1. Introduction

We produce a series of Central Limit Theorems (CLTs) associated to compact metric measure spaces \((K, d, \eta)\), with \(\eta\) a reasonable probability measure. For the first CLT, we can ignore \(\eta\) by isometrically embedding \(K\) into \(\mathcal{C}(K)\), the space of continuous functions on \(K\) with the sup norm, and then applying known CLTs for sample means on Banach spaces (Theorem 3.1). However, the sample mean makes no sense back on \(K\), so using \(\eta\) we develop a CLT for the sample Fréchet mean (Corollary 4.1). This involves working on the closed convex hull of the embedded image of \(K\).

To work in the easier Hilbert space setting of \(L^2(K, \eta)\), we have to modify the metric \(d\) to a related metric \(d_\eta\). We obtain an \(L^2\)-CLT for both the sample mean and the sample Fréchet mean (Theorem 5.1), and we relate the Fréchet sample and population means on the closed convex hull to the Fréchet means on the image of \(K\). Since the \(L^2\) and \(L^\infty\) norms play important roles, in §6 we develop a metric-measure criterion relating \(d\) and \(\eta\) under which all \(L^p\) norms are equivalent.

2. Background Material

Throughout the paper, \((K, d)\) will be a compact metric space. Recall that a \(G\)-valued random variable \(X\) is a function \(X : \Omega \rightarrow G\), where \((\Omega, \mathcal{F}, P)\) is a fixed probability triple. The induced measure/distribution on \(G\) is given by \(X^*(P)\), with

\[ X_*(P)(A) = P(X^{-1}(A)), \quad A \subset G. \]

We recall the setup and statement of a Central Limit Theorem on Banach spaces due to Zinn.

**Definition 2.1.** (i) Let \(G\) be a Banach space. A probability measure \(\gamma\) on \(G\) is a Gaussian Radon measure if for every nontrivial linear functional \(L : G \rightarrow \mathbb{R}\), on \(G\), the pushforward measure \(L_*(\gamma)\) is a non-degenerate Gaussian measure on \(\mathbb{R}\), i.e., a standard Gaussian measure with non-zero variance.

(ii) Let \(X_1, \ldots, X_n, \ldots\) be any set of \(G\)-valued i.i.d. random variables with common distribution \(\mu\). \(\mu\) satisfies the \(G\)-Central Limit Theorem (\(G\)-CLT) on \(G\) if there exists
a Gaussian Radon probability measure $\gamma$ on $G$ such that the distributions, $\mu_n$, of $\frac{X_1 + \ldots + X_n}{\sqrt{n}}$ converge, i.e., for every bounded -continuous real-valued function $f$ on $G$,

$$\int_G f \mu_n \to \int_G f \gamma.$$

(iii) The metric $d$ on $K$ implies Gaussian continuity (or $d$ is CGI) if whenever $\{X(s)\}_{s \in K}$ is a separable Gaussian process such that

$$\mathbb{E} \left[ |X(t) - X(s)|^2 \right] \leq d^2(t, s),$$

then $X$ has continuous sample paths a.s.

Let $H_d(K, \epsilon) = \log(N_d(K, \epsilon))$, where $N_d(K, \epsilon)$ is the smallest number of $d$-balls of diameter at most $2\epsilon$ which cover $K$.

**Proposition 2.1.** [Thm. 3.1] If

$$\int_0^\infty H_d^{1/2}(K, u) du < \infty,$$

then $d$ is GCI.

Let $C(K)$ be the Banach space of continuous functions on the compact metric space $(K, d)$ equipped with the sup-norm $\| \cdot \|_\infty$. $C(K)$ becomes a complete metric space with the induced distance function $d_\infty$ by $d_\infty(f, g) = \| f - g \|_\infty$, $\forall f, g \in C(K)$.

**Definition 2.2.** For the compact metric space $(K, d)$, set

$$\text{Lip}(d) = \{ x \in C(K) : \sup_{t \neq s} \frac{|x(t) - x(s)|}{d(t, s)} < \infty \}.$$

$\text{Lip}(d)$ is nonempty (by letting $x$ be a constant function). We check that $\text{Lip}(d)$ is closed. If $\{x_k\} \in \text{Lip}(d)$ has $\lim_{k \to \infty} x_k = y \in C(K)$, then for any $\epsilon > 0$ and $t \neq s$,

$$\frac{|y(t) - y(s)|}{d(t, s)} \leq \frac{|y(t) - x_j(t)|}{d(t, s)} + \frac{|x_j(t) - x_j(s)|}{d(t, s)} + \frac{|x_j(s) - y(s)|}{d(t, s)} \leq 2\epsilon + \frac{|x_j(t) - x_j(s)|}{d(t, s)},$$

for $j = j(\epsilon) \gg 0$ independent of $t, s$. This implies that $y \in \text{Lip}(d)$.

**Definition 2.3.** A Radon probability measure $\mu$ on the Banach space $(G, \| \cdot \|)$ has zero mean and finite variance, respectively, if

$$\int_G x \mu(dx) = 0, \quad \int_G \|x\|^2 \mu(dx) < \infty,$$

respectively.
Of course, if a sequence of $G$-valued random variables $X_i$ has finite expectation, then the new random variable $X_i - \mathbb{E}[X_i]$ has zero mean.

We can now state Zinn’s CLT.

**Theorem 2.1.** [7] Let $(K, d)$ be a compact metric space with CGI. If $\mu$ is a Radon probability measure on $\text{Lip}(d)$ with zero mean and finite variance, then $\mu$ satisfies the Central Limit Theorem on $(C(K), \| \cdot \|_\infty)$ in the sense of Definition 2.1(ii).

For our main results, we need to define the Fréchet mean with respect to a probability measure $Q$ on $(K, d)$. This generalizes the notion of centroids from vector spaces to metric spaces.

**Definition 2.4.** (i) The Fréchet function $f : K \to \mathbb{R}$ is

$$f(p) = \int_M d^2(p, z)Q(z)dz, p \in M.$$ 

If $f(p)$ has a unique minimizer $\mu = \arg\min_{p \in K} f(p)$, we call $\mu$ the Fréchet mean of $Q$.

(ii) Given an i.i.d. sequence $X_1, \ldots, X_n \sim Q$ on $M$, the empirical Fréchet mean is defined to be

$$\mu_n = \arg\min_{p \in M} \frac{1}{n} \sum_{i=1}^n d^2(p, X_i),$$

provided the argmin is unique.

Unlike centroids in Euclidean space, the uniqueness of Fréchet mean cannot be guaranteed, even in spaces which locally look like Euclidean space.

**Example 2.1.** We parametrize an open cone (minus a line) $Z : x^2 + y^2 = z^2$ of height one by

$$F(u, v) = \left( \frac{1}{\sqrt{2}} u \cos v, \frac{1}{\sqrt{2}} u \sin v, \frac{1}{\sqrt{2}} u \right), \quad (u, v) \in (0, 1) \times (0, 2\pi).$$

There is a Riemannian isometry from the sector $S = \{(r, \theta) \in (0, \sqrt{2}) \times (0, \sqrt{2}\pi)\}$ to $Z$ induced by $\alpha : (r, \theta) \mapsto (u = r/\sqrt{2}, v = \sqrt{2}\theta)$, i.e.,

$$(r, \theta) \mapsto \left( \frac{r}{\sqrt{2}} \cos(\sqrt{2}\theta), \frac{r}{\sqrt{2}} \sin(\sqrt{2}\theta), \frac{r}{\sqrt{2}} \right).$$

Indeed, the first fundamental form of the sector at $(r, \theta)$, respectively the first fundamental form of the cone at $(u, v)$, are

$$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & \frac{u^2}{2} \end{pmatrix},$$
respectively. It is easy to check that the differential $d\alpha$ preserves these inner products. Thus every point $p \in S$ has a neighborhood $U$ such that the usual Euclidean distance between $q_1, q_2 \in U$ equals the geodesic distance between $\alpha(q_1), \alpha(q_2)$.

It is easy to check that for e.g. the uniform distribution on $S$, the Fréchet mean/centroid $(\bar{x}, \bar{y})$ is inside $S$. In contrast, by the rotational symmetry of the geodesic distance function on $Z$, the minima of the Fréchet function on $Z$ form a circle containing $\alpha(\bar{x}, \bar{y})$.

For results on CLTs when the Fréchet mean is not unique, see [2].

3. A CLT for Compact Metric Spaces

In this section, we isometrically embed the compact metric $(K, d)$ into the Banach space $(C(K), d_\infty)$ to obtain a CLT on the image of $K$.

We define $\iota_d : K \rightarrow C(K), x \mapsto f_x := d(x, \cdot)$.

The following proposition is well known.

**Proposition 3.1.** $\iota_d : (K, d) \rightarrow (\iota_d(K), d_\infty)$ is an isometry.

**Proof.** For $x, y \in K$, we have
\[
d_\infty(f_x, f_y) = \|f_x - f_y\|_\infty = \max_{z \in K} |d(x, z) - d(y, z)| \geq |d(x, y) - d(y, y)| = d(x, y).
\]

On the other hand, for $x, y, z \in K$, we have
\[
d(x, z) - d(y, z) \leq d(x, y), d(y, z) - d(x, z) \leq d(x, y) \Rightarrow |d(x, z) - d(y, z)| \leq d(x, y),
\]
so
\[
\max_{z \in K} |d(x, z) - d(y, z)| \leq d(x, y) \Rightarrow d_\infty(f_x, f_y) \leq d(x, y).
\]
Thus $\iota_d$ is an isometry.

It follows that $\iota_d$ is an injection, and $\iota_d(K)$ is a compact subset of $C(K)$.

To obtain a CLT on $\iota_d(K)$ from Theorem 2.1, we need to verify its hypotheses.

**Lemma 3.1.** $\iota_d(K) \subset \text{Lip}(d)$.

**Proof.** For $f_x \in \iota_d(K)$, the triangle inequality for $s, t \in K$ gives
\[
|f_x(t) - f_x(s)| = |d(x, t) - d(x, s)| \leq d(s, t).
\]
It follows that
\[
\sup_{s \neq t} \left| \frac{f_x(s) - f_x(t)}{d(s, t)} \right| \leq 1.
\]

In the following proof, we strongly use the fact that $C(K)$ is a “linearization” of $K$.
Lemma 3.2. The metric $d$ on $K$ is GCI.

Proof. We must verify (2.1) in Proposition 2.1. Equivalently, we will show

$$\int_0^\infty H_{d_{d\infty}}^{1/2}(\nu_d(K), u) du < \infty. \tag{3.1}$$

As a compact set, $\nu_d(K)$ can be covered by $N$ balls of radius 1 for some $N$. Fix any point $x_0 \in \nu_d(K)$, and consider the $d_{d\infty}$-ball $B_{d\infty}(x_0, 1)$ of radius 1 centered at $x_0$. The closure $\overline{B_{d\infty}(x_0, 1)}$ equals $\nu_d(B_d(\nu_d^{-1}(x_0), 1))$ of the corresponding ball in $K$. It follows that $\overline{B_{d\infty}(x_0, 1)}$ is compact, so we can cover $B(x_0, 1)$ by $M$ balls of radius $1/2$ for some $M$.

Since $d_{d\infty}$ is translation invariant, $M$ is independent of the choice of $x_0$. Moreover, $d_{d\infty}$ is homogeneous in the sense that $d_{d\infty}(cf, cg) = |c|d_{d\infty}(f, g)$ for $c \in \mathbb{R}$. Thus for $r > 0$, any $d_{d\infty}$-ball of radius $r$ contained in $\nu_d(K)$ can be covered by $M$ balls of radius $r/2$. Hence

$$N_{d_{d\infty}}(\nu_d(K), 2^{-k}) \leq N \cdot M^{k+1}.$$  

To estimate (2.1), we integrate over $[0, 1]$ and $[1, \infty)$ separately. Since $\nu_d(K)$ is compact, it is covered by a single $d_{d\infty}$-ball $B_{d\infty}(x_0, R)$ for some $R \gg 0$ and a fixed $x_0 \in \nu_d(K)$. Choose $k_0 \in \mathbb{N}$ such that $2^{k_0} \leq R < 2^{k_0+1}$. We have

$$\int_1^\infty H_{d_{d\infty}}^{1/2}(\nu_d(K), u) du = \int_1^\infty \sqrt{\log(N_{d_{d\infty}}(\nu_d(K), u))} du\leq \sum_{k=1}^{k_0+1} \sqrt{\log(N_{d_{d\infty}}(\nu_d(K), 2^k))(2^k - 2^{k-1})}$$

$$\leq \sum_{k=1}^{k_0+1} \sqrt{\log(N_{d_{d\infty}}(\nu_d(K), 2^k))2^{k-1}} < \infty.$$ 

For the region $[0, 1]$, we have

$$\int_0^1 H_{d_{d\infty}}^{1/2}(\nu_d(K), u) du$$

$$= \int_0^1 \sqrt{\log(N_{d_{d\infty}}(\nu_d(K), u))} du \leq \sum_{k=0}^{\infty} \sqrt{\log(N_{d_{d\infty}}(\nu_d(K), 2^{-k-1}))(2^{-k} - 2^{-k-1})}$$

$$\leq \sum_{k=0}^{\infty} \sqrt{\log(N \cdot M^{k+2})} 2^{-k-1} = \sum_{k=0}^{\infty} \sqrt{(k+2) \log M + \log N} 2^{-k-1} < \infty.$$ 

Adding these estimates gives (3.1). \qed

This gives our first Central Limit Theorem on $K$, or really on the isometric space $\nu_d(K)$. 
Theorem 3.1. Let \((K, d)\) be a compact metric space, let \(\mu\) be a Radon probability measure on \(K\) with finite variance and such that \(\iota_d \ast \mu\) has zero mean on \(\iota_d(K)\). Then \(\iota_d \ast \mu\) satisfies the \(G\)-CLT for \(G = (\mathcal{C}(K), \| \cdot \|_\infty)\).

Proof. By Lemmas 3.1, 3.2, the hypotheses of Theorem 2.1 are satisfied for \(\iota_d \ast \mu\). \(\square\)

4. A Fréchet CLT associated to a compact metric space

In the previous section, we found a \(G\)-CLT on Banach space associated with the usual sample mean \(\sqrt{n} \cdot \frac{1}{n} \sum_{i=1}^{n} X_i\) on \(G = \mathcal{C}(K)\). In this section, we prove a \(G\)-CLT on the compact metric space \(K\) (again, really on \(\iota_d(K)\)), endowed with a Radon probability measure \(\eta\), for the sample Fréchet mean

\[
\arg\min_{Y \in \iota_d(K)} \frac{1}{n} \sum_{i=1}^{n} \|X_i - Y\|_2^2, \tag{4.1}
\]

\(X_1, \ldots, X_n\) are i.i.d. \(\iota_d(K)\)-valued random variables, and the \(L^2\) norm is taken with respect to \(\iota_d, \ast \eta\).

Note that we compute the sample Fréchet mean with respect to the \(L^2\)-norm, since we will need a Hilbert space structure below. The minimizer of (4.1) may not exist or be unique, since \(\iota_d(K)\) may be neither closed nor in \(\mathcal{C}(K)\). Instead, we consider the closed convex hull of \(\iota_d(K)\), on which the uniqueness of the Fréchet mean is guaranteed.

Definition 4.1. Let \(\iota_d(K)^c\) be the convex hull of \(\iota_d(K)\), i.e., the intersection of all convex subsets of \(\mathcal{C}(K)\) containing \(\iota_d(K)\), and let

\[
S_d = S_d(K) = \overline{\iota_d(K)^c}
\]

be the closure of \(\iota_d(K)^c\).

By [1] Thm. 5.35, \(S_d\) is a compact subset of \(\mathcal{C}(K)\). It is easy to check that \(S_d(K) \subset \text{Lip}(d)\). As the minimizer of a convex function on a closed convex space, the Fréchet mean is unique.

Example 4.1. To continue with Example 2.1, choose a probability measure \(Q\) on the cone \(Z\). The sample Fréchet mean for \(K\)-valued random variables \(Y_i\) lies in the interior of \(Z\) in \(\mathbb{R}^3\). It is unclear if the sample Fréchet mean for the \(\iota_d(Z)\)-valued random variables \(X_i = \iota_d \circ Y_i\) lies in \(\iota_d(Z)\), but it certainly lies in \(S_d(Z)\). (This example doesn’t really show the strength of embedding \(Z\) into \(\mathcal{C}(Z)\), since \(Z\) lies in a linear space.

Similar remarks apply to the Fréchet minimum. While the Fréchet minimum for the cone \((Z, \eta)\) is not unique, the Fréchet minimum on \((S_d(Z), \iota_d, \ast \eta)\), the closed convex hull of the isometric set \((\iota_d(Z), \iota_d, \ast \eta)\), is unique. (Note that the sector \(S\) in Example 2.1 is only locally isometric to \(Z\).) While we have gained uniqueness, there
is no reason why the Fréchet minimum need be inside \( \iota_d(Z) \), as in Example 2.1. It is shown in [5, Supplement C] that in general the distance from the Fréchet mean in \( S_d(K) \) to \( \iota_d(K) \) is at most twice the diameter of \( K \).

At this point we have the embeddings \( K \hookrightarrow \iota_d(K) \subset S_d(K) \subset C(K) \subset L^2(K) \), where \( L^2(K) \) is taken with respect to a probability measure on \( K \). While \( K \hookrightarrow L^2(K) \) is no longer an isometry, there is a known CLT on \( S_d(K) \) equipped with the \( L^2 \) norm.

**Theorem 4.1.** Let \( \mu \) be a Radon probability measure supported in \( K \) such that \( \iota_d, \ast \mu \) satisfies (2.2). Then \( \iota_d, \ast \mu \) satisfies the G-CLT for \( G = (L^2(K), \| \cdot \|_{2, \eta}) \). The same result holds if the random variables \( \{X_i\} \) in the G-CLT are \( \iota_d(K) \)-valued and/or \( \mu \) has support in \( S_d(K) \).

**Proof.** By [6, Thm. 9.10], the Hilbert space \( L^2(K) \) is of type 2 and cotype 2. The existence of a CLT on spaces of type/cotype 2 follows from [4, Thm. 3.5]. \( \square \)

We also obtain a CLT for the sample Fréchet mean. Here the Hilbert space structure works to our advantage, as the sample Fréchet mean and the usual sample mean coincide.

**Proposition 4.1.** For \( S_d(K) \)-valued random variables \( \{X_i\} \), we have

\[
S_n := \frac{1}{n} \sum_{i=1}^{n} X_i = \arg\min_{Y \in S_d(K)} \frac{1}{n} \sum_{i=1}^{n} \|X_i - Y\|_{2, \eta}^2.
\]

**Proof.** It is well-known that in a finite dimensional Euclidean space, the sample mean coincides with the sample Fréchet mean. For fixed \( x \in K \), the real-valued random variables \( \{X_i(x)\} \) satisfy

\[
\frac{1}{n} \sum_{i=1}^{n} |X_i(x) - S_n(x)|^2 \leq \frac{1}{n} \sum_{i=1}^{n} |X_i(x) - Y(x)|^2, \quad \forall Y \in C(K).
\]

Therefore, for all \( Y \),

\[
\frac{1}{n} \sum_{i=1}^{n} \int_K |X_i(x) - S_n(x)|^2 dQ(x) \leq \frac{1}{n} \sum_{i=1}^{n} \int_K |X_i(x) - Y(x)|^2 dQ(x),
\]

so

\[
\frac{1}{n} \sum_{i=1}^{n} \|X_i - S_n\|_{2, \eta}^2 \leq \frac{1}{n} \sum_{i=1}^{n} \|X_i - Y\|_{2, \eta}^2,
\]

which implies

\[
S_n = \arg\min_{Y \in S_d(K)} \frac{1}{n} \sum_{i=1}^{n} \|X_i - Y\|_{2, \eta}^2.
\]

\( \square \)
Combining the Proposition with Theorem 4.1 gives us a CLT for the sample Fréchet mean. We set \( \| \cdot \|_{2, \eta} \) be the \( L^2 \) norm with respect to a measure \( \mu \), and set \( C_0(X) \) to be the set of bounded continuous functions on a topological space \( X \).

**Corollary 4.1.** (i) Let \( \{ X_i \} \) be i.i.d. \( S_d \)-valued random variables with distribution \( \mu \) a Radon probability measure supported in \( \nu_d(K) \) satisfying (2.3). Then there exists a Gaussian Radon probability measure \( \tilde{\gamma}_2 \) such that the distributions \( \mu_n \) of

\[
\argmin_{Y \in S_d} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \| X_i - Y \|_{2, \eta}^2
\]

converge weakly to \( \tilde{\gamma}_2 \) in the sense of Definition 2.1.

(ii) \( \gamma_2 \) in Theorem 4.1 equals \( \tilde{\gamma}_2 \) in distribution. In particular, for \( f \in C_0(S_d(K)) \),

\[
\int_{S_d(K)} f d\gamma_2 = \int_{S_d(K)} f d\tilde{\gamma}_2.
\]

(iii) Let \( \gamma_1 \) be the limiting measure obtained in Theorem 3.1. Then \( \gamma_1 = \gamma_2 \).

**Proof.** (i) follows from Proposition 4.1. For (ii), the Proposition implies that the distributions of the sample mean and the sample Fréchet mean are the same a.s. For (iii), this seems to be because Theorems 3.1 and 4.1 are essentially the same. \( \square \)

5. \( L^2 \) Techniques and G-CLTs

In this section, we embed the compact metric space \( K \), now equipped with a Radon measure \( \eta \) and a modified metric, into \( L^2(\nu_d(K), \| \cdot \|_{2, \eta}) \) to produce an \( L^2 \) version of a G-CLT. In this Hilbert space setting, we are able to relate the Fréchet means of the closed convex hulls to the Fréchet means on the embedded image of \( K \).

We define a seminorm on \( \nu_d(K) = \{ f_x = d(x, \cdot) : x \in K \} \) by

\[
\| f_x \|_{2, \eta}^2 = \int_{\nu_d(K)} |f_x(y)|^2 d\nu_d, \eta(y) = \int_K d^2(x, y) d\eta(y).
\]

Taking the completion the space of norm zero functions gives the Hilbert space \( (L^2(\nu_d(K)), \| \cdot \|_{2, \eta}) \). More precisely, we will prove \( L^2(\nu_d(K)) \)-CLTs for both the sample mean and the sample Fréchet mean.

The norm \( \| \cdot \|_{2, \eta} \) induces a metric \( d_{2, \eta} \) on \( \nu_d(K) \). Since \( \nu_d : (K, d) \rightarrow (\nu_d(K), d_{2, \eta}) \) is easily continuous, we can pull back \( d_{2, \eta} \) to a metric \( d_\eta := \nu_d^* d_{2, \eta} \) on \( K \):

\[
d_\eta(x, y) = d_{2, \eta}(f_x, f_y) = \left( \int_K \left( d(x, z) - d(y, z) \right)^2 d\eta(z) \right)^{1/2}.
\]

Thus \( \nu_{d_\eta} : (K, d_\eta) \rightarrow (\nu_{d_\eta}(K), d_{2, \eta}) \), defined by \( \nu_{d_\eta}(x) = d_\eta(x, \cdot) \), is an isometry. We interpret \( (K, d_\eta) \) as a modification of \( (K, d) \) which keeps track of the \( L^2 \) information of \( \eta \).
We want to relate the various metrics. Let $d_\infty$ be the metric on $C(K)$ induced by the sup norm $\| \cdot \|_\infty$, and let $[\iota_d(K)]$ be the image of $\iota_d(K)$ in $L^2(K)$. Consider the maps

$$(K, d) \xrightarrow{\iota_d} (\iota_d(K), d_\infty) \xrightarrow{F} ([\iota_d(K)], d_{2,\eta}) \xrightarrow{G} (K, d_\eta)$$

given by $F(f_x) = [f_x]$, where we take the $L^2$ equivalence class, and $G([f_x]) = \iota_{d_\eta}^{-1}(f_x) = \iota_d^{-1}\iota_d(x)$. (We show that $G$ is well-defined below.) $\iota_d$ is an isometry.

In general, $F$ and $G$ are not injective, since equivalence classes in $L^2(K)$ have many representatives, without a restriction on $\eta$.

**Lemma 5.1.** Assume that every $d$-ball $B_\epsilon(x)$ centered at $x \in K$ has $\eta(B_\epsilon(x)) > 0$. Then $F$ is injective, and $G$ is well-defined and injective.

Since $F$ and $G$ are trivially surjective, they are bijective under the Lemma’s hypothesis. Note that for Lebesgue measure and the standard metric on $\mathbb{R}^N$, the hypothesis is satisfied, while delta functions give rise to Radon measures that do not satisfy the hypothesis.

**Proof.** For $F$, it suffices to show that $F \circ \iota_d : x \mapsto [f_x]$ is injective. For $x \neq y$, and $\epsilon < d(x, y)/3$, we have

$$d(y, z) \geq d(x, y) - d(x, z) > 3\epsilon - \epsilon > d(x, z) + \epsilon,$$

for all $z \in B_\epsilon(x)$. Therefore

$$d_{2,\eta}([f_x], [f_y])^2 = \int_K |f_x(z) - f_y(z)|^2 d\eta_z \geq \int_{B_\epsilon(x)} |d(x, z) - d(y, z)|^2 d\eta_z > \epsilon^2 \eta(B_\epsilon(x)) > 0.$$

Thus $[f_x] \neq [f_y]$.

To show that $G$ is well-defined, if $f_x, f_y \in [f_x] \in L^2(K)$, then

$$d_{2,\eta}([f_x], [f_y]) = 0 \Rightarrow \int_K |d(x, z) - d(y, z)|^2 d\eta(z) = 0.$$

As above, this implies that $x = y$, so $[f_x]$ has a unique representative of the form $f_x$. Since $\iota_d, \iota_\eta$ are injective, it follows that $G$ is injective.

We can now state and prove an $L^2$ CLT on $S_{d_\eta}(K)$ for both the sample mean and the sample Fréchet mean. As before, let $S_{d_\eta} = S_{d_\eta}(K)$ be the closed convex hall of $\iota_{d_\eta}(K)$ in the sup norm.

**Theorem 5.1.** Let $\{X_i\}$ be i.i.d. $S_{d_\eta}$-valued random variables with distribution $\mu$ a Radon probability measure supported in $S_{d_\eta}(K)$ satisfying (2.3). Assume that the hypothesis of Lemma 5.1 holds.
(i) There exists a Gaussian Radon probability measure $\gamma$ on $S_{d_\eta}$ such that the distributions $\mu_n$, of $\frac{X_1 + \cdots + X_n}{\sqrt{n}}$, converge to a probability measure $\gamma$ in the sense of Definition 2.1(ii).

(ii) The distributions $\tilde{\mu}_n$ of $\arg\min_{Y \in S_{d_\eta}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \|X_i - Y\|_{2,\eta}^2$ converge in the same sense to the same measure $\gamma$.

(iii) Under the hypothesis in Lemma 5.1, the distributions $\tilde{\mu}_n$ of $\arg\min_{Y \in S_{d_\eta}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \|X_i - Y\|_{2,\eta}^\infty$ converge in the same sense to a Gaussian Radon probability measure $\tilde{\gamma}$.

Proof. (i) Replacing the metric $d$ by $d_\eta$ in Theorem 3.1 gives the CLT for the sample mean.

(ii) Applying Corollary 4.1(i) and (iii) to $d_\eta$ gives the convergence of $\tilde{\mu}_n$ to the same measure $\gamma$.

(iii) The main idea is to use the isometric bijection $\iota_d \circ \iota_d^{-1} : (S_d(K), \| \cdot \|_{2,\eta}) \rightarrow (S_d(K), \| \cdot \|_{2,\eta})$. This extends linearly to an isometric bijection $\iota_{d_\eta} \circ \iota_{d_\eta}^{-1} : (S_{d_\eta}(K), \| \cdot \|_{2,\eta}) \rightarrow (S_{d_\eta}(K), \| \cdot \|_{2,\eta})$.

Set

$$Z_n := \arg\min_{Y \in S_{d_\eta}} \frac{1}{n} \sum_{i=1}^n \|X_i - Y\|_{\infty}^2.$$ 

By Proposition 4.1,

$$Z_n = \arg\min_{Y \in S_{d_\eta}} \frac{1}{n} \sum_{i=1}^n \|\iota_d \circ \iota_{d_\eta}^{-1} X_i - \iota_d \circ \iota_{d_\eta}^{-1} Y\|_{2,\eta}^2$$

$$= \iota_{d_\eta} \circ \iota_d^{-1} \left( \frac{1}{n} \sum_{i=1}^n \|\iota_d \circ \iota_{d_\eta}^{-1} X_i - \iota_d \circ \iota_{d_\eta}^{-1} Y\|_{2,\eta}^2 \right)$$

$$= \iota_{d_\eta} \circ \iota_d^{-1} \left( \frac{1}{n} \sum_{i=1}^n \iota_d \circ \iota_{d_\eta}^{-1} X_i \right).$$

$\{(\iota_d \circ \iota_{d_\eta}^{-1})(X_i)\}$ are $S_d$-valued i.i.d. random variables with common distribution $(\iota_d \circ \iota_{d_\eta}^{-1})_* \mu$. By Theorem 4.1, we obtain a $S_d$-CLT with respect to a Gaussian Radon measure $\gamma'$ on $S_d$. The isometry $\iota_{d_\eta} \circ \iota_d^{-1}$ then gives the $S_{d_\eta}$-CLT with respect to $(\iota_{d_\eta} \circ \iota_d^{-1})_* \gamma'$.

Because we are in a Hilbert space setting, we can prove that the Fréchet sample and population means on $S_{d_\eta}$ and on $\iota_{d_\eta}(K)$ have simple relationships.
Let $S_{d_{\eta}}$ be the closed convex hull of $\iota_{d_{\eta}}(K) := K_0$ in $L^2(K, \eta)$, and let $f^2_y = d_{\eta}(y, \cdot) \in \mathcal{C}(K)$. Let

$$F(\bar{x}) = \int_{S_{d_{\eta}}} d^2_{L^2}(\bar{x}, \bar{y})d\iota_{d_{\eta}, *}\eta(\bar{y}) = \int_{K_0} d^2_{L^2}(\bar{x}, \bar{y})d\iota_{d_{\eta}, *}\eta(\bar{y}) = \int_{K_0} d^2_{L^2}(\bar{x}, f^2_y)d\eta(y),$$

$$F(x) = \int_{K} d^2_{\eta}(x, y)d\eta(y) = \int_{K_0} d^2_{L^2}(f^2_x, f^2_y)d\iota_{d_{\eta}, *}\eta(f^2_y),$$

be the $L^2$ Fréchet functions of $S_{d_{\eta}}$ and $K$, respectively, and let

$$\bar{\mu} = \arg\min_{\bar{x} \in S_{d_{\eta}}} F(\bar{x}), \ \mu = \arg\min_{x \in K} \int_{K} d^2_{\eta}(x, y)d\eta(y)$$

be the population means on $S_{d_{\eta}}$ and $K$, respectively. Set $\mu_0 = \iota_{d_{\eta}}(\mu)$.

We note that as the minimum of a convex function on a convex set, $\bar{\mu}$ is unique. Also, gradients of differentiable functions exist in Hilbert spaces.

**Proposition 5.1.** Assume that (i) $\mu$ is unique, (ii) $K_0$ is the zero set of a Fréchet differentiable function $H : L^2(K, \eta) \rightarrow \mathbb{R}$ with $\nabla H_{\mu_0} \neq 0$. Then $\mu_0$ is the closest point in $K_0$ to $\bar{\mu}$. The same relationship holds for the sample Fréchet means of $K_0$-valued i.i.d. random variables.

**Proof.** Note that $K_0 = \partial K_0$ in $L^2(K, \eta)$, since a compact subset of an infinite dimensional space has no interior. Also, $\bar{\mu} \in K_0$ implies $\bar{\mu} = \mu_0$, so we may assume $\bar{\mu} \notin K_0$.

The method of Lagrangian multipliers is valid in $L^2(K, \eta)$, so there exists $\lambda \in \mathbb{R}$ with $\nabla \hat{F}_{\mu_0} = \lambda \nabla H_{\mu_0}$. The differential $D\hat{F}$ at $\mu_0$ is given by

$$D\hat{F}_{\mu_0}(v) = (d/dt)|_{t=0} \int_{S_{d_{\eta}}} d^2_{L^2}(\mu_0 + tv, f^2_y)d\eta(y)$$

$$= (d/dt)|_{t=0} \int_{K} \langle \mu_0 + tv - f^2_y, \mu_0 + tv - f^2_y \rangle d\eta(y)$$

$$= 2 \left\langle v, \mu_0 - \int_{K} f^2_y d\eta(y) \right\rangle,$$

where the last term equals the Hilbert space integral

$$\int_{K_0} \bar{y}d\iota_{d_{\eta}, *}\eta(\bar{y}) = \int_{S_{d_{\eta}}} \bar{y}d\iota_{d_{\eta}, *}\eta(\bar{y}).$$

Thus $\nabla \hat{F}_{\mu_0} = 2(\mu_0 - \int_{S_{d_{\eta}}} \bar{y}d\iota_{d_{\eta}, *}\eta(\bar{y}))$. Since $\nabla \hat{F}_p = 0$ only at $p = \bar{\mu}$, we see that $\mu = \int_{S_{d_{\eta}}} \bar{y}d\iota_{d_{\eta}, *}\eta(\bar{y})$. (This is the usual statement that the Fréchet mean is the center of mass of a convex set in $\mathbb{R}^n$.) Thus $\nabla \hat{F}_{\mu_0} = 2(\mu_0 - \bar{\mu})$. 
Since $\mu_0 - \bar{\mu}$ is a multiple of $\nabla H_{\mu_0}$, which is perpendicular to the level set $\nu_{d_\eta}(K)$, we have $\mu_0 - \bar{\mu} \perp \partial K_0$. We have not used that $\mu_0$ is a minimum, so the same perpendicularity holds at any critical point of $\bar{F}$ on $\partial K_0$. We can translate in $S^d_\eta$ so that $\bar{\mu} = 0$, in which case $\nabla \bar{F}_p = 2p$ is twice the Euler vector field. The level sets $\bar{F}^{-1}(r)$ are thus spheres centered at the origin in $S^d_\eta$. Since $\mu_0$ is on the lowest level set of any point in $K_0$, $\mu_0$ is closer to the origin than any other critical point of $\bar{F}$ on $\partial K_0$.

If we consider the distance function $D : K_0 \rightarrow \mathbb{R}, D(\bar{x}) = d^2(\mu_0, x)$, then a Lagrangian multiplier argument as above shows that at a critical point $p$ of $D$, we have $\mu_0 - p \perp \partial K_0$. Thus $\mu_0$ is a critical point of $D$, and by the last paragraph $\mu_0$ must be the closest point in $K_0$ to $\mu_0$.

The same argument holds for the sample means. □

If a closest point $p(z) \in K_0$ to each $z \in S^d_\eta$ can be chosen so that $p$ is continuous, as in the unlikely case that $K_0$ is convex, then we get a G-CLT on $K_0$ and hence on $K$ for $p_*\bar{\mu}_n, p_*\gamma$. This would connect Theorem 5.1 and Proposition 5.1.

### 6. Relating $L^p$ norms

We have results for $L^\infty$- and $L^2$-CLTs, so we wish to compare the associated metrics on $\nu_{d}(K)$. Since $\nu_{d}(K)$ is compact, all norms are abstractly equivalent. In this section, we introduce a metric-measure assumption under which the $L^p$ norms on $\nu_{d}(K)$ are explicitly equivalent, i.e., the constants in the norm comparisons are explicit.

Of course, for a probability measure $\eta$ on $K$, we have for $f \in C(K)$ and $p \in [1, \infty)$

$$\|f\|_p^p = \int_K |f|^p d\eta \leq \|f\|_\infty^p \int_K d\eta = \|f\|_\infty^p.$$  

Thus

$$d_p(f, g) \leq d_{\infty}(f, g),$$

for $f, g \in C(K)$ and $d_p, d_{\infty}$, respectively, the $L^p, L^\infty$ metrics, respectively.

We would like a reverse inequality on $\nu_{d}(K)$ in (6.1), which is impossible without any assumptions. In particular, we assume that $f_x = d(x, \cdot) \in \nu_{d}(K)$ is $\eta$-measurable for $x \in K$. We also strengthen the triangle inequality

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

as follows:

**Assumption:** There exist $C, D \in (0, 1)$ such that for all $x, y \in K$

$$\eta\{z \in K : |d(x, z) - d(y, z)| \leq D \cdot d(x, y)\} < C.$$  

(6.2)
The intuition is that for $D = 0$, a workable $\eta$ has $\eta\{z \in K : |d(x, z) - d(y, z)| \leq D \cdot d(x, y)\} = 0$, as for Lebesgue measure. For $D$ close to zero, we demand that
\[
\max_{x, y \in K} \eta\{z \in K : |d(x, z) - d(y, z)| \leq D \cdot d(x, y)\}
\]
should be strictly less than $1 = \eta(K)$. The assumption fails for delta function measures on $\mathbb{R}^n$, but appears to hold for normalized Lebesgue measure on compact subsets of $\mathbb{R}^n$.

**Proposition 6.1.** Assume that $f_x$ is $\eta$-measurable for $x \in K$ and that (6.2) holds. For $p \in [1, \infty), p' \in [1, \infty]$ with $p < p'$, we have
\[
D(1 - C)^{1/p} \cdot d_p(f_x, f_y) \leq d_p(f_x, f_y) \leq d_{p'}(f_x, f_y),
\]
for all $x, y \in K$. Thus the $L^p$ norms are explicitly equivalent.

**Proof.** The fact that $d_p(f_x, f_y) = \|f_x - f_y\|_p \leq \|f_x - f_y\|_{p'} = d_{p'}(f_x, f_y)$ is contained in the Hölder inequality proof that $L^{p'} \subset L^p$ on a finite measure space.

Set $S_{x,y} = \{z \in K : |d(x, z) - d(y, z)| \leq D \cdot d(x, y)\}$. Using (6.2), we have
\[
\|f_x - f_y\|_p^p \geq \int_{K \setminus S_{x,y}} |d(x, z) - d(y, z)|^p d\eta(z) \geq \int_{K \setminus S_{x,y}} D^p \cdot d^p(x, y) d\eta(z) \\
\geq D^p(1 - C)d^p(x, y) = D^p(1 - C)\|f_x - f_y\|_{\infty}^p.
\]

Thus
\[
d_p(f_x, f_y) \geq D(1 - C)^{1/p}d_{\infty}(f_x, f_y) \geq D(1 - C)^{1/p}d_{p'}(f_x, f_y),
\]
by (6.1).

**Remarks 6.1.** (i) The $d_p$ pull back to metrics on $K$, also denoted $d_p$, which by the Proposition are all explicitly equivalent.

(ii) For fixed $x, y$, set $K_D^p = \{z \in K : |d_p(x, z) - d_p(y, z)| \leq D \cdot d_p(x, y)\}$. Thus $K_D^\infty$ is the set in (6.2). From the estimates in the proof above, we obtain
\[
\|f_x - f_z\|_{p'} - \|f_y - f_z\|_{p'} \leq D^{-1}(1 - C)^{-1/p}\|f_x - f_z\|_p - \|f_y - f_z\|_p, \\
D\|f_x - f_y\|_p \leq D\|f_x - f_y\|_{p'}.
\]

Thus
\[
\{z : D^{-1}(1 - C)^{-1/p}\|f_x - f_z\|_p - \|f_y - f_z\|_p \leq D\|f_x - f_y\|_p\} \subset K_D^{p'},
\]
and the same with $x, y$ switched. Since $D^{-1}(1 - C)^{-1/p} > 1$, we obtain
\[
\{z : d_p(x, z) - d_p(y, z) \leq Dd_p(x, y)\}
\]
\[
\subset \{z : D^{-1}(1 - C)^{-1/p}\|f_x - f_z\|_p - \|f_y - f_z\|_p \leq D\|f_x - f_y\|_p\} \subset K_D^{p'},
\]
using $d_p(x, y) = d_p(f_x, f_y)$. The same holds with $x, y$ switched, so $K_D^p \subset K_D^{p'}$ for $p, p'$ as in the Proposition. Thus (6.2) has the greatest chance of holding for $d = d_1$. 

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