Inverse problem and Darboux transformations for two-dimensional finite-difference Schrödinger equation

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A discrete version of the two-dimensional inverse scattering problem is considered. On this basis, algebraic transformations for the two-dimensional finite-difference Schrödinger equation are elaborated. Generalization of the technique of one-dimensional Darboux transformations for the two-dimensional finite-difference Schrödinger equation is presented. Ideas of Bargmann-Darboux transformations for the differential one-dimensional multichannel Schrödinger equation are used for the two-dimensional lattice Schrödinger equation. Analytic relationships are established between different discrete potentials and the corresponding solutions.

I. INTRODUCTION

Study of multidimensional and multi-particle objects is qualitatively more complicated than that of the one-dimensional case. When there is no symmetry in the potentials of interaction between particles, these systems are described by partial differential equations with unseparable variables. Just for this reason the inverse scattering problem in 2- and 3-dimensional spaces was formulated by Faddeev [1] and Newton [2,3], Novikov and Henkin [4] at a much later than the 1-dimensional problem. The foundation for developing the 2-dimensional inverse problem in finite differences was laid by Berezanskii [5] who developed the theory of orthogonal polynomials for the Jacobi infinite matrix. The Darboux transformations in quantum mechanics for the Schrödinger differential and difference 1-dimensional equation have much in common with the spectral transformations for orthogonal polynomials [6]. It is therefore expedient to analyze the spectral inverse problem for the discrete 2-dimensional Schrödinger equation on the basis of the technique of orthogonalization of polynomials. The formulae of the inverse problem with degenerate kernels are closely related to Bargmann-Darboux transformations [7]–[9]. The one-dimensional Darboux transformations has already found wide applications in quantum mechanics and in the theory of nonlinear integrable systems [10,11,6,12].

In the present paper, a discrete version of the Gelfand-Levitan inverse spectral problem is considered for the two-dimensional lattice Schrödinger equation. The two-dimensional finite-difference inverse problem, based on the procedure of orthogonalization of polynomial vectors, is a generalization of the one-dimensional procedure given in [13]. On the basis of the obtained formulae of the inverse problem, the relations are derived for discrete Bargmann-Darboux transformations in two dimensions. The suggested algebraic approach allows one to construct families of discrete potentials in an explicit form and the corresponding solutions.

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II. INVERSE PROBLEM

Consider the Schrödinger equation whose Hamiltonian is tridiagonal in a certain basis with respect to both the coordinate variables \( n \) and \( m \)

\[
(H\psi)_{nm} = a_{nm}\psi(n-1,m) + a_{n+1m}\psi(n+1,m) \\
+ b_{nm}\psi(n,m-1) + b_{nm+1}\psi(n,m+1) + c_{nm}\psi(n,m) = \lambda\psi(n,m).
\]  

(1)

The coefficients \( a_{nm}, b_{nm}, c_{nm} \) are assumed to be real and represent discrete potentials; \( \psi(n,m) \) are discrete wave functions, \((n,m)\) is an integer point of the half-plane, \( n = 0, 1, 2, ..., m = ..., -1, 0, 1, ..., \) \( \lambda \) is a spectral parameter. The index \( n \) can vary from 0 to \( \infty \) and the index \( m \) can vary from \(-\infty \) to \( \infty \) or from 0 to \( \infty \); when \( 0 \leq n \leq N \) and \( 0 \leq m \leq M \), it is a special class of restricted problems. The equation in finite differences (1) can be represented by the expression with operator coefficients

\[
(H\Psi)_n = A_n\Psi(n-1) + V_n\Psi(n) + A_{n+1}\Psi(n+1) = \lambda\Psi(n),
\]

(2)

when we treat one of the variables, \( n \), as the only discrete coordinate; and the other, \( m \), as the channel index. If we set \( \Psi(-1) = 0 \), the action of the Schrödinger discrete operator \( H \) on the vector \( \Psi = \{\Psi(0), \Psi(1), \Psi(2), \ldots, \Psi(n), \ldots\} \) is represented by the action of the Jacobi block matrix \( J \) on \( \Psi \)

\[
(J\Psi)_n = \begin{pmatrix}
V_0 & A_1 & 0 & 0 & 0 & \ldots & 0 \\
A_1 & V_1 & A_2 & 0 & 0 & \ldots & 0 \\
0 & A_2 & V_2 & A_3 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & \ldots & \ldots \\
. & \ldots & 0 & \ldots & 0 & \ldots & \ldots \\
. & \ldots & \ldots & 0 & \ldots & 0 & \ldots \\
. & \ldots & \ldots & \ldots & 0 & \ldots & \ldots \\
. & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\begin{pmatrix}
\Psi(0) \\
\Psi(1) \\
\Psi(2) \\
\Psi(n) \\
\end{pmatrix}
= \lambda
\begin{pmatrix}
\Psi(0) \\
\Psi(1) \\
\Psi(2) \\
\Psi(n) \\
\end{pmatrix}
\]

(3)

whose elements \( V_n \) and \( A_n \) are matrices at each fixed \( n \), and each element \( \Psi(n) \) of the vector \( \Psi \) corresponds to the vector \( \{\psi_m(n)\} \), \( \psi_m(n) \equiv \psi(n, m) \), in the other space variable "\( m \)"

\[
V_n = \begin{pmatrix}
c_{n0} & b_{n1} & 0 & 0 & 0 & \ldots & 0 \\
b_{n1} & c_{n1} & b_{n2} & 0 & 0 & \ldots & 0 \\
0 & b_{n2} & c_{n2} & b_{n3} & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & \ldots \\
. & \ldots & 0 & b_{nm-1} & c_{nm-1} & b_{nm} & . \\
. & \ldots & \ldots & 0 & b_{nm} & c_{nm} & b_{nm+1} \\
. & \ldots & \ldots & \ldots & 0 & \ldots & \ldots \\
. & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

(4)

\[
A_n = \begin{pmatrix}
a_{n0} & 0 & 0 & \ldots & 0 \\
0 & a_{n1} & 0 & \ldots & 0 \\
0 & 0 & a_{n2} & \ldots & 0 \\
0 & 0 & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & a_{nm} & \ldots \\
. & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

It is clear that the operator \( H \) is Hermitian. Note that the coupling of equations (1) or (2) with respect to \( m \) occurs only between immediate neighbours and for this reason
the symmetric coupling matrix \( V_{n;mm'} \equiv V_{mm'}(n) \) in (2) is tridiagonal and relates the functions \( \psi(n,m) \) at \( m, m \pm 1 \), unlike the conventional multichannel case.

From the Jacobi block matrix (3) tridiagonal in the variable \( n \) it is seen that every vector of the solutions \( \Psi(n) \) is connected with the vectors of solutions \( \Psi(n \pm 1) \) at the neighboring points \( n \pm 1 \). As a result, if we take nonhomogeneous boundary conditions at one end of the interval \( 0 \leq n \leq N \), we can obtain the solutions on the whole interval by moving by subsequent steps from that end. It turns out that these vectors of solutions as functions of the spectral parameter \( \lambda \) are polynomials of \( \lambda \) with matrix coefficients [4] which can be orthogonalized in the spectral measure.

The spectral inverse problem is reduced to the construction of potential matrices \( V_n \), \( A_n \) and an unknown system of orthonormal polynomials with the use of the known system of orthonormal polynomials corresponding to the finite-difference equation (2) but with the known matrices \( \hat{V}_n \) and \( \hat{A}_n \).

### A. Orthogonalization of polynomials

Introduce auxiliary solutions \( \varphi_{ms}(n) \equiv \varphi_s(\lambda, n, m) \) and \( \hat{\varphi}_{ms}(n) \) to eq.(1) with the help of the boundary conditions

\[
\varphi_{ms}(-1) = 0; \quad \varphi_{ms}(0) = \delta_{ms}; \\
\hat{\varphi}_{ms}(-1) = 0; \quad \hat{\varphi}_{ms}(0) = \delta_{ms},
\]

where \( s \) is a point on the \( m \) axis. The solutions \( \hat{\varphi}_s \) satisfy the same equation (1) with the known potentials \( \hat{a}_{nm}, \hat{c}_{nm} = \hat{V}_{nm}(n), \hat{b}_{nm+1} = \hat{V}_{nm+1}(n), \hat{b}_{nm} = \hat{V}_{nm-1}(n) \)

\[
\hat{a}_{nm}\hat{\varphi}_{ms}(\lambda, n-1) + \hat{a}_{n+1m}\hat{\varphi}_{ms}(\lambda, n+1) + \sum_{m'=m-1}^{m+1} \hat{V}_{mm'}(n) \hat{\varphi}_{m'n}s(\lambda, n) = \lambda \hat{\varphi}_{ms}(\lambda, n).
\]

As a first boundary condition, we can change the zeroth condition to a more general one corresponding to the homogeneous boundary conditions \((\varphi_{ms}(−1) − \varphi_{ms}(0))/\Delta = D_s\varphi_{ms}(0)\). In what follows we assume the step of finite-difference differentiation \( \Delta \) to equal 1. According to the conditions (3), the functions \( \varphi_{ms}(n) \) and \( \hat{\varphi}_{ms}(n) \) are equal to zero on the lines \((-1, m), (0, m)\) except for the points with \( m = s \) on the line \( n = 0 \) where \( \varphi_{ss}(n) = \hat{\varphi}_{ss}(n) = 1 \). From these values of \( \varphi_s \) and \( \hat{\varphi}_s \) one can find the functions \( \varphi_{ms}(n) \) and \( \hat{\varphi}_{ms}(n) \) at all other points of the coordinate network by using the recurrence equation (1). Since the matrix equation (2) is tridiagonal in the variable \( n \), and owing to the boundary conditions (3), the vectors \( \Phi_s(n) \equiv \{...\varphi_{0s}(n), \varphi_{1s}(n), ...\varphi_{ms}(n), ...\} \) of solutions \( \Phi(n) = (\Phi_0(n), \Phi_1(n), ...\Phi_s(n), ...) \) in the spectral variable \( \lambda \) are polynomials of the \( n \)th degree. At \( n = 1 \) the vector \( \Phi_s(1) \) is a polynomial of the first degree; at \( n = 2 \), the vector \( \Phi_s(2) \) is a polynomial in \( \lambda \) of the second degree, and so on. It appears, however, that different elements \( \varphi_{ms}(n, \lambda) \) of the vector of solutions \( \Phi_s(n) \) have different degrees of polynomials. Because of the three-point coupling with respect to \( m \) the functions \( \varphi_{ms}(n, \lambda) \) are polynomials in \( \lambda \) of the degree \( n − |s − m| \). The maximum \( n \)th degree belongs to the elements with \( m = s \) and the elements \( \varphi_{ms}(n, \lambda) \) vanish at all \( s \) lying out of the values \((n − m, n + m)\). The distribution of polynomial degrees is shown in Fig.1.
FIG. 1. Nonzeroth \( \phi_{ms}(n, \lambda) \) are inside and on boundaries of the cone with its vertex at \((0, s)\); the numbers at the nodes of the net indicate the degrees of \( \varphi_{ms}(n) \) polynomials.

It is clear that the vectors of solutions, whose elements are polynomials in \( \lambda \) can be orthogonalized. Berezanskii has shown [5] that if the elements \( a_{nm}, b_{nm} \) and \( c_{nm} \) in (1) are real and \( a_{nm} > 0 \) \((n = 1, 2, ..., m = ... -1, 0, 1, ...)\), one can introduce polynomials of the first kind 

\[
\varphi_{ms}(\lambda, n) = P^s_n(\lambda)
\]

which are orthogonal with the weight of the spectral matrix \( \rho(\lambda) = \{\rho_{ss}(\lambda)\} \)

\[
\sum_{ss'=\infty} \int_\lambda \varphi_{ms}(\lambda, n) d\rho_{ss'}(\lambda) \varphi_{s'm'}(\lambda, n') = \delta_{nn'} \delta_{mm'}
\]

(7)

and similarly for \( \varphi_{ms}^{\dagger} (\lambda, n) \)

\[
\sum_{ss'} \int \varphi_{ms}^{\dagger} (\lambda, n) d\rho_{ss'}(\lambda) \varphi_{s'm'}^{\dagger} (\lambda, n') = \delta_{nn'} \delta_{mm'}
\]

(8)

The spectral matrix elements \( \rho_{ss'} \) (or \( \varphi_{ss'} \)) are determined by the boundary values of the special solutions \( \psi(n, m) \) (or \( \psi(n', m') \)) obeying the zeroth condition \( \psi(-1, m) = \psi_m(-1) = 0 \) \((m = .. -1, 0, 1...). \) For example, for \( p \) states of the discrete spectrum, the matrix \( \rho(\lambda) \) is the sum of \( p \) terms formed from productions of the column vectors \( \Gamma(\nu) \equiv \{\gamma_s(\lambda_\nu)\} \) and the row vectors \( \Gamma^\dagger(\lambda) \equiv \{\gamma_s(\lambda)\})

\[
\rho_{ss'}(\lambda) = \sum_{\nu=1}^p \theta(\lambda - \lambda_\nu) \gamma_s(\lambda_\nu) \gamma_s'(\lambda_\nu),
\]

where the elements \( \gamma_s(\lambda_\nu) = \psi_s(0, \lambda_\nu) \) are defined by \( \psi(n = 0, m) \) and \( \theta(\lambda - \lambda_\nu) \) is the Heaviside step function equal to 1 at \( \lambda = \lambda_\nu \) and to zero when \( \lambda \neq \lambda_\nu \). A solution \( \psi(n, m) \) to eq. (1) with the zeroth condition (or a more general homogeneous condition ) can be obtained by multiplying the matrix \( \Phi(\lambda, n) \) by the vector \( \Gamma(\lambda) = \Psi(0, \lambda): \)

\[
\psi(n, m, \lambda) = \sum_{s=-\infty}^\infty \varphi_{ms}(n, \lambda) \psi_s(0, \lambda).
\]

(9)

At every fixed \( n \) and \( m \) the summation over \( s \) is finite owing to \( \varphi_{ms}(n, \lambda) = 0 \) outside the interval \((n - m, n + m)\).

\(^\dagger\) Two solutions of the same equation of second order differ from each other by a normalization factor at those points \( \lambda \) where both of them exist when one of the boundary conditions is the same.
B. A discrete version of the Gelfand–Levitan inverse problem

Using the procedure of orthogonalization of polynomials we construct unknown polynomial solutions \( \varphi_{ms}(\lambda, n) \) normalized with the spectral weight \( \rho_{ss'}(\lambda) \) as a linear combination of the known polynomial solutions \( \hat{\varphi}_{ms}(\lambda, n) \), orthogonal with respect to the measure \( \hat{\rho}_{ss'}(\lambda) \)

\[
\varphi_{ms}(\lambda, n) = \sum_{n' = 0}^{n} \frac{m+(n-n')}{m-(n-n')} K(n, m; n', m') \hat{\varphi}_{ms}(\lambda, n').
\]  

(10)

As one can see above, the validity of this relation is a consequence of the fact that both the functions, \( \hat{\varphi}_{ms}(\lambda, n) \) and \( \varphi_{ms}(\lambda, n) \), obey the same discrete Schrödinger equation (1) and the same boundary conditions (2), given on the lines \( n = -1 \) and \( n = 0 \). Therefore, the solutions \( \varphi_{ms}(\lambda, n) \) and \( \hat{\varphi}_{ms}(\lambda, n) \) are polynomials in \( \lambda \) of the same degree \( n - |s - m| \), but with different coefficients. The matrix of the polynomial solutions \( \Phi(\lambda, n) \), which is of degree \( n \), is orthogonal to every polynomial matrix of degree lower than \( n \) and hence to every \( \Phi(\lambda, n') \) for \( n' < n \)

\[
\sum_{ss'} \int \varphi_{ms}(\lambda, n) d\rho_{ss'}(\lambda) \hat{\varphi}_{s'm'}(\lambda, n') = 0.
\]  

(11)

Equation (11) is a discrete analog of the Volterra integral equation, in which the coefficients \( K(n, m; n', m') \) are determined by the condition of orthogonality of the vector-functions \( \Phi(\lambda, n) = (\Phi_1(\lambda, n), \Phi_2(\lambda, n), ..., \Phi_s(\lambda, n), ...) \) orthogonal with the spectral measure \( \rho(\lambda) \) to the functions \( \hat{\Phi}(\lambda, n') \) orthogonal with the weight matrix \( \hat{\rho}(\lambda) \), when \( n' \leq n \)

\[
\sum_{ss'} \int \varphi_{ms}(\lambda, n)(d\hat{\rho}_{ss'}(\lambda) - d\rho_{ss'}(\lambda)) \hat{\varphi}_{s'm'}(\lambda, n') = K(n, m; n', m').
\]  

(12)

The Volterra equations (11) have a triangular form, \( K(n, m; n', m') = 0 \) for \( n' > n \). It is easy to see from (11) and orthogonality of \( \hat{\varphi}_{sm}(\lambda, n) \) (8) that for \( n' < n \)

\[
K(n, m; n', m') = \sum_{ss'} \int \varphi_{ms}(\lambda, n)d\hat{\rho}_{ss'}(\lambda) \hat{\varphi}_{s'm'}(\lambda, n').
\]

Inserting (11) into (12) for \( n' < n \) we obtain the following system of equations for the orthogonalization coefficients \( K(n, m; n', m') \)

\[
K(n, m; n', m') + K(n, m; n, m)Q(n, m; n', m') + \\
+ \sum_{n''=0}^{n-1} \sum_{m''=m-(n-n'')} K(n, m; n', m'')Q(n'', m''; n', m') = 0,
\]  

(13)

where

\[
Q(n, m; n', m') = \sum_{ss'} \int \hat{\varphi}_{ms}(\lambda, n)(d\rho_{ss'}(\lambda) - d\hat{\rho}_{ss'}(\lambda)) \hat{\varphi}_{s'm'}(\lambda, n').
\]  

(14)
Equation (13) is a two-dimensional analog of the Gelfand-Levitan integral equations for finite-difference equation (1). However, this system is not sufficient to determine the coefficients $K$. A supplementary system of equations is obtained upon substituting (14) into the complete relation (7) at $n' = n$ and $m' = m$

$$K^{-2}(n, m; n, m) = Q(n, m; n, m) +$$

$$+ \sum_{n''=0}^{n-1} \sum_{m''=m-(n-n'')} K^{-1}(n, m; n, m) K(n, m; n'', m'') Q(n'', m''; n, m). \quad (15)$$

Now let us find connections between the potential coefficients and orthogonalization coefficients $K(n, m; n', m')$. To this end, we insert expression (14) for the polynomial functions $\varphi_{ms}(\lambda, n)$ in terms of $\varphi\hat{m}'s (\lambda, n)$ into the Schrödinger difference equation (1)

$$a_{nm} \sum_{n'=0}^{n-1} \sum_{m'=m-(n-n')'} K(n - 1, m; n', m') \varphi_{m's} (\lambda, n')$$

$$+ a_{n+1m} \sum_{n'=0}^{n+1} \sum_{m'=m-(n+1-n')'} K(n + 1, m; n', m') \varphi_{m's} (\lambda, n')$$

$$+ \sum_{m''=m-1}^{m} V_{m'm''}(n) \sum_{n'=0}^{n} \sum_{m'=m''-(n-n')'} K(n, m''; n', m') \varphi_{m's} (\lambda, n')$$

$$= \lambda \sum_{n'=0}^{n} \sum_{m'=m-(n-n')'} K(n, m; n', m') \varphi_{m's} (\lambda, n'). \quad (16)$$

We transform the r.h.s. of the above relation by means of the substitution of $\lambda \varphi_{m's} (\lambda, n)$ from the finite difference equation (8) for $\varphi_{ms} (\lambda, n)$ with the known potentials $\hat{a}_{nm}, V_{m'm'} (n), (m' = m - 1, m, m + 1)$

$$\lambda \sum_{n'=0}^{n} \sum_{m'=m-(n-n')'} K(n, m; n', m') \varphi_{m's} (\lambda, n') =$$

$$+ \sum_{n'=0}^{n} \sum_{m'=m-(n-n')'} K(n, m; n', m') (\hat{a}_{n'm'} \varphi_{m's} (\lambda, n' - 1)$$

$$+ \hat{a}_{n'+1m} \varphi_{m's} (\lambda, n' + 1) + \sum_{m''=m-1}^{m'+1} V_{m'm''}(n') \varphi_{m''s} (\lambda, n')). \quad (17)$$

Further, we take advantage of the orthogonality relation (8) for the matrix functions $\hat{\Phi} (\lambda, n)$ orthogonal with the weight matrix $\hat{\rho} (\lambda)$. Multiplying expression (16) with its transformed r.h.s. (17) by $\varphi_{s'm} (\lambda, n + 1)$, integrating over $d \hat{\rho}_{ss'} (\lambda)$, and summing up over the indices $s$ and $s'$, we arrive at the relationship between the potentials $a_{nm}, \hat{a}_{nm}$ and the coefficients $K(n, m; n' m')$

$$a_{n+1m} = \hat{a}_{n+1m} \frac{K(n, m; n, m)}{K(n + 1, m; n + 1, m)}. \quad (18)$$
The relations for the coefficients \( c_{nm} \) and \( b_{nm} \) are established in a similar manner. To determine \( b_{nm+1} = V_{nm+1}(n) \), eq. (16) is multiplied by \( \hat{\varphi}_{s'm+1}(\lambda, n) \) with (17) taken into account and integrated with the weight \( \hat{\varrho}_{ss'}(\lambda) \) by using the orthogonality (8). As a result, we have

\[
b_{nm+1} = b_{nm+1} \frac{K(n, m; n, m)}{K(n, m + 1; n, m + 1)} + \hat{a}_{nm+1} \frac{K(n, m; n - 1, m + 1)}{K(n, m + 1; n, m + 1)} - \hat{a}_{n+1m} \frac{K(n + 1, m; n, m + 1)}{K(n, m + 1; n, m + 1)}. \tag{19}
\]

The relation for \( c_{nm} = V_{nm}(n) \) is derived analogously, only (16) is multiplied by \( \hat{\varphi}_{s'm}(\lambda, n) \)

\[
c_{nm} = c_{nm} + \hat{a}_{nm} \frac{K(n, m; n - 1, m)}{K(n, m; n, m)} - \hat{a}_{n+1m} \frac{K(n + 1, m; n, m)}{K(n, m + 1; n, m + 1)}. \tag{20}
\]

Substituting (18) into (20) we arrive at

\[
c_{nm} = c_{nm} + \hat{a}_{nm} \frac{K(n, m; n - 1, m)}{K(n, m; n, m)} - \hat{a}_{n+1m} \frac{K(n + 1, m; n, m)}{K(n + 1, m; n, m + 1)}. \tag{21}
\]

At \( a_{nm} = \hat{a}_{nm} = 1, b_{nm} = \hat{b}_{nm} = 1 \), the derived generalized expressions turn into more simple ones presented in [7]. The two-dimensional finite-difference inverse problem under consideration is also a generalization of that [15] with the potential coefficients \( a_{nm} \neq 1, b_{nm} \neq 1 \), connected nevertheless in a special way.

In principle, it is easy to formulate the problem of restoring the matrix \( V_{nm'}(n) \) in (2) with all non-zeroth elements, like in a multichannel problem [7]. The latter would correspond to the potential being nonlocal with respect to one of the coordinate variables (in our case “m”). If the consideration were made in the polar coordinate system, nonlocality with respect to discrete angles would occur. In the case of continuous coordinates, the inverse problem for the potential, nonlocal relative to angles, was considered by Kay and Moses [14].

### III. Bargmann–Darboux Transformations for the Two-Dimensional Discrete Schrödinger Equation

In this section, we describe the algebraic procedure by taking into consideration simple kernels \( Q \) in the form of a sum of several terms with a factorized coordinate dependence

\[
Q(n, m; n', m') = \sum_{\mu=1}^{p} \hat{\psi}(\lambda_{\mu}, n, m) \hat{\varphi}(\lambda_{\mu}, n', m') = \sum_{\mu=1}^{p} \sum_{s} \hat{\varphi}_{ms}(\lambda_{\mu}, n) \gamma_{s}(\lambda_{\mu}) \sum_{s'} \gamma_{s'}(\lambda_{\mu}) \hat{\varphi}_{s'm'}(\lambda_{\mu}, n'). \tag{22}
\]

Here the functions \( \hat{\psi}(\lambda_{\mu}, n, m) \equiv \hat{\psi}_{m}(\lambda_{\mu}, n) \) are combined as elements of the vector \( \hat{\psi}(\lambda_{\mu}, n) = (...) \hat{\psi}_{1}(n), \hat{\psi}_{2}(n), ..., \hat{\psi}_{m}(\lambda_{\mu}, n), ... \) obtained as a product of the matrix
solutions $\hat{\Phi}(\lambda_\mu, n)$ taken at eigenvalues $\lambda = \lambda_\mu$ of the reconstructed $H$ and the vector \( \Gamma(\lambda_\mu) \)

\[
\hat{\psi}(\lambda_\mu, n, m) = \sum_s \varphi_{ms}(\lambda_\mu, n) \gamma_s(\lambda_\mu).
\]

The elements $\gamma_s$ form the normalization matrix $C(\lambda_\mu) = \Gamma(\lambda_\mu)\Gamma^\dagger(\lambda_\mu)$ with elements $C_{ss'}(\lambda_\mu) = \gamma_s(\lambda_\mu)\gamma_{s'}(\lambda_\mu)$ corresponding to the bound state $\psi_{\mu}(n, m) \equiv \psi(\lambda_\mu, n, m)$, $\mu = 1, 2, \ldots p$.

Like $Q$, the ortogonalization kernel $K(n, m; n'm')$ is also presented as a sum of several factorized terms. Really, substituting (22) into the Gelfand Levitan equation (13), we obtain

\[
K(n, m; n', m') = -\sum_{\mu=1}^{p} \left( \sum_{n''=0}^{n} \sum_{m''=m-(n-n'')} K(n, m''; n'', m'') \, \hat{\psi}(\lambda_\mu, n'', m'') \right) \hat{\psi}(\lambda_\mu, n', m').
\]

Noting that the expression in braces is the solution $\psi_{\mu}(n, m)$ (14) at $\lambda = \lambda_\mu$ of eq.(11) with the desired potentials $a_{nm}, b_{nm}$ and $c_{nm}$, we immediately receive

\[
K(n, m; n', m') = -\sum_{\mu=1}^{p} \psi_{\mu}(n, m) \hat{\psi}(\lambda_\mu, n', m').
\]

It is evident now that the new wave functions $\varphi_{ms}(\lambda, n)$, determined by (10) with the kernel $K$ taken in the form (24), are related to the old ones $\hat{\varphi}_{ms}(\lambda, n)$ by

\[
\varphi_{ms}(\lambda, n) = -\sum_\mu \psi_{\mu}(n, m) \sum_{n''=0}^{n} \sum_{m''=m-(n-n'')} \hat{\psi}(\lambda_\mu, n'', m'') \hat{\varphi}_{ms'}(\lambda, n').
\]

In view of (24) for $K(n, m; n', m')$ in eqs. (18), (19) and (20), one can immediately write expressions for discrete potentials in the closed form

\[
a_{n+1m} = a_{n, m+1} \frac{\sum_{\mu=1}^{p} \psi_{\mu}(n, m) \hat{\psi}(\lambda_\mu, n, m)}{\sum_{\mu=1}^{p} \psi_{\mu}(n + 1, m) \hat{\psi}(\lambda_\mu, n + 1, m)};
\]

\[
b_{nm+1} = b_{nm} \frac{\sum_{\mu=1}^{p} \psi_{\mu}(n, m) \hat{\psi}(\lambda_\mu, n, m)}{\sum_{\mu=1}^{p} \psi_{\mu}(n, m + 1) \hat{\psi}(\lambda_\mu, n, m + 1)} + a_{n, m+1} \frac{\sum_{\mu=1}^{p} \psi_{\mu}(n, m - 1) \hat{\psi}(\lambda_\mu, n - 1, m + 1)}{\sum_{\mu=1}^{p} \psi_{\mu}(n, m + 1) \hat{\psi}(\lambda_\mu, n, m + 1)} - a_{n+1m} \frac{\sum_{\mu=1}^{p} \psi_{\mu}(n + 1, m) \hat{\psi}(\lambda_\mu, n, m + 1)}{\sum_{\mu=1}^{p} \psi_{\mu}(n, m + 1) \hat{\psi}(\lambda_\mu, n, m + 1)}
\]

and
\[ c_{nm} = c_{nm} + \sum_{\mu=1}^{p} \psi_\mu(n, m) \omega_\mu(n, n-1, m) \]

The solutions \( \psi_\mu(n, m) \) have to be found from the Gelfand-Levitan equations (13) and (15) taking account of (22) and (24). Thus, the operator \( K(n, m; n', m') \) defined by (24) transforms the solutions \( \tilde{\varphi}_{nm} \) of eq. (3) into the solutions \( \varphi_{nm}(n) \) of eq. (1), determined by (10) or (25), with the potentials \( a_{nm}, b_{nm} \) and \( c_{nm} \) defined by (23), (27) and (28).

It is not difficult to see from the definitions (14) and (12) that the kernels \( Q \) and \( K \) like (22) and (24) can be obtained provided that the spectral weight functions \( \rho(\lambda) \) and \( \check{\rho}(\lambda) \) for both the sets of potentials coincide except, for instance, \( p \) eigenvalues at \( \lambda = \lambda_\mu \). This permits one to construct potentials with \( p \) new bound states by using (26), (28) and (27) or generate the family of spectral-equivalent potentials whose spectra coincide \( \lambda_\mu = \lambda_\mu \) and it is only the normalization factors \( C_\mu \neq C_\mu \) that are different. In the latter case \( Q \) is taken in the form

\[ Q(n, m; n', m') = \sum_{\mu=1}^{p} \sum_{ss'} \tilde{\varphi}_{ms}(\lambda_\mu, n)(C_{ss'}(\lambda_\mu) - \tilde{C}_{ss'}(\lambda_\mu)) \tilde{\varphi}_{s'm'}(\lambda_\mu, n') \]

and the above procedure can be used to construct spectral-equivalent operators \( \tilde{H} \) and \( H \). In spite of a complicated form of the expressions for the potentials (26), (27) and (28), they are simplified for a large set of particular cases. For example, if we deal with the free discrete Schrödinger equation as a reference one, fixed by the choice \( c_{nm} = 0, a_{nm} = b_{nm} = 1 \).

Let us now consider another simple case when the Hamiltonians differ only by spectral data at one bound state. In this case, the summation over \( \mu \) in all formulae (25) – (28) vanishes. General solutions \( \varphi_{sm}(\lambda, n) \) from (23) at arbitrary \( \lambda \) can be written as

\[ \varphi_{ms}(\lambda, n) = -\psi(n, m) \sum_{n'=0}^{m} \sum_{m'=m-(n-n')} \tilde{\psi}(n', m') \tilde{\varphi}_{m's}(\lambda, n'), \]

where \( \tilde{\psi}(n, m) \) is a special solution of the Schrödinger equation (3) for the value of the spectral parameter \( \lambda = \mu \) and \( \psi(n, m) \) is the solution of eq. (1) with the same eigenvalue \( \lambda = \mu \). New potentials are expressed in terms of the known old potentials \( a_{nm}, \tilde{c}_{nm}, b_{nm}, \) and \( \tilde{a}_{nm}, \tilde{b}_{nm}, \) functions \( \tilde{\psi}(n, m) \) and functions \( \psi(n, m) \) that can be determined from the second Gelfand-Levitan equation (15)

\[ a_{n+1,m} = \tilde{a}_{n+1,m} \frac{\psi(n, m) \tilde{\psi}(n, m)}{\psi(n, m) \tilde{\psi}(n, m)}, \]

\[ b_{nm+1} = \frac{\psi(n, m)}{\psi(n, m + 1)} \left( b_{nm+1} \frac{\tilde{\psi}(n, m)}{\tilde{\psi}(n, m + 1)} \right) \]
\[
\psi (n - 1, m + 1) - \lambda \psi (n, m + 1) - \lambda \phi (n, m + 1) = 0
\]

and

\[
c_{nm} = c_{nm} + a_{nm} \frac{\psi (n - 1, m)}{\psi (n, m)} - a_{n+1m} \frac{\psi (n, m)}{\psi (n + 1, m)}. \tag{33}
\]

It should be noted that relationships between potentials and functions can be obtained within the Darboux transformation method or factorised method without using formulae of the inverse problem.

**Connection between Darboux transformations and inverse problem ones.** It is interesting to note that transformation \((30)\) with one bound state corresponds to Darboux transformation for the finite-difference equation (1). Indeed, let us search for a solution to the Schrödinger equation \((1)\). Substitute \((30)\) into \((1)\)

\[
\phi = \sum_{m} \sum_{n} a_{nm} \psi (n, m) \phi (n, m). \tag{34}
\]

Transform the r.h.s. of \((34)\) substituting \(\psi (n, m)\) from \((30)\)

\[
\lambda \psi (n, m) \sum_{n'} \sum_{m'} \psi (n', m') \phi (n', m') = \lambda \psi (n, m) \sum_{n'} \sum_{m'} \psi (n', m') \phi (n', m'). \tag{35}
\]

Further, to obtain the relations for potentials \(a_{n+1m}, b_{nm+1} \) and \(c_{nm}\), multiply eq.\((34)\) with its transformed r.h.s. \((35)\) by \(\phi_{m's} (\lambda, n')\), \(\phi_{m's-1} (\lambda, n)\) and \(\phi_{m's} (\lambda, n)\) and take into consideration the completeness relation \((8)\) for the functions \(\phi (\lambda)\). The expressions for \(a_{nm}, b_{nm} \) and \(c_{nm}\) thus derived coincide with formulae \((31)\), \((32)\) and \((33)\), correspondingly, obtained from the formulae of the inverse problem.
IV. CONCLUSION

The Gelfand-Levitan spectral inverse problem for the discrete two-dimensional Schrödinger equation is considered on the basis of the Berezanskii technique of orthogonalization of polynomial matrices. By using the derived formulae of the inverse problem, discrete Bargmann–Darboux transformations in two dimensions are given. Analytic relationships are established between the solutions for two different sets of discrete potentials and the potentials themselves.

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