On solvable Dirac equation with polynomial potentials

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One dimensional Dirac equation is analysed with regard to the existence of exact (or closed-form) solutions for polynomial potentials. The notion of Liouvillian functions is used to define solvability, and it is shown that except for the linear potentials the equation in question is not solvable.

I. INTRODUCTION

The question of solving differential equations in explicit terms has lost some importance with the development of numerical methods, and it is also clear that solvable cases are non-generic in the class of all physical models. Although the linear ordinary differential equations are mostly dealt with by series expansion or even used to define new functions, it is still instructive if an explicit solution can be found. Obviously, exact solutions provide deeper quantitative insight into whole classes of solutions (depending on some parameters); and in case of perturbative approach, where the linear equation is just the first approximation, they allow to shift the numerical approach to the next order. Examples include Schrödinger like equations [1, 2], and perturbation equations in cosmology [3].

When solving the Dirac equation it usually boils down to solving a second order linear differential equation – be it by an appropriate change of variables, or by a particular ansatz. Further assumptions or symmetries lead to special solutions which can be obtained explicitly, most of which are presented in [4]. Depending on the physical theory one starts with, even though the form of the equations stays the same, the results on solvability differ. This happens because of various definitions of the potential – for example, in some supersymmetric theories the potential \( W = V^2 + V' \) is taken to be a generic polynomial, whereas here we take \( V \) to be generic, which leads to restrictions on \( U \). For results on integrability of such equations see for instance [1].

There are also many definitions of “explicit solutions” depending on which special functions are considered to be simple enough. Here we will be using the Liouvillian extensions to construct solutions. Since the equation will be of the form

\[
 f''(x) = r(x)f(x), \tag{1}
\]

where \( r(x) \) is rational, the sought solutions will lie in some extension of the field of rational functions over \( \mathbb{C} \). The extension will be called Liouvillian if it is composed of a finite number of the following steps:

1. Adjoining an element whose derivative is in the field.
2. Adjoining an element whose logarithmic derivative is in the field.
3. Adjoining an element algebraic in the field.

These functions include all the elementary ones (polynomials, exponential, logarithm), and also special cases of transcendental functions. The reason for introducing such class is that it naturally follows from the differential Galois theory [5] and that for the above equation there is an algorithmic approach to checking for solvability [6]. To put it generally, when the solutions are Liouvillian, the differential Galois group is solvable, and there exist invariants of the equation for which one can look. This means that, at least in theory, it is possible to check a given class of equations and rule out the existence of any additional exact solutions – an advantage usually not found in other approaches.

Thus, to put it simply, the aim of this letter is to find Liouvillian solutions of the Dirac equation into which a polynomial potential has been incorporated. As we will see, the conditions for such solvability are almost never met.

II. DIRAC EQUATION WITH A POTENTIAL

We take the one-dimensional form of the Dirac equation

\[
i\partial_t\psi = (\alpha p + \beta m)\psi, \tag{2}\]
where \( p \) is the momentum conjugate to the coordinate \( x \), i.e. \( p = -i\partial_x \), and the Dirac matrices satisfy \( \alpha^2 = \beta^2 = 1 \), \( \{\alpha,\beta\} = 0 \). This form follows from the standard four-dimensional form, when the bispinor \( \psi \) depends on \( t \) and \( x \) only. It can then be taken as a two component vector \( \psi = (\psi_1(t,x),\psi_2(t,x)) \) for simplicity. Also the time derivative can be eliminated by the ansatz \( \psi(t,x) = \exp(-iEt)\psi(x) \), so that the equation is

\[
E\psi = (-i\alpha\partial_x + \beta m)\psi.
\]

(3)

The inclusion of the potential which is specified by only one scalar function can be performed in two ways. The polynomial potential \( V = \lambda x^n + \ldots + \lambda_0 \) can be the time component of a four-vector – like the Coulomb part of the electromagnetic field. The equation would then become

\[
(V + E)\psi = (-i\alpha\partial_x + \beta m)\psi.
\]

(4)

Alternatively, one could introduce scalar coupling, which modifies the \( m \) term to

\[
E\psi = (-i\alpha\partial_x + \beta (V + m))\psi.
\]

(5)

In the first case, the explicit equations are

\[
\begin{align*}
\psi_1' &= (m + V)\psi_1 - E\psi_2, \\
\psi_2' &= -(m + V)\psi_2 + E\psi_1,
\end{align*}
\]

(6)

or, as second order equations,

\[
\begin{align*}
\psi_1'' &= (U' + U^2 - E^2)\psi_1, \\
\psi_2'' &= -(U' + U^2 - E^2)\psi_2,
\end{align*}
\]

(7)

where \( U = m + V \) and the Dirac matrices were taken to be

\[
\alpha = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(8)

The vector coupling, on the other hand, gives

\[
\begin{align*}
\psi_1'' &= (iU' - U^2 + m^2)\psi_1, \\
\psi_2'' &= -(iU' - U^2 + m^2)\psi_2,
\end{align*}
\]

(9)

provided that

\[
\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(10)

and \( U = V + E \). It is obvious that the vector coupling can be transformed into the scalar one with

\[
V \rightarrow -iv, \; E \rightarrow -im, \; m \rightarrow iE.
\]

(11)

### III. THE SOLUTIONS

Let us start with the scalar coupling. Since the spinor components are connected through the first order equations \( \psi' \) it suffices to check just the first of equations \( \psi'' \). For polynomial \( V \), and thus polynomial \( U \) it is straightforward to apply the aforementioned Kovacic algorithm. Because the only coefficient in the equation is an even degree polynomial, only the first case to be considered in depth, and the possible solution will be of the form \( P \exp \int \omega dx \), with a monic polynomial \( P \) and a rational function \( \omega \).

Since the only singular points is the infinity one needs to obtain the expansion \( [\sqrt{r}]_\infty \) of \( r = U' + U^2 - E^2 \). Because only the polynomial part of the expansion matters, it is fully and uniquely determined by comparing the coefficients of \( r \) and \( [\sqrt{r}]_\infty^2 \). Obviously, there are at most \( n + 1 \) terms of the expansion, and the \( U^2 \) term alone fixes the \( n + 1 \) highest terms of \( r \). Thus, \( U \) itself is the solution and

\[
\begin{align*}
\psi_1'' &= (iU' - U^2 + m^2)\psi_1, \\
\psi_2'' &= -(iU' - U^2 + m^2)\psi_2,
\end{align*}
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\]

(11)
If \( n \neq 1 \), the algorithm gives \( \omega = U \), and \( \text{deg}(P) = 0 \) with the additional condition that \( E^2 P = 0 \). In other words, for \( E = 0 \) the solution is

\[
\psi_1 = \exp \int (m + V) dx = \frac{1}{\psi_2},
\]

(13)

If \( n = 1 \), there is only the zeroth order term in (12) and for \( E^2/(2\lambda) \in \mathbb{N} \) we have a whole family of solutions. This is the known case of the so-called Dirac oscillator solvable by Hermite polynomials [4].

For non-zero values of \( E \) no other (non-constant) polynomial potential gives Liouvillian solutions. Thanks to the relations (11), we can see that the same holds for the vector coupling, with the appropriate interchange of \( E \) and \( m \).

Namely, for a linear potential one has the Hermite solutions as above, and for \( m = 0 \) the solution is

\[
\psi_1 = \exp \int i(E + V) dx = \frac{1}{\psi_2},
\]

(14)

The above results can be summarised as follows:

**Theorem 1** The one-dimensional Dirac equation with a polynomial potential of degree \( n > 1 \) only has Liouvillian solutions when \( E = 0 \) (scalar coupling (5)) or \( m = 0 \) (vector coupling (4)), given by formulae (13) and (14) respectively.

For \( n = 1 \) the solutions are expressible by Hermite polynomials given in [4], and \( n = 0 \) is equivalent to the standard Dirac equation.

Two remarks are in order. First, note that in the special cases \( m = 0 \) (\( E = 0 \)), the solutions given above hold for all potentials, not just polynomial ones as can be checked by direct differentiation. Second, the practical consequences of the outcome are that the spinor is not expressible as a polynomial function of position, as is often the case in solvable models of quantum mechanics. This still leaves the possibility of using new transcendental functions or series expansion approach, but precludes the direct (explicit) computation of norms or other general expressions involving \( \psi \).

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