MOMENTS OF DIRICHLET SPLINES AND THEIR APPLICATIONS TO HYPERGEOMETRIC FUNCTIONS

Edward Neuman
Department of Mathematics
Southern Illinois University at Carbondale
Carbondale, IL 62901-4408 U.S.A.
E-mail: GA3856@SIUCVMB.BITNET

*Patrick J. Van Fleet
Department of Mathematics
Vanderbilt University
Nashville, TN 37240 U.S.A.
E-mail: Vanfleet@Athena.Cas.Vanderbilt.Edu

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*Current address: Department of Mathematics, Sam Houston State University, Huntsville, TX 77340 U.S.A., E-mail: Mth_pvf@shsu.edu.
Abstract. Dirichlet averages of multivariate functions are employed for a derivation of basic recurrence formulas for the moments of multivariate Dirichlet splines. An algorithm for computing the moments of multivariate simplex splines is presented. Applications to hypergeometric functions of several variables are discussed.

Introduction.

In [9], H.B. Curry and I.J. Schoenberg have pointed out that univariate $B$-splines can be constructed from volumes of slices of convex polyhedra. An extension of this idea to the case of multivariate splines is due to C. de Boor [1]. Since the geometric construction is too complicated to be used in numerical computations, some basic recurrence formulas for these functions have been found (see [3,8,10,11,14,17,18,19,22,23]). Multivariate $B$-splines (also called simplex splines) have been studied extensively over the past thirteen years by many researchers. These functions have been found useful for some applications of data fitting, computer aided geometric design, and mathematical statistics. In [29] the author addressed some new problems where the simplex splines could play a prominent role.

Further generalizations of multivariate simplex splines appear in [19] and [13]. Here and thereafter we call these functions Dirichlet splines. Our choice of terminology is motivated by the fact that the distributional definition of the Dirichlet splines (see (2.2)) involves the Dirichlet density function. It has been demonstrated that this class of splines is well designed for some problems of mathematical statistics (see [13], [19], and references therein). Applications to the theory of multivariate convex functions are reported in [26, 27].

Recently we have noticed that there is a simple relationship between univariate Dirichlet splines and some special functions such as $R$-hypergeometric functions and confluent hypergeometric functions.
In this paper, we present some results for multivariate Dirichlet splines together with their applications to special functions of several variables.

The outline of this paper is as follows. In Section 2 we introduce notation and definitions which will be used throughout the sequel. In the third section we give a definition and basic properties of a Dirichlet average of a multivariate function. Also, we give a recurrence relation for these averages. In Section 4 we give two results which play a crucial role in our subsequent considerations. Moments of the class of splines under discussion together with two moment generating functions are presented in Section 5. Therein, we also give an algorithm for computing the moments of multivariate simplex splines. Applications to hypergeometric functions, including Appell’s $F_4$ and Lauricella’s $F_{B}$, are discussed in Section 6. In the same section, we give a recurrence formula, two generating functions, and an inequality for Lauricella polynomials.

2. Notation and Definitions.

Let us introduce some notation and definitions which will be used throughout the sequel. By $x, y, \ldots$, we denote elements of Euclidean space $\mathbb{R}^s$ ($s \geq 1$), i.e., $x = (x_1, \ldots, x_s)$. Superscripts are used to number vectors. The inner product (or dot product) of $x, y \in \mathbb{R}^s$ is denoted by $x \cdot y = \sum_{k=1}^{s} x_k y_k$. For a given set $X \subset \mathbb{R}^s$ the symbols $[X]$ and $\text{vol}_s(X)$ mean the convex hull of $X$ and the $s$-dimensional Lebesgue measure, respectively. We use standard multi-index notation, i.e., for $\beta \in \mathbb{Z}_+^s$, $|\beta| = \beta_1 + \cdots + \beta_s$, $\beta! = \beta_1! \cdots \beta_s!$, $\alpha \leq \beta$ means $\alpha_1 \leq \beta_1, \ldots, \alpha_s \leq \beta_s$ ($\alpha \in \mathbb{Z}_+^s$), $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_s^{\beta_s}$. For $r \in \mathbb{Z}_+$, and $\beta \in \mathbb{Z}_+^s$ with $|\beta| = r$, the multinomial coefficient $\binom{r}{\beta}$ is defined
in the usual way
\[
\binom{r}{\beta} = \frac{r!}{\beta!}
\]
By
\[
E_n = \{ t = (t_1, \ldots, t_n) \in \mathbb{R}^n : t_j \geq 0, \text{ for all } j, \sum_{j=1}^{n} t_j \leq 1 \}
\]
we denote the standard \( n \)-simplex. Let \( \mathbb{R}_+ \) represent the set of all positive real numbers and let \( b = (b_0, \ldots, b_n) \in \mathbb{R}_+^{n+1} \). The Dirichlet density function on \( E_n \), denoted by \( \phi_b \), is given by
\[
\phi_b(t) = \frac{\Gamma(c) \prod_{i=0}^{n} \Gamma(b_i)^{-1} t_i^{b_i-1}}{\prod_{i=0}^{n} \Gamma(b_i)}
\]
(2.1)
where \( t \in E_n, t_0 = 1 - t_1 - \cdots t_n \), and \( c = b_0 + \cdots + b_n \).

For \( X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^s \) (\( n \geq s \geq 1 \)) with \( \text{vol}_s([X]) > 0 \), the multivariate Dirichlet spline \( M(\cdot \mid b; X) \) is defined by requiring that
\[
\int_{\mathbb{R}^s} f(x) M(x \mid b; X) dx = \int_{E_n} f(Xt) \phi_b(t) dt
\]
(2.2)
holds for all \( f \in C_0(\mathbb{R}^s) \) – the space of all multivariate continuous functions on \( \mathbb{R}^s \) with compact support (see [13, 19]). Here \( dx = dx_1 \cdots dx_s, \ dt = dt_1 \cdots dt_n, \ Xt = \sum_{i=0}^{n} t_i x^i \). When \( b_0 = \cdots = b_n = 1 \), \( \phi_b(t) = n! \), and the corresponding spline becomes a simplex spline. The latter spline will be denoted by \( M(\cdot \mid X) \). When \( s = 1 \), we will write \( m(\cdot \mid b; Z) \) instead of \( M(\cdot \mid b; X) \), where \( Z = \{z_0, \ldots, z_n\} \subset \mathbb{R} \). In this case (2.2) becomes
\[
\int_{\mathbb{R}} h(u) m(u \mid b; Z) du = \int_{E_n} h(Zt) \phi_b(t) dt
\]
(2.3)
(\( h \in C_0(\mathbb{R}) \)).
3. Dirichlet Averages.

The purpose of this section is two-fold. We give a definition of Dirichlet averages of multivariate functions. Next we prove a recurrence formula for these averages. This result has an immediate application in Section 5. For the reader’s convenience, let us recall a definition of the Dirichlet average of a univariate function $h \in C_0(\mathbb{R})$. Assume that the set $Z = \{z_0, \ldots, z_n\} \subset \mathbb{R}$ is such that $\min\{z_j : 0 \leq j \leq n\} < \max\{z_j : 0 \leq j \leq n\}$. For $b \in \mathbb{R}_n^{n+1}$, the Dirichlet average of $h$, denoted by $H(b; Z)$, is given by

$$H(b; Z) = \int_{E_n} h(Zt)\phi_b(t)dt$$

(3.1)

(see [6]). Comparison with (2.3) yields

$$H(b; Z) = \int_{\mathbb{R}} h(u)m(u|b; Z)du.$$  

(3.2)

We list below some elementary properties of $H(b; Z)$.

(i) $H(b_0, \ldots, b_n; z_0, \ldots, z_n)$ is symmetric in indices $0, 1, \ldots, n$ (see [6, Thm. 5.2–3]).

(ii) A vanishing parameter $b_i$ can be omitted along with the corresponding variable $z_i$ (see [6, (6.3–3)]).

(iii) Equal variables can be replaced by a single variable if the corresponding parameters are replaced by their sum (see [6, Thm. 5.2–4]).

We now introduce the Dirichlet average of $f \in C_0(\mathbb{R}^s)$. For $X \subset \mathbb{R}^s$ with $\text{vol}_s([X]) > 0$ and $b \in \mathbb{R}_s^{n+1}$ ($n \geq s \geq 1$), the Dirichlet average of $f$, denoted by $\mathcal{F}(b; X)$, is given by

$$\mathcal{F}(b; X) = \int_{E_n} f(Xt)\phi_b(t)dt,$$  

(3.3)
where $X_t$ and $\phi_b$ have the same meaning as in Section 2. Comparison with (2.2) shows that

$$F(b; X) = \int_{\mathbb{R}^s} f(x) M(x|b; X) dx. \tag{3.4}$$

It is clear that the properties (i)–(iii) are also valid for the average $F$. In particular, property (iii), when applied to $M(\cdot|b; X)$ yields

$$M(\cdot|b; X) = M(x^0, \ldots, x^n) \quad (b_0 \text{- times}) \quad (b_n \text{- times}) \tag{3.5}$$

provided the $b$’s are positive integers. (See also [13, 19].) The spline on the right hand side of (3.5) is a multivariate simplex spline with coalescent knots (see [14] for a detailed analysis of this class of splines).

Before we state and prove the first result of this section, let us introduce more notation. By $e_j \ (0 \leq j \leq n)$, we denote the $j$th coordinate vector in $\mathbb{R}^{n+1}$. For $f \in C^1_0(\mathbb{R}^s)$ define a function $f_j$ as follows:

$$f_j(x) = D_{x_j} f(x) \tag{3.6}$$

Here $D_y f$ denotes the directional derivative of $f$ in the direction $y \in \mathbb{R}^s$, i.e.,

$$D_y f(x) = \sum_{k=1}^s y_k \frac{\partial}{\partial x_k} f(x).$$

We are now ready to prove the following.

**THEOREM 3.1 ([7]).** Let $X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^s \ (n \geq s \geq 1)$ be such that $\text{vol}_s([X]) > 0$. Further, let $f \in C^1_0(\mathbb{R}^s)$ and let the vector $b \in \mathbb{R}^n_{\geq 1}$ be such that $b_j \geq 1$ for some $0 \leq j \leq n$. Then the following identity

$$(c - 1) F(b; X) = (c - 1) F(b - e_j; X) + F_j(b; X) \tag{3.7}$$

is valid. Here $F_j$ denotes the Dirichlet average of the function $f_j$. 
REMARK. The proof presented below bears no resemblance to what was done in [7, Thm. 3]. In this paper, the author has established (3.7) using generalized Euler-Poisson partial differential equations.

Proof. In order to establish the identity (3.7) we employ the following one

\[(c - 1)H(b; Z) = (c - 1)H(b - e_j; Z) + H_j(b; Z).\] (3.8)

Here \(H_j\) stands for the Dirichlet average of the function \(h_j(u) = (z_j - u)h'(u)\), \(h \in C_0^1(\mathbb{R}^s)\). The relation (3.8) readily follows from (5.6–13) in [6]. Application of (3.2) to (3.8) yields

\[(c - 1)\int_{\mathbb{R}} h(u)m(u|b; Z)du = (c - 1)\int_{\mathbb{R}} h(u)m(u|b - e_j; Z)du \]
\[+ \int_{\mathbb{R}} (z_j - u)h'(u)m(u|b; Z)du.\] (3.9)

We will lift (3.9) to the case of multivariate functions. To this aim we shall employ the following formula

\[\int_{\mathbb{R}} h(u)m(u|b; Z)du = \int_{\mathbb{R}^s} h(\lambda \cdot x)M(x|b; X)dx,\] (3.10)

where now \(Z = \{\lambda \cdot x^0, \ldots, \lambda \cdot x^n\}, \lambda \in \mathbb{R}^s \setminus \{0\}\), and \(h(\lambda \cdot x)\) is a ridge function (or plane wave). Since the proof of (3.10) is similar to that presented in [23], p. 496, we omit further details. Application of (3.10) to (3.9) yields

\[(c - 1)\int_{\mathbb{R}^s} h(\lambda \cdot x)M(x|b; X)dx = (c - 1)\int_{\mathbb{R}^s} h(\lambda \cdot x)M(x|b - e_j; X)dx \]
\[+ \int_{\mathbb{R}^s} (\lambda \cdot x^j - \lambda \cdot x)h'(\lambda \cdot x)M(x|b; X)dx.\] (3.11)

We appeal now to the denseness of ridge functions (these functions form a dense subset in \(C_0^1(\mathbb{R}^s)\)) to conclude that the above identity is valid for any multivariate function \(f \in C_0^1(\mathbb{R}^s)\), see [20]. Substituting \(h(\lambda \cdot x) = f(x)\) into (3.11), we obtain the assertion and the proof is completed. \(\square\)
**COROLLARY 3.2.** Along with the hypotheses of Theorem 3.1, assume that for some $0 \leq i, j \leq n$, $1 \leq k \leq s$, that $x^i_k \neq 0$, $x^j_k \neq 0$, and $b_i \geq 1$, $b_j \geq 1$.

Then

$$(c - 1)[\mathcal{F}(b - e_j; X) - \mathcal{F}(b - e_i; X)] + \mathcal{F}_j(b; X) - \mathcal{F}_i(b; X) = 0 \quad (3.12)$$

and

$$(c - 1)(x^i_k - x^j_k)\mathcal{F}(b; X) = (c - 1)[x^i_k\mathcal{F}(b - e_j; X) - x^j_k\mathcal{F}(b - e_i; X)]$$

$$+ x^i_k\mathcal{F}_j(b; X) - x^j_k\mathcal{F}_i(b; X). \quad (3.13)$$

**REMARK.** (3.12) and (3.13) are generalizations of Exercise 5.9-6 in [6] with the latter being an extension of Zill’s identity for $R$-hypergeometric functions.

Since the proof of (3.12) and (3.13) follows the lines introduced in [6, p. 305], we omit further details.

**4. Auxiliary Results.**

Our first result reads as follows.

**PROPOSITION 4.1.** Let $p(x)$ be an affine function on $\mathbb{R}^s$. Then

$$p(x)M(x|b; X) = \sum_{i=0}^{n} w_i p(x^i)M(x|b + e_i; X) \quad (4.1)$$

provided the splines $M(x|b + e_i; X)$, $0 \leq i \leq n$, are continuous at $x \in \mathbb{R}^s$.

Here $w_i = b_i / c$, $i = 0, 1, \ldots, n$.

**Proof.** We need the following identity for the Dirichlet density function [6, (4.4-8)]:

$$t_i \phi_b(t) = w_i \phi_{b+e_i}(t) \quad (4.2)$$

($t \in E_n; 0 \leq i \leq n$). Since $t_0 + \cdots + t_n = 1$, (4.2) gives

$$\phi_b(t) = \sum_{i=0}^{n} w_i \phi_{b+e_i}(t).$$
Multiplying both sides by \( f(Xt) \) and next integrating over \( E_n \), we obtain by virtue of (2.2)

\[
\int_{\mathbb{R}^s} f(x)M(x|b; X)dx = \int_{\mathbb{R}^s} f(x) \sum_{i=0}^n w_i M(x|b + e_i; X)dx.
\]

Hence, (4.1) follows when \( p(x) = 1 \). To complete the proof we utilize (4.2) again. Multiplying both sides by \( x_l^i \) and next summing over \( i \), we obtain

\[
[(Xt)_l]\phi_b(t) = \sum_{i=0}^n w_i x_l^i \phi_{b+e_i}(t).
\]

Here \((Xt)_l\) denotes the \( l \)th component of \( Xt \). This leads to the following integral relation:

\[
\int_{\mathbb{R}^s} f(x)x_l M(x|b; X)dx = \int_{\mathbb{R}^s} f(x) \left[ \sum_{i=0}^n w_i x_l^i M(x|b + e_i; X) \right] dx,
\]

which proves (4.1) when \( p(x) = x_l, \ (1 \leq l \leq s) \).

Micchelli [22] gave a different proof of (4.1) for simplex splines. For this class of splines, identity (4.1) is called the “degree elevating formula”. A special case of (4.1) appears in [14].

For our further aims, we recall a definition of the \( R \)-hypergeometric function in the real case. Let the set \( Z = \{z_0, \ldots, z_n\} \subset \mathbb{R}^{n+1} \) be such that \( 0 \notin [Z] \). Further, let \( b \in \mathbb{R}_+^{n+1} \). The \( R \)-hypergeometric function \( R_{-a}(b; Z) \), \((a \in \mathbb{R})\), is given by

\[
R_{-a}(b; Z) = \int_{E_n} (Zt)^{-a} \phi_b(t)dt \tag{4.3}
\]

(see [6]). When \(-a \in \mathbb{N}\), the restriction \( 0 \notin [Z] \) can be dropped. Comparison with (3.1) shows that the \( R_{-a} \) is the Dirichlet average of the power function \( u^{-a} \). It is worthy to mention that the Gauss hypergeometric function \( _2F_1 \), Lauricella’s hypergeometric function \( F_D \), the Gegenbauer polynomials, and
the elliptic integrals in the Legendre form can all be represented in terms of
the function $R_{-a}$. Combining (4.3) and (2.3) gives

$$R_{-a}(b; Z) = \int_{\mathbb{R}} u^{-a}m(u|b; Z)du.$$  \hfill (4.4)

For future use, let us record a very useful formula for $R$-hypergeometric functions (see [6], Thm. 6.8–3)

$$R_{-a}(b; Z) = \prod_{j=0}^{n} z_j^{-b_j} R_{a-c}(b; Z^{-1}),$$  \hfill (4.5)

where $Z^{-1} := \{z_0^{-1}, \ldots, z_n^{-1}\}$ ($z_j > 0$, for all $j; c \neq 0, -1, -2, \ldots$). This important result is commonly referred to as Euler’s transformation.

We close this section with the following:

**Proposition 4.2.** Let $a \in \mathbb{R}$ and let the vector $\lambda \in \mathbb{R}^s$ be such that $\lambda \cdot x^j < 1$, $j = 0, \ldots, n$. Then

$$\int_{\mathbb{R}^s} (1 - \lambda \cdot x)^{-a} M(x|b; X) dx = R_{-a}(b; Y),$$  \hfill (4.6)

where

$$Y = 1 - \lambda \cdot X = \{(1 - \lambda \cdot x^0), \ldots, (1 - \lambda \cdot x^n)\}. \hfill (4.7)$$

**Proof.** Substituting $Z = Y$ into (4.4) we obtain

$$R_{-a}(b; Y) = \int_{\mathbb{R}} u^{-a}m(u|b; 1 - \lambda \cdot X)du$$

$$= \int_{\mathbb{R}} (1 - u)^{-a}m(u|b; \lambda \cdot X)$$

$$= \int_{\mathbb{R}^s} (1 - \lambda \cdot X)^{-a} M(x|b; X) dx.$$  \hfill \(\square\)
When \( a = c \), (4.6) becomes Watson’s identity (see [31])

\[
\int_{\mathbb{R}^s} (1 - \lambda \cdot x)^{-c} M(x|b; X) dx = \prod_{j=0}^{n} (1 - \lambda \cdot x^j)^{-b_j}.
\]  

(4.8)

The above identity follows by applying (4.5) to the right side of (4.6) and using \( R_0 = 1 \).

An alternative proof of (4.8) appears in [13] (see also [19] for some comments concerning this identity). Subject: file 3

5. Moments of Multivariate Dirichlet Splines.

A motivation for the investigation of the moments of Dirichlet splines has its origin in two mathematical disciplines. It is well known that the spline \( M(\cdot|b; X) \) is a probability density function on \( \mathbb{R}^s \). We feel that the results of this section can be applied to some problems in mathematical statistics.

A second area of possible applications is the theory of special functions. We have already mentioned that some important special functions can be represented by the \( R \)-hypergeometric functions. For particular values of the \( a \) parameter and the \( b \) parameters in (4.4), this function becomes a complete symmetric function. For particular values of the \( z \)-variables, (4.4) gives an integral formula for the \( q \)-binomial coefficients (Gaussian polynomials). (See [25] for more details).

In this section, we derive recurrence formulas for the moments of multivariate Dirichlet splines. Also, we discuss implementation of these results in the case when \( b = (1, \ldots, 1) \in \mathbb{R}^{n+1} \). For related results when \( s = 1 \), see [24]. We employ the multi-index notation introduced in Section 2.

For \( \beta \in \mathbb{R}^s \), we define the moment of order \(|\beta|\) \((|\beta| = \beta_1 + \cdots + \beta_s)\) of \( M(\cdot|b; X) \) as follows:

\[
m_\beta(b; X) = \int_{\mathbb{R}^s} x^\beta M(x|b; X) dx
\]

(5.1)
provided \( 0_s \notin [X], 0_s \) - the origin in \( \mathbb{R}^s \). When \( \beta \in \mathbb{Z}_+^s \), this restriction is nonessential. In the case of the simplex spline, we shall omit the vector \( b \) and write \( m_\beta(X) \) instead of \( m_\beta(b; X) \). Also, let \( d_l \) stand for the \( l \)th coordinate vector in \( \mathbb{R}^s \).

We are now ready to state and prove the following.

**Theorem 5.1.** Let the weights \( w_0, \ldots, w_n \) be the same as in Proposition 4.1. Then

\[
m_\beta(b; X) = \sum_{i=0}^{n} w_i m_\beta(b + e_i; X),
\]

(5.2)

\[
m_{\beta + d_l}(b; X) = \sum_{i=0}^{n} w_i x_i^l m_\beta(b + e_i; X),
\]

(5.3)

for all \( l = 1, 2, \ldots, s \). Moreover, if \( \text{vol}_s([X]) > 0 \) and \( b_j \geq 1 \), for some \( 0 \leq j \leq n \), then

\[
(c + |\beta| - 1)m_\beta(b; X) = (c - 1)m_\beta(b - e_j; X) + \sum_{l=1}^{s} \beta_l x_i^l m_{\beta - d_l}(b; X).
\]

(5.4)

If for some \( 0 \leq i, j \leq n, 1 \leq k \leq s, x_k^i \neq 0, x_k^j \neq 0 \), and \( b_i \geq 1, b_j \geq 1 \), then

\[
(c - 1)[m_\beta(b - e_j; X) - m_\beta(b - e_i; X)] + \sum_{k=1}^{s} \beta_k (x_k^j - x_k^i) m_{\beta - d_k}(b; X) = 0
\]

(5.5)

and

\[
(c + |\beta| - 1)(x_k^i - x_k^j) m_\beta(b; X) = (c - 1)[x_k^i m_\beta(b - e_j; X) - x_k^j m_\beta(b - e_i; X)]
\]

\[
+ \sum_{l=1}^{s} \beta_l W_{k,l} m_{\beta - d_l}(b; X),
\]

(5.6)

where

\[
W_{k,l} = \det \begin{bmatrix} x_k^i & x_k^j \\ x_l^i & x_l^j \end{bmatrix}.
\]

**Remark.** When \( b_j = 1 \), formula (5.4) holds true provided \( n > s \).

**Proof.** In order to establish the recursion (5.2) and (5.3), we substitute \( p(x) = 1 \) and \( p(x) = x_l \) respectively, into (4.1) and next integrate over \( \mathbb{R}^s \).
For the proof of (5.4), we utilize formula (3.7) with \( f(x) = x^\beta \). The resulting equation, together with (3.6) and (5.1), yields the assertion. Formulas (5.5) and (5.6) follow immediately from Corollary 3.2 with \( f(x) = x^\beta \). □

We now give two moment generating functions. The first generating function involves the confluent hypergeometric function \( S \). Following [6, (5.8-1)], we define

\[
S(b; Z) = \int_{E_n} \exp(Zt)\phi_b(t) \, dt,
\]

(5.7)

\( b \in \mathbb{R}^{n+1}_+, Z = \{z_0, \ldots, z_n\} \). Use of (3.1) and (3.2) gives

\[
S(b; Z) = \int_{\mathbb{R}} \exp(u)m(u|b; Z) \, du.
\]

Letting \( Z = \lambda \cdot X = \{\lambda \cdot x^0, \ldots, \lambda \cdot x^n\}, \lambda \in \mathbb{R}^s \setminus \{0\} \), and next using (3.10), we arrive at

\[
S(b; \lambda \cdot X) = \int_{\mathbb{R}^s} \exp(\lambda \cdot x)M(x|b; X) \, dx.
\]

(5.8)

To obtain the first moment generating function, we expand \( \exp(\lambda \cdot x) \) into a power series. Applying the multinomial theorem to powers of \( \lambda \cdot x \) and next integrating the corresponding power series one term at a time, we obtain by virtue of (5.8) and (5.1)

\[
S(b; \lambda \cdot X) = \sum \lambda^j \frac{m_j(b; X)}{j!}
\]

(5.9)

where the summation extends over all multi-indices \( j \in \mathbb{Z}^s_+ \).

It is worthy to mention that the hypergeometric function \( S(b; \lambda \cdot X) \) can be expressed as a divided difference of \( \exp(z) \) provided that \( b_0, \ldots, b_n \in \mathbb{Z}_+ \). We have,

\[
k! S(b; \lambda \cdot X) = [\lambda \cdot x^0(b_0), \ldots, \lambda \cdot x^n(b_n)]e^z
\]

(5.10)
where $k = b_0 + \cdots b_n - 1$. Here the symbol $\lambda \cdot x^i(b_i)$ means that the knot set $\lambda \cdot x^i$ is repeated $b_i$ times. Formula (5.10) readily follows from (5.8), (5.7), and the Hermite-Genocchi formula for divided differences.

A second generating function is given by

$$ R_{-a}(b; Y) = \sum_{\lambda \cdot x^i} \left( a, |j| \right) \frac{\lambda^j}{j!} m_j(b; X) \quad (5.11) $$

(|$\lambda \cdot x^i$| < 1, for all $i$), where the summation extends over all multi-indices $j \in \mathbb{Z}^+$. Here, $a \in \mathbb{R}$, the set $Y$ is given in (4.7), and $(a, l)$ stands for the Appell symbol, i.e., $(a, 0) = 1$, $(a, l) = a(a+1) \cdots (a+l-1)$, $l \in \mathbb{N}$. In order to establish (5.11) we expand $(1 - \lambda \cdot x)^{-a}$ into a power series and next utilize the multinomial theorem to obtain

$$ (1 - \lambda \cdot x)^{-a} = \sum_{l=0}^{\infty} (a, l) \sum_{|j|=l} \frac{\lambda^j}{j!} x^j \quad (5.12) $$

To complete the proof we substitute (5.12) into (4.6) and next integrate term by term. Applications of (5.9) and (5.11) are discussed in the next section.

We shall now turn our attention to the case of multivariate simplex splines. To this end, let $\beta \in \mathbb{Z}^+$. In the case under discussion, the formulas of (5.3) and (5.4) of Theorem 5.1 take the form

$$ m_{\beta+d_l}(X) = \frac{1}{n+1} \sum_{i=0}^{n} x^i_m \beta(X^i) \quad (5.13) $$

$$(n + |\beta|) m_{\beta}(X) = n[m_{\beta}(X_j)] + \sum_{l=1}^{s} \beta_l x^l m_{\beta-d_l}(X), \quad (5.14)$$

$1 \leq j \leq n; 1 \leq l \leq s$. Here $X^i = X \cup \{x^i\}$, and $X_i = X \backslash \{x^i\}$, $0 \leq i \leq n$. The set $X^i$ appears on the right hand side of (5.13) because of (3.5). A closer look at (5.14) shows that the recursion is in two directions. That is, given $X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^s$ ($n > s$), to compute $m_{\beta}(X)$, we need the moment
of order $|\beta|$ for the knot set consisting of one less vector than $X$, and also $s$ moments of order $|\beta| - 1$.

Define a set $\overline{X}_k = \{x^0, \ldots, x^k\}$, $k = s, s+1, \ldots, n$, and note $m_\beta(\overline{X}_k) = 1$ when $|\beta| = 0$, $k = s, s+1, \ldots, n$. To employ (5.14), we must precompute certain moments of the form $m_\beta(\overline{X}_s)$, $|\beta| > 0$, and $m_{dl}(\overline{X}_k)$, $k = s, \ldots, n$. Let us note that

$$m_{dl}(\overline{X}_k) = \frac{1}{k+1} \sum_{j=0}^{k} x^j$$

follows immediately from the defining equation (2.2).

In order to compute the moments $m_\beta(\overline{X}_s)$, $|\beta| > 0$, we first introduce some new notation and next appeal to the proposition that follows.

Let $t = (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1}$ with $(t_1, \ldots, t_n) \in E_n$, and $t_0 = 1 - \sum_{i=1}^{n} t_i$. Also, let $l = (l_0, \ldots, l_n) \in \mathbb{Z}_+^{n+1}$, with $m = |l|$. Then we define the Bernstein polynomial by

$$B_l^m(t) = \binom{m}{l} t^l.$$

For given coefficients $\{p_l\}$, $|l| = m$, we shall call any polynomial of the form

$$q(t) = \sum_{|l|=m} p_l B_l^m(t)$$

a Bézier polynomial. It is well known that any such $q$ may be stably and efficiently evaluated using deCasteljau’s algorithm (see [2],[12]).

Proposition 5.2 illustrates that we may indeed incorporate deCasteljau’s algorithm when evaluating $m_\beta(\overline{X}_s)$.
PROPOSITION 5.2. Let \( n \geq s \geq 1, \beta \in \mathbb{Z}_+^s \), and \( X = \{x_0, \ldots, x_n\} \subset \mathbb{R}_+^s \) with \( \text{vol}_s([X]) > 0 \). Let \( y^i = (x_0^i, \ldots, x_n^i) \in \mathbb{R}^{n+1}_+ \) and set \( g_i = |y^i| \), \( i = 1, \ldots, s \). Then
\[
\text{m}_\beta(X) = \frac{g^\beta n!}{(|\beta| + n)!} \sum_{|k_1| = \beta_1} B_{k_1} \beta_1 (\tilde{y}_1) \cdots \sum_{|k_s| = \beta_s} B_{k_s} \beta_s (\tilde{y}_s) \eta!, \quad (5.15)
\]
where \( k_i = (k_{i0}, \ldots, k_{in}) \in \mathbb{Z}_+^{n+1} \), \( \eta! = \eta_0! \cdots \eta_n! \), with \( \eta_j = \sum_{i=1}^s k_{ij} \), \( 0 \leq j \leq n \), \( \tilde{y}^i = (1/g_i) y^i \), \( i = 1, \ldots, s \).
\( g = (g_0, \ldots, g_n) \in \mathbb{R}_+^{s+1} \).

Proof. We use the defining relation (2.2) to write:
\[
\text{m}_\beta(X) = n! \int_{E_n} \prod_{j=1}^s (t_0 x_0^j + \cdots + t_n x_n^j)^{\beta_j} \, dt
= n! \sum_{|k_1| = \beta_1} \left( \frac{\beta_1}{k_1} \right) (y^1)^{k_1} \cdots \sum_{|k_s| = \beta_s} \left( \frac{\beta_s}{k_s} \right) (y^s)^{k_s} \int_{E_n} t^n \, dt,
\]
where \( \eta, y^i, k_i, i = 1, \ldots, s \), are given in Proposition 5.2. Using 4.3-4 in [6] to simplify the integral in the above identity, we have
\[
\text{m}_\beta(X) = n! \sum_{|k_1| = \beta_1} \left( \frac{\beta_1}{k_1} \right) (y^1)^{k_1} \cdots \sum_{|k_s| = \beta_s} \left( \frac{\beta_s}{k_s} \right) (y^s)^{k_s} \frac{\eta!}{(|\beta| + n)!}, \quad (5.16)
\]
Now \( \tilde{y}^j, j = 1, \ldots, s \), given in Proposition 5.2 can be viewed as the barycentric coordinates of some point in \( \mathbb{R}^n \). Inserting \( \tilde{y}^j \)'s into (5.16) and scaling by \( g^\beta \) gives the desired result. \( \Box \)

Thus all moments of the form \( \text{m}_\beta(X_s) \) can be expressed as a nested sum of Bézier polynomials and subsequently may be evaluated using de-Casteljau’s algorithm. It should be noted that while both de Casteljau’s algorithm and (5.14) are possible candidates for the task of computing \( \text{m}_\beta(X_k) \), \( k = s + 1, \ldots, n \), the latter scheme requires the evaluation of fewer terms at
each recursion step and is thus the preferred choice. In order to summarize
the procedure for evaluating the moments of simplex splines in Algorithm
5.3, we introduce \( d_0 = (0, \ldots, 0) \in \mathbb{R}^s \).

**Algorithm 5.3.**

Given \( X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^s \), \((n \geq s)\) and \( \beta \in \mathbb{Z}_+^s \) with \(|\beta| > 1\), this
algorithm generates the moment \( m_{\beta}(X) \) of the simplex spline \( M(\cdot|X) \).

1. \( \alpha := d_0 \)
2. For \( k = s \) to \( n \)
   \[ \mapsto m_\alpha(X_k) = 1 \]
3. For \(|\alpha| = 1\) to \(|\beta|\), \( \alpha \in \mathbb{Z}_+^s \), \( \alpha \leq \beta \)
   \[ \mapsto \text{Use (5.15) to express } m_\alpha(X_s) \text{ in terms of Bézier polynomials and evaluate using DeCasteljau’s algorithm.} \]
4. For \( k = s + 1 \) to \( n \)
   \[ \mapsto \text{For } \alpha \in \mathbb{Z}_+^s, \alpha \leq \beta \]
   \[ \mapsto \text{Compute } m_\alpha(X_k) \text{ using (5.14).} \]

We close this section with a remark that this algorithm is numerically
stable if \( x^j > 0 \) for all \( j = 0,1,\ldots,n \).

**6. Applications to Hypergeometric Functions.**

In this section we demonstrate a relationship between Dirichlet splines
and an important class of hypergeometric functions of several variables. We
will deal mainly with Appell’s \( F_4 \) and Lauricella’s \( F_B \). The link between
these classes of functions is provided by another integral average which is
commonly referred to as a double Dirichlet average (see [4] for more de-
tails). Throughout the sequel the double Dirichlet average of a continuous
univariate function \( h \) will be denoted by \( \mathcal{H} \).

Let \( X \in \mathbb{R}^{s \times (n+1)}(n \geq s \geq 1) \). Further, let \( u = (u_1,\ldots,u_s) \) be an
ordered \( s \)-tuple of nonnegative numbers with \( u_1 + \cdots + u_s = 1 \), and similarly
\( v = (v_0, \ldots, v_n) \). We define
\[
u \cdot X v = \sum_{i=1}^{s} \sum_{j=0}^{n} u_i x_i^j v_j,
\]
where \( x_i^j \) stands for the \( i \)-th component of the \( j \)-th column of \( X \). Let \( h \) be a continuous function on \( I = [\text{Min } x_i^j, \text{Max } x_i^j] \). In order to avoid trivialities, we will assume that \( I \) has a nonempty interior. For \( \mathbf{b} = (b_1, \ldots, b_s) \in \mathbb{R}^s \) and \( \mathbf{d} = (d_0, \ldots, d_n) \in \mathbb{R}^{n+1}_+ \), let [4, p. 421]
\[
H(b; X; d) = \int_{E_n} \int_{E_{s-1}} h(u \cdot X v) \phi_b(u) \phi_d(v) \, du \, dv,
\]
\[ du = du_2 \cdots du_s, \, dv = dv_1 \cdots dv_n. \]
Here \( \phi_b \) and \( \phi_d \) are the Dirichlet densities on \( E_{s-1} \) and \( E_n \), respectively (see (2.1)). It is known that for \( \mathbf{b} \in \mathbb{R}^s_+ \),
\[
\mathcal{H}(b; X; d) = \int_{E_{s-1}} H(d; u \cdot X) \phi_b(u) \, du
\]
(see [4, (2.8)]). In (6.1) \( H \) stands for the single Dirichlet average of \( h \) (see (3.1)), \( u \cdot X = \{u \cdot x^0, \ldots, u \cdot x^n\} \), \( x^0, \ldots, x^n \) - the columns of \( X \).

We are in a position to state and prove the following:

**Theorem 6.1.** Let \( \mathbf{d} \in \mathbb{R}^{n+1}_+ \) and let the vector \( \mathbf{b} \in \mathbb{R}^s \) be such that \( c \neq 0, -1, \ldots, (c = b_1 + \cdots + b_s) \). If \( \text{vol}_s([X]) > 0 \), then
\[
\mathcal{H}(b; X; d) = \int_{[X]} M(x|d; X) H(b; x) \, dx,
\]
\[ x = (x_1, \ldots, x_s), \, dx = dx_1 \cdots dx_s. \]

**Proof.** In order to establish (6.2) assume for the moment that \( \mathbf{b} \in \mathbb{R}^s_+ \). Application of (3.2) and (3.10) to (6.1) gives
\[
\mathcal{H}(b; X; d) = \int_{E_{s-1}} \left[ \int_{[X]} h(u \cdot x) M(x|d; X) \, dx \right] \phi_b(u) \, du.
\]
Interchanging the order of integration and next using (3.1), we obtain the assertion provided $b \in \mathbb{R}_{>s}$. This restriction can be dropped because the average $H$ can be continued analytically in the $b$-parameters provided that $c \neq 0, -1, \ldots$, (see [6, Thm. 6.3–7]). This completes the proof. □

Before we state a corollary of Theorem 6.1 let us introduce more notation. For $h(z) = z^{-a}$, $(a \in \mathbb{R})$, the double Dirichlet average of $h$ will be denoted by $\mathcal{R}_{-a}$ (cf. [4]).

**COROLLARY 6.2.** ([7]) Let $d \in \mathbb{R}_{>s}^{n+1}$, $b \in \mathbb{R}^s$, and let the matrix $X$ be such that $0_s \notin [X]$. Then

$$m_{-b}(d; X) = \mathcal{R}_{-c}(b; X; d), \quad (6.3)$$

where $m_{-b}(d; X)$ stands for the moment of order $-c$ of the Dirichlet spline $M(\cdot|d; X)$.

**Proof.** Apply [6,(6.6-5)]

$$R_{-c}(b; X) = \prod_{i=1}^s x_i^{-b_i} \quad (6.4)$$

to (6.2) with $h(t) = t^{-c}$. □

Hereafter, we will deal with the hypergeometric functions and polynomials of several variables. Appell’s hypergeometric function $F_4$ is defined by the double power series [6, Ex. 6.3-5]

$$F_4(\alpha, \beta; \gamma, \delta; x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha, i + j)(\beta, i + j)}{(\gamma, i)(\delta, i)i!j!} x_1^i x_2^j,$$

$\alpha, \beta, \gamma, \delta \in \mathbb{R}, \gamma, \delta \neq 0, -1, \ldots, |x_1|^{\frac{1}{2}} + |x_2|^{\frac{1}{2}} < 1$. The following integral formula [5, p.963]

$$F_4(\alpha, \beta; \gamma, \delta; x_1(1 - x_2), x_2(1 - x_1)) =$$

$$\int_0^1 R_{-\alpha}(d_0, d_1, d_2; u \cdot x^0, u \cdot x^1, u \cdot x^2) \phi_b(u) \, du \quad (6.5)$$
provides the analytic continuation of the $F_4$-series to the region $\Lambda$ defined by

$$\Lambda = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1, x_2 < 1, x_1 + x_2 < 1\}.$$  

In (6.5), $b = (\beta, \gamma - \beta)$, $d_0 = \gamma + \delta - \alpha - 1$, $d_1 = \alpha + \beta - \gamma - \delta + 1$, $d_2 = \delta - \beta$, $\phi_b(u)$ is the Dirichlet density on $E_1$, and $x^0$, $x^1$, and $x^2$ are the columns of $X$, where

$$X = \begin{bmatrix}
(1 - x_1)(1 - x_2) & 1 - x_1 - x_2 & 1 - x_1 \\
1 - x_2 & 1 - x_2 & 1
\end{bmatrix}$$  \hspace{1cm} (6.6)

**Corollary 6.3.** Let $d = (d_0, d_1, d_2) \in \mathbb{R}_+^3$ and let $b = (\beta, \gamma - \beta) \in \mathbb{R}^2$. If $\text{vol}_2([X]) > 0$, then

$$F_4(\alpha; \beta; \gamma, \delta; x_1(1 - x_2), x_2(1 - x_1)) = \int_{[X]} M(y|d; X)R_{-\alpha}(b; y) \, dy$$  \hspace{1cm} (6.7)

(y = (y_1, y_2), dy = dy_1dy_2). Here, $R_{-\alpha}$ is the single Dirichlet average of $h(z) = z^{-\alpha}$ and the matrix $X$ is given in (6.6).

**Proof.** Apply (6.1) to (6.5) and next use (6.2). \hfill \Box

A special case of (6.7) is

$$F_4(\alpha, \beta; \gamma, \delta; x_1(1 - x_2), x_2(1 - x_1)) = m_{-b}(d; X),$$

where now $b = (\beta, \alpha - \beta)$ and $d = (\delta - 1, \beta - \delta + 1, \delta - \beta)$. This follows immediately from (6.7) and (6.4).

We will now deal with Lauricella’s $F_B$ function and Lauricella polynomials. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$ ($\gamma \neq 0, -1, \ldots$), and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, with $|x_i| < 1$, for all $i$. Following [21] we define

$$F_B(\alpha, \beta; \gamma; x) = \sum \frac{(\alpha, k)(\beta, k)}{(\gamma, |k|)k!} x^k,$$  \hspace{1cm} (6.8)
where the summation extends over all multi-indices \( k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n \).

In (6.8) we employ multi-index notation introduced in Section 2. Also,

\[
(\alpha, k) = \prod_{i=1}^{n} (\alpha_i, k_i).
\]

\((\beta, k)\) is defined in an analogous manner. When \( n = 1 \), \( F_B \) becomes Gauss’ \( _2F_1 \) function.

**COROLLARY 6.4.** Let \( d = (\beta, \gamma - |\beta|) \in \mathbb{R}^{n+1}_+ \) and let

\[
X = \begin{bmatrix}
1 - x_1 & 1 & \cdots & 1 & 1 \\
1 & 1 - x_2 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 - x_n & 1
\end{bmatrix}
\] (6.9)

Then

\[
F_B(\alpha, \beta; \gamma; x) = m_{-\alpha}(d; X)
\] (6.10)

provided that \( x_i < 1 \) for all \( i \).

**Proof.** In the stated domain the entries of \( X \) are positive. Thus \( 0 \notin [X] \).

In order to establish (6.10) we utilize \([4,(5.11)]\) to obtain

\[
F_B(\alpha; \beta; \gamma; x) = R_{-c}(\alpha; X; d),
\]

where now \( c = \alpha_1 + \cdots + \alpha_n \). This in conjunction with (6.3) gives the assertion. \(\square\)

Lauricella polynomials \( L_j(x) \), \((j \in \mathbb{Z}_+^n, x \in \mathbb{R}^n)\) are defined in the following way:

\[
L_j(x) = F_B(-j; \beta; \gamma; x).
\]

These polynomials play an important role in the study of coherent states (cf. [16]). On account of (6.10)

\[
L_j(x) = m_j(d; X)
\] (6.11)
where the vector $d$ and the matrix $X$ are the same as in Corollary 6.4.

Two generating functions for the polynomials under discussion can be derived from (5.9) and (5.11). Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $e = (1, \ldots, 1)$, $\lambda, e \in \mathbb{R}^n$, and let $d = (\beta, \gamma - |\beta|) \in \mathbb{R}^{n+1}$. Then

$$\exp(\lambda \cdot e)S(d; -\lambda_1 x_1, \ldots, -\lambda_n x_n, 0) = \sum \frac{\lambda^j}{j!} L_j(x)$$

$j \in \mathbb{Z}_+^n$. If

$$\text{Max}\{|\lambda \cdot e - \lambda_1 x_1|, \ldots, |\lambda \cdot e - \lambda_n x_n|, |\lambda \cdot e|\} < 1,$$

then

$$R_{-a}(d; Y) = \sum \frac{\lambda^j}{j!} (a, |j|)_j L_j(x),$$

$a \in \mathbb{R}$, $j \in \mathbb{Z}_+^n$, $Y = \{1 - \lambda \cdot e + \lambda_1 x_1, \ldots, 1 - \lambda \cdot e + \lambda_n x_n, 1 - \lambda \cdot e\}$.

For the proof of (6.12) we replace $b$ by $d$ in (5.9) and next use (6.11), (6.9), and [6,(5.8-3)]. This gives the desired result provided that $d \in \mathbb{R}^{n+1}$. The latter restriction can be dropped because the $S-$ function can be continued analytically in the $d$-parameters ([6,Corollary 6.3-3]). The generating function (6.13) can be derived from (5.11) by the same means. Feinsilver’s generating function [15] can be obtained from (6.13) by letting $a = \gamma$ and then using (4.5). Recall that $R_0 = 1$. It is not hard to show that the $R$-hypergeometric function $R_{-a}$ in (6.13) is a multiple of the Lauricella function of the fourth kind. We have

$$R_{-a}(d; Y) = (1 - \lambda \cdot e)^{-a} F_D(a, \beta; \gamma; z_1, \ldots, z_n),$$

where

$$z_i = \frac{-\lambda_i x_i}{1 - \lambda \cdot e},$$

$i = 1, 2, \ldots, n$. We omit further details.
Our next goal is to establish a recurrence formula obeyed by Lauricella polynomials

\[(\gamma + |k|)L_{k+d_m}(x) - [\gamma(1 - w_m x_m) + |k|]L_k(x)\]

\[+ \sum_{l=1}^{n} k_l \epsilon_{lm} [L_{k-d_l}(x) - L_{k-d_l+d_m}(x)] = 0\]  \hspace{1cm} (6.14)

\[m = 1, 2, \ldots, n.\] In (6.14), \(\gamma \in \mathbb{R}_+\), \(k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n\), \(d_m\) stands for the \(m\)th coordinate vector in \(\mathbb{R}^n\), similarly \(d_l\), \(w_m = \frac{\beta_m}{\gamma}; \beta_m > 0, 1 \leq m \leq n,\)

\[\epsilon_{lm} = \begin{cases} 
1 & \text{if } l \neq m \\
1 - x_m & \text{if } l = m. 
\end{cases}\]  \hspace{1cm} (6.15)

Here we adopt the convention that \(L_k(x) = 0\) if \(-k_m \in \mathbb{N}\) for some \(m\). In order to establish the recursion (6.14) we derive first a recurrence formula for the moments of multivariate Dirichlet splines with \(s = n\) and \(X\) given in (6.9). We have

\[(c + |\beta|)m_{\beta+d_m}(b; X) - [c(1 - w_m x_m) + |\beta|]m_{\beta}(b; X)\]

\[+ \sum_{l=1}^{n} \beta_l \epsilon_{lm} [m_{\beta-d_l}(b; X) - m_{\beta-d_l+d_m}(b; X)] = 0,\]  \hspace{1cm} (6.16)

\[b \in \mathbb{R}^{n+1}, c = b_1 + \cdots + b_{n+1}, \beta \in \mathbb{R}^n, |\beta| = \beta_1 + \cdots + \beta_n.\] The recursion (6.14) now follows from (6.16) by letting \(b = d, \beta = k \in \mathbb{Z}_+^n\), and using (6.11).

To complete the proof we need to establish (6.16). To this aim we increase the indices of summation in (5.2) and (5.3) by one unit. Next we let \(s = n\) and solve the resulting linear system for \(m_{\beta}(b + e_m; X), 1 \leq m \leq n + 1.\)

Let us note that the assumption \(vol_n([X]) > 0\) is equivalent to \(x_m \neq 0, 1 \leq m \leq n.\) This assures uniqueness of the solution. Subtracting (5.3) from (5.2) we obtain

\[m_{\beta}(b + e_m; X) = \frac{[m_{\beta}(b; X) - m_{\beta+d_m}(b; X)]}{w_m x_m},\]  \hspace{1cm} (6.17)
$1 \leq m \leq n$. The remaining moment $m_\beta(b + e_{n+1}; X)$ can be found using (5.2) and (6.17). To complete the proof of (6.16) we utilize (5.4). Replacing the index $j$ by $m$ and next using (6.17), we can easily obtain the assertion.

We close this section with an inequality for Lauricella polynomials. To this end, let $x_i < 1$, $1 \leq i \leq n$. It follows from (6.9) that $[X] \subset \mathbb{R}^n_+$ in the stated domain. This in conjunction with (6.11) and (5.1) provides

$$L_j(x) = \int_{[X]} y^j M(y|d; X) \, dy > 0$$

$(j \in \mathbb{Z}^n_+, d = (\beta, \gamma - |\beta|) \in \mathbb{R}^{n+1}_+, y = (y_1, \ldots, y_n), dy = dy_1, \ldots dy_n)$. A standard argument applied to the last formula gives

$$[L_j(x)]^2 \leq L_{j-k}(x) L_{j+k}(x),$$

where the vector $k \in \mathbb{Z}^n_+$ is such that $j - k \in \mathbb{Z}^n_+$. In particular, if $k = e_m$ – the $m$th coordinate vector in $\mathbb{R}^n$, then

$$[L_j(x)]^2 \leq L_{j-e_m}(x) L_{j+e_m}(x),$$

provided $j - e_m \in \mathbb{Z}^n_+$. Thus the function $g : \mathbb{Z}^n_+ \to \mathbb{R}$, where

$$g(j_1, \ldots, j_n) = L_{j_1, \ldots, j_n}$$

is log-convex in each variable separately.

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LIST OF SYMBOLS

“Moments of Dirichlet Splines and Their Applications to Hypergeometric Functions”

by

Edward Neuman and Patrick J. Van Fleet

\( \mathbb{R} \) upper case open-face Roman R
\( \geq \) greater than or equal to
\( \in \) is a member of
\( \sum \) summation sign
\( \subset \) is a subset of
\( \mathbb{N} \) upper case open-face Roman N
\( \mathbb{Z} \) upper case open-face Roman Z
\( \beta \) lower case Greek beta
\( ! \) factorial
\( \alpha \) lower case Greek alpha
\( \leq \) less than or equal to
\( > \) greater than
\( \phi \) lower case Greek phi
\( \prod \) product
\( \Gamma \) upper case Greek gamma
\( \int \) integral sign
\( \mathcal{F} \) upper case script F
\( \mathcal{H} \) upper case script H
\( \mathcal{R} \) upper case script R
\( \epsilon \) lower case Greek epsilon
\( \lambda \) lower case Greek lambda
\( \gamma \) lower case Greek gamma
\( \delta \) lower case Greek delta
\( \frac{\partial}{\partial x} \) partial derivative
\( \Lambda \) upper case Greek lambda
\( \eta \) lower case Greek eta