal power series
\[
g(z) = \sum_{n=0}^{\infty} G_{N_n}z^n.
\]

Using the initial condition, we get
\[
g(z) = GN_0 + GN_1z + GN_2z^2 + \sum_{n=3}^{\infty} GN_nz^n
\]
\[
= GN_0 + GN_1z + GN_2z^2 + \sum_{n=3}^{\infty} aGN_{n-2}z^n
\]
\[
+ \sum_{n=3}^{\infty} GN_{n-3}z^n
\]
\[
= GN_0 + GN_1z + GN_2z^2 + \sum_{n=3}^{\infty} aGN_{n-2}z^n
\]
\[
+ \sum_{n=0}^{\infty} GN_nz^n
\]
\[
= GN_0 + GN_1z + GN_2z^2 - aGN_0z^2
\]
\[
+ az^2 \sum_{n=0}^{\infty} G_{N_n}z^n + \sum_{n=0}^{\infty} G_{N_n}z^n
\]
\[
= 1 + bi + (1 + i)z + (1 - a + i(1 - ab))z^2
\]
\[
+ (az^2 + z^3)g(z).
\]

Hence, we obtain
\[
(1 - az^2 - z^3)g(z) = 1 + bi + (1 + i)z
\]
\[
+ (1 - a + i(1 - ab))z^2.
\]

Therefore
\[
g(z) = \frac{1 + bi + (1 + i)z + (1 - a + i(1 - ab))z^2}{1 - az^2 - z^3}.
\]

Thus, this completes the proof.

**Corollary 3.5.** [9] For \(n \in \mathbb{N}\), the generating function of Gaussian Padovan numbers \(G_P_n\) is given by
\[
\sum_{n=0}^{\infty} G_{P_n}z^n = \frac{1 + (1 + i)z + i z^2}{1 - z^2 - z^3},
\]

with
\[
G_{P_n} = S_n(A_3) + (1 + i) S_{n-1}(A_3) + i S_{n-2}(A_3).
\]

* By putting \(a = 1\) and \(b = 0\) in the relationship (3.8), we obtain the following corollary.

**Corollary 3.6.** [9] For \(n \in \mathbb{N}\), the generating function of Gaussian Pell Padovan numbers \(G_{R_n}\) is given by
\[
\sum_{n=0}^{\infty} G_{R_n}z^n = \frac{1 - i + (1 + i)z + (-1 + 3i)z^2}{1 - 2z^2 - z^3},
\]

with
\[
G_{R_n} = (1 - i) S_n(A_3) + (1 + i) S_{n-1}(A_3)
\]
\[
+ (-1 + 3i) S_{n-2}(A_3).
\]
Theorem 3.7. For \( n \in \mathbb{N} \), the new generating function of generalized trivariate Fibonacci polynomials \( W_n(x,y,t) \) is given by

\[
\sum_{n=0}^{\infty} W_n(x,y,t) z^n = \frac{a + (b + (c-a) x + ((d-b) x + (e-a) y + (f-c) x^2) z^2}{1 - xz - yz^2 - tz^3},
\]

with

\[
W_n(x,y,t) = aS_n(A_3) + (b + (c-a) x) S_{n-1}(A_3) + ((d-b) x + (e-a) y + (f-c) x^2) S_{n-2}(A_3).
\]

Proof. The generalized trivariate Fibonacci polynomials can be considered as the coefficients of the formal power series

\[
g(x,y,z) = \sum_{n=0}^{\infty} W_n(x,y,t) z^n.
\]

Using the initial condition, we get

\[
g(x,y,t,z) = W_0(x,y,t) + W_1(x,y,t) z + W_2(x,y,t) z^2 + \sum_{n=3}^{\infty} W_n(x,y,t) z^n
\]

\[
= W_0(x,y,t) + W_1(x,y,t) z + W_2(x,y,t) z^2 + \sum_{n=3}^{\infty} (xW_{n-1}(x,y,t) + yW_{n-2}(x,y,t)) z^n
\]

\[
+ \sum_{n=3}^{\infty} tW_{n-3}(x,y,t) z^n
\]

\[
= W_0(x,y,t) + W_1(x,y,t) z + W_2(x,y,t) z^2 + xz \sum_{n=2}^{\infty} W_n(x,y,t) z^n + yz^2 \sum_{n=1}^{\infty} W_n(x,y,t) z^n
\]

\[
+ tz^3 \sum_{n=0}^{\infty} W_n(x,y,t) z^n
\]

\[
= W_0(x,y,t) + W_1(x,y,t) z + W_2(x,y,t) z^2 - xW_0(x,y,t) + W_1(x,y,t) z - yW_0(x,y,t) z^2
\]

\[
+ (xz + yz^2 + tz^3) \sum_{n=0}^{\infty} W_n(x,y,t) z^n = a + (b + (c-a) x) z + ((d-b) x + (e-a) y + (f-c) x^2) z^2 + (xz + yz^2 + tz^3) g(x,y,t,z).
\]

Hence, we obtain

\[
(1 - xz - yz^2 - tz^3) g(x,y,t,z) = a + (b + (c-a) x) z + ((d-b) x + (e-a) y + (f-c) x^2) z^2.
\]

Therefore

\[
g(x,y,t,z) = a + (b + (c-a) x) z
\]

\[
+ \frac{((d-b) x + (e-a) y + (f-c) x^2) z^2}{1 - xz - yz^2 - tz^3}.
\]

Thus, this completes the proof.

- By setting \( a = c = e = f = 0 \) and \( b = d = 1 \) in the relationship (3.12), we obtain the following corollary.

Corollary 3.8. [9] For \( n \in \mathbb{N} \), the generating function of trivariate Fibonacci polynomials \( H_n(x,y,t) \) is given by

\[
\sum_{n=0}^{\infty} H_n(x,y,t) z^n = \frac{z}{1 - xz - yz^2 - tz^3},
\]

with

\[
H_n(x,y,t) = S_{n-1}(A_3).
\]

- Put \( a = 3, b = d = 0, c = f = 1 \) and \( e = 2 \) in the relationship (3.12), we can state the following corollary.
Corollary 3.9. [9] For \( n \in \mathbb{N} \), the generating function of trivariate Lucas polynomials \( K_n(x,y,t) \) is given by
\[
\sum_{n=0}^\infty K_n(x,y,t)z^n = \frac{3 - 2xz - yz^2}{1 - xz - yz^2 - tz^3},
\]
with
\[
K_n(x,y,t) = 3S_n(A_3) - 2xS_{n-1}(A_3) - yS_{n-2}(A_3).
\]

Case 3. The substitution of
\[
\left\{ \begin{array}{ll}
a_1 + a_2 + a_3 = 0 \\
a_1a_2 + a_1a_3 + a_2a_3 = -ax \\
a_1a_2a_3 = 1
\end{array} \right.
\]
in the relationships (3.1), (3.2) and (3.3), we obtain
\[
\sum_{n=0}^\infty S_n(A_3)z^n = \frac{1}{1 - axz^2 - z^3}, \quad (3.13)
\]
\[
\sum_{n=0}^\infty S_{n-1}(A_3)z^n = \frac{z}{1 - axz^2 - z^3}, \quad (3.14)
\]
\[
\sum_{n=0}^\infty S_{n-2}(A_3)z^n = \frac{z^2}{1 - axz^2 - z^3}, \quad (3.15)
\]
respectively.

Multiplying the equation (3.13) by \((1 + bi)\) and adding it to the equation obtained by (3.14) multiplying by \((1 + i)\) and adding it to the equation (3.15) multiplying by \((1 - ax + i(1 - abx))\), then we obtain
\[
\sum_{n=0}^\infty \left( (1 + bi)S_n(A_3) + (1 + i)S_{n-1}(A_3) + (1 - ax + i(1 - abx))S_{n-2}(A_3) \right)z^n
\]
\[
= \frac{1 + bi + (1 + i)z + (1 - ax + i(1 - abx))z^2}{1 - axz^2 - z^3},
\]
and we have the following theorem.

Theorem 3.10. For \( n \in \mathbb{N} \), the new generating function of generalized Gaussian Padovan polynomials \( GM_n(x) \) is given by
\[
\sum_{n=0}^\infty GM_n(x)z^n = \frac{1 + bi + (1 + i)z + (1 - ax + i(1 - abx))z^2}{1 - axz^2 - z^3}, \quad (3.16)
\]
with
\[
GM_n(x) = (1 + bi)S_n(A_3) + (1 + i)S_{n-1}(A_3) + (1 - ax + i(1 - abx))S_{n-2}(A_3).
\]

Proof. The generalized Gaussian Padovan polynomials can be considered as the coefficients of the formal power series
\[
g(x,z) = \sum_{n=0}^\infty GM_n(x)z^n.
\]
Using the initial condition, we get
\[
g(x,z) = GM_0(x) + GM_1(x)z + GM_2(x)z^2 + \sum_{n=3}^\infty GM_n(x)z^n
\]
\[
= GM_0(x) + GM_1(x)z + GM_2(x)z^2 + axz^2\sum_{n=1}^\infty GM_n(x)z^n + z^3\sum_{n=0}^\infty GM_n(x)z^n
\]
\[
= GM_0(x) + GM_1(x)z + GM_2(x)z^2 + axz^2\sum_{n=1}^\infty GM_n(x)z^n + z^3\sum_{n=0}^\infty GM_n(x)z^n
\]
\[
= 1 + bi + (1 + i)z + (1 - ax + i(1 - abx))z^2 + (axz^2 + z^3)g(x,z).
\]
Hence, we obtain
\[
(1 - axz^2 - z^3)g(x,z) = 1 + bi + (1 + i)z + (1 - ax + i(1 - abx))z^2.
\]
Therefore
\[
g(x,z) = \frac{1 + bi + (1 + i)z + (1 - ax + i(1 - abx))z^2}{1 - axz^2 - z^3}.
\]
Thus, this completes the proof. \(\square\)

- By taking \(a = 1\) and \(b = 0\) in the relationship (3.16), we obtain the following corollary.

Corollary 3.11. [9] For \( n \in \mathbb{N} \), the generating function of Gaussian Padovan polynomials \( GP_n(x) \) is given by
\[
\sum_{n=0}^\infty GP_n(x)z^n = \frac{1 + (1 + i)z + (1 - x + i)z^2}{1 - xz^2 - z^3},
\]
with
\[
GP_n(x) = S_n(A_3) + (1 + i)S_{n-1}(A_3) + (1 - x + i)S_{n-2}(A_3).
\]
- Put \(a = 2\) and \(b = -1\) in the relationship (3.16), we can state the following corollary.
Corollary 3.12. [9] For \( n \in \mathbb{N} \), the generating function of Gaussian Pell Padovan polynomials \( GR_n(x) \) is given by

\[
\sum_{n=0}^{\infty} GR_n(x) z^n = \frac{1 - i + (1 + i) z + (1 - 2x + i(1 + 2x))z^2}{1 - 2xz^2 - z^3},
\]

with

\[
GR_n(x) = (1 - i) S_n(A_3) + (1 + i) S_{n-1}(A_3) + (1 - 2x + i(1 + 2x)) S_{n-2}(A_3).
\]

4. Construction of generating functions of some well-known generalized Vieta polynomials

In this part, we now derive the generating functions of generalized Vieta-Jacobsthal polynomials, generalized Vieta-Jacobsthal-Lucas polynomials, generalized Vieta-Pell polynomials, generalized Vieta-Pell-Lucas polynomials and generalized Vieta-modified Pell polynomials.

- For the case \( A_3 = \{a_1, -a_2, 0\} \) in the relationships (3.1) and (3.2) we deduce the following corollaries.

Corollary 4.1. Given an alphabet \( A_2 = \{a_1, -a_2\} \), we have

\[
\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) z^n = \frac{1}{1 - (a_1 - a_2) z - a_1 a_2 z^2}. \tag{4.1}
\]

Corollary 4.2. Given an alphabet \( A_2 = \{a_1, -a_2\} \), we have

\[
\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) z^n = \frac{z}{1 - (a_1 - a_2) z - a_1 a_2 z^2}. \tag{4.2}
\]

This part consists of two cases.

Case 1. Assuming that

\[
\begin{align*}
  a_1 - a_2 &= 2^k x \\
  a_1 a_2 &= -1
\end{align*}
\]

in the relationships (4.1) and (4.2), we obtain

\[
\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) z^n = \frac{z}{1 - 2^k xz + z^2}, \tag{4.3}
\]

\[
\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) z^n = \frac{z}{1 - 2^k xz + z^2}, \tag{4.4}
\]

respectively, and we have the following corollary.

Corollary 4.3. For \( n \in \mathbb{N} \), the generating function of generalized Vieta-Pell polynomials \( P_{k,n}(x) \) is given by

\[
\sum_{n=0}^{\infty} P_{k,n}(x) z^n = \frac{z}{1 - 2^k xz + z^2}, \tag{4.5}
\]

with

\[
P_{k,n}(x) = S_{n-1}(a_1 + [-a_2]).
\]

Multiplying the equation (4.3) by (2) and adding it to the equation obtained by (4.4) multiplying by \((-2^k x)\), then we have the following proposition and corollary.

Proposition 4.4. For \( n \in \mathbb{N} \), the generating function of generalized Vieta-Pell-Lucas polynomials \( Q_{k,n}(x) \) is given by

\[
\sum_{n=0}^{\infty} Q_{k,n}(x) z^n = \frac{2 - 2^k xz}{1 - 2^k xz + z^2}. \tag{4.6}
\]

Corollary 4.5. The following identity holds true:

\[
Q_{k,n}(x) = 2S_n(a_1 + [-a_2]) - 2^k x S_{n-1}(a_1 + [-a_2]).
\]

- Based on the relationships (4.5) and (4.6) and with \( k = 1 \), we obtain the following corollaries.

Corollary 4.6. For \( n \in \mathbb{N} \), the generating function of Vieta-Pell polynomials \( t_n(x) \) is given by

\[
\sum_{n=0}^{\infty} t_n(x) z^n = \frac{z}{1 - 2xz + z^2}, \tag{4.7}
\]

with

\[
t_n(x) = S_{n-1}(a_1 + [-a_2]).
\]

Corollary 4.7. For \( n \in \mathbb{N} \), the generating function of Vieta-Pell-Lucas polynomials \( s_n(x) \) is given by

\[
\sum_{n=0}^{\infty} s_n(x) z^n = \frac{2 - 2x}{1 - 2xz + z^2}, \tag{4.8}
\]

with

\[
s_n(x) = 2S_n(a_1 + [-a_2]) - 2x S_{n-1}(a_1 + [-a_2]).
\]

Multiplying the equation (4.4) by \((-2^{k-1} x)\) and adding it to the equation (4.3), then we have the following proposition and corollary.

Proposition 4.8. For \( n \in \mathbb{N} \), the generating function of generalized Vieta-modified Pell polynomials \( q_{k,n}(x) \) is given by

\[
\sum_{n=0}^{\infty} q_{k,n}(x) z^n = \frac{1 - 2^{k-1} xz}{1 - 2^k xz + z^2}. \tag{4.9}
\]

Corollary 4.9. The following identity holds true:

\[
q_{k,n}(x) = S_n(a_1 + [-a_2]) - 2^{k-1} x S_{n-1}(a_1 + [-a_2]).
\]

Case 2. By taking

\[
\begin{align*}
  a_1 - a_2 &= 1 \\
  a_1 a_2 &= -2^k x
\end{align*}
\]

in the relationships (4.1) and (4.2), we obtain

\[
\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) z^n = \frac{1}{1 - z + 2^k xz^2}, \tag{4.10}
\]

\[
\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) z^n = \frac{z}{1 - z + 2^k xz^2}, \tag{4.11}
\]

respectively, and we have the following corollary.
Corollary 4.10. For $n \in \mathbb{N}$, the generating function of generalized Vieta-Jacobsthal polynomials $G_{k,n}(x)$ is given by
\[
\sum_{n=0}^{\infty} G_{k,n}(x) z^n = \frac{z}{1-z+2z^2x},
\]
with
\[
G_{k,n}(x) = S_{n-1}(a_1 + [-a_2]).
\]

Multiplying the equation (4.8) by (2) and adding it to the equation obtained by (4.9) multiplying by $(-1)$, then we have the following proposition.

Proposition 4.11. For $n \in \mathbb{N}$, the generating function of generalized Vieta-Jacobsthal-Lucas polynomials $g_{k,n}(x)$ is given by
\[
\sum_{n=0}^{\infty} g_{k,n}(x) z^n = \frac{2-z}{1-z+2z^2},
\]
with
\[
g_{k,n}(x) = 2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2]).
\]

- Based on the relationships (4.10) and (4.11) and with $k = 1$, we obtain the following corollaries.

Corollary 4.12. For $n \in \mathbb{N}$, the generating function of Vieta-Jacobsthal polynomials $G_n(x)$ is given by
\[
\sum_{n=0}^{\infty} G_n(x) z^n = \frac{z}{1-z+2z^2},
\]
with
\[
G_n(x) = S_{n-1}(a_1 + [-a_2]).
\]

Corollary 4.13. For $n \in \mathbb{N}$, the generating function of Vieta-Jacobsthal-Lucas polynomials $g_n(x)$ is given by
\[
\sum_{n=0}^{\infty} g_n(x) z^n = \frac{2-z}{1-z+2z^2},
\]
with
\[
g_n(x) = 2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2]).
\]

5. Conclusion

In this paper, we have generalized the work of Saba, Boussayoud and Abderezzak [9] by introduced the generalizations of some numbers and polynomials. Some important generalizations of generating functions are produced. By making use of theorem (2.8), we have obtained propositions and corollaries which is led to generating function for a class of generalized vieta polynomials.

The results obtained in this work are promising, but there are other perspectives to follow in the field. Future work should be based on the extension of the generating functions of binary products of Gaussian generalized Tribonacci numbers with generalized polynomials of second-order linear recurrence sequences.

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