EXACT STRING SOLUTIONS AND THE OPEN STRING - D-BRANE SYSTEM IN NON-CONSTANT BACKGROUND FIELDS I.

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New exact string solutions in non-constant background fields are found and it is shown that some of them are compatible with the boundary conditions for the open string - D-brane system. Extension of the constraint algebra is proposed and discussed.

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1 Introduction

Obtaining exact solutions of the nonlinear probe string equations of motion and constraints in variable external fields is by all means an interesting task with many possible applications. One such application is connected with the recent investigations of the open string - D-brane system in non-constant background fields [1]-[5]. Of course, in this case, one is forced to use different approximations in order to get explicit results. That is why, it is interesting to see to what extent our knowledge about the existing exact string solutions can help us in considering this dynamical system. More concretely, which kind of solutions are compatible with the mixed boundary conditions characterizing the open string - D-brane system.

In this letter, we consider a few types of exact solutions of the probe string equations of motion and constraints in non-constant background metric and NS-NS two-form gauge field. Then we check their compatibility with the open string - D-brane boundary conditions. It turns out that there exist one type of exact string solutions, which gives also nontrivial solution of the mixed boundary conditions for the open string - D-brane system. After that, we reinterpret the conditions for existence of such solutions as a set of constraints and compute their Poisson bracket algebra.

2 Exact string solutions in non-constant background fields

In this section, our aim is to describe several types of exact solutions for a string moving in background gravitational and NS-NS fields. To this end, we start with the sigma-model action for $D$-dimensional space-time

$$ S_1 = -\frac{T}{2} \int d^2 \xi \left[ \sqrt{-g} \gamma^{mn} \partial_m X^M \partial_n X^N g_{MN}(X) - \varepsilon^{mn} \partial_m X^M \partial_n X^N B_{MN}(X) \right], $$

$$ \partial_m = \partial/\partial \xi^m, \quad \xi^m = (\xi^0, \xi^1) = (\tau, \sigma), \quad m, n = 0, 1, \quad M, N = 0, 1, ..., D - 1, $$

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where \( T = (2\pi \alpha')^{-1} \) is the (fundamental) string tension, \( G_{mn}(X) = \partial_M X^M \partial_N X^N g_{MN}(X) \), \( B_{mn}(X) = \partial_M X^M \partial_N X^N B_{MN}(X) \) are the pullbacks of the background metric and antisymmetric NS-NS tensor to the string worldsheet, and \( \gamma \) is the determinant of the auxiliary metric \( \gamma_{mn} \).

Varying (1) with respect to \( X^M \) and \( \gamma_{mn} \), we obtain the equations of motion

\[
-g_{LK} \left[ \partial_m \left( \sqrt{-\gamma} \gamma^{mn} \partial_n X^K \right) + \sqrt{-\gamma} \gamma^{mn} \Gamma^K_{MN} \partial_M X^M \partial_N X^N \right] = \frac{1}{2} H_{LMN} \varepsilon^{mn} \partial_m X^M \partial_n X^N, \tag{2}
\]

and the constraints

\[
\left( \gamma^{kl} \gamma^{mn} - 2 \gamma^{km} \gamma^{ln} \right) \partial_m X^M \partial_n X^N g_{MN}(X) = 0, \tag{3}
\]

where

\[
\Gamma^K_{MN} = \frac{1}{2} g^{KL} \left( \partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN} \right)
\]

is the connection compatible with the metric \( g_{MN} \) and

\[
H_{LMN} = \partial_L B_{MN} + \partial_M B_{NL} + \partial_N B_{LM}
\]

is the \( B_{MN} \) field strength. From now on, we will work in the gauge \( \gamma_{mn} = \text{constants} \).

First of all, we will try to find a background independent solution of the equations of motion of the type

\[
X^M(\xi) = F^M(a_n \xi^n), \quad a_n = \text{constants}.
\]

It turns out that such solution exists when \( \gamma^{mn} a_m a_n = 0 \). This leads to

\[
X^M(\xi) = F^M(\pm), \quad u_\pm = -\frac{1}{\gamma^{00}} \left( \gamma^{01} \pm \frac{1}{\sqrt{-\gamma}} \right) \xi^0 + \xi^1,
\]

or

\[
X^M(\xi) = F^M(\pm), \quad v_\pm = \xi^0 + \frac{1}{\gamma^{10}} \left( -\gamma^{01} \pm \frac{1}{\sqrt{-\gamma}} \right) \xi^1,
\]

where \( F^M_\pm \) are arbitrary functions of their arguments. The main consequence of the obtained result is that for arbitrary background fields there exist only one solution, \( F^M_+ \) or \( F^M_- \), but not both at the same time. In other words, we have only chiral background independent solutions of the string equations of motion. On the other hand, it must be noted that these are not solutions of the constraints (3). Taking a linear combination of the two independent constraints, we can arrange one of them to be satisfied, but the other one will give restrictions on the metric. However, it can be shown that in the zero tension limit, the background independent solutions of the string equations of motion are also solutions of the corresponding constraints. Moreover, this result extends to arbitrary tensionless \( p \)-branes \[6\]. It corresponds to the limit \((-\gamma)^{-1/2} \to 0\), taken in the expressions for \( u_\pm \) and \( v_\pm \). Let us also note that in conformal gauge, the obtained string solutions \( F^M_\pm(u_\pm) \) and \( F^M_\pm(v_\pm) \) reduce to the solutions \( X^M_\pm(\sigma \pm \tau) \) and \( X^M_\pm(\tau \pm \sigma) \) for left- or right-movers. These are known to be the only background independent non-perturbative solutions for an arbitrary static metric, which are stable and have a conserved topological charge being therefore topological solitons \[7\].
A step further is to search for exact solutions of the string equations of motion and of the constraints of the type

\[ X^M(\xi^m) = F^M_\pm (w_\pm) + y(\xi^0), \quad \text{or} \quad X^M(\xi^m) = F^M_\pm (w_\pm) + z(\xi^1), \]

where \( w_\pm = u_\pm \) or \( w_\pm = v_\pm \). The desired property of such ansatz is to allow for the separation of the variables \( \xi^0 \) and \( \xi^1 \). It can easily be shown that this is achieved only when \( F^M_\pm \) are linear functions of \( w_\pm \)

\[ F^M_\pm (w_\pm) = C^M_\pm w_\pm, \quad C^M_\pm = \text{constants}. \]

In the zero tension limit, the condition for separation of the variables \( \xi^n \) is less restrictive and gives

\[ F^M(w) = C^M F(w), \quad w = u \quad \text{or} \quad w = v, \]

where \( F(w) \) is arbitrary function and we have omitted the subscripts \( \pm \) because in this case \( w_+ = w_- = w \).

As far as every physically relevant background metric has some symmetries, let us split the coordinates \( x^M = (x^\mu, x^a) \), \( \{\mu\} \neq \{0\} \) and let us suppose that there exist an (unspecified) number of independent Killing vectors \( \eta_{\mu} \). Then in appropriate coordinates \( \eta_{\mu} = \partial/\partial x^\mu \) and the metric depends on \( x^a \) only. We also assume that the NS-NS 2-form field has at least the same symmetry, i.e. \( \partial_\mu g_{MN} = \partial_\mu B_{MN} = 0 \). It is natural to expect that this restrictions will give us the possibility to find \( \text{dim} \{\mu\} \) conserved quantities (independent on \( \xi^0 \) or \( \xi^1 \) respectively). It turns out this is indeed the case, when \( C^M_\pm = (C^M_\pm, 0) \). To be more specific, let us use the ansatz

\[ X^\mu(\xi^m) = C^\mu_\pm u_\pm + y^\mu(\xi^0), \]
\[ X^a(\xi^m) = y^a(\xi^0). \]

In order to give a unified description of the tensile and tensionless string solutions, we parameterize the auxiliary metric \( \gamma^{mn} \) as follows:

\[ \gamma^{00} = -1, \quad \gamma^{01} = \lambda^1, \quad \gamma^{11} = (2\lambda^0 T)^2 - (\lambda^1)^2. \]

Then the Euler-Lagrange equations and the independent constraints take the form [8] (the over-dot is used for \( \partial/\partial \xi^0 \))

\[ g_{KL} \ddot{y}^L + \Gamma_{KMN} \dot{y}^M \dot{y}^N + 2\lambda^0 T C^\mu_\pm (H_{K\mu N} \pm 2 \Gamma_{K,\mu N}) \dot{y}^N = 0, \]

\[ g_{MN} (y^a)^\dot{y}^M \dot{y}^N = 0, \quad C^\mu_\pm \left[ g_{\mu N} (y^a) \dot{y}^N \pm 2 \lambda^0 T C^\nu_{\pm} g_{\nu \mu} (y^a) \right] = 0. \]

The corresponding conserved quantities are

\[ g_{\mu N} \dot{y}^N + 2 \lambda^0 T C^\nu_\pm (B_{\mu \nu} \pm g_{\mu \nu}) = B^\pm_\mu = \text{constants}. \]

They are compatible with the constraint [6] when \( B^\pm_\mu C^\mu_\pm = 0 \). Using [7], the equations of motion for \( y^a \) and the other constraint [5] can be transformed into

\[ 2 \left( h_{ab} \dot{y}^b \right) - (\partial_a h_{bc}) \dot{y}^b \dot{y}^c + \partial_a V_B^\pm = 4 \partial_{[a} B^\pm_{bc]} \dot{y}^c, \]
\[ h_{ab} \dot{y}^a \dot{y}^b + V_B^\pm = 0, \]
where
\[ h_{ab} \equiv g_{ab} - g_{a\mu}(g^{-1})^{\mu\nu}g_{\nu b}, \]
\[ V^\pm_B \equiv (2\lambda^0 T)^2 C_{\pm}C_{\pm}^{\mu}g_{\mu\nu} + (B^\pm_{\mu} - 2\lambda^0 TB_{\mu\lambda}C^\lambda_{\pm}) (g^{-1})^{\mu\nu}(B^\pm_{\nu} - 2\lambda^0 TB_{\nu\lambda}C^\lambda_{\pm}), \]
\[ B^\pm_a \equiv g_{a\mu}(g^{-1})^{\mu\nu}B^\pm_{\nu} + 2\lambda^0 T (B_{a\lambda} - g_{a\mu}(g^{-1})^{\mu\nu}B_{\nu\lambda}) C^\lambda_{\pm}. \]

If we restrict the metric \( h_{ab} \) to be a diagonal one, i.e.
\[ g_{ab} = g_{a\mu}(g^{-1})^{\mu\nu}g_{\nu b}, \text{ for } a \neq b, \]
the equations (8) can be further transformed into
\[ \left((h_{aa} y^a)^2\right)' + y^a \partial_a \left(h_{aa} V^+_B\right) + y^a \sum_{b \neq a} \left[ \partial_a \left(\frac{h_{aa}}{h_{bb}}\right) \left(h_{bb} y^b\right)^2 - 4\partial_a B^+_B h_{aa} y^b \right] = 0 \]
(two summations over \( a \)). Now it is evident that we can always integrate these equations, if all coordinates on which the background fields depend, except one, are kept fixed. In the general case, additional restrictions on the metric and on the Kalb-Ramond field will arise, to ensure the separation of the variables \( y^a \) [8]. These conditions are such that the metric and the \( B \)-field are general enough to include many interesting cases of (super)string backgrounds in different dimensions. If the solutions for \( y^a \) are already known, the solutions for \( y^\mu \) are obtainable from (7). Let us finally note that in the framework of this approach, exact \( D \)-string solutions are also found [8].

Our next step is to search for non-chiral solutions of the string equations of motion and constraints, i.e. solutions of the type
\[ X^M(\xi^m) = F^M_+(w_+) + F^M_-(w_-). \]
In the particular case \( B_{MN} = 0 \), and in conformal and light-cone gauges, it is known that such solutions do exist [8]. To describe them, we split the string coordinates
\[ X^i = (X^A, X^\alpha), \quad A = 1, ..., D' - 1, \quad \alpha = D', ..., D - 2, \]
and assume that \( X^A \) are left-movers, and \( X^\alpha \) are right-movers
\[ X^A \equiv X^A_+, \quad X^\alpha \equiv X^\alpha_-, \quad (\partial_0 \mp \partial_1)X_\pm = 0. \]
(10)
The metric is supposed to be of the form
\[ g_{MN} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & g_{AB} & g_{A\beta} \\ 0 & 0 & g_{aB} & g_{a\beta} \end{pmatrix}. \]
Then, the background allowing both movers is given by
\[ g_{AB} = g_{AB}(X^A_+), \quad g_{a\beta} = g_{a\beta}(X^\alpha_-), \quad g_{aB} = \partial_0 \partial_B f(X^A_+, X^\alpha_-), \]
with \( f(X^A_+, X^\alpha_-) \) being an arbitrary function. Obviously, there are no non-chiral coordinates in this solution: the different chirality is associated with different string coordinates.
Now, we are going to show that there exist exact solutions of the string equations of motion and constraints in non-constant background fields $g_{MN}$ and $B_{MN}$, which possess non-chiral coordinates. Putting (1) in the equations of motion (2), we obtain the conditions for the existence of such solutions:

$$(2\Gamma_{L, MN} + H_{LMN}) \frac{dF^M}{dw_+} \frac{dF^N}{dw_-} = 0.$$  \hspace{1cm} (11)

For simplicity, we will consider the case when $g_{MN}$ and $B_{MN}$ depend on only one coordinate, say $r$, and will give the results in conformal gauge.

In our first example, we fix all string coordinates $X^M$ except $X^0$ and $r$ (the remaining coordinates are denoted as $X^\alpha$). Then the conditions (11) and constraints (3) reduce to the system of equations

$$\partial_r g_{00}[(\partial_0 X^0)^2 - (\partial_1 X^0)^2] - \partial_r g_{rr}[(\partial_0 r)^2 - (\partial_1 r)^2] = 0,$$

$$\partial_r B_{0\alpha}(\partial_0 X^0 \partial_1 r - \partial_1 X^0 \partial_0 r) + \partial_r g_{\alpha\alpha}(\partial_0 X^0 \partial_0 r - \partial_1 X^0 \partial_1 r) + \partial_r g_{r\alpha}[(\partial_0 r)^2 - (\partial_1 r)^2] = 0,$$

$$\partial_r g_{00}(\partial_0 X^0 \partial_0 r - \partial_1 X^0 \partial_1 r) + \partial_r g_{0r}[(\partial_0 r)^2 + (\partial_1 r)^2] + 2\partial_r g_{00}(\partial_0 X^0 \partial_0 r + \partial_1 X^0 \partial_1 r) = 0,$$

$$\partial_r g_{00} \partial_0 X^0 + g_{rr} \partial_0 r \partial_1 r + g_{0r}(\partial_0 X^0 \partial_1 r + \partial_1 X^0 \partial_0 r) = 0.$$

Among the nontrivial solutions of the above system, there exist the following non-chiral ones:

$$\frac{\partial_r g_{rr}}{\partial_r g_{00}} = \frac{\partial_r g_{0r}}{\partial_r g_{\alpha\alpha}} = \frac{\partial_r g_{r\alpha}}{\partial_r g_{0\alpha}} = \frac{(\partial_r g_{00})^2}{\partial_r g_{00}}, \quad g_{rr} = \left(2g_{0r} - g_{00} \frac{\partial_r g_{0r}}{\partial_r g_{00}} \right) \frac{\partial_r g_{0r}}{\partial_r g_{00}},$$

$$\frac{\partial_r g_{00}}{\partial_r g_{0r}} = \frac{\partial_r g_{0r}}{\partial_r g_{00}} = \frac{\partial_r g_{r\alpha}}{\partial_r g_{0\alpha}} = \frac{\partial_r g_{0\alpha}}{\partial_r g_{0\alpha}},$$

$$\frac{\partial_r g_{00}}{\partial_r g_{0r}} = -\frac{\partial_r g_{0r}}{\partial_r g_{00}} = \frac{\partial_r g_{r\alpha}}{\partial_r g_{0\alpha}} = -\frac{\partial_r g_{0\alpha}}{\partial_r g_{0\alpha}},$$

$$\partial_r g_{00}(\partial_0 X^0 \partial_0 r - \partial_1 X^0 \partial_1 r) + \partial_r B_{0\alpha}(\partial_0 X^0 \partial_1 r - \partial_1 X^0 \partial_0 r) = 0,$$

$$= -\frac{\partial_r g_{00}(\partial_0 X^0 \partial_0 r - \partial_1 X^0 \partial_1 r)}{\partial_r g_{0r}} \partial_r g_{0r},$$

$$= \partial_r g_{00} \left[ (\partial_0 X^0)^2 - (\partial_1 X^0)^2 \right] \partial_r g_{rr},$$

$$= (\partial_0 r)^2 - (\partial_1 r)^2.$$

In our second example, all string coordinates are kept fixed except $X^0$, $X^1$ and $r$. Only to have readable final expressions, we restrict the metric to be diagonal, and the NS-NS field to be constant. The solutions of the corresponding equations following from (11) and (3) are

$$\partial_0 X^0 = + f \partial_1 r, \quad \partial_1 X^0 = + f \partial_0 r, \quad \partial_0 X^1 = + h \partial_1 r, \quad \partial_1 X^1 = + h \partial_0 r;$$

$$\partial_0 X^0 = + f \partial_1 r, \quad \partial_1 X^0 = + f \partial_0 r, \quad \partial_0 X^1 = - h \partial_1 r, \quad \partial_1 X^1 = - h \partial_0 r;$$

$$\partial_0 X^0 = - f \partial_1 r, \quad \partial_1 X^0 = - f \partial_0 r, \quad \partial_0 X^1 = + h \partial_1 r, \quad \partial_1 X^1 = + h \partial_0 r;$$

$$\partial_0 X^0 = - f \partial_1 r, \quad \partial_1 X^0 = - f \partial_0 r, \quad \partial_0 X^1 = - h \partial_1 r, \quad \partial_1 X^1 = - h \partial_0 r,
where
\[ f(g, \partial g) = \left( \frac{g_{rr} \partial_r g_{11} - g_{11} \partial_r g_{rr}}{g_{11} \partial_r g_{00} - g_{00} \partial_r g_{11}} \right)^{1/2}, \quad h(g, \partial g) = \left( \frac{g_{00} \partial_r g_{rr} - g_{rr} \partial_r g_{00}}{g_{11} \partial_r g_{00} - g_{00} \partial_r g_{11}} \right)^{1/2}. \]

As a consequence, one receives from here that the following equalities are fulfilled
\[ (\partial_0 \pm \partial_1)X^0 = f(g, \partial g)(\partial_0 \pm \partial_1)r, \quad (\partial_0 \pm \partial_1)X^1 = h(g, \partial g)(\partial_0 \pm \partial_1)r. \]

Therefore, we have obtained solutions which allow for all string coordinates to be non-chiral.

Till now, we considered three types of exact string solutions in variable external fields \( g_{MN} \) and \( B_{MN} \). Now, let us see which of them are compatible with the boundary conditions arising in the open string - D-brane system.

### 3 The open string - D-brane system in non-constant background fields

The action for an open string ending on a Dp-brane, in the presence of background gravitational and NS-NS 2-form field, can be written as
\[
S_2 = -\frac{T}{2} \int d^2 \xi \left[ \sqrt{-\gamma} \gamma^{mn} \partial_m X^M \partial_n X^N g_{MN}(X) - \epsilon^{mn} \partial_m X^M \partial_n X^N B_{MN}(X) \right] \tag{12}
\]
where \( Y^\mu(\xi) \) are the coordinates on the D-brane, and \( A_\mu \) is the \( U(1) \) gauge field living on the D-brane worldvolume. In \textit{static gauge} for the D-brane, one identifies \( Y^\mu \) with \( X^\mu \). Then the action (12) acquires the form of the action (1), where instead of \( B_{MN} \) the field \( B'_MN \) is present. The latter is given by the equality:
\[ B'_MN = B_{MN} - \delta_M^\mu \delta_N^\nu F_{\mu \nu}. \]

The replacement \( B_{MN} \to B'_MN \) does not change the equations of motion (13), because \( dF = d^2 A = 0 \). The constraints (3) also remain the same. However, the field \( B'_MN \) explicitly appears in the expressions for the generalized momenta
\[ P_M = -T \left( \sqrt{-\gamma} g_{MN} \gamma^0n \partial_n X^N - B'_MN \partial_1 X^N \right), \tag{13} \]
and in the boundary conditions
\[ \left[ \sqrt{-\gamma} g_{MN} \gamma^0n \partial_n X^\nu + B'_MN \partial_0 X^\nu \right]_{\sigma=0,\pi} = 0, \tag{14} \]
\[ X^\sigma(\tau,0) = X^\sigma(\tau,\pi) = q^a, \quad a = p + 1, ..., D - 1. \tag{15} \]

Here we have split the coordinates \( X^M \) into \( X^\mu \) and \( X^a \), and have denoted the location of the D-brane with \( q^a \).

Thus we saw that the exact string solutions, described in the previous section, are solutions also of the equations of motion and constraints following from the action (12) in static D-brane gauge. Now we are going to check their compatibility with the boundary conditions (14), (15).

We start by considering the case of background independent solutions \( F^M_{\pm}(w_{\pm}) \). On these solutions, the boundary conditions take the form:
\[ \left( g_{MN} \pm B'_MN \right) \frac{dF^\nu}{dw_{\pm}} \bigg|_{\sigma=0,\pi} = 0, \quad \left[ F^a_{\pm} \right]_{\sigma=0,\pi} = q^a. \tag{16} \]
Expanding $F^M_{\mu}$ in Fourier series one easily checks that they do not give nontrivial solution of the equations \( [13] \). It is clear that the same will be true for the solutions \([10]\) with $X^a$ left-movers, and $X^a$ right-movers. As for the solutions of the type \([9]\), it can be shown that the conditions \([13]\) lead to constant background fields, which is not the case under consideration. Therefore, to have the possibility to get nontrivial solutions of the open string - D-brane boundary conditions in non-constant background fields, we need to have at our disposal exact string solutions, for which the coordinates are non-chiral. We know from the previous section that such solutions do exist.

To be able to solve explicitly the boundary conditions, we assume that $g_{MN}$ and $B'_{MN}$ are constant at $\sigma = 0, \pi$. This is automatically achieved if $g_{MN}$ and $B'_{MN}$ depend only on $X^a$, and the $U(1)$ field strength $F_{\mu \nu}$ is constant.

From now on we will work in conformal gauge, and we choose to write down \([3]\) in the form

$$X^M (\tau, \sigma) = X^M_+ (\tau + \sigma) + X^M_-(\tau - \sigma).$$

Using the expansions

$$X^M_\pm (\tau \pm \sigma) = q^M_\pm + \alpha^M_0 (\tau \pm \sigma) + \sum_{k \neq 0} \frac{1}{k} \alpha^M_k e^{-ik(\tau \pm \sigma)},$$

we find the following solution of \([14]\) and \([15]\)

$$X^\mu (\tau, \sigma) = q^\mu + \left[ \delta^\mu_\nu - \left( g^{-1} B' \right)^\mu_\nu (q^a) \right] a^a_0$$
$$+ \sum_{k \neq 0} \frac{e^{-ik\tau}}{k} \left[ i \partial^\nu (k \sigma) - \left(g^{-1} B' \right)^\mu_\nu (q^a) \sin (k \sigma) \right] a^a_k,$$

$$X^a (\tau, \sigma) = q^a + \sum_{k \neq 0} \frac{e^{-ik\tau}}{k} b^a_k \sin (k \sigma),$$

where

$$\left( g^{-1} B' \right)^\mu_\nu = g^{\mu M} B^M_{\mu \nu}, \quad \alpha^M_k = \frac{1}{2} \left( a^M_k \pm b^M_k \right).$$

This result establishes the correspondence with the known solution of the boundary conditions in the case of constant background fields \([3]\).

It is clear that a crucial role in treating the open string - D-brane system in variable external fields is played by the conditions \([13]\), which ensure the existence of nontrivial solutions of the type \([9]\). Actually, \([13]\) are the equations of motion for such type of string solutions. However, they do not contain second derivatives. That is why, we propose to consider them as additional constraints in the Hamiltonian description of the considered dynamical system. So, let us compute the resulting constraint algebra.

Using the manifest expression \([13]\) for the momenta, we obtain the following set of constraints ($\partial X \equiv \partial X / \partial \sigma$)

$$I_0 \equiv g^{MN} P_M P_N - 2T \left(g^{-1} B'\right)^M N P_M \partial X^N + T^2 \left(g - B' g^{-1} B'\right)^M N \partial X^M \partial X^N,$$

$$I_1 \equiv P_N \partial X^N - T g_{MK} \left(g^{-1} B'\right)^K N \partial X^M \partial X^N = P_N \partial X^N,$$

$$I_L \equiv \Gamma_{L,MN} g^{MS} g^{NK} P_S P_K - T \left(2 \Gamma_{L,MN} \left(g^{-1} B'\right)^N K + H_{LMK}\right) g^{MS} P_S \partial X^K$$
$$+ T^2 \left\{ \Gamma_{L,MN} \left(\left(g^{-1} B'\right)^M S \left(g^{-1} B'\right)^N K - \delta^M_S \delta^N_K \right) + H_{LMS} \left(g^{-1} B'\right)^M K \right\} \partial X^S \partial X^K.$$
These constraints have one and the same structure. Namely, all of them are particular cases of the expression

\[ I_J \equiv K^*_J(g, \partial g)P_S P_K + S^S_J(g, \partial g, B', \partial B')P_S \partial X^K + R_{JSDK}(g, \partial g, B', \partial B')\partial X^S \partial X^K, \]

where \( J = (n, L) \), and the coefficient functions \( K^*_J, S^S_J \) and \( R_{JSDK} \) depend on \( X^N \) and do not depend on \( P_N \). The computation of the Poisson brackets, assuming canonical ones for the coordinates and momenta, gives

\[
\{I_J(\sigma_1), I_J(\sigma_2)\} = \left[ M^K_{(J_1 N_{J_2})K}(\sigma_1) + M^K_{(J_1 N_{J_2})K}(\sigma_2) \right] \partial \delta(\sigma_1 - \sigma_2) + C_{[J_1 J_2]} \delta(\sigma_1 - \sigma_2). \tag{17}
\]

Obviously, the algebra does not close on \( I_J \). On the other hand, the right hand side is quadratic with respect to the newly appeared structures \( M^S_J \) and \( N_{JS} \). They are given by

\[
M^S_J = 2K^SN_P + S^S_{JN} \partial X^N, \quad N_{JS} = S^M_{JS} P_M + 2R_{JSM} \partial X^M,
\]

and satisfy the following Poisson brackets among themselves

\[
\left\{ M^S_{J_1}(\sigma_1), M^S_{J_2}(\sigma_2) \right\} = \left[ (K^S_{J_1 N_{J_2}} S^S_{J_2 N} + K^S_{J_2 N_{J_2}} S^S_{J_1 N}) (\sigma_1) + (K^S_{J_1 N_{J_2}} S^S_{J_2 N} + K^S_{J_2 N_{J_2}} S^S_{J_1 N}) (\sigma_2) \right] \partial \delta(\sigma_1 - \sigma_2) + C_{J_1 J_2} S^S_{J_1 N} \delta(\sigma_1 - \sigma_2),
\]

\[
\left\{ N_{J_1 N_{J_2}}(\sigma_1), N_{J_1 N_{J_2}}(\sigma_2) \right\} = \left[ (S^N_{J_1 N_{J_2}} R_{J_2 S_2 N} + S^N_{J_2 S_2 N} R_{J_1 S_1 N}) (\sigma_1) + (S^N_{J_1 N_{J_2}} R_{J_2 S_2 N} + S^N_{J_2 S_2 N} R_{J_1 S_1 N}) (\sigma_2) \right] \partial \delta(\sigma_1 - \sigma_2) + C_{J_1 J_2} S^S_{J_1 N} \delta(\sigma_1 - \sigma_2),
\]

\[
\left\{ M^S_{J_1}(\sigma_1), N_{J_2 S_2}(\sigma_2) \right\} = \left[ (2K^S_{J_1 N_{J_2}} R_{J_2 S_2 N} + \frac{1}{2} S^S_{J_1 N_{J_2}} S^N_{J_2 S_2}) (\sigma_1) + (2K^S_{J_1 N_{J_2}} R_{J_2 S_2 N} + \frac{1}{2} S^S_{J_1 N_{J_2}} S^N_{J_2 S_2}) (\sigma_2) \right] \partial \delta(\sigma_1 - \sigma_2) + C_{J_1 J_2} S^S_{J_1 N} \delta(\sigma_1 - \sigma_2).
\]

\( M^S_J \) and \( N_{JS} \) act on \( P_M \) and \( \partial X^M \) as follows

\[
\left\{ M^S_J(\sigma_1), P_M(\sigma_2) \right\} = S^S_{JM}(\sigma_2) \partial \delta(\sigma_1 - \sigma_2),
\]

\[
\left\{ N_{JS}(\sigma_1), P_M(\sigma_2) \right\} = 2R_{JSM}(\sigma_2) \partial \delta(\sigma_1 - \sigma_2),
\]

\[
\left\{ M^S_J(\sigma_1), \partial X^M(\sigma_2) \right\} = 2K^S_J(\sigma_2) \partial \delta(\sigma_1 - \sigma_2) - 2R_{JSM} \partial X^M \delta(\sigma_1 - \sigma_2),
\]

\[
\left\{ N_{JS}(\sigma_1), \partial X^M(\sigma_2) \right\} = S^M_{JS}(\sigma_2) \partial \delta(\sigma_1 - \sigma_2) - R_{JSM} \partial X^N \delta(\sigma_1 - \sigma_2).
\]

Actually, \( I_J \) can be expressed in terms of \( M^S_J \) and \( N_{JS} \) as

\[
I_J = \frac{1}{2} \left( M^K_J P_K + N_{JK} \partial X^K \right).
\]

Let us now see how from the above open algebra the closed algebra of the constraints arises. For the gauge generators \( I_n \), we have

\[
I_0 : \quad M_0^M = 2 \left[ g^{MN} P_N - T \left( g^{-1} B' \right)^M_N \partial X^N \right],
\]

\[
N_{0M} = 2T \left[ (B' g^{-1})^N_M P_N + T \left( g - B' g^{-1} B' \right)_N M \partial X^N \right] = 0,
\]

\[
I_1 : \quad M_1^M = \partial X^M, \quad N_{1M} = P_M.
\]
Inserting these expressions in (17) one obtains
\[
\{I_0(\sigma_1), I_0(\sigma_2)\} = (2T)^2 [I_1(\sigma_1) + I_1(\sigma_2)] \partial \delta(\sigma_1 - \sigma_2),
\]
\[
\{I_1(\sigma_1), I_1(\sigma_2)\} = [I_1(\sigma_1) + I_1(\sigma_2)] \partial \delta(\sigma_1 - \sigma_2),
\]
\[
\{I_0(\sigma_1), I_1(\sigma_2)\} = [I_0(\sigma_1) + I_0(\sigma_2)] \partial \delta(\sigma_1 - \sigma_2).
\]

In this way, we reproduced an old result stating that the string constraint algebra in a gravitational and 2-form gauge field background coincides with the one in flat space-time [10].

4 Concluding remarks

In this letter we considered the problem of compatibility of the exact probe string solutions in curved backgrounds with torsion, with the mixed boundary conditions arising in the open string-Dp-brane system. We found that there exist solutions of the string equations of motion and constraints, which also solve these boundary conditions non-trivially, and give the known result in the constant background fields limit. We put forward the idea that the conditions for the existence of such solutions can be considered as a set of additional constraints and compute their Poisson bracket algebra.

The second part of this work will be devoted to the Hamiltonian analysis of the open string D-brane system in variable external fields in the framework of Batalin-Fradkin-Vilkovisky approach, including the issue of the appearance of a non-canonical Poisson structure at the string endpoints.

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