Hilbert Space Quantum Mechanics Is Noncontextual

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Abstract

It is shown that quantum mechanics is noncontextual if quantum properties are represented by subspaces of the quantum Hilbert space (as proposed by von Neumann) rather than by hidden variables. In particular, a measurement using an appropriately constructed apparatus can be shown to reveal the value of an observable \(A\) possessed by the measured system before the measurement took place, whatever other compatible ([\(B, A\] = 0) observable \(B\) may be measured at the same time.

1 Introduction

The question of whether quantum mechanics is in some sense “contextual” seems to have first been raised by Bell, Sec. 5 of \([1]\), in the course of a discussion of the possible form which might be taken by hidden variables in quantum theory, i.e., variables which are in addition to the Hilbert space customarily used to discussing quantum mechanics. The basic idea, not altogether easy to extract from Bell’s paper, is the following (Ch. 7 of \([2]\) or Sec. VII of \([3]\)): Let \(A, B, C\) be three observables, represented by Hermitian operators on a finite-dimensional Hilbert space, for some system which is to be measured, and suppose that \(A\) commutes with \(B\) and with \(C\), but \(B\) and \(C\) do not commute with each other. Then according to standard quantum mechanics it is possible in principle to measure \(A\) and \(B\) together in a single experiment, and it is possible to measure \(A\) and \(C\) together, though all three cannot be measured simultaneously. Would a measurement of \(A\) when carried out along with a measurement of \(B\) yield the same result (for \(A\)) as a measurement of \(A\) carried out along with a measurement of \(C\)?

The answer would seem to be an obvious “Yes” if the measurement, however it is carried out, reveals a value of \(A\) that was there before the measurement took place. We shall argue that this is, indeed, correct if the apparatus has been designed and operated by a competent experimentalist. However, a substantial literature has accumulated which would throw doubt on this or suggest the opposite on the basis of various arguments related to the Kochen-Specker\(^2\) theorem \([5]\). A recent discussion by Hermens \([6]\) provides an extensive list of references to work published since 2000, to which could be added much additional material that has appeared in the last two years. How have so many come to this conclusion? By adopting, we shall argue, the view that the real world is classical, contrary to all we have learned from the development of quantum mechanics in the twentieth century. In particular, discussions couched in terms of hidden variables typically assume that they are classical rather than the sort of thing one might expect in a quantum mechanical world.

Such confusion is not unrelated to the failure of introductory quantum mechanics textbooks and courses to provide a proper discussion of what actually happens during a quantum measuring process. Rather than explaining how a piece of apparatus interacts with and thereby extracts and amplifies information

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1The term “contextual” as used here should not be confused with the idea of a “dependent” or “contextual” event as employed in Ch. 14 of \([4]\). The latter makes no reference to measurements, and is a general feature of Hilbert space quantum mechanics when applied to multiple systems described by a tensor product.

2As essentially the same result was obtained and published slightly earlier by Bell in \([1]\), it would be appropriate to refer to the “Bell-Kochen-Specker” theorem. However, in the rest of this paper we follow the common usage in the philosophical literature.
about the measured system, students are given calculational tools, such as wave function collapse, whose connection with the quantities supposedly measured is at best vague and sometimes quite misleading. When Bell introduced the idea of contextuality in 1966 there was no satisfactory theory of quantum measurement processes, and so it was reasonable to ask whether measurement outcomes revealed values of some sort of hidden variable(s). But developments since then, beginning in the 1980s [7–9], allow measurements and other quantum processes to be described in a fully consistent way using only the quantum Hilbert space along with a stochastic quantum dynamics based upon extensions of Born’s rule. A very detailed presentation of this (consistent or decoherent) histories approach will be found in [4], and shorter treatments in [10,12]. Various conceptual issues and criticisms are discussed in [13].

The histories approach is not the only interpretation of quantum mechanics to employ the Hilbert space without any additional hidden variables to represent physical properties. The alternatives include the Everett or “many worlds” interpretation [14–17], various modal interpretations [18], and the spontaneous collapse proposal frequently referred to as GRW [19,20]. So far as the author is aware, none of these interpretations has been developed to the point where it provides a precise and coherent explanation of how a physical measurement process functions, and in particular how the outcome (the traditional “pointer position”) of a measurement is related to the prior state of affairs which the competent experimentalist designed his equipment to study. Thus one must await further developments of these approaches by their proponents before assessing whether they can provide insight into measurement contextuality.

The remainder of this paper is organized as follows. Section 2 is a review of quantum properties and variables as represented in a Hilbert space. Measurements are the subject of Sec. 3. They are discussed using a quite general and fully quantum mechanical measurement model involving no hidden variables, with the conclusion that an apparatus properly designed to reveal the value that the observable A had just before the measurement took place will do just that, and this remains true if the apparatus also measures a compatible (i.e., [B, A] = 0) observable B along with A. It is argued in Sec. 4 that this settles the contextuality question for Hilbert space quantum mechanics. The final Sec. 5 summarizes the result and considers its relation to hidden variables approaches. Some comments on the single framework rule of histories quantum mechanics have been placed in an appendix.

2 Quantum Properties and Variables

We begin with a brief review of how the quantum properties of a system s are represented using a Hilbert space $\mathcal{H}_s$ which (as in Bell’s original discussion) can be conveniently assumed to be finite dimensional. An observable or quantum variable is represented by a Hermitian operator $A = A^\dagger$ on $\mathcal{H}_s$ with spectral representation

$$A = \sum_\alpha a_\alpha P_\alpha,$$

where the $\{a_\alpha\}$ are the eigenvalues of $A$, and the $\{P_\alpha\}$ a collection of projectors that form a decomposition of the identity $I$:

$$I = \sum_\alpha P_\alpha, \quad P_\alpha = P_\alpha^\dagger = P_\alpha^2, \quad P_\alpha P_\alpha' = \delta_{\alpha\alpha'} P_\alpha.$$

We assume that $a_\alpha \neq a_{\alpha'}$ whenever $\alpha \neq \alpha'$, so the decomposition $\{P_\alpha\}$ is uniquely determined by $A$.

A projector $P = P_\alpha^\dagger = P_\alpha^2$ projects onto a linear subspace of $\mathcal{H}_s$ and there is a one-to-one correspondence between projectors and subspaces, so one can speak either of a collection of subspaces or of projectors; we shall generally employ the latter. Following von Neumann, Sec. III.5 of [21], we think of a subspace, or equivalently the associated projector, as the mathematical representation of a quantum property, the quantum analog of a collection of points $E$ in a classical phase space representing a classical property such as “the energy is less than 2 Joules.” The quantum analog of a single point in a classical phase space is a one-dimensional subspace (a ray) consisting of complex multiples of any of its nonzero elements.

Thus for any observable $A$ there is a unique subspace $P_\alpha$ of $\mathcal{H}_s$ where $A$ takes on its eigenvalue $a_\alpha$; it consists of all the eigenvectors of $A$ having this eigenvalue, together with the zero vector. In classical mechanics if a system is described by a point in the phase space inside the set $E$ corresponding to some property we say that the system has or possesses the property $E$. Similarly, in quantum theory it makes sense to say that if a system is described by a nonzero $|\psi\rangle$ in the subspace $P_\alpha$, which is to say

$$P_\alpha |\psi\rangle = |\psi\rangle,$$
then the system has the property $P_\alpha$. (The reason for requiring that $|\psi\rangle$ be nonzero is that the zero ket, which lies in every subspace, is the quantum analog of the the empty set in a classical phase space: it represents a proposition or property which is never true.) If the quantum system is described by a density operator $\rho$ such that

$$P_\alpha \rho = \rho = \rho P_\alpha = P_\alpha \rho P_\alpha,$$

where any one of the equalities implies the others, it again makes sense to say that the system has the property $P_\alpha$. And when the system has the property $P_\alpha$, the observable $A$ has the value $a_\alpha$.

The decomposition $\{P_\alpha\}$ is the quantum analog of a coarse graining of the classical phase space into a collection of nonoverlapping cells, and represents a quantum sample space in the sense employed in probability theory: a set of mutually-exclusive properties ("events"), one and only one of which is true, i.e., actually occurs, at a particular time. Since $A$ is, in effect, a real-valued function on this sample space it can be thought of as a quantum analog of a "random variable" in probability theory, or of a real-valued function on the classical phase space. Thus it is appropriate to refer to a Hermitian operator $A$ as a "physical variable," in addition to using the traditional term "observable."

The quantum-classical analogy cannot be pushed too far, since quantum theory allows for noncommuting observables, whereas in classical mechanics all physical variables (functions on the phase space) commute. We shall call the observables $A$ and $B$ compatible when $[A,B] = 0$; otherwise they are (mutually) incompatible. Similarly, two projectors $P$ and $Q$ are compatible if $[P,Q] = 0$, in which case the product $PQ = QP$ is itself a projector onto a subspace which represents the conjunction $P$ AND $Q$ of the properties $P$ and $Q$. Otherwise they are incompatible, and defining the conjunction $P$ AND $Q$ is problematical. In traditional quantum logic, descending from the work of Birkhoff and von Neumann \[22\], $P$ AND $Q$ is the property associated with the subspace which is the intersection of the two subspaces corresponding to $P$ and $Q$. This, however, requires the use of new logical rules to avoid falling into difficulties—for a simple example see Sec. 4.6 of \[4\]. But the main complaint about quantum logic is that it has not resolved the major conceptual difficulties of quantum mechanics \[23\]-\[24\]. In the histories approach one avoids these difficulties by insisting that $P$ AND $Q$ only makes sense, can only be discussed, when the projectors commute. This is an example of the single framework rule, a principle which can be used to resolve numerous quantum paradoxes, as discussed in Chs. 20 to 25 of \[4\]. See the appendix for some comments on this rule.

Consider an observable $B$ that is compatible with $A$, with a spectral representation

$$B = \sum_\beta b_\beta Q_\beta,$$

where again the decomposition of the identity $\{Q_\beta\}$ is chosen so that $b_\beta \neq b_\beta'$ whenever $\beta \neq \beta'$. The fact that $[A,B] = 0$ implies that $[P_\alpha, Q_\beta] = 0$ for every $\alpha$ and $\beta$, and consequently there is a third decomposition of the identity $\{R_j\}$, the common refinement of $\{P_\alpha\}$ and $\{Q_\beta\}$, consisting of all nonzero projectors of the form $P_\alpha Q_\beta$, such that

$$A = \sum_j a_j R_j, \quad B = \sum_j b_j R_j.$$

The $a_j$ in (6) are of course eigenvalues of $A$, but now two or more of them may have the same value, in contrast to (4), which is the reason for using a distinct subscript; the same comment applies to the $b_j$. Because it consists of products of the form $P_\alpha Q_\beta$, the decomposition $\{R_j\}$ is the smallest collection of projectors such that both $A$ and $B$ can be written in the form (6). If $|\psi\rangle$ belongs to a subspace $R_j$, then since $R_j$ is itself the product $P_\alpha Q_\beta$ for some specific choice of $\alpha$ and $\beta$, it follows that if the quantum system under discussion has the property $R_j$ it also has the property $P_\alpha$ in the sense that (4) is satisfied, and likewise the (compatible) property $Q_\beta$. This is obvious for the analogous situation of a classical phase space, but in the quantum case it only makes sense when all the projectors that we are considering, the $\{P_\alpha\}$, $\{Q_\beta\}$, and $\{R_j\}$, commute with each other; as noted above there are logical difficulties in the noncommuting case.

It should be noted that quantum physical variables (observables) and properties as defined above apply not only to microscopic quantities, but also to macroscopic objects, such as laboratory equipment. There is no experimental evidence for a "classical" world to which quantum principles do not apply. Quantum theory is routinely used when discussing properties of the solid materials out of which experimental apparatus is constructed, the properties of stars, etc. So far as we know at present, it applies "from the quarks to the quasars." To be sure, there are excellent reasons why engineers discuss the motion of automobiles using classical mechanics rather than quantum theory, but this is a practical matter, not one of principle. We now
understand, at least in general terms, how in suitable circumstances classical physics emerges as a good approximation to a more fundamental quantum physics \cite{25,26}, and this justifies the fully quantum mechanical description given below of a physical measurement carried out by a macroscopic piece of apparatus.

3 Measurements

3.1 Introduction

The most direct approach to determining whether quantum theory is or is not contextual is to analyze the process that goes on in a quantum measurement. A measurement, as that term is ordinarily employed in physics, means determining one or more properties possessed by a system at a time just preceding that at which the measurement was carried out. It is in this sense that experimental particle physicists can speak, as they do, of detecting neutrinos arriving from the sun, or measuring the energy of a particle shower that has come to rest in their instrument. The reason for referring to the time just before the measurement is carried out is that the measuring process itself may change the property in question, and sometimes, as in the case of photons, the object being measured actually disappears. There is an important difference between a measurement and a preparation in which the instrument and setup tells one something about the properties of the prepared system after the preparation takes place. Considerable confusion is created in discussions of quantum foundations, and in the minds of students studying quantum theory for the first time, by the common failure to make a clear distinction between measurement and preparation. Part of this goes back to von Neumann’s treatment of measurements, Ch. VI of \cite{21}, in which he employed a very special model that in effect combines a measurement with a preparation. Whatever the utility of this model when quantum theory was first developed, greater clarity is possible nowadays if the distinction between these processes is kept in mind \cite{13}.

3.2 Measurement model

We use a model of the measuring process in which the apparatus itself is a fully quantum mechanical object subject to exactly the same physical laws as the system being measured. Let $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_m$ be the combined Hilbert space of the system $s$ being measuring and the apparatus $m$. The latter can include the environment of the apparatus and whatever else is relevant to the description of the measuring process, enough so that we can consider $\mathcal{H}$ the Hilbert space of a closed system to which Schrödinger’s equation applies. That equation gives rise to a unitary time evolution operator $T(t_2, t_1)$ on $\mathcal{H}$, where $t_1$ is a time just before the measurement takes place and $t_2$ a time just after it is complete, with results indicated by the position of the traditional pointer. Let us assume that at the initial time the system has the property $P_\alpha$ and the measuring apparatus (plus environment, etc.) is in an initial macroscopic “ready” state corresponding to the property or projector $M_0$ onto some subspace (typically of extremely high dimension) of $\mathcal{H}_m$. Thus at $t_1$ the combined system plus apparatus has the property $P_\alpha \otimes M_0$. (The reader who prefers a density operator for the apparatus is welcome to employ that rather than $M_0$ in the following discussion.) Next we assume that there is a decomposition $\{\Pi_\alpha\}$ of the identity $I$ for $\mathcal{H}$ in which the index $\alpha$ labels macrostates corresponding to different possible positions of the pointer indicating the measurement outcome, and in addition to the values in (1) there is another, say $\alpha = 0$, such that

$$\Pi_0 = I - \sum_{\alpha \neq 0} \Pi_\alpha. \quad (7)$$

The projector $\Pi_0$ allows for all other possibilities; e.g., the apparatus was not properly set up in the first place. Next we assume that the unitary dynamics is such that if we define

$$V_\alpha := T(t_2, t_1)(P_\alpha \otimes M_0)T(t_1, t_2), \quad (8)$$

it is the case that

$$\Pi_\alpha V_\alpha = \delta_{\alpha \alpha'} V_\alpha. \quad (9)$$

In words, if the system has property $P_\alpha$ at time $t_1$, then the time-evolved property $V_\alpha$ is contained in the subspace onto which $\Pi_\alpha$ projects, meaning that the pointer is in the position $\alpha$. The $\delta_{\alpha \alpha'}$ implies that the

\footnote{If the total system Hamiltonian $H$ is independent of time, $T(t_2, t_1) = e^{-i(t_2-t_1)H/\hbar}$.}
system property $P_a$ at $t_1$ and the pointer at $\Pi_\alpha$ at $t_2$ are perfectly correlated: from the latter one can infer the former, and vice versa. Notice that the $\Pi_\alpha$ are projectors on $\mathcal{H}$, not on $\mathcal{H}_m$ alone, and are associated with pointer positions. They say nothing at all about the final state of the measured system $s$, which may not even exist; e.g., as when a photon is absorbed by the detector. We are, as noted earlier, discussing a measurement, not a preparation.

The reader unfamiliar with this type of measurement model may find it helpful to consider the special case in which $M_0 = |M_0\rangle\langle M_0| = |M_0\rangle$, where we use the abbreviation $|\psi\rangle = |\psi\rangle\langle\psi|$ for the projector onto the ray containing $|\psi\rangle$, meaning the apparatus is in an initial pure state $|M_0\rangle$, and the system to be measured is in a pure state $|a_\alpha\rangle$ for some $\alpha$, so $P_a = |a_\alpha\rangle$. Then if we define

$$V_\alpha := T(t_2, t_1)(|a_\alpha\rangle \otimes |\Psi_1\rangle),$$

which is analogous to (3): when $t = t_2$ the pointer will have the proper property corresponding to $\Pi_\alpha$.

Obvious this is an idealized description of an actual laboratory measurement, but it is nonetheless accurate in representing the essential features of the latter, in somewhat the same way in which a diagram of a telescope showing rays entering the aperture before bouncing off mirrors and exiting through a lens represents the essential idea of how it operates. It is the task of the competent experimentalist to arrange for the case in which $M = |M_0\rangle\langle M_0|$, where the apparatus is in the ready state $M_0$, and the system $s$ is prepared in one of the states $P_\alpha$, where $\alpha$ varies from run to run. If the final outcome (pointer position) always corresponds to the known initial state of the system, this indicates that (3) is satisfied.

### 3.3 Superposition states

What will occur if the experimenter prepares an initial state

$$|\psi\rangle = (|a_1\rangle + |a_2\rangle)/\sqrt{2},$$

which is a superposition corresponding to distinct eigenvalues of $\hat{a}_1$ and $\hat{a}_2$ of $A$ and then carries out a measurement? The answer is that the outcome will differ from run to run. Sometimes the pointer (or its electronic counterpart in a modern laboratory) will have the proper property corresponding to $\Pi_\alpha$, and sometimes that corresponding to $\Pi_\beta$, with roughly half the runs indicating one and half indicating the other. We agree with Appleby [27] that probabilities can only come from probabilities, so in order to obtain the probabilities of standard quantum mechanics one must introduce a probabilistic axiom. The simplest is the Born rule, which for present purposes can be worded as follows. Suppose that at $t_1$ we assign a property $\hat{P}$ to $s$, where $\hat{P}$ projects on any subspace of $\mathcal{H}_s$, and define

$$\hat{V} := T(t_2, t_1)(\hat{P} \otimes M_0)T(t_1, t_2).$$

At $t_2$ we are interested in a collection of properties of the apparatus represented by the $\Pi_\alpha$, which form a sample space of mutually-exclusive properties, to which Born’s rule assigns conditional probabilities

$$\Pr(\Pi_\alpha | \hat{P}) = \frac{\Tr(\Pi_\alpha \hat{V})}{\Tr(\hat{V})},$$

where the denominator on the right side normalizes the probability. If $\hat{P} = \hat{P}_\alpha$, then $\Pr(\Pi_\alpha | \hat{P}) = \delta_{\alpha \alpha'}$, consistent with the result in Sec. 3.2 but now expressed in the form of a probability.

If, on the other hand, one sets $\hat{P} = |\psi\rangle$, the projector onto $|\psi\rangle$ in (12), the $\hat{V}$ resulting from the time evolution in (13) will be a projector onto a coherent superposition of macroscopically-distinct pointer positions. As long as we are only concerned with calculating the probabilities of the $\Pi_\alpha$, $\hat{V}$ in (14) serves simply as a calculational tool, a pre-probability in the terminology of Sec. 9.4 of [4], and does not count as a physical property. If, on the other hand, one wishes to consider $\hat{V}$ a physical property, a member of the decomposition of the identity formed by the projectors $\hat{V}$ and $I - \hat{V}$, the Born rule assigns a probability $\Pr(\hat{V} | \hat{P}) = 1$: the property $\hat{V}$, a form of “Schrödinger cat”, is certain to occur at $t_2$. How is this to be
reconciled with the previous assignment of probabilities to \( \Pi_1 \) and \( \Pi_2 \)? Observe that just as \( \hat{P} = [\psi] \) does not commute with either \( P_1 \) or \( P_2 \), \( \hat{V} \) does not commute with \( \Pi_1 \) and \( \Pi_2 \), so it makes no sense—it violates the single framework rule—to put all three together into a single discussion. There is no law of nature which identifies the correct framework to be used for a quantum description. However, if one wants to think of the process under discussion as a measurement with some outcome, the way a physicist would wish to talk about it, the useful framework for this purpose is the collection of properties \( \{\Pi_\alpha\} \), not the framework \( \{\hat{V}, I - \hat{V}\} \) which makes it impossible to discuss outcomes.

Thus if one employs the unitarily evolved state \( \hat{V} \) (Schrödinger’s cat) as a physical property at time \( t_2 \) there is no way in which one can assert that the measurement has had a definite outcome, that the apparatus pointer is in a particular state. In this respect the histories approach is distinctly different from the Everett or many-worlds interpretation with its insistence that “the wave function”, in this instance the state onto which \( \hat{V} \) projects, represents fundamental physical reality. From the histories point of view the difficulties which many-worlds advocates have in explaining how ordinary macroscopic physics can be consistent with their perspective is not unrelated to the fact that they are seeking to assign simultaneous reality to properties which in the quantum Hilbert space are represented by incompatible projectors. (For further remarks on the single framework rule see the appendix, and for more details the discussion in Sec. 2.3 of [13].)

But in addition to getting rid of the ghost of Schrödinger’s cat, we still need to show that the measurement apparatus actually carries out a measurement; i.e., the outcome pointer position is properly correlated with a previous property of the measured system. For this purpose we need an extension of Born’s rule that allows probabilities to be assigned to a closed quantum system at three or more times, and this in turn requires the use of consistent (or decoherent) families of histories. As this topic is discussed in considerable detail and with numerous examples in [11], with more abbreviated treatments in [10,11], we omit details and give only the essential results as they apply in the present situation.

To this end it is convenient to introduce a time \( t_0 \) slightly earlier than \( t_1 \), but still later than the completion of the preparation procedure that results in the state \( |\psi\rangle \) in (12). We assume that no significant time development goes on during the short interval from \( t_0 \) to \( t_1 \), and consider a family of histories

\[
|\psi\rangle \otimes M_0 \otimes \{P_\alpha\} \otimes \{\Pi_\alpha\},
\]

where the notation is to be interpreted as follows. The symbol \( \otimes \) separates situations at the different times; it is the counterpart of “\( < \)” in \( t_0 < t_1 < t_2 \). The initial state at time \( t_0 \) is a projector \( |\psi\rangle M_0 \) with \( |\psi\rangle = |\psi\rangle \langle \psi| \) corresponding to the superposition \( |\psi\rangle \) in (12). At \( t_1 \) we consider the collection of properties \( \{P_\alpha\} \) making up the decomposition in (2). (Following the usual convention of physicists, the symbol \( P_\alpha \) should be understood as \( P_\alpha \otimes I_m \) when used in a context in which the full Hilbert space is \( \mathcal{H}_s \otimes \mathcal{H}_m \).) Similarly at time \( t_2 \) the properties under consideration are the pointer positions \( \{\Pi_\alpha\} \). An extension of the Born rule allows the assignment of probabilities to the different histories, each history involving one of the properties in (15) at each time, provided appropriate consistency (or decoherence) conditions are satisfied.

In the present case consistency is a consequence of (9), and the extended Born rule assigns zero probability to each history except for the pair

\[
|\psi\rangle \otimes M_0 \otimes P_1 \otimes \Pi_1, \quad |\psi\rangle \otimes M_0 \otimes P_2 \otimes \Pi_2.
\]

To each of these it assigns a probability of \( 1/2 \). As a consequence the conditional probability of the property \( P_1 \) at time \( t_1 \) given the pointer position \( \Pi_1 \) at \( t_1 \) is equal to 1, and vice versa; and the same for \( P_2 \) and \( \Pi_2 \). Of course there is nothing special about the superposition used in (12); any other would have led to a similar result, which is just the conclusion we reached earlier in Sec. 4.2: there is an exact, one-to-one correspondence between later pointer positions and earlier properties of the system to be measured.

The family of histories in (15) is not the only possible choice for a consistent quantum description. One could instead employ

\[
|\psi\rangle \otimes M_0 \otimes \{|\psi\rangle, I - |\psi\rangle\} \otimes \{\Pi_\alpha\},
\]

with a different choice of properties for \( s \) at the intermediate time \( t_1 \). Indeed, (17) corresponds fairly closely to a systematic quantum description of the narrative students are taught in textbooks, in which there is nothing but unitary time development up to the instant at which a measurement takes place, when by fiat a “collapse” takes place in which pointer positions somehow emerge out of a superposition fog. There is nothing wrong with (17) in terms of fundamental quantum mechanics, but it is not suitable for discussing a measurement, in contrast to a preparation, since the properties which the measurement was designed to
reveal cannot be discussed using this family: the $P_\alpha$ will (in general) not commute with $\lvert \psi \rangle$. Thus the choice between (15) and (17) is not a matter of fundamental quantum theory, where both are equally satisfactory, but instead a matter of utility: only the former allows one to discuss a measurement as a measurement. (For additional remarks on the relationship of incompatible families, see [13].)

In summary, a properly constructed quantum measurement apparatus does just what it was designed to do, measure properties of a system corresponding to a particular decomposition of the identity. It will do this for a system initially prepared in one of the states it was designed to measure, and for one initially prepared in a superposition of those states. To discuss the latter one needs to employ probabilities, in particular the (extended) Born rule, both in order to have a measurement pointer with a well-defined position, and to show that this position is appropriately correlated with one of the properties the apparatus was designed to measure. For details omitted from the preceding discussion the reader is referred to the extensive treatment of measurements in [4], as well as the shorter discussions in [10, 11, 13].

3.4 Measuring compatible observables

Next consider a situation in which two observables, $A$ associated with the decomposition \{\(P_\alpha\)\} and \(B\) with \{\(Q_\beta\)\} commute with each other. As discussed in Sec. 2 there is a decomposition \{\(R_j\)\} of the identity, consisting of products \(P_\alpha Q_\beta\), which allows both \(A\) and \(B\) to be written in the form (9). To determine the values of both of them it suffices (and is also necessary) to carry out a measurement which determines which property \(R_j\) is possessed by the system. To do such a measurement we follow the same strategy as in Sec. 3.2 using an apparatus \(m'\) in an initial state \(M'_0\), and a decomposition \{\(\Xi_j\)\} corresponding to pointer positions replacing the \{\(\Pi_\alpha\)\}. Use operators

\[
W_j := T(t_2, t_1) (R_j \otimes M'_0) T(t_1, t_2)
\]

in place of the \(V_\alpha\), and replace (9) with

\[
\Xi_j W_j' = \delta_{jj'} W_j.
\]

The argument that each outcome \(\Xi_j\) reflects a prior property \(R_j\) is then the same as that used earlier.

A particular \(P_\alpha\) may be equal to one of the \(R_j\), or it may be the sum of two or more. Suppose, for example, that \(P_1 = R_3 + R_4\). If the experimenter prepares the system with a property \(P_1\) then it may have neither of the properties \(R_3\) or \(R_4\). But it will still be the case that as a consequence of the linearity of (19),

\[
(\Xi_3 + \Xi_4)V_1 = V_1,
\]

where \(V_1\) as defined in (8). Hence, again invoking the Born rule, the measurement outcome will be one of the pointer positions corresponding to \(\Xi_1\) or \(\Xi_2\), varying randomly from run to run. These two outcomes are uniquely associated with the property \(P_1\) in that they cannot occur if the initial property is \(P_\alpha\) with \(\alpha \neq 1\). Thus either outcome implies that the system earlier possessed the property \(P_1\). Therefore this new procedure remains valid for measuring \(A\), while at the same time, by the same sort of argument, it provides a measurement of \(B\).

4 Noncontextuality

The preceding argument showing that a joint measurement of an observable \(A\) and a compatible observable \(B\) using a properly constructed apparatus apparatus \(m'\) will reveal a property \(P_\alpha\), corresponding to \(A = a_\alpha\), at a time before the measurement took place, is independent of the choice of \(B\) as long as \([A, B] = 0\). Consequently, if the apparatus, call it \(m''\), is instead designed to measure \(A\) and a different observable \(C\), where \([A, C] = 0\) but \([B, C] \neq 0\), the connection between the measurement outcome and the prior property \(P_\alpha\) corresponding to \(a_\alpha\) is exactly the same. And since—always assuming a properly constructed apparatus—the outcome so far as the value of \(A\) is concerned is in both cases determined by the property \(P_\alpha\) the system \(s\) possessed before the measurement took place, it can make no difference which apparatus, \(m'\) or \(m''\), is used to carry out the measurement. The result will be exactly the same given that the original quantum property of \(s\) was the same. The choice between apparatuses \(m'\) and \(m''\) makes no difference for properties that correspond to some eigenvalue \(a_\alpha\) of \(A\).

Another way of thinking about the matter is to suppose that there is a single measurement apparatus, but it is supplied with a switch that allows it to be changed from a kind that can measure both \(A\) and \(B\)
to a kind that can measure both $A$ and $C$, and the setting of this switch is determined by a coin toss that takes place just prior to the arrival of $s$ at the measuring device. (This can be modeled, if one chooses, using a closed quantum system by employing a quantum coin toss, of the sort discussed in Sec. 19.2 of [4].) Let us suppose that in some particular case the switch setting results in a measurement of $A$ and $B$, with an outcome from which one can, with probability 1, infer that $s$ had, say the property $P_3$ (i.e., $\alpha = 3$) at a time $t_1$ before the quantum coin was tossed. We can then ask the counterfactual question: what would have been the result of the measurement if the coin toss had resulted in the measurement of $A$ and $C$, rather than $A$ and $B$?

Analyzing counterfactual questions is a bit tricky even in situations not entangled with quantum mysteries. There is, however, an approach, see Sec. 19.4 of [4] and later chapters for various applications, which seems to work well in the quantum case, and has withstood (in the author’s opinion; the reader is invited to examine the references) a recent attack by Stapp [28], with response in [29]. It amounts to starting with an event at a later time in the actual world $W$, tracing it backwards in time to some property, referred to as a “pivot,” that $W$ has in common with the counterfactual world $W'$, and then using quantum theory to reason forwards in time to a later property of $W'$. In the present situation—we leave the details of the analysis to the interested reader—if the property $P_3$ at time $t_1$ is used as a pivot, the conclusion is that if $A$ and $C$ had been measured, the result so far as the property $A$ is concerned would have been exactly the same, that is, a pointer position from which the earlier $P_3$ could have been inferred with probability 1.

5 Discussion

The preceding argument establishes the noncontextuality of quantum mechanics when the measured quantities are quantum properties associated with subspaces of the Hilbert space. The essential idea is that apparatus competently designed and constructed to reveal a property which the system of interest had just before the measurement takes place will do exactly that, and it is irrelevant what other properties, if any, the apparatus has been designed to measure. (Of course, incompatible properties cannot be simultaneously measured, because in Hilbert space quantum mechanics there is nothing there to be measured.)

Given this fairly straightforward way of demonstrating that quantum mechanics is noncontextual, it is natural to ask why the topic of contextuality still seems to attract considerable attention, even leading to proposals for experimental tests. In this connection it is helpful to examine the recent paper by Hermens [6] already referred to in Sec. 1. He gives a list of four statements which because of the Kochen-Specker theorem cannot all be true:

- **QM (Quantum mechanics):** Every observable $A$ can be associated with a self-adjoint operator $A$ on some Hilbert space $\mathcal{H}$. The result of a measurement of $A$ is an element of the spectrum $\sigma(A)$ of $A$. Observables whose corresponding operators commute can be measured simultaneously. Moreover, if there is a functional relationship between the associated operators, this relation is preserved in the measurement results.

- **Re (Realism):** Every observable $A$ possesses a certain (real) value $\lambda(A)$ at all times.

- **FM (Faithful measurement):** A measurement of an observable $A$ at a certain time reveals the value $\lambda(A)$ possessed by that observable at that time.

- **CP (Correspondence principle):** There is a bijective correspondence between observables and self-adjoint operators.

To connect these statements with our previous discussion, note that we have used the same symbol $A$ for a physical variable or observable, and the corresponding quantum operator, which is to say we assumed the validity of Hermens’ $CP$ from the very outset: “observable” is just the name associated with self-adjoint or Hermitian (equivalent for finite-dimensional $\mathcal{H}$) operators in the usual textbook approach to quantum theory. Remember that we are employing Hilbert space quantum mechanics, not a Hilbert space augmented with hidden variables, and from this perspective a one-to-one association of self-adjoint operators with quantum observables, or, equivalently, projectors with quantum properties, is entirely reasonable. We also agree with Hermen’s $QM^4$ Furthermore, our measurement model, with $\lambda(A)$ an eigenvalue of $A$, satisfies $FM$, though with the qualification that the measured property was possessed by the system at a time

\footnote{A functional relationship between operators implies a functional relationship between the sets of eigenvalues when both are expressed using a common decomposition of the identity. E.g., one could imagine that there is a (real-valued) function $f(x)$}
just before the measurement took place. There is, to be sure, a subtle point involved in the notion that an observable can “possess” a value, especially since the observable $A$ can in general occur in (i.e., the corresponding decomposition of the identity as in (1)) a variety of different frameworks, including some that are incompatible with others. See the additional remarks in the appendix.

The way our approach avoids any conflict with the Kochen-Specker theorem is by denying $Re$. The claim that every observable possesses a value at every time is, indeed, inconsistent with a representation of quantum properties by subspaces of a Hilbert space. Consider, for example, a spin-half particle. There are distinct rays in the two-dimensional Hilbert space corresponding to $S_x = +1/2$ (in units of $\hbar$) and to $S_x = -1/2$; also rays corresponding to $S_z = +1/2$ and $S_z = -1/2$, but there is no ray that can represent a simultaneous value of $S_x$ and $S_z$. Students are told, correctly, that $S_x$ and $S_z$ cannot be measured simultaneously, and they ought to be told that the reason for this is that there is nothing there to be measured. The projectors corresponding to $S_z$ do not commute with the projectors corresponding to $S_x$, and once one has accepted the connection between quantum properties and Hilbert subspaces proposed by von Neumann it makes no sense to speak of a spin-half system in which, for example, $S_z = +1/2$ at the same time that $S_x = +1/2$. For further discussion of these matters see Sec. 4.6 of [4]. For additional comments on how the histories approach is related to the Kochen-Specker result, see the appendix.

Consequently, three of Hermens’ four statements (perhaps slightly modified) are in agreement with Hilbert space quantum mechanics, but $Re$ is not. It is somewhat odd that this particular principle should be identified with realism, since at the present time all available experimental evidence is in accord with Hilbert space quantum mechanics, and not with classical physics when the two disagree. If a Hilbert space provides the appropriate mathematics to describe everything from the quarks to the quasars, where in the real world, the one we live in, is there any part that satisfies the condition of “realism” given by $Re$? It would be much less confusing if whenever “realism” were used in this way the adjective “classical” were prepended. The hidden variables of typical hidden variable theories are classical hidden variables, and it is for this reason that the attempt to use them for interpreting quantum theory has given rise to numerous conflicts with the latter.

Not only have classical hidden variables given rise to misleading notions about the contextuality of quantum measurements, they have also contributed to the misleading idea that quantum mechanics is nonlocal in the sense that actions at one point in spacetime have a mysterious nonlocal influence on what goes on at other, spatially separated, points in spacetime, contrary to special relativity. For a demonstration that Hilbert space quantum theory contains no such influences, see [11,30]. It is thus surprising that classical hidden variables continue to receive so much attention in discussions of the foundations of quantum mechanics when they are of such little use. Perhaps it is time to pay serious attention to the mathematical properties of Hilbert space, and the representation of physical properties by means of its subspaces.

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A Appendix. The Single Framework Rule

The histories approach to Hilbert space quantum mechanics adopts a precise formulation of quantum probabilities constructed in the following way. A probabilistic theory requires a sample space of mutually exclusive possibilities or events, one and only one of which is true (or occurs), an event algebra, and some way of assigning probabilities to the latter. A quantum sample space in the histories approach is always a projective decomposition of the identity, as in (2). In the case of time-dependent stochastic quantum processes the same idea is used, but the decomposition now refers to projectors on the history Hilbert space, ordered sequences of events at discrete times, which constitute the elementary histories; for details see Ch. 8 of [2]. In either case the event algebra is constituted by sums of collections of projectors from this sample space, and probabilities are assigned to compound events in the usual fashion, as sums of the probabilities such that for every $j$ in (6) $b_j = f(a_j)$. What the measurement actually reveals is the prior property $R_j$, but if one uses it to infer the corresponding eigenvalues the functional relationship between them will, of course, be satisfied.
assigned to the individual events that they contain. Either the original sample space or the event algebra is referred to as a quantum framework.

The single framework rule asserts that any sort of discussion of the quantum system be carried out in some framework of the sort just discussed, which is typically chosen because it has some events which are interesting for some reason or another. The physicist is free to choose any framework he pleases for describing the world; what the single framework rule prohibits is combining frameworks. If two (or more) frameworks are compatible in the sense that all the projectors in one commute with all the projectors in the other one can get around this prohibition by using the common refinement, a sample space formed by nonzero products of the projectors in the two original sample spaces. However, if the frameworks are incompatible, which is to say some projectors from one do not commute with some projectors from the other, a common refinement is no longer possible, and single framework rule forbids using logical reasoning which bridges the two.

The well-known Kochen-Specker paradox is constructed precisely by forming a bridge, or one might better say bridges, between incompatible frameworks in such a way that one eventually ends up with a contradiction. The histories approach disposes of the paradox by declaring the bridging invalid. For more details, including a refutation of the claim by Bassi and Ghirardi [31, 32] that Kochen-Specker invalidates the histories approach, see [33,34] and the discussion in Ch. 22 of [4].

An additional aspect of the single framework rule (or one might say an extension) appears when considering quantum time dependence. In order to extend the Born rule for probabilities in a consistent way to histories involving three or more times it is necessary to impose consistency conditions, see Ch. 10 of [4]. There are examples, of which the best known is the three-box paradox of Aharonov and Vaidman [35]—see the discussion in Sec. 22.5 of [4]—where combining distinct sample spaces of histories is not prohibited by noncommuting projectors, but in which consistency conditions are not longer satisfied by the combination. The (extended) single framework rule then rules out contradictions of the sort claimed by Kent [36] and refuted in [37].

It is important to note that the single-framework rule does not state that there is only one framework which can be used for a valid quantum description of a situation. The quantum physicist is free to choose any framework, consistent with the Hilbert space structure of quantum mechanics (and, in the case of histories, satisfying consistency conditions if the extension of the Born rule is to be used to assign probabilities), in order to describe a quantum system. In some frameworks a particular observable $A$ may possess some value, while in other frameworks it may not. The existence of the latter does not preclude the possibility or the validity of a framework which include the former. The single framework rule is not a restriction on the use of frameworks; it is instead a prohibition against combining incompatible frameworks.

Given the possibility, and indeed validity, of employing a variety of different frameworks the reader might suppose that the histories approach will inevitably produce contradictions. That it does not do so can be seen from the following consistency result discussed in Ch. 16 of [1]. Suppose one wishes to infer from some initial—in the sense that they form the beginning of an argument, not that they are necessarily earlier in time—data that at such-and-such a time a quantum system possesses some property, say a physical variable or observable $A$ has some value. This is done by constructing a framework which includes both the data and the observable $A$ (i.e., the corresponding decomposition of the identity, as in (1)), and then applying the laws of quantum mechanics to reach a conclusion, which in general will be probabilistic. Typically there are a variety of frameworks available for this task, some of which are incompatible with others. The consistency result is that one will reach the same (probabilistic) conclusion in any one of these frameworks. There will also be frameworks in which either the data or the observable $A$ are not represented; these obviously cannot be used to draw such a conclusion, but their existence as (possible) quantum descriptions does not invalidate a conclusion drawn from a framework that includes both the data and $A$.

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