Quasiclassical path-integral approach to quantum mechanics associated with a semisimple Lee algebra

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Abstract

A closed (in terms of classical data) expression for a transition amplitude between two generalized coherent states associated with a semisimple Lee algebra underlying the system is derived for large values of the representation highest weight, which corresponds to the quasiclassical approximation. Consideration is based upon a path-integral formalism adjusted to quantization of symplectic coherent-state manifolds that appear as one-rank coadjoint orbits.

I. Introduction

As is known, quasiclassical quantization of a classical mechanics on a flat configuration space in terms of a Feynman path integral results in a famous van Vleck-Morette formula for a quantum-mechanical transition amplitude (quantum propagator) which follows from an appropriate expansion of an action about classical solutions up to terms coming from the gaussian fluctuations and occurs in powers of the Planck constant $\hbar$, with a classical limit corresponding to $\hbar \to 0$. In a curved configuration space the van Vleck-Morette propagator is generalized to the DeWitt representation, see also Kleinert, and is defined entirely in terms of the classical action, trajectories and metric.

On the other hand, a number of physically relevant models admit a natural representation on phase spaces that appear as orbits of Lie groups in the coadjoint representations (coadjoint orbits). This happens whenever a Hamiltonian can be written in terms of generators of the corresponding Lie algebra, the Hamiltonian is then said to be associated with this algebra. Examples are obvious: spin systems with Hamiltonians being bilinear combinations of the $SU(2)$ generators, models in non-linear quantum optics with the relevant $SU(1,1)$ dynamical symmetry, strongly correlated electron systems (the $t-J$ model, the periodical Anderson model, etc.) with the underlying $SU(2|1)$ supersymmetry. In fact, any homogeneous symplectic manifold that admits a connected group of isometries $G$ is locally homeomorphic to a certain coadjoint orbit of $G$. In view of this, an appropriate quantization of coadjoint orbits of Lie (super)groups seems to provide an adequate basis to treat associated quantum systems.

Path integrals in quantum mechanics fall into two general categories: phase space path integrals and configuration space ones. The most important distinction between them is that a time-lattice image of the phase space integral involves canonical Liouville measure, which makes it possible to employ powerful methods of Hamiltonian (symplectic) geometry. Generic feature of the phase space path-integral representation of a transition amplitude is that it implies that the corresponding initial and final states are to be taken in different polarizations. Phase space path integrals including those on coadjoint group orbits may in turn be divided into two groups depending on whether they are written in real or complex polarizations. A certain polarization of a phase space $M$ is required to be
fixed in order to single out, of an originally $G$-reducible Hilbert space of quantum states, an irreducible component $\mathcal{H}(M)$ which can be identified with a physical state space $\mathcal{H}$.

Path-integral quantization of a group action on the coadjoint orbit in a real polarization has been considered in Refs. [4]. This approach incorporates Kirillov’s two-form $w$ in canonical (action-angle) variables and expresses the purely geometric part of an action in terms of symplectic potential $\theta$, so that locally $w = d\theta$. In practice it is more convenient however to deal with the complex (Kähler) polarization corresponding to the coherent-state representation. Coherent states are particularly convenient since they are in one-to-one correspondence with points of an orbit of the coadjoint representation. Furthermore, they are the coherent states in the Kähler polarization which possess holomorphic properties.

As is known, coherent states form an overcomplete basis in $\mathcal{H}(M)$ whenever $G$ possesses unitary irreducible square integrable representations $U(G)$. It then follows that the corresponding coherent-state manifold $M$ can be viewed as a complex Kähler manifold, that is, its metric, symplectic two-form, etc. can be expressed through a single function on $M$, a Kähler potential $F$. It is important that due to Berezin [4] and Onofri [7] the Kähler potential can be written down explicitly in terms of coherent states. As a result, the path integral is entirely determined by the Kähler potential ($\theta$ that affects a purely geometric part of an action) and a classical Hamiltonian on $M$ (that determines dynamics).

Quantization of a group action on an orbit implies also that the dominant (highest) weight of the corresponding representation comes to play the role of the Planck constant $\hbar$, the large values of the dominant weight corresponding to classical limit. For a spin system with the total spin $s$ quasiclassics occurs provided inequality $2s \gg 1$ holds, whereas in the case of $SU(1,1)$ classical limit corresponds to large values of occupation numbers [9]. In this regard, quasiclassical quantization by the coherent-state path integral may provide some nontrivial asymptotic representations of physical relevance. A conventional configuration space path integral could hardly be used for this purpose, since certain constraints are to be additionally imposed to fix a representation, which usually results in a severe technical problem.

The purpose of the present paper is to derive a closed representation of the coherent-state propagator (matrix element of an evolution operator) for large values of the dominant weight $l$ that specifies the representation $(\mathcal{H}(M), U(G))$. To this end, the corresponding quantization by the coherent-state path integral turns out to be convenient since it properly takes into account the underlying symmetry. The total action turns out to be linear in $l$, which makes it quite reasonable to apply the stationary-phase approximation in evaluating the integral as $l \to \infty$.

As we are interested in the path-integral representation for a transition amplitude, we should care about the so-called boundary term in the continuum action, the latter disappearing in the path integral for a partition function. This term is also expressible via the Kähler potential as has been shown for ordinary coherent states [9] and supercoherent states [10] and is of crucial importance in deriving a correct quasiclassical ($l \to \infty$) coherent-state propagator. Otherwise some difficulties arise, e.g., the so-called overspecification problem [11] and contradictions with the path-integral generalization of the Duistermaat-Heckman (DH) theorem appear. This was just the case in some earlier attempts to derive the quasiclassical $SU(2)$ propagator [12], whereas the correct expression for the $SU(2)$ case has recently been obtained [13].

In the present paper we expound an analogous derivation for a real semisimple Lie group $G$ possessing square integrable representations. As is known, both compact and noncompact groups with the discrete series representations fall into this category, the results we obtain are also equally well applied to the widely used Heisenberg-Weyl coherent states parametrized by points of a complex plane $C$.

The plan of the paper is as follows. In Section II, we present a general coherent-state path integral for a transition amplitude between two coherent states. Section III comprises some preliminaries and a brief review of earlier results on the quasiclassical evaluation of quantum-mechanical propagators. Section IV constitutes the main result, a derivation of the closed quasiclassical formula for the coherent-
state propagator. Relations between various quantization schemes are discussed in Section V. Few examples are gathered in Section VI to illustrate possible applications of the general result. A short summary concludes the paper in Section VII.

II. Coherent-state path integral

As is known, Perelomov’s coherent states for a semisimple group $G$ are points of an orbit of a unitary irreducible representation of $G$ in an abstract Hilbert space $\mathcal{H}$. By choosing an initial state $|0\rangle$ in $\mathcal{H}$, called the fiducial state the vectors of the corresponding $G$ orbit are parametrized by points of a homogeneous space $M = G/G_0$, where $G_0$ is the isotropy subgroup of $|0\rangle$. In the following we will be interested in the case where $|0\rangle$ appears as a dominant weight vector (highest weight vector up to the Weyl transformation), which corresponds to the quantization in the Kähler (holomorphic) polarization. It then follows that a factor space $G/G_0$ appears as a Kähler manifold, the Kähler potential being directly expressible in terms of the coherent states as follows. Given a coherent state $|z\rangle$ where $z$ belongs to $G/G_0$, we define (locally)

$$F(z_1, z_2) = \log \frac{\langle z_1 | z_2 \rangle}{\langle z_1 | 0 \rangle \langle 0 | z_2 \rangle}$$

which can be viewed as an analytic continuation of the real valued function

$$F(z, z) = \log |\langle 0 | z \rangle|^2.$$  

The latter is called the Kähler potential and was introduced in this way by Berezin and Onofri. This function incorporates geometry of the underlying phase space and plays a crucial role in the following.

The phase space $G/G_0$ can be equipped with an invariant supersymplectic 2-form $w$,

$$w = -i\delta \delta F(z, z),$$

where $\delta = dz \otimes \partial/\partial z$ and $\delta = d\bar{z} \otimes \partial/\partial \bar{z}$ such that the exterior derivative $d = \delta + \delta$. A straightforward calculation shows that $w$ is closed, i.e., $dw = 0$, which means that $G/G_0$ is a symplectic manifold, in other words, it may serve as a classical phase space. In terms of $F$, metric and $G$-invariant Liouville measure read

$$ds^2 = g\ dz d\bar{z} = \partial^2_{zz} F\ dz d\bar{z}$$

$$d\mu \sim g\ \frac{dz d\bar{z}}{2\pi i},$$

which yields

$$\int |z\rangle \langle z| d\mu = I,$$

Resolution of unity enters into a path integral as a basic ingredient.

Consider the quantum propagator in the $z$-representation

$$\langle z_F| T \exp -\frac{i}{\hbar} \int_0^\tau H(s) ds |z_I\rangle \equiv \mathcal{P}(z_F, z_I; \tau),$$

which represents Berezin’s covariant symbol of the evolution operator, the Hamiltonian $H(s)$ being a polynomial function of the $G$ generators with time-dependent coefficients. In Eq. $T$ denotes the
time-ordering symbol. It is necessary to include, because the operator $H(s)$ in the exponent does not commute for different values of $s$.

In order to express the transition amplitude by a path integral, we divide the time interval into $N$ small intervals: $\epsilon = \tau/N$ with $N \to \infty$. Let us define

$$s_k = \epsilon k, \quad z_k = z(s_k), \quad 0 \leq k \leq N.$$  

With the aid of the time discretization together with relation (6) the propagator can be written up to first order in $\epsilon$ in the form

$$P = \lim_{N \to \infty} \int_{z_0 = z_I} \prod_{k=1}^{N-1} d\mu_k \langle z_F | z_{N-1} \rangle \langle z_1 | z_I \rangle 
\times \prod_{k=2}^{N-1} \langle z_k | z_{k-1} \rangle \exp \left\{-i \epsilon \sum_{k=1}^{N} H^{cl}(\bar{z}_k, z_{k-1}) \right\},$$  

where

$$H^{cl}(\bar{z}_k, z_{k-1}; s_k) = \frac{\langle z_k | H(s_k) | z_{k-1} \rangle}{\langle z_k | z_{k-1} \rangle}.$$  

The variables $z_N$ and $\bar{z}_0$ do not enter into Eq. (8) at all. In other words, the Euler-Lagrange equations are accompanied by the boundary conditions $z_0 = z_I$ and $\bar{z}_N = \bar{z}_F$, respectively. The term

$$\langle z_F | z_{N-1} \rangle \langle z_1 | z_I \rangle$$

gives rise to the continuum boundary term to be discussed below.

Before proceeding further, the following remark on representation (8) applies. As is known, if a phase space is compact, it cannot be covered by a single chart. On the other hand, every integral in Eq. (8) is written in one local chart. The way out is that the phase space $M$ is $G$-homogeneous (the group $G$ acts on $M$ through holomorphic isometries), so that a full set of local charts is generated by actions of $G$: any two charts are locally related by $z \to gz$, for some $g \in G$. Strictly speaking, any (ordinary) integral in Eq. (8) should have been written by the definition

$$\int_M d\mu (\cdots) = \sum_i \int_{V_i} d\mu (\cdots) \epsilon_i,$$  

where $\{\epsilon_i\}$ is a partition of unity subordinate to the covering $\{V_i\}$. Since each local chart covers $M$ except for a set of measure 0 (with respect to $d\mu$), one may, by employing appropriate $G$-shifts of variables on the lhs of (8) and making use of the $\epsilon$-defining equation $\sum_i \epsilon_i = 1$, restrict the integration to the single local chart $\approx C$.

In the continuum limit Eq. (8) takes on the form

$$P = \int_{\bar{z}(\tau) = \bar{z}_F} D\mu(z) \exp \Phi.$$  

The total action $\Phi$ includes the boundary term $\Gamma$:

$$\Phi = S + \Gamma,$$  

where

$$S = -\frac{1}{2} \int_0^\tau \left( \dot{z} \frac{\partial F}{\partial z} - \dot{\bar{z}} \frac{\partial F}{\partial \bar{z}} \right) ds - \frac{i}{\hbar} \int_0^\tau H^{cl}(\bar{z}, z) ds,$$  

(11)
\[ \Gamma = \frac{1}{2} [F(\bar{z}_F, z(\tau)) + F(\bar{z}(0), z_I) - F(\bar{z}_F, z_F) - F(\bar{z}_I, z_I)]. \]  

These equations coincide up to an obvious change in the notation with those of Ref. [3].

Continuum representation (12) originates from a specific discontinuity of paths \( z(s) \) and \( \bar{z}(s) \) at the relevant end points. For example, let us introduce \( \Delta_k(\epsilon) \equiv z_k - z_{k-1} \). Then one has \( \lim_{\epsilon \to 0} \Delta_k(\epsilon) = 0 \) for all \( k \), except that \( \lim_{\epsilon \to 0} \Delta_N(\epsilon) \neq 0 \), since \( z_N = \bar{z}_F \) and \( z_F \) is an arbitrary complex number. However, instead of explicitly writing out corresponding shifts of the arguments, it is more convenient to consider variables \( \bar{z}(s) \) and \( z(s) \) to be independent. Formally, this amounts to saying that the initial \( |z_I| \) and final \( \langle z_F \rangle \) configurations are in different polarizations [4], which necessitates the appearance of the boundary term. For example, consider a classical system specified by the Hamiltonian function \( h^{cl} \) with initial and final configurations being taken in the polarizations generated by \( \partial/\partial q \) and \( \partial/\partial p \), respectively:

\[ \dot{q} = \frac{\partial h^{cl}(q, p)}{\partial p}, \quad q(\tau) = q_F \]
\[ \dot{p} = -\frac{\partial h^{cl}(q, p)}{\partial q}, \quad p(0) = p_F. \]  

These equations follow from the Hamilton principle of stationary action \( \delta \phi = 0 \), where

\[ \phi = i \int_0^\tau [p\dot{q} - h^{cl}]ds - ip_I[q_F - q(0)]. \]

To specify the classical system associated with \( H \), one needs classical equations of motion. The latter follow from the Hamilton principle \( \delta \Phi = 0 \), which yields

\[ \ddot{z} = i\hbar^{-1}(\partial_{zz} F)^{-1} \partial_z h^{cl} \quad \bar{z}(\tau) = \bar{z}_F, \]
\[ \ddot{\bar{z}} = -i\hbar^{-1}(\partial_{\bar{z}\bar{z}} F)^{-1} \partial_{\bar{z}} h^{cl} \quad \bar{z}(0) = \bar{z}_I. \]  

One sees from (14) that equations of motion are correctly specified by boundary conditions and define a canonical phase flow associated with \( h^{cl} \).

In view of a rather complicated form of the coherent-state path integral, it would be desirable to get simple sufficient criteria for the stationary-phase approximation to be exact. These are provided by the path-integral generalization of the DH theorem which states (omitting some subtleties) that the WKB approximation is exact, provided the Hamiltonian flow leaves a metric of the underlying phase space invariant, that is

\[ \mathcal{L}_{X_H} g = 0, \]  

where \( \mathcal{L}_{X_H} \) stands for a Lie derivative along a Hamiltonian vector field that generates the flow.

To formally apply the DH theorem a kinetic term in an action is required to be of the form \( i \int \theta \), where the symplectic 1-form \( \theta \) determines \( w \) by \( d\theta = w \) [5]. This is pursued for the representation [10][12] as follows. Let us define

\[ \theta = \frac{i}{2} \left[ \delta F(\bar{z}, z) - \delta F(\bar{z}_F, z) - \delta F(\bar{z}, z) + \delta F(\bar{z}, z_I) \right]. \]  

By the very construction, \( d\theta = w \). We recall that \( d = \delta + \bar{\delta} \) and \( \delta^2 = \bar{\delta}^2 = \delta \bar{\delta} + \bar{\delta} \delta = 0 \). A straightforward computation yields

\[ S + \Gamma = i \int \theta - \frac{i}{\hbar} \int_0^\tau H^{cl}ds - \frac{1}{2} [F(\bar{z}_F, z_F) + F(\bar{z}_I, z_I) - 2F(\bar{z}_F, z_I)] \]
\[ \equiv i \int \theta - \frac{i}{\hbar} \int_0^\tau H^{cl}ds + \log(z_F, z_I), \]  

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which results in a desired representation,
\[
\frac{\langle z_F | T \exp - \frac{i}{\hbar} \int H(s) ds | z_I \rangle}{\langle z_F | z_I \rangle} = \left[ i \int_0^\tau \int_0^\tau H^{cl} ds \right].
\]

To avoid a possible confusion, we conclude this section with the following remark. The quantization by means of a path integral depicted above provides an example of the so-called quantization-versus-classical-limit procedures. We start with the quantum Hamiltonian \( H \), evaluate its classical limit through the associated coherent states and then quantize the obtained classical system \( (H^{cl}, G/G_0, w) \) by using the path integral on \( G/G_0 \). At first sight that may seem strange, but the point is that we start with an abstract representation of \( H \) and end up with the explicit one. A few examples to illustrate this situation will be given in section VI.

### III. Quasiclassical approximation: preliminaries

Let \( G \) be a compact simple Lie group. For any unitary irreducible representation \( U^\ell(G) \), its highest weight \( \ell \) is given by a sum of the fundamental weights \( \omega^j \) with the non-negative integer coefficients
\[
\ell = \sum_{j=1}^r l^j \omega^j,
\]
where \( r \) stands for the rank of a Lie algebra of \( G \). It then can be shown that
\[
F^{\ell} = \sum_{j=1}^r l^j F^j,
\]
where \( \{ F^j \} \) represent the fundamental Kähler potentials. The same dependence on \( l^j \) holds for the covariant symbols (coherent state expectation values) of the basic elements of the Lie algebra of \( G \) \[16\]. In the sequel, for the sake of simplicity, we will be solely concerned with the case when only one term in the series contributes to \( \ell \), which corresponds to group orbits of rank \( r = 1 \). Given a group representation \( U^\ell(G) \), the number of the non-zero nodes \( l_j \neq 0 \) in the corresponding Dynkin graph may be called the rank of \( M \) \[16\]. If all \( l_j > 0 \) the corresponding orbit is called a non-degenerate one. This requirement by no means restricts oneself to the groups \( G = SU(2) \) and \( SU(1,1) \) and corresponding homogeneous spaces \( SU(2)/S(U(1) \otimes U(1)) \) and \( SU(1,1)/S(U(1) \otimes U(1)) \), for there exist one-rank manifolds (degenerate orbits of higher rank groups) with complex dimensions \( \text{dim}_c M > 1 \).

In physics the symplectic form \( w \) has the same units as an angular momentum
\[
[w] = [kg \, m^2 \, sec^{-1}] = [\hbar].
\]
Since we assume the coordinates \( z, \bar{z} \) to be dimensionless, the form \( w \) in Eq. \( (3) \) is implicitly understood to be measured in units of \( \hbar. \) It is convenient to introduce a new parameter
\[
\lambda = \hbar l,
\]
that represents a physical quantity whereas \( \hbar \) represents the quantum mechanical yard stick with which to measure \( \lambda \) \[17\]. For instance, in the case \( (M = S^2, l \in N) \) \( \lambda \) represents the intrinsic angular momentum (spin). It is the parameter \( \lambda \) that enters into measurable physical quantities, e.g., energy.
In order to keep them fixed in the limit $l \to \infty$, one should simultaneously imply that $\hbar \to 0$. This notice explains why the large highest weight $l$ corresponds to a quasiclassical region.

In the classical theory, the Lie algebra is represented by covariant symbols or momentum maps which are functions on $M$, with the Lie product being the Poisson brackets given by the Kähler structure. In the limit $l \to \infty$ ($\hbar \to 0$) the algebra of operators (observables) corresponding to the Lie generators of $G$ reduces to the Poisson algebra of functions (momentum maps) on $M$. To put another way, in the classical limit orbits in the coadjoint representations of $G$ emerge, different representations giving rise to different orbits. A method of explicit obtaining classical phase spaces (group orbits) for the case of $G = SO(3)$ has been worked out by Lieb [18] and later on generalized to compact simple groups by Simon [19].

Let $\{L_\alpha\}$ denote a set of the generators of $G$. Consider the Hamiltonian

$$H = \sum_{\alpha=1}^{\dim G} \hbar \omega^{(1)}_\alpha L_\alpha + \sum_{\alpha,\beta=1}^{\dim G} \hbar \omega^{(2)}_{\alpha\beta} \phi^{(2)}_{\alpha\beta}(l)L_\alpha L_\beta + \ldots$$

(18)

where $\omega^{(1)}_\alpha, \omega^{(2)}_{\alpha\beta}, \ldots$ are some frequencies that may explicitly depend on time and functions $\phi^{(2)}_{\alpha\beta}(l), \ldots$ are chosen to ensure that $H^{\ell l}$ linearly depends on $\lambda$. In view of the aforementioned property of momentum maps, this automatically holds for the first term. From Eq. (17) it then follows that all dependence on $l$ is isolated in a single factor of $(l)$ multiplying the total action $\Phi$, which justifies the application of the stationary phase approximation to the path-integral propagator (10).

To conclude this section, a few remarks about already known quasiclassical formulas for quantum-mechanical propagators are to be made. The main result in this respect is due to DeWitt and leads to the short time approximation to a transition amplitude in a configuration space [2].

Numerous spin-spin lattice interactions fall into this category. A simple example of function $\phi^{(2)}$ is provided by

$$H \sim \frac{1}{2j-1}(J^2 + J^2), \quad \langle H \rangle \sim 2j = l, \quad \phi^{(2)} \sim \frac{1}{2j-1}, \quad j > 1/2.$$
where $\mathcal{L}_- = \mathcal{L} + \kappa \hbar^2 R$. As has been shown by Kleinert, the non-classical term in the action proportional to $\hbar^2$ disappears, provided one treats the differences $\Delta q^\mu$ and differentials $dq^\mu$ symmetrically, when transforming the integral measure in the cartesian path integral into the general metric-affine space \cite{3}. This results in a correct energy spectrum of a free particle on surfaces of spheres and on group manifolds obtained from a path integral with a purely classical action in the exponential.

If one neglects the $R$-dependent term $\sim \hbar^2$, Eq. (19) can be regarded as a quasiclassical ($\hbar \to 0$) approximation to the transition amplitude. An inconvenient point in this approach is, however, that there is no simple sufficient criteria for the quasiclassical expression (19) to be exact. Since the configuration space path integral in contrast to the phase space one does not involve any Liouville measure (in a time-lattice discretization), the DH theorem cannot be applied in this case. Yet some "experimental" observation can be made. As has been noted by Schulman \cite{20} and DeWitt \cite{21}, the configuration space path integral is WKB exact if the expression for the finite time propagator concides with that for the short time. As is shown by Dowker \cite{22} and Marinov and Terentyev \cite{23}, this occurs for a free particle propagating on a group manifold. For further development of these ideas the recent papers by Inomata \cite{24} and Junker \cite{25} could be referred to.

The coherent-state path-integral formalism turns out to be more convenient since a powerful machinery of canonical transformations can be employed and a hidden supersymmetry of an action can be revealed, which leads to the path integral generalization of the DH theorem that provides universal simple criteria \cite{13} for the quasiclassics to be exact \cite{15}. Besides that, this method incorporates the underlying symmetry of the problem under consideration, which makes it possible to look for an asymptotic behavior with respect to the representation indices (eigenvalues of Kasimir’s operators).

It was Klauder \cite{11} as well as Klauder and Daubechies \cite{26} who first suggested to use a system of the type (14) to derive the semiclassical approximation for the coherent-state path integral. Identity of the direct semiclassical evaluation of the coherent-state propagator on a complex plane to the stationary phase approximation to the corresponding path integral except for the pre-exponential factors was proved by Weissman \cite{27}. In the important work by Yaffe \cite{28} a general method for finding classical limits in certain quantum theories was developed. This approach is naturally based upon coherent states associated with a symmetry group and is used to explicitly construct a classical phase space, a corresponding coadjoint orbit. Non of the symmetry groups considered in this paper are semisimple, which makes it necessary to distinguish between adjoint and coadjoint orbits. Cadavid and Nakashima \cite{29} studied the coherent-state path integral for semisimple Lie algebras, coherent states being sections of a holomorphic line bundle over $G/G_0$. The semiclassical approximation of the quantum evolution operator via coherent states associated with quantized closed curves on the $SU(2)$ orbits was obtained by Karasev and Kozlov \cite{30}. This method was further extended to semisimple Lie algebras \cite{31} and general Kähler phase spaces \cite{32}.

IV. Quasiclassical approximation: coherent-state propagator

In this section we present a derivation of the quasiclassical coherent-state propagator $\mathcal{P}^{qc}$ by applying the stationary phase approximation to the path integral (11).

We are looking for the representation

$$\mathcal{P} = e^{l(\cdots)} [(\cdots) + o(1)], \quad l \to \infty,$$

(21)

where $\cdots$ stands for $l$-independent functions on a phase space. The quasiclassical propagator is then defined by the leading term of (21)

$$\mathcal{P}^{qc} = e^{l(\cdots)} (\cdots).$$

(22)
We first rewrite (10) to explicitly allow for the normalization:

$$\mathcal{P} = \langle z_F | z_I \rangle \frac{\int D\mu(z) \exp \Phi}{\int D\mu(z) \exp \Phi_0} \Phi_0 \equiv \Phi |_{H=0}.$$  \hspace{1cm} (23)

In order to lift the measure weight factor \(\partial^2_{zz} F\) in an exponential we make use of a trick that consists in the integration over auxiliary anticommuting fields \(\xi(t)\) and \(\bar{\xi}(t)\) (see, e.g., [13]):

$$\mathcal{P} = \langle z_F | z_I \rangle \frac{\int DzD\bar{z}D\xi D\bar{\xi} \exp [\Phi + \int \bar{\xi}(t)(\partial^2_{zz} F)\xi(t) dt]}{\int DzD\bar{z}D\xi D\bar{\xi} \exp [\Phi_0 + \int \bar{\xi}(t)(\partial^2_{zz} F)\xi(t) dt]}.$$ \hspace{1cm} (24)

The quasiclassical \((l \to \infty)\) motion is described by the approximation

$$\Psi \equiv \Phi + \int \bar{\xi}(\partial^2_{zz} F)\xi dt = \Psi |_c + \frac{1}{2} \delta^2 \Psi |_c + ... \simeq \Psi |_c + \frac{1}{2} \delta^2 \Psi |_c, \hspace{0.5cm} \delta \Psi |_c = 0$$ \hspace{1cm} (25)

and

$$\Psi_0 \equiv \Phi_0 + \int \bar{\xi}(\partial^2_{zz} F)\xi dt = \Psi_0 |_c + \frac{1}{2} \delta^2 \Psi_0 |_c, \hspace{0.5cm} \delta \Psi_0 |_c = 0$$ \hspace{1cm} (26)

with the boundary conditions \(z(0) = z_I\) and \(\bar{z}(\tau) = \bar{z}_F\). The subscript \(^c\) denotes a value along the extremals (14).

To proceed further, we introduce variations

$$\delta z \equiv \eta = z - z_c, \hspace{0.5cm} \delta \bar{z} \equiv \bar{\eta} = \bar{z} - \bar{z}_c,$$

which satisfy

$$\eta(0) = 0, \hspace{0.5cm} \bar{\eta}(\tau) = 0.$$

It is clear that \(\bar{\xi} |_c = \xi |_c = 0\) and in view of (11) \(\exp \Phi_0 |_c = \langle z_F | z_I \rangle\). Bearing this in mind we insert expansions (25,26) into Eq. (24) perform integrals over \(\delta \xi, \delta \bar{\xi}\) coming from \(\delta^2 \Psi\) and \(\delta^2 \Psi_0\) that cancel each other and finally arrive at

$$\mathcal{P}^{qc}(\bar{z}_F, z_I; \tau) = \mathcal{P}_{red} \exp \Phi_c,$$ \hspace{1cm} (27)

where the reduced propagator is given by

$$\mathcal{P}_{red} = \int D\eta D\bar{\eta} \exp \left\{ \frac{1}{2} \int_0^\tau (\bar{\eta} \partial_s \eta - \bar{\eta} \eta \partial_s) ds - \frac{i}{2} \int_0^\tau (\bar{\eta}^2 A + \bar{\eta}^2 C + 2 \bar{\eta} \eta B) ds \right\}$$

$$= \left( \frac{\text{Det} K}{\text{Det} K_0} \right)^{-1/2},$$ \hspace{1cm} (28)

with

$$K = \begin{pmatrix} -iA(s) & -iB(s) + \partial_s \\ -iB(s) - \partial_s & -iC(s) \end{pmatrix} \hspace{1cm} \text{and} \hspace{1cm} K_0 = \begin{pmatrix} 0 & \partial_s \\ -\partial_s & 0 \end{pmatrix}.$$

The functions

$$A = \hbar^{-1} \partial_z \left[(\partial^2_{zz} F)^{-1} \partial_z H \right] |_c, \hspace{0.5cm} C = \hbar^{-1} \partial_z \left[(\partial^2_{zz} F)^{-1} \partial_z H \right] |_c,$$

$$B = \frac{1}{2\hbar} \partial_z \left[(\partial^2_{zz} F)^{-1} \partial_z H \right] |_c + \frac{1}{2\hbar} \partial_z \left[(\partial^2_{zz} F)^{-1} \partial_z H \right] |_c$$

are calculated with the aid of the Euler-Lagrange equations (14). We recall that \(\eta\) and \(\bar{\eta}\) are considered to be independent.

\(^3\)The nontrivial measure \(D\mu\) contributes to \(\mathcal{P}\) in higher orders
Our aim now is to express (28) in terms of the classical orbitals. First it is convenient to eliminate the $B$-term from $\delta^2 \Phi_c(\vec{\eta},\eta)$ in (28), which can be achieved by the change
\[
\eta \to \eta \exp(-i \int_0^s B(t)dt).
\]
The result is
\[
\mathcal{P}_{\text{red}} = \int D\eta D\bar{\eta} \exp\left(\frac{1}{2} \delta^2 \bar{\Phi}_c(\bar{\eta},\eta)\right), \tag{29}
\]
where
\[
\delta^2 \Phi_c(\eta,\bar{\eta}) = \int_0^\tau (\dot{\eta}\bar{\eta} - \bar{\eta}\dot{\eta})ds - i \int_0^\tau (\eta^2 \bar{A} + \bar{\eta}^2 \bar{C})ds \tag{30}
\]
and
\[
\bar{A} = A \exp(-2i \int_0^s Bdt), \quad \bar{C} = C \exp(2i \int_0^s Bdt).
\]
The Euler-Lagrange equations for the functional (30) known as the Jacobi equations read
\[
\bar{K} \Upsilon = 0, \tag{31}
\]
where
\[
\bar{K} = \begin{pmatrix} -i\bar{A}(s) & \partial_s \\ -\partial_s & -i\bar{C}(s) \end{pmatrix} \quad \text{and} \quad \Upsilon = \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}, \quad \eta(0) = 0, \quad \bar{\eta}(\tau) = 0.
\]
It is clear that
\[
\mathcal{P}_{\text{red}} = \left(\frac{\text{Det} \bar{K}}{\text{Det} K_0}\right)^{-1/2}. \tag{32}
\]
Let us denote
\[
f(\lambda) = \text{Det} K_0^{-1}(\bar{K} - \lambda),
\]
where $\lambda$ is a spectral parameter (not to be confused with the quasiclassical parameter). It then follows that
\[
\partial_\lambda \log f = -\text{tr}(\bar{K} - \lambda)^{-1}. \tag{33}
\]
To construct the Green function $(\bar{K} - \lambda)^{-1}$, consider the system of two equations
\[
(\bar{K} - \lambda) \Phi = 0, \quad \Phi = \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix}, \quad \phi(0) = 0 \tag{34}
\]
and
\[
(\bar{K} - \lambda) \Psi = 0, \quad \Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad \bar{\psi}(\tau) = 0. \tag{35}
\]
With the aid of (34,35) the Green function is constructed to be
\[
(\bar{K} - \lambda)^{-1}(t,t') = \frac{\theta(t-t')}{w} \begin{bmatrix} \phi(t') \psi(t) & \bar{\phi}(t') \bar{\psi}(t) \\ \phi(t') \bar{\psi}(t) & \bar{\phi}(t') \psi(t) \end{bmatrix} + \frac{\theta(t'-t)}{w} \begin{bmatrix} \phi(t) \psi(t') & \bar{\phi}(t) \bar{\psi}(t') \\ \phi(t) \bar{\psi}(t') & \bar{\phi}(t) \psi(t') \end{bmatrix}. \tag{36}
\]
In the case when the functions Φ and Ψ are linear-independent, i.e., Φ does not coincide with an eigenfunction of \( \tilde{K} \), the Wronskian
\[
w = \begin{vmatrix} \phi(s) & \bar{\phi}(s) \\ \psi(s) & \bar{\psi}(s) \end{vmatrix} = \phi \bar{\psi} - \bar{\phi} \psi
\]
does not depend on \( s \). In view of (34, 35)
\[
w(0) = -\bar{\phi}(0)\psi(0) = -\bar{\phi}(\tau)\psi(\tau) = w(\tau)
\]  
(37)
From Eq.(36) it immediately follows that
\[
tr(\tilde{K} - \lambda)^{-1} = \frac{1}{\omega} \int_0^\tau (\phi \psi + \bar{\phi} \bar{\psi}) dt.
\]  
(38)
This integral can be taken by using the following notice. Instead of system (34, 35) let us consider a slightly changed one:
\[
(\tilde{K} - \lambda')\Phi = 0, \quad (\tilde{K} - \lambda)\Psi = 0
\]
After little algebra one finds
\[
\begin{aligned}
(\lambda' - \lambda)(\phi \psi + \bar{\phi} \bar{\psi}) &= \left[ \bar{\phi} \psi - \bar{\psi} \phi + \bar{\psi} \bar{\phi} - \phi \psi \right] |_{\lambda'=\lambda} \\
+ (\lambda' - \lambda) \left[ \partial_\lambda \bar{\phi} \psi - \bar{\psi} \partial_\lambda \phi + \bar{\psi} \partial_\lambda \bar{\phi} - \bar{\phi} \psi \right] |_{\lambda'=\lambda} + O((\lambda' - \lambda)^2).
\end{aligned}
\]
The first bracket on the rhs equals zero since
\[
\frac{d}{ds} w = 0,
\]
and one finally ends up with
\[
\phi \psi + \bar{\phi} \bar{\psi} = \frac{d}{dt} (\psi \partial_\lambda \bar{\phi} - \bar{\psi} \partial_\lambda \phi),
\]
which yields
\[
tr(\tilde{K} - \lambda)^{-1} = -\frac{\psi(\tau) \partial_\lambda \bar{\phi}(\tau)}{\phi(\tau) \psi(\tau)} + \frac{\psi(0) \partial_\lambda \bar{\phi}(0)}{\phi(0) \psi(0)} = -\partial_\lambda \log \frac{\bar{\phi}(\tau; \lambda)}{\phi(0; \lambda)}.
\]
With the help of this one finally gets
\[
Det \tilde{K}/K_0 = \frac{\phi(\tau; \lambda = 0)}{\phi(0; \lambda = 0)} = \frac{\psi(0; \lambda = 0)}{\psi(\tau; \lambda = 0)}.
\]  
(39)
In the last line the second equality follows from Eq. (37).

Our aim now is to set a connection between Φ and Ψ functions at \( \lambda = 0 \) with the classical orbits (14). The crucial point is that Eqs. (14, 15) at \( \lambda = 0 \) go over to the Jacobi equation (31) except for the boundary conditions. But as is known, the Jacobi equation can be obtained by varying the original Euler-Lagrange equation with respect to the initial data. Let \( \delta z_c \) and \( \delta \bar{z}_c \) be variations of Eqs. (14). Then, the functions
\[
u(s) = \delta z_c (\partial_{z_c}^2 F)_c^{1/2} \exp(+i \int_0^s B dt), \quad \bar{u}(s) = \delta \bar{z}_c (\partial_{\bar{z}_c}^2 F)_{\bar{c}}^{1/2} \exp(-i \int_0^s B dt)
\]  
(40)
are seen to satisfy the Jacobi equation (31).

To identify these with the solutions to Eqs. (14, 15), we put
\[
\phi(s) = (\partial_{z_c}^2 F)_c^{1/2} \exp(i \int_0^s B dt) \frac{\partial z_c(s)}{\partial z_F},
\]
\[
\bar{\phi}(s) = (\partial_{\bar{z}_c}^2 F)_{\bar{c}}^{1/2} \exp(-i \int_0^s B dt) \frac{\partial \bar{z}_c(s)}{\partial \bar{z}_F}
\]  
(41)
and similarly,
\[
\psi(s) = (\partial^2_{zz} F_c)^{1/2} \exp(i \int_0^s B dt) \frac{\partial z_c(s)}{\partial z_I},
\]
\[
\bar{\psi}(s) = (\partial^2_{zz} F_c)^{1/2} \exp(-i \int_0^s B dt) \frac{\partial \bar{z}_c(s)}{\partial z_I}. \tag{42}
\]

In view of Eq. (39) we arrive at
\[
Det \frac{\tilde{K}}{K_0} = \left[ \left(\frac{\partial^2_{zz} F(\bar{z}(\tau), z(\tau))}{\partial^2_{zz} F(\bar{z}(0), z(0))} \right)^{1/2} \exp(-i \int_0^\tau B dt) \left(\frac{\partial z_c(0)}{\partial z_I} \right)^{-1} \right]_{c}. \tag{43}
\]

To end the derivation, we compute
\[
\frac{\partial^2 \Phi_c}{\partial z_F \partial z_I}.
\]
It is noteworthy that we take the derivative of the total classical action including the boundary term. It is the total action \(\Phi_c\) that enters the final answer. A calculation is straightforward and yields
\[
\frac{\partial^2 \Phi_c}{\partial z_F \partial z_I} = \frac{1}{2} \left[ \partial^2_{zz} F_c(\bar{z}(\tau), z(\tau)) \frac{\partial z_c(\tau)}{\partial z_I} + \partial^2_{zz} F_c(0) \frac{\partial \bar{z}_c(0)}{\partial z_F} \right], \tag{44}
\]
where we have set
\[
\partial^2_{zz} F_c(t) \equiv \partial^2_{zz} F(\bar{z}(t), z(t)) \big|_c.
\]
Combining Eqs. (43) and (44) one finally arrives at
\[
\left( \frac{Det \tilde{K}}{K_0} \right)^{-1} = \frac{1}{\partial^2_{zz} F_c(\bar{z}(\tau), z(\tau)) \partial^2_{zz} F_c(0)}^{1/2} \frac{\partial^2 \Phi_c}{\partial z_F \partial z_I} \exp(i \int_0^\tau B dt), \tag{45}
\]
which by virtue of (4) yields
\[
\mathcal{P}^{qc}(\bar{z}_F, z_I; \tau) = \exp \left( \Phi_c + \frac{i}{2} \int_0^\tau B dt \right) \left[ \frac{1}{g(\bar{z}_c(\tau), z_c(\tau)) g(\bar{z}_c(0), z_c(0))}^{1/2} \frac{\partial^2 \Phi_c}{\partial z_F \partial z_I} \right]^{1/2}. \tag{46}
\]

Thus, we have expressed the quasiclassical propagator in terms of the total classical action and classical orbitals, which is similar to the DeWitt result for the short time propagator of a particle in a curved configuration space (13). However, there are some important distinctions. First, the total action \(\Phi\) is involved rather than \(S\). Next, there appears a dependence on the \(B\)-term. The latter plays the role of normalization and is necessary to fix the quantization (by covariant symbols). This term interpolates between the covariant and contravariant quantization schemes and disappears at the point corresponding to the Weyl quantization. With the aid of the Euler-Lagrange equations it can also be represented in terms of the extremals:
\[
B = \frac{i}{2} (\partial_2 \bar{z} - \partial_2 \bar{z}) |_c.
\]
As has already been mentioned, the total action \(\Phi\) and the Kähler potential \(F\) are proportional to \(l\), whereas it is seen that \(B \sim l^0\), which agrees with suggestion (22).

Generalization of the above result to the multidimensional case (provided one-rank orbits are still considered!) is straightforward. For instance, consider the compact degenerate \(U(N)\) orbit, complex
projective space $CP^{N-1} = U(N)/U(N-1) \otimes U(1)$. The complex dimensionality of the manifold is $N-1$ whereas its rank $= 1$. The resulting Kähler potential

$$F = l \log \left[ 1 + \sum_{1}^{N-1} \bar{z}^i z^i \right],$$

where $\{z^i, i = 1, ..., N-1\}$ is a complex vector and a positive integer $l$ is the highest weight specifying a representation. An example is provided by a quantum system with the dynamical $U(N)$ symmetry, e.g., generated by a set of bilinears $a_i^\dagger a_j, \ [a_i, a_j^\dagger] = \delta_{ij}, \ i, j = 1, ... N$, with the quasiclassical parameter being a total number of the field excitations: $l = \sum n_i$.

An obvious modification of Eq. (46) consists in extending Eqs. (23) and (24) to include vector indices and reads

$$\mathcal{P}^{qc} = \exp \left( \Phi_c + \frac{i}{2} \int_0^\tau \text{tr} B dt \right) \left[ \frac{1}{g(\bar{z}_c(\tau), z_c(\tau)) g(\bar{z}_c(0), z_c(0))} \right]^{1/2} \det \left( \frac{\partial^2 \Phi_c}{\partial \bar{z}_c^i \partial z^j} \right)^{1/2}, \quad (47)$$

where $g(\bar{z}, z) = \det \partial^2_{zz} F$.

We conclude this section by the following remark. It seems that the final result (46) is crucially based on the fact whether a path integral (11) exists as a bona-fide integral. The common belief is however that (11) cannot be in general justified as an integral with respect to a certain measure. Moreover, even the justification of the existence of the limit in (8) is rather a nontrivial problem (34), though some exceptions, e.g., the quasiclassical approximation considered above, are possible. The justification of the continuum representation in the quasiclassical domain ($l \to \infty$) this limit under certain restrictions does exist, provided expansion in powers of $1/l$ is first carried out. However, this procedure does not in general lead to a genuine integral with respect to a path measure.

Frequently, the statement occurs (see, e.g., (35)) that once the continuum expression (11) is concerned, only formal calculations are possible. This in turn implies that the continuum form of the path integral may at most provide some hint about an actual answer which nevertheless is to be obtained in a rigorous manner within the time-lattice approximation. In general, this assertion is true, though some exceptions, e.g., the quasiclassical approximation considered above, are possible. The justification of the continuum representation in the semiclassical domain ($l \gg 1$) may be given in two formally distinct but in essence similar ways. One may observe, for instance, that a scalar product $\langle z_k | z_{k-1} \rangle$ entering into Eq. (8) is highly peaked about $\Delta_k(\epsilon) \equiv z_k - z_{k-1} \sim 0$ as $l$ tends to infinity. This implies that once the leading term $\mathcal{P}^{qc}$ is concerned, only terms linear in $\Delta_k$ are to be left in (8), which under some additional mild restrictions on the Hamiltonian function would eventually lead to continuum representation (28), corrections to the leading term coming, in particular, from higher powers of $\Delta_k$.

We prefer, however, to start from the formal continuum representation (11) which makes sense for continuous differentiable paths. Path integral (11) is localized on a Hilbert space of the square integrable paths: $(z, z) < \infty$ (34), so that in general (and actually, almost surely) $z(s)$ has no time derivative. To find a way out, we represent an arbitrary path in the form $z = z_cl(s) + \delta z(s)$ and expand $\delta z(s)$ in a series over an appropriate basis in $[0, \tau]$. The trick due to Berezin consists in retaining only a finite part of the series, thereby dealing with continuous and differentiable paths at all intermediate steps, with the infinite limit being taken only at the final stage. This procedure converges for the gaussian path integral (28) (34), thereby justifying the above continuous calculus. This is however the case, provided we are concerned as before with a calculation of the leading term, $\mathcal{P}^{qc}$.

V. Other quantization schemes

So far we have been concerned with a specific quantization scheme, the quantization by covariant symbols. The next important quantization scheme to be mentioned here is the contravariant

13
The covariant symbol $H^{\text{cov}}$ which we identify with $H^{\text{cl}}$ is related to the contravariant one $H^{\text{ctr}}$ by

$$H^{\text{cov}}(\bar{z}, z) = \int \exp\{\phi(\bar{z}, z | \bar{\xi}, \xi)\} H^{\text{ctr}}(\bar{\xi}, \xi) d\mu(\xi) \equiv (\hat{T} H^{\text{ctr}})(\bar{z}, z),$$

(48)

where

$$\phi(\bar{z}, z | \bar{\xi}, \xi) = F(\bar{\xi}, z) + F(\bar{z}, \xi) - F(\bar{z}, z) - F(\bar{\xi}, \xi)$$

(49)

and a constant in Eq. (48) is fixed by

$$\int e^{-F(\bar{\xi}, \xi)} d\mu(\xi) = 1.$$  

(50)

If the point $(\bar{z}, z)$ is fixed, the potentials $-\phi(\bar{z}, z | \bar{\xi}, \xi)$ and $F(\bar{\xi}, \xi)$ generate the very same metric and are to be related by a group transformation $g_z$:

$$F(g_z \bar{\xi}, g_z \xi) = -\phi(\bar{z}, z | \bar{\xi}, \xi).$$

(51)

Invariance of the measure $d\mu$ upon $g_z$ along with the normalization (50) results in

$$1^{\text{cov}} = 1^{\text{ctr}}$$

as it should be.

In principle, the operator $\hat{T}$, being permutable with a group action, can be expressed via the corresponding Casimir operators. In the case under consideration only the second Casimir operator

$$K_2 \equiv \Delta = (\partial_{\bar{z}z}^2 F)^{-1} \partial_{\bar{z}z}^2,$$

(52)

the Laplace-Beltrami operator with respect to the metric (4), is involved.

As is known, in the flat case

$$\hat{T}(\Delta) = e^\Delta,$$

(53)

whereas for the quantization on a sphere and Lobachevsky plane $\hat{T}(\Delta)$ has been evaluated in a form of infinite products [36].

As is seen from Eq. (46), both the expression in the brackets and the $B$-term are of an order of $O(1)$ whereas $\Phi_c = O(l)$. This means that $H^{\text{ctr}}$ is to be taken with the accuracy up to an order of $O(1)$. Therefore, for our purposes we need an asymptotic relation between symbols rather than the exact one, which is not easily available. This is pursued by the following notice. Being nonpositive, the function $\phi$ reaches zero at the point $(\bar{\xi}, \xi) = (\bar{z}, z)$. As $l$ goes to infinity, the maximum becomes sharper localizing $\phi$ at $(\bar{\xi}, \xi) = (\bar{z}, z)$. Developing then the integrand in powers of $\eta = \xi - z$, $\bar{\eta} = \bar{\xi} - \bar{z}$ one gets

$$H^{\text{cov}}(\bar{z}, z) = \int e^{-\bar{\eta} \eta} \left[ H^{\text{ctr}}(\bar{z}, z) + \Delta H^{\text{ctr}}(\bar{z}, z) \bar{\eta} \eta \right] \frac{d\eta d\bar{\eta}}{2\pi i} + o(1)$$

$$= [1 + \Delta + O(1/l^2)] H^{\text{ctr}}(\bar{z}, z), \quad l \to \infty$$

(54)

In view of this, one may convert Eq. (46) in a form suitable for the quantization by contravariant symbols.

To conclude this section, we will specify Eq. (46) for the flat $(M = C)$ case relevant for the Heisenberg-Weyl coherent states. To avoid confusion with dimensions, we introduce, following Ref. [17], coordinates $x = p/\alpha$ and $y = q/\beta$ and the complex dimensionless coordinate $z = 1/\sqrt{2}(x + iy)$. Constants $\alpha$ and $\beta$ are of dimensions of momentum and position, respectively. It is convenient to introduce the dimensionless constant

$$\gamma = \alpha \beta / \hbar$$

14
which plays the role of the representation index $l$. The classical limit becomes quite transparent in this notation. It means a passage from systems of units to measure $\alpha$ and $\beta$ quite adequate for quantum description to those that are more convenient for the classical one. For instance, if one chooses a "classical scale" $\alpha = 1\text{m}, \beta = 1\text{kg m/sec}$, then $\gamma^{-1} \approx 10^{-34}$, which effectively corresponds to small $\hbar$.

It is just in this sense that one should understand the limit $\hbar \to 0$.

The conventional 2-form $w = dp \wedge dq$ goes over to

$$w = -\alpha \beta dx \wedge dy,$$

so that

$$\frac{w}{\hbar} = i\gamma dz \wedge d\bar{z}.$$  

We introduce a set of the $\gamma$-dependent Heisenberg-Weyl coherent states:

$$|z; \gamma\rangle = \exp\left(-\frac{\gamma}{2} \bar{z} z + \sqrt{\gamma} zd^\dagger\right) |0\rangle,$$  

whence

$$F = \log |\langle 0 | z; \gamma\rangle|^{-2} = \gamma \bar{z} z.$$  

In the flat case covariant symbols are related to those in the $\alpha$-quantization scheme by

$$H_{\text{cov}}(\bar{z}, z) = (\hat{T}_\alpha(\Delta) H^{(\alpha)}(\bar{z}, z)), \quad \hat{T}_\alpha(\Delta) = e^{\alpha \Delta}, \quad \alpha \in [0, 1],$$

the covariant, contravariant and Weyl quantization schemes being specified by $\alpha = 0, 1$ and $1/2$, respectively. As a result one gets

$$H^{\text{cov}} - \frac{1}{2} \Delta H^{\text{cov}} \equiv H^{\text{cov}} - [(\frac{1}{2} - \alpha) + \alpha] \Delta H^{\text{cov}} = H^{(\alpha)} - (\frac{1}{2} - \alpha) \Delta H^{\text{cov}} + O(1/l^2),$$

which yields

$$\mathcal{P}^{(qc)}_{\text{flat}} = \left[\frac{1}{\gamma} \frac{\partial^2 \Phi_c}{\partial z_F \partial z_I}\right]^{1/2} \exp \left\{ \Phi_c + \frac{i}{2} \tau \int_0^\tau B dt \right\},$$

whence

$$\sim \left[\frac{1}{\gamma} \frac{\partial^2 \Phi_c^{(\alpha)}}{\partial z_F \partial z_I}\right]^{1/2} \exp \left\{ \Phi_c^{(\alpha)} + i(\frac{1}{2} - \alpha) \int_0^\tau B^{(\alpha)} dt \right\},$$

where equivalence classes are defined by $f \sim g = \{ f, f/g = 1 + o(1), \quad l \to \infty \}$, so that $H^{(\alpha)} \sim H^{\text{cov}} \equiv H^{cl}$. This result (the first line in Eq. (57)), with the $B$-term however being missed, was derived by Weissman [37] by extending Miller’s semiclassical algebra to the coherent-state setting [4]. Originally, Miller’s formalism incorporated eigenstates of Hermitian operators to relate a quantum mechanical matrix element of a general unitary transformation, in the semiclassical limit, to a generator of a corresponding canonical transformation [38].

It is to be noted that in deriving Eqs. (57,58) original equations of motion (14) that correspond to the covariant quantization have been kept fixed. That is why (58) cannot be regarded as a genuine $\alpha$-representation. To derive the latter, one would have to start with equation $\delta \Phi^{(\alpha \neq 0)} = 0$, whose
solutions in contrast to (14) would be bearing an explicit \( l \)-dependence, namely, \( z^{(\alpha)} = z^d + \mathcal{O}(1/l) \). In that case, however, it would be natural to start, instead of Eq. (5), with the \( \alpha \)-symbol of the evolution operator.

VI. Examples

In this section, we will illustrate the basic result (10) with a few simple but instructive examples. 

\textit{SU}(2) path integral (\( M = \mathbb{C}P^1, \) compact manifold)

Coherent state for the UIR’ of the \( SU(2) \) group is given by

\[
|z; j\rangle = (1 + |z|^2)^{-j} \exp(zJ_+) |j; -j\rangle,
\]

where \( z \in \text{SU}(2)/S(U_1 \times U_1) \simeq \mathbb{C}P^1 \), which can be thought of as an extended complex plane \( \bar{C}^1 \). The operators \( J_\pm, J_0 \) span the \( SU(2) \) algebra

\[
[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0
\]

and the lowest weight state \( |j; -j\rangle \) is annihilated by \( J_- \). From Eq. (2) it follows that

\[
F(\bar{z}, z) = 2j \log(1 + \bar{z}z), \quad l = 2j \in \mathbb{N},
\]

where \( j \) must be a half-integer corresponding to the unitary irreducible representations of \( SU(2) \). From the geometric viewpoint, this requirement is to be imposed in order that a holomorphic prequantum line bundle over \( CP^1 \) can be constructed. The general Eq. (10) reads (\( h = 1 \))

\[
\mathcal{P}_j(\bar{z}_F, z_I; \tau) = \int_{\bar{z}(0) = z_I}^{\bar{z}(\tau) = \bar{z}_F} D\mu(z) \frac{(1 + \bar{z}_Fz(\tau))^j (1 + \bar{z}(0)z_I)^j}{(1 + |z|^2)^j (1 + |z_I|^2)^j} \times \exp\left( j \int_0^\tau \frac{\bar{z}(s)z(s) - \bar{z}(s)\bar{z}(s)}{1 + \bar{z}(s)z(s)} \, ds - i \int_0^\tau H^d(\bar{z}(s), z(s)) \, ds \right),
\]

with the normalization \( \mathcal{P}_j|_{H=0} = \langle z_F; j | z_I; j \rangle \). Here \( D\mu_j(z) \) stands for the infinite pointwise product of the \( SU(2) \) invariant measures

\[
d\mu_j = \frac{2j + 1}{2\pi i} \frac{dzd\bar{z}}{(1 + |z|^2)^2}.
\]

As a simple but nontrivial example that directly demonstrates the usefulness of Eq. (10) consider a system governed by the Hamiltonian (13)

\[
H = 2A(t)J_z + f(t)J_+ + \bar{f}(t)J_-.
\]

The stationary-phase equations read

\[
i\dot{z} = 2A(t)z + f(t)z^2, \quad z(0) = z_I, \quad (61)
\]

\[
-i\dot{\bar{z}} = 2A(t)\bar{z} + \bar{f}(t)\bar{z}^2, \quad \bar{z}(\tau) = \bar{z}_F.
\]

Being of the Riccati type these equations cannot be solved explicitly, but yet some important information is available. Namely, let us look for the solutions to Eqs. (61,62) in the form

\[
z(t) = \frac{a(t)z_I + b(t)}{-b(t)z_I + a(t)} \quad a(0) = 1, b(0) = 0, \quad \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \in SU(2),
\]

\[
(63)
\]
respectively. In view of Eq. (61), the functions $a(t)$ and $b(t)$ satisfy
\[
\dot{a} = -iAa + ifb, \quad \dot{b} = -iAb - ifa,
\]
whereas the functions $d(t)$ and $c(t)$, due to Eq. (62), are to be taken to satisfy
\[
\dot{d} = -iAd + if\bar{c}, \quad \dot{c} = -iAc - if\bar{d}.
\]
The solutions to these equations are related to those of Eqs. (65) by
\[
\tilde{d}(t) = \tilde{a}(t)a(\tau) + \tilde{b}(t)b(\tau), \quad c(t) = -a(t)b(\tau) + b(t)a(\tau).
\]

Despite the fact that we do not have any explicit solutions to Eqs. (61,62), we have managed to explicitly determine their dependence on initial data, $z_I$ and $\tilde{z}_F$. This, in fact, is all we need to apply Eq. (60). After some algebra one gets
\[
\Phi_c = 2j \log[\tilde{a}(\tau) - \bar{b}(\tau)z_I + b(\tau)\tilde{z}_F + a(\tau)\bar{z}_F z_I] - j \log(1 + |z_F|^2)(1 + |z_I|^2),
\]
\[
B = (2A - f\bar{z} - \bar{f}z)|_c = -\frac{i}{d\tau} \log \frac{-\bar{b}(t)z_I + \bar{a}(t)}{-c(t)\bar{z}_F + d(t)},
\]
\[
\frac{\partial^2 \Phi_c}{\partial \bar{z}_F \partial z_I} = \frac{2j}{|\bar{a}(\tau) - b(\tau)z_I + b(\tau)\bar{z}_F + a(\tau)\bar{z}_F z_I|^2}.
\]
By plugging all this into Eq. (60) one finally arrives at
\[
\mathcal{P}_f(\tilde{z}_F, z_I; \tau) = \exp \Phi_c,
\]
which coincides with direct time-lattice calculations. This result agrees with the Diistermaat-Heckman theorem which states that the WKB expansion is exact, provided a Hamiltonian phase flow leaves a metric tensor invariant (15). This is just the case for
\[
H = 2A(t)J_z + f(t)J_+ + \bar{f}(t)J_-
\]
Moreover, the dynamical invariance, i. e., the fact that $H$ belongs to the $SU(2)$ algebra, results in
\[
\mathcal{P}_{\text{red}} = 1,
\]
which is of importance in deriving the generalized Bohr-Sommerfeld quantization conditions (13).

**Path integral for the Heisenberg-Weyl coherent states (M = C, noncompact manifold)**

By virtue of Eq. (23), the general expression (10) reduces to
\[
\mathcal{P}_\gamma = \langle z_F, \gamma | \exp -i \int_0^\tau Hds | z_I, \gamma \rangle = \int Dz D\bar{z} \exp \left\{ \frac{\gamma}{2} \int_0^\tau (z\dot{z} - \bar{z}\dot{\bar{z}}) ds \right\}
\]
\[
- i \int_0^\tau Hcl(\bar{z}, z) ds + \frac{\gamma}{2} |\tilde{z}_F z(\tau) + \bar{z}(0)z_I - |z_F|^2 - |z_I|^2 | \right\}
\]
with the normalization $\mathcal{P}_\gamma(\tilde{z}_F, z_I; \tau) |_{H=0} = \langle z_F, \gamma | z_I, \gamma \rangle$.  

For a harmonic oscillator ($\hbar = 1$) $H = \omega a^\dagger a$ one gets $H^{cl} = H^{(\alpha=0)} = \gamma \omega |z|^2$, $H^{(\alpha)} = H^{cl} - \alpha \omega$ and solutions to Eqs. (14) read

$$z_c(s) = z_I \exp(-i\omega s) \quad \tilde{z}_c(s) = \tilde{z}_F \exp(-i\omega(\tau - s)),$$

which in turn results in

$$\Phi_c^{(\alpha)} = \gamma \tilde{z}_F z_I \exp(-i\omega \tau) - \frac{\gamma}{2}(|z_F|^2 + |z_I|^2) + i\tau \omega, \quad B = \omega.$$

Equation (57) can be applied to yield

$$\mathcal{P}_\gamma = \exp \Phi_c^{(\alpha=0)},$$

as it should be.

For gaussian actions the path integral (67) reduces to Eq. (57). However, in the case when Hamiltonian cannot be cast into a linear combination of the oscillator group generators $a^\dagger a$, $a^\dagger$ and $a$, the quasiclassical propagator (57) does not merely reduce to the simple form

$$\exp \Phi_c.$$

For instance, for the Hamiltonian of a parametric amplifier

$$H = \omega a^\dagger a - \frac{g}{2}[a^2 e^{-2i\omega t} + a^2 e^{2i\omega t}]$$

one gets ($\gamma = 1$)

$$z_c(s) \exp(i\omega s) = \frac{i\tilde{z}_F - z_I \sinh g\tau}{\cosh g\tau} \sinh gs + z_I \cosh gs$$

$$\tilde{z}_c(s) \exp(i\omega(\tau - s)) = \frac{\tilde{z}_F + iz_I \sinh g\tau}{\cosh g\tau} \cosh gs - iz_I \sinh gs$$

$$\Phi_c = \frac{\tilde{z}_F z_I e^{-i\omega \tau}}{\cosh g\tau} + \frac{i}{2} \tanh g\tau (z_F^2 + z_I^2) - \frac{1}{2}(|z_F|^2 + |z_I|^2), \quad B = \omega.$$

Equation (57) is again exact and reads

$$\mathcal{P}(\tilde{z}_F, z_I; \tau) = (\cosh g\tau)^{-1/2} \exp \Phi_c,$$

which coincides with the direct time-lattice calculations [39]. Equation (46) is thus seen to correctly recover the known results for the exact propagators.

**VII. Conclusion**

In the present paper, we have discussed quasiclassical quantization of a classical mechanics on a symplectic group orbit of rank $= 1$ in terms of the relevant coherent-state path integral, which provides an appropriate expansion of quantum-mechanical propagator for large values of the highest weight $l$ that specifies the underlying group representation. The principal result is a new explicit quasiclassical formula for a propagator on a coherent-state manifold which is written entirely in terms of classical data and reveals the leading large $l$ behavior of the propagator. This representation is important since a wealth of physically relevant classical phase spaces admit a natural Kähler polarization, e.g., $S^2 \simeq CP^1$, the classical phase space for a spin or $S^{1,1} \simeq D^1$, a unit disk on a complex plane—a natural phase space for models of quantum optics [24]. In this regard, the quasiclassical representation (46)
can be applied to study, e.g., spin tunneling in the semiclassical limit and related problems as well as the behavior of highly excited field states in quantum optical models.

As for possible generalizations of (46), one might proceed in three directions. First, it would be desirable to extend the result to higher-rank phase spaces. Second, it would be of practical importance to generalize to non-local actions that arise provided certain degrees of freedom in an original Hamiltonian can be integrated out. This is the case for a large class of interactions that involve bilinear combinations of Lie algebra generators and field coordinates. Third, it would be interesting to extend this approach to the supersetting, since even the simplest super phase spaces happen to be relevant to important physics. For instance, one-rank degenerate orbit of the $SU(2\mid 1)$ supergroup can be viewed as a phase space of the $t - J$ model of strongly correlated electrons which is believed to adequately describe a high $T_c$ superconducting state.

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References

[1] J.H. van Vleck, Proc. Natl. Acad. Sci. U.S.A. 14, 178 (1928); C. Morette, Phys. Rev. 81, 848 (1952).
[2] B.S. DeWitt, Rev. Mod. Phys. 29, 377 (1957).
[3] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics* (World Scientific, Singapore, 1995).
[4] A.A. Kirillov, *Elements of the Theory of Representations* (Springer, Berlin, 1976).
[5] A. Alekseev, L. Faddeev, and S. Shatashvili, J. Geom. Phys. 5, 391 (1988); H.B. Nielsen and D. Rohrlich, Nucl. Phys. B299, 471 (1988); M. Stone, Nucl. Phys. B314, 557 (1989).
[6] F.A. Berezin, Math. USSR (Izvestiya) 8, 1109 (1974).
[7] E. Onofri, J. Math. Phys. 16, 1087 (1975).
[8] A.M. Perelomov, *Generalized Coherent States* (Springer-Verlag, Berlin, 1986).
[9] E.A. Kochetov, J. Math. Phys. 36, 1666 (1995).
[10] E.A. Kochetov, Phys. Lett. A217, 65 (1996).
[11] J.R. Klauder, Phys. Rev. D19, 2349 (1979).
[12] H. Kuratsuji and Y. Mizobuchi, J. Math. Phys. 22, 757 (1981).
[13] E.A. Kochetov, J. Math. Phys. 36, 4667 (1995).
[14] D.J. Simms and N. Woodhouse, *Lectures on Geometric Quantization* (Springer, Berlin, 1977).
[15] M. Blau, E. Keski-Vakkuri, and A.J. Niemi, Phys. Lett. 246B, 92 (1990); A.J. Niemi and O. Tirkkonen, Ann. Phys. 235, 318 (1994).
[16] D. Bar-Moshe and M.S. Marinov, J. Phys. A27, 6287 (1994).
[17] G.M. Tuynman, J. Math. Phys. 28, 573 (1987); ibid. 28, 2829 (1987).
[18] E.H. Lieb, Commun. math. Phys. 31, 327 (1973).
[19] B. Simon, Commun. math. Phys. 71, 247 (1980).
[20] L. Schulman, Phys. Rev. 176, 1558 (1968).
[21] B.S. DeWitt, Ann. L’Institut Henri Poincare 11, 152 (1969).
[22] J.S. Dowker, J. Phys. A3, 451 (1970).
[23] M.S. Marinov and M.V. Terent’ev, Fortsch. Phys. 27, 511 (1979).
[24] A. Inomata, H. Kuratsuji, and C.C. Gerry, Path Integrals and Coherent States of SU(2) and SU(1,1) (World Scientific, Singapore, 1992).
[25] G. Junker, J. Phys. A22, L587 (1989).
[26] J. Klauder and I. Daubechies, Phys. Rev. Lett. 52, 1161 (1984); J. Math. Phys. 26, 2239 (1985).
[27] Y. Weissman, J. Phys. A16, 2693 (1983).
[28] L. Yaffe, Rev. Mod. Phys. 54, 407 (1982).
[29] A. Cadavid and N. Nakashima, Lett. Math. Phys. 23, 111 (1991).
[30] M. Karasev and M. Kozlov, J. Math. Phys. 34, 4986 (1993).
[31] M. Karasev and M. Kozlov, Funct. Analysis and its Applications 28, 238 (1994).
[32] M. Karasev, Russ. J. Math. Phys. 3, 393 (1995).
[33] B.M. Levitan and I.S. Sargsian, Introduction to the Spectral Theory (Nauka, Moscow, 1970).
[34] F.A. Berezin, Usp. Fiz. Nauk 132, 497 (1980).
[35] E. Ercolessi, G. Morandi, F. Napoli, and P. Pieri, J. Math. Phys. 37, 535 (1996).
[36] F.A. Berezin, Commun. math. Phys. 40, 153 (1975).
[37] Y. Weissman, J. Chem. Phys. 76, 4067 (1982).
[38] W.H. Miller, in Advances in Chemical Physics ed by I. Prigogine and S.A. Rice, (Wiley, New York, 1974).
[39] M. Hillery and M.S. Zubairy, Phys. Rev. A26, 451 (1982).
[40] J.L. van Hemmen and A. Süto, Physica 141B, 37 (1986); M. Enz and R. Schilling, J. Phys. C19, 1765 (1986); E.M. Chudnovsky and L. Gunther, Phys. Rev. Lett. 60, 661 (1988).
[41] J. Kurchan, P. Leboeuf, and M. Saraceno, Phys. Rev. A40, 6800 (1989).