Nijenhuis geometry II: left-symmetric algebras and linearization problem

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Abstract

A field of endomorphisms $R$ is called a Nijenhuis operator if its Nijenhuis torsion vanishes. In this work we study a specific kind of singular points of $R$ called points of scalar type. We show that the tangent space at such points possesses a natural structure of a left-symmetric algebra (also known as pre-Lie or Vinberg-Kozul algebras). Following Weinstein’s approach to linearization of Poisson structures, we state the linearisation problem for Nijenhuis operators and give an answer in terms of non-degenerate left-symmetric algebras. In particular, in dimension 2, we give classification of non-degenerate left-symmetric algebras for the smooth category and, with some small gaps, for the analytic one. These two cases, analytic and smooth, differ. We also obtain a complete classification of two-dimensional real left-symmetric algebras, which may be an interesting result on its own. This work is the second part of a series of papers on Nijenhuis Geometry started with [19].

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1 Introduction

Let $V$ be a finite dimensional space over the field of real numbers $\mathbb{R}$ and $R : V \rightarrow V$ be a linear operator. We say that $R$ is \textbf{semisimple} if in some basis of $V$ its matrix is diagonal. If all eigenvalues of a semisimple $R$ are pairwise different, we say that $R$ is also \textbf{gl–regular}.

The Nijenhuis torsion \cite{2} of an operator field $R$ on a manifold is a tensor defined on a pair of vector fields $v, w$ as follows:

$$\mathcal{N}_R(v, w) = R[Rv, w] + R[v, Rw] - R^2[v, w] - [Rv, Rw].$$

There is a lot of different definitions of the Nijenhuis torsion \cite{1}. The present work is the second in a series, dedicated to the development of Nijenhuis geometry in general.

An operator field is called Nijenhuis if its Nijenhuis torsion vanishes, that is $\mathcal{N}_R = 0$. We will omit the word "field" and will call such an object simply \textbf{Nijenhuis operator}. Nijenhuis operators play an important role in the theory of complex structures \cite{4}, integrable systems \cite{5} and projectively equivalent metrics \cite{6}.

If not stated otherwise all the objects are defined in $n$–dimensional real ball $U^n(P)$, centered at point $P$. This means, that the coordinates of $P$ are $(0, 0)$. All the coordinate changes we will deal with preserve the coordinates of $P$. 

The Nijenhuis torsion was originally introduced by Albert Nijenhuis in 1951 [2]. In our terms he proved that if a Nijenhuis operator is semisimple with real eigenvalues and \( gl^-\)--regular, then there exists a local coordinate system \( x^1, \ldots, x^n \) such that \( R \) is in a diagonal form. Moreover, the \( i\)th eigenvalue depends only on the \( i\)th coordinate.

Haantjes in 1955 [3] proved a more general result. Let \( R \) be a Nijenhuis operator semisimple at every point. Suppose that \( R \) has \( k \) pairwise different real eigenvalues with constant multiplicities \( m_1, \ldots, m_k \). Then there exists a local coordinate system

\[
x^1, x_1^{m_1}, x_2, \ldots, x_2^{m_2}, \ldots, x_k, \ldots, x_k^{m_k}
\]

such that

\[
R = \begin{pmatrix}
Q_1 & 0 & \cdots & 0 \\
0 & Q_2 & \cdots & 0 \\
& & \ddots & \\
0 & 0 & \cdots & Q_s
\end{pmatrix},
\]

where \( Q_i = \lambda_i(x_1^1, \ldots, x_i^{m_i}) \text{Id} \). If all \( m_i = 1 \), one gets the result by Nijenhuis.

In work [19] the Splitting Theorem is proved.

**Theorem 1.1.** Assume that the spectrum of a Nijenhuis operator \( R \) at a point \( P \) consists of \( k \) real (distinct) eigenvalues \( \lambda_1, \ldots, \lambda_k \) with multiplicities \( m_1, \ldots, m_k \) respectively and \( s \) pairs of complex (non-real) conjugate eigenvalues \( \mu_1, \bar{\mu}_1, \ldots, \mu_s, \bar{\mu}_s \) of multiplicities \( l_1, \ldots, l_s \). Then in a neighborhood of \( P \) there exists a local coordinate system

\[
x_1 = (x_1^1 \ldots x_1^{m_1}), \ldots, x_k = (x_k^1 \ldots x_k^{m_k}), \quad u_1 = (u_1^1 \ldots u_1^{l_1}), \ldots, u_s = (u_s^1 \ldots u_s^{l_s}),
\]

in which \( R \) takes the following block-diagonal form

\[
R = \begin{pmatrix}
Q_1(x_1) & \cdots \\
& Q_k(x_k) \\
& & Q_1^{C}(u_1) \\
& & & \ddots \\
& & & & Q_s^{C}(u_s)
\end{pmatrix},
\]

where each block depends on its own group of variables and is a Nijenhuis operator w.r.t. these variables.

If \( m_i \) are locally constant and \( R \) has only real eigenvalues one gets from this Theorem the mentioned above result by Haantjes.

In the same work [19] the notion of a singular point of a Nijenhuis operator was introduced. In present work we are interested in specific class of singularities called singular
points of scalar type. A point $P$ is called **scalar** for Nijenhuis operator $R$ if at this point $R = \lambda \text{Id}$ for some number $\lambda$.

Singular points of scalar type appear in the theory of the projectively equivalent metrics and are natural singularities to study. If $R$ is semisimple (as in works by Nijenhuis and Haantjes) at $P$, then $P$ is scalar for each $Q_i$.

The original results by Nijenhuis and Haantjes do not work near singular points of scalar type in general.

**Example 1.1.** Consider the linear Nijenhuis operator

$$R = \begin{pmatrix} 0 & -x \\ -x & -2y \end{pmatrix}.$$  

The singular point of scalar type here is the origin $(0,0)$. The eigenvalues of this operator are $\lambda_{1,2} = -y \pm \sqrt{x^2 + y^2}$. At the origin they are not smooth (not, actually, even $C^1$). Therefore there is no smooth coordinate change that transforms $R$ into a diagonal form in the neighbourhood of the origin.

The main tool to study singular points of scalar type of Nijenhuis operators are left-symmetric algebras. These algebras were introduced by Vinberg [7] in his study of homogeneous cones. Later they appeared in different frameworks of geometry, integrable systems and quantum mechanics [8].

Let $a$ be an algebra over $\mathbb{R}$. We denote the multiplication in this algebra by $\ast$. The associator $A$ is the following trilinear operation $A(\xi, \eta, \zeta) = (\xi \ast \eta) \ast \zeta - \xi \ast (\eta \ast \zeta)$, for arbitrary triple $\xi, \eta, \zeta \in a$. An algebra $a$ is called **left-symmetric or LSA** if

$$A(\xi, \eta, \zeta) = A(\eta, \xi, \zeta).$$

In particular every associative algebra is by definition left-symmetric.

Denote the commutator for this algebra by

$$[\xi, \eta] = \xi \ast \eta - \eta \ast \xi.$$ 

The main property of these algebras is as follows: the commutator defines a Lie algebra structure on $a$. We call this algebra the **associated Lie algebra**.

Let us denote by $L_\xi \eta = \xi \ast \eta$ and by $R_\eta \xi = \xi \ast \eta$. The property $A(\xi, \eta, \zeta) - A(\eta, \xi, \zeta) = 0$ can be rewritten in the form:

$$L_\xi L_\eta - L_\eta L_\xi = L_{[\xi, \eta]}.$$  

Arbitrary algebra $a$ over $\mathbb{R}$ has the natural structure of a smooth $n-$dimensional affine manifold. In this case $R_\xi$ defines an operator field as follows: an operator $R_\xi$ at a point $\xi$ applied to a vector $\eta$ is defined as $R_\xi \eta = \eta \ast \xi$. 

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Now fix basis $\xi_i$ in $\mathfrak{a}$ and denote by $a^k_{ij}$ the structure constants of $\mathfrak{a}$. The entries of $R_\xi$ are written as follows $(R_\xi)^i_k = a^k_{is}x^s$ for $\xi = x^s\xi_s$. In particular the entries depend linearly on coordinates.

Moreover, if one has an operator fields $R_\xi$ on an affine space with given coordinates and entries being linear functions the constants $a^k_{ij} = \frac{\partial R^k_i}{\partial x_j}$ define tensor of type $(1, 2)$. Thus, we have a natural bijection between real algebras and operator fields with linear entries on real affine spaces.

The following result was proved in the unpublished preprint by Winterhalder [9] (to keep our work self-sustained we provide our own proof).

**Proposition 1.1.** Let $\mathfrak{a}$ be an algebra over $\mathbb{R}$ of dimension $n$. The following conditions are equivalent:

1. $\mathfrak{a}$ is a left-symmetric algebra
2. $R_\xi$ is a Nijenhuis operator

We call such Nijenhuis operators **linear Nijenhuis operators**. The Proposition establishes the bijection between linear Nijenhuis operators and left-symmetric algebras. For the sake of simplicity in our notations we will omit $\xi$ in $R_\xi$ and just write $R$. We call Nijenhuis operators

**Example 1.2.** Consider one-dimensional algebra with basis $\xi_1$ and structure relation $\xi_1 \ast \xi_1 = \xi_1$. This algebra is left-symmetric and associated Lie algebra is commutative. Let $\mathfrak{a}$ be the direct sum of $n$ such one dimensional algebras. The corresponding linear Nijenhuis operator for $\xi = x^i\xi_i$ has the form:

$$R_\xi = \begin{pmatrix} x^1 & 0 & \ldots & 0 \\ 0 & x^2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & x^n \end{pmatrix}.$$ 

**Example 1.3.** Consider the four-dimensional algebra over $\mathbb{R}$ with a basis $\xi_1 = 1$, $\xi_2 = i$, $\xi_3 = j$, $\xi_4 = k$ and following structure relations:

$$i^2 = a, \quad j^2 = b,$$

$$ij = -ji = k,$$

for arbitrary $a, b \in \mathbb{R}$. $1$ is an identity element.

This algebra is, in fact, associative, and thus left-symmetric. For $a, b \neq 0$ this algebra is isomorphic either to the Hamilton’s quaternions or to $\mathfrak{gl}(2, \mathbb{R})$. The corresponding
linear Nijenhuis operator for $\xi = x^i \xi_i$ has the form:

$$R_\xi = \begin{pmatrix}
  x^1 & x^2 & x^3 & x^4 \\
  x^2 & ax^1 & -x^4 & ax^3 \\
  x^3 & x^4 & bx^1 & bx^2 \\
  x^4 & ax^3 & -bx^2 & -abx^1
\end{pmatrix}.$$ 

Let $R = \lambda \text{Id} + R_1 + R_2 + \ldots$ be a Taylor expansion of $R$ at this point. The entries $R_i$ are homogeneous polynomials of degree $i$. If $R$ is a Nijenhuis operator, then $R - \lambda \text{Id}$ is also a Nijenhuis operator [10]. Under these assumptions linear part of $\mathcal{N}_R$ is $\mathcal{N}_{R_1}$, this $R_1$ is linear Nijenhuis operator. We may ask when is the Nijenhuis operator $R$ equivalent to $R_1$?

We call this the linearization problem for the Nijenhuis operators. Same problem appears in case of vector fields around critical points [18] and Poisson structures around singular points [11].

Let $R$ be an operator field on a manifold, not necessary Nijenhuis. Suppose that $P$ is a singular point of scalar type. Consider a pair of vectors $v_P, w_P$ from the tangent space $T_P$ at this point. Denote by $v, w$ an arbitrary smooth continuation of $v_P, w_P$ in the neighborhood of $P$, that is a pair of vector fields defined on $U(P)$ with the property $v(P) = v_P, w(P) = w_P$. Define the following operation

$$v_P \ast w_P = (\mathcal{L}_w R v)|_P,$$

where $\mathcal{L}_w$ is a Lie derivative along vector field $w$.

It is a simple exercise to show that for a singular point of scalar type $P$ this operation does not depend on the continuation of $v_P$ and $w_P$. That is the tangent space possesses a natural structure of an algebra. We call this algebra the isotropy algebra at point $P$. We denote it as $\mathfrak{a}_R$.

**Proposition 1.2.** Let $R$ be a Nijenhuis operator and $P$ singular point of scalar type. Then

1. The isotropy algebra of $P$ is left-symmetric

2. In local coordinates the structure constants of this algebra are $a^k_{ij} = \frac{\partial R^k}{\partial x^j}|_P$

We say, that a left-symmetric algebra $\mathfrak{a}$ is called non-degenerate if any Nijenhuis operator $R$, whose isotropy left-symmetric algebra $\mathfrak{a}_R$ at a scalar singular point $P$ is isomorphic to $\mathfrak{a}$, is linearizable at this point. We call the corresponding change of coordinates the linearizing change of coordinates. In other words, non-degenerate
left-symmetric algebras are the ones that give stable linear Nijenhuis operators $R_1$. That is every Nijenhuis perturbation $R = R_1 + R_2 + ...$ is equivalent to the $R_1$.

The definition of non-degenerate left-symmetric algebras is almost word-to-word replica of Weinstein’s definition of non-degenerate Lie algebras [15] with left-symmetric algebra replacing Lie algebra and Nijenhuis operator replacing Poisson tensor.

When we talk about the linearization problem, we need to distinguish three cases: smooth, analytic and formal. In his work [11], Weinstein showed that semisimple Lie algebras are non-degenerate in formal category. Later Conn [12], [13] proved that compact Lie algebras are non-degenerate in smooth category, while semisimple Lie algebras are non-degenerate in the analytic category. Weinstein provided an example of semisimple Lie algebra, that is degenerate in the smooth category.

In [19] it is proved that the left-symmetric algebra, described in Example [1.2] is non-degenerate in both formal and analytic category. In this work we give the complete classification of real two-dimensional LSAs in terms of non-degeneracy in the smooth category and an almost complete classification in the real analytic category.

First, we classify all real left-symmetric algebras in dimension two. Until now there has been only a partial classification [14] of the left symmetric algebras over the field of complex numbers.

Theorem 1.2. Up to an isomorphism there are two continuous families and 10 exceptional two dimensional real left-symmetric algebras. The complete list of normal forms is presented in the Table 1 and Table 2 below. For every algebra we give

- All non-zero structure relations for a given basis
- The matrix $L_\xi$ in the same basis
- The matrix $R_\xi$ in the same basis

b stands for algebras with non-commutative associated Lie algebra and c for the algebras
with commutative associated Lie algebra.

**Table 1**

| Name   | Structure relations                     | $L_\xi$          | $R_\xi$          |
|--------|----------------------------------------|------------------|------------------|
| $b_{1,\alpha}$ | $\xi_2 \ast \xi_1 = \xi_1$, $\xi_2 \ast \xi_2 = \alpha \xi_2$ | $(y \ 0)$       | $(0 \ \alpha y)$ |
| $b_2$ | $\xi_2 \ast \xi_1 = \xi_1$, $\xi_2 \ast \xi_2 = \xi_1 + \xi_2$ | $(y \ y)$       | $(0 \ x + y)$    |
| $b_{3,\alpha}$ | $\alpha \neq 0$, $\xi_1 \ast \xi_2 = \xi_1$, $\xi_2 \ast \xi_1 = (1 - \frac{1}{\alpha}) \xi_1$, $\xi_2 \ast \xi_2 = \xi_2$ | $(1 - \frac{1}{\alpha}) y \ x \ y \ y$ | $(0 \ (1 - \frac{1}{\alpha}) x \ y \ y)$ |
| $b_4$ | $\xi_1 \ast \xi_2 = \xi_1$, $\xi_2 \ast \xi_2 = \xi_1 + \xi_2$ | $(0 \ x + y)$   | $(y \ y)$        |
| $b_5^+$ | $\xi_1 \ast \xi_1 = \xi_2$, $\xi_2 \ast \xi_1 = -\xi_1$, $\xi_2 \ast \xi_2 = -2 \xi_2$ | $(-y \ 0)$      | $(0 \ -x)$       |
| $b_5^-$ | $\xi_1 \ast \xi_1 = -\xi_2$, $\xi_2 \ast \xi_1 = -\xi_1$, $\xi_2 \ast \xi_2 = -2 \xi_2$ | $(-y \ 0)$      | $(0 \ -x)$       |

**Table 2**

| Name   | Structure relations                     | $L_\xi = R_\xi$ |
|--------|----------------------------------------|-----------------|
| $c_1$  |                                        | $(0 \ 0)$       |
| $c_2$  | $\xi_2 \ast \xi_2 = \xi_2$           | $(0 \ 0)$       |
| $c_3$  | $\xi_2 \ast \xi_2 = \xi_1$           | $(0 \ y)$       |
| $c_4$  |                                        | $(y \ x)$       |
| $c_5^+$| $\xi_2 \ast \xi_2 = \xi_2$, $\xi_2 \ast \xi_1 = \xi_1$, $\xi_1 \ast \xi_2 = \xi_1$ | $(y \ x)$       |
| $c_5^-$| $\xi_2 \ast \xi_2 = \xi_2$, $\xi_2 \ast \xi_1 = \xi_1$, $\xi_1 \ast \xi_2 = \xi_1$ | $(y \ x)$       |

Now introduce the following subsets of $\mathbb{R}$:
1. \(\Sigma_0 = \{0\}\)
2. \(\Sigma_1 = \{r \mid r \in \mathbb{N}, r \geq 3\}\)
3. \(\Sigma_2 = \{\alpha \mid \alpha \in \mathbb{R}, \alpha < 0\}\)
4. \(\Sigma_3 = \{\frac{1}{r} \mid r \in \mathbb{N}, r \geq 2\}\)

Let \(\Sigma_{sm}\) be a \(\Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3\).

**Theorem 1.3.** The following table provides the complete classification of two-dimensional left-symmetric algebras in terms of non-degeneracy in the smooth category:

| Degenerate LSA | Non-degenerate LSA |
|----------------|-------------------|
| \(c_1, c_2, c_3, c_4,\) \(b_4, b_{3,\alpha}\) \(b_{1,\alpha} \text{ for } \alpha \in \Sigma_{sm}\) | \(b_5^+, b_5^-, c_5^+, c_5^-\) \(b_2, b_{1,\alpha} \text{ for } \alpha \notin \Sigma_{sm}\) |

Let \([q_0, q_1, q_2, \ldots]\) be a decomposition of an irrational \(\alpha\) into the continuous fraction. If the series

\[
B(x) = \sum_{i=1}^{\infty} \frac{\log q_{n+1}}{q_n}
\]

converges, then \(\alpha\) is a **Brjuno number** [16]. We denote by \(\Omega\) the set of negative Brjuno numbers.

Define the following subset of \(\mathbb{R}\): \(\tilde{\Sigma}_2 = \{-\frac{p}{q} \mid p, q \in \mathbb{N}\}\). Let \(\Sigma_{an}\) be a \(\Sigma_0 \cup \Sigma_1 \cup \tilde{\Sigma}_2 \cup \Sigma_3\). Denote by \(\Sigma_u = \{\alpha < 0, \alpha \notin \mathbb{Q}, \alpha \notin \Omega\}\).

**Theorem 1.4.** The following table provides the classification of two-dimensional left-symmetric algebras in terms of non-degeneracy in the analytic case:

| Degenerate LSA | Non-degenerate LSA | Unknown |
|----------------|-------------------|---------|
| \(c_1, c_2, c_3, c_4,\) \(b_4, b_{3,\alpha}, b_{1,\alpha} \text{ for } \alpha \in \Sigma_{an}\) | \(b_5^+, b_5^-, c_5^+, c_5^-\) \(b_2, b_{1,\alpha} \text{ for } \alpha \notin \Sigma_{an} \cup \Sigma_u\) | \(b_{1,\alpha} \text{ for } \alpha \in \Sigma_u\) |

The difference between smooth and analytic cases is only in the left-symmetric algebra \(b_{1,\alpha}\). We explain this difference in detail for dimension two, even though obviously the same effect exists in higher dimensions.

Let \(v\) be a vector field on the real plane with coordinates \(x, y\) and origin \(P\). Assume that \(P\) is a critical point for \(v\), that is \(v = 0\) at this point. We may write a Taylor
decomposition for \( v = v_1 + ... \) at this point. \( v_i \) are homogeneous vector fields of degree \( i \).

Suppose, that linear part \( v_1 \) of \( v \) is \( (x, \alpha y) \) for some real \( \alpha \neq 0 \). One may ask if there exists a linearizing coordinate change for \( v \)? The linearization problem for vector field has long history, see, for example, book [18].

From Theorem by Chen [17] (see section 5) it follows that for \( \alpha \neq 0 \) the smooth linearization exists iff formal linearization exists. All formal normal forms for vector fields for \( v \) are well-known (see Table 5, Section 5). Thus, in smooth case for a given \( \alpha \) we either know that there exists a linearizing coordinate change, or we may write \( v = v_1 + v_2 + ... \) such, that there are no linearizing coordinate change.

The difference between smooth and analytic case for \( v \) is for \( \alpha < 0 \). It follows from the results by Brjuno and Yoccoz [16] [20], [21], [18] that if \( \alpha \) is Brjuno number, than the linearizing coordinate change exists. If \( \alpha \) is not Brjuno, than there exists a vector field \( v_1 + v_2 + ... \) such that there are no linearizing coordinate change.

Basically, we have that all stable (stability is in the same sense as for Nijenhuis operators above) linear vector fields \( v_1 \) are described in formal, smooth and analytic categories.

For \( b_{1,\alpha} \) the linearization problem is reduced to the linearization problem for a vector field with restriction: vector field has specific form \( v = (f(x,y), \alpha y) \) with \( v_1 = (x, \alpha y) \). This means, that we deal with the perturbation of a special kind: only one coordinate is perturbed.

In present work we prove that if there exists a linearizing coordinate change in the form \( u = g(x,y), v = h(x,y) \), then there exists a linearizing coordinate change in the form \( u' = g(x,y), v' = y \). So for \( \alpha \in \Omega \) the existence of the analytic linearizing coordinate change follows from the results of Brjuno and Yoccoz.

We do not know if the opposite is true: given negative irrational number \( \alpha \notin \Omega \) and \( v_1 = (x, \alpha y) \) is there a perturbation of \( v_1 \) in the form \( v = (x + f_2 + ..., \alpha y) \) (\( f_i \) stands for homogeneous polynomials of two variables of degree \( i \)) for which no linearizing coordinate change exists? It seems that this problem wasn’t studied by ODE specialists yet.

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2 Local geometry of Nijenhuis operators in dimension 2

Let $R$ be a Nijenhuis operator. Fix local coordinates $x^1, x^2$. In these coordinates the operator can be written in matrix form as follows

$$R = \begin{pmatrix} R^1_1 & R^1_2 \\ R^2_1 & R^2_2 \end{pmatrix}.$$ 

In this section we study the properties of these equations, we will use later in our work. In particular the important result is that in dimension two in some specific case these equations can be implicitly solved

**Theorem 2.1.** In the case of dimension two the following two conditions are equivalent:
1) Operator $R$ is Nijenhuis;
2) In local coordinates:

$$\det R = \begin{pmatrix} R^2_2 & -R^2_1 \\ -R^2_1 & R^1_1 \end{pmatrix} \, \det R.$$

**Proof.** Vanishing a Nijenhuis torsion in two dimensions gives two PDEs $(\mathcal{N}_R)^1_{12} = 0, (\mathcal{N}_R)^2_{12} = 0$.

The first one in local coordinates has the form

$$0 = (\mathcal{N}_R)^1_{12} = \frac{\partial R^1_1}{\partial x^1} R^1_1 + \frac{\partial R^1_1}{\partial x^2} R^2_1 - \frac{\partial R^1_1}{\partial x^1} R^1_2 - \frac{\partial R^1_1}{\partial x^2} R^2_2 - \frac{\partial R^2_2}{\partial x^1} R^1_1 - \frac{\partial R^2_2}{\partial x^2} R^2_1 + \frac{\partial R^2_1}{\partial x^1} R^1_2 + \frac{\partial R^2_1}{\partial x^2} R^2_2.$$

The first and the fifth terms in this equation cancel out. We add $\frac{\partial R^2_1}{\partial x^2} R^1_1$ to the right side and subtract it. Regrouping the terms we get

$$0 = \frac{\partial R^2_1}{\partial x^2} R^1_1 \left( - R^1_1 R^2_2 + R^2_1 R^1_2 \right) + \frac{\partial R^2_1}{\partial x^1} R^1_1 \left( \frac{\partial R^1_1}{\partial x^2} + \frac{\partial R^2_1}{\partial x^1} \right).$$

In a similar way the equation $(\mathcal{N}_R)^2_{12} = 0$ can be rewritten in the form

$$0 = \frac{\partial R^1_1}{\partial x^1} \left( - R^1_1 R^1_2 + R^2_1 R^2_2 \right) \left( - R^2_1 \left( \frac{\partial R^1_1}{\partial x^1} + \frac{\partial R^1_2}{\partial x^1} \right) + R^1_1 \left( \frac{\partial R^1_1}{\partial x^2} + \frac{\partial R^2_1}{\partial x^2} \right) \right).$$
Note that $R_1^1R_2^2 - R_1^2R_2^1 = \det R$ and $R_1^1 + R_2^2 = \tr R$. The equations 6 and 7 can be rewritten in a matrix form. The Theorem is proved. ■

The matrix in formula (5) is sometimes called a cofactor matrix for a given matrix $R$.

In [19] for Nijenhuis operator $R$ the following formula was proved:

$$R^*d \det R = \det R d \tr R.$$ 

Formula (5) shows that in dimension 2 this formula is, in fact, equivalent.

**Corollary 2.1.** It follows from Theorem 2.1 that

$$R = \begin{pmatrix} g(y) & f(x, y) \\ 0 & g(y) \end{pmatrix},$$

where $f(x, y)$ and $g(y)$ are arbitrary smooth functions of two and one variables respectively, is a Nijenhuis operator.

**Corollary 2.2.** It follows from Theorem 2.1 that

$$R = \begin{pmatrix} 0 & f(x, y) \\ 0 & g(y) \end{pmatrix},$$

where $f(x, y)$ and $g(y)$ are arbitrary smooth functions of two and one variables respectively, is a Nijenhuis operator.

Consider again two PDEs $(\mathcal{N}_R)_1^1 = 0, (\mathcal{N}_R)_2^2 = 0$. They have four functional unknowns: $R_1^1, R_2^1, R_1^2, R_2^2$.

We are interested in the specific class of solutions with $d \tr R \neq 0$ around $P$. We may assume that $\tr R = R_1^1 + R_2^2 = \alpha y$ (we will need $\alpha$ later in our work). We denote the determinant of $R$ by $D$. From (5) we obtain:

$$R_1^2 = -\frac{1}{\alpha} D_x,$$
$$R_1^1 = \frac{1}{y} D_y,$$
$$R_1^1 + R_2^2 = \alpha y,$$
$$R_1^1R_2^2 - R_2^1R_1^2 = D,$$

(8)

We now treat this system as system with a functional parameter $D$. In this case it can be solved in an implicit form.
Corollary 2.3. Every solution of system (8) is written in the form:

\[ R = \begin{pmatrix} \frac{1}{\alpha} D_y & R_2^1 \\ -\frac{1}{\alpha} D_x & \alpha y - \frac{1}{\alpha} D_y \end{pmatrix}, \]  

where \( R_2^1 \) satisfies the following (implicit) condition

\[ \frac{1}{\alpha} D_y(\alpha y - \frac{1}{\alpha} D_y) + \frac{1}{\alpha} D_x R_2^1 - D = 0. \]  

Important thing here is that the condition (10) does not contain any derivatives of \( R_2^1 \). At the same time it provides a non trivial necessary conditions on the partial derivatives of \( D \): if \( D_x = 0 \) at some point \( P \), then \((\alpha y - \frac{1}{\alpha} D_y)D_y - \alpha D = 0\) at the same point. If the left side of this equation is not zero at \( P \), we have that \( R_2^1 ((\alpha y - \frac{1}{\alpha} D_y)D_y - \alpha D)/D_x \to \infty \) as argument approaches \( P \). That is \( R_2^1 \) is not even continuous at this point.

Example 2.1. If \( D = f(y) \), we get \((\alpha y - \frac{1}{\alpha} D_y)D_y - \alpha D = 0\). Differentiating it, we get simpler equation

\[ D_{yy}(\alpha y - \frac{2}{\alpha} D_y) = 0. \]

In this case \( D = \frac{\alpha^2}{2} y^2 + A \) or \( D = By + C \) for some arbitrary constants \( A, B, C \). The corresponding Nijenhuis operator has the form:

\[ R = \begin{pmatrix} \frac{1}{\alpha} D_y & f(x, y) \\ 0 & \alpha y - \frac{1}{\alpha} D_y \end{pmatrix}, \]

where \( f(x, y) \) is an arbitrary function of two variables.

Corollary 2.4. If \( \alpha = 1 \) and \( D \equiv 0 \), then \( R \) has the form

\[ R = \begin{pmatrix} 0 & h(x, y) \\ 0 & y \end{pmatrix}, \]

where \( h(x, y) \) is an arbitrary function.

Proof. It follows from Corollary 2.3.

Corollary 2.5. If \( \alpha = 1 \) and \( D = \frac{y^2}{4} \), then \( R \) has the form

\[ R = \begin{pmatrix} \frac{y}{2} & h(x, y) \\ 0 & \frac{y}{2} \end{pmatrix}, \]

where \( h(x, y) \) is an arbitrary function.

Proof. It follows from Corollary 2.3.
3 The classification of real left-symmetric algebras in dimension 2

Let \( a \) be a left-symmetric algebra. Fix a basis \( \xi_1, \ldots, \xi_n \) and denote structure constants of \( a \) in this basis by \( a^k_{ij} \). For \( \xi = x^i \xi_i \) we have \( L_\xi = a^k_{is}x^s \) and \( R_\xi = a^k_{si}x^s \). In particular, we denote the corresponding maps by \( L, R : a \to \mathfrak{gl}(2, \mathbb{R}) \) respectively.

The images \( \text{Im} L \) and \( \text{Im} R \) of these maps depend upon the choice of a basis in \( a \). Given a change of coordinates \( \xi'_i = C^s_i \xi_s \) the subspaces change as follows: \( \text{Im}' L = C^{-1}_i \text{Im} LC \) and \( \text{Im}' R = C^{-1}_i \text{Im} RC \).

One may see that the basis-independent properties of the subspaces are the ones that are preserved by a conjugation. For example, consider the following property (we will call it the property S): the subspace \( \text{Im} L \) (or \( \text{Im} R \)) contains a non-semisimple element.

Denote by \( \text{tr} R \) the trace of \( R_\xi \), by \( \det R \) the determinant of \( R_\xi \), by \( \text{tr} L \) the trace of \( L_\xi \), by \( \det L \) the determinant of \( L_\xi \). The value of all \( \text{tr} R, \text{tr} L, \det R, \det L \) in \( \xi \) does not depend upon the choice of coordinates that is all four are the functions of \( \xi \).

Lemma 3.1. Consider an arbitrary function \( f \) of four variables and a pair of left-symmetric algebras \( a \) and \( a' \). Suppose that there exists \( \xi \in a' \), such that at this point \( f(\text{tr} L, \det L, \text{tr} R, \det R) \neq 0 \). At the same time suppose, that the same function is zero on the entire \( a \). Then \( a \) and \( a' \) are not isomorphic.

Proof. The proof of the Lemma follows from the definition of the functions. ■

3.1 Proof of Theorem 1.2

We start by showing that left-symmetric algebras in tables with different names and different real parameters are not isomorphic.

First note that algebras from Table 1 and 2 are not isomorphic, as their associated Lie algebras are not isomorphic.

In Table 1 \( \det R \) is zero for \( b_{1,\alpha} \) and \( b_2 \) only, so by Lemma 3.1 these algebras are not isomorphic to any algebra from the rest of Table 1. By Property S for space \( \text{Im} L \) algebras \( b_2 \) and \( b_{1,\alpha} \) are not isomorphic.

Now consider \( b_{1,\alpha} \) and function \( f = \beta \det L - (\text{tr} R)^2 \) for a given constant \( \beta \). This function is identically zero for \( b_{1,\beta} \) and not zero for \( b_{1,\alpha}, \alpha \neq \beta \). By Lemma 3.1 for \( \alpha \neq \beta \) algebras \( b_{1,\alpha} \) and \( b_{1,\beta} \) are not isomorphic.
In Table 1 det $L \equiv 0$ for $b_4$ and $b_{3,1}$ only. By Lemma 3.1 they are not isomorphic to any algebra from the rest of Table 1. By Property S for Im $R$, $b_4$ and $b_{3,1}$ are not isomorphic either.

In Table 1 $(\text{tr } R)^2 - 4 \det R \equiv 0$ for $b_4, b_{3,0}$ and $b_{3,\alpha}$ only. Thus, $b_{3,\alpha}$ is not isomorphic to any of the algebras form the rest of Table 1.

Fix constant $\beta \neq 1$ and consider $\beta \det R - \det L$. It is identically zero on $b_{3,\alpha}$ for $\alpha = \frac{1}{1-\beta}$ and not identically zero otherwise. By Lemma 3.1, algebras $b_{3,\alpha}$ and $b_{3,\alpha'}$ are not isomorphic for $\alpha \neq \alpha'$.

Denote the function
\[ T(x) = \begin{cases} 
0 & x \geq 0, \\
1 & x < 0, 
\end{cases} \]
Take $f = T((\text{tr } R)^2 - 4 \det R)$. $f \equiv 0$ for $b_5$ and $f \not\equiv 0$ for $b_5^\perp$. Once again, by Lemma 3.1 these two algebras are not isomorphic.

Now consider Table 2. We have det $L \equiv 0$ for $c_1, c_2, c_3$. By Lemma 3.1 these algebras are not isomorphic to any of the algebras from the rest of the table. At the same time, $\text{tr } R = 0$ for $c_1, c_3$ and not for $c_2$. By Property S we have that $c_1$ and $c_3$ are not isomorphic. So all three algebras are pairwise non-isomorphic.

The function $f = T(\det R)$ is zero for $c_5^-$ and nonzero for $c_5^+$ and $c_4$. At the same time the function $(\text{tr } R)^2 - 4 \det R$ is zero for $c_4$ and not zero for $c_5^+$ and $c_5^-$. This means, that by Lemma 3.1 these three algebras are pairwise not isomorphic.

Now let us show, that any two-dimentional LSA is isomorphic to an algebra from Table 1 or Table 2.

**Lemma 3.2.** Let $R, Q \in \mathfrak{gl}(n, \mathbb{R})$ and $[R, Q] = \lambda Q$ for $\lambda \neq 0$. Then $Q$ is nilpotent.

**Proof.** An operator $\text{ad}_R : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ is defined by the formula $\text{ad}_R Q = [R, Q] = \lambda Q$. From the properties of the matrix commutator it follows that $\text{ad}_R Q^n = n \lambda Q^n$.

Suppose now that $Q^n \neq 0$ for all $n \in \mathbb{N}$. It means that the finite dimensional operator $\text{ad}_R$ has an infinite set of eigenvectors. This contradiction completes the proof. ■

**Lemma 3.3.** Every two-dimensional commutative subalgebra $\mathfrak{h} \subset \mathfrak{gl}(2, \mathbb{R})$ contains the one-dimensional subspace spanned by the identity matrix.

**Proof.** Suppose that the statement of the Lemma is false and $\mathfrak{h}$ does not contain an identity matrix. We add it to $\mathfrak{h}$ and obtain a commutative subalgebra $\mathfrak{h}'$, that is three-dimensional.

Choose $X \in \mathfrak{h}'$, that is not proportional to the identity matrix. $\mathfrak{h}'$ is contained in the centralizer of $X$. $\mathfrak{gl}(n, \mathbb{R})$ possesses an invariant non-degenerate bilinear form. After
we identify $\mathfrak{gl}(n, \mathbb{R})$ with its dual space, we get that as the dimension of $\mathfrak{gl}(n, \mathbb{R})$ is even, then the dimension of the centralizer is even.

Therefore, the centralizer of $x$ coincides with $\mathfrak{gl}(2, \mathbb{R})$ and $X$ is proportional to the identity matrix. This contradiction completes the proof. ■

The proof of the Classification Theorem is a case-by-case analysis.

$\dim \text{Im} L = 0$. This means that $L_\xi = 0$ for an arbitrary $\xi \in a$. Thus, we get that $a$ is $c_1$.

$\dim \text{Im} L = 1$. Choose a basis $\eta_1, \eta_2$ such that $\eta_1$ spans the kernel of map $L$, that is $L_{\eta_1} = 0$. From Property 2 we have $L_{[\eta_1, \eta_2]} = [L_{\eta_1}, L_{\eta_2}] = 0$.

If $[\eta_1, \eta_2] = 0$, then
\begin{align*}
\eta_1 \ast \eta_1 &= \eta_1 \ast \eta_2 = \eta_2 \ast \eta_1 = 0 \\
\eta_2 \ast \eta_2 &= a \eta_2 + b \eta_1.
\end{align*}

Both $a$ and $b$ can’t be zero at the same time.

If $a \neq 0$, then the change of coordinates $\xi_1 = \eta_1, \xi_2 = \frac{b}{a} \eta_1 + \frac{1}{a} \eta_2$ gives us in the basis $\xi_1, \xi_2$ the structure relations of the left-symmetric algebra $c_2$.

If $a = 0$, then the change of coordinates $\xi_1 = b \eta_1, \xi_2 = \eta_2$ yields $c_3$.

If $[\eta_1, \eta_2] \neq 0$, then without loss of generality we may assume that $[\eta_1, \eta_2] = \eta_1$. The structure relations in this case are
\begin{align*}
\eta_1 \ast \eta_1 &= \eta_1 \ast \eta_2 = 0, \\
\eta_2 \ast \eta_1 &= -\eta_1, \\
\eta_2 \ast \eta_2 &= a \eta_2 + b \eta_1,
\end{align*}

for some constants $a$ and $b$.

If $a \neq -1$, then the change of coordinates $\xi_1 = \eta_1, \xi_2 = \eta_2 + \frac{b}{1+a} \eta_1$ yields $b_{1,a}$.

If $a = -1$ and $b = 0$, we obtain $b_{1,-1}$.

If $a = -1$ and $b \neq 0$, then by the change of coordinates $\xi_1 = b \eta_1, \xi_2 = -\eta_2$ we obtain $b_2$.

$\dim \text{Im} L = 2$. In this case $L$ defines an exact representation of the associated Lie algebra for $a$ in $\mathfrak{gl}(2, \mathbb{R})$.

If the associated Lie algebra is commutative, then, by Lemma 3.3, the image of the representation contains the identity matrix $\text{Id}$. Without loss of generality we may assume that in given basis $\eta_1, \eta_2 L_{\eta_2} = \text{Id}$. The structure relations for the left-symmetric
algebra in this case are:
\[ \eta_1 \ast \eta_1 = a \eta_1 + b \eta_2, \]
\[ \eta_1 \ast \eta_2 = \eta_2 \ast \eta_1 = \eta_1, \]
\[ \eta_2 \ast \eta_2 = \eta_2, \]
for some constants \( a \) and \( b \).

If \( \frac{a^2}{4} + b = 0 \), then after the change of coordinates \( \xi_2 = \eta_2, \xi_1 = \eta_1 - \frac{a}{2} \eta_2 \) algebra \( \mathfrak{a} \) is isomorphic to \( \mathfrak{c}_4 \).

If \( \frac{a^2}{4} + b \neq 0 \), then depending on the sign of \( \frac{a^2}{4} + b \) by the change of coordinates \( \xi_2 = \eta_2, \xi_1 = \frac{1}{\sqrt{|\frac{a^2}{4} + b|}} \eta_1 - \frac{a}{2 \sqrt{|\frac{a^2}{4} + b|}} \eta_2 \) we get structure relations for either \( \mathfrak{c}_5^+ \) or \( \mathfrak{c}_5^- \).

If the associated Lie algebra is not commutative, it is possible to choose such a basis \( \eta_1, \eta_2 \) that
\[ \eta_1 \ast \eta_2 - \eta_2 \ast \eta_1 = \eta_1 \]
and
\[ [L_{\eta_1}, L_{\eta_2}] = L_{\eta_1}. \]

From Lemma 3.2 we get that \( L_{\eta_1} \) is nilpotent. In case of dimension two, the kernel and the image of this operator coincide. Suppose that the image of \( L_{\eta_1} \) is spanned by \( \eta_1 \). That is \( L_{\eta_1} \eta_1 = 0 \) and \( L_{\eta_1} \eta_2 = a \eta_1 \). The matrix \( L_{\eta_1} \) can be written in the following form:
\[ L_{\eta_1} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}. \]

As formula 15 holds, we have, that
\[ L_{\eta_2} = \begin{pmatrix} c - 1 & b \\ 0 & c \end{pmatrix}, \]
for some constants \( b \) and \( c \). From formula 14 we get that \( \eta_1 \ast \eta_2 - \eta_2 \ast \eta_1 = a \eta_1 - (c - 1) \eta_1 = \eta_1 \) and, therefore, \( a = c \).

In basis \( \eta_1, \eta_2 \) the structure relations are:
\[ \eta_1 \ast \eta_1 = 0, \]
\[ \eta_1 \ast \eta_2 = a \eta_1, \]
\[ \eta_2 \ast \eta_1 = (a - 1) \eta_1, \]
\[ \eta_2 \ast \eta_2 = b \eta_1 + a \eta_2. \]

If \( a = 1 \) and \( b \neq 0 \), then by the change of coordinates \( \xi_1 = b \eta_1, \xi_2 = \eta_2 \) we obtain \( \mathfrak{b}_4 \). If \( a = 1 \) and \( b = 0 \), the algebra is \( \mathfrak{b}_{3,1} \).

If \( a \neq 1 \), then by the change of coordinates \( \xi_1 = \eta_1, \xi_2 = \frac{1}{a} \left( \eta_2 + \frac{b}{1 - a} \eta_1 \right) \) we obtain \( \mathfrak{b}_{3,a} \).

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Now suppose that the image of $L_{\eta_1}$ is spanned by $\eta_2 + a\eta_1$. This means that $\eta_1 \ast \eta_1 = b(\eta_2 + a\eta_1)$ for some $b \neq 0$. Let us change the coordinates: $\xi_1 = \frac{1}{|b|}\eta_1$, $\xi_2 = \eta_2 + a\eta_1$.

For these vectors the identity $[\xi_1, \xi_2] = \xi_1$ still holds. At the same time $\xi_1 \ast \xi_1 = \text{sgn}(b)\xi_2$ and $\xi_1 \ast \xi_2 = 0$.

We get that

$$L_{\eta_1} = \begin{pmatrix} 0 & 0 \\ \text{sgn}(b) & 0 \end{pmatrix}. \tag{17}$$

From the relation $[L_{\xi_1}, L_{\xi_2}] = L_{\xi_1}$ we have that the matrix $L_{\eta_2}$ has the following form:

$$L_{\eta_2} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix},$$

for some constant $c$. The structure relations in this basis are:

$$\begin{align*}
\xi_1 \ast \xi_1 &= \text{sgn}(b)\xi_2, \\
\xi_1 \ast \xi_2 &= 0, \\
\xi_2 \ast \xi_1 &= -\xi_1, \\
\xi_2 \ast \xi_2 &= -2\xi_2. \tag{18}
\end{align*}$$

For $\text{sgn}(b) = -1$ and $\text{sgn}(b) = 1$ we get $b_5^+$ and $b_5^-$ respectively. 

Comment 1. Note that the limit of $b_{3,\alpha}$ for $\alpha \to \infty$ is $c_4$.

4 Nijenhuis operators and isotropy left-symmetric algebras

4.1 Proof of Proposition 1.1

Fix a basis $\xi_1, \ldots, \xi_n$ in $\mathfrak{a}$ and take $a_{ij}^k$ to be structure constants for this basis: $\xi_i \ast \xi_j = a_{ij}^k \xi_k$. The property that $\mathfrak{a}$ is left-symmetric can be rewritten as follows: for any three basis vectors $\xi_i, \xi_j, \xi_s$ one has

$$0 = (\xi_i \ast \xi_j) \ast \xi_s - (\xi_j \ast \xi_i) \ast \xi_s + (\xi_j \ast \xi_s) \ast \xi_i + (\xi_j \ast \xi_i) \ast \xi_s + (\xi_i \ast \xi_s) \ast \xi_j + (\xi_i \ast \xi_j) \ast \xi_s =$$

$$= \left( a_{ij}^r a_{ks}^r - a_{ki}^r a_{js}^r - a_{is}^r a_{jr}^k + a_{ir}^k a_{js}^i \right) \xi_k. \tag{18}$$

In these coordinates the operator associated with right action can be written in the form: for $\xi = x^1\xi_1 + \ldots + x^n\xi_n$ we have $R_{\xi_i} = \xi_i \ast \xi = a_{ij}^k x^j \xi_k$. The formula for the
Nijenhuis torsion $\mathcal{N}_R$ in local coordinates is:

\[
(\mathcal{N}_R)^k_{ij} = \frac{\partial R^k_i}{\partial x^r} R^r_j - \frac{\partial R^k_i}{\partial x^r} R^r_j + \frac{\partial R^r_i}{\partial x^j} R^k_r = (a^k_{jr} a^r_{is} - a^k_{ir} a^r_{js} - a^r_{ji} a^k_{rs} + a^r_{ij} a^k_{rs}) x^s.
\]

(19)

The formulas (18) and (19) are equivalent. Thus, the algebra $a$ is left-symmetric iff $\mathcal{N}_R = 0$. □

4.2 Proof of Proposition 1.2

Let us recall that we have a pair of vector fields $v, w$ with the property $v(P) = v_P$ and $w(P) = w_P$, and the operation $v_P * w_P = (\mathcal{L}_w R)v_P$. In local coordinates the Lie derivative of $R$ is given by the formula:

\[
(\mathcal{L}_w R)^k_i = \frac{\partial R^k_i}{\partial x^\alpha} w^\alpha + R^i_r \frac{\partial w^k}{\partial x^\alpha} - R^k_r \frac{\partial w^\alpha}{\partial x^i}.
\]

At the singular point of scalar type $P$ we have $R^p_q = \lambda \delta^p_q$, so we get $(\mathcal{L}_w R)^k_i(P) = \partial R^k_i/\partial x^i w_P$. We can see that the result does not depend on the continuation of $v_P$ and $w_P$ and that the structure constants of the isotropy algebra are $\partial R^k_i/\partial x^i$. So, the second part of the proposition is proved.

If $R$ is Nijenhuis, then $R - \lambda \text{Id}$ is also Nijenhuis. Moreover, $\mathcal{L}_w (R - \lambda \text{Id}) = \mathcal{L}_w R$ for an arbitrary vector field $w$. So, the isotropy algebra for $R - \lambda \text{Id}$ is the same as for $R$. So, without loss of generality, we may consider $R$ to be zero at $P$.

Consider the Taylor expansion $R = R_1 + R_2 + \ldots$. In the corresponding Taylor expansion of $\mathcal{N}_R$ the linear part is exactly $\mathcal{N}_{R_2}$. This means that $R_1$ is a linear Nijenhuis operator and in local coordinates it is defined as $\frac{\partial R^k_i}{\partial x^i} |_{P, x^j}$. By Proposition 1.1 we obtain that $\frac{\partial R^k_i}{\partial x^i} |_{P, x^j}$ are the structure constants of the left-symmetric algebra. □

5 The linearization problem for Nijenhuis operators in dimension 2

in this section for every algebra from the list in Theorem 1.2 we either prove that it is non-degenerate by constructing a linearizing change of coordinates or provide a
higher order perturbation of the corresponding linear Nijenhuis operator $R_\xi$ that is not-linearizable.

For the sake of simplicity we will omit the subscript $\xi$ and will write simply $R$. This is not to be confused with map $R$ we used in section 3.

5.1 Proof of Theorem 1.3 and Theorem 1.4

5.1.1 The algebras $c_1, c_2, c_3, c_4, b_4$

The idea of the proof of degeneracy of $c_1, c_2, c_3, c_4$ is to provide such a function $f(\text{tr } R, \det R)$ (smooth or analytic) that it is identically zero for the linear part, but in every neighbourhood of $P$ there are points with $f \neq 0$. This implies that there is no smooth or analytic linearizing coordinate change.

In all examples the local coordinates are centered at the singular point of scalar type that is $P = (0,0)$. The fact that all operator fields are Nijenhuis is verified either by Formula (5) or Corollaries 2.1 and 2.2.

For $c_1$ we take

$R = \begin{pmatrix} x^2 & 0 \\ 0 & y^2 \end{pmatrix}$.

and the function $f = \text{tr } R = x^2 + y^2$.

For $c_2$ we take

$R = \begin{pmatrix} x^2 & 0 \\ 0 & y \end{pmatrix}$.

and the function $f = \det R = x^2y$.

For $c_3$ we take

$R = \begin{pmatrix} y^2 & y \\ 0 & y^2 \end{pmatrix}$.

and the function $f = \det R = y^4$.

For $c_4$ we take

$R = \begin{pmatrix} y + yx^2 & x + x^3 \\ -xy^2 & y - yx^2 \end{pmatrix}$.

and the function $f = (\text{tr } R)^2 - 4 \det R = -4x^2y^2$.

Take a Nijenhuis operator in the form:

$R = \begin{pmatrix} y & y - x^2 \\ 0 & y \end{pmatrix}$.
Its linear part coincides with a linear Nijenhuis operator corresponding to $b_4$.

Consider the curve $y = x^2$. It consists of all the points, where $R$ is proportional to $\text{Id}$. For linear part consider the curve $y = 0$. It consists of all the points, where $R_1$ is proportional to $\text{Id}$.

Consider now function $\det R$. Its restriction on $y = 0$ is identically zero. At the same time its restriction on the curve $y = x^2$ is zero only at the origin.

Suppose that there exists a linearizing coordinate change $\tilde{x} = g(x, y), \tilde{y} = h(x, y)$. Then in a neighborhood of $P$ the curve $y = x^2$ becomes $\tilde{y} = 0$. At the same time function $\det R$ is invariant, that is its value in a given point is the same for both $x, y$ and $\tilde{x}, \tilde{y}$.

Therefore, if it is not zero on the curve it cannot become zero after the coordinate change. This contradiction completes the proof.

Thus, we have proved that $c_1, c_2, c_3, c_4, b_4$ are degenerate left-symmetric algebras in both analytic and smooth categories.

5.1.2 The algebras $b_5^+, b_5^-, c_5^+, c_5^-$

The following Lemma is well-known, but for self-sufficiency of the paper we provide it with proof.

**Lemma 5.1.** (Parametric Morse Lemma) Consider a smooth (analytic) function $f(x, y)$ with a critical point at $(0, 0)$ and $f''_{xx}(0, 0) = \beta \neq 0$. Then there exists a smooth (analytic) coordinate change $\tilde{x} = h(x, y), \tilde{y} = y$ such that $f = \text{sgn} \beta (\tilde{x})^2 + g(\tilde{y})$.

**Comment 2.** From Lemma 5.1 it follows, that $g''(0) = f''_{yy}(0, 0)$.

**Proof.** By the Implicit Function Theorem, there exists a curve $x(y)$ such that $f_x'(x(y), y) = 0$ and $y(0) = 0$. The function $f(x, y)$ can be written as

$$f(x, y) = f(x(y), y) + \int_{x(y)}^{1} \frac{\partial}{\partial x} f(tx, y) dt = f(x(y), y) + \int_{x(y)}^{1} f_x'(tx, y) dt.$$ 

Applying similar decomposition to $f'(x, y)$ from the definition of the curve $x(y)$ we get:

$$f(x, y) = f(x(y), y) + x^2 \int_{x(y)}^{1} f''_{xx} x(tx, y) dt. \quad (20)$$

Denote the integral from Formula (20) by $F(x, y)$. We have $F(0, 0) = \beta \neq 0$. The coordinate change in this case is $\tilde{y} = y, \tilde{x} = \sqrt{|F(x, y)|}$. Hence the Lemma is proved.
**Proposition 5.1.** The left-symmetric algebra $\mathfrak{c}_5^-$ is non-degenerate in both smooth and analytic categories.

*Proof.* Suppose that we have a Nijenhuis operator $R$ with singular point of scalar type $P$. Without loss of generality assume that $R$ is zero at $P$ and that linear part $R_1$ in $P$ coincides with the linear Nijenhuis operator $R_\xi$ associated with $\mathfrak{c}_5^-$ from Table 2 in Theorem 1.2.

Denote $T$ and $D$ to be trace and determinant of $R$ respectively. We have that $T'_y(P) = 2$ and $P$ is critical for $D$. The quadratic part of the determinant is $x^2 + y^2$. By Lemma 5.1 we may consider a coordinate system with $T = 2y$ and $D = x^2 + g(y)$. We have $g''(P) = 2$.

In these coordinates the entries of the Nijenhuis operator $R^j_i$ satisfy the set of equations (8). From Corollary 2.3 the equation on $R^1_2$ is as follows:

$$\frac{1}{2}g'(y)(2y - \frac{1}{2}g'(y)) - g(y) + xR^1_2 - x^2 = 0. \quad (21)$$

For $x = 0$, we get the differential equation on $g$:

$$\frac{1}{2}g'(y)(2y - \frac{1}{2}g'(y)) - g(y) = 0.$$

Differentiating both sides by $y$ we get

$$\frac{1}{2}g''(y)(2y - g') = 0.$$

As $g''(0) = 2 \neq 0$ and $g(0) = g'(0) = 0$, then $g = y^2$.

In the given coordinates $T = 2y$ and $D = x^2 + y^2$. From Corollary 2.3 we obtain $R^1_2 = x$. Thus $R$ is linear and the coordinate change that is mentioned in Lemma 5.1 is in fact a linearizing change of coordinates for $\mathfrak{c}_5^-$. ■

The proof for $\mathfrak{c}_5^+$ is similar, the only difference being $g(y) = -y^2$.

**Proposition 5.2.** The left-symmetric algebra $\mathfrak{b}_5^+$ is non-degenerate in both analytic and smooth categories.

*Proof.* Like in the previous proposition, we consider a Nijenhuis operator $R$ that is zero at $P$ and its linear part $R_1$ at $P$ coincides with the linear Nijenhuis operator for $\mathfrak{b}_5^+$ from Table 1 from Theorem 1.2.

We again consider the coordinate system $x, y$ with $T = -2y$ and $D = x^2 + g(y)$, with $g(P) = g'(P) = g''(P) = 0$. Applying Corollary 2.3 in the same manner we get, that $g$ satisfies the same equation $\frac{1}{2}g''(y)(2y - g'(y)) = 0.$
Let $F = 2y - g'(y)$. The equation then has the form $g''F = 0$. We have $F(0) = 0$ and $F'(0) = 1$, so in some punctured neighborhood of $y = 0$ the function $F$ is not zero. This means, that $g''(y) \equiv 0$ in the entire neighborhood of $y = 0$. As $g'(0) = g(0) = 0$ we get, that $g \equiv 0$.

Therefore, in these coordinates $T = -2y$ and $D = x^2$. By Corollary 2.3, we have that $R$ is linear in these coordinates and coincides with $R^\epsilon$ for $\mathfrak{b}_5^\pm$. Again, the linearizing coordinate change in this case is the same coordinate change mentioned in Lemma 5.1.

The proof for $\mathfrak{b}_5^-$ is similar, the only difference being $D = -x^2$. So, we have shown that left-symmetric algebras $\mathfrak{b}_5^\pm$, $\mathfrak{c}_5^\pm$, $\mathfrak{c}_5$ are non-degenerate in both smooth and analytic categories.

5.1.3 The algebra $\mathfrak{b}_{3,\alpha}$

Proposition 5.3. The algebra $\mathfrak{b}_{3,\alpha}$ is degenerate in both analytic and smooth category.

Proof. Consider a Nijenhuis operator

\[ R = \begin{pmatrix} y & \beta x + y^2 \\ 0 & y \end{pmatrix}. \]

Its linear part at the origin coincides with the linear Nijenhuis operator corresponding to the left-symmetric algebra $\mathfrak{b}_{3,\alpha}$ from Theorem 1.2. The constants $\beta$ and $\alpha$ are related by $\beta = (1 - \frac{1}{\alpha})$ and we have two cases.

Assume that $\alpha \neq 1$. Consider the curve $y^2 = -\beta x$. It consists of all the singular points of scalar type for $R$ in the neighborhood of $P$. If there exists linearizing change of coordinates, it should transform this curve into the curve $x = 0$. This is exactly the set of singular points of scalar type around $P$ for $R_1$. Note that $x = 0$ is a one-dimensionnal submanifold at the origin, while $y^2 = -\beta x$ is not (at the origin it has a cusp). Thus, there are no linearizing change of coordinates both in smooth and analytic case.

Assume now that $\alpha = 1$. Note, that in this case the entire neighborhood of $P$ consists of singular points of scalar type of $R_1$. At the same time for $-\beta x \neq y^2$ the matrix $R$ is a Jordan block. The Jordan type is preserved under the coordinates changes, thus there is no linearizing coordinate change in both smooth and analytic cases.

5.1.4 The algebra $\mathfrak{b}_{1,\alpha}$

To study non-degeneracy of $\mathfrak{b}_{1,\alpha}$ we need several results about the linearization problem for vector fields in different categories.
Consider $\mathbb{R}^2$ and fix coordinate system. The following statement is from the book by Y. Ilyashenko and S. Yakovenko ([18], section 4.10) deals with formal linearization of vector fields on a plane.

**Theorem 5.1.** Let $v$ be a vector field on $\mathbb{R}^2$ with critical point at the origin. Assume also that the eigenvalues of the linearization matrix $\frac{\partial v_i}{\partial x^j}$ are both real, non-zero and $\lambda_1 \geq \lambda_2$. Then there exists a formal coordinate change, that transforms $v$ into formal normal form from the Table below:

| Type               | Conditions |
|--------------------|------------|
| No resonance       | $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}^-$ or $\lambda_1 = \lambda_2$ |
| Resonant node      | $\frac{\lambda_1}{\lambda_2} = r$ and $r \in \mathbb{N}, r \geq 2$ |
| Resonant saddle    | $\frac{\lambda_1}{\lambda_2} = -\frac{p}{q}, p, q \in \mathbb{N}$ |

The formal normal form for a given vector field $v$ is unique.

Consider $\mathbb{R}^n$ and fix coordinate system. A critical point $P$ of a vector field $v$ is called **elementary** [17][18] if and only if every eigenvalue of linearization matrix $\frac{\partial v_i}{\partial x^j}$ at $P$ has a non-vanishing real part. The following Theorem is due K.T.Chen [17] and deals with the linearization in smooth category.

**Theorem 5.2.** (K.T.Chen, 1963) Let $v$ and $w$ be two $C^\infty$ vector fields having coordinate origin as an elementary critical point. Denote by $v_1 + v_2 + \ldots$ and $w_1 + w_2 + \ldots$ the respective Taylor’s expansions of $v$ and $w$. Then there exists a $C^\infty$ transformation about 0, which carries $v$ to $w$ if and only if there exists a formal transformation which carries the formal vector field $v_1 + v_2 + \ldots$ to $w_1 + w_2 + \ldots$.

Let us now recall the following subsets of $\mathbb{R}$.

1. $\Sigma_0 = \{0\}$
2. $\Sigma_1 = \{r | r \in \mathbb{N}, r \geq 3\}$
3. $\Sigma_2 = \{\alpha | \alpha \in \mathbb{R}, \alpha < 0\}$
4. $\Sigma_3 = \{\frac{1}{r} | r \in \mathbb{N}, r \geq 2\}$
5. $\Sigma_{sm} = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$
In smooth case Theorem 5.2 and Theorem 5.1 yield the following Corollary

**Corollary 5.1.** Consider vector field \( v = (f(x,y), \alpha y) \) with \( v_1 = (x, \alpha y) \) and \( \alpha \notin \Sigma_{sm} \cup \{2\} \). Then there exists a smooth linearizing coordinate change for \( v \).

**Proof.** If \( \alpha \notin \Sigma_{sm} \cup \{2\} \) this means that \( \alpha > 0 \) and \( \alpha \neq n \) or \( \alpha \neq \frac{1}{n} \) for \( n > 1, n \in \mathbb{N} \).

From Theorem 5.1 we get that for given \( \alpha \) the formal normal form is linear, that is there exists a formal linearizing coordinate change.

From Theorem 5.2 it follows, that then there exists a smooth coordinate change that transforms \( v \) into \( v_1 \).

The following theorem deals with \( b_{1,\alpha} \) in the smooth case.

**Theorem 5.3.** The left-symmetric algebra \( b_{1,\alpha} \) is non-degenerate in the smooth category iff \( \alpha \notin \Sigma_{sm} \).

**Proof.** First, we prove that if \( \alpha \in \Sigma_{sm} \), then the algebra \( b_{1,\alpha} \) is degenerate. We split this proof in four lemmas.

**Lemma 5.2.** For \( \alpha \in \Sigma_0 = \{0\} \) the left-symmetric algebra \( b_{1,0} \) is degenerate in both smooth and analytic categories.

**Proof.** Consider a Nijenhuis operator

\[
R = \begin{pmatrix}
y^2 & x \\
0 & y^2
\end{pmatrix}.
\]

For the linear part the function \( \det R \) is zero in the neighborhood of the coordinate origin. For the entire Nijenhuis operator the same function is not zero almost everywhere around \( P \). Thus, there is no linearizing coordinate change in both smooth and analytic categories.

**Lemma 5.3.** For \( \alpha \in \Sigma_1 = \{r| r \in \mathbb{N}, r \geq 3\} \) the left-symmetric algebra \( b_{1,\alpha} \) is degenerate in both smooth and analytic categories.

**Proof.** Consider a Nijenhuis operator

\[
R = \begin{pmatrix}
0 & x \\
x^{\alpha-1} & \alpha y
\end{pmatrix}.
\]

Again, as in the previous Lemma the determinant is not zero almost everywhere for \( R \), while it is zero for linear part. This means that there is no linearizing coordinate change in both smooth and analytic categories.
Lemma 5.4. For $\alpha \in \Sigma_2 = \{\alpha \in \mathbb{R}, \alpha < 0\}$ the left-symmetric algebra $b_{1,\alpha}$ is degenerate in the smooth category.

Proof. Fix the constant $s = -\frac{1}{\alpha} \in \mathbb{R}^+$. Define the function $f$ as follows:

$$h(x, y) = \begin{cases} \exp \left( -\frac{1}{x^2y^2s} \right) & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

Defining all the partial derivatives of $h(x, y)$ as zero on the coordinate cross $xy = 0$ we obtain a function that is smooth in the entire plane.

The partial derivatives of $h(x, y)$ satisfy the following identities:

$$\frac{\partial h}{\partial y}(x, y) = \frac{2s}{x^2y^{2s+1}}h(x, y), \quad \frac{\partial h}{\partial x}(x, y) = \frac{2}{x^3y^{2s}}h(x, y)$$

Thus, for $\alpha < 0$ function $h(x, y)$ defines a smooth integral for linear vector field $v = (x, \alpha y)$. Consider another function:

$$g(x, y) = \begin{cases} h(x, y) \frac{\partial h}{\partial y}(x, y) & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

This function is also smooth and satisfies another identity

$$\frac{\partial h}{\partial x} g = h \frac{\partial h}{\partial y} \quad (22)$$

Take an operator field

$$R = \begin{pmatrix} h(x, y) & x + g(x, y) \\ 0 & \alpha y \end{pmatrix}. \quad (23)$$

From (22) and (5) it follows that this operator is Nijenhuis. The linear part $R_1$ at $P$ is a linear Nijenhuis operator corresponding to $b_{1,\alpha}$.

We have that the function $\det R_1 \equiv 0$ and $\det R = \alpha y h(x, y) \neq 0$ around the coordinate origin. Thus, there is no linearizing coordinate change in smooth case. ■

Lemma 5.5. For $\alpha \in \Sigma_3 = \{\frac{1}{r} | r \in \mathbb{N}, r \geq 2\}$ the left-symmetric algebra $b_{1,\alpha}$ is degenerate in both smooth and analytic categories.

Proof. Consider the operator field

$$R = \begin{pmatrix} 0 & x + \alpha y \frac{1}{r} \\ 0 & \alpha y \end{pmatrix}. \quad (23)$$
It is Nijenhuis with the linear part being $b_{1,\alpha}$.

Suppose, that there exists a linearizing coordinate change $\tilde{x} = f(x, y), \tilde{y} = g(x, y)$ for $R$. $\tr R = \alpha y$ in the old coordinates and $\tr R = \alpha \tilde{y}$ in the new coordinates. Thus, the coordinate change has the form $\tilde{x} = f(x, y), \tilde{y} = y$.

The matrix of an operator is transformed as follows

$$
\begin{pmatrix}
  f_x' & f_y' \\
  0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
  0 & x + \alpha y^{1\alpha} \\
  0 & \alpha y \\
\end{pmatrix}
\begin{pmatrix}
  1 \\
  0 \\
\end{pmatrix}
\begin{pmatrix}
  \frac{f_x}{y} \\
  \frac{f_y}{y} \\
\end{pmatrix}
= 
\begin{pmatrix}
  0 & x \\
  0 & \alpha y \\
\end{pmatrix}.
$$

We get that the coordinate change $\tilde{x} = f(x, y), \tilde{y} = y$ linearizes the vector field $(x + \alpha y^{\frac{1}{\alpha}}, \alpha y)$. At the same time from Table 5 it follows that this vector field is not equivalent to its linear part even in formal category. This contradiction completes the proof. ■

Now we prove that if $\alpha \notin \Sigma_{sm}$, then $b_{1,\alpha}$ is non-degenerate in smooth category. Again we start with several lemmas.

**Lemma 5.6.** Consider $h(t)$ to be a smooth (analytic) function that satisfies differential equation

$$f(t) \cdot \dot{h} - h = 0. \quad (25)$$

Suppose that $f$ is smooth (analytic) around $t = 0$ and $f(0) = 0, \dot{f}(0) = \beta \neq 0$. If $\beta \neq \frac{1}{r}$ for some $r \in \mathbb{N}$, then $h(t) \equiv 0$. In other words, there are no non-zero smooth (analytic) solutions.

**Proof.** As $\beta \neq 0$, then the function $f$ is monotonic. In particular, on $U = (-\epsilon, \epsilon)$ $f(t) = 0$ only for $t = 0$. Denote $U^+ = (0, \epsilon)$ and $U^- = (-\epsilon, 0)$.

Similar to (20) we obtain that $f$ can be written as $f(t) = \beta t + t^2 g(t)$ for smooth (analytic) $g(t)$. We write a solution for equation (25) in the general form for $t \neq 0$:

$$h(t, c) = c \exp\left(\int \frac{1}{f(x)} \, dx\right) = 
= c \exp\left(\int \frac{1}{\beta x + x^2 g(x)} \, dx\right) = 
= c \exp\left(\frac{1}{\beta} \int \frac{1}{x} - \frac{\frac{1}{\beta} g(x)}{1 + x^{\frac{1}{\beta}} g(x)} \, dx\right) = 
= c t^{\frac{1}{\beta}} \exp\left(-\frac{1}{\beta} \int \frac{\frac{1}{\beta} g(x)}{1 + t^{\frac{1}{\beta}} g(x)} \, dx\right) = c t^{\frac{1}{\beta}} F(t). \quad (26)$$

Note, that $F(t)$ does not depend on the constant $c$ and is smooth (analytic) on the entire interval $U$. 

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Consider $h(t)$ to be smooth (analytic) solution of equation (25) on the entire interval $(−ε, ε)$. By the Uniqueness Theorem for the ODE on the intervals $U^+$ and $U^-$ this $h(t)$ coincides with $h(t, c^+)$ and $h(t, c^-)$ respectively.

Suppose that $\frac{1}{\beta} \notin \mathbb{N}$ and $h(t)$ is not zero. Then, without loss of the generality, we may assume, that $c^+ \neq 0$. In this case either $\lim_{t \to 0^+} h(t, c^+) = \infty$ or the limit of the some derivative of $h(t, c^+)$ is $\infty$. This contradiction completes the proof. ■

Lemma 5.7. Consider a smooth (analytic) function $D(x, y)$ with $D(0, 0) = D_x'(0, 0) = D_y'(0, 0) = D_{xx}''(0, 0) = D_{yy}''(0, 0) = 0$. For every $\alpha \neq 0$ there exists smooth (analytic) functions $h$ and $g$ of two and one variable respectively, such that $D$ is written as follows:

$$D = \frac{\alpha^2}{4} y^2 - h^2(x, y) + g(x).$$

Proof. Consider a function $\hat{D} = D - \frac{\alpha^2}{4} y^2$. It has a non-trivial quadratic part $D_2 = -\frac{\alpha^2}{4} y^2$ and $D_{yy}''(0, 0) = -\frac{\alpha^2}{2} < 0$.

Applying Lemma 5.7 to $\hat{D}$ (with $x$ and $y$ interchanged), we get that there exists a coordinate change $\tilde{x} = x, \tilde{y} = h(x, y)$ such that $\hat{D} = -(\tilde{y})^2 + g(x) = -h^2(x, y) + g(x)$. Thus, $D = \frac{\alpha^2}{4} y^2 - h^2(x, y) + g(x)$. The Lemma is proved. ■

Proposition 5.4. Consider a Nijenhuis operator with zero at the coordinate origin. Assume that the linear part of the operator is a linear Nijenhuis tensor corresponding to $b_1 \alpha$. If $\alpha \notin \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ then there exists a smooth (analytic) coordinate change, that transforms $R$ into the following form:

$$R = \begin{pmatrix} 0 & f(x, y) \\ 0 & \alpha y \end{pmatrix}.$$  

In particular, this means that one of the eigenvalues of $R$ is locally constant and equals zero around $P$.

Proof. Suppose that we have $T = \alpha y$. The coordinate origin is a critical point for the determinant $D = \det R$, and $D$ has no quadratic part. Applying Lemma 5.7 we have $D = \frac{\alpha^2}{4} y^2 - h^2(x, y) + g(x)$.

Applying Corollary 2.3 for a given $D$ we get the equation for the entry $R_1^1$ of $R$:

$$- \frac{4}{\alpha^2} (hh_y')^2 + \frac{1}{\alpha} (g' - 2hh_x')R_1^1 + h - g = 0. \quad (27)$$

By definition of $h$ we have $h(0, 0) = 0$. Thus $D_{yy}''(0, 0) = 0 = \frac{\alpha^2}{2} - 2(h_y'(0, 0))^2$ and $h_y'(0, 0) = \pm \frac{\alpha}{2} \neq 0$. By the Implicit Function Theorem there exists a smooth (analytic) curve $y(x)$ passing through the origin such that $h(x, y(x)) \equiv 0$.
Substituting the curve $y(x)$ into (27), we get the following:

$$\frac{1}{\alpha} R_2^1 (x, y(x)) g' - g = 0.$$  

(28)

We get $\frac{\partial R_2^1}{\partial x}(x, y(x)) = \frac{\partial R_1^1}{\partial x} + \frac{\partial R_1^1}{\partial y} y^2$ and $\frac{\partial R_1^1}{\partial x}(0, 0) = 1$.

If $\alpha \notin \Sigma_0 \cup \Sigma_1 \cup \{1, 2\}$ then we apply Lemma 5.6 for $f = \frac{1}{\alpha} R_2^1 (x, y(x))$ and get $g \equiv 0$.

Assume now that $\alpha = 1$. In this case for $x \in (-\epsilon, 0)$ the solution for (28) has the form $g = cx F(x)$ for some constant $c$ and $F(0) \neq 0$. As $D$ has no quadratic part, we have that $0 = g'(0) = \lim_{x \to 0} (cF(x) + cx F'(x)) = cF(0)$. Thus $g \equiv 0$.

Assume now that $\alpha = 2$. In this case for $x \in (-\epsilon, 0)$ the solution for (28) has the form $g = cx^2 F(x)$ for some constant $c$ and $F(0) \neq 0$. As $D$ has no quadratic part from $D = y^2 - h^2(x, y) + g(x)$ we obtain that $D''_{yy}(0, 0) = 2 - 2(h_y'(0, 0))^2 = 0$ and $D''_{xx} = -2(h_x'(0, 0))^2 + 2cF(0) = 0$. As $D''_{xy}(0, 0) = 2h_x'(0, 0)h_y'(0, 0) = 0$ we have $c = 0$ and $g \equiv 0$.

Therefore for $\alpha \notin \Sigma_0 \cup \Sigma_1$ in the given coordinates the determinant has the form $D = \alpha^2 y^2 - h^2(x, y)$. Define a smooth (analytic) function $\mu(x, y) = \frac{1}{2}(\alpha y + 2h(x, y))$. As $\frac{\partial \mu}{\partial y}(0, 0) = \alpha \neq 0$ consider the coordinate change $\tilde{x} = x, \tilde{y} = \frac{1}{\alpha} y$.

The function $\mu$ is an eigenfunction of $R$, that is $\mu$ is an eigenvalue of $R$ at every point. In [19] the following property of the arbitrary eigenfunction is proved

$$R^* d\mu = \mu d\mu.$$  

(29)

Thus, $R^* d\tilde{y} = \alpha \tilde{y} d\tilde{y}$ and in the coordinates $\tilde{x}, \tilde{y}$ the Nijenhuis operator $R$ has the following form:

$$R = \begin{pmatrix} R_1^1 & R_2^1 \\ 0 & \alpha \tilde{y} \end{pmatrix}.$$  



In this form the property $N_R = 0$ is written as follows

$$R_2^1 \frac{\partial R_1^1}{\partial \tilde{x}} + (\alpha \tilde{y} - R_1^1) \frac{\partial R_2^1}{\partial \tilde{y}} = 0.$$  

(30)

Consider a vector field $v(\tilde{x}, \tilde{y}) = (R_2^1, \alpha \tilde{y} - R_1^1)$. The point $P = (0, 0)$ is critical for $v$ and its linear part $v_1 = (\tilde{x}, \alpha \tilde{y})$. Equation (30) has the form $v(R_1^1) = 0$. That is $R_1^1$ is the first integral of the vector field $v$. As $\alpha \notin \Sigma_0 \cup \Sigma_3$, then $P$ is a node and the only first integral is a constant function. Thus $R_1^1 = 0$ and the Proposition is proved. ■

**Lemma 5.8.** Consider a smooth (analytic) function $f(x, y)$ with linear part $x$ at the coordinate origin and a Nijenhuis operator

$$R = \begin{pmatrix} 0 & f(x, y) \\ 0 & \alpha y \end{pmatrix}.$$  

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There exists a smooth (analytic) linearizing coordinate change for a vector field \( v = (f(x,y), \alpha y) \), if and only if there exists a smooth (analytic) linearizing coordinate change for the Nijenhuis operator \( R \).

**Proof.** Assume that there exits a linearizing smooth (analytic) coordinate change \( \tilde{x} = g(x,y), \tilde{y} = h(x,y) \) for the vector field \( v \). The smooth (analytic) functions \( g \) and \( h \) satisfy the following identities \( v(g) = g \) and \( v(h) = \alpha h \). This gives the following identities for the differentials \( dg \) and \( dh \):

\[
(dg)_i = (d(v(g)))_i = \frac{\partial v^s}{\partial x^i},
\]

\[
\frac{\partial g}{\partial x^i} = \frac{\partial^2 g}{\partial x^i \partial x^j}.
\]

(31)

Let us denote by \( J \) the matrix \( \frac{\partial v^s}{\partial x^i} \) at the origin \( P \):

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}.
\]

Recall that the coordinate origin is critical for \( v \), that is \( v(P) = 0 \). Thus, from (31) we get \( J^*d g(P) = dg(P) \) and \( J^*d h(P) = \alpha d h(P) \).

As \( \tilde{x} = g(x,y), \tilde{y} = h(x,y) \) define a coordinate change, then \( d g(P), d h(P) \neq 0 \). In a similar way \( v(y) = \alpha y \) and \( J^*d y = \alpha d y \).

If \( \alpha \neq 1 \) then \( J \) has two different eigenvalues. We have established that covector \( d g(P) \) is an eigenvector of \( J^* \) with eigenvalue \( \lambda = 1 \), that is \((\frac{\partial v^s}{\partial x^i}, \frac{\partial v^s}{\partial y^j}) = J^*d g(P) = (\frac{\partial g}{\partial x^i}(P), \alpha \frac{\partial g}{\partial y^j}(P)) \). Thus, we have \( \frac{\partial g}{\partial x^i}(P) \neq 0, \frac{\partial g}{\partial y^j}(P) = 0 \) and in some neighborhood of \( P \) \( \tilde{x} = g(x,y), \tilde{y} = y \) defines a smooth (analytic) coordinate change.

Under this coordinate change, the operator \( R \) is transformed in the following way

\[
\begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f(x,y) \\ \alpha y \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha} & \frac{\partial g}{\partial x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \alpha y \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \alpha \tilde{y} \end{pmatrix},
\]

as \( \frac{\partial g}{\partial x} f(x,y) + \alpha \frac{\partial g}{\partial y} y = v(g) = g \).

If \( \alpha = 1 \) then eigenvalues of \( J \) coincide. This means that if \( J^*d g(P) = dg(P) \), then in general \( \frac{\partial g}{\partial x^i}(P) \) may be zero. So we have two possibilities. If \( \frac{\partial g}{\partial x^i}(P) \neq 0 \), then we take coordinate change \( \tilde{x} = g(x,y), \tilde{y} = \alpha y \).

If \( \frac{\partial g}{\partial x^i}(P) = 0 \), then by definition of \( g \) and \( h \) as a coordinate change we have \( \frac{\partial h}{\partial x^i}(P) \neq 0 \). So we take different coordinate change \( \tilde{x'} = h(x,y), \tilde{y'} = \alpha y \). Calculations similar to
the (32) show that this is a linearizing coordinate change for $R$ from the statement of the lemma.

Now, suppose that the $\tilde{x} = g(x,y), \tilde{y} = h(x,y)$ is a linearizing coordinate change for the Nijenhuis operator $R$ from the statement of the Lemma. For both $R$ and $R_1$ we have $\text{tr } R = \alpha y = \alpha \tilde{y}$. Therefore, our coordinate change has the form $\tilde{x} = g(x,y), \tilde{y} = y$.

From Formula (32), we get $\frac{\partial}{\partial x} f(x,y) + \alpha \frac{\partial}{\partial y} y = v(g) = g$. At the same time $v(y) = \alpha y$. This means that $\tilde{x}, \tilde{y}$ define the linearizing coordinate change for the vector field $v$. The Lemma is proved.

Now we are ready to prove the non-degenerate part of the Theorem 1.3. Assume that $\alpha \notin \Sigma_{sm}$. In particular, this means that $\alpha \notin \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$.

By the Proposition 5.4 there exists a smooth coordinate change that transforms $R$ into the form:

$$\begin{pmatrix} 0 & f(x,y) \\ 0 & \alpha y \end{pmatrix}.$$ 

From Corollary 5.1 it follows that for $\alpha \notin \Sigma_{sm} \cup \{2\}$ for a vector field $v = (f(x,y), \alpha y)$ there exists a linearizing coordinate change. Thus, by Lemma 5.8 the linearizing coordinate change exists for $R$.

Now consider the case $\alpha = 2$. We need the following lemma.

**Lemma 5.9.** Consider a smooth vector field $v = (f(x,y), 2y)$ with critical point at the coordinate origin $P$ and a linear part $v_1 = (x, 2y)$. Then there exists a smooth function $g$ such that $v(g) = g$ and $dg(P) \neq 0$.

**Proof.** From Theorem 5.1 it follows that in general there are three possible formal normal forms: $(x, 2y + x^2), (x, 2y - x^2)$ and $(x, 2y)$.

From Theorem 5.2 it follows that for every formal normal form there exists a corresponding smooth coordinate change $\tilde{x} = g(x,y), \tilde{y} = h(x,y)$. Note, that in all three cases $v(g) = g$. The Lemma is proved.

Now we have that there exists a smooth function $g$ such that $v(g) = g$. By the definition of $v$ we have $v(y) = 2y$. From identities (31) we get that

$$\begin{pmatrix} \frac{\partial g}{\partial x}(P) \\ \frac{\partial g}{\partial y}(P) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x}(P) \\ \frac{\partial f}{\partial y}(P) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(P) \\ 2 \frac{\partial f}{\partial y}(P) \end{pmatrix}.$$ 

This means, that $\frac{\partial g}{\partial x}(P) \neq 0$ and $\tilde{x} = g(x,y), \tilde{y} = y$ defines a smooth coordinate change in some neighborhood of $P$. From Formula (32) we get that this coordinate change is a linearizing coordinate change and transforms $R$ into:

$$\begin{pmatrix} 0 & \tilde{x} \\ 0 & 2 \tilde{y} \end{pmatrix}.$$
The Theorem 5.3 is proved. ■

Now we consider the linearization problem for $b_1, \alpha$ in the analytic category. We will need some results about linearization problem in analytic category. The following theorem is a corollary of the classical Poincare-Dulac theorem ([18], Theorem 5.5)

**Theorem 5.4.** Consider a vector field $v$ on the plane $\mathbb{R}^2$ with critical point at the origin. Assume, that both eigenvalues of the linearization matrix are real and non-zero. In addition assume that they have the same sign. Denote by $v_{\text{formal}}$ the formal normal form of $v$ from Table 5. Then there exists an analytic coordinate change, that transforms $v$ into $v_{\text{formal}}$.

Let us recall that $[q_0, q_1, q_2, ...]$ denotes a decomposition of an irrational $\alpha$ into a continuous fraction. If the series

$$B(x) = \sum_{i=1}^{\infty} \frac{\log q_{n+1}}{q_n}$$

converges, then $\alpha$ is a Brjuno number and $\Omega$ is the set of negative Brjuno numbers. The next result is a corollary of the main theorem by Brjuno [16].

**Theorem 5.5.** Consider a vector field $v$ on the plane $\mathbb{R}^2$ with critical point at the origin and linear part $v_1 = (x, \alpha y)$. If $\alpha \notin \Sigma_{an} \cup \Sigma_u \cup \{2\}$, then there exists an analytic coordinate change that transforms $v$ into $v_1$.

Recall the following subsets of $\mathbb{R}$:

1. $\hat{\Sigma}_2 = \{-\frac{p}{q} | p, q \in \mathbb{N}\}$
2. $\Sigma_{an} = \Sigma_0 \cup \Sigma_1 \cup \hat{\Sigma}_2 \cup \Sigma_3$
3. $\Sigma_u = \{\alpha < 0, \alpha \notin \mathbb{Q}, \alpha \notin \Omega\}$

Theorems 5.4 and 5.5 yield the following Corollary.

**Corollary 5.2.** Consider a vector field $v = (f(x, y), \alpha y)$ on the plane $\mathbb{R}^2$ with critical point at the origin and linear part $v_1 = (x, \alpha y)$. Assume that $\alpha \notin \Sigma_{an} \cup \Sigma_u \cup \{2\}$. Then there exists an analytic linearizing coordinate change for $v$, that is an analytic coordinate change that transforms $v$ into $v_1$.

**Proof.** If $\alpha \notin \Sigma_{an} \cup \Sigma_u \cup \{2\}$, then by definition either $\alpha > 0$ and $\alpha \neq n, \frac{1}{n}$ for $n \geq 2, n \in \mathbb{N}$ or $\alpha \in \Omega$. 

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If $\alpha > 0$ then in Table 5 the corresponding formal normal forms of $v$ are linear, that is $v_1$. By Theorem 5.4 there exists an analytic coordinate change that transforms $v$ into $v_1$.

If $\alpha < 0$, then it is in $\Omega$. By Theorem 5.5 there exists an analytic linearizing coordinate change for $v$, that is an analytic coordinate change that transforms $v$ into $v_1$. ■

Now we are ready for the following theorem.

**Theorem 5.6.** The algebra $b_{1,\alpha}$ is degenerate in analytic category if $\alpha \in \Sigma_{an}$ and non-degenerate in analytic category if $\alpha \notin \Sigma_{an} \cup \Sigma_u$.

**Proof.** Lemmas 5.2, 5.3, 5.5 are true in both smooth and analytic categories. Thus, if $\alpha \in \Sigma_0 \cup \Sigma_1 \cup \Sigma_3$ then the algebra $b_{1,\alpha}$ is degenerate.

Now, consider a vector field $v = (x + x^p y^q, -\frac{p}{q} y)$. From Theorem 5.1 it follows, that there are no formal (and, therefore, analytic) linearizing coordinate change. From Lemma 5.8 it follows that there are no analytic linearizing coordinate change for a Nijenhuis operator

$$R = \begin{pmatrix} 0 & x + x^p y^q \\ 0 & -\frac{p}{q} y \end{pmatrix}.$$  

Thus, the algebra $b_{1,\alpha}$ is degenerate if $\alpha \in \Sigma_2 = \{-\frac{p}{q} | p, q \in \mathbb{N}\}$.

Therefore, we have shown, that $b_{1,\alpha}$ is degenerate if $\alpha \in \Sigma_{an} = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$.

**Proposition 5.5.** Consider a Nijenhuis operator with zero at the coordinate origin. Assume that the linear part of the operator is a linear Nijenhuis tensor corresponding to $b_{1,\alpha}$. If $\alpha \notin \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ then there exists an analytic coordinate change such that $R$ has the following form:

$$R = \begin{pmatrix} 0 & f(x, y) \\ 0 & \alpha y \end{pmatrix}.$$  

**Proof.** Suppose that we have $T = \alpha y$. The coordinate origin is a critical point for determinant $D = \det R$, and $D$ has no quadratic part. Applying Lemma 5.7 we have $D = \frac{4}{\alpha^2} y^2 - h^2(x, y) + g(x)$, where $h(x, y)$ and $g(x)$ are analytic functions.

Applying Corollary 2.3 for a given $D$ we get the following equation on $R^2$:

$$-\frac{4}{\alpha^2} (hh_y')^2 + \frac{1}{\alpha} (g' - 2hh_x')R^2_1 + h - g = 0. \quad (33)$$

By definition of $h$ we have $h(0, 0) = 0$. Thus $D^\alpha_{yy}(0, 0) = 0 = \frac{4}{\alpha^2} - 2(h_y'(0, 0))^2$ and $h_y'(0, 0) = \pm \frac{\alpha}{2} \neq 0$. By the Implicit Function Theorem we have that there exists an analytic curve $y(x)$ passing through the origin such that $h(x, y(x)) = 0$.  

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Substituting the curve \( y(x) \) into \( (33) \), we get the following:

\[
\frac{1}{\alpha} R_1^1(x, y(x)) g' - g = 0. \tag{34}
\]

We get \( \frac{\partial R_1^1}{\partial x}(x, y(x)) = \frac{\partial R_1^1}{\partial x} + \bar{y} \frac{\partial R_1^1}{\partial y} \) and \( \frac{\partial R_1^1}{\partial x}(0, 0) = 1. \)

If \( \alpha \notin \Sigma_0 \cup \Sigma_1 \cup \{1, 2\} \) then we apply Lemma 5.6 for \( f = \frac{1}{\alpha} R_2^1(x, y(x)) \) and get that \( g = 0. \)

Now we are ready to prove the non-degenerate part of the Theorem 1.4. Assume now that \( \alpha = 1. \) In this case for \( x \in (-\epsilon, 0) \) the solution of \( (34) \) has the form \( g = c x F(x) \) for some constant \( c \) and \( F(0) \neq 0. \) As \( D \) has no quadratic part, we have \( 0 = g'(0) = \lim_{x \to 0} (c F(x) + c x F'(x)) = c F(0). \) Thus, \( g \equiv 0. \)

Assume now that \( \alpha = 2. \) In this case for \( x \in (-\epsilon, 0) \) the solution of \( (34) \) has the form \( g = c x^2 F(x) \) for some constant \( c \) and \( F(0) \neq 0. \) As \( D \) has no quadratic part from \( D = y^2 - h^2(x, y) + g(x) \) we obtain that \( D''_{yy}(0, 0) = 2 - 2(h_y'(0, 0))^2 = 0 \) and \( D''_{xx} = -2(h'_x(0, 0))^2 + 2c F(0) = 0. \) As \( D''_{xy}(0, 0) = 2h_x'(0, 0)h'_y(0, 0) = 0 \) we have that \( c = 0 \) and \( g \equiv 0. \)

Therefore for \( \alpha \notin \Sigma_0 \cup \Sigma_1 \) in the given coordinates the determinant has the form \( D = \alpha^2 y^2 - h^2(x, y). \) Define a analytic function \( \mu(x, y) = \frac{1}{2}(\alpha y + 2h(x, y)) \). As \( \frac{\partial \mu}{\partial y}(0, 0) = \alpha \neq 0 \) consider the analytic coordinate change \( \bar{x} = x, \bar{y} = \frac{1}{\alpha} \mu. \)

The function \( \mu \) is an eigenfunction of \( R \), that is at every point around \( P \) it is an eigenvalue of \( R \) at the same point. By Property (29) we have \( R^* d\bar{y} = \alpha \bar{y} d\bar{y} \) and in the coordinates \( \bar{x}, \bar{y} \) the Nijenhuis operator \( R \) has the following form:

\[
R = \begin{pmatrix} R_1^1 & R_2^1 \\ 0 & \alpha \bar{y} \end{pmatrix}.
\]

In this form the property \( \mathcal{N}_R = 0 \) gives equation \( (30) \). It means that \( R_1^1 \) is an integral of the vector field \( v = (R_2^1, \alpha \bar{y} - R_1^1). \)

We need the followingLemma.

**Lemma 5.10.** Let \( v \) be an analytic vector field with critical point at the origin and linear part \( v_1 \). Assume, that \( v \) has an analytic first integral \( F = F_k + F_{k+1} + \ldots \) with \( k \geq 1. \) Then \( F_k \) is a polynomial first integral for \( v_1. \)

**Proof.** \( F_i \) is a homogeneous polyhomial of degree \( i \). We have that \( v_j(F_i) \) is a homogeneous polynomial of degree \( i + j - 1. \) This means that the first term of \( v(F) \) has degree \( k \) and is \( v_1(F_k). \) The lemma is proved. \( \blacksquare \)
It is easy to check, that \( v_1 = (x, \alpha y) \) has a polynomial first integral for non-zero \( \alpha \) if and only if \( \alpha \) is negative and rational, that is \( \alpha \in \widehat{\Sigma}_2 \).

Thus, by the assumptions of the Proposition 5.5, \( v = (R^1_2, \alpha \tilde{y} - R^1_1) \) has no analytic first integrals and \( R^1_1 \equiv 0 \). The Proposition is proved.

Now, assume that \( \alpha \notin \Sigma_{an} \cup \Sigma_u \). In particular, this means that \( \alpha \notin \Sigma_0 \cup \Sigma_1 \cup \widehat{\Sigma}_2 \). Thus, by Proposition 5.5 there exists an analytic coordinate change that transforms \( R \) into the form:

\[
\begin{pmatrix}
0 & f(x, y) \\
0 & \alpha y
\end{pmatrix}.
\]

For \( \alpha \notin \Sigma_{an} \cup \Sigma_u \cup \{2\} \) by Corollary 5.2 there exists an analytic linearizing coordinate change for vector field \( v = (f(x, y), \alpha y) \).

By Lemma 5.8 in this case there exists an analytic coordinate change that transforms \( R \) into

\[
\begin{pmatrix}
0 & x \\
0 & \alpha y
\end{pmatrix}.
\]

Finally consider \( \alpha = 2 \).

**Lemma 5.11.** Consider a smooth vector field \( v = (f(x, y), 2y) \) with critical point at the coordinate origin \( P \) and a linear part \( v_1 = (x, 2y) \). Then there exists a smooth function \( g \) such that \( v(g) = g \) and \( dg(P) \neq 0 \).

**Proof.** From Theorem 5.1 it follows that in general there are three possible formal normal forms: \((x, 2y + x^2), (x, 2y - x^2)\) and \((x, 2y)\).

From Theorem 5.4 it follows, that for every formal normal form there exists a corresponding analytic coordinate change \( \tilde{x} = g(x, y), \tilde{y} = h(x, y) \). Note that in all three cases \( v(g) = g \). The Lemma is proved.

Now we have that there exists an analytic function \( g \) such that \( v(g) = g \). By the definition of \( v \) we have \( v(y) = 2y \). From identities (31) we get that

\[
\begin{pmatrix}
\frac{\partial g}{\partial x}(P) \\
\frac{\partial g}{\partial y}(P)
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix} \begin{pmatrix}
\frac{\partial g}{\partial x}(P) \\
\frac{\partial g}{\partial y}(P)
\end{pmatrix} = \begin{pmatrix}
\frac{\partial g}{\partial x}(P) \\
2\frac{\partial g}{\partial y}(P)
\end{pmatrix}.
\]

This means, that \( \frac{\partial g}{\partial x}(P) \neq 0 \) and \( \tilde{x} = g(x, y), \tilde{y} = y \) define an analytic coordinate change in some neighborhood of \( P \). From formula (32) we see that this coordinate change is a linearizing coordinate change and transforms \( R \) into:

\[
\begin{pmatrix}
0 & \tilde{x} \\
0 & 2\tilde{y}
\end{pmatrix}.
\]

Theorem 5.6 is proved.
5.1.5 The algebra $b_2$

**Theorem 5.7.** The left-symmetric algebra $b_2$ is non-degenerate in both smooth and analytic category.

**Proof.** Suppose that we have a Nijenhuis operator $R$ with the singular point of scalar type at the origin with its linear part being $b_2$. Choose coordinates such that $T = y$. $D$ has a critical point at $P$ and no quadratic part. Applying Lemma 5.7, we get that without loss of generality we may assume, that $D = \frac{y^2}{4} - h^2(x,y) + g(x)$. From Corollary 2.3 we get the following equation on $h$ and $g$

\[
\left(\frac{y}{2} - 2hh_y'\right)\left(\frac{y}{2} + 2hh_y'\right) + \left(g' - 2hh_y'\right)R_2^1 - \frac{y^2}{4} + h^2 - g = \]

\[
-4(hh_y')^2 + h^2 - 2hh_y'R_2^1 + g'R_2^1 - g = 0.
\]

By definition of $h$, we have that $h_y'(0,0) = \pm \frac{1}{2}$. By the Implicit Function Theorem we can define smooth (analytic) curve $y(x)$ such that $h(x, y(x)) = 0$. We have that $\frac{\partial R_2^1}{\partial x} = \frac{\partial R_3^1}{\partial x} + \tilde{y} \frac{\partial R_3^1}{\partial y}$ and thus $\frac{\partial R_2^1}{\partial x}(0,0) = \frac{\partial R_3^1}{\partial x}(0,0) = 1$ (the linear part of $R_2^1$ is $x + y$).

Substituting $y(x)$ into the (35) we get

\[
R_2^1(x, y(x))g'(x) - g(x) = 0.
\]

Function $R_2^1(x, y(x))$ can be written as $R_2^1 = x(1 + f(x))$ for some smooth (analytic) function $f(x)$. By definition $f(0) = 0$. Solving the equation, we get

\[
g(x) = \exp\left(\int \frac{1}{x(1+f(x))} \, dx\right) = \exp\left(\int \left(\frac{1}{x} - \frac{f}{1+f}\right) \, dx\right) = cxF(x),
\]

where $F(x) = \exp\left(-\int \frac{f}{1+f} \, dx\right)$ is smooth (analytic) function, $F(0) \neq 0$ and $c$ is constant. We have $\lim_{x \to 0} g'(x) = c$. From Lemma 5.7 it follows, that $g'(0) = 0$. Thus $c = 0$ and $g(x) \equiv 0$.

Thus in the given coordinates the determinant has the form $D = \frac{1}{4}y^2 - h^2(x,y)$. Define a smooth (analytic) function $\mu(x,y) = \frac{1}{2}(y + 2h(x,y))$. As $\frac{\partial \mu}{\partial y}(0,0) = 1 \neq 0$ consider smooth (analytic) coordinate change $\tilde{x} = x, \tilde{y} = \frac{1}{\alpha} \mu$.

From (29) we get that in these coordinates $R$ is written in the form:

\[
R = \begin{pmatrix} R_1^1 & R_2^1 \\ 0 & \tilde{y} \end{pmatrix}.
\]

The property $\mathcal{N}_R = 0$ gives an equation:

\[
R_2^1 \frac{\partial R_1^1}{\partial \tilde{x}} + (\tilde{y} - R_1^1) \frac{\partial R_1^1}{\partial \tilde{y}} = 0.
\]
Define in coordinates \( \tilde{x}, \tilde{y} \) vector field \( v = (R_2^1, \tilde{y} - R_1^1) \). Then the property \( N_R = 0 \) can be written as \( v(R_1^1) = 0 \) that is \( R_1^1 \) is a first integral of this vector field.

**Lemma 5.12.** Consider vector field \( v = (x + y, y) \). Assume that smooth (analytic) function \( h \) satisfies equation \( v(h) = h \), then \( h = ay + b \) for some constants \( a, b \).

**Proof.** We have

\[
h'_x(x + y) + h'_y y = h.
\]

Differentiate both sides by \( x \), we obtain the equation \( h''_{xx}(x + y) + h''_{xy} y = v(h'_x) = 0 \). The critical point \((0, 0)\) for \( v \) is a node, so there are no first integrals. In particular, \( h'' = k = \text{const} \).

Thus \( h(x, y) = kx + r(y) \). Rewriting \( v(h) = h \) we get \( ky + r'y = r \). Differentiating it by \( y \), we get \( k + r''y \equiv 0 \). For \( y = 0 \) we get, that \( k = 0 \) and \( r''y \equiv 0 \). Therefore, \( h = ay + v. \)

\( v \) has a critical point at the origin and its linear part is \( v_1 = (\tilde{x} + \tilde{y}, \tilde{y}) \). From Table 5 it follows that it is equivalent to its linear part in both smooth and analytic category. It is node thus there is no analytic or smooth first integral, therefore \( R_1^1 = 0 \).

We have that \( v = (R_2^1, \tilde{y}) \). Suppose, that \( \tilde{x} = g(\tilde{x}, \tilde{y}), \tilde{x} = h(\tilde{x}, \tilde{y}) \) is a linearizing coordinate change for \( v \). This means that there exists a pair of smooth (analytic) functions such that \( v(g) = g + h, v(h) = h \). From Lemma [5.12] we get that in this coordinate change without loss of generality we may assume, that \( \tilde{y} = \tilde{y} \).

The coordinate change \( \tilde{y} = h(\tilde{x}, \tilde{y}), \tilde{y} = \tilde{y} \) yields the linear Nijenhuis operator. This completes the proof of Theorems [1.3] and [1.4] ■

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