Gaussian Limit for Determinantal Random Point Fields *

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Abstract

We prove that, under fairly general conditions, a properly rescaled determinantal random point field converges to a generalized Gaussian random process.

1 Introduction and Formulation of Results

Let $E$ be a locally compact Hausdorff space satisfying the second axiom of countability, $B$ – $\sigma$-algebra of Borel subsets and $\mu$ – a $\sigma$-finite measure on $(E,B)$, such that $\mu(K) < \infty$ for any compact $K \subset E$. We denote by $X$ the space of locally finite configurations of particles in $E$: $X = \{\xi = (x_i)_{i=-\infty}^{\infty} : x_i \in E \ \forall i\}$, and for any compact $K \subset E$ $\#_K(\xi) := \#(x_i : x_i \in K) < +\infty$. 

A $\sigma$-algebra $F$ of measurable subsets of $X$ is generated by the cylinder sets $C_n^B = \{\xi \in X : \#_B(\xi) = n\}$, where $B$ is a Borel set with a compact closure and $n \in \mathbb{Z}_1 = \{0,1,2,\ldots\}$. Let $P$ be a probability measure on $(X,F)$. A triple $(X,F,P)$ is called a random point field (process) (see [DVJ], [Le1-Le3]). In this paper we will be interested in a special class of random point

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fields called determinantal random point fields. It should be noted that most, if not all the important examples of determinantal point fields arise when $E = \bigsqcup_{i=1}^{k} E_i$ (here we use the notation $\bigsqcup$ for the disjoint union), $E_i \cong \mathbb{R}^d$ or $\mathbb{Z}^d$ and $\mu$ is either the Lebesgue or the counting measure. We will, however, develop our results in the general setting (our arguments will not require significant changes).

Let $dx_i, \ i = 1, \ldots, n$ be disjoint infinitesimally small subsets around the $x_i$'s. Suppose that a probability to find a particle in each $dx_i$ (with no restrictions outside of $\bigsqcup_{i=1}^{n} dx_i$) is proportional to $\prod_{i=1}^{n} \mu(dx_i)$, i.e.

$$P(\#(dx_i) = 1, \ i = 1, \ldots, n) = \rho_n(x_1, \ldots, x_n)\mu(dx_1)\ldots\mu(dx_n) \tag{1}$$

The function $\rho_n(x_1, \ldots, x_n)$ is then called the $n$-point correlation function. The equivalent definition is given by the equalities

$$E \prod_{i=1}^{m} \frac{(\#B_i)!}{(\#B_i - n_i)!} = \int_{B_{i_1} \times \ldots \times B_{i_m}} \rho_n(x_1, \ldots, x_n)\mu(dx_1)\ldots\mu(dx_n)$$

where $B_1, \ldots, B_m$ are disjoint Borel sets with compact closures, $m \geq 1, n_i \geq 1, \ i = 1, \ldots, m, n_1 + \cdots + n_m = n$.

A random point field is called determinantal if

$$\rho_n(x_1, \ldots, x_n) = \det(K(x_i, x_j))_{1 \leq i, j \leq n}; \tag{2}$$

where $K(x, y)$ is a kernel of an integral operator $K : L^2(E, d\mu) \to L^2(E, d\mu)$ and $K(x, y)$ satisfies some natural regularity conditions discussed below. Such a kernel $K(x, y)$ is called a correlation kernel.

It follows from 2) and the non-negativity of the $n$-point correlation functions that $K$ must have non-negative minors, and in particular if $K$ is Hermitian it must be a non-negative operator. In this paper we shall always restrict ourselves to the Hermitian case.

Determinantal (also known as fermion) random point fields were introduced by Macchi in the early seventies (see [Ma1], [Ma2], [DVJ]). A recent survey of the subject with applications to random matrix theory, statistical mechanics, quantum mechanics, probability theory and representation theory is given in [So1]. Diaconis and Evans in [DE1] introduced a generalization of determinantal random point processes, called immanantal point processes.

Let $K$ be a Hermitian, locally trace class, integral operator on $L^2(E, d\mu)$. Suppose that we can choose a kernel $K(x, y)$ in such a way that for any Borel
set $B$ with compact closure

$$\text{Tr}(K\mathcal{X}_B) = \int_B K(x, x)dx,$$ (3)

where $\mathcal{X}_B$ denotes the multiplication operator by the indicator of $B$ (≡ projector on the subspace of the functions supported in $B$).

Since it is always true that

$$\text{Tr}(K\mathcal{X}_{B_1} \ldots K\mathcal{X}_{B_n}) = \int_{B_1 \times \ldots \times B_n} K(x_1, x_2)K(x_2, x_3) \ldots K(x_n, x_1)d\mu(x_1) \ldots d\mu(x_n) \quad (3')$$

for $n > 1$ and Borel sets $B_1, \ldots, B_n$ with compact closure, (3) implies that (3') holds for all $n$.

(3) can always be achieved for $E = \mathbb{R}^d$ (see e.g. [So1], Lemmas 1,2). From now on we will assume that both (2) and (3) are satisfied.

The main goal of our paper is to study the behavior of linear statistics

$$S_f(\xi) = \sum_i f(x_i), \; \xi = (x_i),$$

for sufficiently “nice” test functions in a scaling limit. The moments of $S_f$ can be calculated from (2). For instance,

$$E S_f = \int f(x)K(x, x)d\mu(x),$$ (4)

$$\text{Var} S_f = \int f^2(x)K(x, x)d\mu(x) - \int f(x)f(y)|K(x, y)|^2d\mu(x)d\mu(y).$$ (5)

Taking $E = \mathbb{R}^1$ and $K(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$, a so-called sine kernel, we obtain a random point field well known in the theory of random matrices. It can be viewed as a limit $n \to \infty$ of the distribution of the appropriately scaled eigenvalues of $n \times n$ random Hermitian matrices with Gaussian entries (see e.g. [D], chapter 5). It was proven by Spohn in [Sp] (see also [So2]), that if $K$ is the sine kernel and a test function $f$ is sufficiently smooth and fast decaying at infinity, then $\sum_{i=-\infty}^{\infty} f(\xi_i) - L \int_{-\infty}^{\infty} f(x)dx$ converges in distribution to the normal law $N(0, \int_{-\infty}^{\infty} |\hat{f}(k)|^2 \cdot |k|dk)$, where $\hat{f}$ is the Fourier transform of $f$. 

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\( \hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx. \) In other words we can say that the random signed measure
\[
\sum_{i=\infty}^{\infty} \delta(x - \frac{x_i}{L}) - L dx
\]
converges as \( L \to \infty \) to the generalized self-similar Gaussian random process with the spectral density \( |k| \) (see e.g., [Dob1], [Dob2] §3, [S], and, for the introduction to the theory of generalized random processes, [GV]). The fact that the variance of the linear statistics \( \sum_{i=\infty}^{\infty} f(x_i/L) \) does not grow to infinity for Schwartz functions is the manifestation of the strong repulsiveness of the distribution of the eigenvalues of random matrices. Similar results for other ensembles of random matrices have been obtained in [DS], [Jo1], [Jo2], [B], [BW], [So2], [W],[DE2]. The kernels appearing in these ensembles are, in some respect, very much like the sine kernel. In particular, the variance of the number of particles in an interval grows as a logarithm of the mathematical expectation of the number of particles. The following result was established by Costin and Lebowitz for the sine kernel ([CL]): let \( f \) be an indicator of an interval, \( f = \mathcal{X}_I, I = (a,b) \), then
\[
E \sum_{i=\infty}^{\infty} f(x_i/L) = E(\#(x_i : aL < x_i < bL)) = L(b-a),
\]
\[
\text{Var} \left( \sum_{i=\infty}^{\infty} f(x_i/L) \right) = \frac{1}{\pi^2} \log L + O(1),
\]
and
\[
\frac{\#(x_i : aL < x_i < bL) - L(b-a)}{\sqrt{\frac{1}{\pi^2} \log L}}
\]
converges in distribution as \( L \to \infty \) to the normal law \( N(0,1) \). The proof of the Costin-Lebowitz theorem holds, quite remarkably, for arbitrary determinantal random point fields with Hermitian kernel.

**Theorem** ([So3]) Let \((X, \mathcal{F}, P_L), L \geq 0,\) be a family of determinantal random point fields with Hermitian locally trace class kernels \( K_L \) and \( \{I_L\}_{L \geq 0} \) be a family of Borel subsets of \( E \) with compact closure. Then if \( \text{Var}_L \left( \#(x_i : x_i \in I_L) \right) \to \infty, \) the normalized random variable \( \frac{\#(x_i : x_i \in I_L) - E_L \#I_L}{\sqrt{\text{Var}_L \#I_L}} \) converges in distribution to \( N(0,1). \)
Here and below we denote by $E_L$, $\text{Var}_L$ the mathematical expectation and the variance with respect to $P_L$. One can also establish a similar result for the step functions (finite linear combinations of indicators).

**Theorem** Let $(X, \mathcal{F}, P_L)$ be a family of determinantal random point fields with Hermitian locally trace class kernels $K_L$ and $\{I_{L}^{(1)}, \ldots, I_{L}^{(k)}\}_{L \geq 0}$ be a family of Borel subsets of $E$, disjoint for any fixed $L$, with compact closure. Then if for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, the variance of the linear statistics $\sum_{i=-\infty}^{\infty} f_L(x_i)$ with $f_L(x) = \sum_{j=1}^{k} \alpha_j \cdot X_{I_L^{(j)}}(x)$, grows to infinity in such a way that $\text{Var}_L(\#I_L^{(j)}) = O(\text{Var}_L(\sum_{i=-\infty}^{\infty} f_L(x_i)))$ for any $1 \leq j \leq k$, the Central Limit Theorem holds:

$$\frac{\sum_{j=1}^{k} \alpha_j^{(L)} \cdot \#I_L^{(j)} - E_L \left( \sum_{j=1}^{k} \alpha_j \cdot \#I_L^{(j)} \right)}{\sqrt{\text{Var}_L \left( \sum_{j=1}^{k} \alpha_j \cdot \#I_L^{(j)} \right)}} \xrightarrow{w} N(0,1).$$

**Remark 1** We use standard notations $f = O(g)$ and $f = o(g)$ when $\frac{f}{g}$ stays bounded or $\frac{f}{g} \to 0$.

**Remark 2** The last theorem has been explicitly stated in [So3] only in the special case of the Airy and Bessel kernels and the kernels arising in the classical compact groups (see Theorems 1, 2, 4, 6), however the key Lemmas 7 and 8 proven there allow rather straightforward generalization to the case of an arbitrary Hermitian kernel. A result close to our Theorem 6 from [So3] was also established by K. Wieand ([W]).

We recall that a Hermitian kernel $K(x, y)$ defines a determinantal random point field if and only if the integral operator $K$ is non-negative and bounded from above by the identity,

$$0 \leq K \leq \text{Id} \quad \text{(6)}$$

([So1], Theorem 3). For the translation-invariant kernels $K(x - y)$ and $E = \mathbb{R}^d$ or $\mathbb{Z}^d$ this is equivalent to $0 \leq \hat{K}(t) \leq 1$, where

$$K(x) = \int e^{2\pi i(x \cdot t)} \hat{K}(t) dt \quad \text{(7)}$$

The sine kernel $K(x - y) = \frac{\sin \pi(x - y)}{\pi(x - y)}$ corresponds to $\hat{K}(t) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(t)$, the indicator of $[-\frac{1}{2}, \frac{1}{2}]$. It might be worth noting and actually is not very
difficult to see, that the logarithmic rate of the growth of \( \text{Var} \left( \#(x_i : |x_i| \leq L) \right) \) for the sine kernel is the slowest among all translation-invariant kernels corresponding to projectors, \( \hat{K} = \mathcal{X}_B \), for which \( \inf(B) \) and \( \sup(B) \) are the density points of \( B \). For the generic translation-invariant kernel \( K(x - y) \) (\( \hat{K} \) is not an indicator) \( \text{Var} \left( \#(x_i : |x_i| \leq L) \right) \) is proportional to \( \text{Vol} \left( x_i : |x_i| \leq L \right) \) \( \sim E(\#(x_i : |x_i| \leq L)) \) ([So1], section 3).

In our main result we prove CLT for the linear statistics when the variance grows faster than some arbitrary small, but fixed, power of the mathematical expectation.

**Theorem 1** Let \( (X, \mathcal{F}, P_L) \), \( L \geq 0 \) be a family of determinantal random point fields with Hermitian correlation kernels \( K_L \). Suppose that \( f_L \), \( L \geq 0 \) are bounded measurable functions with precompact support (i.e. \( \text{supp}(f_L) \) has a compact closure for any \( L \geq 0 \)), such that

\[
\text{Var}_L S_{f_L} \rightarrow \infty \text{ as } L \rightarrow \infty \tag{8}
\]

and
\[
\sup |f_L(x)| = o(\text{Var}_L)^\epsilon, \quad E_L S_{f_L} = O\left( (\text{Var}_L S_{f_L})^{\delta} \right), \tag{9}
\]

for any \( \epsilon > 0 \) and some \( \delta > 0 \). Then the normalized linear statistics
\[
\frac{S_{f_L} - E_L S_{f_L}}{\sqrt{\text{Var}_L S_{f_L}}}
\]
converges in distribution to the standard normal law \( N(0,1) \).

As a very important special case of Theorem 1 one can consider \( f_L(x) := f(T_L x) \), where \( \{T_L\}, L \in \mathbb{R}^1_+ \), is a one-parameter family of measurable transformations \( T_L : E \rightarrow E \) such that \( T_L^{-1} D \) has compact closure for any compact \( D \). If for a sufficiently rich class of test functions \( f \) (e.g., continuous functions with compact support) (8),(9) are satisfied, and the rate of the growth of \( \text{Var}_L (S_{f_L}) \) is the same,

\[
\text{Var}_L (S_{f_L}) = B(f) \cdot V_L \cdot (1 + o(1)),
\]

where \( B(f) \) is some functional on a space of test functions, Theorem 1 implies that the random signed measure
\[
V_L^{-\frac{1}{2}} \left( \sum_{i=-\infty}^{\infty} \delta(x - T_L x_i) - T_L(K_L(x,x)d\mu(x)) \right)
\]
converges as $L \to \infty$ to the generalized Gaussian process with the correlation functional $B(f, f) = B(f)$ (we denote by $T_L(K_L(x, x)d\mu(x))$ the image of the measure $K_L(x, x)d\mu(x)$ under $T_L$).

Let us consider a Euclidean one-particle space $E = \mathbb{R}^d$, a one-parameter family of dilations $T_L : \mathbb{R}^d \to \mathbb{R}^d$, $T_Lx = x/L$, and a correlation kernel

$$K_L(x, y) = A_L(x - y) + R_L(x, y), \quad \text{(10)}$$

where

$$|R_L(x, y)| \leq Q(|x_{abs} + y_{abs}|), \quad \text{(11)}$$

$x_{abs} = (|x_1|, \ldots, |x_d|)$, $Q \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. It follows from (6), (10) and (11) that $0 \leq A_L \leq \text{Id}$, which implies $0 \leq \hat{A}_L(k) \leq 1$, $0 \leq \int_{\mathbb{R}^d} \hat{A}_L(k) - (\hat{A}_L(k))^2dk = A_L(0) - \int_{\mathbb{R}^d} |A_L(x)|^2dx =: \sigma^2_L$, and $\sigma_L = 0$ if and only if $\hat{A}_L$ is an indicator.

**Theorem 2** Let the kernel $K_L$ satisfy (10), (11) and there exist constants $\text{const}$, $\sigma > 0$ and $\kappa_L \to \infty$ as $L \to \infty$ such that

$$\sigma_L \to \sigma \quad \text{as } L \to \infty,$$

$$|A_L(0)| < \text{const},$$

and

$$\int_{|x| > L/\kappa_L} |A_L(x)|^2dx \to 0.$$

Then for any real-valued function $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ the normalized linear statistics

$$\frac{1}{L^{\frac{d}{2}}\sigma} \left( \sum_{i=\infty}^\infty f \left( \frac{x_i}{L} \right) - A_L(0) \cdot L \int_{\mathbb{R}^d} f(x)dx \right)$$

converges in distribution to the Gaussian random variable $N \left( 0, \int_{\mathbb{R}^d} (f(x))^2dx \right)$.

**Remark 3** Theorem 2 says that under the stated conditions the random signed measure

$$\frac{1}{L^{\frac{d}{2}}\sigma} \left( \sum_{i=\infty}^\infty \delta \left( x - \frac{x_i}{L} \right) - A_L(0) \cdot Ldx \right)$$
converges to the white noise as $L \to \infty$ (for the definition of the white noise see e.g. [H]). Similar results hold in the discrete case.

Let us now restrict our attention to the translation-invariant kernels $K(x, y) = A(x - y)$. We will use the notation

$$m(\lambda) := \int \hat{A}(k) - \hat{A}(k)\hat{A}(k - \lambda)dk.$$ 

Observe that $\sigma^2 = m(0)$ and

$$\text{Var} \left( \sum_{i=-\infty}^{\infty} f(x_i) \right) = \int |\hat{f}(\lambda)|^2 m(\lambda) d\lambda. \quad (12)$$

In particular

$$\text{Var} \left( \#_{[-L,L]^d} \right) = \text{Vol} \left( [-L, L]^d \right) \cdot (m(0) + o(1)). \quad (13)$$

It follows from (12) that the rate of the growth of the variance of $S_{fL}$ depends on the asymptotics of $m(\lambda)$ near the origin. In the next theorem we consider the degenerate case $\sigma^2 = 0$ in one dimension.

**Theorem 3** Let $K(x, y) = A(x - y)$ be a translation-invariant kernel in $\mathbb{R}^1$ and $m(\lambda) = |\lambda|^\alpha \varphi(\lambda)$, where $\varphi(\lambda)$ is a slowly varying function at the origin and $0 < \alpha < 1$. Then for any Schwartz function $f : ES_{fL} = LA(0) \int f(x)dx$, $\text{Var} S_{fL} = L^{1-\alpha} \varphi(L^{-1}) \int |\hat{f}(k)|^2 |k|^\alpha dk(1 + o(1))$, and

$$\frac{S_{fL} - ES_{fL}}{(L^{1-\alpha} \varphi(L^{-1}))^{1/2}}$$

converges in distribution to $N(0, \int |\hat{f}(k)|^2 |k|^\alpha dk)$.

**Remark 5** We recall that $\varphi(\lambda) \geq 0$ is slowly varying at the origin if

$$\lim_{\lambda \to 0} \frac{\varphi(\lambda)}{\varphi(\lambda)^a} = 1$$

for any $a \neq 0$ (see [Se]).

**Remark 6** The result of Theorem 3 can be interpreted as the convergence in distribution of the random signed measure

$$\left( \left( L^{1-\alpha} \varphi(L^{-1}) \right)^{1/2} \left( \sum \delta \left( x - \frac{x_i}{L} \right) - A(0)Ldx \right) \right)$$

to the self-similar (also called automodel in the Russian literature) generalized Gaussian random process with the spectral density $|k|^\alpha$, $0 < \alpha < 1$. 

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Self-similarity means that the distribution of the process is invariant under the action of the renorm-group \( \xi(x) \rightarrow \xi(ax) a^\gamma, \quad \gamma = \frac{1+\alpha}{2} \). The self-similar generalized Gaussian random process corresponding to \( \alpha = 0 \) is exactly the white noise (see Remark 3 above). It was proven by Dobrushin that the only self-similar random processes in \( \mathbb{R}^1 \) are the ones with the spectral density \(|k|^\alpha, \quad 0 \leq \alpha \leq 1\). A self-similar generalized random process with the spectral density \(|k|\) appeared in the Spohn’s results ([Sp]) discussed above after the formulas (4), (5) (see also [Jo1], [B], [So2]). For additional information on self-similar random processes we refer the reader to [Dob1], [Dob2],[S].

**Example** Let \( \hat{A} \) be the indicator of \( \sqcup_{n \geq 1} [n, n + n^{-\beta}] \), \( \beta > 1 \). Then \( m(\lambda) = \text{const} \cdot |\lambda|^{1-\frac{1}{\beta}} (1 + o(1)) \). On the other hand if the length \( l_n \) of the \( n \)-th interval \([n, n + l_n] \) decays sufficiently fast, say \( 0 \leq l_{n+1} \leq l_n^{1+\epsilon}, \quad \epsilon > 0 \) than \( m(\lambda) \) is not regularly varying at the origin.

Finally we consider the case when \( \hat{A} \) is the indicator of a union of \( 1 \leq \ell < \infty \) disjoint intervals. It is straightforward to see that then \( m(\lambda) = \ell |\lambda| \) near the origin.

**Theorem 4** Let \( \hat{A} \) be the indicator of \( I = \sqcup_{i=1}^{\ell} [a_i, b_i], \quad a_1 < b_1 < a_2 < b_2 < \cdots < a_\ell < b_\ell \). Then for any Schwartz function \( f \) \( \sum_{i=-\infty}^{\infty} f \left( \frac{x_i}{\ell} \right) - A(0) \cdot L \int_{-\infty}^{\infty} f(x)dx \) converges in distribution to \( N(0, \ell \int_{-\infty}^{\infty} |\hat{f}(k)|^2 |k| dk) \).

The proofs of Theorems 1–3 will be given in the next three sections. The proof of Theorem 4 is the same, modulo trivial alterations, as the one given for the sine kernel in [So2].

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## 2 Proof of Theorem 1

We are going to prove Theorem 1 by the method of moments. Let us denote by \( C_n(S_f) \) the \( n \)-th cumulant of \( S_f \). We remind the reader that for a random variable \( \eta \) with all finite moments , the cumulants \( C_n(\eta), \quad n = 1, 2, \ldots \) are defined through the Taylor coefficients of the logarithm of the characteristic function:

\[
\log E(\exp(it\eta)) = \sum_{n=1}^{\infty} C_n(\eta)(it)^n / n!.
\]
We show that the $n^{th}$ cumulant of the normalized linear statistics \( \frac{S_{fL} - ES_{fL}}{(Var S_{fL})^{1/2}} \) converges to zero as $L \to \infty$ for sufficiently large $n$ ($n > \max(2\delta, 2)$). The Lemma 3 from the Appendix then asserts that all cumulants of \( \frac{S_{fL} - ES_{fL}}{(Var S_{fL})^{1/2}} \) converge to the cumulants of the standard normal distribution, which implies the weak convergence.

We recall the lemma established in [So2] (see formula (2.7)).

**Lemma 1**

\[
C_n(S_f) = \sum_{m=1}^{n} \sum_{\substack{(n_1, \ldots, n_m) \in \mathbb{N}^m \colon n_1 + \cdots + n_m = n, \ n_1 \geq 1, \ i=1, \ldots, m}} \frac{(-1)^{m-1} n!}{m!} \int f^{n_1}(x_1) K(x_1, x_2) f^{n_2}(x_2) K(x_2, x_3) \cdots f^{n_m}(x_m) K(x_m, x_1) d\mu(x_1) \cdots d\mu(x_m) \tag{14}
\]

Using Lemma 1 we will be able to estimate the cumulants of $S_{fL}$. We claim the following result to be true.

**Lemma 2** Under the assumptions of Theorem 1

\[
C_n(S_{fL}) = O((Var S_{fL})^{\delta+\epsilon}), \ n \geq 1, \tag{15}
\]

where $\epsilon$ is arbitrarily small.

**Proof of Lemma 2** It follows from (14) that $C_n(S_{fL})$ is a linear combination of

\[
\int f^{n_1}_L(x_1) K_L(x_1, x_2) f^{n_2}_L(x_2) K_L(x_2, x_3) \cdots f^{n_m}_L(x_m) K_L(x_m, x_1) d\mu(x_1) \cdots d\mu(x_m) = \text{Tr}(f^{n_1}_L K_L f^{n_2}_L K_L \cdots f^{n_m}_L K_L),
\]

where $n_i \geq 1, \ i = 1, \ldots, m, \ m \geq 1$.

We claim that each term is $O((Var S_{fL})^{\delta+\epsilon})$. Indeed, if $m = 1$, then

\[
|\text{Tr} f^{n}_L K_L| = \int |f^{n}_L(x) K_L(x, x)| d\mu(x) \leq \|f_L\|_{n-1}^{\delta+\epsilon} \int |f_L(x)| K_L(x, x) d\mu(x) = \|f_L\|^{\delta+\epsilon} \left( \text{Var} f_{||L} \right) = O((Var S_{fL})^{\delta+\epsilon}).
\]

If $m > 1$, represent $\text{Tr}(f^{n_1}_L K_L f^{n_2}_L K_L \cdots f^{n_m}_L K_L)$ as a linear combination of $\text{Tr}(f^{n_1}_{\pm,L} K_L f^{n_2}_{\pm,L} K_L \cdots f^{n_m}_{\pm,L} K_L)$, where we use the notations $f_+ = \max(f, 0)$, $f_- = \max(-f, 0)$. Let us fix the choice of $\pm$ in each of the factors. Using the
cyclicity of the trace and the inequality $|\text{Tr}(AB)| \leq (\text{Tr}(AA^*)^{\frac{1}{2}}(\text{Tr}(BB^*))^{\frac{1}{2}}$ for the Hilbert-Schmidt operators ([RS], section VI.6), we obtain

$$\begin{align*}
\text{Tr}(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L\ldots f_{\pm,L}^{n_m}K_L) &= |\text{Tr}(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L\ldots f_{\pm,L}^{n_m}K_Lf_{\pm,L}^{n_1})| \\
&\leq |\text{Tr}((f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L\ldots f_{\pm,L}^{n_m}K_Lf_{\pm,L})^*)|^{\frac{1}{2}}|\text{Tr}((f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L\ldots f_{\pm,L}^{n_m}K_Lf_{\pm,L})^*)|^{\frac{1}{2}}.
\end{align*}$$

(16)

The first factor at the r.h.s. of (16) is equal (again by the cyclicity of the trace) to $[\text{Tr}(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L)]^{\frac{1}{2}}$ (in particular we note that $\text{Tr}(g_1Kg_2K) \geq 0$ for non-negative $g_1, g_2$).

Since

$$\begin{align*}
\text{Tr}((f_{\pm,L}^{n_1} + f_{\pm,L}^{n_2})^2K) - \text{Tr}((f_{\pm,L}^{n_1} + f_{\pm,L}^{n_2})K(f_{\pm,L}^{n_1} + f_{\pm,L}^{n_2})K) = \\
\text{Var}(S_{f_{\pm,L}^{n_1} + f_{\pm,L}^{n_2}}) \geq 0,
\end{align*}$$

we have

$$\begin{align*}
0 \leq \text{Tr}(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L) &\leq \frac{1}{2}\left(\text{Tr}((f_{\pm,L}^{n_1} + f_{\pm,L}^{n_2})^2K) - \text{Tr}(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L) - \\
\text{Tr}(f_{\pm,L}^{n_2}K_Lf_{\pm,L}^{n_1}K_L)\right) &\leq \frac{1}{2}\text{Tr}((f_{\pm,L}^{n_1} + f_{\pm,L}^{n_2})^2K) = O(\text{Tr}(|f_L|K_L))o((\text{Var}_LS_{f_L})^e) \\
&= O((\text{Var}_LS_{f_L})^{\delta + e}).
\end{align*}$$

(17)

As for the second term in (16), one can rewrite $\text{Tr}\left((f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L\ldots f_{\pm,L}^{n_m}K_Lf_{\pm,L}^{n_2}K_Lf_{\pm,L}^{n_1})^*\right)$ as

$$\begin{align*}
\text{Tr}\left((f_{\pm,L}^{n_2}K_Lf_{\pm,L}^{n_1}K_L\ldots f_{\pm,L}^{n_m}K_Lf_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_m})\right) &= \text{Tr}(CDD^*),
\end{align*}$$

(18)

where $C = f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_Lf_{\pm,L}^{n_3}K_L\ldots f_{\pm,L}^{n_m}K_Lf_{\pm,L}^{n_1}$ and $D = f_{\pm,L}^{n_3}K_Lf_{\pm,L}^{n_4}K_L\ldots f_{\pm,L}^{n_m}K_Lf_{\pm,L}^{n_1}$. Note that $C \geq 0$ and $\text{Tr}(C) = \text{Tr}(f_{\pm,L}^{n_3}K_Lf_{\pm,L}^{n_4}K_L) = O((\text{Var}_LS_{f_L})^{\delta + e})$ by arguments similar to (17). Using $|\text{Tr}(CDD^*)| \leq \text{Tr}(C) \cdot \|DD^*\| = \text{Tr}(C) \cdot \|D\|^2$
([RS], Section VI.6) and \( \|D\| \leq \|K\|^m \cdot \|f_L\|_\infty^n \), where \( n = (\sum_{i=1}^m n_1) - n_2 \), we conclude that (18) is \( O((\text{Var}_L S_{f_L})^{\delta+\epsilon}) \). Together with (16) and (17) this concludes the proof of the lemma.

Let us now apply Lemma 2 to estimate the cumulants of the normalized linear statistics. We have

\[
C_1 \left( \frac{S_{f_L} - ES_{f_L}}{\sqrt{\text{Var}_L S_{f_L}}} \right) = 0,
\]

\[
C_2 \left( \frac{S_{f_L} - ES_{f_L}}{\sqrt{\text{Var}_L S_{f_L}}} \right) = 1,
\]

and, for \( n > 2 \),

\[
C_n \left( \frac{S_{f_L} - ES_{f_L}}{\sqrt{\text{Var}_L S_{f_L}}} \right) = \frac{C_n(S_{f_L})}{(\text{Var}_L S_{f_L})^{\frac{n}{2}}} = O \left( E \left( \frac{S_{f_L}}{(\text{Var}_L S_{f_L})^\frac{n}{2}} \right) \right) \tag{19}
\]

It follows from the Lemma 2 and (19) that \( C_n \left( \frac{S_{f_L} - ES_{f_L}}{\sqrt{\text{Var}_L S_{f_L}}} \right) \) goes to zero if \( n > 2\delta \).

Lemma 3 from the Appendix then implies that all cumulants of the normalized linear statistics converge to the cumulants of the standard normal random variable, and weak convergence of the distributions follows.

Theorem 1 is proven. \( \square \)

3 Proof of Theorem 2

Let \((E, d\mu)\) be \((\mathbb{R}^d, dx)\) and \(T_L x = \frac{x}{L} \). Consider a real-valued function \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). The mathematical expectations of \( S_{f_L} \) is equal to

\[
ES_{f_L} = \int_{\mathbb{R}^d} f(x/L)K_L(x,x)dx = \int_{\mathbb{R}^d} f(x/L)A_L(0)dx
\]

\[
+ \int_{\mathbb{R}^d} f(x/L)R_L(x,x)dx = A_L(0) \cdot L^d \int f(x)dx + \int f(x/L)R_L(x,x)dx.
\]
By (11) the absolute value of the second integral is bounded by the sum of
the integrals
\[
\left(\int_{\mathbb{R}_+^d} f^2(\pm x_1/L, \ldots, \pm x_d/L)dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}_+^d} Q(2x)dx \right)^{\frac{1}{2}} = O\left(L^{\frac{d}{2}}\right)
\]
Therefore,
\[
ES_{f_L} = A_L(0) \cdot L^d \cdot \int_{\mathbb{R}^d} f(x)dx + O\left(L^{\frac{d}{2}}\right).
\] (20)

The variance of \(S_{f_L}\) is given by
\[
\text{Var } S_{f_L} = \int f^2(x/L)K_L(x,x)dx - \int f(x/L)f(y/L)|K_L(x,y)|^2dxdy =
A_L(0)L^d \int f^2(x)dx - \int f(x/L)f(y/L)|A_L(x-y)|^2dxdy + r(L),
\] (21)

where
\[
r(L) = \int f^2(x/L)R_L(x,x)dx - 2 \int f(x/L)f(y/L)A_L(x-y)R_L(y,x)dxdy - \int f(x/L)f(y/L)|R_L(x,y)|^2dxdy = r_1(L) + r_2(L) + r_3(L).
\]

It follows from the assumptions of the theorem that the second term at the r.h.s. of (21) is equal to
\[
L^d \int |\hat{f}(k)|^2|\hat{A_L}|^2(k/L)dk = L^d|\hat{A_L}|^2(0) \int |\hat{f}(k)|^2dk \cdot (1 + o(1)) =
L^d \int |A_L(x)|^2dx \int f^2(x)dx(1 + o(1)).
\]

Indeed,
\[
| \int |\hat{f}(k)|^2(|\hat{A_L}|^2(k/L) - |\hat{A_L}|^2(0))dk | \leq
\int_{|k| > \kappa_L} | + \int_{|k| \leq \kappa_L} |.
\]
Since
\[ \| \hat{A}_L(t) \|^2 = \int \hat{A}_L(k) \hat{A}_L(k - t) dk \]
we note that the first integral is bounded from above by
\[ \text{const} \int_{|k| > (\kappa L)^{1/2}} \hat{f}(k)^2 dk \to 0 \text{ as } L \to \infty. \]
To deal with the second integral we estimate from above
\[ \| \hat{A}_L^2(k/L) - \hat{A}_L^2(0) \| \leq \\int |A_L|^2(t) (\exp(2\pi itk/L) - 1) dt \]
\[ = \int_{|t| \geq L/\kappa L} + \int_{|t| < L/\kappa L} |A_L|^2(t) dt + O(1/\sqrt{\kappa L}) = o(1) + O(1/\sqrt{\kappa L}) = o(1). \]
Therefore
\[ \text{Var } S_{f_L} = \left( A_L(0) - \int |A_L(x)|^2 dx \right) L^d \int f^2(x) dx + o(L^d) + r(L) = \]
\[ \sigma^2 L^d \int f^2(x) dx + o(L^d) + r(L). \]
We claim that
\[ r(L) = o(L^d). \]
Consider first \( r_1(L) \). By (11) it is bounded by the integrals
\[ \int_{\mathbb{R}_+^d} f^2(\pm x_1/L, \ldots, \pm x_d/L)Q(2x) dx \]
All of these integrals are estimated in the same way. For example,

\[
\int_{\mathbb{R}^d} f^2(x/L)Q(2x)dx = L^d \int f^2(x)Q(2Lx)dx = \\
L^d \int f^2(x)Q(2Lx)\mathcal{X}_{\{Q(2Lx) > \frac{1}{\sqrt{L}}\}} dx + L^d \int f^2(x)Q(2Lx)\mathcal{X}_{\{Q(2Lx) \leq \frac{1}{\sqrt{L}}\}} dx \\
\leq L^d \|Q\|_{\infty} \int f^2(x)\mathcal{X}_{\{Q(2Lx) > \frac{1}{\sqrt{L}}\}} dx + L^{d-\frac{3}{2}} \int f^2(x)dx = o(L^d),
\]

since

\[
\ell(x : Q(2Lx) > \frac{1}{\sqrt{L}}) \xrightarrow{L \to \infty} 0.
\]

To estimate \(r_3(L)\) we need to estimate the integrals of the form

\[
\int_{\mathbb{R}^d} |f(x/L)| |f(y/L)|Q^2(x + y)dxdy = L^d \int_{\mathbb{R}^d} g(z/L)Q^2(z)dz, \tag{24}
\]

where \(g(z) = \int |f(x)| |f(z - x)| \mathcal{X}_{\mathbb{R}^d_+}(x) \mathcal{X}_{\mathbb{R}^d_+}(z - x)dx\).

Since \(g(z)\) is bounded, continuous, and zero at the origin, we have

\[
(24) = L^d g(0) \int Q^2(z)dz(1 + o(1)) = o(L^d).
\]

Finally,

\[
|r_2(L)| = | \int f(x/L)f(y/L)A_L(x - y)R(y, x)dxdy | \\
\leq \left[ \int |f(x/L)| |f(y/L)| |A_L(x - y)|^2dxdy \right]^\frac{1}{2} \\
\left[ \int |f(x/L)| |f(y/L)| |R_L(y, x)|^2dxdy \right]^\frac{1}{2} = O \left( L^{\frac{d}{2}} \right) o \left( L^{\frac{d}{2}} \right) = o \left( L^d \right).
\]

Combining the above estimates, we prove (23), which implies

\[
\text{Var } S_{fL} = \sigma^2 L^d \int f^2(x)dx(1 + o(1)). \tag{25}
\]

If \(f\) is bounded, the Central Limit Theorem then follows from Theorem 1 (compactness of the support of \(f\) is not needed since \(f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\)).
guarantees that all moments of $S_f$ are finite). The proof in the case of the unbounded $f$ follows by a rather standard approximation argument. We choose $N > 0$ to be sufficiently large and consider a truncated function

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } |f(x)| \leq N \\ N, & \text{if } f(x) > N \\ -N, & \text{if } f(x) < -N. \end{cases}$$

Observe that

$$E \left( \frac{S_f - ES_f}{\sigma L^2} - \frac{S_{\tilde{f}} - ES_{\tilde{f}}}{\sigma L^2} \right)^2 = \frac{\text{Var} S_{\tilde{f}}}{\sigma^2 L^4} = \int_{|x| \geq N} f^2(x) dx + o(1)$$

can be made arbitrarily small by choosing $N$ and $L$ sufficiently large.

Since

$$\frac{S_{\tilde{f}} - ES_{\tilde{f}}}{\sigma L^2} \xrightarrow{w} N \left( 0, \int_{|x| \leq N} f^2(x) dx \right)$$

and

$$\lim_{N \to \infty} \int_{|x| \leq N} f^2(x) dx = \int f^2(x) dx,$$

the result follows.

Theorem 2 is proven. \hfill \Box

4 Proof of Theorem 3

We now turn to the proof of Theorem 3. It is enough to establish that

$$\text{Var} S_f = L^{1-\alpha} \varphi(L^{-1}) \int |\hat{f}(k)|^2 |k|^\alpha dk(1 + o(1)). \quad (26)$$

The result then will follow from Theorem 1. We have (see (12))

$$\text{Var} S_f = \int |\hat{f}(\lambda)|^2 L^2 m(\lambda) d\lambda = L \int |\hat{f}(k)|^2 m(kL^{-1}) dk =$$

$$L \int |\hat{f}(k)|^2 |k|^\alpha L^{-\alpha} \varphi(kL^{-1}) dk = L^{1-\alpha} \varphi(L^{-1})$$

$$\int |\hat{f}(k)|^2 |k|^\alpha \frac{\varphi(kL^{-1})}{\varphi(L^{-1})} dk \quad (27)$$
It was proven by Karamata ([K1], [K2]) that any slowly varying function at
the origin can be represented in some interval \((0, b]\) as
\[
\varphi(x) = \exp\left\{ \eta(x) + \int_{b^{-1}}^{x^{-1}} \frac{\epsilon(t)}{t} dt \right\},
\]  
\(28\)
where \(\eta\) is a bounded measurable function on \((0, b]\), such that \(\eta(x) \to c\)
as \(x \to 0\) \((|c| < \infty)\), and \(\epsilon(x)\) is a continuous function on \((0, b]\) such that
\(\epsilon(x) \to 0\) as \(x \to 0\). (for a modern day reference we refer the reader to [Se],
Theorem 1.2; of course a similar representation holds for \(\varphi\) also on some
interval \([b', 0)\) of the negative semi-axis). In particular
\[
\frac{\varphi\left(\frac{k}{L}\right)}{\varphi\left(\frac{1}{L}\right)} \xrightarrow{L \to \infty} 1
\]  
\(29\)
uniformly in \(k\) on compact subsets of \(\mathbb{R}^1 \setminus \{0\}\), and the following estimates
hold uniformly in \(k\) for sufficiently large \(L\)
\[
\text{const}_1 k^{-n} \leq \varphi(kL^{-1})/\varphi(L^{-1}) \leq \text{const}_2 k^n, \text{ for } 1 \leq k \leq L,
\]  
\(30\)
\[
\text{const}_3 k^{-\frac{1}{2}} \leq \varphi(kL^{-1})/\varphi(L^{-1}) \leq \text{const}_4 k^{\frac{1}{2}}, \text{ for } 0 < k \leq 1,
\]  
\(31\)
where \(\text{const}_i, i = 1, \ldots, 4, n > 0\) are some constants.

The estimates (28)-(31) imply
\[
\int_{-L}^{L} |\hat{f}(k)|^2 |k|^2 \frac{\varphi(kL^{-1})}{\varphi(L^{-1})} dk \xrightarrow{L \to \infty} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 |k|^\alpha dk.
\]
From the other side, the integral over \(|k| \geq L\) is \(o(1)\) since \(f\) is a Schwartz
function and \(m\) is bounded.

Theorem 3 is proven. \(\square\)

Remark 7 We learned very recently that similar results to our Theorem 2
have been independently obtained (in the discrete case) by Tomoyuki Shirai
and Yoichiro Takahashi in the preprint [ST].

Appendix

For the convenience of the reader we give here the proof of a rather standard
fact.
Lemma 3 Let \( \{ \eta_L \} \) be a family of random variables such that \( c_1(\eta_L) = 0 \), \( c_2(\eta_L) = 1 \) and \( c_n(\eta_L) \) converges to zero as \( L \to \infty \) for all \( n \geq N \), where \( N < \infty \). Then \( \lim_{L \to \infty} c_n(\eta_L) = 0 \) for all \( n > 2 \) and \( \eta_L \) converges in distribution to \( N(0,1) \).

Proof Denote \( d_L = \max(|c_j(\eta_L)|^{1/7}, 1 \leq j \leq N - 1) \). It is clear that \( d_L \geq 1 \). Consider the random variable
\[
\tilde{\eta}_L = \eta_L / d_L.
\]
Since \( c_n(\tilde{\eta}_L) = c_n(\eta_L) / d_L^n \) we have \( |c_n(\tilde{\eta}_L)| \leq 1 \) for all \( n \) and \( c_n(\tilde{\eta}_L) \to 0 \) for \( n \geq N \). Consider \((N - 1)\)-dimensional vector \((c_1(\tilde{\eta}_L), \ldots, c_{N-1}(\tilde{\eta}_L))\). Let \((c_1, c_2, \ldots, c_{N-1})\) be a limit point. The Marcinkiewicz theorem (see e.g. [L]) states that if all but a finite number of cumulants of a random variable are non-zero then the random variable must either have a Gaussian distribution or be a constant. In both cases we have \( c_j = 0 \) for \( j > 2 \). Therefore \( d_L = (c_2(\eta_L))^{1/2} = 1 \) for sufficiently large \( L \) and \( c_n(\eta_L) \xrightarrow{L \to \infty} 0 \) for \( n > 2 \).

Convergence of the cumulants of \( \eta_L \) to the cumulants of \( N(0,1) \) is equivalent to the convergence of the moments which in turn implies convergence in distribution.

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