NORM OF THE BERGMAN PROJECTION ONTO THE BLOCH SPACE

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Abstract. We consider weighted Bergman projection $P_\alpha : L^\infty(B) \to B$ where $\alpha > -1$ and $B$ is the Bloch space of the unit ball $B$ of the complex space $\mathbb{C}^n$. We obtain the exact norm of the operator $P_\alpha$ where the Bloch space is observed as a space with norm (and semi-norm) induced from the Besov space $B_p, 0 < p < \infty, (B_\infty = B)$. Our work contains, as a special case, the main results from [3] and [6].

1. Introduction and Preliminaries

Throughout this paper we denote by $\mathbb{C}^n$ complex $n$–dimensional space. Here $n$ is an integer greater than or equal to 1. As usually $\langle \cdot, \cdot \rangle$ represents the inner product in $\mathbb{C}^n$, $\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$, $z, w \in \mathbb{C}^n$, where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are coordinate representations in the standard base $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$. The Euclidean norm in $\mathbb{C}^n$ is given by $|z| = \langle z, z \rangle^{1/2}$.

Let us denote by $B$ the unit ball in $\mathbb{C}^n$, $B = \{z : |z| < 1\}$ and $S$ denotes its boundary. The volume measure $dv$ in $\mathbb{C}^n$ is normalized, i.e. $v(B) = 1$. Also, we are going to treat a class of weighted measures $dv_\alpha$ on $B$, which are defined by $dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dv(z)$, $z \in B$ where $\alpha > -1$, and $c_\alpha$ is a constant such that $v_\alpha(B) = 1$. Direct calculation gives:

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)}.$$

We let $\sigma$ be unitary-invariant positive Borel measure on $S$ for which $\sigma(S) = 1$. The term "unitary-invariant" refers to the unitary transformations of $\mathbb{C}^n$ (see [5]). A special class of automorphism group are involutive automorphisms, which are, for any point $a \in B$, defined as

$$\varphi_a(w) = \frac{a - \frac{(w,a)a}{|a|^2} - \sqrt{1 - |a|^2} (w - \frac{(w,a)a}{|a|^2})}{1 - \langle z, a \rangle}, \ w \in B.$$

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When $a = 0$, we define $\phi_a = -\text{Id}_B$. We should observe that, $\phi_a(0) = a$ and $\phi_a \circ \phi_a = \text{Id}_B$.

In the case when we treat $\mathbb{C}^n$ as the real $2n-$dimensional space $\mathbb{R}^{2n}$, the real Jacobian of $\phi$ is given by

$$(J_R \phi_a)(w) = \left( \frac{1 - |a|^2}{|1 - \langle a, w \rangle|^2} \right)^{n+1}.$$ 

We are going to use the following identities ($a \in \mathbb{B}$)

$$(1) \quad 1 - |\phi_a(w)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle a, w \rangle|^2}, \quad z \in \mathbb{B}$$

and

$$(2) \quad 1 - \langle \phi_a(z), \phi_a(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}, \quad z, w \in \mathbb{B}.$$ 

Traditionally, $H(\mathbb{B})$ denotes the space of all holomorphic functions on $\mathbb{B}$ and the space of all bounded holomorphic functions is denoted by $H^\infty(\mathbb{B})$.

The complex gradient of holomorphic function $f \in H(\mathbb{B})$ is defined as

$$\nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), ..., \frac{\partial f}{\partial z_n}(z) \right), \quad z \in \mathbb{B}.$$ 

The Bloch space $\mathcal{B}$ contains all holomorphic functions in $\mathbb{B}$ with finite semi-norm

$$\|f\|_\beta = \sup_{z \in \mathbb{B}} (1 - |z|^2) |\nabla f(z)|.$$ 

We can obtain proper norm by adding $|f(0)|$, i.e.

$$\|f\|_\mathbb{B} = |f(0)| + \|f\|_\beta.$$ 

The Bloch space is the Banach space in above norm. More information about the Bloch space reader can find in [8].

The Bergman projection operator $P_\alpha$ ($\alpha > -1$) is a central mapping in the study of analytic function spaces and it is defined as follows:

$$P_\alpha f(z) = \int_\mathbb{B} K_\alpha(z, w)f(w)dv_\alpha(w), \quad f \in L^p(\mathbb{B}, dv_\alpha),$$

where $L^p(\mathbb{B}, dv_\alpha)$ is the Lebesgue space of all measurable functions on $\mathbb{B}$ in which modulus with exponent $p$ ($1 \leq p < \infty$) is integrable on $\mathbb{B}$ with respect to the measure $dv_\alpha$. The case $p = \infty$ corresponds to the space of essentially bounded functions in the unit ball. Here

$$K_\alpha(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}$$

is the weighted Bergman kernel. Concerning the Bergman projection, the following two questions are of the main interest for research: establishing the boundedness and determining the exact norm.

Here we want to point out that the Bergman projection $P_\alpha : L^\infty(\mathbb{B}) \to \mathcal{B}$ is bounded and onto (see [8]).

In the case when $n = 1$ for the semi-norm $\|f\| = \sup_{|z| \leq 1} (1 - |z|^2)|f'(z)|$, Perälä (see [9]) determined the norm of the Bergman projection. He obtained that $\|P\| = \sup_{\|f\| \leq 1} \|Pf\| = \frac{\alpha}{2}$. A generalization of this result in the unit ball $\mathbb{B} \subset \mathbb{C}^n$ was done by Kalaj and Marković (see [3]), where it is shown that $\|P\| = \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+2)}$. 

Later, in work of Perälä (see [7]), the author completed his earlier result from [6] and its generalization in [3] by finding the norm of the Bergman projection w.r.t. to the proper norm of the Bloch space. We remark that calculating the exact norm of Bergman projection $P$ on $L^p$ spaces $1 < p < \infty$ is a long-standing problem and only partial results are known, see [9, 1].

There are several ways to define norm on the Bloch space, which makes it the Banach space. To this end, let us recall a definition of the Besov space $B^p_r$ in unit ball $B \subset \mathbb{C}^n$ (for a reference see [8]).

The Besov space $B^p_r$ contains all holomorphic functions $f$ in $B$ such that norm

\begin{equation}
\| f \|_{B^p_r} = \sum_{|m| \leq N-1} \left| \frac{\partial^{|m|} f}{\partial z^m}(0) \right|^p + \sum_{|m| = N} \left( \int_B |(1 - |z|^2)^N \frac{\partial^N f}{\partial z^m}(z) |^p \right)^{1/p},
\end{equation}

is finite, where $N$ is a positive integer such that $pN > n$. The measure $d\tau$ is given by

$$d\tau(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}, \quad z \in B$$

and multi-index $m$, represents $n$-tuples of non-negative integers, $m = (m_1, ..., m_n)$, where $|m| = \sum_{i=1}^n m_i$.

By the semi-norm $\| \cdot \|_{B^p_r}$ in the Besov space $B^p_r$ ($0 < p < \infty$) we imply

$$\| f \|_{B^p_r}^p = \sum_{|m| = N} \left( \int_B |(1 - |z|^2)^N \frac{\partial^N f}{\partial z^m}(z) |^p \right)^{1/p}.$$

When $p = \infty$ the Besov space $B^\infty_r$ is the Bloch space, $B^\infty = B$. We want to define norm (semi-norm) on the Bloch space $B^\infty$ induced from the Besov space $B^p_r$ as $p \to \infty$.

Before we find explicit formula for the norm in the mentioned case, let us state short version of [8, Theorem 3.5].

**Theorem 1.** Suppose $N$ is a positive integer, and $f$ is holomorphic in $B$, then following conditions are equivalent:

1. $f \in B$
2. The functions
   \[(1 - |z|^2)^N \frac{\partial^N f}{\partial z^m}(z), \quad |m| = N,
   \] are bounded in $B$.

Now we prove the following lemma:

**Lemma 2.** Let $B^p_r$, $1 < p < \infty$, be the Besov space and $\| \cdot \|_p$ is the Besov norm defined by (3). Then

$$\| f \|_{B^p_r} \to \| f \|_{B^\infty}, \quad p \to \infty \quad \text{provided} \ f \in B_r \cap B \quad \text{for some} \ r \in (1, \infty),$$

where

\begin{equation}
\| f \|_{B^\infty} = \max_{|m| = N} \sup_{z \in B} (1 - |z|^2)^N \left| \frac{\partial^N f}{\partial z^m}(z) \right|. \quad (1)
\end{equation}
Proof: We will prove the lemma in more general setting. Namely, if \( \{f_k\}_{k=1}^N \) is a sequence of measurable functions on the measure space \((\Omega, \mu)\) such that 
\[
f_k \in L^r(\Omega, \mu) \cap L^\infty(\Omega, \mu), \quad k = 1, \ldots, N,
\]
for some \( r \in (1, \infty) \), then
\[
\left( \sum_{k=1}^N \|f_k\|_p^p \right)^{1/p} \to \max_{1 \leq k \leq N} \|f_k\|_{\infty}.
\]
The last relation is an easy consequence of the following relation \( \lim_{p \to \infty} \|f_k\|_p \to \|f_k\|_{\infty} \) (see e.g. [4, p. 73, Ex. 4]) and the following obvious inequalities
\[
\left( \sum_{k=1}^N \|f_k\|_p^p \right)^{1/p} \leq N^{1/p} \max_k \{\|f_k\|_p\},
\]
and
\[
\left( \sum_{k=1}^N \|f_k\|_p^p \right)^{1/p} \geq \max_k \{\|f_k\|_p\}.
\]
It follows that if \( f \in B_r \cap B \) for some \( r > 1 \), then
\[
\|f\|_B = \lim_{p \to \infty} \|f\|_{B_p} = \max \sup_{|m| = N, z \in B} \left(1 - |z|^2\right)^N \left|\frac{\partial^N f}{\partial z^m}(z)\right|.
\]

Let us notice that in the same way
\[
\left( \sum_{|m| \leq N-1} \left|\frac{\partial^{|m|} f}{\partial z^m}(0)\right|^p \right)^{1/p} \to \max_{|m| \leq N-1} \left|\frac{\partial^{|m|} f}{\partial z^m}(0)\right|, \quad p \to \infty.
\]
Thus, we define the proper norm \( \| \cdot \|_B \) on the Bloch space as follows
\begin{equation}
\|f\|_B = \max_{|m| \leq N-1} \left|\frac{\partial^{|m|} f}{\partial z^m}(0)\right| + \max \sup_{|m| = N, z \in B} \left(1 - |z|^2\right)^N \left|\frac{\partial^N f}{\partial z^m}(z)\right|, \quad f \in B, \quad N \in \mathbb{N},
\end{equation}
and semi-norm \( \| \cdot \|_{\tilde{B}} \) is defined as
\begin{equation}
\|f\|_{\tilde{B}} = \max \sup_{|m| = N, z \in B} \left(1 - |z|^2\right)^N \left|\frac{\partial^N f}{\partial z^m}(z)\right|, \quad f \in B, \quad N \in \mathbb{N}.
\end{equation}

Although in definition (3) of the norm \( \| \cdot \|_p \) for the Besov space \( B_p \), we have condition \( pN > n \), by the formula (4) we can define \( \| \cdot \|_B \) on \( B \) for any \( N \). This is not surprising because \( \infty \cdot N > n \).

The proof of the next lemma is straightforward and we omit the proof.

Lemma 3. The Bloch space \( B \) is a Banach space in the norm (4)

In the sequel, \( \tilde{B} \)-norm and \( B \)-norm of the Bergman projection \( P_\alpha : L^\infty \to B \) are
\begin{equation}
\|P_\alpha\|_{\tilde{B}} = \sup_{\|g\|_{\infty} \leq 1} \|P_\alpha g\|_{\tilde{B}},
\end{equation}
and
\begin{equation}
\|P_\alpha\|_B = \sup_{\|g\|_{\infty} \leq 1} \|P_\alpha g\|_B,
\end{equation}

where \( \tilde{B} \) and \( B \) are defined in (4) and (5).
Thus, we have
\( \|P_\alpha\|_B = \frac{\Gamma(n + N + \alpha + 1)\Gamma(N)}{\Gamma^2(\frac{N}{2} + \frac{n + \alpha + 1}{2})} \).

**Theorem 5.** Let \( P_\alpha \) be the Bergman projection \( P_\alpha : L^\infty(\mathbb{B}) \to \mathcal{B} \), where \( \mathcal{B} \) is Bloch space in norm (4). Then
\[
\|P_\alpha\|_B = \frac{\Gamma(n + N + \alpha + 1)\Gamma(\frac{N}{2})}{\Gamma^2(\frac{N}{2} + \frac{n + \alpha + 1}{2})}, \quad N \in \mathbb{N}.
\]

Let us notice that when \( N = 1 \) for the \( \tilde{B} \)-norm of the Bergman projection we have \( \|P_\alpha\|_B = \frac{\Gamma(n + \alpha + 2)}{\Gamma(\frac{(n + \alpha + 2)^2}{2})} \), and this is one of the main results in \([3]\). For the special case \( n = 1 \), we obtain \( \|P\|_B = \frac{2}{\pi} \), which coincides with the main result of Perälä in \([6]\).

## 2. Proof of Theorem 4 and Theorem 5

Before we start to prove Theorem 4, let us state \([3\text{, Lemma 3.3}]\).

**Lemma 6.** For a multi-index \( m = (m_1, ..., m_n) \in \mathbb{N}_0^n \) we have
\[
\int_{\mathbb{B}} |z^m|d\sigma(z) = \frac{(n-1)!\prod_{i=1}^n \Gamma(1 + \frac{m_i}{2})}{\Gamma(n + \frac{|m|}{2})},
\]
and
\[
\int_{\mathbb{B}} |z^m|d\nu_m(z) = \frac{\Gamma(1 + \alpha + n)}{\Gamma(1 + \alpha + n + \frac{|m|}{2})} \prod_{i=1}^n \Gamma(1 + \frac{m_i}{2}).
\]
Here \( w^m := \prod_{i=1}^n w_i^{m_i} \), and \( |m| = \sum_{i=1}^n m_i \).

**Proof:** Let \( P \) be the Bergman projection, \( P : L^\infty(\mathbb{B}) \to \mathcal{B} \). Since \( P \) is onto, for any \( f \in \mathcal{B} \) there is \( g \in L^\infty(\mathbb{B}) \) such that \( f = Pg \), i.e.
\[
f(z) = \int_{\mathbb{B}} \frac{g(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}d\nu_m(w), \quad z \in \mathbb{B}.
\]

Differenting under integral sign in (10) we obtain
\[
\|P_\alpha g\|_B = \max_{|m|=N} \sup_{z \in \mathbb{B}} (1 - |z|^2)^N \left| \frac{\partial^N f(z)}{\partial z^m} \right|
\]
\[
\leq \|g\|_\infty \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m|=N} \sup_{z \in B_n} (1 - |z|^2)^N \int_{B_n} \frac{|h_m(w)|}{|1 - \langle z, w \rangle|^{n+1+N+\alpha}}d\nu_m(w).
\]
Thus, we have
\[
\|P_\alpha\|_B \leq \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m|=N} \sup_{z \in \mathbb{B}} (1 - |z|^2)^N \int_{B_n} \frac{|h_m(w)|}{|1 - \langle z, w \rangle|^{n+1+N+\alpha}}d\nu_m(w),
\]
where \( h_m(\bar{w}) = \bar{w}^m = (\bar{w}_1)^m \cdots (\bar{w}_n)^m, \sum m_n = N \).

For a fixed \( z \in \mathbb{B} \) let us make the change of variable \( w = \varphi_z(\omega) \). By using the following formula for the real Jacobian

\[
(J_R \varphi_z)(\omega) = \left( \frac{1 - |z|^2}{1 - \langle z, \omega \rangle^2} \right)^{n+1},
\]

and identity (1) we obtain

\[
(13)
\]

\[
dv_\alpha(w) = c_\alpha(1 - |w|^2)^\alpha dv(w) = c_\alpha \left( \frac{1 - |z|^2}{1 - \langle z, \omega \rangle^2} \right)^{n+1} (1 - |\omega|^2)^\alpha (1 - |z|^2)^n \left| \frac{h_m(\varphi_z(\omega))}{1 - \langle z, \omega \rangle^{n+N+\alpha+1}} \right| dv_\alpha(\omega).
\]

By plugging (13) in (12) we obtain

\[
(14)
\]

\[
||P_\alpha||_B \leq \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} \sup_{z \in \mathbb{B}} (1 - |z|^2)^N \int_{\mathbb{B}} \frac{|h_m(w)|}{|1 - \langle z, w \rangle|^{n+N+\alpha+1}} dv_\alpha(w)
= \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} \sup_{z \in \mathbb{B}} \frac{1}{\frac{|h_m(\varphi_z(\omega))|}{|1 - \langle z, \omega \rangle|^{n-N+\alpha+1}}} dv_\alpha(\omega).
\]

Furthermore

\[
(15)
\]

\[
||P_\alpha||_B \leq \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} \sup_{z \in \mathbb{B}} \frac{1}{\frac{|h_m(\varphi_z(\omega))|}{|1 - \langle z, \omega \rangle|^{n-N+\alpha+1}}} dv_\alpha(\omega).
\]

Further, let us note that for every polynomial \( h_m, |h_m(\omega)| \leq 1 \). The maximal value is attained, for example when \( h_{(N,0,\ldots,0)}(\omega) = h_1(\omega) = \omega_1^N \) and \( \omega = e_1 \). So we conclude

\[
(16)
\]

\[
||P_\alpha||_B \leq \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \sup_{z \in \mathbb{B}} \frac{1}{|1 - \langle z, \omega \rangle|^{n-N+\alpha+1}} dv_\alpha(\omega).
\]

Our next goal is to determine maximum of the function \( m(z) \), where

\[
m(z) = \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \int_{\mathbb{B}} \frac{dv_\alpha(\omega)}{|1 - \langle z, \omega \rangle|^{n-N+\alpha+1}}, \; z \in \mathbb{B}.
\]

By using the uniform convergence, the fact that \( \langle z, \omega \rangle^{k_1} \) and \( \langle z, \omega \rangle^{k_2} \) \( (k_1, k_2 \in \mathbb{N}, k_1 \neq k_2) \) are orthogonal in \( L^2(\mathbb{B}, dv_\alpha(\omega)) \), and polar coordinates we obtain the following sequence of equalities

\[
(17)
\]

\[
m(z) = \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \int_{\mathbb{B}} \frac{dv_\alpha(\omega)}{|1 - \langle \zeta, \omega \rangle|^{n-N+\alpha+1}} dv_\alpha(\omega)
= \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \sum_{k = 0}^{\infty} \frac{\Gamma(k + \lambda)}{k! \Gamma(\lambda)} \int_{\mathbb{B}} |\langle z, \omega \rangle|^{2k} dv_\alpha(\omega)
= \frac{2n \Gamma(n + N + \alpha + 1)}{n! \Gamma(\alpha + 1)} \sum_{k = 0}^{\infty} \frac{\Gamma(k + \lambda)}{k! \Gamma(\lambda)} \int_{\mathbb{B}} \int_0^1 r^{2n+2k-1} (1 - r^2)^n dr \int_S |\langle z, \xi \rangle|^{2k} d\sigma(\xi)
\]

where \( \lambda = \frac{n-N+\alpha+1}{2} \) and \( \omega = r\xi, |\xi| = 1 \).
The unitary matrix $U$, $U\xi = \xi' = (\xi'_1, \ldots, \xi'_N) = \frac{(\xi, x)}{|x|}$ constructed in Zhu (see [8, P. 15]) applied on the last surface integral gives

$$m(z) = \frac{\Gamma(n + N + \alpha + 1)}{n!\Gamma(\alpha + 1)} \prod_{k=0}^{\infty} \frac{\Gamma(k + \lambda)}{k!\Gamma(\lambda)} |z|^{2k} = \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} 2F_1(\lambda; \alpha; n + \alpha + 1, |z|^2),$$

where $2F_1(\lambda; \alpha; n + \alpha + 1, |z|^2)$ is the hypergeometric function, i.e. in general

$$2F_1(a; b; c, x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} x^n$$

where $(a)_n = a(a+1) \cdots (a+n-1)$ is the Pochhammer symbol (see [2]).

By using the formula

$$\frac{d}{dx} 2F_1(a, b; c; x) = \frac{ab}{c} 2F_1(a+1, b+1; c+1; x),$$

we conclude that the maximum of $2F_1(\lambda; \alpha; n + \alpha + 1, |z|^2)$ is $2F_1(\lambda; \alpha; n + \alpha + 1, 1)$. So

$$\sup_{z \in \mathbb{B}} m(z) = \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} 2F_1(\lambda; \alpha; n + \alpha + 1, 1) = \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \frac{\Gamma(\alpha + 1)\Gamma(N)}{\Gamma(\lambda + \alpha + 1)} \frac{\Gamma(\frac{N}{2} + \frac{n + \alpha + 1}{2})}{\Gamma(\frac{\lambda}{2} + \frac{n + \alpha + 1}{2})},$$

i.e.

$$\|P_\alpha\|_B \leq \frac{\Gamma(n + N + \alpha + 1)\Gamma(N)}{\Gamma(\frac{N}{2} + \frac{n + \alpha + 1}{2})}, \quad N \in \mathbb{N}$$

In the relation (20) we used the Gauss identity for hypergeometric functions. Namely, for Re$(c - a - b) > 0$, we have

$$2F_1(a; b; c, 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$ 

Let us prove the opposite inequality. Since the function $|h_m(\omega)|$ is subharmonic in $\mathbb{B}$, there exists $\zeta_0 \in S$ such that

$$\max_{|\zeta|=1} |h_m(\zeta)| = |h_m(\zeta_0)|,$$

As we already pointed out if $h_k(\omega) = \omega^N_k$ and $\zeta_0 = e_k$, $(h_k(\omega) = h_{(0, \ldots, N, \ldots, 0)}(w))$, then $|h_k(\zeta_0)| = 1$. We fix $z_r = r\zeta_0$, and the function $g_{z_r}(w) = \frac{(1 - (z_r, w))^{n + N + \alpha + 1}}{|1 - (z_r, w)|^{n + N + \alpha + 1}}$. It
is clear that \( \| g_z \|_\infty = 1 \). Then

\[
\| P_\alpha g_z \|_B = \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} \sup_{z \in B} (1 - |z|^2)^N \left| \int_B \frac{g_z(w) h_m(w) dv_\alpha(w)}{(1 - \langle z, w \rangle)^{n+N+\alpha+1}} \right|
\]

By using the change of variable, \( w \to \varphi_z(w) \), as in the previous case we have

\[
\| P_\alpha g_z \|_B \geq \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} \left| \int_B \frac{h_m(\varphi_z(w)) dv_\alpha(w)}{(1 - \langle z, w \rangle)^{n-N+\alpha+1}} \right|
\]

Since

\[
\left| \int_B \frac{h_m(\varphi_z(\omega)) dv_\alpha(\omega)}{(1 - \langle z, \omega \rangle)^{n-N+\alpha+1}} \right| \leq \int_B \frac{dv_\alpha(\omega)}{(1 - \langle z, \omega \rangle)^{n-N+\alpha+1}} < \infty,
\]

we can apply the Lebesgue dominated convergence theorem in order to obtain

\[
\| P_\alpha \|_B \geq \lim_{r \to 1^-} \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} \left| \int_B \frac{h_m(\varphi_z(w)) dv_\alpha(w)}{(1 - \langle z, w \rangle)^{n-N+\alpha+1}} \right| = \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} \left| \int_B \frac{h_m(w) dv_\alpha(w)}{(1 - \langle \xi, w \rangle)^{n-N+\alpha+1}} \right|
\]

We used in (24) that \( \varphi_{\alpha_0}(w) = \zeta_0 \) when \( |\alpha_0| = 1 \). Finally, from (24) we obtain

\[
\| P_\alpha \|_B \geq \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)} \frac{\| h_m(\zeta_0) \|_B}{\int_B \frac{dv_\alpha(w)}{(1 - \langle \xi, w \rangle)^{n-N+\alpha+1}}}.
\]

Now we prove Theorem 5.

**Proof:** We use the same notation as in the proof of Theorem 4. Let \( f(z) = P_\alpha(g)(z) \), \( z \in B \), where \( g \in L_\infty(B) \), \( f \in B \).

Then

\[
\| P_\alpha g \|_B = \max_{|m| \leq N-1} \left| \frac{\partial^{|m|} f}{\partial z^m}(0) \right| + \max_{|m| = N} \sup_{z \in B} (1 - |z|^2)^N \left| \frac{\partial^{|m|} f}{\partial z^m}(z) \right|
\]

\[
\leq \| g \|_\infty \max_{|m| \leq N-1} \int_B h_m(w) dv_\alpha(w) + \| g \|_\infty \| P \|_B,
\]

i.e.,

\[
\| P \|_B \leq \max_{|m| \leq N-1} \int_B h_m(w) dv_\alpha(w) + \| P \|_B.
\]

By using Lemma 6 and the polar coordinates, we obtain

\[
\| P \|_B \leq \max_{|m| \leq N-1} \frac{\Gamma(|m| + n + \alpha + 1)}{\Gamma(\frac{|m|}{2} + \alpha + n + 1)} \prod_{j=1}^n \frac{\Gamma(1 + \frac{m_j}{2})}{\Gamma(1 + \frac{m_j}{2} + \alpha + n)} + \| P \|_B
\]

\[
= \frac{\Gamma(n + N + \alpha + 1) \Gamma(\frac{1+N}{2})}{\Gamma(\frac{1+N}{2} + \alpha + n)} + \frac{\Gamma(n + N + \alpha + 1) \Gamma(N)}{\Gamma(\frac{N}{2} + \alpha + n + 1)}.
\]
In order to prove the opposite inequality we make use of the functions

\[ g_{z_r}(w) = \frac{(1 - \langle z_r, w \rangle)^{n+N+\alpha+1}}{1 - \langle z_r, w \rangle |z|^{n+N+\alpha+1}} \quad w \in \mathbb{B} \]

which we used in the proof of Theorem 4 to maximize \( \| P_\alpha f \|_g \). We define new test functions \( g^\delta_{z_r} \) with \( \|g^\delta_{z_r}\|_\infty \leq 1 \) as follows:

\[
g^\delta_{z_r}(w) = \begin{cases} 
    g_{z_r}(w), & |w| \geq \delta \\
    \frac{w^{\alpha-1}}{|w|^{\alpha-1}}, & |w| \leq \delta^2 
\end{cases}
\]

and define \( g^\delta_{z_r} \) on \( \{ \delta^2 < |w| < \delta \} \) so that \( g^\delta_{z_r} \) is continuous on \( \mathbb{B} \).

We claim that

\[
(1 - |z_r|^2)^N \max_{|m| = N} \left| \frac{\partial^N P_\alpha g_{z_r}}{z^m} (z_r) \right| \to \| P \|_\tilde{g}, \quad \text{as } r \to 1^-.
\]

Namely, it is clear by the definition of the semi-norm \( \| \cdot \|_g \) that

\[
\limsup_{r \to 1^-} (1 - |z_r|^2)^N \max_{|m| = N} \left| \frac{\partial^N P_\alpha g_{z_r}}{z^m} (z_r) \right| \leq \| P \|_\tilde{g}.
\]

Also, we have shown in the proof of Theorem 4 that

\[
\lim_{r \to 1^-} (1 - |z_r|^2)^N \left| \frac{\partial^N P_\alpha g_{z_r}}{z^m} (z_r) \right| = \| P_\alpha \| \tilde{g}.
\]

Since \( |g_{z_r}(w) - g^\delta_{z_r}(w)| \leq 2 \) on \( \mathbb{B} \) and \( |g_{z_r}(w) - g^\delta_{z_r}(w)| = 0 \) when \( |w| > \delta \), we have

\[
(1 - |z_r|^2)^N \max_{|m| = N} \left| \frac{\partial^N P_\alpha g_{z_r}}{z^m} (z_r) - \frac{\partial^N P_\alpha g^\delta_{z_r}}{z^m} (z_r) \right|
\]

\[
= (1 - |z_r|^2)^N \max_{|m| = N} \left| \frac{\partial^N P_\alpha (g_{z_r} - g^\delta_{z_r})}{z^m} (z_r) \right|
\]

\[
\leq \frac{\Gamma(n + N + \alpha + 1)}{\Gamma(n + \alpha + 1)} \int_{|w| < \delta} \frac{2(1 - |z_r|^2)^N dv_\alpha (w)}{|1 - \langle z_r, w \rangle |n+N+\alpha+1|}.
\]

The right hand side in (28) goes to 0 as \( r \to 1^- \).

Thus

\[
\lim_{r \to 1^-} (1 - |z_r|^2)^N \max_{|m| = N} \left| \frac{\partial^N P_\alpha g^\delta_{z_r}}{z^m} (z_r) \right|
\]

\[
= \left| \frac{\partial^{N-1} P^\delta_{z_r}}{z^{N-1}} \right| (0) \geq \int_{|w| \leq \delta^2} |w|^{N-1}dv_\alpha (w) - \int_{|w| > \delta^2} dv_\alpha
\]

\[
\to \int_{\mathbb{B}} |w|^{N-1}dv_\alpha (w) = \frac{\Gamma(n + N + \alpha + 1) \Gamma(\frac{1+N}{2} + \alpha + n)}{\Gamma(\frac{1+N}{2} + \alpha + n)}.
\]
as $\delta \to 1^-$. It is clear that in (30) we might observe any partial derivative $\frac{\partial^{N-1} P(g_{z_r}^\delta)}{z_r^{N-1}}(0)$, where $k = 1, \ldots, n$.

For given $\epsilon > 0$, we may pick $\delta > 0$ such that

$$\left| \frac{\partial^{N-1} P g_{z_r}^\delta}{z_r^{N-1}}(0) \right| > \frac{\Gamma(n + N + \alpha + 1) \Gamma\left(\frac{1+N}{2}\right)}{\Gamma\left(\frac{1+N}{2} + \alpha + n\right)} - \frac{\epsilon}{2}$$

for every $r \in (0, 1)$. We fix such $\delta$. According to the relation (29), one can pick $r \in (0, 1)$ such that

$$(1 - |z_r|^2)^N \max_{|m| = N} \left| \frac{\partial^N P_\alpha g_{z_r}^\delta}{z^m}(z_r) \right| > \|P_\alpha\|_B - \frac{\epsilon}{2}.$$ 

Then, we can end up with a function $g_{z_r}^\delta$ such that

$$\|P_\alpha\|_B \geq \|P_\alpha g_{z_r}^\delta\|_B \geq \frac{\Gamma(n + N + \alpha + 1) \Gamma\left(\frac{1+N}{2}\right)}{\Gamma\left(\frac{1+N}{2} + \alpha + n\right)} + \|P_\alpha\|_B - \epsilon.$$

Therefore, $\|P_\alpha\|_B \geq \frac{\Gamma(n + N + \alpha + 1) \Gamma\left(\frac{1+N}{2}\right)}{\Gamma\left(\frac{1+N}{2} + \alpha + n\right)} + \|P_\alpha\|_B$, and combining with relation (27) proves the theorem.

**Remark 7.** If $P_\alpha$ is Bergman projection, $P_\alpha : L^\infty(\mathbb{B}) \rightarrow \mathcal{B}$, where $\mathcal{B}$ is Bloch space in the norm (4), then it is easy to find the lower estimate for the $\mathcal{B}$-norm of $P_\alpha$ i.e.

$$\|P_\alpha\|_B \geq \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)}.$$

Namely, we fix $z_0 \in \mathbb{B}$ and we make use of the function $g_{z_0}(w) = \frac{(1 - \langle w, z_0 \rangle)^N}{(1 - \langle w, w \rangle)^N}$. It is clear that $g_{z_0} \in L^\infty$, $\|g_{z_0}\| = 1$.

Hence

$$\|P_\alpha g_{z_0}\|_B = \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} \int_{\mathbb{B}} \frac{g_{z_0}(w)h_m(w)}{(1 - \langle z, w \rangle)^{n+\alpha+1}} \, dv_\alpha(w)$$

$$\geq \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} \int_{\mathbb{B}} \frac{h_m(w)}{(1 - \langle z_0, w \rangle)^{n+\alpha+1}} \, dv_\alpha(w).$$

On the other hand, it holds $\frac{h_m(w)}{(1 - \langle w, z_0 \rangle)^N} \in H^\infty(\mathbb{B})$, and this implies

$$\|P_\alpha\|_B \geq \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)} \max_{|m| = N} |h_m(z)|$$

$$= \frac{\Gamma(N + n + \alpha + 1)}{\Gamma(n + \alpha + 1)}.$$ 

Remark 8. We want to emphasize that on the Bloch space $B$ we may observe the norm

$$\|f\|_B^p = \sum_{|m| \leq N-1} |\frac{\partial^{|m|} f}{\partial z^m}(0)| + \sup_{z \in B} (1 - |z|^2)^N \left( \sum_{|m| = N} |\frac{\partial^N f}{\partial z^m}(z)|^p \right)^{\frac{1}{p}},$$

and the semi-norm

$$\|f\|_{B^p} = \sup_{z \in B} (1 - |z|^2)^N \left( \sum_{|m| = N} |\frac{\partial^N f}{\partial z^m}(z)|^p \right)^{\frac{1}{p}},$$

where $f \in B$, $N \in \mathbb{N}$, $1 \leq p < \infty$.

By the same argument as in the proof of Theorem 4 it can be shown that

$$\|P_\alpha\|_{B^p} \leq \frac{\Gamma(n + N + \alpha + 1)\Gamma(N)}{\Gamma^2\left(\frac{N}{2} + \frac{n + \alpha + 1}{2}\right)} \left( \sum_{|m| = N} \max_{|\zeta| = 1} |h_m(\zeta)|^p \right)^{\frac{1}{p}}.$$ 

In particular, when $p = 2$ and $N = 1$ we have

$$\|f\|_{B_2} = \sup_{z \in B} (1 - |z|^2)^N \left( \sum_{|m| = 1} \left| \frac{\partial f}{\partial z_1}(z) \right|^2 \right)^{\frac{1}{2}}$$

and

$$\|P_\alpha\|_{B_2} \leq \frac{\Gamma(n + N + \alpha + 1)\Gamma(N)}{\Gamma^2\left(\frac{N}{2} + \frac{n + \alpha + 1}{2}\right)} \left( \sum_{|m| = N} \max_{|\zeta| = 1} |h_m(\zeta)|^2 \right)^{\frac{1}{2}} \leq \frac{\Gamma(n + N + \alpha + 1)\Gamma(N)}{\Gamma^2\left(\frac{N}{2} + \frac{n + \alpha + 1}{2}\right)} \sqrt{n}.$$ 

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