A SIMPLE APPROACH TO GEOMETRIC REALIZATION OF SIMPLICIAL AND CYCLIC SETS

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1. Introduction

The theory of simplicial sets and their realization is perhaps the basis to the combinatorial approach to homotopy theory. The construction of the geometric realization is very natural except for one thing: It begins with the postulation of the geometric realization of each of the “standard simplices”, to which no justification is given. It occurred to us that by “explaining” the nature of this realization, the theory could become even more natural. In particular, we wanted to find a completely obvious proof to the fact that geometric realization commutes with products in the right topology.

This explanation is achieved here in the first two sections. We interpret the geometric realization of the standard simplex $\Delta_n$ as an appropriately topologization of the space of order preserving maps from the unit interval to the ordered set $[n] := \{1, \ldots, n\}$.

The definition makes perfect sense for any finite partially ordered set (and in fact more generally for categories). On the other hand, a partially ordered set $P$ gives rise to a natural simplicial set and we show that its realization is the same as that of $P$. On the point set level this boils down to the fact that an order preserving map from the unit interval to $P$ has to factor through a map $[n] \to P$ for some $[n]$. On the point set level it is again clear that for partially ordered set the constructions of geometric realization and of associated simplicial sets commute with products. Once we check that this remains true in topology, the proof that geometric realization of simplicial sets commutes with products is attained.

It turns out that a very similar idea works also for cyclic sets. One needs to introduce a notion that plays the same role in the theory of cyclic sets as that of partially ordered sets in the theory of simplicial sets. This notion, introduced in section 4, is that of a periodic partially ordered set, which is just a partially ordered set with an action of a free abelian group of finite rank, but the definition of morphisms is a
bit awkward. Among other things, we show that the cyclic category of Connes is isomorphic to a certain subcategory of periodic partially ordered set of degree (which is the rank of the acting abelian group) 1.

As the reader will notice, no face or degeneracy maps are mentioned anywhere in the text (except just now). We find this to be one of the more pleasing aspects of the theory.

We would like to thank Ed Efros and Igor Markov for their interest in this work. In particular, the suggestion that the theory should extend to cyclic sets was made by Igor Markov. We would like to thank UCLA, the Max Planck Institute and the Isaac Newton Institute, where part of this work was done.

Note: This work was previously only available on my home page. It has since been cited [2] (who does a similar construction, which as it turns out also appears in [6]) and [1]. In the interest of making it more easily accessible I am submitting it to the archive without any changes. In particular, I have not incorporated suggestions from a referee report that I received for this paper. One important remark that the referee made is that the alternative description of the cyclic category already appeared in [3]

2. Preliminaries

In this section we briefly recall the theory of simplicial sets and their realizations appearing in the standard literature.

\( P \) - the category of finite partially ordered sets with order preserving maps as morphisms.

\([n] \in P\) - the ordered set \(\{0 < 1 < 2 < \cdots < n\}\), for a non negative integer \(n\).

\( I \) - the closed unit interval \([0,1]\).

\( \Delta \) - the subcategory of \( P \) containing all the objects \([n]\).

\( \text{Set} \) - the category of sets.

\( \text{Top} \) - the category of topological spaces.

\( \text{SS} \) - the category of simplicial sets. By definition an object of \( \text{SS} \) is a contravariant functor \( C : \Delta \to \text{Set} \) and morphisms between objects are natural transformations.

\( \Delta_n \) - (for \( n \geq 0 \)) is the standard \(n\)-simplex. This is the simplicial set which is the contravariant functor on \( \Delta \) represented by the object \([n]\). The map \([n] \to \Delta_n\) extends to a functor \( \Delta \to \text{SS} \) and one has \( \text{Hom}_\Delta([n],[m]) \cong \text{Hom}_\text{SS}(\Delta_n,\Delta_m) \) (see below lemma 3.2).
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$|\Delta_n|$ - The topological standard $n$-simplex, defined as

$$|\Delta_n| := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}, \quad x_i \geq 0 \text{ and } \sum x_i = 1\}.$$ 

with the subspace topology. The map $[n] \to |\Delta_n|$ extends to a functor $\Delta \to \text{Top}$ (see [5, p. 3]): For a map $\theta : [n] \to [m]$ corresponds a map $\theta_* : |\Delta_n| \to |\Delta_m|$ defined by $\theta_*(t_0, \ldots, t_n) = (s_0, \ldots, s_m)$ with

$$s_i = \sum_{j=\theta^{-1}(i)} t_j.$$ 

$|\cdot| : \text{SS} \to \text{Top}$ - The functor of geometric realization. We will use the following convenient definition [7, 5]:

$$|C| = \lim_{\Delta_n \to C} |\Delta_n|.$$ 

The limit is taken over the category $\Delta \downarrow C$ whose objects are maps of simplicial sets $f : \Delta_n \to C$ and whose morphisms are commuting triangles

$$\begin{array}{ccc}
\Delta_n & \xrightarrow{f} & \Delta_m \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & C
\end{array}$$

The categories $\mathcal{P}$ and $\text{SS}$ have products - the product of $P, Q \in \mathcal{P}$ is the set $P \times Q$ with the order $(x, y) \geq (z, w)$ if $x \geq z$ and $y \geq w$. The product of the simplicial sets $C$ and $D$ is the functor given on objects by $C \times D([n]) = C([n]) \times D([n])$.

3. Realization of simplicial sets

Definition 3.1. The simplicial realization of a partially ordered set $P$ is the simplicial set $\mathcal{S}(P)$ defined by

$$\mathcal{S}(P)([n]) = \text{Hom}_P([n], P).$$

The assignment $P \to \mathcal{S}(P)$ defines a functor $\mathcal{S} : \mathcal{P} \to \text{SS}$. Clearly $\mathcal{S}([n]) = \Delta_n$. It is also clear that

$$\mathcal{S}(P \times Q) = \mathcal{S}(P) \times \mathcal{S}(Q).$$

Lemma 3.2. The natural map

$$\text{Hom}_P(P, Q) \to \text{Hom}_\text{SS}(\mathcal{S}(P), \mathcal{S}(Q))$$

is a bijection.
Proof. A map $F : \mathcal{S}(P) \to \mathcal{S}(Q)$ associates to every order preserving map $f : [n] \to P$ an order preserving map $g = F(f)$, $g : [n] \to Q$ in such a way that for $\theta : [m] \to [n]$ one has $F(f \circ \theta) = F(f) \circ \theta$. By considering the case $n = 0$ we see that $F$ induces a unique map $	ilde{F} : P \to Q$ such that $F(f) = f \circ \tilde{F}$ for any map $f : [0] \to P$. By considering $\theta_i : [0] \to [n]$ sending 0 to $i \in [n]$ we see that $g_i = \tilde{F}(f(i))$ for all $i$, i.e., that $F(f) = \tilde{F} \circ f$. For $F$ to send an order preserving $f$ to an order preserving $F(f)$ it is necessary and sufficient (consider the case $n = 1$) that $	ilde{F}$ is order preserving. It is clear that $F \to \tilde{F}$ is an inverse to (3.2).

One can define the geometric realization of a partially ordered set $P$ to be the geometric realization of $\mathcal{S}(P)$. We will later show that this is equivalent to the following direct definition:

**Definition 3.3.** Let $P$ be a finite partially ordered set. Consider $P$ as a metric space by the standard discrete metric $d(x, y) = 1 - \delta_{xy}$ for any $x$ and $y$ in $P$. The geometric realization of $P$, denoted $|P|$, is the set of all order preserving upper semi-continuous maps $f : I \to P$. We give $|P|$ the structure of a metric space by the metric

$$d(f, g) = \int_0^1 d(f(t), g(t)) dt.$$  

One checks immediately that $\ | |$ is a functor from $\mathcal{P}$ to Top.

**Lemma 3.4.** For any $n \geq 0$ the geometric realization of $[n]$ is homeomorphic to the geometric realization of $\Delta_n$. The homeomorphism is given by the map

$$f \to (\mu(f^{-1}(0)), \mu(f^{-1}(1)), \ldots, \mu(f^{-1}(n)))$$

with $\mu$ the standard measure on $I$. Moreover, this homeomorphism is compatible with the maps induced by morphisms in $\Delta$: if $\theta : [n] \to [m]$ is a morphism then we have the commutative diagram

$$\begin{array}{ccc}
|[n]| & \longrightarrow & |\Delta_n| \\
|\theta| & \downarrow & \downarrow |\theta_*| \\
|[m]| & \longrightarrow & |\Delta_m|
\end{array}$$

Proof. To check continuity it is enough to note that $\mu(f^{-1}(j)) = 1 - d(f, j)$ where here the last $j$ means the constant function $j$ on $I$. The inverse map is given by $t = (t_0, \ldots, t_n) \to f_{\underline{t}}$, with $f_{\underline{t}}(t) = j$ if $t_0 + \cdots + t_j \leq t < t_0 + \cdots + t_{j+1}$. To see that this inverse map is continuous as
well, suppose we have \( t \) and \( t' \) with \( |t_i - t'_i| < \varepsilon \). Letting \( r_j = t_0 + \ldots + t_j \) and similarly for \( t' \), it follows that
\[
\mu([r_j, r_{j+1}]) - [r'_j, r'_{j+1}]) \leq 2(n+1)\varepsilon,
\]
and since when \( t \in [r_j, r_{j+1}] \), \( f_t \) differs from \( f_{t'} \) only in the above set, we find that \( d(f_t, f_{t'}) \leq 2(n+1)^2\varepsilon \). The compatibility with the morphisms in \( \Delta \) is easily established. \( \square \)

**Proposition 3.5.** There is a functorial isomorphism, for any \( P \in \mathcal{P} \):
\[
|P| \cong |\mathcal{S}(P)|.
\]

*Proof.* For any \( [n] \to P \), we obtain by functoriality a map \( |[n]| \to |P| \) (we remark that this map takes \( (I \to [n]) \in |[n]| \to \text{composition } I \to [n] \to P \). Therefore, there is a map in \( \text{Top} \)
\[
(3.3) \quad \lim_{[n] \to P} |[n]| \to |P|,
\]
where the limit is over the category \( \Delta \downarrow P \) whose object are maps \( [n] \to P \) and whose morphisms are the obvious triangles. It follows from Yoneda’s lemma (or from lemma 3.2) that this last category is isomorphic to \( \Delta \downarrow \mathcal{S}(P) \). We will show that the map (3.3) is a homeomorphism and will therefore be done by lemma 3.4. To construct an inverse to (3.3) we make the key observation that any order preserving map \( f : I \to P \) factors as \( I \xrightarrow{f_n} [n] \xrightarrow{f} P \) for some \( n \). In fact, we may choose a canonical such decomposition: The image of \( f \) in \( P \) is a finite totally ordered set. There is therefore a unique \( n \) and a unique order preserving bijection \( [n] \cong \text{Im}(f) \) which allows us to factor \( f \) through \( [n] \). Let us write this decomposition, by abuse of notation, as \( I \xrightarrow{f_n} \text{Im}(f) \hookrightarrow P \). In this way we obtain from \( f \) an element of \( |\text{Im}(f)| \) and therefore of \( \lim_{[n] \to P} |[n]| \). This gives a well defined map of sets \( |P| \to \lim_{[n] \to P} |[n]| \). It is clear that the composite map \( P \to \lim_{[n] \to P} |[n]| \to |P| \) is the identity map. We show that one gets the identity in the reverse direction as well: Start with a sequence of maps \( I \xrightarrow{u} [n] \xrightarrow{v} P \) representing an object \( [n] \to P \) of \( \Delta \downarrow P \) and an element of \( |[n]| \). If we map this element to \( \lim_{[n] \to P} |[n]| \), then to \( |P| \) and then back to \( \lim_{[n] \to P} |[n]| \), we obtain the image in \( \lim_{[n] \to P} |[n]| \) of the sequence \( I \to \text{Im}(v \circ u) \hookrightarrow P \) and we need to show that this has the same image as the original sequence. This is done by observing that both sequences have the same image in the realization of \( \text{Im}(v) \hookrightarrow P \).

To show that the bijection (3.3) is a homeomorphism, it is enough to show that \( \lim_{[n] \to P} |[n]| \) is compact. This may be achieved by noticing
that the subcategory of $\Delta \downarrow P$ consisting of injective maps $[n] \to P$ is cofinal. This implies by [9, Theorem IX.3.1] that one may take the limit only over this subcategory, which has only a finite number of objects.

\begin{proof}
The projections $P \times Q \to P$ and $P \times Q \to Q$ induce continuous maps $|P \times Q| \to |P|$ and $|P \times Q| \to |Q|$ and therefore a continuous map $|P \times Q| \to |P| \times |Q|$. On the level of sets it is clear that this map is a bijection. Since the two sides are compact by the proof of the previous proposition, we are done.

\end{proof}

Lemma 3.6. For any $P, Q \in P$ one has

\begin{equation}
|P \times Q| \cong |P| \times |Q|.
\end{equation}

\begin{proof}
The projections $P \times Q \to P$ and $P \times Q \to Q$ induce continuous maps $|P \times Q| \to |P|$ and $|P \times Q| \to |Q|$ and therefore a continuous map $|P \times Q| \to |P| \times |Q|$. On the level of sets it is clear that this map is a bijection. Since the two sides are compact by the proof of the previous proposition, we are done.

\end{proof}

Proposition 3.7. If $P, Q \in P$ then $|S(P) \times S(Q)| \cong |S(P)| \times |S(Q)|$.

\begin{proof}
This follows from proposition 3.5, lemma 3.6 and equation (3.1).

\end{proof}

Corollary 3.8. For any non negative integers $n$ and $m$ we have $|\Delta_n \times \Delta_m| \cong |\Delta_n| \times |\Delta_m|$.

\begin{proof}
This is just the last proposition when choosing $[n]$ and $[m]$ for $P$ and $Q$.

\end{proof}

Remark 3.9. It is perhaps instructive to “see” the corollary in the case $n = m = 1$. In this case $P$ and $Q$ are both equal to the ordered set \{0 < 1\} and $P \times Q$ is therefore the poset

\begin{center}
(0, 1)\\
\downarrow \downarrow\\
(0, 0)  (1, 1)\\
\downarrow \downarrow\\
(1, 0)
\end{center}

The image of an order preserving map $I \to P \times Q$ can either be the “lower path” \{(0,0) < (1,0) < (1,1)\}, with boundary cases where the image is a subset of the path, or the “upper path” \{(0,0) < (0,1) < (1,1)\} and its subsets. Each of these paths corresponds to a triangle $||2||$ and the two are glued by the common edge which corresponds to the common subset \{(0,0) < (1,1)\}. The realization is therefore a square.

Remark 3.10. Corollary 3.8 implies by a standard argument (see the proof of [4, Theorem III.3.1]) that for any two simplicial sets $C$ and
$D$ one has $|C \times D| \cong |C| \times |D|$ provided one works in the category of Kelly spaces.

4. Periodic partially ordered sets

In this section we introduce a notion which is the cyclic equivalent of a partially ordered set. This may look a bit unnatural. The usefulness of the definition will be apparent in following sections.

**Definition 4.1.** A periodic partially ordered set (ppset for short), of degree $k > 0$, is a partially ordered set $P$ together with an action of $\mathbb{Z}^k$ by order preserving transformations. In other words, to give a ppset of degree $k$ is to give the poset $P$ together with a collection of commuting order preserving automorphisms $T_1, \ldots, T_k$ (We will call the automorphisms obtained by the action of $\mathbb{Z}^k$ the shifts of $P$). A map $f : P \to Q$ between two ppsets is a function $f$ which is order preserving and satisfies the relation

$$f \circ (1, 1, \ldots, 1) = (1, 1, \ldots, 1) \circ f$$

where the symbol $(1, 1, \ldots, 1)$ represents the action of the appropriate group element (which could of course be different in $P$ and $Q$). We write $\text{Map}(P, Q)$ for the set of maps between ppsets $P$ and $Q$. A morphism between two ppsets $P$ and $Q$ is an equivalence class of maps. The equivalence is given by precomposition with the shifts of $P$ and postcomposition with the shifts of $Q$. The set of morphisms between $P$ and $Q$ is denoted $\text{Hom}(P, Q)$.

Unfortunately the collection of ppsets is not a category with respect to morphisms, because it is easy to see that composition does not preserve the equivalence relation in general. On the other hand, the subcollection of ppsets of degree 1 does become a category this way, because the commutation with $T_1$ implies that it is enough to divide by the shifts on one side.

**Definition 4.2.** The category $\mathcal{PP}_1$ is the category whose objects are degree 1 ppsets and whose morphisms are morphisms of ppsets.

The collection of all ppsets may be given the following formal structure:

**Definition 4.3.** Let $C$ be a category. A module $M$ over $C$ is made up of the following data:

1. A class of objects $\text{Obj}(M)$.
2. For any $P \in C$ and $Q \in M$, a set $\text{Hom}(P, Q)$. 
(3) For any $P, R \in C$ and $Q \in M$ a composition map $\text{Hom}(R, P) \times \text{Hom}(P, Q) \rightarrow \text{Hom}(R, Q)$.

The composition of morphisms should satisfy all the axioms of category theory when applicable. If $M'$ is a $C'$ module then a functor $F$ from $M$ to $M'$ consists of

1. A functor $F : C \rightarrow C'$.
2. a map $F : \text{Obj}(M) \rightarrow \text{Obj}(M')$.
3. For any $P \in C, Q \in M$, a map $F : \text{Hom}(P, Q) \rightarrow \text{Hom}(F(P), F(Q))$ satisfying the obvious compatibilities.

The simplest example of a module occurs when $M$ is a category and $C$ is a subcategory. A structure of a $C$ module on $M$ is obtained by forgetting the morphisms between objects of $M$ and remembering only the ones between objects of $C$ and objects of $M$. The following proposition, whose proof is left to the reader, is the reason for the introduction of the notion of a module:

**Proposition 4.4.** The collection $\mathcal{PP}$ of ppsets of arbitrary degree forms a module over the category $\mathcal{PP}_1$. For $P \in \mathcal{PP}_1$ and $Q \in \mathcal{PP}$, the set $\text{Hom}(P, Q)$ is that of ppset morphisms between $P$ and $Q$ and the composition is induced by the composition of maps of ppsets.

**Definition 4.5.** For $n \geq 0$ the standard ppset $[[n]]$ is the degree 1 ppset defined as follows: As a partially ordered set it is $\mathbb{Z}$ and the generator $T_1$ is translation by $n + 1$.

**Definition 4.6.** The category $\tilde{\nabla}$ is the subcategory of $\mathcal{PP}_1$ whose objects are $[[n]]$ for $n = 0, 1, \ldots$.

Like the case of posets, we need a notion of a product of ppsets. Although the collection of ppsets is not a category, a notion of products still exists as follows:

**Definition 4.7.** Suppose $C$ is a category and $M$ is a $C$-module.

1. When $Q \in M$, the functor $C \rightarrow \text{Set}$ given on objects by $P \rightarrow \text{Hom}(P, Q)$ (visibly seen to be a functor), is called the functor represented by $Q$.
2. An object $Q_3 \in M$ is a product of objects $Q_1$ and $Q_2$ in $M$ if $Q_3$ represents the product of the functors represented by $Q_1$ and $Q_2$ (note that, unlike the case of categories, the product need not be unique).

**Proposition 4.8.** The $\mathcal{PP}_1$ module $\mathcal{PP}$ has products. A product of $P$ and a $\mathbb{Z}^k$ action with $Q$ and a $\mathbb{Z}^m$ action is given by the product poset $P \times Q$ together with the product action of $\mathbb{Z}^k \times \mathbb{Z}^m \cong \mathbb{Z}^{k+m}$. 

Proof. It is immediate to see that, if \( R \in \mathcal{PP} \), then \( \text{Map}(R, P \times Q) \) is naturally isomorphic to \( \text{Map}(R, P) \times \text{Map}(R, Q) \). This isomorphism descends to morphisms if \( R \in \mathcal{PP}_1 \) as

\[
\text{Hom}(R, P \times Q) = \text{Map}(R, P \times Q)/\mathbb{Z}^{k+m} \cong \text{Map}(R, P)/\mathbb{Z}^k \times \text{Map}(R, Q)/\mathbb{Z}^m.
\]

Note that the crucial point here is that we only need to divide on the right side. \( \square \)

5. Cyclic Sets

In this section we want to show that the category \( \tilde{\nabla} \) introduced in the last section is in fact isomorphic to the cyclic category of Connes. We will later use this to provide a new definition of the geometric realization of cyclic sets. We will take [8] as our basic reference on the cyclic category and cyclic sets. A cyclic set is a contravariant functor \( F : \tilde{\Delta} \rightarrow \text{Set} \), where the category \( \tilde{\Delta} \) will be defined below. For each \( [n] \in \Delta \) let \( K([n]) \) be the cyclic group \( \mathbb{Z}/(n + 1) \) thought of as the group of cyclic permutations of \( [n] \).

Definition 5.1. The category \( \tilde{\Delta} \) has the same objects as \( \Delta \). The sets of morphisms are given by

\[
\text{Hom}_{\tilde{\Delta}}([n], [m]) = \text{Hom}_{\Delta}([n], [m]) \times K([n]),
\]

and the composition of \( (\phi, u) \in \text{Hom}_{\Delta}([m], [k]) \times K([m]) \) and \( (\chi, v) \in \text{Hom}_{\Delta}([n], [m]) \times K([n]) \) is defined by

\[
(\phi, u) \circ (\chi, v) = (\phi \circ u \chi, \chi^* u \circ v).
\]

Here, the order preserving map \( u \chi \) and the cyclic permutation \( \chi^* u \) are defined as follows: For \( i \in [m] \) let \( A_i = \chi^{-1}(i) \) and let \( B_{u(i)} = A_i \). Then \( [n] \) can be given a new ordering as the ordered union of the posets \( B_i \). The cyclic permutation \( \chi^* u \) is the unique automorphism of \( [n] \) which is order preserving from \( [n] \) with the new ordering to \( [n] \) with the standard one. Finally \( \chi^* u := u \chi(\chi^* u)^{-1} \).

The following notation and trivial remarks will be useful when we compare this definition with an alternative one: For \( \chi \in \text{Hom}_{\Delta}([n], [m]) \) and \( v \in K([n]) \) we write \( \chi, v \in \text{Hom}_{\tilde{\Delta}}([n], [m]) \) for \( (\chi, \text{id}) \) and \( (\text{id}, v) \) respectively. Then we immediately see that \( (\chi, v) = \chi \circ v \), that composition in \( \tilde{\Delta} \) of morphisms of \( \Delta \) (resp. elements of \( K([n]) \)) is the same as the standard composition and that the composition rule is uniquely determined by the relation

\[
u \circ \chi = u \chi \circ \chi^* u.
\]

We also wish to recall for future use the following observation
Lemma 5.2. With the notation as in definition 5.1, $\chi^*u$ is given by translation by $-k$, where $k = \min B_i$ and $i$ is the smallest index for which $B_i$ is not empty.

Proof. Indeed, this $k$ becomes the smallest element in $[n]$ with the new ordering and is therefore mapped to 0 by $\chi^*u$. □

Theorem 5.3. The categories $\tilde{\nabla}$ and $\tilde{\Delta}$ are isomorphic. The isomorphism is given by the functor $F : \tilde{\Delta} \to \tilde{\nabla}$ defined as follows: On objects $F([n]) = [[n]]$. For $[n]$ and $[m]$ in $\tilde{\Delta}$ and $\chi \in \text{Hom}_\Delta([n],[m]) \subset \text{Hom}_\Delta([n],[m])$, $F(\chi)$ is defined by

$$F(\chi)(a + r \cdot (n + 1)) = \chi(a) + r \cdot (m + 1), \quad 0 \leq a < n + 1, r \in \mathbb{Z}.$$ 

For $u \in K([n])$, $F(u) : [[n]] \to [[n]]$ is translation by $u$. In general, $F(\chi \circ u) = F(\chi) \circ F(u)$.

Proof. Using the remarks after definition 5.1, one easily checks that $F$ is well defined (as a map on morphisms without claiming anything about multiplicativity). We begin the proof by constructing an inverse $G$ to $F$. Clearly on objects we must put $G([[n]]) = [n]$. On morphisms $G$ is defined as follows: Let $f \in \text{Hom}_\nabla([[n]], [[m]])$, represented by $f : \mathbb{Z} \to \mathbb{Z}$ such that $f$ is order preserving and $f(x+n+1) = f(x)+m+1$ for all $x \in \mathbb{Z}$. We set $G(f) = (G_1(f), G_2(f)) = (\chi, v)$, where $v = -\inf f^{-1}\{0, 1, 2, \ldots\}$ and $\chi(x) = f(x-v)$ for $x \in [n]$. If we change $f$ to an equivalent map $f'(x) = f(x) + r \cdot (m+1) = f(x+r \cdot (n+1))$ with $r \in \mathbb{Z}$, then $G_2(f') = G_2(f) - r \cdot (n+1)$ and therefore $G_1(f) = G_2(f)$. It is also easy to see that $\chi$ maps $[n]$ into $[m]$. Indeed, by definition $f(-v) \geq 0$. If $f(n-v) \geq m+1$, then $f(-v-1) = f(n-v) - m-1 \geq 0$, contradicting the minimality of $-v$. Since $f$ is order preserving we have for $i \in [n]$, $0 \leq f(i-v) < m+1$. Therefore, $G$ is well defined.

It is straightforward to check that $G$ is inverse to $F$. It is also clear that $G$ and $F$ respect composition on morphisms in $\Delta$ and in $K([n])$.

To complete the proof, it is enough to show that

$$(5.1) \quad G(F(u) \circ F(\chi)) = u_* \chi \circ \chi^* u.$$ 

Indeed, if this is the case, then since $F$ and $G$ are inverse we will obtain

$$F(u) \circ F(\chi) = F(u_* \chi \circ \chi^* u) = F(u_* \chi) \circ F(\chi^* u),$$ 

and in general we will have

$$F((\phi \circ u) \circ (\chi \circ v)) = F((\phi \circ u_* \chi) \circ (\chi^* u \circ v)) = F(\phi) \circ [F(u_* \chi) \circ F(\chi^* u)] \circ F(v) = F(\phi) \circ F(u) \circ F(\chi) \circ F(v) = F(\phi \circ u) \circ F(\chi \circ v).$$ 

Because of the way $u_* \chi$ is defined in 5.1 it is enough to check the identity (5.1) for the $K([n])$ part, i.e., to show that $G_2(F(u) \circ F(\chi)) =$
\(\chi^*u\). Set \(f = F(u) \circ F(\chi)\). It is clear that \(f(\mathbb{Z}) \cap [m] \neq \emptyset\) hence \(G_2(f) = -\inf(f^{-1}[m])\). It is now easy to check that for \(i \in [m]\)

\[
  f^{-1}(i) = \begin{cases} 
    \chi^{-1}(u^{-1}(i)) & \text{if } i \geq u; \\
    \chi^{-1}(u^{-1}(i)) - n - 1 & \text{if } i < u.
  \end{cases}
\]

From this one sees that, with the notation of definition 5.1, \(\chi^{-1}(u^{-1}(i)) = B_i\). Now one readily sees from the fact that \(f\) is order preserving that the minimal value of \(f^{-1}([m])\) is, modulo \(n + 1\), the smallest element in the \(B_i\) with the smallest index \(i\) for which \(B_i\) is not empty. The theorem now follows from lemma 5.2.

**Remark 5.4.** With the equivalent definition 4.6 of the cyclic category, it is very easy to see that it is self dual: Given \([[n]]\) in \(\tilde{\nabla}\), consider the set \(\text{Map}([[n]], [[0]])\). An element \(f \in \text{Map}([[n]], [[0]])\) satisfies \(f(x + n + 1) = f(x) + 1\) and is therefore surjective and uniquely determined by the number \(i = \inf f^{-1}(0)\). Clearly, the set \(\text{Map}([[n]], [[0]])\) is totally ordered by the relation of inequality of functions and has a shift given by pre or post composition with shifts as usual, making it an object of \(\mathcal{PP}_1\). The unique \(f\) corresponding to \(i\) is the map \(f_i(x) = \lfloor(x - i)/(n + 1)\rfloor\) (here \(\lfloor\rfloor\) denotes the integer part function). This gives a bijection \(\text{Map}([[n]], [[0]]) \to \mathbb{Z}\) which is even order preserving. The shift corresponds to addition of \(n + 1\). Thus, as an object of \(\mathcal{PP}_1\), \(\text{Map}([[n]], [[0]]) \cong [[n]]\). One can easily check that this makes \([[n]] \to \text{Map}([[n]], [[0]])\) into a contravariant isomorphism from \(\tilde{\nabla}\) to itself.

6. Realization of cyclic sets

In this section we wish to mimic the constructions and results of section 3 for cyclic sets. We interpret the unit circle \(S^1\) as an object in \(\mathcal{PP}\). The underlying ordered set is \(\mathbb{R}\) - the set of real numbers. The shift operator is translation by 1.

The analogue of a finite poset is the notion of a compact ppset while that of a totally ordered set is the notion of an archimedean ppset.

**Definition 6.1.** A ppset \(P\) of degree \(k\) is compact if \(\mathbb{Z}^k \setminus P\) is finite.

**Definition 6.2.** A ppset \(P\) is called archimedean if it is of degree 1, totally ordered and for any \(x\) and \(y\) in \(P\) there is some \(n \in \mathbb{Z}\) such that \(T_1^n x > y\).

**Lemma 6.3.** If \(P\) is an archimedean ppset, then either for all \(x \in P\), \(T_1(x) > x\), or for all \(x \in P\), \(T_1(x) < x\).
Proof. Notice that an archimedean ppset must have more than one element. It is enough to prove that it is impossible to have \( x \neq y \in P \) such that \( T_1(x) \geq x \) while \( T_1(y) \leq y \). Suppose that we have these \( x \) and \( y \). Clearly \( T_1^n(x) \geq x \) and \( y \geq T_1^n(y) \) for \( n > 0 \) with the reverse inequalities holding for \( n < 0 \). Suppose \( y > x \). Since \( P \) is archimedean, we have some \( n \) such that \( T_1^n y < x \). We must have \( n > 0 \). But then \( x \geq T_{-n} x > y \) and we arrive at a contradiction. Similarly, if \( x > y \) we have, for some \( n < 0 \), \( T_1^n x < y \) which implies that \( x < T_{-n} y \leq y \). \( \square \)

Definition 6.4. We will say that an archimedean ppset \( P \) is positive (resp. negative) if for one (hence any) \( x \in P \), \( T_1(x) > x \) (resp. \( T_1(x) < x \)).

Definition 6.5. The geometric realization of a compact \( P \in \mathcal{P} \mathcal{P} \) is the topological space \( ||P|| \) defined as followed: As a set \( ||P|| \) is the subset of \( \text{Hom}_{\mathcal{P} \mathcal{P}}(S^1, P) \) of morphisms where the underlying map of sets is upper semicontinuous when \( P \) is taken to have the discrete topology. restriction of maps to \( I \subset \mathbb{R} \) and projection on \( \mathbb{Z}^k \setminus P \) gives an embedding of \( ||P|| \) into a space of maps from \( I \) to \( \mathbb{Z}^k \setminus P \) and the metric on \( ||P|| \) is then defined similarly to the one in definition 3.3.

The precomposition with translation on \( \mathbb{R} \) makes \( ||P|| \) naturally into a space with a circle action and it is immediate that \( || \cdot || \) is a functor from \( \mathcal{P} \mathcal{P} \) to the category of topological spaces with circle action, considered as a module over itself.

Proposition 6.6. The geometric realization of \( [[n]] \) is homeomorphic to the product \( |\Delta_n| \times S^1 \) where the circle action fixes \( |\Delta_n| \) and acts in the obvious way on \( S^1 \).

Proof. This is proved in much the same way as theorem 5.3: To an order preserving map \( f : \mathbb{R} \to \mathbb{Z} \) satisfying \( f(x + 1) = f(x) + n + 1 \), which represents an element of \( ||[[n]]|| \), we associate the pair \((\phi, s) \in |[n]| \times S^1 \) as follows: \( s = -\inf f^{-1}(\{0, 1, 2, \ldots \}) \) and \( \phi \) is given by \( \phi(x) = f(x-s) \).

The inverse map is given by \((\phi, s) \to f(x) = \phi(\{x + s\}) + [x + s] \) where \( \{ \} \) and \( [ \] denote fractional and integer values respectively. One only needs to check that these maps are continuous, which is easy. \( \square \)

The theory can now be developed exactly as in the simplicial case. We only sketch the proofs as needed.

Definition 6.7. The cyclic realization \( \mathcal{C}(P) \) of a ppset \( P \) is the restriction to \( \bar{\nabla} \) of the functor on \( \mathcal{P} \mathcal{P}_1 \) represented by \( P \).

Definition 6.8. The standard cyclic \( n \) simplex is \( \bar{\Delta}_n := \mathcal{C}([[n]]) \).
**Definition 6.9.** The geometric realization of a cyclic set $C$ is given by:

$$||C|| = \lim_{\Delta_n \to C} ||[n]||.$$ 

The limit is taken in the category of topological spaces with circle action.

The analogue of proposition 3.5 is now

**Proposition 6.10.** For any compact ppset $P$ we have $||C(P)|| \cong ||P||$.

**Proof.** The proof is essentially the same as that of proposition 3.5. The only point which is maybe not obvious is the analogue of the fact that the image of $I$ by an order preserving map is isomorphic to $[n]$ for some $n$. This is provided by lemma 6.11 below, together with the obvious fact that the image of $S^1$ under a ppset map is positive archimedean and is compact if the target is. □

**Lemma 6.11.** A positive archimedean compact ppset $P$ is isomorphic to $[[n]]$ where $n + 1$ is the cardinality of $\mathbb{Z} \setminus P$.

**Proof.** Let $\pi : P \to \mathbb{Z} \setminus P$ be the projection. Choose an element in $P$ and call it 0. We can construct a section $s : \mathbb{Z} \setminus P \to P$ to $\pi$ in the following way: to each $\tilde{x} \in \mathbb{Z} \setminus P$, the set $A_{\tilde{x}} := \{x \in \pi^{-1}(\tilde{x}), x \geq 0\}$ has a smallest element. To see this choose $x_0 \in A_{\tilde{x}}$. Then $A_{\tilde{x}} = \{T_1^n(x_0) \geq 0\}$ and the set of $n$ for which $T_1^n(x_0) \geq 0$ is bounded from below. Let $s(\tilde{x})$ be that smallest element. The image set $s(\mathbb{Z} \setminus P)$ is a finite totally ordered set and therefore there is a unique order preserving bijection $t : [n] \to s(\mathbb{Z} \setminus P)$. Note that $t(0) = 0$. We can now construct maps $f : [[n]] \to P$ and $g : P \to [[n]]$ as follows: Let $n_1(i) = [i/(n + 1)]$ and let $n_2(i) = i - (n + 1)n_1(i)$. Then we define

$$f(i) = T_1^{n_1(i)}(t(n_2(i))).$$

Given $x \in P$, there is a unique $m_1(x) \in \mathbb{Z}$ such that $T_1^{m_1(x)}(s\pi(x)) = x$ and we define

$$g(x) = t^{-1}s\pi(x) + (n + 1) \cdot m_1(x).$$

It is easy to see that $f$ and $g$ are order preserving, commute with the shifts and inverse to each other, which completes the proof. □

As for simplicial sets, we may deduce from this a corollary regarding product of realizations. Note that the product in the category of spaces with circle action is given by the product of the underlying spaces together with the diagonal circle action.
Proposition 6.12. If $P$ and $Q$ are compact ppsets, then $||C(P) \times C(Q)|| \cong ||C(P)|| \times ||C(Q)||$.

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