Spectral Properties of Higher Order Anharmonic Oscillators

Bernard Helffer
Mikael Persson

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SPECTRAL PROPERTIES OF HIGHER ORDER ANHARMONIC OSCILLATORS

BERNARD HELFFER AND MIKAEL PERSSON

ABSTRACT. We discuss spectral properties of the self-adjoint operator

$$-\frac{d^2}{dt^2} + \left( \frac{t^{k+1}}{k+1} - \alpha \right)^2$$

in $L^2(\mathbb{R})$ for odd integers $k$. We prove that the minimum over $\alpha$ of the ground state energy of this operator is attained at a unique point which tends to zero as $k$ tends to infinity. Moreover, we show that the minimum is non-degenerate. These questions arise naturally in the spectral analysis of Schrödinger operators with magnetic field. This extends or clarifies previous results by Pan-Kwek [11], Helffer-Morame [8], Aramaki [1], Helffer-Kordyukov [4, 6, 7] and Helffer [3].

1. Introduction

1.1. Definition of $\Omega^{(k)}(\alpha)$ and main result. For any $\alpha \in \mathbb{R}$ we denote by $\lambda_1,\Omega^{(k)}(\alpha)$ the lowest eigenvalue of the self-adjoint second order differential operator

$$\Omega^{(k)}(\alpha) = -\frac{d^2}{dt^2} + \left( \frac{t^{k+1}}{k+1} - \alpha \right)^2.$$

We also denote by $q^{(k)}(\alpha)$ the quadratic form corresponding to $\Omega^{(k)}(\alpha)$,

$$q^{(k)}(\alpha)[u] = \int_{\mathbb{R}} |u'(t)|^2 + \left( \frac{t^{k+1}}{k+1} - \alpha \right)^2 |u(t)|^2 dt.$$

The main result of the present paper is the following theorem.

**Theorem 1.1.** Assume that $k \geq 1$ is an odd integer. There exists a unique $\alpha_{\text{min}}^{(k)}$ such that

$$\inf_{\alpha \in \mathbb{R}} \lambda_1,\Omega^{(k)}(\alpha) = \lambda_1,\Omega^{(k)}(\alpha_{\text{min}}^{(k)}).$$

Moreover, $\alpha_{\text{min}}^{(k)} > 0$ and the minimum is non-degenerate,

$$\partial_{\alpha\alpha}^2 \left( \lambda_1,\Omega^{(k)}(\alpha) \right) |_{\alpha = \alpha_{\text{min}}^{(k)}} > 0.$$

**Theorem 1.2.** Assume that $k$ is even. Then $\alpha = 0$ is a non-degenerate local minimum of $\lambda_1,\Omega^{(k)}(\alpha)$.

**Theorem 1.3.** If $k$ is odd,

$$\lim_{k \to +\infty} \alpha_{\text{min}}^{(k)} = 0.$$
1.2. **Historical context.** The operator $\Omega^{(k)}(\alpha)$ was first introduced in the context of magnetic Schrödinger operators in [10], and was further studied in [8, 11, 4].

The uniqueness of $\alpha^{(k)}_{\min}$ was first observed numerically in [10] for $k = 1$. A proof for $k = 1$ was given in [11], which was completed in [3]. The uniqueness for $k > 1$ (odd) was announced in [1] but the given proof seems incomplete. The non-degeneracy was obtained for $k = 1$ in [3] and conjectured in the general case in [6] and [7]. This conjecture was supported by numerical computations performed by V. Bonnaillie-Noël, see Table 1. The results for large $k$ were announced in [6] and a proof was sketched in [5].

**Table 1.** Numerical values calculated by V. Bonnaillie-Noël with an accuracy of $10^{-2}$.

| $k$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\alpha^{(k)}_{\min}$ | 0.35 | 0   | 0.16 | 0   | 0.10 | 0   | 0.07 | 0   | 0.05 | 0   |
| $\lambda_{1,\Omega^{(k)}(\alpha^{(k)}_{\min})}$ | 0.57 | 0.66 | 0.68 | 0.76 | 0.81 | 0.87 | 0.92 | 0.98 | 1.02 | 1.07 |
| $\lambda_{2,\Omega^{(k)}(\alpha^{(k)}_{\min})}$ | 1.98 | 2.50 | 2.61 | 2.98 | 3.18 | 3.47 | 3.66 | 3.90 | 4.07 | 4.27 |
| $\lambda_{3,\Omega^{(k)}(\alpha^{(k)}_{\min})}$ | 4.11 | 5.24 | 5.68 | 6.52 | 7.03 | 7.69 | 8.16 | 8.70 | 9.12 | 9.57 |

The outline of the paper is the following: In Section 2 we collect some facts about the operator $\Omega^{(k)}(\alpha)$, which we use in Section 3 to prove Theorem 1.1. We prove Theorem 1.2 in Section 4. We consider large values of $k$ in Section 5 and prove Theorem 1.3.

2. **Auxiliary results**

We recall some results about $\Omega^{(k)}(\alpha)$ obtained in [10, 6, 7, 5].

**Lemma 2.1.** It holds that $\lambda_{1,\Omega^{(k)}(\alpha)} \to \infty$ as $|\alpha| \to \infty$.

**Proof.** We first note that if $k$ is odd and $\alpha < 0$, then $q^{(k)}(\alpha)[u] \geq \alpha^2 \|u\|^2$, so $\lambda_{1,\Omega^{(k)}(\alpha)} \geq \alpha^2$. On the other hand, for any integer $k > 0$ one can use semi-classical analysis [13, 9] to show that

$$\lambda_{1,\Omega^{(k)}(\alpha)} \sim (k + 1)^{2k/(k + 1)} \alpha^{k/(k + 1)}, \quad \alpha \to \infty.$$ 

For even $k$ it holds that $\lambda_{1,\Omega^{(k)}(\alpha)} = \lambda_{1,\Omega^{(k)}(-\alpha)}$. \hfill $\Box$

So, it is clear that the smooth function $\lambda_{1,\Omega^{(k)}(\alpha)}$ is lower semi-bounded, and

$$\lambda^{(k)} := \inf_{\alpha \in \mathbb{R}} \lambda_{1,\Omega^{(k)}(\alpha)} > 0$$

and there exists (at least one) $\alpha^{(k)}_{\min} \in \mathbb{R}$ such that $\lambda_{1,\Omega^{(k)}(\alpha^{(k)}_{\min})}$ is minimal,

$$\lambda_{1,\Omega^{(k)}(\alpha^{(k)}_{\min})} = \lambda^{(k)}.$$ 

Let $u_{1,\alpha} \in L^2(\mathbb{R})$ be the $L^2$ normalized strictly positive eigenfunction of the operator $\Omega^{(k)}(\alpha)$ corresponding to the eigenvalue $\lambda_{1,\Omega^{(k)}(\alpha)}$,

$$\Omega^{(k)}(\alpha)u_{1,\alpha} = \lambda_{1,\Omega^{(k)}(\alpha)}u_{1,\alpha}, \quad \|u_{1,\alpha}\| = 1.$$ (2.1)

The function $u_{1,\alpha}$ can be chosen to depend smoothly on $\alpha$.

**Lemma 2.2.** Assume that $k$ is odd. Then it holds that $\alpha^{(k)}_{c} > 0$ for all critical points $\alpha^{(k)}_{c}$ of $\lambda_{1,\Omega^{(k)}(\alpha)}$. In particular, $\alpha^{(k)}_{\min} > 0$.
Proof. Differentiating (2.1) with respect to $\alpha$ and taking the inner product with $u_{1,\alpha}$ we find
\[ \partial_\alpha \lambda_{1,\Omega^{(k)}(\alpha)} = -2 \int_{-\infty}^{\infty} \left( \frac{t}{k+1} - \alpha \right) (u_{1,\alpha})^2 \, dt. \] (2.2)
So, when the derivative is zero, we get
\[ \alpha_c^{(k)} = \int_{-\infty}^{\infty} \frac{t}{k+1} (u_{1,\alpha^{(k)}})^2 \, dt > 0. \] (2.3)

Lemma 2.3. Assume that $\alpha_c^{(k)}$ is a critical point of $\lambda_{1,\Omega^{(k)}(\alpha)}$. If either

(A) $(k+2)\lambda_{1,\Omega^{(k)}(\alpha^{(k)})} > (k+6)\lambda_{1,\Omega^{(k)}(\alpha^{(k)})}$ or

(B) $k$ is odd or $\alpha_c^{(k)} = 0$, and $(k+2)\lambda_{1,\Omega^{(k)}(\alpha^{(k)})} > (k+6)\lambda_{1,\Omega^{(k)}(\alpha^{(k)})}$,

then $\partial_\alpha^2 \lambda_{1,\Omega^{(k)}(\alpha^{(k)})} > 0$. Especially this implies that $\lambda_{1,\Omega^{(k)}(\alpha)}$ has a local minimum at $\alpha_c^{(k)}$ which is non-degenerate.

Proof. We start by assuming that the condition in (A) is fulfilled. The differentiation in the proof of Lemma 2.2 also provides us with a formula for $\partial_\alpha u_{1,\alpha}$,
\[ \partial_\alpha u_{1,\alpha} = -2(\Omega^{(k)}(\alpha) - \lambda_{1,\Omega^{(k)}(\alpha)})^{-1} \left[ \frac{t}{k+1} - \alpha \right] u_{1,\alpha}, \]
where the inverse is the regularized resolvent. Differentiating (2.1) twice, we find
\[ \partial_{\alpha \alpha}^2 \lambda_{1,\Omega^{(k)}(\alpha)} = 2 - 4 \int_{-\infty}^{\infty} \left( \frac{t}{k+1} - \alpha \right) u_{1,\alpha} \partial_\alpha u_{1,\alpha} \, dt \]
By an application of the Cauchy-Schwarz inequality and the bound
\[ \| (\Omega^{(k)}(\alpha) - \lambda_{1,\Omega^{(k)}(\alpha)})^{-1} \| \leq \frac{1}{\lambda_{2,\Omega^{(k)}(\alpha)} - \lambda_{1,\Omega^{(k)}(\alpha)}} \] (2.4)
we find that
\[ \partial_{\alpha \alpha}^2 \lambda_{1,\Omega^{(k)}(\alpha)} \geq 2 - \frac{8}{\lambda_{2,\Omega^{(k)}(\alpha)} - \lambda_{1,\Omega^{(k)}(\alpha)}} \left\| \left( \frac{t}{k+1} - \alpha \right) u_{1,\alpha} \right\|^2. \] (2.5)
To calculate the norm on the right-hand side, we note that the ground state energy of the operator
\[ \Omega^{(k)}(\alpha, \rho) = -\frac{1}{\rho^2} \frac{d^2}{dt^2} + \left( \rho^{k+1} \frac{t}{k+1} - \alpha \right)^2 \]
is independent of $\rho$, i.e.,
\[ -\frac{1}{\rho^2} \frac{d^2}{dt^2} u_{1,\alpha,\rho} + \left( \rho^{k+1} \frac{t}{k+1} - \alpha \right)^2 u_{1,\alpha,\rho} = \lambda_{1,\Omega^{(k)}(\alpha)} u_{1,\alpha,\rho}. \]

Differentiating this identity with respect to $\rho$ and then letting $\rho = 1$ and $\alpha = \alpha_c^{(k)}$, and then taking the inner product with $u_{1,\alpha_c^{(k)}}$, we get
\[ (k+1) \left\| \left( \frac{t}{k+1} - \alpha_c^{(k)} \right) u_{1,\alpha_c^{(k)}} \right\|^2 = \left\| \frac{d}{dt} u_{1,\alpha_c^{(k)}} \right\|^2, \]
and consequently
\[ \left\| \left( \frac{t}{k+1} - \alpha_c^{(k)} \right) u_{1,\alpha_c^{(k)}} \right\|^2 = \frac{1}{k+2} \lambda_{1,\Omega^{(k)}(\alpha_c^{(k)})}. \] (2.6)
If \(u\) is odd, it holds that  

\[
\int_0^\infty \frac{d}{dt} \left( \frac{t^{k+1}}{k+1} - \alpha \right)^2 (u_{0,\alpha})^2 \, dt = (\lambda_{1,\alpha}(\alpha) - \alpha^2) u_{1,\alpha}(0)^2.
\]

For a critical point \(\alpha = \alpha_c^{(k)}\), we get  

\[
\int_0^\infty \left( \frac{t^{k+1}}{k+1} - \alpha \right)(u_{1,\alpha_c^{(k)})}^2 \, dt = 0.
\]

Combining these two formulas, we obtain  

\[
\left( \lambda_{1,\alpha}(\alpha_c^{(k)}) - (\alpha_c^{(k)})^2 \right) u_{1,\alpha_c^{(k)}(0)^2
\]

\[
= 2 \int_0^\infty \left( \frac{t^{k+1}}{k+1} - (\alpha_c^{(k)})^2 \right) (u_{1,\alpha_c^{(k)})^2 \, dt > 0.
\]

If \(u_{1,\alpha_c^{(k)}(0) = 0, then u_{1,\alpha_c^{(k)}} \equiv 0 since \(u'_{1,\alpha_c^{(k)}}(0) = 0, and so (2.7) holds. \]

**Lemma 2.4.** Assume that \(k\) is odd and that \(\alpha_c^{(k)}\) is a critical point of \(\lambda_{1,\alpha}(\alpha)\). Then  

\[
(\alpha_c^{(k)})^2 < \lambda_{1,\alpha}(\alpha_c^{(k)}).
\]

**Proof.** Using the fact that \(u_{1,\alpha}\) is even we get, using integration by parts,  

\[
\int_0^\infty \frac{d}{dt} \left( \frac{t^{k+1}}{k+1} - \alpha \right)^2 (u_{1,\alpha})^2 \, dt = (\lambda_{1,\alpha}(\alpha) - \alpha^2) u_{1,\alpha}(0)^2.
\]

For a critical point \(\alpha = \alpha_c^{(k)}\), we get  

\[
\int_0^\infty \left( \frac{t^{k+1}}{k+1} - \alpha \right)(u_{1,\alpha_c^{(k)}})^2 \, dt = 0.
\]

Combining these two formulas, we obtain  

\[
\left( \lambda_{1,\alpha}(\alpha_c^{(k)}) - (\alpha_c^{(k)})^2 \right) u_{1,\alpha_c^{(k)}(0)^2
\]

\[
= 2 \int_0^\infty \left( \frac{t^{k+1}}{k+1} - (\alpha_c^{(k)})^2 \right) (u_{1,\alpha_c^{(k)})^2 \, dt > 0.
\]

If \(u_{1,\alpha_c^{(k)}(0) = 0, then u_{1,\alpha_c^{(k)}} \equiv 0 since \(u'_{1,\alpha_c^{(k)}}(0) = 0, and so (2.7) holds. \]

3. **Proof of Theorem 1.1**

We will use the lemmas in the previous section to complete the proof. For that, we need an upper bound on \(\lambda_{1,\alpha}(\alpha)\) and a lower bound on \(\lambda_{2,\alpha}(\alpha)\).

3.1. **Upper bound.** In this section we are looking for a good upper bound of \(\lambda_{1,\alpha}(\alpha)\).

**Lemma 3.1.** For all \(k \geq 1\) and \(\alpha > 0\) it holds that  

\[
\lambda_{1,\alpha}(\alpha) \leq \alpha^2 + \frac{\pi^2}{4} \frac{k+2}{k+1} \left( \frac{1}{4} (k+1)(2k+3)(2k+4)(2k+5) \right)^{-1/(k+2)}.
\]

In particular, if \(k\) is odd, it holds that \(\alpha_c^{(k)} \leq \alpha_\ast^{(k)}\) where  

\[
\alpha_\ast^{(k)} = \frac{\pi}{2} \left( \frac{k+2}{k+1} \right)^{1/2} \left( \frac{1}{4} (k+1)(2k+3)(2k+4)(2k+5) \right)^{-1/(2k+4)}.
\]
Proof. We will motivate our choice of trial function, inspired by [5]. For large \( k \), the potential \( \left( \frac{k+1}{k+4} - \alpha \right)^2 \) will look more and more as potential \( p_{\alpha, \infty} \).

\[
p_{\alpha, \infty}(t) = \begin{cases} \alpha^2 & |t| \leq 1 \\ \infty & |t| > 1. \end{cases}
\]

Among the potentials \( p_{\alpha, \infty}, p_{0, \infty} \) is the one that will give the lowest energy, corresponding to the Dirichlet problem of \( -\frac{d^2}{dt^2} \) on \( L^2((-1,1)) \), with eigenvalues

\[
\left\{ \left( \frac{\pi j}{2} \right)^2 \right\}_{j \in \mathbb{N} \setminus \{0\}},
\]

and with first eigenfunction \( \cos(\pi t/2) \). Motivated by this, we introduce a parameter \( \rho > 0 \) and use as a trial function

\[
u(t) = \begin{cases} \cos\left(\frac{\pi t}{2\rho}\right) & |t| \leq \rho, \\ 0 & |t| > \rho. \end{cases}
\]

This function does not belong to the domain of \( \Omega^{(k)}(\alpha) \), but to the form domain of \( q^{(k)}(\alpha) \), which is enough to use the min-max principle. A simple calculation shows that if \( k \) is odd then

\[
\lambda_{1,\Omega^{(k)}(\alpha)} \leq \frac{q^{(k)}(\alpha)[u]}{\|u\|^2} = \alpha^2 - 2\alpha^2 \rho^k + 2 \left( \frac{1}{k+2} + I\left(\frac{k+1}{2}\right) \right) + \frac{\rho^2}{(k+1)^2} \left( \frac{1}{2k+3} + I(k+1) \right) + \frac{\pi^2}{4\rho^2},
\]

where \( I(m) = \int_0^1 s^{2m} \cos(\pi s) \, ds \leq 0 \). By integration by parts we see that

\[
-\frac{1}{2m+1} \leq I(m) \leq -\frac{1}{2m+1} + \frac{\pi^2}{(2m+1)(2m+2)(2m+3)}.
\]

If \( k \) is even the coefficient in front of \( \alpha \) is zero. In any case we get

\[
\lambda_{1,\Omega^{(k)}(\alpha)} \leq \alpha^2 + \frac{\pi^2 \rho^{2k+2}}{(k+1)^2(2k+3)(2k+4)(2k+5)} + \frac{\pi^2}{4\rho^2},
\]

(3.3)

The right-hand side above is clearly minimal for \( \alpha = 0 \). A differentiation in \( \rho \) also shows that it is minimal for

\[
\rho = \rho^{(k)} := \left[ \frac{1}{4} (k+1)(2k+3)(2k+4)(2k+5) \right]^{1/(2k+4)},
\]

and if we put \( \rho^{(k)} \) into (3.3) and simplify we obtain (3.1).

The second statement is an immediate consequence of Lemma 2.4. \( \square \)

Remark. It holds that \( \lim_{\rho \to +\infty} \rho^{(k)} = 1 \), which is coherent with the fact that for the limiting case the first eigenfunction corresponds to \( \rho = 1 \).

3.2. Lower bound on \( \lambda_{3,\Omega^{(k)}(\alpha)} \).

Lemma 3.2. Assume that \( k \geq 3 \) is odd and that \( 0 \leq \alpha \leq \alpha^{(k)}_* \), where \( \alpha^{(k)}_* \) is the constant from (3.2). Then

\[
k + 2 \frac{\lambda_{3,\Omega^{(k)}(\alpha)}}{k+6} \geq \lambda_{1,\Omega^{(k)}(\alpha)}
\]

(3.4)

Proof. We introduce the operator \( \Omega^{(k)}_N(\alpha) \) as the self-adjoint operator in \( L^2(\mathbb{R}^+) \) acting as

\[
\Omega^{(k)}_N(\alpha) = -\frac{d^2}{dt^2} + \left( \frac{k+1}{k+4} - \alpha \right)^2.
\]
and with a Neumann condition at \( t = 0 \). Since it holds that \( \lambda_{2,\Omega_N}^{(k)}(\alpha) = \lambda_{1,\Omega}^{(k)}(\alpha) \) we will work on the half-line with \( \Omega_N^{(k)}(\alpha) \) instead of \( \Omega^{(k)}(\alpha) \), and show the inequality
\[
\frac{k+2}{k+6} \lambda_{2,\Omega_N}^{(k)}(\alpha) \geq \lambda_{1,\Omega}^{(k)}(\alpha).
\]

We introduce constants \( 0 < \varepsilon^{(k)} < 1 \) and \( \hat{\alpha}^{(k)} > 0 \), to be determined in (3.9) and (3.6) below. We also set
\[
\hat{p}^{(k)} = \left((k+1)\hat{\alpha}^{(k)}\right)^{1/(k+1)}.
\]

We claim that if \( 0 < \alpha < \varepsilon^{(k)} \hat{\alpha}^{(k)} < \hat{\alpha}^{(k)} \), then
\[
\left(\frac{k+1}{k+\alpha} - \alpha\right)^2 \geq p(t) := \begin{cases} 2k(1-\varepsilon^{(k)}) (\hat{p}^{(k)})^{2k} (t-\hat{p}^{(k)})^2 & t > \hat{p}^{(k)} \\ 0 & 0 < t < \hat{p}^{(k)}. \end{cases}
\]

This is clear for \( 0 < t \leq \hat{p}^{(k)} \). For \( t > \hat{p}^{(k)} \), we note that the the function \( \hat{p}(t) = \left(\frac{k+1}{k+\alpha} - \alpha\right)^2 - p(t) \) is positive at \( t = \hat{p}^{(k)} \), has a positive derivative at \( t = \hat{p}^{(k)} \),
\[
\hat{p}'(\hat{p}^{(k)}) = 2\left(\frac{(\hat{p}^{(k)})^{k+1}}{k+1} - \alpha\right)(\hat{p}^{(k)})^k > 2\hat{\alpha}^{(k)}(1-\varepsilon^{(k)})(\hat{p}^{(k)})^k > 0
\]

and that \( \hat{p} \) is convex for \( t > \hat{p}^{(k)} \),
\[
\hat{p}''(t) = 2t_{k} + 2\left(\frac{k+1}{k+\alpha} - \alpha\right)kt_{k-1} - \frac{4k(1-\varepsilon^{(k)})}{k+1} (\hat{p}^{(k)})^{2k}
\]
\[
> 2(1-\varepsilon^{(k)})(\hat{p}^{(k)})^{2k} + 2k(\hat{\alpha}^{(k)} - \alpha)(\hat{p}^{(k)})^{k-1} - \frac{4k(1-\varepsilon^{(k)})}{k+1} (\hat{p}^{(k)})^{2k}
\]
\[
> 2k(1-\varepsilon^{(k)})(\hat{p}^{(k)})^{2k} + 2k(1-\varepsilon^{(k)})(\hat{p}^{(k)})^{k-1} - \frac{4k(1-\varepsilon^{(k)})}{k+1} (\hat{p}^{(k)})^{2k}
\]
\[
= 0.
\]

Let us denote by \( b^{(k)} \) the self-adjoint operator in \( L^2(\mathbb{R}^+) \), acting as
\[
b^{(k)} = -\frac{d^2}{dt^2} + p(t),
\]

and with a Neumann condition at \( t = 0 \). Next, we decompose our Hilbert space \( L^2(\mathbb{R}^+) \) as \( L^2((0,\hat{p}^{(k)})) \oplus L^2(\{\hat{p}^{(k)},\infty\}) \) and introduce two new operators \( b_1^{(k)} \) and \( b_2^{(k)} \).

The first one, \( b_1^{(k)} \), is the self-adjoint operator in \( L^2(\{(0,\hat{p}^{(k)})\}) \) acting as
\[
b_1^{(k)} = -\frac{d^2}{dt^2} \quad 0 < t < \hat{p}^{(k)}
\]

with Neumann boundary conditions at \( t = 0 \) and \( t = \hat{p}^{(k)} \). This operator has eigenvalues
\[
\text{Spec}(b_1^{(k)}) = \left\{ \left(\frac{(j-1)\pi}{\hat{p}^{(k)}}\right)^2 \right\}_{j=1}^{\infty}.
\]

The second operator, \( b_2^{(k)} \), is the self-adjoint operator in \( L^2(\{\hat{p}^{(k)},\infty\}) \), acting as
\[
b_2^{(k)} = -\frac{d^2}{ds^2} + \frac{2k(1-\varepsilon^{(k)})}{k+1} (\hat{p}^{(k)})^{2k} (s-\hat{p}^{(k)})^2, \quad t > \hat{p}^{(k)}
\]

with Neumann condition at \( t = \hat{p}^{(k)} \). After translation \( s = t - \hat{p}^{(k)} \) we get
\[
-\frac{d^2}{ds^2} + \frac{2k(1-\varepsilon^{(k)})}{k+1} (\hat{p}^{(k)})^{2k} s^2, \quad s > 0
\]
with Neumann condition at \( s = 0 \). We use a scaling argument and compare with
the harmonic oscillator on the half-line. The result is that the eigenvalues of \( h^{(k)}_2 \)
are
\[
\text{Spec}(h^{(k)}_2) = \left\{ \left[ \frac{2k(1 - \varepsilon^{(k)})}{k + 1} \right]^{1/2} \left( \hat{t}^{(k)} \right)^k (4j - 3) \right\}_{j \in \mathbb{N} \setminus \{0\}}
\]

We clearly have
\[
\lambda_{j,\Omega_N^{(k)}(a)} \geq \lambda_{j,h^{(k)} \oplus h^{(k)}_2}, \quad j \in \mathbb{N} \setminus \{0\},
\]
and \( \text{Spec}(h^{(k)}_1 \oplus h^{(k)}_2) = \text{Spec}(h^{(k)}_1) \cup \text{Spec}(h^{(k)}_2) \).

Next, we choose \( a^{(k)} \) so that the second eigenvalue of \( h^{(k)}_1 \) agrees with the first
one of \( h^{(k)}_2 \), i.e.,
\[
\left( \frac{\pi}{\hat{t}^{(k)}} \right)^2 \left[ \frac{2k(1 - \varepsilon^{(k)})}{k + 1} \right]^{1/2} \left( \hat{t}^{(k)} \right)^k,
\]
This gives
\[
\hat{t}^{(k)} = \left[ \frac{\pi^4(k + 1)}{2k(1 - \varepsilon^{(k)})} \right]^{1/(k + 2)}, \quad \hat{a}^{(k)} = \frac{1}{k + 1} \left[ \frac{\pi^4(k + 1)}{2k(1 - \varepsilon^{(k)})} \right]^{1/(k + 2)}, \quad (3.6)
\]
and the lower bound of \( \lambda_{2,\Omega_N^{(k)}(a)} \) becomes
\[
\lambda_{2,\Omega_N^{(k)}(a)} \geq \pi^2 \left[ \frac{2k(1 - \varepsilon^{(k)})}{\pi^4(k + 1)} \right]^{1/(k + 2)}.
\]
Next we want to choose \( \varepsilon^{(k)} \) in such a way that both
\[
\varepsilon^{(k)} \hat{a}^{(k)} \geq a^{(k)}_\ast,
\]
and
\[
\frac{k + 2}{k + 6} \pi^2 \left[ \frac{2k(1 - \varepsilon^{(k)})}{\pi^4(k + 1)} \right]^{1/(k + 2)} \geq a^2 + (a^{(k)}_\ast)^2, \quad 0 < \alpha \leq a^{(k)}_\ast \quad (3.8)
\]
are satisfied. It is clearly enough to prove the last inequality for \( \alpha = a^{(k)}_\ast \). We let \( \varepsilon^{(k)} \) be given by
\[
\varepsilon^{(k)} = 1 - \frac{2}{k(k + 1)}.
\]
With this choice, \( \hat{t}^{(k)} \) and \( \hat{a}^{(k)} \) reads
\[
\hat{t}^{(k)} = \left[ \frac{\pi^2(k + 1)}{2} \right]^{1/(k + 2)}, \quad \hat{a}^{(k)} = \pi^2 \left[ \frac{2}{\pi^2(k + 1)} \right]^{1/(k + 2)}, \quad (3.10)
\]
and the lower bound of \( \lambda_{2,\Omega_N^{(k)}(a)} \) becomes
\[
\lambda_{2,\Omega_N^{(k)}(a)} \geq \pi^2 \left[ \frac{2}{\pi^2(k + 1)} \right]^{1/(k + 2)}.
\]
We start with (3.7). We claim that \( \varepsilon^{(k)} \hat{a}^{(k)} \) is monotonically increasing for \( k \geq 3 \).
Indeed, both factors are positive, and \( \varepsilon^{(k)} \) is obviously increasing. We differentiate
the expression for \( \hat{a}^{(k)} \) and use the fact that for \( k \geq 3 \)
\[
\log(\pi^2(k + 1)/2) > 2,
\]
to conclude that
\[
\frac{d}{dk} \hat{a}^{(k)} = \hat{a}^{(k)} \left[ \frac{(k + 1) \log(\pi^2(k + 1)/2) - (k + 2)}{(k + 2)^2(k + 1)} \right] > 0.
\]
Moreover, $\varepsilon_{(k)}$ is equal to $2^{-11/5} \times 3^{-1} \times 5\pi^{8/5}$ for $k = 3$. We bound the constants $\alpha_+^{(k)}$ from above as

$$\alpha_+^{(k)} \leq \frac{\pi}{2} \sqrt{\frac{5}{4}} \quad (3.11)$$

for $k \geq 3$. Hence, (3.7) is a consequence of

$$2.26 \approx 2^{-11/5} \times 3^{-1} \times 5\pi^{8/5} > \frac{\pi}{2} \sqrt{\frac{5}{4}} \approx 1.76.$$ 

For inequality (3.8), we note that both sides are positive, so we will show that $A_1(k) \geq 1$ for all $k \geq 3$ with

$$A_1(k) := \frac{k+2}{k+6} \pi^2 \left[ \frac{8(k+1)^2}{\pi^2 (k+1)} \right]^{1/2} = \frac{2(k+1)}{k+6} \left[ \frac{(2k+3)(2k+4)(2k+5)}{\pi^4 (k+1)} \right]^{1/2}. \quad (3.12)$$

A plot of $A_1(k)$ is given in Figure 1. Next, we use the estimate

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{plot.png}
\caption{A plot of $A_1(k)$ for $3 \leq k \leq 50$}
\end{figure}

$$(2k + 3)(2k + 4)(2k + 5) > 8(k + 1)^3,$$

which implies that

$$A_1(k) > \frac{2(k+1)}{k+6} \left( \frac{8(k+1)^2}{\pi^4} \right)^{1/(k+2)}.$$

The first factor is greater than 1 if $k \geq 5$ and the second one is greater than 1 if $k \geq 3$. For $k = 3$ get

$$A_1(3) = 2^{14/5} 3^{-8/5} 5^{4/5} 11^{1/5} \pi^{-4/5} \approx 1.07.$$ 

This finishes the proof of (3.8) and completes the proof.

3.3. End of proof of Theorem 1.1. By Lemmas 2.2 and 2.4 it follows that

$$0 < \alpha^{(k)}_{\min} < \alpha_+^{(k)}.$$ 

However, by Lemmas 2.3 and 3.2, we find that all critical points in this interval must be non-degenerate minima. This clearly implies the uniqueness, and finishes the proof.
4. The case of even $k$

In this section we prove Theorem 1.2.

**Proof** (of Theorem 1.2). The lower bound of $\lambda_{2,n}(\alpha)$ from Lemma 5.1 is no good for small values of $\alpha$. Instead, we use the lower bound

$$\left(\frac{t^{k+1}}{k+1} - \alpha\right) \geq \left(\frac{t^{k+1}}{k+1} - \alpha\right)^2,$$

and then we use that the second eigenvalue corresponding to the potential on the right-hand side on $\mathbb{R}$ is equal to the first eigenvalue of the operator

$$\mathcal{L}_D^{(k)}(\alpha) = -\frac{d^2}{dt^2} + \left(\frac{t^{k+1}}{k+1} - \alpha\right)^2$$

in $L^2(\mathbb{R}^+)$ with a Dirichlet condition at $t = 0$. We use the same type of splitting as in Lemma 3.2,

$$\mathcal{L}_D^{(k)}(\alpha) \geq h_1^{(k)} + h_2^{(k)},$$

and write

$$0 \leq \alpha < \varepsilon(k) \tilde{\alpha}(k) < \tilde{\alpha}(k)^2, \quad \tilde{\alpha}(k) = (\tilde{\alpha}(k)(k+1))^{1/(k+1)},$$

where the constants $\varepsilon(k)$, $\tilde{\alpha}(k)$ and $\tilde{\alpha}(k)$ play the same roles as in the proof of Lemma 3.2 (but, as we will see, they are not the same!). This time the operator $h_1^{(k)}$ is given by

$$h_1^{(k)} = -\frac{d^2}{dt^2}$$

in $L^2((0, \tilde{t}(k)))$ with Dirichlet condition at $t = 0$ and Neumann condition at $t = \tilde{t}(k)$. This operator has eigenvalues

$$\text{Spec}(h_1^{(k)}) = \left\{ \left(\frac{(2j-1)\pi}{2\tilde{t}(k)}\right)^2 \right\}_{j \in \mathbb{N}\setminus\{0\}}.$$

The operator $h_2^{(k)}$ is the same as in the proof of Lemma 3.2, with eigenvalues

$$\text{Spec}(h_2^{(k)}) = \left\{ \left(\frac{2k(1-\varepsilon(k))}{k+1} (\tilde{\alpha}(k))^{2k} \right)^{1/(4j+3)} \right\}_{j \in \mathbb{N}\setminus\{0\}}.$$

As in Lemma 3.2, the best lower bound we can get on $\lambda_{1,n}(\alpha^{(k)}_D)$ is the one we get when the first eigenvalues of $h_1^{(k)}$ and $h_2^{(k)}$ are equal. This determines $\tilde{\alpha}(k)$ as

$$\tilde{\alpha}(k) = \frac{1}{k+1} \left[ \frac{\pi^4(k+1)}{32k(1-\varepsilon(k))} \right]^{\frac{1}{4k+1}}, \quad \tilde{\alpha}(k) = (\tilde{\alpha}(k)(k+1))^{1/(k+1)}.$$

We let $\varepsilon(k) = \frac{1}{2\pi}$. Then the lower bound becomes

$$\lambda_{2,n}(\alpha) \geq \frac{\pi^2}{4} \left[ \frac{32(k-1/2)}{\pi^4(k+1)} \right]^{\frac{1}{2k+2}}.$$

To get the existence of an $\alpha^{(k)}_{\text{max}} > 0$ such that condition (A) in Lemma 2.3 is fulfilled for $\alpha \in (-\alpha^{(k)}_{\text{max}}, \alpha^{(k)}_{\text{max}})$ it is by Lemma 3.1 enough to show that $A_2(k) > 1$ with

$$A_2(k) := k + \frac{\pi^2}{4} \left\{ \frac{32(k-1/2)k+1}{k+6} \right\}^{\frac{1}{2k+2}} = k + \frac{1}{k+6} \left( \frac{8}{\pi^4} (k-1/2)(2k+3)(2k+4)(2k+5) \right)^{1/(k+2)}.$$

See Figure 2 for a plot of $A_2(k)$ for $2 \leq k \leq 50$. We note that $\lim_{k \to \infty} A_2(k) = 1$. By using the estimate

$$(k-1/2)(2k+3)(2k+4)(2k+5) > 16(k+1)^3$$
which is valid for all $k \geq 2$) we find that $A_2(k) > B(k)$ with

$$B(k) := \frac{k+1}{k+6} \left( \frac{128}{\pi^4} (k+1)^3 \right)^{1/(k+2)}.$$

The derivative of $B(k)$ is given by

$$B'(k) = B(k) \frac{8k^2 + 44k + 56 - (k+1)(k+6) \log \left( \frac{128}{\pi^4} (k+1)^3 \right)}{(k+1)(k+2)^2(k+6)}.$$

For $k \geq 14$ it holds that $\log \left( \frac{128}{\pi^4} (k+1)^3 \right) > 8$ and so

$$8k^2 + 44k + 56 - (k+1)(k+6) \log \left( \frac{128}{\pi^4} (k+1)^3 \right) < 8 - 12k \leq -160,$$

which implies that $B'(k) < 0$. Moreover, since $B(14) \approx 1.27$ and $\lim_{k \to \infty} B(k) = 1$$

it follows that $B(k) \geq 1$, and thus $A_2(k) > 1$, for all $k \geq 14$.

For even $2 \leq k \leq 12$, we calculate $A_2(k)$ numerically,

| $k$  | 2  | 4  | 6  | 8  | 10 | 12 |
|------|----|----|----|----|----|----|
| $A_2(k)$ | 1.05 | 1.41 | 1.49 | 1.50 | 1.49 | 1.47 |

which establishes $A_2(k) > 1$ for all even $k \geq 2$.

The proof of the theorem is completed by an application of Lemma 2.3, noting that $\alpha = 0$ is a critical point of $\lambda_{1, \Omega^{(k)}(\alpha)}$ since $\lambda_{1, \Omega^{(k)}(\alpha)}$ is even. \hfill \square

5. The case of large $k$

The goal of this section is to prove Theorem 1.3. It will be done using the ideas from [5].

For even $k \geq 2$ we introduce

$$m_k = \inf_{t \in \mathbb{R}} \frac{t^{k+1} - 1}{t - 1}. \tag{5.1}$$

The constants $m_k$ decrease from $3/4$ for $k = 2$ to $1/2$ as $k \to \infty$.

**Lemma 5.1.** Let $k \geq 2$ be an even integer. With $m_k$ as in (5.1) it holds that

$$\lambda_{j, \Omega^{(k)}(\alpha)} \geq m_k \alpha^{k/(k+1)}(k+1)^{-1/(k+1)} (2j-1), \quad j \in \mathbb{N} \setminus \{0\}. \tag{5.2}$$
Proof. We use a lower bound of the potential
\[
\left(\frac{\eta^{k+1}}{k+1} - \alpha\right)^2 = \alpha^2 \left(\frac{t}{\alpha(k+1)^{(k+1)/2}}\right)^{k+1} - 1 \right)^2 \\
\geq \left(m_k \alpha^{k/(k+1)}(k+1)^{-k/(k+1)}\right)^2 \left(t - (\alpha(k+1)^{1/(k+1)}\right)^2,
\]
and then estimate with the eigenvalues of the harmonic oscillator on the whole line.

Lemma 5.2. Assume that \(k \geq 2\) is an even integer and that \(m_k\) is the constant from Lemma 5.1. Then \(\alpha_{\text{min}}^{(k)} \leq \alpha_{\text{min}}^{(k)}\) where
\[
\alpha_{\text{min}}^{(k)} \leq \left[\frac{(k+1)^{1/(k+1)}}{m_k} \left(\frac{\pi^2 k + 2}{4} k + \frac{1}{4} (k+1)(2k+3)(2k+4)(2k+5)\right)^{-1/(k+2)}\right]^{(k+1)/k}.
\]
In particular, if \(\eta > \frac{\pi^2}{4}\) then there exists \(k_0\) such that, for \(k \geq k_0\), \(k\) even, \(\lambda_{1,\Omega^{(\epsilon)}(\alpha)}\) attains its minimum in \((-\eta, \eta)\).

Proof. Inequality (5.3) follows by combining Lemma 3.1 (with \(\alpha = 0\)) with Lemma 5.1. The second statement is immediate, by letting \(k \to \infty\), and using the fact that \(m_k \geq \frac{1}{k}\) for all \(k\).

Lemma 5.3. Let \(\alpha > 0\). For any \(j \in \mathbb{N} \setminus \{0\}\) it holds that
\[
\lim_{k \to \infty} \lambda_{j,\Omega^{(\epsilon)}(\alpha)} = \alpha^2 + \left(\frac{j\pi}{2}\right)^2
\]
with a uniform control with respect to \(\alpha\) in any compact interval.

This result might be a consequence of \(\Gamma\)-convergence of the Pisa school, except possibly for the uniform control of \(\alpha\). See also [12], in particular Example 4.2. For the sake of completeness, we give a proof inspired by the methods in [2].

Proof. We start with the upper bound, which we prove for \(j \leq 2\) only. The general proof uses the same argument.

For \(j = 1\) the upper bound follows from Lemma 3.1. For \(j = 2\), let us consider the functions
\[
\varphi_1(t) = \begin{cases} \cos\left(\frac{\pi t}{2}\right) & \text{if } |t| \leq 1 \\
0 & \text{if } |t| > 1 \end{cases}, \quad \text{and} \quad \varphi_2(t) = \begin{cases} \sin\left(\frac{\pi t}{2}\right) & \text{if } |t| \leq 1 \\
0 & \text{if } |t| > 1 \end{cases},
\]
They are eigenfunctions of the two lowest eigenvalues of the limiting model \(k \to \infty\), \(-\frac{d^2}{dt^2} + \alpha^2\) in \(L^2((-1, 1))\) with Dirichlet boundary conditions.

Computing the energy of the function \(\mu_1 \varphi_1 + \mu_2 \varphi_2\), \(|\mu_1|^2 + |\mu_2|^2 = 1\), we find a sphere in a two-dimensional space on which the energy is less than \(\mu(k)\), with
\[
\mu(k) = \alpha^2 + \pi^2 + C \frac{1 + |\alpha|}{k+1}.
\]
The upper bound in (5.4) for \(j = 2\) is a consequence of the min-max principle. We continue with the lower bound.

Let \(\varepsilon > 0\) be given. Then, for bounded \(\alpha > 0\), we can choose \(k\) so large that
\[
\left(\frac{t_{k+1}}{k+1} - \alpha\right)^2 \geq p(t) := \begin{cases} \left(\frac{(1+\varepsilon)^{k+1}}{k+1} + \alpha\right)^2, & -\infty < t \leq -1 - \varepsilon, \\
\alpha^2(1-\varepsilon), & -1 - \varepsilon < t \leq 1 - \varepsilon, \\
0, & 1 - \varepsilon < t \leq 1 + \varepsilon, \\
\left(\frac{(1+\varepsilon)^{k+1}}{k+1} - \alpha\right)^2, & 1 + \varepsilon < t < \infty.
\end{cases}
\]
We want to solve the eigenvalue equation

$$-\frac{d^2}{dt^2} u + p(t) u = \lambda u,$$

(5.6)

by solving it for each interval and glue the solutions together as is done in several examples in [2]. We first note that the operator is positive, so we only have to consider $\lambda \geq 0$. Let us introduce the notation

$$A = \left( \frac{(1 + \varepsilon)^{k+1}}{k+1} + \alpha \right)^2, \quad B = \left( \frac{(1 + \varepsilon)^{k+1}}{k+1} - \alpha \right)^2, \quad C = \sqrt{\lambda - \alpha^2(1 - \varepsilon)},$$

$$t_0 = -1 - \varepsilon, \quad t_1 = 1 - \varepsilon, \quad t_2 = 1 + \varepsilon.$$

We may choose $k$ so large that $A > \lambda$ and $B > \lambda$.

If $\lambda > \alpha^2(1 - \varepsilon)$, the square integrable solution to (5.6) is given by

$$u(t) = \begin{cases} 
    a_0 \exp(\sqrt{A - \lambda} t) & -\infty < t \leq t_0, \\
    b_0 \cos(C t) + b_1 \sin(C t) & t_0 < t \leq t_1, \\
    c_0 \cos(\sqrt{A} t) + c_1 \sin(\sqrt{A} t), & t_1 < t \leq t_2, \\
    d_0 \exp(-\sqrt{B - \lambda} t), & t_2 < t < \infty.
\end{cases}$$

(5.7)

Here $a_0$, $b_0$, $b_1$, $c_0$, $c_1$ and $d_0$ are constants that are determined by gluing the solution together. The conditions that both $u$ and $u'$ should coincide at the points $t_0$, $t_1$ and $t_2$ read

$$a_0 \exp(\sqrt{A - \lambda} t_0) = b_0 \cos(C t_0) + b_1 \sin(C t_0),$$

$$a_0 \sqrt{A - \lambda} \exp(\sqrt{A - \lambda} t_0) = -b_0 C \sin(C t_0) + b_1 C \cos(C t_0),$$

$$b_0 \cos(C t_1) + b_1 \sin(C t_1) = c_0 \cos(\sqrt{A} t_1) + c_1 \sin(\sqrt{A} t_1),$$

$$-b_0 \sin(\sqrt{A} t_1) = -c_0 \sqrt{A} \sin(C t_1) + c_1 \sqrt{A} \cos(C t_1),$$

$$c_0 \cos(\sqrt{A} t_2) + c_1 \sin(\sqrt{A} t_2) = d_0 \exp(-\sqrt{B - \lambda} t_2),$$

$$-c_0 \sqrt{A} \sin(\sqrt{A} t_2) + c_1 \sqrt{A} \cos(\sqrt{A} t_2) = -d_0 \sqrt{B - \lambda} \exp(-\sqrt{B - \lambda} t_2).$$

This is a linear system of equations in $a_0$, $b_0$, $b_1$, $c_0$, $c_1$ and $d_0$ which has nontrivial solutions if and only if

$$\frac{1}{\sqrt{C \sqrt{A - \lambda} - C}} \tan(C(t_1 - t_0)) + C = -\frac{1}{\sqrt{\lambda - \alpha^2(1 - \varepsilon)}} \tan(\sqrt{A - \lambda} (t_2 - t_1)) + \frac{\sqrt{C}}{\sqrt{\lambda - \alpha^2(1 - \varepsilon)}} \tan(\sqrt{C} (t_2 - t_1))$$

(5.8)

This is the equation that determines the eigenvalues $\lambda$. For large $k$, the terms $\sqrt{A - \lambda}$ and $\sqrt{B - \lambda}$ are dominating, and we can write (5.8) as

$$\frac{1}{\sqrt{\lambda - \alpha^2(1 - \varepsilon)}} \tan(2 \sqrt{\lambda - \alpha^2(1 - \varepsilon)}) = -\frac{1}{\sqrt{\lambda}} \tan(2 \sqrt{\lambda}) + \mathcal{O}((k+1)(1+\varepsilon)^{-k+1})$$

(5.9)

as $k \to \infty$, where the estimate is uniform for bounded $\alpha$ and $\lambda$. Inserting the values for $t_0$, $t_1$, $t_2$ and $C$, we find that

$$\frac{1}{\sqrt{\lambda - \alpha^2(1 - \varepsilon)}} \tan(2 \sqrt{\lambda - \alpha^2(1 - \varepsilon)}) = -\frac{1}{\sqrt{\lambda}} \tan(2 \sqrt{\lambda}) + \mathcal{O}((k+1)(1+\varepsilon)^{-k+1}),$$

(5.10)

If $0 < \lambda < \alpha^2(1 - \varepsilon)$ then hyperbolic functions appear in the solution of (5.6), and the same type of calculations that resulted in (5.10) this time yield

$$\frac{1}{\sqrt{\alpha^2(1 - \varepsilon) - \lambda}} \tanh(2 \sqrt{\alpha^2(1 - \varepsilon) - \lambda}) = -\frac{1}{\sqrt{\alpha^2(1 - \varepsilon)}} \tan(2 \sqrt{\alpha^2(1 - \varepsilon)}) + \mathcal{O}((k+1)(1+\varepsilon)^{-k+1}).$$
The function
\[
f_1(\lambda) = \begin{cases} 
\frac{1}{\sqrt{\alpha^2(1-\varepsilon) - \lambda}} \tanh(2\sqrt{\alpha^2(1-\varepsilon) - \lambda}), & 0 < \lambda < \alpha^2(1-\varepsilon), \\
\frac{1}{\sqrt{\lambda - \alpha^2(1-\varepsilon)}} \tan(2\sqrt{\lambda - \alpha^2(1-\varepsilon)}), & \alpha^2(1-\varepsilon) < \lambda < \infty,
\end{cases}
\]
is positive for all \(0 \leq \lambda < \alpha^2(1-\varepsilon) + \frac{2}{\pi^2}\), and \(\lim_{\lambda \to \alpha^2(1-\varepsilon) + \frac{2}{\pi^2}} f_1(\lambda) = +\infty\). For larger \(\lambda\) it holds that \(f_1(\lambda)\) is monotonically increasing from \(-\infty\) to \(\infty\) in every interval
\[
\left(\frac{(j-1/2)\pi}{2}\right)^2 + \alpha^2(1-\varepsilon), \left(\frac{(j+1/2)\pi}{2}\right)^2 + \alpha^2(1-\varepsilon). \tag{5.11}
\]
The function
\[
f_2(\lambda) = -\frac{1}{\sqrt{\lambda}} \tan(2\varepsilon\sqrt{\lambda})
\]
is negative for all \(0 \leq \lambda < (\frac{\pi}{4})^2\), and \(\lim_{\lambda \to (\frac{\pi}{4})^2} f_2(\lambda) = -\infty\).

We find that if \(\varepsilon\) satisfies
\[
\left(\frac{\pi}{4}\right)^2 > \left[\frac{(j+1/2)\pi}{2}\right]^2 + \alpha^2(1-\varepsilon)
\]
then there exists a \(k_j(\varepsilon)\) and \(C_j\) such that if \(k \geq k(\varepsilon)\), \(k\) even, it holds that the \(j\)th solution of (5.10) lies in the interval (5.11) and we conclude that
\[
\lambda_{j, \Omega^{(k)}(\alpha)} \geq \left(\frac{(j-1/2)\pi}{2}\right)^2 + \alpha^2(1-\varepsilon) - C_j \varepsilon.
\]
This is not the upper bound we wanted. However, we can do better. There exists a constant \(K_j > 0\) (uniform in \(\alpha, \varepsilon\)) such that
\[
0 < \lambda < \left[\frac{(j+1/2)\pi}{2}\right]^2 + \alpha^2(1-\varepsilon) \implies -K_j \varepsilon < f_2(\lambda) < 0.
\]
This implies that the first \(j\) solutions to (5.10), up to an error of order \(\varepsilon\) coincides with the first \(j\) zeros of the function \(f_1(\lambda)\), i.e., for all \(\varepsilon > 0\) there exist \(\hat{k}_j(\varepsilon)\) and \(\hat{C}_j\) such that for \(k \geq \hat{k}_j(\varepsilon)\), \(k\) even, it holds that
\[
\lambda_{j, \Omega^{(k)}(\alpha)} \geq \left(\frac{j\pi}{2}\right)^2 + \alpha^2(1-\varepsilon) - \hat{C}_j \varepsilon.
\]
This completes the proof of (5.4). \(\square\)

We are now ready to prove Theorem 1.3.

**Proof** (of Theorem 1.3). First, we show (1.3), where we consider odd \(k\) only. We recall the bound (3.11) on \(\alpha^{(k)}_{\min}\), \(0 < \alpha^{(k)}_{\min} < \sqrt{\frac{5}{4} \pi}\), and the formula (2.3) which is valid for \(\alpha^{(k)}_{\min}\), i.e.,
\[
\alpha^{(k)}_{\min} = \int_0^\infty \frac{t^{k+1}}{k+1} (u_{1, \alpha^{(k)}_{\min}})^2 \, dt.
\]
It is enough to show that
\[
\lim_{k \to \infty} \int_1^\infty \frac{t^{k+1}}{k+1} (u_{1, \alpha^{(k)}_{\min}})^2 \, dt = 0.
\]
We first show that, for any \(\varepsilon > 0\) it holds that
\[
\lim_{k \to \infty} \int_{1+\varepsilon}^\infty \frac{t^{k+1}}{k+1} (u_{1, \alpha^{(k)}_{\min}})^2 \, dt = 0. \tag{5.12}
\]
For any $k \geq 3$ and $0 < \alpha < \sqrt{\frac{5}{16}}$ we use Lemma 3.1 to find

$$\int_{1+\varepsilon}^{\infty} \frac{t^{2(k+1)}}{(k+1)^2} (u_{1,\alpha})^2 \, dt \leq 2 \int_{1+\varepsilon}^{\infty} \left( \frac{t^{k+1}}{k+1} - \alpha \right)^2 (u_{1,\alpha})^2 \, dt + 2\alpha^2 \leq 2\lambda_{1,\Omega^{(k)}(\alpha)} + 2\alpha^2 < \frac{15}{8} \pi^2. \quad (5.13)$$

In particular, we get

$$\int_{1+\varepsilon}^{\infty} \frac{t^{k+1}}{k+1} (u_{1,\alpha})^2 \, dt \leq (1+\varepsilon)^{-(k+1)}(k+1)\frac{15}{8} \pi^2,$$

which establishes (5.12). We write the remaining integral as

$$\int_{1}^{1+\varepsilon} \frac{t^{k+1}}{k+1} (u_{1,\alpha})^2 \, dt = \int_{1}^{1+\varepsilon} \left( \frac{t^{k+1}}{k+1} - \alpha \right) (u_{1,\alpha})^2 \, dt + \alpha (u_{1,\alpha})^2 \int_{1}^{1+\varepsilon} (u_{1,\alpha})^2 \, dt,$$

and apply the Cauchy-Schwarz inequality and use (2.6) to conclude that the first integral tends to zero as $k \to \infty$. For the second integral we use the general inequality

$$\int_{a}^{b} u(t)^2 \, dt \leq 4 \int_{a}^{b+\varepsilon} u(t)^2 \, dt + 2(b-a)^2 \int_{a}^{b} u'(t)^2 \, dt$$

with $a = 1$ and $b = 1 + \varepsilon$. We use (5.13) to find that

$$\int_{1+\varepsilon}^{\infty} (u_{1,\alpha})^2 \, dt \leq \left( 1 + \frac{\varepsilon}{2} \right)^{-(k+1)}(k+1)^2 \int_{1+\varepsilon}^{\infty} \frac{t^{2(k+1)}}{(k+1)^2} (u_{1,\alpha})^2 \, dt \leq \frac{15}{8} \pi^2 \left( 1 + \frac{\varepsilon}{2} \right)^{-(k+1)}(k+1)^2.$$

Moreover we use the inequality

$$\int_{1}^{1+\varepsilon} (u_{1,\alpha})^2 \, dt \leq \lambda_{1,\Omega^{(k)}(\alpha)} \leq \frac{5}{16} \pi^2,$$

to get, finally,

$$\int_{1}^{1+\varepsilon} (u_{1,\alpha})^2 \, dt \leq \frac{15}{2} \pi^2 \left( 1 + \frac{\varepsilon}{2} \right)^{-(k+1)}(k+1)^2 + \frac{5}{8} \pi^2 \varepsilon^2.$$

This achieves the proof of (1.3).

We continue with the proof of the second statement. We know that $\alpha = 0$ is a non-degenerate local minima. By Lemma 5.2 it is enough to show that there exists a $k_0$ such that condition (A) in Lemma 2.3 holds for all $k \geq k_0$, $k$ even, and all $0 < \alpha \leq \eta$, where $\eta > \frac{\pi^2}{2}$ is arbitrary. However, it is clear by Lemma 5.3 that this can be done. \hfill \Box

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(Bernard Helffer) DÉPARTEMENT DE MATHÉMATIQUES, BÂTIMENT 425, UNIV PARIS-SUD ET CNRS, F-91405 ORSAY CÉDEX, FRANCE

E-mail address: Bernard.Helffer@math.u-psud.fr

(Mikael Persson) AARHUS UNIVERSITY, DEPARTMENT OF MATHEMATICAL SCIENCES, 1530 NY MUNKEGADE, 8000 AARHUS C, DENMARK

E-mail address: mickep@imf.au.dk