Weak type estimates for intrinsic square functions on the weighted Morrey spaces

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Abstract
In this paper, we will obtain the weak type estimates of intrinsic square functions including the Lusin area integral, Littlewood-Paley g-function and $g_{δ}$-function on the weighted Morrey spaces $L^{1,κ}(w)$ for $0 < κ < 1$ and $w ∈ A_1$.

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1 Introduction and main results
Let $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, ∞)$ and $ϕ_t(x) = t^{-n}ϕ(x/t)$. The classical square function (Lusin area integral) is a familiar object. If $u(x, t) = P_t * f(x)$ is the Poisson integral of $f$, where $P_t(x) = c_{n} t^{n/2} e^{-|x|^2/4t}$ denotes the Poisson kernel in $\mathbb{R}^{n+1}_+$. Then we define the classical square function (Lusin area integral) $S(f)$ by (see [16] and [17])

$$S(f)(x) = \left( \int \int_{Γ(x)} |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2},$$

where $Γ(x)$ denotes the usual cone of aperture one:

$$Γ(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$$

and

$$|\nabla u(y, t)| = \left| \frac{∂u}{∂t} \right|^2 + \sum_{j=1}^{n} \left| \frac{∂u}{∂y_j} \right|^2.$$ 

Similarly, we can define a cone of aperture $β$ for any $β > 0$:

$$Γ_β(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < βt\},$$

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and corresponding square function

\[ S_\beta(f)(x) = \left( \iint_{\Gamma_\beta(x)} |\nabla u(y, t)|^2 t^{1-n} \, dy \, dt \right)^{1/2}. \]

The Littlewood-Paley \(g\)-function (could be viewed as a “zero-aperture” version of \(S(f)\)) and the \(g^*_\lambda\)-function (could be viewed as an “infinite aperture” version of \(S(f)\)) are defined respectively by

\[ g(f)(x) = \left( \int_0^\infty |\nabla u(x, t)|^2 t \, dt \right)^{1/2} \]

and

\[ g^*_\lambda(f)(x) = \left( \int\int_{\mathbb{R}^{n+1}} \left( \frac{t}{t+|x-y|} \right) \lambda^n |\nabla u(y, t)|^2 t^{1-n} \, dy \, dt \right)^{1/2}, \quad \lambda > 1. \]

The modern (real-variable) variant of \(S_\beta(f)\) can be defined in the following way (here we drop the subscript \(\beta\) if \(\beta = 1\)). Let \(\psi \in C_\infty(\mathbb{R}^n)\) be real, radial, have support contained in \(\{x : |x| \leq 1\}\), and \(\int_{\mathbb{R}^n} \psi(x) \, dx = 0\). The continuous square function \(S_{\psi,\beta}(f)\) is defined by (see, for example, [2] and [3])

\[ S_{\psi,\beta}(f)(x) = \left( \iint_{\Gamma_\beta(x)} |f * \psi_t(y)|^2 dy \, dt \right)^{1/2}. \]

In 2007, Wilson [25] introduced a new square function called intrinsic square function which is universal in a sense (see also [26]). This function is independent of any particular kernel \(\psi\), and it dominates pointwise all the above-defined square functions. On the other hand, it is not essentially larger than any particular \(S_{\psi,\beta}(f)\). For \(0 < \alpha \leq 1\), let \(C_\alpha\) be the family of functions \(\varphi\) defined on \(\mathbb{R}^n\) such that \(\varphi\) has support containing in \(\{x \in \mathbb{R}^n : |x| \leq 1\}\), \(\int_{\mathbb{R}^n} \varphi(x) \, dx = 0\), and for all \(x, x' \in \mathbb{R}^n\),

\[ |\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha. \]

For \((y, t) \in \mathbb{R}^{n+1}_+\) and \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\), we set

\[ A_\alpha(f)(y, t) = \sup_{\varphi \in C_\alpha} |f * \varphi_t(y)| = \sup_{\varphi \in C_\alpha} \left| \int_{\mathbb{R}^n} \varphi_t(y-z) f(z) \, dz \right|. \]

Then we define the intrinsic square function of \(f\) (of order \(\alpha\)) by the formula

\[ S_\alpha(f)(x) = \left( \iint_{\Gamma(x)} \left( A_\alpha(f)(y, t) \right)^2 dy \, dt \right)^{1/2}. \]

We can also define varying-aperture versions of \(S_\alpha(f)\) by the formula

\[ S_{\alpha,\beta}(f)(x) = \left( \iint_{\Gamma_\beta(x)} \left( A_\alpha(f)(y, t) \right)^2 dy \, dt \right)^{1/2}. \]
The intrinsic Littlewood-Paley $g$-function and the intrinsic $g^*_\lambda$-function will be given respectively by
\[
g_\alpha(f)(x) = \left( \int_0^{\infty} \left( A_\alpha(f)(x,t) \right)^2 \frac{dt}{t} \right)^{1/2}
\]

and
\[
g^*_{\lambda,\alpha}(f)(x) = \left( \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x-y|} \right)^{\lambda n} \left( A_\alpha(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad \lambda > 1.
\]

In [25] and [26], Wilson has established the following theorems.

**Theorem A.** Let $0 < \alpha \leq 1$, $1 < p < \infty$ and $w \in A_p$ (Muckenhoupt weight class). Then there exists a constant $C > 0$ independent of $f$ such that
\[
\|S_\alpha(f)\|_{L^p_w} \leq C \|f\|_{L^p_w}.
\]

**Theorem B.** Let $0 < \alpha \leq 1$ and $p = 1$. Then for any given weight function $w$ and $\lambda > 0$, there exists a constant $C > 0$ independent of $f$ and $\lambda$ such that
\[
w(\{ x \in \mathbb{R}^n : S_\alpha(f)(x) > \lambda \}) \leq C \int \frac{|f(x)| Mw(x)}{x}\,dx,
\]
where $M$ denotes the standard Hardy-Littlewood maximal operator.

Moreover, in [12], Lerner showed sharp $L^p_w$ norm inequalities for the intrinsic square functions in terms of the $A_p$ characteristic constant of $w$ for all $1 < p < \infty$. For further discussions about the boundedness of intrinsic square functions on some other weighted spaces, we refer the reader to [10, 19, 20, 23, 24].

On the other hand, the classical Morrey spaces $L^{p,\lambda}$ were first introduced by Morrey in [13] to study the local behavior of solutions to second order elliptic partial differential equations. Since then, these spaces play an important role in studying the regularity of solutions to partial differential equations. For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on these spaces, we refer the reader to [11, 4, 15]. For the properties and applications of classical Morrey spaces, see [5, 6, 7] and the references therein.

In 2009, Komori and Shirai [11] first defined the weighted Morrey spaces $L^{p,\kappa}(w)$ which could be viewed as an extension of weighted Lebesgue spaces, and studied the boundedness of the above classical operators on these weighted spaces. Recently, in [15, 19, 21, 22], we have established the continuity properties of some other operators on the weighted Morrey spaces $L^{p,\kappa}(w)$.

In [19], we studied the boundedness properties of intrinsic square functions on the weighted Morrey spaces $L^{p,\kappa}(w)$ for all $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. As a continuation of this work, the main purpose of this paper is to investigate their weak type estimates on the weighted Morrey spaces $L^{1,\kappa}(w)$ when $0 < \kappa < 1$ and $w \in A_1$. Our main results in the paper are formulated as follows.
Theorem 1.1. Let \( 0 < \alpha \leq 1, 0 < \kappa < 1 \) and \( w \in A_1 \). Then there is a constant \( C > 0 \) independent of \( f \) such that
\[
\|S_\alpha(f)\|_{W^{1,\kappa}(w)} \leq C\|f\|_{L^{1,\kappa}(w)}.
\]

Theorem 1.2. Let \( 0 < \alpha \leq 1, 0 < \kappa < 1 \) and \( w \in A_1 \). If \( \lambda > (3n+2\alpha)/n \), then there is a constant \( C > 0 \) independent of \( f \) such that
\[
\|g_{\lambda,\alpha}(f)\|_{W^{1,\kappa}(w)} \leq C\|f\|_{L^{1,\kappa}(w)}.
\]

In [25], Wilson also showed that for any \( 0 < \alpha \leq 1 \), the functions \( S_\alpha(f)(x) \) and \( g_\alpha(f)(x) \) are pointwise comparable. Thus, as a direct consequence of Theorem 1.1, we obtain the following

Corollary 1.3. Let \( 0 < \alpha \leq 1, 0 < \kappa < 1 \) and \( w \in A_1 \). Then there is a constant \( C > 0 \) independent of \( f \) such that
\[
\|g_\alpha(f)\|_{W^{1,\kappa}(w)} \leq C\|f\|_{L^{1,\kappa}(w)}.
\]

2 Notations and definitions

The classical \( A_p \) weight theory was first introduced by Muckenhoupt in the study of weighted \( L^p \) boundedness of Hardy-Littlewood maximal functions in [14]. A weight \( w \) is a nonnegative, locally integrable function on \( \mathbb{R}^n \), \( B = B(x_0, r_B) \) denotes the ball with the center \( x_0 \) and radius \( r_B \). Given a ball \( B \) and \( \lambda > 0 \), \( \lambda B \) denotes the ball with the same center as \( B \) whose radius is \( \lambda \) times that of \( B \). For a given weight function \( w \) and a measurable set \( E \) in \( \mathbb{R}^n \), we also denote the Lebesgue measure of \( E \) by \( |E| \) and the weighted measure of \( E \) by \( w(E) \), where \( w(E) = \int w(x) \, dx \). We say that \( w \) is in the Muckenhoupt class \( A_p \) with \( 1 < p < \infty \), if
\[
\frac{1}{|B|} \int_B w(x) \, dx \left( \frac{1}{|B|} \int_B w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n,
\]
where \( C \) is a positive constant which is independent of the choice of \( B \). For the endpoint case \( p = 1 \), \( w \in A_1 \), if
\[
\frac{1}{|B|} \int_B w(x) \, dx \leq C \cdot \inf_{x \in B} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n.
\]

A weight function \( w \) is said to belong to the reverse Hölder class \( RH_r \) if there exist two constants \( r > 1 \) and \( C > 0 \) such that the following reverse Hölder inequality
\[
\left( \frac{1}{|B|} \int_B w(x)^r \, dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) \, dx \right)
\]
holds for every ball \( B \) in \( \mathbb{R}^n \).

It is well known that if \( w \in A_p \) with \( p = 1 \), then \( w \in A_q \) for all \( q > 1 \). If \( w \in A_p \) with \( 1 \leq p < \infty \), then there exists \( r > 1 \) such that \( w \in RH_r \). We state the following results that will be used in the sequel.
Lemma 2.1 (8). Let $w \in A_p$ with $1 \leq p < \infty$. Then, for any ball $B$, there exists an absolute constant $C > 0$ such that

$$w(2B) \leq C \cdot w(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \leq C \cdot \lambda^n w(B),$$

where $C$ does not depend on $B$ nor on $\lambda$.

Lemma 2.2 (9). Let $w \in RH_r$ with $r > 1$. Then there exists a constant $C > 0$ such that

$$w(E) \frac{w(B)}{w(B)} \leq C \left( \frac{|E|}{|B|} \right)^{(r-1)/r}$$

for any measurable subset $E$ of a ball $B$.

Lemma 2.3 (8). Let $w \in A_q$ with $q > 1$. Then, for all $R > 0$, there exists a constant $C > 0$ independent of $R$ such that

$$\int_{|x| \geq R} \frac{w(x)}{|x|^{nq}} dx \leq C \cdot R^{-nq} w(Q(0, 2R)),$$

where $Q = Q(x_0, \ell)$ denotes the cube centered at $x_0$ with side length $\ell$ and all cubes are assumed to have their sides parallel to the coordinate axes.

Definition 2.4 (11). Let $1 \leq p < \infty$, $0 < \kappa < 1$ and $w$ be a weight function. Then the weighted Morrey space is defined by

$$L^{p, \kappa}(w) = \left\{ f \in L^p_{\text{loc}}(w) : \|f\|_{L^{p, \kappa}(w)} < \infty \right\},$$

where

$$\|f\|_{L^{p, \kappa}(w)} = \sup_B \left( \frac{1}{w(B)^{\kappa/p}} \int_B |f(x)|^p w(x) \, dx \right)^{1/p},$$

and the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

We also denote by $WL^{p, \kappa}(w)$ the weighted weak Morrey spaces of all measurable functions $f$ satisfying

$$\|f\|_{WL^{p, \kappa}(w)} = \sup_{\lambda > 0} \sup_B \frac{1}{w(B)^{\kappa/p} \lambda} \cdot w \left( \{ x \in B : |f(x)| > \lambda \} \right)^{1/p} < \infty.$$

Throughout this paper, the letter $C$ always denote a positive constant independent of the main parameters involved, but it may be different from line to line.
3 Proof of Theorem 1.1

First we note that if \( w \in A_1 \), then \( M(w)(x) \leq C \cdot w(x) \) for a.e. \( x \in \mathbb{R}^n \) by the definition of \( A_1 \) weights. Hence, as a straightforward consequence of Theorem B, we obtain

**Theorem 3.1.** Let \( 0 < \alpha \leq 1 \) and \( w \in A_1 \). Then for any given \( \lambda > 0 \), there exists a constant \( C > 0 \) independent of \( f \) and \( \lambda \) such that

\[
   w\{ x \in \mathbb{R}^n : |S_\alpha(f)(x)| > \lambda \} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| w(x) \, dx.
\]

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( f \in L^{1,\alpha}(w) \). Fix a ball \( B = B(x_0, r_B) \subseteq \mathbb{R}^n \) and decompose \( f = f_1 + f_2 \), where \( f_1 = f \chi_{2B}, \chi_{2B} \) denotes the characteristic function of \( 2B \). Since \( S_\alpha (0 < \alpha \leq 1) \) is a sublinear operator, then for any given \( \lambda > 0 \), we write

\[
   w\{ x \in B : |S_\alpha(f)(x)| > \frac{\lambda}{2} \} + w\{ x \in B : |S_\alpha(f_2)(x)| > \frac{\lambda}{2} \}
\]

\[= I_1 + I_2.\]

Lemma 2.1 and Theorem 3.1 yield

\[
I_1 \leq \frac{C}{\lambda} \int_{2B} |f(y)| w(y) \, dy
\]

\[
\leq \frac{C \cdot w(2B)^\alpha}{\lambda} \|f\|_{L^{1,\alpha}(w)}
\]

\[\leq \frac{C \cdot w(B)^\alpha}{\lambda} \|f\|_{L^{1,\alpha}(w)}.\]

We now turn to estimate the other term \( I_2 \). For any \( \varphi \in C_\alpha, \ 0 < \alpha \leq 1 \) and \((y,t) \in \Gamma(x)\), we have

\[
|f_2 * \varphi_t(y)| = \left| \int_{(2B)^c} \varphi_t(y-z) f(z) \, dz \right|
\]

\[
\leq C \cdot t^{-n} \int_{(2\mathbb{B})^c \cap \{z : |y-z| \leq t\}} |f(z)| \, dz
\]

\[\leq C \cdot t^{-n} \sum_{j=1}^\infty \int_{(2j+1)B(2j) \cap \{z : |y-z| \leq t\}} |f(z)| \, dz. \tag{1}\]

For any \( x \in B, (y,t) \in \Gamma(x) \) and \( z \in (2^{j+1}B \setminus 2^j B) \cap B(y,t) \), then by a direct computation, we can easily see that

\[2t \geq |x-y| + |y-z| \geq |x-z| \geq |z-x_0| - |x-x_0| \geq 2^{j-1} r_B.\]
Thus, by using the above inequality (1) and Minkowski's inequality, we deduce
\[
|S_\alpha(f_2)(x)| = \left( \int_{|x(t)|} \left( \sup_{\varphi \in \mathcal{C}_n} |f_2 \ast \varphi_t(y)| \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}
\]
\[
\leq C \left( \int_{2^{j-2}r_B}^{\infty} \int_{|x-y|<t} \left( \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^jB} |f(z)| \frac{dz}{t^{n+1}} \right)^{1/2} \right)
\]
\[
\leq C \left( \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^jB} |f(z)| \frac{dt}{t^{2n+1}} \right)^{1/2}
\]
\[
\leq C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(z)| \frac{dz}{t^{n+1}}.
\] (2)

It follows directly from the A_1 condition that
\[
\sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(z)| \frac{dz}{t^{n+1}} \leq C \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)} \text{ess inf}_{z \in 2^{j+1}B} w(z) \int_{2^{j+1}B} |f(z)| \frac{dz}{t^{n+1}}
\]
\[
\leq C \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)} \int_{2^{j+1}B} |f(z)| w(z) \frac{dz}{t^{n+1}}
\]
\[
\leq C \|f\|_{L^1(w)} \cdot \frac{1}{w(B)^{1-\kappa}} \sum_{j=1}^{\infty} \frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}}.
\] (3)

Since \( w \in A_1 \), then there exists a number \( r > 1 \) such that \( w \in RH_r \). Consequently, by Lemma 2.2, we obtain
\[
\frac{w(B)}{w(2^{j+1}B)} \leq C \left( \frac{|B|}{|2^{j+1}B|} \right)^{(r-1)/r}.
\] (4)

Hence, for any \( x \in B \),
\[
|S_\alpha(f_2)(x)| \leq C \|f\|_{L^1(w)} \cdot \frac{1}{w(B)^{1-\kappa}} \sum_{j=1}^{\infty} \left( \frac{1}{2^j} \right)^{(1-\kappa)(r-1)/r}
\]
\[
\leq C \|f\|_{L^1(w)} \cdot \frac{1}{w(B)^{1-\kappa}},
\] (5)

where in the last inequality we have used the fact that \((1-\kappa)(r-1)/r > 0\). If \( \{ x \in B : |S_\alpha(f_2)(x)| > \lambda/2 \} = \emptyset \), then the inequality
\[
I_2 \leq \frac{C \cdot w(B)^{\kappa}}{\lambda} \|f\|_{L^1(w)}
\]
holds trivially. Now we may suppose that \( \{ x \in B : |S_\alpha(f_2)(x)| > \lambda/2 \} \neq \emptyset \), then by the pointwise inequality (5), we have
\[
\lambda \leq C \|f\|_{L^1(w)} \cdot \frac{1}{w(B)^{1-\kappa}},
\]

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which is equivalent to
\[
w(B) \leq \frac{C \cdot w(B)^\kappa}{\lambda} \|f\|_{L^{1,\kappa}(w)}.
\]
Therefore
\[
I_2 \leq w(B) \leq C \cdot w(B)^\kappa \|f\|_{L^{1,\kappa}(w)}.
\]
Summing up the above estimates for \(I_1\) and \(I_2\), and then taking the supremum over all balls \(B \subseteq \mathbb{R}^n\) and all \(\lambda > 0\), we complete the proof of Theorem 1.1. \(\square\)

4 Proof of Theorem 1.2

Before proving the main theorem in this section, let us first establish the following results.

**Lemma 4.1.** Let \(0 < \alpha \leq 1\) and \(w \in A_1\). Then for any \(j \in \mathbb{Z}_+\), we have
\[
\|S_{\alpha, 2j}(f)\|_{L^2_w} \leq C \cdot 2^{jn/2} \|S_{\alpha}(f)\|_{L^2_w}.
\]

**Proof.** Since \(w \in A_1\), then by Lemma 2.1, we know that for any \((y, t) \in \mathbb{R}^{n+1}_+\),
\[
w(B(y, 2^j t)) = w(2^j B(y, t)) \leq C \cdot 2^j w(B(y, t)) \quad j = 1, 2, \ldots.
\]
Therefore
\[
\|S_{\alpha, 2j}(f)\|_{L^2_w}^2 = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n+1}_+} \left( A_{\alpha}(f)(y, t) \right)^2 \chi_{|x-y|<2^j t} \frac{dy dt}{t^{n+1}} \right) w(x) dx
\]
\[
\quad \leq C \cdot 2^{jn} \int_{\mathbb{R}^{n+1}_+} \left( \int_{|x-y|<t} w(x) dx \right) \left( A_{\alpha}(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}}
\]
\[
\quad = C \cdot 2^{jn} \|S_{\alpha}(f)\|_{L^2_w}^2.
\]
Taking square-roots on both sides of the above inequality, we are done. \(\square\)

**Theorem 4.2.** Let \(0 < \alpha \leq 1\), \(w \in A_1\) and \(\lambda > (3n + 2\alpha)/n\). Then for any given \(\sigma > 0\), there exists a constant \(C > 0\) independent of \(f\) and \(\sigma\) such that
\[
w \left( \left\{ x \in \mathbb{R}^n : |g_{\lambda, \alpha}(f)(x)| > \sigma \right\} \right) \leq \frac{C}{\sigma} \int_{\mathbb{R}^n} |f(x)|w(x) dx.
\]

**Proof.** First, from the definition of \(g_{\lambda, \alpha}\), we readily see that
\[
g_{\lambda, \alpha}(f)(x)^2 = \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{|x-y|} \right)^{\lambda n} \left( A_{\alpha}(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}}
\]
\[
\quad = \int_{0}^{\infty} \int_{|x-y|<t} \left( \frac{t}{|x-y|} \right)^{\lambda n} \left( A_{\alpha}(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}}
\]
\[ + \sum_{j=1}^{\infty} \int_{0}^{\infty} \int_{2^{j-1} < |x-y| < 2^j t} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} \left( A_\alpha(f)(y,t) \right)^2 \frac{dy}{t^{n+1}} \]

\[ \leq C \left[ S_\alpha(f)(x)^2 + \sum_{j=1}^{\infty} 2^{-j\lambda n} S_{\alpha,2j}(f)(x)^2 \right]. \quad (6) \]

Then for any given \( \sigma > 0 \), it follows from the above inequality (6) that

\[ w\left( \left\{ x \in \mathbb{R}^n : |g_{\sigma,\alpha}(f)(x)| > \sigma \right\} \right) \leq w\left( \left\{ x \in \mathbb{R}^n : |S_\alpha(f)(x)| > \sigma/2 \right\} \right) + w\left( \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{\infty} 2^{-j\lambda n/2} S_{\alpha,2j}(f)(x) > \sigma/2 \right\} \right) = I + II. \]

Using Theorem 3.1, we can get

\[ I \leq \frac{C}{\sigma} \int_{\mathbb{R}^n} |f(x)| w(x) \, dx. \]

In order to estimate the term \( II \), for any fixed \( \sigma > 0 \), we apply the Calderón-Zygmund decomposition of \( f \) at height \( \sigma \) to obtain a sequence of disjoint non-overlapping dyadic cubes \( \{Q_i\} \) such that the following property hold (see [17])

\[ \sigma < \frac{1}{|Q_i|} \int_{Q_i} |f(y)| \, dy < 2^n \sigma. \quad (7) \]

Setting \( E = \bigcup_i Q_i \). Now we define two functions \( g \) and \( b \) as follows:

\[ g(x) = \begin{cases} f(x) & \text{if } x \in E^c, \\ \frac{1}{|Q_i|} \int_{Q_i} |f(y)| \, dy & \text{if } x \in Q_i, \end{cases} \]

and

\[ b(x) = f(x) - g(x) = \sum_i b_i(x), \]

where \( b_i(x) = b(x) \chi_{Q_i}(x) \). Then we have

\[ |g(x)| \leq C \cdot \sigma, \quad \text{a.e. } x \in \mathbb{R}^n \quad (8) \]

and

\[ f(x) = g(x) + b(x). \quad (9) \]

Obviously, \( \text{supp } b \subseteq Q_i \), \( \int_{Q_i} b_i(x) \, dx = 0 \) and \( \|b_i\|_{L^1} \leq 2 \int_{Q_i} |f(x)| \, dx \) by our construction. Now for \( j = 1, 2, \ldots \), since \( S_{\alpha,2j}(f)(x) \leq S_{\alpha,2j}(g)(x) + S_{\alpha,2j}(b)(x) \) by (9), then it follows that

\[ II \leq w\left( \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{\infty} 2^{-j\lambda n/2} S_{\alpha,2j}(g)(x) > \sigma/4 \right\} \right) + w\left( \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{\infty} 2^{-j\lambda n/2} S_{\alpha,2j}(b)(x) > \sigma/4 \right\} \right) = III + IV. \]
Observe that $w \in A_1 \subset A_2$ and $\lambda > 1$. Applying Chebyshev’s inequality, Minkowski’s inequality, Lemma 4.1 and Theorem A, we obtain

$$\text{III} \leq C \sigma^2 \left\| \sum_{j=1}^{\infty} 2^{-j\lambda n/2} S_{\alpha,2^j}(g) \right\|_{L^2_w}^2$$

$$\leq C \sigma^2 \left( \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{jn/2} \left\| S_{\alpha}(g) \right\|_{L^2_w} \right)^2$$

$$\leq C \sigma^2 \left( \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{jn/2} \left\| g \right\|_{L^2_w} \right)^2$$

$$\leq C \sigma^2 \cdot \left\| g \right\|_{L^2_w}^2.$$

Moreover, by the inequality (8) and the $A_1$ condition, we deduce that

$$\left\| g \right\|_{L^2_w}^2 \leq C \cdot \sigma \int_{\mathbb{R}^n} |g(x)|w(x) \, dx$$

$$\leq C \cdot \sigma \left( \int_{E^c} |f(x)|w(x) \, dx + \int_{\bigcup_i Q_i} |g(x)|w(x) \, dx \right)$$

$$\leq C \cdot \sigma \left( \int_{\mathbb{R}^n} |f(x)|w(x) \, dx + \sum_i w(Q_i) \int_{Q_i} |f(y)| \, dy \right)$$

$$\leq C \cdot \sigma \left( \int_{\mathbb{R}^n} |f(x)|w(x) \, dx + \sum_i \text{ess inf} w(y) \int_{Q_i} |f(y)| \, dy \right)$$

$$\leq C \cdot \sigma \left( \int_{\mathbb{R}^n} |f(x)|w(x) \, dx + \int_{\bigcup_i Q_i} |f(y)|w(y) \, dy \right)$$

$$\leq C \cdot \sigma \int_{\mathbb{R}^n} |f(x)|w(x) \, dx.$$

So we have

$$\text{III} \leq C \frac{\sigma}{\sigma} \int_{\mathbb{R}^n} |f(x)|w(x) \, dx.$$

To deal with the last term IV, let $Q_i^* = 2\sqrt{n}Q_i$ be the cube concentric with $Q_i$ such that $\ell(Q_i^*) = (2\sqrt{n})\ell(Q_i)$. Then we can further decompose IV as follows.

$$\text{IV} \leq w \left( \left\{ x \in \bigcup_i Q_i^* : \left| \sum_{j=1}^{\infty} 2^{-j\lambda n/2} S_{\alpha,2^j}(b)(x) \right| > \sigma/4 \right\} \right)$$

$$+ w \left( \left\{ x \notin \bigcup_i Q_i^* : \left| \sum_{j=1}^{\infty} 2^{-j\lambda n/2} S_{\alpha,2^j}(b)(x) \right| > \sigma/4 \right\} \right)$$

$$= \text{IV}^{(1)} + \text{IV}^{(2)}.$$
Since \( w \in A_1 \), then by Lemma 2.1, we can get
\[
IV^{(1)} \leq \sum_i w(Q_i^*) \leq C \sum_i w(Q_i).
\]

Furthermore, it follows from the inequality (7) and the \( A_1 \) condition that
\[
IV^{(1)} \leq C \sum_i \frac{1}{\sigma} \cdot \text{ess inf } w(y) \int_{Q_i} |f(y)| \, dy
\]
\[
\leq \frac{C}{\sigma} \sum_i \int_{Q_i} |f(y)| w(y) \, dy
\]
\[
\leq \frac{C}{\sigma} \int_{Q_i} |f(y)| w(y) \, dy
\]
\[
\leq \frac{C}{\sigma} \int_{\mathbb{R}^n} |f(y)| w(y) \, dy.
\]

Thus, in order to finish our proof, we need only to prove that
\[
IV^{(2)} \leq \frac{C}{\sigma} \int_{\mathbb{R}^n} |f(x)| w(x) \, dx.
\]

Denote the center of \( Q_i \) by \( c_i \). For any \( \varphi \in C_{\alpha}, 0 < \alpha \leq 1 \), by the cancellation condition of \( b_i \), we have that for any \((y, t) \in \Gamma_{2^j}(x)\),
\[
|\langle b_i \ast \varphi_t \rangle(y) \rangle = \left| \int_{Q_i} \left[ \varphi_t(y - z) - \varphi_t(y - c_i) \right] b_i(z) \, dz \right|
\]
\[
\leq \int_{Q_i \cap \{z : |z - y| \leq t\}} \frac{|z - c_i|^\alpha}{t^{n+\alpha}} |b_i(z)| \, dz
\]
\[
\leq C \cdot \ell(Q_i)^\alpha \int_{Q_i \cap \{z : |z - y| \leq t\}} |b_i(z)| \, dz. \quad (10)
\]

In addition, for any \( z \in Q_i \) and \( x \in (Q_i^*)^c \), we have \( |z - c_i| < \frac{|x - c_i|}{2} \). Thus, for all \((y, t) \in \Gamma_{2^j}(x)\) and \( |z - y| \leq t \) with \( z \in Q_i \), we can deduce that
\[
t + 2^j t \geq |x - y| + |y - z| \geq |x - z| \geq |x - c_i| - |z - c_i| \geq \frac{|x - c_i|}{2}. \quad (11)
\]

Hence, for any \( x \in (Q_i^*)^c \), by using the above inequalities (10) and (11), we obtain
\[
|S_{\alpha, 2^j}(b_i)(x)| = \left( \int_{\Gamma_{2^j}(x)} \left( \sup_{\varphi \in C_{\alpha}} |\langle \varphi_t \ast b_i \rangle(y) \rangle \right)^2 \, dy \, dt \right)^{1/2}
\]
\[
\leq C \cdot \ell(Q_i)^\alpha \left( \int_{Q_i} |b_i(z)| \, dz \right) \left( \int_{|z - x| < 2^j t} \frac{dy \, dt}{t^{2(\alpha + \alpha) + n + 1}} \right)^{1/2}
\]
which is just our desired conclusion. Summarizing the estimates for I–IV derived above, we thus complete the proof of Theorem 4.2.

This estimate together with the Chebyshev’s inequality yields

\[ IV^2 \leq \frac{4}{\sigma} \int_{\mathbb{R}^n \setminus \bigcup Q_i} \left| \sum_{j=1}^{\infty} 2^{-j\lambda n/2} S_{\alpha,2}\left( \frac{b_j(x)}{w(x)} \right) \right| w(x) \, dx \]

\[ \leq \frac{4}{\sigma} \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \left( \int_{(Q_i^*)^c} S_{\alpha,2}(b_j(x))w(x) \, dx \right) \]

\[ \leq \frac{C}{\sigma} \left( \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{j(3n+2\alpha)/2} \right) \left( \sum_i \ell(Q_i)^{n/2} \int_{Q_i} |f(z)| \, dz \times \int_{(Q_i^*)^c} \frac{w(x)}{|x-c_i|^{n+\alpha}} \, dx \right) \]

where the last inequality holds under our assumption of \( \lambda > (3n+2\alpha)/n \). On the other hand, since \( w \in A_1 \subset A_{1+\alpha/n} \), then by Lemmas 2.3 and 2.1, we get

\[ \int_{(Q_i^*)^c} \frac{w(x)}{|x-c_i|^{n+\alpha}} \, dx = \int_{|y| \geq \sqrt{n}(Q_i)} \frac{w_1(y)}{|y|^{n+\alpha}} \, dy \]

\[ \leq C \cdot \ell(Q_i)^{-n-\alpha} w_1(Q(0, 2\sqrt{n}\ell(Q_i))) \]

\[ = C \cdot \ell(Q_i)^{-n-\alpha} w(Q(c_i, 2\sqrt{n}\ell(Q_i))) \]

\[ \leq C \cdot \ell(Q_i)^{-n-\alpha} w(Q_i), \]

where \( w_1(x) = w(x + c_i) \) is the translation of \( w(x) \). It is obvious that \( w_1 \in A_1 \) whenever \( w \in A_1 \). Hence, by using the \( A_1 \) condition again, we obtain

\[ IV^2(2) \leq \frac{C}{\sigma} \sum_i \frac{w(Q_i)}{|Q_i|} \int_{Q_i} |f(z)| \, dz \]

\[ \leq \frac{C}{\sigma} \sum_i \text{ess inf}_{z \in Q_i} w(z) \int_{Q_i} |f(z)| \, dz \]

\[ \leq \frac{C}{\sigma} \int_{\mathbb{R}^n} |f(z)| w(z) \, dz, \]

which is just our desired conclusion. Summarizing the estimates for I–IV derived above, we thus complete the proof of Theorem 4.2.

We are now in a position to give the proof of Theorem 1.2.
Proof of Theorem 1.2. Let \( f \in L^{1,\kappa}(w) \). As in the proof of Theorem 1.1, we set \( f = f_1 + f_2 \), where \( f_1 = f\chi_{2B} \). Then for each fixed \( \sigma > 0 \), we can write

\[
\begin{align*}
& w\left( \left\{ x \in B : \left| g_{\lambda,\alpha}^*(f)(x) \right| > \sigma \right\} \right) \\
\leq & w\left( \left\{ x \in B : \left| g_{\lambda,\alpha}^*(f_1)(x) \right| > \sigma/2 \right\} \right) + w\left( \left\{ x \in B : \left| g_{\lambda,\alpha}(f_2)(x) \right| > \sigma/2 \right\} \right) \\
= & J_1 + J_2.
\end{align*}
\]

Theorem 4.2 and Lemma 2.1 imply

\[
J_1 \leq \frac{C}{\sigma} \int_{2B} \left| f(y) \right| w(y) \, dy \\
\leq \frac{C \cdot w(2B)^\kappa}{\sigma} \| f \|_{L^{1,\kappa}(w)} \\
\leq \frac{C \cdot w(B)^\kappa}{\sigma} \| f \|_{L^{1,\kappa}(w)}.
\]

We now turn to deal with the term \( J_2 \). Recall that in the proof of Theorem 1.1, we have already showed that for any \( x \in B \),

\[
\left| S_\alpha(f_2)(x) \right| \leq C \| f \|_{L^{1,\kappa}(w)} \cdot \frac{1}{w(B)^{1-\kappa}}. \tag{12}
\]

On the other hand, for any \( x \in B, (y, t) \in \Gamma_2(x) \) and \( z \in (2^{k+1}B \setminus 2^kB) \cap B(y, t) \), then by a simple calculation, we can easily deduce

\[
t + 2^k t \geq |x - y| + |y - z| \geq |z - x| \geq |x - z| \geq |x - x_0| - |x - x_0| \geq 2^{k-1} r_B.
\]

Hence, it follows from the previous inequality (1) and Minkowski’s inequality that

\[
\left| S_{\alpha,2^k}(f_2)(x) \right| = \left( \int_{\Gamma_2(x)} \left( \sup_{\varphi \in C_{\alpha}} \left| f_2 * \varphi_t(y) \right| \right)^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2} \\
\leq C \left( \int_{2^{k-2}B} \int_{|x-y| < 2^k t} \left| f(z) \right| dz \frac{dy \, dt}{t^{n+1}} \right)^{1/2} \\
\leq C \left( \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^kB} \left| f(z) \right| dz \left( \int_{2(k-2)^r}^{2^k t} dt \right)^{2j_n / 2^{n+1}} \right)^{1/2} \\
\leq C \cdot 2^{3jn / 2} \sum_{k=1}^{\infty} \frac{1}{2^{k+1}B} \int_{2^{k+1}B} \left| f(z) \right| dz.
\]

Furthermore, by using the estimates (3) and (4), we can proceed as in (2) and get

\[
\left| S_{\alpha,2^k}(f_2)(x) \right| \leq C \cdot 2^{3jn / 2} \| f \|_{L^{1,\kappa}(w)} \cdot \frac{1}{w(B)^{1-\kappa}}. \tag{13}
\]

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Therefore, for any $x \in B$, by the inequalities (6), (12) and (13), we have

\[
|g_{\lambda,\alpha}(f_2)(x)| \leq C \left( |S_\alpha(f_2)(x)| + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} |S_{\alpha,2}(f_2)(x)| \right)
\]

\[
\leq C \|f\|_{L^{1,\infty}(w)} \cdot \frac{1}{w(B)^{1-n}} \left( 1 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{3j n/2} \right)
\]

\[
\leq C \|f\|_{L^{1,\infty}(w)} \cdot \frac{1}{w(B)^{1-n}},
\]

where the last series is convergent since $\lambda > (3n + 2\alpha)/n > 3$. The rest of the proof is exactly the same as that of Theorem 1.1, and we finally obtain

\[
J_2 \leq w(B) \leq C \cdot \frac{w(B)^{\kappa}}{\lambda} \|f\|_{L^{1,\infty}(w)}.
\]

Combining the above estimates for $J_1$ and $J_2$, and then taking the supremum over all balls $B \subseteq \mathbb{R}^n$ and all $\sigma > 0$, we conclude the proof of Theorem 1.2.

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