One-parameter Lefschetz class of homotopies on torus

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Abstract. The main result of this paper states that if \( F : T \times I \to T \) is a homotopy on 2-dimensional torus and \( \pi_1(T, x_0) = \langle u, v | [u, v] = 1 \rangle \), then the one-parameter Lefschetz class \( L(F) \) of \( F \) is given by

\[
L(F) = \pm N(F)\alpha,
\]

where \( N(F) \) is the one-parameter Nielsen number of \( F \), and \( \alpha = [u] \in H_1(\pi_1(T), \mathbb{Z}) \).

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1. Introduction

Let \( F : X \times I \to X \) be a homotopy on a finite CW complex and \( G = \pi_1(X, x_0) \). We say that \((x, t) \in X \times I\) is a fixed point of \( F \) if \( F(x, t) = x \). R. Geoghegan and A. Nicas in [5] developed a one-parameter fixed point theory and defined the one-parameter trace \( R(F) \) of \( F \) to study the fixed points of \( F \). From trace \( R(F) \) we define the one-parameter Nielsen number \( N(F) \) of \( F \) and the one-parameter Lefschetz class \( L(F) \). These invariants are computable, depending only on the homotopy class of \( F \) relative to \( X \times \{0, 1\} \), and are strictly analogous to well-known invariants in classical Nielsen fixed point theory.

The study of the fixed points of a homotopy has been considered by many authors, see for example [2,7] and [4]. An important application of the trace \( R(F) \) is the following: Given a smooth flow \( \Psi : M \times \mathbb{R} \to M \) on a closed oriented manifold one may regard any finite portion of \( \Psi \) as a homotopy. Write \( F = \Psi| : M \times [a, b] \to M \). The traces \( L(F) \) and \( R(F) \) recognize dynamical meaning of \( \Psi \). When \( a > 0 \), \( L(F) \) detects the Fuller homology class, derived from Fuller’s index theory. Thus is possible to study periodic orbits of \( \Psi \) using the one-parameter theory, see [6].
In the algebraic setting, the trace $R(F)$ is a 1-chain in $HH_1(ZG, (ZG)\phi)$, Hochschild homology, where the structure of the bimodule $(ZG)\phi$ is given in Sect. 2. This 1-chain gives information about the fixed points of $F$, that is, using $R(F)$ it is possible to define the one-parameter Nielsen number $N(F)$ of $F$ and the one-parameter Lefschetz class $L(F)$ of $F$. $N(F)$ is the number of non-zero $C$-components in $R(F)$, it is a lower bound to the number of path-components in $\text{Fix}(F)$, that is, using $R(F)$ it is possible to define the one-parameter Nielsen number $N(F)$ of $F$ and the one-parameter Lefschetz class $L(F)$ of $F$. $N(F)$ is then the number of non-zero $C$-components in $R(F)$, it is a lower bound to the number of path-components in $\text{Fix}(F)$, fixed point set of $F$.

Consider $X = T$ the 2-dimensional torus and $G = \pi_1(T, x_0) = \langle u, v \mid [u, v] = 1 \rangle$. Let $F : T \times I \to T$ be a homotopy. The main purpose of this paper was to show that $L(F) = \pm N(F)\alpha$, where $\alpha = [u]$ is one generator in $H_1(G)$.

In [1] R.B.S. Brooks et al. showed that if $f : X \to X$ is any map on a $k$-dimensional torus $X$ then $N(f) = |L(f)|$, where $N(f)$ is the Nielsen number and $L(f)$ the Lefschetz number of $f$. In some sense our result is a version of this result for one-parameter case when $k = 2$. In [8] some computations of the one-parameter Nielsen number $N(F)$ for homotopies on the torus were given.

This paper is organized into five sections. In Sect. 2 we present a review of one-parameter fixed point theory. In Sect. 3 we study the semiconjugacy classes on the torus and present a brief review of incidence systems. Section 4 contains the proof of the main result, which is Theorem 4.3.

2. One-parameter fixed point theory

In this section we present to the definition of the one-parameter trace for a homotopy $F : X \times I \to X$, where $X$ is a finite CW complex. For a complete description of the one-parameter fixed point theory see [5].

2.1. Hochschild homology traces

Let $R$ be a ring and $M$ an $R - R$ bimodule, that is, a left and right $R$-module satisfying $(r_1 m)r_2 = r_1 (mr_2)$ for all $m \in M$, and $r_1, r_2 \in R$. The Hochschild chain complex $\{C_\ast(R, M), d\}$ is given by $C_n(R, M) = R^\otimes n \otimes M$ where $R^\otimes n$ is the tensor product of $n$ copies of $R$, taken over the integers, and
\[ d_n(r_1 \otimes \cdots \otimes r_n \otimes m) = r_2 \otimes \cdots \otimes r_n \otimes mr_1 \]
\[ + \sum_{i=1}^{n-1} (-1)^i r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \otimes m \]
\[ + (-1)^n r_1 \otimes \cdots \otimes r_{n-1} \otimes r_n m. \]

The \( n \)-th homology of this complex is the Hochschild homology of \( R \) with coefficient bimodule \( M \), which is denoted by \( HH_n(R, M) \). To compute \( HH_1 \) and \( HH_0 \) we have the formula \( d_2(r_1 \otimes r_2 \otimes m) = r_2 \otimes mr_1 - r_1 r_2 \otimes m + r_1 \otimes r_2 m \) and \( d_1(r \otimes m) = mr - rm \).

**Lemma 2.1.** If \( 1 \in R \) is the unit element and \( m \in M \), then the 1-chain \( 1 \otimes m \) is a boundary.

**Proof.** \( d_2(1 \otimes 1 \otimes m) = 1 \otimes m - 1 \otimes m + 1 \otimes m = 1 \otimes m. \)

The Hochschild homology will arise in the following situation: let \( G \) be a group and \( \phi : G \to G \) an endomorphism. Also denote by \( \phi \) the induced ring homomorphism \( \mathbb{Z}G \to \mathbb{Z}G \). Take the ring \( R = \mathbb{Z}G \) and \( M = (\mathbb{Z}G)^\phi \) the \( \mathbb{Z}G - \mathbb{Z}G \) bimodule whose underlying abelian group is \( \mathbb{Z}G \) and the bimodule structure is given by \( g.m = gm \) and \( m.g = m\phi(g) \).

Two elements \( g_1, g_2 \) in \( G \) are semiconjugate if and only if there exists \( g \in G \) such that \( g_1 = gg_2\phi(g^{-1}) \). We write \( C(g) \) for the semiconjugacy class containing \( g \) and \( G_\phi \) for the set of semiconjugacy classes. Thus, we can decompose \( G \) in the union of its semiconjugacy classes. This partition induces a direct sum decomposition of \( HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \).

In fact, each generating chain \( \gamma = g_1 \otimes \cdots \otimes g_n \otimes m \) can be written in canonical form as \( g_1 \otimes \cdots \otimes g_n \otimes g_{n-1}^{-1} \cdots g_1^{-1} g \) where \( g = g_1 \cdots g_n m \in G \) “marks” a semiconjugacy class. The decomposition \( (\mathbb{Z}G)^\phi \equiv \bigoplus_{C \in G_\phi} \mathbb{Z}C \) as a direct sum of abelian groups determines a decomposition of chains complexes \( C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \equiv \bigoplus_{C \in G_\phi} C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C \), where \( C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C \) is the subgroup of \( C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \) generated by those generating chains whose markers lie in \( C \). Therefore, we have the following isomorphism: \( HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \equiv \bigoplus_{C \in G_\phi} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C \) where \( HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C \) is the summand which corresponds to the homology classes marked by the elements of \( C \). This summand is called the \( C \)-component.

Let \( Z(h) = \{ g \in G | h = gh\phi(g^{-1}) \} \) be the semicentralizer of \( h \in G \). Choosing representatives \( g_C \in C \), we have the following proposition whose proofs is given in [5]:

**Proposition 2.2.** Choosing representatives \( g_C \in C \) then we have
\[ HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G_\phi} H_*(Z(g_C))_C, \]
where \( H_*(Z(g_C))_C \) corresponds to the summand \( HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C \).

Given an \( m \times n \) matrix over \( R \) and an \( n \times m \) matrix over \( M \) we define \( A \otimes B \) to be the \( m \times m \) matrix with entries in \( R \otimes M \) given by \( (A \otimes B)_{ij} = \)
\[ \sum_{k=1}^{n} A_{ik} \otimes B_{kj} \]. The trace of \( A \otimes B \), written \( \text{trace}(A \otimes B) \), is given by
\[ \sum_{i=1}^{n} \sum_{k=1}^{m} A_{ik} \otimes B_{ki} \in C_{1}(R, M) \]. The 1-chain \( \text{trace}(A \otimes B) \) is a cycle if and only if \( \text{trace}(AB) = \text{trace}(B\phi(A)) \), in which case we denote its homology class by \( T_{1}(A \otimes B) \in H_{1}(R, M) \).

### 2.2. One-parameter fixed point theory

Let \( X \) be a finite connected CW complex and \( F : X \times I \to X \) a cellular homotopy. We consider \( I = [0, 1] \) with the usual CW structure and orientation of cells, and \( X \times I \) with the product CW structure, where its cells are given the product orientation. Pick a basepoint \( (v, 0) \in X \times I \), and a basepath \( \tau \) in \( X \) from \( v \) to \( F(v, 0) \). We identify \( \pi_{1}(X \times I, (v, 0)) \equiv G \) with \( \pi_{1}(X, v) \) via the isomorphism induced by projection \( p : X \times I \to X \). We write \( \phi : G \to G \) for the homomorphism:

\[ \pi_{1}(X \times I, (v, 0)) \xrightarrow{F} \pi_{1}(X, F(v, 0)) \xrightarrow{c_{\ast}} \pi_{1}(X, v) \]

We choose a lift \( \tilde{E} \) in the universal cover, \( \tilde{X} \), of \( X \) for each cell \( E \) and we orient \( \tilde{E} \) compatibly with \( E \). Let \( \tilde{\tau} \) be the lift of the basepath \( \tau \) which starts in the basepoint \( \tilde{v} \in \tilde{X} \) and \( \tilde{F} : \tilde{X} \times I \to \tilde{X} \) the unique lift of \( F \) satisfying \( \tilde{F}(\tilde{v}, 0) = \tilde{\tau}(1) \). We can regard \( C_{\ast}(\tilde{X}) \) as a right \( \mathbb{Z}G \) chain complex as follows: if \( \omega \) is a loop at \( v \) which lifts to a path \( \tilde{\omega} \) starting at \( \tilde{v} \), then \( \tilde{E}[\omega]^{-1} = h[\omega](\tilde{E}) \), where \( h[\omega] \) is the covering transformation sending \( \tilde{v} \) to \( \tilde{\omega}(1) \). The homotopy \( \tilde{F} \) induces a chain homotopy \( \tilde{D}_{k} : C_{k}(\tilde{X}) \to C_{k+1}(\tilde{X}) \) given by

\[ \tilde{D}_{k}(\tilde{E}) = (-1)^{k+1} \tilde{F}_{k}(\tilde{E}) \times I \in C_{k+1}(\tilde{X}), \]

for each cell \( \tilde{E} \in \tilde{X} \). This chain homotopy satisfies; \( \tilde{D}(\tilde{E})g = \tilde{D}(\tilde{E}g) \phi(g) \) and the boundary operator \( \tilde{\partial}_{k} : C_{k}(\tilde{X}) \to C_{k-1}(\tilde{X}) \) satisfies; \( \tilde{\partial}(\tilde{E}) = \partial(\tilde{E})g \).

Define endomorphism of \( \oplus_{k} C_{k}(\tilde{X}) \) by \( \tilde{D}_{\ast} = \oplus_{k}(1)^{k+1} \tilde{D}_{k}, \tilde{\partial}_{\ast} = \oplus_{k} \tilde{\partial}_{k}, \tilde{F}_{0\ast} = \oplus_{k}(1)^{k} \tilde{F}_{0k} \) and \( \tilde{F}_{1\ast} = \oplus_{k}(1)^{k} \tilde{F}_{1k} \). We consider trace(\( \tilde{\partial}_{\ast} \otimes \tilde{D}_{\ast} \)) \in \( H_{1}(C_{\ast}(\mathbb{Z}G, (\mathbb{Z}G)\phi)) \). This is a Hochschild 1-chain whose boundary is as follows: trace(\( \tilde{\partial}_{\ast} \phi(\tilde{\partial}_{\ast}) - \tilde{\partial}_{\ast} \tilde{D}_{\ast} \)). We denote by \( G_{\phi}(\partial(F)) \) the subset of \( G_{\phi} \) consisting of semiconjugacy classes associated with fixed points of \( F_{0} \) or \( F_{1} \).

**Definition 2.3.** The one-parameter trace of homotopy \( F \) is as follows:

\[ R(F) \equiv T_{1}(\tilde{\partial}_{\ast} \otimes \tilde{D}_{\ast}, G_{\phi}(\partial(F))) \in \bigoplus_{C \in G_{\phi} \cap G_{\phi}(\partial(F))} H_{1}(\mathbb{Z}G, (\mathbb{Z}G)\phi) \]

\[ \cong \bigoplus_{C \in G_{\phi} \cap G_{\phi}(\partial(F))} H_{1}(\mathbb{Z}(g_{C})). \]

**Definition 2.4.** The \( C \)-component of \( R(F) \) is denoted by \( i(F, C) \in H_{1}(\mathbb{Z}G, (\mathbb{Z}G)\phi) \). We call it the fixed point index of \( F \) corresponding to semiconjugacy class \( C \in G_{\phi} \). A fixed point index \( i(F, C) \) of \( F \) is zero if each cycle in \( i(F, C) \) is homologous to zero.

**Definition 2.5.** Given a cellular homotopy \( F : X \times I \to X \) the one-parameter Nielsen number, \( N(F) \), of \( F \) is the number of components \( i(F, C) \) with nonzero fixed point index \( i(F, C) \).
Definition 2.6. The one-parameter Lefschetz class, \( L(F) \), of \( F \) is defined by
\[
L(F) = \sum_{C \in G \phi - G \phi(\partial F)} j_C(i(F,C))
\]
, where \( j_C : H_1(Z(g_C)) \to H_1(G) \) is induced by the inclusion \( Z(g_C) \subset G \).

From [5] we have the following theorems:

Theorem 2.7. (Invariance) Let \( F,G : X \times I \to X \) be cellular; if \( F \) is homotopic to \( G \) relative to \( X \times \{0,1\} \) then \( R(F) = R(G) \).

Theorem 2.8. (one-parameter Lefschetz fixed point theorem) If \( L(F) \neq 0 \), then every map homotopic to \( F \) relative to \( X \times \{0,1\} \) has a fixed point not in the same fixed point class as any fixed point in \( X \times \{0,1\} \). In particular, if \( F_0 \) and \( F_1 \) are fixed point free, every map homotopic to \( F \) relative to \( X \times \{0,1\} \) has a fixed point.

Theorem 2.9. (one-parameter Nielsen fixed point theorem) Every map homotopic to \( F \) relative to \( X \times \{0,1\} \) has at least \( N(F) \) fixed point classes other than the fixed point classes which meet \( X \times \{0,1\} \). In particular, if \( F_0 \) and \( F_1 \) are fixed point free maps, then every map homotopic to \( F \) relative to \( X \times \{0,1\} \) has at least \( N(F) \) path components.

3. Semiconjugacy classes on torus

In this section we describe some results about the semiconjugacy classes on torus related to a homotopy \( F : T \times I \to T \). We denote by \( \phi = c_r \circ F_\# \) the homomorphism described in Sect. 2.

Let \( T \) be the torus defined as the quotient space \( \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{Z} \). We denote by \((x,y)\) the elements in \( \mathbb{R} \times \mathbb{R} \) and by \([[(x,y)]\) elements in \( T \). We take \( w = [(0,0)] \in T \) and \( G = \pi_1(T,w) = \langle u,v|uvu^{-1}v^{-1} = 1 \rangle \), where \( u \) is the class of the path \([[(x,0)]\], \( 0 \leq x \leq 1 \), and \( v \) the class of the path \([[(0,y)]\], \( 0 \leq y \leq 1 \). Thus, the homomorphism \( \phi : G \to G \) is given by \( \phi(u) = u^{b_1}v^{b_2} \) and \( \phi(v) = u^{b_3}v^{b_4} \). Therefore, \( \phi(u^m v^n) = u^{mb_1+nb_3}v^{mb_2+nb_4} \), for all \( m,n \in \mathbb{Z} \).

We represent this homomorphism by the matrix:
\[
[\phi] = \begin{pmatrix}
    b_1 & b_3 \\
    b_2 & b_4
\end{pmatrix}
\]

Proposition 3.1. Two elements \( g_1 = u^{m_1}v^{n_1} \) and \( g_2 = u^{m_2}v^{n_2} \) in \( G \) belong to the same semiconjugacy class if, and only if, there are integers \( m,n \) satisfying the following equations:

\[
\begin{cases}
    m(b_1 - 1) + nb_3 = m_2 - m_1 \\
    mb_2 + n(b_4 - 1) = n_2 - n_1
\end{cases}
\]

Proof. If there is \( g = u^m v^n \in G \) satisfying \( g_1 = gg_2\phi(g)^{-1} \) then we obtain the equation of the proposition. The other direction is analogous. \( \square \)
We take the isomorphism $\Theta : G \to \mathbb{Z} \times \mathbb{Z}$ such that $\Theta(u^mv^n) = (m, n)$. By Proposition 3.1 two elements $g_1 = u^{m_1}v^{n_1}$ and $g_2 = u^{m_2}v^{n_2}$ in $G$ belong to the same semiconjugacy class if and only if there exists $z \in \mathbb{Z} \times \mathbb{Z}$ satisfying: $([\phi] - I)z = \Theta(g_2g_1^{-1})$, where $I$ is the identity matrix.

**Corollary 3.2.** For each $g \in G$ the semicentralizer $Z(g)$ is isomorphic to the kernel of $[\phi] - I$.

**Lemma 3.3.** The 1-chain, $u^k v^l \otimes u^m v^n$, is a cycle if and only if the element $(k, l) \in \mathbb{Z} \times \mathbb{Z}$ belongs to the kernel of $[\phi] - I$.

**Proof.** If $u^k v^l \otimes u^m v^n$ is a cycle then $0 = d_1(u^k v^l \otimes u^m v^n) = u^m v^n \phi(u^k v^l) - u^k v^l u^m v^n = u^m v^n u^{kb_1 + lb_3} v^{kb_2 + lb_4} - u^k v^l u^m v^n = u^{m+kb_1 + lb_3} v^{kb_2 + lb_4 + n} - u^{k+m+1} v^{l+n}$. This implies $k(b_1 - 1) + lb_3 = 0$ and $kb_2 + l(b_4 - 1) = 0$. The other direction is analogous. \(\square\)

**Corollary 3.4.** If the matrix of the homomorphism $\phi$ is given by

$$[\phi] = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix}$$

with $b_3 \neq 0$ or $b_4 \neq 1$, then the 1-chain, $u^k v^l \otimes u^m v^n$, is a cycle if and only if $l = 0$.

Given a 2-chain $u^s v^t \otimes u^k v^l \otimes u^m v^n \in C_2(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$, by definition of Hochschild boundary operator we have

$$d_2(u^s v^t \otimes u^k v^l \otimes u^m v^n) = u^k v^l \otimes u^{m+sb_1 + lb_3} v^{n+sb_2 + lb_4} - u^{k+s+1} v^{l+t} \otimes u^m v^n + u^s v^t \otimes u^{k+m} v^{l+n}.$$

This expression will be important in the proof of the following result. Henceforth, we suppose that $[\phi]$ satisfies the assumptions of Corollary 3.4.

**Proposition 3.5.** The 1-chain; $u^k \otimes u^m v^n \in C_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ is homologous to the 1-chain; $ku \otimes u^{m+k-1} v^n$, for all $k, m, n \in \mathbb{Z}$.

**Proof.** Note that for $k = 0$ or 1 the proposition is true. We suppose that for some $s > 0 \in \mathbb{Z}$, the 1-chain $u^s \otimes u^m v^n$ is homologous to the 1-chain $su \otimes u^{m+s-1} v^n$; We write $u^s \otimes u^m v^n \sim su \otimes u^{m+s-1} v^n$. Considering the 2-chain, $u^s \otimes u \otimes u^m v^n \in C_2(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$. We have

$$d_2(u^s \otimes u \otimes u^m v^n) = u \otimes u^{m+s} v^n - u^{s+1} \otimes u^m v^n + u^s \otimes u^{1+m} v^n \sim u \otimes u^{m+s} v^n - u^{s+1} \otimes u^m v^n + su \otimes u^{1+m+s-1} v^n = (s+1)u \otimes u^{m+(s+1)-1} v^n - u^{s+1} \otimes u^m v^n.$$

Therefore, $(s+1)u \otimes u^{m+(s+1)-1} v^n \sim u^{s+1} \otimes u^m v^n$. By induction the result follows. The case $k < 0$ the proof is analogous. \(\square\)

**Lemma 3.6.** Each 1-chain $u^{-1} \otimes u^r v^s$, where $r, s \in \mathbb{Z}$ is homologous to the 1-chain $-u \otimes u^r v^s$, for some $\tilde{r}, \tilde{s} \in \mathbb{Z}$. Also, each 1-chain $v^{-1} \otimes u^r v^s, r, s \in \mathbb{Z}$ is homologous to a 1-chain $-v \otimes u^r v^s$. 

Corollary 3.4, then each Hochschild 1-chain of the following form:

$$d_2(u \otimes u^{-1} \otimes u^m v^n) = u^{-1} \otimes u^{m+1} v^n - 1 \otimes u^m v^n + u \otimes u^{m-1} v^n.$$ 

for each $m, n \in \mathbb{Z}$, and

$$d_2(v^{-1} \otimes v \otimes u^m v^n) = v^{-1} \otimes u^{m-b_3} v^{n-b_4} - 1 \otimes u^m v^n + v \otimes u^{m-1} v^{n+1}.$$ 

Proof. This affirmation follows from the next equations and Lemma 2.1:

$$d_2(u \otimes u^{-1} \otimes u^m v^n) = u^{-1} \otimes u^{m+1} v^n - 1 \otimes u^m v^n + u \otimes u^{m-1} v^n.$$ 

Lemma 3.7. Each 1-chain $u^{-1}v^{-1} \otimes u^m v^n$, where $m, n \in \mathbb{Z}$, is homologous to the 1-chain $u^{-1} \otimes u^{m_1} v^{n_1} + v^{-1} \otimes u^{m_2} v^{n_2}$, for some $m_1, n_1, m_2, n_2 \in \mathbb{Z}$.

Proof. Note that

$$d_2(v \otimes u^{-1}v^{-1} \otimes u^m v^n) = u^{-1}v^{-1} \otimes u^{m+b_3} v^{n+b_4} - u^{-1} \otimes u^m v^n + v \otimes u^{m-1} v^{n+1}.$$ 

By Lemma 3.6 the 1-chain $v \otimes u^{-1}m v^{-1+n}$ is homologous to the 1-chain $-v^{-1} \otimes u^m v^\bar{n}$, for some $\bar{m}, \bar{n} \in \mathbb{Z}$.

Proposition 3.8. If the matrix of the homomorphism $\phi$ is given as in the Corollary 3.4, then each Hochschild 1-chain of the following form: $\sum_{i=1}^p v^{-1} \otimes u^{m_i} v^{n_i}$, cannot be a cycle in $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$.

Proof. If $\sum_{i=1}^p v^{-1} \otimes u^{m_i} v^{n_i}$ is a cycle, then we must have

$$d_1 \left( \sum_{i=1}^p v^{-1} \otimes u^{m_i} v^{n_i} \right) = \sum_{i=1}^p u^{m_i-b_3} v^{n_i-b_4} - u^{m_i} v^{n_i-1} = 0 \quad (3.1)$$

The homomorphism $\phi : G = \pi_1(T, w) \to \mathbb{Z} \times \mathbb{Z}$, given by $\phi(u^m v^n) = (m, n)$ induces a homomorphism on the ring group $\phi : \mathbb{Z}[G] \to \mathbb{Z}[\mathbb{Z} \times \mathbb{Z}]$ given by $\overline{\phi}(\sum_{i=1}^t \alpha_i u^{m_i} v^{n_i}) = (\sum_{i=1}^t \alpha_i m_i, \sum_{i=1}^t \alpha_i n_i)$. Applying the homomorphism $\phi$ in equality (3.1) we obtain the following:

$$\left( \sum_{i=1}^p (m_i - b_3 - m_i), \sum_{i=1}^p (n_i - b_4 - (n_i - 1)) \right) = (0, 0),$$

which implies $\sum_{i=1}^p b_3 = 0$ and $\sum_{i=1}^p (b_4 - 1) = 0$, that is, $-pb_3 = 0$ and $-p(b_4 - 1) = 0$. But in the Corollary 3.4 we must have $b_3 \neq 0$ or $b_4 \neq 1$. Therefore, we have a contradiction, that is, the 1-chain $\sum_{i=1}^p v^{-1} \otimes u^{m_i} v^{n_i}$ cannot be a cycle in $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$. 

Proposition 3.9. Each 1-cycle $u^{-1} \otimes u^m v^n \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ is not trivial, that is, is not homologous to zero.

Proof. In fact, we can write $u^{-1} \otimes u^m v^n$ in the following form: $u^{-1} \otimes ug$, where $g = u^{m-1} v^n$. From Proposition 3.1 for each $h = u^k v^l \in G$, the semicentralizer of $h$, $Z(h)$, is given by $Z(h) = \{u^s | s \in \mathbb{Z}\} \cong \mathbb{Z}$. Therefore, $H_1(Z(h)) \cong \mathbb{Z}$. From [5, page 433], there is the following sequence of natural isomorphisms:

$$H_1(Z(h)) \to H_1(G, \mathbb{Z}(G/Z(h))) \to H_1(G, \mathbb{Z}(C(h))) \to HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_{C(h)}.$$
The class of element $u^s$ is sent in the class of the 1-cycle $u^s \otimes u^{-s}g$, which is homologous to a 1-cycle $-su^{-1} \otimes ug = -s(u^{-1} \otimes ug)$. Thus, if the 1-cycle is trivial, then we will obtain $H_1(Z(h)) \cong 0$, which is a contradiction. Therefore, the 1-cycle $u^{-1} \otimes u^m v^n$ cannot be trivial. □

The proof of following results can be found in [8].

**Proposition 3.10.** In the case $b_1 = 1$ and $b_2 = 0$ each 1-cycle $\sum_{i=1}^t a_i u^{k_i} v^{l_i} \otimes u^{m_i} v^{n_i} \in C(ZG, (ZG)^\phi)$ is homologous to a 1-cycle of the following form: $\sum_{i=1}^t \tilde{a}_i u \otimes \tilde{u}^{m_i} \tilde{v}^{n_i}$.

**Corollary 3.11.** Let $\sum_{i=1}^t u \otimes u^{m_i} v^{n_i} \in HH_1(ZG, (ZG)^\phi)$, $m_i, n_i \in \mathbb{Z}$ be a cycle. If the cycles $u \otimes u^{m_i} v^{n_i}$ and $u \otimes u^{m_j} v^{n_j}$ are in different semiconjugacy classes for $i \neq j$, $i, j \in \{1, ..., t\}$, then $\sum_{i=1}^t u \otimes u^{m_i} v^{n_i}$ is a nontrivial cycle. Each cycle $u \otimes u^{m_i} v^{n_i}$ projects to the class $[u]$, that is, one of the two generators of $H_1(G)$.

Next we recall the definition of incidence system given in [9].

### 3.1. Incidence system

An incidence system on the regular cell complex $X$ is a function which assigns to each ordered pair $(E, F)$, $(E$ a $p$-cell of $X$, $F$ a $(p-1)$-cell of $X$) a number $[E : F] = 0$ or $\pm 1$, in such a way that

1. $[E : F] = 0$ if $E$ does not contain $F$;
2. $[E : F] = \pm 1$ if $F \subset E$;
3. Let $G$ be a $(q-2)$-dimensional face of the $q$-cell $E$ ($q \geq 2$) and let $F_1, F_2$ be the $(q-1)$-dimensional faces of $E$ which contain $G$. Then $[E : F_1][F_1 : G] + [E : F_2][F_2 : G] = 0$;
4. Let $E$ be a 1-cell of $X$, $F_1$ and $F_2$ its 0-dimensional faces. Then $[E : F_1] + [E : F_2] = 0$

**Theorem 3.12.** [9, pg 84] Let $X$ be an oriented regular cell complex. If $E, F$ are faces of dimensions $p, p - 1$, respectively, define $[E : F] = [e : f]$, where $e$ and $f$ are the preferred orientations. The function $[\cdot : \cdot]$ so defined is an incidence system on $X$.

This theorem states that each orientation of $X$ determines an incidence system. The converse is also true by the next theorem:

**Theorem 3.13.** [9, pg 85] Let $[\cdot : \cdot]$ be an incidence system on the regular cell complex $X$. Then $X$ can be oriented so that, if $E$ and $F$ are cells of dimensions $q, q - 1$, respectively, and if $e, f$ are their preferred orientations, then $[E : F] = [e, f]$.
4. Homotopies on torus

In this section we prove the main result of this paper, which is Theorem 4.3. In the proof we use the following proposition whose proof is given in [3]:

Proposition 4.1. (D. Dimovski and R. Geoghegan) If $F' : T \times I \to T$ is a homotopy and $P : T \times I \to T$ the projection, then we can deform $F$, relative to $T \times \{0, 1\}$, to homotopy $F'$ such that $\text{Fix}(F)$ is transverse to the projection $P$, that is, $\text{Fix}(F) \cap T \times \{t\}$ is finite for each $t \in I$. $\text{Fix}(F)$ consists of oriented arcs and circles. Let $\text{Fix}(F, \partial)$ be the subset of $\text{Fix}(F)$ consisting of those circles of fixed points which are not in the same fixed point class as any fixed point of $F_0$ or $F_1$; this closed oriented 1-manifold lies in $T \times (0, 1) \subset T \times I$ (Fig. 1).

From Proposition 4.1 if $F : T \times I \to T$ is a homotopy and $P : T \times I \to T$ the projection, then we can choose $F$ such that $\text{Fix}(F)$ is transverse to the projection $P$. Thus, $\text{Fix}(F, \partial)$ is a closed oriented 1-manifold in the interior of $T \times I$. Let $E_F$ be space of all paths $\omega$ in $T \times I \times T$ from the graph $\Gamma_F = \{(x, t, F(x, t)) | (x, t) \in T \times I\}$ of $F$ to the graph $\Gamma_P = \{(x, t, x) | (x, t) \in T \times I\}$ of $P$ with the compact-open topology, that is, maps $\omega : [0, 1] \to T \times I \times T$ such that $\omega(0) \in \Gamma(F)$ and $\omega(1) \in \Gamma(P)$.

Let $C_1, \ldots, C_k$ be isolated circles in $\text{Fix}(F) \cap \text{int}(T \times I)$, oriented by the natural orientations, and $V = \bigcup C_j$. Then $V$ determines a family of circles $V'$ in $E_F$ via constant paths, i.e. each oriented isolated circle of fixed points $C : S^1 \to T \times I$ of $F$ determines an oriented circle $C' : S^1 \to E$ defined by $\text{con}(C(z))$ where $\text{con}(C(z))$ is the constant path at $C(z) = (x, t_0)$, that is, $\text{con}(C(z))(t) = (x, t_0, x)$ for each $t \in [0, 1]$. Therefore, we can write $\sum i(F, C_j)[C_j'] \in H_1(E_F).$ Since $C_j$ is transverse, then $i(F, C_j) = 1$ for all $j$. For more details see [6, page 693]. From [5] we have;

Proposition 4.2. Since $\pi_2(T) = 0$, then there is a isomorphism $\Psi : H_1(E_F) \to HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})$, where $G = \pi_1(T, x_0)$.

Theorem 4.3. (Main Theorem) Let $F : T \times I \to T$ be a homotopy and $\pi_1(T, [(0, 0)]) = < u, v | [u, v] = 1 >$. The one-parameter Lefschetz class $L(F)$
of $F$ satisfies
\[ L(F) = \pm N(F)\alpha \]
where $\alpha = [u]$ in $H_1(\pi_1(T), \mathbb{Z})$.

Proof. Using the previous notation, the proof of Theorem 4.3 will be done in two cases. Case I when $\det([\phi] - I) = L(F|_T) = 0$ and case II when $\det([\phi] - I) = L(F|_T) \neq 0$.

**Case I**

Let us suppose that the homomorphism $\phi$, induced by a homotopy $F$, satisfies $\det([\phi] - I) = 0$. Using the notation from Sect. 3 we can suppose that the matrix $[\phi]$ of the homomorphism $\phi$ is given by
\[ [\phi] = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix} \]
and $[\phi] \neq I \equiv (\text{Identity})$, that is, $b_1 = 1$ and $b_2 = 0$, with $b_3 \neq 0$ or $b_4 - 1 \neq 0$. This is done choosing a base $\{w_1, w_2\}$ for $T = \mathbb{R}^2/\mathbb{Z}^2$, where $w_1$ is a eigenvector of $[\phi]$ associated with 1. We also consider $\pi_1(T, [(0,0)]) = < u, v | uvu^{-1}v^{-1} >$.

Since $T$ is a polyhedron it has a natural structure of a regular CW-complex. From Proposition 4.1 of [5] the trace $R(F)$ is independent of the choice of orientation of cells on $T$. This independence is in terms of homology class. Here, first we define an incidence system $[ : ]$ for $T$ and after we orient $T$ using this incidence system, as in Theorem 3.13.

Define $[ : ]$ in the following way: For each 2-cell $E_k^2$ and each 1-cell $E_j^1$ on $T$ define $[E_k^2 : E_j^1] = 1$ if $E_j^1 \subset E_k^2$ and $[E_k^2 : E_j^1] = 0$ if $E_j^1 \not\subset E_k^2$. Given a 0-cell $G$ such that $G \subset [(1, x)] \subset T$ or $G \subset [(x, 1)] \subset T$, $0 \leq x \leq 1$, for each 1-cell $F$ define $[F : G] = 1$ if $G \subset F$ and $[F : G] = 0$ if $G \not\subset F$. Now define $[ : ]$ for other cells so that $[ : ]$ satisfies the conditions (1), (2), (3) and (4) of the incidence system definition. By Theorem 3.13 there is an orientation for $T$ that is compatible with this incidence system.

Choose a lift, $\tilde{E}_j^i$, in the universal cover $\mathbb{R}^2$ of $T$ for each cell $E_j^i, i = 0, 1, 2$, of $T$. We orient $\tilde{E}_j^i$ compatible with $E_j^i$. We can regard $C_*(\mathbb{R}^2)$ as a right $\mathbb{Z}[\pi_1(T)]$ chain complex, as was defined in Sect. 2.2. Since
\[ \partial_i(e_k^i) = \sum_j [e_k^i : e_j^{i-1}]e_j^{i-1} \]
and $[E_k^i : E_j^{i-1}] = [e_k^i : e_j^{i-1}]$, then, for the choice above, the entries of matrices of the operators $\tilde{\partial}_1$ and $\tilde{\partial}_2$ will be composed by elements $0, \pm 1, u^{-1}, \pm v^{-1}$, that is, the element $u^{-1}$ will appear only with plus sign. This is due to the definition of the incidence system above. It is important to observe that the trace $R(F)$ is algebraic and it depends on the cellular structure of the CW complex. From definition of $R(F)$ we have
\[ R(F) = tr \begin{pmatrix} -\tilde{\partial}_1 \otimes [\tilde{D}_0] & 0 \\ 0 & [\tilde{\partial}_2] \otimes [\tilde{D}_1] \end{pmatrix} \]
where the elements of the matrices $[\hat{\partial}_1]_{ij}$ and $[\hat{\partial}_2]_{kl}$ belong the set $\{0, \pm 1, u^{-1}, \pm v^{-1}\}$. Thus, the general expression of $R(F)$ in $C_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ is:

$$R(F) = -1 \otimes \left( \sum_{i=1}^m g_i \right) + 1 \otimes \left( \sum_{i=1}^m \bar{g}_i \right) + u^{-1} \otimes \left( \sum_{j=1}^n h_j \right) \pm v^{-1} \otimes \left( \sum_{k=1}^p t_k \right) \quad (4.1)$$

where $g_i = u^{m_i}v^{n_i}$, $\bar{g}_i = u^{\bar{n}_i}v^{\bar{n}_i}$, $h_j = u^{x_j}v^{y_j}$, $t_k = u^{z_k}v^{w_k}$, with $m_i, n_i, \bar{m}_i, \bar{n}_i, x_j, y_j, z_k, w_k \in \mathbb{Z}$. From Lemma 2.1 each 1-chain $1 \otimes u^rv^s, r, s \in \mathbb{Z}$ is a boundary. Therefore, the chains $-1 \otimes (\sum_{i=1}^m g_i)$ and $1 \otimes (\sum_{i=1}^m \bar{g}_i)$ are homologous to zero in $C_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$. From Corollary 3.4 each 1-chain $u^{-1} \otimes u^rv^s, r, s \in \mathbb{Z}$ is a cycle. We can see that applying directly the boundary operator $d_1 : C_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \to C_0(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$. Therefore, the 1-chain $u^{-1} \otimes (\sum_{j=1}^n h_j)$ in the equation 4.1 is a cycle.

By definition, the trace $R(F)$ is a 1-cycle in $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$. We know that each 1-chain $-1 \otimes (\sum_{i=1}^m g_i)$, $1 \otimes (\sum_{i=1}^m \bar{g}_i)$, $u^{-1} \otimes (\sum_{j=1}^n h_j)$, in the equation 4.1 is a cycle, but from Proposition 3.8 the 1-chain $\pm v^{-1} \otimes (\sum_{k=1}^p t_k)$, in the equation 4.1 is not a cycle. Therefore, in $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ we can omit the 1-chain $\pm v^{-1} \otimes (\sum_{k=1}^p t_k)$ in the trace $R(F)$. From Lemma 2.1 each 1-chain $1 \otimes u^rv^s, r, s \in \mathbb{Z}$ is a boundary; thus in $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ the general expression of $R(F)$ is

$$R(F) = u^{-1} \otimes \left( \sum_{j=1}^n h_j \right), \quad (4.2)$$

where $h_j = u^{x_j}v^{y_j}$ with $x_j, y_j \in \mathbb{Z}$.

By definition, the isomorphism given in the Proposition 4.2, detects the essential fixed points classes, or the isolated circles, in $\text{Fix}(G) \subset T \times (0, 1)$, for each $G : T \times I \to T$ homotopic to $F$ relative to $T \times \{0, 1\}$. If there exists $G$ homotopic to $F$, relative to $T \times \{0, 1\}$, without isolated circles in $\text{Fix}(G) \subset T \times (0, 1)$, then the terms $u^{-1} \otimes (\sum_{j=1}^n h_j)$ cannot appear in the one-parameter trace of $F$, because from Proposition 3.9 each 1-cycle $u^{-1} \otimes u^mv^n$ is non trivial, and, therefore, represents one nonzero C-component.

Since we have an injection of fixed point classes of $F$ into the set of the semiconjugacy classes, and by isomorphism given in the Proposition 4.2, we can conclude that in $C_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$, $R(F)$ is composed by three types of elements; $1 \otimes u^{m_i}v^{n_i}$ which may represent an inessential fixed point class, $u^{-1} \otimes u^{x_j}v^{y_j}$ which represents a essential fixed point class, that is, a isolated circle in $T \times (0, 1)$, and $-v^{-1} \otimes u^{z_k}v^{w_k}$ which may represent a fixed point class which intersects $T \times \{0, 1\}$, that is, an arc starting in $T \times \{0, 1\}$ and ending in $T \times \{0, 1\}$, see [5] and the figure of Proposition 4.1 for more details.

If each homotopy $G : T \times I \to T$ homotopic to $F$, relative to $T \times \{0, 1\}$, has isolated circles in the $\text{Fix}(G) \subset T \times (0, 1)$, then the one-parameter trace will contain the sum $u^{-1} \otimes (\sum_{j=1}^n h_j)$ which detects exactly these essential fixed point classes. The number of these isolated circles is finite because $T$ is compact.

If there exists $G : T \times I \to T$ homotopic to $F$, relative to $T \times \{0, 1\}$, such that each fixed point class of $G$ is inessential, then $R(F)$ will have the
following form:

\[ R(F) = \sum_{i=1}^{m} 1 \otimes u^{m_i} v^{n_i} \]

where \( m_i, n_i \in \mathbb{Z} \). In \( HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \) we will have \( R(F) = 0 \), because each element \( 1 \otimes u^{m_i} v^{n_i} \) is homologous to zero, and, therefore, \( L(F) = N(F) = 0 \).

Now, if each homotopy \( G : T \times I \rightarrow T \) homotopic to \( F \), relative to \( T \times \{0,1\} \), contains a finite amount of isolated circles in \( Fix(G) \), then the trace \( R(F) \) in \( HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \) will have the following form:

\[ R(F) = \sum_{j=1}^{n} u^{-1} \otimes u^{x_j} v^{y_j} \]

for some positive integer \( n \).

From Proposition 3.9 each 1-cycle \( u^{-1} \otimes u^{x_j} v^{y_j} \) is non trivial, and, therefore, represents a nonzero \( C \)-component. Also, from Proposition 4.2 each nonzero \( C \)-component in \( R(F) \) is represented by a unique isolated circle. Thus, two different elements \( u^{-1} \otimes u^{x_{j_1}} v^{y_{j_1}} \) and \( u^{-1} \otimes u^{x_{j_2}} v^{y_{j_2}} \) in \( R(F) \) belong to different semiconjugacy classes. From Corollary 3.11 the one-parameter Nielsen number of \( F \) will be

\[ N(F) = n. \]

From Sect. 2 the one-parameter Lefschetz class is the image of \( R(F) \) in \( H_1(\pi_1(T), \mathbb{Z}) \) by induced of inclusion \( i : Z(gC) \rightarrow \pi_1(T) \). Thus, each element \( u^{-1} \otimes u^{x_j} v^{y_j} \) is sent to \( H_1(\pi_1(T), \mathbb{Z}) \) in the class \(-[u] \), see page 433 of [5], or the argument used in the proof of the Proposition 3.9. Thus, the image of \( R(F) \) in \( H_1(\pi_1(T), \mathbb{Z}) \) will be

\[ L(F) = \sum_{j=1}^{n} -[u] = -n[u] = -N(F)[u]. \]

Take \( \alpha = -[u] \), which is one of the two generators of \( H_1(\pi_1(T), \mathbb{Z}) \). Is possible choose other incidence system such that \([E_k^2 : E_j^1] = -1\) for each 2-cell \( E_k^2 \) and 1-cell \( E_j^1 \). In this situation the general expression of \( R(F) \) will be \( -u^{-1} \otimes u^{x_j} v^{y_j} \) instead \( u^{-1} \otimes u^{x_j} v^{y_j} \). In the same way we will obtain \( L(F) = N(F)[u] \). Therefore,

\[ L(F) = \pm N(F) \alpha. \]

In the above situation, each fixed point class of \( F|_T, F \) restricted to the torus for each \( t \in [0,1] \), is inessential, because \( N(F|_T) = |\text{det}([\phi] - I)| = 0 \), by hypothesis. But \( F : T \times I \rightarrow T \) may have essential fixed point classes in \( T \times (0,1) \).

In the case \([\phi] = I\) the proof is analogous. In this situation, using the definition of the boundary operator in Hochschild homology, each 1-chain in \( R(F) \) is homologous to a chain of the form \( -v^{-1} \otimes u^{x_k} v^{y_k} \), which cannot be considered in the one-parameter trace of \( F \), in \( HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \), because it is not a cycle by Proposition 3.8.
Case II

In this case we have $\det([\phi] - I) = L(F_{1T}) \neq 0$. Therefore, by Corollary 3.2, for each element $g \in G$ the semicentralizer, $Z(g)$, of $g$ in $G$ is trivial. Thus, $H_1(Z(g_C)) = 0$ for each semiconjugacy class $C$, that is, $HH_1(Z_G, (ZG)^\phi) = 0$ which implies $R(F) = 0$, and, therefore, we have $L(F) = N(F) = 0$. □

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