Existence and Deformation Theory for
Scalar-Flat Kähler Metrics on Compact
Complex Surfaces

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1 Introduction

1.1 Motivation

The classical uniformization theorem provides a complete translation dictionary for the etymologically unrelated languages of complex 1-manifolds and constant curvature Riemannian 2-manifolds. In higher dimensions, there are a number of natural ways in which one might try to generalize this remarkable theorem; unfortunately, these various potential generalizations remain, for the most part, programs rather than established bodies of fact. However, the subject of the present article, namely the existence problem for zero-scalar-curvature Kähler metrics on compact complex 2-manifolds, occupies the cross-roads of several such avenues of research; and by clearing up a substantial piece of this problem, we thereby hope to facilitate the flow of traffic heading on to a number interesting destinations.

Purely in the context of Riemannian geometry, the most optimistic programs to generalize the classical uniformization theorem would try to equip

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every compact smooth manifold of a given dimension with a (small!) class of “optimal” or “canonical” metrics. In dimension four, one of the most natural versions would have us seek extrema (or perhaps just critical points) of the squared $L^2$-norm

$$\mathcal{R}(g) = \int_M \|R\|^2 \, d\mu$$

of the Riemann curvature tensor $R$ over the space of smooth Riemannian metrics $g$ on a given smooth, compact, oriented 4-manifold $M$. Using the Chern-Gauss-Bonnet formulas for the Euler characteristic $\chi$ and signature $\tau$ of our manifold $M$, one may easily show [29] that

$$\mathcal{R}(g) = -8\pi^2(3\tau + \chi) + \int_M (4\|W_+\|^2 + \frac{s^2}{12}) \, d\mu \geq -8\pi^2(3\tau + \chi),$$

with equality iff $W_+ = s = 0$; here $s$ denotes the scalar curvature and $W_+$ the self-dual Weyl curvature (cf. §1.4) of $g$. Metrics with $W_+ = s = 0$, when they exist, are thus absolute minima of $\mathcal{R}$, and it is therefore natural to try to determine which manifolds $M$ can admit such metrics. However, if the intersection form

$$\cup : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \to \mathbb{R}$$

is indefinite, an elementary Weitzenböck argument [25] shows that such a manifold admits an integrable complex structure with respect to which the metric is Kähler; conversely [19], any Kähler manifold of complex dimension 2 with $s \equiv 0$ automatically satisfies $W_+ = 0$ and has indefinite intersection form. Thus the problem of minimizing $\mathcal{R}$ on a smooth manifold leads us quite naturally to the problem of classifying compact Kähler manifolds of complex dimension 2 and scalar curvature zero— henceforth referred to as scalar-flat Kähler surfaces.

A related four-dimensional program would instead seek to optimize the conformal geometry of Riemannian metrics by seeking to minimize the conformally-invariant squared $L^2$-norm

$$\mathcal{W}(g) = \int_M \|W\|^2 \, d\mu$$

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1It should be pointed out that many manifolds which do not admit scalar-flat Kähler metrics nonetheless admit metrics which are absolute minima of $\mathcal{R}$. In particular [9], any Einstein metric on a compact 4-manifold provides an absolute minimum of $\mathcal{R}$. It is this fact which explains much of the current interest in this Riemannian functional.
of the conformal curvature over the space of conformal classes of Riemannian metrics on $M$. Since

$$\mathcal{W}(g) = -12\pi^2 \tau + 2 \int_M \|W_+\|^2 \, d\mu,$$

anti-self-dual metrics (i.e. metrics satisfying $W_+ = 0$) are obviously absolute minima of $\mathcal{W}$, scalar-flat Kähler surfaces again provide examples of absolute minima. Note that while there are strong topological constraints on anti-self-dual metrics with non-negative scalar curvature \cite{12} \cite{15} \cite{25} \cite{26} \cite{36}, the situation is radically different once the scalar curvature condition is dropped; in fact, it has recently been shown \cite{11} that the obstructions to the existence of anti-self-dual metrics on any oriented smooth 4-manifold are so weak that they can always be killed off by “blowing up points,” in the differentiable sense of taking connected sums with enough $\mathbb{CP}^2$’s. Our own results in this article will have something of a similar ring to them—while there are a number of obstructions to the existence of scalar-flat Kähler metrics on a compact complex surface, we will see that all but the crudest can be killed off by blowing up points.

In search of a natural bridge between complex and differential geometry, Calabi has proposed the problem of representing Kähler classes on compact complex manifolds by Kähler metrics of constant scalar curvature. Here again there is a natural variational approach to the problem, since such metrics are absolute minima of the functional

$$\mathcal{C}(g) = \int_M s^2 \, d\mu$$

among metrics in a fixed Kähler class; more generally, critical points of this functional have come to be known \cite{7} \cite{10} as extremal Kähler metrics. However, the existence of constant scalar curvature Kähler metrics is, in general, obstructed \cite{9} \cite{11} \cite{16} \cite{11}, and the known obstructions will necessarily play a central rôle in the present article—although perhaps not quite in the way the reader might expect. It is hoped that our present existence results will provide a useful way station, en route to a more general understanding of Calabi’s problem.

From a quite different perspective, namely that of Hawking’s Euclideanization program in gravitational physics, a fundamental problem is that of classifying compact Riemannian solutions of the Einstein-Maxwell
equations

\[-\frac{1}{2} r^\sharp = \text{trace-free part} \left( F^\sharp \circ F^\sharp \right) \]

dF = d \star F = 0

governing the interaction of the gravitational field, represented by a Riemannian metric \( g \), with the electromagnetic field, represented by a harmonic 2-form \( F \); here the metric has been used to identify the Ricci curvature and electromagnetic field with endomorphisms \( r^\sharp \) and \( F^\sharp \) of the tangent bundle. Any scalar-flat Kähler surface provides a solution of these equations once one sets

\[ F = \rho + \frac{1}{4} \omega , \]

where \( \omega \) and \( \rho \) are respectively the Kähler and Ricci forms\(^2\); conversely [4], these are essentially the only solutions with \( W_+ = 0 \). Thus the classification problem for scalar-flat Kähler surfaces may be seen as part of a quest to classify electro-gravitational instantons.

Finally, the Penrose twistor correspondence [37] gives quite a different way of generalizing the conformal surface/complex curve dictionary to dimension four. If \( M \) is a smooth oriented 4-manifold and

\[ [g] = \{ e^u g \} \]

is a conformal class of Riemannian metrics on \( M \), the space \( Z \) of orthogonal complex structures on \( TM \) compatible with the orientation is an almost-complex manifold of real dimension 6; it then turns out [3] that \( Z \) is a complex manifold iff \((M, [g])\) satisfies \( W_+ = 0 \). In this case, \( Z \) is called the twistor space of \((M, [g])\), and it turns out that both \( M \) and its anti-self-dual conformal structure can be reconstructed from this complex manifold. In particular, every scalar-flat Kähler surface \((M, g, J)\) has associated to it a compact complex 3-fold \( Z \); moreover, \((M, J)\) is naturally a complex submanifold of \( Z \). While this construction has elsewhere served [30][41] primarily as an excellent source of complex 3-folds with various “pathologically” non-Kählerian properties, the deformation theory of \( Z \) will here serve as our guiding light as we trek through the realm of scalar-flat Kähler geometry.

\(^2\)If \( F \) is to be viewed as a \( U(1) \)-gauge field, as it must be in realistic physical theories, our Kähler metric must also be of Hodge type— that is, cohomologous to the metric induced by some projective embedding,
1.2 Outline

We now provide the reader with a statement of the central result of the paper, followed by an indication of the structure of the argument.

Main Theorem Let $M$ be a compact complex surface which admits a Kähler metric whose scalar curvature has integral zero. Suppose $\pi_1(M)$ does not contain an Abelian subgroup of finite index. Then if $M$ is blown up at sufficiently many points, the resulting surface $\tilde{M}$ admits scalar-flat Kähler metrics.

§ 2 We study the behavior of the solution space under small deformations of complex structure of the complex surface in question. Our approach uses the twistor correspondence and a modified version of Kodaira-Spencer theory. This deformation theory is generally obstructed, but we are able to describe the obstructions completely in terms of the Futaki character of the algebra of holomorphic vector fields.

§ 3.1 We compute the Futaki character for all relevant complex surfaces, and use this to show that the deformation theory is unobstructed for all non-minimal surfaces.

§ 3.2 We describe a large class of exact solutions previously found in [27], and an improvement on those results which gives a classification of scalar-flat Kähler surfaces with non-trivial automorphism algebra.

§ 3.3 Using the bimeromorphic classification theory of surfaces, we show that every surface satisfying the hypotheses of the Main Theorem has blow-ups which are arbitrarily small deformations of surfaces on which we have exact solutions. Applying our deformation theory then proves the Main Theorem.

It should be emphasized that the Main Theorem’s fundamental-group hypothesis reflects the limitations of our techniques rather than a known obstruction to the existence of scalar-flat Kähler metrics. In fact, the article concludes with some speculations (Conjecture § 3.3) to the effect that this restriction is essentially superfluous.

Incidental to the main course of the argument, we will also encounter a remarkable empirical relationship, observed (§ 3.2) in two quite different classes
of explicit examples, which seems to link the existence problem for scalar-flat Kähler metrics to the stability of vector bundles with parabolic structure in the sense of Seshadri [39]; it is our belief, as expressed in Conjecture 2, §3.3, that this relationship will actually turn out to hold in complete generality.

1.3 Notation and Conventions

For the purposes of this section, \((M, g)\) will denote an oriented Riemannian 2m-manifold, although in the sequel we will specialize to the case \(m = 2\). We use \(d\mu\) to denote the volume form of \(g\), and \(\nabla\) to denote its Levi-Civita connection. The \(C^\infty\) sections of any smooth vector bundle \(\mathcal{V} \to M\) will be denoted by \(\mathcal{E}(\mathcal{V})\). The operation of raising (lowering) an index will be indicated by \(\#\) (\(\flat\)). We give the curvature tensor \(R\) the usual Riemannian sign:

\[
(\nabla_c \nabla_d - \nabla_d \nabla_c)\xi^a = R^a_{\ bcd} \xi^b \quad \forall \xi \in \mathcal{E}(TM).
\]

The Ricci tensor \(R^c_{\ abc}\) is denoted by \(r_{ab}\) and the scalar curvature \(R^{ab}_{\ ab}\) by \(s\).

The pointwise inner product induced by \(g\) on the tensor bundles will be denoted \((\ , \ )\), whereas the the global \(L^2\) inner product will be denoted by \(\langle \ , \ \rangle\). The formal adjoint of \(d\) with respect to this inner product will be denoted \(\delta\) and is given by the usual formula

\[
\delta = -\star d \star.
\]

The metric \(g\) is said to be Kähler if its holonomy group is conjugate to a subgroup of \(\text{U}(m) \subset \text{O}(2m)\). More concretely, this means that there is a compatible parallel almost-complex structure \(J\):

\[
J^2 = -1, \quad g(J\xi, J\eta) = g(\xi, \eta) \quad \forall \xi, \eta \in TM, \quad \nabla J = 0.
\]

If the holonomy of \(g\) happens to be a proper subgroup of \(\text{U}(m)\), there may be more than one \(J\) which satisfies (1.3); nonetheless, when we speak of a Kähler metric we will henceforth always assume that a particular choice of \(J\) has been made. We therefore have a decomposition \(\mathbb{C} \otimes TM = T^{1,0} \oplus T^{0,1}\) into the \(\pm i\) eigenspaces of \(J\), thereby inducing a decomposition

\[
\bigwedge^r_C = \bigoplus_{p+q = r} \bigwedge^{p,q}
\]
of the bundle of \( r \)-forms into forms of type \((p,q)\), as defined by \( \wedge^{p,q} := (\wedge^p T^{1,0})^* \otimes (\wedge^q T^{0,1})^* \); in particular, \( J \) induces a “standard” orientation of \( M \) by requiring that the \( 2m \)-form \( i_m \phi \wedge \bar{\phi} \) be positive for any non-zero element \( \phi \) of the canonical line bundle \( \kappa = \wedge^m T^{1,0} \). For brevity, we will use \( \mathcal{E}^r \) and \( \mathcal{E}^{p,q} \) to respectively denote \( \mathcal{E}(\wedge^r) \) and \( \mathcal{E}(\wedge^{p,q}) \). Because \( \nabla \) is torsion-free, \([\mathcal{E}(T^{1,0}), \mathcal{E}(T^{1,0})] \subset \mathcal{E}(T^{1,0})\), and the the Newlander-Nirenberg \cite{NewlanderNirenberg} theorem therefore asserts that \( J \) is integrable— that is, there exists a system of local coordinates for which \( J \) becomes the standard almost-complex structure on \( \mathbb{C}^m \), making \( M \) a complex \( m \)-manifold. The Kähler form \( \omega \) and Ricci form \( \rho \) are then defined by the formulae

\[
\omega(X,Y) = g(JX,Y), \quad \rho(X,Y) = r(JX,Y) \quad \forall X,Y \in TM ;
\]

both are real closed forms of type \((1,1)\). Conversely, a closed real \((1,1)\)-form on complex manifold is a Kähler form iff the symmetric form \( g \) it defines implicitly via \((1.5)\) is positive definite. The deRham class \([\omega] \in H^2(M)\) of the Kähler form is called the Kähler class. It is a central fact of Kähler geometry that \( \rho \) is exactly the curvature of the Chern connection on \( \kappa^{-1} = \wedge^m T^{1,0} \); in particular, \( \rho \) is completely determined by \( J \) and \( d\mu \) alone, and the deRham class of \( \rho/2\pi \) is just the first Chern class \( c_1(M) := c_1(T^{1,0}M) = c_1(\kappa^{-1})\).

Composing \( J \) with \( d \) yields a new real operator

\[
d^c := i(\bar{\partial} - \partial).
\]

The formal adjoint of \( d^c \) is denoted by \( \delta^c \) and is given by

\[
\delta^c = - \star d^c \star .
\]

On a Kähler manifold \( d, d^c \) and \( \delta, \delta^c \) are related by the so-called Kähler identities

\[
\delta = [\Lambda, d^c], \quad - \delta^c = [\Lambda, d] \quad \text{(1.8)}
\]

\[
- d = [L, \delta^c], \quad d^c = [L, \delta] \quad \text{(1.9)}
\]

where \( L \) is the algebraic operation

\[
L \phi = \omega \wedge \phi
\]

and \( \Lambda \) is its adjoint (contraction with \( \omega \)). We note

\[
[\Lambda, L] = (m - r)1 \quad \text{(1.11)}
\]
on \(r\)-forms. Finally, the Laplace-Beltrami operator \(\Delta = \delta d\) on functions may be re-expressed in the useful form

\[
\Delta f = -\Lambda dd^c f = - (\omega, dd^c f) .
\]  

(1.12)

If \(\mathcal{V} \to M\) is a holomorphic vector bundle over a complex manifold \(M\), its sheaf of sections will be denoted by \(\mathcal{O}(\mathcal{V})\). We define the projectivization of \(\mathcal{V}\) by \(P(\mathcal{V}) = (\mathcal{V} - \mathbf{0})/\mathbb{C}\times\), where \(\mathbf{0}\) is the zero section; notice that this differs from a competing convention which replaces \(\mathcal{V}\) with its dual \(\mathcal{V}^*\) on the right-hand side. Finally, depending on the context, we will use \(\mathcal{O}(k)\) to denote either the degree \(k\) holomorphic line bundle on \(\mathbb{C}P_m\), or its sheaf of holomorphic sections.

### 1.4 Anti-self-duality

On an oriented Riemannian 4-manifold \((M, g)\), the bundle \(\Lambda^2\) of 2-forms breaks up as

\[
\Lambda^2 = \Lambda^+ \oplus \Lambda^- ,
\]

(1.13)

where \(\Lambda^\pm\) is the eigenspace of the Hodge operator \(*\) with eigenvalue \(\pm 1\). We will call \(\Lambda^+\) the bundle of self-dual (SD) 2-forms and \(\Lambda^-\) the bundle of anti-self-dual (ASD) 2-forms. This decomposition is conformally invariant, in the sense that it is invariant under conformal rescalings \(g \to e^u g\).

The decomposition \((1.13)\) allows us to define differential operators \(d^\pm : \mathcal{E}^1 \to \mathcal{E}(\Lambda^\pm)\) by following the exterior derivative with projection \(\Lambda^2 \to \Lambda^\pm\). Since a closed anti-self-dual form is automatically harmonic, the following useful vanishing result is an immediate consequence of Hodge theory:

**Proposition 1.1** If \(M^4\) is compact and \(\beta \in \mathcal{E}^1(M)\), \(d^+ \beta = 0 \iff d\beta = 0\).

Applying \((1.13)\) to the curvature operator \(R^{ab}_{cd} : \Lambda^2 \to \Lambda^2\) results in a block-matrix decomposition

\[
\begin{array}{c|c}
\Lambda^+ & \Lambda^- \\
\hline
W_+ + \frac{\alpha}{12} & \Phi \\
\hline
\Phi & W_- + \frac{\alpha}{12}
\end{array}
\]
where $W_{\pm}$ are trace-free and $2\Phi$ is the trace-free part of the Ricci curvature $r$. If $W_+ = 0$, the metric $g$ is said to be anti-self-dual, or ASD. Since the Weyl curvature $W = W_+ + W_-$ is precisely the conformally invariant piece of the curvature tensor $R$, the anti-self-duality condition is invariant under conformal rescalings $g \to e^u g$; thus it makes sense to speak of ASD conformal (classes of) metrics.

If the Riemannian manifold $(M, g)$ is actually Kähler, and is given its canonical orientation, the decompositions (1.4) and (1.13) are compatible in the sense that

$$\bigwedge^+ C = C \omega \oplus \bigwedge^{0,2} \oplus \bigwedge^{2,0}$$

(1.14)

and

$$\bigwedge^- C = \bigwedge^0 C^{1,1},$$

(1.15)

where $\bigwedge^0 C^{1,1} = \{ \varphi \in \bigwedge^1, 1 \mid \omega \wedge \varphi = 0 \}$ is the bundle of “primitive” $(1,1)$-forms. But, as a consequence of (1.3), the curvature operator of a Kähler manifold is in $\text{End}(\bigwedge^{1,1})$; thus, the upper left-hand block of the curvature operator must just be a multiple of $\omega \otimes \omega^\sharp$. This immediately leads to the following:

**Proposition 1.2** [14][19] In complex dimension 2, a Kähler metric $g$ is anti-self-dual iff it is scalar-flat ($s \equiv 0$).

We conclude this section with a closely related observation. If $\varphi$ is any form of type $(1,1)$, we can write

$$\varphi = \frac{1}{4} (\Lambda \varphi) \omega + \varphi_0$$

(1.16)

where

$$\Lambda \varphi := (\varphi, \omega)$$

(1.17)

and $\varphi_0 \in \bigwedge^1 0, 1$. Applied to the Ricci form, this yields

$$\rho = \frac{1}{4} s \omega + \rho_0,$$

(1.18)

so that

$$\rho \text{ is ASD } \iff s = 0.$$  

(1.19)

In particular, the Riemannian connections on $\kappa = \bigwedge^{2,0}$ and $\bigwedge^+ \cong \kappa \oplus R$ are ASD iff the Kähler manifold $(M, g)$ is scalar-flat.
1.5 Admissible Kähler Classes

The fact that the Ricci form \( \rho \) of a Kähler manifold represents \( 2\pi c_1 \) in de-Rham cohomology leads to serious constraints on the scalar curvature of Kähler metrics. Indeed, on a compact complex surface with Kähler form \( \omega \), the integral of the scalar curvature \( s \), henceforth called the total scalar curvature, must be given by

\[
\int_M s \, d\mu = 4\pi c_1 \cdot [\omega] \quad (1.20)
\]
as an immediate consequence of (1.18). Since the volume of \( M \) is just \([\omega]^2/2\), we conclude that the average value of the scalar curvature is determined by the Kähler class alone. (In complex dimension \( m \), the total scalar curvature is similarly given by \( \int s \, d\mu = 4\pi c_1 \cdot [\omega]^{m-1}/(m-1)! \), while the volume is \([\omega]^{m}/m!/; thus the average scalar curvature is still completely determined by the Kähler class.) In trying to classify scalar-flat Kähler surfaces, the first logical step is therefore to limit ourselves to those Kähler classes with total scalar curvature zero. This motivates the following definition:

**Definition 1** Let \( M \) be a compact complex surface. A Kähler class \([\omega] \in H^{1,1}(M, \mathbb{R})\) will be said to be admissible iff \( c_1 \cup [\omega] = 0 \), i.e. iff the total scalar curvature \( \int s \, d\mu \) vanishes for Kähler metrics in \([\omega]\). The set of admissible Kähler classes will be denoted by \( \mathcal{A}_M \subset H^{1,1}(M) \).

If \( c_1^R = 0 \), any Kähler class is admissible, and Yau’s solution [43] of the Calabi conjecture asserts that every such class is represented by a unique Ricci-flat metric; moreover, this Ricci-flat metric is the only scalar-flat metric in the class, since (1.18) tells us that \( \rho/2\pi \) is the unique harmonic representative of \( c_1^R = 0 \) when \( s = 0 \). If, on the other hand, \( c_1^R \neq 0 \), the set \( \mathcal{A}_M \) of admissible Kähler classes is evidently an open set in a hyperplane in \( H^{1,1}(M, \mathbb{R}) \). However, this open set is often empty, as is shown by the following arguments of Yau [42]; cf. [18], [20].

**Proposition 1.3** Let \((M, J, \omega)\) be a compact Kähler \( m \)-manifold, and suppose that \( L \to M \) is a holomorphic line bundle such that \( c_1(L) \cup [\omega]^{m-1} = 0 \). Then either \( \Gamma(M, \mathcal{O}(L^\ell)) = 0 \forall \ell \neq 0 \), or else \( c_1^R(L) = 0 \).
Proof. Suppose that \( u \in \Gamma(M, \mathcal{O}(L)) \), \( u \neq 0 \). Using Poincaré duality, the volume of the zero locus \( u = 0 \), counted with multiplicity, must equal 
\[
(c_1(L) \cup [\omega]^{m-1})/(m-1)! = 0.
\]
Thus \( u \neq 0 \), and \( L^\ell \) is trivial.

Corollary 1.4 Let \((M, J, \omega)\) be a compact Kähler manifold of total scalar curvature zero. Then either \( c^R_1 = 0 \), or else \( \Gamma(M, \mathcal{O}(\kappa^\ell)) = 0 \) for all \( \ell \neq 0 \). In particular, the Kodaira dimension of \( M \) is either 0 or \( -\infty \).

Proof. Since the hypothesis may be interpreted as stating that \( [\omega]^{m-1} \cup c_1(\kappa) = 0 \), we may apply Proposition 1.3 with \( L = \kappa \).

Theorem 1.5 \[42\] Let \((M, J)\) be a compact complex surface which carries an admissible Kähler class \([\omega]\). Suppose, moreover, that \( M \) is not covered by a complex torus or a K3 surface. Then \( M \) is a ruled surface. In particular, \( H^{0,2}(M) = H^{2,0}(M) = 0 \), and \( b^+(M) = 1 \).

Proof. From the conclusions of Corollary 1.4, the Kodaira-Enriques classification \[3\] allows us to conclude, in the first instance, that \( M \) is covered by a complex torus or a K3 surface, or that, in the second instance, \( M \) is either a ruled surface—i.e. \( M \) is obtained from \( \Sigma_g \times \mathbb{CP}_1 \) by blowing up and blowing down. To see this, first observe that, since \( E \otimes L \) is generated by its sections when \( L \) is sufficiently positive, there are sections of \( \mathbb{P}(E) \to \Sigma_g \) of a rank 2 holomorphic vector bundle \( E \) over a compact complex curve \( \Sigma_g \) by blowing up \(|\tau(M)|\) points.

While the biregular classification of ruled surfaces over a curve \( \Sigma_g \) involves the classification of holomorphic vector bundles on the curve, the bimeromorphic classification is extremely simple. In fact \[5\], every ruled surface over \( \Sigma_g \) can be obtained from \( \Sigma_g \times \mathbb{CP}_1 \) by blowing up and blowing down. To see this, first observe that, since \( E \otimes L \) is generated by its sections when \( L \) is sufficiently positive, there are sections of \( \mathbb{P}(E) \to \Sigma_g \) passing through any given point. Choose three distinct such sections, and successively blow up each of their points of intersection while at each step blowing down the proper transform of the fiber through it. In finitely many steps this leads us
of a minimal model equipped with a projection to $\Sigma_g$ which admits three disjoint sections, and it is then easy to see that this model must be the product surface $\Sigma_g \times \mathbb{C}P_1$.

Although this knowledge will not prove essential for our purposes, we conclude this section by pointing out that the set of admissible Kähler classes can be described in extremely concrete terms:

**Proposition 1.6** Let $(M, \omega_0)$ be a compact Kähler surface. Then $A_M$ is precisely the set of real $(1,1)$-classes $[\omega] \in H^{1,1}(M, \mathbb{R})$ satisfying the following cohomological conditions:

(i) $[\omega] \cdot c_1 = 0$;
(ii) $[\omega] \cdot [\omega_0] > 0$;
(iii) $[\omega]^2 > 0$; and
(iv) $[\omega] \cdot C > 0$ for every curve $C \subset M$.

**Proof.** If $[\omega]$ satisfies (i) and (iv), the proof of Proposition 1.3 shows that either $H^{2,0} = 0$ or else $\kappa$ is trivial.

If the former happens, $H^{1,1} = H^2$, and hence $H^2(M, \mathbb{Q})$ is dense in $H^{1,1}(M, \mathbb{R})$; in particular, $M$ is algebraic. When $[\omega] \in H^2(M, \mathbb{Q})$, $k[\omega] \in H^2(M, \mathbb{Z})$ for some $k \in \mathbb{N}$, and, by Nakai’s criterion [5], $k[\omega]$ is thus a Kähler class iff (iii) and (iv) hold. Thus the cone of Kähler classes and the cone determined by conditions (iii) and (iv) intersect $H^2(M, \mathbb{Q})$ in the same set. But both these cones are open and convex; since they contain the same set of rational points, and since the open boxes with rational corners form a basis for the topology of the Euclidean space $H^{1,1}(M, \mathbb{R}) = H^2(M, \mathbb{R})$, they must therefore coincide. Conditions (iii) and (iv) are therefore equivalent to the class $[\omega]$ being Kähler, and in particular imply (ii). Condition (i) is thus the only additional condition needed to assure a class is admissible.

If, on the other hand, $\kappa$ is trivial, $M$ is either a torus or a K3 surface. In the torus case every class is uniquely represented by a form with constant coefficients, so that (ii) and (iii) are easily seen to be necessary and sufficient for a class to be Kähler (and automatically admissible). For the K3 case, the claim follows from Todorov’s surjectivity of the refined period map [5].
1.6 Holomorphic Vector Fields and Scalar Curvature

The space \( \Gamma(M, \mathcal{O}(T^{1,0}M)) \) of holomorphic vector fields on a complex manifold \((M, J)\) is equipped by the Lie bracket with the structure of a complex Lie algebra, denoted by \( a(M) \). If \( M \) is compact, this is precisely the Lie algebra of the group of biholomorphisms of \((M, J)\), since a vector field \( \Xi \) of type \((1,0)\) is holomorphic iff \( \mathcal{L}_\Xi J = 0 \), where \( \mathcal{L} \) denotes the Lie derivative. In particular, if \( g \) is a Kähler metric on \( M \), the Lie algebra \( \iota(M, g) \) of real Killing fields is canonically identified with a real sub-algebra of \( a(M) \) because the Kähler form, being the unique harmonic representative of its deRham class, is automatically invariant under the isometry group of \( M \). If, in addition, the scalar curvature of \( M \) is constant, the following remarkable result of Lichnerowicz \[31\], which generalizes work of Matsushima \[32\], says that, modulo parallel fields, \( a(M) \) is in fact just the complexification of \( \iota(M, g) \):

Proposition 1.7 (Matsushima-Lichnerowicz Theorem) If \((M, J, g)\) is a compact Kähler manifold of constant scalar curvature, \( a(M) \) is the direct sum of the space of parallel \((1,0)\)-vector fields and the space of vector fields of the form \((\bar{\partial} f)^\sharp\) where \( f \) is any (complex) solution of the equation

\[
\Delta^2 f + 2(dd^c f, \rho) = 0. \tag{1.21}
\]

Moreover, a solution \( f \) of (1.21) corresponds to a Killing field iff it is purely imaginary (\( \Re f = 0 \)). In particular, the algebra \( a(M) \) is reductive (semisimple plus Abelian), and the identity component of the group of biholomorphisms of \( M \) has a compact real form.

In the above proposition, \( \sharp \) denotes, as always, the inverse of

\[
\flat : C \otimes TM \to C \otimes T^*M : X \mapsto g(X, \cdot),
\]

and in particular induces a complex-linear isomorphism \( T^{0,1}M \to T^{1,0}M \).

Definition 2 Let \( M \) be a compact complex surface. We will say that the Matsushima-Lichnerowicz obstruction vanishes if the identity component of the group of biholomorphisms of \( M \) has a compact real form.
While the Matsushima-Lichnerowicz Theorem gives us an important obstruction to the existence of constant-scalar curvature Kähler metrics on a compact complex manifold \( M \) in terms of the algebra \( a(M) \) of holomorphic vector fields, a more subtle such obstruction was later discovered by Futaki \[16\] \[17\]. The Futaki character \( F(\cdot, [\omega]) : a(M) \to \mathbb{C} \) is defined by

\[
F(\Xi, [\omega]) = \int_M \Xi(\phi_\omega) \, d\mu
\]  
(1.22)

where \( \Xi \) is any holomorphic vector field and \( \phi_\omega \) is the Ricci potential:

\[
\rho = \rho_H + dd^c \phi_\omega
\]  
(1.23)

where \( \rho_H \) is harmonic and \( \phi_\omega \) is \( C^\infty \), real and normalized so that

\[
\int \phi_\omega \, d\mu = 0.
\]  
(1.24)

Notice, by taking the trace of (1.23), that

\[
s = \text{constant} \iff \phi_\omega = 0.
\]  
(1.25)

The most remarkable property of the Futaki character \( F(\cdot, [\omega]) \), implicit in our notation but not evident from the definition, is \[11\] \[16\] that it depends only upon the Kähler class; for this reason it is sometimes referred to as the Futaki invariant. From (1.23), it now follows immediately that the vanishing of \( F(\cdot, [\omega]) \) is a necessary condition for the Kähler class \([\omega]\) to contain a representative with constant scalar curvature.

We now give a way of rewriting the Futaki invariant that is particularly useful when \( \Xi = 2(\bar{\partial} f)^\sharp \) for a complex-valued function \( f \) on \( M \) with \( \int f \, d\mu = 0 \). Our calculations will actually work, however, for an arbitrary holomorphic vector field, provided we define its holomorphy potential \( f \) by

\[
f = \partial^* G \Xi^\flat = \bar{\partial} f \Xi^\flat,
\]
where \( G \) is the Green’s operator of the Hodge Laplacian. Now notice that the Ricci potential can be written in terms of the Green’s operator and the scalar curvature as \( \phi_\omega = -\frac{1}{2} G s \). Hence

\[
F(\Xi, [\omega]) = \int_M \Xi(\phi_\omega) \, d\mu = \langle \Xi^\flat, d\phi_\omega \rangle
\]

\[
= \langle \Xi^\flat, \bar{\partial} \phi_\omega \rangle = \langle \Delta G \Xi^\flat, \bar{\partial} \phi_\omega \rangle
\]

\[
= \langle 2 \bar{\partial} \bar{\partial}^* G \Xi^\flat, \bar{\partial} \phi_\omega \rangle = \langle 2 \bar{\partial} f, \bar{\partial} \phi_\omega \rangle
\]

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\[
\langle f, 2\bar{\partial}\partial\phi_{\omega} \rangle = \langle f, \Delta\phi_{\omega} \rangle \\
= \langle f, -\frac{1}{2}(s - s_H) \rangle
= -\frac{1}{2} \int_M f(s - s_H) \, d\mu
= -\frac{1}{2} \int_M f \, ds \, d\mu
\]

where \( s_H \) is the average value of \( s \) on \((M, g)\). Notice, incidentally, that the conclusion is insensitive to the normalization \( \int f \, d\mu = 0 \) if the total scalar curvature \( \int s \, d\mu = s_H \int d\mu \) happens to vanish.

As a consequence we deduce an innocuous-looking fact (cf. [7], Proposition 2.159) which will later turn out to be surprisingly important:

**Proposition 1.8** Let \((M, \omega)\) be a compact Kähler manifold of constant scalar curvature \( s = c \). Then, for any closed \((1,1)\)-form \( \alpha \) one has

\[
\frac{d}{dt} F(\Xi, [\omega + t\alpha]) \bigg|_{t=0} = \langle f \rho, \alpha_H \rangle,
\]

where \( f \) is the holomorphy potential of \( \Xi \) as defined above, and \( \alpha_H \) is the harmonic part of \( \alpha \).

**Proof.** Since the \( F \) only depends on the Kähler class, we might as well assume that \( \alpha \) is harmonic. Since \( F(\Xi, [\omega + t\omega]) = (1 + t)^m F(\Xi, [\omega]) = 0 \) by the assumption that \( \omega \) has constant scalar curvature, whereas the corresponding right-hand side \( \langle f \rho, \omega \rangle = \frac{1}{2} \int f \, ds \, d\mu = -F(\Xi, [\omega]) \) vanishes for the same reason, we may assume that the harmonic form \( \alpha \) is primitive. This assumption has the effect that the volume form of \( \omega(t) \) is

\[
d\mu(t) = \frac{(\omega + t\alpha)^m}{m!} = d\mu + \frac{t\omega^{m-1} \wedge \alpha}{(m-1)!} + O(t^2) = d\mu + O(t^2),
\]

so that the Ricci form, being determined by the volume form and \( J \), similarly satisfies

\[
\rho(t) = \rho + O(t^2).
\]

The normalization of the holomorphy potential reads \( \int f(t) \, d\mu = O(t^2) \) for the same reason. Hence

\[
\frac{d}{dt} F(\Xi, [\omega + t\alpha]) \bigg|_{t=0} = -\frac{1}{2} \frac{d}{dt} \left[ \int_M f(t) s(t) \, d\mu(t) \right] \bigg|_{t=0}
\]
\[
\begin{align*}
= \left. -\frac{1}{2} \int_{M} \frac{df}{dt} \right|_{t=0} - \frac{1}{2} \Delta f(t) dt \left|_{t=0} \quad 1.7 \text{ Twistor Spaces} \\
= \left. -\frac{1}{2} \int_{M} f \frac{d}{dt} [s(t) d\mu(t)] \right|_{t=0} \\
= \left. -\frac{1}{(m-1)!} \int_{M} f \frac{d}{dt} [\rho(t) \wedge \omega^{m-1}(t)] \right|_{t=0} \\
= \left. -\frac{1}{(m-2)!} \int_{M} f \rho \wedge \omega^{m-2} \wedge \alpha \right|_{t=0} \\
= \int_{M} f \rho \wedge \ast \alpha = \langle f \rho, \alpha_H \rangle.
\end{align*}
\]

The Penrose correspondence \([3][7][37]\) is a dictionary between anti-self-dual conformal Riemannian 4-manifolds and a special class of complex 3-folds. We will begin by briefly explaining how the translation works in each direction.

If \((M, [g])\) is an anti-self-dual Riemannian 4-manifold, its twistor space is a complex 3-manifold \(Z\) whose underlying smooth 6-manifold is the total space of the sphere bundle of the rank-three real vector-bundle of self-dual 2-forms:

\[
S^2 \rightarrow \quad Z := \{\omega \in \wedge^+ \mid \|\omega\| = \sqrt{2}\} \quad \downarrow \phi \\
M
\]

We now give \(Z\) an almost-complex structure \(J : TZ \rightarrow TZ, J^2 = -1\), by first observing that, for each \(x \in M\), there is a natural one-to-one correspondence between \(\phi^{-1}(x)\) and the set of \(g\)-orthogonal complex structures \(j : T_xM \rightarrow T_xM\) inducing the given orientation on \(T_xM\); namely, in the spirit of \((1.3)\), any such \(j\) corresponds to the 2-form \(\omega_j\) defined by

\[
\omega_j(\xi, \eta) = g(j\xi, \eta).
\]
Since the Levi-Civitá connection of $g$ induces a splitting $TZ = H \oplus V$ of the tangent bundle of $Z$ into horizontal and vertical parts, and we have a canonical isomorphism $\varphi_* : H \to \varphi^* TM$, we may define $J_H : H \to H$, $J^2_H = -1$ by $J_H|_\omega := j$. Since the fibers of $\varphi$ are oriented metric 2-spheres, we may also define $J_V : V \to V$, $J^2_V = -1$ to be rotation by $-90^\circ$ in the tangent space of the fiber. Defining $J = J_H \oplus J_V$ then makes $Z$ an almost-complex manifold. Quite remarkably, this almost-complex structure is conformally invariant. Even more remarkably, it is integrable because its Nijenhuis tensor may be identified with $W^+$, and so vanishes precisely by the assumption that $(M, g)$ is anti-self-dual. The fibers of $\varphi$ have become $\mathbb{CP}^1$'s with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ in the complex manifold $Z$, while the fiberwise antipodal map $Z \to Z : \omega \mapsto -\omega$ has become a free anti-holomorphic involution $\sigma : Z \mathcal{O} \to Z$.

Conversely, let $Z$ be a complex 3-fold with free anti-holomorphic involution $\sigma : Z \mathcal{O} \to Z$, $\sigma^2 = \text{id}_Z$, and suppose that there is a smooth $\sigma$-invariant rational curve in $Z$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Let $CM$ denote the connected component of this curve in the space of all $\mathbb{CP}_1 \subset Z$ with normal bundle bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Invoking [22], $CM$ is a complex 4-manifold. Moreover, $\sigma$ induces an anti-holomorphic involution $\hat{\sigma} : CM \to CM$ for which our original curve corresponds to a fixed point; the fixed-point set of $\hat{\sigma}$ is therefore a (non-empty) real-analytic 4-manifold, the obvious connected component of which we denote by $M$. There is then an anti-self-dual conformal class of metrics $[g]$ on $M$ determined by requiring that a complex tangent vector $\xi \in C \otimes TM = TCM|_M$ satisfies $g(\xi, \xi) = 0$ iff the corresponding section of the normal bundle of $\mathbb{CP}_1 \subset Z$ has a zero. If $Z$ actually arises by the construction of the preceding paragraph, and if our initial curve is a fiber of $\varphi$, this exactly reconstructs the given manifold $M$ and conformal structure $[g]$.

Now recall from §1.4 that a Kähler manifold of complex dimension 2 is anti-self-dual iff its scalar-curvature vanishes. (This might be considered rather remarkable insofar as neither the Kähler condition nor the scalar-curvature condition are themselves conformally invariant, and yet their coincidence is reflected by a conformally invariant property.) The Penrose correspondence thus provides an unexpected link between Kähler surfaces and complex 3-folds. As will now be explained, the speciality of the metric being scalar-flat Kähler is echoed by a specialty of the twistor space $Z$ in
a manner simple enough to allow us to study scalar-flat Kähler geometry by using Kodaira-Spencer theory. The central result here is a theorem of Pontecorvo:

**Proposition 1.9** Let $\varphi : Z \to M$ be the twistor fibration of a (perhaps non-compact) anti-self-dual conformal Riemannian 4-manifold $(M, [g])$. Suppose we are given a complex hypersurface $D_1 \subset Z$ for which the restriction $\varphi|_{D_1}$ of the twistor projection is a diffeomorphism onto $M$, and let $J$ denote the complex structure on $M$ given by this section of $Z$. (Thus $\varphi|_{D_1}$ becomes a biholomorphism of $D_1$ and $(M, J)$.) Let $D_2 = \overline{D_1}$ denote the image of $D_1$ under the real structure $\sigma : Z \to Z$, and let $D = D_1 \cup D_2$. Then there exists a metric $g$ in the conformal class $[g]$ such that $(M, g, J)$ is Kähler iff the divisor line bundle of $D$ is isomorphic to the half-anti-canonical line bundle $\kappa^{-1/2}$ of $Z$.

The implication $\Rightarrow$ is relatively straightforward; indeed, the Kähler form $\omega$, being parallel and self-dual, is a solution of the twistor equation, and so defines, via the Penrose transform, a holomorphic section of $\kappa^{-1/2}$ vanishing precisely at $D$. For an analogous proof in the $\Leftarrow$ direction, cf. [28].

This implies the following key result, originally discovered in a somewhat different guise by Boyer [8]:

**Theorem 1.10** Let $\varphi : Z \to M$ be the twistor fibration of a compact anti-self-dual 4-manifold $(M, [g])$, and suppose that $b_1(M)$ is even. Let $D_1 \subset Z$ be a complex hypersurface which meets every fiber in exactly one point. Then, for any $c \in \mathbb{R}^+$, the conformal class $[g]$ contains a unique scalar-flat Kähler metric of volume $c$. Conversely, every scalar-flat Kähler surface arises in this way.

**Proof.** Let $[D]$ denote the divisor line bundle of the hypersurface $D = D_1 \cup \sigma(D_1)$. Then $c_1([D]) = c_1(\kappa^{-1/2})$, so that $[D] \otimes \kappa^{1/2}$ is an element of $\text{Pic}_0(Z)$. By the Penrose transform, and using Proposition 1.1, $H^1(Z, \mathcal{O}) = \{ \beta \in \mathcal{E}^1(M) \mid d^+ \beta = 0 \}/d\mathcal{E}^0(M) = H^1(M, C)$, so that one has $\text{Pic}_0(Z) = H^1(Z, C^\times) = H^1(M, C^\times)$; in other words, every topologically trivial holomorphic line bundle on $Z$ admits a compatible flat connection, and these all come from the base. In particular, $[D] \otimes \kappa^{1/2}$ admits a flat $C^\times$-connection, and is the pull-back of a flat $C^\times$-bundle on $M$. In particular,
$[D] \otimes \kappa^{1/2}$ is trivial on $\varphi^{-1}(U)$ for any sufficiently small open set $U \subset M$. By Proposition 1.9, $g$ is therefore conformal to a Kähler metric on a sufficiently small open set $U$.

We have thus shown that there are locally-defined smooth functions $u \in \mathcal{E}_U$ for which the (1,1)-form $\omega$ associated to $(g, J)$ satisfies $0 = d(e^u \omega) = e^u(du \wedge \omega + d\omega)$. Since any two local choices of $u$ differ by an additive constant, the 1-form $\beta = -du$ is globally defined on $M$, and

$$d\omega = \beta \wedge \omega.$$ 

Because the Fröhlicher spectral sequence of any compact complex surface degenerates \cite{5}, the hypothesis that $b_1(M) \equiv 0 \mod 2$ implies that $H^1_d(M, \mathbb{C}) = H^0(M, \Omega^1) \oplus H^0(M, \Omega^1)$. (A less elementary but deeper explanation of this decomposition stems from the fact \cite{10} that a compact complex surface admits Kähler metrics iff $b_1$ is even.) Thus the closed real 1-form $\beta$ can be written as $\beta = \Re \alpha + df$ for some holomorphic 1-form $\alpha$ and some smooth function $f$. Introducing the conformally rescaled metric $\hat{g} := e^{-f}g$, we now have $d\hat{\omega} = \Re \alpha \wedge \hat{\omega}$. But then

$$0 = \int_M d(\alpha \wedge \hat{\omega}) = -\frac{1}{2} \int_M \alpha \wedge \bar{\alpha} \wedge \hat{\omega} = \frac{i}{2} \|\alpha\|^2_{L^2(\hat{g})},$$

so that $\alpha = 0$, and $\hat{g}$ is Kähler. Since $\hat{g}$ is also ASD, it is automatically scalar-flat by Proposition 1.2.

**1.8 Deformation Problems**

For a compact manifold $M$ which admits a scalar-flat Kähler metric $g$, a number of moduli problems are now obviously of interest:

(a) the moduli of scalar-flat Kähler metrics in the given Kähler class;
(b) the moduli of scalar-flat Kähler metrics for the given complex structure;
(c) the moduli of scalar-flat Kähler metrics, with the complex structure allowed to vary; and
(d) the moduli of ASD conformal structures on $M$.\[19\]
One might also be tempted to add the following:

(b') the moduli of ASD Hermitian conformal structures for a given complex structure;

(c') the moduli of ASD Hermitian conformal structures for some complex structure.

However, as we saw in proving Theorem 1.10, a result of Boyer [8] states that, because $b_1(M)$ is even, (b') and (c') are respectively equivalent to (b) and (c), so nothing is to be gained by considering these problems separately.

Of these problems, (a) can be tackled quite easily within the standard framework of Kähler geometry, but for (b)–(d) very valuable information comes from the twistor description. As we saw in the previous section, the twistor space $Z$ of a scalar-flat Kähler surface $M$ is a complex 3-manifold equipped with a real structure $\sigma$ and a $\sigma$-invariant divisor $D$. The complex structure of $Z$ completely determines the conformal structure of $M$ while the divisor $D$ specifies the given complex structure on $M$. Thus the moduli problems (b)–(d) correspond to the following problems in terms of $(Z,D)$:

(b∗) moduli of complex structures on $Z$ with $D$ as a fixed $\sigma$-invariant divisor;

(c∗) moduli of complex structures on $Z$ which admit a $\sigma$-invariant divisor with divisor line-bundle isomorphic to $\kappa^{-1/2}$;

(d∗) moduli of complex structures on $Z$, which admit a compatible real structure $\sigma$.

Note that this point of view imposes different equivalence relations on the metrics occurring in the different problems; in problems (a)–(b), two metrics of the same total volume will be considered equivalent iff they are literally equal, whereas in problems (c)–(d) two metrics will be equivalent if they are in the same orbit of the diffeomorphism group cross conformal rescalings.

Certainly an advantage of the starred formulation over the original one is that one can appeal to the machinery of Kodaira-Spencer deformation theory to get local information about the moduli spaces in terms of certain sheaf cohomology groups of the twistor spaces. But from our point of view the most significant advantage of this description is that the cohomology groups involved in these distinct problems are related by exact sequences; once problem (b) is thoroughly understood, problems (c) and (d) can also be solved with relatively little further effort.
1.9 Deformation Theory

Let $Z$ be a compact complex manifold. For us, a deformation of $Z$ will consist of the following: a “parameter” manifold $\mathcal{T}$ with basepoint $o$, a smooth manifold $\mathcal{Z}$; a proper submersion $\varpi: \mathcal{Z} \to \mathcal{T}$; an integrable fiber-wise complex structure on $\mathcal{Z}$; and an identification of the central fiber $\pi^{-1}(o)$ with $Z$. If $\mathcal{T}'$ is another manifold, with basepoint $o'$, and $\varphi: \mathcal{T}' \to \mathcal{T}$ is a basepoint-preserving smooth map, there is an induced deformation $\varphi^*(\mathcal{Z}) \to \mathcal{T}'$. The deformation $\varpi: \mathcal{Z} \to \mathcal{T}$ is called complete if any other deformation can be induced from it by a smooth map $\varphi$, versal if, in addition, the derivative of $\varphi$ at $o'$ is always uniquely determined, and universal if, in addition, the inducing $\varphi$ is always unique. When a universal deformation of $Z$ exists, a neighborhood of $o$ in the parameter space $\mathcal{T}$ gives a model for the moduli space of complex structures on $Z$ in a neighborhood of the given structure.

If $\varpi: \mathcal{Z} \to \mathcal{T}$ is any deformation in the above sense, the Kodaira-Spencer map at $o \in \mathcal{T}$ is an $\mathbb{R}$-linear map $ks: T_o \mathcal{T} \to H^1(Z_o, \Theta)$ obtained in Čech cohomology by differentiating the transition functions of a fiber-wise complex coordinate atlas on $\mathcal{Z}$. The first basic result of Kodaira-Spencer theory is that a deformation is complete (respectively, versal) if $ks$ is surjective (respectively, bijective). Notice that, by virtue of its definition, the Kodaira-Spencer map behaves functorially under pull-backs.

The main results of Kodaira-Spencer theory [24] assert that any versal deformation may be made into a holomorphic map $\varpi: \mathcal{Z} \to \mathcal{T}$ between complex manifolds (in an essentially unique manner), and, more importantly, give sufficient conditions [23] for the existence of a versal or universal deformation of $Z$ in terms of the sheaf cohomology groups $H^j(Z, \Theta)$, where $\Theta = \mathcal{O}^\perp(T^{1,0}Z)$ is the sheaf of holomorphic vector fields on $Z$. These results may be summed up as follows:

**Theorem 1.11** Suppose $H^2(Z, \Theta) = 0$. Then a (holomorphic) versal deformation exists, with parameter space $\mathcal{T}$ an open neighborhood of $o = 0$ in $H^1(Z, \Theta)$. This deformation is universal if $H^0(Z, \Theta) = 0$.

Unfortunately, this will not suffice for our purposes, because we will be primarily interested in deformations of complex manifolds with real structure. By a real structure on a compact complex manifold $Z$, we always mean an
anti-holomorphic involution of $Z$—i.e. an anti-holomorphic map $\sigma: Z \rightarrow Z$ such that $\sigma^2 = \text{id}_Z$. We will further assume that $\sigma$ acts freely—i.e. without fixed points. This in particular means that $Z/\sigma$ is a smooth manifold, and while the complex structure tensor $J$ of $Z$ cannot descend to the quotient, the unordered pair $\{J, -J\}$ is globally well-defined downstairs. We therefore introduce the following concept:

**Definition 3** A semi-complex manifold is a smooth manifold $P$, together with a 1-dimensional sub-bundle $L \subset \text{End}(TP)$ such that, in a neighborhood of any point $x \in P$, $L$ is spanned by an integrable complex structure $J$.

Of course, near any point there are then exactly 2 choices of the complex structure spanning $L$—if $J$ is one, $-J$ is the other. If $P^{2m}$ is a semi-complex manifold, we can thus equip $P$ with an atlas for which all the transition functions are either holomorphic or anti-holomorphic diffeomorphisms of domains in $\mathbb{C}^m$; and conversely, any manifold equipped with such an atlas has an induced semi-complex structure.

**Example.** Let $P$ be a smooth, unoriented surface, and let $[g]$ be a conformal class of Riemannian metrics on $P$. Then $[g]$ determines a unique semi-complex structure on $P$.

**Example.** Let $Z$ be the twistor space of a half-conformally-flat Riemannian 4-manifold $(M,g)$, and let $\mathcal{Z}$ be its twistor space. Let $\sigma: Z \rightarrow Z$ be its real structure, acting on the fibers of $\varphi: Z \rightarrow M$ by the antipodal map. Then $P := Z/\sigma$ is a semi-complex manifold. Notice, incidentally, that $P \rightarrow M$ is an $\mathbb{RP}^2$-bundle.

The following observation will be as crucial as it is trivial: every semi-complex manifold $P$ is double-covered by a complex manifold in a manner that makes the non-trivial deck transformation a free anti-holomorphic involution. Indeed, one simply takes the cover to consist of the elements $J \in L \subset \text{End}(TP)$ such that $J^2 = -1$. Thus, our basic example $P = Z/\sigma$ of a semi-complex manifold, where $Z$ is complex and $\sigma: Z \rightarrow Z$ is a free anti-holomorphic involution, actually represents the general case.

If $Z$ is a complex manifold, the sheaf of holomorphic vector fields is, as mentioned above, denoted by $\Theta := \mathcal{O}(T^{1,0}Z)$. However, we may identify the
underlying real vector bundle of $T^{1,0}Z$ with the real tangent bundle $TZ$ by $2\mathcal{R} : T^{1,0}Z \to TZ : \xi \mapsto \xi + \bar{\xi}$, and in the process we identify $\Theta$ with the sheaf

$$\mathcal{R}\Theta := \{\xi \in \mathcal{E}(TZ) \mid \mathcal{L}_\xi J = 0\}$$

of “real holomorphic” vector fields; of course, this only identifies them as sheaves of real Lie algebras. The interesting observation is that $\mathcal{R}\Theta$ is exactly the same for the conjugate complex manifolds $(Z,J)$ and $(Z,-J)$, and is thus even well defined on a semi-complex manifold. This, of course, happens precisely because $\mathcal{R}\Theta$ is the sheaf of infinitesimal automorphisms of the semi-complex structure.

If we repeat our previous definitions of deformations and versality for semi-complex manifolds, with the fiber-wise structures only required to be semi-complex instead of complex, we immediately get the following result:

**Proposition 1.12** Let $P$ be a compact semi-complex manifold such that $H^2(P, \mathcal{R}\Theta) = 0$. Then there exists a versal deformation of $P$ with a neighborhood of $0 \in H^1(P, \mathcal{R}\Theta)$ as parameter space. This deformation is universal if $H^0(P, \mathcal{R}\Theta) = 0$.

**Proof.** The Forster-Knorr power-series proof [13] of Theorem 1.9 goes through without any essential changes.

**Lemma 1.13** Let $Z$ be a complex manifold with free anti-holomorphic involution $\sigma : Z \to Z$, and let $P = Z/\sigma$ be the associated semi-complex manifold. Then $H^j(Z, \Theta) = H^j(P, \mathcal{R}\Theta) \otimes_\mathbb{R} \mathbb{C}$.

**Proof.** There are arbitrarily fine covers $\mathcal{V}$ of $Z$ which are equivariant under $\sigma$, and any such cover descends to a cover $\mathcal{W}$ of $P$. For any such cover $\mathcal{V}$, $\sigma$ acts as on $\hat{H}^j(\mathcal{V}, \Theta)$ via the anti-linear map $\{f_{\alpha\cdots\beta}\} \mapsto \{\sigma^* f_{\alpha\cdots\beta}\}$, and the fixed-point set of this action can be identified with $\hat{H}^j(\mathcal{W}, \mathcal{R}\Theta)$. Hence $\hat{H}^j(\mathcal{V}, \Theta) = \hat{H}^j(\mathcal{W}, \mathcal{R}\Theta) \otimes_\mathbb{R} \mathbb{C}$. The lemma now follows by taking direct limits.

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**Theorem 1.14** Let \( Z \) be a compact complex manifold with \( H^2(Z, \Theta) = 0 \). Suppose that \( \sigma : Z \to Z \) is an anti-holomorphic involution without fixed points. Then \( \sigma \) can be extended as an anti-holomorphic involution \( \sigma_Z : Z \to Z \) of the total space of the versal deformation of \( Z \) which covers an anti-holomorphic involution \( \sigma_T : T \to T \) of the base. The fixed-point set of \( \sigma_T \) is a totally real subspace \( T_\sigma \) of real dimension \( h^1(Z, \Theta) \), and the restriction of \( \varpi : Z \to T \) to this subspace is a versal deformation of \((Z, \sigma)\).

**Remark.** When \( H^0(Z, \Theta) = 0 \), this is an immediate consequence \[12\] of the universal property of the versal deformation.

The importance of real deformations stems from the following observation, the essence of which was discovered by Penrose \[37\]:

**Theorem 1.15** Let \( \varpi : Z \to T \) be a deformation of the twistor space \( Z = Z_o \) of a compact ASD conformal Riemannian 4-manifold \((M, [g])\). Suppose, moreover, that \( Z \) is equipped with a fiber-wise anti-holomorphic involution which restricts to the twistor real structure on \( Z_o \). Then there is a neighborhood \( U \) of \( o \in T \) and a family of ASD Riemannian metrics \( g_t \) on \( M \), depending smoothly on \( t \in U \), such that \( Z_t = \varpi^{-1}(t) \) is biholomorphic to the twistor space of \((M, [g_t])\).

**Proof.** The point is that the normal bundle \( \nu \) of a twistor fiber \( C = \varpi^{-1}(x) \in Z \) is isomorphic to the bundle \( O(1) \oplus O(1) \) on \( \mathbb{CP}_1 \), and so satisfies \( H^1(C, \nu) = 0 \). By Kodaira’s stability theorem \[22\], the complete analytic family generated by the twistor fibers is stable under deformations. Because \( O(1) \oplus O(1) \) is a rigid bundle on a rigid manifold, we have a 4-complex parameter family of \( \mathbb{CP}_1 \)’s with normal bundle \( O(1) \oplus O(1) \) in any small deformation \( Z_t \) of \( Z \); the \( \sigma \)-invariant curves in these families then foliate a manifold containing \( Z_o \) and spread over an open neighborhood of \( Z_o \subset Z \). There is therefore a neighborhood of \( Z_o \subset Z \) foliated by these curves and projecting properly to a neighborhood \( U \) of \( o \in T \). Invoking the inverse twistor correspondence described in \[1.7\] finishes the proof.

In short, the moduli space for problem (d) is locally the same as the moduli space for semi-complex structures on \( Z/\sigma \). In order to attack the
moduli problems (b) and (c), we shall instead require two ‘relative versions’ of the above deformation theory. Suppose we are given a compact complex manifold \( Z \) and a nonsingular complex hypersurface \( D \) in \( Z \). A deformation of \((Z,D)\) is given by the following: a deformation \( \varpi : Z \to \mathcal{T} \) of \( Z \) and a deformation \( \mathcal{D} \to \mathcal{T} \) of \( D \), together with a commutative diagram

\[
\begin{array}{c}
\mathcal{D} \\
\downarrow \\
\mathcal{T} = \mathcal{D}
\end{array}
\]

which restricts to the inclusion of \( D \) in \( Z \) at the central fibers.

A deformation of \((Z,D)\) with fixed divisor \( D \) is a deformation of the above type with \( \mathcal{D} \cong D \times \mathcal{T} \), and \( \varpi|_D \) corresponding to projection on the second factor. The notions of versal and universal deformations exist also for these relative deformations and there are analogues of Theorem 1.11. To state them, let \( \Theta_{Z,D} \) be the sheaf of holomorphic vector fields on \( Z \) that are tangent to \( D \) along \( D \); and let \( \Theta_Z \otimes \mathcal{I}_D \) be the subsheaf of vector fields which vanish along \( D \).

Finally, we can modify all the above definitions so as to replace \( Z \) with a semi-complex manifold \( P \) and \( D \) with a semi-complex submanifold \( M \subset P \) of real codimension 2. The same reasoning as before then yields

**Theorem 1.16** Let \((P,M)\) be a semi-complex manifold with nonsingular semi-complex hypersurface, and let \((Z,D,\sigma)\) be the complex manifold with hypersurface and real structure which covers it.

(i) Suppose that \( H^2(Z,\Theta_{Z,D}) = 0 \). Then there are versal deformations of \((P,M)\) and \((Z,D)\). Moreover, the parameter space for the former deformation is a real slice in that of the latter, which may be taken to be a neighborhood of \( 0 \in H^1(Z,\Theta_{Z,D}) \). These deformations are both universal if \( H^0(Z,\Theta_{Z,D}) = 0 \).

(ii) Suppose that \( H^2(Z,\Theta_Z \otimes \mathcal{I}_D) = 0 \). Then there are versal deformations of \( P \) with fixed divisor \( M \), and of \( Z \) with fixed divisor \( D \). Moreover, the parameter space for the former deformation is a real slice in that of the latter, which may be taken to be a neighborhood of \( 0 \in H^1(Z,\Theta_Z \otimes \mathcal{I}_D) \). These deformations are both universal if \( H^0(Z,\Theta_Z \otimes \mathcal{I}_D) = 0 \).
Notice that these deformation problems are all related in that there exist exact sequences:

\[ 0 \to \Theta_{Z,D} \to \Theta_Z \to N_D \to 0 \]  \hspace{1cm} (1.27)

where \( N_D \) is the normal bundle of \( D \) in \( Z \), and

\[ 0 \to \Theta_Z \otimes \mathcal{I}_D \to \Theta_{Z,D} \to \Theta_D \to 0. \]  \hspace{1cm} (1.28)

Moreover, the induced long-exact sequences exactly intertwine the Kodaira-Spencer maps of the deformation theories involved. In particular, given a deformation \( \varpi : (Z,D) \to \mathcal{T} \) of \((Z,D)\), there is a Kodaira-Spencer map \( \mathbf{ks}_{Z,D} \in \text{Hom}(T_0\mathcal{T}, H^1(Z,\Theta_{Z,D})) \), gotten by differentiating the transition functions of a fiber-wise complex atlas which sends open sets of \( Z_t \) to \( \mathbb{C}^m \) and open sets of \( D_t \) to \( \mathbb{C}^{m-1} \subset \mathbb{C}^m \); this is obviously related to the Kodaira-Spencer maps of \( \varpi : Z \to \mathcal{T} \) and \( \varpi|_D : D \to \mathcal{T} \) by composition with the natural homomorphisms \( H^1(Z,\Theta_{Z,D}) \to H^1(Z,\Theta_Z) \) and \( H^1(Z,\Theta_{Z,D}) \to H^1(D,\Theta_D) \). In particular, if \( H^2(Z,\Theta_Z \otimes \mathcal{I}_D) = 0 \), the resulting surjectivity of \( H^1(Z,\Theta_{Z,D}) \to H^1(D,\Theta_D) \) implies that if \( \varpi : (Z,D) \to \mathcal{T} \) is assumed to be a versal deformation of \((Z,D)\), the induced deformation \( \varpi|_D : D \to \mathcal{T} \) is complete. A similar argument in the semi-complex case will feature prominently in our proof of the Main Theorem.
2 Deformations of Scalar-flat Kähler Surfaces

The standard treatment of these problems is as follows. Given a scalar-flat Kähler metric $g$, Kähler form $\omega$, normalized so that the total volume is 1, we identify the tangent space to the space of volume-1 Kähler forms as

$$K = \left\{ \varphi \in \Lambda^{1,1}(M) : d\varphi = 0 \text{ and } \int (\Lambda\varphi) \, d\mu = 0 \right\}. \tag{2.1}$$

The derivative in the direction $\varphi$ of the scalar curvature is (cf. [7], Lemma 2.158(iii))

$$s'(\varphi) = \Delta(\Lambda\varphi) - 2(\rho, \varphi) \tag{2.2}$$

(where $\rho$ is the Ricci form as in (1.5)).

If it is required to preserve the Kähler class then $\varphi \in K$ is taken to have the form

$$\varphi = -dd^c f \tag{2.3}$$

(for some real $C^\infty$ function $f$) and (2.2) reduces to

$$s'(f) = \Delta^2 f + 2(dd^c f, \rho) \tag{2.4}$$

This we recognize as Lichnerowicz's differential equation (1.21). Invoking Proposition 1.7, we can thus immediately solve problem (a):

**Proposition 2.1** The tangent space to the moduli space of scalar-flat Kähler metrics in a given Kähler class is precisely the space $\iota(M)^{\perp} \subset a(M)$ of holomorphic vector fields orthogonal to the space $\iota(M)$ of Killing fields.

Let us now turn to the more general problem (b). For this, we shall study the equivalent problem of deforming $Z$ with fixed divisor $D$. Referring to Theorem 1.16 we see that the first task is to identify $\Theta_Z \otimes I_D$ and its cohomology groups. Since the ideal sheaf of $D$ is isomorphic to $\mathcal{O}(-2) := \kappa_Z^{1/2}$, we have $\Theta_Z \otimes I_D = \Theta(-2)$. To study the cohomology groups we use the Penrose transform to relate them to data on $M$. A straightforward application of the techniques of [6] or [21] yields:
Proposition 2.2 Let \((M, g)\) be any anti-self-dual manifold. For each \(j = 0, \ldots, 3\), the Penrose transform identifies \(H^j(Z, \Theta(-2))\) with the \(j\)-th cohomology group of the complex

\[
0 \to \wedge^-(M) \xrightarrow{S} \wedge^+(M) \to 0 \tag{2.5}
\]

where

\[
S(\alpha) = d^+ \delta \alpha + \Phi \alpha \tag{2.6}
\]

for \(\alpha \in \wedge^-(M)\). Here \(\Phi : \wedge^- \to \wedge^+\) denotes one-half the trace-free Ricci curvature, acting by \(\alpha_{ab} \mapsto \Phi^c_{[a} \alpha_{bc]}\). 

Remark. The operator \(S\) is conformally invariant, provided that \(\alpha\) is conformal weighted as follows: \(\alpha \mapsto e^{u/2} \alpha\) when \(g \mapsto e^u g\).

Corollary 2.3 Let \((M, g)\) be a scalar-flat Kähler surface. The Penrose transform then identifies \(H^j(Z, \Theta(-2))\) with the \(j\)-th cohomology group of the complex

\[
0 \to \wedge^-(M) \xrightarrow{S} \wedge^+(M) \to 0 \tag{2.7}
\]

where

\[
S(\alpha) = d^+ \delta \alpha - \frac{1}{2} (\rho, \alpha) \omega \tag{2.8}
\]

for \(\alpha \in \wedge^-(M)\).

While the twistor theory predicts that the operator \(S\) completely governs problem (b), it is perhaps not obvious why this is so. Let us therefore digress for a moment in order to observe the kernel of \(S\) can indeed be identified with space of \(\varphi\) given by (2.1) and (2.2).

Theorem 2.4 The map

\[
K \to \wedge^-(M)
\]

given by \(\varphi \mapsto \varphi_0\) (see (1.10)) induces an isomorphism of \(\ker(s')\) with \(\ker(S)\).
Proof. We begin by noticing that, as a consequence of (1.9) and (1.14), the equation

\[ d^+ \delta \alpha = \lambda \omega \]  

(2.9)
is equivalent to the two equations

\[ \Lambda d \delta \alpha = -\Lambda \delta d \alpha = 2\lambda \]  

(2.10)
and

\[ d^+ \Lambda d \alpha = 0. \]  

(2.11)

Now suppose that \( \varphi \) is in \( K \) and write

\[ \varphi = \frac{1}{2} (\Lambda \varphi) \omega + \varphi_0. \]

We have to show that equation (2.2) implies that \( \varphi_0 \) is in the kernel of \( S \). But \( d \varphi = 0 \) is equivalent to

\[ d \varphi_0 = -\frac{1}{2} L d (\Lambda \varphi) \]  

(2.12)
and so to

\[ \Lambda d \varphi_0 = -\frac{1}{2} [\Lambda, L] d \Lambda \varphi = -\frac{1}{2} d (\Lambda \varphi) \]  

(2.13)
by (1.11) (remember \( \Lambda \) of any 1-form is zero). This implies that \( \varphi_0 \) satisfies equation (2.11). On the other hand, by applying \( \Lambda \delta \) to (2.12) we get

\[ \Lambda \delta d \varphi_0 = - (\rho, \varphi_0) \]

because \( \rho \) is ASD (cf. (1.19)), and this is equation (2.10) with \( \lambda = \frac{1}{2} (\rho, \varphi_0) \) as required.

To go in the other direction we suppose \( \alpha \in \ker(S) \) so that it satisfies (2.11) and (2.12) with \( \lambda = \frac{1}{2} (\rho, \alpha) \). Let \( u \) be the unique solution of \( \Delta u = \)
2(\rho, \alpha) with \int u \, d\mu = 0, and put \varphi = \frac{1}{2} u\omega + \alpha. Then \varphi automatically satisfies (2.2): all that remains is to check \( d\varphi = 0 \).

By Proposition 1.1 and equation (2.11), \( \Lambda d\alpha \) is \( d \)-closed. Accordingly its Hodge decomposition takes the form

\[ \Lambda d\alpha = h + dv \]  

(2.14)

where \( h \) is a harmonic 1-form and \( v \) is a \( C^\infty \) function which is unique if we insist that \( \int v \, d\mu = 0 \). We claim that \( h = 0 \). Indeed

\[ ||h||^2 = \langle h, \Lambda d\alpha + dv \rangle = \langle h, -\delta^c \alpha \rangle = -\langle d^c h, \alpha \rangle = 0 \]

where we’ve used the Kähler identity (1.8) and the basic fact that on a Kähler manifold any \( \Delta \)-harmonic form is also \( \Delta^c \)-harmonic. Now if we compare (2.14) with (2.13) and the definition of \( u \) we see that \( d\varphi = 0 \) iff \( v = -\frac{1}{2} u \). To see that this is the case we use (2.14) to compute

\[ \Delta(-2v) = -2\delta \Lambda d\alpha = 2\Lambda d\delta \alpha = 2(\rho, \alpha) = \Delta u \]

as required. In the above we have used the Kähler identities and equation (2.11).

\[ \]  

Remark. Aside from the twistor-theoretic argument, the relevance of the operator \( S \) to problem (b) can best be seen by first restating the problem as problem (b’). One then observes that \( S \) is the linearization of the operator which sends a Hermitian metric \( g \), with associated 2-form \( \omega_g \), to \( W_+g(\omega_g) \in \mathcal{E}(\wedge^+) \). From Boyer’s calculations \cite{8} one then reads off the fact that the kernel of this non-linear operator is precisely the space of ASD Hermitian conformal classes, and, since \( b_1(M) \) is even, these are all represented by unit-volume scalar-flat Kähler metrics.

\[ \]  

Proposition 2.5 The operator \( S \) of (2.8) is elliptic with index equal to \( -\tau(M) \), where \( \tau(M) \) is the signature of \( M \).

\[ \]  

Proof. Both statements depend only on the top-order term \( d^+ \delta \) of \( S \). Now the kernel of \( d^+ \delta \) is the space \( H^- \) of ASD harmonic 2-forms and similarly
the kernel of its adjoint $d^-\delta$ is the space $H^+$ of SD harmonic 2-forms. The proof that the symbol is an isomorphism $\wedge^- \to \wedge^+$ is left to the reader.

We now complete our analysis of problem (b) by identifying the cokernel of $S$.

**Proposition 2.6** Suppose that $M$ is not Ricci-flat. The cokernel of $S$ can then be identified with the space of $C^\infty$ functions $f$ which satisfy the following conditions:

(i) the Lichnerowicz equation $\Delta^2 f = -2(dd^c f, \rho)$;
(ii) the orthogonality conditions $\langle f \rho, \alpha \rangle = 0$ for all ASD harmonic 2-forms $\alpha$.

In particular, if $M$ supports no non-parallel holomorphic vector fields, then $\text{coker}(S) = 0$.

**Proof.** By the Fredholm alternative for elliptic operators, the cokernel of $S$ can be identified with the kernel of the adjoint $S^*$. Now for any $\psi \in \Lambda^+(M)$ and $\alpha \in \wedge^-(M)$,

$$
\langle S^* \psi, \alpha \rangle = \langle \psi, S \alpha \rangle = \langle \psi, d^+d\delta \alpha - \frac{1}{2} (\rho, \alpha) \omega \rangle
= \langle d^-\delta \psi, \alpha \rangle - \frac{1}{2} \langle (\psi, \omega) \rho, \alpha \rangle
$$

so

$$
S^* \psi = d^-\delta \psi - \frac{1}{2} (\psi, \omega) \rho.
$$

(2.15)

To analyze the equation $S^* \psi = 0$, we shall invoke (1.14) to write

$$
\psi = f \omega + \chi
$$

(2.16)

(where $f$ is a $C^\infty$ function and $\chi$ lies in $\wedge^{2,0} \oplus \wedge^{0,2}$). We shall also need to write the operator $d^-\delta$ in terms of $d$ and $d^c$. This is an exercise involving the Kähler identities (1.8). Indeed, as an operator $\wedge^+ \to \wedge^-$,

$$

\begin{align*}
    d^-\delta &= \frac{1}{2} (1 - \ast) d \delta \\
         &= \frac{1}{2} d \delta - \frac{1}{2} \delta d \\
         &= \frac{1}{2} d[\Lambda, d^c] - \frac{1}{2} [\Lambda, d^c] d \\
         &= \frac{1}{2} (d \Lambda d^c + d^c \Lambda d) + \frac{1}{2} (\Lambda dd^c - dd^c \Lambda).
\end{align*}
$$

(2.17)
Consider the second bracketed term in (2.17). For reasons of bidegree, it annihilates χ in (2.16). On the other hand
\[ d^- \delta(f \omega) = d^- \delta L f \]
\[ = -d^- [L, \delta] f \]
\[ = -\frac{1}{2} (1 - \ast) dd^c f \]
\[ = -dd^c f - \frac{1}{2} \omega \Delta f \]
(2.18)
where we have used the Kähler identities, the relation (1.15) to identify the ASD part of dd^c f with its projection perpendicular to ω, and (1.12) to relate this to the Laplacian. Combining (2.16), (2.17) and (2.18), we obtain
\[ S^*(f \omega + \chi) = d^- \delta \chi - dd^c f - \frac{1}{2} \omega \Delta f - \rho f \]
(2.19)
where we can also write
\[ d^- \delta \chi = \frac{1}{2} (d \Lambda d^c + d^c \Lambda d) \chi. \]
(2.20)
Suppose (2.19) vanishes. Applying dd^c (and using (2.20)) we find that f satisfies condition (i) of the Theorem:
\[ 0 = -\frac{1}{2} \omega \wedge dd^c \Delta f - \rho \wedge dd^c f \]
\[ = -\frac{1}{2} (\omega, dd^c \Delta f) d\mu + (dd^c f, \rho) d\mu \]
\[ = \left( \frac{1}{2} \Delta^2 f + (dd^c f, \rho) \right) d\mu \]
(the change of sign in the term in ρ arises because ρ is ASD cf. (1.19)). The orthogonality conditions (ii) are just the conditions that the equation (2.19)
\[ d^- \delta \chi = dd^c f + \frac{1}{2} \omega \Delta f + \rho f \]
(2.21)
be soluble for χ. By the Fredholm alternative this equation is soluble iff the right-hand side is orthogonal to the kernel of the adjoint operator d^+ δ. But we have already identified this kernel in the Proof of Proposition 2.5 with the space H^- of ASD harmonic 2-forms. Since the inner product of the first two terms on the right-hand side with any such form is zero we get condition (ii) of the Theorem. The proof is completed by noting that if χ satisfying (2.21) exists, it is unique. This is because the kernel of d^- δ is H^+ = Cω by Theorem 1.5, and by definition χ is orthogonal to ω.
Definition 4 Let $M$ be a compact complex surface. If $\Xi$ is any holomorphic vector field on $M$, the restricted Futaki invariant of $(M, \Xi)$ is defined to be the map

$$\hat{F}_\Xi : A_M \to \mathbb{C}$$

$$[\omega] \mapsto F(\Xi, [\omega]).$$

Here $A_M := \{[\omega] \in H^{1,1} \mid [\omega] > 0, c_1 \cup [\omega] = 0\}$ again denotes the set of admissible Kähler classes.

Theorem 2.7 Let $(M, \omega)$ be a compact scalar-flat Kähler surface, and let $Z$ be its twistor space. Assume that $M$ is not Ricci-flat. Then the cohomology groups $H^2(Z, \Theta_Z \otimes \mathcal{L}_D)$, $H^2(Z, \Theta_{Z,D})$, and $H^2(Z, \Theta_Z)$ are all equal, and can be identified with the space of holomorphic vector fields $\Xi$ on $M$ such that $d\hat{F}_\Xi|_{[\omega]} = 0$.

Proof. Let us first observe that $M$ cannot carry a non-zero parallel vector field. If it did, $g$ would locally be a Riemannian product of the flat metric and some other Kähler metric on $\mathbb{C}$; and since $s = 0$, the second factor would also have to be flat. Thus $g$ would itself be flat, contradicting the assumption that $\rho \not\equiv 0$. Since $g$ has constant scalar curvature and there are now no parallel vector fields on $M$, we may therefore, by Proposition 1.7, write each holomorphic vector field $\Xi$ in the form $2(\bar{\partial} f)^\sharp$ for a unique $f$ satisfying the Lichnerowicz equation (1.21) and $\int f \, d\mu = 0$.

Now, in accordance with Definition 4 above, the restricted Futaki invariant $\hat{F}_\Xi$ is just the restriction of $F(\Xi, \cdot)$, defined by (1.22), to the admissible Kähler classes $A_M \subset H^{1,1}$. The tangent space of $A_M$ at $[\omega]$ is just the $U$-orthogonal complement of $\rho$ in the harmonic $(1,1)$-forms; but since the Futaki invariant vanishes for all multiples of $[\omega]$, we might as well restrict ourselves to admissible classes of fixed volume, which corresponds to cutting the tangent space down to the $L^2$-orthogonal complement of $\rho$ in the closed ASD 2-forms $H^-$. If $\Xi = 2(\bar{\partial} f)^\sharp$, $\int f \, d\mu = 0$, then, by Proposition 1.8 we have

$$\frac{d}{dt} F(\Xi, [\omega + t\alpha]) = \langle f \rho, \alpha \rangle.$$
However, if $C$ is any constant and $\hat{f} := f + C$, this becomes
\[
\frac{d}{dt} F(\Xi, [\omega + t\alpha]) = \langle \hat{f} \rho, \alpha \rangle - C \langle \rho, \alpha \rangle
\]
so that the right-hand side is independent of the representative $\hat{f}$ provided $\langle \rho, \alpha \rangle = 0$. Thus
\[
d\hat{F}_\Xi(\alpha) = \langle f \rho, \alpha \rangle \ \forall \alpha \in H^- \text{ s.t. } \langle \rho, \alpha \rangle = 0
\]
for any $f$ with $\Xi = 2(\bar{\partial}f)^\sharp$, independent of any statement concerning $\int f \, d\mu$. On the other hand, if $\Xi = 2(\bar{\partial}f)^\sharp$ is a holomorphic vector field with $d\hat{F}_\Xi|_\omega = 0$, there is exactly one $C$ for which $\hat{f} := f + C$ satisfies $\langle \hat{f} \rho, \rho \rangle = 0$, since $\langle \rho, \rho \rangle = -4\pi c_1^2 > 0$. This allows us to identify the space of holomorphic vector fields $\Xi$ satisfying $d\hat{F}_\Xi|_\omega = 0$ with the space of solutions $f$ of the Lichnerowicz equation such that $\langle f \rho, \alpha \rangle = 0$ for all $\alpha \in H^-$. Using Corollary 3.4 and Theorem 2.6, this in turn identifies $H^2(Z, \Theta_Z \otimes I_D)$ with the space of holomorphic vector fields $\Xi$ on $M$ such that $d\hat{F}_\Xi|_\omega = 0$, as promised.

Now, using Serre duality, we observe that, since $M$ is ruled, $H^2(M, \Theta_M) \cong H^0(M, \Omega^1(\kappa_M^{-1})) = 0$ because $\Omega^1(\kappa)$ becomes $\mathcal{O}(-2) \oplus \mathcal{O}(-4)$ when restricted to a smooth rational curve with trivial normal bundle. Since $[\omega]$ has total scalar curvature 0, we also have $H^2(M, \mathcal{O}(\kappa_M^{-1})) \cong H^0(M, \mathcal{O}(\kappa_M^2)) = 0$ by Corollary 1.4. The isomorphisms
\[
H^2(Z, \Theta_Z \otimes I_D) \cong H^2(Z, \Theta_{Z,D}) \cong H^2(Z, \Theta_Z)
\]
now follow immediately from the short exact sequences (1.27) and (1.28), since $D$ consists of $M$, embedded in $Z$ with normal bundle $\kappa_M^{-1}$, together with the image of this surface via the anti-holomorphic map $\sigma$.

**Example.** Let $M = \mathbb{CP}_1 \times \Sigma_g$ be the product of the Riemann sphere with a curve of genus $g \geq 2$. Equip the factors with metrics of curvature $\pm 1$, and let $g$ be the product metric, which is a scalar-flat Kähler metric on $M$. Because there is only one admissible Kähler class on $M$ of a given volume, $\hat{F}_\Xi \equiv 0$ for any holomorphic vector field $\Xi$. Since $a(M) = \text{sl}(2, \mathbb{C})$, we therefore have
\[
h^2(Z, \Theta_Z \otimes I_D) = h^2(Z, \Theta_{Z,D}) = h^2(Z, \Theta_Z) = 3.
\]
On the other hand, if \((M, g)\) is instead the twisted version of the above example constructed on the \(\mathbb{CP}_1\)-bundle associated to any flat connection on a principal \(\text{SU}(2)\)-bundle over \(\Sigma_g\), then, provided that the given flat connection is generic in the sense that its holonomy acts irreducibly on \(\mathfrak{su}(2)\), there are no non-trivial holomorphic vector fields on \(M\), and
\[
h^2(Z, \Theta_Z \otimes \mathcal{I}_D) = h^2(Z, \Theta_{Z,D}) = h^2(Z, \Theta_Z) = 0.
\]

**Remark.** If \((M, g)\) is Ricci-flat, the story is utterly different from that described in Theorem 2.7. Instead, one may immediately read off from Corollary 2.3 that
\[
r \equiv 0 \Rightarrow h^2(Z, \Theta_Z \otimes \mathcal{I}_D) = b^+ \neq 0.
\]

The deformation techniques we are developing here are thus ill-suited to, say, a K3 surface. Instead, in this hyper-Kähler case, when there is more than one choice of parallel complex structure available, an unobstructed deformation theory can be obtained by considering deformations of \(Z\) relative to a fibration over \(\mathbb{CP}_1\). However, in light of the quite definitive theory of Ricci-flat Kähler metrics one obtains from Yau’s solution of the Calabi conjecture \([43]\), there is little reason to pursue this point of view.

As our first application of this result, let \(M\) be a compact Kähler surface with a fixed complex structure \(J\) and \(c_1^2 < 0\). Introduce \(S\), the moduli space of scalar-flat Kähler metrics modulo homothety, and \(\mathcal{A}_M/\mathbb{R}^+\), the projectivized cone of Kähler classes which are \(\cup\)-orthogonal to \(c_1\). There is a natural map \(\mu : S \to \mathcal{A}_M/\mathbb{R}^+\) induced by mapping a metric to its Kähler class.

**Theorem 2.8** Let \(g\) be a scalar-flat Kähler metric on \(M\). Assume that \(g\) is not Ricci-flat.

(i) If \(d\mathcal{F}_\Xi|_{\omega_0} \neq 0\) for every non-zero holomorphic vector field \(\Xi\) on \(M\), the deformation theories for problems (b), (c), and (d) are all unobstructed. In particular, \(g\) is a smooth point of the moduli space \(S\) of problem (b), and \(S\) has dimension \(|\tau(M)|\) near \(g\).

(ii) If \(M\) carries no non-trivial holomorphic vector fields, the moduli space \(S\) is smooth, and \(\mu\) is a local diffeomorphism between \(S\) and \(\mathcal{A}_M/\mathbb{R}^+\). The
set of Kähler classes represented by scalar-flat metrics is therefore open in the space \( \mathcal{A}_M \) of admissible Kähler classes.

**Proof.** The first part is a consequence of the relevant Kodaira-Spencer Theorem 1.16(ii), Corollary 2.3 and Theorem 2.7. The second part follows from Proposition 2.1 and a simple count of dimensions: from its definition, \( \mathcal{A}_M/R^+ \) is a manifold of dimension \( b_2 - 2 \) and this coincides with the dimension \( |\tau(M)| \) by Yau’s Theorem 1.3. \( \square \)
3 Ruled Surfaces

3.1 Computing the Futaki Invariant

Let \((M, J)\) be a compact complex surface with \(c_1^R(M) \neq 0\), and suppose that \([\omega]\) is a Kähler class on \(M\) such that the total scalar curvature vanishes—equivalently, such that \(c_1 \cup [\omega] = 0\). Then, by Theorem 1.5, \(M\) must be a ruled surface, which is to say that \((M, J)\) is obtained from a projectivized rank-2 vector bundle \(\mathbb{P}(E) \to \Sigma_g\) over a compact complex curve \(\Sigma_g\) by blowing up \(m = |\tau(M)|\) points.

As our eventual goal is to study scalar-flat Kähler surfaces, we will only wish to consider surfaces \(M\) with vanishing Matsushima-Lichnerowicz obstruction in the sense of Definition 2. The search is therefore considerably narrowed by the following result:

**Proposition 3.1** Let \(M\) be a compact complex surface with an admissible Kähler class, vanishing Matsushima-Lichnerowicz obstruction, and non-trivial automorphism algebra \(a(M)\). Suppose also that \(M\) is not finitely covered by a complex torus. Then, for some holomorphic line bundle \(\mathcal{L} \to \Sigma_g\) over a compact complex curve \(\Sigma_g\) of genus \(g \geq 2\), \(M\) is obtained from the minimal ruled surface \(\mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma_g\) by blowing up \(|\tau(M)|\) points along the zero section of \(\mathcal{L} \subset \mathbb{P}(\mathcal{L} \oplus \mathcal{O})\). Moreover, unless \(M = \mathbb{C}P_1 \times \Sigma_g\), the space \(a(M)\) of holomorphic vector fields is 1-dimensional, and is spanned by the Euler vector field of \(\mathcal{L}\).

**Proof.** Theorem 1.5 tells us immediately that \(M\) is either ruled or covered by a K3 surface. The latter possibility, however, is excluded because \(\Gamma(K3, \Theta) = 0\).

Since \(c_1 \cdot [\omega] = 0\), \(c_1^R\) is a non-zero primitive class in \(H^{1,1}\), and \(c_1^2 < 0\). Thus, with respect to the complex orientation, \(2\chi + 3\tau < 0\). If the curve \(\Sigma_g\) has genus \(g < 2\), we therefore have an estimate of the number of \((-1)\)-curves contained in \(M\). Specifically, if \(M\) is obtained by blowing up a Hirzebruch surface \(\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}) \to \mathbb{C}P_1\) at \(\ell\) points, we must have \(\ell \geq 9\); and if \(M\) is instead obtained by blowing up a minimal ruled surface \(\tilde{M} \to E\) over an elliptic curve \(E = \mathbb{C}/\Lambda\) at \(\ell\) points, then \(\ell > 0\).

Let \(\Xi \neq 0\) denote a holomorphic vector field on \(M\), and let \(\pi : \tilde{M} \to \Sigma_g\) denote a \(\mathbb{C}P_1\)-bundle from which \(M\) can be obtained by a blow-up \(b : M \to \tilde{M}\).
We then consider the component of $b_\ast \Xi$ normal to the fibers of $\pi$. Since the normal bundle of each such fiber is trivial, this normal component is constant up the fibers, so that $(\pi b)_\ast \Xi$ is a well-defined holomorphic vector field on $\Sigma_g$. If $\Sigma_g$ has genus $> 1$, this vector field must vanish. If, on the other hand, $\Sigma_g$ has genus 1, the fact that $b$ involves blowing up at at least one point forces $b_\ast \Xi$, and hence $(\pi b)_\ast \Xi$, to have at least one zero, implying that $(\pi b)_\ast \Xi \equiv 0$. Finally, if $\Sigma_g$ has genus 0, and if $(\pi b)_\ast \Xi \not\equiv 0$, the $\ell \geq 9$ blown-up points of $\pi : F_k \to \mathbb{CP}^1$ must be located on at most 2 fibers of $\pi$; but then $F_k = P(O \oplus O(k))$ admits sections of $\kappa^{-\ell}$ which vanishes along these two fibers to order $\ell$, and this section lifts to a non-zero element of $H^0(M, O(\kappa^{-\ell}))$, contradicting Corollary 1.4. The vector field $(\pi b)_\ast \Xi$ must therefore vanish identically, and $b_\ast \Xi$ is tangent to the fibers of $\pi$. In short $\mathcal{A}(M)$ consists strictly of vertical vector fields.

By the Matsushima-Lichnerowicz assumption, the identity component of the automorphism group of $M$ is the complexification of a compact group. We therefore have a non-trivial holomorphic vector field $\Xi$ on $M$ whose imaginary part $\xi$ generates an $S^1$-action, and which itself generates a $\mathbb{C}^\times$-action; for brevity’s sake, we shall henceforth refer to any such $\Xi$ as a periodic holomorphic vector field. On the other hand, the minimal model $\pi : \hat{M} \to \Sigma_g$ may be represented in the form $P(E) \to \Sigma_g$ for a rank 2 holomorphic vector bundle $E \to \Sigma_g$ which is completely specified once an arbitrary line bundle $\wedge^2 E$ is chosen, subject to the condition $c_1(E) \equiv w_2(\pi) \mod 2$. The vector field $b_\ast \Xi$ is then uniquely specified by a trace-free holomorphic section $A$ of $\text{End}(E)$. The determinant of $A$ is a holomorphic function on $\Sigma_g$, hence a constant. On the other hand, since $\Xi$ is periodic, $A$ is diagonalizable, and $A$ must be a half-integer multiple of

$$
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
$$

The vector bundle $E$ thus globally splits as a direct sum of the eigenspaces of $A$, and, twisting by a line bundle, we may therefore take $E = \mathcal{L} \oplus \mathcal{O}$, so that $\Xi$ becomes a constant multiple of the Euler vector field on $\mathcal{L}$. We henceforth normalize this constant to be 1. The blown-up points must all occur at zeroes of $\Xi$, namely either at the zero section of $\mathcal{L}$ or at the “infinity section” corresponding to the $\mathcal{O}$ factor. The latter possibility may be reduced to the former by noticing that the proper transform of a fiber through exactly one blown-up point is a (-1)-curve, which may therefore be blown down, thereby
leading to a different minimal model. In our case, iteration of this procedure allows us to replace blown-up points “at the infinity section” by blown-up points “at the zero section,” at the small price of twisting our line bundle $L$ by the divisor of the relevant points of $\Sigma_g$.

The space of vertical vector fields on the minimal model $\tilde{M}$ is now precisely $\Gamma(\Sigma_g, \mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^*)$, with the Lie algebra structure induced by identifying $(u, v, w) \in \Gamma(\Sigma_g, \mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^*)$ with the matrix

$$
\begin{bmatrix}
u & v \\
w & -u
\end{bmatrix}.
$$

If $M = \tilde{M}$, this algebra is itself required to be reductive, implying $H^0(\Sigma_g, \mathcal{L}) \neq 0 \iff H^0(\Sigma_g, \mathcal{L}^*) \neq 0$; we conclude that either $\mathcal{L}$ is trivial and $M = \mathbb{CP}_1 \times \Sigma_g$, or else $a(M)$ is 1-dimensional. If, on the other hand, $M$ is obtained by blowing up points on the zero section of $\mathcal{L}$, the vector field $(u, v, w)$ lifts to $M$ iff $v = 0$, and $a(M) = \{(u, 0, w)\}$; thus, if $\tau(M) \neq 0$, $a(M)$ is reductive iff $\Gamma(\Sigma_g, \mathcal{L}^*) = 0$. Thus, provided that $M \neq \mathbb{CP}_1 \times \Sigma_g$, $a(M)$ is 1-dimensional, with the Euler vector fields $\Xi$ (corresponding to $(u, v, w) = (\frac{1}{2}, 0, 0)$) as a basis.

Finally, we observe that $M$ must have genus $\geq 2$. Indeed, the Euler field $\Xi$ is a vector field on $\tilde{M}$ which vanishes at all the points which are to be blown up. Let $\ell$ be the greatest multiplicity with which any point is to be blown up, and, assuming $g = 0, 1$, let $\Upsilon \neq 0$ be any vector field on $\Sigma_g$. Then $(\Xi \wedge \Upsilon)^{\otimes \ell}$ lifts to $M$ as a non-trivial section of $\kappa^{-\ell}$, contradicting Proposition 1.4. Hence $g \geq 2$.

Our goal is now to calculate the Futaki invariant of $(M, [\omega], \Xi)$, where $M$ is in normal form described in the above Proposition, $[\omega]$ is an admissible Kähler class and $\Xi$ is the Euler vector field. We proceed by a symplectic quotient construction in the spirit of [26].

The invariant we seek to compute is known [11] to be independent of the representative $\omega \in [\omega]$, so we may assume (by averaging) that $\omega$ is invariant under the $S^1$-action generated by $\xi = \Im \Xi$. Since $0 = \mathcal{L}_\xi \omega = d(\xi | \omega)$, we see that $\nu := \omega(\xi, \cdot)$ is closed; and, on the other hand, any real harmonic 1-form on our compact Kähler manifold is the real part of a holomorphic 1-form, and so must everywhere be orthogonal to $\xi = \Im \Xi$, as may either be seen directly from the our explicit description of $(M, \Xi)$, or deduced as a consequence of

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the maximum principle for pluriharmonic functions and the fact that $\Xi$ has zeroes. Harmonic theory therefore yields

$$\nu := (d^* + d^*d)G\nu = d^*G\nu,$$

where $G$ is the Green’s operator, and $R\Xi := \text{grad} f$ for a unique function $f := d^*G\nu$, called the holomorphy potential \[7\] of $\Xi$, such that $\int_M f \, d\mu = 0$. As we saw in §1.6, the Futaki invariant $F(\Xi, [\omega])$ of $(M, [\omega])$ is then given by

$$F(\Xi, [\omega]) = -\frac{1}{2} \int_M f s \, d\mu.$$

The symplectic vector field $\xi = \Re\Xi$ generating the $S^1$-action is now a globally Hamiltonian vector field, meaning that $\omega(\xi, \cdot) = dt$ for a smooth (“Hamiltonian”) function $t : M \to \mathbb{R}$; indeed, anything of the form of the form $t = f + c$ will do. We could, of course, choose our constant $c$ to vanish, but we will instead find it convenient to choose $c$ so that $\max t = -\min t = a$, and $t : M \to [-a, a]$. (As we shall see in a moment, the intrinsic significance of the number $a$ is that $f_F[\omega] = 4\pi a$, where $F$ is any fiber of $M \to \Sigma_g$.) Fortunately, because we have assumed that $c_1 \cup [\omega] = 0$, this will not interfere with our calculation of the Futaki invariant because

$$\int_M ts \, d\mu = \int_M (f + c)s \, d\mu = \int_M fs \, d\mu + c \int_M s \, d\mu = \int_M fs \, d\mu = -2F(\Xi, [\omega]).$$

The only isolated critical points of $t$ occur at those zeroes of $\Xi$ which occur at the intersection of an exceptional curve and the proper transform of a fiber; since such a fixed point is attractive along the exceptional curve and repulsive along the proper transform of a fiber, such a critical point has index 2. On the other hand, the maxima and minima of $t$ occur along a pair of holomorphic curves, $C_0 = t^{-1}(-a)$ and $C_\infty = t^{-1}(a)$, which are just the proper transforms of the “zero” and “infinity” sections of $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \to \Sigma_g$. We now have a projection $\pi : M \to \Sigma_g \times [-a, a]$ given by the product of the ruling $M \to \Sigma_g$ and the Hamiltonian $t$. Because the $C^\infty$-action preserves the ruling, every fiber of $\pi$ consists of exactly one orbit of the $S^1$-action. (This
is really a consequence [2] [4] of the fact that $\Sigma_g$ is both the symplectic and stable quotient of $M$ by the $\mathbb{C}^\times$-action.) Let $q_1, \ldots, q_m \in \Sigma_g \times (-a, a)$ be the images of the isolated fixed points of the action, and let $X := [\Sigma_g \times (-a, a)] - \{q_1, \ldots, q_m\}$ denote the set of regular values of $\pi$. If $Y \subset M$ is the set of regular points, then $\pi : Y \to X$ is a principal $S^1$-bundle, and, by taking the orthogonal complement of the $S^1$ orbits with respect to the Kähler metric, we endow $Y \to X$ with a connection form $\theta$. If $z = x + iy$ is any complex local coordinate on $\Sigma_g$, we may then express the given Kähler metric $g$ on $Y \subset M$ in the form

$$g = vw(dx^{\otimes 2} + dy^{\otimes 2}) + w \ dt^{\otimes 2} + w^{-1}\theta^{\otimes 2},$$

for positive functions $v, w > 0$ on $X$, while the complex structure $J$ is given by

$$
\begin{align*}
dx & \mapsto dy \\
dt & \mapsto w^{-1}\theta 
\end{align*}
$$

so that the Kähler form is given by

$$\omega = dt \wedge \theta + vw \ dx \wedge dy .$$

Since the complex structure $J$ is integrable, the differential ideal

$$\mathcal{J} = \langle dx + idy, wd\!t + i\theta \rangle$$

must satisfy $d\mathcal{J} \subset \mathcal{J}$; explicitly, this means that

$$d(wd\!t + i\theta) = dw \wedge dt + id\theta = \varphi \wedge (dx + idy)$$

for some complex-valued 1-form $\varphi$ on $X$, and, because $\theta$ and $w$ are real, this is in turn equivalent to

$$d\theta \equiv w_x dy \wedge dt + w_y dt \wedge dx \mod dx \wedge dy .$$

The Kähler condition $d\omega = 0$ now reads

$$0 = d(dt \wedge \theta + vw dx \wedge dy) = -dt \wedge d\theta + (vw)_t dt \wedge dx \wedge dy ,$$

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so that the curvature of our $S^1$-connection $\theta$ is now completely determined by $v$ and $w$:

$$d\theta = w_x dy \wedge dt + w_y dt \wedge dx + (vw)_t dx \wedge dy.$$ 

In particular, we conclude that $w_{xx} + w_{yy} + (vw)_{tt} = 0$.

Notice that equation (3.1) says that the metric on any fiber $F$ of $M \to \Sigma_g$ is given by

$$g|_F = w dt^2 + w^{-1} d\vartheta^2,$$

where $\vartheta \in [0, 2\pi]$ is a fiber coordinate in a gauge chosen such that the connection form $\theta$ has no $dt$ component; the area form on $F$ is just

$$\omega|_F = dt \wedge d\vartheta,$$

and the area of $F$ is therefore $4\pi a$. On the other hand, since this metric is smooth at the “south pole” $t = -a$ of the 2-sphere $F$, letting $r$ denote the Riemannian distance from the south pole, we have

$$w dt^2 + w^{-1} d\vartheta^2 = dr^2 + (r^2 + O(r^4)) d\vartheta^2,$$

so that $dt = r(1 + O(r^2)) dr$, $t + a = r^2 + O(r^4)$, and $w^{-1} = 2(t + a) + O((t + a)^2)$. Similarly, $w^{-1} = -2(t - a) + O((t - a)^2)$ near $t = a$. Thus $\ell = w^{-1}$, which is a smooth function on $M$ because it represents the square of the length of the Killing field $\xi$, descends to a differentiable function on $\Sigma_g \times [-a, a]$ which vanishes at the boundary and satisfies $\frac{d\ell}{dt} = \mp 2$ at $t = \pm a$. At the same time, equation (3.1) tells us that $\lim_{t \to -a} vw dx \wedge dy = \omega|_{C_0}$, while $\lim_{t \to a} vw dx \wedge dy = \omega|_{C_\infty}$. Thus $v$ is smooth up to the boundary of $\Sigma_g \times [-a, a]$, and moreover

$$v|_{t=\pm a} = 0, \quad v_t dx \wedge dy|_{t=\pm a} = 2\omega|_{C_0}, \quad v_t dx \wedge dy|_{t=a} = -2\omega|_{C_\infty}.$$

Since $\Xi$ is a holomorphic vector field and the $(2, 0)$-form $\mu := dz \wedge (w dt + i\theta)$ has the property that $\Xi|\mu = 2 dz$ is a holomorphic form, $\mu$ must itself

\footnote{This generalizes to a simple relationship between volumes and moment maps for torus actions that is sometimes called the “Archimedes Principle” [1].}
be holomorphic; thus $\frac{1}{2} \mu \wedge \overline{\mu} = w dx \wedge dy \wedge dt \wedge \theta$ is the volume form of a holomorphic frame. On the other hand, the metric volume form is

$$\frac{1}{2} \omega \wedge \omega = vw \, dx \wedge dy \wedge dt \wedge \theta,$$

so that the Ricci form of $g$ must be

$$\rho = -i \partial \overline{\partial} \log \left( \frac{vw \, dx \wedge dy \wedge dt \wedge \theta}{w \, dx \wedge dy \wedge dt \wedge \theta} \right) = -i \partial \overline{\partial} \log v.$$ 

The scalar curvature density of $g$ thus is given in terms of $u := \log v$ by

$$s \, d\mu = \frac{1}{2} s \, \omega \wedge \omega = 2 \omega \wedge \rho = -2i \omega \wedge \partial \overline{\partial} u = -2i \omega \wedge [\frac{1}{2} dJ du] = - \omega \wedge dJ du = - \omega \wedge d[u_x dx + u_y dy + u_t w^{-1} \theta] = -[dt \wedge \theta + vw \, dx \wedge dy] \wedge d[u_x dy - u_y dx + u_t w^{-1} \theta] = [u_{xx} + u_{yy} + vw(w^{-1}u_t) + (u_t w^{-1}d\theta) \wedge dt \wedge \theta] = [u_{xx} + u_{yy} + (v^t u_t) \wedge dt \wedge \theta] = [u_{xx} + (\log v)_{xx} + (\log v)_{yy} + v^t] \, dx \wedge dy \wedge dt \wedge \theta.$$

In particular,

$$s = \frac{(\log v)_{xx} + (\log v)_{yy} + v^t}{vw}.$$ 

(3.4)

Let us now rephrase the above results in more global terms. Because $g$, $w$ and $dt$ are globally defined, it follows from equation (3.1) that, for $-a < t < a$,

$$g^v(t) := v \,(dx^2 + dy^2)$$

is a well-defined $t$-dependent Kähler metric on $\Sigma_g$, with Kähler form

$$\omega^v(t) := v \, dx \wedge dy.$$ 

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Moreover,
\[ \omega^v|_{t=\pm a} = 0, \]
\[ \frac{d}{dt}\omega^v|_{t=-a} = 2\omega|_{C_0}, \]
\[ \frac{d}{dt}\omega^v|_{t=a} = -2\omega|_{C_{\infty}}. \]

The Ricci form of this metric is
\[ \rho^v(t) = -i(\frac{\partial^2}{\partial z\partial \bar{z}}\log v)dz \wedge d\bar{z}, \]
so that our formula for the density of scalar curvature may be written globally on \( Y \subset M \) as
\[ s \, d\mu = \left[-2\rho^v + \frac{d^2}{dt^2}\omega^v\right] \wedge dt \wedge \theta. \]

Hence
\[ \int_M ts \, d\mu = \int_Y ts \, d\mu \]
\[ = \int_Y t[-2\rho^v + \frac{d^2}{dt^2}\omega^v] \wedge dt \wedge \theta \]
\[ = 2\pi \int_{-a}^{a} t\left[\int_{\Sigma_g} -2\rho^v + \frac{d^2}{dt^2}\omega^v\right] dt \]
\[ = 2\pi \int_{-a}^{a} \frac{dt}{dt^2}\left[\int_{\Sigma_g} \omega^v\right] dt + 2\pi \int_{-a}^{a} t\left[\int_{\Sigma_g} -2\rho^v\right] dt \]
\[ = 2\pi \int_{-a}^{a} \frac{dt}{dt^2}\left[\int_{\Sigma_g} \omega^v\right] dt - 8\pi^2 \chi(\Sigma_g) \int_{-a}^{a} t \, dt \]
\[ = 2\pi \left[\int_{-a}^{a} \frac{dt}{dt^2}\int_{\Sigma_g} \omega^v\right] - 2\pi \left[\int_{-a}^{a} \frac{d}{dt}\int_{\Sigma_g} \omega^v\right] \]
\[ = 2\pi \left[\int_{-a}^{a} \frac{d}{dt}\int_{\Sigma_g} \omega^v\right] - 2\pi \left[\int_{-a}^{a} \omega^v\right] \]
\[ = 2\pi \left[\int_{-a}^{a} \omega^v\right]. \]
\[ = 2\pi a \left[ \int_{\Sigma_g} \frac{d}{dt} \omega^v \bigg|_{t=a} + \int_{\Sigma_g} \frac{d}{dt} \omega^v \bigg|_{t=-a} \right] \]
\[ = 2\pi a \left[ \int_{C_0} 2\omega - \int_{C_\infty} 2\omega \right] = 4\pi a \left[ \int_{C_0} \omega - \int_{C_\infty} \omega \right] \]
\[ = \left[ \int_{C_0} \omega - \int_{C_\infty} \omega \right] \int_F \omega \]

In conclusion, we have

\[ F(\Xi, [\omega]) = -\frac{1}{2} \int_M f s \, d\mu \]
\[ = -\frac{1}{2} \int_M t s \, d\mu \]
\[ = -\frac{1}{2} \left[ \int_{C_0} \omega - \int_{C_\infty} \omega \right] \int_F \omega \]
\[ = \frac{1}{2} \left[ \int_{C_\infty} \omega - \int_{C_0} \omega \right] \int_F \omega \]

where \( F \) is any fiber of \( M \to \Sigma_g \). We have thus proved the following:

**Theorem 3.2** Let \( M \) be any compact complex surface equipped with an admissible Kähler class \([\omega]\) and a holomorphic \( \mathbb{C}^\times \)-action which is free on an open dense set. Assume that \( c_1^R(M) \neq 0 \), and let \( \Xi \in \Gamma(M, \mathcal{O}(TM)) \) denote the holomorphic vector field which generates the action. Thus \( M \) is a ruled surface \( M \to \Sigma_g \) of genus \( g \geq 2 \), the generic fiber \( F \) of which is the closure of an orbit, while the “attractive” and “repulsive” fixed curves \( C_0 \) and \( C_\infty \) of the action are sections of the projection \( M \to \Sigma_g \). The Futaki invariant of \((M, J, [\omega], \Xi)\) is then given by

\[ F(\Xi, [\omega]) = \frac{1}{2} \left[ \int_{C_\infty} \omega - \int_{C_0} \omega \right] \int_F \omega . \]

**Remark.** We have computed the Futaki invariant only for a single vector field \( \Xi \). However, if the Matsushima-Lichnerowicz obstruction vanishes, Proposition \[3.4\] tells us that either \( a(M) \) is spanned by \( \Xi \) or else \( M = \mathbb{CP}_1 \times \Sigma_g \). In the former case, the \( \mathbb{C} \)-linearity of \( F(\cdot, [\omega]) \) tells us
that our computation completely determines the \( F(\cdot, [\omega]) \). In the exceptional case \( M = \mathbb{CP}_1 \times \Sigma_g \), the Futaki character vanishes, since the product of two constant curvature metrics has constant scalar curvature.

Use of the fact that \( c_1 \cdot [\omega] = 0 \) allows one to rewrite the Futaki invariant in interesting equivalent ways. As previously indicated, we will always normalize the minimal model of our ruled surface with holomorphic vector field by putting it in the form \( \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma_g \), where all the blown-up points are on the zero section of \( \mathbb{L} \hookrightarrow \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) : \zeta \mapsto [\zeta, 1] \), and so correspond to fibers of \( \mathcal{O} \subset \mathcal{L} \oplus \mathcal{O} \). For simplicity, let us assume for the moment that the blown-up points in \( \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \) are all distinct, and so give rise to \( m = |\tau(M)| \) exceptional rational curves \( E_j \) of self-intersection \(-1\). Let us associate \( m \) “weights” \( w_j > 0, j = 1, \ldots, m \), to the Kähler class \( [\omega] \) by defining

\[
w_j := \frac{\int_{E_j} \omega}{\int_F \omega}.
\]

The homology classes of \( C_\infty, F, E_1, \ldots, E_m \) form a basis for \( H_2(M) \), and the intersection form of \( M \) is

\[
\begin{bmatrix}
-k & 1 \\
1 & 0 & -1 & \ddots & \ddots \\
& & & -1
\end{bmatrix}
\]

with respect to this basis, where \( k := \deg(\mathcal{L}) \); it follows that the Poincaré dual of \( [\omega] \) can be expressed in this basis as \( A(1, B+k, -w_1, \ldots, -w_m) \), where \( A = \int_F \omega \) and \( AB = \int_{C_\infty} \omega \). Since \( C_\infty \) has genus \( g \) and self-intersection \(-k\), whereas \( F, E_1, \ldots, E_m \) have genus 0 and self-intersection 0, \(-1, \ldots, -1\), the adjunction formula may be used to rewrite the condition \( c_1 \cdot [\omega] = 0 \) as

\[
0 = [2(1 - g) - k, 2, 1, \ldots, 1] \begin{bmatrix} 1 \\ B+k \\ -w_1 \\ \vdots \\ -w_m \end{bmatrix} = 2(1 - g) + k + 2B - \sum_{j=1}^{m} w_j,
\]
so that \( B = [-k + 2(g - 1) + \sum_{j=1}^{m} w_j]/2 \), and
\[
\int_{C_\infty} \omega = \frac{A}{2}[-k + 2(g - 1) + \sum_{j=1}^{m} w_j].
\]

On the other hand, the picture is symmetrical between \( C_\infty \) and \( C_0 \) as long as we remember to replace \( k = -C_\infty \cdot C_\infty \) with \( m - k = -C_0 \cdot C_0 \) and replace the \( E_j \) with new exceptional curves \( \tilde{E}_j \) such that \( [E_j] + [\tilde{E}_j] = [F] \), resulting in a replacement of the weights \( w_j \) by new weights \( 1 - w_j \); the upshot is that
\[
\int_{C_0} \omega = \frac{A}{2}[k - m + 2(g - 1) + \sum_{j=1}^{m} (1 - w_j)]
\]
\[
= \frac{A}{2}[k + 2(g - 1) - \sum_{j=1}^{m} w_j].
\]

The Futaki invariant can therefore be rewritten as
\[
\mathcal{F}(\Xi, [\omega]) = \frac{1}{2} \left[ \int_{C_\infty} \omega - \int_{C_0} \omega \right] \int_F \omega
\]
\[
= -\frac{A^2}{4} \left[ (k + 2(g - 1) - \sum_{j=1}^{m} w_j) - (-k + 2(g - 1) + \sum_{j=1}^{m} w_j) \right]
\]
\[
= -\frac{A^2}{2}[k - \sum_{j=1}^{m} w_j]
\]
\[
= -\frac{A^2}{2}[\deg(L) - \sum_{j=1}^{m} w_j].
\]

We therefore have the following:

**Proposition 3.3** Let \( M \) be obtained from the minimal model \( P(\mathcal{L} \oplus O) \to \Sigma_g \), \( g \geq 2 \), by blowing up \( m \) points on the zero section of \( \mathcal{L} \). Let \( [\omega] \) be a Kähler class on \( M \) such that \( c_1 \cup [\omega] = 0 \), and let the weights \( w_j \in ]0, 1[ \) be defined by \( \int_{E_j} \omega = w_j \int_F \omega \), where \( F \) is a typical fiber of \( M \to \Sigma_g \) and the \( E_j \), \( j = 1, \ldots, m \) are the exceptional curves corresponding to the blown-up points. Let \( \Xi \) denote the vector field on \( M \) corresponding to the Euler field on \( \mathcal{L} \). Then
\[
\mathcal{F}(\Xi, [\omega]) = 0 \iff \sum_{j=1}^{m} w_j = \deg(L).
\]
Proof. We assumed for simplicity in the previous discussion that $M$ is obtained from its minimal model by blowing up distinct points. However, the argument goes through without change provided that, in defining the weights $w_j$, one replaces the integrals of $\omega$ over the exceptional divisors $E_j$ with integrals of $\omega$ over the corresponding homology classes, each of which can be represented by a chain of rational curves with intersection matrix

$$
\begin{pmatrix}
-2 & 1 \\
1 & \ddots & 1 \\
1 & -2 & 1 \\
1 & -1
\end{pmatrix}.
$$

Corollary 3.4 Let $(M, \Xi)$ be as in Proposition 3.3. Then $M$ carries an admissible Kähler class $[\omega]$ for which $\mathcal{F}(\Xi, [\omega]) = 0$ iff one of the following holds:

(a) $0 = \deg(\mathcal{L}) = m$; or

(b) $0 < \deg(\mathcal{L}) < m$.

Proof. The necessity of these conditions is an immediate consequence of Proposition 3.3. For sufficiency, one may either invoke Proposition 1.6, or else wait for the explicit construction in §3.2 below.

Remark. Notice that $m = 1$ is excluded, since $(b) \Rightarrow m \geq 2$. Also notice that the vanishing of the Futaki invariant implies the vanishing of the Matsushima-Lichnerowicz obstruction. Indeed, if $m = 0$ we either have $\mathcal{L}$ is trivial, and $M = \mathbb{CP}_1 \times \Sigma_g$, or else $\Gamma(\Sigma_g, \mathcal{L}) = \Gamma(\Sigma_g, \mathcal{L}^*) = 0$, so that $a(M)$ is generated by the Euler field $\Xi$; if $m > 0$, $\Gamma(\Sigma_g, \mathcal{L}^*) = 0$ and again $a(M)$ is generated by the Euler field $\Xi$.

Corollary 3.5 Suppose that $(M, [\omega])$ is a compact complex surface with admissible Kähler class and vanishing Matsushima-Lichnerowicz obstruction.
Let $\Xi \neq 0$ be any non-trivial holomorphic vector field, and consider the restricted Futaki functional $\hat{F}_\Xi$ of Definition \[.\] Then
\[ d\hat{F}_\Xi|_{[\omega]} = 0 \iff \tau(M) = 0. \]

**Proof.** By Proposition \[3.1\], the only non-rulled surfaces we need consider are tori and their (hyper-elliptic) quotients, for which both $\tau$ and $F$ vanish. For the ruled surfaces, the result follows immediately from Theorem \[3.2\].

Finally, as an aside, let us observe that the condition $F(\Xi, [\omega]) = 0$ can now be restated in terms of parabolic stability in the sense of Seshadri \[3.3\]. We consider the vector bundle $V = L \oplus O$ of our minimal model, equipped with 1-dimensional subspaces $L_j$ in some fibers of $V$ which represent the exceptional divisors $E_1, \ldots, E_m \subset M$. (Thus the subspaces $L_j$ are contained the $O$ factor of $L \oplus O$.) Let $w_1, \ldots, w_m$ denote the weights as before, and let $\alpha_j < \beta_j$ be arbitrary numbers in $[0, 1]$ such that $w_j = \beta_j - \alpha_j$. The criterion
\[ \sum_{j=1}^{m} w_j = \deg(L), \]
is precisely equivalent to the statement that $(V, \{ (L_j, \alpha_j, \beta_j) \})$ is parabolically quasi-stable\[3.4\], in the sense that, for every line sub-bundle $L \subset V$, we have
\[ \text{pardeg}(L) \leq \frac{1}{2} \text{pardeg}(V), \]
with equality iff $L$ is a *direct summand* of $V$; here the *parabolic degree* of a line sub-bundle $L \subset V$ is defined to be
\[ \text{pardeg}(L) := \deg(L) + \sum_{\{ j | L_j \subset L \}} \alpha_j + \sum_{\{ j | L_j \not\subset L \}} \beta_j, \]
whereas
\[ \text{pardeg}(V) := \deg(V) + \sum_{j=0}^{m} \alpha_j + \sum_{j=0}^{m} \beta_j. \]

**Corollary 3.6** The vanishing of the Futaki invariant for a non-trivial $\mathbb{C}^\times$-action on a ruled surface with admissible Kähler class is equivalent to the quasi-stability of the corresponding parabolic bundle.

\[ \text{with this choice of terminology, stable } \Rightarrow \text{ quasi-stable } \Rightarrow \text{ semi-stable.} \]
3.2 Scalar-Flat Ruled Surfaces with Vector Fields

In the last section, we analyzed the Futaki invariant of compact complex surfaces with periodic holomorphic vector fields which admit admissible Kähler classes. (Recall that \textit{admissible} means that the total scalar curvature of a metric in the class vanishes.) Since the Matsushima-Lichnerowicz Theorem and the Futaki invariant are obstructions to the existence of constant scalar curvature Kähler metrics in the given class, this gives us a rough classification of those surfaces with holomorphic vector fields which might admit scalar flat Kähler metrics. In this section we will review a construction \cite{27} \cite{23} of compact scalar-flat Kähler surfaces, and use it to observe that this “rough” classification is in fact perfectly sharp.

The idea is to reverse the symplectic quotient construction of the last section. Let $\Sigma_g$ be any compact complex curve of genus $\geq 2$, and let $h_\Sigma$ be the unique Hermitian metric on $\Sigma_g$ of constant curvature $-1$. We can then give the 3-manifold $\Sigma_g \times (-1,1)$ a hyperbolic structure by introducing the constant curvature $-1$ metric 

$$ h := \frac{h_\Sigma}{(1-t^2)} + \frac{dt^2}{(1-t^2)^2} \ . $$

Let $q_1, \ldots, q_m \in \Sigma_g \times (-1,1)$ be arbitrary points, and associate to each the Green’s function $G_j$, defined by

$$ \Delta G_j = 2\pi \delta_{q_j}, \quad \lim_{t \to \pm 1} G_j = 0 \ , $$

where $\Delta = - \star d \star d$ is the (positive) Laplace-Beltrami operator of $h$. We define $V := 1 + \sum_{j=1}^m G_j$, so that

$$ \Delta V = 2\pi \sum_{j=1}^m \delta_{q_j}, \quad \lim_{t \to \pm 1} V = 1 \ . $$

$V$ extends smoothly to $(\Sigma_g \times [-1,1]) - \{q_1, \ldots, q_m\}$ and satisfies $V \geq 1$.

On $[\Sigma_g \times (-1,1)] - \{q_1, \ldots, q_m\}$, the 2-form

$$ \alpha := \frac{1}{2\pi} \star dV $$

50
is now closed, and its integral on a small sphere around any one of the \( q_j \)'s is \(-1 \). On the other hand, if \( 1 - \epsilon > \max_j t(q_j) \), then one may check \([27]\) that

\[
\int_{\Sigma_g \times \{1 - \epsilon\}} \alpha = - \sum_{j=1}^{m} \left( \frac{1 + t(q_j)}{2} \right),
\]

so that, setting \( w_j := \frac{[1 + t(q_j)]}{2} \), we have

\[
\left[ \frac{1}{2\pi} \star dV \right] \in H^2_d ((\Sigma_g \times (-1, 1)) \setminus \{q_1, \ldots, q_m\}, \mathbb{Z}) \iff \sum_{j=1}^{m} w_j \in \mathbb{Z},
\]

since the second homology of \((\Sigma_g \times (-1, 1)) \setminus \{q_1, \ldots, q_m\}\) is generated by the homology classes of \(\Sigma_g \times \{1 - \epsilon\}\) and \(m\) small spheres centered at the punctures \(q_1, \ldots, q_m\). If we assume this condition is met, the Chern-Weil theorem guarantees that we can then find a principal \(S^1\)-bundle

\[
\pi_0 : M_0 \rightarrow (\Sigma_g \times (-1, 1)) \setminus \{q_1, \ldots, q_m\}
\]

with a connection 1-form \(\theta\) for which the curvature is

\[
d\theta = \star dV.
\]

Notice that, even modulo gauge equivalence, the pair \((M_0, \theta)\) is by no means unique, since our base is not simply connected; instead, the group \(H^1(\Sigma_g, S^1)\) of flat circle bundles on \(\Sigma_g\) acts freely and transitively on the orbits by tensor product. Given a choice of \((M_0, \theta)\) we then equip \(M_0\) with the Riemannian metric

\[
g := (1 - t^2)[V h + V^{-1} \theta^2].
\]

If we identify the universal cover of \(\Sigma_g\) with the upper half-plane \(y = \Re z > 0\) in \(\mathbb{C}\), the metric can be more explicitly written in the form

\[
g = (1 - t^2) \left[ V \frac{dx^2 + dy^2}{y^2(1 - t^2)} + V \frac{dt^2}{(1 - t^2)^2} + V^{-1} \theta^2 \right]
= vw \ (dx^2 + dy^2) + w \ dt^2 + w^{-1} \theta^2,
\]

where

\[
w = \frac{V}{1 - t^2}
\]
and
\[ v = \frac{1 - t^2}{y^2}. \]
Since the equation \( d\theta = *dV \) can now be rewritten as
\[ d\theta = w_x \, dy \wedge dt + w_y \, dt \wedge dx + (vw)_t \, dx \wedge dy, \]
our calculations (3.2) and (3.3) show that \( g \) is Kähler with respect to the integrable complex structure
\[ dx \mapsto dy, \quad dt \mapsto w^{-1} \theta. \]
Moreover, since
\[ (\log v)_{xx} + (\log v)_{yy} + v_{tt} = 0, \]
we conclude from equation (3.4) that \( g \) is scalar-flat.

We can now compactify \( M_0 \) by adding two copies of \( \Sigma_g \), corresponding to \( t = \pm 1 \), and \( m \) isolated points, corresponding to \( q_1, \ldots, q_m \). This compactification \( M \) can then be made into a smooth manifold in such a way that the metric \( g \) and the complex structure \( J \) extend to \( M \), giving us a compact scalar-flat Kähler surface \((M, g)\). The bundle projection \( \pi_0 \) now extends to a smooth map
\[ \pi : M \to \Sigma_g \times [-1, 1], \]
and the original \( S^1 \)-action extends to an action on \( M \) for which \( \pi \) is projection to the orbit space; the points added to \( M_0 \) in order to obtain \( M \) are precisely the fixed points of the action. The tautological projection \( \text{pr}_1 \pi : M \to \Sigma_g \) induced by \( \pi \) and the first-factor projection \( \text{pr}_1 : \Sigma_g \times [-1, 1] \to \Sigma_g \) is now holomorphic, with rational curves as fibers. To get a minimal model for \( M \), we can proceed by observing that, for any “puncture point” \( q_j \), the inverse image \( \pi^{-1}(\{\text{pr}_1(q_j)\} \times [-1, t(q_j)]) \) of the vertical line segment joining the lower boundary of \( \Sigma_g \times [-1, 1] \) to \( q_j \) is a rational curve in \( M \), and, provided the segment does not pass through any other puncture point, this rational curve is smooth, with self-intersection \(-1\). (In the non-generic situation in which several of the puncture points project to the same point of \( \Sigma_g \), the line segment between any two such consecutive points similarly corresponds to a smooth rational curve in \( M \) of self-intersection \(-2\).) By blowing down all such \((-1)\)-curves (and iteratively blowing down the \((-1)\)-curves that then arise
from $(-2)$-curves in the non-generic case) we eventually arrive at a minimal model $\mathbf{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma_g$ with holomorphic vector field, where all the blow-ups occur at the zero section of $\mathcal{L}$. The proper transform $C_0$ of the zero section now corresponds to $t = -1$, whereas the infinity section $C_\infty$ corresponds to $t = +1$. Meanwhile, the line bundle $\mathcal{L}^*$ is exactly the holomorphic line bundle associated to the $U(1)$-connection obtained by restricting $(M_0, \theta)$ to $\Sigma_g \times \{1 - \epsilon\}$, so that equation (3.5) yields

$$\deg(\mathcal{L}) = \sum_{j=1}^{m} w_j. \tag{3.7}$$

But a different choice of $M_0$ would change this $U(1)$-connection by twisting it by an arbitrary flat $U(1)$-connection; since $\text{Pic}_0(\Sigma_g) = H^1(\Sigma_g, \mathcal{O})/H^1(\Sigma_g, \mathbb{Z})$ is canonically identified with $H^1(\Sigma_g, S^1) = H^1(\Sigma_g, \mathbb{R})/H^1(\Sigma_g, \mathbb{Z})$ by the Hodge decomposition, this means that the holomorphic line-bundles which arise for a fixed configuration $q_1, \ldots, q_m$ fill out the entire connected component of $\text{Pic}(\Sigma_g)$ specified by the degree formula (3.7). And since the fiber-wise Kähler form is just

$$\omega|_{\text{fiber}} = dt \wedge \theta,$$

the area of the holomorphic curve $E_j = \pi^{-1}(\{\text{pr}_1(q_j)\} \times [-1, t(q_j)])$ is just

$$\int_{E_j} \omega = 2\pi(t(q_j) - (-1)) = 4\pi w_j.$$

Since, by the same reasoning, the typical fiber $F$ of $M \to \Sigma_g$ has area $4\pi$, the numbers $w_j$ are precisely the weights we associated with the exceptional divisors in §3.1, and equation (3.7) is therefore, by Proposition 3.3, just the assertion that the Futaki invariant vanishes—as of course it must, since our Kähler manifold has constant scalar curvature zero! Since we are free to choose the numbers $w_j$, subject only to the constraint (3.6), and since, by multiplying $g$ by an arbitrary constant, we can make the typical fiber $F$ have any area we choose, the above explicit construction produces a scalar-flat Kähler metric in any Kähler class on $M$ such that both $c_1 \cdot [\omega]$ and the Futaki invariant are zero:

---

5 This is again a manifestation of the “Archimedes Principle” for symplectic torus actions.
Theorem 3.7 Let $M$ be a compact complex surface with $a(M) \neq 0$. Then a Kähler class $[\omega] \in H^{1,1}(M)$ contains a scalar-flat Kähler metric iff the total scalar curvature, the Matsushima-Lichnerowicz obstruction, and the Futaki invariant all vanish. When such a metric exists, it is unique modulo biholomorphisms of $M$.

Proof. For the existence part, it remains only to observe that the only non-ruled cases are tori and hyperelliptic surfaces, and these admit flat metrics in every Kähler class. For the uniqueness result, which we shall never use in this article, we refer the reader to [29], Theorem 3.

The following simple application will prove to be particularly useful:

Corollary 3.8 Let the product surface $\Sigma_g \times \mathbb{CP}_1$ be blown up at any $k > 0$ points along $\Sigma_g \times \{[1 : 0]\}$ and any $\ell > 0$ points along $\Sigma_g \times \{[0 : 1]\}$. (Some or all of the given points are allowed to coincide, but in this case the iterated blow-ups are required to occur along proper transforms of the $\Sigma_g$ or $\mathbb{CP}_1$ factors.) The resulting surface then admits scalar-flat Kähler metrics.

Proof. If the set of blown-up points is

$$\{(p_1, [1 : 0]), \ldots, (p_k, [1 : 0]), (q_1, [0 : 1]), \ldots, (q_\ell, [0 : 1])\},$$

then, letting $L \to \Sigma_g$ denote the divisor line bundle of $\{q_1, \ldots, q_\ell\}$, the surface in question can also be described as the blow-up of $\mathbb{P}(\mathcal{O} \oplus L)$ at the points $\{p_1, \ldots, p_k, q_1, \ldots, q_\ell\}$ on the zero section. Since $0 < \deg(L) = \ell < m = \ell + k$, the result follows from Corollary 3.4 and Theorem 3.7.

In light of [3], the following restatement of the above theorem seems particularly tantalizing:

Corollary 3.9 An admissible Kähler class on a (blown-up) ruled surface with periodic holomorphic vector field contains a scalar-flat Kähler metric iff the corresponding parabolic bundle is quasi-stable.
3.3 Scalar-Flat Metrics on Generic Ruled Surfaces

The key technical result of this paper is as follows:

**Theorem 3.10** Let $\varpi : \mathcal{M} \rightarrow \mathcal{U}$ be a family of non-minimal ruled surfaces of genus $\geq 2$. Suppose that, for some $o \in \mathcal{U}$, the corresponding fiber $M = M_o := \varpi^{-1}(o)$ admits a scalar-flat Kähler metric. Then there is a neighborhood $\tilde{\mathcal{U}} \subset \mathcal{U}$ of $o$ such that $M_t := \varpi^{-1}(t)$ admits a scalar-flat Kähler metric for all $t \in \tilde{\mathcal{U}}$. Moreover, relative to local trivializations of the real-analytic fiber-bundle underlying $\varpi$, these metrics can be chosen so as to depend real-analytically on $t$.

**Proof.** Let $Z$ be the twistor space of $M$, $D = M \coprod \bar{M}$ its standard divisor, and $\sigma$ its real structure. Using Theorem 2.7 and Corollary 3.5, we have $H^2(Z, \Theta_Z \otimes \mathcal{I}_D) = H^2(Z, \Theta_{Z,D}) = 0$. The first statement can be reinterpreted on $Z/\sigma$ as saying that $H^2(Z/\sigma, \mathcal{R}\Theta_Z \otimes \mathcal{I}_D) = 0$, so that the long exact sequence induced by the short exact sequence

$$0 \rightarrow \mathcal{R}[\Theta_P \otimes \mathcal{I}_D] \rightarrow \mathcal{R}\Theta_P \rightarrow \Theta_M \rightarrow 0$$

on $P = Z/\sigma$ predicts that the natural restriction map

$$H^1(Z/\sigma, \mathcal{R}\Theta) \rightarrow H^1(M, \Theta_M)$$

is surjective. Applying this morphism to restrict the Kodaira-Spencer map to $M$, we see that the versal family for $(Z/\sigma, M)$ given by Theorem 1.16 induces a complete deformation of $M$—i.e. a deformation of $M$ which contains a versal deformation as a subspace. Thus any small deformation of $M$ can be extended as a deformation of $(Z, D, \sigma)$. Applying Theorem 1.15 then finishes the proof.

**Remark.** The analogous statement fails for **minimal** ruled surfaces.

We now prove our main result:

**Theorem 3.11** Let $M$ be any ruled surface of genus $\geq 2$. If $M$ is blown up at sufficiently many points, the resulting complex surface $\tilde{M}$ admits scalar-flat Kähler metrics.
Proof. Since any ruled surface \( \Sigma \) is bimeromorphic to a product surface, some blow-up \( \tilde{M} \) of the given surface \( M \) is biregularly equivalent to an iterated blow-up of \( \Sigma_g \times \mathbb{CP}_1 \), \( g \geq 2 \), at \( r_1 \ldots r_m \in \Sigma_g \times \mathbb{CP}_1 \), where repetition of a point indicates that we are also given certain directional information at the multiple point. By blowing up extra points if necessary, we may assume that the projection of \( \{r_1 \ldots r_m\} \) to \( \mathbb{CP}_1 \) consists of more than one point.

By changing to another homogeneous coordinate system \([\zeta_1 : \zeta_2]\) on \( \mathbb{CP}_1 \), we may also assume that \([1 : 0]\) is in the image of \( \{r_1 \ldots r_m\} \). For \( t \in \mathbb{C} \), and if \( r_j \) projects to \([1 : 0]\), set \( r_j(t) := r_j \); otherwise let \( r_j(t) := \mu_t(r_j) \), where

\[
\mu_t : \Sigma_g \times (\mathbb{CP}_1 - \{[1 : 0]\}) \to \Sigma_g \times \mathbb{CP}_1 \quad (p, [\zeta_1 : \zeta_2]) \mapsto (p, [t\zeta_1 : \zeta_2]).
\]

Define a family \( \varpi : \mathcal{M} \to \mathbb{C} \) of complex surfaces by blowing up \( \Sigma_g \times \mathbb{CP}_1 \times \mathbb{C} \) along the graphs of \( t \mapsto r_j(t) \). For \( t \neq 0 \), the manifold \( M_t \) is then biholomorphic to \( \tilde{M} \). However, for \( t = 0 \), the fiber is a ruled surface with a holomorphic vector field, namely the blow-up of \( \Sigma_g \times \mathbb{CP}_1 \) at non-empty collections of points along \( \Sigma_g \times \{[0 : 1]\} \) and \( \Sigma_g \times \{[1 : 0]\} \). (When several of the blown-up points project to the same point of \( \Sigma_g \), one should think of the blow-ups as happening successively rather than simultaneously; for \( t = 0 \), we are, at each stage, blowing up the previous surface at a zero of the vector field \( \Xi = \zeta_1 \partial / \partial \zeta_1 \), and \( \Xi \) therefore lifts to the blow-up.) Corollary 3.8 then asserts that \( M_0 \) admits scalar-flat Kähler metrics. The result therefore follows immediately from Theorem 3.10.

This can be repackaged as follows:

**Corollary 3.12 (Main Theorem)** Let \( M \) be a compact complex surface which admits a Kähler metric whose scalar curvature has integral zero. Suppose \( \pi_1(M) \) does not contain an Abelian subgroup of finite index. Then if \( M \) is blown up at sufficiently many points, the resulting surface \( \tilde{M} \) admits scalar-flat Kähler metrics.

**Proof.** By Theorem 1.3, a complex surface with admissible Kähler class is either ruled or covered by a torus or K3. Thus the fundamental group hypothesis forces \( M \) to be a ruled surface of genus \( \geq 2 \). The statement therefore follows from Theorem 3.11. \( \square \)
We now conclude this article with a pair of conjectures intended to remind the reader that the study of scalar-flat Kähler surfaces is still in its infancy. First, one would like to understand what happens for ruled surfaces of genus 0 and 1. It is our hope and expectation that the genus hypothesis in Theorem 3.11 is actually superfluous:

**Conjecture 1** The blow-up of any ruled surface at sufficiently many points admits scalar-flat Kähler metrics.

On the other hand, one would really like to understand precisely when an admissible class contains a scalar-flat Kähler metric. In light of Corollary 3.9 and known results on relatively minimal ruled surfaces [9], the following would seem very natural:

**Conjecture 2** An admissible Kähler class on a (blown-up) ruled surface of genus $\geq 2$ contains a scalar-flat Kähler metric iff the corresponding parabolic bundle is quasi-stable.

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