Lifshitz black holes in Einstein-Yang-Mills theory

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Abstract

We find that the four dimensional cosmological Einstein-Yang-Mills theory with $SU(2)$ gauge group admits Lifshitz spacetime as a base solution for the dynamical exponent $z > 1$. Motivated by this, we next demonstrate numerically that the field equations admit black hole solutions which behave regularly on the horizon and at spatial infinity for different horizon topologies. The solutions depend on one parameter, the strength of the gauge field at the horizon, which is fine-tuned to capture the Lifshitz asymptotics at infinity. We also discuss the behavior of solutions and the change in Hawking temperature for black holes that are large or small with respect to the length scale $L$, which is itself fixed by the value of the cosmological constant.

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I. INTRODUCTION

The AdS/CFT conjecture has been a strong and versatile tool in the arsenal of high energy theory. The aptly named duality, relating conformal field theories to gravity in higher dimensions, has proven to be a powerful theoretical toolkit and provided great insight in high energy physics. Recently there has been a serious effort to trickle down to the energy scale of condensed matter and make holography accessible to strongly coupled systems which can be realized in experiments [1–4] (and references therein). One of the approaches to achieve such duality is to impose an anisotropic scaling symmetry on the boundary field theory

\[ t \to \lambda^z t, \quad \vec{x} \to \lambda \vec{x}, \quad r \to r \lambda, \]

where \( z \) is called the dynamical exponent. The symmetry algebra of field theories is controlled by \( z \), e.g. \( z = 1 \) generates the Poincaré group with special conformal symmetries, and when \( z > 1 \) one ends up with different scalings for time and space which leads to non-relativistic field theories with Lifshitz symmetries; our main focus in this work. The bulk metric conjuring up these symmetries is found to be

\[ ds^2 = L^2 \left( -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2 \right), \]

with peculiar properties regarding causal structure and geodesics [2, 5]. Einstein gravity with a negative cosmological constant does not admit this type of anisotropic backgrounds as a solution. One should either consider higher derivative theories or matter couplings to source the metric. Once we depart from the Einstein gravity and add the higher curvature corrections, the amended theories begin to accommodate (2) as a solution [6]. On the other hand, the anisotropic backgrounds engineered with various types of matter Lagrangians [7], e.g. string theory motivated \( p \)-form fields [5], massive gauge fields, \( U(1) \) fields with dilatonic-like couplings [8] are better studied models for gravity duals. One of the first examples is the theory considered in [5], which is conjectured to be the gravitational dual of 2+1 dimensional field theories modeling quantum critical behavior in strongly correlated electron systems.

In principle, black hole solutions describe the finite temperature behavior of those dual non-relativistic field theories, which renders them important objects in holography. Curvature corrections open up the way for large families of analytic black holes in different dimensions both for static and stationary Lifshitz spacetimes [6, 9, 11]. However, analytic
black holes with matter fields for generic $z$ are rather rare \cite{7,8}. For a fixed value of $z$, several exact solutions were found \cite{12,15}. On the other hand, different types of numerical solutions were explored with generic $z$ values and for different horizon topologies \cite{16,20} for theories with massive gauge fields and $p$-forms.

The matter Lagrangians with non-abelian gauge fields have been used in holographic superconductor models \cite{21,22}, with AdS/Schwarzschild black hole backgrounds. Recently the effects of Lifshitz scaling on these models have also been considered \cite{23}. In this work we will first focus on a different and a simpler question: whether it is possible at all to support Lifshitz spacetime \cite{2} with non-abelian matter sources. To our knowledge, this has not been addressed previously elsewhere. Having answered the first in the affirmative, the second task we undertake is the dressing up of this background solution with black holes. There is a substantial literature on Einstein-Yang-Mills particle-like and black hole solutions \cite{24,28} both in asymptotically flat and AdS backgrounds with different characteristics. For example, asymptotically flat, colored black holes \cite{25} admit finite range field strength, i.e. there is no global magnetic $SU(2)$ charge that makes them indistinguishable from Schwarzschild at infinity, whereas asymptotically AdS ones can possess global $SU(2)$ magnetic charge \cite{28}. As we will show in what follows, Lifshitz asymptotics are quite different: Fields extend to infinity not only to endow black holes with $SU(2)$ charge but also to support Lifshitz spacetime. By abandoning asymptotic flatness, black holes with non-spherical horizon topologies can be constructed. Accordingly, we will consider three types of event horizon topologies, viz. planar, spherical and hyperbolic, with different gauge field ansätze respecting the corresponding symmetries. For large black holes, these three types have similar behavior but differ significantly in the case of small event horizon radius. Our focus will be on the numerical evidence for the asymptotically Lifshitz black holes in cosmological EYM theory. We will not discuss the relation to the holographic dual field theories, which merits a separate significant problem on its own.

The outline of the paper is as follows: In Section II we start with the equations of motion for the EYM system, state the ansatz for the planar symmetric YM fields and obtain the solution for the background metric \cite{2}. We then set the stage for black hole solutions by dressing up the background metric and gauge fields with suitable functions in Section III. The subsections IV A and IV B are devoted to the series solutions of black holes at infinity and at the horizon, respectively. We next study the numerical black hole solutions of the
theory in subsection IV C. In Section V the Hawking temperature of the solutions we have found are analyzed. Finally we conclude with Section VI.

II. LIFSHITZ ASYMPOTOTICS AND $SU(2)$ GAUGE FIELDS

The gravity theory we consider is the four dimensional cosmological EYM theory for the gauge group $SU(2)$ described by the action

$$S = \int d^4x \sqrt{-g}\left((R - 2\Lambda) - \frac{1}{2g_{YM}^2}\text{Tr} F_{\mu\nu}F^{\mu\nu}\right),$$

where $\Lambda$ is the cosmological and $g_{YM}^2$ is the gauge coupling constant in dimensions of $1/\text{length}^2$. In order to support backgrounds with anisotropic scaling symmetry, a naive approach is to make the coupling constants depend on the geometry, i.e. the parameter $z$. It is worth emphasizing that the path taken here is different from [28, 29], in which AdS is already a vacuum for the gravitational sector and YM field is used only as a hair parameter, not for supporting the AdS geometry. In this work YM field will be used to source the metric [2], so it has to decouple at $z = 1$. Because of this major difference, we will not be able to recover the results of [28] in the conformal limit $z = 1$. As we will show in the discussion below, ours is still an appropriate way to proceed.

Einstein field equations following from the action (3) read

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{1}{g_{YM}^2}T_{\mu\nu},$$

with the traceless YM stress-energy tensor defined as

$$T_{\mu\nu} \equiv \text{Tr} (F_{\mu}{}^{\alpha}F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}),$$

and the YM field equations

$$D_{\mu}F^{\mu\nu} = 0,$$

where the gauge covariant derivative is defined as $D_{\mu} \equiv \nabla_{\mu} - i[A_{\mu}, \ ]$. 

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1 Here $F_{\mu\nu}$ is the gauge field strength $F_{\mu\nu}^a T_a = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - i[A_{\mu}, A_{\nu}]$ and we choose generators $T_a \equiv \tau_a/2$, $a = 1, 2, 3$ with $\tau_a$ denoting Pauli matrices. The commutation relations and the normalization of generators are given as $[T_a, T_b] = i\epsilon_{abc}T_c$ and $\text{Tr} T_a T_b = \delta_{ab}/2$, respectively. Throughout we use the conventions in which the signature of metrics is $(-, +, +, +)$, the Riemann tensor is taken as $R_{\nu\alpha\beta\mu} = \partial_{\nu}\Gamma_{\beta\mu}^{\alpha} - \cdots$ and $R_{\mu\nu} = R^{\alpha}_{\quad \nu\alpha\mu}$. 
The traceless nature of the stress-energy tensor allows us to determine the value of the cosmological constant from Einstein field equations. The trace of (4) when used with the metric (2) yields
\[
\Lambda = -\frac{3 + 2z + z^2}{2L^2}.
\] (7)

The next step is to consider the non-abelian gauge field configuration respecting the symmetry of the plane, which is a subgroup of the Poincaré group and studied extensively in [30, 31]. Additionally, we shall also restrict ourselves to the static and purely magnetic case. This restriction leads to the \( SU(2) \) gauge connection
\[
A_\mu dx^\mu = w(r)T^1 dx_1 + w(r)T^2 dx_2.
\] (8)

For our purposes it is convenient to express the metric (2) in a form which is analogous to the one that is commonly used\(^2\) \[26–28\]
\[
ds^2 = L^2 \left(-S(r)^2 \mu(r) dt^2 + \frac{dr^2}{\mu(r)} + r^2 d\vec{x}^2 \right).
\] (9)

Taking (8), (9) into account, the field equations (4), (6) reduce to the system
\[
S^{-1} S' = \frac{1}{2L^2 g_{YM}^2} \frac{(w')^2}{r},
\] (10)
\[
(\mu w')' = \frac{w^3}{r^2} - \frac{1}{2L^2 g_{YM}^2} \frac{\mu(w')^3}{r},
\] (11)
\[
 r\mu' + \mu + L^2 r^2 \Lambda = -\frac{1}{2g_{YM}^2 L^2} \left( \frac{w^4}{2r^2} + \mu(w')^2 \right),
\] (12)

with prime denoting the ordinary derivative with respect to \( r \).

Plugging in \( S(r) = r^{z-1}, \mu(r) = r^2 \) and using (7), it is straightforward to show that the Lifshitz spacetime (2) is a solution for all \( z > 1 \) provided that the gauge field and the coupling constant are chosen as
\[
w(r) = \pm \sqrt{z + 1} r, \quad g_{YM}^2 = \frac{1}{2L^2} \frac{(z + 1)}{(z - 1)}.
\] (13)

The sign ambiguity of the gauge field can be deduced from the invariance of the field equations (10), (11), (12) under \( w(r) \to -w(r) \), which corresponds to a gauge transformation
\(^2\) Here we are considering the planar case, whereas in the literature the spatial part of (9) is typically spherical, with a different gauge field ansatz. The other cases can also be treated in a similar manner, which will be discussed later in the next section.
Hence, in what follows we will proceed with the positive sign gauge field. The solution we have found is basically a “colorful plane with Lifshitz asymptotics”. Note also that \( z > 1 \) in order to have real gauge fields, which signals the “critical slowing down” of the possible dual field theories.

The conformal limit \( z \to 1 \) of (13) is also peculiar. The YM part decouples from the gravity action and, as well-known, the AdS spacetime is a solution of (4) without matter fields, provided \( \Lambda = -3/L^2 \). Moreover, the decoupled gauge field is a solution to the pure YM part, which is in some sense the AdS analogue of the flatspace solution given in [30, 31].

Having determined that the non-abelian YM matter is suitable for Lifshitz asymptotics, we can now continue and dress up this background geometry to obtain black hole solutions.

III. FIELD EQUATIONS

In this section we first extend the metric and the gauge field ansatz to cover the other types of event horizon topologies, then cast the field equations in a way that is convenient for capturing the Lifshitz asymptotics for both the metric and the gauge field at large spatial distance.

We will control the spatial part of the metric by introducing a parameter \( k \):

\[
ds^2 = L^2 \left( -S(r)^2 \mu(r) dt^2 + \frac{dr^2}{\mu(r)} + r^2 d\Omega_k^2 \right),
\]

where

\[
d\Omega_k^2 \equiv \begin{cases} 
  d\theta^2 + \sin^2 \theta d\phi^2 & ; k = +1 \\
  d\theta^2 + \sinh^2 \theta d\phi^2 & ; k = -1 \\
  d\theta^2 + d\phi^2 & ; k = 0
\end{cases}.
\]

It is clear from this definition that, \( k = 0 \) corresponds to the planar symmetric case we have discussed previously, \( k = 1 \) yields the spherically symmetric metric, and \( k = -1 \) option is invariant under hyperbolic rotations.

The gauge field ansatz will change accordingly by taking into account the symmetries of the metric [15]. The method for constructing symmetric gauge fields is developed in [32]. In addition to the \( E(2) \) symmetric gauge field, we will also consider the following static \( SU(2) \)
connections that are invariant under \(SO(3)\) and the connected part of \(SO(2, 1)\) \[32, 33\]
\[
A = q(r)T^3dt + p(r)T^3dr + (w(r)T^1 + u(r)T^2)d\theta \\
+ (w(r)\Omega_k(\theta)T^2 - u(r)\Omega_k(\theta)T^1 + \tilde{\Omega}_k(\theta)T^3)d\phi,
\] (16)
for \(k = 1, -1, \) where \(\Omega_1(\theta) \equiv \sin \theta, \) \(\Omega_{-1}(\theta) \equiv \sinh \theta, \) \(\tilde{\Omega}_1(\theta) \equiv \cos \theta, \) \(\tilde{\Omega}_{-1}(\theta) \equiv \cosh \theta.\)

This expression still has a \(U(1)\) gauge freedom \[26\], which can be used to set \(u(r) = 0.\) Next, with the help of the field equations, we see that \(p(r) = 0\) provided \(w(r) \neq 0.\) In order to simplify the discussion, we will only consider the gauge field strengths with vanishing electric part, i.e. \(q(r) = 0.\) In fact this choice is rather restrictive. It was shown in \[34\] that, with appropriate asymptotics, the Reissner-Nordström solution is the only static black hole with non-zero YM electric field. However, all of this was for asymptotically flat backgrounds and, obviously these arguments do not necessarily apply for Lifshitz spacetimes. Nevertheless, we shall restrict ourselves to the purely magnetic ansatz in this work.

Taking these considerations into account, we are thus led to
\[
A = \begin{cases} 
  w(r)T^1d\theta + (w(r)\Omega_k(\theta)T^2 + \tilde{\Omega}_k(\theta)T^3)d\phi; & \text{for } k = \pm 1 \\
  w(r)T^1d\theta + w(r)T^2d\phi; & \text{for } k = 0
\end{cases}.
\] (17)

Now utilizing the generalized metric (14) and the gauge field ansatz (17), the equations (10), (11), (12) can be cast into a general form covering all possible cases \[29\]
\[
S^{-1}S' = \frac{1}{2L^2g_{YM}^2} \frac{(w')^2}{r},
\] (18)
\[
(\mu w')' = \frac{w(w^2 - k)}{r^2} - \frac{1}{2L^2g_{YM}^2} \frac{\mu(w')^3}{r},
\] (19)
\[
r\mu' + \mu + L^2r^2\Lambda - k = -\frac{1}{2g_{YM}^2L^2} \left( \frac{(w^2 - k)^2}{2r^2} + \mu(w')^2 \right).
\] (20)

Although this form of the field equations are helpful in exploring the constraints on the functions at the horizon, they are a bit impractical for numerical purposes. It is more appropriate to redefine the metric and the gauge field functions such that the Lifshitz vacuum \[2\] can be explicitly recovered at large radius. One can achieve this with simple redefinitions
\[
w(r) \equiv \sqrt{z + 1} rh(r), \quad \mu(r) \equiv \frac{r^2}{g(r)^2}, \quad S(r) \equiv r^{s-1} f(r)g(r), \quad w'(r) \equiv \sqrt{z + 1} j(r).\] (21)

It is obvious from these definitions that if all the unknown functions \(f(r), g(r), h(r), j(r)\) are unity in the large \(r\) limit, i.e. when \(r \gg 1,\) then we recover the Lifshitz background solution we have constructed for the EYM system with \(k = 0.\)
All these assumptions, identifications and the coupling constants \((7), \ (13)\) yield the following system of equations

\[
rf(r)' = -f(r)\left((z - 1) - \frac{j(r)^2}{2} (z - 1) + \frac{g(r)^2 h(r)^2}{4} (z^2 - 1) - \frac{g(r)^2}{4} (3 + 2z + z^2) + \frac{3}{2}\right)
\]
\[
- kf(r) g(r)^2 \left\{ \frac{k}{4r^4} \frac{(z - 1)}{(z + 1)} - \frac{h(r)^2}{2r^2} (z - 1) - \frac{1}{2r^2} \right\},
\] (22)

\[
rj(r)' = j(r) + g(r)^2 h(r)^3 (z + 1) - \frac{g(r)^2}{2} j(r) (z^2 + 2z + 3) + \frac{g(r)^2 h(r)^2 j(r)}{2} (z^2 - 1)
\]
\[
- k \left\{ \frac{g(r)^2}{r^2} \left( \frac{h(r)^2 j(r)}{r^2} (z - 1) - \frac{g(r)^2}{2} (z + 1) + \frac{j(r)}{r^2} + \frac{h(r)}{r^2} \right) \right\},
\] (23)

\[
rk(r)' = \frac{g(r) j(r)}{2} \left( z - 1 \right) + \frac{g(r)^3 h(r)^4}{4} (z^2 - 1) - g(r)^3 (3 + 2z + z^2) + \frac{3g(r)}{2}
\]
\[
+ k g(r)^3 f \left\{ \frac{k}{4r^4} \frac{(z - 1)}{(z + 1)} - \frac{h(r)^2}{2r^2} (z - 1) - \frac{1}{2r^2} \right\},
\] (24)

\[
rh(r)' = j(r) - h(r).
\] (25)

Several observations are in order here. The highly nonlinear nature of the EYM system makes the analytic study difficult, and despite our efforts, we couldn’t find an exact solution with non-trivial gauge field functions. Yet it is simple enough for working numerically, since we have reduced the system into a system of coupled first order ordinary differential equations with the functions having definite asymptotic values.

Secondly, terms explicitly involving \(1/r^2\) and \(1/r^4\) appear only in spherical and hyperbolic cases \(k = \pm 1\). Exploiting this fact, we will assume that in the large \(r\) limit, the spherical and hyperbolic spatial parts can be replaced in by a flat one \([16], \ [20]\). Thus all of the unknown functions appearing in the numerical solutions will have the same asymptotic behavior, i.e. \(f(r) = g(r) = h(r) = j(r) = 1\).

Note that the three equations \((23)\) to \((25)\) form a closed system on their own, and equation \((22)\) can be considered separately. In addition, the right hand side of \((22)\) is linear in the function \(f(r)\), which makes its normalization undetermined. This leads to a scaling of the initial value of \(f\) at large \(r\), which is essentially a gauge choice, i.e. rescaling of the time coordinate \([5]\). In order to get the correct asymptotics after the numerical integration, proper initial values must be chosen.

There remains now to expand the functions \(f(r), g(r), h(r), j(r)\) at large \(r\) and separately at the horizon, for all possible values of the parameter \(k\) but for a fixed value of \(z\). One can extract a shooting parameter from the asymptotic form of the solutions to \((22), \ (23), \ (24)\).
and (25) provided there is one available with the given boundary conditions, and this is of paramount importance for the numerical study.

IV. SERIES AND NUMERICAL SOLUTIONS

We now describe the results obtained by expanding the functions at large radius and at the horizon whose existence we assume presumably. The series solution will teach a great deal about the initial values and bounds on the functions defined in the previous section. We will then consider the numerical solutions of the system for various cases.

A. Series solution for the large radius

First we look for the series solutions at large $r$, which in principle can confirm the plausibility of the assumption we have made in regards to the employment of the planar background for all horizon types at large $r$. The behavior of solutions is rather interesting for different values of $z$. It turns out that geometries with even integer dynamical exponent $z$ admit only planar solutions. However, all types of geometries are supported when $z$ is chosen to be an odd integer. In order to establish this result, we first fix the value of $z$ in equations (22), (23), (24), (25), then make a simple transformation $r = 1/x$, and finally assume a power series expansion at small $x$

\[
\begin{align*}
    f(r) &= \sum_{n=0}^{\infty} \tilde{f}_n x^n, \\
    g(r) &= \sum_{n=0}^{\infty} \tilde{g}_n x^n, \\
    h(r) &= \sum_{n=0}^{\infty} \tilde{h}_n x^n, \\
    j(r) &= \sum_{n=0}^{\infty} \tilde{j}_n x^n
\end{align*}
\]

with the Lifshitz asymptotics, i.e. $\tilde{f}_0 = \tilde{g}_0 = \tilde{h}_0 = \tilde{j}_0 = 1$. We insert these into the equations of motion (22), (23), (24), (25) and work order by order in $x$. We can summarize our findings as follows\(^3\):

\(^3\) To keep the following discussion simple, we only present our findings for the $z = 2$ and $z = 3$ cases. The generic behavior of the solutions are captured by the $z = 2$ choice for even $z = 4, 6, 8, \cdots$ or by the $z = 3$ choice for odd $z = 5, 7, 9, \cdots$. 
For $z = 2$ and $k = 0$, we find

\begin{align*}
  f(r) &= 1 - \frac{9h_L}{2r^4} - \frac{1557h_L^2}{176 \frac{r^8}{r^8}} + \mathcal{O}(1/r^{16}) + \cdots, \\
  g(r) &= 1 + \frac{6h_L}{r^4} + \frac{1143h_L^2}{22 \frac{r^8}{r^8}} + \mathcal{O}(1/r^{16}) + \cdots, \\
  h(r) &= 1 + \frac{h_L}{r^4} + \frac{405h_L^2}{44 \frac{r^8}{r^8}} + \mathcal{O}(1/r^{16}) + \cdots, \\
  j(r) &= 1 - \frac{3h_L}{r^4} - \frac{2835h_L^2}{44 \frac{r^8}{r^8}} + \mathcal{O}(1/r^{16}) + \cdots.
\end{align*}

(27)\hspace{1cm} (28)\hspace{1cm} (29)\hspace{1cm} (30)

However, for $z = 3$ and with generic $k$, we get

\begin{align*}
  f(r) &= 1 + \frac{k}{2r^2} + \frac{127k^2}{1352 r^4} + \mathcal{O}(1/r^5) + \cdots, \\
  g(r) &= 1 + \frac{23 k^2}{676 r^4} + \frac{12h_L}{r^5} + \mathcal{O}(1/r^6) + \cdots, \\
  h(r) &= 1 - \frac{3 k^2}{338 r^4} + \frac{h_L}{r^5} + \mathcal{O}(1/r^6) + \cdots, \\
  j(r) &= 1 + \frac{9 k^2}{338 r^4} - \frac{4h_L}{r^5} + \mathcal{O}(1/r^6) + \cdots.
\end{align*}

(31)\hspace{1cm} (32)\hspace{1cm} (33)\hspace{1cm} (34)

where we have only one arbitrary parameter $h_L$ characterizing both solutions at large $r$.

Let us emphasize that the discrepancy between even and odd $z$ follows from the expansion (26) we have considered. There may be fractional powers of $x$ in the expansion (26) which can remedy the situation for the even $z$ case. It is also possible that we have made an inappropriate choice of coordinates to discuss the solutions for large $r$. Nevertheless, we fix $z = 3$ in the numerical part of the calculations (see section IV C) for the sake of clarity.

**B. Series solution about the event horizon**

Let us now focus on the series solution about the presumed horizon. In order to have a non-extremal black hole, $g_{tt}$ and $g_{rr}$ components of the metric (14) must have a simple zero and a simple pole [16], [20] at the finite horizon $r = R_0$. This assumption leads to the following horizon expansions of the functions

\begin{align*}
  f(r) &= \sqrt{r - R_0} \sum_{n=0}^{\infty} f_n (r - R_0)^n, \\
  g(r) &= \frac{1}{\sqrt{r - R_0}} \sum_{n=0}^{\infty} g_n (r - R_0)^n.
\end{align*}

(35)\hspace{1cm} (36)
At this stage it is worthwhile to discuss the constraints on the gauge field functions at the horizon in order to construct the series expansion for the functions $h(r)$ and $j(r)$. These constraints can easily be seen from the general form of the field equations (18), (19), (20) we have discussed in section III. This set implies that the gauge field function $w(r)$ and its derivative must be related at the horizon as

$$ w'(R_0) = \left( \frac{w(R_0)(w^2(R_0) - k)}{(kR_0 - \frac{1}{2g_{\gamma\gamma}L^2} (w^2(R_0) - k)^2) - L^2R_0^2\Lambda} \right), $$

which amounts to relating the expansion coefficients on the horizon

$$ j(R_0) = j_0 = \frac{2h_0R_0(h_0^2R_0^2(z + 1) - k)}{2kR_0 + R_0^3(z^2 + 2z + 3) - \frac{(z-1)(k-h_0^2R_0^2(z+1))^2}{R_0(z+1)}}, \text{ for } z > 1 $$

where $w(R_0) = \sqrt{z+1}R_0h_0$ with the definition $h_0 \equiv h(R_0)$. The subtle difference between the planar and the other cases shows itself here. When $k = 0$, the horizon radius cancels out, and $j_0$ depends only on $h_0$ and the dynamical exponent $z$. To make the meaning of $h_0$ clear, consider a non-coordinate basis for the one-forms [5]

$$ \theta_t = Lr^z f(r) dt, \quad \theta_x = Lrdx^i, \quad \theta_r = Lg(r) \frac{dr}{r}, \quad i = 1, 2 $$

in which the planar metric (14) takes the form $ds^2 = \eta^{\mu\nu}d\theta_\mu d\theta_\nu$ with $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The gauge connection simply follows as

$$ A = \frac{\sqrt{z + 1}}{L}h(r)(T^1\theta_1 + T^2\theta_2). $$

This suggests that $h_0$ can be considered as the strength of the gauge field at the horizon, up to some normalization. There is also an upper bound for the gauge field function $w(r)$ for a given horizon radius $R_0$, which follows from the condition for a regular horizon, i.e

$$ \left. \frac{d\mu}{dr} \right|_{r=R_0} > 0. $$

Then, with the help of (20), one finds that

$$ k - \frac{1}{2g_{\gamma\gamma}L^2} \frac{(w^2(R_0) - k)^2}{2R_0^2} - L^2R_0^2\Lambda > 0. $$

In terms of $w(R_0) = \sqrt{z + 1}h_0$, this inequality further simplifies to

$$ \frac{R_0^2(z + 1)(2k + R_0^2(3 + 2z + z^2))}{(z - 1)} > (k - R_0^2(z + 1)h_0^2)^2. $$
The inequality (43) is rather important for numerical purposes. It weakly constrains the strength of the gauge field at the horizon, which in turn reduces the possible values for the shooting parameter \( h_0 \). For \( k = 0 \), \( h_0 \) is solely bounded by the \( z \) value. There is no dependence on the horizon radius; i.e if a numerical solution is found for the system with a fixed value of \( h_0 \), then it will always remain to be a solution for different radii. On the other hand, for the other topologies \( k = \pm 1 \), the gauge field strength changes with the changing horizon radius. The hyperbolic case \( k = -1 \) demands special attention regarding the value of the event horizon radius. By virtue of (42), there is a lower bound on the event horizon radius for fixed \( z \)

\[
|\Lambda| > \frac{1}{L^2 R_0^2} \left( 1 + \frac{1}{4g_{\text{ym}}^2 R_0^2 L^2} \right). \tag{44}
\]

The bound and the relations above can also be extracted from near horizon expansions. Assuming that the functions \( h(r), j(r) \) are finite on the horizon, they read

\[
h(r) = \sum_{n=0}^{\infty} h_n (r - R_0)^n, \tag{45}
\]

\[
j(r) = \sum_{n=0}^{\infty} j_n (r - R_0)^n. \tag{46}
\]

Inserting the expansions (26), (45), (46) into (22), (23), (24), (25), one finds solutions depending on two free parameters \( h_0 \), the strength of the gauge field at the horizon, and \( R_0 \), the horizon radius for a fixed \( z \) value.

As a simple example, for \( z = 2 \) and \( k = 0 \), one gets

\[
g_0 \rightarrow \frac{\sqrt{2} R_0}{\sqrt{11 - 3h_0^4}} \tag{47},
\]

\[
j_0 \rightarrow \frac{6h_0^3}{11 - 3h_0^4} \tag{48},
\]

\[
h_1 \rightarrow \frac{h_0 (3h_0^4 + 6h_0^2 - 11)}{(11 - 3h_0^4) R_0} \tag{49},
\]

\[
g_1 \rightarrow \frac{\sqrt{2} (18h_0^8 + 27h_0^6 - 99h_0^4 + 121)}{(11 - 3h_0^4)^{5/2} \sqrt{R_0}} \tag{50},
\]

\[
f_1 \rightarrow \frac{f_0 (-27h_0^8 + 9h_0^6 + 165h_0^4 - 242)}{(11 - 3h_0^4)^2 R_0}. \tag{51}
\]

Note that, all of the coefficients depend on two parameters \( h_0, R_0 \). Although \( f_0 \) appears to be a free parameter, it is in fact just an overall normalization factor as noted earlier in the
penultimate paragraph of section III. The bound on $h_0$ is now clear. In order to have real values for $g_0$, $h_0$ must be smaller than a value depending on $z$, and for $z = 2$, $k = 0$ the strength of the gauge field must be $h_0^4 < 11/3$ which is consistent with (43). Finally the value of $j_0$ (38) is also recaptured here.

To sum up, we have paved the way for numerical computation, by finding the initial values for functions in terms of $h_0$ and $R_0$. Now fixing one of the two parameters, namely the event horizon radius $R_0$, the shooting method can be used to search for numerical solutions. For a fixed value of $R_0$, we numerically evolve the functions and make them converge to unity at infinity by fine tuning the initial value $h_0$. The behavior of solutions differs considerably for small and large horizon radius values, and it also depends on the topology.

C. Numerical solutions

We begin with the larger black holes, and fix $z = 3$ in order to compare results for different values of $k$. It turns out that there is a unique critical value of $h_0$ within the allowed region described by (43), where we have the desired asymptotics. This is quite different from what was observed in asymptotically flat or AdS analogues of these black holes, where solutions are indexed by an integer $n$ that has the meaning of the node number for the gauge field amplitude $w_n(r)$ [24–28].

Setting $R_0 = 10$, we see from figures 1, 2 that, for large black holes the solutions behave similarly regardless of the topology of the event horizon. Although we plot the functions for all values of $k$, the graphs coalesce into one with a small difference between their shooting parameters $h_0$. The metric functions $f(r)$ and $g(r)$ start from zero and infinity, respectively, then converge to one monotonically. We have the following results for the initial value of the gauge field function, i.e. the shooting parameter

\[
 h_0 = \begin{cases} 
 1.025530137, & \text{for } k = 1, \\
 1.023139854, & \text{for } k = -1, \\
 1.024335678, & \text{for } k = 0. 
\end{cases} \tag{52}
\]

The value of $j_0$ simply follows from (38).

We then fix $R_0 = 0.5$ in order to investigate the smaller black holes. The behavior of the solutions changes drastically. First of all, the functions of spherical and hyperbolic solutions decay appreciably slower, and moreover the shooting parameters i.e. $h_0$ differ considerably.
FIG. 1: The figure plots the metric functions $f(r)$ and $g(r)$ as a function of radius $r$. This is an example of a large black hole with $R_0 = 10$, where the plots overlap for all values of $k$.

From figures 3 and 4 we see that for the spherical case the metric function $f(r)$ makes a peak first and then converges to unity, unlike the planar and hyperbolic cases where the functions monotonically converge to one. The other metric function $g(r)$ reaches a minimum then approaches to one for the spherically symmetric black holes. It turns out that for small black holes we have the following gauge field strengths (see figure 5)

$$h_0 = \begin{cases} 
1.425617169, & \text{for } k = 1, \\
0.278652475, & \text{for } k = -1, \\
1.024335678, & \text{for } k = 0.
\end{cases} \tag{53}$$

Having seen the differences between large and small black holes, let us now compare the analytic bound (43) with the values of $h_0$ for different radii. For planar black holes, a unique value of $h_0$ is sufficient for all event horizon radii. Meanwhile, for the spherical case one needs larger gauge fields for small radii, and hyperbolic ones can support weaker gauge fields as the radius gets smaller. A similar behavior was observed for the abelian field strength in the works of [16, 20]. For clarity, we plot $h_0$ versus $R_0$ both for spherical (figure 6) and
FIG. 2: The figure shows the gauge field functions $h(r)$ and $j(r)$ as a function of radius $r$ with $R_0 = 10$. The initial values of functions for different topologies are very close to each other. Graphs for different topologies merge into one.

hyperbolic (figure 7) cases as well. The solid line depicts the solution of the inequality (43) as a function of $R_0$ and the dashed line is the numerical values obtained from the shooting method. Evidently the bound (43) is saturated as the horizon radius $R_0$ gets smaller. It is worth emphasizing that the lower limit on the horizon radius $R_0$ for $z = 3$ is consistent with the numerical results, i.e. from the figure 7 we see that there is no solution below $R_0 \sim 0.48$.

V. THERMAL BEHAVIOR

Finally, let us compute the temperature and discuss the thermal behavior of these black holes. We resort to the Euclidean metric obtained by a Wick rotation to compute the temperature, which leads to the following expression [17]

$$T = \frac{f_0 R_0^{z+1}}{4\pi g_0},$$  \hspace{1cm} (54)
FIG. 3: A small black hole with $R_0 = 0.5$. Figure shows the metric function $f(r)$ for different cases $k = 1, -1, 0$. The solid line corresponds to $k = 1$, the dashed line to $k = 0$ and dot-dashed line represents $k = -1$, respectively.

where $f_0$, $g_0$ are the expansion coefficients in the near horizon limit. The general expression from the series solution near the horizon determines $g_0$ in terms of $k$, $h_0$, $R_0$ and $z$:

$$g_0 = \frac{\sqrt{2(z+1)R_0^{3/2}}}{(2h_0^2kR_0^2(z-1) + h_0^4R_0^2(1-z)(z+1) - k^2\frac{z-1}{(z+1)} + 2k + R_0^2(3 + 2z + z^2))^{1/2}}.$$

Recall that the coefficient $f_0$ is to be determined from the normalization of the numerical solution, so it depends on the shooting parameter $h_0$. Therefore, fixing $z = 3$, the temperature now depends only on the horizon radius and the topology. After finding several numerical solutions for different $R_0$ values, we plot figure 8 by computing the temperature within the limits of numerical accuracy. It is clear from this figure that as the radius gets smaller, black holes get cooler with different rates. Hyperbolic ones have a higher cooling rate then the planar ones, and the spherical black holes are hotter for small radius. In the large $R_0$ limit, the temperatures become identical just like the solutions. The thermal behavior of these black holes is opposite to their AdS counterparts, where the Hawking temperature increases
FIG. 4: The figure illustrates the metric function \( g(r) \) with a small radius \( R_0 = 0.5 \). The solid line indicates \( k = 1 \), while the \( k = 0 \) and \( k = 1 \) cases are represented by dashed and dot-dashed lines, respectively.

with the ever decreasing radius causing thermal instability. Moreover, it is clear that the EYM black holes do not exhibit Hawking-Page transition. A similar thermal behavior is observed for the Lifshitz black holes supported by abelian \( p \)-forms [16, 20] which indicates that the black holes become extremal i.e. they have zero Hawking temperature in the vanishing black hole size.

VI. CONCLUSIONS

In this work, we have studied the Lifshitz black holes with different horizon topologies in four dimensional cosmological EYM theory. After obtaining the gauge field that supports the Lifshitz spacetime [2], we have found numerical black hole solutions with different horizon topologies by suitably fine-tuning the gauge field strength at the horizon. Through the series solution of the field equations, we have found a quite interesting property: The geometries
FIG. 5: The gauge field function $h(r)$ is displayed on the top and $j(r)$ at the bottom, both as functions of $r$. In both graphs $R_0 = 0.5$. The solid line indicates $k = 1$, while the $k = 0$ and $k = 1$ cases are represented by dashed and dot-dashed lines, respectively.
FIG. 6: The inequality (43) as a function of $R_0$ is plotted with a solid line for $k = 1$. The dashed line corresponds to the numerical values of $h_0$ as a function of $R_0$ for spherically symmetric black holes.

with odd $z$ support black holes with different horizon topologies, whereas for even $z$ only planar ones are supported. Thus we have fixed $z = 3$ in order to investigate all possible scenarios. From numerical results, we have observed that the behavior of solutions for different topologies changes considerably for small black holes whereas it becomes identical for large horizon black holes. We have also analyzed the thermal behavior of the numerical solutions by computing the Hawking temperature for all types of black holes. We have found that there is a rapid decay in temperature as the black hole radius gets smaller, and moreover black holes do not display Hawking-Page transition. In this respect, the EYM black holes and the abelian counterparts [16, 20] have quite similar characteristics, but they both differ considerably from their conformal cousins and some of the Lifshitz black hole solutions to string theory [36].
FIG. 7: The inequality (43) as a function of \( R_0 \) is plotted with a solid line for \( k = -1 \). The dashed line corresponds to the numerical values of \( h_0 \) as a function of \( R_0 \) for hyperbolically symmetric black holes. The lower bound (44) on the horizon radius is apparent.

One of the most important questions to ask is the use of EYM theory in non-relativistic holography. Certainly, Lifshitz spacetimes and black holes with non-abelian matter sources deserve further attention. Although the holographic description of matter Lagrangians with abelian and scalar fields are studied up to some extent, there is not much work done on EYM theory in which these solutions can find a practical application.

A further direction of research would be to consider the extension of these black holes. First, the existence of analogous solutions can be considered by extending the \( SU(2) \) symmetry ansatz to higher spacetime dimensions. It would also be interesting to investigate the generalization of the gauge group \( SU(2) \) to \( SU(N) \). In another vein, here we have only considered a purely magnetic part; One could still extend this ansatz by turning on the function \( q(r) \) in (16) and look for the existence of dyonic black holes. It would certainly be
FIG. 8: Temperature versus horizon radius for $z = 3$. The different topologies are represented by a solid line $k = 0$, by a dashed line $k = 1$ and a dot-dashed line $k = -1$.

of interest if the non-abelian counterparts of Lifshitz solitons [16, 20, 35] could be found.

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