On the transition from crystalline to dynamically triangulated random surfaces

Neil Ferguson\textsuperscript{1} and J.F. Wheater\textsuperscript{2}
Department of Physics
Theoretical Physics
1 Keble Road
Oxford OX1 3NP, UK

Abstract
We consider methods of interpolating between the crystalline and dynamically triangulated random surface models. We argue that actions based on the deviation from six of the coordination number at a site are inadequate and propose an alternative based on Alexander moves. Two simplified models, one of which has a phase transition and the other of which does not, are discussed.
Many apparently disparate physical systems, such as non-critical string theory, 2D quantum gravity, and polymer membranes, are described by the statistical mechanics of random surfaces (for recent reviews see [1]). This letter is concerned with understanding the differences, if any, in the critical behaviour of crystalline surfaces and dynamically triangulated surfaces.

Crystalline surfaces are described by a fixed triangulation $T$ (in continuum terms a fixed intrinsic metric), with the partition function formed by summing over all the possible embeddings of this surface in a $D$-dimensional euclidean space [2]; usually the fixed triangulation is a regular one but this is not essential. If the lattice sites $i$ of $T$ have coordinates $X_i$ in the $D$-dimensional embedding space the action is given by

$$S_T(\kappa, D) = \frac{1}{2} \sum_{<ij> \in T} (X_i - X_j)^2 + \kappa \sum_{\Delta \Delta' \in T} 1 - \hat{n}_\Delta \cdot \hat{n}_{\Delta'}$$

(1)

where $<ij>$ denotes the link from $i$ to $j$ and $\hat{n}_\Delta, \hat{n}_{\Delta'}$ are the unit normal vectors of the triangles on either side of the link. The canonical partition function is

$$Z_T(\kappa, D) = \prod_{i \in T} \int d^D X_i \delta^D \left( \sum_{j \in T} X_j \right) e^{-S_T(\kappa, D)}$$

(2)

where the delta function suppresses the translational zero mode. The first term in the action is the elastic contribution while the second term which is proportional to the extrinsic curvature of the surface inhibits bending. The model exhibits the crumpling transition, characterised by the Hausdorff dimension of the random surface changing from $d_H = 2$ (the smooth phase) to $d_H \to \infty$ (the crumpled phase) at some critical value of the extrinsic curvature coupling. The existence of this transition, which appears to be of second order, is not proven analytically but is well established by the numerical studies [2,3,4,5].

In the case of a dynamically triangulated surface (DTRS), the canonical partition function is formed by summing over all possible embeddings and all possible distinct triangulations $T$ with a fixed number of points $N$ keeping one point marked

$$Z_N(\kappa, D) = \sum_{T:|T|=N} \rho(T) Z_T(\kappa, D)$$

(3)

where $\rho(T)$ is a weighting factor depending only upon the local properties of $T$ [6,7,8]. The grand canonical partition function is then

$$Z(\mu, \kappa, D) = \sum_{N=1}^{\infty} e^{-\mu N} Z_N(\kappa, D)$$

(4)

The sum over triangulations is equivalent to the integral over worldsheet metrics in Polyakov’s ([9]) formulation of string theory; hence these models are used in the numerical study of non-critical strings and 2-D quantum gravity. It is known that in order to obtain an interesting continuum limit in which both the string tension and the mass gap scale to zero it is necessary (but not necessarily sufficient) to have the extrinsic curvature term in the action [10,11]. In the last few years much computational effort has been expended on these models but the results of these simulations are much less clear-cut than for the crystalline surface. The Monte-Carlo simulation of systems with dynamical triangulation is very computationally intensive which limits the size of lattice that can be studied and the difficulties are compounded by the fact that the crumpling transition, if it exists, seems to be rather weak with only a cusping singularity in the specific heat [12,13].

As an alternative to studying the DTRS directly we consider a generalized model which interpolates between it and the better understood crystalline surface. Introduce the notion of a distance, $n(T_0, T)$, between a general triangulation $T$ and reference triangulation $T_0$ (we shall consider shortly what form this distance might take) and consider the canonical partition function for a generalized model

$$Z'_N(\zeta, \kappa, D) = \sum_{T:|T|=N} \rho(T) \exp \left( -\zeta n(T_0, T) \right) Z_T(\kappa, D)$$

(5)
When $\zeta = 0$ this coincides with the DTRS while as $\zeta \to \infty$ all triangulations except $T_0$ are suppressed and we obtain the canonical partition function for the crystalline surface on $T_0$. It is slightly more complicated to define a grand canonical partition function for this new model because not all forms of reference triangulation $T_0$ can be written naturally for all $N$. However, in the cases where a natural definition is possible, such as the example we will discuss later, the grand canonical partition function is given by

$$Z'(\mu, \zeta, \kappa, D) = \sum_{N=1}^{\infty} e^{-\mu N} Z'_N(\zeta, \kappa, D)$$

At large $N$ the asymptotic behaviour of $Z_N$ is expected to be

$$Z_N = N^{\gamma-2} e^{\kappa c N} \left(1 + O \left( \frac{1}{N} \right) \right)$$

Accordingly the grand canonical partition function develops non-analytical behaviour as $\mu \downarrow \mu_c$ and the critical exponent $\gamma$ controls the derivative of $Z$ in this limit through

$$\frac{\partial Z}{\partial \mu} \sim \frac{1}{(\mu - \mu_c)^\gamma}$$

The phase diagram of the generalized model in the $(\kappa, \zeta)$ plane certainly depends on $D$; in practice, most work on the crystalline surface and the DTRS has been done in $D = 3$. It is known that the crystalline surface has a crumpling transition at $\kappa = \kappa_c$; for $\kappa < \kappa_c$ a surface of $N$ sites has mean square radius $R^2 \sim \log N$ whereas for $\kappa > \kappa_c$, $R^2 \sim N$. Since there is no smooth interpolating function between $N$ and $\log N$ when $N \to \infty$ these two regions must be separated by a phase transition. The simplest possibility is that there is a line of transitions (the solid line in fig.1) extending all the way across the diagram and cutting the $\zeta = 0$ axis at $\kappa'_c$; in this case the DTRS also has a crumpling transition. However, there may be a critical point at $X$ so that the order of the transition is different for the DTRS; there is no numerical evidence for a first order transition but a higher order one is certainly possible. Alternatively, the line may stop at an intermediate point $X$ provided that one or both of the lines of transitions $AB$, $CD$ is present to evidence for a first order transition but a higher order one is certainly possible. Alternatively, the line may stop at an intermediate point $X$ provided that one or both of the lines of transitions $AB$, $CD$ is present to evidence for a first order transition but a higher order one is certainly possible. Alternatively, the line may stop at an intermediate point $X$ provided that one or both of the lines of transitions $AB$, $CD$ is present to evidence for a first order transition but a higher order one is certainly possible.

A recent paper [14] proposed using the coordination number of the vertices as a measure of the distance $n(T, T_0)$. In a regular triangulation all (or almost all, depending upon the genus) the vertices have coordination number $q_i = 6$; thus we could choose

$$n(T, T_0) = \sum_{i \in T} |q_i - 6|^\alpha$$

for some positive power $\alpha$ to give the distance of $T$ from a regular triangulation. In [14] it was argued that $\alpha \geq 2$ corresponds to an irrelevant operator in the continuum limit and so $\alpha = 1$ was chosen. However, it is clear that for $\alpha \geq 2$, $n(T, T_0)$ increases faster as $T$ becomes more different from $T_0$ than it does for $\alpha = 1$; thus it is hard to see how $\alpha = 1$ can be more effective at suppressing the irregular triangulations regardless of considerations about the continuum limit. Let us assume that the suppression of irregular triangulations is strong enough that when $\zeta \to \infty$ only the most regular possible triangulations contribute. For genus zero there are many triangulations with $\{q_i = 6, \forall i\}$ differing only by their modular parameter which can vary from $\tau = 1$ (ie the shortest cycle is $O(\sqrt{N}$ links long) to $\tau = O(N)$ (ie the shortest cycle is 3 links long);
they all contribute to $Z_N$ and are not suppressed at all by the $\zeta n(T, T_0)$ term in the action. Thus even at $D = 0$ the action (9) is not suitable to interpolate between the fluid and the crystalline surfaces. At non-zero $D$ the $X$ integral in (2) can be done when $\kappa = 0$ to get \[6,7,8\]

$$ Z_N(0,D) = \sum_{T:|T|=N} \rho(T) \left( \text{Det}' I_T \right)^{-\frac{1}{D}} $$

(10)

where $I_T$ is the incidence matrix of $T$. At large $N$ we know that

$$ \left( \text{Det}' I_{T(\tau=O(N))} \right)^{-1} \sim e^{\beta N} \left( \text{Det}' I_{T(\tau=1)} \right)^{-1} $$

(11)

for finite positive $\beta$ \[15\]. This means that in the thermodynamic limit the ultra-thin configurations will dominate in (10). Hence, even in the limit $\zeta \to 0$, the partition function will be dominated not by a nice regular two-dimensional surface but rather by quasi one-dimensional spaghetti-like objects. We believe that this phenomenon may be responsible, in part, for the observation in \[14\] that, although the intrinslic geometry is smoothed by (9), the extrinsic geometry is not. Clearly the long thin tubes will be suppressed by the extrinsic curvature term when $\kappa > 0$. However, the extrinsic curvature can certainly not be relied upon to suppress all the triangulations except one; in addition the $\kappa$ region where the spaghetti freezes out could get mixed up with the crumpling transition of the $\tau \approx 1$ triangulations and add to the confusion.

To make progress a better definition of $n(T, T_0)$ is needed. It is known that any two triangulations $T_0, T$ with the same number of vertices and the same topology are related by a sequence of link flip operations (known as Alexander moves). Let us define $n(T, T_0)$ to be the minimum number of flips necessary to change $T$ into $T_0$. (It is doubtful that the absolute minimum is really needed; the number of flips taken by any well-defined reproducible procedure will do as well.) The new definition of $n(T, T_0)$ will get rid of the spaghetti. It takes $O(N)$ flips to turn a $\tau = 1$ triangulation into one with $\tau = O(N)$ so the suppression factor $e^{-\zeta n}$ can always be made to overcome the effect of (11) by choosing $\zeta$ large enough. This definition of $n(T, T_0)$ seems much harder to use than (9), especially in a numerical simulation; however there are some special cases (unfortunately in $D = 0$) which can be solved and we now turn to them.

It is convenient to work in terms of the dual graphs rather than the triangulations themselves. We first consider three point functions and take as our reference graph $G_0(N)$ the one shown in fig.2 (note that the actual number of points in the graph is $2N + 1$). It was demonstrated in \[16\] that any three point function graph can be reduced to this form by flips. Of course such a reference graph is very different from the one on which we would like to define the crystalline surface but it serves to illustrate the principle. In the sum over the graphs we include all those which can be constructed from tree graphs in the manner shown in fig.3; $T_k$ denotes the set of rooted trees with $k$ branches and it is known \[16\] that the number of graphs, $T_k$, in this set is given by the $k$’th Catalan number

$$ T_k = \frac{(2k - 2)!}{k!(k - 1)!} $$

(12)

It is easy to see that the rooted tree $T_k$ in a graph of the form shown in fig.3 can be reduced to the form of the reference graph by making $k - 1$ flips, each time flipping the link that is the trunk of a tree. Now introduce the generating function

$$ F_N(t) = \sum_{n=0}^{\text{max}} \Omega_{N,n} t^n $$

(13)

where $\Omega_{N,n}$ is the number of graphs which can be moved to the reference graph $G_0(N)$ by making $n$ flips and $t \equiv e^{-\zeta}$. The graphs obey the Schwinger-Dyson equation shown in fig.4 and hence $F_N(t)$ obeys the recursion relation

$$ F_N(t) = \sum_{k=1}^{N} t^{k-1} T_k F_{N-k}(t) $$

(14)

Multiplying by $z^N$ ($z \equiv e^{-\mu}$) and summing over $N$ gives, after some manipulation,

$$ \mathcal{G}(z, t) = \sum_{N=0}^{\infty} F_N(t) = 1 + t^{-1} \mathcal{G}(z, t) \sum_{k=1}^{\infty} (zt)^k T_k $$

(15)

4
The last sum is just the generating function for the Catalan numbers so we find that the grand canonical partition function \( G(z, t) \) is given by

\[
G(z, t) = \frac{2t - 1 - \sqrt{1 - 4zt}}{2(z - 1 + t)}
\]  

(16)

\( G(z, t) \) displays non-analyticity as we take \( z \to z_c(t) \) to obtain the thermodynamic limit; we find the following behaviour

\[
t < \frac{1}{2}, \quad z_c = 1 - t \quad G \sim (z_c - z)^{-1}
\]

\[
t = \frac{1}{2}, \quad z_c = \frac{1}{2} \quad G \sim (z_c - z)^{-\frac{1}{2}}
\]

\[
t > \frac{1}{2}, \quad z_c = \frac{1}{4t} \quad \frac{\partial G}{\partial z} \sim (z_c - z)^{-\frac{1}{2}}
\]

(17)

The behaviour for \( t > \frac{1}{2} \) corresponds to \( \gamma = \frac{1}{2} \) which is the value expected for a system of branched polymer-like configurations; given the method of construction of our set of graphs that is not surprising. For \( t < \frac{1}{2} \), \( G \) itself diverges; at \( t = 0 \) it is easy to check that we get exactly the partition function expected when only the reference graphs \( G_0(N) \) are included demonstrating that varying \( t \) does indeed allow us to interpolate between an ensemble of graphs and a fixed reference graph. The standard thermodynamic quantities are easy to calculate and we find

\[
\frac{1}{N} \langle n(G, G_0) \rangle = \begin{cases} 
\frac{1}{1 - t} & \text{if } t \leq \frac{1}{2} \\
1 & \text{if } t > \frac{1}{2}
\end{cases}
\]

(18)

and

\[
\frac{1}{N} \langle n(G, G_0)^2 \rangle_C = \begin{cases} 
\frac{t(1-t)}{(1-t)^2} & \text{if } t \leq \frac{1}{2} \\
0 & \text{if } t > \frac{1}{2}
\end{cases}
\]

(19)

There is no divergent specific heat or other behaviour to suggest that a correlation length is diverging at the critical point \( t = \frac{1}{2} \).

The model can be generalized slightly to include all graphs made of \( T_k \) as shown in fig.5 (considering the four point function in this case simplifies the calculation). By making a succession of flips as indicated this can be turned into a graph in which the root of \( T_k \) is isolated as before. The number of flips to do this is \( 2 \min(q-1, k-q) \) so that the factor \( T_k \) in (14) is replaced by

\[
T_k \times \sum_{q=1}^{k} t^{2 \min(q-1, k-q)}
\]

(20)

(In fact this does not give us the minimum number of flips to convert these graphs to \( G_0 \) form but it does yield a soluble model.) Similar manipulations as before lead to the result

\[
G(z, t) = \frac{t^2(1-t^2)}{t^2(1-t^2) - t \left( \sqrt{1 - 4zt^2} - \sqrt{1 - 4zt} \right) + \frac{1}{4}(1-t)^2 \left( \sqrt{1 + 4zt^2} - \sqrt{1 - 4zt^2} \right)}
\]

(21)

Despite its apparent complexity the non-analytic behaviour of \( G \) is the same for all \( t \) and takes the form

\[
G(z, t) \sim \frac{1}{z_c - z}
\]

(22)

There is no phase transition; there is no need for one because \( \gamma \), the only order parameter that could distinguish between \( t = 0 \) and \( t = 1 \) actually takes the same value in each case.

Although our definition of \( n(T, T_0) \) has some appealing properties it is hard to see how it might be implemented efficiently in a Monte Carlo simulation to study the evolution of the crystalline surface into the dynamically triangulated one. To take a random triangulation and “flatten” it into a regular \( T_0 \) by means
of flips in the minimum possible number of moves seems to be a problem of the travelling salesman type. It is easy enough to count flips going away from $T_0$ so long as their concentration is small but this is hardly likely to be the interesting regime for the transition between the two surface models.

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Figure Captions

1) Illustrating possible phase diagrams for the generalized model.
2) The reference dual graph $G_0(N)$.
3) Restricted set of three point graphs constructed from tree graphs.
4) Schwinger-Dyson equation for the restricted set of three point graphs.
5) A larger set of graphs constructed from tree graphs.
\[ \tau_i \tau_j = \text{fig.3} \]

\[ \tau_k = \{ N \}_{k=1}^{N-k} \sum \text{fig.4} \]

\[ = \sum_{k=1}^{N} \tau_k \text{fig.5} \]

\[ k \begin{cases} \tau \end{cases} + k-1 \begin{cases} \tau \end{cases} + \ldots + k-q \begin{cases} \tau \end{cases} \]