Niederer’s transformation, time-dependent oscillators and polarized gravitational waves

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Abstract
It is noted that the Niederer transformation can be used to find the explicit relation between time-dependent linear oscillators, including the most interesting case when one of them is harmonic. A geometric interpretation of this correspondence is provided by certain subclasses of pp-waves; in particular the ones strictly related to the proper conformal transformations. This observation allows us to show that the pulses of plane gravitational wave exhibiting the maximal conformal symmetry are analytically solvable. Particularly interesting is the circularly polarized family for which some aspects (such as the classical cross section, velocity memory effect and impulsive limit) are discussed in more detail. The role of the additional integrals of motion, associated with the conformal generators, is clarified by means of Ermakov–Lewis invariants. Possible applications to the description of interaction of electromagnetic beams with matter are also indicated.

Keywords: plane gravitational waves, Niederer’s transformation, conformal symmetry

1. Introduction

Harmonic oscillators with the time-dependent frequencies appear in many physical contexts. They have been extensively studied at classical and quantum levels. In particular, the possibility of reducing their dynamics to that of an ordinary harmonic oscillator by means of canonical transformations accompanied with a suitable redefinition of the time variable was investigated in some detail, see e.g. [1–5].
The paradigm of such an approach is provided by the Niederer transformation which maps the free motion onto the half-period motion of the harmonic oscillator (also on both levels) [6]. The starting point in the present paper is the observation that, once the Niederer transformation is formulated at the Lagrangian level, it can easily be generalized to the one relating two linear oscillators with time-dependent frequencies connected by a simple formula (see equations (2.4) and (2.5) below). In particular, the latter can be applied when the initial oscillator is harmonic yielding in this way an example of exactly solvable time-dependent oscillator.

In order to gain a deeper understanding of the above described properties of Niederer’s transformation one can refer to a geometric picture. Let us recall that the Niederer transformation has a nice geometric interpretation in terms of Bargmann spaces [7, 8] and the Eisenhart–Duval lift [9, 10]. In the particular case of a 2-dimensional (2D) oscillator considered below one obtains the 4-dimensional Bargmann manifold by adding a new variable which accounts for the nontrivial transformation properties of the action under the Niederer mapping. The basic identity relating the transformed action to the initial one is reformulated as the conformal equivalence of two metrics on the Bargmann manifolds (see equation (2.12) below). This relation admits the immediate generalization, see equation (2.13) below, which provides a geometric picture for the discussed case.

The Bargmann metrics related by Niederer’s transformation may be viewed as describing spacetime in form of (generalized) plane gravitational waves. However, due to the fact that Einstein’s equations are not conformally invariant at most one metric corresponds to a vacuum solution. In spite of this the Niederer transformation seems to be very useful. It allows, for example, to simplify the geodesics equations. To see this let us first note that, being conformally equivalent, both metrics yield the equivalent equations for null geodesics only. However, coming back to the origin of Niederer’s transformation we conclude that the geodesics equations for the transversal variables are of the same form; only the equations corresponding to the longitudinal variables are not equivalent. On the other hand, due to the special form of metrics, the equation for the longitudinal variables decouple and they can easily be solved once the remaining ones are solved (see equations (2.14)–(2.16)).

In view of the above the Niederer transformation can be useful in analysis of some properties of plane gravitational waves. This is interesting for several reasons. First, the plane gravitational waves are solutions to the nonlinear Einstein equations. Second, due to the vast distance from the source they can approximate an arbitrary gravitational wave in a neighbourhood of the detector. Moreover, the Penrose limit of spacetimes yields the (generalized) plane gravitational waves. In consequence, such waves can be used to model and study some more complex problems appearing in physics of the gravitational waves. Among others the gravitational memory is becoming an increasingly important topic, not only because of its potential observability in gravitational waves from astronomical sources [11, 12], but also for its importance in theoretical issues in quantum gravity [13]. The memory effect can be simply stated as displacement (or a residual velocity) noted by freely falling detectors, due to the passage of a pulse of gravitational radiation. This may take the form of a fixed change in position (permanent displacement) or a constant separation velocity. A permanent change in the Minkowski spacetimes which exists before the arrival of the pulse and after its departure was noted in [14] and considered in the context of linearized gravity in [15, 16]. Next it was extended to the full nonlinear theory of general relativity [17]; moreover, it turns out that there is a ‘nonlinear’ contribution to the memory effect, due to the effective stress energy of the gravitational waves transported to null infinity (the latter effect can be much larger than the formerly known ‘linear’ one). Following this idea, it was shown [18–20] that there is a contribution to the memory from other particles having zero rest mass and hence distinguished between the linear and nonlinear contributions as ordinary and null memory; the stress energy travels to null infinity in
the latter case only. Apart from four-dimensional asymptotically flat spacetimes, the memory effect has been studied in other spacetimes as well as in higher dimensions [21–26]; there are also analogous effects for electromagnetic fields [27]. Various extensions of this effect have also been studied in modified gravity and massive gravity theories [28]. For the case of the plane gravitational waves such considerations trace back to the work [29]. However, recently they gained a new impact due to the direct observations of the gravitational waves and its role in understanding inequivalent ground states (vacua) important for certain aspects of soft gravity (asymptotically flat spacetimes before the arrival of the pulse and after its departure are inequivalent and related via transformation which do not tend to the identity at infinity), see e.g. [30–34]. Especially interesting is also the observation that memory is encoded in the Penrose limit of the original gravitational wave spacetime recently noticed in [34]; for null congruences, the Penrose limit is a plane wave. Such analysis enhances the range of applications of existing studies involving geodesic deviation and memory in plane wave spacetimes.

In the current study the reasoning based on the Niederer transformation is applied to some special vacuum metrics distinguished by the existence of proper (special) conformal generators [35, 36]. They describe some linearly and circularly polarized plane gravitational waves. Such waves seem to be of some interest because circularly polarized gravitational waves may be produced by coalescing black holes, neutron star merger or observed from the astrometric data [37–39]; (the linearly polarized case can be also interesting due to its relative simplicity). Using the above mentioned approach we directly obtain the solutions of the geodesics equations for the plane gravitational waves under consideration and explicitly discuss some phenomena such as velocity memory effect, classical scattering and focusing. Furthermore, by taking an appropriate limit we notice that they can be used to model impulsive gravitational waves [40–44] with the Dirac delta profile. Moreover, the role and meaning of the integrals of motion associated with the conformal generators is clarified by means of Ermakov–Lewis invariants.

The paper is organized as follows. In section 2 we describe the Niederer transformation in the above motioned general approach and apply it to obtain solutions of an isotropic time-dependent oscillator related to a special, conformally flat, null fluid solution of Einstein’s equations. Next, in section 3 we explicitly discuss the interaction of the particle with the plane gravitational wave pulses defined by the maximal, 7-dimensional (7D), conformal symmetry; moreover, we give possible applications of the Niederer transformation to the pp-wave spacetimes. The meaning of the integrals of motion associated with conformal generators is discussed in section 4. Finally, section 5 is devoted to some conclusions. In particular, we briefly mention the possible applications of the results obtained here in a construction of electromagnetic backgrounds important for the light-matter interaction. This is motivated by recent results on the classical double copy approach [45–48] which enable one to build some electromagnetic fields from solutions to the Einstein equations. Such an approach, when applied to the families of plane gravitational waves discussed here, may lead to electromagnetic beams similar to the ones appearing in the context of singular optics [49, 50] and intense lasers [51, 52].

2. Niederer’s transformation

In this section we recall the notion of the Niederer transformation and subsequently we extend it to the case of time-dependent linear oscillators; we also notice that a geometric picture of such a situation is a very convenient framework to analyse the geodesic (deviation) equations.
It is well known that the free motion can be mapped into the harmonic one (for our purpose we consider 2D case). Namely, the so-called Niederer transformation\(^2\)

\[
\begin{align*}
  u &= \epsilon \tan(\tilde{u}), \\
  x &= \frac{\epsilon \tilde{x}}{\cos(\tilde{u})},
\end{align*}
\]

relates the free motion \(x = 0\) on the whole real axis \((-\infty < u < \infty)\) to the half of period motion \((-\frac{\pi}{2} < \tilde{u} < \frac{\pi}{2})\) of the attractive harmonic motion \(\tilde{x}'' = -\tilde{x}\); here dot and prime refer to the derivatives with respect to \(u\) and \(\tilde{u}\), respectively. On the level of action functional this basic property of the Niederer transformation is expressed by the following identity

\[
\frac{1}{2} \dot{x}^2 du = \frac{1}{2} (\dot{\tilde{x}}^2 - \tilde{x}^2) \epsilon d\tilde{u} + \epsilon \tan(\tilde{u}) \tilde{x}^2 \epsilon \cos(\tilde{u}).
\]

Furthermore, this equivalence continues to hold at the quantum level \([6]\); namely, if \(\psi(x, u)\) is a solution to the Schrödinger equation for the free particle then

\[
\phi(\tilde{x}, \tilde{u}) = e^{-\frac{\epsilon}{2} \tan(\tilde{u}) \tilde{x}^2 \cos(\tilde{u})} \psi \left( \epsilon \tan(\tilde{u}) \tilde{x} \cos(\tilde{u}) \right),
\]

is a solution to the Schrödinger equation for the harmonic oscillator. The structure of the above map is as follows: first, the arguments of the wave function are replaced by the appropriate functions of the new ones according to the classical formulae; then the two factors are added: the first one accounts for the proper normalization while the other, the phase one, is exactly equal to the function which enters the total time derivative in the transformation rule (2.2).

Of course, the above observations have a local character and their domain of validity must be carefully analysed. However, they reflect a similarity between both systems and bring immediately some useful information. Various local quantities (like symmetry generators) can be directly related; this concerns even the global ones (for instance, Feynman propagators) if sufficient care is exercised (see, e.g. \([6, 53, 54]\)). In particular, the maximal symmetry groups of both systems are isomorphic. Moreover, one obtains the explicit relation between their solutions which enables for a more detailed analysis \([55]\).

Now, let us note that adding the term \(\frac{1}{2} x \cdot H(u) x du\) (where \(H\) is, without loss of generality, a symmetric matrix) to the both sides of identity (2.2) one obtains the following relation

\[
\frac{1}{2} (\dot{x}^2 + x \cdot H(u) x) du = \frac{1}{2} (\dot{\tilde{x}}^2 + \tilde{x} \cdot \tilde{H}(\tilde{u}) \tilde{x}) \epsilon d\tilde{u} + \frac{1}{2} \tan(\tilde{u}) \tilde{x}^2 \epsilon \cos(\tilde{u}),
\]

where

\[
\tilde{H}(\tilde{u}) = \frac{\epsilon^2 H(\epsilon \tan(\tilde{u}))}{\cos^4(\tilde{u})} - I,
\]

which implies that the Niederer mapping connects two linear oscillators with time-dependent frequencies. In particular, if we introduce a matrix \(G\) such that

\[
H(u) = \frac{a}{(\epsilon^2 + u^2)^2} G(u),
\]

then, by virtue of equations (2.4) and (2.5), the equations of motion

\(^2\)For further convenience we adopt, non-standard, \(u\)-notation while bold indices refer to 2D vectors. Let us also note that there exists a hyperbolic counterpart of Niederer’s transformation leading to the repulsive case.
\[ \ddot{x} = H(u)x, \quad (2.7) \]

transform into the following ones

\[ \ddot{x}' = H(\ddot{u})\dot{x} = \left( \frac{a}{\epsilon^2} G(\epsilon \tan(\ddot{u})) - I \right) \dot{x}. \quad (2.8) \]

In consequence we transformed a linear oscillator (2.7) into another one described by equation (2.8); of course the advantage of such result is when the latter one is more tractable or interesting. In what follows we give some example of such situation.

Let us consider a special choice of the matrix \( G \) (equivalently \( H \)) namely, \( G = I \); then (2.7) and (2.8) imply that the (non-singular) time-dependent linear oscillator

\[ \ddot{x} = \frac{a}{(u^2 + \epsilon^2)^2} x, \quad (2.9) \]

where \( a, \epsilon \) are non-zero real numbers, is mapped under (2.1) to

\[ \ddot{x}' = \left( \frac{a}{\epsilon^2} - 1 \right) \dot{x}, \quad (2.10) \]

i.e. to a part of the motion of the harmonic oscillator or even, when \( a = \epsilon \), to the free motion (on the interval \((-\pi/2, \pi/2)\) only, in contrast to standard case), in the agreement with the general theory, see e.g. [4, 5, 7, 10]. In consequence, the Niederer transformation immediately yields the explicit form of the solution to equation (2.9) (see equation (3.7) below for \( i = 1 \)). Moreover, since equation (2.10) describes the isotropic harmonic oscillator one can further use the inverse of the Niederer transformation (with appropriate parameters) to pass at least locally to the free motion.

Of course, a similar reduction (in general to the anisotropic case) holds for an arbitrary constant matrix \( G \) (especially interesting is the case when \( \text{tr}(G) = \text{tr}(H) = 0 \)). This observation not only enables one to understand better some properties of the time-dependent oscillators but it has also a natural geometric counterpart.

To this end let us recall that the Niederer transformation received a nice geometric interpretation in terms of the Bargmann manifolds [7, 8] (see also [9, 10] and references therein). Namely, it is viewed as a chronoprojective transformation (a kind of conformal map) between the Bargmann spacetimes corresponding to the free \((u, x, v)\) and harmonic \((\ddot{u}, \ddot{x}, \ddot{v})\) motions [7]. In this approach the term which appears in form of total time derivative in (2.2) (or the phase factor in (2.3)) enters the transformation rule for the additional, \( v \) and \( \ddot{v} \), variables parametrizing the Bargmann manifolds, i.e.

\[ v = \epsilon \ddot{v} - \frac{\epsilon \tan(\ddot{u})}{2} \dot{x}^2, \quad (2.11) \]

Equation (2.11) together with (2.1) lead to the following identity

\[ \text{d}x^2 + 2\text{d}u\text{d}v = \frac{\epsilon^2}{\cos^4(\ddot{u})} (\text{d}x^2 + 2\text{d}u\ddot{v} - \dot{x}^2\ddot{u}^2), \quad (2.12) \]

between the free and half-oscillatory period Bargmann spaces (see also [56, 57]). Furthermore, the counterpart of the identity (2.4) reads

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3 For \( \alpha = 0 \) this reduces to the standard realization. For the singular cases: \( \epsilon = 0 \) the transformation is of the form \( u = \bar{u}^{-1}, x = \bar{x}u^{-1} \); if the denominator contains \( u^2 - \epsilon^2 \) then one can use the hyperbolic counterpart of Niederer’s map.
\[
g(x, v, \tau) = \frac{e^2(\tau)}{\cos^2(\tau)} (x \cdot \dot{H}(u) x du^2 + 2 du dv + dx^2)
\]
where \(H(u)\) and \(\tilde{H}(\tilde{u})\) are connected by equation (2.5). The metric \(g\) (and, consequently, \(\tilde{g}\)) described in (2.13) belongs to a subclass of pp-waves, the so-called generalized plane gravitational waves. In general, they do not satisfy the vacuum Einstein equations; only scalar curvature is zero while the source is a null fluid (i.e. they are solutions to Einstein’s field equation with the energy-momentum tensor describing some kind of radiation). The weak energy condition implies \(\text{tr}(H) \leq 0\). For \(\text{tr}(H) = 0\) the metric \(g\) satisfies the vacuum Einstein equations and describe the plane gravitational wave (exact gravitational wave).

Now, let us concentrate on the geodesic equations. To this end let us recall that the geodesic equations for the metric \(g\), appearing in equation (2.13), are of the form
\[
\ddot{x} = H x,
\]
\[
\ddot{v} = -\frac{1}{2} x \cdot \dot{H} x - 2 x \cdot \dot{H} x.
\]
Moreover equation (2.14) coincides with the deviation equation. Equation (2.15) can be integrated to the form
\[
v(u) = -\frac{1}{2} x \cdot \dot{x} + C_1 u + C_2,
\]
where \(x\) is a solution of (2.14).

Let us now assume that we are interested in solving the geodesic equations for the metric \(g\); then by applying the Niederer transformation one can associate with \(g\) the new metric \(\tilde{g}\) conformally related to \(g\) (see equation (2.13)). Due to the fact that Einstein’s equations are not conformally invariant at most only one of the metrics \(g\) and \(\tilde{g}\) describes the vacuum solution. However, we are interested in geodesics. Even though the geodesics equations for \(g\) and \(\tilde{g}\) are not equivalent (except those for null geodesics), it follows from the reasoning underlying the derivation of Niederer’s transformation that the equations for \(x\) and \(\tilde{x}\) are equivalent. The equations for \(x\) transform into the following ones
\[
\ddot{\tilde{x}} = \tilde{H}(\tilde{u}) \tilde{x},
\]
which may be simpler in some cases. The non-equivalence of geodesic equations for \(g\) and \(\tilde{g}\) is reduced to the non-equivalence of equations determining \(v\) and \(\tilde{v}\). However, the latter can be easily solved (see equation (2.16)) once \(x(u)\) (or \(\tilde{x}(\tilde{u})\)) are known. In this way we are left with equation (2.17) which can be more tractable than those for the initial metric.

Let us apply the above reasoning; first, for the metric \(g\) with the profile \(H(u)\) given by (2.6) and \(G = I\). Then one obtains the conformally flat metric \(g\); it is worth to notice that this metric is distinguished in the conformally flat subclass of generalized plane waves, it is the only one which admits the so-called proper special conformal vector field (see [35] for more details). Furthermore, the transverse geodesics equations for \(g\) and \(\tilde{g}\) are given by the formulae (2.9) and (2.10), respectively; thus they can be explicitly solved (and consequently (2.15)). Let us also note that if \(H(u)\) is proportional to the identity then the same concerns the matrix \(\tilde{H}(\tilde{u})\). Therefore we do not leave out the conformally flat subclass of the null fluid solutions (in the Eisenhart–Duval lift language the class of isotropic oscillators, including also the free motion).

If \(g\) describes the plane gravitational wave the situation is quite different; then \(\tilde{g}\) is not an exact gravitational wave but a, non-vacuum and non-conformally flat, null fluid solution
(in the Eisenhart–Duval lift language, anisotropic oscillators are transformed into anisotropic ones). However, again for some special plane gravitational waves the geodesic equations for \( \tilde{\mathcal{g}} \) (and consequently for \( g \) too) can be explicitly solved (see below).

3. Polarized plane gravitational waves

3.1. The general discussion

As we noted in the previous section the Niederer map can be used to rewrite the geodesic equations in, perhaps, a simpler form. However, a priori it is not clear for which profiles of plane gravitational waves such a trick should work; the geodesic equations do not distinguish any profiles. Usually some hints concerning the integrability provides the isometry group; however, it is well known that the generic dimension of the isometry group of the gravitational plane waves is five; there are two exceptional classes with the six-dimensional (6D) isometry group, they are extensively studied in the literature: a kind of periodic waves (applied, for example, to describe the gravitational wave which is a sandwich between two Minkowskian regions) and not geodesically complete class (used in the context of the Penrose limit [58]). Thus, in what follows we consider the conformal symmetry.

To this end let us recall that the conformal group of a non-conformally flat spacetime is at most 7D and the maximal dimension is rather rarely attained. If we restrict ourselves to the vacuum solutions then there are only three classes of metrics exhibiting the 7D conformal symmetry [35, 36] (see also [29, 59, 60]); all of them describe plane gravitational waves. Two of them coincide with the two above mentioned classes. We shall discuss the third one which consist of the linearly and circularly polarized families and, in contrast to the previous classes, exhibits a proper (non-homothetic) conformal vector field. Since we are interested in continuous gravitational pulses we concentrate on the geodesically complete cases (with the non-singular metrics). Then the first, linearly polarized, family is defined (up to \( u \) translation) by the metric \( g^{(1)} \) with the profile

\[
H^{(1)}(u) = \frac{a}{(u^2 + \epsilon^2)^2} G^{(1)}(u), \quad G^{(1)}(u) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(3.1)

where \( \epsilon > 0 \) and \( a \) is an arbitrary number (excluding the trivial Minkowski case and redefining, if necessary, \( x^1 \) and \( x^2 \) one can achieve \( a > 0 \)). Moreover, let us note that taking \( a \sim \epsilon^3 \) one obtains the model of an impulsive gravitational wave with the Dirac delta profile (as \( \epsilon \) tends to zero).

The second family \( g^{(2)} \) provides an example of the circularly polarized plane gravitational waves. It is defined by the following profiles

\[
H^{(2)}(u) = \frac{a}{(u^2 + \epsilon^2)^2} G^{(2)}(u), \quad G^{(2)}(u) = \begin{pmatrix} \cos(\phi(u)) & \sin(\phi(u)) \\ \sin(\phi(u)) & -\cos(\phi(u)) \end{pmatrix},
\]

(3.2)

where

\[
\phi(u) = \frac{2\gamma}{\epsilon} \tan^{-1}(u/\epsilon),
\]

(3.3)

and \( \epsilon, \gamma > 0 \) and \( a \) can be chosen as above (for \( \gamma = 0 \) equation (3.2) reduces to the previous case; however, for both physical and mathematical reasons we will consider the linear and circular polarizations separately).
Now, by virtue of (2.5) and (2.13) (see also [35]), the Niederer transformation given by equations (2.1) and (2.11) leads to the following relation
\[ g^{(1,2)} = \epsilon^2 \cos^2(\tilde{u}) \tilde{g}^{(1,2)}, \]  
(3.4)
where the metrics \( \tilde{g}^{(1)} \) and \( \tilde{g}^{(2)} \) are defined by the profiles
\[
\tilde{H}^{(1)}(\tilde{u}) = \left( \begin{array}{cc} \frac{2}{a} - 1 & 0 \\ 0 & -\frac{a}{\epsilon^2} - 1 \end{array} \right), \quad \tilde{H}^{(2)}(\tilde{u}) = \left( \begin{array}{cc} \frac{a \cos(\epsilon \tilde{u})}{\epsilon^2} - 1 & \frac{a \sin(\epsilon \tilde{u})}{\epsilon^2} \\ -\frac{a \sin(\epsilon \tilde{u})}{\epsilon^2} & -\frac{a \cos(\epsilon \tilde{u})}{\epsilon^2} - 1 \end{array} \right),
\]
(3.5)
respectively. Let us stress that, in contrast to \( H^{(1,2)} \), \( \text{tr}(\tilde{H}^{(1)}) = \text{tr}(\tilde{H}^{(2)}) = -2 < 0 \); thus \( \tilde{g}^{(1)} \) and \( \tilde{g}^{(2)} \) describe, non-vacuum, null fluid solutions.

Before we go further let us recall that, in the previous section, we noted that the conformally flat metric with \( G = I \) is related to an \( u \)-dependent isotropic oscillator. In view of the above, the first equation in (3.4) provides the geometric interpretation of the Niederer transformation relating the time-dependent oscillator (described by \( H^{(1)}(u) \)) to the anisotropic harmonic one (described by \( \tilde{H}^{(1)}(\tilde{u}) \)).

Now, on the basis of the above observations we to give a detailed discussion of some phenomena for the considered families of polarized gravitational pulses. The starting point is the observation, due to (2.7) and (2.8) the transversal parts of geodesic equations for the metrics \( g^{(1)} \) and \( g^{(2)} \) take the following form
\[ \tilde{x}'' = \tilde{H}^{(1,2)}(\tilde{u})\tilde{x}. \]
(3.6)
Thus in order to discuss the behaviour of a test particle under the plane gravitational waves \( g^{(1)} \) and \( g^{(2)} \), it is sufficient to consider the above equations with the null fluid spacetimes \( \tilde{g}^{(1)} \) and \( \tilde{g}^{(2)} \), and then go back to the original variables \( x \) (as well as to the \( v \) coordinate, see (2.16)); this is our main idea.

For \( g^{(1)} \), the above procedure gives the solutions of geodesics equations discussed in [61]; namely, in the transverse direction, for \( a < \epsilon^2 \), we have
\[ x_i(u) = C_i \sqrt{u^2 + \epsilon^2 \sin(\sqrt{\Lambda_i} \tan^{-1}(u/\epsilon)) + C_i^2}, \]
(3.7)
where
\[ \Lambda_i = 1 + (-1)^i \frac{a}{\epsilon^2}, \quad i = 1, 2 \]
(3.8)
(for \( a > \epsilon^2 \) the solution \( x^2(u) \) does not change but \( x^1(u) \) is given by replacing the trigonometric functions their hyperbolic counterparts). This allows us to explicitly analyse the focusing and singularity problems. More precisely, imposing the initial conditions
\[ \tilde{x}(-\infty) = 0, \]
(3.9)
supplied by the following one
\[ x(-\infty) = x_0, \]
(3.10)
one concludes that only the second component \( x^2(u) \) exhibits focusing at one point
\[ u_0 = -\epsilon \cot(\frac{\pi}{\sqrt{\Lambda_2}}) > 0. \]
(3.11)
However, for parameters \( a > \epsilon^2 \) the situation changes drastically; namely, for sufficiently large \( \epsilon \) there can be several focusing points [61]. This fact complicates the singularity analysis.
More precisely, so far we discussed the plane gravitational waves in the so-called Brinkmann coordinates, where both the wave and geodesics are global (the metric is non-singular). The Brinkmann coordinates cover the whole plane wave spacetime by a single chart. However, the plane gravitational waves are frequently discussed in the so-called Baldwin–Jeffery–Rosen (BJR) coordinates (see, e.g. [62, 63] and references therein). The BJR coordinates, in contrast to the Brinkmann ones, are typically not global, exhibiting coordinate singularities. This fact is reflected in the transformation rule between the both coordinates. Namely, only a piece of the Brinkmann manifold can be covered by the BJR coordinates and consequently at least two BJR maps are needed to completely describe the interaction (scattering) of particle with the plane gravitational waves. The definition of the BJR coordinates is related to the transverse part of geodesic equations (see equation (3.19) and below) and the minimal number of charts to cover the whole Brinkman manifold is strictly related to the number of focusing points. Moreover, a deeper insight into the structure of BJR coordinates can be profitable since the BJR coordinates seem to be of some importance in understanding inequivalent ground states (vacua) and, consequently, certain aspects of soft gravity [31].

3.2. The circularly polarized case

In view of the above discussion it would be interesting to analyse analytically the behaviour of geodesics in the, physically more interesting, case of the circularly polarized gravitational waves. In what follows we show that this is actually possible for the plane gravitational wave \( g(2) \) (equivalently \( \tilde{g}(2) \), by virtue of Niederer’s transformation). To this end we introduce the new coordinates \( \mathbf{y} \) (see, e.g. [64, 65] and references therein)

\[
\tilde{x} = R(\tilde{u})y,
\]

where

\[
R(\tilde{u}) = \begin{pmatrix} \cos(\omega\tilde{u}) & -\sin(\omega\tilde{u}) \\ \sin(\omega\tilde{u}) & \cos(\omega\tilde{u}) \end{pmatrix}, \quad \omega = \frac{\gamma}{\epsilon}.
\]

Then, the metric \( \tilde{g}(2) \) in terms of \( y \)'s takes the form

\[
\tilde{g}(2) = \left( \frac{a}{\epsilon^2} + \omega^2 - 1 \right)y^2 + \left( -\frac{a}{\epsilon^2} + \omega^2 - 1 \right)y^2 \left( d\tilde{u} \right)^2
\]

\[
+ 2\omega (y^1 dy^2 - y^2 dy^1) d\tilde{u} + (dy^1)^2 + (dy^2)^2 + 2d\tilde{u}d\tilde{v},
\]

which implies the following geodesic equations

\[
\begin{align*}
y^2'' + 2\omega(y^1)' + \Omega_- y^2 &= 0, \\
y^1'' - 2\omega(y^2)' + \Omega_+ y^1 &= 0,
\end{align*}
\]

where

\[
\Omega_\pm = 1 - \omega^2 \mp \Omega, \quad \Omega = \frac{a}{\epsilon^2}.
\]

The above set of equations can be explicitly solved, although the general form of the solution is rather complicated [66, 67]. However, let us note that any solution is a combination of trigonometric functions only or both hyperbolic (or linear) and trigonometric ones depending on the values of parameters. Indeed, the roots of the characteristic determinant (for the functions \( y^{1,2} = A^{1,2} e^{\Omega t} \)) read
\[ t_{1,2} = \pm \sqrt{1 + \omega^2 + \sqrt{4\omega^2 + \Omega^2}}, \]
\[ t_{3,4} = \pm \sqrt{1 + \omega^2 - \sqrt{4\omega^2 + \Omega^2}}, \] (3.18)

thus some \( \hat{x} \) trajectories remain periodic and bounded, while others can spiral outward, see [65]. Thus, by virtue of equation (2.16), we can conclude that the circularly polarized case (3.2) is also explicitly solvable. This fact allows us to get insight into some phenomena, such as the memory effect, focusing and singularity problems of the BJR coordinates as well as to compute the classical cross sections.

To this end let us recall that the heart of these considerations is the matrix \( P(u) \) satisfying the matrix Sturm–Liouville differential equation
\[ \ddot{P} = HP, \] (3.19)
with the constraint
\[ \hat{P}^T P - P^T \hat{P} = 0. \] (3.20)

Since the condition (3.20) is stable against evolution it is sufficient to impose it at one point. Now let us assume that \( x_1(u) \) and \( x_2(u) \) are the solutions to equation (2.14) with the initial conditions (3.9) and the following ones\(^4\)
\[ x_{1,2}(-\infty) = e_{1,2}, \] (3.21)
where \( e_1 \) and \( e_2 \) form the 2D canonical base. Then the matrix
\[ P_{in}(u) = (x_1(u), x_2(u)), \] (3.22)
related to the initial conditions (3.21), is a solution to equation (3.19) and, by (3.21), satisfies the constraint (3.20). As a result, the solutions \( x(u) \) to equation (2.14) with the initial conditions (3.9) and (3.10) can be written as follows
\[ x(u) = P_{in}(u)x_{in}. \] (3.23)

Thus \( \dot{x}(u) = P_{in}(u)x_{in} \) and consequently the Jacobian \( J \) of the transformation from the final velocities \( \dot{x}(\infty) \) to initial positions \( x_{in} \) reads
\[ J = \frac{1}{\det(P_{in}(\infty))}. \] (3.24)

This Jacobian is strictly related to the classical cross section (see [63]),
\[ d\sigma_{\text{classical}} = dx_{in}d\dot{x}_{in}^2 = |J|d\dot{x}_1(\infty)d\dot{x}_2(\infty) = \frac{|J|}{P_{\dot{p}}}d\dot{p}_1(\infty)d\dot{p}_2(\infty) \] (3.25)

where \( p_\ell \) is a constant of motion resulting from the fact the metric does not depend on the coordinate \( v \) (in fact it is the proportionality coefficient between the coordinate \( u \) and the affine parameter). Now, let us recall that the BJR coordinates \( (u, \hat{x}_{in}, \hat{v}_{in}) \) are defined by
\[ x = P_{in}\hat{x}_{in}, \quad \nu = \hat{v}_{in} - \frac{1}{4}\hat{x}_{in} \cdot \dot{H}_{in}\hat{x}_{in}, \quad \dot{H}_{in} = P_{in}^T P_{in}; \] (3.26)
in terms of them the plane gravitational wave takes the form
\[ \dot{g}_{in} = 2d\dot{u}d\hat{v}_{in} + d\hat{x}_{in} \cdot \dot{H}_{in} d\hat{x}_{in}. \] (3.27)

\(^4\)We assume that the gravitational wave vanishes sufficiently fast at null infinities, see also [61].
Let us also note that the solution to the matrix Sturm–Liouville differential equation is strictly related to the Killing vectors and consequently the isometry groups of the plane gravitational waves (see e.g. [30, 31]). The choice (3.21) of the initial conditions implies that the BJR coordinates and the metric (3.27) coincide with the Brinkmann ones at minus infinity (and, due to the asymptotic flatness of gravitational wave, with Minkowski one). On the other hand, the BJR coordinates (metrics) are singular at the point where the matrix $P_{\text{in}}$ vanishes; thus we need several charts to cover the whole Brinkmann manifold (see e.g. [62, 63]). Analogously, one can construct the matrix $P_{\text{out}}$ around plus infinity and consequently the 'out' BJR coordinates; however, as we have indicated above such two maps may not be sufficient to cover the whole Brinkmann manifold (see also [61]).

Now, let us apply the above general considerations to the case of the circularly polarized gravitational waves defined by the profile (3.2). First, we should rewrite the initial conditions in $y$'s variables defined by (3.12). For the transverse velocities, applying L'Hospital's rule, one obtains the relation
\[
\dot{x}(\pm \infty) = \pm R(\pm \pi/2)y(\pm \pi/2).
\] (3.28)

Next, assuming $\dot{x}(-\infty) = 0$, straightforward computations yield
\[
x(-\infty) = \epsilon R(-\pi/2)y'(-\pi/2).
\] (3.29)

Thus the solutions $y_1$ and $y_2$ to equation (3.16) with the initial conditions
\[
y_{1,2}(-\pi/2) = 0, \quad y'_{1,2}(-\pi/2) = \frac{1}{\epsilon}R(\pi/2)e_{1,2},
\] (3.30)
lead directly to the solutions $x_1$ and $x_2$ discussed above and, consequently, to the matrix $P_{\text{in}}$ (see equation (3.22)). Moreover, the determinant of the matrix $P_{\text{in}}$ can be also expressed in terms of $y_{1,2}$, indeed
\[
\det(P_{\text{in}}(\hat{u})) = \frac{\epsilon^2}{\cos^2(\hat{u})} \det(R(\hat{u})y_1(\hat{u}), R(\hat{u})y_2(\hat{u})) = \frac{\epsilon^2}{\cos^2(\hat{u})} \det(y_1(\hat{u}), y_2(\hat{u})).
\] (3.31)

where $\hat{u} = \tan^{-1}(u/\epsilon)$. Finally, due to (3.24) and (3.28) one gets
\[
\frac{1}{J} = \det(P_{\text{in}}(\infty)) = \det(R(\pi/2)y_1(\pi/2), R(\pi/2)y_2(\pi/2)) = \det(y_1(\pi/2), y_2(\pi/2)).
\] (3.32)

In summary, we express the basic quantities corresponding to the circularly polarized plane gravitational waves defined by the profile (3.2), First, we should rewrite the initial conditions in $y$’s variables defined by (3.12). For the transverse velocities, applying L’Hospital’s rule, one obtains the relation
\[
x(\pm \infty) = \pm R(\pm \pi/2)y(\pm \pi/2).
\] (3.28)

Thus the solutions $y_1$ and $y_2$ to equation (3.16) with the initial conditions
\[
y_{1,2}(-\pi/2) = 0, \quad y'_{1,2}(-\pi/2) = \frac{1}{\epsilon}R(\pi/2)e_{1,2},
\] (3.30)
lead directly to the solutions $x_1$ and $x_2$ discussed above and, consequently, to the matrix $P_{\text{in}}$ (see equation (3.22)). Moreover, the determinant of the matrix $P_{\text{in}}$ can be also expressed in terms of $y_{1,2}$, indeed
\[
\det(P_{\text{in}}(\hat{u})) = \frac{\epsilon^2}{\cos^2(\hat{u})} \det(R(\hat{u})y_1(\hat{u}), R(\hat{u})y_2(\hat{u})) = \frac{\epsilon^2}{\cos^2(\hat{u})} \det(y_1(\hat{u}), y_2(\hat{u})),
\] (3.31)

where $\hat{u} = \tan^{-1}(u/\epsilon)$. Finally, due to (3.24) and (3.28) one gets
\[
\frac{1}{J} = \det(P_{\text{in}}(\infty)) = \det(R(\pi/2)y_1(\pi/2), R(\pi/2)y_2(\pi/2)) = \det(y_1(\pi/2), y_2(\pi/2)).
\] (3.32)

In summary, we express the basic quantities corresponding to the circularly polarized plane gravitational wave $g^{(2)}$ in terms of solutions to, explicitly integrable, equation (3.16).

Considering our conclusions, let us analyse the Dirac delta limit. For the case of $g^{(1)}$ such a limit (including the cross section and notion of the conformal group) was discussed in [61]. In the case of $g^{(2)}$ we take $a = 2\omega^3/\pi$ and $\gamma = r\epsilon$, i.e. $\omega = r > 0$. Then in the limit $\epsilon \to 0$ one obtains
\[
H^{(2)}(\hat{u}) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \delta(\hat{u}) \cdot \begin{cases} \frac{\sin(\pi r)}{\pi r(1-r^2)}, & r \neq 1; \\ \frac{1}{2}, & r = 1. \end{cases}
\] (3.33)

Thus for the gravitational wave $g^{(2)}$ the Dirac delta limit reduces, up to a constant, to the one for $g^{(1)}$ (the circular polarization has no time to act), which coincides with the results obtained in [42, 65].
3.3. The explicit example

To illustrate the results from the above subsection let us take the following parameters \( \omega = 1 \), i.e. \( \gamma = \epsilon \) in the circular case. Then \( \Omega_+ = -\Omega_- = -\Omega \) while equations (2.1) and (3.12) give the following relations

\[
\begin{align*}
x^1(\tilde{u}) &= \epsilon y^1(\tan^{-1}(u/\epsilon)) - uy^2(\tan^{-1}(u/\epsilon)), \\
x^2(\tilde{u}) &= uy^1(\tan^{-1}(u/\epsilon)) + \epsilon y^2(\tan^{-1}(u/\epsilon)),
\end{align*}
\]

(3.34)

where \( y \)'s contain both the trigonometric and hyperbolic functions. Explicitly

\[
\begin{align*}
y^1(\tilde{u}) &= (\Omega - \lambda^2_+)(C_1 \sin(\lambda_+ \tilde{u}) + C_4 \cos(\lambda_+ \tilde{u})) + (\lambda^2_+ + \Omega)(C_1 \sinh(\lambda_- \tilde{u}) - C_2 \cosh(\lambda_- \tilde{u})), \\
y^2(\tilde{u}) &= 2\lambda_+(-C_3 \cos(\lambda_+ \tilde{u}) + C_4 \sin(\lambda_+ \tilde{u})) + 2\lambda_-(C_2 \sinh(\lambda_- \tilde{u}) - C_1 \cosh(\lambda_- \tilde{u})),
\end{align*}
\]

(3.35)

where

\[
\lambda_\pm = \sqrt{2 \left( \pm 1 + \sqrt{1 + \Omega^2} \right)}.
\]

(3.36)

From the above, it is clear that freely falling test particles in the neighbourhood of plus null infinity fly apart along straight lines with the constant velocity i.e. they exhibit the so-called velocity memory effect (a test particle initially at rest fly apart with non-vanishing constant velocity after a burst of a gravitational wave has passed, see [30, 31] as well as [65] for the circularly polarized case).

Now, imposing the initial conditions (3.30) on the solutions (3.35) one obtains the following form of \( y_{1,2} \)

\[
(y_1(\tilde{u}), y_2(\tilde{u})) = \frac{1}{\epsilon(\lambda^2_+ + \lambda^2_-)} \begin{pmatrix} -2X(\tilde{u}) & Y(\tilde{u}) \\ Z(\tilde{u}) & -2X(\tilde{u}) \end{pmatrix},
\]

(3.37)

where

\[
\begin{align*}
X(\tilde{u}) &= \cos(\lambda_+ (\tilde{u} + \pi/2)) - \cosh(\lambda_- (\tilde{u} + \pi/2)), \\
Y(\tilde{u}) &= (\lambda_+ - \lambda_-) \sin(\lambda_+ (\tilde{u} + \pi/2)) - (\lambda_+ + \lambda_-) \sinh(\lambda_- (\tilde{u} + \pi/2)), \\
Z(\tilde{u}) &= (\lambda_+ + \lambda_-) \sin(\lambda_+ (\tilde{u} + \pi/2)) + (\lambda_- - \lambda_+) \sinh(\lambda_- (\tilde{u} + \pi/2)).
\end{align*}
\]

(3.38)

Thus by virtue (3.22) and (3.34) one obtains the explicit form of the matrix \( P_m \) and consequently the transformation rule to the BJR coordinates. Moreover one gets

\[
\det(y_1(\tilde{u}), y_2(\tilde{u})) = \frac{\Omega \sin(\lambda_-(\tilde{u} + \pi/2)) \sin(\lambda_+ (\tilde{u} + \pi/2)) - 2 \cos(\lambda_+ (\tilde{u} + \pi/2)) \cosh(\lambda_- (\tilde{u} + \pi/2)) + 2;}{\epsilon^2(4 + \Omega^2)}
\]

(3.39)

thus, due to (3.26), (3.27) and (3.31), \( \det(P_m) \) as well as \( \det(\tilde{g}_{mn}) \) can be directly computed. In consequence the passage to the BJR coordinates is given by the explicit formulae.

Next, by virtue of (3.32) one immediately gets

\[
J = \frac{\epsilon^2(4 + \Omega^2)}{\Omega \sin(\lambda_+ \pi) \sin(\lambda_- \pi) - 2 \cos(\lambda_+ \pi) \cosh(\lambda_- \pi) + 2},
\]

(3.40)

and, consequently, the classical cross section (3.25) (when the final momenta vanish, i.e. the denominator in (3.40) is zero, the notion of cross section should be modified).
In order to get some insight into the focusing and singularity problems let us factorize the determinant (3.39). After some computations one gets

\[ \det(y_1(\tilde{u}), y_2(\tilde{u})) = \frac{V(\tilde{u})W(\tilde{u})}{e^{2(4 + \Omega^2)}}. \] (3.41)

where

\[ V(\tilde{u}) = (\lambda_+ + \lambda_-) \sinh(\frac{\lambda_+}{2}(\tilde{u} + \frac{\pi}{2})) \cos(\frac{\lambda_+}{2}(\tilde{u} + \frac{\pi}{2})) \]
\[ - (\lambda_- - \lambda_+) \cosh(\frac{\lambda_-}{2}(\tilde{u} + \frac{\pi}{2})) \sin(\frac{\lambda_-}{2}(\tilde{u} + \frac{\pi}{2})). \]
\[ W(\tilde{u}) = (\lambda_+ + \lambda_-) \sin(\frac{\lambda_+}{2}(\tilde{u} + \frac{\pi}{2})) \cosh(\frac{\lambda_+}{2}(\tilde{u} + \frac{\pi}{2})) \]
\[ + (\lambda_- - \lambda_+) \cos(\frac{\lambda_-}{2}(\tilde{u} + \frac{\pi}{2})) \sinh(\frac{\lambda_-}{2}(\tilde{u} + \frac{\pi}{2})). \] (3.42)

Thus, in agreement with the general theory [62], the BJR coordinates are singular when either \( V \) or \( W \) vanishes—depending on the wave parameters \( a \) and \( \epsilon \). Furthermore, the focusing holds on a line (since \( X(\tilde{u}) \neq 0 \), the rank of \( P_m \) is at least one). One can also find the kernel of the matrix \( P_m \) describing the initial positions for which the focusing actually appears (by virtue of (3.23) the geodesics corresponding to the initial positions differing by an element of the kernel of \( P_m \) will eventually meet at some point). In our case, \( W(\tilde{u}_0) = 0 \), the kernel of \( P_m \) is spanned by the vector

\[ \left((\lambda_+ + \lambda_-) \sin(\frac{\lambda_+}{2}(\tilde{u}_0 + \frac{\pi}{2})), 2 \cos(\frac{\lambda_+}{2}(\tilde{u}_0 + \frac{\pi}{2})) \right) R(-\tilde{u}_0); \] (3.43)

the case \( V(\tilde{u}_0) = 0 \) can be dealt with analogously.

3.4. Some remarks on the pp-waves solvability

Above we showed that for some special plane gravitational waves there exist explicit solutions to the geodesic equations (such a situation enables one to get some insight into physically important phenomena); moreover, we noted that the notion of the Niederer transformation provides a very convenient framework for the considered cases. Although the explicit solutions are very exceptional (and call for some additional assumptions) the question arises if we can apply this framework to obtain different examples of solvable pp-wave spacetimes. In what follows we give an affirmative answer to this question. More precisely, we show that the Niederer framework can be used to produce some pp-wave spacetimes with analytical solutions of geodesics equations; at least in some directions (the full solvability, as we pointed out above is very exceptional). To this end let us consider the pp-wave metric of the form

\[ g = \mathcal{H}(u, x) du^2 + 2dudv + dx \cdot dx, \] (3.44)

describing, in general, a null fluid solution. Then the geodesic equations can be rewritten in the form

\[ x = \frac{\nabla \mathcal{H}}{2}, \] (3.45)
\[ \dot{v} = -\frac{\partial_v \mathcal{H}}{2} - \nabla \mathcal{H} \cdot x. \] (3.46)
Taking into account $\frac{d^2 u}{\tau^2} = -1$ and the relation $u = \frac{\phi}{\tau}$ one concludes that the last equation can be directly integrated to the form

$$\dot{v} = -\frac{H}{2} - \frac{1}{2} \dot{x}^2 - \frac{m^2}{2p_v^2}.$$  \hspace{1cm} (3.47)

Moreover, when the profile $H$ is a homogenous function of degree two in transverse directions i.e.

$$x \cdot \nabla H = 2H,$$  \hspace{1cm} (3.48)

then equation (3.47) can be directly integrated once more yielding

$$v = -\frac{\dot{x} \cdot x}{2} - \frac{m^2}{2p_v^2}u + C,$$  \hspace{1cm} (3.49)

where $x$ is a solution of equation (3.45). In consequence, for a homogenous function $H$ the solution problem of the geodesic equations for pp-waves reduces to the set of equation (3.45); in general case, there remains one integration of the $v$-coordinate. Thus in what follows we focus on the set of equations given by (3.45).

Let us take an arbitrary function $K(\tilde{x})$ of two variables and consider the following profile of the pp-wave metric

$$H(u, x) = \frac{1}{u^2 + e^2}K \left(\frac{x}{\sqrt{u^2 + e^2}}\right) + \frac{x^2}{2(e^2 + u^2)^2}.$$  \hspace{1cm} (3.50)

Then, after some computations, one can show that equation (3.45) in the new coordinates $\tilde{x}$, defined by equation (2.1), take the following form

$$\tilde{x}'' = \tilde{\nabla}K.$$  \hspace{1cm} (3.51)

The last set of equations is nothing more than the 2D Newton equations for which some explicit solutions are well known; in these cases one immediately obtains, by means of the Niederer transformation, the explicit form of solutions to equation (3.45) with the profile (3.50). For example, let us take

$$K(\tilde{x}) = \frac{a}{(\tilde{x})^2} + \frac{b}{(\tilde{x})^2};$$  \hspace{1cm} (3.52)

then equation (3.51) can be explicitly solved and in consequence the transversal part of the geodesic equation (3.45) with the profile

$$H(u, x) = \frac{a}{(x^1)^2} + \frac{b}{(x^2)^2} + \frac{x^2}{2(e^2 + u^2)^2},$$  \hspace{1cm} (3.53)

is solvable. A similar situation holds for other solvable potentials $K$’s, as the Pöschl–Teller or Morse potentials. However, as we pointed out above there remains one integration of the $v$-coordinate (see equation (3.47)) to obtain the full explicit solvability.

4. Conformal generators and Ermakov–Lewis invariants

To obtain a wider view of the integrability problem of the discussed families of the plane waves let us have a look at them from the symmetry point of view. In the standard approach one considers the isometry group. Then the integrals of motion, associated with the Killing
vectors, are well known and can be used to reduce the solution problem of the geodesic equations to some integrals (though, in general, these integrals cannot be explicitly computed). The possible isometry groups of the vacuum pp-waves were classified, see [35, 36, 60] and references therein. Of course, the most interesting cases should be the ones with the maximal symmetry; such a group is at most 6D. The group of the dimension six is realized by two, thoroughly studied, classes of plane gravitational waves (see section 3.1), whereas the dimension five is realized by the remaining plane gravitational waves. As one can expect for the classes of the dimension six (i.e. the maximal dimension of the isometry group) the geodesic equations can be analytically solved.

Now, let us enlarge the symmetry group by considering the conformal transformations. This is interesting since in the case of pp-waves there can appear additional integrals of motion. For the geodesic $x^\mu(\tau)$ and a conformal vector field $K$, $L_K g = 2\psi g$, the following identity (along the geodesic) holds

$$\frac{d}{d\tau}(K^\mu \frac{dx^\mu}{d\tau}) = \psi \frac{dx^\mu}{d\tau} = -\psi;$$

(4.1)
on the other hand, for the pp-waves the $\tau$ parameter is proportional to $u$. Now, let us consider the homothetic vector field $Y$, then $\psi$ is a constant and thus (4.1) gives a, linearly, $u$-dependent integral of motion. For example, in the case of plane waves there always exists a homothetic vector of the following form

$$Y = 2v \partial_u + x \cdot \nabla.$$

(4.2)

The above vector field is also a homothetic one for a pp-wave spacetime if and only if the condition (3.48) holds, i.e. when equation (3.47) for $v$ can be directly integrated. Thus the existence of the homothetic vector (4.2) ensures the expression of the $v$ coordinate in terms of solutions to equation (3.45) (if the later ones are analytically solvable then one gets the full explicit solvability).

Now, let us focus on the proper conformal transformations. Then, it turns out that for the so-called special conformal vectors of the, non-flat, pp-waves the factor $\psi$ is a linear function of $u$ only (see [60]). Thus again one can use equation (4.1) to find additional integrals of motion containing $u^2$ term. Moreover, in view of the above discussion, it would be desirable to have a rich isometry group as well as a homothetic vector field. In consequence, we should look for pp-waves with the maximal conformal group and admitting a proper conformal transformation. For the vacuum solutions to the Einstein equations there is only one such class and it is precisely given by the considered plane waves, equations (3.1) and (3.2). Let us analyse this situation in some detail.

In the case of the linearly polarized case, defined by the profile $H^{(1)}$ (see (3.1)), the conformal vector field and conformal factor are of the following form

$$K^{(1)} = (u^2 + \epsilon^2) \partial_u - \frac{1}{2} x^2 \partial_v + u x \cdot \nabla, \quad \psi = u.$$

(4.3)

By virtue of (4.1) the vector field (4.3) gives the integral of motion

$$I^{(1)} = (u^2 + \epsilon^2)(x \cdot H^{(1)} x + \dot{x}) - \frac{1}{2} x^2 + u x \cdot \dot{x} + \frac{m^2}{2p_\epsilon} u^2.$$

(4.4)

Using equation (3.47) one gets

$$I^{(1)} = -\frac{m^2 \epsilon^2}{2p_\epsilon} + a \frac{(x_1)^2 - (x_2)^2}{2(u^2 + \epsilon^2)} - (u^2 + \epsilon^2) \frac{\dot{x}^2}{2} - \frac{1}{2} x^2 + u x \cdot \dot{x}.$$  

(4.5)
Now, the key observation is that this integral of motion can be rewritten in terms of the Ermakov–Lewis invariants. To this end let us introduce the function
\[\rho(u) = \frac{\sqrt{u^2 + \epsilon^2}}{\sqrt{\epsilon}}.\] (4.6)

Then the integral of motion \(I^{(1)}\) can be expressed as follows
\[I^{(1)} = -\frac{m^2 \epsilon^2}{2p_v^2} - \epsilon \left[\left(\rho \dot{x}^1 - \dot{\rho} x^1\right)^2 + \frac{\Lambda_1(x^1)^2}{\rho^2}\right] - \frac{\epsilon}{2} \left[(\rho \dot{x}^2 - \dot{\rho} x^2)^2 + \frac{\Lambda_2(x^2)^2}{\rho^2}\right],\] (4.7)

where \(\Lambda_i\) are defined by equation (3.8). Moreover, the function \(\rho\) satisfies the set of the Ermakov–Milne–Pinney equations for the profiles \(H^{(1)}\) and \(\tilde{H}^{(1)}\) (the latter one is defined by the frequencies \(\Lambda_i\), see equations (3.5) and (3.8)), namely
\[\rho H^{(1)} - H^{(1)} \rho = -\tilde{H}^{(1)} \rho.\] (4.8)

Thus the integral of motion defined by the conformal vector field (4.3) is the sum of two Ermakov–Lewis invariants \([1–5]\) with the frequencies \(\Lambda_{1,2}\), respectively. This information can be used to find the solutions of the transverse part of the geodesic equations. According to the general procedure, see e.g. [10], the transformation
\[\frac{\partial \tilde{u}}{\partial u} = \frac{1}{\rho^2(u)}, \quad x = \sqrt{\epsilon \rho(u)} \tilde{x},\] (4.9)

should relate the \(u\)-dependent linear oscillator, defined by \(H^{(1)}\), to the harmonic one with the frequencies \(\Lambda_{1,2}\). In our case, i.e. \(\rho\) given by (4.6), the above formulae yield the Niederer transformation (see equation (2.1)) and consequently the explicit integrability.

Furthermore, it turns out (see [10] and references therein) that the Ermakov–Lewis invariants can be interpreted as the ‘classical’ energy in the new coordinates \(\tilde{x}, \tilde{u}\). In our case this leads to the identity
\[I^{(1)} = -\epsilon^2 \left[\frac{m^2}{2p_v^2} + E^{(1)}\right],\] (4.10)

where
\[E^{(1)} = \frac{1}{2} \tilde{x}^2 - \frac{1}{2} \tilde{x} \cdot \tilde{H}^{(1)} \tilde{x} = \frac{1}{2} \tilde{x}^2 + \frac{\Lambda_1(\tilde{x}^1)^2}{2} + \frac{\Lambda_2(\tilde{x}^2)^2}{2}.\] (4.11)

In consequence, we obtain a more transparent interpretation of the integral of motion associated with the proper conformal generator \(K^{(1)}\).

Now let us consider the second family of more interesting plane gravitational waves which exhibits the maximal conformal symmetry, i.e. defined by \(H^{(2)}\) (equivalently by \(G^{(2)}\) see equation (3.2)). Then, the proper conformal field \(K^{(2)}\) is of the following form
\[K^{(2)} = K^{(1)} - \gamma(x^2 \partial_1 - x^1 \partial_2),\] (4.12)

with the same conformal factor \(\psi = u\). As in the previous case one can express, by means of (4.6), the corresponding integral of motion as follows
\[I^{(2)} = -\frac{m^2 \epsilon^2}{2p_v^2} + a \frac{x \cdot G^{(2)}(u) \tilde{x}}{2(u^2 + \epsilon^2)} - \left(u^2 + \epsilon^2\right) \frac{\tilde{x}^2}{2} - \frac{1}{2} \tilde{x}^2 + a \tilde{x} \cdot x - \gamma \tilde{x} \times x\] (4.13)
\[
\frac{m^2 \epsilon^2}{2p_v^2} - \frac{\epsilon}{2} \left[ (\rho \dot{x} - x \dot{\rho})^2 - \frac{x}{\rho} \cdot \left( \frac{d}{\epsilon^2} G^{(2)}(u) - I \right) \right] - \gamma \dot{\rho} \times \gamma.
\]  
(4.14)

In contrast to the previous case this time the terms in the square brackets do not form the Ermakov–Lewis invariants (since the matrix \( G^{(2)} \) depends explicitly on \( u \)), only together with the last term \( I^{(2)} \) is an integral of motion. However, the function \( \rho \) satisfies Ermakov–Milne–Pinney type equation

\[
\dot{\rho} I - H^{(2)} \rho = -\frac{\dot{H}^{(2)}}{\rho^3};
\]  
(4.15)

thus, in agreement with our previous considerations, it leads to the Niederer transformation and consequently to equation (3.6); the solvability is obtained by further transformation to the \( \gamma \)'s coordinates, see equation (3.12). In view of the above, we expect that the integral of motion \( I^{(2)} \) should be related to total energy for the system defined by the equations of motion (3.16). Indeed, one can check that the integral of motion associated with the conformal vector \( K^{(2)} \) is of the form

\[
I^{(2)} = -\epsilon^2 \left[ \frac{m^2}{2p_v^2} + E^{(2)} \right],
\]  
(4.16)

where

\[
E^{(2)} = \frac{1}{2} (y')^2 + \frac{1}{2} \Omega_+(y^1)^2 + \frac{1}{2} \Omega_-(y^2)^2,
\]  
(4.17)

and \( \Omega_{\pm} \) are given by equation (3.17). In summary, we showed that both homothetic and conformal transformations can be crucial in the solvability of geodesic equations; moreover, the integrals associated with proper conformal generators can be expressed as the total energy of the harmonic oscillators described by the frequencies directly related to parameters defining the gravitational waves.

5. Conclusions and outlook

Let us summarize. In the present work we showed that the Niederer transformation can explicitly connect time-dependent linear oscillators with the ordinary harmonic ones (including isotropic and anisotropic cases). A geometric interpretation of this situation is provided by the special families (strictly related to the proper conformal transformations) of conformally flat generalized plane wave spacetimes in the isotropic case, and by exact gravitational waves in the anisotropic case. Such an observation allows us to show directly that the plane gravitational waves, corresponding to the geodesically complete manifolds and exhibiting the maximal, 7D, conformal symmetry, admit analytical solutions of the geodesics equations. This enables one to understand better the interaction (scattering) of particle with the gravitational fields (the classical cross section, singularity and velocity memory effect). Moreover, in the agreement with results obtained in [65] we showed that the Dirac delta limit of the circularly polarized case reduces to that for the linearly polarized one. Finally, the role and meaning of the additional integrals of motions associated with the conformal generators were discussed by means of the Ermakov–Lewis invariants and their more transparent interpretation was obtained.

The results obtained can be extended in various directions. First, it would be interesting to consider their quantum counterparts, including both the time-dependent linear oscillator as
well as the gravitational case (e.g. [10, 63, 68]). Next, the analytical solutions can give a better insight into singularity problems of the BJR coordinates, e.g. [33, 61]. Such solutions can be also useful in the recent studies concerning some aspects of the energy transfer [69, 70] as well as in the context of some optical effects in nonlinear plane gravitational waves [71, 72]. Furthermore, let us recall that the Penrose limit of spacetimes yields the plane gravitational waves; thus, the question is which spacetimes correspond to the conformally distinguished ones (see [34]). This is of some importance due to the recent result that the memory is encoded in the Penrose plane wave limit of the original gravitational wave.

Moreover, one of the most interesting topics of further investigations is a construction of electromagnetic backgrounds important for the light-matter interaction. This is motivated by the recent results on the classical double copy approach [45–48] which enable one to construct some electromagnetic fields from solutions to the Einstein equations. One of the manifestation of the double copy is the observation that the plane gravitational waves not only satisfy the vacuum Einstein equations but also the wave equation; thus they can be used to construct some potentials of the electromagnetic fields. According to this approach, with the metric $g$, see equation (2.13), one relates the following electromagnetic one-form (in the light-cone coordinates)

$$ A = -x \cdot H(u) x du. $$

(5.1)

When $\text{tr}(H) = 0$, i.e. in the case of the plane gravitational wave, the above potential give the electromagnetic field $\vec{E}$ and $\vec{B}$ which satisfies the vacuum Maxwell equations. However, in contrast to its gravitational counterpart, such a field is not, in general, a plane electromagnetic wave. Furthermore, electromagnetic field obtained in this way is the pure radiation (the energy density and the Poynting vector form a null four-vector or equivalently the square of Riemann–Silberstein vector vanishes). Thus it can describe a vortex of electromagnetic field [73]. An important example of such a situation is a vortex proposed in [49] which can act as a beam guide for charged particles; moreover, it provides analytically solvable example and is an approximation to more realistic beams [74]. Recently, it has been discussed in [75] in the context of the above mentioned gauge-gravity duality. In view of our results one can expect that the electromagnetic backgrounds corresponding to polarized gravitational waves defined by the profiles (3.1) and (3.2) also give analytically solvable electromagnetic backgrounds with vortices (see results of [76, 77]); this is evident as far as the transverse directions are considered because then the Lorentz equations (in the light-cone coordinates) coincide, up to a constant, with the geodesic equations, see e.g. [75]. Moreover, the recently discussed models which are important for the light-matter interaction, in particular for the strong focusing of intense laser pulses [51, 52], should be also obtainable in a similar way; subsequently, they could be extended by adding the new solvable examples or focusing criteria. Some of these problems are under the consideration and preliminary results are presented in [78].

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