Burgers and KP hierarchies: 
A functional representation approach *

Aristophanes Dimakis
Department of Financial and Management Engineering, 
University of the Aegean, 31 Fostini Str., GR-82100 Chios, Greece
dimakis@aegean.gr

Folkert Müller-Hoissen
Max-Planck-Institute for Dynamics and Self-Organization
Bunsenstrasse 10, D-37073 Göttingen, Germany
folkert.mueller-hoissen@ds.mpg.de

Abstract

From a ‘discrete’ functional zero curvature equation, functional representations of (matrix) Burgers and potential KP (pKP) hierarchies (and others), as well as corresponding Bäcklund transformations, can be obtained in a surprisingly simple way. With their help we show that any solution of a Burgers hierarchy is also a solution of the pKP hierarchy. Moreover, the pKP hierarchy can be expressed in the form of an inhomogeneous Burgers hierarchy. In particular, this leads to an extension of the Cole-Hopf transformation to the pKP hierarchy. Furthermore, these hierarchies are solved by the solutions of certain functional Riccati equations.

1 Introduction

It has been noted in [1] (page 119) that any solution of the first two equations of the Burgers hierarchy [2–10] is also a solution of the potential KP (pKP) equation. The generalization to the case where the dependent variables take their values in a matrix (or more generally an associative, and typically noncommutative) algebra $A$ appeared in [11]. By use of functional representations (i.e., generating equations, depending on auxiliary indeterminates) of the corresponding hierarchies, it can easily be shown that indeed any solution of the (‘noncommutative’) Burgers hierarchy also solves the (‘noncommutative’) pKP hierarchy (see section 4). Moreover, it turns out that the pKP hierarchy can be expressed as an ‘inhomogeneous Burgers hierarchy’. This means that there is a functional form of the pKP hierarchy involving a matrix function as an inhomogeneous term. Setting the latter to zero, reduces it to a functional form of the Burgers hierarchy.

Our starting point for the generation of functional representations of integrable hierarchies is a functional zero curvature (Zakharov-Shabat) equation, which we recall in section 2 (see also [12, 13]). Section 3 then treats the simplest non-trivial example, which is a Burgers hierarchy (with dependent variable in $A$). Another version of...
the Burgers hierarchy is dealt with in appendix A. Section 4 addresses the case of the pKP hierarchy and its relations with Burgers hierarchies. In particular, we obtain an extension of the Cole-Hopf transformation from the Burgers to the pKP hierarchy, generalizing a result in [11]. Section 5 shows that there are functional Riccati equations which imply the pKP, respectively Burgers hierarchy. Since such Riccati equations can be solved explicitly, this offers a quick way to exact solutions. Imposing a ‘rank one condition’ (cf. [14] and references therein), these solutions of matrix hierarchies lead to solutions of the corresponding scalar hierarchies.

2 The functional zero curvature condition

The integrability conditions of a linear system

$$\partial_{t_n} \psi = B_n \psi, \quad n = 1, 2, \ldots,$$  \hspace{1cm} (2.1)

with independent variables $t := (t_1, t_2, t_3, \ldots)$, are the Zakharov-Shabat (zero curvature) conditions

$$\partial_{t_n} B_m - \partial_{t_m} B_n = [B_n, B_m].$$  \hspace{1cm} (2.2)

We learned [12, 13] that for several important hierarchies it is more convenient to use instead of the partial derivatives $\partial_{t_n}$ the operators

$$\hat{\chi}_n := p_n(-\bar{\partial}), \quad \bar{\partial} := (\partial_{t_1}, \partial_{t_2}/2, \partial_{t_3}/3, \ldots),$$  \hspace{1cm} (2.3)

where $p_n$ are the elementary Schur polynomials, an insight which can be traced back to [15] (see also [16]). An equivalent form of the above linear system is then

$$\psi_{-\lambda} = E(\lambda) \psi,$$  \hspace{1cm} (2.4)

where $\lambda$ is an indeterminate and $E(\lambda) = \sum_{n \geq 0} \lambda^n E_n$, a formal power series in $\lambda$. Here we use the notation $[\lambda] := (\lambda, \lambda^2/2, \lambda^3/3, \ldots)$ and

$$f_{-\lambda}(t) := f(t - [\lambda]) = \sum_{n=0}^{\infty} \lambda^n \hat{\chi}_n(f)$$  \hspace{1cm} (2.5)

(as a formal power series in $\lambda$), for any object $f$ dependent on $t$. This is sometimes referred to as a Miwa shift. The integrability conditions now read

$$E(\lambda)_{-\mu} E(\mu) = E(\mu)_{-\lambda} E(\lambda),$$  \hspace{1cm} (2.6)

with indeterminates $\lambda, \mu$. Regarding $E(\lambda)$ as a parallel transport operator, (2.6) attains the interpretation of a ‘discrete’ zero curvature condition, as depicted in the following (commutative) diagram.

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1In particular, we have $\hat{\chi}_0 = \text{id}$, $\hat{\chi}_1 = -\partial_{t_2}$, $\hat{\chi}_2 = -\frac{1}{2} \partial_{t_2} + \frac{1}{2} \partial_{t_1}^2$, $\hat{\chi}_3 = -\frac{1}{3} \partial_{t_2} + \frac{1}{2} \partial_{t_2} \partial_{t_1} - \frac{1}{4} \partial_{t_1}^3$, $\hat{\chi}_4 = -\frac{1}{4} \partial_{t_2} + \frac{1}{2} \partial_{t_2} \partial_{t_1} + \frac{1}{6} \partial_{t_1}^3 - \frac{1}{2} \partial_{t_2} \partial_{t_1}^2 + \frac{1}{4} \partial_{t_1}^4$.

2The coefficients $E_n$ can be expressed in terms of the $B_n$ and vice versa. For example, $B_1 = -E_1$, $B_2 = -2E_2 - E_1 t_1 + E_1^2$, $B_3 = -3E_3 - 3E_2 t_1 - E_1 t_1^2 + 2E_1^2 t_1 + E_1 t_1 + 3E_2 E_1 - E_1^3$. We also use ‘positive’ Miwa shifts, $f_{\lambda}(t) := f(t + [\lambda]) = \sum_{n=0}^{\infty} \lambda^n \chi_n(f)$.

3Here ‘discrete’ is used in the sense of [16]. See also [17, 18] for an approach towards integrable equations via discrete zero curvature equations.
Introducing a 'discrete' gauge potential (cf. [19, 20]) via

$$\mathcal{E}(\lambda) = I - \lambda A(\lambda),$$  \hspace{1cm} (2.7)

where $I$ is the unit element of the (typically matrix) algebra from which the coefficients of the formal power series $A(\lambda)$ are taken. (2.6) can be written as

$$\Upsilon(\lambda, \mu) = \Upsilon(\mu, \lambda), \quad \Upsilon(\lambda, \mu) := \mu^{-1}(A(\lambda) - A(\mu)) + A(\lambda - \mu) A(\mu).$$  \hspace{1cm} (2.8)

Equation (2.6) exhibits the following gauge invariance,

$$\lambda^{-1}(\mathcal{B} - \mathcal{B}_{\lambda}) = \mathcal{A}'(\lambda) \mathcal{B}^{-1} = \mathcal{E}'(\lambda)$$  \hspace{1cm} (2.9)

with an invertible $\mathcal{B}$. This originates from the transformation

$$\psi' = \mathcal{B} \psi$$  \hspace{1cm} (2.10)

of the linear system (2.4). In particular, Bäcklund (or Darboux) transformations arise in this way (see also [21], for example). In terms of the gauge potential, (2.9) reads

$$\lambda^{-1}(\mathcal{B} - \mathcal{B}_{\lambda}) = \mathcal{A}'(\lambda) \mathcal{B}^{-1} = \mathcal{E}'(\lambda).$$  \hspace{1cm} (2.11)

In section 3 an Burgers hierarchy results from the simplest non-trivial ansatz for $\mathcal{E}(\lambda)$ (see also appendix A for another version of the matrix Burgers hierarchy). If the gauge potential is linear in the operator of partial differentiation with respect to a variable $x$, we obtain the pKP hierarchy, see section 4. There are more examples (see also [12, 13]) and a generalization of (2.6) which covers multi-component hierarchies.

## 3 The Burgers hierarchy, Cole-Hopf and Bäcklund transformation in functional form

Choosing

$$\mathcal{E}(\lambda) = I - \lambda \phi,$$  \hspace{1cm} (3.1)

so that $\mathcal{A}(\lambda) = \phi$ is independent of $\lambda$, then (2.6) can be expressed as

$$\omega(\lambda) = \omega(\mu), \quad \omega(\lambda) := \lambda^{-1}(\phi - \phi_{\lambda}) + \phi_{-\lambda} \phi.$$  \hspace{1cm} (3.2)

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5More generally, the coefficients of the formal power series $\mathcal{E}(\lambda)$ and $\mathcal{A}(\lambda)$ may be elements of any unital associative algebra $\mathcal{A}$, the elements of which are differentiable with respect to the set of coordinates $t$ (which requires a Banach space structure on $\mathcal{A}$). $\psi$ is then an element of a left $\mathcal{A}$-module.

6(2.9) extends the above planar diagram to a 'commutative cube' where $\mathcal{B}$ acts along the orthogonal bonds.
Since \( \lim_{\lambda \to 0} \omega(\lambda) = \phi_x + \phi^2 \), where \( x := t_1 \), this turns out to be equivalent to
\[
\Omega(\lambda) := \omega(\lambda) - \phi_x - \phi^2 = (\phi - \phi_{-<\lambda>})(\lambda^{-1} - \phi) - \phi_x = 0, \tag{3.3}
\]
which is a functional representation of a (‘noncommutative’) Burgers hierarchy. The first hierarchy equation is the Burgers equation \( \phi_y = \phi_x + 2 \phi_x \phi \), where \( y := t_2 \).

Since the curvature vanishes, we may expect that there is gauge in which the gauge potential \( A \) vanishes. Hence, let us look for an invertible \( f \) such that
\[
f_{-\lambda}^{-1} \mathcal{E}(\lambda) f = I \tag{3.4}
\]
(i.e. \( \mathcal{E}'(\lambda) = I \) and \( B = f^{-1} \) in (2.9)), which is
\[
\lambda^{-1} (f - f_{-\lambda}) = \phi f. \tag{3.5}
\]

**Proposition 3.1** (3.5) is a functional representation of the Cole-Hopf transformation\(^8\)
\[
\phi = f_x f_{-1}, \tag{3.6}
\]
\[
\partial_t f = \partial^3 f \quad n = 2, 3, \ldots. \tag{3.7}
\]

Any invertible \( f \) which solves the linear ‘heat hierarchy’ (3.7) determines via (3.6) a solution of the Burgers hierarchy.\(^9\)

**Proof:** A well-known identity for the elementary Schur polynomials \( p_n \) leads to
\[
n \hat{\chi}_n = -\sum_{k=1}^{n} \partial_{t_k} \hat{\chi}_{n-k} = -\sum_{k=1}^{n-2} \partial_{t_k} \hat{\chi}_{n-k} - \partial_{t_n} + \partial_x \partial_{t_{n-1}} \quad n = 2, 3, \ldots.
\]

With its help one proves by induction that (for an arbitrary integer \( N > 1 \) the first \( N \) of) the equations (3.7) are equivalent to (the first \( N \) of) the equations \( \hat{\chi}_n(f) = 0, \ n = 2, 3, \ldots \). Together with (3.6), these equations are equivalent to (3.5). Furthermore, the integrability condition of (3.5) is the Burgers hierarchy equation (3.2).

**Remark.** Special solutions of the heat hierarchy (3.7) are given by arbitrary linear combinations of the Schur polynomials \( p_n(t) \), \( n = 0, 1, 2, \ldots \), with constant coefficients in \( \hat{\lambda} \). In particular, with constant \( P \in \hat{\lambda} \), the following is a (formal) solution,
\[
\hat{e}^{\xi(P)} = \sum_{n \geq 0} p_n(t) P^n \quad \text{where} \quad \xi(P) := \sum_{m \geq 1} t_m P^m. \tag{3.8}
\]

The transformation equation (2.11) now reads
\[
\lambda^{-1} (B - B_{-\lambda}) = \phi' B - B_{-\lambda} \phi. \tag{3.9}
\]
Taking \( \lambda \to 0 \), this implies
\[
\phi' = B \phi B^{-1} + B_x B^{-1}. \tag{3.10}
\]
\(^8\)This noncommutative version of the Cole-Hopf transformation (see [3, 7, 11, 22–26], for example) for the Burgers equation appeared in [3, 26–28], for instance.
\(^9\)Conversely, if \( \phi \) solves the Burgers hierarchy, choose \( f \) such that \( f_x = \phi f \). Then \( 0 = \Omega(\lambda)f = (\partial_x - \phi_{-\lambda})(\lambda^{-1} (f - f_{-\lambda}) - f_x) \) implies that \( f \) solves the heat hierarchy if \( \partial_x - \phi_{-\lambda} \) is invertible.
Using the last equation to eliminate $\phi'$ from (3.9), yields

$$(B - B_{-}[\lambda])(\lambda^{-1} - \phi) = B_{x}. \quad (3.11)$$

Together with (3.10), this is equivalent to (3.9). Any invertible $B$ which satisfies (3.11), generates via (3.10) a new solution $\phi'$ from a given one $\phi$ of the Burgers hierarchy. Since (3.11) is linear in $B$, linear combinations of solutions (with constant left coefficients) are again solutions of (3.11). Comparison with (3.3) shows that $B = \phi$ is a particular solution. Obviously any constant element $\alpha$ also satisfies (3.11). Hence $B = \alpha + \beta \phi$, with arbitrary constant elements $\alpha, \beta$, satisfies these conditions and (3.10) takes the form

$$\phi' = (\alpha + \beta \phi)(\alpha + \beta \phi)^{-1} + \beta \phi_x (\alpha + \beta \phi)^{-1}, \quad (3.12)$$

assuming the inverse to exist. This covers elementary Bäcklund (or Darboux) transformations obtained in [3, 9, 24, 29] and [30], p.73.

4 The potential KP hierarchy in functional form, and relations with the Burgers hierarchy

Choosing $^{10}$

$$\mathcal{E}(\lambda) = I - \lambda (w(\lambda) + \partial), \quad (4.1)$$

i.e. $A(\lambda) = w(\lambda) + \partial$, where $\partial = \partial_x$, (2.8) leads to the two equations

$$\lambda^{-1}(w(\mu) - w(\mu)_{-}[\lambda]) + w(\mu)_{-}[\lambda] w(\lambda) + w(\lambda)_x \quad (4.2)$$

and

$$\mu^{-1}(w(\lambda) - w(\lambda)_{-}[\mu]) + w(\lambda)_{-}[\mu] w(\mu) + w(\mu)_x \quad (4.3)$$

and

$$w(\lambda) - w(\lambda)_{-}[\mu] = w(\mu) - w(\mu)_{-}[\lambda]. \quad (4.4)$$

The latter is solved by

$$w(\lambda) = \phi - \phi_{-}[\lambda], \quad (4.5)$$

and the first equation then takes the form

$$\omega(\lambda)_{-}[\mu] - \omega(\mu)_{-}[\lambda] = \omega(\lambda) - \omega(\mu) - (\phi_x + \phi^2)_{-}[\lambda] + (\phi_x + \phi^2)_{-}[\mu], \quad (4.6)$$

using the definition in (3.2). Summing this expression three times with cyclically permuted indeterminates $\lambda_1, \lambda_2, \lambda_3$, results in the Bogdanov-Konopelchenko (BK) functional equation $[31, 32]$,

$$\sum_{i,j,k=1}^{3} \epsilon_{ijk} \omega(\lambda_i)_{-}[\lambda_j] = 0 \quad (4.7)$$

$^{10}$Starting instead with $\mathcal{E}(\lambda) = I - \lambda w(\lambda)\partial$, leads in the same way to the modified KP hierarchy $[12]$. The two choices of $\mathcal{E}(\lambda)$ are related by a gauge transformation (Miura transformation).
(where $\epsilon_{ijk}$ is totally antisymmetric with $\epsilon_{123} = 1$). This determines the pKP hierarchy and is equivalent to (4.6). Expanding (4.6) in $\lambda, \mu$, yields $\partial_\phi \phi = \partial_{t_1} \phi$ and

$$\hat{\chi}_n \hat{\chi}_{n+1}(\phi) - \hat{\chi}_{n+1} \hat{\chi}_n(\phi) = \hat{\chi}_n \hat{\chi}_n(\phi) \phi_n - \hat{\chi}_n(\phi) \phi_0 \quad m, n = 1, 2, \ldots . (4.8)$$

An equivalent expression of the pKP hierarchy (in the scalar case) appeared already in [33] (see also [10, 12]). For $m = 1, n = 2$, this yields the pKP equation

$$(4 \phi_t - \phi_{xxx} - 6 \phi_x^2)_{xx} - 3 \phi_{yy} + 6[\phi_x, \phi_y] = 0 , 

(4.9)$$

where $x = t_1, y = t_2, t = t_3$. Comparing (3.2) with (4.6) shows that any solution of the Burgers hierarchy, considered in section 3, also solves the pKP hierarchy.

**Remark.** There is a (Sato-Wilson) pseudo-differential operator $W = I + \sum_{n>0} w_n \partial^{-n}$ such that $B = W^{-1}$ in (2.9) transforms $\hat{\chi}(\lambda)$ to $\hat{\chi}'(\lambda) = I - \lambda \partial$. It is determined (up to multiplication by a constant operator $I + \sum_{n>0} c_n \partial^{-n}$) by $w_1 - w_{1,-[\lambda]} = \phi_{-[\lambda]} - \phi$ and $w_n + 1 - w_{n-1, -[\lambda]} = \lambda^{-1}(w_n - w_{n-1, -[\lambda]}) - w_{n-1} - (\phi - \phi_{-[\lambda]}) w_n$.

### 4.1 The pKP hierarchy as an inhomogeneous Burgers hierarchy

We observe that (4.6) can also be written as

$$\Omega(\mu) - \Omega(\mu+\lambda) = \Omega(\lambda) - \Omega(\lambda+\mu) 

(4.10)$$

where $\Omega(\lambda)$ is the expression defined in (3.3) in terms of $\phi$. As a consequence, the pKP hierarchy takes the form

$$\Omega(\lambda) = \theta - \theta_{-[\lambda]} . 

(4.11)$$

with some $\theta$. If the right hand side vanishes, i.e. if $\theta$ is constant, this is precisely the functional representation (3.3) of the Burgers hierarchy considered in section 3. (4.11) is equivalent to

$$\hat{\chi}_n(\phi) - \hat{\chi}_n(\phi) = \hat{\chi}_n(\theta) \quad n = 1, 2, \ldots . 

(4.12)$$

The first two equations are

$$\phi_y = \phi_{xx} + 2 \phi_x \phi + 2 \theta_x , 

(4.13)$$

$$\phi_t = \phi_{xxx} + 3 \phi_{xx} \phi + 3 \phi_x^2 \phi + 3 \phi_x \phi^2 + 3 \theta_x \phi + \frac{3}{2} \theta_y (\theta_x + \theta_{xx}) , 

(4.14)$$

where we used the first to replace $\phi_y$ in the second equation. For constant $\theta$, these are the first two equations of the Burgers hierarchy. Eliminating $\theta$ from (4.13) and (4.14), we recover the pKP equation (4.9).

Application of a Miwa shift to (4.6) leads to

$$\tilde{\omega}(\lambda) - \tilde{\omega}(\mu) = \tilde{\omega}(\lambda) - \tilde{\omega}(\mu) - (\phi_x + \phi^2)_{[\lambda]} + (\phi_x + \phi^2)_{[\mu]} , 

(4.15)$$

where

$$\tilde{\omega}(\lambda) := \omega(\lambda)_{[\lambda]} = \lambda^{-1}(\phi_{[\lambda]} - \phi) + \phi_{[\lambda]} . 

(4.16)$$

Since this can be written as

$$\tilde{\Omega}(\lambda)_{[\mu]} - \tilde{\Omega}(\lambda) = \tilde{\Omega}(\mu)_{[\lambda]} - \tilde{\Omega}(\mu) 

(4.17)$$

6
with
\[ \hat{\Omega}(\lambda) := \hat{\omega}(\lambda) - \phi_x - \phi^2 = (\lambda^{-1} + \phi)(\phi_{[\lambda]} - \phi) - \phi_x, \]
the pKP hierarchy can also be expressed as
\[ \hat{\Omega}(\lambda) = \hat{\theta}_{[\lambda]} - \hat{\theta} \]
with some \( \tilde{\theta} \). If \( \tilde{\theta} \) is constant, so that the right hand side vanishes, the last equation reduces to the ‘opposite’ Burgers hierarchy (see also appendix A),
\[ (\lambda^{-1} + \phi)(\phi_{[\lambda]} - \phi) = \phi_x, \]
which starts with \( \phi_y = -\phi_{xx} - 2 \phi \phi_x \). In particular, we have the following result.

**Proposition 4.1** Any solution of any of the two Burgers hierarchies also solves the pKP hierarchy.

### 4.2 A Cole-Hopf transformation for the matrix pKP hierarchy

**Theorem 4.1** Let \((\mathbb{A}, \cdot)\) be the algebra of \( M \times N \) matrices of functions of \( t \) with the product
\[ A \cdot B = AQB, \]
where the ordinary matrix product is used on the right hand side, and \( Q \) is a constant \( N \times M \) matrix. Let \( X \) be an invertible \( N \times N \) matrix and \( Y \in \mathbb{A}_x \) such that \( X, Y \) solve the linear heat hierarchy (3.7) and satisfy
\[ X_x = RX + QY, \]
with a constant \( N \times N \) matrix \( R \). The pKP hierarchy in \((\mathbb{A}, \cdot)\) is then solved by
\[ \phi := YX^{-1}. \]

**Proof:** By use of (4.23), the expression \( \Omega(\lambda) \) defined in (3.3) (where because of (4.21) a factor \( Q \) enters the nonlinear term) can be written as
\[ \Omega(\lambda) = (\phi - \phi_{[\lambda]})(X_x - QY)X^{-1} + (\lambda^{-1}(Y - \lambda_{[\lambda]}) - \phi_x)X^{-1} \]
\[ -\phi_{[\lambda]}(\lambda^{-1}(X - \lambda_{[\lambda]}) - \phi_x)X^{-1} \]
If \( X, Y \) solve the heat hierarchy, then \( \hat{x}_n(X) = 0 = \hat{x}_n(Y), n = 2, 3, \ldots \), and thus
\[ \lambda^{-1}(X - \lambda_{[\lambda]}) = X_x, \quad \lambda^{-1}(Y - \lambda_{[\lambda]}) = Y_x. \]
Using these equations, the above expression for \( \Omega(\lambda) \) reduces to
\[ \Omega(\lambda) = (\phi - \phi_{[\lambda]})(X_x - QY)X^{-1}. \]
If \( (X_x - QY)X^{-1} \) is constant, say \( R \), which means that (4.22) holds, this takes the form (4.11) of the pKP hierarchy with \( \theta = \phi R \). Thus \( \phi \) solves the pKP hierarchy.

If \( R = 0 \), and with \( M = N \) and \( Q = I_N \), (4.22) and (4.23) reduce to \( \phi = X_x X^{-1} \), and we recover the Cole-Hopf transformation for the Burgers hierarchy. Note that the conditions imposed on \( X \) already imply \( Q(\lambda^{-1}(Y - \lambda_{[\lambda]}) - \phi_x) = 0 \) and thus \( Y \) automatically satisfies the heat hierarchy if \( Q \) has maximal rank. Furthermore, if we consider \( Q \phi \) instead of \( \phi \), the assumption on \( Y \) can be dropped.

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\(^{11}\tilde{\theta} \) and \( \theta \) are related by \( \tilde{\theta} - \theta = \phi_x + \phi^2 \).

\(^{12}\)Note also that \( \tilde{\theta} = \theta + \phi_x + \phi Q \phi = Y_x X^{-1} \) by use of (4.22).
Corollary 4.1 Let \( X \) solve the heat hierarchy and (4.22) with some \( Y \). Then \( Q\phi \) with \( \phi \) given by (4.23) solves the \( N \times N \) matrix pKP hierarchy with the usual matrix product.

For the case where \( \text{rank}(Q) = 1 \) (cf. [14] and references therein), a similar result appeared already in [11]. Then \( \text{tr}(QA \cdot B) = \text{tr}(QA) \, \text{tr}(QB) \), so that, by use of (4.22),
\[
\varphi := \text{tr}(Q\phi) = -\text{tr}(R) + (\log \tau)_x \quad \text{with} \quad \tau := \det(X) \tag{4.24}
\]
solves the scalar pKP hierarchy.

4.3 Bäcklund and Darboux transformations

Inserting the ansatz \( B = b(t) - \partial \) in (2.11) leads to the two equations
\[
b - \phi' + \phi = (b - \phi' + \phi)_{[-1]} \tag{4.25}
\]
and
\[
\lambda^{-1}(b - b_{[-1]}) - b_x = (\phi' - \phi'_{[-1]})b - b_{[-1]}(\phi - \phi_{[-1]}) + (\phi - \phi_{[-1]})_{x} \tag{4.26}
\]
The solution of (4.25) is
\[
b = \phi' - \phi \tag{4.27}
\]
(absorbing an additive constant into \( \phi' \)). (4.26) can then be written as
\[
\Omega(\lambda) - \Omega'(\lambda) = \Gamma(\phi, \phi') - \Gamma(\phi, \phi'_{[-1]}), \tag{4.28}
\]
where \( \Omega'(\lambda) \) is built with \( \phi' \), and
\[
\Gamma(\phi, \phi') := (\phi' - \phi) \phi - \phi_{x}. \tag{4.29}
\]
This is an elementary Bäcklund transformation (BT) of the pKP hierarchy.\( \text{13} \) Using (4.11), we find
\[
0 = \Gamma(\phi, \phi') + \theta' - \theta = \phi' \phi - \theta' - \tilde{\theta}. \tag{4.30}
\]
Let \( B_{n,m} \) denote the BT taking a pKP solution \( \phi_m \) to a new solution \( \phi_n \). The permutability relation\( \text{14} \) \( \mathcal{B}_{(3,1)}\mathcal{B}_{(1,0)} = \mathcal{B}_{(3,2)}\mathcal{B}_{(2,0)} \) then results in\( \text{15} \)
\[
(\phi_2 - \phi_1)_x = \phi_3 (\phi_2 - \phi_1) + (\phi_2 - \phi_1)\phi_0 + \phi_1^2 - \phi_2^2. \tag{4.31}
\]
This determines algebraically a new solution \( \phi_3 \) in terms of a given solution \( \phi_0 \) and corresponding Bäcklund descendants \( \phi_1, \phi_2 \).

In the case under consideration, the linear system (2.4) takes the form
\[
\lambda^{-1}(\psi - \psi_{[-1]}) - \psi_x = (\phi - \phi_{[-1]})\psi \tag{4.32}
\]
(cf. [15] for an equivalent version in the scalar case). If \( \psi \) is invertible, we obtain
\[
\phi - \phi_{[-1]} = \lambda^{-1}(\psi - \psi_{[-1]})\psi^{-1} - \psi_x \psi^{-1}. \tag{4.33}
\]
\( \text{13} \)Extending the above ansatz for \( B \) to \( n \)th order in \( \partial \) leads to equations which determine \( n \)th order BTs. These are solved by an \( n \)-fold product of elementary BTs.

\( \text{14} \)Note that this is also a discrete zero curvature condition.

\( \text{15} \)In the commutative scalar case, setting \( \phi = \tau_2 / \tau \) with a function \( \tau \), yields \( \tau_0 \tau_3 = \tau_1 \tau_{2,x} - \tau_{1,x} \tau_2 \).
Eliminating $\phi'$ from (4.26) with the help of (4.27), and then $\phi - \phi_{-|\lambda|}$ by use of the last equation, turns it into

\[
(b - \psi_x \psi^{-1})_x + (b - \psi_x \psi^{-1})(b + \lambda^{-1} \psi_{-|\lambda|} \psi^{-1})
- (b_{-|\lambda|} + \lambda^{-1} \psi_{-|\lambda|} \psi^{-1})(b - \psi_x \psi^{-1}) = 0 .
\] (4.34)

This equation is obviously solved by

\[
b = \psi_x \psi^{-1} .
\] (4.35)

Hence, if $\psi_1$ solves the linear system with a solution $\phi$ of the pKP hierarchy, then

\[
\phi' = \phi + \psi_{1,x} \psi_1^{-1}
\] (4.36)

is a new solution of the pKP hierarchy.\textsuperscript{16} This is a Darboux transformation \textsuperscript{[30,34–36].}

\section{Functional Riccati equations associated with KP and Burgers hierarchies, and exact solutions}

Let us consider the BK functional equation (4.7) in the algebra $(\mathbb{A}, \cdot)$, where $\mathbb{A}$ is the set of $M \times N$ matrices of complex functions of $t$, supplied with the product (4.21). The simplest non-trivial equation, which results from this formula by expansion in powers of the indeterminates, is the matrix pKP equation

\[
(4 \phi_t - \phi_{xxx} - 6 \phi_x \phi_x) = 3 \phi_{yy} - 6(\phi_x Q \phi_y - \phi_y Q \phi_x) .
\] (5.1)

As a consequence, $\phi Q$ satisfies the $M \times M$ matrix pKP hierarchy and $Q \phi$ the $N \times N$ matrix pKP hierarchy. Moreover, if $Q = V U^T$, with an $N \times m$ matrix $V$ and an $M \times m$ matrix $U$, then $U^T \phi V$ satisfies the $m \times m$ matrix pKP hierarchy. In particular, for $m = 1$ this becomes the scalar pKP hierarchy. In the latter case, $Q$ has rank one.

The crucial observation now is that the BK functional equation, and thus the pKP hierarchy, is satisfied if $\phi$ solves

\[
\omega(\lambda) = S + L \phi - \phi_{-|\lambda|} R
\] (5.2)

with constant matrices $S, L, R$ of dimensions $M \times N, M \times M$ and $N \times N$, respectively. This is a functional matrix Riccati equation for $\phi$,

\[
\lambda^{-1}(\phi - \phi_{-|\lambda|}) = S + L \phi - \phi_{-|\lambda|} R - \phi_{-|\lambda|} Q \phi .
\] (5.3)

The integrability condition of this functional equation is satisfied\textsuperscript{17}, since

\[
(\phi_{-|\lambda|})_{|\mu|} = [(\lambda^{-1} - L)\phi_{-|\mu|} - S][((\lambda^{-1} - R) - Q \phi_{-|\mu|})]^{-1}
= [(\lambda^{-1} - L)(\mu^{-1} - L)\phi - (\lambda^{-1} - \mu^{-1})S + LS + SR + SQ \phi]
\times[(\lambda^{-1} - R)(\mu^{-1} - R) - (\lambda^{-1} + \mu^{-1})Q \phi + (QR + QL)\phi + QS]^{-1}
\] (5.4)

is symmetric in $\lambda, \mu$ and thus equals $(\phi_{-|\mu|})_{-|\lambda|}$. The Riccati equation implies

\[
\Omega(\lambda) = (\phi - \phi_{-|\lambda|}) R , \quad \hat{\Omega}(\lambda) = L (\phi_{|\lambda|} - \phi) .
\] (5.5)

\textsuperscript{16}Moreover, $\psi' = B \psi = \psi_x - \psi_{1,x} \psi_1^{-1} \psi$ satisfies the linear system with $\phi'$.

\textsuperscript{17}This also follows from our work in [10] and is the reason for the choice of the right hand side of (5.2).
This shows that with $R = 0$ (respectively $L = 0$), any solution of (5.3) also solves the Burgers hierarchy (3.3) (respectively the opposite Burgers hierarchy (4.20)) in $(\lambda, \cdot)$.

It is well-known that matrix Riccati equations can be linearized [37, 38]. In fact, (5.3) can be lifted to a linear equation on the space of $(N + M) \times N$ matrices:

$$\lambda^{-1}(Z - Z_{-[\lambda]}) = HZ$$

with

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad H = \begin{pmatrix} R & Q \\ S & L \end{pmatrix}.$$  \hspace{1cm} (5.7)

Hence

$$\lambda^{-1}(X - X_{-[\lambda]}) = RX + QY, \quad \lambda^{-1}(Y - Y_{-[\lambda]}) = SX + LY.$$  \hspace{1cm} (5.8)

Assuming that $X$ is invertible and setting

$$\phi = Y X^{-1},$$

these equations imply

$$\phi_{-[\lambda]} = Y_{-[\lambda]} X_{-[\lambda]}^{-1} = [\phi - \lambda(S + L\phi)][I_N - \lambda(R + Q\phi)]^{-1},$$

which is (5.3). Thus any solution $Z$ of the linear functional equation (5.6) with invertible $X$ determines via (5.9) a solution of the functional matrix Riccati equation (5.3), and thus a solution of the matrix pKP hierarchy we started with.

**Remark.** The first of equations (5.8) is equivalent to (4.22) and the heat hierarchy for $X$. Since the second of (5.8) implies that also $Y$ has to solve the heat hierarchy, according to theorem 4.1 the $\phi$ determined by (5.9) already solves the pKP hierarchy without use of the additional equation $Y_x = SX + LY$ which results from the second of (5.8). However, this equation helps to single out interesting classes of solutions, see below. In any case, the Riccati approach corresponds to a class of (generalized) Cole-Hopf transformations in the sense of theorem 4.1. Note also that $\tilde{\theta} = S + L\phi$.

The general solution of (5.6) is

$$Z = e^{\xi(H)} Z_0, \quad \xi(H) = \sum_{n \geq 1} H^n t_n, \quad Z_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix},$$

with invertible $X_0$. As a consequence, $Z_{t_n} = H^n Z$. Writing

$$e^{\xi(H)} = \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{pmatrix},$$

we have

$$\phi = (\Xi_{21} + \Xi_{22} \phi_0)(\Xi_{11} + \Xi_{12} \phi_0)^{-1},$$

where $\phi_0 = Y_0 X_0^{-1}$. This is a matrix fractional transformation with coefficients depending on $t$. For any choice of the matrices $S, L, R, Q$, this $\phi$ is a solution of the

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18 This is achieved by regarding $\phi(t)$ as an element of the Grassmannian $G(N, N + M)$ of $N$-dimensional linear subspaces of $\mathbb{C}^{N+M}$ via $\kappa(\phi) = \text{span}(I_N, \phi^T)$, since $\kappa^{-1} : G(N, N + M) \rightarrow \mathbb{C}^{M \times N}$ defines a chart for the manifold $G(N, N + M)$.  

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pKP hierarchy in the matrix algebra with product (4.21). The practical problem is to compute \( e^{\xi(H)} \) explicitly.

**Remark.** With \( Z = e^{\xi(H)}Z_0 \) also \( T^*X \) satisfies (5.6) if \( T \) is constant and commutes with \( H \). In particular, \( T = k_{M+N} + \hat{H} \) with any constant \( k \) induces such a transformation. It results in the matrix fractional transformation (with constant coefficients) \( \phi' = (S + L'F)(R' + QF)^{-1} \) with \( L' := L + k_{M}, \ R' := R + k_{N} \).

**Example 1.** Let \( S = 0 \) and
\[
Q = RK - KL
\]
with a constant \( N \times M \) matrix \( K \). Then we have
\[
H^n = \begin{pmatrix}
R^n & R^n K - KL^n \\
0 & L^n
\end{pmatrix}, \quad \xi(H) = \begin{pmatrix}
\xi(R) & \xi(R)K - K\xi(L) \\
0 & \xi(L)
\end{pmatrix},
\]
and thus
\[
e^{\xi(H)} = \begin{pmatrix}
e^{\xi(R)} & e^{\xi(R)K - K \xi(L)} \\
0 & e^{\xi(L)}
\end{pmatrix},
\]
so that (5.13) becomes
\[
\phi = e^{\xi(L)}\phi_0(I_N + K\phi_0 - e^{-\xi(R)}K e^{\xi(L)}\phi_0)^{-1} e^{-\xi(R)}. \tag{5.17}
\]
If \( Q \) has rank one, then we obtain the following solution of the scalar pKP hierarchy,
\[
\varphi = \text{tr}(Q\phi) = \text{tr} \log(I_N + K\phi_0 - e^{-\xi(R)}K e^{\xi(L)}\phi_0) = (\log \tau)_x, \quad \tau = \det(I_N + K\phi_0 - e^{-\xi(R)}K e^{\xi(L)}\phi_0), \tag{5.19}
\]
which includes well-known formulae for KP multi-solitons [39] and resonances (see e.g. [40, 41] and references therein).

**Example 2.** Let \( M = N \) and
\[
L = S\pi_-, \quad R = \pi_+S, \quad Q = \pi_+S\pi_-, \tag{5.20}
\]
with constant \( N \times N \) matrices \( S, \pi_\pm \) such that \( \pi_+ + \pi_- = I_N \). It is easy to see that
\[
H^n = \begin{pmatrix}
\pi_+ S^n & \pi_+ S^n \pi_- \\
S^n & S^n \pi_-
\end{pmatrix}, \tag{5.21}
\]
As a consequence, we find
\[
e^{\xi(H)} = \begin{pmatrix}
\pi_+ + \pi_+ e^{\xi(S)} & \pi_+ e^{\xi(S)} - I_N \\
e^{\xi(S)} - I_N & \pi_+ + e^{\xi(S)}
\end{pmatrix}, \tag{5.22}
\]
and (5.13) reads
\[
\phi = (-A + e^{\xi(S)}B)(\pi_- A + \pi_+ e^{\xi(S)}B)^{-1}, \tag{5.23}
\]
where \( A := I_N - \pi_+\phi_0, \ B := I_N + \pi_-\phi_0 \). If \( \text{rank}(\pi_+ S\pi_-) = 1 \), then
\[
\varphi = \text{tr}(Q\phi) = -\text{tr}(\pi_+ S) + (\log \tau)_x, \quad \tau = \det(\pi_- A + \pi_+ e^{\xi(S)}B). \tag{5.24}
\]
For example, let $N = m + n$ and choose

$$\pi_- = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_+ = \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}. \tag{5.25}$$

Writing

$$\phi_0 = \begin{pmatrix} \phi_0 \\ \phi_0 \end{pmatrix} - \begin{pmatrix} \phi_0 \\ \phi_0 \end{pmatrix}, \quad S = \begin{pmatrix} S_- & S_+ \\ S_+ & S_+ \end{pmatrix}, \tag{5.26}$$

$\text{rank}(Q) = 1$ means $\text{rank}(S_{+-}) = 1$ (see also [14]) and we find

$$\tau = \det\left( e^{x(S)} \right), \tag{5.27}$$

In particular, if $S$ is the shift operator $Se_i = e_{i+1}$, this determines $\tau$-functions which can be expressed in terms of Schur polynomials. This corresponds to a finite version of the Sato theory, see [14]. For example, if $m = n = 2$ and

$$\phi_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.28}$$

we obtain

$$\tau = 1 + cx + a\left(y + \frac{x^2}{2}\right) + d\left(y - \frac{x^2}{2}\right) + b\left(t - \frac{x^3}{3}\right) + (ad - bc)\left(-xt + y^2 + \frac{x^4}{12}\right). \tag{5.29}$$

**Appendix A: Opposite Burgers hierarchy and beyond**

We generalize the ansatz for $E(\lambda)$ considered in section 3 to

$$E(\lambda) = I - \lambda \sum_{n \geq 0} \lambda^n \phi_n. \tag{A.1}$$

Then (2.8) takes the form

$$\hat{\chi}_n(\phi_m) = \sum_{k=0}^{n} \hat{\chi}_k(\phi_m) \phi_{n-k} - \sum_{k=0}^{m} \hat{\chi}_k(\phi_n) \phi_{m-k}, \tag{A.2}$$

where $m, n = 0, 1, 2, \ldots$. This is an infinite system of coupled equations. As in section 3, we look for a gauge transformation such that $f^{-1}_- E(\lambda) f = I$, which is

$$\lambda^{-1}(f - f_-) = \sum_{n \geq 0} \lambda^n \phi_n f. \tag{A.3}$$

Expanding the left hand side in powers of $\lambda$, this becomes a generalization of the Cole-Hopf transformation,

$$\phi_0 = f_x f^{-1}, \quad \phi_n = -\hat{\chi}_{n+1}(f) f^{-1} \quad n = 1, 2, \ldots. \tag{A.4}$$

By construction, this solves the zero curvature equation and thus the hierarchy (A.2). The gauge transformation (2.11) takes the form

$$\lambda^{-1}(B - B_-) = \sum_{n=0}^{\infty} \lambda^n (\phi_n^* B - B_- \phi_n), \tag{A.5}$$
and thus
\[
\phi_0' = B \phi_0 B^{-1} + B_x B^{-1},
\]
(A.6)
\[
\hat{\chi}_{n+1}(B) = -\phi_n' + \sum_{k=0}^{n} \chi_k(B) \phi_{n-k} \quad n = 1, 2, \ldots.
\]
(A.7)

**Example 1.** Setting \( \phi_n = -\hat{\chi}_n(\phi), n = 0, 1, \ldots \), so that
\[
E(\lambda) = I + \lambda \phi_{-\lambda},
\]
(A.8)
the subsystem of (A.2) for \( m = 0 \) reads
\[
\hat{\chi}_{n+1}(\phi) + \hat{\chi}_n(\phi_x + \phi^2) - \hat{\chi}_n(\phi) = 0 \quad n = 0, 1, \ldots,
\]
(A.9)
which in functional form, and after a Miwa shift, becomes the representation (4.20) of the ‘opposite’ Burgers hierarchy. The remaining equations resulting from (A.2) are
\[
\hat{\chi}_m \hat{\chi}_{n+1}(\phi) - \hat{\chi}_n \hat{\chi}_{m+1}(\phi) = \sum_{k=1}^{m} \hat{\chi}_{m-k} \hat{\chi}_n(\phi) \hat{\chi}_k(\phi) - \sum_{k=1}^{n} \hat{\chi}_n-k \hat{\chi}_m(\phi) \hat{\chi}_k(\phi)
\]
where \( m, n = 1, 2, \ldots \). By use of the Hasse-Schmidt derivation property of the \( \hat{\chi}_n \), this is the form (4.8) of the pKP hierarchy. But we already know that the latter is satisfied as a consequence of the Burgers hierarchy. Equations (A.4) take the form
\[
\phi = -f_x f^{-1}, \quad \hat{\chi}_n(\phi) = \hat{\chi}_{n+1}(f) f^{-1} \quad n = 1, 2, \ldots.
\]
(A.10)
This leads to the linear functional equation
\[
f^{-1}_{\lambda[x]} = f^{-1} + \lambda (f^{-1})_x,
\]
(A.11)
and thus \( \chi_n(f^{-1}) = 0 \) for \( n = 2, 3, \ldots \), which is equivalent to the following version of a linear heat hierarchy,
\[
\partial_{t_n}(f^{-1}) = (-1)^{n+1} \partial_x^n(f^{-1}) \quad n = 2, 3, \ldots.
\]
(A.12)
As a consequence, if \( f^{-1} \) solves the linear hierarchy (A.12), then \( \phi = -f_x f^{-1} \) solves the Burgers hierarchy (4.20) and thus also the pKP hierarchy.

Equations (A.6) and (A.7) are turned into
\[
\phi' = (I + \lambda \phi') B_x B^{-1}, \quad (I + \lambda \phi') B_{\lambda} = B (I + \lambda \phi).
\]
(A.13)
Using the first in the second equation to eliminate \( \phi' \), yields an equation linear in \( B^{-1} \),
\[
(\lambda^{-1} + \phi)(B^{-1}_{\lambda} - B^{-1}) = (B^{-1})_x.
\]
(A.14)
Comparison with the Burgers hierarchy system (4.20) shows that \( B^{-1} = \phi \) is a solution. More generally, \( B^{-1} = \alpha + \phi \beta \) with any constant \( \alpha, \beta \) solves this equation.

**Example 2.** Setting \( \phi_n = 0 \) for \( n > 0 \) and \( \phi := \phi_0 \), reduces the hierarchy (A.2) to the Burgers hierarchy of section 3, and the second of equations (A.4) requires that \( f \) has to solve the linear heat hierarchy. Relaxing the constraint to \( \phi_0 = 0 \) for \( n > 1 \), thus leaving \( \phi_0 \) and \( \phi_1 \) as dependent variables, (A.2) results in
\[
(\hat{\chi}_{n+1}(\phi_0) - \hat{\chi}_n(\phi_0) \phi_0 - \hat{\chi}_{n-1}(\phi_0) \phi_1) \delta_{m,0} + (\hat{\chi}_{n+1}(\phi_1) - \hat{\chi}_n(\phi_1) \phi_0 - \hat{\chi}_{n-1}(\phi_1) \phi_1) \delta_{m,1} = (\hat{\chi}_{m+1}(\phi_0) - \hat{\chi}_m(\phi_0) \phi_0 - \hat{\chi}_{m-1}(\phi_0) \phi_1) \delta_{n,0} + (\hat{\chi}_{m+1}(\phi_1) - \hat{\chi}_m(\phi_1) \phi_0 - \hat{\chi}_{m-1}(\phi_1) \phi_1) \delta_{n,1}.
\]
(A.15)
It is sufficient to consider $m < n$. For $m = 0, n = 1$, this yields
\[ \phi_{0,y} - \phi_{0,xx} - 2\phi_{0,x}\phi_0 = 2\phi_{1,x} + 2[\phi_1, \phi_0] . \quad (A.16) \]
The remaining equations which result from (A.15) are ($m = 0, n > 1$)
\[ \hat{\chi}_{n+1}(\phi_0) - \hat{\chi}_n(\phi_0) \phi_0 - \hat{\chi}_{n-1}(\phi_0) \phi_1 = 0 \quad n = 2, 3, \ldots , \quad (A.17) \]
and ($m = 1, n > 1$)
\[ \hat{\chi}_{n+1}(\phi_1) - \hat{\chi}_n(\phi_1) \phi_0 - \hat{\chi}_{n-1}(\phi_1) \phi_1 = 0 \quad n = 2, 3, \ldots . \quad (A.18) \]
In the case under consideration, equations (A.4) take the form
\[ \phi_0 = f_x f^{-1}, \quad \phi_1 = -\hat{\chi}_2(f) f^{-1} = \frac{1}{2} (f_y - f_{xx}) f^{-1} , \quad (A.19) \]
and
\[ \hat{\chi}_n(f) = 0 , \quad n = 3, 4, \ldots , \quad (A.20) \]
which is not equivalent to the heat hierarchy since $\hat{\chi}_2(f) = 0$ is missing.

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