Robustness of a Network of Networks

Jianxi Gao,1,2 Sergey V. Buldyrev,3 Shlomo Havlin,4 and H. Eugene Stanley1

1Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215 USA
2Department of Automation, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, PR China
3Department of Physics, Yeshiva University, New York, NY 10033 USA
4Minerva Center and Department of Physics, Bar-Ilan University, 52900 Ramat-Gan, Israel

(Dated: 25 October 2010 — gbhs25oct.tex)

Abstract

Almost all network research has been focused on the properties of a single network that does not interact and depends on other networks. In reality, many real-world networks interact with other networks. Here we develop an analytical framework for studying interacting networks and present an exact percolation law for a network of n interdependent networks. In particular, we find that for n Erdős-Rényi networks each of average degree k, the giant component, P∞, is given by

\[ P_\infty = p \left[ 1 - \exp(-kP_\infty) \right] \]

where 1 − p is the initial fraction of removed nodes. Our general result coincides for n = 1 with the known Erdős-Rényi second-order phase transition for a single network. For any n ≥ 2 cascading failures occur and the transition becomes a first-order percolation transition. The new law for P∞ shows that percolation theory that is extensively studied in physics and mathematics is a limiting case (n = 1) of a more general general and different percolation law for interdependent networks.

In recent years dramatic advances in the field of complex networks have occurred [1–13]. The internet, airline routes, and electric power grids are all examples of networks whose function relies crucially on the connectivity between the network components. An important property of such systems is their robustness to node failures. Almost all research has been concentrated on the case of a single or isolated network which does not interact with other networks. Recently, based on the motivation that modern infrastructures are becoming significantly more dependent on each other, a system of two coupled interdependent networks has been studied [14]. A fundamental property of interdependent networks is that when nodes in one network fail, they may lead to the failure of dependent nodes in other networks which may cause further damage in the first network and so on, leading to a global cascade of failures. Buldyrev et al. [14] developed a framework for analyzing robustness of two interacting networks subject to such cascading failures. They found that interdependent networks become significantly more vulnerable compared to their noninteracting counterparts.

For many important examples, more than two networks depend on each other. For example, diverse infrastructures are coupled together, such as water and food supply, communications, fuel, financial transactions, and power stations [15–18]. For further examples see Section I in the Supplementary Information (SI). Understanding the robustness due to such interdependencies is one of the major challenges for designing resilient infrastructures.

We introduce here a model system, comprising a network of n coupled networks, where each network consists of N nodes (See Fig. 1). The N nodes in each network are connected to nodes in neighboring networks by bidirectional dependency links, thereby establishing a one-to-one correspondence as illustrated in Fig. 2 in SI. We develop a mathematical framework to study the robustness of this “network of networks” (NON). We find an exact analytical law for percolation of a NON system composed of n coupled randomly connected networks. Our result generalizes the known Erdős-Rényi (ER) [14–21] result for the giant component of a single network and the n = 2 result found recently [14], and shows that while for n = 1 the percolation transition is a second order transition, for n > 1 cascading failures occur and the transition becomes a first order transition. Our results for n interdependent networks suggest that the classical percolation theory extensively studied in physics and mathematics is a limiting case of a general theory of percolation in NONs, or networks with multiple types of connectivity links. This general theory has many novel features which are not present in classical percolation.

Additionally, we find:

(i) the robustness of NON significantly decreases with n, and

(ii) for a network of n ER networks all with the same average degree k, there exists a minimum degree \( k_{\text{min}}(n) \) increasing with n, below which \( p_c = 1 \), i.e., for \( k < k_{\text{min}} \) the NON will collapse once any finite number of nodes fail. We find an analytical expression for \( k_{\text{min}}(n) \), which generalizes the known result \( k_{\text{min}}(1) = 1 \) for ER below which the network collapses. We also discuss the critical effect of loops in the NON structure.

Real-world interacting networks (See SI for more details) are characterized by complex correlations and a variety of organizational principles governing their internal structures and interdependencies. Once these correlations are quantified from the statistical analysis of actual data bases and the organizational principles are specified from the engineering literature, real world networks can be studied by computer simulations. These simulations will have many parameters and therefore their outcome will also require complex interpretation. It is therefore very important to develop simple analytically tractable
models for the robustness of interdependent networks against which such simulations can be tested. Well-known examples of simplified models that both demonstrate a fundamental phenomenon and significantly advance our knowledge are the Ising model in statistical mechanics and the Erdős-Rényi model in graph theory. This paper presents a simple model that can serve as a benchmark for further studies of NONs.

We assume that a network \( m \) \((m = 1, 2, \ldots, n)\) in the NON is a randomly connected network with a degree distribution \( P_m(k) \). We call a pair of networks \( A \) and \( B \) a fully interdependent pair if it satisfies the following condition: each node \( A_i \) of network \( A \) is connected to one and only one node \( B_i \) in network \( B \) by a bidirectional dependency link such that if node \( A_i \) fails, \( B_i \) also fails and vice versa. Since the number of nodes in each network is the same, these bidirectional links establish a one-to-one correspondence between the nodes in the networks belonging to an interdependent pair. Each node of the NON represents a network and each edge represents a fully interdependent pair of networks. First, we will discuss the case when the NON is a loopless tree of \( n \) networks (Fig. 1). The dependency edges in such a NON establish a unique one-to-one correspondence between the nodes of any two networks not necessarily belonging to the same fully interdependent pair. This one-to-one correspondence established by the interdependency links between the nodes of different networks in the loopless NON uniquely maps any set of nodes in one of the networks to a set of nodes (which we will call an image of the original set) in any other network of the NON (See SI for more details). In principle, the assumption of full interdependence allows one to collapse all the networks of the NON onto a single network with multiple types of links.

We assume that in order to remain functional a node must belong to a sufficiently large mutually connected cluster [14] (See detailed definition in SI). We will show that a large mutually connected cluster which includes a finite fraction of the nodes in each network exists only in networks of sufficiently high mean degree. We call this large mutually connected cluster a mutual giant component, and we postulate that only nodes in the mutual giant component remain functional.

We assume that due to an attack or random failure only a fraction of nodes \( p \) in one particular network which we will call the root of the NON. We can now observe a cascade of failures caused by the failure of the dependent nodes in the networks connected directly to the root by the edges of the NON. The damage will further spread to more distant networks. Moreover, fragmentation of each network caused by the removal of certain nodes will cause malfunction of other nodes which will now belong to small isolated clusters. This malfunction will cause dependent nodes in neighboring networks to malfunction as well. Depending on the time scales of these processes, the damage can spread across the NON back and forth, which we can visualize as cascades of failures, as shown in Fig. 3 of the SI section. At the end of this process only the mutual giant component of the NON, if it exists, will remain functional.

We now introduce generating functions [14, 22–25] of each network, \( G_{m0}(z) = \sum P_m(k) z^k \), and the generating function of the associated branching process, \( G_{m1}(z) = G_{m0}(z)/G_{m0}(1) \). It is known [22, 23] that the generating functions of a randomly connected network once a fraction \( 1 - p \) of nodes has been randomly removed are the same as the generating functions of the original network with the new argument \( 1 - p(1 - z) \). Furthermore it is known [22, 23] that the fraction of nodes in the giant component of a single randomly connected network is \( \mu_{\infty, 1} = pg_m(p) \), where \( g_m(p) = 1 - G_{m0}(1 - p(1 - f_m)) \) and \( f_m \) satisfies a transcendental equation \( f_m(p) = G_{m1}(1 - p(1 - f_m(p))) \).

We next prove that the fraction of nodes, \( \mu_{\infty, n} \), in the mutual giant component of a NON composed of \( n \) networks is the product:

\[
\mu_{\infty, n} = p \prod_{m=1}^{n} g_m(x_m),
\]

where each \( x_m \) satisfies the equation

\[
x_m = \mu_{\infty, n}/g_m(x_m).
\]

The system of \( n + 1 \) equations (1) and (2) defines \( n + 1 \) unknowns: \( \mu_{\infty, n}, x_1, x_2, \ldots, x_n \) as functions of \( p \) and the degree distributions \( P_m(k) \).

We derive Eqs. (1) and (2) by mathematical induction. (An alternative proof is given in the SI). Indeed, for \( n = 1 \), Eqs. (1) and (2) follow directly from the definition of \( \mu_{\infty, 1} \). Assuming that the formulas are valid for a NON of \( n - 1 \) networks we will prove that they are valid also for a NON of \( n \) networks. A loopless NON of \( n \) networks can be always represented as one of its networks connected by a single edge to the other \( n - 1 \) networks in the NON. All the nodes in the \( n \)-th network, which do not belong to the image of the mutual giant component \( \mu_{\infty, n-1} \) of the NON of \( n - 1 \) networks will stop to function. The fraction of the nodes in the image of this mutual giant component onto the \( n \)-th network satisfies the equation \( x_1,n = \mu_{\infty, n-1}(p) \).

The fraction of nodes belonging to the giant component of this dependency image is \( \mu_{1,n} = x_{1,n} g_n(x_{1,n}) \). Only the nodes in the NON of \( n - 1 \) networks which belong to the dependency image of the giant component of the \( n \)-th network will remain functional. Due to the randomness of the connectivity links in different networks, this dependency image can be represented as a random selection of the fraction \( g_n(x_{1,n}) \) out of the originally survived nodes, or as random selection of \( p_1 = pg_n(x_{1,n}) \) fraction of nodes in one of the networks comprising the NON of \( n - 1 \) networks. The fraction of nodes in the new mutual giant component of the NON of \( n - 1 \) networks corresponding to this new random selection will be \( \mu_{\infty, n-1}(p_1) \). The image of this mutual giant component in the \( n \)-th network is equivalent to a random selection of \( x_{2,n} = \mu_{\infty, n-1}(p_1)/g_n(x_{1,n}) \) fraction of nodes out of the
entire \( n \)-th network. As we continue this process, the sequence of giant components \( \mu_{n+1} \) in the \( n \)-th network, randomly selected sets \( x_n \) in the \( n \)-th network and randomly selected sets \( y_n \) in the NON of \( n-1 \) networks will satisfy the recursion relations 
\[
x_{j+1,n} = \frac{\mu_{n+1}}{\mu_{n+1}}(p_{j+1})/g_n(x_{j+1,n}),
\]
\[
\mu_{j+1,n} = x_{j+1,n}g_n(x_{j+1,n}),
\]

In the limit \( j \to \infty \), this process will converge, i.e. all the parameters in the two successive steps will coincide: 
\[
x_{j+1,n} = x_{j,n} \equiv x_n, p_{j+1} = p_{j+1}(x_n) = x_{j,n} \equiv \mu_{n+1}.
\]

Then the failure of single node on node \( x \) will cause an entire cycle to fail. The average size of a cycle is trivial, as shown in Fig. 2(a).

The NON of Erdős-Rényi [19–21] will collapse even if a single node fails. Fig. 2 (b) shows the minimum average degree \( k_{\text{min}} \) as a function of the number of networks \( n \). From Eq. (5) we get the minimum of \( k \) as a function of \( n \)

\[
k_{\text{min}}(n) = \left[ n f_c(1-f_c)^{(n-1)} \right]^{-1},
\]

where the solution can be expressed in terms of the Lambert function \( W(x) \)

\[
f_c = -\left[nW\left(\frac{-e^{-1}}{n}\right)\right]^{-1}.
\]

Once \( f_c \) is known, we obtain \( p_c \) and \( \mu_{\infty,n} = \mu_{\infty} \) by substituting \( k_i = k \) into Eq. (S10) of the SI section

\[
p_c = [nk_f(1-f_c)^{(n-1)}]^{-1},
\]

For \( n = 1 \) we obtain the known result \( p_c = 1/k \) of Erdős-Rényi [19–21]. Substituting \( n = 2 \) in Eqs. (4) and (5), we obtain the exact results of [14].

To analyze \( p_c \) as a function of \( n \), we find \( f_c \) from Eq. (4) and substitute it into Eq. (5), and we obtain \( p_c \) as a function of \( n \) for different values, as shown in Fig. 2(a). It is seen that the NON becomes more vulnerable with increasing \( n \) or decreasing \( k \). This increases when \( n \) increases or \( k \) decreases. Furthermore, for a fixed \( n \), when \( k \) is smaller than a critical number \( k_{\text{min}} \), the NON will collapse even if a single node fails. Fig. 2 (b) shows the minimum average degree \( k_{\text{min}} \) as a function of the number of networks \( n \). From Eq. (6) we obtain the minimum of \( k \) as a function of \( n \)

\[
k_{\text{min}}(n) = \left[ n f_c(1-f_c)^{(n-1)} \right]^{-1},
\]

Note that Eq. (6) together with Eq. (5) yield the value of \( k_{\text{min}}(1) = 1 \), reproducing the known ER result, that \( \langle k \rangle = 1 \) is the minimum average degree needed to have a giant component. For \( n = 2 \), Eq. (6) yields the result obtained in [14], i.e., \( k_{\text{min}} = 2.4554 \).

From Eqs. (1)–(3) we obtain the percolation law for the order parameter, the size of the mutual giant component for all \( p \) values and for all \( k \) and \( n \)

\[
\mu_{\infty,n} = \mu_{\infty} = p[1 - \exp(-kP_{\infty})]^n.
\]

The solutions of equation (7) are shown in Fig. 3 for several values of \( n \). Results are in excellent agreement with simulations. The special case \( n = 1 \) is the known ER percolation law for a single network [19–21]. Note that Eqs. (4)–(7) are based on the assumption that all \( n \) networks have the same average degree \( k \).

In summary, we have developed a framework, Eqs. (1) and (2), for studying percolation of NON from which we derived an exact analytical law, Eq. (7), for percolation in the case of a network of \( n \) coupled ER networks. Equation (7) represents a bound for the case of partially dependent networks [28], which will be more robust. In particular for any \( n \geq 2 \), cascades of failures naturally appear and the phase transition becomes first order transition compared to a second order transition in the classical percolation of a single network. These findings show that the percolation theory of a single network is a limiting case of a more general case of percolation of interdependent networks. Due to cascading failures which increase with \( n \), vulnerability significantly increases with \( n \). We also find that for any loopless network of networks the critical percolation threshold and the mutual giant component depend only on the number of networks and not
on the topology (see Fig. 1(a)). When the NON includes loops, and dependency links are random, $p_c = 1$ and no mutual giant component exists.

---

[1] Watts D. J. & Strogatz S. H. Nature 393, 440-442 (1998).
[2] Albert R., Jeong H. & Barabási A. L. Nature 406, 378-382 (2000).
[3] Cohen R. et al. Phys. Rev. Lett. 85, 4626-4628 (2000).
[4] Callaway D. S. et al. Phys. Rev. Lett. 85, 5468-5471 (2000).
[5] Albert R. & Barabási A. L. Rev. Mod. Phys. 74, 47-97 (2002).
[6] Newman M. E. J. SIAM Review 45, 167-256 (2003).
[7] Dorogovtsev S. N. & Mendes J. F. F. Evolution of Networks: From Biological Nets to the Internet and WWW (Physics) (Oxford Univ. Press, New York, 2003).
[8] Song C. et al. Nature 433, 392-395 (2005).
[9] Satorras R. P. & Vespignani A. Evolution and Structure of the Internet: A Statistical Physics Approach (Cambridge Univ. Press, England, 2006).
[10] Caldarelli G. & Vespignani A. Large scale Structure and Dynamics of Complex Webs (World Scientific, 2007).
[11] Barrát A., Barthélémy M. & Vespignani A. Dynamical Processes on Complex Networks (Cambridge Univ. Press, England, 2008).
[12] Havlin S. & Cohen R. Complex Networks: Structure, Robustness and Function (Cambridge Univ. Press, England, 2010).
[13] Newman M. E. J. Networks: An Introduction (Oxford Univ. Press, New York, 2010).
[14] Buldyrev S. V. et al. Nature 464, 1025-1028 (2010).
[15] Peerenboom J., Fischer R. & Whitfield R. in Proc. CRIS/DRM/IIT/NSF Workshop Mitigat. Vulnerab. Crit. Infrastruct. Catastr. Failures (2001).
[16] Rinaldi S., Peerenboom J. & Kelly T. IEEE Contr. Syst. Mag. 21, 1-25 (2001).
[17] Rosato V. et al. Int. J. Crit. Infrastruct. 4, 63-79 (2008).
[18] Vespignani A. Nature 464, 984-985 (2010).
[19] Erdős P. & Rényi A. I. Publ. Math. 6, 290-297 (1959).
[20] Erdős P. & Rényi A. Publ. Math. Inst. Hung. Acad. Sci. 5, 17-61 (1960).
[21] Bollobás B. Random Graphs (Academic, London, 1985).
[22] Newman M. E. J. Phys. Rev. E 64, 026118 (2001).
[23] Newman M. E. J. Phys. Rev. E 66, 016128 (2002).
[24] Shao J. et al. Europhys. Lett. 84, 48004 (2008).
[25] Shao J. et al. Phys. Rev. E 80, 036105 (2009).
[26] Lambert J. H. Acta Helveticae physico mathematico anatomico botanico medica, Band III, 128-168, (1758).
[27] Corless R. M. et al. Adv. Computational Maths. 5, 329-359 (1996).
[28] Parshani R. et al. Phys. Rev. Lett. 105, 048701 (2010).
FIG. 1: (color online) Three types of loopless networks of networks composed of five coupled networks. All have same percolation threshold and same giant component. The darker green node is the origin network on which failures occur.
FIG. 2: (a) The critical fraction $p_c$ for different $k$ and $n$ and (b) minimum average degree $k_{min}$ as a function of the number of networks $n$. The results of (a) and (b) are obtained using Eqs. (5) and (6) respectively and are in good agreement with simulations. In simulations $p_c$ was calculated from the number of cascading failures which diverge at $p_c$ [28] (see also Fig. 7 in SI).
FIG. 3: For loopless NON, $P_\infty$ as a function of $p$ for $k = 5$ and several values of $n$. The results obtained using Eq. 7 agree well with simulations.