REGULARITY OF THE BOUNDARY OF THE TRAPPED REGION IN ASYMPTOTICALLY EUCLIDEAN RIEMANNIAN MANIFOLDS OF ARBITRARILY LARGE DIMENSIONS

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Abstract. We prove that the boundary of the trapped region in an asymptotically Euclidean Riemannian manifold of dimension at least 3 is a stable smooth minimal hypersurface except for a singular set of codimension at least 8.

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1. INTRODUCTION

Asymptotically Euclidean initial data slices for Einstein’s equation are used to model isolated black holes. In this context it is interesting to know which points are inside the black hole, and one way of thinking about black holes in an initial data setting is to consider the “trapped region”. The conventional way of defining the trapped region in 3-dimensional initial data slices is by using the concept of “outer expansion” to define “weakly outer trapped surfaces”. For details, see [AM09] [Eic10] [AEM11]. The outer expansion of a smooth hypersurface is an analog of the mean curvature in the purely Riemannian setting, and it is defined by modifying the mean curvature by a term from the second fundamental form of the initial data slice. A compact hypersurface is weakly outer trapped if its outer expansion is everywhere nonpositive. The trapped region is then the union of all sets which are enclosed by weakly outer trapped surfaces. From this definition, there is no immediate reason that the boundary of the trapped region, which can be interpreted as an initial-data version of the boundary of the black hole, should have any particular regularity. However, it was proved in work by Andersson, Eichmair, and Metzger [AM09] [Eic10] [AEM11] under the condition that the initial data slice has dimension $3 \leq n \leq 7$, that the boundary of the trapped region is a smooth hypersurface, that it is “marginally outer
trapped" in the sense that its outer expansion is everywhere zero, and that it is stable. A natural question is what can be said about the regularity of the boundary of the trapped region in higher dimensions. The expected answer to this question is that the boundary of the trapped region is marginally trapped, stable, and smooth outside of a singular set of codimension at least 8. This expectation is supported by an analogy with the theory of minimal hypersurfaces. In the case when the second fundamental form of the initial data slice is identically zero, the initial data slice is simply an asymptotically Euclidean Riemannian manifold, and the outer expansion is simply the mean curvature. In this case, the results in dimensions less than 8 say that the boundary of the trapped region is the outermost minimal hypersurface, and that it is stable. It is known from work by Schoen and Simon \[SS81\] that stable minimal hypersurfaces are smooth outside of a singular set of codimension at least 8. One might then hope that this regularity holds for the boundary of the trapped region in all dimensions.

The result of this paper is that this expected regularity indeed holds in all dimensions in the case when the second fundamental form of the initial data slice is identically zero. We expect that this result is known to experts, but we have not been able to find a proof in the literature.

2. Preliminaries

There are several ways of working with nonsmooth analogues of minimal hypersurfaces. We have chosen to primarily use sets of locally finite perimeter, for which \[Giu84\] is a good reference. The minimal hypersurfaces we are interested in are boundaries, and the approach using sets of locally finite perimeter reflects the fact that the sets, and not only their boundaries, are of interest.

2.1. Sets of locally finite perimeter.

Definition 2.1. A subset $E$ of a Riemannian manifold $(M, g)$ with boundary is a set of locally finite perimeter if for every open set $\Omega \subseteq M$ with compact closure it holds that $P(E, \Omega) < \infty$, where

$$P(E, \Omega) = \sup_{X \in \mathcal{X}_0^{C^1}(\Omega), ||X||_g \leq 1} \int_E \text{div } X \, d\mathcal{H}^n.$$  

Here $\mathcal{X}_0^{C^1}(\Omega)$ denotes the set of $C^1$ vector fields which are compactly supported in $\Omega$, and $\mathcal{H}^n$ denotes $n$-dimensional Hausdorff measure on $(M, g)$. The quantity $P(E, \Omega)$ is called the perimeter of $E$ in $\Omega$.

Following \[Giu84, Definition 3.3\], we use $\partial^* E$ to denote the reduced boundary of a set $E$ of locally finite perimeter. Note that a set has locally finite perimeter if and only if its image under any coordinate chart has locally finite perimeter in the Euclidean metric on the coordinate chart. This means
that the compactness theorem for sets of locally finite perimeter in Euclidean space \cite[Theorem 1.19]{Giu84} carries over to Riemannian manifolds.

**Lemma 2.2.** Let \((E_i)_{i=1}^{\infty}\) be a sequence of sets of locally finite perimeter in a Riemannian manifold, and suppose that their perimeters are uniformly bounded. Then there is a subsequence which converges (in \(L^1_{\text{loc}}\)-norm for the indicator functions) to a set of locally finite perimeter.

Similarly, the theorem about semicontinuity of perimeter \cite[Theorem 1.9]{Giu84} also carries over.

**Lemma 2.3.** Let \((E_i)_{i=1}^{\infty}\) be a sequence of sets of locally finite perimeter in a Riemannian manifold which converge in \(L^1\)-norm for the indicator functions to a set \(E\) of locally finite perimeter. Then it holds for every open set \(\Omega\) that

\[
P(E, \Omega) \leq \liminf_{i \to \infty} P(E_i, \Omega).
\]

2.2. Semihorizon domains.

**Definition 2.4.** Let \(M\) be a smooth manifold and let \(S\) be a subset of \(M\). We define \(\text{reg} S\) to be the set of points \(x \in S\) such that there is an open neighborhood \(U \subset M\) of \(x\) such that \(U \cap \overline{S}\) is a connected \(C^2\) hypersurface without boundary. We define \(\text{sing} S = \overline{S} \setminus \text{reg} S\). Note that \(\text{sing} S\) is a closed set.

**Definition 2.5.** Let \((M, g)\) be an asymptotically Euclidean Riemannian manifold. A set \(E \subseteq M\) of locally finite perimeter is a \textit{bounded domain} (with respect to the chosen asymptotically Euclidean end) if

- \(E\) is open,
- the complement of \(E\) is a neighborhood of the chosen asymptotically Euclidean end,
- after compactifying the chosen asymptotically Euclidean end, the complement of \(E\) is compact,
- \(\partial E = \partial^* E\).

**Definition 2.6.** Let \((M, g)\) be an asymptotically Euclidean Riemannian manifold. A bounded domain \(E \subseteq M\) is \textit{outer area minimizing} (with respect to the chosen asymptotically Euclidean end) if \(P(E, M) < \infty\) and there is no bounded domain \(E' \supset E\) such that \(P(E', M) < P(E, M)\).

**Definition 2.7.** Let \((M, g)\) be an \(n\)-dimensional asymptotically Euclidean Riemannian manifold. A bounded domain \(E \subseteq M\) is a \textit{semihorizon domain} (with respect to the chosen asymptotically Euclidean end) if it is outer area minimizing, \(\mathcal{H}^{n-3}(\partial E \setminus \partial^* E) = 0\), and \(\partial^* E\) is a smooth minimal hypersurface.

**Definition 2.8.** Let \((M, g)\) be an asymptotically Euclidean Riemannian manifold. The \textit{trapped region} of \((M, g)\) is the union of all semihorizon domains.
Proposition 2.9. Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 3$. Let $S \subseteq M$ be a set which satisfies $H^{n-1}(\text{sing } S) = 0$. Suppose that there is a constant $\omega$ such that $H^{n-1}(S \cap B(x, r)) < \omega r^{n-1}$ for all $x \in M$ and $r > 0$, where $B(x, r)$ is the $n$-dimensional open ball of radius $r$ around $x$ in $M$. If $S$ is stationary with respect to all variations which are compactly supported outside of a compact set $A$ with $H^{n-2}(A) = 0$, then $S$ is stationary with respect to all compactly supported variations.

Proof. Since $H^{n-1}(\text{sing } S) = 0$, it holds that $S$ is stationary if and only if $\text{reg } S$ is stationary. We may without loss of generality assume that $S = \text{reg } S$, and we will do so for notational convenience. We need to prove that

$$\int_S \text{div}_{\text{reg } S}(X) dH^{n-1} = 0$$

for all compactly supported $C^1$ vector fields $X$ on $M$, where $\text{div}_{S}(X)$ is the divergence of $X$ along the hypersurface $S$. Fix such a vector field $X$.

Since $S$ is stationary with respect to variations which are compactly supported in $M \setminus A$, it holds that

$$\int_S \text{div}_{S}(\eta X) dH^{n-1} = 0$$

if $\eta$ is a smooth function which is zero on a neighborhood of $A$. Let $\pi$ denote the orthogonal projection of $TM$ onto $TS$. Since

$$\text{div}_{S}(\eta X) = \eta \text{div}_{S}(X) + \pi(X)(\eta)$$

we have

$$\left| \int_S \eta \text{div}_{S}(X) dH^{n-1} \right| = \left| \int_S \eta X dH^{n-1} - \int_S \pi(X)(\eta) dH^{n-1} \right|$$

$$= \left| \int_S \pi(X)(\eta) dH^{n-1} \right|$$

$$\leq \left( \sup_M |X| \right) \int_S |\eta| dH^{n-1}.$$  

We will now construct a family of functions $(\eta_\epsilon)_{\epsilon > 0}$ such that

$$\lim_{\epsilon \to 0} \int_S \eta_\epsilon \text{div}_{S}(X) dH^{n-1} = \int_S \text{div}_{S}(X) dH^{n-1}$$

and

$$\int_S |d\eta_\epsilon| dH^{n-1} < \epsilon,$$

thereby proving that

$$\int_S \text{div}_{S}(X) dH^{n-1} = 0.$$
Since $\mathcal{H}^{n-2}(A) = 0$, there is for every $\epsilon > 0$ a cover of $A$ by open balls

$$A \subseteq \bigcup_{i=1}^{\infty} B(x_{\epsilon,i}, r_{\epsilon,i})$$

where

$$\sum_{i=1}^{\infty} r_{\epsilon,i}^{n-2} < \frac{\epsilon}{2^n \omega},$$

and since $A$ is compact, there is a finite subcover

$$A \subseteq \bigcup_{i=1}^{N_{\epsilon}} B(x_{\epsilon,i}, r_{\epsilon,i})$$

where

$$\sum_{i=1}^{N_{\epsilon}} r_{\epsilon,i}^{n-2} < \frac{\epsilon}{2^n \omega}.$$

For each $i$, let $\eta_{\epsilon,i}$ be a smooth function such that

$$\eta_{\epsilon,i}(x) = \begin{cases} 
1 & \text{if } x \in B(x_{\epsilon,i}, r_{\epsilon,i}), \\
0 & \text{if } x \notin B(x_{\epsilon,i}, 2r_{\epsilon,i})
\end{cases}$$

and $||d\eta_{\epsilon,i}||_g < 2/r_{\epsilon,i}$ everywhere. Let

$$\eta_{\epsilon}(x) = 1 - \prod_{i=1}^{N_{\epsilon}} \eta_{\epsilon,i}(x).$$

By the dominated convergence theorem,

$$\lim_{\epsilon \to 0} \int_S \eta_{\epsilon} \text{div}_S(X) \, d\mathcal{H}^{n-1} = \int_S \text{div}_S(X) \, d\mathcal{H}^{n-1}.$$ 

It holds that

$$\int_S |d\eta_{\epsilon}| \, d\mathcal{H}^{n-1} = \int_S \sum_{i=1}^{N_{\epsilon}} \left( \prod_{j \neq i} \eta_{\epsilon,j} \right) |d\eta_{\epsilon,i}| \, d\mathcal{H}^{n-1}$$

$$\leq \sum_{i=1}^{N_{\epsilon}} \int_S \left| \prod_{j \neq i} \eta_{\epsilon,j} \right| |d\eta_{\epsilon,i}| \, d\mathcal{H}^{n-1}$$

$$\leq \sum_{i=1}^{N_{\epsilon}} \int_{|S \cap B(x_{\epsilon,i}, 2r_{\epsilon,i})|} |d\eta_{\epsilon,i}| \, d\mathcal{H}^{n-1}$$

$$= \sum_{i=1}^{N_{\epsilon}} \int_{S \cap B(x_{\epsilon,i},2r_{\epsilon,i})} |d\eta_{\epsilon,i}| \, d\mathcal{H}^{n-1}$$

$$\leq \sum_{i=1}^{N_{\epsilon}} \int_{S \cap B(x_{\epsilon,i},2r_{\epsilon,i})} \frac{2}{r_{\epsilon,i}} \, d\mathcal{H}^{n-1}$$
\[
\sum_{i=1}^{N_\varepsilon} \frac{2}{r_{\varepsilon,i}} \mathcal{H}^{n-1}(S \cap B(x_{\varepsilon,i}, 2r_{\varepsilon,i})) \leq 2^n \omega \sum_{i=1}^{N_\varepsilon} r_{\varepsilon,i}^{n-2} \leq \epsilon.
\]

Hence \( S \) is stationary with respect to all compactly supported variations. \( \square \)

The following lemma is an immediate consequence of the Schoen–Simon regularity theory [SS81] for stable minimal hypersurfaces, which we discuss briefly in Section 2.4.

**Lemma 2.10.** Let \((M, g)\) be an asymptotically Riemannian manifold of dimension \( n \). If \( E \) is a semihorizon domain, then \( \mathcal{H}^{\alpha}(\partial E \setminus \partial^* E) = 0 \) if \( \alpha > n - 8 \) and \( \alpha \geq 0 \).

**Proof.** The minimal hypersurface \( \partial^* E \) is stable since \( E \) is outer area minimizing. Moreover, \( \mathcal{H}^{n-3}(\text{sing} \partial^* E) = 0 \) since \( E \) is a semihorizon domain. Hence Theorem 2.13 is applicable for the constant sequence \( \Sigma_k = \partial^* E \). Since \( \Sigma_k \to \partial^* E \), it follows that \( \mathcal{H}^{\alpha}(\partial E \setminus \partial^* E) = 0 \) if \( \alpha > n - 8 \) and \( \alpha \geq 0 \). \( \square \)

2.3. **The Solomon–White maximum principle.** The central technical tool of this paper is the Solomon–White maximum principle [SW89, Theorem, p. 686]. The version described in [SW89, Additional remarks, pp. 690-691] (see also [Whi10, Theorem 4]) tells us the following:

**Theorem 2.11** (The Solomon–White maximum principle). Let \( U \) be a smooth Riemannian manifold with boundary (not necessarily compact or complete). Let \( T \) be a varifold in \( U \) which is stationary with respect to variations in \( U \). If \( \partial U \) has positive mean curvature with respect to the outward-directed normal, then the support of \( T \) does not intersect \( \partial U \). If \( \partial U \) is a connected minimal hypersurface, then the support of \( T \) contains \( \partial U \).

It is crucial for our application that the theorem is applicable to varifolds which are only stationary with respect to variations in \( U \) and not necessarily stationary with respect to variations in \( U \) (without boundary). See [SW89, Remark (2), p. 691] and the discussion after [Whi10, Theorem 1] for further comments on this point. We will need a slightly stronger version of the theorem, which is not explicitly stated in [SW89], but follows from the proof of [SW89, Theorem, p. 686]:

**Theorem 2.12** (A strengthened version of the Solomon–White maximum principle). Let \( U \) be a smooth Riemannian manifold with boundary (not necessarily compact or complete). Suppose that \( \partial U \) has positive mean curvature with respect to the outward-directed normal. Let \( T \) be a varifold in \( U \) and suppose that the support of \( T \) intersects \( \partial U \) at a point \( x \). Then there is a variation which decreases the area of \( T \) to first order. The variation can be
chosen to be compactly supported in any neighborhood of \( x \). Moreover, the normalized initial velocity of the variation can be made arbitrarily \( C^0 \)-close to the inward-directed unit normal vector field of \( \partial U \).

We can see that the strengthened version of the theorem holds as follows: The proof of the Solomon–White maximum principle given in [SW89] is performed in a manifold without boundary, and the role of \( \partial U \) is played by a smooth hypersurface \( M \). The first step in the proof consists of choosing a point in \( M \), passing to a neighborhood of this point, and replacing \( M \) with a hypersurface with positive mean curvature which intersects the support of \( T \) only in the interior of the chosen neighborhood. This is done by working in coordinates where \( M \) is the graph of a function \( u \), and replacing \( u \) by a function \( u_{s,\tau,\epsilon} \) with certain properties. The second step consists of constructing a vector field orthogonal to the graph of \( u_{s,\tau,\epsilon} \) and proving that a variation of \( T \) by this vector field decreases area to first order. It can be seen by tracing the proof that if \( \epsilon \) and \( s \) are sufficiently small, so that \( u_{s,\tau,\epsilon} \) is sufficiently close to \( u \), then the normalized variation vector field is \( C^0 \)-close to the inward-directed unit normal vector field of \( \partial U \). It can also be seen that we are free to choose \( \epsilon \) and \( s \) arbitrarily small without affecting the proof, since the only requirements for \( \epsilon \) and \( s \) are that they are sufficiently small compared to other quantities. In other words, we may assume that the variation vector field is close to parallel to the inward-directed unit normal vector field of \( \partial U \).

2.4. Convergence of minimal hypersurfaces. The second technical tool in this paper is the convergence theory for minimal hypersurfaces contained in the work of Schoen and Simon in [SS81]. The result we need follows easily from [SS81], but it is not explicitly stated there and we have not been able to find a proof of the exact result we need in the literature. It is stated without proof in [DLT13, Theorem 1.3], and a proof sketch can be found in [DLT16, Theorem 4.2].

**Theorem 2.13** (Schoen–Simon [SS81]). Let \((M, g)\) be a Riemannian manifold of dimension \( n \) and let \( K \subset M \) be compact. Let \((\Sigma_k)_{k=1}^\infty \) be a sequence of smooth (but not necessarily closed) nonempty stable minimal hypersurfaces in \( K \). Suppose that \( \mathcal{H}^{n-3}(\text{sing}(\Sigma_k)) = 0 \) and \( \limsup_{k \to \infty} \mathcal{H}^{n-1}(\Sigma_k) < \infty \). Then there is a subsequence \((\Sigma_{k_i})_{i=1}^\infty \) of \((\Sigma_k)_{k=1}^\infty \) and a nonempty stable minimal hypersurface \( \Sigma_\infty \subset M \) such that

- \( \Sigma_{k_i} \to \Sigma_\infty \) as varifolds,
- \( \mathcal{H}^\alpha(\text{sing}(\Sigma_\infty)) = 0 \) if \( \alpha > n - 8 \) and \( \alpha \geq 0 \),
- for every open set \( \Omega \) with compact closure \( \overline{\Omega} \subset M \setminus \text{sing}(\Sigma_\infty) \)
  - \( \mathcal{H}^{n-1}(\Sigma_\infty \cap \Omega) = \limsup_{k \to \infty} \mathcal{H}^{n-1}(\Sigma_k \cap \Omega) \), and
  - \( (\Sigma_{k_i})_{i=1}^\infty \) converges smoothly to \( \Sigma_\infty \) on \( \Omega \).
3. Proof of the main theorem

Lemma 3.1. Let \((M, g)\) be an asymptotically Euclidean manifold. Then there is a uniform bound for the perimeters of the semihorizon domains in \(M\).

Proof. The chosen asymptotically Euclidean end is foliated by spheres of positive mean curvature. By the Solomon–White maximum principle, no semihorizon domain can contain points in this foliation. Since semihorizon domains are outer area minimizing, their perimeters cannot be larger than the area of any sphere in the foliation. \(\square\)

Proposition 3.2. The union of any two semihorizon domains is contained in a semihorizon domain.

Proof. Let \(E_1\) and \(E_2\) be semihorizon domains. Let \(K\) be the region inside of a large coordinate sphere in the chosen asymptotically Euclidean end such that \(\partial K\) is a sphere of positive mean curvature and the region outside of \(K\) is foliated by spheres of positive mean curvature. Consider the set of bounded domains which are contained in \(K\) and contain \(E_1 \cup E_2\). By applying Lemma 2.2 and Lemma 2.3 to a sequence of such sets for which the perimeter converges to the infimum, we obtain a set of locally finite perimeter \(E\) which minimizes perimeter. It is now sufficient to prove that \(E\) is a semihorizon domain. By possibly replacing \(E\) with another representative of the same equivalence class we can make sure that \(\partial E = \partial^* E\). (See [Giu84, Proposition 3.1 and Theorem 4.4].) By the Solomon–White maximum principle, \(\partial E\) cannot intersect \(\partial K\), and is hence contained in the interior of \(K\). The complement of \(E\) is a neighborhood of the chosen asymptotically Euclidean end, and this complement has compact closure since \(E\) contains a bounded domain. This proves that \(E\) itself is a bounded domain. It holds that \(E\) is outer area minimizing: If some larger bounded domain had strictly smaller perimeter, then such a bounded domain \(E'\) would arise from the minimization problem defining \(E\), possibly with some larger set \(K'\) in place of \(K\). However, \(E'\) is contained in \(K\) by the Solomon–White maximum principle since the region outside of \(K\) is foliated by spheres of positive mean curvature. Hence \(E = E'\), proving that \(E\) is outer area minimizing. We now only need to prove that \(\mathcal{H}^{n-3}(\partial E \setminus \partial^* E) = 0\) and that \(\partial^* E\) is a smooth minimal hypersurface. We will prove this locally. Let \(A = A_1 \cup A_2 \cup A_3\) where

\[
A_1 = \partial E_1 \setminus \partial^* E_1, \\
A_2 = \partial E_2 \setminus \partial^* E_2, \\
A_3 \text{ is the set of points } y \in \partial^* E_1 \cap \partial^* E_2 \text{ such that } \partial^* E_1 \text{ is tangent to } \partial^* E_2 \text{ at } y, \text{ but there is no neighborhood of } y \text{ where } \partial^* E_1 \text{ and } \partial^* E_2 \text{ coincide.}
\]

We will now prove that every point in \(\partial E \setminus A\) has a neighborhood where \(\mathcal{H}^{n-3}(\partial E \setminus \partial^* E) = 0\), \(\partial^* E\) is a smooth minimal hypersurface, and \(\partial E\) is stationary with respect to variations which are compactly supported in the neighborhood. We do this in five cases:
Case I: $x \notin \partial E_1 \cup \partial E_2$

Case II: $x \in \partial^* E_1 \setminus \partial E_2$

Case III: $x \in \partial^* E_2 \setminus \partial E_1$

Case IV: $x \in \partial^* E_1 \cap \partial^* E_2$ and there is a neighborhood of $x$ where $\partial^* E_1$ and $\partial^* E_2$ coincide

Case V: $x \in \partial^* E_1 \cap \partial^* E_2$ and $\partial^* E_1$ is not tangent to $\partial^* E_2$ at $x$

Case II: $x \notin \partial E_1 \cup \partial E_2$

Pick a neighborhood $U$ of $x$ with closure disjoint from $\partial E_1 \cup \partial E_2$. In this neighborhood, $\partial E$ is area minimizing by construction of $E$, and hence it follows from the regularity theory for area minimizers (see for instance [Giu84, Theorem 8.4], [Giu84, Theorem 11.8], [Sim83, Theorem 37.7]) that $\mathcal{H}^{n-3}(\partial E \setminus \partial^* E) = 0$, that $\partial^* E$ is a smooth minimal hypersurface, and that $\partial E$ is stationary with respect to variations which are compactly supported in $U$.

Case III: $x \in \partial^* E_1 \setminus \partial E_2$

By letting $U$ be a sufficiently small neighborhood of $x$, we can ensure that $U \cap \partial E_2 = \emptyset$ and that the connected smooth hypersurface $U \cap \partial^* E_1$ separates $U$ into two components, one of which is $U \setminus (E_1 \cup E_2 \cup \partial^* E_1)$. Then it holds that $N = (U \setminus (E_1 \cup E_2)) \cup \partial^* E_1$ is a smooth manifold with boundary $\partial N = U \cap \partial E_1$. Since $E_1$ is a semihorizon domain, it holds that $\partial N$ is a minimal hypersurface. By the Solomon–White maximum principle it follows that $U \cap \partial E$ contains $U \cap \partial^* E_1$. If it is possible to shrink $U$ so that $U \cap \partial E$ actually coincides with $U \cap \partial^* E_1$, then we are done, since $U \cap \partial^* E_1$ is a smooth minimal hypersurface. If this were not possible, it would hold that $x \in U \cap \partial E \setminus \partial^* E_1$, and we can use an argument from [Whi10, Theorem 4] to obtain a contradiction: Let $W' = (U \cap \partial E) - (U \cap \partial^* E_1)$, where we view the two sets as unit density rectifiable varifolds. Since $U \cap \partial^* E_1$ is stationary and $U \cap \partial E$ minimizes area to first order in the complement of $E_1 \cup E_2$, it holds that $W'$ minimizes area to first order in the complement of $E_1 \cup E_2$. Applying the Solomon–White maximum principle to $W'$ in the manifold with boundary $N$ we see that the support of $W'$ contains $U \cap \partial^* E_1$, which is a contradiction by definition of $W'$. Hence we may shrink $U$ so that $U \cap \partial E = U \cap \partial^* E_1$, proving that $U \cap \partial E = U \cap \partial^* E$ is a smooth minimal hypersurface.

Case III: $x \in \partial^* E_2 \setminus \partial E_1$

This case is analogous to Case II.

Case IV: $x \in \partial^* E_1 \cap \partial^* E_2$ and there is a neighborhood of $x$ where $\partial^* E_1$ and $\partial^* E_2$ coincide

This case is analogous to Case II and Case III.

Case V: $x \in \partial^* E_1 \cap \partial^* E_2$ and $\partial^* E_1$ is not tangent to $\partial^* E_2$ at $x$
We will prove that this case holds vacuously. Suppose for contradiction that \( x \in \partial E \cap (\partial^* E_1 \cap \partial^* E_2) \) and that \( \partial^* E_1 \) is not tangent to \( \partial^* E_2 \) at \( x \). Let \( U \) be a neighborhood of \( x \) such that it holds for \( i \in \{1, 2\} \) that \( U \cap \partial^* E_i \) is connected, diffeomorphic to \( \mathbb{R}^{n-1} \), and separates \( U \) into two components. Let \( \nu_i \) be the outward-directed unit normal vector field of \( U \cap \partial^* E_i \). Let \((\nu_s)_x\) be the unit vector in direction \((\nu_1)_x + (\nu_2)_x\).

This is well-defined since \( \partial^* E_1 \) is not tangent to \( \partial^* E_2 \) at \( x \) so that \((\nu_1)_x + (\nu_2)_x \neq 0\). Then \( g((\nu_s)_x, (\nu_1)_x) > 0 \) for \( i \in \{1, 2\} \).

We need, in a neighborhood of \( x \), a hypersurface \( \Sigma_s \) with nonpositive mean curvature, with \( x \in \Sigma_s \subset \overline{T_1 \cup E_2} \), and with normal vector \((\nu_s)_x\) at \( x \). The intersection of \( \partial^* E_1 \) and \( \partial^* E_2 \) at \( x \) is transverse, so \( I = U \cap \partial^* E_1 \cap \partial^* E_2 \) is a smooth submanifold of codimension 2. Extend \( \nu_s \) by letting it be the unit vector field on \( I \) in direction \( \nu_1 + \nu_2 \). This is well-defined after possibly shrinking \( U \) so that \( \nu_1 + \nu_2 \neq 0 \) on \( I \). Let \( \Sigma_s \subset U \) be a hypersurface which contains \( I \), is orthogonal to \( \nu_s \) along \( I \), and has nonpositive mean curvature. This exists since the mean curvature of \( I \) with respect to the normal vector field \( \nu_s \) can be compensated by the curvature in the direction orthogonal to \( I \) and \( \nu_s \). After possibly shrinking \( U \) if necessary it holds that \( \Sigma_s \subset E_1 \cup E_2 \). Extend \( \nu_s \) to the unit normal vector field on \( \Sigma_s \).

Theorem 2.12 the strengthened version of the Solomon–White maximum principle discussed in Section 2.3 now gives a vector field \( v \), supported in \( U \), which defines a variation which strictly decreases the perimeter of \( E \). This vector field is outward-directed along \( \partial^* E_1 \) and \( \partial^* E_2 \) in some neighborhood of \( x \) since it can be chosen to be arbitrarily close to the normal vector field \( \nu_s \) of \( \Sigma_s \), which is outward-directed along \( \partial^* E_1 \) and \( \partial^* E_2 \) at \( x \). Since \( E \) minimizes perimeter outside of \( E_1 \cup E_2 \) and the variation along \( v \) decreases perimeter, we have a contradiction.

We have proved that every point in \( \partial E \setminus A \) has a neighborhood where \( H^{n-3}(\partial E \setminus \partial^* E) = 0 \), \( \partial^* E \) is a smooth minimal hypersurface, and \( \partial E \) is stationary with respect to variations which are compactly supported outside of \( A \). We will now prove that the set \( A \) is small and compact, which is sufficient for the desired properties to hold on all of \( \partial E \). Since \( E_1 \) and \( E_2 \) are semihorizon domains, it holds that \( H^{n-3}(A_1) = H^{n-3}(A_2) = 0 \). The sets \( A_1 \) and \( A_2 \) are compact since they are closed subsets of the compact sets \( \partial E_1 \) and \( \partial E_2 \). Bounding the dimension of \( A_3 \) is slightly more involved. Consider a point \( x \in A_3 \). Choose a neighborhood \( U \) of \( x \), coordinates \( x_1, x_2, \ldots, x_n \) on \( U \), and a smooth function \( u: \mathbb{R}^{n-1} \to \mathbb{R} \) such that

\[
U \cap \partial^* E_1 = \{(x_1, \ldots, x_n): x_n = 0\}
\]

\[
U \cap \partial^* E_2 = \{(x_1, \ldots, x_n): x_n = u(x_1, \ldots, x_{n-1})\}.
\]

We may choose the coordinates to be normal coordinates along \( \partial^* E_1 \), so that \( g^{nn} = 1 \) and \( g^{ni} = 0 \) for \( i \in \{1, \ldots, n-1\} \). The function \( u \) satisfies the
minimal hypersurface equation

$$\text{div} \left( \frac{\text{grad}(x_n - u)}{||\text{grad}(x_n - u)||} \right) = 0,$$

in other words

$$\text{(1)} \quad \text{div} \left( \frac{\text{grad}(x_n)}{||\text{grad}(x_n - u)||} \right) - \text{div} \left( \frac{\text{grad}(u)}{||\text{grad}(x_n - u)||} \right) = 0,$$

where $u$ and $x_n$ are viewed as functions on $U$. We will now use this equation to construct a linear partial differential equation which is also satisfied by $u$.

Concerning the first term in (1), note that

$$\text{div} \left( \frac{\text{grad}(x_n)}{||\text{grad}(x_n)||} \right) = \text{div} \left( \frac{||\text{grad}(x_n)||}{||\text{grad}(x_n - u)||} \right) \text{grad}(x_n) + \frac{1}{||\text{grad}(x_n - u)||} \text{div} \left( \frac{\text{grad}(x_n)}{||\text{grad}(x_n)||} \right) + 0.$$

The expression $\text{div} \left( \frac{\text{grad}(x_n)}{||\text{grad}(x_n)||} \right)$ gives the mean curvature of level surfaces of $x_n$, and the level surface at level $0$ is $\partial^* E_1$ which is a minimal hypersurface. Hence, for instance by the Malgrange preparation theorem,

$$\frac{1}{||\text{grad}(x_n - u)||} \text{div} \left( \frac{\text{grad}(x_n)}{||\text{grad}(x_n)||} \right) = \phi x_n$$

for some smooth function $\phi$. Evaluated at a point $(x, u(x))$, where $x = (x_1, \ldots, x_{n-1})$, this has the value

$$\phi(x, u(x))u(x).$$

The second term in (1) can be written in coordinates, where $\alpha, \beta, \gamma, \delta$ run through $\{1, \ldots, n\}$ and $i, j, c, d$ run through $\{1, \ldots, n-1\}$, as

$$\text{div} \left( \frac{\text{grad}(u)}{||\text{grad}(x_n - u)||} \right) = \frac{1}{\sqrt{\det g}} \partial_\alpha \left( \sqrt{\det g} g^{\alpha\beta} u_{,\beta} \frac{\text{det} g}{\sqrt{1 + g^{cd} u_{,c} u_{,d}}} \right)$$

$$= \frac{1}{\sqrt{\det g}} \partial_i \left( \sqrt{\det g} \frac{g^{ij} u_{,j}}{\sqrt{1 + g^{cd} u_{,c} u_{,d}}} \right)$$

$$= \frac{g^{ij}}{\sqrt{1 + g^{cd} u_{,c} u_{,d}}} u_{,ij} + \frac{1}{\sqrt{\det g}} \partial_i \left( \frac{g^{ij} \text{det} g}{\sqrt{1 + g^{cd} u_{,c} u_{,d}}} \right) u_{,j}.$$

Introduce functions $a^{ij}$ and $b^j$ defined for $x = (x_1, \ldots, x_{n-1})$ by

$$a^{ij}(x) = -\frac{g^{ij}}{\sqrt{1 + g^{cd} u_{,c} u_{,d}}}.$$
where the components of the metric are evaluated at the point \((x, u(x))\). It now holds that \(v = u\) is a solution to the linear partial differential equation

\[
a^{ij}(x)v_{,ij}(x) + b^j(x)v_{,j}(x) + \phi(x, u(x))v(x) = 0.
\]

The equation is elliptic since \(g\) is a Riemannian metric. Then it holds by [HHOHON99 Corollary 1.1] and [Bär99 Theorem 2] that the set of points \((x_1, \ldots, x_{n-1})\) such that \(u(x_1, \ldots, x_{n-1}) = 0\) and \(du(x_1, \ldots, x_{n-1}) = 0\) has Hausdorff dimension at most \(n - 3\), after possibly shrinking \(U\). Since the function \(u\) is smooth, this means that the set of points in \(U\) where \(\partial^* E_1\) is tangent to \(\partial^* E_2\) has Hausdorff dimension at most \(n - 3\). The set \(A_3\) is closed in \(M \setminus (A_1 \cup A_2)\) since it can be written as the difference between the closed set of points of tangency of \(\partial^* E_1\) and \(\partial^* E_2\), and the open set of points \(x\) where \(\partial^* E_1\) and \(\partial^* E_2\) coincide in some neighborhood of \(x\).

We have now proved that \(A\) is compact and \(\mathcal{H}^{n-2}(A) = 0\). Since \(E\) is outer area minimizing and \(\partial E\) is stationary with respect to all variations which are compactly supported outside of \(A\), it holds that \(\partial E\) satisfies the conditions of Proposition 2.9 which tells us that \(\partial E\) is stationary with respect to all compactly supported variations. Using this fact, we can prove that \(A_3\) is actually empty: Suppose for contradiction that \(x \in A_3\). Since \(\partial^* E_1\) is a minimal hypersurface, it follows from the Solomon–White maximum principle, Theorem 2.11 that \(\partial E\) contains a neighborhood of \(x\) in \(\partial^* E_1\). Analogously, \(\partial E\) contains a neighborhood of \(x\) in \(\partial^* E_2\). Since \(E \supseteq E_1 \cup E_2\), this means that \(\partial^* E_1\) and \(\partial^* E_2\) coincide in a neighborhood of \(x\), which means that \(x \notin A_3\). Hence \(A_3 = \emptyset\). We have now proved that \(\mathcal{H}^{n-3}(A) = \mathcal{H}^{n-3}(A_1 \cup A_2) = 0\). Hence \(\mathcal{H}^{n-3}(\text{sing } \partial E) = 0\), which is one of the conditions needed to apply the Schoen–Simon regularity theory for stable minimal hypersurfaces. We know from the above argument that \(\partial E\) is stationary with respect to all compactly supported variations. It is stable since it is outer area minimizing. The regularity theory from [SSS81] then tells us that \(\mathcal{H}^\alpha(\partial E \setminus \partial^* E) = 0\) if \(\alpha > n - 8\) and \(\alpha \geq 0\). This can be seen as a special case of Theorem 2.13. Finally, \(\partial^* E\) is \(C^\infty\) by standard results on the regularity of \(C^2\) solutions to elliptic partial differential equations. This proves that \(E\) is a semihorizon domain.

\[\tag*{\square}\]

**Proposition 3.3.** The union of a (possibly uncountable) chain of semihorizon domains is a semihorizon domain.

**Proof.** Consider a chain \(\{E_a\}_{a \in A}\) of semihorizon domains, indexed by the totally ordered set \(A\). Then \(\bigcup_{a \in A} E_a\) is an open cover of itself. Smooth manifolds are Lindelöf spaces, so this cover has a countable subcover \(\bigcup_{i=1}^\infty E_{a_i}\). By passing to a subsequence of \((a_i)_{i=1}^\infty\), we may assume that this countable subcover is increasing. The perimeters of the sets in the sequence are uniformly bounded by Lemma 3.1. By the convergence theory for sets of
locally finite perimeter, as stated in Lemma 2.2 and the Schoen–Simon convergence theory of stable minimal hypersurfaces described by Theorem 2.13, it follows that a subsequence \((E_a)_i\) converges to a set \(E\) of locally finite perimeter such that \(\partial^* E\) is a smooth stable minimal hypersurface with \(\mathcal{H}^\alpha(\partial E \setminus \partial^* E) = 0\) if \(\alpha > n-8\) and \(\alpha \geq 0\). Moreover, the perimeter of \(E\) does not exceed the limit inferior of the perimeters of the sets in the subsequence by Lemma 2.3. Since each \(E_a\) is outer area minimizing, it follows from this that \(E\) is outer area minimizing. This proves that \(E\) is a semihorizon domain. Since any subsequence of \((E_a)_i\) is an increasing cover of \(\bigcup_{a \in A} E_a\), it holds that \(E = \bigcup_{a \in A} E_a\).

\[\square\]

**Proposition 3.4.** There is a, necessarily unique, semihorizon domain which contains all other semihorizon domains.

**Proof.** Consider the partially ordered set of semihorizon domains ordered by inclusion. By Proposition 3.3, every chain in this partially ordered set has an upper bound. It follows from Zorn’s lemma that the set has a maximal element. Let \(E\) be such a maximal element. If \(E'\) is any semihorizon domain then it follows from Proposition 3.2 that there is a semihorizon domain \(E''\) which contains \(E \cup E'\). By maximality of \(E\), it holds that \(E'' = E\) so that \(E' \subseteq E\). Hence \(E\) contains all other semihorizon domains.

\[\square\]

**Theorem 3.5.** Let \((M, g)\) be an \(n\)-dimensional asymptotically Euclidean Riemannian manifold with nonempty trapped region. Suppose that \(n \geq 3\). Then the trapped region is a semihorizon domain and contains all other semihorizon domains. In particular, the boundary of the trapped region is a stable smooth minimal hypersurface except for a singular set of codimension at least 8.

**Proof.** Let \(E\) be the unique semihorizon domain which contains all other semihorizon domains, the existence of which was proved in Proposition 3.4. We will prove that \(\mathcal{T} = E\). Since the trapped region is the union of all semihorizon domains, it holds that \(E \subseteq \mathcal{T}\). For the reverse inclusion, let \(x \in \mathcal{T}\). Since \(x \in \mathcal{T}\), there is a semihorizon domain \(E'\) containing \(x\). Since \(E\) contains all semihorizon domains, it holds that \(E' \subseteq E\). Hence \(x \in E' \subseteq E\) proving that \(\mathcal{T} \subseteq E\). This means that \(\mathcal{T} = E\). Hence \(\partial \mathcal{T} = \partial E = \partial^* E\) is the closure of a smooth minimal hypersurface \(\partial^* E\) such that, by Lemma 2.10, \(\mathcal{H}^\alpha(\partial E \setminus \partial^* E) = 0\) if \(\alpha > n-8\) and \(\alpha \geq 0\). It is stable since \(E\) is a semihorizon domain and hence outer area minimizing. This completes the proof.

\[\square\]

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