ON THE CANONICAL FORM OF A PAIR OF COMPATIBLE ANTIBRACKETS

M. A. Grigoriev and A. M. Semikhatov

Tamm Theory Division, Lebedev Physics Institute, Russian Academy of Sciences

In the triplectic quantization of general gauge theories, we prove a ‘triplectic’ analogue of the Darboux theorem: we show that the doublet of compatible antibrackets can be brought to a weakly-canonical form provided the general triplectic axioms of \[2\] are imposed together with some additional requirements that can be formulated in terms of marked functions of the antibrackets. The weakly-canonical antibrackets involve an obstruction to bringing them to the canonical form. We also classify the ‘triplectic’ odd vectors fields compatible with the weakly-canonical antibrackets and construct the Poisson bracket associated with the antibrackets and the odd vector fields. We formulate the \(Sp(2)\)-covariance requirement for the antibrackets and the vector fields; whenever the obstruction to the canonical form of the antibrackets vanishes, the \(Sp(2)\)-covariance condition implies the canonical form of the triplectic vector fields.

1 Introduction

Triplectic quantization of general gauge theories \[1, 2, 3\] was formulated as a generalization of the \(Sp(2)\)-symmetric Lagrangian quantization \[4, 5\], into which the ghost and antighost fields enter in a symmetric way (which is in contrast to the standard BV-formalism \[7\]). In the triplectic formalism, one introduces a pair of antibracket operations (in fact, a pair of odd BV \(\Delta\)-operators) that are required to satisfy certain compatibility conditions. In addition, one introduces two odd vector fields \[5, 1, 6\] which, again, should agree with the antibrackets in a certain sense. All these objects allow one to formulate the triplectic master-equations, whose solutions enter the corresponding path integral.

The starting point of the triplectic quantization prescription is the quantization \[5, 4\] using the coordinates on the field space in which field-antifield identifications are explicit and the antibrackets are written in the ‘canonical’ form. As with the conventional BV formalism, where the covariant formulation is now available \[8, 9, 10\], the aim of the triplectic formalism is to give a formulation that is covariant with respect to changing coordinates on the field space. This amounts to replacing the ‘triplectic phase space’ with a supermanifold \(M\) whose local coordinates are not separated into fields and antifields explicitly. The fundamental objects such as the antibrackets\[1\] are then introduced axiomatically, similarly to how the Poisson bracket is defined on a general Poisson manifold. However, there is a significant gap in the triplectic formulation, which can, potentially, be a source of problems in the formalism. Recall that, for the Poisson manifolds, the Darboux theorem guarantees the existence of local coordinates in which the Poisson bracket takes the canonical form. As regards the antisymplectic manifolds employed in the covariant version of the BV formalism, a similar theorem is usually assumed \[11\]. Thus, the covariant formulation is eventually equivalent to the original formalism \[2\] in the field-antifield space. For the triplectic objects, on the other hand, Darboux-type theorems are not known, and the entire construction of \[2, 3\] relies on the conjecture that some theorem of this kind holds (or, at worst, such a theorem would require a mild modification of the construction).

In this paper, we propose a version of the ‘triplectic Darboux theorem’. In addition to the general axioms formulated in \[2, 3\], the assumptions of the theorem involve conditions that were not specified explicitly in

\[1\]In this paper, we concentrate on the antibrackets rather than on the BV \(\Delta\)-operators and, thus, do not consider the measure on the triplectic manifold.
the formulation of [2, 3]; however, they are valid in the canonical coordinates of [1], which suggests that they are rather natural. The conditions that we impose in order to prove the theorem are formulated in terms of marked functions (or, Casimir functions) of the two antibrackets given on the triplectic manifold. Imposing these conditions allows us to demonstrate the existence of a coordinate system in which one of the antibrackets becomes canonical (just like the antibracket from [1]), while the other assumes the canonical form on a submanifold $\mathcal{L}$ of dimension one third of the dimension of the triplectic manifold $\mathcal{M}$. This form of the antibrackets will be referred to as weakly canonical. We identify the obstruction to reducing the weakly canonical antibrackets to the canonical form — this is a matrix $e_\alpha^i$ whose entries depend only on marked functions $\xi_{2\alpha}$ and $\xi_{1\alpha}$ of the antibrackets $(\ , )^1$ and $(\ , )^2$, respectively; in fact, this matrix relates the vector fields generated by the marked functions: $(\xi_{2\alpha}, \cdot)^2 = (-1)^{(e(i)+1)e(\alpha)}e_\alpha^i (\xi_{1\alpha}, \cdot)^1$.

The submanifold $\mathcal{L}$ where both antibrackets become canonical plays a crucial role in the theory also from the following point of view. The Poisson bracket on $\mathcal{M}$ (in the version given in [3], which is advantageous over the original proposal of [4]) becomes non-degenerate on $\mathcal{L}$ and, thus, makes $\mathcal{L}$ into a symplectic manifold. Then, the boundary conditions on the master-action are imposed on some Lagrangian submanifold $\mathcal{L}_0$ of $\mathcal{L}$, which can be identified as the manifold of fields of the theory, with all the antifields set to zero. Thus, (a Lagrangian submanifold of) the symplectic submanifold is an essential ingredient of the triplectic quantization scheme. In fact, any symplectic leaf of the Poisson bracket may be used as that symplectic submanifold.

The existence of an obstruction to the canonical form in a full-dimensional neighborhood of a point in $\mathcal{M}$ raises the question of whether some further requirements should be imposed on the ‘triplectic data’ or the physics of the quantized gauge system is in some way sensitive only to the symplectic submanifold and to the form the antibrackets take on it. This is left for the further investigations; in this paper, we make one more small step in that direction by classifying the odd vector fields of the form proposed in [3] that are compatible with the antibrackets. We also discuss how the weak canonical form of the antibrackets (and the corresponding odd vector fields) coexist with the requirement that there be an $Sp(2)$ action on the triplectic manifold. The $Sp(2)$ covariance on the triplectic manifold has not been discussed in much detail in the literature, in particular the statement regarding the $Sp(2)$ action in general coordinates was only implicit in [3]. Here, we were not able to classify the weakly canonical antibrackets into those which do, and those which do not, admit an $Sp(2)$ action; however, an infinitesimal analysis suggests that both cases can be realized, and, therefore, the requirement of $Sp(2)$ covariance does not restrict the weakly canonical antibrackets to the canonical ones.

2 Basic definitions

We begin with a brief reminder on the triplectic quantization in the covariant approach. The role of the field-antifield space is played by a $(4N - 2k|N + 2k)$-dimensional supermanifold $\mathcal{M}$. Let $C_\mathcal{M}$ be the algebra of functions on $\mathcal{M}$.\footnote{All of our analysis is local, which we will not stipulate explicitly any more.} An antibracket on $\mathcal{M}$ is a bilinear mapping $(\ , ) : C_\mathcal{M} \times C_\mathcal{M} \to C_\mathcal{M}$ such that $\epsilon((F,G)) = \epsilon(F) + \epsilon(G) + 1$ and

\[
(F, G) = -(-1)^{(e(F)+1)(e(G)+1)} (G, F), \\
(F, GH) = (F, G)H + (-1)^{(e(F)+1)e(G)} G(F, H), \\
(-1)^{(e(F)+1)(e(H)+1)} (F, (G, H)) + \text{cycle}(F, G, H) = 0
\] \hspace{1cm} (2.1)
for all \( F, G, H \in C_M \). The pair of antibrackets \((\ , )^a, a = 1, 2\) is called compatible if
\[
(F, G) = \alpha (\ , )^1 + \beta (\ , )^2
\]
is an antibracket for arbitrary even constants \(\alpha\) and \(\beta\). This is equivalent to
\[
(-1)^{(\epsilon(F)+1)(\epsilon(H)+1)}((F,G)^{(a),H})^b + \text{cycle}(F,G,H) = 0, \tag{2.3}
\]
where the curly brackets stand for symmetrization of indices: \(C^{(a)D^b} = C^aD^b + C^bD^a\). In the local coordinates \(\Gamma^A\) on \(M\), where we can write
\[
E^{aAB} = (\Gamma^A, \Gamma^B)^a, \quad a = 1, 2, \tag{2.4}
\]
the compatibility condition takes the form
\[
(-1)^{(\epsilon(A)+1)(\epsilon(D)+1)}E^{(a)AC}\partial_CE^{b)BD} + \text{cycle}(A, B, D) = 0, \tag{2.5}
\]
The compatibility condition (2.3) (or, (2.5)) is often referred to as the symmetrized Jacobi identity.

In what follows, we use the notion of marked functions (Casimir functions) of an antibracket.

**Definition 2.1**

1. A function \(\varphi \in C_M\) is called a marked function of the antibracket \((\ , )\) if
\[
(F, \varphi) = 0 \tag{2.6}
\]
for any \(F \in C_M\).

2. A collection \(\phi_1, \ldots, \phi_n\) of marked functions of some antibracket \((\ , )\) is called complete if any marked function \(\varphi\) of the antibracket is a function of only the \(\phi_1, \ldots, \phi_n\).

Thus, the marked functions from a complete set generate the algebra of marked functions. A characteristic example is provided by the coordinate functions that are transversal to a symplectic leaf of a chosen (anti)bracket. In what follows, we will always take minimal complete sets (with the minimal possible number of the \(\phi_j\) functions). The number of functions in such a set is then the co-rank of the antibracket (i.e., by definition, the codimension of its symplectic leaf).

In the language of marked functions, one can observe that the ‘canonical’ antibrackets \([\ , ]\), which can be written in the form
\[
(F, G)^a = F \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_{ai}} G - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}(F \leftrightarrow G) \tag{2.7}
\]
in some coordinate system \(\Gamma^A = (x^i, \xi_{ai}), i = 1, \ldots, 2N, a = 1, 2\), satisfy certain properties that have not been explicitly stated before. Thus, the fact that the corresponding matrices \(E^{aAB}\) have no common zero modes, means that the only marked functions shared by the two antibrackets are constants — which we will express by saying that the antibrackets do not have common marked functions. Such antibrackets will be called jointly nondegenerate. Further, the coordinates \(\xi_{1i}\) make up a complete set of marked functions of the second \((a = 2)\) antibracket from (2.7), while \(\xi_{2i}\) are a complete set of marked functions of the first
Thus, if \( \phi_1 \) and \( \psi_1 \) (respectively, \( \phi_2 \) and \( \psi_2 \)) are any two marked functions of the second (resp., the first) antibracket, then
\[
(\phi_1, \psi_1)^1 = 0, \quad (\phi_2, \psi_2)^2 = 0.
\]
(2.8)

The antibrackets whose marked functions satisfy (2.8) will be called \textit{mutually flat}. An important point is that this condition, being formulated in terms of marked functions of two antibrackets, is coordinate-independent. Observe that it is not necessarily fulfilled for marked functions of two compatible antibrackets.

The following fact is a consequence of some simple linear algebra.

**Proposition 2.2** Let two antibrackets be jointly nondegenerate and mutually flat. Then their ranks \( r_a, a = 1, 2 \), satisfy \( r_a \geq 3N \) and \( r_1 + r_2 \geq 8N \), where the dimension of the manifold is \( \dim \mathcal{M} = 6N \).

The triplectic antibrackets, therefore, minimize the quantity \( r_1 + r_2 \) among all the jointly nondegenerate and mutually flat antibrackets.

### 3 Finding weak canonical coordinates

Given two compatible antibrackets that are jointly nondegenerate and mutually flat, we are going to simplify them by choosing an appropriate coordinate system on \( \mathcal{M} \). As in the above, the triplectic manifold \( \mathcal{M} \) is of dimension \( 6N \) and each of the antibrackets is of rank \( 4N \). The mutual flatness condition means that
\[
(\xi_{1i}, \xi_{1j})^1 = 0, \quad (\xi_{2a}, \xi_{2b})^2 = 0, \quad i, j = 1, \ldots, N, \quad a, b = 1, \ldots, N,
\]
(3.1)

where \( \xi_{1i} \) is a full set of marked functions of the second antibracket and \( \xi_{2a} \), those of the first one. By virtue of the assumptions made, there are no common marked functions of the two antibrackets; and, moreover, there exist functions \( x^i, i = 1, \ldots, N \), such that \((x^i, \xi_{1i}, \xi_{2a})\) is a local coordinate system on \( \mathcal{M} \).

We first show that \( x^i \) can be chosen in such a way that \((x^i, \xi_{1j})^1 = \delta^i_j\). Indeed, vector fields
\[
X_i^1 = - (\xi_{1i}, \cdot)^1
\]
(3.2)

are linearly independent at every point (because the antibrackets are jointly nondegenerate) and, moreover, the Jacobi identity for \((\cdot, \cdot)^1\) combined with the first of Eqs. (3.1) show that these vector fields pairwise commute. Let \( \mathcal{L} \) be an integral manifold of the \( X_i^1 \). Making use of the Frobenius theorem for supermanifolds \([1]\), we construct a coordinate system \( x^i, y_A, A = 1, \ldots, 4N, \) on \( \mathcal{M} \) in which \( X_i^1 = \frac{\partial}{\partial x^i} \) (thus, \( \mathcal{L} \) is singled out by the equations \( y_A = \text{const} \)). Then Eqs. (3.1) imply
\[
\frac{\partial}{\partial x^i} \xi_{1j} = \frac{\partial}{\partial x^i} \xi_{2a} = 0,
\]
(3.3)

therefore \( \xi_a = \xi_a(y_A) \), where \( \xi_a = (\xi_{1i}, \xi_{2a}) \). Once all of the functions \( \xi_a \) are independent, we can go over to the coordinate system \((x^i, \xi_{1j}, \xi_{2a})\), where we have \((x^i, \xi_{1j})^1 = \delta^i_j\). Hence, in particular, the Grassmann parities are
\[
\epsilon(x^i) \equiv \epsilon(i), \quad \epsilon(\xi_{1i}) = \epsilon(i) + 1.
\]
(3.4)

For the future use, denote also
\[
\epsilon(\xi_{2a}) \equiv \epsilon(\alpha) + 1.
\]
(3.5)
Having achieved \((x^i,\xi_{1j})^1 = \delta^i_j\), we can simplify the antibrackets \((\ , \ )^1\) and \((\ , \ )^2\) further. Let us look at the analogue of (5.2) for the second antibracket:

\[
X^2_\alpha = -(\xi_{2\alpha}, \cdot)^2 = -(1)^{\epsilon(\alpha)(\epsilon(i)+1)}e^i_\alpha \frac{\partial}{\partial x^i} + A_{\alpha j} \frac{\partial}{\partial \xi_{1 j}} + A_{\alpha \beta} \frac{\partial}{\partial \xi_{2 \beta}},
\]

(3.6)

where we have written the general form involving the \(A_\alpha\) coefficients. The Grassmann parities are \(\epsilon(e^i_\alpha) = \epsilon(i) + \epsilon(\alpha)\). Now,

\[
A_{\alpha j} = X^2_\alpha \xi_{1j} = -(\xi_{2\alpha}, \xi_{1j})^2 = 0,
\]

\[
A_{\alpha \beta} = X^2_\alpha \xi_{2\beta} = -(\xi_{2\alpha}, \xi_{2\beta})^2 = 0.
\]

(3.7)

Therefore, the above submanifold \(\mathcal{L} \subset \mathcal{M}\) is at the same time an integral manifold of the vector fields \(X^2_\alpha\).

In addition, it follows from Eqs. (3.4) that

\[
[X^1_\alpha, X^2_\alpha] = X^1_\alpha X^2_\alpha - (1)^{\epsilon(\alpha)(\epsilon(i)+1)}X^2_\alpha X^1_\alpha = 0
\]

(3.8)

and therefore the functions \(e^i_\alpha\) depend only on \(\xi_{1i}\) and \(\xi_{2\alpha}\).

Denote

\[
\eta^{ij} = (x^i, x^j)^1.
\]

(3.9)

By virtue of the symmetrized Jacobi identities, \(\eta^{ij}\) depend only on \(\xi_{1i}\) and \(\xi_{2\alpha}\) and satisfy the equations

\[
(-1)^{\epsilon(i)(\epsilon(i)+1)(\epsilon(j)+1)} \frac{\partial}{\partial \xi_{1i}} \eta^{jk} + \text{cycle}(i, j, k) = 0,
\]

(3.10)

whence

\[
\eta^{ij}(\xi_1, \xi_2) = \frac{\partial}{\partial \xi_{1i}} f^j(\xi_1, \xi_2) - (1)^{\epsilon(i)(\epsilon(i)+1)(\epsilon(j)+1)} \frac{\partial}{\partial \xi_{1j}} f^i(\xi_1, \xi_2)
\]

(3.11)

for some \(f^i\) (which, again, are functions of only \(\xi_{1i}\) and \(\xi_{2\alpha}\)). Observe that \(f^i\) are defined up to the arbitrariness of the form

\[
 f^i(\xi_1, \xi_2) \rightarrow f^i(\xi_1, \xi_2) + \frac{\partial}{\partial \xi_{1i}} \mathcal{H}(\xi_1, \xi_2).
\]

(3.12)

The functions \(f^i\) can be used to define new coordinates \(\tilde{x}^i = x^i - f^i(\xi_1, \xi_2)\), in which \((\tilde{x}^i, \tilde{x}^j)^1 = 0\). At the same time, antibrackets with the \(\xi_\alpha\) do not change: \((\tilde{x}^i, \xi_{1j})^b = (x^i, \xi_{1j})^b\) and \((\tilde{x}^i, \xi_{2\alpha})^b = (x^i, \xi_{2\alpha})^b\).

Thus, we have found a local coordinate system \((x^i, \xi_{1j}, \xi_{2\alpha})\) in which (removing the tilde)

\[
(x^i, \xi_{1j})^1 = \delta^i_j, \quad (x^i, \xi_{2\alpha})^1 = e^i_\alpha, \\
(x^i, x^j)^1 = 0, \quad (x^i, x^j)^2 = \lambda^{ij},
\]

(3.13)

with all the other pairwise antibrackets of the coordinate functions vanishing. The symmetrized Jacobi identities for the antibrackets of the form (3.13) show that the functions \(\lambda^{ij}\) depend only on \(\xi_{1i}, \xi_{2\alpha}\) and satisfy the equations

\[
(-1)^{\epsilon(i)(\epsilon(i)+1)(\epsilon(k)+1)} \frac{\partial}{\partial \xi_{1i}} \lambda^{jk} + \text{cycle}(i, j, k) = 0,
\]

(3.14)

\[
(-1)^{\epsilon(i)(\epsilon(i)+1)} e^i_\alpha \frac{\partial}{\partial \xi_{2\alpha}} \lambda^{jk} + \text{cycle}(i, j, k) = 0,
\]
while matrix $e^i_\alpha$ satisfies
\[
\frac{\partial}{\partial \xi_{1i}} e^i_\alpha - (-1)^{(\epsilon(i)+1)(\epsilon(j)+1)} \frac{\partial}{\partial \xi_{1j}} e^i_\alpha = 0, \\
\frac{\partial}{\partial \xi_{2\alpha}} e^i_\beta - (-1)^{(\epsilon(i)+1)(\epsilon(j)+1)} \frac{\partial}{\partial \xi_{2\beta}} e^i_\beta = 0.
\] (3.15)

We now use the freedom (3.12) in the definition of $x^i$ in order to make $\lambda^{ij}(\xi_1, \xi_2)$ vanish. Changing the coordinates as
\[
x^i \mapsto x^i - \frac{\partial}{\partial \xi_{1i}} H(\xi_1, \xi_2),
\] (3.16)
we would have $(x^i, x^j)^2 = 0$ whenever $H(\xi_1, \xi_2)$ satisfies the equations
\[
e^i_\alpha \frac{\partial}{\partial \xi_{2\alpha}} H - (-1)^{(\epsilon(i)+1)(\epsilon(j)+1)} e^j_\beta \frac{\partial}{\partial \xi_{1j}} H = \lambda^{ij}.
\] (3.17)
Compatibility conditions for (3.17) are satisfied by virtue of (3.14) and (3.15). We, thus, assume the solution to (3.17) to exist.

To summarize, we have arrived at

**Theorem 3.1** For compatible rank-4N antibrackets that are mutually flat and jointly nondegenerate on the triplectic manifold, there exists a coordinate system in which the antibrackets take the form
\[
(F,G)^1 = F \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_{1i}} G - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)} (F \leftrightarrow G),
\]
\[
(F,G)^2 = F \frac{\partial}{\partial x^i} e^i_\alpha \frac{\partial}{\partial \xi_{2\alpha}} G - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)} (F \leftrightarrow G),
\] (3.18)
where the functions $e^i_\alpha = e^i_\alpha(\xi_1, \xi_2)$ make up a nondegenerate square matrix and satisfy Eqs. (3.15). This form of the antibrackets will be called weakly canonical.

The nondegeneracy of $e^i_\alpha$ follows from the rank assumptions. Note also that the symmetrized Jacobi identities for antibrackets (3.18) are equivalent to equation (3.15). The functions $e^i_\alpha$ are, in general, an obstruction to transforming the triplectic antibrackets to the canonical form of [1].

A more invariant way to look at the $e^i_\alpha$ is to consider them as a matrix relating vector fields (3.12) and (3.6):
\[
(\xi_{2\alpha}, \cdot)^2 = (-1)^{(\epsilon(i)+1)\epsilon(\alpha)} e^i_\alpha (\xi_{1i}, \cdot)^1.
\] (3.19)
It follows from (3.13) that under a change of marked functions $\xi_{1i} \mapsto \theta_{1i}(\xi_1), \xi_{2\alpha} \mapsto \theta_{2\alpha}(\xi_2)$, the quantities $e^i_\alpha$ transform as
\[
(\theta_{2\alpha}, \cdot)^2 = (-1)^{(\epsilon(i)+1)\epsilon(\alpha)} e^i_\alpha (\theta_{1i}, \cdot)^1.
\]
(3.20)
In this sense, the structure $e^i_\alpha$ depends only on the marked functions allowed by the antibrackets. An important consequence of (3.20) is that once $e^i_\alpha$ take the form $e^i_\alpha = \delta^i_\alpha$ for some set of marked functions, then any other choice of marked functions would leave $e^i_\alpha$ in the class of function of the form
\[
e^i_\alpha(\xi_1, \xi_2) = e^{(1)i}_j(\xi_1) \delta^j_\beta e^{(2)\beta}_\alpha(\xi_2).
\] (3.21)
This motivates the following definition.
The structure $e^i_\alpha$ is called reducible if it can be represented in the form (3.21) where the functions $e^{(1)}$ and $e^{(2)}$ depend only on $\xi_1$ and $\xi_2$ respectively.

Conversely, whenever the matrix $e^i_\alpha(\xi_1, \xi_2)$ is reducible, Eqs. (3.13) imply that there exist nondegenerate mappings $\xi_{1i} \mapsto \theta_{1i}(\xi_1)$ and $\xi_{2\alpha} \mapsto \theta_{2\alpha}(\xi_2)$ such that

$$
e^{(1)i}_j = \frac{\partial}{\partial \xi_{1i}} \theta_{1j}, \quad e^{(2)\beta}_\alpha = \frac{\partial}{\partial \theta_{2\beta}} \xi_{2\alpha},$$

choosing which as the new bases of marked functions we bring the $e^i_\alpha$ matrix to the form $e^i_\alpha = \delta^i_\alpha$. We conclude that the condition that $e^i_\alpha$ be reducible is sufficient for the existence of canonical coordinates. However, it remains a problem to formulate the property of antibrackets (and/or their marked functions) on a triplectic manifold that would imply reducibility.

4 Odd vector fields and the Poisson bracket

In addition to the antibrackets subjected to the compatibility condition, the triplectic quantization formalism involves two odd vector fields $V^a$, $a = 1, 2$, (see [3, 2, 6]) that are required to differentiate the antibrackets in following sense:

$$V^{(a}(F,G)^b) = (V^{(a}F,G)^b) + (-1)^{e(F)+1}(F,V^{(a}G)^b)$$

for any two functions $F,G \in C_M$. Further, $V^a$ must obey the condition

$$V^{(a}V^b = 0.$$  

In local coordinates $\Gamma^A$ on $M$, we write $V^a = (-1)^{e(A)}V^aA^A\partial_A$. Following [3], we restrict ourselves to the $V^a$ fields of the following form:

$$V^a = (-1)^{e(C)}E^{aCB}F_B\partial_C = (-1)^{e(B)}F_BE^{aBC}\partial_C, \quad \epsilon(F_A) = \epsilon(\Gamma^A),$$

with some covector field $F = F_Ad\Gamma^A$. Then Eq. (4.1) rewrites as [3]

$$E^{(aAC}F_{CD}(-1)^{e(D)}E^{b)D}B = 0, \quad F_{AB} = \partial_AF_B - (-1)^{e(A)e(B)}\partial_BF_A.$$  

We now investigate the structure of these vector fields (i.e., of the constraints (4.4)) in the coordinates $(x^i, \xi_{1i}, \xi_{2\alpha})$ from the previous section. The respective components of $F$ are then denoted as $F = (F_i, F^{1i}, F^{2\alpha})$. Using the fact that the matrix $e^i_\alpha$ is invertible, we conclude from Eqs. (4.4) that there exist functions $H^1$ and $H^2$ such that

$$F_i = \frac{\partial}{\partial x^i}H^{(1)} = \frac{\partial}{\partial \xi_{1i}}H^{(2)}, \quad F^{1i} = \frac{\partial}{\partial \xi_{1i}}H^{(1)}, \quad F^{2\alpha} = \frac{\partial}{\partial \xi_{2\alpha}}H^{(2)}$$

and the function $H = H^{(2)} - H^{(1)}$ is independent of the $x^i$ coordinates:

$$\frac{\partial}{\partial x^i}H = \frac{\partial}{\partial x^i}(H^{(2)} - H^{(1)}) = 0,$$

while its $\xi$-dependence is governed by

$$e^i_\alpha(\xi_1, \xi_2)\frac{\partial}{\partial \xi_{2\alpha}} \frac{\partial}{\partial \xi_{1j}} H(\xi_1, \xi_2) - (-1)^{(e(i)+1)(e(j)+1)}e^j_\beta(\xi_1, \xi_2)\frac{\partial}{\partial \xi_{2\alpha}} \frac{\partial}{\partial \xi_{1i}} H(\xi_1, \xi_2) = 0.$$

This can be reformulated as follows.
Proposition 4.1 Let a pair of compatible antibrackets be written in the form (3.18), and the vector fields $V^a$ represented as in (4.3) in some local coordinates $x^i, \xi_1^i, \xi_2^\alpha$ on $\mathcal{M}$. The vector fields differentiate the antibrackets if and only if there exist functions $H(\xi_1^i, \xi_2^\alpha)$ and $h(x^i, \xi_1^i, \xi_2^\alpha)$ such that

$$V^1 = (-\frac{1}{2}H, \cdot)^1 + (h, \cdot)^1$$
$$V^2 = (\frac{1}{2}H, \cdot)^2 + (h, \cdot)^2.$$  \hspace{1cm} (4.8)

and the function $H$ satisfies Eq. (4.7).

Thus, two vector fields that differentiate the antibrackets can be split into a ‘Hamiltonian’ or symmetric, part $V^a_S = (h, \cdot)^a$ and an ‘anti-Hamiltonian’ (antisymmetric) part $V^a_A = (-\frac{1}{2}H, \cdot)^a, V^2_A = (\frac{1}{2}H, \cdot)^2$. It is easy to see that functions $H$ and $h$ are defined up to the following transformation:

$$H \rightarrow H + Q^1(\xi_1^i) + Q^2(\xi_2^\alpha),$$
$$h \rightarrow h + \frac{1}{2}Q^1(\xi_1^i) - \frac{1}{2}Q^2(\xi_2^\alpha),$$  \hspace{1cm} (4.9)

where functions $Q^1$ and $Q^2$ depend only on $\xi_1^i$ and $\xi_2^\alpha$ respectively. This arbitrariness, which we will need later, follows from the existence of vector fields $V^a$ that can be represented in the symmetric as well as antisymmetric form.

Note also that the ‘anti-Hamiltonian’ vector fields $V^a_A$ differentiate the antibrackets even before symmetrizing with respect to the $a,b$ indices,

$$V^a_A(F,G)^b = (V^a_A F,G)^b + (-1)^{\epsilon(F)+1}(F,V^a_A G)^b,$$  \hspace{1cm} (4.10)

which is the property postulated in [1, 2].

It was observed in [2] and then developed in [3] that the pair of compatible antibrackets give rise to a Poisson bracket. To this end [3], one defines on $\mathcal{M}$ a tensor field

$$\omega^{AB} = \frac{1}{2} \delta_{ab} (-1)^{\epsilon(D)} E^{aAC} F_{CD} E^{bDB}. \hspace{1cm} (4.11)$$

Due to (4.4), this is (graded) antisymmetric, $\omega^{AB} = -(1)^{\epsilon(A)+1}(\omega^{BA})$. Thus, we have an antisymmetric bracket operation

$$\{F,G\} = F \leftarrow \partial_A \omega^{AB} \rightarrow \partial_B G. \hspace{1cm} (4.12)$$

This structure is invariant under changing $F^a$ from (4.3) to $F^a + \partial^a K$ with an arbitrary $K$.

Proposition 4.2 Under the assumptions of Sect. [3] — i.e., for compatible rank-4N antibrackets that are mutually flat and jointly nondegenerate and for the vector fields of the form (4.3) that differentiate the antibrackets as in (4.4), — $\omega^{AB}$ is a Poisson structure and the integral submanifold $\mathcal{L}$ associated with the antibrackets contains its symplectic leaf.

\footnote{For the general form (4.3), on the other hand, one has to take the ‘Hamiltonian’ $h$ such that $\frac{1}{2}(h,h)^a - V^a_A h = 0$, as in [3].}

\footnote{See also [6, 10], where the Poisson structure was discussed in the context of the standard BV quantization.
Indeed, in the weakly canonical coordinates \((x^i, \xi_{1i}, \xi_{2\alpha})\) from the previous section, the only nonvanishing components of \(\omega^{AB}\) are \(\omega^{ij} = \{x^i, x^j\}\). In particular, therefore, \(\text{rank } \omega^\cdot \cdot \leq 2N\). Using (4.3) for the components of \(F\) in the \((x^i, \xi_{1i}, \xi_{2\alpha})\) coordinates, we arrive at

\[
\omega^{ij} = (-1)^{\epsilon (j)} e^i_\alpha \frac{\partial}{\partial \xi_{2\alpha}} \frac{\partial}{\partial \xi_{1j}} H .
\]  

(4.13)

Thus, (4.12) becomes a Poisson bracket on \(M\) — i.e., satisfies the Jacobi identity — due to the simple fact that \(H\) is independent of \(x^i\). The triplectic manifold \(M\) is then endowed with a Poisson structure.

Observe that, while in general \(\text{rank } \omega^\cdot \cdot \leq 2N\), the physical requirements of quantization (i.e., of the construction of path integral) is such that \(\text{rank } \omega^\cdot \cdot = 2N\), in which case the manifold \(L\), which in the local coordinates is defined by \(\xi_{1i} = \text{const}_i, \xi_{2\alpha} = \text{const}_\alpha\), is a symplectic leaf of the Poisson bracket (4.12). Vice versa, a Lagrangian submanifold \(L_0\) of any symplectic leaf \(L\) of the Poisson bracket can be used in the triplectic quantization in order to impose boundary conditions on the master-action: one identifies this Lagrangian submanifold as the manifold of ‘classical’ fields.

It may also be noted that we have avoided imposing on the vector fields the additional constraints of \([3]\), namely (in terms of the ‘potential’ \(F_A\) from (4.3)), \(F_B E^{aBC} F_C = 0\) and \(F_{AB} E^{aBC} F_{CD} = 0\), which are fulfilled automatically for the ‘anti-Hamiltonian’ part \(V^a_A\) in our approach.

### 5\quad Sp(2)-covariance

Until this moment, we have not discussed the \(Sp(2)\)-covariance of our construction. The standard formulation \([3]\) of the \(Sp(2)\)-symmetric Lagrangian quantization assumes that the \(Sp(2)\) group acts on the phase space coordinates, which are in fact \(Sp(2)\) tensors; the antibrackets and the odd vector fields defined in \([3]\) carry the \(Sp(2)\) vector representation index.

We have to extend the \(Sp(2)\)-covariance requirement to the geometrically covariant formulation.\(^5\) Let us note first of all that the conditions imposed on the antibrackets (that they be compatible, mutually flat and jointly nondegenerate and have rank \(4N\), while the vector fields of the form (4.3) satisfy (1.1) and (4.2)) are preserved by \(Sp(2)\)-transformations acting on the \(a\) index of the antibrackets and the vector fields. Now, this action has to be realized in terms of an \(Sp(2)\)-action on \(M\).

Let \(\phi\) be an \(Sp(2)\) action on \(M\), i.e., to every \(G \in Sp(2)\) there corresponds a mapping \(\phi_G : M \to M\) such that \(\phi_{G_1} \phi_{G_2} = \phi_{G_1 G_2}\). The pullback \(\phi^\#_G\) acts on functions in the standard way:

\[
(\phi^\#_G(f))(p) = f(\phi_G(p)) , \quad p \in M ,
\]  

then \(\phi^\#_{G_1} \phi^\#_{G_2} = \phi^\#_{G_1 G_2}\).

**Definition 5.1** A pair of compatible antibrackets and odd vector fields \(V^a\) on \(M\) are called \(Sp(2)\) covariant if for any \(G \in Sp(2)\) there exists a mapping \(\phi_G : M \to M\) such that

\[
\phi^\#_G((f, g)^a) = G_a^b (\phi^\#_G(f), \phi^\#_G(g))^b , \quad \phi^\#_G(V^a f) = G_a^b V^b (\phi^\#_G(f)) ,
\]  

for all \(f, g \in C_M\).

\(^5\)We thank I. Tyutin for a discussion of this point.
In the infinitesimal form, we have a mapping from the Lie algebra \( sp(2) \) to the algebra of vector fields on \( \mathcal{M} \). Let, in some coordinate system, \( Y = Y^A \partial_A \) be the vector field corresponding to \( g \in sp(2) \). Then the \( sp(2) \) covariance condition takes the following form:

\[
(L_Y E^a)^{AB} = Y^C \partial_C E^a^{AB} - Y^A \partial_C E^a^{CB} - E^a^{AC} \partial_C Y^B = \gamma^b E^b^{AB},
\]

(5.3)

\[
L_Y V^a = [Y, V^a] = \gamma^b V^b^a,
\]

(5.4)

that is, \( Y \) acts by the Lie derivative. The last equation (5.4) imposed on the vector fields \( V^a \) of the form (1.3) one can rewrite as

\[
L_Y F = 0,
\]

(5.5)

where \( F = F_A d\Gamma^A \) is the covector that defines the vector fields. It also follows from the \( Sp(2) \) covariance of the antibrackets and vector fields \( V^a \) of the form (1.3) that the Poisson bracket (4.12) is an \( Sp(2) \) scalar:

\[
L_Y \omega = 0.
\]

(5.6)

Now, we are interested in whether the \( Sp(2) \) covariance is compatible with the weakly canonical for of the antibrackets, i.e., whether there exist vector fields representing the \( sp(2) \) action under which the weakly canonical antibrackets and the odd vector fields \( V^a \) are covariant in the above sense. Let \( Y^\pm, Y^0 \) be the vector fields that implement the infinitesimal action of the respective \( sp(2) \) generators \( J^\pm, J^0 \), respectively. We denote by \( (Y^i, Y_{ij}, Y_{2a}) \) the components in the coordinates \( (x^i, \xi_{11}, \xi_{2a}) \), with \( \epsilon(Y^i) = \epsilon(i), \epsilon(Y_{ij}) = \epsilon(j) + 1, \) and \( \epsilon(Y_{2a}) = \epsilon(a) + 1 \). Then, (5.3) implies that \( Y^i \partial \overleftarrow{\partial}_{\xi_{2j}} = \frac{\partial Y_{ij}}{\partial \xi_{20}} \) for either \( Y^+ \) or \( Y^- \). Moreover, \( Y_{ij}^{\pm,0} \) depend only on \( \xi_a \), which allows us to express \( Y_j^{\pm,0} \) through \( Y_{ij}^{\pm,0} \), while the remaining components must satisfy the following equations:

\[
\frac{\partial}{\partial \xi_{1k}}(Y_{1k}^i e^k_\alpha) + Y_{2j}^\pm \frac{\partial}{\partial \xi_{2\beta}} e^i_\alpha - e^i_\alpha \frac{\partial}{\partial \xi_{2\beta}} Y_{2a}^\pm = 0,
\]

(5.7)

\[
\frac{\partial}{\partial \xi_{2a}} Y_{1j}^+ = 0, \quad \frac{\partial}{\partial \xi_{1i}} Y_{2a}^+ = e^i_\alpha,
\]

and

\[
\frac{\partial}{\partial \xi_{1k}}(Y_{1k}^i e^k_\alpha) + Y_{2j}^- \frac{\partial}{\partial \xi_{2\beta}} e^i_\alpha - e^i_\alpha \frac{\partial}{\partial \xi_{2\beta}} Y_{2a}^- = 0,
\]

(5.8)

\[
e^i_\alpha \frac{\partial}{\partial \xi_{2a}} Y_{1j}^- = \delta^i_j, \quad \frac{\partial}{\partial \xi_{1i}} Y_{2a}^- = 0
\]

(and a similar equation for \( Y^0 \)).

The problem now is whether these equations on the components of \( Y \) can be satisfied for generic \( e^i_\alpha(\xi_1, \xi_2) \) subjected to Eqs. (1.13) or the existence of a solution implies further restrictions on \( e^i_\alpha \). First of all, it is not difficult to check directly that the reducible antibrackets are \( Sp(2) \)-covariant in the sense of the above definitions:

**Proposition 5.2** For a reducible \( e^i_\alpha \), equations (5.7)–(5.8) admit a solution and, therefore, the antibrackets are \( Sp(2) \)-covariant on the triplectic manifold.
Theorem 5.3
If $e^i_\alpha = \delta^i_\alpha$, then Eqs. \((5.3)\)–\((5.8)\) imply that the function $H$ is at most bilinear in $(\xi_1, \xi_2)$. 

To prove this, we employ the general form of the vector fields $Y^{\pm}$ representing generators $J^{\pm}$ (i.e., the general solutions of Eqs. \((5.7)\)–\((5.8)\) with $e^i_\alpha = \delta^i_\alpha$). The $Y$ of this form are then inserted into each of Eqs. \((5.9)\) and the resulting equations are solved making use of the fact that $H$ satisfies Eq. \((4.7)\). Then, if $H$ is homogeneous in $\xi_\alpha$, the desired statement follows immediately, while in the case where $H$ is taken as a series in $\xi_\alpha$ a slightly more involved analysis shows that all of the coefficients except the one in $C^{ij} \xi_1 \xi_2$ vanish as well.

In the reducible case, therefore, the $Sp(2)$-covariance condition for the vector fields implies the following form of the $H$ function:

\[
H = \omega^{ij} \xi_1 \xi_2 + T^{1i} \xi_{1i} + T^{2i} \xi_{2i}.
\]  

(5.11)

Then, using the arbitrariness \((4.9)\) in the definition of ‘anti-Hamiltonian’ vector fields we can reduce the odd vector fields to the form proposed in [4] (the Poisson matrix $\omega^{ij}$ should be nondegenerate in the quantization context, hence, by a linear transformation, it can be brought to the canonical form). We can summarize our results as:

**Theorem 5.4** Let there be given a pair of compatible antibrackets on $\mathcal{M}$ and a pair of odd vector fields $V^a$ of the form \((4.3)\) compatible with the antibrackets. Assume also that

- the antibrackets and the odd vector fields are $Sp(2)$-covariant,
- the antibrackets have the minimal rank, are jointly nondegenerate and mutually flat,
- the $e^i_\alpha$ structure that corresponds to the antibrackets is reducible.

Then the antibrackets, the odd vector fields, and the corresponding Poisson bracket can be brought to the canonical form, which coincides with that proposed in [3] whenever the Poisson bracket is of maximal rank and therefore nondegenerate on $\mathcal{L}$. Note also that in the case of the vector fields, the transformation to the canonical form may involve adding a purely Hamiltonian piece.
6 Concluding remarks

We have shown that the triplectic axioms of [3], together with the additional requirements imposed on the marked functions of the two antibrackets, allow one to find a coordinate system where the antibrackets take the weakly canonical form (3.18). In that formula, the structure $e^i_\alpha(\xi_1, \xi_2)$ considered modulo reducible structures (3.21) is an obstruction to bringing both antibrackets to the canonical form. The weakly-canonical antibrackets become canonical on the symplectic submanifold $L$ of the triplectic manifold. We have also classified the ‘triplectic’ vector fields $V^a$ taken in the framework of the ansatz proposed in [3] that are compatible with the weakly canonical antibrackets. It follows that the conditions imposed on the marked functions imply that the $V^a$ vector fields satisfy the constraints postulated in [3] and, therefore, induce a Poisson bracket on the triplectic manifold $M$. We also have formulated the $Sp(2)$-covariance condition for the antibrackets and the $V^a$ vector fields. For a reducible $e^i_\alpha(\xi_1, \xi_2)$, this condition implies the canonical form of the triplectic vector fields.

It should be recalled that the possibility to transform the antibracket to the canonical form allows one to carry over to the covariant formulation a number of important statements in the theory, such as, for example, the statement regarding the existence and uniqueness of solutions to the master-equation. It, thus, remains to be seen whether working with the weakly-canonical antibrackets allows one to prove the existence of the solution to the triplectic master-equation.

It may be remarked that the weakly canonical form of the antibrackets is somewhat reminiscent of the ‘non-Abelian’ antibrackets of [13], however a significant difference is that the non-Abelian ant bracket involves some functions $u^A_i$ that depend on the $\phi^A$ fields in such a way that the derivations $u^A_i \frac{\partial}{\partial \phi^A}$ make up a Lie algebra, while in our case the antibracket involves functions $e^i_\alpha$ that depend only on the marked functions of the antibrackets and all of the vector fields $e^i_\alpha \frac{\partial}{\partial \xi_2^\alpha}$ pairwise commute.

Acknowledgments We are grateful to K. Bering, O. Khudaverdyan, A. Nersessian, I. Tipunin and, especially, to I. Tyutin, for very useful discussions. We also wish to thank I. Batalin for illuminating discussions on a number of problems in quantization of gauge theories. This work was supported in part by the RFFI grant 96-01-00482 and by grant INTAS-RFBR-95-0829 from the European Community.

Appendix

Here, we show how one can construct a solution for the $e^i_\alpha$ structure that is not necessarily reducible. The general analysis of equations (5.7)–(5.8), which guarantee the $Sp(2)$-covariance of antibrackets (3.18), is very involved. In what follows, we restrict ourselves to the case where $e^i_\alpha$ differs from a constant only infinitesimally:

$$e^i_\alpha(\xi_1, \xi_2) = \delta^i_\alpha + \varepsilon e^i_\alpha(\xi_1, \xi_2). \quad (A.1)$$

Consider then, e.g., Eq. (5.7). It now rewrites in the following terms. We set

$$Y^+ = \overline{Y}^+ + \varepsilon y^+. \quad (A.2)$$

In the zeroth order in $\varepsilon$, we have

$$\overline{Y}^+_{1m} = \xi_1^j a^j_m, \quad \overline{Y}^+_{2m} = \xi_1^m + \xi_2^j a^j_m \quad (A.3)$$
with a constant matrix $a^i_m$. In the first order in $\varepsilon$, then, Eqs. (5.7) take the following form:

$$\frac{\partial}{\partial \xi_2^i} (y^+_{2m} - \xi_{1i} \frac{\partial}{\partial \xi_{1i}} y^+_{2m}) = \frac{\partial}{\partial \xi_{1i}} y^+_{1m} + a^i_k c^k_m - c^i_k a^k_m + \xi_{1i} a^i_k \frac{\partial}{\partial \xi_{1k}} c^i_m + \xi_{2i} a^i_{2k} \frac{\partial}{\partial \xi_{2k}} c^i_m,$$

(A.4)

As we saw in Sec. 5, the equations for $Y$ have a solution when the $\varepsilon^i_k$ structure is reducible. To give an example of a solution which is not reducible, we consider the function $c^i_m$ that is homogeneous in $\xi_1, \xi_2$,

$$c^i_m (\xi_1, \xi_2) = K^i_{pq} \xi_{1p} \xi_{2q}.$$

(A.5)

Inserting $c^i_m$ of this form into (A.4), we can observe that, in addition to functions depending only on $\xi_1$ or on $\xi_2$, homogeneous solutions to that equation must depend linearly on $\xi_2$. In a similar way, we see from the equation ensuring the covariance with respect to the $J^-$ generator that (again, in addition to the ‘trivial’ solutions) the solutions are linear in $\xi_1$. Modulo the $c^i_j$ structures that correspond to reducible $\varepsilon^i_k$, we thus have the following general form of the homogeneous $c^i_j$ functions corresponding to $Sp(2)$-covariant antibrackets:

$$c^i_m (\xi_1, \xi_2) = K^i_{pq} \xi_{1p} \xi_{2q}.$$

(A.6)

In this way, we obtain a vector field on the triplectic manifold which ensures the $Sp(2)$ covariance of the weakly canonical, but not reducible, antibrackets of the form (3.18) in which $\varepsilon^i_\alpha$ has the form (A.1), (A.6). It can also be shown that there exist $Sp(2)$-covariant vector fields $V^a$ that differentiate the antibrackets defined by the above $c^i_j$.

References

[1] I. A. Batalin and R. Marnelius, Phys. Lett. B446 (1995) 44.
[2] I. A. Batalin, R. Marnelius, and A. M. Semikhatov, Nucl. Phys. B446 (1995) 249.
[3] I. A. Batalin and R. Marnelius, Nucl. Phys. B465 (1996) 521.
[4] C.M. Hull, Mod. Phys. Lett. A5 (1990) 1871.
[5] I. A. Batalin, P. M. Lavrov, and I. V. Tyutin, J. Math. Phys. 31 (1990) 1487.
  I. A. Batalin, P. M. Lavrov, and I. V. Tyutin, J. Math. Phys. 32 (1990) 532.
  I. A. Batalin, P. M. Lavrov, and I. V. Tyutin, J. Math. Phys. 32 (1991) 2513.
[6] P. H. Damgaard and A. Nersessian, Phys. Lett. B355 (1995) 150
[7] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B Vol.102 (1981) 27.
  I. A. Batalin and G. A. Vilkovisky, Phys. Rev. D Vol.28 (1983) 2567.
[8] I. A. Batalin and I. V Tyutin, Int. J. Mod. Phys. A8 (1993) 2333.
  I. A. Batalin and I. V Tyutin, Mod. Phys. Lett. A8 (1993) 3673.
  I. A. Batalin and I. V Tyutin, Mod. Phys. Lett. A9 (1994) 1707.
[9] H. Hata and B. Zwiebach, Ann. Phys. 229 (1994) 177.
[10] A. Sen and B. Zwiebach, Commun. Math. Phys.177 (1996) 305.
[11] V. N. Shander, Analogues of the Frobenius and Darboux Theorems for Supermanifold, Comptes Rendus de l’Academie Bulgare des Sciences, Vol.36, n.3, p.309, (1983).
[12] P. H. Damgaard, F. De Jonghe, and K. Bering, Nucl. Phys. B455 (1995) 440.
[13] J. Alfaro and P. H. Damgaard, Phys. Lett. B369 (1996) 289-294.
[14] A. Schwarz, Commun. Math. Phys.155 (1993) 249-260.