A simple higher order shear deformation theory for mechanical behavior of laminated composite plates

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Abstract In the present study, the static, buckling, and free vibration of laminated composite plates is examined using a refined shear deformation theory and developed for a bending analysis of orthotropic laminated composite plates. These models take into account the parabolic distribution of transverse shear stresses and satisfy the condition of zero shear stresses on the top and bottom surfaces of the plates. The most interesting feature of this theory is that it allows for parabolic distributions of transverse shear stresses across the plate thickness and satisfies the conditions of zero shear stresses at the top and bottom surfaces of the plate without using shear correction factors. The number of independent unknowns in the present theory is four, as against five in other shear deformation theories. In the analysis, the equation of motion for simply supported thick laminated rectangular plates is obtained through the use of Hamilton’s principle. The accuracy of the analysis presented is demonstrated by comparing the results with solutions derived from other higher order models and with data found in the literature. It can be concluded that the proposed theory is accurate and simple in solving the static, the buckling, and free vibration behaviors of laminated composite plates.

Keywords Higher-order theories · Shear deformation theory of plates · Laminated composite plate

Introduction

The use of composite material for the structure/component design has grown significantly over the past few decades because their response characteristics can be tailored to meet specific design requirements. Furthermore, composite structures possess high specific stiffness and high specific strength which leads to overall reduction of weight, by increasing the efficiency of the structure. Laminated composite plates are widely used in industry and new fields of technology. Due to the high degrees of anisotropy and the low rigidity in transverse shear of the plates, the Kirchhoff hypothesis as a classical theory is no longer adequate. This hypothesis states that the normal to the midplane of a plate remains straight and normal after deformation because of the negligible transverse shear effects. Refined theories without this assumption have been used recently. The free vibration frequencies calculated by using the classical theory of thin plates are higher than those obtained by the Mindlin theory of plates (Mindlin 1951), in which the transverse shear and rotary inertia effects are included.

A number of shear deformation theories have been proposed to date. The first such theory for laminated isotropic plates was proposed apparently by Stavski (1965). This theory was generalized to laminated anisotropic plates in Yang et al. (1966), Ambartsumyan (1969) and Ambartsumyan and Gnuni (1961). It was shown in Srinivas and Rao (1970), Whitney and Sun (1973) and Bert et al. (1974) that the Yang–Norris–Stavski (YNS) theory (Yang et al. 1966) is adequate for predicting the flexural vibration response of laminated anisotropic plates in the first few modes. In Whitney and Pagano (1970), the YNS theory was employed to study the cylindrical bending of antisymmetric cross-ply and angle-ply plate-strips under sinusoidal loading and the free vibration of antisymmetric
angle-ply plate-strips (see also Fortier and Rossettos 1973; Shinha and Rath 1975). Using the YNS theory, a closed-
form solution for the free vibration of simply supported rectangular plates of antisymmetric angle-ply laminates
was obtained in (Bert and Chen 1978). In Noor (1973) were
also presented exact three-dimensional elasticity solutions
for the free vibration of isotropic and anisotropic composite
laminated plates, which serve as benchmark solutions for
comparison by many researchers. The free vibration of
antisymmetric angle-ply laminated plates, with reference to
transverse shear deformations, was investigated in Reddy
(1979) using the finite-element method. The author also
derived a set of variationally consistent equilibrium equa-
tions for the kinematic models originally proposed by
Levinson and Murthy (Reddy 1984). In Reddy and Khdeir
(1989), analytical and finite-element solutions for the
vibration and buckling of laminated composite plates were
found using various theories of plates to prove the neces-
sity for shear deformation theories to predict the behavior
of composite laminates. Using a higher order shear defor-
mation theory, finite-element solutions for free vibration
analysis of laminated composite plates were also obtained in
(Shankara and Iyengar 1996). The complete set of linear
equations of a second-order theory was derived in Khdeir
and Reddy (1999) to analyze the free vibration behavior of
cross-ply and antisymmetric angle-ply laminated plates. In
Singh et al. (2001), the natural frequencies of composite
plates with random material properties were determined
using a higher order shear deformation theory (including
the rotatory inertia effect). The natural frequencies of
laminated composite plates were also found in Rastgaar
et al. (2006) by employing a third-order shear deformation
theory. In Simsek (2010a), the dynamic deflections and the
stresses of a functionally graded simply supported beam
subjected to a moving mass were investigated using the
Euler–Bernoulli, Timoshenko, and the parabolic shear
deformation theory of beams. In Simsek (2010b), the free
vibration of functionally graded beams with different
boundary conditions was examined by using the classical,
first-order, and different higher order shear deformation
theories of beams. A stress analysis of a functionally gra-
ded plate subjected to thermal and mechanical loads was
performed in Matsunaga (2009) using a two-dimensional
higher order theory. A new trigonometric shear deforma-
tion theory for isotropic and composite laminated and
sandwich plates was developed recently in Mantari et al.
(2012), El and Chulkov (1973), where displacements of the
middle surface were expanded in terms of tangential
trigonometric functions of the thickness coordinate, and the
transverse displacements were assumed to be constant
across the thickness.

In this paper, a refined and simple theory of plates is
presented and applied to the investigation of static,
buckling, and free vibration behavior of laminated com-
posite plates. This theory is based on the assumption that
the in-plane and transverse displacements consist of
bending and shear components where the bending com-
ponents do not contribute to shear forces, and likewise, the
shear components do not contribute to bending moments.
The most interesting feature of this theory is that it allows
for parabolic distributions of transverse shear stresses
across the plate thickness and satisfies zero shear stress
conditions at the top and bottom surfaces of the plate
without using shear correction factors. The equations of
motion are derived using Hamilton’s principle. The funda-
mental frequencies are found by solving an Eigen value
equation. The results obtained by the present method are
compared with solutions and results of the first-order and
the other higher-order theories.

Theoretical formulations

Basic assumptions

Consider a rectangular plate of total thickness h composed
of n orthotropic layers with the coordinate system as shown
in Fig. 1. The assumptions of the refined plate’s theory are
as follows:

- The displacements are small in comparison with the
  plate thickness and, therefore, strains involved are
  infinitesimal.
- The transverse displacement \( w \) includes three compo-
nents of bending \( w_b \) and shear \( w_s \). These components
  are functions of coordinates \( x, y \), and time \( t \) only.

\[
w(x, y, z, t) = w_b(x, y, t) + w_s(x, y, t)
\] (1)
The transverse normal stress $\sigma_z$ is negligible in comparison with in-plane stresses $\sigma_x$ and $\sigma_y$.

The displacements $U$ in $x$-direction and $V$ in $y$-direction consist of extension, bending, and shear components:

$$U = u + u_b + u_s, \quad V = v + v_b + v_s$$

(2)

The bending components $u_b$ and $v_b$ are assumed to be similar to the displacements given by the classical plate theory. Therefore, the expression for $u_b$ and $v_b$ can be given as

$$u_b = -z \frac{\partial w_b}{\partial x}, \quad v_b = -z \frac{\partial w_b}{\partial y}$$

(3)

The shear components $u_s$ and $v_s$ give rise, in conjunction with $w_s$, to the parabolic variations of shear strains $\gamma_{xz}, \gamma_{yz}$ and hence to shear stresses $\sigma_x, \sigma_y$ through the thickness of the plate in such a way that shear stresses $\sigma_x, \sigma_y$ are zero at the top and bottom faces of the plate. Consequently, the expression for $u_s$ and $v_s$ can be given as

$$u_s = f(z) \frac{\partial w_s}{\partial x}, \quad v_s = f(z) \frac{\partial w_s}{\partial y}$$

(4)

Kinematics

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (1)–(4):

$$u(x, y, z, t) = u_0(x, y, t) - z \frac{\partial w_b}{\partial x} - f(z) \frac{\partial w_s}{\partial x},$$

$$v(x, y, z, t) = v_0(x, y, t) - z \frac{\partial w_b}{\partial y} - f(z) \frac{\partial w_s}{\partial y},$$

$$w(x, y, z, t) = w_0(x, y, t) + w_s(x, y, t),$$

where $u_0$ and $v_0$ are the mid-plane displacements of the plate in the $x$ and $y$ direction, respectively; $w_b$ and $w_s$ are the bending and shear components of transverse displacement, respectively, while $f(z)$ represents shape functions determining the distribution of the transverse shear strains and stresses along the thickness. This function ensures zero transverse shear stresses at the top and bottom surfaces of the plate. The parabolic distributions of transverse shear stresses through the plate thickness are taken into account for the analysis, by means of the hyperbolic and exponential function of the assumed displacement field.

Present model 1 HSDT The function $f(z)$ is an hyperbolic shape function (Hassaine Daoudadj et al. 2012, 2013) (Hyperbolic Shear Deformation Theory):

$$f(z) = z \left[ 1 + \frac{3\pi}{2} \sec h^2 \left( \frac{z}{h} \right) \right] - \frac{3\pi}{2} h \tanh \left( \frac{z}{h} \right)$$

(5b)

Present model 2 ESDT The function $f(z)$ is an exponential shape function (Karama et al. 2003) (Exponential Shear Deformation Theory):

$$f(z) = z - ze^{-\left(\frac{z^2}{h^2}\right)}$$

(5c)

The strains associated with the displacements in Eq. (5a), (5b), (5c) are

$$\begin{aligned}
\left\{ \begin{array}{l}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{array} \right\} &= \left\{ \begin{array}{l}
\frac{\partial u_0}{\partial x} \\
\frac{\partial v_0}{\partial y} \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}
\end{array} \right\} + z \left\{ \begin{array}{l}
k^b_x \\
k^b_y \\
k^b_{xy}
\end{array} \right\} + f(z) \left\{ \begin{array}{l}
k^s_x \\
k^s_y \\
k^s_{xy}
\end{array} \right\},
\end{aligned}$$

(6a)

where

$$\begin{aligned}
\left\{ \begin{array}{l}
k^b_x \\
k^b_y \\
k^b_{xy}
\end{array} \right\} &= \begin{pmatrix}
-\frac{\partial^2 w_b}{\partial x^2} \\
-\frac{\partial^2 w_b}{\partial y^2} \\
-2 \frac{\partial^2 w_b}{\partial x \partial y}
\end{pmatrix},
\left\{ \begin{array}{l}
k^s_x \\
k^s_y \\
k^s_{xy}
\end{array} \right\} &= \begin{pmatrix}
\frac{\partial^2 w_s}{\partial x^2} \\
\frac{\partial^2 w_s}{\partial y^2} \\
-2 \frac{\partial^2 w_s}{\partial x \partial y}
\end{pmatrix},
\end{aligned}$$

(6b)

and: $g(z) = 1 - f'(z), f'(z) = \frac{\partial f(z)}{\partial z}$.

Constitutive equations

The stress state in each layer is given by Hooke’s law

$$\left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{array} \right\} = \left[ \begin{array}{lcc}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{44}
\end{array} \right] \left\{ \begin{array}{l}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{array} \right\},$$

(7a)

where $Q_{ij}$ are the stiffnesses, which are defined in terms of engineering constants in the material axes of the layer:

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12} \nu_{21}}, \quad Q_{22} = \frac{E_{22}}{1 - \nu_{12} \nu_{21}}, \quad Q_{12} = \frac{\nu_{12} E_{22}}{1 - \nu_{12} \nu_{21}}, \quad Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13}$$

(7b)

Since the laminate is made of several orthotropic layers with their material axes oriented arbitrarily with respect to laminate coordinates, the constitutive equations of each layer must be transformed to the laminate coordinates $x, y$, and $z$. The stress–strain relations in the laminate coordinates of a $k$th layer are
where $\dot{Q}_{ij}$ are the transformed material constants, which are given in (Karama et al. 2003) as

\[
\begin{align*}
\dot{Q}_{11} &= Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta \\
\dot{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12} (\sin^4 \theta + \cos^4 \theta) \\
\dot{Q}_{22} &= Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta \\
\dot{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta \\
&+ (Q_{12} - 2Q_{66}) \sin \theta \cos \theta \\
\dot{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta \\
&+ (Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta \\
\dot{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{66}) \sin^2 \theta \cos^2 \theta \\
&+ Q_{66} (\sin^4 \theta + \cos^4 \theta) \\
\dot{Q}_{44} &= Q_{44} \cos^2 \theta + Q_{55} \sin^2 \theta \\
\dot{Q}_{45} &= (Q_{55} - Q_{44}) \cos \theta \sin \theta \\
\dot{Q}_{55} &= Q_{55} \cos^2 \theta + Q_{44} \sin^2 \theta
\end{align*}
\]

(7d)

In which $\theta$ is the angle between the global $x$-axis and the local $x$-axis of each layer.

**Governing equations**

Using Hamilton’s energy principle, we derive the equation of motion of the laminated composite plate

\[
\delta \int_{t_1}^{t_2} (U - V - T) \, dt = 0,
\]

(8a)

where $U$ is the strain energy, $T$ is the kinetic energy of the plate, and $V$ is the work of external forces. Employing the principle of minimum total energy leads to the general equation of motion and boundary conditions. Taking the variation of the above equation and integrating by parts, we obtain

\[
\begin{align*}
\int_{t_1}^{t_2} \int_A \left[ (\sigma_{xx} \delta u_x + \sigma_{xy} \delta u_y + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz}) \\
- \rho (\dot{u}_x \delta \omega_x + \dot{u}_y \delta \omega_y + \dot{\gamma}_{xy} \delta \omega_{xy} + \dot{\gamma}_{xz} \delta \omega_{xz} + \dot{\gamma}_{yz} \delta \omega_{yz}) - \rho (\ddot{u}_x \delta \omega_x + \ddot{u}_y \delta \omega_y + \ddot{\gamma}_{xy} \delta \omega_{xy} + \ddot{\gamma}_{xz} \delta \omega_{xz} + \ddot{\gamma}_{yz} \delta \omega_{yz}) \right] dA \\
+ \int_A \left[ \frac{N_{xx}^0 (w_{b,xx} + w_{s,xx}) + N_{yy}^0 (w_{b,yy} + w_{s,yy}) + 2N_{xy}^0 (w_{b,xy} + w_{s,xy}) \delta \omega_x + \delta \omega_y + \delta \omega_{xy}}{2} \\
- \int_A \left[ \delta u_0 (I_1 \dot{u}_0 - I_2 \dot{u}_b - I_3 \dot{u}_s) + \delta v_0 (I_1 \dot{v}_0 - I_2 \dot{v}_b - I_3 \dot{v}_s) \right] dA \\
+ \int_A \left[ \delta w_0 (I_1 \dot{w}_0 + I_2 \dot{w}_b + I_3 \dot{w}_s) + \delta \omega_0 (I_1 \delta \omega_0 - I_2 \delta \omega_b - I_3 \delta \omega_s) \right] dA \right] \, dt = 0
\end{align*}
\]

(9)
The stress resultants $N$, $M$, and $S$ are defined as

$$
(N_x, N_y, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \, dz \\
M^b = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} (\sigma_x, \sigma_y, \tau_{xy}) \, dz \\
(\dot{M}^b, \dot{M}^b, \dot{M}^b) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) z \, dz \\
(M_x, M_y, M_{xy}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} (\sigma_x, \sigma_y, \tau_{xy}) f(z) \, dz \\
(S_{xz}, S_{yz}) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) g(z) \, dz
$$

Inserting Eqs. (7a), (7b), (7c), (7d) into Eqs. (10a), (10b), (10c), (10d) and integrating across the thickness of the plate, the stress resultants are obtained:

$$
\begin{align}
N & = \begin{bmatrix} A & B & B' \end{bmatrix} \begin{bmatrix} A & B & B' \end{bmatrix} \\
M^b & = \begin{bmatrix} B & D & D' \end{bmatrix} \begin{bmatrix} B & D & D' \end{bmatrix} \\
(\dot{M}^b, \dot{M}^b, \dot{M}^b) & = \begin{bmatrix} A_{xx} & A_{xy} & A_{yz} \\
A_{xy} & A_{yy} & A_{xz} \\
A_{yz} & A_{xz} & A_{zz} \end{bmatrix} \begin{bmatrix} \gamma_{xx} \\
\gamma_{xy} \\
\gamma_{xz} \end{bmatrix},
\end{align}
$$

where

$$
\begin{align}
N & = \{N_x, N_y, N_{xy}\}, M^b = \{M_x^b, M_y^b, M_{xy}^b\}, \\
(\dot{M}^b, \dot{M}^b, \dot{M}^b) & = \{M_x^b, M_y^b, M_{xy}^b\},
\end{align}
$$

$$
\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \end{bmatrix}, k^b = \begin{bmatrix} k_{xx}^b & k_{xy}^b & k_{xz}^b \end{bmatrix}, k^\varepsilon = \begin{bmatrix} k_{xx}^\varepsilon & k_{xy}^\varepsilon & k_{xz}^\varepsilon \end{bmatrix}
$$

Collecting the coefficients of $\delta u_0$, $\delta v_0$, $\delta w_b$, and $\delta w_s$ in Eq. (9), the equations of motion are obtained as

$$
\begin{align}
\delta u_0 : N_{xx} + N_{xy} & = I_1 \ddot{u}_0 - I_2 \ddot{w}_b - I_4 \ddot{w}_s \\
\delta v_0 : N_{yx} + N_{yy} & = I_1 \ddot{u}_0 - I_2 \ddot{w}_b - I_4 \ddot{w}_s \\
\delta w_b : M_{xx}^b + 2M_{xy}^b + M_{yy}^b & = q + N \\
\delta w_s : M_{xx}^\varepsilon + 2M_{xy}^\varepsilon + M_{yy}^\varepsilon & = q + N
\end{align}
$$

where $N$ is defined by

$$
N = N_x \dfrac{\partial^2 (w_b + w_s)}{\partial x^2} + N_y \dfrac{\partial^2 (w_b + w_s)}{\partial y^2} + 2N_{xy} \dfrac{\partial^2 (w_b + w_s)}{\partial x \partial y},
$$

Clearly, when the effect of transverse shear deformation is neglected ($w_s = 0$), Eqs. (13a), and (13b) yield the equations of motion of a composite plate based on the classical theory of plates.
Analytical solutions for simply supported rectangular laminates

For antisymmetric cross-ply laminates

The Navier solutions can be developed for rectangular laminates with two sets of simply supported boundary conditions. For antisymmetric cross-ply laminates, the following plate stiffnesses are identically zero:

\[ A_{16} = A_{26} = D_{16} = D_{26} = D_{16}' = D_{26}' = H_{16}' = H_{26}' = 0 \]

\[ B_{12} = B_{26} = B_{16} = B_{66} = B_{12}' = B_{16}' = B_{26}' = B_{66}' = A_{15}' = 0 \]

\[ B_{22} = -B_{11}, B_{22}' = -B_{11}' \]  \hspace{1cm} (14)

The following boundary conditions for antisymmetric cross-ply laminates can be written as:

\[ v(0, y) = w_b(0, y) = w_x(0, y) = \frac{\partial w_b}{\partial y}(0, y) = \frac{\partial w_x}{\partial y}(0, y) = 0 \]

\[ v(a, y) = w_b(a, y) = w_x(a, y) = \frac{\partial w_b}{\partial y}(a, y) = \frac{\partial w_x}{\partial y}(a, y) = 0 \]

\[ N_y(0, y) = M_y^x(0, y) = M_y^x(0, y) = N_y(a, y) = M_y^x(a, y) = 0 \]

\[ u(x, 0) = w_b(x, 0) = w_x(x, 0) = \frac{\partial w_b}{\partial x}(x, 0) = \frac{\partial w_x}{\partial x}(x, 0) = 0 \]

\[ u(x, b) = w_b(x, b) = w_x(x, b) = \frac{\partial w_b}{\partial x}(x, b) = \frac{\partial w_x}{\partial x}(x, b) = 0 \]

\[ N_y(x, 0) = M_y^x(x, 0) = M_y^x(x, 0) = N_y(x, b) = M_y^x(x, b) = 0 \]  \hspace{1cm} (15)

The boundary conditions in Eq. (15) are satisfied by the following expansions:

\[ u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} e^{j\omega t} \cos(\lambda x) \sin(\mu y) \]

\[ v_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} e^{j\omega t} \sin(\lambda x) \cos(\mu y) \]

\[ w_b = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{bmn} e^{j\omega t} \sin(\lambda x) \sin(\mu y) \]

\[ w_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{xmn} e^{j\omega t} \sin(\lambda x) \sin(\mu y) \]  \hspace{1cm} (16)

where \( U_{mn}, V_{mn}, W_{bmn}, \) and \( W_{xmn} \) unknown parameters must be determined, \( \omega \) is the Eigen frequency associated with \((m, n)\) the Eigen-mode, and \( \lambda = \frac{m\pi}{a} \) and \( \mu = \frac{n\pi}{b} \).

The transverse load \( q \) is also expanded in the double-Fourier sine series as follows:

\[ q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \sin(\lambda x) \sin(\mu y) \]  \hspace{1cm} (17)

The coefficients \( Q_{mn} \) are given below for some typical loads:

\[ Q_{mn} = \frac{4}{ab} \int_0^a \int_0^b q(x, y) \sin(\lambda x) \sin(\mu y) dx dy, \]

\[ Q_{mn} = 16q_0 \frac{mn\pi^2}{\pi^2} \]  \hspace{1cm} (18)

Substituting Eqs. (14), (16), and (17) into Eqs. (13a), (13b), the Navier solution of antisymmetric cross-ply laminates can be determined from equations

\[ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} + S & a_{34} + S \\ a_{14} & a_{24} & a_{34} + S & a_{44} + S \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{22} \\ m_{33} \\ m_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ q \end{bmatrix} \]  \hspace{1cm} (19)

where

\[ a_{11} = A_{11} \beta^2 + A_{66} \mu^2, a_{12} = \lambda \mu(A_{12} + A_{66}), \]

\[ a_{13} = -B_{11} \lambda^2, a_{14} = -B_{11}' \lambda^2 \]

\[ a_{22} = A_{66} \lambda^2 + A_{22} \mu^2, a_{23} = B_{11} \mu^2, a_{24} = B_{11}' \mu^2 \]

\[ a_{33} = D_{11} \lambda^4 + 2(D_{12} + 2D_{66}) \lambda^2 \mu^2 + D_{22} \mu^4 \]

\[ a_{44} = H_{11}' \lambda^4 + 2(H_{12} + 2H_{66}) \lambda^2 \mu^2 + H_{22} \mu^4 + A_{44}' \lambda^2 + A_{44}' \mu^2 \]

\[ m_{11} = m_{22}, m_{33} = m_{11} + I_1 (\lambda^2 + \mu^2) \]

\[ m_{34} = I_1 + I_2 (\lambda^2 + \mu^2), m_{44} = I_1 + I_2 (\lambda^2 + \mu^2), S = N_0 \lambda^2 + N_0' \mu^2 \]  \hspace{1cm} (20)

For antisymmetric angle-ply laminates

For antisymmetric angle-ply laminates, the following plate stiffnesses are identically zero:

\[ A_{16} = A_{26} = D_{16} = D_{26} = D_{16}' = D_{26}' = H_{16}' = H_{26}' = 0 \]

\[ B_{11} = B_{12} = B_{22} = B_{66} = B_{11}' = B_{12}' = B_{22}' = B_{66}' = A_{45} = 0 \]  \hspace{1cm} (21)

The following boundary conditions for antisymmetric angle-ply laminates can be written as
The boundary conditions in Eq. (22) are satisfied by the following expansions:

\[ u(0,y) = w_b(0,y) = w_s(0,y) = \frac{\partial w_b}{\partial y}(0,y) = \frac{\partial w_s}{\partial y}(0,y) = 0 \]

\[ u(a,y) = w_s(a,y) = w_s(a,y) = \frac{\partial w_b}{\partial y}(a,y) = \frac{\partial w_s}{\partial y}(a,y) = 0 \]

\[ N_{xy}(0,y) = M_{b}(0,y) = M_{s}(0,y) = N_{xy}(a,y) = M_{b}(a,y) = M_{s}(a,y) = 0 \]

\[ v(x,0) = w_b(x,0) = w_s(x,0) = \frac{\partial w_b}{\partial x}(x,0) = \frac{\partial w_s}{\partial x}(x,0) = 0 \]

\[ v(x,b) = w_b(x,b) = w_s(x,b) = \frac{\partial w_b}{\partial x}(x,b) = \frac{\partial w_s}{\partial x}(x,b) = 0 \]

\[ N_{xy}(x,0) = M_{b}(x,0) = M_{s}(x,0) = N_{xy}(x,b) = M_{b}(x,b) = M_{s}(x,b) = 0. \]

(22)

Numerical results for bending analysis

In this study, various numerical examples are described and discussed for verifying the accuracy of the present models in predicting the static bending, critical buckling load, and free vibration behaviors of simply supported antisymmetric cross-ply and angle-ply laminates. To verify, the results achieved by current models are compared with those of Reddy (1984) and exact solution of elasticity in three dimensions (Pagano 1970). The effectiveness of these present’s theories is the extension component of transverse displacement. The following lamina properties are used:

Material 1 (Noor 1975): \( E_1 = 40E_2, \ G_{12} = G_{13} = 0.6E_1, \ G_{23} = 0.5E_2, \ v_{12} = 0.25 \)

Material 2 (Ren 1990): \( E_1 = 40E_2, \ G_{12} = G_{13} = 0.5E_2, \ G_{23} = 0.6E_2, \ v_{12} = 0.25 \)

Material 3 (Pagano 1970): \( E_1 = 25E_2, \ G_{12} = G_{13} = 0.5E_2, \ G_{23} = 0.2E_2, \ v_{12} = 0.25 \)

For convenience, the following nondimensionalizations are used in presenting the numerical results in graphical and tabular forms:

\[ \vec{u} = \frac{100h^3E_2}{q_0a^4}, \vec{w}(a/2,b/2), \bar{\sigma}_x = \frac{k^2}{q_0a^4}\sigma_x(a/2,b/2), \]

\[ \bar{\tau}_{xy} = \frac{k^2}{q_0a^4}\tau_{xy}(0,0), \bar{\tau}_{xz} = \frac{k}{q_0a^4}\tau_{xz}(0,b/2), \]

\[ \bar{\omega} = \frac{a^2}{2h}\sqrt{\frac{\rho}{E_2}}N = N_{cr} \left( \frac{a^2}{E_2h^3} \right) \]

Numerical results for bending analysis

The static bending solution obtained by setting the time derivative terms and in-plane forces to zero and simplified as

\[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} & a_{14} \\
 a_{12} & a_{22} & a_{23} & a_{24} \\
 a_{13} & a_{23} & a_{33} & a_{34} \\
 a_{14} & a_{24} & a_{34} & a_{44}
\end{bmatrix}
\begin{bmatrix}
 U_{mn} \\
 V_{mn} \\
 W_{nm} \\
 Q_{mn}
\end{bmatrix}
= \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0
\end{bmatrix}
\]

(26)

A simply supported two-layer antisymmetric angle-ply (45°/−45°) laminate under sinusoidal transverse load is considered. Material set 2 is used. The numerical results of nondimensionalized deflection for the square and rectangular plates are shown in Table 1. In the case of thick plates, a considerable difference exists between the results obtained using the various models and the values reported by Ren’s model (Ren 1990). For a /h ratio equal to 4, the deflections predicted by Reddy’s (Reddy 1984), both these theories are 20–25 % lower for a square flat, and 15 % and 20 % lower for a rectangular flat as compared to the values, therefore, obtained by Ren model’s (Ren 1990). The results computed using all the five models are in good agreement with those reported by Reddy (Reddy 1984) and Ren (Ren 1990) for thin plates (a/h = 100). The nondimensionalized deflections of two-layer (45°/−45°) square laminates under sinusoidal transverse load are presented in Fig. 1 for various ratio of modulus \( E_1/E_2 \) (\( G_{12} = G_{13} = 0.5E_2, \ G_{23} = 0.6E_2, m_{12} = 0.25, a/h = 10 \)).
A simply supported two-layer \((0^\circ/90^\circ)\) antisymmetric square laminate under sinusoidal transverse load is considered. The layers have equal thickness. Material set 3 is used. Numerical values of nondimensionalized transverse displacement and in-plane stresses are shown in Table 2. Three-dimensional elasticity results are obtained using the method given by (Pagano 1970). The results clearly indicate that the percentage error with respect to three-dimensional elasticity solution in predicting the transverse displacement and in-plane stresses is very much lesser in the case of present models and the prediction of in-plane normal stresses, \(\sigma_x, \sigma_y\), is very poor.

To further illustrate the accuracy of present theory for wide range of thickness ratio \(a/h\) and material anisotropy \(E_1/E_2\), the variations of dimensionless deflection with respect to thickness ratio and material anisotropy are illustrated in Figs. 2 and 3, respectively. The obtained results are compared with those predicted by (Reddy 1984). Again, the present’s models and existing FSDT give almost identical solutions, whereas CPT underestimates deflections of thick laminates with \(a/h < 20\) due to ignoring shear deformation effects (Fig. 3). The through thickness variations and corresponding values of the in-plane displacement, normal stresses \((\sigma_x, \sigma_y)\), and shear

### Table 1 Nondimensionalized deflections of simply supported two-layer \((45^\circ/-45^\circ)\) square and rectangular laminates under sinusoidal transverse load

| \(a/h\) | Theory | \(\bar{w}\) Square plate \((a = b)\) | \(\bar{w}\) Rectangular plate \((b = 3a)\) |
|--------|--------|-------------------------------|-------------------------------|
| 4      | Model-Ren (Ren 1990) | 1.4471 | 3.9653 |
|        | Model-HSDT (Reddy 1984) | 1.0203 | 3.1560 |
|        | Present model 1 | 1.0220 | 3.0995 |
|        | Present model 2 | 1.0203 | 3.0971 |
| 10     | Model-Ren (Ren 1990) | 0.6427 | 2.3953 |
|        | Model-HSDT (Reddy 1984) | 0.5581 | 2.2439 |
|        | Present model 1 | 0.5583 | 2.2328 |
|        | Present model 2 | 0.5581 | 2.2325 |
| 100    | Model-Ren (Ren 1990) | 0.4676 | 2.0670 |
|        | Model-HSDT (Reddy 1984) | 0.4676 | 2.0670 |

### Table 2 Nondimensionalized deflections and stresses in two-layer \((0^\circ/90^\circ)\) simply supported square laminated plate under sinusoidal transverse load

| \(a/h\) | Theory | \(\bar{w}\) | \(\sigma_x\) | \(\sigma_y\) | \(\tau_{xy}\) |
|--------|--------|-------------|-------------|-------------|-------------|
| 2      | Model-elasticity (Pagano 1970) | 4.9362 | -0.9070 | 1.4480 | -0.0964 |
|        | Model-Reddy (Reddy 1984) | 4.5619 | -1.4277 | 1.4277 | -0.0719 |
|        | Present model 1 | 4.5728 | -1.4256 | 1.4256 | -0.0719 |
|        | Present model 2 | 4.5619 | -1.4277 | 1.4277 | -0.0719 |
| 5      | Model-elasticity (Pagano 1970) | 1.7287 | -0.7723 | 0.8036 | -0.0586 |
|        | Model-Reddy (Reddy 1984) | 1.6670 | -0.8385 | 0.8385 | -0.0558 |
|        | Present model 1 | 1.6680 | -0.8380 | 0.8380 | -0.0558 |
|        | Present model 2 | 1.6670 | -0.8385 | 0.8385 | -0.0558 |
| 10     | Model-elasticity (Pagano 1970) | 1.2318 | -0.7317 | 0.7353 | -0.0540 |
|        | Model-Reddy (Reddy 1984) | 1.2161 | -0.7468 | 0.7468 | -0.0533 |
|        | Present model 1 | 1.2164 | -0.7467 | 0.7467 | -0.0533 |
|        | Present model 2 | 1.2161 | -0.7468 | 0.7468 | -0.0533 |
| 20     | Model-elasticity (Pagano 1970) | 1.1060 | -0.7200 | 0.7206 | -0.0529 |
|        | Model-Reddy (Reddy 1984) | 1.1018 | -0.7235 | 0.7235 | -0.0527 |
|        | Present model 1 | 1.1019 | -0.7235 | 0.7235 | -0.0527 |
|        | Present model 2 | 1.1018 | -0.7235 | 0.7235 | -0.0527 |
| 100    | Model-elasticity (Pagano 1970) | 1.0742 | -0.7219 | 0.7219 | -0.0529 |
|        | Model-Reddy (Reddy 1984) | 1.0651 | -0.7161 | 0.7161 | -0.0525 |
|        | Present model 1 | 1.0651 | -0.7161 | 0.7161 | -0.0525 |
|        | Present model 2 | 1.0651 | -0.7161 | 0.7161 | -0.0525 |
stresses \(\sigma_{xy}, \sigma_{xz}\) are also given in Figs. 4, 5, 6, and 7, respectively, for a moderately thick laminate with \(a/h = 10\). An excellent agreement between the results predicted by the present theories and results of the first-order and the other higher order theories is found in the literature.

**Numerical results for free vibration analysis**

In the case of free vibration, the natural frequencies of the laminates can be obtained by setting the determinant of the coefficient of the following matrix to zero:

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{12} & a_{22} & a_{23} & a_{24} \\
  a_{13} & a_{23} & a_{33} & a_{34} \\
  a_{14} & a_{24} & a_{34} & a_{44}
\end{bmatrix}
\begin{bmatrix}
  m_{11} & 0 & 0 & 0 \\
  0 & m_{22} & 0 & 0 \\
  0 & 0 & m_{33} & m_{34} \\
  0 & 0 & m_{34} & m_{44}
\end{bmatrix}
\]

\[
\begin{align*}
U_{mn} &= 0 \\
V_{mn} &= 0 \\
W_{mn} &= 0
\end{align*}
\]

In Tables 3 and 4, the non-dimensional fundamental frequencies of anti-symmetrically laminated cross-ply plates obtained using different shear deformation theories
are shown for various values of $a/h$ and modules ratios. It can be seen that, in general, the present model gives more accurate results in predicting the natural frequencies than the PSDT (Reddy 1984) and the three-dimensional elasticity solution given in (Noor 1973). It should be noted that unknown functions in the present model are four, while the unknown functions in the higher order shear deformation theories (Reddy 1984) are five. It can be concluded that the present model is not only accurate, but also simple in predicting the natural frequencies of laminated plates.

The variation of natural frequencies with respect to side-to-thickness ratio $a/h$ is presented in Table 5. The natural frequencies obtained using the present model is compared with Reddy’s theory PSDT Reddy (1984), Swaminathan and Patil (2008), and FSDT. In the case of thick plates ($a/h$ ratios 2, 4, 5, and 10) there is a considerable difference between the results computed using the present and the theories of Reddy (1984), Swaminathan and Patil (2008), and (Xiang et al. 2011). The variation of natural frequencies with respect to side-to-thickness ratio $a/h$ for different $E_1/E_2$ ratio is presented. For a four-layered thick plate with $a/h$ ratio equal to 2 and $E_1/E_2$ ratio equal to 3 and 10, the percentage difference in values predicted by present theory is 0.15 and 3.50 % lower as compared to Reddy’s theory PSDT Reddy (1984) and Swaminathan and Patil (2008). At higher range of $E_1/E_2$ ratio equal to 20–40, the percentage difference in values in between both the theories is very much higher, and Reddy’s theory very much overpredicts the natural frequency values. For a four-layered thick plate with $a/h$ ratio equal to 2 and $E_1/E_2$ ratio equal to 20, 30, and 40, the percentage differences in values predicted by present theory are 6, 8, and 9.50 % lower as compared to Reddy’s theory PSDT Reddy (1984) and Swaminathan and Patil (2008) and Xiang et al. (2011). The difference between the models tends to reduce for thin and relatively thin plates. Irrespective of the number of layers, the percentage difference in values between the two theories increases with the increase in the degree of anisotropy. As the number of layer increases, the percentage difference in values between the two theories decreases significantly.

The obtained results of fundamental frequencies are compared with the exact 3D solutions reported by Reddy’s theory (Reddy 1984). Here also the results obtained by the present theories are almost identical with those predicted by existing FSDT. This statement is also firmly demonstrated in Figs. 8 and 9 in which the results obtained by the present theory and FSDT are in excellent agreement for a wide range of thickness ratio $a/h$. According to Table 6 the present results are in good agreement with the results of Reddy PSDT Reddy (1984), Swaminathan and Patil (2008) and Xiang et al. (2011).

**Numerical results for buckling analysis**

For buckling analysis, the applied loads are assumed to be in-plane forces

$$N_x^0 = -N_0, \quad N_y^0 = \gamma N_0, \quad \gamma = \frac{N_x^0}{N_y^0}, \quad N_{xy}^0 = 0. \quad (28)$$

The buckling solution can be obtained from Eq. (19) by setting the time derivative terms and transverse forces to zero:
Table 3  Nondimensional fundamental frequencies of antisymmetric square plates at various values of orthotropy ratio with \( a/h = 5 \)

| No of layers | Theory | \( E_1/E_2 \) |
|--------------|--------|--------------|
|              |        | 3            | 10  | 20  | 30  | 40  |
| \((0^\circ/90^\circ)\) |        |              |     |     |     |     |
| \((0^\circ/90^\circ)\) |        |              |     |     |     |     |
| \((0^\circ/90^\circ)\) |        |              |     |     |     |     |
| \((0^\circ/90^\circ)\) |        |              |     |     |     |     |
| \((0^\circ/90^\circ)\) |        |              |     |     |     |     |
| \((0^\circ/90^\circ)\) |        |              |     |     |     |     |
| \((0^\circ/90^\circ)\) |        |              |     |     |     |     |

Following the condensation of variables procedure to eliminate the in-plane displacements \( U_{nn} \) and \( V_{nn} \), the following system is obtained:

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{12} & a_{22} & a_{23} & a_{24} \\
  a_{13} & a_{23} & a_{33} - N_0(\lambda^2 + \gamma \mu^2) & a_{34} - N_0(\lambda^2 + \gamma \mu^2) \\
  a_{14} & a_{24} & a_{34} - N_0(\lambda^2 + \gamma \mu^2) & a_{44} - N_0(\lambda^2 + \gamma \mu^2)
\end{bmatrix}
\begin{bmatrix}
  U_{nn} \\
  V_{nn} \\
  W_{bmn} \\
  W_{snm}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\]

\[(29)\]

where

\[
\begin{align*}
\bar{a}_{33} &= a_{33} - N_0(\lambda^2 + \gamma \mu^2) \\
\bar{a}_{34} &= a_{34} - N_0(\lambda^2 + \gamma \mu^2) \\
\bar{a}_{43} &= a_{43} - N_0(\lambda^2 + \gamma \mu^2) \\
\bar{a}_{44} &= a_{44} - N_0(\lambda^2 + \gamma \mu^2)
\end{align*}
\]

\[(30)\]

\[
\begin{bmatrix}
  a_{33} - N_0(\lambda^2 + \gamma \mu^2) & a_{34} - N_0(\lambda^2 + \gamma \mu^2) \\
  a_{43} - N_0(\lambda^2 + \gamma \mu^2) & a_{44} - N_0(\lambda^2 + \gamma \mu^2)
\end{bmatrix}
\begin{bmatrix}
  W_{bmn} \\
  W_{snm}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

\[(31)\]
For nontrivial solution, the determinant of the coefficient matrix in Eq. (30) must be zero. This gives the following expression for buckling load:

\[ N_0 = \frac{1}{\lambda^2 + \gamma^2} \times \frac{\tilde{a}_{33} \tilde{a}_{44} - \tilde{a}_{34} \tilde{a}_{43}}{\tilde{a}_{33} + \tilde{a}_{44} - \tilde{a}_{34} - \tilde{a}_{43}} \]  

(32)

A simply supported anti-symmetric cross-ply \((0/90)_n\) square laminate subjected to uniaxial compressive load is considered. Table 7 shows a comparison between the results obtained using the various models and the three-dimensional elasticity solutions given by Noor (1975). The results clearly indicate that the present model gives more accurate results in predicting the buckling loads when compared to Reddy (1984). Compared to the three-dimensional elasticity solution, the buckling loads predicted by present model, Reddy (1984), are 6–7 \%, for four-layer antisymmetric cross-ply \((0/90/0/90)\) square laminates. The effect of side-to-thickness ratio on buckling load of simply supported four-layer \((0/90/0/90)\) square laminates is also presented in Figs. 10 and 11.

In Table 8, a simply supported two-layer anti-symmetric angle-ply \((h/h)\) square laminate subjected to uniaxial compressive is considered. The numerical values of buckling. The results are compared with the values reported by Ren (1990). For all values of side-to-thickness ratio and fiber orientation, the buckling loads predicted by the present model and Reddy (1984) are almost identical. For

**Table 5** Non-dimensionalized fundamental frequencies for a simply supported antisymmetric angle-ply square laminated plate

| No of layers | Theory                        | \(a/h\) |
|--------------|-------------------------------|---------|
|              |                               | 2  | 4  | 5  | 10 | 12.5 | 20 | 25 | 50 | 100 |
| \((45°/-45°)_1\) | Present model 1               | 6.3247 | 9.7517 | 10.8336 | 13.2605 | 13.7058 | 14.2455 | 14.3823 | 14.5722 | 14.6211 |
|              | Present model 2               | 6.1019 | 10.6508 | 12.5342 | 18.3240 | 19.7645 | 21.8072 | 22.3804 | 23.2238 | 24.4508 |
|              | Model-PSDT (Reddy 1984)       | 6.1068 | 10.6508 | 12.5332 | 18.3221 | 19.7621 | 21.8062 | 22.3798 | 23.2237 | 24.4508 |
|              | Model-Swaminathan (Swaminathan and Patil 2008) | 6.1067 | 10.6507 | 12.5331 | 18.3221 | 19.7621 | 21.8063 | 22.3798 | 23.2236 | 24.4507 |
| \((45°/-45°)_2\) | Present model 1               | 6.1608 | 10.9870 | 12.9697 | 19.2659 | 20.8885 | 23.2390 | 23.9092 | 24.9046 | 25.1745 |
|              | Present model 2               | 6.1400 | 10.9906 | 12.9720 | 19.2660 | 20.8885 | 23.2388 | 23.9091 | 24.9046 | 25.1745 |
|              | Model-PSDT (Reddy 1984)       | 6.1286 | 10.9900 | 12.9719 | 19.2659 | 20.8884 | 23.2388 | 23.9091 | 24.9046 | 25.1744 |
|              | Model-Swaminathan (Swaminathan and Patil 2008) | 6.1283 | 10.9871 | 12.9709 | 19.2659 | 20.8885 | 23.2388 | 23.9091 | 24.9046 | 25.1743 |
| \((45°/-45°)_4\) | Present model 1               | 6.2837 | 9.7593 | 10.8401 | 13.2630 | 13.7040 | 14.2463 | 14.3827 | 14.5723 | 14.6214 |
|              | Present model 2               | 6.2837 | 9.7593 | 10.8401 | 13.2630 | 13.7040 | 14.2463 | 14.3827 | 14.5723 | 14.6214 |
|              | Model-PSDT (Reddy 1984)       | 6.2837 | 9.7593 | 10.8401 | 13.2630 | 13.7040 | 14.2463 | 14.3827 | 14.5723 | 14.6214 |
|              | Model-Swaminathan (Swaminathan and Patil 2008) | 6.2837 | 9.7593 | 10.8401 | 13.2630 | 13.7040 | 14.2463 | 14.3827 | 14.5723 | 14.6214 |
The buckling load values predicted by Reddy (1984), and present model are 18–2% lower as compared to the values obtained by Ren (1990). The results computed using all the five models are in a good agreement with those reported by Ren (1990) for thin plates ($a/h = 100$).

The effect of modulus ratio on nondimensionalized uniaxial buckling load of simply supported two-layer $(45/-45)$ square laminate is presented in Fig. 12 ($G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $v_{12} = 0.25$, $a/h = 10$).

### Conclusions

A refined higher order shear deformation theory of plates has been successfully developed for the static, buckling, and free vibration of simply supported laminated plates. The theory allows for a square-law variation in the transverse shear strains across the plate thickness and satisfies the zero-traction boundary conditions on the top and bottom surfaces of the plate without using shear correction factors. The equations of motion were derived from Hamilton’s principle. The accuracy and efficiency of the present models have been demonstrated for static and free vibration behaviors of anti-symmetric cross-ply and angle-ply laminates. The conclusions of this theory are as follows:

- The deflection load obtained using present models (a simpler version of present theory with four unknowns) and other higher-order theories found in the literature (five unknowns) are almost identical.
- Compared to the three-dimensional elasticity solution, the present models give more accurate results of static and dynamic load than the higher order shear deformation theory.
Compared to the three-dimensional elasticity solution, the present theories give more accurate results of deflection and dynamic load than the height order shear deformation theory found in the literature.

The natural frequencies obtained by the proposed model with four unknowns are almost identical to those predicted by the shear deformation theories containing five unknowns.

The buckling load obtained using present’s model (a simpler version of present theory with four unknowns) and height order shear deformation Reddy’s theory (Reddy 1984) (five unknowns) are comparable.

Compared to the three-dimensional elasticity solution, the present model gives more accurate results of buckling load than the height order shear deformation theory.

It can be concluded that the present models proposed prove to be accurate in solving the static, buckling, and dynamic behaviors of anti-symmetric cross-ply and angle-ply laminated composite plates and efficient in predicting the vibration responses of composite plates.

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