Pfister numbers over rigid fields

Nico Lorenz
Fakultät für Mathematik, Ruhr-Universität Bochum, Bochum, Germany

ABSTRACT
For certain types of quadratic forms lying in the $n$-th power of the fundamental ideal, we compute upper bounds and if possible exact values for the minimal number of general $n$-fold Pfister forms, that are needed to write the Witt class of that given form as the sum of the Witt classes of those $n$-fold Pfister forms. We restrict ourselves mostly to the case of so called rigid fields, i.e. fields in which anisotropic binary forms represent at most 2 square classes.

1. Introduction
Throughout this paper, let $F$ be a field of characteristic different from 2. By a quadratic form or just form for short, we will always mean a finite dimensional non-degenerate quadratic form over $F$. We will denote isometry of two forms $\varphi_1, \varphi_2$ by $\varphi_1 \cong \varphi_2$. By abuse of notation, we will denote the Witt class of a quadratic form $\varphi$ again by $\varphi$. An $n$-fold Pfister form for some $n \in \mathbb{N}$ is a form of the shape $\langle \langle a_1, \ldots, a_n \rangle \rangle := \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$ with $a_1, \ldots, a_n \in F^\times$. The set of $n$-fold Pfister forms over $F$ is denoted by $P_n F$, the set of forms that are similar to some $n$-fold Pfister form is denoted by $GP_n F$. By abuse of notation, we will call any form in $GP_n F$ a Pfister form. Both $P_n F$ and $GP_n F$ generate the $n$-th power of the fundamental ideal $IF$, which we denote by $I^n F$, both as an additive group and as an ideal, i.e. any Witt class $\varphi \in I^n F$ can be written as a sum $\varphi = \pi_1 + \cdots + \pi_m$ for suitable $\pi_1, \ldots, \pi_m \in GP_n F$. A frequently asked question in the current research on quadratic forms and the main topic of this article is now to determine the lowest number of Pfister forms that are needed to write a given $\varphi \in I^n F$ as such a sum. This number is called the $n$-Pfister number of $\varphi$. We will collect some known facts about it in the next section.

During this paper we will mostly consider rigid fields. Such fields admit complete discrete valuations, so Section 3 will deal with some basic questions concerning Pfister numbers of forms over complete discrete valuation fields.

In Section 4, we will then introduce rigid fields and study the behavior of quadratic forms over such fields in a more detailed way.

Section 5 is devoted to both 14-dimensional forms in $I^3$ and 8-dimensional forms in $I^2$ over rigid fields. There is a strong connection between forms of these two types and the classification of them over rigid fields will help us to investigate 16-dimensional $I^3$-forms in the following section. We discuss these forms in detail since this is the smallest type of form for which no upper bound on the Pfister number has been established yet. The final part of this article now deals with the growth of Pfister numbers for increasing dimension.
2. Basic results on Pfister numbers

As already mentioned in the introduction, we would like to study the so-called Pfister number, which we will now formally introduce.

**Definition 2.1.** We define the *n-Pfister number* of a quadratic form \( \varphi \in \mathcal{I}^n F \) to be

\[
\text{GPf}_n(\varphi) := \min \{ k \in \mathbb{N} \mid \text{there are } \pi_1, \ldots, \pi_k \in \text{GP}_n F \text{ with } \varphi = \pi_1 + \cdots + \pi_k \in \text{WF} \}.
\]

Additionally, we define

\[
\text{GPf}_n(F, d) := \sup \{ \text{GPf}_n(\varphi) \mid \varphi \in \mathcal{I}^n F, \dim \varphi \leq d \}.
\]

We further define the *unscaled n-Pfister number* of \( \varphi \) to be

\[
\text{Pf}_n(\varphi) := \min \{ k \in \mathbb{N} \mid \text{there are } \varepsilon_1, \ldots, \varepsilon_k \in \{ \pm 1 \} \text{ and } \pi_1, \ldots, \pi_k \in \text{P}_n F \text{ with } \varphi = \varepsilon_1 \pi_1 + \cdots + \varepsilon_k \pi_k \in \text{WF} \}.
\]

If the integer \( n \) is clear from the context, we will often just say (unscaled) Pfister number.

The main task in this article is now to calculate Pfister numbers in terms of invariants of a given form. As this seems to be quite a challenge, we will often be satisfied with upper or lower bounds. We will concentrate on the scaled version, as we have the following correspondence between both versions.

**Proposition 2.2.** For any quadratic form \( \varphi \) over \( F \) and any \( n \in \mathbb{N} \), we have

\[
\text{Pf}_n(\varphi) \leq 2 \cdot \text{GPf}_n(\varphi).
\]

**Proof.** For any \( a, x_1, \ldots, x_n \in F^* \), we have

\[
a \langle x_1, \ldots, x_n \rangle = \langle x_1, \ldots, x_{n-1} \rangle \otimes (a \langle x_n \rangle)
\]

\[
= \langle x_1, \ldots, x_{n-1} \rangle \otimes ((1, a) \perp -(1, ax_n))
\]

\[
= \langle x_1, \ldots, x_{n-1}, -a \rangle \perp -\langle x_1, \ldots, x_{n-1}, -ax_n \rangle
\]

which then readily implies the assertion.

For Pfister numbers in \( \mathcal{I}^2 \), we have the following two results.

**Proposition 2.3** ([11, Chapter X. Exercise 4]). Let \( \varphi \in \mathcal{I}^2 F \) be a form of dimension \( \dim \varphi \in \mathbb{N} \). Then \( \varphi \) is Witt equivalent to a sum of at most \( \frac{\dim \varphi}{2} - 1 \) forms in \( \text{GP}_2 F \).

**Example 2.4** (Parimala, Suresh, Tignol). Let \( K \) be a field and \( F := K((X_1)) \cdots ((X_n)) \) for some \( n \in \mathbb{N} \) with \( n \geq 2 \). According to (the proof of) [12, Theorem 2.2] (in which the assumption that \(-1\) is a square is only needed to assure that the upcoming forms lie in \( \mathcal{I}^2 F \) and can be omitted by adding a sign as below), we see that we have

\[
\text{Pf}^2(\langle 1, X_1, \ldots, X_n, (-1)^{\frac{n+2}{2}} X_1 \cdots X_n \rangle) = n - 1 \text{ if } n \text{ is even}
\]

and

\[
\text{Pf}^2(\langle X_1, \ldots, X_n, (-1)^{\frac{n+1}{2}} X_1 \cdots X_n \rangle) = n - 1 \text{ if } n \text{ is odd}.
\]

Thus, 2.2 implies

\[
\text{GPf}_2(\langle 1, X_1, \ldots, X_n, (-1)^{\frac{n+2}{2}} X_1 \cdots X_n \rangle) \geq \frac{n - 1}{2}
\]

and

\[
\text{GPf}_2(\langle X_1, \ldots, X_n, (-1)^{\frac{n+1}{2}} X_1 \cdots X_n \rangle) \geq \frac{n - 1}{2}.
\]  

(1)
In the case where \( n \) is even, using that the Pfister number is always an integer, we even get
\[
\text{GPf}_2((1, X_1, \ldots, X_n, (-1)^{n/2} X_1 \cdots X_n)) \geq \frac{n}{2}.
\] (2)

The reverse inequalities are covered in 2.3, so we have equalities both in (1) and (2). Of course, since the values of \( \text{GPf}_2 \) are invariant under scaling and since we can redefine the indeterminates, we can restrict ourselves to the case where \( n \) is even and just consider the form
\[
\varphi := (1, X_1, \ldots, X_n, (-1)^{n/2} X_1 \cdots X_n) \in I^2 F
\]
with \( \dim \varphi = n + 2 \) and
\[
\text{GPf}_2(\varphi) = \frac{n}{2},
\]
which is the biggest possible value. This form will also be referred to as the generic (rigid) \( I^2 \)-form of dimension \( n + 2 \).

For further results concerning Pfister numbers, we refer the reader to [2, 6, 8, 9].

### 3. Valuation Theoretic Results

As the yet to be defined rigid fields which we want to study admit valuations, we want to give a short exposition of the main ingredients coming from valuation theory that will be used in the sequel. In this section, we fix a field \( F \) equipped with a complete discrete valuation \( \nu \) with residue class field \( K \) whose characteristic is assumed to be not 2. A quadratic form \( \langle a_1, \ldots, a_n \rangle \) over \( F \) is called unimodular, if each \( a_i \) is a unit in the valuation ring \( \{ x \in F \mid \nu(x) \geq 0 \} \) of \( \nu \), i.e. if we have \( \nu(a_i) = 0 \) for all \( i \in \{1, \ldots, n\} \). We will often use the case of a Laurent series extension \( F = K((t)) \) with the usual valuation given by the least index such that the respective coefficient of the Laurent series is not zero.

In the sequel, we will frequently use the following well known exact sequence.

**Lemma 3.1** ([3, Exercise 19.15]). For all \( n \in \mathbb{N} \) we have a split exact sequence
\[
0 \longrightarrow I^n K \longrightarrow I^n F \longrightarrow I^{n-1} K \longrightarrow 0
\]
where the maps are given by lifting and taking the second residue class form.

**Corollary 3.2.** Let \( \varphi \in I^n F \) be a unimodular form. Then, the \( n \)-Pfister number of \( \varphi \) over \( F \) and of its first residue class form \( \bar{\varphi} \) over \( K \) coincide.

**Proof.** If we have \( \bar{\varphi} = \bar{\pi}_1 + \cdots + \bar{\pi}_k \) for some Pfister forms \( \pi_1, \ldots, \pi_k \in \text{GP}_n K \), we can lift them to get a representation \( \varphi = \pi_1 + \cdots + \pi_k \) by 3.1.

For the converse, we fix a uniformizing element \( t \). Using the isometry \( \langle \langle at, bt \rangle \rangle \cong \langle \langle at, -ab \rangle \rangle \) for all \( a, b \in F^* \), we can find a representation
\[
\varphi = \pi_1 + \cdots + \pi_k + \bar{\pi}_1 \otimes \langle \langle c_1 t \rangle \rangle + \cdots + \bar{\pi}_\ell \otimes \langle \langle c_\ell t \rangle \rangle + t\bar{\pi}_1 + \cdots + t\bar{\pi}_m
\]
with unimodular forms \( \pi_1, \ldots, \pi_k, \bar{\pi}_1, \ldots, \bar{\pi}_m \in \text{GP}_n F \) and \( \bar{\pi}_1, \ldots, \bar{\pi}_\ell \in \text{GP}_{n-1} F \) and \( c_1, \ldots, c_\ell \in F^* \) with \( \nu(c_1) = \cdots = \nu(c_\ell) = 0 \). By comparing both residue class forms, we see that in \( WK \), we have equalities
\[
\bar{\varphi} = \bar{\pi}_1 + \cdots + \bar{\pi}_k + \bar{\pi}_1 + \cdots + \bar{\pi}_\ell
\]
and \( \bar{c}_1 \bar{\pi}_1 + \cdots + \bar{c}_\ell \bar{\pi}_\ell = \bar{\pi}_1 + \cdots + \bar{\pi}_m \).

As the last equality implies
\[
\bar{\pi}_1 + \cdots + \bar{\pi}_\ell = \bar{\pi}_1 + \cdots + \bar{\pi}_m + \langle \langle c_1 \rangle \rangle \otimes \bar{\pi}_1 + \cdots + \langle \langle c_\ell \rangle \rangle \otimes \bar{\pi}_\ell
\]
we have a representation of \( \bar{\varphi} \) of \( k + \ell + m \) \( n \)-fold Pfister forms as needed.
Proposition 3.3. Let \( \psi \in I^{n-1}F \) be a unimodular form and \( \varphi := \langle \langle t \rangle \rangle \otimes \psi \) for some uniformizer \( t \). We then have \( \text{GPf}_{n-1}(\psi) = \text{GPf}_n(\varphi) \).

Proof. The inequality \( \text{GPf}_{n-1}(\psi) \geq \text{GPf}_n(\varphi) \) is clear. For the converse, we consider a representation

\[
\varphi = \pi_1 + \cdots + \pi_k + \tilde{\pi}_1 \otimes \langle \langle c_1 t \rangle \rangle + \cdots + \tilde{\pi}_\ell \otimes \langle \langle c_\ell t \rangle \rangle + t \tilde{\pi}_1 + \cdots + t \tilde{\pi}_m
\]
as above. After comparing residue class forms, we see that we have

\[
\pi_1 + \cdots + \pi_k + \tilde{\pi}_1 + \cdots + \tilde{\pi}_\ell = \psi = -\tilde{\pi}_1 - \cdots - \tilde{\pi}_m + c_1 \tilde{\pi}_1 + \cdots + c_\ell \tilde{\pi}_\ell.
\]

These are representations of \( \psi \) as a sum of \( 2k + \ell \) respectively \( 2m + \ell \) forms in \( \text{GPf}_{n-1}F \). If we had \( k + \ell + m < \text{GPf}_{n-1}(\psi) \) one of the terms \( 2k + \ell \) and \( 2m + \ell \) would also be strictly smaller than \( \text{GPf}_{n-1}(\psi) \), a contradiction. Thus, we have \( \text{GPf}_{n-1}(\psi) \leq \text{GPf}_n(\varphi) \) and the proof is complete.

Proposition 3.4. Let \( \varphi \) be a quadratic form that lies in \( I^nF(X) \) or \( I^nF((t)) \) defined over \( F \). Then, there is a unique preimage \( \psi \in \text{WF} \) under the canonical map \( r_{F(X)/F} \) respectively \( r_{F((t))/F} \) and it fulfills \( \psi \in I^nF \).

Proof. We will denote the map induced by scalar extension in both cases by \( r \). The existence and uniqueness of some \( \psi \in \text{WF} \) with \( r(\psi) = \varphi \) is clear as \( \varphi \) is defined over \( F \) and \( r \) is known to be injective, see e.g. [11, Chapter IX. Lemma 1.1].

As \( \varphi \) has a preimage in \( I^nF \) because of [3, Theorem 21.1, Corollary 21.3] respectively 3.1, the claim follows.

Corollary 3.5. Let \( \varphi \in I^nF \) and \( E \) be a field with \( F(t) \subseteq E \subseteq F((t)) \). We then have \( \text{GPf}_n(\varphi) = \text{GPf}_n(\varphi_E) \).

Proof. As the Pfister number can only decrease when going up to a field extension, it is enough to show the inequality

\[
\text{GPf}_n(\varphi) \leq \text{GPf}_n(\varphi_E)
\]
for \( E = F((t)) \), but this follows directly from 3.2.

The following result should be compared with [14, Lemma 1.5].

Proposition 3.6. Let \( \varphi \in I^nF \) be a quadratic form such that both residue class forms are not hyperbolic. Then there is uniformizer \( t \), unimodular forms \( \sigma \in I^nF \) and \( \tau \in I^{n-1}F \) with \( \varphi = \sigma + \langle \langle -t \rangle \rangle \otimes \tau \in \text{WF} \) and \( \dim \sigma < \dim \varphi \).

Proof. We denote lifts of the first respectively second residue class forms of \( \varphi \) with respect to some uniformizing element \( t \) with \( \varphi_1 \) respectively \( \varphi_2 \). We then have

\[
\varphi \cong \varphi_1 \perp t \varphi_2 = (\varphi_1 \perp -\varphi_2) + (\varphi_2 \perp t \varphi_2) = (\varphi_1 \perp -\varphi_2) + \langle \langle -t \rangle \rangle \otimes \varphi_2.
\]

After multiplying \( t \) with some unit of the valuation ring, i.e. changing the uniformizer, we can assume \( D_F(\varphi_1) \cap D_F(\varphi_2) \neq \emptyset \). Then the form \( \varphi_1 \perp -\varphi_2 \) is isotropic. If we choose

\[
\sigma := (\varphi_1 \perp -\varphi_2)_{an} \text{ and } \tau := \varphi_2,
\]
we have \( \dim \sigma < \dim \varphi \) and \( \tau \in I^{n-1}F \) by 3.1. Finally (3) implies \( \varphi \equiv \varphi_1 - \varphi_2 \mod I^nF \), which then leads to \( \sigma := (\varphi_1 \perp -\varphi_2)_{an} \in I^nF \).

With the above result, we are now in the position to give an upper bound for the Pfister numbers of forms over a complete discrete valuation field in terms of Pfister numbers over the associated residue class field. As a first step, we record the following special case which follows directly from 3.6.
Corollary 3.7. Let \( \varphi \) be as in 3.6. Then its \( n \)-Pfister number is bounded from above by

\[
\GPf_n(K, \dim(\varphi) - 2) + \GPf_{n-1}(K, \frac{1}{2} \dim(\varphi))
\]

Proof. We use the notation as in the proof of 3.6. Since the Pfister number of any form is invariant under scaling, we can assume \( \dim \varphi_2 \leq \frac{1}{2} \dim \varphi \). We thus get \( \sigma \in I^nF \) and \( \tau \in I^{n-1}F \) such that we have a representation \( \varphi = \sigma + \langle -t \rangle \otimes \tau \) in the Witt ring \( WF \) with some suitable uniformizer \( t \) and

\[
\dim \sigma \leq \dim \varphi - 2 \quad \text{and} \quad \dim \tau \leq \frac{1}{2} \dim \varphi,
\]

where the first inequality can be assumed by 3.6 since both residue forms are not hyperbolic. By 3.2 we have

\[
\GPf_n(\langle -t \rangle \otimes \tau) \leq \GPf_{n-1}(K, \frac{1}{2} \dim(\varphi))
\]

and the result now follows.

As the main result of this section, we have the following:

Theorem 3.8. Let \( F \) be complete discrete valuation field such that the characteristic of the residue class field \( K \) is not equal to 2. Then for all \( n \in \mathbb{N} \) and all \( d \in 2\mathbb{N} \), we have

\[
\GPf_n(F, d) \leq \max \left\{ \GPf_n(K, d - 2) + \GPf_{n-1}(K, \frac{d}{2}), \GPf_n(K, d) \right\}.
\]

Proof. For any \( d \)-dimensional quadratic form \( \varphi \in I^nF \), either both of its residue class forms are not hyperbolic or \( \varphi \) is similar to an unimodular form. The claim now follows by 3.7 and 3.2.

We conclude this section with a remark in which we show how we can treat the case of a residue class field that admits a complete discrete valuation itself.

Remark 3.9. Let \( K \) be a field of characteristic not 2, \( F = K(\langle t_1 \rangle) \langle (t_2) \rangle \) and \( E := K(\langle t_2 \rangle)(\langle t_1 \rangle) \). The \( \mathbb{F}_2 \)-linear map \( \Phi : F^* / F^{*2} \to E^* / E^{*2} \) defined by

\[
\begin{align*}
 aF^{*2} &\mapsto aE^{*2} \quad \text{for all } a \in K^*; \\
t_1F^{*2} &\mapsto t_1E^{*2}; \\
t_2F^{*2} &\mapsto t_2E^{*2}
\end{align*}
\]

is a group isomorphism with

\[
\Phi(-1) = -1 \quad \text{and for all } a, b \in F, b \in D_F((1, a)) \iff \Phi(b) \in D_E((1, \Phi(a))).
\]

and thus a so called \( q \)-equivalence, see [11, Chapter XII. Definition 1.1]: it is an isomorphism as if \( \{a_i \mid i \in I\} \) is a system of representatives of \( K^*/K^{*2} \), then \( R := \{a_i, a_it_1, a_it_2, a_it_1t_2 \mid i \in I\} \) is a system of representatives of both \( F^*/F^{*2} \) and \( E^*/E^{*2} \), the first property in (4) is clear and the second one follows readily from [11, Chapter VI. Exercise 3]. Thus \( \Phi \) induces a correspondence between the isometry classes of quadratic forms over \( F \) and the isometry classes of quadratic forms over \( E \). By abuse of notation and identifying an element with its square class, this correspondence is given by \( \langle a_1, \ldots, a_n \rangle \mapsto \langle \Phi(a_1), \ldots, \Phi(a_n) \rangle \), see [11, Chapter XII. Proposition 1.2]. This map preserves orthogonal sums, tensor products and Witt indices, and thus finally induces an isomorphism of the Witt rings \( WF \to WE \) that preserves the Pfister number. Thus, if we have a form over \( F \) whose entries all lie in \( R \), there is form over \( E \) that has the same entries and the same Pfister number which may be easier to calculate.

Further, as the symmetric group \( S_n \) for \( n \geq 2 \) is generated by transpositions of the form \( (k k + 1) \) for all \( k \in \{1, \ldots, n - 1\} \), we can extend the above to find a \( q \)-equivalence between
\[ K(t_1) \cdots (t_n) \text{ and } K(t_{\sigma(1)}) \cdots (t_{\sigma(n)}) \] for every permutation \( \sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \). This \( q \)-equivalence then also induces an isomorphism of the respective Witt rings that we can use as described above.

As a last trick, we would like to mention that we can always change the uniformizing element in some ways. For example, we have \( K(t) = K(at) \) for all \( a \in K^* \) and thus in particular \( K(t_1)(t_2) = K((t_1)(t_1t_2)) \).

We will use these facts without mentioning them explicitly several times in the sequel. The main idea while using this is that quadratic forms are good to manage if they have a well understood subform. It is thus convenient to consider a \( q \)-equivalent field with reordered Laurent variables such that one gets a residue form of small dimension.

### 4. Introduction to the Theory of Rigid Fields

Inspired by the work of M. Raczek [14], we will prove upper bounds for the Pfister number of so called rigid fields. Using similar arguments, we generalize a lot of the arguments used in the just cited article. In the theory of quadratic forms, rigid fields are of interest because of several reasons. Firstly, they have strong connections to other areas of mathematics. As an example, there are a lot of interesting Galois-theoretic results available for rigid fields, see [15]. Furthermore, nonreal rigid fields with a finite number of square classes are examples of the so called \( \mathbb{C} \)-fields. These are extreme examples as these are those fields that have the maximal number of anisotropic quadratic forms that can occur, when considering nonreal fields with finitely many square classes, see [11, Chapter XI., Theorem 7.10, 7.14, Definition 7.16].

**Definition 4.1.** A field \( F \) is called rigid, if, for any binary anisotropic quadratic form \( \beta \) over \( F \), we have \( |DF(\beta)| \leq 2 \).

**Example 4.2.** As the square class groups of finite fields or Euclidean fields consist of only two elements, these fields are rigid. Over a quadratically closed field there are no binary anisotropic forms. Thus quadratically closed fields are rigid as well.

We will now give a characterization of rigid fields that will be useful in the sequel.

**Theorem 4.3 ([15, Theorems 1.5, 1.8, 1.9]).** For a field \( F \) the following are equivalent:

(i) \( F \) is rigid;
(ii) we have an isomorphism \( WF \cong (\mathbb{Z} / n \mathbb{Z})[G] \) with \( n \in \{0, 2, 4\} \) and \( G \) a group of exponent 2;
(iii) we have an isomorphism \( WF \cong (\mathbb{Z} / n \mathbb{Z})[H] \) with either \( n = 2 \) and \( H = F^*/F^{*2} \) or \( n \in \{0, 4\} \) and \( H \subseteq F^*/F^{*2} \) a subgroup with \( -1 \notin H \) and \( [F^*/F^{*2} : H] = 2 \);
(iv) for any anisotropic form \( \varphi \), we have \( |DF(\varphi)| \leq \dim \varphi \);
(v) for any quadratic field extension \( K/F \), the image of the map \( i : F^*/F^{*2} \rightarrow K^*/K^{*2} \) that is induced by inclusion has index \( \leq 2 \).

An important field invariant when studying quadratic forms is the so called level of a field, in symbols \( s(F) \). It is defined as the least number \( n \) of squares such that \(-1\) is a sum of \( n \) squares or \( \infty \) if no such integer exists or equivalently the least integer \( n \) such that \( (n + 1) \times \langle 1 \rangle \) is isotropic. It is well known that the level is either \( \infty \) or a power of 2, see [11, Chapter XI. Pfister’s Level Theorem]. We thus see that rigid fields always have level 1,2, or \( \infty \).

Recall that a field is called pythagorean if any sum of squares is a square. Following [4], we introduce the following name for formally real rigid fields.

**Corollary and Definition 4.4.** If \( F \) is a formally real rigid field, it is pythagorean. A formally real rigid field \( F \) is also called superpythagorean.
Proof. If $F$ is formally real and rigid, its Witt ring is isomorphic to $\mathbb{Z}[G]$ for some group $G$ of exponent 2. We thus have $W_1F = \{0\}$ which is equivalent to $F$ being pythagorean by [11, Chapter VIII., Theorem 4.1 (1)].

The above characterization together with Springer’s theorem for complete discrete valuation fields motivate us to build the following prototypes of rigid fields in which we can calculate reasonably well and such that these fields realize any possible Witt ring of rigid fields.

Corollary 4.5. Let $F$ be a rigid field. Then there is a field $K \in \{\mathbb{F}_3, \mathbb{R}, \mathbb{C}\}$ and an index set $I$ with

$$WF \cong WK((t_i))_{i \in I}.$$

Proof. According to 4.3 (ii), we have $WF \cong \mathbb{Z}/n\mathbb{Z}[G]$ for some $n \in \{0, 2, 4\}$ and some group $G$ of exponent 2.

We choose the field $K$ as shown in the adjacent table:

| $n$ | 0 | 2 | 4 |
|-----|---|---|---|
| $K$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{F}_3$ |

It is well known that we then have $WK \cong \mathbb{Z}/n\mathbb{Z}$. As $G$ is of exponent 2, it can be seen as a vector space over the field with two elements $F_2$ and thus has an $F_2$-basis $(g_i)_{i \in I}$ for some index set $I$. We now consider the field $E := K((t_i))_{i \in I}$. We then have

$$WE \cong \mathbb{Z}/n\mathbb{Z}[G]$$

as in the proof of [15, Lemma 1.6] (this is essentially a direct limit argument using Springer’s Theorem [11, Chapter VI. Theorem 1.4]).

The above result further allows us to always work in explicitly given fields if we want to study rigid fields in general. We will fill in the details in the next remark for future reference.

Remark 4.6. By combining 4.5 with the Harrison-Cordes Theorem [11, Chapter XII. Theorem 1.8], the study of quadratic forms over rigid fields can be restricted to study quadratic forms over fields of the form $K((t_i))_{i \in I}$ for a field $K \in \{\mathbb{F}_3, \mathbb{R}, \mathbb{C}\}$ and some index set $I$, which can be assumed to be well-ordered due to the well-ordering theorem.

If we want to study a concrete form, it is often even possible to only consider the case that $I$ is finite as the direct limit $K((t_i))_{i \in I}$ can be regarded as the union of the fields $K((t_{i_1}) \cdots (t_{i_r}))$ for some $r \in \mathbb{N}_0$ and $i_1, \ldots, i_r \in I$ with $i_1 < \ldots < i_r$, see again the proof of [15, Lemma 1.6]. Thus, if a quadratic form $\varphi$ over $E$ is given, we can take any diagonalization of $\varphi$. In this diagonalization, only finitely many Laurent variables can occur, say these are $t_{j_1}, \ldots, t_{j_m}$ with $j_1 < \cdots < j_m$. Then, $\varphi$ is already defined over $E' := K((t_{j_1}) \cdots (t_{j_m}))$ and we can work over this field. For example, the Pfister number of $\varphi$ over $E'$ is bigger than or equal to the Pfister number of $\varphi$ over $E$ as we have $E' \subseteq E$. Thus the task of finding upper bounds for the Pfister numbers over arbitrary rigid fields is reduced to the task of finding upper bounds for the Pfister numbers over fields of the form $K((t_1) \cdots (t_n))$ for some $n \in \mathbb{N}$ and $K \in \{\mathbb{F}_3, \mathbb{R}, \mathbb{C}\}$.

The following corollary will be the key idea to determine asymptotic upper bounds for the Pfister numbers. Its proof combines the above theory with the tools that were developed before over fields equipped with a discrete valuation.

Corollary 4.7. Let $\varphi \in I^nF$ be a quadratic form over some rigid field $F$ that represents 1 and an element $a \notin \pm D_F(s(F) \times \langle 1 \rangle)$, where we interpret $D_F(\infty \times \langle 1 \rangle)$ as $\bigcup_{n \in \mathbb{N}} D_F(n \times \langle 1 \rangle)$. Then there are quadratic forms $\sigma \in I^nF, \tau \in I^{n-1}F$ with $\dim \sigma < \dim \varphi$ and some $t \in F^n$ with $\varphi = \sigma + \langle \langle t \rangle \otimes \tau$.}

Proof. Using 4.6 and 3.9, we are reduced to the case where we have $F = K((t_1)) \cdots (t_n)$ for some $n \in \mathbb{N}$ with $a = t_n$. But then, the assertion readily follows from 3.6 and 3.1 as both residue class forms for $a = t_n$ are non-hyperbolic by assumption.
We would like to remark that our above result can be applied in particular to rigid fields $F$ with $s(F) = 1$. When specializing to the case $n = 3$, we get the main results from [14, Lemma 1.5], the starting point for the calculation of Pfister numbers in the just cited article.

As usual it may be helpful to study the behavior of a given quadratic form under field extensions. Thus the following result is essential for us.

**Theorem 4.8 ([15, Corollary 2.8]).** Let $F$ be a rigid field and $K/F$ a quadratic field extension. Then $K$ is also a rigid field.

For later reference, we will now discuss the possible diagonalizations of anisotropic binary forms over rigid fields in detail.

**Proposition 4.9.** Let $F$ be a rigid field and $\beta = \langle x, y \rangle$ be an anisotropic binary form over $F$. By abuse of terminology, we say that two diagonalizations of a quadratic form are the same if they only differ by multiplying some entries with a square. We then have one of the following cases:

- $s(F) = 1$, $x, y$ represent different square classes and $\langle x, y \rangle$ and $\langle y, x \rangle$ are the only diagonalizations of $\beta$;
- $s(F) = 2$, $x, y$ represent different square classes and $\langle x, y \rangle$ and $\langle y, x \rangle$ are the only diagonalizations of $\beta$;
- $s(F) = 2$, $x, y$ represent the same square classes and $\langle x, x \rangle$ and $\langle -x, -x \rangle$ are the only diagonalizations of $\beta$;
- $s(F) = \infty$, $x, y$ represent different square classes and $\langle x, y \rangle$ and $\langle y, x \rangle$ are the only diagonalizations of $\beta$;
- $s(F) = \infty$, $x, y$ represent the same square classes and $\langle x, x \rangle$ is the only diagonalization of $\beta$.

**Proof.** We first note that in general, for any $a \in F^*$, we cannot have $a$ and $-a$ in the same diagonalization of an anisotropic quadratic form. In the sequel, we use several times the fact that any entry of a diagonalization is represented by the form. Finally, if $x, y$ represent different square classes, we clearly have $D_F(\beta) = \{x, y\}$ because $F$ is rigid.

If we have $s(F) = 1$ we have $x = -x$ in $F^*/F^*$. It is thus clear that $x, y$ have to represent different square classes. As $F$ is rigid we have $D_F(\beta) = \{x, y\}$ and by the above remarks, this case follows.

For $a \in F^*$, we have $D_F((a, a)) = \{a, -a\}$ if $s(F) = 2$ and $D_F((a, a)) = \{a\}$ if $s(F) = \infty$ by 4.4. Thus, if $x, y$ represent different square classes, they both have to occur in any diagonalization of $\beta$. This readily implies that $(x, y)$ and $(y, x)$ are the only diagonalizations of $\beta$ in the respective cases.

So let now $x, y$ represent the same square class. If we have $s(F) = 2$, it follows by the remarks at the beginning of the proof that $\langle x, x \rangle$ and $\langle -x, -x \rangle$ are the only diagonalizations of $\beta$.

Finally, if we have $s(F) = \infty$, 4.4 implies that $\langle x, x \rangle$ is the only diagonalization of $\beta$.

As a corollary, we will now see what makes the theory of quadratic forms over rigid fields much easier than the general case: if one diagonalization of a given form is known, it is easy to determine all the others.

**Corollary 4.10.** Let $\varphi$ be an anisotropic form over a rigid field $F$. If we have $s(F) \in \{1, \infty\}$ the diagonalization of $\varphi$ is unique up to permuting the entries and multiplying them with squares. If we have $s(F) = 2$, the diagonalization of $\varphi$ is unique up to permuting the entries, multiplying them with squares and replacing subforms of the form $\langle x, x \rangle$ for some $x \in F^*$ with $\langle -x, -x \rangle$.

**Proof.** It is clear that any of the operations in the statement of the proposition describes isometries of quadratic forms. Further it is well known that two quadratic forms are isometric if and only if they are chain equivalent, see [11, Chapter I. Chain Equivalence Theorem 5.2]. The conclusion thus readily follows from 4.9.
Corollary 4.11. Let \( \varphi, \psi \) be quadratic forms over a rigid field \( F \) such that \( \varphi \perp \psi \) is anisotropic. We then have

\[
D_F(\varphi \perp \psi) = \begin{cases} 
D_F(\varphi) \cup D_F(\psi), & \text{if } s(F) \in \{1, \infty\} \\
D_F(\varphi) \cup D_F(\psi) \cup \{ x \in F^* \mid -x \in D_F(\varphi) \cap D_F(\psi) \}, & \text{if } s(F) = 2.
\end{cases}
\]

Proof. It is well known that we have

\[
D_F(\varphi \perp \psi) = \bigcup_{x \in D_F(\varphi), y \in D_F(\psi)} D_F(\langle x, y \rangle),
\]

see for example [11, Chapter I. exercise 20]. As the elements that are represented by a quadratic form are exactly those that can occur in a diagonalization, the claim now readily follows from 4.9.

In the following, we will record some technical results in order to study how hyperbolic planes can occur in the sum of three quadratic forms over rigid fields.

Lemma 4.12. Let \( F \) be a rigid field and \( \varphi_1, \varphi_2, \varphi_3 \) be anisotropic quadratic forms over \( F \) such that \( \varphi_1 \perp \varphi_2 \) is anisotropic as well. Then \( \varphi_1 \perp \varphi_2 \perp \varphi_3 \) is isotropic if and only if one of the following cases occurs:

1. at least one of the forms \( \varphi_1 \perp \varphi_3 \) and \( \varphi_2 \perp \varphi_3 \) is isotropic.
2. we have \( s(F) = 2 \) and \( D_F(\varphi_1) \cap D_F(\varphi_2) \cap D_F(\varphi_3) \neq \emptyset \).

Proof. The form \( (\varphi_1 \perp \varphi_2) \perp \varphi_3 \) is isotropic if and only if there is some \( x \in D_F(\varphi_1 \perp \varphi_2) \cap -D(\varphi_3) \).

As we have determined the value set \( D_F(\varphi_1 \perp \varphi_2) \) in 4.11, the claim readily follows by the validity of the following three easy equivalences for some \( x \) as above:

\[
\begin{align*}
x \in D_F(\varphi_1) \iff \varphi_1 \perp \varphi_3 \text{ is isotropic} \\
x \in D_F(\varphi_2) \iff \varphi_2 \perp \varphi_3 \text{ is isotropic} \\
-x \in D_F(\varphi_1) \cap D_F(\varphi_2) \iff -x \in D_F(\varphi_1) \cap D_F(\varphi_2) \cap D_F(\varphi_3).
\end{align*}
\]

Lemma 4.13. Let \( F \) be a rigid field and \( \varphi_1, \varphi_2 \) be quadratic forms over \( F \) such that the orthogonal sum \( \varphi_1 \perp \varphi_2 \) is anisotropic. Further let \( \psi \leq \varphi_1 \perp \varphi_2 \) be a subform of \( \varphi_1 \perp \varphi_2 \). Then there are quadratic forms \( \psi_1, \psi_2, \psi_3 \) over \( F \) such that we have \( \psi \cong \psi_1 \perp \psi_2 \perp \psi_3 \) and the forms \( \psi_1, \psi_2, \psi_3 \) fulfill the following:

1. \( \psi_1 \leq \psi_1, \psi_2 \leq \varphi_2 \);
2. \( D_F(\varphi_1) \cup D_F(\varphi_2) \cap D_F(\psi_3) \neq \emptyset \);
3. if we have \( s(F) \neq 2 \), we further have \( \psi_3 = 0 \);
4. for any \( x \in F^* \), the form \( \langle x, x \rangle \) is not a subform of \( \psi_3 \).

Proof. We prove the assertion by induction on \( \dim \psi \), the initial step \( \dim \psi = 0 \) being trivial. We thus assume \( \dim \psi > 0 \) in the following. We will first show that we can decompose \( \psi \cong \psi_1 \perp \psi_2 \perp \psi_3 \) such that (a), (b) and (c) are fulfilled and finally that any such decomposition fulfills (d) as well.

If we have

\[
D_F(\psi) \cap (D_F(\varphi_1) \cup D_F(\varphi_2)) = \emptyset,
\]

we must have \( s(F) = 2 \) by 4.11 and we can put \( \psi_3 = \psi \) and \( \psi_1 = 0 = \psi_2 \).

Otherwise we choose an arbitrary \( x \in D_F(\psi) \cap (D_F(\varphi_1) \cup D_F(\varphi_2)) \) and write \( \psi \cong \langle x \rangle \perp \psi' \) for some suitable form \( \psi' \) over \( F \). After renumbering we can assume without loss of generality that we have \( x \in D_F(\varphi_1) \). In particular there is a form \( \varphi'_1 \) such that we have \( \varphi_1 \cong \langle x \rangle \perp \varphi'_1 \). Using Witt’s Cancellation Theorem, we see that \( \psi' \) is a subform of \( \varphi'_1 \perp \varphi_2 \).

By induction hypothesis there are quadratic forms \( \psi'_1 \leq \varphi'_1, \psi'_2 \leq \varphi_2 \) and \( \psi'_3 \) with \( (D_F(\varphi'_1) \cup D_F(\varphi_2)) \cap D_F(\psi'_3) = \emptyset \), such that we have \( \psi' \cong \psi'_1 \perp \psi'_2 \perp \psi'_3 \).
We now put
\[ \psi_1 := \psi_1' \perp \langle x \rangle, \quad \psi_2 := \psi_2', \quad \psi_3 := \psi_3'. \]

Obviously, we have \( \psi \cong \psi_1 \perp \psi_2 \perp \psi_3 \) and \( \psi_1 \subseteq \varphi_1 \) and \( \psi_2 \subseteq \varphi_2 \). We will now prove \( (D_F(\varphi_1) \cup D_F(\varphi_2)) \cap D_F(\psi_3) = \emptyset \).

At first, we note that we have
\[ DF(\varphi_1) = \begin{cases} 
DF(\varphi_1') \cup \{ x \}, & \text{if } s(F) = 1 \\
DF(\varphi_1') \cup \{ -x \}, & \text{if } s(F) = 2 \text{ and } x \notin DF(\varphi_1') \\
DF(\varphi_1') \cup \{ x \}, & \text{if } s(F) = \infty \text{ and } x \notin DF(\varphi_1') \\
DF(\varphi_1'), & \text{if } s(F) = \infty \text{ and } x \in DF(\varphi_1'). 
\end{cases} \]

As we have \( (DF(\varphi_1') \cup DF(\varphi_2)) \cap DF(\psi_3') = \emptyset \) by induction hypothesis, the last case is clear. Since \( \psi \cong \psi_1 \perp \psi_2 \perp \psi_3 \) with \( x \in DF(\psi_1') \) is anisotropic, we further cannot have \( -x \in DF(\psi_3) \). Thus, the first and the third case are done.

For the remaining two cases, we have to exclude \( x \in DF(\psi_3) \). Assume the contrary. Since we have \( \psi_3 = \psi_3' \), the induction hypothesis yields \( x \notin DF(\varphi_1') \cup DF(\varphi_2') \). But \( \psi_3 = \psi_3' \) is a subform of \( \psi_1' \perp \varphi_2 \) so we have \( x \in DF(\varphi_1' \perp \varphi_2) \). As \( F \) is rigid, this is only possible if we have \( s(F) = 2 \) and additionally \( -x \in DF(\varphi_1' \perp \varphi_2) \), see 4.11. But this is impossible since then, \( \varphi_1 = \langle x \rangle \perp \varphi_1' \) would be isotropic. Thus (b) holds.

To prove (c), we now assume \( s(F) \neq 2 \). It is then enough to remark that we have \( DF(\varphi_1 \perp \varphi_2) = DF(\varphi_1) \cup DF(\varphi_2) \) by 4.11. Thus the first case in the induction step never occurs and we get \( \psi_3 = 0 \) automatically by proceeding as described above.

Finally, for (d), we can assume that we have \( s(F) = 2 \) according to (c). If we had \( \langle z, z \rangle \subseteq \psi_3 \) for some \( z \in F^* \), we would have \( z, -z \in DF(\psi_3) \subseteq DF(\psi) \subseteq DF(\varphi_1 \perp \varphi_2) \). As we have
\[ DF(\varphi_1 \perp \varphi_2) = DF(\varphi_1) \cup DF(\varphi_2) \cup \{ -x \mid x \in DF(\varphi_1) \cap DF(\varphi_2) \} \]
by 4.11, this would contradict the fact that we have
\[ (DF(\varphi_1) \cup DF(\varphi_2)) \cap DF(\psi_3) = \emptyset \]
and the conclusion follows.

As a strengthening of the above results, we get the following consequence which gives us a precise description of how three quadratic forms over a rigid field have to be related such that their sum has a prescribed Witt index.

**Corollary 4.14.** Let \( F \) be a rigid field and \( \varphi_1, \varphi_2, \varphi_3 \) be anisotropic forms over \( F \) such that \( \varphi_1 \perp \varphi_2 \) is anisotropic as well. Further let \( m \in \mathbb{N} \) be an integer. We then have \( i_W(\varphi_1 \perp \varphi_2 \perp \varphi_3) \geq m \) if and only if one of the following cases holds:

- we have \( s(F) \neq 2 \) and there are quadratic forms \( \psi_1 \subseteq \varphi_1, \psi_2 \subseteq \varphi_2 \) over \( F \) such that
  \[ \dim(\psi_1 \perp \psi_2) \geq m \quad \text{and} \quad -\psi_1 \perp -\psi_2 \subseteq \varphi_3; \]

- or

- we have \( s(F) = 2 \) and there are quadratic forms \( \psi_1 \subseteq \varphi_1, \psi_2 \subseteq \varphi_2 \) over \( F \) and \( x_1, \ldots, x_r \in F^* \setminus (DF(\varphi_1) \cup DF(\varphi_2)) \) representing pairwise different square classes such that
  \[ -\psi_1 \perp -\psi_2 \perp -(x_1, \ldots, x_r) \subseteq \varphi_3 \]
  \[ (x_1, \ldots, x_r) \subseteq (\varphi_1 \perp -\varphi_1)_{an} \perp (\varphi_2 \perp -\varphi_2)_{an}, \]
  \[ \dim \psi_1 + \dim \psi_2 + r \geq m. \]
**Proof.** By an easy induction on the integer \( m \) using the uniqueness of the Witt decomposition and the anisotropy of \( \varphi_1 \perp \varphi_2 \), we have \( h_W(\varphi_1 \perp \varphi_2) \geq m \) if and only if there is some quadratic form \( \psi \) over \( F \) of dimension at least \( m \) such that we have \( -\psi \not\subseteq \varphi_3 \) and \( \psi \subseteq \varphi_1 \perp \varphi_2 \).

Thus, to show the if part, it is enough to remark that we can choose

\[
\psi := \begin{cases} 
\varphi_1 \perp \varphi_2, & \text{if } s(F) \neq 2 \\
\varphi_1 \perp \varphi_2 \perp \langle x_1, \ldots, x_r \rangle, & \text{if } s(F) = 2 
\end{cases}
\]

as such a form. To show the only if part, let \( \psi \) be given as above. We separate the cases \( s(F) \neq 2 \) and \( s(F) = 2 \). If we have \( s(F) \neq 2 \), \[4.13\] yields that we have a decomposition \( \psi = \varphi_1 \perp \varphi_2 \) and for these \( \varphi_1, \varphi_2 \), the requirements are obviously fulfilled.

So let now \( s(F) = 2 \). We apply \[4.13\] again and get a decomposition \( \psi = \varphi_1 \perp \varphi_2 \perp \varphi_3 \), where we can write \( \varphi_3 = \langle x_1, \ldots, x_r \rangle \) for some \( r \in \mathbb{N} \) and \( x_1, \ldots, x_r \in F^* \) representing different square classes. As the other properties are readily seen to be satisfied, it remains to show that we have \( \langle x_1, \ldots, x_r \rangle \subseteq (\varphi_1 \perp -\varphi_1)_{\text{an}} \perp (\varphi_2 \perp -\varphi_2)_{\text{an}} \). As \( \varphi_i \) is a subform of \( \varphi_i \) for \( i \in \{1, 2\} \) and \( \varphi_1 \perp \varphi_2 \) is anisotropic, the latter form is isometric to \( (\varphi_1 \perp \varphi_2 \perp -\varphi_1 \perp -\varphi_2)_{\text{an}} \). Since we have

\[
\psi = \varphi_1 \perp \varphi_2 \perp \langle x_1, \ldots, x_r \rangle \subseteq \varphi_1 \perp \varphi_2
\]

we get the desired subform relation as an easy consequence of Witt’sCancellation Theorem.

**Lemma 4.15.** Let \( F \) be a rigid field of level \( s(F) = 2 \) and let \( x_1, \ldots, x_r \in F^* \) represent pairwise different square classes such that the quadratic form \( \langle x_1, \ldots, x_r \rangle \) is anisotropic. Further, let \( \varphi, \psi \) be quadratic forms over \( F \) such that \( \varphi \perp \psi \) is anisotropic and such that we have \( x_i \notin D_F(\varphi) \cup D_F(\psi) \), but \( x_i \in D_F(\varphi \perp \psi) \) for all \( i \in \{1, \ldots, r\} \). We then have both

\[
-\langle x_1, \ldots, x_r \rangle \subseteq \varphi \quad \text{and} \quad -\langle x_1, \ldots, x_r \rangle \subseteq \psi.
\]

**Proof.** As we have \( \langle x_1, \ldots, x_r \rangle \subseteq \varphi \perp \psi \) but \( x_i \notin D_F(\varphi) \cup D_F(\psi) \) for all \( i \in \{1, \ldots, r\} \), \[4.11\] implies \( -x_i \in D_F(\varphi) \cap D_F(\psi) \). Thus, the induction base is clear by the Representation Criterion. So let now \( r \geq 2 \). By the above, we further have representations

\[
\varphi = \langle -x_1 \rangle \perp \varphi' \quad \text{and} \quad \psi = \langle -x_1 \rangle \perp \psi'.
\]

We thus have

\[
\varphi \perp \psi \cong \langle x_1, x_1 \rangle \perp \varphi' \perp \psi'
\]

and \[4.11\] then implies that we have a disjoint union

\[
D_F(\varphi \perp \psi) = D_F(\langle x_1, x_1 \rangle \perp \varphi' \perp \psi') = \{ \pm x_1 \} \cup D_F(\varphi' \perp \psi').
\]

Since the form \( \langle x_1, \ldots, x_r \rangle \) is anisotropic and the \( x_i \) represent different square classes, we have \( x_2, \ldots, x_r \notin \{ \pm x_1 \} \). We thus have \( x_2, \ldots, x_r \in D_F(\varphi' \perp \psi') \).

It is clear that we still have \( x_i \notin D_F(\varphi') \cup D_F(\psi') \) for all \( i \in \{2, \ldots, r\} \) as these are subforms of \( \varphi \) respective \( \psi \). By induction hypothesis, we have

\[
-\langle x_2, \ldots, x_r \rangle \subseteq \varphi' \quad \text{and} \quad -\langle x_2, \ldots, x_r \rangle \subseteq \psi'
\]

which then implies the assertion.

### 5. 14-dimensional \( I^3 \)-forms and 8-dimensional \( I^2 \)-forms

From [6] and [8], it is known that there is a deep connection between 14-dimensional \( I^3 \)-forms and 8-dimensional \( I^2 \)-forms. In this section, we will study both types over rigid fields since the results obtained here will help us to classify 16-dimensional forms in the third power of the fundamental ideal. In this context, it is convenient to introduce the following notation.
Definition 5.1. A field $F$ is called a $D(8)$-field, if any 8-dimensional form in $I^2 F$ whose Clifford invariant has index 4 is isometric to a sum of two forms in $GP_2 F$.

The field $F$ is called a $D(14)$-field if any 14-dimensional form in $I^3 F$ is Witt equivalent to a sum of two forms in $GP_3 F$.

We will see that rigid fields fulfill both $D(8)$ and $D(14)$. Before proving this, we repeat the classification theorem for 14-dimensional $I^3$-forms.

Proposition 5.2 ([6, Proposition 2.3] or [8, Proposition 17.2]). Let $\varphi \in I^3 F$ be a quadratic form over $F$ with $\dim \varphi = 14$. Then $\varphi$ is Witt equivalent to a sum of 3 $GP_3$-forms. Further the following are equivalent:

(i) there are $\tau_1, \tau_2 \in P_3 F$ and $s_1, s_2 \in F^*$ such that $\varphi$ is Witt equivalent to $s_1 \tau_1 \perp s_2 \tau_2$;

(ii) there are $\tau_1, \tau_2 \in P_3 F$ and $s \in F^*$ such that $\varphi$ is isometric to $s(\tau_1^* \perp -\tau_2^*)$;

(iii) there is some $\sigma \in GP_2 F$ with $\sigma \subseteq \varphi$.

As finite fields, the reals and the complex numbers all have Hasse invariant at most 2 and each rigid field is $q$-equivalent to a field of iterated Laurent series over one of these fields by 4.5, [6, Corollary 5.1] implies the following:

Proposition 5.3. Each rigid field fulfills both $D(8)$ and $D(14)$.

### 6. 16-dimensional $I^3$-forms

We are able to classify those 16-dimensional forms in $I^3 F$ for rigid fields that are Witt equivalent to a sum of at most three forms in $GP_3 F$. At the end of the section, we will see that any 16-dimensional form in $I^3 F$ satisfies the following equivalent conditions.

Proposition 6.1. Let $F$ be a rigid field and $\varphi \in I^3 F$ be an anisotropic quadratic form with $\dim \varphi = 16$. Then the following are equivalent:

(i) $\varphi$ is isometric to a sum of 4 forms in $GP_2 F$;

(ii) $\varphi$ contains a subform in $GP_2 F$;

(iii) $\varphi$ is Witt equivalent to a sum of at most 3 forms in $GP_3 F$.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. For the implication (ii) $\Rightarrow$ (iii), we write $\varphi = \sigma \perp \langle w \rangle \perp \psi$ for some $\sigma \in GP_2 F$, some $w \in F^*$ and a suitable quadratic form $\psi$ of dimension 11 over $F$. We can find $x, y, z \in F^*$ such that we have $\sigma \perp \langle w, x, y, z \rangle \in GP_3 F$. In $WF$ we thus have

$$\varphi = \sigma \perp \langle w \rangle \perp \psi = \sigma \perp \langle w, x, y, z \rangle \perp \psi \perp \langle -x, -y, -z \rangle.$$

We further have $\dim(\psi \perp \langle -x, -y, -z \rangle) = 14$ so that this form is Witt equivalent to a sum of at most two $GP_3 F$-forms by 5.3, so this implication is done.

For the implication (iii) $\Rightarrow$ (i), let now $\varphi = \pi_1 + \pi_2 + \pi_3 \in WF$ with $\pi_1, \pi_2, \pi_3 \in GP_3 F$. We further assume that $\dim(\pi_1 \perp \pi_2)_{an}$ is minimal under all such representations. We will distinguish between the possible dimensions that can occur. As we have $(\pi_1 \perp \pi_2)_{an} \in I^3 F$, the Gap Theorem implies $\dim(\pi_1 \perp \pi_2)_{an} \in \{0, 8, 12, 14, 16\}$.

- $\dim = 0$: This contradicts the fact that we have $\dim \varphi = 16$.
- $\dim = 8$: If we have $\dim(\pi_1 \perp \pi_2)_{an} = 8$ then $(\pi_1 + \pi_2)_{an}$ is isometric to some $\pi \in GP_3 F$ according to the Arason-Pfister Hauptsatz. Thus, we have $\varphi \cong \pi \perp \psi$ and the conclusion follows.
- $\dim = 12$: If we have $\dim(\pi_1 \perp \pi_2)_{an} = 12$, then $(\pi_1 + \pi_2)_{an}$ is divisible by a binary form $\langle a \rangle$ due to [13, Satz 14, Zusatz]. Thus, we have $i_W(\varphi_{F, \sqrt{a}}) \geq 4$ and we can write $\varphi \cong \langle a \rangle \otimes \sigma \perp \psi$ with some
4-dimensional form \( \sigma \) and some 8-dimensional form \( \psi \). According to [10, Example 9.12] \( \langle a \rangle \otimes \sigma \) is an 8-dimensional form in \( I^2 F \), whose Clifford invariant has index at most 2. In \( WF \) we therefore have
\[
\psi = \varphi - \langle a \rangle \otimes \sigma \in I^2 F
\]
which then implies
\[
c(\psi) = c(\varphi) c(\langle a \rangle \otimes \sigma) = c(\langle a \rangle \otimes \sigma).
\]

Using [10, Example 9.12] again, we see that \( \psi \) is divisible by a binary form as well. As 8-dimensional forms that are divisible by a binary form are isometric to a sum of two forms in \( GP_2 F \), we are done in this case.

\[\text{dim} = 14: \text{So let now} \dim(\pi_1 \perp \pi_2)_{an} = 14. \text{According to 5.2 we can assume we have} \ (\pi_1 \perp \pi_2)_{an} \cong \pi_1' \perp -\pi_2', \text{possibly after a scaling. We further have} \ i_W(\pi_1' \perp -\pi_2' \perp \pi_3) = 3, \text{such that there is some} \ 3\text{-dimensional form} \ \psi \subseteq \pi_1' \perp -\pi_2' \text{with} \ -\psi \subseteq \pi_3. \]

We decompose \( \psi = \psi_1 \perp \psi_2 \perp \psi_3 \) as in 4.13. We then have \( \dim \psi_1, \dim \psi_2 \leq 1 \) since if we had say \( \dim \psi_1 \geq 2 \), we would have
\[
\dim(\pi_1 \perp \pi_3)_{an} \leq \dim \pi_1 + \dim \pi_3 - 2 \dim \psi_1 \leq 8 + 8 - 2 \cdot 2 = 12,
\]
contradicting the minimality of \( \dim(\pi_1 \perp \pi_2)_{an} \). As the dimensions of \( \psi_1, \psi_2 \) and \( \psi_3 \) have to sum up to 3, we have \( \dim \psi_3 \geq 1 \) and thus \( s(F) = 2 \). We will now show that we have \( \dim \psi_3 = 1 \). Otherwise according to 4.13 (b) and (d) there would be
\[
x, y \in D_F(\pi_1' \perp -\pi_2') \setminus (D_F(\pi_1') \cup D_F(-\pi_2'))
\]
that represent different square classes and are represented by \( \psi_3 \). Now 4.15 implies
\[
-(x, y) \subseteq \pi_1' \quad \text{and} \quad (x, y) \subseteq \pi_2'.
\]
This implies that both \( \pi_1 \) and \( \pi_2 \) become isotropic (hence hyperbolic) over \( F(\sqrt{-xy}) \). Since this is equivalent to \( \pi_1, \pi_2 \) having a common slot and since \( \pi_1 \perp \pi_2 \) is isotropic, this would imply \( \dim(\pi_1 \perp \pi_2)_{an} < 14 \) by [11, Chapter X, Theorem 5.16], a contradiction. We therefore obtain
\[
\dim \psi_1 = \dim \psi_2 = \dim \psi_3 = 1.
\]

Thus \( \varphi \) contains a 10-dimensional subform that is the orthogonal sum of a 5-dimensional subform of \( \pi_1 \) and a 5-dimensional subform of \( \pi_2 \). Both of these forms are Pfister neighbors that contain a subform in \( GP_2 F \) according to [11, Chapter X, Proposition 4.19]. Thus \( \varphi \) has a decomposition \( \varphi = \sigma \perp \tau \), where \( \sigma \) is isometric to a sum of two forms in \( GP_2 F \). We thus have \( \sigma \in I^2 F \) and the Clifford invariant of \( \sigma \) has index at most 4. As in the case \( \dim = 12 \), these properties also hold for \( \tau \). Applying 5.3 now gives us that \( \tau \) is also isometric to a sum of two forms in \( GP_2 F \) which finishes this case.

\[\text{dim} = 16: \text{Here, we are reasoning just as in the latter case and use the same terminology for all upcoming forms etc. We have} \ \dim \psi = 4. \text{Because of the minimality of} \ \dim(\pi_1 \perp \pi_2)_{an}, \text{we even have} \ \psi_1 = 0 = \psi_2. \text{As in the case} \ \dim = 14 \ \text{above, we see that the Pfister forms that are similar to} \ \pi_1 \ \text{respectively} \ \pi_2 \ \text{have a common slot, so that} \ \pi_1 \perp \pi_2 \ \text{is divisible by a binary form} \ \langle a \rangle. \text{Now the conclusion follows as in the case} \ \dim = 12. \]

Our next goal is to study 16-dimensional form in \( I^3 F \) in more detail in order to prove that each such form satisfies the equivalent conditions of 6.1. To do so, we need the next technical lemma.

**Lemma 6.2.** Let \( F \) be a rigid field and \( \varphi_1, \varphi_2 \) be two anisotropic quadratic forms over \( F \), such that \( \varphi_1 \perp \varphi_2 \) is an anisotropic form in \( I^3 F \) of dimension 14. Then, for any \( t \in F^* \), the form \( \varphi_1 \perp t \varphi_2 \) contains a subform in \( GP_2 F \).

**Proof.** We show that one of the forms \( \varphi_1 \) and \( \varphi_2 \) already contains a subform in \( GP_2 F \) or that there is some binary form that is similar to both a subform of \( \varphi_1 \) and a subform of \( \varphi_2 \). This obviously implies the assertion.
Since $F$ is a rigid field, $F$ is a $D(14)$-field by 5.3. Therefore, after a possible scaling, we may assume that we have $\pi_1, \pi_2 \in \mathcal{P}_3 F$ with

$$\varphi_1 \perp \varphi_2 \cong \pi_1' \perp -\pi_2'.$$

We remark that $\pi_1, \pi_2$ cannot have a common slot.

As $\pi_1, \pi_2$ are 3-fold Pfister forms, we can choose $a, a' \in F^*$ and 3-dimensional forms $\sigma, \sigma'$ over $F$ such that we have

$$\psi := \langle \langle a \rangle \rangle \otimes \sigma \subseteq \pi_1', \ \psi' := \langle \langle a' \rangle \rangle \otimes \sigma' \subseteq -\pi_2',$$

see [5, Theorem 4.1]. In particular $\psi \perp \psi'$ is also a subform of $\varphi_1 \perp \varphi_2$. We now decompose $\psi \cong \psi_1 \perp \psi_2 \perp \psi_3$ and $\psi' \cong \psi_1' \perp \psi_2' \perp \psi_3'$ according to 4.13. We will now proof the assertion while distinguishing the possible dimensions of these subforms:

Case 1: $\dim \psi_3 = 0$ or $\dim \psi_3' = 0$.

According to the symmetry of the statement, it is enough to consider the case $\dim \psi_3 = 0$. Further we can assume $\dim \psi_1 \geq \dim \psi_2$, possibly after renumbering the $\varphi_i$. As we clearly have $\dim \psi_1 + \dim \psi_2 = 6$ the latter implies $\dim \psi_3 \geq 3$.

If we have $\dim \psi_1 \geq 5$, it follows readily that $\psi_1$ already contains a four dimensional subform that is divisible by $\langle \langle a \rangle \rangle$, i.e. a form in $\text{GP}_2 F$.

If we have $\dim \psi_1 = 4$ we can use the same arguments as above to get that $\psi_1$ becomes isotropic over $F(\sqrt{a})$ which then implies that $\psi_1$ is similar to $\langle \langle a \rangle \rangle \perp \beta$ with some quadratic form $\beta$ of dimension 2.

Thus, $\beta \perp \psi_2$ is divisible by $\langle \langle a \rangle \rangle$, which implies $\beta \perp \psi_2 \in \text{GP}_2 F$. Using [11, Chapter X. Corollary 5.4] one readily sees that this is only possible if $\beta$ and $\psi_2$ are similar which concludes this case.

If $\dim \psi_1 = 3$ and $\psi_1$ becomes isotropic over $F(\sqrt{a})$, then so does $\psi_2$ as $\psi$ becomes hyperbolic over $F(\sqrt{a})$. Thus, both $\psi_1$ and $\psi_2$ contain a subform similar to $\langle \langle a \rangle \rangle$ and this case is done.

Otherwise $\psi_1$ and $\psi_2$ are quadratic forms of dimension 3 that stay anisotropic over $K := F(\sqrt{a})$ but fulfill $(\psi_1)_K \cong - (\psi_2)_K$. By 4.8 $K$ is a rigid field, too. Using 4.10 we see that the diagonalization of $(\psi_1)_K$ is either unique up to multiplying its entries with squares and permuting the entries or we have $s(K) = 2$ (and thus also $s(F) = 2$ as can readily seen using [15, Theorem 2.7]) and $(\psi_1)_K = \langle x, y, z \rangle$ for some $x, y, z \in F^*$.

In the first case, we write $(\psi_1)_K = \langle x, y, z \rangle$ for suitable $x, y, z \in F^*$ representing pairwise different square classes in $K$. Using [11, Chapter VII. Theorem 3.8], we see that we have

$$\psi_1 = \langle a^i x, a^j y, a^k z \rangle$$

and $\psi_2 = - \langle a^{i2} x, a^{j2} y, a^{k2} z \rangle$

for some $i_1, i_2, j_1, j_2, k_1, k_2 \in \{0, 1\}$. After renaming $x, y, z$, the pigeon hole principle implies that we have either $i_1 = i_2$ and $j_1 = j_2$ or $i_1 \neq i_2$ and $j_1 \neq j_2$. Both the equalities and the inequalities imply $\langle a^1 x, a^2 y \rangle$ and $- \langle a^{i2} x, a^{j2} y \rangle$ to be similar so that this case is done.

In the second case we argue the same way. We see that $\psi_1$ is isometric to one of the following forms on the left for some $i \in \{0, 1\}$ and $\psi_2$ is isometric to one of the forms on the right for some $j \in \{0, 1\}$:

$$\langle x, x, a^i y \rangle \cong \langle -x, -x, a^j y \rangle$$

$$\langle a x, a x, a^i y \rangle \cong \langle - a x, - a x, a^j y \rangle$$

$$\langle x, x, a^i y \rangle$$

Thus a binary form that is similar to both a subform of $\psi_1$ and a subform of $\psi_2$ can be found in the upcoming table in which all cases with $\psi_1 \not\cong - \psi_2$ (that case being clear) are considered.
Case 2: $\dim \psi_3 \geq 2$ or $\dim \psi'_3 \geq 2$

It is again enough to consider the case $\dim \psi_3 \geq 2$. Because of 4.13 (d) there are $x, y \in F^*$ representing different square classes with $\psi_3 = (x, y, \ldots)$. Because of 4.13 (b) we have $x, y \in D_F(\psi_3) \subseteq D_F(\varphi_1 \perp \varphi_2)$ but $x, y \notin D_F(\varphi_1) \cup D_F(\varphi_2)$. Now, 4.15 implies both $\varphi_1 = (-x, -y, \ldots)$ and $\varphi_2 = (-x, -y, \ldots)$. According to the statement at the beginning of the proof, this case is done.

Case 3: $\dim \psi_3 = 1 = \dim \psi'_3$

If we have $\psi_3 = \langle x \rangle \not\equiv \langle y \rangle = \psi'_3$ for some $x, y \in F^*$, we can argue as in the last case using 4.15 to get $\varphi_1 = (-x, -y, \ldots)$ and $\varphi_2 = (-x, -y, \ldots)$ and we are done.

Otherwise we have $\psi_3 = \langle x \rangle = \psi'_3$, so we can write $\varphi_1 = v_1 \perp (-x)$ and $\varphi_2 = v_2 \perp (-x)$. We further choose orthogonal complements of $\langle x \rangle$ in $\pi'_1$ respectively $-\pi'_2$. As in the beginning of the proof, we can write them as a product of a Pfister form and a ternary form, i.e., we have

$$\pi'_1 = \langle \langle b \rangle \otimes \tau \perp \langle x \rangle \rangle \quad \text{and} \quad -\pi'_2 = \langle \langle b' \rangle \otimes \tau' \perp \langle x \rangle \rangle,$$

for some ternary forms $\tau, \tau'$ and $b, b' \in F^*$. We have a chain of isometries

$$v_1 \perp \langle x \rangle \perp v_2 \perp \langle x \rangle \equiv v_1 \perp \langle -x \rangle \perp v_2 \perp \langle -x \rangle \equiv \varphi_1 \perp \varphi_2 \equiv \pi'_1 \perp -\pi'_2 \equiv \langle \langle b \rangle \otimes \tau \perp \langle x \rangle \rangle \otimes \langle b' \rangle \otimes \tau' \perp \langle x \rangle \rangle.$$

Witt’s cancellation law now implies $\langle \langle b \rangle \otimes \tau \perp \langle b' \rangle \otimes \tau' \equiv v_1 \perp v_2$.

We now apply the above argument for $\langle \langle b \rangle \otimes \tau \rangle$ and $\langle b' \rangle \otimes \tau'$ as subforms of $v_1 \perp v_2$. Note that all arguments used above stay valid as we did not use any specific information on $\varphi_1, \varphi_2$ but only of the chosen subforms $\psi, \psi'$. If we are in case 1 or 2 for $b, b', \tau, \tau', v_1, v_2$ we are done as we have already seen. If we are again in case 3 for $b, b', \tau, \tau', v_1, v_2$, we get the existence of some $y \in F^*$ represented by both $\pi'_1$ and $-\pi'_2$. This would imply $\pi_1$ and $\pi_2$ to have $-xy$ as a common slot similar as in the case $\dim = 14$ in 6.1, which we excluded at the beginning of the proof. Thus we are done.

**Theorem 6.3.** Let $F$ be a rigid field and $\varphi \in I^3 F$ be an anisotropic quadratic form over $F$ of dimension 16. Then $\varphi$ is Witt equivalent to a sum of at most three forms in $GP_3 F$.

**Proof.** We will show that $\varphi$ contains a subform in $GP_2 F$ so that the conclusion then follows by 6.1. After scaling, we can assume $1 \in D_F(\varphi)$. If $\varphi$ is isometric to $16 \times \langle 1 \rangle$ (which is only possible if $F$ is superpythagorean), the assertion is clear. Otherwise there is some $n \in \mathbb{N}$ such that we can assume $\varphi$ to be defined over the field $K(\langle t_1 \rangle) \cdots (\langle t_n \rangle)$ and that $\varphi$ has a decomposition into residue class forms $\varphi \cong \varphi_1 \perp t_n \varphi_2$ such that both residue class forms have positive dimension. As mentioned in 3.9 we can replace the uniformizer $t_n$ with $at_n$ for any $a \in K(\langle t_1 \rangle) \cdots (\langle t_{n-1} \rangle)^*$. By doing so, we also get $a \varphi_2$ as the second residue class form instead of $\varphi_2$. We may thus assume $D_F(\varphi_1) \cap D_F(\varphi_2) = \emptyset$, i.e., $\sigma := (\varphi_1 \perp -\varphi_2)_n$ has dimension at most 14. If we have $\dim \sigma \leq 12$, there is some binary form $\beta$ that is a subform of both $\varphi_1$ and $\varphi_2$, so that $\beta \otimes (1, t_n) \in GP_2 F$ is a subform of $\varphi$.

If we have $\dim \sigma = 14$, there is some $x \in F^*$ and quadratic forms $\psi_1, \psi_2$ such that we have

$$\varphi_1 \cong \langle x \rangle \perp \psi_1 \quad \text{and} \quad \varphi_2 \cong \langle x \rangle \perp \psi_2.$$

As in the proof of 3.6, we have $\sigma \cong \psi_1 \perp -\psi_2 \in I^3 F$ (in fact, our $\sigma$ here has exactly the same role as the $\sigma$ in the above mentioned result). As we have $\dim \sigma = 14$, it contains a subform lying in $GP_2 F$ according to 5.3. By 6.2 the form $\psi_1 \perp t_n \psi_2$ also contains a $GP_2$-subform, which then trivially implies

$$\varphi \cong \psi_1 \perp \langle x \rangle \perp t_n \psi_2 \perp \langle x \rangle,$$

to have a subform in $GP_2 F$, which concludes the proof.

**Example 6.4.** The bound in 6.3 is sharp as the following example shows. Let $K \in \{ \mathbb{F}_3, \mathbb{R}, \mathbb{C} \}$ and $F = K(\langle (a) \rangle (\langle b \rangle) (\langle c \rangle) (\langle d \rangle) (\langle e \rangle) (\langle f \rangle))$. We first construct an 8-dimensional form in $I^3 F$ that is not Witt equivalent
to a sum of 2 forms in $GP_2 F$. To do so, we can consider
\[ \psi := \langle 1, a, b, c, d, e, f, abcdef \rangle \in I^2 F, \]
which is the generic 8-dimensional form in $I^2 F$ and fulfills $GP_{2}(\psi) = 3$ by 2.4. Then, $\varphi := \psi \otimes \langle t \rangle \in I^3 F(t)$ fulfills $GP_{3}(\varphi) = 3$ by 3.3.

Another common way to measure the complexity of a quadratic form is to study its splitting behavior over multiquadratic field extensions. There are 16-dimensional $I^3$-forms over non-rigid fields that do not split over multiquadratic extensions of degree \( \leq 8 \), see [9, Theorem 2.1]. For rigid fields, the situation is much less involved.

**Proposition 6.5.** Let $\varphi$ be a 16-dimensional form in $I^3 F$ with $F$ rigid. Then $\varphi$ splits over some biquadratic extension of $F$, i.e. there are $a, b \in F^*$ such that $\varphi_{F(\sqrt{a}, \sqrt{b})}$ is hyperbolic.

**Proof.** According to 6.3 and 6.1 we can write $\varphi = \psi \perp \sigma$ where we have $\sigma \in GP_2 F$. We choose $a \in F$ such that $\sigma_{F(\sqrt{a})}$ is isotropic hence hyperbolic. If $\psi_{F(\sqrt{a})}$ is isotropic then it is hyperbolic or Witt equivalent to a form in $GP_3 F(\sqrt{a})$ that is defined over $F$ as quadratic extensions are excellent, see [11, Chapter XII, Proposition 4.4]. In both cases the assertion is clear.

Otherwise $\psi_{F(\sqrt{a})}$ is an anisotropic, 12-dimensional form in $I^3 F(\sqrt{a})$ and hence divisible by a binary Pfister form $\langle b \rangle$ for some $b \in F(\sqrt{a})^*$. By [15, Theorem 1.9], the square class of $b$ in $F(\sqrt{a})$ has a representative of the form $z$ or $z\sqrt{a}$ for some $z \in F^*$. We are done if we can exclude the latter case. As $F(\sqrt{a})$ is also a rigid field by 4.8, we know how two diagonalizations of the same form can differ by 4.10. As $\psi$ is defined over $F$, we can thus deduce that we must have $b \in F^*$.

**Example 6.6.** 6.5 is sharp in the sense that in general, forms of dimension 16 in $I^3 F$ over a rigid field $F$ will not split over a quadratic extension. As an example, we can consider the 16-dimensional form $\langle a, b, c \rangle \perp \langle d, e, f \rangle$ over the field $F := K(\langle a \rangle) \langle b \rangle \langle c \rangle \langle d \rangle \langle e \rangle \langle f \rangle$ where we can choose $K \in \{R, C, F_3\}$.

We can show that the characterization in 6.1 does not generalize to arbitrary fields. To do so, we need the following result.

**Proposition 6.7.** Let $\varphi \in I^3 F$ be an anisotropic quadratic form with $\dim \varphi = 16$. We further presume the existence of some $\sigma, \tau \in GP_2 F$ with $\sigma \perp \tau \subseteq \varphi$. Then $\varphi$ is Witt equivalent to a sum of at most three elements in $GP_3 F$.

**Proof.** By our assumption, we have $\varphi \cong \sigma \perp \tau \perp \langle w \rangle \perp \psi$ for some $w \in F^*$ and a 7-dimensional quadratic form $\psi$ over $F$. We choose $x, y, z \in F^*$ such that $\langle w, x, y, z \rangle$ is similar to $\sigma$. This implies in particular $\sigma \perp \langle w, x, y, z \rangle \in GP_3 F$. In $WF$ we thus have
\[ \varphi = \sigma + \tau + \langle w \rangle + \psi = \left( \sigma \perp \langle w, x, y, z \rangle \right) + \left( \tau \perp \langle -x, -y, -z \rangle \perp \psi \right). \]
Since we have $\varphi, \sigma \perp \langle w, x, y, z \rangle \in I^3 F$, we also have $\tau \perp \langle -x, -y, -z \rangle \perp \psi \in I^3 F$. Further we have $\dim(\tau \perp \langle -x, -y, -z \rangle \perp \psi) = 14$ and this form contains $\tau \in GP_2 F$ as a subform. Thus $\tau \perp \langle -x, -y, -z \rangle \perp \psi$ is Witt equivalent to a sum of at most two $GP_3 F$-forms by 5.2 and the conclusion follows.

The rest of this section is devoted to the construction of an example that shows, that the characterization in 6.1 does not hold over non-rigid fields.

**Proposition 6.8.** Let $F$ be a field that does not satisfy $D(8)$. Then there is a 16-dimensional form $\varphi \in I^3 F(t_1) \langle t_2 \rangle$ that is a sum of three elements in $GP_3 F$ but is not isometric to a sum of four elements in $GP_2 F$. 


Proposition 7.1. lowerboundsascanbeseenintheupcomingProposition. assumption to talk about meaningful lower bounds, we will allow rigid field extensions while finding fields. As a fixed field can be too small to have anisotropic forms of all dimensions, which is a necessary In this section, we will study the growth of Pfister numbers for forms of increasing dimension over rigid fields. As a fixed field can be too small to have anisotropic forms of all dimensions, which is a necessary assumption to talk about meaningful lower bounds, we will allow rigid field extensions while finding lower bounds as can be seen in the upcoming Proposition.

Proposition 7.1. Let $F$ be a rigid field. Then, there is some field extension $E/F$ such that $E$ is a rigid field and for any integer $d \geq 8$, we have

$$\text{GPf}_3(E, d) \geq \left\lceil \frac{d}{4} \right\rceil - 1.$$  (5)
Proof. As the term on the right sight of (5) increases monotonously when \(d\) grows, we may assume that \(d\) is even. According to 4.5 and passing to a field extension, we may further assume \(F = K((t_i))_{i \in I}\) for some algebraically closed field \(K\) and some infinite index set \(I\). To simplify notation, we assume \(\mathbb{N} \subseteq I\).

We define the integer \(n\) to be

\[
n := 2 \cdot \left\lfloor \frac{d}{4} \right\rfloor - 2 = \begin{cases} \frac{d}{2} - 2, & \text{if } d \equiv 0 \pmod{4} \\ \frac{d}{2} - 3, & \text{if } d \equiv 2 \pmod{4} \end{cases}.
\]

Note that \(n\) is even in both cases. By 2.4, using 3.5 and induction (recall the definition of \(K((t_i))_{i \in I}\) as a direct limit, see 4.5 again), for \(\psi := (1, t_1, \ldots, t_n, (-1)^{\frac{n+2}{2}} t_1 \cdot \ldots \cdot t_n) \in I^2 F\), we have

\[
GP_f^2(\psi) = \frac{n}{2}.
\]

Now, for the form \(\varphi := (t_{n+1}) \otimes \psi \in I^3 F\), which is of dimension

\[
2(n + 2) \leq 2 \left( \frac{d}{2} - 2 + 2 \right) = d,
\]

we have

\[
GP_f^3(\varphi) = \frac{n}{2} = \left\lfloor \frac{d}{4} \right\rfloor - 1.
\]

by 3.3 and the conclusion follows.

Furthermore we are already in a good position to determine an upper bound for the 3-Pfister number over rigid fields that generalizes [14, Theorem 1.13]. Our main ingredient is 4.7, which was proved with valuation theory.

**Theorem 7.2.** Let \(F\) be a rigid field. For all even \(d \in \mathbb{N}_0\), we have

\[
GP_f^3(F, d) \leq \frac{d^2}{16}.
\]

If we further have \(d \geq 16\), we even have

\[
GP_f^3(F, d) \leq \frac{d^2}{16} - \frac{d}{2} - \frac{82 - 2 \cdot (-1)^{\frac{d}{2}}}{16}.
\]

Proof. We will implicitly use that the functions \(d \mapsto GP_f^3(F, d)\) and \(d \mapsto \frac{d^2}{16} - \frac{d}{2} - \frac{82 - 2 \cdot (-1)^{\frac{d}{2}}}{16}\) are monotonically increasing on the set of even integers \(\geq 16\) without referring to this fact explicitly. The proof is by induction on \(d\). We already know the following inequalities

\[
GP_f^3(F, d) = 0 \text{ for all even } d < 8, \quad GP_f^3(F, 8) = GP_f^3(F, 10) = 1,
\]

\[
GP_f^3(F, 12) = 2, \quad GP_f^3(F, 14) = 2, \quad GP_f^3(F, 16) = 3,
\]

that are all compatible with the assertion. As we obviously have the inequality

\[
\frac{d^2}{16} - \frac{d}{2} - \frac{82 - 2 \cdot (-1)^{\frac{d}{2}}}{16} \leq \frac{d^2}{16}
\]

for \(d \geq 16\), we only have to show the second bound.

If a form \(\varphi \in I^3 F\) of dimension \(d \geq 16\) is similar to \(d \times (1)\) it is Witt equivalent (in fact even isometric) to a sum of \(\frac{d}{2}\) elements in \(GP_f^3 F\) and we are done. Otherwise we can bound \(GP_f^3(\varphi)\) according to 4.7 by

\[
GP_f^3(F, d - 2) + GP_f^2(F, k),
\]
where $k$ is the biggest integer $\leq \frac{d}{2}$ that is divisible by 2, i.e. we have
$$k = \frac{d}{2} - \frac{1}{2} + (-1)^{\frac{d}{2}} \cdot \frac{1}{2} = 2 \cdot \left\lfloor \frac{d}{4} \right\rfloor,$$
as we can assume the form $\tau$ in 4.7 to be of dimension at most $\leq \frac{d}{2}$ after possibly scaling with a uniformizer (note that $\tau$ is the second residue class form). By 2.3 we thus know
$$\text{GPf}_2(F, k) \leq \text{GPf}_2\left(F, \frac{d}{2} - \frac{1}{2} + (-1)^{\frac{d}{2}} \cdot \frac{1}{2}\right)$$
$$= \frac{d}{2} - \frac{1}{2} + (-1)^{\frac{d}{2}} \cdot \frac{1}{2} - 1 = \frac{d}{4} - \frac{5}{4} + (-1)^{\frac{d}{2}} \cdot \frac{1}{4},$$
which leads to
$$\text{GPf}_3(F, d) \leq \text{GPf}_3(F, d - 2) + \frac{d}{4} - \frac{5}{4} + (-1)^{\frac{d}{2}} \cdot \frac{1}{4}. \quad (6)$$

We now put $n := \frac{d}{2} - 8$, which is equivalent to $d = 2n + 16$, and consider for $n \in \mathbb{N}$ the recurrence relation
$$a_n = a_{n-1} + \frac{n}{2} + \frac{11}{4} + (-1)^n \cdot \frac{1}{4},$$
which was built by replacing the inequality with an equality in (6). For $a_0 = 3$ (corresponding to $\text{GPf}_3(F, 16) = 3$) this relation has the unique solution
$$a_n = \frac{1}{8} (2n(n + 12) + (-1)^n + 23) = \frac{d^2}{16} - \frac{d}{2} - \frac{82}{16} \cdot (-1)^{\frac{d}{2}}.$$  

By construction this is an upper bound for $\text{GPf}_3(\varphi)$ and the proof is complete.

Remark 7.3. For non-rigid fields, the 3-Pfister number of quadratic forms may grow exponentially in terms of the dimension, see [2, Theorem 1.1] (with 2.2 in mind).

We can use the above result with an induction to also get upper bounds for the $n$-Pfister numbers of forms in $I^nF$ for any $n \geq 4$. This time, we will estimate a little bit coarser to get more succinct bounds. We will further use the following number theoretic result due to Jacob I. Bernoulli [1].

Theorem 7.4. [7, Chapter 15, Theorem 1] Let $m \in \mathbb{N}$ be an integer. Then there is some polynomial $p \in \mathbb{Q}[X]$ of degree $\deg(p) = m + 1$ such that
$$1^m + 2^m + \cdots + n^m = p(n)$$
for all $n \in \mathbb{N}$.

Using the distributive rule and the above result several times, we immediately get the following consequence:

Corollary 7.5. Let $q \in \mathbb{Q}[X]$ be a polynomial of degree $\deg(q) = m$. Then there is some polynomial $p \in \mathbb{Q}[X]$ of degree $m + 1$ such that we have
$$q(1) + q(2) + \cdots + q(n) = p(n)$$
for all $n \in \mathbb{N}$.

The main result of this chapter is the following which states that Pfister numbers over all rigid fields can only increase polynomially. For non-rigid fields, it is not even known if the Pfister numbers are finite, see [2, Remark 4.3].
**Theorem 7.6.** Let \( n \geq 3 \) be an integer. Then there is some polynomial \( p \in \mathbb{Q}[X] \) of degree \( n - 1 \) whose associated function \( \mathbb{R}_{\geq 0} \to \mathbb{R} \) is increasing, nonnegative and fulfills
\[
\text{GPf}_n(F, d) \leq p(d)
\]
for all rigid fields \( F \) and all even integers \( d \geq 2^n \).

**Proof.** We prove this by induction on \( n \), where the induction base \( n = 3 \) is covered by 7.2. So let now \( n \geq 4 \) and let \( q_{n-1} \in \mathbb{Q}[X] \) be the polynomial as described in the statement for \( n - 1 \) that exists due to the induction hypothesis and let \( p_{n-1} \in \mathbb{Q}[X] \) be the polynomial of degree \( n - 1 \) with
\[
q_{n-1}(1) + \cdots + q_{n-1}(k) = p_{n-1}(k)
\]
for all \( k \in \mathbb{N} \) that exists by 7.5. Obviously, the function
\[
\mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto p_{n-1}(x)
\]
is increasing and nonnegative as the function defined by \( q_{n-1} \) is so.

Just as in the proof of 7.2 we have
\[
\text{GPf}_n(F, d) \leq \text{GPf}_n(F, d - 2) + \text{GPf}_{n-1} \left( F, \frac{d}{2} - \frac{1}{2} + \frac{1}{2} \cdot (-1)^{\frac{d}{2}} \right)
\]
which is - using the same argument again - lower than or equal to
\[
\text{GPf}_n(F, d - 4) + \text{GPf}_{n-1} \left( F, \frac{d}{2} - \frac{1}{2} + \frac{1}{2} \cdot (-1)^{\frac{d}{2}} \right) + \text{GPf}_{n-1} \left( F, \frac{d}{2} - \frac{1}{2} + \frac{1}{2} \cdot (-1)^{\frac{d}{2}} \right).
\]

Iterating this process, we get a sum of expressions of the form \( \text{GPf}_{n-1}(F, k) \) with \( 2^{n-1} \leq k \leq \frac{d}{2} \) - each of these summands occuring at most 2 times - and one summand of the form \( \text{GPf}_n(F, 2^n) \).

As we have \( \text{GPf}_n(F, 2^n) = 1 \) according to the Arason-Pfister Hauptsatz, we thus get the upper bound
\[
1 + 2 \sum_{k=2^{n-1}, 2|k}^{\lfloor \frac{d}{2} \rfloor} \text{GPf}_{n-1}(F, k) \leq 1 + 2 \sum_{k=2^{n-1}, 2|k}^{\lfloor \frac{d}{2} \rfloor} q_{n-1}(k)
\]
\[
\leq 1 + 2 \sum_{k=1}^{\lfloor \frac{d}{2} \rfloor} q_{n-1}(k)
\]
\[
= 1 + 2p_{n-1} \left( \frac{d}{2} \right) \leq 1 + 2p_{n-1} \left( \frac{d}{2} \right).
\]

It is thus easy to see that the polynomial \( p_n(X) := 1 + 2p_{n-1} \left( \frac{X}{2} \right) \) does the job.

**Acknowledgments**

The results contained in this paper are part of the PhD-Thesis of the author. He would like to thank Detlev Hoffmann for supervising this work and giving some very useful hints and corrections in the process.

**Disclosure statement**

The author reports there are no competing interests to declare.

**References**

[1] Bernoulli, J. (1713). *Ars conjectandi, opus posthumum. Accedit Tractatus de seriebus infinitis, et epistola Gallicè scripta de ludo pilae reticularis.* Basel: Impensis Thurnisiorum, Fratrum.
[2] Brosnan, P., Reichstein, Z., Vistoli, A. (2010). Essential dimension, spinor groups, and quadratic forms. *Ann. Math.* 171:1587–1600.

[3] Elman, R., Karpenko, N., Merkurjev, A. (2008). *The Algebraic and Geometric Theory of Quadratic Forms*, volume 56 of American Mathematical Society Colloquium Publications. Providence, RI: American Mathematical Society. DOI: 10.1090/coll/056.

[4] Elman, R., Lam, T. Y. (1972). Quadratic forms over formally real fields and pythagorean fields. *Amer. J. Math.* 94:1155–1194. DOI: 10.2307/2373568.

[5] Hoffmann, D. W. (1998). Splitting patterns and invariants of quadratic forms. *Math. Nachr.* 190:149–168. DOI: 10.1002/mana.19981900108.

[6] Hoffmann, D. W., Tignol, J.-P. (1998). On 14-dimensional quadratic forms in $I^3$, 8-dimensional forms in $I^2$, and the common value property. *Doc. Math.* 3:189–214.

[7] Ireland, K., Rosen, M. (1990). *A Classical Introduction to Modern Number Theory*, volume 84 of Graduate Texts in Mathematics, 2nd ed. New York: Springer-Verlag. DOI: 10.1007/978-1-4757-2103-4.

[8] Izhboldin, O. T., Karpenko, N. A. (2000). Some new examples in the theory of quadratic forms. *Math. Z.* 234(4):647–695. DOI: 10.1007/s002090050003.

[9] Karpenko, N. A. (2017). Around 16-dimensional quadratic forms in $I^4_q$. *Math. Z.* 285(1–2):433–444. DOI: 10.1007/s00209-016-1714-x.

[10] Knebusch, M. (1977). Generic splitting of quadratic forms. II. *Proc. London Math. Soc. (3)* 34(1):1–31. DOI: 10.1112/plms/s3-34.1.1.

[11] Lam, T. Y. (2005). *Introduction to Quadratic Forms Over Fields*, volume 67 of Graduate Studies in Mathematics. Providence, RI: American Mathematical Society.

[12] Parimala, R., Suresh, V., Tignol, J.-P. (2009). On the Pfister number of quadratic forms. In Baeza, R., Chan, W. K., Hoffmann, D. W., Schulze-Pillot, R., eds. *Quadratic Forms—Algebra, Arithmetic, and Geometry*, volume 493 of Contemporary Mathematics. Providence, RI: American Mathematical Society, pp. 327–338. DOI: 10.1090/conm/493/09677.

[13] Pfister, A. (1966). Quadratische Formen in beliebigen Körpern. *Invent. Math.* 1:116–132. DOI: 10.1007/BF01389724.

[14] Raczek, M. (2013). On the 3-Pfister number of quadratic forms. *Commun. Algebra* 41(1):342–360. DOI: 10.1080/00927982.2011.630709.

[15] Ware, R. (1978). When are Witt rings group rings? II. *Pacific J. Math.* 76(2):541–564.