Offensive Alliances in Graphs

Ajinkya Gaikwad* and Soumen Maity**

Indian Institute of Science Education and Research, Pune, India
ajinkya.gaikwad@students.iiserpune.ac.in; soumen@iiserpune.ac.in

Abstract. The Offensive Alliance problem has been studied extensively during the last twenty years. A set \( S \subseteq V \) of vertices is an offensive alliance in an undirected graph \( G = (V, E) \) if each \( v \in N(S) \) has at least as many neighbours in \( S \) as it has neighbours (including itself) not in \( S \). We study the classical and parameterized complexity of the Offensive Alliance problem, where the aim is to find a minimum size offensive alliance. We enhance our understanding of the problem from the viewpoint of parameterized complexity by showing that (1) the problem is \( W[1] \)-hard parameterized by a wide range of fairly restrictive structural parameters such as the feedback vertex set number, treewidth, pathwidth, and treedepth of the input graph; we thereby resolve an open question stated by Bernhard Bliem and Stefan Woltran (2018) concerning the complexity of Offensive Alliance parameterized by treewidth, (2) unless ETH fails, Offensive Alliance problem cannot be solved in time \( O^*(2^{o(k \log k)}) \) where \( k \) is the solution size, (3) Offensive Alliance problem does not admit a polynomial kernel parameterized by solution size and vertex cover of the input graph. On the positive side we prove that (4) Offensive Alliance can be solved in time \( O^*\left(\text{vc}(G)^{O(\text{vc}(G))}\right) \) where \( \text{vc}(G) \) is the vertex cover number of the input graph. In terms of classical complexity, we prove that (5) Offensive Alliance cannot be solved in time \( 2^{o(n)} \) even when restricted to bipartite graphs, unless ETH fails, (6) Offensive Alliance cannot be solved in time \( 2^{o(\sqrt{n})} \) even when restricted to apex graphs, unless ETH fails. We also prove that (7) Offensive Alliance is NP-complete even when restricted to bipartite, chordal, split and circle graphs.

Keywords: Defensive and Offensive alliance · Parameterized Complexity · FPT · \( W[1] \)-hard · treewidth · ETH

1 Introduction

This paper studies the Offensive Alliance problem: given an undirected graph \( G \) and a positive integer \( r \), determine whether \( G \) admits an offensive

* The first author gratefully acknowledges support from the Ministry of Human Resource Development, Government of India, under Prime Minister’s Research Fellowship Scheme (No. MRF-192002-211).

** The second author’s research was supported in part by the Science and Engineering Research Board (SERB), Govt. of India, under Sanction Order No. MTR/2018/001025.
alliance of size at most \( r \). It is not surprising that the complexity of Offensive Alliance and several of its variants has been studied extensively by the theory community in the past years. The decision version for several types of alliances have been shown to be NP-complete. For an integer \( \ell \), a nonempty set \( S \subseteq V(G) \) is a defensive \( \ell \)-alliance if for each \( v \in S \), \( d_S(v) \geq d_S^c(v) + \ell \). A set is a defensive alliance if it is a defensive \((-1)\)-alliance. A defensive \( \ell \)-alliance \( S \) is global if \( S \) is a dominating set. The defensive \( \ell \)-alliance problem is NP-complete for any \( \ell \) \([28]\). The defensive alliance problem is NP-complete even when restricted to split, chordal and bipartite graph \([18]\). For an integer \( \ell \), a nonempty set \( S \subseteq V(G) \) is an offensive \( \ell \)-alliance if for each \( v \in N(S) \), \( d_S(v) \geq d_S^c(v) + \ell \). An offensive 1-alliance is called an offensive alliance. An offensive \( \ell \)-alliance \( S \) is global if \( S \) is a dominating set. Fernau et al. showed that the offensive \( \ell \)-alliance and global offensive \( \ell \)-alliance problems are NP-complete for any fixed \( \ell \) \([12]\). They also proved that for \( \ell > 1 \), \( \ell \)-offensive alliance is NP-hard, even when restricted to \( r \)-regular planar graphs. There are polynomial time algorithms for finding minimum alliances in trees \([3,18]\). A polynomial time algorithm for finding minimum defensive alliance in series parallel graph is presented in \([17]\).

Fernau and Raible showed in \([10]\) that the defensive, offensive and powerful alliance problems and their global variants are fixed parameter tractable when parameterized by solution size \( r \). Kiyomi and Otachi showed in \([21]\), the problems of finding smallest alliances of all kinds are fixed-parameter tractable when parametered by the vertex cover number. The problems of finding smallest defensive and offensive alliances are also fixed-parameter tractable when parametered by the neighbourhood diversity \([14]\). Enciso \([8]\) proved that finding defensive and global defensive alliances is fixed parameter tractable when parameterized by domino treewidth. Bliem and Woltran \([2]\) proved that deciding if a graph contains a defensive alliance of size at most \( r \) is W[1]-hard when parameterized by treewidth of the input graph. This puts it among the few problems that are FPT when parameterized by solution size but not when parameterized by treewidth (unless FPT=W[1]).

**Contribution:** The parameterized complexity of Offensive Alliance parameterized by several structural parameters has remained unexplored. We resolve the problem with most of these parameters. We mostly discuss the parameters that deal with sparseness of graph. We show that the problem is W[1]-hard parameterized by any of the following parameters: the feedback vertex set number, treewidth, pathwidth, and treedepth of the input graph. Interestingly, our result is significantly stronger since we show that hardness even applies in the case that the remaining parts, after deleting the feedback vertex set, are trees of height at most seven. Next, we turn our attention to parameters vertex cover number and solution size. As mentioned before, it is already proved that Offensive Alliance problem admits FPT algorithms parameterized by each of these parameters individually. As there is no hope to get FPT algorithms with small structural parameters, we need to make the most out of these two parameters by obtaining efficient algorithms and kernels. The algorithm mentioned in \([8]\),
Offensive Alliances in Graphs

has a running time $O^*(2^{O(r \log r)})$. The first question that arises from here is that whether we can get a single exponential algorithm? We answer this question in a negative way by proving that unless ETH fails, **Offensive Alliance problem** cannot be solved in time $O^*(2^{o(r \log r)})$. For the parameter vertex cover number, the algorithm mentioned in [21] has running time $O^*((2^{\text{vc}(G)})^{O(\text{vc}(G))})$. In this case, we improve the running time to $O^*(\text{vc}(G)^{O(\text{vc}(G))})$ where $\text{vc}(G)$ is the vertex cover number of the input graph. Finally, we show that it is unlikely to get polynomial kernel when parameterized by both of these parameters combined.

In search of efficient algorithms, alliance problems have been studied on special graph classes. There are polynomial time algorithms for finding minimum alliances in trees [3,18]. A polynomial time algorithm for finding minimum defensive alliance in series parallel graph is given in [17]. But still, alliance problems remained unexplored on special classes of intersection graphs such as interval graphs, circle graphs, circular arc graphs, unit disk graphs etc. We show that alliances in trees [3,18]. A polynomial time algorithm for finding minimum defensive alliance in series parallel graph is given in [17]. But still, alliance problems remained unexplored on special classes of intersection graphs such as interval graphs, circle graphs, circular arc graphs, unit disk graphs etc. We show that the problem remains NP-hard even when restricted to bipartite, chordal, split and circle graphs. We also prove that the known algorithms on general graphs and apex graphs are unlikely to improve. This is done by showing that the problem cannot be solved in $2^{\Omega(n)}$ time even when restricted to bipartite graphs and also the problem cannot be solved in $2^{\Omega(\sqrt{n})}$ time even when restricted to apex graphs, unless ETH fails.

### 2 Preliminaries

In real life, an alliance is a collection of people, groups, or states such that the union is stronger than individual. The alliance can be either to achieve some common purpose, to protect against attack, or to assert collective will against others. This motivates the definitions of defensive and offensive alliances in graphs. The properties of alliances in graphs were first studied by Kristiansen, Hedetniemi, and Hedetniemi [23]. They introduced defensive, offensive and powerful alliances. An alliance is global if it is a dominating set. The alliance problem has been studied extensively during last twenty years [13,29,4,27,30], and generalizations called $r$-alliances are also studied [28]. Throughout this article, $G = (V,E)$ denotes a finite, simple and undirected graph of order $|V| = n$. The subgraph induced by $S \subseteq V(G)$ is denoted by $G[S]$. For a vertex $v \in V$, we use $N_G(v) = \{u : (u,v) \in E(G)\}$ to denote the (open) neighbourhood of vertex $v$ in $G$, and $N_G[v] = N_G(v) \cup \{v\}$ to denote the closed neighbourhood of $v$. The degree $d_G(v)$ of a vertex $v \in V(G)$ is $|N_G(v)|$. For a subset $S \subseteq V(G)$, we define its closed neighbourhood as $N_G[S] = \bigcup_{v \in S} N_G[v]$ and its open neighbourhood as $N_G(S) = N_G[S] \setminus S$. For a non-empty subset $S \subseteq V$ and a vertex $v \in V(G)$, $N_S(v)$ denotes the set of neighbours of $v$ in $S$, that is, $N_S(v) = \{u \in S : (u,v) \in E(G)\}$. We use $d_S(v) = |N_S(v)|$ to denote the degree of vertex $v$ in $G[S]$. The complement of the vertex set $S$ in $V$ is denoted by $S^c$.

**Definition 1.** A non-empty set $S \subseteq V$ is a **defensive alliance** in $G$ if $d_S(v) + 1 \geq d_{S^c}(v)$ for all $v \in S$. 
Since each vertex in a defensive alliance $S$ has at least as many vertices from its closed neighbor in $S$ as it has in $S^c$, by strength of numbers, we say that every vertex in $S$ can be defended from possible attack by vertices in $S^c$.

**Definition 2.** A non-empty set $S \subseteq V$ is an offensive alliance in $G$ if $d_S(v) \geq d_{S^c}(v) + 1$ for all $v \in N(S)$.

Since each vertex in $N(S)$ has more neighbors in $S$ than in $S^c$, we say that every vertex in $N(S)$ is vulnerable to possible attack by vertices in $S$. Equivalently, since an attack by the vertices in $S$ on the vertices in $V \setminus S$ can result in no worse than a “tie” for $S$, we say that $S$ can effectively attack $N(S)$.

**Definition 3.** A non-empty set $S \subseteq V$ is a strong offensive alliance in $G$ if $d_S(v) \geq d_{S^c}(v) + 2$ for all $v \in N(S)$.

In this paper, we consider **Offensive Alliance** and **Strong Offensive Alliance** problems under structural parameters. We define these problems as follows:

| **Offensive Alliance** |
|------------------------|
| **Input:** An undirected graph $G = (V, E)$ and an integer $r \geq 1$. |
| **Question:** Is there an offensive alliance $S \subseteq V(G)$ such that $1 \leq |S| \leq r$? |

| **Strong Offensive Alliance** |
|-----------------------------|
| **Input:** An undirected graph $G = (V, E)$ and an integer $r \geq 1$. |
| **Question:** Is there a strong offensive alliance $S \subseteq V(G)$ such that $1 \leq |S| \leq r$? |

For standard notations and definitions in graph theory, we refer to West [31]. For the standard concepts in parameterized complexity, see the recent textbook by Cygan et al. [5]. The graph parameters we explicitly use in this paper are feedback vertex set number, pathwidth, treewidth and treedepth.

**Definition 4.** For a graph $G = (V, E)$, the parameter *feedback vertex set* is the cardinality of a smallest set $S \subseteq V(G)$ such that the graph $G - S$ is a forest and it is denoted by $fvs(G)$.

We now review the concept of a tree decomposition, introduced by Robertson and Seymour in [26]. Treewidth is a measure of how “tree-like” the graph is.

**Definition 5.** [7] A *tree decomposition* of a graph $G = (V, E)$ is a tree $T$ together with a collection of subsets $X_t$ (called bags) of $V$ labeled by the vertices $t$ of $T$ such that $\bigcup_{t \in T} X_t = V$ and (1) and (2) below hold:

1. For every edge $uv \in E(G)$, there is some $t$ such that $\{u, v\} \subseteq X_t$.
2. (Interpolation Property) If $t$ is a vertex on the unique path in $T$ from $t_1$ to $t_2$, then $X_{t_1} \cap X_{t_2} \subseteq X_t$. 
Definition 6. [7] The width of a tree decomposition is the maximum value of $|X_t| - 1$ taken over all the vertices $t$ of the tree $T$ of the decomposition. The treewidth of a graph $G$ is the minimum width among all possible tree decompositions of $G$.

Example 1. Figure 1 gives an example of a tree decomposition of width 2.

Definition 7. If the tree $T$ of a tree decomposition is a path, then we say that the tree decomposition is a path decomposition, and use pathwidth in place of treewidth.

A rooted forest is a disjoint union of rooted trees. Given a rooted forest $F$, its transitive closure is a graph $H$ in which $V(H)$ contains all the nodes of the rooted forest, and $E(H)$ contain an edge between two vertices only if those two vertices form an ancestor-descendant pair in the forest $F$.

Definition 8. The treedepth of a graph $G$ is the minimum height of a rooted forest $F$ whose transitive closure contains the graph $G$. It is denoted by $td(G)$.

2.1 Parameterized Complexity

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where $\Sigma$ is a fixed, finite alphabet. For an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, $k$ is called the parameter. A parameterized problem $P$ is fixed-parameter tractable (FPT in short) if a given instance $(x, k)$ can be solved in time $f(k) \cdot |(x, k)|^c$ where $f$ is some (usually computable) function, and $c$ is a constant. Parameterized complexity classes are defined with respect to fpt-reducibility. A parameterized problem $P$ is fpt-reducible to $Q$ if in time $f(k) \cdot |(x, k)|^c$, one can transform an instance $(x, k)$ of $P$ into an instance $(x', k')$ of $Q$ such that $(x, k) \in P$ if and only if $(x', k') \in Q$, and $k' \leq g(k)$, where $f$ and $g$ are computable functions depending only on $k$. Owing to the
Fig. 2. Relationship between vertex cover (vc), neighbourhood diversity (nd), twin cover (tc), modular width (mw), cluster vertex deletion number (cvd), feedback vertex set (fvs), pathwidth (pw), treewidth (tw) and clique width (cw). Note that $A \rightarrow B$ means that there exists a function $f$ such that for all graphs, $f(A(G)) \geq B(G)$. It also gives an overview of the parameterized complexity landscape for the Offensive Alliance problem with general thresholds. The problem is FPT parameterized by blue colored parameters and W[1]-hard when parameterized by red colored parameters. The problem remains unsettled when parameterized by twin cover, modular width and cluster vertex deletion number.
definition, if \( \mathcal{P} \) \textit{fpt-reduces} to \( \mathcal{Q} \) and \( \mathcal{Q} \) is fixed-parameter tractable then \( \mathcal{P} \) is fixed-parameter tractable as well.

What makes the theory more interesting is a hierarchy of intractable parameterized problem classes above FPT which helps in distinguishing those problems that are not fixed parameter tractable. Central to parameterized complexity is the following hierarchy of complexity classes, defined by the closure of canonical problems under \textit{fpt-reductions}: \( \text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \ldots \subseteq \text{XP} \). All inclusions are believed to be strict. In particular, \( \text{FPT} \neq \text{W}[1] \) under the Exponential Time Hypothesis \cite{16}. The class \( \text{W}[1] \) is the analog of NP in parameterized complexity. A major goal in parameterized complexity is to distinguish between parameterized problems which are in FPT and those which are \( \text{W}[1] \)-hard, i.e., those to which every problem in \( \text{W}[1] \) is \textit{fpt-reducible}. There are many problems shown to be complete for \( \text{W}[1] \), or equivalently \( \text{W}[1]-complete \), including the Multi-Colored Clique (MCC) problem \cite{7}.

Closely related to fixed-parameter tractability is the notion of preprocessing. A \textit{reduction to a problem kernel}, or equivalently, \textit{problem kernelization} means to apply a data reduction process in polynomial time to an instance \((x, k)\) such that for the reduced instance \((x', k')\) it holds that \((x', k')\) is equivalent to \((x, k)\), \(|x'| \leq g(k)\) and \(k' \leq g(k)\) for some function \(g\) only depending on \(k\). Such a reduced instance is called a \textit{problem kernel}. It is easy to show that a parameterized problem is in FPT if and only if there is kernelization algorithm. A \textit{polynomial kernel} is a kernel, whose size can be bounded by a polynomial in the parameter. We refer to \cite{5,7} for further details on parameterized complexity.

## 3 \textit{W}[1]-Hardness Parameterized by Structural Parameters

In this section we show that \textsc{Offensive Alliance} is \textit{W}[1]-hard parameterized by a vertex deletion set to trees of height at most seven, that is, a subset \(D\) of the vertices of the graph such that every component in the graph, after removing \(D\), is a tree of height at most seven. On the way towards this result, we provide hardness results for several interesting versions of the \textsc{Offensive Alliance} problem which we require in our proofs. The problem \textsc{Offensive Alliance} is a generalization of \textsc{Offensive Alliance} where some vertices are forced to be outside the solution; these vertices are called forbidden vertices. This variant can be formalized as follows:

\begin{center}
\textbf{Offensive Alliance}\textsuperscript{F}\\
\textbf{Input}: An undirected graph \(G = (V, E)\), an integer \(r\) and a set \(V_{\Box} \subseteq V\) of forbidden vertices such that each degree one forbidden vertex is adjacent to another forbidden vertex and each forbidden vertex of degree greater than one is adjacent to a degree one forbidden vertex.\\
\textbf{Question}: Is there an offensive alliance \(S \subseteq V\) such that (i) \(1 \leq |S| \leq r\), and (ii) \(S \cap V_{\Box} = \emptyset\)?
\end{center}

\textsc{Strong Offensive Alliance}\textsuperscript{FN} is a generalization of \textsc{Strong Offensive Alliance}...
This variant can be formalized as follows:

**Strong Offensive Alliance**

**Input:** An undirected graph \( G = (V, E) \), an integer \( r \), a set \( V_\triangle \subseteq V \), and a set \( V_\square \subseteq V \) of forbidden vertices such that each degree one forbidden vertex is adjacent to another forbidden vertex and each forbidden vertex of degree greater than one is adjacent to a degree one forbidden vertex.

**Question:** Is there a strong offensive alliance \( S \subseteq V \) such that (i) \( 1 \leq |S| \leq r \), (ii) \( S \cap V_\square = \emptyset \), and (iii) \( V_\triangle \subseteq S \)?

While the Offensive Alliance problem asks for offensive alliance of size at most \( r \), we also consider the Exact Offense Alliance problem that concerns offensive alliance of size exactly \( r \). Analogously, we also define exact versions of Strong Offensive Alliance presented above. To prove Lemma 2, we consider a variant of the following problem:

**Multidimensional Subset Sum (MSS)**

**Input:** An integer \( k \), a set \( S = \{s_1, \ldots, s_n\} \) of vectors with \( s_i \in \mathbb{N}^k \) for every \( i \) with \( 1 \leq i \leq n \) and a target vector \( t \in \mathbb{N}^k \).

**Parameter:** \( k \)

**Question:** Is there a subset \( S' \subseteq S \) such that \( \sum_{s \in S'} s = t \)?

In the Multidimensional Relaxed Subset Sum (MRSS) problem, an additional integer \( k' \) is given (which will be part of the parameter) and we ask whether there is a subset \( S' \subseteq S \) with \( |S'| \leq k' \) such that \( \sum_{s \in S'} s \geq t \). More formally,

**Multidimensional Relaxed Subset Sum (MRSS)**

**Input:** Two integers \( k \) and \( k' \), a set \( S = \{s_1, \ldots, s_n\} \) of vectors with \( s_i \in \mathbb{N}^k \) for every \( i \) with \( 1 \leq i \leq n \) and a target vector \( t \in \mathbb{N}^k \).

**Parameter:** \( k + k' \)

**Question:** Is there a subset \( S' \subseteq S \) with \( |S'| \leq k' \) such that \( \sum_{s \in S'} s \geq t \)?

**Lemma 1.** [15] MRSS is \( W[1] \)-hard when parameterized by the combined parameter \( k + k' \), even if all integers in the input are given in unary.

We now show that the Strong Offensive Alliance problem is \( W[1] \)-hard parameterized by the size of a vertex deletion set into trees of height at most 5, via a reduction from MRSS.

**Lemma 2.** The Strong Offensive Alliance problem is \( W[1] \)-hard when parameterized by the size of a vertex deletion set into trees of height at most 5.

**Proof.** To prove this we reduce from MRSS, which is known to be \( W[1] \)-hard when parameterized by the combined parameter \( k + k' \), even if all integers
in the input are given in unary \( \mathbb{1} \). Let \( I = (k, k', S, t) \) be an instance of MRSS. We construct an instance \( I' = (G, r, V_\triangle, V_\square) \) of Strong Offensive Alliance in the following way. See Figure 3 for an illustration. First, we introduce a set of \( k \) forbidden vertices \( U = \{u_1, u_2, \ldots, u_k\} \). For each vector \( s = (s(1), s(2), \ldots, s(k)) \in S \), we introduce a tree \( T_s \) into \( G \). We define \( \max(s) = \max \{s(i)\} \). The vertex set of tree \( T_s \) is defined as follows:

\[
V(T_s) = A_s \cup B_s \cup A_s^{\triangle} \cup B_s^{\triangle} \cup C_s \cup Z_s \cup \{x_s, y_s, z_s\}
\]

where \( A_s = \{a_1^s, \ldots, a_{\max(s)+1}^s\} \), \( B_s = \{b_1^s, \ldots, b_{\max(s)+1}^s\} \), \( A_s^{\triangle} = \{a_1^{\triangle}, \ldots, a_{\max(s)+1}^{\triangle}\} \), \( B_s^{\triangle} = \{b_1^{\triangle}, \ldots, b_{\max(s)+1}^{\triangle}\} \) and \( C_s = \{c_1^s, \ldots, c_{2\max(s)+2}^s\} \) are five sets of ver-

![Graph](image-url)
Observe that $N_G$ is a strong offensive alliance of $S$. Towards showing the forward direction, let $A_k = \sum_{i=1}^{k} \{a_i \Delta, z_i\}$ containing three necessary vertices and one forbidden vertex. Make $A_k$ a component of the resulting graph is a tree with height at most 5. Note that, next we introduce a vertex $u_i$ and a set of four vertices $A = \{a_1 \Delta, a_2 \Delta, a_3 \Delta, a_4 \Delta\}$ containing three necessary vertices and one forbidden vertex. We make $u_i$ adjacent to exactly $s(i)$ vertices of $A$, in arbitrary manner. For each $s \in S$, we make a set adjacent to all the vertices of $A_s \cup B_s \cup C_s$. For every $u_i \in U$, we create a set $V_{u_i, \Delta}$ of $2 \sum_{s \in S} s(i) - 2t(i) + 2$ vertices of $A_s \cup B_s \cup C_s$. For every $u_i \in U$, we create a set $V_{u_i, \Delta}$ of $2 \sum_{s \in S} s(i) - 2t(i) + 2$ necessary vertices; and make $u_i$ adjacent to every vertex of $V_{u_i, \Delta} \cup V_{u_i, \Delta}$. We define

$$V_{u_i, \Delta} = \bigcup_{i=1}^{k} \left( V_{u_i, \Delta} \cup A \setminus \{a_i \Delta\} \cup Z \setminus \{z_i\} \cup \sum_{s \in S} s(i) \right)$$

and

$$V_{u_i, \Delta} = U \cup \{a_i \Delta\} \cup \bigcup_{i=1}^{k} \left( V_{u_i, \Delta} \cup A \setminus \{a_i \Delta\} \cup B \setminus \{z_i\} \cup \sum_{s \in S} s(i) \right).$$

We set $r = \sum_{i=1}^{k} 2 \left( \sum_{s \in S} s(i) - 2t(i) + 1 \right) + \sum_{s \in S} 2(\max(s) + 1) + 5n + 3 + k$. Observe that if we remove the set $U \cup \{a\}$ of $k + 1$ vertices from $G$, each connected component of the resulting graph is a tree with height at most 5. Note that, $I'$ can be constructed in polynomial time. The reason is this. As all integers in $I$ are bounded by a polynomial in $n$, the number of vertices in $G$ is also polynomially bounded in $n$.

It remains to show that $I$ is a yes instance if and only if $I'$ is a yes instance. Towards showing the forward direction, let $S'$ be a subset of $S$ such that $|S'| \leq k'$ and $\sum_{s \in S'} s \geq t$. We claim

$$R = V_{u_i, \Delta} \bigcup_{s \in S'} A_s \cup B_s \cup \{x_s\} \bigcup_{s \in S \setminus S'} C_s$$

is a strong offensive alliance of $G$ such that $|R| \leq r$, $V_{u_i, \Delta} \subseteq R$, and $V_{u_i, \Delta} \cap R = \emptyset$. Observe that $N_G(R) = U \cup \{a\} \cup \{z_s\} \cup \{y_s\} \cup \{z_i\} \cup \{a_i \Delta\}$. Let $u_i \in U$, then we
show that $d_R(u_i) \geq d_{R^c}(u_i) + 2$. As $\sum_{s \in S'} s(i) - t(i) \geq 0$, we get

$$d_R(u_i) = \sum_{s \in S'} s(i) + |V_{u_i\triangle}|$$

$$= \sum_{s \in S'} s(i) + 2 \sum_{s \in S} s(i) - 2t(i) + 2$$

$$= \left( \sum_{s \in S'} s(i) - t(i) \right) + \sum_{s \in S} s(i) - t(i) + \sum_{s \in S} s(i) + 2$$

$$\geq \sum_{s \in S} s(i) - t(i) + \sum_{s \in S} s(i) + 2$$

$$= \sum_{s \in S \setminus S'} s(i) + \left( \sum_{s \in S'} s(i) - t(i) \right) + \sum_{s \in S} s(i) + 2$$

$$\geq \sum_{s \in S \setminus S'} s(i) + \sum_{s \in S} s(i) + 2 = \sum_{s \in S \setminus S'} s(i) + |V_{u_i\square}| + 2$$

$$= d_{R^c}(u_i) + 2.$$

For the remaining vertices $x$ in $N(R)$, it is easy to see that $d_R(x) \geq d_{R^c}(x) + 2$. Therefore, $R$ is a strong offensive alliance.

Towards showing the reverse direction of the equivalence, suppose $G$ has a strong offensive alliance $R$ of size at most $r$ such that $V_\triangle \subseteq R$ and $V_\square \cap R = \emptyset$. From the definition of $V_\triangle$ and $V_\square$, it is easy to note that $U \subseteq N(R)$. We know $V_\triangle$ contains $\sum_{i=1}^k \left( \sum_{s \in S} 2s(i) - 2t(i) \right) + 5n + 3$ vertices; thus besides the vertices of $V_\triangle$, there are at most $\sum_{s \in S} 2(\max(s) + 1) + k'$ vertices in $R$. Since $a \in N(V_\triangle)$ and $d_G(a) = \sum_{s \in S} 4(\max(s) + 1) + 4$ where $a$ is adjacent to three necessary vertices, it must have at least $\sum_{s \in S} 2(\max(s) + 1)$ many neighbours in $R$ from the set $\bigcup_{s \in S} (A_s \cup B_s \cup C_s)$. It is to be noted that if a vertex from the set $A_s \cup B_s$ is in the solution then the whole set $A_s \cup B_s \cup \{x_s\}$ lie in the solution. Otherwise $v \in A_{s_0} \subseteq N(R)$ will have $d_R(v) < d_{R^c}(v) + 2$ which is a contradiction as $R$ is a strong offensive alliance. This shows that at most $k'$ many sets of the form $A_s \cup B_s \cup \{x_s\}$ contribute to the solution as otherwise the size of solution exceeds $r$. Therefore, any strong offensive alliance $R$ of size at most $r$ can be transformed to another strong offensive alliance $R'$ of size at most $r$ as follows:

$$R' = V_\triangle \cup \bigcup_{x_s \in R} A_s \cup B_s \cup \{x_s\} \cup \bigcup_{x_s \in V(G) \setminus R} C_s.$$

We define a subset $S' = \left\{ s \in S \mid x_s \in R' \right\}$. Clearly, $|S'| \leq k'$. We claim that $\sum_{s \in S'} s(i) \geq t(i)$ for all $1 \leq i \leq k$. Assume for the sake of contradiction that
\[
\sum_{s \in S'} s(i) < t(i) \text{ for some } i \in \{1, 2, \ldots, k\}. \text{ Then, we have }
\]
\[
d_{R'}(u_i) = \sum_{s \in S'} s(i) + |V_{u_i\Delta}|
\]
\[
= \sum_{s \in S'} s(i) + 2 \sum_{s \in S} s(i) - 2t(i) + 2
\]
\[
= \sum_{s \in S'} s(i) - t(i) + \sum_{s \in S} s(i) - t(i) + \sum_{s \in S} s(i) + 2
\]
\[
< \sum_{s \in S} s(i) - t(i) + \sum_{s \in S} s(i) + 2
\]
\[
= \sum_{s \in S \setminus S'} s(i) + \left( \sum_{s \in S'} s(i) - t(i) \right) + \sum_{s \in S} s(i) + 2
\]
\[
< \sum_{s \in S \setminus S'} s(i) + \sum_{s \in S} s(i) + 2 = \sum_{s \in S \setminus S'} s(i) + |V_{u_i\square}| + 2
\]
\[
= d_{R'}(u_i) + 2
\]
and we also know \(u_i \in N(R')\), which is a contradiction to the fact that \(R'\) is a strong offensive alliance. This shows that \(I\) is a yes instance. \(\square\)

We have the following corollaries from Lemma 2.

**Corollary 1.** The **Strong Offensive Alliance**\(^{\text{FN}}\) problem is \(W[1]\)-hard when parameterized by the size of a vertex deletion set into trees of height at most 5, even when \(|V_{\Delta}| = 1.\)

**Proof.** Given an instance \(I = (G, r, V_{\Delta}, V_{\square})\) of **Strong Offensive Alliance**\(^{\text{FN}}\), we construct an equivalent instance \(I' = (G', r', V_{\Delta}', V_{\square}')\) with \(|V_{\Delta}'| = 1.\) See Figure 4 for an illustration.

![Fig. 4. An illustration of the gadget used in the proof of Corollary 1.](image)

Let \(v_1, v_2, \ldots, v_{\ell}\) be vertices of \(V_{\Delta}\) where we assume that \(\ell > 1.\) We introduce two vertices \(x\) and \(y\) where \(x\) is a forbidden vertex and \(y\) is a necessary vertex;
and make $x$ and $y$ adjacent. We make $x$ adjacent to all the vertices in $V_\triangle$. We
also introduce a set $V_\square$ of $\ell - 1$ forbidden vertices and make them adjacent to
$x$. Set $r' = r + 1$. Define $V'_\triangle = \{y\}$ and $V'_\square = \{x\} \cup V_x \cup V_\square$. We also define $G'$
as follows
\[ V(G') = V(G) \cup \{x, y\} \cup V_\square \]
and
\[ E(G') = E(G) \cup \{(x, y), (x, \alpha), (x, \beta) \mid \alpha \in V_x, \beta \in V_\triangle\}. \]
Let $H$ be a vertex deletion set of $G$ into trees of height at most 5. Clearly, if $H$
has at most $k$ vertices then the set $H \cup \{x\}$ has at most $k + 1$ vertices and is a
vertex deletion set of $G'$ into trees of height at most 5. It is easy to see that $I$
and $I'$ are equivalent instances.

We can get an analogous result for the exact variant.

**Corollary 2.** The *Exact Strong Offensive Alliance* problem is W[1]-
hard when parameterized by the size of a vertex deletion set into trees of height
at most 5 even when $|V_\triangle| = 1$.

Next, we give an FPT reduction that eliminates necessary vertices.

**Lemma 3.** The *Offensive Alliance* problem is W[1]-hard when parametrize
by the size of a vertex deletion set into trees of height at most 5.

**Proof.** To prove this we reduce from the *Strong Offensive Alliance* problem,
which is W[1]-hard when parameterized by the size of a vertex deletion
set into trees of height at most 5, even when $|V_\triangle| = 1$. See Corollary 1. Given
an instance $I = (G, r, V_\triangle = \{x\}, V_\square)$ of *Strong Offensive Alliance*, we
construct an instance $I' = (G', r', V'_\square)$ of *Offensive Alliance* the following way. See Figure 5 for an illustration. Let $n$ be the number of vertices in $G$ and

![Fig. 5. The reduction from *Strong Offensive Alliance* to *Offensive Alliance* in Lemma 3. Note that the set $\{v_1, \ldots, v_n\}$ may contain forbidden vertices of degree greater than one.](image)
let \( V(G) = \{x, v_1, v_2, \ldots, v_n, f_1, \ldots, f_{n'}\} \) where \( F = \{f_1, \ldots, f_{n'}\} \) is the set of degree one forbidden vertices in \( V(G) \). We introduce two vertices \( t^\Box, x^\Box \) into \( G' \). We create a set \( V_t^\Box = \{t_1^\Box, \ldots, t_n^\Box\} \) of 4n forbidden vertices into \( G' \) and make them adjacent to \( t^\Box \). We introduce a set \( V_x^\Box \) of \( n \) forbidden vertices and make them adjacent to \( x^\Box \). Finally we create a set \( T = \{t_1, \ldots, t_{4n}\} \) of 4n vertices and make the vertices in \( T \) adjacent to \( t^\Box \) and \( x^\Box \), and make the vertices in \( V(G) \setminus F \) adjacent to \( x^\Box \). We also add an edge \((x, t^\Box)\). Set \( r' = r + 4n \). We define \( G' \) as follows:

\[
V(G') = V(G) \cup T \cup V_t^\Box \cup V_x^\Box \cup \{t^\Box, x^\Box\}
\]

and

\[
E(G') = E(G) \cup \{(t^\Box, \alpha) : \alpha \in T \cup V_t^\Box \cup \{x\}\}
\]

\[
\cup \{(x^\Box, \beta) : \beta \in T \cup V_x^\Box \cup V(G) \setminus F\}
\]

We define \( V'_t = V_t \cup V_t^\Box \cup V_x^\Box \cup \{t^\Box, x^\Box\} \). Observe that there exists a set of at most \( k + 2 \) vertices in \( G' \) whose deletion makes the resulting graph a forest containing trees of height at most 5. We can find such a set because there exists a vertex deletion set \( H \) of \( G \) into trees of height at most 5. We just add \( \{x^\Box, t^\Box\} \) to the set \( H \), then the resulting set is of size \( k + 2 \) whose deletion makes the resulting graph a forest containing trees of height at most 5.

We now claim that \( I \) is a yes-instance if and only if \( I' \) is a yes-instance. Assume first that \( R \) is a strong offensive alliance of size at most \( r \) in \( G \) such that \( \{x\} \subseteq R \) and \( V_t \cap R = \emptyset \). We claim \( R' = R \cup T \) is an offensive alliance of size at most \( r + 4n \) in \( G' \) such that \( V_t' \cap R' = \emptyset \). Clearly, \( N(R') = \{t^\Box, x^\Box\} \cup N(R) \). For each \( v \in N(R) \), we know that \( d_{R'}(v) \geq d_{R}(v) + 2 \) in \( G \). Therefore in graph \( G' \), we get \( d_{R'}(v) \geq d_{R}(v) + 1 \) for each \( v \in N(R) \) due to the vertex \( x^\Box \). For \( v \in \{x^\Box, t^\Box\} \), it is clear that \( d_{R'}(v) \geq d_{R}(v) + 1 \). This shows that \( I' \) is a yes instance.

To prove the reverse direction of the equivalence, suppose \( R' \) is an offensive alliance of size at most \( r' = r + 4n \) in \( G' \) such that \( R' \cap V_t' = \emptyset \). We claim that \( T \cup \{x\} \subseteq R' \). Since \( R' \) is non empty, it must contain a vertex from the set \( T \cup V(G) \setminus F \). Then \( x^\Box \in N(R') \) and it satisfies the condition \( d_{R'}(x^\Box) \geq d_{R}(x^\Box) + 1 \). Due to \( n \) forbidden vertices in the set \( V_x^\Box \), node \( x^\Box \) must have at least \( n \) neighbours in \( R' \). This implies that \( R' \) contains at least one vertex from \( T \). Then \( t^\Box \in N(R') \) and it satisfies the condition \( d_{R'}(t^\Box) \geq d_{R}(t^\Box) + 1 \). Since \( |V_t'| = 4n \), the condition \( d_{R'}(t^\Box) \geq d_{R}(t^\Box) + 1 \) forces the set \( \{x\} \cup T \) to be inside the solution. Consider \( R = R' \cap V(G) \). Clearly \( |R| \leq r \), \( x \in R \), \( R \cap V_t = \emptyset \) and we show that \( R \) is a strong offensive alliance in \( G \). For each \( v \in N(R') \cap V(G) = N(R) \), we have \( N_{R'}(v) \geq N_{R'}(v) + 1 \) in \( G' \). Notice that we do not have \( x^\Box \) in \( G \) which is adjacent to all vertices in \( N(R) \). Thus for each \( v \in N(R) \), we get \( N_R(v) \geq N_{R'}(v) + 2 \) in \( G \). Therefore \( R \) is a strong offensive alliance of size at most \( r \) in \( G \) such that \( x \in R \) and \( R \cap V_t = \emptyset \). This shows that \( I \) is a yes instance. \( \square \)
Corollary 3. The exact offensive alliance problem is $W[1]$-hard when parameterized by the size of a vertex deletion set into trees of height at most 5.

We are now ready to show our main hardness result for offensive alliance using a reduction from offensive alliance.

Theorem 1. The offensive alliance problem is $W[1]$-hard when parameterized by the size of a vertex deletion set into trees of height at most 7.

Proof. We give a parameterized reduction from offensive alliance which is $W[1]$-hard when parameterized by the size of a vertex deletion set into trees of height at most 5. Let $I = (G, r, V)$ be an instance of offensive alliance. Let $n = |V(G)|$. We construct an instance $I' = (G', r')$ of offensive alliance the following way. We set $r' = r$. Recall that each degree one forbidden vertex is adjacent to another forbidden vertex and each forbidden vertex of degree greater than one is adjacent to a degree one forbidden vertex. Let $u$ be a degree one forbidden vertex in $G$ and $u$ is adjacent to another forbidden vertex $v$. For each degree one forbidden vertex $u \in V$, we introduce a tree $T_u$ rooted at $u$ of height 2 as shown in Figure 6. The forbidden vertex $v$ has additional neighbours from the original graph $G$ which are not shown in the figure. We define $G'$ as follows:

$$V(G') = V(G) \cup \{V(T_u) \mid u \text{ is a degree one forbidden vertex in } G\}$$

and

$$E(G') = E(G) \cup \bigcup_{u' \in V} E(T_u).$$

We claim $I$ is a yes instance if and only if $I'$ is a yes instance. It is easy to see that if $R$ is an offensive alliance of size at most $r$ in $G$ such that $R \cap V = \emptyset$, then it is also an offensive alliance of size at most $r' = r$ in $G'$.

To prove the reverse direction of the equivalence, suppose that $G'$ has an offensive alliance $R'$ of size at most $r' = r$. We claim that no vertex from the
set \( V \bigcup_{u \in V} V(T_u) \) is part of \( R' \). It is easy to see that if any vertex from the
set \( V \bigcup_{u \in V} V(T_u) \) is in \( R' \) then the size of \( R' \) exceeds \( 2r \). This implies that
\( R = R' \cap G \) is an offensive alliance such that \( R \cap V = \emptyset \) and \( |R| \leq r \). This shows
that \( I \) is a yes instance.

We have the following consequences.

**Corollary 4.** The Exact Offensive Alliance problem is \( W[1] \)-hard when parameterized by the size of a vertex deletion set into trees of height at most 7.

Clearly trees of height at most seven are trivially acyclic. Moreover, it is easy
to verify that such trees have pathwidth \( 22 \) and treedepth \( 25 \) at most seven, which implies:

**Theorem 2.** The Offensive Alliance and Exact Offensive Alliance
problems are \( W[1] \)-hard when parameterized by any of the following parameters:

- the feedback vertex set number,
- the treewidth and pathwidth of the input graph,
- the treedepth of the input graph.

4 FPT Lower Bound Parameterized by Solution Size

We know that Offensive Alliance admits an FPT algorithm when param-
eterized by the solution size \( [8] \). The algorithm in \( [8] \) uses branching technique and solves the problem in \( O^*(2^{O(k \log k)}) \) time. It appears that this running time is essentially optimal assuming ETH, which is proved in the following theo-
rem. Hardness for Offensive Alliance follows from a reduction from \( k \times k \) (Permutation) Hitting Set with Thin Sets. In the \( k \times k \) (Permutation) Hitting Set problem, we are given a family \( F \) of subsets of \([k] \times [k]\), and we would like to find a set \( X \), consisting of one vertex from each row and induces a
permutation of \([k]\), such that \( X \cap F \neq \emptyset \) for each \( F \in F \). In the thin set variant we assume that each \( F \in F \) contains at most one vertex from each row. In the proof, we will use the fact that \( k \times k \) (Permutation) Hitting Set with Thin Sets cannot be solved in time \( 2^{o(k \log k)} \), unless ETH fails \( [21] \).

**Theorem 3.** Unless ETH fails, Offensive Alliance cannot be solved in time \( O^*(2^{o(k \log k)}) \), where \( k \) is the solution size.

**Proof.** We provide a polynomial-time algorithm that takes an instance \((F, k)\)
of \( k \times k \) (Permutation) Hitting Set with Thin Sets, and outputs an
equivalent instance \((G, r)\) of Offensive Alliance with \( r = 5k \). We construct
\( G \) the following way.

1. For every \( F \in F \), we introduce a vertex \( v_F \) into \( G \). Let \( V_F = \{v_F \mid F \in F\} \).
   We also introduce a set of \( k^2 \) vertices \( W = \{w_{i,j} : i \in [k], j \in [k]\} \). Make
   \( v_F \) adjacent to \( w_{i,j} \) if \((i, j) \in F\).
2. We introduce a clique $D\triangle$ of size $4k$ into $G$. For every $d \in D\triangle$, we add a set of $10k$ vertices and make them adjacent to $d$. For every $F \in \mathcal{F}$, we make $v_F$ adjacent to every vertex of $D\triangle$.

3. We introduce another clique $D\Box$ of size $12k + 1$ into $G$. Let $d_W(v_F) = d_F$.

As we are dealing with thin sets, we have $d_F \leq k$ for all $F \in \mathcal{F}$. For every $F \in \mathcal{F}$, we make $v_F$ adjacent to any $4k - d_F + 1$ vertices of $D\Box$.

4. For every row $i \in [k]$, create a vertex $r_i$ into $G$ and make $r_i$ adjacent to $w_{ij}$ for all $j \in [k]$. Let $R = \{r_1, \ldots, r_k\}$. For every $r \in R$, make $r$ adjacent to every vertex of $D\triangle$ and any $3k + 1$ vertices of $D\Box$.

5. For every column $j \in [k]$, create a vertex $c_j$ into $G$ and make $c_j$ adjacent to $w_{ij}$ for all $i \in [k]$. Let $C = \{c_1, \ldots, c_k\}$. For every $c \in C$, make $c$ adjacent to every vertex of $D\triangle$ and any $3k + 1$ vertices of $D\Box$.

This completes the construction of $G$. Set $r = 5k$. We now formally argue that instances $(\mathcal{F}, k)$ and $(G, r)$ are equivalent. Assume first that $X$ is a solution to the instance $(\mathcal{F}, k)$. For each $i \in [k]$, let $j_i$ be the unique index such that $(i, j_i) \in X$. We claim that the set

$$S = D\triangle \cup \{w_{1j_1}, w_{2j_2}, \ldots, w_{kj_k}\}$$

is an offensive alliance of size exactly $5k$ in $G$. We see that $N(S) = R \cup C \cup \{v_F : F \in \mathcal{F}\}$. Let $v$ be an arbitrary element of $N(S)$. We need to prove
that $d_S(v) \geq d_S(v) + 1$ for all $v \in N(S)$. If $v$ is an element of $R$ or $C$, the neighbours of $v$ in $S$ are the elements of $D^\triangle$ and one element from $W$. Thus we have $d_S(v) = 4k + 1$.

The neighbours of $v$ in $S'$ are $3k + 1$ elements of $D^\square$ and $k - 1$ elements of $W$; therefore we have $d_S(v) = 4k$. If $v$ is an element of \{ $v_F : F \in \mathcal{F}$ \}, the neighbours of $v$ in $S$ are the elements of $D^\triangle$ and at least one element from $W$ as $X$ is a hitting set; thus we have $d_S(v) \geq 4k + 1$. The neighbours of $v$ in $S'$ are $4k - d + 1$ elements of $D^\square$ and at most $d - 1$ elements from $W$; thus we have $d_S(v) \leq (4k - d + 1) + (d - 1) = 4k$. This shows that $S$ is indeed an offensive alliance.

In the reverse direction, let $S$ be an offensive alliance of size at most $5k$ in $G$. First we show that it can be assumed that $N(S) \cap D^\square = \emptyset$. Suppose, for the sake of contradiction, that $v \in N(S) \cap D^\square$. Then $v$ must satisfy the condition $d_S(v) \geq d_S(v) + 1$. As $v$ has degree at least $12k$, in order to satisfy the condition $d_S(v) \geq d_S(v) + 1$, the size of $S$ must be at least $6k$, a contradiction to the assumption that the size of $S$ is at most $5k$. Next we show that it can be assumed that $S \cap D^\triangle = \emptyset$. Suppose that $S$ contains an element of $D^\square$. As $D^\square$ is a clique and $D^\square \cap N(S) = \emptyset$, if $S$ contains one element of $D^\square$ then $D^\square \subseteq S$. This is not possible as $D^\square$ has $12k + 1$ elements and $S$ has at most $5k$ elements. Therefore, we may assume that $(S \cup N(S)) \cap D^\square = \emptyset$. This in turn implies that that $(R \cup C \cup V_F) \cap S = \emptyset$. As the offensive alliance $S$ is non-empty, we have $S \cap D^\triangle \neq \emptyset$ or $S \cap W \neq \emptyset$. Note that in either case, we get $N(S) \cap (R \cup C \cup V_F) \neq \emptyset$.

Let $u$ be an arbitrary element of $N(S) \cap (R \cup C \cup V_F)$. If $S \cap D^\triangle = \emptyset$ then clearly we have $d_S(u) < d_S(u) + 1$ which is not possible. Therefore $S \cap D^\triangle \neq \emptyset$. We observe that if $S$ contains one element of $D^\triangle$ then it contains all elements of $D^\triangle$, that is, $D^\triangle \subseteq S$. As $D^\triangle \subseteq S$ and $(R \cup C \cup V_F) \cap S = \emptyset$, we get $(R \cup C \cup V_F) \subseteq N(S)$. As $S$ is an offensive alliance, every element $u$ of $N(S)$ has to satisfy the condition $d_S(u) \geq d_S(u) + 1$. Consider an arbitrary vertex $r_i$ of $R$. If $S \cap \{w_{ij} : j \in [k]\} = \emptyset$ then $d_S(r_i) = 4k$ and $d_S(r_i) = 4k$ which is not possible as $r_i$ does not satisfy the condition $d_S(r_i) \geq d_S(r_i) + 1$. This implies that, for each $i \in [k]$, $S$ contains at least one element from $\{w_{ij} : j \in [k]\}$ but since $|S| \leq 5k$ and $D^\triangle \subseteq S$, $S$ contains exactly one element from $\{w_{ij} : j \in [k]\}$. Using the same argument for an arbitrary vertex $c_j \in C$, we get that $S$ contains exactly one element from $\{w_{ij} : i \in [k]\}$. We claim that $X = \{(i, j) : w_{ij} \in S\}$ is a permutation hitting set of size $k$. Let us assume that there exists a set $F \in \mathcal{F}$ which is not hit by $X$. In that case, we have $d_S(v_F) = 4k$ and $d_S(v_F) = 4k$. This means that $d_S(v_F) < d_S(v_F) + 1$ which is a contradiction. Therefore, $X$ is a hitting set of size $k$. As $X$ has exactly one element in each row and in each column, $X$ is a permutation hitting set.

An algorithm solving Offense Alliance in time $2^{o(k \log k)}$ would therefore translate into an algorithm running in time $2^{o(k \log k)}$ for $k \times k$ (Permutation) Hitting Set with Thin Sets and contradicts the ETH.

**Corollary 5.** Unless ETH fails, Exact Offense Alliance problem cannot be solved in time $O^*(2^{o(k \log k)})$, where $k$ is the solution size.
5 No polynomial kernel parameterized by solution size and vertex cover

Parameterized by the solution size, the problem is FPT and in this section we show the following kernelization hardness of OFFENSIVE ALLIANCE.

**Theorem 4.** OFFENSIVE ALLIANCE parameterized by the solution size and vertex cover combined does not admit a polynomial compression unless coNP ⊆ NP/poly.

To prove Theorem 4 we give a polynomial parameter transformation (PPT) from CLOSEST STRING to OFFENSIVE ALLIANCE parameterized by the solution size. In the CLOSEST STRING problem, we are given a set of k strings $\mathcal{X} = \{x_1, x_2, \ldots, x_k\}$, each string over an alphabet $\Sigma$ and of length $n$, and an integer $d$. The objective is to check whether there exists a string $y$ of length $n$ over $\Sigma$ such that $d_H(y, x_i) \leq d$ for all $i \in \{1, 2, \ldots, k\}$. Here $d_H(x, y)$ is the Hamming distance between strings $x$ and $y$, that is, the number of places where strings $x$ and $y$ differ. We call any such string $y$ a central string. Let $x$ be a string over alphabet $\Sigma$. We denote the letter on the $p$-th position of $x$ as $x[p]$.

The following theorem is known:

**Theorem 5.** \[ (d, n) \] CLOSEST STRING does not admit a polynomial kernel unless NP ⊆ coNP/poly.

They also observe that the kernelization lower bound for CLOSEST STRING works for any fixed alphabet $\Sigma$ of size at least two. Therefore, without loss of generality, we assume that $\Sigma = \{A_1, A_2\}$.

5.1 Proof of Theorem 4

We give a PPT from the CLOSEST STRING problem. Given an instance $(\mathcal{X}, d)$ of the CLOSEST STRING problem, we construct an instance $(G, r)$ of OFFENSIVE ALLIANCE the following way.

1. For every $x \in \mathcal{X}$, we introduce a vertex $v_x$ into $G$. Let $V_{\mathcal{X}} = \{v_x \mid x \in \mathcal{X}\}$.
   We also introduce a set of $2n$ vertices $W = \{w_{i,j} : i \in [n], j \in [2]\}$. Make $v_x$ adjacent to $w_{i,1}$ if the letter on the $i$th position of $x$ is $A_1$; make $v_x$ adjacent to $w_{i,2}$ if the letter on the $i$th position of $x$ is $A_2$.
2. We introduce a clique $D^\triangle$ of size $3n + 2d + 1$ into $G$. For every $d \in D^\triangle$, we add a set $V_d$ of $12n$ vertices and make them adjacent to $d$. For every $v_x \in V_{\mathcal{X}}$, we make $v_x$ adjacent to every vertex of $D^\triangle$.
3. We introduce another clique $D^{\square}$ of size $12n + 1$ into $G$. For every $v_x \in V_{\mathcal{X}}$, we make $v_x$ adjacent to any $4n$ vertices of $D^{\square}$.
4. For every row $i \in [n]$, create a vertex $r_i$ into $G$ and make $r_i$ adjacent to $w_{i,1}$ and $w_{i,2}$. Let $R = \{r_1, \ldots, r_n\}$. For every $r \in R$, make $r$ adjacent to any three vertices of $D^\triangle$ and any two vertices of $D^{\square}$. 

Fig. 8. Example of reduction of Theorem 4 applied to an instance \((\chi, d)\) of Closest String where \(\chi\) contains three strings \(x_1 = 1011100, x_2 = 1101010, x_3 = 1110001\) and \(d = 3\). A solution string \(y = 1000000\) is shown in red circles.

This completes the construction of \(G\). Note that the set \(R \cup W \cup D^\square \cup D^\triangle\) forms a vertex cover of \(G\) of size \(18n + 2d + 2\). We set \(r = 4n + 2d + 1\). It is easy to see that the above construction takes polynomial time. We now formally argue that instances \((\mathcal{X}, d)\) and \((G, r)\) are equivalent. Assume first that \(y\) is a solution to the instance \((\mathcal{X}, d)\), that is, \(d_H(x_i, y) \leq d\) for all \(i \in \{1, 2, \ldots, k\}\). We claim that \(S = D^\triangle \cup \{w_{ij} \mid y[i] = A_j \text{ for } i = 1, 2, \ldots, n\}\) is an offensive alliance of size at most \(r\). We see that \(N(S) = R \cup V_X\). We need to prove that \(d_S(v) \geq d_{S^c}(v) + 1\) for every \(v \in R \cup V_X\). If \(v\) is an element of \(R\), the neighbours of \(r\) in \(S\) are three elements of \(D^\triangle\) and one element from \(W\). Thus we have \(d_S(r) = 4\). The neighbours of \(r\) in \(S^c\) are two elements of \(D^\square\) and one elements of \(W\); therefore we have \(d_{S^c}(r) = 3\) and \(r\) satisfies the required condition. If \(v_x\) is an element of \(V_X\), the neighbours of \(v_x\) in \(S\) are \(3n + 2d + 1\) elements of \(D^\triangle\) and \(n - d_H(x, y)\) element from \(W\). Thus we have \(d_S(v_x) \geq (3n + 2d + 1) + (n - d) = 4n + d + 1\). The neighbours of \(v_x\) in \(S^c\) are \(4n\) elements of \(D^\square\) and \(d_H(x, y)\) element from \(W\); hence \(d_{S^c}(v_x) = 4n + d_H(x, y) \leq 4n + d\). Therefore \(v_x\) satisfies the required condition.

In the reverse direction, let \(S\) be an offensive alliance of size at most \(4n + 2d + 1\) in \(G\). First we show that it can be assumed that \(N(S) \cap D^\square = \emptyset\). Suppose, for the sake of contradiction, that \(v \in N(S) \cap D^\square\). Then \(v\) must satisfy the condition \(d_S(v) \geq d_{S^c}(v) + 1\). As \(v\) has degree at least \(12k\), in order to satisfy the condi-
Therefore, \( d_S(v) \geq d_{S^c}(v) + 1 \), the size of \( S \) must be at least \( 6k \), a contradiction to the assumption that the size of \( S \) is at most \( 4n + 2d + 1 \). Next we show that it can be assumed that \( S \cap D^\square = \emptyset \). Suppose that \( S \) contains an element of \( D^\square \). As \( D^\square \) is a clique and \( D^\square \cap N(S) = \emptyset \), if \( S \) contains one element of \( D^\square \) then \( D^\square \subseteq S \). This is not possible as \( D^\square \) has \( 12n + 1 \) elements and \( S \) has at most \( 4n + 2d + 1 \) elements. Therefore, we may assume that \( (S \cup N(S)) \cap D^\square = \emptyset \). This in turn implies that \( S \) does not contain any element from \( V_X \cup R \). As the offensive alliance \( S \) is non-empty, we have \( S \cap D^\Delta \neq \emptyset \) or \( S \cap W \neq \emptyset \). Note that in either case, we get \( N(S) \cap (R \cup V_X) \neq \emptyset \). Let \( u \) be an arbitrary element of \( N(S) \cap (R \cup V_X) \).

If \( S \cap D^\Delta = \emptyset \) then clearly we have \( d_S(u) < d_{S^c}(u) + 1 \) which is not possible. Therefore \( S \cap D^\Delta \neq \emptyset \). We observe that if \( S \) contains one element of \( D^\Delta \) then it contains all elements of \( D^\Delta \), that is, \( D^\Delta \subseteq S \). As \( D^\Delta \subseteq S \) and \( (R \cup V_X) \cap S = \emptyset \), we get \( (R \cup V_X) \subseteq N(S) \). As \( S \) is an offensive alliance, every element \( u \) of \( N(S) \) has to satisfy the condition \( d_S(u) \geq d_{S^c}(u) + 1 \). Consider an arbitrary vertex \( r_i \) of \( R \). If \( S \cap \{w_1, w_2\} = \emptyset \) then \( d_S(r_i) = 3 \) and \( d_{S^c}(r_i) = 4 \) which is not possible as \( r_i \) does not satisfy the condition \( d_S(r_i) \geq d_{S^c}(r_i) + 1 \). This implies that, for each \( i \in [n] \), \( S \) contains at least one element from \( \{w_1, w_2\} \). Since \( |S| \leq 4n + 2d + 1 \) and \( D^\Delta \subseteq S \), \( S \) contains exactly one element from \( \{w_1, w_2\} \) for each \( i \). Define a string \( y = y[1]y[2] \ldots y[n] \), where \( y[i] = 1 \) if \( w_1 \in S \) and \( y[i] = 2 \) if \( w_2 \in S \). We claim \( y \) is a central string. Assume, for the sake of contradiction, that there exists a string \( x_i \) such that \( d_H(x_i, y) > d \). In this case, we see that \( d_S(v_{x_i}) < (3n + 2d + 1) + n - d \leq 4n + d \) and \( d_{S^c}(v_{x_i}) > 4n + d \). Therefore, \( d_S(v_{x_i}) < d_{S^c}(v_{x_i}) + 1 \), which is a contradiction. \( \square \)

6 Faster FPT algorithms parameterized by vertex cover number

We know that both Offensive Alliance and Strong Offensive Alliance admit FPT algorithms \([21]\) when parameterized by the vertex cover number of the input graph. The algorithms in \([21]\) use Integer Linear Programming, and thus their dependency on the parameter may be gigantic. The reason is this. The Offensive Alliance problem is mapped to an ILP with at most \( 2^{vc(G)} \) many variables where \( vc(G) \) is the vertex cover number of the input graph. It is proved in \([1]\) that \( p \)-Variable Integer Linear Programming Optimization (\( p \)-Opt-ILP) can be solved using \( O(p^{2.5p + o(p)} \cdot L \cdot \log(MN)) \) arithmetic operations and space polynomial in \( L \). Thus the algorithm in \([21]\) requires \( O^*(2^{vc(G)})^O(2^{vc(G)}) \) time. The natural question would be whether they admit \( O^*(vc(G))^{O(vc(G))} \) time algorithm. We answer this question with the following theorem.

**Theorem 6.** Offensive Alliance can be solved in time \( O^*(vc(G))^{O(vc(G))} \), where \( vc(G) \) is the vertex cover number of the input graph \( G \).

**Proof.** Let \( C \) be a vertex cover of \( G \) of size \( vc(G) \). Note that \( C \) forms an offensive alliance. This is because \( N(C) \) is an independent set and every vertex of \( N(C) \)
has no neighbours in \( C^c \) and has at least one neighbour in \( C \). Therefore we have \( d_{C^c}(v) \geq d_{C^c}(v) + 1 \) for all \( v \in N(C) \), and hence \( C \) is an offensive alliance. This implies that the size of minimum offensive alliance is at most \( \text{vc}(G) \). In [11], it was proved using branching technique that \text{Offensive Alliance} problem parameterized by solution size admits a \( O(n^2 k(2k)^{k-1}) \). This implies that, we have an algorithm with running time \( O^*(\text{vc}(G))^\text{vc}(G) \).

The arguments in the proof of Theorem [6] is also applicable to \textbf{Strong Offensive Alliance} as long as the input graph \( G \) has minimum degree at least two. As a direct consequence of Theorem [6] we have the following corollary.

**Corollary 6.** \textbf{Strong Offensive Alliance} can be solved in time \( O^*(\text{vc}(G)^{\text{vc}(G)}) \) where \( \text{vc}(G) \) is the vertex cover number of the input graph \( G \) with \( \delta(G) \geq 2 \).

### 7 Classical lower bounds under ETH

In this section, we prove lower bound based on ETH for the time needed to solve the \text{Offensive Alliance} problem. In order to prove that a too fast algorithm for \text{Offensive Alliance} contradicts ETH, we give a reduction from \text{Vertex Cover} in graphs of maximum degree 3 and argue that a too fast algorithm for \text{Offensive Alliance} would solve \text{Vertex Cover} in graphs of maximum degree 3 in time \( 2^{o(n)} \). Johnson and Szegedy [19] proved that, assuming ETH, there is no algorithm with running time \( 2^{o(n)} \) to compute a minimum vertex cover in graphs of maximum degree 3.

**Theorem 7.** Unless ETH fails, \text{Offensive Alliance} does not admit a \( 2^{o(n)} \) algorithm, even when restricted to bipartite graphs.

**Proof.** We give a linear reduction from \text{Vertex Cover} in graphs of maximum degree 3 to \text{Offensive Alliance}, that is, a polynomial-time algorithm that takes an instance of \text{Vertex Cover} and outputs an equivalent instance of \text{Offensive Alliance} whose size is bounded by \( O(n) \). Let \( (G, k) \) be an instance of \text{Vertex Cover}, where \( G = (V, E) \) has maximum degree 3. We construct an equivalent instance \( (G', k') \) of \text{Offensive Alliance} the following way. See Figure [6] for an illustration. Take two distinct copies \( V_0, V_1 \) of \( V = \{v_1, v_2, \ldots, v_n\} \), and let \( v' \) be the copy of \( v \in V \) in \( V_1 \). We introduce the vertex set \( E_0 \) into \( G' \), where \( E_0 = \{e_1, \ldots, e_m\} \), the edge set of \( G \). We make \( v'_0 \) adjacent to \( e_j \) if and only if \( v_i \) is an endpoint of \( e_j \) in \( G \). We make \( v'_0 \) adjacent to \( v'_1 \) for all \( i \). Next, introduce five new vertices \( a, b, c, d, e \). For each \( x \in \{a, b, c, d, e\} \), introduce a set \( V_x \) of \( 4k' \) vertices and make \( x \) adjacent to every vertex of \( V_x \). Moreover, we make vertex \( a \) and \( e \) adjacent to every vertex of \( E_0 \) and make \( b \) and \( c \) adjacent to every vertex of \( V_1 \). We also make \( d \) adjacent to every vertex of \( \{a, b, c\} \). Note that \( G' \) is a bipartite graph with bipartition \( \{d\} \cup V_1 \cup E_0 \cup \bigcup_{x \in \{a, b, c\}} V_x \) and \( \{a, b, c, e\} \cup V_d \cup V_0 \). We set \( k' = k + 5 \). Clearly, the size of \( G' \) is bounded by \( O(n) \).
We claim that \((G, k)\) is a yes-instance of \textsc{Vertex Cover} if and only if \((G', k')\) is a yes-instance of \textsc{Offensive Alliance}. Suppose \(G\) has a vertex cover \(S\) of size at most \(k\). We show that \(D = \{v^0 \in V_0 \mid v \in S\} \cup \{a, b, c, d, e\}\) is an offensive alliance of size at most \(k'\) in \(G'\). We see \(N(D) = E_0 \cup V_1 \cup \bigcup_{x \in \{a, b, c, d\}} V_x\). It is clear that for each \(v \in V_1 \cup \bigcup_{x \in \{a, b, c, d\}} V_x\), we have \(d_D(v) \geq d_{D_1}(v) + 1\). Each \(v \in E_0\) has at least three neighbours in \(D\), more precisely, \(a, e\) and at least one neighbour in \(V_0\). This implies that for each \(v \in E_0\), we have \(d_D(v) \geq d_{D_1}(v) + 1\).

Conversely, assume that \(G'\) admits an offensive alliance \(D\) of size at most \(k' = k + 5\). Note that the vertices \(a, b, c, d\) and \(e\) cannot be part of the set \(N(D)\) as each of them has degree \(4k'\); otherwise the size of \(D\) will exceed \(k'\). We now show that \(\{a, b, c, d, e\} \subseteq D\). By definition, offensive alliance cannot be empty; therefore it must contain a vertex from \(V(G')\).

**Case 1.** Suppose \(D\) contains an arbitrary vertex of \(\bigcup_{x \in \{a, b, c, d\}} V_x\). Without loss of generality, assume that \(D\) contains an arbitrary vertex of \(V_a\). Since \(a \not\in N(D)\), it implies that \(a \in D\). Since \(a \in D\), we get \(b, c, d\) and \(e\) also lie in \(D\) as otherwise \(\{b, c, d, e\} \subseteq N(D)\). Therefore, if any vertex from the set \(\{a, b, c, d, e\}\) is in \(D\), it implies that the whole set is in \(D\).

**Case 2.** Suppose \(D\) contains an arbitrary vertex of \(V_1\). It implies that \(b, c\) are in \(D\) and therefore \(\{a, b, c, d, e\} \subseteq D\).

**Case 3.** Suppose \(D\) contains an arbitrary vertex of \(E_0\). It implies that \(\{a, e\} \subseteq D\) and therefore \(\{a, b, c, d, e\} \subseteq D\).

**Case 4.** Suppose \(D\) contains \(v^0\) from \(V_0\). This implies that \(v^1 \in V_1\) is in \(N(D)\). If both \(b\) and \(c\) are outside \(D\) then we have \(d_D(v^1) < d_{D_1}(v^1) + 1\), which is a contradiction. This implies that either \(b\) or \(c\) is in \(D\) and therefore \(\{a, b, c, d, e\} \subseteq D\).
Now since \( \{a, b, c, d, e\} \subseteq D \), a vertex \( e \) in \( E_0 \) will be either in \( N(D) \) or \( D \). If \( e \in D \), then we pick an arbitrary neighbour of \( e \) in \( V_0 \) and put it in \( D \) and remove \( e \) from \( D \). Therefore, starting with an arbitrary offensive alliance \( D \), we can transform it into another offensive alliance such that \( D \cap E_0 = 0 \), that is, \( E_0 \subseteq N(D) \). As each \( e \in E_0 \) has to satisfy the condition \( d_S(e) \geq d_{S'}(e) + 1 \), we must have a set \( S \subseteq V_0 \) of size at most \( k \) in \( D \) such that every vertex in \( E_0 \) has at least one neighbour in \( S \). This implies that \( S \) is a vertex cover of size at most \( k \) in \( G \). \( \square \)

7.1 Offensive Alliance on apex graphs

**Theorem 8.** Unless ETH fails, the Strong Offensive Alliance problem does not admit a \( 2^{|V(G)|} \) algorithm even when restricted to apex graphs.

**Proof.** We give a linear reduction from Planar Dominating set to Strong Offensive Alliance, that is, a polynomial-time algorithm that takes an instance of Planar Dominating set on \( n \) vertices and \( m = O(n) \) edges, and outputs an equivalent instance of Strong Offensive Alliance whose size is bounded by \( O(n) \). Let \((G, k)\) be an instance of Planar Dominating set. Without loss of generality, we assume that \( G \) is connected. We construct an equivalent instance \((G', k')\) of Strong Offensive Alliance in the following way. See Figure 10 for an illustration. To construct graph \( G' \), we start with graph \( G \). Now for every edge \( e \in E(G) \), we add one parallel edge \( e' \) with same endpoints. Now, we take subdivision of newly added parallel edges and denote the edge vertex by \( v_e \) corresponding to parallel edge of \( e \). Next, we make every edge vertex \( v_e \) adjacent to three new vertices \( \{1^e, h_2^e, h_3^e\} \). We introduce two new vertices \( x \) and \( x' \). Finally, we add a set \( v_x \) of 6n vertices and make all of them adjacent to both \( x \) and \( x' \). Lastly, we make \( x \) adjacent to all the vertices in the set \( \bigcup_{e \in E(G)} \{v_e, h_1^e, h_2^e, h_3^e\} \). This completes the construction of \( G' \). We set \( k' = m + k + 2 \) We observe that deleting vertex \( x \) makes the graph \( G' \) planar. Therefore, \( G' \) is an apex graph.

Formally, we claim that \( G \) has a dominating set of size at most \( k \) if and only if \( G' \) has an strong offensive alliance of size at most \( k' \). Let us assume that \( G \) admits a dominating set \( D \) of size at most \( k \). We claim that \( S = D \cup \{x, x'\} \cup \bigcup_{e \in E(G)} \{v_e\} \) is an strong offensive alliance of size at most \( k' \). It is easy to see that \( |S| \leq k' \). First we observe that \( N(S) = V(G') \setminus S \cup V_x \). We see that for all vertices \( v \in \bigcup_{e \in E(G)} \{h_1^e, h_2^e, h_3^e\} \), we have \( N_S(v) \geq N_{S'}(v) + 2 \). It is also easy to see that same inequality is true for all the vertices in the set \( V_x \) as well. We observe that if a vertex \( v \in V(G) \) have degree \( d \) in \( G \) then \( d_{G'}(v) = 2d \). Since \( D \) is a dominating set, the vertex \( v \) has at least \( d + 1 \) neighbours in \( S \) and at most \( d - 1 \) neighbours outside \( S \). This implies that \( N_S(v) \geq N_{S'}(v) + 2 \). Therefore \( S \) is an strong offensive alliance.
Theorem 9. The **Strong Offensive Alliance** problem admits a \( O^*(2^{O(\sqrt{n \log n})}) \) algorithm even when restricted to apex graphs.

Note that the treewidth of any apex graph with \( n \) vertices is bounded by \( O(\sqrt{n}) \). In [14], a polynomial time algorithm is given to solve offensive alliance problem on bounded treewidth graphs with running time \( O^*(2^\omega n^{\omega}) \) where \( \omega \) denotes the treewidth of the input graph. This algorithm can be used to obtain
an algorithm with running time $O^*(2^{O(\sqrt{\log n})})$ for apex graphs.

8 NP-completeness results

In this section, we prove that the Offensive Alliance problem is NP-complete, even when restricted to split, chordal and circle graphs.

8.1 Split and Chordal Graphs

A graph $G$ is called chordal if it does not contain any chordless cycle of length at least four. Split graphs are a subclass of chordal graphs, where the vertex set can be partitioned into an independent set and a clique. We now prove the following theorem.

Theorem 10. The Offensive Alliance problem is NP-complete, even when restricted to split or chordal graphs.

Proof. It is easy to see that the problem is in NP. To show that the problem is NP-hard we give a polynomial reduction from Vertex Cover in graphs of maximum degree 3. Let $(G, k)$ be an instance of Vertex Cover, where $G$ has maximum degree 3. We construct an equivalent instance $(G', k')$ of Offensive Alliance the following way. See Figure 10 for an illustration.

Fig. 10. An illustration of the reduction from Vertex Cover to Offensive Alliance in Theorem 10.

We construct $G'$ with vertex sets $V$ and $V_e$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $V_e = E(G) = \{e_1, e_2, \ldots, e_m\}$, the edge set of $G$. We make $v_i$ adjacent to $e_j$ if
and only if \(v_i\) is an endpoint of \(e_j\). Make all the vertices of \(V_e\) pairwise adjacent. We further add a set \(Y = \{y_1, y_2, \ldots, y_{m+1}\}\) of \(m + 1\) vertices; the vertices in \(Y\) are adjacent to every element of \(V_e\) and the vertices in \(Y\) are pairwise adjacent. Note that the vertices of \(V_e \cup Y\) form a clique of size \(2m + 1\). Finally we introduce a set \(X = \{x_1, \ldots, x_{4(n+m)}\}\) of \(4(n+m)\) vertices. We make every vertex of \(X\) adjacent to every vertex of \(Y\). Note that \(V \cup X\) forms an independent set where as the vertices in \(V_e \cup Y\) form a clique. Therefore, \(G\) is a split graph. We set \(k' = k + m + 1\).

Formally, we claim that \(G\) has a vertex cover of size at most \(k\) if and only if \(G'\) has an offensive alliance of size at most \(k'\). Assume first that \(G\) admits a vertex cover \(S\) of size at most \(k\). Consider \(D = S \cup Y\). Clearly, \(|D| \leq k'\). We claim that \(D\) is an offensive alliance in \(G'\). Note that \(N(D) = V_e \cup X\). For each \(x \in X\), we have \(d_D(x) \geq d_{D'}(x) + 1\) as all its neighbours are inside \(D\). Each \(e \in V_e\) has at least \(m + 2\) neighbours in \(D\) and at most \(m + 1\) neighbours, including itself, outside \(D\). This implies that \(D\) is an offensive alliance of size at most \(k'\) in \(G'\).

For the reverse direction, let \(D\) be an offensive alliance of size at most \(k'\) in \(G'\). We first show that \(Y \subseteq D\). It is easy to note that \(Y \cap N(D) = \emptyset\) as otherwise each \(v \in Y \cap N(D)\) has to satisfy the condition \(d_D(v) \geq d_{D'}(v) + 1\) which requires more than \(k'\) vertices in \(D\). Since \(D\) is a non-empty offensive alliance, it must contain a vertex from the set \(V \cup V_e \cup X \cup Y\).

**Case 1:** Suppose \(D\) contains a vertex from \(V_e \cup X \cup Y\). As \(Y \cap N(D) = \emptyset\), therefore we get \(Y \subseteq D\).

**Case 2:** Suppose \(D\) contains a vertex \(v\) from \(V\). Let \(e \in V_e\) be a neighbour of \(v\) in \(G'\). Then \(e\) could be either in \(D\) or in \(N(D)\). If \(e\) is in \(D\), then Case 1 implies that \(Y \subseteq D\). Suppose \(e\) is in \(N(D)\). Then \(e\) has to satisfy the condition \(d_{D'}(e) \geq d_{D'}(v) + 1\), which requires at least one vertex from \(Y\) inside \(D\). Again Case 1 implies that \(Y \subseteq D\).

Since the size of \(D\) is at most \(m + k + 1\) and \(Y \subseteq D\), it can contain at most \(k\) vertices besides the vertices in \(Y\). Given an offensive alliance \(D\), we can construct another offensive alliance \(D'\) such that \(|D'| \leq |D|\) and \(D' \cap (V_e \cup X) = \emptyset\), in the following way. For each \(e \in V_e \cap D\), we replace \(e\) by an arbitrary neighbour of \(e\) in \(V_e\). If a neighbour of \(e\) is already present in \(D\) then just remove \(e\) and do not add any new vertex. We also remove all the vertices of \(X\) from \(D\). The modified \(D\) is our \(D'\). Next we argue that \(D'\) is an offensive alliance. Since \(Y \subseteq D'\) and \(D' \cap (V_e \cup X) = \emptyset\), we have \(N(D') = V_e \cup X\). It is easy to see that each \(x \in X\) satisfies the condition \(d_{D'}(x) \geq d_{D'}(x) + 1\). We know that \(V_e \subseteq D \cup N(D)\). We observe that for the vertices in set \(N(D) \cap V_e\), we only increase their number of neighbours in \(D'\). Therefore, the vertices \(v\) in \(N(D) \cap V_e\) satisfy the condition \(d_{D'}(v) \geq d_{D'}(v) + 1\). We see that for the vertices in \(V_e \cap D\), we have at least one neighbour from \(V\) inside \(D'\) by the construction of \(D'\). Clearly, each vertex in \(V_e \cap D\) has at least \(m + 2\) neighbours inside \(D'\) and at most \(m + 1\) (including itself) outside \(D'\). This shows that \(D'\) is an offensive alliance. We also know
that $V_e \subseteq N(D')$ and therefore each vertex $e \in V_e$ has to satisfy the condition $d_{D'}(e) \geq d_{D''}(e) + 1$ which requires at least one neighbour from $V$ inside $D'$. Therefore $D' \cap V$ forms a vertex cover of size at most $k$ in $G$. This proves that $(G, k)$ is a yes instance.

8.2 Circle graphs

A circle graph is an undirected graph whose vertices can be associated with chords of a circle such that two vertices are adjacent if and only if the corresponding chords cross each other. Here, we prove that the Offense Alliance problem is NP-complete even when restricted to circle graphs, via a reduction from Dominating Set. It is known that the Dominating Set problem on circle graphs is NP-hard [20].

**Theorem 11.** The Offense Alliance problem on circle graphs is NP-complete.

**Proof.** It is easy to see that the problem is in NP. To show that the problem is NP-hard we give a polynomial reduction from Dominating Set on circle graphs. Let $(G, k)$ be an instance of Dominating Set, where $G$ is a circle graph. Suppose we are also given the circle representation $C$ of $G$. Without loss of generality, we assume that there are no one degree vertices in $G$. We construct an instance of $(G', r)$ of Offense Alliance as follows (see Figure 11). Set $r = 2m + k$, where $m$ is the number of edges in $G$. For every $v \in V(G)$, we

![Fig. 11. (a) Graph $G$ and its circle representation. (b) The graph $G'$ produced by the reduction algorithm. Note that every orange vertex is adjacent to a set of $2r$ vertices, which are not shown here.](image-url)
introduce two cliques \( C^1_v \) and \( C^2_v \) where \( C^1_v \) has \( \lfloor \frac{d(v)}{2} \rfloor \) nodes and \( C^2_v \) has \( \lceil \frac{d(v)}{2} \rceil \) nodes; make \( v \) adjacent to every vertex of \( C^1_v \) and \( C^2_v \); for every \( x \in C^1_v \cup C^2_v \), introduce a set \( V_{v,x}^{\square} \) of \( 2r \) new vertices and make \( x \) adjacent to every vertex of \( V_{v,x}^{\square} \). We start at an arbitrary vertex of the circle representation \( C \) of \( G \) and then traverse the circle in a clockwise direction. We record the sequence in which the chords are visited. For example, in Figure 11 if we start at the red vertex on the circle, then the sequence in which the chords are visited, is \( a, b, d, c, a, d, b, c \). Note that every vertex appears twice in the sequence as every chord is visited twice while traversing the circle. Thus we get a sequence \( S \) of length \( 2n \) where \( n \) is the number of chords in \( C \). We use the sequence to connect \( 2n \) newly added cliques. For every consecutive pair \( (u, v) \) in the sequence \( S \), put an edge between a vertex of \( C^1_u \) (resp. \( C^2_u \)) and a vertex of \( C^1_v \) (resp. \( C^2_v \)) if both \( u, v \) appear for the first time (resp. second time) in the sequence; put an edge between a vertex of \( C^1_u \) and a vertex of \( C^2_v \) if \( u \) appears for the first time and \( v \) appears for the second time in the sequence. These edges are shown in orange in Figure 11. This completes the construction of graph \( G' \). Now we show that \( G' \) is indeed a circle graph.

In the reduction algorithm, we have three operations: (i) For every \( v \in V(G) \), we introduce two cliques \( C^1_v \) and \( C^2_v \) and make \( v \) adjacent to every vertex of \( C^1_v \) and \( C^2_v \). This operation can be incorporated in the circle representation by introducing \( \lfloor \frac{d(v)}{2} \rfloor \) intersecting chords at one end of the chord corresponds to \( v \) and \( \lceil \frac{d(v)}{2} \rceil \) intersecting chords at the other end of the chord corresponds to \( v \). See Figure 12 for an illustration. (ii) For every vertex \( x \) in clique, we introduce a set of \( 2r \) new vertices and make \( x \) adjacent to each of them. This operation can be easily incorporated in the circle representation by introducing \( 2r \) parallel chords intersecting the chord corresponds to \( x \). See Figure 12. (iii) For every consecutive pair \( (u, v) \) in the sequence \( S \), we put an edge between a vertex of \( C^1_u \) and a vertex of \( C^1_v \). This is incorporated in the circle representation by making

Fig. 12. (a) The circle representation for the first operation. Let \( d(v) = 6 \). For \( v \), we introduce \( C^1_v \) in \( G' \), and make \( v \) adjacent with every vertex of \( C^1_v \). The circle representation of \( v \), \( C^1_v \) and their adjacency are shown here. (b) The circle representation for the second operation with \( r = 1 \).
the last chord (in clockwise direction) of $C^{1}_{v}$ intersect with the first chord (in clockwise direction) of $C^{1}_{u}$. This is demonstrated in Figure 13.

![Figure 13](image)

**Fig. 13.** A circle representation of the graph $G'$ in Figure 11. We do not shown the parallel chords correspond to $2r$ vertices adjacent to every vertex in each clique.

Formally, we claim that $G$ has a dominating set of size at most $k$ if and only if $G'$ has an offensive alliance of size at most $r$. Assume first that $G$ admits a dominating set $S$ of size at most $k$. Consider

$$D = \bigcup_{v \in V(G)} V(C^{1}_{v}) \cup V(C^{2}_{v}) \cup S.$$  

Clearly $|D| \leq r$. We claim that $D$ is an offensive alliance in $G'$. Clearly

$$N(D) = (V(G) \setminus S) \bigcup \bigcup_{v \in V(G)} \bigcup_{x \in C^{1}_{v} \cup C^{2}_{v}} V_{v,x}.$$  

It is easy to see that each $u \in \bigcup_{v \in V(G)} \bigcup_{x \in C^{1}_{v} \cup C^{2}_{v}} V_{v,x}$ satisfies $d_{D}(u) \geq d_{D'}(u) + 1$. For $u \in V(G) \setminus S$, if $d_{G}(u) = d$ then $d_{D}(u) \geq d + 1$ and $d_{D'}(u) \leq d - 1$ in $G'$. Thus $D$ is an offensive alliance of size at most $r$ in $G'$.

Conversely, suppose $G'$ admits an offensive alliance $D$ of size at most $r$. First, we show that

$$\bigcup_{v \in V(G)} V(C^{1}_{v}) \cup V(C^{2}_{v}) \subseteq D.$$  

It is to be noted that any offensive alliance $D$ of size at most $r$ cannot contain a vertex of degree more than $2r$ in its neighbourhood $N(D)$. This implies that $N(D) \bigcap \bigcup_{v \in V(G)} V(C^{1}_{v}) \cup V(C^{2}_{v}) = \emptyset$. Since $D$ is a non-empty offensive alliance, it must contain a vertex from $G'$. Suppose $D$ contains a vertex from $\bigcup_{v \in V(G)} V(C^{1}_{v}) \cup V(C^{2}_{v})$. Since $N(D)$ cannot contain an element of $\bigcup_{v \in V(G)} V(C^{1}_{v}) \cup V(C^{2}_{v})$, it implies that $\bigcup_{v \in V(G)} V(C^{1}_{v}) \cup V(C^{2}_{v}) \subseteq D$. Suppose $D$ contains an element from $V(G) \bigcup \bigcup_{v \in V(G)} \bigcup_{x \in C^{1}_{v} \cup C^{2}_{v}} V_{v,x}$ then it is clear from the
above argument that \( \bigcup_{v \in V(G)} V(C^1_v) \cup V(C^2_v) \subseteq D \). It is to be noted that the total number of vertices in \( \bigcup_{v \in V(G)} V(C^1_v) \cup V(C^2_v) \) is \( \sum_{v \in V} d(v) = 2m \), where \( m \) is the number of edges in \( G \). We have added \( 2m \) vertices inside the solution. Apart from these vertices, we can add \( k \) more vertices inside the solution. We observe that the vertices in \( V(G) \) are either in \( D \) or \( N(D) \). Each \( v \in V(G) \cap N(D) \) needs exactly one neighbour from \( V(G) \cap D \), in addition to its neighbours in \( C^1_v \cup C^2_v \), in order to satisfy the condition \( d_D(v) \geq d_D^c(v) + 1 \). Therefore, \( D \cap V(G) \) is a dominating set of size at most \( k \) in \( G \).

9 Conclusion and Future Directions

In this work we proved that the Offensive Alliance problem is NP-complete even when restricted to bipartite, chordal, split and circle graphs. We proved that the Offensive Alliance problem is \( W[1] \)-hard parameterized by a wide range of fairly restrictive structural parameters such as the feedback vertex set number, treewidth, pathwidth, and treedepth of the input graph thus not FPT (unless \( FPT = W[1] \)). We thereby resolved an open question stated by Bernhard Bliem and Stefan Woltran (2018) concerning the complexity of Offensive Alliance parameterized by treewidth. This is especially interesting because most “subset problems” that are FPT when parameterized by solution size turned out to be FPT for the parameter treewidth [1], and moreover Offensive Alliance is easy on trees. We gave lower bound based on ETH for the time needed to solve the Offensive Alliance problem; we proved that it cannot be solved in time \( 2^{o(n)} \) even when restricted to bipartite graphs, unless ETH fails. We list some natural questions that arise from the results of this study:

- Does Offensive Alliance parameterized by vertex cover number admit a single exponential algorithm or can one show a lower bound with matching time complexity?
- Does Offensive Alliance admits polynomial-time algorithms on some special classes of intersection graph family such as interval graphs, circular arc graphs, unit disk graphs, etc?
- Determine parameterized complexity of Offensive Alliance problem when parameterized by other structural parameters such as twin cover, cluster vertex deletion number and modular-width.

References

1. M. Basavaraju, F. Panolan, A. Rai, M. S. Ramanujan, and S. Saurabh. On the kernelization complexity of string problems. Theor. Comput. Sci., 730:21–31, 2018.
2. B. Bliem and S. Woltran. Defensive alliances in graphs of bounded treewidth. Discrete Applied Mathematics, 251:334 – 339, 2018.
3. C.-W. Chang, M.-L. Chia, C.-J. Hsu, D. Kuo, L.-L. Lai, and F.-H. Wang. Global defensive alliances of trees and cartesian product of paths and cycles. Discrete Applied Mathematics, 160(4):479 – 487, 2012.
4. M. Chellali and T. W. Haynes. Global alliances and independence in trees. *Discuss. Math. Graph Theory*, 27(1):19–27, 2007.
5. M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.
6. M. Dom, D. Lokshtanov, S. Saurabh, and Y. Villanger. Capacitated domination and covering: A parameterized perspective. In M. Grohe and R. Niedermeier, editors, *Parameterized and Exact Computation*, pages 78–90, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.
7. R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer, 2012.
8. R. Enciso. *Alliances in graphs: Parameterized algorithms and on partitioning series-parallel graphs*. PhD thesis, University of Central Florida, USA, 2009.
9. M. R. Fellows, D. Lokshtanov, N. Misra, F. A. Rosamond, and S. Saurabh. Graph layout problems parameterized by vertex cover. In S.-H. Hong, H. Nagamochi, and T. Fujunaga, editors, *Algorithms and Computation*, pages 294–305, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.
10. H. Fernau and D. Raible. Alliances in graphs: a complexity-theoretic study. In *Proceeding Volume II of the 33rd International Conference on Current Trends in Theory and Practice of Computer Science*, 2007.
11. H. Fernau and D. Raible. Alliances in graphs: a complexity-theoretic study. In J. van Leeuwen, G. F. Italiano, W. van der Hoek, C. Meinel, H. Sack, F. Plasil, and M. Bieliková, editors, *SOFSEM 2007: Theory and Practice of Computer Science, 33rd Conference on Current Trends in Theory and Practice of Computer Science, Harrachov, Czech Republic, January 20-26, 2007, Proceedings Volume II*, pages 61–70. Institute of Computer Science AS CR, Prague, 2007.
12. H. Fernau, J. A. Rodríguez, and J. M. Sigarreta. Offensive r-alliances in graphs. *Discrete Applied Mathematics*, 157(1):177 – 182, 2009.
13. G. Fricke, L. Lawson, T. Haynes, M. Hedetniemi, and S. Hedetniemi. A note on defensive alliances in graphs. *Bulletin of the Institute of Combinatorics and its Applications*, 38:37–41, 2003.
14. A. Gaikwad, S. Maity, and S. K. Tripathi. Parameterized complexity of defensive and offensive alliances in graphs. In D. Goswami and T. A. Hoang, editors, *Distributed Computing and Internet Technology*, pages 175–187, Cham, 2021. Springer International Publishing.
15. R. Ganian, F. Klute, and S. Ordyniak. On structural parameterizations of the bounded-degree vertex deletion problem. *Algorithmica*, 2020.
16. R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001.
17. L. H. Jamieson. *Algorithms and Complexity for Alliances and Weighted Alliances of Various Types*. PhD thesis, Clemson University, USA, 2007.
18. L. H. Jamieson, S. T. Hedetniemi, and A. A. McRae. The algorithmic complexity of alliances in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 68:137–150, 2009.
19. D. S. Johnson and M. Szegedy. What are the least tractable instances of max independent set? In *Proceedings of the Tenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’99, page 927–928, USA, 1999. Society for Industrial and Applied Mathematics.
20. J. Keil. The complexity of domination problems in circle graphs. *Discrete Applied Mathematics*, 42(1):51–63, 1993.
21. M. Kiyomi and Y. Otachi. Alliances in graphs of bounded clique-width. *Discrete Applied Mathematics*, 223:91 – 97, 2017.
22. T. Kloks. *Treewidth, Computations and Approximations*, volume 842 of *Lecture Notes in Computer Science*. Springer, 1994.
23. P. Kristiansen, M. Hedetniemi, and S. Hedetniemi. Alliances in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 48:157–177, 2004.
24. D. Lokshtanov, D. Marx, and S. Saurabh. Slightly superexponential parameterized problems. *SIAM Journal on Computing*, 47(3):675–702, 2018.
25. J. Nesetril and P. O. de Mendez. *Sparsity: Graphs, Structures, and Algorithms*. Springer Publishing Company, Incorporated, 2014.
26. N. Robertson and P. Seymour. Graph minors. iii. planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49 – 64, 1984.
27. J. Rodríguez-Velázquez and J. Sigarreta. Global offensive alliances in graphs. *Electronic Notes in Discrete Mathematics*, 25:157 – 164, 2006.
28. J. Sigarreta, S. Bermudo, and H. Fernau. On the complement graph and defensive k-alliances. *Discrete Applied Mathematics*, 157(8):1687 – 1695, 2009.
29. J. Sigarreta and J. Rodríguez. On defensive alliances and line graphs. *Applied Mathematics Letters*, 19(12):1345 – 1350, 2006.
30. J. Sigarreta and J. Rodríguez. On the global offensive alliance number of a graph. *Discrete Applied Mathematics*, 157(2):219 – 226, 2009.
31. D. B. West. *Introduction to Graph Theory*. Prentice Hall, 2000.