DYNAMICS OF LARGE BOSON SYSTEMS WITH ATTRACTIVE INTERACTION AND A DERIVATION OF THE CUBIC FOCUSING NLS IN $\mathbb{R}^3$

JACKY JIA WEI CHONG

Abstract. We consider a system of $N$ bosons where the particles experience a short range two-body interaction given by $N^{-1}v_N(x) = N^{3\beta - 1}v(N^\beta x)$ where $v \in C_c^\infty(\mathbb{R})$, without a definite sign on $v$, for some range of the scaling parameter $\beta$. We extend the results of [GM13b, Kuz17] regarding the second-order correction to the mean-field evolution of systems with repulsive interaction to systems with attractive interaction for $0 < \beta < \frac{1}{2}$. The two key ingredients used in the extension are the proofs of the uniform global well-posedness of solutions to a family of Hartree-type equations and the corresponding uniform $L^\infty$-decay estimates of the solutions. Inspired by the recent works of X. Chen & J. Holmer in [CH16a, CH16c], we also provide a derivation of the focusing nonlinear Schrödinger equation (NLS) in 3D from the many-body boson system and its rate of convergence toward mean-field. In particular, we give two derivations of the focusing NLS, one by the Fock space method and the other via a method introduced by Pickl. For the latter method, we prove a regularity condition for the solution to the NLS that, subsequently, feeds into the works of P. Pickl in [Pic11, Pic10] which gives a rigorous derivation of the focusing cubic NLS in $\mathbb{R}^3$ for $0 < \beta < \frac{1}{5}$. The techniques used in this article are standard in the literature of dispersive PDEs. Nevertheless, the derivation of the focusing NLS had only previously been considered in [CH16a, CH16c, CH17] for the 1D & 2D cases and conditionally answered for the 3D case in [Pic10].

1. Introduction

Bose-Einstein condensation is a physical phenomenon which occurs when a dilute gas\(^1\) of indistinguishable integer-spin particles\(^2\) undergoes extreme cooling. Under the extreme condition, the gas of particles experiences a phase transition in which a macroscopic fraction of the particles coalesce into a single quantum state, the ground state.

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\(^1\)Dilute means the average particle separation distance is much bigger than the scattering length.

\(^2\)In relativistic quantum mechanics, bosons are classified, by the Spin-Statistic theorem, to be particles with integer intrinsic spin. However, in this paper, we work in the realm of non-relativistic quantum physics where the bosonic property of particles is captured by the symmetric structure of the wave-function.
Historically, Bose-Einstein condensate (BEC) was predicted by Albert Einstein in the 1920s long before its first realization in atomic gases by a series of experiments conducted on vapors of alkali metals in 1995. On June of 1995, for the first time, the JILA group led by Eric Cornell and Carl Wieman at the University of Colorado at Boulder NIST-JILA lab was able to achieve a condensation limit in a gas of rubidium, $^{87}\text{Rb}$, inside a magnetic trap by using a combination of laser cooling and evaporative cooling techniques to lower the temperature of the substance to a mere 20 nK (nano-kelvin). Shortly after the publication of the results of the JILA group, a group at MIT led by Wolfgang Ketterle was able to exhibit BEC using sodium, $^{23}\text{Na}$, but with many times more atoms then the experiment by the JILA group. In effect, Ketterle’s group also demonstrated and measured many important properties of BEC. Subsequently, the demonstrations by the two groups greatly increased both the experimental and theoretical activities in the field of large boson systems.

In recent years, many mathematicians have made vigorous attempts at tackling the theory of many-body quantum mechanical systems in order to understand the evolution problem of condensates in the absolute zero temperature regime. One of the difficulties of modeling BEC is, in fact, the size of the condensate. Since a system of $N$ interacting bosons is modeled by a symmetric wave-function with $3N + 1$ variables, then the study of the solution to the evolution of the wave-function becomes impractical when $N$ is large, say $N \sim 10^3$. Thus, it is favorable to try to find a more effective description for the dynamics of the large interacting boson system in a lower dimension space. Informally, we would like to project the original linear PDE of $3N + 1$ variables to a nonlinear PDE with much lesser variables, say $3+1$. The desire to find an effective description for the $N$-body boson dynamics leads to the studies of mean-field approximation to the evolution of large particle systems.

Despite the simplicity of the idea of trying to find an effective description for the dynamics of a large particle system, a rigorous justification for the the mean-field approximation is rather involved. In particular, the problem of finding an effective description for the evolution of BEC was only first studied systematically in a series of papers by Erdős and Yau, Elgart, Erdős, Schlein and Yau, and Erdős, Schlein and Yau, [EY01, EESY06, ESY06, ESY07a, ESY07b, ESY10, ESY09]. Using the formalism of quantum BBGKY hierarchy, they were able to extract the mean-field limit as the
number of particles tends to infinity and show that the limit satisfies the defocusing cubic NLS. Furthermore, this series of works drew the attention of the PDE community. Due to the complexity of the historical development of the studies of infinite hierarchies from the point of view of dispersive PDEs, we refer the interested reader to a list of articles \[KM08, KSSS11, CPT11, CPT14, CHe12, CHe13, CH16a, CH16c, CH17, CH16b, CH15, GSS14, Soh16, Soh15\] for a more in-depth view of the subject; the list is not intended to be comprehensive overview of the subject.

Let us briefly discuss the mathematical setting for our problem. Consider a $N$-body boson system in $\mathbb{R}^3$ whose dynamics is governed by the $N$-body linear Schrödinger equation

$$\frac{1}{i} \frac{\partial}{\partial t} \Psi_N = \left( \sum_{j=1}^{N} \Delta_{x_j} - \frac{1}{N} \sum_{i<j} v_N(x_i - x_j) \right) \Psi_N$$

with factorized initial datum, i.e. $\Psi_N(0, x_1, \ldots, x_N) = \prod_{j=1}^{N} \phi_0(x_j)$. This setting provides us with an appropriate model for studying the evolution of BEC.

Formally, we say a $N$-body boson system exhibits the complete BEC property provided the one-particle marginal density operator, $\gamma^{(1)}_N$, factorizes in trace norm as $N \to \infty$, i.e.

$$\text{Tr} \left| \gamma^{(1)}_N - \left| \phi \right\rangle \left\langle \phi \right| \right| \to 0 \quad \text{as} \quad N \to \infty$$

for some $\phi$. Let us note the kernel of $\gamma^{(1)}_N$ is given by

$$\gamma^{(1)}_N(x, x') = \int d\mathbf{x} \Psi_N^*(x, \mathbf{x}) \Psi_N(x', \mathbf{x}) \quad x, x' \in \mathbb{R}^3 \quad \text{and} \quad \mathbf{x} \in \mathbb{R}^{3(N-1)}.$$

Indeed, using this definition, one can show that the evolution of BEC can be effectively approximated by a one-body mean-field equation; see \[EY01, EESY06, ESY06, ESY07a, ESY10, ESY09\].

Then a natural question one could ask is whether the above statement holds true in state space. More specifically, if we start with a factorized initial condition then is it true that under time evolution the many-body wavefunction can be approximated by

$$\Psi_N(t, x_1, \ldots, x_N) \sim \prod_{j=1}^{N} \phi(t, x_j) e^{i\chi(t)}$$

in $L^2(\mathbb{R}^{3N})$ as $N \to \infty$ for some phase $\chi(t)$. Unfortunately, the answer is negative. Thus, one of our goals is to see what modification is required in order for us to get a better understanding of the structure of $\Psi_N$ in state space.

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6It should be noted that the ground state of the system, in general, cannot be approximated by a factorized state. However,

7Cf. chapter 1.3 and 7 of \[LSSY05\].
2. Earlier Results and Main Statements

2.1. Background and Earlier Results. This section provides a brief summary of the results obtained in [GM13a, GM13b, GM17, Kuz15] along with some background materials for the convenience of the reader.

Let us introduce the mathematical setting for our article. Our one-particle base space, denote by \( h := L^2(\mathbb{R}^3, dx) \), is a complex separable Hilbert space endowed with the inner product \( \langle \cdot, \cdot \rangle_h \) which is linear in the second variable and conjugate linear (or anti-linear) in the first variable.\(^8\)

We define the \textit{bosonic Fock space over} \( h \) to be the closure of \( \mathcal{F}_s(h) = \mathcal{F}_s := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \text{Sym}(h^{\otimes n}) \) with respect to the norm induced by the Fock inner product

\[
\langle \varphi, \psi \rangle_{\mathcal{F}} = \overline{\varphi_0} \psi_0 + \sum_{n=1}^{\infty} \langle \varphi_n, \psi_n \rangle_h^{\otimes n}
\]

where \( \varphi = (\varphi_0, \varphi_1, \ldots), \psi = (\psi_0, \psi_1, \ldots) \in \mathcal{F}_s(h) \). For convenience, we shall refer to \( \mathcal{F}_s \) simply as the Fock space henceforth. The \textit{vacuum}, denote by \( \Omega \), is defined to be the Fock vector \((1, 0, 0, \ldots) \in \mathcal{F}_s \).

For every field \( \phi \in h \) we can define the associated creation and annihilation operators on \( \mathcal{F}_s \), denote respectively by \( a^\dagger(\phi) \) and \( a(\bar{\phi}) \), as follow

\[
(a^\dagger(\phi)\psi)_n(x_1, \ldots, x_n) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \phi(x_j) \psi_{n-1}(x_1, \ldots, \hat{x}_j, \ldots, x_n) \tag{3a}
\]

\[
(a(\bar{\phi})\psi)_n(x_1, \ldots, x_n) := \sqrt{n+1} \int dx \bar{\phi}(x) \psi_{n+1}(x, x_1, \ldots, x_n). \tag{3b}
\]

with the property that \( a(\phi)\Omega = 0 \). We can also define the corresponding creation and annihilation distribution-valued operators associated to (3a) and (3b), denote by \( a^\dagger_x \) and \( a_x \), as follow

\[
(a^\dagger_x \psi)_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(x - x_j) \psi_{n-1}(x_1, \ldots, \hat{x}_j, \ldots, x_n) \tag{4a}
\]

\[
(a_x \psi)_n := \sqrt{n+1} \psi_{n+1}(x, x_1, \ldots, x_n). \tag{4b}
\]

In short, we have the relations

\[
a(\phi) = \int dx \{ \phi(x) a^\dagger_x \} \quad \text{and} \quad a(\bar{\phi}) = \int dx \{ \bar{\phi}(x) a_x \}.
\]

Let us note that the creation and annihilation operators \( a(\bar{\phi}) \) and \( a^\dagger(\phi) \) associated to the field \( \phi \) are unbounded, densely defined, closed operators.\(^8\)

\(^8\)This is the physicists’ inner product.
Moreover, one can easily verify, formally, \((a_x^\dagger, a_x)\) satisfy the canonical commutation relation (CCR): 
\[ [a_x, a_y^\dagger] = \delta(x - y), \quad [a_x, a_y] = [a_x^\dagger, a_y^\dagger] = 0, \]
and the number operator defined by
\[ \mathcal{N} := \int dx \ a_x^\dagger a_x \tag{5} \]
is a diagonal operator on \(\mathcal{F}\) that counts the number of particles in each sector.

As mention in the introduction we are interested in studying the time evolution of the coherent state in Fock space. Before doing so, let us define the initial datum, the coherent state and the Fock Hamiltonian. For each \(\phi \in \mathfrak{h}\), we associate the corresponding unique closure of the operator
\[ A(\phi) = a(\bar{\phi}) - a^\dagger(\phi) \tag{6} \]
then the Weyl operator\(^{10}\) is defined to be
\[ e^{-\sqrt{\mathcal{N}}A(\phi)}. \tag{7} \]

Let us note the operator \(A(\phi)\) is a skew-Hermitian unbounded operator which means the corresponding Weyl operator is unitary. The coherent state associated to \(\phi\) is given by
\[ \psi(\phi) := e^{-\sqrt{\mathcal{N}}A(\phi)}\Omega. \tag{8} \]
Using the Baker-Campbell Hausdorff formula, one can show
\[ e^{-\sqrt{\mathcal{N}}A(\phi)}\Omega = (\ldots, c_n\phi^\otimes n, \ldots) \quad \text{where} \quad c_n = \left( e^{-N\|\phi\|^2} N^n / n! \right). \]
For a fixed \(N \in \mathbb{N}\), we defined the Fock Hamiltonian associated to \(N\), denote by \(H_N\), to be the diagonal operator on the Fock space given by
\[ (H_N\psi)_n = \left( \sum_{j=1}^n \Delta x_j - \frac{1}{N} \sum_{i<j}^n v_N(x_i - x_j) \right) \psi_n =: H_{N,n}\psi_n. \]

\(^{9}\)The reader should note for any \(f, g \in \mathfrak{h}\) the CCR for \(a^\dagger(f)\) and \(a(g)\) are not well defined since there are domain issues that need to be resolved for the given unbounded operators. For an exotic example of an ill-defined commutator of unbounded operators, we refer the reader to Chapter VIII.5 of [RS80].

\(^{10}\)To avoid the unfavorable technicality associated with the unbounded natural of our creation and annihilation operators one often choose to work with the corresponding Weyl algebra, the \(C^*\)-algebra generated by the exponential of \(A(\phi)\) where \(\phi \in \mathfrak{h}\) (cf. chapter 9 of [DG13] and chapter 5.2 of [BR13]).
where \( v_N(x) = N^{3\beta} v(N^\beta x) \). Rewrite \( H_N \) using creation and annihilation operators we get

\[
H_N := H_1 - \frac{1}{N} V
\]

\[
H_1 := \int dx dy \left\{ \Delta_x \delta(x - y) a_x^\dagger a_y \right\} \text{ and}
\]

\[
V := \frac{1}{2} \int dx dy \left\{ v_N(x - y) a_x^\dagger a_y^\dagger a_y a_x \right\}.
\]

In light of (9a), we are interested in the solution to the following Cauchy problem in Fock space

\[
\frac{1}{i} \frac{\partial}{\partial t} \psi = H_N \psi \quad \text{with initial datum} \quad \psi_0 = e^{-\sqrt{NA}(\phi_0)} \Omega
\]

which we shall write it as

\[
\psi_{\text{exact}} = e^{itH_N} e^{-\sqrt{NA}(\phi_0)} \Omega.
\]

An important fact to note about the Fock Hamiltonian is its action on the \( N \)th sector of the Fock space. There, the Fock Hamiltonian acts as a mean-field Hamiltonian for the \( N \)-particle system, that is

\[
(H_N \psi)_N = \left( \sum_{j=1}^{N} \Delta_{x_j} - \frac{1}{N} \sum_{i < j}^{N} v_N(x_i - x_j) \right) \psi_N =: H_{mf} \psi_N.
\]

Since the \( N \)th coefficient \( c_N \) could be approximated by \( N^{-1/4} \) using Stirling’s formula and the coherent state is a \( N \)-tensor of \( \phi \) in the \( N \) sector, then heuristically we see how by understanding the evolution of the coherent state we would also understand the mean-field evolution of the \( N \)-particle factorized state.

Based on the earlier works of Hepp and Ginibre & Velo in [Hep74, GV79a, GV79b], Rodnianski and Schlein in [RS09] studies the one-particle Fock marginal, which is defined as follows: for every \( \psi \in \mathcal{F}_s \) the one-particle Fock marginal of \( \psi \), denote by \( \Gamma^{(1)}(\psi) \), is a positive trace class integral operator on \( \mathfrak{h} \) with kernel given by

\[
\Gamma^{(1)}(x, y) = \frac{\langle \psi, a_x^\dagger a_y \psi \rangle_F}{\langle \psi, \mathcal{N} \psi \rangle_F}.
\]

They were about to show that the one-particle Fock marginal with an initial coherent state converges to the Hartree dynamics in trace norm for the case \( \beta = 0 \). Furthermore, they were also able to obtain a rate of convergence

\[
\text{Tr} \left| \Gamma^{(1)}_{N,t} - \langle \phi_t \rangle \langle \phi_t \rangle \right| \lesssim e^{Kt} \frac{1}{N}
\]

where \( \Gamma^{(1)}_{N,t} \) denotes the one particle Fock marginal for \( \psi_{\text{exact}} \) and \( \phi_t \) satisfies the Hartree equation. Later, Kuz in [Kuz15] improved the estimate
substantially in time and obtain the estimate

$$\text{Tr} \left[ \Gamma_{N,t}^{(1)} - |\phi_t\rangle\langle\phi_t| \right] \lesssim \frac{t}{N}. \quad (14)$$

Unlike the approach of Rodnianski and Schlein which uses the mean-field approximation of the form

$$\psi_{mf} = e^{-\sqrt{N}A(\phi_t)}\Omega = e^{-\sqrt{N}A(t)}\Omega, \quad (15)$$

Kuz uses the method of second-order correction introduced in the works of Grillakis, Machedon and Margetis in [GMM10, GMM11, GM13a] to establish (14), which relies on tracking the exact dynamics of the evolution of the coherent state in Fock space.

To track the exact dynamics in Fock space, we need to introduce the pair excitation function, $k(x,y) = k(y,x)$, and its corresponding quadratic operator $B(k)$ with kernel

$$B(k_t)(x,y) = \mathcal{B}(t)(x,y) = \int dx dy \left\{ k(t,x,y)a_xa_y - k(t,x,y)a_x^+a_y^+ \right\}. \quad (16)$$

From the pair excitation we concoct a new approximation scheme, which is a second order correction to the mean field (15), given by

$$\psi_{\text{approx}} = e^{i\chi(t)} e^{-\sqrt{N}A(t)} e^{-\mathcal{B}(t)} \Omega \quad (17)$$

where $\chi(t)$ is some phase factor to be determined. With some appropriate choice of evolution equations for $\phi$ and $k$ we will later see that (17) will indeed allow use to track the exact dynamics of the evolution of coherent state in Fock space.

Incidentally one could show via a Lie algebra isomorphism argument established in [GM13a, GM13b, GM17] that the evolution of $k$ could be described by some nonlinear evolution equation of

$$\text{sh}(k) := k + \frac{1}{3!} k \circ \bar{k} \circ k + \frac{1}{5!} k \circ \bar{k} \circ k \circ \bar{k} \circ k + \ldots$$

$$\text{ch}(k) := \delta + \frac{1}{2!} \bar{k} \circ k + \frac{1}{4!} \bar{k} \circ k \circ \bar{k} \circ k + \ldots$$

where $\circ$ denotes the composition of operators. Moreover, in [GM13b], Grillakis and Machedon show by using a specific coordinate the nonlinear equation of the pair excitation could be express as a system of coupled linear equations in $\text{sh}(2k)$ and $\text{ch}(2k)$.

\[In the mathematical physics literature, $e^B$ is called the infinite dimensional Segal-Shale-Weil Representation of the double cover of the group of symplectic matrices of integral operators. The elements of the corresponding $C^*$-algebra are called Bogoliubov transformations (cf. chapter 4 of [Fol89] and chapter 11 of [DG13]).\]
Let us introduce some notation to help us compactly write out the evolution equations for φ and k
\[
\begin{align*}
g_N(t, x, y) & := - \Delta_x \delta(x - y) + (v_N * |\phi|^2)(t, x) \delta(x - y) \\
& \quad + v_N(x - y) \bar{\phi}(t, x) \phi(t, y) \\
m_N(t, x, y) & := - v_N(x - y) \phi(t, x) \phi(t, y) \\
S(s) & := \frac{1}{i} \partial_t s + g_N^T \circ s + s \circ g_N \quad \text{(Schrödinger-type operator)} \\
W(p) & := \frac{1}{i} \partial_t p + [g_N^T, p] \quad \text{(Wigner-type operator)}
\end{align*}
\]
then the desired evolution equations of φ and k are given by
\[
\begin{align*}
\frac{1}{i} \partial_t \phi - \Delta_x \phi + (v_N * |\phi|^2) \phi & = 0 \quad \text{(Hartree-type equation)} \quad (18a) \\
S(\text{sh}(2k)) & = m_N \circ \text{ch}(2k) + \text{ch}(2k) \circ m_N \quad (18b) \\
W(\text{ch}(2k)) & = m_N \circ \text{sh}(2k) - \text{sh}(2k) \circ m_N. \quad (18c)
\end{align*}
\]
The system equations (18) is referred to as the **uncoupled system** in contrast to the coupled system introduced in [GM13a, GM17] where the equation for φ and the pair excitation equations are coupled.

Now, let us summarize the results in [GM13b, Kuz15], which built on earlier works by Grillakis, Machedon and Margetis in [GMM10, GMM11].

**Theorem 2.1.** Let \( v \in C^1_c(\mathbb{R}^3) \) and \( v \geq 0 \). Assume φ and k satisfy (18) with initial conditions \( \phi(0, \cdot) = \phi_0 \in L^2(\mathbb{R}^3) \cap W^{m,1}(\mathbb{R}^3) \) for some sufficiently large \( m \) and \( k(0, \cdot) = 0 \). If \( \psi_{\text{exact}} \) and \( \psi_{\text{approx}} \) are defined by (11) and (17) respectively, then we have the following estimate
\[
\| \psi_{\text{exact}}(t) - \psi_{\text{approx}}(t) \|_{\mathcal{F}} \lesssim \frac{(1 + t) \log^4(1 + t)}{N^{(1-\beta)/2}} \quad (19)
\]
provided \( 0 < \beta < \frac{1}{4} \). Moreover, if \( (\partial_t \text{sh}(2k))(0, \cdot) \) is sufficiently regular, then for any \( \epsilon > 0 \) and \( j \) a positive integer, we have
\[
\| \psi_{\text{exact}}(t) - \psi_{\text{approx}}(t) \|_{\mathcal{F}} \lesssim t^{\frac{4j+3}{2}} \log^6(1 + t) \cdot \left\{ \begin{array}{ll}
N^{-1/2+\beta(1+\epsilon)} & \frac{1}{3} \leq \beta < \frac{2j}{(1-2\epsilon+4j)}, \\
N^{-3+\beta \frac{j}{2}+j(1+\epsilon)} & \frac{2j}{(1-2\epsilon+4j)} \leq \beta < \frac{1+2j}{3+4j}.
\end{array} \right. \quad (20)
\]

**Remark 2.2.** It should be noted that the assumption \( (\partial_t \text{sh}(2k))(0, \cdot) \) must be sufficiently regular imposes a restriction on the form of the initial condition; in particular, \( k(0, \cdot) \) cannot be zero. Due to the restriction, we could not choose the coherent state as our initial condition since \( e^{-\sqrt{N}A_\theta e^{-B_\theta} \Omega} \) is a coherent state if and only if \( k(0, \cdot) = 0 \).

2.2. **Main Statements.** One of the main purposes of this article is to extend the results in [GM13b, Kuz15] to the case of arbitrary \( v \in C^0_\infty \) with sufficiently small \( L^1 \)-norm allowing non-positive \( v \). Let us state the first main statement.
**Theorem 2.3.** Let \( v \in C_0^\infty(\mathbb{R}^3) \). Assume \( \phi \) and \( k \) satisfy (18) with initial conditions \( \phi_0 \in L^2(\mathbb{R}^2) \cap W^{m,1}(\mathbb{R}^3) \) for some sufficiently large \( m \) and sufficiently small \( \dot{H}^{1/2}_x \)-norm, depending on \( v \), and \( k(0, \cdot) = 0 \). If \( \psi_{\text{exact}} \) and \( \psi_{\text{approx}} \) are defined by (17) and (11) respectively, then we have the following estimate

\[
\| \psi_{\text{exact}}(t) - \psi_{\text{approx}}(t) \|_F \lesssim \frac{t}{N^{(1-3\beta)/2}}
\]

provided \( 0 < \beta < \frac{1}{7} \). Moreover, if \( (\partial_t \text{sh}(2k))(0, \cdot) \) is sufficiently regular, then for any \( \epsilon > 0 \) and \( j \) a positive integer, we have

\[
\| \psi_{\text{exact}}(t) - \psi_{\text{approx}}(t) \|_F \lesssim t^{\frac{1}{2j+3}} \log^6(1+t) \cdot \begin{cases} N^{-1/2+\beta(1+\epsilon)} & 0 < \beta < \frac{2j}{(1-2\epsilon+4j)}, \\ N^{3+7\beta/(j-1)(-1+2\beta)} & \frac{2j}{(1-2\epsilon+4j)} \leq \beta < \frac{1+2j}{3+4j}. \end{cases}
\]

**Remark 2.4.** Let us note that there is a trade off between size of the data \( \phi_0 \) and size of the interaction potential \( v \). Due to the nature of our proof, if we want to assume \( v \) to be large, then we need to restrict the size the \( \dot{H}^{1/2}_x \) norm of \( \phi_0 \), and vice versa.

**Remark 2.5.** A similar result was obtained in [NM17] for the case of repulsive interaction. As stated in remark 4 in [NM17], their method also extends to the case of attractive interaction provided the uniform in \( N \) well-posedness and decay estimates for the corresponding Hartree equations hold, which we will show in the next section.

**Remark 2.6.** The second estimate in Theorem 2.3 could be improved. In particular, we can get rid of the logarithmic terms. However, to keep the organization of the article simple, we decided to keep the logarithmic terms. Nevertheless, we have included a proof of how to remove the logarithmic terms in §4.

The second purpose of the article is to derive the focusing cubic NLS in \( \mathbb{R}^3 \) from a many-body boson system as in [CH16a, CH17, CH16c]. For this purpose, we assume \( v \leq 0 \), i.e. the interaction is attractive. In this case, we have the following statement.

**Theorem 2.7.** (Factorized Initial Condition) Assume \( v \in C_0^\infty(\mathbb{R}^3) \) and \( v \leq 0 \). Suppose \( \Psi_N(t, x) \) solves the initial value problem

\[
\frac{1}{i} \partial_t \Psi_N(t, x) = H_{mf} \Psi_N(t, x), \quad \Psi_N(0, \cdot) = \phi_0 \otimes \phi_0
\]

where \( \phi_0 \) satisfies the same conditions as in Theorem 2.3 and \( \| \phi_0 \|_{L_x^2} = 1 \). Denote the one-particle density associated to \( \Psi_N(t, x) \) by \( \gamma^{(1)}_{N,t} \). Then we have the estimate

\[
\text{Tr} \left| \gamma^{(1)}_{N,t}(t, \cdot) - |\phi_t\rangle \langle \phi_t| \right| \lesssim N^\delta
\]

for some \( \delta < 0 \) provided \( 0 < \beta < \frac{1}{6} \).
Remark 2.8. The reader should note that Theorem 2.7 only addresses the derivation of the focusing NLS for a system of weakly-interacting dense Bose gas since $\beta \in (0, \frac{1}{6})$.

Remark 2.9. As pointed out by the referee, the case at hand deals with the situation where (18a) does not exhibit soliton solutions. C.f. Remark 3.3 and Remark 4.5.

3. Estimates for the Solution to the Hartree Equation

Let us consider the following family of Hartree-type PDEs

$$\frac{1}{i} \partial_t \phi - \Delta_x \phi + (v_N * |\phi|^2) \phi = 0$$

$$\phi(0, \cdot) = \phi_0 \quad \phi_0 \in H^{3/2}(\mathbb{R}^3)$$

where $v_N(x) = N^{3\beta}v(N^\beta x)$ for $0 \leq \beta \leq 1$ and $v \in C_0^\infty(\mathbb{R}^3)$ is not necessary nonnegative. In this section we prove the uniform in $N$ well-posedness of the Hartree-type equation for small data and the corresponding decay estimates.

3.1. Global Wellposedness. In this subsection we prove the uniform in $N$ global well-posedness of (23) assuming small data. Let us recall the Strichartz norm. We said a pair of numbers $(q, r)$ is admissible provided $q, r \geq 2$ and

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}.$$

Then the Strichartz norm is defined by

$$\| \phi \|_{S^0} := \sup_{(q,r) \text{ admissible}} \| \phi \|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^3)}.$$

Proposition 3.1 (a-priori estimates). Let $\phi$ be a solution to (23), then we have the estimate

$$\| \nabla^{1/2} \phi \|_{S^0} \lesssim \| \phi_0 \|_{\dot{H}^{3/2}_x} + \| v \|_{L^1_x} \| \nabla^{1/2} \phi \|_{S^0}^3$$

which is independent of $N$. Moreover, if $\| \phi_0 \|_{\dot{H}^{3/2}_x}$ is sufficiently small then we obtain the estimate

$$\| \nabla^{1/2} \phi \|_{S^0} \lesssim 1$$

which depends only on $\| \phi_0 \|_{\dot{H}^{3/2}_x}$ and independent of $N$. Similar estimates holds for time and higher spatial derivatives, that is

$$\| \partial_t^m \nabla^s_x \phi \|_{S^0} \lesssim 1$$

where the estimate only depends on $m, s$ and the initial datum.

Proof. Similar to the local estimate, we begin by differentiating (23)

$$\frac{1}{i} \partial_t \nabla^{1/2} \phi - \Delta_x \nabla^{1/2} \phi + \nabla^{1/2}((v_N * |\phi|^2) \cdot \phi) = 0$$
where
\[ \nabla_x^{1/2}((v_N * |\phi|^2) \cdot \phi) = (v_N * |\phi|^2) \cdot \nabla_x^{1/2}\phi + (v_N * \nabla_x^{1/2}|\phi|^2) \cdot \phi \]
+ “lower order” terms.

Applying the $L^2L^{6/5}_x$ endpoint Strichartz estimate of \[KT98\] and the fractional Leibniz rule, we obtain the following estimate
\[
\| \nabla_x^{1/2}\phi \|_{S^0} \lesssim \| \phi \|_{\dot{H}^{1/2}_x} + \| v_N * |\phi|^2 \|_{L^2_x L^3_t} \| \nabla_x^{1/2}\phi \|_{L^\infty_t L^2_x} \\
+ \| v_N * \nabla_x^{1/2}|\phi|^2 \|_{L^2_x L^3_t} \| \phi \|_{L^\infty_t L^6_x}.
\]

For the first forcing term we have the estimate
\[
\| v_N * |\phi|^2 \|_{L^2_x L^3_t} \| \nabla_x^{1/2}\phi \|_{L^\infty_t L^2_x} \lesssim \| v \|_{L^2_t L^6_x} \| \phi \|_{L^4_t L^6_x} \| \nabla_x^{1/2}\phi \|_{L^\infty_t L^2_x} \\
\lesssim \| v \|_{L^2_t} \| \nabla_x^{1/2}\phi \|_{L^4_t L^6_x} \| \nabla_x^{1/2}\phi \|_{L^\infty_t L^2_x} \\
\lesssim \| v \|_{L^2_t} \| \nabla_x^{1/2}\phi \|_{S^0}^3.
\]

The other term can be estimated in a similar fashion. Moreover, estimate (20) follows from the observation
\[
\| \partial_t^m \nabla_x^s \phi \|_{S^0} \lesssim \| \partial_t^m \nabla_x^s \phi \|_{L^2_x} + \| v \|_{L^2_t} \| \partial_t^m \nabla_x^s \phi \|_{S^0} \| \nabla_x^{1/2}\phi \|_{S^0}^2.
\]

\[ \square \]

As an immediate corollary of Proposition 3.1, we have

\textbf{Corollary 3.2} (Uniform in $N$ global well-posedness). Let $v \in C^\infty_c(\mathbb{R}^3)$. Then there exists $\varepsilon = \varepsilon(\| v \|_{L^1_x}) > 0$, independent of $N$, such that for any $\varphi_0 \in \{ \varphi \in \dot{H}^{1/2}_x \ | \ \| \varphi \|_{\dot{H}^{1/2}_x} < \varepsilon \}$ there exists a unique solution to \[23\] with initial data $\varphi_0$ satisfying $\varphi_t \in C([0, \infty) \to \dot{H}^{1/2}_x) \cap S^0$.

\textbf{Remark 3.3.} In this paper, we always consider sufficiently smooth initial data. In particular, we could take any $\phi_0 \in H^1_x$ and obtain a uniform in $N$ local well-posedness of solutions to the family of Hartree-type equations. Of course, the tradeoff is that we can only have uniform in $N$ local well-posedness for short time. C.f. chapter 3.3 Proposition 3.19 in \[Tao06\].

3.2. \textbf{Decay Estimates.} In this subsection we prove the uniform in $N$ decay estimates for $\phi_t$ following the approach in \[GM13\], which is in the spirit of \[LS78\]. Before we begin let us make a note on the notation used in this section. The notation $\alpha \pm \varepsilon$ means $\alpha \pm \varepsilon$ for some fixed $0 < \varepsilon \ll 1$.

\textbf{Proposition 3.4.} Suppose $\phi_0 \in W^{k,1}_x$ for some sufficiently large $k$. Let $\phi$ be a solution to \[23\] with small $\dot{H}^{1/2}_x$ data $\phi_0$. Then we have the decay estimate
\[
\| \phi(t, \cdot) \|_{L^\infty_x} \lesssim \frac{1}{1 + t^{3/2}}
\]
which only depends on $\| \phi_0 \|_{W^{k,1}_x}$ and independent of $N$.  

Let us first prove the following lemmas.

**Lemma 3.5.** Assuming the same conditions as in Proposition 3.4. Then \( \| \phi(t, \cdot) \|_{L^\infty_x} \to 0 \) as \( t \to \infty \).

**Proof of lemma 3.5.** By Proposition 3.1 and Sobolev embedding, we have the estimates
\[
\| \phi \|_{L^{10/3}_{t, x}} \leq C \quad \text{and} \quad \| \phi \|_{L^{\infty}_{t, x}(\mathbb{R}^3 \times \mathbb{R}^3)} \lesssim \| \partial_t \nabla_x \phi \|_{L^3_t L^6_x} \leq C.
\]
Hence by interpolation we have
\[
\| \phi \|_{L^p((n, n+1); L^p_x)} \leq \| \phi \|_{L^{10/3}_{t, x} L^{10/3}_{x}} \| \phi \|_{L^\infty_{t, x} L^n_x}^{1-p}
\]
\[
\leq \| \phi \|_{L^{10/3}_{t, x} L^{10/3}_{x}} \| \partial_t \nabla_x \phi \|_{L^2_{t} L^6_x} \to 0
\]
as \( n \to \infty \) for all \( 10/3 < p < \infty \). Letting \( p \to \infty \) yields the desired result.

**Remark 3.6.** The slight analytic gymnastic is a consequence of the fact that we do not have the endpoint Sobolev estimate.

**Lemma 3.7.** Assuming the same conditions as in Proposition 3.4. There exist \( k \in L^1([0, \infty)) \) and \( \delta > 0 \) such that
\[
\| e^{i(t-s)\Delta} (v_N | \phi|^2 \cdot \phi(s)) \|_{L^\infty_x} \leq k(t-s) \| \phi(s, \cdot) \|_{L^2_x}^{1+\delta}. \quad (27)
\]

**Proof of Lemma 3.7.** Using the \( L^\infty L^1 \)-decay and conservation of probability, we have
\[
\| e^{i(t-s)\Delta} (v_N | \phi|^2 \cdot \phi(s)) \|_{L^\infty_x} \lesssim \frac{1}{|t-s|^{3/2}} \| v_N | \phi|^2 \cdot \phi(s) \|_{L^1_x}
\]
\[
\lesssim \frac{1}{|t-s|^{3/2}} \| \phi(s, \cdot) \|_{L^\infty_x} \quad (28)
\]
On the other hand, applying Sobolev embedding, \( L^{3+} L^{3/2-} \) decay estimate and interpolation yields
\[
\| e^{i(t-s)\Delta} (v_N | \phi|^2 \cdot \phi(s)) \|_{L^\infty_x} \lesssim \| \nabla_x e^{i(t-s)\Delta} (v_N | \phi|^2 \cdot \phi(s)) \|_{L^3_x}
\]
\[
\lesssim \frac{1}{|t-s|^{1/2}} \| \nabla_x \phi \|_{L^2_x} \| \phi \|_{L^4_x}^{2}
\]
\[
\lesssim \frac{1}{|t-s|^{1/2}} \| \phi \|_{L^\infty_x}^{3/9} \quad (29)
\]
In the case \( |t-s| < 1 \), we could simply take \( k(t-s) = |t-s|^{1/2} \). In the case \( |t-s| \geq 1 \) we interpolate estimates (28) and (29).

**Proof of Proposition 3.4.** Let \( \phi_0 \) be a test function and write (for \( t > 0 \))
\[
\phi(t) = e^{it\Delta} \phi_0 - i \left[ \int_0^{t/2} + \int_{t/2}^t \right] e^{i(t-\tau)\Delta} (v_N | \phi(\tau)|^2) \phi(\tau) \ d\tau.
\]
Taking the $L^\infty$ norm yields
\[
\| \phi(t) \|_{L^\infty_x} \lesssim \frac{\| \phi_0 \|_{L^1_x}}{t^{3/2}} + \left[ \int_0^{t/2} + \int_{t/2}^t \right] \| e^{i(t-\tau)\Delta} (v_N * |\phi|^2) \phi(\tau) \|_{L^\infty_x} \, d\tau
\]
where the first term is a consequence of the $L^\infty L^1$-decay estimate for the free evolution. For the second term, we apply the $L^\infty L^1$-decay estimate and Young’s convolution estimate to get
\[
\int_0^{t/2} \| e^{i(t-\tau)\Delta} (v_N * |\phi|^2) \phi(\tau) \|_{L^\infty_x} \, d\tau \leq \int_0^{t/2} \left\| (v_N * |\phi|^2) \phi(\tau) \right\|_{L^3} \, d\tau
\]
\[
\lesssim \frac{1}{t^{3/2}} \int_0^{t/2} \| \phi(\tau) \|_{L^\infty_x} \, d\tau.
\]
Lastly, by Lemma 3.7 there exists $k \in L^1([0, \infty])$ and $\delta > 0$ such that
\[
\int_{t/2}^t \| e^{i(t-\tau)\Delta} (v_N * |\phi|^2) \phi(\tau) \|_{L^\infty_x} \, d\tau \lesssim \int_{t/2}^t k(t-\tau) \| \phi(\tau) \|_{L^\infty_x}^{1+\delta} \, d\tau.
\]
Combining all the estimates, we have
\[
\| \phi(t) \|_{L^\infty_x} \lesssim \frac{\| \phi_0 \|_{L^1_x}}{1 + t^{3/2}} + \int_0^{t/2} \frac{\| \phi(\tau) \|_{L^\infty_x}}{1 + t^{3/2}} \, d\tau + \int_{t/2}^t k(t-\tau) \| \phi(\tau) \|_{L^\infty_x}^{1+\delta} \, d\tau
\]
which holds for all $t > 0$.

Since we care about large time behavior we may assume $t \geq 1$. In particular, we get the equivalent estimate
\[
\| \phi(t) \|_{L^\infty_x} \lesssim \frac{\| \phi_0 \|_{L^1_x}}{1 + t^{3/2}} + \int_0^{t/2} \frac{\| \phi(\tau) \|_{L^\infty_x}}{1 + t^{3/2}} \, d\tau + \int_{t/2}^t k(t-\tau) \| \phi(\tau) \|_{L^\infty_x}^{1+\delta} \, d\tau.
\]
(30)

Multiply estimate (30) by $1 + t^{3/2}$ yields
\[
(1 + t^{3/2}) \| \phi(t) \|_{L^\infty_x} \lesssim \| \phi_0 \|_{L^1_x} + \int_0^{t/2} \| \phi(\tau) \|_{L^\infty_x} \, d\tau
\]
\[
+ (1 + t^{3/2}) \int_{t/2}^t k(t-\tau) \| \phi(\tau) \|_{L^\infty_x}^{1+\delta} \, d\tau
\]
\[
\lesssim \| \phi_0 \|_{L^1_x} + \int_0^{t/2} \| \phi(\tau) \|_{L^\infty_x} \, d\tau
\]
\[
+ \sup_{t/2 \leq s \leq t} (1 + s^{3/2}) \| \phi(s) \|_{L^\infty_x}^{1+\delta}
\]
since $k \in L^1([0, \infty])$. Next, by Lemma 3.5, there exists $T > 0$ such that
\[
(1 + t^{3/2}) \| \phi(t) \|_{L^\infty_x} \leq c \| \phi_0 \|_{L^1_x} + c \int_0^{t/2} \| \phi(\tau) \|_{L^\infty_x} \, d\tau
\]
\[
+ \frac{1}{2} \sup_{t/2 \leq s \leq t} (1 + s^{3/2}) \| \phi(s) \|_{L^\infty_x}
\]
whenever \( t \geq 2T \) for some constant \( c > 0 \).

Let \( M(t) := \sup_{T \leq s \leq (1 + s^{3/2})} \| \phi(s) \|_{L^\infty_x} \) and \( C := \sup_{0 \leq s \leq 2T} (1 + s^{3/2}) \| \phi(s) \|_{L^\infty_x} \)
then for all \( t \geq T \) we have either

\[
(1 + t^{3/2}) \| \phi(t) \|_{L^\infty_x} \leq c \| \phi_0 \|_{L^1_x} + c \int_0^{t/2} \frac{M(\tau)}{1 + \tau^{3/2}} \, d\tau + \frac{1}{2} M(t)
\]
or \( M(t) \leq C \). Note, for all \( T < s < t \) we also have the following estimate

\[
(1 + s^{3/2}) \| \phi(s) \|_{L^\infty_x} \leq \max \left( c \| \phi_0 \|_{L^1_x} + c \int_0^{t/2} \frac{M(\tau)}{1 + \tau^{3/2}} \, d\tau + \frac{1}{2} M(t), C \right).
\]

Hence it following

\[
M(t) \leq \max \left( c \| \phi_0 \|_{L^1_x} + c \int_0^{t/2} \frac{M(\tau)}{1 + \tau^{3/2}} \, d\tau + \frac{1}{2} M(t), C \right)
\]
for all \( t \geq T \). Then by Gronwall’s inequality, it follows

\[
M(t) \lesssim \max \left( \| \phi_0 \|_{L^1_x} \exp \left( \int_0^t \frac{d\tau}{1 + \tau^{3/2}} \right), C \right) \lesssim 1.
\]

Thus, we have proved

\[
\sup_{0 \leq s \leq t} (1 + s^{3/2}) \| \phi(s) \|_{L^\infty_x} \lesssim \max(M(t), C) \lesssim 1.
\]

\[\square\]

**Corollary 3.8.** Assume the same conditions as Proposition 3.4 then there exists a constant \( C \) depending only on \( \| \phi_0 \|_{W^{k,1}_x} \) and \( \| \partial_t \phi_0 \|_{W^{k,1}_x} \) such that

\[
\| \partial_t \phi(t, \cdot) \|_{L^\infty_x} \lesssim \frac{1}{1 + t^{3/2}}.
\]

(31)

**Proof.** Begin by taking the time derivative of (23)

\[
\frac{1}{i} \partial_t \partial_t \phi - \Delta_x \partial_t \phi + \partial_t (v_N \ast |\phi|^2) \phi = 0.
\]

Then applying the \( L^\infty L^1 \) decay estimate yields \( (t \geq 1) \)

\[
\| \partial_t \phi(t) \|_{L^\infty_x} \lesssim \left\| \frac{\partial_t \phi(t)}{t^{3/2}} \right\|_{L^1_x} + \int_0^{t/2} \| e^{i(t-\tau)} \partial_t (v_N \ast |\phi|^2) \phi(\tau) \|_{L^\infty_x} \, d\tau
\]

\[
+ \int_{t/2}^t \| e^{i(t-\tau)} \partial_t (v_N \ast |\phi|^2) \phi(\tau) \|_{L^\infty_x} \, d\tau.
\]
For the first integral, we shall apply the $L^\infty L^1$-decay estimate and Proposition 3.4 to get
\[
\int_0^{t/2} \| e^{i(t-\tau)\Delta} \partial_x (v_N * |\phi|^2) \phi(\tau) \|_{L^\infty_x} \, d\tau \lesssim \int_0^{t/2} \frac{\| \partial_x (v_N * |\phi|^2) \phi(\tau) \|_{L^1_x}}{|t - \tau|^{3/2}} \, d\tau
\]
\[
\lesssim \frac{1}{1 + t^{3/2}} \int_0^{t/2} \| \phi(\tau) \|_{L^\infty_x} \| \phi(\tau) \|_{L^2_x} \| \partial_x \phi(\tau) \|_{L^2_x} \, d\tau
\]
\[
\lesssim \frac{1}{1 + t^{3/2}} \int_0^{t/2} \frac{d\tau}{1 + \tau^{3/2}} \lesssim \frac{1}{1 + t^{3/2}}.
\]
Note we have used the fact $\| \partial_t^n \nabla_x \phi \|_{S^0} \lesssim_{m,s} 1$.

For the second integral, we use Sobolev embedding and $L^{3+} L^{3/2-}$ decay estimate to obtain the bound
\[
\int_{t/2}^t \| e^{i(t-\tau)\Delta} \partial_x [(v_N * |\phi|^2) \phi(\tau)] \|_{L^\infty_x} \, d\tau
\]
\[
\lesssim \int_{t/2}^t \frac{1}{|t - \tau|^{1/2+}} \| \nabla_x \partial_t \phi \|_{L^2_x} \| \phi \|_{L^{5/3-}_x} \, d\tau
\]
\[
+ \int_{t/2}^t \frac{1}{|t - \tau|^{1/2+}} \| \partial_t \phi \|_{L^{1/3+}_x} \| \partial_t \phi \|_{L^{2/3-}_x} \| \nabla_x \phi \|_{L^2_x} \| \phi \|_{L^{\infty}_x} \, d\tau.
\]
Note the last inequality is a consequence of Hölder inequalities and space interpolation. Since $\partial_t \phi$ is bounded, then by Proposition 3.4 it follows
\[
\int_{t/2}^t \| e^{i(t-\tau)\Delta} \partial_x (v_N * |\phi|^2) \phi(\tau) \|_{L^\infty_x} \, d\tau \lesssim \int_{t/2}^t \frac{1}{|t - \tau|^{1/2+}} \| \phi(\tau) \|_{L^\infty_x} \, d\tau
\]
\[
\lesssim \frac{1}{1 + t^{3/2}} \int_{t/2}^t \frac{1}{|t - \tau|^{1/2+}} \, d\tau \lesssim \frac{1}{1 + t^{3/2}}.
\]

4. Estimates for the Pair Excitations

There are two goals in this section. The first goal is to extend the estimates for $\text{sh}(2k)$ to the case of non-positive interacting potential, which we will see only depends on the decay estimate of $\phi$. The other goal is to provide a way to improve the estimate in Theorem 2.6 which we have mentioned in Remark 3.4. However, for the sake of simplicity, we will not propagate the improvement to the rest of the paper and happily leave it as an exercise(s) for the interested reader.

Let us define the shorthand notation $\text{ch}(k) := \delta + p_1$, $\text{sh}(k) := s_1$, and also $\text{ch}(2k) := \delta + p_2$, $\text{sh}(2k) := s_2$.

**Proposition 4.1.** Assume $\phi_0 \in W^{k,1}$ for $k$ sufficiently large. The following estimates hold:
\[
\| s_2(t, \cdot) \|_{L^2_{t,x}} + \| p_2(t, \cdot) \|_{L^2_{t,x}} \lesssim 1
\]
where the estimate only depends on $\|\phi_0\|_{W^{k,1}}$ for some $k$.

To prove the above proposition, we begin by proving a few preliminary lemmas.

**Lemma 4.2.** Let $m_N(t, x, y) := -v_N(x - y)\phi(t, x)\phi(t, y)$. Then we have the following estimates

$$
\int \frac{|m_N(t, \xi, \eta)|^2}{(|\xi|^2 + |\eta|^2)^2} \, d\xi d\eta \lesssim \|\phi(t, \cdot)\|_{L^4_x}^4 \tag{32}
$$

and

$$
\int_{|\xi - \eta| > 1} \frac{|\partial_t m_N(t, \xi, \eta)|^2}{(|\xi|^2 + |\eta|^2)^2} \, d\xi d\eta \lesssim \|\partial_t \phi(t, \cdot)\|_{L^4_x}^2 \|\phi(t, \cdot)\|_{L^4_x}^2. \tag{33}
$$

**Proof.** The proof of the first estimate can be found in [GM13b]. We shall focus on the proof of the second estimate where the proof is a slight modification of the first. First, observe

$$
v_N(x - y)\phi(x)\phi(y) = \int \delta(x - y - z)v_N(z)\phi(x)\phi(y) \, dz
$$

then the Fourier transform of $\delta(x - y - z)\phi(x)\phi(y)$ is given by

$$
\int e^{-i(x\eta + y\xi)}\delta(x - y - z)\phi(x)\phi(y) \, dx dy
$$

$$
= \int e^{-i(x\eta + (y-z)\xi)}\delta(x - y)\phi(x)\phi(y - z) \, dx dy
$$

$$
= e^{iz\xi} \int e^{-ix(\eta + \xi)}\phi(x)\phi(x - z) \, dx = e^{iz\xi} \hat{\phi}(t, \eta + \xi)
$$

which means

$$
|\partial_t m_N(t, \eta, \xi)|^2 = \left| \int e^{iz\xi}v_N(z)\partial_t(\hat{\phi}(t, \eta + \xi)) \, dz \right|^2
$$

$$
\lesssim \|v\|_{L^1_t} \int |v_N(z)||\partial_t(\hat{\phi}(t, \eta + \xi))|^2 \, dz.
$$

Then it follows

$$
\int_{|\xi - \eta| > 1} \frac{|\partial_t m_N(t, \eta, \xi)|^2}{(|\eta|^2 + |\xi|^2)^2} \, d\eta d\xi \lesssim \int |v_N(z)| \int_{|\xi - \eta| > 1} \frac{|\partial_t(\hat{\phi}(t, \eta + \xi))|^2}{(|\eta|^2 + |\xi|^2)^2} \, d\eta d\xi dz
$$

$$
\lesssim \int |v_N(z)| \int_{|\eta'| > 1} \frac{|\partial_t(\hat{\phi}(t, \xi'))|^2}{(|\eta'|^2 + |\xi'|^2)^2} \, d\eta' d\xi' dz
$$

$$
\lesssim \int |v_N(z)||\partial_t(\hat{\phi}(t, \xi'))|^2 \, d\xi' dz
$$

$$
\lesssim \|\partial_t \phi(t, \cdot)\|_{L^4_x}^2 \|\phi(t, \cdot)\|_{L^4_x}^2.
$$

$\square$
Lemma 4.3. Let \( s_0^0 \) be the solution to
\[ \left( \frac{1}{i} \frac{\partial}{\partial t} - \Delta_x - \Delta_y \right) s_0^0(t, x, y) = 2m_N(t, x, y), \quad s_0^0(0, x, y) = 0. \]
Then it follows
\[ \| s_0^0(t, \cdot) \|_{L^2_{x,y}} \lesssim 1 \tag{34} \]
where the estimate only depends on \( \| \phi_0 \|_{W^{k,1}} \).

Proof. Using Duhamel’s principle, we have
\[ \| s_0^0(t, \cdot) \|_{L^2_{x,y}} = 2 \left\| \int_0^t e^{i(t-s)\Delta} m_N(s, \cdot) \, ds \right\|_{L^2_{x,y}} \]
\[ \lesssim \left\| P_{|\xi-\eta| \leq 1} \int_0^t e^{i(t-s)\Delta} m_N(s, \cdot) \, ds \right\|_{L^2_{x,y}} \]
\[ + \left\| P_{|\xi-\eta| > 1} \int_0^t e^{i(t-s)\Delta} m_N(s, \cdot) \, ds \right\|_{L^2_{x,y}}. \]

For the first term we shall directly apply Minkowski’s inequality to get
\[ \left\| P_{|\xi-\eta| \leq 1} \int_0^t e^{i(t-s)\Delta} m_N(s, \cdot) \, ds \right\|_{L^2_{x,y}} \]
\[ \lesssim \int_0^t \left[ \int_{|\xi-\eta| \leq 1} |\tilde{m}(s, \xi, \eta)|^2 \, d\xi d\eta \right]^{1/2} \, ds \]
\[ \lesssim \int_0^t \left[ \int |v_N(z)| \int_{|\eta'| \leq 1} |\tilde{\phi}_x(s, \xi')|^2 \, d\xi' dz \right]^{1/2} \, ds \]
\[ \lesssim \int_0^t \| \phi(s, \cdot) \|^2_{L^2_x} \, ds. \]
Using the decay estimate, we have that the first term is bounded. For the second term we have
\[ \left\| P_{|\xi-\eta| > 1} \int_0^t e^{i(t-s)\Delta} m_N(s, \cdot) \, ds \right\|_{L^2_{x,y}} \]
\[ = \left\| \chi_{|\xi-\eta| > 1} \int_0^t \partial_s e^{i(t-s)(|\eta|^2 + |\xi|^2)} \frac{\tilde{m}(s, \xi, \eta)}{|\eta|^2 + |\xi|^2} \, ds \right\|_{L^2_{x,y}} \]
\[ \lesssim \left\| \frac{\tilde{m}(0, \xi, \eta)}{|\eta|^2 + |\xi|^2} \right\|_{L^2_{x,\eta}} + \left\| \frac{\tilde{m}(t, \xi, \eta)}{|\eta|^2 + |\xi|^2} \right\|_{L^2_{x,\eta}} \]
\[ + \left\| \chi_{|\xi-\eta| > 1} \int_0^t e^{i(t-s)(|\eta|^2 + |\xi|^2)} \frac{\partial_s \tilde{m}(s, \xi, \eta)}{|\eta|^2 + |\xi|^2} \, ds \right\|_{L^2_{x,\eta}}. \]
It’s clear the first two terms are bounded by the previous lemma. For the last term, using Minkowski’s and the previous lemma we have
\[
\left\| \chi_{|\xi-\eta|>1} \int_0^t e^{i(t-s)(|\eta|^2+|\xi|^2)} \frac{\partial \tilde{m}(s,\eta,\xi)}{|\eta|^2+|\xi|^2} \, ds \right\|_{L^2_{\xi,\eta}} \leq \int_0^t \| \partial_t \phi(s,\cdot) \|_{L^4_x} \| \phi(s,\cdot) \|_{L^4_x} \, ds.
\]
Again by the decay estimate, the second term is also bounded. \[\square\]

**Lemma 4.4.** Let \( s_a \) be a solution to
\[
S(s_a) = 2m_N(t,x,y), \quad s_a(t,\cdot) = 0.
\]
Then
\[
\| s_a(t,\cdot) \|_{L^2_{x,y}} \lesssim 1
\]
where the estimate depends only on the \( \| \phi_0 \|_{W^{k,1}} \).

**Sketch of the Proof.** The idea is essential to decompose the solution into a two parts
\[
s_a = s^0_a + s^1_a
\]
where \( s^0_a \) satisfies the equation in the previous lemma and \( s^1_a \) solves
\[
S(s^1_a) = -V(s^0_a(t,\cdot)).
\]
By the previous lemma, we know the \( L^2 \)-norm of \( s^0_a \) is uniformly bounded in time. Next, we shall cite \[\text{[GM13b, lemma 4.5]}, \text{for the proof that the}
\]
\( L^2 \)-norm of \( s^1_a \) is also uniformly bounded in time. \[\square\]

**Sketch of the Proof of Proposition 4.1.** The proof is the same as the proof of Theorem 4.1 in \[\text{[GM13b, Kuz17]}. \text{Again, the only difference comes from the replacement of the estimate to the solution of S(s_a) = 2m_N by the result of the previous lemma.} \[\square\]

**Remark 4.5.** Following Remark 3.3 if we consider the subcritical uniform in \( N \) well-posedness for \( H^1_x \) data, we would obtain the estimate \( \sup_{t \in [0,T]} (\| s_2(t,\cdot) \|_{L^2_{x,y}} + \| p_2(t,\cdot) \|_{L^2_{x,y}}) \lesssim 1 \) for some small time \( T \) and independent of \( N \).

5. **Proof of Theorem 2.3**

The proof of Theorem 2.3 is essentially the same as the proof given in \[\text{[GM13b, Kuz17]}. \text{provided we have established the decay estimate for } \phi. \text{For}
\]
the sake of simplicity, we shall only provide a complete proof of the first part of Theorem 2.3 since the second part of the theorem is significantly lengthier to present. We shall refer the interested reader to \[\text{[Kuz17]}. \text{for a}
\]
complete proof of the second part of Theorem 2.3.
The quadratic term is given by

\[ E(t) = e^{B(\mathcal{A}, \mathcal{V}) + N^{-1/2}V)}e^{-B} \] (35)

where \([\mathcal{A}, \mathcal{V}]\) and \(N^{-1/2}V\) are cubic and quartic polynomials in \((a_x, a_x^\dagger)\) respectively. Using the conjugation formulae

\[ e^{B}a_xe^{-B} = \int dy \, \{ \text{ch}(y, x)a_y + \text{sh}(y, x)a_y^\dagger \} \] (36a)

\[ e^{B}a_x^\dagger e^{-B} = \int dy \, \{ \text{sh}(y, x)a_y + \text{ch}(y, x)a_y^\dagger \} \] (36b)

we could further expand the error terms into another fourth-order polynomial in \((a_x^\dagger, a_x)\).

5.1. List of Error Terms. For convenience, we shall include the list of error terms which were explicitly computed in §5 of [GM13b].

Recall the error terms are defined to be

\[ \mathcal{E}(t) = e^{B(\mathcal{A}, \mathcal{V}) + N^{-1/2}V)}e^{-B} \]

The following is the result of expanding \(\mathcal{E}(t)\). First, let us list all the error terms of \(N^{-1/2}e^{B}V)e^{-B}\) which is a fourth-order polynomial in \((a_x^\dagger, a_x)\) with no linear nor cubic terms. The quadratic term is given by

\[ \frac{1}{2N} \int dy_1dy_2dy_3dy_4 \left\{ v_N(y_1 - y_2) \text{sh}(k)(y_3, y_1) \text{sh}(k)(y_2, y_4) + \right. \]

\[ \int dx \, \{ \bar{p}(y_2, x)v_N(y_1 - x) \text{sh}(k)(x, y_4) \} \text{sh}(k)(y_3, y_1) + \]

\[ \int dx \, \{ \bar{p}(y_1, x)v_N(x - y_2) \text{sh}(k)(y_3, x) \} \text{sh}(k)(y_2, y_4) + \]

\[ \int dx_1dx_2 \{ \bar{p}(y_1, x_1)p(x_2, y_2)v_N(x_1 - x_2) \text{sh}(k)(y_3, x_1) \text{sh}(k)(x_2, x_4) \} \]

\[ \} a_{y_1}^\dagger a_{y_2}^\dagger a_{y_3}^\dagger a_{y_4}^\dagger. \]

The quadratic term is given by

\[ \frac{1}{2N} \int dy_1dy_2dx_1dx_2 \left\{ \right. \]

\[ \frac{\text{ch}(k)(y_1, x_2) \text{sh}(k)(x_2, y_2) (\text{sh}(k) \circ \text{sh}(k))(x_1, x_1)v_N(x_1 - x_2)}{\text{ch}(k)(y_1, x_2) \text{sh}(k)(x_1, y_2) (\text{sh}(k) \circ \text{sh}(k))(x_1, x_2)v_N(x_1 - x_2)} + \]

\[ \frac{\text{ch}(k)(y_1, x_1) \text{sh}(k)(x_2, y_2) (\text{sh}(k) \circ \text{sh}(k))(x_1, x_2)v_N(x_1 - x_2)}{\text{ch}(k)(y_1, x_1) \text{sh}(k)(x_1, y_2) (\text{sh}(k) \circ \text{sh}(k))(x_1, x_2)v_N(x_1 - x_2)} + \]

\[ \frac{\text{sh}(k)(y_1, x_1) \text{sh}(k)(x_2, y_2) (\text{sh}(k) \circ \text{sh}(k))(x_1, x_2)v_N(x_1 - x_2)}{\text{ch}(k)(y_1, x_1) \text{ch}(k)(x_1, y_2) (\text{ch}(k) \circ \text{sh}(k))(x_1, x_2)v_N(x_1 - x_2)} + \]

\[ \frac{\text{ch}(k)(y_1, x_1) \text{ch}(k)(x_2, y_2) (\text{ch}(k) \circ \text{sh}(k))(x_1, x_2)v_N(x_1 - x_2)}{\text{ch}(k)(y_1, x_1) \text{ch}(k)(x_2, y_2) (\text{ch}(k) \circ \text{sh}(k))(x_1, x_2)v_N(x_1 - x_2)} \} a_{y_1}^\dagger a_{y_2}^\dagger \] (38f)
Lastly the linear term is given by

\[
\frac{1}{2N} \int dx_1 dx_2 \left\{ \begin{array}{l}
(sh(k) \circ sh(k)) (x_1, x_2) v_N (x_1 - x_2) (sh(k) \circ sh(k)) (x_1, x_2) + \\
(sh(k) \circ sh(k)) (x_1, x_2) v_N (x_1 - x_2) (sh(k) \circ sh(k)) (x_2, x_2) + \\
(sh(k) \circ ch(k)) (x_1, x_2) v_N (x_1 - x_2) (ch(k) \circ sh(k)) (x_1, x_2) \end{array} \right\}. \quad (39a)
\]

\[
\frac{1}{2N} \int dx_1 dx_2 \left\{ \begin{array}{l}
(sh(k) \circ ch(k)) (x_1, x_2) v_N (x_1) (sh(k) \circ sh(k)) (x_1, x_2) + \\
(sh(k) \circ ch(k)) (x_1, x_2) v_N (x_2) (sh(k) \circ sh(k)) (x_1, x_2) + \\
(sh(k) \circ ch(k)) (x_1, x_2) v_N (x_1 - x_2) (sh(k) \circ sh(k)) (x_1, x_2) \end{array} \right\}. \quad (39b)
\]

\[
\frac{1}{2N} \int dx_1 dx_2 \left\{ \begin{array}{l}
(sh(k) \circ ch(k)) (x_1, x_2) v_N (x_1) (sh(k) \circ sh(k)) (x_1, x_2) + \\
(sh(k) \circ ch(k)) (x_1, x_2) v_N (x_1 - x_2) (sh(k) \circ sh(k)) (x_1, x_2) + \\
(sh(k) \circ ch(k)) (x_1, x_2) v_N (x_1 - x_2) (sh(k) \circ sh(k)) (x_1, x_2) \end{array} \right\}. \quad (39c)
\]

In the case of \( e^{\mathcal{B}[A, V]} e^{-\mathcal{B}} \) we have a cubic polynomial in \((a^+_x, a_x)\) with no quadratic nor zeroth-order terms. The cubic term is given by

\[
\frac{1}{\sqrt{N}} \int dy_1 dy_2 dy_3 \left\{ \begin{array}{l}
v_N (y_1 - y_2) \phi(y_2) sh(k)(y_3, y_1) + \\
\int dx \left\{ v_N (y_1 - x) \phi(x) sh(k)(x, y_3) \right\} sh(k)(y_2, y_1) + \\
\int dx \left\{ \bar{p}(y_1, x) v_N (y_2 - y_3) sh(k)(y_3, x) \right\} \phi(y_2) + \\
\int dx \left\{ \bar{p}(y_2, x) v_N (y_1 - x) \phi(x) \right\} sh(k)(y_3, y_1) + \\
\int dx_1 dx_2 \left\{ \bar{p}(y_1, x_1) v_N (x_1 - x_2) \phi(x_2) sh(k)(y_2, x_1) sh(k)(x_2, y_3) \right\} + \\
\int dx_1 dx_2 \left\{ \bar{p}(y_1, x_1) p(x_2, y_2) v_N (x_1 - x_2) \phi(x_2) sh(k)(y_3, x_1) \right\} \right\} a^+_y a^+_x a^+_y. \quad (40f)
\]

Lastly the linear term is given by

\[
\frac{1}{\sqrt{N}} \int dy dx_1 dx_2 \left\{ \begin{array}{l}
sh(k)(y, x_2) (sh(k) \circ sh(k)) (x_1, x_1) \phi(x_2) v_N (x_1 - x_2) + \\
sh(k)(y, x_1) (sh(k) \circ sh(k)) (x_1, x_2) \phi(x_2) v_N (x_1 - x_2) + \\
ch(k)(y, x_1) (ch(k) \circ sh(k)) (x_1, x_2) \phi(x_2) v_N (x_1 - x_2) + \\
ch(k)(y, x_1) (ch(k) \circ sh(k)) (x_1, x_2) \phi(x_2) v_N (x_1 - x_2) + \\
sh(k)(y, x_1) (sh(k) \circ ch(k)) (x_1, x_2) \phi(x_2) v_N (x_1 - x_2) + \\
sh(k)(y, x_1) (sh(k) \circ ch(k)) (x_1, x_2) \phi(x_2) v_N (x_1 - x_2) \right\} a^+_y. \quad (40f)
\]

5.2. Estimates for the Error Terms. To prove theorem 2.3 it suffices to establish the following estimates on \( \mathcal{E}(t) \).
Proposition 5.1. For the two error terms we have the following estimates

\[
\frac{1}{\sqrt{N}} \| e^B [A, V] e^{-B} \Omega \|_F \lesssim \frac{N^{3\beta-1}}{1 + t^{3/2}} \tag{42}
\]

and

\[
\frac{1}{N} \| e^B V e^{-B} \Omega \|_F \lesssim N^{3\beta-1}. \tag{43}
\]

Proof. Since many of the terms are similar, without loss of generality, we shall pick representatives in each category and prove the bound holds for the representatives.

First, let us look at the quartic term. The two representatives are (37a) and (37d), since (37b) and (37c) could be handled similarly by the techniques in bounding (37d). In the case (37a), we see

\[
\frac{1}{N} \| v_N(y_1 - y_2) \text{sh}(k)(y_3, y_1) \text{sh}(k)(y_2, y_4) \|_{L^2_{y_1,y_2,y_3,y_4}} \lesssim \frac{1}{N} \| v_N \|_{L^\infty_x} \| \text{sh}(k) \|_{L^2_{x,y}} \lesssim N^{3\beta-1}.
\]

where we used the fact Proposition 4.1. For (37d), we have

\[
\frac{1}{N} \left\| dx_1 dx_2 \{ \bar{p}(y_1, x_1)p(x_2, y_2)v_N(x_1 - x_2) \text{sh}(k)(y_3, x_1) \text{sh}(k)(x_2, x_4) \} \right\|_{L^2_{y_1,y_2,y_3,y_4}} \lesssim \frac{1}{N} \| v_N \|_{L^\infty_x} \| \text{sh}(k) \|_{L^2_{x,y}} \lesssim N^{3\beta-1}.
\]

For the quadratic term, the worse term is given by (38a) due to the \(\delta\) function contribution. Looking term with the most \(\delta\) function contribution, we have

\[
\frac{1}{N} \| \text{sh}(k)(y_1, y_2)v_N(y_1 - y_2) \|_{L^2_{y_1,y_2}} \lesssim \frac{1}{N} \| v_N \|_{L^\infty_x} \| \text{sh}(k) \|_{L^2_{x,y}} \lesssim N^{3\beta-1}.
\]

For the cubic term, we shall consider (40a) and (40b). In the case of (40a), we have

\[
\frac{1}{\sqrt{N}} \| v_N(y_1 - y_2)\phi(y_2) \text{sh}(k)(y_3, y_1) \|_{L^2_{y_1,y_2,y_3}} \lesssim \frac{1}{\sqrt{N}} \| \phi \|_{L^\infty} \| v_N \|_{L^2} \| \text{sh}(k) \|_{L^2_{x,y}} \lesssim \frac{N^{(3\beta-1)/2}}{1 + t^{3/2}}.
\]

And for (40b), it follows

\[
\frac{1}{\sqrt{N}} \left\| dx_1 dx_2 \{ \bar{p}(y_1, x_1)p(x_2, y_2)v_N(x_1 - x_2)\phi(x_2) \text{sh}(k)(y_3, x_1) \} \right\|_{L^2_{y_1,y_2,y_3}} \lesssim \frac{1}{\sqrt{N}} \| \phi \|_{L^\infty} \| p(k) \|_{L^2_{x,y}} \| \text{sh}(k) \|_{L^2_{x,y}} \| v_N \|_{L^2} \lesssim \frac{N^{(3\beta-1)/2}}{1 + t^{3/2}}.
\]
Lastly, for the linear term, we shall consider (41c). Again, consider the term with the $\delta$ contribution, we have

$$\frac{1}{\sqrt{N}} \left\| \int dx_2 \text{sh}(k)(y, x_2)\tilde{\phi}(x_2)v_N(y - x_2) \right\|_{L^2_y} \lesssim \frac{1}{\sqrt{N}} \| \phi \|_{L^\infty_x} \| v_N \|_{L^2_x} \| \text{sh}(k) \|_{L^2_{x,y}} \lesssim \frac{N^{(3\beta-1)/2}}{1 + t^{3/2}}.$$ \hspace{1cm} \Box

6. Application: Derivation of The Focusing NLS in $\mathbb{R}^3$

We provide two derivation of the focusing nonlinear Schrödinger equation. For the first derivation we will use the method of pair excitation developed in the previous sections and the second derivation will be via a method introduced by Pickl in \cite{Pic11, Pic10}.

6.1. Pair Excitation Method. In this section, we provide the Fock space method\textsuperscript{12} for analyzing the rate of convergence of the one-particle marginal toward mean field. However, in the next subsection, we shall provide Pickl’s method which offers an error bound which will be independent of time. Nevertheless, the purpose of this section is to show that one could still derive the focusing NLS from the pair excitation method developed thus far in the article.

Let us recall a couple results proven in \cite{Kuz15}:

**Lemma 6.1.** Let $k(x, y) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ symmetric in $(x, y)$. Then the following operator inequality holds

$$e^{B(k)}N e^{-B(k)} \lesssim N + 1$$ \hspace{1cm} (44)

uniformly in time.

**Lemma 6.2.** We define the reduced dynamics, denoted by $\psi_{\text{red}}$, to be

$$\psi_{\text{red}}(t) := e^{B(k_t)} e^{\sqrt{N} \mathcal{A}(\phi_t)} e^{it\mathcal{H}_N} e^{-\sqrt{N} \mathcal{A}(\phi_0)} \Omega.$$ \hspace{1cm} (45)

Then we have the following estimates

$$\|Ne^{-B}\psi_{\text{red}}\|_F \lesssim \sqrt{N} \left\| (N + 1)^{1/2}\psi_{\text{red}} \right\|_F \quad \text{and}$$

$$\|N^{1/2}\psi_{\text{red}}\|_F \lesssim \sqrt{N} \| \psi_{\text{exact}} - \psi_{\text{approx}} \|_F.$$ \hspace{1cm} (46)

\textsuperscript{12} The pair excitation method is also referred to as the Fock space method.
Following [RS09] and [Kuz15], we rewrite the one-particle marginal density as follows

\[ \gamma_{N,t}^{(1)}(t, x, y) \]
\[ = \frac{1}{c_N N} \left( e^{itH} P_N e^{-\sqrt{N}A} \phi, a_x^+ a_y e^{itH} e^{-\sqrt{N}A} \Omega \right) \]
\[ = \frac{1}{c_N N} \left( e^{itH} P_N e^{-\sqrt{N}A} \phi, e^{-\sqrt{N}A} a_x^+ a_y e^{-\sqrt{N}A} e^{itH} e^{-\sqrt{N}A} \Omega \right) \]
\[ = \frac{1}{c_N N} \left( e^{itH} P_N e^{-\sqrt{N}A} \phi, e^{-\sqrt{N}A} a_x^+ a_y e^{-\sqrt{N}A} e^{itH} e^{-\sqrt{N}A} \Omega \right) \]
\[ + \frac{\phi(t, x)}{c_N^2 \sqrt{N}} \left( e^{itH} P_N e^{-\sqrt{N}A} \phi, e^{-\sqrt{N}A} a_y e^{-\sqrt{N}A} e^{itH} e^{-\sqrt{N}A} \Omega \right) \]
\[ + \frac{\phi(t, y)}{c_N^2 \sqrt{N}} \left( e^{itH} P_N e^{-\sqrt{N}A} \phi, e^{-\sqrt{N}A} a_x^+ e^{-\sqrt{N}A} e^{itH} e^{-\sqrt{N}A} \Omega \right) + \phi(t, x) \tilde{\phi}(t, y). \]

Here \( P_N \) is the projection operator onto the \( N \)th sector of the Fock space. Moreover, the identities

\[ e^{\sqrt{N}A} a_x^+ e^{-\sqrt{N}A} = a_x^+ + \sqrt{N} \phi \] \hspace{1cm} (48a)
\[ e^{\sqrt{N}A} a_x e^{-\sqrt{N}A} = a_x + \sqrt{N} \phi \] \hspace{1cm} (48b)

are direct consequences of the Lie-type identity used in [GM13b].

Using the above calculation, we have

\[ \left| \gamma_{N,t}^{(1)}(t, x, y) - \phi_N(t, x) \bar{\phi}(t, y) \right| \leq \frac{1}{c_N N} \left\| a_x^+ a_y e^{\sqrt{N}A} e^{itH} P_N e^{-\sqrt{N}A} \Omega \right\|_F \]
\[ + \frac{\phi_N(t, x)}{c_N \sqrt{N}} \left\| a_x e^{\sqrt{N}A} e^{itH} P_N e^{-\sqrt{N}A} \Omega \right\|_F \]
\[ + \frac{\phi_N(t, y)}{c_N \sqrt{N}} \left\| a_y e^{\sqrt{N}A} e^{itH} P_N e^{-\sqrt{N}A} \Omega \right\|_F \]

which means

\[ \int dx dy \left| \gamma_{N,t}^{(1)}(t, x, y) - \phi_N(t, x) \bar{\phi}(t, y) \right|^2 \]
\[ \lesssim \frac{1}{c_N N^2} \left\| Ne^{\sqrt{N}A} e^{itH} e^{-\sqrt{N}A} \Omega \right\|_F^2 + \frac{1}{c_N N} \left\| N^{1/2} e^{\sqrt{N}A} e^{itH} e^{-\sqrt{N}A} \Omega \right\|_F^2 \]
\[ = \frac{1}{c_N^2 N^2} \left\| Ne^{-B} \psi_{\text{red}} \right\|_F^2 + \frac{1}{c_N N} \left\| N^{1/2} e^{-B} \psi_{\text{red}} \right\|_F^2. \]
Applying lemma (6.1) and (6.2), we get

\[
\int dx dy \left| \gamma_N^{(1)}(t, x, y) - \phi_N(t, x) \overline{\phi_N(t, y)} \right|^2 \lesssim \frac{1}{N^{3/2}} \left\| N e^{-B} \psi_{\text{red}} \right\|_F^2 + \frac{1}{\sqrt{N}} \left\| N^{1/2} \psi_{\text{red}} \right\|_F^2
\]

\[
\lesssim \sqrt{N} \left\| \psi_{\text{exact}} - \psi_{\text{approx}} \right\|_F^2.
\]

Finally, by the appendix in [Kuz15] and remark 1.4 in [RS09], we have provided both a derivation of the focusing Schrödinger equation and a rate of convergence of the \(N\) body interacting bosonic system toward mean field for \(\beta\) in the range \(0 < \beta < \frac{1}{6}\).

**Remark 6.3.** One should note we could only use part one of Theorem 2.3 for our derivation of the focusing NLS since we are considering evolution of coherent states, i.e. \(k(0, \cdot) = 0\).

### 6.2. Pickl’s Method.

Following closely the presentation in [Pic10], we consider the quantities

**Definition 6.4.** Let \(\phi \in L^2(\mathbb{R}^3)\)

(a) For each \(1 \leq j \leq N\) we define the projectors \(p^\phi_j : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})\) and \(q^\phi_j : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})\) given by

\[
p^\phi_j \Psi_N(x_1, \ldots, x_N) = \phi(x_j) \int \phi^*(x'_j) \Psi_N(x_1, \ldots, x'_j, \ldots, x_N) dx_j
\]

and \(q^\phi_j = 1 - p^\phi_j\) respectively.

(b) Furthermore, for any \(1 \leq k \leq N\) we defined \(P^\phi_k : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})\) given by

\[
P^\phi_k := \sum_{a \in A_k} \prod_{\ell=1}^N (p^\phi_{a_\ell})^{1-a_\ell} (q^\phi_{a_\ell})^{a_\ell}
\]

where

\[A_k = \{(a_1, \ldots, a_N) \mid a_i \in \{0, 1\} \text{ and } \sum_{i=1}^N a_i = k\}\]

(c) Assume \(0 < \lambda \leq 1\). Let us define the function \(m^\lambda : \{1, \ldots, N\} \to \mathbb{R}_{\geq 0}\) given by

\[
m^\lambda(k) := \begin{cases} 
k/N^\lambda, & \text{for } k \leq N^\lambda, \\ 1, & \text{otherwise} \end{cases}
\]
and a corresponding functional $\alpha_N^\lambda : L^2(\mathbb{R}^{3N}) \times L^2(\mathbb{R}^3) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$\alpha_N^\lambda(\Psi_N, \phi) := \langle \Psi_N, \sum_{k=1}^{N} m^\lambda(k) P^\phi_j \Psi_N \rangle$$

$$= \langle \Psi_N, \hat{m}^\lambda \phi \Psi_N \rangle = \| (\hat{m}^\lambda \phi)^{1/2} \Psi_N \|^2_{L^2_x}.$$  

For convenience, we shall use the notation $\alpha_N$ instead of $\alpha_N^1$. 

As a direct consequence of the definitions, one could verify the following

$$\alpha_N(\Psi_N, \phi) = \| q_1^\phi \Psi_N \|^2_{L^2_x} \leq \alpha_N^\lambda(\Psi_N, \phi)$$

for $0 < \lambda < 1$. Again, by the definition, we could derive an error bound for the rate of convergence of the one particle density towards the mean field limit

$$\| \gamma_N^{(1)} - |\phi\rangle \langle \phi| \| \leq \| p_1^\phi \Psi_N \|^2_{L^2_x} - 1 \| |\phi\rangle \langle \phi| \| + 2 \| q_1^\phi \Psi_N \|_{L^2_x} \| p_1^\phi \Psi_N \|_{L^2_x} + \| q_1^\phi \Psi_N \|^2_{L^2_x}$$

$$\leq \| p_1^\phi \Psi_N \|^2_{L^2_x} - 1 + 2 \| q_1^\phi \Psi_N \|_{L^2_x} \| p_1^\phi \Psi_N \|_{L^2_x}$$

$$\leq \| q_1^\phi \Psi_N \|^2_{L^2_x} + \| q_1^\phi \Psi_N \|_{L^2_x}.$$  

Since $|\phi\rangle \langle \phi|$ is a rank one projection operator, by remark 1.4 in [RS09] the trace norm is two times the operator norm, i.e., $2 \| \gamma_N^{(1)} - |\phi\rangle \langle \phi| \|_{op} = \text{Tr} |\gamma_N^{(1)} - |\phi\rangle \langle \phi| |$. Then it follows from the above estimates

$$\text{Tr} |\gamma_N^{(1)} - |\phi_\lambda\rangle \langle \phi_\lambda| | \leq \alpha_N(\Psi_N, \phi_t) + \sqrt{\alpha_N^\lambda(\Psi_N, \phi_t)}.$$  

Thus, to obtain a rate of convergence for the error it suffices to prove an estimate for $\alpha_N^\lambda(\Psi_N, \phi)$. Let us now state the main theorem in [Pic10] which we will use to derive the focusing NLS:

**Theorem 6.5.** Assume $0 < \lambda, \beta < 1$ and $\nu_N$ satisfies the same conditions as before. Assume for every $N \in \mathbb{N}$ there exists a solution to the linear $N$-body Schrödinger equation $\Psi_N(t, x)$ and a $L^\infty$ solution of the mean field equation $\psi_t$ on some interval $[0, T)$ with $T \in \mathbb{R}_{>0} \cup \{\infty\}$. Then for any $t \in [0, T)$

$$\alpha_N^\lambda(\Psi_N, \psi_t) \leq \exp \left( \int_0^t C_v \| \phi_s \|_{L^\infty_x}^2 \, ds \right) \alpha_N^\lambda(\Psi_N, \phi_0)$$

$$+ \left[ \exp \left( C_v \int_0^t \| \phi_s \|_{L^\infty_x}^2 \, ds \right) - 1 \right] \sup_{0 \leq s \leq t} K^\phi v N^{\delta_\lambda}$$

where $\delta_\lambda = \frac{1}{4} \max \{ 1 - \lambda - 4\beta, 3\beta - \lambda, -1 + \lambda + 3\beta \}$, $C_v$ is some constant depending only on $v$ and

$$K^\phi := C_v (\| \Delta |\phi| \|^2_{L^2_x} + \| \phi \|_{L^\infty_x} + 1 \| \phi \|_{L^\infty_x^2}).$$
Proof of Theorem 2.7. Note if $\Psi_N(0, x) = \phi^{\otimes N}$ then $\alpha_N^\lambda(\phi^{\otimes N}, \phi) = 0$. Hence combining with our above decay result for $\phi$ satisfying the Hartree equation

$$\frac{1}{i} \partial_t \phi - \Delta \phi + (\int v) |\phi|^2 \phi = 0$$

we have that

$$\alpha_N^\lambda(\Psi_{N,t}, \phi_t) \leq \exp \left( C_v \int_0^t \| \phi_s \|_{L^\infty_x}^2 ds \right) \sup_{0 \leq s \leq t} K^{\phi_s} N^{\delta^\lambda}$$

where

$$K^{\phi_t} = C_v (\| \Delta |\phi_t|^2 \|_{L^2_x} + \| \phi_t \|_{L^\infty_x} + 1) \| \phi_t \|_{L^\infty_x}$$

$$\lesssim (\| \nabla_x |\phi_t|^2 \|_{L^2_x} + \| \phi_t \Delta \phi_t \|_{L^2_x} + \| \phi_t \|_{L^\infty_x} + 1) \| \phi_t \|_{L^\infty_x}$$

$$\lesssim \frac{1}{1 + t^{3/2}}.$$

Thus, it follows

$$\text{Tr} \left| \gamma_{N,t}^{(1)} - |\phi_t\rangle \langle \phi_t| \right| \lesssim \sqrt{\alpha_N^\lambda(\Psi_{N,t}, \phi_t)} \lesssim N^{\delta^\lambda/2}.$$

By remark 1 in [Pic10], we see there is a choice of $\lambda$ such that $\delta^\lambda < 0$ when $0 < \beta < \frac{1}{6}$. \hfill $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742

E-mail address: jwchong@math.umd.edu