SOME UPPER BOUNDS FOR THE $A$-NUMERICAL RADIUS OF $2 \times 2$ BLOCK MATRICES

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Abstract. Let $A = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ be the $2 \times 2$ diagonal operator matrix determined by a positive bounded operator $A$. For semi-Hilbertian operators $X$ and $Y$, we first show that

$$w_A^2 \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \leq \frac{1}{4} \max \{ \|X^T A + Y^T A Y\|_A, \|X^T A X + Y^T A Y\|_A \} + \frac{1}{2} \max \{ w_A(X Y), w_A(Y X) \},$$

where $w_A(\cdot), \| \cdot \|_A$ and $w_A(\cdot)$ are the $A$-numerical radius, $A$-operator seminorm and $A$-numerical radius, respectively. We then apply the above inequality to find some upper bounds for the $A$-numerical radius of certain $2 \times 2$ operator matrices. In particular, we obtain some refinements of earlier $A$-numerical radius inequalities for semi-Hilbertian operators. An upper bound for the $A$-numerical radius of $2 \times 2$ block matrices of semi-Hilbertian space operators is also given.

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle )$ be a complex Hilbert space equipped with the norm $\| \cdot \|$ and let $I$ stand for the identity operator on $\mathcal{H}$. If $\mathcal{M}$ is a linear subspace of $\mathcal{H}$, then $\overline{\mathcal{M}}$ stands for its closure in the norm topology of $\mathcal{H}$. We denote the orthogonal projection onto a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ by $P_\mathcal{M}$. Let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For every operator $T \in \mathbb{B}(\mathcal{H})$ its range is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$, and its adjoint by $T^*$. Throughout this paper, we assume that $A \in \mathbb{B}(\mathcal{H})$ is a positive operator, which induces a positive semidefinite sesquilinear form $\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$. We denote by $\| \cdot \|_A$ the seminorm induced by $\langle \cdot, \cdot \rangle_A$, that is, $\|x\|_A = \sqrt{\langle x, x \rangle_A}$ for every $x \in \mathcal{H}$. Observe that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. Then $\| \cdot \|_A$ is a norm if and only if $A$ is one-to-one, and the seminormed space $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. For $T \in \mathbb{B}(\mathcal{H})$, an operator $R \in \mathbb{B}(\mathcal{H})$ is called an $A$-adjoint operator of $T$ if for every $x, y \in \mathcal{H}$, we have $\langle Tx, y \rangle_A = \langle x, Ry \rangle_A$, that is, $\mathcal{R} = T^* A$. Generally, the existence of an $A$-adjoint operator is not guaranteed. The set of all operators that admit $A$-adjoints is denoted by $\mathbb{B}_A(\mathcal{H})$. By Douglas’ Theorem [8], we have

$$\mathbb{B}_A(\mathcal{H}) = \{ T \in \mathbb{B}(\mathcal{H}) : \mathcal{R}(T^* A) \subseteq \mathcal{R}(A) \}.$$
If \( T \in \mathcal{B}_A(\mathcal{H}) \), then the reduced solution to the equation \( AX = T^*A \) is denoted by \( T^\dagger \), which is called the distinguished \( A \)-adjoint operator of \( T \). It verifies that \( AT^\dagger = T^*A \), \( \mathcal{R}(T^\dagger) \subseteq \overline{\mathcal{R}(A)} \) and \( \mathcal{N}(T^\dagger) = \mathcal{N}(T^*A) \). Note that \( T^\dagger = AT^*A \), where \( A^\dagger \) is the Moore–Penrose inverse of \( A \), which is the unique linear mapping from \( \mathcal{R}(A) \oplus \overline{\mathcal{R}(A)}^\perp \) into \( \mathcal{H} \) satisfying the Moore–Penrose equations:

\[
AXA = A, \quad XAX = X, \quad XA = P_{\overline{\mathcal{R}(A)}} \text{ and } AX = P_{\mathcal{R}(A)} T P_{\overline{\mathcal{R}(A)}}.
\]

In general, \( A^\dagger \not\in \mathcal{B}(\mathcal{H}) \). Indeed, \( A^\dagger \in \mathcal{B}(\mathcal{H}) \) if and only if \( A \) has closed range; see, for example, [14]. Notice that if \( T \in \mathcal{B}_A(\mathcal{H}) \), then \( T^\dagger \in \mathcal{B}_A(\mathcal{H}) \), \( (T^\dagger)^{\dagger} = T^\dagger \) and \( (T^\dagger)^{\sharp} = T^\dagger \). Also, for \( T, S \in \mathcal{B}_A(\mathcal{H}) \), it is easy to see that \( TS \in \mathcal{B}_A(\mathcal{H}) \) and \( (TS)^{\sharp} = S^{\sharp} T^{\sharp}A \). An operator \( U \in \mathcal{B}_A(\mathcal{H}) \) is called \( A \)-unitary if \( \| Ux \|_A = \| U^{\dagger}x \|_A = \| x \|_A \) for all \( x \in \mathcal{H} \). It should be mentioned that, an operator \( U \in \mathcal{B}_A(\mathcal{H}) \) is \( A \)-unitary if and only if \( U^{\dagger}U = U^{\dagger}U^{\dagger}A = P_{\overline{\mathcal{R}(A)}} \). The set of all operators admitting \( A^{1/2} \)-adjoints is denoted by \( \mathcal{B}_{A^{1/2}}(\mathcal{H}) \). Again, by applying Douglas’ Theorem, we obtain

\[
\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : \exists c > 0; \| Tx \|_A \leq c \| x \|_A, \forall x \in \mathcal{H} \}.
\]

Clearly, \( \langle \cdot, \cdot \rangle_A \) induces a seminorm on \( \mathcal{B}_{A^{1/2}}(\mathcal{H}) \). Indeed, if \( T \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \), then it holds that

\[
\| T \|_A := \sup_{0 \neq x \in \mathcal{R}(A)} \frac{\| Tx \|_A}{\| x \|_A} = \sup \{ \| Tx \|_A : x \in \mathcal{H}, \| x \|_A = 1 \} < +\infty.
\]

It can be verified that, for \( T \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \), we have \( \| Tx \|_A \leq \| T \|_A \| x \|_A \) for all \( x \in \mathcal{H} \). This implies that, for \( T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \), we have \( \| TS \|_A \leq \| T \|_A \| S \|_A \). Notice that it may happen that \( \| T \|_A = +\infty \) for some \( T \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{B}_{A^{1/2}}(\mathcal{H}) \). For example, let \( A \) be the diagonal operator on the Hilbert space \( \ell^2 \) given by \( A e_n = \frac{e_n}{n!} \), where \( \{ e_n \} \) denotes the canonical basis of \( \ell^2 \) and consider the left shift operator \( T \in \mathcal{B}(\ell^2) \). An operator \( T \in \mathcal{B}(\mathcal{H}) \) is called \( A \)-positive if \( AT \) is positive. Note that if \( T \) is \( A \)-positive, then \( \| T \|_A = \sup \{ \langle Tx, x \rangle_A : x \in \mathcal{H}, \| x \|_A = 1 \} \). An operator \( T \in \mathcal{B}(\mathcal{H}) \) is said to be \( A \)-selfadjoint if \( AT \) is selfadjoint, that is, \( AT = T^*A \). Obviously, an \( A \)-positive operator is always an \( A \)-selfadjoint operator. Observe that if \( T \) is \( A \)-selfadjoint, then \( T \in \mathcal{B}_A(\mathcal{H}) \). However, it does not hold, in general, that \( T = T^{\dagger} \). More precisely, if \( T \in \mathcal{B}_A(\mathcal{H}) \), then \( T = T^{\dagger} \) if and only if \( T \) is \( A \)-selfadjoint and \( \mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)} \). Note that for \( T \in \mathcal{B}_A(\mathcal{H}) \), \( T^{\dagger}A \) and \( TT^{\dagger} \) are \( A \)-selfadjoint and so

\[
\| TT^{\dagger} \|_A = \| T \|_A^2 = \| T \|_A^2 = \| T^{\dagger} \|_A^2.
\] (1.1)

For an account of results, we refer the reader to [1, 2, 13, 19].

The \( A \)-numerical radius of \( T \in \mathcal{B}(\mathcal{H}) \) (see [3] and the references therein) is defined by

\[
w_A(T) = \sup \{ \| \langle Tx, x \rangle_A \| : x \in \mathcal{H}, \| x \|_A = 1 \}.
\]
It was recently shown in [20, Theorem 2.5] that if $T \in \mathbb{B}_A(\mathcal{H})$, then
\[
 w_A(T) = \sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta} T + (e^{i\theta} T)^*}{2} \right\|_A.
\]
Moreover, it is known that if $T$ is $A$-selfadjoint, then $w_A(T) = \|T\|_A$. One of the important properties of $w_A(\cdot)$ is that it is weakly $A$-unitarily invariant (see [5]), that is,
\[
 w_A(U^* A T U) = w_A(T),
\]
for every $A$-unitary $U \in \mathbb{B}_A(\mathcal{H})$. Another basic fact about the $A$-numerical radius is the power inequality (see [16]), which asserts that
\[
 w_A(T^n) \leq w_A(T)^n \quad (n \in \mathbb{N}).
\]
Notice that it may happen that $w_A(T) = +\infty$ for some $T \in \mathbb{B}(\mathcal{H})$. Indeed, one can take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In fact, if $T \in \mathbb{B}(\mathcal{H})$ is such that $T(\mathcal{N}(A)) \notin \mathcal{N}(A)$, then $w_A(T) = +\infty$. However, $w_A(\cdot)$ is a seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ which is equivalent to the $A$-operator seminorm $\| \cdot \|_A$. Namely, for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have
\[
 \frac{1}{2} \|T\|_A \leq w_A(T) \leq \|T\|_A.
\]
For proofs and more facts about $A$-numerical radius of operators, we refer the reader to [3, 4, 5, 6, 9, 10, 16, 20].

Now, let $\mathbb{A}$ be the $2 \times 2$ diagonal operator matrix whose diagonal entries are $A$, that is, $\mathbb{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. Clearly, $\mathbb{A} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})^+$ and so $\mathbb{A}$ induces the semi-inner product $\langle x, y \rangle_\mathbb{A} = \langle \mathbb{A} x, y \rangle = \langle x_1, y_1 \rangle_A + \langle x_2, y_2 \rangle_A$, for all $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$ and $y = (y_1, y_2) \in \mathcal{H} \oplus \mathcal{H}$. Very recently, inspired by the numerical radius equalities and inequalities for operator matrices in [11, 12, 15, 17, 18], some inequalities for the $A$-numerical radius of $2 \times 2$ operator matrices have been computed in [5, 6]. This paper is also devoted to the study of the $A$-numerical radius of $2 \times 2$ block matrices. More precisely, we first derive an upper bound for the $A$-numerical radius of the off-diagonal operator matrix $\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ defined on $\mathcal{H} \oplus \mathcal{H}$. We then find some upper bounds for the $A$-numerical radius of certain $2 \times 2$ block matrices. In particular, we obtain a refinement on the second inequality (1.4). Finally, we compute a new upper bound for the $A$-numerical radius of $2 \times 2$ operator matrices. Some of the obtained results are new even in the case that the underlying operator $A$ is the identity operator.

2. Results

In order to achieve the goal of this section, we need the following lemmas. The first lemma was recently given in [5, 6].

Lemma 2.1. Let $T, S, X, Y \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then
\(\begin{align*}
&(i) \begin{bmatrix} T & X \\ Y & S \end{bmatrix}^{Z_A} = \begin{bmatrix} T^{Z_A} & Y^{Z_A} \\ X^{Z_A} & S^{Z_A} \end{bmatrix}.\\
&(ii) \left\| \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right\|_A = \left\| \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right\|_A = \max \{\|X\|_A, \|Y\|_A\}.\\
&(iii) w_A\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{w_A(X), w_A(Y)\}.\\
&(iv) w_A\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) = \max\{w_A(X + Y), w_A(X - Y)\}.
\end{align*}\)

In particular, \(w_A\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = w_A(Y)\).

The second lemma is stated as follows.

**Lemma 2.2.** Let \(x, y, z \in \mathcal{H} \oplus \mathcal{H}\) with \(\|z\|_A = 1\). Then
\[
|\langle x, z\rangle_A \langle z, y\rangle_A| \leq \frac{1}{2} (\|x\|_A \|y\|_A + |\langle x, y\rangle_A|).
\]

**Proof.** Notice first that, by \([7]\), we have
\[
|\langle a, c\rangle \langle c, b\rangle| \leq \frac{1}{2} (\|a\| \|b\| + |\langle a, b\rangle|), \tag{2.1}
\]
for every \(a, b, c \in \mathcal{H} \oplus \mathcal{H}\) with \(\|c\| = 1\). Now, let \(x, y, z \in \mathcal{H} \oplus \mathcal{H}\) with \(\|z\|_A = 1\). Since \(\|A^{1/2}\| = 1\), then by using \((2.1)\), we see that
\[
|\langle x, z\rangle_A \langle z, y\rangle_A| = |\langle A^{1/2}x, A^{1/2}z\rangle \langle A^{1/2}z, A^{1/2}y\rangle| \\
\leq \frac{1}{2} (\|A^{1/2}x\| \|A^{1/2}y\| + |\langle A^{1/2}x, A^{1/2}y\rangle|) \\
= \frac{1}{2} (\|x\|_A \|y\|_A + |\langle x, y\rangle_A|).
\]
This proves the desired result. \(\square\)

Now, we are in the position to state an upper bound for the \(A\)-numerical radius of the off-diagonal part \(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\) of a \(2 \times 2\) operator matrix \(\begin{bmatrix} T & X \\ Y & S \end{bmatrix}\) defined on \(\mathcal{H} \oplus \mathcal{H}\).

**Theorem 2.3.** Let \(X, Y \in \mathcal{B}_{A^{1/2}}(\mathcal{H})\). Then
\[
\begin{align*}
\frac{w_A^2\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)}{4} &\leq \frac{1}{4} \max\{\|X^{Z_A}X + YY^{Z_A}\|_A, \|XX^{Z_A} + Y^{Z_A}Y\|_A\} + \frac{1}{2} \max\{w_A(XX), w_A(YY)\}. \\
\end{align*}
\]

**Proof.** Put \(M = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\), \(N = \begin{bmatrix} XX^{Z_A} & 0 \\ 0 & YY^{Z_A} \end{bmatrix}\), \(P = \begin{bmatrix} Y^{Z_A} & 0 \\ 0 & X^{Z_A} \end{bmatrix}\) and \(Q = \begin{bmatrix} XY & 0 \\ 0 & YX \end{bmatrix}\), to simplify the writing. Then
\[
P + N = \begin{bmatrix} XX^{Z_A} + Y^{Z_A}Y & 0 \\ 0 & X^{Z_A}X + YY^{Z_A} \end{bmatrix}.
\]
Furthermore, by Lemma 2.1(i), we get

$$MM^{z_A} = N, \quad M^{z_A}M = P \quad \text{and} \quad M^2 = Q.$$  \hspace{1cm} (2.3)

Now, let \( z \in \mathcal{H} \oplus \mathcal{H} \) with \( \|z\|_A = 1 \). Since \( \langle Mz, z \rangle_A = \langle z, M^{z_A}z \rangle_A \), we have

$$2\|Mz, z \rangle_A^2 = 2\|Mz, z \rangle_A \|M^{z_A}z \rangle_A \| \langle Mz, M^{z_A}z \rangle_A \| \leq \|Mz\|_A \|M^{z_A}z\|_A + \|Mz, z \rangle_A \| \langle Mz, M^{z_A}z \rangle_A \| \leq \sqrt{\|Mz, z \rangle_A \langle M^{z_A}z, M^{z_A}z \rangle_A + \|Mz, z \rangle_A \| \langle M^{z_A}z, M^{z_A}z \rangle_A} = \sqrt{\|Pz, z \rangle_A \langle Nz, z \rangle_A + \|Qz, z \rangle_A} \leq \frac{1}{2} \left(Pz, z \rangle_A + (Nz, z \rangle_A \right) + w_A(Q) \leq \frac{1}{2} \left(P + N\right) + \max \{w_A(XY), w_A(YX)\} \leq \frac{1}{2} \left\|P + N\right\|_A + \max \{w_A(XY), w_A(YX)\} \quad (\text{since } P + N \text{ is an } A\text{-positive operator})

= \frac{1}{2} \max \left\{\|XX^{z_A} + YY^{z_A}\|_A, \|XX^{z_A} + Y^{z_A}Y\|_A\right\} + \max \{w_A(XY), w_A(YX)\}. \quad (\text{by Lemma 2.1(ii) and (2.2)})$$

Hence

$$\|Mz, z \rangle_A^2 \leq \frac{1}{4} \max \left\{\|XX^{z_A} + YY^{z_A}\|_A, \|XX^{z_A} + Y^{z_A}Y\|_A\right\} + \frac{1}{2} \max \{w_A(XY), w_A(YX)\}.$$ 

Taking the supremum in the above inequality over \( z \in \mathcal{H} \oplus \mathcal{H} \) with \( \|z\|_A = 1 \), we deduce the desired inequality.

**Remark 2.4.** Let \( X, Y \in \mathbb{B}_{A^{1/2}}(\mathcal{H}) \). By Theorem 2.3, (1.1) and (1.4), we have

$$w_A \left( \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right) \leq \frac{1}{4} \max \left\{\|XX^{z_A} + YY^{z_A}\|_A, \|XX^{z_A} + Y^{z_A}Y\|_A\right\} + \frac{1}{2} \max \{w_A(XY), w_A(YX)\} \leq \frac{1}{4} \max \left\{\|XX^{z_A}\|_A, \|YY^{z_A}\|_A, \|XX^{z_A}\|_A, \|YY^{z_A}\|_A\right\} + \frac{1}{2} \max \{\|XY\|_A, \|YX\|_A\} \leq \frac{1}{4} \left(\|X\|_A^2 + \|Y\|_A^2\right) + \frac{1}{2} \|XX\|_A \|YY\|_A \leq \frac{\|X\|_A + \|Y\|_A}{2} \|X\|_A \|YY\|_A \leq \frac{\|X\|_A + \|Y\|_A}{2} \|X\|_A \|YY\|_A \leq \frac{\|X\|_A + \|Y\|_A}{2}.$$

and hence

$$w_A \left( \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right) \leq \frac{\|X\|_A + \|Y\|_A}{2}. \quad (2.4)$$
On the other hand, since \( \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}^2 = \begin{bmatrix} XY & 0 \\ 0 & YX \end{bmatrix} \), by the power inequality (1.3) and Lemma 2.1(iii), we have
\[
\begin{align*}
  w^2_A \left( \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right) &\geq w_A \left( \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}^2 \right) = w_A \left( \begin{bmatrix} XY & 0 \\ 0 & YX \end{bmatrix} \right) = \max \{ w_A(XY), w_A(YX) \}
\end{align*}
\]
and so
\[
\max \{ w^{1/2}_A(XY), w^{1/2}_A(YX) \} \leq w_A \left( \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right).
\]
Therefore, from (2.4) and (2.5) it follows that
\[
\max \{ w^{1/2}_A(XX), w^{1/2}_A(YX) \} \leq w_A \left( \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right) \leq \| X \|_A + \| Y \|_A.
\]

As an immediate consequence of the preceding theorem, we give an improvement of the second inequality in (1.4).

Corollary 2.5. [20, Theorem 2.11] Let \( X \in \mathbb{B}_{A^{1/2}}(\mathcal{H}) \). Then
\[
w_A(X) \leq \frac{1}{2} \sqrt{\| X^{2,1}X + XX^{2,1} \|_A} + 2w_A(X^2).
\]

Proof. By letting \( Y = X \) in Theorem 2.3, we have
\[
w^2_A \left( \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \right) \leq \frac{1}{4} \| X^{2,1}X + XX^{2,1} \|_A + \frac{1}{2} w_A(X^2).
\]
Now, by Lemma 2.1 (iv), we deduce the desired result. \( \square \)

The following results are another consequences of Theorem 2.3 for certain \( 2 \times 2 \) operator matrices.

Corollary 2.6. Let \( X \in \mathbb{B}_{A^{1/2}}(\mathcal{H}) \). Then
\[
w_A \left( \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) = w_A \left( \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix} \right) = \frac{1}{2} \| X \|_A.
\]

Proof. By letting \( Y = 0 \) in Theorem 2.3, we have
\[
w^2_A \left( \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) \leq \frac{1}{4} \| X^{2,1}X \|_A,
\]
wherefrom, by (1.1) we obtain
\[
w_A \left( \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) \leq \frac{1}{2} \| X \|_A.
\]
Also, by (1.4) and Lemma 2.1 (ii), we have
\[
\frac{1}{2} \| X \|_A = \frac{1}{2} \left\| \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right\|_A \leq w_A \left( \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right).
\]
Thus \( w_A \left( \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) = \frac{1}{2} \| X \|_A. \) \( \square \)

Corollary 2.7. Let \( T, S, X, Y \in \mathbb{B}_{A^{1/2}}(\mathcal{H}) \). Then
(i) \( w_A \left( \begin{bmatrix} T & X \\ 0 & 0 \end{bmatrix} \right) \leq w_A(T) + \frac{1}{2} \|X\|_A. \)

(ii) \( w_A \left( \begin{bmatrix} 0 & 0 \\ Y & S \end{bmatrix} \right) \leq w_A(S) + \frac{1}{2} \|Y\|_A. \)

**Proof.** By the triangle inequality for \( w_A(\cdot) \), Lemma 2.1 (iii) and Corollary 2.6 it follows that

\[
w_A \left( \begin{bmatrix} T & X \\ 0 & 0 \end{bmatrix} \right) = w_A \left( \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) \leq w_A \left( \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \right) + w_A \left( \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) = w_A(T) + \frac{1}{2} \|X\|_A.
\]

Hence \( w_A \left( \begin{bmatrix} T & X \\ 0 & 0 \end{bmatrix} \right) \leq w_A(T) + \frac{1}{2} \|X\|_A. \)

By a similar argument, we get the inequality (ii). \( \square \)

**Corollary 2.8.** Let \( T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H}) \). Then

(i) \( w_A \left( \begin{bmatrix} T & S \\ T & S \end{bmatrix} \right) \leq w_A(T - S) + \frac{1}{2} \|T + S\|_A. \)

(ii) \( w_A \left( \begin{bmatrix} T & S \\ -T & -S \end{bmatrix} \right) \leq w_A(T + S) + \frac{1}{2} \|T - S\|_A. \)

**Proof.** (i) Let \( U = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} I \\ -I \end{array} \right] \). Then \( U \) is \( A \)-unitary. Consequently, by the identity (1.2) and Corollary 2.7, we have

\[
w_A \left( \begin{bmatrix} T & S \\ T & S \end{bmatrix} \right) = w_A \left( U^* A U \begin{bmatrix} T & S \\ T & S \end{bmatrix} \right) = w_A \left( \begin{bmatrix} 0 & 0 \\ T - S & T + S \end{bmatrix} \right) \leq w_A(T + S) + \frac{1}{2} \|T - S\|_A,
\]

and hence \( w_A \left( \begin{bmatrix} T & S \\ T & S \end{bmatrix} \right) \leq w_A(T - S) + \frac{1}{2} \|T + S\|_A. \)

(ii) The proof is similar to (i) and so we omit it. \( \square \)

**Corollary 2.9.** Let \( X \in \mathbb{B}_{A^{1/2}}(\mathcal{H}) \). Then

(i) \( w_A \left( \begin{bmatrix} 0 & X \\ X^* A & 0 \end{bmatrix} \right) = \|X\|_A. \)

(ii) \( \max \left\{ \|X^* A X + X A X^* A\|_A, \|X^* A X + (X^* A)^2 A A\|_A \right\} \leq 2 \|X\|^2_A. \)

**Proof.** First, note that \( XX^* A \) and \( X^* A X \) are \( A \)-selfadjoint, and then by (1.1) we have

\[
w_A(XX^* A) = \|XX^* A\|_A = \|X\|^2_A = \|XX^* A\|_A = w_A(XX^* A).
\]

Also, since \( \left[ \begin{bmatrix} 0 & X \\ X^* A & 0 \end{bmatrix} \right] = \left[ \begin{bmatrix} 0 & X \end{bmatrix} \right] \), by Lemma 2.1 (iii) and (2.6) we obtain
Further, by the power inequality (1.3), Theorem 2.3, (1.1) and (1.4) we have

\[ \|X\|_A^2 = w_A \left( \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \right)^2 \leq w_A \left( \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \right) \]

\[ \leq \frac{1}{4} \max \left\{ \|X A X + X^* A (X^* A)^*\|_A, \|X A X + (X^* A)^* A X^* A\|_A \right\} \]

\[ + \frac{1}{2} \max \{ w_A(X A X^*), w_A(X^* A X) \} \]

\[ \leq \frac{1}{2} \|X\|_A^2 + \frac{1}{2} \|X\|_A^2 = \|X\|_A^2. \]

Therefore, \( w_A^2 \left( \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \right) = \|X\|_A^2 \) and

\[ \frac{1}{4} \max \left\{ \|X A X + X^* A (X^* A)^*\|_A, \|X A X + (X^* A)^* A X^* A\|_A \right\} = \frac{1}{2} \|X\|_A^2. \]

Now, we deduce the desired results. \( \square \)

In the following theorem, we establish an upper bound for the \( A \)-numerical radius of \( 2 \times 2 \) block matrices of semi-Hilbertian space operators.

**Theorem 2.10.** Let \( T, S, X, Y \in \mathbb{B}_{A^{1/2}}(\mathcal{H}) \). Then

\[ w_A^2 \left( \begin{bmatrix} T & X \\ Y & S \end{bmatrix} \right) \leq w_A^2 \left( \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right) + w_A \left( \begin{bmatrix} 0 & X S \\ Y T & 0 \end{bmatrix} \right) \]

\[ + \max \{ w_A(T), w_A(S) \} + \frac{1}{2} \max \left\{ \|T A T + X A X^*\|_A, \|S A S + Y A Y^*\|_A \right\}. \]

**Proof.** We will assume that \( M = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}, N = \begin{bmatrix} X A X^* & 0 \\ 0 & Y A Y^* \end{bmatrix}, P = \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}, \)

\( Q = \begin{bmatrix} T A T & 0 \\ 0 & S A S \end{bmatrix} \) and \( R = \begin{bmatrix} 0 & X S \\ Y T & 0 \end{bmatrix}, \) to simplify notations.

Therefore,

\[ MP = R \quad \text{and} \quad Q + N = \begin{bmatrix} T A T + X A X^* & 0 \\ 0 & S A S + Y A Y^* \end{bmatrix}. \quad (2.7) \]

Also, by Lemma 2.1(i), we obtain

\[ MM^2 = N \quad \text{and} \quad P^2 = Q. \quad (2.8) \]

Now, let \( z \in \mathcal{H} \oplus \mathcal{H} \) with \( \|z\|_A = 1 \). Since \( \begin{bmatrix} T & X \\ Y & S \end{bmatrix} = P + M, \) we have
\[
\left|\begin{bmatrix} T & X \\ Y & S \end{bmatrix}\right|^2 = \left|\begin{bmatrix} (P + M)z, z \end{bmatrix}_A\right|^2 \\
= \left|\begin{bmatrix} Pz, z \end{bmatrix}_A + \begin{bmatrix} Mz, z \end{bmatrix}_A\right|^2 \\
\leq \left(\left|\begin{bmatrix} Pz, z \end{bmatrix}_A\right| + \left|\begin{bmatrix} Mz, z \end{bmatrix}_A\right|\right)^2 \\
= \left|\begin{bmatrix} Pz, z \end{bmatrix}_A\right|^2 + \left|\begin{bmatrix} Mz, z \end{bmatrix}_A\right|^2 + 2\left|\begin{bmatrix} Pz, z \end{bmatrix}_A\right|\left|\begin{bmatrix} Mz, z \end{bmatrix}_A\right| \\
\leq w^2_A(P) + w^2_A(M) + 2\left|\begin{bmatrix} Pz, z \end{bmatrix}_A\right|\left|\begin{bmatrix} Mz, z \end{bmatrix}_A\right| \\
\leq w^2_A(P) + w^2_A(M) + \|Pz\|_A\|Mz\|_A \quad \text{(by Lemma 2.2)} \\
= w^2_A(P) + w^2_A(M) + \|MPz, z\|_A + \sqrt{(Pz, Pz)_A(Mz, Mz)_A} \\
= w^2_A(P) + w^2_A(M) + \|Rz, z\|_A + \sqrt{(Pz, Pz)_A(Mz, Mz)_A} \quad \text{(by (2.7))} \\
= w^2_A(P) + w^2_A(M) + \|Rz, z\|_A + \sqrt{(Qz, z)_A(Nz, z)_A} \quad \text{(by (2.8))} \\
\leq w^2_A(P) + w^2_A(M) + w_A(R) + \frac{1}{2}\left(\|Q\|_A + \|N\|_A\right) \\
\quad \text{(by the arithmetic-geometric mean inequality)} \\
= w^2_A(P) + w^2_A(M) + w_A(R) + \frac{1}{2}\|Q + N\|_A. \\
\] 

Thus

\[
w^2_A\left(\begin{bmatrix} T & X \\ Y & S \end{bmatrix}\right) \leq w^2_A(P) + w^2_A(M) + w_A(R) + \frac{1}{2}\|Q + N\|_A.
\]

Now, by (2.7) and Lemma 2.1(ii)-(iii), we deduce the desired result. \qed

As an application of Theorem 2.10, we obtain the following result.

**Corollary 2.11.** Let \(T, X \in \mathbb{B}_{A^{1/2}}(\mathcal{H})\). Then

\[
\max\{w_A(T), w_A(X)\} + \frac{1}{2}|w_A(T + X) - w_A(T - X)| \\
\leq \max\{w_A(T + X), w_A(T - X)\} \\
\leq \sqrt{w^2_A(X) + w_A(AXT) + w^2_A(T) + \frac{1}{2}\|XX^A + T^2AT\|_A}.
\]

**Proof.** By letting \(S = T\) and \(Y = X\) in Theorem 2.10, and using Lemma 2.1 (iv), we have

\[
w^2_A\left(\begin{bmatrix} T & X \\ X & T \end{bmatrix}\right) \leq w^2_A\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right) + w_A\left(\begin{bmatrix} 0 & XT \\ XT & 0 \end{bmatrix}\right) \\
+ \max\{w^2_A(T), w^2_A(T)\} + \frac{1}{2}\max\{\|T^2AT + XX^A\|_A, \|T^2AT + XX^A\|_A\} \\
= w^2_A(X) + w_A(AXT) + w^2_A(T) + \frac{1}{2}\|T^2AT + XX^A\|_A.
\]

Thus

\[
w^2_A\left(\begin{bmatrix} T & X \\ X & T \end{bmatrix}\right) \leq \sqrt{w^2_A(X) + w_A(AXT) + w^2_A(T) + \frac{1}{2}\|XX^A + T^2AT\|_A}.
\]
Thus we have
\[
\begin{bmatrix} T & X \\ X & T \end{bmatrix} \leq w_A^2(X) + w_A(XT) + w_A^2(T) + \frac{1}{2} \| T^* T + XX^* T \|_A.
\]

Since by Lemma 2.1 (iv) we have
\[
w_A \begin{bmatrix} T \\ X \\ T \end{bmatrix} = \max \{ w_A(T+X), w_A(T-X) \},
\]
therefore by the above inequality we get
\[
\max \{ w_A(T + X), w_A(T - X) \} \leq \sqrt{w_A^2(X) + w_A(XT) + w_A^2(T) + \frac{1}{2} \| T^* T + XX^* T \|_A}.
\]
(2.9)

On the other hand, by the triangle inequality for \( w_A(\cdot) \), we have
\[
\max \{ w_A(T + X), w_A(T - X) \} = \frac{w_A(T + X) + w_A(T - X)}{2} + \frac{1}{2} | w_A(T + X) - w_A(T - X) | \\
\geq \frac{w_A(T + X + T - X)}{2} + \frac{1}{2} | w_A(T + X) - w_A(T - X) |\\
= w_A(T) + \frac{1}{2} | w_A(T + X) - w_A(T - X) |.
\]

Thus
\[
w_A(T) + \frac{1}{2} | w_A(T + X) - w_A(T - X) | \leq \max \{ w_A(T + X), w_A(T - X) \}.
\]
(2.10)

Similarly,
\[
w_A(X) + \frac{1}{2} | w_A(T + X) - w_A(T - X) | \leq \max \{ w_A(T + X), w_A(T - X) \}.
\]
(2.11)

So, by (2.10) and (2.11), we obtain
\[
\max \{ w_A(T), w_A(X) \} + \frac{1}{2} | w_A(T + X) - w_A(T - X) | \leq \max \{ w_A(T + X), w_A(T - X) \}.
\]
(2.12)

Finally, the result follows from the inequalities (2.9) and (2.12).

\[\square\]

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