Light from Schwarzschild black holes in de Sitter expanding universe

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Abstract

A new method is applied for deriving the redshift and shadow of a Schwarzschild black hole moving freely in the de Sitter expanding universe as recorded by a co-moving remote observer. This method is mainly algebraic focusing on the transformation of the conserved quantities under the de Sitter isometry relating the black hole proper co-moving frame to observer's one. Hereby one extracts the general expressions of the redshifts and shadows of the black holes having peculiar velocities but their expressions are too extended to be written down here. Therefore, only some particular cases and intuitive expansions are presented while the complete results are given in an algebraic code on computer [1].

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1 Introduction

The light emitted by cosmic objects is one of the principal sources of empirical data in astrophysics. An important accessible observable is the redshift which encapsulates information about the cosmic expansion and possible peculiar velocity of the observed object. For separating these two contributions one combined so far the Lemaitre rule [2, 3] of Hubble’s law [4], governing the cosmological effect, [5–7] with the usual theory of the Doppler effect of special relativity [8] even though there are evidences that our universe is expanding.

Recently we proposed an improvement of this approach replacing the special relativity with our de Sitter relativity [9, 10] where local charts of the same type, playing the role of inertial frames, are related among themselves through de Sitter isometries. In the case of the longitudinal Doppler effect, when a point-like source is moving along the axis observer-source, we obtained a rdshift formula having a new term combining the cosmological and kinetic contributions in a non-trivial manner [11].

The next step is to extend this method to the black holes which, in general, are no point-like sources. The light emitted by a black hole comes from an apparent source situated on a sphere, surrounding the black hole, which is observed as the black hole shadow. When this is not negligible, as in the case of the object M87 [12, 13], and the black hole may have a peculiar velocity, the Doppler effect is no longer longitudinal such that the transverse contributions due to the black hole shadow must be evaluated. This can be done only by studying simultaneously the Doppler effect and black hole shadow in the same theoretical framework.

The black hole shadow was extensively studied by many authors which used a geometrical method which exploits the tangent to the null geodesic in the point where this meets the observer. Thus the shadows of the non-rotating Schwarzschild black holes were studied in Minkowski space-time [14] or in expanding universes [15–20] while for the rotating black holes the Kerr [21, 22] or Kerr-de Sitter [23–26] metrics were considered. Other recent studies focused on more complex models which were studied with the same geometric method [27–42] but which is not suitable for studying the Doppler effect.

As in this paper we would like to study simultaneously the redshifts and shadows of the Schwarzschild black holes having peculiar velocities in de Sitter expanding universe we give up the geometric method adopting the same algebraic approach as in Ref. [11]. This is based on our de Sitter relativity we use now for relating the moving black hole proper frames to those of remote co-moving observers freely falling in the de
Sitter expanding universe.

We start supposing that our expanding universe is satisfactory described by the expanding portion of a \((1 + 3)\)-dimensional de Sitter manifold. As the actual observations show with reasonable accuracy that this universe is spatially flat, we consider only local charts with Painlevé coordinates \[43\] since these have flat space sections. In these local charts, we call from now co-moving frames, the coordinates we use are the cosmic time and Cartesian or spherical space coordinates. These frames may carry observers related among themselves through the de Sitter isometries which transform simultaneously the coordinates and the conserved quantities \[9, 10\].

The Schwarzschild black holes in de Sitter expanding universe are usually considered in proper frames with static coordinates and Kottler metric \[44\], known mainly as the Schwarzschild-de Sitter proper frames. However, here we give up the static frames preferring the corresponding co-moving frames with Painlevé coordinates whose metrics have the same asymptotic behaviour as the metric of the observer co-moving frame. Then for a remote observer the black hole co-moving frame appears as an empty de Sitter one which can be related to observer’s frame through a de Sitter isometry, in accordance with the relative motion of the black hole with respect to observer.

We assume that, at the initial moment when the photon is emitted by black hole, this is translated and has a relative velocity with respect to the remote observer. Then a suitable isometry will give the conserved quantities measured in observer’s frame we need for extracting physical results without resorting to geodesics or other geometric quantities. For this reason we say that our method is algebraic. We use this method for deriving the redshift and shadow of a non-rotating Schwarzschild black hole freely moving in the de Sitter expanding universe.

We start in the second section with a brief review of the metrics of the black hole and observer co-moving frames with Cartesian or spherical coordinates, revisiting the equation giving the geodesic shapes in the black hole co-moving frames. Starting with this equation we present in the next section the null geodesics around the black hole determining its shadow. The fourth section is devoted to the de Sitter isometries relating the conserved quantities measured in different co-moving frames. In the next section we obtain our new results assuming that for a remote observer the light is emitted by an apparent source on a rectilinear de Sitter geodesic whose conserved quantities can be determined. Furthermore, by using an isometry formed by a translation followed by a Lorenzian isometry we obtain the conserved quantities in observer’s co-moving frame.
from which we extract the observed redshift and shadow of the moving black hole. More specific, the redshift results from the observed energy while the angular radius of the black hole shadow is derived from the photon angular momentum which must vanish when the photon meets the origin of observer’s co-moving frame. These results are elementary but with a large number of terms that cannot be written here in the general case of a moving black hole. Consequently, we restrict ourselves to present here their series expansions with respect to a small parameter, the particular case when the relative velocity vanishes and the flat limit. The complete results which cannot be written down here are given in an algebraic code on computer \[1\]. In order to convince oneself that the flat limit is correct we derive in App. B the results that can be obtained by applying our algebraic method to a Schwarzschild black hole in Minkowski flat space-time. Finally we present some concluding remarks.

As our approach may be applied even in quantum theory we introduce a special notation denoting by \( \omega_H = \sqrt{\frac{\Lambda}{3}c} \) the de Sitter Hubble constant (frequency) since \( H \) is reserved for the Hamiltonian operator \[45\]. Moreover, the Hubble time \( t_H = \frac{1}{\omega_H} \) and the Hubble length \( l_H = \frac{c}{\omega_H} \) will have the same form in the natural Planck units with \( c = \hbar = G = 1 \) we use here.

## 2 Co-moving frames

The proper frames of the spherically symmetric static systems in a (1+3)-dimensional isotropic Riemannian manifold, \((M, g)\), are static local charts \(\{x\}\) with spherical symmetry whose coordinates \(x^\mu (\alpha, \mu, \nu, ... = 0, 1, 2, 3)\) can be chosen in different manners. In what follows we consider either Cartesian space coordinates \(x=(x^1, x^2, x^3)\) or associated spherical ones \((r, \theta, \phi)\) with \(r = |x|\). The traditional static proper frames, \(\{t_s, x\}\) or \(\{t_s, r, \theta, \phi\}\), depend on the static time \(t_s\) having the line elements

\[
\begin{align*}
    ds^2 &= f(|x|)dt_s^2 + \left(\frac{x \cdot dx}{|x|}\right)^2 \left(1 - \frac{1}{f(|x|)}\right) - dx \cdot dx \\
    &= f(r) dt_s^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2,
\end{align*}
\]
where \(d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2\). These line elements can be put at any time in Painlevé forms [43],

\[
\begin{align*}
ds^2 &= f(|x|) dt^2 + 2 \frac{1}{|x|} \sqrt{1 - f(|x|)} x \cdot dx \, dt - dx \cdot dx, \quad (3) \\
&= f(r) dt^2 + 2 \sqrt{1 - f(r)} \, dt \, dr - dr^2 - r^2 d\Omega^2, \quad (4)
\end{align*}
\]

substituting in Eqs. (1) and (2)

\[
t_s = t + \int dr \frac{\sqrt{1 - f(r)}}{f(r)}, \quad (5)
\]

where \(t\) represents the cosmic (or proper) time of the proper frames having flat space sections with Cartesian, \(\{t, x\}\), or spherical, \(\{t, r, \theta, \phi\}\), coordinates.

Here we focus on a Schwarzschild black hole of mass \(M\) embedded in the de Sitter expanding universe for which the metric (2) of its static frame has the Kottler [44] (or Schwarzschild-de Sitter) form with \(f(r) = 1 - \frac{2M}{r} - \omega^2_H r^2\).

where, as mentioned before, \(\omega_H\) is the de Sitter Hubble constant in our notation. The corresponding frames with Painlevé coordinates have the asymptotic behaviour of the de Sitter co-moving frames with \(f(r) \to f_0(r) = 1 - \omega^2_H r^2\). For this reason we say that the black hole proper frames with Painlevé coordinates, denoted by \(\{t, x\}_{BH}\) and \(\{t, r, \theta, \phi\}_{BH}\), are the co-moving proper frames of the Schwaezschild black hole in de Sitter expanding universe.

The proper frames of the remote observers, \(\{t, x\}\) and \(\{t, r, \theta, \phi\}\), located in the asymptotic zone, are genuine de Sitter co-moving frames where the astronomical observations are performed and recorded. The observers stay at rest in origins of their own proper frames evolving along the unique time-like Killing vector field of the de Sitter geometry which is not time-like everywhere but has this property just in the null cone where the observations are allowed [45].

Here we use simultaneously Cartesian and spherical coordinates since the Cartesian coordinates are suitable for studying the conserved quantities and the transformation rules under isometries while the spherical coordinates help one to integrate the geodesic equations. For example, the spherical symmetry is obvious in Cartesian coordinates where the metrics (1) and (3) are invariant under the global rotations \(x^i \to R^i_j x^j\) such that we can use the vector notation. On the other hand, only in
spherical coordinates one can integrate the geodesics equations in the black hole co-moving frame we revisit briefly in the next.

In the frame \( \{ t, r, \theta, \phi \}_{\text{BH}} \) with the line element \( g \) the conserved quantities along geodesics are the energy \( E \) and angular momentum \( L \). These give rise to the prime integrals of a geodesic of a particle of mass \( m \) moving in the equatorial plane of the black hole (with fixed \( \theta = \frac{\pi}{2} \)) as

\[
E = f(r) \dot{t} + \sqrt{1 - f(r)} \dot{r}, \\
L = r^2 \dot{\phi},
\]

(7)

(8)

where 'dot' denotes the derivatives with respect to the affine parameter \( \lambda \) which satisfies \( ds = m \, d\lambda \). The third prime integral comes from the line element in the equatorial plane that reads,

\[
f(r) \dot{t}^2 + 2 \sqrt{1 - f(r)} \dot{t} \dot{r} - \dot{r}^2 - r^2 \dot{\phi}^2 = m^2.
\]

(9)
as it results from Eq. (4). Hereby one may derive the function \( r(\phi) \) substituting

\[
r \to r(\phi), \quad \dot{r} \to \frac{dr(\phi)}{d\phi} \dot{\phi} = \frac{L}{r(\phi)^2} \frac{dr(\phi)}{d\phi}.
\]

(10)

After a little calculation, combining the above prime integrals, one obtains the well-known equation

\[
\left( \frac{dr(\phi)}{d\phi} \right)^2 - r(\phi)^4 \frac{E^2}{L^2} + r(\phi)^2 f(r(\phi)) \left( 1 + r(\phi)^2 \frac{m^2}{L^2} \right) = 0,
\]

(11)
giving the geodesic shapes but which is not enough for finding the time behaviour of the functions \( \phi(t) \) and \( r(t) = r[\phi(t)] \) for which one must apply special methods \[46\].

Note that Eq. (11) derived in the co-moving frame is the same as that of the static frame since this equation is static giving only the shape of trajectory in the same space coordinates. In fact, the time evolution on geodesics is quite different in the static and co-moving frames.

3 Light around black holes

The problem of the gravitational lensing which has a long history \[47\] was studied in general relativity first by Einstein and Eddington \[48\] but was solved by Darwin \[49, 50\] which derived the null geodesics around the
photon sphere of a Schwarzschild black hole in the flat Minkowski spacetime. Recent studies focused on the general properties of the photon spheres \[51\] and their role in gravitational lensing \[52–56\].

Bellow we restrict ourselves to inspect briefly the null geodesics in the co-moving frame \(\{t, r, \theta, \phi\}_{BH}\) of the Schwarzschild-de Sitter system. The shapes of these geodesics are given by the functions \(r(\phi)\) which satisfy Eq. (11) with \(m = 0\). We assume first that there exists at least a null geodesic where the conditions

\[
\frac{dr(\phi)}{d\phi} = \frac{d^2 r(\phi)}{d\phi^2} = 0, \tag{12}
\]

are satisfied simultaneously for any \(\phi\). After a little algebra it results that this happens only on the photon sphere of radius \(r_{ph} = 3M\) where the energy and angular momentum must satisfy

\[
L = \frac{3\sqrt{3}ME}{\sqrt{1 - 27\omega_H^2 M^2}}, \tag{13}
\]

as derived in Ref. \[15\]. On the photon sphere the photons are trapped on circular orbits of radius \(r_{ph}\) without escaping outside such that all the observed photons must have trajectories outside this sphere.

Looking for all the null geodesics whose energies and angular momenta are related in a similar manner, we substitute the angular momentum \(\omega_H\) in Eq. (11) with \(m = 0\) relaxing the condition \(r = 3M\).
We obtain thus the new equation

\[ 27M^2 \left( \frac{dr(\phi)}{d\phi} \right)^2 - 57M^3 r(\phi) + 27M^2 r(\phi)^2 - r(\phi)^4 = 0, \]  

(14)

which is independent on the Hubble de Sitter constant \( \omega_H \). This equation can be analytically solved finding, apart from the circular orbits of photon sphere, two spiral solutions of the form

\[ r(\pm)(\phi) = 3M \frac{(6Me^{\mp\phi} + 1)^2}{(6Me^{\mp\phi} + 1)^2 - 36Me^{\mp\phi}} \]  

(15)

which are determined up to a rotation, \( \phi \rightarrow \phi - \phi_0 \), fixing the origin of this angular coordinate. For example, if we translate the arguments of the functions \( r(\pm) \) as \( \phi \rightarrow \phi^\pm = \phi \pm \ln 6M \) then we recover the elegant Darwin solution \[49\]

\[ \frac{1}{r(\pm)(\phi^\pm)} = -\frac{1}{6M} + \frac{1}{2M} \tanh^2 \left( \frac{\phi^\pm}{2} \right), \]  

(16)

of the spiral geodesics. Thus we may conclude that the presence of the de Sitter gravity is encapsulated only in Eq. (13) while the photon sphere and the shapes of the spiral geodesics remain the same as in Minkowski flat space-time (when \( \omega_H = 0 \)).

The spiral geodesics are symmetric in the sense that \( r(\pm)(\phi) = r(\mp)(-\phi) \), having vertical asymptotes at

\[ \phi_{(\pm)1,2} = -\ln \frac{2 \mp \sqrt{3}}{6M}, \quad \phi_{(-)1,2} = \ln \frac{2 \mp \sqrt{3}}{6M}, \]  

(17)

In Fig. 1, we see that the functions \( r(\pm)(\phi) \) are defined on the domains \( (-\infty, \phi_{(\pm)1}) \cup (\phi_{(\pm)2}, \infty) \) since between the vertical asymptotes their values are negative having no physical meaning. It is interesting that this opaque window is independent on the black hole mass,

\[ |\phi_{(\pm)2} - \phi_{(\pm)1}| = \ln \frac{2 + \sqrt{3}}{2 - \sqrt{3}} \simeq 0.8384 \pi. \]  

(18)

On the physical domain these trajectories remain outside the photon sphere, \( r(\pm)(\phi) > 3M \), but approaching to this for large \( |\phi| \) since

\[ \lim_{|\phi| \to \infty} r(\pm)(\phi) = 3M. \]  

(19)

This gives us the image of the spiral geodesics rolled out around the photon sphere (as in Fig. 2.) escaping outside only when \( \phi \) is approaching
Figure 2: The functions \( r(\pm)(\phi) \) of the spiral photon geodesics rolling out and escaping from the photon sphere of radius \( 3M \).

to the values (17) where the functions \( r(\pm) \) can take larger values near singularities.

The geodesics \( r(\pm)(\phi) \) are the closest trajectories to the photon sphere of the first photons that can be observed at the limit of the black hole shadow. Therefore, for studying this shadow and the associated redshift we have to consider only these photons.

4 de Sitter isometries

The de Sitter co-moving frames play the role of inertial frames being related among themselves through de Sitter isometries as in our de Sitter relativity [9, 10]. Moreover, the system black hole-observers has the asymptotic de Sitter symmetry which governs the relative motion of the black hole with respect to different observers such that we may use these isometries for relating the observers co-moving frames to the black hole one.

The de Sitter isometries can be studied easily since this manifold is a hyperboloid of radius \( 1/\omega_H \) embedded in the five-dimensional flat space-time \((M^5, \eta^5)\) of coordinates \( z^A \) (labelled by the indices \( A, B, ... = 0, 1, 2, 3, 4 \)) and metric \( \eta^5 = \text{diag}(1, -1, -1, -1, -1) \). Any local charts can be introduced giving the set of functions \( z^A(x) \) which solve the hy-
perboloid equation,
\[ \eta^{5}_{AB} z^A(x) z^B(x) = -\frac{1}{\omega_{H}^2}, \]
(20)
giving the line element
\[ ds^2 = \eta^{5}_{AB} dz^A(x) dz^B(x) = g_{\mu\nu} dx^\mu dx^\nu. \]
(21)
The functions that introduce our Painlevé coordinates are
\[ z^0(x) = \frac{1}{2\omega_{H}^2} \left[ e^{\omega_{H}t} - e^{-\omega_{H}t}(1 - x^2) \right], \]
\[ z^i(x) = x^i, \]
\[ z^4(x) = \frac{1}{2\omega_{H}^2} \left[ e^{\omega_{H}t} + e^{-\omega_{H}t}(1 - x^2) \right]. \]
(22)

The de Sitter isometry group is just the stable group \( SO(1,4) \) of the embedding manifold \((M^5, \eta^5)\) that leave invariant its metric and implicitly Eq. (20). Therefore, given a system of coordinates defined by the functions \( z = z(x) \), each transformation \( g \in SO(1,4) \) gives rise to the isometry \( x' \rightarrow x = \phi_g(x') \) derived from the system
\[ z[\phi_g(x')] = g z(x'). \]
(23)
The local charts related through these isometries play the same role as the inertial frames of special relativity.

The classical conserved quantities under de Sitter isometries are given by the Killing vectors \( k_{(AB)} \) of the de Sitter manifold [45] that are related to those of \((M^5, \eta^5)\) as
\[ K^{(AB)}_C dz^C = z^A dz^B - z^B dz^A = k_{(AB)}^\mu dx^\mu, \]
(24)
allowing us to derive the covariant components of the Killing vectors in an arbitrary chart \( \{x\} \) of the de Sitter space-time as
\[ k_{(AB)\mu} = \eta^5_{AC} \eta^5_{BD} k_{(CD)}^\mu = z_A \partial_{\mu} z_B - z_B \partial_{\mu} z_A, \]
(25)
where \( z_A = \eta_{AB} z^B \). The conserved quantities along the time-like geodesic of a particle of mass \( m \) have the general form \( K_{(AB)}(x, P) = \omega_{H} k_{(AB)\mu} \dot{x}^\mu \).
The conserved quantities with physical meaning are the energy \( E \), momentum \( P \), angular momentum \( L \) and a specific vector \( Q \), we called the adjoint momentum [45]. A geodesic in the co-moving frame \( \{t, x\} \) [57],
\[ x(t) = x_0 e^{\omega_{H}(t-t_0)} + \frac{P}{P^2} \left( \sqrt{m^2 + P^2} e^{\omega_{H}(t-t_0)} - \sqrt{m^2 e^{2\omega_{H}(t-t_0)} + P^2} \right), \]
(26)
depends only on the momentum \( P = \lvert P \rvert \) and the initial condition \( x(t_0) = x_0 \) fixed at the time \( t_0 \). The conserved quantities in an arbitrary point \((t, x(t))\) of this geodesic read [9, 57],

\[
E = \omega_H x(t) \cdot P e^{-\omega_H(t-t_0)} + \sqrt{m^2 + P^2 e^{-2\omega_H(t-t_0)}},
\]

\[
L = x(t) \wedge P,
\]

\[
Q = 2\omega_H x(t) E e^{-\omega_H(t-t_0)} + P e^{-2\omega_H(t-t_0)} \left[ 1 - \omega_H^2 x(t)^2 \right].
\]

satisfying the obvious identity

\[
E^2 - \omega_H^2 L^2 - P \cdot Q = m^2
\]

corresponding to the first Casimir invariant of the \( so(1, 4) \) algebra [45].

In the flat limit, when \( \omega_H \to 0 \), we have \( Q \to P \) such that this identity becomes just the usual mass-shell condition, \( E^2 - P^2 = m^2 \), of special relativity.

The conserved quantities \( E, P \) and the new ones,

\[
K = -\frac{1}{2\omega_H} (P - Q), \quad R = -\frac{1}{2\omega_H} (P + Q),
\]

form a skew-symmetric tensor on \( M^5 \),

\[
\mathcal{K}(x, P) = \begin{pmatrix}
0 & \omega_H K_1 & \omega_H K_2 & \omega_H K_3 & E \\
-\omega_H K_1 & 0 & \omega_H L_3 & -\omega_H L_2 & \omega_H R_1 \\
-\omega_H K_2 & -\omega_H L_3 & 0 & \omega_H L_1 & \omega_H R_2 \\
-\omega_H K_3 & \omega_H L_2 & -\omega_H L_1 & 0 & \omega_H R_3 \\
E & -\omega_H R_1 & -\omega_H R_2 & -\omega_H R_3 & 0
\end{pmatrix},
\]

whose components transform under the isometries \( x' \to x = \phi_g(x') \) defined by Eq. (23) as

\[
\mathcal{K}(t, x, P) = \bar{g} \mathcal{K}'(t', x', P') \bar{g}^T,
\]

where \( \bar{g} = \eta^5 g \eta^5 \) [9].

Summarizing, we can say that the de Sitter isometries are generated globally by the \( SO(1, 4) \) transformations which determine the transformations of the coordinates and conserved quantities. We have thus a specific relativity on the de Sitter space-time allowing us to study different relativistic effects in the presence of the de Sitter gravity. In what follows we use the Lorentzian isometries defined in Ref. [9] and the translations presented in the App. A.
Figure 3: The geodesic $r_{(+)}(\phi)$ observed as a rectilinear de Sitter one of a photon of momentum $k$ emitted by the apparent source $S$ situated on the sphere of radius $r_S > 3\sqrt{3}M$.

5 Observing light from black holes

Let us consider now a mobile black hole in its proper co-moving frame $\{t', x'\}_{BH}$ with the origin in $O_{BH}$ and a fixed remote observer in his own co-moving frame $\{t, x\}$ having the origin in $O$. We consider that the space Cartesian axes of these frames are parallel with the basis of unit vectors $(e_1, e_2, e_3)$ such that the geodesic of the emitted photon remains in the plane $(e_1, e_2)$. In this geometry we assume that the photon is emitted at the initial moment $t = t' = 0$ when the origin $O_{BH}$ is translated with $d$ and has the relative velocity $V$ with respect to $O$.

5.1 Related conserved quantities

A remote observer sees the photon of momentum $k = n_k k$, energy $E = k = |k|$ and angular momentum (13), as emitted from an apparent source $S$ of position vector $n_S r_S$ on the sphere of radius

$$r_S = \frac{L}{k} = \frac{3\sqrt{3}M}{\sqrt{1 - 2\omega_H^2 M^2}},$$

which is the apparent radius of the black hole shadow. Therefore, $r_S$ is the radius of the sphere hosting different photon sources $S$ that can be
observed nearest to the black hole shadow (as in Fig. 3). When $\omega_H \to 0$ this becomes just the shadow radius $3\sqrt{3}M$ derived in special relativity [49].

The apparent trajectory of the emitted photon is a de Sitter null geodesic of momentum $k$ that, according to Eq. (26), reads

$$x_{ph}(t) = n_S r_S e^{\omega_H t} + n_k (e^{\omega_H t} - 1),$$

(35)

complying with the initial condition $x_{ph}(0) = n_S r_S$. This geodesic depends on the orthogonal unit vectors that can be represented as

$$n_k = -e_1 \cos \alpha - e_2 \sin \alpha,$$

(36)

$$n_S = -e_1 \sin \alpha + e_2 \cos \alpha,$$

(37)

where the angle $\alpha$, giving the apparent direction of the photon, will depend on observer’s position. We have thus the opportunity of defining the de Sitter conserved quantities on this geodesic as in an apparent de Sitter empty co-moving frame $\{t', x'\}$ associated to $\{t', x'\}_BH$. One vector is missing, namely the adjoint momentum $Q'$ that can be derived simply at the time $t = 0$ according to Eq. (29). We complete thus the set of conserved quantities,

$$E' = k,$$

(38)

$$P' = n_k k,$$

(39)

$$L' = e_3 r_S k,$$

(40)

$$Q' = n_S 2\omega_H r_S k + n_k k (1 - \omega_H^2 r_S^2),$$

(41)

which satisfy the condition (30).

Now we can deduce how these conserved quantities are measured by the fixed observer $O$ since the observer frame $\{t, x\}$ and the apparent black hole one, $\{t', x'\}$, are related through an isometry, $x = \phi_g(x')$, of the de Sitter relativity [9]. According to our hypotheses, this is generated by the $\text{SO}(1,4)$ transformation,

$$g = g(V)g(a),$$

(42)

formed by a translation (A.1) of parameter $a = e_1 d = (d, 0, 0)$, having the form [58, 9],

$$g(a) = \begin{pmatrix}
1 + \frac{1}{2} \omega_H^2 d^2 & \omega_H d & 0 & 0 & \frac{1}{2} \omega_H^2 d^2 \\
\omega_H d & 1 & 0 & 0 & \omega_H d \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-\frac{1}{2} \omega_H^2 d^2 & -\omega_H d & 0 & 0 & 1 - \frac{1}{2} \omega_H^2 d^2
\end{pmatrix},$$

(43)
followed by the Lorentz boost

\[ \mathbf{V} = (V, 0, 0) \to \mathbf{g}(\mathbf{V}) = \begin{pmatrix}
\frac{1}{\sqrt{1-V^2}} & \frac{V}{\sqrt{1-V^2}} & 0 & 0 & 0 \\
\frac{V}{\sqrt{1-V^2}} & \frac{1}{\sqrt{1-V^2}} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad (44) \]

of the particular Lorenzian isometry we need here [9].

Applying then the transformation (33) with \( \mathbf{g} \) given by Eq. (42) we obtain the conserved quantities observed by \( O \). This calculation is elementary but complicated involving many terms that can be manipulated only by using suitable algebraic codes on computer. For presenting the final result it is convenient to introduce the notation

\[ \delta = \omega_H \frac{d}{l_H}, \quad \xi = \frac{r_S}{d}, \quad (45) \]

which allows us to write down the conserved quantities in observer’s frame as

\[ E = \frac{k}{\sqrt{1-V^2}} \left[ 1 - V \cos \alpha + \delta (V - \cos \alpha - \xi \sin \alpha) - \frac{\delta^2}{2} (1 - \xi^2) V \cos \alpha \right], \quad (46) \]

\[ P^1 = k \left( \frac{1}{\sqrt{1-V^2}} - 1 \right) \left[ \delta (1 - \xi \sin \alpha) - \frac{\delta^2}{2} (1 - \xi^2) \cos \alpha \right] + \frac{k}{\sqrt{1-V^2}} [V (1 - \delta \cos \alpha) - \cos \alpha], \quad (47) \]

\[ P^2 = k \left( \frac{1}{\sqrt{1-V^2}} - 1 \right) \left[ \delta \xi (1 - \cos \alpha) - \frac{\delta^2}{2} (1 + \xi^2) \sin \alpha \right] - k \sin \alpha + \frac{k \delta V}{\sqrt{1-V^2}} (\xi - \sin \alpha), \quad (48) \]

\[ Q^1 = k \left( \frac{1}{\sqrt{1-V^2}} + 1 \right) \left[ \delta (1 - \xi \sin \alpha) - \frac{\delta^2}{2} (1 - \xi^2) \cos \alpha \right] + \frac{k}{\sqrt{1-V^2}} [V (1 - \delta \cos \alpha) - \cos \alpha], \quad (49) \]
\[ Q^2 = k \left( \frac{1}{\sqrt{1-V^2}} + 1 \right) \left[ -\delta \xi (\delta - \cos \alpha) + \frac{\delta^2}{2} (1 + \xi^2) \sin \alpha \right] \\
- k \sin \alpha - \frac{k \delta V}{\sqrt{1-V^2}} (\xi - \sin \alpha), \quad (50) \]

\[ L_3 = \frac{k \delta}{\omega_H \sqrt{1-V^2}} \left[ \xi (1 - V \cos \alpha) - \sin \alpha + \delta V \\
- \frac{\delta V}{2} (1 + \xi^2) \sin \alpha \right] \quad (51) \]

while \( L_1 = L_2 = P^3 = Q^3 = 0 \). As expected, these quantities satisfy the invariant identity (30).

In other respects, we observe that all the vector components we derived above can take any real values in contrast with the energy which must remain positive definite. This condition is fulfilled only if

\[ V < V_{lim} = \frac{2(1 - \delta \cos \alpha)}{\delta^2 (1 - \xi^2) \cos \alpha - 2 \delta (1 - \xi \sin \alpha) + 2 \cos \alpha} \quad (52) \]

This is in fact the mandatory condition for observing the photon in \( O \) at finite time. When the relative velocity \( V \) exceeds this limit then the photon cannot arrive in \( O \) at finite time because of the background expansion. Thus \( V_{lim} \) defines a new velocity horizon restricting the velocities such that for very far sources with \( \alpha = 0 \) and \( \delta = \omega_H d \sim 1 \) this limit vanishes.

### 5.2 Shadow and redshift

The angle \( \alpha \) which depends on the relative position between black hole and observer can be found simply imposing the condition \( L_3 = 0 \) when the photon trajectory is passing through the point \( O \). Solving this equation for \( L_3 \) given by Eq. (51) we find

\[ \tan \frac{\alpha}{2} = \frac{\delta V (1 + \xi^2) + 2 - \sqrt{[V \delta (1 - \xi^2) + 2]^2 + 4 \xi^2 (V^2 - 1)}}{2\xi(V\delta + V + 1)}. \quad (53) \]

Substituting then this angle in Eqs. (46), (51) and (52) we obtain as final result all the conserved quantities measured by \( O \) and the maximal velocity \( V_{lim} \).

Hereby we can extract the quantities of general interest namely \( \sin \alpha \), measured in the black hole proper frame, the angular radius

\[ \sin \alpha_{obs} = \frac{|P^2|}{P}, \quad P = \sqrt{P^1^2 + P^2^2}, \quad (54) \]
of the shadow, measured in observer’s frame, and the redshift \( z \) defined as

\[
1 + z = \frac{E'}{E} = \frac{k}{E}.
\]  

(55)

Unfortunately, the exact expressions of these quantities have a huge number of terms that cannot be written here but can be manipulated on computer [1] for extracting significant particular cases or intuitive approximations.

The simplest particular case is when the black hole does not have an initial relative velocity with respect to \( O \). Then by setting \( V = 0 \) we obtain the simple formulas

\[
\sin \alpha_{\text{obs}} = \sin \alpha = \xi,
\]

(56)

\[
\frac{1}{1 + z} = 1 - \omega_H d \sqrt{1 - \xi^2} = 1 - \omega_H d \cos \alpha,
\]

(57)

showing how the observations of the shadow and redshift are related each other.

In the general case of \( V \neq 0 \), we observe that the expansions around \( \xi = 0 \) are useful since this is the only parameter which remains very small in astronomical observations as long as the other ones have larger ranges, \( 0 < \delta = \omega_H d < 1 \) and \( 0 < V < V_{\text{lim}} \). Computing these series we write down here only the lowest terms

\[
\sin \alpha = \frac{2(1 + \omega_H d V - V)}{2 + \omega_H d V} \xi + \mathcal{O}(\xi^3),
\]

(58)

\[
\sin \alpha_{\text{obs}} = \frac{2 \xi \sqrt{1 - V^2}}{2 + \omega_H d V (1 - \xi^2)} = \frac{2 \sqrt{1 - V^2}}{2 + \omega_H d V} \xi + \mathcal{O}(\xi^3),
\]

(59)

\[
\frac{1}{1 + z} = \sqrt{\frac{1 - V}{1 + V}} \left( 1 - \omega_H d - \frac{\omega_H^2 d^2}{2} \frac{V}{1 - V} \right) + \mathcal{O}(\xi^2),
\]

(60)

which are still comprehensible and can be interpreted. Similarly, for the velocity limit we have

\[
V_{\text{lim}} = \frac{2(1 - \omega_H d)}{1 + (1 - \omega_H d)^2} + \mathcal{O}(\xi^2).
\]

(61)

The principal novelty here is that \( \sin \alpha_{\text{obs}} \) depends on the relative velocity \( V \) as in Eq. (59) just from the first order of the expansion which may be observationally accessible. In contrast, the expansion (60) has a first term independent on \( \xi \) recovering just the redshift formula we derived recently for a point-like source moving along the \( \mathbf{e}_1 \) axis [III]. Therefore, the influence of the black hole dimensions could be observed only when the second term of the order \( \mathcal{O}(\xi^2) \) could be measured with a satisfactory accuracy.
5.3 Flat limit

The limit of the de Sitter relativity when the de Sitter-Hubble constant \( \omega_H \) vanishes is just the usual special relativity. Then all the co-moving frames of the de Sitter relativity become inertial frames in Minkowski space-time without affecting the black hole geometry in its proper frame. Now a remote observer sees a photon emitted in \( S \) as having an apparent rectilinear trajectory with momentum \( k \) and energy \( E = k \). Then all the measured quantities can be obtained from the de Sitter ones in the limit \( \omega_H \to 0 \). In this limit our principal parameters becomes

\[
 r_S \to \hat{r}_S = 3\sqrt{3}M, \quad \xi \to \hat{\xi} = \frac{3\sqrt{3}M}{d}
\]

while the other quantities have the limits

\[
 \sin \hat{\alpha} = \lim_{\omega_H \to 0} \sin \alpha = (1 - V) \hat{\xi} \Delta, \quad \hat{\alpha}_{obs} = \lim_{\omega_H \to 0} \sin \alpha_{obs} = \sqrt{1 - V^2} \hat{\xi}, \quad \frac{1}{1 + \hat{\eta}} = \lim_{\omega_H \to 0} \frac{1}{1 + \hat{\xi}^2} = \sqrt{\frac{1 - V}{1 + V}} \Delta,
\]

where

\[
 \Delta = \frac{1 + V}{1 + V \cos \hat{\alpha}_{obs}} = \frac{1 + V}{1 + V \sqrt{1 - \hat{\xi}^2(1 - V^2)}}
\]

\[
 = 1 + V(1 - V) \hat{\xi}^2 + O(\hat{\xi}^4).
\]

As expected, hereby we recover the usual aberration formulas

\[
 \sin \hat{\alpha} = \frac{\sqrt{1 - V^2} \sin \hat{\alpha}_{obs}}{1 + V \cos \hat{\alpha}_{obs}} \Leftrightarrow \sin \hat{\alpha}_{obs} = \frac{\sqrt{1 - V^2} \sin \hat{\alpha}}{1 - V \cos \hat{\alpha}}.
\]

In addition, we obtain the limit velocity

\[
 \hat{V}_{lim} = \lim_{\omega_H \to 0} V_{lim} = \frac{1}{\cos \hat{\alpha}} \geq 1,
\]

does not make sense exceeding the speed of light when \( \hat{\alpha} \to 0 \). Thus the limitation of velocities disappears together with the de Sitter event horizon.

For interpreting these results a good choice is \( \Delta \approx 1 \) as the parameter \( \xi \) remains very small. Then Eq. (65) is just the redshift due to the Doppler effect in special relativity. Moreover, we may convince ourselves that all the limits derived above are just the results that may be obtained by applying our method in special relativity, as presented briefly in the App. B.
6 Concluding remarks

We derived the shadow and redshift of a Schwarzschild black hole with a peculiar velocity in the de Sitter expanding universe by using an algebraic method offered by our de Sitter relativity. This focuses only on the conserved quantities, applying suitable isometries for transforming the conserved quantities of the photon, emitted by a moving black hole, into those recorded by a co-moving remote observer.

The principal advantage of this method is the control upon the isometries relating the black hole and observer frames that now have a precise physical meaning. The choice of the co-moving frames with Painlevé coordinates is crucial since then we can use the translation (A.2) which does not affect the time. In contrast, when we use static coordinates, defined by Eq. (A.4), the same translation gives the transformation (A.6) which affects the time such that it is difficult to synchronise the clocks by setting common initial conditions.

In other respects, the conserved quantities are not enough for understanding the entire information carried out by the light emitted by moving black holes. There are important observable quantities resulted from the coordinate transformations under isometries as, for example, the photon propagation time or the real distance between observer and black hole at the time when the photon is measured. In Ref. [11] we derived such quantities in the longitudinal case of a point-like source moving along the direction observer-black hole. Therefore, when we apply the algebraic method the coordinate transformations under isometries or other geometric methods may complete our investigation.

Another problem is the relation between the algebraic and geometric methods. In the simplest classical example of $\omega_H = 0$ and $V = 0$ we obtain from Eq. (64) the rule $\sin \hat{\alpha}_{\text{obs}} = \xi$ which is the result of Ref. [11] up to the factor $\sqrt{1 - \frac{2M}{d}}$ whose second term does not contribute in the asymptotic zone. This means that some minor differences may appear between these two methods since in the algebraic approach the observers are in the asymptotic zone where $d \gg 2M$. However, now it is premature to say more about this relationship before analysing many examples in de Sitter relativity.

We hope that the algebraic method proposed here will complete the general geometric approach for getting over the difficulties in analysing the light emitted by various cosmic objects moving in the de Sitter expanding universe.
A Translations

The space translations of the $SO(1, 4)$ group are less used in applications such that it is worth reviewing briefly their action in different coordinates. A space translation of parameters $a = (a^1, a^2, a^3)$ is the isometry $x' \rightarrow x = \phi_{g(a)}(x')$ generated by the $SO(1, 4)$ transformation \[58, 9],

$$g(a) = \begin{pmatrix} 1 + \frac{1}{2} \omega_H^2 a^2 & \omega_H a^1 & \omega_H a^2 & \omega_H a^3 & \frac{1}{2} \omega_H^2 a^2 \\ \omega_H a^1 & 1 & 0 & 0 & \omega_H a^1 \\ \omega_H a^2 & 0 & 1 & 0 & \omega_H a^2 \\ \omega_H a^3 & 0 & 0 & 1 & \omega_H a^3 \\ -\frac{1}{2} \omega_H^2 a^2 & -\omega_H a^1 & -\omega_H a^2 & -\omega_H a^3 & 1 - \frac{1}{2} \omega_H^2 a^2 \end{pmatrix},$$  

(A.1)

according to Eq. (23) where now $g = g(a)$. Solving this equation in the co-moving frames with Painlevé coordinates where the $z$-functions have the form \[22\] we find the transformation

$$t = t', \quad x = x' + a e^{\omega_H t},$$  

(A.2)

which does not affect the time but is not static. The only genuine static translations transform the conformal coordinates

$$t_c = -\frac{1}{\omega_H} e^{-\omega_H t}, \quad x_c = x_c e^{-\omega_H t}, \quad \rightarrow \quad t_c = t_c', \quad x_c = x_c' + a.$$  

(A.3)

The static coordinates $(t_s, x)$ where

$$t_s = t - 2 \ln \chi(r), \quad \chi(r) = \sqrt{1 - \omega_H^2 r^2},$$  

(A.4)

with $r = |x|$, as defined by Eq. \[5\], can be introduced by the functions

$$z^0(t_s, x) = \frac{1}{\omega_H} \chi(r) \sinh \omega_H t_s,$$

$$z^i(t_s, x) = x^i,$$

$$z^4(t_s, x) = \frac{1}{\omega_H} \chi(r) \cosh \omega_H t_s.$$  

(A.5)

Solving again Eq. (23) for these functions and the transformation (A.1) we find the transformation rules under translations,

$$t_s = \frac{1}{2} t_s' + \frac{1}{2 \omega_H} \ln \left| \frac{\chi(r')}{\omega_H^2 a^2 \chi(r') e^{\omega_H t_s'} + 2 \omega_H a \cdot x' - \chi(r') e^{-\omega_H t_s'}} \right|,$$

$$x = x' + a \chi(r') e^{\omega_H t_s'}.$$  

(A.6)

we present here for the first time.
B  Moving black holes in special relativity

Let us consider now a Schwarzschild black hole embedded in the Minkowski flat space-time where the remote observers stay in inertial frames. We apply our algebraic method assuming that at the initial time $t = 0$ the black hole origin $O_{BH}$ is translated with $d$ and moves with the relative velocity $V = (V, 0, 0)$ with respect to $O$. At the same time, the apparent source $S$ of coordinates

$$x' = (0, -\hat{r}_S \sin \hat{\alpha}, -\hat{r}_S \cos \hat{\alpha}, 0)^T$$ (B.1)

emits a photon of energy-momentum

$$p' = (k, -k \cos \hat{\alpha}, -k \sin \hat{\alpha}, 0)^T,$$ (B.2)

which has to be observed in the fixed origin $O$.

The black hole and observer frames, $O_{BH}$ and $O$, are related through the isometry $\Lambda = L(V)T(d)$ formed by the translation $T(d) : x'^1 \to x^1 = x'^1 + d$ followed by a Lorentz transformation $L(V)$, which is just the four-dimensional restriction of the matrix (44). Performing this isometry we obtain the four-vectors $x = \Lambda x'$ and $p = \Lambda p'$ whose components give the energy and angular momentum observed in $O$ as

$$E = p^0 = \frac{k(1 - V \cos \hat{\alpha})}{\sqrt{1 - V^2}},$$ (B.3)

$$L_3 = x^1 p^2 - x^2 p^1 = \frac{kd [\xi (1 - V \cos \hat{\alpha}) - \sin \hat{\alpha}]}{\sqrt{1 - V^2}}.$$ (B.4)

The condition $L_3 = 0$ of the photon passing through $O$ gives angle $\alpha$ in the black hole frame,

$$\tan \frac{\hat{\alpha}}{2} = \frac{1 - \sqrt{1 - \xi^2 (1 - V^2)}}{\xi (1 + V)}.$$ (B.5)

In observer’s frame the black hole shadow is seen under the angle $\hat{\alpha}_{\text{obs}}$ defined as

$$\sin \hat{\alpha}_{\text{obs}} = \frac{|p^2|}{p}, \quad p = \sqrt{p_{12}^2 + p_{22}^2}.$$ (B.6)

Finally, calculating $\sin \hat{\alpha}$ and $\cos \hat{\alpha}$ and substituting these values in Eqs. (B.3) and (B.6) we verify that the limits (63-68) are correct.
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