PENETRATION OF ROTATING CONVECTION

K. C. Augustson and S. Mathis

Abstract. A simplified model for stellar and planetary convection is derived for the magnitude of the rms velocity, degree of superadiabaticity, and characteristic length scale with Rossby number as well as with thermal and viscous diffusivities. Integrating the convection model into a linearization of the dynamics in the transition region between convectively unstable and stably-stratified region yields a Rossby number, diffusivity, and pressure scale height dependent convective penetration depth into the stable region. This may have important consequences for mixing along the evolution of rotating stars.

Keywords: Instabilities – Turbulence – Stars: convection, evolution, rotation

1 Introduction

The secular impacts of rotation and magnetic fields on stellar and planetary evolution are of keen interest within the astrophysical community (e.g., Maeder 2009; Mathis 2013). As expounded upon in Stevenson (1979), a surprisingly effective approach to including rotation in MLT is to hypothesize a convection model where the convective length-scale, degree of superadiabaticity, and velocity are governed by the linear mode that maximizes the convective heat flux. This model of rotating convection has its origins in the principle of maximum heat transport proposed by Malkus (1954). In that principle, an upper limit for a boundary condition dependent turbulent heat flux is established that depends upon the smallest Rayleigh unstable convective eddy. The size of this eddy is determined with a variational technique that is similar to that developed in Chandrasekhar (1961) for the determination of the Rayleigh number, which is the ratio of the buoyancy force to the viscous force multiplied by the ratio of the thermal to viscous diffusion timescales. This technique then permits the independent computation of the rms values of the fluctuating temperature and velocity amplitudes. Numerical simulations have lent some credence to this simple model (Käpylä et al. 2005; Barker et al. 2014). In particular, those simulations indicate that the low Rossby number scaling regime established in Stevenson (1979) appears to hold up well for three decades in Rossby number (Ro is the ratio of inertial and Coriolis forces) and for about one decade in Nusselt number (Nu is the ratio of the convective and conductive fluxes). What remains to be shown is how such a model of convection can impact the mixing for intermediate Rossby numbers while including diffusion and the depth of convective penetration.

2 General Framework

The heuristic model will be considered to be local such that the length scales of the flow are much smaller than either density or pressure scale heights. This is equivalent to ignoring the global dynamics and assuming that the convection can be approximated as local at each radius and colatitude in a star or planet. As such, one may consider the dynamics to be Boussinesq. In other words, the model consists of an infinite Cartesian plane of a nearly incompressible fluid with a small thermal expansion coefficient $\alpha_T = -\partial \ln \rho / \partial T |_P$ that is confined between two impenetrable plates differing in temperature by $\Delta T$ and separated by a distance $\ell_0$. As seen in many papers regarding Boussinesq dynamics (e.g., Chandrasekhar 1961), the linearized Boussinesq equations can be reduced to a single third-order in time and eighth-order in space equation for the vertical velocity. The difference here and in the work of Stevenson (1979) is that the state that the system is being linearized

1 AIM, CEA, CNRS, Université Paris-Saclay, Université Paris Diderot, Sorbonne Paris Cité, F-91191 Gif-sur-Yvette Cedex, France

© Société Française d’Astronomie et d’Astrophysique (SF2A) 2018
about is nonlinearly saturated, meaning that the potential temperature gradient is given by the Malkus-Howard convection theory (e.g., [Malkus 1954, Howard 1963]). Together, these equations provide a dispersion relationship on which the convection model can be constructed. The details of how this model can be constructed are given in [Augustson & Mathis 2018]. The parametric quantities needed to see how this model can be leveraged to give estimates of the rotational and diffusive influence are

\[
z^3 = \frac{k^2}{k_z^2}, \quad O = q\sqrt{\frac{3 \cos \theta}{2 N_0 R_0}} = qO_0, \quad K = q\frac{k^2}{N_0} = qK_0, \quad V = q\frac{\nu k^2}{N_0} = qV_0, \tag{2.1}
\]

where \( k \) is the magnitude of the wavevector that characterizes the mode that maximizes the heat flux, \( k_z \) is its vertical component, and \( \theta \) is the latitude. Note that the variation of the superadiabaticity for this system is

\[
\epsilon = H_P \beta / T, \quad N^2 = g\alpha T \epsilon / H_P, \quad \beta / T = \frac{\partial \theta}{\partial z}, \quad \kappa = \frac{\partial q}{\partial z} = \frac{\partial q}{\partial z}.
\]

The convective Rossby number is \( \text{Ro} = v_0 / (2 \Omega_0 \ell_0) \), where \( \Omega_0 \) is the constant angular velocity of the system and the characteristic velocity \( v_0 \) is easily derived from the nonrotating and nondiffusive case as \( v_0 = \frac{v_3}{\ell_0} = \sqrt{v_0} \ell_0 / (5 \pi) \). Two relevant equations are the dispersion relationship linking \( \sigma \) to \( q \) and \( z \), and the heat flux \( F \) to be maximized with respect to \( z \)

\[
F = \frac{1}{q} \frac{\sigma^3}{q^2} \left( \frac{qz}{\sigma} + V_0 q \sigma^2 \right). \tag{2.3}
\]

To assess the scaling of the superadiabaticity, the velocity, and the horizontal wavevector, a further assumption must be made in which the maximum heat flux is invariant to any parameters, namely that \( \max[F] = F_0 \) so the heat flux is equal to the maximum value \( F_0 \) obtained in the Malkus-Howard turbulence model for the nonrotating case. Therefore, building this convection model consists of three steps: deriving a dispersion relationship that links \( \sigma \), to \( q \) and \( z \), maximizing the heat flux with respect to \( z \), and assuming an invariant maximum heat flux that then closes these three variable system.

3 Convection Model

In the case of planetary and stellar interiors, the viscous damping timescale is generally longer than the convective overturning timescale (e.g., \( V_0 \ll N_0 \)). Thus, the maximized heat flux invariance is much simpler to treat. In particular, the flux invariance condition under this assumption is then

\[
\max [F] = \frac{\sigma^3}{q^2 z^3} + \frac{V_0 \sigma^2}{q^2} \max \approx \frac{\sigma^3}{q^2 z^3} \max = 1 \implies \sigma = qz + O(v_0/N_0). \tag{3.1}
\]

One primary assumption of this convection model is that the magnitude of the velocity is defined as the ratio of the maximizing growth rate and wavevector. With the above approximation, the velocity amplitude can be defined generally. The velocity relative to the nondiffusive and nonrotating case scales as

\[
\frac{V}{v_0} = \left( \frac{5}{2} \right)^{\frac{3}{2}} \frac{\sigma}{q^3 z^3/2} = \left( \frac{5}{2} \right)^{\frac{3}{2}} z^{-\frac{1}{2}}, \tag{3.2}
\]

To find the scaling of the heat flux maximizing wavevector \( k = z^{3/2} \) and the superadiabaticity \( \epsilon / \ell_0 = q^{-2} \), one may find the implicit wavevector derivative of the growth rate \( \sigma \) from Equation 2.2 and equate it to the derivative of the heat flux \( \partial F / \partial z = \sigma / z \), which neglects the heat flux arising from the viscous effects. Using the heat-flux invariance, e.g., letting \( \sigma = qz \), the constraining dispersion relationship (Equation 2.2) can be manipulated to solve for \( q \) as a function of \( z \). Substituting this solution into the equation resulting from the flux maximization yields an equation solely for the wavevector \( z \):

\[
z^3 (V_0 z^2 + 1)^2 \left[ 3 V_0 K_0 z^4 (2 z^3 - 3) + z^2 (V_0 + K_0) (4 z^3 - 7) + 2 z^3 - 5 \right] - 6 \cos^2 \theta \frac{25 \pi^2 R_0^2}{25 \pi^2 R_0^2} \left[ K_0 (3 V_0 z^5 + z^3 + 2) + V_0 (5 z^3 - 2) + 3z \right] = 0, \tag{3.3}
\]

whereas the superadiabaticity is found to be

\[
\frac{\epsilon}{\ell_0} = \left( \frac{2}{5} \right)^{\frac{3}{2}} \left( 1 + K_0 z^2 \right) \left( 25 \pi^2 R_0^2 z^3 (1 + V_0 z)^2 + 6 \cos^2 \theta \right) \frac{25 \pi^2 R_0^2 (z^3 - 1) (1 + V_0 z^2)}{25 \pi^2 R_0^2 (z^3 - 1) (1 + V_0 z^2)}. \tag{3.4}
\]
4 Convective Penetration

The Zahn (1991) model of convective penetration is built upon a linearization of the thermodynamics with respect to the vertical displacement, which permits the equation of motion to be integrated in depth from the point where the convective flux vanishes to the point where the velocity vanishes. This yields an estimate of the depth of penetration $L_P$ of a fluid element that depends upon the boundary value of the convective velocity. Following Zahn (1991) as closely as possible, one may consider the system at the pole so that the direct effects of the local Coriolis acceleration $2f_0 \sin \theta v_z$ may be neglected. Instead, the Coriolis effect implicitly influences the penetration depth by modifying the upper boundary value of the velocity. From Equation 3.9 of Zahn (1991), the penetration depth scales as

$$L_P \propto \frac{\left(\frac{2}{3} \frac{(1-f)}{g \alpha \kappa, \chi, L_P, v_z}\right)^{\frac{1}{2}}}{\frac{3}{5} g \alpha \kappa, \chi, L_P, v_z}$$

(4.1)

where $v_z$ is the boundary value of the velocity given by the convection model derived above, $\nabla_{ad}$ is the adiabatic temperature gradient and $\chi = \partial \ln \kappa / \partial \ln P|_{S}$ is the adiabatic logarithmic derivative of the radiative conductivity with respect to pressure. It is assumed that only downward penetrating flows are effective at carrying enthalpy. This asymmetry between upflows and downflows is parameterized through the filling factor $f$. Note that the adiabatic temperature gradient $\nabla_{ad} = dT/\ln_{ad} + \epsilon$. However, a basic assumption of the model is that the superadiabaticity $\epsilon$ does not grow large enough to modify the background temperature gradient in a steady state. Thus, the ratio of the penetration depth with rotation and diffusion to the nonrotating inviscid value for convective penetration into a stable layer either above or below a convection zone therefore scales as

$$\frac{L_P}{L_{P,0}} = \left(\frac{v}{v_0}\right)^{3/2} = \left(\frac{5}{2}\right)^{\frac{1}{2}} \frac{v}{v_0}$$

(4.2)

As seen in the previous section, the velocity amplitude of the mode that maximizes the heat flux decreases with lower diffusivities and lower Rossby numbers. Therefore, the penetration depth necessarily must decrease when the Rossby number is decreased. This behavior follows intuitively given that the reduced vertical momentum of the flows implies that the temperature perturbations are also reduced. Thus, due to the decreased buoyant thermal equilibrium time and the reduced inertia of the flow the penetration depth must decrease. In contrast, the velocity and the horizontal scale of the flow increase with greater diffusivities in order to offset the reduced temperature perturbations in the case of a larger thermal conductivity. In the case of a larger viscosity, the horizontal scale of the velocity field is increased, whereas, for a fixed thermal conductivity, the thermal perturbations are of a smaller scale. Thus, to maintain the heat flux, the amplitude of the velocity must increase in order to compensate for the reduced correlations between the two fields. The scaling behaviors of the penetration depth are illustrated as a function of diffusivities and Rossby number in Figure 4(a).

In the 3D f-plane simulations of rotating convection described in Brummell et al. (2002), it is found that the penetration depth into a stable layer below a convective region scales as $L_P \propto R_o^{1.4}$, due primarily to a reduction in the flow amplitude. In a similar suite of f-plane simulations examined in Pal et al. (2007), it is found that there is a decrease in the penetration depth with increasing rotation rate that scales as $L_P \propto R_o^{0.2}$ at the pole and to $L_P \propto R_o^{0.4}$ at mid-latitude. The depth of convective penetration as assessed in those numerical simulations appears to be roughly consistent with the heuristic model derived above, where $L_P/L_{P,0} \propto R_o^{3/10}$, which follows from $v/v_0 \propto R_o^{1/5}$ in the nondiffusive and low Rossby number limit of the convection model.

5 A Diffusive Approach

A diffusive parameterization of mixing processes has been extensively examined in many stellar settings. One such model has been established through an extreme-value statistical analysis of 3D penetrative convection simulations (Pratt et al. 2017b), permitting the construction of a model for a turbulent diffusion based upon the Gumbel distribution (Pratt et al. 2017a). Using the above extension of the Zahn (1991) model, one can estimate both the penetration depth and the level of turbulent diffusion as a function of the Rossby number and diffusivities of the convection model. Doing so yields the following description of the radial dependence of the diffusion coefficient

$$D(r) = \left(\frac{5}{2}\right)^{\frac{1}{2}} \frac{\alpha H P v_z}{3 \sqrt{2}} \left\{1 - \exp[-\exp((r - r_c) / \lambda L_P + \mu / \lambda)]\right\}$$

(5.1)
Fig. 1. Rossby and Prandtl number dependencies of the convective penetration depth $L_P$ at the pole ($\theta = 0$). (a) Scaling of $L_P$ with viscosity $V_0$ at a fixed thermal diffusivity $K_0 = 10^{-5}$. (b) The radial dependence of the vertical mixing length diffusion coefficient for a solar-like model for the inviscid convection model, showing the dual effects of decreased diffusion with decreasing Rossby number and the increasing lower radial limit of the diffusion coefficient due to the decreasing depth of penetration.

where $v_c$ and $r_c$ are the velocity and radius at the base of the convection zone and where $\mu$ and $\lambda$ are the empirically determined parameters from [Pratt et al. 2017a]. An illustrative example of the scaling behavior of $D_c$ for a solar-like star where the transition region begins around $r \approx 0.7R_\odot$ is shown in Figure 1(b). The radial structure of the diffusion coefficient follows from the scaling of the velocity, namely the diffusion will globally decrease with decreasing Rossby number. The depth of penetration is perhaps most notable, in that its strong rotational dependence can lead to severe restrictions on the region in which the diffusion acts. This potentially has strong implications for mixing in rotating stars (e.g., Jørgensen & Weiss 2018).

6 Summary

A simple model of rotating convection originating with Stevenson (1979) has been extended to include thermal and viscous diffusion for any convective Rossby number. Moreover, a systematic means of developing such models has been developed for an arbitrary dispersion relationship. An explicit expression is given for the scaling of the horizontal wavenumber in terms of the Rossby number and diffusion coefficients (Equation 3.3), from which a similar scaling of the velocity and superadiabaticity is derived (Equations 3.2 and 3.4). Utilizing the linearized model of Zahn (1991), this rotating convection model is employed to assess the scaling of the depth of convective penetration with Rossby number and diffusivities. The turbulent diffusivity arising from that penetrating convection is then estimated utilizing the statistical model found in 3D simulations (Pratt et al. 2017a,b).

The authors acknowledge support from the ERC SPIRE 647383 grant and PLATO CNES grant at CEA/DAp-AIM.

References

Augustson, K. C. & Mathis, S. 2018, ApJ, 869, 92
Barker, A. J., Dempsey, A. M., & Lithwick, Y. 2014, ApJ, 791, 13
Brummell, N. H., Clune, T. L., & Toomre, J. 2002, ApJ, 570, 825
Chandrasekhar, S. 1961, Hydrodynamic and hydromagnetic stability
Howard, L. N. 1963, Journal of Fluid Mechanics, 17, 405
Jørgensen, A. C. S. & Weiss, A. 2018, MNRAS, 481, 4389
Käpylä, P. J., Korpi, M. J., Stix, M., & Tuominen, I. 2005, A&A, 438, 403
Maeder, A. 2009, Physics, Formation and Evolution of Rotating Stars
Malkus, W. V. R. 1954, Proceedings of the Royal Society of London Series A, 225, 196
Mathis, S. 2013, in Lecture Notes in Physics, Berlin Springer Verlag, Vol. 865, Lecture Notes in Physics, Berlin Springer Verlag, ed. M. Goupil, K. Belkacem, C. Neiner, F. Lignières, & J. J. Green, 23
Pal, P. S., Singh, H. P., Chan, K. L., & Srivastava, M. P. 2007, Ap&SS, 307, 399
Pratt, J., Baraffe, I., Goffrey, T., et al. 2017a, A&A, 604, A125
Pratt, J., Busse, A., Müller, W.-C., Watkins, N. W., & Chapman, S. C. 2017b, New Journal of Physics, 19, 065006
Stevenson, D. J. 1979, Geophysical and Astrophysical Fluid Dynamics, 12, 139
Zahn, J.-P. 1991, A&A, 252, 179