Gauge systems and functions, hermitian operators and clocks as conjugate functions for the constraints

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Abstract. We work with gauge systems and using gauge invariant functions we study its quantum counterpart and we find if all these operators are self adjoint or not. Our study is divided in two cases, when we choose clock or clocks that its Poisson brackets with the set of constraints is one or it is different to one. We show some transition amplitudes.

1. Introduction
The study of gauge systems or systems with first class constraints is an outstanding branch of theoretical physics and mathematics, its importance lies in the fact that a major number of physical systems have first class constraints, including parametric systems, quantum electrodynamics, the standard model, general relativity and a lot of systems with a finite number of degrees of freedom and so on. A special case of these theories are the covariant systems. In this case the canonical Hamiltonian vanishes and the system is invariant under the reparametrization of the coordinates. In consequence the time is not a priori defined.

Let us consider a phase space with coordinates $\left(q_1, \ldots, q_n, p^1, \ldots, p^n\right)$ and first class constraints $\gamma_a$, where the set of constraints obeys,

$$\{\gamma_a, \gamma_b\} = C_{ab}^c \gamma_c,$$

and $a = 1, \ldots, m$ and $m$ is the number of first class constraints. We take the action principle,

$$S[q, p, \lambda^a] = \int_{\sigma_1}^{\sigma_2} \left(p_i q^i - \lambda^a \gamma_a\right) d\sigma,$$

then if we vary the action we can obtain the equations of motion for our system. Now, we define a gauge invariant function $R$ or a complete observable as a phase space function such that the Poisson brackets with the full set of constraints is zero, i. e. $\{R, \gamma_a\} = 0$. Then, all the phase space functions such that its Poisson bracket with all the constraints is not zero are called partial observables. If we take $m$ partial observables as clocks $T_1, \ldots, T_m$, and a partial observable $f$, then we can find a complete observable $F$ or a gauge invariant function (see [1] and [2] for instance). Now, our aim is to consider
what happen to the quantum level. In that case we study two problems, the first problem is to define our set of clocks all along the real line and the second is the self adjoint character for the $\hat{F}$ operator, where $F$ is a complete observable. If $\text{Dim} [\text{Ker}(\hat{F} + i)] = \text{Dim} [\text{Ker}(\hat{F} - i)]$, then $\hat{F}$ is self adjoint; if $\text{Dim} [\text{Ker}(\hat{F} + i)] \neq \text{Dim} [\text{Ker}(\hat{F} - i)]$, then $\hat{F}$ is not self adjoint and it has not self adjoint extensions (see [3] for instance). In the covariant systems we have two problems first for arbitrary defined clock the

$$\text{Det} \left[ \left( \begin{array}{cc} \{T_i, T_j\} & \{T_i, \gamma_j\} \\ \{\gamma_i, T_j\} & \{D_i, D_j\} \end{array} \right) \right], \quad (3)$$

is zero for some regions at the phase space and then clocks are not globally well defined, the second problem is that $\hat{F}$ is not self adjoint, if we find one or two of the previous problems we propose that the clocks must be selected in such a way that $\{T_i, \gamma_j\} = \delta_{ij}$ (see [4] for instance) and we show in this work that this form of selection for the time solve the problems for several systems.

2. Non relativistic parametric free particle
In this case, the constraint for the one dimensional non-relativistic free particle is,

$$D = p_0 + \frac{p^2}{2m} = 0, \quad (4)$$

and the phase space coordinates are $(x^0, x, p, p_0)$.

2.1. Example 1
For the first example, we can take the $T$ clock like,

$$T = x^0 - ax, \quad (5)$$

and the partial observable like $f = x$, if we make that we will obtain the complete observable:

$$F = \frac{q + \frac{p}{m} \tau}{1 - \frac{ap}{m}}, \quad q = x - \frac{px^0}{m}. \quad (6)$$

However, we find a first problem for the present system, in this case we have

$$\text{Det} \left[ \left( \begin{array}{cc} \{T, T\} & \{T, D\} \\ \{D, T\} & \{D, D\} \end{array} \right) \right] = \left( \begin{array}{cc} 0 & 1 - \frac{ap}{m} \\ \frac{ap}{m} - 1 & 0 \end{array} \right)$$

$$= \left( 1 - \frac{ap}{m} \right)^2, \quad (7)$$

and our choice for the time is not defined all along the real line.

Now, we consider the pair of equations $(\hat{F} + i)\psi_+ = 0$ and $(\hat{F} - i)\psi_- = 0$ to determine if $\hat{F}$ is a self adjoint operator or it is not. In this case the solutions are:

$$\psi_\pm = r_\pm \sqrt{1 - \frac{ap}{m}} \exp \left[ \frac{\hbar^2 r}{2m} \mp \left( p - \frac{ap^2}{2m} \right) \right], \quad (8)$$

where $r_+$ and $r_-$ are constants, then,

$$|\psi_\pm|^2 = |r_\pm|^2 \left( 1 - \frac{ap}{m} \right) \exp \left[ \mp \left( 2p - \frac{ap^2}{m} \right) \right]; \quad (9)$$

$\psi_+$ is not square integrable and $\psi_-$ is square integrable, then $\text{Dim} [\text{Ker}(\hat{F} + i)] = 0$, $\text{Dim} [\text{Ker}(\hat{F} + i)] = 1$ and we have found a second problem: the $\hat{F}$ operator is not self adjoint (see [3] for instance).
2.2. Example 2

We will study our previous system and we correct the problems of the previous example, we will use $T$ such that $\{T, D\} = 1$ (see [4] for instance) and the general solution for $T$ will be:

$$ T = \frac{mx}{p} + f \left( p, x^0 - \frac{mx}{p}, p_0 \right), $$

(10)
then

$$ Det \left[ \begin{pmatrix} \{T, T\} & \{T, D\} \\ \{D, T\} & \{D, D\} \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1, $$

and our $T$ clock is well defined all along the real line.

To simplify we consider the clock,

$$ T = \frac{mx}{p} + \alpha p + \beta \left( x^0 - \frac{mx}{p} \right) + \rho p_0, $$

(11)
with $\{T, D\} = 1$ and for the partial observable $f = x$ we obtain,

$$ F = \frac{pr}{m} - \frac{\alpha p^2}{2m} + \frac{\beta x}{m} - \frac{\rho p_0^2}{m}. $$

(12)
To determine the self adjointness for $\hat{F}$ we must find

$$ \text{Dim} \left[ \text{Ker} \left( \hat{F} + i \right) \right], \text{Dim} \left[ \text{Ker} \left( \hat{F} - i \right) \right] $$

and compare our results, with the previous purpose we consider

$$ (\hat{F} + i) \psi = 0 \quad \text{and} \quad (\hat{F} - i) \psi = 0. $$

We take,

$$ \psi = \delta \left( p_0 + \frac{p^2}{2m} \right), $$

(13)
and the solutions for $(\hat{F} \pm i) \psi = 0$ are:

$$ g_\pm = r_\pm \exp \left( \mp \frac{p^2}{2m} + i \frac{p^2 \tau}{2m \beta} - \frac{\alpha p^3}{3m \beta} + \frac{\rho p_0^2}{8m^2 \beta} \right) - \frac{p^2 \rho(2mp_0 + p_0^2)}{4m^2 \beta}, $$

(14)
in this case we define,

$$ g(p) = g \left( p, -\frac{p^2}{2m} \right) $$

(15)
and,

$$ \langle \psi_1 | \psi_2 \rangle_{F_{1s}} = \int_{-\infty}^{\infty} dp' \tilde{g}_1(p) \tilde{g}_2(p), $$

(16)
$\hat{F}$ is self adjoint and we have not the second problem of the first example.

Now, we take the eigenvalue equation $\hat{X} \psi = x_1 \psi$, then we take:

$$ \psi(p, p_0) = \delta \left( p_0 + \frac{p^2}{2m} \right) \frac{g(p)}{r_1}, $$

(17)
and,

$$ g(p) = r_1 \exp \left( \frac{\rho p_0^2}{8m^2 \beta} - \frac{\alpha p^3}{3m \beta} + \frac{p^2 \tau}{2m \beta} - \frac{i x_1 p}{\beta} \right), $$

(18)
using the results above we obtain:

$$ \langle \psi_{x', \tau'} | \psi_{x, \tau} \rangle = |r_1|^2 \sqrt{\frac{2m \pi \beta}{\tau - \tau'}} \exp \left( \frac{im(x' - x_1)^2}{2\beta(\tau - \tau')} \right), $$

(19)
and in consequence we obtain the correct amplitude for the free particle (see [3] for instance).
3. Two constraints
Now, we will extend our results for covariant systems with more constraints,

\[
D_1 = \frac{1}{2}[-(p_1)^2 + (p_2)^2 + (p_3)^2],
\]

\[
D_2 = -\frac{1}{2}[q_1 p_1 + q_2 p_2 + q_3 p_3],
\]

the phase space coordinates are \((q_1, q_2, q_3, p_1, p_2, p_3)\) and,

\[
\{D_1, D_2\} = D_1.
\]

We take similar restrictions like our previous example, in this way we define clocks all along the real line and the quantum operator associated with a complete observable is self adjoint, with the restrictions:

\[
\{T_1, D_1\} = 1, \quad \{T_2, D_2\} = 1, \quad \{T_2, D_1\} = 0,
\]

we have,

\[
\text{Det} \begin{pmatrix}
{T_1, T_1} & {T_1, T_2} & {T_1, D_1} & {T_1, D_2} \\
{T_2, T_1} & {T_2, T_2} & {T_2, D_1} & {T_2, D_2} \\
{D_1, T_1} & {D_1, T_2} & {D_1, D_1} & {D_1, D_2} \\
{D_2, T_1} & {D_2, T_2} & {D_2, D_1} & {D_2, D_2}
\end{pmatrix} = 1,
\]

and the general solutions for \(T_1\) and \(T_2\) are:

\[
T_1 = -\frac{q_1}{p_1} + f \left( p_1, p_2, q_2 + \frac{q_1 p_2}{p_1}, p_3, q_3 + \frac{q_1 p_3}{p_1} \right),
\]

\[
T_2 = 2 \ln(p_1) + g \left( \frac{p_2}{p_1}, \frac{p_1}{p_2}, (p_1 q_2 + q_1 p_2), \frac{p_3}{p_1}, \right.
\]

\[
\left. \frac{p_1}{p_2} (-p_3 q_2 + p_2 q_3) \right),
\]

and our clocks are defined all along the real line.

3.1. Example 1
We can take as clocks,

\[
T_1 = -\frac{q_1}{p_1}, \quad T_2 = 2 \ln(p_1),
\]

and \(f = q_2\) as partial observable, then the complete observable will be:

\[
F = (q_1 p_2 + q_2 p_1) \exp\left( -\frac{\tau_2}{2} \right) + \frac{p_2}{p_1} \tau_2 \exp\left( \frac{\tau_2}{2} \right),
\]

we consider \((\hat{F} + i)\psi_+ = 0\) and we obtain the solutions:

\[
\psi_+ = p_1^{\tau_1} (p_1 + p_2)^{\tau_2} \left( \frac{\tau_2}{2} \right) (p_2 - p_1) g_+(p_3),
\]

then we have,

\[
|\psi_+|^2 = |p_1 + p_2|^{\tau_2} \exp\left( \frac{\tau_2}{2} \right) (p_2 - p_1)^2 |g_+(p_3)|^2,
\]

and we can choose \(g_+(p_3)\) such that \(\psi_+\) and \(\psi_-\) are square integrable; \(\text{Dim}[\text{Ker}(\hat{F} + i)] = 1\), \(\text{Dim}[\text{Ker}(\hat{F} - i)] = 1\) and \(\hat{F}\) will be self adjoint.
3.2. Example 2
We use now,
\[ T_1 = -\frac{q_1}{p_1}, \quad T_2 = \ln(p_2), \]  
(28)
as clocks and \( f = q_1 \) like partial observable, then the complete observable will be:
\[ F = -\frac{p_1}{p_2} \tau_1 \exp \left( \frac{T_2}{2} \right), \]  
(29)
we must solve the equations \( (\hat{F} \pm i) \psi_\pm = 0 \) to determine if \( \hat{F} \) is self adjoint or it is not, we find the solutions:
\[ \psi_\pm = \exp \left( \mp \frac{q_1 p_2}{\tau_1} \exp \left( -\frac{T_2}{2} \right) \right) g_\pm(p_3), \]  
(30)
and later we obtain,
\[ |\psi_\pm|^2 = \exp \left( \mp \frac{2q_1 p_2}{\tau_1} \exp \left( -\frac{T_2}{2} \right) \right) |g_\pm(p_3)|^2. \]  
(31)
We can choose \( g_\pm(p_3) \) such that \( \text{Dim}(\text{Ker}(\hat{F} + i)) = \text{Dim}(\text{Ker}(\hat{F} - i)) = 0 \) and \( \hat{F} \) will be self adjoint.

4. Conclusions and perspectives
To conclude our work we make the following discussion. In the first example, for the one dimensional non-relativistic free particle we found two problems: the first problem was that,
\[ \text{Det} \left( \begin{array}{cc} \{T, T\} & \{T, D\} \\ \{D, T\} & \{D, D\} \end{array} \right) = 0, \]

for a selection for \( T \) and our clock was not defined all along the real line, the second problem was that the operator associated with our complete observable \( F \) was not self adjoint, we correct both problems when we choose our clock \( T \) like a phase space function such that the Poisson bracket with the constraint is one \( \{T, D\} = 1 \). Now, in this case,
\[ \text{Det} \left( \begin{array}{cc} \{T, T\} & \{T, D\} \\ \{D, T\} & \{D, D\} \end{array} \right) = 1, \]

and our clock is well defined all along the real line and our operator associated with our complete observable \( F \) is self adjoint.

For the case of two constraints, we begin with clocks that are conjugate to the set of constraints and we did not find problems with the operators associated with the complete observable.

For the model with two constraints we studied some partial observables with different selections for the clocks in such a way that,
\[ \text{Det} \left( \begin{array}{cc} \{T_i, T_j\} & \{T_i, D_j\} \\ \{D_i, T_j\} & \{D_i, D_j\} \end{array} \right) = 1, \]

and we found that \( \hat{F} \) is self adjoint, where \( F \) is a complete observable constructed for a partial observable \( f \) and the clocks are globally well defined. However, with a different choice for the previous determinant, we could have the problem that our clocks are not defined all along the real line and the problem to the self adjointness for the operator associated with a complete observable could be present.

The following general study can be done: if you have a phase space with coordinates \((q_1, \ldots, q_n, p^1, \ldots, p^n)\) and first class constraints \( \gamma_a \), in such a way that,
\[ \{\gamma_a, \gamma_b\} = C_{ab}^{\phantom{a}c} \gamma_c, \]  
(32)
where $a = 1, \ldots, m$ and $m$ is the number of first class constraints. Then, we can choose $m$ clocks in such a way that $\{T_i, \gamma_j\} = \delta_{ij}$ and a first problem is to determine if,

$$\text{Det} \left[ \begin{pmatrix} \{T_i, T_j\} & \{T_i, \gamma_j\} \\ \{\gamma_i, T_j\} & \{D_i, D_j\} \end{pmatrix} \right] \neq 0,$$

for all the values of $T_i$, if you have that, the clocks are defined all along the real line.

A second general problem with our previous conditions is to determine if the operator $\hat{F}$ associated with the complete observable $F$ is self adjoint, where $F$ is obtained of a partial observable $f$. If our method fails in some cases, we must determine these reasons in future studies.

The definition of clocks and self adjoint operators associated with complete observables or gauge functions is open for complex systems, including general relativity.

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