Greatest Common Divisor and Least Common Multiple Matrices on Factor Closed Sets in a Principal Ideal Domain

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Abstract: Problem statement: Let T be a set of n distinct positive integers, x_1, x_2, ..., x_n. The n×n matrix [T] having (x_i, x_j), the greatest common divisor of x_i and x_j, as its (i,j)-entry is called the greatest common divisor (GCD) matrix on T. The matrix [[T]] whose (i,j)-entry is [x_i, x_j], the least common multiple of x_i and x_j, is called the least common multiple (LCM) matrix on T. Many aspects of arithmetics in the domain of natural integers can be carried out to Principal Ideal Domains (PID). In this study, we extend many recent results concerning GCD and LCM matrices defined on Factor Closed (FC) sets to an arbitrary PID such as the domain of Gaussian integers and the ring of polynomials over a finite field. Approach: In order to extend the various results, we modified the underlying computational procedures and number theoretic functions to the arbitrary PIDs. Properties of the modified functions and procedures were given in the new settings. Results: Modifications were used to extend the major results concerning GCD and LCM matrices defined on FC sets in PIDs. Examples in the domains of Gaussian integers and the ring of polynomials over a finite field were given to illustrate the new results. Conclusion: The extension of the GCD and LCM matrices to PIDs provided a larger class for such matrices. Many of the open problems can be investigated in the new settings.

Key words: GCD matrix, lcm matrix, factor-closed sets, principal ideal domain

INTRODUCTION

Let T = {x_1, x_2, ..., x_n} be a set of n distinct positive integers. The n×n matrix [T] having (x_i, x_j), the greatest common divisor of x_i and x_j, as its (i,j)-entry is called the Greatest Common Divisor (GCD) matrix on T. The matrix [[T]] whose (i,j)-entry is [x_i, x_j], the least common multiple of x_i and x_j, is called the least common multiple (LCM) matrix on T. The set T is said to be factor closed (FC) if it contains every divisor of x for any x∈T. In 1876, Smith showed that the determinant of the GCD matrix [T] on a FC set T is the product \( \prod_{i=1}^{n} \phi(x_i) \), where \( \phi \) is Euler's totient function. Moreover, Smith considered the determinant of the LCM matrix on a FC set and showed that it is equal to the product \( \prod_{i=1}^{n} \phi(x_i) \pi(x_i) \), where \( \pi \) is a multiplicative function defined for a prime power \( p^r \) by \( \pi(p^r) = -p \). Since then many papers related to Smith's results have been published. Recently, this field has been studied intensively. This new inspiration started in by Beslin and Ligh.

In [3], Beslin and Ligh obtained a structure theorem for GCD matrices and showed that, if S is FC, then det[T] = \( \prod_{i=1}^{n} \phi(x_i) \). They conjectured that the converse is true. In [10], Li proved the converse and provided a formula for the determinant of an arbitrary GCD matrix. Beslin and Ligh generalized these results by extending the FC sets to a larger class of sets, gcd-closed sets. In [11], a structure theorem for [[T]] was obtained from the structure of the reciprocal GCD matrix 1/[[T]], the (i,j)-entry of which is 1/(x_i, x_j). Given a FC set T, Bourque and Ligh calculated the inverses of [T] and [[T]] and showed that [[T]][T]^{-1} is an integral matrix. In that study, they stated their famous conjecture that the LCM matrix on any gcd-closed set is invertible. Bourque and Ligh investigated the structures, the determinants and the inverses associated with classes of arithmetical functions. For a brief
review of papers relating to Smith’s determinant, we refer to[6]. Using the language of posets, the authors gave a common structure that is present in many extensions of Smith’s determinants. Beslin and El-Kassar[2] extended the results in[3] to unique factorization domains.

The purpose of this study is to extend many of the recent results concerning GCD and LCM matrices defined on factor-closed sets to arbitrary Principal Ideal Domains (PID) such as the domain of Gaussian integers and the ring of polynomials over a finite field.

MATERIALS AND METHODS

Let S be a PID and let a, b ∈ S. We say that a and b are associates and write a ∼ b, if a = ub for some unit element u in S. If b is a nonzero nonunit element, then b has a unique factorization, up to associates, into prime elements in S. That is, b = up11...pn n, where the p1...N are distinct primes in S. Also, every finite set {b1, b2, ..., bn} admits, up to associates, a greatest common divisor. For a nonzero element b in S, define q(b) to be |S/<b>^|, the order of the quotient ring S/<b>, where <b> is the principal ideal generated by b. Note that q(u) = 1, for any unit u. Also note that in Z, Z[i] and Z[x], q(b) is finite ∀ b ≠ 0. Throughout the following we consider S to be a PID having the property that q(b) is finite ∀ b ≠ 0. It can be shown that q(ab) = q(a)q(b). Hence, if b = up11...pn n, then q(b) = (q(p1))^n (q(p2))^n ...(q(pn))^n. For a positive integer n, q(n) = n and for a Gaussian integer a + bi, q(a+bi) = a²+b². Also, if f(x) is a polynomial of degree n in Z[x], then q(f(x)) = p^n. If f(x) is a polynomial of degree n in Z[x], then q(f(x)) = p^n.

Define φ(b) = |U(S/<b>)^|, the order of the group of units U(S/<b>), then φ(b) ≥ 1 and the equality holds iff b is a unit. Also, φ(b) = q(b)^−1 if p is prime in S. It can be shown that φ(b) is multiplicative and if b = up11...pn n, then:

φ(b) = (q(p1)−1)(q(p2)−1)...(q(pn)−1)

Now, if S = Z, then φ(b) becomes Euler’s phi-function. Also, if f(x) = (p1(x))^n (p2(x))^m...(pn(x))^k is a polynomial of degree n in S = Z[x], a product of powers of distinct irreducible polynomials p1(x), 1 ≤ j ≤ n, then:

φ(f(x)) = \prod_{1 \leq j \leq n}(p_j - 1)

where, p_j(x) is of degree r_j. For example, if f(x) = (x^2+1)^3(x+2)^2(x^2+4) in S = Z[x], then φ(f(x)) = 3^i . 2^7 . 13. Now, if β = u + ib, 1 ≤ j ≤ n, a product of distinct Gaussian primes β_j = a_j + ib_j, then:

φ(β) = \prod_{j=1}^n (a_j^2 + b_j^2 - 1)

For example, if β = 6+2i, then φ(β) = 640.

Let S be a PID and let T = {t1, t2, ..., tn} be a set of nonzero nonassociate elements in S. Define a linear ordering < on T according to the following scheme: If t1 < t2 then t1 and t2 have the order in S. Define t1 < t2 if and only if t1 < t2 and t1 < t2, then define t1 < t2 whenever b < d. In the case S = Z[i] and S = Z[x], then t1 = t2 whenever u0φ(t1) < φ(t2). If the set T is ordered so that t1 = t2,...,tn, we say that T is a linear ordering.

Theorem 1: Let S be a PID and let b be a nonzero element in S. If E(b) is a complete set of distinct nonassociate divisors of b in S, then E(a)∩E(b) = E((a, b)) and E((p^n)) = {1, p, p^2, ..., p^n}. Note that if t1 < t2, then q(t1) = q(t2) and q(t1) = q(t2). Also, ∑_{d|(x)}φ(d) = ∑_{d|(x)}φ(d) whenever T < T'.

Proof: Let b ≠ 0. The result is true when b is a unit. Suppose that b is a nonunit so that b = up11...pn n. Since φ(b) is multiplicative, the function f(b) = ∑_{d|(b)}φ(d) is also multiplicative. For any prime element p_j, (1) gives:

f(p^n_j) = ∑_{d|(p^n_j)}φ(d) = φ(p^1_j)φ(p^2_j)...φ(p^n_j)

= 1 + (q(p^1_j)−1)(q(p^2_j)−1)...(q(p^n_j)−1)

= (q(p^1_j))^i = q(p^i_j)
By the multiplicativity of \( f(b) \), we have
\[
q(b) = \sum_{d \in E(b)} \varphi(d).
\]

**Corollary 1:** (Euler's) If \( n \) is a positive integer, then
\[
n = \sum_{d \mid n} \varphi(d).
\]

**RESULTS**

**GCD matrices on FC sets in a PID:** Throughout the following, we consider \( T = \{t_1, t_2, \ldots, t_n\} \) to be a \( q \)-ordered set of nonzero nonassociate elements of a PID \( S \).

Define the GCD matrix on \( S \) to be the \( n \times n \) matrix \([T]\) whose ordered set of nonzero nonassociate elements of a PID \( S \).

The set \( T \) is said to be a factor-closed (FC) in \( S \) iff \( t_i \) is factor-closed or it is contained in a factor-closed set \( D \).

**Theorem 2:** The GCD matrix \([T]\) can be decomposed into a product of an \( n \times m \) matrix \( A \) and an \( m \times n \) matrix \( B \), for some \( m \geq n \). The nonzero entries of \( A \) are equal to \( \varphi(d) \) for some \( d \) in a FC set \( D \) containing \( T \) and \( B \) is an incidence matrix.

**Proof:** Let \( D = \{d_1, d_2, \ldots, d_m\} \) be a FC set containing \( T \) in the PID \( S \). Define the \( n \times m \) matrix \( A = (a_{ij}) \) by
\[
a_{ij} = \begin{cases} 
\varphi(d_i) & \text{if } d_i \in E(t_j), \\
0 & \text{otherwise}
\end{cases}
\]
and let \( B = (b_{ij}) \) be the incidence matrix corresponding to the transpose of \( A \), where
\[
b_{ij} = \begin{cases} 
1 & \text{if } a_{ij} \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]
Hence, the product \( AB \) is given by:
\[
(AB)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = \sum_{d_k \in E(t_i), d_k \in E(t_j)} \varphi(d_k) = \varphi((t_i, t_j)).
\]

**Example 1:** Let \( T = \{1, 1+x, 1+x^3, (1+x)^3\} \) in \( \mathbb{Z}_2[x] \).

Then \([T] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 8 \\ 1 & 2 & 8 \end{bmatrix}\). Let \( D = \{1, 1+x, (1+x)^3\}, 1+x+x^2, 1+x^3, (1+x)^3\} \). Then, \([T]_{4\times4} = A_{4\times4} B_{4\times4} \) where:
\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 3 \\ 1 & 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]

**Example 2:** Let \( T = \{1, 2, 5\} \). In \( \mathbb{Z}[i] \), \([T] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 25 \end{bmatrix} \).

Note that \( T \) is not FC in \( \mathbb{Z}[i] \). Select \( D = \{1, 1+i, 2, 2+i, 1+2i, 5\} \). Then \( A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 4 & 16 \end{bmatrix} \) and
\[
B = \begin{bmatrix} 1 & 1 \\ 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

**Theorem 3:** The GCD matrix \([T]\) is the product of an \( n \times m \) matrix \( A \) and its transpose \( A^T \). The nonzero entries of \( A \) are of the form \( \sqrt{\varphi(d)} \) for some \( d \) in a FC set \( D \) containing \( T \).

**Proof:** Let \( D = \{d_1, d_2, \ldots, d_m\} \) be a FC set containing \( T \). Define the \( n \times m \) matrix \( A = (a_{ij}) \) by
\[
a_{ij} = \begin{cases} 
\sqrt{\varphi(d_i)} & \text{if } d_i \in E(t_j), \\
0 & \text{otherwise}
\end{cases}
\]
Hence, the product \( A^T B \) is given by:
\[
(AA^T)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = \sum_{d_k \in E(t_i), d_k \in E(t_j)} \varphi(d_k) = \varphi((t_i, t_j)),
\]
where \( \varphi((t_i, t_j)) \). Note that Theorem 2 and 3 hold even if \( T \) is not \( q \)-ordered. In the case when both \( T \) and \( D \) are \( q \)-ordered, \( B \) becomes in raw-echelon form.

**Corollary 2:** (Smith's Determinant over a PID) If \( T \) is FC in \( S \), then \( \det[T] = \prod_{i=1}^{n} \varphi(t_i) \).

**Proof:** Let \( T \) be a FC set. Choose \( D \) to be \( q \)-ordered and \( D \sim T \). From Theorem 2, the GCD matrix \([T] = AB \), where \( A \) is an \( n \times n \) lower triangular matrix and \( B \) is an upper triangular matrix such that \( a_{ii} = \varphi(t_i) \) and \( b_{ij} = 1, 1 \leq i \leq n \). Therefore, \( \det[T] = \det[AB] = \det[A]\det[B] = \varphi(t_1)\varphi(t_2)\cdots\varphi(t_n) \).

We note that if \( T' = \{t_1, t_2, \ldots, t_n\} \) is any arrangement of the elements of \( T = \{t_1, t_2, \ldots, t_n\} \) in \( S \), then \( \det[T] = \det[T'] \). This can be verified as follows.
The matrix \([T]\) can be obtained from \([T']\) by switching the rows and the columns of \([T']\). Thus, \([T] = E_1E_2\ldots E_{n}[T']\), where the \(E_i's\) are elementary matrices with \(\det[E_i] = \pm 1, 1 \leq j \leq n\). Hence, \([T]\) and \([T']\) are similar matrices and \(\det[T] = \det[T']\).

Next, we consider the converse of Corollary 2. Let \(S\) be a PID and let \(T = \{t_1, t_2, \ldots, t_n\}\) be a nonempty set of nonzero nonassociate elements in \(S\) with \(\det[T] = \prod_{i=1}^{n} \varphi_\ast(t_i)\). Is it true that \(T\) is factor-closed in \(S\)?

Consider a minimal FC set \(D = \{t_1, t_2, \ldots, t_m, t_{m+1}, \ldots, t_{n+1}\}\) containing \(T = \{t_1, t_2, \ldots, t_n\}\) with \(t_1 < t_2 < \ldots < t_n\) and \(t_{m+1} < t_{m+2} < \ldots < t_{n+1}\). Define an \(n \times (n+r)\) matrix \(A\) by

\[
(A)_{ij} = \sqrt[\varphi_\ast(t_i)]{\varphi(t_j)},
\]

where \(e_{ij} = 1\) if \(t_i \in E(t_j)\) and 0 otherwise. Denote the matrix \((e_{ij})_{n \times n}\), by \(E\), a \([0, 1]\)-matrix. Note that the matrix \(A\) is the same matrix \(A\) defined in Theorem 3.

For an \(n \times m\) matrix \(M, n > m\) and any set of indices \(k_1, k_2, \ldots, k_n\) with \(1 \leq k_1 < k_2 < \ldots < k_n \leq m\), let \(M_{k_1, k_2, \ldots, k_n}\) denote the submatrix consisting of \(k_1\), \(k_2\), \ldots, \(k_n\) columns of \(M\).

**Theorem 4:** Let \(D = \{t_1, t_2, \ldots, t_m, t_{m+1}, \ldots, t_{n+1}\}\) be a minimal FC set containing \(T = \{t_1, t_2, \ldots, t_n\}\) in \(S\), where \(t_1 < t_2 < \ldots < t_n\) and \(t_{m+1} < t_{m+2} < \ldots < t_{n+1}\). Then:

\[
\det[T] = \sum_{k_1 < k_2 < \ldots < k_n} \prod_{i=1}^{n} \varphi_\ast(t_i)^2 \varphi(t_{k_1}) \varphi(t_{k_2}) \cdots \varphi(t_{k_n})
\]

**Proof:** Since \([T] = AA^T\), Cauchy-Binet formula gives that:

\[
\det[T] = \det[AA^T] = \sum_{k_1 < k_2 < \ldots < k_n} \prod_{i=1}^{n} \varphi(t_{k_i}) \varphi(t_{k_i}) \cdots \varphi(t_{k_i})
\]

The result follows from the fact that:

\[
\det[A_{k_1, k_2, \ldots, k_n}] = \det[E_{k_1, k_2, \ldots, k_n}] \varphi(t_{k_1}) \varphi(t_{k_2}) \cdots \varphi(t_{k_n})
\]

Corollary 3: Let \([T]\) be the GCD matrix defined on \(T\) in \(S\). Then, \(\det[T] \geq \varphi_\ast(t_1)\varphi_\ast(t_2)\cdots\varphi_\ast(t_n)\).

**Proof:** The terms in the summation of Theorem 4 are nonnegative. Since the submatrix \(E_{k_1, k_2, \ldots, k_n}\) is lower triangular with diagonal elements equal to 1, we have that the term corresponding to \((k_1, k_2, \ldots, k_n) = (1, 2, \ldots, n)\) is \(\det(E(1, 2, \ldots, n))\varphi_\ast(t_1)\varphi_\ast(t_2)\cdots\varphi_\ast(t_n)\). Therefore, \(\det[T] \geq \varphi_\ast(t_1)\varphi_\ast(t_2)\cdots\varphi_\ast(t_n)\).

**Theorem 5:** Let \([T]\) be the GCD matrix defined on \(T\) in \(S\). Then, \(\det[T] = \varphi_\ast(t_1)\varphi_\ast(t_2)\cdots\varphi_\ast(t_n)\) if and only if \(T\) is factor-closed in \(S\).

**Proof:** The sufficient condition holds from Corollary 2. Conversely, suppose that \(\det[T] = \varphi_\ast(t_1)\varphi_\ast(t_2)\cdots\varphi_\ast(t_n)\). For contradiction purposes, suppose that \(T\) is not FC. Let \(D = \{t_1, t_2, \ldots, t_m, t_{m+1}, \ldots, t_{n+1}\}\) be a minimal FC set containing \(T\) in \(S\) such that \(t_1 < t_2 < \ldots < t_m\) and \(t_{m+1} < t_{m+2} < \ldots < t_{n+1}\). Since \(T\) is not FC, \(D\) is not associate to \(T\) in \(S\). Then, \(t_{m+1}\) is in \(D\) but not in \(T\) and \(t_{m+1} \in E(t)\) for some \(t\) in \(T\). Now, let \(t\) be the first element in \(T\) such that \(t_{m+1} \in E(t)\). Then, the submatrix \(A_{(1,2,...,r-1,m+1,...,n)}\) consisting of the \(1^{\text{st}}, 2^{\text{nd}}, \ldots, (r-1)^{\text{th}}, (n+1)^{\text{th}}, (r+1)^{\text{th}}, \ldots, n^{\text{th}}\) columns of \(A_{1,2,...,n}\) is a \([0, 1]\)-matrix whose diagonal elements are equal to 1. Since \(A_{1,2,...,r-1,m+1,...,n}\) can be obtained from \(E_{1,2,...,r-1,m+1,...,n}\) by performing a certain numbers of successive column permutations, \(\det[E_{1,2,...,r-1,m+1,...,n}] = \pm \det[E_{1,2,...,r-1,m+1,...,n}] = \pm 1\). From Theorem 4, we have:

\[
\det[T] = \sum_{k_1 < k_2 < \ldots < k_n} \prod_{i=1}^{n} \varphi(t_{k_i}) \varphi(t_{k_i}) \cdots \varphi(t_{k_i})
\]

This contradicts the necessary condition that the equality holds.

**Inverses of GCD matrices in a PID:** Let \(t\) be any nonzero element in \(S\). The generalized Mobius function over \(S\) is defined by:

\[
\mu_\ast(t) = \begin{cases} 1 & \text{if } t \text{ is a unit} \\ (-1)^m & \text{if } t \text{ is the product of } m \text{ nonassociate primes} \\ 0 & \text{otherwise} \end{cases}
\]

Note that:

\[
\sum_{d \mid t} \mu_\ast(d) = \begin{cases} 1 & \text{if } t \text{ is a unit} \\ 0 & \text{otherwise} \end{cases}
\]

(3)

**Corollary 4:** Let \([T]\) be the GCD matrix defined on \(T\) in \(S\). Then, \([T]\) is invertible and its inverse \([T]\^{-1} = (r_{ij})\) is given by:
\[ r_{ij} = \sum_{t_i \in E(t_i)} \frac{1}{q(t_i)} \mu(t_i / t) \mu(t_i / t_j) \]

**Proof:** Define the n×n matrices \( E = (e_{ij}) \) and \( U = (u_{ij}) \) as follows: \( e_{ij} = \begin{cases} 1 & \text{if } t_i \in E(t_j) \\ 0 & \text{otherwise} \end{cases} \) and \( u_{ij} = \begin{cases} \mu(t_i / t_j) & \text{if } t_i \in E(t_j) \\ 0 & \text{otherwise} \end{cases} \). Then, \( (EU)_{ij} = \sum_{t_k \in E(t_i)} e_{ik} u_{kj} = \sum_{t_k \in E(t_i)} \mu(t_k / t_j) \mu(t_i / t_j) \).

**Example 3:** Let \( S = \mathbb{Z}_2[x] \) and let \( T = \{1, 1+x, 1+x^2, (1+x)^2, 1+x+x^2\} \), which is a \( q \)-ordered FC set of nonzero nonassociate elements in \( \mathbb{Z}_2[x] \). Then,

\[
[T] = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 1 \\
1 & 2 & 4 & 4 & 1 \\
1 & 1 & 1 & 1 & 4
\end{pmatrix}
\]

By corollary 4, \([T]^{-1}\) is obtained as follows:

\[
\begin{align*}
a_{ij} &= \begin{cases} 1 & \text{if } \phi_q(t_i) + \phi_q(t_j) + \phi_q(1+x) = 1+1+1 = 3 \\
-1 & \text{if } \phi_q(1+x^2) = 1 \\
0 & \text{if } \phi_q((1+x)^2) = 1 \\
1/3 & \text{if } \phi_q(1+x+x^2) = 1/3 \\
0 & \text{if } a_{ij} = 0
\end{cases} \\
\end{align*}
\]

and so forth. Therefore:

\[
[T]^{-1} = \begin{pmatrix}
7/3 & -1 & 0 & 0 & -1/3 \\
-1 & 3/2 & -1/2 & 0 & 0 \\
0 & -1/2 & 3/4 & -1/4 & 0 \\
0 & 0 & -1/4 & 1/4 & 0 \\
-1/3 & 0 & 0 & 0 & 1/3
\end{pmatrix}
\]

**Reciprocal GCD Matrices in a PID:** The reciprocal GCD matrix on \( T \) in \( S \) is the n×n matrix \( 1/[T] \) whose \((i,j)\)-entry is \( 1/q((t_i, t_j)) \). It is clear that \( 1/[T] \) is symmetric. Furthermore, permutations of the elements of \( T \) yield similar reciprocal GCD matrices. For a nonzero element \( t \) in \( T \), define the function \( \xi(t) \) by \( \xi(t) = \frac{1}{q(t)} \sum_{d \in E(t)} q(d) \mu(t, d) \). A generalized version of the Mobius inversion formula can be used to show that \( \frac{1}{q(t)} = \sum_{d \in E(t)} \xi(t) \). Since \( \xi(t) \) is the product of two multiplicative functions \( \frac{1}{q(t)} \) and \( \chi(t) = \sum_{d \in E(t)} q(d) \mu(t, d) \), we have that \( \xi(t) \) is itself multiplicative. Moreover, if \( p \) is prime in \( S \), then \( \chi(p) = 1q(p) \). Hence, \( \xi(p^k) = \frac{1}{q(p)^k} \). Therefore:

\[
\xi(t) = \frac{1}{q(t)} \prod_{p \mid t} (1 - q(p)) = \frac{\phi_q(t)}{(q(t))^2} \prod_{p \mid t} (-q(p)) \quad (4)
\]

where, the product runs over all prime divisors \( p \) of \( t \) in \( E(t) \).

In the following two theorems we obtain two factorizations for the reciprocal GCD matrices.

**Theorem 6:** Let \( D = \{d_1, d_2, \ldots, d_m\} \) be a FC set containing \( T \) in \( S \). The reciprocal GCD matrix defined on \( T \) is the product of an \( n \times n \) matrix \( A = [a_{ij}] \), defined by \( a_{ij} = \begin{cases} \xi(d_{ij}) & \text{if } d_{ij} \in E(t_i) \\
0 & \text{otherwise} \end{cases} \) and an \( m \times n \) incidence matrix \( B \) corresponding to \( A^T \).

**Proof:** Let \( A \) be as defined and let \( B \) be the \( m \times n \) matrix with \( b_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0 \\
0 & \text{if } a_{ij} = 0 \end{cases} \). Then:

\[
(AB)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = \sum_{d_{ij} \in E(t_i)} \xi(d_{ij}) = \sum_{d_{ij} \in E(t_i)} \frac{1}{q((t_i, t_j))}
\]

In a similar manner we prove the second factorization given in the following theorem.

**Theorem 7:** Let \( D = \{d_1, d_2, \ldots, d_m\} \) be a FC set containing \( T \) in \( S \) and let \( C \) be the the \( n \times n \) matrix given by \( a_{ij} = \begin{cases} \sqrt{\xi(d_{ij})} & \text{if } d_{ij} \in E(t_i) \\
0 & \text{otherwise} \end{cases} \). Then \( 1/[T] = CC^T \).

The proof of the following theorem is similar to those of Theorems 4 and 5.
Theorem 8: Let T be a set in S. Then, \( \det(1/[T]) = \xi(t_1)\xi(t_2)\ldots\xi(t_n) \) iff T is factor-closed in S.

**LCM Matrices on FC Sets in a PID:** The least common multiple (LCM) matrix defined on T in S is the n×n matrix \([T] = \{t_i\}\), where \(t_j = q(t_1,t_j)\) and \([t_i,t_j]\) is the least common multiple of \(t_i\) and \(t_j\) in S.

**Theorem 9:** If T is FC in S, then
\[
\det([T]) = \prod_{i=1}^{n} \left( \phi(t_i) \prod_{p \in \mathbb{P}, t_i \mid p} (q(p)) \right).
\]

Proof: Since \([t_i,t_j]/(t_i,t_j)\), we have and \(q([t_i,t_j]) = q(t_i)q(t_j)/q((t_i,t_j))\). Now \(q(t_i)\) can be factored out from the \(i\)th row and \(q(t_j)\) from the \(j\)th column to obtain \(1/[T]\). Hence, \([T] = D.1/[T].D\), where D is the n×n diagonal matrix with diagonal entries \(q(t_1), q(t_2), ..., q(t_n)\). From 4, we have that:
\[
\det([S]) = \det(D.1/[T].D) = \det(D)=\det(1/[T])n = \prod_{i=1}^{n} \left( \phi(t_i) \prod_{p \in \mathbb{P}, t_i \mid p} (q(p)) \right).
\]

Cauchy Binet formula yields a formula for the determinant of the LCM matrix defined on a set T which is not necessarily FC. The formula is given by:
\[
\det([T]) = \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq n} \det(E_{(i_1,k_1), \ldots, (i_n,k_n)})^2 (q(t_{(i_1,k_1)})q(t_{(i_2,k_2)})\ldots q(t_{(i_n,k_n)}))^2 \xi(t_{(i_1,k_1)})\xi(t_{(i_2,k_2)})\ldots\xi(t_{(i_n,k_n)}).
\]

From Theorem 9, we have that the determinant of the GCD matrix on S divides the determinant of the LCM matrix whenever T is FC in S.

**DISCUSSION**

Most of the existing results related to GCD and LCM matrices are obtained in the domain of natural integers. The results are based on certain number theoretic functions such as Euler’s phi function and the Mobius function. These function and their properties can be generalized to principal ideal domains. By describing the underlying computational procedures and the various properties in the new settings, the existing results related to GCD and LCM defined on factor closed sets are extended to PIDs. This provides a large class of such matrices where many new examples can be constructed. In particular, examples in the domains of Gaussian integers and the ring of polynomials over a finite field may give new insight to some open problems.

**CONCLUSION**

The extension of the GCD and LCM matrices to PIDs provide a larger class for such matrices. Many of the open problems can be investigated in the new settings. For future study, we suggest the problem of extending GCD and LCM matrices defined on gcd closed sets to PIDS.

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