Critical statistics in a power–law random banded matrix ensemble

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(March 24, 2022)

We investigate the statistical properties of the eigenvalues and eigenvectors in a random matrix ensemble with $H_{ij} \sim |i-j|^{-\mu}$. It is known that this model shows a localization-delocalization transition (LDT) as a function of the parameter $\mu$. The model is critical at $\mu = 1$ and the eigenstates are multifractals. Based on numerical simulations we demonstrate that the spectral statistics at criticality differs from semi–Poisson statistics which is expected to be a general feature of systems exhibiting a LDT or ‘weak chaos’.

In a recent paper Bogomolny \textit{et al.}\textsuperscript{1} have investigated a number of dynamical systems and found remarkable similarities between the spectral statistics of pseudointegrable billiards and the critical statistics in the Anderson model at the mobility edge. The latter has been investigated numerically by a number of authors\textsuperscript{2,3,4}. At the metal–insulator transition (MIT) properties intermediate between those predicted by random matrix theory (RMT)\textsuperscript{5} and those of uncorrelated spectra with Poisson statistics were found. On one hand the spectral statistics for small differences in energy is reminiscent of the universal level repulsion in metals, i.e. the near–dimensionality of the spectra and of the eigenstates are linked to the spectral statistics at criticality differs from semi–Poisson statistics which is expected to be a general feature of systems exhibiting a LDT or ‘weak chaos’.

The energy level statistics is universal at the MIT in the sense that the level number variance $\Sigma^2(\bar{N}) = \langle (\delta N)^2 \rangle$ is proportional to the mean number of levels $\bar{N} \gg 1$ and the coefficient $\chi$ is independent of the BCs. Dependence of the $P(s)$ function on the BCs in $d = 2$ quantum–Hall systems\textsuperscript{6} and appearance of the semi–Poisson statistics for a $d = 2$ Anderson model with symplectic symmetry\textsuperscript{7} obtained as an average over the BCs have also been reported since then. Interestingly similar spectral statistics has been obtained for the case of two interacting particles in a one dimensional disordered system at the interaction strength producing maximal mixing of the noninteracting basis.

At the transition point (MIT) the statistical properties of the spectra and of the eigenstates are linked to one another. A remarkable relation between the level compressibility $\chi$ and the density correlation dimension of the eigenstates $D_2$ has been derived in\textsuperscript{8,9}:

$$\chi = \frac{1}{2} \left( 1 - \frac{D_2}{d} \right).$$

(2)

The dimension $D_2$ describes how the fourth moment of the components of an eigenfunction $\psi$ scales with the linear length of the system\textsuperscript{10}. Generally, the $2p$-th moment scales as $\sum_i \langle |\psi_i|^{2p} \rangle \propto L^{-D_p(2p-1)}$, where in the case of multifractality $D_p$ is a nonlinear function of $p$.

The same parameter $D_2$ describes the scaling of the probability overlap of two states with an energy separation substantially exceeding the mean level spacing; hence the name of density correlation dimension. It has been shown\textsuperscript{11} that the heavily fluctuating local densities still produce a considerable overlap so that at the MIT level repulsion is still present.

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In Ref.\textsuperscript{12} quantum chaotic systems were numerically compared to one of the simplest models that provides semi–Poissonian statistics: the short–range plasma model (SRPM). The model describes $N$ levels that repel each other logarithmically as in conventional RMT\textsuperscript{5}, but with the interaction restricted to nearest neighbors only. For this SRPM many quantities can be computed analytically in the large $N$ limit. The spacing distribution is given by Eq. (3), the two–level correlation function reads

$$R(s) = 1 - e^{-4s}.$$
By allowing the logarithmic repulsion to act without limits in the SRPM one recovers the RMT result in which the level spacing distribution is well approximated by the Wigner–surmise

\[ P(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4} s^2}, \quad (5) \]

and the two–level correlation function is given by\[ R(s) = 1 - c^2(s) - \frac{dc(s)}{ds} \int_s^\infty c(s')ds', \quad (6) \]

with \(c(s) = \sin(\pi s)/\pi s\). From this correlation function the level number variance of standard RMT (up to \(1/L\) corrections) follows

\[ \Sigma^2(L) = \frac{2}{\pi^2} \left[ \ln(2\pi L) + 1 + \frac{\pi^2}{8} \right] \quad (7) \]

where \(\gamma = 0.5772\ldots\) denotes Euler’s constant. The level compressibility vanishes; the levels can be thought of as particles of an incompressible fluid\[ . \]

In the present paper we investigate the critical spectral statistics and the multifractality of the eigenstates in a random matrix model originally proposed by Mirlin \textit{et al.}\[ ] and later discussed in\[ .\] The \(N \times N\) matrices in this model are real symmetric and all entries are drawn from a normal distribution with mean zero, \(\langle H_{ij}\rangle = 0\). The variance depends on the distance of the matrix element from the diagonal\[ .\]

\[ \langle (H_{ij})^2 \rangle = [1 + (|i - j|/B)^{2\mu}]^{-1}. \quad (8) \]

In Ref\[ .\] it has been shown using field theoretical methods that for a fixed \(B \gg 1\) the statistical properties of such matrices for \(\mu < 1\) resemble those of RMT. On the other hand values \(\mu > 1\) in the limit \(N \to \infty\) lead to uncorrelated spectra similarly as in the case of banded random matrices\[ .\] The case \(\mu = 1\) was proven to be of special importance for it produces critical (multifractal) eigenstates and critical statistics. Due to the simplicity of the basic model and the possibility of analytical treatment of a MIT further details have been revealed recently by Kravtsov and Mirlin\[ .\]

Here we compare the statistical properties of energy spectra and eigenfunction properties of this random matrix model at \(\mu = 1\) with the semi–Poisson statistics of the SRPM in a regime so far inaccesable to field theoretical methods, at \(B = 1\). We study the shape of the nearest neighbor spacing distribution \(P(s)\), the two–level correlation function \(R(s)\) and obtain the spectral compressibility \(\chi\) from the asymptotic behavior of the number variance \(\Sigma^2\). We also show the multifractality of the eigenstates and provide the correlation dimension \(D_2\). We will show that for \(B = 1\) relation (\[\]) is satisfied provided that the spectra and the eigenstates are limited to a small portion around the middle of the band.

In our numerical investigation we have collected the spectra of \(N \times N\) matrices for \(N = 800\ldots4800\). The power law nature of the problem did not allow the use of efficient algorithms (e.g. the Lanczos–algorithm) usually applied for the study of the MIT in the Anderson model. We have performed a very careful unfolding since the \(N \to \infty\) and \(L \to \infty\) limits are approached very slowly. Based on the values of \(N\) we considered, we were able to extrapolate the expected behavior of several quantities in the \(N \to \infty\) limit. In all of our subsequent discussion we limit ourselves to the middle half of the spectra both for the eigenvalues and for the eigenvectors.

First the density of states of the model is plotted in Fig.\[ .\] The rescaled function is different from the semi–circle law. In Fig.\[ ] we show that the \(P(s)\) is independent of \(N\) and shows a non–negligible deviation from the semi–Poissonian form. The deviation is seen both for the low–\(s\) and the large–\(s\) part of the function. The low–\(s\) part of the \(P(s)\) is of the form \(P(s) \sim s\), but with a
different prefactor. The large–$s$ behavior differs considerably from the $\ln P(s) \sim -2s$ form expected from the semi–Poisson distribution $[1]$. The deviation from the semi–Poisson statistics seems to persist in the $N \rightarrow \infty$ limit. The ‘peakedness’ parameter $q = (s^2)^{-1}$ of the $P(s)$ converges to a value $q = 0.7342 \pm 0.0009$ larger than that calculated for the semi–Poisson distribution $[1]$ for which $q = 2/3$. We see that the shape of the $P(s)$ is indeed intermediate between the semi–Poisson and the Wigner–surmise. The spacing distribution in a similar random matrix model has been studied by Nishigaki $[2]$, whose result applied for our case is $q = 0.7624$ which is close to our value.

Fig. 3 shows that the level number variance $\Sigma^2(L)$ converges to a function that has a linear part, $\Sigma^2(L) \sim \chi L$ with a slope less than unity, $\chi < 1$, showing a nonzero compressibility of the spectra. However, this compressibility seems to converge to a value smaller than the $\chi = 0.5$ of the SRPM. This behavior is reflected also in the two–level correlation function that shows deviations from Eq. (3), while it is better described by the function calculated in $[2]$. It is of the form of $[2]$ with $c(s) = 0.25 \sin(\pi s)/\sinh(\pi s/4)$. The same kernel applies also for models discussed in Refs. $[3][4]$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Level number variance $\Sigma^2(N)$ as a function of the unfolded $N = L$. The continuous line shows the RMT case, Eq. (1), and the dashed line represents a semi–Poisson behavior of the form Eq. (1). In the insert we compare the two–level correlation function with the one obtained from the SRPM Eq. (3) (dashed line). The data are a little bit better described by the function obtained by Kravtsova. The RMT function, Eq. (1), is given as a dashed–dotted curve.}
\end{figure}

We have checked the validity of the relation $[3]$ by comparing the level compressibility obtained from the energy spectra with the value extracted from the multifractal dimension $D_2$ of the eigenstates. The results are shown in Fig. 4. As $N \rightarrow \infty$ the data extrapolate nicely to a common value of $\chi = 0.169 \pm 0.019$ that is less than the value 0.5 expected from the SRPM and is also smaller than the value 0.27 found at the Anderson transition. Our result is in full agreement with the recent analytical estimate of $\chi = 1/(2\pi)$ obtained in Refs. $[2][3]$ and also earlier in an analogous random matrix model $[4]$. Note that the $D_2$ value is obtained as an average of the states in the middle of the band. The inset of Fig. 4 shows that indeed the variance of $D_2$ decreases with increasing $N$ while taking the full set of eigenstates results in a distribution of these exponents that remains broad even in the $N \rightarrow \infty$ limit a phenomenon already noticed in $[2]$ and discussed in $[4]$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Level compressibility as a function of the logarithm of inverse matrix size, calculated from the eigenvalue spectra (open squares) and from the multifractal properties of the eigenstates (crosses) [c.f. Eq. (2)]. The dashed–dotted line stands for the horizontal line corresponds to the theoretical estimate of $\chi = 1/(2\pi) = 0.159154 \ldots$. The errorbars are calculated from the variance of the correlation dimension, $\sigma(D_2)$, the errorbar of the data from the spectrum is smaller than the size of the symbol. The insert shows the $N$–dependence of $\sigma(D_2)$ if the averaging is over the full band (circles) or the middle half of the band (crosses).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Distribution function of the eigenvector components $P(y)$ where $y = (\ln Q - m)/\sqrt{\sigma}$, $m$ is the mean and $\sigma$ is the variance of $\ln Q$. The dashed–dotted line stands for the power law asymptotics with a power of 1.6. The inset shows a close to Gaussian form for $P(y)$.}
\end{figure}

In Table I we collected the relevant quantities obtained from extrapolations to the thermodynamical limit. The
values of $D_2$ and $\alpha_0$ were obtained as an average of the corresponding exponent over the ensemble of states in the middle half of the spectrum. The $D_2$ and $\alpha_0$ for each state were obtained using the standard box counting algorithm.

A further evidence of the multifractal behavior of the eigenstates is also presented in the distribution function of the eigenvector components $P(Q)$ with $Q \equiv |\psi|^2$. It is broad at all length scales, i.e. a close to log-normal form is expected. In Fig. 8 the inset shows a very nice normal distribution of the variable ln $Q$ plotted after rescaling with the mean and the variance of it. However, the figure clearly shows deviation from the log-normal form especially for the low-$y$ part of the distribution that may be described with a power law tail of the form $y^{1.6}$. Similar deviations have already been detected and studied at the quantum–Hall transition [27] and are clearly seen at the Anderson–transition in $d = 3$ systems [28] as well.

In summary we have investigated the spectral properties of a random matrix ensemble with entries decaying away from the diagonal in a power–law fashion. We found that the spacing distribution function is similar to our numerical results the correlation dimension $\tilde{D}_2$ and $\alpha_0$ are characterized by a correlation dimension $\tilde{\chi}$. We also confirm the existence of multifractal states that are especially for the low-$\chi$ part of the distribution that may be described with a power law tail of the form $y^{1.6}$. Sim-ilar deviations have already been detected and studied at the quantum–Hall transition [27] and are clearly seen at the Anderson–transition in $d = 3$ systems [28] as well.

Acknowledgment: We are indebted to F. Evers, Y. Fyodorov, M. Janssen, V.E. Kravtsov, A.D. Mirlin, T.H. Seligman, and P. Shukla for stimulating discussions and financial support of I.V. from the Alexander von Humboldt Foundation and from Országos Tudományos Kutatási Alap (OTKA), Grant Nos. T029813, T024136 and F024135 are gratefully acknowledged.

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| TABLE I. Relevant parameters obtained in the $N \to \infty$ limit from the spectral and eigenvector statistics of the states in the middle half of the band. |
|-------------------------------------------------|
| quantity | origin | value |
|----------|--------|-------|
| eigenvalues $\xi$ | $\langle s^x \rangle^{-1}$ | $0.7342 \pm 0.0009$ |
| $\chi$ | $\Sigma^2(L) \to a + \chi L$ | $0.1732 \pm 0.0029$ |
| eigenvectors $D_2$ | $\langle D_2 \rangle$ | $0.6617 \pm 0.0389$ |
| $\chi$ | $(1 - D_2)/2$ | $0.1691 \pm 0.0195$ |