QUANTUM EFFECTS IN THE SPACETIME OF A MAGNETIC FLUX COSMIC STRING

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Abstract

In this work we compute the vacuum expectation values of the energy-momentum tensor and the average value of a massive, charged scalar field in the presence of a magnetic flux cosmic string for both zero- and finite-temperature cases.

1 Introduction

In the General Relativity framework, a straight, infinite, static string lying on the z-axis is described by a static metric with cylindrical symmetry

\[ ds^2 = -dt^2 + dz^2 + d\rho^2 + B^2\rho^2 d\varphi^2 \] (1)

where \( \rho \geq 0 \) and \( 0 \leq \varphi < 2\pi \) and the constant \( B \) is related to the linear mass density \( \mu \) as \( B = 1 - 4\mu \).

Metric (1) is locally but not globally flat. The presence of the string leads to an azimuthal deficit angle equal to \( 8\pi\mu \) and as a result this spacetime has a conical singularity. Placing the origin of the polar coordinate system on the string axis, one reveals a deficit \( 2\pi(1 - B) \) of the polar angle \( \varphi \). Thus, near the string world sheet \( \Sigma \) the space looks like the direct product \( C_B \times \Sigma \) where \( C_B \) is the conical space with the corresponding ranging of the angle \( 0 \leq \varphi \leq 2\pi B \).

Quantum effects such as the vacuum polarization effect arise in a flat spacetime whenever the topology is non-trivial or boundaries are presented. A typical example is the Casimir effect in which the nonzero expectation values of the energy-momentum tensor owes its existence to the presence of boundaries in Minkowski spacetime. In the present case, the manifold is complete, without boundaries, but topologically different from Minkowski space.

The phenomenon of vacuum polarization of a quantum field by a cosmic string carrying a internal magnetic flux can be understood as a realization in
cosmology of the Aharonov-Bohm effect in electromagnetism. Indeed, a quantum field placed in the exterior of the string acquires an additional phase shift proportional to the magnetic flux even though there is no magnetic field outside the string. Our purpose in this paper is to report and to show how to use some results obtained in [4] concerning the vacuum polarization effect of a massive, charged scalar field in the spacetime (1) for zero- and finite-temperature cases. This problem has been treated by many authors [5]. The main contribution is to use a specific method for computing the Green’s functions in spacetime (1) in which the renormalization procedure is straightforward.

Throughout this paper we adopt the system of units where \( G = c = \hbar = 1. \)

2 General Framework

In order to study the vacuum polarization effect we need first to compute the scalar Green’s function in spacetime (1). Once it is more convenient to work in the Euclidean approach to quantum theory we will in practice compute the Euclidean Green’s function for the scalar field. Therefore, let us consider the metric

\[
d s^2 = d\tau^2 + dz^2 + d\rho^2 + B^2\rho^2 d\varphi^2
\]

which is obtained from the metric (1) by a Wick rotation \( t = -i\tau \) in the coordinate \( t. \)

The Euclidean Green’s function of a charged massive scalar field, \( G_E^{(\gamma)}(x, x_0; m) \) is a solution of the covariant Laplace equation in the space (1)

\[
\left[ \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{B^2\rho^2} \frac{\partial^2}{\partial \varphi^2} - m^2 \right] G_E^{(\gamma)}(x, x_0; m) = \frac{1}{B} \delta^{(4)}(x, x_0)
\]

and satisfies the following boundary conditions

\[
G_E^{(\gamma)}(\tau, z, \rho, \varphi + 2\pi) = e^{2i\pi\gamma} G_E^{(\gamma)}(\tau, z, \rho, \varphi), \quad (4)
\]

\[
\frac{\partial}{\partial \varphi} G_E^{(\gamma)}(\tau, z, \rho, \varphi + 2\pi) = e^{2i\pi\gamma} \frac{\partial}{\partial \varphi} G_E^{(\gamma)}(\tau, z, \rho, \varphi) \quad (5)
\]

where \( \gamma \) is the fractional part of \( \Phi/\Phi_0 \), \( \Phi_0 \) being the flux quantum \( 2\pi/e \) and lies in the interval \( 0 \leq \gamma < 1 \). The case where \( \gamma = 0 \) corresponds to a vanishing flux whereas the case where \( \gamma = 1/2 \) corresponds to a twisted field around the axis \( \rho = 0 \). In addition to Eqs. (3-5), we also require that \( G_E^{(\gamma)}(x, x_0; m) \) vanishes when \( x \) and \( x_0 \) are infinitely separated.

In order to solve Eq. (3), we will make a convenient change of coordinate \( \theta = B\varphi \) in such a way that now we have to determine the Green’s function in the subset of a Euclidean space covered by the coordinate system \( (\tau, z, \rho, \theta) \) with \( 0 \leq \theta < 2\pi B \). Eq. (3) becomes the usual Laplace equation
\((\Delta - m^2)G_k^{(2)}(x, x_0; m) = -\delta^{(2)}(x, x_0)\). \hspace{1cm} (6)

It is easy to see that the boundary conditions (4) and (5) will also modify under this new coordinate change.

The important thing to notice here is that we can reduce our problem to 2-dimensions by means of a recurrence relation between the Green’s functions of spaces of different dimensions

\[ G_N^{(\gamma)}(x^1, ..., x^{N-2}, \rho, \theta; m) = \frac{1}{2} \int_{-\infty}^{\infty} G_{N-1}^{(\gamma)}(x^1, ..., x^{N-3}, \rho, \theta; \sqrt{m^2 + \lambda^2}) \times \cos[\lambda(x^{N-2} - x_0^{N-2})] d\lambda. \hspace{1cm} (7) \]

After setting the relevant preliminary steps, we are now in a position to compute the Green’s function for a massive scalar field. We start by noting that the solution of Eq. (6) is the usual Green’s function

\[ \frac{1}{2} K_0(m r_2), \hspace{1cm} (8) \]

where \(r_2\) is the Euclidean distance between the two points \((\rho, \theta)\) and \((\rho_0, \theta_0)\) and \(K_0\) denotes the modified Bessel function of second kind. However, clearly, this solution does not satisfy the boundary conditions (4) and (5). To solve this problem we propose a generalisation of the integral expression of the Bessel function in the form

\[ G^{(2)}_{(\gamma)}(x^1, ..., x^{N-2}; \rho, \theta; m) = \int_{-\infty}^{\infty} K_{i\nu}(m \rho) K_{i\nu}(m \rho_0) e^{\pi\nu U_{\nu}(\theta)} d\nu, \hspace{1cm} (9) \]

where \(U_{\nu}\) must satisfy the boundary conditions (4) and (5). Therefore, we find that

\[ U_{\nu}(\theta) = \frac{e^{2i\gamma} \sinh(\nu | \theta - \theta_0 |) - \sinh(\nu | \theta - \theta_0 | - 2\pi\nu B)}{\cosh(2\pi\nu B) - \cos(2\pi\gamma)}. \hspace{1cm} (10) \]

In this way, the Green’s function \(G^{(2)}_{(\gamma)}\) with \(U_{\nu}\) is a solution of the Laplace equation \(\Delta = 0\) and satisfies the boundary conditions (4) and (5). In principle, our problem is solved and it would be enough to apply twice the recurrence relation. However, we can still improve the expression of the Green’s function by manipulating with its integral expressions. By doing this (for detail of this calculations, we refer the reader to Refs. 4) and by applying successively the recurrence relation, we get a local form

\[ G^{(4)}_{(\gamma)}(x, x_0; m) = \frac{m K_1(m r_4)}{4\pi^2 r_4} + G^{(4)}_{(\gamma)}(x, x_0; m) \hspace{1cm} (11) \]

valid in the domain \(\pi/B - 2\pi < \varphi - \varphi_0 < 2\pi - \pi/B\) for \(B > 1/2\). The term \(G^{(4)}_{(\gamma)}(x, x_0; m)\) appearing in Eq. (11) is a regular term which is given by
\[ G_{(\gamma)}^{(4)}(x,x_0;m) = \frac{m}{8\pi^3 B} \int_0^\infty \frac{K_1[mR_4(u)]}{R_4(u)} F_B^{(\gamma)}(u,\varphi - \varphi_0) du , \]  

with \( R_4(u) = \sqrt{(\tau - \tau_0)^2 + (z - z_0)^2 + \rho^2 + \rho_0^2 + 2\rho\rho_0 \cosh u} \) and the function \( F_B^{(\gamma)}(u,\varphi - \varphi_0) \) contains the contributions coming from the Aharonov-Bohm and the non-trivial gravitational interactions between the scalar field and the cosmic string.

\[ F_B^{(\gamma)}(u,\psi) = i e^{i(\psi + \pi/B)\gamma} \frac{\cosh[u(1 - \gamma)/B] - e^{-i(\psi + \pi/B)(1-\gamma)} \cosh[u\gamma/B]}{\cosh(u/B) - \cos(\psi + \pi/B)} \]  

\[ -i e^{i(\psi - \pi/B)\gamma} \frac{\cosh[u(1 - \gamma)/B] - e^{-i(\psi - \pi/B)(1-\gamma)} \cosh[u\gamma/B]}{\cosh(u/B) - \cos(\psi - \pi/B)} \]  

with \( \psi \equiv \varphi - \varphi_0 \).

### 3 Vacuum Polarization Effect

The renormalization of the vacuum expectation values (V.E.V.) of some physical quantities is performed in a straightforward way and consists solely in removing the usual Green’s function from Eq. (11). In this section, we will compute the V.E.V. of the average value and the energy-momentum tensor of the scalar field. We will treat both the zero- and finite-temperature cases.

Let us suppose that the scalar field is in thermal equilibrium with finite temperature \( T \). In this case, its (thermal) Green’s function is a solution of Eq. (6), satisfies boundary conditions (4) and (5) and, in addition, is periodic in the coordinate \( \tau \) with a period equal to \( \beta = 1/kT \), where \( k \) is the Boltzmann constant.

Metric (2) is ultrastatic (static and \( g_{00} = 1 \)). Therefore, we can apply de Schwinger-deWitt formalism [7]

\[ G_{ET}^{(\gamma)}(x,x_0;m) = \int_0^\infty K_{ET}^{(\gamma)}(x,x_0;s) ds \]  

and we can derive the Euclidean thermal heat kernel \( K_{ET}^{(\gamma)}(x,x_0;s) \) from its corresponding Euclidean zero-temperature heat kernel \( K_E^{(\gamma)}(x,x_0;s) \) becomes [8]

\[ K_{ET}^{(\gamma)}(x,x_0;s) = \Theta_3 \left( i \frac{\beta(\tau - \tau_0)}{4s} \right) K_E^{(\gamma)}(x,x_0;s), \]  

where \( \Theta_3 \) is defined as in Ref. 6. To simplify our problem, we will compute the Euclidean thermal Green’s function for a massless scalar field \( \nu_{ET}^{(\gamma)}(x,x_0) \).
Also, we will skip here any details of the calculations and we will give the final results directly. Therefore, we have

\[
D^{(\gamma)}_{ET}(x, x_0) = \frac{1}{4\pi\beta d} \frac{\sinh(2\pi/\beta) d}{\cosh(2\pi/\beta) d - \cos(2\pi/\beta)(\tau - \tau_0)} + \frac{1}{8\pi^2 B \beta} \int_0^\infty \frac{\sinh[(2\pi/\beta)D(u)]\mathcal{F}^{(\gamma)}(u, \psi)du}{D(u)[\cosh(2\pi/\beta)D(u) - \cos(2\pi/\beta)(\tau - \tau_0)]}
\]

where \( d = \frac{1}{\beta^2}[(z - z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos \psi]^{1/2} \) and \( D(u) = \frac{1}{\beta^2}[(z - z_0)^2 + \rho^2 \rho_0^2 + 2\rho\rho_0 \cosh u]^{1/2} \).

Using (16), we can compute the (thermal) average \(<\phi^2(x)>_\beta \) and the (thermal) V.E.V. of the energy-momentum tensor for a massless scalar field. We can give only analytic results in the asymptotic limits. Therefore, in the limit \( \epsilon \rightarrow \infty \) (or, equivalently \( T \rightarrow 0 \)), we have:

\[
<\phi^2(x)>_\beta \rightarrow \infty \sim \frac{\omega_2(\gamma)}{2\rho^2},
\]

(17)

\[
<T_{\mu\nu}>_\beta \rightarrow \infty = \left[ \omega_4(\gamma) - \frac{1}{3} \omega_2(\gamma) \right] \frac{1}{\rho^4} \text{diag}(1, 1, 1, -3) + 2(\xi - \frac{1}{6})\omega_2(\gamma) \frac{1}{\rho^4} \text{diag}(1, 1, -\frac{3}{2}, \frac{3}{2}).
\]

(18)

In the limit \( \beta \rightarrow 0 \), we have:

\[
<\phi^2(x)>_\beta \rightarrow 0 \sim \frac{1}{12\beta^2} + \frac{M^{(\gamma)}}{\beta \rho^3},
\]

(19)

\[
<T^t_t>_\beta \rightarrow 0 \sim -\frac{\pi^2}{15\beta^3} + (2\xi - 1/2)\frac{M^{(\gamma)}}{\beta \rho^3},
\]

(20)

\[
<T^z_z>_\beta \rightarrow 0 \sim -\frac{\pi^2}{45\beta^4} + \frac{N^{(\gamma)}}{\beta \rho^3} + (2\xi - 1/2)\frac{M^{(\gamma)}}{\beta \rho^3},
\]

(21)

\[
<T^\rho_\rho>_\beta \rightarrow 0 \sim -\frac{\pi^2}{45\beta^4} + \frac{N^{(\gamma)}}{\beta \rho^3} - 2\xi \frac{M^{(\gamma)}}{\beta \rho^3},
\]

(22)

\[
<T^\varphi_\varphi>_\beta \rightarrow 0 \sim -\frac{\pi^2}{45\beta^4} - 2\frac{N^{(\gamma)}}{\beta \rho^3} + 4\xi \frac{M^{(\gamma)}}{\beta \rho^3},
\]

(23)

with \( \omega_2(\gamma), \omega_4(\gamma) \), evaluated numerically, and \( M^{(\gamma)}, N^{(\gamma)} \), given in integral forms, respectively:

\[
\omega_2(\gamma) = -\frac{1}{8\pi^2} \left\{ \frac{1}{3} - \frac{1}{2}B^2 [4(\gamma - \frac{1}{2})^2 - \frac{1}{3}] \right\},
\]

(24)

\[
\omega_4(\gamma) = -\frac{1}{720\pi^4} \left\{ 11 - \frac{15}{B^4} [4(\gamma - \frac{1}{2})^2 - \frac{1}{3}] \right\},
\]
\[ M^{(\gamma)} = \frac{1}{16\pi^2B}\int_0^\infty \frac{F_B^{(\gamma)}(u,0)}{\cosh u/2} \, du \] (26)
\[ N^{(\gamma)} = \frac{1}{32\pi^2B}\int_0^\infty \frac{F_B^{(\gamma)}(u,0)}{\cosh^{3/2} u/2} \, du \] (27)

4 Concluding Remarks

This work considered the vacuum polarization effect of a scalar field in the spacetime generated by a cosmic string. Due to the particular properties of this spacetime we could apply a method which allows one to write the Green’s functions in a local form as a sum of the usual Green’s function in Minkowski spacetime and a regular term which is responsible for the vacuum polarization effect. Although here we considered the metric of a cosmic string in General Relativity, we point out that this method can be extended to the case of a scalar-tensor theory of gravity as well. [9]

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