AMGCL: an Efficient, Flexible, and Extensible Algebraic Multigrid Implementation

Denis Demidov

Kazan Branch of Joint Supercomputer Center, Scientific Research Institute of System Analysis, the Russian Academy of Sciences, Lobachevsky st. 2/31, 420111 Kazan, Russian Federation

The paper presents AMGCL – an opensource C++ library implementing the algebraic multigrid method (AMG) for solution of large sparse linear systems of equations, usually arising from discretization of partial differential equations on an unstructured grid. The library supports both shared and distributed memory computation, allows to utilize modern massively parallel processors via OpenMP, OpenCL, or CUDA technologies, has minimal dependencies, and is easily extensible. The design principles behind AMGCL are discussed and it is shown that the code performance is on par with alternative implementations.

I. INTRODUCTION

The ability to solve large sparse linear system of equations that arise from discretizations of partial differential equations on either structured or unstructured grids is extremely important in modern numerical methods. Direct methods fail to scale beyond a certain size, typically of the order of a few millions of unknowns [22, 23], due to their intrinsic memory requirements and shear computational cost. This makes preconditioned iterative methods the only viable approach for solution of large scale problems.

The combination of a Krylov subspace method [27] with algebraic multigrid (AMG) as a preconditioner is considered to be one of the most effective choices for solution of such systems [6, 25, 33]. The AMG can be used as a black box solver for various computational problems, since it does not require any information about the underlying geometry, and is known to be robust and scalable [8]. There are several well-known AMG implementations available today. Notable examples are Trilinos ML package [19], BoomerAMG from Hypre [17], and GAMG from PETSC [3]. These packages are provided as parts of complex frameworks targeting large distributed memory machines, have a steep learning curve in general, and may be difficult to compile or distribute. Another problem is that most of the available packages are licensed under GPL, and so are usually deemed inappropriate for use in commercial software. Being as large and inert as they are, the frameworks are usually slow to implement support for modern hardware, such as CUDA of OpenCL based GPUs. There are smaller packages that address this issue. For example, the CUSP library provides implementation of smoothed aggregation multigrid [5, 12], but only supports NVIDIA hardware. The ViennaCL compute library [26] is another example of GPGPU (general purpose GPU) AMG implementation, which also supports OpenMP and OpenCL standards. However, neither of the GPGPU libraries provide support for distributed memory clusters.

The AMGCL C++ library (published at https://github.com/ddemidov/amgcl) provides an AMG implementation while trying to address the above issues. It has minimal set of dependencies, is published under permissive MIT license, targets both shared and distributed memory machines, and supports modern many-core architectures. The library allows to utilize user-defined data structures and operations, thus making it easy to integrate AMGCL into existing software with large and stable codebase. This paper serves as an introduction to the library, discusses its design principles, and compares performance of the provided algorithms with alternative implementations.

II. AMG OVERVIEW

This section outlines the basic principles behind the algebraic multigrid method [6, 22]. Consider a system of linear algebraic equations in the form

\[ Au = f, \]  

(1)

where \( A \) is a square matrix. Multigrid methods are based on recursive application of a two-grid scheme, which combines relaxation and coarse grid correction. Relaxation, or smoothing iteration, is a simple iterative method, such as Jacobi

*E-mail: dennis.demidov@gmail.com*
or Gauss–Seidel iteration \[4\]. Coarse grid correction solves the residual equation on a coarser grid, and improves the fine-grid approximation with the interpolated coarse-grid solution. Transfer between grids is described with transfer operators \(P\) (prolongation or interpolation) and \(R\) (restriction).

In geometric multigrid methods the matrices \(A_i\) and operators \(P_i\) and \(R_i\) are usually supplied by the user based on the problem geometry. In algebraic multigrid methods the grid hierarchy and the transfer operators are in general constructed automatically, based only on the algebraic properties of the matrix \(A\). The setup phase of a generic AMG algorithm may be described as follows:

**Algorithm 1 AMG setup**

Start with a system matrix \(A_1 \leftarrow A\).

while the matrix \(A_i\) is too big to be solved directly do

Introduce prolongation operator \(P_i\) and restriction operator \(R_i\).

Construct the coarse system using Galerkin operator: \(A_{i+1} \leftarrow R_i A_i P_i\).

end while

Construct a direct solver for the coarsest system \(A_L\).

Note that in order to construct the next level in the AMG hierarchy, one only needs to define transfer operators \(P\) and \(R\). Also, a common choice for the restriction operator \(R\) is a transpose of the prolongation operator: \(R = P^T\). After the AMG hierarchy is constructed, it is used to solve the system \(1\) as follows:

**Algorithm 2 AMG solve**

Start at the finest level with initial approximation \(u_1\).

while not converged do

\{ V-cycle: \}

for each level of the hierarchy, finest-to-coarsest do

Apply a couple of smoothing iterations (pre-relaxation) to the current solution:

\(u_i \leftarrow S_i(A_i, f_i, u_i)\)

Find residual \(e_i \leftarrow f_i - A_i u_i\) and restrict it to the right-hand side on the coarser level:

\(f_{i+1} \leftarrow R_i e_i\)

end for

Solve the coarsest system directly: \(u_L \leftarrow A_L^{-1} f_L\).

for each level of the hierarchy, coarsest-to-finest do

Update the current approximation with the interpolated solution from the coarser level:

\(u_i \leftarrow u_i + P_i u_{i+1}\)

Apply a couple of smoothing iterations (post-relaxation) to the updated solution:

\(u_i \leftarrow S_i(A_i, f_i, u_i)\)

end for

end while

Usually AMG is not used standalone, but as a preconditioner with a Krylov subspace iterative solver. In this case single V-cycle is used as a preconditioning step.

### III. AMGCL DESIGN PRINCIPLES

The most algorithmically challenging part of the AMG is the setup phase, or, more specifically, finding a set of transfer operators that would work well for the problem at hand. Most of the approaches here are intrinsically serial and are hard to parallelize. On the contrary, once the hierarchy is constructed, the solution step is mostly trivial and may be expressed through several well-defined and easily parallelizable primitives such as sparse matrix-vector product or linear vector combinations. In terms of machine time, the setup usually takes minor part of the overall algorithm. Usually this is no more than 30%, often much smaller, depending on the convergence rate of the problem. Also, since the setup phase only depends on the system matrix, it is possible to reuse the constructed hierarchy with multiple right-hand sides. This is especially important for solution of nonstationary problems \[13\].

This difference between algorithmic and computational complexity of the setup and the solution phases of the AMG method is the reason for the most important design decision behind the AMGCL library. Namely, the AMG hierarchy is always constructed on the main processor (CPU), using the internal data structures and employing the OpenMP parallelization standard where appropriate. The constructed hierarchy is then transferred into one of the provided (or user-defined) backends for the solution phase. The solution step, depending on the selected backend, may utilize various parallelization technologies, such as OpenMP \[11\], OpenCL \[31\], or CUDA \[29\].
Listing 1: An example of AMGCL solver declaration.

```cpp
#include <amgcl/backend/builtin.hpp>
#include <amgcl/make_solver.hpp>
#include <amgcl/amg.hpp>
#include <amgcl/coarsening/smoothed_aggregation.hpp>
#include <amgcl/relaxation/spai0.hpp>
#include <amgcl/solver/bicgstab.hpp>

typedef amgcl::backend::builtin<double> Backend;

typedef amgcl::make_solver<
    amgcl::amg<
        Backend,
        amgcl::coarsening::smoothed_aggregation,
        amgcl::relaxation::spai0
    >,
    amgcl::solver::bicgstab<Backend>
> Solver;
```

AMGCL is implemented as a header-only C++ library, and uses the compile-time policy-based design [1], which allows its users to compose their own customized version of the AMG from the algorithmic components described below.

**Backends.** Backend is a class that defines matrix and vector types and defined parallel primitives that are utilized during the solution phase of the algorithm. This level of abstraction enables transparent acceleration of the solution with OpenMP, OpenCL, or CUDA technologies, and also allows users of the library to employ their own data structures either to avoid unnecessary data copies or to use faster algorithms (for example, the ones that are customized for a specific hardware).

**Value types.** Value types allow to generalize AMGCL algorithms onto complex or non-scalar systems. A value type defines a number of overloads for common math operations, and is used as a template parameter for a backend. Most often, a value type is simply a builtin `double` or `float`, but it is also possible to use small statically sized matrices as value type, which may increase cache locality, or convergence ratio, or both, when the system matrix has a block structure.

**Coarsening strategies.** These are various options for creating coarse levels in the AMG hierarchy. A coarsening strategy takes the system matrix $A$ at the current level, and returns prolongation operator $P$ and the corresponding restriction operator $R$. AMGCL provides classical Ruge–Stuben strategy, smoothed and non-smoothed aggregation, and smoothed aggregation with energy minimization [25, 28, 34].

**Relaxation methods.** These are the smoothers that are used on each level of the AMG hierarchy during solution phase and are expressed in terms of parallel primitives defined by the current backend. Supported options include damped Jacobi iteration, sparse approximate inverse, Gauss–Seidel, incomplete LU decomposition, and Chebyshev relaxation [4, 7, 27].

**Iterative solvers.** Krylov subspace methods that may be combined with the AMG preconditioner to solve the linear system (1). AMGCL provides implementation for several variants of CG, BiCGStab, GMRES, and IDR(s) [2, 18, 27, 30]. Similarly to a smoother, an iterative solver is expressed in terms of the parallel primitives defined by the backend.

Listing 1 shows an example of a solver declaration that uses a BiCGStab iterative method preconditioned with a smoothed aggregation AMG. A sparse approximate inverse method (SPAI-0) is used as a smoother on each level of the hierarchy. The solver uses the builtin backend (accelerated with OpenMP) and double precision scalars as value type.

### A. Algorithm parameters

Each component in AMGCL defines its own parameters by declaring a `param` subtype with some meaningful default values. When a class is composed from several subclasses, it includes parameters of its children into its own `param`
Listing 2: Runtime definition of an AMGCL solver.

```cpp
#include <amgcl/backend/builtin.hpp>
#include <amgcl/make_solver.hpp>
#include <amgcl/amg.hpp>
#include <amgcl/coarsening/runtime.hpp>
#include <amgcl/relaxation/runtime.hpp>
#include <amgcl/solver/runtime.hpp>

typedef amgcl::backend::builtin<double> Backend;

typedef amgcl::make_solver<
    amgcl::amg<
        Backend,
        amgcl::runtime::coarsening::wrapper,
        amgcl::runtime::relaxation::wrapper
    >,
    amgcl::runtime::solver::wrapper<Backend>
> Solver;

boost::property_tree::ptree prm;

prm.put("solver.type", "bicgstab");
prm.put("solver.tol", 1e-6);
prm.put("precond.coarsening.type", "smoothed_aggregation");
prm.put("precond.relax.type", "spai0");

Solver solve(A, prm);
```

This allows to provide a unified interface to the various algorithmic components in AMGCL. For example, parameters for the `make_solver<Precond, Solver>` class from Listing 1 are declared as

```cpp
template <class Precond, class Solver>
struct make_solver {
    struct params {
        // Preconditioner parameters:
        typename Precond::params precond;
        // Iterative solver parameters:
        typename Solver::params solver;
    };
};
```

B. Runtime interface

The compile-time configuration of AMGCL solvers is efficient, but is not always convenient, especially if the solvers are used inside a software package or another library and it is not possible to choose an optimal configuration a priori. The runtime interface allows to postpone some of the configuration decisions until the program is running and the type of problem being solved is known. The classes inside `amgcl::runtime` namespace correspond to their compile-time alternatives, but their only template parameter is the backend to use.

Since there is no way to know the parameter structure at compile time, the runtime classes accept parameters as instances of `boost::property_tree::ptree` class provided as part of Boost.PropertyTree library[36]. The actual components of the method are set through the property tree as well. For example, Listing 2 shows the runtime alternative to the solver declared in Listing 1.

C. Extensibility

Compile-time strategy-based design of the library allows its maintainers and users to customize the existing components and introduce new functionality by creating new classes and passing those as template parameters (strategies) to AMGCL templated algorithms. Also, the unified initialization interface of AMGCL building blocks means it is easy to reuse existing functionality in order to come up with new preconditioning techniques. Notable examples
included into the library codebase are CPR \cite{21} and Schur complement pressure correction \cite{20,35} preconditioners. CPR works best in fully implicit black-oil simulations, and Schur complement pressure correction is well-suited for Navier–Stokes-like problems. Both of these are field-split, two-step type preconditioners, allowing to efficiently solve problems arising from discretizations of coupled systems of PDEs, which are otherwise converge slowly or do not converge at all if treated as monolithic systems. The pressure subblock of the system matrix is preconditioned with AMG, and both preconditioners reuse core components of the library.

IV. PERFORMANCE AND SCALABILITY

In this section the performance and scalability of the shared and distributed memory versions of the library algorithms are tested using two sample problems in the three dimensional space. The source code for the benchmarks is available at https://github.com/ddemidov/amgcl\_benchmarks.

The first example is a classical 3D Poisson problem:

\[-\Delta u = 1\]  \hspace{1cm} (2)

defined in the unit cube \( \Omega = [0,1]^3 \) with homogeneous Dirichlet boundary conditions \( u = 0 |_{\partial \Omega} \). The problem is discretized using finite difference method on a uniform structured mesh.

The second example is an incompressible 3D Navier–Stokes problem:

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \mathbf{b}, \hspace{1cm} (3a) \\
\nabla \cdot \mathbf{u} &= 0. \hspace{1cm} (3b)
\end{align*}
\]

The problem is discretized using an equal-order tetrahedral finite elements, stabilized employing an ASGS-type (algebraic subgrid-scale) approach \cite{16}. This results in a discretized linear system of equations with a block structure of the type

\[
\begin{bmatrix}
K & G \\
D & S
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix} =
\begin{bmatrix}
b_u \\
p
\end{bmatrix},
\]

where each of the matrix subblocks is a large sparse matrix, and the blocks \( G \) and \( D \) are non-square. The overall system matrix for the problem was assembled in the Kratos\cite{37} Multi-Physics package \cite{9,10} developed at CIMNE, Barcelona.

There are at least two ways to solve the Navier–Stokes problem. First, one can treat the system as a monolithic one, and provide some minimal help to the preconditioner by supplying it with the near null space vectors. Second option is to employ the knowledge about the problem structure, and to combine separate preconditioners for individual fields (in this particular case, for pressure and velocity). In case of AMGCL both options are available and were tested: the monolithic system was solved with static \( 4 \times 4 \) matrices as value type, and the field-split approach was implemented using the Schur complement pressure correction preconditioner, where the pressure part of the system was preconditioned with smoothed aggregation AMG, and the velocity part was preconditioned with SPAI-0. Trilinos ML only provides the first option: PETSC implements both, but we only show results for the second, superior option of field-split preconditioner. CUSP library does not provide field-split preconditioner and nor does it allow to specify near null space vectors, so it was not tested with the Navier–Stokes problem.

A. Shared memory benchmarks

The performance of the shared memory (single compute node) solution was tested on a dual socket system with two Intel Xeon E5-2640 v3 CPUs. The system also had an NVIDIA Tesla K80 GPU installed, which was used for testing the GPU based versions. The results were compared to those of the PETSC \cite{3} and the Trilinos ML \cite{19} distributed memory libraries and the CUSP GPGPU library \cite{12}.

Figure 1 presents the multicore scalability of the Poisson problem \cite{2}. The problem was discretized on a \( 150^3 \) uniform mesh and the resulting system matrix contained 3,375,000 unknowns and 23,490,000 nonzero elements. The figure has separate plots for the total solution time, the setup step time, the solve step time, and the number of iterations made until convergence. Here PETSC and Trilinos versions use MPI for parallelisation within the single compute node, AMGCL/OpenMP and AMGCL/CUDA use OpenMP and CUDA backends correspondingly. Both AMGCL/CUDA and CUSP versions employ the NVIDIA Tesla K80 GPU. All libraries use the conjugate gradient (CG) iterative
method preconditioned with the smoothed aggregation AMG. Trilinos and PETSC use their corresponding defaults for smoothers (symmetric Gauss–Seidel and damped Jacobi accordingly), AMGCL uses SPAI-0, and CUSP uses Gauss–Seidel.

The CPU-based results show that AMGCL performs on par with Trilinos, and both outperform PETSC by a large margin. Also, AMGCL is able to setup the solver about 20–100% faster than Trilinos, and 4–7 times faster than PETSC. This is probably due to the fact that both Trilinos and PETSC target distributed memory machines and thus have an overhead not present in the shared memory version of AMGCL.

The CUDA backend of AMGCL during solution phase performs slightly better than the CUSP version, and the setup time for AMGCL is significantly lower than that of CUSP. The main difference between the libraries is that AMGCL constructs the AMG hierarchy completely on the CPU, and the setup step in CUSP is done on the GPU. When taking into consideration just the solution time (without setup), then both AMGCL/CUDA and CUSP are able to outperform CPU-based versions (with full utilization of the CPU cores) by a factor of 3–4. The total solution time for AMGCL/CUDA is only 30% better than that of AMGCL/OpenMP or Trilinos. This is explained by the fast convergence rate of the Poisson problem, which means the solution step of AMGCL/CUDA is not able to amortize the relatively expensive setup step.

Figure 2 shows the results for the Navier–Stokes problem. The system matrix has 713,456 unknowns and 41,277,920 nonzeros. The assembled problem files may be downloaded at [15]. Here the ‘block’ versions correspond to the solution of the monolithic system with a smoothed aggregation AMG as preconditioner. The ‘split’ versions correspond to using a field-split preconditioner better suited for the Navier–Stokes problem. The ‘split’ variants of both AMGCL and PETSC are predictably able to outperform the ‘block’ versions. AMGCL/block version with
FIG. 2: Shared memory solution of the Navier–Stokes problem \([3]\). The system matrix has 713,456 unknowns.

The scalability of the distributed memory (MPI based) solutions was tested on a SuperMUC cluster located at the Leibniz Supercomputing Centre of the Bavarian Academy of Sciences and Humanities. Each compute node was equipped with two 14 core Intel Haswell Xeon E5-2697 v3 CPUs, and 64 GB of RAM. The obtained results were compared to those of the Trilinos ML package.

Figure 3 shows both weak and strong scalability tests of the Poisson problem solution on the SuperMUC cluster. Weak scaling results are presented on Figure 3(a). Here the problem size grows proportionally to the number of MPI processes so that each process owns approximately 100^3 unknowns. As in the shared memory tests, both AMGCL and Trilinos versions use conjugate gradient method preconditioned with smoothed AMG. Trilinos uses symmetric Gauss–Seidel, and AMGCL uses SPAI-0 for smoothing on each level of the AMG hierarchy. In case of ideal weak scaling the total solution time should be invariant with respect to the number of MPI processes. However, both Trilinos and AMGCL demonstrate around 20% weak scaling efficiency (defined as \(T_1/T_{1792}\)) at 1792 MPI processes. Overall, both libraries show very similar weak scalability.
Figure 3(b) depicts strong scalability of the Poisson problem. Here the problem size is fixed at 256³ unknowns. In case of ideal scaling (presented as dotted line on the figure) the solution time should decrease with increasing number of MPI processes. Here AMGCL demonstrates better scalability starting at about 100-400 MPI processes.

Scalability of the distributed memory Navier–Stokes solution is presented on Figure 4. The system matrix here has 4,773,588 unknowns and 281,089,456 nonzeros. The assembled problem is available for download at [14]. The fact that the problem size is fixed means that this is basically a strong scalability test. AMGCL version uses field split approach with Schur complement pressure correction preconditioner. Trilinos ML uses the nonsymmetric smoothed aggregation variant (NSSA) recommended by the user manual for this type of problems [19]. The tests use default NSSA parameters. AMGCL scales much better for the Navier–Stokes problem, which is mostly due to the better choice of preconditioner rather than any omissions in Trilinos implementation.

V. CONCLUSION

AMGCL is a header only C++ library that provides an efficient, scalable, and customizable implementation of algebraic multigrid method for solution of large sparse linear systems of equations arising from discretization of partial differential equations on structured or unstructured grids. The library supports parallelization with modern hardware using OpenMP, OpenCL, or CUDA technologies, and may work in shared or distributed memory mode.

The paper provides an overview of design decisions behind the library and demonstrates the library performance and scalability in comparison with PETSc, Trilinos ML, and CUSP packages. Numerical experiments presented in Section IV of the paper show that AMGCL performs on par or better than the alternatives on the examples of Poisson and Navier–Stokes problems.

Among AMGCL advantages are the liberal MIT license it is published under, possibility to customize, extend, and reuse the library components, and easy adoption to user data types, which makes integration of AMGCL into existing projects straightforward. Examples of projects (known to the author) that are currently using AMGCL are the Kratos Multi-Physics package [10] developed at CIMNE, Barcelona, and the MATLAB Reservoir Simulation Toolbox (MRST) [24] developed by the Computational Geosciences group in the Department of Mathematics and Cybernetics at SINTEF Digital. Kratos uses AMGCL as the default linear solver, and MRST provides a MATLAB interface to linear AMGCL solvers preconditioned with AMG or CPR.

The library source code is available for download at https://github.com/ddemidov/amgcl.

VI. ACKNOWLEDGMENTS

The development of the AMGCL library was partially funded by the state assignment to the Joint supercomputer center of the Russian Academy of sciences for scientific research. Work on field-split type preconditioners was partially funded by the RFBR grant Nos 18-07-00964, 18-47-160010. Access to the SuperMUC cluster was provided with support of the PRACE program, project 2010PA4512.

[1] Andrei Alexandrescu, Modern C++ design: generic programming and design patterns applied, Addison-Wesley, 2001.
[2] Allison H. Baker, Elizabeth R. Jessup, and Thomas Manteuffel, "A technique for accelerating the convergence of restarted GMRES", SIAM Journal on Matrix Analysis and Applications 26 (4), 962–984 (2005).
[3] Satish Balay, Shrirang Abhyankar, Mark F. Adams, Jed Brown, Peter Brune, Kris Buschelman, Lisandro Dalcin, Victor Eijkhout, William D. Gropp, Dinesh Kaushik, Matthew G. Knepley, Lois Curfman McInnes, Karl Rupp, Barry F. Smith, Stefano Zampini, Hong Zhang, and Hong Zhang. PETSc users manual, Technical Report ANL-95/11 - Revision 3.7, Argonne National Laboratory, 2016.
[4] Richard Barrett, Michael Berry, Tony F. Chan, James Demmel, June Donato, Jack Dongarra, Victor Eijkhout, Roldan Pozo, Charles Romine, and Henk Van der Vorst, Templates for the solution of linear systems: building blocks for iterative methods, SIAM, 1994.
[5] Nathan Bell, Steven Dalton, and Luke N Olson, "Exposing fine-grained parallelism in algebraic multigrid methods", SIAM Journal on Scientific Computing 34 (4), C123–C152 (2012).
[6] A. Brandt, S. McCormick, and J. Hult, "Algebraic multigrid (AMG) for sparse matrix equations", Sparsity and its Applications, 257 (1985).
[7] Oliver Bröker and Marcus J. Grote, "Sparse approximate inverse smoothers for geometric and algebraic multigrid", Applied numerical mathematics 41 (1), 61–80 (2002).
(a) Weak scaling. Approximately $100^3$ unknowns per MPI process.

(b) Strong scaling. The system matrix has $256^3$ unknowns.

FIG. 3: Scaling of the Poisson problem on the SuperMUC cluster.
FIG. 4: Scaling of the Navier–Stokes problem (3) on the SuperMUC cluster. The system matrix has 4 773 588 unknowns.

[8] Andrew J. Cleary, Robert D. Falgout, Van Emden Henson, Jim E. Jones, Thomas A. Manteuffel, Stephen F. McCormick, Gerald N. Miranda, and John W. Ruge, "Robustness and scalability of algebraic multigrid". *SIAM Journal on Scientific Computing* 21 (5), 1886–1908 (2000).

[9] P. Dadvand, R. Rossi, M. Gil, X. Martorell, J. Cotela, E. Juanpere, S. R. Idelsohn, and E. Oñate, "Migration of a generic multi-physics framework to HPC environments", *Computers and Fluids* 80 (1), 301–309 (2013).

[10] P. Dadvand, R. Rossi, and E. Oñate, "An object-oriented environment for developing finite element codes for multidisciplinary applications", *Archives of Computational Methods in Engineering* 17 (3), 253–297 (2010).

[11] Leonardo Dagum and Ramesh Menon. "OpenMP: an industry standard API for shared-memory programming", *IEEE Computational Science and Engineering* 5 (1), 46–55 (1998).

[12] Steven Dalton, Nathan Bell, Luke Olson, and Michael Garland, "Cusp: Generic parallel algorithms for sparse matrix and graph computations", [http://cusplibrary.github.io](http://cusplibrary.github.io) 2014. Version 0.5.0.

[13] D. E. Demidov and D. V. Shevchenko, "Modification of algebraic multigrid for effective GPGPU-based solution of non-stationary hydrodynamics problems", *Journal of Computational Science* 3 (6), 460–462 (2012).

[14] Denis Demidov and Riccardo Rossi, "Navier–Stokes problem for distributed memory AMGCL benchmarks", https://doi.org/10.5281/zenodo.1231818, April 2018.

[15] Denis Demidov and Riccardo Rossi, "Navier–Stokes problem for shared memory AMGCL benchmarks", https://doi.org/10.5281/zenodo.1231961, April 2018.

[16] J. (Jean) Donèa and Antonio Huerta, *Finite element methods for flow problems*. Wiley, 2003.

[17] Robert D. Falgout and Ulrike Meier Yang. hypre: A library of high performance preconditioners. In *International Conference on Computational Science*, 632–641, Springer, 2002.

[18] Diederik R. Fokkema, *Enhanced implementation of BiCGstab (l) for solving linear systems of equations*. Universiteit Utrecht, 1996.

[19] M. W. Gee, C. M. Siebert, J. J. Hu, R. S. Tuminaro, and M. G. Sala, "ML 5.0 Smoothed Aggregation User’s Guide", Technical Report SAND2006-2649, Sandia National Laboratories, 2006.

[20] Björn Gmeiner, Markus Huber, Lorenz John, Ulrich Rüde, and Barbara Wohlmuth, "A quantitative performance study for stokes solvers at the extreme scale". *Journal of Computational Science* 17, 509–521 (2016).

[21] Sebastian Gries, Klaus Stüben, Geoffrey L. Brown, Dingjun Chen, David A. Collins, et al. "Preconditioning for efficiently applying algebraic multigrid in fully implicit reservoir simulations", *SPE Journal* 19 (04), 726–736 (2014).
[22] Pascal Hénon, Pierre Ramet, and Jean Roman, "PASTIX: a high-performance parallel direct solver for sparse symmetric positive definite systems", *Parallel Computing* **28** (2), 301–321 (2002).
[23] Jonathan Hogg and Jennifer Scott, "New parallel sparse direct solvers for multicore architectures", *Algorithms* **6** (4), 702–725 (2013).
[24] Knut-Andreas Lie, *An introduction to reservoir simulation using MATLAB: user guide for the Matlab Reservoir Simulation Toolbox (MRST)*. SINTEF ICT, Norway, 2016.
[25] John W Ruge and Klaus Stüben, *Algebraic multigrid*. In *Multigrid methods*, 73–130. SIAM, 1987.
[26] K. Rupp, F. Rudolf, and J. Weinbub, *ViennaCL - A High Level Linear Algebra Library for GPUs and Multi-Core CPUs*, In *Intl. Workshop on GPUs and Scientific Applications*, 51–56, 2010.
[27] Yousef Saad, *Iterative methods for sparse linear systems*. SIAM, 2003.
[28] Marzio Sala and Raymond S. Tuminaro, "A new Petrov–Galerkin smoothed aggregation preconditioner for nonsymmetric linear systems", *SIAM Journal on Scientific Computing* **31** (1), 143–166 (2008).
[29] Jason Sanders and Edward Kandrot, *CUDA by example: an introduction to general-purpose GPU programming*. Addison-Wesley Professional, 2010.
[30] Peter Sonneveld and Martin B. Van Gijzen, "IDR(s): A family of simple and fast algorithms for solving large nonsymmetric systems of linear equations", *SIAM Journal on Scientific Computing* **31** (2), 1035–1062 (2008).
[31] John E. Stone, David Gohara, and Guochun Shi, "OpenCL: A parallel programming standard for heterogeneous computing systems", *Computing in Science & Engineering* **12** (3), 66–73 (2010).
[32] K. Stuben, "Algebraic multigrid (AMG): an introduction with applications", GMD Report 70, GMD, Sankt Augustin, Germany, 1999.
[33] U. Trottenberg, C. Oosterlee, and A. Schüller, *Multigrid*. Academic Press, London, 2001.
[34] Petr Vaněk, Jan Mandel, and Marian Brezina. "Algebraic multigrid by smoothed aggregation for second and fourth order elliptic problems", *Computing* **56** (3), 179–196 (1996).
[35] Rudiger Verfurth, "A combined conjugate gradient-multi-grid algorithm for the numerical solution of the Stokes problem", *IMA Journal of Numerical Analysis* **4** (4), 441–455 (1984).
[36] https://www.boost.org
[37] http://www.cimne.com/kratos
[38] https://github.com/ddemidov/vexcl