On the Convexity of Independent Set Games*

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Abstract

This paper investigates independent set games (introduced by Deng et al., Math. Oper. Res., 24:751-766, 1999 [4]), which belong to cooperative profit games. Let $G = (V, E)$ be an undirected graph and $\alpha(G)$ be the size of maximum independent sets in $G$. For any $F \subseteq E$, $V(F)$ denotes the set of vertices incident only to edges in $F$, and $G[V(F)]$ denotes the induced subgraph on $V(F)$. An independent set game on $G$ is a cooperative game $\Gamma_G = (E, \gamma)$, where $E$ is the set of players and $\gamma : 2^E \rightarrow \mathbb{R}$ is the characteristic function such that $\gamma(F) = \alpha(G[V(F)])$ for any $F \subseteq E$.

Independent set games were first studied by Deng et al. [4], where the algorithmic aspect of the core was investigated and a complete characterization for the core non-emptiness was presented. In this paper, we focus on the convexity of independent set games, since convex games possess many nice properties both economically and computationally. We propose the first complete characterization for the convexity of independent set games, i.e., every non-pendant edge is incident to a pendant edge in the underlying graph. Our characterization immediately yields a polynomial time algorithm for recognizing convex instances of independent set games. We also introduce two relaxations of independent set games and characterize their convexity respectively.

Keywords: cooperative game, independent set, convexity.
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1 Introduction

Cooperative games form an important class of problems in game theory, which have a lot of applications in economics, computer science, and mathematics. Formally, a cooperative game $\Gamma$ is a pair

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\((N, \gamma)\), where \(N\) is the set of players and \(\gamma : 2^N \rightarrow \mathbb{R}\) is the characteristic function with \(\gamma(\emptyset) = 0\). A subset \(S\) of \(N\) is called a coalition, and \(N\) is called the grand coalition. For each coalition \(S\), \(\gamma(S)\) represents the profit received (resp. cost paid) by the players in \(S\). When the value of \(\gamma\) represents profits (resp. costs), we call \(\Gamma\) a cooperative profit (resp. cost) game. Throughout this paper, we concentrate on cooperative profit games. One of the major problems in cooperative profit games is to allocate the profit among players. There are many criteria for evaluating how “good” or “reasonable” an allocation is, such as fairness, stability, and so on. Emphases on different criteria lead to different solution concepts, e.g., the core, the Shapley value, the nucleolus, the bargaining set, and the von Neumann-Morgenstern solution \([17]\). Among those solution concepts, the core which addresses the issue of stability is one of the most attractive solution concepts. The core is the set of profit allocations where no coalition has an incentive to split off from the grand coalition, and does better on its own. Hence a cooperative game with a non-empty core is especially interesting, and such a game is called balanced. Moreover, a cooperative game is called totally balanced if every subgame is balanced. Clearly, a totally balanced game is balanced. A special subclass of totally balanced games is formed by convex games.

Convex (resp. concave) games, introduced by Shapley \([16]\), are cooperative games whose characteristic functions are supermodular (resp. submodular). Convex (or concave) games exhibit many desirable properties in cooperative game theory. In particular, (i) the core is always non-empty and a core element can be found in polynomial time \([16]\); (ii) testing whether an allocation belongs to the core can be performed in polynomial time \([5]\); (iii) computing the nucleolus can be done in polynomial time \([11]\); (iv) there is an appealing snowball effect, i.e., the incentive to join a coalition increases as the coalition grows larger \([16]\). We refer to \([13, 16]\) for many interesting properties of convex (or concave) games involving other solution concepts. Hence the convexity (or concavity) of cooperative games has attracted a lot of research efforts. Many results on the convexity (or concavity) concern combinatorial optimization games, which are cooperative games arising from combinatorial optimization problems, e.g., airport games, bankruptcy games, communication games, traveling salesman games, and spanning tree games. However, only a few combinatorial optimization games are universally convex (or concave) \([2, 3, 7, 12]\). Hence one working direction is to characterize the condition of combinatorial optimization games being convex (or concave).

There is a line of research in which the convexity (or concavity) of a combinatorial optimization game is characterized by the property of its underlying graph. Van den Nouweland and Borm \([18]\) showed that communication vertex games are convex if and only if the underlying graph is cycle-complete and communication arc games are convex if and only if the underlying graph is cycle-free. Herer and Penn \([8]\) showed that Steiner traveling salesman games are concave if the underlying graph is a 1-sum of \(K_4\) and outerplanar graphs. Hamers \([6]\) showed that Chinese
postman games are concave if the underlying graph is weakly cyclic. Okamoto [14] showed that vertex cover games are concave if and only if the underlying graph is \((K_3, P_3)\)-free, and coloring games are concave if and only if the underlying graph is complete multipartite. Based on the result of Hamers [6], Albizuri and Hamers [1] characterized the concavity of some variants of Chinese postman games. Kobayashi and Okamoto [9] initialized the study for the concavity of spanning tree games, where a sufficient condition and a necessary condition were given separately. Koh and Sanità [10] proposed the first complete characterization for the concavity of spanning tree games, which efficiently recognizes concave instances of spanning tree games. Platz [15] gave a complete characterization for the concavity of multi-depot Steiner traveling salesman games.

In this work, we focus on characterizing the convexity of independent set games. The paper is structured as follows. In Section 2 we introduce definitions, notations and related results. In Section 3 we present a complete characterization for the convexity of independent set games. Our characterization immediately yields a polynomial time algorithm for recognizing convex instances of independent set games. In Section 4 we introduce two relaxations of independent set games and present complete characterizations for their convexity respectively. Section 5 concludes the results in this paper and discusses the directions of future work.

2 Preliminaries

2.1 Graphs

We assume that the readers have a moderate familiarity with graphs. However, some notions and notations used in this paper should be clarified before proceeding. Throughout, a graph is always finite, undirected and simple. For \(n \in \mathbb{N}\), we use \(K_n\) to denote the complete graph with \(n\) vertices, use \(K_{1,n}\) to denote the graph which is a star, i.e., a complete bipartite graph where one partition has 1 vertex and the other partition has \(n\) vertices, use \(C_n\) to denote the graph which is a cycle with \(n\) edges, and use \(P_n\) to denote the graph which is a path with \(n\) edges. Since \(K_2\) is isomorphic to \(K_{1,1}\) and \(P_2\) is isomorphic to \(K_{1,2}\), both \(K_2\) and \(P_2\) are stars. Let \(H\) be a graph. We use \(V(H)\) to denote the vertex set of \(H\) and use \(E(H)\) to denote the edge set of \(H\). A graph is said \(H\)-free if it contains no subgraph isomorphic to \(H\). Let \(G = (V, E)\) be a graph. For any \(v \in V\), \(N_G(v)\) denotes the set of vertices adjacent to \(v\), \(\delta_G(v)\) denotes the set of edges incident to \(v\), and \(\delta_G(v)\) denotes the degree of \(v\). A vertex is isolated if it is a vertex with degree zero, i.e., it is not an endpoint of any edge. A vertex is pendant if it is a vertex with degree one. An edge is pendant if it is incident to a pendant vertex. For any \(U \subseteq V\), \(G[U]\) denotes the induced subgraph of \(G\). In particular, \(G[\emptyset]\) is an empty graph which has no vertex. For any \(F \subseteq E\), \(V(F)\) denotes the set of vertices incident only to edges in \(F\), \(V(F)\) denotes the set of vertices incident to edges in \(F\), and
denotes the edge-induced subgraph \((V[F], F)\) of \(G\), i.e., the subgraph of \(G\) spanned by \(F\). An independent set of \(G\) is a vertex set \(U \subseteq V\) such that \(G[U]\) has no edge. We use \(\alpha(G)\) to denote the size of maximum independent sets in \(G\). An edge cover of \(G\) is an edge set \(F \subseteq E\) such that every vertex of \(G\) is incident to at least one edge in \(F\). We use \(\rho(G)\) to denote the size of minimum edge covers in \(G\). It is well known that \(\alpha(G) \leq \rho(G)\) for any graph \(G\).

### 2.2 Cooperative games

Let \(\Gamma = (N, \gamma)\) be a cooperative game, where \(N\) is the set of players and \(\gamma : 2^N \to \mathbb{R}\) is the characteristic function with \(\gamma(\emptyset) = 0\). A subset \(S \subseteq N\) is called a coalition, and \(N\) is called the grand coalition. For each coalition \(S\), \(\gamma(S)\) represents the profit received by the players in \(S\). We call \(\Gamma\) convex if for any \(S, T \subseteq N\),

\[
\gamma(S) + \gamma(T) \leq \gamma(S \cap T) + \gamma(S \cup T),
\]

or equivalently, for any \(i \in N\) and any \(S \subseteq T \subseteq N \setminus \{i\}\),

\[
\gamma(S \cup \{i\}) - \gamma(S) \leq \gamma(T \cup \{i\}) - \gamma(T).
\]

Moreover, we call \(\Gamma\) concave if the reverse inequality holds in (2.1) or (2.2), and linear if the equality holds in (2.1) or (2.2).

A profit allocation of \(\Gamma\) is a vector \(x \in \mathbb{R}^{\mid N\} which consists of proposed amounts to be received by players in \(N\). A profit allocation \(x\) is called efficient if \(\sum_{i \in N} x_i = \gamma(N)\), and called group rational if \(\sum_{i \in S} x_i \geq \gamma(S)\) for any \(S \subseteq N\). In particular, \(x\) is called individual rational if \(x_i \geq \gamma(\{i\})\) for any \(i \in N\). An imputation of \(\Gamma\) is a profit allocation that is efficient and individual rational. The core of \(\Gamma\) is the set of imputations that are group rational. The core of cooperative games may be empty. We call \(\Gamma\) balanced if the core is non-empty, and totally balanced if each subgame is balanced. Clearly, totally balanced games are balanced. Moreover, totally balanced games contain convex games as a special subclass.

**Theorem 2.1 (Shapley [16]).** Convex cooperative games are totally balanced.

### 2.3 Independent set games

Let \(G = (V, E)\) be a graph. An independent set game on \(G\) is a cooperative game \(\Gamma_G = (E, \gamma)\), where \(E\) is the set of players and \(\gamma : 2^E \to \mathbb{N}\) is the characteristic function such that \(\gamma(F) = \alpha(G[V \langle F \rangle])\) for any \(F \subseteq E\). Hence \(\gamma(F)\) is the size of maximum independent sets in induced subgraph \(G[V \langle F \rangle]\), where \(V \langle F \rangle\) is the set of vertices incident only to edges in \(F\). Clearly, \(\gamma(\emptyset) = \alpha(G[\emptyset]) = 0\). To make sure the independent set game is well-defined, we always assume the underlying graph has no isolated vertex.
Deng, Ibaraki and Nagamochi [4] introduced independent set games, studied the algorithmic aspect of the core, and gave a complete characterization for the balancedness.

**Theorem 2.2** (Deng et al. [4]). Let $G = (V, E)$ be an undirected graph without isolated vertices and $\Gamma_G = (E, \gamma)$ be an independent set game on $G$. Then $\Gamma_G$ is balanced if and only if the size of maximum independent sets is equal to the size of minimum edge covers in $G$, i.e., $\alpha(G) = \rho(G)$.

3 Convexity of independent set games

**Theorem 3.1.** Let $G = (V, E)$ be an undirected graph without isolated vertices and $\Gamma_G = (E, \gamma)$ be the independent set game on $G$. Then $\Gamma_G$ is convex if and only if every non-pendant edge is incident to a pendant edge in $G$.

We remark that (non-)pendant vertices (resp. edges) always refer to vertices (resp. edges) in the underlying graph $G$ of independent set game $\Gamma_G$ in this section. Before proving Theorem 3.1, we investigate some properties of independent set games first. Let $G = (V, E)$ be a graph without isolated vertices and $\Gamma_G = (E, \gamma)$ be an independent set game on $G$. Take an edge $e = \{u_1, u_2\} \in E$ and an edge set $F \subseteq E \setminus \{e\}$. For simplicity, denote $F \cup \{e\}$ by $F'$. We have the following observations for the independent set game $\Gamma_G$ on $G$.

**Lemma 3.1.** $V\langle F \rangle \cap \{u_1, u_2\} = \emptyset$.

**Proof.** Since $e \notin F$, the endpoints $u_1$ and $u_2$ of $e$ are not in $V\langle F \rangle$.

**Lemma 3.2.** $V\langle F \rangle \subseteq V\langle F' \rangle \subseteq V\langle F \rangle \cup \{u_1, u_2\}$.

**Proof.** The inclusion $V\langle F \rangle \subseteq V\langle F' \rangle$ is trivial, since every vertex incident only to edges in $F$ is also incident only to edges in $F'$. It remains to show that $V\langle F' \rangle \subseteq V\langle F \rangle \cup \{u_1, u_2\}$. Take a vertex $v \in V\langle F' \rangle \setminus V\langle F \rangle$. It follows that $v$ is incident only to edges in $F'$, but incident to an edge outside $F$. Since $F' = F \cup \{e\}$, $v$ is incident to $e$, i.e., $v \in \{u_1, u_2\}$. Therefore, $V\langle F' \rangle \subseteq V\langle F \rangle \cup \{u_1, u_2\}$ follows.

**Lemma 3.3.**

(i) For any independent set $I_F$ in $G[V\langle F \rangle]$, $I_F$ is also an independent set in $G[V\langle F' \rangle]$.

(ii) For any independent set $I_{F'}$ in $G[V\langle F' \rangle]$, $I_{F'} \setminus \{u_1, u_2\}$ is an independent set in $G[V\langle F \rangle]$. Moreover, if $I_{F'}$ is a maximum independent set in $G[V\langle F' \rangle]$ such that $I_{F'} \subseteq V\langle F \rangle$, then $I_{F'}$ is also a maximum independent set in $G[V\langle F \rangle]$.
Proof. (i) Let $I_F$ be an independent set in $G[V(F)]$. Notice that any two vertices in $I_F$ are not adjacent in $G[V(F')]$, since otherwise they are also adjacent in $G[V(F)]$. Hence $I_F$ is also an independent set in $G[V(F')]$.

(ii) Let $I_{F'}$ be an independent set in $G[V(F')]$. Notice that any two vertices in $I_{F'} \setminus \{u_1, u_2\}$ are not adjacent in $G[V(F)]$, since otherwise they are also adjacent in $G[V(F')]$. Hence $I_{F'} \setminus \{u_1, u_2\}$ is an independent set in $G[V(F')]$.

Now let $I_{F'}$ be a maximum independent set in $G[V(F')]$ such that $I_{F'} \subseteq V(F)$. It follows that $I_{F'}$ is an independent set in $G[V(F)]$. It remains to show that $I_{F'}$ is maximum in $G[V(F)]$. Assume to the contrary that there is an independent set $I_F$ in $G[V(F)]$ with $|I_F| > |I_{F'}|$. By Lemma 3.3(i), $I_F$ is also an independent set in $G[V(F')]$, which contradicts the maximality of $I_{F'}$. □

Lemma 3.4. $\gamma(F') \geq \gamma(F)$.

Proof. By Lemma 3.3 any independent set in $G[V(F)]$ is an independent set in $G[V(F')]$. Hence $\gamma(F') \geq \gamma(F)$ follows. □

Lemma 3.5. If $V(F') = V(F)$, then $\gamma(F') = \gamma(F)$.

Proof. When $V(F') = V(F)$, we have $G[V(F')] = G[V(F)]$, implying that $\gamma(F') = \gamma(F)$. □

Lemma 3.6. If $e$ is a pendant edge in $G$, then $\gamma(F') = \gamma(F) + 1$.

Proof. Let $I_F$ (resp. $I_{F'}$) be a maximum independent set in $G[V(F)]$ (resp. $G[V(F')]$). It suffices to show that $|I_{F'}| = |I_F| + 1$. Since $e$ is a pendant edge, we may assume that $u_1$ is a pendant vertex to which $e$ is incident. It follows that $u_1 \in V(F')$. By Lemma 3.1 $V(F) \cap \{u_1, u_2\} = \emptyset$. By Lemma 3.2 $V(F) \subseteq V(F') \subseteq V(F) \cup \{u_1, u_2\}$. Hence $V(F) \cup \{u_1\} \subseteq V(F') \subseteq V(F) \cup \{u_1, u_2\}$ follows. In the following, we distinguish two cases of $V(F')$.

Case 1: $V(F') = V(F) \cup \{u_1\}$. By Lemma 3.3 $I_F$ is also an independent set in $G[V(F')]$. Since $u_2 \notin V(F')$, $u_1$ is an isolated vertex in $G[V(F')]$. It follows that $I_F \cup \{u_1\}$ is an independent set in $G[V(F')]$. We claim that $I_F \cup \{u_1\}$ is maximum in $G[V(F')]$. Assume to the contrary that $|I_{F'}| > |I_F \cup \{u_1\}| = |I_F| + 1$. Since $u_1$ is an isolated vertex in $G[V(F')]$, it follows that $u_1 \in I_{F'}$. By Lemma 3.3 $I_{F'} \setminus \{u_1\}$ is an independent set in $G[V(F')]$. Due to the maximality of $I_F$, we have $|I_F| \geq |I_{F'} \setminus \{u_1\}| = |I_{F'}| - 1$, which contradicts $|I_{F'}| > |I_F| + 1$. Hence $I_F \cup \{u_1\}$ is a maximum independent set in $G[V(F')]$, implying that $|I_{F'}| = |I_F| + 1$.

Case 2: $V(F') = V(F) \cup \{u_1, u_2\}$. Notice that $I_{F'} \cap \{u_1, u_2\} \neq \emptyset$, since otherwise $I_{F'} \cup \{u_1\}$ is an independent set in $G[V(F')]$, which contradicts the maximality of $I_{F'}$. Besides, $I_{F'}$ contains at most one of $u_1$ and $u_2$, as they are adjacent in $G[V(F')]$. Since $u_1$ is a pendant vertex in $G$, we may always assume that $u_1 \in I_{F'}$ (by replacing $u_2$ with $u_1$ in $I_{F'}$ when necessary). By Lemma 3.3 $I_{F'} \setminus \{u_1\}$ is
an independent set in $G[V(F)]$. We claim that $I_{F'} \setminus \{u_1\}$ is maximum in $G[V(F)]$. Assume to the contrary that $|I_F| > |I_{F'} \setminus \{u_1\}| = |I_{F'}| - 1$. Since $u_1, u_2 \notin I_F$ and $N_G(u_1) = \{u_2\}$, $I_F \cup \{u_1\}$ is an independent set in $G[V(F')]$. Due to the maximality of $I_{F'}$, we have $|I_{F'}| \geq |I_F \cup \{u_1\}| = |I_F| + 1$, which contradicts $|I_F| > |I_{F'}| - 1$. Hence $I_{F'} \setminus \{u_1\}$ is a maximum independent set in $G[V(F')]$, implying that $|I_{F'}| = |I_F| + 1$.

**Lemma 3.7.** If $e$ is not a pendant edge in $G$, then $\gamma(F) \leq \gamma(F') \leq \gamma(F) + 1$.

**Proof.** By Lemma 3.3, $\gamma(F) \leq \gamma(F')$ follows directly. It remains to show that $\gamma(F') \leq \gamma(F) + 1$. Without loss of generality, assume $V(F') \neq V(F)$. Otherwise, by Lemma 3.3, $\gamma(F') = \gamma(F)$ follows. Let $I_F$ (resp. $I_{F'}$) be a maximum independent set in $G[V(F)]$ (resp. $G[V(F')]$). It suffices to show that $|I_{F'}| \leq |I_F| + 1$. We may further assume that $I_{F'} \cap \{u_1, u_2\} \neq \emptyset$. Otherwise, by Lemma 3.3, $I_{F'}$ is also a maximum independent set in $G[V(F)]$, implying that $|I_{F'}| = |I_F|$. Now since $I_{F'} \cap \{u_1, u_2\} \neq \emptyset$ and $e$ is not a pendant edge, we may always denote the endpoint of $e$ in $I_{F'}$ by $u_1$, i.e., $u_1 \in I_{F'}$. By Lemma 3.3, $I_{F'} \setminus \{u_1\}$ is an independent set in $G[V(F')]$. Due to the maximality of $I_F$, we have $|I_F| \geq |I_{F'} \setminus \{u_1\}| = |I_{F'}| - 1$. Therefore, $\gamma(F') \leq \gamma(F) + 1$ follows.

With all these preparations above, we are ready to prove our main theorem.

**Proof of Theorem 3.1** We first prove the “only if” part. Assume to the contrary that there is a non-pendant edge $e = \{u_1, u_2\}$ in $G$ which is not incident to a pendant edge, i.e., $d_G(v) \geq 2$ for any $v \in N_G(u_1) \cup N_G(u_2)$. Let $W_{u_1u_2}$ denote the set of vertices from $N_G(u_1) \cap N_G(u_2)$ with degree two, i.e., $W_{u_1u_2} = \{w \in N_G(u_1) \cap N_G(u_2) : d_G(w) = 2\}$. We proceed by distinguishing two cases of the cardinality of $W_{u_1u_2}$.

**Case 1:** $|W_{u_1u_2}| \leq 1$. Denote $\delta_G(u_1)$ by $S$ and $\delta_G(u_2)$ by $T$. Since $d_G(v) \geq 2$ for any $v \in N_G(u_1) \cup N_G(u_2)$, we have $\gamma(S) = \gamma(T) = \alpha(K_1) = 1$ and $\gamma(S \cap T) = \gamma(\{e\}) = \alpha(K_0) = 0$. Moreover, $\gamma(S \cup T) = \alpha(K_2) = 1$ if $|W_{u_1u_2}| = 0$, and $\gamma(S \cup T) = \alpha(K_3) = 1$ if $|W_{u_1u_2}| = 1$. It follows that $\gamma(S) + \gamma(T) > \gamma(S \cap T) + \gamma(S \cup T)$, which contradicts the convexity of $\Gamma_G$.

**Case 2:** $|W_{u_1u_2}| \geq 2$. Take a vertex $w^*$ from $W_{u_1u_2}$. Clearly, $N_G(w^*) = \{u_1, u_2\}$. We remark that the following arguments hold for $i = 1, 2$. Since $d_G(v) \geq 2$ for any $v \in N_G(u_i) \cup N_G(w^*)$, $\{u_i, w^*\}$ is a non-pendent edge which is not incident to a pendant edge. Let $W_{u_iw^*}$ be the vertex set defined analogously to $W_{u_1u_2}$, i.e., $W_{u_iw^*} = \{w \in N_G(u_i) \cap N_G(w^*) : d_G(w) = 2\}$. Instead of $W_{u_1u_2}$, we turn to consider the cardinality of $W_{u_iw^*}$. Clearly, $W_{u_iw^*} \subseteq N_G(w^*)$. It follows that $W_{u_iw^*} \subseteq \{u_1, u_2\}$. However, $|W_{u_1u_2}| \geq 2$ implies that $d_G(u_i) \geq 3$. It turns out that $W_{u_iw^*} = \emptyset$. Then the cardinality discussion of $W_{u_iw^*}$ boils down to Case 1, which contradicts the convexity of $\Gamma_G$. 

7
Therefore, either case leads to a contradiction. We conclude that every non-pendant edge is incident to a pendant edge in $G$.

Now we prove the “if” part. Assume to the contrary that $\Gamma_G$ is not convex. Then there is an edge $e = \{u_1, u_2\}$ and two edge sets $S \subseteq T \subseteq E \setminus \{e\}$ such that

$$\gamma(S \cup \{e\}) - \gamma(S) > \gamma(T \cup \{e\}) - \gamma(T).$$

(3.1)

For simplicity, denote $S \cup \{e\}$ (resp. $T \cup \{e\}$) by $S'$ (resp. $T'$). Then inequality (3.1) becomes

$$\gamma(S') - \gamma(S) > \gamma(T') - \gamma(T).$$

(3.2)

By Lemma 3.4, $\gamma(T') - \gamma(T) \geq 0$. By Lemma 3.5, $V(S') \neq V(S)$, since otherwise $\gamma(T') - \gamma(T) < \gamma(S') - \gamma(S) = 0$. By Lemma 3.6, $e$ is not a pendant edge in $G$, since otherwise $\gamma(S') - \gamma(S) = \gamma(T') - \gamma(T) = 1$. By Lemma 3.7, inequality (3.2) implies

$$\gamma(S') - \gamma(S) = 1$$

(3.3)

and

$$\gamma(T') - \gamma(T) = 0,$$

(3.4)

which are crucial ingredients in our proof. The remainder of our proof is twofold: (a) we prove that an endpoint of $e$ is not adjacent to a pendant vertex, and (b) we prove the convexity of $\Gamma_G$ by exploiting the endpoint of $e$ which is not adjacent to a pendant vertex.

(a) Let $I_{S'}$ be a maximum independent set in $G[V(S')]$. We prove by contradiction that $I_{S'}$ contains an endpoint of $e$ which is not adjacent to a pendant vertex. In particular, we derive a contradiction by constructing an independent set of $G[V(S)]$ from $I_{S'}$, the size of which is as large as that of $I_{S'}$. Our discussion in this part is mainly based on equality (3.3).

- $u_1 \in I_{S'}$.

Since $\gamma(S') - \gamma(S) = 1$, $I_{S'} \cap \{u_1, u_2\} \neq \emptyset$ follows. Otherwise, by Lemma 3.3, $I_{S'}$ is also a maximum independent set in $G[V(S)]$, which contradicts $\gamma(S') - \gamma(S) = 1$. Since $e$ is not a pendant edge, we may always denote the endpoint of $e$ in $I_{S'}$ by $u_1$.

- $\delta_G(u_1) \subseteq S'$.

Since $u_1 \in I_{S'}$, $u_1 \in V(S')$ follows. Thus, we have $\delta_G(u_1) \subseteq S'$.

- $u_1$ is not adjacent to a pendant vertex.

We show that $d_G(v) \geq 2$ for any $v \in N_G(u_1)$. Assume to the contrary that there is a vertex $v^*$ in $N_G(u_1)$ such that $d_G(v^*) = 1$. It follows that $\{u_1, v^*\}$ is a pendant edge. Since $e$ is not a pendant edge, $v^*$ is not incident to $e$. Thus, $\{u_1, v^*\}$ is an edge in $\delta_G(u_1) \setminus \{e\}$. Further notice that $\delta_G(u_1) \setminus \{e\} \subseteq S$ which follows from $\delta_G(u_1) \subseteq S'$. Hence $\{u_1, v^*\}$ is an edge in $S$. It follows that
\( v^* \in V(S) \), as \( v^* \) is a pendant vertex. Since \( N_G(v^*) = \{u_1\} \) and \( u_1 \notin V(S) \), \( v^* \) is an isolated vertex in \( G[V(S)] \). Let \( I'_S \) denote the vertex set obtained from \( I_S \) by removing \( u_1 \), i.e., \( I'_S = I_S \setminus \{u_1\} \). Clearly, \( I'_S \subseteq V(S) \). By Lemma 3.3, \( I'_S \) is an independent set in \( G[V(S)] \) such that \( |I'_S| = |I_S| - 1 \).

Since \( u_1 \) and \( v^* \) are adjacent in \( G[V(S')] \), \( v^* \notin I'_S \) follows from \( u_1 \in I'_S \). Since \( v^* \) is an isolated vertex in \( G[V(S)] \), \( I'_S \cup \{v^*\} \) yields an independent set in \( G[V(S)] \) with cardinality \( |I'_S| \), which contradicts \( \gamma(S') - \gamma(S) = 1 \). Therefore, \( d_G(v) \geq 2 \) for any \( v \in N_G(u_1) \).

(b) Let \( I_T \) be a maximum independent set in \( G[V(T)] \). We prove by contradiction that \( \Gamma_G \) is convex. In particular, we derive a contradiction by constructing an independent set of \( G[V(T')] \) with \( u_1 \) from \( I_T \), the size which is larger than that of \( I_T \). Our discussion in this part is mainly based on equality (3.1).

- \( u_1 \notin I_T \).
  
  By Lemma 3.1, \( u_1 \notin V(T) \) follows. Since \( I_T \subseteq V(T) \), we have \( u_1 \notin I_T \).

- \( \delta_G(u_1) \subseteq T' \).

  Since \( \delta_G(u_1) \subseteq S' \) and \( S' \subseteq T' \), \( \delta_G(u_1) \subseteq T' \) follows directly.

- \( I_T \cap N_G(u_1) \neq \emptyset \), i.e., \( u_1 \) is adjacent to some vertex in \( I_T \).

  By Lemma 3.3, \( I_T \) is also an independent set in \( G[V(T')] \). If \( I_T \cap N_G(u_1) = \emptyset \), i.e., \( u_1 \) is not adjacent to any vertex in \( I_T \), then \( I_T \cup \{u_1\} \) yields an independent set in \( G[V(T')] \) with cardinality \( |I_T| + 1 \), which contradicts \( \gamma(T') = \gamma(T) \).

  - Every vertex in \( I_T \cap N_G(u_1) \) is adjacent to a pendant vertex in \( V(T) \).

    Since \( u_1 \) is not adjacent to a pendant vertex, every vertex in \( I_T \cap N_G(u_1) \) is adjacent to a pendant vertex. Further notice that every pendant vertex which is adjacent to a vertex in \( V(T) \) is also a vertex in \( V(T) \). Hence every pendant vertex which is adjacent to a vertex in \( I_T \cap N_G(u_1) \) is a vertex in \( V(T) \).

- \( \Gamma_G \) is convex.

  We prove the convexity of \( \Gamma_G \) by contradiction, where we construct an independent set of \( G[V(T')] \) with \( u_1 \), the size which is larger than that of \( I_T \). Let \( I'_T \) denote the vertex set obtained from \( I_T \) by replacing each vertex in \( I_T \cap N_G(u_1) \) with a pendant vertex in \( V(T) \) adjacent to it. It follows that \( I'_T \subseteq V(T) \). Moreover, \( I'_T \) remains a maximum independent set in \( G[V(T)] \). By Lemma 3.3, \( I'_T \) is also an independent set in \( G[V(T')] \). Since \( u_1 \) is not adjacent to any vertex in \( I'_T \), \( I'_T \cup \{u_1\} \) yields an independent set in \( G[V(T')] \) with cardinality \( |I_T| + 1 \), which contradicts \( \gamma(T') = \gamma(T) \). Therefore, \( \Gamma_G \) is convex.

Recall that we propose to call a cooperative game that is both convex and concave a linear game. It turns out that Theorem 3.1 can be strengthened to obtain a complete characterization for the linearity of independent set games.
Theorem 3.2. Let $G = (V, E)$ be an undirected graph without isolated vertices and $\Gamma_G = (E, \gamma)$ be the independent set game on $G$. Then $\Gamma_G$ is linear if and only if every non-pendant vertex is adjacent to a pendant vertex.

Proof. Notice that if every non-pendant vertex is adjacent to a pendant vertex, then every non-pendant edge is incident to a pendant edge. Hence to prove the linearity of $\Gamma_G$, it suffices to show that $\Gamma_G$ is concave if and only if every non-pendant vertex is adjacent to a pendant vertex in $G$.

We first show that if $\Gamma_G$ is concave, then every non-pendant vertex is adjacent to a pendant vertex. Assume to the contrary that there is a non-pendant vertex $v^* \in V$ with $d_G(v) \geq 2$ for any $v \in N_G(v^*)$. Let $S, T \subseteq \delta_G(v^*)$ be two non-empty edge sets such that $S \cap T = \emptyset$ and $S \cup T = \delta_G(v^*)$. It follows that $\gamma(S) = \gamma(T) = \alpha(K_0) = 0$, $\gamma(S \cap T) = \gamma(\emptyset) = 0$ and $\gamma(S \cup T) = \alpha(K_1) = 1$. Therefore, we have

$$\gamma(S) + \gamma(T) < \gamma(S \cap T) + \gamma(S \cup T),$$

which contradicts the concavity of $\Gamma_G$.

Now we show that if every non-pendant vertex is adjacent to a pendant vertex, then $\Gamma_G$ is concave. Assume to the contrary that $\Gamma_G$ is not concave. Then there is an edge $e = \{u_1, u_2\}$ and two edge sets $S \subseteq T \subseteq E \setminus \{e\}$ such that

$$\gamma(S') - \gamma(S) < \gamma(T') - \gamma(T),$$

where $S'$ (resp. $T'$) denotes $S \cup \{e\}$ (resp. $T \cup \{e\}$). With a similar argument as in the proof of Theorem 3.1, we conclude that $e$ is not a pendant edge and that $\gamma(T') - \gamma(T) = 1$. Moreover, a similar discussion as in the proof of Theorem 3.1 on $\gamma(T') - \gamma(T) = 1$ shows that an endpoint of $e$ is not adjacent to a pendant vertex, which contradicts our basic assumption.

Therefore, $\Gamma_G$ is concave if and only if every non-pendant vertex is adjacent to a pendant vertex. Combining characterizations for the convexity and the concavity, we conclude that $\Gamma_G$ is linear if and only if every non-pendant vertex is adjacent to a pendant vertex.

Since pendant edges and pendant vertices can be recognized in polynomial time by checking the degree of every vertex, Theorem 3.1 and 3.2 immediately yield polynomial time algorithms for recognizing convex instances and linear instances of independent set games.

Corollary 3.3. Let $G = (V, E)$ be an undirected graph without isolated vertices and $\Gamma_G = (E, \gamma)$ be the independent set game on $G$. Then both convexity and linearity of $\Gamma_G$ can be determined in polynomial time.
4 Convexity of two relaxations of independent set games

The independent set game introduced by Deng et al. [4] is restrictive in the sense that each coalition cannot make use of vertices which are incident to players (edges) outside the coalition. Here we introduce two relaxations of independent set games, namely, the intermediate independent set game and the relaxed independent set game.

The intermediate independent set game on \( G = (V, E) \) is a cooperative game \( \tilde{\Gamma}_G = (E, \tilde{\gamma}) \), where \( E \) is the set of players and \( \tilde{\gamma} : 2^E \to \mathbb{N} \) is the characteristic function such that \( \tilde{\gamma}(F) = \alpha(G[V[F]]) \) for any \( F \subseteq E \). We remark that \( \tilde{\gamma}(F) \) is the size of maximum independent sets in induced subgraph \( G[V[F]] \), where \( V[F] \) is the set of vertices incident to edges in \( F \).

The relaxed independent set game on \( G = (V, E) \) is a cooperative game \( \hat{\Gamma}_G = (E, \hat{\gamma}) \), where \( E \) is the set of players and \( \hat{\gamma} : 2^E \to \mathbb{N} \) is the characteristic function such that \( \hat{\gamma}(F) = \alpha(G[F]) \) for any \( F \subseteq E \). We remark that \( \hat{\gamma}(F) \) is the size of maximum independent sets in edge-induced subgraph \( G[F] \), i.e., the subgraph spanned by \( F \).

Observe that any independent set in \( G[V(F)] \) is also an independent set in \( G[V[F]] \), and that any independent set in \( G[V[F]] \) is also an independent set in \( G[F] \). It follows that

\[
\gamma(F) \leq \tilde{\gamma}(F) \leq \hat{\gamma}(F) \tag{4.1}
\]

for any \( F \subseteq E \). It turns out that the two relaxations of independent set games share the same structure for convexity. Our proof relies on the following lemma, which was first proved in [14]. For the sake of completeness, we also include the proof here.

Lemma 4.1 ([14]). Let \( G = (V, E) \) be an undirected graph. Then \( G \) is \( (K_3, P_3) \)-free if and only if each connected component of \( G \) is a star.

Proof. Necessity. We show that any connected component which is not a star contains a subgraph isomorphic to \( K_3 \) or \( P_3 \). Let \( H \) be a connected component which is not a star. Since both \( K_2 \) and \( P_2 \) are stars, \( |E(H)| \geq 3 \) follows. If \( H \) contains a cycle, then a cycle of length 3 is isomorphic to \( K_3 \) and a cycle of length larger than 3 contains a part isomorphic to \( P_3 \). Otherwise, \( H \) is acyclic, implying that \( H \) contains a subgraph isomorphic to \( P_3 \).

Sufficiency. It is trivial, since any subgraph isomorphic to \( K_3 \) or \( P_3 \) gives rise to a connected component which is not a star.

Theorem 4.1. Let \( G = (V, E) \) be an undirected graph, \( \tilde{\Gamma}_G = (E, \tilde{\gamma}) \) be the intermediate independent set game on \( G \), and \( \hat{\Gamma}_G = (E, \hat{\gamma}) \) be the relaxed independent set game on \( G \). Then the following statements are equivalent.
(i) $G$ is $(K_3, P_3)$-free.

(ii) $\hat{\Gamma}_G$ is convex.

(iii) $\tilde{\Gamma}_G$ is convex.

**Proof.** We prove the implications $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$ in part (a), and prove the implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ in part (b). As we shall see, our proof implies stronger results.

(a) We show that neither $\hat{\Gamma}_G$ nor $\tilde{\Gamma}_G$ is convex when $G$ is not $(K_3, P_3)$-free.

We first prove that any subgraph isomorphic to $K_3$ gives rise to non-convex instances of $\hat{\Gamma}_G$ and $\tilde{\Gamma}_G$. Let $H$ be a subgraph isomorphic to $K_3$. Clearly, $|E(H)| = 3$. Let $S, T \subseteq E(H)$ be two non-empty edge sets such that $S \cup T = E(H)$ and $S \cap T = \emptyset$, i.e., $S$ and $T$ form a nontrivial partition of $E(H)$. Without loss of generality, assume $|S| = 2$ and $|T| = 1$. Notice that $\tilde{\gamma}(S) = \alpha(K_3) = 1$ and $\tilde{\gamma}(T) = \alpha(P_2) = 2$. Moreover, $\tilde{\gamma}(T) = \tilde{\gamma}(T) = \alpha(K_2) = 1$, $\tilde{\gamma}(S \cap T) = \tilde{\gamma}(S \cap T) = \tilde{\gamma}(\emptyset) = 0$ and $\tilde{\gamma}(S \cup T) = \tilde{\gamma}(S \cup T) = \alpha(K_3) = 1$. It follows that

$$\tilde{\gamma}(S) + \tilde{\gamma}(T) > \tilde{\gamma}(S \cap T) + \tilde{\gamma}(S \cup T) \quad (4.2)$$

and

$$\tilde{\gamma}(S) + \tilde{\gamma}(T) > \tilde{\gamma}(S \cap T) + \tilde{\gamma}(S \cup T), \quad (4.3)$$

which contradict the convexity of $\hat{\Gamma}_G$ and $\tilde{\Gamma}_G$ respectively.

Now we prove that any subgraph isomorphic to $P_3$ results in non-convex instances of $\hat{\Gamma}_G$ and $\tilde{\Gamma}_G$. We may further assume that $G$ has no subgraph isomorphic to $K_3$. Let $H$ be a subgraph isomorphic to $P_3$. Let $S \subseteq E(H)$ be an edge set consisting of two incident edges in $H$, and let $T$ be the complement of $S$ in $E(H)$. Since $G[V[S \cup T]]$ is either isomorphic to $P_3$ or isomorphic to $C_4$ and $\alpha(P_3) = \alpha(C_4) = 2$, we have $\tilde{\gamma}(S \cup T) = 2$. Notice that $G[S \cup T]$ is isomorphic to $P_3$. Thus we have $\tilde{\gamma}(S \cup T) = \alpha(P_3) = 2$. Moreover, $\tilde{\gamma}(S) = \tilde{\gamma}(S) = \alpha(P_2) = 2$, $\tilde{\gamma}(T) = \tilde{\gamma}(T) = \alpha(K_2) = 1$, $\tilde{\gamma}(S \cap T) = \tilde{\gamma}(\emptyset) = 0$, and $\tilde{\gamma}(S \cap T) = \tilde{\gamma}(\emptyset) = 0$. Then inequalities (4.2) and (4.3) follow, which contradict the convexity of $\hat{\Gamma}_G$ and $\tilde{\Gamma}_G$ respectively.

(b) We show that both $\hat{\Gamma}_G$ and $\tilde{\Gamma}_G$ are convex when $G$ is $(K_3, P_3)$-free.

Let $H_1, \ldots, H_r$ be the connected components of $G$. Clearly, $\alpha(G) = \sum_{i=1}^{r} \alpha(H_i)$. By Lemma 4.1 each connected component is star, we have $\alpha(H_i) = |E(H_i)|$ for $i = 1, \ldots, r$. It follows that $\alpha(G) = \sum_{i=1}^{r} |E(H_i)| = |E|$. Furthermore, $\tilde{\gamma}(F) = \tilde{\gamma}(F) = \sum_{i=1}^{r} |E(H_i) \cap F| = |F|$ for any $F \subseteq E$. Take an edge $e \in E$ and any two edge sets $S \subseteq T \subseteq E \setminus \{e\}$. Denote $S \cup \{e\}$ (resp. $T \cup \{e\}$) by $S'$ (resp. $T'$). It follows that

$$\tilde{\gamma}(S') - \tilde{\gamma}(S) = \tilde{\gamma}(S') - \tilde{\gamma}(S) = |S'| - |S| = 1 \quad (4.4)$$
and

\[ \tilde{\gamma}(T') - \tilde{\gamma}(T) = \hat{\gamma}(T') - \hat{\gamma}(T) = |T'| - |T| = 1. \quad (4.5) \]

Therefore, both \( \tilde{\Gamma}_G \) and \( \hat{\Gamma}_G \) are convex.

From our proof above, Theorem 4.1 can be strengthened as follows.

**Theorem 4.2.** Let \( G = (V, E) \) be an undirected graph, \( \tilde{\Gamma}_G = (E, \tilde{\gamma}) \) be the intermediate independent set game on \( G \), and \( \hat{\Gamma}_G = (E, \hat{\gamma}) \) be the relaxed independent set game on \( G \). Then the following statements are equivalent.

(i) \( G \) is \((K_3, P_3)\)-free.

(ii) \( \tilde{\Gamma}_G \) is linear.

(iii) \( \hat{\Gamma}_G \) is linear.

Since every \((K_3, P_3)\)-free graph can be recognized in polynomial time, Theorem 4.1 and 4.2 immediately yield polynomial time algorithms for recognizing convex (actually linear) instances of intermediate and relaxed independent set games respectively.

**Corollary 4.3.** Let \( G = (V, E) \) be an undirected graph, \( \tilde{\Gamma}_G = (E, \tilde{\gamma}) \) be the intermediate independent set game on \( G \), and \( \hat{\Gamma}_G = (E, \hat{\gamma}) \) be the relaxed independent set game on \( G \). Then convexity (actually linearity) of \( \tilde{\Gamma}_G \) and \( \hat{\Gamma}_G \) can be determined in polynomial time.

### 5 Conclusions

Efficient characterizations for the convexity of several independent set games are given in this paper. We show that an independent set game is convex if and only if every non-pendant edge is incident to a pendant edge in the underlying graph. Our characterization yields an efficient algorithm for recognizing the convexity of independent set games. We also introduce two relaxations of independent set games and characterize their convexity respectively.

Notice that the independent set game introduced by Deng et al. [4] admits complete characterizations on the balancedness (cf. Theorem 2.2) and on the convexity (cf. Theorem 3.1) respectively. A possible direction for future work is to give a complete characterization on the total balancedness. Characterizations on the (total) balancedness of intermediate and relaxed independent set games will also be interesting.
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