Abstract. We present some streamlined proofs of some of the basic results in Aubry-Mather theory (existence of quasi-periodic minimizers, multiplicity results when there are gaps among minimizers) based on the study of hull functions. We present results in arbitrary number of dimensions.

We also compare the proofs and results with those obtained in other formalisms.

Quasi-periodic solutions, variational methods, Aubry-Mather theory, hull functions

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1. Introduction

Many problems in dynamics and in solid state physics lead to the study of minimizers and other critical points of (formal) variational problems. One wants to establish existence and geometric properties of these minimizers and critical points.

For example, orbits of a twist map are critical points of the action (see [Gol01]). In other physical problems (e.g. motion of dislocations, spin waves, etc.) the interpretation of the variational principle is energy and the critical points are equilibrium states (see [ALD83, GHM08]), whereas minimizers are ground states.

The theory of critical points for such functionals was studied by mathematicians very intensely since the early 80’s due to the systematic work of Aubry ([ALD83]) and Mather ([Mat82a]), (but there are precedents in the mathematical work of Morse and Hedlund in the 30’s [Mor24, Mor73, Hed32] and much more work by physicists [BK04]).

From the point of view of analysis, one of the problems of the theory is that the variational problems are formal and that therefore, one cannot use a straightforward approach to the calculus of variations. Also, to look for quasi-periodic solutions, one has to deal with functionals in $\ell^\infty = \{ \{u_i\}_{i \in \mathbb{Z}^d} \mid \| u \|_{\ell^\infty} = \sup_{i \in \mathbb{Z}^d} |u_i| < \infty \}$ which is a notoriously ill-behaved space.

For example, we will be dealing with the variational problem for “configurations” i.e $u : \mathbb{Z}^d \to \mathbb{R}$

$$\mathcal{L}(u) = \sum_{i \in \mathbb{Z}^d} \sum_{j=1}^{d} H_j(u_i, u_{i+e_j})$$ (1)
The case \( d = 1 \) corresponds to twist mappings. When we are looking for quasi-periodic solutions, the sums in (1) are clearly, not meant to converge but there are ways of associating well defined variational problems to the formal functionals (1).

There are many standard ways of dealing with such problems. The two main ones are: A) To work in spaces of sequences defining precisely what one means by minimizers, critical values of the action, etc. This is what was done in the classical calculus of variations starting with [Mor24]. B) We assume that \( u \) are parameterized by a function \( h \) – the hull function – and a frequency \( \omega \in \mathbb{R}^d \) such that

\[
(2) \quad u_i = h(\omega \cdot i)
\]

and derive a variational principle for \( h \).

1.1. **Heuristic derivation of the Percival Lagrangian.** The heuristic derivation of the variational principle in B) is as follows [Per79]. If we assume solutions of the form (2), considering a big box and normalizing the Lagrangian (which does not change the minima or critical points), we are led to considering

\[
\mathcal{L}_{N,\omega}(u) = \frac{1}{N^d} \sum_{i \in \mathbb{Z}^d, |i| \leq N} \sum_{j=1}^d H_j(h(\omega \cdot i), h(\omega \cdot i + \omega_j))
\]

Heuristically, for \( N \to \infty \), \( \mathcal{L}_{N,\omega} \to \mathcal{P}_\omega \) where

\[
(3) \quad \mathcal{P}_\omega(h) = \sum_{j=1}^d \int_0^1 H_j(h(\theta), h(\theta + \omega_j))
\]

This heuristic derivation shows that given a solution \( h \) of \( \omega \)'s of the form (2), \( \mathcal{P}_\omega(h) \) has a direct physical interpretation as the energy per volume.

In a similar heuristic way, we can argue that the Euler-Lagrange equations for \( \mathcal{P}_\omega \) are obtained by computing

\[
\mathcal{P}_\omega(h + \varepsilon \eta) - \mathcal{P}_\omega(h)
\]

\[
= \varepsilon \int_0^1 d\theta \sum_j \partial_1 H_j(h(\theta), h(\theta + \omega_j))\eta(\theta) + \partial_1 H_j(h(\theta), h(\theta + \omega_j))\eta(\theta + \omega_j) + O(\varepsilon^2)
\]

\[
= \int_0^1 d\theta \left[ \sum_j \partial_1 H_j(h(\theta), h(\theta + \omega_j)) + \partial_2 H_j(h(\theta - \omega_j), h(\theta)) \right] \eta(\theta) + O(\varepsilon^2)
\]

If \( \eta \) is arbitrary, the Euler-Lagrange equations should be:

\[
(4) \quad X(h) \equiv \sum_j \partial_1 H_j(h(\theta), h(\theta + \omega_j)) + \partial_2 H_j(h(\theta - \omega_j), h(\theta)) = 0
\]

Of course, the above heuristic derivation is rather imprecise since, depending on the space of \( h \)'s we consider the variations allowed may not be arbitrary and minimizers may not satisfy Euler-Lagrange equation.

Besides the heuristic derivation, the paper [Per79] used this formalism as a very effective numerical method to compute quasi-periodic solutions.
We also note that this formalism can be used as the basis of KAM theory to produce smooth solutions under some assumptions (Diophantine properties of the frequencies that the system is close to integrable, etc.) (see \cite{SZ89, CdlL09}).

The rigorous study of \((\ref{eq:3})\) that we will pursue here entails

- I) To identify appropriate spaces in which one can study \(P_\omega\) and show that it has a minimizer satisfying geometric properties.
- II) To show that the minimizer of \(P_\omega\) satisfies the Euler-Lagrange equations.
- III) To show that the minimizers thus obtained, correspond to minimizers in the formalism A).
- IV) To show existence of other critical points and their properties provided there are two minimizers that are essentially different.

We point out that there are different tradeoffs. If we choose a very restrictive space on which to consider the minimization problem (i.e. a space of functions enjoying many properties), then it becomes hard to show that the minimizer exists and that it satisfies the Euler-Lagrange equations. On the other hand, if we choose very general spaces, the minimizers may become useless. One has to consider spaces general enough so that minimizers exist and satisfy Euler-Lagrange equations, but restrictive enough so that they satisfy enough properties that can be bootstrapped. This compromise is, of course, far from unique and we will make a point of showing several such compromises.

Another point to keep in mind is that the problem of existence of minimizers can be approximated by simpler ones (under rather soft assumptions the limit of minimizers of a sequence of problems is a minimizer of the limiting problem \cite{Mor73}). On the other hand, passing to the limit on multiplicity results is difficult because the limits of two different solutions of the approximating problems could be the same.

Even if the results that we will obtain have already been obtained (in the \(d = 1\) case), the spaces that we choose are different and we obtain shorter proofs. For the proof in IV) we use a gradient flow approach.

We refer to \cite{Ban88, For96, Gol01} for surveys of the classical results. Notably, the results of existence of minimizers were obtained by the hull function approach in \cite{Mat82a}, the critical points in \cite{Mat86}. Besides the fact that we deal with \(d > 1\) and more general lattices, we think it is worthwhile to present the arguments in a coherent way. It is also interesting to compare the approach presented in this paper with that in \cite{dlLV07a} which covers similar ground (it includes weak twist, and long range interactions) using methods based on orbit spaces.

Of course, Aubry-Mather theory has grown well beyond the results that we consider here and now includes studies of other objects such as Mather measures, Mañé critical values, which lead to applications to construction of connecting orbits, viscosity solutions, transport theory, multi-bump solutions etc. Some surveys on these more recent aspects are \cite{Man91, Man96b, CI99, Fat97, Fig08}.
Remark 1. It is very important to note that the variational principles are degenerate in the sense that minimizers will not be unique. We note that if \( u_i \) is a ground state (resp. a critical point) of \( \mathcal{L} \), then, for any \( k \in \mathbb{Z}^d \) so is \( \tilde{u} \) defined by \( \tilde{u}_i = u_{i+k} \).

Similarly, if \( u \) is a minimizer (resp. a critical point) of \( \mathcal{P}_\omega \), so is \( \tilde{h} \) defined by \( \tilde{h}(\theta) = h(\theta + a) \).

Note that, in the hull function formalism, the symmetries are continuous symmetries indexed by the real number \( a \) whereas in the orbit formalism, the symmetries are indexed by \( k \in \mathbb{Z}^d \).

Later on, we will see that our assumptions on \( H \) will imply other symmetries of the problem.

Remark 2. We note that the methods we consider can be extended with only typographical changes in the formulas to interactions of infinite range and involving many bodies

\[
\mathcal{L}(u) = \sum_{L \in \mathbb{N}} \sum_{i \in \mathbb{Z}^d} H_L(T_i u)
\]

where \( H_L(u) \) depends only \( \{u_j\}_{j \leq L} \) and \( T_i \) is the translation. Of course, one needs to assume that the interactions decrease fast enough with the distance \( L \).

It is easy to see that when we consider (5), the corresponding variational principle for the hull functions

\[
\mathcal{P}_\omega(h) = \int_0^1 d\theta \sum_{L \in \mathbb{N}} H_L(h)(\theta)
\]

where \( H_L(h) \) is the function obtained replacing \( u_j \) by \( h(\theta + \omega \cdot j) \).

Remark 3. In Appendix A, we will show how the method of hull functions can be extended to study configurations \( u : \Lambda \to \mathbb{R} \) when \( \Lambda \) is, e.g. the Bethe lattice.

This is somewhat surprising because the heuristic derivation outlined in Section 1.2 uses that \( \mathbb{Z}^d \) is amenable and the Bethe lattice is not amenable.

Remark 4. Note that in point III) we show that the minimizers of the hull function approach give rise to minimizing sequences, which also satisfy some growth properties at \( \infty \) and several monotonicity properties.

It seems to be an open question to decide whether there are converses to III). That is, whether the minimizers satisfying several order and growth properties are of the form (2), in particular, they depend on just one variable.

There are several versions of these questions formulated for PDE’s in Mos86, Ban89. In JGV09 one can find that there is a close relation between these questions and a famous conjecture by De Giorgi. Indeed in JGV09, FV11 one can find counterexamples to the PDE version in high enough dimension.

It would be interesting to study these questions in the setting considered in the present paper.

We recall that De Giorgi conjecture asks whether solutions \( u(x_1, \ldots, x_d) \) of \( \Delta u = u - u^3 \) which are monotone in \( x_d \), are indeed functions of \( \omega \cdot x \), for some \( \omega \in \mathbb{R}^d \) (at least in dimension \( d \leq 8 \)).
In Aubry-Mather theory, one considers periodic potentials and one allows instead of $\triangle$ an elliptic operator with periodic potentials.

The fact that minimizers are functions of $\omega \cdot x$ is quite analogous to the fact that they are given by a hull function.

Remark 5. The approach based on the Lagrangian (3) has been shown to be a very effective numerical tool [Per79]. It has also been used as the basis of a KAM theory [LM01, dlL08, CdL09, SdlL11].

In Section 2, we will recall the standard definitions in the calculus of variations adapted to our situation. In Section 1.2, we will detail the assumptions of our Theorems which we will state and prove in Section 3 (existence of minimizers), Section 4 (minimizers are ground states) and Section 5 (existence of other critical points).

1.2. Standing assumptions on the $H_j$. In order to implement the above program, we will use several assumptions on the variational principle.

$H_j : \mathbb{R}^2 \to \mathbb{R}$ satisfies periodic condition (H1) and negative twist condition (H2):

(H1): $H_j(u + 1, v + 1) = H_j(u, v) \quad \forall \; u, v \in \mathbb{R}, \; j = 1, \ldots, d$

(H2): $H_j \in C^2$ and $\partial_1 \partial_2 H_j \leq c < 0, \; j = 1, 2, \ldots, d$.

(H3): $H_j, j = 1, \ldots, d$ have a lower bound.

These assumptions are very representative of the assumptions customary in Aubry-Mather theory, even if they can be weakened slightly.

As a consequence of (H1), we see that the functional $\mathcal{P}_\omega$ has the following symmetries:

• (a) $\mathcal{P}_\omega(h) = \mathcal{P}_\omega(h + 1)$;

• (b) $\mathcal{P}_\omega(h) = \mathcal{P}_\omega(h \circ T_a)$ where $T_a(x) = x + a$.

Note that these symmetries make the variational problem “degenerate”. As often used in the calculus of variations one can overcome this degeneracy by formulating the problem in appropriate quotient spaces (see [Pal79]) for a discussion of these questions.

Definition 1. We call $\omega$ non-resonant if $\omega_1, \omega_2, \ldots, \omega_d, 1$ are rationally independent and resonant otherwise.

2. Preliminaries

In this section, we collect some standard definitions from the calculus of variations that we will use. This section contains only standard definitions and elementary results and should be used only as reference.

2.1. Basic definitions in classical calculus of variations. We start by summarizing the main concepts in the sequences approach. This is not the basis of our approach, but eventually, we will show that the solutions obtained by the hull function approach lead to sequences which are minimizers in the sense of calculus of variations.

According to [Mor24].
Definition 2. A configuration \( u : \mathbb{Z}^d \to \mathbb{R} \) is called a class-A minimizer for (1) when for every \( \varphi : \mathbb{Z}^d \to \mathbb{R} \) with \( \varphi_i = 0 \) when \(|i| \geq N\), we have

\[
\sum_{i \in \mathbb{Z}^d, |i| \leq N} \sum_{j=1}^d H_j(u_i + \varphi_i, u_{i+e_j} + \varphi_{i+e_j}) \geq \sum_{i \in \mathbb{Z}^d, |i| \leq N} \sum_{j=1}^d H_j(u_i, u_{i+e_j})
\]

The equation (7) can be interpreted heuristically as saying \( \mathcal{L}(u + \varphi) \geq \mathcal{L}(u) \) after we cancel the terms on both sides that are identical.

Class-A minimizers are also called ground states in the mathematical physics literature and local minimizers in the calculus of variations literature.

Definition 3. We say that a configuration is a critical point of the action whenever it satisfies the Euler-Lagrange equations for every \( i \in \mathbb{Z}^d \)

\[
\sum_j \partial_1 H_j(u_i, u_{i+e_j}) + \partial_2 H_j(u_{i-e_j}, u_i) = 0
\]

The equations (8) are heuristically \( \partial_{u_i} \mathcal{L}(u) = 0 \). Note that, even if the sum in (1) is purely formal, the system of equations (8) is well defined. For every \( i \in \mathbb{Z}^d \), equation (8) involves only a finite sum of terms.

By considering \( \varphi_i = \epsilon \delta_{i,j} \) where \( \delta_{i,j} \) is the Kronecker delta, it is easy to see that if \( u \) is a ground state, then, it satisfies the Euler-Lagrange equations (8). The converse is certainly not true.

2.2. Order properties of configurations. Order properties of configurations play a very important role in Aubry-Mather theory.

The following is a standard definition.

Definition 4. We say that \( u : \mathbb{Z}^d \to \mathbb{R} \) is a Birkhoff configuration if for every \( k \in \mathbb{Z}^d, l \in \mathbb{Z} \), we have either

\[
u_{i+k} + l \geq u_i \quad \forall \ i \in \mathbb{Z}^d
\]

or

\[
u_{i+k} + l \leq u_i \quad \forall \ i \in \mathbb{Z}^d
\]

In other words, the graph of \( u \) does not intersect its horizontal or vertical translations by integer vectors.

We also have

\[
\omega_j = \lim_{n \to \pm \infty} \frac{1}{n}(u_{i+ne_j} - u_i)
\]

and the limit is reached uniformly in \( i \).

The notion of Birkhoff configurations was introduced in [Mat82a]. The name Birkhoff configurations appeared in [Kat83]. These configurations are also called self-conforming or non-self-intersecting. Their relevance to classical problems in calculus of variation was emphasized in [Mos86].

Birkhoff order properties are closely related to hull functions.

We note that if \( h \) is monotone, \( h(\theta + 1) = h(\theta) + 1 \) and \( \omega \in \mathbb{R}^d \), then

\[
u_i = h(i \cdot \omega).
\]
Then
\[ u_{i+k} + l = h(i \cdot \omega + k \cdot \omega) + l = h(i \cdot \omega + k \cdot \omega + l) \]
and if \( k \cdot \omega + l \geq 0 \), then \( h(i \cdot \omega + k \cdot \omega + l) \geq h(i \cdot \omega) = u_i \).

Therefore, configurations given by hull functions satisfy the following

**Definition 5.** Let \( \omega \in \mathbb{R}^d \). We say that \( u : \mathbb{Z}^d \to \mathbb{R} \) is \( \omega \)-Birkhoff if

\[ \omega \cdot k + l \geq 0, \ k \in \mathbb{Z}^d, \ l \in \mathbb{Z} \]

implies
\[ u_{i+k} + l \geq u_i \quad \forall \ i \in \mathbb{Z}^d. \]

Equivalently,
\[ \omega \cdot k + l \leq 0, \ k \in \mathbb{Z}^d, \ l \in \mathbb{Z} \]

implies
\[ u_{i+k} + l \leq u_i \quad \forall \ i \in \mathbb{Z}^d. \]

Clearly, \( \omega \)-Birkhoff configurations are Birkhoff. That is why [dlLV07a, dlLV10] formulated existence and multiplicity results for \( \omega \)-Birkhoff orbits.

The converse is close to being true, but it is not exactly true.

First, we note that, given \( u \) Birkhoff, there is one and only one candidate for \( \omega \) which would make it \( \omega \)-Birkhoff (analogue of rotation number). If this \( \omega \) turns out to be irrationally related, (i.e., \( \omega \cdot k + l = 0, \ k \in \mathbb{Z}^d, \ l \in \mathbb{Z} \implies k = 0, \ l = 0 \)) then, \( u \) is \( \omega \)-Birkhoff. If \( \omega \) has some relations, in Remark 7 we will present examples of Birkhoff orbits with \( \omega \) rotation vector which are not \( \omega \)-Birkhoff.

**Proposition 1.** Assume \( u \) is Birkhoff. Then, there exists \( \omega \in \mathbb{R}^d \) such that
\[
\lim_{n \to \infty} \frac{1}{n} [u_{i+n} - u_i] = \omega \cdot k.
\]

Furthermore,
\[ |u_{i+n} - u_i - n \omega \cdot k| \leq 2 \]
and, if \( \omega \cdot k + l > 0 \) (resp. \( \omega \cdot k + l < 0 \)) for \( k \in \mathbb{Z}^d, \ l \in \mathbb{Z} \), we have
\[ u_{i+k} + l > u_i \quad \forall \ i \in \mathbb{Z}. \]

(resp. \( u_{i+k} + l < u_i \quad \forall \ i \in \mathbb{Z} \))

We note that given a Birkhoff configuration, the sets
\[ A_{\geq} = \{(k, l) \in \mathbb{Z}^d \times \mathbb{Z} \mid u_{i+k} + l \geq u_i \}, \]
\[ A_{\leq} = \{(k, l) \in \mathbb{Z}^d \times \mathbb{Z} \mid u_{i+k} + l \leq u_i \} \]
and
\[ A_{=} = \{(k, l) \in \mathbb{Z}^d \times \mathbb{Z} \mid u_{i+k} + l = u_i \} \]
are respectively cones and subspaces. If \( (k_1, l_1), \ (k_2, l_2) \in A_{\geq} \), then \( \forall i \in \mathbb{Z}^d \)
\[ u_{i+(k_1+k_2)} + l_1 + l_2 \geq u_{i+k_1} + l_1 \geq u_i \]
Since \( u \) is Birkhoff \( A_{\geq} \cup A_{\leq} = \mathbb{Z}^d \times \mathbb{Z} \) and \( A_{\geq} \cap A_{\leq} = A_{=} \) (it could be open).
Proceeding as in the theory of Dedekind cuts, we can find a unique \( \omega \in \mathbb{R}^d \) such that

\[
\begin{align*}
A_\geq &= \{ \omega \cdot k + l \geq 0 \} \\
A_\leq &= \{ \omega \cdot k + l \leq 0 \} \\
A_\equiv &= \{ \omega \cdot k + l = 0 \}.
\end{align*}
\]

The proof of the existence of the limit can be done exactly as in the proof of the rotation number in [Poi85]. (See [Kra96] for a proof in the context of commuting diffeomorphisms or [CdIL98].)

If there exists \( i \in \mathbb{Z}^d \) such that

\[
u_i + k + l \geq \nu_i.
\]

Because \( u \) is Birkhoff, we should have the inequality of all \( i \) having

\[
u_{i+n} + n \cdot l \geq \nu_i
\]

and, taking limits \( \omega \cdot k + l \leq 0 \). Similarly, we have that if \( \nu_{i+k} + l \leq \nu_i \), \( \omega \cdot k + l \geq 0 \).

Therefore, we see, comparing with \( \nu_0 \) that

\[
|\nu_i - \nu_0 - \omega \cdot k| \leq 1.
\]

This, of course establishes that the limit defining the rotation number is reached uniformly.

**Remark 6.** We note that Proposition 7 uses essentially the fact that we are considering configurations on \( \mathbb{Z}^d \), which is a commutative group. If we consider configurations in non-commutative groups, it is not clear that for configurations satisfying Definition 2 we have that the limit (9) exists and has good properties. Therefore in [dILV07a, dILV10], the Birkhoff orbits are defined as those which satisfy the conclusions of Proposition 7. In our context, both are equivalent. There are definitions of \( \omega \)-Birkhoff orbits for more general latices in Appendix A.

**Remark 7.** In view of Proposition 7 the main difference between \( \omega \)-Birkhoff and Birkhoff is that when \( \omega \cdot k + l = 0 \), Birkhoff only claims that we can compare \( u_{i+k} + l \) and \( u_i \) with the same sign. The \( \omega \)-Birkhoff claims that since we have both inequalities \( u_{i+k} + l = u_i \).

This shows how to construct solutions which are Birkhoff with rotation vector \( \omega \) but not \( \omega \)-Birkhoff. For example in \( d = 1 \), let \( f \) be an orientation preserving diffeomorphism of the circle with rotation number \( = \frac{1}{2} \) with isolated periodic points of period 2. Any orbit is a Birkhoff sequence, but only the periodic orbits are \( \frac{1}{2} \)-Birkhoff.

**Proposition 2.** Assume that \( u_i \) is an \( \omega \)-Birkhoff configuration. Then, there exists \( h : \mathbb{R} \to \mathbb{R}, \ h(\theta + 1) = h(\theta) + 1 \) monotone such that (2) holds.

**Proof.** The definition of \( \omega \)-Birkhoff shows that

\[
u_i + l - \nu_0
\]

satisfies the same order relation \( \omega \cdot i + l \).
Therefore, if we write \( u_i + l - u_0 \) as a function of \( \omega \cdot i + l \), we will obtain a monotone function \( h \) defined on the set \( \{ \omega \cdot i + l \}_{i \in \mathbb{Z}^d, l \in \mathbb{Z}} \). It can be extended to a monotone function on \([0, 1]\).

We also note that because of the way that \( l \) enters, we obtain \( h(\theta + 1) = h(\theta) + 1 \). Hence, we can extend the function \( h \) to a hull function. □

2.3. Spaces for hull functions, topology and order. As we indicated, we will present two proofs of Theorem 1. The main trade-off is between establishing the validity of the Euler-Lagrange equations and establishing properties of the minimizers. If we include spaces of functions that incorporate many properties, then these properties are, of course, true for the spaces, but, then, it is hard to establish the Euler-Lagrange equations because we may be at the boundary of the spaces.

We will start by indicating two different spaces.

2.3.1. Two spaces of hull functions. We define the space of functions

\[
Y = \{ h | h \text{ monotone}, h(\theta + 1) = h(\theta) + 1, h(\theta-) = h(\theta) \}
\]

This is the space of functions which are monotone – and therefore have at most countably many points of discontinuity – we assume that the functions are continuous on the left.

Now, we turn to give \( Y \) a topology and collect some of the properties.

We first define

\[
\text{graph}(h) = \{(\theta, y) \in \mathbb{R}^2 : h(\theta) \leq y \leq h(\theta+)\}.
\]

If \( h, \tilde{h} \in Y \) we define the distance as the Hausdorff distance of the graphs.

\[
d(h, \tilde{h}) = \max\{ \sup_{\xi \in \text{graph}(h)} \rho(\xi, \text{graph}(\tilde{h})), \sup_{\eta \in \text{graph}(\tilde{h})} \rho(\eta, \text{graph}(h)) \}
\]

where \( \rho(\cdot, \cdot) \) is the Euclidean distance from a point to a set, \( \rho(x, S) = \inf_{y \in S} |y - x| \). Note that the graph topology is weaker than the \( L^\infty \) topology.

It is a standard result that the functions \( h \in Y \) can be identified with non-negative periodic Borel probability measures times the reals by \( h(\theta) = \mu([0, x]) + h(0) \). The topology induced by the distance in (11) is the same as the topology induced by the weak-* convergence in the unit interval. In dynamics, the measures associated to \( h \)’s that satisfy the Euler-Lagrange equation (4) are called Mather measures and are the basic objects for extending Aubry-Mather theory to higher codimension in [Mat89, Mat91, Man96a, CT99].

Another space that we will consider is \( Y_\ast^N = \{ h \in L^\infty_{\text{loc}} | h(\theta + N) = h(\theta) + N \} \) for any \( N \in \mathbb{Z} \).

We consider it endowed with the topology of pointwise convergence. By Tikhonov theorem, subsets of \( Y_\ast^N \equiv Y_1^\ast \) which are bounded in \( \| \cdot \|_{L^\infty} \) are precompact.
Compared with $Y$, the space $Y^*$ is more flexible because it does not have the constraint of monotonicity.

2.3.2. Order properties. Also, we endow $L^\infty \supseteq Y$ with a partial order given by $h < \tilde{h} \iff h(\theta) \leq \tilde{h}(\theta)$ for all $\theta \in \mathbb{R}$ and $h \neq \tilde{h}$. We write $h \ll \tilde{h}$ to denote $h(\theta) < \tilde{h}(\theta)$ for all $\theta \in \mathbb{R}$.

A small corollary is that, given two functions $h_\leq \leq h_\geq$, $\{ h \in Y \mid h_\leq \leq h \leq h_\geq \}$ is compact with the graph topology. It is clear that it is a closed set of a compact set.

The analogous set in $Y^*$ $\{ h \in Y^* \mid h_\leq \leq h \leq h_\geq \}$ is also compact for the pointwise convergence topology.

2.3.3. Some background in lattice theory.

**Definition 6** (Lattice). A lattice is a partially ordered set any two of whose elements have a greatest lower bound and a least upper bound.

**Definition 7** (Complete lattice). A lattice $\Lambda$ is complete if each $X \subseteq \Lambda$ has a least upper bound and a greatest lower bound in $\Lambda$.

The set $Y^*_n$ has a natural lattice structure induced by the canonical lattice operations on the real line, i.e.

$$ h \lor \tilde{h}(\theta) = \max\{h(\theta), \tilde{h}(\theta)\}, \quad h \land \tilde{h}(\theta) = \min\{h(\theta), \tilde{h}(\theta)\}, $$

where $h, \tilde{h} \in Y^*_n$. It is easily seen that $h \lor \tilde{h}, h \land \tilde{h} \in Y^*_n$. $Y^*_1$ is not complete because it includes an $\mathbb{R}$ factor but the subspace $Y/\sim$ is.

**Remark 8.** The space $Y/\sim$ is the basis of [Mat82b]. The paper [Mat82a] uses the space

$$ X = \{ h \in Y \mid h(\theta) \geq 0 \text{ for } \theta > 0 \text{ and } h(\theta) \leq 0 \text{ for } \theta \leq 0 \}. $$

The space $X$ is also compact as shown in [Mat82a].

The idea is somewhat similar. The reason why $Y$ is not compact is because it contains an $\mathbb{R}$ factor. (The Borel measure factor is compact by Banach-Alaoglu theorem.)

Because of the symmetries (a),(b) of the variational principle, we can formulate the variational problem on “normalized” $h$’s. If we use (a) to normalize $h$ by adding integers we are led to $Y/\sim$. If we use (b) to normalize the $h$ by composing with a translation $T_a$ where $a = \inf\{x \mid h(x) \geq 0\}$, we are led to $X$.

In a complete lattice $\Lambda$, we define the order-converge of any net $\{h_\alpha\} \subseteq \Lambda$. We say that $h_\alpha$ order converges when

$$ \liminf h_\alpha = \limsup h_\alpha $$

where $\liminf h_\alpha \equiv \sup_\beta\{\inf_{\alpha \geq \beta} h_\alpha\}$ and $\limsup h_\alpha \equiv \inf_\beta\{\sup_{\alpha \geq \beta} h_\alpha\}$.

**Definition 8.** A real-valued function $\mathcal{P}$ on a complete lattice $\Lambda$ is called lower semi-continuous if

$$ \mathcal{P}\left(\lim_{j \to \infty} h_j\right) \leq \liminf_{j \to \infty} \mathcal{P}(h_j) $$

whenever the limit exists in $\Lambda$ with respect to the order-convergence.
Definition 9 (Sub-modular). \( \mathcal{P} \) is called sub-modular if for all \( h, \tilde{h} \in \Lambda \) it satisfies the following inequality:

\[
\mathcal{P}(h \lor \tilde{h}) + \mathcal{P}(h \land \tilde{h}) \leq \mathcal{P}(h) + \mathcal{P}(\tilde{h})
\]

where \( \lor \) and \( \land \) are the abstract lattice operations.

For example Percival’s Lagrangian \( \mathcal{P}_\omega \) is lower semi-continuous and sub-modular (see Lemma 3).

3. Existence of minimizers and their properties

In this section, we construct minimizers of \( \mathcal{P}_\omega \) in (3) and show that they are solutions of Euler-Lagrange equation (4).

We present two different functional approaches. One is based on \( Y \), the space of monotone functions, and another one is based on \( Y_1^+ \) the space of measurable functions and we will show that they coincide. Later, in Section 4 we will show that the configurations generated by \( h \) according to (2) are indeed ground states.

3.1. A treatment of minimizers based on compactness.

Theorem 1. Given \( \omega \in \mathbb{R}^d \), the Percival Lagrangian \( \mathcal{P}_\omega \) reaches a minimum in \( Y \).

Any minimizer satisfies the Euler-Lagrange equation (4).

Proof. From (H1), the definition of \( \mathcal{P}_\omega(h) \), and \( h(\theta + 1) = h(\theta) + 1 \), it follows that \( \mathcal{P}_\omega(h) \) is translation invariant (a), (b).

To prove the existence of the minimizer, it suffices to prove the continuity of \( \mathcal{P}_\omega \) on \( Y \). If it is true, we can obtain a minimal point on the compact subset \( C = \{ 0 \leq h(\theta) \leq 2 \} \).

This minimizer will also be a minimizer in \( Y \) because, given \( h \in Y \), we can find \( a \in \mathbb{R}, n \in \mathbb{Z} \) such that \( h \circ T_a + n \in C \) and using (a),(b), \( \mathcal{P}_\omega(h) \circ T_a + n = \mathcal{P}_\omega(h) \).

In fact, let

\[
M = \max\{1, \max_{j} \sup_{x, x' \leq 2} |\partial_1 H_j(x, x')|, \max_{j} \sup_{x, x' \leq 2} |\partial_2 H_j(x, x')|\}.
\]

Since \( H_j(u + 1, v + 1) = H_j(u, v) \forall u, v \in \mathbb{R} \), we will get \( \partial_1 H_j(u + 1, v + 1) = \partial_1 H_j(u, v) \), \( \partial_2 H_j(u + 1, v + 1) = \partial_2 H_j(u, v) \). It follows that \( M \leq 1 \).

From the definition of \( \mathcal{P}_\omega \) and the mean value theorem, it follows that

\[
(12) \quad |\mathcal{P}_\omega(h) - \mathcal{P}_\omega(\tilde{h})| \leq \int_0^1 [dM|h(\theta) - \tilde{h}(\theta)| + M \sum_{j=1}^d |h(\theta + \omega_j) - \tilde{h}(\theta + \omega_j)|]d\theta.
\]

Let \( 0 < \epsilon \leq 1 \). Let \( \delta = \delta(\epsilon) = \frac{\epsilon^2}{10000dM^3} \). Suppose \( d(h, \tilde{h}) < \delta < \frac{1}{10000} \), i.e. for any \( \theta \in \mathbb{R} \), there exists \( (\tilde{\theta}, \tilde{y}) \in \text{graph}(\tilde{h}) \) such that

\[
|((\theta, h(\theta)) - (\tilde{\theta}, \tilde{y}))| < \delta,
\]

which implies

\[
|h(\theta) - \tilde{h}(\theta)| < 1 + \delta < 2.
\]
Suppose \( a \in \mathbb{R} \). Let \( \pi_a = \{ \theta \in (a, a+1) \mid |h(\theta) - \tilde{h}(\theta)| \geq \frac{\epsilon}{5dM} \} \). From the assumption that \( d(h, \tilde{h}) < \delta \), i.e. for any \( \theta \in \mathbb{R} \), there exists \((\tilde{\theta}, \tilde{\gamma}) \in \text{graph}(h)\) such that
\[
\begin{align*}
&|\theta - \tilde{\theta}| < \delta \\
&|\tilde{h} - \tilde{\gamma}| < \delta
\end{align*}
\]
we obtain
\[
(13) \quad h(\theta + \delta) \geq \tilde{h}(\theta) - \delta \geq h(\theta) + \frac{\epsilon}{5dM} - \delta \geq h(\theta) + \frac{199\epsilon}{1000dM}
\]
in the case \( \tilde{h}(\theta) \geq h(\theta) + \frac{\epsilon}{5dM} \) and we obtain similarly
\[
(14) \quad h(\theta - \delta) \leq \tilde{h}(\theta) - \delta \leq h(\theta) - \frac{\epsilon}{5dM} + \delta \leq h(\theta) - \frac{199\epsilon}{1000dM}
\]
in the case \( \tilde{h}(\theta) \leq h(\theta) - \frac{\epsilon}{5dM} \).

Let \( \pi'_a \) (resp. \( \pi''_a \)) denote the set of \( \theta \in (a, a+1) \) where (13) (resp. (14)) holds. Then
\[
\pi_a \subseteq \pi'_a \cup \pi''_a.
\]

At any point \( \theta \in \pi'_a \) the variation of \( h \) over the interval \( [\theta, \theta + \delta] \) is \( \geq \frac{\epsilon}{5dM} - \delta \). Since the total variation of \( h \) over \( (a, a+1) \) is \( \leq 1 \), it follows that \( \pi'_a \) can be covered by at most \( \lceil \frac{1000dM}{199\epsilon} \rceil + 1 \leq \frac{2dM}{\epsilon} \) intervals of length \( \delta \). Hence the measure of \( \pi'_a \) is at most \( \frac{2dM}{\epsilon} \leq \frac{\delta}{50dM} \). Similarly, the measure of \( \pi''_a \) is \( \leq \frac{4\delta}{50dM} \). Hence the measure of \( \pi_a \) is \( \leq \frac{\delta}{50dM} \).

Since \( |h(\theta) - \tilde{h}(\theta)| \leq 2 \) for all \( \theta \in \mathbb{R} \) and \( |h(\theta) - \tilde{h}(\theta)| \leq \frac{\epsilon}{5dM} \) for \( \theta \in (0, 1) - \pi_0 \) and for \( \theta \in (\omega_j, \omega_j) - \pi_{\omega_j} \), we obtain from (12) that
\[
|\mathcal{P}_{\omega}(h) - \mathcal{P}_{\omega}(\tilde{h})| \leq dM \cdot (\frac{4\epsilon}{5dM} + \frac{\epsilon}{5dM}) < \epsilon.
\]

This completes the proof of the existence of the minimizer.

Now we go into the proof that the minimizer satisfies the Euler-Lagrange equation. The proof below is similar to [Mat82a]. The key point in the proof is that given a minimizer we can find enough deformations that do not leave the space so that we can conclude that the Euler-Lagrange equations hold. These arguments are sometimes called in the calculus of variations deformation lemmas, a name which is used with another meaning in other fields.

**Lemma 1.** Suppose \( a \leq 0 \leq b \) and \( a < b \). Suppose an element \( h_s \) of \( Y \) is given for \( a \leq s \leq b \), \( h_s(\theta) \) is \( C^2 \) function of \( s \) for each fixed \( \theta \), and \( \frac{\partial}{\partial s} h_s(\theta), \frac{\partial^2}{\partial s^2} h_s(\theta) \) are uniformly bounded and measurable for \( a \leq s \leq b, \theta \in \mathbb{R} \). Then
\[
(15) \quad \frac{d}{ds} \mathcal{P}_\omega(h_s)|_{s=0} = \int_0^1 X(h) \cdot \dot{h}(\theta)d\theta,
\]
where \( \dot{h}_s(\theta) = \frac{\partial}{\partial s} h_s(\theta), \dot{h}(\theta) = h_0(\theta) \) and \( h = h_0 \).

Clearly, if \( h \) is a minimizer and \( h_s \) is a deformation, \( h_0 = h \), we have \( \frac{d}{ds} \mathcal{P}_\omega(h_s)|_{s=0} = 0 \). Using (15) we obtain that \( \int_0^1 X(h) \cdot \dot{h} = 0 \). To conclude that \( X(h) \) is identically zero, we have to argue that we can obtain enough deformations \( \dot{h}(\theta) \) that force that...
X(h) is zero in the neighborhood of any point \( \theta \in \mathbb{T}^1 \). We will generate deformations by solving the ordinary differential equation:

\[
\begin{aligned}
\frac{d}{ds} u_s(\theta) &= \rho \circ \pi \circ u_s(\theta) \\
u_0 &= \text{id},
\end{aligned}
\]

where \( \pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \) is the projection map and \( \rho \) which has values in \([0, 1]\) will be decided later. We will consider every continuous point \( \theta_0 \) of \( h \) first and then take the limit to approximate the discontinuous ones due to the fact that \( h \) is monotone. We simplify the formula in the above lemma in the two cases below:

1. When \( h^{-1} \circ h(\theta_0) \) is a single point, we define \( h_s = u_s \circ h \) and get

\[
\frac{d}{ds} \mathcal{P}_\omega(h_s)_{s=0} = \int_0^1 X(h) \cdot \rho \circ \pi \circ h \; d\theta.
\]

2. When \( h^{-1} \circ h(\theta_0) \) is an interval, there exists \( \theta_1 > \theta_0 \) such that

- we define
  \[
  \psi_s(\theta) = \begin{cases} 
  u_s \circ h(\theta) & \text{if } \exists \ n \in \mathbb{Z} \text{ such that } \theta_0 + n < \theta \leq \theta_1 + n \\
  h(\theta) & \text{otherwise}
  \end{cases}
  \]
  and get
  \[
  \frac{d}{ds} \mathcal{P}_\omega(\psi_s)_{s=0} = \int_{\theta_0}^{\theta_1} X(h) \cdot \rho \circ \pi \circ h \; d\theta.
  \]
- we define
  \[
  \xi_s(\theta) = \begin{cases} 
  h(\theta) & \text{if } \exists \ n \in \mathbb{Z} \text{ such that } \theta_0 + n < \theta \leq \theta_1 + n \\
  u_s \circ h(\theta) & \text{otherwise}
  \end{cases}
  \]
  and get
  \[
  \frac{d}{ds} \mathcal{P}_\omega(\xi_s)_{s=0} = \int_{\theta_0}^{\theta_1} X(h) \cdot \rho \circ \pi \circ h \; d\theta.
  \]

For case (1), provided that \( \rho \) has support in a sufficiently small neighborhood of \( \pi \circ h(\theta_0) \), we have \( h_s \in Y \) for \( s \) sufficiently small.

The hypothesis that \( \mathcal{P}_\omega \) takes its minimum at \( h = h_0 \) implies \( \frac{d}{ds} \mathcal{P}_\omega(h_s)_{s=0} = 0 \).

Since \( X(h) \) is continuous at \( \theta = \theta_0 \), and \( \theta_0 = h^{-1}(\theta_0) \), the fact that \( \int_0^1 X(h) \cdot \rho \circ \pi \circ h \; d\theta = 0 \) for all \( \rho \) of the type we consider, implies \( X_{\theta_0}(h) = 0 \). (Here we write explicitly the dependence of \( X(h) \) on the point \( \theta_0 \) we choose.)

For case (2), let \( \alpha \) and \( \beta \) be endpoints of \( h^{-1}(\theta_0) \) with \( \alpha < \beta \). \( X(h) \) is a decreasing function of \( \theta \in (\alpha, \beta) \) by (H2). It is easy to see that if \( \rho \) has support in a sufficiently small neighborhood of \( \pi \circ h(\theta) \), then \( \psi_s \in Y \) for \( s \geq 0 \) sufficiently small and \( \xi_s \in Y \) for \( s \leq 0 \) sufficiently small. The assumption that \( \mathcal{P}_\omega \) takes its minimum at \( h = \psi_0 = \xi_0 \) implies \( \frac{d}{ds} \mathcal{P}_\omega(\psi_s)_{s=0} \geq 0 \) and \( \frac{d}{ds} \mathcal{P}_\omega(\xi_s)_{s=0} \leq 0 \). In view of the fact that \( X(h) \) is a decreasing function on \((\alpha, \beta)\), we have \( X(h) \geq 0 \) and \( X(h) \leq 0 \). Hence \( X_{\theta_0}(h) = 0 \). This completes the proof of the second part of Theorem 1.

3.2. Existence of minimizers based on order properties. In the following we present another approach to the same problem based on different spaces. Basically, we show that \( \mathcal{P}_\omega \) reaches a minimum on \( Y^* \).

**Theorem 2.** Under our standing assumptions, there is a minimizer of \( \mathcal{P}_\omega \) over \( Y^* \). Any minimizer on \( Y^* \) satisfies the Euler-Lagrange equations.

There is one minimizer which lies on \( Y \).
Of course, once we prove that there is one minimizer in the whole space $Y^*$ which actually lies in $Y$ we conclude that $\inf_{h \in Y} \mathcal{P}_\omega(h) = \inf_{h \in Y} \mathcal{P}_\omega(h)$ and, therefore that all the minima in $Y$ are also minima in $Y^*$.

The main advantage of this argument is that, since $Y^*$ does not involve any constraints, the deformation lemmas are almost trivial and, therefore it is easy to show that the minimizers satisfy the Euler-Lagrange equations (4).

3.3. Proof of Theorem 2. We use the following basic lemma in [For96]:

Lemma 2. Let $\mathcal{P}$ be a real-valued function on a complete lattice $\Lambda$. Suppose $\mathcal{P}$ is sub-modular, lower semi-continuous and bounded from below. Then $\mathcal{P}$ has a minimum on $\Lambda$.

Proof. Since $\mathcal{P}$ is bounded from below, $\beta = \inf_{\Lambda} \mathcal{P}$ is a real number. Suppose given any sequence of positive real numbers $(\epsilon_j)_{j \in \mathbb{N}}$ converging to zero, there exists a sequence $(h_j)_{j \in \mathbb{N}} \subseteq \Lambda$ such that:

$$\beta \leq \mathcal{P}(h_j) \leq \beta + \epsilon_j$$

By the sub-modularity, since $\mathcal{P}(h_j \vee h_{j+1}) \geq \beta$, we have

$$\mathcal{P}(h_j \wedge h_{j+1}) \leq \beta + \epsilon_j + \epsilon_{j+1}$$

and by induction, we have

$$\mathcal{P}(h_j \wedge h_{j+1} \wedge \cdots \wedge h_{j+k}) \leq \beta + \epsilon_j + \epsilon_{j+1} + \cdots + \epsilon_{j+k}$$

Define $\tilde{h}_{j,k} \equiv h_j \wedge h_{j+1} \wedge \cdots \wedge h_{j+k}$ for all $j, k \geq 1$. By construction, it is a non-increasing sequence included in $\Lambda$ with respect to $k$. Due to the completeness of $\Lambda$, we can define $\tilde{h}_j = \lim_{k \to \infty} \tilde{h}_{j,k} \in \Lambda$. Consequently, one gets:

$$\mathcal{P}(\tilde{h}_j) = \mathcal{P}(\lim_{k \to \infty} \tilde{h}_{j,k}) \leq \liminf_{k \to \infty} \mathcal{P}(\tilde{h}_{j,k}) \leq \beta + \epsilon_j$$

by the above inequality and the lower semi-continuity of $\mathcal{P}$, where $\epsilon_j = \sum_{k \geq j} \epsilon_k$ is finite and converges to zero, as $j \to \infty$ if we choose $(\epsilon_j)_{j \in \mathbb{N}}$ such that

$$\sum_{j \geq 0} \epsilon_j < \infty.$$ 

On the other hand, since $\tilde{h}_{j,k} \leq \tilde{h}_{j+1,k-1}$ for all $j, k \geq 1$, then $(\tilde{h}_j)_{j \in \mathbb{N}}$ is a non-decreasing sequence, hence it has a limit $\tilde{h} \in \Lambda$. By the lower semi-continuity and the choice of the sequence $(\epsilon_j)_{j \in \mathbb{N}}$, (16) implies:

$$\mathcal{P}(\tilde{h}) \leq \beta + \liminf_{j \to \infty} \epsilon_j = \beta$$

which concludes the proof, by showing that $\tilde{h}$ is a minimum point for $\mathcal{P}$. □

Now we turn to show how the concrete functional $\mathcal{P}_\omega$ in (3) satisfies the assumptions of the abstract results.

Lemma 3 (Fundamental Inequality in Aubry-Mather theory). If $h$, $\tilde{h} \in Y^*$, then

$$\mathcal{P}_\omega(h \vee \tilde{h}) + \mathcal{P}_\omega(h \wedge \tilde{h}) \leq \mathcal{P}_\omega(h) + \mathcal{P}_\omega(\tilde{h})$$

(17)
Proof. Using the fundamental theorem of calculus, we have
\[ H_j(h(\theta) \land \tilde{h}(\theta), h(\theta + \omega_j) \land \tilde{h}(\theta + \omega_j)) + H_j(h(\theta) \lor \tilde{h}(\theta), h(\theta + \omega_j) \lor \tilde{h}(\theta + \omega_j)) - H_j(h(\theta), h(\theta + \omega)) - H_j(\tilde{h}(\theta), \tilde{h}(\theta + \omega)) = \int_{[h(\theta) \land \tilde{h}(\theta)] \leq [h(\theta + \omega_j) \land \tilde{h}(\theta + \omega_j)]} \partial_1 \partial_2 H_j(x, y) \, dx \, dy. \]

Adding over \( j \) and integrating with respect to \( \theta \), we obtain
\[
\mathcal{P}_\omega(h \lor \tilde{h}) + \mathcal{P}_\omega(h \land \tilde{h}) - \mathcal{P}_\omega(h) - \mathcal{P}_\omega(\tilde{h}) = \frac{1}{n} \sum_{j=1}^d \int_0^1 d\theta \int_{[h(\theta) \land \tilde{h}(\theta)] \leq [h(\theta + \omega_j) \land \tilde{h}(\theta + \omega_j)]} \partial_1 \partial_2 H_j(x, y) \, dx \, dy \leq 0.
\]

The last inequality holds because of (H2). Hence we obtain \( \mathcal{P}_\omega(\tilde{h}) = \mathcal{P}_\omega(h) \). \( \square \)

We had defined before two spaces \( Y^*_1 \) (see Section 2.3.1) and \( X \) (see Remark 8). We have \( X \subseteq Y^*_1 \) (roughly \( X \) is a subset of functions in \( Y^*_1 \) with some monotonicity properties). Since \( X \subseteq Y^*_1 \) it is clear that
\[
\inf_{h \in Y^*_1} \mathcal{P}_\omega(h) \leq \inf_{h \in X} \mathcal{P}_\omega(h).
\]

In Lemma 4 we show that both minima are actually equal.

This is useful because it is easier to show that minimizers in \( Y^*_1 \) satisfy the Euler-Lagrange equation. (Since \( Y^*_1 \) has less properties, it is easy to construct deformations that do not leave the space.)

**Lemma 4.** Let \( \tilde{h} \in Y^*_1 \). Then there exists \( h \in X \) such that
\[
\mathcal{P}_\omega(h) \leq \mathcal{P}_\omega(\tilde{h}).
\]

**Proof.** Let \( Y_A(\tilde{h}) \subseteq Y^*_1 \) be the complete lattice generated by the set \( \{ \tilde{h} \circ T_d : 0 \leq a \leq A \} \), which exists since \( \tilde{h} \) is locally bounded, i.e. \( Y_A(\tilde{h}) \) is the smallest complete lattice that includes the set \( \{ \tilde{h} \circ T_d : 0 \leq a \leq A \} \). Applying Lemma 2 there exists \( h_A \in Y_A(\tilde{h}) \) for each \( A \geq 0 \) which minimizes \( \mathcal{P}_\omega \) over \( Y_A(\tilde{h}) \). Next, we will consider the quotient set \( \Lambda_A(\tilde{h}) \equiv Y_A(\tilde{h})/\mathbb{R} \), obtained by projecting the sub-lattices \( Y_A(\tilde{h}) \) into the quotient space \( Y^*_1/\mathbb{R} \). Since \( \tilde{h} \) is locally bounded and satisfies \( \tilde{h}(\theta + 1) = \tilde{h}(\theta) + 1 \), the sets \( \Lambda_A(\tilde{h}) \) stabilize as \( A \to \infty \), i.e., there exists \( M > 0 \) such that
\[
\Lambda_A(\tilde{h}) = \Lambda_M(\tilde{h}) \text{ if } A \geq M.
\]

Hence, due to the translation invariance of \( \mathcal{P}_\omega \), it is possible to choose \( h \in Y_M(\tilde{h}) \) for each \( A \geq M \) that minimizes \( \mathcal{P}_\omega \) over \( Y_M(\tilde{h}) \). From the translation invariance and sub-modularity property of \( \mathcal{P}_\omega \), we obtain
\[
\mathcal{P}_\omega(h \land h \circ T_d) = \mathcal{P}_\omega(h \lor h \circ T_d) = \mathcal{P}_\omega(h) = \mathcal{P}_\omega(h \circ T_d),
\]
since \( h \) minimizes \( \mathcal{P}_\omega \) over \( Y_A(\tilde{h}) \) if \( A \) is sufficiently large. Let \( \{ a_i \}_{i \in \mathbb{N}} \) be an enumeration of all the positive rational numbers and \( \tilde{h}_m = h \land h \circ T_{a_1} \land \ldots \land h \circ T_{a_m} \).

Repeating the argument above \( m \) times, we get
\[
\mathcal{P}_\omega(\tilde{h}_m) = \mathcal{P}_\omega(h).
\]
We have $h \geq \bar{h}_1 \geq \ldots \geq \bar{h}_m \geq \ldots$ and $\bar{h}_m|_{[a,b]} \geq C = \inf \{h(s) : a \leq s \leq a + 1\}$ for all $m$ and each finite interval $[a, b]$ since $h$ is locally bounded and $h(\theta + 1) = h(\theta) + 1$. Consequently, $\bar{h}_\infty(\theta) = \lim_{m \to \infty} \bar{h}_m(\theta)$ exists for all $\theta \in \mathbb{R}$ and by the lower semi-continuity of $\mathcal{P}_\omega$ we get

$$\mathcal{P}_\omega(\bar{h}_\infty) \leq \lim_{m \to \infty} \mathcal{P}_\omega(\bar{h}_m) = \mathcal{P}_\omega(h). \quad (18)$$

It is sufficient to prove that $\bar{h}_\infty$ is order-preserving almost everywhere (i.e. that it agrees with an order-preserving function almost everywhere). We adapt an argument in [Mat85, Lemma 7.2].

We know that $\bar{h}_\infty(\theta) = \inf \{h(\theta + a) : a \text{ is a positive rational number}\}$ is order-preserving except on a set of zero measure. In fact, for a positive rational number $a$, we have $\bar{h}_\infty \circ T_a \geq \bar{h}_\infty$ by the definition of $\bar{h}_\infty$. Since $\bar{h}_\infty \in L^\infty_{\text{loc}}(\mathbb{R})$, i.e. is measurable and bounded on bounded sets, we have $\int [\bar{h}_\infty \circ T_a - \bar{h}_\infty \circ T_b] \to 0$ as $a \to b$, for any finite interval $I$. Therefore, if $b > 0$, we obtain $\bar{h}_\infty \circ T_b \geq \bar{h}_\infty$ almost everywhere. In other words, for each $b > 0$, $\text{Leb}\{\theta \in \bar{h}_\infty(\theta) > \bar{h}_\infty(\theta + b)\} = 0$ where $\text{Leb}$ is the Lebesgue measure. We obtain

$$\text{Leb}\{(\theta, s) \in \mathbb{R}^2 : (\theta - s)(\bar{h}_\infty(\theta) - \bar{h}_\infty(s)) < 0\} = 0$$

by Fubini’s theorem. So there exists a set $E \subseteq \mathbb{R}$ and $\text{Leb}(E) = 0$ such that if $\theta \notin E$, we have $\text{Leb}\{s \in \mathbb{R} : (\theta - s)(\bar{h}_\infty(\theta) - \bar{h}_\infty(s)) < 0\} = 0$. Hence for $\theta, s \notin E$, $\theta < s$, and a.e. $\theta < u < s \bar{h}_\infty(\theta) \leq \bar{h}_\infty(u)$ and $\bar{h}_\infty(u) \leq \bar{h}_\infty(s)$, that is, $\bar{h}_\infty(\theta) \leq \bar{h}_\infty(s)$ holds for a.e. $\theta, s \notin E$. (It is easy to see that $\bar{h}_\infty(\theta) = \text{ess. inf}_{s \geq \theta} h(s)$ holds a.e. $\theta$.)

Take $\bar{h} \in Y$ such that $\bar{h}(\theta) = \bar{h}_\infty(\theta)$ a.e. $\theta \in \mathbb{R}$. We have $\mathcal{P}_\omega(\bar{h}) = \mathcal{P}_\omega(\bar{h}_\infty) \leq \mathcal{P}_\omega(h)$ due to (18). The final step is to choose $a$ such that $h_0 = \bar{h} \circ T_a \in X$. This choice is explained at the end of Remark 8. Then, we have

$$\mathcal{P}_\omega(h_0) = \mathcal{P}_\omega(\bar{h}) \leq \mathcal{P}_\omega(h) \leq \mathcal{P}_\omega(h).$$

where the first equality is a consequence of the translation invariance of $\mathcal{P}_\omega$, the next inequality follows from the property of $\bar{h}$ and the last inequality holds because $h$ minimizes $\mathcal{P}_\omega$ over $Y_A(h)$ for $A$ sufficient large. This completes the proof. \hfill \Box

4. Minimizers of the Percival Lagrangian give rise to ground states.

In this section, we prove that the minimizers of $\mathcal{P}_\omega$ give rise to ground states when $\omega$ is both non-resonant and resonant.

**Theorem 3.** Let $h_\omega$ be a minimizer of $\mathcal{P}_\omega$ as in Theorem 2. The configuration $u_t = h_\omega(\theta + \omega \cdot t)$ when $\omega$ is non-resonant is an $\omega$-Birkhoff ground state.

Out of Theorem 3, we can obtain several results using approximation arguments. We present two representative results, Corollary 1 (based on approximation in the orbit formalism) and Corollary 2 (based on the hull function formalism). Since the result of Corollary 1 is based on choices of approximating subsequences, it is not clear that the orbits produced are the same.

**Corollary 1.** Given any frequency $\omega \in \mathbb{R}^d$, there is a Birkhoff ground state of frequency $\omega$.  


It is amusing to note that in the orbit based approach, it is more convenient to construct ground states of non-resonant frequencies approximating them by ground states of rational frequencies. Now, we find it more convenient to construct ground states of non-resonant frequencies and use an approximation argument to get those of rational frequencies.

**Proof.** The proof of the corollary is very simple. We observe that given any \( \omega \), we can find a sequence \( \omega_n \) of nonresonant vectors such that \( \lim_{n \to \infty} \omega_n = \omega \).

Denote by \( u^n \), the ground states corresponding to this sequence. By the invariance of the action under addition of integers we can assume that that \( u^n_i \in [0, 1) \) and by the Birkhoff property, \( |u^n_i - \omega_n \cdot i| \leq 2 \). It follows that, using the diagonal trick, we can assume that \( \lim_{n \to \infty} u^n_i = u^*_i \) exists for all \( i \in \mathbb{Z}^d \).

Then, it is a classical argument in [Mor73] to show that \( u^* \) is a ground state. Suppose by contradiction that we could find \( \varphi \) such that \( \varphi_i = 0, \left| i \right| \geq N - 1 \) and that \( \mathcal{L}_N(u^*) - \mathcal{L}_N(u^* + \varphi) \geq \delta > 0 \). Since \( \mathcal{L}_N \) involves only finitely many sites, we can find \( n^* \) such that \( \mathcal{L}_N(u^{n^*}) - \mathcal{L}_N(u^{n^*} + \varphi) \geq \delta/2 > 0 \). This is a contradiction with \( u^{n^*} \) being a ground state.

To finish the argument, we show that the limiting sequence is Birkhoff. Fixed \( k \in \mathbb{Z}^d, l \in \mathbb{Z}, \) we can find an infinite sequence of \( n \)'s in which the comparison between \( (\tau_k \circ R_l)u^n \) (where \( \tau_k \) and \( R_l \) are the horizontal and vertical translations respectively) and \( u^n \) has the same sign. Therefore, the limit of \( (\tau_k \circ R_l)u^n \) can be compared with \( u^* \).

Of course, it is perfectly possible that for each of the two possible comparison signs between \( (\tau_k \circ R_l)u^n \) and \( u^n \), there are infinitely many \( n \)'s. In this case, \( u^* \) would satisfy both comparisons.

**Corollary 2.** \( u_i = h_{i,\omega}(\theta + \omega \cdot i) \) is an \( \omega \)-Birkhoff ground state for any rotation vector \( \omega \in \mathbb{R}^d \).

The proof of Corollary uses the fact that the hull function we obtain satisfies the non-symmetry breaking property, which means \( \mathcal{P}_\omega \) reach the same minimum over \( Y_n^* \) and \( Y_n^* \) for any \( n \in \mathbb{Z} \) (see [Mat85] Lemma 7.3). Since the technique of the proof of Corollary is very similar to that of Theorem we postpone it.

**Remark 9.** In fact, Corollary implies Corollary The non-symmetry breaking property plays an important role here.

**4.1 Proof of Theorem**. We use arguments inspired by [Mat85] but we require some more detailed computations.

Suppose \( u_i \) is not a ground state, so there exists a configuration \( \tilde{u}_i \) and \( K \in \mathbb{Z}^+ \) such that \( \tilde{u}_i = u_i \) if \( |i| \geq K \) and \( \mathcal{L}_K(\tilde{u}) < \mathcal{L}_K(u) \). Let \( 1 \gg \delta > 0 \) and set

\[
\begin{align*}
\tilde{h}(\theta) &= \tilde{u}_i, \quad \text{if } |i| \leq K, t + \omega \cdot i - \delta \leq \theta \leq t + \omega \cdot i, \\
\tilde{h}(\theta + 1) &= \tilde{h}(\theta) + 1, \text{ for all } \theta, \text{ and } \\
\tilde{h}(\theta) &= h_{i,\omega}(\theta), \text{ whenever } \tilde{h}(\theta) \text{ is not defined by the previous two conditions.}
\end{align*}
\]
Since $\omega$ is non-resonant and $\delta$ is small, there is no contradiction between the first two conditions. Consequently, $\tilde{h}$ is well-defined and $\tilde{h} \in Y_1^*$. 

$$\mathcal{P}_\omega(h_\omega) - \mathcal{P}_\omega(\tilde{h}) = \sum_{j=1}^{d} \int_{\omega}^{\omega+1} [H_j(h_\omega(\theta), h_\omega(\theta + \omega j)) - H_j(\tilde{h}(\theta), \tilde{h}(\theta + \omega j))]d\theta$$

$$= \int_{\omega-\delta}^{\omega} [A(\theta) + B(\theta) + C(\theta)]d\theta$$

where

$$A(\theta) = \mathcal{L}_K(h_\omega(\theta + \omega \cdot i)) - \mathcal{L}_K(\tilde{u})$$

$$B(\theta) = \sum_{j=1}^{d} \left[ \sum_{i \in \mathbb{Z}, |i| = K} H_j(h_\omega(\theta + \omega \cdot i), h_\omega(\theta + \omega \cdot i + \omega j)) - \sum_{i \in \mathbb{Z}, |i| = K} H_j(h_\omega(t + \omega \cdot i), h_\omega(\theta + \omega \cdot i + \omega j)) \right]$$

$$C(\theta) = \sum_{j=1}^{d} \left[ \sum_{i \in \mathbb{Z}, |i| = K} H_j(h_\omega(\theta + \omega \cdot i - \omega j), h_\omega(\theta + \omega \cdot i)) - \sum_{i \in \mathbb{Z}, |i| = K} H_j(h_\omega(\theta + \omega \cdot i - \omega j), h_\omega(t + \omega \cdot i)) \right]$$

Clearly, $A(\theta) \rightarrow A(i) = \mathcal{L}_K(u) - \mathcal{L}_K(\tilde{u}) > 0$, $B(\theta) \rightarrow 0$ and $C(\theta) \rightarrow 0$ as $\theta \uparrow i$. Consequently, $\mathcal{P}_\omega(h) - \mathcal{P}_\omega(\tilde{h}) > 0$ for $\delta > 0$ small enough. But this contradicts the fact that $h$ minimizes $\mathcal{P}_\omega$ over $Y_1^*$ if we already know the fact that $h_\omega$ minimizes $\mathcal{P}_\omega$ over $Y_1^*$. □

**Proof of Corollary** Suppose $u_i$ is not a ground state, so there exists a configuration $\tilde{u}_i$ and $K \in \mathbb{Z}^+$ such that $\tilde{u}_i = u_i$ if $|i| \geq K$ and $\mathcal{L}_K(\tilde{u}) < \mathcal{L}_K(u)$. Let $1 \gg \delta > 0$ and set

$$\tilde{h}(\theta) = \tilde{u}_i, \quad \text{if } |i| \leq K, \ t + \omega \cdot i - \delta \leq \theta \leq t + \omega \cdot i,$$

$$\tilde{h}(\theta + N) = \tilde{h}(\theta) + N, \ for \ all \ \theta, \ and$$

$$\tilde{h}(\theta) = h_\omega(\theta), \ whenever \ \tilde{h}(\theta) \ is \ not \ defined \ by \ the \ previous \ two \ conditions.$$  

Consequently, $\tilde{h}$ is well-defined and $\tilde{h} \in Y_N^*$ for some sufficiently large $N$. By non-symmetry breaking property, we get the same contradiction. □

**5. Existence of non-minimal critical points**

We will refer to the functions given in the form as “quasi-periodic”. In some literature, the term quasi-periodic is reserved for situations when $h$ is smooth, whereas we will accept $h$ which are discontinuous. In some literature, these functions are given the name “almost-automorphic” in [Ell69]. We will follow the
customary notation in the calculus of variations. One of the most interesting phenomena in Aubry-Mather theory is that the quasi-periodic solutions obtained may be discontinuous.

We note that the discontinuity of the minimizers has profound physical and dynamical interpretations. In the solid state physical interpretation, if $h \circ T_a$ is a continuous family of critical points, the physical system can “slide” whereas if $h \circ T_a$ involves discontinuity, the system is “pinned”. In the case of twist maps, that $h_\omega$ is continuous corresponds to an invariant orbit, which is a complete barrier for transport.

In this section, we study the situation when there are two minimizers which are comparable. Similar results in PDE were studied in [dlLV07b]. There are other more delicate results that show that if there are gaps in the range of $h$, then there is another minimizing sequence [Mat86]. In [dlLV07b], one can find a proof using the gradient flow approach in spaces of sequences. We do not present these results here. Indeed we do not know how do they fit in the hull function approach, except in the rational frequency case.

**Theorem 4.** Suppose $h^- < h^+$ are both minimizers of $P_\omega$ on $Y$ with frequency vector $\omega$ not completely resonant (not all the components of $\omega$ are rational numbers). Then,

1. $h^- \ll h^+$;
2. There exists a critical point $h^0$ of $P_\omega$ such that $h^- \ll h^0 \ll h^+$ holds;

To prove Theorem 4, we will use the gradient flow method (see [KdlLR97, Gol01]) for $Y \subseteq L^\infty$.

**Lemma 5.** Assume $\partial_1 H_j, \partial_2 H_j$ are uniformly $C^r$, $r \geq 1$. The infinite system of ODE’s:

\[
\begin{align*}
\frac{d}{dt} h^t &= -X(h^t) \equiv -\sum_{j=1}^d [\partial_1 H_j(h^t, h^t \circ T_{\omega_j}) + \partial_2 H_j(h^t \circ T_{-\omega_j}, h^t)] \\
h^0 &= h_0
\end{align*}
\]

(19)

defines a $C^r$ flow $\Phi^t$ on $L^\infty$. The rest points of $\Phi^t$ correspond to critical points of the Percival Lagrangian $P_\omega$.

By ODE theory in Banach space (see [Hal80]), it is easy to see that the gradient flow $\Phi^t$ is well-defined for $t \geq 0$ since the vector field $-X(h^t)$ is globally Lipschitz. From the gradient flow equation itself, we can get some simple properties.

**Proposition 3.** $[\Phi^t(h_0)] \circ T_a = \Phi^t(h_0 \circ T_a)$; $\Phi^t(h_0 + m) = \Phi^t(h_0) + m$ for any $a \in \mathbb{R}, m \in \mathbb{Z}$. 
Proof. For the first equality, we differentiate its left hand side with respect to $t$, and get:

$$\frac{d}{dt}[\Phi'(h_0)] \circ T_a = -X(\Phi'(h_0)) \circ T_a$$

$$= -\sum_{j=1}^{d} [\partial_1 H_j(\Phi'(h_0), \Phi'(h_0) \circ T_{\omega_j}) + \partial_2 H_j(\Phi'(h_0) \circ T_{-\omega_j}, \Phi'(h_0))] \circ T_a$$

$$= -\sum_{j=1}^{d} [\partial_1 H_j(\Phi'(h_0)) \circ T_a, \Phi'(h_0) \circ T_{\omega_j} \circ T_a) + \partial_2 H_j(\Phi'(h_0) \circ T_{-\omega_j} \circ T_a, \Phi'(h_0) \circ T_a)]$$

$$= -X(\Phi'(h_0)) \circ T_a = \frac{d}{dt}\Phi'(h_0 \circ T_a)$$

This means that there exists some $C$ independent of $t$ such that $[\Phi'(h_0)] \circ T_a = \Phi'(h_0 \circ T_a) + C$. Take $t = 0$. We have $C = 0$, i.e. the first equality holds.

For the second equality, we just consider the case when $m = 1$ and observe some symmetry of the gradient flow equation. Due to (H1), we know that $\Phi'(h_0)$ is also a solution of

$$\{\begin{array}{l}
\frac{d}{dt}(h' + 1) = -X(h' + 1) \\
\eta^0 + 1 = h_0 + 1.
\end{array}\}$$

This means $\Psi'(h_0 + 1) = \Phi'(h_0) + 1$ is a solution of

$$\{\begin{array}{l}
\frac{d}{dt}(h') = -X(h') \\
\eta^0 = h_0 + 1.
\end{array}\}$$

Moreover, by comparing equation (20) with equation (19), we have $\Psi'(h_0 + 1) = \Phi'(h_0 + 1)$. This finishes the proof. \qed

One of the key properties of $\Phi'$ which was first observed by S. B. Angenent in the case of standard map ([Ang88]) is that it is strictly monotone, i.e.

**Lemma 6** (Strong Comparison Principle). If $h, \bar{h} \in Y$ and $h < \bar{h}$ and $\omega$ is not completely resonant, we have $\Phi'(h) \ll \Phi'(\bar{h})$ for any $t > 0$.

**Proof.** We use that the flow in the Banach space is differentiable (see [KdlLR97] for details). By the general theory of ODE, we also have that the derivative satisfies the equations of variation

$$DX(h) \cdot \eta = \sum_{j=1}^{d} [(\partial_{11} H_j(h, h \circ T_{\omega_j}) + \partial_{22} H_j(h \circ T_{-\omega_j}, h)) \cdot \eta +$$

$$\partial_{12} H_j(h \circ T_{-\omega_j}, h) \cdot \eta \circ T_{-\omega_j} + \partial_{12} H_j(h, h \circ T_{\omega_j}) \cdot \eta \circ T_{\omega_j}]$$

for any $\eta \in L^\infty$

Let $M'(h_0) = D\Phi'(h_0) : L^\infty \to L^\infty$ is a linear operator which satisfies the operator equation below (often called variational equation [Hal80] even if they do not have much to do with calculus of variations):

$$\{\begin{array}{l}
\frac{d}{dt} M' = -DX(\Phi'(h_0)) \cdot M' \\
M'^0 = id
\end{array}\}$$
To prove Lemma 6 due to the fact that $M'$ is a linear operator, it suffices to prove that the solution is strictly positive on $Y$. That is, $0 < v = \bar{h} - v$ implies $0 < v' \equiv M'(h_0) \cdot v$. In fact, let $h_0 = h + s \cdot (\bar{h} - h)$ for $0 \leq s \leq 1$. By the fundamental theorem of calculus, we have $\Phi'(\bar{h}) - \Phi'(h) = \int_0^1 D\Phi'(h + s \cdot (\bar{h} - h)) \cdot (\bar{h} - h) \, ds = \int_0^1 D\Phi'(h_0) \cdot v \, ds \geq 0$ for any $\theta \in \mathbb{R}$, i.e. $\Phi'(h) \ll \Phi'(\bar{h})$.

Let $u' = -\sum_{j=1}^d (\partial_{11} H_j(\Phi'(h_0), \Phi'(h_0) \circ T_{\omega_j}) + \partial_{12} H_j(\Phi'(h_0) \circ T_{-\omega_j}, \Phi'(h_0))) \cdot v' \cdot e^{-\int_0^t u' \, ds}$ with $\partial_{12} H_j(\Phi'(h_0), \Phi'(h_0) \circ T_{\omega_j})$. By the fundamental theorem of calculus, we have $\Phi'(h) = \Phi'(h_0) \cdot v$. We get:

$$\frac{d}{dt} W' = e^{-\int_0^t u' \, ds} \cdot u' \cdot v + e^{-\int_0^t u' \, ds} \cdot \frac{d}{dt} v'$$

$$= -u' \cdot W' - \sum_{j=1}^d (\partial_{11} H_j(\Phi'(h_0), \Phi'(h_0) \circ T_{\omega_j}) + \partial_{12} H_j(\Phi'(h_0) \circ T_{-\omega_j}, \Phi'(h_0))) \cdot v' \cdot e^{-\int_0^t u' \, ds}$$

$$- \sum_{j=1}^d \partial_{12} H_j(\Phi'(h_0), \Phi'(h_0) \circ T_{\omega_j}) \cdot v' \cdot T_{\omega_j} \cdot e^{-\int_0^t u' \, ds}$$

$$- \sum_{j=1}^d \partial_{12} H_j(\Phi'(h_0), \Phi'(h_0) \circ T_{\omega_j}) \cdot W' \cdot T_{-\omega_j} \cdot e^{-\int_0^t u' \, ds}$$

By using Euler method, for $t$ small enough,

$$W' = v + t \cdot \frac{d}{dt} W' \cdot v + O(t^2).$$

Since $0 \leq v \in Y$, there exists a small interval $[\alpha, \beta]$ such that $v|_{[\alpha, \beta]} > 0$. Since $\omega$ is not completely resonant, we can find some component $\omega_m$ which is irrational for some $m \in \{1, \ldots, d\}$. Due to (H2) and Picard’s iteration, $W'^1|_{[\alpha+\omega_j, \beta+\omega_j]} > 0$ for sufficiently small $t_1$ and $j = 1, \ldots, d$. In particular, $W'^1|_{[\alpha+\omega_m, \beta+\omega_m]} > 0$. Repeating $k$ times, we get $W'^2|_{[\alpha+k \omega_m, \beta+k \omega_m]} > 0$ for small $t_2 > t_1$. Due to the compactness of interval $[0, T]$ and the fact $\omega_m$ is irrational, this leads to $0 \ll W'$ for any $t \in (0, T]$. Therefore $0 \ll v'$ holds for any $t > 0$. This finishes the proof. \hfill $\Box$

**Proposition 4.** $Y$ is invariant under the gradient flow $\Phi'$, that is, $\Phi'(Y) \subseteq Y$ for $t \geq 0$.

**Proof:** For any $h \in Y$ and $t > 0$, since $h < h \circ T_a$ if $a > 0$, the fact that $\Phi'(h) \ll \Phi'(h \circ T_a)$ is just an immediate consequence of Lemma 6. We already know that $\Phi'(h) \circ T_1 = \Phi'(h) + 1$ by Proposition 3. The left continuity of $\Phi'(h)$ is from
continuity of the gradient flow with respect the initial data and the definition of $h$.

\textbf{Lemma 7.} If $h^- < h^+$ are both critical points of $\mathcal{P}_o$ on $Y$, then $h^- \ll h^+$.

\textit{Proof.} Due to $h^- < h^+ \in Y$ and Lemma\textsuperscript{[6]} we have $\Phi'(h^-) \ll \Phi'(h^+)$. On the other hand, since $h^-$ and $h^+$ are both critical points of $\mathcal{P}_o$, $\Phi'(h^-) = h^-$ and $\Phi'(h^+) = h^+$ hold by Lemma\textsuperscript{[5]} This finishes the proof. \hfill \qed

\textit{Proof of Theorem\textsuperscript{[7]}} (1) is an immediate consequence of Lemma\textsuperscript{[7]}

In order to prove (2), we follow the method used by [dlLV07a]. We define the compact set $\mathcal{K} \equiv \{ h \in Y : h^- \leq h \leq h^+ \}$. Due to the compactness of $\mathcal{K}$, the topology induced by $L^\infty$ norm and the topology induced by the Hausdorff metric are equivalent on $\mathcal{K}$. For any $h \in \mathcal{K}$, we know that $h^- = \Phi'(h^-) \leq \Phi'(h) \leq \Phi'(h^+) = h^+$ by Lemma\textsuperscript{[6]} and the definition of $h^-$ and $h^+$. This means that $\Phi'(\mathcal{K}) \subseteq \mathcal{K}$ due to Proposition\textsuperscript{[4]} Let $h^s = s \cdot h^+ + (1-s) \cdot h^-$ for any $s \in [0,1]$. We have

\begin{equation}
\frac{d}{dt} \mathcal{P}_o(\Phi'(h^s)) = - \int_0^1 |X(\Phi'(h^s))|^2 d\theta \leq 0,
\end{equation}

i.e. $\mathcal{P}_o(\Phi'(h^s))$ is decreasing with respect to $t$ for any fixed $s \in [0,1]$. Since $\mathcal{P}_o|_{\mathcal{K}}$ is bounded and $\frac{d}{dt} \mathcal{P}_o(\Phi'(h^s))$ is bounded from above, $\lim_{t \to \infty} \mathcal{P}_o(\Phi'(h^s))$ exists and $\lim_{t \to \infty} \frac{d}{dt} \mathcal{P}_o(\Phi'(h^s)) = 0$.

Let

$$\mathcal{B}_o = \max_{s \in [0,1]} \inf_{t \geq 0} \mathcal{P}_o(\Phi'(h^s)) \equiv \max_{s \in [0,1]} \lim_{t \to \infty} \mathcal{P}_o(\Phi'(h^s)) \geq \mathcal{P}_o(h^-).$$

There are two possibilities $\mathcal{B}_o > \mathcal{P}_o(h^-)$ or $\mathcal{B}_o = \mathcal{P}_o(h^-)$. We will show that the conclusion holds in each of the two cases.

- If $\mathcal{B}_o > \mathcal{P}_o(h^-)$, there exists $s_0 \in (0,1)$ such that

$$\lim_{t \to \infty} \mathcal{P}(\Phi'(h^{s_0})) = \mathcal{B}_o$$

and

$$\lim_{t \to \infty} \frac{d}{dt} \mathcal{P}(\Phi'(h^{s_0})) = 0.$$  

Due to the compactness of $\mathcal{K}$, we can extract a subsequence $t_n \to \infty$ such that $\Phi'(h^{s_0}) \to h^* \in \mathcal{K}$. This leads to $\mathcal{P}_o(h^*) = \mathcal{B}_o$ which means that $h^*$ is different from $h^-$ and $h^+$. In the other hand, due to (21), we have $\lim_{t_n \to \infty} \int_0^1 |X(\Phi'(h^{s_0}))|^2 d\theta = \int_0^1 |X(h^*)|^2 d\theta = 0$. Since $h^*$ is left-continuous, we get $X(h^*) = 0$ which means $h^*$ is a critical point of $\mathcal{P}_o$. This finishes the proof when $\mathcal{B}_o > \mathcal{P}_o(h^-)$.

- If $\mathcal{B}_o = \mathcal{P}_o(h^-)$, we have $\inf_{s \geq 0} \mathcal{P}(\Phi'(h^s)) \leq \mathcal{B}_o = \mathcal{P}_o(h^-)$. This means $\inf_{s \geq 0} \mathcal{P}(\Phi'(h^s)) = \mathcal{P}_o(h^-)$ for any $s \in [0,1]$. We now argue by contradiction and assume that no other critical point (and so a fortiori no minimizer) but $h^-$ and $h^+$ in $\mathcal{K}$. We have two alternatives, both of which lead to contradictions with the non-existence of other critical points.
(a) One is that the omega limit set of $\Phi'(h^t)$ contains both $\{h^-, h^+\}$. Let
\[
B_r(h^-) \equiv \{ h \in \mathcal{K} : d(h, h^-) < r \},
B_r(h^+) \equiv \{ h \in \mathcal{K} : d(h, h^+) < r \}
\]
denote the $r$-ball of $h^-$ and $h^+$ respectively in $\mathcal{K}$. Take $0 < r < \frac{1}{2}d(h^-, h^+)$ sufficiently small such that $B_r(h^-) \cap B_r(h^+) = \emptyset$. Thus there exists an $M_0(r)$ in this case, such that $\Phi'(h^t) \in B_r(h^-) \cup B_r(h^+)$ for any $t > M_0(r)$. Let $D^- \equiv \{ t > M_0(r) : \Phi'(h^t) \in B_r(h^-) \}$ and $D^+ \equiv \{ t > M_0(r) : \Phi'(h^t) \in B_r(h^+) \}$ which are nonempty. We know $D^- \cap D^+ = \emptyset$. By the continuity of $\Phi'(h^t)$ with respect to $t$ these two sets are open. This means that two nonempty disjoint open sets $D^-$ and $D^+$ cover a connected open interval $(M_0(r), \infty)$ which is a contradiction.

(b) The other is that the omega limit set of $\Phi'(h^t)$ has only one point either $\{h^-\}$ or $\{h^+\}$. Let $E^- \equiv \{ s \in (0, 1) : \lim_{t \to \infty} \Phi'(h^s) = h^- \}$ and $E^+ \equiv \{ s \in (0, 1) : \lim_{t \to \infty} \Phi'(h^s) = h^+ \}$ which are nonempty open sets due to the continuous dependence of $\Phi'$ on initial data. This is a contradiction by the same trick used in (a).

This completes the proof of (2). \qed

Appendix A. Hull function approach to general lattices

The method of hull functions can be extended to more general lattices.

For simplicity, we discuss only when the place of $\mathbb{Z}^d$ is taken by a finitely generated group $G$ and the interaction is invariant under the action of $G$, as well as by addition of 1 to the configurations. See [Dilv10] for more general lattices.

Because of the translation invariance, we consider variational principles

\begin{equation}
\mathcal{L}(u) = \sum_{\substack{B \subseteq G \\text{finite} \\\ 0 \in B \\text{ finite}}} S_{B \cdot \omega}(u)
\end{equation}

where $S_B$ depends only on $u|_B$.

We recall that $\omega : G \to \mathbb{R}$ is a cocycle when $\omega(g \cdot \tilde{g}) = \omega(g) + \omega(\tilde{g})$.

Given a cocycle $\omega$ we seek configurations:

\begin{equation}
x_g = h_\omega(\omega \cdot g)
\end{equation}

for $h_\omega : \mathbb{R} \to \mathbb{R}$ monotone, $h_\omega(t + 1) = h_\omega(t) + 1$.

It is immediate that all configurations \((23)\) satisfy for all $k, g \in G, l \in \mathbb{Z}$

\begin{equation}
x_{g+k} + l \leq x_g \iff \omega(k) + l \leq 0
\end{equation}

which is an analogue of the $\omega$-Birkhoff property. Similarly, one can easily see that the $\omega$-Birkhoff property \((24)\) implies the existence of a hull function.
Given \( B = \{ s_0 = 0, s_1, \ldots, s_n \} \subseteq G \) we can write \( S_B(u) = S_B(u_0, u_1, \ldots, u_n) \).

Given a variational principle (22) we can associate the Percival variational principle (25)

\[
\mathcal{P}_\omega(h) = \int_0^1 d\theta \sum_{\substack{B \subseteq G \\ \#B \text{ finite}}} S_B(h(\theta), h(\theta + \omega(s_1)), \ldots, h(\theta + \omega(s_n))).
\]

We use the same procedure as the commutative group case \((\mathbb{Z}^d)\) to prove the existence of the minimal configurations generated by hull functions approach. Namely:

- The minimizers (resp. critical points) of (22) give via (23) class-A (resp. critical) configurations.
- For every cocycle \( \omega \), there exists a class-A minimizer.
- For every \( \omega \) there are at least two different critical points. If there are two minimizers, then one gets a circle of critical points.

It is easy to get the following theorem.

**Theorem 5.** Under the assumptions as in [CdIL98] for general lattices, there is a minimizer \( h_\omega \) of \( \mathcal{P}_\omega \) over \( Y \) or \( Y^* \). \( x_\omega = h_\omega(\omega \cdot g) \) is a \( \omega \)-Birkhoff ground state of cocycle \( \omega \). In addition, if both \( h^- < h^+ \) are minimizers of \( \mathcal{P}_\omega \) on \( Y \) then \( h^- \prec h^+ \) and there is a critical point in between.

**Proof.** We give a sketch of proof. We assume that the sum

\[
\sum_{\substack{B \subseteq G \\ \#B \text{ finite}}} S_B(h(\theta), h(\theta + \omega(s_1)), \ldots, h(\theta + \omega(s_n))).
\]

converges uniformly. We first check the symmetries and obtain

\[
\mathcal{P}_\omega(h \circ T_a) = \sum_{\substack{B \subseteq G \\ \#B \text{ finite}}} \int_0^1 S_B(h(\theta+a), h(\theta+a+\omega(s_1)), \ldots, h(\theta+a+\omega(s_n))) = \mathcal{P}_\omega(h),
\]

and

\[
\mathcal{P}_\omega(h+1) = \int_0^1 d\theta \sum_{\substack{B \subseteq G \\ \#B \text{ finite}}} S_B(h(\theta)+1, h(\theta+\omega(s_1))+1, \ldots, h(\theta+\omega(s_n))+1) = \mathcal{P}_\omega(h).
\]

In addition, we assume that \( S_B \) satisfies the weak twist condition (see [CdIL98])

\[
\sum_{B \ni q} \frac{\partial^2}{\partial p \partial q} S_B(u) \leq 0
\]

for any \( p \neq q \). The twist condition implies the rearrangement inequality:

\[
\mathcal{P}(h \wedge \mathring{h}) + \mathcal{P}(h \lor \mathring{h}) \leq \mathcal{P}(h) + \mathcal{P}(\mathring{h}).
\]

\( \square \)
Remark 10. The heuristic argument for (25) is that, even if the group \( G \) is not amenable since we consider only configurations which depend only on the value of the cocycle and which transform well, we only need to average over the values of the cocycle.

One can make assumptions that argue that the sum (22) converges. For example \( S_B = 0 \) when \( \text{diam}_B \geq R \) (finite range). We have not explored what are the optimal assumptions.

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