Tetraquark-adequate formulation of QCD sum rules

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We study details of QCD sum rules à la Shifman-Vainshtein-Zakharov for exotic tetraquark states. We point out that duality relations for correlators involving exotic currents have fundamental differences compared with the duality relations for the correlators of bilinear quark currents: namely, the $O(1)$ and $O(\alpha_s)$ terms in the OPE for the exotic correlators exactly cancel against the contributions of the two-meson states on the phenomenological side of QCD sum rules. As a result, the tetraquark properties turn out to be related to the specific non-factorizable parts of the OPE for the exotic Green functions; the relevant non-factorizable diagrams start at order $O(\alpha_s^2)$. Moreover, we show that all appropriate diagrams may be easily obtained from those Feynman diagrams for the four-point function of bilinear quark currents which contain four-quark $s$-channel singularities.

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1. MOTIVATION

Motivated by the increasing experimental evidence for narrow near-threshold hadron resonances with a favourable interpretation as tetraquark and pentaquark hadrons (i.e., hadrons with minimal parton configurations consisting of four and five quarks, respectively) \cite{1,2}, extensive theoretical studies of such objects have been carried out. This letter focuses on the subtleties of the description of tetraquark mesons within the method of Shifman-Vainshtein-Zakharov (SVZ) sum rules in QCD \cite{4}; we demonstrate that some essential criteria for selecting QCD diagrams appropriate for the tetraquark properties in the method of QCD sum rules have not been properly taken into account.

For a proper analysis of possible tetraquarks in QCD and for the selection of the appropriate Feynman diagrams, the understanding of the four-quark singularities of Feynman diagrams plays a crucial role. In a number of publications \cite{5,6,7,8,9,10,11,12}, the four-quark singularities of Feynman diagrams describing the four-point functions of bilinear quark-antiquark interpolating currents have been carefully studied. In recent papers \cite{13,14}, the notion of tetraquark-phile diagrams ($T$-phile diagrams) has been introduced: by definition, the $T$-phile diagrams are those Feynman QCD diagrams that have four-quark singularities in the appropriate kinematic variable. For the four-point function, those diagrams that contain at least two gluon exchanges with a special topology have been shown to belong to the set of $T$-phile diagrams.

Independently of this line of research, numerous works deal with the analysis of tetraquark states within the method of QCD sum rules (see the review papers \cite{15,16} and references therein). The emphasis has been laid on two-point functions $\Pi_{\theta \theta}(x) = \langle T\{\theta(x)\theta(0)\} \rangle$ of the tetraquark interpolating currents $\theta(x) = \bar{q}(x)q(x)\bar{q}(x)q(x)$, and three-point functions, $\Gamma_{\theta jj}(0|x,y) = \langle T\{\theta(0)\theta(x)j(y)\} \rangle$, involving one tetraquark current $\theta$ and two ordinary bilinear quark currents $j(x) = \bar{q}(x)q(x)$. (The quark flavour content of the currents will be specified in the forthcoming sections.) All these applications of SVZ sum rules (SR) to exotic states share one common feature: they calculate the leading-order $O(1)$ diagrams (and in some cases also radiative corrections) and power corrections induced by these leading-order diagrams, and borrow exactly the same criteria for continuum subtraction as prescribed by the SVZ sum rules for the ordinary mesons \cite{4}. As a result, the tetraquark contribution is obtained to be dual to the low-energy spectral integral of the QCD diagrams for the corresponding Green function. In particular, the tetraquark receives substantial contributions of the $O(1)$ and $O(\alpha_s)$ QCD diagrams.

One may easily see that the procedures adopted in the SR analyses of exotic states \cite{13,14} in fact do not properly take into account the properties of four-quark singularities of Feynman diagrams. To show this, we make two almost self-evident observations:

(i) It is sufficient to consider tetraquark interpolating currents composed as the product of two colorless bilinear quark currents \cite{17}. All other structures of the tetraquark currents are reduced to the product of colorless bilinears by performing Fierz transformations. Next, a tetraquark interpolating current may be consistently defined as the product of two point-splitted bilinear quark currents by sending the displacement parameter $\delta$ to zero: $\theta(x) = \lim_{\delta \to 0} j(x)j(x + \delta)$. Then any diagram describing $\Pi_{\theta \theta}$ and $\Gamma_{\theta jj}$ may be obtained from the diagrams of the 4-point function $\Gamma_{jj} \equiv \langle T\{jjjj\} \rangle$ by merging two pairs of vertices (in the case of $\Pi_{\theta \theta}$) or one pair of vertices (in the case of $\Gamma_{\theta jj}$).
(ii) The procedures adopted in the method of QCD sum rules relate the tetraquark properties to those contributions to $\Pi_{\theta\theta}$ and $\Gamma_{\theta j j}$ that are obtained by merging vertices in non-$T$-philic diagrams of $\Gamma_{\theta j}$. Recall that such contributions to $\Gamma_{\theta j}$ have no four-quark cuts \cite{18} and therefore may not be related to tetraquark properties \cite{10–12}.

In this paper, we show that no contradiction emerges as soon as one makes use of quark-hadron duality relations for correlation functions involving exotic tetraquark currents in a proper way: one observes an exact cancellation of the $O(1)$ and $O(\alpha_s)$ contributions on the OPE side against the two-meson contribution on the phenomenological side of the SVZ sum rule \cite{4}, due to quark-hadron duality relations for correlation functions of ordinary bilinear quark currents describing properties of ordinary mesons. Consequently, a consistent QCD sum rule for any exotic state should be formulated in the following way: on the OPE side for $\Pi_{\theta\theta}$ and $\Gamma_{\theta j j}$, one takes into account only $T$-philic diagrams, i.e., those diagrams which are obtained from $T$-philic diagrams for $\Gamma_{\theta j}$; on the phenomenological side, one has the suspected tetraquark pole and the interacting mesons. One may then assume, similarly to the conventional procedures of QCD sum rules for ordinary correlators, that the tetraquark contribution is dual to the low-energy part of the $T$-philic contributions to $\Pi_{\theta\theta}$ and $\Gamma_{\theta j j}$.

2. DIRECT GREEN FUNCTIONS INVOLVING TETRAQUARK CURRENTS

Let us discuss tetraquarks involving two quarks of flavours $a$ and $c$ and two antiquarks of flavours $b$ and $d$. We therefore consider interpolating currents with two different flavour structures, $\theta_{\bar{a}b\bar{c}d} = \bar{j}_{\bar{a}b}j_{\bar{c}d}$ and $\theta_{\bar{a}d\bar{c}b} = \bar{j}_{\bar{a}d}j_{\bar{c}b}$, with $j_{\bar{a}d} = \bar{q}_a q_b$. We do not specify here the Dirac structure of the currents, so we do not explicitly write the appropriate combinations of $\gamma$ matrices between the quark fields. This technical complication does not change the argument developed in this paper.

As mentioned above, an appropriate definition of the four-quark operator may be given by point-splitting in the product of bilinear quark current operators. From this perspective, any diagram involving a tetraquark interpolating current may be obtained from the four-point function of bilinear quark currents studied in detail in \cite{11}. One should distinguish between the diagrams where quark flavours in the initial state and the final state are combined in the same way (direct diagrams, this section) and in a different way (recombination diagrams, the next section). Feynman diagrams for the corresponding four-point functions have different topologies and different structures of four-quark singularities. Accordingly, the duality relations for these correlators should be discussed separately.

A. Two-point function $\Pi_{\theta\theta}^{\text{dir}}$

Figure \ref{fig:1} shows the direct four-point function $\Gamma_{\theta j j}^{\text{dir}}$ and the corresponding two-point function of the tetraquark currents. In Fig. \ref{fig:1} only diagrams (c) are $T$-philic diagrams, so we expect that the r.h.s. diagrams (a,b) should drop out from the tetraquark sum rule. Diagrams with one-gluon exchanges between the disconnected loops are null.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Fig1.png}
\caption{Feynman diagrams for a direct two-point function of tetraquark currents, $\Pi_{\theta\theta}^{\text{dir}}$, as obtained by merging vertices in the direct four-point function of bilinear quark currents. In the left column, diagrams (a) and (b) do not contain four-quark singularities in the $s$ channel, whereas diagram (c) is the lowest-order diagram that contains the four-quark $s$ cut and is thus the only $T$-philic diagram.}
\end{figure}

Fig. 1: Feynman diagrams for a direct two-point function of tetraquark currents, $\Pi_{\theta\theta}^{\text{dir}}$, as obtained by merging vertices in the direct four-point function of bilinear quark currents. In the left column, diagrams (a) and (b) do not contain four-quark singularities in the $s$ channel, whereas diagram (c) is the lowest-order diagram that contains the four-quark $s$ cut and is thus the only $T$-philic diagram.
Phenomenological side of SR

Theoretical side of SR

Fig. 2: QCD sum rule for the two-point function $\Pi_{\theta\theta}^{\text{dir}}$: the l.h.s. gives the OPE (for each QCD diagram the related power corrections are not shown explicitly); the r.h.s. gives the corresponding representation using meson degrees of freedom and includes a tetraquark pole. The first line on both sides of the sum rule shows diagrams factorizable into two parts, separated by the horizontal red dash-dotted line; the second line on both sides shows non-factorizable contributions. Diagrams in the first line on both sides are equal to each other by virtue of QCD sum rules for $\Pi_{jj}$, Fig. 3.

Fig. 3: The conventional QCD sum rule for the two-point function of ordinary bilinear quark currents $\Pi_{jj}^{\text{dir}}$: the l.h.s. gives the OPE (the related power corrections are not shown explicitly; the diagram with two gluon lines with the dots in-between represents the sum of diagrams with an arbitrary number of gluon exchanges, starting with the quark loop with no gluons); the r.h.s. gives the corresponding representation using meson degrees of freedom.

Fig. 4: The final tetraquark QCD sum rule which relates the non-factorizable terms in the OPE for the two-point function $\Pi_{\theta\theta}^{\text{dir}}$ (including also not explicitly shown power corrections generated by the QCD diagram on the l.h.s.) to the sum of the tetraquark pole and the non-factorizable meson-interaction diagrams. The dots in the meson diagrams denote the sum of meson diagrams of the same topology (i.e., in this figure the sum of non-factorizable meson diagrams).

To show that this is indeed the case, let us start by writing the OPE (theoretical side of the sum rule) and the hadron representation (phenomenological side of the sum rule) for the two-point direct correlation function $\Pi_{\theta\theta}$, Fig. 2. There is an infinite subset of diagrams in the OPE for $\Pi_{\theta\theta}^{\text{dir}}$ that factorize in coordinate space into two parts separated by the red dash-dotted line. The $O(1)$ and $O(\alpha_s)$ diagrams belong to this subset. On the phenomenological side, there is also an infinite subset of meson contributions that factorize in coordinate space. It is straightforward to check that the factorizable subset of diagrams on the OPE side is exactly equal to the factorizable subset on the phenomenological side as soon as the QCD sum rule for the two-point function of ordinary bilinear of quark currents $\Pi_{jj}$ (Fig. 3) is used. After cancelling out the equal factorizable parts on both sides of the sum rule of Fig. 2, we arrive at the tetraquark sum rule given by Fig. 4. Now, similarly to the case of ordinary mesons, we consider a single spectral representation in the variable $p^2$ of the QCD diagrams in the l.h.s. of Fig. 4 and introduce the effective threshold $s_{\text{eff}}$ such that the contribution of the QCD diagrams in the l.h.s. of Fig. 4 above $s_{\text{eff}}$ cancels the non-factorizable meson-meson interaction diagrams. Then, after Borel transformation, we obtain the tetraquark sum rule

$$
(f_{\bar{T}}^{abcd})^2 \exp(-M_{T}^2\tau) = \int_{(4m_q)^2}^{s_{\text{eff}}} ds \exp(-s\tau)\rho_T^{\text{dir}}(s) + \text{power corrections},
$$

(2.1)

where $4m_q \equiv m_a + m_b + m_c + m_d$, $\rho_T^{\text{dir}}$ is the spectral density in the variable $s$ of the r.h.s. diagram of Fig. 4(c) with two-gluon exchanges of order $O(\alpha_s^2)$. Power corrections in the r.h.s. of Eq. (2.1) correspond to condensate insertions in the diagram of Fig. 4(c). Let us emphasize that power corrections generated by the r.h.s. diagrams in Fig. 4(a,b) do not contribute to the tetraquark sum rule (2.1), as they have cancelled against the factorizable meson-meson contributions. Here, $M_T$ is the tetraquark mass and

$$f_{\bar{T}}^{abcd} = \langle T|\theta_{\bar{a}bc\bar{d}}|0\rangle.
$$

(2.2)

As argued above, only the $T$-phile diagram of Fig. 4(c) and the corresponding power corrections contribute to the tetraquark sum rule (2.1).
B. Three-point function $\Gamma_{\theta jj}^{\text{dir}}$

Similarly to $\Pi_{\theta \theta}^{\text{dir}}$, the direct Green functions $\Gamma_{\theta jj}^{\text{dir}}$ may be obtained from the direct Green functions $\Gamma_{4j}^{\text{dir}}$ by merging in the latter only one (say, the left) pair of coordinates. The corresponding procedure is shown in Fig. 5 here, only diagram (c) is the $T$-phile diagram, so the r.h.s. diagrams (a) and (b) should not contribute to the tetraquark coupling to two mesons. For the direct three-point function this is very easy to show: the tetraquark would lead to the pole $1/(p^2 - M_T^2)$ in the full Green function $\Gamma_{\theta jj}^{\text{dir}}$ ($p$ is the total momentum of the currents), and the residue in this pole would be related to the coupling $T \to M_{\bar{a}b}M_{\bar{d}c}$. It is clear that the diagrams of Figs. 5(a) and (b) cannot contribute to the pole, as their dependence on the variable $p^2$ is at most polynomial, due to the traces over quark loops. Taking the Borel transform of $\Gamma_{\theta jj}^{\text{dir}}$ in $p^2$, $p^2 \to \tau$, the Borel image of the r.h.s. diagrams of Figs. 5(a) and (b) vanishes. So the r.h.s. diagram of Fig. 5(c) is the lowest-order diagram that gives a nontrivial contribution to the tetraquark pole. Introducing the effective continuum threshold in the $p^2$ channel and performing the Borel transform $p^2 \to \tau$, we end up with the following sum rule:

$$f_T^{\bar{a}b\bar{c}d} \exp(-M_T^2\tau)A(T \to j_{\bar{a}b}\bar{j}_{c\bar{d}}) = \int_{(4m_q)^2}^{s_{\text{eff}}} ds \exp(-s\tau)\Delta_{\text{dir}}^T(s) + \text{power corrections},$$

(2.3)

where $\Delta_{\text{dir}}^T(s)$ is the spectral density in the variable $s$ of the r.h.s. diagram (c) in Fig. 5 and the power corrections correspond to condensate insertions in this diagram. Let us emphasize that power corrections generated by non-$T$-phile diagrams do not appear in the sum rule for the tetraquark state.

This completes, for the direct Green function, the proof of the statement that only diagrams obtained from the set of $T$-phile diagrams of $\Gamma_{4j}$ contribute to the tetraquark sum rule.

3. RECOMBINATION GREEN FUNCTIONS INVOLVING TETRAQUARK CURRENTS

We now briefly discuss diagrams with a different – “recombination” – topology (Figs. 6 and 7), where the quark color singlets in the initial and final currents have different flavours structures. Similarly to the case of the direct diagrams, only the diagrams of Figs. 6(c) and 7(c) are $T$-phile diagrams and contribute to the sum rules for the tetraquark properties. The proof of this statement is technically not as simple as for the direct diagrams, where the contribution of $O(1)$ and $O(\alpha_s)$ diagrams just dropped out from the tetraquark sum rule after the duality relations for the correlators of bilinear currents have been taken into account. Nevertheless, in [11] it was shown that the recombination Green functions (a) and (b) on the l.h.s. of Fig. 6 and Fig. 7 do not contribute to the tetraquark poles and are related to specific meson amplitudes without $s$-channel four-quark singularities. This property also holds after merging the initial and/or final vertices. Thus, the r.h.s. diagrams (a) and (b) of Fig. 6 and Fig. 7 still do not contribute to the $T$-pole properties. In the end, only the diagrams in Fig. 6(c) and Fig. 7(c) are $T$-phile diagrams and...
appear in the tetraquark sum rules, which take the forms

\begin{align}
 f_T^{abcd} f_T^{debc} \exp(-M_T^2 \tau) &= \int \frac{ds}{(4m_q)^2} ds \exp(-s \tau) \tilde{\rho}_{T}^{cc}(s) + \text{power corrections}, \quad (3.1) \\
 f_T^{abcd} A(T \to j_{\hat{a}d} j_{\hat{e}b}) \exp(-M_T^2 \tau) &= \int \frac{ds}{(4m_q)^2} ds \exp(-s \tau) \Delta_T^{rec}(s) + \text{power corrections}, \quad (3.2)
\end{align}

where $4m_q \equiv m_a + m_c + m_d$ and $\tilde{\rho}_{T}^{cc}(s)$ and $\Delta_T^{rec}(s)$ are the spectral densities in the variable $s$ of the $O(\alpha_s^2)$ diagrams with two-gluon exchanges, shown in the r.h.s. of Fig. 6(c) and Fig. 7(c), respectively; power corrections in the r.h.s. of Eqs. (3.1) and (3.2) correspond to condensate insertions in these diagrams. The coupling constants are defined as

\begin{align}
 f_T^{abcd} = \langle T | \theta_{abcd} | 0 \rangle, \quad f_T^{debc} = \langle T | \theta_{debc} | 0 \rangle. \quad (3.3)
\end{align}

**Fig. 6:** Feynman diagrams for a recombination two-point function of tetraquark currents as obtained by merging vertices in the recombination four-point function of bilinear quark currents. Diagrams (a) and (b) on the l.h.s. do not contain four-quark singularities in the $s$ channel. Diagram (c) on the l.h.s. is the lowest-order diagram that contains a four-quark $s$ cut; thus, only diagram (c) on the l.h.s. is the $T$-phile diagram. Also, among the diagrams on the r.h.s., only diagram (c) contributes to the theoretical side of the tetraquark SVZ sum rule (3.1).

**Fig. 7:** Feynman diagrams for a recombination three-point function as obtained by merging vertices in the four-point function of bilinear quark currents. Diagrams (a) and (b) on the l.h.s. have at most a polynomial dependence on the variable $p^2$ and have no four-quark cut in $p^2$. Therefore, (a) and (b) are not related to tetraquark properties. Only the diagram with the two-gluon exchange (c) is the $T$-phile diagram, and therefore contributes to the tetraquark SVZ sum rule (3.2).

### 4. CONCLUSIONS

We have scrutinized the derivation of SVZ sum rules for “exotic” correlation functions involving tetraquark currents $\theta$, namely, the two-point function $\Pi_{\theta \theta}$, and the three-point function $\Gamma_{\theta j j}$ involving one tetraquark current and two ordinary bilinear quark currents. Our main findings may be summarized as follows:

(i) We have demonstrated that the duality relations for such exotic correlators have fundamentally different properties compared with the duality relations for the correlators of bilinear quark currents: the $O(1)$ and $O(\alpha_s)$ terms (including power corrections generated by these diagrams) in the OPE for the exotic correlators exactly cancel against the contributions of the two-meson states on the phenomenological side of the QCD sum rules. As a result, the properly formulated SVZ sum rules for the tetraquark states, Eqs. (4.1), (4.2), (4.3), and (4.4), relate the tetraquark properties to the specific $T$-phile non-factorizable parts of the OPE for the exotic Green functions; the corresponding non-factorizable diagrams start at order $O(\alpha_s^2)$.

(ii) The set of $T$-phile diagrams that contribute to the properties of a given exotic state involves only those diagrams that can be obtained from the set of $T$-phile diagrams of the four-point functions of ordinary bilinear quark currents by merging the appropriate vertices. (In particular, power corrections generated by $T$-phile diagrams are the only power corrections that contribute to the properties of the exotic states.) This observation reduces the analysis of the duality relations for SVZ sum rules for correlation functions involving tetraquark currents to the analysis of four-quark singularities in the four-point function of ordinary bilinear quark currents.
In summary, a proper application of QCD sum rules to tetraquarks requires the calculation of the presently unknown non-factorizable $O(\alpha_s^2)$ radiative corrections and calls for further efforts in order to obtain reliable conclusions about the specific tetraquark candidates.

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