CONSTRUCTION OF SPECTRAL INVARIANTS
OF HAMILTONIAN PATHS ON
CLOSED SYMPLECTIC MANIFOLDS

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Abstract. In this paper, we develop a mini-max theory of the action functional over
the semi-infinite cycles via the chain level Floer homology theory and construct spec-
tral invariants of Hamiltonian diffeomorphisms on arbitrary, especially on non-exact
and non-rational, compact symplectic manifold \((M, \omega)\). To each given time depen-
dent Hamiltonian function \(H\) and quantum cohomology class \(0 \neq a \in QH^*(M)\), we
associate an invariant \(\rho(H; a)\) which varies continuously over \(H\) in the \(C^0\)-topology.
This is obtained as the mini-max value over the semi-infinite cycles whose homology
class is ‘dual’ to the given quantum cohomology class \(a\) on the covering space \(\tilde{\Omega}_0(M)\)
of the contractible loop space \(\Omega_0(M)\). We call them the Novikov Floer cycles. We
apply the spectral invariants to the study of Hamiltonian diffeomorphisms in sequels
of this paper.

We assume that \((M, \omega)\) is strongly semi-positive in this paper, which will be
removed in a sequel to this paper.

Dedicated to Alan Weinstein in honor of his 60-th birthday

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§1. Introduction and the main results

The group $\mathcal{H}am(M, \omega)$ of (compactly supported) Hamiltonian diffeomorphisms of the symplectic manifold $(M, \omega)$ carries a remarkable invariant norm defined by

$$
\| \phi \| = \inf_{H \mapsto \phi} \| H \|
$$

$$
\| H \| = \int_0^1 (\max H_t - \min H_t) \, dt
$$

(1.1)

which was introduced by Hofer [Ho]. Here $H \mapsto \phi$ means that $\phi$ is the time-one map $\phi^1_H$ of the Hamilton’s equation $\dot{x} = X_H(x)$ of the Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$, where the Hamiltonian vector field is defined by

$$
\omega(X_H, \cdot) = dH.
$$

(1.2)

This norm can be easily defined on arbitrary symplectic manifolds although proving non-degeneracy is a non-trivial matter (See [Ho], [Po1] and [LM] for its proof of increasing generality. See also [Ch] for a Floer theoretic proof and [Oh3] for a simple proof of the non-degeneracy in tame symplectic manifolds).

On the other hand Viterbo [V] defined another invariant norm on $\mathbb{R}^{2n}$. This was defined by considering the graph of the Hamiltonian diffeomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and compactifying the graph in the diagonal direction in $\mathbb{R}^{4n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ into $T^*S^{2n}$. He then applied the critical point theory of generating functions of the Lagrangian submanifold graph $\phi \subset T^*S^{2n}$ which he developed on the cotangent bundle $T^*N$ of the arbitrary compact manifold $N$. To each cohomology class $a \in H^*(N)$, Viterbo associated certain homologically essential critical values of generating functions of any Lagrangian submanifold $L$ Hamiltonian isotopic to the zero section of $T^*N$ and proved that they depend only on the Lagrangian submanifold but not on the generating functions, at least up to normalization.

The present author [Oh1,2] and Milinković [MO1,2, M] developed a Floer theoretic approach to construction of Viterbo’s invariants using the canonically defined action functional on the space of paths, utilizing the observation made by Weinstein [W] that the action functional is a generating function of the given Lagrangian submanifold defined on the path space. This approach is canonical including normalization and provides a direct link between Hofer’s geometry and Viterbo’s invariants in a transparent way. One of the key points in our construction in [Oh2] is the emphasis on the usage of the existing group structure on the space of Hamiltonians defined by

$$
(H, K) \mapsto H \# K := H + K \circ (\phi_H^1)^{-1}
$$

(1.3)

in relation to the pants product and the triangle inequality. However we failed to fully exploit this structure and fell short of proving the triangle inequality at the time of writing [Oh1,2].
This construction can be carried out for the Hamiltonian diffeomorphisms as long as the action functional is single valued, e.g., on weakly-exact symplectic manifolds. Schwartz [Sc] carried out this construction in the case of symplectically aspherical \((M,\omega)\), i.e., for \((M,\omega)\) with \(c_1|_{\pi_2(M)} = \omega|_{\pi_2(M)} = 0\). Among other things he proved the triangle inequality for the invariants constructed using the notion of Hamiltonian fibration and (flat) symplectic connection on it. It turns out that the proof of this triangle inequality [Sc] is closely related to the notion of the \(K\)-area of the Hamiltonian fibration [Po2] with connections [GLS], [Po2], especially to the one with fixed monodromy studied by Entov [En1]. In this context, the choice of the triple \((H,K;H\#K)\) we made in [Oh2] can be interpreted as the one which makes infinity the \(K\)-area of the corresponding Hamiltonian fibration over the Riemann surface of genus zero with three punctures equipped with the given monodromy around the punctures. Entov [En1] develops a general framework of Hamiltonian connections with fixed boundary monodromy and relates the \(K\)-area with various quantities of the given monodromy which are of the Hofer length type. This framework turns out to be particularly useful for our construction of spectral invariants in the present paper.

On non-exact symplectic manifolds, the action functional is not single valued and the Floer homology theory has been developed as a circle-valued Morse theory or a Morse theory on a covering space \(\tilde{\Omega}_0(M)\) of the space \(\Omega_0(M)\) of contractible (free) loops on \(M\) in the literature related to Arnold's conjecture which was initiated by Floer himself [Fl]. The Floer theory now involves quantum effects and uses the Novikov ring in an essential way [HoS]. The presence of quantum effects and denseness of the action spectrum in \(\mathbb{R}\) (as in non-rational symplectic manifolds), had been the most serious obstacle that plagued the study of family of Hamiltonian diffeomorphisms, until the author [Oh4] developed a general framework of the min-max theory over natural semi-infinite cycles on the covering space \(\tilde{\Omega}_0(M)\) which we call the Novikov Floer cycles. In the present paper, we will exploit the ‘finiteness’ condition in the definitions of the Novikov ring and the Novikov Floer cycles in a crucial way for the proofs of various existence results of pseudo-holomorphic curves that are needed in the proofs of the axioms of spectral invariants and nondegeneracy of the norm that we construct [Oh8]. Although the Novikov ring is essential in the definition of the Floer homology and the quantum cohomology in the literature, as far as we know, it is the first time for the finiteness condition to be explicitly used beyond the purpose of giving the definition of the quantum cohomology and the Floer homology.

A brief description of the setting of the Floer theory [HoS] is in order, partly to fix our convention: Let \((\gamma,w)\) be a pair of \(\gamma \in \Omega_0(M)\) and \(w\) be a disc bounding \(\gamma\). We say that \((\gamma,w)\) is \(\Gamma\)-equivalent to \((\gamma,w')\) iff

\[
\omega([w'\#\overline{w}]) = 0 \quad \text{and} \quad c_1([w'\#\overline{w}]) = 0
\]

where \(\overline{w}\) is the map with opposite orientation on the domain and \(w'\#\overline{w}\) is the obvious glued sphere. Here \(\Gamma\) stands for the group

\[
\Gamma = \frac{\pi_2(M)}{\ker (\omega|_{\pi_2(M)}) \cap \ker (c_1|_{\pi_2(M)})}
\]

We denote by \([\gamma,w]\) the \(\Gamma\)-equivalence class of \((\gamma,w)\) and by \(\tilde{\Omega}_0(M)\) the set of \(\Gamma\)-equivalence classes. Let \(\pi : \tilde{\Omega}_0(M) \to \Omega_0(M)\) the canonical projection. We call
\( \tilde{\Omega}_0(M) \) the \( \Gamma \)-covering space of \( \Omega_0(M) \). The action functional \( A_0 : \tilde{\Omega}_0(M) \to \mathbb{R} \) is defined by

\[
A_0([\gamma, w]) = -\int w^* \omega. \tag{1.5}
\]

Two \( \Gamma \)-equivalent pairs \( (\gamma, w) \) and \( (\gamma, w') \) have the same action and so the action is well-defined on \( \tilde{\Omega}_0(M) \). When a one-periodic Hamiltonian \( H : (\mathbb{R}/\mathbb{Z}) \times M \to \mathbb{R} \) is given, we consider the functional \( A_H : \tilde{\Omega}(M) \to \mathbb{R} \) by

\[
A_H([\gamma, w]) = -\int w^* \omega - \int H(t, \gamma(t)) dt. \tag{1.6}
\]

Our convention is chosen to be consistent with the classical mechanics Lagrangian on the cotangent bundle with the symplectic form

\[
\omega_0 = -d\theta, \quad \theta = \sum_i p_i dq^i
\]

when (1.2) is adopted as the definition of Hamiltonian vector field. See the remark in the end of this introduction on other conventions in the symplectic geometry. The conventions in the present paper coincide with our previous papers [Oh1,2,4] and Entov’s [En1,2] but different from many other literature on the Floer homology one way or the other. (There was a sign error in [Oh1,2] when we compare the Floer complex and the Morse complex for a small Morse function, which was rectified in [Oh4]. In our convention, the positive gradient flow of \( \epsilon f \) corresponds to the negative gradient flow of \( A_{\epsilon f} \).)

The mini-max theory of this action functional on the \( \Gamma \)-covering space has been implicitly used in the proof of Arnold’s conjecture. Recently the present author has further developed this mini-max theory via the Floer homology and applied it to the study of Hofer’s geometry of Hamiltonian diffeomorphism groups [Oh4]. We also outlined construction of spectral invariants of Hamiltonian diffeomorphisms of the type [V], [Oh2], [Sc] on arbitrary non-exact symplectic manifolds for the classical cohomological classes. The main purpose of the present paper is to further develop the chain level Floer theory introduced in [Oh4] and to carry out construction of spectral invariants for arbitrary quantum cohomology classes. The organization of the paper is now in order.

In \( \S 2 \), we briefly review various facts related to the action functional and its action spectrum. Some of these may be known to the experts, but precise details for the action functional on the covering space \( \tilde{\Omega}_0(M) \) of general \( (M, \omega) \) first appeared in our paper [Oh5] especially concerning the normalization and the loop effect on the action spectrum: We define the action spectrum of \( H \) by

\[
\text{Spec}(H) := \{ A_H([z, w]) \in \mathbb{R} \mid [z, w] \in \tilde{\Omega}_0(M), \ dA_H([z, w]) = 0 \}
\]

i.e., the set of critical values of \( A_H : \tilde{\Omega}_0(M) \to \mathbb{R} \). In [Oh5], we have shown that once we normalize the Hamiltonian \( H \) on compact \( M \) by

\[
\int_M H_t \ d\mu = 0
\]
with $d\mu$ the Liouville measure, Spec$(H)$ depends only on the equivalence class $\bar{\phi} = [\phi, H]$ (see §2 for the definition) and so Spec$(\bar{\phi}) \subset \mathbb{R}$ is a well-defined subset of $\mathbb{R}$ for each $\bar{\phi} \in \hat{\text{Ham}}(M, \omega)$. Here

$$\pi : \hat{\text{Ham}}(M, \omega) \to \text{Ham}(M, \omega)$$

is the universal covering space of $\text{Ham}(M, \omega)$. This kind of normalization of the action spectrum is a crucial point for systematic study of the spectral invariants of the Viterbo type in general. Schwarz [Sc] previously proved that in the aspherical case where the action functional is single valued already on $\Omega_0(M)$, this normalization can be made on $\text{Ham}(M, \omega)$, not just on $\hat{\text{Ham}}(M, \omega)$.

In §3, we review the quantum cohomology and its Morse theory realization of the corresponding complex. We emphasize the role of the Novikov ring in relating the quantum cohomology and the Floer homology and the reversal of upward and downward Novikov rings in this relation. In §4, we review the standard operators in the Floer homology theory and explain the filtration naturally present in the Floer complex and how it changes under the Floer chain map. In §5, we give the definition of our spectral invariants for the Hamiltonian functions $H$, and prove finiteness of the mini-max values $\rho(H; a)$. In §6, we prove all the basic properties of the spectral invariants. We summarize these into the following theorem. We denote by $C^\infty_m([0, 1] \times M)$ the set of normalized continuous functions on $[0, 1] \times M$.

Noting that there is a one-one correspondence between the set $C^\infty_m([0, 1] \times M)$ and the set of Hamiltonian paths

$$\lambda = \phi_H : t \in [0, 1] \mapsto \phi^t_H \in \text{Ham}(M, \omega),$$

one may equally consider $\rho(H; a)$ as an invariant attached to the Hamiltonian path $\phi_H$.

**Theorem I.** Let $(M, \omega)$ be arbitrary closed symplectic manifold. For any given quantum cohomology class $0 \neq a \in QH^*(M)$, we have a continuous function denoted by $\rho = \rho(H; a) : C^\infty_m([0, 1] \times M) \times QH^*(M) \to \mathbb{R}$

such that they satisfy the following axioms: Let $H, F \in C^\infty_m([0, 1] \times M)$ be smooth Hamiltonian functions and $a \neq 0 \in QH^*(M)$. Then $\rho$ satisfies the following axioms:

1. (Projective invariance) $\rho(H; \lambda a) = \rho(H; a)$ for any $0 \neq \lambda \in \mathbb{Q}$.
2. (Normalization) For $a = \sum_{A \in \Gamma} a_A q^{-A}$, we have $\rho(0; a) = v(a)$ where $0$ is the zero function and $v(a) := \min \{\omega(-A) \mid a_A \neq 0\} = -\max \{\omega(A) \mid a_A \neq 0\}$. \hspace{1cm} (1.7)

is the (upward) valuation of $a$.
3. (Symplectic invariance) $\rho(\eta^* H; a) = \rho(H; a)$ for any symplectic diffeomorphism $\eta$.
4. (Triangle inequality) $\rho(H \# F; a \cdot b) \leq \rho(H; a) + \rho(F; b)$
5. ($C^0$-continuity) $|\rho(H; a) - \rho(F; a)| \leq \|H \# F\| = \|H - F\|$ where $\| \cdot \|$ is the Hofer’s pseudo-norm on $C^\infty_m([0, 1] \times M)$. In particular, the function $\rho_a : H \mapsto \rho(H; a)$ is $C^0$-continuous.
We will call the set
\[ \text{spec}(H) := \{ \rho(H; a) \mid a \in QH^*(M) \} \] (1.8)
the \textit{essential spectrum} of \( H \).

Most of the properties stated in this theorem are direct analogs to the ones in [Oh1,2] and [Sc]. \textit{Except for the proof of finiteness of} \( \rho(H; a) \), proofs of all of the properties are refinements of the arguments used in [Oh2,4], [Sc]. In addition, the proof of the triangle inequality uses the concept of Hamiltonian fibration with fixed monodromy and the \( K \)-area [Po2], [En1], which is an enhancement of the arguments used in [Oh2], [Sc].

In the classical mini-max theory for the \textit{indefinite} functionals [Ra], [BnR], there was implicitly used the notion of ‘semi-infinite cycles’ to carry out the mini-max procedure. There are two essential ingredients needed to prove existence of actual critical values out of the mini-max values: one is the finiteness of the mini-max value, or the \textit{linking property} of the (semi-infinite) cycles associated to the class \( a \) and the other is to prove that the corresponding mini-max value is indeed a critical value of the action functional. \textit{When the global gradient flow of the action functional exists} as in the classical critical point theory [BnR], this point is closely related to the well-known Palais-Smale condition and the deformation lemma which are essential ingredients needed to prove the criticality of the mini-max value. Partly because we do not have the global flow, we need to geometrize all these classical mini-max procedures. It turns out that the Floer homology theory in the chain level is the right framework for this purpose.

In section 7, we will restrict to the \textit{rational} case and prove the following additional property of spectral invariants, \textit{the spectrality axiom}. We will study the non-rational cases elsewhere for which we expect the same property holds, at least for the nondegenerate Hamiltonian functions, but its proof seems to be much more nontrivial.

We now recall the definition of rational symplectic manifolds: Denote
\[ \Gamma_{\omega} := \{ \omega(A) \mid A \in \pi_2(M) \} = \omega(\Gamma) \subset \mathbb{R} \]
and
\[ \text{Spec}(H) = \cup_{z \in \text{Per}(H)} \text{Spec}(H; z). \]
Recall that \( \Gamma_{\omega} \) is either a discrete or a countable dense subset of \( \mathbb{R} \).

\textbf{Definition 1.1.} A symplectic manifold \((M, \omega)\) is called \textit{rational} if \( \Gamma_{\omega} \) is discrete.

\textbf{Theorem II. (Spectrality Axiom)} Suppose that \((M, \omega)\) be rational. Then \( \rho \) satisfies the following additional properties:

1. For any smooth one-periodic Hamiltonian function \( H : S^1 \times M \to \mathbb{R} \), we have
   \[ \rho(H; a) \in \text{Spec}(H) \]
   for each given quantum cohomology class \( 0 \neq a \in QH^*(M) \).
2. For two smooth functions \( H \sim K \) we have
   \[ \rho(H; a) = \rho(K; a) \] (1.9)
   for all \( a \in QH^*(M) \).
In particular, \( \rho \) can be pushed down to the ‘universal covering space’ \( \widetilde{\Ham}(M, \omega) \) of \( \Ham(M, \omega) \) by putting \( \rho(\tilde{\phi}; a) \) to be the this common value for \( \tilde{\phi} = [h] \). We call the subset \( \text{spec}(\tilde{\phi}) \subset \text{Spec}(\tilde{\phi}) \) defined by

\[
\text{spec}(\tilde{\phi}) = \{ \rho(\tilde{\phi}; a) \mid a \in QH^*(M) \}
\]

the (homologically) essential spectrum of \( \tilde{\phi} \). Then we have the following refined version of Theorem II for the rational cases.

**Theorem III.** Let \((M, \omega)\) be rational and define the map

\[
\rho : \widetilde{\Ham}(M, \omega) \times QH^*(M) \to \mathbb{R}
\]

by \( \rho(\tilde{\phi}; a) := \rho(H; a) \). Let \( \tilde{\phi}, \tilde{\psi} \in \widetilde{\Ham}(M, \omega) \) and \( a \neq 0 \in QH^*(M) \). Then \( \rho \) satisfies the following axioms:

1. **(Spectrality)** For each \( a \in QH^*(M) \), \( \rho(\tilde{\phi}; a) \in \text{Spec}(\tilde{\phi}) \).
2. **(Projective invariance)** \( \rho(\tilde{\phi}; \lambda a) = \rho(\tilde{\phi}; a) \) for any \( 0 \neq \lambda \in \mathbb{Q} \).
3. **(Normalization)** For \( a = \sum_{A \in F} a_A q^{-A} \), we have \( \rho(\emptyset; a) = v(a) \) where \( \emptyset \) is the identity in \( \Ham(M, \omega) \) and

\[
v(a) := \min\{\omega(-A) \mid a_A \neq 0\} = -\max\{\omega(A) \mid a_A \neq 0\}.
\]

4. **(Symplectic invariance)** \( \rho(\eta \tilde{\phi} \eta^{-1}; a) = \rho(\tilde{\phi}; a) \) for any symplectic diffeomorphism \( \eta \).
5. **(Triangle inequality)** \( \rho(\tilde{\phi} \cdot \tilde{\psi}; a \cdot b) \leq \rho(\tilde{\phi}; a) + \rho(\tilde{\psi}; b) \)
6. **(C^0-continuity)** \( |\rho(\tilde{\phi}; a) - \rho(\tilde{\psi}; a)| \leq \| \tilde{\phi} \circ \tilde{\psi}^{-1} \| \) where \( \| \cdot \| \) is the Hofer’s pseudo-norm on \( \Ham(M, \omega) \). In particular, the function \( \rho_a : \tilde{\phi} \mapsto \rho(\tilde{\phi}; a) \) is \( C^0 \)-continuous.
7. **(Monodromy shift)** Let \([h, \tilde{h}] \in \pi_0(\tilde{G})\) act on \( \widetilde{\Ham}(M, \omega) \times QH^*(M) \) by the map

\[
(\tilde{\phi}, a) \mapsto (h \cdot \tilde{\phi}, \tilde{h}^* a)
\]

where \( \tilde{h}^* a \) is the image of the (adjoint) Seidel’s action \([Se]\) by \([h, \tilde{h}]\) on the quantum cohomology \( QH^*(M) \). Then we have

\[
\rho([h, \tilde{h}] \cdot (\tilde{\phi}, a)) = \rho(\tilde{\phi}; a) + L_\omega([h, \tilde{h}])
\]

It would be an interesting question to ask whether these axioms characterize the spectral invariants \( \rho \). It is related to the question whether the graph of the sections

\[
\rho_a : \tilde{\phi} \mapsto \rho(\tilde{\phi}; a) ; \quad \widetilde{\Ham}(M, \omega) \to \text{Spec}(M, \omega)
\]

can be split into other ‘branch’ in a way that the other branch can also satisfy all the above axioms or not. Here the action spectrum bundle \( \mathcal{S}pec(M, \omega) \) is defined by

\[
\mathcal{S}pec(M, \omega) := \bigcup_{\tilde{\phi} \in \widetilde{\Ham}(M, \omega)} \text{Spec}(\tilde{\phi}) \subset \widetilde{\Ham}(M, \omega) \times \mathbb{R}.
\]
We will investigate this question elsewhere.

To get the main stream of ideas in this paper without getting bogged down
with technicalities related with transversality question of various moduli spaces, we
assume in this paper that \((M, \omega)\) is strongly semi-positive in the sense of \([Se], [En1]\):
A closed symplectic manifold is called \emph{strongly semi-positive} if there is no spherical
homology class \(A \in \pi_2(M)\) such that
\[
\omega(A) > 0, \quad 2 - n \leq c_1(A) \leq 0.
\]

Under this condition, the transversality problem concerning various moduli spaces
of pseudo-holomorphic curves is standard. We will not mention this generic transversality
question at all in the main body of the paper unless it is absolutely necessary.
In §7, we will briefly explain how this general framework can be incorporated in
our proofs in the context of Kuranishi structure [FOn] all at once. In Appendix, we
introduce the notion of \emph{continuous quantum cohomology} and explain how to extend
our definition of spectral invariants to the continuous quantum cohomology classes.

The present work is originated from a part of our paper entitled “Mini-max the-
ory, spectral invariants and geometry of the Hamiltonian diffeomorphism group”
[Oh6] that has been circulated since July, 2002. We isolate and streamline the
construction part of spectral invariants from [Oh6] in the present paper with some
minor corrections and addition of more details. In particular, we considerably sim-
plify the definition of \(\rho(H; a)\) from [Oh6] here. We leave the application part
of [Oh6] to a separate paper [Oh8] in which we construct the homological norm of
Hamiltonian diffeomorphism and apply them to the study of geometry of Hamil-
tonian diffeomorphisms on general compact symplectic manifolds.

Another application of the spectral invariants to the study of length minimizing
property of Hamiltonian paths is given by the author [Oh7,8]. See also [En2],
[EnP] for other interesting applications of spectral invariants. In another sequel
to this paper, we will provide a description of spectral invariants in terms of the
Hamiltonian fibration.

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**Convention.**

1. The Hamiltonian vector field \(X_f\) associated to a function \(f\) on \((M, \omega)\) is
defined by \(df = \omega(X_f, \cdot)\).

2. The addition \(F \# K\) and the inverse \(K\) on the set of time periodic Hamilto-
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$nians C^\infty(M \times S^1)$ are defined by

\[ F \# G(x, t) = F(x, t) + G((\phi^t_F)^{-1}(x), t) \]
\[ G(x, t) = -G(\phi^t_G(x), t). \]

There is another set of conventions which are used in the literature (e.g., in [Po3]):

1) $X_f$ is defined by $\omega(X_f, \cdot) = -df$
2) The action functional has the form

\[ A_H([z, w]) = -\int w^*\omega + \int H(t, z(t)) dt. \] (1.12)

Because our $X_f$ is the negative of $X_f$ in this convention, the action functional is the one for the Hamiltonian $-H$ in our convention. While our convention makes the positive Morse gradient flow correspond to the negative Cauchy-Riemann flow, the other convention keeps the same direction. The reason why we keep our convention is that we would like to keep the definition of the action functional the same as the classical Hamilton’s functional

\[ \int pdq - H dt \] (1.13)

on the phase space and to make the negative gradient flow of the action functional for the zero Hamiltonian become the pseudo-holomorphic equation.

It appears that the origin of the two different conventions is the choice of the convention on how one defines the canonical symplectic form on the cotangent bundle $T^*N$ or in the classical phase space: If we set the canonical Liouville form

\[ \theta = \sum_i p_idq^i \]

for the canonical coordinates $q^1, \cdots, q^n, p_1, \cdots, p_n$ of $T^*N$, we take the standard symplectic form to be

\[ \omega_0 = -d\theta = \sum dq^i \wedge dp_i \]

while the people using the other convention (see e.g., [Po3]) take

\[ \omega_0 = d\theta = \sum dp_i \wedge dq^i. \]

As a consequence, the action functional (1.12) in the other convention is the negative of the classical Hamilton’s functional (1.13). It seems that there is not a single convention that makes everybody happy and hence one has to live with some nuisance in this matter one way or the other.

§2. The action functional and the action spectrum

Let $(M, \omega)$ be any compact symplectic manifold and $\Omega_0(M)$ be the set of contractible loops and $\tilde{\Omega}_0(M)$ be its the covering space mentioned before. We will always consider normalized functions $f : M \to \mathbb{R}$ by

\[ \int_M f d\mu = 0 \] (2.1)
where \( d\mu \) is the Liouville measure of \((M, \omega)\).

When a periodic normalized Hamiltonian \( H : M \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{R} \) is given, we consider the action functional \( A_H : \tilde{\Omega}(M) \to \mathbb{R} \) by

\[
A_H([\gamma, w]) = -\int w^* \omega - \int H(\gamma(t), t) dt
\]

We denote by \( \text{Per}(H) \) the set of periodic orbits of \( X_H \).

**Definition 2.1.** We define the action spectrum of \( H \), denoted as \( \text{Spec}(H) \subset \mathbb{R} \), by

\[
\text{Spec}(H) := \{ A_H(z, w) \in \mathbb{R} \mid [z, w] \in \tilde{\Omega}_0(M), z \in \text{Per}(H) \},
\]

i.e., the set of critical values of \( A_H : \tilde{\Omega}(M) \to \mathbb{R} \). For each given \( z \in \text{Per}(H) \), we denote

\[
\text{Spec}(H; z) = \{ A_H(z, w) \in \mathbb{R} \mid (z, w) \in \pi^{-1}(z) \}.
\]

Note that \( \text{Spec}(H; z) \) is a principal homogeneous space modelled by the period group of \((M, \omega)\)

\[
\Gamma := \{ \omega(A) \mid A \in \pi_2(M) \} = \omega(\Gamma) \subset \mathbb{R}
\]

and

\[
\text{Spec}(H) = \cup_{z \in \text{Per}(H)} \text{Spec}(H; z).
\]

The following was proven in [Oh4].

**Lemma 2.2.** For any closed symplectic manifold \((M, \omega)\) and for any smooth Hamiltonian \( H \), \( \text{Spec}(H) \) is a measure zero subset of \( \mathbb{R} \) for any \( H \).

For given \( \phi \in \text{Ham}(M, \omega) \), we denote \( F \mapsto \phi \) if \( \phi^1 F = \phi \), and denote

\[
\mathcal{H}(\phi) = \{ F \mid F \mapsto \phi \}.
\]

We say that two Hamiltonians \( F \) and \( K \) are equivalent and denote \( F \sim K \) if they are connected by one parameter family of Hamiltonians \( \{ F_s \}_{0 \leq s \leq 1} \) such that \( F_s \mapsto \phi \) for all \( s \in [0, 1] \). We write \([F]\) for the equivalence class of \( F \). Then the universal covering space \( \tilde{\text{Ham}}(M, \omega) \) of \( \text{Ham}(M, \omega) \) is realized by the set of such equivalence classes. Note that the group \( G := \Omega(\text{Ham}(M, \omega), id) \) of based loops naturally acts on the loop space \( \tilde{\Omega}(M) \) by

\[
(h \cdot \gamma)(t) = h(t)(\gamma(t))
\]

where \( h \in \Omega(\text{Ham}(M, \omega)) \) and \( \gamma \in \Omega(M) \). An interesting consequence of Arnold’s conjecture is that this action maps \( \Omega_0(M) \) to itself (see e.g., [Lemma 2.2, Se]). Seidel [Lemma 2.4, Se] proves that this action can be lifted to \( \tilde{\Omega}_0(M) \). The set of lifts \((h, \tilde{h})\) forms a covering group \( \tilde{G} \to G \)

\[
\tilde{G} \subset G \times \text{Homeo}(\tilde{\Omega}_0(M))
\]

whose fiber is isomorphic to \( \Gamma \). Seidel relates the lifting \((h, \tilde{h})\) of \( h : S^1 \to \text{Ham}(M, \omega) \) to a section of the Hamiltonian bundle associated to the loop \( h \) (see §2 [Se]).
When a Hamiltonian $H$ generating the loop $h$ is given, the assignment

$$z \mapsto h \cdot z$$

provides a natural one-one correspondence

$$h : \text{Per}(F) \mapsto \text{Per}(H\#F)$$

(2.2)

where $H\#F = H + F \circ (\phi_h')^{-1}$. Let $F, K$ be normalized Hamiltonians with $F, K \mapsto \phi$ and $H$ be the Hamiltonian such that $K = H\#F$, and $f_t, g_t$ and $h_t$ be the corresponding Hamiltonian paths as above. In particular the path $h = \{h_t\}_{0 \leq t \leq 1}$ defines a loop. We also denote the corresponding action of $h$ on $\Omega_0(M)$ by $\tilde{h}$. Let $\tilde{h}$ be any lift of $h$ to $\text{Homeo}(\tilde{\Omega}_0(M))$. Then a straightforward calculation shows (see [Oh5])

$$\tilde{h}^*(dA_F) = dA_K$$

(2.3)

as a one-form on $\tilde{\Omega}_0(M)$. In particular since $\tilde{\Omega}_0(M)$ is connected, we have

$$\tilde{h}^*(A_F) - A_K = C(F, K, \tilde{h})$$

(2.4)

where $C = C(F, K, \tilde{h})$ is a constant a priori depending on $F, K, \tilde{h}$.

**Theorem 2.3 [Theorem II, Oh5].** Let $h$ be the loop as above and $\tilde{h}$ be a lift. Then the constant $C(F, K, \tilde{h})$ in (2.4) depends only on the homotopy class $[h, \tilde{h}] \in \pi_0(\tilde{G})$. In particular if $F \sim K$, we have $A_F \circ \tilde{h} = A_K$ and hence

$$\text{Spec } F = \text{Spec } K$$

as a subset of $\mathbb{R}$. For any $\tilde{\phi} \in \tilde{\text{Ham}}(M, \omega)$, we define

$$\text{Spec } (\tilde{\phi}) := \text{Spec } F$$

for a (and so any) normalized Hamiltonian $F$ with $[\phi, F] = \tilde{\phi}$.

**Definition 2.4 [Action Spectrum Bundle].** We define the action spectrum bundle of $(M, \omega)$ by

$$\mathcal{Spec}(M, \omega) = \bigcup_{\tilde{\phi} \in \tilde{\text{Ham}}(M, \omega)} \mathcal{Spec}_{\tilde{\phi}}(M, \omega) \subset \tilde{\text{Ham}}(M, \omega) \times \mathbb{R}$$

where

$$\mathcal{Spec}_{\tilde{\phi}}(M, \omega) = \{A_F([z, w]) \mid dA_F([z, w]) = 0, \quad \tilde{\phi} = [F] \} \subset \mathbb{R}$$

and denote by $\pi : \mathcal{Spec}(M, \omega) \to \tilde{\text{Ham}}(M, \omega)$ the natural projection.
§3. Quantum cohomology in the chain level

We first recall the definition of the quantum cohomology ring $QH^*(M)$. As a module, it is defined as

$$ QH^*(M) = H^*(M, \mathbb{Q}) \otimes \Lambda^+ $$

where $\Lambda^+$ is the (upward) Novikov ring

$$ \Lambda^+ = \left\{ \sum_{A \in \Gamma} a_A q^{-A} \mid a_A \in \mathbb{Q}, \#\{ A \mid a_i \neq 0, \int_{-A} \omega < \lambda \} < \infty, \forall \lambda \in \mathbb{R} \right\}. $$

Due to the finiteness assumption on the Novikov ring, we have the natural (upward) valuation $v : QH^*(M) \rightarrow \mathbb{R}$ defined by

$$ v( \sum_{A \in \Gamma} a_A q^{-A} ) = \min\{ \omega(-A) : a_A \neq 0 \} $$

which satisfies that for any $a, b \in QH^*(M)$

$$ v(a + b) \geq \min\{ v(a), v(b) \}. $$

**Definition 3.1.** For each homogeneous element

$$ a = \Sigma_{A \in \Gamma} a_A q^{-A} \in QH^k(M), \quad a_A \in H^*(M, \mathbb{Q}) $$

of degree $k$, we also call $v(a)$ the *level* of $a$ and the corresponding term in the sum the *leading order term* of $a$ and denote by $\sigma(a)$. Note that the leading order term $\sigma(a)$ of a homogeneous element $a$ is unique among the summands in the sum by the definition (1.4) of $\Gamma$.

The product on $QH^*(M)$ is defined by the usual quantum cup product, which we denote by “$\cdot$” and which preserves the grading, i.e, satisfies

$$ QH^k(M) \times QH^\ell(M) \rightarrow QH^{k+\ell}(M). $$

Often the homological version of the quantum cohomology is also useful, sometimes called the quantum homology, which is defined by

$$ QH_*(M) = H_*(M, \mathbb{Q}) \otimes \Lambda_+ $$

where $\Lambda_+$ is the (downward) Novikov ring

$$ \Lambda_+ = \left\{ \sum_{B_j \in \Gamma} b_j q^{B_j} \mid b_j \in \mathbb{Q}, \#\{ B_j \mid b_j \neq 0, \int_{B_j} \omega > \lambda \} < \infty, \forall \lambda \in \mathbb{R} \right\}. $$

We define the corresponding (downward) valuation by

$$ v(\sum_{B \in \Gamma} a_B q^B) = \max\{ \omega(B) : a_B \neq 0 \} $$

(3.3)
which satisfies that for \( f, g \in \text{QH}_*(M) \)

\[
v(f + g) \leq \max\{v(f), v(g)\}.
\]

We like to point out that the summand in \( \Lambda \downarrow \omega \) is written as \( b_B q^B \) while the one in \( \Lambda \uparrow \omega \) as \( a_A q^{-A} \) with the minus sign. This is because we want to clearly show which one we use. Obviously \(-v\) in (3.1) and \(v\) in (3.3) satisfy the axiom of non-Archimedean norm which induce a topology on \( \text{QH}^*(M) \) and \( \text{QH}_*(M) \) respectively. In each case the finiteness assumption in the definition of the Novikov ring allows us to numerate the non-zero summands in each given Novikov chain (3.2) so that

\[
\lambda_1 > \lambda_2 > \cdots > \lambda_j > \cdots
\]

with \( \lambda_j = \omega(B_j) \) or \( \omega(A_j) \).

Since the downward Novikov ring appears mostly in this paper, we will just use \( \Lambda \downarrow \omega \) or \( \Lambda \) for \( \Lambda \downarrow \omega \), unless absolutely necessary to emphasize the direction of the Novikov ring. We define the level and the leading order term of \( b \in \text{QH}_*(M) \) similarly as in Definition 3.1 by changing the role of upward and downward Novikov rings. We have a canonical isomorphism

\[
\flat : \text{QH}^*(M) \to \text{QH}_*(M); \quad \sum a_i q^{-A_i} \to \sum \text{PD}(a_i) q^{A_i}
\]

and its inverse

\[
\sharp : \text{QH}_*(M) \to \text{QH}^*(M); \quad \sum b_j q^B_j \to \sum \text{PD}(b_j) q^{-B_j}.
\]

We denote by \( a^\flat \) and \( b^\sharp \) the images under these maps.

There exists the canonical non-degenerate pairing

\[
\langle \cdot, \cdot \rangle : \text{QH}^*(M) \otimes \text{QH}_*(M) \to \mathbb{Q}
\]

defined by

\[
\langle \sum a_i q^{-A_i}, \sum b_j q^B_j \rangle = \sum (a_i, b_j) \delta_{A_i, B_j}
\]

where \( \delta_{A_i, B_j} \) is the delta-function and \( (a_i, b_j) \) is the canonical pairing between \( H^*(M, \mathbb{Q}) \) and \( H_*(M, \mathbb{Q}) \). Note that this sum is always finite by the finiteness condition in the definitions of \( \text{QH}^*(M) \) and \( \text{QH}_*(M) \) and so is well-defined. This is equivalent to the Frobenius pairing in the quantum cohomology ring. However we would like to emphasize that the dual vector space \((\text{QH}_*(M))^*\) of \( \text{QH}_*(M) \) is not isomorphic to \( \text{QH}^*(M) \) even as a \( \mathbb{Q} \)-vector space. Rather the above pairing induces an injection

\[
\text{QH}^*(M) \hookrightarrow (\text{QH}_*(M))^*
\]

whose images lie in the set of continuous linear functionals on \( \text{QH}_*(M) \) with respect to the topology induced by the valuation \( v \). (3.3) on \( \text{QH}_*(M) \). We refer to [Br] for a good introduction to non-Archimedean analytic geometry. In fact, the description of the standard quantum cohomology in the literature is not really a ‘cohomology’ but a ‘homology’ in that it never uses linear functionals in its definition. To keep our exposition consistent with the standard literature in the Gromov-Witten invariants and the quantum cohomology, we prefer to call them
the quantum cohomology rather than the quantum homology as some authors did (e.g., [Se]) in the symplectic geometry community. In Appendix, we will introduce a genuinely cohomological version of quantum cohomology which we call continuous quantum cohomology using the continuous linear functionals on the quantum chain complex below with respect to the topology induced by the valuation \( v \).

Let \((C_\ast, \partial)\) be any chain complex on \( M \) whose homology is the singular homology \( H_\ast(M) \). One may take for \( C_\ast \) the usual singular chain complex or the Morse chain complex of a Morse function \( f : M \to \mathbb{R} \), \((C_\ast(-\epsilon f), \partial_{-\epsilon f})\) for some sufficiently small \( \epsilon > 0 \). However since we need to take a non-degenerate pairing in the chain level, we should use a model which is finitely generated. We will always prefer to use the Morse homology complex because it is finitely generated and avoids some technical issue related to singular degeneration problem of the type studied in [FOh1,2]. The negative sign in \((C_\ast(-\epsilon f), \partial_{-\epsilon f})\) is put to make the correspondence between the Morse homology and the Floer homology consistent with our conventions of the Hamiltonian vector field (1.2) and the action functional (1.6). In our conventions, solutions of negative gradient of \(-\epsilon f\) correspond to ones for the negative gradient flow of the action functional \( A_{-\epsilon f} \). We denote by

\[
(C_\ast^\ast(-\epsilon f), \delta_{-\epsilon f})
\]

the corresponding cochain complex, i.e,

\[
C^k := \text{Hom}(C_k, \mathbb{Q}), \quad \delta_{-\epsilon f} = \partial_{-\epsilon f}^\ast.
\]

Now we extend the complex \((C_\ast(-\epsilon f), \partial_{-\epsilon f})\) to the quantum chain complex, denoted by

\[
(CQ_\ast(-\epsilon f), \partial_Q)
\]

\[
\begin{align*}
CQ_\ast(-\epsilon f) &:= C_\ast(-\epsilon f) \otimes \Lambda_\omega, \\
\partial_Q &:= \partial_{-\epsilon f} \otimes \Lambda_\omega.
\end{align*}
\]

This coincides with the Floer complex \((CF_\ast(\epsilon f), \partial)\) as a chain complex if \( \epsilon \) is sufficiently small. Similarly we define the quantum cochain complex \((CQ_\ast^\ast(-\epsilon f), \delta^Q)\) by changing the downward Novikov ring to the upward one. In other words, we define

\[
CQ_\ast^\ast(-\epsilon f) := CM_{2m-\ast}(-\epsilon f) \otimes \Lambda^+_\omega, \quad \delta^Q := \partial_{\epsilon f} \otimes \Lambda^+_\omega.
\]

Again we would like to emphasize that \( CQ_\ast^\ast(-\epsilon f) \) is not isomorphic to the dual space of \( CQ_\ast(-\epsilon f) \) as a \( \mathbb{Q} \)-vector space. We refer to Appendix for some further discussion on this issue.

It is well-known that the corresponding homology of this complex is independent of the choice \( f \) and isomorphic to the above quantum cohomology (resp. the quantum homology) as a ring (see [PSS], [LT2], [Lu] for its proof). This isomorphism however plays no significant role in the current paper, except for the purpose of bookkeeping the family of invariants \( \rho(H; a) \) that we associate to each quantum cohomology class \( a \in \text{QH}_\ast(M) \) later (See section 5.1 for more explanation on this point). To emphasize the role of the Morse function in the level of complex, we denote the corresponding homology by \( HQ_\ast(-\epsilon f) \cong \text{QH}_\ast(M) \).

With these definitions, we have the obvious non-degenerate pairing

\[
CQ_\ast^\ast(-\epsilon f) \otimes CQ_\ast(-\epsilon f) \to \mathbb{Q}
\]
in the chain level which induces the pairing (3.4) above in homology.

We now choose a generic Morse function $f$. Then for any given homotopy $\mathcal{H} = \{H^s\}_{s \in [0,1]}$ with $H^0 = \epsilon f$ and $H^1 = H$, we denote by

$$h_\mathcal{H} : CQ_*(-\epsilon f) = CF_{* -n}(\epsilon f) \rightarrow CF_{* -n}(H) \quad (3.7)$$

the standard Floer chain map from $\epsilon f$ to $H$ via the homotopy $\mathcal{H}$. This induces a homomorphism

$$h_\mathcal{H} : HQ_*(-\epsilon f) \cong HF_{* -n}(\epsilon f) \rightarrow HF_{* -n}(H). \quad (3.8)$$

Although (3.7) depends on the choice $\mathcal{H}$, (3.8) is canonical, i.e., does not depend on the homotopy $\mathcal{H}$. One confusing point in this isomorphism is the issue of grading. See the next section for a review of the construction of this chain map and the issue of grading of $HF_*(H)$.

§4. Filtered Floer homology

For each given generic non-degenerate $H : S^1 \times M \rightarrow \mathbb{R}$, we consider the free $\mathbb{Q}$ vector space over

$$\text{Crit} \mathcal{A}_H = \{[z,w] \in \overline{\Omega}_0(M) \mid z \in \text{Per}(H)\}. \quad (4.1)$$

To be able to define the Floer boundary operator correctly, we need to complete this vector space downward with respect to the real filtration provided by the action $\mathcal{A}_H([z,w])$ of the element $[z,w]$ of (4.1). More precisely,

**Definition 4.1.** We call the formal sum

$$\beta = \sum_{[z,w] \in \text{Crit} \mathcal{A}_H} a_{[z,w]} [z,w], \quad a_{[z,w]} \in \mathbb{Q} \quad (4.2)$$

a **Novikov chain** if there are only finitely many non-zero terms in the expression (4.2) above any given level of the action. We denote by $CF_*(H)$ the set of Novikov chains. We call those $[z,w]$ with $a_{[z,w]} \neq 0$ **generators** of the chain $\beta$ and just denote as $[z,w] \in \beta$

in that case. Note that $CF_*(H)$ is a graded $\mathbb{Q}$-vector space which is infinite dimensional in general, unless $\pi_2(M) = 0$.

As in [Oh4], we introduce the following notion which is a crucial concept for the mini-max argument we carry out later.

**Definition 4.2.** Let $\beta$ be a Novikov chain in $CF_*(H)$. We define the **level** of the cycle $\beta$ and denote by $\lambda_H(\beta) = \max_{[z,w]} \{\mathcal{A}_H([z,w]) \mid a_{[z,w]} \neq 0 \text{ in } (4.2)\}$

if $\beta \neq 0$, and just put $\lambda_H(0) = -\infty$ as usual. We call the unique critical point $[z,w]$ that realizes the maximum value $\lambda_H(\beta)$ the **peak** of the cycle $\beta$, and denote it by $\text{pk}(\beta)$. 
We briefly review construction of basic operators in the Floer homology theory [Fl]. Let $J = \{J_t\}_{0 \leq t \leq 1}$ be a periodic family of compatible almost complex structure on $(M, \omega)$.

For each given pair $(J, H)$, we define the boundary operator

$$\partial : CF_*(H) \to CF_*(H)$$

considering the perturbed Cauchy-Riemann equation

$$\begin{cases}
\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\
\lim_{\tau \to -\infty} u(\tau) = z^-, \lim_{\tau \to \infty} u(\tau) = z^+
\end{cases} \quad (4.3)$$

This equation, when lifted to $\tilde{\Omega}_0(M)$, defines nothing but the negative gradient flow of $A_H$ with respect to the $L^2$-metric on $\tilde{\Omega}_0(M)$ induced by the family of metrics on $M$

$$g_{J_t} = (\cdot, \cdot)_{J_t} := \omega(\cdot, J_t \cdot) :$$

This $L^2$-metric is defined by

$$\langle \xi, \eta \rangle_{J_t} := \int_0^1 \langle \xi, \eta \rangle_{J_t} \, dt.$$ 

We will also denote

$$\|v\|^2_{J_0} = (v, v)_{J_0} = \omega(v, J_0 v) \quad (4.4)$$

for $v \in TM$.

For each given $[z^-, w^-]$ and $[z^+, w^+]$, we define the moduli space

$$M(H, J; [z^-, w^-], [z^+, w^+])$$

of solutions $u$ of (4.3) wit finite energy

$$E_J(u) = \frac{1}{2} \int_{\mathbb{R} \times S^1} \left( \frac{\partial u}{\partial \tau} \right)^2_{J_t} + \left| \frac{\partial u}{\partial t} - X_H(u) \right|^2_{J_t} \, dt \, d\tau < \infty$$

and satisfying

$$w^- \# u \sim w^+. \quad (4.5)$$

$\partial$ has degree $-1$ and satisfies $\partial \circ \partial = 0$.

When we are given a family $(j, \mathcal{H})$ with $\mathcal{H} = \{H^s\}_{0 \leq s \leq 1}$ and $j = \{J^s\}_{0 \leq s \leq 1}$, the chain homomorphism

$$h(j, \mathcal{H}) : CF_*(H^0) \to CF_*(H^1)$$

is defined by the non-autonomous equation

$$\begin{cases}
\frac{\partial u}{\partial \tau} + J^s_{\rho_1(\tau)} \left( \frac{\partial u}{\partial t} - X_{H^s_{\rho_2(\tau)}}(u) \right) = 0 \\
\lim_{\tau \to -\infty} u(\tau) = z^-, \lim_{\tau \to \infty} u(\tau) = z^+
\end{cases} \quad (4.6)$$

also with the condition (4.5). Here $\rho_i, i = 1, 2$ is the cut-off functions of the type $\rho : \mathbb{R} \to [0, 1]$,

$$\rho(\tau) = \begin{cases} 
0 & \text{for } \tau \leq -R \\
1 & \text{for } \tau \geq R
\end{cases} \quad \rho'(\tau) \geq 0$$
for some $R > 0$. $h_{(j,H)}$ has degree 0 and satisfies

$$\partial_{(j^1,H^1)} \circ h_{(j,H)} = h_{(j,H)} \circ \partial_{(j^0,H^0)}.$$  

Two such chain maps for different homotopies $(j^1,H^1)$ and $(j^2,H^2)$ connecting the same end points are also known to be chain homotopic [Fl2].

Finally when we are given a homotopy $(j,H)$ of homotopies with $j = \{ j_\kappa \}$, $H = \{ H_\kappa \}$, consideration of the parameterized version of (4.6) for $0 \leq \kappa \leq 1$ defines the chain homotopy map

$$H_H : CF_* (H^0) \to CF_* (H^1)$$  

which has degree +1 and satisfies

$$h_{(j_1,H_1)} - h_{(j_0,H_0)} = \partial_{(j^1,H^1)} \circ H_H + H_H \circ \partial_{(j^0,H^0)}.$$  

(4.8)

By now, construction of these maps using these moduli spaces has been completed with rational coefficients (See [FO1], [LT1] and [Ru]) using the techniques of virtual moduli cycles. We will suppress this advanced machinery from our presentation throughout the paper. The main stream of the proof is independent of this machinery except that it is implicitly needed to prove that various moduli spaces we use are non-empty. Therefore we do not explicitly mention these technicalities in the main body of the paper until §8, unless it is absolutely necessary. In §8, we will provide justification of this in the general case all at once.

The following upper estimate of the action change can be proven by the same argument as that of the proof of [Ch], [Oh1,4]. We would like to emphasize that in general there does not exist a lower estimate of this type. The upper estimate is just one manifestation of the ‘positivity’ phenomenon in symplectic topology through the existence of pseudo-holomorphic curves that was first discovered by Gromov [Gr]. On the other hand, the existence of lower estimate is closely tied to some nontrivial homological property of (Floer) cycles, and best formulated in terms of Floer cycles instead of individual critical points $[z,w]$ for the nondegenerate Hamiltonians. However, we would like to point out that the equations (4.3), (4.6) themselves or the numerical estimate of the action changes for solutions $u$ with finite energy can be studied for any $H$ or $(H,j)$ which are not necessarily non-degenerate or generic, although the Floer complex or the operators may not be defined for such choices.

Proposition 4.3. Let $H,K$ be any Hamiltonian not necessarily non-degenerate and $j = \{ J^s \}_{s \in [0,1]}$ be any given homotopy and $H^{lin} = \{ H^s \}_{0 \leq s \leq 1}$ be the linear homotopy $H^s = (1 - s)H + sF$. Suppose that (4.6) has a solution satisfying (4.5). Then we have the identity

$$A_F ([z^+,w^+]) - A_H ([z^-,w^-])$$

$$= - \int \left( \frac{\partial u}{\partial t} \right)^2_{J^s(\tau)} - \int_{-\infty}^{\infty} \rho'(\tau) \left( F(t,u(\tau,t)) - H(t,u(\tau,t)) \right) dt d\tau$$

(4.9)

$$\leq - \int \left( \frac{\partial u}{\partial t} \right)^2_{J^s(\tau)} + \int_0^1 - \min_{x \in M} (F_1 - H_t) dt$$

(4.10)

$$\leq \int_0^1 - \min_{x \in M} (F_1 - H_t) dt$$

(4.11)

By considering the case $K = H$, we immediately have
Corollary 4.4. For a fixed $H$ and for a given one parameter family $j = \{J^s\}_{s \in [0,1]}$, let $u$ be as in Proposition 4.3. Then we have
\[
A_H([z^+, w^+]) - A_H([z^-, w^-]) = -\int \left| \frac{\partial u}{\partial \tau} \right|_{J^s(\tau)}^2 \leq 0.
\] (4.12)

Remark 4.5. We would like to remark that similar calculation proves that there is also an uniform upper bound $C(j, H)$ for the chain map over general homotopy $(j, \mathcal{H})$ or for the chain homotopy maps (4.7). In this case, the identity (4.9) is replaced by
\[
A_F([z^+, w^+]) - A_H([z^-, w^-])
\leq -\int \left| \frac{\partial u}{\partial \tau} \right|_{J^s(\tau)}^2 + \int_0^1 - \min_{x \in \mathcal{M}} \left( \frac{\partial H^s}{\partial s} \right) dt
\leq \int_0^1 - \min_{x \in \mathcal{M}} \left( \frac{\partial H^s}{\partial s} \right) dt
\] (4.13)

This upper estimate is also crucial for the construction of these maps. This upper estimate depends on the choice of homotopy $(j, \mathcal{H})$ and is related to the curvature estimates of the relevant Hamiltonian fibration (see [Po2], [En1]).

Now we recall that $CF_*(H)$ is also a $\Lambda$-module: each $A \in \Gamma$ acts on $\text{Crit}A_H$ and so on $CF_*(H)$ by “gluing a sphere”

\[ A : [z, w] \rightarrow [z, w \# A]. \]

Then $\partial$ is $\Lambda$-linear and induces the standard Floer homology $HF_*(H; \Lambda)$ with $\Lambda$ as its coefficients (see [HoS] for a detailed discussion on the Novikov ring and on the Floer complex as a $\Lambda$-module). However the action does not preserve the filtration we defined above. Whenever we talk about filtration, we will always presume that the relevant coefficient ring is $\mathbb{Q}$.

For a given nondegenerate $H$ and an $\lambda \in \mathbb{R} \setminus \text{Spec}(H)$, we define the relative chain group
\[ CF^\lambda_k(H) := \{ \beta \in CF_k(H) \mid \lambda_H(\beta) < \lambda \}. \]
Corollary 4.4 impies that between the two chain complexes $(CF_k(H), \partial_{(H,J)})$ and $(CF_k(H), \partial_{(H,J')})$, there is a canonical filtration preserving chain isomorphism
\[ h_{(j,H)} : (CF_k(H), \partial_{(H,J)}) \rightarrow (CF_k(H), \partial_{(H,J')}) \]
where $j$ is any homotopy from $J$ and $J'$, and $\mathcal{H} \equiv H$ is the constant homotopy of $H$. Therefore from now on, we suppress $J$-dependence on the Floer homology in our exposition unless it is absolutely necessary.

For each given pair of real numbers $\lambda, \mu \in \mathbb{R} \setminus \text{Spec}(H)$ with $\lambda < \mu$, we define
\[ CF^{(\lambda, \mu)}_*(H) := CF^\mu(H)/CF^{\lambda}(H). \]
Then for each triple \( \lambda < \mu < \nu \) where \( \lambda = -\infty \) or \( \nu = \infty \) are allowed, we have the short-exact sequence of the complex of graded \( \mathbb{Q} \) vector spaces

\[
0 \to CF_k^{[\lambda, \mu]}(H) \to CF_k^{[\lambda, \nu]}(H) \to CF_k^{(\mu, \nu)}(H) \to 0
\]

for each \( k \in \mathbb{Z} \). This then induces the long exact sequence of graded modules

\[
\cdots \to HF_k^{[\lambda, \mu]}(H) \to HF_k^{[\lambda, \nu]}(H) \to HF_k^{(\mu, \nu)}(H) \to HF_{k-1}^{[\lambda, \mu]}(H) \to \cdots
\]

whenever the relevant Floer homology groups are defined.

We close this section by fixing our grading convention for \( HF_*(H) \). This convention is the analog to the one we use in [Oh1,2] in the context of Lagrangian submanifolds. We first recall that solutions of the negative gradient flow equation of \( -f \), (i.e., of the positive gradient flow of \( f \))

\[
\dot{\chi} - \text{grad} \ f(\chi) = 0
\]

corresponds to the negative gradient flow of the action functional \( A_{\epsilon f} \). This gives rise to the relation between the Morse indices \( \mu_{\text{Morse}} - \epsilon f(p) \) and the Conley-Zehnder indices \( \mu_{\text{CZ}}([p, \hat{p}]; H) \) (see [Lemma 7.2, SZ] but with some care about the different convention of the Hamiltonian vector field. Their definition of \( X_H \) is \( -X_H \) in our convention):

\[
\mu_{\text{CZ}}([p, \hat{p}]; \epsilon f) = \mu_{\text{Morse}} - \epsilon f(p) - n
\]

in our convention. On the other hand, obviously we have

\[
\mu_{\text{Morse}} - \epsilon f(p) - n = (2n - \mu_{\text{Morse}} - \epsilon f(p)) - n = \mu_{\text{Morse}} - \epsilon f(p)
\]

We will always grade \( HF_*(H) \) by the Conley Zehnder index

\[
k = \mu_H([z, w]) := \mu_{\text{CZ}}([z, w]; H).
\]

This grading convention makes the degree \( k \) of \([q, \hat{q}]\) in \( CF_k(\epsilon f) \) coincides with the Morse index of \( q \) of \( \epsilon f \) for each \( q \in \text{Crit} f \). Recalling that we chose the Morse complex

\[
CM_*(-\epsilon f) \otimes \Lambda^l
\]

for the quantum chain complex \( CQ_*(-\epsilon f) \), it also coincides with the standard grading of the quantum cohomology via the map

\[
\flat : QH^k(M) \to QH_{2n-k}(M).
\]

Form now on, we will just denote by \( \mu_H([z, w]) \) the Conley-Zehnder index of \([z, w]\) for the Hamiltonian \( H \). Under this grading, we have the following grading preserving isomorphism

\[
QH^{n-k}(M) \to QH_{n+k}(M) \cong HQ_{n+k}(-\epsilon f) \to HF_k(\epsilon f) \to HF_k(H).
\]

We will also show in §6 that this grading convention makes the pants product, denoted by \(*\), has the degree \(-n\):

\[
* : HF_k(H) \otimes HF_l(K) \to HF_{(k+l)-n}(H \# K)
\]

which will be compatible with the degree preserving quantum product

\[
\cdot : QH^a(M) \otimes QH^b(M) \to QH^{a+b}(M).
\]
§5. Construction of the spectral invariants of Hamiltonian functions

In this section, we associate some homologically essential critical values of the action functional $A_H$ to each Hamiltonian functions $H$ and quantum cohomology class $a$, and call them the spectral invariants of $H$. We denote this assignment by

$$\rho : C^\infty_m([0,1] \times M) \times QH^*(M) \rightarrow \mathbb{R}$$

as described in the introduction of this paper. Before launching our construction, some overview of our construction of spectral invariants is necessary.

5.1. Overview of the construction

We recall the canonical isomorphism

$$h_{\alpha\beta} : HF_*(H_\alpha) \rightarrow HF_*(H_\beta)$$

which satisfies the composition law

$$h_{\alpha\gamma} = h_{\alpha\beta} \circ h_{\beta\gamma}.$$  

We denote by $HF_*(M)$ the corresponding model $\mathbb{Q}$-vector space. We also note that $HF_*(H)$ is induced by the filtered chain complex $(CF^\lambda_\ast(H), \partial)$ where

$$CF^\lambda_\ast(H) = \text{span}_\mathbb{Q}\{\alpha \in CF_\ast(H) \mid \lambda_H(\alpha) \leq \lambda\}$$

i.e., the sub-complex generated by the critical points $[z,w] \in \text{Crit}A_H$ with $A_H([z,w]) \leq \lambda$.

Then there exists a canonical inclusion

$$i_\lambda : CF^\lambda_\ast(H) \rightarrow CF^\infty_\ast(H) := CF_\ast(H)$$

which induces a natural homomorphism $i_\lambda : HF^\lambda_*(H) \rightarrow HF_*(H)$. For each given element $\ell \in FH_*(M)$ and Hamiltonian $H$, we represent the class $\ell$ by a Novikov cycle $\alpha$ of $H$ and measure its level $\lambda_H(\alpha)$ and define

$$\rho(H; \ell) := \inf\{\lambda \in \mathbb{R} \mid \ell \in \text{Im} \ i_\lambda\}$$

or equivalently

$$\rho(H; \ell) := \inf_{\alpha; i_\lambda(\alpha) = \ell} \lambda_H(\alpha).$$

The crucial task then is to prove that for each (homogeneous) element $\ell \neq 0$, the value $\rho(H; \ell)$ is finite, i.e., “the cycle $\alpha$ is linked and cannot be pushed away to infinity by the negative gradient flow of the action functional”. In the classical critical point theory (see [BnR] for example), this property of semi-infinite cycles is called the linking property. We like to point out that there is no manifest way to see the linking property or the criticality of the mini-max value $\rho(H; \ell)$ out of the definition itself.
We will prove this finiteness first for the Hamiltonian $\epsilon f$ where $f$ is a Morse function and $\epsilon$ is sufficiently small. This finiteness strongly relies on the facts that the Floer boundary operator $\partial_{\epsilon f}$ in this case has the form

$$\partial_{\epsilon f} = \partial_{\text{Morse}} - \epsilon f \otimes \Lambda$$

i.e., “there is no quantum contribution on the Floer boundary operator”, and that the classical Morse theory proves that $\partial_{\text{Morse}}$ cannot push down the level of a non-trivial cycle more than $-\epsilon \max f$ (see [Oh4]).

Once we prove the finiteness for $\epsilon f$, then we consider the general nondegenerate Hamiltonian $H$. We compare the cycles in $CF_\ast(H)$ and the transferred cycles in $CF_\ast(\epsilon f)$ by the chain map $h_{\text{H}}^{-1} : CF_\ast(H) \to CF_\ast(\epsilon f)$ where $\mathcal{H}$ is a homotopy connecting $\epsilon f$ and $H$. The change of the level then can be measured by judicious use of (4.7) and Remark 4.5 which will prove the finiteness for any $H$.

After we prove finiteness of $\rho(H; a)$ for general $H$, we study the continuity property of $\rho(H; a)$ under the change of $H$. This will be done, via the equation (4.6), considering the level change between arbitrary pair $(H, K)$.

Finally we prove that the limit

$$\lim_{\epsilon \to 0} \rho(\epsilon f; \ell)$$

exists and is independent of the choice of Morse function $f$. If the Floer homology class $\ell$ is identified with $a^\flat$ for a quantum cohomology class $a \in QH^\ast(M)$ under the PSS-isomorphism [PSS], then this limit is nothing but the valuation $v(a)$.

In this procedure, we can avoid considering the ‘singular limit’ of the ‘chains’ (See the [section 8, Oh8] for some illustration of the difficulty in studying such limits). We only need to consider the limit of the values $\rho(H; \ell)$ as $H \to 0$ which is a much simpler task than considering the limit of chains which involves highly non-trivial analytical work (we refer to the forthcoming work [FOh2] for the consideration of this limit in the chain level).

5.2. Finiteness; the linking property of semi-infinite cycles

With this overview, we now start with our construction. We first recall the natural pairing

$$\langle \cdot, \cdot \rangle : CQ^\ast(-\epsilon f) \otimes CQ_\ast(-\epsilon f) \to \mathbb{Q} :$$

where we have

$$CQ_k(-\epsilon f) := (CM_k(-\epsilon f), \partial_{\epsilon f}) \otimes \Lambda^\downarrow$$
$$CQ^k(-\epsilon f) := (CM_{2n-k}(\epsilon f), \partial_{\epsilon f}) \otimes \Lambda^\uparrow.$$
See Appendix for more discussions on this aspect. We would like to emphasize that (5.2) is well-defined because of the choice of directions of the Novikov rings $\Lambda^\uparrow$ and $\Lambda^\downarrow$. In general, the map (5.2) is injective but not an isomorphism. Polterovich [Po4], [EnP] observed that this point is closely related to certain failure of “Poincaré duality” of the Floer homology with Novikov rings as its coefficients.

Now we are ready to give the definition of our spectral invariants. Previously in [Oh4], the author outlined this construction for the classical cohomology class in $H^*(M) \subset QH^*(M)$.

**Definition 5.2.** Let $H$ be a generic non-degenerate Hamiltonian. For each given $a \in QH^k(M) \cong HQ^k(-\epsilon f)$, we define

$$\rho(H,a) = \inf_{\alpha} \{ \lambda_H(\alpha) \mid [\alpha] = a^\flat, \alpha \in CF_k(H) \}. \quad (5.3)$$

**Theorem 5.3.** Let $0 \neq a \in QH^*(M)$.

1. Let $H$ be a generic non-degenerate Hamiltonian. Then $\rho(H,a)$ is finite.
2. For any pair of generic nondegenerate Hamiltonians $H, K$, we have the inequality

$$\int_0^1 - \max(K - H) \, dt \leq \rho(K,a) - \rho(H,a) \leq \int_0^1 - \min(K - H) \, dt. \quad (5.4)$$

In particular, the function $H \mapsto \rho(H,a)$ continuously extends to $C_0^0([0,1]^\times M)$.

**Proof.** We will prove the finiteness in two steps: first we prove the finiteness for $\epsilon f$ for sufficiently small $\epsilon > 0$ for any given Morse function $f$, and then prove it for general $H$ using this finiteness for $\epsilon f$. After then we will prove the inequality (5.4).

**Step 1: The finiteness of for $\epsilon f$.** Let $f$ be any fixed Morse function and fix $\epsilon > 0$ so small that there is no quantum contribution for the Floer boundary operator $\partial_{(\epsilon f, J_0)}$ for a time independent family $J_t \equiv J_0$ for any compatible almost complex structure $J_0$, i.e., we have

$$\partial_{(\epsilon f, J_0)} \simeq \partial_{-\epsilon f} \Morse \otimes \Lambda^\downarrow. \quad (5.5)$$

It is well-known ([Fl], [FOn], [LT1]) that this is possible. Fixing such $\epsilon$ and $J_0$, we just denote

$$\partial_{\epsilon f} = \partial_{(\epsilon f, J_0)}.$$

Then by considering the Morse homology of $-\epsilon f$ with respect to the Riemannian metric $g_{J_0} = \omega(\cdot, J_0 \cdot)$, we have the identity

$$QH^*(M) \cong \ker \partial_{\epsilon f}^{\Morse} \otimes \Lambda^\uparrow / \text{Im} \partial_{\epsilon f}^{\Morse} \otimes \Lambda^\downarrow = HM_*(\epsilon f) \otimes \Lambda^\uparrow$$

$$QH_*^*(M) \cong \ker \partial_{-\epsilon f}^{\Morse} \otimes \Lambda^\downarrow / \text{Im} \partial_{-\epsilon f}^{\Morse} \otimes \Lambda^\downarrow = HM_*(-\epsilon f) \otimes \Lambda^\downarrow.$$

Recalling

$$CF_k(\epsilon f) \cong CQ_{n+k}(-\epsilon f),$$
from (5.5), we represent \( a^p \in QH_{n+k}(M) \) by a Novikov cycle of \( \epsilon f \) where

\[
\alpha = \sum_A a_p \otimes q^A
\]

with \( a_p \in \mathbb{Q} \) and \( p \in \text{Crit}_*(-\epsilon f) \) and

\[ n + k = \mu_{\epsilon f}(p \otimes q^A) \]  \hspace{1cm} (5.6)

where \( \mu_{\epsilon f}(p \otimes q^A) \) is the Conley-Zehnder index of the element \( p \otimes q^A = [p, \tilde{p} \# A] \).

We recall the general index formula

\[ \mu_H([z, w \otimes A]) = \mu_H([z, w]) + 2c_1(A) \]

in our convention (see [Oh9] for the proof of this index formula). Applying this to \( H = \epsilon f \), we have obtained

\[ \mu_{\epsilon f}([p, \tilde{p} \# A]) = \mu_{\epsilon f}([p, \tilde{p}]) + 2c_1(A). \]

Combining this with

\[ \mu_{-\epsilon f}^\text{Morse}(p) = \mu_{\epsilon f}([p, \tilde{p}]) + n \]

we derive that (5.6) is equivalent to

\[ \mu_{-\epsilon f}^\text{Morse}(p) = n + k - 2c_1(A). \]

Next we see that \( \alpha \) has the level

\[ \lambda_{\epsilon f}(\alpha) = \max\{-\epsilon f(p) - \omega(A) \mid a_p \otimes q^A \neq 0\} \]  \hspace{1cm} (5.7)

because \( A_{\epsilon f}([p, \tilde{p} \# A]) = -\epsilon f(p) - \omega(A) \). Now the most crucial point in our construction is to prove the finiteness

\[ \inf_{[\alpha] = a^p} \lambda_{\epsilon f}(\alpha) > -\infty. \]  \hspace{1cm} (5.8)

The following lemma proves this linking property. We first like to point out that the quantum cohomology class

\[ a = \sum_A a_A q^{-A} \]

uniquely determines the set

\[ \Gamma(a) := \{ A \in \Gamma \mid a_A 
eq 0 \}. \]

By the finiteness condition in the formal power series, we can enumerate \( \Gamma(a) \) so that

\[ \lambda_1 > \lambda_2 > \lambda_3 > \cdots \]  \hspace{1cm} (5.9)

without loss of generality. In particular, we have

\[ v(a) = -\omega(A_1) = \lambda_1. \]  \hspace{1cm} (5.10)
Lemma 5.4. Let $a \neq QH^k(M)$ and $a^b \in QH_{n+k}(M)$ be its dual. Suppose that
\[ a^b = \sum_j a_j q^{A_j} \]
with $0 \neq a_j \in H_{n+k+2c_1(A_j)}(M)$ where $\lambda_j = -\omega(A_j)$ are arranged as in (5.9). Denote by $\gamma$ a Novikov cycle of $\epsilon f$ with $[\gamma] = a^b \in HF_k(\epsilon f) \cong QH_{n+k}(M)$ and define the ‘gap’
\[ c(a) := \lambda_1 - \lambda_2. \]
Then we have
\[ v(a) - \frac{1}{2} c(a) \leq \inf_{[\gamma]} \{ \lambda_{\epsilon f}(\gamma) | [\gamma] = a^b \} \leq v(a) + \frac{1}{2} c(a) \quad (5.11) \]
for any sufficiently small $\epsilon > 0$ and in particular, (5.8) holds. We also have
\[ \lim_{\epsilon \to 0} \inf_{[\gamma]} \{ \lambda_{\epsilon f}(\gamma) | [\gamma] = a^b \} = v(a) \quad (5.12) \]
and so the limit is independent of the choice of Morse functions $f$.

Proof. We represent $a^b$ by a Novikov cycle
\[ \gamma = \sum \gamma_{A} q^{A}, \quad \gamma_j \in CM_{*}(-\epsilon f) \]
of $\epsilon f$. It follows from (5.3) that if $A \in \Gamma(a)$, all the coefficient Morse chains in this sum must be cycles, and if $A \not\in \Gamma(a)$, the corresponding coefficient cycle must be a boundary. Therefore we can decompose $\gamma$ as
\[ \gamma = \gamma_{\Gamma(a)} + \gamma_{\Gamma(a)^{c}} \quad (5.13) \]
where
\[ \gamma_{\Gamma(a)} := \sum_{A \in \Gamma(a)} \gamma_{A} q^{A} \]
\[ \gamma_{\Gamma(a)^{c}} := \sum_{B \not\in \Gamma(a)} \gamma_{B} q^{B} \]
and we have
\[ \gamma_{\Gamma(a)^{c}} = \partial_{\epsilon f}(\nu) \]
for some Floer chain $\nu$ of $\epsilon f$. Since the summands in $\gamma_{\Gamma(a)^{c}}$ cannot cancel those in $\gamma_{\Gamma(a)}$, we have
\[ \lambda_{\epsilon f}(\gamma) \geq \lambda_{\epsilon f}(\gamma_{\Gamma(a)}) = \lambda_{\epsilon f} \left( \sum_{A \in \Gamma(a)} \gamma_{A} q^{A} \right). \]
Therefore by removing the exact term $\partial_{\epsilon f}(\gamma)$ when we take the infimum over the cycles $\gamma$ with $[\gamma] = a^b$ for the definition of $\rho(\epsilon f; a)$, we may always assume that $\gamma$ has the form
\[ \gamma = \sum_j \gamma_j q^{A_j} \]
with $A_j \in \Gamma(a)$. Then again by (5.3), we have

$$[\gamma_j] = a_j \in H_*(M).$$

Furthermore we note that we have

$$-\omega(A_j) - \max(\epsilon f) \leq \lambda_{\epsilon f}(\gamma_j q^{A_j}) \leq -\omega(A_j) - \min(\epsilon f).$$

Therefore if we choose $\epsilon > 0$ so small that

$$\epsilon(\max f - \min f) \leq c(a) = \lambda_1 - \lambda_2,$$

then we have

$$\lambda_{\epsilon f}(\gamma_1 q^{A_1}) \geq \lambda_{\epsilon f}(\gamma_j q^{A_j})$$

for all $j = 1, 2, \cdots$ and so

$$\lambda_{\epsilon f}(\gamma) = \lambda_{\epsilon f}(\gamma_1 q^{A_1}).$$

Combining these, we derive

$$-\omega(A_1) - \epsilon \max f \leq \lambda_{\epsilon f}(\gamma) \leq -\omega(A_1) + \epsilon \max f. \quad (5.14)$$

(5.11) follows from (5.14) if we choose $\epsilon$ so that $\epsilon(\max f - \min f) < c(a)/2$. (5.12) also immediately follows from (5.14). □

**Step 2: The finiteness for general $H$.** Now we consider generic nondegenerate $H$’s. We fix $f$ be any Morse function and $\epsilon > 0$ as in Lemma 5.4. Let $\alpha \in CF_*(H)$ be a Floer cycle of $H$ with $[\alpha] = a^k$, and $\mathcal{H} = \mathcal{H}_{lin}$ the linear homotopy

$$\mathcal{H}_{lin} : s \mapsto (1 - s)(\epsilon f) + sH.$$

Applying (4.12) to the ‘inverse’ linear homotopy

$$\mathcal{H}_{lin}^{-1} : s \mapsto (1 - s)H + s(\epsilon f).$$

we obtain the inequality

$$\lambda_{\epsilon f}(h_{\mathcal{H}_{lin}^{-1}}(\alpha)) \leq \lambda_{\mathcal{H}}(\alpha) + \int_0^1 -\min(\epsilon f - H) \, dt : \quad (5.15)$$

More precisely, it follows from the definition of the chain map $h_{\mathcal{H}_{lin}^{-1}}$ that for any generator $[z', w']$ of the cycle $h_{\mathcal{H}_{lin}^{-1}}(\alpha)$ of $\epsilon f$, there is a generator $[z, w]$ of the cycle $\alpha$ such that the equation (4.6) has a solution. Then we derive from (4.11)

$$\mathcal{A}_{\epsilon f}([z', w']) \leq \mathcal{A}_{\mathcal{H}}([z, w]) + \int_0^1 -\min(\epsilon f - H) \, dt$$

$$\leq \lambda_{\mathcal{H}}(\alpha) + \int_0^1 -\min(\epsilon f - H) \, dt.$$  

Since this holds for any generator $[z', w']$ of $\mathcal{H}_{lin}^{-1}(\alpha)$, we obtain

$$\lambda_{\epsilon f}(\mathcal{H}_{lin}^{-1}(\alpha)) \leq \lambda_{\mathcal{H}}(\alpha) + \int_0^1 -\min(\epsilon f - H) \, dt. \quad (5.16)$$
On the other hand, it follows from \( [\mathcal{H}^{-1}_{\text{lin}}(\alpha)] = a^b \), that we have 
\[ \rho(a;\epsilon f) \leq \lambda_{\epsilon f}(\mathcal{H}^{-1}_{\text{lin}}(\alpha)). \]
Combining this with (5.16), we derive
\[ \lambda_\mathcal{H}(\alpha) \geq \rho(a;\epsilon f) - \left( \int_0^1 - \min_{x \in \mathcal{M}} (\epsilon f - H_t) \, dt \right). \tag{5.17} \]
Since this holds for any cycle \( \alpha \) of \( \mathcal{H} \) with \([\alpha] = a^b\), by taking the infimum over all such \( \alpha \) in (5.17), we have finally obtained
\[ \rho(\mathcal{H};a) \geq \rho(a;\epsilon f) - \left( \int_0^1 - \min_{x \in \mathcal{M}} (\epsilon f - H_t) \, dt \right). \tag{5.18} \]
Since Lemma 5.4 shows that \( \rho(a;\epsilon f) > -\infty \), this in particular implies that \( \rho(\mathcal{H};a) > -\infty \) and so \( \rho(\mathcal{H};a) \) is finite.

**Step 3: Proof of (5.4).** Finally we prove the inequality (5.4). For this purpose, we consider general generic nondegenerate pairs \( \mathcal{H}, \mathcal{K} \). Let \( \delta > 0 \) be any given number. We choose a cycle \( \alpha \) of \( \mathcal{H} \) respectively so that \([\alpha] = a^b\) and 
\[ \lambda_\mathcal{H}(\alpha) \leq \rho(\mathcal{H};a) + \delta. \tag{5.19} \]
We would like to emphasize that this is possible, because we have already shown that \( \rho(\mathcal{H};a) > -\infty \).

By considering the linear homotopy \( h_{\mathcal{H}\mathcal{K}}^{\text{lin}} \) from \( \mathcal{H} \) to \( \mathcal{K} \), we derive
\[ \lambda_\mathcal{K}(h_{\mathcal{H}\mathcal{K}}^{\text{lin}}(\alpha)) \leq \lambda_\mathcal{H}(\alpha) + \int \min_{x} (K_t - H_t) \, dt. \tag{5.20} \]
On the other hand (5.19) implies
\[ \lambda_\mathcal{H}(\alpha) + \int \min_{x} (K_t - H_t) \, dt \leq \rho(\mathcal{H};a) + \delta + \int \min_{x} (K_t - H_t) \, dt. \tag{5.21} \]
Since \([h_{\mathcal{H}\mathcal{K}}^{\text{lin}}(\alpha)] = a^b\), we have 
\[ \lambda_\mathcal{K}(h_{\mathcal{H}\mathcal{K}}^{\text{lin}}(\alpha)) \geq \rho(\mathcal{K};a) \] by the definition of \( \rho(\mathcal{K};a) \). Combining (5.20)-(5.22), we have derived
\[ \rho(\mathcal{K};a) - \rho(\mathcal{H};a) \leq \delta + \int \min_{x} (K_t - H_t) \, dt. \]
Since this holds for arbitrary \( \delta \), we have derived
\[ \rho(\mathcal{K};a) - \rho(\mathcal{H};a) \leq \int \min_{x} (K_t - H_t) \, dt. \]
By changing the role of \( \mathcal{H} \) and \( \mathcal{K} \), we also derive
\[ \rho(\mathcal{H};a) - \rho(\mathcal{K};a) \leq \int \min_{x} (H_t - K_t) \, dt = \int \max_{x} (K_t - H_t) \, dt \]
Hence, we have the inequality
\[ \int \max_{x} (K_t - H_t) \, dt \leq \rho(\mathcal{K};a) - \rho(\mathcal{H};a) \leq \int \min_{x} (K_t - H_t) \, dt \]
which is precisely (5.4). Obviously the inequality (5.4), enables us to extend the definition of \( \rho \) by continuity to arbitrary \( C^0 \)-Hamiltonians. This finishes the proof of Theorem 5.3. \( \square \)
§6. Basic properties of the spectral invariants

In this section, we will prove all the remaining properties stated in Theorem I in the introduction. We first re-state the main axioms of the spectral invariants.

**Theorem 6.1.** Let $H$, $F$ be arbitrary smooth Hamiltonian functions, and $a \neq 0 \in QH^*(M)$ and let

$$\rho : C^\infty([0, 1] \times M) \times QH^*(M) \to \mathbb{R}$$

be as defined in §5. Then $\rho$ satisfies the following properties:

1. **(Projective invariance)** $\rho(H; \lambda a) = \rho(H; a)$ for any $0 \neq \lambda \in \mathbb{Q}$.
2. **(Normalization)** For $a = \sum_{A \in \Gamma} a_A \otimes q_A^\lambda$, $\rho(\mathbf{0}; a) = v(a)$, the valuation of $a$.
3. **(Symplectic invariance)** $\rho(\eta^* H; a) = \rho(H; a)$ for any symplectic diffeomorphism $\eta$.
4. **(Triangle inequality)** $\rho(H \# F; a \cdot b) \leq \rho(H; a) + \rho(F; b)$.
5. **($C^0$-Continuity)** $|\rho(H; a) - \rho(F; a)| \leq \|H - F\|$ and in particular $\rho(\cdot; a)$ is continuous with respect to the $C^0$-topology of Hamiltonian functions.

We have already proven the properties of normalization and $C^0$ continuity in the course of proving the linking property of the Novikov Floer cycles in §5. The remaining parts of the proofs deal with the symplectic invariance and the triangle inequality.

**6.1. Proof of symplectic invariance.**

We consider the symplectic conjugation

$$\phi \mapsto \eta^{-1} \phi \eta; \quad \mathcal{H}am(M, \omega) \to \mathcal{H}am(M, \omega)$$

for any symplectic diffeomorphism $\eta : (M, \omega) \to (M, \omega)$. Recall that the pull-back function $\eta^* \mathcal{H}$ given by

$$\eta^* \mathcal{H}(t, x) = \mathcal{H}(t, \eta(x))$$

(6.1)

generates the conjugation $\eta^{-1} \phi \eta$ when $H \mapsto \phi$.

We summarize the basic facts on this conjugation relevant to the filtered Floer homology here:

1. when $H \mapsto \phi$, $\eta^* \mathcal{H} \mapsto \eta \phi \eta^{-1}$,
2. if $H$ is nondegenerate, $\eta^* H$ is also nondegenerate,
3. if $(J, H)$ is regular in the Floer theoretic sense, then so is $(\eta^* J, \eta^* H)$,
4. there exists natural bijection $\eta_* : \Omega_0(M) \to \Omega_0(M)$ defined by

$$\eta_*([z, w]) = ([\eta \circ z, \eta \circ w])$$

under which we have the identity

$$A_H([z, w]) = A_{\eta^* H}(\eta_* [z, w]).$$

(6.2)

5. the $L^2$-gradients of the corresponding action functionals satisfy

$$\eta_* (\text{grad}_J A_H)([z, w]) = \text{grad}_{\eta^* J} (\mathcal{A}_{\eta^* H})(\eta_* ([z, w]))$$

(6.3)

6. if $u : \mathbb{R} \times S^1 \to M$ is a solution of perturbed Cauchy-Riemann equation for $(J, H)$, then $\eta \circ u$ is a solution for the pair $(\eta^* J, \eta^* H)$. In addition, all the Fredholm properties of $(J, H, u)$ and $(\eta^* J, \eta^* H, \eta_* u)$ are the same.
These facts imply that the conjugation by $\eta$ induces the canonical filtration preserving chain isomorphism

$$\eta_* : (CF^*_{\lambda}(H), \partial_{(H,J)}) \to (CF^*_{\lambda}(\eta^* H), \partial_{(\eta^* H, \eta^* J)})$$

for any $\lambda \in \mathbb{R} \setminus \text{Spec}(H) = \mathbb{R} \setminus \text{Spec}(\eta^* H)$. In particular it induces a filtration preserving isomorphism

$$\eta_* : HF^*_{\lambda}(H, J) \to HF^*_{\lambda}(\eta^* H, \eta^* J).$$

in homology. The symplectic invariance is then an immediate consequence of our construction of $\rho(H; a)$.

6.2. Proof of the triangle inequality

To start with the proof of the triangle inequality, we need to recall the definition of the “pants product”

$$HF_*(H) \otimes HF_*(F) \to HF_*(H \# F). \quad (6.4)$$

We also need to straighten out the grading problem of the pants product.

For the purpose of studying the effect on the filtration under the product, we need to define this product in the chain level in an optimal way as in [Oh2], [Sc] and [En1]. For this purpose, we will mostly follow the description provided by Entov [En1] with few notational changes and different convention on the grading. As pointed out before, our grading convention satisfies (4.17) under the pants product. Except the grading convention, the conventions in [En1,2] on the definition of Hamiltonian vector field and the action functional coincide with our conventions in [Oh1-3,5] and here.

Let $\Sigma$ be the compact Riemann surface of genus 0 with three punctures. We fix a holomorphic identification of a neighborhood of each puncture with either $[0, \infty) \times S^1$ or $(-\infty, 0] \times S^1$ with the standard complex structure on the cylinder. We call punctures of the first type negative and the second type positive. In terms of the “pair-of-pants” $\Sigma \setminus \cup_i D_i$, the positive puncture corresponds to the outgoing ends and the negative corresponds to the incoming ends. We denote the neighborhoods of the three punctures by $D_i$, $i = 1, 2, 3$ and the identification by

$$\varphi_i^+ : D_i \to (-\infty, 0] \times S^1$$

for positive punctures and

$$\varphi_3^- : D_3 \to [0, \infty) \times S^1$$

for negative punctures. We denote by $(\tau, t)$ the standard cylindrical coordinates on the cylinders.

We fix a cut-off function $\rho^+ : (-\infty, 0] \to [0, 1]$ defined by

$$\rho = \begin{cases} 1 & \tau \leq -2 \\ 0 & \tau \geq -1 \end{cases}$$

and $\rho^- : [0, \infty) \to [0, 1]$ by $\rho^-(\tau) = \rho^+(-\tau)$. We will just denote by $\rho$ these cut-off functions for both cases when there is no danger of confusion.
We now consider the (topologically) trivial bundle \( P \to \Sigma \) with fiber isomorphic to \((M, \omega)\) and fix a trivialization
\[
\Phi_i : P_i := P|_{D_i} \to D_i \times M
\]
on each \( D_i \). On each \( P_i \), we consider the closed two form of the type
\[
\omega_{P_i} := \Phi_i^*(\omega + d(pH_i dt)) \tag{6.5}
\]
for a time periodic Hamiltonian \( H : [0, 1] \times M \to \mathbb{R} \). The following is an important lemma whose proof we omit (see [En1]).

**Lemma 6.2.** Consider three normalized Hamiltonians \( H_i, i = 1, 2, 3 \). Then there exists a closed 2-form \( \omega_P \) such that

1. \( \omega_P|_{P_i} = \omega_{P_i} \)
2. \( \omega_P \) restricts to \( \omega \) in each fiber
3. \( \omega_P^{n+1} = 0 \)

Such \( \omega_P \) induces a canonical symplectic connection \( \nabla = \nabla_{\omega_P} \) [GLS], [En1]. In addition it also fixes a natural deformation class of symplectic forms on \( P \) obtained by those
\[
\Omega_{P, \lambda} := \omega_P + \lambda \omega_{\Sigma}
\]
where \( \omega_{\Sigma} \) is an area form and \( \lambda > 0 \) is a sufficiently large constant. We will always normalize \( \omega_{\Sigma} \) so that \( \int_\Sigma \omega_{\Sigma} = 1 \).

Next let \( \tilde{J} \) be an almost complex structure on \( P \) such that

1. \( \tilde{J} \) is \( \omega_P \)-compatible on each fiber and so preserves the vertical tangent space
2. the projection \( \pi : P \to \Sigma \) is pseudo-holomorphic, i.e., \( d\pi \circ \tilde{J} = j \circ d\pi \).

When we are given three \( t \)-periodic Hamiltonian \( H = (H_1, H_2, H_3) \), we say that \( \tilde{J} \) is \((H, J)\)-compatible, if \( J \) additionally satisfies

3. For each \( i \), \( (\Phi_i)_* \tilde{J} = j \oplus J_{H_i} \) where
\[
J_{H_i}(\tau, t, x) = (\phi^{J}_{H_i})^* J
\]
for some \( t \)-periodic family of almost complex structure \( J = \{ J_t \}_{0 \leq t \leq 1} \) on \( M \) over a disc \( D_i' \subset D_i \) in terms of the cylindrical coordinates. Here \( D_i' = \varphi_i^{-1}((-\infty, -K_i] \times S^1), \) \( i = 1, 2 \) and \( \varphi_i^{-1}([K_i, \infty) \times S^1) \) for some \( K_i > 0 \). Later we will particularly consider the case where \( J \) is in the special form adapted to the Hamiltonian \( H \). See (6.23).

The condition (3) implies that the \( \tilde{J} \)-holomorphic sections \( v \) over \( D_i' \) are precisely the solutions of the equation
\[
\frac{\partial u}{\partial \tau} + J_i \left( \frac{\partial u}{\partial t} - X_{H_i}(u) \right) = 0 \tag{6.6}
\]
if we write \( v(\tau, t) = (\tau, t, u(\tau, t)) \) in the trivialization with respect to the cylindrical coordinates \((\tau, t)\) on \( D_i' \) induced by \( \phi^{J}_{H_i} \) above.

Now we are ready to define the moduli space which will be relevant to the definition of the pants product that we need to use. To simplify the notations, we denote
\[
\widehat{z} = [z, u]
\]
in general and \( \widehat{z} = (\widehat{z}_1, \widehat{z}_2, \widehat{z}_3) \) where \( \widehat{z}_i = [z_i, u_i] \in \text{Crit}A_{H_i} \) for \( i = 1, 2, 3 \).
**Definition 6.3.** Consider the Hamiltonians \( H = \{ H_i \}_{1 \leq i \leq 3} \) with \( H_3 = H_1 \# H_2 \), and let \( \bar{J} \) be an \( H \)-compatible almost complex structure. We denote by \( \mathcal{M}(H, \bar{J}; \bar{z}) \) the space of all \( \bar{J} \)-holomorphic sections \( u : \Sigma \to P \) that satisfy

1. The maps \( u_i := u \circ (\phi_i^{-1}) : (\infty, K_i] \times S^1 \to M \) which are solutions of (6.6), satisfy
   \[
   \lim_{\tau \to -\infty} u_i(\tau, \cdot) = z_i, \quad i = 1, 2
   \]
   and similarly for \( i = 3 \) changing \( -\infty \) to \( +\infty \).
2. The closed surface obtained by capping off \( \text{pr}_M \circ u(\Sigma) \) with the discs \( w_i \) taken with the same orientation for \( i = 1, 2 \) and the opposite one for \( i = 3 \) represents zero (mod) \( \mathbb{Z} \)-torsion elements.

Note that \( \mathcal{M}(H, \bar{J}; \bar{z}) \) depends only on the equivalence class of \( \bar{z} \)'s: we say that \( \bar{z}' \sim \bar{z} \) if they satisfy

\[
\begin{align*}
   z_i' &= z_i, \\
   w_i' &= w_i \# A_i
\end{align*}
\]

for \( A_i \in \pi_2(M) \) and \( \sum_{i=1}^3 A_i \) represents zero (mod) \( \mathbb{Z} \)-torsion elements. The (virtual) dimension of \( \mathcal{M}(H, \bar{J}; \bar{z}) \) is given by

\[
\dim \mathcal{M}(H, \bar{J}; \bar{z}) = 2n - (\mu_{H_1}(z_1) + n) - \mu_{H_2}(z_2) + n - (\mu_{H_3}(z_3) + n) - (\mu_{H_1}(z_1) + n) - (\mu_{H_2}(z_2)).
\]

Note that when \( \dim \mathcal{M}(H, \bar{J}; \bar{z}) = 0 \), we have

\[
n = -\mu_{H_3}(\bar{z}_3) + \mu_{H_1}(\bar{z}_1) + \mu_{H_2}(\bar{z}_2)
\]

which is equivalent to

\[
\mu_{H_3}(\bar{z}_3) = (\mu_{H_1}(\bar{z}_1) + \mu_{H_2}(\bar{z}_2)) - n
\]

which provides the degree of the pants product (4.17) in our convention of the grading of the Floer complex we adopt in the present paper. Now the pair-of-pants product \( * \) for the chains is defined by

\[
\bar{z}_1 * \bar{z}_2 = \sum_{\bar{z}_3} \#(\mathcal{M}(H, \bar{J}; \bar{z})) \bar{z}_3
\]

for the generators \( \bar{z}_i \) and then by linearly extending over the chains in \( CF_*(H_1) \otimes CF_*(H_2) \). Our grading convention makes this product is of degree zero. Now with this preparation, we are ready to prove the triangle inequality.

**Proof of the triangle inequality.** Let \( \alpha \in CF_*(H) \) and \( \beta \in CF_*(F) \) be Floer cycles with \( [\alpha] = [\beta] = a^\gamma \) and consider their pants product cycle \( \alpha * \beta := \gamma \in CF_*(H \# F) \). Then we have

\[
[\alpha * \beta] = (a \cdot b)^\gamma
\]

and so

\[
\rho(H \# F; a \cdot b) \leq \lambda_{H \# F}(\alpha * \beta).
\]

(6.10)
Let $\delta > 0$ be any given number and choose $\alpha \in CF_*(H)$ and $\beta \in CF_*(F)$ so that
\[
\lambda_H(\alpha) \leq \rho(H; a) + \frac{\delta}{2} \quad \text{and} \quad \lambda_H(\beta) \leq \rho(F; b) + \frac{\delta}{2}.
\] (6.11)

Then we have the expressions
\[
\alpha = \sum_i a_i[z_i, w_i] \text{ with } A_H([z_i, w_i]) \leq \rho(H; a) + \frac{\delta}{2}
\]
and
\[
\beta = \sum_j a_j[z_j, w_j] \text{ with } A_H([z_j, w_j]) \leq \rho(F; b) + \frac{\delta}{2}.
\]

Now using the pants product (6.9), we would like to estimate the level of the chain $\alpha \# \beta \in CF_*(H \# F)$. The following is a crucial lemma whose proof we omit but refer to Sect. 4.1, Sc or Sect. 5, En1.

**Lemma 6.4.** Suppose that $\mathcal{M}(H, \tilde{J}; \tilde{z})$ is non-empty. Then we have the identity
\[
\int v^* \omega_P = -A_{H_1 \# H_2}([z_3, w_3]) + A_{H_1}([z_1, w_1]) + A_{H_2}([z_2, w_2])
\] (6.12)
for any $\in \mathcal{M}(H, \tilde{J}; \tilde{z})$.

Now since $\tilde{J}$-holomorphic and $\tilde{J}$ is $\Omega_{P, \lambda}$-compatible, we have
\[
0 \leq \int v^* \Omega_{P, \lambda} = \int v^* \omega_P + \lambda \int v^* \omega_{\Sigma} = \int v^* \omega_P + \lambda.
\]

**Lemma 6.5 [Theorem 3.6.1 & 3.7.4, En1].** Let $H_i$’s be as in Lemma 6.2. Then for any given $\delta > 0$, we can choose a closed 2-form $\omega_P$ so that $\Omega_{P, \lambda} = \omega_P + \lambda \omega_{\Sigma}$ becomes a symplectic form for all $\lambda \geq \delta$. In other words, the size $\text{size}(H)$ (see Definition 3.1, En1) is $\infty$.

We recall that from the definition of $\ast$ that for any $[z_3, w_3] \in \alpha \ast \beta$ there exist $[z_1, w_1] \in \alpha$ and $[z_2, w_2] \in \beta$ such that $\mathcal{M}(\tilde{J}, H; \tilde{z})$ is non-empty with the asymptotic condition
\[
\tilde{z} = ([z_1, w_1], [z_2, w_2]; [z_3, w_3]).
\]

Applying this and the above two lemmata to $H$ and $F$ for $\lambda$ arbitrarily close to 0, and also applying (6.10) and (6.11), we immediately derive
\[
A_{H \# F}([z_3, w_3]) \leq A_H([z_1, w_1]) + A_F([z_2, w_2]) + \delta \leq \lambda_H(\alpha) + \lambda_F(\beta) + \delta \leq \rho(H; a) + \rho(F; b) + 2\delta
\] (6.13)
for any $[z_3, w_3] \in \alpha \ast \beta$. Combining (6.10), (6.11) and (6.13), we derive
\[
\rho(H \# F; a \cdot b) \leq \rho(H; a) + \rho(F; b) + 2\delta
\]
Since this holds for any $\delta$, we have proven
\[
\rho(H \# F; a \cdot b) \leq \rho(H; a) + \rho(F; b).
\]
The triangle inequality mentioned in Theorem 6.1 immediately follows from the definition $\rho(\phi; a) = \rho(H; a)$ in Theorem 5.5. □
§7. The rational case; proof of the spectrality

In this section, we will prove the spectrality for the rational symplectic manifolds: we recall that a symplectic manifold \((M, \omega)\) rational if the period group \(\Gamma_\omega\) is discrete. We will further study the spectrality property on general symplectic manifolds elsewhere, which turns out to be much more nontrivial to prove.

**Theorem 7.1.** Suppose that \((M, \omega)\) is rational. Then for any smooth one-periodic Hamiltonian function \(H : S^1 \times M \to \mathbb{R}\), we have

\[
\rho(H; a) \in \text{Spec}(H)
\]

for each given quantum cohomology class \(0 \neq a \in QH^*(M)\).

**Proof.** We need to show that the mini-max value \(\rho(H; a)\) is a critical value, i.e., that there exists \([z, w] \in \tilde{\Omega}_0(M)\) such that

\[
\mathcal{A}_H([z, w]) = \rho(H; a) \quad \text{and} \quad d\mathcal{A}_H([z, w]) = 0, \quad \text{i.e.,} \quad \dot{z} = X_H(z).
\]  

(7.1)

We have already shown the finiteness of the value \(\rho(H; a)\) in section 5. If \(H\) is nondegenerate, we just use the fixed Hamiltonian. If \(H\) is not nondegenerate, we approximate \(H\) by a sequence of nondegenerate Hamiltonians \(H_j\) in the \(C^2\) topology.

Let \([z_j, w_j] \in \text{Crit} \mathcal{A}_{H_j}\) be the peak of a Floer cycle \(\alpha_j \in CF_*(H_j)\), such that

\[
\lim_{j \to \infty} \mathcal{A}_{H_j}([z_j, w_j]) = \rho(H; a).
\]  

(7.2)

Such a sequence can be chosen from the definition of \(\rho(\cdot; a)\) and the finiteness thereof.

Since \(M\) is compact and \(H_j \to H\) in the \(C^2\) topology, and \(\dot{z}_j = X_{H_j}(z_j)\) for all \(j\), it follows from the standard boot-strap argument that \(z_j\) has a subsequence, which we still denote by \(z_j\), converging to \(z_\infty\) which solves \(\dot{z} = X_H(z)\). Now we show that \([z_j, w_j]\) themselves are pre-compact on \(\tilde{\Omega}_0(M)\). Since we fix the quantum cohomology class \(0 \neq a \in QH^*(M)\) (or more specifically since we fix its degree) and the Floer cycle satisfies \([\alpha_j] = a\), we have

\[
\mu_{H_j}([z_j, w_j]) = \mu_{H_j}([z_i, w_i]).
\]  

(7.3)

**Lemma 7.2.** When \((M, \omega)\) is rational, \(\text{Crit} \mathcal{A}_K \subset \tilde{\Omega}_0(M)\) is a closed subset of \(\mathbb{R}\) for any smooth Hamiltonian \(K\), and is locally compact in the subspace topology of the covering space

\[
\pi : \tilde{\Omega}_0(M) \to \Omega_0(M).
\]

**Proof.** First note that when \((M, \omega)\) is rational, the covering group \(\Gamma_\omega\) of \(\pi\) above is discrete. Together with the fact that the set of solutions of \(\dot{z} = X_K(z)\) is compact (on compact \(M\)), it follows that

\[
\text{Crit} \mathcal{A}_K = \{[z, w] \in \tilde{\Omega}_0(M) \mid \dot{z} = X_K(z)\}
\]

is a closed subset which is also locally compact. \(\square\)
Now consider the bounding discs of $z_\infty$

$$w'_j = w_j \# u^\text{can}_j$$

for all sufficiently large $j$, where $u^\text{can}_j$ is the homotopically unique thin cylinder between $z_j$ and $z_\infty$; more precisely, $u^\text{can}_j$ is given by the formula

$$u^\text{can}_j(s, t) = \exp_{z_j}(s \xi_j(t)), \quad \xi_j(t) = (\exp_{z_j(t)})^{-1}(z_\infty(t)) \quad (7.4)$$

where exp is the exponential map with respect to a fixed metric $g_{J, ref} = \omega(\cdot, J_{ref} \cdot)$ for a fixed compatible almost complex structure. We note that as $j \to \infty$ the geometric area of $u^\text{can}_j$ converges to 0.

We compute the action of the critical points $[z_\infty, w'_j] \in \text{Crit} A_H$, 

$$A_H([z_\infty, w'_j]) = -\int_{w'_j} \omega - \int_0^1 H(t, z_\infty(t)) \, dt \quad (7.5)$$

$$= -\int_{w_j} \omega - \int_{u^\text{can}_j} - \int_0^1 H(t, z_\infty(t)) \, dt$$

$$= \left( -\int_{w_j} \omega - \int_0^1 H_j(t, z_j(t)) \, dt \right)$$

$$- \left( \int_0^1 H(t, z_\infty(t)) - \int_0^1 H_j(t, z_j(t)) \right) - \int_{u^\text{can}_j} \omega.$$ 

From the explicit expression (7.4), it follows that

$$\lim_{j \to \infty} \int_{u^\text{can}_j} \omega = 0 \quad (7.6)$$

since the geometric area of $u^\text{can}_j$ converges to zero and we have $\text{Area}(u^\text{can}_j) \geq |\int_{u^\text{can}_j} \omega|$. Since $z_j$ converges to $z_\infty$ uniformly and $H_j \to H$, we have

$$- \left( \int_0^1 H(t, z_\infty(t)) - \int_0^1 H(t, z_j(t)) \right) \to 0. \quad (7.7)$$

Therefore combining (7.2), (7.6) and (7.7), we derive

$$\lim_{j \to \infty} A_H([z_\infty, w'_j]) = \rho(H; \alpha).$$

In particular $A_H([z_\infty, w'_j])$ is a Cauchy sequence, which implies

$$\left| \int_{w'_j} \omega - \int_{w'_i} \omega \right| = \left| A_H([z_\infty, w'_j]) - A_H([z_\infty, w'_i]) \right| \to 0$$
i.e.,

\[ \int_{w_j^\prime \not\equiv w_i^\prime} \omega \to 0. \]

Since \( \Gamma_\omega \) is discrete and \( \int_{w_j^\prime \not\equiv w_i^\prime} \omega \in \Gamma_\omega \), this indeed implies that

\[ \int_{w_j^\prime \not\equiv w_i^\prime} \omega = 0 \quad (7.8) \]

for all sufficiently large \( i, j \). Since the set \( \{ \int_{w_j^\prime} \omega \} \) is bounded, these imply that the sequence \( \int_{w_j^\prime} \omega \) eventually stabilizes. Going back to (7.5), we have proven that the actions

\[ A_H([z_\infty, w_j^\prime]) \]

themselves stabilize and so we have

\[ A_H([z_\infty, w_N^\prime]) = \lim_{j \to \infty} A_H([z_\infty, w_j^\prime]) = \rho(H; a) \]

for a fixed sufficiently large \( N \in \mathbb{Z}_+ \). This proves that \( \rho(H; a) \) is indeed a critical value of \( A_H \) at the critical point \([z_\infty, w_N^\prime]\]. This finishes the proof. \( \square \)

We now state the following theorem.

**Theorem 7.3.** Let \((M, \omega)\) be rational and \( C^\infty_m(M \times [0, 1], \mathbb{R}) \) be the set of normalized \( C^\infty \)-Hamiltonians on \( M \). We denote by \( \rho_a : C^\infty_m(M \times [0, 1], \mathbb{R}) \to \mathbb{R} \) the extended continuous function defined by \( \rho_a(H) = \rho(H; a) \).

1. The image of \( \rho_a \) depends only on the homotopy class \( \bar{\phi} = [\phi, H] \). Hence \( \rho_a \) pushes down to a well-defined function

\[ \rho : \text{Ham}(M, \omega) \times QH^*(M) \to \mathbb{R}; \quad \rho(\bar{\phi}; a) := \rho(H; a) \quad (7.9) \]

for any \( H \) with \( \bar{\phi} = [\phi, H] \).
2. We have the formula

\[ \rho(H; a) = \inf_{\lambda} \{ \lambda \mid a^\prime \in \text{Im} (i_{\lambda} : HF^\lambda_*(H) \to HF_*(H)) \}. \quad (7.10) \]

**Proof.** We have shown in Theorem 7.1 that \( \rho(H; a) \) is indeed a critical value of \( A_H \), i.e., lies in \( \text{Spec}(H) \). With this fact in our disposition, the well-definedness of the definition (7.9), i.e., independence of \( H \) with \( \bar{\phi} = [\phi, H] \) is an immediate consequence of combination of the following results:

1. \( H \mapsto \rho(H; a) \) is continuous,
2. \( \text{Spec}(H) \) is of measure zero subset of \( \mathbb{R} \) (Lemma 2.2),
3. \( \text{Spec}(H) = \text{Spec}(\bar{\phi}) \) depends only on its homotopy class \([H] = \bar{\phi}\) and so fixed as long as \([H] = \bar{\phi}\) (Theorem 2.3),
4. the only real-valued continuous function from a connect space (e.g., \([0, 1]\)) whose image has measure zero in \( \mathbb{R} \), is a constant function.
(7.10) is just a rephrasing of the definition of $\rho(H; a)$. This finishes the proof of Theorem 7.3. □

One more important property concerns the effect of $\rho$ under the action of $\pi_0(\widetilde{G})$. We first explain how $\pi_0(\widetilde{G})$ acts on $\widetilde{\text{Ham}}(M, \omega) \times QH^*(M)$ following (and adapting into cohomological version) Seidel’s description of the action on $QH_*(M)$. According to [Se], each element $[h, \widetilde{h}] \in \pi_0(\widetilde{G})$ acts on $QH_*(M)$ by the quantum product of an even element $\Psi([h, \widetilde{h}])$ on $QH_*(M)$. We take the adjoint action of it on $a \in QH^*(M)$ and denote it by $\tilde{h}^*a$. More precisely, $\tilde{h}^*a$ is defined by the identity

$$\langle \tilde{h}^*a, \beta \rangle = \langle a, \Psi([h, \widetilde{h}]) \cdot \beta \rangle$$

(7.11)

with respect the non-degenerate pairing $\langle \cdot , \cdot \rangle$ between $QH^*(M)$ and $QH_*(M)$.

**Theorem 7.4. (Monodromy shift)** Let $\pi_0(\widetilde{G})$ act on $\widetilde{\text{Ham}}(M, \omega) \times QH^*(M)$ as above, i.e,

$$[h, \widetilde{h}] \cdot ([\phi, a]) = (h \cdot \widetilde{\phi}, \tilde{h}^*a)$$

(7.12)

Then we have

$$\rho([h, \widetilde{h}] \cdot ([\phi, a])) = \rho([\phi, a]) + I_\omega([h, \widetilde{h}]).$$

**Proof.** This is immediate from the construction of $\Psi([h, \widetilde{h}])$ in [Se]. Indeed, the map

$$[h, \widetilde{h}]_* : CF_*(F) \mapsto CF_*(H\#F)$$

(7.13)

is induced by the map

$$[z, w] \mapsto \tilde{h}([z, w])$$

and we have

$$A_{H\#F}(\tilde{h}([z, w])) = A_F([z, w]) + I_\omega([h, \widetilde{h}])$$

by (2.5). Furthermore the map (7.12) is a chain isomorphism whose inverse is given by $([h, \widetilde{h}]^{-1})_*$. This immediately implies the theorem from the construction of $\rho$. □

**Remark 7.5.** Strictly speaking, $\tilde{h}^*a$ may not lie in the standard quantum cohomology $QH^*(M)$ because it is defined as the linear functional on the complex $CQ_*(M)$ that is dual to the Seidel element $\Psi([h, \widetilde{h}]) \in CQ_*(M)$ under the canonical pairing between $CQ_*(M)$ and $CQ^*(M)$. A priori, the bounded linear functional

$$\tilde{h}^*a = \langle \Psi([h, \widetilde{h}]), \cdot \rangle$$

may not lie in the image of $\sharp : QH_*(M) \to QH^*(M)$, mentioned in section 3, in general. In that case, one should consider $\tilde{h}^*a$ as a continuous quantum cohomology class in the sense of Appendix. We refer readers to Appendix for the explanation on how to extend the definition of our spectral invariants to the continuous quantum cohomology classes.

Now we can define

$$\rho : \widetilde{\text{Ham}}(M, \omega) \times QH^*(M)$$
by putting
\[ \rho(\tilde{\phi}; a) := \rho(H; a) \]
for any \( H \mapsto \tilde{\phi} \) with \( [H] = \tilde{\phi} \) when \( \tilde{\phi} \) is nondegenerate, and then extending to arbitrary \( \phi \) by continuity.

Then by the spectrality of \( \rho(\tilde{\phi}; a) \) for each \( a \in QH^*(M) \), we have constructed continuous ‘sections’ of the action spectrum bundle
\[ \mathcal{S} \text{pec}(M, \omega) \to \tilde{\text{Ham}}(M, \omega) \]

We define the essential spectrum of \( \tilde{\phi} \) by
\[ \text{spec}(\tilde{\phi}) := \{ \rho(\tilde{\phi}; a) | 0 \neq a \in QH^*(M) \} \]
\[ \text{spec}_k(\tilde{\phi}) := \{ \rho(\tilde{\phi}; a) | 0 \neq a \in QH^k(M) \} \]

and the bundle of essential spectra by
\[ \text{spec}(M, \omega) = \bigcup_{\tilde{\phi} \in \tilde{\text{Ham}}(M, \omega)} \text{spec}(\tilde{\phi}) \]

and similarly for \( \text{spec}_k(M, \omega) \).

§8. Remarks on the transversality

Our construction of various maps in the Floer homology works as they are in the previous section for the strongly semi-positive case [Se], [En1] by the standard transversality argument. On the other hand in the general case where constructions of operations in the Floer homology theory requires the machinery of virtual fundamental chains through multi-valued abstract perturbation [FOn], [LT1], [Ru], we need to explain how this general machinery can be incorporated in our construction. The full details will be provided elsewhere. We will use the terminology ‘Kuranishi structure’ adopted by Fukaya and Ono [FOn] for the rest of the discussion.

One essential point in our proofs is that various numerical estimates concerning the critical values of the action functional and the levels of relevant Novikov cycles do not require transversality of the solutions of the relevant pseudo-holomorphic sections, but depends only on the non-emptiness of the moduli space
\[ \mathcal{M}(H, \tilde{J}; \tilde{z}) \]
which can be studied for any, not necessarily generic, Hamiltonian \( H \). Since we always have suitable a priori energy bound which requires some necessary homotopy assumption on the pseudo-holomorphic sections, we can compactify the corresponding moduli space into a compact Hausdorff space, using a variation of the notion of stable maps in the case of non-degenerate Hamiltonians \( H \). We denote this compactification again by
\[ \mathcal{M}(H, \tilde{J}; \tilde{z}). \]

This space could be pathological in general. But because we assume that the Hamiltonians \( H \) are non-degenerate, i.e, all the periodic orbits are non-degenerate, the
moduli space is not completely pathological but at least carries a Kuranishi structure in the sense of Fukaya-Ono [FOn] for any $H$-compatible $\tilde{J}$. This enables us to apply the abstract multi-valued perturbation theory and to perturb the compactified moduli space by a Kuranishi map $\Xi$ so that the perturbed moduli space

$$\mathcal{M}(H, \tilde{J}; \tilde{z}, \Xi)$$

is transversal in that the linearized equation of the perturbed equation

$$\tilde{\partial}_J(v) + \Xi(v) = 0$$

is surjective and so its solution set carries a smooth (orbifold) structure. Furthermore the perturbation $\Xi$ can be chosen so that as $\|\Xi\| \to 0$, the perturbed moduli space $\mathcal{M}(H, \tilde{J}; \tilde{z}, \Xi)$ converges to $\mathcal{M}(H, \tilde{J}; \tilde{z})$ in a suitable sense (see [FOn] for the precise description of this convergence).

Now the crucial point is that non-emptiness of the perturbed moduli space will be guaranteed as long as certain topological conditions are met. For example, the followings are the prototypes that we have used in this paper:

1. $h_{H_1}: CF_0(\tilde{f}) \to CF_0(H)$ is an isomorphism in homology and so $[h_{H_1}(1^*)] \neq 0$. This is immediately translated as an existence result of solutions of the perturbed Cauchy-Riemann equation.

2. The definition of the pants product $\ast$ and the identity

$$[\alpha \ast \beta] = (a \cdot b)^b$$

in homology guarantee non-emptiness of the relevant perturbed moduli space $\mathcal{M}(H, \tilde{J}; \tilde{z}, \Xi)$ for $\alpha \in CF_*(H_1), \beta \in CF_*(H_2)$ with $[\alpha] = a^b$ and $[\beta] = b^b$ respectively.

Once we prove non-emptiness of $\mathcal{M}(H, \tilde{J}; \tilde{z}, \Xi)$ and an a priori energy bound for the non-empty perturbed moduli space and if the asymptotic conditions $\tilde{z}$ are fixed, we can study the convergence of a sequence $v_j \in \mathcal{M}(H, \tilde{J}; \tilde{z}, \Xi_j)$ as $\Xi_j \to 0$ by the Gromov-Floer compactness theorem. However a priori there are infinite possibility of asymptotic conditions for the pseudo-holomorphic sections that we are studying, because we typically impose that the asymptotic limit lie in certain Novikov cycles like

$$\tilde{z}_1 \in \alpha, \tilde{z}_2 \in \beta, \tilde{z}_3 \in \alpha \ast \beta$$

Because the Novikov Floer cycles are generated by an infinite number of critical points $[z, w]$ in general, one needs to control the asymptotic behavior to carry out compactness argument. For this purpose, we need to establish a lower bound for the actions which will enable us to consider only finite possibilities for the asymptotic conditions because of the finiteness condition in the definition of Novikov chains. We would like to emphasize that obtaining a lower bound is the heart of matter in the classical mini-max theory of the indefinite action functional which requires a linking property of semi-infinite cycles. On the other hand, obtaining upper bound is usually an immediate consequence of the identity like (4.10).

With such a lower bound for the actions, we may then assume, by taking a subsequence if necessary, that the asymptotic conditions are fixed when we take the limit and so we can safely apply the Gromov-Floer compactness theorem to
produce a (cusp)-limit lying in the compactified moduli space $\mathcal{M}(H, \tilde{J}; \tilde{\mathcal{Z}})$. This will then justify all the statements and proofs in the previous sections for the complete generality.

**Appendix: Continuous quantum cohomology**

In this appendix, we define the genuinely cohomological version of the quantum cohomology and explain how we can extend the definition of the spectral invariants to the classes in this cohomological version.

We call this *continuous quantum cohomology* and denote by

$$QH^*_{\text{cont}}(M).$$

In this respect, we call the usual quantum cohomology ring $QH^*(M) = H^*(M) \otimes \Lambda^\uparrow$ the *finite quantum cohomology*. We call elements in $QH^*_{\text{cont}}(M)$ and $QH^*(M)$ continuous (resp. finite) quantum cohomology classes.

We first define the chain complex associated to $QH^*_{\text{cont}}(M)$. Let $f$ be a Morse function and consider the complex of Novikov chains

$$CQ_{2n-k}(-\epsilon f) = CM_{2n-k}(-\epsilon f) \otimes \Lambda^\downarrow(= CF_k(\epsilon f)).$$  \hspace{1cm} (A.1)

On non-exact symplectic manifolds, this is typically infinite dimensional as a $Q$-vector space. Therefore it is natural to put some topology on it rather than to consider it just as an algebraic vector space. For this purpose, we recall the definition of the level $\lambda(\alpha) = \lambda_{\epsilon f}(\alpha)$ of an element

$$\lambda(\alpha) = \max\{\lambda_{\epsilon f}(\alpha_A q^A) \mid \alpha_A \neq 0\}$$

$$= \max\{\lambda_{Morse}(\alpha_A) - \omega(A)\}.$$

As we saw before, this level function satisfies the inequality

$$\lambda(\alpha + \beta) \leq \max\{\lambda(\alpha), \lambda(\beta)\}$$  \hspace{1cm} (A.2)

and provides a natural filtration on $CQ_{2n-k}(-\epsilon f)$, which defines a *Non-Archimedean topology*. We refer to [Br] for a nice exposition to the Non-Archimedean topology and geometry.

**Definition & Proposition A.1.** For each degree $*$, consider the collection

$$\mathcal{B} = \bigcup_{\alpha \in CQ_*(\epsilon f), R \in \mathbb{R}} \{U(\alpha, R) \subset CQ_*(\epsilon f)\}$$

of the subsets $U(\alpha, R)$ that is defined by

$$U(\alpha, R) = \{\beta \in CQ_*(\epsilon f) \mid \lambda(\beta - \alpha) < R\}.$$
Then \( \mathcal{B} \) satisfies the properties of a basis of topology. We equip \( CQ_*(-\epsilon f) \) with the topology generated by the basis \( \mathcal{B} \).

**Proof.** We need to prove that for any given \( U(\alpha_1, R_1) \) and \( U(\alpha_2, R_2) \) with \( U(\alpha_1, R_1) \cap U(\alpha_2, R_2) \neq \emptyset \) and for any \( \alpha \in U(\alpha_1, R_1) \cap U(\alpha_2, R_2) \), there exists \( R_3 \) such that

\[
U(\alpha, R_3) \subset U(\alpha_1, R_1) \cap U(\alpha_2, R_2). \tag{A.3}
\]

Let \( \beta \in U(\alpha_1, R_1) \cap U(\alpha_2, R_2) \). Then \( \beta \) satisfies

\[
\lambda(\beta - \alpha_i) < R_i, \quad i = 1, 2 \tag{A.4}
\]

Suppose \( \gamma \in U(\beta, R) \) where \( R \) is to be determined. Then we derive from (A.2)

\[
\lambda(\gamma - \alpha_i) \leq \max\{\lambda(\gamma - \beta), \lambda(\beta - \alpha_i)\} = \max\{R, R_i\} \tag{A.5}
\]

Therefore if we choose \( R \leq \min\{R_1, R_2\} \), then we will have

\[
U(\beta, R) \subset U(\alpha_1, R_1) \cap U(\alpha_2, R_2)
\]

which finishes the proof of the fact that \( \mathcal{B} \) really defines a basis of topology. \( \square \)

By the Non-Archimedean triangle inequality (A.2), it follows that the basis element \( U(\alpha, R) \) is nothing but the affine subspace

\[
U(\alpha, R) = CQ^R_*(\epsilon f) + \alpha = CF^R_{2n-*}(\epsilon f) + \alpha
\]

where \( CF^R_\ast \) is defined as in section 4.

The following is an easy consequence of the definition of the boundary operator.

**Lemma A.2.** The boundary operator

\[
\partial_{\epsilon f} = \partial_{-\epsilon f}^{Morse} \otimes \Lambda : CQ_{2n-k}(\epsilon f) \to CQ_{2n-k-1}(\epsilon f)
\]

is continuous with respect to this topology.

**Proof.** Let \( U(\alpha, R) \) be a basis element and consider the preimage

\[
(\partial_{\epsilon f})^{-1}(U(\alpha, R)).
\]

Suppose \( \beta \in (\partial_{\epsilon f})^{-1}(U(\alpha, R)) \), i.e., \( \partial_{\epsilon f}(\beta) \in U(\alpha, R) \) and so

\[
\lambda(\partial_{\epsilon f}(\beta) - \alpha) < R. \tag{A.6}
\]

Recall that

\[
\lambda(\partial_{\epsilon f}(\delta)) \leq \lambda(\delta) \tag{A.7}
\]

for any Novikov Floer chain \( \delta \). Now we consider the basis element \( U(\beta, R) \). Then if \( \gamma \in U(\beta, R) \), we have

\[
\lambda(\partial_{\epsilon f}(\gamma) - \alpha) \leq \max\{\lambda(\partial_{\epsilon f}(\gamma - \beta)), \lambda(\partial_{\epsilon f}(\beta) - \alpha)\} \\
\leq \max\{\lambda(\gamma - \beta), \lambda(\partial_{\epsilon f}(\beta) - \alpha)\} \leq \max\{R, R_i\} = R \tag{A.8}
\]

where the second inequality comes from (A.7). This finishes the proof of \( \partial_{\epsilon f}(U(\beta, R)) \subset U(\alpha, R) \) i.e., \( U(\beta, R) \subset (\partial_{\epsilon f})^{-1}(U(\alpha, R)) \) for any \( \beta \in U(\alpha, R) \). Hence the proof. \( \square \)

Now we define
Definition A.3. A linear functional
\[ \mu : CQ_{2n-k}(-\epsilon f) \to \mathbb{Q} \]
is called continuous (or bounded) if it is continuous with respect to the topology induced by the above filtration. We denote by \( CQ^f_{\text{cont}}(-\epsilon f) \) the set of continuous linear functionals on \( CQ_{2n-k}(-\epsilon f) \).

The following is easy to see from the definition of Novikov chains.

Lemma A.4. A linear functional \( \mu \) is continuous if and only if there exists \( \lambda_\mu \in \mathbb{R} \) such that
\[ \mu(\alpha_A q^A) = 0 \] (A.9)
for all \( A \) with \( -\omega(A) \leq \lambda_\mu \).

Proof. The sufficiency part of the proof is easy and so we will focus on the necessary condition. We will prove this by contradiction. Suppose that \( \mu : CQ_{2n-k}(-\epsilon f) \to \mathbb{Q} \) is a continuous linear functional, but there exists a sequence of \( A_j \) with
\[ -\omega(A_j) \to -\infty, \quad i.e., \quad \omega(A_j) \to +\infty \] (A.10)
and \( \alpha_j \in CM_{A}(-\epsilon f) \) such that
\[ \mu(\alpha_j q^{A_j}) \neq 0. \] (A.11)

Now consider the sequence of Novikov chains
\[ \beta_N = \sum_{j=1}^{N} \alpha_j q^{A_j}. \] (A.12)
It is easy to check from (A.10) that \( \beta_N \) converges to the Novikov chain
\[ \beta = \sum_{j=1}^{\infty} \alpha_j q^{A_j} \]
in the given Non-Archimedean topology on \( CQ_{A}(-\epsilon f) \). In fact, this convergence holds for the sequence
\[ \beta_{c,N} = \sum_{j=1}^{N} (c_j \alpha_j) q^{A_j} \] (A.13)
for any given sequence \( c = \{c_j \in \mathbb{Q}\}_{1 \leq j < \infty} \). We choose \( c_j \)'s so that
\[ c_j = \frac{1}{\mu(\alpha_j q^{A_j})} \]
which is well-defined by (A.11). However we then have
\[ \mu(\beta_{c,N+1}) - \mu(\beta_{c,N}) = \mu(c_{N+1} \alpha_{N+1} q^{A_{N+1}}) = c_{N+1} \mu(\alpha_{N+1} q^{A_{N+1}}) = 1 \]
for all $N$. This proves that $\mu$ cannot be continuous, a contradiction. This finishes the proof. □

It then follows that

$$\partial_Q = \partial^*_{\epsilon f} : (CQ_{\ell}(-\epsilon f))^* \to (CQ_{\ell+1}(-\epsilon f))^*$$

maps continuous linear functionals to continuous ones and so defines the canonical complex

$$(CQ^*_{\text{cont}}(-\epsilon f), \partial_Q^*)$$

and hence defines the homology

$$QH^\ell_{\text{cont}}(M) := H^\ell(CQ^*_{\text{cont}}(-\epsilon f), \partial_Q^*).$$

We recall the canonical embedding

$$\sigma : CQ^\ell(-\epsilon f) = CM_{2n-\epsilon \ell f} \otimes \Lambda^\ell \hookrightarrow CQ^\ell_{\text{cont}}(-\epsilon f); a \mapsto \langle a, \cdot \rangle$$

mentioned in Remark 5.1. We have the following proposition which is straightforward to prove. We refer to the proof of [Proposition 2.2, Oh2] for the details.

**Proposition A.5.** The map $\sigma$ in (A.14) is a chain map from $(CQ^\ell(-\epsilon f), \delta^Q)$ to $(CQ^\ell_{\text{cont}}(-\epsilon f), \partial_Q^*)$. In particular we have a natural degree preserving homomorphism

$$\sigma : QH^*_{\text{cont}}(M) \cong HQ^*(-\epsilon f) \to HQ^*_{\text{cont}}(-\epsilon f) \cong QH^*_{\text{cont}}(M).$$

(A.15)

Now we can define the notion of continuous Floer cohomology $HF^*_{\text{cont}}(H)$ for any given Hamiltonian in a similar way. Then the co-chain map

$$(h_H)^* : CF^k(H) \to CF^k(\epsilon f)$$

restricts to the co-chain map

$$(h_H)^* : CF^k_{\text{cont}}(H) \to CF^k_{\text{cont}}(\epsilon f).$$

Once we have defined the continuous quantum cohomology and the continuous Floer cohomology, it is straightforward to define the spectral invariants for the continuous cohomology class in the following way.

**Definition A.6.** Let $\mu \in QH^\ell_{\text{cont}}(M)$. Then we define

$$\rho(H; \mu) := \inf \{ \lambda \in \mathbb{R} \mid \mu \in \text{Im} \ i^\lambda \}$$

(A.16)

Now it is straightforward to generalize all the axioms in Theorem I to the continuous quantum cohomology class. The only non-obvious axiom is the triangle inequality. But the proof will be a verbatim modification of [Theorem II (5), Oh2] incorporating the argument in the present paper that uses the Hamiltonian fibration and pseudo-holomorphic sections. We leave the details to the interested readers. We hope to investigate further properties of the continuous quantum cohomology and its applications elsewhere.
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