DECAY RATES FOR SECOND ORDER EVOLUTION EQUATIONS IN HILBERT SPACES WITH NONLINEAR TIME-DEPENDENT DAMPING

JUN-REN LUO AND TI-JUN XIAO

Shanghai Key Laboratory for Contemporary Applied Mathematics
School of Mathematical Sciences, Fudan University
Shanghai 200433, China

(Communicated by Alain Haraux)

Abstract. The paper is concerned with the Cauchy problem for second order hyperbolic evolution equations with nonlinear source in a Hilbert space, under the effect of nonlinear time-dependent damping. With the help of the method of weighted energy integral, we obtain explicit decay rate estimates for the solutions of the equation in terms of the damping coefficient and two nonlinear exponents. Specialized to the case of linear, time-independent damping, we recover the corresponding decay rates originally obtained in [3] via a different way. Moreover, examples are given to show how to apply our abstract results to concrete problems concerning damped wave equations, integro-differential damped equations, as well as damped plate equations.

1. Introduction. In this paper we are concerned with the abstract second order evolution equation

\[
\begin{aligned}
    &u''(t) + Au(t) + \gamma(t)g(u'(t)) + \nabla F(u(t)) = 0, \quad \forall \ t \geq 0, \\
    &u(0) = u_0, \quad u'(0) = u_1,
\end{aligned}
\]

where \(\gamma(t)g(u')\) is a nonlinear time-dependent damping term, \(A\) a nonnegative self-adjoint linear operator on a Hilbert space \(H\), and \(\nabla F\) a nonlinear source term.

Energy decay of the solutions for second order semilinear or nonlinear damped hyperbolic equations has been extensively studied, when the linear part is governed by a strongly positive operator \(A\); see, e.g., [1, 11, 12, 13, 15, 17]. When \(A\) has a nontrivial kernel (like negative Neumann Laplacian), the situation is quite tricky and different. In [3], Ghisi, Gobbino, and Haraux proved, for the case of linear time-independent damping, namely, for the equation

\[
    u''(t) + Au(t) + u'(t) + \nabla F(u(t)) = 0,
\]

2000 Mathematics Subject Classification. 35B40, 35L70, 35L90.

Key words and phrases. Nonlinear hyperbolic equations, nonlinear time-dependent damping, nonlinear source, decay rates; second order evolution equations, Hilbert space.

The work was supported partly by the NSF of China (11771091, 11831011), the Fudan University (IDH 1411016), and the Shanghai Key Laboratory for Contemporary Applied Mathematics (08DZ2271900).

Corresponding author: Ti-Jun Xiao.
that the solutions decay at least as fast as $t^{-1/p}$, if $\nabla F(u)$ meets a sign condition and its norm is comparable to $|u|^{p+1}$, and this decay rate is optimal (see also [4, 6] for more refined analysis).

In the work, we study the decay property for (1.1) subject to a nonlinear, time-dependent damping, and obtain uniform decay rates of the solutions and energies. Specialized to the equation (1.2), this recovers the rates in [3, Theorem 2.2]. While a modified Lyapunov functional was used in the proof of [3, Theorem 2.2], we exploit a different approach, rather focusing on the original energy, showing that some of its power fulfills a weighted integral inequality, and so achieving the final decay estimates. On the other hand, how to show the existence of slow solutions (and so the power fulfills a weighted integral inequality, and so achieving the final decay estimates. On the other hand, how to show the existence of slow solutions (and so extend the main result of [3]) still remains to be explored.

For related information, we refer to [14], which handles the equation

$$u''(t) + Au(t) + \gamma(t)u'(t) + \nabla F(u(t)) = 0,$$

with the time-dependent linear damping $\gamma(t)u'(t)$, both $\gamma$ and $\nabla F$ being monotone. Also, we refer to [7, 8, 10] for the case of $F$ being analytical or satisfying a gradient inequality of Lojasiewicz type.

The outline of this paper is the following. In Section 2 we state notations and assumptions, and present our decay rate theorem, as well as a result of wellposedness. In Section 3 we prove our results. Finally, in Section 4 we give examples to show how to apply our abstract results to concrete problems concerning damped wave equations, integro-differential damped equations, as well as damped plate equations.

2. Preliminaries and the main result. We assume that $H$ is a real Hilbert space, denote by $\langle v, w \rangle$ the inner product of two vectors $v, w$ in $H$, and by $\|v\|$ the $H$-norm of $v$. Since $A$ is nonnegative (that is $\langle Av, v \rangle \geq 0$ for every $v \in D(A)$), the power $A^{1/2}$ is well defined, and we define

$$H_1 = D(A^{1/2}) \quad \text{with norm } \|v\|_{H_1} := \left(\|v\|^2 + \|A^{1/2}v\|^2\right)^{1/2}.$$

The following is the basic assumptions on the damping and source terms.

**Assumption 2.1.**
1. $\gamma \in W^{1,\infty}_{{\text{loc}}}(R^+) \text{ is bounded and nonnegative on } R^+$,
2. $g : H \to H \text{ satisfies } \langle g(v), v \rangle \geq 0 \text{ for } v \in H, \text{ and is locally Lipschitz continuous, i.e.,}$

$$\|g(v) - g(w)\| \leq L(\|v\|, \|w\|)|v - w|, \quad \forall \ v, w \in H,$$

for some positive function $L$ on $R^+ \times R^+$, which is bounded on bounded sets.

**Assumption 2.2.**
1. $F : H_1 \to R$ is nonnegative and differentiable.
2. The gradient $\nabla F : H_1 \to H$ is locally Lipschitz continuous, i.e.

$$\|\nabla F(v) - \nabla F(w)\| \leq L_1(\|v\|_{H_1}, \|w\|_{H_1})\|v - w\|_{H_1}, \quad \forall \ v, w \in H_1,$$

for some positive function $L_1$ on $R^+ \times R^+$, which is bounded on bounded sets.

Next, we define the notion of mild and strong solutions to (1.1), as usual. A function $u \in C([0, +\infty); H_1) \cap C^1([0, +\infty); H)$ is called a *mild solution* of problem (1.1), if it satisfies the integral equation

$$u(t) = S'(t)u_0 + S(t)u_1 - \int_0^t S(t - \tau)[\gamma(\tau)g(u'(\tau)) + \nabla F(u(\tau))]d\tau, \quad t \geq 0.$$

Here $S(\cdot) : [0, +\infty) \to \mathcal{L}(H)$ (the space of bounded linear operators on $H$) is a solution operator for the linear equation
\[ u''(t) + Au(t) = 0, \quad t \geq 0, \]

with \( S(0) = 0 \) and \( S'(0) = I \) (the identity). It is known (cf., e.g., [2, 5]) that \( \{S(t)\}_{t \geq 0} \) is a sine operator function, which satisfies

\[ S(t)v \in C^1([0, +\infty); H) \]

for each \( v \in H \),

and

\[ S(t)v \in C^2([0, +\infty); H) \]

for each \( v \in H_1 \);

\( \{S'(t)\}_{t \geq 0} \) is a cosine operator function, the derivative being in the sense of strong topology (rather than uniform operator topology).

In particular, \( u \) is called a strong solution if \( u \in C^1([0, +\infty); H_1) \cap C^2([0, +\infty); H) \) and (1.1) holds.

The following is the result of wellposedness.

**Proposition 2.3.** Suppose that Assumptions 2.1 and 2.2 are satisfied. Then, for every \((u_0, u_1) \in H_1 \times H\), problem (1.1) admits a unique global mild solution

\[ u \in C([0, +\infty); H_1) \cap C^1([0, +\infty); H), \quad (2.1) \]

which depends continuously on the initial data. In particular, \( u \) is a strong solution if \((u_0, u_1) \in D(A) \times H_1\). Moreover, defining the energy

\[ E(t) := \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \|A^{1/2}u(t)\|^2 + F(u(t)), \quad (2.2) \]

one has

\[ \frac{dE(t)}{dt} = -\langle \gamma g(u'), u' \rangle \quad (2.3) \]

(for strong solutions).

In view of Proposition 2.3, we know that \( E(t) \) is nonincreasing because of the properties of \( \gamma \) and \( g \). Hence,

\[ \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \|A^{1/2}u(t)\|^2 \leq E(t) \leq E(0). \]

Now we present our theorem on decay rates.

**Theorem 2.4.** Let Assumptions 2.1 and 2.2 hold. Assume that

\[ \langle \nabla F(v), v \rangle \geq cF(v), \quad \forall \ v \in H_1, \quad (2.4) \]

for some constant \( c > 0 \),

\[ \|v\|^{p+2} \leq M_0(\|A^{1/2}v\|) \left( \|A^{1/2}v\|^2 + F(v) \right), \quad (2.5) \]

\[ \|v\|, \ |g(v)| \leq M_1(\|v\|) (g(v), v)^{1/(k+1)} \quad (2.6) \]

for \( v \in H_1 \), with constants \( p > 0, \ k \geq 1 \), and some positive functions \( M_0, M_1 \) on \( R^+ \), which are bounded on bounded sets. Moreover, we assume that

\[ \gamma' \in L^1(R^+), \quad \int_0^\infty \gamma(t)dt = +\infty. \quad (2.7) \]

Let \((u_0, u_1) \in H_1 \times H\), and let \( u(t) \) be the unique global mild solution of problem (1.1). Then

\[ E(t) \leq M(E(0)) \left( \frac{1}{1 + \int_0^t \gamma(\tau)d\tau} \right)^{\frac{k+2}{kp+k+1}}, \quad \forall \ t \geq 0, \]
\[ \|u(t)\| \leq M(E(0)) \left( \frac{1}{1 + \int_0^t \gamma(\tau)d\tau} \right)^{\frac{1}{n+1}}, \quad \forall \ t \geq 0, \]

for some positive function \( M \) on \( \mathbb{R}^+ \), which is bounded on bounded sets.

**Remark 2.1.** (1). Specialized to the case of \( \gamma(t) = 1 \), \( g(v) = v \) (and so \( k = 1 \)), the decay rates of the energy and solution, given in Theorem 2.4, recover those in [3, Theorem 2.2].

(2). A canonical example of \( \gamma(t) \) meeting the conditions in Theorem 2.4 will be given in Example 4.1 below.

In [9], asymptotic stability for second-order evolution equations (with the linear parts governed by self-adjoint and coercive operators) is established, when the feedbacks act intermittently. Theorem 2.4 here suits some special cases of intermittent feedbacks. For instance, consider a sequence of intervals in \([1, +\infty)\):

\[ (a_n, b_n), \quad n = 1, 2, 3, \cdots, \]

where \( b_n + 2 \leq a_{n+1} \) for each \( n \), and \( \sum_{n=1}^{\infty} n^{-2}(b_n - a_n) = +\infty \). We define

\[ \gamma(t) = \begin{cases} 
    n^{-2}, & \text{if } t \in [a_n, b_n], \\
    n^{-2}(t + 1 - a_n), & \text{if } t \in [a_n - 1, a_n], \\
    n^{-2}(b_n + 1 - t), & \text{if } t \in [b_n, b_n + 1], \\
    0, & \text{otherwise.}
\end{cases} \]

Then

\[ \int_0^{\infty} \gamma(t)dt = \sum_{n=1}^{\infty} n^{-2}(b_n - a_n + 1) = +\infty, \]

\[ \int_0^{\infty} |\gamma'(t)|dt = \sum_{n=1}^{\infty} 2n^{-2} < +\infty, \]

and so \( \gamma(t) \) satisfies the conditions in Theorem 2.4. Clearly, the associated feedback acts intermittently, yet the lengths of the damping-effect intervals growing to infinity.

3. **Proofs.**

3.1. **Proof of Proposition 2.3.**

*Proof.* Let \( \mathcal{H} := H_1 \times H \), endowed with the inner product

\[ \langle (u, v), (u_1, v_1) \rangle_{\mathcal{H}} = \langle u, u_1 \rangle + \langle A^{1/2}u, A^{1/2}u_1 \rangle + \langle v, v_1 \rangle. \]

Define two operators by

\[ A_0(u, v) := (-v, Au + u), \quad \forall \ (u, v) \in D(A_0) := D(A) \times H_1, \]

\[ A(u, v) := (-v, Au), \quad \forall \ (u, v) \in D(A) := D(A_0), \]

And for \( t \geq 0 \), set the operator

\[ \mathcal{F}(t, (u, v)) := (0, -\nabla F(u) - \gamma(t)g(v)), \quad \forall \ (u, v) \in \mathcal{H}. \]
Clearly, problem (1.1) can be rewritten as
\[ U'(t) + A U(t) = F(t, U(t)), \quad t \geq 0, \] (3.1)
with initial data \( U(0) = U_0 := (u_0, u_1) \), where \( U(t) := (u(t), u'(t)) \).
Observe
\[ \langle A_0(u, v), (u, v) \rangle_H = 0, \quad \forall (u, v) \in D(A_0), \]
and it is easy to check that
\[ A_0D(A_0) = H. \]
Thus, \(-A_0\) is the generator of a strongly continuous contraction semigroup on \( H \), according to the Lumer-Phillips theorem (cf. Section 4.1 of [16]). Therefore, \(-A\) (as a bounded perturbation of \(-A_0\)) generates a strongly continuous semigroup \( \{ T(t) \} \) on \( H \), which is given by
\[ T(t) = \begin{pmatrix} S'(t) & S(t) \\ S^0(t) & S'(t) \end{pmatrix} \]
(cf., e.g., [2, 5]).
In addition, given \( T_0 > 0 \), we observe that for \( t, s \in [0, T_0] \),
\[ \| F(t, U_1) - F(s, U_2) \|_H \]
\[ = \| \nabla F(u_2) - \nabla F(u_1) + \gamma(s)g(v_2) - \gamma(t)g(v_1) \|_H \]
\[ \leq \| \nabla F(u_2) - \nabla F(u_1) \|_H + \| \gamma(s)g(v_2) - \gamma(t)g(v_1) \|_H \]
\[ \leq L_1(\| u_1 \|_H, \| u_2 \|_H) \| u_1 - u_2 \|_H + \| \gamma \|_{L^\infty(0, T_0)} L(\| v_1 \|, \| v_2 \|) \| v_2 - v_1 \|_H \]
\[ + \| \gamma' \|_{L^\infty(0, T_0)} L(\| v_2 \|) \| t - s \| \]
\[ \leq L_2(\| U_1 \|_H, \| U_2 \|_H) \| t - s \| + \| U_1 - U_2 \|_H, \quad U_i = (u_i, v_i) \in H \quad (i = 1, 2) \]
for some positive function \( L_2 \) on \( R^+ \times R^+ \) that is bounded on bounded sets by the properties of \( \gamma, g \) and \( \nabla F \). Therefore, \( F \) is a locally Lipschitz continuous operator.
Thus, according to Section 6.1 of [16], we know that equation (3.1) with the initial datum \( U(0) \in H \) has a unique mild solution \( U(\cdot) = (u(\cdot), v(\cdot)) \) in a maximal interval \([0, T_*)\), with either \( T_* = +\infty \), or
\[ \limsup_{t \to T_*} \| U(t) \|_H \to +\infty; \]
namely, \( U(\cdot) \) is the solution of the integral equation
\[ U(t) = T(t)U_0 + \int_0^t T(t - \tau)F(\tau, U(\tau))d\tau, \quad t \in [0, T_*). \] (3.2)
Also, we know that the solution depends continuously on initial data; moreover, \( U(\cdot) \) is a strong solution whenever \( U(0) \in D(A) \).
Letting \( E(t) \) be as in (2.2) and using (3.1), we get, for strong solutions,
\[ E'(t) = -\gamma \langle g(u'(t)), u'(t) \rangle \leq 0, \quad \text{a.e. on } [0, T_*), \]
and hence,
\[ E(t) \leq E(0), \quad \forall \ t \in [0, T_*). \]
The latter holds too for mild solutions, by the continuous dependence of solutions on initial data.
From the nonincreasing character of \( E(t) \) and our assumption that \( F(u) \geq 0 \), we see that \( \| U(t) \|_H \) can be controlled by \( E(t) \), which means that \( T_* = +\infty \).
Rewrite (3.2) as
\[
\begin{aligned}
    u(t) &= S'(t)u_0 + S(t)u_1 - \int_0^t S(t-\tau)\gamma(\tau)g(v(\tau)) + \nabla F(u(\tau))d\tau, \quad t \geq 0, \\
    v(t) &= S''(t)u_0 + S'(t)u_1 - \int_0^t S'(t-\tau)\gamma(\tau)g(v(\tau)) + \nabla F(u(\tau))d\tau, \quad t \geq 0.
\end{aligned}
\]

We see that for any \((u_0, u_1) \in H \times H\), problem (1.1) admits a unique mild solution \(u(\cdot)\), noting \(v(t) = u'(t)\). Also, when \((u_0, u_1) \in D(A) \times H\), \((u(\cdot), v(\cdot))\) satisfies (3.1), and so \(u(\cdot)\) is a strong solution of (1.1). Therefore, the proof is finished.

3.2. Proof of Theorem 2.4. First we recall the following lemma from [12], which will be used in the proof of Theorem 2.4.

Lemma 3.1. Let \(E : \mathbb{R}^+ \to \mathbb{R}^+\) be a nonincreasing function and \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) a strictly increasing \(C^1\)-function such that
\[
\phi(0) = 0, \text{ and } \phi(t) \to +\infty \text{ as } t \to +\infty.
\]

Assume that there exist \(q \geq 0\) and \(\omega > 0\) such that
\[
\int_{S}^{+\infty} E(t)^{q+1} \phi'(t)dt \leq \frac{1}{\omega} E(0)^q E(S), \quad \forall \ S \geq 0.
\]

Then, for \(t \geq 0\),
\[
E(t) \leq E(0) \left( \frac{1 + q}{1 + q\omega \phi(t)} \right)^{\frac{1}{q}} \quad \text{if } q > 0,
\]
and
\[
E(t) \leq E(0)e^{1-\omega \phi(t)} \quad \text{if } q = 0.
\]

Throughout the proof, \(C, C_1, C_2, C_3, C_\epsilon\) denote generic positive constants, which may depend on \(E(0)\) and may vary from line to line. Moreover, we only need to treat the strong solution case. The general case can be handled by a density argument, with the aid of the continuous dependence of solutions on initial data.

We divide the proof into three steps.

Step 1. Estimate the weighted integral \(\int_{S}^{T} E^{q+1}\gamma dt\), for \(T > S \geq 0\) and
\[
q \geq \frac{p}{2(p+2)}. \tag{3.3}
\]

Let us take the inner product of both sides of the equation in (1.1) with \(E^{q}\gamma u\), and integrate over \([S, T]\). Then we have
\[
0 = \int_{S}^{T} E^{q}\gamma(u'' + Au + \gamma g(u') + \nabla F, u)dt. \tag{3.4}
\]

Integrating by parts yields
\[
\begin{align*}
\int_{S}^{T} E^{q}\gamma(u'', u)dt &= \left[ E^{q}\gamma(u', u) \right]_{S}^{T} - \int_{S}^{T} E^{q}\gamma u''^2 dt - \int_{S}^{T} qE^{q-1}E'\gamma + E^{q}\gamma'(u', u)dt.
\end{align*}
\]

Also,
\[
\int_{S}^{T} E^{q}\gamma(Au, u)dt = \int_{S}^{T} E^{q}\gamma A^{1/2}u^2 dt.
\]
Accordingly, we obtain
\[
0 = \left[ E^q \gamma \langle u', u \rangle \right]_S^T - \int_S^T E^q \| u' \|^2 dt - \int_S^T (qE^{q-1}E^q \gamma + E^q \gamma')(u', u) dt \\
+ \int_S^T E^q \| A^{1/2} u \|^2 dt + \int_S^T E^q \gamma^2(g(u'), u) dt + \int_S^T E^q \langle \nabla F(u), u \rangle dt.
\]
Hence, from (2.4) and (2.2) it follows that
\[
\min(c, 2) \int_S^T E^{q+1} \gamma dt \leq \left[ E^q \gamma \langle u', u \rangle \right]_S^T + \int_S^T (qE^{q-1}E^q \gamma + E^q \gamma')(u', u) dt \\
+ 2 \int_S^T E^q \| u' \|^2 dt - \int_S^T E^q \gamma^2(g(u'), u) dt.
\]
(3.5)
Noting that \( E \) is nonincreasing and \( \gamma \) bounded, by (2.5) we deduce that
\[
\left[ E^q \gamma \langle u', u \rangle \right]_S^T \leq CE^q(S)(\| u'(T) \|u(T) + \| u'(S) \|u(S) \|)
\]
\[
\leq CE^q(S)E^{\frac{q}{2}}(S)E^{\frac{q}{p+2}}(S) = CE(S)^{q+\frac{1}{2}+\frac{2}{p+2}};
\]
and furthermore,
\[
\left| \int_S^T (qE^{q-1}E^q \gamma + E^q \gamma')(u', u) dt \right|
\leq C \int_S^T -E^qE^{\frac{q}{p+2}} \gamma' dt + C \int_S^T E^{q+\frac{1}{2}+\frac{2}{p+2}} \gamma' dt
\]
\[
\leq CE(S)^{q+\frac{1}{2}+\frac{2}{p+2}},
\]
since \( \gamma' \in L^1(R^+) \) (by (2.7)), and
\[
q - \frac{1}{2} + \frac{1}{p+2} \geq 0
\]
(by (3.3)). Also we get
\[
\left| \int_S^T E^q \gamma^2 \langle u, g(u') \rangle dt \right| \leq \int_S^T E^q \gamma^2 \| u \| \| g(u') \| dt
\]
\[
\leq C \int_S^T E^{q+\frac{2}{p+2}} \gamma \| g(u') \| dt.
\]
Accordingly,
\[
\int_S^T E^{q+1} \gamma dt \leq CE(S)^{q+\frac{1}{2}+\frac{2}{p+2}} + C \int_S^T E^q \| u' \|^2 dt + C \int_S^T E^{q+\frac{2}{p+2}} \gamma \| g(u') \| dt,
\]
(3.6)
for \( T > S \geq 0 \).

**Step 2.** Estimate \( \int_S^T E^q \gamma(t) \| u' \|^2 dt \) and \( \int_S^T E^{q+\frac{2}{p+2}} \gamma \| g(u') \| dt \).

**Case I :** \( k = 1 \).

By (2.6), we have
\[
\| u' \|^2 + \| g(u') \|^2 \leq C \langle g(u'), u' \rangle,
\]
(3.7)
which gives that for any $T > S \geq 0$,
\[
\int_{S}^{T} E^{q}\gamma\|u'\|^2 dt \leq C \int_{S}^{T} E^{q}(\gamma g(u'), u') dt \leq CE(S)^{q+1}
\] (3.8)
due to (2.3).

As for $\int_{S}^{T} E^{q+\frac{2}{p+2}}\gamma\|g(u')\| dt$, we infer that by using Young’s inequality and (3.7),
\[
\int_{S}^{T} E^{q+\frac{2}{p+2}}\gamma\|g(u')\| dt \leq \varepsilon \int_{S}^{T} E^{2(q+\frac{2}{p+2})}\gamma dt + \frac{1}{\varepsilon} \int_{S}^{T} \gamma\|g(u')\|^2 dt
\]
\[
\leq \varepsilon \int_{S}^{T} E^{2(q+\frac{2}{p+2})}\gamma dt + C_{\varepsilon} \int_{S}^{T} \langle \gamma g(u'), u' \rangle dt
\]
and so
\[
\int_{S}^{T} E^{q+\frac{2}{p+2}}\gamma\|g(u')\| dt \leq \varepsilon \int_{S}^{T} E^{2(q+\frac{2}{p+2})}\gamma dt + C_{\varepsilon} E(S)
\] (3.9)
for any $T > S \geq 0$ and $\varepsilon > 0$.

Case II: $k > 1$.

It follows from (2.6) that
\[
\|u'\|^2 + \|g(u')\|^2 \leq C\langle g(u'), u' \rangle^{\frac{k}{k+1}}.
\] (3.10)
This and (2.3), together with Young’s inequality, yield that
\[
\int_{S}^{T} E^{q}\gamma\|u'\|^2 dt \leq C \int_{S}^{T} E^{q-1}\gamma^{\frac{k-1}{k+1}} \langle \gamma g(u'), u' \rangle^{\frac{k}{k+1}} dt
\]
\[
\leq C \int_{S}^{T} E^{q}\gamma^{1-\frac{2}{p+1}} (-E')^{\frac{2}{p+1}} dt
\]
\[
\leq \varepsilon^{\frac{k+1}{k+2}} \int_{S}^{T} \left( E^{q}\gamma^{\frac{k-1}{k+1}} \right)^{\frac{k+1}{k} \frac{k+1}{k+2}} dt + \frac{C}{\varepsilon^{(k+1)/2}} \int_{S}^{T} (-E')^{\frac{k+1}{k+2}} dt.
\]
Therefore
\[
\int_{S}^{T} E^{q}\gamma\|u'\|^2 dt \leq C_{\varepsilon} E(S) + \varepsilon^{\frac{k+1}{k+2}} \int_{S}^{T} E^{q+\frac{k+1}{k+2}}\gamma dt
\] (3.11)
for $T > S \geq 0$ and $\varepsilon > 0$.

For $\int_{S}^{T} E^{q+\frac{2}{p+2}}\gamma\|g(u')\| dt$, by using (3.10), (2.3), and Young’s inequality again we infer that
\[
\int_{S}^{T} E^{q+\frac{2}{p+2}}\gamma\|g(u')\| dt \leq C \int_{S}^{T} E^{q+\frac{2}{p+2}}\gamma \langle \gamma g(u'), u' \rangle^{\frac{1}{k+1}} dt
\]
\[
\leq C \int_{S}^{T} E^{q+\frac{2}{p+2}}\gamma \left( \frac{\langle \gamma g(u'), u' \rangle}{\gamma} \right)^{\frac{1}{k+1}} dt
\]
\[
\leq \varepsilon^{\frac{k+1}{k+2}} \int_{S}^{T} \left( E^{q+\frac{2}{p+2}}\gamma^{1-\frac{1}{k+1}} \right)^{\frac{k+1}{k} \frac{k+1}{k+2}} dt + \frac{C}{\varepsilon^{(k+1)/2}} \int_{S}^{T} (-E') dt.
\]
Hence,
\[
\int_{S}^{T} E^{q+\frac{2}{p+2}}\gamma\|g(u')\| dt \leq C_{\varepsilon} E(S) + \varepsilon^{\frac{k+1}{k+2}} \int_{S}^{T} E^{(q+\frac{2}{p+2})}\frac{k+1}{k} \gamma dt
\] (3.12)
for $T > S \geq 0$ and $\varepsilon > 0$.

Step 3. Obtain the final energy estimates.
For the case $k > 1$, combining (3.6), (3.11) with (3.12) together, we obtain
\[
\int_{S}^{T} E^{q+1} \gamma dt \leq C E(S)^{q+\frac{1}{p} + \frac{q}{p+2}} + C_{2} E(S) + C \varepsilon^{\frac{k+1}{p+2}} \int_{S}^{T} E^{q+\frac{1}{p+2}} \gamma dt + C \varepsilon^{\frac{k}{p+2}} \int_{S}^{T} E^{q+\frac{1}{p+2}} \gamma dt, \tag{3.13}
\]
for $T > S \geq 0$ and $\varepsilon > 0$.

Choose a suitable $q$ such that
\[
\begin{cases}
q \geq \frac{k-1}{2}, \\
(q + \frac{1}{p+2}) \frac{k+1}{k} \geq 1 + q,
\end{cases}
\tag{3.14}
\]
which is equivalent to
\[
\begin{cases}
q \geq \frac{k-1}{2}, \\
q \geq k - \frac{k+1}{p+2}.
\end{cases}
\]
Since $k > 1$ and $p > 0$, we have
\[
k - \frac{k+1}{p+2} > \frac{p}{p+2}, \quad k - \frac{k+1}{p+2} > \frac{k-1}{2}.
\]
Take
\[
q = k - \frac{k+1}{p+2}, \tag{3.15}
\]
which definitely satisfies (3.14). Then, we get
\[C \int_{S}^{T} E^{q+\frac{1}{p+2}} \gamma dt, \quad C \int_{S}^{T} E^{q+\frac{1}{p+2}} \gamma dt \leq C_{1} \int_{S}^{T} E^{q+1} \gamma dt.
\]
Let $\varepsilon$ be small enough such that
\[
C_{1} \varepsilon^{\frac{k+1}{p+2}} + C_{1} \varepsilon^{\frac{k}{p+2}} < \frac{1}{2}.
\]
Then, (3.13) implies that
\[
\int_{S}^{T} E^{q+1} \gamma dt \leq C_{2} E(S)^{q+\frac{1}{p} + \frac{q}{p+2}} + C_{2} E(S).
\]
Hence,
\[
\int_{S}^{T} E^{1+q} \gamma dt \leq C_{2} E(S), \quad T > S \geq 0. \tag{3.16}
\]

As for the case $k = 1$, we use (3.6), (3.11) and (3.12) to deduce that
\[
\int_{S}^{T} E^{q+1} \gamma dt \leq C E(S)^{q+\frac{1}{p} + \frac{q}{p+2}} + C E(S)^{1+q} + C_{2} E(S) + C_{2} \int_{S}^{T} E^{2(q+\frac{1}{p+2})} \gamma dt.
\]
Take
\[
q = \frac{p}{p+2}, \tag{3.17}
\]
which satisfies
\[
q + 1 = 2 \left( q + \frac{1}{p+2} \right).
\]
Then, we can choose a small $\varepsilon$ ensuring that (3.16) holds as well in this case.
Finally, notice that
\[
q + \frac{1}{2} + \frac{1}{p+2} = \frac{kp+k-1}{p+2} + \frac{1}{2} + \frac{1}{p+2} = \frac{k(p+1)}{p+2} + \frac{1}{2} > 1, \quad \forall \, k \geq 1,
\]
by (3.17) and (3.15). We obtain by (3.16),
\[
\int_{S}^{+\infty} E^{1+q} \gamma \, dt \leq C_2 E(S), \quad S \geq 0.
\]
Applying Lemma 3.1 with
\[
\phi(t) := \int_{0}^{t} \gamma(s) \, ds, \quad q := \frac{kp+k-1}{p+2}, \quad \omega := \frac{E(0)^q}{C_2},
\]
we have
\[
E(t) \leq E(0) \left( \frac{1 + q}{1 + qC_2^{-1} E(0)^q \int_{0}^{t} \gamma(s) \, ds} \right)^{\frac{1}{q}}
\leq \left( \frac{1 + q}{E(0)^{-q} + qC_2^{-1} \int_{0}^{t} \gamma(s) \, ds} \right)^{\frac{1}{q}}
\leq C_3 \left( \frac{1}{1 + \int_{0}^{t} \gamma(\tau) \, d\tau} \right)^{\frac{1}{q}}, \quad t \geq 1.
\]
This, combined with the fact that
\[
E(t) \leq E(0), \quad t \in [0,1],
\]
and (2.5), completes the proof.

4. Applications. Throughout this section, \( \Omega \subset \mathbb{R}^n \) (\( n \) a natural number) is a bounded open connected set with smooth boundary \( \partial \Omega = \Gamma \) and \( \nu \) is the unit outward normal on \( \partial \Omega \). Points in \( \Omega \) will be denoted by \( x \).

Our first example concerns a nonautonomous, nonlinear damped wave equation with the Neumann boundary condition.

**Example 4.1.** Let us consider the following problem:

\[
\begin{aligned}
&u_{tt}(t,x) - \Delta u(t,x) + \gamma(t) h(u_t) + f(u) = 0, \quad \text{in } [0, +\infty) \times \Omega, \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \\
&\frac{\partial u(t,x)}{\partial \nu} = 0, \quad \text{on } \Gamma \times (0, \infty).
\end{aligned}
\]

Assume that
\begin{itemize}
  \item \( \gamma \) is a locally absolutely continuous, bounded, and nonnegative function on \( \mathbb{R}^+ \) such that \( \gamma' \in L^1(\mathbb{R}^+) \), and
  \[
  \gamma(t) \geq c_0 (t+1)^{-\beta}, \quad \text{with some constant } \beta \in [0,1];
  \]
  \item \( h : \mathbb{R} \to \mathbb{R} \) is a Lipschitz continuous function satisfying
    \[
    h(s)s > 0 \quad \text{for } s \neq 0,
    \]
    \[
    c_1 |s|^k \leq |h(s)| \leq c_2 |s| \quad \text{for } |s| \leq 1, \quad \text{with some constant } k \geq 1,
    \]
\end{itemize}
and
\[ c_1 s^2 \leq h(s) s \leq c_2 s^2 \quad \text{for} \ |s| > 1; \] (4.3)

• \( f \in W^{1,1}_{\text{loc}}(\mathbb{R}) \), and for \( s \in \mathbb{R} \),
\[ c_1 |s|^{p+2} \leq f(s) s \leq c_2 |s|^{p+2}, \] (4.4)
\[ |f'(s)| \leq c_1 + c_2 |s|^p, \quad \text{with some positive constant} \ p \ \text{which satisfies} \ (n-2)p \leq 2; \] (4.5)

here \( c_0, c_1 \) and \( c_2 \) are positive constants.

We can rewrite (4.1) as an abstract problem of the type (1.1). In fact, let \( H = L^2(\Omega) \) be endowed with the usual inner product and norm, and define the operator \( A : D(A) \subset H \to H \) by
\[ D(A) = \left\{ u \in H^2(\Omega); \ \frac{\partial u}{\partial v} = 0 \text{ on} \ \partial \Omega \right\}, \]
\[ Av(x) = -\Delta v(x), \quad v \in D(A), \ x \in \Omega \ a.e. \]

It is clear that \( A \) is a self-adjoint nonnegative operator on \( H \), and \( H_1 := D(A^{1/2}) = H^{1}(\Omega) \).

Set
\[ g(v)(x) = h(v(x)), \quad x \in \Omega, \ v \in H. \]

Then, it is obvious that \( g \) is Lipschitz continuous on \( H \), and \( \langle g(v), v \rangle \geq 0 \) for \( v \in H \).

So we just need to show (2.6).

We observe from (4.2) that for \( v \in H \),
\[ h^2(v(x)) = |h^k(v(x))h(v(x))|^{\frac{1}{p+1}} \leq C \left( v(x)h(v(x)) \right)^{\frac{1}{p+1}, \}
\[ v^2(x) \leq C \left( |v(x)|^{\frac{1}{p+1}} |h(v(x))|^{\frac{1}{p+1}} \right)^2 = C \left( v(x)h(v(x)) \right)^{\frac{1}{p+1}} \]

(here and in the sequel, \( C > 0 \) denote a generic constant), whenever \( |v(x)| \leq 1. \) Moreover, by (4.3) we know that
\[ h^2(v(x)), \ v^2(x) \leq C v(x)h(v(x)), \quad \text{if} \ |v(x)| \geq 1. \]

Therefore,
\[ \|v\| + \|g(v)\| = \left( \int_{\Omega} |v(x)|^2 dx \right)^{1/2} + \left( \int_{\Omega} |h(v(x))|^2 dx \right)^{1/2} \]
\[ \leq C \left( \int_{\Omega} v(x)h(v(x))dx \right)^{1/2} + C \left( \int_{\Omega} |v(x)h(v(x))|^{\frac{2}{p+1}} dx \right)^{\frac{1}{2}} \]
\[ \leq C\langle g(v), v \rangle^{1/2} + C \left( \int_{\Omega} |v(x)h(v(x))|^{\frac{2(p+1)}{p+2}} dx \right)^{\frac{1}{2}} \]
\[ = C\langle g(v), v \rangle^{1/2} + C\langle g(v), v \rangle^{\frac{1}{p+1}}. \]

Noting that
\[ \langle g(v), v \rangle^{1/2} \leq \left( \|g(v)\| \|v\| \right)^{\frac{1}{2(p+1)}} \langle g(v), v \rangle^{\frac{1}{p+1}} \leq L_h^{\frac{1}{p+1}} \|v\|^{\frac{p+1}{p+2}} \langle g(v), v \rangle^{\frac{1}{p+1}}, \]

where \( L_h \) is the Lipschitz constant of \( h \), we see that (2.6) is satisfied.
For the nonlinear term $f(u)$, we define

$$F(v) := \int_{\Omega} \left( \int_{0}^{v(x)} f(s) ds \right) dx, \quad v \in H_{1}.$$ 

Apparently $F$ is nonnegative and differentiable at any point $u \in H_{1}$, and

$$(\nabla F(v))(x) = f(v(x)), \quad v \in H_{1}.$$ 

From (4.5) we infer that for $v, w \in H_{1}$,

$$\| \nabla F(v) - \nabla F(w) \|^2 = \int_{\Omega} \left( f(v(x)) - f(w(x)) \right)^2 dx$$

$$\leq C \int_{\Omega} \left( |v(x)| + |w(x)| \right)^{2p} |v(x) - w(x)|^2 dx + C\|v - w\|^{2}_{L^{2}}$$

$$\leq C \left( 1 + \|v\|^{2p}_{L^{2}(p+1)} + \|w\|^{2p}_{L^{2}(p+1)} \right) \cdot \|v - w\|^{2}_{L^{2}(p+1)} \quad \text{(by Holder’s inequality)}$$

$$\leq C \left( 1 + \|v\|^{2p}_{H^{1}} + \|w\|^{2p}_{H^{1}} \right) \cdot \|v - w\|^{2}_{H^{1}} \quad \text{(by Sobolev’s inequality)} \quad (4.6)$$

This shows local Lipschitz continuity of $\nabla F$. Moreover, by (4.4) we have

$$\frac{c_{1}}{p + 2} |r|^{p+2} \leq \int_{0}^{r} f(s) ds \leq \frac{c_{2}}{p + 2} |r|^{p+2}, \quad r \in R,$$

and so

$$\frac{c_{1}}{p + 2} \int_{\Omega} |v(x)|^{p+2} dx \leq F(v) \leq \frac{c_{2}}{p + 2} \int_{\Omega} |v(x)|^{p+2} dx, \quad v \in H_{1}.$$ 

Thus, (2.4) and (2.5) are easily verified. It is obvious that $\gamma(t) := (t + 1)^{-\beta}$ satisfies condition (2.7).

Therefore, by Proposition 2.3 and Theorem 2.4 we conclude that for $(u_{0}, u_{1}) \in H^{1}(\Omega) \times L^{2}(\Gamma)$, the problem (4.1) admits a unique mild solution

$$u \in C([0, \infty); H^{1}(\Omega)) \cap C^{1}([0, \infty); L^{2}(\Omega)), $$

such that the energy

$$E_{u}(t) := \frac{1}{2} \int_{\Omega} (|u_{t}|^{2} + |\nabla u|^{2}) dx + \frac{1}{p + 2} \int_{\Omega} |u(x)|^{p+2} dx$$

and the solution satisfy the following decay estimates that for $t \geq 0$,

$$E(t), \quad \|u(t)\|^{p+2} \leq \frac{M(E(0))}{(t + 1)^{(1-\beta)/(p+2)/(kp+k-1)}} \quad \text{if } \beta \in [0, 1), \quad (4.7)$$

$$E(t), \quad \|u(t)\|^{p+2} \leq \frac{M(E(0))}{(\ln t + 1)^{(p+2)/(kp+k-1)}} \quad \text{if } \beta = 1, \quad (4.8)$$

with some positive function $M$ on $R^{+}$ that is bounded on bounded sets.
Example 4.2. Consider the initial-boundary value problem for an integro-differential damped hyperbolic equation

\[
\begin{aligned}
  u_{tt}(t, x) - \Delta u(t, x) - \lambda_1 u(t, x) + \gamma(t) h(u_t) + \left( \int_{\Omega} |u(t, x)|^2 dx \right)^{p/2} \\
  u(t, x) = 0, \text{ in } [0, +\infty) \times \Omega, \\
  u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega, \\
  u(t, x) = 0, \quad \text{on } \Gamma \times (0, \infty),
\end{aligned}
\]

(4.9)

where \( p \geq 1 \), \( \lambda_1 \) is the first eigenvalue of the negative Dirichlet-Laplacian on \( \Omega \), and \( \gamma, h \) are as in Example 4.1.

Take \( H = L^2(\Omega) \), and define operator \( A \) by

\[
Av(x) = -\Delta v(x) - \lambda_1 v(x), \quad x \in \Omega \text{ a.e.}
\]

with \( v \in D(A) = H^2(\Omega) \cap H^1_0(\Omega) \). It is known that \( A \) is a self-adjoint nonnegative operator on \( H \), and \( H_1 := D(A^{1/2}) = H^1_0(\Omega) \).

Let \( g \) be as in Example 4.1. Also, we set, for \( v \in H_1 \),

\[
F(v) := \frac{1}{p+2} \left( \int_{\Omega} |v(x)|^2 dx \right)^{p/2} = \frac{1}{p+2} \|v\|^{p+2}.
\]

Then it is obvious that

\[
[\nabla F(v)](x) = \left( \int_{\Omega} |v(x)|^2 dx \right)^{p/2} v(x) = \|v\|^p v(x).
\]

Moreover, for \( v, w \in H_1 \),

\[
\|\nabla F(v) - \nabla F(w)\|^2 = \int_{\Omega} \left( \|v\|^p v(x) - \|w\|^p w(x) \right)^2 dx \leq \|v\|^2 \left( \|v\|^p - \|w\|^p \right)^2 + \|v\|^p \|v - w\|^2 \leq \left[ \|v\|^p + p^2 (\|v\| + \|w\|)^{2(p-1)} \right] \|v - w\|^2.
\]

So \( \nabla F \) is Locally Lipschitz continuous.

Consequently, an application of Proposition 2.3 and Theorem 2.4 shows that the energy

\[
E_u(t) := \frac{1}{2} \int_{\Omega} \left( |u_t|^2 + |
abla u|^2 - \lambda_1 |u|^2 \right) dx + \frac{1}{p+2} \left( \int_{\Omega} |u(t, x)|^2 dx \right)^{p/2}
\]

and the solution for problem (4.9) satisfy the same estimates as in (4.7) and (4.8).

Our last example is related to a plate model, subject to guided boundary conditions.

Example 4.3. Of concern is

\[
\begin{aligned}
  u_{tt}(t, x) + \Delta^2 u(t, x) + \gamma(t) h(u_t) + f(u) = 0, \quad \text{in } [0, +\infty) \times \Omega, \\
  u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega, \\
  \frac{\partial u(t, x)}{\partial \nu} = \frac{\partial (\Delta u(t, x))}{\partial \nu} = 0, \quad \text{on } \Gamma \times (0, \infty),
\end{aligned}
\]

(4.10)
where $\gamma$, $h$ and $f$ are as in Example 4.1, with $(n-2)p \leq 2$ replaced by $(n-4)p \leq 4$.

Let $H = L^2(\Omega)$, and consider the operator $A$ defined by

$$D(A) = \left\{ u \in H^4(\Omega); \frac{\partial u}{\partial \nu} = \frac{\partial (\Delta u(t, x))}{\partial \nu} = 0 \text{ on } \partial \Omega \right\},$$

$Au(x) = \Delta^2 v(x), \quad v \in D(A), \ x \in \Omega \text{ a.e.}$

We know that $A$ is a nonnegative selfadjoint operator on $H$, and $H_1 := D(\sqrt{A}) = \left\{ u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \right\}$. Define $g, F$ as in Example 4.1. We observe (as in (4.11)) that for $v, w \in H_1$,

$$\| \nabla F(v) - \nabla F(w) \|^2 \leq C \left( 1 + \| v \|_{L^{2p}(\Omega)}^{2p} + \| w \|_{L^{2p}(\Omega)}^{2p} \right) \cdot \| v - w \|_{H^2}^2.$$

Therefore, the conditions stated in Proposition 2.3 and Theorem 2.4 are satisfied. Thus, for $(u_0, u_1) \in H^2(\Omega) \times L^2(\Omega)$ with \( \frac{\partial u_0}{\partial \nu} \big|_{\partial \Omega} = 0 \), the problem (4.10) admits a unique mild solution

$$u \in C([0, \infty); H^2(\Omega)) \cap C^1([0, \infty); L^2(\Omega)),$$

such that its energy

$$E_u(t) := \frac{1}{2} \int_{\Omega} \left( |u_t|^2 + |\Delta u|^2 \right) dx + \frac{1}{p + 2} \int_{\Omega} |u(x)|^{p+2} \, dx$$

and solution decay as in (4.7) and (4.8).

**Acknowledgments.** The authors would like to thank the reviewers very much for valuable comments and suggestions.

**REFERENCES**

[1] M. Daoulatli, Rates of decay for the wave systems with time-dependent damping, *Discrete Contin. Dyn. Syst.*, 31 (2011), 407–443.

[2] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Mathematics Studies, 108. North-Holland Publishing Co., Amsterdam, 1985.

[3] M. Ghisi, M. Gobbino and A. Haraux, Optimal decay estimates for the general solution to a class of semil-linear dissipative hyperbolic equations, *J. Eur. Math. Soc. (JEMS)*, 18 (2016), 1961–1982.

[4] M. Ghisi, M. Gobbino and A. Haraux, Finding the exact decay rate of all solutions to some second order evolution equations with dissipation, *J. Funct. Anal.*, 271 (2016), 2359–2395.

[5] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1985.

[6] A. Haraux, Slow and fast decay of solutions to some second order evolution equations, *J. Anal. Math.*, 95 (2005), 297–321.

[7] A. Haraux and M. A. Jendoubi, Asymptotics for a second order differential equation with a linear, slowly time-decaying damping term, *Evolution Equations and Control Theory*, 2 (2013), 461–470.

[8] A. Haraux and M. A. Jendoubi, *The Convergence Problem for Dissipative Autonomous Systems, Classical Methods and Recent Advances*, BCAM SpringerBriefs. Springer, Cham; BCAM Basque Center for Applied Mathematics, Bilbao, 2015.

[9] A. Haraux, P. Martinez and J. Vancostenoble, Asymptotic stability for intermittently controlled second-order evolution equations, *SIAM J. Control Optim.*, 43 (2005), 2089–2108.
[10] Z. Jiao and T.-J. Xiao, Convergence and speed estimates for semilinear wave systems with nonautonomous damping, *Math. Methods Appl. Sci.*, 39 (2016), 5465–5474.

[11] K.-P. Jin, J. Liang and T.-J. Xiao, Coupled second order evolution equations with fading memory: Optimal energy decay rate, *J. Differential Equations*, 257 (2014), 1501–1528.

[12] P. Martinez, A new method to obtain decay rate estimates for dissipative systems, *ESAIM Control Optim. Calc. Var.*, 4 (1999), 419–444.

[13] P. Martinez, Precise decay rate estimates for time-dependent dissipative systems, *Israel J. Math.*, 119 (2000), 291–324.

[14] R. May, Long time behavior for a semilinear hyperbolic equation with asymptotically vanishing damping term and convex potential, *J. Math. Anal. Appl.*, 430 (2015), 410–416.

[15] M. Nakao, On the time decay of solutions of the wave equation with a local time-dependent nonlinear dissipation, *Adv. Math. Sci. Appl.*, 7 (1997), 317–331.

[16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences 44, Springer-Verlag, New York, 1983.

[17] T.-J. Xiao and J. Liang, Coupled second order semilinear evolution equations indirectly damped via memory effects, *J. Differential Equations*, 254 (2013), 2128–2157.

Received January 2019; 1st revision March 2019; 2nd revision April 2019.

E-mail address: tjxiao@fudan.edu.cn
E-mail address: 15110180021@fudan.edu.cn