Global regularity for a model Navier-Stokes equations on $\mathbb{R}^3$

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Abstract

We study a nonlinear parabolic system for a time dependent solenoidal vector field on $\mathbb{R}^3$. The nonlinear term of this new model equations is obtained slightly modifying that of the Navier-Stokes equations. The system has the same scaling property and the Galileian invariance as the Navier-Stokes equations. For such system we prove the global regularity for a smooth initial data.

AMS Subject Classification Number: 35K55, 35B05, 76A02

1 Introduction

We consider the incompressible Navier-Stokes equations in $\mathbb{R}^3$,

\begin{equation}
\begin{aligned}
    v_t + v \cdot \nabla v &= -\nabla p + \Delta v, \quad (x, t) \in \mathbb{R}^3 \times [0, \infty) \\
    \nabla \cdot v &= 0, \quad (x, t) \in \mathbb{R}^3 \times [0, \infty) \\
    v(x, 0) &= v_0(x), \quad x \in \mathbb{R}^3
\end{aligned}
\end{equation}

(\text{NS})

where $v = (v_1, v_2, v_3), v_j = v_j(x, t), j = 1, 2, 3$, is the velocity field, and $p = p(x, t)$ is the pressure. For simplicity we consider the case of zero external force. The problem of global regularity/finite time singularity of solutions to
(NS) for a smooth initial data \( v_0 \) is an outstanding open problem. We know the local in time well-posedness for smooth initial data \([14, 10]\), the global existence of weak solutions \([13, 9]\), and the partial regularity of suitable weak solutions \([18, 1]\). For comprehensive studies of the Cauchy problem of (NS) we refer \([11, 12]\). Using the vector identity

\[
v \cdot \nabla v = -v \times \text{curl} v + \frac{1}{2} \nabla |v|^2,
\]

One can rewrite the system (NS) in an equivalent form,

\[
(NS)_1 \begin{cases}
v_t - v \times \omega &= -\nabla (p + \frac{1}{2} |v|^2) + \Delta v, \\
\nabla \cdot v &= 0, \quad \omega = \nabla \times v \\
v(x, 0) &= v_0(x).
\end{cases}
\]

For further discussion let us recall the definition of the Riesz transform \([19]\) in \( \mathbb{R}^n, n \in \mathbb{N} \). For \( f \in L^p(\mathbb{R}^n), p \in [1, \infty) \), the Riesz transform \( R_j \) of \( f \) is given by

\[
R_j(f)(x) = c_n \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y) dy, \quad j = 1, \cdots, n,
\]

where

\[
c_n = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{(n+1)/2}}.
\]

Let \( \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x) dx \) be the Fourier transform of \( f \). Then, the Riesz transform is more conveniently defined by the Fourier transform as

\[
\widehat{(R_j f)}(\xi) = i \frac{\xi_j}{|\xi|} \hat{f}(\xi), \quad i = \sqrt{-1}.
\]

Let \( R_1, R_2, R_3 \) be the Riesz transforms in \( \mathbb{R}^3 \), and \( u = (u_1, u_2, u_3) \) be a vector field on \( \mathbb{R}^3 \). Then, we define for scalar function \( f, R(f) := (R_1(f), R_2(f), R_3(f)) \), and for a vector field \( u = (u_1, u_2, u_3) \),

\[
R \cdot u := R_1(u_1) + R_2(u_2) + R_3(u_3),
\]

and

\[
R \times u := (R_2 u_3 - R_3 u_2, R_3 u_1 - R_1 u_3, R_1 u_2 - R_2 u_1).
\]
Then, using the fact $R_1^2 + R_2^2 + R_3^2 = -I$, which follows immediately from (1.1), we have for a vector field $F$, satisfying $\text{div} F = 0$,

$$R \times R \times F = R^2 F - R(R \cdot F) = -F.$$ 

Since

$$R \times R \times \{\nabla(p + \frac{1}{2}|v|^2)\} = 0, \quad R \times R \times \{v_t - \Delta v\} = -v_t + \Delta v$$

for $v$ with $\text{div} v = 0$, the system $(NS)_1$ can be written as the following equivalent system.

$$(NS)_2 \begin{cases}
v_t + R \times R \times (v \times \omega) = \Delta v, \\
v(x, 0) = v_0(x), \quad \nabla \cdot v_0 = 0.
\end{cases}$$

The implication that $(NS)_2$ follows from $(NS)_1$ is obvious by taking $R \times R \times$ on $(NS)_1$. For the reverse direction we note that the first equation of $(NS)_2$ can be written as

$$R \times R \times (v_t - v \times \omega - \Delta v) = 0,$$

from which it follows that there exists a scalar function $p = p(x, t)$ such that

$$v_t - v \times \omega - \Delta v = -\nabla(p + \frac{1}{2}|v|^2).$$

Note that the divergence free condition of the initial data is preserved by the equations in $(NS)_2$ for smooth solutions, and the solution satisfies $\text{div} v = 0$ automatically. We also observe that in the formulation $(NS)_2$ there exists no pressure term, although the nonlocality is now moved to the nonlinear term via the Riesz transforms.

We expect that the problem of the global regularity/finite time singularity for the system $(NS)_2$ has the similar difficulty as the original Navier-Stokes equations $(NS)$. Instead of $(NS)_2$ we consider its modified version:

$$(mNS) \begin{cases}
v_t + R \times (v \times \omega) = \Delta v, \\
v(x, 0) = v_0(x), \quad \nabla \cdot v_0 = 0,
\end{cases}$$

which is obtained by omitting one nonlocal operation $R \times$ in the nonlinear term in the system $(NS)_2$. Another interpretation of $(mNS)$ is its relation with the following Hall equations, which is obtained from the Hall
magnetohydrodynamics (Hall-MHD) equations by setting the velocity \(= 0\), and which represent the major difficult part of the whole system (see \([2, 3, 4, 5]\) for more detailed studies of the Hall-MHD system).

\[
\begin{align*}
\text{(Hall)} \quad & \begin{cases}
B_t + \nabla \times (B \times (\nabla \times B)) = \Delta B, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \\
B(x, 0) = B_0(x), & \nabla \cdot B_0 = 0, \quad x \in \mathbb{R}^3
\end{cases}
\end{align*}
\]

where \(B = (B_1, B_2, B_3)\), \(B_j = B_j(x, t), j = 1, 2, 3\) is the magnetic field. Comparing with (Hall), we find that the nonlinear term of \((mNS)\) is regularized by one derivative, in the sense that \(\nabla \times\) in front of the nonlinear term (Hall) is replaced by \(R \times = \nabla (\Delta)^{-1/2} \times\) in (mMS). We note that the symmetry properties (say, the Galilean invariance and the scaling symmetry) of \((mNS)\) are the same as the original Navier-Stokes equations. Our result in the following theorem shows that for such modified system \((mNS)\) we could show the global regularity for a given smooth initial data.

**Theorem 1.1** Let \(v_0 \in H^m(\mathbb{R}^3)\) with \(m > 5/2\). Then, for all \(T \in (0, \infty)\) there exists a unique solution \(v \in C([0, T); H^m(\mathbb{R}^3)) \cap L^2(0, T; H^{m+1}(\mathbb{R}^3))\) to the system \((mNS)\). Moreover, the solution satisfies the inequality:

\[
\begin{align*}
\sup_{0 < t < T} \|v(t)\|^2_{H^m} + \int_0^T \|Dv(s)\|^2_{H^m} ds \\
&\leq \|v_0\|^2_{H^m} \exp \left\{ T\|v_0\|^2_{L^2} \exp(C\|\Lambda^{\frac{1}{2}}v_0\|^2_{L^2}) \right\} \times \\
&\times \exp \left[ \|\Lambda v_0\|^2_{L^2} \exp \left(C\|\Lambda^{\frac{1}{2}}v_0\|^2_{L^2}\right) \|\Lambda v_0\|^2_{L^2} \exp(C\|\Lambda^{\frac{1}{2}}v_0\|^2_{L^2}) \right]
\end{align*}
\]

(1.2)

for all \(T > 0\).

**Remark 1.1** The result shows that the simple estimates for the nonlinear term of \((NS)_2\) is not enough to deduce correct result for the problem of the global regularity/finite time singularity. We mention here that there are also studies of the other model equations of the Navier-Stokes system, where the authors show the finite time blow-up (see e.g. \([6, 7, 8, 16, 17, 20]\)).

## 2 Proof of Theorem 1.1

Let us set the multi-index \(\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n\) with \(|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n\). Then, \(D^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}\), where \(D_j = \partial/\partial x_j\),
\( j = 1, 2, \cdots, n \). Given \( m \in \mathbb{N} \cup \{0\} \) the Sobolev space, \( H^m(\mathbb{R}^n) \) is the Hilbert space of functions consisting of functions \( f \in L^2(\mathbb{R}^n) \) such that

\[
\|f\|_{H^m} := \left( \sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} |D^\alpha f(x)|^2 dx \right)^{\frac{1}{2}} < \infty,
\]

where the derivatives are in the sense of distributions. Given \( s \in \mathbb{R} \), we use the notation \( \Lambda^s(f) = (-\Delta)^{\frac{s}{2}} f \), which is defined by its Fourier transform as

\[
\hat{\Lambda^s(f)}(\xi) = |\xi|^s \hat{f}(\xi).
\]

Then, we observe that

\[
\hat{R_j(f)} = i \frac{\xi_j}{|\xi|} \hat{f} = \partial_j \Lambda^{-1} f(\xi).
\]

We use the energy method for the proof of Theorem 1.1 (see e.g. [15]).

**Proof of Theorem 1.1** Since the local well-posedness in the Sobolev space \( H^m(\mathbb{R}^3) \) for \( m > 5/2 \) is standard, we proceed directly to the global in time a priori estimate. We take \( L^2 \) inner product \((mNS)\) by \( \Lambda v \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^{\frac{1}{2}} v\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} v\|_{L^2}^2 = - \int_{\mathbb{R}^3} R \times (v \times \omega) \cdot \Lambda v \, dx
\]

\[
= - \int_{\mathbb{R}^3} (v \times \omega) \cdot \Lambda R \times v \, dx = - \int_{\mathbb{R}^3} (v \times \omega) \cdot \omega \, dx = 0.
\]

We have therefore

\[
\frac{1}{2} \|\Lambda^{\frac{1}{2}} v(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}} v(s)\|_{L^2}^2 ds = \frac{1}{2} \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2
\]

Next we take \( L^2 \) inner product \((mNS)\) by \( v \) to deduce

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = - \int_{\mathbb{R}^3} v \times \omega \cdot R \times v \, dx
\]

\[
\leq \int_{\mathbb{R}^3} |v| |\omega| |Rv| \, dx \leq \|v\|_{L^6} \|\omega\|_{L^3} \|Rv\|_{L^2}
\]

\[
\leq C \|\nabla v\|_{L^2} \|\Lambda^{\frac{3}{2}} v\|_{L^2} \|v\|_{L^2}
\]

\[
\leq \frac{1}{2} \|\nabla v\|_{L^2}^2 + C \|\Lambda^{\frac{3}{2}} v\|_{L^2}^2 \|v\|_{L^2}^2.
\]
Hence, we have
\[ \|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v\|_{L^2}^2 ds \leq \|v_0\|_{L^2}^2 \exp \left( C \int_0^t \|\Lambda^{3/2} v\|_{L^2}^2 ds \right) \]
\[ \leq \|v_0\|_{L^2}^2 \exp(C\|\Lambda^{3/2} v_0\|_{L^2}^2) \] (2.2)
where we used the estimate (2.1). We now take \( L^2 \) inner product \((mNS)\) by \( \Delta v \) to have
\[ \frac{1}{2} \frac{d}{dt} \|\Lambda v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 = -\int_{\mathbb{R}^3} v \times \omega \cdot R \times \Delta v \, dx \]
\[ \leq \|v\|_{L^6} \|\omega\|_{L^3} \|R\Delta v\|_{L^2} \leq C \|\nabla v\|_{L^2} \|\Lambda^{3/2} v\|_{L^2} \|\Delta v\|_{L^2} \]
\[ \leq \frac{1}{2} \|\Delta v\|_{L^2}^2 + C \|\Lambda^{3/2} v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 , \]
from which we obtain
\[ \|\Lambda v(t)\|_{L^2}^2 + \int_0^t \|\Delta v\|_{L^2}^2 ds \leq \|\Lambda v_0\|_{L^2}^2 \exp \left( C \int_0^t \|\Lambda^{3/2} v\|_{L^2}^2 ds \right) \]
\[ \leq \|\Lambda v_0\|_{L^2}^2 \exp(C\|\Lambda^{3/2} v_0\|_{L^2}^2) \] (2.3)
by (2.1). We operate \( D^2 \) on \((mNS)\), and take \( L^2 \) inner product of it by \( D^2 v \) to obtain
\[ \frac{1}{2} \frac{d}{dt} \|D^2 v\|_{L^2}^2 + \|D^3 v\|_{L^2}^2 = -\int_{\mathbb{R}^3} D(R \times (v \times \omega)) \cdot D^3 v \, dx \]
\[ = -\int_{\mathbb{R}^3} (Dv \times \omega + v \times D\omega) \cdot R \times D^3 v \, dx \]
\[ \leq (\|Dv\|_{L^6} \|\omega\|_{L^3} + \|v\|_{L^\infty} \|D\omega\|_{L^2}) \|D^3 v\|_{L^2} \]
\[ \leq C(\|\Lambda^{3/2} v\|_{L^2}^2 + \|v\|_{L^\infty}) \|D^2 v\|_{L^2} \|D^3 v\|_{L^2} \]
\[ \leq \frac{1}{2} \|D^3 v\|_{L^2}^2 + C(\|\Lambda^{3/2} v\|_{L^2}^2 + \|v\|_{L^\infty}) \|D^2 v\|_{L^2}^2 \]
Hence,
\[ \|D^2 v(t)\|_{L^2}^2 + \int_0^t \|D^3 v\|_{L^2}^2 ds \leq \|D^2 v_0\|_{L^2}^2 \exp \left\{ C \int_0^t (\|\Lambda^{3/2} v\|_{L^2}^2 + \|v\|_{L^\infty}^2) ds \right\} \]
\[ \leq \|\Lambda v_0\|_{L^2}^2 \exp \left\{ C \int_0^t (\|v(s)\|_{L^2}^2 + \|\Delta v(s)\|_{L^2}^2) ds \right\} , \]
\[ \leq \|\Lambda v_0\|_{L^2}^2 \exp \left\{ C \|v_0\|_{L^2}^2 \exp(C\|\Lambda^{3/2} v_0\|_{L^2}^2) \right\} \|\Lambda v_0\|_{L^2}^2 \exp\left(CH\|\Lambda^{3/2} v_0\|_{L^2}^2 \right) \] (2.4)
where we used the estimates (2.2) and (2.3). Let \( m > 5/2 \). Operating \( D^\alpha \) on \((mNS)\) and taking \( L^2 \) inner product of it by \( D^\alpha v \), and summing over \(|\alpha| \leq m\), one has

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{H^m}^2 + \|Dv\|_{H^m}^2 = - \sum_{|\alpha| \leq m} \int_{\mathbb{R}^3} D^\alpha (v \times \omega) \cdot R \times D^\alpha v \, dx
\]

\[
= \sum_{|\beta|=|\alpha|+1 \leq m} \int_{\mathbb{R}^3} D^\beta (v \times \omega) \cdot R \times D^\beta Dv \, dx - \int_{\mathbb{R}^3} (v \times \omega) \cdot R \times v \, dx
\]

\[
\leq C \|v \times \omega\|_{H^{m-1}} \|RDv\|_{H^m} \leq C(\|v\|_{L^\infty} + \|\nabla v\|_{L^\infty}) \|v\|_{H^m} \|Dv\|_{H^m}
\]

\[
\leq \frac{1}{2} \|Dv\|_{H^m}^2 + C(\|v\|_{L^\infty} + \|\nabla v\|_{L^\infty}) \|v\|_{H^m}^2,
\]

and therefore

\[
\|v(t)\|_{H^m}^2 + \int_0^t \|Dv(s)\|_{H^m}^2 \, ds \leq \|v_0\|_{H^m}^2 \exp \left( C \int_0^t (\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^\infty}^2) \, ds \right)
\]

\[
\leq \|v_0\|_{H^m}^2 \exp \left\{ C \int_0^t (\|v\|_{L^2}^2 + \|D^3 v\|_{L^2}^2) \, ds \right\}
\]

\[
\leq \|v_0\|_{H^m}^2 \exp \left\{ t \|v_0\|_{L^2}^2 \exp \left( C \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2 \right) \right\} \times \exp \left[ \|\Lambda v_0\|_{L^2}^2 \exp \left\{ C \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2 \right\} \right],
\]

where we used the estimates (2.2) and (2.3). \( \square \)

**Acknowledgements**

This work was supported partially by NRF Grant no. 2006-0093854 and 2009-0083521.

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