Abstract

In this paper, we study the structure of finite groups $G = AB$ which are a weakly mutually $sn$-permutable product of the subgroups $A$ and $B$, that is, $A$ permutes with every subnormal subgroup of $B$ containing $A \cap B$ and $B$ permutes with every subnormal subgroup of $A$ containing $A \cap B$. We obtain generalisations of known results on mutually $sn$-permutable products.

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Let $G = AB$ be the mutually sn-permutable product of the subgroups $A$ and $B$, where $A$ is supersoluble and $B$ is nilpotent. If $B$ permutes with each Sylow subgroup of $A$, then the group $G$ is supersoluble.

Following [8], we say that a subgroup $H$ of a group $G$ is $\mathcal{P}$-subnormal in $G$ whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for every $i = 1, \ldots, n$. It turns out that supersoluble groups are exactly those groups in which every subgroup is $\mathcal{P}$-subnormal. Having in mind this result and the influence of the embedding of Sylow subgroups on the structure of a group, the following extension of the class of supersoluble groups introduced in [8] seems to be natural.

**Definition 1.3.** A group $G$ is called widely supersoluble, w-supersoluble for short, if every Sylow subgroup of $G$ is $\mathcal{P}$-subnormal in $G$.

The class of all finite w-supersoluble groups, denoted by $wU$, is a saturated formation of soluble groups containing $U$, the class of all supersoluble groups, which is locally defined by a formation function $f$, such that for every prime $p$, $f(p)$ is composed of all soluble groups $G$ whose Sylow subgroups are abelian of exponent dividing $p - 1$ [8, Theorems 2.3 and 2.7]. Not every group in $wU$ is supersoluble [8, Example 1]. However, every group in $wU$ has an ordered Sylow tower of supersoluble type [8, Proposition 2.8].

In [4], mutually sn-permutable products in which the factors are w-supersoluble are analysed. The following extension of Theorem 1.2 holds.

**Theorem 1.4 [4, Theorem 4].** Let $G = AB$ be the mutually sn-permutable product of the subgroups $A$ and $B$, where $A$ is w-supersoluble and $B$ is nilpotent. If $B$ permutes with each Sylow subgroup of $A$, then the group $G$ is w-supersoluble.

Assume that $G = AB$ is the mutually sn-permutable product of the subgroups $A$ and $B$. Then, by [3, Proposition 4.1.16 and Corollary 4.1.17], $A \cap B$ is subnormal in $G$ and permutes with every subnormal subgroup of $A$ and $B$. Assume now that $G = AB$ and $A \cap B$ satisfy this condition. Then $G$ is the mutually sn-permutable product of $A$ and $B$ if and only if $A$ permutes with every subnormal subgroup $V$ of $B$ such that $A \cap B \leq V$ and $B$ permutes with every subnormal subgroup $U$ of $A$ such that $A \cap B \leq U$. This motivates the following definition.

**Definition 1.5.** Let $A$ and $B$ be two subgroups of a group $G$ such that $G = AB$. We say that $G$ is the weakly mutually sn-permutable product of $A$ and $B$ if $A$ permutes with every subnormal subgroup $V$ of $B$ such that $A \cap B \leq V$ and $B$ permutes with every subnormal subgroup $U$ of $A$ such that $A \cap B \leq U$.

Obviously, mutually sn-permutable products are weakly mutually sn-permutable, but the converse is not true in general, as the following example shows.

**Example 1.6.** Let $G = \Sigma_4$ be the symmetric group of degree 4. Consider a maximal subgroup $A$ of $G$ which is isomorphic to $\Sigma_3$, and $B = A_4$, the alternating group of
degree 4. Then $G = AB$ is the weakly mutually $sn$-permutable product of the subgroups $A$ and $B$. However, the product is not mutually $sn$-permutable because $A$ does not permute with a subnormal subgroup of order 2 of $B$.

The first goal of this paper is to prove weakly mutually $sn$-permutable versions of the aforesaid theorems. We show that Theorem 1.4 holds for weakly mutually $sn$-permutable products.

**Theorem A.** Let $G = AB$ be the weakly mutually $sn$-permutable product of the subgroups $A$ and $B$, where $A$ is $w$-supersoluble and $B$ is nilpotent. If $B$ permutes with each Sylow subgroup of $A$, then the group $G$ is $w$-supersoluble.

The next corollary follows from the proof of Theorem A and generalises Theorem 1.2.

**Corollary B.** Let $G = AB$ be the weakly mutually $sn$-permutable product of the subgroups $A$ and $B$, where $A$ is supersoluble and $B$ is nilpotent. If $B$ permutes with each Sylow subgroup of $A$, then the group $G$ is supersoluble.

The second part of the paper is concerned with weakly mutually $sn$-permutable products with nilpotent derived subgroups. Our starting point is the following extension of a classical result of Asaad and Shaalan [2].

**Theorem 1.7 [1, Theorem C].** Let $G = AB$ be the mutually $sn$-permutable product of the supersoluble subgroups $A$ and $B$. If the derived subgroup $G'$ of $G$ is nilpotent, then $G$ is supersoluble.

A natural question is whether this result is true for weakly mutually $sn$-permutable products under the same conditions. The following example answers this question negatively.

**Example 1.8.** Let $G = \langle a, b, c : a^5 = b^5 = c^6 = 1, ab = ba, a^c = a^2b^3, b^c = a^{-1}b^{-1} \rangle \cong [C_5 \times C_5]C_6$. Then $G = AB$ is the weakly mutually $sn$-permutable product of $A = \langle c \rangle$ and $B = \langle (a) \times (b) \rangle \langle c^3 \rangle$. Note that $B$ is a normal subgroup of $A$; therefore, it permutes with every subgroup of $A$. Moreover, $A \cap B = \langle c^3 \rangle$ and the unique subnormal subgroup of $B$ containing $A \cap B$ is the whole of $B$. It is not difficult to see that $B$ is supersoluble. Therefore, $A$ and $B$ are supersoluble and $G'$ is nilpotent. Moreover, $A$ is nilpotent and $B$ is a normal subgroup of $G$. Thus, in particular, it permutes with every Sylow subgroup of $A$.

However, an additional assumption allows us to get supersolubility.

**Theorem C.** Let $G = AB$ be the weakly mutually $sn$-permutable product of the supersoluble subgroups $A$ and $B$. If $B$ permutes with each Sylow subgroup of $A$, $A$ permutes with every Sylow subgroup of $B$ and the derived subgroup $G'$ of $G$ is nilpotent, then $G$ is supersoluble.
By [7, Theorem 2.6], a group $G$ is w-supersoluble if and only if every metanilpotent subgroup of $G$ is supersoluble. In particular, if we have a group with $G'$ nilpotent, every w-supersoluble subgroup is supersoluble. Therefore, the following result is clear.

**Corollary D.** Let $G = AB$ be the weakly mutually sn-permutable product of the w-supersoluble subgroups $A$ and $B$. If $B$ permutes with each Sylow subgroup of $A$, $A$ permutes with every Sylow subgroup of $B$ and the derived subgroup $G'$ of $G$ is nilpotent, then $G$ is w-supersoluble.

### 2. Preliminary results

In this section we will prove some results needed for the proofs of our main results. We start by showing that factor groups of weakly mutually sn-permutable products are also weakly mutually sn-permutable products.

**Lemma 2.1.** Let $G = AB$ be the weakly mutually sn-permutable product of $A$ and $B$, and let $N$ be a normal subgroup of $G$. Then $G/N = (AN/N)(BN/N)$ is the weakly mutually sn-permutable product of $AN/N$ and $BN/N$.

**Proof.** Let us consider $G/N = (AN/N)(BN/N)$. Suppose that $HN/N$ is a subnormal subgroup of $AN/N$ such that $AN/N \cap BN/N \leq HN/N$. Note that $U = HN \cap A$ is a subnormal subgroup of $A$ such that $UN = HN$ and $A \cap B \leq U$. Since $U$ permutes with $B$, it follows that $HN = UN$ permutes with $BN$.

Interchanging $A$ and $B$ and arguing in the same manner proves the result. □

**Lemma 2.2.** Let $G = AB$ be the weakly mutually sn-permutable product of $A$ and $B$.

(a) If $H$ is a subnormal subgroup of $A$ such that $A \cap B \leq H$, then $HB$ is a weakly mutually sn-permutable product of $H$ and $B$.

(b) If $A \cap B = 1$, then $G = AB$ is a totally sn-permutable product of $A$ and $B$.

**Proof.** Since every subnormal subgroup of $H$ is a subnormal subgroup of $A$, it follows that $B$ permutes with every subnormal subgroup $L$ of $H$ such that $A \cap B \leq L$. On the other hand, let $M$ be a subnormal subgroup of $B$ such that $A \cap B \leq M$. Then we have $HM = H(A \cap B)M = (A \cap HB)M = AM \cap HB = MA \cap BH = M(A \cap BH) = M(A \cap B)H = MH$. Hence $HB$ is a weakly mutually sn-permutable product of $H$ and $B$.

For (b), every subnormal subgroup of $A$ permutes with $B$ by (a) and every subnormal subgroup of $B$ permutes with $A$. So $G = AB$ is the mutually sn-permutable product of $A$ and $B$. Hence $G = AB$ is the totally sn-permutable product of $A$ and $B$ since $A \cap B = 1$. □

Observe that Lemma 2.2 implies that if $G = AB$ is the weakly mutually sn-permutable product of $A$ and $B$, $H$ is a subnormal subgroup of $A$ such that $A \cap B \leq H$ and $K$ is a subnormal subgroup of $B$ such that $A \cap B \leq K$, then $HK$ is a weakly mutually sn-permutable product of $H$ and $K$. In the next result we analyse the
behaviour of minimal normal subgroups of weakly mutually $sn$-permutable products containing the intersection of the factors.

**Lemma 2.3.** Let $G = AB$ be the weakly mutually $sn$-permutable product of $A$ and $B$. If $N$ is a minimal normal subgroup of $G$ such that $A \cap B \leq N$, then either $A \cap N = B \cap N = 1$ or $N = (N \cap A)(N \cap B)$.

**Proof.** Observe that $N$ is a normal subgroup of $A$ such that $A \cap B \leq A \cap N$ and consequently $H = (A \cap N)B$ is a subgroup of $G$. Note that $N \cap H = N \cap (A \cap N)B = (A \cap N)(B \cap N)$. Since $N \cap H$ is a normal subgroup of $H$, it follows that $B$ normalises $N \cap H = (A \cap N)(B \cap N)$.

By the same argument, $K = A(B \cap N)$ is a subgroup of $G$ such that $K \cap N = A(B \cap N) \cap N = (A \cap N)(B \cap N)$. Moreover, $A$ normalises $K \cap N = (A \cap N)(B \cap N)$. It follows that $(A \cap N)(B \cap N)$ is a normal subgroup of $G$. By the minimality of $N$, $A \cap N = B \cap N = 1$ or $N = (N \cap A)(N \cap B)$ as required. □

**Lemma 2.4.** Let $G = AB$ be the weakly mutually $sn$-permutable product of the subgroups $A$ and $B$, where $B$ is nilpotent. If $B$ permutes with each Sylow subgroup of $A$, then $A \cap B$ is a nilpotent subnormal subgroup of $G$.

**Proof.** It is clear that $A \cap B$ is nilpotent. The Sylow subgroups of $B$ are normal in $B$, so $A \cap B$ permutes with every Sylow subgroup of $B$. Let $A_q$ be a Sylow subgroup of $A$, with $q$ a prime dividing $|A|$. Since $B$ permutes with every Sylow subgroup of $A$, it follows that $BA_q$ is a subgroup of $G$. Hence $BA_q \cap A = A_q(A \cap B)$. Therefore $A \cap B$ permutes with every Sylow subgroup of $A$. Applying [3, Theorem 1.2.14(3)], $A \cap B$ is a subnormal subgroup of both $A$ and $B$. By [3, Theorem 1.1.7], $A \cap B$ is a subnormal subgroup of $G$. □

**Lemma 2.5.** Let $G = AB$ be the weakly mutually $sn$-permutable product of the subgroups $A$ and $B$, where $A$ is soluble and $B$ is nilpotent. If $B$ permutes with each Sylow subgroup of $A$, then the group $G$ is soluble.

**Proof.** Suppose that the theorem is false, and let $G$ be a minimal counterexample. If $N$ is a minimal normal subgroup of $G$, then $G/N = (AN/N)(BN/N)$ is the weakly mutually $sn$-permutable product of the subgroups $AN/N$ and $BN/N$ by Lemma 2.1. Since $BN/N$ permutes with each Sylow subgroup of $AN/N$, it follows that $G/N$ is soluble by the minimality of $G$. Let $N_1$ and $N_2$ be two minimal subgroups of $G$. Then $G \cong G/(N_1 \cap N_2)$ is soluble, a contradiction. Hence $G$ has a unique minimal normal subgroup $N$ of $G$ and we may assume that $N$ is nonabelian. This means that $F(G) = 1$.

On the other hand, $A \cap B \leq F(G)$ using Lemma 2.4. Therefore $A \cap B = 1$ and then $G = AB$ is the totally $sn$-permutable product of $A$ and $B$. The result then follows by applying [5, Theorem 6]. □

**Lemma 2.6** [1, Lemma 3]. Let $G$ be a primitive group and let $N$ be its unique minimal normal subgroup. Assume that $G/N$ is supersoluble. If $N$ is a $p$-group, where $p$ is the largest prime dividing $|G|$, then $N = F(G) = O_p(G)$ is a Sylow $p$-subgroup of $G$. 


3. Main results

We are ready to prove our main results.

**Proof of Theorem A.** Suppose the theorem is not true and let $G$ be a minimal counterexample. We shall prove our theorem in five steps.

(a) $G$ is a primitive soluble group with a unique minimal normal subgroup $N$ and $N = C_G(N) = F(G) = O_p(G)$ for a prime $p$. Let $N$ be a minimal normal subgroup of $G$. By Lemma 2.1, $G/N = (AN/N)(BN/N)$ is a weakly mutually $sn$-permutable product of $AN/N$ and $BN/N$ and it is clear that $BN/N$ permutes with every Sylow subgroup of $AN/N$. Moreover, $AN/N$ is $w$-supersoluble and $BN/N$ is nilpotent. By the minimality of $G$, it follows that $G/N$ is $w$-supersoluble. Note that $wU$, the class of finite $w$-supersoluble groups, is a saturated formation of soluble groups by [8, Theorems 2.3 and 2.7]. This implies that $G$ is a primitive soluble group and so $G$ has a unique minimal normal subgroup $N$ with $N = C_G(N) = F(G) = O_p(G)$ for some prime $p$ as required.

(b) The subgroup $BN$ is $w$-supersoluble and $1 \neq A \cap B \leq N$. If $A \cap B = 1$, then the result follows by Lemma 2.2 and Theorem 1.4. Applying Lemma 2.4, it follows that $A \cap B$ is a nilpotent subgroup of $G$. Therefore $1 \neq A \cap B \leq F(G) = N$ and so $N = (N \cap A)(N \cap B)$ by Lemma 2.3. Hence $NB = (N \cap A)(N \cap B)B = (N \cap A)B$ is a weakly mutually $sn$-permutable product of $N \cap A$ and $B$. Also note that $B$ permutes with every Sylow subgroup of $N \cap A$ (there is only one Sylow subgroup of $N \cap A$, which is $N \cap A$). If $NB < G$, then $NB$ is $w$-supersoluble by the choice of $G$. So we may assume that $G = NB$. In this case, consider a subgroup $N_1 \leq A \cap B \leq N$. Note that $N_1$ is normal in $N$ since $N$ is abelian. Hence $N = N_1^G = N_1^{NB} = N_1^B \leq B$ and $G = B$, a contradiction. Hence the result follows.

(c) $N$ is the Sylow $p$-subgroup of $G$ and $p$ is the largest prime dividing $|G|$. Let $q$ be the largest prime dividing $|G|$ and suppose that $q \neq p$. Suppose first that $q$ divides $|BN|$. Since $BN$ has a Sylow tower of supersoluble type, $BN$ has a unique Sylow $q$-subgroup, say $(BN)_q$. This means that $(BN)_q$ centralises $N$. Thus $(BN)_q = 1$, since $C_G(N) = N$, a contradiction.

We may assume that $q$ divides $|A|$ but does not divide $|BN|$. Since $A$ has a Sylow tower of supersoluble type, $A$ has a unique Sylow $q$-subgroup, $A_q$ say. This means that $A_q$ is normalised by $N \cap A$. Consider $A_q(N \cap B) = A_q(A \cap B)(N \cap B)$, a weakly mutually permutable product of $A_q(A \cap B)$ and $N \cap B$ by Lemma 2.2. Also $N \cap B$ permutes with each Sylow subgroup of $A_q(A \cap B)$. Suppose that $A_q(N \cap B) < G$. Then $A_q(N \cap B)$ is $w$-supersoluble by the choice of $G$. It follows that $A_q(N \cap B)$ has a unique Sylow $q$-subgroup since it has a Sylow tower of supersoluble type. In other words, $A_q$ is normalised by $N \cap B$. Hence $A_q$ is normalised by $(N \cap A)(N \cap B) = N$. This means that $A_q$ centralises $N$, a contradiction. We may assume that $A_q(N \cap B) = G$. Then $N \cap B = B$ and so $B$ is an elementary abelian $p$-group. Moreover, $A = A_q(A \cap B)$. Since $A \cap B$ is a Sylow $p$-subgroup of $A$ which is subnormal in $A$, it is normal in $A$. Hence $A \cap B$ is normal in $G$ because $A \cap B$ is normal in the abelian group $B$. By the
minimality of $N$, it follows that $N = A \cap B$, that is, $G = A_q(N \cap B) = A_q(A \cap B) = A$, a contradiction. Therefore $p$ is the largest prime dividing $|G|$. 

We now prove that $N$ is the Sylow $p'$-subgroup of $G$. Since $G$ is a primitive soluble group, $G = NM$, where $M$ is a maximal subgroup of $G$ and $N \cap M = 1$. Then $M \cong G/N$ is $p'$-supersoluble. By [6, Theorem A.15.6], $O_p(M) = 1$. If $p$ divides $|M|$, then since $M$ has a Sylow tower of supersoluble type, $O_p(M) \neq 1$, a contradiction. Hence $p$ does not divide $|M|$ and therefore $N$ is the unique Sylow $p'$-subgroup of $G$.

(d) $N$ is contained in $A$ and $N$ is not contained in $B$. Suppose that $B$ is a $p'$-group. Then $G = AN$. Let $N_1 \leq A \cap B$. Since $B$ is abelian, $N \leq N^G_1 = N^A_1 \leq A$ and so $G = AN = A$, a contradiction. We may assume that $B$ is not a $p'$-group. If $N$ is contained in $B$, then since $B$ is nilpotent and $N = C_B(N)$, it follows that $B$ is a $p'$-group, a contradiction. Therefore $N$ is not contained in $B$. Hence $B$ has a nontrivial Hall $p'$-subgroup, $B_{p'}$, which is normal in $B$. Consequently, $AB_{p'} = A(A \cap B)B_{p'}$ is a subgroup of $G$. Then $1 \neq B_{p'} = AB_{p'}$ and so $N \leq AB_{p'}$. Hence $N \leq A$ as required.

(e) Final contradiction. Let $A_{p'}$ be a Hall $p'$-subgroup of $A$. If $A_{p'} = 1$, then $G = BN$ is $p'$-supersoluble by (b), a contradiction. Hence $A_{p'} \neq 1$. Since $B$ permutes with every Sylow subgroup of $A$, it follows that $A_{p'}B$ is a subgroup of $G$. But $N$ is not contained in $B$, so $A_{p'}B$ is a proper subgroup of $G$. Since $NA_{p'}B = G$, it follows that $N \cap A_{p'}B = N \cap B$ is normal in $G$. The minimality of $N$ implies that $N = N \cap B$ or $N \cap B = 1$. If $N = N \cap B$, we get a contradiction with (d). Therefore $N \cap B = 1$, and then $A \cap B = N \cap B = 1$, contradicting (b).

PROOF OF THEOREM C. Assume the result is not true and let $G$ be a minimal counterexample. It is clear that $G' \neq 1$, $A$ and $B$ are proper subgroups of $G$, and $G$ is a primitive soluble group. Hence there exists a unique minimal normal subgroup $N$ of $G$, such that $N = F(G) = C_G(N)$. Moreover, $G' = N$. We may assume that $A' \neq 1$ and $B' \neq 1$, otherwise $A$ or $B$ is nilpotent and the result follows from Corollary B. If $A \cap B = 1$, then $G$ is the mutually $sn$-permutable product of $A$ and $B$. By [1, Theorem C], the group is supersoluble, a contradiction. Thus we may assume $A \cap B \neq 1$. Since $A$ permutes with every Sylow subgroup of $B$ and $B$ permutes with every Sylow subgroup of $A$, it follows that $A \cap B$ permutes with every Sylow subgroup of $A$ and every Sylow subgroup of $B$. Hence $A \cap B$ is subnormal in $A$ and it is a subnormal subgroup of $B$. Let $N_1$ denote a minimal normal subgroup of $A$ such that $N_1 \leq A'$. Since $A$ is supersoluble, it is clear that $|N_1| = p$. Note that $N_1(A \cap B)$ is a subnormal subgroup of $A$. Therefore $BN_1(A \cap B) = BN_1$ is a subgroup of $G$. Now $N \neq N^G_1 = N^B_1 \leq BN_1$. Hence $N \leq BN_1$ and then $N = N_1(N \cap B)$. Consequently, either $N_1 \leq N \cap B$ or $N_1 \leq N \cap B$. Denote $T = BN$. If $N_1 \leq N \cap B$, then $T = B$ is a supersoluble normal subgroup of $G$. Assume $N_1 \cap (N \cap B) = 1$. Then $N \cap B$ is a maximal subgroup of $N$ and so $T$ is the weakly mutually $sn$-permutable product of $B$ and $N$. Consequently, $T$ satisfies the hypotheses of the theorem. If $T$ is a proper subgroup of $G$, then $T = BN$ is supersoluble. Assume that $G = BN$. Then $B$ is a maximal subgroup of $G$ such that $B \cap N = 1$, $B' \leq N \cap B = 1$ and $B$ is nilpotent. By Corollary B, $G$ is supersoluble, contrary to assumption. Hence either $B$ is a normal subgroup of $G$ or $BN$ is a supersoluble normal subgroup of $G$. 


Arguing in an analogous manner with $A$ shows that if $AN$ is a proper subgroup of $G$, then it is supersoluble. Consequently if $BN$ and $AN$ are both proper subgroups of $G$, then $G$ is the product of two supersoluble normal subgroups with $G'$ nilpotent. Then $G$ is supersoluble, a contradiction. Therefore we may assume that $G = BN$ or $G = AN$. Suppose without loss of generality that $G = BN$. Then $N \cap B$ is a normal subgroup of $G$. If $N \cap B = N$, then $G = B$, a contradiction. Hence $N \cap B = 1$. Now $B' \leq N \cap B = 1$ and $B$ is nilpotent, the final contradiction. □

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