ON THE RELATION BETWEEN ADM AND BONDI ENERGY-MOMENTA III – PERTURBED RADIATIVE SPATIAL INFINITY

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ABSTRACT. In a vacuum spacetime equipped with the Bondi’s radiating metric which is asymptotically flat at spatial infinity including gravitational radiation (Condition D), we establish the relation between the ADM total energy-momentum and the Bondi energy-momentum for perturbed radiative spatial infinity. The perturbation is given by defining the “real” time the sum of the retarded time, the Euclidean distance and certain function $f$.

1. INTRODUCTION

It is a fundamental problem in gravitational radiation what the relation is between the ADM energy-momentum and the Bondi energy-momentum. Under certain asymptotic flatness conditions at spatial infinity, it was shown the ADM energy-momentum at spatial infinity is the past limit of the Bondi energy-momentum in certain vacuum spacetimes [2, 7, 10, 11, 4, 13]. However, it is presumably believed the assumed asymptotic flatness at spatial infinity in all above works precludes gravitational radiation, at least near spatial infinity.

In [13], the second author introduced a weaker assumption of asymptotic flatness at spatial infinity in Bondi’s radiating spacetimes (Condition D). This condition can be viewed as Sommerfeld’s radiation condition at spatial infinity [13]. And it should not preclude gravitational radiation. Under this condition, it is found that the ADM total energy of a $t$-slice is no longer the past limit of the Bondi mass, and they differ by a quantity related to the news functions. In [8], the authors established the relation between the ADM total linear momentum and the past limit of the Bondi “momentum” under Condition D. It is surprising that in this case the second fundamental form $h$ on the $t$-slice falls off as $O(\frac{1}{r})$, however, some nice cancellation occurs and the ADM total linear momentum is still finite. The ADM total linear momentum at spatial infinity is no longer the past limit of the Bondi “momentum”, and the difference is calculated.

The definition of the “real” time $t$ is essential in Bondi’s radiating spacetimes and the spatial infinity is defined as a $t$ slice. In the works of [13, 8],

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the “real” time $t$ is defined as the sum of the retarded coordinate $u$ and the Euclidean distance $r$. However, this definition is rather restricted, it does not hold true even in the Schwarzschild spacetime. In this paper, we will perturb the “real” time $t$ by adding certain function $f$. The spatial infinity is also defined in terms of this new $t$-slice. We establish the relation between the ADM energy-momentum and the Bondi energy-momentum under Condition D.

We do not consider the polyhomogeneous Bondi expansions in the present paper, as studied in [6]. In [6] (Appendix D), the logarithmic singularities at null infinity can be removed in the axisymmetric case if the free function $\gamma_2(u, x^a)$ is chosen to be zero and $\gamma_{3, 1}(u_0, x^a)$ is chosen to be zero for some $u_0$. We believe this is a generic property that under certain class of Bondi’s radiating metrics which make the ADM total energy-momentum and the Bondi energy-momentum well-defined, the standard Bondi expansions without any logarithmic singularity at null infinity can always be achieved.

The paper is organized as follows: In Section 2, we state some well-known results of Bondi, van der Burg, Metzner and Sachs. In Section 3, we define the “real” time $t$ via adding a perturbation $f$. We study the asymptotic behaviors of the induced metric and the the second fundamental form of a $t$-slice. In Section 4, we derive the difference between the ADM total energy and the past limit of the Bondi mass. Under certain conditions, we can find a good perturbation such that the ADM total energy equals the past limit of the Bondi mass. In Section 5, we derive the difference between the ADM total linear momentum and the past limit of the Bondi “momentum”. We also find certain condition ensuring the existence of good perturbation such that the ADM total linear momentum equals the past limit of the Bondi “momentum”. However, we find, in general, it is impossible to perturb spatial infinity so that both the ADM total energy and the ADM total linear momentum are the past limit of the Bondi energy-momentum.

2. Bondi’s radiating vacuum spacetimes

The Bondi’s radiating vacuum spacetime $(L^{3,1}, \tilde{g})$ is a vacuum spacetime equipped with the following metric

$$\tilde{g} = \left( \frac{V}{r} e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta + r^2 e^{-2\gamma} W^2 \cosh 2\delta \right) du^2 - 2e^{2\beta} dudr$$

$$+ 2r^2 UW \sinh 2\delta dud\theta$$

$$- 2r^2 \left( e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta \right) dud\psi$$

$$- 2r^2 \left( e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta \right) \sin \theta dud\psi$$

$$+ r^2 \left( e^{2\gamma} \cosh 2\delta d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2 \theta d\psi^2 \right)$$

$$+ 2 \sinh 2\delta \sin \theta d\theta d\psi \right), \quad (2.1)$$
where $\beta, \gamma, \delta, U, V, W$ are functions of 

$$x^0 = u, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \psi,$$

$u$ is a retarded coordinate, $r$ is Euclidean distance, $\theta$ and $\psi$ are spherical coordinates, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$. We assume that $\tilde{g}$ satisfies the outgoing radiation condition.

The metric (2.1) was studied by Bondi, van der Burg, Metzner and Sachs in the theory of gravitational waves in general relativity [3, 9, 12]. They proved that the following asymptotic behavior holds for $r$ sufficiently large if the spacetime satisfies the outgoing radiation condition [12]:

\begin{align*}
\gamma &= \frac{c(u, \theta, \psi)}{r} + \frac{C(u, \theta, \psi) - \frac{1}{6} c^3 - \frac{2}{3} c d^2}{r^3} + O\left(\frac{1}{r^4}\right), \\
\delta &= \frac{d(u, \theta, \psi)}{r} + \frac{H(u, \theta, \psi) + \frac{1}{2} c^2 d - \frac{1}{3} d^3}{r^3} + O\left(\frac{1}{r^4}\right), \\
\beta &= \frac{-c^2 + d^2}{4r^2} + O\left(\frac{1}{r^4}\right), \\
U &= -\frac{l(u, \theta, \psi)}{r^2} + \frac{p(u, \theta, \psi)}{r^3} + O\left(\frac{1}{r^4}\right), \\
W &= -\frac{\bar{l}(u, \theta, \psi)}{r^2} + \frac{\bar{p}(u, \theta, \psi)}{r^3} + O\left(\frac{1}{r^4}\right), \\
V &= -r + 2M(u, \theta, \psi) + \frac{\bar{M}(u, \theta, \psi)}{r} + O\left(\frac{1}{r^2}\right),
\end{align*}

where

\begin{align*}
l &= c_2 + 2c \cot \theta + d_3 \csc \theta, \\
\bar{l} &= d_2 + 2d \cot \theta - c_3 \csc \theta, \\
p &= 2N + 3(cc_2 + dd_2) + 4(c^2 + d^2) \cot \theta - 2(c_3 d - cd_3) \csc \theta, \\
\bar{p} &= 2P + 2(c_2 d - cd_2) + 3(cc_3 + dd_3) \csc \theta, \\
M &= N_2 + \cot \theta + P_3 \csc \theta - \frac{c^2 + d^2}{2} \\
&\quad - [(c_2)^2 + (d_2)^2] - 4(cc_2 + dd_2) \cot \theta - 4(c^2 + d^2) \cot^2 \theta - [(c_3)^2 + (d_3)^2] \csc^2 \theta \\
&\quad + 4(c_3 d - cd_3) \csc \theta \cot \theta + 2(c_3 d_2 - c_2 d_3) \csc \theta.
\end{align*}

(We denote $f_\nu = \frac{\partial f}{\partial x_\nu}$ for $\nu = 0, 1, 2, 3$ throughout the paper.) $M$ is the mass aspect and $c_{0, \nu}, d_{0, \nu}$ are the news functions and they satisfy the following equation [12]:

\begin{equation}
M_{\nu} = -\left[(c_0)^2 + (d_0)^2\right] + \frac{1}{2} \left(l_2 + l \cot \theta + \bar{l}_3 \csc \theta\right),
\end{equation}

(2.2)
Let \( N_{u_0} \) be a null hypersurface which is given by \( u = u_0 \) at null infinity. The Bondi energy-momentum of \( N_{u_0} \) is defined by \[ m_\nu(u_0) = \frac{1}{4\pi} \int_{S^2} M(u_0, \theta, \psi) n^\nu dS \]
where \( \nu = 0, 1, 2, 3 \), \( S^2 \) is the unit sphere,
\[ n^0 = 1, \quad n^1 = \sin \theta \cos \psi, \quad n^2 = \sin \theta \sin \psi, \quad n^3 = \cos \theta. \]
And \( m_0 \) is the Bondi mass, \( m_i \) is the Bondi momentum.

Denote by \( \{ \tilde{e}^i \} \) the coframe of the standard flat metric \( g_0 \) on \( \mathbb{R}^3 \) in polar coordinates,
\[ \tilde{e}^1 = dr, \quad \tilde{e}^2 = r d\theta, \quad \tilde{e}^3 = r \sin \theta d\psi. \]
Denote by \( \{ \tilde{\omega}_i \} \) the dual frame \((i = 1, 2, 3)\). The connection 1-form \( \{ \tilde{\omega}_{ij} \} \) is defined by
\[ d\tilde{e}^i = -\tilde{\omega}_{ij} \wedge \tilde{e}^j. \]
It is easy to find that
\[ \tilde{\omega}_{12} = -\frac{1}{r} \tilde{e}^2, \quad \tilde{\omega}_{13} = -\frac{1}{r} \tilde{e}^3, \quad \tilde{\omega}_{23} = -\frac{\cot \theta}{r} \tilde{e}^3. \]
The Levi-Civita connection \( \tilde{\nabla} \) of \( g_0 \) is given by
\[ \tilde{\nabla} \tilde{e}_i = -\tilde{\omega}_{ij} \otimes \tilde{e}_j. \]
We denote \( \tilde{\nabla}_i \equiv \tilde{\nabla}_{\tilde{e}_i} \) for \( i = 1, 2, 3 \) throughout the paper.

Define \( \mathcal{C}_{\{a_1, a_2, a_3\}} \) the space of smooth functions in the spacetime which satisfies the following asymptotic behavior at spatial infinity
\[ \mathcal{C}_{\{a_1, a_2, a_3\}} = \left\{ f : \lim_{r \to \infty} \lim_{u \to -\infty} r^{a_1} f = O(1), \quad \lim_{r \to \infty} \lim_{u \to -\infty} r^{a_2} \tilde{\nabla}_i f = O(1), \quad \lim_{r \to \infty} \lim_{u \to -\infty} r^{a_3} \tilde{\nabla}_i \tilde{\nabla}_j f = O(1) \right\}. \tag{2.3} \]
In \[ 3 \], the following four conditions were introduced:

**Condition A:** Each of the six functions \( \beta, \gamma, \delta, U, V, W \) together with its derivatives up to the second orders are equal at \( \psi = 0 \) and \( 2\pi \).

**Condition B:** For all \( u \),
\[ \int_0^{2\pi} c(u, 0, \psi) d\psi = 0, \quad \int_0^{2\pi} c(u, \pi, \psi) d\psi = 0. \]

**Condition C:** \( \gamma \in \mathcal{C}_{\{1, 2, 3\}}, \quad \delta \in \mathcal{C}_{\{1, 2, 3\}}, \quad \beta \in \mathcal{C}_{\{2, 3, 4\}}, \quad U \in \mathcal{C}_{\{2, 3\}}, \quad W \in \mathcal{C}_{\{2, 3, 4\}}, \quad V + r \in \mathcal{C}_{\{0, 1, 2\}}. \)

**Condition D:** \( \gamma \in \mathcal{C}_{\{1, 1, 1\}}, \quad \delta \in \mathcal{C}_{\{1, 1, 1\}}, \quad \beta \in \mathcal{C}_{\{2, 2, 2\}}, \quad U \in \mathcal{C}_{\{2, 2, 2\}}, \quad W \in \mathcal{C}_{\{2, 2, 2\}}, \quad V + r \in \mathcal{C}_{\{0, 0, 0\}}. \)
**Condition A** and **Condition B** ensure that the metric \((2.1)\) is regular, also ensure the following Bondi mass loss formula:

\[
\frac{d}{du} m_\nu = -\frac{1}{4\pi} \int_{S^2} [(c,0)^2 + (d,0)^2] n^\nu dS.
\]

**Condition C** ensures the Schoen-Yau’s positive mass theorem at spatial infinity. However, it precludes gravitational radiation. **Condition D** should include gravitational radiation. It indicates that, for \(r\) sufficiently large,

\[
\lim_{u \to -\infty} \left\{ M, c, d, M_0, c_0, d_0, M_A, c_A, d_A \right\} = O(1)
\]

where \(2 \leq A \leq 3\). We refer to [13] for some physical interpretation of **Condition D**.

### 3. Initial data sets

From now on, we assume the “real” time \(t\) is defined as

\[
t = u + r + f(r, \theta, \psi)
\]

for \(r\) sufficiently large, where \(f\) is smooth function which has the following asymptotic behavior

\[
f = a_1 \ln r + a_2 (\theta, \psi) + a_3 (r, \theta, \psi)
\]

for \(r\) sufficiently large, where \(a_1\) is constant, \(a_2, a_3\) are smooth functions which satisfy

\[
a_i \big|_{\psi=0} = a_i \big|_{\psi=2\pi}, \quad a_i, A \big|_{\psi=0} = a_i, A \big|_{\psi=2\pi}, \quad a_i, AB \big|_{\psi=0} = a_i, AB \big|_{\psi=2\pi}
\]

for \(i = 2, 3\), \(A, B = 2, 3\). Moreover,

\[
a_3 = O\left(\frac{1}{r^2}\right), \quad \nabla_k a_3 = O\left(\frac{1}{r^2}\right), \quad \nabla_l \nabla_k a_3 = O\left(\frac{1}{r^3}\right).
\]

for \(r\) sufficiently large.

Substituting (3.1) into (2.1), we obtain the the spacetime metric

\[
\tilde{g} = \tilde{g}_{tt} dt^2 + 2\tilde{g}_{ti} dt dx^i + \tilde{g}_{ij} dx^i dx^j
\]

\((1 \leq i, j \leq 3)\). An initial data set \((N_{t_0}, g, h)\) is a spacelike hypersurface in \(L^{3,1}\) which is given by \(\{ t = t_0 \}\). Here \(g\) is the induced metric of \(\tilde{g}\) and \(h\) is the second fundamental form. The lapse \(\mathcal{N}\) and the shift \(X_i\) \((i = 1, 2, 3)\) of the spacelike hypersurface \(N_{t_0}\) are

\[
\mathcal{N} = \left( -\tilde{g}^{tt} \right)^{-\frac{1}{2}} = \left( -\tilde{g}_{tt} + \tilde{g}_{ti} \tilde{g}_{ij} g^{ij} \right)^{-\frac{1}{2}}, \quad X_i = \tilde{g}_{ti}.
\]

The second fundamental form is then given by

\[
h_{ij} = \frac{1}{2\mathcal{N}} \left( \nabla_i X_j + \nabla_j X_i - \partial_i \tilde{g}_{ij} \right)_{t = t_0}.
\]
With the help of asymptotic behavior of $\beta, \gamma, \delta, U, V, W, f$ and Mathematica 5.0, we obtain the asymptotic expansion of $g_{ij}$ and $h_{ij}$:

\[
\begin{align*}
  g_{11} &= 1 + \frac{2M}{r} + \frac{1}{r^2} \left[ -\frac{c^2}{2} - \frac{d^2}{2} + \ell^2 + \bar{\ell}^2 + M - a_1^2 - 4Ma_1 \right] \\
  &\quad + O\left(\frac{1}{r^3}\right), \\
  g_{22} &= r^2 + 2rc + 2c^2 + 2d^2 + 2la_{2,2} - a_{2,2}^2 + O\left(\frac{1}{r}\right), \\
  g_{33} &= \left( r^2 - 2rc + 2c^2 + 2d^2 \right) \sin^2 \theta + 2\bar{\ell} \sin \theta a_{2,3} - a_{2,3}^2 + O\left(\frac{1}{r}\right), \\
  g_{12} &= -l + \frac{1}{r} \left( -2cl - 2d\bar{\ell} + p + a_1 l - a_{1}a_{2,2} - 2Ma_{2,2} \right) + O\left(\frac{1}{r^2}\right), \\
  g_{13} &= -\bar{\ell} \sin \theta + \frac{1}{r} \left[ \left( 2c\bar{\ell} - 2dl + \bar{p} \right) \sin \theta + \bar{\ell} \sin \theta a_1 - a_{1}a_{2,3} - 2Ma_{2,3} \right] \\
  &\quad + O\left(\frac{1}{r^2}\right), \\
  g_{23} &= 2rd \sin \theta + la_{2,3} + \bar{\ell} \sin \theta a_{2,2} - a_{2,2}a_{2,3} + O\left(\frac{1}{r}\right), \\
  h_{11} &= \frac{M_0}{r} + \frac{1}{r^2} \left[ 2M - MM_0 + \frac{cc_0}{2} + \frac{dd_0}{2} + bl_0 + M_0 \right] \\
  &\quad + a_1 - 3M_0a_1 + \bar{\ell}_0 \csc \theta a_{2,3} \right) + O\left(\frac{1}{r^3}\right), \\
  h_{22} &= -rc_0 - 2M + Mc_0 - 2cc_0 - 2dd_0 + l_2 - a_1 - a_{2,2} + c_0a_1 \\
  &\quad + O\left(\frac{1}{r}\right), \\
  h_{33} &= \left( rc_0 - 2M - Mc_0 - 2cc_0 - 2dd_0 + l \cot \theta + \bar{l}_3 \csc \theta \right) \sin^2 \theta \\
  &\quad - \sin^2 \theta a_1 - a_{2,33} - \sin \theta \cos \theta a_{2,2} - c_0 \sin^2 \theta a_1 + O\left(\frac{1}{r}\right), \\
  h_{12} &= \frac{1}{r} \left[ -M_2 - l + c_0l + d_0\bar{l} - (c_0 + M_0 - 1)a_{2,2} - d_0 \csc \theta a_{2,3} \right] \\
  &\quad + O\left(\frac{1}{r^2}\right), \\
  h_{13} &= \frac{1}{r} \left[ (-M_3 \csc \theta - \bar{l} - c_0\bar{l} + d_0l) \sin \theta + (c_0 - M_0 + 1)a_{2,3} \\
  &\quad - d_0 \sin \theta a_{2,2} \right] + O\left(\frac{1}{r^2}\right), \\
  h_{23} &= \left[ -rd_0 + Md_0 + \frac{1}{2} (l_2 - l \cot \theta + \bar{l}_3 \csc \theta) \right] \sin \theta - a_{2,23} \\
  &\quad + d_0 \sin \theta a_1 + \cot \theta a_{2,3} + O\left(\frac{1}{r}\right). 
\end{align*}
\]
The trace of the second fundamental form is

\[
tr_g(h) = \frac{M_0}{r} + \frac{1}{r^2} \left[ -2M - 3MM_0 + l_2 + l \cot \theta + \bar{l}_3 \csc \theta \\
+ \frac{cc,0}{2} + \frac{dd,0}{2} + l l_0 + \frac{M_0}{2} - (a_1 + 3M_0a_1 + a_{2,22} \\
+ \csc^2 \theta a_{2,33} + \cot \theta a_{2,2}) + \bar{l}_0 \csc \theta a_{2,3} \right] + O\left(\frac{1}{r^3}\right).
\]

4. ADM and Bondi total energy

In [13, 8], the authors derived the relation between the ADM total energy, the total linear momentum and the Bondi energy-momentum under Condition A, Condition B and Condition D and under the assumption \( t = u + r \). In this section, we study the relation between them under these conditions and under (3.1) with condition (3.2). Let Euclidean coordinates

\[
y^1 = r \sin \theta \cos \psi, \quad y^2 = r \sin \theta \sin \psi, \quad y^3 = r \cos \theta.
\]

In polar coordinates, the ADM total energy \( E \) and the ADM total linear momentum \( P_k \) are [1, 13, 8]

\[
E = \frac{1}{16\pi} \lim_{r \to \infty} \int_{S_r} \left[ \nabla^i g(\hat{e}_i, \hat{e}_j) - \nabla_1 tr_{g_0}(g) \right] \hat{e}^2 \wedge \hat{e}^3,
\]

\[
P_k = \frac{1}{8\pi} \lim_{r \to \infty} \int_{S_r} \left[ h \left( \frac{\partial}{\partial y^k}, \frac{\partial}{\partial r} \right) - g \left( \frac{\partial}{\partial y^k}, \frac{\partial}{\partial r} \right) tr_g(h) \right] \hat{e}^2 \wedge \hat{e}^3.
\]

**Theorem 4.1.** Let \( E(t_0) \) be the ADM total energy of the initial data set \((N_{t_0}, g, h)\) where \( t \) is given by (3.1) with condition (3.2). Under Condition A, Condition B and Condition D, we have

\[
E(t_0) = m_0(-\infty) + \frac{1}{2\pi} \lim_{u \to -\infty} \int_{0}^{\pi} \int_{0}^{2\pi} (cc,0 + dd,0) \sin \theta d\psi d\theta \\
+ \frac{1}{16\pi} \lim_{u \to -\infty} \int_{0}^{\pi} \int_{0}^{2\pi} (a_{2,2} \sin \theta l_0 + a_{2,3}l_0) d\psi d\theta.
\]
**Proof:** Under **Condition A**, **Condition B** and **Condition D**, we have, as $r \to \infty$, (or $u \to -\infty$),

\[
\mathcal{\hat{V}}^j g(\mathcal{\hat{e}}_1, \mathcal{\hat{e}}_j) - \mathcal{\hat{V}}^1 tr_{g_0}(g) = \mathcal{\hat{e}}_j\left(g(\mathcal{\hat{e}}_1, \mathcal{\hat{e}}_j)\right) - \mathcal{\hat{e}}_1 tr_{g_0}(g) - g(\mathcal{\hat{e}}_j, \mathcal{\hat{e}}_i)\mathcal{\hat{\omega}}_{ij}(\mathcal{\hat{e}}_j) - g(\mathcal{\hat{e}}_1, \mathcal{\hat{e}}_i)\mathcal{\hat{\omega}}_{ij}(\mathcal{\hat{e}}_j)
\]

\[
= \frac{4M}{r^2} - \frac{l_2}{r^2} - \frac{l \cot \theta}{r^2} - \frac{\bar{l}_3}{r^2 \sin \theta} - 4\mathcal{\hat{e}}_1 \left(\frac{c^2 + d^2}{r^2}\right) + \frac{a_{2,2} l_0 + a_{2,3} \csc \theta \bar{l}_{i,0}}{r^2} + O\left(\frac{1}{r^3}\right)
\]

Therefore the theorem is a direct consequence of integrating it over $S_r$ and using that for fixed $t = t_0$, $r \to \infty$ is equivalent to $u \to -\infty$. Q.E.D.

**Remark 4.1.** If $a_2$ is chosen to be a constant, the difference $E(t_0) - m_0(-\infty)$ is independent on the choice of $f$, which is invariant in the perturbed class \([3.2]\) that $a_1$, $a_2$ are constant.

The following theorem indicates that, under certain conditions, we can choose suitable $f$ such that, at spatial infinity defined by \([3.2]\), the ADM total energy is the past limit of the Bondi mass.

**Theorem 4.2.** Suppose that **Condition A**, **Condition B** and **Condition D** hold in the Bondi’s radiating spacetime \([2.1]\). If

\[
\lim_{u \to -\infty} c_0|_{\theta=0} = \lim_{u \to -\infty} c_0|_{\theta=\pi} = 0, \quad (4.1)
\]

\[
\lim_{u \to -\infty} d_{30}|_{\theta=0} = \lim_{u \to -\infty} d_{30}|_{\theta=\pi} = 0, \quad (4.2)
\]

\[
\lim_{u \to -\infty} \left[M_0 + (c_0)^2 + (d_0)^2\right] \neq 0, \quad (4.3)
\]

then there exists $f_0$ such that

\[
E_{f_0}(t_0) = m_0(-\infty)
\]

where $E_{f_0}(t_0)$ is the ADM total energy of the initial data set $(N_{t_0}, g, h)$ where $t$ is given by \([3.1]\) and $f = f_0$ given by \([3.2]\) with

\[
a_2 = \lim_{u \to -\infty} \frac{4cc_0 + 4dd_0}{M_0 + (c_0)^2 + (d_0)^2}. \quad (4.4)
\]
Proof: Note that

\[
\lim_{u \to -\infty} \int_0^{\pi} a_{2,2} \sin \theta l_0 d\theta = \lim_{u \to -\infty} a_2 \left( c_{20} \sin \theta + 2c_{0,0} \cos \theta + d_{30} \right) \bigg|_0^\pi \\
- \lim_{u \to -\infty} \int_0^{\pi} a_2 \left( \sin \theta l_0 \right)^2 d\psi d\theta \\
= - \lim_{u \to -\infty} \int_0^{\pi} a_2 \left( \cos \theta l_0 + \sin \theta l_{02} \right) d\theta,
\]

\[
\lim_{u \to -\infty} \int_0^{2\pi} a_{2,3} \bar{l}_{0} d\psi = \lim_{u \to -\infty} a_2 \left( d_{20} + 2d_{0,0} \cot \theta - c_{30} \csc \theta \right) \bigg|_0^{2\pi} \\
- \lim_{u \to -\infty} \int_0^{2\pi} a_2 \bar{l}_{03} d\psi \\
= - \lim_{u \to -\infty} \int_0^{2\pi} a_2 \bar{l}_{03} d\psi.
\]

We obtain

\[
\lim_{u \to -\infty} \int_0^{\pi} \int_0^{2\pi} (a_{2,2} \sin \theta l_0 + a_{2,3} \bar{l}_0) d\psi d\theta \\
= - \lim_{u \to -\infty} \int_0^{\pi} \int_0^{2\pi} a_2 \left( l_{2,0} + \cot \theta + \bar{l}_{03} \csc \theta \right) \sin \theta d\psi d\theta \\
= -2 \lim_{u \to -\infty} \int_0^{\pi} \int_0^{2\pi} a_2 \left[ M_0 + (c_{0,0})^2 + (d_{0,0})^2 \right] \sin \theta d\psi d\theta.
\]

Therefore the theorem is a direct consequence of Theorem 4.1. Q.E.D.

5. ADM and Bondi total momenta

In this section, we study the relation between the ADM total linear momentum and the Bondi momentum.

**Theorem 5.1.** Let \( P_k(t_0) \) be the ADM total linear momentum of the initial data set \((N_{t_0}, g, h)\) where \( t \) is given by \( 3.1 \) with condition \( 3.2 \). Under Condition A, Condition B and Condition D, we have

\[
P_k(t_0) = m_k(-\infty) + \frac{1}{8\pi} \lim_{u \to -\infty} \int_0^{\pi} \int_0^{2\pi} \bar{P}_k d\psi d\theta
\]
for \( k = 1, 2, 3 \), where

\[
\mathcal{P}_1 = \left[(c_0 + M_0) \cos \theta \cos \psi - d_0 \sin \psi\right] (l - a_{2,2}) \sin \theta + \left[(c_0 - M_0) \sin \psi + d_0 \cos \theta \cos \psi\right] (l \sin \theta - a_{2,3}),
\]
\[
\mathcal{P}_2 = \left[(c_0 + M_0) \cos \theta \sin \psi + d_0 \cos \theta \cos \psi\right] (l - a_{2,2}) \sin \theta - \left[(c_0 - M_0) \cos \psi - d_0 \cos \theta \sin \psi\right] (l \sin \theta - a_{2,3}),
\]
\[
\mathcal{P}_3 = -(c_0 + M_0) (l - a_{2,2}) \sin^2 \theta - d_0 (l \sin \theta - a_{2,3}) \sin \theta.
\]

**Proof**: Denote \( \mathbb{K}_k = h(\frac{\partial}{\partial y}, \frac{\partial}{\partial r}) - g(\frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}) \text{tr}_3(h) \). Using the asymptotic expansions of \( g_{ij} \) and \( h_{ij} \), we obtain

\[
\mathbb{K}_1 = \frac{1}{r^2} \left\{ 4M \sin \theta \cos \psi - M_2 \cos \theta \cos \psi + M_3 \csc \theta \sin \psi - l_2 \sin \theta \cos \psi - l_3 \cos \psi + \left[(c_0 + M_0 - 2) \cos \theta \cos \psi - d_0 \sin \psi\right] (l - a_{2,2}) + \left[(c_0 - M_0 + 1) \sin \psi + d_0 \cos \theta \cos \psi\right] (l - \csc \theta a_{2,3}) + 2 \sin \theta \cos \psi a_1 + \csc \theta \cos \psi a_{2,3,3} + \sin \theta \cos \psi a_{2,22} \right\} + O \left( \frac{1}{r^3} \right),
\]
\[
\mathbb{K}_2 = \frac{1}{r^2} \left\{ 4M \sin \theta \sin \psi - M_2 \cos \theta \sin \psi - M_3 \csc \theta \cos \psi - l_2 \sin \theta \sin \psi - l_3 \sin \psi + \left[(c_0 + M_0 - 2) \cos \theta \sin \psi + d_0 \cos \psi\right] (l - a_{2,2}) - \left[(c_0 - M_0 + 1) \cos \psi - d_0 \cos \theta \sin \psi\right] (l - \csc \theta a_{2,3}) + 2a_1 \sin \theta \sin \psi + \csc \theta \sin \psi a_{2,3,3} + \sin \theta \sin \psi a_{2,22} \right\} + O \left( \frac{1}{r^3} \right).
\]
\[
\mathbb{K}_3 = \frac{1}{r^2} \left\{ 4M \cos \theta - M_2 \sin \theta - l_2 \cos \theta - l_3 \cot \theta - \left[(c_0 + M_0 - 2) \sin \theta + \csc \theta\right] l - d_0 (l \sin \theta - a_{2,3}) + 2a_1 \cos \theta + \cos \theta \cot \theta a_{2,22} + \cot \theta \csc \theta a_{2,3,3} + (c_0 + M_0 - 1) \sin \theta a_{2,2} \right\} + O \left( \frac{1}{r^3} \right).
\]

Since \( a_1 \) is constant,

\[
\int_{S^2} a_1 n^i dS = 0
\]

for \( i = 1, 2, 3 \), then the theorem is a direct consequence of integrating \( \mathbb{K}_k \) over \( S_r \) and using that for fixed \( t = t_0 \), \( r \to \infty \) is equivalent to \( u \to -\infty \). Q.E.D.
Remark 5.1. If $a_2$ is chosen to be a constant, then the difference $P_k(t_0) - m_k(-\infty)$ are independent on the choice of $f$, which are invariant in the perturbed class (3.2) that $a_1, a_2$ are constant.

Theorem 5.2. Suppose that Condition A, Condition B and Condition D hold in the Bondi’s radiating spacetime (2.1). If
\[
\lim_{u \to -\infty} (\bar{l}_2 + \bar{l} \cot \theta - l_3 \csc \theta) = 0,
\]
then there exists $f_1$ such that
\[
P_{k,f_1}(t_0) = m_k(-\infty)
\]
where $P_{k,f_1}(t_0)$ is the ADM total linear momentum of the initial data set $(N_{t_0}, g, h)$ where $t$ is given by (3.1) and $f = f_1$ given by (3.2) with $a_2$ satisfying
\[
a_{2,2} = l, \quad a_{2,3} = \bar{l} \sin \theta.
\]

Proof: The condition (5.1) ensures the existence of (5.2). Therefore the theorem follows.

Remark 5.2. From Theorem 4.2 and Theorem 5.2, we know that, in general, it is impossible to perturb spatial infinity via the function $f$ so that both the ADM total energy and the ADM total linear momentum are the past limit of the Bondi energy-momentum.

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