Fixed boundary Grad-Shafranov solver using finite difference method in nonhomogeneous meshgrid

To cite this article: J E López et al 2019 J. Phys.: Conf. Ser. 1159 012017

View the article online for updates and enhancements.
Fixed boundary Grad-Shafranov solver using finite difference method in nonhomogeneous meshgrid

J E López¹, E A Orozco¹ and V D Dougar-Zhabon¹

¹ Universidad Industrial de Santander, Bucaramanga, Colombia

E-mail: jelopez2663@gmail.com, eaorozco@uis.edu.co

Abstract. In this work we present a numerical scheme to solve the Grad-Shafranov equation which correspond to magnetohydrodynamic equilibrium equation for a two-dimensional plasma. A typical case are the toroidal plasma in magnetic confinement devices used in thermonuclear fusion well known as Tokamaks. The proposed numerical scheme is based on the finite-difference method in nonhomogeneous meshgrid, which is adjusted to the fixed plasma boundary with “D-shape”. The solution of the Grad-Shafranov equation is obtained using the successive over-relaxation method, usually applied to solve Poisson equation’s problems. The values of the total plasma current and pressure in the magnetic axis are conserved in each iteration of the convergence process. The scheme is validated by direct comparison with the analytical result obtained by Soloviev.

1. Introduction

In thermonuclear fusion, magnetohydrodynamic equilibrium equation (MHD) equilibrium is the starting point to understand MHD instabilities [1]. Many numerical and analytical studies of confined plasmas in devices based on closed magnetic field lines as Tokamak have focused on this aspect [2, 3]. Usually, solvers have two ways to find the numerical equilibrium. In the first one way there are the “fixed boundary solvers”, where the equilibrium state is obtained when the plasma boundary is analytically know [3, 4]. Then, the magnetic field, pressure and current density inside the plasma region is full characterized and the external region is not relevant [3,5]. In the second one way there are the “free boundary solvers”, which consider that plasma doesn’t have a well defined boundary, therefore it is necessary calculate profiles both inside and outside the plasma region [6]. In this kind of solvers, the border varies until its change is small compared to an established tolerance.

In the present paper, we show a computational Grad-Shafranov (GS) solver based on fixed plasma boundary with D-shape. This solver uses the finite differences method (FDM) in nonhomogeneous meshgrid and the successive over-relaxation scheme. Details of the meshgrid generation, the Grad-Shafranov equation in a finite difference scheme, and the conservation of the total plasma current and pressure on the magnetic axis are shown. We start from the main ideas presented in the Stephen’s Jardin book [6]. The proposed solver is validated by direct comparison with the analytical result obtained by Soloviev [7].
2. Theoretical formalism

2.1. Grad-Shafranov equation

Ideal magnetohydrodynamics is the simplest MHD formulation where the plasma is considered as a charged fluid with very small resistivity (perfect conductor fluid) [6, 8]. In the static and stationary equilibrium, such formulation leads to:

\[ \nabla p = \vec{J} \times \vec{B} \]  
\[ \nabla \times \vec{B} = \mu_0 \vec{J} \]  
\[ \nabla \cdot \vec{B} = 0 \]  

Where \( p \), \( \vec{J} \) and \( \vec{B} \) are the pressure, current density and magnetic field, respectively.

Introducing the magnetic vector potential in cylindrical coordinates for the azimuthally symmetric system and defining the functions:

\[ \psi = r A_{\phi} \]  
\[ g = r B_{\phi} = r \left[ \partial_z (A_r) - \partial_r (A_z) \right] \]

The magnetic field can be written as:

\[ \vec{B} = -\nabla \phi \times \nabla \psi + g \nabla \phi. \]  

Combining Equations (6) and (2) and then (1); and taken into account that \( \vec{B} \cdot \nabla p = \vec{B} \cdot (\vec{J} \times \vec{B}) = 0 \) and \( \vec{J} \cdot \nabla p = \vec{J} \cdot (\vec{J} \times \vec{B}) = 0 \) it lead to the Grad-Shafranov equation, the most famous equilibrium expression in plasma fusion physics [1,9–11]:

\[ \Delta^* \psi = -\mu_0 r^2 \frac{dp}{d\psi} - g \frac{dg}{d\psi} \]

Where \( \Delta^* \psi \) is the elliptical operator, defined as:

\[ \Delta^* \psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} \]

Because \( \psi \) has a close relationship with the poloidal magnetic flux, \( \psi \) is usually called the “poloidal flux function”. Pressure \( p(\psi) \) and the “poloidal current function” \( g(\psi) \) are two free functions whose boundary conditions together with the corresponding boundary conditions for the poloidal flux determine the plasma equilibrium profile.

Grad-Shafranov equation is a nonlinear elliptic partial differential equation in \((r, z)\) plane. Only for the particular case (Soloviev equilibrium):

\[ \frac{dp}{d\psi} = -\frac{c_1}{\mu_0}, \quad g \frac{dg}{d\psi} = -c_2 R_o^2 \]

Analytical solution can be found [7]

\[ \psi(r, z) = \frac{1}{2} \left( c_2 R_o^2 + c_o r^2 \right) z^2 + \frac{1}{8} (c_1 - c_o) \left( r^2 - R_o^2 \right)^2 \]

Where \( c_o, c_1, c_2 \) are constants and \( R_o \) is the major radius. For Tokamak devices, usually the next set of values are used: \( c_o = B_o/(R_o^2 \kappa_o q_o) \), \( c_1 = B_o (\kappa_o^2 + 1)/(R_o^2 \kappa_o q_o) \) and \( c_2 = 0 \); where \( B_o \), \( \kappa_o \) and \( q_o \) are the magnetic field on the axis, the ellipticity and safety factor value on the axis respectively.
2.2. D-shape plasma boundary

There are two ways to define a good D-shape boundary in a toroidal plasma. The first one way use a parametric equations [12]. The second one way is based on Soloviev equilibrium, where the contour levels define a D-shape, given by the expression:

\[
z_b = \pm \frac{1}{r} \sqrt{\frac{2R_2^2 \kappa_o q_0}{B_o} \psi_b - \frac{\kappa_o^2}{4} (r^2 - R_0^2)^2}
\]  

(11)

3. Numerical scheme

3.1. Finite difference scheme

The Grad-Shafranov equation in a dimensionless form can be written as:

\[
\begin{align*}
\hat{\Delta} \psi & = -\hat{r} \hat{J}_\phi, & \hat{J}_\phi &= \hat{r} \frac{d\hat{p}}{d\hat{r}} + \hat{g} \frac{d\hat{g}}{d\psi} \\
\end{align*}
\]

Where: \( \hat{\psi} = \psi / \psi_c, \hat{g} = g / g_c, \hat{p} = p / p_c, \hat{J} = J / J_c, \hat{r} = r / R_o, \hat{\psi} = \psi / \psi_c, \hat{I} = I / I_p. \)

The subscript \( c \) refers to characteristic value for each variable, \( I_p \) is the total plasma current and finally: \( J_c = I_p / R_o, \psi_c = \mu_o J_c R_o^3, p_c = \mu_o I_p^2 R_o^2 \) and \( g_c = \sqrt{\mu_o B_o R_o} \).

The GS equation can be expressed in a finite difference scheme by using a nonhomogeneous meshgrid, being \( h_i = x_{i+1} - x_i \) the variable size step. For this case, using this meshgrid, expressions for the first and second derivative in the second-order approximation are given by:

\[
\begin{align*}
 f'(i) &= \frac{h_{i+1}^2 (f_{i+1} - f_i) - h_i^2 (f_{i-1} - f_i)}{h_{i-1}^2 h_i + h_i^2 h_{i+1}}, & f''(i) &= \frac{h_{i-1} (f_{i+1} - f_i) + h_i (f_{i-1} - f_i)}{\frac{1}{2} (h_{i-1}^2 h_i^2 + h_i^2 h_{i+1}^2)} \\
\end{align*}
\]

(13)

Respectively, which can be used to express the GS Equation (12) in a finite difference way as:

\[
\begin{align*}
\hat{\psi}(i, j) &= C_1 \hat{\psi}(i + 1, j) + C_2 \hat{\psi}(i - 1, j) + C_3 \hat{\psi}(i, j + 1) + C_4 \hat{\psi}(i, j - 1) + C_5 \hat{J}_\phi(i + 1, j) \\
\end{align*}
\]

(14)

Where: \( C_1 = A_r / C, C_2 = B_r / C, C_3 = A_z / C, C_4 = B_z / C, C_5 = \hat{r}_i / C, A_r = \frac{hr_{i+1}}{D_{1r}} - \frac{hr_{i-1}}{D_{2r}}, B_r = \frac{hr_{i+1}}{D_{1r}} + \frac{hr_i^2}{r_i D_{2r}}, \)

\( A_z = \frac{hz_{j+1}}{D_{1z}}, B_z = \frac{hz_j^2}{D_{2z}}, C = \frac{hr_{i+1}}{D_{1r}} + \frac{hr_i r_{i+1}}{r_i D_{2r}} - \frac{hr_{i-1}}{D_{2r}} + \frac{hr_i r_{i+1}}{r_i D_{2r}} + \frac{hz_{j+1}}{D_{1z}} + \frac{hz_j^2}{D_{2z}} \)

and \( D_{1r} = \frac{1}{2} (hr_{i-1} r_{i+1}^2 + hr_i r_{i+1}^2), D_{2r} = hr_{i-1}^2 hr_i + hr_i^2 hr_{i+1}, D_{1z} = \frac{1}{2} (hz_{j-1} h_{z+1}^2 + h_{z+1} h_{z+1}^2). \)

Note that \( \hat{\psi}(i, j) \) depends on the current density while current density depend on the \( p(\hat{\psi}) \) and \( g(\hat{\psi}) \) functions. From these expressions and choosing the adequate functions to \( p \) and \( g \) we can build a Poisson solver to find \( \hat{\psi} \) inside the plasma.

With the aim of keep some parameters constant, we introduce a new function:

\[
\psi_n = \frac{\tilde{\psi}_l - \tilde{\psi}_o}{\tilde{\psi}_l - \tilde{\psi}_o}
\]

(15)

here, \( \tilde{\psi}_l \) and \( \tilde{\psi}_o \) are the value of \( \hat{\psi} \) at the boundary and magnetic axis respectively, so, \( \psi_n \) is between 0 and 1 for all \( \hat{\psi} \) values inside the plasma. Now is possible write to \( p \) function as \( p(\hat{\psi}) = p_0 \hat{\psi}(\psi_n) \), being \( p_0 \) the pressure on the magnetic axis. If we choose \( \hat{p}(\psi_n) \) in such way that its value in the plasma edge is 0, and equal to 1 on the magnetic axis, then the pressure on the magnetic axis is thereby held fixed. For the \( g \) function is appropriate to use
\[ \frac{1}{2}g^2(\tilde{\psi}) = \frac{1}{2}g_0^2[1 + \alpha_g \tilde{g}(\psi_n)], \]

where \( \alpha_g \) is a constant which is adjusted in each iteration to keep constant the plasma current value; which is given by:

\[ I_p = \sum_{i,j} \tilde{J}_\phi(i,j)hr_ihz_j = \sum_{i,j} \left[ -\frac{\tilde{p}_o}{\Delta \psi} \tilde{g}'(\psi) - \frac{\alpha g_0^2}{2r \Delta \psi} \tilde{g}'(\psi_n) \right] hr_ihz_j \]  

\( \alpha_g = -\frac{2\left[ I_p * \Delta \tilde{\psi} + p_o \sum_{i,j} \tilde{r} \tilde{p}' \right]}{g_0^2 \sum_{i,j} \tilde{g}'/\tilde{r}} \) \hspace{1cm} (17)

Where: \( \Delta \tilde{\psi} = \tilde{\psi}_l - \tilde{\psi}_o \), \( \tilde{g}(0) = 0 \) and \( \tilde{g}(1) = 1 \) are the values of \( \tilde{g} \) at the border and on the axis, respectively.

3.2. Adaptable meshgrid

To calculate numerically \( \tilde{\psi}(i,j) \) from Equation (14), a simplest method to create a non homogeneous meshgrid which is fitted to the fixed plasma boundary was used. This method is described below.

We define 4 regions on the plasma border \( \mathcal{R}_1 : [R_{min}, R_o - a\delta] \times [0, z_{top}] \), \( \mathcal{R}_2 : [R_o - a\delta, R_{max}] \times [0, z_{top}] \), \( \mathcal{R}_3 : [R_o - a\delta, R_{max}] \times [-z_{top}, 0] \) and \( \mathcal{R}_4 : [R_{min}, R_o - a\delta] \times [-z_{top}, 0] \). To locate meshgrid points on the plasma boundary, first we consider \( \mathcal{R}_1 \) to solve the Equation (18) by using any root solver, e.g., the bisection method. For this case \( r_o \) and \( z_o \) starts in the values \( R_{min} \) and 0.0, respectively.

\[ f(r_i) = (r_o - r_i)^2 + (z_o - z_i(r_i))^2 - (ds)^2 = 0 \] \hspace{1cm} (18)

\( ds \) is the distance between the points \( (r_o, z_o) \) and \( (r_i, z_i) \), which is the order of \( 10^{-2} \). Once the first \( r_i \) value is found then \( z_i \) is obtained from \( z_i(r_i) \). Next, the values of \( r_o \) and \( z_o \) are updated from \( r_i \) and \( z_i \), respectively. Then the process start again to obtain the following meshgrid point value and so on. It should be noted that \( z_i(r_i) \) is the function defining the plasma boundary with D-shape, given by Equation (11) in Soloviev equilibrium (see Figure 1).

![Figure 1](image1.png) Geometrical parameters in a D-shape plasma boundary.

![Figure 2](image2.png) Meshgrid points generated inside plasma region with D-shape by analytical Soloviev equilibrium.

![Figure 3](image3.png) Meshgrid points (red) adjusted to the plasma border and computational molecule scheme.
In $\mathbb{R}_2$ the same idea is used; however in this region $z_i$ is not a function of $r_i$. Therefore we must use the $z_i$ values obtained in $\mathbb{R}_1$. The next equation is used to fit the meshgrid point which comes from Equation (11):

$$ f(r_i) = (z_b r_i)^2 - \frac{2 R_o^2 \kappa_o q_o}{B_o} \psi + \frac{\kappa_o^2}{4} \left(r_i^2 - R_o^2\right)^2 = 0 $$

(19)

Similarly than $\mathbb{R}_1$, the bisection method is used to solve it. Because the symmetry, the meshgrid points in $\mathbb{R}_3$ and $\mathbb{R}_4$ are obtained in straightforward from those previously obtained in both $\mathbb{R}_1$ and $\mathbb{R}_2$, respectively. Figures 2 and 3 show the the meshgrid points obtained from this method.

4. Results

To check the proposed solver, we compare the obtained results with those obtained from the Soloviev’s solution. To do this, a mesh with Soloviev’s contours is constructed (See Equation (11)). We choose the next parameters for the test: $B_o = 0.5T$, $R_o = 0.95m$, $a = 0.6m$, $\kappa_o = 2.2$, $q_o = 1.1$, $\psi_b = 0.09T/m^2$. In this calculation we use a meshgrid of $241 \times 241$ points.

Figures 4(a) and 4(b) show the poloidal flux in the $(r, z)$ plane obtained from both the Soloviev’s solution and the numerical solution, respectively. These graphics show an excellent agreement. The error along the $r$ axis does not exceed $10^{-5}$ (See Figure 4(c)).

![Figure 4](image.png)

**Figure 4.** (a) Poloidal flux by analytical Soloviev equilibrium, (b) poloidal flux obtained from proposed scheme and (c) numerical and exact equilibrium and absolute error along $r$ axis ($z = 0$).

5. Conclusions

A fixed Grad-Shafranov solver to obtain equilibrium profiles in toroidal devices with azimuthal symmetry was both presented and validated. The maximum error obtained in Soloviev’s equilibrium is about $10^{-6}$. The method presented in this paper doesn’t require any approximation for $p$ and $g$ functions and hold fixed both the pressure on the magnetic axis and the total plasma current.

References

[1] K Miyamoto 2005 *Plasma physics and controlled nuclear fusion* vol 38 (Berlin: Springer-Verlag)
[2] M Ariola and P Alfredo 2008 *Magnetic control of tokamak plasmas* (London: Springer-Verlag)
[3] Y Jeon 2015 Development of a free-boundary tokamak equilibrium solver for advanced study of tokamak equilibria *Journal of the Korean Physical Society* **67** 843-853

[4] A Pataki and A Cerfon 2013 A fast, high-order solver for the GradShafranov equation *Journal of Computational Physics* **243** 28-45

[5] R Srinivasan L Lao and M Chu 2010 Analytical description of poloidally diverted tokamak equilibrium with linear stream functions *Plasma Physics and Controlled Fusion* **52**(3) 035007 1-12

[6] S Jardin 2010 *Computational Methods in Plasma Physics* (Boca Raton: Taylor & Francis)

[7] T Tatsuoki and T Shinji 1991 Computation of MHD equilibrium of tokamak plasma *Journal of Computational Physics* **93** 1-107

[8] J P Freidberg 2014 *Ideal MHD* (Cambridge: Cambridge University Press)

[9] Grad H and Rubin H 1958 Hydromagnetic Equilibria and Force-Free Fields *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* **31** (Geneva: United Nations) 190-197

[10] V Shafranov 1966 Plasma equilibrium in a magnetic field *Reviews of plasma physics* **2** 103

[11] Erwan Deriaz, Bruno Despres, Gloria Faccanoni, Kirill Pichon Gostaf, Lise-Marie Imbert-Gérard, Georges Sadaka, Remy Sart 2011 Magnetic equations with FreeFem++: the Grad-Shafranov equation & the current hole * ESAIM: Proceedings* **32** 76-94

[12] F Troyon, R Gruber, H Saurenmann, S Semenzato and S Succi 1984 MHD-Limits to Plasma Confinement *Plasma Physics and Controlled Fusion* **26** 209-215