Quantum communication with $SU(2)$ invariant separable $2 \times N$ level systems

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Abstract

Information is encoded in a qubit in the form of its Bloch vector. In this paper, we propose protocols for remote transfers of information in a known and an unknown qubit to qudits using $SU(2)$-invariant $2 \times N$-level separable discordant states as quantum channels. These states have been identified as separable equivalents of the two-qubit entangled Werner states in Bharath and Ravishankar (Phys Rev A 89:062110, 2014). Due to $SU(2) \times SU(2)$ invariance of these states, the remote qudit can be changed by performing appropriate measurements on the qubit. We also propose a protocol for transferring information of a family of unknown qudits to remote qudits using $2 \times N$-level states as channels. Finally, we propose a protocol for swapping of quantum discord from $2 \times N$-level systems to $N \times N$-level systems. All the protocols proposed in this paper involve separable states as quantum channels.

Keywords Equivalent state · Remote state preparation · Quantum communication · Quantum discord · Quantum discord swapping

1 Introduction

Recent times have witnessed a surge of interest in the study of novel and distinctive features of different quantum correlations, e.g. quantum nonlocality [1], quantum entanglement [2], steering [3], quantum discord [4], etc. The interest owes to several nonclassical tasks in which they act as resources. Examples include quantum teleportation [5], superdense coding [6], remote state preparation [7], device-independent quantum cryptography [8] and entanglement swapping [9]. In these protocols, either only entangled states are resourceful or they yield an advantage, manifested as a better performance of the protocol (e.g. teleportation fidelity [10]).

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Promising though these applications are, the challenges in the generation of entangled states and sustaining them for a long duration act as restrictions in these tasks [11–13]. This has led to the study of nonclassical communication tasks that can be performed using nonclassical correlations beyond entanglement, viz., quantum discord [14,15], geometric discord, etc. Yet, in all these approaches, there is a sharp decline in fidelity of the transmitted state with the input state, as compared to the scenario in which an entangled state is put to use [10,16].

In a parallel approach, there have been many theoretical proposals and successful attempts to simulate (or perform) quantum computation [17], quantum search algorithm [18], quantum information processing [19] and quantum random walk [20] with classical waves. The employment of classical waves for their implementation overcomes the problem of sustaining quantum coherence for a long duration. In spite of these advances, a major glitch is that quantum nonlocality cannot be simulated in the classical domain. It restricts the implementation of quantum communication protocols involving nonlocal states using classical waves.

These difficulties have led us to ask the question: do there exist separable resource states in $2 \times N$ dimensions, which can be used for transferring the information encoded in a qubit to a remote qudit? The rationale underlying this question are: (i) in order to transfer information encoded in a qubit to a remote qudit, the minimal requirement is that the latter should have a nonzero vector polarisation in the spin operator basis that can be changed in a controlled manner, and, (ii) generation and manipulation of separable quantum states are easier than entangled quantum states. Interestingly, the answer to this question is in the affirmative. Such states have been identified in the study made in [21], in which the concept of equivalent states has been introduced. An emergent feature of this concept is the notion of classical simulation of entangled states. To make the discussion easier, we have used the spin operator basis. The analysis is, however, applicable to any arbitrary system.

Two quantum states belonging to Hilbert spaces of different dimensions are termed as equivalent if their Q-representations are the same. Operationally, the crux of the formalism involving equivalent states lies in the fact that the Q-representation of a lower-dimensional noisy entangled state is the same as those of higher-dimensional noisy separable states. That is to say, the so-called lower-dimensional mixed entangled states admit classical simulation in higher dimensions. The mixed separable equivalent states of the two-qubit entangled Werner states have been obtained in [21]. In a subsequent study [22], the separable equivalents of $SU(2)$ invariant $3 \times N$ states have been identified. The higher-dimensional equivalent states are not only separable but highly mixed also. Thus, this formalism provides a niche for the processing of quantum information with highly mixed states as well.

In this paper, we propose various communication protocols, using higher-dimensional separable equivalents of $2 \times 2$ entangled Werner states, identified in [21]. Motivated by the fact that information is encoded in a qubit in the form of its Bloch vector, we propose protocols for remote transfer of information in a known and an unknown qubit to a remote qudit using $2 \times N$-dimensional discordant states. We also propose a protocol for transferring information encoded in the polarisation vector of a qudit to a remote qudit. Finally, we propose a protocol for swapping of quantum
discord from two $2 \times N$-dimensional discordant states to a $N \times N$-dimensional state. A ubiquitous feature of all these protocols is that none of them employ entangled states.

There has been considerable improvement in the generation and manipulation of higher-dimensional separable orbital angular momentum states (see, e.g. [23,24] and references therein). Thus, we believe that employing the protocols proposed in this paper, one can avoid the need of entangled states for certain quantum communication tasks.

The plan of the paper is as follows: We briefly review the formalism to be employed in Sect. (2). Thereafter, we turn our attention to applications of the formalism in Sect. (3), which is central to the paper. In Sects. (3.1) and (3.2), the protocols for transfer of information encoded in an unknown and a known qubit to remote qudits are presented, respectively. In Sect. (3.3), the protocol for transferring information from an unknown spin-$S$ state (having only nonzero vector polarisation) by employing $\frac{1}{2} \otimes S$ separable equivalent of a $\frac{1}{2} \otimes \frac{1}{2}$ entangled Werner state is presented. In Sect. (3.4), the protocol for swapping of quantum discord from two $\frac{1}{2} \otimes S$ states to a single $S \otimes S$ state has been proposed. Finally, we discuss future prospects and possible advantages of quantum communication with equivalent states in Sect. (3.5). Section (4) concludes the paper with closing remarks.

2 Formalism

In this section, we briefly recapitulate the formalism involving equivalent states. For a detailed discussion, one can refer to [21].

2.1 Spin coherent state and the Q-representation

The spin-coherent state (SCS) $|\hat{n}(\theta, \phi)\rangle$ for a spin-$S$ particle may be generated by the action of the rotation group ($SU(2)$) on the state with the highest weight (stretched case), $|S_z = +S\rangle$ [25],

$$|\hat{n}(\theta, \phi)\rangle \equiv e^{-iS_z\phi} e^{-iS_y\theta} e^{-iS_z\psi} |S\rangle. \quad (1)$$

Spin-coherent states have the following properties:

$$\langle \hat{n}(\theta, \phi)|\hat{S}|\hat{n}(\theta, \phi)\rangle = \hat{n}(\theta, \phi),$$

$$|\langle \hat{n}|\hat{n}'\rangle|^2 = \left(\frac{1 + \hat{n} \cdot \hat{n}'}{2}\right)^{2S}, \quad (2)$$

where $\hat{S} = \frac{S}{\hat{S}}$. The set of all SCSs $\{|\hat{n}(\theta, \phi)\rangle\}$ forms an overcomplete set

$$\frac{2S + 1}{4\pi} \int \sin \theta d\theta d\phi |\langle \hat{n}(\theta, \phi)\rangle| = 1. \quad (3)$$
Being overcomplete, SCSs allow any state to be *completely* expressed in terms of its diagonal elements

\[
F(\hat{n}) \equiv \frac{2S + 1}{4\pi} \langle \hat{n} | \rho | \hat{n} \rangle.
\]  

(4)

As an example, the Q-representation of a spin-$\frac{1}{2}$ state

\[
\rho = \frac{1}{2}(\openone + \sigma \cdot p),
\]  

(5)

is given by

\[
F(\hat{n}) = \frac{1}{4\pi} (1 + \hat{n} \cdot p).
\]  

(6)

Similarly, the Q-representation of a two-qubit state,

\[
\rho_{12} = \frac{1}{4} \{ 1 + \sigma_1 \cdot P_1 + \sigma_2 \cdot P_2 + \sigma_1 i \sigma_2 j \Pi_{ij} \},
\]  

(7)

is given by

\[
F(\hat{m}, \hat{n}) = \frac{4}{(4\pi)^2} \langle \hat{m} \otimes \hat{n} | \rho_{12} | \hat{m} \otimes \hat{n} \rangle \\
\equiv \frac{1}{(4\pi)^2} \{ 1 + P_1 \cdot \hat{m} + P_2 \cdot \hat{n} + \Pi_{ij} m_i n_j \}.
\]  

(8)

In what follows, we show that Q-representation serves to identify equivalent states.

### 2.2 Equivalent states

Since there is a bijective mapping between a state $\rho$ and its Q-representation, the idea is to take a lower-dimensional state as an abstract entity and to look for all physical manifests (higher-dimensional states) that yield the same expectation values for certain observables.

In order to elucidate the concept of equivalent states, let there be two Hilbert spaces, $\mathcal{H}^{d_1}, \mathcal{H}^{d_2}$ of dimensions $d_1$ and $d_2$, respectively ($d_1 < d_2$). If a state $\rho \in \mathcal{H}^{d_1}$ has an equivalent state $\rho' \in \mathcal{H}^{d_2}$, it implies that for all the operators $\hat{O} \in \mathcal{H}^{d_1}$, there exist operators $\hat{O}' \in \mathcal{H}^{d_2}$ such that

\[
\text{Tr}(\rho \hat{O}) = \text{Tr}(\rho' \hat{O}').
\]  

(9)

Operationally, equivalent states reproduce the same expectation values for all the observables, after suitable rescalings. The equivalence does not extend to properties...
such as rank and purity of the state. For example, the spin-$S$ equivalent of a spin-$\frac{1}{2}$ state, $\rho = \frac{1}{2}(\mathbb{1} + \sigma \cdot P)$, is given by

$$\rho^S(P) = \frac{1}{2S+1}(\mathbb{1} + \hat{S} \cdot P) \in \mathcal{H}^{2S+1}; \hat{S} = \frac{S}{S}; S \neq 0. \quad (10)$$

The nonnegativity of the eigenvalues mandates $|P| \leq 1$, for all $S$. These states form a family of equivalent states all yielding the same probability distribution function, $F(\hat{n}) = \frac{1}{4\pi}(1 + \mathbf{P} \cdot \hat{n})$. While for spin-$\frac{1}{2}$, $|P| = 1$ corresponds to a pure state, it is mixed for all other spins $S \neq \frac{1}{2}$. The crucial point is that all these states contain the same information which is encoded in their polarisation vector. We next discuss how to retrieve this information.

### 2.2.1 Retrieval of information from equivalent state of a qubit

The observables required for retrieval of information encoded in $\rho^S(P)$ (given in equation (10)) are given by

$$\hat{O}^S(P) = \frac{3S}{S+1}\hat{S} \cdot \hat{P}. \quad (11)$$

For $S = \frac{1}{2}$, the observable is simply given by $\sigma \cdot \hat{P}$, whereas for higher $S$ values (close to 20), it is approximately given by $3\hat{S} \cdot \hat{P}$. An analogy may be drawn with the energy level splitting of a particle with a nonzero magnetic moment in the magnetic field (Zeeman effect). If the magnetic moment $\mu$ is very small, a large enough magnetic field $B$ will provide the same level splitting ($\propto \mu \cdot B$) as is obtained with a low magnetic field and higher magnetic moment.

The equivalence of the expectation values of $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ of a spin-$\frac{1}{2}$ quantum state and of its spin-$S$ equivalent is shown pictorially in Fig. 1.

### 2.3 Classical simulation of entangled states

With this concept of equivalent states, the emergent exquisite feature is the notion of classical simulation of entangled states [21]. Classical simulation implies that a lower-dimensional entangled state can be mimicked by higher-dimensional separable states, as they share the same Q-representation. For example, consider the family of $2 \times 2$ Werner states [26]

$$\rho^{1/2\otimes 1/2}[\alpha] = \frac{1}{4}(\mathbb{1} - \alpha \sigma_1 \cdot \sigma_2); \quad -\frac{1}{3} \leq \alpha \leq 1. \quad (12)$$

The $(2S_1 + 1) \times (2S_2 + 1)$-dimensional equivalent state of $\rho^{1/2\otimes 1/2}[\alpha]$ is

$$\rho^{S_1\otimes S_2}[\alpha] = \frac{1}{(2S_1 + 1)(2S_2 + 1)}\{\mathbb{1} - \alpha \hat{S}_1 \cdot \hat{S}_2\}. \quad (13)$$
The set of $2 \times (2S + 1)$-dimensional equivalent separable states of two-qubit entangled Werner states has been identified in [21]. It has been shown that the state $\rho^{S/2 \otimes S}[\alpha]$ is separable in the range $|\alpha| \leq S_{min}/(S + 1)$. For any given $\alpha$, let $S_{min}$ be the minimum value of $S$, for which the state $\rho^{S/2 \otimes S}[\alpha]$ is separable. Its value is then given by the smallest half-integer greater than or equal to $|\alpha|/(1 - |\alpha|)$. Clearly, the value of $S_{min}$ increases with an increase in the entanglement of two-qubit Werner states.

Evidently, the separable equivalent of a pure two-qubit singlet state ($\alpha = 1$) does not exist in any finite dimension, so it is termed as exceptional. We next show that the separable equivalent states of noisy entangled two-qubit Werner states are discordant.

### 2.4 Separable equivalent states of two-qubit Werner states have nonzero discord

The $2 \times (2S + 1)$-dimensional equivalent state of the two-qubit Werner state $\rho^{S/2 \otimes S}[\alpha] = \frac{1}{4}(1 - \alpha \hat{\sigma}_1 \cdot \hat{\sigma}_2)$ is given by

$$\rho^{S/2 \otimes S}[\alpha] = \frac{1}{2(2S + 1)}(1 - \alpha \hat{\sigma}_1 \cdot \hat{S}_2).$$

The state $\rho^{S/2 \otimes S}[\alpha]$ is separable in the region $|\alpha| \leq S_{min}/(S + 1)$. We prove that it has a nonzero quantum discord in this range for $\alpha \neq 0$ by using its expansion in terms of separable states, given in [21].

Consider the following state:

$$\rho_z \equiv \frac{1}{2}(1 - \beta \sigma_1 z) \otimes |S\rangle \langle S|; \quad 0 < \beta \leq 1.$$
In the irreducible tensor basis, it has the form:

$$\rho_z = \frac{1}{2} (1 - \beta \sigma_{1z}) \otimes \frac{1}{2S + 1} \left\{ \mathbb{1} + \frac{3S}{S + 1} \hat{S}_{2z} + \sum_{k=2}^{2S} q^k S^{(k)} \right\}. \quad (16)$$

Obviously, the quantisation axis has been chosen to be the $z$ axis. The irreducible tensor operators are defined by,

$$S^{(k)}_z = C_k (S_2 \cdot \nabla)^k \ell Y_{k0}(\ell).$$

The normalisation factors, $C_k$, can be fixed conveniently.

Unlike $\rho_{1/2} \otimes \rho_1$, $\rho_z$ admits polarisations of all ranks $k \leq 2S$ and is anisotropic. We now construct similar states with the quantisation axes along the $x$ and the $y$ directions and denote them by $\rho_x, \rho_y$, respectively. The separable state $\bar{\rho} = \frac{1}{3} (\rho_x + \rho_y + \rho_z)$ obtained by their incoherent superposition has the form

$$\bar{\rho} = \frac{1}{3} (\rho_x + \rho_y + \rho_z) = \frac{1}{2(2S + 1)} \left\{ \mathbb{1} - \frac{S}{S + 1} \sigma_1 \cdot \hat{S}_2 \right\} + \cdots$$

$$= \frac{1}{2(2S + 1)} \left\{ \mathbb{1} - \alpha \sigma_1 \cdot \hat{S}_2 \right\} + \cdots; \quad \alpha = \frac{S}{S + 1}.$$

where the anisotropic terms are indicated by ellipses. To eliminate the unwanted anisotropic terms, we perform a uniformisation by averaging over the full sphere, which of course leaves the isotropic terms unchanged, and yields

$$\bar{\rho} \to \frac{1}{4\pi} \int \rho d\omega \equiv \rho^{1/2} \otimes \rho^{1/2}[\alpha]. \quad (18)$$

Evidently, the state $\rho^{1/2} \otimes \rho^{1/2}[\alpha]$ cannot be written as $\sum_i |\phi_{1i}\rangle \langle \phi_{1i}| \otimes \rho_{2i}$ or $\sum_j \rho_{1j} \otimes |\psi_{2j}\rangle \langle \psi_{2j}|$, where the sets $\{|\phi_{1i}\rangle\}$ and $\{|\psi_{2j}\rangle\}$ represent orthonormal bases in the spaces of the first and the second subsystems, respectively. Thus, $\rho^{1/2} \otimes \rho^{1/2}$ has a nonzero quantum discord, for all nonzero values of $\alpha$.

### 3 Applications

In this section, we propose various applications which use $2 \times N$-dimensional separable equivalent state $\rho^{1/2} \otimes \rho^{1/2}$, as a resource instead of the two-qubit entangled Werner state $\rho^{1/2} \otimes \rho^{1/2}$. As stated before, information is encoded in a qubit in the form of its Bloch vector. So, to transfer the encoded information to a remote qudit, its vector polarisation (in the spin operator basis) has to be suitably changed. After this, the remote party may employ equivalent observables given in [21] and also discussed in Sect. (2.2.1) to retrieve the encoded information. With this dictum, we propose various protocols as follows. The practical advantage that the proposed protocols serve is that
of encoding information in quantum states quite close to the maximally mixed state,\(^1\) which have hitherto been relegated from an information-theoretic viewpoint. We start with a protocol for remotely transferring information in an unknown qubit to a qudit.

### 3.1 Transfer of information from an unknown qubit to a remote qudit

In this section, we present a protocol for the transfer of information in an unknown qubit to a remote qudit. In the protocol, the separable state,

\[
\rho_{AB}^{(\alpha)} = \frac{1}{2S+1} (1 - \alpha \sigma_2 \cdot \hat{S}_3),
\]

(19)

is shared between Alice and Bob and thus acts as a quantum channel for communication. For a given \(\alpha\), the minimum value of \(S\), for which the state \(\rho_{AB}^{(\alpha)}\) is separable, is given by the smallest half-integer greater than \(\frac{|\alpha|}{1-|\alpha|}\).

Let \(\rho_A^1 = \frac{1}{2} (1 + \sigma_1 \cdot \mathbf{p})\) be the unknown qubit with Alice whose equivalent is to be remotely prepared. The superscripts of a state indicate the parties having or sharing the state, e.g. \(\rho_{AB}\) represents a bipartite state shared between Alice and Bob. The protocol is as follows:

1. Alice has two qubits:
   - a qubit, unknown to her, whose information is to be transferred remotely,
   - the qubit of \(\rho_{AB}^{(\alpha)}\), which has a shared correlation with the qudit at Bob.

   Thus, the combined state of Alice and Bob is given as

\[
\rho_1^A \otimes \rho_{AB}^{(\alpha)} = \frac{1}{2^2S+1} (1 + \sigma_1 \cdot \mathbf{p}) \otimes (1 - \alpha \sigma_2 \cdot \hat{S}_3).
\]

(20)

2. Alice performs a measurement in the Bell basis on her two qubits. She gets one of the four Bell states given below, with equal probability of \(1/4\):

\[
\rho_1^{AA} = \frac{1}{4} (1 - \sigma_1 \cdot \sigma_2);
\]

\[
\rho_2^{AA} = \frac{1}{4} (1 - \sigma_1 \sigma_2 + \sigma_1 \sigma_2 + \sigma_1 \sigma_2);
\]

\[
\rho_3^{AA} = \frac{1}{4} (1 + \sigma_1 \sigma_2 + \sigma_2 \sigma_1 + \sigma_1 \sigma_2);
\]

\[
\rho_4^{AA} = \frac{1}{4} (1 + \sigma_1 \sigma_2 + \sigma_2 \sigma_1 - \sigma_1 \sigma_2).
\]

(21)

3. She sends the information about her measurement outcome through a classical channel to Bob. Depending upon the information received, Bob performs a corresponding rotation on his state to retrieve the equivalent state of the qubit.

\(^1\) For \(S = 20\), the equivalent state \(\frac{1}{2^{S+1}} (1 + \frac{S}{S})\) of a pure single-qubit state \(\frac{1}{2} (1 + \sigma_z)\) has fidelity 0.876 with the completely mixed state \(\frac{1}{2^{S+1}} \mathbb{1}\).
Table 1 Post-measurement state of Alice and the respective transformation to be applied by Bob

| Post-measurement state of Alice | Collapsed state of Bob | Transformation to be applied by Bob |
|---------------------------------|------------------------|-------------------------------------|
| $\rho_1^{AA}$                  | $\frac{1}{2S+1}(1 + \alpha \hat{S}_3 \cdot \hat{p})$ | $\mathbb{I}$                        |
| $\rho_2^{AA}$                  | $\frac{1}{2S+1}(1 + \alpha (\hat{S}_{3x} p_{3x} - \hat{S}_{3y} p_{3y} - \hat{S}_{3z} p_{3z}))$ | $R_x(\pi)$                        |
| $\rho_3^{AA}$                  | $\frac{1}{2S+1}(1 + \alpha (-\hat{S}_{3x} p_{3x} + \hat{S}_{3y} p_{3y} - \hat{S}_{3z} p_{3z}))$ | $R_y(\pi)$                        |
| $\rho_4^{AA}$                  | $\frac{1}{2S+1}(1 + \alpha (-\hat{S}_{3x} p_{3x} - \hat{S}_{3y} p_{3y} + \hat{S}_{3z} p_{3z}))$ | $R_z(\pi)$                        |

Fig. 2 Pictorial representation of protocol for remote transfer of information from an unknown qubit to a qudit

4. If Alice obtains $\rho_1^{AA}, \rho_2^{AA}, \rho_3^{AA}, \rho_4^{AA}$, the transformations that Bob has to apply on his qudit are $\mathbb{I}, R_x(\pi), R_y(\pi), R_z(\pi)$, respectively. The symbols $R_x(\pi), R_y(\pi), R_z(\pi)$ represent Wigner rotation matrices of spin-$S$ about $x, y$, and $z$ axes through an angle $\pi$. After performing the rotations, the state at Bob is $\tilde{\rho}_B^B(\alpha) \equiv \frac{1}{2S+1}(1 + \alpha \hat{S}_3 \cdot \hat{p})$. Different transformations to be applied by Bob depending on the measurement outcomes of Alice are summarised in Table (1).

Thus, following the procedure, the spin-$S$ equivalent state of an unknown qubit is prepared at Bob’s end. The calculations for all the four possibilities (in which Alice gets $\rho_1^{AA}, \cdots, \rho_4^{AA}$) are shown below.
3.1.1 Calculation

We start with the case when the post-measurement state of Alice is the singlet state out of the four Bell states:

\[
\frac{1}{(2S + 1)^2} \text{Tr}_{12} \left\{ \left( \mathbb{1} - \sigma_1 \cdot \sigma_2 \right) \otimes \left( \mathbb{1} + \sigma_1 \cdot \hat{p} \right) \otimes \left( \mathbb{1} - \alpha \sigma_2 \cdot \hat{S}_3 \right) \right\}
\]

Measurement operator
Qubit whose polarisation is to be transferred remotely
State shared between Alice and Bob

\[
= \frac{1}{4} \times \frac{1}{2S + 1} \left( \mathbb{1} + \alpha \hat{S}_3 \cdot \hat{p} \right).
\]

(22)

Similarly, the post-measurement states corresponding to the other three Bell states are shown below:

\[
\frac{1}{(2S + 1)^2} \text{Tr}_{12} \left\{ \left( \mathbb{1} - (\sigma_{1x} \sigma_{2x} - \sigma_{1y} \sigma_{2y} - \sigma_{1z} \sigma_{2z}) \right) \right\}
\]

\[
\times (\mathbb{1} + \sigma_1 \cdot \hat{p}) \otimes (1 - \alpha \sigma_2 \cdot \hat{S}_3)
\]

\[
= \frac{1}{4} \times \frac{1}{2S + 1} \left( \mathbb{1} + \alpha (-\hat{S}_{3x} p_{3x} - \hat{S}_{3y} p_{3y} - \hat{S}_{3z} p_{3z}) \right);
\]

(23)

\[
\frac{1}{(2S + 1)^2} \text{Tr}_{12} \left\{ \left( \mathbb{1} - (\sigma_{1x} \sigma_{2x} + \sigma_{1y} \sigma_{2y} - \sigma_{1z} \sigma_{2z}) \right) \right\}
\]

\[
\times (\mathbb{1} + \sigma_1 \cdot \hat{p}) \otimes (1 - \alpha \sigma_2 \cdot \hat{S}_3)
\]

\[
= \frac{1}{4} \times \frac{1}{2S + 1} \left( \mathbb{1} + \alpha (-\hat{S}_{3x} p_{3x} + \hat{S}_{3y} p_{3y} - \hat{S}_{3z} p_{3z}) \right);
\]

(24)

\[
\frac{1}{(2S + 1)^2} \text{Tr}_{12} \left\{ \left( \mathbb{1} - (\sigma_{1x} \sigma_{2x} - \sigma_{1y} \sigma_{2y} + \sigma_{1z} \sigma_{2z}) \right) \right\}
\]

\[
\times (\mathbb{1} + \sigma_1 \cdot \hat{p}) \otimes (1 - \alpha \sigma_2 \cdot \hat{S}_3)
\]

\[
= \frac{1}{4} \times \frac{1}{2S + 1} \left( \mathbb{1} + \alpha (-\hat{S}_{3x} p_{3x} - \hat{S}_{3y} p_{3y} + \hat{S}_{3z} p_{3z}) \right).
\]

(25)

Consequently, Bob will be in possession of the states given in the equations (22), (23), (24) and (25) with equal probability of \(\frac{1}{4}\). The advantage of the proposed protocol is that it does not require an entangled state as a quantum channel. Furthermore, mixed separable (but discordant) states act as a quantum channel which are relatively easier to produce. Detailed calculations are given in Appendix (A).

3.2 Remote transfer of information of a known qubit to qudit

In this section, we lay down a protocol for remote transfer of information of a known qubit to a qudit. The quantum channel \(\rho^{AB} (\alpha)\), as before, is the \(2 \times (2S + 1)\)-
dimensional separable equivalent of $2 \times 2$-entangled Werner state

$$\rho^{AB}(\alpha) = \frac{1}{2(2S+1)}(\mathbb{1} - \alpha \sigma_1 \cdot \hat{S}_2).$$

Let us assume Alice wants to transfer the information of her qubit $\frac{1}{2}(\mathbb{1} - \sigma_1 \cdot \hat{m})$ to Bob’s qudit. The steps to be performed by Alice are as follows:

1. Alice measures $\sigma_1 \cdot \hat{m}$ on her qubit. She obtains either $+1$ or $-1$ with equal probability of $1/2$.
2. If she obtains $\pm 1$ as the measurement outcome, her subsystem collapses to the state $\pi_{\hat{m}}^\pm = \frac{1}{2}(\mathbb{1} \pm \sigma_1 \cdot \hat{m})$. The normalised post-measurement state of Bob’s subsystem is

$$\rho_{pm}^B = \frac{1}{2S+1}\{\mathbb{1} \mp \alpha \hat{S}_2 \cdot \hat{m}\}.$$  

(27)

3. If Alice obtains $+1$, she sends a classical message to Bob to do no transformation on his state. If Alice gets $-1$, she asks Bob to apply a rotation of $\pi$ about an axis perpendicular to $\hat{m}$ (such that $\hat{S}_2 \cdot \hat{m} \to -\hat{S}_2 \cdot \hat{m}$). This is because the state $\frac{1}{2S+1}\{\mathbb{1} + \alpha \hat{S}_2 \cdot \hat{m}\}$ can be changed to $\frac{1}{2S+1}\{\mathbb{1} - \alpha \hat{S}_2 \cdot \hat{m}\}$ by this transformation. Thus, Alice can prepare the requisite state at Bob’s end by sending one bit of classical information.

The pictorial representation of the proposed protocol is shown in Fig. 3.

### 3.2.1 Evaluation of the performance of the protocols

Obviously, the performance of the protocols proposed in Sects. (3.1) and (3.2) depends on the closeness of the qudit state prepared at Bob with the equivalent state of the qubit at Alice. We employ two distance measures, viz., fidelity, and Hilbert–Schmidt distance for analysing the performance of the protocol.

#### 3.2.2 Fidelity

Fidelity of two quantum states $\rho_1$ and $\rho_2$ is given by [27]

$$F = \left(\text{Tr}\sqrt{\rho_1 \rho_2 \sqrt{\rho_1}}\right)^2.$$  

(28)

In order to assess the performance of the protocols, we plot the fidelity of the remotely prepared qudit at Bob with the equivalent qudit of Alice’s qubit as a function of $\alpha$ in Fig. 4. For a given value of $\alpha$, we have taken the minimum value of $S$, corresponding to which the separable equivalent of the two-qubit Werner state exists (for details, see Sect. (2.3)). On the same graph, we also plot the fidelity of $\frac{1}{2}(\mathbb{1} + \sigma_z)$ with $\frac{1}{2}(\mathbb{1} + \alpha \sigma_z)$ that is prepared using an entangled two-qubit Werner state as a quantum channel in the quantum teleportation protocol [5].
Evidently, the plots show that it is easier to transfer information of a qubit to a remote qudit due to the following two reasons:

1. For transfer of information from a qubit to a qudit, only a discordant state needs to be shared, and,
2. For a given value of \( \alpha \), the fidelity of the remotely prepared qudit with the equivalent qudit of the qubit (at Alice) is higher than that of the remotely prepared qubit with itself.

### 3.2.3 Hilbert–Schmidt distance

The Hilbert–Schmidt distance between two quantum states \( \rho_1 \) and \( \rho_2 \) is given by [28]

\[
d = \sqrt{\text{Tr}[(\rho_1 - \rho_2)^2]}.
\]  

(29)
For the proposed protocol, we require the distance between \( \rho_1 = \frac{1}{2S+1}(\mathbb{1} + \hat{S} \cdot \hat{p}) \) (equivalent qudit of Alice’s qubit) and \( \rho_2 = \frac{1}{2S+1}(\mathbb{1} + \alpha \hat{S} \cdot \hat{p}) \) (Bob’s qudit at the end of the protocol), which is given as

\[
d = (1 - \alpha)\sqrt{\frac{S + 1}{3S(2S + 1)}}.
\] (30)

If the quantum channel between Alice and Bob is completely mixed, Bob has a completely mixed state with him irrespective of the Bloch vector of Alice’s qubit. In this case, the distance between (i) equivalent qudit of Alice’s qubit, and, (ii) the qudit prepared at Bob, which is the maximally mixed state, is given as

\[
d_0 = \sqrt{\frac{S + 1}{3S(2S + 1)}}.
\] (31)

Thus, the difference of Eqs. (31) and (30), termed as relative distance,

\[
\mathcal{D} \equiv d_0 - d = \alpha \sqrt{\frac{S + 1}{3S(2S + 1)}},
\] (32)

may be used for quantification of performance of the protocol. This is because it quantifies the reduction in the distance between the qudit prepared at Bob for a given value of \( \alpha \), and, that prepared at Bob for \( \alpha = 0 \) (which corresponds to the completely noisy channel). The derivative of the relative distance \( \mathcal{D} \) (given in equation (32)) with respect to \( S \) is

\[
\mathcal{D}'(S) = -\frac{\alpha(2S^2 + 4S + 1)}{2\sqrt{3}(S + 1)S^{3/2}(2S + 1)^{3/2}} < 0,
\] (33)

which is negative for any nonzero \( \alpha \). Clearly, for a given \( \alpha \), the relative distance \( \mathcal{D} \) decreases as \( S \) increases. The plot of the relative distance \( \mathcal{D} \) with \( S \) is shown in Fig. 5 for \( \alpha = 0.9 \).

For large values of \( S \), the asymptotic values of \( \mathcal{D} \) and \( \mathcal{D}'(S) \) (given in equations (32) and (33), respectively) are \( \frac{\alpha}{\sqrt{6S}} \) and \( -\frac{\alpha}{2\sqrt{6S^{3/2}}} \), both of which tend to zero as \( S \to \infty \).

### 3.2.4 Efficiency of the proposed protocols

In this section, we show that the proposed protocols may be implemented efficiently, thanks to the separability of shared channels and availability of higher-dimensional orbital angular momentum states. The yield, in these protocols, will be high since there is no need for a nonlinear process to generate separable states. In contrast, entangled states, which are generated by, say, spontaneous parametric down conversion, have a very low yield with a reported efficiency of \( 4 \times 10^{-6} \) [29]. The techniques available for generation, manipulation and detection of orbital angular momentum states are...
well-developed (see, for example, [30,31] and references therein). Hence, they can be employed in the implementation of these protocols. Finally, apart from being merely discordant, the resource states employed in the protocols presented in this paper are mixed, which further reduces the burden of preparation.

3.3 Transfer of information of an unknown qudit to a remote qudit with separable equivalent of Werner state

In this section, we propose a protocol for the transfer of information of vector polarisation in the spin operator basis of an unknown qudit to a remotely located qudit. As before, the $1/2 \otimes S$ separable equivalent of the $1/2 \otimes 1/2$ Werner state,

$$\rho^{AB}(\alpha)^{1/2 \otimes 1/2} \approx \rho^{AB}(\alpha)^{1/2 \otimes S} = \frac{1}{2(2S+1)} (\mathbb{1} - \alpha \sigma_2 \cdot \hat{S}_3), \quad (34)$$

acts as a quantum channel in this protocol. The steps of the protocol are as follows:

1. Let $\rho^A = \frac{1}{2S+1} (\mathbb{1} + \hat{S}_1 \cdot \sigma)\mathbb{1}$ be the $(2S+1)$-dimensional state whose information is to be transferred by Alice. Thus, Alice has two subsystems; (i) the unknown state whose information she wants to transfer to a remote qudit, (ii) the other qubit which has shared correlation with Bob’s qudit.

2. Alice performs a measurement of $\hat{S}_1 \cdot \sigma_2$, which is a dichotomic observable having eigenvalues $+1$ and $-\frac{S}{S+1}$, The two eigenvalues are degenerate with respective degeneracies of $2(S+1)$ and $2S$, respectively. The corresponding eigenprojections are

$$\Pi_1 = \frac{S+1}{2S+1} (\mathbb{1} + \frac{S}{S+1} \hat{S}_1 \cdot \sigma_2),$$

$$\Pi_2 = \frac{S}{2S+1} (\mathbb{1} - \hat{S}_1 \cdot \sigma_2). \quad (35)$$
3. Upon measurement of $\Pi_1$ and $\Pi_2$ by Alice, the respective post-measurement states at Bob are \[ \frac{1}{2S+1} \left( \mathbb{1} - \frac{\alpha}{3} \hat{S}_3 \cdot \hat{p} \right) \] and \[ \frac{1}{2S+1} \left( \mathbb{1} + \alpha \frac{S+1}{3S} \hat{S}_3 \cdot \hat{p} \right) \] with probabilities $\frac{S+1}{2S+1}$ and $\frac{S}{2S+1}$, respectively.

4. After performing the measurement, Alice sends Bob information about her measurement result. If Alice obtains $+1$, the direction of polarisation vector of Bob’s qudit is opposite to that of Alice’s qudit. On the other hand, if Alice gets $-\frac{S+1}{S}$, the direction of polarisation vector of Bob’s qudit is the same as that of Alice’s qudit. Since Bob has the knowledge of the channel, he knows that in both the cases, the polarisation vector of the qudit obtained by him has shrunk by a factor of approximately $\frac{1}{3}$ as compared to the qudit at Alice.

5. As Bob is interested in the information encoded in the polarisation vector, he needs to measure equivalent observables, proposed in [21]; given as $\frac{S}{S+1} \hat{S}_3 \cdot \hat{m}$, where $\hat{m} \in \{ \hat{x}, \hat{y}, \hat{z} \}$. As explained in the protocol, the magnitude of the polarisation vector is diminished by a constant factor $\frac{1}{3}$ which is a characteristic of the channel used. Therefore, it can be compensated by an extra factor of $3$ in the observable to be measured for retrieval of information (as explained in (2.2.1) for the example of magnetic moment). Thus, Bob can retrieve full information about the polarisation vector by measurement of appropriate observables.

The calculations for this are shown in appendix (B). The protocol is schematically shown in Fig. 6.

The advantage of this protocol is that in place of an entangled state, a mixed separable (but discordant) state acts as a quantum channel. The above protocol can be used...
to send only those qudit states which have only nonzero vector polarisation. However, this restriction can be relaxed if one uses the equivalent of higher-dimensional entangled states.

### 3.3.1 Evaluation of the performance of the protocol

We employ the difference in the Hilbert–Schmidt distances of remotely prepared qudits with a nonzero value of $\alpha$ and with $\alpha = 0$ as the figure of merit. This difference, for the outcomes $+1$ and $-\frac{S+1}{S}$ for $\hat{S}_1 \cdot \sigma_2$, is given by the following expressions, respectively:

$$D_1 = \frac{\alpha}{3} \sqrt{\frac{S+1}{3S(2S+1)}},$$

$$D_2 = \frac{\alpha(S+1)}{3S} \sqrt{\frac{S+1}{3S(2S+1)}}. \quad (36)$$

For large values of $S$, the asymptotic values of $D_1$ and $D_2$ are the same, given by $\frac{\alpha}{3\sqrt{6S}}$, both of which tend to 0 as $S \to \infty$.

### 3.4 Swapping of quantum discord

Swapping of quantum correlations has been an interesting area of study, with the celebrated protocol proposed in [9] for entanglement swapping. Remote transfer of Gaussian quantum discord and swapping of quantum correlations between two-qubit Werner states has been earlier studied in [32] and [33], respectively.

In this section, we describe a protocol for swapping of quantum discord from $1/2 \otimes S$ systems to $S \otimes S$ systems. For this, let there be four parties named as Alice, Bob, Charlie and David. Let the states shared between the pairs (Alice, Bob) and (Charlie, David) be

$$\frac{1}{2^{(2S+1)}} (1 - \alpha \sigma_1 \cdot \hat{S}_2)$$

and

$$\frac{1}{2^{(2S+1)}} (1 - \beta \hat{S}_3 \cdot \sigma_4),$$

respectively. These states are separable equivalents of

$$\frac{1}{4} (1 - \sigma_1 \cdot \sigma_2)$$

and

$$\frac{1}{4} (1 - \sigma_3 \cdot \sigma_4),$$

respectively. Note that the minimum value of $S$ gets fixed according to the values of $\alpha$ or $\beta$, as discussed in the Sect. (2.3). Obviously, the states shared between the pairs (Alice, Bob) and (Charlie, David) are discordant states. There is no correlation between the pairs (Alice, David) and (Bob, Charlie). In this protocol, we show that quantum discord can be generated between Bob and Charlie, which are a-priori uncorrelated, by appropriate measurements.

The combined state of all the four-parties is as follows:

$$\frac{1}{2^{(2S+1)}} (1 - \alpha \sigma_1 \cdot \hat{S}_2)(1 - \beta \hat{S}_3 \cdot \sigma_4). \quad (37)$$

If a measurement in the Bell basis is performed by Alice and David, the resultant state of Bob and Charlie will be discordant. For example, if a measurement of the singlet state is performed on the first and the fourth party, i.e. $\frac{1}{4} (1 - \sigma_1 \cdot \sigma_4)$ is measured,
Table 2 Measurement performed by Alice and David and corresponding transformation to be applied by Bob

| Measurement performed by Alice and David | Transformation to be applied by Bob |
|------------------------------------------|------------------------------------|
| $\rho_{1AD}^{AD} \equiv \frac{1}{4}(\mathbb{1} - \sigma_1 \cdot \sigma_4)$ | $\mathbb{1}$ |
| $\rho_{2AD}^{AD} \equiv \frac{1}{4}(1 - \sigma_1 \cdot \sigma_4 + \sigma_1 \cdot \sigma_4 + \sigma_1 \cdot \sigma_4)$ | $R_x(\pi)$ |
| $\rho_{3AD}^{AD} \equiv \frac{1}{4}(1 + \sigma_1 \cdot \sigma_4 + \sigma_1 \cdot \sigma_4 - \sigma_1 \cdot \sigma_4)$ | $R_z(\pi)$ |
| $\rho_{4AD}^{AD} \equiv \frac{1}{4}(1 + \sigma_1 \cdot \sigma_4 - \sigma_1 \cdot \sigma_4 + \sigma_1 \cdot \sigma_4)$ | $R_y(\pi)$ |

then

$$
\text{Tr}_{14} \frac{1}{2^4(2S+1)^2} \left\{ (\mathbb{1} - \sigma_1 \cdot \sigma_4)(\mathbb{1} - \alpha \sigma_1 \cdot \hat{S}_2)(\mathbb{1} - \beta \hat{S}_3 \cdot \sigma_4) \right\} = \\
= \frac{1}{2^4(2S+1)^2} \text{Tr}_{14} \left( \mathbb{1} - \alpha \beta \sigma_1 \cdot \hat{S}_2 \sigma_1 \cdot \sigma_4 \hat{S}_3 \cdot \sigma_4 \right) = \\
= \frac{1}{4} \times \frac{1}{(2S+1)^2}(\mathbb{1} - \alpha \beta \hat{S}_2 \cdot \hat{S}_3). 
$$

Similar calculations for the other three Bell states are shown in Appendix (C). The transformation to be applied by Bob corresponding to different measurements performed by Alice and David are shown in table (2).

Thus, by having two lower-dimensional discordant states \( \frac{1}{2(2S+1)}(1 - \alpha \sigma_1 \cdot \hat{S}_2) \) and \( \frac{1}{2(2S+1)}(1 - \beta \hat{S}_3 \cdot \sigma_4) \), a higher-dimensional discordant state \( \frac{1}{(2S+1)^2}(1 - \alpha \beta \hat{S}_2 \cdot \hat{S}_3) \) can be produced. The schematic representation of the protocol is given in Fig. 7.

The protocols proposed in Sects. (3.1) and (3.4) involve Bell state measurements. Fortunately, due to their vital role, Bell state measurements have been studied extensively [34–38]. These studies would play a crucial role in the experimental implementation of the protocols.

3.5 Future prospects

At this stage, it is worthwhile to consider the future prospects of quantum communication with equivalent states and possible advantages that they would afford. Firstly, generalisations of our protocols to higher dimensional and multi-party systems constitute an interesting area of study. In fact, our protocols can be immediately generalised to remote transfer of information encoded in qutrits to remote qudits. This would require employment of the separable equivalents of \( SU(2) \)-invariant \( 3 \times N \) level systems, as presented in [22]. Secondly, these protocols provide a motivation for the identification of higher-dimensional separable equivalent states of multi-party entangled systems, which is an interesting theoretical problem by itself. Those equivalent states can be employed for distribution of quantum information over a quantum network. In fact, a number of possibilities emerge when we seek to determine \( k \)-separable higher-dimensional equivalent states of an \( N \)-party lower-dimensional fully entangled state.
The condition, $k = N$, implies a fully separable state, whereas $k = 1, \cdots, (N - 1)$ represent partially entangled states, which are equivalent to a fully entangled state in lower dimensions. Finally, this work shows the existence of quantum communication protocols in which we can dispense with entanglement.

**4 Conclusion**

In summary, this paper proposes various quantum communication protocols using $\frac{1}{2} \otimes S$ discordant states, identified in [21], as separable equivalents of entangled $\frac{1}{2} \otimes \frac{1}{2}$ Werner states. We have shown that remote transfer of equivalent state of qubit can be performed with approximately the same fidelity as noisy entangled state. Thus, the requirement of noisy entangled states, whose generation is very difficult, can be removed for transfer of information of polarisation vector in the spin operator basis. We have also shown how quantum discord in higher-dimensional states can be generated from two lower-dimensional discordant states by appropriate measurements.

Finally, the protocols based on separable equivalents of two-qutrit entangled states constitute an interesting study which will be undertaken elsewhere.

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**Author Contributions**  All the authors have contributed equally in all respects.
A Transfer of information from an unknown qubit to a remote qudit

Detailed calculations leading to Eqs. (22), (23), (24) and (25) are as follows. They essentially employ three properties, viz., tracelessness of Pauli matrices, \( \text{Tr}(\sigma \cdot \hat{m} \sigma \cdot \hat{n}) = 2\hat{m} \cdot \hat{n} \), and, \( \text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B) \). It is sufficient to illustrate it for one case. The same method works for the other three equations as well.

We start with the case when Alice measures a singlet state out of the four Bell states, i.e. \( \rho^{AA}_1 = \frac{1}{4}(1 - \sigma_1 \cdot \sigma_2) \), which leads to equation (22), as follows:

\[
\frac{1}{(2S + 1)^2} \text{Tr}_{12} \left\{ \left( 1 - \sigma_1 \cdot \sigma_2 \right) \left( 1 + \sigma_1 \cdot \hat{p} \right) \otimes \left( 1 - \alpha \sigma_2 \cdot \hat{S}_3 \right) \right\} \\
= \frac{1}{(2S + 1)^2} \text{Tr}_{12} \left\{ \left( 1 - \sigma_1 \cdot \sigma_2 \right) \left( 1 + \sigma_1 \cdot \hat{p} - \alpha \sigma_2 \cdot \hat{S}_3 - \alpha \sigma_1 \cdot \hat{p} \otimes \sigma_2 \cdot \hat{S}_3 \right) \right\} \\
= \frac{1}{(2S + 1)^2} \text{Tr}_{12} \left\{ 1 + \sigma_1 \cdot \hat{p} - \alpha \sigma_2 \cdot \hat{S}_3 - \alpha \sigma_1 \cdot \hat{p} \otimes \sigma_2 \cdot \hat{S}_3 \right. \\
\left. - \sigma_1 \sigma_2 (1 + \sigma_1 \cdot \hat{p} - \alpha \sigma_2 \cdot \hat{S}_3 - \alpha \sigma_1 \cdot \hat{p} \otimes \sigma_2 \cdot \hat{S}_3) - \sigma_1 \sigma_2 (1 + \sigma_1 \cdot \hat{p} - \alpha \sigma_2 \cdot \hat{S}_3 - \alpha \sigma_1 \cdot \hat{p} \right. \\
\otimes \sigma_2 \cdot \hat{S}_3 \right\} \\
= \frac{1}{(2S + 1)^2} \left\{ 4(1 + 4p_x \hat{S}_3x + 4p_y \hat{S}_3y + 4p_z \hat{S}_3z) \right\} \\
= \frac{1}{4} \times \frac{1}{2S + 1} (1 + \alpha \hat{S}_3 \cdot \hat{p}), \quad (39)
\]

which is the same as the equation (22).

B Transfer of information from an unknown qudit to a remote qudit

When Alice measures \( \hat{S}_1 \cdot \sigma_2 \), she gets the states \( \frac{1}{2S + 1} \left( 1 - \frac{\alpha}{3} \hat{S}_3 \cdot \hat{p} \right) \) and \( \frac{1}{2S + 1} \left( 1 + \frac{\alpha}{3} \hat{S}_3 \cdot \hat{p} \right) \) with respective probabilities of \( \frac{S + 1}{2S + 1} \) and \( \frac{S}{2S + 1} \). The detailed calculations are as follows, which essentially employ the following two properties:

1. The spin operators \( S_x, S_y, S_z \) are traceless, i.e. \( \text{Tr} S_i = 0 \) and \( \text{Tr} \sigma_j = 0 \), \( i, j \in x, y, z \).
2. Trace of tensor product of two operators is equal to the product of the trace of operators, i.e. \( \text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B) \).

Case I When Alice’s state is projected into the eigenstate corresponding to the eigenvalue +1, which occurs with a probability \( \frac{S + 1}{2S + 1} \), then

\[
\frac{1}{2(2S + 1)^2} \text{Tr}_{12} \left\{ \left( 1 + \frac{S}{S + 1} \hat{S}_1 \cdot \sigma_2 \right) \left( 1 + \hat{S}_1 \cdot \hat{p} \right) \otimes \left( 1 - \alpha \sigma_2 \cdot \hat{S}_3 \right) \right\}
\]
\[
\frac{1}{2(2S+1)^2} \text{Tr}_{12}\left\{ \left(1 + \frac{S}{S+1} \hat{S}_1 \cdot \sigma_2 \right) \times \left(1 + \hat{S}_1 \cdot p - \alpha \sigma_2 \cdot \hat{S}_3 - \alpha \hat{S}_1 \cdot p \otimes \sigma_2 \cdot \hat{S}_3 \right) \right\}
\]
\[
= \frac{1}{2(2S+1)^2} \text{Tr}_{12}\left\{ \hat{S}_1 \cdot p - \alpha \sigma_2 \cdot \hat{S}_3 - \alpha \hat{S}_1 \cdot p \otimes \sigma_2 \cdot \hat{S}_3 \right\}
\]
\[
\geq \frac{S}{S+1} \hat{S}_1 \cdot \sigma_2 + \frac{S}{S+1} (\hat{S}_1 \cdot \sigma_2)(\hat{S}_1 \cdot p) - \alpha \frac{S}{S+1} (\hat{S}_1 \cdot \sigma_2)(\sigma_2 \cdot \hat{S}_3)
\]
\[
- \alpha \frac{S}{S+1} (\hat{S}_1 \cdot \sigma_2)(\sigma_2 \cdot \hat{S}_3)
\]
\[
= \frac{1}{2(2S+1)^2} \left\{ \hat{S}_1 \cdot p - \alpha \sigma_2 \cdot \hat{S}_3 - \alpha (\hat{S}_1 \cdot p) \otimes \sigma_2 \cdot \hat{S}_3 \right\}
\]
\[
= \frac{1}{2S+1} \left(1 - \frac{\alpha}{3} \hat{S}_3 \cdot p \right).
\]

The formula that we have used \(\text{Tr}(\hat{S}_i \hat{S}_j) = \frac{(S+1)(2S+1)}{3S} \delta_{ij}\) can be proved as follows. We have

\[
S_x^2 + S_y^2 + S_z^2 = S(S+1)I.
\]

Taking trace of both sides and employing the property that \(\text{Tr}S_x^2 = \text{Tr}S_y^2 = \text{Tr}S_z^2\), we obtain

\[
3\text{Tr}S_x^2 = S(S+1)\text{Tr}I = S(S+1)(2S+1)
\]
\[
\Rightarrow \text{Tr}S_x^2 = \frac{S(S+1)(2S+1)}{3}.
\]

Similarly, \(\text{Tr}S_y^2 = \text{Tr}S_z^2 = \frac{S(S+1)(2S+1)}{3} \). Furthermore, we have the following expression of invariance of trace under a unitary transformation,

\[
\text{Tr}(S_x S_y) = \text{Tr}(U S_x S_y U^\dagger),
\]

where \(U\) is a unitary transformation. Consider a rotation about the \(X\) axis such that \(\hat{y} \rightarrow -\hat{y}\) and \(\hat{z} \rightarrow -\hat{z}\). Thus, under the effect of this transformation

\[
\text{Tr}(S_x S_y) = \text{Tr}(-S_x S_y) \quad \Rightarrow \quad \text{Tr}(S_x S_y) = 0.
\]
Both of these results can be combined to the following equation:

\[
\text{Tr}(\hat{S}_i \hat{S}_j) = \frac{(S + 1)(2S + 1)}{3S} \delta_{ij}.
\]

(45)

**Case II** When Alice’s state is projected to the eigenstate corresponding to the eigenvalue \(-\frac{S+1}{S}\), which occurs with a probability \(\frac{S}{2S+1}\), then

\[
\frac{1}{2(2S+1)^2} \text{Tr}_{12} \left\{ (\mathbb{I} - \hat{S}_1 \cdot \sigma_2)(\mathbb{I} + \hat{S}_1 \cdot \mathbf{p}) \otimes (1 - \alpha \sigma_2 \cdot \hat{S}_3) \right\} = \frac{1}{2S + 1} \left( \mathbb{I} + \alpha \frac{(S + 1)}{3S} \hat{S}_3 \cdot \mathbf{p} \right).
\]

(46)

This result is calculated following the same methodology as done for the Case I.

**C Swapping of quantum discord**

In this appendix, we present those cases of swapping of quantum discord, proposed in Sect. (3.4), in which the remaining three Bell states are measured. The calculations for rest of the three projection operators of Bell basis are as follows. They essentially employ three properties, \textit{viz.}, tracelessness of Pauli matrices, \(\text{Tr}(\sigma \cdot \hat{m} \sigma \cdot \hat{n}) = 2\hat{m} \cdot \hat{n}\), and, \(\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)\).

1. When the measurement of \(\frac{1}{4} \left( \mathbb{I} - (\sigma_{1x} \sigma_{4x} - \sigma_{1y} \sigma_{4y} - \sigma_{1z} \sigma_{4z}) \right)\) is made, the post-measurement state is

\[
\frac{1}{2^4(2S+1)^2} \text{Tr}_{14} \left\{ (\mathbb{I} - (\sigma_{1x} \sigma_{4x} - \sigma_{1y} \sigma_{4y} - \sigma_{1z} \sigma_{4z}))(\mathbb{I} - \alpha \sigma_1 \cdot \hat{S}_2) \right\}
\]

\[
= \frac{1}{2^4(2S+1)^2} \text{Tr}_{14} \left\{ (\mathbb{I} - \beta \hat{S}_3 \cdot \sigma_4) \right\}
\]

\[
= \frac{1}{2^4(2S+1)^2} \text{Tr}_{14} \left\{ (\mathbb{I} - \alpha \beta \sigma_1 \cdot \hat{S}_2(\sigma_{1x} \sigma_{4x} - \sigma_{1y} \sigma_{4y} - \sigma_{1z} \sigma_{4z}) \hat{S}_3 \cdot \sigma_4) \right\}
\]

\[
= \frac{1}{2^4(2S+1)^2} \text{Tr}_{14} \left\{ (\mathbb{I} - \alpha \beta \sigma_1 \cdot \hat{S}_2(\sigma_{1x} \sigma_{4x} \hat{S}_{3x} \sigma_{4x}) \right. \\
+ \sigma_{1x} \sigma_{4x} \hat{S}_{3z} \sigma_{4y} + \sigma_{1x} \sigma_{4x} \hat{S}_{3z} \sigma_{4z} \\
- \sigma_{1y} \sigma_{4y} \hat{S}_{3x} \sigma_{4x} - \sigma_{1y} \sigma_{4y} \hat{S}_{3y} \sigma_{4y} - \sigma_{1y} \sigma_{4y} \hat{S}_{3z} \sigma_{4z} - \sigma_{1z} \sigma_{4z} \hat{S}_{3x} \sigma_{4x} \\
- \sigma_{1z} \sigma_{4z} \hat{S}_{3y} \sigma_{4y} - \sigma_{1z} \sigma_{4z} \hat{S}_{3z} \sigma_{4z} \right\}
\]

\[
= \frac{1}{2^4(2S+1)^2} \text{Tr}_{14} \left\{ (\mathbb{I} - \alpha \beta \sigma_1 \cdot \hat{S}_2(2\sigma_{1x} \hat{S}_{3x} - 2\sigma_{1y} \hat{S}_{3y} - 2\sigma_{1z} \hat{S}_{3z}) \right. \\
= \frac{1}{2^3(2S+1)^2} \text{Tr}_{1} \left\{ (\mathbb{I} - \alpha \beta \sigma_1 \cdot \hat{S}_2(\sigma_{1x} \hat{S}_{3x} - \sigma_{1y} \hat{S}_{3y} - \sigma_{1z} \hat{S}_{3z}) \right\}
\]
\[
\begin{align*}
= \frac{1}{2^3(2S+1)^2} \text{Tr}_1 \left( \mathbb{1} - \alpha \beta (\sigma_{1x} \hat{S}_{2x} (\sigma_{1x} \hat{S}_{3x} - \sigma_{1y} \hat{S}_{3y} - \sigma_{1z} \hat{S}_{3z}) \\
+ \sigma_{1y} \hat{S}_{2y} (\sigma_{1x} \hat{S}_{3x} - \sigma_{1y} \hat{S}_{3y} - \sigma_{1z} \hat{S}_{3z}) \\
+ \sigma_{1z} \hat{S}_{2z} (\sigma_{1x} \hat{S}_{3x} - \sigma_{1y} \hat{S}_{3y} - \sigma_{1z} \hat{S}_{3z}) \right) \\
= \frac{1}{2^3(2S+1)^2} \text{Tr}_1 \left( \mathbb{1} - \alpha \beta (\sigma_{1x} \hat{S}_{2x} (\sigma_{1x} \hat{S}_{3x} - \sigma_{1y} \hat{S}_{3y} - \sigma_{1z} \hat{S}_{3z}) \\
+ \sigma_{1y} \hat{S}_{2y} (\sigma_{1x} \hat{S}_{3x} - \sigma_{1y} \hat{S}_{3y} - \sigma_{1z} \hat{S}_{3z}) \\
+ \sigma_{1z} \hat{S}_{2z} (\sigma_{1x} \hat{S}_{3x} - \sigma_{1y} \hat{S}_{3y} - \sigma_{1z} \hat{S}_{3z}) \right) \\
= \frac{1}{2^3(2S+1)^2} \left( 2 \mathbb{1} - \alpha \beta (2 \hat{S}_{2x} \hat{S}_{3x} - 2 \hat{S}_{2y} \hat{S}_{3y} - 2 \hat{S}_{2z} \hat{S}_{3z}) \right) \\
= \frac{1}{4} \times \frac{1}{(2S+1)^2} \left( \mathbb{1} - \frac{\alpha \beta}{S^2} (S_{2x} S_{3x} - S_{2y} S_{3y} - S_{2z} S_{3z}) \right). \tag{47}
\end{align*}
\]

The factor of \( \frac{1}{4} \) represents the probability with which this state is prepared. Other calculations are performed using the same approach.

2. When the measurement of \( \frac{1}{4} \left( \mathbb{1} - (-\sigma_{1x} \sigma_{4x} - \sigma_{1y} \sigma_{4y} + \sigma_{1z} \sigma_{4z}) \right) \) is made

\[
\text{Tr}_{14} \frac{1}{2^4(2S+1)^2} \left\{ \left( \mathbb{1} - (-\sigma_{1x} \sigma_{4x} - \sigma_{1y} \sigma_{4y} + \sigma_{1z} \sigma_{4z}) \right) \times (\mathbb{1} - \alpha \sigma_1 \cdot \hat{S}_2)(\mathbb{1} - \beta \hat{S}_3 \cdot \sigma_4) \right\} \\
= \frac{1}{2^4(2S+1)^2} \text{Tr}_{14} \left( \mathbb{1} - \alpha \beta \sigma_1 \cdot \hat{S}_2 (-\sigma_{1x} \sigma_{4x} - \sigma_{1y} \sigma_{4y} + \sigma_{1z} \sigma_{4z}) \hat{S}_3 \cdot \sigma_4 \right) \\
= \frac{1}{4} \times \frac{1}{(2S+1)^2} \left( \mathbb{1} - \frac{\alpha \beta}{S^2} (-S_{2x} S_{3x} - S_{2y} S_{3y} + S_{2z} S_{3z}) \right). \tag{48}
\]

The factor of \( \frac{1}{4} \) represents the probability with which this state is prepared.

3. When the measurement of \( \frac{1}{4} \left( \mathbb{1} - (-\sigma_{1x} \sigma_{4x} + \sigma_{1y} \sigma_{4y} - \sigma_{1z} \sigma_{4z}) \right) \) is made

\[
\text{Tr}_{14} \frac{1}{2^4(2S+1)^2} \left\{ \left( \mathbb{1} - (-\sigma_{1x} \sigma_{4x} + \sigma_{1y} \sigma_{4y} - \sigma_{1z} \sigma_{4z}) \right) (\mathbb{1} - \alpha \sigma_1 \cdot \hat{S}_2) \\
(\mathbb{1} - \beta \hat{S}_3 \cdot \sigma_4) \right\} \\
= \frac{1}{2^4(2S+1)^2} \text{Tr}_{14} \left( \mathbb{1} - \alpha \beta \sigma_1 \cdot \hat{S}_2 (-\sigma_{1x} \sigma_{4x} + \sigma_{1y} \sigma_{4y} - \sigma_{1z} \sigma_{4z}) \hat{S}_3 \cdot \sigma_4 \right) \\
= \frac{1}{4} \times \frac{1}{(2S+1)^2} \left( \mathbb{1} - \frac{\alpha \beta}{S^2} (-S_{2x} S_{3x} + S_{2y} S_{3y} - S_{2z} S_{3z}) \right). \tag{49}
\]

The factor of \( \frac{1}{4} \) represents the probability with which this state is prepared.
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