Expansion properties of finite simple groups

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by
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This work was carried out under the supervision of Prof. Alex Lubotzky
To my mother, who taught me that talent is useless without the ability to carry it out.

To my father, who showed me that imagination has no boundaries, and whom I deeply miss.
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Chapter 1

Abstract

1.1 Diameter and Growth of Cayley graphs

A family of finite groups \( \{G_n\}_{n \in \mathbb{N}} \) is said to have poly-logarithmic diameter if for some absolute constants \( C, d > 0 \), for every \( G_n \) and every subset \( S_n \subseteq G_n \) generating \( G_n \), we have

\[
\text{diam}(\text{Cay}(G_n, S_n)) \leq C \log^d(|G_n|),
\]

where \( \text{diam}(\text{Cay}(G, S)) \) is the diameter of the Cayley graph of \( G \) with respect to \( S \).

A well known conjecture of Babai [BS2] asserts that all the non-abelian finite simple groups have poly-logarithmic diameter. In this work we investigate the family of groups \( \text{SL}_2 \) (and \( \text{PSL}_2 \)) over finite fields, and we prove the conjecture for this family of groups.

In fact, we investigate a stronger Growth property that would imply in particular the poly-logarithmic diameter bounds. By this, we extend the techniques that were developed by Helfgott [He] who dealt with the family
of groups $\text{SL}_2$ (and $\text{PSL}_2$) over finite fields of prime order.

1.2 The main results

Our main result asserts that the family

$$\{\text{SL}_2(\mathbb{F}_{p^n}) : p \text{ prime; } n \in \mathbb{N}\}$$

has poly-log diameter. Note that this result holds uniformly for all finite fields regardless of their characteristic. This result holds also for the family $\text{PSL}_2$ over finite fields.

By using results from Additive Combinatorics, we proved the following stronger Growth property:

There exists $\varepsilon > 0$ such that the following holds for any finite field $\mathbb{F}_q$. Let $G$ be the group $\text{SL}_2(\mathbb{F}_q)$ (or $\text{PSL}_2(\mathbb{F}_q)$) and let $A$ be a generating set of $G$. Then we have,

$$|A \cdot A \cdot A| \geq \min\{|A|^{1+\varepsilon}, |G|\}.$$

Our work extends the work of Helfgott [He] who proved similar results for the family $\{\text{SL}_2(\mathbb{F}_p) : p \text{ prime}\}$. 
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Chapter 2

Introduction

2.1 Background

Let us define the directed diameter of a finite group $G$ with respect to a set of generators $S$ to be the minimal number $l$ for which any element in $G$ can be written as a product of at most $l$ elements in $S$. We denote this number by $\text{diam}^+(G, S)$. Define the (undirected) diameter of a finite group $G$ with respect to a set of generators $S$ to be $\text{diam}(G, S) := \text{diam}^+(G, S \cup S^{-1})$.

The diameter of groups has many applications. Aside from group theory (see [BKL, La, LS]) and combinatorics (see [Di2, ER, ET1, ET2]) the diameter of groups shows up in computer science areas such as communication networks (see [Sto, PV]), generalizations of Rubik’s puzzles (see [DF, McK]), algorithms and complexity (see [EG, Je]). For a detailed review see [BHKLS].

Since we are interested in the “worst case generators”, we define

$$\text{diam}(G) := \max\{\text{diam}(G, S) : G = \langle S \rangle\}.$$ 

A family of finite groups $\{G_n : n \in \mathbb{N}\}$ is said to have poly-log diameter
(resp. log diameter) if for any $n \in \mathbb{N}$ we have

$$\text{diam}(G_n) \leq C \log^d(|G_n|)$$

for some constants $C, d > 0$ (resp. for $d = 1$).

In [Di1], the author shows (with an effective algorithm) that for any fixed $p, m \in \mathbb{N}$ with $p$ a prime and $p > m \geq 2$, the family

$$G_{m,p} := \{\text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) : n \in \mathbb{N}\}$$

has poly-log diameter. Abert and Babai [AB] showed that for any fixed prime $p_0$, the family $\{C_{p_0} \wr C_p : p \text{ prime; } p \neq p_0\}$ has logarithmic diameter.

A long standing conjecture of Babai [BS2] asserts that the family of non-abelian finite simple groups has a poly-logarithmic diameter. Very little is known about this conjecture. See [BS1] and [BS2] for some partial results concerning the alternating groups.

A breakthrough result of Helfgott [He] proves the conjecture for the family $\{\text{SL}_2(\mathbb{F}_p) : p \text{ prime}\}$. The main goal of this paper is to extend Helfgott work to the family $\{\text{SL}_2(\mathbb{F}_{p^n}) : p \text{ prime; } n \in \mathbb{N}\}$. We follow the basic strategy of Helfgott (with some short cuts following [BG2]) and in particular we also appeal to additive combinatorics and sum-product theorems. The new difficulty is that unlike fields of prime order, general finite fields have subfields, and subsets which are “almost” subfields - which are “almost” stable with respect to sum and product.

### 2.2 Main results

Our main results are the following.
Theorem 2.2.1 (See Theorem 7.2.2 in §7.2). There exists \( \varepsilon \in \mathbb{R}_+ \) such that the following holds for any finite field \( \mathbb{F}_q \). Let \( G \) be the group \( \text{SL}_2(\mathbb{F}_q) \) and let \( A \) be a generating set of \( G \). Then we have\(^1\),

\[
|A \cdot A \cdot A| \geq \min\{|A|^{1+\varepsilon}, |G|\}.
\]

From this we easily get the following.

Corollary 2.2.2 (See corollary 7.2.3 in §7.2). There exist \( C, d \in \mathbb{R}_+ \) such that the following holds for any finite field \( \mathbb{F}_q \). Let \( A \) be a subset of generators of \( G = \text{SL}_2(\mathbb{F}_q) \). Then we have,

\[
\text{diam}^+(G, A) < C \log^d(|G|)
\]

and for any \( \delta \in \mathbb{R}_+ \) we have,

\[
|A| > |G|^\delta \Rightarrow \text{diam}^+(G, A) < C \left( \frac{1}{\delta} \right)^d.
\]

2.3 Organization of the manuscript

The manuscript is organized as follows: In §3 we bring notations and definitions, which are required for this work, as well as mathematical background. In §4 we collect useful facts from Additive Combinatorics to be used later. In §5 we prove some useful facts about \( \text{SL}_2(\mathbb{F}_q) \). In §6 we extend few of the main ingredients from the proof of Helfgott, from \( \text{SL}_2(\mathbb{F}_p) \) to \( \text{SL}_2(\mathbb{F}_q) \). In §7 we show how to use all the previous sections in order to prove the main results of this manuscript. In §8 we present some questions/conjectures.

\(^1\)The same assertion holds for \( \text{PSL}_2(\mathbb{F}_q) \).
Chapter 3

Preliminaries

3.1 Notations

We will use the following notations. \( \log x \) will stand for \( \log_2 x \), \log in the base 2. We will always use \( p \) for a prime number and \( q \) for a prime power. For a subset \( A \subseteq B \) and \( x \in B \) denote for short \( A\setminus\{x\} \) by \( A\setminus x \) and similarly \( A \cup x := A \cup \{x\} \). For a field \( \mathbb{F} \), denote by \( \overline{\mathbb{F}} \) some fixed algebraic closure of \( \mathbb{F} \). We denote

\[
(G, \cdot)
\]

a multiplicative group which is not necessarily commutative and

\[
(G, +)
\]

will stand for a commutative additive group.

Definition 3.1.1. Let \( G \) be a group and let \( A, B, A_1, \ldots, A_n \subseteq G \) be non-
empty subsets of $G$. For $k \in \mathbb{Z}$ denote

$$A^k := \{ a^k : a \in A \}$$

$$A^{\pm 1} := A \cup A^{-1}$$

Define the product-set,

$$A \cdot B := \{ a \cdot b : a \in A, b \in B \}$$

and for $x \in G$ define $x \cdot A := \{ x \} \cdot A$ and $A \cdot x := A \cdot \{ x \}$. Denote the product set of $A_1, \ldots, A_n$ by

$$\prod_{i=1}^{n} A_i := \{ a_1 \cdots a_n : \forall 1 \leq i \leq n, a_i \in A_i \}$$

and the product set of one set with itself $n$-times by

$$A^{(n)} := \prod_{i=1}^{n} A.$$

The most important notations in this manuscript will be

$$A^{[0]} := \{ 1 \}$$

$$A^{[1]} := A^{\pm} \cup 1$$

$$A^{[n]} := (A^{[1]})^{(n)}$$

the set of words of length at most $n$ in the letters $A^{\pm} := A \cup A^{-1}$. Note that in general we have only the containments

$$A^n \subseteq A^{(n)} \subseteq A^{[n]}.$$

**Simple Fact 3.1.2.** Since we have three possible operations on the subsets\(^1\), $A^{[m]}$, $A^{(n)}$ and $A^k$, we use the following “group action” notation $A^{gh} = (A^g)^h$.

\(^1\)Note the these operations on subsets of $G$ are not induced from operations on elements of $G$.\]
For example,

\[ A^{(n)[m]} := (A^{(n)})^{[m]} \]
\[ A^{[m](n)} := (A^{[m]})^{(n)}. \]

Similarly

\[ A^{xyz} := ((A^x)^y)^z \]

when \( x, y, z \) is any of these operation e.g., \( A^{k(n)[m]} := ((A^k)^{(n)})^{[m]} \). Note the these operations on subsets is associative

\[ A^{(xy)z} = A^{x(yz)} = ((A^x)^y)^z. \]

Note that in general we have only the containments,

\[ A^{(n)[m]} \subseteq A^{[m](n)} = A^{[nm]}. \]

We can write these properties as a table of relations between the operations as \([n][m] = [nm]\) and \((n)(m) = (nm)\) and \([mn] = [m](n) \neq (n)[m]\).

Note that if \( \langle A \rangle \) is abelian then \( A^{k(n)} = A^{(n)k} \) and similarly

\[ A^{k[m]} = A^{[m]k}. \]

**Definition 3.1.3.** Let \( G \) be a group and let \( g, h \in G \). We will denote by

\[ g^h := h^{-1}gh \]
\[ [g, h] := g^{-1}h^{-1}gh \]

For subsets \( A, B \subseteq G \) we denote by

\[ A^B := \{ a^b : a \in A, b \in B \} \]
and \( x^B := \{ x \}^B \) for short. For commutator of two subsets we will write

\[
[A, B]_{\text{set}} := \{ [a, b] : a \in A; b \in B \}.
\] (3.1.3a)

Note that we have only containments

\[
[A, B]_{\text{set}} \subseteq A^{-1}A^B \subseteq A^{-1}B^{-1}AB.
\]

**Definition 3.1.4.** Let \( G \) be a group and let \( A, B \subseteq G \). Define

\[
C_B(A) := \{ b \in B : a^b = a \text{ for all } a \in A \}.
\]

**Simple Fact 3.1.5.** Note that using these notations we always have \( g(n) = (n)g \) for \( g \in G \) and \( n \in \mathbb{N} \). I.e.,

\[
A^{g(n)} = A^{(n)g} = (A^g)^{(n)} = (A^{(n)})^g
\]

and similarly \( g[m] = [m]g \) and \( kg = gk \). So conjugation (or any other automorphism) commutes with the operations \( A^{[m]}, A^{(n)}, A^k \).

**Definition 3.1.6.** We will use the generation notation \( \langle A \rangle \) depending on the category we are working. The categories that will be involved in the manuscript will be groups and rings.

**Definition 3.1.7.** Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{F}) \). Denote,

\[
\text{Prod}(g) := a \cdot d
\]

\[
\text{Diag}(g) := (a, d).
\]

Extend these functions to \( \text{Prod}(V) \) and \( \text{Diag}(V) \) for subset \( V \subseteq \text{SL}_2(\mathbb{F}) \).
Definition 3.1.8. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{F})$ and $x \in \mathbb{F}^\times$. Denote,

- $D_g := \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$
- $D_{(a,d)} := \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$
- $D_x := \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$

Extends these notations to subsets in the obvious way $D_X := \{ D_x : x \in X \}$ where $X$ is either $X \subseteq \mathbb{F}^\times$ or $X \subseteq \mathbb{F} \times \mathbb{F}$ or $X \subseteq \text{SL}_2(\mathbb{F})$.

Definition 3.1.9. For positive real-valued functions, we write $f \ll g$ if $f = O(g)$. Similarly we write $f \gg g$ if $g \ll f$, and $f \approx g$ if $f \ll g \ll f$. Similarly we will use the dual notation $f = \Omega(g)$ for $g = O(f)$. Denote also $f \sim g \iff \frac{1}{2} f \leq g \leq 2f$.

Simple Fact 3.1.10. Let $\varepsilon \in \mathbb{R}_+$ be real number with $\varepsilon < \frac{1}{2}$. Then we have

\[ 1 - \varepsilon < \frac{1}{1+\varepsilon} < 1 - \frac{1}{2}\varepsilon \]
\[ 1 + \varepsilon < \frac{1}{1-\varepsilon} < 1 + 2\varepsilon. \]

Therefore for any $X, Y \in \mathbb{R}_+$ we have in the $\Omega$-language:

$X \ll Y^{1+O(\varepsilon)} \iff X^{1-O(\varepsilon)} \ll Y$

and similarly

$X^{1+\Omega(\varepsilon)} \ll Y \iff X \ll Y^{1-\Omega(\varepsilon)}$. 

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Definition 3.1.11. Let $R$ be a ring (not necessarily commutative) and let $a \in R$. Define the endomorphisms $L_a$ and $R_a$ by $L_a(b) = ab$ and $R_a(b) = ba$. Then $L_a$ is endomorphism of the right\footnote{the action of the scalars is from the right.} $R$-module $R$ and $R_a$ is endomorphism of the left\footnote{the action of the scalars is from the left.} $R$-module $R$. Denote the right ideal $\ker(L_a)$ and the left ideal $\ker(R_a)$ by

$$
\ker(L_a) := \{ b : ab = 0 \}
$$

$$
\ker(R_a) := \{ b : ba = 0 \}.
$$

Now suppose $R$ is commutative ring. Denote the set of non zero-divisors in $R$ by $R^\times$:

$$
a \in R^\times \iff L_a \text{ is injective} \iff \ker(L_a) = 0.
$$

If $A$ is a subset of a commutative ring $R$ we will need different notations to distinguish the product-set $A \cdot A = \{ ab : a, b \in A \}$ and the sum-set $A + A = \{ a + b : a, b \in A \}$. Therefore we will need in some situations the following definitions.

Definition 3.1.12. Let $A$ be a subset of an additive (semi) group $G$ and let $n \in \mathbb{N}$. Denote by

$$
\sum_n A := \{ a_1 + \ldots + a_n : \forall i, a_i \in A \}.
$$

Definition 3.1.13. Let $\Gamma \subseteq X \times Y$ be a directed graph. Denote the inverse (opposite) graph $\Gamma^{-1} \subseteq Y \times X$ (or $\Gamma^{\text{op}}$) by

$$
\Gamma^{-1} := \{ (y, x) : (x, y) \in \Gamma \}.
$$
Let $A \subseteq X$ and $a \in X$. Denote
\[
\Gamma_a := \{y \in Y : (a, y) \in \Gamma\}
\]
\[
\Gamma(A) := \bigcup_{a \in A} \Gamma_a
\]
Denote
\[
\deg(\Gamma) := \max_{x \in X}\{|\Gamma_x|\}.
\]
Clearly $\deg(\Gamma) \leq d \Rightarrow |\Gamma(A)| \leq d|A|$ for any $A \subseteq X$. We will say that $\Gamma$ is $d$-regular\(^4\), if $|\Gamma_x| = d$ for all $x \in X$. We define the multiplicity of $\Gamma$ to be
\[
\text{mult}(\Gamma) := \deg(\Gamma^{-1}).
\]

We will use the previous definition with the following simple observations.

**Simple Fact 3.1.14.** A function $f \in Y^X \subseteq X \times Y$ is a directed graph which is 1-regular graph. Therefore we get,
\[
\text{mult}(f) \leq n \quad \Rightarrow \quad |f(A)| \geq |A|/n \text{ for any } A \subseteq X.
\]

For example, any one variable polynomial $0 \neq f(x) \in \mathbb{F}[x]$ of degree $d$ defines a substitution map $f_s : \mathbb{F} \to \mathbb{F}$ such that
\[
\text{mult}(f_s) \leq \deg(f).
\]

Similarly if $0 \neq f(x, x^{-1}) \in \mathbb{F}[x, x^{-1}]$ with $\deg_x(f) + \deg_{x^{-1}}(f) = d$ then $\text{mult}(f_s) \leq d$ where
\[
f_s : \mathbb{F}^{\times} \to \mathbb{F}.
\]
E.g., $f(x) = x^2 + x^{-3}$ has multiplicity $\leq 5$. By abuse of notation we will write for $f \in \mathbb{F}[x, x^{-1}]$,
\[
\text{mult}(f) := \text{mult}(f_s).
\]
3.2 Uniform poly-logarithmic diameter bounds

Definition 3.2.1. For a finite (undirected) graph \( \Gamma = (V, E) \) define \( \text{diam}(\Gamma) \), the diameter of the graph \( \Gamma \), to be the minimal \( l \) such that any two vertices are connected by a path with at most \( l \) edges. Set the diameter to be \( \text{diam}(\Gamma) = \infty \) if the graph is not connected.

Definition 3.2.2. For a finite group \( G \) and a subset \( S \) of \( G \), define
\[
\text{diam}(G, S) := \text{diam}(\text{Cay}(G, S)).
\]
For a finite group \( G \) and a set of generators \( S \) of \( G \), we have
\[
\text{diam}(G, S) = \min \{ k : S^k = G \}
\]
Define the maximal diameter of \( G \) to be
\[
\text{diam}_{\text{max}}(G) := \max \{ \text{diam}(G, S) : S \subseteq G, \langle S \rangle = G \}
\]
or just for short \( \text{diam}(G) = \text{diam}_{\text{max}}(G) \). For a finite group \( G \) and a set of generators \( S \) of \( G \), define
\[
\text{diam}^+(G, S) := \min \{ k : (S \cup \{1\})^k = G \}.
\]
Remark. It is easy to see that for a set of generators \( S \) of \( G \) with \( s := |S \cup S^{-1}| \) we have
\[
\log_s |G| - 1 \leq \text{diam}(G, S) \leq |G| - 1
\]
by a simple count of words in \( G \) with the letters \( S \cup S^{-1} \). Still there is an exponential gap between these two bounds. So usually the goal is to find either an upper logarithmic or a poly-logarithmic diameter bound for \( \text{diam}_{\text{max}}(G) \). This bound is of interest when each group, in the family of groups, can be generated by a subset of bounded size.
A well known conjecture of Babai asserts the following (cf. [BS1, BS2]).

**Conjecture 1** (Babai). There exist $C, d \in \mathbb{R}_+$ such that for any non-abelian finite simple group $G$ we have

$$\text{diam}_{\max}(G) \leq C \log^d |G|.$$ 

This bound may even be true for $d = 2$, but not $d < 2$, as the groups $\text{Alt}(n)$ demonstrate.

The first step towards proving Babai’s conjecture was made by Helfgott (cf. [He, §1.2 Main Theorem]).

**Theorem 3.2.3** (Helfgott). Denote the family of groups

$$\mathcal{G} = \{\text{SL}_2(\mathbb{F}_p) : p \text{ prime}\}.$$ 

There exist $C, d \in \mathbb{R}_+$ such that for any $G \in \mathcal{G}$ we have,

$$\text{diam}_{\max}(G) \leq C \log^d |G|.$$ 

We extend this theorem to all finite fields to get the following.

**Theorem 3.2.4** (See corollary 7.2.3 in §7.2). Denote the family of groups

$$\mathcal{G} = \{\text{SL}_2(\mathbb{F}_{p^n}) : p \text{ prime}; n \in \mathbb{N}\}.$$ 

There exist $C, d \in \mathbb{R}_+$ such that for any $G \in \mathcal{G}$ we have,

$$\text{diam}_{\max}^+(G) \leq C \log^d |G|.$$ 

The main idea in Helfgott’s work is to show an expansion property of subsets w.r.t the product operation in the group. For this he reduced the problem
to an expansion property of the addition and multiplication operations in the underline fields. One advantage of these results is their, relatively, elementary proofs. One disadvantage of these results is that they do not supply an algorithm/method for actually calculating such short paths/products in the graphs/groups.
Chapter 4

Tools from Additive combinatorics

4.1 The fundamental tools

Ruzsa triangle inequality

The following useful lemma of Ruzsa allows one to pass from control of sum-set to control of minus-sets (cf. [TV, Lemma 2.6] and [He, §2.3 Lemma 2.1]).

**Lemma 4.1.1** (Ruzsa). Let $G$ be a group and let $A, B, C \subseteq G$ be finite subsets. Then we have,

$$|AB||C| \leq |AC^{-1}||CB|.$$  \hfill (4.1.1a)

**Proof.** Define the product map $p : AC^{-1} \times CB \to G$ by $p(x, y) = xy$. Then for any $a \in A$ and $b \in B$ we have,

$$p^{-1}(ab) \supseteq \{(ac^{-1}, cb) : c \in C\}$$
so \(|p^{-1}(ab)| \geq |C|\). Therefore \(|AC^{-1}||CB| \geq |p^{-1}(AB)| \geq |C||AB|\) so we are done. 

In particular by taking \(B = C = A^{-1}\) we get the following Corollary.

**Corollary 4.1.2.** Let \(G\) be a group and let \(A \subseteq G\) be a finite subset. For any \(1 \leq K \in \mathbb{R}\) we have,

\[
|A \cdot A| \leq K|A| \Rightarrow |AA^{-1}| \leq K^2|A|.
\]

**Proof.** By lemma 4.1.1 we get

\[
\frac{|A \cdot A^{-1}|}{|A|} \leq \frac{|A \cdot A| |A^{-1} \cdot A^{-1}|}{|A| |A|} = \left( \frac{|A \cdot A|}{|A|} \right)^2 \leq K^2. 
\]

**Definition 4.1.3.** Let \(G\) be a group and let \(A, B \subseteq G\) be finite non empty subsets. Define,

\[
D(A, B) := \frac{|AB^{-1}|}{|A|^{1/2}|B|^{1/2}}.
\]

Define the **Ruzsa distance** between \(A\) and \(B\) to be

\[
d(A, B) := \log(D(A, B)).
\]

It is easy to see that the following properties hold.

**Simple Fact 4.1.4.** Let \(G\) be a group and let \(\emptyset \neq A, B \subseteq G\) be finite subsets. Then for any \(x, y \in G\) we have,

\[
d(A, B) = d(B, A) = d(xA, yB) = d(Ax, Bx) \geq 0.
\]
As an immediate consequence of lemma 4.1.1 we get,

\[ D(A, B) \leq D(A, C)D(B, C) \]

therefore we get the following Triangle inequality.

**Simple Fact 4.1.5.** Let \( G \) be a group and let \( \emptyset \neq A, B, C \subseteq G \) be finite subsets. Then we have,

\[ d(A, B) \leq d(A, C) + d(B, C). \]

Therefore \( d(A, B) \) is quasi-metric\(^1\) on the set of finite subsets of \( G \).

**Plünnecke-Ruzsa inequality**

The following theorem of Plünnecke-Ruzsa allows one to pass from control of sum-set to control of iterated sum-sets (cf. [TV, §6.5, Corollary 6.29]).

**Theorem 4.1.6** (Plünnecke-Ruzsa). Let \((G, +)\) be an additive group and let \( A, B \subseteq G \) be finite subsets. Suppose

\[ |A + B| \leq K|B| \] (4.1.6a)

for some \( 1 \leq K \in \mathbb{R} \). Then for any \( n, m \in \mathbb{N} \) we have,

\[ |\sum_{n} A| \leq K^n|B| \quad \text{and} \quad |\sum_{n} A - \sum_{m} A| \leq K^{n+m}|B|. \] (4.1.6b)

In particular we get the following.

**Corollary 4.1.7.** Let \((G, +)\) be an additive group and let \( A \subseteq G \) be finite subset. Then for any \( 1 \leq K \in \mathbb{R} \) we have,

\[ |A - A| \leq K|A| \quad \Rightarrow \quad |A + A| \leq K^2|A|. \] (4.1.7a)

\(^1\)actually \( d(A, B) = 0 \iff A, B \) are both left cosets of some finite subgroup \( H \leq G \) (see [TV, Proposition 2.38]).
Proof. By taking $B = -A$, we are done by theorem 4.1.6. 

Another special case of theorem 4.1.6 is the following result.

Corollary 4.1.8. Let $R$ be a commutative ring and $A \subseteq R$ a finite subset and let $b \in R^\times$. Suppose

$$|A + bA| \leq K|A|$$

for some $1 \leq K \in \mathbb{R}$. Then we have,

$$|A + A| \leq K^2|A| \quad \text{and} \quad |A - A| \leq K^2|A|.$$  \hspace{1cm} (4.1.8a)

Proof. By taking $B = bA$, we are done by theorem 4.1.6. 

Remark. Note that actually we only used the fact that the addition in $R$ is commutative and that $|A| = |bA|$. Therefore this statement is true also for non-commutative rings provided that $L_b$ is injective.

From large growth to large tripling

In corollary 4.1.7 one cannot drop the additive assumption to get polynomial bound like (4.1.7a) (cf. [He, §2]). However, one can deduce easily from Lemma 4.1.1 the following result.

Lemma 4.1.9. ([He, §2.3 Lemma 2.2]) Let $G$ be a group and let $A \subseteq G$ be a finite subset.

For any $1 \leq K \in \mathbb{R}$ and $x_1, x_2, x_3 \in \{\pm 1\}$ we have,

$$|A^{(3)}| \leq K|A| \quad \Rightarrow \quad |A^{x_1}A^{x_2}A^{x_3}| \leq K^3|A|.$$  \hspace{1cm} (4.1.9a)
For any $3 \leq n \in \mathbb{N}$ and $1 \leq K \in \mathbb{R}$ we have,

$$|A^{[3]}| \leq K|A| \quad \Rightarrow \quad |A^{[n]}| \leq K^{n-2}|A|.$$  \hfill (4.1.9b)

In particular for any $3 \leq n \in \mathbb{N}$ and $1 \leq K \in \mathbb{R}$ we get,

$$|A^{[n]}| > K|A| \quad \Rightarrow \quad |A^{(3)}| > \frac{1}{2} \sqrt[n]{K}|A|.$$  \hfill (4.1.9c)

**Proof.** By the assumption,

$$|A^{-1}A^{-1}A^{-1}| = |AAA| \leq K|A|.$$  

Therefore by Lemma 4.1.1 we get,

$$|AAA^{-1}| \overset{(4.1.1a)}{\leq} \frac{1}{|A|}|AAA||A^{-1}A^{-1}|$$

$$\leq \frac{1}{|A|}|AAA||A^{-1}A^{-1}A^{-1}|$$

$$= \left(\frac{|AAA|}{|A|}\right)^2 |A|$$

$$\leq K^2|A|$$

Therefore we get also,

$$|AA^{-1}A^{-1}| = |AAA^{-1}| \leq K^2|A|.$$  \hfill (4.1.9.1)

By repeating the previous argument but now with $A = A^{-1}$ (i.e., $A^{-1}$ in the roll of $A$) we get

$$|A^{-1}A^{-1}A|, |A^{-1}AA| \leq K^2|A|.$$  

On the other hand,

$$|A^{-1}AA^{-1}| = |AA^{-1}A|$$

$$\overset{(4.1.1a)}{\leq} \frac{1}{|A|}|AA^{-1}A^{-1}||AA|$$

$$\leq \frac{|AA^{-1}A^{-1}| |AAA|}{|A| |A||A|}$$

$$\overset{(4.1.9.1)}{\leq} K^3|A|.$$  

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Therefore we are done with the bound (4.1.9a). √

By induction for \( n \geq 3 \) we get from Lemma 4.1.1 that,

\[
|A^{[n+1]}| \overset{\text{(4.1.1a)}}{\leq} \frac{1}{|A|} |A^{[n-1]}A||A^{-1}A^{[2]}| \\
\leq \frac{|A^{[n]}| |A^{[3]}|}{|A|} |A| \\
\leq K^{n-1}|A|.
\]

so we are done with (4.1.9b). √

If we combine (4.1.9a) and (4.1.9b) we get for any \( n \geq 3 \),

\[
|A^{(3)}| \leq K|A| \\
\implies |A^{[3]}| < (2K)^3|A| \\
\implies |A^{[n]}| \leq (2K)^{3(n-2)}|A| \\
\implies |A^{[n]}| < (2K)^{3n}|A|.
\]

Therefore by negating the inequalities we get,

\[
|A^{[n]}| \geq K|A| \implies |A^{(3)}| > \frac{1}{2} K^{1/3n}|A|
\]

so we are done with (4.1.9c). √

\[\square\]

### 4.2 Expansion properties in fields

When dealing with fields one can use the following Sum-Product theorem (cf. [TV, §2.8]) which is a slight improvement of [BKT, BK].

**Theorem 4.2.1.** ([TV, Theorem 2.52]) *There exists an absolute \( C > 0 \) such that the following holds for any \( 1 \leq K \in \mathbb{R} \) and any field \( \mathbb{F} \). Let \( A \subseteq \mathbb{F} \) be a finite subset and suppose*

\[
|A + A| + |A \cdot A| \leq K|A|.
\]
Then either $|A| < CK^C$ or for some subfield $E \leq F$ and $x \in F^\times$ we have,

$$|E| \leq CK^C|A| \quad \text{and} \quad |A \setminus xE| \leq CK^C.$$  

The power of this quantitative theorem is that if a set is almost stable under the two field’s operations then as a set it is almost a field, up to a polynomial lost. We will be interested in subsets with large growth:

$$\max \{|A + A|, |A \cdot A|\} \sim |A + A| + |A \cdot A| \gg |A|^{1+\varepsilon}.$$  

Therefore we will use the following definition.

**Definition 4.2.2** (Almost fields). Let $F$ be a field and let $A \subseteq F$ be a finite subset and let $\varepsilon \in \mathbb{R}_+$. We will say that $A$ is $\varepsilon$-almost field, or $\varepsilon$-field for short, if for some subfield $E \leq F$ and $x \in F^\times$ we have,

$$|E| \leq |A|^{1+\varepsilon} \quad \text{and} \quad |A \setminus xE| \leq |A|^{\varepsilon}. \quad (4.2.2a)$$  

If the above holds then we will say that that $A$ is $\varepsilon$-field $E$. Define $A$ to be pure $\varepsilon$-field if

$$|E| \leq |A|^{1+\varepsilon} \quad \text{and} \quad A \subseteq E. \quad (4.2.2b)$$  

If (4.2.2a) holds but (4.2.2b) does not hold then we will say that $A$ is an impure $\varepsilon$-field. In other words, $A$ is impure $\varepsilon$-field if (4.2.2a) holds and also

$$|A \setminus E| > 0. \quad (4.2.2c)$$

**Definition 4.2.3** (Almost stable subsets). Let $F$ be a field, $A \subseteq F$ be a finite set and let $\varepsilon \in \mathbb{R}_+$. We will say that $A$ is $\varepsilon$-close, or $\varepsilon$-stable, if

$$|A \cdot A| + |A + A| \leq |A|^{1+\varepsilon}. \quad (4.2.3a)$$  

Otherwise, we will say that $A$ has $\varepsilon$-expansion, or $\varepsilon$-growth.
Let’s restate Theorem 4.2.1 using this terminology.

**Theorem 4.2.4.** There exists $C > 0$ such that the following holds for any $\varepsilon \in \mathbb{R}_+$ with $\varepsilon < \frac{1}{C}$. Let $\mathbb{F}$ be a field and let $A \subseteq \mathbb{F}$ be a finite subset of size $|A| > C^{1/\varepsilon}$. Then we have,

\begin{align*}
A \text{ is } \varepsilon\text{-field} & \implies A \text{ is } C\varepsilon\text{-stable.} \quad (4.2.4a) \\
A \text{ is } \varepsilon\text{-stable} & \implies A \text{ is } C\varepsilon\text{-field.} \quad (4.2.4b)
\end{align*}

**Remark.** The statement $(4.2.4a)$ is trivial, as we shall see in the proof below. The important part of the theorem is $(4.2.4b)$. The theorem can be stated as follows: For any $\varepsilon > 0$ which is small enough, if $A$ is big enough (depending on $\varepsilon$), both $(4.2.4a)$ and $(4.2.4b)$ hold.

**Proof.** Suppose $A$ is $\varepsilon$-field. Therefore by 4.2.2 we get,

\[ |E| \leq |A|^{1+\varepsilon} \quad \text{and} \quad |A \setminus xE| \leq |A|^{\varepsilon}. \]

Denote $X := A \setminus xE$ and so we get,

\begin{align*}
|A + A| & \leq |(xE \cup X) + (xE \cup X)| \\
& \leq |E| + |E||X| + |X|^2 \\
& \leq 3|A|^{1+\varepsilon}
\end{align*}

and similarly the same bound for $|A \cdot A|$. Therefore if $|A|^{\varepsilon} \geq 6$ we get

\begin{align*}
|A + A| + |A \cdot A| & \leq 6|A|^{1+\varepsilon} \\
& \leq |A|^{1+2\varepsilon}
\end{align*}

so we are done with $(4.2.4a)$.$\square$
Now suppose $A$ is $\varepsilon$-stable. Denote $K := |A|^\varepsilon$ so by 4.2.3 we get,

$$|A \cdot A| + |A + A| \leq K|A|.$$ 

Therefore by Theorem 4.2.1 the following holds for some absolute\textsuperscript{2} $C_1 > 0$.

Either

$$|A| < C_1 K^{C_1} \quad (4.2.4.1)$$

or for some subfield $E \leq \mathbb{F}$ and $x \in \mathbb{F}^\times$ we have,

$$|E| \leq C_1 K^{C_1} |A| \quad \text{and} \quad |A \setminus xE| \leq C_1 K^{C_1}. \quad (4.2.4.2)$$

Therefore if $\varepsilon$ is small enough, say $\varepsilon < \frac{1}{2C_1}$, and $|A|^\varepsilon$ is big enough, say $|A|^\varepsilon > C_1$, then

$$C_1 K^{C_1} = C_1 |A|^{C_1 \varepsilon} < |A|^{2C_1 \varepsilon} < |A|.$$ 

Therefore (4.2.4.1) does not hold and from (4.2.4.2) we get that $A$ is $2C_1 \varepsilon$-field, so we are done with (4.2.4b). \hfill \square

We can state the non trivial part of theorem (4.2.4b) in the $\Omega$-language as follows:

**Corollary 4.2.5.** There exists $C > 0$ such that for any $\varepsilon > 0$ small enough the following hold for any finite subset $A \subseteq \mathbb{F}$ which is big enough.

$$A \text{ is not } \varepsilon\text{-field} \quad \implies \quad A \text{ has } \Omega(\varepsilon)\text{-growth.}$$

### 4.3 Expansion functions in fields

We begin by introducing some new notations.

\textsuperscript{2}The constant $C > 0$ from Theorem 4.2.1 is absolute and do not depend in $\varepsilon$.
**Definition 4.3.1.** Let $\mathbb{F}$ be a field and let $g \in \text{GL}_n(\mathbb{F})$. Define,

$$\text{Tr}_g(A, B) := \text{Tr}(AB^g)$$

for any $A, B \in M_n(\mathbb{F})$ and denote for $V \subseteq M_n(\mathbb{F}),$

$$\text{Tr}_g(V) := \{\text{Tr}(AA^g) : A \in V\}.$$  

**Definition 4.3.2.** Let $\mathbb{F}$ be a field, $x, y \in \mathbb{F}^\times$ and $g \in \text{SL}_2(\mathbb{F})$. Define

$$\text{tr} : \mathbb{F}^\times \to \mathbb{F}$$
$$\text{tr}_g : \mathbb{F}^\times \times \mathbb{F}^\times \to \mathbb{F}$$

by

$$\text{tr}(x) := \text{Tr}(D_x)$$
$$\text{tr}_g(x, y) := \text{Tr}(D_x(D_y)^g).$$

Extend these definitions to $\text{tr}(X)$ and $\text{tr}_g(X, Y)$ for subsets $X, Y \subseteq \mathbb{F}^\times$.

We immediately get the following equivalent definition.

**Simple Fact 4.3.3.** Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{F})$ and $x, y \in \mathbb{F}^\times$. Then we have,

$$\text{tr}(x) = x + x^{-1}$$
$$\text{tr}_g(x, y) = ad \cdot \text{tr}(xy) - bc \cdot \text{tr}(x/y).$$

**Definition 4.3.4.** Let $x, y \in \mathbb{F}^\times$ and let $t \in \mathbb{F}$. Define,

$$\text{tr}_t(x, y) := t \cdot \text{tr}(xy) + (1 - t) \cdot \text{tr}(x/y)$$

As a consequence of 4.3.3 and 4.3.4, we immediately deduce the following.
Simple Fact 4.3.5. Let \( x, y \in \mathbb{F}^\times \) and \( g \in \text{SL}_2(\mathbb{F}) \) with \( t = \text{Prod}(g) \). Then we have,

\[
\text{Tr}(D_x(D_y)^g) = \text{tr}_g(x, y) = \text{tr}_t(x, y) = t \cdot \text{tr}(xy) + (1 - t) \cdot \text{tr}(x/y).
\]

(4.3.5a)

Remark. In particular from (4.3.5a), we get that

\[
\text{Prod}(g) = 1 \Rightarrow \text{tr}_g(x, y) = \text{tr}(xy)
\]

\[
\text{Prod}(g) = 0 \Rightarrow \text{tr}_g(x, y) = \text{tr}(x/y).
\]

Note that

\[
\text{Prod}(g) = 1 \iff g \text{ is triangular}
\]

i.e., \( g \) is either upper triangular or lower triangular.

We make the following easy observations in any field \( \mathbb{F} \).

Simple Fact 4.3.6. Let \( \mathbb{F} \) be a field and let \( G = \mathbb{F}^\times \) be its multiplicative group. Let \( x, y \in \mathbb{F}^\times \) and \( X, Y \subseteq \mathbb{F}^\times \). Then we have,

\[
\text{tr}(x) \text{tr}(y) = \text{tr}(xy) + \text{tr}(xy^{-1})
\]

(4.3.6a)

and therefore,

\[
\text{tr}(X) \text{tr}(Y) \subseteq \text{tr}(XY) + \text{tr}(XY^{-1}).
\]

and in particular,

\[
\text{tr}(X) \text{tr}(X) \subseteq \text{tr}(X^2) + \text{tr}(X^2).
\]

(4.3.6b)
Proof. (4.3.6a) is trivial from the definition of $\text{tr}(x) = x + x^{-1}$. The two other equations immediately follow from (4.3.6a).

The following striking reduction of Helfgott\(^3\) allows one to gain large expansion from the non commutativity in the group by twisting properly some commutative sets (cf. [He, §3] and [BG2, §4]).

**Theorem 4.3.7 (Helfgott).** There exists $C > 0$ such that the following holds for any field $\mathbb{F}$. Let $X \subseteq \mathbb{F}^\times$ be a finite subset and suppose

$$|\{a_1 \cdot \text{tr}(xy) + a_2 \cdot \text{tr}(xy^{-1}) : x, y \in X^{|d|}\}| < K|\text{tr}(X)|.$$  \hspace{1cm} (4.3.7a)

for some $1 \leq K \in \mathbb{R}$ and $a_1, a_2 \in \mathbb{F}^\times$.

Then we have,

$$|\text{tr}(X^2) \text{tr}(X^2)| + |\text{tr}(X^2) + \text{tr}(X^2)| < CK|\text{tr}(X)|.$$  \hspace{1cm} (4.3.7b)

Let $V \subseteq \text{SL}_2(\mathbb{F})$ be a finite subset of diagonal matrices and suppose

$$|\text{Tr}(V^{|d|}.V^{|g|})| < |\text{Tr}(V)|^{1+\varepsilon}$$  \hspace{1cm} (4.3.7c)

for some $g \in \text{SL}_2(\mathbb{F})$ with\(^4\) $\text{Prod}(g) \notin \{0, 1\}$ and some $\varepsilon \in \mathbb{R}_+$.

Then we have,

$$|\text{Tr}(V^2).\text{Tr}(V^2)| + |\text{Tr}(V^2) + \text{Tr}(V^2)| < C|\text{Tr}(V^2)|^{1+C\varepsilon}.$$  \hspace{1cm} (4.3.7d)

**Proof.** Denote $N := |\text{tr}(X)|$ and for $x, y \in \mathbb{F}^\times$ denote

$$\text{tr}_{(a_1,a_2)}(x,y) := a_1 \cdot \text{tr}(xy) + a_2 \cdot \text{tr}(xy^{-1}).$$

\(^3\)The following proof is due to Helfgott and is different from his original proof.

\(^4\)i.e., $g$ has no zero entries.
By the assumption (4.3.7a) we get,

\[ |\text{tr}_{(a_1, a_2)}(X^{[4]}, X^{[4]})| = |\text{tr}_{(1, a_1/a_2)}(X^{[4]}, X^{[4]})| < K|\text{tr}(X)| = KN. \]  

(4.3.7.1)

Now for any subset \( Y \subseteq \mathbb{F}^x \) we get the following. For any \( z, w \in Y \) we have \( x := zw, y := zw^{-1} \in Y^{[2]} \) and \( t := xy = z^2, s := xy^{-1} = w^2 \in Y^2 \). Therefore we get,

\[ \{(t, s) : t, s \in Y^2\} \subseteq \{(xy, xy^{-1}) : x, y \in Y^{[2]}\}. \]  

(4.3.7.2)

Now set \( Y := X^{[2]} \) which satisfy \( X^{[2]} = Y^2 \subseteq Y^{[2]} = Y^{[4]} = X^{[4]} \).

Therefore by (4.3.7.1) and (4.3.7.2) we get,

\[ |\{\text{tr}(t) + a \cdot \text{tr}(s) : t, s \in Y^2\}| < KN. \]

Denote \( Z := Y^2 = X^{[2]} \) so we get \( |\text{tr}(Z) + a \cdot \text{tr}(Z)| < K N. \)

Since \( \text{mult}(\text{tr}(x^2)) \leq 4 \), we have

\[ N = |\text{tr}(X)| \leq |X| \leq |Y^2| \leq 4|\text{tr}(Y^2)| = 4|\text{tr}(Z)|. \]

Therefore \( |\text{tr}(Z)| \leq |\text{tr}(Z) + a \cdot \text{tr}(Z)| < K N \leq 4K|\text{tr}(Z)|. \)  

(4.3.7.3)

Therefore, by Plünnecke-Ruzsa (4.1.8a) with \( A = B = \text{tr}(Z) = \text{tr}(X^{[2]}). \)
we get

\[
| \text{tr}(X^2) + \text{tr}(X^2) | \leq | \text{tr}(X^{2[2]} + \text{tr}(X^{[2]2}) | \\
< 4^2 K^2 | \text{tr}(X^{[2]2}) | \\
= 2^4 K^2 | \text{tr}(Z) |
\]

(4.3.7.4)

Now by fact 4.3.6 applied to \( W = X^2 \) we get that

\[
| \text{tr}(X^2) \cdot \text{tr}(X^2) | \leq | \text{tr}(X^{[2]2}) + \text{tr}(X^{[2]2}) |.
\]

But since \( X \subseteq F \) we have \( Z = X^{[2][2]} = X^{[2]2} \) we get by (4.3.7.4) that

\[
| \text{tr}(X^2) \cdot \text{tr}(X^2) | \leq | \text{tr}(X^{[2]2}) + \text{tr}(X^{[2]2}) | \\
= | \text{tr}(X^{[2]2}) + \text{tr}(X^{[2]2}) | \\
\leq 2^4 K^2 | \text{tr}(Z) |.
\]

(4.3.7.5)

Therefore by combing (4.3.7.5) and (4.3.7.4) we get

\[
| \text{tr}(X^2) + \text{tr}(X^2) | + | \text{tr}(X^2) \cdot \text{tr}(X^2) | \leq 2^5 K^2 | \text{tr}(Z) | \\
\leq 2^5 K^3 | \text{tr}(X) | \\
\leq 2^5 K^3 N
\]

(4.3.7.3)

so we are done with (4.3.7b).

Set \( X := \{ x \in F : D_x \in V \} \) (i.e., \( V = D_X \)). By the assumption (4.3.7c) and by fact 4.3.5 we get

\[
| \{ ad \cdot \text{tr}(xy) - bc \cdot \text{tr}(xy^{-1}) : x, y \in X^{[4]} \} | \\
(4.3.5a) = | \text{Tr}(V^{[4]}V^{[4]}^T) | \\
(4.3.7c) \leq | \text{Tr}(V) |^{1+\epsilon} \\
= | \text{tr}(X) |^{1+\epsilon}
\]

Therefore by (4.3.7b), we have,

\[
| \text{tr}(X^2) \text{tr}(X^2) | + | \text{tr}(X^2) + \text{tr}(X^2) | \ll | \text{tr}(X) |^{1+O(\epsilon)}.
\]

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In other words we have,
\[ |\text{Tr}(V^2) \text{Tr}(V^2)| + |\text{Tr}(V^2) + \text{Tr}(V^2)| \ll |\text{Tr}(V)|^{1+O(\epsilon)} \]
so we are done with (4.3.7d). √

Now let us see some very simple observations that we will use later.

**Lemma 4.3.8.** There exists \( c > 0 \) such that the following holds. Let \( \mathbb{F} \) be a field and let \( g \in \text{SL}_2(\mathbb{F}) \). Let \( V \subseteq \text{SL}_2(\mathbb{F}) \) be a finite subset of diagonal matrices. Suppose \( \text{Tr}(V) \subseteq \mathbb{E} \) for some subfield \( \mathbb{E} \leq \mathbb{F} \).

If \( \text{Prod}(g) \notin \mathbb{E} \) then we have,
\[ |\text{Tr}(V^{[4]}V^{[4]}g)| > c|\text{Tr}(V)|^2. \] (4.3.8a)

If \( \text{Prod}(g) \neq 1 \) then we have,
\[ |\text{Tr}([V, g])| > c|\text{Tr}(V)|. \] (4.3.8b)

**Proof.** Denote \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Set \( X := \{ x \in \mathbb{F} : D_x \in V \} \) and set
\[ T = \text{Tr}(V^{[4]}V^{[4]}g) \]
\[ \text{by (4.3.5a)} \]
\[ = \{ ad \cdot \text{tr}(xy) - bc \cdot \text{tr}(x/y) : x, y \in X^{[4]} \}. \]

Therefore we get,
\[ T' := \{ ad \cdot \text{tr}(t) - bc \cdot \text{tr}(s) : t, s \in X^{[2]} \} \subseteq T. \]

Set \( f(z, w) := ad \cdot z + (1 - ad) \cdot w \) and since \( ad - bc = 1 \) we get
\[ T' = f(\text{tr}(X^{[2]}), \text{tr}(X^{[2]})). \]
Now if \( \text{Prod}(g) = \text{ad} \notin \mathbb{E} \) then \( f|_{\mathbb{E} \times \mathbb{E}} \) is injective. Indeed: if we set \( t = \text{ad} \) then by solving \( tz + (1-t)w = tz' + (1-t)w' \), we get that \( t(z-z') = (1-t)(w'-w) \). Since \( t \neq 0,1 \) we get that either \( z - z' = w' - w = 0 \) or \( \frac{1-t}{t} = t^{-1} - 1 \in \mathbb{E} \) which contradicts our assumption that that \( t = \text{ad} \notin \mathbb{E} \). Note that by the same way \( f|_{x\mathbb{E} \times x\mathbb{E}} \) is injective for any coset of \( \mathbb{E} \). By the assumption

\[
\text{tr}(X^{[2]^2}) \subseteq \text{tr}(X^{[4]}) = \text{Tr}(V^{[4]}) \subseteq \mathbb{E}
\]

therefore

\[
|T| \geq |T'| = |\text{tr}(X^{[2]^2})|^2 \geq |\text{tr}(X^2)|^2 \geq \left( \frac{1}{4} |\text{tr}(X)| \right)^2
\]

so we are done with (4.3.8a).√

Now if \( \text{Prod}(g) = \text{ad} \neq 1 \) then we get by fact 4.3.5 that,

\[
|\text{Tr}([V^{[4]},g])| = \left| \left\{ \text{Tr}(v^{-1}v^g) : v \in V^{[4]} \right\} \right| = \left| \left\{ 2\text{ad} + (1 - \text{ad}) \text{tr}(x^2) : x \in X^{[4]} \right\} \right|
\]

\[
= \left| \text{tr}(X^{[4]^2}) \right|
\]

\[
\geq \frac{1}{4} |X^{[4]}|
\]

so we are done with (4.3.8b).√

\[\square\]

**Simple Fact 4.3.9.** Let \( V \) and \( g \) be as in Lemma 4.3.8 and let \( x, y \in \mathbb{F}^\times \). Then we have,

\[
\text{tr}(xy) = \text{tr}(x/y) \iff \text{either } x^2 = 1 \text{ or } y^2 = 1 \iff \text{either } x = \pm 1 \text{ or } y = \pm 1.
\]

(4.3.9a)

If \( \text{Tr}(V^{[2]}) \subseteq \mathbb{E} \) and \( V \notin \{ \pm I \} \) then,

\[
\text{Prod}(g) \in \mathbb{E} \iff \text{Tr}(VV^g) \subseteq \mathbb{E}.
\]

(4.3.9b)
Proof. Note that
\[
\text{tr}(x) = 2 \iff x = 1
\]
and \(\text{tr}(x) = -2 \iff x = -1\). Moreover for any \(c \neq \pm 2\),
\[
|\text{tr}^{-1}(c)| = 2
\]
since \(\text{tr}(x) = \text{tr}(x^{-1})\) and \(x \neq x^{-1}\). Therefore
\[
\text{tr}(x) = \text{tr}(y) \iff x \in \{ y^{\pm 1} \}
\]
\[
\iff xy = 1 \quad \text{or} \quad x/y = 1.
\]

In particular (4.3.9a) follows.\(\sqrt{38}\)

By fact 4.3.5 we get that
\[
\text{Tr}(D_xD_y^g) \overset{(4.3.5a)}{=} \text{Prod}(g)(\text{tr}(xy) - \text{tr}(x/y)) + \text{tr}(x/y).
\]

Therefore if \(D_x \neq \pm I\) and \(D_y \neq \pm I\) and \(\text{tr}(xy), \text{tr}(x/y) \in \mathbb{E}\) then
\[
\text{Tr}(D_xD_y^g) \in \mathbb{E} \iff \text{Prod}(g) \in \mathbb{E}.
\]

Therefore we immediately get (4.3.9b).\(\sqrt{38}\)
Chapter 5

Useful properties of $\text{SL}_2(\mathbb{F})$

5.1 Bounded generation of large subsets

In the following section, we will prove few Growth properties of large subsets of finite (quasi) simple groups. First we give some background concerning the regular representation (and the convolution of functions). We will follow the techniques which were developed by Gowers (cf. [G]) and later were expanded by Babai, Nikolov and Pyber (cf. [BNP1, NP]).

The spectral decomposition

**Definition 5.1.1.** Let $G$ be a finite group. We identify the group ring $\mathbb{C}[G]$ with $\mathbb{C}^G$ so instead of writing $\sum a_g g \in \mathbb{C}[G]$ we write $X \in \mathbb{C}^G$ with $X(g) = a_g$ for all $g \in G$.

On the other hand we identify subsets $A \subseteq G$ as the indicators functions $1_A \in \mathbb{C}^G$ and similarly elements $g \in G$ as the indicators functions $1_g \in \mathbb{C}^G$.

In the algebra $\mathbb{C}[G]$ we have the usual inner product and convolution
product. For \(X, Y \in \mathbb{C}[G]\) we have

\[
\langle X, Y \rangle = \sum_{g} X(g)\overline{Y(g)}
\]

and the (convolution) product \(X \ast Y\), or for short just \(X.Y\), is defined by,

\[
(X.Y)(g) = \langle X.Y, g \rangle = \sum_{xy=g} X(x)Y(y).
\]

**Simple Fact 5.1.2.** Let \(G\) be a finite group and let \(X, Y, Z \in \mathbb{C}[G]\). Define \(X^T \in \mathbb{C}[G]\) by

\[
X^T(x) := X(x^{-1})
\]

and \(X^* \in \mathbb{C}[G]\) by

\[
X^*(x) = \overline{X(x^{-1})}.
\]

We will be interested mainly in functions in \(\mathbb{R}[G]\) so there will be no difference in these notations. Then we have,

\[
\langle X.Y, Z \rangle = \langle Y, X^*Z \rangle = \langle X, ZY^* \rangle.
\]

**Proof.** For any \(x, y, z \in G\) we have,

\[
\langle xy, z \rangle = \langle y, x^{-1}z \rangle = \langle x, zy^{-1} \rangle
\]

therefore by linearity we get,

\[
\langle X.Y, Z \rangle = \langle Y, X^*Z \rangle = \langle X, ZY^* \rangle.
\]

\[
\Box
\]

**Definition 5.1.3.** Let \(G\) be a finite group and let \(X, Y \in \mathbb{C}[G]\). Let \(L(\cdot)\) and \(R(\cdot)\) be the left and the right regular representations of \(G\),

\[
L(X)(Y) := X.Y
\]

\[
R(X)(Y) := Y.X^*.
\]
**Simple Fact 5.1.4.** Let $G$ be a finite group, $X, Y \in \mathbb{C}[G]$ and let $L(\cdot)$ and $R(\cdot)$ be the left and the right regular representations of $G$. Then we have,

$$
L(X.Y) = L(X)L(Y)
$$
$$
R(X.Y) = R(X)R(Y).
$$

Clearly $L(\cdot)$ and $R(\cdot)$ commutes,

$$
L(X)R(Y) = R(Y)L(X).
$$

Moreover, we have

$$
L(X)^* = L(X^*)
$$
$$
R(X)^* = R(X^*).
$$
i.e.,

$$
\langle X.u, v \rangle = \langle L(X)u, v \rangle = \langle u, L(X)^*v \rangle = \langle u, X^*.v \rangle
$$

for any $u, v \in \mathbb{C}[G]$ (and similarly for $R(X)$).

*Proof.* All follows immediately from the definitions of $L(\cdot)$ and $R(\cdot)$ in 5.1.3 and fact 5.1.2. \qed

**Simple Fact 5.1.5.** Let $V = \mathbb{C}[G]$ and denote by $U(V)$ the group of unitary transformations of $V$. Then $L(G)$ and $R(G)$, the left and the right regular representations, and also $X \mapsto X^T$, are all in $U(\mathbb{C}[G])$.

*Proof.* Clearly for any $g \in G$, $L(g)$ and $R(g)$ and $X \mapsto X^T$, are linear maps which permute the orthonormal basis $\{h : h \in G\}$. \qed

**Simple Fact 5.1.6.** Let $G$ be a finite group of size $N$ and $X \in \mathbb{C}[G]$. Then we have,

$$
X(1) = \frac{1}{N} \text{Tr}(L(X)) = \frac{1}{N} \text{Tr}(R(X^*)).
$$
Proof. For any $g \in G$ we have,

$$X(1) = \langle X.1, 1 \rangle = \langle X.g, g \rangle = \langle g.X, g \rangle$$

Therefore $L(X)$ and $R(X^*)$ have the same diagonal with respect to the orthonormal basis $\{g : g \in G\}$. \hfill \square

**Definition 5.1.7.** Let $G$ be a finite group of size $N$ and let $X \in \mathbb{C}[G]$. Denote,

$$\text{Tr}(X) := \text{Tr}(L(X)).$$

Therefore by 5.1.6 we get,

$$X(1) = \frac{1}{N} \text{Tr}(X).$$

**Simple Fact 5.1.8.** Let $G$ be a finite group of size $N$ and $X \in \mathbb{C}[G]$. Then,

$$\|X\|^2 = X^*X(1) = X.X^*(1)$$

and

$$\|X\|^2 = \frac{1}{N} \text{Tr}(X^*.X) = \frac{1}{N} \text{Tr}(X.X^*).$$

Proof. We have,

$$\|X\|^2 = \langle X.1, X.1 \rangle = X^*.X(1) = X.X^*(1)$$

Therefore by 5.1.7 with $X = X.X^*$ (and $X = X^*.X$) we are done. \hfill \square
Theorem 5.1.9 (SD\(^1\) of real symmetric endomorphism). Let \( G \) be a finite group of size \( N \) and let \( A \in \text{End}(\mathbb{R}[G]) \). Suppose \( A \) is be a symmetric endomorphism i.e.\(^2\), \( A = A^T \). Then there exist an orthonormal basis \( \mathbf{\alpha} = (\alpha_i) \) of \( \mathbb{R}[G] \), and \( \lambda_1 \geq \lambda_2 \ldots \geq \lambda_N \) in \( \mathbb{R} \) such that

\[
\langle A\alpha_i, \alpha_j \rangle = \delta_{ij} \lambda_i
\]  

(5.1.9a)

for any \( 1 \leq i, j \leq N \).

Proof. This is a standard theorem in linear algebra for symmetric matrix \( T \in M_n(\mathbb{R}) \). \( \Box \)

Corollary 5.1.10 (Rayley inequality). Let \( G \) be a finite group of size \( N \) and let \( A \in \text{End}(\mathbb{R}[G]) \) (not necessarily symmetric). Then there exist orthonormal basis \( \mathbf{\beta} \) of \( \mathbb{R}[G] \), and \( \lambda_1 \geq \lambda_2 \ldots \geq \lambda_N \geq 0 \) in \( \mathbb{R} \) such that

\[
\langle A\beta_i, A\beta_j \rangle = \delta_{ij} \lambda_i^2
\]  

(5.1.10a)

for any \( 1 \leq i, j \leq N \). Let \( 1 \leq k \leq N \) and suppose \( v \in \mathbb{C}[G] \) with \( v \perp \beta_i \) for all \( i < k \). Then we have,

\[
\|Av\| \leq \lambda_k \|v\|.
\]  

(5.1.10b)

Proof. Since \( AA^T, A^TA \in \text{End}(\mathbb{R}[G]) \) are symmetric we can decompose \( A^TA \) and \( AA^T \) by theorem 5.1.9. Moreover \( AA^T, A^TA \geq 0 \) (i.e., they are positive-semidefinite) therefore they have the same, non negative, eigen values. Therefore there exist orthonormal basis \( \mathbf{\beta} \) of \( \mathbb{R}[G] \), and \( \lambda_1 \geq \lambda_2 \ldots \geq \lambda_N \geq 0 \) in \( \mathbb{R} \) such that

\[
\langle A^TA\beta_i, \beta_j \rangle = \delta_{ij} \lambda_i^2
\]
for any $1 \leq i, j \leq N$. So we are done with (5.1.10a). √

Let $1 \leq k \leq N$ and $v \in \mathbb{C}[G]$ and suppose $v \perp \beta_i$ for all $i < k$. Then we have,

\[
\|Av\|^2 = \langle Av, Av \rangle = \langle A^T Av, v \rangle = \left\langle \sum_i \langle v, \beta_i \rangle A^T A \beta_i, \sum_j \langle v, \beta_j \rangle \beta_j \right\rangle = \sum_{1 \leq i, j \leq N} \langle A^T A \beta_i, \beta_j \rangle \langle v, \beta_i \rangle \langle v, \beta_j \rangle \tag{5.1.10a} \]

\[
\leq N \sum_{i=k}^N \lambda_i^2 \|v\|^2
\]

so we are done. □

**Definition 5.1.11.** Let $G$ be a finite group of size $N$ and let $A \in \text{End} (\mathbb{R}[G])$. By corollary 5.1.10 there exist orthonormal basis $\overline{\beta}$ of $\mathbb{R}[G]$, and $0 \leq \lambda_i \in \mathbb{R}$, in decreasing order, s.t.

\[
\langle A^T A \beta_i, \beta_j \rangle = \delta_{ij} \lambda_i^2.
\]

Denote $\lambda_i(A) := \lambda_i$ and by $m_i(A)$ the multiplicity of $\lambda_i(A)$. I.e.,

\[
m_i(A) := \dim(\text{Ker}(A^T A - \lambda_i^2 \text{Id})).
\]

Denote $\lambda(X) := \lambda_2(X)$ and $m(X) := m_2(X)$. 

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Definition 5.1.12. Let $G$ be a group and let $\mathbb{F}$ be a field and let $(\rho, V)$ be
finite dimensional representation of $G$. Denote the fix points of $(\rho, V)$ by
\[
\text{Fix}(\rho(G)) := \{ v \in V : \rho(g)v = v \text{ for any } g \in G \}
\]
\[
:= \bigcap_{g \in G} \text{Ker}(\rho(g) - \text{Id})
\]
and if the action is clear from the context we will abbreviate and write Fix$(G)$. We will say that $(\rho, V)$ is a trivial representation if \[ \text{Fix}(G) = V. \]

Definition 5.1.13. Let $G$ be a finite group and let $\mathbb{F}$ be a field. Define
\[
M(G, \mathbb{F}) := \min \{ \text{deg}(\rho) : \rho \text{ is a non-trivial irreducible } \mathbb{F}\text{-representation of } G \}.
\]
Since $M(G, \mathbb{C})$ and $M(G, \mathbb{R})$ will be more relevant for our purposes when
investigating finite groups, we abbreviate
\[
M(G) := M(G, \mathbb{R})
\]
the minimal degree of non-trivial real representation of it.

Definition 5.1.14. Denote by Prob[$G$] the elements $X \in \mathbb{R}[G]$ with $X(g) \geq 0$ for any $g \in G$ and with $\|X\|_1 = 1$. Denote by $U_X$ the uniform probability on the support of $X$. I.e., if $A = \text{supp}(X)$ then $U_X = \frac{1}{|A|} 1_A$. Denote by $U = U_G \equiv \frac{1}{N}$ the uniform probability on $G$.

Simple Fact 5.1.15. Let $G$ be a group of size $N$ and let $Y \in \text{Prob}[G]$. Then we have,
\[
\|Y - U\|^2 = \|Y\|^2 - \frac{1}{N}.
\]
In particular

\[ \|Y\|^2 \geq \frac{1}{N} \]

with equality if and only if \( Y = U \). Moreover

\[ \|Y\|^2 \geq \frac{1}{|\text{supp}(Y)|} \]

with equality if and only if \( Y = U_Y \).

**Proof.** Since \( Y - U \perp U \) and \( Y - U_Y \perp U_Y \) we get

\[
\|Y\|^2 = \|Y - U\|^2 + \|U\|^2 = \|Y - U_Y\|^2 + \|U_Y\|^2
\]

therefore the claim follows. \( \square \)

**Proposition 5.1.16** (Young inequality). Let \( 1 \leq r, p, q \leq \infty \) and suppose \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \). Let \( G \) be a finite group and let \( X, Y \in \mathbb{C}[G] \). Then we have,

\[ \|X.Y\|_r \leq \|X\|_p \|Y\|_q. \quad (5.1.16a) \]

We will call such a triple \((r, p, q)\) a **Young triple**.

**Definition 5.1.17.** Let \( G \) be a finite group and \( A \in \text{End}(\mathbb{C}[G]) \). For any \( p, q \geq 1 \) denote the **operator norm** \( \|A\|_{p,q} \) by

\[
\|A\|_{p,q} = \max_{v \neq 0} \frac{\|A(v)\|_p}{\|v\|_q} = \max_{\|v\|_q = 1} \|A(v)\|_p
\]

Denote by \( \Lambda(A) \) the **spectrum** of \( A \) and by \( \rho(A) \) the **spectral radius** of \( A \).
Simple Fact 5.1.18. For any $X \in \mathbb{R}[G]$, the operators

$$L(X), L(X^T), L(X^*), R(X), R(X^T), R(X^*)$$

have the same spectral radius, the same spectrum and the same operator norms. Therefore we write for short

$$\|X\|_{p,q} = \|L(X)\|_{p,q}$$

and

$$\rho(X) = \rho(L(X))$$

and the spectrum of $L(X)$ by

$$\Lambda(X) = \Lambda(L(X)).$$

In particular, for any Young triple $(r, p, q)$ and $X \in \mathbb{C}[G]$ we get

$$\rho(X) \leq \|X\|_{r,p} \leq \|X\|_{q}.$$  

Simple Fact 5.1.19. Let $G$ be a group of size $N$ and let $X, Y \in \text{Prob}[G]$. Then we have,

$$\|X.Y - U\| \leq \lambda(X) \|Y - U\|$$

$$\|X.Y - U\| \leq \lambda(Y) \|X - U\|$$

(5.1.19a)

Proof. On the one hand

$$U.g = g.U = U$$

for any $g \in G$ so we get that

$$X.U = U.X = U$$

and so $\lambda_1(X) \geq 1$. 

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On the other hand by Young inequality 5.1.16 with \((r, p, q) = (2, 1, 2)\) we get, \(\|X.Y\| \leq \|Y\|\) for any \(Y \in \mathbb{C}[G]\) i.e.,

\[
\rho(X) \leq \|X\|_{2,2} \leq \|X\|_1 = 1,
\]

therefore

\[
\lambda_1(X) = 1,
\]

and by corollary 5.1.10, without loss of generality we can assume \(\beta_1 \equiv \frac{1}{\sqrt{N}}\).

Now since \(X \perp Y - U\), we get by Rayley inequality 5.1.10 that

\[
\|X.Y - U\| = \|X.(Y - U)\| \leq \lambda_2(X) \|Y - U\|
\]

so we are done with the first inequality of (5.1.19a).

Now since

\[
\|X.Y - U\| = \|Y^T.X^T - U\|
\]

we can apply the first bound with \(X = Y^T\) and \(Y = X^T\), so we are done with the second inequality of (5.1.19a).

**Simple Fact 5.1.20.** Let \(G\) be a finite group. Then

\[
\text{Fix}(L(G)) = \text{Fix}(R(G)) = \text{span}(U).
\]

In other words for any linear subspace \(0 \neq W \leq \mathbb{C}[G]\) we have,

\[
G.W \subseteq W \\
\iff \ W.G \subseteq W \\
\iff \ W = \text{span}(U).
\]
Proof. Since for any $g, h \in G$ we have

$$
X(hg^{-1}) = \langle X, hg^{-1} \rangle = g.X(h) = X.g(h)
$$

we get that

$$
X \in \text{Fix}(G) \iff X \equiv X(1).
$$

**Simple Fact 5.1.21.** Let $G$ be a finite group of size $N$ with $M = M(G)$ and let $X \in \text{Prob}[G]$. Then for any $1 < i \leq N$ we have,

$$
m_i(X) \geq M.
$$

Proof. Set $A := L(X^T.X)$ and for any $1 \leq i \leq N$ set

$$
A_i := A - \lambda_i^2 \text{Id}
$$

$$
V_i := \text{Ker}(A_i).
$$

Since $A_i$ commutes with all the elements of $R(G)$ and $A_i \in \text{End}(\mathbb{R}[G])$ we get that $V_i$ is a real representation of $G$ (with the right action of $G$ on $V_i$).

If $i \neq 1$ then $\beta_i \perp U$ and so by 5.1.20, $V_i$ is non trivial real representation of $G$ so

$$
m_i = \dim(V_i) \geq M.
$$

Remark. Note that there is no a priori assumption that $\lambda_2(X) \neq 1$. Actually if $\lambda_2(X) = 1$ then by the same argument we get that

$$
m(X) = m_2(X) \geq M + 1.
$$
Simple Fact 5.1.22. Let $G$ be a group of size $N$ and let $M = M(G)$. Then for any $Y \in \text{Prob}[G]$ we have,

$$\lambda(Y) \leq \sqrt{\frac{N}{M}} \|Y - U\|.$$  

(5.1.22a)

**Proof.** Since $m_2(Y) \geq M$ and $Y - U \perp U$ we get

$$\|Y - U\|^2 = \|Y\|^2 - \|U\|^2 = \frac{1}{N}(\text{Tr}(Y^T.Y) - 1) = \frac{1}{N} \sum_{i=2}^{N} \lambda_i^2(Y) \geq \frac{M}{N} \lambda^2(Y).$$  

□

**Corollaries**

The following Corollary is a slight modification of an argument of [NP, BNP1, BNP2] (which followed and extended results of [G]).

**Corollary 5.1.23. ([BNP2, Theorems 2.1 and Corollary 2.2])** Let $G$ be a group of size $N$ with $M = M(G)$ and let $X, Y \in \text{Prob}[G]$. Then we have,

$$\|X.Y - U\| \leq \sqrt{\frac{N}{M}} \|Y - U\| \|X - U\|.$$  

(5.1.23a)

Inductively we get for any $n \in \mathbb{N}$ and $X_1, \ldots, X_{n+1} \in \text{Prob}[G],$

$$\|X_1 \ldots X_{n+1} - U\| \leq \left(\frac{N}{M}\right)^{n/2} \prod_{i=1}^{n+1} \|X_i - U\|.$$  

(5.1.23b)

**Proof.** By facts 5.1.19 and 5.1.22 we get,

$$\|X.Y - U\|^2 \leq \lambda(X) \lambda(Y) \|Y - U\| \|X - U\| \leq \frac{N}{M} \|Y - U\|^2 \|X - U\|^2$$  

□
Corollary 5.1.24. ([BNP2, Corollary 2.3]) Let \( G \) be a group of size \( N \) with \( M = M(G) \) and let \( X, Y, Z \in \text{Prob}[G] \). Then we have,

\[
\|X.Y.Z - U\|_\infty < \frac{\sqrt{N}}{M} \|X\| \|Y\| \|Z\|. \tag{5.1.24a}
\]

Inductively we get for any \( n \in \mathbb{N} \) and \( X_1, \ldots, X_{n+2} \in \text{Prob}[G] \) that,

\[
\|X_1 \ldots X_{n+2} - U\|_\infty < \left(\frac{N}{M}\right)^{n/2} \prod_{i=1}^{n+2} \|X_i\|. \tag{5.1.24b}
\]

Proof. By proposition 5.1.16 with \((r, p, q) = (\infty, 2, 2)\) and corollary 5.1.23 we get,

\[
\|X.Y.Z - U\|_\infty = \|(X.Y - U).Z\|_\infty \\
\overset{(5.1.16a)}{\leq} \|X.Y - U\| \|Z\| \\
\overset{(5.1.23a)}{\leq} \frac{\sqrt{N}}{M} \|X - U\| \|Y - U\| \|Z\| \\
< \frac{\sqrt{N}}{M} \|X\| \|Y\| \|Z\|. \tag*{\qed}
\]

Now let us the implications of the properties above (cf. [BNP2, Corollaries 2.5 and 2.6 and Theorem 2.14]).

Theorem 5.1.25 (Babai-Nikolov-Pyber). Let \( G \) be a finite group of size \( N \) with \( M = M(G) \). Let \( A_1, \ldots, A_t \subseteq G \) be subsets of size \( |A_i| = K_i \frac{N}{M} \) where \( K_i \in \mathbb{R}_+ \). Then we have,

\[
|A_1 A_2| > \frac{1}{2} \min\{K_1 K_2 \frac{N}{M}, N\} \tag{5.1.25a}
\]

and if \( t \geq 3 \) then we have\(^3\),

\[
\prod_{i=1}^{t} K_i \geq M^2 \implies \prod_{i=1}^{t} A_i = G. \tag{5.1.25b}
\]

\(^3\)The case \( t = 3 \) was proved in [G, Theorem 3.3].
Proof. For any $1 \leq i \leq t$ set $X_i \in \text{Prob}[G]$ by

$$X_i := U_{A_i} = \frac{1}{|A_i|} 1_{A_i}.$$ 

Since $^4$

$$\text{supp}(X_1 \ldots X_t) = \text{supp}(X_1) \cdots \text{supp}(X_t),$$

we get by corollary 5.1.23 that,

$$\frac{1}{|A_1 A_2|} = \frac{1}{|\text{supp}(X_1 X_2)|} \leq \|X_1 X_2\|^2$$

$$= \|X_1 X_2 - U\|^2 + \|U\|^2$$

$^{(5.1.23a)}$

$$\leq \frac{N}{M} \|X_1 - U\|^2 \|X_2 - U\|^2 + \frac{1}{N}$$

$$< \frac{N}{M} \|X_1\|^2 \|X_2\|^2 + \frac{1}{N}$$

$$= \frac{N}{M} \left( \frac{1}{|A_1| |A_2|} + \frac{1}{N} \right)$$

$$= \frac{M}{N} \left( \frac{1}{K_1 K_2} + \frac{1}{M} \right)$$

$$\leq 2 \frac{M}{N} \max \left\{ \frac{1}{K_1 K_2}, \frac{1}{M} \right\}$$

Therefore by rearranging the inequalities we are done with (5.1.25a).

Now by corollary 5.1.24 we get,

$$\|X_1 X_2 X_3 - U\|_\infty \overset{(5.1.24a)}{<} \sqrt{\frac{N}{M}} \|X_1\| \|X_2\| \|X_3\|$$

$$= \sqrt{\frac{N}{M}} (|A_1| |A_2| |A_3|)^{-1/2}$$

$$= \frac{M}{N} (K_1 K_2 K_3)^{-1/2}.$$

$^4$One can denote the convolution either as $X_1 \ast \ldots \ast X_t$ or $X_1 \ldots X_t$ or just by $X_1 \cdots X_t$ since this is the product in the algebra $\mathbb{C}[G]$. We use in this manuscript the middle way.
Therefore if $K_1 K_2 K_3 \geq M^2$ then $\|X_1 . X_2 . X_3 - U\|_\infty < \frac{1}{N}$ so

\[ A_1 \cdot A_2 \cdot A_3 = \text{supp}(X_1 . X_2 . X_3) = G \]

so we are done with (5.1.25b) for $t = 3$.

Similarly by corollary 5.1.24 we get for any $n \in \mathbb{N}$,

\[
\|X_1 \ldots X_{n+2} - U\|_\infty < \left( \frac{N}{M} \right)^{n/2} \prod_{i=1}^{n+2} \|X_i\|
\]

\[ = \frac{M^{n+2}}{N} \left( \prod_{i=1}^{n+2} K_i \right)^{-1/2} \]

so we are done with (5.1.25b) for $t \geq 3$. \qed

As a special case of the previous we get immediately the following Corollary.

**Corollary 5.1.26.** ([BNP2, Corollary 2.11]) Let $G$ be a finite group of size $N$ with $M = M(G)$. Let $A \subseteq G$ be a subset of size $|A| = K \frac{N}{M}$. Then we have,

\[ |A^{(2)}| > \frac{1}{2} \min\{N, K|A|\} \].

And for any $t \geq 3$ we have,

\[ |K| \geq M^{2t} \implies A^{(t)} = G. \]

**Theorem 5.1.27.** There exist $C \in \mathbb{R}_+$ such that the following holds. Let $\mathbb{F}_q$ be a finite field and let $A$ be a subset of $G = \text{SL}_2(\mathbb{F}_q)$. Then we have,

\[ |A| > C q^{2 \frac{2}{3}} \Rightarrow A^{(3)} = \text{SL}_2(\mathbb{F}_q). \]  \hfill (5.1.27a)

For any $3 \leq m \in \mathbb{N}$ we have,

\[ |A| > C q^{2 \frac{2}{3}} \Rightarrow A^{(m)} = \text{SL}_2(\mathbb{F}_q). \]  \hfill (5.1.27b)
For any $0 < \delta \leq \frac{1}{2}$ we have,

$$|A| > q^{2+\delta} \Rightarrow |A^{(2)}| > \frac{1}{C} q^{2+2\delta}.$$ (5.1.27c)

**Proof.** By a well known fact (which was first proved by Frobenius) for any finite field $\mathbb{F}_q$ and $G = \text{SL}_2(\mathbb{F}_q)$ we have,

$$M(G, \mathbb{R}) = \frac{1}{2}(q - 1) \geq \frac{1}{2}q(1 - o(1)) \gg q.$$  

Therefore if $N = |G| = q(q^2 - 1)$ and $M = M(G)$ then

$$\frac{N}{M} = 2q(q + 1) \leq 2q^2(1 + o(1)) \ll q^2$$

and for any $m \geq 3$,

$$\frac{N}{M^{1-2/m}} \ll q^{2\frac{2}{m}}.$$  

Therefore the claim follows immediately by corollary 5.1.26. \(\Box\)

**Remark.** In particular the theorem guarantee bounded generation for any large subset $A$ of $G = \text{SL}_2(\mathbb{F}_q)$. In particular, any subgroup $H < G$ has large index

$$[G : H] \gg q.$$
5.2 Symbolic generation of traces

The Invariant theory of tuples of matrices under various actions was developed over fields of zero characteristic. We will actually be interested in the positive characteristic (cf. [P], [CP], [Do]).

**Definition 5.2.1.** For \( m \geq 2 \) denote by \( R_{2,m} \) the ring of invariants of \( m \)-tuples of \( 2 \times 2 \) generic matrices \((X_1, \ldots, X_m)\) over a infinite field \( F \) under the simultaneous conjugation action of the general linear group. To be precise, we have \( 4m \) variables \( x_1, y_1, z_1, w_1, \ldots, x_m, y_m, z_m, w_m \) which we denote by \( \overline{X}_i = (x_i, y_i, z_i, w_i) \) and \( \overline{X} = (\overline{X}_1, \ldots, \overline{X}_m) \). Each matrix \( X_i = \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix} \) is a formal matrix with four variables \( \overline{X}_i \) for \( 1 \leq i \leq m \). We define an action of \( g \in \text{GL}_2(F) \) on \( f(X_1, \ldots, X_m) \in F[\overline{X}] \) by

\[
f^g(X_1, \ldots, X_m) := f(X_1^g, \ldots, X_m^g).
\]

We define the algebra of invariants of this polynomial ring under the action of \( \text{GL}_2(F) \) by

\[
R_{2,m}(F) := \{ f \in F[\overline{X}] : f^g = f \text{ for any } g \in \text{GL}_2(F) \}.
\]

We will use the following results of Procesi and Domokos-Kuzmin-Zubkov (cf. [P1] and [DKZ, §4]).

**Theorem 5.2.2.** ([DKZ, Corollary 4.1])

*If \( \text{char}(F) \neq 2 \) then,*

\[
\{ \text{det}(X_i), \text{tr}(X_{i_1} \cdots X_{i_s}) : 1 \leq i \leq m; 1 \leq s \leq 3; 1 \leq i_1 < \ldots < i_s \leq m \}
\]

*is a minimal system of generators of \( R_{2,m}(F) \).*

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If \( \text{char}(\mathbb{F}) = 2 \) then,

\[
\{\det(X_i), \text{tr}(X_{i_1} \cdots X_{i_s}) : 1 \leq i, s \leq m; 1 \leq i_1 < \ldots < i_s \leq m\}
\]

is a minimal system of generators of \( R_{2,m}(\mathbb{F}) \).

From this we get immediately the following result.

**Lemma 5.2.3 (Trace generation).** Let \( \mathbb{F} \) be a field and let \( A \subseteq \text{SL}_2(\mathbb{F}) \) be a subset of size \( 2 \leq |A| \leq m \). Then we have the ring generation,

\[
\langle \text{Tr}(A^{[m]}) \rangle = \langle \text{Tr}(\langle A \rangle) \rangle.
\]

Moreover if \( \text{char}(\mathbb{F}) \neq 2 \) then we have the ring generation,

\[
\langle \text{Tr}(A^{[3]}) \rangle = \langle \text{Tr}(\langle A \rangle) \rangle.
\]

In particular we get the following.

**Corollary 5.2.4.** Let \( \mathbb{F} \) be a finite field and let \( A \subseteq \text{SL}_2(\mathbb{F}) \) be a subset of size \( |A| \leq m \). Suppose \( \langle A \rangle = \text{SL}_2(\mathbb{F}) \). Then we have,

\[
\langle \text{Tr}(A^{[m]}) \rangle = \mathbb{F}
\]

and if \( \text{char}(\mathbb{F}) \neq 2 \)

\[
\langle \text{Tr}(A^{[3]}) \rangle = \mathbb{F}.
\]

The same assertion holds under the weaker assumption

\[
\langle \text{Tr}(\langle A \rangle) \rangle = \mathbb{F}.
\]

Similarly if \( \mathbb{E} \) is a subfield of \( \mathbb{F} \) then

\[
\langle \text{Tr}(\langle A \rangle) \rangle = \mathbb{E} \implies \langle \text{Tr}(A^{[m]}) \rangle = \mathbb{E}.
\]
Remark. There are various possible generations types, depending on the category of objects which are involved: groups, rings, algebras, vector spaces, modules and fields. In the invariant context, rings and groups operations are involved. E.g., generation as $\mathbb{F}_p$ vectors spaces is stronger then rings and for finite fields there is no difference between ring generation and field generation. Here the meaning is ring generation in the outer bracket and group generation in the internal bracket. Explicitly: $\langle \text{Tr}(A^{[m]}) \rangle_{\text{ring}} = \langle \text{Tr}(\langle A \rangle_{\text{group}}) \rangle_{\text{ring}}$.

5.3 Size of Minimal generating sets of $\text{PSL}_2(\mathbb{F}_q)$

By Lemma 5.2.3 we got that for any finite field $\mathbb{F} = \mathbb{F}_q$ with char($\mathbb{F}$) $\neq 2$ and any subset of generators $\langle A \rangle = \text{SL}_2(\mathbb{F}_q)$ we have a “Bounded Generation of Trace Generators” i.e.,

$$\langle \text{Tr}(A^{[n]}) \rangle = \mathbb{F}.$$ 

In this section we want to extend it to char($\mathbb{F}$) = 2 as well. The main theorem of this section, and the only part that we will use later, is Theorem 5.3.4 which asserts,

$$\langle \text{Tr}(A^{[6]}) \rangle = \mathbb{F}.$$ 

Definition 5.3.1. Let $G$ be a finitely generated group. Let us call a subset $A$ of a group $G$ a minimal generating set if $\langle A \rangle = G$ but for any proper subset $A' \subset A$ we have $\langle A' \rangle \neq G$. Let us call a subgroup $H$ of $\text{PSL}_2(\mathbb{F}_q)$ a subfield subgroup if $H \cong \text{PSL}_2(q')$ for some subfield $\mathbb{F}_{q'}$ of $\mathbb{F}_q$.

Saxl and Whiston proved the following result about the size of minimal generating sets of $\text{PSL}_2(\mathbb{F}_q)$ (cf. [SW, Theorem 3 and Theorem 7 with its proof]).
Theorem 5.3.2. ([SW, Theorems 3,7]) Let $G = \text{PSL}_2(\mathbb{F}_q)$ with $q = p^r$ a prime power and let $A = \{g_1, \ldots, g_m\}$ be a minimal set of generators of $G$.

If $r = 1$ then $|A| \leq 4$. If $r > 1$ let $r = p_1^{e_1} \cdots p_n^{e_n}$ be the prime decomposition of $r$ and let

$$A_i := A \setminus g_i \quad \text{and} \quad H_i := \langle A_i \rangle.$$

If $|A| > 6$ then up to some reordering of the $g_i$’s and the $p_j$’s one of the following hold.

1. For any $i \geq 3$, $H_i$ is a subfield subgroup and there exists a unique $j$ for which

$$H_i \leq G_j \cong \text{PSL}_2(p^{e_j/p_j}).$$

2. For any $i \geq 2$, $H_i$ is a subfield subgroup. For any $1 \leq j \leq n$, let $S_j$ be the set of subfield subgroups $H_i$ for which $j$ is minimal subject to $H_i \leq G_j \cong \text{PSL}_2(p^{e_j/p_j})$. Then $|S_1| \leq 2$ and $|S_j| \leq 1$ for any $j \geq 2$.

3. For any $i \geq 1$, $H_i$ is a subfield subgroup. For any $1 \leq j \leq n$, let $S_j$ be the set of subfield subgroups $H_i$ for which $j$ is minimal subject to $H_i \leq G_j \cong \text{PSL}_2(p^{e_j/p_j})$. Then $|S_1| \leq 3$ and $|S_j| \leq 1$ for any $j \geq 2$.

As an immediate corollary we get the following claim.

Corollary 5.3.3. Let $q$ be a prime power and $G = \text{PSL}_2(\mathbb{F}_q)$ and $A = \{g_1, \ldots, g_m\}$ be a minimal set of generators of $G$. Let $H_i := \langle A \setminus \{g_i\} \rangle$.

If $|A| \geq 7$ then the subgroups $H_i$ which are subfield subgroups

$$H_i \cong \text{PSL}_2(\mathbb{F}_{q_i})$$

satisfy that their underlying fields $\mathbb{F}_{q_i}$ are generating the whole field $\mathbb{F}_q$. 

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Proof. Let us use the same notations of the previous theorem. Let \( q = p^r \) and \( r = p_1^{e_1} \ldots p_n^{e_n} \) be the prime decomposition of \( r \). By the previous theorem we have three cases to consider. In all the cases we get that for any \( S_j \) there exist \( i = i_j \) and \( H_i \) and \( r_i \) such that

\[
H_i \cong \text{PSL}_2(p^{r_i}) \notin S_j.
\]

In other words for any \( 1 \leq j \leq n \), \( r_i \nmid (r/p_j) \). Therefore the l.c.m. of these \( r_i \)'s is

\[
\text{lcm}(r_{i_1}, \ldots, r_{i_n}) = r
\]

so we are done.

Now let us use this corollary to prove the following Theorem.

**Theorem 5.3.4.** Let \( \mathbb{F}_q \) be a finite field of order \( q \), \( G = \text{SL}_2(\mathbb{F}_q) \) and \( A \) be a set of generators of \( G \). Then we have,

\[
\langle \text{Tr}(A^{[6]} \rangle = \mathbb{F}_q.
\]

**Proof.** By Lemma 5.2.3 we got that if \( \text{char}(\mathbb{F}) \neq 2 \) then

\[
\langle \text{Tr}(A^{[3]} \rangle = \mathbb{F},
\]

so we are only left with the case that \( \text{char}(\mathbb{F}) = 2 \) and

\[
G = \text{SL}_2(\mathbb{F}_q) = \text{PSL}_2(\mathbb{F}_q)
\]

with \( q = 2^r \). By taking a subset \( A' \) of \( A \) if needed, without loss of generality \( A \) is minimal generating set. If \( |A| \leq 6 \) then by Lemma 5.2.3 we get \( \langle \text{Tr}(A^{[6]} \rangle = \mathbb{F}_q \).
Now by induction on \( r \), and the previous theorem, if \( r = 1 \) then \( |A| \leq 4 \) and so
\[
\langle \operatorname{Tr}(A^{[4]}) \rangle = \mathbb{F}_q.
\]
Otherwise, let \( r = p_1^{e_1} \ldots p_n^{e_n} \) be the prime decomposition of \( r \). Now if \( |A| \geq 7 \) then by the previous corollary we get proper subfield subgroups
\[
H_i \cong \text{SL}_2(2^{r_i})
\]
such that the subfields \( \mathbb{F}_{2^{r_i}} \) generate \( \mathbb{F}_2^r \). By the induction hypothesis on these \( H_i \) which are generated by \( A_i = A \setminus g_i \), we get
\[
\langle \operatorname{Tr}(A_i^{[6]}) \rangle = \mathbb{F}_{2^{r_i}}.
\]
Therefore \( \langle \operatorname{Tr}(A^{[6]}) \rangle = \mathbb{F}_q \) as we wanted. \( \Box \)

### 5.4 Avoiding certain traces

We first start with a useful identity that we will use many times.

**Lemma 5.4.1.** Let \( \mathbb{F} \) be a field and \( g, h \in \text{SL}_2(\mathbb{F}) \). Then we have,
\[
\operatorname{Tr}(g) \operatorname{Tr}(h) = \operatorname{Tr}(gh) + \operatorname{Tr}(gh^{-1}).
\] (5.4.1a)

**Proof.** From the Cayley-Hamilton identity \( h^2 - \operatorname{Tr}(h)h + I = 0 \), we get by multiplying by \( gh^{-1} \), the matrix identity
\[
gh - \operatorname{Tr}(h)g + gh^{-1} = 0.
\]

Therefore by taking the trace and reordering the identity we are done. \( \Box \)
**Definition 5.4.2.** Let $G$ be a linear group and let $A \subseteq G(\mathbb{F})$ and let $X \subseteq \mathbb{F}$. Denote,

$$A|_X := \{ g \in A : \text{Tr}(g) \in X \}$$

$$A\not|_X := \{ g \in A : \text{Tr}(g) \notin X \}$$

As usual, when $X = \{x\}$ is singleton we will write just $x$ instead of $X$ and we write $\pm x$ instead of $\{\pm x\}$. I.e.,

$$A|_x := A|_{\{x\}}$$

$$A|_{\pm x} := A|_{\{\pm x\}}$$

and similarly for $A\not|_x$ and $A\not|_{\pm x}$.

**Definition 5.4.3.** Let $\mathbb{F}$ be a field and let $V(\mathbb{F}) = \mathbb{F}^2 \setminus \{(0,0)\}$. Let

$$P(\mathbb{F}) := \overline{V(\mathbb{F})} = V(\mathbb{F})/\sim$$

be the **projective line** over $\mathbb{F}$ where for any $u, v \in V(\mathbb{F})$,

$$\overline{u} = \overline{v} \iff u \sim v \iff \text{span}(u) = \text{span}(v).$$

Now let $V = V(\mathbb{F}) = \mathbb{F}^2 \setminus \{(0,0)\}$ and let $G = \text{SL}_2(\mathbb{F})$ act on $V$ by left multiplication. We will be interested in the action of $G$ on $P(\mathbb{F})$ which is induced from the action of $G$ on $V$. Note that

$$g\overline{u} = \overline{v} \iff gv = \lambda v \text{ for some } \lambda \in \mathbb{F}^\times.$$

For $g \in G$ denote,

$$\text{Fix}(g) := \{ \overline{u} \in P(\mathbb{F}) : g\overline{u} = \overline{u} \},$$

the **fix points** of $g$ with respect to the action on $P(\mathbb{F})$.  

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The following simple fact is stated also as a definition.

**Simple Fact 5.4.4.** Let \( G = SL_2(F) \). Denote by \( G_u \) the non trivial \( \pm \) unipotent elements in \( G \):

\[
\begin{align*}
u \in G_u & \iff \text{there exist } w \in SL_2(F) \text{ and } a \in \{\pm 1\} \text{ and } x \in F^\times \text{ such that} \\
u^w &= a \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = a(I + xE_{12}).
\end{align*}
\]

If we denote the two columns of \( w \) by \( w = (w_1, w_2) \) then

\[
\text{Fix}(u) = \{w_1\}.\]

We have,

\[
G_u = G_{|_{\pm 2}} \setminus \{\pm I\} \\
= \{u \in G : |\text{Fix}(u)| = 1\}.
\]

In other words\(^5\)

\[
G_u = \{u \in G : \text{Tr}(u) = \pm 2\} \setminus \{\pm I\},
\]

are the elements with exactly one fix point in \( \mathbb{P}(F) \). For \( A \subseteq G \) denote

\[
A_u := A \cap G_u.
\]

The following simple fact is stated also as a definition.

**Simple Fact 5.4.5.** Let \( G = SL_2(F) \). Denote by \( G_s \) the semi simple elements in \( G \):

\[
s \in G_s \iff \text{there exist } w \in SL_2(F) \text{ and } y \in F \setminus \{\pm 1\} \text{ such that} \\
u^w &= D_y = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}.
\]

\(^5\)We write for short \( x = \pm y \iff x \in \{\pm y\}.\)
If we denote the two columns of \( w \) by \( w = (w_1, w_2) \) then

\[
\text{Fix}(s) = \{\overline{w_1}, \overline{w_2}\}.
\]

We have,

\[
G_s = G^s_{|s|_2}
= \{s \in G : |\text{Fix}(s)| = 2\}.
\]

In other words\(^6\)

\[
G_s = \{s \in G : \text{Tr}(u) \neq \pm 2\},
\]

are the elements with exactly two fix points in \( \mathbb{P}(\mathbb{F}) \). For \( A \subseteq G \) denote

\[
A_s := A \cap G_s.
\]

**Definition 5.4.6.** For \( A \subseteq G \) we denote for short,

\[
\begin{align*}
C(A) &= C_G(A) = \{g \in G : a^g = a \text{ for any } a \in A\} \\
N(A) &= N_G(A) = \{g \in G : A^g = A\}.
\end{align*}
\]

**Simple Fact 5.4.7.** Let \( G = \text{SL}_2(\mathbb{F}) \) and let \( s \in G_s \) and \( u \in G_u \). Then we have,

\[
\begin{align*}
C(s) &\subseteq G_s \cup \{\pm I\} \\
C(u) &\subseteq G_u \cup \{\pm I\}
\end{align*}
\]

In fact,

\[
\begin{align*}
C(s) &= \{s' \in G : \text{Fix}(s') = \text{Fix}(s)\} \cup \{\pm I\} \\
C(u) &= \{u' \in G : \text{Fix}(u') = \text{Fix}(u)\} \cup \{\pm I\}.
\end{align*}
\]

\(^6\)We write for short \( x \neq \pm y \iff x \notin \{\pm y\} \).
Simple Fact 5.4.8. Let $G = \text{SL}_2(\mathbb{F})$ and let $s \in G_s$ and $u \in G_u$. Then we have,

$$N(C(s)) = \{g \in G : g(\text{Fix}(s)) = \text{Fix}(s)\}$$

$$N(C(u)) = \{b \in G : \text{Fix}(u) \subseteq \text{Fix}(b)\}.$$ 

In other words if $\text{Fix}(s) = \{\overline{w}_1, \overline{w}_2\}$ then

$$g \in N(C(s)) \iff \text{either } g \text{ fix both } \overline{w}_i \text{ or } g \text{ flips between them.}$$

Similarly if $\text{Fix}(u) = \{\overline{w}_1\}$ then

$$g \in N(C(u)) \iff g \text{ fix } \overline{w}_1.$$ 

Definition 5.4.9. For a subset $V \subseteq \text{SL}_2(\mathbb{F})$ denote

$$\text{Fix}(V) := \bigcap_{g \in V} \text{Fix}(g).$$

The following Lemma is a slight modification of an argument of Helfgott for producing many semi-simple elements (cf. [He, §4.1 Lemma 4.2]).

Lemma 5.4.10 (Helfgott). Let $\mathbb{F}$ be a field, let $G = \text{SL}_2(\mathbb{F})$ and let $A \subseteq G$ be a finite subset. Suppose $\langle A \rangle$ is a non-abelian subgroup\(^7\) of $G$. Then we have,

$$|A^3 \cap G_s| \geq \frac{1}{4}|A|.$$ 

Proof. Let $A' := A \setminus \{\pm I\}$. Then $A' = A_u \cup A_s$ and by the assumption $|A'| \geq 2$. If $A_u = \emptyset$ then $|A_s| \geq 2$ so $|A_s| \geq \frac{1}{2}|A|$ so we are done. Otherwise let $g \in A_u$ and set $C = C_G(g)$ and $B = A \setminus C$. By the assumption, $B \neq \emptyset$. 

\(^7\)or we could write for short $[A,A] \neq 1$. 

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If $h \in B_u$ then for some $x, y \in \mathbb{F}^\times$ and $w \in \text{SL}_2(\mathbb{F})$ and $a, b \in \{\pm 1\}$ we have

$$g^w = \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} = aI + xE_{12}$$

$$h^w = \begin{pmatrix} b & 0 \\ y & b \end{pmatrix} = bI + yE_{21}$$

and $(h^{-1})^w = bI - yE_{21}$. Therefore

$$\text{Tr}(gh^{\pm 1}) = 2ab \pm xy.$$ 

Now if char($\mathbb{F}$) = 2 then both $gh, gh^{-1}$ are semi simple elements and if char($\mathbb{F}$) \(\neq 2\) then at least one of $gh, gh^{-1}$ is semi simple.

Therefore we get that for any $h \in B$, either $h \in G_s$ or $gh \in G_s$ or $gh^{-1} \in G_s$. Therefore $A^{[2]}$ contains at least $\frac{1}{2}|B|$ semi-simple elements so

$$|A^{[2]} \cap G_s| \geq \frac{1}{2}|B| = \frac{1}{2}(|A| - |C_A(g)|).$$  \hspace{1cm} (5.4.10.1)

On the other hand, if $h \in A \setminus C$ then $hC_A(g) \subseteq A^{[2]} \setminus C$. Set $B' = A^{[2]} \setminus C$ and so $|B'| \geq |C_A(g)|$. Therefore by applying the previous argument (5.4.10.1) with $B = B'$ we get that,

$$|A^{[3]} \cap G_s| \geq \frac{1}{2}|B'| \geq \frac{1}{2}|C_A(g)|.$$  \hspace{1cm} (5.4.10.2)

Putting together (5.4.10.1) and (5.4.10.2) we get,

$$|A^{[3]} \cap G_s| \geq \frac{1}{4} \max\{|A| - |C_A(g)|, |C_A(g)|\} \geq \frac{1}{4}|A|. \hspace{1cm} \Box$$

The following Lemma is a slight variant of Lemma 5.4.10.
Lemma 5.4.11. Let $\mathbb{F}$ be a finite field. Let $G = \text{SL}_2(\mathbb{F})$ and let $A \subseteq G$ and suppose $\langle A \rangle = G$. Then we have,

$$|A^{[3]}|_0 \geq \frac{1}{4}|A|.$$

Proof. If $\text{char}(\mathbb{F}) = 2$ then $G_s = G \setminus 0$ so we are done by Lemma 5.4.10. Otherwise $\text{char}(\mathbb{F}) \neq 2$ and therefore $G|_0 \subseteq G_s$. If $0 \notin \text{Tr}(A^{[3]})$ then we are done. Otherwise fix $g \in A^{[3]}|_0$ and let $\omega \in \overline{\mathbb{F}}$ with $\omega^2 = -1$. Therefore $\Lambda(g) = \text{Spec}_{\overline{\mathbb{F}}}(g) = \{\pm \omega\}$.

Note that

$$\text{Tr}(g) = 0 \iff g^2 = -I \iff g^{-1} = -g.$$ 

Denote $C = C_G(g)$ and $N = N_G(C)$. By the assumption and by fact 5.4.8

$$A \notin N.$$ 

Set $B = A \setminus N \neq \emptyset$ and let $h \in B$. If $\text{Tr}(h) = 0$ then

$$\text{Tr}(gh) = 0 \iff ghgh = -I \iff gg^h = I \iff g^h = g^{-1}.$$ 

Therefore $\text{Tr}(gh) = 0 \Rightarrow h \in N \Rightarrow \text{ contradiction! }$ (since we we took $h \notin N$). Therefore we got that either $\text{Tr}(h) \neq 0$ or $\text{Tr}(gh) \neq 0$. So

$$|A^{[3]}|_0 \geq \frac{1}{2}|B| = \frac{1}{2}(|A| - |A \cap N|). \quad (5.4.11.1)$$

On the other hand if $h \in A \setminus N$ then $h(A \cap N) \subseteq A^{[2]} \setminus N$ therefore,

$$|A^{[2]} \setminus N| \geq |A \cap N|.$$ 

\footnote{We denote $\text{Spec}_{\overline{\mathbb{F}}}(g)$ to emphasize that we take all the eigen values in $\overline{\mathbb{F}}$.}
Therefore by applying the previous argument (5.4.11) with \( B = B' = A^{[2]} \setminus N \) we get that
\[
|A^{[3]}|_0 \geq \frac{1}{2} |B'| \geq \frac{1}{2} |A \cap N|.
\] (5.4.11.2)
Combining (5.4.11.1) and (5.4.11.2) we get
\[
|A^{[3]}|_0 \geq \frac{1}{4} |A|.
\]

Lemma 5.4.12. Let \( \mathbb{F} \) be a finite field and let \( G = \text{SL}_2(\mathbb{F}) \). Suppose \( A \subseteq G \) with \( \langle A \rangle = G \) and let \( \mathbb{E} < \mathbb{F} \) be a proper subfield. Then we have,
\[
|A|_\mathbb{E} > 0 \quad \Rightarrow \quad |A^{[4]}|_\mathbb{E} \geq \frac{1}{12} |A|.
\]

Proof. Denote \( B = A^{[3]} \). If \( |B|_\mathbb{E} \geq \frac{1}{12} |A| \) then we are done so assume
\[
|B|_\mathbb{E} < \frac{1}{12} |A|.
\]
From Lemma 5.4.11 we get that
\[
|B|_0 \geq \frac{1}{4} |A|.
\]
Therefore
\[
|B|_\mathbb{E} > \left( \frac{1}{4} - \frac{1}{12} \right) |A| = \frac{1}{6} |A|.
\]
From Lemma 5.4.1 if \( g \in G|_\mathbb{E} \) and \( h \in G|_{\mathbb{E}^\times} \) then,
\[
\text{either} \quad \text{Tr}(gh^{-1}) \notin \mathbb{E} \quad \text{or} \quad \text{Tr}(gh) \notin \mathbb{E}.
\]
By the assumption there is \( g \in A|_\mathbb{E} \) therefore we get \( B' := gB \subseteq A^{[4]} \) and so
\[
|A^{[4]}|_\mathbb{E} \geq |B'|_\mathbb{E} \geq \frac{1}{2} |B|_\mathbb{E} \times | \geq \frac{1}{12} |A|.
\]
\[\square\]
Therefore we get immediately the following result.

**Corollary 5.4.13.** Let $\mathbb{F}$ is a finite field and let $G = \text{SL}_2(\mathbb{F})$. Let $A \subseteq G$ and suppose $\langle A \rangle = G$ and $\langle \text{Tr}(A) \rangle = \mathbb{F}$. Then for any proper subfield $E < \mathbb{F}$ we have,

$$|A^{[4]}_{|E}| \geq \frac{1}{12}|A|.$$

**Corollary 5.4.14.** Let $\mathbb{F}$ is a finite field and let $G = \text{SL}_2(\mathbb{F})$. Let $A \subseteq G$ and suppose $\langle A \rangle = G$. Then for any proper subfield $E < \mathbb{F}$ we have,

$$|A^{[9]}_{|E}| \geq \frac{1}{12}|A|.$$

**Proof.** By Lemma 5.3.4, $\langle \text{Tr}(A^{[6]}) \rangle = \mathbb{F}$ therefore

$$|A^{[6]}_{|E}| > 0.$$

Now as in the proof of Lemma 5.4.12 we get that either $|A^{[3]}_{|E}| \geq \frac{1}{12}|A|$ (and then we are done) or

$$|A^{[8]}_{|E^\times}| > \frac{1}{6}|A|.$$

Therefore if take $b \in A^{[6]}_{|E}$ and $B' := A^{[3]}_{|E^\times}$ and $B'' := bB' \subseteq A^{[9]}$ then we get

$$|A^{[9]}_{|E}| \geq |B''_{|E}| \geq \frac{1}{12}B' > \frac{1}{12}|A|. \blackslug
Chapter 6

Growth properties of $\text{SL}_2(\mathbb{F}_q)$

6.1 Some useful Growth properties

Definition 6.1.1. Let $G$ be a group and $g, h \in G$. Define the conjugacy class equivalence by

$$g \sim h \iff g^G = h^G.$$  

I.e., $g \sim h \iff g^x = x^{-1}gx = h$ for some $x \in G$. Given a subset $A \subseteq G$ denote

$$\tilde{A} = A/\sim.$$  

By abuse of notation we will view $\tilde{A} \subseteq A$ as a set of representatives so:

$$\forall a \in A, \exists! b \in \tilde{A} \text{ such that } a \sim b.$$  

The following useful Lemma connects growth and commutativity.

Lemma 6.1.2. ([He, §4.1 Proposition 4.1]) Let $G$ a finite group and let $\emptyset \neq A \subseteq G$. Then there exists $a \in A$ such that,

$$|C_{A^{-1}A}(a)| \geq \frac{[\tilde{A}]|A|}{|A^{-1}AA|}.$$  

(6.1.2a)
If $\langle A \rangle = G$ then for any proper subgroups $H, K < G$ we have,

$$|A^4 \setminus (H \cup K)| > \frac{1}{4}|A|.$$  \hspace{1cm} (6.1.2b)

Proof. Let $a, b \in A$ and $g \in G$ and suppose $g^a = g^b$. Then we have,

$$ba^{-1} \in C_{Aa^{-1}}(g) \subseteq C_{AA^{-1}}(g).$$

Therefore we get,

$$b \in C_{Aa^{-1}}(g)a \subseteq C_{AA^{-1}}(g)a.$$

Therefore for any $g \in A$ we get that $gA \subseteq A^{-1}AA$ and

$$\frac{|A|}{|C_{AA^{-1}}(g)|} \leq |gA|. \hspace{1cm} (6.1.2.1)$$

On the other hand if we denote $\Lambda = \tilde{A}$ then

$$\frac{1}{|\Lambda|} \sum_{g \in \Lambda} |gA| = \frac{|A^4|}{|\Lambda|} \leq \frac{|A^{-1}AA|}{|A|}. \hspace{1cm} (6.1.2.2)$$

Therefore there exists $g \in \tilde{A} \subseteq A$ s.t.

$$\frac{|A|}{|C_{AA^{-1}}(g)|} \leq |gA| \leq \frac{|A^{-1}AA|}{|A|} \hspace{1cm} (6.1.2.1)$$

so by arranging the inequality we are done with (6.1.2a).\(\Box\)

Now suppose $\langle A \rangle = G$. Since $A \setminus H \neq \emptyset$ we get that for $a \in A \setminus H$, $a(A \cap H) \subseteq A^2 \setminus H$ therefore

$$|A^2 \setminus H| = \max\{|A \setminus H|, |A \cap H|\} \geq \frac{1}{2}|A|.$$

If $H = K$ then we are done. If $A \subseteq H \cup K$ then there exists $a, a' \in A$ such that $a \in H \setminus K$ and $a' \in K \setminus H$ and therefore $aa' \in A^2 \setminus (H \cup K)$. In any case there exists $b \in A^2 \setminus (H \cup K)$. Denote $B = A^2 \setminus H$ so $b \in B \setminus K$ therefore $b(B \cap K) \subseteq A^4 \setminus K$ therefore

$$|A^4 \setminus (H \cup K)| = \max\{|B \setminus K|, |B \cap K|\} \geq \frac{1}{2}|B| \geq \frac{1}{4}|A|$$

so we are done with (6.1.2b).\(\Box\)
Corollary 6.1.3. Let $\mathbb{F}$ be a field. Let $G$ be a subgroup of $\text{GL}_n(\mathbb{F})$ and let $A \subseteq G$ be a finite subset. Let $B \subseteq A$ with $|B| \geq c|A|$ for some $c \in \mathbb{R}_+$. Then there exists $b \in B$ such that,

$$\left| C_{AA^{-1}}(b) \right| \geq c \frac{|\text{Tr}(B)||A|}{|A^{-1}AA|}. \quad (6.1.3a)$$

Proof. Since conjugate elements have the same trace we get,

$$|\tilde{A}| \geq |\text{Tr}(A)|.$$  

Therefore by Lemma 6.1.2 there exists $a \in A$ such that,

$$\left| C_{AA^{-1}}(a) \right| \geq \left(6.1.2a\right) \frac{|\text{Tr}(A)||A|}{|A^{-1}AA|}.$$  

Therefore if $B \subseteq A$ and $|B| \geq c|A|$ then there exists $b \in B$ such that\footnote{I want to thank H.Helfgott for helpful discussion concerning this variant.},

$$\left| C_{AA^{-1}}(b) \right| \geq \left| C_{BB^{-1}}(b) \right| \geq \frac{|\text{Tr}(B)||B|}{|B^{-1}BB|} \geq c \frac{|\text{Tr}(B)||A|}{|A^{-1}AA|}.$$

\[\square\]

A variant of the following Lemma was proved in [He, Proposition 4.10]. Here, we will show another way of proving it.

Lemma 6.1.4. Let $\mathbb{F}$ be a field and let $G = \text{SL}_2(\mathbb{F})$. Let $g \in G_s$ be a semi simple element. Let $h \in G$ and suppose $\text{Fix}(h) \setminus \text{Fix}(g) \neq \emptyset$. Define the function $F : \text{SL}_2(\mathbb{F}) \rightarrow \mathbb{F}^3$ by

$$F(b) = (\text{Tr}(b), \text{Tr}(gb), \text{Tr}(hb)).$$

Then $\text{mult}(F) \leq 2$. In particular, for any subset $B \subseteq G$,

$$\frac{1}{2}|B| \leq |F(B)| \leq |\text{Tr}(B)||\text{Tr}(gB)||\text{Tr}(hB)|. \quad (6.1.4a)$$
Proof. There exists \( w \in \text{SL}_2(\mathbb{F}) \) such that
\[
g = \begin{pmatrix} \alpha & a \\ 0 & \alpha^{-1} \end{pmatrix}^w
\]
\[
h = \begin{pmatrix} \beta & 0 \\ b & \beta^{-1} \end{pmatrix}^w
\]
with \( b \in \mathbb{F}^\times \) and \( \alpha \notin \{\pm 1\} \). Let
\[
g' = \begin{pmatrix} x & y \\ z & w \end{pmatrix}^w \in \text{SL}_2(\mathbb{F}).
\]
We need to show that for any \( c_1, c_2, c_3 \) there are at most two \( g' \) with
\[
\begin{cases}
\det(g') = 1 \\
F(g') = (\text{Tr}(g'), \text{Tr}(gg'), \text{Tr}(hg')) = (c_1, c_2, c_3)
\end{cases}
\]
By opening trace equalities we get the linear system
\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
\alpha & \alpha^{-1} & 0 & a \\
\beta & \beta^{-1} & b & 0
\end{pmatrix}
\begin{pmatrix}
x \\
w \\
y \\
z
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix}
\]
Denote \( A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \alpha & \alpha^{-1} & 0 & a \\ \beta & \beta^{-1} & b & 0 \end{pmatrix} \) and \( \overline{x} = \begin{pmatrix} x \\ w \\ y \\ z \end{pmatrix} \) and \( \overline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \). Therefore,
\[
\text{rank}(A) = 3
\]
so the set of solutions \( A^{-1}(\overline{c}) \) is either empty, or a one dimensional affine linear subspace\(^2\) of \( \mathbb{F}^4 \). Note that for any \( z \) there is exactly one triple \( (x, w, y) \)
\[\text{i.e., } A^{-1}(\overline{c}) \text{ is a dilation of a one dimensional linear subspace of } \mathbb{F}^4.\]
such that $g'$ is a solution. On the other hand, $g' \in \operatorname{SL}_2(\mathbb{F})$ so $xw - yz = 1$ and therefore there at most two solutions $g'$ on the affine line $A^{-1}(\mathfrak{c})$ with $\det(g') = 1$. In other words

$$|A^{-1}(\mathfrak{c}) \cap \operatorname{SL}_2(\mathbb{F})| \leq 2.$$

\[\qed\]

6.2 Avoiding subvarieties

**Definition 6.2.1.** Let $\mathbb{F}$ be a field. Let $G$ be a group and let $(V, \rho)$ be a finite dimensional representation of $G$ over $\mathbb{F}$. When the action will be clear from the context we will write the linear action on $V$ simply by $gv$ instead of $\rho(g)v$. Let $W_1, \ldots, W_m < V$ be proper subspaces of $V$ and let

$$W = \bigcup_{i=1}^{m} W_i.$$

We will assume that the above union is non trivial in the sense that

$$W_i \leq W_j \Rightarrow i = j.$$

We will call $W$ a linear variety with decomposition\(^3\) $W = \bigcup_{i=1}^{m} W_i$. Denote

$$\operatorname{Stab}_G(W) = \{g \in G : gW = W\}.$$  

We will sometimes abbreviate and write

$$G_W = \operatorname{Stab}(W) = \operatorname{Stab}_G(W)$$

\(^3\)if the union is non trivial then the decomposition is unique.
when the group $G$ is clear from the context. Denote,

$$\dim(W) := \max_i \{\dim(W_i)\}$$

$$\deg_d(W) := |\{i : \dim(W_i) = d\}|$$

$$\deg(W) := \deg_{\dim(W)}(W).$$

The following “escaping Lemma” will be useful. The following proof is a slight modification of [He, §4.2 Lemma 4.4].

**Lemma 6.2.2 (Helfgott).** For any $n, m \in \mathbb{N}_+$ there exists $k \in \mathbb{N}_+$ such that the following holds. Let $G$ be a group and let $(V, \rho)$ be a finite dimensional representation of $G$ over a field $\mathbb{F}$. Let $W_1, \ldots, W_m \leq V$ be subspaces of $V$ and suppose $W = \bigcup_i W_i$ is a linear variety with $\dim(W) \leq n$. Let $A$ be a subset of generators of $G$. Let $0 \neq w \in V$ and denote the orbit of $w$ by $O := Gw$ and

$$V_w := \mathbb{F}[G]w = \text{span}(O).$$

Suppose $O \not\subseteq W$.

Then for any $0 \neq w' \in V_w$ there exists $g \in A^{|k|}$ such that $gw' \not\in W$. In particular for any $w' \in O$ there exists $g \in A^{|k|}$ such that $gw' \not\in W$.

**Proof.** Note that the claim is trivially true for $w' \in V_w \setminus W$ so we need to prove it for $0 \neq w' \in V_w \cap W$. In particular, if $V_w \cap W = 0$ we are done.

Without loss of generality $W = \bigcup_i W_i$ is the decomposition of $W$ as a union of spaces. Set for $1 \leq i \leq m$, $O_i := O \cap W_i$ and $V_i := V_w \cap W_i$ and

$$W(0) := V_w \cap W = \bigcup_{i=1}^m V_i.$$

By the assumption for any $i \leq m$, $O \not\subseteq W_i$. Therefore

$$V_w = \text{span}(O) \not\subseteq W_i.$$
so $V_i < V_w$ and $V_i \leq W_i$. Now for any $g \in G$,

$$gO_i = g(O \cap W_i) = O \cap gW_i$$

and $gV_i = g(V_w \cap W_i) = V_w \cap gW_i$. Note that $O_i = \emptyset \iff V_i = 0$.

If $V_i = 0$ then for any $g \in G$,

$$0 = V_i \cap gV_i = V_w \cap W_i \cap gW_i < W_i.$$

Now suppose $O_i = O \cap W_i \neq \emptyset$ for some $i \leq m$ and let

$$x_i \in O_i \subseteq V_i.$$

Since $Gx_i = O \not\subseteq W_i$ we get that there exists $g_i \in G$ such that $g_i x_i \not\in W_i$ so

$$g_i V_i \not\subseteq V_i.$$

In other words $Stab(V_i) \neq G$. Therefore $V_i \cap g_i V_i < V_i$ so

$$V_w \cap W_i \cap g_i W_i < V_w \cap W_i \leq W_i.$$

Since $\langle A \rangle = G$ we can choose $g_i$ to be $g_i \in A$.

Therefore if $\dim(W) > 0$ and $\dim(W_1) = \dim(W)$ then there exists $a_1 \in A$ such that

$$V_i \cap a_1 V_i = V_w \cap W_i \cap a_1 W_i < W_i$$

and for all other $j \leq m$, $V_j \cap a_1 V_j \leq W_j$. Set for any $1 \leq j \leq m$, $W_{1j} := V_j \cap a_1 V_j = V_w \cap W_j \cap a_1 W_j$ and

$$W_{(1)} := W_{(0)} \cap a_1 W_{(0)} = \bigcup_{j=1}^{m} W_{1j}.$$

Therefore

$$W_{(1)} \subset W_{(0)} \subset W$$

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so either
\[ \dim(W(1)) < \dim(W(0)) \leq \dim(W) \]
or
\[ \deg(W(1)) < \deg(W(0)) \leq \deg(W). \]

Therefore by iterating the previous step either \( W(1) = 0 \) or we can find \( a_2 \in A \) such that for \( W(2) := V_w \cap W(1) \cap a_2 W(1) \) we get
\[ \text{either } \dim(W(2)) < \dim(W(1)) \text{ or } \deg(W(2)) < \deg(W(1)). \]

Therefore for some \( k \leq mn \) we get that \( W(k) = 0 \) therefore
\[ \bigcap_{g \in A^{[k]}} g(V_w \cap W) = 0. \]

Therefore for any \( 0 \neq w' \in V_w \cap W \) there exists \( g \in A^{[k]} \) such that \( w' \notin W \) so we are done. \( \square \)

Now we will prove the following result.

**Corollary 6.2.3.** There exists \( k \in \mathbb{N}_+ \) such that the following holds for any finite field \( \mathbb{F} \) of size \( |\mathbb{F}| > 3 \), and for any subset of generators \( A \) of \( SL_2(\mathbb{F}) \).

For any \( u \in GL_2(\mathbb{F}) \), there exists \( a \in A^{[k]} \), such that \( a^u \) has no zero entries.

**Proof.** Denote \( G := SL_2(\mathbb{F}) \) and \( V := M_2(\mathbb{F}) \) and for \( 1 \leq i, j \leq 2 \)
\[ W_{ij} := \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in V : a_{ij} = 0 \right\} \]
and \( W = \bigcup_{i,j} W_{ij} \). Equivalently, if \( g = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix} \in V \) then
\[ a_{ij} = 0 \iff ge_j = \lambda e_i \text{ for some } \lambda \in \mathbb{F} \]
Now we are going to use Lemma 6.2.2 with the group $G^u$ and the orbit $O = G^u$ of $w' = I$ and the linear variety $W$. We can use Lemma 6.2.2 if we show that $G^u \not\subseteq W$. We will show that $|G^u \cap W| < |G|$ so $G^u \not\subseteq W$.

Let $u = (u_1, u_2)$ where $u_i$ are the columns of $u$. Therefore for any $g \in G^u \cap W$ there exist $1 \leq i, j \leq 2$ such that $g u_i = u_j$. I.e., $gu_i = \lambda u_j$ for some $\lambda \in \mathbb{F}^\times$. Denote

$$G_{ij} := \{g \in G : gu_i = u_j\}.$$ 

So $G^u \cap W = \bigcup_{i,j} G_{ij}$. In order to prove $|G^u \cap W| < |G|$ we will bound $|\bigcup_{i,j} G_{ij}|$ from above.

Let us choose for any $i \in \{1, 2\}$ some $u'_i \in \mathbb{F}^2 \setminus \{0\}$ such that $u_i, u'_i$ are linear independent. Now if $g, g' \in G_{ij}$ then $gu_i = \lambda u_j$ and $g'u_i = \lambda' u_j$ for some $\lambda, \lambda' \in \mathbb{F}$. Note that knowing $gu'_i$ and $gu_i$ determine $g$ therefore if $g, g' \in G_{ij}$ and $gu'_i = g'u'_i \in \mathbb{F}^2 \setminus \{0\}$ then we must have $\lambda = \lambda'$ since $\det(g) = \det(g') = 1$. Therefore we conclude that for any $i, j$ we have $|G_{ij}| \leq |\mathbb{F}|^2 - 1$. Therefore $|G^u \cap W| = |\bigcup G_{ij}| \leq 4(|\mathbb{F}|^2 - 1) - 1$ since $I \in G_{11} \cap G_{22}$. So if $|\mathbb{F}| = q \geq 4$ then

$$|G^u| = |SL_2(\mathbb{F})| = q(q^2 - 1) > 4(q^2 - 1) - 1 \geq |\bigcup G_{ij}|$$

so in particular $G^u \not\subseteq W$.

Therefore we can apply Lemma 6.2.2 to get the following. For any $u \in GL_2(\mathbb{F})$ there exist $a \in A^k$ such that

$$a^u$$

has no zero entries. \hfill \square
6.3 Reduction from matrices to traces

**Definition 6.3.1.** Let $\mathbb{F}$ be a field and let $g, h \in \text{SL}_2(\mathbb{F})$. We will say the $g$ and $h$ are **entangled** (or **simultaneously triangular**) if

\[
\text{either } \text{Fix}(h) \subseteq \text{Fix}(g) \text{ or } \text{Fix}(g) \subseteq \text{Fix}(h).
\]

The following Lemma will be useful later (cf. [He, Lemmas 4.7, 4.9]).

**Lemma 6.3.2** (Helfgott). There exists $C > 0$ such that the following properties hold for any field $\mathbb{F}$. Let $g, h \in \text{SL}_2(\mathbb{F})$ and suppose they are not entangled. Then there exists $w \in \text{SL}_2(\mathbb{F})$ such that

\[
g^w = \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad h^w = \begin{pmatrix} b & 0 \\ y & b^{-1} \end{pmatrix}. \tag{6.3.2a}
\]

Moreover if $g \in G_u$ then $a = \pm 1$ and $x \neq 0$ (and similarly for $h \in G_u$).

Let $V \subseteq \text{SL}_2(\mathbb{F})$ be a finite subset of diagonal matrices and suppose $V \not\subseteq \{\pm I\}$. Let $g \in \text{SL}_2(\mathbb{F})$. If $g$ has no zero entries\(^4\) then we have,

\[
|VgVg^{-1}V| \geq \frac{1}{C}|V|^3. \tag{6.3.2b}
\]

If $U \subseteq \text{SL}_2(\mathbb{F})$ is a finite non empty subset which has no triangular matrices\(^5\) then we have,

\[
|\text{Tr}(UU^{-1})| \geq \frac{1}{C}\frac{|U|}{|\text{Diag}(U)|}. \tag{6.3.2c}
\]

**Proof.** By taking the two eigen vectors $w_1, w_2 \in \mathbb{F}^2$ of $g$ and $h$ respectively such that $\overrightarrow{w_1} \in \text{Fix}(g) \setminus \text{Fix}(h)$ and $\overrightarrow{w_2} \in \text{Fix}(h) \setminus \text{Fix}(g)$, and normalize them if needed, we get (6.3.2a). \(\square\)

---

\(^4\)i.e., $abcd \neq 0$ where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

\(^5\)i.e., $bc \neq 0$ where $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. 

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Suppose $V = D_S$ i.e., $S := \left\{ s \in \mathbb{F} : \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \in V \right\}$. For any $g' = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix}$ we get $Vg'V = \left\{ \begin{pmatrix} stx' & s^{-1}y' \\ s^{-1}t' & s^{-1}t w' \end{pmatrix} : s, t \in S \right\}$. Therefore $\text{Prod}(Vg'V) = \text{Prod}(g')$. Moreover, we see that unless $g'$ is diagonal or anti-diagonal\(^6\) we can recover from any element of $Vg'V$ the values $s^2, t^2$ so $|Vg'V| \geq \frac{1}{4}|V|^2$.

Now let $g' \in V^g$ so

$$g' = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}^{(a b \ c \ d)} = \begin{pmatrix} ads - s^{-1}bc & (s - s^{-1})db \\ (s^{-1} - s)ac & ads^{-1} - sbc \end{pmatrix}.$$ 

Therefore if $s \neq \pm 1$ then $x'y'z'w' \neq 0$ so in particular $g'$ is neither diagonal nor anti diagonal. Altogether we get that

$$|VV^gV| \geq \frac{1}{4}|V|^2|V \setminus \{\pm I\}| \geq \frac{1}{12}|V|^3$$

so we are done with (6.3.2b).

For any $g \in U$ denote by $U_g$ the subset of all $g' \in U$ with the same diagonal as $g$. Consider the trace map $\text{Tr} : g(U_g)^{-1} \rightarrow \text{Tr}(UU^{-1})$. By calculating the trace $\text{Tr}(gg'^{-1})$ one see that each fiber is of size at most 2. Therefore for any $g \in U$ we have $|\text{Tr}(UU^{-1})| \geq \frac{1}{2}|U_g|$. Since there exists $g$ with

$$|U_g| \geq \frac{|U|}{|\text{Diag}(U)|}$$

\(^6\)i.e., has the form $\begin{pmatrix} \neq 0 & 0 \\ 0 & \neq 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \neq 0 \\ \neq 0 & 0 \end{pmatrix}$. 

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we get,

$$|\text{Tr}(UU^{-1})| \geq \frac{1}{2}|U_g| \geq \frac{1}{2}\frac{|U|}{|\text{Diag}(U)|}$$

so we are done with (6.3.2c).}

The following Lemma is the corner stone which connects the Growth of matrices and the Growth of traces (cf. [He, Propositions 4.8, 4.10]).

**Lemma 6.3.3** (Helfgott). There exist $k \in \mathbb{N}_+$ and $C \in \mathbb{R}_+$ such that the following holds for any finite field $\mathbb{F}$. Let $G = \text{SL}_2(\mathbb{F})$ and let $A \subseteq G$ be a subset of generators of $G$. Then we have,

$$|\text{Tr}(A^k)| > \frac{1}{C}|A|^{1/3} \quad (6.3.3a)$$

There exist $V \subseteq A^k$ and $w \in \text{SL}_2(\mathbb{F})$ such that $V^w$ are diagonal and

$$|V| \geq \frac{1}{C}\frac{|\text{Tr}(A)||A|}{|A^k|}. \quad (6.3.3b)$$

We also have,

$$|\text{Tr}(A)| \leq C\frac{|A^k|^{4/3}}{|A|} \quad (6.3.3c)$$

**Proof.** By Lemma 5.4.10 there exists $k_0 \in \mathbb{N}_+$ such that for $A_0 := A^{[k_0]}$ we have

$$|A_0 \cap G_s| \gg |A|.$$  

Let $h \in A_0 \cap G_s$ be a semi simple element in $A_0$ and let $\{v, w\} = \text{Fix}(h)$ be its two fix points in $\mathbb{P}(\mathbb{F})$. Without loss of generality $(v, u) \in \text{SL}_2(\mathbb{F})$ and let us write from now the $\text{SL}_2(\mathbb{F})$ elements with respect to the basis\(^7 (v, u)$ of $\mathbb{F}^2$.

---

\(^7\)We denote a basis of a space as a tuple of vectors and not as a set of vectors. Therefore the notation $(v, u)$ has a double meaning either as matrix (a tuple of columns) or as tuple of vectors.
Denote by $H$ and $K$ the stabilizers of these points

$$H := \{ g \in G : g\bar{v} = \bar{v} \} \quad \text{and} \quad K := \{ g \in G : g\bar{u} = \bar{u} \}.$$ 

By Lemma 6.1.2 there exists $k_1 \in \mathbb{N}_+$ such that for $A_1 := A_0^{[k_1]}$ and $U := A_1 \setminus (H \cup K)$ we have,

$$|U| \quad \text{(6.1.2b)} \gg |A|.$$ 

Since $U$ has no triangular matrices we get by Lemma 6.3.2 some $k_2 \in \mathbb{N}_+$ such that for $A_2 := A_1^{[k_2]}$ and $D := \text{Diag}(U)$ we have,

$$|\text{Tr}(A_2)| \geq |\text{Tr}(UU^{-1})| \quad \text{(6.3.2c)} \quad \frac{|U|}{|D|} \gg \frac{|A|}{|D|}.$$ 

In order to complete the proof of (6.3.3a) we will show that $|D| \leq |\text{Tr}(A_2)|^2$. Now for any $t \in \text{Tr}(U)$ denote by $S_t$ the elements in $D$ with this sum and by $U_t$ the elements in $U$ with this trace. Therefore for some $t \in \text{Tr}(U)$ we have,

$$|U_t| \geq |S_t| \geq \frac{|D|}{|\text{Tr}(U)|} \geq \frac{|D|}{|\text{Tr}(A_1)|}.$$ 

Therefore in order to complete the proof of (6.3.3a) we will show that for any $t \in \text{Tr}(U)$,

$$|U_t| \leq |\text{Tr}(A_2)|$$

Indeed since $h = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \in A_0$ then for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_t$ we have

$$\text{Tr}(hg) = ra + r^{-1}d = (r - r^{-1})a + r^{-1}t$$

therefore the trace map $\text{Tr} : hU_t \to \mathbb{F}$ is injective so

$$|U_t| = |\text{Tr}(hU_t)| \leq |\text{Tr}(A_2)|$$

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so we are done with (6.3.3a). √

Denote by $B := A_0 \cap G_s$ the semi simple elements in $A_0$. As we seen before previously $|B| \gg |A|$. By corollary 6.1.3 there exists $b \in B$ such that for $A_3 := A_0^{[3]}$ and $V := C_{A_3}(b)$ we get,

$$|V| = |C_{A_3}(b)|$$

$$\geq |C_{BB^{-1}}(b)|$$

$$\geq \frac{|\text{Tr}(B)||A_0|}{|A_0^{-1}A_0A_0|}$$

$$\gg \frac{(|\text{Tr}(A_0)| - 2)|A_3|}{|A_3|}$$

$$\gg \frac{|\text{Tr}(A)||A|}{|A_3|}.$$  

(6.3.3.1)

Since $b$ is semi-simple there exists $w' \in \text{SL}_2(\overline{F})$ such that

$$V^{w'}$$

are diagonal and $V \subseteq A_3$  

(6.3.3.2)

so we are done with (6.3.3b). √

By (6.3.3.1) and (6.3.3.2) there exists a basis $w' \in \text{SL}_2(\overline{F})$ and $V \subseteq A_3$ such that $V^{w'}$ are all diagonal and

$$|\text{Tr}(A)|^{(6.3.3.1)} \ll |V|\frac{|A_3|}{|A|}$$

where $A_3 = A^{[k_3]}$ and $k_3 := 3k_0$.

By corollary 6.2.3 there exists $k_4 \in \mathbb{N}_+$ and $g \in A_4 := A^{[k_4]}$ such that $g^{w'}$ has no zero entries. Now we set $k_5 := 5 \max\{k_4, k_3\}$. Therefore $VV^gV \subseteq A_5 := A^{[k_5]}$. Therefore by Lemma 6.3.2 we get,

$$|A_5| \geq |VV^gV|$$

$$\gg |V|^3.$$  

(6.3.3.3)
Therefore we conclude,

\[ |\text{Tr}(A)| \ll |V| \frac{|A_3|}{|A|} \]

(6.3.3.3) \[ \ll \frac{|A_5|^{4/3}}{|A|} \]

so we are done with (6.3.3c). √

### 6.4 Corollaries

Let us collect the properties that we will exploit soon.

**Theorem 6.4.1** (Helfgott). There exist \( k \in \mathbb{N}_+ \) and \( C \in \mathbb{R}_+ \) such that the following holds for any finite field \( \mathbb{F} \). Let \( A \) be a subset of generators of \( \text{SL}_2(\mathbb{F}) \). Then we have,

\[ |\text{Tr}(A^{[k]})| > \frac{1}{C} |A|^{1/3} \] (6.4.1a)

\[ |\text{Tr}(A)| < C \frac{|A^{[k]}|^{4/3}}{|A|} \] (6.4.1b)

\[ |A^{[k]} \cap G_s| > \frac{1}{C} |A| \] (6.4.1c)

**Proof.** Parts (6.4.1a) and (6.4.1b) were proved in Lemma 6.3.3 parts (6.3.3a) and (6.3.3c). Part (6.4.1c) was proved in Lemma 5.4.10. ∎

Now let’s see how Helfgott managed to reduce the Growth of \( A^{[k]} \) to the Growth of \( \text{Tr}(A^{[k]}) \) and then to reduce the Growth of traces to the Growth of eigenvalues (cf. [He, §3 Proposition 3.3 and §4.4]).

**Theorem 6.4.2** (Helfgott). There exist \( k \in \mathbb{N}_+ \) and \( C \in \mathbb{R}_+ \) such that for any \( \varepsilon \in \mathbb{R}_+ \) that following holds. Let \( \mathbb{F} \) be a finite field and let \( A \) be a subset of generators of \( \text{SL}_2(\mathbb{F}) \). Denote \( A_1 = A^{[k]} \) and \( A_2 = A^{[k^2]} \). Suppose

\[ |A_2| < |A|^{1+\varepsilon}. \] (6.4.2a)
Then we have,

$$\frac{1}{C}|A|^{1/3} < |\text{Tr}(A_1)| < C|A|^{1/3+C\varepsilon}$$  \hspace{1cm} (6.4.2b)

and there exists a semi simple element \( g \in A_1 \cap G_s \) and \( V := C_{A_2}(g) \) with

$$|V| > \frac{1}{C}|\text{Tr}(A_1)|^{1-C\varepsilon}.$$  \hspace{1cm} (6.4.2c)

Moreover, if

$$|A_2| < |A|^{1+\varepsilon} \quad \text{and} \quad |\text{Tr}(A_2)| < |\text{Tr}(A_1)|^{1+\varepsilon}$$  \hspace{1cm} (6.4.2d)

then there exists a semi simple element

$$g \in A_1 \cap G_s \quad \text{and} \quad V := C_{A_2}(g)$$

such that (6.4.2c) holds and also

$$|\text{Tr}(V^2) \cdot \text{Tr}(V^2)| + |\text{Tr}(V^2) + \text{Tr}(V^2)| < C|\text{Tr}(V^2)|^{1+C\varepsilon}.$$  \hspace{1cm} (6.4.2e)

Proof. In the following proof we will use always the notation \( A_i = A^{[k]} \) but we will change few times the value of \( k \) itself. We will always increase its value so to fit to all the properties that we will need. All the properties of subsets that we will use are “monotone increasing” in the sense that if \( A^{[k]} \) has the property \( \mathcal{P} \) and \( k \leq k' \) then \( A^{[k']} \in \mathcal{P} \) as well. Note that the hypothesis depend also in the value of \( k \) (and they are “monotone decreasing” properties).

By theorem 6.4.1 there exists \( k_1 \in \mathbb{N}_+ \) such that the following holds.
From the assumption (6.4.2a) for $k \geq k_1$ we get,

$$|A|^{1/3} \overset{(6.4.1a)}{\ll} |\operatorname{Tr}(A_1)|$$

$$(6.4.1b) \quad \frac{|A_1^{[k_1]}|^{4/3}}{|A_1|} \leq \frac{|A_2|^{4/3}}{|A|} \leq |A|^{(4/3)(1+\varepsilon)-1}$$

$$(6.4.2a) \quad < |A|^{4/3} \overset{(6.4.2.1)}{\ll} |A|^{1/3 + O(\varepsilon)}.$$

so we are done with (6.4.2b).

$\sqrt{\text{We also get by theorem 6.4.1 that,}}$

$$|A_1 \cap G_s| \overset{(6.4.1c)}{\gg} |A|.$$  

Therefore by corollary 6.1.3 there exists $k_2 \in \mathbb{N}_+$ such that the following holds. If $k \geq \max\{k_1, k_2\}$ then there exists a semi simple $g \in B := A_1 \cap G_s$ with large centralizer:

$$|C_{A_2}(g)| \overset{(6.1.3a)}{\gg} |\operatorname{Tr}(B)| \frac{|A_1|}{|A_2|}$$

$$(6.4.2a) \quad \gg |\operatorname{Tr}(A_1)| |A|^{-\varepsilon}$$

$$(6.4.2.1) \quad \gg |\operatorname{Tr}(A_1)|^{1-O(\varepsilon)}$$

so we are done with (6.4.2c). $\sqrt{\text{Let } g \in A_1 \cap G_s \text{ be as in (6.4.2.2) with large centralizer}}$

$$|\operatorname{Tr}(A_1)| \overset{(6.4.2.2)}{\ll} |C_{A_2}(g)|^{1+O(\varepsilon)} = |V|^{1+O(\varepsilon)} \quad (6.4.2.3)$$

where $V := C_{A_2}(g)$. By conjugating $g$ with $u \in \text{SL}_2(\mathbb{F}_{q^2})$ such that $g^u$ is diagonal we get that all $V^u$ are diagonal, since $g$ is regular semi simple. By
corollary 6.2.3 there exists $k_3 \in \mathbb{N}_+$ such that if $k \geq k_3$ then there exists $a \in A_1$ such that $a^u$ has no zero entries. Therefore if $k \geq \max\{k_1, k_2, k_3\}$ then

$$V^{[4]}V^α^{[4]} := V^{[4]}a^{-1}V^{[4]}a \subseteq A_2^{[10]} = A^{[10k^2]} \subseteq A^{[(10k)^2]}.$$ 

Therefore if we take $k = 10 \max\{k_1, k_2, k_3\}$ we get that

$$V^{[4]}V^α^{[4]} \subseteq A_2.$$

Now suppose (6.4.2d) holds with $k = 10 \max\{k_1, k_2, k_3\}$. Therefore we get,

$$|\text{Tr}(V^u^{[4]}(a^u)^{-1}V^u^{[4]}a^u)| = |\text{Tr}(V^{[4]}V^α^{[4]})| \leq |\text{Tr}(A_2)| \leq |\text{Tr}(A_1)|^{1+\epsilon} \lesssim |V|^{1+O(\epsilon)} \lesssim |\text{Tr}(V)|^{1+O(\epsilon)}$$

By applying theorem 4.3.7 with $V^u$ and $a^u$ we get,

$$|\text{Tr}(V^2)\cdot\text{Tr}(V^2)| + |\text{Tr}(V^2) + \text{Tr}(V^2)| \lesssim C|\text{Tr}(V^2)|^{1+C\epsilon}$$

so we are done with (6.4.2e). $\Box$
Chapter 7

Main results

7.1 From matrices to traces and back in finite fields

Proposition 7.1.1. There exists $C \in \mathbb{R}_+$ such that the following holds. Let $F$ be a finite field and $G = SL_2(F)$ and let $\varepsilon \in \mathbb{R}_+$ with $\varepsilon < \frac{1}{C}$. Let $V \subseteq SL_2(F)$ be a subset of diagonal matrices of size $|V| > C$.

Suppose

$$\text{Tr}(V) \text{ is an impure } \varepsilon\text{-field} \quad (7.1.1a)$$

and

$$|\text{Tr}(V^{|k|})| < |\text{Tr}(V)|^{1+\varepsilon}. \quad (7.1.1b)$$

Then we have,

$$\text{Tr}(V^{|k|}) \text{ is not } \varepsilon\text{-field.} \quad (7.1.1c)$$

Proof. Set $N := |\text{Tr}(V)|$. By the assumption (7.1.1a) there is some proper
subfield $\mathbb{E} < \mathbb{F}$ and some $x \in \mathbb{F}^\times$ such that we have

$$|\text{Tr}(V) \setminus x\mathbb{E}| < N^\varepsilon \quad \text{and} \quad |\mathbb{E}| < N^{1+\varepsilon}.$$ 

By the assumption (7.1.1a), $\text{Tr}(V)$ is an impure subfield so $|\text{Tr}(V) \setminus \mathbb{E}| > 0$. There are two cases to consider: either (1) $x \in \mathbb{E}$ or (2) $x \notin \mathbb{E}$.

Case (1): Suppose

$$x \in \mathbb{E} \quad \text{and} \quad 0 < |\text{Tr}(V) \setminus \mathbb{E}| < N^\varepsilon$$

and let $g \in V$ with $\text{Tr}(g) \notin \mathbb{E}$. Since $g(V|_{\mathbb{E}^\times}) \subseteq V^{[2]}$ we get by Lemma 5.4.1 that,

$$|V^{[2]}|_{|\mathbb{E}|} \geq |(g(V|_{\mathbb{E}^\times}))|_{|\mathbb{E}|} \quad (7.1.1.1)$$

$$\geq \frac{1}{2} |V|_{|\mathbb{E}^\times|}$$

$$\geq \frac{1}{2} (|V|_{|\mathbb{E}|} - 2)$$

$$\gg |V|_{|\mathbb{E}|}$$

$$\gg N - N^\varepsilon$$

$$\gg N.$$

By the assumption $|\text{Tr}(V^{[4]})| \leq N^{1+\varepsilon}$ so

$$\text{Tr}(V^{[2]}) \text{ cannot be } \varepsilon\text{-field.} \quad (7.1.1.2)$$

Indeed: the bound (7.1.1.1) exclude the possibility of $\text{Tr}(V^{[2]})$ to be $\mathbb{E}'$-field for $\mathbb{E}' = \mathbb{E}$ or any other coset $\mathbb{E}' = x\mathbb{E}$ of $\mathbb{E}$. Now for any other field $\mathbb{E}' \neq \mathbb{E}$ if $|\mathbb{E}'| \leq |\text{Tr}(V^{[2]})|^{1+\varepsilon}$ then

$$|\mathbb{E}'| \leq N^{1+O(\varepsilon)}$$
since $|\text{Tr}(V)| \leq N^{1+\varepsilon}$. Therefore the intersection of the field $E$ with any coset $x'E'$ is

$$|E \cap x'E'| \leq |E \cap E'| \leq N^{O(\varepsilon)}.$$ 

So the intersection is too small to contain $\text{Tr}(V|_E)$, since

$$|\text{Tr}(V|_E)| \geq \frac{1}{2}|V| \geq N - N^{\varepsilon} \gg N.$$ 

Therefore we are done with (7.1.1.2).

Case (2): Suppose

$$\text{Tr}(V) \subseteq xE \text{ with } |E| \leq N^{1+\varepsilon} \text{ and } x \not\in E.$$ 

This case is proved in a similarly to Case (1): By multiplying by some $g \in V|_x$ we get by Lemma 5.4.1 that at least $\frac{1}{2}|V|_{x(x^\times)}$ elements in $V^{[2]}$ have trace not in $xE$. Therefore, as was proved in (7.1.1.2) in Case (1), we find that $\text{Tr}(V^{[2]})$ cannot be $\varepsilon$-field.

In both cases we get that $\text{Tr}(V^{[2]})$ cannot be $\varepsilon$-field so we are done with (7.1.1c).

\[\square\]

**Proposition 7.1.2.** There exist $C \in \mathbb{R}_+ \text{ and } k \in \mathbb{N}_+ \text{ with } k > C$ such that the following holds. Let $F$ be a finite field, $G = \text{SL}_2(F)$ and let $\varepsilon \in \mathbb{R}_+$ with $\varepsilon < \frac{1}{C}$.

Let $E < F$ be a proper subfield, $A \subseteq \text{SL}_2(F)$ with $\langle A \rangle = G$. For $1 \leq i \leq 2$ denote $A_i = A^{[k_i]}$ and suppose

$$|A_3| < |A|^{1+\varepsilon}. \quad (7.1.2a)$$

Then there exists a semi simple element

$$g \in A_1 \cap G \text{ and } V \subseteq C_{A_2}(g)$$
such that

\[ |\text{Tr}(V)| > \frac{1}{C} |\text{Tr}(A_2)|^{1-C\varepsilon} \tag{7.1.2b} \]

\[ \text{Tr}(V) \subseteq \mathbb{F} \setminus \mathbb{E} \tag{7.1.2c} \]

**Proof.** In order to make the notations in the proof simpler, we will use the notation \( A_i := A^{[k_i]} \) and we will increase, during the proof, the value of \( k \).

By Lemma 5.4.14 there exists \( k_1 \in \mathbb{N}_+ \) such that for \( k \geq k_1 \) and \( B := A_{11}^{k_1} \) we have

\[ |B| = |A_{11}^{k_1}| \gg |A|. \tag{7.1.2.1} \]

Now let \( g \in A_1 \cap G_s \) be a semi simple element with

\[ \text{Fix}(g) = \{x_1, x_2\} \subseteq \mathbb{P}(\mathbb{F}). \]

Suppose that for any \( h \in A \) we have \( \text{Fix}(h) \subseteq \text{Fix}(g) \). Since \( \langle A \rangle = G \) we have \( \text{Fix}(A) = \emptyset \) so we can find \( h_1, h_2 \in A \) such that \( \text{Fix}(h_i) = \{x_i\} \) so \( \text{Fix}(h_1h_2) \cap \text{Fix}(g) = \emptyset \). In any case there exist \( h \in A^{[2]} \) such that

\[ \text{Fix}(h) \setminus \text{Fix}(g) \neq \emptyset. \]

Therefore by Lemma 6.1.4 we get that if \( k \geq \max\{k_1, 2\} \) then

\[ |B| \stackrel{(6.1.4a)}{\ll} |\text{Tr}(A)| |\text{Tr}(gB)||\text{Tr}(hB)| \tag{7.1.2.2} \]

\[ \leq |\text{Tr}(B)||\text{Tr}(A_2)|^2. \]

Now by by theorem 6.4.1 there exists \( k_2 \in \mathbb{N}_+ \) such that if \( k \geq \max\{2, k_1, k_2\} \) then we have,

\[ |\text{Tr}(A_2)| \stackrel{(6.4.1b)}{\ll} \frac{|A_2^{[k_2]}|^{4/3}}{|A_2|} \]

\[ \leq \frac{|A_3|^{4/3}}{|A|} \]

\[ \stackrel{(7.1.2a)}{\ll} |A|^{1/3+O(\varepsilon)}. \tag{7.1.2.3} \]
Therefore we conclude,

\[ |\text{Tr}(A_2)|^{3-O(\varepsilon)} \leq |A| \]
\[ \leq |B| \]
\[ \leq |\text{Tr}(B)||\text{Tr}(A_2)|^2 \]
\[ \leq |\text{Tr}(A_2)|^3 \]

Therefore we get

\[ |\text{Tr}(B)| \gg |\text{Tr}(A_2)|^{1-O(\varepsilon)} \]  \hspace{1cm} (7.1.2.4)
\[ |\text{Tr}(A_2)| \gg |A|^{1/3}. \]  \hspace{1cm} (7.1.2.5)

Now suppose

\[ k \geq \max\{3, k_1, k_2\}. \]

Therefore by corollary 6.1.3 there exists \( b \in B \) s.t.

\[ |C_{B^{-1}B}(b)| \overset{(6.1.3a)}{\geq} \frac{|\text{Tr}(B)||A_1|}{|A_1^{-1}A_1 A_1|} \]
\[ \overset{(7.1.2a)}{\geq} \frac{|\text{Tr}(B)||A|}{|A_3|} \]
\[ \overset{(7.1.2.6)}{\gg} |\text{Tr}(B)||A|^{-\varepsilon} \]
\[ \overset{(7.1.2.5)}{\gg} |\text{Tr}(B)||\text{Tr}(A_2)|^{-O(\varepsilon)} \]
\[ \overset{(7.1.2.4)}{\gg} |\text{Tr}(A_2)|^{1-O(\varepsilon)}. \]
Let $b$ be as in (7.1.2.6) and set\(^1\)

\[
C' := C_{B^{-1}B}(b) \\
C'' := C' \cup bC' \\
V := C''|_E.
\]

Note that $\text{Tr}(b) \notin E$ so $b$ is semi simple therefore we get that $C_G(b)$ are simultaneously diagonalizable therefore $|C'| \geq |C| - 2$ and $|\text{Tr}(V)| \geq \frac{1}{2}|V|$. Now by Lemma 5.4.1 we get that for any $c \in C''$ that,

either $\text{Tr}(c) \notin E$ or $\text{Tr}(bc) \notin E$ or $\text{Tr}(bc^{-1}) \notin E$.

Altogether we get that $V \subseteq C_{A_2}(b)$, since $V \subseteq A_1^{[3]} \subseteq A_2$, and

\[
|\text{Tr}(V)| \gg |V| \geq \frac{1}{2}|C'| \geq \frac{1}{2}(|C| - 2) \gg |C| \gg |\text{Tr}(A_2)|^{1-O(\varepsilon)}.
\]

\(\square\)

### 7.2 Conclusions

We extend the following key proposition of Helfgott (cf. [He, “Key Proposition” in §1.2]).

\(^1\)I want to thank H.Helfgott for a very fruitful discussions related the following argument.
Theorem 7.2.1 (Helfgott). For any \( \delta \in \mathbb{R}_+ \) there exist \( \varepsilon \in \mathbb{R}_+ \) such that for any finite field \( \mathbb{F}_p \) of prime order and any subset of generators \( A \) of \( G = \text{SL}_2(\mathbb{F}_p) \) we have,

\[
|A| < |G|^{1-\delta} \Rightarrow |A^{(3)}| > |A|^{1+\varepsilon}.
\]

Moreover, there exist absolute \( k \in \mathbb{N} \) and \( \delta_0 \in \mathbb{R}_+ \) such that

\[
|A| > |G|^{1-\delta_0} \Rightarrow A^{[k]} = G.
\]

The main result of this manuscript is the following extension of the theorem above.

Theorem 7.2.2 (Theorem 2.2.1 from the Introduction). There exists \( \varepsilon \in \mathbb{R}_+ \) such that the following holds for any finite field \( \mathbb{F}_q \). Let \( G \) be the group \( \text{SL}_2(\mathbb{F}_q) \) and let \( A \) be a generating set of \( G \). Then we have,

\[
|A^{(3)}| \geq \min\{|A|^{1+\varepsilon}, |G|\}. \tag{7.2.2a}
\]

Proof. By theorem 5.1.27 there exists \( C_0, \delta_0 \in \mathbb{R}_+ \) such that

\[
|A| \geq C_0|G|^{1-\delta_0} > C_0 q^{2^\frac{2}{3}} \quad (5.1.27a) \quad A^{(3)} = G.
\]

Therefore if \( |A| \geq C_0|G|^{1-\delta_0} \) we are done with (7.2.2a). So we will assume from now

\[
|A| \ll |G|^{1-\delta_0}.
\]

Let \( 3 \leq k \in \mathbb{N}_+ \), \( \varepsilon_0 \in \mathbb{R}_+ \) and \( c_0 \in \mathbb{R}_+ \) with \( c_0 \leq 1 \). By Lemma 4.1.9 the following holds with \( \varepsilon' = \frac{\varepsilon}{2\delta} \) and \( c' = \frac{c}{2} \). For any group \( G \) and any finite subset \( A \subseteq G \) we have,

\[
|A^{[k]}| > c_0|A|^{1+\varepsilon_0} \quad (4.1.9c) \quad |A^{(3)}| > c'|A|^{1+\varepsilon'}.
\]
Now if $|A|^{ε'/2} < \frac{1}{2}$ then $A$ is bounded but if $A$ is a subset of generators we get that

$$|A^{(3)}| \geq |A| + 2 \geq |A|^{1+ε''}$$

for some $ε'' \in \mathbb{R}_+$. Therefore for any $ε < \min\{ε'/2, ε''\}$ we get that,

$$|A^{[k]}| > c_0 |A|^{1+ε_0} \implies |A^{(3)}| > |A|^{1+ε}.$$

Therefore in order to prove (7.2.2a) it is enough to prove

$$|A^{[k]}| > c_0 |A|^{1+ε_0}$$

for some absolute $3 \leq k \in \mathbb{N}_+$ and $c_0, ε_0 \in \mathbb{R}_+$.

We will use the notation

$$A_i := A^{[k_i]}$$

and we will prove that there exists $C \in \mathbb{R}_+$ and $i \in \mathbb{N}_+$ such that the following holds. There exists $k \in \mathbb{N}$ and $ε \in \mathbb{R}_+$ with $k > C$ and $ε < \frac{1}{C}$ such that we have (provided $|A| \leq C|G|^{1-δ_0}$)

$$|A_i| = |A^{[k_i]}| > \frac{1}{C}|A|^{1+ε_i}.$$

By Lemma 5.3.4 there exists $k_0 \in \mathbb{N}_+$ such that if $k > k_0$ then $\text{Tr}(A_1)$ is not contained in any subfield i.e.,

$$\langle \text{Tr}(A_1) \rangle = \mathbb{F}_q.$$

Set $ε_1 := \frac{1}{2}$. Note that if $0 < f < ε_1$, then

$$1 - f < \frac{1}{1+f} < 1 - \frac{1}{2}f < 1 - Ω(f)$$

and similarly $1 + f < \frac{1}{1-f} < 1 + 2f < 1 + O(f)$. 

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By theorem 6.4.2 (6.4.2b) there exists $k_1 \in \mathbb{N}_+$ (and implicit $C_1 > 0$) such that for any $\varepsilon \in \mathbb{R}_+$ and $k > \max\{k_0, k_1\}$ we have either

$$ |A_2| \geq |A|^{1+\varepsilon} $$

(so we are done) or

$$ |A|^{1/3} \ll |\text{Tr}(A_1)| \ll |A|^{1/3+O(\varepsilon)} \quad (7.2.2.1) $$

(explicitly: $\frac{1}{C_1} |A|^{1/3} < |\text{Tr}(A_1)| < C_1 |A|^{1/3+C_1\varepsilon}$). By applying again theorem 6.4.2 (6.4.2b) now for $A_1$, for any $k > \max\{k_0, k_1\}$ we have either

$$ |A_3| \geq |A_1|^{1+\varepsilon^2} $$

(so we are done) or

$$ |A_1|^{1/3} \ll |\text{Tr}(A_2)| \ll |A_1|^{1/3+O(\varepsilon^2)}. \quad (7.2.2.2) $$

Now if $|A_3| < |A|^{1+\varepsilon^2}$ and in addition

$$ |\text{Tr}(A_1)|^{1+\varepsilon} \leq |\text{Tr}(A_2)| $$

then both (7.2.2.1) and (7.2.2.2) hold and we get,

$$ |A| \overset{(7.2.2.1)}{\ll} |\text{Tr}(A_1)|^3 
< |\text{Tr}(A_2)|^{3(1-\frac{1}{2}\varepsilon)} 
\overset{(7.2.2.2)}{\ll} |A_1|^{(1-\frac{1}{2}\varepsilon)(1+O(\varepsilon^2))} 
< |A_1|^{1-\frac{1}{2}\varepsilon+O(\varepsilon^2)} 
\leq |A_1|^{1-\Omega(\varepsilon)}. $$

In other words $|A|^{1+\Omega(\varepsilon)} \ll |A_1|$ so we are done with (7.2.2a).
Restating the conclusion explicitly, there exist $C_2 = k_2 \in \mathbb{N}_+$ and $\varepsilon_2 = \frac{1}{k_2}$ such that if $k > \max\{k_i\}$ and $\varepsilon < \min\{\varepsilon_i\}$ then

$$|A| \ll |A_1|^{1-\frac{1}{2}\varepsilon + O(\varepsilon^2)} \leq |A_1|^{1-\frac{1}{4}\varepsilon}$$

i.e., $\frac{1}{C_2}|A|^{1+\frac{1}{4}\varepsilon} < |A_1|$.

Note that so far we have changed $k$ and $\varepsilon$ independently to get the required growth property (7.2.2a). We can summarize what have proved so far as follows. There exist some $C \in \mathbb{R}_+$ such that for any $\varepsilon < \frac{1}{C}$ and for any $k > C$ we have,

$$|\text{Tr}(A_2)| \geq |\text{Tr}(A_1)|^{1+\varepsilon} \Rightarrow |A_3| \geq \frac{1}{C}|A|^{1+\varepsilon_2}.$$ We can restate this in the $\Omega$-language as follows,

$$|\text{Tr}(A_2)| \gg |\text{Tr}(A_1)|^{1+\Omega(\varepsilon)} \Rightarrow |A_3| \gg |A|^{1+\Omega(\varepsilon^2)}.$$ (7.2.2.3)

Therefore in order to complete the proof, we can assume from now

$$|A_3| < |A|^{1+\varepsilon_2} \quad \text{and} \quad |\text{Tr}(A_2)| < |\text{Tr}(A_1)|^{1+\varepsilon_2}.$$ (7.2.2.4)

In particular,

$$|A_2| < |A_1|^{1+\varepsilon^2}$$ (7.2.2.5)

$$|\text{Tr}(A_2)| < |\text{Tr}(A_1)|^{1+\varepsilon^2}$$

so we can apply theorem 6.4.2 (6.4.2d). Therefore there exists a semi simple element $g \in A_1 \cap G_s$ and $V := C_{A_2}(g)$ with

$$|V| > \frac{1}{C_1} |\text{Tr}(A_1)|^{1-C_1\varepsilon^2}$$

$$> \frac{1}{C_1} |\text{Tr}(A_1)|^{1-C_1\varepsilon}.$$ (7.2.2.6)

$^2$Note that there was nothing special in choosing $\varepsilon^2$ above, and we can replace $\varepsilon^2$ with any $f = o(\varepsilon)$.  

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and in addition
\[ |\text{Tr}(V^2)\cdot\text{Tr}(V^2)| + |\text{Tr}(V^2) + \text{Tr}(V^2)| < C_1|\text{Tr}(V^2)|^{1+C_1\varepsilon^2} \]
\[ < C_1|\text{Tr}(V^2)|^{1+C_1\varepsilon}. \]  
\text{(7.2.2.7)}

Therefore we get,
\[ |\text{Tr}(V)| \geq \frac{1}{2}|V| \]
\text{(7.2.2.6)}
\[ \Rightarrow |\text{Tr}(A_1)|^{1-O(\varepsilon^2)} \]
\text{(7.2.2.5)}
\[ \Rightarrow |\text{Tr}(A_2)|^{1-O(\varepsilon^2)(1-\varepsilon^2)} \]
\text{(7.2.2.8)}
\[ \Rightarrow |\text{Tr}(A_2)|^{1-O(\varepsilon^2)} \]
\text{(7.2.2.9)}

and also
\[ |\text{Tr}(V^2)| \geq \frac{1}{4}|V| \]
\text{(7.2.2.8)}
\[ \Rightarrow |\text{Tr}(A_2)|^{1-O(\varepsilon^2)} \]
\text{(7.2.2.2)}
\[ \Rightarrow |A_1|^{1/3-O(\varepsilon^2)} \]
\[ \geq k^{1/3-O(\varepsilon^2)}. \]
\text{(7.2.2.9)}

Denote
\[ V_1 := V \]
\[ U_1 := \text{Tr}(V_1^2) \]
\[ K_1 := C_1|U_1|^{C_1\varepsilon}. \]

Therefore we get
\[ |U_1\cdot U_1| + |U_1 + U_1|^{(7.2.2.7)} < K_1|U_1| \]
\text{(7.2.2.10)}

and for some absolute \( C_3 \in \mathbb{R}_+ \)
\[ |U_1| \geq \frac{1}{C_3}k^{1/3-C_3\varepsilon}. \]
Therefore by theorem 4.2.1 there exists (an absolute) $C \in \mathbb{R}_+$ such that either

$$|U_1| < CK_1^C$$

or for some subfield $E_1 \leq F$ and $x_1 \in F^*$ we have,

$$|U_1 \setminus x_1 E_1| \leq CK_1^C \quad \text{and} \quad |E| \leq CK_1^C |U_1|.$$  \hspace{1cm} (7.2.2.11)

Now set $C_4 := 2CC_1$ and $\varepsilon_3 = \frac{1}{\varepsilon_1 CC_4}$. Since $CK_1^C = CC_1 |U_1|^{CC_3}$ we get that for any $\varepsilon < \min \{\varepsilon_i\}$ there exists $k > \max \{k_i\}$ such that

$$CK_1^C < |U_1|^{C_4 \varepsilon} < |U_1|.$$ 

Therefore the alternative (7.2.2.11) must hold and we get

$$|U_1 \setminus x_1 E_1| \leq |U_1|^{C_4 \varepsilon} \quad \text{and} \quad |E| \leq |U_1|^{1+C_4 \varepsilon}. \hspace{1cm} (7.2.2.12)$$

In particular we get

$$|A_2| \overset{(7.2.2.2)}{\gg} \text{Tr}(A_2)^{3-O(\varepsilon^2)} \overset{(7.2.2.12)}{\geq} |U_1|^{3-O(\varepsilon^2)} \geq |E|^{3-O(\varepsilon^2)}.$$

Therefore for any $\delta_0 \in \mathbb{R}_+$ we can find $C_5 \in \mathbb{R}_+$ large enough\footnote{such that $C_5 > \max \{k_i\}$ and $\frac{1}{C_5} < \min \{\varepsilon_i\}$.} and we can find $\varepsilon < \frac{1}{C_5}$ and $k > C_5$ such that

$$|A_2| > |E|^{3-\delta_0}.$$  

Therefore if $E = F$ then we are done by theorem 5.1.27 which guarantee bounded generation for large subsets of $\text{SL}_2(F)$. 

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Therefore in order to complete the proof of (7.2.2a) we are left to treat the case that for some proper subfield $\mathbb{E} < \mathbb{F}$

$$\text{Tr}(V^2) \text{ is } C_4\varepsilon\text{-field.} \quad (7.2.2.13)$$

Suppose first that,

$$\text{Tr}(V^2) \text{ is an impure } O(\varepsilon)\text{-field.} \quad (7.2.2.14)$$

By proposition 7.1.1 (7.1.1c) we get that

$$\text{Tr}(V^{2[4]}) \text{ is not } C_4\varepsilon\text{-field.}$$

Denote $V_2 := V_1^{[4]}$ and $U_2 := \text{Tr}(V_2^2)$ and $K_2 := |U_2|^{C_4\varepsilon}$ and

$$K'_2 := (K_2/C)^{1/C} \gg |U_2|^{2C_4\varepsilon}$$

Therefore by theorem 4.2.1 we get

$$|\text{Tr}(V_2^2) \cdot \text{Tr}(V_2^2)| + |\text{Tr}(V_2^2) + \text{Tr}(V_2^2)| \gg K'_2 |\text{Tr}(V_2^2)| \gg |\text{Tr}(V_2^2)|^{1+2C_4\varepsilon} \quad (7.2.2.15)$$

Now by corollary 6.2.3 there exists $k_5 \in \mathbb{N}_+$ such that the following hold for any $k > k_5$. For any $w \in \text{GL}_2(\overline{\mathbb{F}})$ there exists $g \in A_1$ such that $g^w$ has no zero entries. In particular we can apply this for the basis $v \in \text{GL}_2(\overline{\mathbb{F}})$ for which $V^v$ are simultaneously diagonalizable.

Therefore by the bound (7.2.2.15) we can apply theorem 4.3.7 and we get
that for some absolute $C_6 = k_6 \in \mathbb{N}_+$ and for $k > \max \{k_i\}$ we have

\[
|\text{Tr}(A_3)| \gg |\text{Tr}(V_2^{[4]}V_2^{[4]})| \quad (4.3.7d) \\
\gg |\text{Tr}(V_2)|^{1+2C_1\varepsilon/C_6} \\
\gg |\text{Tr}(V_2)|^{1+\Omega(\varepsilon)} \\
\geq |\text{Tr}(V_1^2)|^{1+\Omega(\varepsilon)} \quad (7.2.2.9) \\
\gg |\text{Tr}(A_1)|^{(1+\Omega(\varepsilon))(1-O(\varepsilon^2))} \\
\gg |\text{Tr}(A_1)|^{1+\Omega(\varepsilon)}.
\]

Therefore we get

\[
|\text{Tr}(A_3)| \gg |\text{Tr}(A_1)|^{1+\Omega(\varepsilon)} \quad (7.2.2.16)
\]

and this imply that

\[
\text{either} \quad |\text{Tr}(A_2)| \gg |\text{Tr}(A_1)|^{1+\Omega(\varepsilon)} \quad \text{or} \quad |\text{Tr}(A_3)| \gg |\text{Tr}(A_2)|^{1+\Omega(\varepsilon)}.
\]

Therefore by what we have proved in (7.2.2.3) we get that

\[
\text{either} \quad |A_3| \gg |A|^{1+\Omega(\varepsilon^2)} \quad \text{or} \quad |A_4| \gg |A_1|^{1+\Omega(\varepsilon^2)}
\]

In other words by (7.2.2.3) we get

\[
|\text{Tr}(A_3)| \gg |\text{Tr}(A_1)|^{1+\Omega(\varepsilon)} \Rightarrow |A_4| \gg |A|^{1+\Omega(\varepsilon^2)} \quad (7.2.2.17)
\]

Therefore if (7.2.2.14) holds (the case of impure proper almost subfield) then by (7.2.2.16) we are done with the proof of (7.2.2a).

Therefore we are left to treat the second subcase of (7.2.2.13) that

\[
\text{Tr}(V^2) \quad \text{is pure} \quad O(\varepsilon)\text{-field} \quad (7.2.2.18)
\]

for some proper subfield $E < F$. Note that if $\text{Tr}(V^{[4]}) \not\subseteq E$ then we are done with (7.2.2a) by a similar argument to (7.2.2.14) which treated the case of impure $O(\varepsilon)$-field.
Therefore in order to complete (7.2.2a) we can assume in addition to (7.2.2.18) that

\[
\text{Tr}(V) \subseteq \text{Tr}(V^4) \subseteq E
\]

and

\[
|E| \ll \left| \frac{\text{Tr}(V)}{1+O(\varepsilon)} \right| \ll \left| \frac{\text{Tr}(A_1)}{1+O(\varepsilon)} \right|.
\]

(7.2.2.19)

Now suppose we can find \( g \in A_1 \) such that \( 4^{\text{Prod}(g^v)} \notin E \), then by Lemma 4.3.8 we get

\[
|\text{Tr}(V)|^{2-O(\varepsilon)} \ll |\text{Tr}(VV^g)| \ll |\text{Tr}(A_3)|
\]

(4.3.8a)

(7.2.2.20)

so \( |\text{Tr}(V)| \ll |\text{Tr}(A_3)|^{1+O(\varepsilon)} \). On the other hand

\[
|\text{Tr}(V)| \geq |\text{Tr}(A_1)|^{1-O(\varepsilon)}
\]

(7.2.2.9)

(7.2.2.21)

therefore

\[
|\text{Tr}(A_1)| \ll |\text{Tr}(A_3)|^{1+O(\varepsilon)}
\]

(7.2.2.17)

\[
|\text{Tr}(A_1)| \ll |\text{Tr}(A_3)|^{1+O(\varepsilon)}(1+O(\varepsilon))
\]

\[
|\text{Tr}(A_3)|^{1+O(\varepsilon)}
\]

\[
|\text{Tr}(A_3)|^{1+O(\varepsilon)}
\]

Therefore by (7.2.2.17) we are done with the case (7.2.2.18).

Therefore we are left to treat the case that (7.2.2.18) and (7.2.2.19) hold and \( \text{Prod}(g^v) \in E \) for any \( g \in A_1 \). Therefore by fact 4.3.5 we get for any \( g \in A_1 \) that

\[
\text{Tr}(VV^g) \subseteq E.
\]

(4.3.5a)

(7.2.2.21)

In particular by definition 3.1.3 we get,

\[
\text{Tr}([V, A_1]_{set}) \subseteq \text{Tr}(VV'A_1) \subseteq E.
\]

(3.1.3a)

(7.2.2.21)

\( ^4v \) was a basis that \( V^v \) were diagonal.
Therefore the only case that we are left to resolve, in order to complete (7.2.2a), is:

\[
\text{Tr}(VV^A) \subseteq \mathbb{E} \quad (7.2.2.21)
\]

\[
|\mathbb{E}| \ll |\text{Tr}(V)|^{1+O(\varepsilon)} \quad (7.2.2.22)
\]

\[
|\text{Tr}(V)| \gg |\text{Tr}(A_2)|^{1-O(\varepsilon)}.
\]

Now by proposition 7.1.2 there exists \( C_7 \in \mathbb{R}_+ \) such that the following holds with \( k_7 = C_7 \) and \( \varepsilon_7 = \frac{1}{C_7} \). Assume \( k > \max\{k_i\} \) and \( \varepsilon < \min\{\varepsilon_i\} \). Since \( |A_3| < |A|^{1+o(\varepsilon)} < |A|^{1+\varepsilon} \) we get by proposition 7.1.2 that there exists a semi simple element \( h \in A_1 \cap G_s \) and \( U \subseteq C_{A_2}(h) \) with

\[
|\text{Tr}(U)| \gg |\text{Tr}(A_2)|^{1-O(\varepsilon)}
\]

\[
\text{Tr}(U) \subseteq \mathbb{F} \setminus \mathbb{E}.
\]

Therefore there exists \( u \in \text{SL}_2(\mathbb{F}_{q^2}) \) such that \( U^u \) are diagonal and

\[
\text{Tr}(U) \cap \mathbb{E} = \emptyset. \quad (7.2.2.23)
\]

By repeating all the cases before (7.2.2.22), but now with \( \text{Tr}(U) \) instead of \( \text{Tr}(V) \), we get that the only case that we need to treat, in order to complete the theorem, is:

\( \text{Tr}(U) \) is \( O(\varepsilon) \)-field

for some proper field \( \mathbb{E}' < \mathbb{F} \) and also,

\[
\text{Tr}(UU^A) \subseteq \mathbb{E}'
\]

\[
|\mathbb{E}'| \ll |\text{Tr}(U)|^{1+O(\varepsilon)} \quad (7.2.2.24)
\]

\[
|\text{Tr}(U)| \gg |\text{Tr}(A_2)|^{1-O(\varepsilon)}.
\]
Let us check what we got so far. Denote

\[ N := |\text{Tr}(A_2)| \quad \text{and} \quad E'' := E \cap E'. \]

By the construction of \( U \) in (7.2.2.23) we get that \( E \neq E' \) and

\[ N^{1-O(\varepsilon)} \ll \min\{|E|, |E'|\} < \max\{|E|, |E'|\} \ll N^{1+O(\varepsilon)}. \]

Therefore we get,

\[ |E''| \ll N^{O(\varepsilon)}. \tag{7.2.2.25} \]

In particular by combining (7.2.2.22) and (7.2.2.24) we get that

\[ \text{Tr}([U, V]_{set}) \subseteq E''. \tag{7.2.2.26} \]

Now if \( V \) and \( U \) do not have a common fix point\(^5\), then by Lemma 4.3.8 we get,

\[ |\text{Tr}([U, V]_{set})| \overset{(4.3.8b)}{\gg} |\text{Tr}(V)| \gg N^{1-O(\varepsilon)}. \]

Therefore by (7.2.2.26) and (7.2.2.25) we get a contradiction for \( \varepsilon \) small enough.

On the other hand, if \( V \) and \( U \) do have a common fix point, then denote their eigen values \( X \) and \( Y \) respectively. So \( \text{tr}(X^{[4]}) \subseteq E \) and \( \text{tr}(Y^{[4]}) \subseteq E' \) and \( X \subseteq K \) and \( Y \subseteq K' \) where \( K \) and \( K' \) are the two quadratic extensions of \( E \) and \( E' \) respectively. Denote \( K'' = K \cap K' \) so we get

\[ |K''| = |E''|^2 \ll N^{O(\varepsilon)}. \]

\(^5\text{i.e., Fix}(g) \cap \text{Fix}(h) = \text{Fix}(U) \cap \text{Fix}(V) = \emptyset.\)
Therefore we get,

\[
| \text{Tr}(A_3) | \geq | \text{Tr}(A_2(2)) | \\
\geq | \text{Tr}(UV) | \\
= | \text{tr}(XY) | \\
\geq \frac{1}{2} |XY| \\
\geq \frac{1}{2} |X||Y| \\
\geq \frac{1}{2} |K''| \\
\gg N^{2-O(\varepsilon)} \\
\gg | \text{Tr}(A_2)|^{2-O(\varepsilon)}.
\]

Therefore by (7.2.2.17) we are done with the case (7.2.2.18). So the proof is complete.

\[\Box\]

**Corollary 7.2.3** (Corollary 2.2.2 from the Introduction). There exist \( C, d \in \mathbb{R}_+ \) such that the following holds for any finite field \( \mathbb{F}_q \). Let \( A \) be a subset of generators of \( G = \text{SL}_2(\mathbb{F}_q) \). Then we have,

\[ \text{diam}^+(G, A) < C \log^d(|G|) \quad (7.2.3a) \]

and for any \( \delta \in \mathbb{R}_+ \) we have,

\[ |A| > |G|^\delta \Rightarrow \text{diam}^+(G, A) < C \left( \frac{1}{\delta} \right)^d. \quad (7.2.3b) \]

**Proof.** First suppose \( \langle A \rangle = G \) and \( |A| > |G|^\delta \). Then by theorem 7.2.2 we get for some absolute \( \varepsilon_0 \in \mathbb{R}_+ \) that

\[ |A|^{(3)} \geq \min\{ |A|^{1+\varepsilon_0}, |G| \} \geq |G|^{\min\{\delta(1+\varepsilon_0), 1\}} \quad (7.2.3.1) \]

By iterating (7.2.3.1) we get for any \( i \geq 0 \) that

\[ A^{(3^i)} \geq |G|^{\min\{\delta(1+\varepsilon_0)^i, 1\}}. \]
Therefore by taking $i$ such that $1 - \delta_0 < \delta(1 + \varepsilon_0)^i$ we get that

$$A^{(3i+1)} = G.$$

Now if we take $i$ such that,

$$(1 - \delta_0)\frac{1}{\delta} < (1 + \varepsilon_0)^i < (1 + \varepsilon_0)^{i+1} \leq 4(1 - \delta_0)\frac{1}{\delta}$$

then for $d := \log_{1+\varepsilon_0}(3)$ and $C_1 := (4(1 - \delta_0))^d$ we get

$$3^{i+1} = (1 + \varepsilon_0)^{(i+1)d} \leq C_1(\frac{1}{\delta})^d$$

and $A^{(3i+1)} = G$ so we are done with $(7.2.3b)$. √

Now for arbitrary subset of generators we have $|A| \geq 2$ therefore

$$|A^{[3]}| \geq \min\left\{2^{(1+\varepsilon_0)^i}, |G|\right\}.$$

Therefore if we take $\delta = \frac{1}{2}$ and $i$ such that

$$\delta \log(|G|) \leq (1 + \varepsilon_0)^i \leq 2\delta \log(|G|)$$

then for $C_2 := (2\delta)^d$ we get,

$$3^i = (1 + \varepsilon_0)^{id} \leq C_2 \log^d(|G|) \quad (7.2.3.2)$$

and $|A^{(3i)}| \geq |G|^\delta$. Denote $m_1 := 3^i \leq C_2 \log^d(|G|)$ such that $(7.2.3.2)$ holds. Therefore if we apply the first argument to $A_1 = A^{(m_1)}$ we get that there exists $m_2 \leq C_1(\frac{1}{\delta})^d$ with $A_1^{[m_2]} = A^{[m_1m_2]} = G$. Therefore

$$m_1m_2 \leq C_3 \log^d(|G|)$$

where $C_3 = C_1C_2(\frac{1}{\delta})^d$, so we are done with $(7.2.3a)$. √
Chapter 8

Further conjectures and questions

We include here some interesting questions that we encountered during this work.

8.1 Trace generation

In the proof of theorem 7.2.2 we have seen in lines (7.2.2.20), that if one can prove that the set of products\(^1\) Prod\((A^k)\) is not contained in any subfield, then by Lemma 4.3.8 (4.3.8a), we could complete the proof with a much simpler argument.

If it would be true, then we will get (in particular) that this property is preserved under conjugation, since the generation property of the matrices is preserved. Is it true for any subset of generators?

\(^1\)see definition 3.1.7.
**Question 1.** Does there exists an absolute $k \in \mathbb{N}_+$ such that for any finite field $\mathbb{F}_q$ and $A \subseteq \text{SL}_2(\mathbb{F}_q)$ we have

$$\langle A \rangle = \text{SL}_2(\mathbb{F}_q) \implies \langle \text{Prod}(A^k) \rangle = \mathbb{F}_q.$$ 

By using the invariant argument of Lemma 5.2.3 we have seen,

$$\langle A \rangle = \text{SL}_2(\mathbb{F}_q) \implies \langle \text{Tr}(A^6) \rangle = \mathbb{F}_q.$$ 

Can one use this property of $\text{Tr}(A^k)$ in order to prove question 1?

### 8.2 Avoiding proper subfields

We have seen in corollary 5.4.14 how to escape from one subfield. I.e., there are at least $c|A|$ elements in $A^k$ with trace outside this field, where $k$ and $c$ are absolute constants. Clearly we can always assume the subfield is a maximal subfield. More precisely: for any proper $E < \mathbb{F}_q$ we have\(^2\),

$$\langle A \rangle = \text{SL}_2(\mathbb{F}_q) \implies |A^k|_E \gg |A|.$$ 

Therefore if $\mathbb{F}_{p^n}$ has only one maximal subfield (i.e., $n$ is a prime power), then we could complete the proof of theorem 7.2.2 with a much simpler argument:

First, in the steps after equation (7.2.2.4) we would invoke proposition 7.1.2, instead of theorem 6.4.2 (6.4.2d). By this proposition we would get that $\text{Tr}(V^2) \subseteq \mathbb{F}\setminus E$ so in particular $\text{Tr}(V^2)$ cannot be a pure $O(\varepsilon)$ field. Under these terms, the proof will be much shorter since its second half (7.2.2.18), which deals with the case of a pure $O(\varepsilon)$-fields, is no longer needed.

\(^2\)See definition 5.4.2 of $A|_E$. 

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Now suppose $\mathbb{F}_{p^n}$ and $n = p_1^{n_1} \cdots p_m^{n_m}$. Therefore if we could escape from all the $m$ maximal subfields \textit{simultaneously} i.e., finding $c|A|$ elements in $A^{[k]}$ with traces which are primitive generators of $\mathbb{F}_q$, we could simplify the proof of theorem 7.2.2 as above. Is it possible? Is it possible to avoid with the traces simultaneity a \textit{bounded} number of subfields?

\textbf{Question 2.} Let $\mathbb{F}_q$ be a finite field and let $\mathbb{E}_1, \ldots, \mathbb{E}_m < \mathbb{F}_q$ be proper subfields of $\mathbb{F}$ and denote

$$\mathbb{W} = \bigcup_{1 \leq i \leq m} \mathbb{E}_i.$$ 

Is is possible to find $k \in \mathbb{N}_+$ and $c \in \mathbb{R}_+$ (which may depend in $m$) such that

$$\langle A \rangle = \text{SL}_2(\mathbb{F}_q) \Rightarrow |A^{[k]}|_{\mathbb{W}} \geq c|A|?$$ 

Is is possible to find an \textit{absolute} $k \in \mathbb{N}_+$ and $c \in \mathbb{R}_+$ as above?

\section*{8.3 Growth of trace functions}

In the proof of theorem 7.2.2 in equations (7.2.2.19) and (7.2.2.24), we got two subsets $V_i \subseteq \text{SL}_2(\mathbb{F})$ and two $v_i \in \text{SL}_2(\mathbb{F})$ such that $V_i^{v_i}$ are diagonal matrices. Denote the eigen values of $V_i$ by $X_i$. I.e., $X_i$ are the diagonal entries of $V_i^{v_i} = D_{X_i}$. Denote $T_i := \text{Tr}(V_i)$ and note that $|T_i| \sim |X_i|$. In the course of the proof of theorem 7.2.2 we found two proper subfields $\mathbb{E}_i < \mathbb{F}$.
such that,

\[ N^{1-O(\epsilon)} \ll |T_1| \leq |T_2| = N \]

\[ T_i \subseteq E_i \]

\[ X_i \subseteq K_i \]

\[ |E_i| \ll N^{1+O(\epsilon)} \]

and \( E_1 \neq E_2 \)

where \( K_i \) are the quadratic extension of \( E_i \) respectively. Therefore,

\[ |T_1 \cap T_2| \leq |E_1 \cap E_2| \ll N^{O(\epsilon)} \]

\[ |X_1 \cap X_2| \leq |K_1 \cap K_2| \ll N^{O(\epsilon)} \]

and so we have,

\[ |X_1 X_2|, |X_1 + X_2|, |X_1 T_2|, |X_1 + T_2| \ldots \gg N^{2-O(\epsilon)}. \]

Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and denote\(^3\),

\[ D_x := \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \]

\[ \text{tr}(x) := \text{Tr}(D_x) = x + x^{-1} \]

\[ \text{tr}_g(x, y) := \text{Tr}(D_x(D_y)^g) \]

\[ = ad \cdot \text{tr}(xy) - bc \cdot \text{tr}(x/y). \]

\(^3\)See definitions 4.3.1, 4.3.2 and facts 4.3.6, 4.3.5 in §4.3.
Now set $u_i := v_i^{-1}$ and $h := u_2gv_1$ and define,

$$\text{Tr}_g(g_1, g_2) := \text{Tr}((g_1)(g_2)^g)$$

$$= \text{Tr}((D_{x_1})^{u_1} (D_{x_2})^{u_2g})$$

$$= \text{Tr}\left(\begin{pmatrix} x_1 & 0 \\ 0 & x_1^{-1} \end{pmatrix} \begin{pmatrix} x_2 & 0 \\ 0 & x_2^{-1} \end{pmatrix}^{u_2gv_1}\right)$$

$$= \text{tr}_h(x_1, x_2).$$

Now define $T_g : X_1 \times X_2 \to \mathbb{F}$ by

$$T_g(x_1, x_2) := \text{tr}_{u_2gv_1}(x_1, x_2).$$

Therefore if we denote $t = \text{Prod}(h)$ then,

$$\text{Im}(T_g) = T_g(X_1, X_2)$$

$$= \text{Tr}(V_1 V_2^g)$$

$$= \{ t \cdot \text{tr}(x_1x_2) + (1 - t) \cdot \text{tr}(x_1/x_2) : x_i \in X_i \}$$

Now in the proof of theorem 7.2.2 we have seen that Trace Growth imply Growth of Matrices. Therefore if one can prove that

$$|\text{Im}(T_g)| = |\text{Tr}(V_1 V_2^g)| \gg N^{1+\Omega(\epsilon)}$$

then we could simplify the arguments in the proof of theorem 7.2.2.

**Question 3.** Can one prove for some absolute $\delta > 0$ that

$$|\text{Im}(T_g)| \geq N^{1+\delta - O(\epsilon)}$$

with $T_g$ and $N$ as defined above.

Now denote

$$N_i(c) := |\{(x, y) : T_g(x, y) = c\}|$$
the number of solution for \( T_g(x, y) = c \) where

\[
T_g(x, y) = t \cdot \text{tr}(xy) + (1 - t) \cdot \text{tr}(x/y)
\]

as was defined above.

What is the best upper bound \( N_t(c) \) for a general \( t \) ?

**Question 4.** Can one prove for some absolute \( \delta > 0 \) we have,

\[
t \in \mathbb{K}_1 \mathbb{K}_2 \setminus \{0, 1\} \implies |N_t(c)| \ll N^{1-\delta+O(\varepsilon)}.
\]

If the answer is affirmative then

\[
|\text{Im}(T_g)| \gg |X_1||X_2| \implies \frac{N^{2-O(\varepsilon)}}{N^{1-\delta+O(\varepsilon)}} \gg N^{1+\delta-O(\varepsilon)}
\]

so question 3 is resolved also. Note that we have seen in Lemma 4.3.8,

\[
t \notin \mathbb{K}_1 \mathbb{K}_2 \overset{(4.3.8a)}{\implies} \text{mult}(T_g) = 1 \implies |\text{Im}(T_g)| = |X_1||X_2| \gg N^{2-O(\varepsilon)}.
\]

And since \( |X_1X_2|, |X_1X_2^{-1}| \gg N^{2-O(\varepsilon)} \) we also have

\[
t \in \{0, 1\} \implies |\text{Im}(T_g)| \gg N^{2-O(\varepsilon)}.
\]
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