Willmore deformations between minimal surfaces in $\mathbb{H}^{n+2}$ and $\mathbb{S}^{n+2}$

Changping Wang · Peng Wang

Received: 15 November 2021 / Accepted: 20 October 2022 / Published online: 7 December 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
In this paper we show that locally there exists a Willmore deformation between minimal surfaces in $\mathbb{S}^{n+2}$ and minimal surfaces in $\mathbb{H}^{n+2}$, i.e., there exists a smooth family of Willmore surfaces $\{y_t : \tilde{U} \to \mathbb{S}^{n+2}, t \in [0, 2\pi]\}$ such that $(y_t)|_{t=0}$ is conformally equivalent to a minimal surface in $\mathbb{S}^{n+2}$ and $(y_t)|_{t=\pi/2}$ is conformally equivalent to a minimal surface in $\mathbb{H}^{n+2}$. Here $\tilde{U}$ is a simply connected open subset of the surface $M$. For some cases the deformations are global. By the Willmore deformations of the Veronese two-sphere and its generalizations in $\mathbb{S}^4$, for any positive number $W_0 \in \mathbb{R}^+$, we construct complete minimal surfaces in $\mathbb{H}^4$ with Willmore energy being equal to $W_0$. An example of complete minimal Möbius strip in $\mathbb{H}^4$ with Willmore energy $\frac{6\sqrt{5}}{5} \approx 10.733\pi$ is also presented. We also show that all isotropic minimal surfaces in $\mathbb{S}^4$ admit Jacobi fields different from Killing fields, i.e., they are not “isolated”.

Keywords Minimal surfaces · Minimal Möbius strip · $K^C$—Dressing · Willmore energy · Willmore two-spheres

Mathematics Subject Classification 53A31 · 53A10 · 53C40 · 58E20

1 Introduction

Minimal surfaces in $\mathbb{H}^n$ are important geometric objects in geometry [3] and mathematical physics [1, 2, 25, 42] and attract many attentions from different kind of directions ([16, 17, 20, 21, 22, 23]).

CPW was partly supported by the Project 11831005 of NSFC. PW was partly supported by the Project 11971107 of NSFC. The authors are thankful to Prof. Josef Dorfmeister, Prof. Shimpei Kobayashi, Prof. Xiang Ma, Prof. Zhenxiao Xie and Prof. Nan Ye for valuable discussions.

Peng Wang
pengwang@fjnu.edu.cn; netwangpeng@hotmail.com

Changping Wang
cpwang@fjnu.edu.cn

1 School of Mathematics and Statistics, FJKLMAA, Fujian Normal University, Fuzhou 350117, People’s Republic of China
For instance, in [1] it is shown by Alexakis and Mazzeo that the renormalized area introduced by Maldacena in [42] can be expressed as the Willmore functional of minimal surfaces in $\mathbb{H}^n$. Moreover, minimal surfaces in $\mathbb{H}^n$ are special kind of Willmore surfaces, which are critical surfaces of the Willmore functional. As a consequence, in [2] Alexakis and Mazzeo discussed in details the geometry and analysis of complete Willmore surfaces in $\mathbb{H}^3$ which meet the infinity boundary $\partial_\infty \mathbb{H}^3$ orthogonally. Therefore it is natural to consider the above surfaces under the (conformal) framework of Willmore surfaces. In [20, 21], Dorfmeister and Wang started to consider the global geometry of Willmore surfaces in terms of the harmonic conformal Gauss maps and the DPW method. Such an idea was first introduced by Hélein in [30] and generalized by Xia-Shen [58]. In particular, descriptions of minimal surfaces in space forms as special Willmore surfaces are presented in [30, 55, 58].

In this paper, we continue to study minimal surfaces in $\mathbb{H}^n$ and $S^n$ along this direction. To begin with, we first recall the characterization of minimal surfaces in space forms [55] briefly. Roughly speaking, the DPW method gives a representation of Willmore surfaces in terms of some Lie-algebra-valued meromorphic 1-form called normalized potential [19–21, 30]. Then a Willmore surface being minimal in some space form is equivalent to the Lorenzian orthogonality of some (non-zero) constant real vector $v$ with some part of the normalized potential [55]. The vector $v$ being lightlike, timelike or spacelike corresponds to the space form $\mathbb{R}^{n+2}$, $S^{n+2}$ or $\mathbb{H}^{n+2}$ respectively (See [55] or Theorem 2.5 of Sect. 2; Compare also [30, 58] for a slightly different treatment, where a different harmonic map introduced by [30] is used).

A key observation due to this paper is that the Lorenzian orthogonality is preserved by some complex group action, while the minimality in space forms could be changed. This makes it possible to deform minimal surfaces in $S^{n+2}$ into non-minimal Willmore surfaces and furthermore into minimal surfaces in $\mathbb{H}^{n+2}$ or conversely:

**Theorem 1.1** (See Theorem 4.2 for a full version) Let $y : U \to S^{n+2}$ be a minimal surface from a simply connected open subset $U \subset M$. There exists a family of Willmore surfaces $y_t : \tilde{U} \subset U \to S^{n+2}$, $t \in [0, 2\pi)$, such that $y_t|_{t=0} = y$, and the surfaces $y_t|_{t=\pi/2}$ and $y_t|_{t=3\pi/2}$ are conformally equivalent to some minimal surfaces in $\mathbb{H}^{n+2}$. Here $\tilde{U}$ is an open subset of $U$.

Such a phenomenon is new to the authors’ best knowledge. Note that in [10, 13], dressing actions of Willmore surfaces are discussed. But they are slightly different from the actions discussed here since we just use elements in the complexified subgroup $K^C$. For a general discussion of dressing actions, we refer to [26, 51, 52].

One of the simplest minimal surfaces in $S^{n+2}$ is the Veronese two-sphere in $S^4$. We show explicitly the Willmore deformations for the Veronese two-sphere in $S^4$. Moreover, we obtain a lot of explicit examples of complete minimal disks in $\mathbb{H}^4$ which are deformed from the Veronese two-sphere and its generalizations:

---

1 It is natural to compare this correspondence with the famous Lawson correspondence [35]. A crucial difference is that from a minimal surface in $S^n$, one can obtain a lot of non-isometric minimal surfaces in $\mathbb{H}^n$. See Sect. 5.
**Example 1.2** (of Proposition 5.10) Set

\[
Y_t = \begin{pmatrix}
    y_0 \\
    y_1 \\
    y_2 \\
    y_3 \\
    y_4 \\
    y_5
\end{pmatrix} = \begin{pmatrix}
    (k - 1)(e^{2t}r^{2k+2} + 1) + (k + 1)(r^{2k} + e^{2t}r^2) \\
    -(k - 1)(e^{2t}r^{2k+2} + 1) + (k + 1)(r^{2k} + e^{2t}r^2) \\
    ie'\sqrt{k^2 - 1}(1 + r^{2k})(z - \bar{z}) \\
    e'\sqrt{k^2 - 1}(1 + r^{2k})(z + \bar{z}) \\
    i\sqrt{k^2 - 1}(1 - e^{2t}r^2)(z^k - \bar{z}^k) \\
    -\sqrt{k^2 - 1}(1 - e^{2t}r^2)(z^k + \bar{z}^k)
\end{pmatrix}.
\]  

(1.1)

The equation \( y_1 = 0 \) gives two circles of \( S^2 \), which divide \( S^2 \) into three parts. On each part of them,

\[
y_t = \frac{1}{y_1} \left( y_0 \ y_2 \ y_3 \ y_4 \ y_5 \right)
\]

provides a proper, complete minimal surface in \( \mathbb{H}^4 \) with finite Willmore energy. Moreover, for any number \( W_0 \in \mathbb{R}^+ \), there exist some \( k \in \mathbb{Z}^+ \setminus \{1\} \) and \( t' \in \mathbb{R} \) such that one of the above three minimal surfaces, has Willmore energy \( W_0 \). Note that when \( k = 2 \), \( y_t \) is in the Willmore deformation family of the Veronese sphere in \( S^4 \).

**Remark 1.3**

1. This is different from the value distribution of Willmore two-spheres in \( S^4 [9, 45] \), where the Willmore energy is always \( 4\pi k \) for some \( k \in \mathbb{Z}^+ \cup \{0\} \). Note that different from the cases discussed in [1, 2], the examples constructed here do not intersect the infinite boundary \( S^3_\infty = \partial_\infty \mathbb{H}^4 \) orthogonally, since they are equivariant and not rotationally symmetric. But they do intersect the infinite boundary \( S^3_\infty = \partial_\infty \mathbb{H}^4 \) with a constant angle.

2. By embedding \( \mathbb{H}^4 \) conformally into \( S^4 \) via the canonical map (see e.g. [4, 12, 54])

\[
x = (x_0, x_1, \ldots, x_4) \mapsto \frac{1}{x_0} (1, x_1, \ldots, x_4),
\]

the three minimal surfaces form a Willmore immersion from \( S^2 \) to \( S^4 \) by crossing the infinite boundary of \( \mathbb{H}^4 \), which gives an explicit illustration of Balch and Bobenko’s famous construction of Willmore tori (with umbilical circles) in \( S^3 \) via gluing complete minimal surfaces in \( \mathbb{H}^3 \) at the infinite boundary of \( \mathbb{H}^3 \) in [4]. A slight difference is that, although here the intersection of these surfaces with the infinite boundary \( S^3_\infty \) is not orthogonal, the whole surface stays smooth. We refer to Section 5.3 for more details.

We also obtain a complete minimal Möbius strip in \( \mathbb{H}^4 \) with Willmore energy \( \frac{6\sqrt{5}\pi}{5} \approx 10.733\pi \) (see Section 5.5). It can be extended as above to obtain a branched Willmore \( \mathbb{R}P^2 \) in \( S^4 \) (Compare [31]). It is natural to ask the infimum of the Willmore energy of non-oriented complete minimal surfaces in \( \mathbb{H}^n \), in comparison with the famous Willmore conjecture, which is proved by Marques and Neves [43] for the case of \( S^3 \). This example shows that the infimum is \( \leq \frac{6\sqrt{5}\pi}{5} \).

By use of some special \( K^C \) dressing actions, for each isotropic minimal surface in \( S^4 \), we can construct concretely a smooth family of isotropic minimal surfaces, which means that such surfaces are not “isolated”.

This paper is organized as follows: in Sect. 2 we will review the basic theory of Willmore surfaces and loop group description of them in terms of their conformal Gauss map. Then in Sect. 3 we will discuss in details of the \( K^C \)-dressing of Willmore surfaces in \( S^{n+2} \), as well as applications to minimal surfaces in \( S^{n+2} \) and \( \mathbb{H}^{n+2} \). In Sect. 4 we describe two kinds of one
parameter group dressing actions on minimal surfaces in $S^{n+2}$ and $H^{n+2}$. Then in Sect. 5 we will focus on examples of complete minimal surfaces in $H^4$ with bounded Gauss curvature and finite Willmore energy. In Sect. 6 we show that isotropic minimal surfaces in $S^4$ admit non-trivial minimal deformations. The paper is ended by an appendix for the technical proof of a lemma.

2 Surface theory of Willmore surfaces and the DPW constructions

In this section we will first recall the basic theory about Willmore surfaces in $S^{n+2}$. Then we will introduce the basic DPW theory for harmonic maps in symmetric space and its applications to Willmore surfaces.

2.1 Willmore surfaces in $S^{n+2}$

Here we will follow the treatment for Willmore surfaces in $[12, 20, 21, 40]$. Note that in $[30, 58]$, different frames are used in the spirits of $[9]$ and $[54]$ respectively. Let $\mathbb{R}^{n+4}$ be the Lorentz-Minkowski space with the Lorentzian metric $\langle x, y \rangle = -x_0 y_0 + \sum_{j=1}^{n+3} x_j y_j = x^t I_{1,n+3} y$, for all $x, y \in \mathbb{R}^{n+4}$.

Here $I_{1,n+3} = diag (-1, 1, \ldots, 1)$. Let $C^{n+3}_+ = \{ x \in \mathbb{R}^{n+4} \mid \langle x, x \rangle = 0, x_0 > 0 \}$ be the forward light cone. Let $Q^{n+2}_+ = C^{n+3}_+/\mathbb{R}^+$ be the projective light cone. For a point $Y \in C^{n+3}_+$, we denote by $[Y]$ its projection in $Q^{n+2}_+$. Then we can identify $S^{n+2}$ with $Q^{n+2}_+$ by setting $y \in S^{n+2}$ to $[Y] = (1, y) \in Q^{n+2}_+$. Let $y : M \to S^{n+2}$ be a conformal immersion from a Riemann surface $M$. Let $z$ be a local complex coordinate on $U \subset M$ with $e^{2\omega} = 2\langle y_z, y_z \rangle$. We have a canonical lift $Y = e^{-\omega}(1, y)$ into $C^{n+3}_+$ with respect to $z$ since $|Y_z|^2 = \frac{1}{2}$. Moreover, there exists a global bundle decomposition $M \times \mathbb{R}^{n+4} = V \oplus V^\perp$. Here $V_p = \text{Span}\{ Y, RY_z, \text{Im} Y_z, Y_z \}_p$ for $p \in M$, and $V^\perp|_p$ is the orthogonal complement of $V_p$ in $\mathbb{R}^{n+4}$. Note that $V_p$ is a 4-dimensional Lorentzian subspace and $V^\perp|_p$ is an $n$–dimensional Euclidean subspace. Denote by $V_C$ and $V_C^\perp$ the complexifications of $V$ and $V^\perp$ respectively. Let $\{ Y, Y_z, Y_{zz}, N \}$ be a frame of $V_C$ such that $\langle N, Y_z \rangle = \langle N, Y_{zz} \rangle = \langle N, N \rangle = 0$, $\langle N, Y \rangle = -1$. Let $D$ be the normal connection on $V_C^\perp$, and $\psi \in \Gamma(V_C^\perp)$ be an arbitrary section of $V_C^\perp$. Then we have:

$$
\begin{align*}
Y_{zz} &= -\frac{s}{2} Y + \kappa, \\
Y_{z\bar{z}} &= \langle \kappa, \bar{\kappa} \rangle Y + \frac{s}{2} N, \\
N_z &= -2\langle \kappa, \bar{\kappa} \rangle Y_z - s Y_{z\bar{z}} + 2D_z \kappa, \\
\psi_z &= D_z \psi + 2\langle \psi, D_z \kappa \rangle Y - 2\langle \psi, \kappa \rangle Y_{z\bar{z}}.
\end{align*}
$$

Structure equations. (2.1)

Here $\kappa$ and $s$ are named as the conformal Hopf differential and the Schwarzian of $y$ respectively $[12]$. The conformal Hopf differential $\kappa dz^3/2 dz^{-1/2}$ is a global $V_C^\perp$–valued quadratic differential on $M$ (See (23) of $[12]$). The integrability conditions are as follows:

$$
\begin{align*}
\frac{1}{2} s_z &= 3\langle \kappa, D_z \bar{\kappa} \rangle + \langle D_z \kappa, \bar{\kappa} \rangle, \\
\text{Im}(D_z D_{\bar{z}} \kappa + \frac{s}{2} \kappa) &= 0, \\
R^D_z \psi &= D_z D_z \psi - D_z D_{\bar{z}} \psi = 2\langle \psi, \kappa \rangle \bar{\kappa} - 2\langle \psi, \bar{\kappa} \rangle \kappa.
\end{align*}
$$

Gauss, Codazzi, Ricci eqs. (2.2)
The Willmore energy of $y$ is defined to be
\[
W(y) = \frac{i}{2} \int_M |\kappa|^2 dz \wedge d\bar{z}.
\]

Let $H$ and $K$ denote the mean curvature and Gauss curvature of $y$ in $\mathbb{S}^{n+2}$ respectively. We have
\[
W(y) = \int_M (H^2 - K + 1) dM.
\]

Note that in many cases the Willmore energy is also defined as
\[
\tilde{W}(y) = \int_M (H^2 + 1) dM = W(y) + \int_M K dM.
\]

In particular, for an oriented closed surface $M$ with Euler number $\chi(M)$,
\[
\tilde{W}(y) = W(y) + 2\pi \chi(M).
\]

For compact surfaces with boundary, to get a conformal invariant functional, one needs to use $W(y)$ instead of $\tilde{W}(y)$ (See e.g. [1, 2, 49]).

For a surface in hyperbolic space $x : M \to \mathbb{H}^{n+2}$, with or without boundary, the conformal invariant Willmore energy is defined to be (See e.g. [1, 2, 49]).
\[
W(x) = \int_M (H^2 - K - 1) dM. \tag{2.3}
\]

By the Gauss equation of $x$ one has
\[
H^2 - K - 1 = \frac{1}{2} (S - 2H^2),
\]
where $S$ is the square of the length of the second fundamant form of $x$ (Compare Theorem 1.2 of [1]). For the case of surfaces in $S^{n+2}$, see (1.2) and (2.8) of [37].

It is well-known that Willmore surfaces can be characterized as follows.

**Theorem 2.1** [9, 12, 29]: $y$ is a Willmore surface if and only if the Willmore equation holds
\[
D_{\bar{z}} D_z \kappa + \frac{\bar{s}}{2} \kappa = 0; \tag{2.4}
\]
if and only if the conformal Gauss map $Gr : M \to Gr_{1,3}(\mathbb{R}^{n+4}) = SO^{+}(1, n + 1)/SO^{+}(1, 3) \times SO(n)$ of $y$ is harmonic. Here $Gr$ is defined as
\[
Gr := Y \wedge Y_u \wedge Y_v \wedge N = -2i \cdot Y \wedge Y_{\bar{z}} \wedge Y_{\bar{\bar{z}}} \wedge N.
\]

A local lift of $Gr$ into $SO^{+}(1, n + 3)$ can be chosen as
\[
F := \left( \frac{1}{\sqrt{2}} (Y + N), \frac{1}{\sqrt{2}} (-Y + N), e_1, e_2, \psi_1, \ldots, \psi_n \right) : U \to SO^{+}(1, n + 1) \tag{2.5}
\]
with Maurer-Cartan form
\[
\alpha = F^{-1} dF = \begin{pmatrix}
A_1 \\
-B_1^t I_{1,3} A_2 \\
B_1 \\
\bar{A}_1 \\
\bar{B}_1^t I_{1,3} \bar{A}_2
\end{pmatrix} dz + \begin{pmatrix}
\bar{A}_1 \\
\bar{B}_1^t I_{1,3} \bar{A}_2
\end{pmatrix} d\bar{z},
\]
and
\[
B_1 = \begin{pmatrix}
\sqrt{2} \beta_1 & \cdots & \sqrt{2} \beta_n \\
-\sqrt{2} \beta_1 & \cdots & -\sqrt{2} \beta_n \\
-k_1 & \cdots & -k_n \\
-i k_1 & \cdots & -i k_n
\end{pmatrix}.
\]
(2.6)

Here \( \{ \psi_j \} \) is an orthonormal basis of \( V^\perp \) and 
\( \kappa = \sum_j k_j \psi_j, \ D \kappa = \sum_j \beta_j \psi_j, \ k = \sqrt{\sum_j |k_j|^2} \).

Finally we recall that for a surface \( y \) in \( S^4 \), it is called isotropic if and only if its Hopf differential satisfies (see [14, 29, 45, 46])
\[
\langle \kappa, \kappa \rangle \equiv 0.
\]

This is a conformal invariant condition and it plays important roles in the classification of minimal two-spheres [14] and Willmore two-spheres in \( S^4 \) [29, 45, 46]. It is well-known that if \( y \) is an isotropic surface in \( S^4 \), then it is Willmore [29].

2.2 The DPW construction of Willmore surfaces in \( S^{n+2} \) via conformal Gauss maps

2.2.1 The DPW construction of harmonic maps

We will recall the basic theory of the DPW methods (See [19, 21] for more details). Let \( G/K \) be a symmetric space defined by the involution \( \sigma : G \rightarrow G, \) with \( G^\sigma \supset K \supset (G^\sigma)_0 \), and Lie algebras \( g = \text{Lie}(G), \mathfrak{k} = \text{Lie}(K) \). Then \( g = \mathfrak{t} \oplus \mathfrak{p}, \ [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t}, \ [\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}, \ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t} \).

Let \( f : M \rightarrow G/K \) be a harmonic map. Let \( z \) be a complex coordinate on \( U \subset M \). Then there exists a frame \( F : U \rightarrow G \) of \( f \) with Maurer-Cartan form \( F^{-1}dF = \alpha \). The Maurer-Cartan equation reads \( d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \). Decompose it with respect to the Cartan decomposition, we obtain \( \alpha = \alpha_0 + \alpha_1 \) with \( \alpha_0 \in \Gamma(\mathfrak{t} \otimes T^*M), \ \alpha_1 \in \Gamma(\mathfrak{p} \otimes T^*M) \). Decompose \( \alpha_1 \) further into the \((1,0)\)–part \( \alpha_1' \) and the \((0,1)\)–part \( \alpha_1'' \). Introducing \( \lambda \in S^1 \), set
\[
\alpha_\lambda = \lambda^{-1} \alpha_1' + \alpha_0 + \lambda \alpha_1'', \ \lambda \in S^1.
\]
(2.7)

It is well known ([19]) that the map \( f : M \rightarrow G/K \) is harmonic if and only if
\[
d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \quad \text{for all} \ \lambda \in S^1.
\]

**Definition 2.2** Let \( F(z, \lambda) \) be a solution to the equation \( dF(z, \lambda) = F(z, \lambda)\alpha_\lambda \). \( F(0, \lambda) = F(0) \). Then \( F(z, \lambda) \) is called the extended frame of the harmonic map \( f \). Moreover,
\[
f(z, \lambda) := F(z, \lambda) \mod K
\]
are harmonic maps in \( G/K \) for all \( \lambda \in S^1 \), called the associated family of \( f \). Note that \( f(z, \lambda)|_{\lambda=1} = f(z) \) and \( F(z, \lambda)|_{\lambda=1} = F(z) \).

So far we have related harmonic maps with maps into loop groups. Moreover, we need the Iwasawa and Birkhoff decompositions for loop groups. Let \( \hat{G}^C \) be the complexified Lie group of \( G \). Extend \( \sigma \) to an inner involution of \( \hat{G}^C \) with \( F i \sigma \hat{G}^C = \hat{G}^C \). Let \( \Lambda \hat{G}^C_\sigma \) be the group of loops in \( \hat{G}^C \) twisted by \( \sigma \). Let \( \Lambda \hat{G}^C_\sigma \) be the group of loops that extends holomorphically into \( \infty \) and take values \( I \) at \( \infty \).
Theorem 2.3 [19], [20]

1. (Iwasawa decomposition): There exists a closed, connected solvable subgroup $S \subseteq K^C$ such that the multiplication $\Lambda G^0_\sigma \times \Lambda_S^+ G^C_\sigma \to \Lambda G^C_\sigma$ is a real analytic diffeomorphism onto the open subset $\Lambda G^0_\sigma \cdot \Lambda_S^+ G^C_\sigma = T^U_\sigma \subseteq (\Lambda G^C_\sigma)^0$, with $\Lambda_S^+ G^C_\sigma := \{ \gamma \in \Lambda^+ G^S_\sigma \mid \gamma|\lambda = 0 \in S \}$.

2. (Birkhoff decomposition): The multiplication $\Lambda^- G^C_\sigma \times \Lambda_S^+ G^C_\sigma \to \Lambda G^C_\sigma$ is an analytic diffeomorphism onto the open, dense subset $\Lambda^- G^C_\sigma \cdot \Lambda_S^+ G^C_\sigma$ of $\Lambda G^C_\sigma$ (the big Birkhoff cell), with $\Lambda_S^+ G^C_\sigma := \{ \gamma \in \Lambda^+ G^C_\sigma \mid \gamma|\lambda = 0 \in (K^C)^0 \}$.

The well-known DPW construction for harmonic maps can be stated as follows

Theorem 2.4 [19] Let $\mathbb{D} \subset \mathbb{C}$ be a disk or $\mathbb{C}$ with complex coordinate $z$.

1. Let $f : \mathbb{D} \to G/K$ denote a harmonic map with an extended frame $F(z, \tilde{z}, \lambda) \in \Lambda G_\sigma$ and $F(0, 0, \lambda) = I$. Then there exists a Birkhoff decomposition of $F(z, \tilde{z}, \lambda) : F_-(z, \lambda) = F(z, \tilde{z}, \lambda) F_+(z, \tilde{z}, \lambda)$, with $F_+$ taking values in $\Lambda_S^+ G^C_\sigma$, such that $F_-(z, \lambda) : \mathbb{D} \to \Lambda^- G^C_\sigma$ is meromorphic. Moreover, the Maurer-Cartan form of $F_-$ is the form

$$\eta = F_-^{-1}dF_- = \lambda^{-1}\eta_{-1}(z)dz,$$

called the normalized potential of $f$, with $\eta_{-1} : \mathbb{D} \to \mathfrak{p} \otimes \mathbb{C}$ independent of $\lambda$.

2. Let $\eta$ be a $\lambda^{-1} : \mathfrak{p} \otimes \mathbb{C}$-valued meromorphic 1-form on $\mathbb{D}$. Let $F_-(z, \lambda)$ be a solution to $F_-^{-1}dF_- = \eta$, $F_-(0, \lambda) = I$. Then there exists an Iwasawa decomposition

$$F_-(0, \lambda) = \tilde{F}(z, \tilde{z}, \lambda) \tilde{F}^+(z, \tilde{z}, \lambda),$$

with $\tilde{F} \in \Lambda G_\sigma$, $\tilde{F} \in \Lambda_S^+ G^C_\sigma$ on an open subset $\mathbb{D}_3 \subset \mathbb{D}$. Moreover, $(\tilde{F}(z, \tilde{z}, \lambda)$ is an extended frame of some harmonic map from $\mathbb{D}_3$ to $G/K$ with $\tilde{F}(0, 0, \lambda) = I$. All harmonic maps can be obtained in this way, since the above two procedures are inverse to each other if the normalization at some base point is fixed.

2.2.2 Normalized potentials of Willmore surfaces in $\mathbb{S}^{n+2}$

For simplicity let us restrict to the case for Willmore surfaces [20, 21, 55]. In this case, $G = SO^+(1, n + 3)$, $K = SO^+(1, 1) \times SO(n)$, and $\mathfrak{g} = so(1, n + 3) = \{ X \in \mathfrak{gl}(n + 4, \mathbb{R}) | X^T I_{1,n+3} + I_{1,n+3}X = 0 \}$. The involution $\sigma$ is given by $\sigma : SO^+(1, n + 3) \to SO^+(1, n + 3)$, $\sigma(A) := DAD^{-1}$, with $D = diag\{-I_4, I_n\}$. We also have $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, with

$$\mathfrak{t} = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} | A_1 I_{1,3} + I_{1,3} A_1 = 0, A_2 + A_2^T = 0 \right\}, \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & B_1 \\ -B_1^T I_{1,3} & 0 \end{pmatrix} \right\}.$$

Let $G^C = SO^+(1, 1, \mathbb{C}) = \{ X \in SL(n+4, \mathbb{C}) | X^T I_{1,n+4} X = I_{1,n+4} \}$ with Lie algebra $so(1, n + 3, \mathbb{C})$. Extend $\sigma$ to an inner involution of $SO^+(1, n + 3, \mathbb{C})$ with fixed point group $K^C = S(O^+(1, 1, \mathbb{C}) \times O(n, \mathbb{C}))$.

Since a Willmore surface corresponds uniquely to its oriented conformal Gauss map [20, 29, 40], we will use the normalized potential for a Willmore surface directly. For later applications, we recall the description of minimal surfaces in space forms in terms of the normalized potentials.

Theorem 2.5 [55] (compare also [8, 30, 58]) Let $y$ be a Willmore surface in $\mathbb{S}^{n+2}$, with its normalized potential being of the form

$$\eta = \lambda^{-1}\eta_{-1}dz = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^T I_{1,3} & 0 \end{pmatrix} dz, \quad \text{and} \quad \hat{B}_1^T I_{1,3} \hat{B}_1 = 0.$$
Then \( y \) is conformally equivalent to some minimal surface in \( \mathbb{R}^{n+2} \), \( S^{n+2} \) or \( \mathbb{H}^n \) if and only if there exists a non-zero, real, constant vector \( v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4 \) such that

\[
v' I_{1,3} \hat{B}_1 \equiv 0.
\] (2.8)

Moreover,
1. the space form is \( \mathbb{R}^{n+2} \) if and only if \( \langle v, v \rangle = v' I_{1,3} v = 0 \);
2. the space form is \( S^{n+2} \) if and only if \( \langle v, v \rangle = v' I_{1,3} v < 0 \);
3. the space form is \( \mathbb{H}^n \) if and only if \( \langle v, v \rangle = v' I_{1,3} v > 0 \).

Note that there are some different treatments of minimal surfaces in \( \mathbb{H}^3 \) via loop group methods in [7, 18].

### 3 \( K^C \)–dressing actions on Willmore surfaces

In this section, we will use the dressing actions on harmonic maps by \( K^C \) for Willmore surfaces. We refer to [13, 26, 36, 51, 52] for more details on dressing actions and their applications to other geometric problems. Note that here we use the elements in \( K^C \) instead of the loop group elements.

#### 3.1 \( K^C \)–dressing actions

**Definition 3.1** Let \( k \in K^C \). Let \( f : \mathbb{D} \to G/K \) be a harmonic map with an extended frame \( F(z, \lambda) \), based at \( z_0 \) such that \( F(z_0, \lambda) = e \in G \). A dressing action by \( k \) on \( f \) is defined by the harmonic map

\[
k \# f := \hat{F} \mod K,
\]

where \( \hat{F} : \mathbb{D} \to \Lambda G_{\sigma} \) is given by the following

\[
\hat{F} = k F(z, \lambda) \hat{V}_+ \text{, with } \hat{V}_+ \in \Lambda^+ G^C_{\sigma}.
\] (3.1)

From the definition it is obvious that

**Corollary 3.2** \( \bar{f} = k \# f \) if \( f = k^{-1} \# \bar{f} \).

The following result is well-known to the experts. Here we include a proof for readers’ convenience.

**Proposition 3.3** Let \( \eta \) and \( \hat{\eta} \) be the normalized potentials of \( f \) and \( k \# f \) respectively. Assume that the extended frames \( F \) and \( \hat{F} \) of \( f \) and \( k \# f \) are both based at \( z_0 \). Then

\[
\hat{\eta} = k \eta k^{-1}.
\] (3.2)

Conversely, assume that \( \eta \) and \( \hat{\eta} \) satisfy (3.2), and their integrations have the same initial conditions. Then their corresponding harmonic maps \( f \) and \( \hat{f} \) satisfy \( \hat{f} = k \# f \).

**Proof** By Theorem 2.4, we have

\[
F_- = F F_+ \text{ and } \eta = F_-^{-1} dF_- , \quad \hat{F}_- = \hat{F} F_+ \text{ and } \hat{\eta} = \hat{F}_-^{-1} d\hat{F}_-.
\]

From (3.1), we also have \( \hat{F} = k F(z, \lambda) \hat{V}_+ \). So

\[
\hat{F}_- = k F(z, \lambda) \hat{V}_+ F_+ = k F_- F_+^{-1} \hat{V}_+ = k F_- k^{-1} k F_+^{-1} \hat{V}_+.
\]
Together with the assumption of having same initial conditions, we obtain that $\hat{F}_- = kF_-k^{-1}$, and (3.2) follows directly.

Concerning the converse part, first by the assumptions we have $\hat{F}_- = kF_-k^{-1}$. So

$$\hat{F} = \hat{F}_- \hat{F}_+^{-1} = kF_-k^{-1} \hat{F}_+^{-1} = kF\hat{V}_+$$

with $\hat{V}_+ = F_+^{-1}k^{-1}\hat{F}_+^{-1}$, that is, $\hat{f} = k\hat{\eta}f$.

\[ \square \]

Applying to Willmore surfaces, we obtain

**Proposition 3.4** Let $T = T_1 \times T_2 \in SO(1, 3, \mathbb{C}) \times SO(n, \mathbb{C})$. Let $f$ be a harmonic map with the normalized potential

$$\eta = \lambda^{-1}\eta_{-1}dz = \lambda^{-1}\begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix}dz, \text{ with } \hat{B}_1^t I_{1,3} \hat{B}_1 = 0. \quad (3.3)$$

Then the normalized potential $\eta_T$ of $T^\sharp f$ has the form

$$\eta_T = T\eta T^{-1} = \lambda^{-1}\begin{pmatrix} 0 & T_1 \hat{B}_1 T_2^t \\ -T_2 \hat{B}_1^t I_{1,3} T_1^t & 0 \end{pmatrix}dz. \quad (3.4)$$

We define the space of the conformal Gauss maps of minimal surfaces in three space forms:

$$\mathcal{M}_0 := \{ f | f \text{ is the conformal Gauss map of a minimal surface in } \mathbb{R}^{n+2}, \} \quad (3.5)$$

$$\mathcal{M}_1 := \{ f | f \text{ is the conformal Gauss map of a minimal surface in } S^{n+2}, \} \quad (3.6)$$

$$\mathcal{M}_{-1} := \{ f | f \text{ is the conformal Gauss map of a minimal surface in } \mathbb{H}^{n+2}. \} \quad (3.7)$$

We also define the space $\mathcal{M}_L$ and its subset $\widetilde{\mathcal{M}}_0$ as below

$$\mathcal{M}_L := \{ f | \text{The normalized potential of } f \text{ satisfies } v^t I_{1,3} \hat{B}_1 = 0 \text{ for some } v \in \mathbb{C}^4 \setminus \{0\}, \} \quad (3.8)$$

$$\widetilde{\mathcal{M}}_0 := \{ f | f \in \mathcal{M}_L, \text{ with } v \text{ satisfying } v^t I_{1,3} v = 0 \} \quad (3.9)$$

Note that $\mathcal{M}_0 \subsetneq \widetilde{\mathcal{M}}_0$ (See [55] for example). In [55], it is shown that up to a conjugation, for any $f \in \widetilde{\mathcal{M}}_0$, the normalized potential of $f$ has the form ((1) of [55])

$$\hat{B}_1 = \begin{pmatrix} \hat{f}_{11} & \hat{f}_{12} & \cdots & \hat{f}_{1,n} \\ -\hat{x}_{11} & -\hat{x}_{12} & \cdots & -\hat{x}_{1,n} \\ \hat{f}_{13} & \hat{f}_{32} & \cdots & \hat{f}_{3n} \\ i\hat{f}_{13} & i\hat{f}_{32} & \cdots & i\hat{f}_{3n} \end{pmatrix}$$

with $\hat{f}_{ij}$ being meromorphic functions.

Set $K^C = SO(1, 3, \mathbb{C}) \times SO(n, \mathbb{C})$ and we define

$$K^C_{\sharp j} \mathcal{M}_j := \{ T^\sharp f | T \in K^C, f \in \mathcal{M}_j \}, \quad j = 0, 1, -1;$$

$$K^C_{\sharp} \widetilde{\mathcal{M}}_0 := \{ T^\sharp f | T \in K^C, f \in \widetilde{\mathcal{M}}_0 \};$$

$$K^C_{\sharp} \mathcal{M}_L := \{ T^\sharp f | T \in K^C, f \in \mathcal{M}_L \}. \quad (3.10)$$
3.2 $K^C$—dressing actions preserve minimal surfaces in $\mathbb{R}^{n+2}$

**Theorem 3.5** Let $f$ be the oriented conformal Gauss map of a minimal surface in $\mathbb{R}^{n+2}$ and $T \in K^C$. Then $T^\circ f$ is the oriented conformal Gauss map of a minimal surface in $\mathbb{R}^{n+2}$, i.e.,

$$K^C \pi \mathcal{M}_0 = \mathcal{M}_0.$$ 

**Proof** To show that $K^C \pi \mathcal{M}_0 = \mathcal{M}_0$, we need to show that $K^C \pi \mathcal{M}_0 \setminus \mathcal{M}_0 = \emptyset$. Otherwise assume that $\tilde{f} \in K^C \pi \mathcal{M}_0 \setminus \mathcal{M}_0$. Then there exists $f \in \mathcal{M}_0$ and $T \in K^C$ such that $T^\circ f = \tilde{f}$. So $T^{-1} \pi \tilde{f} = f$. Assume that $\tau = \text{diag}(T_1, T_2)$ and the normalized potential of $\tilde{f}$ is given by $\tilde{B}_1$. Then the normalized potential of $f$ is given by $T_1 \tilde{B}_1 T_2^{-1}$ by (3.4).

Since $\tilde{f} \notin \mathcal{M}_0$, by [55] we have that either $T_1 \tilde{B}_1 T_2^{-1}$ has rank 2 or $\tilde{f}$ reduces to a map into $SO(n + 2)/SO(2) \times SO(n)$ or $SO(1, 1)/SO(1, 1) \times SO(n)$. If $\tilde{B}_1$ has maximal rank 2, then $\tilde{B}_1$ also has maximal rank 2, which is not possible since $\tilde{B}_1$ has maximal rank 1 due to the assumption $f \in \mathcal{M}_0$. If $\tilde{f}$ reduces to a map into $SO(n + 2)/SO(2) \times SO(n)$ or $SO(1, 1)/SO(1, 1) \times SO(n)$, then we can assume w.l.g. that

$$T_1 \tilde{B}_1 T_2^{-1} = \begin{pmatrix} \tilde{f}_{11} & \tilde{f}_{12} & \cdots & \tilde{f}_{1,n} \\ -\tilde{f}_{11} & -\tilde{f}_{12} & \cdots & -\tilde{f}_{1,n} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \tilde{f}_{13} & \tilde{f}_{32} & \cdots & \tilde{f}_{3n} \\ i \tilde{f}_{13} & i \tilde{f}_{32} & \cdots & i \tilde{f}_{3n} \end{pmatrix}.$$

So

$$\tilde{B}_1 = T_1^{-1} \begin{pmatrix} \tilde{f}_{11} & \tilde{f}_{12} & \cdots & \tilde{f}_{1,n} \\ -\tilde{f}_{11} & -\tilde{f}_{12} & \cdots & -\tilde{f}_{1,n} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} T_2 \text{ or } \tilde{B}_1 = T_1^{-1} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \tilde{f}_{13} & \tilde{f}_{32} & \cdots & \tilde{f}_{3n} \\ i \tilde{f}_{13} & i \tilde{f}_{32} & \cdots & i \tilde{f}_{3n} \end{pmatrix} T_2.$$

Consider in the first case the constant vector

$$v^* = T^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}_1^4.$$ 

Apparently the action of $T_2$ does not change its form. So we can assume without lose of generality $T_2 = I$. Note by construction $v^*$ stays an isotropic vector, i.e., $(v^*)' I_{1,3} v^* = 0$. So there exists some real $\tilde{T}_1 \in SO(1, 3)$ such that

$$\tilde{T}_1 T^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \text{ or } a \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix},$$

depending on whether $\text{Re} v^*$ and $\text{Im} v^*$ are linear dependent or not. Here $a$ is a constant number. So up to an action $\tilde{T} = \text{diag}(\tilde{T}_1, I)$, $f$ reduces to a harmonic map into $SO(1, n + 1)/SO(1, 1) \times SO(n)$, which is contradicted to the assumption $f \in \mathcal{M}_0$, since harmonic maps into $SO(1, n + 1)/SO(1, 1) \times SO(n)$ does not produce minimal surfaces in $\mathbb{R}^{n+2}$. Similarly, the second case produces a harmonic map into $SO(n + 2)/SO(2) \times SO(n)$, which also does not give minimal surfaces in $\mathbb{R}^{n+2}$. Hence $K^C \pi \mathcal{M}_0 = \mathcal{M}_0$. □

**Remark 3.6** In [36], it is shown that the simple dressing actions preserve minimal surfaces in $\mathbb{R}^4$. Our result here shows that $K^C$—dressing actions preserve minimal surfaces in $\mathbb{R}^{n+2}$. © Springer
3.3 $K^C$—dressing actions on minimal surfaces in $S^{n+2}$ and $H^{n+2}$

**Theorem 3.7** 1. Let $f \in M_1$. Then $T^n f \in M_L \setminus \hat{M}_0$ for any $T \in K^C$. Conversely, let $\hat{f} \in M_L \setminus \hat{M}_0$. Then there exists some $\tilde{T} \in K^C$ such that $\tilde{T}^n \hat{f} \in M_1$, that is,

$$K^C \hat{f} \in M_1 = M_L \setminus \hat{M}_0.$$  \hspace{1cm} (3.11)

2. Let $f \in M_{-1}$. Then $T^n f \in M_L \setminus \hat{M}_0$ for any $T \in K^C$. Conversely, let $\hat{f} \in M_L \setminus \hat{M}_0$. Then there exists some $\tilde{T} \in K^C$ such that $\tilde{T}^n \hat{f} \in M_{-1}$, that is,

$$K^C \hat{f} \in M_{-1} = M_L \setminus \hat{M}_0.$$  \hspace{1cm} (3.12)

3. In particular, for any $f \in M_1$, there exists some $T \in K^C$ such that $\tilde{T}^n f \in M_{-1}$. For any $f \in M_{-1}$, there exists $T \in K^C$ such that $\tilde{T}^n f \in M_1$, that is,

$$M_{-1} \subseteq K^C \hat{f} M_1, \quad M_1 \subseteq K^C \hat{f} M_{-1}.$$

**Proof** (1) Let $f \in M_1$ with its normalized potential given by $\hat{B}_1$. Then by Theorem 2.5, there exists some $v \in \mathbb{R}_1^4$ such that

$$v^t I_{1,3} \hat{B}_1 \equiv 0, \quad \langle v, v \rangle = v^t I_{1,3} v < 0.$$

So for any $T \in K^C$, the normalized potential of $T^n f$ is given by $\tilde{B}_1 = T_1 \hat{B}_1 T_2^{-1}$, where $T = \text{diag}(T_1, T_2)$. Then $\tilde{v} = T_1 v$ is the vector such that $\tilde{v}^t I_{1,3} T_1 \hat{B}_1 T_2^{-1} = 0$. So $T^n f \in M_L$. Since $\tilde{v}^t I_{1,3} \tilde{v} = v^t I_{1,3} v < 0$, $T^n f \in M_L \setminus \hat{M}_0$.

Now let $\hat{f} \in M_L \setminus \hat{M}_0$ with its normalized potential given by $\hat{B}_1$. Then there exists some $v$ such that $v^t I_{1,3} \hat{B}_1 \equiv 0$, $\langle v, v \rangle = v^t I_{1,3} v \neq 0$. Set $v_1 = \frac{i}{\sqrt{v^t I_{1,3} v}} v$. There exists some $v_j$, $j = 1, 2, 3, 4$, such that

$$T_1 = (v_1, v_2, v_3, v_4)^t \in SO(1, 3, \mathbb{C})$$

Set $\tilde{T} = \text{diag}(T_1, I_n)$ and

$$v_0 = T_1 v_1 = (-1 0 0 0)^t.$$

We see that $\tilde{T}^n \hat{f}$ has its normalized potential satisfying $\hat{B}_1 = T_1 \hat{B}_1$ and

$$v_0^t I_{1,3} \hat{B}_1 = v_0^t T_1^t I_{1,3} T_1 \hat{B}_1 = v_0^t T_1^t I_{1,3} \hat{B}_1 = v_0^t I_{1,3} \hat{B}_1 = \frac{i}{\sqrt{v^t I_{1,3} v}} v^t I_{1,3} \hat{B}_1 = 0.$$

Since $v_0 \in \mathbb{R}_1^4$ and $v_0^t I_{1,3} v_0 \prec 0$, $\tilde{T}^n \hat{f} \in M_1$.

The proof of (2) is similar to (1) and we leave it for interested readers. (3) is a corollary of (1) and (2). \hfill $\Box$

4 On $K^C$—dressing actions of minimal surfaces in $S^{n+2}$ & $H^{n+2}$

We will first discuss the general $K^C$—dressing actions briefly. Then we will consider concretely two kinds of 1-parameter subgroups of $SO(1, 3, \mathbb{C})$ and their dressing actions on minimal surfaces in $S^{n+2}$ and $H^{n+2}$, that is, dressing actions on surfaces by elements of these two subgroups. One of the subgroups changes the minimality and builds a local Willmore deformation between minimal surfaces in $S^{n+2}$ and $H^{n+2}$. And the other one keeps the minimality and gives a family of minimal surfaces in $S^{n+2}(H^{n+2})$. \hfill \textcopyright Springer
4.1 \(K^C\) — dressing actions of minimal surfaces in \(S^{n+2}\) & \(\mathbb{H}^{n+2}\)

It is direct to have the following proposition by Theorem 2.5 and Proposition 3.4.

**Proposition 4.1** 1. The dimension of non-trivial \(K^C\) — dressing actions of a Willmore surfaces in \(S^{n+2}\) is less or equal to

\[
\dim SO(1, 3, \mathbb{C}) \times SO(n, \mathbb{C}) - \dim SO(1, 3) \times SO(n) = \frac{n(n-1)}{2} + 6. \quad (4.1)
\]

2. The dimension of non-trivial \(K^C\) — dressing actions of a minimal surface in \(S^{n+2}\) preserving minimality infinitesimally is less or equal to

\[
\dim SO(3, \mathbb{C}) \times SO(n, \mathbb{C}) - \dim SO(3) \times SO(n) = \frac{n(n-1)}{2} + 3. \quad (4.2)
\]

3. The dimension of non-trivial \(K^C\) — dressing actions of a minimal surface in \(\mathbb{H}^{n+2}\) preserving minimality infinitesimally is less or equal to

\[
\dim SO(1, 2, \mathbb{C}) \times SO(n, \mathbb{C}) - \dim SO(1, 2) \times SO(n) = \frac{n(n-1)}{2} + 3. \quad (4.3)
\]

The above spaces of the non-trivial \(K^C\) — dressing actions can be locally expressed (near \(I\)) as \(\exp \mathcal{G}, \exp \mathcal{G}_1\) and \(\exp \mathcal{G}_{-1}\) respectively, where

\[
\mathcal{G} = \{A \in \mathfrak{s}\mathfrak{o}(1, 3, \mathbb{C}) \times \mathfrak{s}\mathfrak{o}(n, \mathbb{C}) | A = -\hat{A}\},
\]

\[
\mathcal{G}_1 = \{A \in \mathfrak{s}\mathfrak{o}(3, \mathbb{C}) \times \mathfrak{s}\mathfrak{o}(n, \mathbb{C}) | A = -\hat{A}\},
\]

\[
\mathcal{G}_{-1} = \{A \in \mathfrak{s}\mathfrak{o}(1, 2, \mathbb{C}) \times \mathfrak{s}\mathfrak{o}(n, \mathbb{C}) | A = -\hat{A}\}.
\]

Here \(\mathfrak{s}\mathfrak{o}(3, \mathbb{C})\) and \(\mathfrak{s}\mathfrak{o}(1, 2, \mathbb{C})\) are viewed as subsets of \(\mathfrak{s}\mathfrak{o}(1, 3, \mathbb{C})\) naturally.

4.2 On some \(S^1\) — dressing actions of minimal surfaces in \(S^{n+2}\) & \(\mathbb{H}^{n+2}\)

In this subsection, we discuss some special \(S^1\) — dressing actions which build a smooth local Willmore deformations between minimal surfaces in \(S^{n+2}\) and \(\mathbb{H}^{n+2}\). Set

\[
T_{1,t} = \begin{pmatrix}
\cos t & i \sin t & 0 & 0 \\
i \sin t & \cos t & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in SO(1, 3, \mathbb{C}), \ t \in [0, 2\pi].
\]

Then \(T_{1,t}, t \in [0, 2\pi]\), is a circle subgroup of \(SO(1, 3, \mathbb{C})\). We see that \(T_{1,t} \in SO(1, 3) \cap SO(1, 3, \mathbb{C})\) if and only if \(t = 0, 2\pi\). And \(T_{1,t} \in (i \cdot O(1, 3)) \cap SO(1, 3, \mathbb{C})\) if and only if \(t = \frac{\pi}{2}, \frac{3\pi}{2}\).

First, assume without lose of generality that the normalized potential of a Willmore surface in \(S^{n+2}\) has the form

\[
\eta = \lambda^{-1} \begin{pmatrix} \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} \end{pmatrix} dz, \quad \hat{B}_1 = (h_1 \cdots h_n).
\]
By Theorem 2.5, we can assume without lose of generality that the normalized potential of a minimal surface in $\mathbb{S}^{n+2}$ has the form
\[
h_j = h_{0j} \begin{pmatrix} 0 \\ \tilde{h}_2 \\ \tilde{h}_3 \\ \tilde{h}_4 \end{pmatrix}, \quad (\tilde{h}_2)^2 + (\tilde{h}_3)^2 + (\tilde{h}_4)^2 = 0, \quad j = 1, \ldots, n, \quad (4.4)
\]
where $\tilde{h}_2, \tilde{h}_3$ and $\tilde{h}_4$ are linear independent meromorphic functions.

Theorem 4.2 The normalized potential
\[
\eta_t = \lambda^{-1} \begin{pmatrix} 0 & T_{1t} \hat{B}_1 \\ -\hat{B}^t_1 T_{1t}^t, I_{1,3} & 0 \end{pmatrix} d\bar{z},
\]
with $\hat{B}_1 = (h_1 \cdots h_n)$ and all of $\{h_j, 1 \leq j \leq n\}$ being of the form $(4.4)$, locally gives a family of Willmore surfaces $y_t, \ t \in [0, 2\pi)$, such that $(y_t)|_{t=0}, (y_t)|_{t=\pi}$ are conformally equivalent to minimal surfaces in $\mathbb{S}^{n+2}$ and $(y_t)|_{t=\frac{\pi}{2}}, (y_t)|_{t=\frac{3\pi}{2}}$ are conformally equivalent to minimal surfaces in $\mathbb{H}^{n+2}$, and for each other $t$, $y_t$ is a Willmore surface in $\mathbb{S}^{n+2}$ not minimal in any space form.

Proof It is direct to see that
\[
v_t = (\cos t \ i \sin t \ 0 \ 0)^t = T_{1t}^t (1 \ 0 \ 0 \ 0)^t
\]
satisfies
\[
v_t^t I_{1,3} T_{1t} \hat{B}_1 \equiv 0.
\]
So when $t = 0$ or $\pi$, one obtains minimal surfaces in $\mathbb{S}^{n+2}$. So when $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, one obtains minimal surfaces in $\mathbb{H}^{n+2}$.

For other $t$, assume $y_t$ is conformal to some minimal surface in space forms. Then there exists a real vector $v \in \mathbb{R}^4$ such that $v I_{1,3} T_{1t} \hat{B}_1 \equiv 0$. So w.l.g. we can assume $v = (a \ b \ c \ 0)^t$. So we have
\[-ia\tilde{h}_2 \sin t + b\tilde{h}_2 \cos t + c\tilde{h}_3 = 0
\]
Since $a, b \in \mathbb{R}$, we see that $c \neq 0$ and $\tilde{h}_3 = -\frac{b-ia}{c} \tilde{h}_2$, which contradicts to the assumption that $\tilde{h}_2$ and $\tilde{h}_2$ are linear independent. This finishes the proof. \hfill \Box

Similarly, by Theorem 2.5 we can assume without lose of generality that the normalized potential of a minimal surface in $\mathbb{H}^{n+2}$ has the form
\[
h_j = h_{0j} \begin{pmatrix} \tilde{h}_1 \\ 0 \\ \tilde{h}_3 \\ \tilde{h}_4 \end{pmatrix}, \quad -(\tilde{h}_1)^2 + (\tilde{h}_3)^2 + (\tilde{h}_4)^2 = 0, \quad j = 1, \ldots, n. \quad (4.5)
\]

Theorem 4.3 The normalized potential
\[
\eta_t = \lambda^{-1} \begin{pmatrix} 0 & T_{1t} \hat{B}_1 \\ -\hat{B}^t_1 T_{1t}^t, I_{1,3} & 0 \end{pmatrix} d\bar{z},
\]
with $\hat{B}_1 = (h_1 \cdots h_n)$ and all of $(h_j, 1 \leq j \leq n)$ being of the form (4.5), locally gives a family of Willmore surfaces $y_t$, $t \in [0, 2\pi)$, such that $(y_t)|_{t=0}$, $(y_t)|_{t=\pi}$ are conformally equivalent to minimal surfaces in $\mathbb{H}^{n+2}$ and $(y_t)|_{t=\frac{\pi}{2}}$, $(y_t)|_{t=\frac{3\pi}{2}}$ are conformally equivalent to minimal surfaces in $\mathbb{S}^{n+2}$, and for all other $t$, $y_t$ is a non-minimal Willmore surface in $\mathbb{S}^{n+2}$.

**Proof** The proof is the same as above theorem. So we omit it. □

**Remark 4.4** Comparing (4.4) and (4.5), we see that the loop group data of minimal surfaces in $\mathbb{S}^{n+2}$ and $\mathbb{H}^{n+2}$ differ essentially by some shifting and multiplying some $i$ for some terms, which can achieved of the above dressing action. This is the key observation & motivation of the $K^C$ dressing actions.

### 4.3 On some $\mathbb{R} -$ dressing actions preserving minimal surfaces in $\mathbb{S}^{n+2}$ & $\mathbb{H}^{n+2}$

Set

$$T_{2,t} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh t & i \sinh t \\ 0 & 0 & -i \sinh t & \cosh t \end{pmatrix} \in SO(1, 3, \mathbb{C}), \quad t \in \mathbb{R}.$$ 

Then $T_{2,t}, t \in \mathbb{R}$, is a $\mathbb{R} -$ subgroup of $SO(1, 3, \mathbb{C})$. Note that $T_{2,t} \in SO(1, 3) \cap SO(1, 3, \mathbb{C})$ if and only if $t = 0$.

**Theorem 4.5** 1. The normalized potential

$$\eta_t = \lambda^{-1} \left( \begin{array}{c} 0 \\ -\hat{B}_1 T_{2,t} \hat{B}_1 I_{1,3} \end{array} \right) dz,$$

with $\hat{B}_1 = (h_1 \cdots h_n)$ and $h_j$ being of the form (4.4), locally gives a family of Willmore surfaces $y_t$ conformally equivalent to minimal surfaces in $\mathbb{S}^{n+2}$.

2. The normalized potential

$$\eta_t = \lambda^{-1} \left( \begin{array}{c} 0 \\ -\hat{B}_1 T_{2,t} \hat{B}_1 I_{1,3} \end{array} \right) dz,$$

with $\hat{B}_1 = (h_1 \cdots h_n)$ and $h_j$ being of the form (4.5), locally gives a family of Willmore surfaces $y_t$ conformally equivalent to minimal surfaces in $\mathbb{H}^{n+2}$.

**Proof** By Theorem 2.5 and setting $\mathbf{v} = (1 0 0 0)^t$ and $(0 1 0 0)^t$ respectively, we obtain (1) and (2) respectively. □

**Remark 4.6** Note that the $T_{2,t}$ action on (4.4), is used exactly as the famous Lopez-Ros deformation for minimal surfaces in $\mathbb{R}^3$ [39]. We refer to [36] for the simple factor dressing expression of the Lopez-Ros deformation for minimal surfaces in $\mathbb{R}^3$, which is different from the action considered in this paper.
5 Examples of minimal surfaces in $\mathbb{H}^4$

In this section, we will illustrate the $K^C$—dressing actions for isotropic minimal surfaces in $S^4$ in terms of the formula in the previous section. $K^C$—dressing actions of the Veronese 2-spheres give many explicit examples of Willmore two-spheres in $\mathbb{H}^4$. In particular, we obtain many examples of complete minimal surfaces in $S^4$. Let us first recall a Weierstrass type formula for isotropic (Willmore) surfaces in $S^4$ [56]. Then we will discuss in details two kinds of one-parameter group action on isotropic surfaces in $S^4$. With the help of the formula, we derive many explicit examples with expected properties in Section 5.3-5.6.

We refer to [29, 45, 46, 56] for more discussions of isotropic Willmore surfaces.

5.1 The Weierstrass formula for isotropic surfaces in $S^4$

The following formula provides all explicit examples $^2$ in this paper. So we include it here for readers’ convenience.

**Theorem 5.1** [56] Let $M$ be a Riemann surface, and let

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1 I_{1,3} & 0 \end{pmatrix} \, dz,$$

with $\hat{B}_1 = (h \, ih) = \frac{1}{2} \begin{pmatrix} i(h'_3 - h'_4) - (h'_3 - h'_4) \\ i(h'_3 + h'_2) - (h'_3 + h'_2) \\ h'_4 - h'_1 \\ i(h'_4 - h'_1) \\ i(h'_4 + h'_1) - (h'_4 + h'_1) \end{pmatrix}.$$

(5.1)

Here $h_j$ are meromorphic functions on $M$ satisfying $h'_1 h'_4 + h'_2 h'_3 = 0$, and $h'_1 \neq 0$. Then the corresponding Willmore surface $[Y_\lambda]$ is of the form $Y_\lambda = R_s Y_1$, with

$$Y_1 = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = |h'_1|^2 \begin{pmatrix} (1 + |h_2|^2 + |h_4|^2) \\ 1 - |h_2|^2 + |h_4|^2 \\ -i(-\bar{h}_2 h_4 + h_2 \bar{h}_4) \\ (-h_2 h_4 + h_4 \bar{h}_4) \\ i(h_3 - h_2) \\ (h_2 + h_4) \end{pmatrix} + |h'_2|^2 \begin{pmatrix} (1 + |h_1|^2 + |h_3|^2) \\ -(1 + |h_1|^2 - |h_3|^2) \\ i(-\bar{h}_1 h_3 + h_1 \bar{h}_3) \\ (\bar{h}_1 h_3 + h_3 \bar{h}_1) \\ i(h_3 - \bar{h}_3) \\ (-h_3 + \bar{h}_3) \end{pmatrix} + h'_1 h'_2 \begin{pmatrix} -\bar{h}_1 h_2 + \bar{h}_3 h_4 \\ \bar{h}_1 h_2 + \bar{h}_3 h_4 \\ -(1 + \bar{h}_1 h_4 + h_2 \bar{h}_3) \\ -\bar{h}_1 h_4 + h_2 \bar{h}_3 \\ i(-\bar{h}_1 + h_4) \\ -(\bar{h}_1 + h_4) \end{pmatrix} + h'_1 h'_2 \begin{pmatrix} -\bar{h}_1 h_2 + \bar{h}_3 h_4 \\ \bar{h}_1 h_2 + \bar{h}_3 h_4 \\ i(-\bar{h}_1 + h_4) \\ -(\bar{h}_1 + h_4) \end{pmatrix},$$

(5.2)

$^2$ Note that the above examples are all of finite uniton numbers [11, 52], that is, the extended frame (of each example) which takes value $e$ at the base point is a Laurant polynomial in $\lambda$ so that one can carry out an Iwasawa factorization explicitly by hands [11, 15, 21, 52, 56].
and

\[
R_\lambda = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \lambda + \frac{1}{2}
\end{pmatrix} \quad (5.3)
\]

\([Y_\lambda]\) is an (possibly branched) isotropic Willmore surface in \(\mathbb{S}^4\).
Moreover, a lift \(\hat{Y}_1\) of the dual surface of \(y_1 = [Y_1]\) is of the form

\[
\hat{Y}_1 = \begin{pmatrix}
\hat{y}_0 \\
\hat{y}_1 \\
\hat{y}_2 \\
\hat{y}_3 \\
\hat{y}_4 \\
\hat{y}_5
\end{pmatrix} = |h_1'|^2 \begin{pmatrix}
(1 + |h_3|^2 + |h_4|^2) \\
-(1 - |h_3|^2 + |h_4|^2) \\
-i(\bar{h}_3 h_4 - h_3 \bar{h}_4) \\
i(-h_3 + h_3) \\
-\bar{h}_3 + h_3
\end{pmatrix} + |h_3'|^2 \begin{pmatrix}
(1 + |h_1|^2 + |h_2|^2) \\
1 + |h_1|^2 - |h_2|^2 \\
i(-\bar{h}_1 h_2 + h_1 \bar{h}_2) \\
i(h_2 - \bar{h}_2) \\
h_2 + \bar{h}_2
\end{pmatrix} + h_1' \bar{h}_1'
\]

\[
= \begin{pmatrix}
\bar{h}_1 h_3 + h_2 h_4 \\
\bar{h}_1 h_3 - \bar{h}_2 h_4 \\
i(1 + h_1 h_4 + \bar{h}_2 h_3) \\
i(h_1 - h_4) \\
\bar{h}_1 + h_4
\end{pmatrix} \quad (5.4)
\]

and \(\hat{Y}_\lambda = R_\lambda \hat{Y}_1\).
Moreover we have

1. \(\hat{Y}_\lambda\) reduces to a point and \([Y_\lambda]\) is conformally equivalent to an isotropic minimal surface in \(\mathbb{R}^4\), if and only if \(h_3' = h_4' = 0\);

2. Both \([Y_\lambda]\) and \([\hat{Y}_\lambda]\) are conformally equivalent to (full) isotropic minimal surfaces in \(\mathbb{S}^4\), if and only if there exists a non-zero, real, constant vector \(v = (v_1, v_2, v_3, v_4)^t \in \mathbb{R}_1^4\) with \(v' I_{1,3} v = -1\), such that

\[
(-v_3 + iv_4)h_1' + (v_1 + iv_2)h_2' + (-v_1 + iv_2)h_3' + (v_3 + iv_4)h_4' = 0; \quad (5.5)
\]

3. Both \([Y_\lambda]\) and \([\hat{Y}_\lambda]\) are conformally equivalent to (full) isotropic minimal surfaces in \(\mathbb{H}^4\), if and only if there exists a non-zero, real, constant vector \(v = (v_1, v_2, v_3, v_4)^t \in \mathbb{R}_1^4\) with \(v' I_{1,3} v = 1\), such that

\[
(-v_3 + iv_4)h_1' + (v_1 + iv_2)h_2' + (-v_1 + iv_2)h_3' + (v_3 + iv_4)h_4' = 0. \quad (5.6)
\]

### 5.2 \(S^1\)—dressing action on isotropic surfaces in \(\mathbb{S}^4\)

Recall that for an isotropic surface in \(\mathbb{S}^4\), its normalized potential has the form [56]

\[
\hat{B}_1 = (h \ i h), \quad \text{with} \quad h = \frac{1}{2} \begin{pmatrix}
i(h_3' - h_2') \\
i(h_3' + h_2') \\
h_4' - h_1' \\
i(h_4' + h_1')
\end{pmatrix} \quad (5.7)
\]
Then we have

\[ T_{1,1}h = \tilde{h} = \begin{pmatrix} i(\tilde{h}_3' - \tilde{h}_2') \\ i(\tilde{h}_3' + \tilde{h}_2') \\ h_4' - h_1' \\ i(h_4' + h_1') \end{pmatrix}, \quad \text{with } \tilde{h}_2 = e^{-it}h_2, \tilde{h}_3 = e^{it}h_3. \]

By Theorem 4.2 and 4.3, when \( h_2 = h_3 \), we obtain a minimal surface in \( S^4 \). When \( \tilde{h}_2 = -\tilde{h}_3 \) we obtain a minimal surface in \( \mathbb{H}^4 \).

**Proposition 5.2** We retain the notations in Theorem 5.1. Assume furthermore that \( h_2 = h_3 \) in (5.7) and set \( y_t = T_{1,2}y \), with \( T_{1,2} = \text{diag}(T_{1,1}, I_2) \). Then

1. \( y_t \) is conformally equivalent to a minimal surface in \( S^4 \) if and only if \( t = 0 \) or \( \pi \);
2. \( y_t \) is conformally equivalent to a minimal surface in \( \mathbb{H}^4 \) if and only if \( t = \frac{\pi}{2} \) or \( \frac{3\pi}{2} \);
3. \( y_t \) is not conformally equivalent to any minimal surface in any space form for any \( t \in (0, 2\pi) \) and \( t \notin \{ \frac{\pi}{2}, \pi, \frac{3\pi}{2} \} \).

**Proposition 5.3** We retain the notations in Theorem 5.1 and set \( T_t = \text{diag}(T_{2,1}, I_2) \).

1. Assume that \( h_2 = h_3 \) and \( y_t = T_{1,2}y \). Then \( y_t \) is conformally equivalent to a minimal surface in \( S^4 \) for any \( t \in \mathbb{R} \).
2. Assume that \( h_2 = -h_3 \) and \( y_t = T_{1,2}y \). Then \( y_t \) is conformally equivalent to a minimal surface in \( \mathbb{H}^4 \) for any \( t \in \mathbb{R} \).

### 5.3 The Veronese sphere and its Willmore deformations

Applying to the Veronese surface in \( S^4 \), we obtain many new Willmore two-spheres in \( S^4 \) with the same Willmore energy. Moreover, we also obtain many examples of minimal surfaces in \( \mathbb{H}^4 \) with Willmore energy taking every value in \( (0, 2\pi) \).

#### 5.3.1 The Veronese sphere and its \( S^1 \)—Willmore deformations

**Proposition 5.4** Let \( z = re^{it} \). Set

\[
    h_1 = -2\bar{z}z^2, \quad h_2 = \sqrt{3}i\bar{z}^2, \quad h_3 = \sqrt{3}iz^2, \quad h_4 = -2z, \quad (5.8)
\]

in (5.7). Let \( [Y] \) be the corresponding Willmore surface in \( S^4 \). Set \( Y_t = Y_t \bar{Z}Y \) with \( T_t = \text{diag}(T_{1,1}, I_2) \). Then

\[
    Y_t = \begin{pmatrix} r^4 + 2r^2 + 1 \\ -r^4 + 4r^2 - 1 \\ \sqrt{3}(ze^{-it} + \bar{z}e^{it} - r^2(ze^{it} + \bar{z}e^{-it})) \\ -i\sqrt{3}(ze^{-it} - \bar{z}e^{it} - r^2(ze^{it} - \bar{z}e^{-it})) \\ \sqrt{3}(\bar{z}e^{-it} + z\bar{e}^{it} + r^2(\bar{z}e^{it} + ze^{-it})) \\ i\sqrt{3}(\bar{z}e^{-it} - z\bar{e}^{it} - r^2(\bar{z}e^{it} - ze^{-it})) \\ 1 + r^2 \end{pmatrix} = \begin{pmatrix} r^4 + 2r^2 + 1 \\ -r^4 + 4r^2 - 1 \\ 2\sqrt{3}r(\cos(\theta-t) - r^2 \cos(\theta+t)) \\ 2\sqrt{3}r(\sin(\theta-t) - r^2 \sin(\theta+t)) \\ 2\sqrt{3}r^2(\cos(2\theta-t) + r^2 \cos(2\theta+t)) \\ -2\sqrt{3}r^2(\sin(2\theta-t) + r^2 \sin(2\theta+t)) \\ 1 + r^2 \end{pmatrix}. \quad (5.9)
\]
and \( y_t = [Y_t] : S^2 \to S^4 \) is an isotropic Willmore immersion with

\[
y_t = \frac{1}{(r^2 + 1)^3} \begin{pmatrix}
-r^6 + 3r^4 + 3r^2 - 1 \\
2\sqrt{3}r (\cos(\theta - t) - r^4 \cos(\theta + t)) \\
2\sqrt{3}r (\sin(\theta - t) - r^4 \sin(\theta + t)) \\
2\sqrt{3}r^2 (\cos(2\theta - t) + r^2 \cos(2\theta + t)) \\
-2\sqrt{3}r^2 (\sin(2\theta - t) + r^2 \sin(2\theta + t))
\end{pmatrix}
\]  

(5.10)

and

\[
|dy_t|^2 = \frac{12(r^8 + 4r^6 + 6r^4 \cos 2t + 4r^2 + 1)}{(r^2 + 1)^6} |dz|^2.
\]  

(5.11)

1. \( W([Y_t]) = 8\pi \) for all \( t \in [0, 2\pi] \). \([Y_t]\) is conformally equivalent to \([Y_{t+\pi}]\) for all \( t \in [0, \pi] \). And for any \( t_1, t_2 \in [0, \pi] \), \([Y_{t_1}]\) is conformally equivalent to \([Y_{t_2}]\) if and only if \( t_1 = t_2 \) or \( t_1 + t_2 = \pi \).

2. \([Y_t]\) is conformally equivalent to the Veronese surface in \( S^4 \) when \( t = 0 \) and \([Y_t]\) is conformally equivalent to three complete minimal surfaces in \( H^4 \) on three open subsets of \( S^2 \) when \( t = \frac{\pi}{2} \). For any other \( t \in (0, \pi) \), \([Y_t]\) is a Willmore surface in \( S^4 \) not minimal in any space form.

3. When \( t = \frac{3\pi}{2} \), consider the projection of \( [(Y_t)|_{\sigma = \frac{3\pi}{2}}] \) into \( H^4 \) w.r.t. \( (0, 1, 0, 0, 0, 0)^t \in \mathbb{R}^6 \).

\[
\hat{y} = \frac{-1}{(1 + r^2)(r^4 - 4r^2 + 1)} \begin{pmatrix}
(1 + r^2)^3 \\
\sqrt{3}i(z - \bar{z})(1 + r^4) \\
\sqrt{3}(z + \bar{z})(1 + r^4) \\
\sqrt{3}i(z^2 - \bar{z}^2)(1 - r^2) \\
\sqrt{3}(z^2 + \bar{z}^2)(1 - r^2)
\end{pmatrix}.
\]  

(5.12)

It has metric

\[
|d\hat{y}|^2 = \frac{12(r^8 + 4r^6 - 6r^4 + 4r^2 + 1)}{(r^2 + 1)^2(r^4 - 4r^2 + 1)^2} |dz|^2
\]  

and Gauss curvature

\[
K = -1 - \frac{2}{3} \frac{(r^2 + 1)^4(r^4 - 4r^2 + 1)^4}{(r^8 + 4r^6 - 6r^4 + 4r^2 + 1)^3}
\]  

(5.13)

on \( S^2 \setminus \{|z| = r_1\} \cup \{|z| = r_2\} \). Here \( r_1 = \sqrt{\frac{6 - \sqrt{3}}{2}} \) and \( r_2 = \sqrt{\frac{6 + \sqrt{3}}{2}} \). Set

\[
M_1 = \{z \in \mathbb{C} : |z| < r_1\}, \ M_2 = \{z \in \mathbb{C} : r_1 < |z| < r_2\}, \ M_3 = \{z \in \mathbb{C} : |z| > r_2\}.
\]

1. Set \( \mu(z) := -\frac{1}{z} \) on \( S^2 \). Then

\[
\hat{y} \circ \mu = R\hat{y}, \text{ with } R = \text{diag}(1, -1, -1, 1, 1).
\]

2. \( \hat{y}|_{M_1} : M_1 \to H^4 \) is a proper, complete minimal disk with finite Willmore energy \((4 - 2\sqrt{3})\pi\). Its Gauss curvature takes value in \([-\frac{5}{3}, -1]\). In particular, it has bounded Gauss curvature. And \( \hat{y}|_{M_3} \) is congruent to \( \hat{y}|_{M_1} \) in the sense \( \hat{y}|_{M_3} = R(\hat{y} \circ \sigma)|_{M_1} \).

3. \( \hat{y}|_{M_2} : M_2 \to H^4 \) is a proper, complete minimal annulus with finite Willmore energy \(4\sqrt{3}\pi\). Its Gauss curvature takes value in \([-\frac{11}{3}, -1]\). In particular, it has bounded Gauss curvature.

4. Each of the three minimal surfaces intersects the infinite boundary of \( H^4 \) with a constant angle \(< \frac{\pi}{2}\). The circles \( r = \sqrt{\frac{6 \pm \sqrt{3}}{2}} \) are the umbilical sets of the Willmore immersion \( [Y_t]|_{t = \frac{3\pi}{2}} \).
To be concrete, we have $Y_t$, the Veronese two-sphere, which is not possible. Therefore, given by \[ (5.9) \]

By Theorem 4.2, we see that (2) holds. From (5.9) we see that each $Y_t$ admits an $S^1$-symmetry given by

$$\mathbb{R}_t = \text{diag}(I_2, R_t, R_{2t}), \quad R_t = \begin{pmatrix} \cos \tilde{t} - \sin \tilde{t} \\ \sin \tilde{t} \cos \tilde{t} \end{pmatrix}, \quad R_{2t} = \begin{pmatrix} \cos 2\tilde{t} & \sin 2\tilde{t} \\ -\sin 2\tilde{t} & \cos 2\tilde{t} \end{pmatrix}.$$

To be concrete, we have $Y_t(ze^{i\tilde{t}}, \bar{z}e^{-i\tilde{t}}) = \mathbb{R}_t Y_t$. Moreover, for any $t \in (0, \pi)$, $[Y_t]$ does not admit another $S^1$-symmetry. Otherwise, we will see that $[Y_t]$ is a homogeneous Willmore two-sphere since it has two different $S^1$-symmetry. By [23, 41], it is conformally equivalent to the Veronese two-sphere, which is not possible. Therefore, $[Y_{t_1}]$ is conformally equivalent to $[Y_{t_2}]$ only if $y_{t_1}$ is isometric to $y_{t_2}$, which by (5.11), if and only if $t_1 = t_2$ or $t_1 + t_2 = \pi$. By (5.9), $[Y_{t_1}]$ is conformally equivalent to $[Y_{t_2}]$ if $t_1 + t_2 = \pi$. This finishes (1). Then (2) comes from Theorem 4.2.

Then (3) comes from a lengthy but straightforward computation. Note that the properness of $\tilde{y}|_{M_j}$, $j = 1, 2, 3$ comes from the fact that they have smooth boundary curves at infinity.

\[\Box\]

**Remark 5.5**

1. Note that $K$ attains maximal value $-1$ at $r = \sqrt{6+\sqrt{2}}$ and attains minimal value $-\frac{11}{3}$ at $r = 1$ (1). This means that the two circles $r = \sqrt{6+\sqrt{2}}$ on $S^2 = \mathbb{C}$ are exactly the umbilical sets on the Willmore surface $[Y_t]|_{t = \frac{3\pi}{2}}$ (Compare also [4]).

2. The surface $[\tilde{Y}_{t = \frac{3\pi}{2}}]$ can be looked as a combination of three complete minimal surfaces $\tilde{y}|_{M_j}$ in $\mathbb{H}^4$, with $j = 1, 2, 3$. To be concrete, when $|z| < \sqrt{6+\sqrt{2}}$, the surface takes values in the upper connected component of $\mathbb{H}^4$, and tends to the boundary of $\mathbb{H}^4$ when $|z| \to \sqrt{6+\sqrt{2}}$ from the left side. When $|z| = \sqrt{6-\sqrt{2}}$, it takes values at the boundary of $\mathbb{H}^4$. When $\sqrt{6-\sqrt{2}} < |z| < \sqrt{6+\sqrt{2}}$, it takes values in the lower connected component of $\mathbb{H}^4$, and tends to the boundary of $\mathbb{H}^4$ again when $|z| \to \sqrt{6+\sqrt{2}}$ from the right side. When $|z| = \sqrt{6+\sqrt{2}}$, it takes values at the boundary of $\mathbb{H}^4$ again. When $|z| > \sqrt{6+\sqrt{2}}$, it again takes values in the upper connected component of $\mathbb{H}^4$. When viewing the surface in $\mathbb{H}^4$, it blows up at the points $\sqrt{6+\sqrt{2}}$. If we embed $\mathbb{H}^4$ conformally into $S^4$, the surface will be a smooth immersion on the whole $S^2$. This is the well-known construction of compact Willmore surfaces due to Babich and Bobenko [4] for minimal surfaces in $\mathbb{H}^3$, where they constructed successfully Willmore tori with a umbilical line in $S^3$ via this way. It is hence not surprising that similar construction also works for Willmore two-spheres. To the authors’ best knowledge, the example in Proposition 5.4 should be the first explicit example of Willmore two-sphere in $S^4$ which is conformally equivalent to some minimal surface in $\mathbb{H}^4$ on an open subset of $S^2$ (Note that this is not possible for Willmore two-spheres in $S^3$ except the round sphere, due to Bryant’s famous classification theorem of Willmore two-spheres in $S^3$ [9]).

3. In [45], it is shown that all Willmore two-spheres with $W([Y_t]) = 8\pi$ are expressed as twistor deformations of the Veronese surface in $S^4$. Here we derive some explicit
examples. Moreover, the generating curve of the $S^1$–equivariant Willmore two-sphere $y_t, t \in (0, \frac{\pi}{2})$, in $S^4$ is

$$
\gamma_t = \frac{1}{(r^2 + 1)^3} \begin{pmatrix}
-r^6 + 3r^4 + 3r^2 - 1 \\
2\sqrt{3}r (1 - r^4) \cos t \\
-2\sqrt{3}r (1 + r^4) \sin t \\
2\sqrt{3}r^2 (1 + r^2) \cos t \\
2\sqrt{3}r^2 (1 - r^2) \sin t
\end{pmatrix}.
$$

So $\gamma_t$ is full in $S^4$ for all $t \in (0, \frac{\pi}{2})$ and $\gamma_t$ takes value in some $S^2 \subset S^4$ when $t = 0, \frac{\pi}{2}$. This indicates that in general, $S^1$–equivariant Willmore two-spheres in $S^4$ have more complicated structures than $S^1$–equivariant minimal two-spheres in $S^4$ [28].

4. Different from the case of complete minimal surfaces in $\mathbb{H}^3$ with finite Willmore energy, which always intersect the infinity boundary orthogonally as shown in [2], here the complete minimal surface $\tilde{y}$ intersect the infinity boundary with a constant angle not equal to $\frac{\pi}{2}$.

5.3.2 $\mathbb{R}$–minimal deformations of the minimal surface $\tilde{y}$

Let us consider the $\mathbb{R}$–minimal deformations of the minimal surface $\tilde{y}$ in $\mathbb{H}^4$ given in (5.12), by the $\mathbb{R}$–minimal deformations, we obtain a lot of (non-congruent) complete minimal surfaces in $\mathbb{H}^4$.

**Proposition 5.6** Let $z = re^{i\theta}$. Set

$$
h_1 = -2z^3, \quad h_2 = \sqrt{3}z^2, \quad h_3 = -\sqrt{3}z^2, \quad h_4 = -2z.
$$

(5.14)
in (5.7). Let \([Y]\) be the corresponding Willmore surface in \(S^4\). Set \(Y_t = T_t Y\) with \(T_t = \text{diag}(T_{2,t}, I_2)\). Then

\[
Y_t = \begin{pmatrix}
0 \\
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5
\end{pmatrix} = \begin{pmatrix}
e^{2t}r^6 + 3r^4 + 3e^{2t}r^2 + 1 \\
-e^{2t}r^6 + 3r^4 + 3e^{2t}r^2 - 1 \\
l\sqrt{3}e^t (1 + r^4) (z - \bar{z}) \\
l\sqrt{3}e^t (1 + r^4) (z + \bar{z}) \\
l\sqrt{3}(1 - e^{2t}r^2) (z^2 + \bar{z}^2) \\
-l\sqrt{3}(1 - e^{2t}r^2) (z^2 + \bar{z}^2)
\end{pmatrix}.
\]

(5.15)

1. For every \(t \in \mathbb{R}\), \([Y_t]\) is a Willmore immersion from \(S^2\) to \(S^4\) with \(W([Y_t]) = 8\pi\) and \([Y_t]\) is oriented for all \(t \in \mathbb{R}\). Moreover, \([Y_t(z, \bar{z})]\) is conformally equivalent to \([Y_{-t}(\frac{1}{t}, \frac{1}{t})]\) for all \(t \in \mathbb{R}\).

2. Set

\[
y_t = \frac{1}{y_1} \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}.
\]

Then \(y_t\) is minimally immersed into \(\mathbb{H}^4\) on the points where \(y_0 \neq 0\), with metric

\[
|\langle y_t, \bar{z}\rangle| = \frac{6(e^{2t}r^6 + 4e^{4t}r^6 - 6e^{2t}r^4 + 4r^2 + e^{2t})}{(e^{2t}r^6 - 3r^4 - 3e^{2t}r^2 + 1)^2}
\]

and curvature

\[
K = -1 - \frac{2e^{2t}(e^{2t}r^6 - 3r^4 - 3e^{2t}r^2 + 1)^4}{3(e^{2t}r^6 + 4e^{4t}r^6 - 6e^{2t}r^4 + 4r^2 + e^{2t})^3}
\]

In particular, set

\[
M_{t,1} = \{z \in \mathbb{C} | |z| < r_1\},
\]

\[
M_{t,2} = \{z \in \mathbb{C} | r_1 < |z| < r_2\},
\]

\[
M_{t,3} = \{z \in \mathbb{C} | |z| > r_2\}.
\]

Here we denote by \(r_1\) and \(r_2\) the two positive solutions to \(e^{2t}r^6 - 3r^4 - 3e^{2t}r^2 + 1 = 0\) with \(0 < r_1 < r_2\):\(^3\)

\[
r_1^2 = \sqrt{1 + e^{-4t}} \left(\cos 3\theta_0 - 2 \cos(\theta_0 + \frac{\pi}{3})\right), \quad r_2^2 = \sqrt{1 + e^{-4t}} \left(\cos 3\theta_0 + 2 \cos \theta_0\right).
\]

Here \(\theta_0 = \frac{1}{3} \arccos \frac{1}{\sqrt{1+e^{-4t}}}\). Then we obtain two complete minimal disks \(M_{t,1}, M_{t,3}\) and one complete minimal annulus \(M_{t,2}\) in \(\mathbb{H}^4\).

3. \([Y_t]\vert_{M_{t,1}}\) and \([Y_t]\vert_{M_{t,3}}\) are conformally equivalent to complete immersed, isotropic minimal disks \(y_{t,1}\) and \(y_{t,3}\) in \(\mathbb{H}^4\). Moreover, \(y_{t,1}\) and \(y_{t,3}\) are isometrically congruent if and only if \(t = 0\). \([Y_t]\vert_{M_{t,2}}\) is conformally equivalent to an immersed, complete, isotropic minimal annulus \(y_{t,2}\) in \(\mathbb{H}^4\).

4. When \(t \to +\infty\), \([Y_t]\) tends to a branched double cover of a totally geodesic surface \(y_\infty\) in \(S^4\) which is orthogonal to the equator \(S^3_0 = \{x \in S^4 | x \perp (1, 0, 0, 0, 0)\}\).

\(^3\) Note that \(\cos 3\theta_0 - 2 \cos \left(\theta_0 + \frac{\pi}{3}\right) = 2 \sin \theta_0 \sin \frac{\pi}{3} - \sin 2\theta_0 > 0\) since \(0 < \theta_0 < \pi/6\) for all \(t \in \mathbb{R}\).
5. When \( t = 0 \), \( W([Y_t]|_{M_{t,1}}) = W([Y_t]|_{M_{t,3}}) = (4 - 2\sqrt{3})\pi \). When \( t \to +\infty \), \( W([Y_t]|_{M_{t,1}}) \to 0 \). There exists \( t \in \mathbb{R}^+ \) such that \( W([Y_t]|_{M_{t,1}}) > 1.9999\pi \). Hence for every \( c_0 \in (0, 1.9999\pi] \), there exists some \( t \in \mathbb{R} \) such that \( W([Y_t]|_{M_{t,1}}) = c_0 \).

**Proof** (1) and (2) come from direct computations, as shown in the proposition. (3) is obvious. Now let’s consider (4). When \( t \to +\infty \), from (5.15) it is direct to see that \( y_t \) tends to

\[
\frac{1}{3 + r^4} \begin{pmatrix} -r^4 + 3 \\ 0 \\ -i\sqrt{3}(z^2 - z^2) \\ \sqrt{3}(z^2 + z^2) \end{pmatrix},
\]

which is exactly a branched double covering of a totally geodesic surface \( \gamma_\infty \) orthogonal to the infinity boundary of \( \mathbb{H}^4 \). Moreover, \( [Y_t]|_{M_{t,1}} \) tends to the branched point \( p_0 = (1, 0, 0, 0)' \). The equator \( S_0^3 \) divides \( \gamma_\infty \) into two parts: \( \gamma_\infty^+ \) (containing \( p_0 \)) and \( \gamma_\infty^- \). Therefore \( [Y_t]|_{M_{t,2}} \) tends to \( \gamma_\infty^- \setminus \{p_0\} \) and \( [Y_t]|_{M_{t,3}} \) tends to \( \gamma_\infty^+ \).

Finally, let’s consider (5). First we note that the Willmore energy of \( [Y_t]|_{M_{t,j}} \) are

\[
W(M_{t,j}) = 16\pi \int_{r_{j-1}}^{r_j} \frac{e^{2t}(e^{2t}r^6 - 3r^4 - 3e^{2t}r^2 + 1)^2}{(e^{2t}r^8 + 4e^{4t}r^6 - 6e^{2t}r^4 + 4r^2 + e^{2t}r^2)rdr}, \quad j = 1, 2, 3,
\]

with \( r_0 = 0, r_3 = +\infty \) and \( r_1 \) and \( r_2 \) as shown in the proposition.

Since

\[
\lim_{t \to +\infty} r_1 = 0 \quad \text{and} \quad \lim_{t \to +\infty} e^{2t}r_1 = 1,
\]

when \( t \to +\infty \) we have for \( 0 \leq r \leq r_1 \)

\[
(e^{2t}r^8 + 4e^{4t}r^6 - 6e^{2t}r^4 + 4r^2 + e^{2t})^2 \geq e^{4t}, \quad (e^{2t}r^6 - 3r^4 - 3e^{2t}r^2 + 1)^2 < 1.
\]

So when \( t \to +\infty \),

\[
\int_0^{r_1} \frac{e^{2t}(e^{2t}r^6 - 3r^4 - 3e^{2t}r^2 + 1)^2}{(e^{2t}r^8 + 4e^{4t}r^6 - 6e^{2t}r^4 + 4r^2 + e^{2t}r^2)rdr} \leq \int_0^{r_1} e^{-2t}rdr = 2e^{-2t}r_1^2.
\]

So

\[
\lim_{t \to +\infty} W(M_{t,1}) = 0.
\]

On the other hand, numerical computation shows when \( t = \ln 0.000039 \),

\[
W(M_{t,2}) \approx 6.000089931\pi, \quad W(M_{t,1}) \approx 1.999910062\pi.
\]

Since \( W(M_1) \) depends continuously on \( t \), we see that for any number \( c_0 \in (0, 1.9999\pi] \), there exists some \( t_0 \in \mathbb{R} \) such that \( W(M_1) = c_0 \) for \( t = t_0 \). This finishes the proof.

**Remark 5.7** It is interesting to ask whether there exists a complete minimal annulus \( x \) in \( \mathbb{H}^4 \) with \( W(x) \leq 6\pi \). Moreover, what is the infimum of the Willmore energy of a complete minimal annulus \( x \) in \( \mathbb{H}^4 \)?
5.3.3 \( \mathbb{R} \)–minimal deformations of the Veronese two-sphere in \( S^4 \)

Similarly we can construct a family of minimal two-spheres in \( S^4 \) via the \( \mathbb{R} \)–action on the Veronese two-sphere in \( S^4 \).

**Proposition 5.8** Let \( z = re^{i\theta} \). Set

\[
\begin{align*}
    h_1 &= -2z^3, \\
    h_2 &= \sqrt{3}iz^2, \\
    h_3 &= \sqrt{3}iz^2, \\
    h_4 &= -2z.
\end{align*}
\]

Set \( Y_t = T_t Y \) with \( T_t = \text{diag}(T_{2,t}, I_2) \). Then

\[
Y_t = \begin{pmatrix}
    y_0 \\
    y_1 \\
    y_2 \\
    y_3 \\
    y_4 \\
    y_5
\end{pmatrix} = \begin{pmatrix}
    e^{2tr_8'} + 3r^4 + 3e^{2t}r^2 + 1 \\
    -e^{2tr_8'} + 3r^4 + 3e^{2t}r^2 - 1 \\
    \sqrt{3}e^{(1 - t^4)}(z + \bar{z}) \\
    -i\sqrt{3}e^{(1 - t^4)}(z - \bar{z}) \\
    \sqrt{3}(1 + e^{2tr^2})(z^2 + \bar{z}^2) \\
    i\sqrt{3}(1 + e^{2tr^2})(z^2 - \bar{z}^2)
\end{pmatrix}.
\]

1. For every \( t \in \mathbb{R} \), \( [Y_t] \) is conformally equivalent to an immersed isotropic minimal two-sphere \( y_t = \frac{1}{y_0}(y_1 y_2 y_3 y_4 y_5) \) in \( S^4 \) with \( W([Y_t]) = 8\pi \),

\[
|d(y_t)|^2 = \frac{12(e^{2tr_8'} + 4e^{4tr_8'} + 6e^{2tr^2} r^4 + 4r^2 + e^{2tr})}{(e^{2tr_8'} + 3r^4 + 3e^{2tr^2} + 1)^2}|dz|^2
\]

and

\[
K_t = 1 - \frac{2e^{2t}(e^{2tr_8'} + 3r^4 + 3e^{2tr^2} + 1)^4}{3(e^{2tr_8'} + 4e^{4tr_8'} + 6e^{2tr^2} + 4r^2 + e^{2tr})}.
\]

2. \( [Y_t] \) descend to a minimal \( \mathbb{R}P^2 \) if and only if \( t = 0 \).
3. When \( t \to \infty \), \( y_t \) tends to a branched double covering of a totally geodesic round two-sphere of \( S^4 \).

5.4 \( S^1 \)–deformation of generalizations of Veronese two-sphere in \( S^4 \)

In [24], some generalizations of Veronese two-sphere in \( S^4 \) are discussed. Here we consider the \( S^1 \)–deformation of them, which will give more examples of complete minimal surfaces in \( \mathbb{H}^4 \). They are important in Willmore energy estimates of complete minimal surfaces in \( \mathbb{H}^4 \).

**Proposition 5.9** Let \( z = re^{i\theta} \). Set

\[
\begin{align*}
    h_1 &= -kz^{k+1}, \\
    h_2 &= i\sqrt{k^2 - 1}z^k, \\
    h_3 &= i\sqrt{k^2 - 1}z^k, \\
    h_4 &= -kz^{k-1},
\end{align*}
\]

in (5.7). Let \([\hat{Y}]\) be the corresponding Willmore surface in \( S^4 \). Set \( \hat{Y}_t = T_t \hat{Y} \) with \( T_t = \text{diag}(T_{1,t}, I_2) \). Then

\[
\hat{Y}_t = \begin{pmatrix}
    \hat{y}_0 \\
    \hat{y}_1 \\
    \hat{y}_2 \\
    \hat{y}_3 \\
    \hat{y}_4 \\
    \hat{y}_5
\end{pmatrix} = \begin{pmatrix}
    (k - 1)(r^{2k+2} + 1) + (k + 1)(r^{2k} + r^2) \\
    -k(1)(r^{2k+2} + 1) + (k + 1)(r^{2k} + r^2) \\
    \sqrt{k^2 - 1}((z^{k}e^{-it} + \bar{z}^{k}e^{it}) - r^{2k}(z^{k}e^{it} + \bar{z}^{k}e^{-it})) \\
    i\sqrt{k^2 - 1}((z^{k}e^{-it} + \bar{z}^{k}e^{it}) - r^{2k}(z^{k}e^{it} - \bar{z}^{k}e^{-it})) \\
    \sqrt{k^2 - 1}((z^{k}e^{-it} + \bar{z}^{k}e^{it}) + r^{2}(z^{k}e^{it} + \bar{z}^{k}e^{-it})) \\
    i\sqrt{k^2 - 1}((z^{k}e^{-it} - \bar{z}^{k}e^{it}) + r^{2}(z^{k}e^{it} - \bar{z}^{k}e^{-it}))
\end{pmatrix}.
\]
1. For every $t \in [0, 2\pi)$, $[\hat{Y}_t]$ is an oriented Willmore immersion from $S^2$ to $\mathbb{S}^4$ with Willmore energy $4\pi k$. $[\hat{Y}_t]$ is conformally equivalent to $[\hat{Y}_{t+\pi}]$ for all $t \in [0, \pi]$. And for any $t_1, t_2 \in [0, \pi)$, $[\hat{Y}_{t_1}]$ is conformally equivalent to $[\hat{Y}_{t_2}]$ if and only if $t_1 = t_2$ or $t_1 + t_2 = \pi$.

2. $[\hat{Y}_t]$ is conformally equivalent to a minimal two-sphere in $\mathbb{S}^4$ when $t = 0$ and $[Y_t]$ is conformally equivalent to three complete minimal surfaces in $\mathbb{H}^4$ on open subsets of $S^2$ when $t = \frac{\pi}{2}$. For any other $t \in (0, \pi)$, $[Y_t]$ Willmore surfaces in $\mathbb{S}^4$ not minimal in any space form.

3. $[\hat{Y}_t]$ reduces to a non-oriented Willmore surface from $\mathbb{R}P^2 = S^2 / \mu$, if and only if $t = 0$ or $\pi$, and $k = 2\tilde{k}$ for some $\tilde{k} \in \mathbb{Z}^+$. Here $\mu(z) = -\frac{1}{z'}$.

**Proof** The equation (5.21) comes from direct computations. We need only to show that $W([\hat{Y}_t]) = 4\pi k$, since proofs of the rest of (1) and (2) are the same as Proposition 5.4. Since the Willmore energy of $[Y_t]$ depends smoothly on $t$ and the Willmore energy of a Willmore two-sphere is $4\pi m$ for some $m \in \mathbb{Z}$ [45], we have $\text{Area}(\hat{Y}_t) = \text{Area}(\hat{Y})$. By Theorem 3.1 of [28] (see also [5]), $\text{Area}([\hat{Y}_t]) = \text{Area}(\hat{Y}) = 4\pi (k + 1)$ since the equivariant action here is $(m_1, m_2) = (1, k)$.

Substituting $\mu$ into (5.21) shows that $[\hat{Y}_t \circ \mu] = [\hat{Y}_t]$ if and only if $k$ is even and $t = 0$ or $\pi$, which finishes the proof of (3).

---

### 5.5 $\mathbb{R}$—minimal deformations of another type of minimal surfaces in $\mathbb{H}^4$

It is natural to show the existence of complete minimal surfaces in $\mathbb{H}^4$ with any Willmore energy $W_0 \in \mathbb{R}^+ \cup \{0\}$ by further generalization of the above examples.

**Proposition 5.10** Let $z = re^{i\theta}$. Let $[Y] = \hat{Y}_{t=\frac{\pi}{2}}$. Then its normalized potential can be given by setting

$$h_1 = -kz^{k+1}, \quad h_2 = \sqrt{k^2 - 1}z^k, \quad h_3 = -\sqrt{k^2 - 1}z^k, \quad h_4 = -kz^{k-1}, \quad k \geq 2, \quad (5.22)$$

in (5.7). Set $Y_t = T_t \hat{Y}$ with $T_t = \text{diag}(T_{t_1}, T_2)$. Then

$$Y_t = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} (k-1)(e^{2r}r^{2k+2}+1)+(k+1)(r^{2k}+e^{2r}r^2) \\ -(k-1)(e^{2r}r^{2k+2}+1)+(k+1)(r^{2k}+e^{2r}r^2) \\ ie^t(\sqrt{k^2-1}(1+r^{2k})(z-\bar{z})) \\ e^t(\sqrt{k^2-1}(1+r^{2k})(z+\bar{z})) \\ i\sqrt{k^2-1}(1-e^{2r}r^2)(z^k-\bar{z}^k) \\ -\sqrt{k^2-1}(1-e^{2r}r^2)(z^k+\bar{z}^k) \end{pmatrix}. \quad (5.23)$$

1. For every $t \in \mathbb{R}$, $[Y_t]$ is an oriented Willmore immersion from $S^2$ to $\mathbb{S}^4$ with Willmore energy $4\pi k$ and $[Y_t(z, \bar{z})]$ is conformally equivalent to $[Y_{t=\frac{\pi}{2}}(\frac{1}{z}, \frac{1}{z})]$.

2. Set

$$y_t = \frac{1}{y_1} \begin{pmatrix} y_0 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}'.$$

Then $y_t$ is minimally immersed into $\mathbb{H}^4$ on the points where $y_0 \neq 0$, with metric

$$|dy_t|^2 = \frac{4(k^2-1)(e^{2r}(1+r^{2k})^2+k^2r^{2k-2}(1-e^{2r}r^2)^2)}{(k-1)(e^{2r}r^{2k+2}+1)-(k+1)(r^{2k}+e^{2r}r^2)^2}|dz|^2.$$
and curvature
\[ K = -1 - \frac{k^2 e^{2t} r^{2k-4} ((k-1)(e^{2t} r^{2k+2} + 1) - (k+1)(r^{2k} + e^{2t} r^2))^4}{2(k^2 - 1) \left( e^{2t} (1 + r^{2k})^2 + k^2 r^{2k-2} (1 - e^{2t} r^2)^2 \right)^3} \]

In particular, set
\[ M_{t,1} = \{ z \in \mathbb{C} \mid |z| < r_1 \}, \]
\[ M_{t,2} = \{ z \in \mathbb{C} \mid r_1 < |z| < r_2 \}, \]
\[ M_{t,3} = \{ z \in \mathbb{C} \mid |z| > r_2 \}. \]

Here we denote by \( r_1 \) and \( r_2 \) the two positive solutions to
\[ (k-1)(e^{2t} r^{2k+2} + 1) - (k+1)(r^{2k} + e^{2t} r^2) = 0 \]
with \( 0 < r_1 < r_2 \). Then we obtain two complete minimal disks \( M_{t,1}, M_{t,3} \) and one complete minimal annulus \( M_{t,2} \) in \( \mathbb{H}^4 \).

3. \([Y_t]\mid_{M_{t,1}} \text{ and } [Y_t]\mid_{M_{t,3}}\) are conformally equivalent to complete immersed, isotropic minimal disks \( y_{t,1} \) and \( y_{t,3} \) in \( \mathbb{H}^4 \). Moreover, \( y_{t,1} \) and \( y_{t,3} \) are isometrically congruent if and only if \( t = 0 \). \([Y_t]\mid_{M_{t,2}}\) is conformally equivalent to an immersed, complete, isotropic minimal annulus \( y_{t,2} \) in \( \mathbb{H}^4 \).

4. For every fixed \( k \), when \( t \to +\infty \), \([Y_t]\mid_{M_{t,1}}\) tends to a branched \( k \)-cover of a totally geodesic surface \( y_{\infty} S^4 \) which is orthogonal to the equator \( S^3_0 = \{ x \in S^4 \mid x \perp (1, 0, 0, 0, 0)^t \} \).

5. When \( t \to +\infty \), \( W([Y_t]\mid_{M_{t,1}}) \to 0 \).

6. Set \( t_0 = \frac{1-k}{2} \ln k \). Then when \( k \) is large enough,
\[ W([Y_{t_0}]\mid_{M_{t_0,1}}) \geq \frac{(k-1)\pi}{3}. \] (5.24)

Moreover, when \( k \to +\infty \), \( W([Y_{t_0}]\mid_{M_{t_0,1}}) \to +\infty \). In particular for every \( W_0 \in \mathbb{R}^+ \), there exists some \( k \in \mathbb{Z}^+ \) with \( k > 2 + \frac{3W_0}{\pi} \), and \( t' \in \mathbb{R} \), such that \( W([Y_{t'}]\mid_{M_{t',1}}) = W_0 \).

**Proof** (1). By Proposition 5.9, we have \( W([Y_t]) = W([Y]) = 4\pi k \).

The proof of (2)-(4) is the same as Proposition 5.6. So let’s focus on (5) and (6). First we note that the Willmore energy of \([Y_t]\mid_{M_{t,j}}\) are (Here \( b = \frac{k+1}{k-1} \))
\[ W(M_{t,j}) = 4\pi \int_{r_{j-1}}^{r_j} \frac{k^2 (k-1)^2 e^{2t} r^{2k-3} (e^{2t} r^{2k+2} + 1 - b(r^{2k} + e^{2t} r^2))^2}{(e^{2t} (1 + r^{2k})^2 + k^2 r^{2k-2} (1 - e^{2t} r^2)^2)^2} \, dr, \quad j = 1, 2, 3, \] (5.25)

with \( r_0 = 0, r_3 = +\infty \) and \( r_1 \) and \( r_2 \) as shown in the proposition.

It is direct to check that
\[ \lim_{t \to +\infty} r_1 = 0 \quad \text{and} \quad e^{2t} r_1^2 \leq 1. \]

When \( t \to +\infty \) we have for \( 0 \leq r \leq r_1 \)
\[ e^{2t} (1 + r^{2k})^2 + k^2 r^{2k-2} (1 - e^{2t} r^2)^2 \geq e^{2t} \left( e^{2t} r^{2k+2} + 1 - b(r^{2k} + e^{2t} r^2) \right) < 1. \]
So when $t \to +\infty$,
\[
\int_0^{r_1} \frac{k^2 e^{2t} r^{2k-3} (e^{2t} r^{2k+2} + 1 - b(r^{2k} + e^{2t} r^2))^2}{(e^{2t} (1 + r^{2k})^2 + k^2 r^{2k-2}(1 - e^{2t} r^2)^2)^2} \, dr \leq \int_0^{r_1} \frac{2k^2 e^{2t} r^{2k-3} dr}{(e^{2t})^2 r^{2k-3} dr} = \frac{2k^2 e^{-2t} r^{2k-2}}{2k - 2} \to 0.
\]
So for every fixed $k$, $\lim_{t \to +\infty} W([Y_t]|_{M_{1,1}}) = 0$.

The key point of (6) is the technical estimate (5.24). We will leave the proof of it for the appendix.

\[ \square \]

### 5.6 Non-oriented examples of minimal Moebius strips in $\mathbb{H}^4$

In this subsection, we consider some non-oriented minimal surfaces in $\mathbb{H}^4$, which is based on the work of [22] and [56].

Set

\[
h_1 = \frac{3}{2} z^5, \quad h_2 = -h_3 = \frac{\sqrt{5}}{2} z^3, \quad h_4 = \frac{3}{2} z.
\]

(5.26)

We have

\[
Y = \begin{pmatrix}
y_0 \\
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5
\end{pmatrix}
= \begin{pmatrix}
(r^{10} + 5r^6 + 5r^4 + 1) \\
-(r^{10} - 5r^6 - 5r^4 + 1) \\
\sqrt{5}(1 + r^6)(z^2 - z^5) \\
\sqrt{5}(1 + r^6)(z^2 + z^5) \\
-\sqrt{5}(1 - r^4)(z^3 - z^5) \\
\sqrt{5}(1 - r^4)(z^3 + z^5)
\end{pmatrix}.
\]

(5.27)

with

\[ |dY|^2 = 40r^2(4r^4 - 7r^2 + 4)(r^2 + 1)^4|dz|^2. \]

So $Y$ has exactly two branched points 0 and $\infty$. Consider $\mu(z) = -\frac{1}{z}$, we have

\[ [Y(\mu(z))] = [Y(z)]. \]

As a consequence, $[Y]$ induces a branched Willmore $\mathbb{R}P^2$: $[Y] : S^2/\mu = \mathbb{R}P^2 \to S^4$ is a Willmore $\mathbb{R}P^2$ with Willmore energy $12\pi$ and one branched point at $z = 0$. For more discussions on singularities and branched points of Willmore surfaces, see [6, 32–34, 44, 48].

Set $r_1 = \frac{\sqrt{5} - 1}{2}$ and

\[
M_1 = \{ z \in \mathbb{C} | 0 \leq r < r_1 \}, \quad M_2 = \left\{ z \in \mathbb{C} | r_1 < r < \frac{1}{r_1} \right\}, \quad M_3 = \left\{ z \in \mathbb{C} | r > \frac{1}{r_1} \right\}.
\]

Set $\tilde{y} = \frac{1}{r_1}(y_0, y_2, y_3, y_4, y_5)^t$. We see that

1. $\tilde{y}|_{M_{2/\mu}}$ is a complete minimal Moebius strip in $\mathbb{H}^4$ with $W(\tilde{y}) = \frac{6\sqrt{5}\pi}{5} \approx 10.733\pi$.
2. $\tilde{y}|_{M_3} = (\tilde{y} \circ \mu)|_{M_3}$ is a branched minimal disk in $\mathbb{H}^4$ with Willmore energy $W(\tilde{y}) = 12\pi(1 - 2\sqrt{5}/5) \approx 1.267\pi$ and one branched point $z = 0$.

It is natural to ask whether the complete minimal Moebius strip $\tilde{y}|_{M_2}$ takes uniquely the minimum of the Willmore energy among all complete minimal Moebius strips in $\mathbb{H}^n, n \geq 4$. 

\[ \text{Springer} \]
6 Remarks on the non-rigidity of isotropic surfaces in $\mathbb{S}^4$

Finally we would like to discuss briefly some simple applications of the W-deformations on the study of stability problems of Willmore surfaces and minimal surfaces. More detailed study will be done in a separate publication, since it will involve many other independent calculations. We refer to [47, 50, 53, 57] for more details on this topics, in particular Theorem 3.3.1 and Corollary 3.3.1 of [50].

Since for isotropic surfaces in $\mathbb{S}^4$, we have an explicit W-representation formula, we see that W-deformations are globally defined if the surfaces are globally defined. From this we see immediately that they are Willmore non-rigid since they admit non-trivial Willmore deformations.

**Theorem 6.1** Let $y : M \rightarrow \mathbb{S}^4$ be an isotropic (hence Willmore) surface from a closed Riemann surface $M$ with its conformal Gauss map in $\mathcal{M}_L$. Then $y$ is Willmore non-rigid. That is, it admits conformal Jacobi fields different from the conformal Killing fields which come from conformal transformations of $\mathbb{S}^4$.

**Proof** We first consider the case that $y$ is not conformally equivalent to a minimal surface in $\mathbb{S}^4$. By Theorem 3.7, the condition that the conformal Gauss map of $y$ is in $\mathcal{M}_L$, is equivalent to saying that it is coming from a $K^C_{-d}$-dressing of some minimal surface in $\mathbb{S}^4$. Therefore by Theorem 5.1, there exists a family of Willore surfaces $y_t$ such that $y_t$ is real analytic in $t$ and $y_t|_{t=0} = y$ and $y_t|_{t=t_0}$ is a minimal surface in $\mathbb{S}^4$. So $\{y_t\}$ does not come from any conformal transformations of $\mathbb{S}^4$ and the Jacobi field of $y_t$ is not a conformal Killing field.

Now consider the case that $y$ is conformally equivalent to a minimal surface in $\mathbb{S}^4$. Without lose of generality, we assume $y$ has the potential as the form in Proposition 5.2. By Proposition 5.2 and Theorem 5.1, there exists globally a family of Willmore surfaces $y_t$ such that $y_t$ is real analytic in $t$ and $y_t$ is not conformally equivalent to any minimal surface in $\mathbb{S}^4$ when $0 < t < \pi/2$. So $\{y_t\}$ does not come from any conformal transformations of $\mathbb{S}^4$ and the Jacobi field of $y_t$ is not a conformal Killing field. $\square$

For minimal surfaces in $\mathbb{S}^4$, we also have the following

**Theorem 6.2** Let $y : M \rightarrow \mathbb{S}^4$ be an isotropic minimal surface from a closed Riemann surface $M$. Then $y$ is non-rigid, that is, it admits Jacobi fields different from the Killing fields which come from isometric transformations of $\mathbb{S}^4$.

**Proof** Assume without loss of generality the normalized potential of $y$ is of the form (2.6) with $h_2 = h_3$. Let

$$\hat{T}_t = \begin{pmatrix} I_4 & 0 & 0 \\ 0 & \cosh t & i \sinh t \\ 0 & -i \sinh t & \cosh t \end{pmatrix}$$

be a one-parameter subgroup of $KC$. The one-parameter family of normalized potentials $\eta_t$ has the same form as $y$ in (5.1), except the functions $\{h_j\}$ becomes $\{e^t h_j\}$. Substituting $\{e^t h_j\}$ into (5.2), we obtain the Willmore family $y_t = \frac{1}{\gamma_0}(y_{1t}, y_{2t}, y_{3t}, y_{4t}, y_{5t})$ derived by $\eta_t$. We have that $y_t$ is real analytic in $t$ and for every $t$, $y_t$ is a minimal surface in $\mathbb{S}^4$.

Let $t$ tends to $+\infty$. We have that $y_t$ tends to a conformal map into $S^2$. As a consequence, $y_t$ can not be derived by an isometric transformations of $\mathbb{S}^4$. Hence the Jacobi field of $y_t$ is not a Killing field of $y$. $\square$
We refer to Ejiri’s interesting paper [27] for the discussion of the index of minimal two-spheres in $S^{2m}$. Note that the Willmore deformations contribute explicitly to the index of the isotropic minimal surfaces in $S^4$, with eigenvalue $-2$ of the Jacobi operator [27, 47, 57].

**Appendix: Proof of (5.24)**

Set $a = e^{2\gamma_0} = k^{-(k-1)}$, $\rho = r^2$. Set $L = a\rho^{k+1} + 1 - b\rho^k - ab\rho$ with $b = \frac{k+1}{k-1}$. Let $\rho_1 \in (0, 1)$ and $\rho_2 \in (1, +\infty)$ be the two solutions to

$$L(\rho) = a\rho^{k+1} + 1 - b\rho^k - ab\rho = 0.$$ 

We can rewrite $W(M_{0,1})$ as

$$W(M_{0,1}) = 2\pi \int_0^{\rho_1} \frac{ak^2(k-1)^2\rho^{k-2}(a\rho^{k+1} + 1 - b\rho^k - ab\rho)^2}{(a + \rho^k)^2 + k^2\rho^{k-1}(1 - a\rho)^2} \, d\rho.$$ 

Then (5.24) follows from the following Lemma.

**Lemma 7.1** 1. When $k \to +\infty$, $\rho_1 > e^{-3/k^2}$; In particular

$$\lim_{k \to \infty} \rho_1 = \lim_{k \to \infty} (\rho_1)^k = 1.$$

2. On $[0, \rho_1]$, $L(\rho) \geq \rho_1^{-1}(\rho_1 - \rho)$. When $k \to +\infty$,

$$W(M_{0,1}) \geq \frac{2\pi k^2(k-1)^2}{\rho_1^2} I_1, \text{ with } I_1 = \int_0^{\rho_1} \frac{a\rho^{k-1}((\rho_1 - \rho)^2}{(2a + k^2\rho^{k-1})^2} \, d\rho. \quad (7.1)$$

3. Set $\varphi = \rho/\rho_1$. Then $I_1$ is tending to

$$I_2 = \int_0^1 \frac{a\varphi^{k-1}(1 - \varphi)^2}{(2a + k^2\varphi^{k-1})^2} \, d\varphi$$

when $k \to +\infty$.

4. $I_2 > \frac{1}{3} R(a, k)$ with $\delta = (\frac{a}{k^2})^{\frac{1}{k-1}} \in (0, 1)$ and

$$R(a, k) = \frac{1}{k^2} \left( \frac{2}{k-1} \right) \frac{2\delta}{k} + \frac{\delta^2}{k+1} + \frac{1}{k^2} \left( - \frac{2\delta}{k-2} + \frac{\delta^2}{k-3} \right)$$

$$- \frac{a}{k^4} \left( \frac{1}{k-1} - \frac{2}{k-2} + \frac{1}{k-3} \right).$$

Moreover, when $k \to +\infty$, $R(a, k) = \frac{2}{k^2(k-1)} + o \left( \frac{1}{k^3} \right)$.

5. When $k$ is large enough, $W(M_{0,1}) > \frac{1}{3} (k-1)\pi$.

**Proof** (1) From $0 < \rho_1 < 1$ and $a\rho_1^{k+1} + 1 - b\rho_1^k - ab\rho_1 = 0$, we have

$$(\rho_1)^{k+1} = \frac{1 - ab\rho_1}{b\rho_1^k - a} \geq \frac{1 - ab}{b - a} = 1 + \frac{(1 - b)(1 + a)}{b - a} = 1 + \frac{1 + a - 2}{b - a} = 1.$$ 

From this, $\lim_{k \to \infty} \rho_1 = \lim_{k \to \infty}(\rho_1)^k = 1$. 

Springer
(2) Since \( L'(\rho) = a(k+1)\rho^k - bk\rho^{k-1} - ab \), \( L''(\rho) = ak(k+1)\rho^{k-1} - bk(k-1)\rho^{k-2} = ak(k+1)\rho^{k-2}(\rho-1) \). So on \((0, \rho_1)\), \( L'(\rho) < 0 \), from which we have \( L(\rho) \geq \rho_1^{-1}(\rho_1 - \rho) \). And (7.1) follows from this and the fact that \( a(1 + \rho^k)^2 + k^2\rho^{k-1}(1 - a\rho)^2 < 2a + k^2\rho^{k-1} \).

(3) Since \( \rho = \rho_1\varphi \), we have

\[
I_1 = \rho_1^{k+1} \int_0^1 \frac{a\varphi^{k-1}(1 - \varphi)^2}{(2a + k^2\rho_1^{k-1}\varphi^{k-1})^2} d\varphi.
\]

Since \( \lim_{k \to +\infty} \rho_1 = \lim_{k \to +\infty} \rho_1^{k+1} = \lim_{k \to +\infty} \rho_1^{k-1} = 1 \), we have

\[
1 < \frac{2a + k^2\varphi^{k-1}}{2a + k^2\rho_1^{k-1}\varphi^{k-1}} < \frac{1}{\rho_1^{k-1}} \to 1
\]
as \( k \to +\infty \). (3) follows from this.

(4) First we have \( k^2\delta^{k-1} = a \).

So

\[
2a + k^2\varphi^{k-1} < 3a, \quad \forall \varphi \in (0, \delta); \quad 2a + k^2\varphi^{k-1} < 3k^2\varphi^{k-1}, \quad \forall \varphi \in (\delta, 1).
\]

By substituting \( \delta^{k-1} = \frac{a}{k^2} \), we have

\[
I_2 > \int_0^\delta \frac{a\varphi^{k-1}(1 - \varphi)^2}{9a^2} d\varphi + \int_\delta^1 \frac{a\varphi^{k-1}(1 - \varphi)^2}{9k^4\varphi^{2(k-1)}} d\varphi
\]

\[
= \frac{\delta^{k-1}}{9a} \left( \frac{1}{k-1} - \frac{2\delta}{k} + \frac{\delta^2}{k+1} \right) + \frac{a\delta^{1-k}}{9k^4} \left( \frac{1}{k-1} - \frac{2\delta}{k} + \frac{\delta^2}{k-3} \right)
\]

\[
- \frac{a}{9k^4} \left( \frac{1}{k-1} - \frac{2}{k-2} + \frac{1}{k-3} \right)
\]

\[
= \frac{1}{9}R(a, k).
\]

When \( k \to +\infty \), \( \delta = k^{-1} \frac{a}{k^2} \to 0 \) and hence \( R(a, k) = \frac{2}{k^4(k-1)} + o\left(\frac{1}{k^2}\right) \). This finishes (4).

(5) As a consequence, we have

\[
W(M_{0,1}) \geq \frac{2\pi k^2(k-1)^2}{\rho_1^3} I_1 \geq \frac{(4 - \varepsilon)\pi}{9}(k-1),
\]

for some \( \varepsilon \in (0, 1/2) \) when \( k \to +\infty \), which finishes (5).

\( \square \)

References

1. Alexakis, S., Mazzeo, R.: Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds. Comm. Math. Phys. 297(3), 621–651 (2010)
2. Alexakis, S., Mazzeo, R.: Complete Willmore surfaces in \( \mathbb{H}^3 \) with bounded energy: boundary regularity and bubbling. J. Diff. Geom. 101(3), 369–422 (2015)
3. Anderson, M.: Complete Minimal Varieties in hyperbolic space. Invent. Math. 69, 477–494 (1982)
4. Babich, M., Bobenko, A.: Willmore tori with umbilic lines and minimal surfaces in hyperbolic space. Duke Math. J. 72(1), 151–185 (1993)
5. Barbosa, J.L.: On minimal immersions of \( S^2 \) into \( S^2 \). Trans. Amer. Math. Soc. 210, 75–106 (1975)
6. Bernard, Y.: Rivière, T. Singularity removability at branch points for Willmore surfaces. Pacific J. Math. 265(2), 257–311 (2013)
7. Bobenko, A., Heller, S., Schmitt, N.: Minimal n-Noids in Hyperbolic and Anti-de Sitter 3-Space. Proc. Royal Soc A Math. Phys. Eng. Sci. 475(2227), 20190173 (2019)
8. Brander, D., Wang, P.: On the Björling problem for Willmore surfaces. J. Diff. Geom. 108(3), 411–457 (2018)
9. Bryant, R.: A duality theorem for Willmore surfaces. J. Diff. Geom. 20, 23–53 (1984)
10. Burstall, F., Ferus, D., Leschke, K., Pedit, F., Pinkall, U.: Conformal geometry of surfaces in $S^4$ and quaternions. Lecture Notes in Mathematics, vol. 1772. Springer, Berlin (2002)
11. Burstall, F.E., Guest, M.A.: Harmonic two-spheres in compact symmetric spaces, revisited. Math. Ann. 309, 541–572 (1997)
12. Burstall, F., Quintino, A.: Dressing transformations of constrained Willmore surfaces. Commun. Anal. Geom. 22, 469–518 (2014)
13. Burstall, F., Pedit, F., Pinkall, U.: Schwarzian derivatives and flows of surfaces, Contemporary Mathematics 308, 39–61, Providence. Amer. Math. Soc, RI (2002)
14. Calabi, E.: Minimal immersions of surfaces in Euclidean spheres. J. Diff. Geom. 1, 111–125 (1967)
15. Clancey, K.F., Gohberg, I.: Factorization of matrix functions and singular integral operators. Operator Theory: Advances and Applications, 3. Birkhäuser Verlag, Basel-Boston, Mass., (1981)
16. Coskunuzer, B.: Minimal planes in hyperbolic space. Comm. Anal. Geom. 12(4), 821–836 (2004)
17. Coskunuzer, B.: Generic uniqueness of least area planes in hyperbolic space. Geom. Top. 10, 401–412 (2006)
18. Dorfmeister, J.F., Inoguchi, J., Kobayashi, S.: Constant mean curvature surfaces in hyperbolic 3-space via loop groups. J. Reine Angew. Math. 686, 1–36 (2014)
19. Dorfmeister, J., Pedit, F., Wu, H.: Weierstrass type representation of harmonic maps into symmetric spaces. Comm. Anal. Geom. 6, 633–668 (1998)
20. Dorfmeister, J., Wang, P.: Willmore surfaces in spheres: the DPW approach via the conformal Gauss map. Abh. Math. Semin. Univ. Hambg. 89(1), 77–103 (2019)
21. Dorfmeister, J., Wang, P.: On symmetric Willmore surfaces in spheres II: the orientation reversing case. Diff. Geom. Appl. 69, 101606 (2020)
22. Dorfmeister, J., Wang, P.: Classification of homogeneous Willmore surfaces in $S^n$, Osaka. J. Math., Vol. 57 No. 4 (2020)
23. Dorfmeister, J., Wang, P.: Classification of equivariant Willmore $\mathbb{RP}^2$ in $S^4$, in preparation
24. Drukker, N., Gross, D., Ooguri, H.: Wilson Loops and Minimal Surfaces. Phys. Rev. D (3) 60(12), 125006 (1999)
25. Guest, M.A., Ohnita, Y.: Group actions and deformations for harmonic maps. J. Math. Soc. Japan 45(4), 671–704 (1993)
26. Ejiri, N.: The index of minimal immersions of $S^2$ into $S^{2n}$. Math. Z. 184(1), 127–132 (1983)
27. Ejiri, N.: Equivariant minimal immersions of $S^2$ into $S^{2m}(1)$. Trans. Amer. Math. Soc. 297(1), 105–124 (1986)
28. Ejiri, N.: Willmore surfaces with a duality in $S^n(1)$. Proc. London Math. Soc. 57(2), 383–416 (1988)
29. Hélein, F.: Willmore immersions and loop groups. J. Differ. Geom. 50, 331–385 (1998)
30. Ishihara, T.: Harmonic maps of nonorientable surfaces to four-dimensional manifolds. Tohoku Math. J. 45(1), 1–12 (1993)
31. Kuwert, E.: Willmore surfaces in $S^n$: transforms and vanishing theorems, dissertation, Technischen Universität Berlin (2005)
41. Ma, X., Pedit, F., Wang, P.: Möbius homogeneous Willmore 2-spheres. Bull. Lond. Math. Soc. 50(3), 509–512 (2018)
42. Maldacena, J.: Wilson loops in Large N field theories. Phys. Rev. Lett. 80, 4859 (1998)
43. Marques, F., Neves, A.: Min-Max theory and the Willmore conjecture. Ann. Math. 179(2), 683–782 (2014)
44. Michelat, A., Rivière, T.: The Classification of Branched Willmore Spheres in the 3-Sphere and the 4-Sphere, arXiv:1706.01405
45. Montiel, S.: Willmore two spheres in the four-sphere. Trans. Amer. Math. Soc. 352(10), 4469–4486 (2000)
46. Musso, E.: Willmore surfaces in the four-sphere. Ann. Global Anal. Geom. 8(1), 21–41 (1990)
47. Ndiaye, C.M., Schätzle, R.: Explicit conformally constrained Willmore minimizers in arbitrary codimension. Calc. Var. Partial Differential Equations 51(1–2), 291–314 (2014)
48. Rivière, T.: Analysis aspects of Willmore surfaces. Invent. Math. 174(1), 1–45 (2008)
49. Schätzle, R.: The Willmore boundary problem. Cal. Var. PDE 37, 275–302 (2010)
50. Simons, J.: Minimal varieties in Riemannian manifolds. Ann. Math. 88(2), 62–105 (1968)
51. Terng, C., Uhlenbeck, K.: Bäcklund transformations and loop group actions. Commun. Pure Appl. Math. 53(1), 1–75 (2000)
52. Uhlenbeck, K.: Harmonic maps into Lie groups (classical solutions of the chiral model). J. Diff. Geom. 30, 1–50 (1989)
53. Urbano, F.: Minimal surfaces with low index in the three-dimensional sphere. Proc. Amer. Math. Soc. 108(4), 989–992 (1990)
54. Wang, C.P.: Moebius geometry of submanifolds in $S^{n+2}$. Manuscripta Mathematica 96(4), 517–534 (1998)
55. Wang, P.: Willmore surfaces in spheres via loop groups III: on minimal surfaces in space forms. Tohoku Math. J. 69(2), 141–160 (2017)
56. Wang, P.: A Weierstrass type representation of isotropic Willmore surfaces in $S^4$, in preparation
57. Weiner, J., On a Problem of Chen, Willmore, et al.: Indiana Univ. Math. J. 27(1), 19–35 (1978)
58. Xia, Q.L., Shen, Y.B.: Weierstrass Type Representation of Willmore Surfaces in $S^{n+2}$. Acta Math. Sinica, Vol. 20, No. 6, 1029-1046

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.