Research Article

The Global Weak Solution for a Generalized Camassa-Holm Equation

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Received 25 October 2012; Accepted 24 December 2012

Academic Editor: Yong Hong Wu

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An online generalization of the famous Camassa-Holm model is investigated. Provided that initial value \( u_0 \in H^s_\infty \) and \((1 - \delta^2_\text{u})u_0\) satisfies an associated sign condition, it is shown that there exists a unique global weak solution to the equation in space \( H^s_\infty \) in the sense of distribution, and \( u_x \in L^\infty([0, +\infty) \times R) \).

1. Introduction

In recent years, a lot of works have been carried out to investigate the Camassa-Holm equation [1],

\[
u_t - u_{xxx} + ku_x + 3uu_x = 2uu_x + uu_{xx},
\]

where \( m \geq 0 \) is a natural number. Obviously, (2) reduces to (1) if \( m = 0 \). The authors applied the pseudoparabolic regularization technique to build the local well-posedness for (2) in Sobolev space \( H^s(R) \) with \( s > 3/2 \) via a limiting procedure. Provided that the initial value \( u_0 \) satisfies a sign condition and \( u_0 \in H^s(R) \), it is shown that there exists a unique global strong solution for (2) in space \( C([0, \infty); H^s(R)) \cap C^i([0, \infty); H^{s+i}(R)) \). However, the existence and uniqueness of the global weak solution for (2) is not investigated in [20].

The objective of this paper is to establish the well-posedness of global weak solutions for (2). Using the estimates in \( H^q(R) \) with \( 0 \leq q \leq 1/2 \), which are derived from the equation itself, we prove that there exists a unique global weak solution to (2) in space \( H^s(R) \) with \( 1 \leq s \leq 3/2 \) if \( u_0 \in H^s(R) \), and \((1 - \delta^2_\text{x})u_0\) satisfies an associated sign condition.

Section 2 gives some notations. Firstly, we give some notations.

The space of all infinitely differentiable functions \( \phi(t, x) \) with compact support in \([0, +\infty) \times R\) is denoted by \( C^\infty_0 \).

\[
L^p = L^p(R) \quad (1 \leq p < +\infty)
\]

is the space of all measurable functions \( h \) such that \( \|h\|_{L^p}^p = \int_R |h(t, x)|^p dx < \infty \). We define \( L^\infty = L^\infty(R) \) with the standard norm.
\[ \|h\|_{L^\infty} = \inf_{\varepsilon > 0} \sup_{x \in R} |h(t, x)|. \] For any real number \( s \), we let \( H^s = H^s(R) \) denote the Sobolev space with the norm defined by
\[ \|h\|_{H^s} = \left( \int_R \left( 1 + |\xi|^2 \right)^s |\hat{h}(t, \xi)|^2 \, d\xi \right)^{1/2} < \infty, \] (3)
where \( \hat{h}(t, \xi) = \int_R e^{-ix\xi} h(t, x) \, dx \).

For \( T > 0 \) and nonnegative number \( s \), let \( C([0, T]; H^s(R)) \) denote the Frechet space of all continuous \( H^s \)-valued functions on \([0, T]\). We set \( \Lambda = (1 - \partial_x^2)^{1/2} \).

Defining \( \phi(x) = \begin{cases} e^{1/(2x^2)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \) (4)
and letting \( \phi_\varepsilon(x) = e^{-(1/\varepsilon)} \phi(\varepsilon^{1/4} x) \) with \( 0 < \varepsilon < 1/4 \) and \( u_{\varepsilon 0} = \phi_\varepsilon * u_0 \) (convolution of \( \phi_\varepsilon \) and \( u_0 \)), we know that \( u_{\varepsilon 0} \in C^\infty \) for any \( u_0 \in H^s \) with \( s > 0 \). Notation \((1 - \partial_x^2)u + k/2(m+1) \in N^s(R)\) (or equivalently \( (1 - \partial_x^2)^{1/2} u + k/2(m+1) \in N^{s/2}(R)\)) means that \((1 - \partial_x^2)^{1/2} u + \phi_\varepsilon + k/2(m+1) \geq 0 \) (or equivalently \( (1 - \partial_x^2)^{1/2} u + \phi_\varepsilon + k/2(m+1) \leq 0 \)) for an arbitrary sufficiently small \( \varepsilon > 0 \).

For the equivalent form of (2), we consider its Cauchy problem
\[ u_t - u_{txx} = -\frac{k}{m+1} (u^{m+1})_x - \frac{m+3}{m+2} (u^{m+2})_x + \frac{1}{m+2} \partial_x^3 (u^m)_x - (m+1) \partial_x (u^m u_x^2) \] (5)
and
\[ u(0,x) = u_0(x). \]

**Definition 1.** A function \( u(t,x) \in L^2([0,\infty), H^s(R)) \) is called a global weak solution to problem (5) if for every \( T > 0 \), \( u(t,x) \in H^s(R) \), \( u(t,x) \in H^{s-1}(R) \), and all \( \psi(t,x) \in C^\infty \), it holds that
\[ \int_0^T \int_R \left[ u_t - u_{txx} + ku^m u_x + (m+3) u^{m+1} u_x - (m+2) u^m u_x u_{xx} - u^{m+1} u_{xxxx} \right] \psi(t,x) \, dx \, dt = 0 \] (6)
with \( u(0,x) = u_0(x) \).

Now, we give the main result of this work.

**Theorem 2.** Let \( u_0(x) \in H^s(R), 1 \leq s \leq 3/2, (1 - \partial_x^2) u_0 + k/2(m+1) \in N^s(R) \), and \( k \geq 0 \) (or equivalently \( (1 - \partial_x^2)^{1/2} u_0 + k/2(m+1) \in N^{s/2}(R) \), \( k \leq 0 \)). Then, problem (5) has a unique global weak solution \( u(t,x) \in L^2([0,\infty), H^s(R)) \) in the sense of distribution, and \( u_x \in L^\infty([0,\infty) \times R) \).

**3. Several Lemmas**

**Lemma 3** (see [20]). Let \( u_0(x) \in H^s(R) \) with \( s > 3/2 \). Then, the Cauchy problem (5) has a unique solution
\[ u(t,x) \in C \left( [0,T) ; H^s(R) \right) \bigcap C^1 \left( [0,T) ; H^{s-1}(R) \right), \] (7)
where \( T > 0 \) depends on \( \|u_0\|_{H^s(R)} \).

**Lemma 4** (see [20]). Let \( u_0(x) \in H^s, s > 3/2, \) and \( k \geq 0, (1 - \partial_x^2) u_0 + k/2(m+1) \geq 0 \) (or equivalently \( k \leq 0, (1 - \partial_x^2)^{1/2} u_0 + k/2(m+1) \leq 0 \)). Then, problem (5) has a unique solution satisfying
\[ u(t,x) \in C \left( [0,\infty), H^s(R) \right) \bigcap C^1 \left( [0,\infty) ; H^{s-1}(R) \right). \] (8)

Using the first equation of system (5) derives
\[ \frac{d}{dt} \int_R \left( u^2 + u_x^2 \right) \, dx = 0, \] (9)
from which one has the conservation law
\[ \int_R \left( u^2 + u_x^2 \right) \, dx = \int_R \left( u_0^2 + u_{\varepsilon 0 x}^2 \right) \, dx. \] (10)

**Lemma 5** (see [20]). Let \( s > 3/2 \), and the function \( u(t,x) \) is a solution of problem (5) and the initial data \( u_0(x) \in H^s \). Then, the following inequality holds:
\[ \|u\|_{H^s}^2 \leq \int_R \left( u^2 + u_x^2 \right) \, dx = \int_R \left( u_0^2 + u_{\varepsilon 0 x}^2 \right) \, dx. \] (11)

For \( q \in (0, s - 1) \), there is a constant \( c \) such that
\[ \int_R \left( A^{q+1} u \right)^2 \, dx \leq \int_R \left( A^{q+1} u_0 \right)^2 \, dx + c \int_0^t \left( \|u\|_{L^{2m+1}} \right)^m \, d\tau. \] (12)

For \( q \in [0, s - 1] \), there is a constant \( c \) such that
\[ \|u\|_{L^{2m+1}} \leq c \|u\|_{L^{2m+1}} \left( \|u\|_{L^m}^{m+1} + \|u\|_{L^\infty}^{m+1} \right)^2 + \|u\|_{L^\infty} \|u_x\|_{L^\infty}^2. \] (13)

For (2), consider the problem
\[ p_t = u^{m+1} (t, p), \quad t \in [0,T), \quad p(0,x) = x. \] (14)

**Lemma 6** (see [20]). Let \( u_0 \in H^s, s \geq 3, \) and let \( T > 0 \) be the maximal existence time of the solution to problem (5). Then, problem (14) has a unique solution \( p \in C^1([0,T) \times R) \). Moreover, the map \( p(t, \cdot) \) is an increasing diffeomorphism of \( R \) with \( p_x(t,x) > 0 \) for \( (t,x) \in [0,T) \times R \).
Differentiating (14) with respect to $x$ yields
\[
\frac{d}{dt} p_x = (m+1) u^{m_u} p_x (t, p_x), \quad t \in [0, T),
\]
(15)
which leads to
\[
p_x (t, x) = \exp \left( \int_0^t (m+1) u^{m_u} (\tau, p (\tau, x)) d\tau \right).
\]
(16)

The next lemma is reminiscent of a strong invariance property of the Camassa-Holm equation (the conservation of momentum [21]).

Lemma 7 (see [20]). Let $u_0 \in C^2$ with $s \geq 3$, and let $T > 0$ be the maximal existence time of the problem (5). It holds that
\[
y (t, p (t, x)) = y_0 (x) e^{\int_0^t m_u u_x d\tau},
\]
(17)
where $(t, x) \in [0, T) \times R$ and $y := u - u_{xx} + k/2(m+1)$.

Lemma 8. If $u_0 \in C^3$, $s \geq 3$, such that $(1-\partial_x^2)u_0 + k/2(m+1) \geq 0$, $k \geq 0$ (or equivalently, $(1-\partial_x^2)u_0 + k/2(m+1) \leq 0$, $k \leq 0$), then the solution of problem (5) satisfies
\[
\|u_x\|_{C^1} \leq \|u_0\|_{C^1} + \frac{|k|}{2(m+1)} \leq c.
\]
(18)

Proof. Using $u_0 - u_0^{xx} + k/2(m+1) \geq 0$, it follows from Lemma 7 that $u - u_{xx} + k/2(m+1) \geq 0$. Letting $Y_1 = u - u_{xx}$, we have
\[
u = \frac{1}{2} e^{-x} \int_0^x e^{\eta Y_1 (t, \eta)} d\eta + \frac{1}{2} e^{x} \int_0^\infty e^{-\eta Y_1 (t, \eta)} d\eta,
\]
(19)
from which we obtain
\[
\partial_x u (t, x)
= -\frac{1}{2} \left( e^{-x} \int_0^x e^{\eta Y_1 (t, \eta)} d\eta + e^{x} \int_x^\infty e^{-\eta Y_1 (t, \eta)} d\eta \right)
+ e^{x} \int_0^\infty e^{\eta Y_1 (t, \eta)} d\eta
= -u (t, x) + e^{x} \int_0^\infty e^{-\eta Y_1 (t, \eta)} d\eta
\]
\[
= -u (t, x) + e^{x} \int_0^\infty e^{-\eta} \left( Y_1 (t, \eta) + \frac{k}{2(m+1)} \right) d\eta
- \frac{k}{2(m+1)} e^{x} \int_0^\infty e^{-\eta} d\eta
\]
\[
= -u (t, x) + e^{x} \int_0^\infty e^{-\eta} (y (t, \eta)) d\eta - \frac{k}{2(m+1)}
\geq -u (t, x) - \frac{k}{2(m+1)},
\]
(20)

On the other hand, we have
\[
\partial_x u (t, x)
= \frac{1}{2} \left( e^{-x} \int_0^x e^{\eta Y_1 (t, \eta)} d\eta + e^{x} \int_0^\infty e^{-\eta Y_1 (t, \eta)} d\eta \right)
- e^{-x} \int_0^x e^{\eta Y_1 (t, \eta)} d\eta
= u (t, x) - e^{-x} \int_0^x e^{\eta Y_1 (t, \eta)} d\eta
= u (t, x) - e^{-x} \int_0^x e^{\eta} \left( Y_1 (t, \eta) + \frac{k}{2(m+1)} \right) d\eta
+ \frac{k}{2(m+1)} e^{x} \int_0^\infty e^{\eta} d\eta
\]
\[
= u (t, x) - e^{-x} \int_0^x e^{\eta} y (t, \eta) d\eta + \frac{k}{2(m+1)}
\leq u (t, x) + \frac{k}{2(m+1)}.
\]
(21)

The inequalities (19), (20), and (21) derive that inequality (18) is valid. Similarly, if $(1-\partial_x^2)u_0 + k/2(m+1) \leq 0$, $k \leq 0$, we still know that (18) is valid.

Lemma 9. For $s > 0$, $u_0 \in C^3$, it holds that
\[
\|u_{\epsilon x}\|_{L^\infty} \leq c\|u_{0 x}\|_{L^\infty} + \frac{|k|}{2(m+1)} \leq c,
\]
\[
\|u_{\epsilon}\|_{H^s L^t} \leq c, \quad \text{if } q \leq s,
\]
\[
\|u_{\epsilon x}\|_{H^s L^t} \leq c e^{-q/4}, \quad \text{if } q > s,
\]
\[
\|u_{\epsilon x} - u_{0 x}\|_{H^s L^t} \leq c e^{-q/4}, \quad \text{if } q \leq s,
\]
\[
\|u_{\epsilon x} - u_{0 x}\|_{H^s L^t} = o (1),
\]
(22)
where $c$ is a constant independent of $\epsilon$.

The proof of this lemma can be found in Lai and Wu [15].

From Lemma 3, it derives that the Cauchy problem
\[
u_t - u_{t xx} = -m + \frac{3}{m+2} (u^{m+2})_x + \frac{1}{m+2} \partial_x (u^{m+2})
- (m+1) \partial_x (u^{m+2})_x + u^m u_{xx}
\]
(23)
has a unique solution $u$ depending on the parameter $x$. We write $u_{\epsilon x} (t, x)$ to represent the solution of problem (23). Using Lemma 3 derives that $u_{\epsilon x} (t, x) \in C^0 ([0, T), H^\infty (R))$ since $u_{\epsilon x} (x) \in C^0 (R)$. 

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Lemma 10. Provided that \( u_0 \in H^s \), \( 1 \leq s \leq 3/2 \), \( k \geq 0 \), and 
\((1 - \partial_x^2)u_0 + k/(m+1) \in N^\gamma(R) \) (or equivalently \( (1 - \partial_x^2)u_0 + k/(m+1) \in N^\gamma(R) \), \( k \leq 0 \)), then there exists a constant \( c_0 > 0 \) independent of \( \varepsilon \) such that the solution of problem (23) satisfies

\[
\left\| u_{\varepsilon t} \right\|_{L^\infty} \leq \left\| u_0 \right\|_{H^s} + \frac{|k|}{2(m+1)} \leq c_0. \tag{24}
\]

Proof. Using identity (10) and Lemma 9, if \( u_0 \in H^s(R) \) with \( 1 \leq s \leq 3/2 \), we have

\[
\left\| u_t \right\|_{L^\infty} \leq \left\| u_0 \right\|_{H^s} \leq c, \tag{25}
\]

where \( c \) is independent of \( \varepsilon \).

From Lemma 8, we have

\[
\left\| u_{\varepsilon x} \right\|_{L^\infty} \leq \left\| u_0 \right\|_{H^s} + \frac{|k|}{2(m+1)} \leq c + \frac{|k|}{2(m+1)}, \tag{26}
\]

which completes the proof. \( \square \)

Lemma 11. For any \( f_1, f_2 \in L^\infty \), \( f_2 \in H^z \) with \( z \leq 0 \), it holds that

\[
\left\| f_1 f_2 \right\|_{H^z} \leq c \left\| f_1 \right\|_{L^\infty} \left\| f_2 \right\|_{H^z} \quad \text{for any } z \leq 0. \tag{27}
\]

The proof of this lemma can be found in [15].

4. Existence and Uniqueness of Global Weak Solution

Provided that \( 1 \leq s \leq 3/2 \), for problem (23), applying Lemmas 5, 9, and 10, and the Gronwall’s inequality, we obtain the inequalities

\[
\left\| u_t \right\|_{H^s} \leq \left\| u_0 \right\|_{H^s} \leq c, \tag{28}
\]

\[
\left\| u_x \right\|_{H^s} \leq c \left\| u_0 \right\|_{H^s} \exp \left[ \int_0^t \left( \left\| u_x \right\|_{L^\infty} + \left\| u_{xx} \right\|_{L^\infty} \right) d\tau \right] \leq ce^d, \tag{29}
\]

\[
\left\| u_{xx} \right\|_{H^{s-1}} \leq \left\| u_t \right\|_{H^s} \leq \left( 1 + e^{d+e^c} \right) \leq \left( 1 + e^c \right), \tag{30}
\]

where \( q \in (0, s) \), \( r \in [0, s-1] \), and \( c \) is a constant independent of \( \varepsilon \).

It follows from the Aubin’s compactness theorem that there is a subsequence of \( \{u_n\} \), denoted by \( \{u_{e_n}\} \), such that \( \{u_{e_n}\} \) and their temporal derivatives \( \{u_{e_n}\} \) are weakly convergent to a function \( u(t, x) \) and its derivative \( u_t \) in \( L^2([0, T], H^s) \) and \( L^2([0, T], H^{s-1}) \), respectively, where \( T \) is an arbitrary fixed positive number. Moreover, for any real number \( R_1 > 0 \), \( \{u_{e_n}\} \) is convergent to the function \( u \) strongly in the space \( L^2([0, T], H^s(-R_1, R_1)) \) for \( q \in (0, s) \) and \( \{u_{e_n}\} \) converges to \( u_t \) strongly in the space \( L^2([0, T], H^s(-R_1, R_1)) \) for \( r \in [0, s-1] \).

4.1. The Proof of Existence for Global Weak Solution. For an arbitrary fixed \( T > 0 \), from Lemma 10, we know that \( \{u_{e_n}\}(\varepsilon_n \to 0) \) is bounded in the space \( L^\infty \). Thus, the sequences \( \{u_{e_n}\}, \{u_{e_nx}\}, \{u_{e_nxx}\}, \) and \( \{u_{e_nxxx}\} \) are weakly convergent to \( u, u_x, u_{xx}, \) and \( u_{xxx} \) in \( L^2([0, T], H^s(-R_1, R_1)) \) for any \( r \in [0, s-1] \), separately. Using \( u_t(u_{xx}^2) = (u^m u_x^2)_x - (u^m)_x u_{xx}^2 \), we know that \( u_t \) satisfies the equation

\[
- \int_0^T \int_0^T u_t (g_x - g_{xxx}) \, dx \, dt = \int_0^T \int_R^T \left( \frac{m+3}{m+2} u_{mm}^2 + \frac{m+1}{2} u_{mm}^2 \right) g_x \, dx \, dt - \frac{1}{m+2} u_{mm}^2 g_{xxx} - \frac{1}{2} u_{mm}^2 g_x
\]

\[
- \frac{m}{2} u_{mm}^2 u_{xx}^2 \right) \, dx \, dt,
\]

with \( u(0, x) = u_0(x) \) and \( g \in C_0^\infty \). Since \( X = L^1([0, T] \times R) \) is a separable Banach space and \( \{u_{e_n}\} \) is a bounded sequence in the dual space \( X^* = L^\infty([0, T] \times R) \) of \( X \), there exists a subsequence of \( \{u_{e_n}\} \), still denoted by \( \{u_{e_n}\} \), weakly star convergent to a function \( u \) in \( L^\infty([0, T] \times R) \). As \( \{u_{e_n}\} \) weakly converges in \( L^2([0, T] \times R) \), it results that \( u_x = \nu \) almost everywhere. Thus, we obtain \( u_x \in L^\infty([0, T] \times R) \). Since \( T > 0 \) is an arbitrary number, we complete the global existence of weak solutions to problem (5).

Proof of Uniqueness. Suppose that there exist two global weak solutions \( u(t, x) \) and \( v(t, x) \) to problem (5) with the same initial value \( u_0(x) \in H^s(R) \), \( 1 \leq s \leq 3/2 \), we consider its associated regularized problem (23). Letting \( w_{e_n} = u_{e_n}(t, x) - v_{e_n}(t, x) \), from Lemma 10, we get \( \left\| \partial u_{e_n(t,x)}/\partial x \right\|_{L^\infty} \leq c \) and \( \left\| \partial w_{e_n(t,x)}/\partial x \right\|_{L^\infty} \leq c \) which is independent of \( \varepsilon \). Still denoting \( u = u_{e_n}, v = v_{e_n}, \) and \( w = w_{e_n}, \) it holds that

\[
\omega_t = (\partial_x^2)^{-1} [-\partial_x (u^{mm} - v^{mm}) - \partial_x (u^{mm}) \partial_x w + \partial_x (u^{mm} - v^{mm}) \partial_x v + [u^{mm} u_{xx} - u^{mm} v_{xx}]],
\]

\[
\omega(0, x) = 0.
\]
Multiplying both sides of (30) by \( w \), we get
\[
\frac{1}{2} \frac{d}{dt} \int_R w^2 \, dx \leq c \left| \int_R w \left( u^{m+2} - u^{m+2} \right)_x \, dx \right|
+ \int_R w \Lambda^{-2} \left( u^{m+2} - u^{m+2} \right)_x \, dx
+ \int_R w \Lambda^{-2} \left[ \partial_x \left( u^{m+1} \right) \partial_x w \right] \, dx
+ \int_R w \Lambda^{-2} \left[ \partial_x \left( u^{m+1} \right) \partial_x v \right] \, dx
+ \int_R w \Lambda^{-2} \left[ u^m u_{xx} - u^m v_x v_x \right] \, dx
= I_1 + I_2 + I_3 + I_4 + I_5.
\]
(31)

Using \( \|u\|_{L^\infty} \leq c \), \( \|v\|_{L^\infty} \leq c \), \( \|u_x\|_{L^\infty} \leq c \), \( \|v_x\|_{L^\infty} \leq c \), we have
\[
I_1 \leq c \int_R \left[ w \sum_{j=0}^{m+1} u^j u^{m+1-j} \right] \cdot \left( u^{m+1} \right)_x \, dx
= \int_R \left[ w \sum_{j=0}^{m+1} u^j u^{m+1-j} \right] \cdot \left( u^{m+1} \right)_x \, dx
= \int_R \left( \frac{1}{2} \right) \left( u^{m+1} \right)_x \sum_{j=0}^{m+1} u^j u^{m+1-j} \, dx
= \int_R \left( \frac{1}{2} \right) \left( u^{m+1} \right)_x \sum_{j=0}^{m+1} u^j u^{m+1-j} \, dx
\leq c \|w\|_{L^2}^2 \sum_{j=0}^{m+1} \left( \left( u^j u^{m+1-j} \right) \right)_{L^\infty}
\leq c \|w\|_{L^2}^2.
\]
(32)

Applying Lemma 11 repeatedly, we have
\[
I_2 \leq c \|w\|_{L^2}^2 \left( \Lambda^{-2} \left( u^{m+2} - u^{m+2} \right)_x \right)_{L^2}
\leq c \|w\|_{L^2}^2 \left( \sum_{j=0}^{m+1} u^j u^{m+1-j} \right)_{L^2}
\leq c \|w\|_{L^2}^2 \sum_{j=0}^{m+1} \|u^j u^{m+1-j}\|_{L^\infty}^2
\leq c \|w\|_{L^2}^2.
\]

For \( I_5 \), using Lemma 11 derives
\[
I_5 \leq c \|w\|_{L^2} \left( \left( u^m - u^m \right) \left( v_x^2 \right) \right) + u^m \left[ u_x^2 - v_x^2 \right]_{H^{-1}}
\leq c \|w\|_{L^2} \left( \left( u^m - u^m \right) \left( v_x^2 \right) \right)_{H^{-1}} + u^m \left[ u_x^2 - v_x^2 \right]_{H^{-1}}
\leq c \|w\|_{L^2} \left( \left( u^m - u^m \right) \left( v_x^2 \right) \right)_{H^{-1}} + u^m \left[ u_x^2 - v_x^2 \right]_{H^{-1}}
\leq c \|w\|_{L^2} \left( \left( u^m - u^m \right) \left( v_x^2 \right) \right)_{H^{-1}} + u^m \left[ u_x^2 - v_x^2 \right]_{H^{-1}}
\leq c \|w\|_{L^2} \left( \left( u^m - u^m \right) \left( v_x^2 \right) \right)_{H^{-1}} + u^m \left[ u_x^2 - v_x^2 \right]_{H^{-1}}
\leq c \|w\|_{L^2} \left( \left( u^m - u^m \right) \left( v_x^2 \right) \right)_{H^{-1}} + u^m \left[ u_x^2 - v_x^2 \right]_{H^{-1}}
\leq c \|w\|_{L^2}.
\]
(34)

Using (32)--(34), we get
\[
\frac{1}{2} \frac{d}{dt} \int_R w^2 \, dx \leq c \|w\|_{L^2}^2.
\]
(35)

Applying \( w(0) = 0 \) results in \( \|w\|_{L^2}^2 = 0 \). Consequently, we know that the global weak solution is unique.

\[ \square \]

**Acknowledgment**

This work is supported by the Fundamental Research Funds for the Central Universities (JBK120504).
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