TWO DETECTION RESULTS OF KOVANOV HOMOLOGY ON LINKS

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Abstract. We prove that Khovanov homology with $\mathbb{Z}/2$-coefficients detects the link $L_7n_1$, and the union of a trefoil and its meridian.

1. Introduction

Given an oriented link $L$ in $S^3$ and a commutative ring $R$, Khovanov homology $\text{Kh}(L; R)$ is a bi-graded $R$-module that assigns information about the link. In 2011, Kronheimer and Mrowka [KM11] proved that Khovanov homology detects the unknot. Since then, many other detection results of Khovanov homology have been obtained. It is now known that Khovanov homology detects the unlink [BS15, HN13], the trefoil [BS18], the Hopf link [BSX19], the forest of unknots [XZ19], the splitting of links [LS19], and the torus link $T(2, 6)$ [Mar20].

In [XZ20], a classification is given for all links $L$ such that $\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) \leq 8$ and all 3-component links $L$ such that $\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) \leq 12$. By [Shu14, Corollary 3.2.C], $\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 2 \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2)$, where $\text{Khr}$ denotes the reduced Khovanov homology. Moreover, the parity of $\text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2)$ is invariant under crossing changes and hence is always even for 2-component links (as is the case for the 2-component unlink). Therefore $\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2)$ is always a multiple of 4. As a consequence, if a 2-component link $L$ satisfies $\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) > 8$, then $\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) \geq 12$.

This paper studies 2-component links $L$ such that $\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 12$. Among 2-component links with crossing numbers less than or equal to 7, there are four links (up to mirror images) satisfying $\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 12$. These links are:

1. $L_7n_1$ in the Thistlethwaite Link Table,
2. $L_6a3$ in the Thistlethwaite Link Table,
3. the disjoint union of a trefoil and an unknot,
4. the union of a trefoil and its meridian.

Question 1.1. Suppose $L$ is a 2-component link with $\text{rank}_{\mathbb{Z}/2}(L; \mathbb{Z}/2) = 12$, is it true that $L$ must be isotopic (up to mirror image) to one of the links listed above?

Instead of giving a full answer to the question above, we show that Khovanov homology (with the bi-grading) detects the link $L_7n_1$, and the union of a trefoil with its meridian, from the list above. Since [XZ19] proved that Khovanov homology detects the disjoint union of a trefoil and an unknot, and Martin [Mar20] recently proved that Khovanov homology detects $L_6a3$, we conclude that Khovanov homology detects all the links on the list.

In the following, we will call the link $L_7n_1$ as $L_1$, and the union of a trefoil with a meridian $L_2$. Moreover, we fix the chirality and orientation of these two links by...
Figure 1. The two links \( L_1 \) and \( L_2 \)

Figure 1. Notice that the link \( L_1 \) can also be described as the closure of the 2-braid \( \sigma_1^3 \) together with an axis unknot.

Recall that the internal grading of Khovanov homology is defined by \( h - q \) in [BS15], where \( h \) is the homological grading and \( q \) is the quantum grading. The precise statement of our detection result is given as follows.

**Theorem 1.2.** Let \( L_1, L_2 \subseteq S^3 \) be the oriented links as shown in Figure 1, and let \( i \in \{1, 2\} \). Suppose \( L \subseteq S^3 \) is a 2-component oriented link, such that

\[
\text{Kh}(L; \mathbb{Z}/2) \cong \text{Kh}(L_i; \mathbb{Z}/2)
\]

as abelian groups equipped with the internal gradings, then \( L \) is isotopic to \( L_i \) as oriented links.

The proof of Theorem 1.2 depends on a rank inequality between reduced Khovanov homology and knot Floer homology by Dowlin [Dow18], and a braid detection property of link Floer homology by Martin [Mar20]. The main ingredient of the proof of Theorem 1.2 is the following proposition, which is established in Section 3.

**Proposition 1.3.** Let \( L = K \cup U \) be a link such that \( U \) is an unknot and \( K \) is either an unknot or a trefoil. Let \( l = |\text{lk}(K, U)| \) be the linking number of \( K \) and \( U \). Suppose \( l > 0 \), and

\[
\dim_\mathbb{Q} \mathcal{HFK}(L; \mathbb{Q}) \leq 12, \tag{1.1}
\]

where \( \mathcal{HFK} \) is the knot Floer homology defined in [Ras03, OS04]. Then at least one of the following holds:

1. \( K \) is the closure of an \( l \)-braid with axis \( U \).
2. \( l = 1 \), \( K \) is an unknot.
3. \( l = 1 \), \( K \) is a trefoil, \( U \) is the meridian of \( K \).

Recall that a 2-component link \( K_1 \cup K_2 \) is said to be exchangeably braided (or mutually braided) if both \( K_1 \) and \( K_2 \) are unknots, \( K_1 \) is a braid closure with axis \( K_2 \), and \( K_2 \) is a braid closure with axis \( K_1 \). The concept of exchangeably braided links was introduced and studied by Morton [Mor85]. We will also need the following result from [XZ20].
Proposition 1.4 ([XZ20, Corollary 3.9]). Suppose \( L \) is an exchangeably braided link with linking number \( l \geq 3 \), then we have \( \text{rank}_Q \widehat{HF}(L; \mathbb{Q}) \geq 12 \). Moreover, if \( l > 3 \), then \( \text{rank}_Q \widehat{HF}(L; \mathbb{Q}) > 12 \).

We will prove Theorem 1.2 in Section 4 as a consequence of Proposition 1.3, Proposition 1.4, Dowlin’s rank inequality [Dow18, Corollary 1.7], and Batson-Seed’s spectral sequence [BS15].

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2. Link Floer homology

This section reviews the basic properties of link Floer homology and proves a result on the rank of link Floer homology that will play an important role in the proof of Proposition 1.3.

The link Floer homology was originally defined for \( \mathbb{Z}/2 \)-coefficients by Ozsváth and Szabó in [OS04], and was generalized to \( \mathbb{Z} \)-coefficients in [Sar11]. We will work with \( \mathbb{Q} \)-coefficients in order to invoke Dowlin’s spectral sequence [Dow18]. For the rest of this section, all Floer homology groups are with \( \mathbb{Q} \)-coefficients and it will be omitted from the notation.

Given an oriented \( n \)-component link \( L \subset S^3 \), its link Floer homology \( \widehat{HF}(L) \) carries a homological grading over \( \mathbb{Z} \) and \( n \) Alexander gradings associated to the \( n \) components of \( L \). The Alexander grading associated to the \( i \)-th component \( K_i \) takes values in either \( \mathbb{Z} \) or \( \mathbb{Z} + \frac{1}{2} \), which depends on the parity of the linking number \( \text{lk}(K_i, L - K_i) \).

By [OS08a], when \( n \geq 2 \), the link Floer homology recovers the multi-variable Alexander polynomial in the following sense:

\[
\sum_{a_1, \ldots, a_n} \chi(\widehat{HF}(L, a_1, \ldots, a_n)) \cdot T_1^{a_1} \cdots T_n^{a_n} = (T_1^{1/2} - T_1^{-1/2}) \cdots (T_n^{1/2} - T_n^{-1/2}) \Delta_L(T_1, \ldots, T_n),
\]

where \( \widehat{HF}(L, a_1, \ldots, a_n) \) is the component of \( \widehat{HF}(L) \) with multi-Alexander grading \( (a_1, \ldots, a_n) \), and \( \chi(\cdot) \) denotes the Euler characteristic with respect to the homological grading. The notation “\( \cong \)” means that the two sides are equal up to a multiplication by \( \pm T_1^{b_1} \cdots T_n^{b_n} \) for some \( b_1, \ldots, b_n \in \frac{1}{2} \mathbb{Z} \). There is also a symmetry

\[
\widehat{HF}(L, a_1, \ldots, a_n) \cong \widehat{HF}(L, -a_1, \ldots, -a_n).
\]

The following proposition is a special case of the Thurston norm detection property of link Floer homology.

Proposition 2.1 ([OS08b, Theorem 1.1]). Suppose \( L = K \cup U \subset S^3 \) is a 2-component link with an unknotted component \( U \) and \( l = |\text{lk}(K, U)| > 0 \). Then the top Alexander grading of \( \widehat{HF}(L) \) associated to \( U \) is \( \frac{1}{2} \) if and only if \( U \) has a Seifert disk that intersects \( K \) transversely at \( l \) points.

Remark 2.2. The proof of [OS08b, Theorem 1.1] was originally given for \( \mathbb{Z}/2 \)-coefficients, but the same argument applies to \( \mathbb{Q} \)-coefficients. Alternatively, a similar norm-detection property for instanton Floer homology was established by [GL19]
using sutured manifold decompositions and the formal properties of Floer homology, and the same argument can be carried over to Heegaard Floer homology with $\mathbb{Q}$–coefficients.

The following is a weaker version of a result from [Mar20].

**Proposition 2.3** ([Mar20, Corollary 2]). Let $L = K \cup U \subset S^3$ be a 2-component link such that $U$ an unknot and $l = |\text{lk}(K, U)| > 0$. Then $K$ is the closure of a braid with axis $U$ if and only if the dimension of $\widehat{HFL}(L)$ is 2 at the top Alexander grading associated to $U$.

The link Floer homology $\widehat{HFL}$ can be interpreted by sutured Floer homology using the following proposition. Here we use SFH to denote the sutured Floer homology defined by Juhász in [Juh06].

**Proposition 2.4** ([Juh06, Proposition 9.2]). Suppose $L = K_1 \cup \ldots \cup K_n$ is an oriented link, and let $S^3 - N(L)$ be the link complement. Let $\gamma$ be a suture on $\partial(S^3 - N(L))$ which consists of two meridians of each $K_i$. Then there is an isomorphism

$$\widehat{HFL}(L) \cong \text{SFH}(S^3 - N(L), \gamma).$$

Moreover, the Alexander grading associated to $K_i$ corresponds to the grading induced by a Seifert surface of $K_i$ on $\text{SFH}(S^3 - N(L), \gamma)$.

**Remark 2.5.** The original statement is for $\mathbb{Z}/2$–coefficients, but the proof is done by examining the Heegaard diagrams, which also works for $\mathbb{Q}$–coefficients.

We also need the following proposition from [Juh10].

**Proposition 2.6** ([Juh10, Proposition 9.2]). Suppose $(M, \gamma)$ is a balanced sutured manifold. Suppose $\gamma_0$ is a component of $\gamma$ that is homologically essential on $\partial M$. Let $\gamma'$ be a suture on $\partial M$ obtained by adding two parallel copies of $\gamma_0$ to $\gamma$. Then we have

$$\text{SFH}(M, \gamma') \cong \text{SFH}(M, \gamma) \otimes \mathbb{Q}^2.$$

The main result of this section is the following proposition.

**Proposition 2.7.** Suppose $L = K \cup U \subset S^3$ is a 2-component link with an unknotted component $U$ and $l = |\text{lk}(K, U)| > 0$, and suppose $U$ has a Seifert disk $D$ that intersects $K$ transversely at $l$ points. Then

$$\dim_{\mathbb{Q}} \widehat{HFL}(L, \frac{l}{2}) \equiv 2 \mod 4,$$

where $\widehat{HFL}(L, l/2)$ is the component of $\widehat{HFL}(L)$ with degree $l/2$ on the Alexander grading associated to $U$.

In order to prove Proposition 2.7, we need to establish the following property of sutured Floer homology.

**Proposition 2.8.** Let $l \in \mathbb{Z}^+$, let $T \subset [-1, 1] \times D^2$ be a tangle given by $T = \alpha_1 \cup \ldots \cup \alpha_l$, where $\alpha_i$ is an arc connecting $\{-1\} \times D^2$ and $\{1\} \times D^2$ for all $i$. Let $M_T = [-1, 1] \times D^2 - N(T)$, let $\gamma_T \subset \partial M_T$ be a suture on $M_T$ with $(l + 1)$ components: one meridian component on each one of $\partial N(\alpha_1), \ldots, \partial N(\alpha_l)$, and a component on $[-1, 1] \times \partial D^2$ given by $\{\text{pt}\} \times \partial D^2$. Then $\dim_{\mathbb{Q}} \text{SFH}(M_T, \gamma_T)$ is odd.

We start the proof of Proposition 2.8 by verifying the trivial case.
Lemma 2.9. If $T$ is a product tangle, i.e., there are points $p_1, ..., p_n \subset \text{int}(D^2)$ so that $\alpha_i = [-1,1] \times \{p_i\}$ for all $i$, then $\dim Q \text{SFH}(M_T, \gamma_T)$ is odd.

Proof. When $T$ is a product tangle, $(M_T, \gamma_T)$ is a product sutured manifold. Hence it follows from [Juh06] that the dimension of $\text{SFH}(M_T, \gamma_T)$ is one. □

Let $T$ be the tangle in Proposition 2.8. Orient $T$ so that each $\alpha_i$ goes from $\{-1\} \times D^2$ to $\{1\} \times D^2$. Fix a diagram on $[-1,1] \times [-1,1]$ that represents the tangle $T$. We will also denote the diagram by $T$ when there is no source of confusion. For a positive crossing of $T$, we can perform surgeries along the curve $\beta$ as depicted in Figure 2. Let $M_T$, $-1$ be the manifold obtained by performing the $(-1)$–surgery along $\beta$, and let $M_T$, $0$ be the manifold obtained by performing the 0–surgery along $\beta$. Let $T_-$ be the tangle that only differs from $T$ at the crossing linked by $\beta$ as depicted in Figure 2. It straightforward to show that $(M_T$, $-1$, $\gamma_T) \sim (M_T$, $-1$, $\gamma_T)$. □

Definition 2.10. We call the operation of switching from $T$ to $T_-$ or from $T_-$ to $T$ a crossing change.

Lemma 2.11. For any vertical tangle $T \subset [-1,1] \times D^2$, there is a finite sequence of crossing changes that takes $T$ to the product tangle. □

Now we study the sutured manifold $(M_{T,0}, \gamma_T)$. Inside $[-1,1] \times D^2$, the circle $\beta$ bounds a disk $D$ that intersects the tangle $T$ twice. After performing the 0–surgery, the boundary $\partial D$ can be capped by a meridian disk in the surgery solid torus, and hence we obtain a 2–sphere $S$ that intersects the tangle $T$ twice. The intersection of $S$ and $M_{T,0}$ is a properly embedded annulus $A_\beta \subset M_{T,0}$. We can pick the suture $\gamma_T$ so that one boundary component of $A_\beta$ lies in $R_+ (\gamma_T)$ and the other lies in $R_- (\gamma_T)$. Then there is a sutured manifold decomposition

$$(M_{T,0}, \gamma_T) \overset{A_\beta}{\sim} (M', \gamma').$$

From [KM10, Section 3.1], we know that $M' \cong M_{T_0} := [-1,1] \times D^2 - N(T_0)$, where $T_0$ is another tangle on $[-1,1] \times D^2$, possibly having closed components, such that $T_0$ only differs from $T$ near the crossing linked by $\beta$ as depicted in Figure 2.

Definition 2.12. We say that $T_0$ is obtained from $T_+$ by an oriented smoothing.

Lemma 2.13. We have $\text{SFH}(M_{T,0}, \gamma_T) \cong \text{SFH}(M_{T_0}, \gamma')$, and $\dim Q \text{SFH}(M_{T_0}, \gamma')$ is even.
Proof. The isomorphism

\[ \text{SFH}(M_{T,0}, \gamma_T) \cong \text{SFH}(M_{T_0}, \gamma') \]

follows from [Juh08, Lemma 8.9]. For the parity statement, we argue in two cases.

Case 1. The crossing linked by $\beta$ involves two different components of $T$. Without loss of generality we can assume that they are $\alpha_1$ and $\alpha_2$. See Figure 3. Recall we have an annulus $A_\beta \subset M_{T,0}$ after performing the 0-surgery along $\beta$. To make sure that the two boundary components of $A_\beta$ lie in two different component of $R(\gamma_T)$, the suture $\gamma_T$ must be arranged as in one of the two possibilities shown in Figure 3. After performing the sutured manifold decomposition along $A_\beta$, the new tangle $T_0$ has two new arcs $\alpha'_1$ and $\alpha'_2$. For $i = 1, 2$, let $C'_i = \partial N(\alpha'_i) - \{-1, 1\} \times D^2$. It is straightforward to check that, after the sutured manifold decomposition along $A_\beta$, one and exactly one of the following two possibilities happens, as shown in Figure 3:

- $\gamma' \cap C'_1$ consists of three parallel copies of meridians of $\alpha'_1$
- $\gamma' \cap C'_2$ consists of three parallel copies of meridians of $\alpha'_2$.

Without loss of generality, we assume that the first possibility happens, i.e., $\gamma'$ contains three copies meridians of $\alpha'_1$. Removing two such copies, we obtain a new sutured manifold $(M_{T_0}, \gamma_{T_0})$, and by Proposition 2.6 we have

\[ \dim \mathbb{Q} \text{SFH}(M_{T_0}, \gamma') = 2 \dim \mathbb{Q} \text{SFH}(M_{T_0}, \gamma_{T_0}). \]
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As a result, \( \dim \mathbb{Q} \text{SFH}(M_{T_0}, \gamma') \) is even.

**Case 2.** The crossing linked by \( \beta \) involves only one component of \( T \). Without loss of generality, we assume it is \( \alpha_1 \), see Figure 4. Let \( C_i = \partial N(\alpha_i) - \{ -1, 1 \} \times D^2 \) for all \( i \). To make sure that the two boundary components of \( A_\beta \) lie in two different components of \( \partial N(\gamma_T) \), we must have the suture on \( C_1 \) to be in the position as depicted in Figure 4. After the decomposition along \( A_\beta \), the new tangle \( T_0' \) now has a closed component, which we call \( \alpha'_0 \), and an arc that we call \( \alpha'_1 \). Let \( C'_0 = \partial N(\alpha'_0) \) and \( C'_1 = \partial N(\alpha'_1) - \{ -1, 1 \} \times D^2 \). Note that \( C'_0 \) is a torus while \( C'_1 \) is an annulus.

The suture \( \gamma' \) contains two meridians on \( C'_0 \), and one meridian on \( C'_1 \) (and one meridian on every other \( C_i \)).

Recall that our goal is to show that \( \dim \mathbb{Q} \text{SFH}(M_{T_0}, \gamma') \) is even. Write \( T_0' = T_0 \setminus \alpha'_0 \).

**Case 2.1.** When \( \alpha'_0 \) is split from \( T_0' \), i.e., there is a 3-ball \( B^3 \subset (-1, 1) \times D^2 \) so that

\[ B^3 \cap T_0 = \alpha'_0. \]

In this case, we know that \((M_{T_0}, \gamma')\) is a connected sum:

\[ (M_{T_0}, \gamma') \cong (M_{T_0'}, \gamma' - B^3) \# (S^3(\alpha'_0), \gamma' \cap B^3). \]

Here \( S^3(\alpha'_0) \) is the knot complement of the knot \( \alpha'_0 \subset B^3 \subset S^3 \). It then follows from [Juh06, Proposition 9.15] that the dimension of \( \text{SFH}(M_{T_0}, \gamma') \) is even.

**Case 2.2.** When \( \alpha'_0 \) is not split from \( T_0' \). Suppose there is a positive crossing of \( T_0 \) involving both \( \alpha_0' \) and \( T'_0 \). Pick the circle \( \theta \) as depicted in Figure 5. Suppose the component of \( T'_0 \) involved in the crossing is \( \alpha' \). There is a surgery exact triangle associated to \( \theta \):

\[ \text{SFH}(M_{T_0}, \gamma') \rightarrow \text{SFH}(M_{T_0}, -, \gamma') \rightarrow \text{SFH}(M_{T_0}, -, \gamma'_0) \rightarrow \text{SFH}(M_{T_0, 0}, \gamma'_0) \rightarrow \text{SFH}(M_{T_0}, \gamma') \]

As above, \( T_{0, -} \) is obtained from \( T_0 \) by a crossing change and \((M_{T_0, -}, \gamma'_0)\) is the corresponding sutured manifold. The tangle \( T_{0, 0} \) is obtained from \( T_0 \) by an oriented smoothing. As in Figure 5, \( \alpha'_0 \) and \( \alpha' \) merge into a single component \( \alpha''_0 \subset T_{0, 0} \). It is then straightforward to check that the new suture \( \gamma'_0 \) consists of five meridians of \( \alpha''_0 \). The two meridians of \( \alpha'_0 \) and one meridian of \( \alpha' \) all survive, and there are...
two more meridians coming from the decomposition along an annulus \( A_\theta \) (similar to the annulus \( A_\beta \) above). By Proposition 2.6, we know that the dimension of \( \text{SFH}(M_{T_0,0}, \gamma_0') \) is even, and hence

\[
\dim \mathbb{Q} \text{SFH}(M_{T_0,0}, \gamma_0') \equiv \dim \mathbb{Q} \text{SFH}(M_{T_0,-}, \gamma_0'-) \mod 2. \tag{2.3}
\]

However, \( T_0 \) and \( T_0,- \) only differ by a crossing change, and following the same line of Lemma 2.11, there is a finite sequence of such crossing changes that makes \( \alpha_0' \) split from \( T_0' \). Hence, it follows from Case 2.1 and (2.3) that the dimension of \( \text{SFH}(M_{T_0,0}, \gamma_0') \) must be even. This concludes the proof of Lemma 2.13. \( \square \)

**Proof of Proposition 2.8.** There is a surgery exact triangle associated to \( \beta \):

\[
\text{SFH}(M_{T}, \gamma_T) \rightarrow \text{SFH}(M_{T,-}, \gamma_{-T}) \rightarrow \text{SFH}(M_{T,0}, \gamma') \rightarrow \text{SFH}(M_{T,-}, \gamma_{-T})
\]

Therefore Lemma 2.13 implies \( \dim \mathbb{Q} \text{SFH}(M_{T}, \gamma_T) \equiv \dim \mathbb{Q} \text{SFH}(M_{T,-}, \gamma_{-T}) \mod 2 \). Proposition 2.8 then follows from Lemma 2.9 and Lemma 2.11. \( \square \)

**Remark 2.14.** The statement and the proof of Proposition 2.8 can be applied to sutured monopole theory and sutured instanton theory as well (with suitable choices of coefficients).

**Proof of Proposition 2.7.** Take a pair of oppositely oriented meridional sutures to each boundary component of \( S^3 - N(L) \), then \( S^3 - N(L) \) becomes a balanced sutured manifold.

Decompose \( S^3 - N(L) \) along the disk \( D \), we obtain a sutured manifold \((M, \gamma)\). The manifold \( M \) is given by \([-1,1] \times D^2 - N(T)\), where \( T \) is a tangle in \([-1,1] \times D^2\). Since the linking number of \( K \) and \( U \) is equal to \(|K \cap U|\), we have \( T = \alpha_1 \cup \cdots \cup \alpha_l \) where \( \alpha_i \) is an arc from \( \{1\} \times D^2 \) to \( \{-1\} \times D^2 \) for each \( i \). The suture \( \gamma \) consists of \((l+3)\) components: one meridian on each of \( \partial N(\alpha_1), \cdots \partial N(\alpha_{l-1}) \), three parallel meridians on \( \partial N(\alpha_l) \), and one component on \([-1,1] \times \partial D^2 \) given by \( \{\text{pt}\} \times \partial D^2 \). We have

\[
\widehat{\text{HFL}}(L, \frac{l}{2}) \cong \text{SFH}(M, \gamma).
\]
Removing two sutures from $\partial N(\alpha_t)$, we obtain the sutured manifold $(M_T, \gamma_T)$ as in Proposition 2.8. By Proposition 2.6, we have

$$\dim_q \text{SFH}(M_T, \gamma) = 2 \dim_q \text{SFH}(M_T, \gamma_T).$$

Therefore the desired result follows from Proposition 2.8.

3. Proof of Proposition 1.3

The strategy of our proof of Proposition 1.3 is to exploit the properties of the multi-variable Alexander polynomial so that we can apply the braid detection property of link Floer homology by Martin [Mar20, Corollary 2]. The link Floer homology and the multi-variable Alexander polynomial are related by (2.1).

Suppose $L$ is a 2-component link, let $\Delta_L(x, y) \in \mathbb{Z}[x, y, x^{-1}, y^{-1}]$ be the multi-variable Alexander polynomial of $L$. Then $\Delta_L(x, y)$ is a priori only well-defined up to a multiplication by $\pm x^a y^b$. It is possible to normalize the Alexander polynomial, for example, using Equation (2.1). However, the Alexander polynomial normalized by (2.1) can be a Laurent polynomial with half-integer exponents. For our purpose, it is more convenient to take $\Delta_L(x, y)$ as Laurent polynomial with integer exponents, and therefore we will not normalize $\Delta_L(x, y)$.

For $f_1, f_2 \in \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, we write $f_1 \equiv f_2$ if and only if there exists a multiplicative unit $\epsilon$ such that $f_1 = \epsilon f_2$.

For $f \in \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, we use $\|f\|$ to denote the sum of the absolute values of the coefficients of $f$. By (2.1), we have

$$\text{rank}_q \widehat{\text{HFK}}(L; \mathbb{Q}) = \text{rank}_q \widehat{\text{HFL}}(L; \mathbb{Q}) \geq \|(1 - x)(1 - y)\Delta_L(x, y)\|.$$

We need the following result.

**Theorem 3.1** ([Tor53]). Suppose $L = K_1 \cup K_2$ is a 2-component link with multi-variable Alexander polynomial $\Delta_L(x, y)$, where $x, y$ are the variables associated to $K_1, K_2$ respectively. Then we have

$$\Delta_L(x, 1) \equiv \frac{1 - x^l}{1 - x} \Delta_{K_1}(x),$$

where $\Delta_{K_1}(x)$ is the Alexander polynomial of $K_1$ and $l = \text{lk}(K_1, K_2)$.

From now on, let $L = K \cup U$ be a 2-component link such that

1. $U$ is an unknot,
2. $K$ is either a trefoil or an unknot,
3. the linking number $l = \text{lk}(K, U)$ is positive.

Let $\Delta_L(x, y)$ be the multi-variable Alexander polynomial of $L$, where $x, y$ are the variables corresponding to $K$ and $U$ respectively. Define

$$F(x, y) = (1 - x)(1 - y)\Delta_L(x, y).$$

By Theorem 3.1, we have

$$\Delta_L(1, y) \equiv (1 + y + \cdots + y^{l-1})\Delta_U(y) = 1 + y + \cdots + y^{l-1}. \quad (3.1)$$

Write

$$(1 - y)\Delta_L(x, y) =: \sum_{m=-\infty}^{+\infty} g_m(x)y^m, \quad (3.2)$$
then by definiton,
\[ F(x, y) = \sum_m (1 - x)g_m(x)y^m. \] (3.3)

By (3.1), have
\[ (1 - y)\Delta_L(1, y) = (1 - y)(1 + y + \cdots + y^{l-1}) = 1 - y^l. \]
Therefore, after multiplying \( \Delta_L(x, y) \) by \( \pm y^a \), we may assume without loss of generality that
\[ g_0(1) = -g_l(1) = 1, \quad g_m(1) = 0 \text{ for all } m \neq 0, l. \] (3.4)

We establish the following two technical lemmas, which allow us to deduce topological properties of \( L \) from the sequence of Laurent polynomials \( \{g_m(x)\}_{m \in \mathbb{Z}} \).

**Lemma 3.2.** Let \( L, \{g_m(x)\}_{m \in \mathbb{Z}} \) be as above. If \( g_m(x) = 0 \) for all \( m \neq 0, l \), then we have \( l = 1 \).

**Proof.** By the assumption and (3.2),
\[ (1 - y) \Delta_L(x, y) = g_0(x) + g_l(x)y^l. \]
Plugging in \( y = 1 \), we have \( g_l(x) = -g_0(x) \), therefore
\[ \Delta_L(x, y) = \frac{(1 - y^l)g_0(x)}{1 - y} = (1 + y + \cdots + y^{l-1})g_0(x), \]
and hence
\[ \Delta_L(x, 1) = l g_0(x). \]
On the other hand, by Theorem 3.1,
\[ \Delta_L(x, 1) = (1 + x + \cdots + x^{l-1})\Delta_K(x). \]
Recall that \( l \) is assumed to be positive. Comparing the two equations above, we have
\[ \frac{\Delta_K(x)}{l} \in \mathbb{Z}[x, x^{-1}]. \]
Since \( \Delta_K(1) = \pm 1 \), this implies \( l = 1 \). \( \square \)

Recall that for a Laurent polynomial \( f \), we use \( \|f\| \) to denote the sum of the absolute values of the coefficients of \( f \).

**Lemma 3.3.** Let \( L, \{g_m(x)\}_{m \in \mathbb{Z}} \) be as above. Suppose the following two conditions hold:

1. There exists \( k \in \mathbb{Z}^+ \), such that \( g_m(x) = 0 \) for all \( m \neq 0, l, -k, l + k \),
2. \( \|(1 - x)g_0(x)\| = \|(1 - x)g_l(x)\| = 2 \),
then \( l = 1 \) and \( K \) is an unknot.

**Proof.** By Condition (1),
\[ (1 - y)\Delta_L(x, y) = g_{-k}(x)y^{-k} + g_0(x) + g_l(x)y^l + g_{l+k}(x)y^{l+k}. \] (3.5)
By Condition (2), \( \|(1 - x)g_0(x)\| = 2 \). Hence there exists an integer \( s > 0 \) such that \( (1 - x)g_0(x) \doteq 1 - x^s \), thus
\[ g_0(x) \doteq 1 + \cdots + x^{s-1}. \]
By (3.4), this implies \( s = 1 \), therefore \( g_0(x) \doteq 1 \). Similarly \( g_l(x) \doteq 1 \). By (3.4), there exist integers \( a, b \), such that
\[ g_0(x) = x^a, g_l(x) = -x^b. \]
Plugging in $y = 1$ to (3.5), we obtain

$$g_{l+k}(x) = x^b - x^a - g_{-k}(x), \quad (3.6)$$

and hence

$$(1 - y)\Delta_L(x, y) = g_{-k}(x)y^{-k} + x^a - x^b y^l + (x^b - x^a - g_{-k}(x))y^{l+k}$$

$$= g_{-k}(x)(y^{-k} - y^{l+1}) + x^a(1 - y^{l+1}) - x^b(y^l - y^{l+k}).$$

Therefore

$$\Delta_L(x, 1) = \lim_{y \to 1} \left( \frac{g_{-k}(x)(y^{-k} - y^{l+1}) + x^a(1 - y^{l+1}) - x^b(y^l - y^{l+k})}{1 - y} \right)$$

$$= (l + 2k)g_{-k}(x) + (l + k)x^a - kx^b. \quad (3.7)$$

By (3.7), if $\Delta_L(x, 1)$ has more than two terms, then at least one of its coefficients is a multiple of $(l + 2k)$. On the other hand, by Theorem 3.1,

$$\Delta_L(x, 1) = (1 + x + \cdots + x^{l-1})\Delta_K(x),$$

and hence (recall we have assumed that $K$ is either a trefoil or an unknot)

$$\Delta_L(x, 1) = \begin{cases} 
1 + x + \cdots + x^{l-1} & \text{if } K \text{ is an unknot}, \\
1 - x + x^2 & \text{if } K \text{ is a trefoil and } l = 1, \\
1 + \sum_{k=2}^{l-1} x^k + x^{l+1} & \text{if } K \text{ is a trefoil and } l \geq 2.
\end{cases} \quad (3.8)$$

In particular, all the coefficients of $\Delta_L(x, 1) \in \mathbb{Z}[x, x^{-1}]$ are ±1. Since $l + 2k \geq 3$, the previous argument implies that $\Delta_L(x, 1)$ has at most two terms, and hence there are three possibilities:

1. $K$ is an unknot, $l = 1$,
2. $K$ is an unknot, $l = 2$,
3. $K$ is a trefoil, $l = 2$.

To eliminate the second and third possibilities, notice that in these cases, (3.7) and (3.8) yield

$$(2k + 2)g_{-k}(x) + (k + 2)x^a - kx^b \equiv x + 1 \text{ or } x^3 + 1.$$ 

By the assumptions, we have $k \in \mathbb{Z}^+$, and hence

$$(k + 2)x^a - kx^b \equiv x + 1 \text{ or } x^3 + 1 \mod (2k + 2),$$

therefore we have

$$a \neq b,$$

and

$$k + 2 \equiv \pm 1, k \equiv \mp 1 \mod (2k + 2),$$

which imply $k = 1$, thus $g_{-k}(x) = -x^a$. This yields a contradiction to (3.4). \hfill \square

Proof of Proposition 1.3. Without loss of generality, we assume $l = \text{lk}(K, U) > 0$. By (1.1), we have

$$\dim_{\mathbb{Q}} \widehat{HFL}(L; \mathbb{Q}) = \dim_{\mathbb{Q}} \widehat{HF}(L; \mathbb{Q}) \leq 12. \quad (3.9)$$
For a ∈ 1/2Z, we use $\widehat{HFL}(L; a; \mathbb{Q})$ to denote the component of $\widehat{HFL}(L; \mathbb{Q})$ with degree $a$ on the Alexander grading associated to $U$. Let $\Delta_L(x, y)$, $F(x, y)$, $g_m(x)$ be as above, and we choose $\Delta_L(x, y)$ such that (3.4) holds.

Recall that by (2.1), the coefficients of $F(x, y)$ are the bi-graded Euler characteristics of $\widehat{HFL}(L; \mathbb{Q})$. Since $F(1, y) = 0$, we have $\dim_{\mathbb{Q}} \widehat{HFL}(L, a; \mathbb{Q})$ is even for all $a$. By (2.2), we have

$$\dim_{\mathbb{Q}} \widehat{HFL}(L, a; \mathbb{Q}) = \dim_{\mathbb{Q}} \widehat{HFL}(L, -a; \mathbb{Q}).$$

(3.10)

Since $0 \neq \Delta_L(x, y) = \Delta_L(x^{-1}, y^{-1})$, there is a unique $(a, b) \in 1/2 \mathbb{Z} \times 1/2 \mathbb{Z}$, such that $\hat{F}(x, y) := x^a y^b F(x, y)$ satisfies $\hat{F}(x, y) = \pm \hat{F}(x^{-1}, y^{-1})$. Write

$$\hat{F}(x, y) = \sum_{m \in 1/2 \mathbb{Z}} \hat{f}_m(x) y^m,$$

then by (3.4), we have $\hat{f}_{l/2}(x) = \pm \hat{f}_{-l/2}(x^{-1}) \neq 0$. Therefore by (2.1) and (2.2),

$$\dim_{\mathbb{Q}} \widehat{HFL}(L, l/2; \mathbb{Q}) = \dim_{\mathbb{Q}} \widehat{HFL}(L, -l/2; \mathbb{Q}) \neq 0.$$  

(3.11)

Let $s \in 1/2 \mathbb{Z}$ be the maximum degree such that $\dim_{\mathbb{Q}} \widehat{HFL}(L, s; \mathbb{Q}) \neq 0$, then $s \geq l/2 > 0$. Since $\dim_{\mathbb{Q}} \widehat{HFL}(L, s; \mathbb{Q})$ is even, by (3.9) and (3.10), we have

$$\dim_{\mathbb{Q}} \widehat{HFL}(L, s; \mathbb{Q}) = 2, 4, \text{ or } 6.$$

We discuss four cases.

**Case 1.** $\dim_{\mathbb{Q}} \widehat{HFL}(L, s; \mathbb{Q}) = 2$. By Proposition 2.3, $K$ is a braid closure with axis $U$, therefore Case (1) of the proposition holds.

**Case 2.** $\dim_{\mathbb{Q}} \widehat{HFL}(L, s; \mathbb{Q}) = 4$, and $s = \frac{l}{2}$. By Proposition 2.1, $U$ has a Seifert disk that intersects $K$ transversely at $l$ points, therefore this assumption contradicts Proposition 2.7.

**Case 3.** $\dim_{\mathbb{Q}} \widehat{HFL}(L, s; \mathbb{Q}) = 4$, and $s > \frac{l}{2}$. By (3.9) and (3.11),

$$\dim_{\mathbb{Q}} \widehat{HFL}(L, \pm s; \mathbb{Q}) = 4, \dim_{\mathbb{Q}} \widehat{HFL}(L, \pm \frac{l}{2}; \mathbb{Q}) = 2,$$

and $\widehat{HFL}(L, a; \mathbb{Q})$ vanishes at all the other degrees. By (2.1), $\{g_m(x)\}_{m \in \mathbb{Z}}$ satisfies the assumption of Lemma 3.3, therefore $l = 1$ and $K$ is an unknot, and hence Case (2) holds.

**Case 4.** $\dim_{\mathbb{Q}} \widehat{HFL}(L, s; \mathbb{Q}) = 6$. By (3.9) and (3.11), we have $s = \frac{l}{2}$. By Proposition 2.1, there is a Seifert disk of $U$ that intersects $K$ transversely at $l$ points. By (2.1), $F(x, y)$ is supported at only two degrees in $y$, and hence $\{g_m(x)\}_{m \in \mathbb{Z}}$ satisfies the assumption of Lemma 3.2. By Lemma 3.2, we have $l = 1$, therefore $U$ is a meridian of $K$. If $K$ is an unknot, then $L$ is the Hopf link, which satisfies Case (1). Otherwise, $K$ is a trefoil, and hence Case (3) holds. \hfill \Box

### 4. Proof of the main theorem

In this section, we prove that Khovanov homology detects the links $L_1$ and $L_2$ given by Figure 1.

Recall that the internal grading of the Khovanov homology of a link $L$ is introduced in [BS15, Section 2] as $h - q$, where $h$ is the homological grading, and $q$ is the
Theorem 4.1 ([BS15, Corollary 4.4]). Suppose \( L = K_1 \cup K_2 \) is a 2-component oriented link. Then we have

\[
\text{rank}_{\mathbb{F}} \text{Kh}(L; \mathbb{F}) \geq \text{rank}_{\mathbb{F}}^{l+2\text{lk}(K_1;K_2)}(\text{Kh}(K_1; \mathbb{F}) \otimes \text{Kh}(K_2; \mathbb{F}))
\]

for all \( l \in \mathbb{Z} \), where \( \mathbb{F} \) is an arbitrary field and rank\(^k \) denotes the rank of the summand with internal grading \( k \).

Let \( \mathbb{F} = \mathbb{Z}/2 \) from now on. We have

\[
\text{Kh}(U_2; \mathbb{F}) = F_{(-2)} \oplus F_{(0)} \oplus F_{(2)},
\]

where \( U_2 \) is the 2-component unlink and the subscripts denote the internal gradings. Let \( T \) be the left-handed trefoil, and let \( U \) be the unknot, we have

\[
\text{Kh}(T; \mathbb{F}) \otimes \text{Kh}(U; \mathbb{F}) = F_{(0)} \oplus F_{(2)}^{(3)} \oplus F_{(3)}^{(4)} \oplus F_{(5)}^{(2)} \oplus F_{(6)} \oplus F_{(7)}
\]

(4.1)

Let \( \hat{T} \) be the right-handed trefoil, we have

\[
\text{Kh}(\hat{T}; \mathbb{F}) \otimes \text{Kh}(U; \mathbb{F}) = F_{(0)} \oplus F_{(-2)}^{(3)} \oplus F_{(-3)}^{(4)} \oplus F_{(-4)}^{(2)} \oplus F_{(-5)} \oplus F_{(-6)} \oplus F_{(-7)}
\]

(4.2)

Recall that the link \( L_1 = L_{7n1} \) can be described as \( \hat{\sigma}^3_1 \cup U \) where \( \hat{\sigma}^3_1 \) is the closure of the 2-braid \( \sigma^3_1 \) with axis unknot \( U \), and we choose the orientation as given by Figure 1 and hence the linking number is 2. The link \( L_2 \) is given by \( T \cup U \) where \( U \) is a meridian of \( T \) and the orientation is chosen so that the linking number is 1.

We have

\[
\text{Kh}(L_1; \mathbb{F}) = F_{(4)} \oplus F_{(6)}^{(3)} \oplus F_{(7)}^{(8)} \oplus F_{(9)}^{(2)} \oplus F_{(10)} \oplus F_{(11)};
\]

(4.3)

\[
\text{Kh}(L_2; \mathbb{F}) = F_{(2)} \oplus F_{(4)}^{(5)} \oplus F_{(5)}^{(6)} \oplus F_{(7)}^{(2)} \oplus F_{(8)} \oplus F_{(9)}.
\]

(4.4)

Besides the above links, we define the link \( L_3 = \sigma_1^3 \cup U \), which is the union of the closure of the 3-braid \( \sigma_1^3 \) and its axis unknot \( U \). This is the torus link \( T(2,6) \), which is denoted by \( L_{6a3} \) in the Thistlethwaite Link Table. We pick the orientation properly so that the linking number is positive, then

\[
\text{Kh}(L_3; \mathbb{F}) = F_{(4)} \oplus F_{(6)}^{(2)} \oplus F_{(7)}^{(8)} \oplus F_{(9)}^{(2)} \oplus F_{(10)}^{(2)} \oplus F_{(11)} \oplus F_{(12)}.
\]

(4.5)

We now prove Theorem 1.2.

Theorem 1.2. Suppose \( L = K_1 \cup K_2 \) is a 2-component oriented link and \( i \in \{1, 2\} \). If \( \text{Kh}(L; \mathbb{F}) \cong \text{Kh}(L_i; \mathbb{F}) \) \((i = 1, 2)\) as \( l \)-graded abelian groups, then \( L \) is isotopic to \( L_i \) as oriented links.

Proof. Recall that \( \mathbb{F} = \mathbb{Z}/2 \). By the assumptions, we have

\[
\text{rank}_{\mathbb{F}} \text{Kh}(L; \mathbb{F}) = 12.
\]

(4.6)

By [Dow18, Corollary 1.7], we have

\[
\text{rank}_Q \text{HFK}(L; \mathbb{Q}) \leq 2 \text{rank}_Q \text{Khr}(L; \mathbb{Q}) \leq 2 \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) = 12.
\]

(4.7)

Theorem 4.1 yields

\[
\text{rank}_K \text{Khr}(K_i; \mathbb{F}) = \frac{1}{2} \text{rank}_K \text{Kh}(K_i; \mathbb{F}) \leq \frac{12}{2} = 3.
\]

Therefore \( K_i \) \((i = 1, 2)\) is either the unknot or a trefoil according to [KM11, BS18]. By Theorem 4.1 again, we have at least one of \( K_1 \) and \( K_2 \) is the unknot. Without
loss of generality we assume $K_2$ is an unknot, and we discuss two cases.

**Case 1.** $K_1$ is also an unknot. We show that this is contradictory to the assumptions. In fact, Theorem 4.1 and (4.1), (4.4), (4.5) imply that $l = \text{lk}(K_1, K_2)$ is no less than 2. Therefore $K_1$ is the closure of an $l$-braid with axis $K_2$ by Proposition 1.3. Switching the role of $K_1$ and $K_2$ we obtain that $K_2$ is the closure of an $l$-braid with axis $K_1$. By Proposition 1.4, we have $l \leq 3$. If $l = 2$, then $L = \delta^+ \cup U$, which is the link $L_{4a1}$ in the Thistlethwaite Link Table. This contradicts (4.7) because $\text{rank}_y \text{Kh}(L_{4a1}; F) = 8$. If $l = 3$, since the only 3-braid representations of the unknot are given by $\sigma_1^\pm \sigma_2^\pm$ and $\sigma_1^\pm \sigma_2^\pm$, we further divide into two cases:

**Case 1.1.** $L = \sigma_1 \sigma_2 \cup U = L_3$ or $L = \sigma_1^{-1} \sigma_2^{-1} \cup U = \hat{L}_3$. Recall that $\text{Kh}(L_3; F)$ is given by (4.6). Changing the orientation or taking the mirror image will shift the $l$-grading or change the sign of the $l$-grading, respectively. In any case, the $l$-graded Khovanov homology of $L$ cannot be isomorphic to $\text{Kh}(L_i; F)$ ($i \in \{1, 2\}$), contradicting the assumptions.

**Case 1.2.** $L = \sigma_1 \sigma_2^{-1} \cup U$ or $L = \sigma_1^{-1} \sigma_2 \cup U$. In this case $L$ is the link $L_{6a2}$ (or its mirror image) in the Thistlethwaite Link Table. We have $\text{rank}_y \text{Kh}(L; F) = 20$, which is not the same as $L_1$ and $L_2$.

In conclusion, $K_1$ cannot be an unknot.

**Case 2.** $K_1$ is a trefoil. There are two cases.

**Case 2.1.** $K_1$ is the right-handed trefoil $\hat{T}$. Then Theorem 4.1 and (4.3), (4.4), (4.5) yield a contradiction.

**Case 2.2.** $K_1$ is the left-handed trefoil $T$

If $\text{Kh}(L; F) \cong \text{Kh}(L_1; F)$, then by Theorem 4.1, we have $\text{lk}(K_1, K_2) = 2$. By Proposition 1.3, the knot $K_1$ is the closure of a 2-braid in $S^3 - N(K_2)$. A 2-braid representing the left-handed trefoil can only be $\sigma_1^3$. Therefore $L$ is isotopic $L_1$.

If $\text{Kh}(L; F) \cong \text{Kh}(L_2; F)$, then by Theorem 4.1, we have $\text{lk}(K_1, K_2) = 1$. By Proposition 1.3, the knot $K_2$ is a meridian of the left-handed trefoil $K_1$. Therefore $L$ is isotopic to $L_2$.

**Remark 4.2.** The argument above gives an alternative proof of Martin’s theorem that Khovanov homology detects the torus link $T(2, 6)$ [Mar20, Theorem 4]. In fact, the link $T(2, 6)$ is detected by Case 1.1 in the argument above.

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