Correlation Clustering and Biclustering with Locally Bounded Errors

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Abstract

We consider a generalized version of the correlation clustering problem, defined as follows. Given a complete graph $G$ whose edges are labeled with $+$ or $-$, we wish to partition the graph into clusters while trying to avoid errors: $+$ edges between clusters or $-$ edges within clusters. Classically, one seeks to minimize the total number of such errors. We introduce a new framework that allows the objective to be a more general function of the number of errors at each vertex (for example, we may wish to minimize the number of errors at the worst vertex) and provide a rounding algorithm which converts “fractional clusterings” into discrete clusterings while causing only a constant-factor blowup in the number of errors at each vertex. This rounding algorithm yields constant-factor approximation algorithms for the discrete problem under a wide variety of objective functions.

1 Introduction

Correlation clustering is a clustering model first introduced by Bansal, Blum, and Chawla [5, 6]. The basic form of the model is as follows. We are given a collection of objects and, for some pairs of objects, we are given a judgment of whether the objects are similar or dissimilar. This information is represented as a labeled graph, with edges labeled $+$ or $-$ according to whether the endpoints are similar or dissimilar. Our goal is to cluster the graph so that $+$ edges tend to be within clusters and $-$ edges tend to go across clusters. The number of clusters is not specified in advance; determining the optimal number of clusters is instead part of the optimization problem.

Given a solution clustering, an error is a $+$ edge whose endpoints lie in different clusters or a $-$ edge whose endpoints lie in the same cluster. In the original formulation of the correlation clustering, the goal is to minimize the total number of errors; this formulation of the optimization problem is called MINDISAGREE. Finding an exact optimal solution is NP-hard even when the input graph is complete [5, 6]. Furthermore, if the input graph is allowed to be arbitrary, the best known approximation ratio is $O(\log n)$, obtained by [9, 10, 14]. Assuming the Unique Games Conjecture of Khot [16], no constant-factor approximation for MINDISAGREE on arbitrary graphs is possible; this follows from the results of [11, 21] concerning the minimum multicut problem and the connection between correlation clustering and minimum multicut described in [9, 10, 14].

Since theoretical barriers appear to preclude constant-factor approximations on arbitrary graphs, much research has focused on special graph classes such as complete graphs and complete bipartite graphs, which are the graph classes we consider here. Ailon, Charikar, and Newman [2, 3] gave a very simple randomized 3-approximation algorithm for MINDISAGREE on complete graphs. This algorithm was derandomized by van Zuylen and Williamson [24], and a parallel version of the algorithm was studied by Pan, Papailiopoulos, Recht, Ramchandran, and Jordan [20]. More recently, a 2.06-approximation algorithm was announced by Chawla, Makarychev, Schramm and Yaroslavtsev [12]. Similar results have been obtained for complete bipartite graphs. The first constant approximation algorithm for correlation clustering on complete bipartite graphs was described by Amit [4], who gave an 11-approximation algorithm. This ratio was improved by Ailon, Avigdor-Elgrabli, Liberty and van Zuylen [1], who obtained a 4-approximation algorithm. Chawla, Makarychev, Schramm and Yaroslavtsev [12] announced a 3-approximation algorithm for correlation clustering on complete $k$-partite graphs, for arbitrary $k$, which includes the complete bipartite case. Bipartite clustering has also been studied, outside the correlation-clustering context, by Lim, Chen, and Xu [19].
We depart from the classical correlation-clustering literature by considering a broader class of objective functions which also cater to the need of many community-detection applications in machine learning, social sciences, recommender systems and bioinformatics [13, 22, 15]. The technical details of this class of functions can be found in Section 2. As a representative example of this class, we introduce minimax correlation clustering.

In minimax clustering, rather than seeking to minimize the total number of errors, we instead seek to minimize the number of errors at the worst-off vertex in the clustering. Put more formally, if for a given clustering each vertex \( v \) has \( y_v \) incident edges that are errors, then we wish to find a clustering that minimizes \( \max_v y_v \).

Minimax clustering, like classical correlation clustering, is NP-hard on complete graphs, as we prove in Appendix C. To design approximation algorithms for minimax clustering, it is necessary to bound the growth of errors locally at each vertex when we round from a fractional clustering to a discrete clustering: this introduces new difficulties in the design and analysis of our rounding algorithm. These new technical difficulties cause the algorithm of [2, 3] to fail in the minimax context, and there is no obvious way to adapt that algorithm to this new context; this phenomenon is explored further in Appendix A.

Minimax correlation clustering on graphs is relevant in detecting communities, such as gene, social network, or voter communities, in which no antagonists are allowed. Here, an antagonist refers to an entity that has properties inconsistent with a large number of members of the community. Alternatively, one may view the maximin constraint as enabling individual vertex quality control within the clusters, which is relevant in biclustering applications such as collaborative filtering for recommender systems, where minimum quality recommendations have to be ensured for each user in a given category. As an illustrative example, one may view a complete bipartite graph as a preference model in which nodes on the left represent viewers and nodes on the right represent movies. A positive edge between a user and a movie indicates that the viewer likes the movie, while a negative edge indicates that they do not like or have not seen the movie. We may be interested in finding communities of viewers for the purpose of providing them with joint recommendations. Using a minimax objective function here allows us to provide a uniform quality of recommendations, as we seek to minimize the number of errors for the user who suffers the most errors.

A minimax objective function for a graph partitioning problem different from correlation clustering was previously studied by [7]. In that paper, the problem under consideration was to split a graph into \( k \) roughly-equal-sized parts, minimizing the total number of edges leaving any part. Thus, the minimum in [7] is being taken over the parts of the solution, rather than minimizing over vertices as we do here.

Another idea slightly similar to minimax clustering has previously appeared in the literature on fixed-parameter tractability of the Cluster Editing problem, which is an equivalent formulation of Correlation Clustering. In particular, Komusiewicz and Uhlmann [17] proved that the following problem is fixed-parameter tractable for the combined parameter \((d, t)\):

\[
(d, t)\text{-Constrained-Cluster Editing}
\]

**Input:** A labeled complete graph \( G \), a function \( \tau : V(G) \rightarrow \{0, \ldots, t\} \), and nonnegative integers \( d \) and \( k \).

**Question:** Does \( G \) admit a clustering into at most \( d \) clusters with at most \( k \) errors such that every vertex \( v \) is incident to at most \( \tau(v) \) errors?

(Here, we have translated their original formulation into the language of correlation clustering.) Komusiewicz and Uhlmann also obtained several NP-hardness results related to this formulation of the problem. While their work involves a notion of local errors for correlation clustering, their results are primarily focused on fixed-parameter tractability, rather than approximation algorithms, and are therefore largely orthogonal to the results of this paper.

The contributions of this paper are organized as follows. In Section 2, we introduce and formally express our framework for the generalized version of correlation clustering, which includes both classical clustering and minimax clustering as special cases. In Section 3, we give a rounding algorithm which allows the development of constant-factor approximation algorithms for the generalized clustering problem. In Section 4, we give a version of this rounding algorithm for complete bipartite graphs.

In Appendix A, we discuss minimax clustering in more detail, and show that algorithms similar to the Ailon–Charikar–Newman algorithm fail in the minimax context. In Appendix B, we discuss the approximation properties of the MaxAgree formulation of minimax clustering, where the objective is to maximize the
number of correct edges, rather than minimize the number of incorrect edges, at the worst vertex. In Appendix [C] and Appendix [D] we prove that the minimax correlation clustering problem is NP-hard on complete graphs and complete bipartite graphs, respectively. Appendix [E] contains technical details for various proofs.

2 Framework and Formal Definitions

In this section, we formally set up the framework we will use for our broad class of correlation-clustering objective functions.

**Definition 1.** Let $G$ be an edge-labeled graph. A discrete clustering (or just a clustering) of $G$ is a partition of $V(G)$. A fractional clustering of $G$ is a vector $x$ indexed by $\binom{V(G)}{2}$ such that $x_{uv} \in [0, 1]$ for all $uv \in \binom{V(G)}{2}$ and such that $x_{vz} \leq x_{uv} + x_{wz}$ for all distinct $v, w, z \in V(G)$.

If $x$ is a fractional clustering, we can view $x_{uv}$ as a “distance” from $u$ to $v$; the constraints $x_{vz} \leq x_{uv} + x_{wz}$ are therefore referred to as triangle inequality constraints. We also adopt the convention that $x_{uu} = 0$ for all $u$.

In the special case where all coordinates of $x$ are 0 or 1, the triangle inequality constraints guarantee that the relation defined by $u \sim v$ if $x_{uv} = 0$ is an equivalence relation. Such a vector $x$ can therefore naturally be viewed as a discrete clustering, where the clusters are the equivalence classes under $\sim$. By viewing a discrete clustering as a fractional clustering with integer coordinates, we see that fractional clusterings are a continuous relaxation of discrete clusterings, which justifies the name. This gives a natural notion of the total weight of errors at a given vertex.

**Definition 2.** Let $G$ be an edge-labeled complete graph, and let $x$ be a fractional clustering of $G$. The error vector of $x$ with respect to $G$, written $\err(x)$, is a real vector indexed by $V(G)$ whose coordinates are defined by

$$\err(x)_v = \sum_{w \in N^+(v)} x_{uv} + \sum_{w \in N^-(v)} (1 - x_{uv}).$$

If $C$ is a clustering of $G$ and $x^C$ is the natural associated fractional clustering, we define $\err(C)$ as $\err(x^C)$.

We are now prepared to formally state the optimization problem we wish to solve. Let $\mathbb{R}^n_{\geq 0}$ denote the set of vectors in $\mathbb{R}^n$ with all coordinates nonnegative. Our problem is parameterized by a function $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$.

**f-Correlation Clustering**

**Input:** A labeled graph $G$.

**Output:** A clustering $C$ of $G$.

**Objective:** Minimize $f(\err(C))$.

In order to approximate $f$-Correlation Clustering, we introduce a relaxed version of the problem.

**Fractional f-Correlation Clustering**

**Input:** A labeled graph $G$.

**Output:** A fractional clustering $x$ of $G$.

**Objective:** Minimize $f(\err(x))$.

If $f$ is convex on $\mathbb{R}^n_{\geq 0}$, then using standard techniques from convex optimization [5], the Fractional $f$-Correlation Clustering problem can be approximately solved in polynomial time, as the composite function $f \circ \err$ is convex and the constraints defining a fractional clustering are linear inequalities in the variables $x_e$. When $G$ is a complete graph, we then employ a rounding algorithm based on the algorithm of Charikar, Guruswami, and Wirth [9, 10] to transform the fractional clustering into a discrete clustering. Under rather modest conditions on $f$, we are able to obtain a constant-factor bound on the error growth, that is, we can produce a clustering $C$ such that $f(\err(C)) \leq cf(\err(x))$, where $c$ is a constant not depending on $f$ or $x$. In particular, we require the following assumptions on $f$.

**Assumption A.** We assume that $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ has the following properties.

3
(1) \( f(cy) \leq cf(y) \) for all \( c \geq 0 \) and all \( y \in \mathbb{R}^n \), and

(2) If \( y, z \in \mathbb{R}_{\geq 0}^n \) are vectors with \( y_i \leq z_i \) for all \( i \), then \( f(y) \leq f(z) \).

Under Assumption \( \Xi \), the claim that \( f(\text{err}(C)) \leq cf(\text{err}(x)) \) follows if we can show that \( \text{err}(C)_v \leq c \text{err}(x)_v \) for every vertex \( v \in V(G) \). This is the property we prove for our rounding algorithms.

We will slightly abuse terminology by referring to the constant \( c \) as an approximation ratio for the rounding algorithm; this notation is motivated by the fact that when \( f \) is linear, the Fractional \( f \)-Correlation Clustering problem can be solved exactly in polynomial time, and applying a rounding algorithm with constant \( c \) to the fractional solution yields a \( c \)-approximation algorithm to the (discrete) \( f \)-Correlation Clustering problem. In contrast, when \( f \) is nonlinear, we may only be able to obtain a \((1 + \epsilon)\)-approximation for the Fractional \( f \)-Correlation Clustering problem, in which case applying the rounding algorithm yields a \( c(1 + \epsilon) \)-approximation algorithm for the discrete problem.

A natural class of convex objective functions obeying Assumption \( \Xi \) is the class of \( \ell^p \) norms. For all \( p \geq 1 \), the \( \ell^p \)-norm on \( \mathbb{R}^n \) is defined by

\[
\ell^p(x) = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}.
\]

As \( p \) grows larger, the \( \ell^p \)-norm puts more emphasis on the coordinates with larger absolute value. This justifies that definition of the \( \ell^\infty \)-norm as

\[
\ell^\infty(x) = \max\{x_1, \ldots, x_n\}.
\]

Classical correlation clustering is the case of \( f \)-Correlation Clustering where \( f(x) = \frac{1}{n} \ell^1(x) \), while minimax correlation clustering is the case of \( f \)-Correlation Clustering where \( f(x) = \ell^\infty(x) \).

Our emphasis on convex \( f \) is due to the fact that convex programming techniques allow the Fractional \( f \)-Correlation Clustering problem to be approximately solved in polynomial time when \( f \) is convex. However, the correctness of our rounding algorithm does not depend on the convexity of \( f \), only on the properties listed in Assumption \( \Xi \). If \( f \) is nonconvex and obeys Assumption \( \Xi \) and we produce a “good” fractional clustering \( x \) by some means, then our algorithm still produces a discrete clustering \( C \) with \( f(\text{err}(C)) \leq cf(\text{err}(x)) \).

### 3 A Rounding Algorithm for Complete Graphs

We now describe a rounding algorithm to transform an arbitrary fractional clustering \( x \) of a labeled complete graph \( G \) into a clustering \( C \) such that \( \text{err}(C)_v \leq c \text{err}(x)_v \) for all \( v \in V(G) \).

Our rounding algorithm is based on the algorithm of Charikar, Guruswami, and Wirth [9, 10] and is shown in Algorithm 1. The main difference between Algorithm 1 and the algorithm of [9, 10] is the new strategy of choosing a pivot vertex that maximizes \( \text{cluster-cost} \) of an edge \( uv \). This requires a different analysis for each vertex \( v \) and for all clustering errors incident to \( v \) by charging to the LP cost incident to \( v \). In particular, every clustering error must now be paid for at each of its endpoints, while in [9, 10], it was enough to pay for each clustering error at one of its endpoints. For edges which cross between a cluster and its complement, this requires a different analysis.
with $0 < \gamma < \alpha < 1/2$.

Let $S = V(G)$.

\begin{algorithm}
\begin{algorithmic}
\State \textbf{while} $S \neq \emptyset$ \textbf{do}
\State \hspace{1em} For each $u \in S$, let $T_u = \{ w \in S - \{u\} : x_{uw} \leq \alpha \}$ and let $T_u^* = \{ w \in S - \{u\} : x_{uw} \leq \gamma \}$.
\State \hspace{1em} Choose a pivot vertex $u \in S$ that maximizes $|T_u^*|$.
\State \hspace{1em} Let $T = T_u$.
\State \hspace{1em} \textbf{if} $\sum_{w \in T} x_{uw} \geq \alpha |T|/2$ \textbf{then}
\State \hspace{2em} Output the cluster \{u\}. \{Type 1 cluster\}
\State \hspace{2em} Let $S = S - \{u\}$.
\State \hspace{1em} \textbf{else}
\State \hspace{2em} Output the cluster \{u\} $\cup$ $T$. \{Type 2 cluster\}
\State \hspace{2em} Let $S = S - (\{u\} \cup T)$.
\State \textbf{end if}
\State \textbf{end while}
\end{algorithmic}
\end{algorithm}

at each endpoint, a difficulty which was not present in \[9,10\]. Our proof emphasizes the solutions to these new technical problems; the parts of the proof that are technically nontrivial but follow earlier work are omitted due to space constraints but can be found in Appendix E.

**Observation 4.** Let $x$ be a fractional clustering of a graph $G$, and let $w, z \in V(G)$. For any vertex $u$, we have $x_{u z} \geq x_{u w} - x_{uw}$ and $1 - x_{u z} \geq 1 - x_{u w} - x_{uw}$.

**Theorem 5.** Let $G$ be a labeled complete graph, let $\alpha$ and $\gamma$ be parameters with $0 < \gamma < \alpha < 1/2$, and let $x$ be any fractional clustering of $G$. If $C$ is the clustering produced by Algorithm 4 with the given input, then for all $v \in V(G)$ we have $\text{err}(C)_v \leq c \text{err}(x)_v$, where $c$ is a constant depending only on $\alpha$ and $\gamma$.

**Proof.** Let $k_1, k_2, k_3$ be constants to be determined, with $1/2 < k_1 < 1$ and $0 < 2k_2 \leq k_3 < 1/2$. Also assume that $k_3 \alpha > \gamma$ and that $k_2 \alpha \leq 1 - 2\alpha$.

To prove the approximation ratio, we consider the cluster-costs incurred as each cluster is output, splitting into cases according to the type of cluster. In our analysis, as the algorithm runs, we will mark certain vertices as “safe”, representing the fact that some possible future clustering costs have been paid for in advance. Initially, no vertex is marked as safe.

**Case 1:** A Type 1 cluster is output. Let $X = S \cap N^+(u)$, with $S$ as in Algorithm 4. The new cluster-cost incurred at $u$ is $|X|$, and for each $v \in X$, a new cluster-cost of 1 is incurred at $v$.

First we pay for the new cluster cost incurred at $u$. For each edge $uv$ with $v \in T$, we have $x_{uv} \leq \alpha$ and so $1 - x_{uv} \geq 1 - \alpha \geq x_{uw}$. Thus, the total LP cost of edges $uv$ with $v \in T$ is at least $\sum_{v \in T} x_{uv}$, which is at least $\alpha |T|/2$ since \{u\} is output as a Type 1 cluster. Thus, charging each edge $uv$ with $v \in T$ a total of $2/\alpha$ times its LP-cost pays for the cluster-cost of any positive edges from $u$ to $T$. On the other hand, if $uv$ is a positive edge with $v \in S - T$, then since $v \notin T$, we have $x_{uv} \geq \alpha$. Hence, the LP-cost of $uv$ is at least $\alpha$, and charging $1/\alpha$ times the LP-cost of $uv$ pays for the cluster-cost of this edge.

Now let $v \in X$; we must pay for the new cluster cost at $v$. If $x_{u v} \geq k_1 \alpha$, then the edge $uv$ already incurs LP cost at least $k_2 \alpha$, so the new cost at $v$ is only $1/(k_2 \alpha)$ times the LP-cost of the edge $uv$. So assume $x_{uv} < k_2 \alpha$. In this case, we say that $u$ is a bad pivot for $v$.

First suppose that $v$ is not safe (as is initially the case). We will make a single charge to the edges incident to $v$ that is large enough to pay for both the edge $uv$ and for all possible future bad pivots, and then we will mark $v$ as safe to indicate that we have done this. The basic idea is that if $v$ has many possible bad pivots, then since $x_{uv}$ is “small”, all of these possible bad pivots are also close to $u$, thus included in $T_u$. Since $\sum_{w \in T_u} x_{uw} \geq \alpha |T_u|/2$, there is a large set $B \subseteq T_u$ of vertices that are “moderately far” from $u$, and therefore moderately far from $v$. The number of these vertices grows with the number of bad pivots, so charging all the edges $vz$ for $z \in B$ is sufficient to pay for all bad pivots.

We now make this argument rigorous. Let $P_v$ be the set of potential bad pivots for $v$, defined by

\[ P_v = \{ p \in S : x_{vp} < k_2 \alpha \}. \]
Note that \( u \in P_v \). Since \( k_2 < 1/4 \), we have \( x_{uv} \leq x_{uv} + x_{vp} < \alpha/2 \) for all \( p \in P_v \); hence \( P_v \subseteq T \). Define the vertex set \( B \) by
\[
B = \{ z \in T : x_{uz} > k_3 \alpha \}.
\]
Since \( x_{uz} \leq \alpha \) for all \( z \in T \), we see that
\[
\sum_{z \in T} x_{uz} \leq k_3 \alpha |T - B| + \alpha |B|.
\]
On the other hand, since \( \{ u \} \) is output as a Type 1 cluster, we have
\[
\sum_{z \in T} x_{uz} \geq \alpha |T|/2.
\]
Combining these inequalities and rearranging, we obtain \( |B| \geq (1 - 2k_3) |T - B| \). For each vertex \( z \in B \), we have \( x_{uz} \geq x_{uv} - x_{uv} \geq (k_3 - k_2) \alpha \); in particular, since \( k_3 \geq 2k_2 \), we have \( x_{uz} \geq 2k_2 \alpha \), so that \( z \notin P_v \). Hence \( \max \{ |B|, |T - B| \} \geq \max \{ |P_v|, \alpha \} \), and we have \( |B| \geq (1 - 2k_3) \alpha |P_v| \).

On the other hand, for \( z \in B \) we also have \( 1 - x_{uz} \geq 1 - x_{uv} - x_{uv} \geq 1 - (1 + k_2) \alpha \). It follows that each edge \( vz \) for \( z \in B \) has LP-cost at least \( \min((k_3 - k_2) \alpha, 1 - (1 + k_2) \alpha) \), independent of whether \( vz \) is positive or negative. It is easy to check that since \( \alpha < 1/2 \) and \( k_2 < 1 \), this minimum is always achieved by \( (k_3 - k_2) \alpha \). Therefore, we can pay for the (possible) Type-1-cluster cost of all edges \( vp \) for \( p \in P_v \) by charging each edge \( vz \) with \( z \in B \) a total of
\[
\frac{1}{(1 - 2k_3)(k_3 - k_2) \alpha}
\]
times its LP-cost. We make all these charges when the cluster \( \{ u \} \) is created and put them in a “bank account” to pay for later Type-1-cluster costs for \( v \). Then we mark \( v \) as safe. The total charge in the bank account is at least \( |P_v| \), which is enough to pay for all bad pivots for \( v \).

We have just described the case where \( u \) is a bad pivot and \( v \) is not safe. On the other hand, if \( u \) is a bad pivot and \( v \) is safe, then \( v \) already has a bank account large enough to pay for all its bad pivots, and we simply charge 1 to the account to pay for the edge \( uv \).

**Case 2:** A Type 2 cluster \( \{ u \} \cup T \) is output. The negative edges within \( \{ u \} \cup T \) are easy to pay for: if \( vw \) if a negative edge inside \( \{ u \} \cup T \), then we have \( 1 - x_{uv} \geq 1 - x_{uv} - x_{uv} \geq 1 - 2\alpha \), so we can pay for each of these edges by charging a factor of \( \frac{1}{2\alpha} \) times its LP-cost.

Thus, we consider edges joining \( \{ u \} \cup T \) with \( S - \{ u \} \cup T \). We call these edges cross-edges for their endpoints. A standard argument (see Appendix 3) shows that for \( z \in S - \{ u \} \cup T \), the total cluster-cost of the cross-edges for \( z \) is at most \( \max \{ 1/((1 - 2\alpha), 2/\alpha) \} \) times the LP-cost of those edges, so the vertices outside \( \{ u \} \cup T \) can be dealt with easily.

However, we also must bound the cluster-cost at vertices inside \( \{ u \} \cup T \). This is where we use the maximality of \( |T_u^*| \).

Let \( w \in \{ u \} \cup T \). First consider the positive cross-edges \( wz \) such that \( x_{wz} \geq \gamma \). Any such edge has cluster-cost 1 and already has LP-cost at least \( \gamma \), so charging \( 1/\gamma \) times the LP-cost to such an edge pays for its cluster cost. Now let \( X = \{ z \in S - \{ u \} \cup T : x_{wz} < \gamma \} \); we must pay for the edges \( wz \) with \( z \in X \).

If \( x_{uw} \leq k_1 \alpha \), which includes the case \( u = w \), then for all \( z \in X \), we have \( x_{wz} \geq x_{uw} - x_{uw} \geq \alpha - k_1 \alpha = (1 - k_1) \alpha \). Hence, for any positive edge \( wz \) with \( z \in X \), the LP-cost of \( wz \) is at least \((1 - k_1) \alpha \), and so the cluster cost of the edge \( wz \) is at most \( 1/((1 - k_1) \alpha) \) times the LP cost. Charging this factor to each cross-edge pays for the cluster-cost of each cross-edge.

Now suppose \( x_{uw} > k_1 \alpha \). Since \( k_1 \alpha > \gamma \), this implies \( w \notin T_u^* \). In this case, it is possible that \( w \) may have many positive neighbors \( z \in X \) for which \( x_{wz} \) is quite small, so we cannot necessarily pay for the cluster-cost of the edges joining \( w \) and \( X \) by using their LP-cost. Instead, we charge their cluster-cost to the LP-cost of edges within \( T \).

Observe that \( X \subseteq T_u^* \), and hence \( |T_u^*| \geq |X| \). By the maximality of \( |T_u^*| \), this implies that \( |T_u^*| \geq |X| \). Now for any \( v \in T_u^* \), we have the following bounds:
\[
\begin{align*}
x_{uv} \geq x_{uw} - x_{uv} \geq k_1 \alpha - \gamma, \\
1 - x_{uv} \geq 1 - x_{uw} - x_{uv} \geq 1 - \alpha - \gamma.
\end{align*}
\]
Since \( \alpha < 1/2 \) and \( k_1 \leq 1 \), we have \( k_1 \alpha \leq \alpha < 1 - \alpha \), so these lower bounds imply that each edge \( vw \) with \( v \in T_u^* \) has LP-cost at least \( k_1 \alpha - \gamma \), independent of whether \( vw \) is a positive or negative edge. Thus, the total LP cost of edges joining \( w \) to \( T_u^* \) is at least \( (k_1 \alpha - \gamma) |T_u^*| \).

Since the total cluster-cost of edges joining \( w \) and \( X \) is at most \( |X| \) and since \( |T_u^*| \geq |X| \), we can pay for these edges by charging each edge \( vw \) with \( v \in T_u^* \) a factor of \( \frac{1}{k_1 \alpha - \gamma} \) times its LP-cost.

Having paid for all cluster-costs, we now look at the total charge accrued at each vertex. Fix any vertex \( v \) and an edge \( vw \) incident to \( v \). We bound the total amount charged to \( vw \) by \( v \) in terms of the LP-cost of \( vw \). There are three distinct possibilities for the edge \( vw \): either \( vw \) ended inside a cluster, or \( v \) was clustered before \( w \), or \( w \) was clustered before \( v \).

**Case 1: \( vw \) ended within a cluster.** In this case, \( v \) may have made the following charges:

- A charge of \( \frac{1}{(1 - 2k_3)(k_3 - k_2)\alpha} \) times the LP-cost, to pay for a “bank account” for \( v \),
- A charge of \( \frac{1}{1 - 2\alpha} \) times the LP-cost, to pay for \( vw \) itself if \( vw \) is a negative edge,
- A charge of \( \frac{1}{k_1 \alpha - \gamma} \) times the LP-cost, to pay for positive edges leaving the \( v \)-cluster.

Thus, in this case the total cost charged to \( vw \) by \( v \) is at most \( c_1 \) times the LP-cost of \( vw \), where

\[
c_1 = \frac{1}{(1 - 2k_3)(k_3 - k_2)\alpha} + \frac{1}{1 - 2\alpha} + \frac{1}{k_1 \alpha - \gamma}.
\]

**Case 2: \( v \) was clustered before \( w \).** In this case, \( v \) may have made the following charges:

- A charge of \( \frac{1}{(1 - 2k_3)(k_3 - k_2)\alpha} \) times the LP-cost, to pay for a “bank account” for \( v \),
- A charge of at most \( \frac{2}{\alpha} \) times the LP-cost, to pay for all cross-edges if \( v \) was output as a Type 1 cluster,
- A charge of at most \( \max \left\{ \frac{1}{(1 - k_1)\alpha}, \frac{1}{\gamma} \right\} \) times the LP-cost, to pay for \( vw \) if \( v \) was output in a Type 2 cluster.

Note that \( k_1 > 1/2 \) implies that \( \frac{1}{k_1 \alpha} \geq \frac{2}{\alpha} \), so we may disregard the case where \( v \) is output as a Type 1 cluster. Thus, in this case the total cost charged to \( vw \) by \( v \) is at most \( c_2 \) times the LP-cost of \( vw \), where

\[
c_2 = \frac{1}{(1 - 2k_3)(k_3 - k_2)\alpha} + \max \left\{ \frac{1}{(1 - k_1)\alpha}, \frac{1}{\gamma} \right\}.
\]

**Case 3: \( w \) was clustered before \( v \).** In this case, \( v \) may have made the following charges:

- A charge of at most \( \frac{1}{(1 - 2k_3)(k_3 - k_2)\alpha} \) times the LP-cost, to pay for a “bank account” for \( v \),
- A charge of at most \( \frac{2}{\alpha} \) times the LP-cost, to pay for the cluster-cost of \( vw \) if \( vw \) is a positive edge and \( w \) was output as a Type 1 cluster,
- A charge of at most

\[
\max \left\{ \frac{1}{1 - 2\alpha}, \frac{2}{\alpha} \right\}
\]

times the LP-cost, to pay for \( vw \) if \( w \) was output in a Type 2 cluster.

Clearly \( vw \) cannot receive both the second and third types of charge. Furthermore, since \( k_2 \leq 1/4 \), we have \( \frac{1}{k_2 \alpha} \geq \frac{2}{\alpha} \). Since \( k_2 \alpha \leq 1 - 2\alpha \), we see that \( \frac{1}{k_2 \alpha} \) is the largest charge that \( vw \) could receive from either the second or third type of charge. Thus, in this case the total cost charged to \( vw \) by \( v \) is at most \( c_3 \) times the LP-cost, where

\[
c_3 = \frac{1}{(1 - 2k_3)(k_3 - k_2)\alpha} + \frac{1}{k_2 \alpha}.
\]

Thus, the approximation ratio of the algorithm is at most \( \max\{c_1, c_2, c_3\} \). We wish to choose the various parameters to make this ratio as small as possible, subject to the various assumptions on the parameters
required for the correctness of the proof. It seems difficult to obtain an exact solution to this optimization problem. Solving the problem numerically, we obtained the following values for the parameters:

\[
\begin{align*}
\alpha &= 0.465744 & \gamma &= 0.0887449 \\
k_1 &= 0.767566 & k_2 &= 0.117219 & k_3 &= 0.308433.
\end{align*}
\]

These parameters yield an approximation ratio of roughly 48.

4 A Rounding Algorithm for One-Sided Biclustering

In this section, we consider a version of the \(f\)-Correlation Clustering problem on complete bipartite graphs. Let \(G\) be a complete bipartite graph with edges labeled + and −, and let \(V_1\) and \(V_2\) be its partite sets. We will obtain a rounding algorithm that transforms any fractional clustering \(x\) into a discrete clustering \(C\) such that \(\text{err}(C)_v \leq \text{cerr}(x)_v\) for all \(v \in V_1\). Our algorithm is shown in Algorithm 2.

Our algorithm does not guarantee any upper bound on \(\text{err}(C)_v\) for \(v \in V_2\): as the algorithm treats the sides \(V_1\) and \(V_2\) asymmetrically, it is difficult to control the per-vertex error at \(V_2\). Nevertheless, an error guarantee for the vertices in \(V_1\) suffices for some applications. Our approach is motivated by applications in recommender systems, where vertices in \(V_1\) correspond to users, while vertices in \(V_2\) correspond to objects to be ranked. In this context, quality of service conditions only need to be imposed for users, and not for objects.

**Algorithm 2** Round fractional clustering to obtain a discrete clustering, using threshold parameters \(\alpha, \gamma\) with \(\alpha < 1/2\) and \(\gamma < \alpha\).

```plaintext
Let \(S = V(G)\).

while \(V_1 \cap S \neq \emptyset\) do
    For each \(u \in V_1 \cap S\), let \(T_u = \{w \in S - \{u\} : x_{uw} \leq \alpha\}\) and let \(T_u^* = \{w \in V_2 \cap S : x_{uw} \leq \gamma\}\).
    Choose a pivot vertex \(u \in V_1 \cap S\) that maximizes \(|T_u^*|\).
    Let \(T = T_u\).
    if \(\sum_{w \in V_2 \cap T} x_{uw} \geq \alpha |V_2 \cap T| / 2\) then
        Output the singleton cluster \(\{u\}\). \{Type 1 cluster\}
        Let \(S = S - \{u\}\).
    else
        Output the cluster \(\{u\} \cup T\). \{Type 2 cluster\}
        Let \(S = S - (\{u\} \cup T)\).
    end if
end while
Output each remaining vertex of \(V_2 \cap S\) as a singleton cluster.
```

**Theorem 6.** Let \(G\) be a labeled complete bipartite graph with partite sets \(V_1\) and \(V_2\), let \(\alpha, \gamma\) be parameters as described in Algorithm 2 and let \(x\) be any fractional clustering of \(G\). If \(C\) is the clustering produced by Algorithm 2 with the given input, then for all \(v \in V_1\) we have \(\text{err}(C)_v \leq \text{cerr}(x)_v\), where \(c\) is a constant depending only on \(\alpha\) and \(\gamma\).

We note that the proof of Theorem 6 is actually simpler than the proof of Theorem 5, because the focus on errors only at \(V_1\) eliminates the need for the “bad pivots” argument used in Theorem 5. This also leads to a smaller value of \(c\) in Theorem 6 than we were able to obtain in Theorem 5.

**Proof.** As before, we make charges to pay for the new cluster costs at each vertex of \(V_1\) as each cluster is output, splitting into cases according to the type of cluster. Let \(k_1\) be a constant to be determined, with \(k_1 \alpha > \gamma\).

**Case 1:** A Type 1 cluster \(\{u\}\) is output. In this case, the only cluster costs incurred are the positive edges incident to \(u\), all of which have their other endpoint in \(V_2\). The averaging argument used in Case 1 of Section 3 shows that charging every edge incident to \(u\) a factor of \(2/\alpha\) times its LP cost pays for the cluster cost of all such edges.
Case 2: A Type 2 cluster \( \{u\} \cup T \) is output. Negative edges within the cluster are easy to pay for: if \( u_1u_2 \) is a negative edge within the cluster, with \( u_i \in V_i \), then we have

\[
1 - x_{u_1u_2} \geq 1 - x_{u_1} - x_{u_2} \geq 1 - 2\alpha,
\]

so we can pay for the cluster-cost of such an edge by charging it a factor of \( 1/(1 - 2\alpha) \) times its LP-cost.

We still must pay for positive edges joining the cluster with the rest of \( S \); we call such edges cross-edges. Each such edge must be paid for at its endpoint in \( V_1 \).

If \( z \in V_1 \) is a vertex outside the cluster, then a standard argument (see Appendix E) shows that the cross-edges for \( z \) can be paid for by charging each such edge a factor of \( \max\{1/(1 - 2\alpha), 2/\alpha\} \) times its LP cost.

Now let \( w \in V_1 \) be a vertex inside the cluster. We must pay for the cross-edges incident to \( w \) using the LP-cost of the edges incident to \( w \). First consider the positive edges from \( w \) to vertices \( z \) outside the cluster such that \( x_{uz} \geq \gamma \). Any such edge has cluster-cost 1 and LP-cost at least \( \gamma \), so charging each such edge a factor of \( 1/\gamma \) times its LP-cost pays for its cluster cost. Let \( X = \{ z \in (S \cap V_1) \setminus T : x_{uz} < \gamma \} \); we must pay for the edges \( wz \) for \( z \in X \). Note that \( x_{uz} \geq \alpha \) for all \( z \in X \), since \( z \in X \) implies \( z \notin T \).

If \( x_{uw} \leq k_1\alpha \), then for all \( z \in X \), we have

\[
x_{wz} \geq x_{uz} - x_{uw} \geq (1 - k_1)\alpha.
\]

Hence, for any positive cross-edge \( wz \) with \( z \in X \), the LP-cost of \( wz \) is at least \( (1 - k_1)\alpha \), and so we can pay for the cluster-cost of \( wz \) by charging \( wz \) a factor of \( 1/(1 - k_1)\alpha \) times its LP-cost.

Now suppose \( x_{uw} > k_1\alpha \). As before, we pay for the cross-edges by charging the edges inside the cluster. Observe that \( |T^*_u| \geq |X| \). Since \( u \) was chosen to maximize \( |T^*_u| \), this implies that \( |T^*_u| \geq |X| \). For any \( v \in T^*_u \), we have

\[
x_{uv} \geq x_{uw} - x_{uv} \geq k_1\alpha - \gamma.
\]

On the other hand, for any \( v \in T^*_u \) we also have

\[
1 - x_{uv} \geq 1 - x_{uw} - x_{uv} \geq 1 - \alpha - \gamma \geq \alpha - \gamma.
\]

Since \( k_1 \leq 1 \), it follows that the edge \( uv \) has LP-cost at least \( k_1\alpha - \gamma \) independent of whether \( uv \) is positive or negative. Thus, the total LP cost of edges joining \( w \) to \( T^*_u \) is at least \( (k_1\alpha - \gamma) |T^*_u| \).

Since the total cluster-cost of the cross-edges joining \( w \) and \( X \) is at most \( |X| \) and since \( |T^*_u| \geq |X| \), we can pay for the cross-edges by charging each edge \( uv \) with \( v \in T^*_u \) a factor of \( 1/(k_1\alpha - \gamma) \) times its LP-cost.

Having paid for all cluster-costs, we now look at the total charge accrued at each vertex. Fix a vertex \( v \in V_1 \) and an edge \( vw \) incident to \( v \). We bound the total amount charged to \( vw \) by \( v \) in terms of the LP-cost of \( vw \). There are three distinct possibilities for the edge \( vw \): either \( vw \) ended inside a cluster, or \( v \) was clustered before \( w \), or \( w \) was clustered before \( v \).

Case 1: \( vw \) ended within a cluster. In this case, \( v \) may have made the following charges:

- A charge of at most \( 1/(1 - 2\alpha) \) times the LP cost, to pay for \( vw \) itself if \( vw \) is a negative edge,

- A charge of \( 1/(k_1\alpha - \gamma) \) times the LP-cost, to pay for positive edges leaving the \( v \)-cluster.

Thus, in this case the total cost charged to \( vw \) by \( v \) is at most \( c_1 \) times the LP-cost of \( vw \), where

\[
c_1 = \frac{1}{1 - 2\alpha} + \frac{1}{k_1\alpha - \gamma}.
\]

Case 2: \( v \) was clustered before \( w \). In this case, \( v \) may have made the following charges:

- A charge of \( 2/\alpha \) times the LP cost, to pay for \( vw \) if \( v \) was output as a singleton,

- A charge of \( \max\{1/(1 - k_1\alpha), 1/\gamma\} \) times the LP cost, to pay for \( vw \) if \( v \) was output in a nonsingleton cluster,
Since $v$ makes at most one of the charges above, the total cost charged to $vw$ by $v$ is at most $c_2$ times the LP-cost of $vw$, where

$$c_2 = \max \left\{ \frac{1}{1 - k_1 \alpha}, \frac{2}{\gamma}, \frac{1}{\alpha} \right\}.$$

**Case 3: $w$ was clustered before $v$.** In this case, $v$ may have made the following charges:

- A charge of at most $\max\left\{ \frac{1}{1 - 2\alpha}, \frac{2}{\gamma} \right\}$ times the LP cost, to pay for cross-edges at $v$ if $w$ is output in a nonsingleton cluster.

Thus, in this case the total cost charged to $vw$ by $v$ is at most $c_3$ times the LP-cost of $vw$, where

$$c_3 = \max \left\{ \frac{1}{1 - 2\alpha}, \frac{2}{\alpha} \right\}.$$

The approximation ratio is $\max\{c_1, c_2, c_3\}$. Numerically, we obtain an approximation ratio of at most 10 by taking the following parameter values:

$$\alpha = 0.377 \quad \gamma = 0.102 \quad k_1 = 0.730$$

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A Minimax Clustering and the Failure of Pivoting Algorithms

In this appendix, we consider minimax clustering, which is the special case of $f$-Correlation Clustering where $f(y) = \max_{e \in E(V(G))} y_v$. Thus, in minimax clustering, we seek to minimize the number of errors at the worst vertex in the clustering. Equivalently, we are trying to minimize the $\ell^\infty$-norm of the error vector, in contrast to classical correlation clustering, where we are trying to minimize the $\ell^1$-norm.

Minimax clustering is a representative example of the difficulties which arise in moving from classical correlation clustering to the more general $f$-Correlation Clustering problem. We will show that some techniques which work well for the classical correlation clustering problem break down in the minimax context.

Ailon, Charikar, and Newman \cite{AilonCharikarNewman08, AilonCharikarNewman10} gave a beautifully simple randomized 3-approximation algorithm for classical correlation clustering on complete graphs. Their algorithm is shown in Algorithm 3. Since our rounding clustering in Section 3 is based on the Charikar–Guruswami–Wirth algorithm with a modified pivoting rule, it is natural to ask whether a similar modification to the Ailon–Charikar–Newman algorithm also yields a constant-factor approximation algorithm for minimax clustering.

Algorithm 3 Ailon–Charikar–Newman algorithm \cite{AilonCharikarNewman08, AilonCharikarNewman10}.

Let $S = V(G)$.

while $S \neq \emptyset$ do
    Pick $v \in S$ uniformly at random.
    Let $T = \{v\} \cup N^+(v) \cap S$.
    Output the cluster $T$.
    Let $S = S - T$.
end while

Unfortunately, it seems that there are severe obstacles to modifying the ACN algorithm in this manner. For any positive integer $t$, let $M_t$ be a graph on $2t$ vertices consisting of $t$ pairwise disjoint edges, and let $G_t$ be the labeling of $K_{2t}$ in which the edges of $M_t$ are labeled $-$ and all other edges are labeled $+$.

Clearly, if all vertices of $G_t$ are placed in the same cluster (the “giant clustering”), then there is only 1 error at each vertex of $G_t$. We show that all other clusterings of $G_t$ have many more errors at some vertex.

Lemma 7. If $\mathcal{C}$ is a clustering of $G_t$ with more than 1 cluster, then some vertex of $G_t$ has at least $t - 1$ errors in $\mathcal{C}$.

Proof. Let $X$ be the smallest cluster in $\mathcal{C}$. Since $\mathcal{C}$ has at least 2 clusters, we have $|X| \leq t$. For any $v \in X$, there is at most one $w \notin X$ such that $vw$ is a negative edge. Hence, each $v \in X$ has at least $t - 1$ incident errors.

By Lemma 7 any constant-factor randomized algorithm for minimax clustering must return the giant clustering for $G_t$ with probability $1 - O(1/t)$. On the other hand, if we modify Algorithm 3 by changing the rule for choosing the pivot vertex $v$, the resulting algorithm still cannot produce the giant clustering. It is difficult to see how Algorithm 3 could sensibly be modified in order to return the giant clustering for $G_t$ with high enough probability.

We now consider the behavior of Algorithm 3 on the graph $G_t$. While the minimax objective function is not linear in the variables $x_{uv}$, we can still model the $f$-Fractional Correlation Clustering problem using the linear program $L$ shown in Figure 1.

Since the algorithm presented in Section 3 yields a constant-factor approximation algorithm for minimax clustering, and since every clustering of $G_t$ other than the giant clustering has $t - 1$ errors at some vertex, it is necessary that our rounding algorithm, applied to an optimal solution of $L$, returns the giant clustering for all sufficiently large $t$. This follows immediately from the following result.

Proposition 8. Let $L$ be the linear program shown in Figure 1 as formulated for $G_t$. If $t \geq 3$, then the unique optimal solution to $L$ has $x_{uv} = 0$ for all $uv \in E(G)$.

Proof. The dual program to $L$ is shown in Figure 2 with the following variables:

- For each $v \in V(G_t)$, a variable $\pi_v$ corresponding to the constraint $\sum_{w \in N^+(v)} x_{vw} + \sum_{w \in N^-(v)} (1 - x_{vw}) \leq M$,
minimize $M$, subject to:

$$\sum_{w \in N^+(v)} x_{uv} + \sum_{w \in N^-(v)} (1 - x_{uw}) \leq M \quad \text{(for all } v \in V(G))$$

$$0 \leq x_e \leq 1 \quad \text{(for all } e \in E(G))$$

$$M \in \mathbb{R}$$

Figure 1: LP formulation $L$ of $f$-Fractional Correlation Clustering, where $f(y) = \max_{v \in V(G)} y_v$.

maximize $\sum_{v \in V(G)} d^-(v) \pi_v$, subject to:

$$-\pi_u - \pi_v + \hat{\sigma}_{u,v} \leq 0 \quad \text{(for all } uv \in E^+(G))$$

$$\pi_u + \pi_v + \hat{\sigma}_{u,v} \leq 0 \quad \text{(for all } uv \in E^-(G))$$

$$\sum_v \pi_v \leq 1$$

$$\pi_z, \sigma_{u,v} \geq 0 \quad \text{(for all } z \in V(G) \text{ and all } u,v \in E(G)).$$

Figure 2: Dual of $L$.

• For each ordered triple $(u,v,z)$ where $u,v,z$ are distinct vertices of $V(G_t)$, a variable $\sigma_{(u,v,z)}$ corresponding to the constraint $x_{uv} \leq x_{uz} + x_{zv}$.

For convenience of notation, we also introduce the abbreviation $\hat{\sigma}_{u,v}$ to stand for $\sum_{z \in V(G) - \{u,v\}} (-\sigma_{u,v,z} - \sigma_{v,u,z} + \sigma_{z,u,v} + \sigma_{z,v,u} + \sigma_{u,z,v} + \sigma_{v,z,u})$. Observe that there are exactly $2t - 2$ choices of $z$ to sum over.

Now we define a dual solution. Let $u'u''$ be an edge of the negative matching. Consider the dual solution defined below:

$$\pi_u = \pi_{u''} = 1/2,$$

$$\pi_v = 0 \text{ for all } v \notin \{u', u''\},$$

$$\sigma_{u',u,v,z} = 1/(2t - 2) \text{ for all } z \notin \{u', u''\},$$

$$\sigma_{u,v,z} = 0 \text{ if } (u,v) \neq (u', u'').$$

Clearly this solution has an objective value of 1; we check that it is feasible for $t \geq 2$. If $uv$ is an edge containing neither of $\{u', u''\}$, then $\pi_u = \pi_v = 0$ and $\hat{\sigma}_{u,v} = 0$, since every term of $\hat{\sigma}_{u,v}$ is 0. The edge $u'u''$ is a negative edge with $\pi_{u'} = \pi_{u''} = 1/2$, and after eliminating all the zero terms, we have

$$\hat{\sigma}_{u'u''} = \sum_{z \in V(G) - \{u,v\}} (-\sigma_{u',u'',z}) = -\sum_{z \in V(G) - \{u,v\}} \frac{1}{2t - 2} = -1.$$

Thus, $\pi_{u'} + \pi_{u''} + \hat{\sigma}_{u',u''} \leq 0$, as required. Finally, if $uv$ is a positive edge with $u \in \{u', u''\}$, say if $u = u'$, then the only nonzero term of $\hat{\sigma}_{u,v}$ is $\sigma_{u',u''} = 1$, and we have $-\pi_u - \pi_v + \hat{\sigma}_{u,v} = -1/2 + 1/(2t - 2) \leq 0$ as required. The same argument holds if $u = u''$.

Since this solution has an objective value of 1, matching the primal objective when $x_{uv} = 0$ everywhere, it is clearly optimal. Furthermore, if $t \geq 3$, then for each positive edge incident to $u'$ or $u''$, there is slack in the corresponding constraint of the dual problem. By complementary slackness, this implies that in any optimal solution to $L$, we have $x_{u'v} = x_{u''v} = 0$ for all $v \in V(G) - \{u', u''\}$. The triangle inequality constraints in $L$ then imply that in an optimal primal solution, $x_{uv} = 0$ for all $uv \in E(G)$.

B MaxAgree for Classical and Minimax Clustering

In this paper, we have mainly focused on studying the MINDISAGREE formulation of $f$-Correlation Clustering, where we seek to minimize an objective function related to the clustering errors in a candidate solution, and
where a $c$-approximation algorithm is an algorithm whose total error weight is at most $c$ times the optimal weight.

An alternative formulation to MINDISAGREE is MAXAGREE, where we instead seek to maximize some function related to the edges that are not errors. In classical correlation clustering, this means that we want to maximize the number of edges which are correct. In minimax clustering, we wish to minimize the number of correct edges at the vertex with the fewest correct edges. In both cases, an optimal solution to MINDISAGREE is also an optimal solution to MAXAGREE, but their approximation properties differ.

In the classical case, there is a trivial 2-approximation algorithm for MAXAGREE on arbitrary graphs: we can simply choose the better of clustering with all vertices in separate clusters and the clustering with all vertices in the same cluster. All negative edges are correct in the first clustering and all positive edges are correct in the second clustering, so taking the better of the two yields a clustering with at least half the edges correct, which is clearly at least half the value of an optimal clustering. Less trivially, Bansal, Blum, and Chawla [5, 6] gave a PTAS for MAXAGREE, so that any approximation ratio greater than 1 is achievable.

It is natural to ask whether some algorithm can also be found to approximate MAXAGREE in the minimax context. The trivial 2-approximation algorithm no longer works, since if $G$ both has vertices of high positive degree and high negative degree, then each of the “extreme” clusterings will cause a large number of errors at some vertex. We have not been able to find any constant-factor approximation algorithm for the MAXAGREE formulation of minimax clustering, even with the additional assumption that $G$ is a labeled complete graph.

We now construct a graph which seems to be a good example of the difficulties in designing an algorithm for this problem. For any $n$, let $G_n$ be the complete graph on $n + 1$ vertices, and fix some vertex $u^* \in V(G_n)$. All edges incident to $u^*$ are labeled $+$, while all other edges are labeled $-$. Thus, $u^*$ has positive degree $n$, while all other vertices have positive degree 1.

The following result demonstrates why algorithms based on LP rounding are likely to have trouble finding a good clustering of $G_n$ under the MAXAGREE objective. We reuse the LP formulation of MINDISAGREE shown in Figure 1; this is valid because when we seek an exact solution, minimizing $M$ in Figure 1 is equivalent to maximizing $|V(G)| - 1 - M$, the weight of the correct edges at the worst vertex.

**Proposition 9.** Let $L$ be the linear program shown in Figure 1 as formulated for $G_n$. If $n \geq 2$, then the unique optimal solution to $L$ has $x_{u^*v} = 1/3$ for all $v \neq u^*$ and $x_{vw} = 2/3$ for all $vw \in E(G_n - u^*)$.

**Proof.** In the proposed solution, we have $M = n/3$. To show that this solution is optimal and unique, we construct a solution to the dual program shown in Figure 2 as in the proof of Proposition 8. Consider the dual solution defined by

$$
\pi_{u^*} = 1 - \frac{n}{3(n-1)}, \quad \pi_v = \frac{1}{3(n-1)} \text{ for all } v \neq u^*,
$$

$$
\sigma_{v,w,u^*} = \frac{1}{3(n-1)} \text{ for all } vw \in E(G_n - u^*), \quad \sigma_{v,w} = 0 \text{ if } z \neq u^*.
$$

Since $d^-(u^*) = 0$ and $d^-(v) = n - 1$ for all $v \neq u^*$, the objective value of this solution is $n/3$. Thus, if this solution is feasible, then it is optimal.

To see that this solution is feasible, we observe that for $v, w \neq u^*$, we have $\hat{\sigma}_{v,w} = -\sigma_{v,w,u^*} - \sigma_{w,v,u^*} = -\pi_v - \pi_w$, so that $\pi_v + \pi_w + \hat{\sigma}_{v,w} \leq 0$ for all negative edges $vw$, as needed. On the other hand, for $v \neq u^*$ we have

$$
\hat{\sigma}_{u^*,v} = \sum_{z \notin \{u^*,v\}} (\sigma_{v,z,u^*} + \sigma_{z,v,u^*}) = 2(n-1)\pi_v.
$$

Since $\pi_{u^*} = 1 - n\pi_v$, this implies that

$$
-p\pi_{u^*} - \pi_v + \hat{\sigma}_{u^*,v} = -(1 - n\pi_v) - \pi_v + 2(n-1)\pi_v = (n-1)\pi_v - 1 + 2(n-1)\pi_v = 0,
$$

where $a \cdot \text{weight}$.
so that $-\pi_{u^*} - \pi_v + \hat{\sigma}_{w,v} \leq 0$ for all positive edges $u^*v$, as needed. Since also $\sum_v \pi_v = 1$, we see that the proposed dual solution is feasible, so the given primal solution is optimal.

Now we argue that the given primal solution is the unique optimal solution. Let $x$ be any optimal primal solution. For each edge $vw \in E(G_n - u^*)$, the dual variable $\sigma_{v,w,u^*}$ is nonzero in the dual solution above, so by complementary slackness we have $x_{vw} = x_{u^*v} + x_{u^*w}$. Furthermore, since each $\pi_v > 0$, each $v \neq u^*$ must have total error weight equal to $M$, again by complementary slackness. Therefore, for each $v \neq u^*$, we have

$$M = \sum_{w \in N^+(v)} x_{vw} + \sum_{w \in N^-(v)} (1 - x_{vw}) = x_{u^*v} + \sum_{w \notin \{v,u^*\}} (1 - (x_{u^*v} + x_{u^*w}))$$

$$= (n - 1) - (n - 3)x_{u^*v} - \sum_{w \neq u^*} x_{u^*w}.$$

This implies that $x_{u^*v} = x_{u^*w}^*$ for all $v \neq w$. Letting $p$ denote this common value, we have $M = (n - 1) - (n - 3)p - np = (n - 1) - (2n - 3)p$. On the other hand, since $\pi_{u^*} > 0$, we also have

$$M = \sum_{w \in N^+(u^*)} x_{u^*w}^* = np.$$

Thus, $(n - 1) - (2n - 3)p = np$, which implies that $p = 1/3$. Hence, in any optimal solution we have $x_{u^*v}^* = 1/3$ for all $v \neq u^*$ and $x_{vw}^* = 2/3$ for all $vw \in E(G_n - u^*)$, as desired.

Thus, the only optimal solution to the natural LP rounding is highly symmetric, but the natural symmetric clusterings of $G_n^\pi$ into either all singletons or into one giant cluster – both have at most 1 correct edge at the worst vertex, which is far short of the optimum value of $\lceil n/2 \rceil$ correct edges. We note that this does not pose a problem for the MinDisagree formulation: in a c-approximation for MinDisagree, we only promise that the generated clustering has at most $c \lceil n/2 \rceil$ errors at its worst vertex, and if $c > 2$, then any clustering at all meets this guarantee.

C NP-Completeness of Minimax Clustering on Complete Graphs

To show that minimax clustering is NP-hard on complete graphs, we use a reduction from the Partition-into-Triangles problem, originally stated in [15] and attributed to Schaefer.

**Partition into Triangles**

**Input:** A graph $G$ with $|V(G)| = 3q$ for some integer $q$.

**Question:** Is there a partition of $V(G)$ into $q$ sets $V_1, \ldots, V_q$ such that each set $V_i$ induces a triangle in $G$?

Specifically, we reduce from the 4-regular case:

**Theorem 10** (van Rooij, van Kooten Nickerk, Bodlaender [23]). *Partition into Triangles on 4-regular graphs is NP-complete.*

(Although this is not explicitly stated in [23], it follows immediately from two of their results: that the problem is NP-hard on graphs of maximum degree at most 4, and that every partition-into-triangles instance with maximum degree at most 4 can be transformed in polynomial time into an equivalent 4-regular instance.)

To prove that minimax clustering is NP-hard, we use the following reformulation, which is more convenient for our purposes.

**t-Perfect Clustering**

**Input:** A labeled complete graph $G$ together with a tolerance $t_v \in \mathbb{Z}^+$ for each $v \in V(G)$.

**Question:** Does $G$ admit a $t$-perfect clustering, that is, a clustering such that each vertex $v$ has at most $t_v$ incident mistakes?
Taking $\lambda_v = 1/t_v$, we see that $G$ has a $t$-perfect clustering if and only if the minimax-clustering value of the resulting weighted graph is at most 1.

Our NP-completeness proof mimics the proof given by Bansal, Blum, and Chawla for the classical correlation clustering problem. Let $G$ be a 4-regular graph on $n$ vertices, where $n \geq 7$, and let $G'$ be the labeled complete graph on the same vertex set whose positive edges are exactly the edges of $G$. Observe that $G$ has a partition into triangles if and only if $G'$ has a clustering with all clusters of size at most 3 and exactly 2 mistakes at each vertex. The idea is to expand $G'$ into a larger labeled complete graph $H$ such that in an optimal clustering of $H$, every cluster has at most three $G'$-vertices.

We use essentially the same construction as Bansal-Blum-Chawla. Let $H$ consist of $G'$, augmented as follows. For every 3-set $\{u, v, w\} \subseteq V(G')$, add to $H$ a clique $C_{uvw}$ with 7 vertices. All edges within $C_{uvw}$ are positive, all edges from $C_{uvw}$ to the vertices $\{u, v, w\}$ are positive, and all other edges incident to $C_{uvw}$ are negative.

We assign the following tolerances: each original vertex $u \in G'$ has $t_u = 7\left(\binom{n-1}{2}\right) - 1 + 2$, and each added vertex $v \in H - G'$ has $t_v = 3$.

**Lemma 11.** If $H$ has a $t$-perfect clustering $C$, then every cluster of $C$ contains at most three vertices of $G'$, and every cluster of $C$ contains vertices from at most exactly one clique of $H - G'$.

**Proof.** First suppose that $C$ has a cluster $X$ containing vertices from two different cliques of $H - G'$. Let $v_1, v_2$ belong to the cliques $C_1, C_2$ respectively. If $|X \cap C_1| > 3$, then $v_2$ has more than 3 incident mistakes, which exceeds its tolerance. On the other hand, if $|X \cap C_1| \leq 3$, then since $|C_1| = 7$, we have $|C_1 - X| \geq 4$, so $v_1$ has at least 4 incident mistakes, which again exceeds its tolerance. Thus, if $C$ is $t$-perfect, then every cluster contains vertices from at most one clique.

Now suppose that $C$ has a cluster $X$ that does not contain vertices from any clique of $H - G'$. Since clusters are nonempty, $X$ contains a vertex $v \in V(G')$. Since $v$ has $7\left(\binom{n-1}{2}\right)$ neighbors in $V(H - G')$ and is not clustered with any of them, $v$ has at least $7\left(\binom{n-1}{2}\right)$ incident mistakes, which exceeds its tolerance of $7\left(\binom{n-1}{2}\right) - 5$.

Finally, suppose that $C$ has some cluster $X$ with at least four $G'$-vertices. Since $X$ contains vertices from at most one clique of $H - G'$, there is some vertex $v \in V(G') \cap X$ does not have any positive neighbors in $X \cap V(H - G')$. Since $v$ has a total of $7\left(\binom{n-1}{2}\right)$ positive neighbors in $H - G'$, it again follows that $v$ has at least $7\left(\binom{n-1}{2}\right)$ incident mistakes, exceeding its tolerance.

**Corollary 12.** $H$ has a $t$-perfect clustering if and only if $G$ has a partition into triangles.

**Proof.** First suppose that $V_1, \ldots, V_k$ is a partition of $G$ into triangles. Cluster $H$ as follows: for $i \in [k]$, let $X_i = V_i \cup C_{V_i}$, where $C_{V_i}$ is the clique of $H$ with vertex set $V_i$. For every clique $C$ that is not equal to some $V_i$, cluster $C$ on its own.

Each $v \in V(G')$ has exactly $7\left(\binom{n-1}{2}\right) - 1 + 2$ mistakes: among the $7\left(\binom{n-1}{2}\right)$ positive edges to vertices of $H - G'$, it is clustered with exactly 7 of them, and among its 4 positive neighbors in $G$, it is clustered with exactly 2 of them (and with no negative neighbors), since $V_1, \ldots, V_k$ is a partition of $G$ into triangles. Furthermore, each $v \in V(H - G')$ has at most 3 mistakes, since this clustering has no mistakes within $H - G'$ and does not cluster any $v \in V(C_{xyz})$ with a vertex outside of $\{x, y, z\}$. Thus, the clustering is $t$-perfect.

Now suppose that $H$ has a $t$-perfect clustering $C$. By Lemma 11, every cluster of $C$ contains at most three vertices of $G$ and contains vertices from exactly one cluster $C_{uvw}$ of $V(H - G')$. We claim that the restriction of $C$ to $V(G')$ is a partition of $G$ into triangles. If not, some vertex $v \in V(G')$ is clustered with fewer than 2 of its positive neighbors, and therefore has at least 3 incident mistakes in $G'$. Since the cluster containing $v$ contains vertices from only one of the cliques containing $v$, we see that $v$ also has at least $7\left(\binom{n-1}{2}\right) - 1$ incident mistakes to vertices of $V(H' - G)$, for at total of at least $7\left(\binom{n-1}{2}\right) - 1 + 3$ incident mistakes. This exceeds its tolerance, contradicting the hypothesis that $C$ is $t$-perfect.

**D  NP-Completeness on Complete Bipartite Graphs**

In this section, we show that “one-sided” minimax clustering on complete bipartite graphs is NP-hard. This complements the approximation algorithm given in Section 4 for the same problem. Our proof is similar
to the proof of Amit [4] which shows that biclustering with the classical objective function is NP-hard, but requires significant modifications to accommodate the new objective function. The proof uses a reduction from the 3-cover problem, which is well-known to be NP-complete [13].

3-Cover

**Input:** A ground set \( U = \{u_1, \ldots, u_{3n}\} \) and a family of subsets \( S = \{S_1, \ldots, S_p\} \) with each \( |S_i| = 3 \).

**Question:** Is there a subfamily \( S' \subseteq S \) such that each \( u_i \) lies in exactly one element of \( S' \)?

Given an instance of 3-cover, we construct an instance of the following problem:

**One-Sided \( t \)-perfect Biclustering**

**Input:** A labeled complete bipartite graph \( G \) with partite sets \( V_1, V_2 \) and a tolerance \( t_v \in \mathbb{Z}^+ \) for each \( v \in V_1 \).

**Question:** Does \( G \) have a clustering such that each vertex \( v \in V_1 \) has at most \( t_v \) incident edges that are errors?

By the same argument used in Appendix C, any algorithm which exactly determines the optimal one-sided minimax clustering for complete bipartite graphs would also solve the \( t \)-perfect biclustering problem. Hence, it suffices to show that \( t \)-perfect biclustering is NP-hard. Note also that one-sided minimax clustering can be viewed as the special case of (two-sided) minimax clustering for which \( t_v = |V_1| \) for all \( v \in V_2 \); thus, the reduction in this section also shows that the two-sided version of the problem is NP-hard.

Given a nontrivial instance of 3-cover (that is, an instance with \( n, p \geq 1 \)), we construct an instance of \( t \)-perfect biclustering as follows. For each \( u \in U \), construct a pair of vertices \( x_i, y_i \in V_1 \). Call these vertices **ground vertices**. Each edge \( x_i y_j \) is positive if \( u_i = u_j \) or if \( u_i \) and \( u_j \) lie in some common triplet of \( S \), and negative otherwise.

For each \( S_i \in S \), we create a vertex \( x(S_i) \in V_1 \) and \( m \) vertices \( y_1(S_i), \ldots, y_m(S_i) \in V_2 \), where each \( x_j(S_i) \in V_1 \) and \( y_j(S_i) \in V_2 \), where \( m \geq 6n + 3p \) is some fixed constant. Call these vertices **triplet vertices**, and let \( B_i = \{x(S_i)\} \cup \{y_j(S_i) : j \in \{1, \ldots, m\}\} \). All edges \( x(S_i)y_k(S_j) \) for a fixed \( i \) are positive, and all edges \( x(S_i)y_k(S_j) \) for \( i \neq \ell \) are negative. For \( u_i \in U \), if \( u_i \in S_j \), then the edges \( x_i y_k(S_j) \) and \( y_i x(S_j) \) are positive, and otherwise these edges are negative.

Finally, let \( Z = \{z_1, \ldots, z_{3n}\} \) be new \( V_2 \)-vertices, and for each \( z_i \in Z \), add positive edges to all ground-vertices in \( V_1 \) and negative edges to all triplet-vertices in \( V_1 \). Call these vertices **dummy vertices**.

Next we determine the tolerances \( t_u \). For \( S_i \in S \), let \( t_x(S_i) = 3 \). For \( u \in U \), the corresponding tolerances are computed more intricately. Let \( d(u_i) \) be the number of triplets \( S_j \in S \) containing \( u_i \) and let \( c(u_i) \) be the number of \( u_j \in U - \{u_i\} \) such that \( u_j \) and \( u_i \) lie in some common triplet \( S_j \). We define

\[
t_{x_i} = m(d(u_i) - 1) + (c(u_i) - 2) + (|Z| - 3).
\]

It is clear that \( G \) and \( t \) can be constructed in polynomial time.

**Lemma 13.** Suppose that \( G \) has a \( t \)-perfect clustering \( C \). For any \( S_i, S_j \in S \) with \( i \neq j \), the vertices \( x(S_i) \) and \( x(S_j) \) lie in different clusters.

**Proof.** Suppose that \( x(S_i) \) and \( x(S_j) \) lie in the same cluster \( X \). Since \( t_{x(S_i)} = 3 \), we see that \( X \) contains at least \( m - 3 \) vertices from \( y_1(S_i), \ldots, y_m(S_i) \). Since \( x(S_j) \) has negative edges to all these vertices, it follows that \( x(S_j) \) has at least \( m - 3 \) incident errors. Since \( m - 3 > 3 = t_{x(S_j)} \), this contradicts the fact that \( C \) is \( t \)-perfect.

**Lemma 14.** Suppose that \( G \) has a \( t \)-perfect clustering \( C \). For any \( u_j \in U \), there is a unique \( S_i \in S \) such that \( x_j \) is clustered with \( x(S_i) \). Furthermore, this \( S_i \) has the following properties:

1. \( u_j \in S_i \), and
2. \( x_j \) is clustered with each vertex \( y_{\ell} \) such that \( u_{\ell} \in S_i \).
Proof. First we prove the existence of a unique \( S_i \) such that \( x_j \) is clustered with \( x(S_i) \), then we show that \( S_i \) has the desired properties.

If \( y_k(S_i) \) is a triplet \( V_2 \)-vertex not clustered with \( x(S_i) \), call \( y_k(S_i) \) a rogue vertex. It is immediate from the definition of \( t \) that in a \( t \)-perfect clustering, each \( B_i \) contains at most 3 rogue vertices.

To prove that \( x_j \) is clustered with some \( x(S_i) \), it suffices to show that \( x_j \) is clustered with some triplet \( V_2 \)-vertex that is not a rogue vertex. Since each \( B_i \) contains at most 3 rogue vertices, there are at most \( 3p \) rogue vertices in total, where \( p = |S'|. \) If all triplet vertices clustered with \( x_j \) are rogue vertices, then since \( x_j \) has \( md(u_j) \) positive edges to triplet vertices, it follows that \( x_j \) has at least \( md(u_j) - 3p \) incident errors. Now we have

\[
x_j = m(d(u_j) - 1) + (c(u_j) - 2) + (|Z| - 3) < md(u_j) - m + 6n \leq md(u_j) - 3p,
\]

where the last inequality follows from \( m \geq 6n + 3p \). Thus, there are more than \( t_{x_j} \) errors at \( x_j \), contradicting the assumption that \( C \) is \( t \)-perfect. Thus, \( x_j \) is clustered with some \( x(S_i) \). Uniqueness of \( S_i \) follows immediately from Lemma 10.

To see that \( u_j \in S_i \), suppose that \( u_j \notin S_i \). Then \( x_j \) is clustered with at most 3 triplet-vertices that are its positive neighbors, and therefore there is at least \( md(u_j) - 3 \) incident errors. Since \( md(u_j) - 3 > t_{x_j} \), this contradicts the assumption that \( C \) is \( t \)-perfect.

Next we prove (2). Let \( B = N^+(x_j) - N^+(x(S_i)) \). Since \( t_{x(S_i)} = 3 \), the cluster containing \( x_j \) contains at most 3 vertices from \( B \). Thus, there are at least \( |B| - 3 \) errors from \( x \) to the vertices of \( B \), where

\[
|B| - 3 = |Z| + m(d(u_j) - 1) + (c(u_j) - 2) - 3 = t_{x_j}.
\]

Thus, for \( C \) to be \( t \)-perfect, it is necessary that all errors incident to \( x_j \) are edges from \( x \) to \( B \). In particular, \( x_j \) is clustered with all vertices in \( N^+(x_j) \cap N^+(x(S_i)) \), so that \( x_j \) is clustered with all \( y_{\ell} \) such that \( y_{\ell} \in S_i \).

Corollary 15. \( G \) has a \( t \)-perfect clustering if and only if \( S' \) has a 3-cover.

Proof. Given any \( t \)-perfect clustering, let \( S' \) be the family of triplets \( S_i \) such that some vertex of \( B_i \) is clustered with some \( V_1 \)-ground-vertex \( x_j \). Lemma 14 immediately implies that these triplets cover all of \( u \). Furthermore, Lemma 14 implies that these triplets are pairwise disjoint: if \( S'_1 \) and \( S'_2 \) are triplets of \( S' \) that both contain \( u_j \), then Lemma 14 would force each \( x(S'_1) \) and \( x(S'_2) \) to both be clustered with \( y_j \) and hence to be clustered together, which contradicts Lemma 13. Hence, \( S' \) is a 3-cover.

Conversely, let \( S' \) be a 3-cover in \( S \). We define a clustering of \( G \). Since \( S' \) is a 3-cover, we have \( |S'| = n \). Let \( Z_{S_1}, \ldots, Z_{S_n} \) be a partition of \( Z \) into \( n \) disjoint sets of size 3, indexed by the sets of \( S' \). Now for each \( S_i \in S \), define a cluster \( X_i \) by

\[
X_i = \begin{cases} 
B_i \cup \{x, y_j : u_j \in S_i\} \cup Z_{S_i}, & \text{if } S_i \in S', \\
B_i, & \text{otherwise}.
\end{cases}
\]

Since \( S' \) is a 3-cover, the clusters \( X_i \) are pairwise disjoint and cover the vertices of \( G \). We claim that this clustering is \( t \)-perfect. If \( x(S_i) \) is a triplet vertex corresponding to some \( S_i \notin S' \), then \( x(S_i) \) has exactly 3 incident errors, namely its edges to the ground-vertices \( y_j \) with \( u_j \in S_i \). On the other hand, if \( x(S_i) \) is a triplet vertex corresponding to some \( S_i \in S' \), then \( x(S_i) \) again has exactly 3 incident errors, namely its edges to the dummy-vertices in \( Z_{S_i} \).

If \( x_j \) (or \( y_j \)) is a ground vertex, then \( x_j \) has \( m(d(u_j) - 1) \) incident errors which are positive edges to triplet-vertices, \( c(u_j) - 2 \) incident errors which are positive edges to ground-vertices, and \( |Z| - 3 \) incident errors which are positive edges to dummy-vertices. This is a total of exactly \( t_{x_j} \) incident errors. Hence the clustering is \( t \)-perfect.

E Technical Details

Lemma 16. Suppose a Type 2 cluster \( \{u\} \cup T \) has just been output in Algorithm 1. For any \( z \in S - (\{u\} \cup T) \), the total cluster-cost of the cross-edges for \( z \) is at most \( \max\{1/(1 - 2\alpha), 2/\alpha\} \) times the total LP-cost of the cross-edges for \( z \).
Proof. This is essentially the same proof given by Charikar, Guruswami, and Wirth [9,10]; we repeat it here to keep the paper self-contained. If \( x_{uz} \geq 1 - \alpha \), then for each \( w \in \{u\} \cup T \), we have

\[
x_{wz} \geq x_{uz} - x_{uw} \geq 1 - 2\alpha.
\]

If there are \( p \) positive cross-edges, this implies that the total LP-cost of the cross-edges for \( z \) is at least 

\[
(1 - 2\alpha)p. 
\]

Since the total cluster-cost of the cross-edges for \( z \) is \( p \), the claim holds.

Now consider \( x_{uz} \in (\alpha, 1 - \alpha) \). Let \( P = N^+(z) \cap (\{u\} \cup T) \) and let \( Q = N^-(z) \cap (\{u\} \cup T) \); the total cluster-cost of the cross-edges for \( z \) is just \( |P| \). We have the following lower bound on the total LP-cost of the cross-edges for \( z \):

\[
\sum_{w \in P} x_{wz} + \sum_{w \in N} (1 - x_{wz}) \geq \sum_{w \in P} (x_{uz} - x_{uw}) + \sum_{w \in Q} (1 - x_{uz} - x_{uw}) \\
= |P| x_{uz} + |Q| (1 - x_{uz}) - \sum_{w \in \{u\} \cup T} x_{uw} \\
\geq |P| x_{uz} + |Q| (1 - x_{uz}) - \alpha (|P| + |Q|),
\]

where in the last line we used the inequality \( \sum_{w \in \{u\} \cup T} x_{uw} \leq \frac{\alpha |\{u\} \cup T|}{2} \). This lower bound is linear in \( x_{uz} \), so we study its behavior at the endpoints of \( (\alpha, 1 - \alpha) \). When \( x_{uz} = \alpha \), the lower bound rearranges as follows:

\[
\alpha |P| + (1 - \alpha) |Q| - \frac{\alpha (|P| + |Q|)}{2} = \frac{\alpha}{2} |P| + (1 - \frac{3\alpha}{2}) |Q| \geq \frac{\alpha}{2} |P|.
\]

When \( x_{uz} = 1 - \alpha \), the lower bound rearranges as follows:

\[
(1 - \alpha) |P| + \alpha |Q| - \frac{\alpha (|P| + |Q|)}{2} = (1 - \frac{3\alpha}{2}) |P| + \frac{\alpha}{2} |Q| \geq \frac{\alpha}{2} |P|.
\]

In both cases, we used the assumption \( \alpha < 1/2 \), which implies \( 1 - \frac{3\alpha}{2} \geq \frac{\alpha}{2} \). It follows that charging \( \frac{2}{\alpha} \) times the LP-cost of each cross-edge yields enough charge to pay for the cluster-cost of all cross-edges.

Lemma 17. Suppose that a Type 2 cluster \( C \) has just been output in Algorithm 2. For any vertex \( z \in V_1 - C \), the total cluster-cost of the cross-edges for \( z \) is at most \( \max\{1/(1 - 2\alpha), 2/\alpha\} \) times the total LP-cost of the cross-edges for \( z \).

Proof. We essentially repeat the proof of Lemma 16. If \( x_{uz} \geq 1 - \alpha \), then for each \( w \in \{u\} \cup T \), we have

\[
x_{wz} \geq x_{uz} - x_{uw} \geq 1 - 2\alpha.
\]

If there are \( p \) positive cross-edges, this implies that the total LP-cost of the cross-edges for \( z \) is at least

\[
(1 - 2\alpha)p. 
\]

Since the total cluster-cost of the cross-edges for \( z \) is \( p \), the claim holds.

Now consider \( x_{uz} \in (\alpha, 1 - \alpha) \). Let \( P = N^+(z) \cap (\{u\} \cup T) \) and let \( Q = N^-(z) \cap (\{u\} \cup T) \); the total cluster-cost of the cross-edges for \( z \) is just \( |P| \). We note that \( P \cup Q = V_2 \cap T \). We have the following lower bound on the total LP-cost of the cross-edges for \( z \):

\[
\sum_{w \in P} x_{wz} + \sum_{w \in N} (1 - x_{wz}) \geq \sum_{w \in P} (x_{uz} - x_{uw}) + \sum_{w \in Q} (1 - x_{uz} - x_{uw}) \\
= |P| x_{uz} + |Q| (1 - x_{uz}) - \sum_{w \in V_2 \cap T} x_{uw} \\
\geq |P| x_{uz} + |Q| (1 - x_{uz}) - \frac{\alpha}{2} (|P| + |Q|),
\]

where in the last line we used the inequality \( \sum_{w \in V_2 \cap T} x_{uw} \leq \frac{\alpha}{2} |\{u\} \cup T| \). This lower bound is linear in \( x_{uz} \), so we study its behavior at the endpoints of \( (\alpha, 1 - \alpha) \). When \( x_{uz} = \alpha \), the lower bound rearranges as follows:

\[
\alpha |P| + (1 - \alpha) |Q| - \frac{\alpha (|P| + |Q|)}{2} = \frac{\alpha}{2} |P| + (1 - \frac{3\alpha}{2}) |Q| \geq \frac{\alpha}{2} |P|.
\]
When $x_{uz} = 1 - \alpha$, the lower bound rearranges as follows:

$$(1 - \alpha) |P| + \alpha |Q| - \frac{\alpha}{2}(|P| + |Q|) \geq (1 - \alpha - \frac{\alpha}{2}) |P| + \frac{\alpha}{2} |Q| \geq \frac{\alpha}{2} |P|.$$

In both cases, we used the assumption $\alpha < 1/2$. It follows that when $x_{uz} \in (\alpha, 1 - \alpha)$, charging $1/(\alpha - \beta)$ times the LP-cost of each cross-edge yields enough charge to pay for the cluster-cost of all cross-edges. \qed