GRADED DECOMPOSITION NUMBERS FOR BLOCKS OF SMALL WEIGHT

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Abstract. We present some blocks of Ariki-Koike algebras $H_{n,r}$ where the decomposition matrices are independent of the characteristic of the underlying field. We complete the description of the graded decomposition numbers for blocks of Ariki-Koike algebras of weight at most two, which consists of analysing the indecomposable core blocks at level $r = 3$, and give a closed formula for the decomposition numbers in this case. We then look in detail at core blocks in higher levels whose weight graphs are trees. We end with some graded decomposition numbers for such blocks when $r = 4$.

1. Introduction

The Ariki-Koike algebras $H_{n,r}$ arise naturally in multiple contexts. Originally introduced by Ariki and Koike [3] as a simultaneous generalisation of the Hecke algebras of types $A$ and $B$, they also correspond to the Hecke algebras of the complex reflection groups of type $G(r,1,n)$ [4] and can be seen in the work of Cherednik [9]; furthermore, the symmetric group algebra $F\Sigma_n$ occurs as an example of such an algebra. Recently, they have attracted attention due to their relevance within the categorification program. It was shown in the 1990s [1, 21] that the finite dimensional $H_{n,r}$-modules for all $n \geq 0$ categorify the irreducible highest weight module $V(\Lambda)$ of a certain Kac-Moody algebra $g$. More recently, a result of Brundan and Kleshchev [5] showed that the Ariki-Koike algebras are isomorphic to certain $\mathbb{Z}$-graded algebras introduced independently by Khovanov and Lauda [26, 27] and Rouquier [36]. This defines a grading on $H_{n,r}$ and gives rise to a categorification of $V(\Lambda)$ over the quantized enveloping algebra $U_v(g)$.

The most important open problem in the study of the Ariki-Koike algebras is to determine the decomposition matrices, that is, find the composition factors of the Specht modules $S^\lambda$. It was shown by Brundan, Kleshchev and Wang [7] that the Specht modules are graded, and so we define graded decomposition numbers. Over fields of characteristic 0, these numbers are the polynomials arising from the LLT conjecture [30] and so can, in principle, be computed. In practice, this computation is not possible, except for small values of $n$ and some specific cases where there exist closed formulae.

Graded decomposition matrices for Ariki-Koike algebras defined over a field of positive characteristic are related to those over a field of characteristic 0 by (graded) adjustment matrices whose entries come from $\mathbb{N}[v^{-1} + v]$; in particular, the graded decomposition numbers in characteristic 0 give a lower bound for those in arbitrary characteristic [5]. James’ conjecture [24], for Hecke algebras of type $A$, was that when the characteristic of the field is less than the weight of the block, the adjustment matrix is the identity matrix. However, as this paper was being written, Williamson announced counterexamples to Lusztig’s conjecture and, consequently, to James’ conjecture [35].

The question of when an adjustment matrix is the identity matrix is now wide open, for all $r \geq 1$, and it remains to be seen what role the weight of the block plays. If $r = 2$ and $e = 0$ or $e > n$, the decomposition numbers are independent of the characteristic of the field [8, 23, Corollary A.5], regardless of the weight of the block. In this paper, we present certain other blocks, when $r = 3$ and when $r = 4$, in which the decomposition numbers are also independent of the characteristic of the field. We first complete the description of the decomposition matrices for blocks of weight at most two. The blocks that we need to consider are indecomposable core blocks, with $r = 3$. We have the following simple description of the corresponding decomposition numbers. If $\mu$ and $\lambda$ are multipartitions in the same block, write $\mu \sim \lambda$ if $\lambda$ is formed from $\mu$ by removing a single rim hook from component $k$ and attaching it to component $k + 1$, where $k \in \{1, 2, \ldots, r - 1\}$. Write $\mu \sim \lambda$ if $\mu \sim \lambda$ and the leg lengths of these two hooks are equal. If $\lambda$ is
a Kleshchev multipartition, let $\tilde{\lambda} = (\lambda^c)^t$ denote the conjugate of the image of $\lambda$ under the generalisation of the Mullineux involution. (See, for example, [6, §3.10].)

**Theorem 1.** Suppose that $\lambda$ and $\mu$ are 3-multipartitions in an indecomposable core block of weight 2 and that $\lambda$ is a Kleshchev multipartition. Then

$$d_{\mu\lambda}(v) = \begin{cases} 1, & \mu = \lambda, \\ v, & \tilde{\lambda} \triangleright \mu \triangleright \lambda \text{ and } (\mu \preceq \lambda \text{ or } \tilde{\lambda} \preceq \mu), \\ v^2, & \mu = \tilde{\lambda}, \\ 0, & \text{otherwise.} \end{cases}$$

This formula is close in spirit to the formulae given by Richards [35, Theorem 4.4] for decomposition numbers of weight 2 blocks of Hecke algebras of type $A$ and Fayers [19, Theorem 3.18] for certain weight 2 blocks of Hecke algebras of type $B$. A faster way to compute the decomposition matrix for our blocks is given in Theorem 3.4 where, for each Kleshchev multipartition $\lambda$, we list the multipartitions $\mu$ such that $D^\lambda$ is a composition factor of $S^\mu$. In fact, the proof of Theorem 1 follows from case-by-case analysis of Theorem 3.4.

For each multipartition in a core block, there is a natural way of associating a weight graph on $r$ vertices. A natural progression would be to consider core blocks for arbitrary values of $r$ in which the weight graphs of the corresponding multipartitions are trees; this is indeed a block invariant property. We classify the blocks containing multipartitions whose weight graphs are trees and look in more detail at such blocks when $r = 4$.

2. **Preliminaries**

2.1. **The Ariki-Koike algebras.** Let $F$ be a field of arbitrary characteristic. Pick $r > 0$ and $n \geq 0$. Pick $q \in F$ with $q \neq 0$, and let $e$ be minimal such that $1 + q + \cdots + q^{e-1} = 0$, or 0 if no such value exists. Choose nonzero parameters $Q_1, \ldots, Q_s \in F$. The Ariki-Koike algebra $H_{n,r}$ is the $F$-algebra with generators $T_0, \ldots, T_{n-1}$ and relations

$$(T_i + q)(T_i - 1) = 0 \quad \text{for } 1 \leq i \leq n - 1,$$

$$T_iT_j = T_jT_i \quad \text{for } 1 \leq i, j \leq n - 1, |i - j| > 1,$$

$$T_iT_{i+1}T_i = T_{i+1}T iT_{i+1} \quad \text{for } 1 \leq i \leq n - 2,$$

$$(T_0 - Q_1)(T_0 - Q_s) = 0,$$

$$T_0T_1T_0T_1 = T_1T_0T_1T_0.$$

We say two parameters $Q_s$ and $Q_t$ are $q$-connected if $Q_s = q^kQ_t$, for some $k \in Z$. A result of Dipper and Mathas [10] states that each Ariki-Koike algebra $H_{n,r}$ is Morita equivalent to a direct sum of tensor products of smaller algebras whose parameters are all $q$-connected. In view of this result, we may assume that our parameters are $q$-connected, in fact, that they are powers of $q$. If we set $I = Z$ if $e = 0$, and $I = \{0, 1, \ldots, e - 1\}$ otherwise then there exists a unique $a = (a_1, \ldots, a_r) \in I^r$ such that $Q_s = q^{a_s}$ for all $1 \leq s \leq r$. We call $a$ the multicharge.

The algebras $H_{n,r}$ are cellular algebras [20, 11], with the cell modules indexed by $r$-multipartitions of $n$. (We provide a definition of multipartitions and Kleshchev multipartitions in the next section.) The simple modules arise as the heads of certain cell modules which are indexed by a subset of multipartitions known as Kleshchev multipartitions. This allows us to define the decomposition matrix of $H_{n,r}$ to be the matrix recording the multiplicity of a simple module $D^\lambda$ as a composition factor of a cell module $S^\mu$. However, this is not the end of the story. In [5], Brundan and Kleshchev showed that the algebras $H_{n,r}$ are isomorphic to certain $Z$-graded algebras defined by Khovanov and Lauda [20, 27] and Rouquier [30]. The definition of these cyclotomic quiver Hecke algebras is not necessary for this paper; we refer the reader to the survey paper [23] for more information. We have, therefore, a grading on $H_{n,r}$. (By grading, we will always mean $Z$-grading.) Moreover, the cell modules $S^\mu$ admit a grading [17] and so we may talk about graded decomposition numbers. For a graded algebra $A$, we let $\text{Rep}(A)$ denote the category of finite-dimensional graded right $A$-modules. Recall that if $M = \oplus_{d \in Z} M_d \in \text{Rep}(A)$, then for $k \in Z$ we define $M(k)$ to be the $A$-module isomorphic to $M$ but with grading shifted by $k$, that is, $M(k)_d = M_{d-k}$. Given $M, L \in \text{Rep}(A)$ with $L$ irreducible, we define
the graded decomposition number

\[ |M : L|_v = \sum_{k \in \mathbb{Z}} |M : L(k)| v^k \]

where \( v \) is an indeterminate over \( \mathbb{Z} \) and \( |M : L(k)| \) is the graded multiplicity of \( L(k) \) in \( M \). To see that this is well-defined, we refer the reader to [34] (but see also [28]).

Our aim is to compute certain graded decomposition numbers. To this end, we must first introduce some combinatorics.

2.2. Partitions and abacus displays. A partition \( \lambda \) of \( n \) is a weakly decreasing sequence of non-negative integers \( (\lambda_1, \lambda_2, \ldots) \), where \( |\lambda| = \sum_{i=1}^{\infty} \lambda_i = n \). An \( r \)-multipartition of \( n \) (usually called a multipartition) is an \( r \)-tuple \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}) \) of partitions such that \( |\lambda^{(1)}| + |\lambda^{(2)}| + \ldots + |\lambda^{(r)}| = n \). Let \( \mathcal{P}_n^r \) denote the set of \( r \)-multipartitions of \( n \) and set \( \mathcal{P}^r = \bigcup_{n \geq 0} \mathcal{P}_n^r \). We define a partial order \( \supseteq \) on \( \mathcal{P}_n^r \) by saying that \( \mu \supseteq \lambda \) if and only if

\[ \sum_{i=1}^{t-1} |\lambda^{(\sigma(i))}| + \sum_{i=0}^{z} \mu_x^{(t)} \geq \sum_{i=1}^{t-1} |\lambda^{(\sigma(i))}| + \sum_{i=0}^{z} \lambda_x^{(t)} \quad \text{for all} \quad 1 \leq t \leq r \text{ and } z \geq 0, \]

and we say that \( \mu \supset \lambda \) if \( \mu \supseteq \lambda \) and \( \mu \neq \lambda \). We make the usual identification of \( \lambda \) with its Young diagram: the collection of nodes

\[ \{(s, x, y) \mid 1 \leq s \leq r, 1 \leq y \leq \lambda_x^{(s)} \} \subset \{1, 2, \ldots, r\} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}. \]

More generally, we refer to any \( (s, x, y) \in \{1, 2, \ldots, r\} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \) as a node and say that \( (s, x, y) \) is above \( (s', x', y') \) if and only if \( s < s' \) or \( s = s' \) and \( x < x' \). A node is a removable/addable node of \( \lambda \) if removing it from \( \lambda \) yields the Young diagram of another multipartition.

Recall that we have fixed a multicharge \( a = (a_1, a_2, \ldots, a_r) \in I^r \). Define the residue of a node \( A = (s, x, y) \) by \( \text{res}(A) = a_s - x + y \mod e \), if \( e > 0 \), and \( \text{res}(A) = a_s - x + y \) if \( e = 0 \). We refer to nodes of residue \( i \) as \( i \)-nodes and write \( c_i(\lambda) \) for the number of \( i \)-nodes in \( \lambda \). If \( \lambda \) is formed from \( \sigma \) by adding \( k > 0 \) nodes, all of residue \( i \), then we write \( \sigma \xrightarrow{i} \lambda \) and set

\[ N_i(\sigma, \lambda) = \sum_{\gamma \in \lambda^{(i)}} \# \{\gamma' \mid \gamma' \text{ an addable } i \text{-node of } \lambda \text{ below } \gamma\} - \# \{\gamma' \mid \gamma' \text{ a removable } i \text{-node of } \sigma \text{ below } \gamma\}. \]

A removable \( i \)-node \( A \) of \( \lambda \) is normal if whenever \( B \) is an addable \( i \)-node below \( A \) there are more removable \( i \)-nodes between \( A \) and \( B \) than addable \( i \)-nodes. The highest normal \( i \)-node in \( \lambda \) – if such a thing exists – is said to be good. A multipartition \( \lambda \) is called Kleshchev if it is empty, or if another Kleshchev multipartition can be obtained from it by removing a good node (of any residue). If \( r = 1 \), Kleshchev is equivalent to \( e \)-restricted. As previously mentioned, the Kleshchev multipartitions will index the irreducible \( H_{n,r} \)-modules.

It will be convenient to describe multipartitions in terms of abacus configurations. Suppose that \( \lambda \) is a partition and let \( a \in I \). For \( j \geq 1 \), set \( \beta_j = \lambda_j - j + a \) and define

\[ \beta_a(\lambda) = \{ \beta_j \mid j \geq 1 \} \]

to be the set of \( \beta \)-numbers associated with the partition \( \lambda \) and the charge \( a \). Take an abacus with runners indexed from left to right by the elements of \( I \) with possible bead positions indexed by the elements of \( \mathbb{Z} \) from top to bottom and left to right, such that for \( i \in I \), the integers \( k \equiv i \mod e \) appear on runner \( i \). The abacus configuration of \( \lambda \) with respect to \( a \) is then the abacus configuration with a bead at position \( \beta_j \) for each \( j \geq 1 \). If \( e = 0 \), runners are indexed by the elements of \( \mathbb{Z} \), with one bead appearing on runner \( i \) for each \( i \in \beta_a(\lambda) \).

Example 1. Suppose that \( e = 4 \). Let \( a = 2 \) and suppose that \( \lambda = (10^2, 5, 3^3, 2, 1^3) \). Then

\[ \beta_2(\lambda) = \{11, 10, 4, 1, 0, -1, -3, -5, -6, -7, -9, -10, \ldots\} \]

and the abacus configuration has the form:
For a multipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$, the abacus configuration of $\lambda$ is the $r$-tuple of abacuses in which abacus $s$ represents the $\beta$-numbers $\beta_{\mu}(\lambda^{(s)})$. Note that increasing (resp. decreasing) a $\beta$-number by one in any of these sets corresponds to moving a bead across from a runner $i$ to runner $i - 1$ (resp. from runner $i$ to runner $i - 1$), considered modulo $e$, and that this corresponds to adding (resp. removing) a node of residue $i$ from the Young diagram of $\lambda$.

Defined thus, when $e \geq 2$ each runner of the abacus contains an infinite ‘sea’ of beads that does not actually encode any information about the shape of the corresponding partition. If we consider a truncated abacus configuration to be one with only finitely many beads on each runner, then we can associate it with a partition by filling in all the rows above its highest bead with other beads. Conversely, if we pick $N$ to be maximal so that $x \in \beta(\lambda)$ whenever $x < Ne$, we can define a canonical finite abacus configuration for $\lambda$ to be the one corresponding to the set $\beta_0(\lambda) \cap \{Ne, Ne + 1, \ldots\}$. Considering this truncated abacus amounts to ignoring all the rows that do not affect the shape of the corresponding partition.

Having set up our combinatorics, we may now return to the algebra $\mathcal{H}_{n,r}$.

2.3. Specht modules, simple modules, and blocks. The Ariki-Koike algebras are cellular [20, 11], with cell modules, called Specht modules, indexed by $r$-multipartitions of $n$, and denoted $S^\lambda$ for $\lambda \in \mathcal{P}^r_n$. By the general theory of cellular algebras, each $S^\lambda$ comes equipped with an $\mathcal{H}_{n,r}$-invariant bilinear form, and every simple $\mathcal{H}_{n,r}$-module occurs as the quotient of some unique $S^\lambda$ by the radical of this form. Write $D^\lambda$ for the module obtained from $S^\lambda$ in this way.

**Theorem 2.1** (2). Let $\lambda \in \mathcal{P}^r_n$. Then $D^\lambda \neq 0$ if and only if $\lambda$ is a Kleshchev multipartition. Hence, by the properties of cellular algebras,

$$\{D^\lambda \mid \lambda \text{ is a Kleshchev multipartition}\}$$

forms a complete set of pairwise non-isomorphic irreducible $\mathcal{H}_{n,r}$-modules.

It is known that the Specht modules $S^\lambda$ are graded [7] and therefore the (non-zero) quotient modules $D^\lambda$ inherit this grading. Given a multipartition $\mu$ and a Kleshchev multipartition $\lambda$, define $d_{\mu\lambda}(v)$ to be the graded multiplicity of $D^\lambda$ as a composition factor of $S^\mu$. The most important open problem in the representation theory of $\mathcal{H}_{n,r}$ is to determine these composition multiplicities. Recall that $d_{\mu\lambda}(v) \in \mathbb{N}[v, v^{-1}]$, where we assume $0 \in \mathbb{N}$.

**Theorem 2.2** (20, 11, 7). Suppose that $\mu$ is a multipartition and $\lambda$ a Kleshchev multipartition of $n$. Then the following results hold.

1. We have $d_{\lambda\lambda}(v) = 1$ and if $d_{\mu\lambda}(v) \neq 0$ then $\mu \triangleright \lambda$.
2. Suppose that $\mathbb{F} = \mathbb{C}$. If $\mu \neq \lambda$ then $d_{\mu\lambda}(v) \in v\mathbb{N}[v]$.

We note that if $\text{char}(\mathbb{F}) \neq 0$ then there exist graded decomposition numbers that do not lie in $\mathbb{N}[v]$. The smallest known such example, due to Anton Evseev [12], is when $r = 1, n = 8$ and $e = p = 2$, so for the symmetric group algebra $\mathbb{F}_2S_8$. Setting $\lambda = (1^8)$ and $\mu = (3, 2^2, 1)$ we have $d_{\mu\lambda}(v) = v^{-1} + v$.

One way of computing decomposition numbers is by using $i$-induction. We note that Proposition 2.3 below, while sufficient for our purposes, is only an application of some far more general theory. Assume $v$ to be an indeterminate over $\mathbb{C}$ and define the Fock space $\mathcal{F}(\mathfrak{a})$ to be the $\mathbb{C}[v]$-vector space with basis $\mathcal{P}^r$. Then
\(F(a)\) becomes a module for the quantized enveloping algebra \(U_q(\mathfrak{g})\) where \(\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})\) if \(e > 0\) and \(\mathfrak{g} = \mathfrak{sl}_\infty(\mathbb{C})\) if \(e = 0\). We describe the action of the divided powers of the Chevalley generators \(F_i\), where \(i \in \mathbb{Z}/e\mathbb{Z}\). If \(\sigma \in \mathcal{P}_r\) then for \(k > 0\), we have

\[
F_i^{(k)} \sigma = \sum_{\sigma \vdash \lambda} v^{N_i(\sigma \vDash \lambda)} \lambda.
\]

For a Kleshchev multipartition \(\lambda \in \mathcal{P}_r\), define \(P(\lambda) = \sum_{\mu \in \mathcal{P}_r} a_{\mu \lambda}(v) \mu\).

**Proposition 2.3.** Suppose that \(P(\sigma) = \sigma\) and that there exist \(i_1, i_2, \ldots, i_m \in \mathbb{Z}/e\mathbb{Z}\) and \(k_1, k_2, \ldots, k_m > 0\) such that

\[
F_i^{(k_m)} \ldots F_{i_2}^{(k_2)} F_{i_1}^{(k_1)} \sigma = \lambda + \sum_{\mu \neq \lambda} a_{\mu \lambda}(v) \mu
\]

where \(a_{\mu \lambda}(v) \in v\mathbb{N}[v]\) for all \(\mu\). Then \(\lambda\) is a Kleshchev multipartition and \(d_{\mu \lambda}(v) = a_{\mu \lambda}(v)\) for all \(\mu \neq \lambda\); or equivalently,

\[
P(\lambda) = F_i^{(k_m)} \ldots F_{i_2}^{(k_2)} F_{i_1}^{(k_1)} \sigma.
\]

**Proof.** When \(F = \mathbb{C}\), these polynomials agree with those computed when applying the LLT conjecture. By [6], these agree with the decomposition numbers. If \(F\) has positive characteristic, the proposition then follows from the theory of graded adjustment matrices, again by [6].

Since \(\mathcal{H}_{n,r}\) is cellular, all composition factors of a Specht modules lie in the same block, so to classify the blocks of the \(\mathcal{H}_{n,r}\), it is sufficient to decide when two Specht modules lie in the same block.

**Proposition 2.4 ([22]).** Two Specht modules \(S^{\lambda}\) and \(S^{\mu}\) lie in the same block if and only if \(c_i(\lambda) = c_i(\mu)\) for all \(i \in \mathbb{Z}/e\mathbb{Z}\).

If \(r = 1\), then an alternative description of the blocks can be given as follows. If \(e = 0\), the algebra is semisimple, so assume \(e > 0\). Define the weight (more precisely, \(e\)-weight) of a partition \(\lambda\) to be the number of \(e\)-hooks that can be removed from it in succession. The partition left afterwards is called the core of \(\lambda\), and hence partitions from which no \(e\)-hooks can be removed are called cores. Since removing an \(e\)-rim hook corresponds to moving a bead up one position on the abacus, a core corresponds to an abacus configuration in which no bead can be moved any higher on its runner. Then two Specht modules \(S^{\lambda}\) and \(S^{\mu}\) lie in the same block if and only if \(\lambda\) and \(\mu\) have the same core and the same weight.

When \(r > 1\), the natural generalization of this construction is incorrect. Say that a multipartition is a multicore if \(e = 0\) of if \(e \geq 2\) and the multipartition does not have any removable \(e\)-rim hooks. Note that if \(S^{\lambda}\) and \(S^{\mu}\) lie in the same block, it is possible to have \(\lambda\) be a multicore, but \(\mu\) not. We define a core block to be a block in which all the Specht modules correspond to multicores; note that if \(e = 0\) then every block is a core block. The following definition is due to Fayers [14]. We define the weight of a multipartition \(\lambda\) by

\[
\text{wt}(\lambda) = \sum_{s=1}^{\lambda} c_{a_s}(\lambda) + \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \left( c_i(\lambda) - c_{i+1}(\lambda) \right)^2.
\]

It follows from Proposition 2.4 that if \(S^{\lambda}\) and \(S^{\mu}\) lie in the same block then \(\text{wt}(\lambda) = \text{wt}(\mu)\) and so we will talk about the weight of a block. This definition of ‘weight’ agrees with the definition of the ‘defect’ of the corresponding block of the cyclotomic quiver Hecke algebra [25 Eqn. (3.10)].

Fayers’ ingenious definition of the weight of a multipartition is not so easy to read off the abacus; in particular, it is not generally true that multicore have weight zero. The following facts about the weight function are proved in [14].

**Proposition 2.5.** Let \(\lambda \in \mathcal{P}_r\).

1. We have \(\text{wt}(\lambda) \geq 0\) and \(\text{wt}(\lambda) = 0\) if and only if \(P(\lambda) = \lambda\).
2. If \(r = 1\), \(\text{wt}(\lambda)\) agrees with the usual definition of weight for partitions.
3. Let \(\mu\) be obtained from \(\lambda\) by removing a rim \(e\)-hook from any component – equivalently, by moving a bead one space up in any component – of \(\lambda\). Then \(\text{wt}(\mu) = \text{wt}(\lambda) - r\).
(4) If $\lambda$ is a multicore then
\[
\text{wt}(\lambda) = \sum_{1 \leq s < t \leq r} \text{wt}(\lambda^{(s)}, \lambda^{(t)}).
\]

Using these results, the computation of $\text{wt}(\lambda)$ reduces to the case where $\lambda$ is a bicore, which is also described in [13]. Note that if $S^{\lambda}$ lies in a block which is not a core block then $\text{wt}(\lambda) \geq r$. Hence blocks of small weight often correspond to core blocks. In fact, it is possible to further simplify the decomposition matrices corresponding to core blocks. Say $\lambda \in \mathcal{P}^r$ is decomposable if there exist proper subsets $S$ and $T$ partitioning $\{1, \ldots, r\}$ in such a way that $\text{wt}(\lambda^{(s)}, \lambda^{(t)}) = 0$ whenever $s \in S$ and $t \in T$. If $S = \{s_1, \ldots, s_n\}$ define $\lambda^S = (\lambda^{(s_1)}, \ldots, \lambda^{(s_n)})$ and define $\lambda^T$ analogously.

In view of the following results of Fayers, which are proved in section 3.5 of [17], it makes sense to talk about decomposable blocks.

**Proposition 2.6.** Let $\lambda$ be a decomposable multipartition with $S$ and $T$ as in the definition above.

1. We have that $\lambda$ lies in a core block.
2. If $S^\mu$ lies in the same block as $S^\lambda$ then $\mu$ is decomposable relative to the same choice of $S$ and $T$.
3. The partition $\lambda$ is Kleshchev if and only if $\lambda^S$ and $\lambda^T$ are both Kleshchev.

For $\lambda$ lying in a core block, define the weight graph $G(\lambda)$ of $\lambda$ to be the undirected graph with vertices $\{1, 2, \ldots, r\}$ and with $\text{wt}(\lambda^{(s)}, \lambda^{(t)})$ edges between vertices $s$ and $t$. From the definition above, it is clear that $\text{wt}(\lambda)$ is the number of edges in $G(\lambda)$ and that $\lambda$ is indecomposable if and only if $G(\lambda)$ is connected.

We are now in a position to consider what is known about blocks of small weight. If a block has weight 0, it contains one cell module $S^\lambda$ with $S^\lambda \simeq D^{\lambda}$. A block of weight 1 must either satisfy $r = 1$ or be a core block, where either $r = 2$ or the block is decomposable with the weight graph consisting of a single edge between two vertices. The decomposition numbers in the former case are well-known [24]. The case that $r = 2$ is dealt with explicitly in [13]; however the results may easily be seen to extend to $r > 2$.

A block of weight 2 must either have $r \leq 2$ or be a core block. The former cases have been dealt with in [35] [13] [15]. If it is a decomposable core block, then the techniques in [14] [15] enable us to determine the decomposition matrices. The remaining open case, which we consider in the next section, is when we have an indecomposable core block, that is, when $r = 3$ and the weight graph is a line.

### 2.4 Core blocks

We finish this section with some remarks on core blocks. Recall that $I = \{0, 1, \ldots, e - 1\}$ if $e \geq 2$ and $I = \mathbb{Z}$ if $e = 0$.

Let $\mathfrak{M}$ be the set of matrices $M$ with entries in $\{0, 1\}$ with $r$ rows labelled 1, 2, $\ldots$, $r$ and with columns labelled by the ordered elements of $I$, where if $e = 0$ then for all $1 \leq s \leq r$, we have $M(s, i) = 1$ and $M(s, j) = 0$ for all $i < 0$ and all $j > 0$. Let $\mathfrak{M} \subset \mathfrak{M}$ be the subset of $\mathfrak{M}$ consisting of matrices with the further property that if $e \geq 2$ then each column contains at least one 0.

If $e \geq 2$, let $\mathfrak{B}$ denote the set of tuples $B = (b_0, \ldots, b_{e-1}) \in \mathbb{N}^e$ such that $b_i = 0$ for at least one value of $i$ and if $e = 0$, let $\mathfrak{B} = \{0\}$. We call $\mathfrak{B}$ the set of base tuples. For $e \geq 2$, let $\mathfrak{O}$ denote the zero tuple $(0, \ldots, 0)$. Our first step is to set up a function
\[
\mathfrak{B} \times \mathfrak{M} \to I^r \times \mathcal{P}^r
\]
which identifies a base tuple $B$ and a matrix $M \in \mathfrak{M}$ with a multicharge $a$ and a partition $\lambda$ in a core block (with respect to that multicharge). If we restrict to $\mathfrak{B} \times \mathfrak{M}$, the map will be an injection.

Suppose first that $e = 0$ so that all multipartitions lie in core blocks. Then there is a bijection between $I^r \times \mathcal{P}^r$ and $\mathfrak{B} \times \mathfrak{M}$: namely, given $a = (a_1, a_2, \ldots, a_r) \in I^r$ and $\lambda \in \mathcal{P}^r$, we identify them with the unique base tuple $0 \in \mathfrak{B}$ and the matrix $M \in \mathfrak{M}$ given by
\[
M(s, i) = \begin{cases} 1, & i \in \beta_{a_s}(\lambda^{(s)}), \\ 0, & \text{otherwise}, \end{cases}
\]
for all $1 \leq s \leq r$ and $i \in I$. We write $\lambda = \text{Pt}(0, M)$ (and note that $\lambda$ is identified with a Young diagram along with its collection of residues, and therefore incorporates complete information about the multicharge). Let $\prec$ be the total order on $\mathbb{Z}$, which in this case agrees with the usual total order $\prec$.

Now suppose $e \geq 2$. For $B \in \mathfrak{B}$, we define a total order $\prec$ on $\{0, 1, \ldots, e - 1\}$ by saying that
\[
i < j \iff b_i < b_j \text{ or } b_i = b_j \text{ and } i < j
\]
and set $\pi = \pi(B)$ to be the permutation of $\{0, 1, \ldots, e-1\}$ such that
$$
\pi(0) \prec \pi(1) \prec \ldots \prec \pi(e-1).
$$
For $M \in \mathcal{M}$ and $B = (b_0, \ldots, b_{e-1}) \in \mathcal{B}$, we define an $r$-tuple of truncated abaci by saying
$$
\# \{ \text{beads on runner } i \text{ of abacus } s \} = b_i + M(s, \pi^{-1}(i)).
$$
If $\lambda$ is the multipartition associated to this configuration and $\mathbf{a}$ is the corresponding multicharge then we write $\lambda = \text{Pt}(B, M)$ (again on the understanding that information about the multicharge is incorporated in $\lambda$). By $M(s, \pi^{-1}(i))$ we mean the entry in the row indexed by $s$ and the column indexed by $\pi^{-1}(i)$, where $1 \leq s \leq r$ and $0 \leq \pi^{-1}(i) \leq e-1$.

We now go back to the general case where $e = 0$ or $e \geq 2$.

**Lemma 2.7.** Fix a multicharge $\mathbf{a}$ and suppose $\lambda \in \mathcal{P}^r_n$. The Specht module $S^\lambda$ lies in a core block if and only $\lambda = \text{Pt}(B, M)$ for a unique $B \in \mathcal{B}$ and $M \in \mathcal{M}$.

**Proof.** See Theorem 3.1 in [17].

**Example 2.** Let $e = 5$, let $r = 3$ and let $\mathbf{a} = (4,1,0)$. Let $\lambda = ((4,2^3,1^4), (7,5,4,2^4), (3,1^2))$. Then we have the abacus diagram

```
1 1 0 0 0
1 0 0 1 1
0 1 0 0 0
```

Hence $\lambda = \text{Pt}(B, M)$ where $B = (1,0,3,2,2)$, the total order $\prec$ is given by $1 \prec 0 \prec 3 \prec 4 \prec 2$ and $M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$. In particular, note that the order of the columns of $M$ depends on the order $\prec$.

We now describe how to calculate the weight of a core block, and, given $S^\lambda$ in a core block, how to describe the other Specht modules in the block. First suppose $M \in \mathcal{M}$. For $1 \leq s, t \leq r$ define
$$
\delta^M(t,s) = \# \{ 0 \leq i \leq e-1 \mid M(t,i) = 1 \text{ and } M(s,i) = 0 \},
$$
$$
\delta^M(t,s) = \# \{ 0 \leq i \leq e-1 \mid M(t,i) = 0 \text{ and } M(s,i) = 1 \},
$$
$$
\text{wt}^M(t,s) = \text{Min} \{ \delta^M(t,s), \delta^M(t,s) \},
$$
and define
$$
\text{wt}(M) = \sum_{1 \leq s < t \leq r} \text{wt}^M(t,s).
$$

**Lemma 2.8.** Suppose $\lambda = \text{Pt}(B, M)$. Then $\text{wt}(\lambda^{(t)}, \lambda^{(s)}) = \text{wt}^M(t,s)$ for all $1 \leq s < t \leq r$ and therefore $\text{wt}(\lambda) = \text{wt}(M)$.

**Proof.** Allowing for the change of notation, this follows from [14].

Let $M \in \mathcal{M}$. Suppose there exist $i,j \in I$ and $1 \leq s, t \leq r$ with
$$
M(s,i) = M(t,j) = 1, \quad M(s,j) = M(t,i) = 0.
$$

Let $M'$ be the matrix with
$$
M'(s,i) = M'(t,j) = 0, \quad M'(s,j) = M'(t,i) = 1, \quad M'(u,k) = M(u,k) \text{ otherwise}.
$$
Say that $M'$ is obtained from $M$ by a *bead swap*, and define the equivalence relation $\sim_b$ on $\mathcal{M}$ to be the equivalence relation generated by all bead swaps. The next result comes from [17].
Lemma 2.9. Suppose that $\lambda = \text{Pt}(B, M)$. If $\mu$ is a multipartition then $S^\mu$ belongs to the same core block as $S^\lambda$ if and only if $\mu = \text{Pt}(B, K)$ for some $K \sim_0 M$.

We may therefore identify the Specht modules in a core block with the elements in an equivalence class of $\mathcal{M}/ \sim_0$. The next results suggest that this will be a useful identification.

Proposition 2.10. Let $e \geq 2$. Suppose that $\lambda = \text{Pt}(B, M)$ is a partition in a core block. Then $\lambda$ is Kleshchev if and only if $\mu = \text{Pt}(0, M)$ is Kleshchev.

Proof. It suffices to prove that we can reduce from $\lambda$ to $\mu$ by removing a sequence of good nodes. Clearly, if the highest $s$ nodes of a given residue in $\lambda$ are all normal, then the multipartition $\sigma$ obtained from $\lambda$ by removing them all will be Kleshchev if and only if $\lambda$ is Kleshchev. Writing $B = (b_0, \ldots, b_{e-1})$ as usual, we will proceed by induction on $\sum_{i=0}^{e-1} b_i = 0$, with the base case being trivial.

We begin by considering the case when $b_i < b_{i+1}$ for some $i$. Then it is not possible for there to be any addable $i + 1$-nodes in $\lambda$, and so all removable $i + 1$-nodes are normal. The multipartition $\sigma$ obtained by removing all the removable $i + 1$-nodes satisfies $\sigma = \text{Pt}(B', M)$, where $B' = (b_0, \ldots, b_{i+1}, b_{i+1}, \ldots, b_{e-1})$ (observe that $i + 1$ swap places in the $\prec$ ordering as we pass from $\lambda$ to $\sigma$, so $M$ does not change) and $\lambda$ is Kleshchev if and only if $\sigma$ is Kleshchev. By repeatedly applying this argument, we may assume that

$$b_0 = \cdots = b_j > b_{j+1} = \cdots > b_{k-1} > b_k = \cdots = b_{e-1} = 0$$

for some $0 \leq j < k \leq e - 1$. Now there are two cases to consider.

Firstly, if $b_0 > 1$ then there are removable 0-nodes in $\lambda$ but no addable 0-nodes, and so we can consider the multipartition $\sigma$ obtained by removing all removable 0-nodes from $\lambda$ as above. Again, $\sigma$ will be Kleshchev if and only if $\lambda$ is, and since $0$ and $e - 1$ swap places in the $\prec$ ordering we pass from $\lambda$ to $\sigma$ we can write $\sigma = \text{Pt}(B', M)$. Now, if $k < e - 1$, so that $b_{k-2} = 0$, then $B' = (1, b_1, \ldots, b_{e-2}, b_0 - 1)$ and we can return to the previous case, but with fewer entries of the multicharge equal to 0, so we may repeat the argument until we obtain $k = e - 1$. If $k = e - 1$, then $b_{e-2} > 0$, and (since the definition of $B$ requires at least one entry to be zero) we have $B' = (0, b_1 - 1, \ldots, b_{e-2} - 1, b_0 - 2)$ and we can apply the inductive hypothesis to $\sigma$.

Secondly (and finally) we have the case in which $b_0 = 1$ and $j = k - 1$, so that $B = (1, \ldots, 1, 0, \ldots, 0)$, with the final 1 occurring in the $j$th position. Here we can reduce to the base case: this time, let $\sigma$ be obtained from $\lambda$ by moving every bead in $\lambda$ $j$ positions to the left. Then $\sigma = \text{Pt}(0, M) = \mu$, since the columns corresponding to the ones in $\lambda$ now correspond to the rightmost zeros in $\mu$, so $M$ remains unchanged. Now, since the only difference between $\mu$ and $\lambda$ is that the multicharge associated to $\sigma = \mu$ is obtained by subtracting $k \mod e$ from each entry in the multicharge associated to $\lambda$, it follows that $\lambda$ is Kleshchev if and only if $\mu$ is. □

Define the set $\text{Ind}(e)$ by

$$\text{Ind}(e) = \begin{cases} \mathbb{Z}, & e = 0, \\ \{1, 2, \ldots, e - 1\}, & e \geq 2. \end{cases}$$

Now let $L \in \mathcal{M}$. Suppose that we have $i \in \text{Ind}(e)$ and $k > 0$. Write $L \xrightarrow{i,k} M$ if $M \in \mathcal{M}$ can be formed by choosing distinct rows $r_1, r_2, \ldots, r_k$ of $L$ with $L(r_a, i - 1) = 1$ and $L(r_a, i) = 0$ for all $1 \leq a \leq k$ and setting

$$M(s,j) = \begin{cases} 0, & j = i - 1, s = r_a \text{ for some } 1 \leq a \leq k, \\ 1, & j = i, s = r_a \text{ for some } 1 \leq a \leq k, \\ (L(s,j)), \text{ otherwise}. \end{cases}$$

We can think of this as moving $k$ ones from column $i - 1$ to column $i$ (and $k$ zeros back). If $L \xrightarrow{i,k} M$, define

$$N_i(L, M) = \sum_{a=1}^{k} \#\{s \mid s > r_a \text{ and } M(s, i - 1) = 1 \text{ and } M(s, i) = 0\} - \#\{s \mid s > r_a \text{ and } L(s, i - 1) = 0 \text{ and } L(s, i) = 1\}.$$ 

Consider the $\mathbb{C}(\nu)$-vector space with basis the elements of $\mathcal{M}$. For $L \in \mathcal{M}$, $i \in \text{Ind}(e)$ and $k > 0$, define

$$G_i^{(k)} L = \sum_{L \xrightarrow{i,k} M} \nu^{N_i(L, M)} M.$$
and extend linearly to give an action on the whole space.

**Proposition 2.11.** Suppose that $\sigma = \text{Pt}(B, L)$. Let $i \in \text{Ind}(e)$ and $k > 0$ and suppose that

$$G_i^{(k)} L = \sum_{L : \overset{\leftarrow}{i} \mapsto M} v_i^{N_i(L, M)} M.$$ 

Then there exist $i_1, i_2, \ldots, i_{m'} \in \mathbb{Z}/\epsilon\mathbb{Z}$ and $k_1, k_2, \ldots, k_{m'} > 0$ such that

$$F_{k_1, \ldots, k_{m'}}^{(i_1)} \cdots F_{k_2, \ldots, k_{m'}}^{(i_2)} F_{k_1}^{(i_1)} \sigma = \sum_{L : \overset{\leftarrow}{i} \mapsto M} v_i^{N_i(L, M)} \text{Pt}(B, M).$$

**Proof.** If $e = 0$ then this follows immediately from the definitions, so suppose that $e \geq 2$. Let $\prec$ be the total order on $I = \{0, 1, \ldots, e - 1\}$ associated with the base tuple $B = (b_0, b_1, \ldots, b_{e-1})$ and $τ$ the corresponding permutation. Let $j_0 = \pi(i - 1)$ and let $j_1 = \pi(i)$ so that $j_0 < j_1$ and there does not exist $j' \in I$ such that $j_0 < j' < j_1$. To minimise notation, we will assume that $j_0 = 0$; for the more general case, either make some slight modifications to the argument below or note that the combinatorics do not change if we shift the multicharge so that this holds. Now observe that the base tuple $B$ satisfies the following properties:

- $b_{j_0} \leq b_{j_1}$.
- If $j_0 < j' < j_1$ then $b_{j'} < b_{j_0}$ or $b_{j'} > b_{j_1}$.
- If $j_0 < j' \leq e - 1$ then $b_{j'} < b_{j_0}$ or $b_{j'} \geq b_{j_1}$.

Let $\Delta = b_{j_1} - b_{j_0}$. Suppose that $l_0 < l_1 < \ldots < l_m$ are the elements of the set

$$\{ l \in I \mid l < j_1 \text{ and } b_l > b_{j_1} \} \cup \{ l \in I \mid l \geq j_1 \text{ and } b_l \geq b_{j_1} \} \cup \{ j_0 \},$$

so that $j_0 = l_0$ and $j_1 = l_y$ for some $y$. We claim that

$$F_l^{(k(\Delta+1))} \cdots F_{l_0+2}^{(k(\Delta+1))} F_{l_0+1}^{(k(\Delta+1))} \cdots F_{l_y+2}^{(k(\Delta+1))} F_{l_y+1}^{(k(\Delta+1))} (F_{l_y+2}^{(k(\Delta+1))} F_{l_y+1}^{(k(\Delta+1))}) (F_{l_y}^{(k(\Delta+1))} \cdots F_{l_{m-1}+2}^{(k(\Delta+1))} F_{l_{m-1}+1}^{(k(\Delta+1))}) \sigma = \sum_{L : \overset{\leftarrow}{i} \mapsto M} v_i^{N_i(L, M)} \text{Pt}(B, M).$$

Before continuing with this proof, the reader might like to examine Example 3 below.

We first investigate which abacus configurations can occur as a result of this induction sequence. Consider

$$(F_{l_y+1}^{(k(\Delta+1))} \cdots F_{l_{m-1}+2}^{(k(\Delta+1))} F_{l_{m-1}+1}^{(k(\Delta+1))})(F_{l_y}^{(k(\Delta+1))} \cdots F_{l_{m-1}+2}^{(k(\Delta+1))} F_{l_{m-1}+1}^{(k(\Delta+1))}) \sigma.$$ 

With the restrictions on the abacus configuration for $\sigma$, the abacus configurations that occur in this sum are those that can be obtained by choosing $k\Delta$ beads each of which lies on runner $j_1$ of some abacus and moving each of them forward $e - j_1$ positions to an empty position on runner 0. Now let $τ$ be such a configuration, where $δ_σ$ such beads have been moved on abacus $s$, for $1 \leq s \leq r$, and consider

$$(F_{l_y+1}^{(k(\Delta+1))} \cdots F_{l_{m-1}+2}^{(k(\Delta+1))} F_{l_{m-1}+1}^{(k(\Delta+1))})(F_{l_y}^{(k(\Delta+1))} \cdots F_{l_{m-1}+2}^{(k(\Delta+1))} F_{l_{m-1}+1}^{(k(\Delta+1))}) \tau.$$ 

Again, this must consist of all abacus configurations that can be obtained by choosing $k(\Delta + 1)$ beads each of which lies on runner 0 of some abacus and moving each of them forward $j_1$ positions to an empty position on runner $j_1$. On abacus $s$, we can move at most $δ_σ + 1$ such beads if $L(s, i - 1) = 1$ and $L(s, i) = 0$ and $δ_σ = \Delta$; otherwise we can move at most $δ_σ$ beads. If it is possible to move $k(\Delta + 1) = k + \sum_{i=1}^s δ_i$ beads, that is if Equation 2.2 is not zero, there must be $k$ values of $1 \leq s \leq r$ satisfying $L(s, i - 1) = 1$ and $L(s, i) = 0$ and $δ_σ = \Delta$.

So suppose that $τ$ is such that we have $1 \leq r_1 < r_2 < \ldots < r_k \leq r$ with $L(r_{a_1}, i - 1) = 1$ and $L(r_{a_1}, i) = 0$ and $δ_{s_a} = \Delta$ for all $1 \leq a \leq k$. Then $τ$ is formed from $σ$ by moving the last $\Delta$ beads on runner $j_1$ of abacuses $r_1, r_2, \ldots, r_k$ along $e - j_1$ positions to runner 0. The only multicharge that then occurs in Equation 2.2 corresponds to the multipartition formed by then moving the last $\Delta + 1$ beads on runner 0 of abacus $r_a$ along $j_1$ positions to runner $j_1$ for each $1 \leq a \leq k$.

In short, the multicharges which occur on the right hand side of Equation 2.1 are exactly those formed by choosing $1 \leq r_1 < r_2 < \ldots < r_k \leq r$ with $L(r_{a_1}, i - 1) = 1$ and $L(r_{a_1}, i) = 0$ and moving a bead in each abacus configuration $σ^{(r_a)}$ from the last full position on runner 0 to the first empty position on runner $j_1$. Hence we have shown that Equation 2.1 is equal to $\sum_{L : \overset{\leftarrow}{i} \mapsto M} v_i^{N_i(L, M)} \text{Pt}(B, M)$ for some $α(L, M) \in \mathbb{Z}$ and so
it only remains to check that \( \alpha(L, M) = N_i(L, M) \) for each \( L \overset{i,k}{\rightarrow} M \). Choose such an \( M \), let \( \lambda = \text{Pt}(B, M) \) and suppose that \( \lambda \) was formed from \( \sigma \) by moving beads in components \( r_1, r_2, \ldots, r_k \). Over the course of the induction, the addable and removable nodes that contribute to \( s(L, M) \) almost all cancel. If \( s \neq r_{a'} \) for any \( a' \) and \( L(s, 0) = 1 \) and \( L(s, j_1) = 1 \) then each \( 1 \leq a \leq r \) with \( r_a < s \) contributes \(+1\) towards \( \alpha(L, M) \). If \( s \neq r_{a'} \) for any \( a' \) and \( L(s, 0) = 0 \) and \( L(s, j_1) = 0 \) then each \( 1 \leq a \leq r \) with \( r_a < s \) contributes \(-1\) towards \( \alpha(L, M) \). All other contributions match up to sum to zero. Hence \( \alpha(L, M) = N_i(L, M) \) as required. \( \square \)

**Example 3.** Let \( B = (1, 0, 3, 2, 2) \) and \( M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \) as in Example 2 and let \( \sigma = \text{Pt}(B, M) \). Let \( i = 2 \) and \( k = 1 \) so that \( j_0 = 0 \) and \( j_1 = 3 \). In terms of matrices, the induction is given by

\[
G_2^{(1)} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} + v \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix},
\]

Then \( F_2^{(2)} F_1^{(2)} F_3^{(2)} F_4^{(1)} F_0^{(1)} \sigma \) is given by

\[
\begin{array}{c}
+ v \\
\end{array}
\]

**Corollary 2.12.** Suppose that \( \sigma = \text{Pt}(B, L) \) and that \( \text{wt}(\sigma) = 0 \). Suppose that there exist \( i_1, i_2, \ldots, i_m \in \text{Ind}(c) \) and \( k_1, k_2, \ldots, k_m \) with \( k_l > 0 \) for all \( l \) such that

\[
G_{i_m}^{(k_m)} \cdots G_{i_1}^{(k_1)} L = K + \sum_{M \in \mathfrak{M} \setminus \{K\}} a_M(v) M
\]

where \( a_M(v) \in v \mathbb{N} \). Then \( \lambda = \text{Pt}(B, K) \) is a Kleshchev multipartition and if \( \mu \neq \lambda \) then

\[
d_{\lambda, \mu}(v) = \begin{cases} a_M(v), & \mu = \text{Pt}(B, M) \text{ for some } M \in \mathfrak{M}, \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** This follows from Proposition 2.11 and Proposition 2.11. \( \square \)

Above, we introduced the sets \( \mathfrak{M} \) and \( \bar{\mathfrak{M}} \). By Lemma 2.7 and Lemma 2.9, the core blocks may be described using only the matrices in \( \mathfrak{M} \). However, in order to state and apply Proposition 2.11, we need to use the matrices in \( \bar{\mathfrak{M}} \). We now show that when inducing, as in Corollary 2.12 or otherwise, we can ignore columns of \( L \) in which all entries are the same. This is intuitively obvious when the column consists of zeros, and not difficult to show when there is a column consisting of ones.

**Lemma 2.13.** Suppose that \( L \in \mathfrak{M} \). Let \( i_1, i_2, \ldots, i_m \in \text{Ind}(c) \) and \( k_1, k_2, \ldots, k_m \) with \( k_l > 0 \) for all \( l \) and suppose

\[
G_{i_m}^{(k_m)} \cdots G_{i_1}^{(k_1)} L = \sum_{M \in \mathfrak{M}} a_M(v) M
\]

where \( a_M(v) \in \mathbb{N}[v^{-1}, v] \). Let \( \mathfrak{M} \) be the set of matrices obtained by adding a column consisting entirely of zeros or entirely of ones to the same position in every matrix in \( \mathfrak{M} \), with the bijection \( K \leftrightarrow \bar{K} \) for \( K \in \mathfrak{M} \).
If \( e \geq 2 \) (resp. \( e = 0 \)) there exist \( \tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_m \in \text{Ind}(e + 1) \) (resp. \( \text{Ind}(e) \)) and \( \bar{k}_1, \bar{k}_2, \ldots, \bar{k}_m \) with \( \bar{k}_l > 0 \) for all \( l \) such that

\[
G_{i_m}^{(k_m)} \ldots G_{i_2}^{(k_2)} G_{i_1}^{(k_1)} L = \sum_{M \in \mathfrak{M}} a_M(v) M.
\]

Proof. Let \( 1 \leq l \leq m \). If the column added is before column \( i_l - 1 \), replace \( G_{i_l}^{(k_l)} \) with \( G_{i_l+1}^{(k_l)} \). If the column added is between columns \( i_l - 1 \) and \( i_l \) then replace \( G_{i_l}^{(k_l)} \) with \( G_{i_l+1}^{(k_l)} G_{i_l}^{(k_l)} \) if we add a column of zeros and with \( G_{i_l}^{(k_l)} G_{i_l+1}^{(k_l)} \) if we add a column of ones. \( \square \)

If we can find decomposition numbers using Corollary 2.12, then they are independent of the characteristic of the field. In the next sections, we demonstrate some core blocks for which this is the case. It is not true that decomposition numbers for core blocks are always independent of the field. For example, if \( e = 0 \) then all blocks are core blocks. However, if \( e = 0 \), \( r = 8 \), \( a = (4,4,3,3,2,2,1,1) \) and \( \lambda = ((2),(2),(1),(1),(3),(3),(2),(2)) \) then \( \dim(B^\lambda) \) is smaller in characteristic 2 than in characteristic 0 \[37\]. (We would like to thank Andrew Mathas for his translation of Williamson’s example into the notation used in this paper.)

3. Indecomposable weight blocks of weight two

In this section, we assume that \( r = 3 \) and consider the indecomposable core blocks of weight 2. We begin by describing these blocks, and then use Corollary 2.12 to compute the decomposition matrices. If \( \lambda = \text{Pt}(B,M) \) recall the relation between \( \text{wt}(M) \) and \( \text{wt}(\lambda) \) described in Lemma 2.8.

**Proposition 3.1.** Suppose that \( e \geq 2 \). Let \( \lambda = \text{Pt}(B,M) \) be such that \( S^\lambda \) belongs to an indecomposable core block of weight 2, where we may assume that \( M \in \mathfrak{M} \). Then \( M \) has one of the following forms, for some \( x, y, z \geq 0 \) satisfying \( x + y + z + 2 = e \):

\[
\gamma^1 = \begin{array}{ccc}
x & y & z \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \alpha^1 = \begin{array}{ccc}
x & y & z \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \beta^1 = \begin{array}{ccc}
x & y & z \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{array} \quad ,
\]

\[
\gamma^2 = \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
\end{array} \quad , \quad \beta^2 = \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
\end{array} \quad , \quad \alpha^2 = \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
\end{array} \quad ,
\]

\[
\gamma^3 = \begin{array}{ccc}
x & y & z \\
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \alpha^3 = \begin{array}{ccc}
x & y & z \\
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \beta^3 = \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
\end{array} \quad ,
\]

\[
\gamma^4 = \begin{array}{ccc}
x & y & z \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \beta^4 = \begin{array}{ccc}
x & y & z \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \alpha^4 = \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
\end{array} \quad ,
\]

\[
\gamma^5 = \begin{array}{ccc}
x & y & z \\
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \alpha^5 = \begin{array}{ccc}
x & y & z \\
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \beta^5 = \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{array} \quad ,
\]

\[
\gamma^6 = \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \beta^6 = \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \quad , \quad \alpha^6 = \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \quad .
\]
The notation is taken to mean that \( M \) is a \( 3 \times (x + y + z + 2) \)-matrix, with \( x \) columns equal to the entry below \( x \) and so on, and columns appearing in any order. The subscripts on \( 1, \alpha \) and \( 2 \) should be ignored for the moment. Note that the matrices indexed by high superscripts are obtained from matrices directly above by rearranging the rows.

**Proof.** Suppose that \( \lambda = (\lambda(1), \lambda(2), \lambda(3)) \) satisfies \( \text{wt}(\lambda(1), \lambda(2)) = \text{wt}(\lambda(2), \lambda(3)) = 1 \), whence \( \text{wt}(\lambda(1), \lambda(3)) = 0 \). The latter requirement means that there cannot simultaneously be columns of the form \((1 * 0)^T\) and \((0 * 1)^T\), which taken in conjunction with the former requirement means that there must be \( r \) columns of the form \((1 0 1)^T\) and \( s \) columns of the form \((0 1 0)^T\), with \( r, s \geq 1 \) and either \( r = 1 \) or \( s = 1 \). If \( r = s = 1 \) then the condition above shows that we end up with either \( \gamma^1 \) or \( \gamma^6 \). If \( r > 1 \) then we cannot have any columns of the form \((0 1 1)^T\) or \((1 1 0)^T\) and we cannot have both \((1 0 0)^T\) and \((0 0 1)^T\) occurring, so we end up with either \( \alpha^2 \) or \( \alpha^4 \). Similarly if \( s > 1 \) we end up with either \( \alpha^3 \) or \( \alpha^5 \).

All other cases now follow by permuting the rows of the matrices determined above. \( \square \)

It is straightforward to give an analogue of Proposition 3.1 for \( e = 0 \): For \( \delta \in \{\alpha, \beta, \gamma\} \) and \( 1 \leq u \leq 6 \), replace matrices of the form \( \delta^u \) with matrices whose rows are indexed by the elements of \( Z \) in the obvious way. For example, replace \( \gamma^1 \) with a matrix which has \( z \geq 0 \) columns equal to \((0 0 1)^T\), \( y \geq 0 \) columns equal to \((0 1 1)^T\) and one column equal to \((0 1 0)^T\) and to \((1 0 1)^T\); and where all other columns are equal to \((0 0 0)^T\) or \((1 1 1)^T\) with all columns of sufficiently high index containing the former and all columns of sufficiently low index containing the latter.

Fix once and for all a base tuple \( B \in \mathcal{B} \). From now on, we do not differentiate between a matrix \( M \in \mathfrak{M} \) and the partition \( \text{Pt}(B, M) \). We will, for example, refer to a matrix \( M \) as being Kleshchev whenever \( \text{Pt}(B, M) \) is Kleshchev; by Proposition 2.10 this is independent of the choice of \( B \).

Suppose we have integers \( y, z \geq 0 \) and a partition

\[ I = X \sqcup X' \sqcup Y \sqcup Z \]

such that
- \( |Y| = y + 1 \) and \( |Z| = z + 1 \).
- If \( e \geq 2 \) then \( X' = \emptyset \).
- If \( e = 0 \) then \( i \notin X' \) for all \( i \gg 0 \) and \( i \in X' \) for all \( i \ll 0 \).

Let \( Y = \{i_1, i_2, \ldots, i_{y+1}\} \) and \( Z = \{j_1, j_2, \ldots, j_{z+1}\} \) where we assume that \( i_1 < i_2 \ldots < i_{y+1} \) and \( j_1 < j_2 < \ldots < j_{z+1} \). We now use the notation of Proposition 3.1. Let \( 1 \leq u \leq 6 \) and \( \delta \in \{\alpha, \beta, \gamma\} \). If \( e \in \{x, y, z\} \), let \( \epsilon(\delta^u) \) be the column below \( e \) in \( \gamma^u \). If \( e \in \{1, 1_y, 1_z\} \), set \( \epsilon(\delta^u) \) be the column below \( e \) in \( \delta^u \). Set \( x(\delta^u) = (1 1 1)^T \). We define the following elements of \( \mathfrak{M} \).

1. For \( 1 \leq k \leq y + 1 \), let \( \alpha_k^u = \alpha_k^u(X, X', Y, Z) \) be the matrix where for \( c \in I \),

\[
\begin{align*}
\text{column } c \text{ is equal to} & \\
1(\alpha^u), & c = i_k, \\
y(\alpha^u), & c \in Y \text{ and } c \neq i_k, \\
z(\alpha^u), & c \in Z, \\
x(\alpha^u), & c \in X, \\
x'(\alpha^u), & c \in X'.
\end{align*}
\]

2. For \( 1 \leq l \leq z + 1 \), let \( \beta_l^u = \beta_l^u(X, X', Y, Z) \) be the matrix where for \( c \in I \),

\[
\begin{align*}
\text{column } c \text{ is equal to} & \\
y(\beta^u), & c \in Y, \\
1(\beta^u), & c = j_l, \\
z(\beta^u), & c \in Z \text{ and } c \neq j_l, \\
x(\beta^u), & c \in X, \\
x'(\beta^u), & c \in X'.
\end{align*}
\]
(3) For $1 \leq k \leq y + 1$ and $1 \leq l \leq z + 1$, let $\gamma^n_{kl} = \gamma^n_{kl}(X, X', Y, Z)$ be the matrix where for $c \in I$,

$$
column c is equal to

$$
\begin{cases}
1_y(\gamma^n), & c = i_k, \\
y(\gamma^n), & c \in Y and c \neq i_k, \\
1_z(\gamma^n), & c = j_l, \\
z(\gamma^n), & c \in Z and c \neq j_l, \\
x(\gamma^n), & c \in X, \\
x'(\gamma^n), & c \in X'.
\end{cases}
$$

Proposition 3.2. Suppose that we have an indecomposable core block of weight 2 with $r = 3$. Then the Specht modules are indexed by the matrices

$$
\{a^n_k | 1 \leq k \leq y + 1\} \cup \{\beta^n_l | 1 \leq l \leq z + 1\} \cup \{\gamma^n_{kl} | 1 \leq k \leq y + 1, 1 \leq l \leq z + 1\}
$$

for some fixed $1 \leq u \leq 6$ and partition $I = X \cup X' \cup Y \cup Z$ as above.

Proof. By Lemma 2.9, this is equivalent to showing that the equivalence classes generated by bead swaps matrices $M$ is identical to

Similarly if $1 \leq l \leq z + 1$ then performing a single bead swap on the matrix $\beta^n_l$ gives us the matrices

Finally if $1 \leq k \leq y + 1$ and $1 \leq l \leq z + 1$ then performing a single bead swap on the matrix $\gamma^n_{kl}$ gives us the matrices

$$
\{\gamma^n_{kl} | 1 \leq k' \leq y + 1\} \cup \{\gamma^n_{kl'} | 1 \leq l' \leq z + 1\} \cup \{a^n_k, \beta^n_l \}
$$

Henceforth we fix a partition $I = X \cup X' \cup Y \cup Z$ satisfying the conditions overleaf. Let $Y = \{i_1, i_2, \ldots, i_{y+1}\}$ and $Z = \{j_1, j_2, \ldots, j_{z+1}\}$ where $i_1 < i_2 < \ldots < i_{y+1}$ and $j_1 < j_2 < \ldots j_{z+1}$. For $1 \leq u \leq 6$ define

$$
D^u = \{a^n_k | 1 \leq k \leq y + 1\} \cup \{\beta^n_l | 1 \leq l \leq z + 1\} \cup \{\gamma^n_{kl} | 1 \leq k \leq y + 1, 1 \leq l \leq z + 1\}
$$

to be the set of matrices indexing the Specht modules lying in the corresponding block.

Proposition 3.3. Suppose $M \in D^u$. Then $M$ is Kleshchev if and only if it is one of the following:

$$
\begin{align*}
\{\gamma^1_{kl} | i_k < j_l and k \neq z + 1\}, & \quad \{\gamma^1_{kl} | j_l < i_k and k \neq y + 1\}, & \quad \{\alpha^n_k | i_k < j_{z+1}\}, & \quad \{\beta^n_l | j_l < i_{y+1}\}, \\
\{\gamma^n_{kl} | j_l < i_k or (k \neq 1 and l \neq z + 1)\}, & \quad \{\alpha^n_k | k \neq 1 and i_k < j_{z+1}\}, & \quad \{\beta^n_l | i_1 < j_l\}, & \quad \text{when } u = 2, 3, \\
\{\gamma^n_{kl} | i_k < j_l or (l \neq 1 and k \neq y + 1)\}, & \quad \{\alpha^n_k | k \neq y + 1 and j_1 < j_k\}, & \quad \{\beta^n_l | j_l < i_{y+1}\}, & \quad \text{when } u = 4, 5, \\
\{\gamma^6_{kl} | i_k < j_l and k \neq 1\}, & \quad \{\gamma^6_{kl} | j_l < i_k and l \neq 1\}, & \quad \{\alpha^6_k | j_1 < i_k\}, & \quad \{\beta^6_l | i_1 < j_l\}.
\end{align*}
$$

Proof. In view of Proposition 2.9, it is sufficient to consider the case in which $B = 0$, so that the relation $<$ is identical to $<$, and there is a maximum of one bead on each runner. This means that if $c \geq 2$, there are no removable 0-nodes in $\lambda = Pt(0, M)$, since it is a multicore. Any removable i-node of $\lambda$ corresponds to an entry 1 in column $i$ of $M$ that has an entry 0 immediately to its left (call this an “01”), and any addable i-node corresponds to a 0 in column $i$ that has a 1 immediately to its left (likewise, a “10”). We can read off normality from $M$: the 1 in an “01” corresponds to an abnormal node if and only if there is no “01” below it but at least one “10”, or there is an “01” below it with a “10” in between them. At any given point and for any $i$, the good i-node is simply the normal i-node corresponding to a 1 in the highest possible row. Removing a node simply corresponds to moving a 1 one space to the left, and $\lambda$ is Kleshchev if and only if we can move all the 1s all the way to the left without ever removing any abnormal nodes.
We will describe the $\alpha_k^1$, $\beta_l^1$, and $\gamma_{kl}^1$ cases in detail; the others vary only in the role played by the different rows in the matrix.

Consider the matrix $\alpha_k^1$: the condition in the proposition amounts to the requirement that there be at least one $z(\alpha^1)$ column to the right of the $1(\alpha^1)$ column. The only potentially abnormal nodes correspond to the 1s in the $1(\alpha^1)$ column, with the abnormality occurring at the stage at which we attempt to move them past either a $y(\alpha^1)$ or a $z(\alpha^1)$ column to its left—and since there must be at least one $z(\alpha^1)$ column somewhere in $M$, this will definitely happen if the required condition is not met. To get rid of this abnormality, we need to fill the $1(\alpha^1)$ column with 1s, which is to say we need to be able to move a 1 on the bottom row to the left, into the $1(\alpha^1)$ column. If we can do this, we can then move all the 1s all the way to the left, by moving leftmost columns first and working from the top row to the bottom.

For the matrix $\beta_l^1$, initially, the only potentially abnormal node corresponds to the 1 entry in the $1(\beta^1)$ column. The condition in the proposition is equivalent to the assertion that there is a $y(\beta^1)$ column somewhere to the right of the $1(\beta^1)$ column. If this is the case, then we can fill the $1(\alpha^1)$ column with 1s by removing normal nodes, and from that configuration all the 1s can be moved all the way to the left, as in the previous paragraph. On the other hand, if $y_{l+1} < j_l$, so there is no $y(\beta^1)$ column to the right of the $1(\beta^1)$ column but at least one to its left, then it is impossible to move the 1 in the top row all the way to the left—there is always a “10” in the middle row to block it.

For the matrix $\gamma_{kl}^1$, the condition in the proposition says that if the $1_z(\gamma^1)$ column is to the right of the $1_y(\gamma^1)$ column then there must be at least one $z(\gamma^1)$ column further to its right, and if the $1_y(\gamma^1)$ column is to the right of the $1_z(\gamma^1)$ column then there must be a $y(\gamma^1)$ column further to its right. Either way, the $1_y(\gamma^1)$ and $1_z(\gamma^1)$ columns contain the 1s corresponding to the only potentially abnormal nodes, and $\lambda$ will once again be Kleshchev if we can manage to fill whichever of these columns is leftmost with 1s. If the $1_y(\gamma^1)$ column occurs to the right of the $1_z(\gamma^1)$ column then at some stage we need to move an abnormal 1 in the middle row; provided that there is a 1 in the bottom row somewhere to its right, we can move it until it is below the abnormal 1, at which point the previously addable node that was causing the abnormality will no longer be addable. (That is, the “10” in the bottom row will have become a “11.”) On the other hand, if the $1_z(\gamma^1)$ column occurs to the right of the $1_y(\gamma^1)$ column then at some stage we first to move an abnormal 1 in the middle row—provided there is a $z(\gamma^1)$ somewhere to its right, we can move both its 1s over until the abnormal 1 is rendered normal, then we need to move the 1 in the bottom row, which cannot be abnormal (but will only be good once the 1 in the top row is dealt with). Either way, once the leftmost of the $1_y(\gamma^1)$ and $1_z(\gamma^1)$ columns has been filled with 1s, we can proceed as in the $\alpha_k^1$ and $\beta_l^1$ cases.

\[ \square \]

Remark. Combining Proposition 3.2 and Proposition 3.3, a simple counting argument shows that there are $(y + 2)(z + 2) - 1$ Specht modules and $(y + 1)(z + 1)$ simple modules in each block.

We are now ready to compute the decomposition matrices. Recall that if $\lambda$ is a Kleshchev multipartition, we defined

\[ P(\lambda) = \sum_{\mu \in \mathcal{P}^n} d_{\mu\lambda}(v) \mu. \]

If $e \geq 2$, set $i_0 = j_0 = -1$ and $i_{y+2} = j_{z+2} = e$. If $e = 0$, choose $i_0, j_0 \ll 0$ and $i_{y+2}, j_{z+2} \gg 0$. We make these definitions so that the indices $k, l$ below always exist.

**Theorem 3.4.** We now find $P(\delta)$ for all Kleshchev multipartitions $\delta \in \mathcal{D}^n$, for each $1 \leq u \leq 6$.

1. Suppose that $\alpha_k^1$ is Kleshchev. Let $\tilde{k}$ be minimal such that $i_k < j_k$. Then

\[ P(\alpha_k^1) = \begin{cases} \alpha_k^1 + v_{1\tilde{k}} \alpha_{k+1}^1 + v^2 \beta_{\tilde{k}}^1, & j_{\tilde{k}} < i_{k+1}, \\ \alpha_k^1 + v \alpha_{k+1}^1 + v^2 \gamma_{k,k}^1 + v^2 \gamma_{k+1,k}^1, & i_{k+1} < \tilde{j}_k. \end{cases} \]

2. Suppose that $\beta_l^1$ is Kleshchev. Let $\tilde{l}$ be minimal such that $j_l < i_l$. Then

\[ P(\beta_l^1) = \begin{cases} \beta_l^1 + v \gamma_{1l}^1 + v^2 \alpha_l^1, & \tilde{i}_l < j_{l+1}, \\ \beta_l^1 + v \beta_{l+1}^1 + v \gamma_{1l}^1 + v^2 \gamma_{1l+1}^1, & j_{l+1} < \tilde{i}_l. \end{cases} \]
Suppose that $\gamma^u_{k,l}$ is Kleshchev. Then

$$P(\gamma^u_{k,l}) = \begin{cases} 
\gamma^1_{k,l} + v\gamma_{k,l+1}^1 + v^2\gamma_{k+1,l+1}^1, & i_{k+1} < j_i \text{ or } j_i+1 < i_k, \\
\gamma^2_{k,l} + v\gamma_{k,l+1}^2 + v\gamma_{k+1,l+1}^2 + v^2\gamma_{k+1,l+1}^2, & i_k < j_i < i_{k+1} < j_{i+1}, \\
\gamma^3_{k,l} + v\gamma_{k+1,l+1}^2 + v\gamma_{k+1,l+1}^2 + v^2\gamma_{k+1,l+1}^2, & i_k < j_{i+1} < i_{k+1}, \\
\gamma^4_{k,l} + v\gamma_{k+1,l+1}^2 + v\gamma_{k+1,l+1}^2 + v^2\gamma_{k+1,l+1}^2, & j_i < i_k < j_{i+1} < i_{k+1}.
\end{cases}$$

(2) Suppose that $u = 2$ or $u = 3$.

- Suppose that $\alpha^u_k$ is Kleshchev. Let $\tilde{k}$ be minimal such that $i_k < j_{\tilde{k}}$. Then

$$P(\alpha^u_k) = \begin{cases} 
\alpha^u_k + v\beta^u_{k-1} + v\gamma_{k,k+1}^u + v^2\gamma_{k,k+1}^u, & i_{k-1} < j_{\tilde{k}} - 1, \\
\alpha^u_k + v\gamma_{k,k+1}^u + v^2\gamma_{k,k+1}^u, & j_{\tilde{k}} - 1 < i_{k-1}.
\end{cases}$$

- Suppose that $\beta^u_i$ is Kleshchev. Let $\tilde{i}$ be maximal such that $i_{\tilde{i}} < j_i$. Then

$$P(\beta^u_i) = \begin{cases} 
\beta^u_i + v\alpha^u_{i-1} + v^2\gamma_{i-1,i}^u, & j_{i-1} < i_{\tilde{i}}, \\
\beta^u_i + v\gamma_{i-1,i}^u + v^2\gamma_{i-1,i}^u, & i_{\tilde{i}} < j_{i-1}.
\end{cases}$$

- Suppose that $\gamma^u_{k,l}$ is Kleshchev. Then

$$P(\gamma^u_{k,l}) = \begin{cases} 
\gamma^u_{k,l} + v\gamma_{k,l+1}^u + v^2\gamma_{k+1,l+1}^u, & j_{\tilde{k}} + 1 < i_{k+1}, \\
\gamma^u_{k,l} + v\gamma_{k,l+1}^u + v\gamma_{k+1,l+1}^u + v^2\gamma_{k+1,l+1}^u, & i_k < j_i < i_{k+1} < j_{i+1}, \\
\gamma^u_{k,l} + v\gamma_{k+1,l+1}^u + v\gamma_{k+1,l+1}^u + v^2\gamma_{k+1,l+1}^u, & i_k < j_{i+1} < i_{k+1}, \\
\gamma^u_{k,l} + v\gamma_{k+1,l+1}^u + v\gamma_{k+1,l+1}^u + v^2\gamma_{k+1,l+1}^u, & j_i < i_k < j_{i+1} < i_{k+1}.
\end{cases}$$

(3) Suppose that $u = 4$ or $u = 5$.

- Suppose that $\alpha^u_k$ is Kleshchev; note that this implies that $k \neq y + 1$. Let $\tilde{k}$ be maximal such that $j_{\tilde{k}} < i_k$. Then

$$P(\alpha^u_k) = \begin{cases} 
\alpha^u_k + v\beta^u_{k-1} + v\gamma_{k,k+1}^u + v^2\gamma_{k,k+1}^u, & j_{\tilde{k}} < i_{k+1}, \\
\alpha^u_k + v\alpha^u_{k+1} + v\gamma_{k,k+1}^u + v^2\gamma_{k,k+1}^u, & i_{k+1} < j_{\tilde{k}} + 1.
\end{cases}$$

- Suppose that $\beta^u_i$ is Kleshchev. Let $\tilde{i}$ be minimal such that $j_{\tilde{i}} < i_{\tilde{i}}$. Then

$$P(\beta^u_i) = \begin{cases} 
\beta^u_i + v\alpha^u_{i-1} + v^2\gamma_{i-1,i}^u, & i_{\tilde{i}} < j_{i+1}, \\
\beta^u_i + v\beta^u_{i-1} + v\gamma_{i-1,i}^u + v^2\gamma_{i-1,i}^u, & j_{i+1} < i_{\tilde{i}}.
\end{cases}$$

- Suppose that $\gamma^u_{k,l}$ is Kleshchev.

$$P(\gamma^u_{k,l}) = \begin{cases} 
\gamma^u_{k,l} + v\gamma_{k,l+1}^u + v\gamma_{k+1,l+1}^u + v^2\gamma_{k+1,l+1}^u, & i_{k+1} < j_{i-1} \text{ or } j_i < i_k, \\
\gamma^u_{k,l} + v\gamma_{k+1,l+1}^u + v\gamma_{k+1,l+1}^u + v^2\gamma_{k+1,l+1}^u, & i_k < j_{i-1} < i_{k+1} < j_i, \\
\gamma^u_{k,l} + v\gamma_{k+1,l+1}^u + v\gamma_{k+1,l+1}^u + v^2\gamma_{k+1,l+1}^u, & i_k < j_i < i_{k+1} < j_{i-1}, \\
\gamma^u_{k,l} + v\gamma_{k+1,l+1}^u + v\gamma_{k+1,l+1}^u + v^2\gamma_{k+1,l+1}^u, & j_i < i_k < j_{i-1} < i_{k+1},
\end{cases}$$

(4) Suppose that $\alpha^6_k$ is Kleshchev. Let $\tilde{k}$ be maximal such that $j_{\tilde{k}} < i_k$. Then

$$P(\alpha^6_k) = \begin{cases} 
\alpha^6_k + v\alpha^6_k + v^2\beta^6_k, & i_{\tilde{k}} < j_{\tilde{k}}, \\
\alpha^6_k + v\alpha^6_k + v\alpha^6_k + v^2\gamma_{k,k+1}^6, & j_{\tilde{k}} < i_k.
\end{cases}$$
• Suppose that $\beta^6_l$ is Kleshchev. Let $\bar{l}$ be maximal such that $i_{\bar{l}} < j_l$. Then

$$P(\beta^6_l) = \begin{cases} 
\beta^6_l + v_0 \gamma^6_{l,1} + v^2 \alpha^6_l, & j_l - 1 < i_{\bar{l}}, \\
\beta^6_l + v_0 \beta^6_{l-1} + v_0 \gamma^6_{l,1} & i_{\bar{l}} < j_l - 1.
\end{cases}$$

• Suppose that $\gamma^6_{k,l}$ is Kleshchev. Then

$$P(\gamma^6_{k,l}) = \begin{cases} 
\gamma^6_{k,l} + v_0 \gamma^6_{k,l} - 1 + v_0 \gamma^6_{k-1,l} + v^2 \gamma^6_{k-1,l-1}, & i_k < j_l - 1 \ or \ j_l < i_{k-1}, \\
\gamma^6_{k,l} + v_0 \gamma^6_{k,l} - 1 + v_0 \gamma^6_{k-1,l} + v^2 \gamma^6_{k-1,l-1}, & i_k < j_l - 1 < i_k < j_l, \\
\gamma^6_{k,l} + v_0 \gamma^6_{k,l} - 1 + v_0 \gamma^6_{k-1,l} + v_0 \gamma^6_{k-1,l-1}, & i_k < j_l - 1 < i_k < j_l, \\
\gamma^6_{k,l} + v_0 \gamma^6_{k,l} - 1 + v_0 \gamma^6_{k-1,l} + v_0 \gamma^6_{k-1,l-1}, & j_l - 1 < i_k - 1 < j_l < i_k.
\end{cases}$$

Before proving this theorem, we give an example. Recall that when we change the values of $u$, we essentially permute the rows of the matrix and hence the order of the terms in the multicharge and the components in the multipartitions. From the example below, we can see that this leads to a non-trivial rearrangement of the decomposition matrices.

**Example 4.** Suppose that $e = 4$. Take the base tuple to be $B = 0$. Below, for $u = 1, 3, 5$ we have taken $Y = \{1, 3\}$ and $Z = \{0, 2\}$ and for $u = 2, 4, 6$ we have taken $Y = \{0, 2\}$ and $Z = \{1, 3\}$. Note that the multicharge is determined by the partitions. The block decomposition matrices are then as follows, where the dimensions of the simple modules are given in the top row and the dimensions of the Specht modules are given in the last column. For reasons of space, we have removed the outer brackets on each partition and used the symbol $|$ to indicate a new component.

\begin{align*}
\alpha = (0, 1, 2) & & \begin{array}{cccc}
40 & 20 & 60 & 40
\end{array} & \begin{array}{cccc}
\beta^1_1 & (\emptyset,2,1,1^3) & 1 & 40 \\
\gamma^1_1 & (1,2,1^3) & v & 1 & 60 \\
\alpha^1_1 & (1,2,1^2,1^2) & v^2 & v & 1 & 120 \\
\gamma^1_2 & (1,2^2,1^3) & v & 60 \\
\beta^2_1 & (2,2,1,1) & v^2 & v & 1 & 120 \\
\gamma^2_1 & (3,\emptyset,1^3) & v & 20 \\
\gamma^2_2 & (3,1^2,1) & v^2 & v & 60 \\
\alpha^2_1 & (3,2,1,\emptyset) & v^2 & 40
\end{array} \\
\alpha = (2, 1, 0) & & \begin{array}{cccc}
40 & 20 & 60 & 40
\end{array} & \begin{array}{cccc}
\beta^2_1 & (\emptyset,2,1,3) & 1 & 40 \\
\gamma^2_1 & (1,1^2,3) & v & 1 & 60 \\
\alpha^2_1 & (1,1^2,1,2) & v^2 & v & 1 & 120 \\
\gamma^2_2 & (1,2^2,1) & v & 60 \\
\beta^1_1 & (1,2^2,1^2,1^1) & v^2 & v & 1 & 120 \\
\gamma^1_2 & (1,3^3,1^3) & v & 20 \\
\gamma^1_1 & (1,3^2,1) & v^2 & v & 60 \\
\alpha^1_1 & (1,3,2,1,\emptyset) & v^2 & 40
\end{array} \\
\alpha = (2, 1, 0) & & \begin{array}{cccc}
40 & 20 & 60 & 40
\end{array} & \begin{array}{cccc}
\beta^5_1 & (\emptyset,1^3,2,1) & 1 & 40 \\
\gamma^1_1 & (1,1^2,1,2) & v & 1 & 60 \\
\alpha^1_1 & (1,1^2,1,2) & v^2 & v & 1 & 120 \\
\gamma^1_2 & (1,1^2,1,2) & v & 60 \\
\beta^1_1 & (1,1^2,1,2) & v^2 & v & 1 & 120 \\
\gamma^2_1 & (1,3^3,1^3) & v & 20 \\
\gamma^1_1 & (1,3^2,1) & v^2 & v & 60 \\
\alpha^1_1 & (1,3,2,1,\emptyset) & v^2 & 40
\end{array} \\
\alpha = (0, 2, 1) & & \begin{array}{cccc}
40 & 20 & 60 & 40
\end{array} & \begin{array}{cccc}
\beta^2_1 & (\emptyset,3,2,1) & 1 & 40 \\
\gamma^2_1 & (1,1^2,2,1) & v & 1 & 60 \\
\alpha^2_1 & (1,2^2,1,2) & v^2 & v & 1 & 120 \\
\gamma^2_2 & (1,3,1^2,2) & v & 60 \\
\beta^2_1 & (1,3,1^2,2) & v^2 & v & 1 & 120 \\
\gamma^1_2 & (1,3^3,1^3) & v & 20 \\
\gamma^1_1 & (1,3^2,1) & v^2 & v & 60 \\
\alpha^1_1 & (1,3,2,1,\emptyset) & v^2 & 40
\end{array}
\end{align*}
\[ \alpha = (1, 2, 0) \]

| \( \alpha^{(3)} \) | (0\{1\}1\}3 | 1 | 20 | \( \alpha^{(4)} \) | (0\{1\}1\}3 | 1 | 20 |
| --- | --- | --- | --- | --- | --- | --- | --- |
| \( \gamma^{(3)}_{21} \) | (1\{1\}1\}3 | 1 | 60 | \( \gamma^{(3)}_{22} \) | (1\{1\}1\}3 | v | 60 |
| \( \gamma^{(3)}_{22} \) | (2\{1\}1\}3 | v | 60 | \( \gamma^{(4)}_{22} \) | (2\{1\}1\}3 | v | 60 |
| \( \gamma^{(3)}_{11} \) | (2\{1\}1\}3 | v | 60 | \( \alpha^{(4)}_{1} \) | (2\{1\}1\}3 | v | 40 |
| \( \alpha^{(3)}_{2} \) | (2\{1\}1\}3 | v | 40 | \( \beta^{(4)}_{1} \) | (2\{1\}1\}3 | v | 120 |
| \( \beta^{(3)}_{2} \) | (2\{1\}1\}3 | v | 120 | \( \alpha^{(4)}_{1} \) | (2\{1\}1\}3 | v | 60 |
| \( \alpha^{(4)}_{2} \) | (2\{1\}1\}3 | v | 120 | \( \beta^{(4)}_{1} \) | (2\{1\}1\}3 | v | 120 |
| \( \gamma^{(3)}_{12} \) | (2\{1\}1\}3 | v | 40 | \( \beta^{(4)}_{1} \) | (2\{1\}1\}3 | v | 40 |
| \( \gamma^{(4)}_{13} \) | (2\{1\}1\}3 | v | 60 |

**Proof of Theorem 3.4**. The proof of the main theorem is given by a case-by-case analysis in which we apply Corollary 2.12. Rather than go through each one, we look in detail at some cases when \( j < i_k \). Suppose now that \( j \leq i_k \). Then \( j_k + 1 = i_k \) and part of the matrix for \( \gamma^{0}_{h} \) is given by

\[
\begin{bmatrix}
... & j_k & i_k & ... \\
1 & 0 & & \\
1 & 0 & & \\
0 & 1 & & \\
\end{bmatrix}
\]

Let

\[
L = \begin{bmatrix}
... & j_k & i_k & ... \\
1 & 0 & & \\
1 & 0 & & \\
1 & 0 & & \\
\end{bmatrix}
\]

with all other columns as in \( \alpha^{0}_{h} \). Then \( \text{wt}(L) = 0 \) and

\[
G_{i_k}^{(1)} L = \begin{bmatrix}
1 & 0 & & v & 0 & 0 \\
1 & 0 & & v & 0 & 0 \\
0 & 1 & & & & \\
\end{bmatrix} = \alpha^{0}_{h} + v \gamma^{0}_{i_k} + v^2 \beta^{0}_{i_k}.
\]

Suppose now that \( j_k < i_k \). Then part of the matrix for \( \gamma^{0}_{i_k} \) is given by

\[
\begin{bmatrix}
... & j_k & i_k-1 & i_k & ... \\
1 & 1 & ... & 0 & \\
1 & 0 & ... & 0 & \\
0 & 0 & ... & 0 & 1 \\
\end{bmatrix}
\]

Let

\[
L = \begin{bmatrix}
... & j_k & i_k-1 & i_k & ... \\
1 & 1 & ... & 0 & \\
1 & 0 & ... & 0 & \\
1 & 0 & ... & 0 & 0 \\
\end{bmatrix}
\]

with all other columns as in \( \alpha^{0}_{h} \). Then \( \text{wt}(L) = 0 \) and

\[
G_{i_k}^{(1)} G_{j_k+1}^{(1)} G_{j_k+2}^{(1)} L = \begin{bmatrix}
1 & 1 & ... & 1 & 0 & \\
1 & 0 & ... & 0 & 0 & \\
0 & 0 & ... & 0 & 1 & \\
\end{bmatrix} + v \begin{bmatrix}
1 & 1 & ... & 0 & 1 & \\
1 & 0 & ... & 0 & 0 & \\
0 & 0 & ... & 0 & 1 & \\
\end{bmatrix}
\]
The case where $β^6_i$ is Kleshchev is very similar, so we omit it here and finish by looking in detail at the Kleshchev multipartitions $γ^6_{kl}$ where $i_k < j_i$ and $k \neq 1$. First suppose that $j_{i-1} < i_k < j_i$. Then part of the matrix for $γ^6_{kl}$ is given by

\[
\begin{array}{cccccc}
... & i_{k-1} & i_k & ... & j_i & ... \\
1 & 0 & 1 & ... & 1 & 1 \\
0 & 1 & 0 & ... & 0 & 0 \\
0 & 0 & 0 & ... & 0 & 1 \\
\end{array}
\]

Let

\[
L = \begin{array}{cccccc}
... & i_{k-1} & i_k & ... & j_i & ... \\
1 & 0 & 1 & ... & 1 & 1 \\
1 & 0 & 0 & ... & 0 & 0 \\
1 & 0 & 0 & ... & 0 & 0 \\
\end{array}
\]

with all other columns as in $γ^6_{kl}$. Then $\text{wt}(L) = 0$ and

\[
G^{(1)}_{j_1} \cdots G^{(1)}_{i_{k+1}} G^{(2)}_{i_k} L = \begin{array}{cccccc}
... & i_{k-1} & i_k & ... & j_i & ... \\
1 & 0 & 1 & ... & 1 & 1 \\
0 & 1 & 0 & ... & 0 & 0 \\
0 & 0 & 0 & ... & 0 & 1 \\
\end{array} + v
\]

\[
= α^6_k + vα^6_{k-1} + vγ^6_{k-1} + v^2γ^6_{k-1} \cdot
\]

Now suppose that $i_{k-1} < j_{i-1} < i_k < j_i$. Then part of the matrix for $γ^6_{kl}$ is given by

\[
\begin{array}{cccccc}
... & i_{k-1} & i_k & ... & j_i & ... \\
1 & 1 & ... & 1 & 0 & 1 \\
0 & 1 & ... & 1 & 0 & 0 \\
0 & 0 & ... & 0 & 0 & 1 \\
\end{array}
\]

Let

\[
L = \begin{array}{cccccc}
... & i_{k-1} & i_k & ... & j_i & ... \\
1 & 1 & ... & 1 & 0 & 1 \\
1 & 1 & ... & 1 & 0 & 0 \\
0 & 0 & ... & 1 & 0 & 0 \\
\end{array}
\]

with all other columns as in $γ^6_{kl}$. Then $\text{wt}(L) = 0$ and

\[
G^{(1)}_{i_{k-1+1}} \cdots G^{(1)}_{j_{i-1-1}} G^{(1)}_{i_1} G^{(1)}_{j_1} \cdots G^{(1)}_{i_{k+1}} G^{(2)}_{i_k} L = \begin{array}{cccccc}
... & i_{k-1} & i_k & ... & j_i & ... \\
1 & 1 & ... & 1 & 0 & 1 \\
0 & 1 & ... & 1 & 0 & 0 \\
0 & 0 & ... & 0 & 0 & 1 \\
\end{array} + v
\]

\[
= γ^6_{kl} + vα^6_k + vγ^6_{k-1} + v^2α^6_{k-1}.
\]
Finally, suppose that $i_{k-1} < i_k < j_{t-1} < j_t$. Then part of the matrix for $\gamma_{k\lambda}^6$ is given by

$$
\begin{pmatrix}
... & i_{k-1} & ... & j_{t-1} & i_k & ... & j_t & ... \\
1 & 1 & ... & 0 & 1 & 1 & ... & 1 \\
0 & 1 & ... & 1 & 0 & 0 & ... & 0 \\
0 & 0 & ... & 1 & 0 & 0 & ... & 0 \\
0 & 0 & ... & 1 & 0 & 0 & ... & 0 \\
\end{pmatrix} + v^2
$$

$= \gamma_{k\lambda}^6 + v\alpha_{k}^6 + v\gamma_{k-1\lambda}^6 + v^2\gamma_{k-1\lambda}^6 + v^2\gamma_{k-1\lambda}^6.$

We now use Theorem 3.4 to prove Theorem 1. Recall that if $\mu$ and $\lambda$ belong to the same block then $\mu \sim \lambda$ if $\lambda$ is formed from $\mu$ by removing a single rim hook from component $k$ and attaching it to component $k + 1$, where $k \in \{1, 2, \ldots, r - 1\}$, and if the leg lengths of these two hooks are equal then we write $\mu \lessdot \lambda$.

**Lemma 3.5.** Suppose that $\lambda = \text{Pt}(B, M)$ and $\mu = \text{Pt}(B, L)$ are $3$-multipartitions in an indecomposable core block of weight 2.

1. $\mu \sim \lambda$ if and only if there exist $i, j \in I$ with $i < j$ and $k \in \{1, 2\}$ such that

   $$
   L(k, i) = L(k + 1, j) = M(k, j) = M(k + 1, i) = 0,
   $$

   $$
   L(k, j) = L(k + 1, i) = M(k, i) = M(k + 1, j) = 0,
   $$

   and $L(k', i') = M(k', i')$ for all possible $k' \neq k, k + 1$ and $i' \neq i, j$.

2. $\mu \lessdot \lambda$ if and only if $\mu \sim \lambda$ and, for the $i$ and $j$ specified in part (i), we have

   $$
   |\{x \mid i < x < j, L(k, x) = 1\}| = |\{x \mid i < x < j, L(k + 1, x) = 1\}|.
   $$

**Proof.** The condition in part (i) says that row $k + 1$ of $M$ is obtained from row $k + 1$ of $L$ by moving a $1$ to the right from column $i$ to column $j$, into a position occupied by $0$. In terms of abaci, this means removing a bead from the bottom of one runner of $\mu^{(k+1)}$ and adding a bead to the bottom of another runner. Although the ordering $\prec$ will determine precisely which runners of the abacus display are affected, the condition $i < j$ does require that the bead that is added will have a higher $\beta$-number. Consequently, one can regard $\lambda^{(k+1)}$ as being obtained from $\mu^{(k+1)}$ by moving a bead a number of spaces to the right (with the usual convention that moving a bead to the right from runner $c - 1$ takes it to runner $0$ of the next row down). This corresponds precisely to adding a rim hook to $\mu^{(k+1)}$: the number of boxes added is equal to the number of spaces the bead is moved (which will depend on $B$). Similarly, the condition in part (i) says that $\lambda^{(k)}$ is obtained from $\mu^{(k)}$ by removing a rim hook, so if the condition is satisfied then $\mu \sim \lambda$.

In general, if $\lambda$ is obtained from $\mu$ by removing a rim hook from $\mu^{(k)}$ and adding a rim hook of the same size to $\mu^{(k+1)}$, then the corresponding abaci could be related by moving beads between different runners in $\mu^{(k)}$ than $\mu^{(k+1)}$ (that is, by moving $1$s between different columns in row $k$ of $L$ than in row $k + 1$ or by moving beads that are not the lowest beads on their runners (in which case $\lambda$ would no longer be a multicore). However, since we are assuming that $\lambda$ and $\mu$ lie in the same block, the bead shift in component $k + 1$ must involve the same columns as the bead shift in component $k$ in order to satisfy Proposition 2.4. Moreover,
Theorem 3.6. Suppose that $\lambda$ and $\mu$ are 3-multipartitions in an indecomposable core block of weight 2 and that $\lambda$ is a Kleshchev multipartition. Then

$$d_{\mu\lambda}(v) = \begin{cases} 1, & \mu = \lambda, \\ \lambda \triangleright \mu \triangleright \lambda \text{ and } (\mu \sim \lambda \text{ or } \lambda \sim \mu), \\ \lambda = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

The result will follow from a case-by-case analysis of Theorem 3.4. First recall the following result.

Theorem 3.7 (18, Corollary 2.4, 6, Remark 3.19). Suppose that $\lambda$ and $\mu$ are multipartitions in the same block of weight $w$ and that $\lambda$ is a Kleshchev multipartition.

1. We have $d_{\lambda\lambda}(v) = 1$ and $d_{\lambda\lambda}(v) = v^w$.
2. We have $d_{\mu\lambda}(v) = 0$ unless $\lambda \triangleright \mu \triangleright \lambda$.

So suppose that $\lambda$ is a Kleshchev 3-multipartition in an indecomposable core block of weight 2. From Theorem 3.7 and Theorem 3.4 we can read off $\lambda$ and we know that $d_{\mu\lambda}(v) = 0$ unless $\lambda \triangleright \mu \triangleright \lambda$, where $d_{\lambda\lambda}(v) = 1$ and $d_{\lambda\lambda}(v) = v^w$. So to prove Theorem 3.6 we take a set containing all partitions $\tau$ such that $\lambda \triangleright \tau \triangleright \lambda$ and use Lemma 3.5 to show that the partitions $\mu$ in that set satisfy either $\mu \sim \lambda$ or $\lambda \sim \mu$ if and only if the coefficient of $\mu$ in $P(\lambda)$ is $v$. We remark that it turns out to be the case that if $\mu \sim \lambda$ and $\lambda \sim \mu$ then $\mu \sim \lambda$ if and only if $\lambda \sim \mu$.

Below, we give the details for the cases $u = 2$ and $u = 3$. The notation is that used in Theorem 3.4. We leave verification of the other four cases as an exercise to the reader.

**Proof of Theorem 3.7 when $u = 2$ and $u = 3$**. Suppose that $u = 2$ or $u = 3$. Recall that the partitions are given by:

$$\gamma^2 = \begin{array}{cccc} x & y & z & \lambda_y \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}, \quad \beta^2 = \begin{array}{cccc} x & y & z & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}, \quad \alpha^2 = \begin{array}{cccc} x & y & z & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}, \quad \beta^3 = \begin{array}{cccc} x & y & z & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}$$

The relations in the following two lemmas are easy to check.
Lemma 3.8. We have the following relations:

- \( \alpha^u_k \triangleright \alpha^u_{k+1} \) and \( \beta^u_k \triangleright \beta^u_{k+1} \).
- If \( i_k < j_l \) then \( \alpha^u_k \triangleright \beta^u_k \) and if \( j_l < i_k \) then \( \beta^u_k \triangleright \alpha^u_k \).
- \( \gamma^u_{k+1} \triangleright \gamma^u_{k,l} \triangleright \gamma^u_{k+1,l} \) if \( k < j_l \).
- If \( \gamma^u_{k,l} \triangleright \alpha^u_k \) then \( k \leq v \) and if \( \alpha^u_k \triangleright \gamma^u_{k,l} \) then \( v \leq k \).
- If \( \gamma^u_{k,l} \triangleright \alpha^u_k \) then \( i_k < j_l \) and if \( \alpha^u_k \triangleright \gamma^u_{k,l} \) then \( j_l < i_k \).
- If \( \gamma^u_{k,l} \triangleright \beta^u_w \) then \( i_k < j_w \) and if \( \beta^u_w \triangleright \gamma^u_{k,l} \) then \( j_w < i_k \).

Lemma 3.9. We have the following relations:

- If \( k < k' \) then \( \alpha^u_k \sim \alpha^u_{k'} \) and if \( k < k' \) then \( \beta^u_k \sim \beta^u_{k'} \).
- If \( i_k < j_l \) then \( \alpha^u_k \sim \beta^u_k \) and if \( j_l < i_k \) then \( \beta^u_k \sim \alpha^u_k \).
- We have \( \gamma^u_{k,l} \sim \alpha^u_k \) if and only if \( v = k \) and \( i_k < j_l \) and we have \( \alpha^u_k \sim \gamma^u_{k,l} \) if and only if \( v = k \) and \( j_l < i_k \).
- If \( l < l' \) then \( \gamma^u_{k,l} \sim \gamma^u_{k,l'} \) and if \( k < k' \) then \( \gamma^u_{k,l} \sim \gamma^u_{k,l'} \).
- We have \( \gamma^u_{k,l} \not\triangleright \beta^u_w \) and \( \beta^u_w \not\triangleright \gamma^u_{k,l} \) in all cases.

We now look at all the Kleshchev multipartitions in the block.

1. Suppose that \( \lambda = \alpha^u_i \) is Kleshchev. Let \( \bar{k} \) be minimal such that \( i_k < j_{\bar{k}} \).

- Suppose \( j_{\bar{k}} < j_k \). Then \( \bar{\lambda} = \gamma^u_{k-1,\bar{k}} \). From Lemma 3.8 the only partitions that can have the property that \( \bar{\lambda} \triangleright \mu \triangleright \lambda \) belong to the set
  \[ \{ \alpha_{k-1} \} \cup \{ \beta_l \mid i_k < j_l < i_k \} \cup \{ \gamma_{k-1,l} \mid l < \bar{k} \} \cup \{ \gamma_{k,l} \mid i_k < j_l \} \].

2. Suppose that \( \lambda = \alpha^u_k \) is Kleshchev. Let \( \bar{\lambda} \) be maximal such that \( i_k < j_l \).

- Suppose \( j_{\bar{k}} < i_k \). Then \( \bar{\lambda} = \gamma^u_{\bar{k},l} \). From Lemma 3.8 the only partition that can have the property that \( \bar{\lambda} \triangleright \mu \triangleright \lambda \) is \( \beta^u_l \).

\[ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\mu & \alpha^u_k & \beta^u_k & \gamma^u_{k-1,k} & \gamma^u_{k-1,k} & \gamma^u_{k-1,k} & \beta^u_w & \gamma^u_{k-1,k} & \gamma^u_{k-1,k} & \gamma^u_{k-1,k} & \gamma^u_{k-1,k} & \gamma^u_{k-1,k} & \gamma^u_{k-1,k} \\
\hline
\alpha_{k-1} & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\beta_{k-1} & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\gamma_{k-1,l} & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\gamma_{k,l} & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\gamma_{k,k} & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\hline
\end{array} \]

(2) Suppose that \( \lambda = \beta^u_l \) is Kleshchev. Let \( \bar{l} \) be maximal such that \( j_l < j_l \).

- Suppose \( j_l < j_{\bar{l}} \). Then \( \bar{\lambda} = \gamma^u_{l,\bar{l}} \). From Lemma 3.8 the only partition that can have the property that \( \bar{\lambda} \triangleright \mu \triangleright \lambda \) is \( \alpha^u_l \).

\[ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\mu & \alpha^u_l & \beta^u_l & \gamma^u_{l,\bar{l}} & \gamma^u_{l,\bar{l}} & \gamma^u_{l,\bar{l}} & \gamma^u_{l,\bar{l}} & \gamma^u_{l,\bar{l}} & \gamma^u_{l,\bar{l}} & \gamma^u_{l,\bar{l}} & \gamma^u_{l,\bar{l}} & \gamma^u_{l,\bar{l}} \\
\hline
\alpha^u_l & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\beta^u_l & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\gamma^u_{l,\bar{l}} & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\gamma^u_{l,\bar{l}} & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\gamma^u_{l,\bar{l}} & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\gamma^u_{l,\bar{l}} & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\hline
\end{array} \]
\( \alpha_k^u \)
\( \beta_w^u \)
\( \gamma_k^u, w \)

(3) Suppose that \( \lambda = \gamma_{k,l}^u \) is Kleshchev.

- Suppose that \( i_{k-1} < i_k < j_i < j_{i+1} \). Then \( \tilde{\lambda} = \gamma_{k-1,l+1}^u \). From Lemma 3.8, the only partitions that can have the property that \( \tilde{\lambda} \succ \mu \succ \lambda \) belong to the set

\[ \{ \alpha_k^u \} \cup \{ \beta_w^u \mid i_{k-1} < j_w < i_k \} \cup \{ \gamma_{k-1,l+1}^u,w \mid w < l+1 \} \cup \{ \gamma_{k,w} \mid l < w \}. \]

However, further investigation shows that if \( i_{k-1} < j_w < i_k \) then \( \beta_w \not\succ \gamma_{k,l} \).

- Suppose that \( j_i < j_{i+1} < i_{k-1} < i_k \). Then \( \tilde{\lambda} = \gamma_{k-1,l+1}^u \). From Lemma 3.8, the only partitions that can have the property that \( \tilde{\lambda} \succ \mu \succ \lambda \) belong to the set

\[ \{ \alpha_k^u \} \cup \{ \beta_w^u \mid i_{k-1} < j_w < i_k \} \cup \{ \gamma_{k-1,l+1}^u,w \mid w < l+1 \} \cup \{ \gamma_{k,w} \mid l < w \}. \]

However, further investigation shows that if \( i_{k-1} < j_w < i_k \) then \( \gamma_{k-1,l+1} \not\succ \beta_w \).

- Suppose that \( j_i < i_{k-1} < j_{i+1} < i_k \). Then \( \tilde{\lambda} = \alpha_{k-1} \). From Lemma 3.8, the only partitions that can have the property that \( \tilde{\lambda} \succ \mu \succ \lambda \) belong to the set

\[ \{ \alpha_k^u \} \cup \{ \beta_w^u \mid i_{k-1} < j_w < i_k \} \cup \{ \gamma_{k-1,l+1}^u,w \mid j_w < i_{k-1} \} \cup \{ \gamma_{k,w} \mid l < w \}. \]

- Suppose that \( j_i < i_{k-1} < j_{i+1} < i_k \). Then \( \tilde{\lambda} = \alpha_{k-1} \). From Lemma 3.8, the only partitions that can have the property that \( \tilde{\lambda} \succ \mu \succ \lambda \) belong to the set

\[ \{ \alpha_k^u \} \cup \{ \gamma_{k-1,l+1}^u,w \mid j_w < i_{k-1} \} \cup \{ \gamma_{k,w} \mid l < w \}. \]
Suppose that $i_{k-1} < j_l < j_{l+1} < i_k$. Then $\lambda = \beta_l$. From Lemma 3.8 the only partitions that can have the property that $\lambda \triangleright \mu \triangleright \lambda$ belong to the set

$\{\alpha^a_k\} \cup \{\beta^a_w \mid j_l < j_w < i_k\} \cup \{\gamma_{k,w} \mid l < w\}$.

Suppose that $i_{k-1} < j_l < i_k < j_{l+1}$. Then $\lambda = \beta_l$. From Lemma 3.8 the only partitions that can have the property that $\lambda \triangleright \mu \triangleright \lambda$ belong to the set

$\{\alpha^a_k\} \cup \{\gamma_{k,w} \mid l < w\}$.

This concludes the proof of Theorem 3.6 when $u = 2$ and $u = 3$. □

4. Core blocks whose weight graphs are trees

Understanding decomposition numbers for core blocks would imply an understanding of decomposition numbers when $e = 0$: these numbers are known when $r \leq 2$ [11] [8] (or, trivially when $r = 1$). As a next step, we start to investigate indecomposable core blocks where the weight graphs of the Specht modules are trees. Recall that if $S^\lambda$ is in a core block then the weight graph $G(\lambda)$ has vertices $\{1, 2, \ldots, r\}$ with $\text{wt}(\lambda(s), \lambda(t))$ edges between vertices $s$ and $t$. To each $\lambda$ with $S^\lambda$ in an indecomposable core block we associated $\lambda = \text{Pt}(B, M)$ for $B \in \mathfrak{B}$ and $M \in \mathfrak{M}$; set $G(M) = G(\lambda)$. Using Lemma 2.8 we can read $G(M)$ from $M$ without referring to $\lambda$: we defined

$$\delta^M(t, s) = \# \{0 \leq i \leq e - 1 \mid M(t, i) = 1 \text{ and } M(s, i) = 0\},$$

$$\delta^M(t, s) = \# \{0 \leq i \leq e - 1 \mid M(t, i) = 0 \text{ and } M(s, i) = 1\},$$

$$\text{wt}^M(t, s) = \text{Min}\{\delta^M(t, s), \delta^M(t, s)\},$$

so that $G(M)$ has $\text{wt}^M(t, s)$ edges between vertices $t$ and $s$.

If $S^\lambda$ is in a core block and $G(\lambda)$ is a tree then it is a connected graph on $r$ vertices with $r - 1$ edges. If $S^\mu$ is in the same block as $S^\lambda$ then, by Proposition 2.6 $G(\mu)$ is also connected with $r - 1$ edges, so whether the weight graph is a tree is indeed a block invariant property.

We begin by describing the matrices $M \in \mathfrak{M}$ which have the property that $G(M)$ is a tree. We say that two matrices are row equivalent if one can be obtained from another by rearranging the rows; clearly if $M'$ is obtained from $M$ by such a rearrangement then $G(M)$ is a tree if and only if $G(M')$ is a tree.

We start by defining some column vectors. Note that our convention is to read vectors from top to bottom. Suppose that $0 \leq a \leq r$. Define $I(a)$ to be the column vector of length $r$ given by

$$I(a)_s = \begin{cases} 1, & 1 \leq s \leq a, \\ 0, & a + 1 \leq s \leq r. \end{cases}$$
Fix a composition \( \mathbf{y} = (y_1, y_2, \ldots, y_{r-1}) \) such that \( y_1, y_{r-1} \geq 1 \). For \( 1 \leq t \leq r-1 \), let \( P_t(\mathbf{y}) \) be the set of pairs \((i,j)\) with \( i = (i_1, i_2, \ldots, i_t)\) and \( j = (j_1, j_2, \ldots, j_{t+1})\) such that
\[
1 = j_1 \leq i_1 < j_2 \leq i_2 < \cdots < i_t < j_{t+1} = r,
\]
and if \( a \in i \) then \( y_a \geq 1 \). For \( 1 \leq m \leq t \), and \((i,j) \in P_t(\mathbf{y})\), define \( c_m(i,j) \) to be the column vector of length \( r \) given by
\[
c_m(i,j)_s = \begin{cases} 1 & 1 \leq s \leq j_m - 1 \text{ or } j_m + 1 \leq s \leq i_m \text{ or } s = j_{m+1}, \\ 0, & \text{otherwise.} \end{cases}
\]
A more intuitive description of \( c_m(i,j) \) is that it is formed by taking the vector \( I(i_m) \) and swapping the entries in rows \( j_m \) and \( j_{m+1} \).

Let \( 1 \leq t \leq r-1 \) and suppose \((i,j) \in P_t(\mathbf{y})\). Let \( \mathcal{T} \mathcal{Y}(i,j) \) be the set of matrices lying in \( \mathfrak{M} \) which have the further properties that:

- \( y_a - 1 \) columns equal to \( I(a) \) for \( 1 \leq a \leq r - 1 \) and \( a \in \{i_1, i_2, \ldots, i_t\} \).
- \( y_a \) columns equal to \( I(a) \) for \( 1 \leq a \leq r - 1 \) and \( a \notin \{i_1, i_2, \ldots, i_t\} \).
- One column equal to \( c_m(i,j) \) for each \( 1 \leq m \leq t \).
- All other columns equal to \( I(0) \) or \( I(r) \).

**Proposition 4.1.** Let \( M \in \mathfrak{M} \). Then \( G(M) \) is a tree if and only if there exist a composition \( \mathbf{y} = (y_1, y_2, \ldots, y_{r-1}) \) with \( y_1, y_{r-1} \geq 1 \) and \((i,j) \in P_t(\mathbf{y})\) for some \( 1 \leq t \leq r - 1 \) such that \( M \) is row equivalent to a matrix in \( \mathcal{T} \mathcal{Y}(i,j) \).

Before proving Proposition 4.1, we describe the equivalence classes under \( \sim_b \) whose elements have weight graphs which are trees. Let \( \Lambda \) be the set of sequences of sets \( \mathbf{Y} = (Y_0, Y_1, \ldots, Y_r) \) with the properties that
\[
I = Y_0 \sqcup Y_1 \sqcup \ldots \sqcup Y_{r-1} \sqcup Y_r
\]
such that

- \( |Y_s| < \infty \) for \( 1 \leq s \leq r-1 \).
- \( Y_1, Y_{r-1} \neq \emptyset \).
- If \( e \geq 2 \) then \( Y_e = \emptyset \).
- If \( e = 0 \) then \( i \notin Y_r \) for all \( i \gg 0 \) and \( i \in Y_r \) for all \( i \ll 0 \).

For \( \mathbf{Y} \in \Lambda \), define \( \mathbf{y} = (y_1, y_2, \ldots, y_{r-1}) \) where \( y_s = |Y_s| \) for all \( 1 \leq s \leq r-1 \). If \( 1 \leq t \leq r - 1 \) let \( P_t(\mathbf{Y}) = P_t(\mathbf{y}) \) and for \((i,j) \in P_t(\mathbf{Y})\), let
\[
\mathbf{W}(i,j) = \{ \mathbf{w} = (w_1, w_2, \ldots , w_t) | 1 \leq w_u \leq y_{w_u} \text{ for } 1 \leq u \leq t \}. \]

For \( 1 \leq s \leq r - 1 \), suppose that \( Y_s = \{ \alpha_1^s, \alpha_2^s, \ldots , \alpha_{v_s}^s \} \) where \( \alpha_u^s < \alpha_{u+1}^s \) for \( 1 \leq u < v \leq y_s \). We define the matrix \( \Delta(\mathbf{Y}, (i,j), \mathbf{w}) \) to be the matrix in \( \mathfrak{M} \) in which the column indexed by \( c \in I \) is equal to

- \( c_{w_u}(i,j) \) if \( c = \alpha_{w_u}^s \) for \( 1 \leq u \leq t \).
- \( I(a) \) if \( c \in Y_a \) where \( a = i_u \in I \) and \( c \neq \alpha_{w_u}^a \).
- \( I(a) \) if \( c \in Y_a \) where \( a \notin I \).

For \( \pi \in \mathfrak{S}_r \), let \( \Delta^\pi(\mathbf{Y}, (i,j), \mathbf{w}) \) be the matrix obtained by permuting the rows of \( \Delta(\mathbf{Y}, (i,j), \mathbf{w}) \) by the permutation \( \pi \). For fixed \( \mathbf{Y} \in \Lambda \) and \( \pi \in \mathfrak{S}_r \), we define
\[
\mathcal{R}(\mathbf{Y}, \pi) = \{ \Delta^\pi(\mathbf{Y}, (i,j), \mathbf{w}) | 1 \leq t \leq r - 1, (i,j) \in P_t(\mathbf{Y}), \mathbf{w} \in \mathbf{W}(i,j) \}. \]

**Proposition 4.2.** Let \( \mathcal{R} \) be an equivalence class of \( \mathfrak{M}/ \sim_b \) whose elements have weight graphs which are trees. Then there exists \( \mathbf{Y} \in \Lambda \) and \( \pi \in \mathfrak{S}_r \) such that
\[
\mathcal{R} = \mathcal{R}(\mathbf{Y}, \pi).\]
Proof. Let \( M \in \mathcal{T}^r(i,j) \). Suppose \( 1 \leq s < u \leq r \). There exists a column in \( M \) with 0 in row \( s \) and 1 in row \( u \) if and only if the column is equal to \( c_m(i,j) \) for some \( 1 \leq m \leq t \) and

- \( s = j_m \) and \( j_m + 1 \leq u \leq i_m \); or
- \( i_m + 1 \leq s \leq j_m + 1 - 1 \) and \( u = j_m + 1 \); or
- \( s = j_m \) and \( u = j_m + 1 \).

This shows that \( wt(s,u) \leq 1 \) in each of the cases above, and \( wt(s',u') = 0 \) for all other \( 1 \leq s' < u' \leq r \). Hence \( G(M) \) has no cycles or multiple edges and it remains only to show that it is connected. For \( 1 \leq m \leq t - 1 \), we have

\[
c_m(i,j)_{j_m} = 0 = c_{m+1}(i,j)_{j_{m+1}}, \quad c_m(i,j)_{j_{m+1}} = 1 = c_{m+1}(i,j)_{j_m},
\]

so that \( wt(j_m,j_{m+1}) = 1 \). For \( 1 \leq m \leq t - 1 \) and \( i_m + 1 \leq s \leq j_{m+1} - 1 \), we have

\[
c_m(i,j)_s = 0 = c_{m+1}(i,j)_{j_{m+1}}, \quad c_m(i,j)_{j_{m+1}} = 1 = c_{m+1}(i,j)_s,
\]

so that \( wt(s,j_{m+1}) = 1 \). For \( 2 \leq m \leq t \) and \( j_m + 1 \leq s \leq i_m \) we have

\[
c_{m-1}(i,j)_s = 0 = c_m(i,j)_{j_m}, \quad c_{m-1}(i,j)_{j_m} = 1 = c_m(i,j)_s,
\]

so that \( wt(j_m,s) = 1 \). This proves that vertex \( s \) is connected in the graph \( G(M) \) to vertex \( j_1 = 1 \) in all cases except where \( 2 = j_1 + 1 \leq s \leq i_1 \) and \( i_1 + 1 \leq s \leq j_{1+1} - 1 = r - 1 \). If there exists a vertex \( s \) with \( 2 \leq s \leq i_1 \) then \( i_1 > 1 \), that is, \( 1 \notin i \). By the assumption that \( y_1 \geq 1 \), there must exist a column in \( M \) equal to \( I(1) \). Then for \( 2 \leq s \leq i_1 \),

\[
c_1(i,j)_1 = 0 = I(1)_s, \quad c_1(i,j)_{1} = I(1)_1,
\]

and so vertex \( s \) is connected to vertex \( 1 \) in \( G(M) \). A similar argument shows that if there exists \( s \) with \( i_t + 1 \leq s \leq r - 1 \) then vertex \( s \) is connected to vertex \( r \). This completes the proof that \( G(M) \) is a tree. \( \Box \)

Remark. The tree \( G(M) \) where \( M \in \mathcal{T}^r(i,j) \) has the form:

\[
\begin{array}{c}
\text{Remark. The tree } G(M) \text{ where } M \in \mathcal{T}^r(i,j) \text{ has the form:}
\end{array}
\]

Lemma 4.4. Suppose that \( M \in \mathcal{M} \) and that \( G(M) \) is a tree. Then there exist \( y' = (y'_1, y'_2, \ldots, y'_{r-1}) \) with \( y'_1, y'_{r-1} \geq 1 \) and \( (i',j') \in P_r(y') \) for some \( 1 \leq t' \leq r - 1 \) such that \( M \) is row equivalent to a matrix in \( \mathcal{T}^{t'}(i',j') \).

Proof. We use induction on \( r \). If \( r = 2 \) and \( M \in \mathcal{M} \) is such that \( G(M) \) is a tree then \( M \) is row equivalent to a matrix which has one column equal to \((0,1)^T\), \( k \geq 1 \) columns equal to \((1,0)^T\) and all other columns equal to \((0,0)^T\) or \((1,1)^T\) and which therefore satisfies the conditions of Lemma 4.3. So suppose that \( r > 2 \) and that Lemma 4.3 holds for \( r - 1 \). Suppose that \( M \in \mathcal{M} \) and that \( G(M) \) is a tree. By reordering the rows of the matrix, we may assume that vertex \( r \) in the graph \( G(M) \) has degree 1 so that the matrix \( M \) consisting of the first \( r - 1 \) rows of \( M \) is also such that \( G(M) \) is a tree. By the induction hypothesis, and by reordering the rows, we may assume that there exist a composition \( y = (y_1, y_2, \ldots, y_{r-2}) \) with \( y_1, y_{r-2} \geq 1 \), an integer \( 1 \leq t \leq r - 2 \) and \( (i,j) \in P_r(y) \) such that \( M \in \mathcal{T}^r(i,j) \).

If \( c \in I \), we will write \( \text{col}(c) = I(a) \) (resp. \( \text{col}(c) = c_m(i,j) \)) to mean that the column of \( M \) indexed by \( c \) is equal to \( I(a) \) (resp. \( c_m(i,j) \)).

Without loss of generality, assume that for \( 1 \leq m < t \), we have \( \text{col}(m) = c_m(i,j) \). Suppose first that there exists \( 1 \leq m < t \) with \( M(r,m) \neq M(r,m+1) \). Since we also have

\[
M(j_m,m) = M(j_{m+2},m) = 0, \quad M(j_m,m+1) = M(j_{m+2},m+1) = 1,
\]

we must have \( M(r,m) = 0 \) and \( M(r,m+1) = 1 \), otherwise the degree of the vertex \( r \) in \( G(M) \) is at least 2. This argument further implies that

\[
M(r,a) = \begin{cases} 0, & 1 \leq a \leq m, \\ 1, & m + 1 \leq a \leq t. \end{cases}
\]
Moreover,

\[ M(j_{m+1}, m) = M(r, m+1) = 1, \quad M(r, m) = M(j_{m+1}, m + 1) = 0, \]

so that in the graph \( G(M) \), vertex \( r \) is adjacent to vertex \( j_{m+1} \).

Now consider the columns of the matrix \( M \) which are equal to \( I(b) \) for some \( 0 \leq b \leq r - 1 \). Suppose that there exist \( c, c' \in I \) with \( \text{col}(c) = b \) and \( \text{col}(c') = b' \) and where \( 0 \leq b' < b \leq r - 1 \). If \( M(r, c') = 1 \) and \( M(r, c) = 0 \) then \( w_t^M(r, s) \geq 1 \) for all \( b' < s \leq b \), giving a contradiction unless \( b = j_{m+1} \) and \( b' = j_{m+1} - 1 \). But if \( b = j_{m+1} \) and \( b' = j_{m+1} - 1 \) then \( w_t^M(j_{m+1}, r) \geq 2 \), also giving a contradiction. Hence if \( \text{col}(c) = I(b) \) and \( \text{col}(c') = I(b') \) with \( 0 \leq b' < b \leq r - 1 \) then \( M(r, c') \leq M(r, c) \).

Suppose that there exists \( c \in I \) such that \( \text{col}(c) = I(b) \) where \( b < i_m \) and that \( M(c, r) = 1 \). If \( i_m \neq j_m \) then \( M(m, i_m) = 1 \) and \( M(m, r) = 0 \), giving a contradiction, since \( w_t^M(i_m, r) = 0 \). If \( i_m = j_m \) and \( m \neq 1 \) then \( M(m - 1, i_m) = 1 \) and \( M(m - 1, r) = 0 \), again giving a contradiction. So we must have \( m = 1 \) and \( i_1 = j_1 = 1 \) so that \( b = 0 \). Furthermore, if \( \text{col}(c') = I(b') \) for \( b' > 0 \) then \( M(r, c') = 1 \), since otherwise \( w_t^M(1, r) = 0 \). So \( M(1, r) = 0 \) and if \( M(c, r) = 0 \) for \( c \neq 1 \) then \( \text{col}(c) = I(0) \). Suppose there are \( u \geq 1 \) columns \( d \) with \( \text{col}(d) = I(0) \) and \( M(d, r) = 1 \). If we consider the matrix \( \hat{M} \) obtained from \( M \) by moving row \( r \) before row 1 then \( \hat{M} \in T^Y(I', j') \) where

\[ I' = (i_1, i_2 + 1, \ldots, i_t + 1), \quad j' = (j_1 + 1, j_2 + 1, \ldots, j_{t+1} + 1) \]

and \( y' = (u, y_1, y_2, \ldots, y_{r-2}) \).

A similar argument takes care of the case when there exists \( c \in I \) with \( \text{col}(c) = I(b) \) where \( b > i_{m+1} \) and \( M(c, r) = 0 \).

So assume that if \( \text{col}(c) = I(b) \) then \( M(c, r) = 0 \) if \( b < i_m \) and \( M(c, r) = 1 \) if \( b > i_{m+1} \). Then there exists some (possibly not unique) \( i_m \leq g \leq i_{m+1} \) such that \( M(r, c) = 0 \) whenever column \( c \) is equal to \( I(a) \) and \( a < g \) and \( M(r, c) = 1 \) whenever column \( c \) is equal to \( I(a) \) and \( a > g \). Suppose that there are \( u \) columns \( d \) with \( \text{col}(d) = I(g) \) and \( M(d, r) = 0 \) and \( v \) columns \( d' \) with \( \text{col}(d') = I(g) \) and \( M(d', r) = 1 \). Consider the matrix \( \hat{M} \) formed by moving row \( r \) of \( M \) in between rows \( g \) and \( g + 1 \). Then \( \hat{M} \) lies in \( T^Y(I', j') \) where

\[ i = (i_1, \ldots, i_m, i_m + 1, \ldots, i_t + 1), \quad j' = (j_1, \ldots, j_m, j_m + 1 + \epsilon, j_m + 2 + 1, \ldots, j_{t+1} + 1) \]

where \( \epsilon = 0 \) if \( j_{m+1} \leq g \) and \( \epsilon = 1 \) and where \( y' = (y_1', y_2', \ldots, y_{r-1}') \) is given by

\[ y'_s = \begin{cases} 
  y_s, & 1 \leq s < g, \\
  u, & s = g \neq i_m, \\
  u + 1, & s = g = i_m, \\
  v, & s = g + 1 \neq i_{m+1} + 1, \\
  v + 1, & s = g + 1 = i_{m+1} + 1, \\
  y_{s-1}, & g + 2 \leq s \leq r - 1.
\end{cases} \]

Note that these conditions imply that \( y_1, y_{r-1} \geq 1 \).

We secondly consider the case where \( M(r, m) = M(r, m+1) \) for all \( 1 \leq m \leq t \). Suppose that \( M(r, m) = 0 \) for all \( 1 \leq m \leq t \). There must exist a column \( c \in I \) with \( \text{col}(c) = I(g) \) for some \( 0 \leq g < r - 1 \) with \( M(r, c) = 1 \), so suppose \( c \) is chosen such that \( g \) is minimal. Then

\[ M(r - 1, t) = M(r, c) = 1, \quad M(r - 1, c) = M(r, t) = 0, \]

so that \( w_t^M(r - 1, 1) = 1 \). An argument similar to that given above shows that \( g \geq i_t \) or \( t = 1 \), \( i_t = j_t = 1 \) and \( g = 0 \). In the latter case, suppose that there are \( u \geq 1 \) columns \( d \) with \( \text{col}(d) = I(0) \) and \( M(d, r) = 1 \). If we consider the matrix \( \hat{M} \) obtained from \( M \) by moving row \( r \) before row 1 then \( \hat{M} \in T^Y(I', j') \) where

\[ I' = (1), \quad j' = (1, r), \quad y' = (u + 1, y_1, y_2, \ldots, y_{r-2}). \]

So suppose that \( g \geq i_t \). Suppose there exists \( c' \in I \) with \( \text{col}(c') = I(b) \) and with \( g < b \leq r - 1 \) and \( B(r, c') = 1 \). Then \( g = r - 2 \) and \( b = r - 1 \). The matrix \( \hat{M} \) then lies in \( T^Y(I', j') \) where

\[ I' = (i_1, \ldots, i_t, r - 1), \quad j' = (j_1, \ldots, j_{t+1}, r), \quad y' = (u + 1, y_1, y_2, \ldots, y_{r-2}, u) \]

where \( u \) is the number of columns with \( r - 1 \) entries equal to 1; by assumption, this is at least 1.
So assume $c' \in I$ with $\text{col}(c') = I(b)$ and with $g < b \leq r - 1$ where $M(r, c') = 1$. Suppose that there are $u$ columns $d$ with $\text{col}(d) = I(g)$ and $M(r, d) = 0$ and $v$ columns $d'$ with $\text{col}(d') = I(g)$ and $M(r, d') = 1$. Consider the matrix formed by moving row $r$ of $M$ in between rows $g$ and $g + 1$. Then this matrix lies in $\mathcal{T} Y'(i, j')$ where

$$i = (i_1, \ldots, i_t), \quad j' = (j_1, \ldots, j_t, j_{t+1} + 1),$$

$$y'_s = \begin{cases} y_s, & 1 \leq s < g, \\ u, & s = g \neq i_t, \\ u + 1, & s = g = i_t, \\ v, & s = g + 1, \\ y_{s-1}, & g + 2 \leq s \leq r - 1. \end{cases}$$

Note that these conditions imply that $y_1, y_{r-1} \geq 1$ since $v \geq 1$.

The argument is symmetrical if $B(r, m) = 1$ for all $1 \leq m \leq t$. This completes the proof of Lemma 4.4. □

Combining Lemma 4.3 and Lemma 4.4, we get the proof of Proposition 4.1 in fact we have shown that if $M \in \mathcal{R}$ then $G(M)$ is a tree if and only if $M \in \mathcal{R}(Y, \pi)$ for some $Y \in \Lambda$ and $\pi \in \mathcal{G}_r$. We now consider Proposition 4.2. For $Y \in \Lambda$ and $\pi \in \mathcal{G}_r$, let $N(Y, \pi) = \Delta^e(Y, ((1), (1, r)), 1)$ be the matrix in which the column indexed by $c \in I$ is equal to $I(a)$ if $c \in Y_s$, unless $c = \alpha_1^s$ in which case it has the entry in row $r$ equal to 1 and all other entries equal to 0.

**Lemma 4.5.** Suppose that $M \in \mathcal{R}(Y, \pi)$ for some $Y \in \Lambda$ and $\pi \in \mathcal{G}_r$. Then $M \sim_b N(Y, \pi)$.

**Proof.** Since $\pi$ only permutes the rows of the matrices, we may assume that $\pi = \text{Id}$. Since $M \in \mathcal{R}(Y, \pi)$, there exist $1 \leq t \leq r - 1$, $(i, j) \in P_t(Y)$ and $w \in \mathcal{W}(i, j)$ such that $M = \Delta(Y, (i, j), w)$. Define a sequence of matrices

$$M = N_1, N_2, \ldots, N_{t+1}$$

as follows. If $i_1 = 1$ and $w_1 = 1$, let $N_2 = M$. Otherwise define $N_2$ to be the matrix obtained from $M$ by making the bead swap that involves rows $j_1$ and $j_2$ and columns $\alpha_1^1$ and $\alpha_{w_1}^1$. For $2 \leq s \leq t$, let $N_{s+1}$ be the matrix obtained from $N_s$ by making the bead swap that involves rows $j_s$ and $j_{s+1}$ and columns $\alpha_1^s$ and $\alpha_{w_s}^s$. Then $N_{t+1} = N(Y, \pi)$ and hence $M \sim_b N(Y, \pi)$. □

**Lemma 4.6.** Suppose that $M \in \mathcal{R}(Y, \pi)$ for some $Y \in \Lambda$ and $\pi \in \mathcal{G}_r$. If $L \in \mathcal{R}$ is such that $M \sim_b L$ then $L \in \mathcal{R}(Y, \pi)$.

**Proof.** As above, we may assume that $\pi = \text{Id}$. Suppose that $M = \Delta(Y, (i, j), w)$ where $(i, j) \in P_t(Y)$ and we assume the usual notation. We show that if $N$ is formed from $M$ by a bead swap then $N \in \mathcal{R}(Y, \pi)$. As in the proof of Lemma 4.3, any such swap must involve a column which contains a column vector $c_m(i, j)$ for some $1 \leq m \leq t$. The other column involved must either contain $I(a)$ for some $1 \leq a \leq r - 1$ or $c_{m+1}(i, j)$. We first consider the latter case. For $t > 1$, consider the following transposed picture, with the rows corresponding to the columns containing $c_{m-1}(i, j)$ and $c_m(i, j)$ for some $2 \leq m \leq t$.

| $j_{m-1}$ | $j_m$ | $j_{m+1}$ |
|----------|-------|-----------|
| 1        | 0     | 0         |
| 0        | 0     | 0         |
| 0        | 0     | 0         |
| 0        | 0     | 0         |
| 0        | 0     | 0         |
| 0        | 0     | 0         |
| 0        | 0     | 0         |

Any bead swap involving these two columns must involve row $j_m$. We now look at the other possible rows.

- If $N$ is formed from $M$ by a bead swap involving the columns equal to $c_{m-1}(i, j)$ and $c_m(i, j)$ and the rows $j_{m-1}$ and $j_m$ then $N = \Delta(Y, (i', j'), w')$ where
  
  $$i' = (i_1, \ldots, i_{m-2}, i_m, \ldots, i_t),$$
  
  $$j' = (j_1, \ldots, j_{m-1}, j_{m+1}, \ldots, j_{t+1}),$$
  
  $$w' = (w_1, \ldots, w_{m-2}, w_m, \ldots, w_t).$$
If $N$ is formed from $M$ by a bead swap involving the columns equal to $c_{m-1}(i, j)$ and $c_m(i, j)$ and the rows $s$ and $j_m$ where $i_{m-1} + 1 \leq s < j_m$ or $j_m < s \leq i_m$ then $N = \Delta(Y, (i, j'), w)$ where 

$$j' = (j_1, \ldots, j_{m-1}, s, j_{m+1}, \ldots, i_t).$$

If $N$ is formed from $M$ by a bead swap involving the columns equal to $c_{m-1}(i, j)$ and $c_m(i, j)$ and the rows $j_m$ and $j_{m+1}$ then $N = \Delta(Y, (i', j'), w')$ where 

$$i' = (i_1, \ldots, i_{m-1}, a, i_{m+1}, \ldots, i_t),$$

$$j' = (j_1, \ldots, j_{m-1}, j_m, j_{m+1}, \ldots, j_{t+1}),$$

$$w' = (w_1, \ldots, w_{m-1}, b, w_{m+1}, \ldots, w_t),$$

where the second column involved in the swap is indexed by $a^g_b$ for some $1 \leq b \leq y_a$.

If $N$ is formed from $M$ by a bead swap involving columns equal to $c_m(i, j)$ and $I(a)$ and the rows $j_m$ and $j_{m+1}$ then $N = \Delta(Y, (i', j'), w')$ where 

$$i' = (i_1, \ldots, i_{m-1}, a, i_{m+1}, \ldots, i_t),$$

$$j' = (j_1, \ldots, j_{m-1}, j_m, j_{m+1}, \ldots, j_{t+1}),$$

$$w' = (w_1, \ldots, w_{m-1}, b, w_{m+1}, \ldots, w_t),$$

where the second column involved in the swap is indexed by $a^g_b$ for some $1 \leq b \leq y_a$.

If $N$ is formed from $M$ by a bead swap involving columns equal to $c_m(i, j)$ and $I(a)$ and the rows $j_{m+1}$ and $s$ where $i_m + 1 \leq s < j_{m+1}$ then $s \leq a < j_{m+1}$ and $N = \Delta(Y, (i', j'), w')$ where 

$$i' = (i_1, \ldots, i_{m-1}, a, i_{m+1}, \ldots, i_t),$$

$$j' = (j_1, \ldots, j_{m-1}, j_m, j_{m+1}, \ldots, j_{t+1}),$$

$$w' = (w_1, \ldots, w_{m-1}, b, w_{m+1}, \ldots, w_t),$$

where the second column involved in the swap is indexed by $a^g_b$ for some $1 \leq b \leq y_a$.

Combining Lemma 4.5 and Lemma 4.6 we get the proof of Proposition 4.2.

Appendix A. When $r = 4$

In this appendix we look in detail at certain indecomposable core blocks of weight 3 when $r = 4$. Fix 

$$\mathbf{Y} = (Y_0, Y_1, Y_2, Y_3, Y_4) \in \Lambda$$

and let 

$$\mathfrak{R} = \{\Delta(Y, (i, j), w) \mid 1 \leq t \leq r - 1, (i, j) \in P_t(\mathbf{Y}), w \in \mathbf{W}(i, j), \pi \in \mathfrak{S}_r\}.$$  

Since $\mathbf{Y}$ is fixed, we will write $\Delta((i, j), w) = \Delta(Y, (i, j), w)$ for each matrix in $\mathfrak{R}$. Fix $B \in \mathfrak{B}$. From now on, we do not distinguish between a matrix $M \in \mathfrak{R}$ and the partition $\lambda = P_t(B, M)$. Assume that 

$$Y_1 = \{i_1, i_2, \ldots, i_{y_1}\}, \quad Y_2 = \{j_1, j_2, \ldots, j_{y_2}\}, \quad Y_3 = \{k_1, k_2, \ldots, k_{y_3}\},$$

where we have $i_1 < i_2 < \ldots < i_{y_1}$; similarly for $Y_2$ and $Y_3$. If $e \geq 2$ set $i_0 = j_0 = k_0 = 0$ and $i_{y_1+1} = j_{y_2+1} = k_{y_3+1} = e$; if $e = 0$ choose $i_0, j_0, k_0 < 0$ and $i_{y_1+1}, j_{y_2+1}, k_{y_3+1} \gg 0$.

To help with notation, we relabel the elements of $\mathfrak{R}$. The indices $f, g, h$ below are all assumed to satisfy $1 \leq f \leq y_1, 1 \leq g \leq y_2$ and $1 \leq h \leq y_3$. Write 

$$\alpha^1_f = \Delta(((1), (1, 4)), (f)), \quad \alpha^2_g = \Delta(((2), (1, 4)), (g)), \quad \alpha^3_h = \Delta(((3), (1, 4)), (h)),$$

$$\beta^2_{fg} = \Delta(((1, 2), (1, 2, 4)), (f, g)), \quad \beta^3_{fh} = \Delta(((1, 3), (1, 3, 4)), (f, h)).$$
The columns that arise in these matrices are, in some order, the columns below, with the entry in the top row describing how many of those entries occur.

\[
\begin{align*}
\beta_{fh}^{13} & = \Delta((1,3),(1,2,4)), (f,h), \\
\beta_{gh}^{23} & = \Delta((2,3),(1,3,4)), (g,h), \\
\gamma_{fgh}^{123} & = \Delta((1,2,3),(1,2,3,4)), (f,g,h)).
\end{align*}
\]

We can now describe the columns of the decomposition matrix for \(\mathfrak{N}\). As in Section 3, the proofs of Propositions A.2 to A.4 follow from case by case analysis using Corollary 2.12. The data given here were produced by computer.

**Proposition A.1.** The Kleshchev multipartitions in \(\mathfrak{N}\) are the following, where each \(f,g,h\) is assumed to satisfy \(1 \leq f \leq y_1\), \(1 \leq g \leq y_2\), \(1 \leq h \leq y_3\).

\[
\begin{align*}
\alpha_1^f & : \{\alpha_1^f \mid k_1 < i_f\}, \\
\alpha_2^g & : \{\alpha_2^g \mid i_1, k_1 < j_g\}, \\
\alpha_3^h & : \{\alpha_3^h \mid i_1 < k_h\}, \\
\beta_{fg}^{12} & : \{\beta_{fg}^{12} \mid k_1 < j_g < i_f\}, \\
\beta_{fh}^{13} & : \{\beta_{fh}^{13} \mid i_1 < j_g < k_h\}, \\
\beta_{gh}^{23} & : \{\beta_{gh}^{23} \mid k_1, i_f < j_g \text{ and } f \neq 1\}, \\
\gamma_{fgh}^{123} & : \{\gamma_{fgh}^{123} \mid i_1, k_1 < j_g \text{ and } g \neq 1\}, \\
\gamma_{gh}^{12} & : \{\gamma_{gh}^{12} \mid j_1 < k_h < i_f\}, \\
\gamma_{fh}^{23} & : \{\gamma_{fh}^{23} \mid i_1 < j_g < k_h \text{ and } f \neq 1\}, \\
\gamma_{gh}^{13} & : \{\gamma_{gh}^{13} \mid i_1 < j_g \text{ and } h \neq 1\}. \\
\end{align*}
\]
Proposition A.2. Suppose that \( \alpha_1 \) is Kleshchev. Define \( \tilde{g} \) and \( \tilde{h} \) respectively to be maximal such that \( j_{\tilde{g}}, k_{\tilde{h}} < i_f \). Then \( P(\alpha_1) \) is given by one of the following possibilities.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\alpha_1 & \alpha_1 & \beta^{12}_{f_{\tilde{g}}} & \beta^{13}_{f_{\tilde{h}}} & \beta^{12}_{f_{\tilde{h}}-1} & \beta^{13}_{f_{\tilde{h}}-1} & \beta^{12}_{f_{\tilde{h}}-1} & \beta^{13}_{f_{\tilde{h}}-1} & \gamma^{123}_{f_{\tilde{g}}f_{\tilde{h}}} & \gamma^{123}_{f_{\tilde{h}}-1f_{\tilde{h}}} \\
\hline
1 & v & v & v & v^2 & v^3 & i_{f-1}, j_{\tilde{g}} < k_{\tilde{h}} \\
1 & v & v & v & v^2 & v^3 & k_{\tilde{h}}, i_{f-1} < j_{\tilde{g}} \\
1 & v & v & v & v^2 & v^3 & j_{\tilde{g}} < k_{\tilde{h}} < i_{f-1} \\
1 & v & v & v & v^2 & v^3 & k_{\tilde{h}} < j_{\tilde{g}} < i_{f-1} \\
\hline
\end{array}
\]

Suppose that \( \alpha_2 \) is Kleshchev. Define \( \tilde{f} \) and \( \tilde{h} \) respectively to be maximal such that \( i_{\tilde{f}}, k_{\tilde{h}} < j_{\tilde{g}} \). Then \( P(\alpha_2) \) is given by one of the following possibilities.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\alpha_2 & \alpha_2 & \beta^{12}_{f_{\tilde{g}}} & \beta^{13}_{f_{\tilde{h}}} & \beta^{12}_{f_{\tilde{h}}-1} & \beta^{13}_{f_{\tilde{h}}-1} & \beta^{12}_{f_{\tilde{h}}-1} & \beta^{13}_{f_{\tilde{h}}-1} & \gamma^{123}_{f_{\tilde{g}}f_{\tilde{h}}} & \gamma^{123}_{f_{\tilde{h}}-1f_{\tilde{h}}} \\
\hline
1 & v & v & v^2 & v^3 & i_{\tilde{f}} < i_{f} \text{ and } j_{f-1} < i_{\tilde{f}} \\
1 & v & v & v^2 & v^3 & i_{\tilde{f}} < k_{\tilde{h}} \text{ and } j_{f-1} < k_{\tilde{h}} \\
1 & v & v & v^2 & v^3 & i_{\tilde{f}}, k_{\tilde{h}} < j_{g-1} \\
\hline
\end{array}
\]

Suppose that \( \alpha_3 \) is Kleshchev. Define \( \tilde{f} \) and \( \tilde{g} \) respectively to be maximal such that \( i_{\tilde{f}}, j_{\tilde{g}} < k_{\tilde{h}} \). Then \( P(\alpha_3) \) is given by one of the following possibilities.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\alpha_3 & \alpha_3 & \beta^{12}_{f_{\tilde{g}}} & \beta^{13}_{f_{\tilde{h}}} & \beta^{21}_{f_{\tilde{h}}-1} & \beta^{21}_{f_{\tilde{h}}-1} & \beta^{12}_{f_{\tilde{h}}-1} & \beta^{13}_{f_{\tilde{h}}-1} & \gamma^{123}_{f_{\tilde{g}}f_{\tilde{h}}} & \gamma^{123}_{f_{\tilde{h}}-1f_{\tilde{h}}} \\
\hline
1 & v & v & v^2 & v^3 & i_{\tilde{g}}, k_{\tilde{h}} < i_{f} \\
1 & v & v & v^2 & v^3 & i_{\tilde{f}}, k_{h-1} < j_{\tilde{g}} \\
1 & v & v & v^2 & v^3 & j_{\tilde{g}} < i_{f} < k_{h-1} \\
1 & v & v & v^2 & v^3 & i_{f}, j_{\tilde{g}} < k_{h-1} \\
\hline
\end{array}
\]
Proposition A.3. Suppose that $\beta^{12}_{fg}$ is Kleshchev. Define $\tilde{h}$ to be maximal such that $k_{\tilde{h}} < j_{g}$. Then $P(\beta^{12}_{fg})$ is given by one of the following possibilities.

| $\beta$ | $\alpha$ | $\beta$ | $\overline{\beta}$ | $\gamma$ |
|---------|-----------|---------|------------------|---------|
| $f_{g}$ | $f^{-1}g^{-1}h^{-1}$ | $f_{g}^{-1}f^{-1}g^{-1}h^{-1}$ | $f_{h}^{-1}h^{-1}$ | $f_{g}h^{-1}f^{-1}g^{-1}h^{-1}$ |
| 1       | $v v v^{2}$ | $v v^{2} v^{3}$ | $i_{f}, k_{h} < j_{g} - 1$ or $k_{h} < j_{g} - 1 < j_{g} < i_{f} - 1$ |
| 1       | $v$       | $v^{2} v^{3}$ | $i_{f}, j_{g} - 1 < k_{h} < j_{g}$ or $j_{g} - 1 < k_{h} < j_{g} < i_{f} - 1$ |
| 1       | $v v$     | $v^{2} v^{2}$ | $i_{f} - 1, k_{h} < j_{g} - 1 < i_{f}$ |
| 1       | $v v^{2}$ | $v v^{2} v^{3}$ | $i_{f} - 1, k_{h} < j_{g} - 1 < j_{g} < i_{f}$ |
| 1       | $v v^{2}$ | $v^{2} v^{3}$ | $i_{f} - 1, j_{g} - 1 < k_{h} < i_{f}$ |
| 1       | $v v^{3}$ | $v^{2} v^{3}$ | $j_{g} - 1, k_{h} < i_{f} < j_{g}$ |
| 1       | $v v$     | $v^{2} v^{3}$ | $j_{g} - 1 < k_{h} < i_{f} < j_{g} < i_{f}$ |

Suppose that $\beta^{23}_{gh}$ is Kleshchev. Define $\tilde{j}$ to be maximal such that $i_{j} < j_{g}$. Then $P(\beta^{23}_{gh})$ is given by one of the following possibilities.

| $\beta$ | $\alpha$ | $\beta$ | $\overline{\beta}$ | $\gamma$ |
|---------|-----------|---------|------------------|---------|
| $g_{h}$ | $g^{-1}h^{-1}g^{-1}h^{-1}$ | $g_{h}^{-1}h^{-1}g^{-1}h^{-1}$ | $g_{h}^{-1}g^{-1}h^{-1}g_{h}^{-1}h^{-1}g^{-1}h^{-1}$ |
| 1       | $v v v^{2}$ | $v v^{2} v^{3}$ | $i_{j}, k_{h} < j_{g} - 1$ or $i_{j} < j_{g} - 1 < j_{g} < k_{h} - 1$ |
| 1       | $v$       | $v^{2} v^{3}$ | $j_{g} - 1, k_{h} < i_{j} < j_{g}$ or $j_{g} - 1 < i_{j} < j_{g} < k_{h} - 1$ |
| 1       | $v v^{2}$ | $v v^{2} v^{3}$ | $j_{g} - 1, k_{h} - 1 < i_{j}$ |
| 1       | $v v^{2}$ | $v v^{2} v^{3}$ | $k_{h} - 1, i_{j} < j_{g} - 1 < j_{g} < k_{h}$ |
| 1       | $v v^{3}$ | $v v^{2} v^{3}$ | $j_{g} - 1, k_{h} - 1 < i_{j} < j_{g} < k_{h}$ |
| 1       | $v v^{2}$ | $v v^{2} v^{3}$ | $i_{j}, j_{g} - 1 < k_{h} - 1 < k_{h} < j_{g}$ |
| 1       | $v v$     | $v^{2} v^{3}$ | $i_{j} < j_{g} - 1 < k_{h} - 1 < j_{g} < k_{h}$ |
| 1       | $v v$     | $v^{2} v^{3}$ | $j_{g} - 1 < i_{j} < k_{h} - 1 < j_{g} < k_{h}$ |
Suppose that $\beta_{fh}^{13}$ is Kleshchev. Define $\tilde{g}$ to be maximal such that $j_{\tilde{g}} < i_f$. Then $P(\beta_{fh}^{13})$ is given by one of the following possibilities.

| $\beta_{fh}^{13}$ | $\alpha_1^1$ | $\alpha_2^2$ | $\alpha_3^3$ | $\beta_{fh}^{12}$ | $\beta_{fh}^{13}$ | $\beta_{fh}^{13}$ | $\beta_{fh}^{23}$ | $\gamma_{fh}^{123}$ | $j_{\tilde{g}} < i_{f-1} < i_f < k_{h-1}$ or $k_h < j_{\tilde{g}} < i_{f-1} < i_f$ | $i_{f-1}, k_h < j_{\tilde{g}} < i_f$ or $i_{f-1} < j_{\tilde{g}} < k_{h-1}$ | $i_{f-1} < j_{\tilde{g}} < k_{h-1} < i_f < k_h$ | $i_{f-1}, k_h < j_{\tilde{g}} < i_f < k_h$ | $i_{f-1}, k_h < j_{\tilde{g}} < i_f$ | $j_{\tilde{g}} < i_{f-1} < i_f < k_h$ | $j_{\tilde{g}}, i_{f-1} < k_{h-1} < k_h < i_f$ | $i_{f-1}, k_h < j_{\tilde{g}} < i_f < k_h$ | $j_{\tilde{g}} < k_{h-1} < i_{f-1} < i_f < k_h$ | $k_{h-1} < j_{\tilde{g}} < i_{f-1} < k_h < i_f$ | $j_{\tilde{g}} < k_{h-1} < k_h < i_{f-1}$ |
|------------------|---------------|---------------|---------------|------------------|------------------|------------------|------------------|------------------|---------------------------------|-------------------------------------------------|---------------------------------|---------------------------------|------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 1                |               |               |               | 1               | v                | v                | v                |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
| 1                |               |               |               | 1               | v                | v                | v                |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
| 1                | v              | v^2            | v              | v                | v                | v^2              | v                |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
| 1                | v              | v^2            | v              | v^2              | v^3              | v                | v                |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
| 1                | v              | v^2            | v              | v^2              | v^3              | v^2              | v                |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
| 1                | v              | v^2            | v              | v^2              | v^3              | v^2              | v^2              |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
| 1                | v              | v^2            | v              | v^2              | v^3              | v^2              | v^2              |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
| 1                | v              | v^2            | v              | v^2              | v^3              | v^2              | v^2              |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
| 1                | v              | v^2            | v              | v^2              | v^3              | v^2              | v^2              |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
| 1                | v              | v^2            | v              | v^2              | v^3              | v^2              | v^2              |               |                                |                                           |                                |                                | 1                 |                                |                                |                                |                                |                                |                                |                                | 1                 |
Suppose that $\beta_{fh}^{13}$ is Kleshchev. Define $\tilde{g}$ to be maximal such that $j_{\tilde{g}} < k_h$. Then $P(\beta_{fh}^{13})$ is given by one of the following possibilities.

| $\beta_{fh}^{13}$ | $\alpha^3$ | $\alpha^2$ | $\alpha^1$ | $\tilde{g}$ | $\tilde{g}h$ | $\tilde{g}h^{-1}$ | $\beta_{fh}^{13}$ | $\beta_{fh}^{13}$ | $\gamma_{123}^{13}$ |
|-------------------|------------|------------|------------|-------------|-------------|-------------|----------------|----------------|----------------|
| $\beta_{fh}^{13}$ | $\alpha^3$ | $\alpha^2$ | $\alpha^1$ | $\tilde{g}$ | $\tilde{g}h$ | $\tilde{g}h^{-1}$ | $\beta_{fh}^{13}$ | $\beta_{fh}^{13}$ | $\gamma_{123}^{13}$ |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |
| 1                 | v          | v          | v          | v           | v           | v           | v              | v              | v              |

$\bar{\beta}_{fh}$ is Kleshchev. Define $\tilde{g}$ to be maximal such that $j_{\tilde{g}} < k_h$. Then $P(\bar{\beta}_{fh}^{13})$ is given by one of the following possibilities.
Proposition A.4. Suppose that $\gamma_{fgh}^{123}$ is Kleshchev. Then $P(\gamma_{fgh}^{123})$ is given by one of the following possibilities.

| $\gamma_{fgh}^{123}$ | $\alpha^1$ | $\alpha^2$ | $\alpha^3$ | $\beta_{fgh}^{12}$ | $\beta_{fgh}^{13}$ | $\beta_{fgh}^{23}$ | $\gamma_{fgh}^{123}$ |
|----------------------|-------------|-------------|-------------|------------------|------------------|------------------|------------------|
| $f^{g}_{h}$ | $f$ | $f^{-1}$ | $g^{-1}$ | $h^{-1}$ | $g^{f-1}-1_{h^{-1}}$ | $g^{-1}_{h^{-1}}$ | $f^{-1}_{g^{-1}}h^{-1}$ | $f^{-1}_{g^{-1}}h^{-1}$ |
| 1 | $v$ | $v^2$ | $v^3$ | $v$ | $v^2$ | $v^2$ | $i_{f-1},j_{g-1} < i_f < j_g < j_h$ |
| 1 | $v$ | $v^2$ | $v^3$ | $v$ | $v^2$ | $v^2$ | $i_{f-1},j_{g-1} < i_f < j_g < j_h$ |
| 1 | $v$ | $v^2$ | $v^3$ | $v$ | $v^2$ | $v^2$ | $i_{f-1},j_{g-1} < i_f < j_g < j_h$ |
| 1 | $v$ | $v^2$ | $v^3$ | $v$ | $v^2$ | $v^2$ | $i_{f-1},j_{g-1} < i_f < j_g < j_h$ |
| 1 | $v$ | $v^2$ | $v^3$ | $v$ | $v^2$ | $v^2$ | $i_{f-1},j_{g-1} < i_f < j_g < j_h$ |

| $\gamma_{fgh}^{123}$ | $\alpha^1$ | $\alpha^2$ | $\alpha^3$ | $\beta_{fgh}^{12}$ | $\beta_{fgh}^{13}$ | $\beta_{fgh}^{23}$ | $\gamma_{fgh}^{123}$ |
|----------------------|-------------|-------------|-------------|------------------|------------------|------------------|------------------|
| $f^{g}_{h}$ | $f$ | $f^{-1}$ | $g^{-1}$ | $h^{-1}$ | $g^{f-1}-1_{h^{-1}}$ | $g^{-1}_{h^{-1}}$ | $f^{-1}_{g^{-1}}h^{-1}$ | $f^{-1}_{g^{-1}}h^{-1}$ |
| 1 | $v^2$ | $v^3$ | $v$ | $v^2$ | $v$ | $i_{f-1},k_{h-1} < i_f < j_g < j_h$ |
| 1 | $v^2$ | $v^3$ | $v$ | $v^2$ | $v^2$ | $i_{f-1},j_{g-1} < i_f < j_g < j_h$ |
| 1 | $v^2$ | $v^3$ | $v$ | $v^2$ | $v^2$ | $i_{f-1},j_{g-1} < i_f < j_g < j_h$ |
| 1 | $v^2$ | $v^3$ | $v$ | $v^2$ | $v^2$ | $i_{f-1},j_{g-1} < i_f < j_g < j_h$ |
| 1 | $v^2$ | $v^3$ | $v$ | $v^2$ | $v^2$ | $i_{f-1},j_{g-1} < i_f < j_g < j_h$ |
| $\gamma_{23}$ | $\alpha^3$ | $\alpha^2$ | $\alpha^1$ | $\beta_{23}$ | $\beta_{13}$ | $\beta_{12}$ | $\beta_{13}$ | $\gamma_{123}$ |
|------|------|------|------|------|------|------|------|------|
| $f$ | $g$ | $h$ | $g$ | $h$ | $g$ | $h$ | $g$ | $h$ |
| 1 | $v$ | $v^2$ | $v^2$ | $v^3$ | $v$ | $v$ | $v^2$ | $i_{f-1}, j_{g-1} < k_h - 1 < k_h < i_f < j_g$ |
| 1 | $v$ | $v$ | $v^2$ | $v^3$ | $v$ | $v$ | $v^2$ | $i_{f-1}, k_{h-1} < j_{g-1} < k_h < i_f < j_g$ |
| 1 | $v$ | $v$ | $v^2$ | $v^3$ | $v$ | $v$ | $v^2$ | $i_{f-1}, k_{h-1}, k_h < j_{g-1} < i_f < j_g$ |
| 1 | $v$ | $v$ | $v^2$ | $v^3$ | $v$ | $v$ | $v^2$ | $j_{g-1}, k_{h-1} < i_{f-1} < k_h < i_f < j_g$ |
| 1 | $v$ | $v$ | $v^2$ | $v^3$ | $v$ | $v$ | $v^2$ | $j_{g-1}, k_{h-1}, k_h < i_{f-1} < i_f < j_g$ |

| $\gamma_{123}$ | $\alpha^3$ | $\alpha^2$ | $\alpha^1$ | $\beta_{23}$ | $\beta_{13}$ | $\beta_{12}$ | $\beta_{13}$ | $\gamma_{123}$ |
|------|------|------|------|------|------|------|------|------|
| $f$ | $g$ | $h$ | $g$ | $h$ | $g$ | $h$ | $g$ | $h$ |
| 1 | $v^2$ | $v^3$ | $v$ | $v^2$ | $v$ | $v$ | $i_{f-1}, k_{h-1} < j_{g-1} < j_g < k_h < i_f$ |
| 1 | $v^2$ | $v^2$ | $v$ | $v^2$ | $v$ | $v$ | $k_{h-1} < j_{g-1} < i_{f-1} < j_g < k_h < i_f$ |
| 1 | $v^2$ | $v^2$ | $v$ | $v^2$ | $v$ | $v$ | $j_{g-1}, k_{h-1} < i_{f-1} < j_g < j_{g-1} < k_h$ |
| 1 | $v^2$ | $v^2$ | $v$ | $v^2$ | $v$ | $v$ | $j_{g-1}, k_{h-1} < i_{f-1} < j_g < k_h < i_f$ |
| 1 | $v^2$ | $v^2$ | $v$ | $v^2$ | $v$ | $v$ | $k_{h-1} < j_{g-1} < i_{f-1} < j_g < k_h < i_f$ |
| 1 | $v^2$ | $v^2$ | $v$ | $v^2$ | $v$ | $v$ | $j_{g-1}, k_{h-1} < i_{f-1} < j_g < k_h < i_f$ |
| $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\beta_{12}$ | $\beta_{13}$ | $\beta_{23}$ | $\gamma_{123}$ |
|-----------|-----------|-----------|-------------|-------------|-------------|--------------|
| $v_1$     | $v_2$     | $v_3$     | $v_1$       | $v_2$       | $v_3$       | $v_1$        |
| $v_1$     | $v_2$     | $v_3$     | $v_1$       | $v_2$       | $v_3$       | $v_1$        |
| $v_1$     | $v_2$     | $v_3$     | $v_1$       | $v_2$       | $v_3$       | $v_1$        |
| $v_1$     | $v_2$     | $v_3$     | $v_1$       | $v_2$       | $v_3$       | $v_1$        |
| $v_1$     | $v_2$     | $v_3$     | $v_1$       | $v_2$       | $v_3$       | $v_1$        |
| $v_1$     | $v_2$     | $v_3$     | $v_1$       | $v_2$       | $v_3$       | $v_1$        |
| $v_1$     | $v_2$     | $v_3$     | $v_1$       | $v_2$       | $v_3$       | $v_1$        |
| $v_1$     | $v_2$     | $v_3$     | $v_1$       | $v_2$       | $v_3$       | $v_1$        |

These are examples of table data.
| $\gamma^{123}$ | $\alpha^3$ | $\alpha^2$ | $\alpha^1$ | $\beta^{23}$ | $\beta^{13}$ | $\gamma^{123}$ |
|---|---|---|---|---|---|---|
| $f$ | $h$ | $v^2$ | $v^3$ | $v$ | $v^2$ | $v$ |
| $h$ | $g$ | $f$ | $f$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v^2$ | $v^3$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v$ | $v^2$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v$ | $v^2$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v$ | $v^2$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v$ | $v^2$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v$ | $v^2$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v$ | $v^2$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v$ | $v^2$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v$ | $v^2$ | $v$ | $v^2$ | $v$ |
| 1 | 1 | $v$ | $v^2$ | $v$ | $v^2$ | $v$ |

$\gamma^{123}$: $\gamma^{123} = k_{h_1, j_{g-1}} < i_{f-1} < i_f < j_g < k_h$.

$\beta^{23}$: $\beta^{23} = k_{h_1, j_{g-1}} < i_{f-1} < i_f < j_g < k_h$.

$\beta^{13}$: $\beta^{13} = k_{h_1, j_{g-1}} < i_{f-1} < i_f < j_g < k_h$.

$\gamma^{123}$: $\gamma^{123} = k_{h_1, j_{g-1}} < i_{f-1} < i_f < j_g < k_h$.
When we act on the elements of \( \mathfrak{M} \) by a permutation \( \pi \in \mathfrak{S}_r \), we reorder the multicharge. Although the decomposition matrices depend on the ordering of the multicharge, we do have the following result.

**Corollary A.5.** The decomposition numbers for indecomposable core blocks of weight 3 when \( r = 4 \) are independent of the characteristic of the field and the ungraded decomposition numbers are either 0 or 1.

We would like to be able to give a closed formula, similar to that given in Theorem I for these decomposition numbers. In all the cases that we have looked at, if \( d_{\mu \lambda}(v) \neq 0 \) then either two components of \( \mu \) and \( \lambda \) are equal or two components of \( \tilde{\lambda} \) and \( \tilde{\mu} \) are equal. Although we have not been able to check all (or even most) cases, it seems plausible that a closed formula exists and may even extend to decomposition numbers for core blocks whose weight graphs are trees in higher levels.

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