Global Convergence of Stochastic Gradient Descent for Some Nonconvex Matrix Problems

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Abstract

The Burer-Monteiro decomposition \( X = YY^T \) with stochastic gradient descent is commonly employed to speed up and scale up matrix problems including matrix completion, subspace tracking, and SDP relaxation. Although it is widely used in practice, there exist no known global convergence results for this method. In this paper, we prove that, under broad sampling conditions, a first-order rank-1 stochastic gradient descent (SGD) matrix recovery scheme converges globally from a random starting point at a \( O(\epsilon^{-1} n \log n) \) rate with constant probability. We demonstrate our method experimentally.

1 Introduction

In a variety of machine learning, combinatorics, and optimization applications including matrix completion [2], general data analysis [3], and subspace tracking [4], our input data is a symmetric data matrix \( A \). For some integer \( p \geq 1 \), our goal is to solve the problem

\[
\begin{align*}
\text{minimize} & \quad \|A - X\|_F^2 \\
\text{subject to} & \quad X \in \mathbb{R}^{n \times n}, \text{rank}(X) \leq p, X \succeq 0.
\end{align*}
\]

The solution to this problem is the matrix formed by zeroing out all but the largest \( k \) eigenvalues of the matrix \( A \). The standard way to solve this problem is to compute a singular value decomposition (SVD). If the data matrix \( A \) is completely observed and of modest size, the SVD performs well. However, there are some conditions under which the SVD sometimes is not the method of choice.

1. The data matrix \( A \) could be only partially known, such that we are actually operating on a series of random samples which are unbiased estimates of \( A \). In this case, we can only solve (1) stochastically, which SVD doesn’t provide a framework for.

2. Alternatively, we could only know that \( A \) is in some set \( \mathcal{A} \), and so we are actually trying to solve the problem

\[
\begin{align*}
\text{minimize} & \quad \|A - X\|_F^2 \\
\text{subject to} & \quad A \in \mathcal{A}, X \in \mathbb{R}^{n \times n}, \text{rank}(X) \leq p, X \succeq 0.
\end{align*}
\]

In this case, the SVD does not provide the best solution because we must “guess” (usually incorrectly) at a particular \( A \) before running the algorithm.

3. The dimensions of \( A \) could be such that computing the SVD could be intractable. Standard SVD runs in \( O(n^3) \) time and \( O(n^2) \) space, which is too much for even moderately-large sparse matrices.
To handle these conditions, researchers have proposed methods based on stochastic gradient descent (SGD) that scale to large data sets [5][6]. This works directly under Condition 1 above, by using the random samples in the algorithm. Furthermore, under Condition 2 or 3, we can “convert” the problem into a stochastic one by simply sampling the matrix $A$ ourselves as part of the algorithm.

Probably the most common SGD algorithms are based on the work of Burer and Monteiro [1]. The idea is to substitute $X = YY^T$ (a quadratic substitution) and solve the equivalent problem

$$\text{minimize } \| A - YY^T \|_F^2 \quad \text{subject to } Y \in \mathbb{R}^{n \times p}. \tag{2}$$

By construction, the solution $YY^T$ to (2) is positive semidefinite and of rank at most $p$. In order to solve this problem using SGD, we let $A_k$ be the unbiased sample of $A$ we choose at timestep $k$, and let $\tilde{f}_k(Y) = \| \tilde{A}_k - YY^T \|_F^2$ be the sampled objective function at that timestep. Then for some sequence of step sizes $\alpha_k$, stochastic gradient descent has the update rule

$$Y_{k+1} = Y_k + \alpha_k \nabla \tilde{f}_k(y_k) = Y_k + 2\alpha_k \left( Y_k Y_k^T Y_k - \tilde{A}_k Y_k \right). \tag{3}$$

The benefits of using the Burer-Monteiro substitution for SGD are readily apparent: not only do we get to store the matrix $Y$ (of size $pN$) rather than the large, dense matrix $X$ (of size $n^2$), but we are also free from any need to project onto the low-rank constraint set. This decomposition is the basis of many scalable algorithms for matrix completion problems [5][7], subspace tracking problems [4], and a range of general purpose optimization problems [8][9][10].

Unfortunately, this reformulation of the problem is not convex, because the substitution results in a quartic, rather than quadratic, objective function. In place of a convex function’s single global minimum, this formulation adds several additional unstable points of inflection to the objective, each corresponding to a non-dominant eigenvector of the data matrix $A$. Furthermore, it introduces a non-convex set of equivalent solutions, since $YY^T = YUU^T Y^T$ for any $U$ such that $UU^T = I$. That is, if $Y^*$ is a solution to the problem, then so will $Y^*U$ for any orthogonal matrix $U$. This complicates matters as the solutions $Y$ form a manifold that is neither well-separated nor (for $Y^*$ full rank) connected in the space $\mathbb{R}^{n \times p}$ [11]. To remove this symmetry, one approach is to operate in the space “quotiented out” by the symmetry. The structure of the resulting space is a Reimannian manifold (where we quotient $\mathbb{R}^{n \times p}$ by the orthogonal group $O(p)$)

$$\tilde{M} = \mathbb{R}^{n \times p}/O(p).$$

Absil’s text on manifold optimization gives a range of approaches to solving problem in manifolds. The monotonicity of first-order (gradient descent) approaches guarantees convergence to a stationary point—but not to a global minimum.

Previous work has achieved many interesting results, including local convergence results of SGD [9], and global convergence of alternating minimization [2][12] and phase retrieval [13]. However, no existing work has achieved a proof of global convergence for SGD applied to a Burer-Monteiro decomposition. In particular, Burer’s original result, which showed that all local minima of the transformed objective function are global minima of the original problem, does not have a global convergence guarantee. For example, if the matrix $A$ is fully known, and we initialize with a vector orthogonal to the solution, SGD will never recover the solution (see Appendix A). In spite of this, many people use SGD methods because of its scalable, fast implementations [14][5]. It remains an open question, then: Does this algorithm converge globally? And if so, how fast does it converge?

Our main result is to answer these questions in the case of a rank-1 ($p = 1$) SGD algorithm. We exhibit Alecton, an algorithm based on Burer-Monteiro SGD for a particular step-size scheme, and show that it
converges globally from a random start point in worst-case $O(\epsilon^{-1} n \log n)$ time with constant probability. Additionally, Alecton accesses information about the matrix $A$ only through a series of unbiased samples $\tilde{A}$, and the convergence proof depends only on the variance of these samples. As a result, our result applies to a variety of sampling and noise models, and we exhibit two applications:

- **matrix completion** in which entries of $A$ are observed one at a time (Section 2.1), and
- **phase retrieval** in which one observes $\text{tr}(u^T Av)$ for randomly selected $u, v$ (Section 2.2).

We also show that we can recover higher-rank matrices using multiple iterations of the Alecton algorithm.

**Main Technique** The standard proof of SGD’s convergence, in which we choose a convex Lyapunov function and show that this function’s expectation decreases with time, cannot work in this case. This is because, if such a Lyapunov function were to exist, it would show that no matter where we initialize the iteration, convergence will still occur rapidly; this cannot be possible due to the presence of the unstable fixed points (eigenvectors of $A$) of the iteration introduced by the quadratic substitution. Therefore, a standard statement of global convergence, that convergence occurs uniformly regardless of initial condition, cannot hold.

We therefore invoke martingale-based methods of showing convergence. Specifically, our attack involves defining a process $x_k$ with respect to the natural filtration $\mathcal{F}_k$ of the iteration, such that $x_k$ is a supermartingale, that is $E [x_{k+1} | \mathcal{F}_k] \leq x_k$. We then use the **optional stopping theorem** to bound both the probability and rate of convergence of $x_k$, from which we derive convergence of the original algorithm. This technique is a standard martingale approach: for example it can be used to find the expected stopping time of random walks and the probability of stopping at either boundary.

Another hurdle to global convergence is the choice of a step size for gradient descent. The classical constant step-size scheme is prone to difficulty. For example, if we naively use a step size $\alpha_k$ independently of $Y_k$ in (3), then for sufficiently large magnitudes of the iterate the algorithm will diverge to infinity with high probability (see Appendix A). This is one of the issues caused by “falling off” the intended manifold during iteration. Previous approaches [4, 9] have attempted to correct for falling off the manifold using Reimannian retractions, geodesic steps, or projections back onto the manifold. However, we show that a simple modification of the step size rule to separate the radial and angular components of the iteration is sufficient to resolve these problems.

Our primary contribution is this set of theoretical results that generalize several approaches in the literature [5, 8, 1, 9]. We also empirically validate that Alecton converges globally in practice, and runs efficiently on large datasets.

## 2 Statement of Results

We focus on the rank-1 quadratically decomposed matrix completion problem

\[
\begin{aligned}
\text{minimize} & \quad f(y) = \|y\|^4 - 2y^T Ay + \|A\|^2_F \\
\text{subject to} & \quad n \in \mathbb{R}^n.
\end{aligned}
\]

where we require $A$ to be a positive semidefinite matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ and corresponding unit eigenvectors $u_1, u_2, \ldots, u_n$. We apply stochastic gradient descent to this problem, resulting in the rank-1 version of the update rule from above,

\[
y_{k+1} = y_k - \alpha_k \nabla \hat{f}_k(y_k) = y_k - \alpha_k \left(\|y_k\|^2 y_k - \hat{A}_k y_k\right).
\]
A critical idea that we borrow from manifold optimization techniques \[11\] is that, when performing gradient descent in a manifold, we must take the manifold structure into account when computing the appropriate step. In particular, rather than using a constant a priori step size scheme, it behooves us to factor observable information into the choice at each step. In the rank-1 case, the only observable is the magnitude of the iterate \(|y(k)|\). If we incorporate our observable manifold information by choosing step size \(\alpha_k = \eta_k \left(1 + \eta_k |y(k)|^2\right)^{-1}\), for some constant-a-priori sequence of new step sizes \(\eta_k\), then

\[
y(k+1) = y(k) - \eta_k \left(1 + \eta_k |y(k)|^2\right)^{-1} \left(|y(k)|^2 y(k) - \hat{A}_k y(k)\right) = \left(1 + \eta_k |y(k)|^2\right)^{-1} \left(I + \eta_k \hat{A}_k\right) y(k).
\]

The reason why selecting this step size is useful is that the iteration now satisfies

\[
\frac{y(k+1)}{|y(k+1)|} = \left(I + \eta_k \hat{A}_k\right) \frac{y(k)}{|y(k)|}.
\]  

(4)

If we conceptualize \(\frac{y(k)}{|y(k)|}\) as the “angular component” of the iterate, then we can compute the angular component at the next timestep independently of \(|y(k)|\) (the “radial component”). Intuitively, we have transformed an optimization problem operating in the whole space \(\mathbb{R}^n\) to one operating on the unit hypersphere; this prevents the problems with divergence that can happen with other step size schemes. We can therefore use (4) to compute the angular component of the solution first, before then computing the radial component. Doing this corresponds to Algorithm 1, Alecton. Notice that, unlike most iterative algorithms for matrix recovery, Alecton does not require any special initialization phase and can be initialized randomly.

\begin{algorithm}
\textbf{Algorithm 1} Alecton: Solve stochastic rank-1 matrix completion problem
\begin{algorithmic}
\REQUIRE \(\eta_k \in \mathbb{R}, K \in \mathbb{N}, L \in \mathbb{N}\), and an unbiased sampling distribution \(\mathcal{A}\)
\\\\n\textbf{\triangleright Angular component (eigenvector) estimation phase}
Select \(y_0\) uniformly as a unit vector in \(\mathbb{R}^n\)
\\FOR \(k = 0\) to \(K - 1\)
\\STATE Select \(\hat{A}_k\) uniformly and independently at random from the sampling distribution \(\mathcal{A}\).
\\STATE \(y_{k+1} \leftarrow y_k + \eta_k \hat{A}_k y_k\) Periodically do \(y_{k+1} \leftarrow y_{k+1} / |y_{k+1}|\) to avoid arithmetic overflow.
\ENDFOR
\\STATE \(\hat{y} \leftarrow y_K / |y_K|\)
\\\\\\\triangleright Radial component (eigenvalue) estimation phase
\\STATE \(r_0 \leftarrow 0\)
\\FOR \(l = 0\) to \(L - 1\)
\\STATE Select \(\hat{A}_l\) uniformly and independently at random from the sampling distribution \(\mathcal{A}\).
\\\STATE \(r_{l+1} \leftarrow r_l + \hat{y}^T \hat{A}_l \hat{y}\)
\ENDFOR
\\STATE \(\bar{r} \leftarrow r_L / L\)
\RETURN \(\bar{r} \hat{y}\)
\end{algorithmic}
\end{algorithm}

We prove three statements about the convergence of this algorithm. First, we prove that, for the angular component, convergence occurs with some constant probability. Next, we establish a rate of convergence for the angular component. Finally, we provide a rate of convergence of the radial component. Since we do not reuse samples in Alecton, our rates do not differentiate between sampling and computational complexity, unlike many other algorithms (see Appendix B).

In order to prove convergence, we need some bounds on the sampling distribution \(\mathcal{A}\). We are able to show rates with only the following bound on the variance of \(\mathcal{A}\).

\footnote{This is equivalent to knowing the Riemannian metric (curvature) of the manifold, which here is \(|y|^2\).}
Definition 1 (Alecton Variance Condition). An unbiased sampling distribution \( \mathcal{A} \) with expected value \( A \) satisfies the Alecton Variance Condition (AVC) with parameters \( (\sigma_a, \sigma_r) \) if and only if for any \( y \), if \( \tilde{A} \) is sampled from \( \mathcal{A} \), the following two conditions hold:

\[
E \left[ (u_1^T \tilde{A} y)^2 + n^{-1} \| \tilde{A} y \|^2 \right] \leq \sigma_a^2 \| y \|^2 ,
\]

and

\[
E \left[ (y^T \tilde{A} y)^2 \right] \leq \sigma_r^2 \| y \|^4 .
\]

To measure the angular closeness of our iterate to the optimal value, we define a quantity \( \rho_{1,k} \) such that

\[
\rho_{1,k} = (u_1^T y_k)^2 \| y_k \|^{-2} .
\]

Informally, \( \rho_{1,k} \) represents the fraction of the “power” of the iterate that lies in the dominant eigenspace. In particular, if \( \rho_{1,k} \) is high, then \( y_k \) will be angularly close to the solution \( u_1 \).

We prove these results for a constant step size \( \eta_k \). In particular, the theorems hold when the step size is smaller than some function of the problem parameters. This means that, even if we do not know these parameters exactly, we can still choose a feasible step size as long as we can lower bound them. (However, smaller step sizes imply slower convergence, so it is a good idea to choose \( \eta \) as large as possible.)

Theorem 1. Assume that we run Alecton on a matrix \( A \) with sampling distribution \( \mathcal{A} \) that satisfies AVC with parameters \( (\sigma_a, \sigma_r) \). For any \( 0 < \delta < 1 \) and \( 0 < \epsilon < 1 \), let \( T \) be the first time during the angular phase of Alecton at which either \( \rho_{1,k} \geq 1 - \epsilon \) (success condition) or \( \rho_{1,k} \leq \delta n^{-1} \) (failure condition). Further assume that for some constant \( 0 < \chi \leq 1 \), for all \( k \) we use constant step size

\[
\eta_k = \frac{\chi(\lambda_1 - \lambda_2)\epsilon}{\sigma_a^2 n (1 + \epsilon)} ,
\]

and we initialize \( y_0 \) uniformly, then it holds that

\[
P (\rho_{1,T} > 1 - \epsilon) \geq \frac{1}{3} - \delta .
\]

That is, the angular phase of Alecton succeeds with probability at least \( 1/3 - \delta \). The expected time until this occurs is

\[
E [T] \leq \frac{2\sigma_a^2 n (1 + \epsilon) (\log(n) - \log(\epsilon))}{\chi (\lambda_1 - \lambda_2) \epsilon^2 \delta} .
\]

Finally, assume the radial phase of Alecton starts after the angular phase succeeded, such that \( (u_1^T \tilde{A} y)^2 \geq 1 - \epsilon \). Then if we iterate in the second phase for \( L \) steps, for any constant \( \psi \) such that \( \psi \geq 2\lambda_1^2 \epsilon^2 \), it holds that

\[
P \left( (\tilde{r} - \lambda_1)^2 \geq \psi \right) \leq \frac{2\sigma_r^2}{\psi L} .
\]

In particular, if \( \sigma_a, \sigma_r, \) and \( \lambda_1 - \lambda_2 \) do not vary with \( n \), this theorem implies convergence with constant probability in \( O(\epsilon^{-1} n \log n) \) samples and in the same amount of time.

A proof for this theorem will appear in Appendix C of this document, but since the method is nonstandard, we will outline it here. First, we construct a sequence \( \tau_k \) such that, whenever our convergence criteria are not met (i.e. \( k < T \) where \( T \) is the time defined above), \( \tau_k \) satisfies

\[
E [\tau_{k+1} | \mathcal{F}_k] \geq \tau_k (1 + R (1 - \tau_k)) \tag{5}
\]

for some constant \( R \), where \( \mathcal{F}_k \) denotes the filtration at time \( k \), which contains all the events that have occurred up to time \( k \). This implies that \( \tau_k \) is a submartingale. We now invoke the optional stopping Theorem [15, p. 59] (here we state a somewhat restricted version).
Definition 2 (Stopping Time). A random variable $T$ is a stopping time with respect to a filtration $\mathcal{F}_k$ if and only if $T \leq k \in \mathcal{F}_k$ for all $k$. That is, we can tell whether $T \leq k$ using only events that have occurred up to time $k$.

Theorem 2 (Optional Stopping Theorem). If $x_k$ is a martingale (or submartingale) with respect to a filtration $\mathcal{F}_k$, and $T$ is a stopping time with respect to the same filtration, then $x_{k \wedge T}$ is also a martingale (resp. submartingale) with respect to the same filtration, where $k \wedge T$ denotes the minimum of $k$ and $T$. In particular, for bounded submartingales, this implies that $\mathbb{E} \left[ x_0 \right] \leq \mathbb{E} \left[ x_T \right]$.

Applying this to the submartingale $\tau_k$ (here, $T$ is obviously a stopping time since it depends only on $\rho_{1,k}$) results in
$$\mathbb{E} \left[ \tau_0 \right] \leq \mathbb{E} \left[ \tau_T \right] \leq \delta + P \left( \rho_T \geq 1 - \epsilon \right).$$

This isolates the probability of successful convergence and so allows us to prove the first part of Theorem 1. Next, subtracting 1 from both sides of (5) and taking the logarithm results in
$$\mathbb{E} \left[ \log \left( 1 - \tau_{k+1} \right) \mid \mathcal{F}_k \right] \leq \log \left( 1 - \tau_k \right) + \log \left( 1 - R \tau_k \right) \leq \log \left( 1 - \tau_k \right) - R \delta.$$ 

So, if we let $W_k = \log \left( 1 - \tau_k \right) + R \delta k$, then $W_k$ is a supermartingale. We again apply the optional stopping theorem to produce
$$\mathbb{E} \left[ W_0 \right] \geq \mathbb{E} \left[ W_T \right] = \mathbb{E} \left[ \log \left( 1 - \tau_T \right) \right] + R \delta \mathbb{E} \left[ T \right] \geq \mathbb{E} \left[ \log \left( n^{-1} \epsilon \right) \right] + R \delta \mathbb{E} \left[ T \right].$$

This isolates the expected stopping time and so allows us to prove the second part of Theorem 1. Finally, the third part of the theorem follows from an application of Chebychev’s inequality to the average of $L$ samples of $\hat{y}^T \hat{A} \hat{y}$.

2.1 Entrywise Sampling

One sampling distribution that arises in many applications (most importantly, matrix completion [16]) is entrywise sampling. This occurs when the samples are independently chosen from the entries of $A$. Specifically,
$$\hat{A} = n^2 e_i e_i^T A e_j e_j^T,$$

where $i$ and $j$ are each independently drawn from $[1, n]$. This distribution is unbiased since
$$\mathbb{E} \left[ \hat{A} \right] = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n n^2 e_i e_i^T A e_j e_j^T = \left( \sum_{i=1}^n e_i e_i^T \right) A \left( \sum_{j=1}^n e_j e_j^T \right) = I A I = A.$$

Unfortunately, there are situations in which using this distribution will cause Alecton to converge poorly. For example, consider the case where $A$ is a diagonal matrix with a single nonzero entry. Unless this entry is chosen (which will not happen for $O(n^2)$ time), the algorithm will be totally unable to converge to the optimal solution. In order to prevent situations like this, we introduce a matrix coherence bound, which has been used in the past [2] for entrywise sampling.

Definition 3. A matrix $A \in \mathbb{R}^{n \times n}$ is incoherent with parameter $\mu$ if and only if for all eigenvectors $u_i$ of the matrix, and for all standard basis vectors $e_j$,
$$|e_j^T u_i| \leq \mu n^{-\frac{1}{2}}.$$

Using this definition, we can provide constraints on the second moment of $\hat{A}$.
Lemma 1. If $A$ is incoherent with parameter $\mu$, and $\tilde{A}$ is sampled uniformly from the entries of $A$, then the distribution of $\tilde{A}$ satisfies the Alecton variance condition with parameters $\sigma_a^2 = \mu^2(1 + \mu^2) \|A\|^2_F$ and $\sigma_r^2 = \mu^4 \text{tr} (A)^2$.

This lemma applied to Theorem 1 above produces a more specialized result.

Corollary 1 (Convergence of Alecton for Entrywise Sampling). Assume that we run Alecton using entrywise sampling on a matrix $A$ such that $A$ is incoherent with parameter $\mu$. As above, we let $T$ be the first time at which either $\rho_{1,k} \geq 1 - \epsilon$ or $\rho_{1,k} \leq \delta n^{-1}$, and for all $k$, we use constant step size

$$\eta_k = \frac{\chi(\lambda_1 - \lambda_2)\epsilon}{\mu^2(1 + \mu^2)n \|A\|^2_F (1 + \epsilon)}.$$  

Then as long as we initialize $y_0$ uniformly, it will hold that

$$P (\rho_T > 1 - \epsilon) \geq \frac{1}{3} - \delta.$$  

The expected time until this occurs will be

$$\mathbb{E}[T] \leq \frac{2\mu^2(1 + \mu^2)n \|A\|^2_F (1 + \epsilon) (\log(n) - \log(\epsilon))}{\chi(\lambda_1 - \lambda_2)^2\delta\epsilon}.$$  

Finally, the radial phase will be characterized by

$$P \left( (\tilde{r} - \lambda_1)^2 \geq \psi \right) \leq \frac{2\mu^4 \text{tr} (A)^2}{\psi L}.$$  

Notice that, if the eigenvalues of $A$ are independent of $n$, this corollary implies $O(\epsilon^{-1} n \log n)$ convergence.

2.2 Trace Sampling

Another common sampling distribution arises from the matrix sensing problem [2]. In this problem, we are given the value of $v^T A w$ (or, equivalently $\text{tr} (A v^T w)$) for unit vectors $v$ and $w$ selected uniformly at random. (This problem has been handled for the more general complex case in [13] using Wirtinger flow.) Using a trace sample, we can construct an unbiased sample

$$\tilde{A} = n^2 vv^T A w w^T.$$  

This is unbiased because $\mathbb{E} [vv^T] = n^{-1} I$. We can also prove the following facts about the second moment of $\tilde{A}$:

Lemma 2. If $n > 50$, and $v$ and $w$ are sampled uniformly from the unit sphere in $\mathbb{R}^n$, then for any positive semidefinite matrix $A$, if we let $\tilde{A} = n^2 vv^T A w w^T$, then the distribution of $\tilde{A}$ satisfies the Alecton variance condition with parameters $\sigma_a^2 = 20 \|A\|^2_F$ and $\sigma_r^2 = 16 \text{tr} (A)^2$.

As in the entrywise sampling case, we can now proceed to specialize our main theorems.

Corollary 2 (Convergence of Alecton for Trace Sampling). Assume that we run Alecton with trace sampling for a rank-$m$ matrix $A$. As usual, we let $T$ be the first time at which either $\rho_{1,k} \geq 1 - \epsilon$ or $\rho_{1,k} \leq \delta n^{-1}$, and for all $k$, we use constant step size

$$\eta_k = \frac{\chi(\lambda_1 - \lambda_2)\epsilon(20)^{-1} n^{-1} \|A\|^2_F (1 + \epsilon)^{-1}}{\mu^4 \text{tr} (A)^2}.$$  

Then as long as we initialize $y_0$ uniformly, it will hold that:

$$P (\rho_T > 1 - \epsilon) \geq \frac{1}{3} - \delta.$$ 

The expected value of the stopping time will be

$$E [T] \leq \frac{40n \|A\|_F^2 (1 + \epsilon) (\log(n) - \log(\epsilon))}{\chi (\lambda_1 - \lambda_2)^2 \delta \epsilon}.$$ 

Finally, the radial phase will be characterized by

$$P \left( (\bar{r} - \lambda_1)^2 \geq \psi \right) \leq \frac{16 \|A\|_F^2}{\psi L}.$$ 

Notice that, as in the entrywise sampling case, if the eigenvalues of $A$ are independent of $n$, this corollary implies $O(\epsilon^{-1} n \log n)$ convergence.

In some cases of the trace sampling problem, instead of being given samples of the form $u^T Av$, we know $u^T Au$. In this case, we need to use two independent samples $u_1^T A u_1$ and $u_2^T A u_2$, and let $u \propto u_1 + u_2$ and $v \propto u_1 - u_2$ be two unit vectors which we will use in the above sampling scheme. Notice that since $u_1$ and $u_2$ are independent and uniformly distributed, $u$ and $v$ will also be independent and uniformly distributed (by the spherical symmetry of the underlying distribution). Furthermore, we can compute

$$u^T Av = (u_1 + u_2)^T A(u_1 - u_2) = u_1^T A u_1 - u_2^T A u_2.$$ 

This allows us to use our above trace sampling scheme even with samples of the form $u^T Au$.

### 2.3 Recovering Additional Eigenvectors

It is possible to use multiple iterations of Alecton to recover additional eigenvalue/eigenvector pairs of the data matrix $A$ one-at-a-time. For example, consider the situation where we have already recovered a rank-$p$ estimate $\hat{A}_p$ of $A$ using Alecton. Using this estimate and our original sampling scheme, we can produce unbiased samples of the matrix $A - \hat{A}_p$. (For most sampling distributions, this will have similar second-moment bounds as the original distribution $\tilde{A}$.) Then, we can use another invocation of Alecton to recover the largest eigenvector of $A - \hat{A}_p$, which will be the $(p + 1)$-th eigenvector of $A$. This strategy allows us to recover the largest $p$ eigenvectors of $A$ using $p$ executions of Alecton. If the eigenvalues of the matrix are independent of $n$ and $p$, we will be able to accomplish this in $O(\epsilon^{-1} pn \log n)$ steps.

### 2.4 Experiments

First, we verify our main claim, that Alecton does converge for practical datasets within $O(n)$ time. Then, we demonstrate that, for the entrywise sampling case, a parallel version of Alecton without any locking (based on the Hogwild! algorithm) performs similarly to the sequential version, while allowing for a linear parallel speedup.

All experiments were run on a machine with two sockets, each with twelve cores (Intel Xeon E5-2697, 2.70GHz), and 256 GB of shared memory. The experiments were run in Julia, and used twenty-four worker processes for parallelism.

The first experiments were run on two randomly-generated rank-$m$ data matrices $A \in \mathbb{R}^{n \times n}$. Each was generated by selecting a random orthogonal matrix $U \in \mathbb{R}^{n \times n}$, then independently selecting a diagonal matrix $\Lambda$ with $m$ positive nonzero eigenvalues, and constructing $A = U \Lambda U'$. Figure 1a illustrates the convergence of three methods on a dataset with $n = 10^4$: Alecton with trace sampling, Alecton with
Figure 1: This figure illustrates that convergence occurs in $O(n)$ steps, and that the parallel Hogwild! version has slightly better algorithmic performance to the sequential version.

entrywise sampling, and a parallel, lockfree (Hogwild!) version of Alecton with entrywise sampling. Figure 1b compares the performance of Alecton with a Hogwild! version of Alecton on the same entrywise distribution. Not only does the Hogwild! version perform slightly better algorithmically, but the parallel speedup also greatly decreases the total runtime (the sequential version of this experiment took 1324 seconds to run, while the Hogwild! version took only 274 seconds, including overhead). The experiments only consider the angular component of the iteration, since the radial component behaves as an averaging process and so is already well understood.

Figure 1c demonstrates similar convergence results on real data from the Netflix Prize problem. This problem involves recovering a matrix with 480,189 columns and 17,770 rows from a training dataset containing 110,198,805 revealed entries. We used the standard block matrix approach to put the data matrix into symmetric form, then ran Alecton with $\eta = 10^{-12}$ for ten million iterations to recover the most significant singular vector (Figure 1c). For this dataset, the sequential version ran in 6.7 seconds while the Hogwild! version ran in 2.5 seconds. Next, we used the method outlined above (section 2.3) to recover additional singular vectors of the matrix by subtracting our current estimate at each rank from the samples. The absolute runtime (sequential Julia) and RMS errors versus both the training and test datasets are plotted in Figure 1d.
This plot illustrates that the one-at-a-time algorithm converges rapidly, even for recovering less-significant eigenvectors — in particular, the runtime does not degrade significantly as the model expands.

3 Conclusion

This paper introduced the first theoretical result that a simple stochastic gradient descent scheme, applied to a rank-1 Burer-Monteiro decomposition, converges with constant probability in $O(\epsilon^{-1} n \log n)$ steps. Furthermore, this is true under broad sampling conditions that include both matrix completion and matrix sensing, and is also able to take noisy samples into account.

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A Interesting Negative Results

Pathological Step-Size Example Here, we observe what happens when we naively choose a constant step size for stochastic gradient descent for quartic objective functions. Consider the simple optimization problem of minimizing

$$ f(x) = \frac{1}{4}x^4. $$

This function will have gradient descent update rule

$$ x_{k+1} = x_k - \alpha_k x_k^3 = (1 - \alpha_k x_k^2) x_k. $$

We now prove that, for any reasonable step size rule chosen independently of $x_k$, there is some initial condition such that this iteration diverges to infinity.

**Proposition 1.** Assume that we iterate using the above rule, for some choice of $\alpha_k$ that is not super-exponentially decreasing; that is, for some $C > 1$ and some $\alpha > 0$, $\alpha_k \geq \alpha C^{-2k}$ for all $k$. Then, if $x_0^2 \geq \alpha^{-1}(C + 1)$, for all $k$

$$ x_k^2 > \alpha^{-1}C^{2k}(C + 1). $$

**Proof.** We will prove this by induction. The base case follows directly from the assumption, while under the inductive case, if the proposition is true for $k$, then

$$ \alpha_k x_k^2 \geq \alpha C^{-2k} \alpha^{-1} C^{2k}(C + 1) = C + 1. $$

Therefore,

$$ x_{k+1}^2 = (\alpha_k x_k^2 - 1)^2 x_k^2 $$

$$ \geq C^2 x_k^2 $$

$$ \geq C^2 \alpha^{-1} C^{2k}(C + 1) $$

$$ = \alpha^{-1} C^{2(k+1)}(C + 1). $$

This proves the statement.

This proof shows that, for some choice of $x_0$, $x_k$ will diverge to infinity exponentially quickly. Furthermore, no reasonable choice of $\alpha_k$ will be able to halt this increase for all initial conditions. We can see the effect of this in stochastic gradient descent as well, where there is always some probability that, due to an unfortunate series of gradient steps, we will enter the zone in which divergence occurs. On the other hand, if we chose step size $\alpha_k = \gamma_k x_k^{-2}$, for some $0 < \gamma_k < 2$, then

$$ x_{k+1} = (1 - \gamma_k) x_k, $$

which converges for all starting values of $x_k$. This simple example is what motivates us to take $\|y_k\|$ into account when choosing the step size for Alecton.

Manifold Optimization Example We now consider a case where SGD on a manifold using a Burer-Monetteiro decomposition doesn’t converge to the global optimum for a particular starting point. Let matrix $A \in \mathbb{R}^{2 \times 2}$ be the diagonal matrix with diagonal entries 4 and 1. Further, let’s assume that we are trying to minimize the decomposed objective function

$$ f(y) = \|A - yy^T\|_F = \|y\|^4 - 2y^T Ay + \|A\|_F^2. $$

If our stochastic samples of $A$ are simply all of $A$ (i.e. we use a perfect estimator), then the SGD update rule is

$$y_{k+1} = y_k - \alpha_k \nabla f(y_k) = y_k - 4\alpha_k \left(y_k \|y_k\|^2 - Ay_k\right).$$

Now, we know that $e_1$ is the most significant eigenvector of $A$, and that $y = 2e_1$ is the global solution to the problem. However,

$$e_1^T y_{k+1} = e_1^T y_k - 4\alpha_k \left( e_1^T y_k \|y_k\|^2 - e_1^T Ay_k \right) = \left( 1 - 4\alpha_k \left( \|y_k\|^2 - 4 \right) \right) (e_1^T y_k).$$

This implies that if $e_1^T y_0 = 0$, then $e_1^T y_k = 0$ for all $k$, which means that convergence to the global optimum cannot occur. This illustrates that global convergence does not occur for all manifold optimization problems using a Burer-Monteiro decomposition and for all starting points.

**Another Manifold Optimization Example** We now consider another case where the manifold optimization results could be misleading. For any graph with node set $N$ and edge set $E$, the MAXCUT problem on the graph requires us to solve

$$\begin{align*}
\text{minimize} & \quad \sum_{(i,j) \in E} y_i y_j \\
\text{subject to} & \quad y_i \in \{-1, 1\}.
\end{align*}$$

Equivalently, if we let $A$ denote the edge-matrix of the graph, we can represent this as a matrix problem

$$\begin{align*}
\text{minimize} & \quad y^T Ay \\
\text{subject to} & \quad y_i \in \{-1, 1\}.
\end{align*}$$

We relax this problem to

$$\begin{align*}
\text{minimize} & \quad y^T Ay \\
\text{subject to} & \quad -1 \leq y_i \leq 1.
\end{align*}$$

Notice that, since the diagonal of $A$ is zero, if we fix all but one of the entries of $y$, the objective function will have an affine dependence on that entry. In particular, this means that a global minimum of the problem must occur on the boundary where $y_i \in \{-1, 1\}$, which implies that this problem has the same global solution as the original MAXCUT problem. Furthermore, for sufficiently large values of $\sigma$, the problem

$$\begin{align*}
\text{minimize} & \quad \|y\|^4 + 2\sigma y^T Ay + \sigma^2 \|A\|^2_F \\
\text{subject to} & \quad -1 \leq y_i \leq 1
\end{align*}$$

will also have the same solution. But, this problem is in the same form as a Burer-Monteiro transformation of

$$\begin{align*}
\text{minimize} & \quad \|X + \sigma A\|^2_F \\
\text{subject to} & \quad X_{ii} \leq 1, X \succeq 0, \text{rank} (X) = 1
\end{align*}$$

where $X = yy^T$. Since MAXCUT is NP-complete, it can’t possibly be the case that SGD applied to the Burer-Monteiro decomposition converges quickly to the global optimum, because that would imply an efficient solution to this NP-complete problem. This implies that, at least in the presence of constraints, manifold-based SGD can not converge quickly in general.
B Comparison with Other Methods

There are several other algorithms that solve similar matrix recovery problems in the literature. Here, we list some other algorithms, and their convergence rates, in terms of both number of samples required (sampling complexity) and number of iterations performed (computational complexity). For this table, the data is assumed to be of dimension $n$, and the rank (where applicable) is assumed to be $m$. (In order to save space, factors of $\log \log \epsilon^{-1}$ have been omitted from some formulas.)

| Algorithm                        | Sampling Scheme | Complexity |
|----------------------------------|-----------------|------------|
|                                  |                 | Sampling   | Computational |
| Alecton                          | Any             | $O(\frac{\epsilon^{-1}mn \log n}{n})$         |              |
| SVD                              | Various         | $o(mn)$    | $O(n^3)$      |
| Spectral Matrix Completion [18]  | Elementwise     | $o(mn)$    | $O(m^2n \log n)$ |
| PhaseLift [19]                   | Phase Retrieval | $o(n)$     | $O(\epsilon^{-1}n^3)$ |
| Alternating Minimization [20]    | Phase Retrieval | $o(n \log(\epsilon^{-1}))$ | $O(n^2 \log^2(\epsilon^{-1}))$ |
| Wirtinger Flow [13]              | Phase Retrieval | $o(n \log^2 n)$ | $O(mn \log(\epsilon^{-1}))$ |

C Proofs of Main Results

Notation In this section, we require that $A \succeq 0$ and let $\lambda_i$ and $u_i$ denote its eigenvalues and eigenvectors, respectively. We assume that at each timestep, we obtain an unbiased sample $\tilde{A}_k$ of $A$. Furthermore, we let $\beta_{i,k} = u_i^T y_k$, $B_{i,k} = u_i^T \tilde{A}_k y_k$, and $\rho_{i,k} = \beta_{i,k}^2 \| y_k \|^2$. Abstractly, $\rho_{i,k}$ represents the fraction of the mass that is currently in the $i$th eigenspace of $A$. In particular, $\rho_{1,k}$ represents the mass that is in the dominant eigenspace. We also let $e_i$ denote the $i$th standard orthonormal basis element within the appropriate vector space. When we are analyzing martingale processes, we let $F_k$ denote the appropriate filtration at timestep $k$.

In this appendix, we will prove the results stated in the paper. In order to accomplish the proof, we will consider, instead of the variable $\rho_{1,k}$, the alternate variable

$$\tau_k = \frac{n \rho_{1,k}}{(n-1) \rho_{1,k} + 1}.$$ 

Using this process, we will proceed to prove Theorem 1 in four steps:

- First, we will prove a lemma (Lemma 5) about the evolution of $\tau_k$ that both linearizes the iteration for small values of $\eta_k$, and uses the distribution of $\tilde{A}_k$ to bound $\tau_k$.

- Next, we will specialize this lemma to prove a stronger statement (Lemma 6) for our particular constant choice of $\eta_k$ and our particular stopping time $T$.

- Then, we will leverage this lemma to prove the first and second parts of Theorem 1 using the optional stopping theorem. This establishes convergence of the angular phase of Algorithm SC1.

- Finally, we will use Chebyshev’s inequality to prove the third part of Theorem 1. This establishes convergence of the radial phase of Algorithm SC1.
C.1 Preliminaries

First, we state some lemmas we will need in the following proofs. We defer proofs of the lemmas themselves to a later section. First, we state a lemma about quadratic rational functions that we will need in the next section.

Lemma 3 (Quadratic rational lower bound). For any \( a, b, c \), and \( x \) in \( \mathbb{R} \), if \( 1 + bx + cx^2 > 0 \), then

\[
\frac{(1 + ax)^2}{1 + bx + cx^2} \geq 1 + (2a - b)x - cx^2.
\]

Next, a lemma about the expected value of a particular function of a normal random variable:

Lemma 4. If \( y_0 \) is chosen from a standard normal random variable (equivalently, initialized uniformly on the unit hypersphere), then

\[
\mathbb{E}[\tau_0] \geq \frac{1}{3}.
\]

Now, we prove our main lemma about the evolution of \( \tau_k \).

Lemma 5 (Dominant Mass Bound). If we run Alecton on a matrix \( A \) with unbiased sampling distribution \( \mathcal{A} \) that satisfies the Alecton Variance Condition with parameters \((\sigma_x, \sigma_y)\), then

\[
\mathbb{E}[\tau_{k+1}|\mathcal{F}_k] \geq \tau_k \left(1 + \eta_k \left(2(\lambda_1 - \lambda_2) - \eta_k \sigma_x^2 n\right) \left(1 - \tau_k\right) - \eta_k^2 \sigma_y^2\right).
\]

Proof. Evaluating \( \tau \) at the next timestep results in

\[
\tau_{k+1} = \frac{\beta^2_{1,k+1}}{(1-n^{-1})\beta^2_{1,k+1} + n^{-1}||y_{k+1}||^2}
\]

\[
= \frac{(\beta_{1,k} + \eta_k B_{1,k})^2}{(1-n^{-1})\beta^2_{1,k} + \eta_k A_{1,k}y_k + n^{-1}||y_k||^2}
\]

\[
= \tau_k \frac{1 + 2\eta_k (1-n^{-1})\beta_{1,k}B_{1,k} + n^{-1}y_{k}^TA_{1,k}y_k}{(1-n^{-1})\beta^2_{1,k} + \eta_k^2 (1-n^{-1})B^2_{1,k} + n^{-1}||y_k||^2},
\]

Applying Lemma 3,

\[
\tau_{k+1} \geq \tau_k \left(1 + 2\eta_k R_k - \eta_k^2 Q_k\right)
\]

for sequences \( R_k \) and \( Q_k \). Now, we recall that

\[
\mathbb{E}[B_{1,k}|\mathcal{F}_k] = \mathbb{E}\left[u_1^T \tilde{A}_k y_k | \mathcal{F}_k\right]
\]

\[
= u_1^T A y_k
\]

\[
= \lambda_1 \beta_{1,k}.
\]
So, taking the expected value of $R_k$ with respect to the filtration,

$$
\mathbb{E}[R_k | \mathcal{F}_k] = \mathbb{E} \left[ \frac{B_{1,k} - (1 - n^{-1})\beta_{1,k} B_{1,k} + n^{-1}y_k^T \bar{A}_k y_k}{(1 - n^{-1})\beta_{1,k}^2 + n^{-1}\|y_k\|^2} \bigg| \mathcal{F}_k \right]
$$

$$
= \lambda_1 - \frac{(1 - n^{-1})\lambda_1 \beta_{1,k}^2 + n^{-1}y_k^T \bar{A}_k y_k}{(1 - n^{-1})\beta_{1,k}^2 + n^{-1}\|y_k\|^2}
$$

$$
= n^{-1}\lambda_1 \|y_k\|^2 - n^{-1}y_k^T \bar{A}_k y_k
$$

Now, since $A \preceq (\lambda_1 - \lambda_2)u_1 u_1^T + \lambda_2 I$,

$$
\mathbb{E}[R_k | \mathcal{F}_k] \geq \frac{n^{-1}\lambda_1 \|y_k\|^2 - n^{-1}y_k^T ((\lambda_1 - \lambda_2)u_1 u_1^T + \lambda_2 I) y_k}{(1 - n^{-1})\beta_{1,k}^2 + n^{-1}\|y_k\|^2}
$$

$$
= \frac{n^{-1}\lambda_1 \|y_k\|^2 - n^{-1}\left((\lambda_1 - \lambda_2)\beta_{1,k}^2 + \lambda_2 \|y_k\|^2\right)}{(1 - n^{-1})\beta_{1,k}^2 + n^{-1}\|y_k\|^2}
$$

$$
= \frac{(\lambda_1 - \lambda_2)n^{-1}\|y_k\|^2 - n^{-1}\beta_{1,k}^2}{(1 - n^{-1})\beta_{1,k}^2 + n^{-1}\|y_k\|^2}
$$

$$
= (\lambda_1 - \lambda_2) \frac{n^{-1}\|y_k\|^2 + (1 - n^{-1})\beta_{1,k}^2 - \beta_{1,k}^2}{(1 - n^{-1})\beta_{1,k}^2 + n^{-1}\|y_k\|^2}
$$

$$
= (\lambda_1 - \lambda_2) (1 - \tau_k).
$$

Taking the expected value of $Q_k$ with respect to the filtration,

$$
\mathbb{E}[Q_k | \mathcal{F}_k] = \mathbb{E} \left[ \frac{(1 - n^{-1})B_{1,k}^2 + n^{-1}\|\bar{A}_k y_k\|^2}{(1 - n^{-1})\beta_{1,k}^2 + n^{-1}\|y_k\|^2} \bigg| \mathcal{F}_k \right].
$$

Applying the Alecton Variance Bound,

$$
\mathbb{E}[Q_k | \mathcal{F}_k] \leq \mathbb{E} \left[ \frac{\sigma_a^2 \|y_k\|^2}{(1 - n^{-1})\beta_{1,k}^2 + n^{-1}\|y_k\|^2} \bigg| \mathcal{F}_k \right]
$$

$$
= \sigma_a^2 \frac{(n - 1)\beta_{1,k}^2 + \|y_k\|^2 - (n - 1)\beta_{1,k}^2}{(1 - n^{-1})\beta_{1,k}^2 + n^{-1}\|y_k\|^2}
$$

$$
= \sigma_a^2 \frac{(n - (n - 1)\tau_k)}{1 + (n - 1)(1 - \tau_k)}
$$

$$
\leq \sigma_a^2 \frac{(1 + n(1 - \tau_k))}{(1 + n(1 - \tau_k))}.
$$

Substituting this back into our original equation,

$$
\mathbb{E}[\tau_{k+1} | \mathcal{F}_k] \geq \tau_k \left(1 + 2\eta_k \mathbb{E}[R_k | \mathcal{F}_k] - \eta_k^2 \mathbb{E}[Q_k | \mathcal{F}_k]\right)
$$

$$
\geq \tau_k \left(1 + 2\eta_k (\lambda_1 - \lambda_2) (1 - \tau_k) - \eta_k^2 \sigma_a^2 (1 + n(1 - \tau_k))\right)
$$

$$
= \tau_k \left(1 + \eta_k \left(2(\lambda_1 - \lambda_2) - \eta_k \sigma_a^2 n\right)(1 - \tau_k) - \eta_k^2 \sigma_a^2\right),
$$

as desired.
Next, we analyze what happens if we iterate with constant parameter $\eta_k$. We first specialize the above lemma to the specific stopping time $T$ and step size that we intend to use. This lemma will establish that $\tau_k$ is increasing in expectation, and further provides a rate at which it is increasing.

**Lemma 6.** If we iterate using Algorithm SCI for a matrix $A$ with unbiased sampling distribution $A$ that satisfies the Alecton Variance Condition with parameters $(\sigma_a, \sigma_r)$, and for some $k$ we use step size

$$\eta_k = \frac{\chi(\lambda_1 - \lambda_2)e}{\sigma^2 n(1+\epsilon)}$$

for some $0 < \chi < 1$, and furthermore at this $k$ we have $\rho_{1,k} \geq 1 - \epsilon$, then

$$E[\tau_{k+1}|F_k] \geq \tau_k (1 + \eta_k(\lambda_1 - \lambda_2)(1 - \tau_k)).$$

**Proof.** First, if $1 - \rho_{1,k} \geq \epsilon$, then

$$1 - \tau_k \geq 1 - \frac{\rho_{1,k}}{(1 - n^{-1})\rho_{1,k} + n^{-1}} = \frac{\rho_{1,k} - n^{-1} \rho_{1,k} + n^{-1}}{\rho_{1,k} - n^{-1} \rho_{1,k} + n^{-1}} = \frac{n^{-1}(1 - \rho_{1,k})}{1 - (1 - n^{-1})(1 - \rho_{1,k})} \geq n^{-1}(1 - \rho_{1,k}) \geq n^{-1}\epsilon.$$

Now, from the result of Lemma 5

$$E[\tau_{k+1}|F_k] \geq \tau_k (1 + \eta_k(2(\lambda_1 - \lambda_2) - \eta_k\sigma^2 n) (1 - \tau_k) - \eta_k^2\sigma^2).$$

Substituting our step size in some locations,

$$E[\tau_{k+1}|F_k] \geq \tau_k (1 + \eta_k(2(\lambda_1 - \lambda_2) - \frac{\chi(\lambda_1 - \lambda_2)e}{\sigma^2 n(1+\epsilon)}\sigma^2_n) (1 - \tau_k) - \eta_k \frac{\chi(\lambda_1 - \lambda_2)e}{\sigma^2 n(1+\epsilon)}\sigma^2_n).$$

Since $\chi < 1$, it follows that

$$(\lambda_1 - \lambda_2) - \frac{\chi(\lambda_1 - \lambda_2)e}{1+\epsilon} \geq (\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_2) = 0.$$ 

Combining this with the fact that $1 - \tau_k > n^{-1}\epsilon$,

$$E[\tau_{k+1}|F_k] \geq \tau_k (1 + \eta_k(\lambda_1 - \lambda_2)(1 - \tau_k) + \eta_k (\lambda_1 - \lambda_2) - \frac{\chi(\lambda_1 - \lambda_2)e}{1+\epsilon} n^{-1}\epsilon - \eta_k \frac{\chi(\lambda_1 - \lambda_2)e}{n(1+\epsilon)} n^{-1}\epsilon)$$

$$= \tau_k (1 + \eta_k(\lambda_1 - \lambda_2)(1 - \tau_k) + \eta_k(\lambda_1 - \lambda_2)n^{-1}\epsilon (1 - \eta))$$

$$\geq \tau_k (1 + \eta_k(\lambda_1 - \lambda_2)(1 - \tau_k)),$$

as desired. 

\[\square\]
C.2 Proof of Main Theorem

Now, we use the fact that the above lemma shows that $\tau_k$ is a supermartingale to bound the probability that convergence occurs.

Proof of First Part of Theorem 1. First, if $k < T$, it follows that $\rho_{1,k} \leq 1 - \epsilon$, so we can apply Lemma 6, which results in

$$\mathbb{E} [\tau_{k+1} | \mathcal{F}_k] \geq \tau_k (1 + \eta_k (\lambda_1 - \lambda_2) (1 - \tau_k)) \geq \tau_k.$$ 

Therefore $\tau_k$ is a supermartingale for $k < T$. So, we can apply the optional stopping theorem, which results in

$$\mathbb{E} [\tau_0] \leq \mathbb{E} [\tau_T].$$

From Lemma 4 we know that $\mathbb{E} [\tau_0] \geq \frac{1}{3}$. Therefore, by the law of total expectation,

$$\frac{1}{3} \leq \mathbb{E} [\tau_T | \rho_{1,T} > 1 - \epsilon] P (\rho_{1,T} > 1 - \epsilon) + \mathbb{E} [\tau_T | \rho_{1,T} \leq 1 - \epsilon] P (\rho_{1,T} \leq 1 - \epsilon).$$

Now, if $\rho_{1,T} \leq 1 - \epsilon$, it follows from the way we defined the stopping time $T$ that $\rho_{1,T} \leq n^{-1}\delta$. Therefore,

$$\tau_T = \frac{\rho_{1,T}}{(1 - n^{-1})\rho_{1,T} + n^{-1}} \leq \frac{n^{-1}\delta}{(1 - n^{-1})n^{-1}\delta + n^{-1}} = \frac{(1 - n^{-1})\delta + 1}{(1 - n^{-1})\delta + 1} \leq \delta.$$

So,

$$\frac{1}{3} \leq \mathbb{E} [\tau_T | \rho_{1,T} > 1 - \epsilon] P (\rho_{1,T} > 1 - \epsilon) + \mathbb{E} [\tau_T | \rho_{1,T} \leq 1 - \epsilon] P (\rho_{1,T} \leq 1 - \epsilon) \leq (1) P (\rho_{1,T} > 1 - \epsilon) + (\delta)(1).$$

Therefore,

$$P (\rho_{1,T} > 1 - \epsilon) \geq \frac{1}{3} - \delta,$$

as desired. \qed

Next, we establish a rate of convergence.

Proof of Second Part of Theorem 1. First, as above if $k < T$, then $\rho_{1,k} \leq 1 - \epsilon$, and we can apply Lemma 6 which results in

$$\mathbb{E} [\tau_{k+1} | \mathcal{F}_k] \geq \tau_k (1 + \eta_k (\lambda_1 - \lambda_2) (1 - \tau_k)) = \tau_k + \eta_k (\lambda_1 - \lambda_2) \tau_k (1 - \tau_k).$$

Therefore,

$$\mathbb{E} [1 - \tau_{k+1} | \mathcal{F}_k] \leq (1 - \tau_k) - \eta_k (\lambda_1 - \lambda_2) \tau_k (1 - \tau_k).$$
Now, if \( k > T \), then \( \rho_{1,k} \geq n^{-1}\delta \), and so

\[
\tau_k = \frac{\rho_{1,k}}{1 - n^{-1}\rho_{1,k} + n^{-1}} \geq \frac{n^{-1}\delta}{1 - n^{-1}\delta + n^{-1}} = \frac{(1 - n^{-1})\delta + 1}{\delta} \geq \frac{\delta}{2}.
\]

Substituting this into the previous expression results in

\[
E\left[1 - \tau_{k+1} | \mathcal{F}_k\right] \leq (1 - \tau_k) - \frac{1}{2} \eta_k (\lambda_1 - \lambda_2) \delta (1 - \tau_k) = (1 - \tau_k) \left( 1 - \frac{1}{2} \eta_k (\lambda_1 - \lambda_2) \delta \right).
\]

Now, since the logarithm function is concave, by Jensen’s inequality,

\[
E\left[\log(1 - \tau_{k+1}) | \mathcal{F}_k\right] \leq \log E\left[1 - \tau_{k+1} | \mathcal{F}_k\right],
\]

and thus by transitivity

\[
E\left[\log(1 - \tau_{k+1}) | \mathcal{F}_k\right] \leq \log(1 - \tau_k) - \frac{1}{2} \eta_k (\lambda_1 - \lambda_2) \delta.
\]

Substituting the step size,

\[
E\left[\log(1 - \tau_{k+1}) | \mathcal{F}_k\right] \leq \log(1 - \tau_k) - \frac{\chi(\lambda_1 - \lambda_2)^2\delta\epsilon}{2\sigma_a^2n(1 + \epsilon)}.
\]

Now, we define a new process \( W_k \) as

\[
W_k = \log(1 - \tau_k) + \frac{\eta(\lambda_1 - \lambda_2)^2\delta\epsilon k}{2\sigma_a^2n(1 + \epsilon)}.
\]

Using this definition, for \( k < T \),

\[
E\left[ W_{k+1} | \mathcal{F}_k\right] = E\left[ \log(1 - \tau_{k+1}) | \mathcal{F}_k\right] + \frac{\eta(\lambda_1 - \lambda_2)^2\delta\epsilon(k + 1)}{2\sigma_a^2n(1 + \epsilon)} \leq \log(1 - \tau_k) - \frac{\eta(\lambda_1 - \lambda_2)^2\delta\epsilon}{2\sigma_a^2n(1 + \epsilon)} + \frac{\eta(\lambda_1 - \lambda_2)^2\delta\epsilon(k + 1)}{2\sigma_a^2n(1 + \epsilon)} = \log(1 - \tau_k) + \frac{\eta(\lambda_1 - \lambda_2)^2\delta\epsilon k}{2\sigma_a^2n(1 + \epsilon)} = W_k.
\]

Therefore \( W_k \) is a supermartingale for \( k < T \), so we can apply the optional stopping theorem, which states that

\[
E\left[ \log(1 - \tau_0) \right] = E\left[ W_0 \right] \geq E\left[ W_T \right].
\]
Since $1 - \tau_0 < 1$, it follows that $\log(1 - \tau_0) < 0$. Therefore,

$$0 \geq E[W_T] = E[\log(1 - \tau_T)] + \frac{\eta(\lambda_1 - \lambda_2)^2\delta\epsilon}{2\sigma_n^2n(1 + \epsilon)E[T]}.$$  

Because we defined our stopping time such that $1 - \rho_{1,k} > \epsilon$, it follows that

$$1 - \tau_T \geq 1 - \frac{\rho_{1,T}}{(1 - n^{-1})\rho_{1,T} + n^{-1}}$$

$$= \frac{\rho_{1,T} - n^{-1}\rho_{1,T} + n^{-1}}{\rho_{1,T} - n^{-1}\rho_{1,T} + n^{-1}}$$

$$= n^{-1}(1 - \rho_{1,T})$$

$$\geq n^{-1}\epsilon.$$  

Therefore $\log(1 - \tau_T) \geq \log(\epsilon) - \log(n)$, and so

$$0 \geq \log(\epsilon) - \log(n) + \frac{\eta(\lambda_1 - \lambda_2)^2\delta\epsilon}{2\sigma_n^2n(1 + \epsilon)E[T]}.$$  

Solving for $E[T]$ results in

$$E[T] \leq \frac{2\sigma_n^2n(1 + \epsilon)(\log(n) - \log(\epsilon))}{\eta(\lambda_1 - \lambda_2)^2\delta\epsilon},$$

as desired. \qed

Lastly, we prove that the radial phase of Algorithm SC1 converges.

**Proof of Third Part of Theorem** Recall that, from the description of Algorithm SC1, $\bar{r}$ is defined as

$$\bar{r} = \frac{1}{L} \sum_{l=0}^{L-1} \hat{y}^T \hat{A}_l \hat{y}.$$  

Therefore, taking the expected value,

$$E[\bar{r}] = \frac{1}{L} \sum_{l=0}^{L-1} \hat{y}^T E[\hat{A}_l] \hat{y}$$

$$= \frac{1}{L} \sum_{l=0}^{L-1} \hat{y}^T A\hat{y}$$

$$= \hat{y}^T A\hat{y}.$$  

Now, since $\lambda_1$ is the largest eigenvalue of $A$, and $|\|\hat{y}\| = 1$, it follows immediately that $E[\bar{r}] \leq \lambda_1$. Furthermore, since $(u_1^T \hat{y})^2 \geq 1 - \epsilon$,

$$E[\bar{r}] = \hat{y}^T A\hat{y}$$

$$\geq \lambda_1 \hat{y}^T u_1^T u_1 \hat{y}$$

$$\geq \lambda_1 (1 - \epsilon).$$
Now, computing the variance of $\bar{r}$,

$$\text{Var}(\bar{r}) = \text{Var}\left( \frac{1}{L} \sum_{l=0}^{L-1} \hat{y}^T \hat{A}_l \hat{y} \right).$$

Since the $\hat{A}_l$ are independently sampled,

$$\text{Var}(\bar{r}) = \frac{1}{L^2} \sum_{l=0}^{L-1} \text{Var}(\hat{y}^T \hat{A}_l \hat{y}) \leq \frac{1}{L^2} \sum_{l=0}^{L-1} \mathbb{E}\left[ (\hat{y}^T \hat{A}_l \hat{y})^2 \right].$$

Applying the Aleckton Variance Condition, and recalling that $\|\hat{y}\| = 1$,

$$\text{Var}(\bar{r}) \leq \frac{1}{L^2} \sum_{l=0}^{L-1} \sigma_r^2 \leq \frac{\sigma_r^2}{L}. \leq \frac{\sigma_r^2}{L^2}L^{-1}. \leq \frac{\sigma_r^2}{L}.$$ 

So, we can apply Chebyshev’s inequality to the variable $\bar{r}$. This results in, for any constant $\gamma > 0$,

$$\gamma^{-2} \geq P\left( |\bar{r} - \mathbb{E}[\bar{r}]| \geq \gamma \sqrt{\text{Var}(\bar{r})} \right).$$

Applying the bound on the variance,

$$\gamma^{-2} \geq P\left( |\bar{r} - \mathbb{E}[\bar{r}]| \geq \gamma \sigma_r \sqrt{L^{-1}} \right) \geq P\left( |\bar{r} - \lambda_1| - |\lambda_1 - \mathbb{E}[\bar{r}]| \geq \gamma \sigma_r \sqrt{L^{-1}} \right).$$

Applying the bound that $\lambda_1(1 - \epsilon) < \mathbb{E}[\bar{r}] < \lambda_1$,

$$\gamma^{-2} \geq P\left( |\bar{r} - \lambda_1| - \lambda_1 \epsilon \geq \gamma \sigma_r \sqrt{L^{-1}} \right) = P\left( |\bar{r} - \lambda_1| \geq \gamma \sigma_r \sqrt{L^{-1}} + \lambda_1 \epsilon \right) = P\left( (\bar{r} - \lambda_1)^2 \geq \left( \gamma \sigma_r \sqrt{L^{-1}} + \lambda_1 \epsilon \right)^2 \right).$$

Now, we choose some $\psi \geq 2\lambda_1^2 \epsilon^2$ and substitute the following value for gamma:

$$\gamma = \frac{\sqrt{\psi} - \lambda_1 \epsilon}{\sigma_r \sqrt{L^{-1}}}.$$ 

This results in

$$P\left( (\bar{r} - \lambda_1)^2 \geq \psi \right) \leq \frac{\sigma_r^2}{(\sqrt{\psi} - \lambda_1 \epsilon)^2 L} \leq \frac{2\sigma_r^2}{\psi L},$$

as desired.
D Proofs of Lemmas

First, we prove the lemmas used above to demonstrate the general result.

Proof of quadratic rational lower bound lemma (Lemma 3). Expanding the product results in
\[
(1 + bx + cx^2)(1 + (2a - b)x - cx^2) = 1 + ((2a - b) + b)x + (c - c + (2a - b)b)x^2 + ((2a - b)c - bc)x^3 - c^2x^4
\]
\[
= 1 + 2ax + (2ab - b^2)x^2 + 2(a - b)cx^3 - c^2x^4
\]
\[
= 1 + 2ax + a^2x^2 - (a^2 - 2ab + b^2)x^2 + 2(a - b)cx^3 - c^2x^4
\]
\[
= 1 + 2ax + a^2x^2 - x^2 ((a - b)^2 - 2(a - b)cx + c^2x^2)
\]
\[
= (1 + ax)^2 - x^2((a - b) - cx)^2
\]
\[
\leq (1 + ax)^2.
\]
Dividing both sides by \(1 + bx + cx^2\) (which we can do since this is assumed to be positive) reconstructs the desired identity. \(\square\)

Proof of Lemma 4. First, let \(x = u^T_1 y_0 = \beta_1, 0\), and let \(z \in \mathbb{R}^{n-1}\) be the orthogonal components of \(y_0\) in some basis. Then, \(\|y_0\|^2 = x^2 + \|z\|^2\), and so
\[
\tau_0 = \frac{\beta^2_{1,0}}{(1 - n^{-1})\beta^2_{1,0} + n^{-1} \|y_0\|^2} = \frac{x^2}{x^2 + n^{-1} \|z\|^2}.
\]
Now, since the above function is convex with respect to \(\|z\|^2\), we can apply Jensen’s inequality, which results in
\[
E[\tau_0 | x] = E \left[ \frac{x^2}{x^2 + n^{-1} \|z\|^2} \bigg| x \right] \geq \frac{x^2}{x^2 + n^{-1} E [\|z\|^2 | x]}.
\]
Since \(z\) is normally distributed independently from \(x\), it follows that \(E[\|z\|^2] = n - 1\), and so
\[
E \left[ \frac{x^2}{x^2 + n^{-1} \|z\|^2} \bigg| x \right] \geq \frac{x^2}{x^2 + n^{-1}(n - 1)} \geq \frac{x^2}{x^2 + 1}.
\]
Therefore, by the law of total probability,
\[
E[\tau_0] = E[E[\tau_0 | x]] \geq E \left[ \frac{x^2}{x^2 + 1} \right] = 1 - E \left[ \frac{1}{x^2 + 1} \right].
\]
Now, by the definition of expected value, since \(x\) is normally distributed,
\[
E \left[ \frac{1}{x^2 + 1} \right] = \int_{-\infty}^{\infty} \left( \frac{1}{x^2 + 1} \right) \left( \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \right) dx.
\]
If we let \(\mathcal{F}\) denote the fourier transform, then
\[
\mathcal{F} \left[ \frac{1}{x^2 + 1} \right] = \sqrt{2\pi} \exp \left( -|\omega| \right).
\]
Furthermore, since the Gaussian functions are eigenfunctions of the Fourier transform, we know that
\[
\mathcal{F} \left[ \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \right] = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\omega^2}{2} \right).
\]
And so, by Parseval’s theorem,
\[ E \left[ \frac{1}{x^2 + 1} \right] = \int_{-\infty}^{\infty} \mathcal{F} \left[ \frac{1}{x^2 + 1} \right] \mathcal{F} \left[ \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \right] d\omega \\
= \int_{-\infty}^{\infty} \sqrt{2\pi} \exp (-|\omega|) \left( \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\omega^2}{2} \right) \right) d\omega \\
= \int_{0}^{\infty} \exp \left( -\omega - \frac{\omega^2}{2} \right) d\omega \\
= \sqrt{\pi} \int_{0}^{\infty} \exp \left( -\frac{1}{2} - \omega - \frac{\omega^2}{2} \right) d\omega. \]

Letting \( u = \frac{\omega + 1}{\sqrt{2}} \) and \( d\omega = \sqrt{2} du \), so
\[ E \left[ \frac{1}{x^2 + 1} \right] = \sqrt{\pi} \int_{\frac{1}{\sqrt{2}}}^{\infty} \exp (-u^2) \sqrt{2} du \\
= \sqrt{\pi} e \frac{\sqrt{\pi}}{2} \text{erfc} \left( \frac{1}{\sqrt{2}} \right) \\
= \sqrt{\pi} e \frac{\sqrt{\pi}}{2} \text{erfc} \left( \frac{1}{\sqrt{2}} \right). \]

Evaluating this, we arrive at
\[ E \left[ \frac{1}{x^2 + 1} \right] \approx 0.65568 \leq \frac{2}{3}. \]

Therefore,
\[ E [\tau_0] \geq 1 - E \left[ \frac{1}{x^2 + 1} \right] \geq 1 - \frac{2}{3} = \frac{1}{3}, \]
as desired. \( \square \)

### D.1 Proofs of Alecton Variance Condition Lemmas

Next, we prove the lemmas about the second moment of \( \tilde{A} \) for the distributions mentioned in the paper, which are used to specialize the general result.

#### D.1.1 Entrywise Sampling

**Proof of the \( \sigma_a \) bound part of Lemma**

By the definition of expected value,
\[ E \left[ \left( u_i^T \tilde{A} y \right)^2 + n^{-1} \| \tilde{A} y \|^2 \right] = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( (u_i^T (n^2 e_j e_i^T A e_i e_i^T) y)^2 + n^{-1} \| (n^2 e_j e_i^T A e_i e_i^T) y \|^2 \right) \\
= n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} (e_j^T A e_i)^2 (e_i^T y)^2 \left( (u_1 e_j)^2 + n^{-1} \right). \]

\[ \]
Applying the coherence bound to $u_i^T e_j$ results in

$$
E \left[ (u_i^T \tilde{A} y)^2 + n^{-1} \| \tilde{A} y \|^2 \right] \leq n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} (e_j^T A e_i)^2 (e_i^T y)^2 (\mu^2 n^{-1} + n^{-1})
$$

$$
= (\mu^2 + 1)n \sum_{i=1}^{n} \sum_{j=1}^{n} (e_j^T A e_i)^2 (e_i^T y)^2
$$

$$
= (\mu^2 + 1)n \sum_{i=1}^{n} e_i^T A \left( \sum_{j=1}^{n} e_j^T e_j \right) A e_i (e_i^T y)^2
$$

$$
= (\mu^2 + 1)n \sum_{i=1}^{n} e_i^T A^2 e_i (e_i^T y)^2
$$

$$
= (\mu^2 + 1)n \sum_{i=1}^{n} e_i^T \left( \sum_{k=1}^{n} \lambda_k^2 u_k u_k^T \right) e_i (e_i^T y)^2
$$

$$
= (\mu^2 + 1)n \sum_{i=1}^{n} \sum_{k=1}^{n} \lambda_k^2 (e_i^T u_k)^2 (e_i^T y)^2.
$$

Applying the coherence bound again,

$$
E \left[ (u_i^T \tilde{A} y)^2 + n^{-1} \| \tilde{A} y \|^2 \right] \leq \mu^2 (\mu^2 + 1) \sum_{i=1}^{n} \sum_{k=1}^{n} \lambda_k^2 (e_i^T y)^2
$$

$$
= \mu^2 (\mu^2 + 1) \| A \|_F^2 \| y \|^2.
$$

This is the desired result. \qed

Proof of the $\sigma_r$ bound part of Lemma[7] By the definition of expected value,

$$
E \left[ (y^T \tilde{A} y)^2 \right] = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (y^T n^2 e_i^T A e_j e_j^T y)^2
$$

$$
= n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} (e_i^T A e_j)^2 (e_i^T y)^2 (e_j^T y)^2
$$

$$
= n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left( e_i^T \left( \sum_{k=1}^{n} \lambda_k u_k u_k^T \right) e_j \right)^2 (e_i^T y)^2 (e_j^T y)^2
$$

$$
= n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \lambda_k (e_i^T u_k) (u_k^T e_j) \right)^2 (e_i^T y)^2 (e_j^T y)^2.
$$
Now, applying the coherence bound results in
\[
E \left[ \left( y^T A y \right)^2 \right] \leq n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \lambda_k \left( \mu \sqrt{\frac{n}{n}} \right) \right)^2 (e_i^T y)^2 (e_j^T y)^2
\]
\[
= \mu^4 \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \lambda_k \right)^2 (e_i^T y)^2 (e_j^T y)^2
\]
\[
= \mu^4 \left( \sum_{k=1}^{n} \lambda_k \right)^2 \left( \sum_{i=1}^{n} (e_i^T y)^2 \right) \left( \sum_{j=1}^{n} (e_j^T y)^2 \right)
\]
\[
= \mu^4 \text{tr} (A)^2 \|y\|^4,
\]
as desired.

\[\square\]

### D.1.2 Trace Sampling

In order to prove our second moment lemma for the trace sampling case, we must first derive some lemmas about the way this distribution behaves.

**Lemma 7 (Sphere Component Fourth Moment).** If \( n > 50 \), and \( w \in \mathbb{R}^n \) is sampled uniformly from the unit sphere, then for any unit vector \( y \in \mathbb{R}^n \),
\[
E \left[ (y^T w)^4 \right] \leq \frac{4}{n^2}.
\]

**Proof.** Let \( x \) be sampled from the standard normal distribution in \( \mathbb{R}^n \). Then, by radial symmetry,
\[
E \left[ (y^T w)^4 \right] = E \left[ (y^T x)^4 \| x \|^4 \right].
\]

If we let \( u \) denote \( y^T x \), and \( z \) denote the components of \( x \) orthogonal to \( y \), then \( \| x \|^2 = u^2 + \| z \|^2 \). Furthermore, by the properties of the normal distribution, \( u \) and \( z \) are independent. Therefore,
\[
E \left[ (y^T w)^4 \right] = E \left[ \frac{u^4}{u^2 + \| z \|^2} \right]
\]
\[
\leq E \left[ \frac{u^4}{\| z \|^2} \right]
\]
\[
= E \left[ u^4 \right] E \left[ \| z \|^{-4} \right].
\]

Now, \( E \left[ u^4 \right] \) is the fourth moment of the normal distribution, which is known to be 3. Furthermore, \( E \left[ \| z \|^{-4} \right] \) is the second moment of an inverse-chi-squared distribution with parameter \( n - 1 \), which is also a known result. Substituting these in,
\[
E \left[ (y^T w)^4 \right] \leq 3 \left( \left( n - 3 \right)^{-2} + 2 \left( n - 3 \right)^{-2} \left( n - 5 \right)^{-1} \right)
\]
\[
= 3 \left( n - 3 \right)^{-2} \left( 1 + 2 \left( n - 5 \right)^{-1} \right).
\]
This quantity has the asymptotic properties we want. In particular, applying the constraint that \( n > 50 \),

\[
E \left[ (y^T w)^4 \right] \leq \frac{4}{n^2}.
\]

This is the desired result. \( \square \)

**Lemma 8 (Sphere Component Fourth Moment Matrix).** If \( n > 50 \), and \( w \in \mathbb{R}^n \) is sampled uniformly from the unit sphere, then for any unit vector \( y \in \mathbb{R}^n \),

\[
E \left[ ww^T yy^T ww^T \right] \preceq 4n^{-2} I.
\]

**Proof.** For any unit vector \( z \),

\[
z^T E \left[ ww^T yy^T ww^T \right] z = E \left[ (z^T w)^2 (y^T w)^2 \right].
\]

By Cauchy-Schwarz,

\[
z^T E \left[ ww^T yy^T ww^T \right] z \leq \sqrt{E \left[ (z^T w)^4 \right] E \left[ (y^T w)^4 \right]}.
\]

Applying Lemma 7 results in

\[
z^T E \left[ ww^T yy^T ww^T \right] z \leq \sqrt{(4n^{-2})(4n^{-2})} = 4n^{-2}.
\]

Since this is true for any unit vector \( z \), by the definition of the positive semidefinite relation,

\[
E \left[ ww^T yy^T ww^T \right] \preceq 4n^{-2} I,
\]

as desired. \( \square \)

Now, we prove the AVC lemma for this distribution.

**Proof of \( \sigma_a \) bound part of Lemma 2** We start with the expression we want to bound:

\[
E \left[ \left( u_1^T \tilde{A} y \right)^2 + n^{-1} \left\| \tilde{A} y \right\|^2 \right] = n^4 E \left[ \left( u_1^T v v^T A y w w^T y \right)^2 + n^{-1} \left\| v v^T A w w^T y \right\|^2 \right]
\]

\[
= n^4 E \left[ y^T w w^T A v v^T u_1 u_1^T v v^T A w w^T y + n^{-1} y^T w w^T A v v^T A w w^T y \right].
\]

Evaluating the expected value with respect to \( v \), and applying Lemma \( \tilde{8} \) results in

\[
E \left[ \left( u_1^T \tilde{A} y \right)^2 + n^{-1} \left\| \tilde{A} y \right\|^2 \right] \leq n^4 E \left[ y^T w w^T A (4n^{-2} I) A w w^T y + n^{-1} y^T w w^T A (n^{-1} I) A w w^T y \right]
\]

\[
= 5n^2 E \left[ y^T w w^T A^2 w w^T y \right]
\]

\[
= 5n^2 \text{tr} \left( A^2 E \left[ w w^T y y^T w w^T \right] \right).
\]

Again applying Lemma \( \tilde{8} \)

\[
E \left[ \left( u_1^T \tilde{A} y \right)^2 + n^{-1} \left\| \tilde{A} y \right\|^2 \right] \leq 5n^2 \text{tr} \left( A^2 (4n^{-2} I) \right) = 20 \left\| A \right\|_F^2,
\]

as desired. \( \square \)
Proof of $\sigma$ bound part of Lemma 2. Evaluating the expected value of the expression we want to bound,

\[
\mathbb{E}\left[ (y Ay)^2 \right] = n^4 \mathbb{E}\left[ (y vv^T A w w^T y)^2 \right] \\
= n^4 \mathbb{E}\left[ \text{tr} (A v v^T y y^T A w w^T y y) \right] \\
= n^4 \text{tr} \left( A \mathbb{E}[v v^T y y^T v v^T] A \mathbb{E}[w w^T y y^T w w^T] \right).
\]

Applying Lemma 8 to this results in

\[
\mathbb{E}\left[ (y Ay)^2 \right] \leq n^4 \text{tr} \left( A (4n^{-2} I) A (4n^{-2} I) \right) = 16 \|A\|_F^2,
\]

as desired. $\square$

E Lower Bound on Convergence Rate of Entrywise SGD

In this section, we prove a rough lower bound on the rate of convergence of an Entrywise SGD algorithm. While this algorithm is slightly different from Alecton, it is still a Burer-Monteiro SGD algorithm. First, a simple lemma that we’ll need later:

Lemma 9. For any $\alpha, x > 0$, and $y > 0$,

\[
(1 - \alpha)^2 x + \alpha^2 y \geq \frac{xy}{x + y}.
\]

Proof.

\[(1 - \alpha)^2 x + \alpha^2 y (x + y) = (1 - \alpha)^2 x^2 + (1 - \alpha)^2 xy + \alpha^2 xy + \alpha^2 y^2 = (1 - \alpha)^2 x^2 + (1 - 2\alpha + 2\alpha^2)xy + \alpha^2 y^2 = (1 - \alpha)^2 x^2 - 2\alpha(1 - \alpha)xy + \alpha^2 y^2 + xy = (1 - \alpha)x + \alpha y + xy \geq xy.\]

Now we prove that a stochastic gradient descent algorithm applied to this problem can’t converge any faster than $1/K$ because the non-dominant components are lower bounded in variance by a sequence asymptotically decreasing at a $1/K$ rate. This result provides a specific adaptation of the general $1/K$ rate observed in SGD to this particular problem.

Lemma 10. If we iterate using an Entrywise SGD algorithm with the update rule

\[y_{k+1} = (1 - \alpha_k)y_k + \alpha_k A_k y_k / \|y_k\|^2\]

for a matrix $A$ with unbiased independent sampling distribution $A$, then for any vector $u$ in the null space of $A$ such that for some $\sigma_u$ and for any vector $y$,

\[\mathbb{E}\left[ (A y)^2 \right] \geq \sigma_u^2 \|y\|^2,\]
then if $E \left[ \| y_k \|^{-2} \right] \leq M$ over all iterations, it follows that

$$E \left[ (u^T y_K)^2 \right] \geq \frac{\sigma_u^2 M E \left[ (u^T y_0)^2 \right]}{\sigma_u^2 M + K E \left[ (u^T y_0)^2 \right]}.$$ 

In particular, notice that this result does not depend on the choice of $\alpha_k$ (i.e. the lower bound is independent of step size).

**Proof.** We start with the expression we would like to bound:

$$E \left[ (u^T y_{k+1})^2 \right] = (1 - \alpha_k)^2 E \left[ (u^T y_k)^2 \right] + 2\alpha_k (1 - \alpha_k) E \left[ u^T y_k u^T \tilde{A}_k y_k \| y_k \|^{-2} \right] + \alpha_k^2 E \left[ (u^T \tilde{A}_k y_k)^2 \| y_k \|^{-4} \right]$$

Applying our assumptions,

$$E \left[ (u^T y_{k+1})^2 \right] \geq (1 - \alpha_k)^2 E \left[ (u^T y_k)^2 \right] + 2\alpha_k (1 - \alpha_k) E \left[ u^T y_k u^T \tilde{A}_k y_k \| y_k \|^{-2} \right] + \alpha_k^2 \sigma_u^2 E \left[ \| y_k \|^2 \| y_k \|^{-4} \right]$$

$$= (1 - \alpha_k)^2 E \left[ (u^T y_k)^2 \right] + \alpha_k^2 \sigma_u^2 \left[ \| y_k \|^2 \| y_k \|^{-2} \right]$$

$$\geq (1 - \alpha_k)^2 E \left[ (u^T y_k)^2 \right] + \alpha_k^2 \sigma_u^2 M.$$

Applying Lemma[9] results in

$$E \left[ (u^T y_{k+1})^2 \right] \geq \frac{\sigma_u^2 M E \left[ (u^T y_k)^2 \right]}{E \left[ (u^T y_k)^2 \right] + \sigma_u^2 M}.$$

Taking the inverse of both sides,

$$E \left[ (u^T y_{k+1})^2 \right] \leq \frac{E \left[ (u^T y_k)^2 \right] + \sigma_u^2 M}{\sigma_u^2 M E \left[ (u^T y_k)^2 \right]}$$

$$= \frac{1}{E \left[ (u^T y_k)^2 \right]} + \frac{1}{\sigma_u^2 M}.$$

Summing this up to $K$,

$$E \left[ (u^T y_K)^2 \right] \leq \frac{1}{E \left[ (u^T y_0)^2 \right]} + \frac{K}{\sigma_u^2 M},$$

and taking the inverse again produces

$$E \left[ (u^T y_K)^2 \right] \geq \frac{\sigma_u^2 M E \left[ (u^T y_0)^2 \right]}{\sigma_u^2 M + K E \left[ (u^T y_0)^2 \right]}.$$ 

This is the desired result.
F Handling Constraints

Algorithm SC1 can easily be adapted to solve the problem of finding a low-rank approximation to a matrix under a spectrahedral constraint. That is, we want to solve the problem

\[
\begin{align*}
\text{minimize} & \quad \|A - X\|_F^2 \\
\text{subject to} & \quad X \in \mathbb{R}^{N \times N}, \quad \text{tr}(X) = 1, \quad \text{rank}(X) \leq 1, \quad X \succeq 0.
\end{align*}
\]

This is equivalent to the decomposed problem

\[
\begin{align*}
\text{minimize} & \quad \|y\|_4^4 - 2y^T Ay + \|A\|_F^2 \\
\text{subject to} & \quad y \in \mathbb{R}^N, \quad \|y\|_2^2 = 1,
\end{align*}
\]

which is itself equivalent to:

\[
\begin{align*}
\text{minimize} & \quad 1 - 2y^T Ay + \|A\|_F^2 \\
\text{subject to} & \quad y \in \mathbb{R}^N, \quad \|y\|_2^2 = 1.
\end{align*}
\]

Obviously, this will have a minimum when \(y = u_1\). We can therefore solve the problem using only the angular phase of Alecton, which recovers the vector \(u_1\). The same convergence analysis described above still applies.

For an example of a constrained problem that Alecton cannot handle, because it is NP-hard, see the elliptope-constrained MAXCUT embedding in Appendix A. This shows that constrained problems can’t be solved efficiently by SGD algorithms in all cases.

G Towards a Linear Rate

In this section, we consider a special case of the matrix recovery problem: one in which the samples we are given would allow us to exactly recover \(A\). That is, for some linear operator \(\Omega : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^s\), we are given the value of \(\Omega(A)\) as an input, and we know that the unique solution of the optimization problem

\[
\begin{align*}
\text{minimize} & \quad \|\Omega(X - A)\|_F^2 \\
\text{subject to} & \quad X \in \mathbb{R}^{n \times n}, \quad \text{rank}(X) \leq p, \quad X \succeq 0
\end{align*}
\]

is \(X = A\). Performing a rank-\(p\) quadratic substitution on this problem results in

\[
\begin{align*}
\text{minimize} & \quad \|\Omega(Y Y^T - A)\|_F^2 \\
\text{subject to} & \quad Y \in \mathbb{R}^{n \times p}.
\end{align*}
\]

The specific case we will be looking at is where the operator \(\Omega\) satisfies the \(p\)-RIP constraint [2].

**Definition 4** (Restricted isometry property). A linear operator \(\Omega : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^s\) satisfies \(p\)-RIP with constant \(\delta\) if for all \(X \in \mathbb{R}^{n \times n}\) of rank at most \(p\),

\[
(1 - \delta) \|X\|_F^2 \leq \|\Omega(X)\|_2^2 \leq (1 + \delta) \|X\|_F^2.
\]

This definition encodes the notion that \(\Omega\) preserves the norm of low-rank matrices under its transformation. We can prove a simple lemma that extends this to the inner product.

**Lemma 11.** If \(\Omega\) is \((p + q)\)-RIP with parameter \(\delta\), then for any symmetric matrices \(X\) and \(Y\) of rank at most \(p\) and \(q\) respectively,

\[
\Omega(X)^T \Omega(Y) \geq \text{tr}(XY) - \delta \|X\|_F \|Y\|_F.
\]
Proof. For any $a \in \mathbb{R}$, since $\Omega$ is linear,
\[
\text{tr} (\Omega(X)\Omega(Y)) = \frac{1}{4a} \left(\|\Omega(X) + a\Omega(Y)\|^2 - \|\Omega(X) - a\Omega(Y)\|^2\right)
\]
\[
= \frac{1}{4a} \left(\|\Omega(X + aY)\|^2 - \|\Omega(X - aY)\|^2\right).
\]
Since $\text{rank}(X - aY) \leq \text{rank}(X) + \text{rank}(Y) \leq p + q$, we can apply our RIP inequalities, which produces
\[
\text{tr} (\Omega(X)\Omega(Y)) \geq \frac{1}{4a} \left( (1 - \delta) \|X + aY\|^2_F - (1 + \delta) \|X - aY\|^2_F \right)
\]
\[
\geq \frac{1}{4a} \left( -2\delta \|X\|^2_F + 4a\text{tr}(XY) - 2\delta a^2 \|Y\|^2_F \right)
\]
\[
= \text{tr}(XY) - \delta \frac{\|X\|^2_F + a^2 \|Y\|^2_F}{2a}.
\]
Substituting $a = \frac{\|X\|^2_F}{\|Y\|^2_F}$ results in
\[
\text{tr} (\Omega(X)\Omega(Y)) \geq \text{tr}(XY) - \delta \frac{\|X\|_F \|Y\|_F},
\]
as desired. \qed

Finally, we prove our main theorem that shows that the quadratically transformed objective function is strongly convex in a ball about the solution.

**Theorem 3.** If we define $f(Y)$ as the objective function of the above optimization problem, that is for $Y \in \mathbb{R}^{n \times p}$ and $A \in \mathbb{R}^{n \times n}$ symmetric of rank no greater than $p$,
\[
f(Y) = \|\Omega(YY^T - A)\|^2_F,
\]
and $\Omega$ is $3p$-RIP with parameter $\delta$, then for all $Y$, if we let $\lambda_p$ denote the smallest positive eigenvalue of $A$ then
\[
\nabla_Y^2 f(Y) \geq 2 \left( (1 - \delta)\lambda_p - (3 + \delta) \|YY^T - A\|_F \right) I.
\]

Proof. The directional derivative of $f$ along some direction $V$ will be, by the product rule,
\[
\nabla_V f(Y) = 2\Omega(YY^T - A)^T \Omega(YV^T + VY^T).
\]
The second derivative along this same direction will be
\[
\nabla_V^2 f(Y) = 4\Omega(YY^T - A)^T \Omega(VV^T) + 2\Omega(YV^T + VY^T)^T \Omega(YV^T + VY^T)
\]
\[
= 4\Omega(YY^T - A)^T \Omega(VV^T) + 2 \|\Omega(YV^T + VY^T)\|^2_F.
\]
To this, we can apply the definition of RIP, and the corollary lemma, which results in
\[
\nabla_V^2 f(Y) \geq 4\text{tr} \left( (YY^T - A)(UU^T) \right) - 4\delta \|YY^T - A\|_F \|UU^T\|_F + 2(1 - \delta) \|YY^T + UY^T\|_F^2.
\]
By Cauchy-Schwarz,
\[
\nabla_V^2 f(Y) \geq -4 \|YY^T - A\|_F \text{tr} \left( UU^T \right) - 4\delta \|YY^T - A\|_F \text{tr} \left( UU^T \right) + 2(1 - \delta)\lambda_{\text{min}}(Y^TY) \text{tr} \left( UU^T \right)
\]
\[
= 2 \left( (1 - \delta)\lambda_{\text{min}}(Y^TY) - 2(1 + \delta) \|YY^T - A\|_F \right) \text{tr} \left( UU^T \right).
\]

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Now, since at the optimum, \( \lambda_{\text{min}}(Y^TY) = \lambda_p \), it follows that for general \( Y \),
\[
\lambda_{\text{min}}(Y^TY) \geq \lambda_p - \|YY^T - A\|_F.
\]
Substituting this in to the previous expression,
\[
\nabla^2_V f(Y) \geq 2 ((1 - \delta)(\lambda_p - \|YY^T - A\|_F) - 2(1 + \delta)\|YY^T - A\|_F) \text{tr}(U^TU) \\
= 2 ((1 - \delta)\lambda_p - (3 + \delta)\|YY^T - A\|_F) \|U\|^2_F.
\]
Since this is true for an arbitrary direction vector \( U \), it follows that
\[
\nabla^2_V f(Y) \succeq 2 ((1 - \delta)\lambda_p - (3 + \delta)\|YY^T - A\|_F) I,
\]
which is the desired result.

This theorem shows that there is a region of size \( O(1) \) (i.e. not dependent on \( n \)) within which the above problem is strongly convex. So, if we start within this region, any standard convex descent method will converge at a linear rate. In particular, coordinate descent will do so. Therefore, we can imagine doing the following:

- First, use Alecton to, with high probability, recover an estimate \( Y \) that for which \( \|YY^T - A\|_F \) is sufficiently small for the objective function to be strongly convex with some probability. This will only require \( O(n \log n) \) steps of the angular phase of the algorithm per iteration of Alecton, as stated in the main body of the paper. We will need \( p \) iterations of the algorithm to recover a rank-\( p \) estimate, so a total \( O(np \log n) \) iterations will be required.

- Use a descent method, such as coordinate descent, to recover additional precision of the estimate. This method is necessarily more heavyweight than an SGD scheme (see Section\[E\] for the reason why an SGD scheme cannot achieve a linear rate), but it will converge monotonically at a linear rate to the exact solution matrix \( A \).

This hybrid method is in some sense a best-of-both worlds approach. We use fast SGD steps when we can afford to, and then switch to slower coordinate descent steps when we need additional precision.

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