ORTHOGONAL APARTMENTS IN HILBERT GRASSMANNIANS.
FINITE-DIMENSIONAL CASE

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Abstract. Let $H$ be a complex Hilbert space of finite dimension $n \geq 3$. Denote by $G_k(H)$ the Grassmannian consisting of $k$-dimensional subspaces of $H$. Every orthogonal apartment of $G_k(H)$ is defined by a certain orthogonal base of $H$ and consists of all $k$-dimensional subspaces spanned by subsets of this base. For $n \neq 2k$ (except the case when $n = 6$ and $k$ is equal to 2 or 4) we show that every bijective transformation of $G_k(H)$ sending orthogonal apartments to orthogonal apartments is induced by an unitary or conjugate-unitary operator on $H$. The second result is the following: if $n = 2k \geq 8$ and $f$ is a bijective transformation of $G_k(H)$ such that $f$ and $f^{-1}$ send orthogonal apartments to orthogonal apartments then there is an unitary or conjugate-unitary operator $U$ such that for every $X \in G_k(H)$ we have $f(X) = U(X)$ or $f(X)$ coincides with the orthogonal complement of $U(X)$.

1. Introduction and statement of results

Let $H$ be a complex Hilbert space of finite or infinite dimension. For every natural $k < \dim H$ we denote by $G_k(H)$ the Grassmannian consisting of $k$-dimensional subspaces of $H$. For every orthogonal base of $H$ the associated orthogonal apartment of $G_k(H)$ consists of all $k$-dimensional subspaces spanned by subsets of this base. Orthogonal apartments of Hilbert Grassmannians were introduced in [8]. This notion comes from the theory of Tits buildings [11]. Grassmannians related to a building of type $A_{n-1}$ are the Grassmannians $G_k(V), k \in \{1, \ldots, n-1\}$ formed by $k$-dimensional subspaces of an $n$-dimensional vector space $V$ (see [6, 9] for the details) and every apartment of $G_k(V)$ consists of all $k$-dimensional subspaces spanned by subsets of a certain base of $V$.

Recall that two closed subspaces $X, Y \subset H$ are compatible if there exist closed subspaces $X', Y'$ such that $X \cap Y, X', Y'$ are mutually orthogonal and

$$X = X' + (X \cap Y), \quad Y = Y' + (X \cap Y).$$

The concept of orthogonal apartment is interesting for the following reason: orthogonal apartments can be characterized as maximal subsets of mutually compatible elements of $G_k(H)$ [3, Proposition 1]. Note that the compatibility relation is defined for any logic, i.e. a lattice with an addition operation known as the negation. In classical logics any two elements are compatible and quantum logics contain non-compatible elements.

Consider the logic $L(H)$ whose elements are closed subspaces of $H$ and the negation is the operation of orthogonal complementary. In the case when $H$ is infinite-dimensional and separable, this logic is exploited in mathematical foundations of quantum theory (see, for example, [12]). Every automorphism of $L(H)$

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is induced by an unitary or conjugate-unitary operator on $H$. It follows from [5, Theorem 2.8] that for every bijective transformation $f$ of $L(H)$ preserving the compatibility relation in both directions, i.e. $f$ and $f^{-1}$ send compatible elements to compatible elements, there is an unitary or conjugate-unitary operator $U$ such that for every $X \in L(H)$ we have $f(X) = U(X)$ or $f(X)$ coincides with the orthogonal complement of $U(X)$. Note that if closed subspaces $X$ and $Y$ are compatible then the orthogonal complement of $X$ is compatible to $Y$.

If $H$ is infinite-dimensional and $f$ is a bijective transformation of $G_k(H)$ such that $f$ and $f^{-1}$ send orthogonal apartments to orthogonal apartments (equivalently, $f$ preserves the compatibility relation in both directions) then $f$ is induced by an unitary or conjugate-unitary operator [8, Theorem 1]. In this paper, a similar result will be obtained for the case when $H$ is finite-dimensional. As in [8], we will use complementary subsets of orthogonal apartments, but the arguments will be more complicated.

**Theorem 1.** Suppose that $\dim H = n$ is finite and not less than 3. Let $f$ be a bijective transformation of $G_k(H)$ sending orthogonal apartments to orthogonal apartments and let $n \neq 2k$. We also require that $k$ is not equal to 2 or 4 if $n = 6$. Then $f$ is induced by an unitary or conjugate-unitary operator on $H$.

In Theorem 1 we do not require that the inverse transformation sends orthogonal apartments to orthogonal apartments. If $H$ is finite-dimensional and $f$ is a bijective transformation of $G_k(H)$ sending compatible elements to compatible elements then $f$ transfers orthogonal apartments to orthogonal apartments. This is a simple consequence of the following facts: the class of orthogonal apartments coincides with the class of maximal subsets of mutually compatible elements of $G_k(H)$, all orthogonal apartments in $G_k(H)$ are of the same finite cardinality.

**Theorem 2.** Suppose that $\dim H = 2k \geq 8$. Let $f$ be a bijective transformation of $G_k(H)$ such that $f$ and $f^{-1}$ send orthogonal apartments to orthogonal apartments, in other words, $f$ preserves the compatibility relation in both directions. Then there is an unitary or conjugate-unitary operator $U$ on $H$ such that for every $X \in G_k(H)$ we have $f(X) = U(X)$ or $f(X)$ coincides with the orthogonal complement of $U(X)$.

**Remark 1.** Apartments preserving transformations of Grassmannians corresponding to buildings of classical types are described in [6]. Also, there is a characterization of isometric embeddings of Grassmann graphs in terms of “generalized” apartments [7, Chapter 5].

## 2. Grassmann graphs

To prove Theorems 1 and 2 we need some properties of Grassmann graphs. Suppose that $\dim H = n$ is finite. The Grassmann graph $\Gamma_k(H)$, $1 < k < n - 1$ is the graph whose vertex set is $G_k(H)$ and two $k$-dimensional subspaces are adjacent vertices of this graph if their intersection is $(k-1)$-dimensional. In what follows we say that two $k$-dimensional subspaces of $H$ are *adjacent* if they are adjacent vertices of $\Gamma_k(H)$.

If $S$ and $N$ are subspaces of $H$ such that $\dim S < k$ and $\dim N > k$ then we denote by $[S]_k$ and $[N]_k$ the sets consisting of $k$-dimensional subspaces containing $S$ and contained in $N$, respectively. There are precisely the following two types of maximal cliques in $\Gamma_k(H)$:
• the stars \([S]_k\), \(S \in \mathcal{G}_{k-1}(H)\);
• the tops \((N)_k\), \(N \in \mathcal{G}_{k+1}(H)\).

See, for example, [7 Section 3.2].

Let \(\mathcal{A}\) be the orthogonal apartment of \(\mathcal{G}_k(H)\) defined by an orthogonal base \(B\).

The restriction of \(\Gamma_k(H)\) to \(\mathcal{A}\) is isomorphic to the Johnson graph \(J(n,k)\). As above, we have two types of maximal cliques:

• the stars \(\mathcal{A} \cap [S]_k\),
• the tops \(\mathcal{A} \cap (N)_k\),

where \(S \in \mathcal{G}_{k-1}(H)\) and \(N \in \mathcal{G}_{k+1}(H)\) are spanned by subsets of \(B\). Stars and tops of \(\mathcal{A}\) contain precisely \(n - k + 1\) and \(k + 1\) elements, respectively.

By classical Chow’s theorem [2], every automorphism of the graph \(\Gamma_k(H)\) is induced by an invertible semilinear operator if \(n \neq 2k\). In the case when \(n = 2k\), the automorphism group of \(\Gamma_k(H)\) is generated by the automorphisms induced by invertible semilinear operators and the mapping \(X \to X^\perp\), where \(X^\perp\) is the orthogonal complement of \(X\). Huang’s result [4] (see also [7 Section 3.10]) says that every bijective transformation of \(\mathcal{G}_k(H)\) sending adjacent elements to adjacent elements is an automorphism of \(\Gamma_k(H)\).

3. COMPLEMENTARY SUBSETS IN ORTHOGONAL APARTMENTS

Let \(\mathcal{A}\) be the orthogonal apartment of \(\mathcal{G}_k(H)\) defined by an orthogonal base \(\{e_i\}_{i=1}^n\) and let \(1 < k < n - 1\). For every \(i \in \{1, \ldots, n\}\) we denote by \(\mathcal{A}(+i)\) and \(\mathcal{A}(-i)\) the sets consisting of all elements of \(\mathcal{A}\) which contain \(e_i\) and do not contain \(e_i\), respectively. For any distinct \(i, j \in \{1, \ldots, n\}\) we define

\[
\mathcal{A}(+i, +j) := \mathcal{A}(+i) \cap \mathcal{A}(+j), \\
\mathcal{A}(+i, -j) := \mathcal{A}(+i) \cap \mathcal{A}(-j), \\
\mathcal{A}(-i, -j) := \mathcal{A}(-i) \cap \mathcal{A}(-j).
\]

A subset of \(\mathcal{A}\) is called inexact if there is an orthogonal apartment distinct from \(\mathcal{A}\) and containing this subset. By [8 Lemma 1], every maximal inexact subset is

\[\mathcal{A}(+i, +j) \cup \mathcal{A}(-i, -j).\]

We say that a subset \(\mathcal{C} \subset \mathcal{A}\) is complementary if \(\mathcal{A} \setminus \mathcal{C}\) is a maximal inexact subset. Then

\[\mathcal{A} \setminus \mathcal{C} = \mathcal{A}(+i, +j) \cup \mathcal{A}(-i, -j)\]

and

\[\mathcal{C} = \mathcal{A}(+i, -j) \cup \mathcal{A}(+j, -i).\]

This complementary subset is denoted by \(\mathcal{C}_{ij}\). Observe that \(\mathcal{C}_{ij} = \mathcal{C}_{ji}\).

Remark 2. Suppose that \(\mathcal{A}\) is an arbitrary (not necessarily orthogonal) apartment of \(\mathcal{G}_k(H)\). A subset of \(\mathcal{A}\) is inexact if there is an apartment distinct from \(\mathcal{A}\) and containing this subset. Every maximal inexact subset is of type \(\mathcal{A}(+i, +j) \cup \mathcal{A}(-i)\) and the corresponding complimentary subset is \(\mathcal{A}(+i, -j)\), see [7 Section 5.2].

Lemma 1. Let \(f\) be a bijective transformation of \(\mathcal{G}_k(H)\) sending orthogonal apartments to orthogonal apartments and let \(\mathcal{A}\) be an orthogonal apartment of \(\mathcal{G}_k(H)\). Then \(\mathcal{C} \subset \mathcal{A}\) is a complementary subset if and only if \(f(\mathcal{C})\) is a complementary subset of \(f(\mathcal{A})\).
Proof. It is clear that \( f \) transfers inexact subsets of \( \mathcal{A} \) to inexact subsets of \( f(\mathcal{A}) \). An inexact subset is maximal if and only if it contains

\[
\binom{n-2}{k-2} + \binom{n-2}{k}
\]
elements. This implies that maximal inexact subsets of \( \mathcal{A} \) go to maximal inexact subsets of \( f(\mathcal{A}) \). Since \( \mathcal{A} \) and \( f(\mathcal{A}) \) have the same number of such subsets, \( \mathcal{X} \) is a maximal inexact subset of \( \mathcal{A} \) if and only if \( f(\mathcal{X}) \) is a maximal inexact subset of \( f(\mathcal{A}) \). This gives the claim. \( \square \)

Lemma 2. Let \( X, Y \) be distinct elements of an orthogonal apartment \( \mathcal{A} \subset G_k(H) \). If \( \dim(X \cap Y) = m \) then there are precisely

\[
c(m) = (k-m)^2 + m(n-2k+m)
\]
distinct complementary subsets of \( \mathcal{A} \) containing this pair.

Proof. Since \( \dim(X \cap Y) = m \), we have \( \dim(X + Y) = 2k - m \). Let \( \{e_i\}_{i=1}^n \) be one of the orthogonal bases associated to \( \mathcal{A} \). If the complementary subset \( C_{ij} \) contains both \( X \) and \( Y \) then one of the following possibilities is realized:

1. one of \( e_i, e_j \) belongs to \( X \setminus Y \) and the other to \( Y \setminus X \),
2. one of \( e_i, e_j \) belongs to \( X \cap Y \) and the other is not contained in \( X + Y \).

There are precisely \( (k-m)^2 \) and \( m(n-2k+m) \) distinct \( C_{ij} = C_{ji} \) satisfying (1) and (2), respectively. \( \square \)

Remark 3. In the case when \( H \) is infinite-dimensional, there is the following characterization of the orthogonality relation \( \mathbb{S} \) Lemma 2; two elements in an orthogonal apartment \( \mathcal{A} \) are orthogonal if and only if there is only a finite number of complementary subsets of \( \mathcal{A} \) containing this pair. This implies that every bijective transformation of \( G_k(H) \) preserving the class of orthogonal apartments in both directions preserves also the orthogonality relation in both directions (see \( \mathbb{S} \) for the details). By \( \mathbb{S} \mathbb{U} \), the latter condition guarantees that the bijection is induced by an unitary or conjugate-unitary operator.

4. Proof of Theorem \( \mathbb{U} \)

Suppose that \( \dim H = n \) is finite and not less then 3. Let \( f \) be a bijective transformation of \( G_k(H) \) sending orthogonal apartments to orthogonal apartments.

4.1. Preliminary remarks. The mapping \( X \to X^\perp \) is a bijection between \( G_k(H) \) and \( G_{n-k}(H) \). It transfers every orthogonal apartment of \( G_k(H) \) to the orthogonal apartment of \( G_{n-k}(H) \) defined by the same orthogonal base. Thus \( X \to f(X^\perp)^\perp \) is a bijective transformation of \( G_{n-k}(H) \) sending orthogonal apartments to orthogonal apartments. If it is induced by an unitary or conjugate-unitary operator then \( f \) is induced by the same operator. For this reason it is sufficiently to prove Theorem \( \mathbb{U} \) only in the case when \( k \leq n - k \).

Suppose that \( k = 1 \). Then \( f \) transfers orthogonal elements of \( G_1(H) \) to orthogonal elements. For any 2-dimensional subspace \( Y \subset H \) we take orthogonal 1-dimensional subspaces \( X_1, X_2 \subset Y \) and extend them to an orthogonal apartment \( \{X_i\}_{i=1}^n \in G_1(H) \). If \( X \) is a 1-dimensional subspace of \( Y \) then \( f(X) \) is orthogonal to \( f(X_i) \) for every \( i \geq 3 \), i.e. \( f(X) \) is contained in \( f(X_1) + f(X_2) \). So, \( f \) sends all lines of the projective space over \( H \) to subsets of lines and, by the Fundamental Theorem of Projective Geometry \( \mathbb{I} \), \( f \) is induced by an invertible semilinear operator. This
operator transfers orthogonal vectors to orthogonal vectors. An easy verification shows that it is a scalar multiple of an unitary or conjugate-unitary operator $U$. It is clear that $f$ is induced by $U$.

From this moment we will suppose that $1 < k \leq n - k$.

4.2. The case $n \neq 2k+2$. Following Lemma 2, we consider the quadratic function

$$c(x) = (k - x)^2 + x(n - 2k + x) = 2x^2 - (4k - n)x + k^2.$$ 

It takes the minimal value on $x = \frac{4k - n}{4}$. This implies that

$$c(k - 1) > c(m)$$

for all $m \in \{0, 1, \ldots, k - 2\}$ if

$$k - 1 > \frac{4k - n}{2}$$

or, equivalently, if

$$k < \frac{n - 2}{2}.$$ 

By our assumption, $n = 2k + l$ for some natural $l \geq 0$ and the latter inequality fails only in the case when $l \in \{0, 1, 2\}$.

If $n = 2k + 2$ then $k - 1$ is equal to $(4k - n)/2$ which means that

$$c(k - 1) = c(0)$$

and the latter number is greater than $c(m)$ for any $m \in \{1, \ldots, k - 2\}$.

If $n = 2k + 1$ then $k - 1$ is less than $(4k - n)/2$ which implies that $c(k - 1) < c(0)$, but we have

$$c(k - 1) \neq c(m)$$

for every $m \in \{0, 1, \ldots, k - 2\}$. Indeed, if $c(k - 1) = c(x)$ and $x \neq k - 1$ then an easy calculation shows that $x = 1/2$.

Lemma 2 together with the above arguments give the following characterization of adjacent elements in orthogonal apartments.

**Lemma 3.** Suppose that $n$ is not equal to $2k$ or $2k + 2$. Let $A$ be an orthogonal apartment of $G_k(H)$. Then $X, Y \in A$ are adjacent if and only if there are precisely $c(k - 1)$ distinct complementary subsets of $A$ containing this pair.

**Lemma 4.** As in the previous lemma, we suppose that $n$ is not equal to $2k$ or $2k + 2$ and $A$ is an orthogonal apartment of $G_k(H)$. Then $X, Y \in A$ are adjacent if and only if $f(X)$ and $f(Y)$ are adjacent; moreover, $f$ transfers stars of $A$ to stars of $f(A)$.

**Proof.** Using Lemmas 1 and 3 we show that $X, Y \in A$ are adjacent if and only if the same holds for $f(X)$ and $f(Y)$. Then $f$ transfers every star of $A$ to a star or a top of $f(A)$. Stars and tops contain $n - k + 1$ and $k + 1$ elements, respectively. Since $n \neq 2k$, these numbers are distinct and the image of every star is a star. □

We prove Theorem 1 for $n \neq 2k + 2$.

Let $X$ and $Y$ be adjacent elements of $G_k(H)$ which are not contained in an orthogonal apartment. Denote by $N$ the orthogonal complement of $X \cap Y$. The dimension of $N$ is equal to $n - k + 1$. Let $S$ be the intersection of $X + Y$ with $N$. This is a 2-dimensional subspace. Then

$$\dim(S^\perp \cap N) = n - k - 1 \geq 2.$$
induced by $U$

The case 4.3.

Precisely $C_A \subset G$

A multiple of an unitary or conjugate-unitary operator $U$.

This operator sends orthogonal vectors to orthogonal vectors. Hence it is a scalar and $Y$ an automorphism of $\Gamma_k$.

So, $f$ sends adjacent vertices of $\Gamma_k(H)$ to adjacent vertices which implies that $f$ is an automorphism of $\Gamma_k(H)$.

Then $f$ is induced by an invertible semilinear operator.

This operator sends orthogonal vectors to orthogonal vectors. Hence it is a scalar multiple of an unitary or conjugate-unitary operator $U$.

The transformation $f$ is induced by $U$.

4.3. The case $n = 2k + 2$. Suppose that $n = 2k + 2$ and consider an orthogonal apartment $A \subset G_k(H)$. By the previous subsection, if $X, Y \in A$ and there are precisely $c(k - 1)$ distinct complementary subsets of $A$ containing this pair then $X$ and $Y$ are adjacent or $\dim(X \cap Y) = 0$ and they are orthogonal.

We say that two distinct complementary subsets $C_{ij}$ and $C_{i'j'}$ are adjacent if $\{i, j\} \cap \{i', j'\} \neq \emptyset$. In this case, we have

$$|C_{ij} \cap C_{i'j'}| = \binom{n - 3}{k - 1} + \binom{n - 3}{k - 2} = \binom{n - 2}{k - 1} = \binom{2k}{k - 1}.$$

otherwise, we get

$$|C_{ij} \cap C_{i'j'}| = 4 \binom{n - 4}{k - 2} = 4 \binom{2k - 2}{k - 2}.$$

The equality

$$\binom{2k}{k - 1} = 4 \binom{2k - 2}{k - 2}$$

holds only for $k = 2$, i.e. $n = 6$. Therefore, for $n \neq 6$ two complementary subsets $C, C' \subset A$ are adjacent if and only if $f(C)$ and $f(C')$ are adjacent complementary subsets of $f(A)$.

If $X, Y \in A$ are orthogonal then for every complementary subset $C \subset A$ containing this pair there is a complementary subset of $A$ containing $X, Y$ and adjacent to $C$. In the case when $X, Y \in A$ are adjacent, there is the unique complementary subset of $A$ containing $X, Y$ and non-adjacent to any other complementary subset of $A$ containing $X, Y$.

Using the latter observation, we establish the direct analogue of Lemma 4 for $n = 2k + 2 \neq 6$. As in the previous subsection, we show that $f$ is induced by an unitary or conjugate-unitary operator.

Remark 4. Consider the case when $n = 6$ and $k = 2$. If $A$ is an orthogonal apartment of $G_k(H)$ then any distinct $X, Y \in A$ are contained in precisely 4 distinct complementary subsets of $A$. The intersection of any two distinct complementary subsets consists of 3 elements. Thus the dimension of the intersection of $X, Y \in A$ cannot be determined in terms of complementary subsets.
5. Proof of Theorem 2

Suppose that \( \dim H = 2k \geq 8 \) and \( f \) is a bijective transformation of \( G_k(H) \) such that \( f \) and \( f^{-1} \) send orthogonal apartments to orthogonal apartments.

In this case we have

\[
c(m) = (k - m)^2 + m^2.
\]

It is easy to see that

\[
c(0) > c(m)
\]

for every \( m \in \{1, \ldots, k-1\} \) and

\[
c(m) = c(m')
\]

only in the case when \( m' = m \) or \( m' = k - m \). The standard arguments give the following.

**Lemma 5.** Let \( X, Y \) be distinct elements in an orthogonal apartment \( A \subset G_k(H) \). If \( \dim(X \cap Y) = m \) then \( \dim(f(X) \cap f(Y)) \) is equal to \( m \) or \( k - m \).

Since \( \dim H = 2k \), the orthogonal complement \( X^\perp \) is the unique element of \( G_k(H) \) orthogonal to \( X \in G_k(H) \). Any pair \( X, X^\perp \) is contained in a certain orthogonal apartment of \( G_k(H) \). In the previous lemma we put \( m = 0 \) and get the following.

**Lemma 6.** For every \( X \in G_k(H) \) we have \( f(X^\perp) = f(X)^\perp \).

Let \( G' \) be the set of all 2-element subsets \( \{X, X^\perp\} \subset G_k(H) \). By Lemma 6, \( f \) induces a bijective transformation of \( G' \). We denote this transformation by \( f' \).

Consider the graph \( \Gamma' \) whose vertex set is \( G' \) and subsets

\[
\{X, X^\perp\}, \{Y, Y^\perp\} \subset G'
\]

are adjacent vertices of this graph if \( X \) is adjacent to \( Y \) or \( Y^\perp \). The latter conditions guarantees that \( X^\perp \) is adjacent to \( Y^\perp \) or \( Y \), respectively. Two elements of \( G' \) will be called adjacent if they are adjacent vertices of \( \Gamma' \).

For every \( (k - 1) \)-dimensional subspace \( S \subset H \) we denote by \( C(S) \) the set of all \( \{X, X^\perp\} \subset G' \) such that \( X \) or \( X^\perp \) contains \( S \) (then the \( (k+1) \)-dimensional subspace \( S^\perp \) contains \( X^\perp \) or \( X \), respectively). This is a maximal clique of the graph \( \Gamma' \).

**Lemma 7.** If \( C \) is a maximal clique of \( \Gamma' \) then \( C = C(S) \) for a certain \( (k - 1) \)-dimensional subspace \( S \).

**Proof.** Let \( \{X, X^\perp\} \in C \). Denote by \( X \) the set of all \( Y \in G_k(H) \) adjacent to \( X \) and such that \( \{Y, Y^\perp\} \in C \). It is clear that \( X \) is a maximal clique of \( \Gamma_k(H) \). If this is the star corresponding to a \( (k - 1) \)-dimensional subspace \( S \) then \( C = C(S) \). If \( X \) is the top defined by a \( (k+1) \)-dimensional subspace \( N \) then \( N^\perp \) is \( (k - 1) \)-dimensional and \( C = C(N^\perp) \).

**Lemma 8.** If \( X, Y, Z \) are mutually adjacent elements of \( G_k(H) \) contained in an orthogonal apartment then there is the unique maximal clique of \( \Gamma' \) containing the corresponding elements of \( G' \).

**Proof.** For \( S = X \cap Y \cap Z \) and \( N = X + Y + Z \) one of the following possibilities is realized:

1. \( \dim S = k - 1 \) and \( \dim N = k + 2 \),
2. \( \dim S = k - 2 \) and \( \dim N = k + 1 \).
The required maximal clique is $C(S)$ or $C(N^\perp)$, respectively.

**Lemma 9.** The transformation $f'$ is an automorphism of the graph $\Gamma'$.

**Proof.** Let $\{X, X^\perp\}$ and $\{Y, Y^\perp\}$ be adjacent elements of $\mathcal{G}'$. We need to show that $f'$ transfers them to adjacent elements. It is sufficient to restrict ourself to the case when $X$ and $Y$ are adjacent.

Suppose there is an orthogonal apartment of $\mathcal{G}_k(H)$ containing $X$ and $Y$. Note that $X^\perp$ and $Y^\perp$ also belong to this apartment. By Lemma 5, $f(X)$ and $f(Y)$ are adjacent or

$$\dim(f(X) \cap f(Y)) = 1.$$ 

The latter equality implies that $f(X)$ is adjacent to $f(Y^\perp)$. Therefore, $f'$ transfers $\{X, X^\perp\}$ and $\{Y, Y^\perp\}$ to adjacent elements of $\mathcal{G}'$.

Consider the case when there is no orthogonal apartment containing $X$ and $Y$. Let $N$ be the orthogonal complement of $X \cap Y$ and let $S$ be the intersection of $X + Y$ with $N$. The dimensions of $N$ and $S$ are equal to $k + 1$ and 2, respectively. Also, we have

$$\dim(S^\perp \cap N) = k - 1 \geq 3.$$ 

This means that $N$ contains three mutually orthogonal 1-dimensional subspaces $P, Q, T$ which are orthogonal to $S$. We set

$$X' = P + (X \cap Y), \quad Y' = Q + (X \cap Y), \quad Z' = T + (X \cap Y).$$

Note that $X, Y, X', Y', Z'$ are mutually adjacent. Since $X, X', Y', Z'$ are mutually compatible, there is an orthogonal apartment of $\mathcal{G}_k(H)$ containing them. The same holds for $Y, X', Y', Z'$. Then there is an orthogonal apartment containing

$$f(X), f(X'), f(Y'), f(Z')$$

and, by the above arguments, the corresponding elements of $\mathcal{G}'$ are mutually adjacent. Lemma 8 implies the existence of the unique maximal clique

$$\mathcal{C}(M), \ M \in \mathcal{G}_{k-1}(H)$$

containing them. Similarly, there is the unique maximal clique

$$\mathcal{C}(N), \ N \in \mathcal{G}_{k-1}(H)$$

which contains the elements of $\mathcal{G}'$ corresponding to

$$f(Y), f(X'), f(Y'), f(Z').$$

Then $\mathcal{C}(M) \cap \mathcal{C}(N)$ contains at least 3 elements and Lemma 8 guarantees that $M = N$. So, the elements of $\mathcal{G}'$ corresponding to $f(X)$ and $f(Y)$ belong to a certain maximal clique of $\Gamma'$, in other words, they are adjacent.

Similarly, we show that $f^{-1}$ sends adjacent elements to adjacent elements. \qed

It follows from Lemma 8 that $f'$ sends maximal cliques of $\Gamma'$ to maximal cliques and there is a transformation $g$ of $\mathcal{G}_{k-1}(H)$ such that

$$f'(\mathcal{C}(S)) = \mathcal{C}(g(S))$$

for every $S \in \mathcal{G}_{k-1}(H)$. This transformation is bijective.

**Lemma 10.** The transformation $g$ sends orthogonal apartments to orthogonal apartments.
Proof. Let $B$ be an orthogonal base of $H$ and let $A$ be the associated orthogonal apartment of $G_k(H)$. We take any orthogonal base $B'$ corresponding to the orthogonal apartment $f(A)$. An easy verification shows that $g$ transfers the orthogonal apartment defined by $B$ to the orthogonal apartment defined by $B'$.

By Theorem 1, the transformation $g$ is induced by an unitary or conjugate-unitary operator $U$. This operator satisfies the required condition.

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