LONG-TIME BEHAVIOUR FOR A NON-AUTONOMOUS KLEIN-GORDON-ZAKHAROV SYSTEM

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Abstract. The aim of this paper is to study the long-time dynamics of solutions of the evolution system
\[
\begin{align*}
\frac{u_{tt}}{t} - \Delta u + u + \eta (-\Delta)^{\frac{1}{2}} u_t + a_{\varepsilon}(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), & \quad (x, t) \in \Omega \times (\tau, \infty), \\
\frac{v_{tt}}{t} - \Delta v + \eta (-\Delta)^{\frac{1}{2}} v_t - a_{\varepsilon}(t)(-\Delta)^{\frac{1}{2}} u_t = 0, & \quad (x, t) \in \Omega \times (\tau, \infty),
\end{align*}
\]
subject to boundary conditions
\[
u = v = 0, \quad (x, t) \in \partial \Omega \times (\tau, \infty),
\]
where \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^n\), \(n \geq 3\), with the boundary \(\partial \Omega\) assumed to be regular enough, \(\eta > 0\) is constant, \(a_{\varepsilon}\) is a Hölder continuous function and \(f\) is a dissipative nonlinearity. This problem is a non-autonomous version of the well known Klein-Gordon-Zakharov system. Using the uniform sectorial operators theory, we will show the local and global well-posedness of this problem in \(H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)\). Additionally, we prove existence, regularity and upper semicontinuity of pullback attractors.

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1. Introduction

In this paper, we study a non-autonomous version of the well known Klein-Gordon-Zakharov system. We consider the following initial-boundary value problem
\[
\begin{align*}
\frac{u_{tt}}{t} - \Delta u + u + \eta (-\Delta)^{\frac{1}{2}} u_t + a_{\varepsilon}(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), & \quad (x, t) \in \Omega \times (\tau, \infty), \\
\frac{v_{tt}}{t} - \Delta v + \eta (-\Delta)^{\frac{1}{2}} v_t - a_{\varepsilon}(t)(-\Delta)^{\frac{1}{2}} u_t = 0, & \quad (x, t) \in \Omega \times (\tau, \infty),
\end{align*}
\]
where \(\eta\) is a positive constant, subject to boundary conditions
\[
u = v = 0, \quad (x, t) \in \partial \Omega \times (\tau, \infty),
\]
and initial conditions
\[
\begin{align*}
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\end{align*}
\]
where \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^n\) with \(n \geq 3\), and the boundary \(\partial \Omega\) is assumed to be regular enough.

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In the case that \( \epsilon(t) \equiv a \), the system (1.1) represents the autonomous version of the Klein-Gordon-Zakharov system. Within the autonomous case, if \( n = 3 \) then the Klein-Gordon-Zakharov system arises to describe the interaction of a Langmuir wave and an ion acoustic wave in a plasma, see [4, 15, 24] and references therein.

These types of systems have been considered by many researchers in recent years. In what follows, we recall some related results for these kinds of systems. In [24], the authors considered the following system (in dimension 2 and 3)

\[
\begin{align*}
    u_{tt} - \Delta u + u + vu &= 0, \\
    v_{tt} - c_0^2 \Delta v &= \Delta(|u|^2),
\end{align*}
\]

and they proved instability of solutions in the sense that small perturbations of the initial data can make the perturbed solution blow up in finite time.

In [1], it is considered the following coupled system of wave equations:

\[
\begin{align*}
    u_{tt} - \Delta u + \int_0^{+\infty} g(s) \Delta u(t - s) ds + \alpha v &= 0, \\
    v_{tt} - \Delta v + \alpha u &= 0,
\end{align*}
\]

where the authors showed the dissipativeness of this system, and, moreover, they proved that the associated semigroup is not exponentially stable. Later in [20], the authors studied a more general and abstract version of the previous system presented in [1]. In fact, they obtained existence of solutions and an optimal energy decay estimate for the following coupled system of second order abstract evolution equations:

\[
\begin{align*}
    u_{tt}(t) + A_1 u(t) - \int_0^{+\infty} g(s) Au(t - s) ds + B v(t) &= 0, \\
    v_{tt}(t) + A_2 v(t) + B u(t) &= 0,
\end{align*}
\]

where \( A, A_1 \) and \( A_2 \) are positive self-adjoint linear operators in a Hilbert space \( H \), \( B \) is a positive self-adjoint bounded linear operator in \( H \), and \( g \) is a non-increasing function satisfying some properties. With this formulation, this system covers the well-known Timoshenko system, which appears in mechanics and thermoelasticity, and models the transverse vibrations of a beam.

For a deeper and more detailed discussion about systems consisting of wave equations and other types of physical models, we refer to [16], [17], [22], [23] and [27].

The main purpose of this paper is to show the global well-posedness and to study the long-time dynamics of solutions of the evolution system (1.1). In order to do that, we shall use the uniform sectorial operators theory to show the local and global well-posedness of system (1.1) and we will use the abstract evolution processes theory to prove existence, regularity and upper semicontinuity of pullback attractors.

We emphasize that, in general, to obtain existence of attractors we need some type of “dissipation” and “compactness” for the dynamical system associated with the problem. In the literature, for non-autonomous problems, the “compactness” is the so called pullback asymptotic compactness, and this is obtained by decomposing the nonlinear process into two parts, where one part decays to zero and the other one is compact. See [5], [6], [11], [8] and [9] for more details. However, in this paper, we establish the compactness of the nonlinear process in a direct way, see Proposition 5.4.

This paper is organized as follows: The main results are presented in Section 2. We recall some concepts associated to the theory of pullback attractors in Section 3. In Section 4 we obtain the global well-posedness of solutions. Section 5 is devoted to the
existence of pullback attractors. Regularity of pullback attractors is obtained in Section 6. Finally in Section 7 we study the upper semicontinuity of pullback attractors.

2. Main Results

In this section, we present the statement of the main results which will be proved in the next sections. We start by presenting the general conditions to obtain the local and global well-posedness of the problem (1.1) − (1.3) in some appropriate space which will be specified later. Assume that the function \( a_\epsilon : \mathbb{R} \to (0, \infty) \) is continuously differentiable in \( \mathbb{R} \) and satisfies the following condition:

\[
0 < a_0 \leq a_\epsilon(t) \leq a_1,
\]

for all \( \epsilon \in [0, 1] \) and \( t \in \mathbb{R} \), with positive constants \( a_0 \) and \( a_1 \), and we also assume that the first derivative of \( a_\epsilon \) is uniformly bounded in \( t \) and \( \epsilon \), that is, there exists a constant \( b_0 > 0 \) such that

\[
|a_\epsilon'(t)| \leq b_0 \quad \text{for all} \quad t \in \mathbb{R}, \quad \epsilon \in [0, 1].
\]

Furthermore, we assume that \( a_\epsilon \) is \((\beta, C)\)-Hölder continuous, for each \( \epsilon \in [0, 1] \); that is,

\[
|a_\epsilon(t) - a_\epsilon(s)| \leq C|t - s|^\beta
\]

for all \( t, s \in \mathbb{R} \) and \( \epsilon \in [0, 1] \). Concerning the nonlinearity \( f \), we assume that \( f \in C^1(\mathbb{R}) \) and it satisfies the dissipativeness condition

\[
\limsup_{|s| \to \infty} \frac{f(s)}{s} \leq 0,
\]

and also satisfies the subcritical growth condition given by

\[
|f'(s)| \leq c(1 + |s|^{p-1}),
\]

for all \( s \in \mathbb{R} \), where \( 1 < p < \frac{n}{n-2} \), with \( n \geq 3 \), and \( c > 0 \) is a constant.

In order to formulate the non-autonomous problem (1.1) − (1.3) in a nonlinear evolution process setting, we introduce some notations. Let \( X = L^2(\Omega) \) and denote by \( A : D(A) \subset X \to X \) the negative Laplacian operator, that is, \( Au = (-\Delta)u \) for all \( u \in D(A) \), where \( D(A) = H^2(\Omega) \cap H_0^1(\Omega) \). Thus \( A \) is a positive self-adjoint operator in \( X \) with compact resolvent and, therefore, \(-A\) generates a compact analytic semigroup on \( X \). Following Henry [19], \( A \) is a sectorial operator in \( X \). Now, denote by \( X^\alpha, \alpha > 0 \), the fractional power spaces associated with the operator \( A \); that is, \( X^\alpha = D(A^\alpha) \) endowed with the graph norm. With this notation, we have \( X^{-\alpha} = (X^\alpha)' \) for all \( \alpha > 0 \), see [2].

In this framework, the non-autonomous problem (1.1) − (1.3) can be rewritten as an ordinary differential equation in the following abstract form

\[
\begin{aligned}
W_t + A(t)W &= F(W), \quad t > \tau, \\
W(\tau) &= W_0, \quad \tau \in \mathbb{R},
\end{aligned}
\]

where \( W = W(t) \), for all \( t \in \mathbb{R} \), and \( W_0 = W(\tau) \) are respectively given by

\[
W = \begin{bmatrix} u \\ u_1 \\ v \\ v_1 \end{bmatrix} \quad \text{and} \quad W_0 = \begin{bmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \end{bmatrix},
\]
and, for each \( t \in \mathbb{R} \), the unbounded linear operator \( \mathcal{A}(t) : D(\mathcal{A}(t)) \subset Y \to Y \) is defined by

\[
(2.7) \quad \mathcal{A}(t) \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 0 & -I & 0 & 0 \\ A + I & \eta A^\frac{\alpha}{2} & 0 & a_\epsilon(t)A^\frac{\alpha}{2} \\ 0 & 0 & 0 & -I \\ 0 & -a_\epsilon(t)A^\frac{\alpha}{2} & A & \eta A^\frac{\alpha}{2} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} -v \\ (A + I)u + \eta A^\frac{\alpha}{2}v + a_\epsilon(t)A^\frac{\alpha}{2}z \\ -z \\ -a_\epsilon(t)A^\frac{\alpha}{2}v + Aw + \eta A^\frac{\alpha}{2}z \end{bmatrix}
\]

for each \( [u, v, w, z]^T \) in the domain \( D(\mathcal{A}(t)) \) defined by the space

\[
(2.8) \quad D(\mathcal{A}(t)) = X^1 \times X^\frac{\alpha}{2} \times X^1 \times X^\frac{\alpha}{2},
\]

where

\[
Y = Y_0 = X^\frac{\alpha}{2} \times X \times X^\frac{\alpha}{2} \times X
\]

is the phase space of the problem \((1.1) - (1.3)\). The nonlinearity \( F \) is given by

\[
(2.9) \quad F(W) = \begin{bmatrix} 0 \\ f^\epsilon(u) \\ 0 \\ 0 \end{bmatrix},
\]

where \( f^\epsilon(u) \) is the Nemitskiı̆ operator associated with \( f(u) \); that is,

\[
f^\epsilon(u)(x) = f(u(x)), \quad \text{for all } x \in \Omega.
\]

Now, we observe that the norms

\[
\|(x, y, z, w)\|_1 = \|x\|_{X^\frac{\alpha}{2}} + \|y\|_{X^\frac{\alpha}{2}} + \|z\|_{X^\frac{\alpha}{2}} + \|w\|_X
\]

and

\[
\|(x, y, z, w)\|_2 = (\|x\|^2_{X^\frac{\alpha}{2}} + \|y\|^2_{X^\frac{\alpha}{2}} + \|z\|^2_{X^\frac{\alpha}{2}} + \|w\|^2_X)^\frac{1}{2}
\]

are equivalent in \( Y_0 \). In this way, we shall use the same notation \( \|(x, y, z, w)\|_{Y_0} \) for both norms and the choice will be as convenient.

In the next lines, we describe the main results of this paper.

Let \( \alpha \in (0, 1) \) and consider \( 1 < \rho < \frac{n + 2(1 - \alpha)}{2} \). Recall that \( Y_{\alpha - 1} = [Y_{-1}, Y_0]_\alpha \), where \( Y_{-1} \) denotes the extrapolation space of \( Y_0 \) and \([\cdot, \cdot]_\alpha\) denotes the complex interpolation functor, see [26]. Under these conditions, we obtain the following result on well-posedness, which is proved in Section 4.

**Theorem 2.1. [Well-Posedness]** Let \( f \in C^1(\mathbb{R}) \) be a function satisfying \((2.4) - (2.5)\), assume conditions \((2.1) - (2.3)\) hold and let \( F : Y_0 \to Y_{\alpha - 1} \) be defined in \((2.9)\). Then for any initial data \( W_0 \in Y_0 \) the problem \((2.6)\) has a unique global solution \( W(t) \) such that

\[
W(t) \in C([\tau, \infty), Y_0).
\]

Moreover, such solutions are continuous with respect to the initial data on \( Y_0 \).

The existence of a pullback attractor, see Theorem 2.2 below, is presented in Section 5.

**Theorem 2.2. [Pullback Attractors]** Under the conditions of Theorem 2.1, the problem \((1.1) - (1.3)\) has a pullback attractor \( \{A(t) : t \in \mathbb{R}\} \) in \( Y_0 \) and

\[
\bigcup_{t \in \mathbb{R}} A(t) \subset Y_0
\]

is bounded.
Theorem 2.3 deals with the regularity of the pullback attractor obtained in Theorem 2.2. This result is proved in Section 6.

Theorem 2.3. [Regularity of Pullback Attractors] Assume that \( \frac{n-1}{n-2} \leq \rho < \frac{n}{n-2} \). The pullback attractor \( \{A(t) : t \in \mathbb{R}\} \) for the problem (1.1) – (1.3), obtained in Theorem 2.2, lies in a more regular space than \( Y_0 \). More precisely,

\[
\bigcup_{t \in \mathbb{R}} A(t)
\]

is a bounded subset of \( X^1 \times X^\frac{1}{2} \times X^1 \times X^\frac{1}{2} \).

Lastly, in Section 7, we show a result on the upper semicontinuity of the pullback attractor, which is stated in Theorem 2.4.

Theorem 2.4. [Upper Semicontinuity] For each \( \eta > 0 \) and \( \varepsilon \in [0, 1] \), let \( W^{(\varepsilon)}(\cdot) = S(\varepsilon)(\cdot, \tau)W_0 \) be the solution of (1.1) in \( Y_0 \). Assume that \( \|a_\varepsilon - a_0\|_{L^\infty(\mathbb{R})} \to 0 \) as \( \varepsilon \to 0^+ \). Then, for each \( T > 0 \), \( W^{(\varepsilon)} \) converges to \( W^{(0)} \) in \( C([0, T], Y_0) \) as \( \varepsilon \to 0^+ \). Moreover, the family of pullback attractors \( \{A_{(\varepsilon)}(t) : t \in \mathbb{R}\} \) is upper semicontinuous at \( \varepsilon = 0 \).

3. Preliminaries

The main purpose of this section is to briefly introduce the reader to some terminology and facts about the theory of abstract parabolic problems, as well as to present definitions and existence results that appear in the study of pullback attractors for non-autonomous dynamical systems. Let \( X \) be a Banach space. As a standard notation, we denote by \( L(X) \) the space of all bounded linear operators from \( X \) into itself. Let \( \{B(t) : t \in \mathbb{R}\} \) be a family of unbounded closed linear operators, where each \( B(t) \) has the same dense subspace \( D \) of \( X \) as domain.

3.1. Non-autonomous abstract linear problem. Consider the singularly non-autonomous abstract linear parabolic problem of the form

\[
\begin{cases}
\frac{du}{dt} = -B(t)u, & t > \tau, \\
u(\tau) = u_0 \in D.
\end{cases}
\]

The term singularly non-autonomous is used to evidence the fact that the unbounded operator \( B(t) \) has explicit dependence with the time. When it comes to the parabolic structure of the above problem, we assume the following conditions:

(A1) The family of operators \( B(t) : D \subset X \to X \) is uniformly sectorial in \( X \); that is, \( B(t) \) is closed and densely defined for every \( t \in \mathbb{R} \), with domain \( D \) fixed, and for all \( T \in \mathbb{R} \) there exists a constant \( C_1 > 0 \), independent of \( T \), such that

\[
\|(B(t) + \lambda I)^{-1}\|_{L(X)} \leq \frac{C_1}{|\lambda| + 1}
\]

for all \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) \geq 0 \) and for all \( t \in [-T, T] \).

(A2) The map \( \mathbb{R} \ni t \mapsto B(t) \) is uniformly Hölder continuous in \( X \); that is, for all \( T \in \mathbb{R} \) there are constants \( C_2 > 0 \) and \( 0 < \varepsilon_0 \leq 1 \), independent of \( T \), such that

\[
\|(B(t) - B(s))B^{-1}(\tau)\|_{L(X)} \leq C_2|t - s|^\varepsilon
\]

for every \( t, s, \tau \in [-T, T] \).
Denote by $\mathcal{B}_0$ the operator $\mathcal{B}(t_0)$ for some $t_0 \in \mathbb{R}$ fixed. If $X^\alpha$ denotes the domain of $\mathcal{B}_0^\alpha$, $\alpha > 0$, with the graph norm, and $X^0 = X$, then $\{X^\alpha : \alpha \geq 0\}$ is the fractional power scale associated with $\mathcal{B}_0$. For more details about fractional powers of operators, see Henry [19].

From (A1), $-\mathcal{B}(t)$ is the generator of an analytic semigroup $\{e^{-t\mathcal{B}(t)} : \tau \geq 0\} \subset \mathcal{L}(X)$. Using this and the fact that $0 \in \rho(\mathcal{B}(t))$, one can obtain a constant $C > 0$ such that the following estimates hold:

$$\|e^{-t\mathcal{B}(t)}\|_{\mathcal{L}(X)} \leq C, \tau \geq 0, \ t \in \mathbb{R},$$

and

$$\|\mathcal{B}(t)e^{-t\mathcal{B}(t)}\|_{\mathcal{L}(X)} \leq C\tau^{-1}, \tau > 0, \ t \in \mathbb{R}.$$

For a given bounded set $I \subset \mathbb{R}^2$, it follows from (A2), that there exists a constant $K = K(I) > 0$ such that

$$\|\mathcal{B}(t)\mathcal{B}^{-1}(\tau)\|_{\mathcal{L}(X)} \leq K,$$

for all $(t, \tau) \in I$.

Also, the semigroup $\{e^{-t\mathcal{B}(t)} : \tau \geq 0\}$ satisfies

$$\|e^{-t\mathcal{B}(t)}\|_{\mathcal{L}(X^\alpha, X^\alpha)} \leq C(\alpha, \beta)\tau^{\beta-\alpha}, \tau > 0, \ t \in \mathbb{R},$$

where $0 \leq \beta \leq \alpha < 1 + \epsilon_0$.

3.2. Abstract results on pullback attractors. In this subsection, we will present basic definitions and results of the theory of pullback attractors for nonlinear evolution processes. For more details we refer to [9], [11] and [13].

Let $(Z, d)$ be a metric space. An evolution process in $Z$ is a two-parameter family $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ of maps from $Z$ into itself such that:

(a) $S(t, t) = I$ for all $t \in \mathbb{R}$, ($I$ is the identity operator in $Z$),

(b) $S(t, \tau) = S(t, s)S(s, \tau)$ for all $t \geq s \geq \tau$, and

(c) the map $\{(t, \tau) \in \mathbb{R}^2 : t \geq \tau\} \times Z \ni (t, \tau, x) \mapsto S(t, \tau)x \in Z$ is continuous.

If $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\} \subset \mathcal{L}(X)$ is an evolution process, then we will call this process as a linear evolution process.

**Remark 3.1.** If the operator $\mathcal{B}(t) : D \subset X \to X$ of equation (3.1) is uniformly sectorial and uniformly Hölder continuous, then there exists a linear evolution process $\{L(t, \tau) : t \geq \tau \in \mathbb{R}\}$ associated with $\mathcal{B}(t)$, which is given by

$$L(t, \tau) = e^{-(t-\tau)\mathcal{B}(\tau)} + \int_{\tau}^{t} L(t, s)[\mathcal{B}(\tau) - \mathcal{B}(s)]e^{-(s-\tau)\mathcal{B}(\tau)}ds, \quad t \geq \tau.$$

The process $\{L(t, \tau) : t \geq \tau \in \mathbb{R}\}$ satisfies the following condition:

$$\|L(t, \tau)\|_{\mathcal{L}(X^\alpha, X^\alpha)} \leq C(\alpha, \beta)(t-\tau)^{\beta-\alpha},$$

where $0 \leq \beta \leq \alpha < 1 + \epsilon_0$. The reader may consult [12] and [25] for more details.

Now, let us consider the following singularly non-autonomous abstract parabolic problem

$$\begin{cases}
\frac{du}{dt} = -\mathcal{B}(t)u + g(u), & t > \tau, \\
u(\tau) = u_0 \in D,
\end{cases}$$

where the operator $\mathcal{B}(t) : D \subset X \to X$ is uniformly sectorial and uniformly Hölder continuous, and the nonlinearity $g$ satisfies some suitable conditions that will be specified.
later. The nonlinear evolution process \( \{S(t, \tau) : t \geq \tau \in \mathbb{R} \} \) associated with \( \mathcal{B}(t) \) is given by
\[
S(t, \tau) = L(t, \tau) + \int_{\tau}^{t} L(t, s) g(S(s, \tau)) ds, \quad t \geq \tau.
\]

**Definition 3.2.** Let \( g : X^\alpha \to X^\beta, \alpha \in [\beta, \beta + 1), \) be a continuous function. A continuous function \( u : [\tau, \tau + t_0] \to X^\alpha \) is said to be a local solution of the problem (3.2), starting at \( u_0 \in X^\alpha, \) if the following conditions hold:
- (a) \( u \in C([\tau, \tau + t_0], X^\alpha) \cap C^1((\tau, \tau + t_0], X^\alpha); \)
- (b) \( u(\tau) = u_0; \)
- (c) \( u(t) \in D(\mathcal{B}(t)) \) for all \( t \in (\tau, \tau + t_0]; \)
- (d) \( u(t) \) satisfies (3.2) for all \( t \in (\tau, \tau + t_0]. \)

Now we state the following abstract local well-posedness result, whose proof can be found in [8]. The reader may consult [12] for a more general version that includes the critical growth case.

**Theorem 3.3.** Assume that the family of operators \( \{\mathcal{B}(t) : t \in \mathbb{R}\} \) is uniformly sectorial and uniformly Hölder continuous in \( X^\beta. \) If \( g : X^\alpha \to X^\beta, \alpha \in [\beta, \beta + 1), \) is a Lipschitz continuous map in bounded subsets of \( X^\alpha, \) then given \( r > 0 \) there exists a time \( t_0 > 0 \) such that for all \( u_0 \in B_{X^\alpha}(0, r) \) there exists a unique solution of the problem (3.2) starting in \( u_0 \) and defined on \( [\tau, \tau + t_0]. \) Moreover, such solutions are continuous with respect to the initial data in \( B_{X^\alpha}(0, r). \)

In the sequel, we present the concepts concerning the theory of pullback attractors which we will use in this work. Further details can be found in [9], [11] and [13]. Recall that, if \( (Z, d) \) is a metric space, then the Hausdorff semidistance between two nonempty subsets \( A \) and \( B \) of \( Z \) is defined by
\[
d_H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).
\]

**Definition 3.4.** Let \( \{S(t, \tau) : t \geq \tau \in \mathbb{R}\} \) be an evolution process in \( Z. \) Given \( t \in \mathbb{R} \) and \( A, B \) subsets of \( Z, \) we say that \( A \) pullback attracts \( B \) at time \( t \) if
\[
\lim_{\tau \to -\infty} d_H(S(t, \tau)B, A) = 0,
\]
where \( S(t, \tau)B = \{S(t, \tau)x : x \in B\}. \) The set \( A \) pullback attracts bounded sets at time \( t, \) if (3.3) holds for every bounded subset \( B \) of \( Z. \) Moreover, we say that a time-dependent family \( \{A(t) : t \in \mathbb{R}\} \) of subsets of \( Z \) pullback attracts bounded subsets of \( Z, \) if \( A(t) \) pullback attracts bounded sets at time \( t, \) for each \( t \in \mathbb{R}. \)

**Definition 3.5.** A family of compact subsets \( \{A(t) : t \in \mathbb{R}\} \) of \( Z \) is a pullback attractor for the evolution process \( \{S(t, \tau) : t \geq \tau \in \mathbb{R}\} \) if
- (i) \( \{A(t) : t \in \mathbb{R}\} \) is invariant; that is, \( S(t, \tau)A(\tau) = A(t) \) for all \( t \geq \tau, \)
- (ii) \( \{A(t) : t \in \mathbb{R}\} \) pullback attracts bounded subsets of \( Z, \) in the sense of Definition 3.4 and
- (iii) \( \{A(t) : t \in \mathbb{R}\} \) is the minimal family of closed sets satisfying property (ii).

**Definition 3.6.** An evolution process \( \{S(t, \tau) : t \geq \tau \in \mathbb{R}\} \) in \( Z \) is said to be pullback asymptotically compact if, for each \( t \in \mathbb{R}, \) each sequence \( \{\tau_k\}_{k \in \mathbb{N}} \) with \( \tau_k \leq t \) for all \( k \in \mathbb{N} \) and \( \tau_k \xrightarrow{k \to \infty} -\infty, \) and each bounded sequence \( \{x_k\}_{k \in \mathbb{N}} \subset Z, \) then the sequence \( \{S(t, \tau_k)x_k\}_{k \in \mathbb{N}} \) has a convergent subsequence.
Definition 3.7. We say that a set $B \subset Z$ pullback absorbs bounded sets at time $t \in \mathbb{R}$ if, for each bounded subset $D$ of $Z$, there exists a time $T = T(t, D) \leq t$ such that $S(t, \tau)D \subset B$ for all $\tau \leq T$. Moreover, we say that a time-dependent family $\{B(t) : t \in \mathbb{R}\}$ of subsets of $Z$ pullback absorbs bounded sets of $Z$, if $B(t)$ pullback absorbs bounded sets at time $t$, for each $t \in \mathbb{R}$.

Definition 3.8. We say that an evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ in $Z$ is:

(i) pullback strongly bounded if, for each bounded subset $B$ of $Z$ and each $t \in \mathbb{R}$, then the set $\bigcup_{s \leq t} \gamma_p(B, s)$ is bounded, where $\gamma_p(B, t) = \bigcup_{\tau \leq t} S(t, \tau)B$ is the pullback orbit of $B \subset Z$ at time $t \in \mathbb{R}$.

(ii) pullback strongly bounded dissipative if, for each $t \in \mathbb{R}$, then there is a bounded subset $B(t)$ of $Z$ which pullback absorbs bounded subsets of $Z$ at time $s$ for each $s \leq t$; that is, given a bounded subset $D$ of $Z$ and $s \leq t$, there exists $\tau_0(s, D)$ such that $S(s, \tau)D \subset B(t)$ for all $\tau \leq \tau_0(s, D)$.

Theorem 3.9. If an evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ in $Z$ is pullback strongly bounded dissipative and pullback asymptotically compact, then $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ has a pullback attractor $\{A(t) : t \in \mathbb{R}\}$, such that $\bigcup_{\tau \leq t} A(\tau)$ is bounded for each $t \in \mathbb{R}$.

Definition 3.10. A global solution for an evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ in $Z$ is a function $\xi : \mathbb{R} \to Z$ such that $S(t, \tau)\xi(\tau) = \xi(t)$ for all $t \geq \tau$.

It is well-known that if a semigroup has a global attractor, then it is characterized as the union of all bounded global solutions. In the non-autonomous case, an equivalent characterization is given by the following result.

Theorem 3.11. If a pullback attractor $\{A(t) : t \in \mathbb{R}\}$ is bounded in the past then

$$A(t) = \{\xi(t) : \xi(\cdot) \text{ is a backward-bounded global solution}\}.$$ 

We end this section with the concept of upper semicontinuity.

Definition 3.12. A family $\{A_\epsilon(t)\}_{\epsilon \in [0, 1]}$, $t \in \mathbb{R}$, of subsets of $Z$ is upper semicontinuous at $\epsilon = 0$ if, for each $t \in \mathbb{R}$, then

$$\lim_{\epsilon \to 0^+} d_H(A_\epsilon(t), A_0(t)) = 0.$$ 

4. LOCAL AND GLOBAL WELL-POSEDNESS

This section concerns the investigation of the existence of global solutions for (2.6). We are going to present auxiliary results to conclude Theorem 2.1. We start by obtaining some spectral properties for the unbounded linear operator $A(t)$ given in (2.7) and (2.8).

It is not difficult to see that $\det(A(t)) = A(A + I)$, and therefore that $0 \in \rho(A(t))$, for all $t \in \mathbb{R}$. Moreover, for each $t \in \mathbb{R}$, the operator $A^{-1}(t) : Y_0 \to Y_0$ is defined by

$$(4.1) \quad A^{-1}(t) \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} I & A_{1/2} & A_{1/2} & 0 \\ -I & 0 & 0 & 0 \\ -a_{1/2}(t) & 0 & \eta A_{1/2} & A^{-1} \\ 0 & 0 & -I & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}. $$

Proposition 4.1. For each fixed $t \in \mathbb{R}$, the operator $A(t)$ defined in (2.7) — (2.8) is maximal accretive.
Proposition 4.2. If \( D(A) \) is fixed but arbitrary, and let \( x = [u \ v \ w \ z]^T \in D(A(t)) \). At first, we note that \( \langle v, u \rangle_X^{1/2} = \left( (A + I)^{1/2}v, (A + I)^{1/2}u \right)_X \), because from Corollary 1.3.5, we have \( D((A + I)^{1/2}) = D(A^{1/2}) \). Thus,

\[
\langle A(t)x, x \rangle_{Y_0} = \langle -v, u \rangle_X^{1/2} + \langle (A + I)u + \eta A^{1/2}v + a_e(t)A^{1/2}z, v \rangle_X + \langle -z, w \rangle_X^{1/2}
\]

\[
= \langle (A + I)^{1/2}z, (A + I)^{1/2}v \rangle_X - \langle (A + I)^{1/2}v, (A + I)^{1/2}u \rangle_X
\]

\[
+ a_e(t) \left( \langle A^{1/2}z, v \rangle_X - \langle v, A^{1/2}z \rangle_X \right)
\]

\[
+ \langle Aw, z \rangle_X - \langle z, Aw \rangle_X + \eta \|A^{1/2}v\|_X^2 + \eta \|A^{1/2}z\|_X^2.
\]

Hence,

\[
\text{Re}(\langle A(t)x, x \rangle_{Y_0}) = \eta \|A^{1/2}v\|_X^2 + \eta \|A^{1/2}z\|_X^2 \geq 0,
\]

which proves the accretivity of \( A(t) \).

\[ \square \]

Proposition 4.2. If \( Y_{-1} \) denotes the extrapolation space of \( Y_0 = X^{1/2} \times X \times X^{1/2} \times X \) generated by the operator \( A^{-1}(t) \), then

\[
Y_{-1} = X \times X^{-1/2} \times X \times X^{-1/2}.
\]

Proof. This proof is analogous to the proof of [9, Proposition 2] and so we omit it. See also [9].

\[ \square \]

Proposition 4.3. The operator \( A^{-1}(t) \) given in (1.1) is a compact map for each \( t \in \mathbb{R} \).

Proof. Let \( B \subset Y_0 \) be a bounded set and denote \( Y_1 = X \times X^{1/2} \times X \times X^{1/2} \). At first, note that since \( (A + I)(A + I)^{-1} = I \) and \( A \) is uniformly sectorial, we have

\[
\|A(A + I)^{-1}\|_{\mathcal{L}(X)} \leq 1 + \|A + I\|_{\mathcal{L}(X)} \leq 1 + M,
\]

for some constant \( M > 0 \). Thus, for \( x = [u \ v \ w \ z]^T \in B \), we have

\[
\|A^{-1}(t)x\|_{Y_1} = \|\eta A^{1/2}(A + I)^{-1}u + (A + I)^{-1}v + a_e(t)A^{1/2}(A + I)^{-1}w\|_{X^1} + \| - u \|_{X^{1/2}}
\]

\[
+ \| - a_e(t)A^{-1/2}u + \eta A^{-1/2}w + A^{-1}z\|_{X^1} + \| - w \|_{X^{1/2}}
\]

\[
\leq \eta \|A + I\|_{\mathcal{L}(X)} \|A^{1/2}u\|_X + \|A(A + I)^{-1}\|_{\mathcal{L}(X)} \|v\|_X
\]

\[
+ a_1 \|A(A + I)^{-1}\|_{\mathcal{L}(X)} \|A^{1/2}w\|_X + (1 + a_1) \|u\|_{X^{1/2}} + (1 + \eta) \|w\|_{X^{1/2}} + \|z\|_X
\]

\[
\leq [\eta(1 + M) + 1 + a_1] \|u\|_{X^{1/2}} + (1 + M) \|v\|_X + [a_1(1 + M) + 1 + \eta] \|w\|_{X^{1/2}} + \|z\|_X
\]

\[
\leq C \left( \|u\|_{X^{1/2}} + \|v\|_X + \|w\|_{X^{1/2}} + \|z\|_X \right),
\]

where \( C \) is a positive constant, that is,

\[
\|A^{-1}(t)x\|_{Y_1} \leq C \|x\|_{Y_0}.
\]

Thus, \( A^{-1}(t)B \) is bounded in \( Y_1 \). Using the compact embedding \( Y_1 \hookrightarrow Y_0 \), we conclude that the operator \( A^{-1}(t) \) is compact. 

\[ \square \]
Proposition 4.4. The family of operators \( \{ A(t) : t \in \mathbb{R} \} \), defined in (2.7) – (2.8), is uniformly Hölder continuous in \( Y_{-1} \).

**Proof.** Using (2.3), this result follows immediately from (2.7) and (2.8). \( \square \)

The next step is to show the analyticity of the semigroup \( \{ e^{-tA(t)} : \tau \geq 0 \} \). For that, we will make use of the following result whose proof can be found in [21].

Theorem 4.5. Let \( \{ T(\tau) : \tau \geq 0 \} \) be a \( C_0 \)-semigroup of contractions in a Hilbert space \( H \) with infinitesimal generator \( \mathcal{B} \). Suppose that \( i \mathbb{R} \subset \rho(\mathcal{B}) \). Then \( \{ T(\tau) : \tau \geq 0 \} \) is analytic if, and only if

\[
\limsup_{|\beta| \to \infty} \| \beta (i \beta I - \mathcal{B})^{-1} \|_{\mathcal{L}(H)} < \infty.
\]

The next lemma shows that \( i \mathbb{R} \subset \rho(-A(t)) \) for all \( t \in \mathbb{R} \).

**Lemma 4.6.** The semigroup \( \{ e^{-tA(t)} : \tau \geq 0 \} \), generated by \( -A(t) \), satisfies

\[
i \mathbb{R} \subset \rho(-A(t))
\]

for all \( t \in \mathbb{R} \).

**Proof.** Arguing by contradiction, suppose that there exists \( 0 \neq \beta \in \mathbb{R} \) such that \( i \beta \) is in the spectrum of \( -A(t) \) for some \( t \in \mathbb{R} \). Then \( i \beta \) must be an eigenvalue of \( -A(t) \), since the operator \( A^{-1}(t) \) is compact. Consequently, there exists

\[
U = [u \ v \ w \ z]^T \in D(A(t)), \quad \| U \|_{Y_0} = 1,
\]

such that \( i \beta U - (-A(t))U = 0 \) or, equivalently,

\[
i \beta u - v = 0,
\]

\[
i \beta v + Au + u + \eta A_t^1 v + \alpha(t) A_t^1 z = 0,
\]

\[
i \beta w - z = 0,
\]

\[
i \beta z - \alpha(t) A_t^1 v + Aw + \eta A_t^1 z = 0.
\]

Now, taking the real part of the inner product of \( i \beta U + A(t)U \) with \( U \) in \( Y_0 \), we have

\[
\langle i \beta U + A(t)U, U \rangle_{Y_0} = \langle 0, U \rangle_{Y_0} = 0 \implies i \beta \| U \|_{Y_0}^2 + \langle A(t)U, U \rangle_{Y_0} = 0 \implies \Re(\langle A(t)U, U \rangle_{Y_0}) = 0 \implies \eta \| A_t^1 v \|_X^2 + \eta \| A_t^1 z \|_X^2 = 0 \implies \| A_t^1 v \|_X^2 = \| A_t^1 z \|_X^2 = 0 \implies v = z = 0.
\]

Consequently, \( u = w = 0 \). Therefore, \( U = 0 \), which is a contradiction. This proves our claim. \( \square \)

Now, we are in position to prove that the semigroup generated by \( -A(t) \) is analytic.

**Theorem 4.7.** The semigroup \( \{ e^{-\tau A(t)} : \tau \geq 0 \} \), generated by \( -A(t) \), is analytic for each \( t \in \mathbb{R} \).

**Proof.** We are going to use Theorem [4.3]. Let \( t \in \mathbb{R} \). In view of Lemma [4.6] it is enough to prove that there exists a positive constant \( C \) such that

\[
| \beta | \| U \|_{Y_0} \leq C \| F \|_{Y_0},
\]

for all \( t \in \mathbb{R} \). This is the task of the next chapter.
for all $F \in Y_0$ and all $\beta \in \mathbb{R}$, where
\[ U = (i \beta I + \mathcal{A}(t))^{-1}F \in D(\mathcal{A}(t)). \]

In fact, denoting $U = [u \ v \ w \ z]^T$ and $F = [f \ g \ h \ k]^T$, we can write the resolvent equation
\[ (i \beta I + \mathcal{A}(t))U = F \]
in $Y_0$ in terms of its components, obtaining the following scalar equations
\[ i \beta u - v = f, \]
\[ Au + u + i \beta v + \eta A^{\frac{1}{2}}v + a_c(t) A^{\frac{1}{2}}z = g, \]
\[ i \beta w - z = h, \]
\[ -a_c(t) A^{\frac{1}{2}}v + Aw + i \beta z + \eta A^{\frac{1}{2}}z = k. \]

Taking the inner product of (4.3) with $U$ in $Y_0$, we obtain
\[ i \beta \|U\|_{Y_0}^2 + \langle \mathcal{A}(t)U, U \rangle_{Y_0} = \langle F, U \rangle_{Y_0}. \]

By the proof of Proposition 4.1, see (4.2), we get
\[ \Re(\langle \mathcal{A}(t)U, U \rangle_{Y_0}) = \eta \|A^{\frac{1}{2}}v\|^2_X + \eta \|A^{\frac{1}{2}}z\|^2_X \geq 0. \]

It follows by the Cauchy-Schwartz inequality that
\[ \eta \|A^{\frac{1}{2}}v\|^2_X + \eta \|A^{\frac{1}{2}}z\|^2_X = |\Re(\langle \mathcal{A}(t)U, U \rangle_{Y_0})| \leq |\Re(\langle F, U \rangle_{Y_0})| \leq \|F\|_{Y_0} \|U\|_{Y_0} \]
and, therefore, we obtain
\[ \|A^{\frac{1}{2}}v\|^2_X \leq \frac{1}{\eta} \|F\|_{Y_0} \|U\|_{Y_0} \quad \text{and} \quad \|A^{\frac{1}{2}}z\|^2_X \leq \frac{1}{\eta} \|F\|_{Y_0} \|U\|_{Y_0}. \]

Now, taking the inner product of (4.3) with $x_1 = [A^{-\frac{1}{2}}v \ 0 \ 0 \ 0]^T$ in $Y_0$, it leads to
\[ \langle (i \beta I + \mathcal{A}(t))U, x_1 \rangle_{Y_0} = \langle F, x_1 \rangle_{Y_0} \iff \langle i \beta u - v, A^{-\frac{1}{2}}v \rangle_{X^{\frac{1}{2}}} = \langle f, A^{-\frac{1}{2}}v \rangle_{X^{\frac{1}{2}}} \iff \langle A^{\frac{1}{2}}u, -i \beta v \rangle_{X} - \|A^{\frac{1}{2}}v\|^2_X = \langle A^{\frac{1}{2}}f, v \rangle_{X} \]
and then, using (4.3), we conclude that
\[ \langle A^{\frac{1}{2}}u, Au + u + \eta A^{\frac{1}{2}}v + a_c(t) A^{\frac{1}{2}}z - g \rangle_{X} - \|A^{\frac{1}{2}}v\|^2_X = \langle A^{\frac{1}{2}}f, v \rangle_{X}. \]

Thus, from Cauchy-Schwartz and Young inequalities and (4.7), we obtain
\[ \|A^{\frac{1}{2}}u\|^2_X = -\|A^{\frac{1}{2}}u\|^2_X - \eta \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v \rangle_{X} - a_c(t) \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}z \rangle_{X} + \langle A^{\frac{1}{2}}u, g \rangle_{X} + \langle A^{\frac{1}{2}}f, v \rangle_{X} + \|A^{\frac{1}{2}}v\|^2_X \leq \eta \|A^{\frac{1}{2}}u\|_X \|A^{\frac{1}{2}}v\|_X + a_1 \|A^{\frac{1}{2}}u\|_X \|A^{\frac{1}{2}}z\|_X + \|A^{\frac{1}{2}}u\|_X \|g\|_X + \|A^{\frac{1}{2}}f\|_X \|v\|_X + \|A^{\frac{1}{2}}v\|^2_X \leq \frac{e_1}{2} \eta \|A^{\frac{1}{2}}u\|^2_X + \frac{1}{2 \epsilon_1} \|A^{\frac{1}{2}}v\|^2_X + \frac{e_2}{2} \|A^{\frac{1}{2}}u\|^2_X + \frac{1}{2 \epsilon_2} \|A^{\frac{1}{2}}z\|^2_X + \frac{1}{\eta} \|F\|_{Y_0} \|U\|_{Y_0} \leq \left( \frac{e_1}{2} \eta^2 + \frac{e_2}{2} a_1^2 \right) \|A^{\frac{1}{2}}u\|^2_X + \left( \frac{1}{2 \eta \epsilon_1} + \frac{1}{2 \eta \epsilon_2} + \frac{1}{\eta} + 2 \right) \|F\|_{Y_0} \|U\|_{Y_0}, \]
for all \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \). Now, it is enough to choose \( \epsilon_1 = \frac{1}{2\eta} \) and \( \epsilon_2 = \frac{1}{2\eta^2} \), and so we get
\[
\|A^\frac{4}{3} u\|_X^2 \leq \left( 2\eta + \frac{2(a_1^2 + 1)}{\eta} + 4 \right) \|F\|_{Y_0} \|U\|_{Y_0}.
\]

Next, taking the inner product of (4.3) with \( x_2 = [0 0 0 A^\frac{4}{3} w]^T \), we have
\[
\langle (i\beta I + A(t))U, x_2 \rangle_{Y_0} = \langle F, x_2 \rangle_{Y_0} \iff \langle -a_s(t)A^\frac{4}{3} v + A w + i\beta z + \eta A^\frac{4}{3} z, A^\frac{4}{3} w \rangle_X = \langle k, A^\frac{4}{3} w \rangle_X.
\]

That is,
\[
am_s(t)\langle A^\frac{4}{3} v, A^\frac{4}{3} w \rangle_X + \|A^\frac{4}{3} w\|_X^2 + \langle A^\frac{4}{3} z, -i\beta w \rangle_X + \eta \langle A^\frac{4}{3} z, A^\frac{4}{3} w \rangle_X = \langle k, A^\frac{4}{3} w \rangle_X
\]

and then, using (4.3), we have
\[
\|A^\frac{4}{3} w\|_X^2 = a_s(t)\langle A^\frac{4}{3} v, A^\frac{4}{3} w \rangle_X - \eta \langle A^\frac{4}{3} z, A^\frac{4}{3} w \rangle_X + \|A^\frac{4}{3} z\|_X^2 + \langle A^\frac{4}{3} z, h \rangle_X + \langle k, A^\frac{4}{3} w \rangle_X.
\]

Using again the Cauchy-Schwarz and Young inequalities, and (4.7), we obtain
\[
\|A^\frac{4}{3} w\|_X^2 \leq a_1 \|A^\frac{4}{3} v\|_X \|A^\frac{4}{3} w\|_X + \eta \|A^\frac{4}{3} z\|_X \|A^\frac{4}{3} w\|_X + \|A^\frac{4}{3} z\|_X^2 + \|z\|_X \|A^\frac{4}{3} h\|_X + \|k\|_X \|A^\frac{4}{3} w\|_X
\]
\[
\leq \frac{\epsilon_3}{2} a_1^2 \|A^\frac{4}{3} w\|_X^2 + \frac{1}{2\epsilon_3} \|A^\frac{4}{3} v\|_X^2 + \eta^2 \|A^\frac{4}{3} w\|_X^2 + \frac{1}{2\epsilon_4} \|A^\frac{4}{3} z\|_X^2 + \left( \frac{1}{\eta} + 2 \right) \|F\|_{Y_0} \|U\|_{Y_0}
\]
\[
\leq \left( \frac{\epsilon_3}{2} a_1^2 + \frac{\epsilon_4}{2} \eta^2 \right) \|A^\frac{4}{3} w\|_X^2 + \left( \frac{1}{2\epsilon_3} + \frac{1}{2\epsilon_4} + \frac{1}{\eta} + 2 \right) \|F\|_{Y_0} \|U\|_{Y_0},
\]

for all \( \epsilon_3 > 0 \) and \( \epsilon_4 > 0 \). Choosing \( \epsilon_3 = \frac{1}{2\eta^2} \) and \( \epsilon_4 = \frac{1}{2\eta^2} \), we get
\[
\|A^\frac{4}{3} w\|_X^2 \leq \left( 2\eta + \frac{2(a_1^2 + 1)}{\eta} + 4 \right) \|F\|_{Y_0} \|U\|_{Y_0}.
\]

By [4.3], Corollary 1.3.5], we have \( D((A + I)^{\frac{4}{3}}) = D(A^\frac{4}{3}) \), consequently,
\[
\langle A^\frac{4}{3} u, A^\frac{4}{3} v \rangle_X = \langle u, v \rangle_{X^\frac{4}{3}} = \left\langle (A + I)^{\frac{4}{3}} u, (A + I)^{\frac{4}{3}} v \right\rangle_X.
\]

Using this fact and the proof of Proposition 4.1.1 we obtain
\[
\langle A(t)U, U \rangle_{Y_0} = \langle A^\frac{4}{3} u, A^\frac{4}{3} v \rangle_X - \langle A^\frac{4}{3} v, A^\frac{4}{3} u \rangle_X
\]
\[
+ a_s(t) \left( \langle A^\frac{4}{3} z, v \rangle_X - \langle v, A^\frac{4}{3} z \rangle_X \right)
\]
\[
+ \langle Aw, z \rangle_X - \langle z, Aw \rangle_X + \eta \|A^\frac{4}{3} v\|_X^2 + \eta \|A^\frac{4}{3} z\|_X^2,
\]

and, taking the imaginary part, we have
\[
\text{Im}(\langle A(t)U, U \rangle_{Y_0}) = 2\text{Im}(\langle A^\frac{4}{3} u, A^\frac{4}{3} v \rangle_X) + 2a_s(t)\text{Im}(\langle A^\frac{4}{3} z, A^\frac{4}{3} v \rangle_X)
\]
\[
+ 2\text{Im}(\langle A^\frac{4}{3} w, A^\frac{4}{3} z \rangle_X)
\]
\[
= 2\text{Im}(\langle A^\frac{4}{3} u, A^\frac{4}{3} v \rangle_X) + 2a_s(t)\text{Im}(\langle A^\frac{4}{3} z, A^\frac{4}{3} v \rangle_X)
\]
\[
+ 2\text{Im}(\langle A^\frac{4}{3} w, A^\frac{4}{3} z \rangle_X).
With this last equality and taking the imaginary part in (4.10), it follows by the Cauchy-Schwarz and Young inequalities that

\[ \beta \| U \|^2_{Y_0} = \text{Im}(\langle F, U \rangle_{Y_0}) - \text{Im}(\langle A(t)U, U \rangle_{Y_0}) \leq \| F \|_{Y_0} \| U \|_{Y_0} + 2 \| A^{1/2}u \|_X \| A^{1/2}v \|_X + 2a_i \| A^{1/2}z \|_X + 2 \| A^{1/2}w \|_X \| A^{1/2}z \|_X \]

\[ \leq \| F \|_{Y_0} \| U \|_{Y_0} + \| A^{3/2}u \|_X^2 + (1 + a_i) \| A^{3/2}v \|_X^2 + (1 + a_i) \| A^{3/2}w \|_X^2 \]

and, using the estimates obtained in (4.7), (4.8) and (4.9), we get

\[ \beta \| U \|^2_{Y_0} \leq \left( 1 + 2 \left( 2\eta + \frac{2(a_i^2 + 1)}{\eta} + 4 \right) + \frac{2a_i + 2}{\eta} \right) \| F \|_{Y_0} \| U \|_{Y_0}, \]

that is, there exists a positive constant C, independent of \( \beta \), such that

\[ \beta \| (i\beta I + A(t))^{-1}F \|_{Y_0} \leq C \| F \|_{Y_0} \]

for all \( F \in Y_0 \) and all \( \beta \in \mathbb{R} \). Since this holds for \( \beta \in \mathbb{R} \) arbitrary,

\[ \| \beta \| (i\beta I + A(t))^{-1} \|_{\mathcal{L}(Y_0)} \leq C, \quad \text{for all } \beta \in \mathbb{R}, \]

and, therefore, we conclude that

\[ \limsup_{|\beta| \to +\infty} \| \beta (i\beta I + A(t))^{-1} \|_{\mathcal{L}(Y_0)} < \infty. \]

By Theorem 4.5, the semigroup \( \{ e^{-\tau A(t)} : \tau \geq 0 \} \) is analytic. \( \square \)

**Remark 4.8.** We have the following description of the fractional power scale for the operator \( A(t) \), given as follows

\[ Y_0 \hookrightarrow Y_{\alpha-1} \hookrightarrow Y_{-1}, \quad \text{for all } 0 < \alpha < 1, \]

where

\[ Y_{\alpha-1} = [Y_{-1}, Y_0]: [X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{3}{2}}, X^{\frac{3}{2}} \times X \times X^{\frac{1}{2}} \times X]_{\alpha} \]

\[ = [X, X^{\frac{3}{2}}]_{\alpha} \times [X^{-\frac{1}{2}}, X]_{\alpha} \times [X, X^{\frac{3}{2}}]_{\alpha} \times [X^{-\frac{3}{2}}, X]_{\alpha} \]

\[ = X^{\frac{3}{2}} \times X^{\frac{3}{2}} \times X^{\frac{3}{2}} \times X^{\frac{3}{2}}, \]

where \([\cdot, \cdot]_{\alpha}\) denotes the complex interpolation functor, see \cite{26}. The first equality follows from Proposition 4.7 (recall that \( 0 \in \rho(A(t)) \)), see \cite{2} Example 4.7.3 (b) and the others equalities follow from \cite{10} Proposition 2.

Proposition 4.9 gives us sufficient conditions for \( F : Y_0 \to Y_{\alpha-1} \) to be Lipschitz continuous in bounded subsets of \( Y_0 \). For a proof, the reader may consult \cite{2}, Corollary 2.7 and \cite{10} Corollary 3.6.

**Proposition 4.9.** Assume that \( 1 < \rho < \frac{n+2(1-\alpha)}{n-2}, \) with \( \alpha \in (0, 1) \). Then the map \( F : Y_0 \to Y_{\alpha-1} \), defined in (2.9), is Lipschitz continuous in bounded subsets of \( Y_0 \).

Proposition 4.9 and Theorem 3.3 ensure the local well-posedness of (2.6) in the phase space \( Y_0 \), and this allows us to establish the following existence result.

**Corollary 4.10.** Let \( 1 < \rho < \frac{n+2(1-\alpha)}{n-2}, \) with \( \alpha \in (0, 1), f \in C^1(\mathbb{R}) \) be a function satisfying (2.4)–(2.5), assume conditions (2.11)–(2.13) hold and let \( F : Y_0 \to Y_{\alpha-1} \) be defined in (2.9). Then given \( r > 0 \), there exists a time \( t_0 = t_0(r) > 0 \) such that for all \( W_0 \in B_{Y_0}(0, r) \), there exists a unique solution \( W : [\tau, \tau + t_0) \to Y_0 \) of the problem (2.6) starting in \( W_0 \). Moreover, such solutions are continuous with respect to the initial data in \( B_{Y_0}(0, r) \).
Before to present the proof of Theorem 2.11, we give an auxiliary result.

**Lemma 4.11.** Let \( f \in C^1(\mathbb{R}) \) be a function satisfying (2.4)–(2.5). The following conditions hold:

(i) There is a constant \( k > 0 \) such that \( |f(s)| \leq |f(0)| + k(|s| + |s|^p) \) for all \( s \in \mathbb{R} \). Consequently, \( |f(s)| \leq c(1 + |s|^p) \) for all \( s \in \mathbb{R} \) and some \( c > 0 \).

(ii) Given \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that

\[
\int_\Omega f(u) dx \leq C_\delta + \delta \|u\|_{\dot{X}^1}^2 \quad \text{and} \quad \int_\Omega \int_0^u f(s) ds dx \leq C_\delta + \delta \|u\|_{\dot{X}^1}^2,
\]

for all \( u \in X \).

(iii) Given \( r > 0 \), there exist constants \( C_r > 0 \) and \( C \geq 0 \) (which does not depend on \( r \)) such that

\[
\left| \int_\Omega f(u) dx \right| \leq C_r \|u\|_{\dot{X}^1}^2 + C \quad \text{and} \quad \left| \int_\Omega \int_0^u f(s) ds dx \right| \leq C_r \|u\|_{\dot{X}^1}^2 + C
\]

for all \( u \in \dot{X}^1 \) with \( \|u\|_{\dot{X}^1} \leq r \). If \( f(0) = 0 \) then the constant \( C \) can be chosen zero.

**Proof.** Condition (i) is a consequence of (2.5) and [7, Lemma 2.4]. Condition (ii) follows by the ideas presented in [18] (see page 76).

Let us prove condition (iii). Let \( u \in \dot{X}^1 \). Using the Hölder’s inequality, the Poincaré inequality \( \|u\|_{\dot{X}^1} \leq \lambda_1^{-1}\|u\|_{\dot{X}^1}^2 \) (\( \lambda_1 > 0 \) is the first eigenvalue of the negative Laplacian operator with homogeneous Dirichlet boundary condition) and item (i), we have

\[
\left| \int_\Omega f(u) dx \right| \leq \left( \int_\Omega |u|^2 dx \right) \frac{1}{2} \left( \int_\Omega |f(u)|^2 dx \right) \frac{1}{2} \\
\leq \kappa_0 \|u\|_{\dot{X}^1} \left( \int_\Omega (|f(0)|^2 + |u|^2 + |u|^{2p}) dx \right) \frac{1}{2} \\
\leq \kappa_1 \|u\|_{\dot{X}^1} \left( |f(0)||\Omega|^\frac{1}{2} + \|u\|_{X} + \|u\|_{L^{2p}(\Omega)}^p \right) \\
\leq \kappa_1 \left( |f(0)|^2|\Omega| + \|u\|_{\dot{X}^1}^2 + \|u\|_{L^{2p}(\Omega)}^{2p} \right) \frac{1}{2} \\
\leq \kappa_2 \left( |f(0)|^2|\Omega| + \lambda_1^{-1}\|u\|_{\dot{X}^1}^2 + \|u\|_{L^{2p}(\Omega)}^{2p} \right)
\]

with \( \kappa_2 > 0 \) being a constant. Thanks to our assumption on the exponent \( \rho \), we have \( 2\rho < \frac{2n}{n-2} \) and, moreover, since we know that the embedding \( H^1(\Omega) \hookrightarrow L^p(\Omega) \) holds if and only if \( p \leq \frac{2n}{n-2} \), it follows that \( H^1(\Omega) \hookrightarrow L^{2p}(\Omega) \). Thus, there exists \( \kappa_3 > 0 \) such that \( \|u\|_{L^{2p}(\Omega)} \leq \kappa_3 \|u\|_{\dot{X}^1}^\frac{2p}{p} \) and, hence,

\[
\left| \int_\Omega f(u) dx \right| \leq \kappa_2 \left( |f(0)|^2|\Omega| + \lambda_1^{-1}\|u\|_{\dot{X}^1}^2 + \kappa_3^{2p}\|u\|_{\dot{X}^1}^{2p} \right).
\]

Now, given \( r > 0 \), if \( \|u\|_{\dot{X}^1} \leq r \), then we get

\[
(4.10) \quad \left| \int_\Omega f(u) dx \right| \leq \kappa_2(\lambda_1^{-1} + \kappa_3^{2p}r^{2p-2})\|u\|_{\dot{X}^1}^2 + \kappa_2|f(0)|^2|\Omega|.
\]
Next, we show the other inequality. At first, note that
\[
\left| \int_0^s f(\theta) d\theta \right| \leq |f(0)||s| + k \left( \frac{|s|^2}{2} + \frac{|s|^{\rho+1}}{\rho + 1} \right) \quad \text{for all } s \in \mathbb{R}.
\]
Now, let \( u \in X^{\frac{1}{2}} \). Using the Poincaré inequality \( \|u\|_X^2 \leq \lambda_1^{-1}\|u\|_{X^{\frac{1}{2}}}^2 \), we obtain
\[
\left| \int_0^u f(s) ds \right| \leq \int_0^u \left[ \frac{|f(0)|^2}{2} + \frac{|u|^2}{2} + k \left( \frac{|u|^2}{2} + \frac{|u|^{\rho+1}}{\rho + 1} \right) \right] dx
\leq \kappa_4 \left( |f(0)|^2|\Omega| + \|u\|_X^2 + \|u\|_{L^{\rho+1}(\Omega)}^{\rho+1} \right)
\leq \kappa_4 \left( |f(0)|^2|\Omega| + \kappa_5 \|u\|_{X^{\frac{1}{2}}}^2 + \|u\|_{L^{\rho+1}(\Omega)}^{\rho+1} \right)
\leq \kappa_5 \left( |f(0)|^2|\Omega| + \|u\|_{X^{\frac{1}{2}}}^2 + \|u\|_{L^{\rho+1}(\Omega)}^{\rho+1} \right),
\]
with \( \kappa_5 > 0 \) being a constant. Since \( 1 < \rho < \frac{n}{n-2} \), with \( n \geq 3 \), we have \( 2 < \rho + 1 < \frac{2n-2}{n-2} < \frac{2n}{n-2} \), which ensures that \( H^1(\Omega) \hookrightarrow L^{\rho+1}(\Omega) \). Thus, there exists a constant \( \kappa_6 > 0 \) such that \( \|u\|_{L^{\rho+1}(\Omega)} \leq \kappa_6 \|u\|_{X^{\frac{1}{2}}} \) and, hence,
\[
\left| \int_0^u f(s) ds \right| \leq \kappa_5 \left( |f(0)|^2|\Omega| + \|u\|_{X^{\frac{1}{2}}}^2 + \kappa_6^{\rho+1} \|u\|_{L^{\rho+1}(\Omega)}^{\rho+1} \right).
\]
Now, for a given \( r > 0 \), if \( \|u\|_{X^{\frac{1}{2}}} \leq r \), then we get
\[
(4.11) \quad \left| \int_0^u f(s) ds \right| \leq \kappa_5 |f(0)|^2|\Omega| + \kappa_5 \left( 1 + \kappa_6^{\rho+1} r^{\rho-1} \right) \|u\|_{X^{\frac{1}{2}}}^2.
\]
Therefore, we conclude that, for all \( r > 0 \) given, and for all \( u \in X^{\frac{1}{2}} \) with \( \|u\|_{X^{\frac{1}{2}}} \leq r \), taking
\[
C_r = \max \left\{ \kappa_2 (\lambda^{-1} + \kappa_3^{2p-2}), \kappa_5 (1 + \kappa_6^{\rho+1} r^{\rho-1}) \right\} > 0 \quad \text{and} \quad C = |\Omega||f(0)|^2 \max \{\kappa_2, \kappa_5\} \]
it follows by (4.10) and (4.11) that
\[
\left| \int_\Omega f(u) du \right| \leq C_r \|u\|_{X^{\frac{1}{2}}}^2 + C \quad \text{and} \quad \left| \int_0^u f(s) ds \right| \leq C_r \|u\|_{X^{\frac{1}{2}}}^2 + C.
\]

Proof of Theorem 2.1. By Corollary 4.10 the problem (1.1)–(1.3) has a local solution \((u(t), u_t(t), v(t), v_t(t)) \) in \( Y_0 \) defined on some interval \([\tau, \tau + t_0] \). Consider the original system (1.1). Multiplying the first equation in (1.1) by \( u_t \), and the second by \( v_t \), we obtain
\[
(4.12) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega |u_t|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 dx + \eta \langle (-\Delta)^{\frac{1}{2}} u_t, u_t \rangle_X = \frac{d}{dt} \int_0^u f(s) ds dx,
\]
and
\[
(4.13) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega |v_t|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla v|^2 dx + \eta \langle (-\Delta)^{\frac{1}{2}} v_t, v_t \rangle_X - a_c(t) \langle (-\Delta)^{\frac{1}{2}} u_t, v_t \rangle_X = 0,
\]
for all \( \tau < t \leq \tau + t_0 \). Combining (4.12) and (4.13), we get
\[
\frac{d}{dt} \mathcal{E}(t) = -\eta \|(-\Delta)^{\frac{1}{2}} u(t)\|_X^2 - \eta \|(-\Delta)^{\frac{1}{2}} v(t)\|_X^2
\]
for all \( \tau < t \leq \tau + t_0 \), where
\[
\mathcal{E}(t) = \frac{1}{2}\|u(t)\|_X^2 + \frac{1}{2}\|v(t)\|_X^2 + \frac{1}{2}\|u(t)\|_X^2 + \frac{1}{2}\|v(t)\|_X^2 + \frac{1}{2}\|v(t)\|_X^2
\]
(4.15)
\[
- \int_\Omega \int_0^u f(s)dsdx
\]
is the total energy associated with the solution \((u(t), u(t), v(t), v(t))\) of the problem (1.1)–(1.3) in \( Y_0 \). The identity (4.14) means that the map \( t \mapsto \mathcal{E}(t) \) is monotone decreasing along solutions. Moreover, using the property \( \mathcal{E}(t) \leq \mathcal{E}(\tau) \) for all \( \tau < t \leq \tau + t_0 \), we can obtain a priori estimate of the solution \((u(t), u(t), v(t), v(t))\) in \( Y_0 \). In fact, given \( \delta > 0 \), it follows by Lemma 4.11 item (ii), that there is \( C_\delta > 0 \) such that
\[
\int_\Omega \int_0^u f(s)dsdx \leq C_\delta + \delta \|u\|^2_X.
\]
Thus, for all \( \tau < t \leq \tau + t_0 \), we have
\[
\|u\|_X^2 + \|u(t)\|_X^2 + \|v\|_X^2 + \|u(t)\|_X^2 + \|v\|_X^2 + \|u(t)\|_X^2 + \|v\|_X^2 + \|v(t)\|_X^2
\]
(4.16)
\[
= 2\mathcal{E}(\tau) + 2\int_\Omega \int_0^u f(s)dsdx \leq 2\mathcal{E}(\tau) + 2(\delta \|u\|^2_X + C_\delta)
\]
\[
\leq 2(\mathcal{E}(\tau) + C_\delta) + 2\delta\lambda_1^{-1}\|u\|^2_X + \|u(t)\|^2_X + \|v\|^2_X + \|v(t)\|^2_X,
\]
where we have used the Poincaré inequality (recall that \( \lambda_1 > 0 \) is the first eigenvalue of the negative Laplacian operator with homogeneous Dirichlet boundary condition).

Now, choosing \( \delta = \frac{\lambda_1}{4} \), we get
\[
\|u\|_X^2 + \|u(t)\|_X^2 + \|v\|_X^2 + \|v(t)\|_X^2 \leq 4 \left( \mathcal{E}(\tau) + C_\frac{\lambda_1}{4} \right),
\]
that is,
\[
\|(u(t), u(t), v(t), v(t))\|_{Y_0}^2 \leq 4 \left( \mathcal{E}(\tau) + C_\frac{\lambda_1}{4} \right).
\]
This ensures that the problem (1.1) – (1.3) has a global solution \( W(t) \) in \( Y_0 \), which proves the result.

Since the problem (1.1) – (1.3) has a global solution \( W(t) \) in \( Y_0 \), we can define an evolution process \( \{S(t, \tau) : t \geq \tau \in \mathbb{R} \} \) in \( Y_0 \) by
\[
S(t, \tau)W_0 = W(t), \quad t \geq \tau \in \mathbb{R},
\]
(4.16)

By (12), we have
\[
S(t, \tau)W_0 = L(t, \tau)W_0 + U(t, \tau)W_0, \quad t \geq \tau \in \mathbb{R},
\]
(4.17)
where \( \{L(t, \tau) : t \geq \tau \in \mathbb{R} \} \) is the linear evolution process in \( Y_0 \) associated with the homogeneous problem
\[
\begin{cases}
W_t + A(t)W = 0, & t > \tau, \\
W(\tau) = W_0, & \tau \in \mathbb{R},
\end{cases}
\]
(4.18)
and
\begin{equation}
U(t, \tau)W_0 = \int_{\tau}^{t} L(t, s)F(S(s, \tau)W_0)ds.
\end{equation}

5. Existence of the pullback attractor

In this section, we prove the existence of the pullback attractor of the problem (1.1)-(1.3). To this end, we need to make a modification on the energy functional. More precisely, for \( \gamma_1, \gamma_2 \in \mathbb{R}_+ \), let us define \( L_{\gamma_1, \gamma_2} : Y_0 \rightarrow \mathbb{R} \) by the map
\begin{equation}
L_{\gamma_1, \gamma_2}(\phi, \varphi, \psi, \Phi) = \frac{1}{2} \| \phi \|_X^2 + \frac{1}{2} \| \varphi \|_X^2 + \frac{1}{2} \| \psi \|_X^2 + \frac{1}{2} \| \Phi \|_X^2 + \gamma_1 (\phi, \varphi) + \gamma_2 (\psi, \Phi) - \int_{\Omega} \int_{0}^{\phi} f(s)dsdx.
\end{equation}

We start by noting that if \( \gamma_i < \frac{1}{2} \) and \( \frac{\gamma_i}{\lambda_1^{-1}} < \frac{1}{4} \), \( i = 1, 2 \), then
\begin{equation}
\frac{1}{4} \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2 \leq L_{\gamma_1, \gamma_2}(\phi, \varphi, \psi, \Phi) + \int_{\Omega} \int_{0}^{\phi} f(s)dsdx
\leq \frac{3}{4}(1 + \lambda_1^{-1}) \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2.
\end{equation}

Indeed, using the Cauchy-Schwartz and Young inequalities, we obtain
\begin{align*}
|\gamma_1 (\phi, \varphi) + \gamma_2 (\psi, \Phi)| & \leq \gamma_1 \|\phi\|_X \|\varphi\|_X + \gamma_2 \|\psi\|_X \|\Phi\|_X \\
& \leq \frac{\gamma_1}{2} (\|\phi\|_X^2 + \|\varphi\|_X^2) + \frac{\gamma_2}{2} (\|\psi\|_X^2 + \|\Phi\|_X^2) \\
& \leq \frac{\gamma_1}{2} \lambda_1^{-1} \|\phi\|_X^2 + \frac{\gamma_1}{2} \|\varphi\|_X^2 + \frac{\gamma_2}{2} \lambda_1^{-1} \|\psi\|_X^2 + \frac{\gamma_2}{2} \|\Phi\|_X^2 \\
& \leq \frac{1}{4} \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2,
\end{align*}

which leads to
\begin{equation}
\frac{1}{4} \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2 \leq \frac{1}{2} \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2 + \gamma_1 (\phi, \varphi) + \gamma_2 (\psi, \Phi) - \int_{\Omega} \int_{0}^{\phi} f(s)dsdx \leq \frac{3}{4} \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2.
\end{equation}

Consequently,
\begin{align*}
\frac{1}{4} \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2 & \leq L_{\gamma_1, \gamma_2}(\phi, \varphi, \psi, \Phi) + \int_{\Omega} \int_{0}^{\phi} f(s)dsdx \leq \frac{3}{4} \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2 + \frac{1}{2} \|\phi\|_X^2.
\end{align*}

But since \( \|\phi\|_X^2 \leq \lambda_1^{-1} \|\phi\|_X^2 \), we have
\begin{equation}
\frac{3}{4} \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2 + \frac{1}{2} \|\phi\|_X^2 \leq \frac{3(1 + \lambda_1^{-1})}{4} \|(\phi, \varphi, \psi, \Phi)\|_{Y_0}^2,
\end{equation}
and the claim is proved.

**Theorem 5.1.** There exists \( R > 0 \) such that for any bounded subset \( B \subset Y_0 \) one can find \( t_0(B) > 0 \) satisfying
\[ \|(u, u_t, v, v_t)\|_{Y_0}^2 \leq R \quad \text{for all} \quad t \geq \tau + t_0(B). \]
In particular, the evolution process \( \{S(t, \tau): t \geq \tau \in \mathbb{R}\} \) defined in (4.14) is pullback strongly bounded dissipative.

**Proof.** At first, note that we can differentiate the expression (5.1) along the solution \( W(t) = (u(t), u_t(t), v(t), v_t(t)) \) and, using (4.14) and (4.15), we get

\[
\frac{d}{dt} L_{\gamma_1, \gamma_2} (u, u_t, v, v_t) = \frac{d}{dt} E(t) + \gamma_1 (u, u_t)_X + \gamma_1 (u, u_t)_X + \gamma_2 (v, v_t)_X + \gamma_2 (v, v_t)_X
\]

\[
= -\eta \|A^\frac{1}{2} u_t\|_X^2 - \eta \|A^\frac{1}{2} v_t\|_X^2 + \gamma_1 \|u_t\|_X^2 + \gamma_1 (u, -Au - u - \eta A^\frac{1}{2} u_t - a_1(t) A^\frac{1}{2} v_t + f(u))_X
\]

\[
+ \gamma_2 \|v_t\|_X^2 + \gamma_2 (v, -Av - \eta A^\frac{1}{2} v_t + a_2(t) A^\frac{1}{2} u_t)_X
\]

\[
= -\eta \|u_t\|_X^2 + \eta \|v_t\|_X^2 + \gamma_1 \|u_t\|_X^2 - \gamma_1 (\|u\|_X^2 + \|v\|_X^2) - \gamma_1 \eta (A^\frac{1}{2} u, u)_X - \gamma_1 a_1(t) (A^\frac{1}{2} u, v_t)_X + \gamma_2 \|v_t\|_X^2 - \gamma_2 \|v\|_X^2
\]

Now, if \( c > 0 \) is the embedding constant of \( X^\frac{1}{2} \hookrightarrow X \), then one has

\[
(5.5) \quad -\eta \| \cdot \|_X^2 \leq -\eta \frac{1}{c^2} \| \cdot \|^2_X.
\]

Moreover, by Lemma 4.11 item (ii), for each \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that

\[
\int_\Omega f(u) u dx \leq \delta \|u\|_X^2 + C_\delta,
\]

which implies

\[
(5.6) \quad \gamma_1 (u, f(u))_X \leq \gamma_1 \delta \|u\|_X^2 + \gamma_1 C_\delta \leq \gamma_1 \delta \lambda_1^{-1} \|u\|_X^2 + \gamma_1 C_\delta.
\]

Thus, using (5.5), (5.6) and the Cauchy-Schwartz and Young inequalities, we have

\[
\frac{d}{dt} L_{\gamma_1, \gamma_2} (u, u_t, v, v_t)
\]

\[
\leq -\gamma_1 (1 - \delta \lambda_1^{-1}) \|u\|_X^2 + \gamma_1 \delta \|v\|_X^2 - \gamma_2 \|v_t\|_X^2 - \left( \frac{\eta}{c^2} - \gamma_1 \right) \|u_t\|_X^2 - \left( \frac{\eta}{c^2} - \gamma_2 \right) \|v_t\|_X^2
\]

\[
+ \gamma_2 a_1 \left( \frac{\eta}{c^2} \|v\|_X^2 + \frac{\eta}{c^2} \|u\|_X^2 \right) + \gamma_2 a_1 \left( \frac{\eta}{c^2} \|v\|_X^2 + \frac{\eta}{c^2} \|v_t\|_X^2 \right)
\]

\[
= -\gamma_1 \left( 1 - \delta \lambda_1^{-1} - \eta \frac{\epsilon_1}{c^2} - \gamma_1 \frac{\eta}{c^2} - \gamma_1 \gamma_2 \frac{\eta}{c^2} \right) \|u\|_X^2 + \left( \frac{\eta}{c^2} - \gamma_2 \right) \|v_t\|_X^2
\]

\[
- \gamma_2 \left( 1 - \eta \frac{\epsilon_1}{c^2} - \gamma_2 \right) \|v\|_X^2 + \left( \frac{\eta}{c^2} - \gamma_2 \right) \|v_t\|_X^2
\]

for all \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0 \). Choosing \( \delta = \frac{\lambda_1}{8}, \epsilon_1 = \epsilon_4 = \frac{1}{9} \) and \( \epsilon_2 = \epsilon_3 = 2a_1 \), we obtain

\[
\frac{d}{dt} L_{\gamma_1, \gamma_2} (u, u_t, v, v_t) \leq -\frac{1}{8} \gamma_1 \|u\|_X^2 - \left( \frac{\eta}{c^2} - \gamma_1 \left( 1 + \frac{\eta^2}{2} \right) - \gamma_2 a_1 \right) \|u_t\|_X^2 - \frac{1}{4} \gamma_2 \|v\|_X^2
\]

\[
- \left( \frac{\eta}{c^2} - \gamma_2 a_1 \right) \|v_t\|_X^2 + \gamma_1 C_\delta.
\]
We may choose \( \gamma_i > 0, i = 1, 2 \), sufficiently small such that
\[
\gamma_i < \frac{\eta}{4c^2} \min \left\{ \frac{1}{a_i^2}, \left( 1 + \frac{\eta^2}{2} \right)^{-1} \right\}, \quad i = 1, 2.
\]

Now, taking
\[
C_1 = \min \left\{ \frac{1}{8} \gamma_1, \frac{\eta}{c^2}, \gamma_1 \left( 1 + \frac{\eta^2}{2} \right) - \gamma_2 a_1^2, \frac{1}{4} \gamma_2, \frac{\eta}{c^2} - \gamma_1 a_1^2 - \gamma_2 \left( 1 + \frac{\eta^2}{2} \right) \right\} > 0,
\]
and \( C_2 = \gamma_1 C_{\frac{1}{\delta}} > 0 \), we obtain
\[
\frac{d}{dt} L_{\gamma_1, \gamma_2}(u, u_t, v, v_t) \leq -C_1 \|(u, u_t, v, v_t)\|_{Y_0}^2 + C_2.
\]
Note that \( C_1 \) and \( C_2 \) are independent of \( B \).

We claim that there exists \( K > 0 \) such that \( L_{\gamma_1, \gamma_2}(u, u_t, v, v_t) \geq \frac{1}{8} \|(u, u_t, v, v_t)\|_{Y_0}^2 - K \).

In fact, by Lemma 4.11 item (ii), given \( \tilde{\delta} > 0 \), there exists a constant \( C_{\tilde{\delta}} > 0 \) such that
\[
\int_0^\infty f(s) ds dx \leq \tilde{\delta} \|u\|_{X_\frac{1}{2}}^3 + C_{\tilde{\delta}},
\]
which, together with \( \|u\|_{X_\frac{1}{2}} \leq \lambda_1^{-1} \|u\|_{X_{\frac{1}{2}}^\frac{1}{2}} \), implies
\[
L_{\gamma_1, \gamma_2}(u, u_t, v, v_t) \geq \frac{1}{4} \|(u, u_t, v, v_t)\|_{Y_0}^2 - \int_0^\infty f(s) ds dx
\]
\[
\geq \frac{1}{4} \|(u, u_t, v, v_t)\|_{Y_0}^2 - \tilde{\delta} \lambda_1^{-1} \|u\|_{X_{\frac{1}{2}}^\frac{1}{2}}^2 - C_{\tilde{\delta}}
\]
\[
\geq \left( \frac{1}{4} - \tilde{\delta} \lambda_1^{-1} \right) \|(u, u_t, v, v_t)\|_{Y_0}^2 - C_{\tilde{\delta}}.
\]

Choosing \( \tilde{\delta} = \frac{\lambda_1}{8} \), we get
\[
L_{\gamma_1, \gamma_2}(u, u_t, v, v_t) \geq \frac{1}{8} \|(u, u_t, v, v_t)\|_{Y_0}^2 - K,
\]
where \( K = C_{\lambda_1} > 0 \), which proves the claim.

Now, define the set
\[
\ell_r = \sup \{ \|(u, u_t, v, v_t)\|_{Y_0}^2 : t \geq \tau, \|(u(\tau), u_t(\tau), v(\tau), v_t(\tau))\|_{Y_0} \leq r \}.
\]
Note that \( \ell_r < \infty \) for each \( r > 0 \). In fact, by the proof of Theorem 2.1 we have
\[
\|W(t)\|_{Y_0}^2 = \|(u(t), u_t(t), v(t), v_t(t))\|_{Y_0}^2 \leq 4 \left( \mathcal{E}(\tau) + C_{\lambda_1} \right), \quad t \geq \tau,
\]
where
\[
\mathcal{E}(\tau) = \frac{1}{2} \|W(\tau)\|_{Y_0}^2 + \frac{1}{2} \|u(\tau)\|_{X_\frac{1}{2}}^2 - \int_0^{\|W(\tau)\|_{Y_0}^2} f(s) ds dx
\]
\[
\leq \frac{1}{2} \|W(\tau)\|_{Y_0}^2 + \lambda_1^{-1} \|u(\tau)\|_{X_{\frac{1}{2}}^\frac{1}{2}}^2 + \left| \int \int f(s) ds dx \right|
\]
\[
\leq \left( \frac{1}{2} + \lambda_1^{-1} \right) \|W(\tau)\|_{Y_0}^2 + C_r \|u(\tau)\|_{X_{\frac{1}{2}}^\frac{1}{2}}^2 + C
\]
\[
\leq \left( \frac{1}{2} + \lambda_1^{-1} \right) r + C_r r + C.
\]
with $C_r$ and $C$ given by Lemma 4.11 item (iii). This shows that $\ell_r < \infty$.

Now, we claim that given a bounded set $B \subset Y_0$ there exists $t_0(B) > 0$ such that

$$\|(u, u_t, v, v_t)\|_{Y_0}^{w_0} \leq \max \left\{ 8K, \ell_{C_2+1} \right\} \quad \text{for all} \quad t \geq \tau + t_0(B).$$

In fact, let $B \subset Y_0$ be a bounded set. Let $r_0 > 0$ be such that $B \subset B_{Y_0}(0, r_0)$. By (5.2) and Lemma 4.11 we obtain

$$L_{\gamma_1, \gamma_2}(u(\tau), u_t(\tau), v(\tau), v_t(\tau)) \leq \frac{3}{4} (1 + \lambda_1^{-1}) r_0 + r_0 C_{r_0} + C = T_{r_0},$$

for all $(u(\tau), u_t(\tau), v(\tau), v_t(\tau)) \in B$.

Let $(u(\tau), u_t(\tau), v(\tau), v_t(\tau)) \in B$ be arbitrary. If $\|(u, u_t, v, v_t)\|_{Y_0}^{w_0} > \frac{C_2+1}{C_1}$ for all $t \geq \tau$ then

$$\frac{d}{dt} L_{\gamma_1, \gamma_2}(u, u_t, v, v_t) \leq -C_1 \|(u, u_t, v, v_t)\|_{Y_0}^{w_0} + C_2 \leq -1 \quad \text{for all} \quad t \geq \tau,$$

which implies

$$L_{\gamma_1, \gamma_2}(u, u_t, v, v_t) \leq L_{\gamma_1, \gamma_2}(u(\tau), u_t(\tau), v(\tau), v_t(\tau)) - (t - \tau) \quad \text{for all} \quad t \geq \tau.$$

Thus, $L_{\gamma_1, \gamma_2}(u, u_t, v, v_t) \leq 0$ for all $t \geq \tau + T_{r_0}$. Consequently, using (5.3), we have

$$\|(u, u_t, v, v_t)\|_{Y_0}^{w_0} \leq 8K \quad \text{for all} \quad t \geq \tau + T_{r_0}.$$

On the other hand, if there exists $t_u \geq \tau$ such that $\|(u(t_u), u_t(t_u), v(t_u), v_t(t_u))\|_{Y_0}^{w_0} \leq \frac{C_2+1}{C_1}$ (take the smallest $t_u$ with this property) then

$$\|(u, u_t, v, v_t)\|_{Y_0}^{w_0} \leq \ell_{C_2+1} \frac{C_2+1}{C_1} \quad \text{for all} \quad t \geq t_u.$$

Set

$$B^u = \left\{ w_0 \in B : \text{there exists } t_u^{w_0} > \tau \text{ such that } \|W(t_u^{w_0})w_0\|_{Y_0}^{w_0} = \frac{C_2+1}{C_1} \right\}.$$

We claim that $T_u(B) = \sup\{t_u^{w_0} : w_0 \in B^u\} < \infty$. In fact, suppose to the contrary that there exists a sequence $\{w_0^n\}_{n \in \mathbb{N}} \subset B^u$ such that $t_u^{w_0^n} \to \infty$ as $n \to \infty$. Since $\|W(t)w_0^n\|_{Y_0}^{w_0^n} \geq \frac{C_2+1}{C_1}$ for all $t \leq t_u^{w_0^n}$, we conclude that

$$L_{\gamma_1, \gamma_2}(W(t)w_0^n) \leq L_{\gamma_1, \gamma_2}(w_0^n) - (t - \tau) \leq T_{r_0} - t + \tau \quad \text{for all} \quad \tau \leq t \leq t_u^{w_0^n}.$$

This implies that $\lim_{n \to \infty} L_{\gamma_1, \gamma_2}(W(t_u^{w_0^n})w_0^n) = -\infty$. But, using (5.2), we obtain

$$\frac{1}{4}\|W(t_u^{w_0^n})w_0^n\|_{Y_0}^{w_0^n} \leq L_{\gamma_1, \gamma_2}(W(t_u^{w_0^n})w_0^n) + \int_0^{t_u^{w_0^n}} f(s)ds \int_0 \int_0 f(s)dsdx$$

$$\leq L_{\gamma_1, \gamma_2}(W(t_u^{w_0^n})w_0^n) + \left| \int_0^{t_u^{w_0^n}} f(s)ds \int_0 \int_0 f(s)dsdx \right|$$

$$\leq L_{\gamma_1, \gamma_2}(W(t_u^{w_0^n})w_0^n) + C_{C_2+1} \frac{\|u(t_u^{w_0^n})\|_{Y_0}^{w_0^n}}{A} + C$$

$$\leq L_{\gamma_1, \gamma_2}(W(t_u^{w_0^n})w_0^n) + C_{C_2+1} \frac{\|W(t_u^{w_0^n})w_0^n\|_{Y_0}^{w_0^n}}{A} + C$$

$$= L_{\gamma_1, \gamma_2}(W(t_u^{w_0^n})w_0^n) + C_{C_2+1} \frac{C_2+1}{C_1} + C.$$
where contradicts the fact that \(\lim_{n\to\infty} L_{\gamma_1,\gamma_2}(W(t^n_0))w_0^n = -\infty\).

Taking \(t_0(B) = \max\{T_u(B), T_{\tau_0}\}\), we conclude that
\[
\|(u, u_t, v, v_t)\|_{Y_0}^2 \leq \max \left\{ 8K, \ell \epsilon_{c+1} \right\} \quad \text{for all} \quad t \geq \tau + t_0(B).
\]

This shows that, if \(s \leq t\) and \(B \subset Y_0\) is a bounded set then
\[
S(s, \tau)B \subset B_{Y_0}(0, R) \quad \text{for all} \quad \tau \leq \tau_0(s, B),
\]
where \(\tau_0(s, B) = s - t_0(B)\) and \(R = \max \left\{ 8K, \ell \epsilon_{c+1} \right\}\). Therefore, the process given by (4.16) is pullback strongly bounded dissipative. \(\Box\)

Next, we prove that the solutions of problem (2.6) are uniformly exponentially dominated when the initial data are in bounded subsets of \(Y_0\).

**Theorem 5.2.** Let \(B \subset Y_0\) be a bounded set. If \(W: [\tau, \infty) \to Y_0\) is the global solution of (2.6) starting at \(W_0 \in B\), then there are positive constants \(\sigma = \sigma(B), K_1 = K_1(B)\) and \(K_2 = K_2(B)\) such that
\[
\|W(t)\|_{Y_0}^2 \leq K_1 e^{-\sigma(t-\tau)} + K_2, \quad t \geq \tau.
\]

**Proof.** Let \(r > 0\) be such that \(B \subset B_{Y_0}(0, r)\). We claim that there is \(M_r > 0\) and \(C > 0\) such that \(L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) \leq M_r \|(u, u_t, v, v_t)\|_{Y_0}^2 + C\) for all \(t \geq \tau\). In fact, by (5.2), we have
\[
L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) + \int_\Omega \int_0^u f(s)d\sigma dx \leq \frac{3}{4}(1 + \lambda_1^{-1}) \|(u, u_t, v, v_t)\|_{Y_0}^2.
\]

By the proof of Theorem 5.1, the set
\[
\ell_r = \sup \{ \|(u, u_t, v, v_t)\|_{Y_0}^2; \quad t \geq \tau, \quad \|(u(\tau), u_t(\tau), v(\tau), v_t(\tau))\|_{Y_0}^2 \leq r \} < \infty.
\]

Now, using Lemma 4.11 condition (iii), there are constants \(C_{\ell_r} > 0\) and \(C\) such that
\[
\left| \int_\Omega \int_0^u f(s)d\sigma dx \right| \leq C_{\ell_r} \|u\|_{X_1}^2 + C
\]
whenever \(\|(u(\tau), u_t(\tau), v(\tau), v_t(\tau))\|_{Y_0}^2 \leq r\). Hence, if \(\|(u(\tau), u_t(\tau), v(\tau), v_t(\tau))\|_{Y_0}^2 \leq r\), then
\[
L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) \leq \frac{3}{4}(1 + \lambda_1^{-1}) \|(u, u_t, v, v_t)\|_{Y_0}^2 - \int_\Omega \int_0^u f(s)d\sigma dx
\]
\[
\leq \frac{3}{4}(1 + \lambda_1^{-1}) \|(u, u_t, v, v_t)\|_{Y_0}^2 + C_{\ell_r} \|u\|_{X_1}^2 + C
\]
\[
\leq M_r \|(u, u_t, v, v_t)\|_{Y_0}^2 + C,
\]
where \(M_r = \frac{3}{4}(1 + \lambda_1^{-1}) + C_{\ell_r} > 0\), which proves the claim.

Using the proof of Theorem 5.1, it follows by (5.7) that
\[
\frac{d}{dt} L_{\gamma_1,\gamma_2}(W(t)) \leq -\frac{C_1}{M_r} L_{\gamma_1,\gamma_2}(W(t)) + \frac{CC_1}{M_r} + C_2, \quad t \geq \tau,
\]
which implies
\[
L_{\gamma_1,\gamma_2}(W(t)) \leq L_{\gamma_1,\gamma_2}(W(\tau)) e^{-\frac{C_1}{M_r}(t-\tau)} + \left( C_2 + \frac{CC_1}{M_r} \right) \frac{M_r}{C_1}, \quad t \geq \tau,
\]
and, using the fact that \( \frac{1}{8} \|W(t)\|_{Y_0}^2 - K \leq L_{\gamma_1, \gamma_2}(W(t)) \) (see (5.8)), we conclude that
\[
\|W(t)\|_{Y_0}^2 \leq 8L_{\gamma_1, \gamma_2}(W(\tau))e^{-\frac{C_2}{M_1}(t-\tau)} + 8 \left( C_2 \frac{M_1}{C_1} + C + K \right), \quad t \geq \tau.
\]
Since \( L_{\gamma_1, \gamma_2}(W(\tau)) \leq K, M_r + C \), we get
\[
\|W(t)\|_{Y_0}^2 \leq 8(K, M_r + C)e^{-\frac{C_2}{M_1}(t-\tau)} + 8 \left( C_2 \frac{M_1}{C_1} + C + K \right), \quad t \geq \tau,
\]
and the result follows by taking \( \sigma = \frac{C_2}{M_1} \), \( K_1 = 8(K, M_r + C) \) and \( K_2 = 8 \left( C_2 \frac{M_1}{C_1} + C + K \right) \).

**Theorem 5.3.** Let \( B \subset Y_0 \) be a bounded set and denote by \( L: [\tau, \infty) \to Y_0 \) the solution of the homogeneous problem (4.18) starting in \( W_0 \in B \). Then there exist positive constants \( K = K(B) \) and \( \zeta \) such that
\[
\|L(t)\|_{Y_0}^2 \leq Ke^{-\zeta(t-\tau)}, \quad t \geq \tau.
\]

**Proof.** The proof is analogous to the proof of Theorem 5.2 taking \( f \equiv 0 \). \(\square\)

**Proposition 5.4.** For each \( t > \tau \in \mathbb{R} \), the evolution process \( S(t, \tau): Y_0 \to Y_0 \) given in (4.16) is a compact map.

**Proof.** Using the identity (4.14), the energy functional (4.15) and the Cauchy-Schwartz and Young inequalities, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{X^\frac{1}{2}}^2 + \|u_l\|_{X^\frac{1}{2}}^2 + \|v\|_{X^\frac{1}{2}}^2 + \|v_l\|_{X^\frac{1}{2}}^2 \right) + \eta \|u\|_{X^\frac{1}{2}}^2 + \eta \|v\|_{X^\frac{1}{2}}^2 
\leq \frac{1}{2c} \|f(u)\|_{X}^2 + \frac{\epsilon^2}{2} \|u_l\|_{X^\frac{1}{2}}^2,
\]
for all \( \epsilon > 0 \), where \( \tilde{c} > 0 \) is the embedding constant of \( X^\frac{1}{2} \hookrightarrow X \). Choosing \( \epsilon = \frac{2c}{M_1} \), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{X^\frac{1}{2}}^2 + \|u_l\|_{X^\frac{1}{2}}^2 + \|v\|_{X^\frac{1}{2}}^2 + \|v_l\|_{X^\frac{1}{2}}^2 \right) + \frac{\eta}{2} \|u\|_{X^\frac{1}{2}}^2 + \eta \|v\|_{X^\frac{1}{2}}^2 
\leq \frac{c^2}{2\eta} \|f(u)\|_{X}^2.
\]

Now, knowing that the embedding \( X^\frac{1}{2} \hookrightarrow L^{2\rho}(\Omega) \) holds for \( 1 < \rho \leq \frac{2}{n-2} \), and using Lemma 4.11 condition (i), we get
\[
\|f(u)\|_{X}^2 \leq \int_{\Omega} [c(1 + |u|^\rho)]^2 dx \leq c_1 \int_{\Omega} (1 + |u|^{2\rho}) dx 
= c_1 |\Omega| + c_1 \|u\|_{L^{2\rho}(\Omega)}^{2\rho} \leq c_1 |\Omega| + c_2 \|u\|_{X^\frac{1}{2}}^{2\rho} \leq c_1 |\Omega| + c_2 \|W\|_{Y_0}^{2\rho},
\]
where \( c_1, c_2 \) are positive constants and \( W(t) = (u(t), u_l(t), v(t), v_l(t)) \). Thus, combining (5.9) and (5.10), we obtain
\[
\frac{d}{dt} \left( \|W\|_{Y_0}^2 + \|u\|_{X^\frac{1}{2}}^2 \right) + \eta \|u_l\|_{X^\frac{1}{2}}^2 + 2\eta \|v_l\|_{X^\frac{1}{2}}^2 \leq \frac{c^2 c_1 |\Omega|}{\eta} + \frac{c^2 c_2}{\eta} \|W\|_{Y_0}^{2\rho}.
\]
Integrating the previous inequality from $\tau$ to $t$, we obtain

\[
\|W(t)\|_{\mathcal{Y}_0}^2 + \|u(t)\|_{X^\frac{3}{4}}^2 + \eta \int_{\tau}^{t} \|u_r(r)\|_{X^\frac{3}{4}}^2 dr + 2\eta \int_{\tau}^{t} \|v_t(r)\|_{X^\frac{3}{4}}^2 dr
\eta \int_{\tau}^{t} \|W(r)\|_{X^\frac{3}{4}}^2 dr + \|W(\tau)\|_{\mathcal{Y}_0}^2 + \|u(\tau)\|_{X^\frac{3}{4}}^3
\leq \frac{\beta c_1|\Omega|}{\eta} (t - \tau) + \frac{\beta c_2}{\eta} \int_{\tau}^{t} \|W(r)\|_{X^\frac{3}{4}}^2 dr + \|W(\tau)\|_{\mathcal{Y}_0}^2 + \|u(\tau)\|_{X^\frac{3}{4}}^3
\leq \frac{\beta c_1|\Omega|}{\eta} (t - \tau) + \frac{\beta c_2}{\eta} \int_{\tau}^{t} \|W(r)\|_{X^\frac{3}{4}}^2 dr + \frac{1 + \lambda_1^{-1}}{\eta} \|W(\tau)\|_{\mathcal{Y}_0}^2,
\]

where we have used the Poincaré inequality $\|u(\tau)\|_{X^\frac{3}{4}}^3 \leq \lambda_1^{-1}\|u(\tau)\|_{X^\frac{3}{4}}^2$. Also, note that inequality (5.11) implies

\[
\int_{\tau}^{t} \|u_t(r)\|_{X^\frac{3}{4}}^2 dr + \int_{\tau}^{t} \|v_t(r)\|_{X^\frac{3}{4}}^2 dr
\leq \frac{\beta c_1|\Omega|}{\eta^2} (t - \tau) + \frac{\beta c_2}{\eta^2} \int_{\tau}^{t} \|W(r)\|_{X^\frac{3}{4}}^2 dr + \frac{1 + \lambda_1^{-1}}{\eta^2} \|W(\tau)\|_{\mathcal{Y}_0}^2.
\]

Now, consider the original system (1.1). By taking the inner product of the first equation in (1.1) with $A^\frac{1}{2}u$, and also the inner product of the second equation in (1.1) with $A^\frac{1}{2}v$, and noticing the identity

\[
\langle u_t, A^\frac{1}{2}u \rangle_X = \frac{d}{dt} \langle u, A^\frac{1}{2}u \rangle_X - \|u_t\|_{X^\frac{3}{4}}^2,
\]

we obtain,

\[
\frac{d}{dt} \langle u_t, A^\frac{1}{2}u \rangle_X - \|u_t\|_{X^\frac{3}{4}}^2 + \|u\|_{X^\frac{3}{4}}^2 + \frac{\eta}{2} \frac{d}{dt} \|u\|_{X^\frac{3}{4}}^2 + a_4(t) \langle A^\frac{1}{2}v, A^\frac{1}{2}u \rangle_X + \frac{\eta}{2} \frac{d}{dt} \|v\|_{X^\frac{3}{4}}^2 - a_4(t) \langle A^\frac{1}{2}u_t, A^\frac{1}{2}v \rangle_X = \langle f(u), A^\frac{1}{2}u \rangle_X.
\]

Once again, using the Cauchy-Schwarz and Young inequalities, we have

\[
\frac{d}{dt} \left( \langle u_t, A^\frac{1}{2}u \rangle_X + \langle v_t, A^\frac{1}{2}v \rangle_X \right) + \frac{\eta}{2} \frac{d}{dt} \left( \|u\|_{X^\frac{3}{4}}^2 + \|v\|_{X^\frac{3}{4}}^2 \right) + \|u\|_{X^\frac{3}{4}}^2 + \|u\|_{X^\frac{3}{4}}^2 + \|v\|_{X^\frac{3}{4}}^2
\leq \|u_t\|_{X^\frac{3}{4}}^2 + \|v_t\|_{X^\frac{3}{4}}^2 + \|v_t\|_{X^\frac{3}{4}}^2 \|u\|_{X^\frac{3}{4}}^2 + \|u_t\|_{X^\frac{3}{4}}^2 \|v\|_{X^\frac{3}{4}}^2 + \|f(u)\|_X \|u\|_{X^\frac{3}{4}}^2
\leq \left( 1 + \frac{1}{2\epsilon_2} \right) \|u_t\|_{X^\frac{3}{4}}^2 + \left( 1 + \frac{1}{2\epsilon_1} \right) \|v_t\|_{X^\frac{3}{4}}^2 + \frac{\epsilon_1}{2} \|u\|_{X^\frac{3}{4}}^2 + \frac{\epsilon_2}{2} \|v\|_{X^\frac{3}{4}}^2
\]

for all $\epsilon_1, \epsilon_2 > 0$. Choosing $\epsilon_1 = \epsilon_2 = \frac{1}{a_1}$, and using (5.10), we get

\[
\frac{d}{dt} \left( \langle u_t, A^\frac{1}{2}u \rangle_X + \langle v_t, A^\frac{1}{2}v \rangle_X \right) + \frac{\eta}{2} \frac{d}{dt} \left( \|u\|_{X^\frac{3}{4}}^2 + \|v\|_{X^\frac{3}{4}}^2 \right) + \frac{1}{2} \|u\|_{X^\frac{3}{4}}^2 + \frac{1}{2} \|v\|_{X^\frac{3}{4}}^2
\leq \frac{2 + a_1^2}{2} \|u_t\|_{X^\frac{3}{4}}^2 + \frac{2 + a_1^2}{2} \|v_t\|_{X^\frac{3}{4}}^2 + \frac{c_1|\Omega|}{2} \|W\|_{\mathcal{Y}_0}^2 + \frac{c_2}{2} \|W\|_{\mathcal{Y}_0}^2 + \frac{1}{2} \|W\|_{\mathcal{Y}_0}^2.
\]

Integrating the previous inequality from $\tau$ to $t$, and using (5.12), we obtain
\[
\frac{\eta}{2} \left( \|u(t)\|^2_{X^{\frac{1}{2}}} + \|v(t)\|^2_{X^{\frac{1}{2}}} \right) + \frac{1}{2} \int_{\tau}^{t} \|u(r)\|^2_{X^{\frac{1}{2}}} dr + \frac{1}{2} \int_{\tau}^{t} \|v(r)\|^2_{X^{\frac{1}{2}}} dr \leq \frac{2 + a_1^2}{2} \left( \int_{\tau}^{t} \|u_t(r)\|^2_{X^{\frac{1}{2}}} dr + \int_{\tau}^{t} \|v_t(r)\|^2_{X^{\frac{1}{2}}} dr \right) + \frac{c_1|\Omega|}{2} (t - \tau) + \frac{c_2}{2} \int_{\tau}^{t} \|W(r)\|^2_{Y_0} dr + \frac{\eta}{2} \left( \|u(t)\|^2_{X^{\frac{1}{2}}} + \|v(t)\|^2_{X^{\frac{1}{2}}} \right)
\]

Integrating the previous inequality from \( \tau \) to \( t \), for \( \tau < r < t \), we have

\[
\frac{d}{dt} \left( \|u(t)\|^2_{X^{\frac{1}{2}}} + \|u\|^2_{X^{\frac{1}{2}}} + \|u_t\|^2_{X^{\frac{1}{2}}} + \|v_t\|^2_{X^{\frac{1}{2}}} + \|v\|^2_{X^{\frac{1}{2}}} \right) \leq \frac{c_1|\Omega|}{\eta} + \frac{c_2}{\eta} \|W\|^2_{Y_0}.
\]

Integrating the previous inequality from \( r \) to \( t \), for \( \tau < r < t \), we have

\[
\|u(t)\|^2_{X^{\frac{1}{2}}} + \|u(t)\|^2_{X^{\frac{1}{2}}} + \|u(t)\|^2_{X^{\frac{1}{2}}} + \|v(t)\|^2_{X^{\frac{1}{2}}} + \|v(t)\|^2_{X^{\frac{1}{2}}} \leq \frac{c_1|\Omega|}{\eta} (t - r) + \frac{c_2}{\eta} \int_{r}^{t} \|W(s)\|^2_{Y_0} ds + \|u(r)\|^2_{X^{\frac{1}{2}}} + \|u(r)\|^2_{X^{\frac{1}{2}}} + \|v_t(r)\|^2_{X^{\frac{1}{2}}} + \|v_t(r)\|^2_{X^{\frac{1}{2}}}.
\]
consequently,
\[
\|u(t)\|_{X^\frac{1}{2}}^2 + \|u_t(t)\|_{X^\frac{1}{2}}^2 + \|v(t)\|_{X^\frac{1}{2}}^2 + \|v_t(t)\|_{X^\frac{1}{2}}^2 \\
\leq \frac{c_1|\Omega|}{\eta} (t - r) + \frac{c_2}{\eta} \int_r^t \|W(s)\|_{Y_0}^2 ds + \hat{k}\|W(r)\|_{Y_0}^2 + \|u(r)\|_{X^\frac{1}{2}}^2 \\
+ \|u_t(r)\|_{X^\frac{1}{2}}^2 + \|v(r)\|_{X^\frac{1}{2}}^2 + \|v_t(r)\|_{X^\frac{1}{2}}^2,
\]
(5.14)

where we have used the embedding \( X^\frac{1}{2} \hookrightarrow \tilde{X}^\frac{1}{2} \), i.e., \( \|u(r)\|_{\tilde{X}^\frac{1}{2}}^2 \leq \hat{k}\|u(r)\|_{X^\frac{1}{2}}^2 \).

Now, by integrating inequality (5.14), with respect to \( t \), from \( \tau \) to \( t \), we obtain
\[
(t - \tau) \left( \|u(t)\|_{X^\frac{1}{2}}^2 + \|u_t(t)\|_{X^\frac{1}{2}}^2 + \|v(t)\|_{X^\frac{1}{2}}^2 + \|v_t(t)\|_{X^\frac{1}{2}}^2 \right) \\
\leq \frac{c_1|\Omega|}{2\eta} (t - \tau)^2 + \frac{c_2}{\eta} \int_\tau^t \|W(s)\|_{Y_0}^2 ds + \hat{k} \int_\tau^t \|W(r)\|_{Y_0}^2 dr \\
+ \int_\tau^t \|u(r)\|_{X^\frac{1}{2}}^2 dr + \int_\tau^t \|u_t(r)\|_{X^\frac{1}{2}}^2 dr \\
+ \int_\tau^t \|v(r)\|_{X^\frac{1}{2}}^2 dr + \int_\tau^t \|v_t(r)\|_{X^\frac{1}{2}}^2 dr.
\]
(5.15)

Combining the inequalities obtained in (5.12), (5.13) and (5.15), we get
\[
\|u(t)\|_{X^\frac{1}{2}}^2 + \|u_t(t)\|_{X^\frac{1}{2}}^2 + \|v(t)\|_{X^\frac{1}{2}}^2 + \|v_t(t)\|_{X^\frac{1}{2}}^2 \\
\leq \frac{\hat{c}^2 c_1|\Omega|(3 + \alpha^2)}{\eta^2} + c_1|\Omega| + \frac{c_1|\Omega|}{2\eta} (t - \tau) + \frac{c_2}{\eta(t - \tau)} \int_\tau^t \|W(s)\|_{Y_0}^2 ds dr \\
+ \frac{1}{t - \tau} \left( \frac{\hat{c}^2 c_2(3 + \alpha^2)}{\eta^2} + c_2 \right) \int_\tau^t \|W(r)\|_{Y_0}^2 dr + \frac{\hat{k} + 1}{t - \tau} \int_\tau^t \|W(r)\|_{Y_0}^2 dr \\
+ \frac{1}{t - \tau} \|W(t)\|_{Y_0}^2 + \frac{(1 + \lambda_1^{-1})(3 + \alpha^2) + \eta(1 + \eta)}{\eta(t - \tau)} \|W(\tau)\|_{Y_0}^2.
\]
(5.16)

Now, if the global solution \( W(t) = (u(t), u_t(t), v(t), v_t(t)) \) of the problem (1.1) - (1.3) starts in a bounded subset \( B \) of \( Y_0 \), then
\[
\|W(\tau)\|_{Y_0} \leq M
\]
for some positive constant \( M \). Moreover, remember that from Theorem 5.2 there exist positive constants \( \sigma = \sigma(B), K_1 = K_1(B) \) and \( K_2 = K_2(B) \) such that
\[
\|W(t)\|_{Y_0}^2 \leq K_1e^{-\sigma(t-\tau)} + K_2, \quad t \geq \tau.
\]

With this, we can handle with the three integrals that appear on the right hand side of inequality (5.16). In fact, first note that
\[
\int_\tau^t \|W(r)\|_{Y_0}^2 dr \leq \int_\tau^t [K_1e^{-\sigma(r-\tau)} + K_2] dr \leq \frac{K_1}{\sigma} + K_2(t - \tau)
\]
and
\[
\int_\tau^t \|W(r)\|_{Y_0}^{2\rho} dr \leq \int_\tau^t [\hat{K}_1e^{-\rho\sigma(r-\tau)} + \hat{K}_2] dr \leq \frac{\hat{K}_1}{\rho\sigma} + \hat{K}_2(t - \tau),
\]
where $\tilde{K}_1, \tilde{K}_2$ are positive constants. For the last integral remaining, note that
\[
\int_{\tau}^{t} \|W(s)\|_{Y_0}^{2\rho} ds \leq \int_{\tau}^{t} \left[ \tilde{K}_1 e^{-\rho \sigma (s-\tau)} + \tilde{K}_2 r \right] ds \leq \frac{\tilde{K}_1}{\rho \sigma} e^{-\rho \sigma (s-\tau)} + \tilde{K}_2 (t - r),
\]
for positive constants $\tilde{K}_1$ and $\tilde{K}_2$, and then it follows that
\[
\int_{\tau}^{t} \int_{\tau}^{r} \|W(s)\|_{Y_0}^{2\rho} ds dr \leq \int_{\tau}^{t} \left[ \frac{\tilde{K}_1}{\rho \sigma} e^{-\rho \sigma (s-\tau)} + \tilde{K}_2 (t - r) \right] dr
\leq \frac{\tilde{K}_1}{\rho \sigma^2} + \frac{\tilde{K}_2}{2} (t - \tau)^2.
\]

Finally, combining all the estimates in (5.16) – 5.17, we conclude that there exist positive constants $k_1, k_2, k_3, k_4, k_5$ such that
\[
\|u(t)\|_{X_T^4} + \|u^t(t)\|_{X_T^4} + \|v(t)\|_{X_T^4} + \|v^t(t)\|_{X_T^4}
\leq k_1 + k_2 (t - \tau) + \frac{1}{t - \tau} [k_3 e^{-k_4 (t - \tau)} + k_5].
\]

Hence, $S(t, \tau) B$ is bounded in $X_T^4 \times X_T^4 \times X_T^4 \times X_T^4$. Since $X_T^4 \times X_T^4 \times X_T^4 \times X_T^4 \rightarrow Y_0$, and this embedding is compact, we conclude that $S(t, \tau): Y_0 \rightarrow Y_0$, given in (4.16), is compact for each $t > \tau$.

We end this section with the proof of Theorem 2.2.

**Proof of Theorem 2.2.** Theorem 5.1 assures that the evolution process $S(t, \tau): Y_0 \rightarrow Y_0$ given by (4.16) is pullback strongly bounded dissipative. Additionally, it follows by Proposition 5.3 that $S(t, \tau): Y_0 \rightarrow Y_0$ is compact, and, consequently, it is pullback asymptotically compact. Now the result is a simple consequence of Theorem 3.9.

6. Regularity of the pullback attractor

The purpose of this section is to show that the regularity of the pullback attractor can be improved, using energy estimates and progressive increases of regularity.

**Proof of Theorem 2.3.** Let $\xi: \mathbb{R} \rightarrow Y_0$ be a bounded global solution for the system (1.1). Since $\bigcup_{t \in \mathbb{R}} \mathbb{A}(t)$ is bounded in $Y_0$ (see Theorem 2.2), we have $\{\xi(t): t \in \mathbb{R}\}$ is a bounded subset of $Y_0$ by Theorem 3.1. Moreover, $\xi(\cdot) = (\mu(\cdot), \mu(\cdot), \nu(\cdot), \nu(\cdot)): \mathbb{R} \rightarrow Y_0$ is such that $\xi(t) \in \mathbb{A}(t)$ for all $t \in \mathbb{R}$, and by (1.17),
\[
\xi(t) = L(t, \tau) \xi(\tau) + \int_{\tau}^{t} L(t, s) F(\xi(s)) ds, \quad t \geq \tau.
\]

Using the decay of $L(\cdot, \cdot)$, which was established in Theorem 5.3 and letting $\tau \rightarrow -\infty$, we get
\[
\xi(t) = \int_{-\infty}^{t} L(t, s) F(\xi(s)) ds, \quad t \in \mathbb{R}.
\]

Now, for $\tau \in \mathbb{R}$ fixed, we write $W_0 = (u_0, u_1, v_0, v_1) = \xi(\tau)$ and consider
\[
(u(t), u^t(t), v(t), v^t(t)) = U(t, \tau) W_0 = \int_{\tau}^{t} L(t, s) F(S(s, \tau) W_0) ds,
\]
where \( U(\cdot, \cdot) \) is defined as in (6.19). Note that \((u(\cdot), v(\cdot))\) solves the system

\[
\begin{cases}
u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}} u_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}} v_t = f(u(t, \tau; u_0)), & (x, t) \in \Omega \times (\tau, \infty), \\
u_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}} v_v - a_\epsilon(t)(-\Delta)^{\frac{1}{2}} u_v = 0, & (x, t) \in \Omega \times (\tau, \infty),
\end{cases}
\]

with

\[
u(\tau, x) = 0, \ v(\tau, x) = 0, \ x \in \Omega.
\]

To estimate the solution of (6.1) - (6.2) for \((u_0, u_1, v_0, v_1)\) in a bounded subset \(B \subset Y_0\), we again consider the maps

\[
E(t) = \frac{1}{2} \left\| u(t) \right\|^2_{X^\frac{1}{2}} + \frac{1}{2} \left\| u(t) \right\|^2_X + \frac{1}{2} \left\| u(t) \right\|^2_{X^\frac{3}{2}} + \frac{1}{2} \left\| v(t) \right\|^2_{X^\frac{1}{2}} + \frac{1}{2} \left\| v(t) \right\|^2_X - \int_0^t \int_\Omega f(s)dsdx,
\]

and

\[
L(t) = \frac{1}{2} \left\| u(t) \right\|^2_{X^\frac{1}{2}} + \frac{1}{2} \left\| u(t) \right\|^2_X + \frac{1}{2} \left\| u(t) \right\|^2_{X^\frac{3}{2}} + \frac{1}{2} \left\| v(t) \right\|^2_{X^\frac{1}{2}} + \frac{1}{2} \left\| v(t) \right\|^2_X + \gamma_1 \left\langle u(t), u(t) \right\rangle_X + \gamma_2 \left\langle v(t), v(t) \right\rangle_X
\]

with \( \gamma_1, \gamma_2 \in \mathbb{R}^+ \). Using (6.1), we can write

\[
\frac{d}{dt} L(t) = \frac{d}{dt} \left( E(t) + \gamma_1 \left\langle u(t), u(t) \right\rangle_X + \gamma_2 \left\langle v(t), v(t) \right\rangle_X + \int_0^t \int_\Omega f(s)dsdx \right)
= -\eta \left\| u(t) \right\|^2_{X^\frac{1}{2}} - \eta \left\| v(t) \right\|^2_{X^\frac{1}{2}} + \gamma_1 \left\| u(t) \right\|^2_X - \gamma_1 \left\| u(t) \right\|^2_{X^\frac{3}{2}} + \left( \gamma_1 \left\| u(t) \right\|^2_X + \left\| u(t) \right\|^2_{X^\frac{3}{2}} \right) - \gamma_1 \eta \left\langle A^\frac{1}{2} u, u(t) \right\rangle_X
- \gamma_1 a_\epsilon(t) \left\langle A^\frac{1}{2} u, u(t) \right\rangle_X + \gamma_1 \left\langle u(t), f(u(t)) \right\rangle_X + \gamma_2 \left\| v(t) \right\|^2_X - \gamma_2 \left\| v(t) \right\|^2_{X^\frac{1}{2}} - \gamma_2 \eta \left\langle A^\frac{1}{2} v, v(t) \right\rangle_X
+ \gamma_2 a_\epsilon(t) \left\langle A^\frac{1}{2} v, v(t) \right\rangle_X + \left\langle f(u(t)), u(t) \right\rangle_X.
\]

In the first place, let’s deal with the nonlinearity \( f \). By Lemma 4.11 it follows that for each \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that

\[
\int_\Omega f(u)udx \leq \delta \left\| u \right\|^2_X + C_\delta.
\]

Further, once the condition \( 1 < \frac{n+1}{n-2} \leq \rho < \frac{n}{n-2} \) implies \( X^\frac{1}{2} \hookrightarrow L^{2\rho}(\Omega) \), and using again Lemma 4.11 condition (i), we have

\[
\left\| f(u) \right\|_X \leq \left( \int_\Omega [c(1 + \left| u \right|^\rho)]^2 dx \right)^{\frac{1}{2}} \leq \tilde{c} \left( \left| \Omega \right|^\frac{1}{\rho} + \int_\Omega \left| u \right|^{2\rho} dx \right)^{\frac{1}{2}}
\leq \tilde{c} \left( \left| \Omega \right|^\frac{1}{\rho} + \left\| u \right\|^\rho_{L^{2\rho}(\Omega)} \right) \leq \tilde{c} \left\| u \right\|^\rho_{X^\frac{1}{2}} + \tilde{c} \left| \Omega \right|^\frac{1}{\rho} \leq \tilde{c} \left| \Omega \right|^\frac{1}{\rho} + \tilde{c} \left| \Omega \right|^\frac{1}{\rho} = \tilde{c},
\]

whenever \( \left\| u \right\|_{X^\frac{1}{2}} \leq r. \)
Hence, using the Poincaré and Young Inequalities, one can obtain
\[
\frac{d}{dt} \mathcal{L}(t) \leq -\eta \frac{1}{c^2} \|u_t\|_{X^\frac{1}{2}}^2 - \frac{\eta}{c^2} \|v_t\|_{X^\frac{1}{2}}^2 + \gamma_1 \|u_t\|_{X^\frac{1}{2}}^2 - \gamma_1 \|u_t\|_{X^\frac{1}{2}}^2 + \gamma_1 \eta \left( \frac{\epsilon_1}{2} \|u_t\|_{X^\frac{1}{2}}^2 + \frac{1}{2\epsilon_1} \|u_t\|_{X^\frac{1}{2}}^2 \right) \\
+ \gamma_1 a_1 \left( \frac{1}{2\epsilon_2} \|u_t\|_{X^\frac{1}{2}}^2 + \frac{\epsilon_2}{2} \|v_t\|_{X^\frac{1}{2}}^2 \right) + \gamma_1 \lambda \left( \frac{1}{2} \|u_t\|_{X^\frac{1}{2}}^2 + \gamma_1 C \right) + \gamma_2 \|v_t\|_{X^\frac{1}{2}}^2 - \gamma_2 \|v_t\|_{X^\frac{1}{2}}^2 \\
+ \gamma_2 a_1 \left( \frac{1}{2\epsilon_3} \|v_t\|_{X^\frac{1}{2}}^2 + \frac{\epsilon_3}{2} \|u_t\|_{X^\frac{1}{2}}^2 \right) + \gamma_2 \eta \left( \frac{\epsilon_4}{2} \|v_t\|_{X^\frac{1}{2}}^2 + \frac{1}{2\epsilon_4} \|v_t\|_{X^\frac{1}{2}}^2 \right) + \frac{1}{2\epsilon_5} \|f(u_t)\|_{X^\frac{1}{2}}^2 + \frac{\epsilon_5}{2} \|u_t\|_{X^\frac{1}{2}}^2 \\
\leq -\gamma_1 \left( 1 - \delta \lambda \frac{1}{2} - \eta \frac{\epsilon_1}{2} - a_1 \frac{1}{2\epsilon_2} \right) \|u_t\|_{X^\frac{1}{2}}^2 - \frac{\eta}{2c} - \gamma_1 \eta \frac{1}{2\epsilon_1} - \gamma_2 a_1 \frac{\epsilon_3}{2} - \frac{\epsilon_5}{2} \|u_t\|_{X^\frac{1}{2}}^2 \\
- \gamma_2 \left( 1 - a_1 \frac{1}{2\epsilon_3} - \eta \frac{\epsilon_4}{2} \right) \|v_t\|_{X^\frac{1}{2}}^2 - \frac{\eta}{2c} - \gamma_2 a_1 \frac{\epsilon_2}{2} - \gamma_2 \eta \frac{1}{2\epsilon_4} \|v_t\|_{X^\frac{1}{2}}^2 + \frac{1}{2\epsilon_5} \|v_t\|_{X^\frac{1}{2}}^2 + \frac{\epsilon_5}{2} \|v_t\|_{X^\frac{1}{2}}^2 + \gamma_1 C \delta
\]
for all \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5 > 0 \), where \( c > 0 \) is the embedding constant of \( X^\frac{1}{2} \hookrightarrow X \). Choosing
\[
\delta = \frac{\lambda_1}{8}, \quad \epsilon_1 = \frac{1}{\eta}, \quad \epsilon_2 = 2a_1, \quad \epsilon_3 = 2a_1, \quad \epsilon_4 = \frac{1}{\eta}, \quad \epsilon_5 = \frac{\eta}{c^2},
\]
it follows that
\[
\frac{d}{dt} \mathcal{L}(t) \leq -\frac{1}{8} \gamma_1 \|u_t\|_{X^\frac{1}{2}}^2 - \frac{\eta}{2c^2} - \gamma_1 \left( 1 + \eta \frac{\epsilon_1}{2} \right) \|u_t\|_{X^\frac{1}{2}}^2 - \frac{\eta}{4} \gamma_2 \|v_t\|_{X^\frac{1}{2}}^2 \\
- \frac{\eta}{2c^2} - \gamma_2 \left( 1 + \eta \frac{\epsilon_2}{2} \right) \|v_t\|_{X^\frac{1}{2}}^2 + \frac{\epsilon_5^2 \gamma_1 C}{2} + \gamma_1 C \delta.
\]
Now, note that one can take \( \gamma_1 > 0, i = 1, 2 \), sufficiently small such that
\[
\gamma_i < \frac{\eta}{8c^2} \min \left\{ \frac{1}{a_i^2}, \left( 1 + \eta \frac{\epsilon_1}{2} \right) \right\}, \quad i = 1, 2.
\]
Setting
\[
C_1 = \min \left\{ \frac{1}{8} \gamma_1, \frac{\eta}{2c^2} - \gamma_1 \left( 1 + \eta \frac{\epsilon_1}{2} \right) - \gamma_2 a_1^2, \frac{1}{4} \gamma_2, \frac{\eta}{2c^2} - \gamma_1 a_1^2 - \gamma_2 \left( 1 + \eta \frac{\epsilon_2}{2} \right) \right\} > 0
\]
and \( C_2 = \frac{\epsilon_5^2 \gamma_1 C}{2} + \gamma_1 C \delta > 0 \), we obtain
\[
\frac{d}{dt} \mathcal{L}(t) \leq -C_1 \|u, u_t, v, v_t\|_{Y_0}^2 + C_2.
\]
Using (5.3) and (5.4), we get
\[
\frac{1}{4} \|(u, u_t, v, v_t)\|_{Y_0}^2 \leq \mathcal{L}(t) \leq \frac{3(1 + \lambda_1^{-1})}{4} \|(u, u_t, v, v_t)\|_{Y_0}^2,
\]
and putting \( C_3 = C_1 \left( \frac{3(1 + \lambda_1^{-1})}{4} \right)^{-1} \), one has
\[
\frac{1}{4} \|(u, u_t, v, v_t)\|_{Y_0}^2 \leq \mathcal{L}(t) \leq \mathcal{L}(t)e^{-C_3(t-t)} + \frac{C_2}{C_3}, \quad t \geq \tau.
\]
From this, we obtain
\[
\bigcup_{\tau \leq s \leq t} U(s, \tau)B \text{ is a bounded subset of } Y_0.
\]
On the other hand, note that \((\phi, \varphi) = (u_t, v_t)\) solves the system

\[
\begin{aligned}
\phi_t - \Delta \phi + \phi + \eta(-\Delta)^{\frac{1}{2}} \varphi_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}} \varphi + a_\epsilon'(t)(-\Delta)^{\frac{1}{2}} \phi &= f'(u)\phi, \\
\varphi_t - \Delta \varphi + \eta(-\Delta)^{\frac{1}{2}} \varphi_t - a_\epsilon(t)(-\Delta)^{\frac{1}{2}} \phi - a_\epsilon'(t)(-\Delta)^{\frac{1}{2}} \phi &= 0.
\end{aligned}
\]

(6.3)

We want to estimate \((\phi, \varphi_t, \varphi, \varphi_t)\) in \(Y_0\), but our solutions are not regular enough for this to be done in a direct way. Thus, instead, the process will be done by progressive increases of regularity. For \(\alpha > 0\), let us consider the fractional power spaces \(X^\alpha = D(A^\alpha)\) endowed with the graph norm, and let \(X^{-\alpha} = (X^\alpha)'\). For

\[
(\phi, \varphi_t, \varphi, \varphi_t) \in X^{\frac{1}{2} - \frac{\alpha}{2}} \times X^{-\frac{\alpha}{2}} \times X^{\frac{1}{2} - \frac{\alpha}{2}} \times X^{-\frac{\alpha}{2}};
\]

let us define

\[
\mathcal{L}_\alpha(t) = \frac{1}{2} \left( \|\phi(t)\|_{X^{\frac{1}{2} - \frac{\alpha}{2}}}^2 + \|\phi(t)\|_{X^{-\frac{\alpha}{2}}}^2 + \|\varphi(t)\|_{X^{\frac{1}{2} - \frac{\alpha}{2}}}^2 + \|\varphi(t)\|_{X^{-\frac{\alpha}{2}}}^2 \right)
+ \gamma_1 \langle \phi(t), \varphi_t(t) \rangle_{X^{-\frac{\alpha}{2}}} + \gamma_2 \langle \varphi(t), \varphi_t(t) \rangle_{X^{-\frac{\alpha}{2}}},
\]

with \(\gamma_1, \gamma_2 \in \mathbb{R}^+\). Using (6.3), we obtain

\[
\frac{d}{dt} \mathcal{L}_\alpha(t) = \langle \phi, \varphi_t \rangle_{X^{\frac{1}{2} - \frac{\alpha}{2}}} + \langle \phi, \varphi \rangle_{X^{-\frac{\alpha}{2}}} + \langle \varphi_t, \varphi \rangle_{X^{\frac{1}{2} - \frac{\alpha}{2}}} + \langle \varphi_t, \varphi_t \rangle_{X^{-\frac{\alpha}{2}}}
+ \gamma_1 \langle \phi_t, \phi_t \rangle_{X^{-\frac{\alpha}{2}}} + \gamma_1 \langle \phi, \varphi_t \rangle_{X^{-\frac{\alpha}{2}}} + \gamma_2 \langle \varphi_t, \varphi_t \rangle_{X^{-\frac{\alpha}{2}}}
+ \gamma_2 \langle \varphi, \varphi_t \rangle_{X^{-\frac{\alpha}{2}}}
+ \gamma_1 \langle \phi, f'(u) \rangle_{X^{-\frac{\alpha}{2}}} + \langle f'(u), \phi \rangle_{X^{-\frac{\alpha}{2}}}.
\]

Next, we shall estimate the terms that appear on the right hand side of the above expression, beginning with those in which the nonlinearity \(f'\) is explicit. To do this, consider

\[
\alpha_1 = \frac{(\rho - 1)(n - 2)}{2}.
\]

Since \(\frac{n - \alpha_1}{n - 2} \leq \rho < \frac{n}{n - 2}\), we have \(\frac{1}{2} \leq \alpha_1 < 1\).

Noticing that

(6.4)

\[
\|f'(u)\phi, \varphi_t\|_{X^{-\frac{\alpha}{2}}} \leq \|f'(u)\phi\|_{X^{-\frac{\alpha}{2}}} \|\phi_t\|_{X^{-\frac{\alpha}{2}}}
\]

and that the embedding \(X^{-\frac{\alpha}{2}} = H^\alpha(\Omega) \hookrightarrow L^p(\Omega)\) or, equivalently, \(L^{\frac{2n}{n-2}}(\Omega) \hookrightarrow X^{-\frac{\alpha}{2}}\), holds for any \(2 \leq p \leq \frac{2n}{n-2}\), one can obtain an estimate for the term \(\|f'(u)\phi\|_{X^{-\frac{\alpha}{2}}}\) using Hölder’s Inequality and the growth condition, in the following way:

\[
\|f'(u)\phi\|_{X^{-\frac{\alpha}{2}}} \leq c_1 \|f'(u)\phi\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq c_1 \|\phi\|_{L^2(\Omega)} \|f'(u)\|_{L^{\frac{2n}{n-2}}(\Omega)}
\]

\[
\leq c_1 \|\phi\|_{X} \left( \int_\Omega [c(1 + |u|^{\rho-1})]^{\frac{n}{n-2}} dx \right)^{\frac{n}{n-2}} \leq c_2 \|\phi\|_{X} \left( |\Omega| + \int_\Omega |u|^{\frac{(\rho-1)n}{2}} dx \right)^{\frac{n}{n-2}}
\]

\[
\leq c_3 \|\phi\|_{X} \left( 1 + \|u|^{\rho-1})L^{\frac{(\rho-1)n}{2}}(\Omega) \right).
\]
Now, once the embedding \( H^1(\Omega) \hookrightarrow L^{\frac{(\rho-1)n}{\alpha}}(\Omega) \) holds, if and only if \( \alpha \geq \frac{(\rho-1)(n-2)}{2} \) and \( \frac{(\rho-1)n}{\alpha} \geq 2 \), that is, \( \leq \frac{(\rho-1)n}{\alpha} \leq \frac{(\rho-1)(n-2)}{2} \), then for \( \alpha = 1 \) we have

\[
(6.5) \quad \|f'(u)\phi\|_{X^{-\frac{\alpha}{2}}} \leq c_3\|\phi\|_X \left( 1 + \|u\|^\frac{(\rho-1)n}{\alpha} \right) \leq c_5\|\phi\|_X \left( 1 + \|u\|^\frac{(\rho-1)n}{\alpha} \right) \leq c_6,
\]

since \( u \) and \( \phi \) remain in bounded subsets of \( X^\frac{\alpha}{2} \) and \( X \), respectively. Hence, from Young’s inequality, and using (6.1) and (6.5), we get

\[
\langle f'(u)\phi, \phi_t \rangle_{X^{-\frac{\alpha}{2}}} \leq \frac{1}{2\epsilon_0} \|f'(u)\phi\|^2_{X^{-\frac{\alpha}{2}}} + \frac{\epsilon_0}{2} \|\phi_t\|^2_{X^{-\frac{\alpha}{2}}} \leq \frac{1}{2\epsilon_0} c_1^2 + \frac{\epsilon_0}{2} \|\phi_t\|^2_{X^{-\frac{\alpha}{2}}}
\]

for all \( \epsilon_0 > 0 \). With this in mind, it is possible to obtain an estimate for the other term that has the nonlinearity \( f' \), that is,

\[
\gamma_1 \langle \phi, f'(u)\phi \rangle_{X^{-\frac{\alpha}{2}}} \leq \frac{\epsilon_1}{2} \|\phi\|^2_{X^{-\frac{\alpha}{2}}} + \frac{\epsilon_1}{2} \|f'(u)\phi\|^2_{X^{-\frac{\alpha}{2}}} \leq \frac{\epsilon_1}{2} \|\phi\|^2_{X^{-\frac{\alpha}{2}}} + \frac{1}{2\epsilon_1} \gamma_1^2 c_1^2
\]

for all \( \epsilon_1 > 0 \).

Next, from Cauchy-Schwartz and Young inequalities, we have

\[
- \eta \gamma_1 \langle \phi, A^\frac{\alpha}{2} \phi_t \rangle_{X^{-\frac{\alpha}{2}}} \leq \eta \gamma_1 \frac{\epsilon_2}{2} \|\phi\|^2_{X^{-\frac{\alpha}{2}}} + \frac{1}{2\epsilon_2} \|\phi_t\|^2_{X^{-\frac{\alpha}{2}}},
\]

\[
- a_c(t) \gamma_1 \langle \phi, A^\frac{\alpha}{2} \psi_t \rangle_{X^{-\frac{\alpha}{2}}} \leq a_c(t) \gamma_1 \frac{\epsilon_3}{2} \|\phi\|^2_{X^{-\frac{\alpha}{2}}} + \frac{1}{2\epsilon_3} \|\psi_t\|^2_{X^{-\frac{\alpha}{2}}},
\]

\[
- \eta \gamma_2 \langle \psi, A^\frac{\alpha}{2} \psi_t \rangle_{X^{-\frac{\alpha}{2}}} \leq \eta \gamma_2 \frac{\epsilon_4}{2} \|\psi\|^2_{X^{-\frac{\alpha}{2}}} + \frac{1}{2\epsilon_4} \|\psi_t\|^2_{X^{-\frac{\alpha}{2}}}
\]

and

\[
- a_c(t) \gamma_2 \langle \varphi, A^\frac{\alpha}{2} \varphi_t \rangle_{X^{-\frac{\alpha}{2}}} \leq a_c(t) \gamma_2 \frac{\epsilon_5}{2} \|\varphi\|^2_{X^{-\frac{\alpha}{2}}} + \frac{1}{2\epsilon_5} \|\varphi_t\|^2_{X^{-\frac{\alpha}{2}}},
\]

for all \( \epsilon_2 > 0, \epsilon_3 > 0, \epsilon_4 > 0 \) and \( \epsilon_5 > 0 \).

Since \( \frac{1}{2} \leq \alpha_1 < 1 \), we have the embedding \( X \hookrightarrow X^{\frac{1-\alpha_1}{4}} \), that is,

\[
\| \cdot \|_{X^{\frac{1-\alpha_1}{4}}} \leq \tilde{c} \| \cdot \|_X
\]

for some constant \( \tilde{c} > 0 \). From this, and by condition (2.2), and also using the fact that \( \varphi \) remains in a bounded subset of \( X \), we get

\[
- a_c(t) \langle A^\frac{\alpha}{2} \varphi, \phi_t \rangle_{X^{-\frac{\alpha}{2}}} \leq b_0 \|\varphi\|_{X^{\frac{1-\alpha_1}{4}}} \|\phi_t\|_{X^{\frac{1-\alpha_1}{4}}} \leq b_0 \frac{1}{2\epsilon_6} \|\varphi\|^2_{X^{\frac{1-\alpha_1}{4}}} + b_0 \frac{\epsilon_6}{2} \|\phi_t\|^2_{X^{\frac{1-\alpha_1}{4}}} \leq b_0 \frac{1}{2\epsilon_6} c_2^2 \|\varphi\|^2_{X} + b_0 \frac{\epsilon_6}{2} \|\phi_t\|^2_{X^{\frac{1-\alpha_1}{4}}} \leq \frac{1}{2\epsilon_6} b_0 c_7 + b_0 \frac{\epsilon_6}{2} \|\phi_t\|^2_{X^{\frac{1-\alpha_1}{4}}}
\]

for all \( \epsilon_6 > 0 \),

\[
- a_c(t) \langle A^\frac{\alpha}{2} \varphi, \varphi_t \rangle_{X^{-\frac{\alpha}{2}}} \leq b_0 \frac{1}{2\epsilon_7} \|\varphi\|^2_{X^{\frac{1-\alpha_1}{4}}} + b_0 \frac{\epsilon_7}{2} \|\varphi_t\|^2_{X^{\frac{1-\alpha_1}{4}}} \leq b_0 \frac{1}{2\epsilon_7} \|\varphi\|^2_{X} + b_0 \frac{\epsilon_7}{2} \|\varphi_t\|^2_{X^{\frac{1-\alpha_1}{4}}} \leq \frac{1}{2\epsilon_7} b_0 c_8 + b_0 \frac{\epsilon_7}{2} \|\varphi_t\|^2_{X^{\frac{1-\alpha_1}{4}}}.
\]
for all \( \epsilon_7 > 0 \),

\[
-a'(t) \gamma_1(\phi, A^\frac{1}{2} \varphi)_{X^{-\frac{a_1}{2}}} \leq b_0 \gamma_1 \frac{\epsilon_8}{2} \| \phi \|_{X^{\frac{a_1}{2}}}^2 + b_0 \gamma_1 \frac{1}{2 \epsilon_8} \| \varphi \|_{X^{\frac{1}{2}}}^2
\]

\[
\leq \frac{b_0 \gamma_1 \epsilon_8 c^2}{2} \| \phi \|_{X}^2 + \frac{b_0 \gamma_1 c^2}{2 \epsilon_8} \| \varphi \|_{X}^2 \leq c_9
\]

and

\[
a'(t) \gamma_2(\varphi, A^\frac{1}{2} \phi)_{X^{-\frac{a_1}{2}}} \leq b_0 \gamma_2 \frac{\epsilon_9}{2} \| \varphi \|_{X^{\frac{1}{2}}}^2 + b_0 \gamma_2 \frac{1}{2 \epsilon_9} \| \phi \|_{X^{\frac{1}{2}}}^2
\]

\[
\leq \frac{b_0 \gamma_2 \epsilon_9 c^2}{2} \| \varphi \|_{X}^2 + \frac{b_0 \gamma_2 c^2}{2 \epsilon_9} \| \phi \|_{X}^2 \leq c_{10}
\]

for some constants \( c_9 > 0 \) and \( c_{10} > 0 \).

Finally, combining all the estimates obtained before, we get

\[
\frac{d}{dt} L_{\alpha_1}(t) \leq - \left( \gamma_1 - \eta \gamma_1 \frac{\epsilon_2}{2} - a_1 \gamma_1 \frac{\epsilon_3}{2} \right) \| \phi \|_{X^{\frac{1}{2}}}^2
\]

\[- \left( -\gamma_1 - \eta \gamma_1 \frac{\epsilon_2}{2} - a_1 \gamma_1 \frac{\epsilon_5}{2} - \frac{\epsilon_0}{2} \right) \| \phi_t \|_{X^{-\frac{a_1}{2}}}^2
\]

\[- \left( -\gamma_2 - \eta \gamma_2 \frac{\epsilon_4}{2} - a_1 \gamma_2 \frac{\epsilon_5}{2} \right) \| \varphi \|_{X^{\frac{1}{2}}}^2
\]

\[- \left( -\gamma_2 - a_1 \gamma_1 \frac{\epsilon_3}{2} - \eta \gamma_1 \frac{\epsilon_4}{2} \right) \| \varphi_t \|_{X^{-\frac{a_1}{2}}}^2
\]

\[- \left( \frac{\epsilon_1}{2} \right) \| \phi \|_{X^{-\frac{a_1}{2}}}^2 + \frac{b_0 \frac{\epsilon_6}{2}}{2} \| \phi_t \|_{X^{\frac{1}{2}}}^2 + \frac{b_0 \frac{\epsilon_7}{2}}{2} \| \varphi_t \|_{X^{\frac{1}{2}}}^2
\]

\[+ \frac{1}{2 \epsilon_0} \gamma_1^2 c_6^2 + \frac{1}{2 \epsilon_1} \gamma_1^2 c_6^2 + \frac{1}{2 \epsilon_6} b_0 \epsilon_7 + \frac{1}{2 \epsilon_7} b_0 c_8 + c_9 + c_{10}.
\]

Now, by choosing \( \epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0, \epsilon_4 > 0, \epsilon_5 > 0, \epsilon_6 > 0 \) and \( \epsilon_7 > 0 \), respectively, such that

\[\epsilon_1 = 2 \gamma_1, \quad \epsilon_2 = \frac{1}{2 \eta}, \quad \epsilon_3 = \frac{1}{2 a_1}, \quad \epsilon_4 = \frac{1}{2 \eta}, \quad \epsilon_5 = \frac{1}{2 a_1}, \quad \epsilon_6 = \frac{\eta}{b_0} \quad \text{and} \quad \epsilon_7 = \frac{3 \eta}{2 b_0},\]

we obtain

(6.6)

\[
\frac{d}{dt} L_{\alpha_1}(t) \leq - \frac{1}{2} \gamma_1 \| \phi \|_{X^{\frac{1}{2}}}^2 - \left( -\gamma_1 + \eta \gamma_1 - a_1 \gamma_2 \frac{\epsilon_2}{2} - \frac{\epsilon_0}{2} \right) \| \phi_t \|_{X^{-\frac{a_1}{2}}}^2 - \frac{1}{2} \gamma_2 \| \varphi \|_{X^{-\frac{a_1}{2}}}^2
\]

\[- \left( -\gamma_2 - a_1 \gamma_1 - \eta \gamma_2 \frac{\epsilon_4}{2} \right) \| \varphi_t \|_{X^{\frac{1}{2}}}^2 - \frac{\eta}{2} \| \phi_t \|_{X^{\frac{1}{2}}}^2 - \frac{\eta}{4} \| \varphi_t \|_{X^{\frac{1}{2}}}^2
\]

\[+ \frac{1}{2 \epsilon_0} \gamma_1^2 c_6^2 + \frac{1}{4} \gamma_1^2 c_6^2 + \frac{b_0^2 \epsilon_7}{2 \eta} + \frac{b_0^2 c_8}{3 \eta} + c_9 + c_{10}.
\]

As \( \frac{1}{2} \gamma_1 - \eta \gamma_1 \frac{\epsilon_2}{2} - a_1 \gamma_2 \frac{\epsilon_0}{2} > - \frac{a_1}{2} \), we have the embedding \( X^{\frac{1}{2}} \hookrightarrow X^{\frac{a_1}{2}} \), and so

\[\| \cdot \|_{X^{-\frac{a_1}{2}}} \leq \tilde{c} \| \cdot \|_{X^{\frac{1}{2}}}^{1-\frac{a_1}{2}}\]

for some constant \( \tilde{c} > 0 \), which implies

(6.7)

\[- \| \cdot \|_{X^{\frac{1}{2}}}^{1-\frac{a_1}{2}} \leq - \frac{1}{\tilde{c}^2} \| \cdot \|_{X^{-\frac{a_1}{2}}}^{\frac{1}{2}} \cdot \| \cdot \|_{X^{-\frac{a_1}{2}}}.\]
Hence, combining (6.6) and (6.7), we get
\[
\frac{d}{dt} \mathcal{L}_{o_1}(t) \leq -\frac{1}{2} \gamma_1 \|\phi\|_{X^{\frac{1}{2},\gamma_1}}^2 - \left( \frac{\eta}{4c^2} - \gamma_1 - \eta^2 \gamma_1 - a_1^2 \gamma_2 - \frac{\epsilon_0}{2} \right) \|\varphi\|_{X^{\frac{1}{2},\gamma_2}}^2
\]
\[
- \frac{1}{2} \gamma_2 \|\varphi\|_{X^{\frac{1}{2},\gamma_1}}^2 - \left( \frac{\eta}{4c^2} - \gamma_2 - a_1^2 \gamma_1 - \eta^2 \gamma_2 \right) \|\varphi\|_{X^{\frac{1}{2},\gamma_2}}^2
\]
\[
+ \frac{1}{2} \epsilon_0 c^2 + \frac{1}{4} \gamma_1 c_6^2 + \frac{b_0^2 c_7}{2 \eta} + \frac{b_0^2 c_8}{3 \eta} + c_9 + c_{10}.
\]
(6.8)

At last, choosing $\epsilon_0 > 0$ such that $\epsilon_0 = \frac{\eta}{2c^2}$, expression (6.8) turns into
\[
\frac{d}{dt} \mathcal{L}_{o_1}(t) \leq -\frac{1}{2} \gamma_1 \|\phi\|_{X^{\frac{1}{2},\gamma_1}}^2 - \left( \frac{\eta}{4c^2} - (1 + \eta^2) \gamma_1 - a_1^2 \gamma_2 \right) \|\varphi\|_{X^{\frac{1}{2},\gamma_2}}^2
\]
\[
- \frac{1}{2} \gamma_2 \|\varphi\|_{X^{\frac{1}{2},\gamma_1}}^2 - \left( \frac{\eta}{4c^2} - a_1^2 \gamma_1 - (1 + \eta^2) \gamma_2 \right) \|\varphi\|_{X^{\frac{1}{2},\gamma_2}}^2
\]
\[
+ \frac{\gamma_1}{\eta} c_6^2 + \frac{1}{4} \gamma_1 c_6^2 + \frac{b_0^2 c_7}{2 \eta} + \frac{b_0^2 c_8}{3 \eta} + c_9 + c_{10}.
\]

Now, taking $\gamma_i > 0, i = 1, 2$, sufficiently small such that
\[
\gamma_i < \min \left\{ \frac{1}{2k}, \frac{\eta}{16c^2}, \frac{1}{1 + \eta^2}, \frac{1}{16c^2 a_1^2} \right\}, \quad i = 1, 2,
\]
where $\tilde{k} > 0$ is the embedding constant of $X^{\frac{1}{2},\gamma_1} \hookrightarrow X^{-\frac{3}{4},2}$ and taking
\[
M_1 = \min \left\{ \frac{1}{2} \gamma_1, \frac{\eta}{4c^2} - (1 + \eta^2) \gamma_1 - a_1^2 \gamma_2, \frac{1}{2} \gamma_2, \frac{\eta}{4c^2} - a_1^2 \gamma_1 - (1 + \eta^2) \gamma_2 \right\} > 0
\]

and $M_2 = \frac{\gamma_1}{\eta} c_6^2 + \frac{1}{4} \gamma_1 c_6^2 + \frac{b_0^2 c_7}{2 \eta} + \frac{b_0^2 c_8}{3 \eta} + c_9 + c_{10} > 0$, it follows that
\[
(6.9) \quad \frac{d}{dt} \mathcal{L}_{o_1}(t) \leq -M_1 \left( \|\phi\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\phi_t\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\varphi\|_{X^{\frac{1}{2},\gamma_2}}^2 + \|\varphi_t\|_{X^{\frac{1}{2},\gamma_2}}^2 \right) + M_2.
\]

Observe that
\[
|\gamma_1 \langle \phi, \phi_t \rangle_{X^{-\frac{3}{4},2}} + \gamma_2 \langle \varphi, \varphi_t \rangle_{X^{-\frac{3}{4},2}}| \leq \frac{1}{4} \left( \|\phi\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\phi_t\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\varphi\|_{X^{\frac{1}{2},\gamma_2}}^2 + \|\varphi_t\|_{X^{\frac{1}{2},\gamma_2}}^2 \right).
\]

In this way, using a similar argument as in (5.3) and (5.4), we get
\[
\frac{1}{4} \left( \|\phi\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\phi_t\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\varphi\|_{X^{\frac{1}{2},\gamma_2}}^2 + \|\varphi_t\|_{X^{\frac{1}{2},\gamma_2}}^2 \right)
\]
\[
\leq \mathcal{L}_{o_1}(t) \leq \frac{3(1 + \tilde{k}^2)}{4} \left( \|\phi\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\phi_t\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\varphi\|_{X^{\frac{1}{2},\gamma_2}}^2 + \|\varphi_t\|_{X^{\frac{1}{2},\gamma_2}}^2 \right).
\]

This estimate together with (6.9) implies that
\[
\|\phi\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\phi_t\|_{X^{\frac{1}{2},\gamma_1}}^2 + \|\varphi\|_{X^{\frac{1}{2},\gamma_2}}^2 + \|\varphi_t\|_{X^{\frac{1}{2},\gamma_2}}^2 \leq 4\mathcal{L}_{o_1}(t) e^{-M_3(t-\tau)} + M_4,
\]
with positive constants $M_3$ and $M_4$. This assures that $(\phi, \phi_t, \varphi, \varphi_t)$ is bounded in the space $X^{\frac{1}{2},\gamma_1} \times X^{-\frac{3}{4},2} \times X^{\frac{1}{2},\gamma_1} \times X^{-\frac{3}{4},2}$. But we want to conclude that $\bigcup_{t \in \mathbb{R}} A(t)$ is bounded in $X^{2-\frac{4}{n},2} \times X^{\frac{1}{2},\gamma_1} \times X^{\frac{1}{2},\gamma_1} \times X^{1,\gamma_1}$. We already know that $u_t$ and $v_t$ are bounded in
\( X^{1-\alpha_1} \). Now, to show that \( u \in X^{2-\alpha_1} \) and it is bounded, it is enough to show that \( \|Au\|_{X^{2-\alpha_1}} \leq C_1 \) for some constant \( C_1 > 0 \), since

\[
\|Au\|_{X^{2-\alpha_1}} = \|A^{2-\alpha_1}u\|_X = \|u\|_{X^{2-\alpha_1}}.
\]

Indeed, note that

\[
\| - Au\|_{X^{2-\alpha_1}} - \|u + \eta A^{1/2}u_t + a_c(t)A^{1/2}v_t - f(u)\|_{X^{2-\alpha_1}}
\]

\[
\leq \| - Au - u - \eta A^{1/2}u_t - a_c(t)A^{1/2}v_t + f(u)\|_{X^{2-\alpha_1}}
\]

\[
= \|u\|_{X^{2-\alpha_1}} = \|\phi\|_{X^{2-\alpha_1}} \leq k_1,
\]

which yields

\[
\|Au\|_{X^{2-\alpha_1}} \leq k_1 + \|u\|_{X^{2-\alpha_1}} + \eta \|A^{1/2}u_t\|_{X^{2-\alpha_1}} + a_1 \|A^{1/2}v_t\|_{X^{2-\alpha_1}} + \|f(u)\|_{X^{2-\alpha_1}}.
\]

Thus, we need to obtain estimates for the terms that are on the right hand side of the above inequality. Using the embedding \( L^{2n/(n-2\rho)}(\Omega) \hookrightarrow X^{1/2} \) and Lemma 4.11, condition (i), we have

\[
\|f(u)\|_{X^{2-\alpha_1}} \leq c_1 \|f(u)\|_{L^{2n/(n-2\rho)}(\Omega)} \leq c_1 \left( \int_{\Omega} [c(1 + |u|^p)]^{n/(n+2\alpha_1)} dx \right)^{n+2\alpha_1/(2n)}
\]

\[
\leq c_2 \left( |\Omega| + \int_{\Omega} |u|^{2n/(n+2\alpha_1)} dx \right)^{n/(n+2\alpha_1)} \leq c_3 \left( 1 + \|u\|^p_{L^{2n/(n-2\rho)}(\Omega)} \right).
\]

Since the embedding \( H^1(\Omega) \hookrightarrow L^{p}(\Omega) \) holds, if and only if \( p \leq \frac{2n}{n-2} \), and

\[
(n-2)\rho + 2 > (n-2)\rho \implies \frac{2n\rho}{(n-2)\rho + 2} < \frac{2n\rho}{(n-2)\rho} = \frac{2n}{n-2},
\]

it follows that

\( H^1(\Omega) \hookrightarrow L^{2n/(n-2\rho)+2}(\Omega) \)

and, therefore,

\[
\|f(u)\|_{X^{1/2}} \leq c_3 \left( 1 + \|u\|^p_{L^{2n/(n-2\rho)+2}(\Omega)} \right) \leq c_5 \left( 1 + \|u\|^p_{X^{1/2}} \right) \leq k_2.
\]

For the remaining terms, note that

\[
\|u\|_{X^{2-\alpha_1}} \leq \tilde{c} \|u\|_{X^{1/2}} \leq k_3,
\]

since \( X^{1/2} \hookrightarrow X^{1/2-} \), and, moreover,

\[
\eta \|A^{1/2}u_t\|_{X^{2-\alpha_1}} = \eta \|\phi\|_{X^{1-\alpha_1}} \leq k_4,
\]

and

\[
a_1 \|A^{1/2}v_t\|_{X^{2-\alpha_1}} = a_1 \|\varphi\|_{X^{1-\alpha_1}} \leq k_5.
\]

Therefore, we conclude that

\[
\|Au\|_{X^{2-\alpha_1}} \leq k_1 + k_2 + k_3 + k_4 + k_5 = C_1,
\]

as desired.
Now, to show that $v \in X^{\frac{2-n}{2}}$ and it is bounded, the idea is similar, because

$$
\| - Av \|_{X^{\frac{a_1}{2}}} - \| \eta A^\frac{1}{2} v_t - a_c(t) A^\frac{1}{2} u_t \|_{X^{\frac{a_1}{2}}}
\leq \| - Av - \eta A^\frac{1}{2} v_t + a_c(t) A^\frac{1}{2} u_t \|_{X^{\frac{a_1}{2}}}
= \| v_t \|_{X^{\frac{a_1}{2}}} = \| \varphi_t \|_{X^{\frac{a_1}{2}}} \leq k_6,
$$

which implies

$$
\| v \|_{X^{\frac{2-n}{2}}} = \| Av \|_{X^{\frac{a_1}{2}}} \leq k_6 + \eta \| A^\frac{1}{2} v_t \|_{X^{\frac{a_1}{2}}} + a_1 \| A^\frac{1}{2} u_t \|_{X^{\frac{a_1}{2}}}
= k_6 + \eta \| \varphi \|_{X^{\frac{a_1}{2}}} + a_1 \| \varphi \|_{X^{\frac{a_1}{2}}} \leq C_2,
$$

with $C_2 > 0$ being constant.

From the previous observations and from the fact that $\mathbb{A}(t) = \{ \xi(t) : \xi(t) \text{ is a bounded global solution} \}$, we conclude that

$$
(6.10) \quad \bigcup_{t \in \mathbb{R}} \mathbb{A}(t) \text{ is bounded in } X^{\frac{2-n}{2}} \times X^{\frac{1-a_1}{2}} \times X^{\frac{2-n}{2}} \times X^{\frac{1-a_1}{2}}.
$$

Now, we turn our attention once again to the term $\| f'(u) \phi \|_{X^{\frac{1}{2}}} \frac{2}{n}$ that appears in (6.4). Note that the embedding $X^{\frac{1-a_1}{2}} = H^{1-a_1} (\Omega) \hookrightarrow L^p (\Omega)$ holds, if and only if $p \leq \frac{2n}{n-2(1-a_1)}$.

Hence, using (6.10), the Hölder’s inequality and the growth condition (2.5), we have

$$
\| f'(u) \phi \|_{X^{\frac{1}{2}}} \frac{2}{n} \leq c_1 \| f'(u) \phi \|_{L^{\frac{2n}{n-2(1-a_1)}} (\Omega)} \leq c_1 \| u_t \|_{L^{\frac{2n}{n-2(1-a_1)+1}} (\Omega)} \| f'(u) \|_{L^{\frac{2}{n-2(1-a_1)+1}} (\Omega)}
\leq c_2 \| u_t \|_{X^{\frac{1-a_1}{2}}} \left( \int_{\Omega} [c(1 + |u|^{\rho-1})]^{\frac{n}{1-a_1+n}} \right) \frac{1}{n-2(1-a_1)}
\leq c_3 \| u_t \|_{X^{\frac{1-a_1}{2}}} \left( |\Omega| + \int_{\Omega} |u|^{\frac{(\rho-1)n}{1-a_1+n}} \right) \frac{1}{n-2(1-a_1)}
\leq c_4 \| u_t \|_{X^{\frac{1-a_1}{2}}} \left( |\Omega| \frac{1}{n-2(1-a_1)} + \| u \|^{\rho-1}_{L^{\frac{(\rho-1)n}{1-a_1+n}} (\Omega)} \right)
\leq c_5 \| u_t \|_{X^{\frac{1-a_1}{2}}} \left( 1 + \| u \|^{\rho-1}_{L^{\frac{(\rho-1)n}{1-a_1+n}} (\Omega)} \right).
$$

Now, note that the embedding $X^{\frac{2-a_1}{2}} = H^{2-a_1} (\Omega) \hookrightarrow L^{\frac{(\rho-1)n}{1-a_1+n}} (\Omega)$ holds, if and only if $(2-a_1) - \frac{n}{2} \geq -\frac{1-a_1}{(\rho-1)}$ and $\frac{(\rho-1)n}{1-a_1+n} \geq 2$, that is

$$
\frac{(\rho-1)(n-2)}{2} + \rho(\alpha_1 - 1) \leq \alpha \leq \frac{(\rho-1)n}{2} + \alpha_1 - 1.
$$

If $\frac{(\rho-1)(n-2)}{2} + \rho(\alpha_1 - 1) = \alpha_1 + \rho(\alpha_1 - 1) \geq 0$, then using (6.10) and restarting the whole process from (6.4) with $\alpha_2 = \alpha_1 + \rho(\alpha_1 - 1)$, we will get

$$
\bigcup_{t \in \mathbb{R}} \mathbb{A}(t) \text{ is bounded in } X^{\frac{2-n}{2}} \times X^{\frac{1-n_2}{2}} \times X^{\frac{2-n_3}{2}} \times X^{\frac{1-n_4}{2}}.
$$

We continue with this iterative process getting $\alpha_{k+1} = \alpha_1 + \rho(\alpha_k - 1)$ for $k \geq 1$ while $\alpha_k \geq 0.$
There will be an integer $k_0 \geq 1$ such that $\alpha_{k_0} \geq 0$ and $\alpha_{k_0+1} < 0$. Thus, we obtain
\[
\bigcup_{t \in \mathbb{R}} A(t) \text{ is bounded in } X^{\frac{2-\alpha_{k_0}}{2}} \times X^{\frac{1-\alpha_{k_0}}{2}} \times X^{\frac{2-\alpha_{k_0+1}}{2}} \times X^{\frac{1-\alpha_{k_0+1}}{2}},
\]
but we cannot assure the boundedness in $X^{\frac{2-\alpha_{k_0+1}}{2}} \times X^{\frac{1-\alpha_{k_0+1}}{2}} \times X^{\frac{2-\alpha_{k_0+1}}{2}} \times X^{\frac{1-\alpha_{k_0+1}}{2}}$ because of the embeddings. Here, we set $\alpha = 0$ and we restart the whole process from (6.4), with the obvious adaptations using the boundedness in $X^{\frac{2-\alpha_{k_0}}{2}} \times X^{\frac{1-\alpha_{k_0}}{2}} \times X^{\frac{2-\alpha_{k_0+1}}{2}} \times X^{\frac{1-\alpha_{k_0+1}}{2}}$, and we conclude that
\[
\bigcup_{t \in \mathbb{R}} A(t) \text{ is bounded in } X^{1} \times X^{\frac{3}{2}} \times X^{1} \times X^{\frac{3}{2}},
\]
and the proof is complete. \hfill \qed

7. Upper semicontinuity of pullback attractors

This last section is devoted to study the upper semicontinuity of pullback attractors with respect to the functional parameter $a_{\epsilon}$. To this end, we will use the regularity result obtained in the previous section. Let $\{a_{\epsilon} : \epsilon \in [0, 1]\}$ be a family of real valued functions of one real variable satisfying (2.1). For each $\epsilon \in [0, 1]$ denote by $\{S_{(\epsilon)}(t, \tau) : t \geq \tau \in \mathbb{R}\}$ and $\{A_{(\epsilon)}(t) : t \in \mathbb{R}\}$, respectively, the evolution process and its pullback attractor associated with the problem (1.1)-(1.3).

Moreover, we will assume that $\|a_{\epsilon} - a_0\|_{L^\infty(\mathbb{R})} \to 0$ as $\epsilon \to 0^+$.

Proof of Theorem 2.4: Let $W = W^{(\epsilon)} - W^{(0)}$, where
\[
W^{(\epsilon)} = (u^{(\epsilon)}, u_t^{(\epsilon)}, v^{(\epsilon)}, v_t^{(\epsilon)}) \quad \text{and} \quad W^{(0)} = (u^{(0)}, u_t^{(0)}, v^{(0)}, v_t^{(0)}),
\]
with $u = u^{(\epsilon)} - u^{(0)}$ and $v = v^{(\epsilon)} - v^{(0)}$. From this, we have
\[
\begin{align*}
\begin{cases}
 u_{tt} - \Delta u + u + \eta(-\Delta)^\frac{1}{2} u + a_\epsilon(t)(-\Delta)^\frac{1}{2} v_t^{(\epsilon)} - a_0(t)(-\Delta)^\frac{1}{2} v_t^{(0)} = f(u^{(\epsilon)}) - f(u^{(0)}), \\
v_{tt} - \Delta v + \eta(-\Delta)^\frac{1}{2} v - a_\epsilon(t)(-\Delta)^\frac{1}{2} u_t^{(\epsilon)} + a_0(t)(-\Delta)^\frac{1}{2} u_t^{(0)} = 0,
\end{cases}
\end{align*}
\]
for all $t > \tau$ and $x \in \Omega$. Taking the inner product of the first equation with $u_t$, and also the inner product of the second equation with $v_t$, we get
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_t|^2 dx &+ \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 dx + \eta \|(-\Delta)^\frac{1}{2} u_t\|^2_X \\
&+ a_\epsilon(t)\langle (-\Delta)^\frac{1}{2} v_t^{(\epsilon)}, u_t^{(\epsilon)} \rangle_X - a_0(t)\langle (-\Delta)^\frac{1}{2} v_t^{(0)}, u_t^{(0)} \rangle_X \\
&- a_\epsilon(t)\langle (-\Delta)^\frac{1}{2} v^{(\epsilon)}, u_t^{(\epsilon)} \rangle_X + a_0(t)\langle (-\Delta)^\frac{1}{2} v^{(0)}, u_t^{(0)} \rangle_X \\
&= \int_\Omega [f(u^{(\epsilon)}) - f(u^{(0)})]u_t dx,
\end{align*}
\]
and
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega |v_t|^2 dx &+ \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla v|^2 dx + \eta \|(-\Delta)^\frac{1}{2} v_t\|^2_X \\
&- a_\epsilon(t)\langle (-\Delta)^\frac{1}{2} u_t^{(\epsilon)}, v_t^{(\epsilon)} \rangle_X + a_0(t)\langle (-\Delta)^\frac{1}{2} u_t^{(0)}, v_t^{(0)} \rangle_X - a_\epsilon(t)\langle (-\Delta)^\frac{1}{2} u^{(\epsilon)}, v_t^{(\epsilon)} \rangle_X + a_0(t)\langle (-\Delta)^\frac{1}{2} u^{(0)}, v_t^{(0)} \rangle_X = 0,
\end{align*}
\]
and combining these two last equations, it follows that
\[
\frac{d}{dt} \left( \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u|^2 \, dx + \int_\Omega |u_t|^2 \, dx + \int_\Omega |\nabla v|^2 \, dx + \int_\Omega |v_t|^2 \, dx \right)
+ 2\eta \|A^\frac{1}{2}u_t\|_X^2 + 2\eta \|A^\frac{1}{2}v_t\|_X^2 + 2(a_e - a_0)(t) \langle A^\frac{1}{2}v_t, A^\frac{1}{2}u_t \rangle_X
+ 2(a_e - a_0)(t) \langle A^\frac{1}{2}v_t, A^\frac{1}{2}v_t \rangle_X + 2 \int_\Omega [f(u^\epsilon) - f(u^{0\epsilon})] u_t \, dx
\leq 2\|a_e - a_0\|_\infty \left( \frac{1}{2} \|u_t\|^2_{X^{\frac{1}{2}}} + \frac{1}{2} \|v_t\|^2_{X^{\frac{1}{2}}} \right)
\leq C\|a_e - a_0\|_\infty \left( \|u_t\|^2_{X^{\frac{1}{2}}} + \|v_t\|^2_{X^{\frac{1}{2}}} \right) + 2 \int_\Omega [f(u^\epsilon) - f(u^{0\epsilon})] u_t \, dx.
\]

(7.1)

On the other hand, from Theorem 2.3 we know that $W^\epsilon(t)$ and $W(t)$ are bounded in $X^1 \times X^\frac{1}{2} \times X^1 \times X^\frac{1}{2}$. In particular, there exists a constant $C > 0$, independent of $\epsilon$, such that
\[
\|u_t^\epsilon\|_{X^{\frac{1}{2}}}, \|u_t^{0\epsilon}\|_{X^{\frac{1}{2}}}, \|v_t^\epsilon\|_{X^{\frac{1}{2}}}, \|v_t^{0\epsilon}\|_{X^{\frac{1}{2}}} \leq C.
\]

(7.2)

Therefore, from (7.1), (7.2), and the embedding $X^{\frac{1}{2}} \hookrightarrow X^\frac{1}{2}$, we obtain
\[
\frac{d}{dt} \left( \|u\|^2_{X^{\frac{1}{2}}} + \|u_t\|^2_X + \|v_t\|^2_{X^{\frac{1}{2}}} + \|v_t\|^2_X \right)
\leq C \|a_e - a_0\|_\infty + 2 \int_\Omega [f(u^\epsilon) - f(u^{0\epsilon})] u_t \, dx,
\]

(7.3)

where $C' > 0$ is independent of $\epsilon$.

By the Mean Value Theorem, there exists $\sigma \in (0, 1)$ such that
\[
|f(u^\epsilon) - f(u^{0\epsilon})| = |f'(\sigma u^\epsilon + (1 - \sigma)u^{0\epsilon})| |u^\epsilon - u^{0\epsilon}| = |f'(\sigma u^\epsilon + (1 - \sigma)u^{0\epsilon})| |u_t|,
\]

and so
\[
\int_\Omega |f(u^\epsilon) - f(u^{0\epsilon})| u_t \, dx = \int_\Omega |f'(\sigma u^\epsilon + (1 - \sigma)u^{0\epsilon})| |u| u_t \, dx.
\]
As in the proof of Theorem 2.3, the condition $1 < \rho \leq \frac{n}{n-2}$ implies $X^{\frac{1}{2}} \hookrightarrow L^{2\rho}(\Omega)$. Since $\frac{(\rho-1)}{2\rho} + \frac{1}{2\rho} + \frac{1}{2} = 1$, then Hölder’s inequality gives us

\begin{equation}
\int_{\Omega} \left| f(u^{(c)}) - f(u^{(0)}) \right| u_t \, dx \leq \| f'(\sigma u^{(c)} + (1 - \sigma)u^{(0)}) \|_{L^{2\rho}(\Omega)} \| u \|_{L^{2\rho}(\Omega)} \| u_t \|_{L^2(\Omega)};
\end{equation}

but note that

\begin{align}
\| f'(\sigma u^{(c)} + (1 - \sigma)u^{(0)}) \|_{L^{2\rho}(\Omega)} &\leq \left( \int_{\Omega} | f(1 + |\sigma u^{(c)} + (1 - \sigma)u^{(0)}|^{\rho-1}) |^{\frac{2\rho}{\rho-1}} \, dx \right)^{\frac{\rho-1}{2\rho}} \\
&\leq \frac{\hat{c}}{2} \left( |\Omega| \int_{\Omega} |\sigma u^{(c)} + (1 - \sigma)u^{(0)}|^{2\rho} \, dx \right)^{\frac{\rho-1}{2\rho}} \\
&\leq \frac{\hat{c}}{2} \left( [1 + (\|\sigma u^{(c)}\|_{L^{2\rho}(\Omega)} + \|1 - \sigma\|_{L^{2\rho}(\Omega)})^{\rho-1}] \right) \\
&\leq \frac{\hat{c}}{2} \left( 1 + \|u^{(c)}\|_{X^{\frac{1}{2}}}^{\rho-1} + \|u^{(0)}\|_{X^{\frac{1}{2}}}^{\rho-1} \right) \leq C_0,
\end{align}

where $C_0 > 0$ is independent of $\epsilon$. Thus, combining (7.4), (7.3) and the Young’s inequality, we obtain

\begin{align}
\int_{\Omega} \left| f(u^{(c)}) - f(u^{(0)}) \right| u_t \, dx &\leq C_0 \| u \|_{L^{2\rho}(\Omega)} \| u_t \|_{L^2(\Omega)} \leq \hat{c} \| u \|_{X^{\frac{1}{2}}} \| u_t \|_X \\
&\leq \frac{\hat{c}}{2} \left( \| u \|_{X^{\frac{1}{2}}}^2 + \| u_t \|_X^2 \right) \leq \frac{\hat{c}}{2} \left( \| u \|_{X^{\frac{1}{2}}}^2 + \| u_t \|_X^2 + \| u_t \|_X^2 + \| v_t \|_X^2 \right).
\end{align}

Now, denoting $G(t) = \| u(t) \|_{X^{\frac{1}{2}}}^2 + \| u(t) \|_X^2 + \| u_t(t) \|_{X}^2 + \| v(t) \|_{X}^2 + \| v_t(t) \|_X^2$, from (7.3) and (7.6), it follows that

\begin{align}
\frac{d}{dt} G(t) &\leq C' \| a_e - a_0 \|_{L^\infty(\mathbb{R})} + \hat{c} G(t) \leq C \| a_e - a_0 \|_{L^\infty(\mathbb{R})} + \overline{C} \overline{G}(t),
\end{align}

where $\overline{C} = \max \{ C', \hat{c} \}$. Since this holds for all $t > \tau$, and noticing that $G(\tau) = 0$, we get

\begin{align}
G(t) e^{-\overline{C}(t-\tau)} \leq \| a_e - a_0 \|_{L^\infty(\mathbb{R})} e^{-\overline{C}(t-\tau)} + \| a_e - a_0 \|_{L^\infty(\mathbb{R})}, \quad t > \tau,
\end{align}

that is,

\begin{align}
\| u \|_{X^{\frac{1}{2}}}^2 + \| u \|_X^2 + \| u_t \|_X^2 + \| v \|_{X^{\frac{1}{2}}}^2 + \| v_t \|_X^2 \leq e^{-\overline{C}(t-\tau)} \| a_e - a_0 \|_{L^\infty(\mathbb{R})} \to 0
\end{align}

as $\epsilon \to 0^+$ with $t, \tau$ in compact subsets of $\mathbb{R}$, and uniformly for $W_0$ in bounded subsets of $Y_0$. This proves the first part of the result.

In order to show the upper semicontinuity of the family of pullback attractors $\{ \mathcal{A}_e(t) : t \in \mathbb{R} \}$ at $\epsilon = 0$, let $\delta > 0$ be given. Let $t \in \mathbb{R}$ be fixed but arbitrary and

$B \supset \bigcup_{s \leq t} \mathcal{A}_e(s)$

be a bounded set in $Y_0$, whose existence is guaranteed by Theorem 3.9. Now, let $\tau \in \mathbb{R}$, $\tau < t$, be such that

$\delta \left( \frac{\delta}{2} \right).$
Using the convergence obtained in the first part of this proof, there exists $\epsilon_0 > 0$ such that
\[
\sup_{u_\epsilon \in \mathcal{A}(\epsilon)(\tau)} \| S(\epsilon)(t, \tau) u_\epsilon - S(0)(t, \tau) u_\epsilon \|_{Y_0} < \frac{\delta}{2}
\]
for all $\epsilon < \epsilon_0$. Finally, for $\epsilon < \epsilon_0$, we have
\[
d_H(\mathcal{A}(\epsilon)(t), \mathcal{A}(0)(t))
\leq d_H(S(\epsilon)(t, \tau) \mathcal{A}(\epsilon)(\tau), S(0)(t, \tau) \mathcal{A}(\epsilon)(\tau)) + d_H(S(0)(t, \tau) \mathcal{A}(\epsilon)(\tau), \mathcal{A}(0)(t))
\]
\[
= \sup_{u_\epsilon \in \mathcal{A}(\epsilon)(\tau)} \| S(\epsilon)(t, \tau) u_\epsilon - S(0)(t, \tau) u_\epsilon \|_{Y_0} + d_H(S(0)(t, \tau) \mathcal{A}(\epsilon)(\tau), \mathcal{A}(0)(t))
\]
\[
< \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]
which proves the upper semicontinuity of the family of pullback attractors. $\square$

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