A Lagrangian perspective on nonautonomous advection-diffusion processes in the low-diffusivity limit

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Abstract

We study mass preserving transport of passive tracers in the low-diffusivity limit using Lagrangian coordinates. Over finite-time intervals, the solution-operator of the nonautonomous diffusion equation is approximated by that of a time-averaged diffusion equation. We show that leading order asymptotics that hold for functions [Krol, 1991] extend to the dominant nontrivial singular value. This answers questions raised in [Karrasch & Keller, 2020]. The generator of the time-averaged diffusion/heat semigroup is a Laplace operator associated to a weighted manifold structure on the material manifold. We show how geometrical properties of this weighted manifold directly lead to physical transport quantities of the nonautonomous equation in the low-diffusivity limit.

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1 Introduction

We begin by sketching the outlines of this paper, a more comprehensive introduction with more detailed references is given in section 2. We are concerned with the problem of transport and mixing in nonautonomous advection-diffusion processes in the vanishing-diffusivity limit. Such processes are, in the simplest case, described by the \textit{advection-diffusion equation},

$$\partial_t u_\varepsilon(x,t) + \text{div}(u_\varepsilon(x,t) V(x,t)) = \varepsilon \Delta u_\varepsilon(x,t),$$  \hspace{1cm} (1)

where $V$ is a time-dependent, smooth velocity field, $u_\varepsilon$ the density of a weakly diffusive passive scalar, and $\varepsilon > 0$ is referred to as the (strength of) \textit{diffusivity}. We sometimes omit the explicit $\varepsilon$ in our notation when referring to $u_\varepsilon$ for the sake of clarity. In this work, we are interested in the \textit{finite-time} setting, i.e., without loss $t \in I = [0,1]$.

Lagrangian coordinates can be obtained from the advection-only version of eq. (1) with $\varepsilon = 0$. With $p$ denoting an arbitrary point in these coordinates, it is known that eq. (1) takes the form of a time-dependent diffusion (or heat) equation

$$\partial_t u_\varepsilon(p,t) = \varepsilon \Delta_t u_\varepsilon(p,t).$$  \hspace{1cm} (2)

The smoothly varying family of operators $(\Delta_t)_{t \in I}$ may be viewed as Laplace operators on a suitably defined time-dependent family of weighted manifolds. We want to compare the solution $u_\varepsilon$ of eq. (2) to the solution $\overline{u}_\varepsilon$ of the \textit{time-averaged} equation

$$\partial_t \overline{u}_\varepsilon(p,t) = \varepsilon \overline{\Delta} u_\varepsilon(p,t),$$  \hspace{1cm} (3)

as $\varepsilon \to 0$ at the final time $t = 1$. To the best of our knowledge, an averaging approach like this has been first taken in [34], albeit in an infinite-time setting. The operator $\overline{\Delta}$ has also been introduced by Froyland in his recent work on \textit{dynamic} isoperimetry [10]. Consistently with his work, we will refer to $\overline{\Delta}$ as the \textit{dynamic Laplacian}.

In the present work, we prove two results in the spirit of averaging theory, whose precise formulation we defer to section 3. First, for fixed initial condition $u_0$ the final density $u_\varepsilon(1, \cdot)$ (of eq. (2)) is uniformly approximated by $\overline{u}_\varepsilon(1, \cdot)$ (of eq. (3)) in leading order as $\varepsilon \to 0$; see theorem 3.1. This result follows directly from prior work by Krol [24] on the averaging method.
applied to time-periodic advection-diffusion equations, in which, by the way, the transformation to \textit{standard averaging form} is what is known as the transformation from Eulerian to Lagrangian coordinates in continuum mechanics. Second, we show that the largest (nontrivial) singular value/vector of the time-1 solution operator converge in a suitable sense to the largest (nontrivial) eigenvalue/eigenfunction of the averaged heat semigroup defined by eq. \textit{3}; see theorem \textit{3.5}.

In section \textit{4}, we work towards a geometric interpretation of our averaging results within the framework of the \textit{geometry of mixing}, as introduced in \textit{[22]}. This leads to a strengthened version of Froyland’s dynamic Cheeger inequality \textit{[10]}; we also draw a connection to the notion of \textit{material barriers to diffusive transport} as developed in \textit{[19] \textit{20]}. A by-product of our averaging result is an alternative and simplified proof of a low-diffusivity approximation result for the diffusive transport across boundaries of full-dimensional material subsets; see eq. \textit{(6)} of \textit{[19]} and corollary \textit{4.2}. Diffusive flux or material leakage has long been implicit in different approaches in finding so-called \textit{Lagrangian coherent structures (LCSs)}; see, for instance, \textit{[18] \textit{10] [16]} and \textit{17]} for a general review. It has been identified as the potentially unifying perspective on LCSs as \textit{diffusion barriers} in \textit{[22]}, and finally became the central object in the variational approach to \textit{material barriers to diffusive transport} in \textit{[19] \textit{20]}.

Our main motivation stems from transport and mixing problems as studied in physical oceanography and the atmospheric sciences. There, a typical problem is that presumably purely advective transport processes are observed only up to some finite scale. The effect of unresolved (small) scales is then often modelled via a weak diffusion with spatiotemporal inhomogeneity; see, for instance, \textit{[43]}. To address such problems, we treat advection-diffusion processes on (compact) smooth manifolds, and include general time-dependent, spatially inhomogeneous and anisotropic diffusion.

We would like to emphasize that we are interested in the details of spatial inhomogeneity of mixing, that would allow to explain significant differences in the mixing ability of different flow regions (transport/diffusion barriers vs. enhancers). This is in contrast to asymptotic or statistical information, like decay rates to equilibrium or spatially homogeneous effective diffusion tensors, typically obtained in homogenization theory; see, for instance, \textit{[8] [32]}. The advection-diffusion equation \textit{(2)} has been extensively studied in the literature. The time-periodic case has been investigated in the low-diffusivity limit by Krol \textit{[24]}, cf. also \textit{[38] [44]}. Time-periodic advection-diffusion prob-
lems have also been studied by Liu & Haller \[30\] from the Eulerian perspective. They developed mathematical theory for observed time-periodic patterns, *strange eigenmodes*, in periodically driven advection-diffusion processes. The time-periodic setting is closely related to this work, as one may construct a time-periodic advection-diffusion process from the finite-time setting by appending its adjoint equation (which is again of advection-diffusion type) and time-periodic extension. This resulting equation is then periodic with continuous coefficients. Such a time-periodic extension procedure has been employed recently by Froyland et al. \[11\] to the Fokker–Planck equation associated to a stochastic differential equation, in order to find approximations to Eulerian, spatiotemporal sets with small exponential escape rates of stochastic trajectories.

In the *autonomous* case—where $V$ in eq. (2) does not depend on time—semi-group theory may be applied, and many results have been obtained in this case. For instance, Kifer \[23, \text{Chapter III}\] studies asymptotics of spectra in the low-diffusivity limit; see also \[9, \text{Chapter 6.7}\]. Further autonomous, non-finite-time results were obtained in \[3, 6\].

\section{Diffusion-induced Lagrangian geometries}

This section is meant to be both a motivation and a gentle recall of the geometric interpretation of advection-diffusion processes, as developed in \[22, 19\]. For a recall of fundamental differential geometry concepts and notation used, we refer to appendix \[A\].

\subsection{Advection processes}

We recall some properties of advection processes that preserve mass; see also \[20\]. These generalize the notion of volume-preservation to vector fields whose flows do not preserve volume; this is done by constructing a time-dependent volume-form $\varphi$, the (fluid) mass form, that has precisely the property that it is preserved by the flow of a time-dependent velocity field $V$. Readers who are only interested in volume-preserving flows (with respect to a volume $\omega$, such as the usual Euclidean volume) may set $\varphi(t, x) \equiv \omega(x)$ everywhere below.

Recall that, in a fixed spatial frame, the evolution of the passively advected mass-form $\varphi$, with initial value $\omega$, is given by the advection equa-
tion/conservation law
\[
\partial_t \varrho = -d(V \varrho) = -\mathcal{L}_V \varrho \tag{4}
\]
\[
\varrho(0, \cdot) = \omega. \tag{5}
\]
Here, \( V \) is the sufficiently regular time-dependent fluid (bulk) velocity. We consider eq. (4) on an orientable smooth manifold \( M \), potentially with a sufficiently regular boundary, over a (finite subset of the) finite time interval \( I \subseteq \mathbb{R} \). For notational simplicity, we assume w.l.o.g. that \( I = [0, 1] \).

Equation (4) is well-known as a hyperbolic partial differential equation (PDE) that can be solved by the method of lines/characteristics. That is, consider the associated ordinary differential equation (ODE)
\[
\dot{x} = v(t, x), \quad x(0) = x_0 \in M, \tag{6}
\]
on \( I \). Let \( \phi \) denote the flow map associated to eq. (6), i.e., \( t \mapsto \phi_t(x_0) \) is the unique solution of eq. (6) satisfying the initial condition \( \phi_0(x_0) = x_0 \). The solutions of eq. (6) are then known as the characteristics of eq. (4), and each characteristic is also referred to as the trajectory of a (fluid) particle. Now, with the formula for time-derivatives along trajectories, \cite[Chapter V, Prop. 5.2]{27}, eq. (4) becomes
\[
(\tau \mapsto (\phi_t^0)^* \varrho)'(t) = (\phi_t^0)^*(\mathcal{L}_V \varrho) + (\phi_t^0)^* \partial_t \varrho = 0,
\]
where \( (\phi_t^0)^* \) is the pullback by \( \phi_t^0 \). For its push-forward \( (\phi_t^0)_* \) this implies
\[
\rho(t, \cdot) = (\phi_t^0)_* \omega. \tag{7}
\]
In particular, the mass-form along a trajectory is uniquely determined by its value anywhere on the trajectory.

In addition to the mass form \( \varrho \), we would like to model the evolution of a passive tracer that is advected by the fluid. This passive tracer is described by a (time-dependent) function/density \( u \) such that the volume form \( u \varrho \), integrated against any (measurable) \( S \subseteq M \), returns the total amount of the tracer in \( S \). Here,
\[
\partial_t u = -d(u(V)) = -\mathcal{L}_V u, \quad u(0, \cdot) = u_0. \tag{8}
\]
As above, along characteristics we obtain
\[
\frac{d}{dt} ((\phi_t^0)^* u) = (\phi_t^0)^*(\mathcal{L}_V u) + (\phi_t^0)^* \partial_t u = 0,
\]
therefore \( u \) is constant along characteristics.

One important consequence is the following intimate relation between the
PDE formulation of transport, eqs. (4) and (5), and its ODE formulation,
eq. (6). For any (measurable) set \( S \subset M \) and any (measurable) initial scalar
density \( u_0 \) one has

\[
\int_S u_0 \omega = \int_{\phi_0^t(S)} u(t, \cdot) \varrho(t, \cdot).
\]

(9)

Note that eq. (9) contains both the densities \( \varrho \) and \( u \) and the flow map \( \phi \),
which otherwise do not occur simultaneously in eqs. (4), (3) and (8).

Next, assume the scalar is confined to some set \( S \subset M \), e.g., \( u_0 = 1_S \).
Then, as a direct consequence of eq. (9), we have

\[
\int_{\phi_0^t(S)^c} u(t, \cdot) \varrho(t, \cdot) = 0,
\]

(10)

where \( A^c \) denotes the complement of \( A \) (in \( M \)). In other words, none of \( u \) leaks out of the spatiotemporal tube \( \bigcup_{t \in [0,1]} \phi_0^t(S) \). For later reference
and in accordance with continuum mechanics, we call any flow-invariant spa-
tiotemporal set \( S = \bigcup_{t \in [0,1]} \phi_0^t(S_0) \) a material set. So far, all considerations
were relative to some spatial or, synonymously, some Eulerian frame. Be-
sides different spatial frames, however, which can be related to different ob-
servers of the physical transport process, there exists the Lagrangian frame
that is related only to the characteristics/particles of the underly-
ing process. Changing from some Eulerian to the Lagrangian frame is essentially applying
the method of lines, where one additionally declares the initial conditions of
eq. (6), i.e., the particles, as coordinates, and represents all physical equations
w.r.t. those.

Briefly, in Lagrangian coordinates that are co-moving with the trajecto-
ries, eq. (4) becomes

\[
\partial_t \varrho = 0, \quad \partial_t u = 0;
\]

eq. (3) reads as

\[
\dot{x} = 0,
\]

(11)

and, as a consequence, the “flow map” is the identity map for all times. Equation (11) then states that no scalar mass leaks out of any material
set into the respective complementary material set; likewise the Lagrangian
advective transport through any material surface vanishes.
2.2 Advection-diffusion processes

In the following, we will consider advection-diffusion processes and re-inspect our above considerations in this framework.

Recall that, in a fixed spatial frame, the evolution of a weakly diffusive scalar, given by its density \( u \), passively advected by a (possibly compressible) fluid described with mass form \( \varrho \) is given by the advection-diffusion equation \[25, 41\]

\[
\begin{align*}
\partial_t u &= -\mathcal{L}_V u + \varepsilon \text{div}_\varrho(D du), \\
\partial_t \varrho &= -\mathcal{L}_V \varrho.
\end{align*}
\]

(12a) \hspace{1cm} (12b)

Here, \( \varepsilon > 0 \) is the diffusivity (or the inverse Péclet number in non-dimensionalized units), which is assumed to be small, and \( D: T^*M \to TM \) is a (possibly time-dependent) bundle morphism satisfying the following property: for given \( (t, x) \in \mathcal{I} \times M \) identify \( D \) with a bilinear form \( \tilde{g}_t^{-1} \) on \( T^*M \), then this bilinear form is symmetric and positive-definite. In particular, \( D \) gives rise to a dual metric, inducing a Riemannian metric \( \tilde{g}_t \) on \( M \). Viewed in this sense, \( D \) is a diffusion tensor field, modeling possibly (spatially and temporally) inhomogeneous, anisotropic diffusion. It is also necessary to impose suitable boundary conditions in the case that the manifold \( M \) has nonempty boundary \( \partial M \). We will focus on homogeneous boundary conditions of Dirichlet—, and for only some of our results, Neumann form.

Taking a closer look at eq. (12a), we directly recognize \( D du \) as the gradient of \( u \) w.r.t. the metric \( \tilde{g} \). As a consequence, the diffusion term can then be elegantly represented via the Laplace operators on the family of weighted manifolds \((M, \tilde{g}_t, \theta)\),

\[
\begin{align*}
\partial_t u &= -\mathcal{L}_V u + \varepsilon \Delta_{\tilde{g}_t} u, \\
\partial_t \varrho &= -\mathcal{L}_V \varrho,
\end{align*}
\]

(13a) \hspace{1cm} (13b)

where \( \tilde{\varrho}_t \) is the density of \( \varrho \) w.r.t. \( d\tilde{g}_t \), i.e., \( \tilde{\varrho}_t d\tilde{g}_t = \varrho(t, \cdot) \).

In stark contrast to the advection equations, eqs. (4), (12b) and (13b), the advection-diffusion equation, eq. (13a), is not amenable to the method of characteristics, and, therefore does not introduce a concept of particles, trajectories, or Lagrangian coordinates for the scalar \( u \). On the other hand, it is a singular perturbation of a hyperbolic PDE: namely eq. (13a) with \( \varepsilon = 0 \) as considered before. Hence, we may introduce Lagrangian coordinates based on the characteristics of its singular limit, or, equivalently, based on eq. (13b).
In these Lagrangian coordinates, the advective terms in eq. (13) vanish as in section 2.1 and we obtain from the well-known pullback transformation rules

\[ \partial_t u = \varepsilon \Delta_{\theta_t, g_t} u = \varepsilon \text{div}_\omega (g_t^{-1} du) , \]  

(14)

which is an evolution equation on the material manifold \(M\). Here, \(g_t := (\phi^t_0)*\tilde{g}_t\) is the diffusion-adapted pullback metric on \(M\) and \(\theta_t = (\phi^t_0)*\theta\). As a consequence of mass preservation, the volume form \(\omega\)—w.r.t. which we compute the divergence—does not depend on time. Henceforth, we write \(\text{div}\) without a subscript whenever we refer to \(\text{div}_\omega\). Moreover, let \(\Delta_t := \text{div}(g_t^{-1} du)\), then with this notation eq. (14) simplifies to

\[ \partial_t u = \varepsilon \Delta_t u . \]  

(15)

The lack of characteristics for the advection-diffusion equation has another, crucial consequence: given a (proper) material subset \(S \subset M\), the amount of \(u\) is no longer constant over time, or, equivalently

\[ T^t_t(S, u_0) := \int_S (u_0 - u(t, \cdot)) \omega \neq 0 . \]  

(16)

In simple words, there is leakage of \(u\) out of or into material sets. For given scalar fields and material subsets, the associated scalar leakage is an non-trivial and interesting quantity when regarded as a function of material sets \(S\), see [19, 20].

In Lagrangian coordinates, eq. (16) reads as

\[ T^t_t(S, u_0) = \int_S (u_0 - u(t, \cdot)) \omega . \]  

(17)

Furthermore, assuming for the moment that all involved functions are sufficiently smooth, differentiating with respect to \(t\) and applying the fundamental theorem of calculus yields

\[ T^t_0(S, u_0) = -\varepsilon \int_0^t \left( \int_S \Delta_\tau u(\tau, \cdot) \omega \right) d\tau . \]  

(18)

Heuristically, for very small \(\varepsilon\) we have \(u \approx u_0\) which suggests that

\[ \frac{T^t_0(S, u_0)}{\varepsilon} \approx -\int_0^t \int_S \Delta_\tau u_0 \omega \ d\tau =: \mathcal{L}^t_t(S, u_0) . \]  

(19)
Indeed, it was shown in [19, 20] for the case that $M \subset \mathbb{R}^n$ and homogeneous Neumann boundary condition that

$$T_0^t(S, u_0) = \varepsilon T_0^t(S, u_0) + o(\varepsilon), \quad \varepsilon \to 0.$$  (20)

In section 3 we develop an alternative proof of eq. (20) on compact manifolds with Dirichlet boundary; see corollary 3.4.

By Fubini’s theorem, we have that

$$T_0^t(S, u_0) = -\int_S \int_0^t \Delta \tau u_0 d\tau \omega = -\int_S \int_0^t \Delta \tau d\tau u_0 \omega.$$  

For $t = 1$, this suggests the definition

$$\Delta := \int_0^1 \Delta \tau dt.$$  (21)

This operator was recently introduced in [10, 12] and coined *dynamic Laplacian*. With this notation, eq. (20) reads as

$$T_0^1(S, u_0) = -\varepsilon \int_S \Delta u_0 \omega + o(\varepsilon),$$

and combines mathematical tools from recent work on material surfaces that extremize diffusive flux [19, 20] on the one hand, and dynamic isoperimetry [10, 12] on the other hand. A goal of this work is to investigate these connections rigorously; see corollary 4.2.

### 2.3 The geometry of mixing and diffusive permeability

Our study is centered around the *geometry of mixing* as induced by $\Delta$ and introduced in [22]. There, it was observed that $\Delta$ is the Laplace operator of a specific *weighted* (Riemannian) manifold. With the above notation, let us define

$$\bar{g} = \left( \int_x g_t^{-1} dt \right)^{-1}.$$  (22)

**Lemma 2.1 ([22 Prop. 3])**. The dynamic Laplacian $\Delta$ is the Laplace operator associated to the weighted manifold $(M, \bar{g}, \theta)$, where $\theta d\bar{g} = \omega$. 

9
As in [22], we refer to the material manifold $M$, equipped with the metric $\bar{g}$ and density $\theta$ as the geometry of mixing. This, together with appendix A, shows that the geometry of mixing is constructed to have the following elegant properties: (i) volume/fluid mass is given by $\omega$, the differential form preserved by the flow, and (ii) diffusion is given by averaged pullback diffusion tensors as featured in the dynamic Laplacian. It was further observed in [22] that
\[ \partial_t \bar{u} = \varepsilon \Delta \bar{u}, \quad \bar{u}(0, \cdot) = u_0, \tag{23} \]
is an averaged (cf. [37, 32]) form of eq. (15). It was conjectured that eq. (23) approximates eq. (15) in the vanishing diffusivity limit, leaving open the concrete nature of the approximation and the required assumptions. We prove this in section 3 building on a similar result in the classic, i.e., time-periodic, averaging context [24]. A by-product of our averaging result is a new proof of eq. (20), as mentioned above. We also prove that the convergence extends to singular values/vectors, addressing an open question from [22].

To summarize the previous sections: eq. (20) shows that in leading order as $\varepsilon \to 0$, the diffusive transport out of a material set is determined by
\[ \mathcal{T}_0^1(S, u_0) = -\int_S \Delta u_0 \omega. \]
By the divergence theorem, we have
\[ \mathcal{T}_0^1(S, u_0) = -\int_{\partial S} d\nu_0(\bar{\nu}) d\bar{A}, \tag{24} \]
where $\bar{\nu}$ is the outward-pointing $\bar{g}$-unit normal vector field on $\partial S$ and $d\bar{A} = \theta dA_{\bar{g}}$ is the induced area form on $\partial S$ in the geometry of mixing.

Of course, $\mathcal{T}_0^1(S, u_0)$ could be represented similarly in other weighted geometries on $M$: choose any metric $\tilde{g}$, compute the density $\tilde{\theta}$ of the fluid mass relative to the induced volume $dg$, denote the induced area form and the $\tilde{g}$-unit normal vector field on $\partial S$ by $d\tilde{A}$ and $\tilde{\nu}$, respectively, then one obtains analogously to eq. (24)
\[ \mathcal{T}_0^1(S, u_0) = -\int_{\partial S} du_0(H\tilde{\nu}) d\tilde{A}, \tag{25} \]
where $H = \bar{g}^{-1}\tilde{g}$ is a tangent space isomorphism; cf. also section 4.1 below and references [19, 20], in which $\mathcal{T}_0^1(S, u_0)$ is represented in the usual Euclidean/physical geometry, and $H$ is coined the transport tensor (denoted by
...T^i_{t_0}$ there). It is exactly the absence of any additional tensor in eq. (24) that makes arguably the geometry of mixing the “best-adapted” or “most natural” geometry in which to look at leading-order diffusive flux in Lagrangian coordinates.

Equation (24) emphasizes that, in leading order, the diffusive transport $T^1_0(S, u_0)$ out of a material set $S$ depends on (i) the differential/gradient of the initial concentration $u_0$ along $\partial S$, and (ii) properties of the geometry of mixing via the surface measure $d\mathcal{A}$ and unit normal vector field $\mathbf{v}$. As argued in [22], $d\mathcal{A}$ is particularly interesting as an intrinsic measure of the “diffusive permeability” of the material boundary $\partial S$. In many physical applications, it is of great interest to diagnose the mixing structure of an advection-dominated transport process independent of any specific scalar quantity; cf. the discussion in [19].

3 Finite-time averaging of the advection-diffusion equation

We now show that in the setting of the advection-diffusion equation, the diffusion process induced by the dynamic Laplacian approximates the diffusion of the advection-diffusion equation in Lagrangian coordinates, in the limit of vanishing diffusivity.

In this section, we restrict to those $M$ that are compact manifolds whose boundary, if it is nonempty, is smooth. The proof can be extended to other classes of manifolds also, provided there is a suitable maximum principle.

3.1 Uniform convergence

Let $u_\varepsilon : M \times [0, 1] \to \mathbb{R}$ solve the advection-diffusion equation in Lagrangian coordinates for diffusivity $\varepsilon$ with initial condition $u_0 : M \to \mathbb{R}$ with—if there is a boundary—homogeneous Dirichlet or Neumann boundary conditions. Thus, in the interior of $M$, $u_\varepsilon$ satisfies

$$\partial_t u_\varepsilon = \varepsilon \Delta_t u_\varepsilon.$$  

\footnote{Given a metric for each time $t \in [0, 1]$, we also have a $g_t$ unit-normal vector $\nu_t$ field on $\partial M$ for each $t \in [0, 1]$. The natural homogeneous Neumann condition is thus $du_\varepsilon(t, \cdot)(\nu_t) = 0$ for each $t \in [0, 1]$ on $\partial M$.}
Similarly, let \( \overline{u}_\varepsilon : M \times [0,1] \to \mathbb{R} \) be the solution of the heat flow generated by the dynamic Laplacian \( \overline{\Delta} \), with initial condition \( u_0 \) and diffusivity \( \varepsilon \), i.e.,

\[
\partial_t \overline{u}_\varepsilon = \varepsilon \overline{\Delta} \overline{u}_\varepsilon
\]

and—if there is a boundary—homogeneous Dirichlet or Neumann\(^2\) boundary conditions of the corresponding type. We will focus mainly on the case of Dirichlet boundary, and refer to appendix\(^\text{D}\) for a recall of results regarding existence, uniqueness and regularity of solutions. We expect analogous existence, uniqueness and regularity results to hold in the Neumann case on manifolds, but could not find a reference.

**Definition.** Depending on the boundary condition used, we call an initial value \( u_0 \in C^\infty(M) \) admissible if (i) \( u_0 \) is compactly supported in the interior of \( M \) (Dirichlet case), (ii) if \( u_0 \) is constant in a neighborhood of the spatial boundary \( \partial M \) (Neumann case).

This definition is motivated by the fact that the time-dependent parabolic eqs. \((26)\) and \((27)\) may not be smooth at \( t = 0 \) if the initial value \( u_0 \) does not satisfy certain compatibility conditions at the boundary; see [7, Sect. 7.1, Thm. 6]; cf. also [24]. These may differ between eq. \((26)\) and eq. \((27)\). Our definition of admissibility guarantees that the compatibility conditions of both the time-dependent and the averaged equations are satisfied simultaneously.

**Theorem 3.1.** With \( u_\varepsilon \) and \( \overline{u}_\varepsilon \) as above, let \( u_0 \) be an admissible initial value. Then

\[
u_\varepsilon(1, x) = \overline{u}_\varepsilon(1, x) + O(\varepsilon^2), \quad \varepsilon \to 0,
\]

uniformly in \( x \).

**Proof.** The proof is a simplification of the one given in [24]. Let

\[
\tilde{u}_\varepsilon = u_0 + \varepsilon \int_0^t \Delta \tau u_0 d\tau.
\]

We start with the Dirichlet boundary condition case. Let \( \mathcal{L}_\varepsilon \coloneqq \partial_t - \varepsilon \Delta_t \) and observe that \( \mathcal{L}_\varepsilon \tilde{u}_\varepsilon = -\varepsilon^2 \int_0^t \Delta_t \Delta_s u_0 ds \). As \( u_0 \) is smooth and \( M \) compact,

\[
C \coloneqq \sup_{t \in [0,1]} \left\| \left( \int_0^t \Delta_t \Delta_s u_0 d\tau \right) \right\|_{L^\infty(M)} < \infty.
\]

\(^2\)Here, we require \( d\overline{u}_\varepsilon(t, \cdot)(\overline{\nu}) = 0 \) independent of \( t \), where \( \overline{\nu} \) is unit normal field for \( \overline{\eta} \) on \( \partial M \).
By definition, $L^\varepsilon u^\varepsilon = 0$. Thus, $L^\varepsilon (\tilde{u}^\varepsilon - u^\varepsilon - C\varepsilon^2 t) \leq 0$; $u^\varepsilon$ and $\tilde{u}^\varepsilon$ agree at $t = 0$; and (by the admissibility of the initial value) both satisfy Dirichlet boundary conditions. The weak maximum principle (lemma D.4) therefore yields that

$$\max_{[0,1] \times M} (\tilde{u}^\varepsilon - u^\varepsilon - C\varepsilon^2 t) = \max_{[0,1] \times \partial M \cup \{0\} \times M} (-C\varepsilon^2 t) \leq 0.$$  

As a consequence, we have $\max_{[0,1] \times M} (\tilde{u}^\varepsilon - u^\varepsilon - C\varepsilon^2 t) \leq C\varepsilon^2$.

One may prove $(u^\varepsilon - \tilde{u}^\varepsilon) \leq C\varepsilon^2$ along the same lines. Thus, $\|u^\varepsilon - \tilde{u}^\varepsilon\|_{L^\infty([0,1] \times M)} = O(\varepsilon^2)$. For $t = 1$, this implies the uniform expansion

$$u^\varepsilon(1,\cdot) = \tilde{u}^\varepsilon(1,\cdot) + O(\varepsilon^2) = u_0 + \varepsilon \Delta u_0 + O(\varepsilon^2).$$  

The right-hand side coincides up to second order with the expansion of $\pi_\varepsilon(1,\cdot)(= \exp(\varepsilon \Delta)u_0)$ which yields the claim.

For Neumann boundary conditions, the proof goes along the same lines, where the weak maximum principle must be augmented with the parabolic Hopf boundary point lemma (cf. [35, Chapter 3, Thm. 6]) to ensure that a strict maximum cannot be achieved at positive time.

We restate eq. (29) for further reference, and also observe that it can be interpreted as the time-continuous generalization of [10, Thm. 5.1].

**Corollary 3.2.** Under the assumptions of theorem 3.1,

$$u^\varepsilon(1,x) = u_0(x) + \varepsilon \Delta u_0(x) + O(\varepsilon^2),$$  

uniformly in $x$.

**Corollary 3.3.** Under the assumptions of theorem 3.1,

$$u^\varepsilon(1,\cdot) = u_0 + \varepsilon \Delta u_0 + O(\varepsilon^2),$$  

in $L^p(M,\omega)$ for all $p \in [1,\infty]$.

**Proof.** For $p = \infty$, our claim corresponds to corollary 3.2. For $p \in [1,\infty)$, the natural injection $L^\infty(M,\omega) \hookrightarrow L^p(M,\omega)$ is well-defined and continuous since $\omega(M)$ is finite, which yields the claim.

**Corollary 3.4.** Under the assumptions of theorem 3.1,

$$T^1_0(S,u_0) = \int_S u_0(x) \omega - \int_S u^\varepsilon(x,1) \omega = -\varepsilon \int_S \Delta u_0 \omega + O(\varepsilon^2).$$

**Proof.** This follows by integrating eq. (29) over $S$. 


3.2 Convergence of singular values

We denote by $P_\varepsilon$ and $P_\varepsilon(t)$ the time-$t$ solution operators of eqs. (26) and (27), respectively, i.e., $u_\varepsilon(t, \cdot) = P_\varepsilon(t) u_0$ and $\overline{u}_\varepsilon(t, \cdot) = \overline{P}_\varepsilon u_0$. To reduce notational clutter, we write $P_\varepsilon := P_\varepsilon^1$ and $\overline{P}_\varepsilon := \overline{P}_1$. We only treat homogeneous Dirichlet boundary in this section.

The previous section dealt with the relationship of $P_\varepsilon$ and $\overline{P}_\varepsilon$ in the limit $\varepsilon \to 0$. In particular, theorem 3.1, with this notation, is

$$\| (P_\varepsilon - \overline{P}_\varepsilon) u_0 \|_{L^\infty(M)} = O(\varepsilon^2) \quad \text{for all } u_0 \in C_c^\infty(\tilde{M}). \quad (32)$$

Recall that as $P_\varepsilon$ is compact (cf. appendix D), the first singular value of $P_\varepsilon$ is given by the operator norm of $P_\varepsilon : L^2(M, \omega) \to L^2(M, \omega)$. By lemma C.2, $P_\varepsilon$ is a contraction on $L^2(M, \omega)$, hence $\| P_\varepsilon \| \leq 1$.

If $M$ is boundaryless, then $P_\varepsilon 1_M = 1_M$, and, as a consequence, $\| P_\varepsilon \| = 1$ for any $\varepsilon > 0$. Since the subspace of constant functions is a trivial invariant subspace, we restrict the domain of $P_\varepsilon$ to its orthogonal complement, the space of mean-free functions. If $M$ has a boundary, we consider $P_\varepsilon$ with its domain the entire $L^2(M, \omega)$.

With these preparations, we denote the largest nontrivial singular value by $\sigma_\varepsilon$, and a corresponding (normalized) left singular vector by $v_\varepsilon$, i.e.,

$$\| v_\varepsilon \|_{L^2(M, \omega)} = 1, \quad \| P_\varepsilon \| = \| P_\varepsilon v_\varepsilon \|_{L^2(M, \omega)} = \sigma_\varepsilon.$$

For the sake of brevity, let $\| \cdot \|_0 := \| \cdot \|_{L^2(M, \omega)}$ and $\langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle_{L^2(M, \omega)}$. Equation (32) suggests the conjecture that

$$\| (P_\varepsilon - \overline{P}_\varepsilon) u_0 \|_{L^\infty(M)} = O(\varepsilon^2)$$

(33)

where the norm is the operator norm. By the spectral mapping theorem (see, for instance, [33 Sect. 1, Thm. 2.4(c)]),

$$\| \overline{P}_\varepsilon \| = e^{\varepsilon \overline{\lambda}} = 1 + \varepsilon \overline{\lambda} + o(\varepsilon),$$

where $\overline{\lambda} < 0$ is the largest, i.e., smallest in absolute value, nontrivial eigenvalue of the dynamic Laplacian. Thus, eq. (33) is equivalent to

$$\| P_\varepsilon \| = 1 + \varepsilon \overline{\lambda} + o(\varepsilon). \quad (34)$$

It can be interpreted as an expansion of the first singular value of $P_\varepsilon$ in $\varepsilon$, in analogy to the expansion obtained in corollary 3.2. We will prove the following equivalent statement.
Theorem 3.5. With the above notation and assuming a Dirichlet boundary, one has
\[
\lim_{\varepsilon \to 0} \frac{\sigma^\varepsilon - 1}{\varepsilon} = \overline{\lambda}.
\] (35)

Proof. We split the proof into several steps.

Step 1: We start by proving the lower bound
\[
\liminf_{\varepsilon \to 0} \frac{\sigma^\varepsilon - 1}{\varepsilon} \geq \overline{\lambda}.
\] (36)
To this end, the operator-norm definition of \(\sigma^\varepsilon\) shows that
\(\sigma^\varepsilon \geq \|P^\varepsilon u\|_0\) for all \(u \in C^\infty_c(\mathbb{R})\) in the domain of \(P^\varepsilon\) which have \(\|u\|_0 = 1\). Applying corollary 3.3 to such \(u\) yields \(P^\varepsilon u = u + \varepsilon \Delta u + o(\varepsilon)\) in \(L^2(M, \omega)\). Therefore, we also have \(\|P^\varepsilon u\|_0^2 = \|u\|_0^2 + 2\varepsilon \langle u, \Delta u \rangle + o(\varepsilon)\). Since \(\|u\|_0 = 1\), we obtain
\[
\liminf_{\varepsilon \to 0} \frac{(\sigma^\varepsilon)^2 - 1}{\varepsilon} \geq 2\langle u, \Delta u \rangle.
\] The right hand side can be made arbitrarily close to \(2\overline{\lambda}\), which shows
\[
\liminf_{\varepsilon \to 0} \frac{(\sigma^\varepsilon)^2 - 1}{\varepsilon} \geq 2\overline{\lambda}.
\] (37)
From lemma C.2 it follows that \(0 \leq \sigma^\varepsilon \leq 1\). Thus \(\sigma^\varepsilon \to 1\) for \(\varepsilon \to 0\). Finally, as \((\sigma^\varepsilon)^2 - 1 = (\sigma^\varepsilon - 1)(\sigma^\varepsilon + 1)\), we deduce eq. (36) from eq. (37).

Step 2: We now prove the upper bound,
\[
\limsup_{\varepsilon \to 0} \frac{\sigma^\varepsilon - 1}{\varepsilon} \leq \overline{\lambda},
\] (38)
which is somewhat more involved. It is based on the identity:
\[
\frac{(\sigma^\varepsilon - 1)(\sigma^\varepsilon + 1)}{\varepsilon} = \frac{\|P^\varepsilon v^\varepsilon\|_0^2 - \|v^\varepsilon\|_0^2}{\varepsilon} = 2 \int_0^1 \langle v^\varepsilon(t), \Delta_t v^\varepsilon(t) \rangle_0 \, dt,
\] (39)
where the first equality is satisfied as \(v^\varepsilon\) is first non-trivial singular vector, \(v^\varepsilon(t) := P^\varepsilon_t(v^\varepsilon)\), and the second equality is a direct consequence of the fundamental theorem of calculus applied to \(f^\varepsilon(t) := \langle v^\varepsilon(t), v^\varepsilon(t) \rangle_0\). To connect eq. (39) to the theory of elliptic partial differential equations, we introduce the bilinear form
\[
a_t(u, w) := -\langle u, \Delta_t w \rangle,
\]
defined (by unique continuous extension) for \(u, w \in H^1_0(M, g, \omega)\), where \(g\) is an arbitrary fixed metric (e.g. \(g_0\)) used to measure lengths and angles. The
Sobolev space \( H^1_0(M, g, \omega) \subset L^2(M, \omega) \) is defined as the Hilbert space with norm \( \| \cdot \|_1^2 := \| \cdot \|_0^2 + \| \cdot \|_1^2 \), here \( \| \cdot \|_1 \) is induced by the bilinear form

\[
\langle u, v \rangle_1 := \int_M g(\text{grad}_g u, \text{grad}_g v) \omega
\]

using the metric \( g \) and volume-form \( \omega \). This norm is equivalent to the usual \( H^1(M, g, dg) \) Sobolev norm since \( \omega \) is smooth and nonvanishing on the compact manifold \( M \). As usual, \( H^1_0(M, g, \omega) \) is defined as the completion of \( C_\infty^c(\bar{M}) \) w.r.t. the norm \( \| \cdot \|_1 \). We have shown in Step 1 that \( \sigma^\varepsilon \to 1 \) for \( \varepsilon \to 0 \). As a consequence, eq. (39) is equivalent to

\[
\beta := \liminf_{\varepsilon \to 0} \frac{1 - \sigma^\varepsilon}{\varepsilon} = \liminf_{\varepsilon \to 0} \int_0^1 a_t(v^\varepsilon(t), v^\varepsilon(t)) \, dt.
\]

Equation (41) is the negative of the left hand side of eq. (38). The bilinear form \( a_t(\cdot, \cdot) \) on \( H^1_0(M, g, \omega) \) is positive, continuous and coercive (cf. lemma C.1), and thus induces a norm \( \| \cdot \|_{a_t} \) that is equivalent to \( \| \cdot \|_1 \). In particular, \( \| \cdot \|_{a_t} \)-continuous functionals are \( \| \cdot \|_1 \)-continuous functionals and vice versa. Therefore, the weak topologies for these norms coincide. The Banach-Steinhaus theorem, with the norm \( \| \cdot \|_{a_t} \), thus states that if \( u_n \to u \) weakly in \( H^1_0(M, g, \omega) \), then

\[
a_t(u, u) \leq \liminf_{n \to 0} a_t(u_n, u_n).
\]

We are now in a position to prove eq. (35) by contradiction. To do so, we will employ a construction similar to the “direct method” from the calculus of variations; cf. [14]. To this end, we take a null sequence \( (\varepsilon_n)_{n \in \mathbb{N}} \) for which \( \int_0^1 a_t(v^{\varepsilon_n}(t), v^{\varepsilon_n}(t)) \, dt \) converges to \( \beta \). Assume, for the sake of contradiction, that

\[
\beta = \lim_{n \to \infty} \int_0^1 a_t(v^{\varepsilon_n}(t), v^{\varepsilon_n}(t)) < -\lambda.
\]

We will use a claim whose proof we defer:

**Claim.** There exist \( v \in H^1_0(M, g, \omega) \) with \( \| v \|_0 = 1 \) and a subsequence of \( (\varepsilon_n) \), for simplicity again denoted by \( (\varepsilon_n) \), for which the sequences \( (v^{\varepsilon_n}(t))_n \) converge weakly in \( H^1_0(M, g, \omega) \) to \( v \) for every \( t \in [0, 1] \).
For this specific $v$, Fatou’s lemma and eq. (42) imply that
\[
\int_0^1 a_t(v, v) \, dt \leq \liminf_{n \to \infty} \int_0^1 a_t(v^{\varepsilon_n}(t), v^{\varepsilon_n}(t)) \, dt = \beta < -\lambda. \tag{44}
\]
The left hand side, in a weak sense, is equal to $-\langle v, \Delta v \rangle_0$, the bilinear form associated to the weak form of the dynamic Laplacian. It is well-known that the Rayleigh quotient $v \mapsto -\langle v, \Delta v \rangle_0 / \langle v, v \rangle_0$ is minimized by $-\lambda$ on $H^1_0(M, g, \omega)$; see, for instance, [7, Sect. 6.5, Thm. 2]. With $\|v\|_0 = 1$, eq. (44) states that $v$’s Rayleigh quotient is strictly lower, hence a contradiction.

It follows that $\beta \geq -\lambda$, we conclude using eq. (41) that
\[
\liminf_{\varepsilon \to 0} \frac{1 - \sigma}{\varepsilon} \geq -\lambda,
\]
which proves Step 2.

**Step 3, proof of claim:** Our proof requires that there exists $\varepsilon_0 > 0$ such that $C := \sup_{0 < \tau < \varepsilon_0, t \in [0, 1]}|v^{\varepsilon}(t)|_1$ is finite, this part is done in appendix C.

Assuming that $C$ is finite, the Rellich-Kondrachev theorem [40, Sect. 4, Prop. 3.4], states that $v^{\varepsilon_n}(0) \to v$ in $L^2(M, \omega)$ (up to passing to a subsequence if necessary), and therefore $\|v\|_0 = 1$. After again passing to a subsequence if necessary, we may assume $v^{\varepsilon_n}(0) \to v \in H^1_0(M, g, \omega)$ weakly in $H^1(M, g, \omega)$ by the (sequential) Banach-Alaoglu theorem; see, for instance, [5, Thm. 3.2.1].

To show that this limit is attained by $v^{\varepsilon}(t)$ also for $t \neq 0$ as $\varepsilon \to 0$, we differentiate $h^{\varepsilon}(t) := \|v^{\varepsilon}(t) - v^{\varepsilon}(0)\|_0^2$, and apply the fundamental theorem of calculus to yield
\[
\|v^{\varepsilon}(t) - v^{\varepsilon}(0)\|_0^2 = 2\varepsilon \left| \int_0^t a_\tau(v^{\varepsilon}(\tau), v^{\varepsilon}(\tau) - v^{\varepsilon}) \, d\tau \right| \leq 4\varepsilon C^2 C',
\]
where
\[
C' := \sup_{t \in [0, 1], u, w \in H^0_0(M, g, \omega)} |a_\tau(u, w)|/(|u|_1|w|_1) < \infty;
\]
see lemma C.1. We may apply the fundamental theorem of calculus due to the absolute continuity ensured by [7 Sect. 5.9, Thm. 3], see also appendix D. As $v^{\varepsilon_n}(0) \to v$, it follows that $v^{\varepsilon_n}(t) \to v$ in $L^2(M, \omega)$ for all $t \in [0, 1]$. In particular, $v$ is the only $L^2(M, g, \omega)$ accumulation point in the set $F := \{v^{\varepsilon_n}(t)\}_{n \in \mathbb{N}, t \in [0, 1]}$, therefore also the only weak $H^1(M, g, \omega)$ accumulation point. The sequential Banach-Alaoglu theorem guarantees that the
set $F$ is weakly sequentially compact in $H^1(M, g, \omega)$. Combining this with the fact that $v$ is its only accumulation point yields weak convergence of $v^\varepsilon(t) \to v$ in $H^1(M, g, \omega)$ for all $t \in [0, 1]$. This finishes the proof of theorem 3.5.

3.2.1 Convergence of eigenvectors

The proof of theorem 3.5 also shows that the corresponding eigenvectors must convergence in $L^2$ (in fact, even weakly in $H^1$). Since, in general, the singular vectors of $P^\varepsilon$ satisfy different compatibility conditions at the boundary to those of $\overline{P}^\varepsilon$, this is somewhat surprising.

4 Diffusive transport and surface area

In this section, we look at properties of the surface area form $d\mathcal{A}$ in the geometry of mixing, and how it relates to other, similar, area forms obtained from different types of averaging.

In the setting of the advection-diffusion equation, we have assumed that the time set $\mathcal{I}$ is the unit interval equipped with the Lebesgue measure. For the purpose of this section (only), we may weaken this assumption towards $(\mathcal{I}, dt)$ being a probability space, such as a finite set of numbers equipped with the normalized counting measure, or a compact interval equipped with the Lebesgue measure normalized by the interval’s length. By the term surface, we refer to a smooth, oriented, embedded (codimension-1) submanifold.

4.1 Surface area in the geometry of mixing

Let $g$ be any metric on the material manifold $M$, we call $g$ the “reference metric”. This could be, for instance, some “universal” spatial metric (the way we measure lengths and volume), defined on $M$, or any of the diffusion-adapted metrics from $(g_t)_{t \in \mathcal{I}}$. The choice of $g$ is in analogy to the choice of local coordinates in differential geometry – we will derive expressions for various quantities in terms of $g$. The metric $g$ is in no way required to be related to the physical transport process under consideration. In particular, if $g$ is the Euclidean metric in some coordinate chart, we obtain coordinate representations in that chart.
As before, define a mass-induced surface area form \( dA \) on any surface \( \Gamma \subset M \) via \( \iota_\nu \omega \), where \( \nu \) is the \( g \)-unit normal vector field\(^3\). With this notation, \( \overline{\nu} := \overline{g}^{-1}g \) and \( C_t := g_t^{-1}g \) are tangent bundle isomorphisms, i.e., \( \overline{\nu}, C_t : TM \to TM \). Then
\[
\overline{\nu} = \left( \int_I g_t^{-1} \, dt \right) g = \int_I C_t \, dt .
\]
For \( v \in T_x M \subset TM \), we have that
\[
\|\overline{\nu}v\|^2 = \overline{g}(\overline{\nu}v, \overline{\nu}v) = \left[ \overline{g}(\overline{\nu}v) \right](\overline{\nu}v) = \left[ \overline{g} \overline{g}^{-1}g \right](\overline{\nu}v) = g(v)(\overline{\nu}v) = g(v, \overline{\nu}v) .
\]

Denote by \( \nu_t, t \in [0, 1] \), and \( \nu \) the unit normal vector fields w.r.t. \( g_t \) and \( \overline{g} \) on \( \Gamma \). As with the reference metric, we define
\[
dA_t := \iota_{\nu_t} \omega , \quad d\overline{A} := \iota_{\nu} \omega . \tag{45}
\]
In other words, corresponding to the three types of metrics—reference \( g \), time-dependent \( (g_t)_t \), and time-averaged \( \overline{g} \)—we derive three area forms \( (dA, dA_t, and d\overline{A}) \) from the mass form.

We now show how to relate to each other area form that are induced by the mass form via different metrics on a surface \( \Gamma \).

**Lemma 4.1.** Let \( g, \overline{g} \) be metrics on \( M \). Let \( \Gamma \) be a surface in \( M \), \( \overline{C} := \overline{g}^{-1}g \), and \( \nu \) and \( \overline{\nu} \) their respective (consistently oriented) unit normal vector fields on \( \Gamma \). Then
\[
\iota_{\overline{\nu}} \omega = g(\nu, \overline{\nu}) \iota_{\nu} \omega = g(\nu, \overline{C}\nu)^{1/2} \iota_{\nu} \omega .
\]

**Proof.** The first equality is trivial, because we may represent \( \overline{\nu} \) as the linear combination of \( g(\nu, \overline{\nu})\nu \) and its projection onto \( T_p \Gamma \). But the latter does not contribute to the result. It remains to show \( g(\nu, \overline{\nu}) = g(\nu, \overline{C}\nu)^{1/2} \). To this end, we show that \( \overline{\nu} = g(\nu, \overline{C}\nu)^{-1/2} \overline{C}\nu \). First, observe that \( \overline{C}\nu \) is \( \overline{g} \)-normal to \( T_p \Gamma \), since for any \( v \in T_p \Gamma \) we have
\[
\overline{g}(\overline{C}\nu, v) = (\overline{g}\overline{g}^{-1}g\nu)(v) = g(\nu, v) = 0 .
\]

\(^3\)If \( \Gamma \) is the boundary of a full-dimensional submanifold, we take the outward-pointing unit normal
Now, $\|\tilde{C}\nu\|^2_g = \tilde{g}\left(\tilde{C}\nu, \tilde{C}\nu\right) = g\left(\nu, \tilde{C}\nu\right)$, which means that $g(\nu, \tilde{C}\nu)^{-1/2} \tilde{C}\nu$ is also $\tilde{g}$-normalized. Finally, $g(\nu, \tilde{C}\nu)^{-1/2} \tilde{C}\nu = \tilde{\nu}$ necessarily as they share the same orientation: $g(\nu, \tilde{\nu}) = \tilde{g}(\tilde{C}\nu, \tilde{C}\nu) > 0$.  

Applying lemma 4.1 to the metrics $g$ and $g_t$, we obtain

$$dA_t = \sqrt{g(\nu, C_t\nu)} dA,$$

(46)

and for $g$ and $\tilde{g}$,

$$dA = \sqrt{g (\nu, C\nu)} dA.$$

(47)

By combining lemma 4.1 with corollary 3.4, we obtain the approximation result for accumulated diffusive flux through boundaries of full-dimensional material submanifolds.

**Corollary 4.2** ([19, eq. (6)]). Let $S \subset M$ be a full-dimensional submanifold with smooth boundary, and $u_0$ an admissible initial condition. Then

$$T_0^1(S, u_0) = -\varepsilon \int_0^1 \int_{\partial S} du_0(C_t\nu) dA dt + O(\varepsilon^2).$$

**Proof.** We calculate with lemma 4.1

$$\varepsilon \int_0^1 \int_{\partial S} du_0(C_t\nu) dA dt = \varepsilon \int_0^1 \int_{\partial S} du_0(\nu_t) g(\nu, C_t\nu)^{1/2} dA dt,$n

$$= \varepsilon \int_0^1 \int_{\partial S} du_0(\nu_t) dA_t dt,$$

and conclude with the divergence theorem and Fubini’s theorem

$$= \varepsilon \int_0^1 \int_S \Delta_t u_0 \omega dt = \varepsilon \int_S \Delta u_0 \omega .$$

The claim now follows from corollary 3.4.  

---

4Recall that $C_t = g_t^{-1}g$, where $g$ is here the Euclidean metric on the flat state space, and corresponds to the transport tensor in [19, 20]; and $\nu$ is the outward-pointing $g$-unit normal vector field on $\partial S$. In [19], material surfaces are considered that are not necessarily the boundary of a full-dimensional set. In case they are, [19, Eq. (6)] measures the influx, which explains the opposite sign to ours.
Using the transformation rules for normal vectors and surface forms from lemma [4.1] we can find the representation of (the negative of) the leading-order total diffusive transport through a material boundary w.r.t. an arbitrary weighted manifold structure on the material manifold \((M, ˜g, \omega d ˜g)\):

\[
-\frac{1}{\varepsilon} T_0^1(S, u_0) = -\mathbf{T}_0^1(S, u_0) = \int_0^1 \int_{\partial S} du_0(C_i \nu) \, dA \, dt = \int_{\partial S} du_0(\mathbf{C} \nu) \, dA = \int_{\partial S} du_0(\mathbf{C} \mathbf{C}^{-1} \tilde{\nu}) \, d\tilde{A} = \int_{\partial S} \tilde{g} (\text{grad}_g u_0, H \tilde{\nu}) \, d\tilde{A},
\]

where \(H = \bar{g}^{-1} \hat{g}\) as claimed in eq. (25).

### 4.2 Relations to other dynamic surface areas

On a surface \(\Gamma \subset M\) with \(g\)-unit normal vector field \(\nu\), we compute

\[
d\bar{A} = \sqrt{g(\nu, \mathbf{C} \nu)} \, dA = g(\nu, \left(\int_I C_i \, dt \nu\right) \right)^{1/2} \, dA = \left(\int_I g(\nu, C_i \nu) \, dt\right)^{1/2} \, dA.
\]

Plugging in eq. (46) gives:

\[
d\bar{A} = \left(\int_I \left(\frac{dA_t}{dA}\right)^2 \, dt\right)^{1/2} \, dA.
\]

This shows that the density of the surface element in the geometry of mixing w.r.t. \(dA\) is an \(L^2\)-average of the densities of the time-\(t\) surface elements. Relating this with the interpretation in terms of diffusive transport, this is consistent with the observation made in [13, Sect. III.A], “that the rate of mass transport from an element of a material interface is related to the square of the relative change of the surface area”.

**Proposition 4.3** (Comparison to averages of surface areas). Let \(\Gamma\) be a compact surface, and \(d\bar{A}(\Gamma)\) and \(dA_t(\Gamma)\) be its surface area as measured by \(d\bar{A}\) and \(dA_t\), respectively; i.e.,

\[
d\bar{A}(\Gamma) = \int_{\Gamma} d\bar{A}, \quad dA_t(\Gamma) = \int_{\Gamma} dA_t.
\]
Then
\[ d\overline{A}(\Gamma) \geq \left( \int_t I d^2 (\Gamma) dt \right)^{\frac{1}{2}} \geq \int_t I dA_t (\Gamma) dt =: d\overline{A}_t (\Gamma). \]

**Proof.** For convenience, we denote \( \xi(t, p) = \frac{dA_t(p)}{dA} \) and compute
\[
d\overline{A}(\Gamma) = \int_\Gamma \left( \int_t I (\xi(t, p))^2 dt \right)^{1/2} dA(p) = \sqrt{\int_\Gamma \| \xi(\cdot, p) \|_{L^2(\mathcal{I})} dA(p)} \]
\[
\geq \sqrt{\int_\Gamma \| \xi(\cdot, p) \|_{L^2(\mathcal{I})}^2 dt} = \left( \int_I \left( \int_t I \xi(t, p) dA(p) \right)^2 dt \right)^{1/2},
\]
where eq. (49) is the triangle inequality for Banach-space valued maps (e.g. for the Bochner integral see [26, Sect. VI]). The second claimed inequality is a direct consequence of Jensen’s inequality applied to the expression in eq. (49). \( \square \)

Notably, \( d\overline{A}_t(\Gamma) \) appears in the definition of the **dynamic Cheeger constant** in [10 Eq. (4)]. Moreover, by means of the Cheeger inequality for weighted manifolds, proposition B.1 in appendix B, we may strengthen the dynamic Cheeger inequality [10 Thm. 3.2], where it was shown that
\[ \inf_{\Gamma} \frac{dA_t(\Gamma)}{\min\{\omega(M_1), \omega(M_2)\}} \leq 2\sqrt{-\lambda_2}, \]
for the case that \( \omega = dg_t \) for all \( t \in \mathcal{I} \). In this case, \( dA_t \) is the \( g_t \)-Riemannian area. Flat Riemannian manifolds were considered in [10], an extension to more general geometries was made in [12]. The left hand side was coined **dynamic Cheeger constant** in [10].

**Corollary 4.4 (Strong dynamic Cheeger inequality).** It holds
\[ \inf_{\Gamma} \frac{dA_t(\Gamma)}{\min\{\omega(M_1), \omega(M_2)\}} \leq \inf_{\Gamma} \frac{d\overline{A}(\Gamma)}{\min\{\omega(M_1), \omega(M_2)\}} \leq 2\sqrt{-\lambda_2}, \]
where \( \inf_\Gamma \) denotes the infimum over all dividing surfaces \( \Gamma \) that split \( M \) into two sets \( M_1 \) and \( M_2 \), and \( \lambda_2 < 0 \) is the first non-trivial eigenvalue of \( \Delta \).

**Proof.** The first estimate follows from proposition 4.3, the second from proposition B.1, since \( d\overline{A}(\Gamma)/\min\{\omega(M_1), \omega(M_2)\} \) is the Cheeger constant for the geometry of mixing. \( \square \)
4.3 Relation to total Lagrangian diffusive transport

The authors of [19] establish the approximation of the total diffusive flux as in corollary 3.4 in order to define a measure of diffusive permeability for a generic material surface $\Gamma$. Here, the (“diffusive transport”) response $T^1_0(\Gamma, u_0)$ to a “diffusion stress” given by some virtual initial condition $u_0$—of which $\Gamma$ is supposed to be a level set—is computed. As a consequence, the gradient of $u_0$ along $\Gamma$ is normal to $\Gamma$.

To make this construction comparable among different surfaces, they require that the norm is uniformly constant along the entire $\Gamma$, which specifies $u_0$ in a neighborhood of $\Gamma$ to first order. It remains to choose a norm w.r.t. which to measure the gradient and w.r.t. which require constancy. Since the response depends linearly on this constant in the stress, one may take this constant to be equal to 1 without loss of generality. The requirement on $u_0$ then reads as $\text{grad}_g u_0 = \nu$, with $\nu$ the $g$-unit normal along $\Gamma$.

We set

$$T^1_0(\Gamma; g) := -\int_\Gamma g(\nu, \nabla \nu) \, dA,$$

where, notionally, we replace the dependence of $T^1_0$ on $u_0$ by a dependence on the metric $g$ which determines (i) the gradient of $u_0$; (ii) the unit normal vector; and (iii) the area element $dA$. By corollary 4.2, the previous definition equals the leading-order coefficient of $T^1_0(S, u_0)$ in the case that $\partial S = \Gamma$ and $u_0$ is chosen as described above.

In [19], the reference metric $g$ is chosen as the one induced by the initial spatial configuration of the fluid. For this choice, the norm of the gradient of $u_0$ is constant as measured in the spatial metric. This choice suggests itself, but is by no means natural. For instance, if at the initial time instance the diffusion is not spatially homogeneous (along $\Gamma$), a $u_0$ chosen with constant gradient measured w.r.t. $g$ may have non-constant gradient w.r.t. $g_0$, the initial, diffusion-adapted metric. As a consequence, it will have non-constant instantaneous diffusive flux, which puts different diffusion stress on different subsets of $\Gamma$, and hence makes them incomparable.

Alternatively, one could argue that the gradient should be measured in the “effective” diffusion-adapted norm $\mathbf{7}$, the norm in the geometry of mixing, and request uniform constancy w.r.t. this norm; i.e. $\text{grad}_7 u_0 = \mathbf{7}$. The diffusive transport represented in the geometry of mixing ($g = \mathbf{7}$), where $H$
is the identity (see eq. (48)), reduces to
\[ T_0^1(\Gamma; \varrho) = -\int_\Gamma \varrho(\nabla, \nabla) \, dA = -\int_\Gamma dA, \]  
(51)
the (negative of the) surface area of \( \Gamma \) in the geometry of mixing. For comparison, we represent the surface area in the geometry of the initial configuration using lemma 4.1 and obtain
\[ T_0^3(\Gamma; \varrho) = -\int_\Gamma \sqrt{\varrho(\nu, \nabla \nu)} \, dA. \]  
(52)

and find that the different uniformization choices for (the gradient of) \( u_0 \) lead to integrands that are the square and square root of each other, respectively.

Noticeably, within the \( T_0^1(\Gamma; \varrho) \) setting, the problem of finding closed material surfaces that minimize leading-order diffusive transport normalized by the enclosed fluid mass is exactly the isoperimetric problem posed in the geometry of mixing; cf. [10, 12] for a related but different approach (recall also corollary 4.4 and the surrounding discussion).

## 5 Conclusions

In the above, we have investigated the \( O(\varepsilon) \) asymptotics of finite-time, time-dependent heat flow on manifolds as the diffusivity \( \varepsilon \) goes to zero. Such time-dependent heat flows arise naturally when studying (possibly time-dependent) advection-diffusion equations in Lagrangian coordinates. When the initial concentration \( u_0 \) is smooth with support compactly contained in \( M \), the behaviour of the advection-diffusion equation in leading order is described by the time-averaged heat equation or, equivalently, the heat flow in the geometry of mixing.

The advection-diffusion equation remains well-defined even with non-smooth initial data \( u_0 \). In particular, it seems natural to investigate \( T_0^1(S, 1_S) \), the diffusive transport out of a material set \( S \) when the initial density is uniformly distributed on \( S \). The theory developed in this work does not apply to this quantity. Here, the leading order asymptotics is no longer of order \( \varepsilon \), as even in the autonomous heat flow context \( T_0^1(S, 1_S) \) is of order \( \varepsilon^{1/2} \); see, e.g., [42, 39]. There, the leading-order coefficient is proportional to the surface area of the boundary of \( S \). In the time-dependent, finite-time heat
flow case, a similar result can be shown, where the relevant surface area is the one in the geometry of mixing. This will be published in forthcoming work.

Acknowledgements

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A Differential geometric preliminaries

In this section, we briefly recall some fundamental concepts from differential geometry and fix our notation. General references include [29, 15]. Throughout, let $M$ be a smooth, oriented, compact manifold of dimension $\dim M = n$, possibly with smooth boundary.

A (Riemannian) metric $g$ on $M$ is a symmetric, positive-definite, contravariant tensor field of rank 2, i.e., $g: TM \times TM \to \mathbb{R}$. For any tangent vector $v \in T_xM$, a metric $g$ induces a linear form $g_x(v, \cdot)$ on $T_xM$. Correspondingly, for any vector field $v$, the metric $g$ induces a one-form on $M$. Henceforth, we identify a metric $g$ with its interpretation as the linear transformation $g_x: T_xM \to T_x^*M, v \mapsto (w \mapsto g_x(v, w))$, often referred to as the canonical/musical isomorphism between $T_xM$ and $T_x^*M$. Moreover, we will often suppress the subscript $x$ and regard $g$ as a vector-bundle morphism $g: TM \to T^*M$; cf., for instance, [27].

Non-degeneracy of $g$ implies its invertibility, and we may interpret its inverse $g^{-1}: T^*M \to TM$ by a similar identification as above with a symmetric, positive-definite, covariant tensor field of rank 2, i.e., $g^{-1}: T^*M \times T^*M \to \mathbb{R}$. This can be interpreted as an inner product on one-forms, and is known in the literature as the dual metric (to $g$).

With this notation, the gradient (induced by $g$) $\text{grad}_g f$ is defined as the vector field (a section of $TM$) obtained from transforming the one-form $df$ by $g^{-1}: T^*M \to TM$,

$$\text{grad}_g f = g^{-1} df .$$ (53)

The induced (Riemannian) volume element is the unique top-degree form, denoted by $dg$ (the $d$ here does not refer to the exterior derivative we used before), that returns 1 when applied to an oriented, orthonormal set of tangent
vectors $v_1, \ldots, v_n \in T_x M$. Furthermore, let $\Gamma$ be an oriented codimension-1 surface in $M$, then the metric $g$ induces a surface element $dA_g$ on $\Gamma$ via the volume element on $M$ as follows. For given, oriented linearly independent $v_1, \ldots, v_{n-1} \in T_p \Gamma$, let $\nu_g \perp_g \text{span}\{v_1, \ldots, v_{n-1}\}$ with $\|\nu_g\|_g = 1$ be such that $(\nu_g, v_1, \ldots, v_{n-1})$ is positively oriented in $M$. We call such $\nu$ the unit normal vector to $\Gamma$ at $p$. Then the action of the surface element is given by
\begin{equation}
    dA_g(v_1, \ldots, v_{n-1}) = d\langle\nu, v_1, \ldots, v_{n-1}\rangle, \quad v_1, \ldots, v_{n-1} \in T_p \Gamma.
\end{equation}

Intuitively, $(v_1, \ldots, v_{n-1})$ span a parallelepiped of area 1 if, when expanded by the unit normal $\nu$, the resulting parallelepiped has volume 1. By construction, a surface $\Gamma$ has non-negative surface area
\[ dA_g(\Gamma) := \int_\Gamma dA_g. \]

The surface element $dA_g$ is a top-degree form on $\Gamma$ and can be hence regarded as the volume element there.

For any volume form $\omega$ on $M$, the induced divergence $\text{div}_\omega$ of a smooth vector field $F: M \to T M$ is defined via
\[ (\text{div}_\omega F)\omega := d(\iota_F\omega) = L_V\omega, \]
where $\text{div}_\omega F \in C^\infty(M)$, and $L$ is the Lie-derivative. Here, $\iota$ refers to the contraction operation on forms, i.e.,
\[ (\iota_F\alpha)(v_1, \ldots, v_{k-1}) = \alpha(F, v_1, \ldots, v_{k-1}), \]
for any $k$-form $\alpha$. It holds that $\text{div}_{dg}$ is the usual Riemannian divergence.

A natural differential operator on Riemannian manifolds is the Laplace-Beltrami operator $\Delta_g$ defined as
\[ \Delta_g := \text{div}_{dg} \circ \text{grad}_g. \]

It will turn out that for an elegant description and study of a suitably general class of advection-diffusion processes, weighted manifolds (also known as manifolds with density [3]) are very helpful. A weighted manifold $(M, g, \theta)$ is a Riemannian manifold $(M, g)$, on which the volume form and—as a consequence—the induced surface area forms are weighted by a (strictly) positive smooth function $\theta: M \to \mathbb{R}$ w.r.t. the canonical volume
dg or surface area dA_g forms. For the induced surface area the same intuition and formalism applies: measure the volume of a higher-dimensional parallelepiped as obtained by expansion with a suitably oriented unit normal vector, and the result is the area of the base parallelepiped.

On a weighted manifold \((M, g, \theta)\), the Laplace operator \(\Delta_{\theta,g}\) is defined analogously to the classic Riemannian case by composition of the associated divergence and gradient,

\[
\Delta_{\theta,g} := \text{div}_{\theta dg} \circ \text{grad}_g.
\]

## B  Cheeger inequality on weighted manifolds

**Proposition B.1** (Cheeger inequality for weighted manifolds). Let \((M, g, \theta)\) be a compact weighted manifold with Laplace operator \(\Delta\). We denote the (weighted) volume form by \(\omega := \theta dg\), the (weighted) surface measure by \(dA\), and the first nontrivial eigenvalue of \(\Delta\) by \(\lambda\). Furthermore \(\text{grad} := \text{grad}_g\), and \(\|\cdot\| := \|\cdot\|_g\). Then the Cheeger inequality holds:

\[
h := \inf_{\Gamma \text{ disconnects } M} \frac{dA(\Gamma)}{\min\{\omega(M_1), \omega(M_2)\}} \leq 2\sqrt{-\lambda}.
\]  

(55)

**Proof.** The proof follows exactly the lines of the proof for the classical Cheeger inequality (see, e.g., the treatment in \[28\] which we follow in the proof below). Without loss of generality, we can scale \(\theta\) so that \(\omega(M) = 1\). First, from the co-area formula follows that for any sufficiently smooth positive \(\varphi\) one has that

\[
\int_M \|\text{grad} \varphi\| \omega = \int_0^\infty (dA(\varphi^{-1}(\{s\}))) ds
\]

\[
\geq h \int_0^\infty \min\{\omega(\varphi \geq s), 1 - \omega(\varphi \geq s)\} ds.
\]  

(56)

Let \(f \in C^\infty(M)\) and denote by \(m\) a median of \(f\), that is \(\omega(f \geq m) \geq \frac{1}{2}\) and \(\omega(f \leq m) \geq \frac{1}{2}\). Let \(f^+, f^-\) be the positive and negative part, respectively, of \(f - m\). From the definition of the median, we know that for any \(s > 0\) holds \(\omega((f^+)^2 \geq s) \leq \frac{1}{2}\), and similarly \(\omega((f^-)^2 \geq s) \leq \frac{1}{2}\). It follows that

\[
\min\{\omega((f^+)^2 \geq s), 1 - \omega((f^+)^2 \geq s)\} = \omega((f^+)^2 \geq s),
\]
and likewise for $f^-$. Thus, by eq. (56) applied to $(f^+)^2$ and $(f^-)^2$,
\[
\int_M \|\text{grad}(f-m)^2\| \omega = \int_M \|\text{grad}(f^+)^2\| \omega + \int_M \|\text{grad}(f^-)^2\| \omega \\
\geq h \int_0^\infty \omega ((f^+)^2 \geq s) + \omega ((f^-)^2 \geq s) \, ds \\
= h \int_M ((f^+)^2 + (f^-)^2) \omega = h \int_M (f-m)^2 \omega.
\]
We can rewrite the left hand side and apply Cauchy-Schwarz:
\[
\int_M \|\text{grad}(f-m)^2\| \omega = \int_M \|2(f-m) \text{grad} f\| \omega \\
\leq 2 \left( \int_M \|\text{grad} f\|^2 \omega \right)^{1/2} \left( \int_M (f-m)^2 \omega \right)^{1/2}.
\]
Thus, independently from the median $m$ we have
\[
h \leq 2 \left( \frac{\int \|\text{grad} f\|^2 \omega}{\int_M (f-m)^2 \omega} \right)^{1/2}. \tag{57}
\]
Note that the inequality $\int_M (f - \text{mean}(f))^2 \omega \leq \int_M (f-m)^2 \omega$ implies $\int_M f^2 \omega \leq \int_M (f-m)^2 \omega$ for mean-free functions, which gives
\[
h \leq 2 \left( \frac{\int \|\text{grad} f\|^2 \omega}{\int_M f^2 \omega} \right)^{1/2}. \tag{58}
\]
if $f$ is mean-free. The theorem now follows from the variational characterization of the eigenvalue $\lambda$ as the minimum of the right hand side of eq. (58).

\section*{C Spectral convergence}

Recall that we used the notation $\langle \cdot, \cdot \rangle_0$ for the $L^2(M, \omega)$ scalar product and $\langle \cdot, \cdot \rangle_1$ for the $H^1(M, g, \omega)$ scalar product; furthermore, we introduced $g$ as some reference metric on $M$ and $\text{grad} = \text{grad}_g$. For later reference, we first prove estimates on solutions.

\begin{lemma}[Uniform parabolicity] \label{lem:C.1}
There exist constants $C_1, C_2 > 0$ independent of $t$ so that $C_1 |u|_1^2 \leq -\langle u, \Delta_t u \rangle_0 \leq C_2 |u|_1^2$ for all $u \in H^1_0(M, g, \omega)$. Moreover, for $u_1, u_2 \in H^1_0(M, g, \omega)$ it holds that $\langle u_1, \Delta_t u_2 \rangle \leq C_2 |u_1|_1 |u_2|_1$.
\end{lemma}

\section*{References}

\begin{thebibliography}{10}

\bibitem{bib:C.1}
B. A. Schr"{o}dinger, "{O}ver die {K}inetische {E}nergie des {W}ellen-{P}aktes
\int M \|\text{grad} \omega = 
\int M \|\text{grad}(f^+)^2\| \omega + \int_M \|\text{grad}(f^-)^2\| \omega \\
\geq h \int_0^\infty \omega ((f^+)^2 \geq s) + \omega ((f^-)^2 \geq s) \, ds \\
= h \int_M ((f^+)^2 + (f^-)^2) \omega = h \int_M (f-m)^2 \omega.
\]
We can rewrite the left hand side and apply Cauchy-Schwarz:
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\end{lemma}

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\bibitem{bib:C.1}
B. A. Schr"{o}dinger, "{O}ver die {K}inetische {E}nergie des {W}ellen-{P}aktes
\int M \|\text{grad} \omega = 
\int M \|\text{grad}(f^+)^2\| \omega + \int_M \|\text{grad}(f^-)^2\| \omega \\
\geq h \int_0^\infty \omega ((f^+)^2 \geq s) + \omega ((f^-)^2 \geq s) \, ds \\
= h \int_M ((f^+)^2 + (f^-)^2) \omega = h \int_M (f-m)^2 \omega.
\]
We can rewrite the left hand side and apply Cauchy-Schwarz:
\[
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\]
Thus, independently from the median $m$ we have
\[
h \leq 2 \left( \frac{\int \|\text{grad} f\|^2 \omega}{\int_M (f-m)^2 \omega} \right)^{1/2}. \tag{57}
\]
Note that the inequality $\int_M (f - \text{mean}(f))^2 \omega \leq \int_M (f-m)^2 \omega$ implies $\int_M f^2 \omega \leq \int_M (f-m)^2 \omega$ for mean-free functions, which gives
\[
h \leq 2 \left( \frac{\int \|\text{grad} f\|^2 \omega}{\int_M f^2 \omega} \right)^{1/2}. \tag{58}
\]
if $f$ is mean-free. The theorem now follows from the variational characterization of the eigenvalue $\lambda$ as the minimum of the right hand side of eq. (58).
Proof. This follows directly from uniform ellipticity of the smooth, $t$-dependent family of operators $\Delta_t$, defined on the compact $[0,1]$, which are in divergence form w.r.t. the volume form $\omega$. \hfill \square

For reference, we state the following well-known result.

**Lemma C.2** ($L^2$ contractivity; [7, cf. Sect. 7.1, Thm. 2]). Let $u_0 \in L^2(M, \omega)$. Then $\|P^\varepsilon_t u_0\|_0 \leq \|u_0\|_0$.

Proof. To see this, note that

$$\partial_t \|P^\varepsilon_t u\|_0^2 = 2\varepsilon \langle P^\varepsilon_t u, \Delta_t P^\varepsilon_t u \rangle_0 \leq 0,$$

since $\Delta_t$ is non-positive. Absolute continuity of $t \mapsto \|P^\varepsilon_t u\|_0^2$ is established in lemma D.1, appendix D. \hfill \square

**Lemma C.3** (Uniform $H^1$ boundedness; cf. [30, Prop. 2(iii)]). For $t \in [0,1]$ and $u_0 \in H^1_0$, we have $|P^\varepsilon_t u_0|_1 \leq C_3|u_0|_1$ for some constant $C_3$ that does not depend on $u_0$, $t$ or $\varepsilon$.

Proof. Our proof conceptually follows [30, App. B], which is given in Eulerian coordinates, and therefore takes a seemingly different form because of the presence of the advection term in the evolution PDE.

We start with the case that $u_0$ is in the domain of $\Delta_0$. By uniform parabolicity it suffices to find bounds on $f^\varepsilon(t) := -\langle u_\varepsilon(t), \Delta_t u_\varepsilon(t) \rangle_0$. Using lemma D.2 see see that $f^\varepsilon(t)$ is absolutely continuous, and moreover

$$\partial_t \langle u_\varepsilon(t), \Delta_t u_\varepsilon(t) \rangle_0 = 2\varepsilon \langle \Delta_t u_\varepsilon(t), \Delta_t u_\varepsilon(t) \rangle_0 + \langle u_\varepsilon(t), \partial_t(\Delta_t) u_\varepsilon(t) \rangle_0 \\
\geq \langle u_\varepsilon(t), \partial_t(\Delta_t) u_\varepsilon(t) \rangle_0.$$

The operator $\partial_t(\Delta_t)$ is given via its action on $u \in C^\infty(M)$ as

$$\partial_t(\Delta_t) u = \text{div}_\omega(\partial_t(g_t^{-1})du),$$

recalling that $\Delta_t u = \text{div}_\omega(g_t^{-1}du)$ is the action of $\Delta_t$. Hence, $\partial_t(\Delta_t)$ is a well-defined second-order partial differential operator with smooth coefficients. Arguments as in the proof of lemma C.1 yield that

$$C := \sup_{u \in H^1_0(M,\omega,g)} |\langle \partial_t(\Delta_t) u, u \rangle_0| |u|_1^2 < \infty.$$
Therefore
\[-\partial_t \langle u_\varepsilon(t), \Delta t u_\varepsilon(t) \rangle_0 \leq C |u_\varepsilon(t)|_1^2.\]
Due to uniform parabolicity we have that \(|u_\varepsilon(t)|_1^2 \leq C_1^{-1} f_\varepsilon(t)\). Hence by Grönwall’s lemma ([7, Appendix B.2]), \(f_\varepsilon(t) \leq e^{CC_1^{-1}} f_\varepsilon(0)\), which finishes the proof. Since the domain of \(\Delta_0\) is dense in \(H^1_0\), the general result is a consequence of this special case (using the results in cref:parabolicpdes).

Recall the well-known fact that the \(L^2\)-adjoint of \(P_\varepsilon\) is the time-1 solution operator associated to the Lagrangian advection-diffusion equation with the same Dirichlet boundary conditions, but with reversed time\(^5\), i.e.,
\[\partial_t u(t, x) = \varepsilon \Delta_{(1-t)} u(t, x).\]

The range of \((P_\varepsilon)^*\) is a subset of \(H^1_0(M, g, \omega)\); see appendix [D]. Therefore, the left singular vectors of \(P_\varepsilon\), or equivalently the eigenvectors of \((P_\varepsilon)^*P_\varepsilon\), are in \(H^1_0(M, g, \omega)\). Recall that the constant \(C_3\) from lemma [C.3] depends (i) on the uniform parabolicity bounds \(C_1\) and \(C_2\) from lemma [C.1] and (ii) on bounds on \(\partial_t(\Delta_t)\). All of these bounds equally apply to eq. (59). Therefore, we conclude with lemma [C.3] that
\[|P^*u|_1 \leq C_3 |u|_1,\]
for \(u \in H^1_0(M, g, \omega)\). Furthermore, the same estimate
\[|P_{t,1}u|_1 \leq C_3 |u|_1,\]
applies to the solution operator (from time \(t\) to time 1) of the Lagrangian advection-diffusion equation, considered on the time interval \([t, 1]\). By construction,
\[P_{t,1}^\varepsilon = P_t^\varepsilon P_{t,1}^\varepsilon\]
for any \(t \in (0, 1)\).

**Lemma C.4.** There exists \(C_4 > 0\), independent from \(\varepsilon\), satisfying
\[|v^\varepsilon|_1 \leq \max_{t \in [0,1]} |v^\varepsilon(t)|_1 \leq C_4 \min_{t \in [0,1]} |v^\varepsilon(t)|_1 \leq C_4 |v^\varepsilon|_1\]
for sufficiently small \(\varepsilon\) and the singular vector \(v^\varepsilon\). Recall that \(v^\varepsilon(t) := P_t^\varepsilon v^\varepsilon\).

---

\(^5\)See, for example, the proof of [I Prop. 2.9] and appendix [D]

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Proof. The rightmost and leftmost inequalities are trivial. For the middle inequality, by lemma C.3 we have

$$\max_{t \in [0, 1]} |v^\varepsilon(t)|_1 \leq C_3 |v^\varepsilon|_1.$$  \hfill (62)

Thus it is enough to show

$$|v^\varepsilon|_1 \leq C |v^\varepsilon(t)|_1$$  \hfill (63)

for all $t \in [0, 1]$, with some $C > 0$ independent from $t$ or $\varepsilon$. Since the square of singular values of $P^\varepsilon$ are eigenvalues of $(P^\varepsilon)^* P^\varepsilon$, we have

$$(\sigma^\varepsilon)^2 v^\varepsilon(0) = (P^\varepsilon)^* P^\varepsilon v^\varepsilon(0) = (P^\varepsilon)^* v^\varepsilon(1).$$  \hfill (64)

Applying eq. (61) to $v^\varepsilon(1) = P^\varepsilon(t) v^\varepsilon(t)$ yields $|v^\varepsilon(1)|_1 \leq C_3 |v^\varepsilon(t)|_1$. Equations (60) and (64) yield that $(\sigma^\varepsilon)^2 |v^\varepsilon|_1 \leq C^2_3 |v^\varepsilon(t)|_1$. Combining these inequalities, we obtain $|v^\varepsilon(0)|_1 \leq (C^2_3)^{-1} C^2_3 |v^\varepsilon(t)|_1$. We know that (step 1 of theorem 3.5) $\sigma^\varepsilon \to 1$ for $\varepsilon \to 0$, and is thus bounded away from zero for sufficiently small $\varepsilon$. This proves eq. (63), and the claim is shown.

Lemma C.5 ($H^1$ bound on singular vectors). There exists a constant $C > 0$, independent of $\varepsilon$ and $t$, for which $|v^\varepsilon(t)|_1 \leq C$ holds for $t \in [0, 1]$ and sufficiently small $\varepsilon$.

Proof. With eq. (39) and lemmas C.1 and C.4 we obtain for any $t \in (0, 1]$ and sufficiently small $\varepsilon$ that:

$$\frac{(1 - (\sigma^\varepsilon)^2)}{2\varepsilon} = -\int_0^1 \langle v^\varepsilon(t), \Delta_t v^\varepsilon(t) \rangle_0 \, dt$$

$$\geq \min_{t \in [0, 1]} -\langle v^\varepsilon(t), \Delta_t v^\varepsilon(t) \rangle_0$$

$$\geq C_1 \min_{t \in [0, 1]} |v^\varepsilon(t)|_1^2$$

$$\geq C_1 C_4^{-1} |v^\varepsilon(0)|^2_1.$$

In eq. (37) we have already shown that the limit superior of the left hand side is less than or equal to $-\overline{\lambda}$, and, therefore, it may be bounded from above by, say, $-2\overline{\lambda}$ for sufficiently small $\varepsilon$. This shows that $|v^\varepsilon|^2 \leq -2\overline{\lambda} C_4 C_1^{-1}$, proving the claim for $t = 0$. The case $t \neq 0$ is now a consequence of lemma C.3. \hfill \square
D Parabolic PDEs on compact manifolds with boundary

We briefly collect some properties of second-order parabolic PDEs on compact and orientable smooth Riemannian manifolds with (potentially empty) $C^2$ boundary. These properties are well known when the domain is an open subset of $\mathbb{R}^n$ [7, 36] and the straightforward extension to compact manifolds seems to be folklore knowledge, though rarely explicitly treated; see [2].

Let $\omega$ be a smooth, nonvanishing volume-form on $M$. For convenience, we will use a metric $g$ such that $dg = \omega$. The metric may be constructed by any metric on $M$ after suitable rescaling. We need this metric only for defining a norm on $H^1(M, g)$, given by

$$\|u\|_{H^1(M, g)}^2 := \int_M g(\text{grad}u, \text{grad}u) \omega + \int_M |u|^2 \omega,$$

where grad $u$ is interpreted in a suitably weak sense. Since $M$ is compact, the specific choice of $g$ will not affect the topology of $H^1(M, g)$. The space $H^1_0(M, g)$ is defined as the completion of $C_\infty^\infty(\bar{M})$ w.r.t. $\|\cdot\|_{H^1(M, g)}$ [15, 21].

We will describe the parabolic PDE theory needed for the equation

$$\partial_t u = \text{div}_\omega(D(t)du),$$

with $D: [0,1] \times T^*M \to TM$ a smooth—including at the boundary—family of nonvanishing bundle morphisms that are symmetric in the sense that $D(t, u)(v) = D(t, v)(u)$ for $t \in [0,1]$ and all vector fields $u$ and $v$. Let $L(t) := \text{div}_\omega(D(t)du)$. The tensor field $D$ is bounded—due to its smoothness and compactness of $M$—and nonvanishing, hence the operator $\partial_t - L$ is uniformly parabolic, i.e., there exists $\alpha > 0$ such that for any $v \in H^1_0(M, g)$

$$\alpha^{-1}\|\text{grad}_g v\|_{L^2(M, \omega)}^2 \leq -\langle v, L(t)v \rangle_{L^2(M, \omega)} \leq \alpha\|\text{grad}_g v\|_{L^2(M, \omega)}^2.$$  

For what follows, we require the well-known theory of vector-valued Sobolev spaces, and our notation essentially follows [36]; see also [1, Appendix A] for proofs of fundamental results. For a Hilbert space $X$, we write $X^*$ for its dual, and $H^{-1}(M, g) := H^1_0(M, g)^*$. As in [36], to each $t \in [0,1]$ we associate an operator $L(t): H^1_0(\omega) \mapsto H^{-1}(M, g)$, defined by

$$(L(t)u)v = \langle du, D(t)dv \rangle_{L^2(M, \omega)} = \int_M du(D(t)dv) \omega.$$
The space $L^2(M, \omega)$ embeds continuously into $H^{-1}$ by the identification of a function $f \in L^2(M, \omega)$ with the functional $\langle f, \cdot \rangle_{L^2(M, \omega)}$. By a slight abuse of notation, for $f \in H^{-1}(M, g)$ and $g \in H^1_0(M, g)$ we will write $(f, g)_{L^2(M, \omega)} := f(g)$, even if $f$ is not contained in the image of the embedding.

**Lemma D.1.** Equation (65) has a unique weak solution

$$u \in C([0, 1]; L^2(M, \omega)) \cap L^2((0, 1); H^1_0(M, g)),$$

given an initial value $u(0, \cdot) \in L^2(M, \omega)$. Moreover, the function $t \mapsto \|u(t, \cdot)\|_{L^2(M, \omega)}^2$ is absolutely continuous, with

$$\frac{d}{dt} \|u(t)\|_{L^2(M, \omega)}^2 = 2\langle L(t)u(t), u(t)\rangle_{L^2(M, \omega)} \tag{67}$$

for almost all $t$, where the right hand side must be interpreted in a weak sense.

**Proof.** The $L^2$-Galerkin approach described in [7, Sect. 7.1, Thms. 3 & 4] yields existence and uniqueness for the compact manifold case just like for $M \subset \mathbb{R}^n$ compact. Theorem 3 in [7, Sect. 5.9] proves the remaining claims. \qed

These arguments show that for $t \in [0, 1]$, the time-$t$ solution operator to eq. (65) is well-defined when viewed as an operator $P_t : L^2 \to L^2$. Arguments as in lemma C.2 establish its continuity.

Let the domain of $L(t)$ be the collection of $f \in H^1_0(M, g) \subset L^2(M, \omega)$ satisfying $L(t)f \in L^2(M, \omega)$. As a consequence of elliptic regularity theory and the fact that we are working with homogeneous Dirichlet boundary (cf. [7, Chapters 6.3 & 7.4]), this function space does not depend on $t$. By arguments as in [33, Chapter 7], one sees that for $t \in (0, 1]$, the image of the time-$t$ solution operator $P_t$ is in the domain of $L(t)$. Hence, the image of $P_t$ is in $H^1_0(M, g)$ for all $t \in (0, 1]$. Thus, the operator $P_t : L^2(M, \omega) \to H^1_0(M, g)$ is well-defined, and by the closed graph theorem it is continuous. By the Rellich-Kondrachev theorem, $P_t$ is therefore compact when viewed as an operator from $L^2(M, \omega)$ to itself.

**Lemma D.2.** Provided that the initial value $u_0$ is in the domain of $L(0)$, the solution from lemma D.1 is sufficiently regular such that

(i) $u \in H^1((0, 1); H^1_0(M, g))$, 33
(ii) \( L(t)u \in H^1((0,1); H^{-1}(M,g)) \), and

(iii) \( L(t)u \in C([0,1]; L^2(M,\omega)) \).

The function \( \langle u, L(t)u \rangle_{L^2(M,\omega)} \) is absolutely continuous with

\[
\frac{d}{dt} \langle u(t), L(t)u(t) \rangle_{L^2(M,\omega)} = 2\langle L(t)u(t), L(t)u(t) \rangle_{L^2(M,\omega)} + \langle L'(t)u(t), u(t) \rangle_{L^2(M,\omega)} \tag{68}
\]

for almost all \( t \in [0,1] \).

**Proof.** Proceed as in [36, Sect. 11.1.4], for the last statement a result like [4, Cor. A.4] is required.

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**Lemma D.3.** If \( u_0 \in C^\infty_c(\bar{M}) \), then the solution \( u \) from lemma [D.1] is in \( C^\infty([0,1] \times M) \).

**Proof.** Certainly \( u_0 \) is in the domain of all powers of \( L(t) \). Iterating the construction of [36, Sect. 11.1.4] together with Sobolev embedding and elliptic-regularity results yields the claim. See also [7, Sect. 7.1, Thm. 7] for a proof of the nonautonomous case on open subsets of \( \mathbb{R}^n \).

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We conclude with a well-known property of smooth solutions to parabolic equations.

**Lemma D.4** (Weak maximum principle on manifolds; [21 Thm. A.3.1] or [7 Sect. 7.1, Thm. 8]). Let \( u \in C^{1,2}([0,1] \times M) \cap C([0,1] \times \overline{M}) \). If \( L^\varepsilon u \leq 0 \) on \([0,1] \times \text{int}(M)\), then for the “parabolic boundary” \( B := [0,1] \times \partial M \cup \{0\} \times M \) one has

\[
\max_{[0,1] \times M} u = \max_{B} u. \tag{69}
\]

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**References**

[1] P. Acquistapace, F. Flandoli, and B. Terreni. Initial boundary value problems and optimal control for nonautonomous parabolic systems. *SIAM Journal on Control and Optimization*, 29(1):89–118, 1991. [doi:10.1137/0329005]
[2] H. Amann. *Parabolic Equations on Uniformly Regular Riemannian Manifolds and Degenerate Initial Boundary Value Problems*, pages 43–77. Springer Basel, 2016. doi:10.1007/978-3-0348-0939-9_4

[3] H. Berestycki, F. Hamel, and N. Nadirashvili. Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena. *Communications in Mathematical Physics*, 253(2):451–480, 2005. doi:10.1007/s00220-004-1201-9

[4] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces du Hilbert*, volume 5 of *North-Holland Mathematics Studies*. North Holland Publishing Company Amsterdam, 1973.

[5] T. Bühler and D. A. Salamon. *Functional Analysis*, volume 191 of *Graduate Studies in Mathematics*. American Mathematical Society, 2018.

[6] P. Constantin, A. Kiselev, L. Ryzhik, and A. Zlatoš. Diffusion and mixing in fluid flow. *Annals of Mathematics*, 168(2):643–674, 2008. URL: http://www.jstor.org/stable/40345422

[7] L.C. Evans. *Partial Differential Equations*, volume 19 of *Graduate studies in mathematics*. American Mathematical Society, 2nd edition, 2010.

[8] A. Fannjiang and G. Papanicolaou. Convection enhanced diffusion for periodic flows. *SIAM Journal on Applied Mathematics*, 54(2):333–408, 1994. doi:10.1137/S0036139992236785

[9] M. I. Freidlin and A. D. Wentzell. *Random Perturbations of Dynamical Systems*, volume 260 of *Grundlehren der mathematischen Wissenschaften*. Springer, 3rd edition, 2012. doi:10.1007/978-3-642-25847-3

[10] G. Froyland. Dynamic isoperimetry and the geometry of Lagrangian coherent structures. *Nonlinearity*, 28(10):3587–3622, 2015. doi:10.1088/0951-7715/28/10/3587

[11] G. Froyland, P. Koltai, and M. Stahn. Computation and optimal perturbation of finite-time coherent sets for aperiodic flows without trajectory integration. *SIAM Journal on Applied Dynamical Systems*, 19(3):1659–1700, 2020. doi:10.1137/19M1261791
[12] G. Froyland and E. Kwok. A Dynamic Laplacian for Identifying Lagrangian Coherent Structures on Weighted Riemannian Manifolds. *Journal of Nonlinear Science*, 30:1889–1971, 2020. doi:10.1007/s00332-017-9397-y

[13] M. M. Fyrillas and K. K. Nomura. Diffusion and Brownian motion in Lagrangian coordinates. *The Journal of Chemical Physics*, 126(16), 2007. doi:10.1063/1.2717185

[14] I.M. Gelfand and S.V. Fomin. *Calculus of variations*. Prentice-Hall, Inc., 1963.

[15] A. Grigor’yan. *Heat Kernel and Analysis on Manifolds*. Number 47 in Studies in Advanced Mathematics. AMS/IP, 2009.

[16] A. Hadjighasem, D. Karrasch, H. Teramoto, and G. Haller. Spectral-clustering approach to Lagrangian vortex detection. *Phys. Rev. E*, 93:063107, 2016. doi:10.1103/PhysRevE.93.063107

[17] G. Haller. Langrangian Coherent Structures. *Annual Review of Fluid Mechanics*, 47(1):137–162, 2015. doi:10.1146/annurev-fluid-010313-141322

[18] G. Haller and F. J. Beron-Vera. Coherent Lagrangian vortices: the black holes of turbulence. *Journal of Fluid Mechanics*, 731:R4, 2013. doi:10.1017/jfm.2013.391

[19] G. Haller, D. Karrasch, and F. Kogelbauer. Material barriers to diffusive and stochastic transport. *Proceedings of the National Academy of Sciences*, 115(37):9074–9079, 2018. doi:10.1073/pnas.1720177115

[20] G. Haller, D. Karrasch, and F. Kogelbauer. Barriers to the Transport of Diffusive Scalars in Compressible Flows. *SIAM Journal on Applied Dynamical Systems*, 19(1):85–123, 2020. doi:10.1137/19M1238666

[21] J. Jost. *Riemannian Geometry and Geometric Analysis*. Universitext. Springer, 6 edition, 2011. doi:10.1007/978-3-642-21298-7

[22] D. Karrasch and J. Keller. A Geometric Heat-Flow Theory of Lagrangian Coherent Structures. *Journal of Nonlinear Science*, 30(4):1849–1888, 2020. doi:10.1007/s00332-020-09626-9
[23] Y. Kifer. *Random Perturbations of Dynamical Systems*, volume 16 of *Progress in Probability and Statistics*. Birkhäuser Boston, 1988. doi:10.1007/978-1-4615-8181-9

[24] M. S. Krol. On the Averaging Method in Nearly Time-Periodic Advection-Diffusion Problems. *SIAM Journal on Applied Mathematics*, 51(6):1622–1637, 1991. doi:10.1137/0151083

[25] L. D. Landau and E.M. Lifshitz. *Fluid Mechanics*, volume 6 of *Course of Theoretical Physics*. Pergamon Press, 2nd edition, 1987.

[26] S. Lang. *Real and Functional Analysis*, volume 142 of *Graduate Texts in Mathematics*. Springer Berlin, 3rd edition, 1993. doi:10.1007/978-1-4612-0897-6

[27] S. Lang. *Differential and Riemannian Manifolds*, volume 160 of *Graduate Texts in Mathematics*. Springer New York, 1995. doi:10.1007/978-1-4612-4182-9

[28] M. Ledoux. A simple analytic proof of an inequality by P. Buser. *Proceedings of the American Mathematical Society*, 121(3):951–959, 1994. doi:10.1090/S0002-9939-1994-1186991-X

[29] J. M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, 2nd edition, 2013. doi:10.1007/978-1-4419-9982-5

[30] W. Liu and G. Haller. Strange eigenmodes and decay of variance in the mixing of diffusive tracers. *Physica D: Nonlinear Phenomena*, 188(1–2):1–39, 2004. doi:10.1016/S0167-2789(03)00287-2

[31] F. Morgan. Manifolds with density. *Notices of the AMS*, 52(8):853–858, 2005.

[32] G. A. Pavliotis and A. M. Stuart. *Multiscale Methods: Averaging and Homogenization*, volume 53 of *Texts in Applied Mathematics*. Springer New York, 2008. doi:10.1007/978-0-387-73829-1

[33] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences*. Springer New York, 1983. doi:10.1007/978-1-4612-5561-1
[34] W. H. Press and G. B. Rybicki. Enhancement of Passive Diffusion and Suppression of Heat Flux in a Fluid with Time Varying Shear. *The Astrophysical Journal*, 248:751–766, 1981. doi:10.1086/159199.

[35] M. H. Protter and H. F. Weinberger. *Maximum Principles in Differential Equations*. Springer New York, 1984. doi:10.1007/978-1-4612-5282-5.

[36] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations*, volume 13 of *Texts in Applied Mathematics*. Springer New York, 2nd edition, 2004. doi:10.1007/b97427.

[37] J. A. Sanders, F. Verhulst, and J. Murdock. *Averaging Methods in Nonlinear Dynamical Systems*, volume 59 of *Applied Mathematical Sciences*. Springer, 2nd edition, 2007. doi:10.1007/978-0-387-48918-6.

[38] T. Schäfer, A. C. Poje, and J. Vukadinovic. Averaged dynamics of time-periodic advection diffusion equations in the limit of small diffusivity. *Physica D*, 238(3):233–240, 2009. doi:10.1016/j.physd.2008.10.015.

[39] N. Schilling. Short-time heat content asymptotics via the wave and eikonal equations. *The Journal of Geometric Analysis*, 2020. doi:10.1007/s12220-020-00416-z.

[40] M. E. Taylor. *Partial Differential Equations I*, volume 115 of *Applied Mathematical Sciences*. Springer New York, 2nd edition, 2011. doi:10.1007/978-1-4419-7055-8.

[41] J.-L. Thiffeault. Advection–diffusion in Lagrangian coordinates. *Physics Letters A*, 309(5–6):415 – 422, 2003. doi:10.1016/S0375-9601(03)00244-5.

[42] M. van den Berg and P. Gilkey. Heat flow out of a compact manifold. *The Journal of Geometric Analysis*, 25:1576–1601, 2015.

[43] E. van Sebille, S. M. Griffies, R. Abernathey, T. P. Adams, P. Berloff, A. Biastoch, B. Blanke, E. P. Chassignet, Y. Cheng, C. J. Cotter, E. Deleersnijder, K. Döös, H. F. Drake, S. Drijfhout, S. F. Gary, A. W. Heemink, J. Kjellsson, I. M. Koszalka, M. Lange, C. Lique, G. A. MacGilchrist, R. Marsh, C. G. M. Adame, R. McAdam, F. Nencioli, C. B. Paris, M. D. Piggott, J. A. Polton, S. Rühs, S. H.A.M. Shah,
[44] J. Vukadinovic, E. Deditis, A.C. Poje, and T. Schäfer. Averaging and spectral properties for the 2d advection–diffusion equation in the semi-classical limit for vanishing diffusivity. *Physica D*, 310:1 – 18, 2015. [doi:10.1016/j.physd.2015.07.011]