Weak amenability of Coxeter groups

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Abstract

Let \((G, S)\) be a Coxeter group. We construct a continuation, to the open unit disc, of the unitary representations associated to the positive definite functions \(g \mapsto r^{l(g)}\). (Here \(0 < r < 1\), and \(l\) denotes the length function with respect to the generating set \(S\).

The constructed representations are uniformly bounded and we prove that this implies the weak amenability of the group \(G\).

Résumé

Soit \((G, S)\) un groupe de Coxeter. Nous construisons une extension, au disque unité ouvert, de la série des représentations unitaires associées aux fonctions définies positives \(g \mapsto r^{l(g)}\). (Ici, \(0 < r < 1\), et \(l\) désigne la fonction longueur par rapport aux générateurs \(S\).

Les représentations ainsi construites sont uniformément bornées et nous démontrons que ceci implique que le groupe \(G\) est faiblement moyenable.

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1 Introduction

Let $G$ be a locally compact group. On the space $L^2(G)$ of, with respect to a left invariant Haar measure, square integrable functions the left regular representation of $G$ is defined by the unitary operators $\lambda(g), g \in G$, where those are given by

$$\lambda(g)f(h) = f(g^{-1}h), \quad h \in G, \quad f \in L^2(G).$$

The reduced von Neumann algebra $VN_r(G) \subset B(L^2(G))$ is the weak operator topology closure of the linear span of $\{\lambda(g) : g \in G\}$.

A complex valued function $\phi$ on $G$ is called positive definite whenever for finite subsets $g_1, \ldots, g_n \in G, c_1, \ldots, c_n \in \mathbb{C}$:

$$\sum_{i,j=1}^n c_i \overline{c_j} \phi(g_i^{-1}g_j) \geq 0.$$  

We denote $A(G)$ the Fourier algebra of $G$ as defined by Eymard and recall that on one hand it is the linear span of the positive definite functions $g \mapsto \langle \lambda(g)f, f \rangle$, $f \in L^2(G)$, which actually is an algebra of continuous functions on $G$, and on the other hand is naturally identified with the predual of $VN_r(G)$.

In terms of the Fourier algebra amenability of $G$ can be characterized by the existence of a $A(G)$–norm bounded approximate identity in $A(G)$. That is, there exist a constant $C > 0$ and a net $(m_i)_{i \in I} \in A(G)$ such that

$$\lim_i \|m_i\varphi - \varphi\|_{A(G)} = 0 \quad \forall \varphi \in A(G)$$

$$\sup_i \|m_i\|_{A(G)} = C.$$  

In their work on multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups de Canni`ere and Haagerup started to investigate more general approximate identities of the Fourier algebra than norm bounded ones.

They introduced the concept of completely bounded multipliers in that paper:

A function $m$ on the locally compact group $G$ is a multiplier of the Fourier algebra $A(G)$, if for any $\varphi \in A(G)$ the pointwise product of functions $M_m(\varphi) = m \cdot \varphi \in A(G)$ again. Now, the dual to this multiplication operator acts on the reduced von Neumann algebra $VN_r(G) \subset B(L^2(G))$. From this last inclusion one can let act the $n \times n$ matrices with entries from $VN_r(G)$ on $l^2_n \otimes_2 L^2(G)$ in the canonical way and norm $M_n(VN_r(G))$ accordingly, for each $n \in \mathbb{N}$.

The action of a linear operator on $VN_r(G)$ can be extended to a linear action on $M_n(VN_r(G))$ simply by letting it act on each matrix entry (of course we think of $M^*_n$). The multiplier $m$ of $A(G)$ is called a completely bounded multiplier of the Fourier algebra if those extensions are bounded, uniformly in $n \in \mathbb{N}$. Clearly, taking the smallest possible bound defines a norm $\|m\|_{M_nA(G)}$ on a subspace of all multipliers of the Fourier algebra.

A locally compact group is weakly amenable, if there exist a constant $C < \infty$ and a net $(m_i)_{i \in I} \in A(G)$ such that

$$\lim_i \|m_i\varphi - \varphi\|_{A(G)} = 0 \quad \forall \varphi \in A(G)$$

$$\sup_i \|m_i\|_{M_nA(G)} = C.$$  

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The least such constant is called the Cowling Haagerup constant of \( G \).

Since \( \| M_{h,A(G)} \| \leq \| A(G) \| \), an amenable group is weakly amenable too, but on the other hand there are non–amenable groups which are weakly amenable.

It turned out [3] that the space of completely bounded multipliers of the Fourier algebra had been discussed in harmonic analysis by Herz [11] as a generalisation of the Fourier Stieltjes algebra, the algebra of coefficients of continuous unitary representations. In their above cited paper, de Canni`ere and Haagerup showed that the Fourier algebras of finite extensions of the Lie groups \( SO_0(n, 1) \), \( n \geq 2 \) indeed contain completely bounded approximate identities. This result transfers to the discrete subgroups of those groups too. A special example is the free group on two generators, \( \mathbb{F}_2 \).

Subsequently, this result was extended on one hand to simple Lie groups of real rank one by Cowling and Haagerup [7] and, on the other hand, to amalgamated products by Bo˙ zejko and Picardello [8] and to groups acting on trees by Szwarc [15] and Valette [16].

The construction of such an approximate identity relies on the existence of a path of uniformly bounded representations which connect, in some way, the regular representation to the trivial representation of the group in question. Moreover, and we see no way around this difficulty, one has to extend this path, in some way continuously, to some complex region, still preserving the uniform boundedness of the representations. We note that, for non-amenable groups, it is not possible to deal here only with unitary representations. Whereas the original path of representations might consist of unitary ones. (Of course if the group has the Kazdhan property then this is not possible either.)

In this paper we shall continue to consider some new examples from the variety of discrete groups. In fact we shall deal with Coxeter systems \((G, S)\), which we, by abuse of language call Coxeter groups, the generating set is always understood.

For Coxeter groups it is known from the work of Bo˙ zejko, Januszkiewicz and Spatzier [4] that the length function, with respect to \( S \) is negative definite, where a function \( \phi : G \to \mathbb{C} \) is called negative definite, whenever for finitely many \( g_1, \ldots, g_n \in G \) and \( c_1, \ldots, c_n \in \mathbb{C} \) with \( \sum_{i=1}^{n} c_i = 0 \):

\[
\sum_{i,j=1}^{n} c_i \overline{c}_j \phi(g_i^{-1} g_j) \leq 0.
\]

Hence, by a theorem of Schoenberg (see e. g. [1]), for \( 0 < r < 1 \),

\[
g \mapsto r^{\ell(g)} \quad g \in G,
\]

is a positive definite function and the associated representations form a suitable path of unitary representations.

Starting from this we consider the problem of extending this series of representations to a complex parameter \( z \in D = \{ z \in \mathbb{C} : |z| < 1 \} \).

The author knows of a manuscript of T. Januszkiewicz in which it is proved that for all finitely generated Coxeter groups the functions \( g \mapsto z^{\ell(g)} \), \( z \in \mathbb{C} \), \( 0 \leq |z| < 1 \), are coefficients of uniformly bounded representations [12]. We learned a geometric Lemma from this, which is used in work of Millson, see Lemma 2.1 of [13]. Millson attributes it to Jaffe. For the readers convenience we shall state it as Lemma 3 and prove it in a formulation convenient for us.
The paper is organised as follows. After this introduction in section 2 we define some positive definite kernels and discuss domination properties between them, which we shall apply to prove bounds on representations of the Coxeter group. In section 3 we give an introduction to the standard geometrical representation of a Coxeter group and show that the geometry implies the positive definiteness of certain kernels related to the length function of the group. These results are then used to prove uniform bounds on representations constructed by modifying the Gelfand-Naimark-Segal representation associated to the positive definite functions \( g \mapsto r^l(g), \ 0 < r < 1 \) in section 4. Section 5 finally contains a proof of the weak amenability.

2 Some analytical tools

Denoting \( \mathbb{R}_+ \) the non-negative reals and \( s \land t = \min(s, t) \), \( s, t \in \mathbb{R} \) the minimum of two reals we define for \( \alpha > 1 \) a kernel \( k_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
k_\alpha(s, t) := \alpha^{s \land t} \quad s, t \in \mathbb{R}_+
\]

and a corresponding sesquilinear form on the space \( \mathcal{F}(\mathbb{R}_+) \) of complex valued compactly supported locally Lebesgue integrable functions defined on \( \mathbb{R}_+ \):

\[
[f, g]_\alpha := \int_0^\infty \int_0^\infty \alpha^{s \land t} f(s) \overline{g(t)} \, ds \, dt \quad f, g \in \mathcal{F}(\mathbb{R}_+).
\]

It is then easy to see that the sesquilinear form is non-negative definite. For, if \( s, t \geq 0 \), then

\[
s \land t = \int_0^\infty \chi_{[0, s)}(v) \chi_{[0, t)}(v) \, dv,
\]

where \( \chi_{[0, s)} \) denotes the indicator function of the interval \([0, s)\). Thus, \((s, t) \mapsto s \land t\) is a positive definite and hence \((s, t) \mapsto -s \land t\) is a negative definite kernel on \( \mathbb{R}_+ \). Since \( \log(\alpha) > 0 \), we obtain by Schönberg’s theorem [1] that

\[
k_\alpha : (s, t) \mapsto \alpha^{s \land t} = e^{-\log(\alpha)(-s \land t)} \quad s, t \in \mathbb{R}_+
\]

is positive definite. Moreover, from its series development

\[
k_\alpha(s, t) = 1 + \sum_{n=1}^\infty \frac{1}{n!} (\log(\alpha) \, s \land t)^n \quad s, t \in \mathbb{R},
\]

we see that even \( k_\alpha - 1 \) is positive definite, if \( \alpha > 1 \).

We denote further, for \( \theta = e^{i\psi} \) with \( \psi \in [-\pi, \pi) \), by \( D_\theta \) the multiplication operator

\[
D_\theta : \mathcal{F}(\mathbb{R}_+) \to \mathcal{F}(\mathbb{R}_+)
\]

defined by

\[
D_\theta f(t) = e^{it\psi} f(t) \quad t \geq 0.
\]

As a theorem we state that the multiplication operators defined above act boundedly with respect to our sesquilinear form:
Theorem 1 For all \( f \in \mathcal{F}(\mathbb{R}_+) \) we have

\[
\int_0^\infty \int_0^\infty \alpha^{s+\lambda} D_\theta f(s) \overline{D_\theta f(t)} \, ds \, dt \leq C^2 \int_0^\infty \int_0^\infty \alpha^{s+\lambda} f(s) \overline{f(t)} \, ds \, dt,
\]

where \( C = 1 + \frac{2|\psi|}{\log(\alpha)}. \)

Proof: Since for \( x \in \mathbb{R} \)

\[
\alpha^x = 1 + \log \alpha \int_0^x \alpha^u \, du,
\]

it follows that

\[
[f, f]_\alpha = \int_0^\infty f(t) \, dt |^2 + \log \alpha \int_0^\infty \int_0^\infty \alpha^u \, ds \, du f(s) \overline{f(t)} \, ds \, dt
\]

\[
= \int_0^\infty f(t) \, dt |^2 + \log \alpha \int_0^\infty \alpha^u \int_0^\infty \chi_{[0, s \wedge t]}(u) f(s) \overline{f(t)} \, ds \, dt du
\]

\[
= \int_0^\infty f(t) \, dt |^2 + \log \alpha \int_0^\infty \alpha^u \int_u^\infty f(t) \, dt |^2 \, du.
\]

Now, for arbitrary \( \eta > 0, u \geq 0 \):

\[
\left| \int_u^\infty \left( e^{it\psi} - e^{iu\psi} \right) f(t) \, dt \right|^2
\]

\[
= \left| \int_u^\infty e^{it\psi} - e^{iu\psi} \right|^2 f(t) \, dt \right|^2
\]

\[
\leq (1 + \eta) \int_u^\infty f(t) \, dt |^2 + (1 + \frac{1}{\eta}) \int_u^\infty \left( e^{it\psi} - e^{iu\psi} \right) f(t) \, dt |^2.
\]

The last integral can be estimated using an arbitrary \( q > 1 \):

\[
\left| \int_u^\infty \left( e^{it\psi} - e^{iu\psi} \right) f(t) \, dt \right|^2
\]

\[
= \left| \int_u^\infty i \psi \left\{ \int_0^t e^{is\psi} \, ds \right\} f(t) \, dt \right|^2
\]

\[
= |\psi|^2 \int_u^\infty e^{is\psi} \left\{ \int_s^\infty f(t) \, dt \right\} ds |^2
\]

\[
\leq |\psi|^2 \int_u^\infty q^{-2s} \, ds \int_u^\infty q^{2s} \left| \int_s^\infty f(t) \, dt \right|^2 \, ds
\]

\[
= |\psi|^2 \int_u^\infty \frac{1}{2} q^{-2u} \int_u^\infty q^{2s} \left| \int_s^\infty f(t) \, dt \right|^2 \, ds.
\]

Hence, if \( \eta > 0 \) and \( 1 < q < \sqrt{\alpha} \) then

\[
[D_\theta f, D_\theta f]_\alpha = \int_0^\infty D_\theta f(t) \, dt |^2 + \log \alpha \int_0^\infty \alpha^u \int_u^\infty D_\theta f(t) \, dt |^2 \, du
\]

\[
\leq (1 + \eta) \left( \int_0^\infty f(t) \, dt |^2 + \log \alpha \int_0^\infty \alpha^u \int_u^\infty f(t) \, dt |^2 \, du \right) +
\]

\[
+ (1 + \frac{1}{\eta}) |\psi|^2 \int_0^\infty q^{-2u} \left( \int_0^\infty q^{2s} \left| \int_s^\infty f(t) \, dt \right|^2 \, ds \right)
\]

\[
+ \log \alpha \int_0^\infty \alpha^u q^{-2u} \int_u^\infty q^{2s} \left| \int_s^\infty f(t) \, dt \right|^2 \, ds \, du.
\]
\[ (1 + \eta) [f, f]_\alpha + (1 + \frac{1}{\eta}) |\psi|^2 \frac{1}{2\log q} \left( \int_0^\infty q^{2s} \left| \int_s^\infty f(t) \, dt \right|^2 \, ds + \right. \]
\[ \left. \int_0^s q^{2s} \log \alpha \int_0^u \alpha^u q^{-2u} du \left| \int_s^\infty f(t) \, dt \right|^2 \, ds \right) \]
\[ = (1 + \eta) [f, f]_\alpha + \]
\[ + (1 + \frac{1}{\eta}) |\psi|^2 \frac{1}{2\log q} \int_0^\infty \left( q^{2s} + \log \alpha \frac{\alpha^s - q^{2s}}{\log \alpha - 2\log q} \right) \left| \int_s^\infty f(t) \, dt \right|^2 \, ds \]
\[ \leq (1 + \eta) [f, f]_\alpha + \]
\[ + (1 + \frac{1}{\eta}) |\psi|^2 \frac{1}{2\log q} \frac{\log \alpha}{\log \alpha - 2\log q} \left( \alpha^s \right) \left| \int_s^\infty f(t) \, dt \right|^2 \, ds. \]

Choosing \( q = \alpha^{\frac{1}{4}} \) we have

\[ [D_\theta f, D_\theta f]_\alpha \leq (1 + \eta) [f, f]_\alpha + \]
\[ + (1 + \frac{1}{\eta}) |\psi|^2 \frac{4}{\log \alpha} \left( \alpha^s \right) \left| \int_s^\infty f(t) \, dt \right|^2 \, ds \]
\[ \leq \left( (1 + \eta) + (1 + \frac{1}{\eta}) |\psi|^2 \frac{4}{\log^2 \alpha} \right) [f, f]_\alpha. \]

By minimising this in \( \eta > 0 \) we obtain finally

\[ [D_\theta f, D_\theta f]_\alpha \leq (1 + 2 |\arg \theta|) \left( \frac{2}{\log \alpha} \right)^2 [f, f]_\alpha. \]

We shall actually need to apply a discrete version of the above theorem, which we want to formulate next. To do this we define for finitely supported functions

\[ g, f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} \]

an, again positive definite, sesquilinear form

\[ [f, g]_\alpha^o = \sum_{k,l=0}^\infty \alpha^{k+l} f_k g_l \]

and we let the multiplication operator \( D_\theta \) be defined by

\[ (D_\theta f)_n = \theta^n f_n \quad n \in \mathbb{Z}_+. \]

**Theorem 2** For all finitely supported functions \( f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} \):

\[ [D_\theta f, D_\theta f]_\alpha^o \leq \left( 1 + \frac{2 |\arg \theta|}{\log \alpha} \right) [f, f]_\alpha^o. \]

**Proof:** For a function \( f : \mathbb{Z} \to \mathbb{C} \) and for \( n \in \mathbb{N} \) let \( \Phi_n(f) \) be the function on \( \mathbb{R} \) defined by

\[ \Phi_n(f)(x) = \sum_{k \in \mathbb{Z}} f_k n \chi_{[k,k+\frac{1}{n})}(x) \quad x \in \mathbb{R}. \]
Here $\chi_{[a,b)}$ denotes the indicator function of the interval $[a, b)$.

Then, if $f$ is supported on $\mathbb{Z}_+$:

$$[f, f]_\alpha^0 = \lim_{n \to \infty} [\Phi_n(f), \Phi_n(f)]_\alpha.$$

And, if $\theta = e^{it}$, then

$$\|\Phi_n(D_\theta f) - D_\theta \Phi_n(f)\|_1 \leq \int_0^\infty \sum_{k \in \mathbb{Z}_+} |f_k| e^{it\psi} - e^{it\psi} n \chi_{[k,k+\frac{1}{n})}(t) \, dt$$

$$\leq \sum_{k \in \mathbb{Z}_+} |f_k| n \int_0^1 |1 - e^{it\psi}| \, dt$$

$$\leq \sum_{k \in \mathbb{Z}_+} |f_k| n \int_0^{\frac{1}{n}} |\psi| \frac{1}{n} \, dt,$$

which converges to zero as $n \to \infty$.

Since for a finitely supported $f$ all the above functions $\Phi_n(D_\theta f), \Phi_n(f)$... have their support in a compact set on whose Cartesian product the kernel $k_\alpha$ remains bounded we obtain

$$\lim_{n \to \infty} [\Phi_n(D_\theta f), \Phi_n(D_\theta f)]_\alpha = \lim_{n \to \infty} [D_\theta \Phi_n(f), D_\theta \Phi_n(f)]_\alpha.$$  

Then, of course

$$[D_\theta f, D_\theta f]_\alpha^0 = \lim_{n \to \infty} [\Phi_n(D_\theta f), \Phi_n(D_\theta f)]_\alpha$$

$$= \lim_{n \to \infty} [D_\theta \Phi_n(f), D_\theta \Phi_n(f)]_\alpha$$

$$\leq C_{\theta,\alpha} \lim_{n \to \infty} [\Phi_n(f), \Phi_n(f)]_\alpha$$

$$= C_{\theta,\alpha} [f, f]_\alpha^0.$$

**Remark 1** Similarly to the above, one can extend Theorem 1 to finite Borel measures which are supported in bounded subsets of $\mathbb{R}_+$.

**Corollary 1** If

$$(k, l) \mapsto \Lambda_{k,l}$$

is a positive definite kernel with finite support, then with the same constant $C_{\theta,\alpha}$ as in the above theorems:

$$\sum_{k,l} \theta^k \bar{\theta}^l \alpha^{k \wedge l} \Lambda_{k,l} \leq C_{\theta,\alpha} \sum_{k,l} \alpha^{k \wedge l} \Lambda_{k,l}.$$  



\(^0\text{positive definite only means positive semi-definite\)
Proof: Since $\Lambda$ is positive definite with finite support, there exists a finite dimensional Hilbert space $\mathcal{H}$ and a finite sequence $(\xi_k)_{K \geq 0}$ of its elements, such that $\Lambda_{k,l} = \langle \xi_k, \xi_l \rangle$. If $(e_i)_{i \in I}$ is an orthonormal basis of $\mathcal{H}$, then

$$
\sum_{k,l} \theta^k \theta^l \alpha^{k \wedge l} \Lambda_{k,l} = \sum_{i \in I} \sum_{k,l=0}^K \theta^k \theta^l \alpha^{k \wedge l} \langle \xi_k, e_i \rangle \langle e_i, \xi_l \rangle \leq C_{\theta,\alpha} \sum_{k,l=0}^K \alpha^{k \wedge l} \Lambda_{k,l}.
$$

In the rest of this section we state and prove, for later use, two lemmas from Fourier analysis on the abelian groups $\mathbb{Z}$ and $\mathbb{Z}^k$.

**Lemma 1** For $0 < q < 1$ there exists $\mu = \mu(q) > 0$ such that

$$
m \mapsto q^{\left| m \right|^2 - \mu |m|} \quad m \in \mathbb{Z}
$$

is a positive definite function.

Proof: The orthogonality relations for the characters of the torus, the dual group of $\mathbb{Z}$ show that it suffices to prove for sufficiently small $\mu > 0$ the non-negativity of the Fourier series

$$
\Psi_\mu(t) = \sum_{j \in \mathbb{Z}} q^{\left| j \right|^2 - \mu |j|} e^{-ijt} \quad t \in [0, 2\pi).
$$

To this end let $\tau = -\log q$ and notice that for $j \in \mathbb{Z}$:

$$
q^{\left| j \right|^2} = e^{-\tau j^2} = \frac{1}{2\sqrt{\pi \tau}} \int_{\mathbb{R}} e^{-\frac{x^2}{4\tau}} e^{ixj} dx = \frac{1}{2\sqrt{\pi \tau}} \int_{[0, 2\pi]} \sum_{k \in \mathbb{Z}} e^{-\frac{(x+2\pi k)^2}{4\tau}} e^{ixj} dx.
$$

Thus, by the uniqueness of Fourier inversion:

$$
\Psi_0(t) = \sum_{j \in \mathbb{Z}} q^{\left| j \right|^2} e^{-ijt} = \sqrt{\frac{\pi}{\tau}} \sum_{k \in \mathbb{Z}} e^{-\frac{(t+2\pi k)^2}{4\tau}} \quad t \in [0, 2\pi),
$$

which takes a strictly positive minimum $c_\tau > 0$ on $[0, 2\pi)$. But uniformly in $t \in [0, 2\pi)$:

$$
|\Psi_\mu(t) - \Psi_0(t)| \leq \sum_{j \in \mathbb{Z}} |q^{-\mu |j|} - 1| q^{\left| j \right|^2},
$$

which for $\mu \searrow 0$ tends to $0$.

The next lemma is prove similarly.

**Lemma 2** For $0 < q < 1$ and $k \in \mathbb{N}$ there exists $\mu' = \mu'(q, k) > 0$ such that

$$(m_1, \ldots, m_k) \mapsto q^{\left| m_1 \right|^2 + \ldots + \left| m_k \right|^2 - \mu' |m_1 + \ldots + m_k|}, \quad m = (m_1, \ldots, m_k) \in \mathbb{Z}^k$$

is a positive definite function on $\mathbb{Z}^k$. 

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Proof: This time we note first that

$$\Phi(t) = \sum_{n=-\infty}^{\infty} q^{|n|} e^{-\text{int}} = \frac{1 - q^2}{1 - 2q \cos t + q^2}$$

is strictly positive on the one dimensional torus \([0,2\pi)\).

Hence on the \(k\)-dimensional torus

$$\Psi_0(t_1, \ldots, t_k) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_k=-\infty}^{\infty} q^{(|n_1| + \ldots + |n_k|) e^{(n_1 t_1 + \ldots + n_k t_k)}} = \prod_{l=1}^{k} \Phi(t_l)$$

is strictly positive too.

Denote

$$\Psi_{\mu'}(t_1, \ldots, t_k) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_k=-\infty}^{\infty} q^{(|n_1| + \ldots + |n_k|) - \mu' |n_1 + \ldots + n_k| e^{(n_1 t_1 + \ldots + n_k t_k)}}.$$

Then, as \(\mu' \searrow 0\)

$$\Psi_{\mu'}(t_1, \ldots, t_k) \to \Psi_0(t_1, \ldots, t_k)$$

uniformly in \((t_1, \ldots, t_k) \in [0,2\pi)^k\). In fact, the absolute value of the difference is dominated uniformly by

$$\sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_k=-\infty}^{\infty} q^{(|n_1| + \ldots + |n_k|)} \left( q^{-\mu' |n_1| + \ldots + n_k|} - 1 \right).$$

The only point to note is that this series is finite for small \(\mu' > 0\). But, by the triangle inequality:

$$-\mu' |n_1 + \ldots + n_k| \geq -\mu' (|n_1| + \ldots + |n_k|),$$

and since \(q < 1\):

$$q^{(|n_1| + \ldots + |n_k|)} q^{-\mu' |n_1 + \ldots + n_k|} \leq q^{(1-\mu)(|n_1| + \ldots + |n_k|)}.$$

From this we infer for \(\mu' < 1\) the summability of the geometric series.

3 Coxeter groups, their Cayley graph and their geometrical representation

A group \(G\) together with a finite generating set \(S = \{s_1, \ldots, s_n\}\) is called a Coxeter system, or Coxeter group, if it has a presentation

$$s_i^2, \ldots, s_n^2, (s_is_j)^{m_{ij}} \quad i,j \in \{1, \ldots, n\}, i \neq j.$$

Here \(m_{ij} \in \{2, 3, \ldots, \infty\}\) denotes the order of the product of the two generators \(s_i, s_j \in S, i \neq j\), setting \(m_{ij} = \infty\) if \(s_is_j\) has no finite order. (If \(s, s'\) are elements of \(S\), \(s = s_i\) and \(s' = s_j\) for some \(i, j \in \{1, \ldots, n\}\) we shall write \(m(s, s')\) instead of \(m_{ij}\).) The cardinality \(n\) of \(S\) is called the rank of the Coxeter group.
The Cayley graph \( C(G, S) \) is the graph with vertices just the group elements \( V = \{ g : g \in G \} \). Two vertices \( u, v \in V \) are connected by an edge if \( u = vs \) for some \( s \in S \). We denote \( E = \{ \{ u, v \} : u = vs \text{ for some } s \in S \} \) the set of edges and notice that an edge connecting the vertices \( g \in V \) and \( gs \in V \) is canonically labeled by a generator \( s \in S \).

We shall identify every edge with an image of the unit interval \([0, 1] \subset \mathbb{R}\) obtaining thus a connected metric space \( (C(G, S), d) \). A path in \( C(G, S) \) then is a rectifiable map \( p : I \rightarrow C(G, S) \) defined on some closed interval \( I \subset \mathbb{R} \) into \( C(G, S) \).

If we denote by \( l : G \rightarrow \mathbb{N} \) the length function with respect to the generating set, i.e. if:

\[
l(g) = \inf\{ m : g = s_{i_1} \cdots s_{i_m}, \; s_{i_j} \in S \cup S^{-1} \} \quad g \in G
\]

then clearly this relates to the distance in \( C(G, S) \) by:

\[
d(g, h) = l(g^{-1}h) \quad g, h \in G.
\]

It is obvious from the above definitions that the action of the group \( G \) on itself by left multiplication, \( g : h \mapsto gh, \; g, h \in G, \) extends to an action of \( G \) by isometries of the metric space \( (C(G, S), d) \).

In an equation \( g = w_1 \ldots w_m \), we shall call the right hand side a reduced representation of \( g \), if \( w_i \in S, \; i = 1, \ldots, m \) and \( l(g) = m \). A product \( u_1 \ldots u_k \) will be called reduced if \( l(u_1 \ldots u_k) = l(u_1) + \ldots + l(u_k) \). The void word represents the identity \( e \) of \( G \).

A useful tool for a finitely generated Coxeter group is its representation as a discrete subgroup of the general linear group of a finite dimensional real vector space \( E \) of dimension \( \# S \) (see chapter V of [4]). We shall denote it:

\[
\sigma : G \rightarrow \text{Gl}(E).
\]

A self inverse element \( t \in G, \; t \neq e \) will be called a reflection if \( \sigma(t) \), or equivalently \( \sigma^*(t) \), is a reflection. The corollaire in 3.2, chapter V of [2], asserts that any reflection is conjugate to some generator. (\( T = \{ g^{-1}sg : s \in S, \; g \in G \} \) will denote the set of all reflections). We shall denote for \( g \in G \):

\[
N_g = \{ t \in T : l(tg) < l(g) \}
\]

**Remark 2**  (i) The length of a group element \( g \in G \) is given by:

\[
l(g) = \# \{ t \in T : l(tg) < l(g) \}.
\]

(ii) If \( g = w_1 \ldots w_n \) is a reduced representation, then

\[
N_g = \{ w_1, w_1w_2w_1, \ldots, w_1 \ldots w_{n-1}w_nw_{n-1} \ldots w_1 \}.
\]

The next theorem is a reformulation, in our setting, of one of [9].

**Theorem 3** For \( g, h \in G \):

\[
d(g, h) = \# N_g \triangle N_h = \sum_{t \in T} |\chi_{N_g}(t) - \chi_{N_h}(t)|^2.
\]

Hence \( g \mapsto l(g) \) is a negative definite function.
We shall actually need some consequences of the action of $G$ by means of the contragradient representation

$$\sigma^*: G \to \text{Gl}(E^*)$$
on the Tits cone $U$, see n° 4.6. of [2]

**Remark 3** (i) If for $t, t' \in T$ the hyperplanes stabilised by $\sigma^*(t)$, respectively $\sigma^*(t')$, intersect inside the Tits cone $U$, then the product $tt'$ has finite order. (This is contained in exercise 2c) and 2d) of [2])

(ii) $G$ contains a normal torsion free subgroup $\Gamma$ of finite index. (see loc. cit. exercise 9).

Now $G$, and hence $\Gamma$, act on the set of reflections $T$ by conjugation. Let

$$T = T_1 \dot{\cup} T_2 \dot{\cup} \ldots \dot{\cup} T_\Lambda$$

be a decomposition into $\Gamma$–orbits.

**Lemma 3** If $t, t' \in T$ belong to the same $\Gamma$–orbit, then either $t = t'$ or the hyperplanes stabilised by $\sigma^*(t)$, and $\sigma^*(t')$ respectively, do not intersect inside the Tits cone $U$.

**Proof:** If we assume that the stabilised hyperplanes intersect inside $U$, then $tt'$ has a finite order. On the other hand, if $t$ and $t'$ both belong to the same $\Gamma$–orbit, then, for some $t_0 \in T$ and some $\gamma, \gamma' \in \Gamma$:

$$t = \gamma^{-1}t_0\gamma \quad \text{and} \quad t' = \gamma'^{-1}t_0\gamma'$$

Since $t_0^2 = e$, and $\Gamma$ is normal

$$tt' = \gamma^{-1}t_0\gamma\gamma'^{-1}t_0\gamma' \in \Gamma.$$ 

Because $\Gamma$ is torsion–free it follows that this element of finite order equals the identity.

For $g \in G$ we momentarily fix a reduced decomposition and order $N_g$ according to ([4]). Further we endow the sets

$$N^i_g = N_g \cap T_i, \quad i = 1, \ldots, \Lambda$$

with the order inherited as subsets of $N(g)$. In fact, the order obtained on the subsets $N^i_g$ does not depend on the reduced decomposition chosen.

**Lemma 4** Let $g \in G$ be given. Then for any $u \in G$ and for $i \in \{1, \ldots, \Lambda\}$

$$N^i_g \cap N_u$$

is an initial segment of $N^i_g$.

**Proof:** We have to show, that $t \in N^i_g \cap N_u$ and $t' \in N^i_g$, $t' < t$ imply $t' \in N_u$.

Denote $H_t, H_{t'}$ the hyperplanes in $E^*$ fixed by $\sigma^*(t)$, respectively by $\sigma^*(t')$. Then $t \in N_g$ means, that $H_t$ separates the fundamental chamber $C$ from $\sigma^*(g)C$, similarly for $u$. Moreover $t' < t$ shows, that there is a point in $H_t \cap U$ separated from $C$ by $H_{t'}$. Since $H_t$ and $H_{t'}$ do not intersect inside $U$, we conclude that all of $H_t \cap U$ and $C$ lie on different sides of $H_{t'}$. Since $\sigma^*(g)C$ and $\sigma^*(u)C$ are separated from $C$ by $H_t$.
we conclude that any line segment from $\sigma^*(u)C$ to $C$ must intersect $H_{t'}$, meaning that $t' \in N_u$.

Let us denote

$$N^g = \{N_g \cap N_u : u \in G\}.$$ 

**Proposition 1** For $0 < r < 1$ there exists a constant $\kappa$ such that

$$(U, V) \mapsto r^{\kappa(#U \wedge #V) + (#U \triangle V)}$$

is a positive definite kernel on $N^g \times N^g$.

**Proof:** For $U = N_u \cap N_g \in N^g$ we denote $U_j = T_j \cap U$, $j = 1, \ldots, \Lambda$; similarly for $V \in N^g$. Further denote $u_j = \#U_j$ and $v_j = \#V_j$, the respective cardinalities.

Lemma 2 provides a constant $\mu' > 0$, such that

$$(U, V) \mapsto r^{-\mu' \sum |u_j - v_j| + \sum |u_j - v_j|}$$

is positive definite. Since

$$2(#U \wedge #V) = \#U + \#V - |#U - #V|$$

$$= \#U + \#V - \sum (u_j - v_j),$$

and since

$$(U, V) \mapsto r^{\mu' (#U + #V)} = r^{\mu' #U} \cdot r^{\mu' #V}$$

is positive definite, it suffices to show

$$#(U \triangle V) = \sum |u_j - v_j|.$$ 

Because then

$$r^{2\mu' (#U \wedge #V) + (#U \triangle V)} = r^{\mu' (#U + #V)} \cdot r^{-\mu' \sum |u_j - v_j| + \sum |u_j - v_j|}$$

is positive definite as a product of positive definite kernels. Now,

$$#(U \triangle V) = \sum_{t \in N_g} |\chi_U(t) - \chi_V(t)|^2$$

$$= \sum_{j} \sum_{t \in N_g \cap T_j} |\chi_U(t) - \chi_V(t)|^2.$$ 

Because $\chi_U - \chi_V$ takes only values in $\{-1, 0, 1\}$ we can omit the squares. Moreover, since the sets $U_j$ and $V_j$ contain all predecessors of their elements, $\chi_{U_j} - \chi_{V_j}$ is either a non-negative or a non-positive function. Thus we may continue:

$$#(U \triangle V) = \sum_j \sum_{t \in N_g} |\chi_{U_j}(t) - \chi_{V_j}(t)|$$

$$= \sum_j |\sum_{t \in N_g} \chi_{U_j}(t) - \chi_{V_j}(t)|$$

$$= \sum_j |u_j - v_j|.$$ 

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4 Construction of a series of representations

Theorem 3 shows that for any Coxeter group
\[ d(.,.) : G \times G \rightarrow \mathbb{Z}_+ \]
is a negative definite kernel on \( G \). By Schoenberg’s theorem (see e.g. [1]), then, for \( r \in (0,1) \),
\[ g \mapsto r^{l(g)} \quad g \in G \]
is a positive definite function.

We are going to consider modifications of the Gelfand–Naimark–Segal representation constructed from this positive definite functions. So let
\[ \mathcal{F} = \{ f : G \rightarrow \mathbb{C} \text{ with finite support} \} \]
\[ < f, h >_r = \sum_{u,v \in G} r^{d(u,v)} f(u) \overline{h(v)} \quad f, h \in \mathcal{F} \]
\[ \| f \|_r = ( < f, f >_r )^{\frac{1}{2}} \quad f \in \mathcal{F} \]
\[ \pi_r(g)f(u) = f(g^{-1}u) \quad g, u \in G, f \in \mathcal{F}. \]

**Proposition 2** The kernel of \( \| . \|_r \) on \( \mathcal{F} \) equals \( \{ 0 \} \).

*Proof:* We shall show, that for a finite set \( u_1, \ldots, u_n \in G \) the matrix
\[ (r^{d(u_i,u_j)})_{i,j=1}^n \]
is non-degenerate. Denote \( \tau = -\log r \) and consider the functions \( \chi_i = \sqrt{\tau} \chi_{N_{u_i}} \) as elements of the Hilbert space \( l^2(T) \). By Theorem 3
\[ r^{d(u_i,u_j)} = \exp(\| \chi_i - \chi_j \|_2^2) \]
\[ = \exp(\| \chi_i \|_2^2) \exp(-2 < \chi_i, \chi_j >) \exp(\| \chi_j \|_2^2). \]

Proposition 2.2 of [10] implies that
\[ (\exp(-2 < \chi_i, \chi_j >))_{i,j=1}^n \]
is non-degenerate. But then, from (4), we see that \( (r^{d(u_i,u_j)})_{i,j=1}^n \) is non-degenerate too.

For a given \( \theta \in \mathbb{C}, |\theta| = 1 \) of absolute value one we define an equivariant cocycle
\[ c_\theta : G \times G \rightarrow \text{Gl}(\mathcal{F}) \]
for \( \pi_r \) by:
\[ c_\theta(u,g)f(v) = \theta^{l(g^{-1}v)-(l^{-1}v)} f(v) \quad f \in \mathcal{F}, u, v, g \in G. \]

For any \( r \in (0,1) \) the cocycle equalities
\[ c_\theta(u,u) = \text{id}, \]
\[ c_\theta(u,g)c_\theta(g,v) = c_\theta(u,v), \]
and the equivariance
\[ \pi_r(g) c_\theta(v, u) = c_\theta(gv, gu) \pi_r(g) \]
are easily checked.

Thus for \( z \in \mathbb{C} \) with \(|z| < 1\), which we write as \( z = \theta r, \ r \in (0, 1), \ |\theta| = 1 \), we may define a representation of \( G \) on \( \mathcal{F} \) by
\[ \pi_z(g)f = c_\theta(e, g) \circ \pi_r(g)f \quad f \in \mathcal{F}, \ g \in G. \]

We are about to formulate a criterion for the boundedness of this representation on the semi–normed space \( \mathcal{F}, \| \cdot \|_r \). To do this let me denote, for \( g \in G \):
\[ N^g = \{N_g \cap N_u : u \in G\}. \]

This is a subset of the potency set of \( N_g \). In the cases which are of interest to us its cardinality will be much smaller than \( 2^{l(g)} \).

**Theorem 4** Let \((G, S)\) be a Coxeter group. Then for \( z = \theta r \in D = \{z \in \mathbb{C} : |z| < 1\} \) there exists a uniformly bounded representation \((\pi_z, H_r)\) such that for some \( \xi_0 \in H_r \):
\[ \langle \pi_z(g)\xi_0, \xi_0 \rangle_r > z^l(g) \quad g \in G. \]

Moreover, for some constant \( \kappa \), depending only on \((G, S)\):
\[ \sup_{g \in G} \| \pi_z(g) \| \leq 1 + \frac{2|\text{arg}(z^2)|}{\kappa |\log r|}. \]

**Proof:** Since \( \pi_r(g) : G \to \text{Gl}(\mathcal{F}) \) is a representation by invertible isometries, it suffices to show, that \( c_\theta(e, g) \) is bounded by \( 1 + \frac{2|\text{arg}(z^2)|}{\kappa |\log r|} \).

We note that for \( u, v \in G \)
\[ d(u, v) = \sum_{t \in N_g} |\chi_{N_u}(t) - \chi_{N_v}(t)|^2 + \sum_{t \in N_g^c} |\chi_{N_u}(t) - \chi_{N_v}(t)|^2 \]
\[ = \#((N_g \cap N_u) \triangle (N_g \cap N_v)) + d'(u, v), \]
where \( N_g^c \) denotes the complement of \( N_g \) in the set of all reflections and
\[ d'(u, v) = \sum_{t \in N_g^c} |\chi_{N_u}(t) - \chi_{N_v}(t)|^2 \]
is a negative definite kernel on \( G \). Thus for \( f \in \mathcal{F} \)
\[ (U, V) \mapsto \sum_{u \in U, v \in V} r^{d'(u, v)} f(u) \overline{f(v)} \]
is positive definite on \( N^g \). For \( k \in \mathbb{Z}_+ \) let \( E_k = \{U \in N^g : \#U = k\} \). From the assumption on \( g \) and \( \kappa \) it follows, using Schur’s theorem (see e.g. [1]), that
\[ (k, l) \mapsto \Lambda_{(k, l)} = \sum_{U \in E_k, V \in E_l} r^{\#U \cap \#V + \#(U \triangle V)} \sum_{u \in U, v \in V} r^{d'(u, v)} f(u) \overline{f(v)} \]
is a positive definite kernel on $\mathbb{Z}_+$. In Theorem 2 we let $\alpha = r^{-\kappa}$ and conclude

$$\sum_{k,l=0}^{\infty} \theta^{2k-2l} \alpha^{k\wedge l} \Lambda_{k,l} \leq C_{\theta^2,\alpha} \sum_{k,l=0}^{\infty} \alpha^{k\wedge l} \Lambda_{k,l}.$$ 

The left hand side of this inequality computes to $\|c_\theta(e,g)f\|^r_\pi$ and the right one to $C_{\theta,r^{-\kappa}} \|f\|^r_\pi$.

In fact, we notice that

$$1(u) + 1(g) - 1(g^{-1}u) = \#N_u + \#N_g - \#N_g \Delta N_u = 2\#(N_g \cap N_u)$$

and compute:

$$\sum_{k,l=0}^{\infty} \theta^{2k-2l} \alpha^{k\wedge l} \Lambda_{k,l} =$$

$$= \sum_{k,l=0}^{\infty} \theta^{2k-2l} \alpha^{k\wedge l} \sum_{U \in E_k, V \in E_l} r^{\kappa(#U \wedge #V)} r^{\#(U \Delta V)} \sum_{u \in U, v \in V} r^{d'(u,v)} f(u) \overline{f(v)}$$

$$= \sum_{k,l=0}^{\infty} \theta^{2k-2l} \alpha^{k\wedge l} \sum_{u \in E_k', v \in E_l'} r^{d(u,v)} f(u) \overline{f(v)}$$

$$= \sum_{u \in G, v \in G} \theta^{2\#((N_g \cap N_u) \Delta (N_g \cap N_v))} r^{2d(u,v)} f(u) \overline{f(v)}$$

$$= \sum_{u \in G, v \in G} \theta^{(1(u) - 1(g^{-1}u))} r^{(1(v) - 1(g^{-1}v))} f(u) \overline{f(v)},$$

where, for $k \in \mathbb{Z}_+$, we denoted $E_k' = \{u \in G : \#(N_g \cap N_u) = k\}$. The right hand side is computed similarly and checking for the constant finishes the proof.

We shall denote $H_r$ the Hilbert space obtained by completing $F$ with respect to $\| \cdot \|_r$. The above gives the announced bound on the norm of the operators $\pi_z(g)$, $z \in D, g \in G$.

If we denote for $u \in G$ by $\delta_u$ the point mass one at $u$, then

$$\pi_z(g) \delta_u (x) = c_\theta(e,g) \delta_u(g^{-1}.) (x)$$

$$= \theta^{I(x) - I(g^{-1}x)} \delta_u (g^{-1}x)$$

$$= \theta^{I(gu) - I(u)} \delta_{gu} (x) \quad x \in G.$$ 

Hence,

$$< \pi_z(g) \delta_e, \delta_e >_r = \theta^{I(g)} < \delta_g, \delta_e >_r$$

$$= \theta^{I(g)} r^{I(g)}.$$ 

**Remark 4** Assume that the conditions of the corollary are satisfied.
(1) For any $u, v \in G$ the map $\theta \mapsto c_{\theta}(u, v)$ is a group homomorphism from the circle group (the torus) into the bounded invertible operators on $H_r$. It is continuous for the strong operator topology on $B(H_r)$.

(2) If $\chi : g \mapsto (-1)^{l(g)}$, then $\chi$ is a character of the Coxeter group $G$. For $z \in D$ the tensor product representation $\chi \otimes \pi_z$ is canonically isomorphic to $\pi_{-z}$.

(3) Complex conjugation of functions in $\mathcal{F}$ defines a conjugate linear intertwining operator between $\pi_z$ and $\pi_{\overline{z}}$.

5 Weak amenability

To prove the weak amenability of $G$ we can not directly apply the Theorem of [17], since our series of representations is not realized on one Hilbert space. But in the proof of the mentioned theorem it is only used that $z \mapsto \varphi_z$ is analytic as a function from $D$ to the completely bounded multipliers of $A(G)$. This can be proved by a method of Pytlik and Szwarc [14], see also [15].

For this we denote by $\chi_n$ the characteristic function of the set $\{ g \in G : l(g) = n \}$ of group elements of length $n \in \mathbb{N}$.

Proposition 3 For some constant $\kappa$, depending only on $(G, S)$:

$$\sup_{g \in G} \| \chi_n \|_{M_0 A(G)} \leq 2\pi e (1 + \frac{4\pi}{\kappa} n).$$

Proof:

Let for $0 < r < 1$ denote $\tilde{\pi}_r : G \to B(L^2([0, 2\pi], H_r))$ the direct integral representation on $L^2([0, 2\pi], H_r)$ defined by:

$$\tilde{\pi}_r(g)f)(t) = \pi_z(g)(f(t)), \text{ where } z = re^{it}, f \in L^2([0, 2\pi], H_r), g \in G.$$

Now, let $f, h \in L^2([0, 2\pi], H_r)$ denote the square integrable $H_r$ valued functions:

$$f : t \mapsto \delta_e \text{ and } h : t \mapsto e^{int}\delta_e.$$

Then, for any $g \in G$:

$$< \tilde{\pi}_r(g)f, h > = \int_0^{2\pi} < \pi_{r e^{it}} \delta_e, \delta_e > \ e^{-int} \ dt$$

$$= \int_0^{2\pi} r^{l(g)} e^{i(l(g))t} e^{-int} \ dt$$

$$= \begin{cases} 2\pi r^{l(g)} & \text{if } n = l(g), \\ 0 & \text{if } n \neq l(g) \end{cases}$$

$$= 2\pi r^n \chi_n(g).$$

Since $\| f \| = \| h \| = \sqrt{2\pi}$, we infer from Theorem [3]:

$$\| \chi_n \|_{M_0 A(G)} \leq r^{-n} \sup_{g \in G} \| \tilde{\pi}_r(g) \| \| f \| \| h \| \leq 2\pi r^{-n} (1 + \frac{4\pi}{\kappa |\log r|}).$$
Here we may take \( r = e^{-\frac{1}{n}} \) on the right hand side and obtain the constant given in the statement.

**Corollary 2** The function \( z \mapsto z^{l(z)} \) is analytic on \( D \).

**Proof:** We just note that the series

\[
z^{l(z)} = \sum_{n=0}^{\infty} \chi_n z^n
\]

is norm convergent in \( M_0 A(G) \), on the whole open disk \( D \).

Arguing either as in [17] or as in the proof of Theorem 6 in [15], see also the article of de Cannière and Haagerup [6], we obtain:

**Theorem 5** A Coxeter group \( (G, S) \) is weakly amenable with Cowling–Haagerup constant one.

The proof of the theorem is immediate from the following lemma, which we state and prove for the readers convenience.

**Lemma 5** Let \( G \) be a locally compact group, and \( z \mapsto \varphi_z \) an analytical map from the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) to \( M_0 A(G) \), such that

(i) \( \varphi_z = \sum_{n=0}^{\infty} \psi_n z^n \), fore some \( \psi_n \in A(G) \),

(ii) \( \varphi_r \) is an element of the unit sphere of the Fourier–Stieltjes algebra, for \( r \in [0, 1) \),

(iii) locally uniformly on \( G \):

\[
\lim_{r \to 1} \varphi_r = 1.
\]

Then \( G \) is weakly amenable with Cowling–Haagerup constant 1.

**Proof:** From theorem B_2 of [9] we infer that for all \( \psi \in A(G) \):

\[
\| \varphi_r \psi - \psi \|_{A(G)} \to 0, \quad \text{as } r \to 1.
\]

(5)

Now taking the Fejer kernel on the torus \( \mathbb{T} \):

\[
F_N(e^{it}) = \sum_{|k| \leq N} \left( 1 - \frac{|k|}{N + 1} \right) e^{ikt},
\]

we have

1) \( F_N \geq 0 \) and \( \frac{1}{2\pi} \int_0^{2\pi} F_N(e^{it}) \, dt = 1 \),

2) for all \( f \in C(\mathbb{T}) \): \( F_N * f(\theta) \to f(\theta), \quad \forall \theta \in \mathbb{T} \) as \( N \to \infty \).
Let
\[ \psi_{N,r} = \frac{1}{2\pi} \int_0^{2\pi} F_N(e^{it}) \varphi_{re^{it}} dt. \]
Then \( \psi_{N,r} \in M_0A(G) \) and
\[ \| \psi_{N,r} - \varphi_r \|_{M_0A(G)} \leq \frac{1}{2\pi} \int_0^{2\pi} F_N(e^{it}) \| \psi_{N,r} - \varphi_{re^{it}} \|_{M_0A(G)} dt, \]
which, since \( z \mapsto \varphi_z \) is continuous on \( D \), converges to 0 as \( N \to \infty \), by 2).

On the other hand \( \psi_{N,r} \in A(G) \), since
\[ \psi_{N,r}(g) = \frac{1}{2\pi} \int_0^{2\pi} F_N(e^{it}) \sum_n \psi_n(g) (re^{it})^n dt \]
\[ = \sum_{n=0}^N (1 - \frac{n}{N+1}) \psi_n(g). \]

Now from [3] and [8] it is easy to construct an approximate unit in \( A(G) \) with its completely bounded multiplier norm bounded by 1.

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