SERRE MULTIPLICITY QUESTION AND MUKAI PAIRING

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Abstract. The Serre conjecture on intersection multiplicity in proper intersections over general regular rings, is still a challenging open question. We show some connections of this conjecture to Hodge theory, and Riemann-Hodge bilinear relations, using Fourier-Mukai integral transform and Gamma class. Some connections with Grothendieck Standard conjectures have been discussed.

Introduction

In 1950’s J. Serre made a definition of Intersection multiplicity of two finitely generated modules over a regular local ring $A$ by a type of Euler characteristic, namely Tor-formula. He proved the non-negativity of this Euler characteristic in several special cases that were enough for the purpose of geometers, for instance in the case where the ring $A$ is essentially of finite type over a filed or a discrete valuation ring $k$. Serre conjectured the vanishing of multiplicity in non-proper intersections, and its positivity in the proper case, for general regular local rings, [S]. The vanishing part was proved by H. Gillet and C. Soule, [GS1], and also independently by P. Roberts, [RO1]. A proof of non-negativity was given by O. Gabber [RO2]. However, the positivity remained open. I try to investigate some relations between the positivity and Hodge theory, using Fourier-Mukai pairing of categories and Hochschild homology.

Serre intersection multiplicity definition, naturally lifts to a product on $K_0(A)$. That the $K$-theory of the regular ring $A$ concerns an intersection theory. In this way the intersection multiplicity can be written as a cup product in $K_0(A)$. The chern character $ch$ and the Riemann-Roch map transform this product into the Chow ring of $A$, where one may use Kodaira vanishing theorem in order to establish the vanishing of multiplicity.

One step further is to try to push the above product into cohomology of ambient space by integral transform. This allows some more flexible theory to discuss about positivity. However, it still does not provide a complete answer, because one needs a type of positivity of Poincare product in some Weil cohomology theory of a general
regular scheme. In this article we investigate this connection in characterisitic 0, using de Rham cohomology and its cup product. In characteristic \( p > 0 \), over a field, we make a discussion related to Grothendieck Standard Conjectures.

The Mukai Pairing generally is a non-degenerate pairing on \( HH_*(X) \). However it may also be formulated at the level of de Rham cohomologies. Mukai transform is what that makes the Riemann-Roch map a homomorphism. It is the toll we have used to express the Serre (Cartan-Eilenberg) Euler characteristic as a Hodge theory product. To do this one needs to modify Mukai vector by another class namely Gamma class. The Gamma class appears in the context of Mirror symmetry as a perturbative correction term. The original definition of Mukai vector reflects the Calabi-Yau case, where \( \Gamma = 0 \).

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1. Serre Multiplicity Conjecture

Let \( A \) be a regular local ring, and \( M, N \) finitely generated \( A \)-modules such that \( M \otimes_A N \) has finite length. J. P. Serre \cite{S}, defines the intersection multiplicity as

\[
\chi^A(M, N) := \sum (-1)^i l(Tor^A_i(M, N))
\]

He proves the basic fact that in this case:

\[
\dim M + \dim N \leq \dim A
\]

will hold and makes the following question, known as Serre Multiplicity conjecture.

1. If \( \dim M + \dim N < \dim A \), then \( \chi^A(M, N) = 0 \)

2. In case \( \dim M + \dim N = \dim A \), called proper intersection, \( \chi^A(M, N) > 0 \).

The quantity

\[
\xi_A(M, N) := \sum (-1)^i l(Ext^A_i(M, N))
\]
called the Euler form is equivalent to $\chi$ by;

$$\chi^A(M, N) = (-1)^{\dim N} \xi_A(M, N)$$

Both $\chi^A, \xi_A$ are additive functions in each variable on exact sequences of $A$-modules, [CHA].

**Theorem 1.1.** (Vanishing, Roberts [RO1], Gillet and Soule [GS1]) Assume $M, N$ are finitely generated modules over a complete intersection (hence regular) $A$, such that $M, N$ both have finite projective dimension (modules over regular rings always satisfy this) and $M \otimes_A N$ has finite length. If $\dim M + \dim N < \dim A$, then $\chi^A(M, N) = 0$.

The condition, $M, N$ both have finite projective dimensions implies that the sums in (1) have finitely many terms. Also the condition, $M \otimes_A N$ has finite length, implies, all the $\text{Tor}_i^A(M, N)$ and hence all $\text{Ext}_i^A(M, N)$ have finite length. This makes the former criteria meaningful.

One may explain the Euler characteristic in terms of projective resolutions. By a perfect $A$-complex we mean a bounded complex of finitely generated free (Projective) $A$-modules. The support of such complex would be the closed subspace $\text{Supp}(G_\bullet)$ where the localization $(G_\bullet)_p$ has non-trivial homology. Then the dimension of the complex is defined to be $\dim \text{Supp}(G_\bullet)$.

If $E_\bullet$ and $F_\bullet$ be free resolutions of the $A$-modules $M, N$ (which may be taken to be finite, by the regularity of $A$), then

$$\chi^A(M, N) = \chi(E_\bullet \otimes F_\bullet) = (-1)^{\text{codim } M} \chi(E_\bullet^* \otimes F_\bullet)$$

where the right hand side is the usual Euler characteristic of the complex $E_\bullet \otimes F_\bullet$. The latter makes sense for the complex is supported on the maximal ideal of $A$.

**2. RIEMANN-ROCH FOR LOCAL CHERN CHARACTER**

The local chern character is an analogue of the usual chern character, on the $K$-theory of perfect $A$-complexes. We only need the formal properties of this theory to explain the vanishing part of the conjecture. The local Chern character of a perfect complex may be written as
\[
\text{ch}(G_ullet) = \text{ch}_0(G_ullet) + \text{ch}_1(G_ullet) + \ldots
\]

where

\[(6) \quad \text{ch}_i(G_ullet) : \text{CH}_k(\text{Spec}A)_\mathbb{Q} \to A_{k-i}(\text{Supp}(G_ullet))_\mathbb{Q}\]

The Euler characteristic and local Chern character are related by

\[(7) \quad \chi(G_ullet) = \text{ch}(G_ullet) \cdot \text{td}(A)\]

which is called local Riemann-Roch theorem. When \(G_ullet\) is supported at the maximal ideal \((7)\) becomes the simple one

\[
\chi(G_ullet) = \text{ch}(G_ullet)[\text{Spec}(A/m)]
\]

The correct language to work with characteristic classes is to consider them as maps on the \(K_0\) groups, or rings. Then the Chern character becomes a ring homomorphisms between \(K_0\) and \(\text{CH}^*(A) = \text{CH}(\text{Spec}(A))\).

In geometry we replace \(A\) with a noetherian scheme \(X\). For \(Y \subset X\) a closed subset, \(K_0^Y(X)\) may be defined similar to the usual \(K_0(X)\) for bounded complexes having support in \(Y\). Then we will have the natural product;

\[(8) \quad \cup : K_0^Y(X) \otimes K_0^Z(X) \to K_0^{Y \cap Z}(X)\]

given by \([E_ullet] \otimes_{\mathcal{O}_X} [F_ullet]\). So if we set:

\[(9) \quad K_0^\sigma(X) = \bigoplus_{Y \subset X} K_0^Y(X)\]

then we obtain a ring structure with unit the complex \([\mathcal{O}_X]\). The \(K\)-theory with support satisfies all the natural functorial properties with respect to flat pull-backs or proper push-forwards.
Let $K'_0$ be the $K$-theory of finitely generated modules (probably over a singular space or ring-sometimes denoted by $G_0$). If we define,

$$ \cap : K^Y_0(X) \times K'_0(Z) \to K'_0(Y \cap Z) $$

$$ [E_\bullet] \cap [M] = \sum_{i \geq 0} (-1)^i [H_i(E_\bullet \otimes_{\sigma_X} M)] $$

then for $f$ a flat proper morphism, we have the familiar projection formula

$$ f_*(f^*(\beta) \cap \alpha) = \beta \cap f_*(\alpha) $$

where

$$ f_*[M] = \sum_{i \geq 0} (-1)^i [R^i f_* M] $$

with appropriate $\beta, \alpha$.

If $X$ is a regular scheme, then the map

$$ K^Y_0(X) \to K'_0(Y) $$

defined by

$$ [E_\bullet] \mapsto \sum_i (-1)^i [H^i(E_\bullet)] $$

would be an isomorphism. In this case if $Y, Z$ be closed subsets then the product structure on $K_0^Y(X)$ induces the following pairing

$$ K'_0(Y) \otimes K'_0(Z) \to K'_0(Y \cap Z) $$

$$ [E_\bullet] \otimes [F_\bullet] \mapsto \sum_i (-1)^i [\text{Tor}^\bullet_{\sigma_X}(E_\bullet, F_\bullet)] $$
Theorem 2.1. [GS1] The exterior powers endow \( K_0^\sigma(X) = \bigoplus_{Y \subset X} K_0^Y(X) \) with a \( \lambda \)-ring structure.

This means that there exists a collection of maps \( \{ \lambda^k : K_0^\sigma(X) \to K_0^\sigma(X) \}_{k \geq 0} \) given by exterior powers, satisfying some combinatorial conditions, reflecting the simplicial structure of these rings, [GS1]. Then we have a collection of ring homomorphisms

\[
\{ \psi_k : K_0^\sigma(X) \to K_0^\sigma(X) \},
\]

called Adams operations. They are defined by certain axiomatic properties similar to chern classes. The restrictions,

\[
\psi_k : K_0^Y(X) \to K_0^Y(X)
\]

are group homomorphisms. If \( X = \text{Spec}(A) \), with \( A \) noetherian they are defined by

\[
\psi_k[K(a)] = k[K(a)]
\]

where \( K(a) \) is the single Koszul complex \( A \to_A A, a \in A \). The single Koszul complexes are considered as the building blocks of the \( \lambda \)-ring \( K_0^\sigma(X) \), [GS1].

If \( F^m \) is the filtration by co-dimension of support, i.e.

\[
F^m K_0^Y(X) := \lim_{\text{codim}_X(Z) \geq m} \text{Im}(K_0^Z(X) \to K_0^Y(X))
\]

then

\[
\text{Gr}_F^i \psi_k : \text{Gr}_F^i K_0^Y(X)_Q \to \text{Gr}_F^i K_0^Y(X)_Q
\]

is just multiplication by \( k^i \), [GS1]. We have the product

\[
F^m K_0^Y(X)_Q \otimes F^n K_0^Y(X)_Q \to F^{m+n} K_0^Y(X)_Q
\]

this automatically implies that when

\[
\text{codim}(\text{Supp}(M)) + \text{codim}(\text{Supp}(N)) > d = \text{dim} A
\]

then
(17) \[ \chi(M, N) = ([M] \cup [N]) \cap [\mathcal{O}_X] = \sum (-1)^i l(\text{Tor}^A_i(M, N)) = 0 \]

In this way the intersection multiplicity can be written as a cup product on the \( K \)-theory of perfect \( A \)-complexes which is the same as usual \( K_0(A) \) when \( A \) is regular, [GS1].

3. **Mukai Pairing and Hochschild Homology**

Let \( k \) be a commutative ring. The Hochschild homology and cohomology of an \( k \)-algebra \( A \) are defined by,

\[
HH_k^i(A) := \text{Tor}_k^i(A, A), \quad HH_k^0(A) := \text{Ext}_{A^e}^0(A, A)
\]

where \( A^e = A^\text{op} \otimes_k A \). Note that in case \( A \) is commutative \( A^\text{op} = A \). The Hochschild homology and cohomology of a regular scheme is a sheafification of this definition using the structure sheaves \( \mathcal{O}_X \). The Denis trace map

\[ (18) \quad \text{Den} : K_0(X) \to HH_0(X) \]

would play the role of classical chern character, whose composition with Hochschild-Konstant-Rosenberg homomorphism induces the usual chern character,

\[ (19) \quad K_0(X) \xrightarrow{\text{ch}} HH_0(X) \xrightarrow{\text{HKR}} \bigoplus_i H^i(X, \Omega^i_X) \]

The Denis trace can be defined in more general context, even for non-smooth \( X \).

**Remark 3.1.** The Denis trace is specifically defined by

\[ ch([e]) = \text{Tr}(\hat{e}) \]

in terms of the idempotent \( e \in M_n(A) \), where

\[ \hat{e} = e + \sum_{n \geq 1} \frac{(2n)!}{(n!)^2} (e - \frac{1}{2}) (de)^{2n} \]
As an element of the completion, \( \hat{\Omega}(A) = \prod \Omega^i(A) \), [L], [CST].

We only consider regular rings \( A \) and regular schemes. Therefore, we have \( HH_\bullet(A) = HH_\bullet(\text{perf}(A)) \). Thus we identify the theory of Chern characters for \( K \)-theory of sheaves or their perfect complexes. The H-K-R would be the isomorphism induced by the map;

\[
\begin{align*}
(20) \quad b_0 \otimes b_1 \otimes \ldots \otimes b_r &\to \frac{1}{r!} b_0. db_1 \wedge \ldots \wedge db_r
\end{align*}
\]

when \( X \) is smooth, [RA].

The philosophy of Mukai-pairing is to modify \( \text{ch} \), by cup product with a class \( \sqrt{\text{td}_X} \) such that we obtain a homomorphism in Riemann-Roch theorem. We have

\[
\text{td}_X^{1/2} \in \bigoplus_i H^i(X, \Omega^i_X).
\]

Let \( X, Y \) be complex manifolds, and let \( E \in D^b(X \times Y) \). Let \( \pi_X, \pi_Y \) be the projections. Define the integral transform with kernel \( E \) by;

\[
(21) \quad \Phi^E_{X \to Y} : D^b(X) \to D^b(Y), \quad \Phi^E_{X \to Y}(.) = \pi_{Y,*}(\pi_X^*(.) \otimes E)
\]

Similarly for \( \mu \in H^*(X \times Y, \mathbb{Q}) \)

\[
(22) \quad \Phi^\mu_{X \to Y} : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}), \quad \Phi^\mu_{X \to Y}(.) = \pi_{Y,*}(\pi_X^*(.) \otimes \mu)
\]

called the integral transform in cohomology associated to \( \mu \). The association between objects of \( D^b(X \times Y) \) or \( H^*(X \times Y) \) is functorial. In order to relate the above two functors we use chern character and Riemann-Roch theorem. The Riemann-Roch theorem states that, if \( \pi : X \to Y \) is a local complete intersection morphism;

\[
(23) \quad \pi_*(\text{ch}(\bullet) \text{td}(X)) = \text{ch}(\pi_*(\bullet)) \text{td}(Y)
\]

This suggest to define the Mukai vector of \( E \) as follows,

\[
(24) \quad v : D^b(X) \to H^*(X, \mathbb{Q}), \quad v(.) = \text{ch}(\bullet). \sqrt{\text{td}(X)}
\]
Then the commutativity of the following diagram is straight-forward;

\[
\begin{array}{ccc}
D^b(X) & \xrightarrow{\Phi_X^{\mathcal{E} \to Y}} & D^b(Y) \\
v \downarrow & & \downarrow v \\
H^*(X, \mathbb{Q}) & \xrightarrow{\phi_{\mathcal{E} \to Y}^*} & H^*(Y, \mathbb{C})
\end{array}
\]

We will denote \( \Phi_* = \Phi_{\mathcal{E} \to Y}^* \), where \( \Phi = \Phi_{\mathcal{E} \to Y} \), and it satisfies the associativity and functorial properties naturally. In case \( \Phi = \Phi_{\mathcal{E} \to Y} \) be an equivalence of categories, \( \Phi_* = \Phi_{\mathcal{E} \to Y}^* \) would be an isomorphism, [CA1], [CA2], [CA3].

**Remark 3.2.** When \( X \) is a projective smooth manifold, the map \( \Phi_* \) does respects the columns of Hodge diamond;

\[
\Phi_* = \phi_{\mathcal{E} \to Y}^* : \bigoplus_{p-q=i} H^{p,q}(X) \to \bigoplus_{p-q=i} H^{p,q}(X)
\]

This is for the class \( v(\mathcal{E}) \) is a Hodge class.

Let's define \( \tau : H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C}) \) by

\[
(26) \quad \tau(v_0, v_1, \ldots, v_{2n}) = (v_0, iv_1, -v_2, \ldots, i^{2n}v_{2n})
\]

and set,

\[
(27) \quad \vee : H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C}), \quad v^\vee = \tau(v) \cdot \frac{1}{\sqrt{ch(\omega_X)}}
\]

For

\[
\text{td}(T_X^\vee) = \text{td}(T_X) \cdot \exp(-c_1(T_X)) = \text{td}(T_X) \cdot \text{ch}(\omega_X)
\]

If \( X \) is proper and smooth, there is a natural isomorphism

\[
(28) \quad HH_\bullet(X) \cong HH_\bullet(\text{perf}(X)).
\]
When $Y$ is of the same type, an object $\Phi \in \text{perf}(X \otimes Y)$, may be considered as the kernel of an integral transform $\text{perf}(X) \to \text{perf}(Y)$. Then we would have the induced map

$$\Phi_* : HH_*(X) \to HH_*(Y)$$

Using Kunneth quasi-isomorphism

$$HH_*(\text{perf}(X)) \otimes HH_*(\text{perf}(Y)) \to HH_*(\text{perf}(X \times Y))$$

$O_\Delta := R\Delta_* O_X$ is a perfect complex, and we may regard it as the kernel of an integral transform from $X \times X \to \mathbb{C}$. Then by composition we get a pairing

$$HH_*(\text{perf}(X)) \otimes HH_*(\text{perf}(X)) \to HH_*(\text{perf}(X \times X)) \stackrel{\Delta}{\to} HH_*\text{perf}(\mathbb{C}) = \mathbb{C}$$

Which is given by shuffle products, [RAI]. A non-trivial fact is that the above induced map $\Phi_*$ will become equivalent to the integral transform induced by $\Phi$.

The Mukai Pairing can be generalized to Hochschild homology as

$$\langle \cdot, \cdot \rangle_M : HH_*(X) \otimes HH_*(X) \to \mathbb{C}$$

called generalized Mukai pairing. This generalization can be easily written using the isomorphism

$$D : R\text{Hom}(\Delta_! O_X, \Delta_* O_X) \cong R\text{Hom}(\Delta_* O_X, \Delta_* \omega_X)$$

where $\Delta_! O_X \cong \Delta_* \omega_X^{-1}$ and $\omega_X$ is the dualizing sheaf. Then, the Mukai pairing is

$$v \otimes w \to \text{tr}_{X \times X}(D(v) \circ w)$$

where $\text{tr}$ is Serre duality trace. If

$$^\vee : HH_*(X) \to HH_*(X)$$

is the involution induced through H-K-R isomorphism by the similar one to be $(-1)^p$ on $H^q(X, \Omega^p)$, as defined before. Then we would have
Theorem 3.3. [RA1] Suppose $X$ is smooth, then

\[ \langle b^\vee, a \rangle_M = \langle a, b \rangle_M, \quad a, b \in HH_* (X) \]

Theorem 3.4. [RA1] If $X$ is smooth, the generalized Mukai pairing on the Hochschild homology of $X$ satisfies

\[ \langle a, b \rangle_M = \int_X I(a)^\vee I(b).td_X, \quad a, b \in HH_* (X) \]

where $I$ is the H-K-R isomorphism.

The Euler pairing on $K_0 (X)$ is defined by

\[ \chi(\mathcal{E}, \mathcal{F}) := \sum_i (-1)^i \dim Ext_X^i (\mathcal{E}, \mathcal{F}) \]

Assume $H^* (X)$ is equipped with the pairing

\[ \langle x, y \rangle := (x \cup y \cup td_X) \cap [X] \]

Then the Riemann-Roch theorem states that, the chern character $ch : K_0 \to H^* (X)$ is map of inner product spaces. The same fact is true for Hochschild homology and Denis trace map, where we have the compatibility of the two chern character by H-K-R homomorphism, [CA1].

Theorem 3.5. [CA1], [CA2], [CA3] The chern character

\[ ch : K_0 \to HH_0 (X) \]

is a map of inner product spaces, in other words for $\mathcal{E}, \mathcal{F} \in D(X)$, we have

\[ \langle ch(E), ch(F) \rangle_M = \chi(\mathcal{E}, \mathcal{F}) \]

By the way, the intersection multiplicity of Serre may be written in the form,

\[ \chi(M, N) = \int_X ch(M)^\vee.ch(N).td_X \]
This is a first step toward our strategy in explaining the positivity of intersection multiplicity. We transformed the intersection product as one on $K_0(X)$ to some product on cohomology cycles, and more specific on Chern characters. This is one step more frontier than the work of P. Roberts, H. Gillet and C. Soule. Working in cohomology of a projective manifold allows to better analyse positivity or vanishing of the cup product.

4. **The Gamma-class**

A modification of Mukai pairing is to use the $\hat{\Gamma}_X$-class instead of the square $\sqrt{td_X}$. Then we replace the Mukai vector by the vector

$$E \mapsto (2\pi i)^{\deg(E)/2} \frac{1}{(2\pi)^{d/2}} \Gamma(X) \wedge ch(E)$$

The cohomology class $\hat{\Gamma}_X$ is defined via the identity,

$$\frac{z}{1 - e^{-z}} = e^{i\pi z} \Gamma(1 - x)\Gamma(1 + x)$$

used to share the two factors of $\sqrt{td_X}$ with the other chern classes in the Mukai pairing. It explicitly is given by the formula,

$$\hat{\Gamma}_X = \exp(C.ch_1(TX) + \sum_{n \geq 2} \frac{\zeta(n)}{n} ch_n(TX))$$

where

$$C = \lim_{n \to \infty} (1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n))$$

is the Euler constant, $\zeta$ is the Riemann zeta. Let’s write,

$$H^*(X, \mathbb{Q}) \xrightarrow{\vartheta} H^*(X, \mathbb{C}) \xrightarrow{\hat{\Gamma}_X \wedge (\bullet)} H^*(X, \mathbb{C})$$

where

$$\vartheta := (2\pi i)^{\deg(\cdot)/2} \frac{1}{(2\pi)^{d/2}}$$

Previously, we defined the Mukai vector as $\nu(\mathcal{E}) = ch(\mathcal{E}) \wedge \sqrt{td_X}$, and defined the pairing
\[ \langle v, w \rangle = \int_X v^\vee \wedge w, \quad v^\vee := \frac{\tau(v)}{\sqrt{ch(\omega_X)}} \]

This vector may also be more modified by setting

\[ \mu_\Lambda(E) := ch(E) \sqrt{td_X}. \exp(i\Lambda) \]

where \( \tau(\Lambda) = -\Lambda \). Thus the former Mukai vector is the special case \( \Lambda = 0 \). Then

\[ \mu_\Lambda(E)^\vee = ch(E)^\vee. \sqrt{td_X}. \exp(-i\Lambda) \]

Knowing \( \tau(td_X) = ch(\omega_X)td_X \), and we would still have

\[ \langle \mu_\Lambda(E), \mu_\Lambda(F) \rangle = \int \mu_\Lambda(E)^\vee \mu_\Lambda(F) = \int ch(E)^\vee ch(F).td_X \]

In this way the replacement for the square root of \( td_X \) is a multiplicative characteristic class, namely complex Gamma class, \([D], [HJLM]\),

\[ \hat{\Gamma}_X^C = \sqrt{td_X} \exp(i\Lambda_X) \]

**Remark 4.1.** *The Gamma class can also be defined via the chern roots \( \lambda_i \) of \( T_X \), by*

\[ \hat{\Gamma}(T_X) := \prod_{i=1}^d \Gamma(1 + \lambda_i) \]

*where the \( \Gamma \) on the right hand side is the usual Gamma function.*

The complex conjugate of a form \( \omega \) may be computed as,

\[ \omega \mapsto ch^{-1}(\omega/\hat{\Gamma}_X^C) \mapsto ch^{-1}(\bar{\omega}/\bar{\hat{\Gamma}_X^C}) \mapsto \bar{\omega}(\hat{\Gamma}_X^C/\bar{\hat{\Gamma}_X^C}) \]

The \( \Gamma \)-class plays an important role in mirror symmetry, concerning the rational structure of the Hodge structure on the mirror family. Essentially this class measures this issue.
5. SERRE MULTIPLICITY QUESTION OVER $\mathbb{C}$

By the machinery introduced in the former Sections we may easily study the positivity of Serre multiplicity over $\mathbb{C}$. First we write the Serre-Cartan-Eilenberg Euler characteristic as,

$$\chi(E, F) = \int_X \mu(E)^\vee \wedge \mu(F),$$

(30)

In this way we need to study the positivity of the right hand side using Riemann-Hodge bilinear relations for pure Hodge structures on $H^*(X, \mathbb{C})$.

Remark 5.1. The Mukai pairing $\langle v, w \rangle = \int_X v^\vee \wedge w$ appears in the context of Mirror Symmetry as a mirror to polarization form of a PVHS. This means it is a polarization of the mirror manifold or the PVHS we already have.

The positivity of intersection multiplicity proved in [S], shows when $Y$ and $Z$ are projective sub-varieties of $X$, then

$$\int_X \mu(O_Y)^\vee \wedge \mu(O_Z) \geq 0,$$

(31)

The aforementioned procedure, i.e expressing the Cartan-Eilenberg Euler characteristic as Hermitian type cup product suggests the idea to prove positivity in the serre multiplicity conjecture by Hodge theory. Formulas (30) and (31) directly relate the intersection multiplicity of algebraic cycles in the chow ring to cup product in homology, and thus to polarization of Hodge structures. Thus it provides a way to determine the positivity on both sides.

6. HODGE FILTRATION ON HOCHSCHILD HOMOLOGY

Let $k$ be a field and assume the $k$-algebra $A$ is commutative with a unit. The complex

$$A = \Omega^0_{A|k} \rightarrow \Omega^1_{A|k} \rightarrow \cdots \rightarrow \Omega^n_{A|k} \rightarrow \Omega^{n+1}_{A|k} \rightarrow \cdots,$$

$$d(a_0 \ da_1 \ldots \ da_n) = da_0 \ da_1 \ldots \ da_n$$
is called de Rham complex of \( A \), denoted \((\Omega^*_{A[k]}, d)\) and its cohomologies are called de Rham cohomologies of \( A \) over \( k \). \((\Omega^*_{A[k]}, d)\) is a (differential graded) DG-algebra. The anti-symmetrization map (20) defines a quasi-isomorphism of complexes

\[
C_\bullet(A) \to (\Omega^*_{A[k]}, 0)
\]

\[
CC^- \to (\Omega^*_{A[k]}[[u]], ud)
\]

\[
C_\bullet(A) \to (\Omega^*_{A[k]}[[u, u^{-1}]]/u\Omega_{A[k]}[[U]], ud)
\]

\[
C^{\text{per}}_\bullet(A) \to (\Omega^*_{A[k]}[u^{-1}, u], ud)
\]

of Hochschild, negative cyclic, cyclic and periodic cyclic complexes.

**Remark 6.1.** Periodic cyclic homology is total homology of the (horizontally) unbounded cyclic bi-complex. The usual cyclic homology (resp. negative cyclic homology) is the homology of the positive sub-bicomplex (resp. the negative part) of the periodic bi-complex.

If the ground field contains \( \mathbb{Q} \) and if \( A \) is commutative, then the Hochschild homology groups split functorially into smaller pieces,

\[
HH_n(A) = HH_n^{(1)}(A) \oplus HH_n^{(2)}(A) \oplus ... \oplus HH_n^{(n)}(A)
\]

namely the Hodge decomposition of Hochschild homologies, [L], [W].

**Theorem 6.2.** [L] Suppose \( k \) contains \( \mathbb{Q} \), \( A \) is commutative. Then there exists idempotents \( e_n^i \in \text{End}_k(A), i \geq 0 \) which naturally split the Hochschild bi-complex into a sum of sub-complexes \( C^i_\bullet(A) \), whose homologies are denoted by \( H_n^i(A) \). Therefore

\[
H_0(A) = H_0^{(0)}(A)
\]

\[
H_n(A) = H_n^{(1)} \oplus ... \oplus H_n^{(n)}, \quad n \geq 1
\]

These idempotents play a crucial role in explaining \( \lambda \) operations. There is a similar decomposition for cyclic homology.

**Remark 6.3.** [L] Suppose \( k \) contains \( \mathbb{Q} \), \( A \) is commutative, Then the cyclic bi-complex breaks into a sum of sub-complexes whose homologies are denoted by \( HC_n^i(A) \). Therefore,
The Hochschild or cyclic complexes may be sheafified so that one can define Hochschild and cyclic homology of schemes as one associated to their structure sheaves. Then similar Hodge decomposition still hold for smooth schemes in char = 0. This splittings is compatible with the S-B-I sequence.

\[ ... \rightarrow HH_n^{(i)}(X) \xrightarrow{I} HC_n^{(i)} \xrightarrow{S} HC_{n-2}^{(i-1)} \xrightarrow{B} ... \]

Moreover, If \( X \) is smooth

\[ HH_n^{(i)} = H^{i-n}(X, \Omega_X^n) \]

given by anti-symmetrization map, [W].

**Remark 6.4.** When \( X \) is a quasi-projective scheme over a field \( k \) of characteristic 0, the periodic cyclic homology of \( X \) is naturally isomorphic to crystalline cohomology of \( X \):

\[ HP_n^{(i)}(X) \cong H_{cris}^{2i-n}(X/k) \]

In particular, if \( k = \mathbb{C} \), it is isomorphic to the betti-cohomology of \( X(\mathbb{C}) \).

\[ HP_n^{(i)}(X) \cong H_{top}^{2i-n}(X(\mathbb{C}), \mathbb{C}) \]

The Hodge decomposition of the Hochschild homology of a smooth projective manifold concerns the columns of the Hodge diamond, [W]. This provides some explanation of the pairings that we explained in the former sections. The Mukai pairing may be written in the following form,

\[ \langle \bullet, \bullet \rangle_M = HH_0^{(k)} \otimes HH_0^{(n-k)} \rightarrow \mathbb{C} \]

\[ \langle a, b \rangle_M = a^{\vee} \wedge b \]

where \( \vee : HH_0^{(k)} \rightarrow HH_0^{(n-k)} \). This pairing together with H-K-R isomorphism shows that a type of Riemann-Hodge bilinear relations may be considered for the Hochschild homology of smooth projective spaces. Moreover, the positivity in R-H bilinear relations of the Hodge structure of the cohomology of projective varieties implies that the Mukai pairing satisfies a Riemann-Hodge positivity (using compatibility of
chern characters in Hochschild and de Rham cohomologies via H-K-R). We assume to know this fact that chern characters define primitive cohomology classes.

7. Multiplicity question over arbitrary field

Accordingly a motivation to prove the positivity of Serre multiplicity conjecture would be to investigate a type of positivity on the actions of correspondences in arbitrary characteristic. This leads to Grothendieck standard conjectures. Let \( X \) be a smooth projective variety \( /k \) of dimension \( d \), and \( L \) an ample divisor class. \( L \) acts on etale cohomology of \( X \) and by hard Lefschetz,

\[
L^j : H^{n-j}(X(\bar{k}), \mathbb{Q}_l) \cong H^{n+j}(X(\bar{k}), \mathbb{Q}_l)
\]

which implies

\[
H^{n-j}(X(\bar{k}), \mathbb{Q}_l) = \oplus_k L^k H^{j-2k}(X(\bar{k}), \mathbb{Q}_l)_{\text{prim}}
\]

that induces a morphism

\[
\Lambda : H^j(X(\bar{k}), \mathbb{Q}_l) \to H^{j-2}(X(\bar{k}), \mathbb{Q}_l(-1))
\]

such that for \( m \in H^j(X(\bar{k}), \mathbb{Q}_l)_{\text{prim}} \), \( \Lambda(L^k m) = L^{k-1} m \), if \( k > 0 \) and 0 otherwise. The standard conjecture \( B \) asserts that \( \Lambda \) is defined algebraically as the action of a correspondence.

If \( A^j(X) \) is the co-image of the cycle map

\[
cl : CH^j(X)_\mathbb{Q} \to H^{2j}(X(\bar{k}), \mathbb{Q}_l(j))
\]

The standard conjecture \( A \) asserts that the morphisms

\[
L^{n-2j} : A^i(X) \cong A^{n-i}(X), \quad i < n/2,
\]

are isomorphisms. This follows from Conjecture \( B \), that says the Lefschetz decomposition is compatible with \( A^j \)'s,
\[
A^j(X) = \bigoplus_k L^k . A^{j-k}(X)^{prim}
\]

The conjecture I asserts that the pairing,

\[
(40) \quad (-1)^j \langle L^{n-2j} a, b \rangle, \quad a, b \in A^j(X)^{prim}
\]

is positive definite for \( j \leq n/2 \). If we assume I, then \( A \) would be equivalent to \( D \) stating the equivalence of numerical and homological equivalence for cycles on \( X \), [SA].

By the Lefschetz decomposition we have an isomorphism,

\[
* : H^{n+j}(X_k, \mathbb{Q}_l) \to H^{n-j}(X_k, \mathbb{Q}_l)
\]

such that for \( m \in H^i(X_k, \mathbb{Q}_l)^{prim} \), we have

\[
*(L^k m) = (-1)^{i(i+1)/2} L^{n-i-k} m
\]

Combined with Poincare duality this defines a pairing on \( H^*(X, \mathbb{Q}_l) \) defined by \((m, *n)\).

For a non-zero correspondence \( \lambda \in A^n(X \times_k X) \subset \operatorname{End}(H^*(X, \mathbb{Q}_l)) \) we consider the transpose \( \lambda' \) w.r.t this pairing. Then, if the standard conjecture B and I are satisfied,

\[
\chi = Tr(\lambda' \circ \lambda) > 0
\]

The action of the correspondences always determine the pairing on the homologies, by composing the action of diagonal \( \Delta \) with the product structure. In this way the positivity of the above trace always implies positivity of the Mukai pairing and therefore, the intersection multiplicity. This shows: \textit{If the Grothendieck Standard conjecture I are satisfied then the intersection multiplicity \( \chi(M, N) \) is strictly positive on proper intersections for projective varieties /k, [SA].}

8. ARAKELOV THEORY AND MUKAI INTEGRAL TRANSFORM

We conclude with an interesting generalization of integral transforms to arithmetic varieties. Let \( X \) be an arithmetic variety, i.e. with \( s : X \to \operatorname{Spec}(\mathbb{Z}) \). An arithmetic
cycle is a pair \((Z, g)\) where \(Z\) is an algebraic cycle, and \(g\) is a green current for \(Z\), i.e. \(dd^cg + \delta_Z = \omega\), with \(\omega\) a smooth form. Then the arithmetic Chow groups \(\widehat{CH}^p(X)\) are defined as the quotient of the free abelian group of arithmetic cycles by appropriate equivalence relation. The arithmetic Chow groups are functorial with respect to flat pull backs and proper push forwards. Then we will have similar intersection theory for arithmetic Chow groups

\[
\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X)
\]

The arithmetic K-group, \(\hat{K}_0(X)\) can be defined similarly for hermitian vector bundles on an arithmetic variety. Analogously we will have the Chern character \(\hat{ch}(E)\) for hermitian vector bundles \(E\) on \(X\),

\[
\hat{ch} : \hat{K}_0(X) \cong \bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}
\]

It induces an isomorphism on the eigen-spaces of Adams operations on \(\hat{K}_0(X)\), as

\[
\hat{ch} : \hat{K}_0(X)_{\mathbb{Q}}^{(p)} \cong \widehat{CH}^p(X)_{\mathbb{Q}}
\]

There exists an arithmetic analogue of Todd class of tangent bundle of \(X\), denoted \(\hat{td}_X\) which fulfills the arithmetic Riemann-Roch theorem, \([GS2]\).}

\[
\begin{align*}
\hat{K}_0(X)_{\mathbb{Q}} & \xrightarrow{\hat{td}_X \cdot \hat{ch}(\cdot)} \bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}} \\
K_0(Y)_{\mathbb{Q}} & \xrightarrow{\hat{td}_Y \cdot \hat{ch}(\cdot)} \bigoplus_{p \geq 0} \widehat{CH}^p(Y)_{\mathbb{Q}}
\end{align*}
\]

The cycle class map takes values in the Deligne-Beilinson cohomology of \(X_{\mathbb{R}}\),

\[
cl(Z) = [(\omega, g)] \in H_{D}^{2p}(X, \mathbb{R}(p))
\]

Let \(\pi : X \to S\) be a proper map between non-singular arithmetic varieties, with generic smooth fibers. if \(d = \dim(X) - \dim(S)\), and \(p + q = d + 1\), then we have a pairing

\[
\langle , \rangle : \widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^1(S)_{\mathbb{Q}}
\]
which is given by composition with the direct image map \( \pi_* : \hat{CH}^{d+1}(X) \to \hat{CH}^1(S)_\mathbb{Q} \). When \( X \) is projective non-singular of dimension \( d + 1 \), as \( \hat{CH}^1(Z) = \mathbb{R} \)

\[
\hat{CH}^p(X) \otimes \hat{CH}^q(X) \to \mathbb{R}
\]

Restriction of this pairing to the null homologous divisors gives Beilinson height pairing

\[
\hat{CH}^p_0(X) \otimes \hat{CH}^q_0(X) \to \mathbb{R}
\]

which is a generalization of Neron-Tate height pairing, [GS2].

Suppose \( \mathcal{E} \) is a sheaf on \( X \times Y \) it induces a

\[
\begin{array}{ccc}
\hat{K}_0(X)_\mathbb{Q} & \xrightarrow{\Phi^\mathcal{E}_K} & \hat{K}_0(Y)_\mathbb{Q} \\
\mu & \downarrow & \mu \\
\bigoplus_{p \geq 0} \hat{CH}^p(X)_\mathbb{Q} & \xrightarrow{\phi_{\mathcal{E}}^H} & \bigoplus_{p \geq 0} \hat{CH}^p(Y)_\mathbb{Q} \\
cl_X & \downarrow & cl_Y \\
H^*_D(X) & \xrightarrow{\Phi^\mathcal{E}_H} & H^*_D(X)
\end{array}
\] (42)

In this way one may define the Mukai product on the category of hermitian bundles in the form

\[
\langle E, F \rangle_M = \int_X \mu(E)^\vee \mu(F)
\]

Some analogues of standard conjecture A and I is valid for arithmetic intersections in \( \mathbb{P}^n \) with Fubini-Study metric. First lets state a general conjecture.

**Conjecture 8.1.** \( A_p(X, \mathcal{O}) \). (H. Gillet, C. Soule) [KK] Let \( \mathcal{O} \) be an ample line bundle on \( X \), that carries a positive hermitian metric \( \| \| \). Let \( \bar{X} = (X, h) \) be the associated Arakelov variety. Then if \( 2p \leq n + 1 \) the followings are satisfied,

- The operator

\[
L_{\mathcal{O}, \| \|}^{n+1-2p} : CH^p(\bar{X})_\mathbb{R} \to CH^{n+1-p}(\bar{X})_\mathbb{R}
\]
is an isomorphism.

- If $x \in CH^p(\bar{(X)})_R$, $x \neq 0$ and $L^{n+2-2p}_\varphi(x) = 0$, then

$$(-1)^{p}\deg(x.L^{n+1-2p}_\varphi(x)) > 0$$

**Theorem 8.2.** [KK] (cor. 2.2) Conjecture $A_p(\mathbb{P}^n, \mathcal{O}(1))$ holds for all $p$ with the Fubini-Study metric $\parallel \parallel$.

As a motivation, if some one may test a Weil cohomology theory defined over regular schemes, and satisfying a type of Hodge positivity, then the Serre conjecture of positivity may be proved using Mukai transforms. This may also answer similar question of intersections over special type of schemes that their cohomology or even the Chow ring has a positivity property.

**Appendix: Grothendieck Standard Conjectures**

We list the Grothendieck Standard conjectures, [CH], [GR]:

- **A**: Hard Lefschetz on cycles

(43) \[ A(X) : L^{n-2k} : CH^r(X) \cong CH^{n-r}(X) \]

- **B**: Lefschetz type Standard Conjecture

(44) \[ B(X) : \ast L : \bigoplus_l r, H^l(X)(r) \to \bigoplus_l r, H^l(X)(r) \text{ is algebraic.} \]

- **C**: Kunneth type Standard Conjecture

(45) \[ C(X) : \pi^i_X : H^\ast(X) \to H^i(X) \hookrightarrow H^\ast(X) \text{ is algebraic} \]

- **D**: Homological and numerical equivalence coincide

(46) \[ D(X) : \sim_{hom, \mathbb{Q}} \rightleftharpoons \sim_{num, \mathbb{Q}} \]
• I : Hodge type Standard conjecture

\[ I(X) : \text{the } \mathbb{Q}-\text{valued quadratic form } \alpha \mapsto \langle \alpha, \ast L(\alpha) \rangle \text{ on } Z_{\text{hom}}^*(X) \mathbb{Q} \text{ is positive definite.} \]

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