A note on star-like configurations in finite settings

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May 11, 2014

Abstract

Given $E \subset \mathbb{F}_q^d$, we show that certain configurations occur frequently when $E$ is of sufficiently large cardinality. Specifically, we show that we achieve the statistically number of $k$-stars $\left| \left\{ (x, x^1, \ldots, x^k) \in E^{k+1} : \|x - x^i\| = t_i \right\} \right|$ when $|E| \gg_k q^{\frac{d+1}{2}}$. This result can be thought of as a natural generalization of the Erdős-Falconer distance problem. Our result improves on a pinned-version of our theorem which implied the above result, but only in the range $|E| \gg q^{\frac{d+k}{2}}$.

As an immediate corollary, this demonstrates that when $|E| \gg c_k q^{\frac{d+1}{2}}$, then $E$ determines a positive proportion of all $k$-stars. Our results also extend to the setting of integers mod $q$.

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1 Background

Given $E \subset \mathbb{R}^d$, let $\Delta(E) = \{|x-y| : x, y \in E\}$, where $|\cdot|$ is the usual Euclidean distance. The classical Erdős-distance problem ([6]) is as follows.

**Conjecture 1.1.** If $E \subset \mathbb{R}^d$ is a finite set with cardinality $|E| = n$, then,

$$ |\Delta(E)| \geq \begin{cases} cn^{1-\frac{d}{2}} & d = 2 \\ cn^{\frac{d}{2}} & d \geq 3 \end{cases} $$

Guth-Katz ([11]) showed the conjecture holds for $d = 2$. See [9, 14] for more details and the best known results in higher dimensions. In [7], Falconer considered a continuous version of the problem.

**Conjecture 1.2.** If $E \subset [0,1]^d$ has Hausdorff dimension $\dim_H(E) > \frac{d}{2}$, then $\Delta(E)$ has positive Lebesgue measure.

See [5, 7, 8, 15] for more information on the problem including the best known bounds to date. In [13], Iosevich-Rudnev considered a finite field analogue of the problem. $\mathbb{F}_q^d$ will denote the $d$-dimensional vector space over the finite field with $q$ elements, and for $E \subset \mathbb{F}_q^d$, we define

$$ \Delta(E) = \{|\|x-y\| : x, y \in E\}, $$

where for $x = (x_1, \ldots, x_d) \in \mathbb{F}_q^d$ we put $\|x\| = x^t x = x_1^2 + \cdots + x_d^2$. Clearly $\|\cdot\|$ is not a metric, but this notion of distance is preserved under orthogonal transformations. If $O$ is a $d \times d$ orthogonal matrix over $\mathbb{F}_q$, then $\|Ox\| = \|x\|$ for all $x \in \mathbb{F}_q^d$. Once we have defined a suitable notion of distance in $\mathbb{F}_q^d$, the problem proceeds directly as before. The so-called Erdős-Falconer distance problem in $\mathbb{F}_q^d$ asks one to show that if $q$ is odd and $E \subset \mathbb{F}_q^d$ is a set of sufficiently large cardinality, then $\Delta(E) = \mathbb{F}_q$. The first step in this direction was made by Iosevich-Rudnev ([13]):

**Theorem 1.3.** Let $E \subset \mathbb{F}_q^d$ with $|E| \gg q^{\frac{d+1}{2}}$. Then $\Delta(E) = \mathbb{F}_q$.

The notation $X \gg Y$ will be used throughout to mean that $Y = o(X)$. Theorem 1.3 is sharp, at least in odd dimensions, as was shown in [12]. The problem is open in even dimensions:

**Conjecture 1.4.** Suppose that $E \subset \mathbb{F}_q^d$, where $d \geq 2$ is even. If $|E| \gg q^d$, then $\Delta(E) = \mathbb{F}_q$.

The Erdős-Falconer distance problem has also been considered in $\mathbb{Z}_q^d$, the free module of rank $d$ over the ring of integers modulo $q$. In this setting, for
$E \subset \mathbb{Z}_q^d$, define $\Delta(E)$ as in the finite field case. It has been shown ([4]) that if $q = p^\ell$, where $p \neq 2$ and $E \subset \mathbb{Z}_q^d$ has size $|E| \gg \ell(q^{d(2\ell-1)+1})$, then $\Delta(E) \supset \mathbb{Z}_q^d$, where $\mathbb{Z}_q^d$ is the group of units in $\mathbb{Z}_q$.

Generalizations of the distance problem can be extended in a few directions. One direction which has received considerable attention is the study of finite point configurations. Let $T_k(E)$ denote the set of $k$-simplices with vertices in $E$, where a $k$-simplex is a set of $(k+1)$-points spanning a $k$-dimensional subspace. It can be shown (see [2], for example) that the set of $k$-simplices can be viewed as a $(k+1)$-dimensional set. In [2], it was shown that if $E \subset \mathbb{F}_q^d$ with $|E| \gg q^{\frac{d+1}{2}}$, then $|T_k(E)| \geq cq^{k+1}$. Their proof relied on the following observation.

Theorem 1.5 ([2], Theorem 2.12). For $x^1, \ldots, x^k \in E$, define

$$\Delta_{x^1, \ldots, x^k}(E) = \left\{ \left( \|x - x^1\|, \ldots, \|x - x^k\| \right) : x \in E \right\} \subset \mathbb{F}_q^k.$$  

Then, for $|E| \gg q^{\frac{d+1}{2}}$, we have

$$|E|^{-k} \sum_{x^1, \ldots, x^k \in E} |\Delta_{x^1, \ldots, x^k}(E)| \geq cq^k.$$

The case $d = k$ is notoriously difficult, and only a few results are known in the case $d = k = 2$ ([1, 10]). Rather than studying full $k$-simplices, it is sometimes advantageous to study certain subsets of $k$-simplices.

1.1 Results

Theorem 1.6. Let $E \subset \mathbb{F}_q^d$, and suppose $t_i \in \mathbb{F}_q \setminus \{0\}$ for $i = 1, \ldots, k$. For $T = (t_1, \ldots, t_k)$, let

$$\nu_k(T) = \left| \left\{ (x, x^1, \ldots, x^k) \in E^{k+1} : \|x - x^i\| = t_i, \quad i = 1, \ldots, k \right\} \right|.$$  

Then,

$$\nu_k(T) = \frac{|E|^{k+1}}{q^k} (1 + o(1))$$

when $|E| \gg c_k q^{\frac{d+1}{2}}$, where the constant $c_k$ depends only on $k$.

Corollary 1.7. If $|E| \gg c_k q^{\frac{d+1}{2}}$, then $E$ determines a positive proportion of all $k$-stars. More precisely, let $S_k(E)$ denote the set of $k$-stars determined by $E$:

$$S_k(E) = \left\{ \left( \|x - x^1\|, \ldots, \|x - x^k\| \right) : x, x^i \in E \right\} \subset \mathbb{F}_q^k.$$  

Then, $|S_k(E)| \geq cq^k$. 

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Remark 1.8. Note that Corollary 1.7 differs from Theorem 1.5 in that the quantity $\Delta_{x_1, \ldots, x_k}(E)$ prescribes the vectors $x^i \in E$ and only $x$ runs through $E$, whereas $S_k(E)$ is simply the set of all possible $k$-stars. That is, in the quantity $S_k(E)$, the vectors $x, x^1, \ldots, x^k$ all run through $E$.

Theorem 1.6 also extends previous results relying on $k$-star estimates. Specifically, our result gives the following improvement of Theorem 1.1 from [3].

Corollary 1.9. Let $T_k(E)$ be the set of $k$-simplices determined by $E$ as defined above. Let $E \subset \mathbb{F}_q^d$ with $|E| \geq \rho q^d$ for $q\frac{d-1}2 \ll \rho \leq 1$. Then, there exists an absolute constant $c > 0$ such that

$$|T_d(E)| \geq c \rho^{d-1} q^{\frac{d+1}2}.$$ 

See Remark 1.2 in [3] for more details.

The methods used to establish our results also extend to $\mathbb{Z}_q$, the integers modulo $q$.

Theorem 1.10. Suppose that $q = p^\ell$ for odd $p$, and let $t_i \in \mathbb{Z}_q^\times$ be a unit for $i = 1, \ldots, k$. For $T = (t_1, \ldots, t_k)$ and $E \subset \mathbb{Z}_q^d$, let

$$\nu_k(T) = \left| \left\{ (x, x^1, \ldots, x^k) \in E^{k+1} : \|x - x^i\| = t_i, \quad i = 1, \ldots, k \right\} \right|,$$

where as before, $\|x\| = x_1^2 + \cdots + x_d^2$. Then,

$$\nu_k(T) = \frac{|E|^{k+1}}{|\mathbb{Z}_q^d|} (1 + o(1))$$

when $|E| \gg c(k, \ell) q^{\frac{d(2\ell-1)+1}{2\ell}}$, where the constant $c(k, \ell)$ depends only on $k$ and $\ell$.

Corollary 1.11. If $|E| \gg c(k, \ell) q^{\frac{d(2\ell-1)+1}{2\ell}}$, then $E$ determines a positive proportion of the $k$-stars. That is, for the range of $E$ above, $|S_k(E)| \geq cq^k$ for some absolute constant $c$ if $q$ is sufficiently large.

1.2 Fourier analysis

Let $G$ denote either $\mathbb{F}_q$ or $\mathbb{Z}_q$. Let $f : G^d \to \mathbb{C}$. Then the Fourier transform of $f$ is defined as

$$\hat{f}(m) = |G|^{-d} \sum_{x \in G^d} f(x) \chi(-x \cdot m),$$

where $\chi(x)$ is the canonical additive character in $\mathbb{F}_q$, and where $\chi(x) = \exp(2\pi i x/q)$, when $G = \mathbb{Z}_q$. Note that $\hat{f}(0) = |G|^{-d} \|f\|_{L^1(G)}$ is the average value of $f$. The Fourier transform also has the properties of inversion and Plancherel.
Lemma 1.12.  
\[ f(x) = \sum_{x \in G^d} \hat{f}(m) \chi(x \cdot m) \]
\[ \sum_{m \in G^d} |\hat{f}(m)|^2 = |G|^{-d} \sum_{x \in G^d} |f(x)|^2 \]

These properties are easy to prove using orthogonality of characters, and we leave the proofs to the reader.

2 Proof of Theorem 1.6

For ease of exposition we supply the proof of Theorem 1.6 for the case \( t_1 = \cdots = t_k = 1 \). The reader can easily verify that the proof goes through unchanged so long as \( t_i \neq 0 \). For simplicity, let \( \nu_k(1, \ldots, 1) = \nu_k \). We will rely on the following well known results.

Lemma 2.1. Let \( S_t = \{ x \in \mathbb{F}_q^d : \|x\| = t \} \). Then for \( t \neq 0 \), we have
\[ |S_t| = q^{d-1}(1 + o(1)). \]
If \( t \neq 0 \) and \( m \in \mathbb{F}_q^d \setminus \{0\} \), then
\[ |\hat{S}_t(m)| \leq 2q^{-\frac{d+1}{2}}. \]

We refer the interested reader to [13] for a proof of the Lemma. We put \( S = S_1 \), and we identify \( S \) and \( E \) with their indicator functions, so that, for example, \( E(x) = 1 \) for \( x \in E \) and is zero otherwise. Then, applying Fourier inversion we see
\[ \nu_k = \sum_{x, x^1, \ldots, x^k} E(x)E(x^1) \cdots E(x^k)S(x - x^1) \cdots S(x - x^k) \]
\[ = q^{kd} \sum_{x, m^1, \ldots, m^k} E(x) \chi(x \cdot (m^1 + \cdots + m^k))\hat{E}(m^1) \cdots \hat{E}(m^k)\hat{S}(m^1) \cdots \hat{S}(m^k) \]
\[ = q^{(k+1)d} \sum_{m^1, \ldots, m^k} \hat{E}(-m^1 - \cdots - m^k)\hat{E}(m^1) \cdots \hat{E}(m^k)\hat{S}(m^1) \cdots \hat{S}(m^k) \]
\[ = M + \sum_{j=0}^{k-1} \binom{k}{j} R_j, \]
where
\[ M = q^{(k+1)d} \left( \hat{E}(0) \right)^{k+1} \left( \hat{S}(0) \right)^k = \frac{|E|^{k+1}}{q^k} (1 + o(1)), \]
by Lemma 2.1, and where
\[ R_j = q^{(k+1)d} \sum_{m^{j+1} \ldots m^k \neq 0} \hat{E}(-m^{j+1} - \ldots - m^k) \hat{E}(m^{j+1}) \ldots \hat{E}(m^k) \hat{S}(m^{j+1}) \ldots \hat{S}(m^k). \]

It should be noted that the quantity \( \binom{k}{j} \) arises since the terms in the remainders \( R_j \) are symmetric in every variable, and \( \binom{k}{j} \) is the number of ways to choose \( m^1, \ldots, m^k \) such that \( j \) of the terms are zero. However, since \( \binom{k}{j} \leq \binom{k}{[k/2]} \leq 2^k \) for all \( j \), these binomial coefficients can be absorbed into the constant \( c_k \). Also, the constant \( c_k \) will change from line to line, but ultimately depends only on \( k \). Now, we have
\[ R_j = q^{(k+1)d} \left( \hat{E}(0) \hat{S}(0) \right)^j \sum_{m^{j+1} \ldots m^k \neq 0} \hat{E}(-m^{j+1} - \ldots - m^k) \hat{E}(m^{j+1}) \ldots \hat{E}(m^k) \hat{S}(m^{j+1}) \ldots \hat{S}(m^k). \]

We will deal separately with the remainders \( R_{k-1} \) and \( R_j \) for \( 0 \leq j \leq k-2 \). First we deal with the case \( j = k-1 \). In this case, we have
\[ R_{k-1} = q^{(k+1)d} q^{-2d(k-1)} |E|^{k-1} |S|^{k-1} \sum_{m^k \neq 0} \hat{E}(-m^k) \hat{E}(m^k) \hat{S}(m^k). \]

Bringing absolute values inside the sum, and then applying Lemma 2.1 followed by Lemma 1.12, we see that
\[ |R_{k-1}| \leq c_k q^{(k+1)d} q^{-2d(k-1)} |E|^{k-1} q^{(d-1)(k-1)} q^{-\frac{d+1}{2}} q^{-d} |E| \]
\[ = c_k q^{kd+d-2dk+2d+dk-k-d+1-\frac{d+1}{2}-d} |E|^{k} \]
\[ = c_k q^{\frac{d+1}{2}-k} |E|^k. \]

When \( j \leq k-2 \), we have
\[ |R_j| \leq c_k q^{(k+1)d} q^{-2dj} |E|^j |S|^j q^{-\frac{(k-j)(d+1)}{2}} \sum_{m^{j+1} \ldots m^k \neq 0} \left| \hat{E}(-m^{j+1} - \ldots - m^k) \right| \left| \hat{E}(m^{j+1}) \right| \left| \hat{E}(m^k) \right| \]
\[ \leq c_k q^{(k+1)d} q^{-2dj} |E|^j |S|^j q^{-\frac{(k-j)(d+1)}{2}} (q^{-d} |E|^j)^{k-j-1} \sum_{m^{k-1} \ldots m^k \neq 0} \left| \hat{E}(m^{k-1}) \right| \left| \hat{E}(m^k) \right| \]
\[ \leq c_k q^{(k+1)d} q^{-2dj} |E|^j |S|^j q^{-\frac{(k-j)(d+1)}{2}} (q^{-d} |E|^j)^{k-j-1} \left( \sum_{m^{k-1}} \left| \hat{E}(m^{k-1}) \right|^2 \right)^{1/2} \left( \sum_{m^k} \left| \hat{E}(m^k) \right|^2 \right)^{1/2} \]
\[ \leq c_k q^{(k+1)d} q^{-2dj} |E|^j (q^{d-1})^j q^{-\frac{(k-j)(d+1)}{2}} (q^{-d} |E|^j)^{k-j} \]
\[ = c_k q^{2d-k-kd+j(d-1)} |E|^k \]

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where we pushed absolute values inside the sum, applied Lemma 2.1, applied the trivial bound $|\hat{E}(\eta)| \leq q^{-d}|E|$, and applied Cauchy-Schwarz followed by Plancherel. Note that we have exhibited the bound

$$|R_j| \leq c_k q^{2d-k-kd+1(d-1)} |E|^k$$

for all $j$ including $j = k - 1$. Since $|R_j| \leq |R_{k-1}|$ for all $0 \leq j \leq k - 1$, we see that

$$|\nu_k| = M + \sum_{j=0}^{k-1} \binom{k}{j} R_j = \frac{|E|^{k+1}}{q^k} (1 + o(1)) + O \left( c_k q^{d+1-k} |E|^k \right)$$

which implies that

$$|\nu_k| = \frac{|E|^{k+1}}{q^k} (1 + o(1))$$

when

$$|E| \gg c_k q^{d+1},$$

as claimed.

3 Proof of Theorem 1.10

The Proof of Theorem 1.10 is exactly the same as that of Theorem 1.6, except that we rely on different estimates. We leave many of the details to the reader. The following results originally appeared in [4].

Lemma 3.1. Let $S_t = \{x \in \mathbb{Z}_q^d : \|x\| = t\}$. For $t \in \mathbb{Z}_q^\times$, we have

$$|S_t| = q^{d-1} (1 + o(1))$$

If $t \in \mathbb{Z}_q^\times$ and $m \in \mathbb{Z}_q^d \setminus \{0\}$, then

$$|\hat{S}_t(m)| \leq \ell(\ell + 1) q^{-\frac{d^{d-1}}{2}}.$$

To prove Theorem 1.10, we write

$$\nu_k = q^{(k+1)d} \sum_{m^1, \ldots, m^k} \hat{E}(-m^1 - \cdots - m^k) \hat{E}(m^1) \cdots \hat{E}(m^k) \hat{S}(m^1) \cdots \hat{S}(m^k)$$

$$= M + \sum_{j=0}^{k-1} \binom{k}{j} R_j$$
Now, \( M = \frac{|E|^{k+1}}{q^k}(1 + o(1)) \), and
\[
R_j = q^{(k+1)d} \sum_{\substack{m^1, \ldots, m^j = 0 \\ m^{j+1}, \ldots, m^k \neq 0}}  \hat{E}(-m^1 - \cdots - m^k) \hat{E}(m^1) \cdots \hat{E}(m^k) \hat{S}(m^1) \cdots \hat{S}(m^k).
\]

Using precisely the same strategy as before save that of applying Lemma 3.1 rather than Lemma 2.1, we see that
\[
|R_j| \leq c(k, \ell)q^{(k+1)d}q^{-2d}q^{(d-1)j}q^{-\frac{(k-j)(d+2\ell-1)}{2\ell}}q^{-d(k-j)}|E|^k
= c(k, \ell)q^{2d-kd+k+2\ell j+(d-1)}|E|^k
\]
so that
\[
|R_j| \leq |R_{k-1} | \leq c(k, \ell)q^{\frac{d(2\ell-1)+1}{2\ell}}|E|^k.
\]
This immediately gives that
\[
\nu_k = \frac{|E|^{k+1}}{q^k}(1 + o(1))
\]
whenever
\[
|E| \gg c(k, \ell)q^{\frac{d(2\ell-1)+1}{2\ell}}.
\]

References

[1] M. Bennett, A. Iosevich, J. Pakianathan, Three point configurations in two-dimensional vector spaces over finite fields via the Elekes-Sharir paradigm, to appear, Combinatorica, (2013).

[2] J. Chapman, M. B. Erdogan, D. Hart, A. Iosevich, D. Koh, Pinned distance sets, k-simplices, Wolff’s exponent in finite fields and sum-product estimates, Math Z. 271 (2012), 63–93.

[3] D. Covert, D. Hart, A. Iosevich, S. Senger, I. Uriarte-Tuero, A Furstenberg-Katznelson-Weiss type theorem on \((d+1)\)-point configurations in sets of positive density in finite field geometries (to appear, Discrete Mathematics).

[4] D. Covert, A. Iosevich, and J. Pakianathan, Geometric configurations in the ring of integers modulo \(p^\ell\), Indiana University Mathematics Journal, to appear, (2013).

[5] B. Erdogan, A bilinear Fourier extension theorem and applications to the distance set problem, Int. Math. Res. Not. (2005), 23, 1411–1425.
[6] P. Erdős, *Integral distances*, Bull. Amer. Math. Soc. **51** (1946) 996.

[7] K. Falconer *On the Hausdorff dimensions of distance sets*, Mathematika **32** (1986), 206–212.

[8] K. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics, **85** Cambridge Univ. Pr., Cambridge (1986).

[9] J. Garibaldi, A. Iosevich, and S. Senger, *The Erdős distance problem*, AMS Student Library Series, **56**, (2011).

[10] A. Greenleaf and A. Iosevich, *Three points configuration in the plane, a bilinear operator and applications to discrete geometry*, Analysis & PDE **5-2** (2012), 397-409.

[11] L. Guth and N. Katz, *On the Erdős distinct distance problem in the plane*, (preprint) http://arxiv.org/pdf/1011.4105.

[12] D. Hart, A. Iosevich, D. Koh, M. Rudnev, *Averages over hyperplanes, sum-product theory in finite fields, and the Erdős-Falconer distance conjecture*, Transactions of the AMS, **363** (2011) 3255-3275.

[13] A. Iosevich and M. Rudnev *Erdős distance problem in vector spaces over finite fields*, Transactions of the AMS, (2007).

[14] J. Solymosi and V. Vu, *Distinct distances in high dimensional homogeneous sets*, Towards a theory of geometric graphs, 259-268, Contemp. Math., **342**, Amer. Math. Soc., Providence (2004).

[15] T. Wolff, *Decay of circular means of Fourier transforms of measures*, Int. Math. Res. Not. **10** (1999), 547–567.