Determination of velocities of wave propagation in some media through the eigenvalues of the material tensors

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Abstract. It is known that the eigenvalues of the tensor and the tensor-block matrix are invariant quantities. Therefore, in this work, our goal is to find the expression for the velocities of wave propagation of certain media through the eigenvalues of the material tensors. In particular, we consider materials with the anisotropy symbol $\{1.5\}$ and $\{5.1\}$, as well as isotropic materials, and for them we determine the expressions for the velocities of wave propagation.

In addition, we obtained expressions for the velocities of wave propagation for materials of cubic syngony with the anisotropy symbol $\{1,2,3\}$ (the matrix of the elastic modulus tensor components has three independent components), hexagonal system (transversal isotropy) with anisotropy symbol $\{1,1,2,2\}$ (the matrix of the elastic modulus tensor components has five independent components), trigonal system with anisotropy symbol $\{1,1,2,2\}$ (the matrix of the elastic modulus tensor components has six independent components), tetragonal system with anisotropy symbol $\{1,1,1,2,1\}$ (the matrix of the elastic modulus tensor components has six independent components).

We also obtained the expressions for the velocities of wave propagation for a micro-polar medium with the anisotropy symbol $\{1.5.3\}$ and $\{5.1.3\}$, and for an isotropic micro-polar material.

1. Kinematic and dynamic conditions on the surface of a strong discontinuity in micro-polar mechanics

Consider a moving regular surface in an unbounded space, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$, whose equation in a fixed Cartesian coordinate system is given by

$$\psi(x, t) = 0. \quad (1)$$

The regularity of the equation (1) means the existence of the unit normal vector $n(x, t)$ at
each point of the surface (1) at time $t$:  
\[ n(x, t) = \frac{\nabla_x \psi}{|\nabla_x \psi|}, \quad \nabla_x \psi = k_i \partial_i \psi, \quad i = 1, 2, 3, \]
as well as the required number of times continuously differentiable functions $\psi$ of $x$. For our purpose it is sufficient the existence of the unit normal, i.e. $\psi \in C^1$. Here $k_i$ is an orthonormal basis of the Cartesian coordinate system, and $C^1$ is the set of continuously differentiable functions.

**Definition 1.** If the vectors $u$ and $\varphi$ are continuous when passing through the surface (1), and the first derivatives $\partial_k u$, $\partial_t u$, $\partial_k \varphi$, $\partial_t \varphi$ undergo a discontinuity such that on each side of this surface they take different finite values, then the surface (1) is called the **surface of a strong discontinuity (or a stress wave and a moment stress)**.

**Definition 2.** If the vectors $u$, $\partial_k u$, $\partial_t u$, $\varphi$, $\partial_k \varphi$ and $\partial_t \varphi$ are continuous when passing through the surface (1), and the second derivatives of the vectors $u$ and $\varphi$ with respect to $x_k$ and $t$ undergo a discontinuity such that on each side of this surface they take different finite values, then the surface (1) is called the **surface of a weak discontinuity (or an acceleration wave)**.

We introduce the notation: $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$. Next, the surface of a strong discontinuity is studied.

As in [16–19], the velocity of moving an arbitrary point of the surface (1) in the direction of the normal of this surface at this point, we denote by $c$:

\[ c = -\frac{1}{|\nabla_x \psi|} \partial_t \psi, \quad \nabla_x \psi = k_i \partial_i \psi. \quad (2) \]

If the wavefront (the surface of strong discontinuity) of (1) moves in a medium having a velocity field $v(x, t)$, then the velocity of the wavefront relative to the particles of the medium will be determined by the formula

\[ \theta = c - v_n, \quad v_n = n \cdot v. \quad (3) \]

**1.1. Kinematic conditions on the surface of a strong discontinuity**

The kinematic conditions on the surface of strong discontinuity can be obtained in the same way as is done, for example, in [16–18] for the classical case. In this case, the difference lies in that instead of a motion vector should be considered two independent vectors $u(x, t)$, $\varphi(x, t)$, and they have the form

\[ c[\partial_k u] + n_i[\partial_i u] = 0, \quad c[\partial_k \varphi] + n_i[\partial_i \varphi] = 0, \quad (4) \]

where $c$ is determined by the first formula (2), $n_i$ are the components of the unit normal $n$ to the wave front. Writing $[w] = w^+ - w^-$, where $w = u$ or $w = \varphi$, means a jump in the value of $w$ relative to the wave front, $w^-$ (or $w^+$) is the limiting value of $w$ when the arbitrarily chosen point is in front (behind) the wave front to the point at the wave front. The square brackets $[ ]$ will be called the jump operator.

**1.2. Laws of conservation of mass and the tensor of moments of inertia at the wave front**

Applying the law of conservation of mass and the law of the tensor of moments of inertia

\[ \frac{d}{dt} \int \rho dV = 0, \quad \frac{d}{dt} \int \mathbf{j} dV = 0, \]

1 We use the usual rules of tensor calculus [1–6]. We mainly preserve the notation and conventions of the previous works, capital Latin indices assume the values 1, 2. Over repeated indices there is a summation.
where $\rho$ is a material density, and $\mathbf{J}$ is the density of the inertia tensor (a special dynamic characteristic of the medium) of the particles of the medium [11–13], to the elementary cylinder isolated in the medium. After simple transformations analogous to the classical case [16–18], we obtain the required laws

$$[\theta \rho] = 0, \quad [\theta \mathbf{J}] = 0, \quad (5)$$

where $\theta$ is defined in (3).

1.3. Dynamic conditions on the wave front

Dynamic conditions on the wave front can be easily obtained with the help of the law on the change of momentum and of the theorem on the change in the angular momentum of the internal rotational motions of the particles of the medium, which are defined as follows:

$$\frac{d}{dt} \int_{\mathcal{V}} \rho \mathbf{v} dV = \int_{\mathcal{V}} \rho \mathbf{F} dV + \int_{\Sigma} \mathbf{P}_{(n)} d\Sigma, \quad \frac{d}{dt} \int_{\mathcal{V}} \mathbf{J} \cdot \mathbf{\omega} dV = \int_{\mathcal{V}} (\rho \mathbf{m} + \mathbf{C} \otimes \mathbf{P}) dV + \int_{\Sigma} \mathbf{\mu}_{(n)} d\Sigma, \quad (6)$$

where $\mathbf{F}$ is the mass force, $\mathbf{m}$ is the mass moment, $\mathbf{P}_{(n)}$ and $\mathbf{\mu}_{(n)}$ are the voltage and moment stress vectors on the area with a unit vector of the normal $\mathbf{n}$ respectively, $\mathcal{V}$ is the volume of the body, $\Sigma$ is the boundary of the body, $\mathbf{v} = \mathbf{u}$, $\mathbf{\omega} = \mathbf{\varphi}$, where the dot over the letter indicates the time derivative. Like the classical case [16–18], we apply (6) to the above-mentioned cylinder and, taking into account (5), we obtain the relations

$$\rho \frac{d}{dt} [\mathbf{v}] = -[\mathbf{P}_{(n)}] = -\mathbf{n} \cdot [\mathbf{P}], \quad \theta \mathbf{J} \cdot [\mathbf{\omega}] = -[\mathbf{\mu}_{(n)}] = -\mathbf{n} \cdot [\mathbf{\mu}], \quad (7)$$

where $\rho \theta = \rho^{-} \theta^{-} = \rho^{+} \theta^{+}$, $\theta \mathbf{J} = \theta^{-} \mathbf{J}^{-} = \theta^{+} \mathbf{J}^{+}$.

We note that the relations (7), in which the Cauchy formulas are taken into account $\mathbf{P}_{(n)} = \mathbf{n} \cdot \mathbf{P}$ and $\mathbf{\mu}_{(n)} = \mathbf{n} \cdot \mathbf{\mu}$, they are also valid for any medium. We note that the conditions (7) and their analogs for different media are derived in [12].

2. Determination of wave propagation velocities in an infinite micro-polar solid

Having kinetic (4) and dynamic (7) conditions on the front of the wave, it is easy to find an equation for determining the propagation velocities of waves in any infinite micro-polar medium, including in an infinite micro-polar solid. Consider a micro-polar solid body, the defining relations of which are represented in the form (see, [8,15,20])

$$\mathbf{P} = \mathbf{A} \otimes \nabla \mathbf{u} + \mathbf{B} \otimes \nabla \varphi - \mathbf{A} \otimes \mathbf{C} \cdot \mathbf{\varphi} - \mathbf{b} \vartheta, \quad (8)$$

$$\mathbf{\mu} = \mathbf{C} \otimes \nabla \mathbf{u} + \mathbf{D} \otimes \nabla \varphi - \mathbf{C} \otimes \mathbf{C} \cdot \mathbf{\varphi} - \mathbf{\beta} \vartheta,$$

where $\mathbf{b} = \mathbf{A} \otimes \mathbf{a} + \mathbf{B} \otimes \mathbf{d}$, $\mathbf{B} = \mathbf{C} \otimes \mathbf{a} + \mathbf{D} \otimes \mathbf{d}$. Here $\otimes$ is an inner 2-product, $\mathbf{C} = \mathbf{B}^T$, $\mathbf{A}$, $\mathbf{D}$ are the material tensors (tensors of the elastic moduli) of the 4th rank, $\vartheta = \mathbf{T} - \mathbf{T}_{0}$ is the temperature drop, $\mathbf{b}$ and $\mathbf{\beta}$ are the tensors of thermomechanical properties, and $\mathbf{T}$ in the upper corner of the tensor means the sign of transposition.

It is known [12,13], that $\mathbf{B}$ is an asymmetric tensor, as mentioned above. In particular, in [14] it is proved that $\mathbf{B}$ is a symmetric tensor, and $\mathbf{C} = \mathbf{B}$. Here we derive these relations when $\mathbf{C} = \mathbf{B}^T$, since it is easy to obtain from them the corresponding relations for the case $\mathbf{C} = \mathbf{B}$.

Next, for simplicity, let us consider isothermal processes, i.e. we assume that $\vartheta = 0$. Then assuming that the material tensors do not undergo a discontinuity when passing through the front of the wave and applying the jump operator to (8), we have
\[ [\mathbf{P}] = \mathbf{A} \otimes [\nabla \mathbf{u}] + \mathbf{B} \otimes [\nabla \varphi], \quad [\mathbf{\mu}] = \mathbf{C} \otimes [\nabla \mathbf{u}] + \mathbf{D} \otimes [\nabla \varphi]. \] (9)

Multiplying (4) by \( k_i \) with subsequent summation over \( i \) (hereinafter, we omit the index \( x \) of the operator \( \nabla_x \)), we obtain
\[ [\nabla \mathbf{u}] = -\frac{1}{c} n [\mathbf{v}], \quad [\nabla \varphi] = -\frac{1}{c} n [\omega]. \] (10)

From (9) taking into account (10) we find
\[ [\mathbf{P}] = -\frac{1}{c} (\mathbf{A} \otimes n [\mathbf{v}] + \mathbf{B} \otimes n [\omega]), \quad [\mathbf{\mu}] = -\frac{1}{c} (\mathbf{C} \otimes n [\mathbf{v}] + \mathbf{D} \otimes n [\omega]), \]
with the help of which from the dynamic conditions (7) we arrive at the relations
\[ (n \cdot \mathbf{A} \otimes n \mathbf{E}) \cdot [\mathbf{v}] + (n \cdot \mathbf{B} \otimes n \mathbf{E}) \cdot [\mathbf{\omega}] = c \rho \theta \mathbf{E} \cdot [\mathbf{v}], \]
\[ (n \cdot \mathbf{C} \otimes n \mathbf{E}) \cdot [\mathbf{v}] + (n \cdot \mathbf{D} \otimes n \mathbf{E}) \cdot [\mathbf{\omega}] = c \theta \mathbf{J} \cdot [\mathbf{\omega}], \] (11)
where \( \mathbf{E} \) is the unit tensor of the second rank.

We note that, in view of the laws of conservation of mass and the tensor of moments of inertia, (5), the expressions \( \rho \theta \) and \( \theta \mathbf{J} \) in (11) can be replaced by \( \rho^+ \theta^+ \) and \( \theta^+ \mathbf{J}^+ \) respectively, since in front of the wave front \( \rho^+ \) and \( \mathbf{J}^+ \), and also \( v^+_n \) can be considered known, but to simplify the recording we will not do this.

We introduce the notation
\[ \mathbf{A} = (1/\rho) n \cdot \mathbf{A} \otimes n \mathbf{E}, \quad \mathbf{B} = (1/\rho) n \cdot \mathbf{B} \otimes n \mathbf{E}, \quad \mathbf{C} = \mathbf{J}^{-1} \cdot (n \cdot \mathbf{C} \otimes n \mathbf{E}), \quad \mathbf{D} = \mathbf{J}^{-1} \cdot (n \cdot \mathbf{D} \otimes n \mathbf{E}), \quad \mathbb{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \quad [\mathbf{V}] = \begin{pmatrix} [\mathbf{v}] \\ [\mathbf{\omega}] \end{pmatrix}, \] (12)
where \( \mathbb{M} \) is called the tensor-block matrix (TBM), \( \mathbf{V} \) is a vector column of vectors of linear and angular velocities, and \( [\mathbf{V}] \) is the jump of this vector column, (11) can be represented in the form
\[ \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \cdot \begin{pmatrix} [\mathbf{v}] \\ [\mathbf{\omega}] \end{pmatrix} = \lambda \begin{pmatrix} [\mathbf{v}] \\ [\mathbf{\omega}] \end{pmatrix}, \quad \lambda = c \theta \] (13)
or, in short,
\[ \mathbb{M} : [\mathbf{V}] = \lambda [\mathbf{V}]. \] (14)

The equality (14) (see also (13)) represents a homogeneous system of six algebraic equations with respect to six unknowns (two vectors \([\mathbf{v}]\) and \([\mathbf{\omega}]\) having a nontrivial solution). In order for this system to have a nontrivial solution it is necessary and sufficient that its determinant be zero. Since the determinant is of the 6th order, and \( \theta^+ = c - v^+_n \), then from the equality to zero of this determinant we obtain an algebraic equation of degree 6 with respect to \( c^2 \), which is the desired dispersion equation for determining the velocities of waves in an infinite anisotropic micro-polar body and their number in a given direction.

So, we got the eigenvalue problem in the form of (14) for TBM. Obviously, \( \lambda = c \theta \) is an eigenvalue, and \([\mathbf{V}]\) is its corresponding jump of the vector column. By virtue of (14) the dispersion equation (the characteristic equation for \( \mathbb{M} \)) can be written in the form
\[ \det(\mathbb{M} - \lambda \mathbb{E}) = 0, \] (15)
where \( \mathbb{E} \) is a unit TBM of the 2nd rank. In the expanded form, the equation (15) can be written as follows:
\[ \lambda^6 - I_1(\mathbb{M}) \lambda^5 + I_2(\mathbb{M}) \lambda^4 - I_3(\mathbb{M}) \lambda^3 + I_4(\mathbb{M}) \lambda^2 - I_5(\mathbb{M}) \lambda + I_6(\mathbb{M}) = 0, \quad I_6(\mathbb{M}) = \det(\mathbb{M}). \] (16)
It is seen that the dispersion equation (16) is an algebraic equation of the 6th degree and must have six roots (eigenvalues), counting each root as many times as its multiplicity. Each multiple root determines the square of the velocity of one wave. Hence, in an arbitrary anisotropic infinite micropolar medium, in the general case, no more than six waves can arise in each direction. Note that, based on the dispersion equation of the form (16), it is easy to establish the number of waves arising in a micropolar elastic medium for different anisotropy. So, it is sufficient to find the invariants of the TBM that appear in (16), and then solve the equation itself (16). Invariants are easily found through the first invariants of powers \( M \). We have ([4,5,7,9,10])

\[
S_k = I_k(M) = \frac{1}{k!} \begin{vmatrix}
1 & 0 & \cdots & 0 \\
0 & s_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & s_k
\end{vmatrix}, \quad k = 1,6,
\]

where \( I_k(M) \), \( k = 1,6 \), denote the invariants of TBM \( M \). In this case, the inverse relations to (17) are represented in the form

\[
s_k = I_1(M^k), \quad k = 1,6, \quad M^k = M \cdot M \cdot \cdots \cdot M,
\]

If the material has a center of symmetry, then \( B = C^T = 0 \). Then \( B = 0 \) and \( C = 0 \), and the TBM \( M \) becomes a diagonal TBM. For such a matrix, the characteristic equation and eigenvalues (wave velocities) are easily found. We have

\[
\det(M - \lambda\mathbf{E}) = \det \begin{pmatrix}
\mathbf{A} - \lambda\mathbf{E} & \mathbf{0} \\
\mathbf{0} & \mathbf{D} - \lambda\mathbf{E}
\end{pmatrix} = \det(\mathbf{A} - \lambda\mathbf{E}) \det(\mathbf{D} - \lambda\mathbf{E}) = 0,
\]

where \( \mathbf{0} \) is the zero tensor of rank 2. The equation (18) is equivalent to the following equations:

\[
\lambda^3 - I_1(\mathbf{A})\lambda^2 + I_2(\mathbf{A})\lambda - I_3(\mathbf{A}) = 0, \quad \lambda^3 - I_1(\mathbf{D})\lambda^2 + I_2(\mathbf{D})\lambda - I_3(\mathbf{D}) = 0.
\]

So, if the material has a center of symmetry, then in this case to determine the wave velocities we have two cubic equations (19), which are easily solved. Based on (19) we can conclude that in an arbitrary anisotropic micro-polar elastic medium with a center of symmetry, no more than six waves appear in each direction.

We note that in the case of a classical medium, we have one cubic equation analogous to the first of the equations (19) provided that \( \mathbf{A} \) is determined using the first relation (12), where \( \mathbf{A} \) is the elastic modulus tensor. Consequently, in an arbitrary anisotropic classical elastic medium, no more than three waves appear in each direction. Note also that in the case of a micro-polar medium, it is reasonable to call \( \mathbf{M} \) the dispersion TBM, in the case of the classical medium \( \mathbf{A} \) is the dispersion tensor, since their characteristic equations are the dispersion equations.

Having the dispersion equations derived above for the micro-polar (classical) medium, we can find the propagation velocities of the waves in the media under consideration for any anisotropy. Of course, the required velocities can be found both in the traditional representation of material objects (TBM, tensors of elastic moduli) and with the help of their eigenvalues.
solving in advance the eigenvalue problems of the corresponding object in the case under consideration. The eigenvalue problems for the tensor and TBM of any even rank are solved in the papers [4,5,7,9] in which the notion of the anisotropy (structure) symbol of the material is introduced and the classifications of micro–polar and classical media are given.

Therefore, for each material included in these classifications, on the basis of the corresponding dispersion equation obtained for the micro-polar (classical) medium, one can obtain a dispersion equation for the micro-polar (classical) medium under consideration, and then determine the number of waves and their propagation velocity.

**Definition 3.** The symbol \{\alpha_1,\alpha_2,\ldots,\alpha_k\}, where \(k\) is the number of different eigenvalues of the TBM (tensor), and \(\alpha_i\) is the multiplicity of the eigenvalue \(\lambda_i, i = 1, 2, \ldots, k\), is called the *symbol of the anisotropy (structure)* of the TBM (tensor).

Note that the *symbol of the anisotropy* of the material TBM (material tensor) is called also the symbol of anisotropy of the material.

Next, we consider some particular cases of materials and find the propagation velocities of the waves in them using the eigenvalues of the corresponding tensor objects.

### 3. Classical materials with anisotropy symbols \{1,5\} and \{5,1\}

For materials with anisotropy symbols \{1,5\} and \{5,1\} the elastic modulus tensor is represented in the form

\[
\mathbf{A} = (\lambda_1 - \lambda_2)\mathbf{a}_1\mathbf{a}_1 + \lambda_3 \mathbf{E}, \quad \mathbf{B} = \mu_1 \mathbf{E} - (\mu_1 - \mu_6)\mathbf{b}_6\mathbf{b}_6.
\]

\[
(20)
\]

Here \(\mathbf{E} = \sum_{k=1}^{6} \mathbf{a}_k\mathbf{a}_k = \sum_{k=1}^{6} \mathbf{b}_k\mathbf{b}_k = (1/2)(\mathbf{C}^{(2)} + \mathbf{C}^{(3)})\) is the unit tensor of rank 4, \(\mathbf{a}_k\) and \(\mathbf{b}_k\), \(k = 1, 6\), are complete orthonormal systems of proper tensors for tensors \(\mathbf{A}\) and \(\mathbf{B}\) respectively. \(\mathbf{C}^{(2)}\) and \(\mathbf{C}^{(3)}\) are isotropic tensors of rank 4, \(\lambda_1, \lambda_2, \lambda_3\), and \(\mu_1, \mu_6\) are eigenvalues, \(\mathbf{a}_1 = \mathbf{a}_1^T\) and \(\mathbf{b}_6 = \mathbf{b}_6^T\) are proper tensors corresponding to eigenvalues \(\lambda_1\) and \(\mu_6\) respectively.

Since \(\mathbf{a}_1\) (\(\mathbf{b}_6\)) is a symmetric tensor satisfying the orthonormality condition \(\mathbf{a}_1 \otimes \mathbf{a}_1 = 1\) (\(\mathbf{b}_6 \otimes \mathbf{b}_6 = 1\)), then in a basis constructed using a basis of an arbitrary coordinate system, it is characterized by five components, and in the main basis for \(\mathbf{a}_1\) (\(\mathbf{b}_6\)) - two components. Hence, the tensor \(\mathbf{A}\) (\(\mathbf{B}\)) in a basis formed by using a basis to an arbitrary coordinate system is characterized by seven parameters: two eigenvalues and five components of the tensor \(\mathbf{a}_1\) (\(\mathbf{b}_6\)), and in the basis formed by means of the canonical basis for \(\mathbf{a}_1\) (\(\mathbf{b}_6\)), four parameters: two eigenvalues and two components of the tensor \(\mathbf{a}_1\) (\(\mathbf{b}_6\)).

**Proposition.** Let \(\mathbf{a} = \mathbf{a}^T\). Then \(\mathbf{aa}\) is an isotropic tensor of rank 4 if and only if \(\mathbf{a}\) is a spherical tensor. In this case, if \(\mathbf{a}^2 = \mathbf{a} = 1\), then \(\mathbf{a} = [(\pm \sqrt{3})/3] \mathbf{E}\), and \(\mathbf{aa} = (1/3)\mathbf{C}^{(1)}\).

Here \(\mathbf{C}^{(1)} = \mathbf{EE}\) is the first of three isotropic tensors of rank 4. From this statement it follows that the tensors \(\mathbf{A}\) and \(\mathbf{B}\) of (20) will be traditionally isotropic if and only if, when \(\mathbf{a}_1\mathbf{a}_1 = \mathbf{b}_6\mathbf{b}_6 = (1/3)\mathbf{C}^{(1)}\). In this case, they can be written in the form

\[
\mathbf{A} = (1/3)(\lambda_1 - \lambda_2)\mathbf{C}^{(1)} + \lambda_3 \mathbf{E} = \lambda_3 \mathbf{E} + 2\mu_1 \mathbf{E}, \quad \mathbf{B} = \mu_1 \mathbf{E} - (1/3)(\mu_1 - \mu_6)\mathbf{C}^{(1)}.
\]

\[
(21)
\]

We note that an isotropic material whose properties are characterized by the tensor \(\mathbf{A}\) (\(\mathbf{B}\)) has a positive (negative) Poisson’s ratio [4,5,9]. In addition, the tensor \(\mathbf{A}\) of (21) is presented in the traditional form (by the Lamé coefficients \(\lambda = 1/3(\lambda_1 - \lambda_2), \mu = 1/2\lambda_2\)).

It is easy to see that as tensors \(\mathbf{A}\) and \(\mathbf{B}\) of (20), and the corresponding dispersion tensors \(\mathbf{A}\) and \(\mathbf{B}\) have the same structure and are represented as

\[
\mathbf{A} = (1/\rho)\mathbf{n} \cdot \mathbf{A} \cdot \mathbf{n} = a(\mathbf{E} + \mathbf{nn}) + b\mathbf{u}, \quad \mathbf{B} = (1/\rho)\mathbf{n} \cdot \mathbf{B} \cdot \mathbf{n} = f(\mathbf{E} + \mathbf{nn}) + g\mathbf{y}.
\]

\[
(22)
\]
\[ a = \lambda_2/(2\rho) = \mu/\rho > 0, \quad b = (\lambda_1 - \lambda_2)/\rho = (3\lambda)/\rho > 0, \quad u = \mathbf{n} \cdot \mathbf{a}_1 \mathbf{n} \cdot \mathbf{a}_1, \]
\[ f = \mu_1/(2\rho) > 0, \quad g = (\mu_6 - \mu_1)/\rho < 0, \quad v = \mathbf{n} \cdot \mathbf{b}_1 \mathbf{n} \cdot \mathbf{b}_1. \]

Assuming in (22) \( \mathbf{a}_1 = \mathbf{b}_6 = (\pm \sqrt{3}/3)\mathbf{E} \), we obtain the dispersion tensors \( \mathbf{A} \) and \( \mathbf{B} \), corresponding to isotropic tensors (21),
\[ \mathbf{A} = a \mathbf{E} + b_1 \mathbf{n} \mathbf{n}, \quad \mathbf{B} = f \mathbf{E} + g_1 \mathbf{n} \mathbf{n}, \]
\[ a = \lambda_2/(2\rho) = \mu/\rho > 0, \quad b_1 = (2\lambda_1 + \lambda_2)/(6\rho) = (\lambda + \mu)/\rho > 0, \quad u = \mathbf{n} \cdot \mathbf{a}_1 \mathbf{n} \cdot \mathbf{a}_1, \]
\[ f = \mu_1/(2\rho) > 0, \quad g_1 = (\mu_1 + 2\mu_6)/(6\rho) > 0, \quad v = \mathbf{n} \cdot \mathbf{b}_1 \mathbf{n} \cdot \mathbf{b}_1. \]

Obviously, the tensors (24), which are special cases of tensors (22), similarly to the latter have the same structure. In this regard, below we will consider the first tensor (22), and for the rest of tensors (22) and (24) we will obtain the corresponding relations by an appropriate renaming of the coefficients and tensors.

So, we find the propagation velocities of the waves in the material \{1,5\} whose dispersion tensor has the form (22), and the characteristic equation is represented as the first equation (19). To do this, we first find \( I_1(\mathbf{A}^k) \), \( k = 1, 2, 3 \), and then using the formula (17), which is true for a tensor of the corresponding rank, we find \( I_k(\mathbf{A}) \), \( k = 1, 2, 3 \). After simple calculations, we have
\[ I_1(\mathbf{A}) = 4a + bI_1(\mathbf{u}), \quad I_1(\mathbf{A}^2) = 6a^2 + 2ab[I_1(\mathbf{u}) + I^2] + b^2I_1^2(\mathbf{u}), \quad I_1(\mathbf{u}) = \mathbf{n} \cdot \mathbf{a}_1^2 \cdot \mathbf{n}, \]
\[ I_1(\mathbf{A}^3) = 10a^3 + 3a^2b[I_1(\mathbf{u}) + 3I^2] + 3ab^2I_1(\mathbf{u})[I_1(\mathbf{u}) + I^2] + b^3I_1^3(\mathbf{u}), \quad I = \mathbf{a}_1^2 \mathbf{n} \mathbf{n}; \]
\[ I_2(\mathbf{A}) = 5a^2 + 3abI_1(\mathbf{u}) - abI^2, \quad I_3(\mathbf{A}) = 2a^3 + 2a^2bI_1(\mathbf{u}) - a^2bI^2. \]

Further, by virtue of the corresponding invariants (26), constructing the characteristic equation from eqref dyn20 for the considered tensor \( \mathbf{A} \) and solving it, we obtain the following expressions for the roots:
\[ \eta_1 = a, \quad \eta_{2,3} = \frac{1}{2} \left[ 3a + bI_1(\mathbf{u}) \pm \sqrt{(a - bI_1(\mathbf{u}))^2 + 4abI^2} \right]. \]

**Theorem.** The dispersion tensor is positive definite; as well as the dispersion TBM is positively determined.

It follows from this theorem that the eigenvalues of the dispersion tensor and the dispersion tensor-block matrix are positive. It is easy to prove that the eigenvalues (27) obtained above are positive. Knowing the roots of the characteristic equation of the tensor \( \mathbf{A} \) (see, (27)), it is easy to find the propagation velocities of waves in an initially resting medium. We have
\[ c_1 = \sqrt{\eta_1}, \quad c_2 = \sqrt{\eta_2}, \quad c_3 = \sqrt{\eta_3}. \]

Thus, in an initially resting medium with the structure symbol \{1,5\} using the formulas (28), we can find the propagation velocities of the waves in an arbitrary direction. In the general case, their number is not more than three. Note that if the medium does not rest, then to determine the wave velocities instead of (28) we will have formulas
\[ c_1(c_1 - v^+_1) = \eta_1, \quad c_2(c_2 - v^+_1) = \eta_2, \quad c_3(c_3 - v^+_1) = \eta_3. \]

It is seen that each relation from (29) is a square equation with respect to the wave propagation velocity. Solving them we obtain explicit expressions for the propagation velocities of waves in the initially disturbing medium with the structure symbol \{1,5\}. It can be assumed that the
number of waves arising in such a medium in an arbitrary direction will be no more than six, but not less than three. To conduct this study, and similar to the above for $\mathbf{A}$ (1st eq. of (22)) and for $\mathbf{B}$ (2nd eq. of (22)) will not be difficult. Obviously in the latter case, it is sufficient in the above formulas for $\mathbf{A}$, as it was said above, to replace $a, b, A$ and $u$ by $f, g, B$ and $v$ respectively. In connection with this simplicity, we will not dwell on these questions.

Next, consider the first dispersion tensor (24), which corresponds to the first tensor (21), characterizing the properties of a traditionally isotropic material and being a particular case of the first tensor (22). It is easy to see that in this case analogous (26) relations are represented in the form

\[ I_1(\mathbf{A}) = 3a + b_1, \quad I_1(\mathbf{A}^2) = 2a^2 + (a + b_1)^2, \quad I_1(\mathbf{A}^3) = 2a^3 + (a + b_1)^3, \]
\[ I_2(\mathbf{A}) = a(3a + 2b_1), \quad I_3(\mathbf{A}) = a^2(a + b_1). \]

(30)

Taking into account the invariants corresponding to (30), from the first equation (19) we obtain the characteristic equation for the investigated dispersion tensor $\mathbf{A}$. Solving it, we get

\[ \eta_1 = \eta_2 = a = \frac{\lambda_2}{2\rho} = \frac{\mu}{\rho}, \quad \eta_3 = a + b_1 = \frac{\lambda_1 + 2\lambda_2}{3\rho} = \frac{\lambda + 2\mu}{\rho}. \]

(31)

The velocities of the propagation of waves in an initially resting infinite isotropic elastic medium by virtue of (31) are determined by formulas

\[ c_1 = \sqrt{\eta_1} = \sqrt{\frac{\lambda_2}{2\rho}} = \sqrt{\frac{\mu}{\rho}}, \quad c_2 = \sqrt{\eta_3} = \sqrt{\frac{\lambda_1 + 2\lambda_2}{3\rho}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \]

(32)

It can be seen that by the formulas (32) the propagation velocities of the waves are expressed both in terms of the eigenvalues and the Lamé parameters. In the case under consideration, writing out the analogous (29) relations and investigating them is not difficult, so, we shall not dwell on this.

From the first tensor (24) we see that any vector perpendicular to $\mathbf{n}$ and located in the tangent plane to the wave surface, is an eigenvector of the dispersion tensor $\mathbf{A}$, and that $\mathbf{n}$ is its eigenvector. Consequently, the system of vectors ($\mathbf{s}, \mathbf{l}, \mathbf{n}$), where $\mathbf{s}$ and $\mathbf{l}$ are mutually perpendicular unit tangent vectors to the wave surface, is the complete orthonormal system of eigenvectors of the considered dispersion tensor $\mathbf{A}$ (1st tensor of (24)). The eigenvectors of the tensor $\mathbf{A}$ can be found by solving the system of equations corresponding to these vectors, but in the case under consideration there is no such need, since they can be easily guessed. Consequently, the canonical representation of the dispersion tensor $\mathbf{A}$ in view of what has been said above, has the form

\[ \mathbf{A} = \eta_1(\mathbf{ss} + \mathbf{ll}) + \eta_3\mathbf{nn} = \eta_1\mathbf{I} + \eta_3\mathbf{nn}, \quad \mathbf{l} = \mathbf{ss} + \mathbf{ll}. \]

(33)

It is easy to see that by virtue of (32) from (33) for the initially resting medium, we obtain

\[ \mathbf{A} = c_1^2(\mathbf{ss} + \mathbf{ll}) + c_2^2\mathbf{nn} = c_1^2\mathbf{I} + c_2^2\mathbf{nn} = \mathbf{v}_s\mathbf{v}_s + \mathbf{v}_l\mathbf{v}_l + \mathbf{v}_n\mathbf{v}_n, \]

(34)

and we also have

\[ \mathbf{V} = \sqrt{\mathbf{A}} = c_1(\mathbf{ss} + \mathbf{ll}) + c_2\mathbf{nn} = c_1\mathbf{I} + c_2\mathbf{nn} = \mathbf{v}_s\mathbf{s} + \mathbf{v}_l\mathbf{l} + \mathbf{v}_n\mathbf{n} = \mathbf{s}\mathbf{v}_s + \mathbf{l}\mathbf{v}_l + \mathbf{n}\mathbf{v}_n, \]

(35)

where $\mathbf{v}_s = c_1\mathbf{s}$, $\mathbf{v}_l = c_1\mathbf{l}$ and $\mathbf{v}_n = c_2\mathbf{n}$.

From (33), (34) and (35) it is seen that the tensors $\mathbf{A}$ and $\mathbf{V}$ are represented by the sum of two or three orthogonal tensors. In addition, it is seen from (32) that the eigenvalues of the tensor
Further, the 2-index notation of the 4th rank tensor is mainly applied (see (40) and (41)). At the same time, if the components of the tensor \( A \) (1st eq. (24)) can be written in the form
\[
\{(a - \eta)I + (b_1 + a - \eta)nn\} \cdot [\mathbf{v}] = 0. \tag{36}
\]
Consequently, for an arbitrary motion of the medium, the expression \([\mathbf{v}]\) can be represented in the form
\[
[\mathbf{v}] = [\mathbf{v}_r] + [\mathbf{v}_n], \quad \mathbf{v}_r = v_r \mathbf{r}, \quad \mathbf{v}_n = v_n \mathbf{n}, \quad \mathbf{r} \perp \mathbf{n}. \tag{37}
\]
Taking into account (37), from (36) we have
\[
\{(a - \eta)I\} \cdot [\mathbf{v}_r] + \{(b_1 + a - \eta)nn\} \cdot [\mathbf{v}_n] = 0. \tag{38}
\]

Note that the following theorem holds, which can be proved by the kinematic conditions (1st eq. (4)).

**Theorem.** \( rot \mathbf{u} = \nabla \times \mathbf{u} = 0 \) if and only if \([\mathbf{v}] \parallel \mathbf{n}\), and \( div \mathbf{u} = \nabla \cdot \mathbf{u} = 0 \) if and only if \([\mathbf{v}] \perp \mathbf{n}\).

Note also that if \([\mathbf{v}] \parallel \mathbf{n}\) or \([\mathbf{v}] \perp \mathbf{n}\), then the wave is called **longitudinal** or **transverse**, respectively.

If now, the motion of the medium is such that \([\mathbf{v}] \parallel \mathbf{n}\) (\([\mathbf{v}] \perp \mathbf{n}\)), i.e. \( \mathbf{v} = \mathbf{v}_n = \mathbf{v} \mathbf{n} \), \( \mathbf{v} = \mathbf{v}_r = \mathbf{v} \mathbf{r} \), the from (38) we get
\[
\{(b_1 + a - \eta)nn\} \cdot [\mathbf{v}_n] = 0 \quad (\{(a - \eta)I\} \cdot [\mathbf{v}_r] = 0), \tag{39}
\]
and hence it is easy to obtain the propagation velocities of longitudinal (transverse) waves. Above they were written out (32) (see also (29) (31)), therefore, here we will not write them out.

Note that an analogous to the above study can be carried out in a more general case, but for brevity we shall not dwell on this.

Further, before considering the materials of other structures, we note that for a 4th rank tensor we apply the 4-index and 2-index representations (\([4, 5, 7, 9]\))

\[
A = A_{ijkl} e_i e_j e_k e_l = \sum_{m=1}^{9} \sum_{n=1}^{9} A_{mn} e_m e_n = A_{mn} e_m e_n, \quad i, j, k, l = 1, 2, 3, \quad m, n = \overline{1, 9},
\]
\[
e_i \cdot e_j = \delta_{ij}, \quad i, j = 1, 2, 3; \quad e_1 = e_1 e_1, \quad e_2 = e_2 e_2, \quad e_3 = e_3 e_3, \quad e_4 = \frac{1}{\sqrt{2}}(e_1 e_2 + e_1 e_2), \tag{40}
\]
\[
e_5 = \frac{1}{\sqrt{2}}(e_2 e_3 + e_3 e_2), \quad e_6 = \frac{1}{\sqrt{2}}(e_3 e_1 + e_1 e_3), \quad e_7 = \frac{1}{\sqrt{2}}(e_1 e_2 - e_1 e_2),
\]
\[
e_8 = \frac{1}{\sqrt{2}}(e_2 e_3 - e_3 e_2), \quad e_9 = \frac{1}{\sqrt{2}}(e_3 e_1 - e_1 e_3), \quad e_m \otimes e_m = \delta_{mn}, \quad m, n = \overline{1, 9}.
\]

At the same time, if the components of the tensor \( A \) have symmetries \( A_{ijkl} = A_{klij} = A_{jikl} \), then we have
\[
A = A_{ijkl} e_i e_j e_k e_l = \sum_{m=1}^{6} \sum_{n=1}^{6} A_{mn} e_m e_n = A_{mn} e_m e_n, \quad i, j, k, l = 1, 2, 3, \quad m, n = \overline{1, 6}. \tag{41}
\]

Further, the 2-index notation of the 4th rank tensor is mainly applied (see (40) and (41)).
4. Classical material with the anisotropy symbol \{1,2,3\} (cubic symmetry)

In this case we have the following canonical representation of the elastic modulus tensor \( \mathbf{A} \):

\[
\mathbf{A} = (\lambda_1 - \lambda_4) \mathbf{W}_1 \mathbf{W}_1 + (\lambda_2 - \lambda_4) (\mathbf{W}_2 \mathbf{W}_2 + \mathbf{W}_3 \mathbf{W}_3) + \lambda_4 \mathbf{E} \quad (\mathbf{E} = \sum_{m=1}^{6} \mathbf{W}_m \mathbf{W}_m),
\]

where the eigenvalues and proper tensors are represented in the form

\[
\lambda_1 = A_{11} + 2A_{12}, \quad \lambda_2 = \lambda_3 = A_{11} - A_{12}, \quad \lambda_4 = \lambda_5 = \lambda_6 = A_{44},
\]

\[
\mathbf{W}_1 = \frac{\pm \sqrt{3}}{3} (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \frac{\pm \sqrt{3}}{3} \mathbf{E}, \quad \mathbf{W}_2 = \frac{\pm \sqrt{2}}{2} (\mathbf{e}_1 - \mathbf{e}_2),
\]

\[
\mathbf{W}_3 = \frac{\pm \sqrt{6}}{6} (\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3), \quad \mathbf{W}_4 = \mathbf{e}_4, \quad \mathbf{W}_5 = \mathbf{e}_5, \quad \mathbf{W}_6 = \mathbf{e}_6.
\]

Taking into account the expressions for the intrinsic tensors (43), from (42) we get

\[
\mathbf{A} = \frac{1}{3} (\lambda_1 - \lambda_2) \mathbf{C}_1 + (\lambda_2 - \lambda_4) \sum_{k=1}^{3} \mathbf{e}_k \mathbf{e}_k + \lambda_4 \mathbf{E}, \quad \mathbf{E} = \frac{1}{2} (\mathbf{C}_2 + \mathbf{C}_3).
\]

By virtue of (43), for the dispersion tensor \( \mathbf{A} \) we have the expression

\[
\mathbf{A} = \alpha \mathbf{n} + \beta \mathbf{E} + d \sum_{k=1}^{3} n_k^2 \mathbf{e}_k.
\]

Based on (45) easy to find \( I_k (\mathbf{A}) \), \( k = 1,2,3 \), and then using formulas (17), which are also true for \( \mathbf{A} \), to find \( I_k (\mathbf{A}) \), \( k = 2,3 \). In fact, after simple calculations we find

\[
I_1 (\mathbf{A}) = a + 3b + d, \quad I_1 (\mathbf{A}^2) = a(a + 2b) + 2bd + 3b^2 + (2ad + d^2) \frac{3}{3} \sum_{k=1}^{3} n_k^4,
\]

\[
I_2 (\mathbf{A}) = a(a^2 + 3ab + b^2) + 3b^2(b + d) + d[3a(a + 2b) + 3bd] \frac{3}{3} \sum_{k=1}^{3} n_k^4 + d^2(3a + d) \frac{3}{3} \sum_{k=1}^{3} n_k^6,
\]

\[
I_3 (\mathbf{A}) = \frac{1}{2} \left[ 6b^2 + 4ab + 2ad + 4bd + d^2 - d(2a + d) \frac{3}{3} \sum_{k=1}^{3} n_k^4 \right], \quad I_3 (\mathbf{A}) = \frac{1}{3} [6a^2b + 3ad^2 + 3bd + 3d^2 + 6b^2 + 6bd + 6d^2 + 3d^3 - 9ad^2 + 6abd + 3bd^2 + 3d^3] \frac{3}{3} \sum_{k=1}^{3} n_k^6.
\]

Knowing the invariants \( I_k (\mathbf{A}) \), \( k = 1,2,3 \), in view of, for example, the first equation of (19) we make the characteristic equation for the tensor under consideration, which is an algebraic equation of the third degree and always has three positive roots, counting each root as many times as its multiplicity. Thus, in the case under consideration, for any direction, depending on the multiplicity of the roots, one can determine the number of waves and find their propagation velocities. Consider, for example, three mutually perpendicular directions of wave propagation in the medium under consideration, which are determined by the following values of the components of the normal: \( n_1 = \delta_{11} \), \( n_2 = \delta_{11} \), \( n_3 = \delta_{13} \), \( i = 1,2,3 \). Note that for each of these directions the first invariant of the tensor \( \mathbf{A} \) does not depend on the direction \( \mathbf{n} \), and the second and third invariants have the same values. In particular,

\[
I_1 (\mathbf{A}) = a + 3b + d, \quad I_2 (\mathbf{A}) = b^2(2a + 3b + 2d), \quad I_3 (\mathbf{A}) = b^2(a + b + d).
\]

By (47) the dispersion equation for each of these directions is the same and has roots

\[
\mu_1 = \frac{\lambda_1 + 2\lambda_2}{3}, \quad \mu_2 = \mu_3 = \frac{\lambda_4}{2}.
\]

Consequently, in each of these directions, in an infinite medium, in the case of cubic symmetry, two waves arise and on the basis of (48) their velocities are given by formulas

\[
c_1 = \sqrt{(\lambda_1 + 2\lambda_2)/(3\rho)}, \quad c_2 = \sqrt{\lambda_4/(2\rho)}.
\]
5. Classical material with the anisotropy symbol \{1,1,2,2\} (transversal isotropy)

In this case, the canonical representation of the elastic modulus tensor \( \mathbf{A} \) has the form

\[
\mathbf{A} = \mu_1 \mathbf{w}_1 \mathbf{w}_1 + \mu_2 \mathbf{w}_2 \mathbf{w}_2 + \mu_3 (\mathbf{w}_3 \mathbf{w}_3 + \mathbf{w}_4 \mathbf{w}_4) + \mu_5 (\mathbf{w}_5 \mathbf{w}_5 + \mathbf{w}_6 \mathbf{w}_6),
\]

where the eigenvalues are given by formulas

\[
\mu_1 = \frac{1}{2} (A_{11} + A_{12} + A_{33} - \sqrt{(A_{11} + A_{12} - A_{33})^2 + 8A_{13}^2}), \quad \mu_3 = \mu_4 = A_{11} - A_{12},
\]

\[
\mu_2 = \frac{1}{2} (A_{11} + A_{12} + A_{33} + \sqrt{(A_{11} + A_{12} - A_{33})^2 + 8A_{13}^2}), \quad \mu_5 = \mu_6 = A_{55},
\]

and the proper tensors are represented in the form

\[
\begin{align*}
\mathbf{w}_1 &= -\frac{\sqrt{2}}{2} \sin \alpha (\mathbf{e}_1 + \mathbf{e}_2) + \cos \alpha \mathbf{e}_3 = -\frac{\sqrt{2}}{2} \sin \alpha \mathbf{I} + \cos \alpha \mathbf{e}_3, \\
\mathbf{w}_2 &= \frac{\sqrt{2}}{2} \cos \alpha (\mathbf{e}_1 + \mathbf{e}_2) + \sin \alpha \mathbf{e}_3 = \frac{\sqrt{2}}{2} \cos \alpha \mathbf{I} + \sin \alpha \mathbf{e}_3, \\
\mathbf{w}_3 &= \frac{\sqrt{2}}{2} (\mathbf{e}_1 - \mathbf{e}_2), \quad \mathbf{w}_4 = \mathbf{e}_1, \quad \mathbf{w}_5 = \mathbf{e}_5, \quad \mathbf{w}_6 = \mathbf{e}_6, \quad \operatorname{tg} 2 \alpha = \frac{2\sqrt{2}A_{13}}{A_{11} + A_{12} - A_{33}}.
\end{align*}
\]

Note that transversely isotropic materials according to the classification adopted in [4, 5, 9], can be of the following types: \{1,1,2,2\}, \{1,2,1,2\}, \{1,2,2,1\}, \{2,1,1,2\}, \{2,1,2,1\}, \{2,2,1,1\}.

Given (51), the tensor (50) can be written in the form

\[
\mathbf{A} = a_2 \mathbf{C}_{(1)} + (a_1 - a_2) \mathbf{E} + a_3 (\mathbf{e}_3 e_3 + \mathbf{e}_3) + a_4 \mathbf{e}_3 e_3 + \frac{1}{2} a_5 (\mathbf{e}_4 e_3 e_4 e_4 + \mathbf{e}_4 e_4),
\]

\[
\mathbf{C}_{(1)} = \mathbf{I}, \quad \mathbf{E} = \frac{1}{2} (\mathbf{C}_{(2)} + \mathbf{C}_{(3)}) = \mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2 + \mathbf{e}_4 \mathbf{e}_4, \quad \mathbf{I} = \mathbf{e}_1 + \mathbf{e}_2,
\]

\[
\begin{align*}
a_1 &= \frac{1}{2} (\mu_1 \sin^2 \alpha + \mu_2 \cos^2 \alpha + \mu_3), \quad a_2 = \frac{1}{2} (\mu_1 \sin^2 \alpha + \mu_2 \cos^2 \alpha - \mu_3), \\
a_3 &= \frac{\sqrt{2}}{2} (\mu_2 - \mu_1) \sin \alpha \cos \alpha, \quad a_4 = \mu_1 \cos^2 \alpha + \mu_2 \sin^2 \alpha, \quad a_5 = \mu_5.
\end{align*}
\]

The dispersion tensor for (52) after simple calculations is written as follows:

\[
\mathbf{A} = A_1 \mathbf{m} + A_2 \mathbf{I} + A_3 (\mathbf{e}_3 e_3 + \mathbf{e}_3),
\]

\[
\mathbf{m} = n_1 \mathbf{e}_1, \quad A_1 = \frac{1}{\rho} (a_1 + \frac{1}{2} a_2), \quad A_2 = \frac{1}{\rho} (a_3 m^2 + a_5 n_3^2), \quad A_3 = \frac{1}{\rho} (a_5 + a_5) n_3, \quad A_4 = \frac{1}{\rho} (a_4 n_3^2 + a_5 m^2).
\]

Then it's easy to find \( \mathbf{A}^2 \) and \( \mathbf{A}^3 \). By virtue of (54) we have

\[
\begin{align*}
\mathbf{A}^2 &= B_1 \mathbf{m} + B_2 \mathbf{I} + B_3 (\mathbf{e}_3 e_3 + \mathbf{e}_3), \\
\mathbf{A}^3 &= C_1 \mathbf{m} + C_2 \mathbf{I} + C_3 (\mathbf{e}_3 e_3 + \mathbf{e}_3),
\end{align*}
\]

\[
\begin{align*}
B_1 &= A_1^2 m^2 + 2A_2 A_4 + A_3^3, \quad B_2 = A_2^2, \quad B_3 = A_1 A_3 m^2 + A_2 A_3 + A_3 A_4, \\
B_4 &= A_2^2 m^2 + A_3^3, \quad C_1 = A_1 m^4 + (3A_1^2 A_3 + 2A_1 A_3^2) m^2 + 3A_1 A_3^2 + 2A_2 A_3^2 + A_3 A_4, \\
C_3 &= A_1 A_3 m^4 + (2A_1 A_3 A_4 + A_1 A_3 A_4 + A_3 A_4^2) m^2 + A_1 A_3 A_4 + A_3 A_4^2, \\
C_2 &= A_3^3, \quad C_4 = A_1 A_3^2 m^4 + (2A_3 A_4 + A_2 A_3^2) m^2 + A_3 A_4^2.
\end{align*}
\]
From (54) and (56), we find $I_1(\mathbf{A}^k)$, $k = 1, 2, 3$, and, by virtue of (17) we get $I_m(\mathbf{A})$, $m = 2, 3$. We have

$$I_1(\mathbf{A}) = A_1 m^2 + 2A_2 + A_4, \quad I_1(\mathbf{A}^2) = B_1 m^2 + 2B_2 + B_4, \quad I_1(\mathbf{A}^3) = C_1 m^2 + 2C_2 + C_4,$$

$$I_2(\mathbf{A}) = \frac{1}{2} \{ A_1^2 m^4 + 2A_1(2A_2 + A_4) - B_1 m^2 + (2A_2 + A_4)^2 - 2B_2 - B_4 \},$$

$$I_3(\mathbf{A}) = \frac{1}{3!} \{ A_1^3 m^6 + 3A_1^2 (2A_2 + A_4) - A_1 B_1 m^4 + [3A_1 (2A_2 + A_4)^2 - 3A_1 (2B_2 + B_4) - 3B_1 (2A_2 + A_4) + 2C_1] m^2 + (2A_2 + A_4)^3 - 3(2A_2 + A_4)(2B_2 + B_4) + 2(2C_2 + C_4) \}.$$  \hspace{1cm} (58)

Knowing $I_m(\mathbf{A})$, $m = 1, 2, 3$ (see (58)), from the dispersion equation for any direction $\mathbf{n}$ we find the eigenvalues $\mathbf{A}$, and then the wave velocities.

Let us find, for example, the wave velocities in the following directions: $n_i = \delta_{i1}$, $n_i = \delta_{i2}$ and $n_i = \delta_{i3}$, $i = 1, 2, 3$. Note that from the expression for $I_m(\mathbf{A})$, $m = 1, 2, 3$ (see (58)) follows that for directions $n_i = \delta_{i1}$ and $n_i = \delta_{i2}$, $i = 1, 2, 3$, they will accept the same values. In fact, by virtue of the corresponding formulas (55) and (58) for these directions we will have

$$I_1(\mathbf{A}) = \frac{a_1 + a_2}{\rho} + \frac{a_2}{2\rho} + \frac{a_5}{\rho}, \quad I_2(\mathbf{A}) = \frac{a_1 + a_2 a_2}{2\rho} + \frac{a_1 + a_2 a_5}{\rho} + \frac{a_2 a_5}{2\rho},$$

$$I_3(\mathbf{A}) = \frac{a_1 + a_2 a_5}{\rho} + \frac{a_2}{2\rho}.$$  \hspace{1cm} (59)

From (58) we see that the characteristic equation of the tensor $\mathbf{A}$ has roots

$$\mu_1 = \frac{a_1 + a_2}{\rho}, \quad \mu_2 = \frac{a_2}{2\rho}, \quad \mu_3 = \frac{a_5}{\rho}.$$  \hspace{1cm} (60)

Hence, for the wave velocities in the initially resting medium, we have the following values:

$$c_1 = \sqrt{\frac{a_1 + a_2}{\rho}}, \quad c_2 = \sqrt{\frac{a_2}{2\rho}}, \quad c_3 = \sqrt{\frac{a_5}{\rho}}.$$  \hspace{1cm} (61)

For the direction $n_i = \delta_{i3}$, $i = 1, 2, 3$, similar to (59) – (61), the relations have the form

$$I_1(\mathbf{A}) = \frac{a_4}{\rho} + \frac{2a_5}{\rho}, \quad I_2(\mathbf{A}) = 2 \frac{a_4 a_5}{\rho} + \left( \frac{a_5}{\rho} \right)^2, \quad I_3(\mathbf{A}) = \frac{a_4}{\rho} \left( \frac{a_5}{\rho} \right)^2,$$

$$\mu_1 = \frac{a_4}{\rho}, \quad \mu_2 = \frac{a_5}{\rho}, \quad c_1 = \sqrt{\frac{a_4}{\rho}}, \quad c_2 = \sqrt{\frac{a_5}{\rho}}.$$  \hspace{1cm}

Note that the consideration of the material with the symbol of the structure \{1,1,2,2\} of the trigonal syngonies (6 essential components) reduces to the previous case.

6. Micro-polar material with a center of symmetry and the anisotropy symbol \{1,5,3\}

In this case, the properties of the medium are characterized by two tensors of the 4th rank $\mathbf{A}$ and $\mathbf{D}$, which have the same structure. Therefore, it suffices to consider one of these tensors, since all the relations obtained for one tensor are obtained in a completely analogous way for the other (see (19)). Consider, for example, the tensor $\mathbf{A}$. In this case, its canonical representation has the form

$$\mathbf{A} = (\lambda_1 - \lambda_2) \mathbf{u}_1 \mathbf{u}_1 + \lambda_2 \mathbf{E} + (\lambda_7 - \lambda_2) \sum_{m=7}^9 \mathbf{u}_m \mathbf{u}_m, \quad \mathbf{E} = C(2) = \sum_{m=1}^9 \mathbf{u}_m \mathbf{u}_m.$$  \hspace{1cm} (62)
By virtue of (62), it’s easy to find \( I_1(\mathbf{A}^k) \), \( k = 1, 2, 3 \), and then by virtue of (17) get \( I_m(\mathbf{A}) \), \( m = 2, 3 \). Knowing \( I_m(\mathbf{A}) \), \( m = 1, 2, 3 \), we can write the characteristic equation \( \mathbf{A} \) and find its roots, and then determine the desired velocity of the waves. Formulas are cumbersome, but the main thing is that they can be obtained in an explicit form. Note that in the case of the medium \{5,1,3\} a similar consideration of \{1,5,3\}. Therefore, in the general case we will not stop. Next we consider the case of a micro-polar isotropic elastic medium with a center of symmetry and the symbol of structure \{1,5,3\}. In this case, the tensors \( \mathbf{A} \) and \( \mathbf{D} \) have representations

\[
\mathbf{A} = a_1 \mathbf{C}(1) + a_2 \mathbf{C}(2) + a_3 \mathbf{C}(3), \quad \mathbf{D} = d_1 \mathbf{C}(1) + d_2 \mathbf{C}(2) + d_3 \mathbf{C}(3),
\]

(63)

where the coefficients \( a_i, d_i, i = 1, 2 \), are determined in terms of the eigenvalues of these tensors by the formulas:

\[
a_1 = \frac{1}{3}(\lambda_1 - \lambda_2), \quad a_2 = \frac{1}{2}(\lambda_2 + \lambda_7), \quad a_3 = \frac{1}{2}(\lambda_2 - \lambda_7),
\]

\[
d_1 = \frac{1}{3}(\mu_1 - \mu_2), \quad d_2 = \frac{1}{2}(\mu_2 + \mu_7), \quad d_3 = \frac{1}{2}(\mu_2 - \mu_7).
\]

It is easy to see that in the present case the dispersion tensors \( \mathbf{A} \) and \( \mathbf{D} \), corresponding to tensors \( \mathbf{\bar{A}} \) and \( \mathbf{\bar{D}} \) of (63) respectively have the form

\[
\mathbf{A} = \frac{1}{\rho}[(a_1 + a_3)\mathbf{n}\mathbf{n} + a_2\mathbf{E}], \quad \mathbf{D} = \frac{1}{\rho}[(d_1 + d_3)\mathbf{n}\mathbf{n} + d_2\mathbf{E}].
\]

(64)

From (64) we see that \( \mathbf{A} \) and \( \mathbf{D} \) are similar to the tensors considered above from (24). Introducing the notation

\[
a = \frac{a_2}{\rho}, \quad b = \frac{a_1 + a_3}{\rho}, \quad f = \frac{d_2}{\rho}, \quad g = \frac{d_1 + d_3}{\rho},
\]

the tensors (64) can be written as follows:

\[
\mathbf{A} = a\mathbf{E} + b\mathbf{n}\mathbf{n}, \quad \mathbf{D} = f\mathbf{E} + g\mathbf{nn}.
\]

(65)

Then, similarly to (31) and (32), the roots of the dispersion equations of the tensors \( \mathbf{A} \) and \( \mathbf{D} \) and the wave velocities in the initially at rest media are determined by the formulas

\[
\eta_1 = \eta_2 = a = \frac{\lambda_2 + \lambda_7}{2\rho}, \quad \eta_3 = a + b = \frac{\lambda_1 + 2\lambda_2}{3\rho} = \frac{\lambda + 2\mu}{\rho},
\]

\[
\eta_4 = \eta_5 = f = \frac{\mu_2 + \mu_7}{2\rho}, \quad \eta_6 = f + g = \frac{\mu_1 + 2\mu_2}{3\rho},
\]

\[
c_1 = \sqrt{\frac{\lambda_2 + \lambda_7}{2\rho}}, \quad c_2 = \sqrt{\frac{\lambda_1 + 2\lambda_2}{3\rho}}, \quad c_3 = \sqrt{\frac{\mu_2 + \mu_7}{2\rho}}, \quad c_4 = \sqrt{\frac{\mu_1 + 2\mu_2}{3\rho}}.
\]

(66)

(67)

Thus, in a micro-polar isotropic, infinite, initially resting medium, four waves appear in each direction, the velocities of which are calculated from formulas (67).

Note that to determine the direction of propagation of these waves it is necessary to find a complete system of eigenvectors and give a canonical representation of the dispersion tensors. Then, carry out the investigation in the same way as was done above in the case of a classical isotropic material. Such an investigation can always be carried out also in the case of an anisotropy of a more general nature.

The approach proposed in this article was developed in the articles of the authors, and the results obtained were presented in [21], [22].

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References
[1] Vekua I N 1978, Fundamentals of tensor analysis and covariant theory. Nauka, Moscow (in Russian)
[2] Pobedrya B E 1986, Lectures in tensor analysis. Moscow State Univ., Moscow, (in Russian).
[3] Lur’e A I 1980, Nonlinear elasticity. Nauka, Moscow, (in Russian).
[4] Nikabadze M U 2017, "Topics on tensor calculus with applications to mechanics", J. Math. Sci., 225:1, 194 pp. DOI: 10.1007/s10958-017-3467-4
[5] Nikabadze M U 2015, On Several Issues of Tensor Calculus with Applications to Mechanics. Contemporary Mathematics. Fundamental Directions. 55, 3–194, (in Russian).
http://istina.msu.ru/media/publications/book/e25/00c/10117043/M.U.Nikabadze.pdf
[6] Pobedrya B E 1995, Numerical methods in the theory of elasticity and plasticity. 2nd ed. Moscow State Univ., Moscow, 366 p. (in Russian).
[7] Nikabadze M U 2014, "Construction of Eigentensor Columns in the Linear Micropolar Theory of Elasticity", Moscow Univ. Mech. Bull. 69:1, 1–9. DOI: 10.3103/S0027133014010014
[8] Nikabadze M U 2014, Development of the Method of Orthogonal Polynomials in the Classical and Micropolar Mechanics of Elastic Thin Bodies. Moscow University Press, 515 p. (in Russian).
http://istina.msu.ru/media/publications/book/707/ea1/6738800/Monographiya.pdf
[9] Nikabadze M U 2016, Eigenvalue Problems of a Tensor and a Tensor-Block Matrix (TMB) of Any Even Rank with Some Applications in Mechanics, 279–317, in: Generalized Continua as Models for Classical and Advanced Materials, Advanced Structured Materials, Vol. 42, Eds.: H. Altenbach and S. Forest, Springer. DOI 10.1007/978-3-319-31721-2_14
[10] Mikhail U. Nikabadze 2017, "To the problem of decomposition of the initial boundary value problems in mechanics", J. Phys.: Conf. Series, 739:1, 11 p. DOI 10.1088/1742-6596/936/1/012056
[11] Kupradze V D, Gegelia T G, Basheleishvili M O, and Burchuladze T V 1976, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. Nauka, Moscow, 664 p. (in Russian).
[12] Eringen A C 1999, Microcontinuum Field Theories. 1. Foundation and solids. Springer-Verlag. 341 p.
[13] Nowacki W 1975, Theory of Elasticity. Moscow, Mir. 872 p.
[14] Baskakov V A, Bestuzheva N P and Konchakova N A 2001, Linear dynamic theory of thermoelastic medium with microstructure. Voronezh. Voronezh University Press. 162 p.
[15] Nikabadze M U and Ulukhanyan A R 2016, Analytical Solutions in the Theory of Thin Bodies, 319–361, in: Generalized Continua as Models for Classical and Advanced Materials, Advanced Structured Materials, Vol. 42, Eds.: H. Altenbach and S. Forest, Springer. DOI 10.1007/978–3–9381721–2_15
[16] Petraschen G I 1978, Foundations of the mathematical theory of elastic wave propagation. Questions of the dynamic theory of propagation of seismic waves. 18. Leningrad: Nauka, 248 p. (in Russian).
[17] Petraschen G I 1980, Propagation of waves in anisotropic elastic media. Leningrad: Nauka, 280 p. (in Russian).
[18] Sagomonyan A Ya 1985, Stress waves in continuous media., Moscow Univ.;, 416 p. (in Russian).
[19] Poruchikov V B 1986, Methods of dynamic elasticity theory. Moscow: Nauka, 328 p. (in Russian).
[20] Nikabadze M U 2007, Some issues concerning a version of the theory of thin solids based on expansions in a system of Chebyshev polynomials of the second kind. Mech. Solids 42:3, 391–421.
[21] Hovik Matevossian, Mikhail Nikabadze, Armine Ulukhanian 2017, On solutions of biharmonic problems, Mathematical and Numerical Aspects of Dynamical Systems Analysis. DSTA’2017: Abstracts. 14th International Conference on ”Dynamical Systems – Theory and Applications” (Lodz, Poland, December 11-14, 2017), 369–380. ISBN 978-83-935312-6-4
[22] Hovik A. Matevossian, Mikhail U. Nikabadze and Armine R. Ulukhanian 2018, On solutions of some biharmonic problems and their applications, IJNNS Int. J. Nonlin. Sci. Numer. Simul., 10 p. (to appear).