Disentanglement is the process which transforms a state $\rho$ of two subsystems into an unentangled state, while not affecting the reduced density matrices of each of the two subsystems. Recently Terno [1] showed that an arbitrary state cannot be disentangled into a tensor product of its reduced density matrices. In this letter we present various novel results regarding disentanglement of states. Our main result is that there are sets of states which cannot be successfully disentangled (not even into a separable state). Thus, we prove that a universal disentangling machine cannot exist.

Definition.— Disentanglement into a tensor product state is the process that transforms a state of two (or more) subsystems into a tensor product of the two reduced density matrices.

We noticed that according to these definitions, when a successful disentanglement is applied onto any pure product state, the state must be left unmodified. That is, $|00\rangle \rightarrow |00\rangle$ (in an appropriate basis). This fact proved very useful in the analysis we report here.

The main goal of this letter is to show that a universal disentangling machine cannot exist. A universal disentangling machine is a machine that could disentangle any state which is given to it as an input. In order to prove that such a machine cannot exist, it is enough to find one set of states that cannot be disentangled if the data (regarding which state is used) is not available.

To analyze the process of disentanglement consider the following experiment involving two subsystems “X” and “Y”, and a sender who sends both systems to the receiver who wishes to disentangle the state of these two subsystems: Let the sender (Alice) and the disentangler (Eve) define a finite set of states $|\psi_i\rangle$; let Alice choose one of the states at random, and let it be the input of the disentangling machine designed by Eve. Eve does not get from Alice the data regarding which of the states Alice chose, so Eve’s aim is to design a machine that will succeed to disentangle any of the possible states $|\psi_i\rangle$.

In the same sense that an arbitrary state cannot be cloned (a universal cloning machine does not exist [2]), it was recently shown by Terno [1] that an arbitrary state cannot be disentangled into a tensor product of its reduced density matrices. Note that this novel result of [1] proves that universal disentanglement into product states is impossible, and it leaves open the more general question of whether a universal disentanglement is impossible (that is, disentanglement into separable states).

We extend the investigation of the process of disentanglement well beyond Terno’s novel analysis in several
ways. First, we find a larger class (then the one found by Terno) of states which cannot be disentangled into product states. Then, we show that there are non-trivial sets of states that can be disentangled. In particular, we present a set of states that cannot be disentangled into tensor product states, but can be disentangled into separable states. Finally, we present our most important result: a set of states that cannot be disentangled. The existence of such a set of states proves that a universal disentangling machine cannot exist. Using the terminology of Ref. we can say that our letter shows that a single quantum cannot be disentangled.

Consider a set of states containing only one state. Since the state is known, obviously it can be disentangled. E.g., it is replaced by the appropriate tensor product state.

We first prove that there are infinitely many sets of states that cannot be disentangled into product states. Our proof here follows from Terno’s method, with the addition of using the Schmidt decomposition to analyze a larger class of states. The most general form of two entangled states can always be presented (by an appropriate choice of bases) as:

\[
|\psi_0\rangle = \cos \phi_0 |00\rangle + \sin \phi_0 |11\rangle \\
|\psi_1\rangle = \cos \phi_1 |00\rangle + \sin \phi_1 |11\rangle .
\]

(4)

To prove that there are states for which disentangling into tensor product states is impossible, let us restrict ourselves to the simpler subclass

\[
|\psi_0\rangle = \cos \phi |00\rangle + \sin \phi |11\rangle \\
|\psi_1\rangle = \cos \phi |00\rangle + \sin \phi |11\rangle .
\]

(5)

There exists some basis

\[
|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

(6)

such that the bases vectors $|0\rangle ; |1\rangle$ and $|0\rangle ; |1\rangle$ become

\[
|0\rangle = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} ; |1\rangle = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix},
\]

(7)

and

\[
|0\rangle = \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix} ; |1\rangle = \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix},
\]

(8)

respectively, in that basis. The states [1] are now

\[
|\psi_0\rangle = c_0 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + s_0 \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix},
\]

(9)

with $c_0 \equiv \cos \phi$, etc. The overlap of the two states is $\langle \psi_0 | \psi_1 \rangle \equiv \cos^2 \theta + \sin 2\phi \sin^2 \theta$. The reduced states are given by

\[
\hat{\rho}_0 = c_0^2 \begin{pmatrix} \cos^2 \theta \\ -\sin \theta \sin \theta \\ \cos \theta \cos \theta \end{pmatrix} + s_0^2 \begin{pmatrix} \sin^2 \theta \\ -\sin \theta \sin \theta \\ \cos \theta \cos \theta \end{pmatrix},
\]

(10)

and

\[
\hat{\rho}_1 = c_0^2 \begin{pmatrix} \cos^2 \theta \\ -\sin \theta \sin \theta \\ \cos \theta \cos \theta \end{pmatrix} + s_0^2 \begin{pmatrix} \sin^2 \theta \\ -\sin \theta \sin \theta \\ \cos \theta \cos \theta \end{pmatrix}.
\]

(11)

Thus, the state after the disentangling into tensor product states is $(\hat{\rho}_{\text{disent}}) = \hat{\rho}_0 \hat{\rho}_0 - \hat{\rho}_1 \hat{\rho}_1$.

The minimal probability of error for distinguishing two states [1] is given by PE = $\frac{1}{2} - \frac{1}{4} Tr[\rho_0 - \rho_1]$. For two pure states there is a simpler expression: PE = $\frac{1}{2} - \frac{1}{2} \sqrt{1 - OL^2}$. Thus,

\[
PE_{\text{ent}} = \frac{1}{2} - \frac{1}{2} \sqrt{1 - (c_0^2 + s_0 \sin 2\phi)^2}
\]

(12)

for the two initial entangled states. This probability of error is minimal, hence it cannot be reduced by any physical process. Therefore, if, for some $\theta$ and $\phi$, the disentanglement into the tensor product states reduces the PE, then that process is non-physical.

The difference of the states obtained after disentangling into tensor product states is $\Delta_{\text{disent}} = \rho_0 \rho_0 - \rho_1 \rho_1$ This matrix is

\[
\Delta_{\text{disent}} = \cos 2\phi \sin 2\theta \begin{pmatrix} a & a & 0 \\ a & 0 & b \\ 0 & b & 0 \end{pmatrix},
\]

(13)

with $a = \cos^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta$ and $b = \cos^2 \phi \sin^2 \theta + \sin^2 \theta$. After diagonalization, we can calculate the Trace-Norm, so finally we get

\[
PE_{\text{disent}} = \frac{1}{2} - \frac{1}{\sqrt{2}} \sin 2\theta \cos 2\phi \sqrt{a^2 + b^2}
\]

\[
= \frac{1}{2} - \frac{1}{2} \sqrt{2} \sqrt{\sin^2 \theta \cos^2 \phi} (1 + c_0^2 s_0^2).
\]

(14)

We can now observe that there are values of $\theta$ and $\phi$, e.g., $\theta = \phi = \pi/8$, for which the outcomes of the disentanglement process are illegitimate since they satisfy $PE_{\text{disent}} < PE_{\text{ent}}$. Once these outcomes are illegitimate the disentanglement process leading to these outcomes is non-physical, proving that a disentangling machine which disentangles the states $|\psi_0\rangle$ and $|\psi_1\rangle$ cannot exist for these values of $\theta$ and $\phi$. Therefore, this analysis provides a proof (similar to Terno’s proof [1]) that a universal machine performing disentanglement into tensor product states cannot exist.

The following set of states can easily be disentangled:

\[
|\psi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) ; \\
|\psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle).
\]

(15)
To disentangle them, Eve’s machine uses an ancilla which is another pair of particles in a maximally entangled state (e.g., the singlet state) in any basis. Eve’s machine swaps one of the above particles with one of the members of the added pair, and traces out the ancillary particles. As a result, the state of the remaining two particles (one from each entangled pair) is

\[(1/4)|00⟩⟨00| + |01⟩⟨01| + |10⟩⟨10| + |11⟩⟨11|, \quad (16)\]

the completely mixed state in four dimensions. This set provides a trivial example of the ability to perform the disentanglement process. It is a trivial case of disentanglement, since the two states are orthogonal: they can first be measured and distinguished, and then, once the state is known, clearly it can be disentangled.

However, exactly the same disentanglement process can be used to successfully disentangle a non-trivial set of states. Let the basis used for the two states be a different basis (and not the same basis), so the first state is still \(|ψ_0⟩\), and the second state is

\[|ψ'_1⟩ = \frac{1}{\sqrt{2}}(|0′0′⟩ - |1′1′⟩). \quad (17)\]

The same process of disentanglement still works, while now the states are non-orthogonal, and cannot always be successfully distinguished. Hence, this disentanglement process is non-trivial. Note that the same process successfully works also when more than two maximally entangled states are used as the possible inputs.

Before we continue, let us recall some proofs of the no-cloning argument, since the methods we use here are quite similar the those used in the no-cloning argument. Let the cloner obtain an unknown state and try to clone it. To prove that this is impossible, it is enough to provide one set of states for which the cloner cannot clone any state in this set. Let the sender (Alice) and the cloner (Eve) use three states \(|0⟩, |1⟩, \text{and } |+⟩ = (1/√2)(|0⟩ + |1⟩).\) The most general process which can be used here in the attempt of cloning the unknown state from this set is to attach an ancilla in an arbitrary dimension and in a known state (say \(|E⟩\)), to transform the entire system using an arbitrary unitary transformation, and to trace out the unrequired parts of the ancilla. In order to clone the states \(|0⟩\) and \(|1⟩\) the transformations are restricted to be

\[|E0⟩ → |E00⟩ ; \quad |E1⟩ → |E11⟩ \quad (18)\]

and once the remaining ancilla is traced out, the cloning process is completed. Due to linearity, this fully determine the transformation of the last state to be

\[|E+⟩ → \frac{1}{\sqrt{2}}(|E00⟩ + |E11⟩), \quad (19)\]

while a cloning process should yield

\[|E+⟩ → |E++⟩. \quad (20)\]

The contradiction is clearly apparent since, once the remaining ancilla is traced out, the second expression has a non-zero amplitude for the term \(|01⟩\) while the first expression does not. The conventional way [3] of proving the no-cloning theorem (using only two states, say \(|0⟩\) and \(|+⟩\)) is to compare the overlap before and after the transformation (it must be equal due to the unitarity of quantum mechanics): We obtain that \(⟨E|E⟩⟨0|+⟩ = ⟨E0|E+⟩⟨0|+⟩. \) Hence \(1 = ⟨E0|E+⟩⟨0|+⟩\) which is obviously wrong since all the terms on the right hand side are smaller than one.

We shall now use the linearity of quantum mechanics to show that there are states that cannot be disentangled into tensor product states, but can only be disentangled into a mixture of tensor product states. Surprisingly, our proof is mainly based on the disentanglement of product states, that is, on the disentanglement of states which are anyhow not entangled even before the disentanglement process. The reason for the usefulness of such states is that they provide strict restrictions on the allowed transformations.

The following set of states cannot be disentangled into product states:

\[|ψ_0⟩ = |00⟩\]
\[|ψ_1⟩ = |11⟩\]
\[|ψ_2⟩ = |00⟩ + |11⟩ \quad (21)\]

We shall assume that these states can be disentangled into product states and we shall reach a contradiction. Note that the resulting states should be \(|ψ_0⟩\) and \(|ψ_1⟩\) in the first two cases (see Eq. [3]), and the resulting state should be the completely mixed state (in 4 dimensions) in the last case (see Eq. [11]).

The most general process which can be used here is to attach an ancilla in an arbitrary dimension and in a known state (say \(|E⟩\)), to transform the entire system using an arbitrary unitary transformation, and to trace out the ancilla. In order to avoid changing the states \(|ψ_0⟩\) and \(|ψ_1⟩\) the transformations are restricted to be

\[|Eψ_0⟩ = |E00⟩ → |E000⟩\]
\[|Eψ_1⟩ = |E11⟩ → |E111⟩. \quad (22)\]

As in the no-cloning argument, these transformations fully determine the transformation of the last state to be

\[|Eψ_2⟩ → \frac{1}{\sqrt{2}}(|E00⟩ + |E11⟩). \quad (23)\]

Once we trace out the ancilla, the resulting state is still entangled unless \(|E0⟩\) and \(|E1⟩\) are orthogonal. The proof of that statement is as follows: Without loss of generality the states \(|E0⟩\) and \(|E1⟩\) can be written as \(|E0⟩ = |0⟩\)
and $|E_1\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. Thus, $|E\psi_2\rangle \rightarrow \frac{1}{\sqrt{2}}\biggl[|0\rangle(00) + \alpha|11\rangle + \frac{\beta}{\sqrt{2}}|11\rangle\biggr]$. When the ancilla is traced out the remaining state is

$$
\begin{pmatrix}
1/2 & 0 & 0 & \alpha^*/2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\alpha/2 & 0 & 0 & 1/2
\end{pmatrix}.
$$

(24)

The resulting state is entangled unless $\alpha = 0$. Thus, in a successful disentanglement process $\alpha = 0$ and hence, $|E_1\rangle = e^{i\theta}|1\rangle$. This state, however, is not a tensor product state, thus the above set of states cannot be disentangled into tensor product states.

At the same time, this example also shows that the above set of states can be disentangled into a mixture of tensor product states. The resulting state (24) still has the correct reduced density matrices for each subsystem—the completely mixed state in two dimensions. With $\alpha = 0$, the resulting state is $\frac{1}{2}\bigl[[|11\rangle \langle 00| + |11\rangle \langle 11|]\bigr]$, so we succeeded in showing an example where the states can only be disentangled into a separable state, but not into a tensor product state.

Our result resembles a result regarding two commuting mixed states [7]; these states cannot be cloned, but can be broadcast. That is, the resulting state of the cloning device cannot be a tensor product of states which are equal to the original states, but can be a separable state whose reduced density matrices are equal to the original states [8].

At that stage, the main question (raised by [1] and [3]) is still left open: Can there be a universal disentangling machine? That is, can there exist a machine that disentangles any set of states into separable states? We shall now show that such a machine cannot exist.

Our result is obtained by combining several of the previous techniques: the use of linearity, unitarity, and the disentanglement of product state.

Consider the following set of states

$$
\begin{align*}
|\psi_0\rangle &= |00\rangle \\
|\psi_1\rangle &= |11\rangle \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}\bigl(|00\rangle + |11\rangle\bigr) \\
|\psi_3\rangle &= |++\rangle
\end{align*}
$$

(25)

in which we added the states $|\psi_3\rangle$ to the previous set. This set of states cannot be disentangled.

The allowed transformations are now more restricted since, in addition to (Eq. 24), the state $|\psi_3\rangle$ must also not be changed by the disentangling machine:

$$
|E\psi_3\rangle = |E++\rangle \rightarrow |E_++\rangle.
$$

(26)

Due to unitarity, $\langle E|E\rangle (0+)\langle 0+| = \langle E_0|E_+\rangle (0+)\langle 0+|$, and also $\langle E|E\rangle (1+)\langle 1+| = \langle E_1|E_+\rangle (1+)\langle 1+|$. These expressions yield $1 = \langle E_0|E_+\rangle$, and $1 = \langle E_1|E_+\rangle$, from which we must conclude that $|E_+\rangle = |E_0\rangle = |E_1\rangle$. Recall that we already found that $|E_0\rangle = |0\rangle$ and $|E_1\rangle = e^{i\theta}|1\rangle$, due to the disentanglement of $|\psi_2\rangle$. Since the two requirements contradict each other, the proof that the above set of states cannot be disentangled (not even to a separable state) is completed. Thus, we have proved that a universal disentangling machine cannot exist. In other words—a single quantum cannot be disentangled.

This result resembles a result regarding two noncommuting mixed states [7]; these states cannot be cloned, and furthermore, they cannot be broadcast.

To summarize, we provided a thorough analysis of disentanglement processes, and we proved that a single quantum cannot be disentangle. Interestingly, we used a set of four states to prove this, but we conjecture that there are smaller sets that could be used to establish the same conclusion.

The no-cloning of states of composite systems were investigated recently [9,10], and it seems that several interesting connections between these works and the idea of disentanglement can be further explored. For instance, one can probably find systems where the states can only be disentangled (or only be disentangled into product states) if the two subsystems are available together, but cannot be disentangled if the subsystems are available one at a time (with similarity to [11]), or cannot be disentangled if only bilocal superoperators can be used for the disentanglement process (with similarity to [10]).

I would like to thank Charles Bennett, Oscar Boykin, and Danny Terno for very helpful remarks and discussions.

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