Abstract We prove that the basic intersection cohomology $\mathbb{H}^p_\pi(M/F)$, where $F$ is the singular foliation determined by an isometric action of a Lie group $G$ on the compact manifold $M$, is finite dimensional.

Keywords Basic intersection cohomology · Lie group actions

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1 Introduction

This paper deals with an action $\Phi: G \times M \rightarrow M$ of a Lie group on a compact manifold preserving a riemannian metric on it. The orbits of this action define a singular foliation $F$ on $M$. Putting together the orbits of the same dimension we get a stratification of $M$. This structure is still very regular. The foliation $F$ is in fact a conical foliation and we can define the basic intersection cohomology $\mathbb{H}^p_\pi(M/F)$ (cf. [10]). This invariant becomes the basic cohomology $H^\pi_\pi(M/F)$ when the action $\Phi$ is almost free, and the intersection cohomology $\mathbb{H}^p_\pi(M/G)$ when the Lie group $G$ is compact.

The aim of this work is to prove that this cohomology $\mathbb{H}^p_\pi(M/F)$ is finite dimensional. This result generalizes [3] (almost free case), [11] (abelian case) and [10] (compact case).
The paper is organized as follows. In Sect. 2 we present the foliation \( \mathcal{F} \). The basic intersection cohomology \( \mathbb{H}_p^*(M/\mathcal{F}) \) associated to this foliation is studied in Sect. 3. Twisted products are studied in Sect. 4. The finiteness of \( \mathbb{H}_p^*(M/\mathcal{F}) \) is proved in Sect. 5.

In the sequel \( M \) is a connected, second countable, Hausdorff, without boundary and smooth (of class \( C^\infty \)) manifold of dimension \( m \). All the maps are considered smooth unless something else is indicated.

2 Killing foliations determined by isometric actions

We study in this work the foliations induced by isometric actions: the Killing foliations. These foliations are examples of the conical foliations for which the basic intersection cohomology has been defined (see [10,11]). We present this geometrical framework in this section.

2.1 Killing foliations

A smooth action \( \Phi: G \times M \rightarrow M \) of a Lie group \( G \) on a manifold \( M \) is a isometric action when there exists a riemannian metric \( \mu \) on \( M \) preserved by \( G \).

The connected components of the orbits of the action \( \Phi \) determine a partition \( \mathcal{F} \) on \( M \). In fact, this partition is a singular riemannian foliation that we shall call Killing foliation (cf. [7]). Notice that \( \mathcal{F} \) is also a conical foliation in the sense of [10,11]. Classifying the points of \( M \) following the dimension of the leaves of \( \mathcal{F} \) one gets the stratification \( S_\mathcal{F} \) of \( \mathcal{F} \). It is determined by the equivalence relation \( x \sim y \iff \dim G_x = \dim G_y \). The elements of \( S_\mathcal{F} \) are called strata.

In the particular case where the closure of \( G \) in the isometry group of \((M, \mu)\) is a compact Lie group we shall say that the action \( \Phi \) is a tame action. In fact, a smooth action \( \Phi: G \times M \rightarrow M \) is tame if and only if it extends to a smooth action \( \Phi: K \times M \rightarrow M \) where \( K \) is a compact Lie group containing \( G \) (cf. [6]). The group \( K \) is not unique, but we always can choose \( K \) in such a way that \( G \) is dense in \( K \). We shall say that \( K \) is a tamer group. Here the strata of \( S_\mathcal{F} \) are \( K \)-invariant closed submanifolds of \( M \).

Since the aim of this work is the study of \( \mathcal{F} \) and not the action \( \Phi \) itself, we can consider that the Lie group \( G \) is connected. Let us see that.

**Proposition 1** Let \( \Phi: G \times M \rightarrow M \) be a tame action. Let \( G_0 \) be the connected component of \( G \) containing the unity element. The Killing foliation defined by the restriction \( \Phi: G_0 \times M \rightarrow M \) is also \( \mathcal{F} \).

**Proof** The partition \( \mathcal{F} \) is defined by this equivalence relation:

\[ x \sim y \iff \exists \text{ continuous path } \alpha: [0, 1] \rightarrow G(x) \text{ such that } \alpha(0) = x \text{ and } \alpha(1) = y. \]

Since the map \( \Delta: G \rightarrow G(x) \), defined by \( \Delta(g) = \Phi(g, x) = g \cdot x \), is a submersion (see for example [2]) then

\[ x \sim y \iff \exists \text{ continuous path } \beta: [0, 1] \rightarrow G \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y, \]

and by definition of \( G_0 \)

\[ x \sim y \iff \exists \text{ continuous path } \beta: [0, 1] \rightarrow G_0 \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y. \]

This gives the result.  

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1 This is always the case when the manifold \( M \) is a compact.
When $G$ is connected, the tamer group $K$ has richer properties.

**Proposition 2** Let $G$ be a connected Lie subgroup of a compact Lie group $K$. If $G$ is dense in $K$ then $G \triangleleft K$ and the quotient group $K / G$ is commutative.

**Proof** The Lie algebra $\mathfrak{g}$ is $\text{Ad}_G$-invariant and hence, by density, $\text{Ad}_K$-invariant. Then $\mathfrak{g}$ is an ideal of $\mathfrak{k}$. The connectedness of $G$ gives that $G$ is a normal subgroup of $K$. Since $\text{Ad}_G$ acts trivially on $\mathfrak{k}/\mathfrak{g}$, $\mathfrak{k}/\mathfrak{g}$ is abelian (see for example [8, p. 628]). \qed

2.2 Particular tame actions

A **trio** is a triple $(K, G, H)$, with $K$ a compact Lie group, $G$ a normal subgroup of $K$ and $H$ a closed subgroup of $K$. We present now some tame actions associated to a trio $(K, G, H)$. They are going to be intensively used in this work. First of all we need some definitions.

- The action $\Phi_l : K \times K \to K$ is defined by $\Phi_l(g, k) = g \cdot k$. For each element $u$ of the Lie algebra $\mathfrak{t}$ of $K$, we shall write $X_u$ the associated (right invariant) vector field. It is defined by $X_u(k) = L_k(u)$ where $L_k : K \to K$ is given by $L_k(\ell) = \ell \cdot k$.

- The action $\Phi_r : K \times K \to K$ is defined by $\Phi_r(g, k) = k \cdot g^{-1}$. For each element $u \in \mathfrak{t}$ of $K$, we shall write $X_u$ the associated (left invariant) vector field. It is defined by $X_u(k) = \mathfrak{t}_e L_k(u)$ where $L_k : K \to K$ is given by $L_k(\ell) = k \cdot \ell$.

- The action $\Psi : K \times K / H \to K / H$ is defined by $\Psi(g, k H) = (g \cdot k) H$. For each element $u \in \mathfrak{t}$, we shall write $Y_u$ the associated vector field. Since the canonical projection $\pi : K \to K / H$ is a $K$-equivariant map, then we have $\pi_\ast X_u = Y_u$ for each $u \in \mathfrak{t}$.

- The action $\Gamma : H \times H \to H$ is defined by $\Gamma(g, h) = g \cdot h$. For each element $u$ of the Lie algebra $\mathfrak{h}$ of $H$ we write $Z_u$ the associated (right invariant) vector field.

The associated actions we are going to use are the following.

(a) The restriction $\Phi_l : G \times K \to K$, which induces the regular Killing foliation $\mathcal{K}$.

(b) The restriction $\Phi_r : G \times K \to K$, which induces the regular Killing foliation $\mathcal{K}$.

Since $G \triangleleft K$, the foliation $\mathcal{K}$ is determined by the family of vector fields $\{X_u / u \in \mathfrak{g}\}$, where $\mathfrak{g}$ is the Lie algebra of $G$, and also by the family $\{X_u / u \in \mathfrak{g}\}$. The orbits $G(k) = Gk = kG$ have the same dimension $\dim G$.

(c) The restriction $\Psi : G \times K / H \to K / H$, which induces the regular Killing foliation $\mathcal{D}$.

The foliation $\mathcal{D}$ is determined by the family of vector fields $\{Y_u / u \in \mathfrak{g}\}$. The orbits $G(k H)$ have the same dimension $\dim G - \dim (G \cap H)$.

(d) The restriction $\Gamma : (G \cap H) \times H \to H$, which induces the regular Killing foliation $\mathcal{C}$.

The foliation $\mathcal{C}$ is determined by the family of vector fields $\{Z_u / u \in \mathfrak{g} \cap \mathfrak{h}\}$. The orbits $(G \cap H)(k)$ have the same dimension $\dim (G \cap H)$.

(e) The restriction $\Phi_r : G H \times K \to K$, which induces the regular Killing foliation $\mathcal{E}$.

Notice that $G H$ is a Lie group since $G$ is normal in $K$. The foliation $\mathcal{E}$ is, in fact, determined by the vector fields $\{X_u / u \in \mathfrak{g} + \mathfrak{h}\}$. The orbits $(G H)(k)$ have the same dimension $\dim G + \dim H - \dim (G \cap H)$.
2.3 Twisted product

In order to prove the finiteness of the basic intersection cohomology we decompose the manifold in a finite number of simpler pieces. These are the twisted products we introduce now.

We fix a trio \((K, G, H)\) and a smooth action \(\Theta : H \times N \to N\) of \(H\) on the manifold \(N\). The twisted product is the quotient \(K \times_H N\) of \(K \times N\) by the equivalence relation \((k, z) \sim (k \cdot h^{-1}, \Theta(h, z) = h \cdot z)\). The element of \(K \times_H N\) corresponding to \((k, z) \in K \times N\) is denoted by \(\langle k, z \rangle\). This manifold is endowed with the tame action

\[
\Phi : G \times (K \times_H N) \longrightarrow (K \times_H N),
\]

defined by \(\Phi(g, \langle k, z \rangle) = \langle g \cdot k, z \rangle\). We denote by \(\mathcal{W}\) the induced Killing foliation.

We also use the following tame action, namely, the restriction

\[
\Theta : (G \cap H) \times N \to N
\]

whose induced Killing foliation is denoted by \(\mathcal{N}\).

The canonical projection \(\Pi : K \times N \to K \times_H N\) relates the involved foliations as follows:

(a) \(\Pi_\mathcal{W}(K \times \mathcal{I}) = \mathcal{W}\), where \(\mathcal{I}\) is the pointwise foliation (since the map \(\Pi\) is \(G\)-equivariant).

(b) \(S_{\mathcal{W}} = \{\Pi(K \times S) / S \in S_{\mathcal{N}}\} = \Pi([K] \times S_{\mathcal{N}})\) (since \(G_{<k,z>} = k(G \cap H)_z k^{-1}\)).

3 Basic intersection cohomology

In this section we recall the definition of the basic intersection\(^2\) cohomology and we present the main properties we are going to use in this work. For the rest of this section, we fix a conical foliation \(\mathcal{F}\) defied on a manifold \(M\). The associated stratification is \(S_{\mathcal{F}}\). The regular stratum of is denoted by \(R_{\mathcal{F}}\). We shall write \(m = \dim M, r = \dim \mathcal{F}\) and \(s = m - r = \codim_M \mathcal{F}\).

We are going to deal with differential forms on a product \((\text{manifold}) \times [0, 1]^p\), they are restrictions of differential forms defined on \((\text{manifold}) \times \} - 1, 1]^p\).

3.1 Perverse forms

Recall that a conical chart is a foliated diffeomorphism \(\varphi : (\mathbb{R}^{m-n-1} \times cS^n, \mathcal{H} \times cG) \to (U, \mathcal{F}_U)\) where \((\mathbb{R}^{m-n-1}, \mathcal{H})\) is a simple foliation and \((S^n, G)\) is a conical foliation without 0-dimensional leaves. We also shall denote this chart by \((U, \varphi, S)\) where \(S\) is the stratum of \(S_{\mathcal{F}}\) verifying \(\varphi(\mathbb{R}^{m-n-1} \times \{\emptyset\}) = U \cap S\).

The differential complex \(\Omega^\ast_{\mathcal{F}}(M \times [0, 1]^p)\) of perverse forms of \(M \times [0, 1]^p\) is introduced by induction on depth \(S_{\mathcal{F}}\). When this depth is 0 then

\[
\Omega^\ast_{\mathcal{F}}(M \times [0, 1]^p) = \Omega^\ast(M \times [0, 1]^p).
\]

Consider now the generic case. A perverse form of \(M \times [0, 1]^p\) is first of all a differential form \(\omega \in \Omega^\ast(R_{\mathcal{F}} \times [0, 1]^p)\) such that,

\[
\begin{cases}
\text{the pull-back} & (\varphi \times \mathbb{I}_{[0,1]^p})^\ast \Omega \in \Omega^\ast(\mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times [0, 1[^p] \\
\text{extends to} & \omega_\varphi \in \Omega^\ast_{\mathcal{H} \times S^n}(\mathbb{R}^{m-n-1} \times S^n \times [0, 1[^{p+1}])
\end{cases}
\]

\(^2\) We refer the reader to [10,11] for details.
Finally, we define the perverse degree $\Pi^{|\cdot|}$ local perverse degree space. A differential form $\omega$ is bounded by the perversity $\pi$ local perverse degree space of the complex of basic perverse forms whose perverse degree (and that of the their derivative) $\omega$ is $\pi$, with $\omega_\pi(v_0, \ldots, v_p, -) \equiv 0$ where the vectors $\{v_0, \ldots, v_p\}$ are tangent to the fibers of $P_\pi : \mathbb{R}^{m-1} \times R^*_\pi \times \{0\} \to U \cap S^4$.

This number does not depend on the choice of the conical chart (cf. [11, Proposition 1.3.1]). Finally, we define the perverse degree $||\omega||_S$ by

$$||\omega||_S = \sup \{ ||\omega||_U \text{ / (U, } \varphi, S \text{ conical chart }) \}.$$  

The perverse degree of $\omega \in \Omega^*(M)$ verifies $||\omega||_S \leq 0$ for any singular stratum $S \in S_{\mathcal{F}}$ (cf. 3.1).

3.3 Basic cohomology

The basic cohomology of the foliation $\mathcal{F}$ is an important tool to study its transversal structure and plays the role of the cohomology of the orbit space $M/\mathcal{F}$, which can be a wild topological space. A differential form $\omega \in \Omega^*(M)$ is basic if $i_X \omega = i_X d\omega = 0$, for each vector field $X$ on $M$ tangent to the foliation $\mathcal{F}$. For example, a function $f$ is basic iff $f$ is constant on the leaves of $\mathcal{F}$. We shall write $\Omega^*(M/\mathcal{F})$ for the complex of basic forms. Its cohomology $H^*(M/\mathcal{F})$ is the basic cohomology of $(M, \mathcal{F})$. We also use the relative basic cohomology $H^*(M, F)/\mathcal{F}$, that is, the cohomology computed from the complex of basic forms vanishing on the saturated set $F \subset M$. The basic cohomology does not use the stratification $S_{\mathcal{F}}$.

3.4 Basic intersection cohomology

A perversity is a map $\overline{\pi} : S^\sigma_{\mathcal{F}} \to \mathbb{Z} \cup \{-\infty, \infty\}$, where $S^\sigma_{\mathcal{F}}$ is the family of singular strata. The constant perversity $\iota$ is defined by $\iota(S) = \iota$, where $\iota \in \mathbb{Z} \cup \{-\infty, \infty\}$.

The basic intersection cohomology appears when one considers basic perverse forms whose perverse degree is controlled by a perversity. We shall put

$$\Omega^\iota_{\overline{\pi}}(M/\mathcal{F}) = \left\{ \omega \in \Pi^*(M) \text{ / } \omega \text{ is basic and } \max \{ ||\omega||_S, ||d\omega||_S \} \leq \overline{\pi}(S) \ \forall S \in S^\sigma_{\mathcal{F}} \right\}$$

the complex of basic perverse forms whose perverse degree (and that of the their derivative) is bounded by the perversity $\overline{\pi}$. The cohomology $H^\iota_{\overline{\pi}}(M/\mathcal{F})$ of this complex is the basic intersection cohomology$^5$ of $(M, \mathcal{F})$ relatively to the perversity $\overline{\pi}$.

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$^3$ Through the restriction $\omega \mapsto \omega_{\mathcal{F}}$.

$^4$ The map $P_\varphi : \mathbb{R}^{m-1} \times \mathbb{S}^n \times [0, 1] \to U$ is defined by $P_\varphi(x, y, t) = \varphi(x, [y, t])$.

$^5$ BIC for short.
Consider a twisted product $K \times_{\mu} N$. Perversities on $K \times_{\mu} N$ and $K \times N$ are determinate by perversities on $N$ by the formula (cf. 2.3 (b)):

$$\overline{p}(K \times S) = \overline{p}(\Pi(K \times S)) = \overline{p}(S).$$

(1)

3.5 Mayer-Vietoris

This is the technique we use in order to decompose the manifold in nicer pieces. An open covering $\{U, V\}$ of $M$ by saturated open subsets is a basic covering. It possesses a subordinated partition of the unity made up of basic functions defined on $M$ (see [9]). For a such covering we have the Mayer-Vietoris short sequence

$$0 \to \Omega^*(M/\mathcal{F}) \to \Omega^*(U/\mathcal{F}) \oplus \Omega^*(V/\mathcal{F}) \to \Omega^*((U \cap V)/\mathcal{F}) \to 0,$$

where the map are defined by $\omega \mapsto (\omega, \omega)$ and $(\alpha, \beta) \mapsto \alpha - \beta$. The third map is onto since the elements of the partition of the unity are controlled functions, i.e., elements of $\Omega^0_\pi(-)$ (cf. 3.2). Thus, the sequence is exact. This result is not longer true for more general coverings.

We shall use in this work the two following local calculations (see [11, Proposition 3.5.1 and Proposition 3.5.2] for the proofs).

**Proposition 3** Let $((\mathbb{R}^k, \mathcal{H})$ be a simple foliation. Consider $\overline{p}$ a perversity on $M$ and define the perversity $\overline{p}$ on $(\mathbb{R}^k \times M) \times (\mathbb{R}^k \times M) = \overline{p}(S)$. The canonical projection $pr : \mathbb{R}^k \times M \to M$ induces the isomorphism

$$\mathbb{H}_p^*(M/\mathcal{F}) \cong \mathbb{H}_\pi^*(\mathbb{R}^k \times M / \mathcal{H} \times \mathcal{F}).$$

**Proposition 4** Let $\mathcal{G}$ be a conical foliation without 0-dimensional leaves on the sphere $\mathbb{S}^n$. A perversity $\overline{p}$ on $c\mathbb{S}^n$ gives the perversity $\overline{p}$ on $\mathbb{S}^n$ defined by $\overline{p}(S) = \overline{p}(S \times \{0, 1\})$. The canonical projection $pr : \mathbb{S}^n \times \{0, 1\} \to \mathbb{S}^n$ induces the isomorphism

$$\mathbb{H}_p^j(c\mathbb{S}^n / c\mathcal{G}) = \begin{cases} \mathbb{H}_p^j(\mathbb{S}^n / \mathcal{G}) & \text{if } i \leq \overline{p}(\{0\}) \\ 0 & \text{if } i > \overline{p}(\{0\}) \}. $$

In the next section we shall need the following technical Lemma.

**Lemma 1** Let $\Phi : K \times M \to M$ be a smooth action, where $K$ is a compact Lie group, and let $V$ be a fundamental vector field of this action. Consider a normal subgroup $G$ of $K$ and write $\mathcal{F}$ the associated conical foliation on $M$. Then, the interior operator $i_{\nu} : \Omega^*(M/\mathcal{F}) \to \Omega^{*-1}_\pi(M/\mathcal{F})$ is well defined, for any perversity $\overline{p}$.

**Proof** Since the question is a local one, then it suffices to consider the case where $M$ is a twisted product $K \times_{\mu} N$.

6 Notice that the blow up $\Pi : \mathbb{R}_{\mathcal{F}}^n \to M$ is a $K$-equivariant map relatively to the action $\ell \cdot (k, z) = (\ell \cdot k, z)$. This gives $\Pi_u(X^u, 0) = V$ for some $u \in \mathcal{F}$. From Lemma 2 we know that it suffices to prove that the operator

$$i_{(X^u, 0)} : \Omega^*_\pi(K \times N / G \times N) \to \Omega^{*-1}_\pi(K \times N / G \times N)$$

is well defined. Since $G < K$ then the vector field $X^u$ preserves the foliation $\mathcal{K}$. So, it suffices to prove that the operator

$$i_{(X^u, 0)} : \Omega^*_\pi(K \times N) \to \Omega^{*-1}_\pi(K \times N)$$

In fact, $N$ is an euclidean space $\mathbb{R}^d$ et $\Theta$ is an orthogonal action.
is well defined. This comes from the fact that $X^u$ acts on the $K$-factor while the perversions conditions are measured on the $N$-factor (cf. (1)).

4 The BIC of a twisted product

We compute now the BIC of a twisted product $K \times_H N$ (cf. 2.3) for a perversity $\Pi$ (cf. (1)).

Lemma 2 The natural projection $\Pi: K \times N \to K \times_H N$ induces the differential monomorphism

$$\Pi^*: \Omega^*_\Pi(K \times_H N / W) \longrightarrow \Omega^*_\Pi(K \times N / K \times N).$$

Moreover, given a differential form $\omega$ on $K \times_H R_W$, we have:

$$\Pi^* \omega \in \Omega^*_\Pi(K \times N / K \times N) \iff \omega \in \Omega^*_\Pi(K \times_H N / W).$$

Proof Notice that the injectivity of $\Pi^*$ comes from the fact that $\Pi$ is a surjection. For the rest, we proceed in several steps.

(a) A foliated atlas for $\pi: K \to K / H$.

Since $\pi: K \to K / H$ is a $H$-principal bundle then it possesses an atlas $A = \{ \varphi: \pi^{-1}(U) \to U \times H \}$ made up with $H$-equivariant charts: $\varphi(k \cdot h^{-1}) = (\pi(k), h \cdot h_0)$ if $\varphi(k) = (\pi(k), h_0)$. We study the foliation $\mathcal{F}_\pi K$. This equivariance property gives $\varphi_* X_u = (0, Z^u)$ for each $u \in g \cap H$. Thus, the trace of the foliation $\mathcal{F}_\pi K$ on the fibers of the canonical projection $pr: U \times H \to U$ is $\mathcal{C}$. On the other hand, since the map $\pi$ is a $G$-equivariant map then $\pi_* \mathcal{K} = \mathcal{D}$, which gives $\varphi_* \mathcal{K} = \mathcal{D}$. We conclude that $\mathcal{F}_\pi \subset \mathcal{D} \times \mathcal{C}$. By dimension reasons we get $\mathcal{F}_\pi \subset \mathcal{D} \times \mathcal{C}$. The atlas $A$ is an $H$-equivariant foliated atlas of $\pi$.

(b) A foliated atlas for $\Pi: K \times N \to K \times_H N$.

We claim that $A_\Pi = \{ \overline{\varphi}: \pi^{-1}(U) \times_H N \to U \times N / (U, \varphi) \in A \}$ is a foliated atlas of $K \times_H N$ where the map $\overline{\varphi}$ is defined by $\overline{\varphi}(< k, z >) = (\pi(k), (\Theta((\varphi^{-1}(\pi(k), e))^{-1} \cdot k), z))$. This map is a diffeomorphism whose inverse is $\overline{\varphi^{-1}}(u, z) = < \varphi^{-1}(u, e), z >$. It verifies

$$\overline{\varphi_\pi N} \overset{2.3(a)}{=} \overline{\varphi_* \Pi_{\mathcal{K}} (K \times \mathcal{I})} = \overline{\varphi_* \Pi_{\mathcal{K}} (\varphi^{-1} \times \mathcal{I} N)} (\mathcal{D} \times \mathcal{C} \times \mathcal{I}).$$

A straightforward calculation shows $\overline{\theta_{\mathcal{C}}(\varphi^{-1} \times \mathcal{I} N)} = (\mathcal{I} U \times \Theta)$. Since $\mathcal{C}$ is defined by the action $\Gamma$ then $\theta_{\mathcal{C}}(\mathcal{I} \times \mathcal{I} = \mathcal{N}$. Finally we obtain $\overline{\varphi_* N} = \mathcal{D} \times \mathcal{N}$.

(c) Last Step. Given $(U, \varphi) \in A_\Pi$, we have the commutative diagram

$$\begin{array}{ccc}
U \times H \times N & \xrightarrow{\varphi^{-1} \times \mathcal{I} N} & K \times N \\
\downarrow Q & & \downarrow \Pi \\
U \times N & \xrightarrow{\overline{\varphi^{-1}}} & K \times_H N
\end{array}$$

where $Q(u, h, z) = (u, h^{-1} \cdot z)$. $\Pi^{-1} (\text{Im } \overline{\varphi^{-1}}) = \text{Im } (\varphi^{-1} \times \mathcal{I} N)$ and the rows are foliated imbeddings. Now, since (2) and (3) are local questions then it suffices to prove that
4.2 Two actions of $H$ with pr 0 defined by

This comes from the fact that the map

\[ \nabla : (U \times H \times N, D \times C \times N) \rightarrow (U \times H \times N, D \times C \times N), \]

defined by $\nabla (u, h, z) = (u, h, h^{-1} \cdot z)$, is a foliated diffeomorphism and $Q = \text{pr}_0 \circ \nabla$, with $\text{pr}_0 : U \times H \times N \rightarrow U \times N$ canonical projection (cf. Proposition 3).

4.1 The Lie algebra $\mathfrak{t}$

We suppose in this paragraph that $G$ is also dense on $K$. Choose $\nu$ a bi-invariant riemannian metric on $K$, which exists by compactness. Consider

\[ B = \{ u_1, \ldots, u_d, u_{a+1}, \ldots, u_{b}, u_{b+1}, \ldots, u_{c}, u_{c+1}, \ldots, u_f \} \]

an orthonormal basis of the Lie algebra $\mathfrak{t}$ of $K$ with \{u_1, \ldots, u_b\} basis of the Lie algebra $\mathfrak{g}$ of $G$ and \{u_{a+1}, \ldots, u_c\} basis of the Lie algebra $\mathfrak{h}$ of $H$. For each indice $1 \leq i \leq f$ we shall write $X_i \equiv X_{ui}$ and $X^i \equiv X^{ui}$ (cf. 2.2).

Let $\gamma_l \in X^i (K)$ be the dual form of $X_i$, that is, $\gamma_l = i_{X_i} \nu$. Notice that $\delta_{ij} = \gamma_j (X_i)$. These forms are invariant by the left action of $K$. Since the flow of $X^j$ is the multiplication on the left by $\exp(tu_j)$ then $L_{X_j} \gamma_l = 0$ for each $1 \leq j \leq f$.

For the differential, we have the formula $d\gamma_l = \sum_{1 \leq i < j \leq f} C_{ij}^l \gamma_i \wedge \gamma_j$, where $[X_i, X_j]$ is $\sum_{l=1}^f C_{ij}^l X_l$, and $1 \leq i, j, l \leq f$. We have several restrictions on these coefficients. Since $G \triangleleft K$ then $\mathfrak{g}$ is an ideal of $\mathfrak{t}$ and therefore we have

\[ C_{ij}^l = 0 \quad \text{for} \quad i \leq b < l. \]

Since $K/G$ is an abelian group (cf. Proposition 2) then the induced bracket on $\mathfrak{t}/\mathfrak{g}$ is zero and therefore we have

\[ C_{ij}^l = 0 \quad \text{for} \quad b < i, j, l \leq f. \]

These equations imply that

\[ d\gamma_l = 0 \quad \text{for each} \quad b < l. \]

The $\mathcal{E}$-basic differential forms in $\bigwedge^*(\gamma_1, \ldots, \gamma_f)$ are exactly $\bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f)$ since they are cycles and the family $\{X_1, \ldots, X_c\}$ generates the foliation $\mathcal{E}$. This gives

\[ H^*(K/\mathcal{E}) = \bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f). \]

4.2 Two actions of $H/H_0$

The Lie group $H$ preserves the foliation $\mathcal{N}$ since the Lie group $G \cap H$ is a normal subgroup of $H$. Put $H_0$ the connected component of $H$ containing the unity element. Since it is a connected compact Lie group then a standard argument shows that

\[ \left( \Omega^*_\mathfrak{t}(N/\mathcal{N}) \right)^{H_0} = H^* \left( \Omega^*_\mathfrak{h}(N/\mathcal{N}) \right)^{H_0} = \Omega^*_\mathfrak{t}(N/\mathcal{N}) \]

\[ \text{Since } G \cap H \triangleleft H. \]
(cf. [5, Theorem I, Ch. IV, vol. II]). We conclude that the finite group $H/H_0$ acts naturally on $H^*_p(N/N)$.

Since $H_0$ is a connected Lie subgroup of $GH$ then $(H^*(K/E))^{H_0} = H^*(K/E)$. We conclude that the finite group $H/H_0$ acts naturally on $H^*(K/E)$.

**Proposition 5** Let $(K, G, H)$ be a trio with $G$ connected and dense in $K$. Then

$$H^*_p(K \times H N/W) = \left( H^*(K/E) \otimes H^*_p(N/N) \right)^{H/H_0}.$$  

**Proof** Using the blow up $\Pi : K \times N \to K \times H N$, the computation of $H^*_p(K \times H N/W)$ can be done by using the complex $\text{Im} \left\{ \Pi^* : \Omega^*_p(K \times H N/F) \to \Omega^*_p(K \times N/K \times N) \right\}$ (cf. Lemma 2). We study this complex in several steps. We fix $B = \{ u_1, \ldots, u_f \}$ an orthonormal basis of $\mathfrak{t}$ as in 4.1.

(i) **Description of $\Omega^*_K(K \times R_N)$**.
A differential form $\omega \in \Omega^*_K(K \times R_N)$ is of the form

$$\eta + \sum_{1 \leq i_1 < \cdots < i_\ell \leq f} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \ldots, i_\ell},$$

where the forms $\eta, \eta_{i_1, \ldots, i_\ell} \in \Omega^*(K \times R_N)$ verify $i_{x_1} \eta = i_{x_1} \eta_{i_1, \ldots, i_\ell} = 0$ for each $1 \leq j \leq f$ and each $1 \leq i_1 < \cdots < i_\ell \leq f$.

(ii) **Description of $\Pi^*_K(K \times N)$**.
Since the foliation $\mathcal{K}$ is regular then we always can construct a conical chart of the form $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ where $(U_1, \varphi_1)$ is a foliated chart of $(K, \mathcal{K})$ and $(U_2, \varphi_2)$ is a conical chart of $(N, \mathcal{N})$. The local blow up of the chart $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ is constructed from the second factor without modifying the first one. So, the differential forms $\gamma_i$ are always perverse forms and a differential form $\omega \in \Pi^*_K(K \times N)$ is of the form (7) where $\eta, \eta_{i_1, \ldots, i_\ell} \in \Pi^*_K(K \times N)$ verify $i_{x_j} \eta = i_{x_j} \eta_{i_1, \ldots, i_\ell} = 0$ for each $1 \leq j \leq f$ and each $1 \leq i_1 < \cdots < i_\ell \leq f$.

(iii) **Description of $\Omega^*(K \times R_N/\mathcal{K} \times N)$**.
Take $\omega \in \Omega^*_K(K \times R_N/\mathcal{K} \times N)$. Since $\mathcal{K}$ is generated by the family $\{ X_j / 1 \leq j \leq b \}$ then $L_{X_j} \omega = 0$ for any $1 \leq j \leq b$, or equivalently, $R_{X_j} \omega = \omega$ for each $g \in G$ since $G$ is connected. By density, $R_{X_j} \omega = \omega$ for each $k \in K$ and therefore $L_{X_j} \omega = 0$ for any $1 \leq j \leq f$ since $K$ is connected. We conclude that $L_{X_j} \eta = L_{X_j} \eta_{i_1, \ldots, i_\ell} = 0$ for any $1 \leq j \leq f$ and each $1 \leq i_1 < \cdots < i_\ell \leq f$. This gives $\omega \in \bigwedge^* (\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega^*(R_N)$. The $\mathcal{N}$-basic differential forms of $\Omega^*(R_N)$ are exactly $\Omega^*(R_N/\mathcal{N})$. The $\mathcal{K}$-basic differential forms of $\bigwedge^* (\gamma_{b+1}, \ldots, \gamma_f)$ are exactly $\bigwedge^* (\gamma_{b+1}, \ldots, \gamma_f)$ (cf. (4)). From these two facts, we get

$$\Omega^*(K \times R_N/\mathcal{K} \times N) = \bigwedge^* (\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega^*(R_N/\mathcal{N})$$

as differential graduate commutative algebras.

(iv) **Description of $\Omega^*_p(K \times N/\mathcal{K} \times N)$**.
From (ii) and (iii) it suffices to control the pervers degree of the forms

$$\sum_{b+1 \leq i_1 < \cdots < i_\ell \leq f} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \ldots, i_\ell} \in \bigwedge^* (\gamma_{b+1}, \ldots, \gamma_f) \otimes \Pi^*_N(N).$$
Consider $S$ a stratum of $S_N^\prime$. From $||\gamma^i||_{K \times S} = 0$ and $||\eta||_{K \times S} = ||\eta||_S$, we get $||\gamma^i \wedge \ldots \gamma^i \wedge \eta_{i, \ldots, i}||_{K \times S} = ||\eta_{i, \ldots, i}||_S$. We conclude that

$$\Omega^*_{\mathcal{P}}(K \times N/K \times N) \cong \bigwedge^*(\gamma^b_1, \ldots, \gamma^f) \otimes \Omega^*_{\mathcal{P}}(N/N)$$

(cf. 2.3 (b)).

(iii) Description of $\text{Im} \left\{ \Pi^* : \Omega^*_{\mathcal{P}}(K \times N/F) \rightarrow \Omega^*_{\mathcal{P}}(K \times N/K \times N) \right\}$.

We denote by $\{W_{a+1}, \ldots, W_c\}$ the fundamental vector fields of the action $\Theta : H \times N \rightarrow N$ associated to the basis $\{u_{a+1}, \ldots, u_c\}$. Consider now the action $\Upsilon : H \times (K \times N) \rightarrow (K \times N)$ defined by $\Upsilon(h, (k, z)) = (k \cdot h^{-1}, \Theta(h, z))$. Its fundamental vector fields associated to the basis $\{u_{a+1}, \ldots, u_c\}$ are $\{(X_{a+1}, W_{a+1}), \ldots, (X_c, W_c)\}$. Given $h \in H$, we take $\Upsilon_h : K \times N \rightarrow K \times N$ the map defined by $\Upsilon_h(k, z) = \Upsilon(h, (k, z))$. Then, we have

$$\text{Im} \Pi^* = \left\{ \omega \in \bigwedge^*(\gamma^b_1, \ldots, \gamma^f) \otimes \Omega^*_{\mathcal{P}}(N/N) \begin{array}{c}
(i) i_{X_i} \omega = -i_{W_i} \omega \text{ if } a < i \leq c \\
(ii) L_{\gamma_i} \omega = -L_{W_i} \omega \text{ if } a < i \leq c, \\
(iii) (\Upsilon_h)^* \omega = \omega \text{ for } h \in H.
\end{array} \right\}$$

Let $H_0$ be the unity connected component of $H$. Recall that the subgroup $H_0$ is normal in $H$ and that the quotient $H/H_0$ is a finite group. Condition (ii) gives that $\omega$ is $H_0$-invariant. So, condition (iii) can be replaced by: (iv) $(\Upsilon_h)^* \omega = \omega$ for $h \in H/H_0$. Therefore

$$\text{Im} \Pi^* = \left\{ \omega \in \bigwedge^*(\gamma^b_1, \ldots, \gamma^f) \otimes \Omega^*_{\mathcal{P}}(N/N) \begin{array}{c}
(i) i_{X_i} \omega = -i_{W_i} \omega \text{ if } a < i \leq c \\
(ii) L_{\gamma_i} \omega = -L_{W_i} \omega \text{ if } a < i \leq c, \\
\end{array} \right\}.$$  

Since the group $H/H_0$ is a finite one, we get that the cohomology $H^*(\text{Im} \Pi^*)$ is isomorphic to $\left( H^*(A^\prime) \right)^{H/H_0}$, where $A^\prime$ is the differential complex

$$\left\{ \omega \in \bigwedge^*(\gamma^b_1, \ldots, \gamma^f) \otimes \Omega^*_{\mathcal{P}}(N/N) \begin{array}{c}
(i) i_{X_i} \omega = -i_{W_i} \omega \text{ if } a < i \leq c, \\
(ii) L_{\gamma_i} \omega = -L_{W_i} \omega \text{ if } a < i \leq c, \\
\end{array} \right\}.$$  

So, it remains to compute $H^*(A^\prime)$. This computation can be simplified by using these three facts:

- $i_{W_i} \omega = L_{W_i} \omega = 0$ for each $a < i \leq b$, since the foliation $\mathcal{N}$ is defined by the action of $G \cap H$.
- $i_{X_i} \gamma_j = \delta_{ij}$ for all $i$, $j$ (cf. 4.1).
- $d^\gamma_j = 0$ for $b < j$ (cf. (4)).
We get that $A^*$ is the differential complex

$$\left\{ \omega \in \bigwedge^*(\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega^*_p(N/N) \mid \begin{array}{l}
  (i) \ i_X^* \omega = -i_{W^i} \omega \text{ if } b < i \leq c \\
  (ii) \ 0 = L_{W^i} \omega \text{ if } b < i \leq c
\end{array} \right\}$$

where $A^*$ is isomorphic to

$$\bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f) \otimes \Omega^*_p(N/N)$$

A straightforward computation gives that a form $\omega \in \bigwedge^*(\gamma_{b+1}, \ldots, \gamma_c) \otimes \Omega^*_p(N/N)$ verifying (i) is in fact

$$\omega = \omega_0 + \sum_{b < i_1 < \cdots < i_c \leq c} (-1)^i \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_c} \wedge (i_{W_{i_1}} \cdots i_{W_{i_c}} \omega)$$

for some $\omega_0 \in \Omega^*_p(N/N)$ (cf. Lemma 1).

Consider now $b < i, j \leq c$. Since $K/G$ is an abelian group (cf. Proposition 2) and $H$ is a Lie group then $[W_i, W_j] = \sum_{i=a+1}^b C_{ij} W_i$. Then, $i_{[W_i, W_j]} \omega_0 = 0$ since the foliation $\mathcal{N}$ is defined by the action of $G \cap H$. So, the canonical writing of a form $\omega \in B^*$ is (8) for some $\omega_0 \in \left\{ \eta \in \Omega^*_p(N/N) \mid L_{W_i} \eta = 0 \text{ if } b < i \leq c\right\} = \left( \Omega^*_p(N/N) \right)_{H_0}^H$.

Then, the operator $\Delta : B^* \rightarrow \left( \Omega^*_p(N/N) \right)_{H_0}^H$, defined by $\Delta(\omega) = \omega_0$, is a differential isomorphism. We conclude that the differential complex $A^*$ is isomorphic to $\bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f) \otimes \left( \Omega^*_p(N/N) \right)_{H_0}^H$ and therefore $H^*(A^*) \cong H^* (K/\mathcal{E}) \otimes \mathbb{H}^*_p(N/N)$ (cf. (5) and (6)). Since the operator $\Delta$ is $(H/H_0)$-equivariant (cf. 4.2) then we get

$$H^*_p(K \times H N/\mathcal{W}) = H^* (\text{Im } \Pi^*) = \left( H^* (A^*) \right)^{H/H_0} = \left( H^* (K/\mathcal{E}) \otimes \mathbb{H}^*_p(N/N) \right)^{H/H_0}.$$ 

This ends the proof. \qed

4.3 Remarks

(a) When the Lie group $G$ is commutative then $K$ is also commutative. Differential forms $\gamma^*_\bullet$ are $K$-invariants on the left and on the right, so $\left( H^* (K/\mathcal{E}) \right)^H = H^* (K/\mathcal{E})$ and therefore

$$H^*_p(K \times H N/\mathcal{W}) = H^* (K/\mathcal{E}) \otimes \left( H^*_p(N/N) \right)^{H/H_0} = H^* (K/\mathcal{E}) \otimes \left( \mathbb{H}^*_p(N/N) \right)^{H}$$

as it has been proved in [11, Proposition 3.8.4].

(b) Since the foliation $\mathcal{E}$ is a riemannian foliation defined on a compact manifold then we know that the cohomology $H^* (K/\mathcal{E})$ is finite (cf. [4]). So, the finiteness of $H^*_p(K \times H N/\mathcal{W})$ depends on the finiteness of $H^*_p(N/N)$. 

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5 Finiteness of the BIC

We prove in this section that the BIC of a Killing foliation on a compact manifold is finite dimensional. First of all, we present two geometrical tools we shall use in the proof: the isotropy type stratification and the Molino’s blow up.

We fix an isometric action $\Phi : G \times M \rightarrow M$ on the compact manifold $M$. We denote by $\mathcal{F}$ the induced Killing foliation. For the study of $\mathcal{F}$ we can suppose that $G$ is connected (see Lemma 1). We fix $K$ a tamer group. Notice that the group $G$ is normal in $K$ and the quotient $K/G$ is commutative (cf. Proposition 2).

5.1 Isotropy type stratification

The isotropy type stratification $S_{K,M}$ of $M$ is defined by the equivalence relation:

$$x \sim y \Leftrightarrow K_x \text{ is conjugated to } K_y.$$ 

When depth $S_{K,M} > 0$, any closed stratum $S \in S_{K,M}$ is a $K$-invariant submanifold of $M$ and then it possesses a $K$-invariant tubular neighborhood $(T, \tau, S, \mathbb{R}^m)$ whose structural group is $O(m)$. Recall that there are the following smooth maps associated with this neighborhood:

1. The radius map $\rho : T \rightarrow [0, 1]$ defined fiberwise from the assignation $[x, t] \mapsto t$. Each $t \neq 0$ is a regular value of the $\rho$. The pre-image $\rho^{-1}(0)$ is $S$. This map is $K$-invariant, that is, $\rho(k \cdot z) = \rho(z)$.
2. The contraction $H : T \times [0, 1] \rightarrow T$ defined fiberwisely from $([x, t], r) \mapsto [x, rt]$. The restriction $H_t : T \rightarrow T$ is an embedding for each $t \neq 0$ and $H_0 \equiv \tau$. We shall write $H(t, t) = t \cdot z$. This map is $K$-invariant, that is, $t \cdot (k \cdot z) = k \cdot (t \cdot z)$.

The hyper-surface $D = \rho^{-1}(1/2)$ is the tube of the tubular neighborhood. It is a $K$-invariant submanifold of $T$. Notice that the map

$$\nabla : D \times [0, 1[ \longrightarrow T,$$

defined by $\nabla (z, t) = (2t \cdot z)$ is a $K$-equivariant smooth map, where $K$ acts trivially on the $[0, 1]$-factor. Its restriction $\nabla : D \times ]0, 1[ \longrightarrow T \setminus S$ is a $K$-equivariant diffeomorphism.

Denote $S_{min}$ the union of closed (minimal) strata and choose $T_{min}$ a disjoint family of $K$-invariant tubular neighborhoods of the closed strata. The union of associated tubes is denoted by $D_{min}$. Notice that the induced map $\nabla_{min} : D_{min} \times ]0, 1[ \longrightarrow T_{min} \setminus S_{min}$ is a $K$-equivariant diffeomorphism.

5.2 Molino’s blow up

The Molino’ blow up [7] of the foliation $\mathcal{F}$ produces a new foliation $\tilde{\mathcal{F}}$ of the same kind but of smaller depth. We suppose depth $S_{K,M} > 0$. The blow up of $M$ is the compact manifold

$$\tilde{M} = \left\{ ((D_{min} \times]0, 1[) \cup \left( (M \setminus S_{min}) \times \{-1, 1\} \right) \right\} / \sim,$$

where $(z, t) \sim (\nabla_{min}(z, |t|), t/|t|)$, and the map $\mathcal{L} : \tilde{M} \rightarrow M$ defined by

$$\mathcal{L}(v) = \begin{cases} 
\nabla_{min}(z, |t|) & \text{if } v = (z, t) \in D_{min} \times ]0, 1[ \\
\quad \text{or } v = (z, j) \in (M \setminus S_{min}) \times \{-1, 1\}.
\end{cases}$$

For notions related with compact Lie group actions, we refer the reader to [1].

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Notice that $\mathcal{L}$ is a continuous map whose restriction $\mathcal{L} : \hat{M} \setminus \mathcal{L}^{-1}(S_{min}) \to M \setminus S_{min}$ is a $K$-equivariant smooth trivial 2-covering.

Since the map $\nabla_{min}$ is $K$-equivariant then $\Phi$ induces the action $\hat{\Phi} : K \times \hat{M} \to \hat{M}$ by saying that the blow-up $\mathcal{L}$ is $K$-equivariant. The open submanifolds $\mathcal{L}^{-1}(T_{min})$ and $\mathcal{L}^{-1}(T_{min} \setminus S_{min})$ are clearly $K$-diffeomorphic to $D_{min} \times [1 - 1, 1]$ and $D_{min} \times [0, 1]$ respectively.

The restriction $\hat{\Phi} : G \times \hat{M} \to \hat{M}$ is an isometric action with $K$ as a tamer group. The induced Killing foliation is $\hat{\Phi}$. Foliations $\mathcal{F}$ and $\hat{\Phi}$ are related by $\mathcal{L}$ which is a foliated map. Moreover, if $S$ is a not minimal stratum of $S_{K,M}$ then there exists an unique stratum $S' \in S_{K,\hat{M}}$ such that $\mathcal{L}^{-1}(S) \subset S'$. The family $\{S' / S \in S_{K,M}\}$ covers $\hat{M}$ and verifies the relationship: $S_1 < S_2 \iff S'_1 < S'_2$. We conclude the important property

$$\text{depth } S_{K,\hat{M}} < \text{depth } S_{K,M}. \quad (9)$$

5.3 Finiteness of a tubular neighborhood

We suppose depth $S_{K,M} > 0$. Consider a closed stratum $S \in S_{K,M}$. Take $(T, \tau, S, \mathbb{R}^m)$ a $K$-invariant tubular neighborhood. We fix a base point $x \in S$. The isotropy subgroup $K_x$ acts orthogonally on the fiber $\mathbb{R}^m = \tau^{-1}(x)$. So, the induced action $\Lambda_x : G_x \times \mathbb{R}^m \to \mathbb{R}^m$ is an isometric action, it gives the Killing foliation $\mathcal{N}$ on $\mathbb{R}^m$.

**Proposition 6** If the BIC of $(\mathbb{R}^m, \mathcal{N})$ is finite dimensional then the BIC of $(T, \mathcal{F})$ is also finite dimensional.

**Proof** We proceed in two steps.

(a) $K_y = K_x$ for each $y \in S$.

The canonical projection $\pi : S \to S/K$ is an homogeneous bundle with fiber $K/K_x$. For any open subset $V \subset S/K$ the pull back $\tau^{-1}\pi^{-1}(V)$ is a $K$-invariant subset of $T$, then we can apply the Mayer-Vietoris technics to this kind of subsets (cf. 3.5).

Since the manifold $S/K$ is a compact one then we can find a finite good covering $\{U_i / i \in I\}$ of it (cf. [2]). An inductive argument on the cardinality of $I$ reduces the proof of the Lemma to the case where $T = \tau^{-1}\pi^{-1}(V)$, where $V$ is a contractible open subset of $S/K$.

Here, the manifold $T$ is $K$-equivalently diffeomorphic to $V \times (K \times_{K_x} \mathbb{R}^m)$, where $K$ does not act on the first factor. So, the natural retraction of $V$ to a point gives a $K$-equivariant retraction of $T$ to the twisted product $K \times_{K_x} \mathbb{R}^m$. Now the result comes directly from 4.3(b) since $(K, G, K_x)$ is a trio.

(b) **General case.**

The stratum $S$ is $K$-equivariantly diffeomorphic to the twisted product $K \times_{N(K_x)} F$ where $N(K_y)$ is the normalizer of $K_x$ on $K$ and $F = S^{K_x}$. So, the tubular neighborhood $T$ is $K$-equivariantly diffeomorphic to the twisted product $K \times_{N(F)} N$ where $N$ is the manifold $\tau^{-1}(F)$. The previous case gives that the BIC of $(N, \mathcal{F}_N)$ is finite dimensional. Now the result comes directly from 4.3(b) since $(K, G, N(K_x))$ is a trio.

The main result of this work is the following

**Theorem 1** The BIC of the foliation determined by an isometric action on a compact manifold is finite dimensional.
Proof Let $\mathcal{F}$ be a Killing foliation defined on a compact manifold $M$ induced by an isometric action $\Phi: G \times M \to M$ where $G$ is a Lie group. Without loss of generality we can suppose that the Lie group $G$ is a connected one (cf. Lemma 1). We fix a tamer group $K$. We know that $G$ is normal in $K$ and the quotient group $K/G$ is commutative (cf. Proposition 2).

Let us consider the following statement

$$\mathfrak{A}(U, \mathcal{F}) = \text{"The BIC } H^*_\mathcal{F}(U/\mathcal{F}) \text{ is finite dimensional for each perversity } \mathcal{P},\text{"}$$

where $U \subset M$ is a $K$-invariant submanifold. We prove $\mathfrak{A}(M, \mathcal{F})$ by induction on $\dim M$. The result is clear when $\dim M = 0$. We suppose $\mathfrak{A}(W, \mathcal{F})$ for any $K$-invariant compact submanifold $W$ of $M$ with $\dim W < \dim M$ and we prove $\mathfrak{A}(M, \mathcal{F})$. We proceed in several steps.

First step: 0-depth. Let us suppose depth $S_{K,M} = 0$. Since $G < K$ and $K_x$ is conjugated to $K_y$ then $G_{x,y}$ is conjugated to $G_y, \forall x, y \in M$. We get that the foliation $\mathcal{F}$ is a (regular) riemannian foliation (cf. [7]). Its BIC is just the basic cohomology (cf. 3.3). Then $\mathfrak{A}(M, \mathcal{F})$ comes from [4].

Second step: Inside $M$. Let us suppose depth $S_{K,M} > 0$. The family $\{M \backslash S_{min}, T_{min}\}$ is a basic covering of $M$ and we get the exact sequence (cf. 3.5)

$$0 \to \Omega^*_p(M/\mathcal{F}) \to \Omega^*_p((M \backslash S_{min})/\mathcal{F}) \oplus \Omega^*_p(T_{min}/\mathcal{F}) \to \Omega^*_p(T_{min} \backslash S_{min}/\mathcal{F}) \to 0.$$

The Five Lemma gives

$$\mathfrak{A}(T_{min} \backslash S_{min}, \mathcal{F}), \mathfrak{A}(T_{min}, \mathcal{F}) \text{ and } \mathfrak{A}(M \backslash S_{min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

Since $T_{min} \backslash S_{min}$ is $K$-diffeomorphic to $D_{min} \times ]0, 1[$ (cf. (5.1)) then $\mathfrak{A}(D_{min}, \mathcal{F}) \implies \mathfrak{A}(T_{min} \backslash S_{min}, \mathcal{F}).$ The inequality $\dim D_{min} < \dim M$ gives

$$\mathfrak{A}(T_{min}, \mathcal{F}) \text{ and } \mathfrak{A}(M \backslash S_{min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

In order to prove $\mathfrak{A}(T_{min}, \mathcal{F})$ it suffices to prove $\mathfrak{A}(T, \mathcal{F})$ where $(T, \tau, S, \mathbb{R}^m)$ a $K$-invariant tubular neighborhood of closed stratum $S$ of $S_{K,M}$. Following Proposition 6 we have

$$\mathfrak{A}(\mathbb{R}^m, \mathcal{N}) \implies \mathfrak{A}(T, \mathcal{F}) \implies \mathfrak{A}(T_{min}, \mathcal{F}).$$

Consider the orthogonal decomposition $\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, where $\mathbb{R}^{m_1} = (\mathbb{R}^m)^{G_x}$. The only fixed point of the restriction $A_x: G_x \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_2}$ is the origin. So, there exists a Killing foliation $G$ on the sphere $S^{m_2-1}$ with $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}, \mathcal{F}) = (\mathbb{R}^{m_1} \times cS^{m_2-1}, \mathcal{I} \times c\mathcal{G}).$ Propositions 3 and 4 give:

$$\mathfrak{A}(S^{m_2-1}, \mathcal{G}) \implies \mathfrak{A}(\mathbb{R}^{m_1} \times cS^{m_2-1}, \mathcal{I} \times c\mathcal{G}) \implies \mathfrak{A}(\mathbb{R}^m, \mathcal{N}).$$

Finally, since $\dim S^{m_2-1} < m \leq \dim T \leq \dim M$ we have

$$\mathfrak{A}(M \backslash S_{min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

Third step: Blow-up. Let us suppose depth $S_{K,M} > 0$. The family $\{\mathcal{L}^{-1}(M \backslash S_{min}), \mathcal{L}^{-1}(T_{min})\}$ is a basic covering of $\hat{M}$ and we get the exact sequence (cf. 3.5)

$$0 \to \Omega^*_p(\hat{M}/\mathcal{F}) \to \Omega^*_p(\mathcal{L}^{-1}(M \backslash S_{min})/\mathcal{F}) \oplus \Omega^*_p(\mathcal{L}^{-1}(T_{min})/\mathcal{F}) \to \Omega^*_p(\mathcal{L}^{-1}(T_{min} \backslash S_{min})/\mathcal{F}) \to 0.$$

Following 5.2 we have that

\[9\] It is given by the orthogonal action $A_x: G_x \times S^{m_2-1} \to S^{m_2-1}$. 

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Finiteness of the basic intersection cohomology of a killing foliation

- $L^{-1} (M \setminus S_{\text{min}})$ is $K$-diffeomorphic to two copies of $M \setminus S_{\text{min}},$
- $L^{-1} (T_{\text{min}})$ is $K$-diffeomorphic to $D_{\text{min}} \times [-1, 1[,$
- $L^{-1} (T_{\text{min}} \setminus S_{\text{min}})$ is $K$-diffeomorphic to $D_{\text{min}} \times (-1, 0 \cup ]0, 1[).$

Now, the Five Lemma gives

$$\mathfrak{a}(D_{\text{min}}, \hat{\mathcal{F}}) \text{ and } \mathfrak{a}(\hat{M}, \hat{\mathcal{F}}) \implies \mathfrak{a}(M \setminus S_{\text{min}}, \mathcal{F}).$$

But, the inequality $\dim D_{\text{min}} < \dim M$ gives

$$\mathfrak{a}(\hat{M}, \hat{\mathcal{F}}) \implies \mathfrak{a}(M \setminus S_{\text{min}}, \mathcal{F}).$$

**Forth step: Final blow-up.** When depth $S_{K,M} = 0$ we get $\mathfrak{a}(M, \mathcal{F})$ from the First step. Let us suppose depth $S_{K,M} > 0.$ From (10) and (11) we get

$$\mathfrak{a}(\hat{M}, \hat{\mathcal{F}}) \implies \mathfrak{a}(M, \mathcal{F}).$$

with depth $S_{K,\hat{M}} < depth S_{K,M}$ (cf. (9)). By iterating this procedure we get

$$\mathfrak{a}(\hat{M}, \hat{\mathcal{F}}) = \mathfrak{a}(\hat{\hat{M}}, \hat{\mathcal{F}}) \implies \cdots \implies \mathfrak{a}(\hat{M}, \hat{\mathcal{F}}) \implies \mathfrak{a}(M, \mathcal{F}),$$

with depth $S_{K,\hat{M}} = 0.$ We finish the proof by applying again the First Step.$\blacksquare$

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