A Generalized Class of Estimators for Finite Population Mean Using Two Auxiliary Variables in Sample Surveys

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Abstract

In this paper we have suggested a generalized class of estimators for estimating the finite population mean \( \bar{Y} \) of the study variable \( y \) using information on two auxiliary variables \( x \) and \( z \). We have studied the properties of the proposed generalized class of estimators in simple random sampling without replacement scheme and in stratified random sampling up to the first order of approximation. It is shown that the suggested class of estimators is more efficient than the conventional unbiased estimator, ratio estimator, product estimator, traditional difference estimator, Srivastava (1967) estimator, Ray et al. (1979) estimator, Vos (1980) estimator, Upadhyaya et al. (1985) estimator, Rao (1991) estimator and Gupta and Shabbir (2008) estimator. Theoretical results are well supported through an empirical study.

Keywords: Auxiliary variable, study variable, bias, mean squared error, simple random sampling, stratified random sampling.

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1 Introduction

In sample surveys, the use of auxiliary variable(s) at the estimation stage played a prominent role in improving the precision of an estimate of the population mean. Various authors have paid their attention towards the estimation of population mean $\bar{Y}$ of the study variable $y$ using information on a single auxiliary variable $x$ and suggested a large number of estimators along with their properties in simple random sampling without (or with) replacement schemes for instance, see Murthy (1967), Singh (1986, 2003) and the references cited therein. In many survey situations of practical importance, adequate information on more than one of auxiliary variables is available. In such a situation Olkin (1958) was first to introduce multivariate ratio estimator for population mean $\bar{Y}$ of the study variable $y$ using information on $p(>1)$ auxiliary variables. Later many authors including Raj (1965), Rao and Mudholkar (1967), Singh (1967, 1969), Shukla (1966), Srivastava (1965, 1967, 1971), Singh and Tailor (2005), Gupta and Shabbir (2008), Singh et al. (2009), Swain (2013) and Sharma and Singh (2014, 2015) etc. have developed estimators which utilize data from $p(>1)$ auxiliary variables. The properties of the estimators studied under simple random sampling with (or without) replacement i.e. $SRSWR$ (or $SRSWOR$) scheme.

It is well established fact that the simple random sampling ($SRS$) procedure is employed when the population is homogeneous. However in practice, the populations encountered are not homogeneous (i.e. populations are heterogeneous). Thus in such a situation $SRS$ procedure does not provide a sample which is good representative of the entire population. Hence we can say that when the population is heterogeneous, $SRS$ procedure does not provide better estimate of the population mean $\bar{Y}$. To cope up with this situation, we use stratified random sampling for selecting a good sample from the target population. Thus when the population is heterogeneous stratified random sampling is more appropriate and gives better estimate of the population mean. In a stratified random sampling design, we divide the population into groups known as strata and samples are selected from each group with pre-determined sample size. Several authors including Diana (1993), Kadilar and Cingi (2003), Shabbir and Gupta (2005), Singh and Vishwakarma (2008), Singh et al. (2008), Koyuncu and Kadilar (2009, 2010), Koyuncu (2013), Yadav et al. (2015a, 2015b) and Koyuncu (2016) etc. have suggested estimators for population mean $\bar{Y}$ of $y$ using information on single auxiliary variable $x$ in stratified random sampling. It is further noticed that various authors including Koyuncu and Kadilar (2009), Tailor et al. (2012),
Singh and Kumar (2012), Olufadi (2013), Tailor and Chouhan (2014), Verma et al. (2015), Shabbir and Gupta (2015, 2016), Muneer et al. (2016), Malik and Singh (2017), Mishra et al. (2017) and Shabbir (2018) etc. have suggested several estimators for population mean $\bar{Y}$ of $y$ using two auxiliary variables $x$ and $z$ in stratified random sampling.

In this paper we have made an effort to develop a generalized class of estimators for population mean $\bar{Y}$ of $y$ using information on two auxiliary variables $x$ and $z$. The properties of the suggested class of estimators are studied up to the first order of approximation in SRSWOR scheme as well as in stratified random sampling. Numerical examples are given in support of the present study.

2 Some Existing Estimators of SRS

Consider a finite population $\Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_N\}$ of $N$ units. Let $y$ and $(x, z)$ be study variable and auxiliary variables respectively. Let $y_i$ and $(x_i, z_i)$ be the values of study variable $y$ and auxiliary variables $(x, z)$ on the $i$th unit $\Omega_i$ of the population $\Omega$. Suppose a sample of size $n$ is drawn by using SRSWOR scheme from the population $\Omega$ for estimating the population mean $\bar{Y}$ of the study variable $y$. Let $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$ and $(\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \bar{z} = \frac{1}{n} \sum_{i=1}^{n} z_i)$ be the unbiased estimators of the population means $\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} y_i$ and $(\bar{X} = \frac{1}{N} \sum_{i=1}^{N} x_i, \bar{Z} = \frac{1}{N} \sum_{i=1}^{N} z_i)$ respectively.

It is assumed that the population means $(\bar{X}, \bar{Z})$ of $(x, z)$ are known. Further we denote

- $C_y = \frac{S_y}{\bar{Y}}$: the population coefficient of variation of $y$,
- $C_x = \frac{S_x}{\bar{X}}$: the population coefficient of variation of $x$,
- $C_z = \frac{S_z}{\bar{Z}}$: the population coefficient of variation of $z$,
- $\rho_{yx} = \frac{S_{yx}}{S_y S_x}$: the population correlation coefficient between $y$ and $x$,
- $\rho_{yz} = \frac{S_{yz}}{S_y S_z}$: the population correlation coefficient between $y$ and $z$,
- $\rho_{xz} = \frac{S_{xz}}{S_x S_z}$: the population correlation coefficient between $x$ and $z$,
- $S_{yx} = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X})$: the population covariance between $y$ and $x$,
- $S_{yz} = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{Y})(z_i - \bar{Z})$: the population covariance between $y$ and $z$,
- $S_{xz} = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{X})(z_i - \bar{Z})$: the population covariance between $x$ and $z$,
\[ S_y^2 = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2: \] the population mean square of \( y \),
\[ S_x^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{X})^2: \] the population mean square of \( x \),
\[ S_z^2 = \frac{1}{N-1} \sum_{i=1}^{N} (z_i - \bar{Z})^2: \] the population mean square of \( z \).

\[ K_{yx} = \rho_{yx} C_y C_x, \quad K_{yz} = \rho_{yz} C_y C_z, \quad K_{xz} = \rho_{xz} C_x C_z, \quad K_{zx} = \rho_{zx} C_z C_x \] and \( f = \frac{n}{N} \): is the sampling fraction.

Now we review some existing estimators.

The usual unbiased estimator for population mean \( \bar{Y} \) is given by

\[ \hat{Y}_0 = \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \] (1)

The variance/mean squared error (MSE) under SRSWOR is given by

\[ \text{Var}(\hat{Y}_0) = \text{MSE}(\hat{Y}_0) = \left( \frac{1-f}{n} \right) \bar{Y}^2 C_y^2 = \left( \frac{1-f}{n} \right) S_y^2 \] (2)

The usual ratio and product estimators for population mean \( \bar{Y} \) are respectively defined by

\[ \hat{Y}_R = \bar{y} \left( \frac{\bar{X}}{\bar{x}} \right) \] (3)

\[ \hat{Y}_P = \bar{y} \left( \frac{\bar{x}}{\bar{X}} \right) \] (4)

To the first degree of approximation (fda), the MSEs of the estimators \( \hat{Y}_R \) and \( \hat{Y}_P \) are respectively given by

\[ \text{MSE}(\hat{Y}_R) = \left( \frac{1-f}{n} \right) \bar{Y}^2 [C_y^2 + C_x^2 (1 - 2 K_{yx})] \] (5)

\[ \text{MSE}(\hat{Y}_P) = \left( \frac{1-f}{n} \right) \bar{Y}^2 [C_y^2 + C_x^2 (1 + 2 K_{yx})] \] (6)

The generalized version of the estimators \( \hat{Y}_0, \hat{Y}_R \) and \( \hat{Y}_P \) due to Srivastava (1967) is given by

\[ \hat{Y}_{\alpha_1} = \bar{y} \left( \frac{\bar{x}}{\bar{X}} \right)^{\alpha_1} \] (7)

where \( \alpha_1 \) is a suitably chosen constant.
To the \textit{fda}, the \( \text{MSE}(\hat{Y}_{\alpha_1}) \) is given by

\[
\text{MSE}(\hat{Y}_{\alpha_1}) = \frac{(1-f)}{n} \bar{Y}^2 [C_y^2 + \alpha_1 C_x^2 (\alpha_1 + 2K_{yx})]
\]

which is minimum when

\[
\alpha_1 = -K_{yx} \alpha_{(opt)}, \text{ say}
\]

Substitution of (9) in (8) yields the minimum \text{MSE} of \( \hat{Y}_{\alpha_1} \) as

\[
\text{MSE}_{\text{min}}(\hat{Y}_{\alpha_1}) = \frac{(1-f)}{n} \bar{Y}^2 C_y^2[1 - \rho_{yx}^2] = \frac{(1-f)}{n} S_y^2[1 - \rho_{yx}^2]
\]

The traditional difference estimator for \( \bar{Y} \) is defined by

\[
\hat{Y}_{D1} = \bar{y} + d_0(\bar{X} - \bar{x}),
\]

where \( d_0 \) is a suitably chosen constant to be determined such that the \text{MSE} of \( \hat{Y}_{D1} \) is minimum.

The minimum \text{MSE} of \( \hat{Y}_{D1} \) is given by

\[
\text{MSE}_{\text{min}}(\hat{Y}_{D1}) = \left( \frac{1-f}{n} \right) \bar{Y}^2 C_y^2(1 - \rho_{yx}^2) = \left( \frac{1-f}{n} \right) S_y^2(1 - \rho_{yx}^2)
\]

Upadhyaya et al. (1985) suggested a class of estimators for population mean \( \bar{Y} \) as

\[
\hat{Y}_{USV} = w_0 \bar{y} + w_1 \bar{y} \left( \frac{\bar{x}}{\bar{X}} \right)^{\alpha_1}
\]

where \( w_0 \) and \( w_1 \) are suitably chosen weights whose sum need not be ‘unity’ and \( \alpha_1 \) is a design parameter.

The \text{MSE} of \( \hat{Y}_{USV} \) to the \textit{fda} is given by

\[
\text{MSE}(\hat{Y}_{USV}) = \bar{Y}^2 [1 + w_0^2 A_{0(srs)} + w_1^2 A_{1(srs)} + 2 w_0 w_1 A_{3(srs)} - 2 w_0 - 2 w_1 A_{6(srs)}]
\]
where

\[ A_{0(srs)} = \left[ 1 + \left( \frac{1 - f}{n} \right) C_y^2 \right] \]
\[ A_{1(srs)} = \left[ 1 + \left( \frac{1 - f}{n} \right) \left\{ C_y^2 + 4\alpha_1 \rho_y C_y C_x + \alpha_1 (2\alpha_1 - 1) C_x^2 \right\} \right] \]
\[ A_{3(srs)} = \left[ 1 + \left( \frac{1 - f}{n} \right) \left\{ C_y^2 + 2\alpha_1 \rho_y C_y C_x + \frac{\alpha_1 (\alpha_1 - 1)}{2} C_x^2 \right\} \right] \]
\[ A_{6(srs)} = \left[ 1 + \left( \frac{1 - f}{n} \right) \left\{ \alpha_1 \rho_y C_y C_x + \frac{\alpha_1 (\alpha_1 - 1)}{2} C_x^2 \right\} \right] \]

The best values of \((w_0, w_1)\) for which the MSE of \(\hat{Y}_{USV}\) is minimized, are given by

\[ w_0 = \frac{\Delta^*_0}{\Delta^*}, \quad w_1 = \frac{\Delta^*_1}{\Delta^*} \]  \hspace{1cm} (15)

where

\[ \Delta^* = \begin{vmatrix} A_{0(srs)} & A_{3(srs)} \\ A_{3(srs)} & A_{1(srs)} \end{vmatrix} = (A_{0(srs)} A_{1(srs)} - A_{3(srs)}^2) \]
\[ \Delta^*_0 = \begin{vmatrix} 1 & A_{3(srs)} \\ A_{0(srs)} & A_{1(srs)} \end{vmatrix} = (A_{1(srs)} - A_{3(srs)} A_{0(srs)}) \]
\[ \Delta^*_1 = \begin{vmatrix} A_{0(srs)} & 1 \\ A_{3(srs)} & A_{0(srs)} \end{vmatrix} = (A_{0(srs)} A_{6(srs)} - A_{3(srs)} A_{0(srs)}) \]

Thus the minimum MSE of \(\hat{Y}_{USV}\) is given by

\[ MSE_{\text{min}}(\hat{Y}_{USV}) = \bar{y}^2 \left[ 1 - \left\{ \frac{A_{1(srs)} - 2A_{3(srs)} A_{6(srs)} + A_{0(srs)} A_{2(srs)}}{(A_{0(srs)} A_{1(srs)} - A_{3(srs)}^2)} \right\} \right] \]  \hspace{1cm} (16)

If we set \(w_0 + w_1 = 1 \Rightarrow w_1 = (1 - w_0)\) in (13) we get an estimator for population mean \(\bar{Y}\) of \(y\) as

\[ \hat{Y}^* = w_0 \bar{y} + (1 - w_0) \bar{y} \left( \frac{\bar{x}}{\bar{X}} \right)^{\alpha_1} \]  \hspace{1cm} (17)

which includes the estimators due to Srivastava (1967), Ray et al. (1979) and Vos (1980).
Putting $w_1 = (1 - w_0)$ in (14) we get the $MSE$ of $\hat{Y}^*$ to the fda as

$$MSE(\hat{Y}^*) = \bar{Y}^2 \left[ 1 + A_1(srs) - 2A_6(srs) + w_0^2 (A_0(srs) + A_1(srs) - 2A_3(srs)) - 2w_0 (1 + A_1(srs) - A_3(srs) - A_6(srs)) \right]$$

Equation (18)

which is minimized for

$$w_0 = \frac{(1 + A_1(srs) - A_3(srs) - A_6(srs))}{(A_0(srs) + A_1(srs) - 2A_3(srs))} = \left( 1 + \frac{K_{yx}}{\alpha_1} \right) = w_0(\text{opt}), \text{ say}$$

Equation (19)

Thus the minimum $MSE$ of $\hat{Y}^*$ is given by

$$MSE_{\text{min}}(\hat{Y}^*)$$

$$= \bar{Y}^2 \left[ 1 + A_1(srs) - 2A_6(srs) - \frac{(1 + A_1(srs) - A_3(srs) - A_6(srs))^2}{(A_0(srs) + A_1(srs) - 2A_3(srs))} \right]$$

$$= \left( 1 - \frac{f}{n} \right) \bar{Y}^2 C_y^2 (1 - \rho_{yx}^2) = \left( 1 - \frac{f}{n} \right) S_y^2 (1 - \rho_{yx}^2)$$

Equation (20)

Rao (1991) suggested difference-type estimator for population mean $\bar{Y}$ as

$$\hat{Y}_{Rao} = \alpha_1 \bar{y} + \alpha_2 (\bar{X} - \bar{x})$$

Equation (21)

where $\alpha_1$ and $\alpha_2$ are constants to be determined such that $MSE$ of $\hat{Y}_{Rao}$ is minimum.

The bias and $MSE$ of $\hat{Y}_{Rao}$ are respectively given by

$$B(\hat{Y}_{Rao}) = \bar{Y} (\alpha_1 - 1),$$

Equation (22)

$$MSE(\hat{Y}_{Rao}) = \bar{Y}^2 \left[ 1 + \alpha_1^2 \left( 1 + \left( \frac{1 - f}{n} \right) C_y^2 \right) + \alpha_2^2 \left( \frac{1 - f}{n} \right) C_x^2 + \frac{K_{yx} C_y^2}{R^2} \right]$$

Equation (23)

$$- 2\alpha_1 \alpha_2 \left( \frac{1 - f}{n} \right) \frac{K_{yx} C_y^2}{R} - 2\alpha_1$$

where $R = \frac{\bar{Y}}{\bar{X}}$. 


The $\text{MSE}(\hat{Y}_{\text{Rao}})$ at (23) is minimized for

$$
\alpha_1 = \left\{ 1 + \left( \frac{1 - f}{n} \right) \left( C_y^2 - K_{yx}^2 C_x^2 \right) \right\}^{-1} = \alpha_{1(\text{opt})}, \text{ say}
$$

$$
\alpha_2 = -RK_{yx} \left\{ 1 + \left( \frac{1 - f}{n} \right) \left( C_y^2 - K_{yx}^2 C_x^2 \right) \right\}^{-1} = \alpha_{2(\text{opt})}, \text{ say}
$$

Thus the minimum MSE of $\hat{Y}_{\text{Rao}}$ is given by

$$
\text{MSE}_{\text{min}}(\hat{Y}_{\text{Rao}}) = \frac{(1 - f) \bar{Y}^2 (C_y^2 - K_{yx}^2 C_x^2)}{1 + \left( \frac{1 - f}{n} \right) (C_y^2 - K_{yx}^2 C_x^2)}
$$

(24)

Gupta and Shabbir (2008) proposed the following class of estimators for population mean $\bar{Y}$ as

$$
\hat{Y}_{GS} = [\alpha_3 \bar{y} + \alpha_4 (\bar{X} - \bar{x})] \left( \frac{\bar{X}}{\bar{x}} \right)
$$

(25)

where $(\alpha_3, \alpha_4)$ are suitably chosen constants such that the MSE of $\hat{Y}_{GS}$ is minimum.

To the fda, the bias and MSE of $\hat{Y}_{GS}$ are respectively given by

$$
\text{B}(\hat{Y}_{GS}) = \bar{Y} \left[ \alpha_3 \left\{ 1 + \left( \frac{1 - f}{n} \right) C_x^2 (1 - K_{yx}) \right\} + \frac{\alpha_4}{R} C_x^2 - 1 \right]
$$

(26)

$$
\text{MSE}(\hat{Y}_{GS}) = \bar{Y}^2 \left[ 1 + \alpha_3^2 \left\{ 1 + \left( \frac{1 - f}{n} \right) \left[ C_y^2 + C_x^2 (3 - 4K_{yx}) \right] \right\} + \frac{\alpha_4 (1 - f) C_x^2}{n R^2} + \frac{2 \alpha_3 \alpha_4 (1 - f)}{n R} C_x^2 (2 - K_{yx}) - 2 \alpha_3 \left\{ 1 + \left( \frac{1 - f}{n} \right) C_x^2 (1 - K_{yx}) \right\} - 2 \alpha_4 \left( \frac{1 - f}{n} \right) \frac{C_x^2}{R} \right]
$$

(27)
The $MSE(\hat{Y}_{GS})$ at (27) is minimum when

$$\alpha_3 = \frac{(a_2a_4 - a_3a_5)}{(a_1a_2 - a_3^2)} = \alpha_{3(\text{opt})}, \text{ say}$$

$$\alpha_4 = \frac{(a_1a_5 - a_3a_4)}{(a_1a_2 - a_3^2)} = \alpha_{4(\text{opt})}, \text{ say}$$

where

$$a_1 = \left[ 1 + \left( \frac{1-f}{n} \right) \left\{ C_y^2 + C_x^2 (3 - 4K_{yx}) \right\} \right]$$

$$a_2 = \left( \frac{1-f}{n} \right) \frac{C_x^2}{R^2}$$

$$a_3 = \left( \frac{1-f}{n} \right) \frac{C_x^2}{R} (2 - K_{yx})$$

$$a_4 = \left[ 1 + \left( \frac{1-f}{n} \right) C_x^2 (1 - K_{yx}) \right]$$

$$a_5 = \left( \frac{1-f}{n} \right) \frac{C_x^2}{R}$$

Thus the minimum $MSE$ of $\hat{Y}_{GS}$ is given by

$$MSE_{\text{min}} (\hat{Y}_{GS}) = \bar{Y}^2 \left\{ 1 - \frac{(a_2a_4 - 2a_3a_4a_5 + a_1a_5^2)}{(a_1a_2 - a_3^2)} \right\}$$

(28)

The traditional difference estimator for population mean $\bar{Y}$ using two auxiliary variables $x$ and $z$ is defined by

$$\hat{Y}_{D2} = \{ \bar{y} + k_1 (\bar{X} - \bar{x}) + k_2 (\bar{Z} - \bar{z}) \}$$

(29)

where $k_1$ and $k_2$ are constants to be determined such that $MSE(\hat{Y}_{D2})$ is minimum.

It is obvious from (29) that the difference estimator $\hat{Y}_{D2}$ is unbiased for population mean $\bar{Y}$. 
The variance/MSE of the estimator $\hat{Y}_{D2}$ is given by

$$\text{Var}(\hat{Y}_{D2}) = \text{MSE}(\hat{Y}_{D2}) = \left(1 - \frac{f}{n}\right) \bar{Y}^2 \left[C_y^2 + k_1 C_x^2 \frac{C_y}{R^2} + k_2 C_z^2 \frac{C_y}{R^2} + 2k_1k_2 \frac{K_{xz}C_y^2}{R^*}\right]$$

$$-2k_1 \frac{K_{yx}C_y^2}{R} - 2k_2 \frac{K_{yz}C_z^2}{R^*} \right]$$

(30)

where $R^* = \frac{\bar{Y}}{\bar{Z}}$.

The $\text{MSE}(\hat{Y}_{D2})$ at (30) is minimized for

$$k_1 = \frac{RC_y (\rho_{yx} - \rho_{yz} \rho_{xz})}{C_x (1 - \rho_{xz}^2)} = k_{1(\text{opt})}, \text{ say}$$

$$k_2 = \frac{R^* C_y (\rho_{yz} - \rho_{yx} \rho_{xz})}{C_z (1 - \rho_{xz}^2)} = k_{2(\text{opt})}, \text{ say}$$

Thus the minimum $\text{MSE}$ of $\hat{Y}_{D2}$ is given by

$$\text{MSE}_{\text{min}}(\hat{Y}_{D2}) = \left(1 - \frac{f}{n}\right) \bar{Y}^2 C_y^2 (1 - R_{y.xz}^2) = \left(1 - \frac{f}{n}\right) S_y^2 (1 - R_{y.xz}^2)$$

(31)

where $R_{y.xz}^2 = \frac{\rho_{yx}^2 + \rho_{yz}^2 - 2 \rho_{yx} \rho_{xz} \rho_{xz}}{1 - \rho_{xz}^2}$ is the multiple correlation coefficient.

3 Suggested Generalized Class of Estimators in Simple Random Sampling

Motivated by Upadhyaya et al. (1985) we propose a generalized class of estimators based on two auxiliary variables $x$ and $z$ for population mean $\bar{Y}$ of $y$ as

$$t = w_0 \bar{y} + w_1 \bar{y} \left(\frac{\bar{x}}{X}\right)^{\alpha_1} + w_2 \bar{y} \left(\frac{\bar{z}}{Z}\right)^{\alpha_2}$$

(32)

where $(w_0, w_1, w_2)$ are weights whose sum need not be ‘unity’ and $(\alpha_1, \alpha_2)$ are design parameters. The constants $(\alpha_1, \alpha_2)$ may take positive (+, +) or negative (−, −) or positive-negative (+, −) or negative-positive (−, +) values to form product-type or ratio-type or product-cum-ratio-type or ratio-cum-product-type estimator.
To obtain the bias and MSE of the propounded estimator \( t \), we write
\[
\bar{y} = \bar{Y}(1 + e_0), \quad \bar{x} = \bar{X}(1 + e_1) \quad \text{and} \quad \bar{z} = \bar{Z}(1 + e_2)
\]
such that
\[
E(e_0) = E(e_1) = E(e_2) = 0
\]
\[
E(e_0^2) = \left( \frac{1 - f}{n} \right) C_y^2, \quad E(e_1^2) = \left( \frac{1 - f}{n} \right) C_x^2,
\]
\[
E(e_2^2) = \left( \frac{1 - f}{n} \right) C_z^2
\]
\[
E(e_0e_1) = \left( \frac{1 - f}{n} \right) \rho_{yx} C_y C_x = \left( \frac{1 - f}{n} \right) K_{yx} C_x^2,
\]
\[
E(e_0e_2) = \left( \frac{1 - f}{n} \right) \rho_{yx} C_y C_z = \left( \frac{1 - f}{n} \right) K_{yx} C_z^2
\]
\[
E(e_1e_2) = \left( \frac{1 - f}{n} \right) \rho_{xz} C_x C_z = \left( \frac{1 - f}{n} \right) K_{xz} C_z^2.
\]
Expressing (32) in terms of \( e \)’s we have
\[
t = \bar{Y} \left[ w_0 (1 + e_0) + w_1 (1 + e_0)(1 + e_1)^{\alpha_1} + w_2 (1 + e_0)(1 + e_2)^{\alpha_2} \right]
\]
\[
(33)
\]
We suppose that \(|e_i| \ll 1\) so that \((1 + e_i)^{\alpha_i}, i = 1, 2\) are expandable. Expanding the right hand side of (33), multiplying out and ignoring terms of \( e \)’s having power greater than two, we have
\[
t \cong \bar{Y} \left[ w_0 (1 + e_0) + w_1 \left\{ 1 + e_0 + \alpha_1 e_1 + \alpha_1 e_0 e_1 + \frac{\alpha_1 (\alpha_1 - 1)}{2} e_1^2 \right\} \right.
\]
\[
+ w_2 \left\{ 1 + e_0 + \alpha_2 e_2 + \alpha_2 e_0 e_2 + \frac{\alpha_2 (\alpha_2 - 1)}{2} e_2^2 \right\} \right]
\]
or
\[
(t - \bar{Y}) \cong \bar{Y} \left[ w_0 (1 + e_0)
\right.
\]
\[
+ w_1 \left\{ 1 + e_0 + \alpha_1 e_1 + \alpha_1 e_0 e_1 + \frac{\alpha_1 (\alpha_1 - 1)}{2} e_1^2 \right\}
\]
\[
+ w_2 \left\{ 1 + e_0 + \alpha_2 e_2 + \alpha_2 e_0 e_2 + \frac{\alpha_2 (\alpha_2 - 1)}{2} e_2^2 \right\} - 1
\]
\[
(34)
\]
Taking expectation of both sides of (34) we get the bias of $t$ to the fida as

$$B(t) = \bar{Y} \left[ w_0 + w_1 \left\{ 1 + \left( \frac{1 - f}{n} \right) \frac{a_1}{2} (a_1 + 2K_{yx} - 1) C_x^2 \right\} + w_2 \left\{ 1 + \left( \frac{1 - f}{n} \right) \frac{a_2}{2} (a_2 + 2K_{yz} - 1) C_z^2 \right\} - 1 \right]$$

$$= \bar{Y} \left[ w_0 + w_2 A_{6(srs)} + w_3 A_{7(srs)} - 1 \right]$$  \quad (35)

where

$$A_{6(srs)} = \left[ 1 + \left( \frac{1 - f}{n} \right) \frac{a_1}{2} (a_1 + 2K_{yx} - 1) C_x^2 \right]$$

$$A_{7(srs)} = \left[ 1 + \left( \frac{1 - f}{n} \right) \frac{a_2}{2} (a_2 + 2K_{yz} - 1) C_z^2 \right]$$

Squaring both sides of (34) and ignoring terms of $e$’s having power greater than two we have

$$(t - \bar{Y})^2 = 2w_0$$

$$= \begin{bmatrix}
1 + w_0^2 (1 + 2e_0 + e_0^2) \\
+ w_1^2 \left\{ 1 + 2e_0 + 2a_1 e_1 \\
+ e_0^2 + 4a_1 e_0 e_1 \\
+ a_1 (2a_1 - 1) e_1^2 \right\} \\
+ w_2^2 \left\{ 1 + 2e_0 + 2a_2 e_2 \\
+ e_0^2 + 4a_2 e_0 e_2 \\
+ a_2 (2a_2 - 1) e_2^2 \right\} \\
+ 2w_0 w_1 \left\{ 1 + 2e_0 + a_1 e_1 + e_0^2 \\
+ 2a_1 e_0 e_1 + \frac{a_1 (a_1 - 1)}{2} e_1^2 \right\} \\
+ 2w_0 w_2 \left\{ 1 + 2e_0 + a_2 e_2 + \frac{a_2 (a_2 - 1)}{2} e_2^2 \right\} \\
1 + 2e_0 + a_1 e_1 + a_2 e_2 \\
+ e_0^2 + 2a_1 e_0 e_1 \\
+ 2a_2 e_0 e_2 + a_1 a_1 e_1 e_0 \\
+ \frac{a_1 (a_1 - 1)}{2} e_1^2 \right\} \\
+ \frac{a_2 (a_2 - 1)}{2} e_2^2 \right\} \\
1 + e_0 + a_1 e_1 + a_2 e_2 \\
+ \frac{a_1 (a_1 - 1)}{2} e_1^2 \right\} \\
+ \frac{a_2 (a_2 - 1)}{2} e_2^2 \right\} \\
\end{bmatrix}$$  \quad (36)
Taking expectation of both sides of (36) we get the MSE of $t$ to the fda as

$$MSE(t) = \bar{Y}^2 \left[ 1 + w_0^2 A_{0(srs)} + w_1^2 A_{1(srs)} + w_2^2 A_{2(srs)} + 2w_0w_1 A_{3(srs)} + 2w_0w_2 A_{4(srs)} + 2w_1w_2 A_{5(srs)} - 2w_0 - 2w_1 A_{6(srs)} - 2w_2 A_{7(srs)} \right]$$

where

$$A_{2(srs)} = 1 + \left( \frac{1 - f}{n} \right) \left\{ C_y^2 + 4\alpha_2 \rho_{yz} C_y C_z + \alpha_2 (2\alpha_2 - 1) C_z^2 \right\}$$

$$A_{4(srs)} = 1 + \left( \frac{1 - f}{n} \right) \left\{ C_y^2 + 2\alpha_2 \rho_{yz} C_y C_z + \frac{\alpha_2 (2\alpha_2 - 1)}{2} C_z^2 \right\}$$

$$A_{5(srs)} = 1 + \left( \frac{1 - f}{n} \right) \left\{ C_y^2 + 2\alpha_1 \rho_{xz} C_x C_z + 2\alpha_2 \rho_{yz} C_y C_z + \frac{\alpha_1 (\alpha_1 - 1)}{2} C_x^2 + \frac{\alpha_2 (2\alpha_2 - 1)}{2} C_z^2 \right\}$$

$(A_{0(srs)}, A_{1(srs)}, A_{3(srs)}, A_{6(srs)} \text{and } A_{7(srs)})$ are same as defined earlier.

Minimization of $MSE(t)$ at (37) with respect to $(w_0, w_1, w_2)$ yields

$$\begin{bmatrix} A_{0(srs)} & A_{3(srs)} & A_{4(srs)} \\ A_{3(srs)} & A_{1(srs)} & A_{5(srs)} \\ A_{4(srs)} & A_{5(srs)} & A_{2(srs)} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ A_{6(srs)} \\ A_{7(srs)} \end{bmatrix}$$

(38)

After simplification of (38), we get the optimum values of $(w_0, w_1, w_2)$ respectively as

$$w_{00} = \frac{\Delta_0}{\Delta}, \quad w_{10} = \frac{\Delta_1}{\Delta}, \quad w_{20} = \frac{\Delta_2}{\Delta};$$

where

$$\Delta = \begin{vmatrix} A_{0(srs)} & A_{3(srs)} & A_{4(srs)} \\ A_{3(srs)} & A_{1(srs)} & A_{5(srs)} \\ A_{4(srs)} & A_{5(srs)} & A_{2(srs)} \end{vmatrix}$$

$$= A_{0(srs)}(A_{1(srs)} A_{2(srs)} - A_{5(srs)}^2) - A_{3(srs)}(A_{2(srs)} A_{3(srs)} - A_{4(srs)} A_{5(srs)} - A_{1(srs)} A_{4(srs)})$$
Substitution of (39) in (37) yields the minimum $MSE$ of $t$ as

$$MSE_{\text{min}}(t) = \bar{Y}^2 \left[ 1 - \frac{\Delta_0}{\Delta} - \frac{A_6^{(srs)} \Delta_1}{\Delta} - \frac{A_7^{(srs)} \Delta_2}{\Delta} \right]$$

(40)

Now we state the following theorem.

**Theorem-3.1** – The minimum $MSE$ of $t$ is greater than or equal to $MSE(t)$ i.e.

$$MSE(t) \geq MSE_{\text{min}}(t) = \bar{Y}^2 \left[ 1 - \frac{\Delta_0 + A_6^{(srs)} \Delta_1 + A_7^{(srs)} \Delta_2}{\Delta} \right]$$

(41)

with equality holding if

$$w_{0i} = \frac{\Delta_i}{\Delta}, \quad i = 0, 1, 2.$$

**Remark-3.1** – It is to be mentioned that the class of estimators ‘$t$’ at (32) will attained its minimum $MSE$ at (40) only when the optimum
values \((w_{00}, w_{10}, w_{20})\) at (39) of the weights \((w_0, w_1, w_2)\) are known exactly, but in practice the exact values of the population parameters \((C_y, C_x, C_z, \rho_{yx}, \rho_{yz}, \rho_{xz})\) are rarely available. However in repeated surveys or studies based on multiphase sampling, where information regarding the same variates is gathered on several occasions, it is possible to guess quite precisely the values of certain population parameters such as \((C_y, C_x, C_z, \rho_{yx}, \rho_{yz}, \rho_{xz})\). Further we mention that the good guess values of these population parameters can also be obtained from the past data or the experience gathered in due course of time or through a pilot sample survey. This problem has been discussed among others by Murthy (1967, pp. 96–99), Searls (1964), Srivastava (1966), Gleser and Healy (1976), Das and Tripathi (1978), Reddy (1978), Tripathi et al. (1983) and Srivenkataramana and Tracy (1984). Thus the values of such population parameters \((C_y, C_x, C_z, \rho_{yx}, \rho_{yz}, \rho_{xz})\) can be known exactly. We recall that the scalars \((\alpha_1, \alpha_2)\) are real. The values of the scalars \((\alpha_1, \alpha_2)\) are known (or can be known by the experimental practitioner) as the values of \((\alpha_1, \alpha_2)\) yield the form of the estimator. Thus the optimum values \((w_{00}, w_{10}, w_{20})\) of the corresponding constants \((w_0, w_1, w_2)\) can be obtained quite accurately. Hence we conclude that in practice, an operational estimator can be derived from the suggested class of estimators ‘t’ with mean squared error smaller than the conventional estimators.

On the other hand if the values of the population parameters such as \((C_y, C_x, C_z, \rho_{yx}, \rho_{yz}, \rho_{xz})\) are not known (or cannot be made known) at all. In such situations, the practical utility of such estimators is limited. So in such circumstances one can estimate the value of these population parameters by their corresponding sample statistics. Hence the estimates \((\hat{w}_{00}, \hat{w}_{10}, \hat{w}_{20})\) say, of the corresponding optimum values \((w_{00}, w_{10}, w_{20})\) can be obtained. Thus this also suggests that one can also obtain the operational (feasible) estimator from the proposed class of estimators ‘t’ having mean squared error fewer than the usual estimators.

4 Efficiency Comparison

It is observed from (10), (12) and (20) that the common minimum \(MSE\)s of the estimators \(\hat{Y}_{\alpha_1}, \hat{Y}_{D_1}\) and \(\hat{Y}^*\) is same, i.e.

\[
MSE_{\min} \left( \hat{Y}_{\alpha_1} \right) = MSE_{\min} \left( \hat{Y}_{D_1} \right) = MSE_{\min} \left( \hat{Y}^* \right) = \left( 1 - \frac{f}{n} \right) \bar{Y}^2 \left( C_y^2 - K_{yx}^2 \right) C_z^2
\]  

(42)
Now we compare the efficiency of traditional difference estimator with usual unbiased estimator \( \bar{Y}_0 = \bar{y} \), ratio estimator \( \hat{Y}_R \) and product estimator \( \hat{Y}_P \).

From (2), (5), (6) and (12), we have,

\[
\text{MSE} \left( \hat{Y}_0 = \bar{y} \right) - \text{MSE}_{\text{min}} \left( \hat{Y}_{D1} \right) = \left( \frac{1 - f}{n} \right) \bar{Y}^2 K_{y|x}^2 C_x^2 \geq 0 \quad (43)
\]

\[
\text{MSE} \left( \hat{Y}_R \right) - \text{MSE}_{\text{min}} \left( \hat{Y}_{D1} \right) = \left( \frac{1 - f}{n} \right) \bar{Y}^2 C_x^2 (1 - K_{y|x})^2 \geq 0 \quad (44)
\]

\[
\text{MSE} \left( \hat{Y}_P \right) - \text{MSE}_{\text{min}} \left( \hat{Y}_{D1} \right) = \left( \frac{1 - f}{n} \right) \bar{Y}^2 C_x^2 (1 + K_{y|x})^2 \geq 0 \quad (45)
\]

It follows from (42), (43), (44) and (45) that the estimators \( \hat{Y}_{\alpha_1}, \hat{Y}_{D1} \) and \( \hat{Y}^* \) are more efficient than usual unbiased estimator \( \bar{y} \), ratio estimator \( \hat{Y}_R \) and product estimator \( \hat{Y}_P \).

From (23) and (42), we have,

\[
\left[ \text{MSE}_{\text{min}} \left( \hat{Y}_{\alpha_1} \right) = \text{MSE}_{\text{min}} \left( \hat{Y}_{D1} \right) = \text{MSE}_{\text{min}} \left( \hat{Y}^* \right) \right] - \text{MSE}_{\text{min}} \left( \hat{Y}_{Rao} \right)
= \left( \frac{1 - f}{n} \right)^2 \bar{Y}^2 \frac{\left( C_y^2 - K_{y|x}^2 C_x^2 \right)^2}{1 + \left( \frac{1 - f}{n} \right) \left( C_y^2 - K_{y|x}^2 C_x^2 \right)}
\geq 0 \quad (46)
\]

It follows from (42), (43), (44), (45) and (46) that the estimator \( \hat{Y}_{Rao} \) due to Rao (1991) is more efficient than \( \bar{y}, \hat{Y}_R, \hat{Y}_P, \hat{Y}_{\alpha_1}, \hat{Y}_{D1}, \hat{Y}_{\alpha_1} \) and \( \hat{Y}^* \).

The minimum \( \text{MSE} \) of the difference estimator \( \hat{Y}_{D1} \) given by (12) can be expressed as

\[
\text{MSE}_{\text{min}} \left( \hat{Y}_{D1} \right)
= \bar{Y}^2 \left[ 1 + A_{1(srs)} - 2A_{1(srs)} - \frac{\left( 1 + A_{1(srs)} - A_{3(srs)} - A_{4(srs)} \right)^2}{A_{0(srs)} + A_{1(srs)} - 2A_{3(srs)}} \right] \quad (47)
\]
From (16) and (47), we have

$$MSE_{min} \left( \hat{\bar{Y}}_{D1} \right) - MSE_{min} \left( \hat{\bar{Y}}_{USV} \right)$$

$$= \bar{Y}^2 \left[ \frac{A_1(srs) \left( 1 - A_0(srs) \right) + A_3(srs) \left( A_3(srs) - 1 \right)}{A_0(srs) A_1(srs) - A_3(srs)} \right]^2 \geq 0$$

(48)

From (12) and (31) we have

$$MSE_{min} \left( \hat{\bar{Y}}_{D1} \right) - MSE_{min} \left( \hat{\bar{Y}}_{D2} \right) = \frac{1 - f_n}{n} \bar{Y}^2 C_y^2 (\rho_{yz} - \rho_{yx}^2 \rho_{xz})^2 \left( 1 - \rho_{xz}^2 \right)$$

which shows that the traditional estimator \( \hat{Y}_{D1} \) is better than \( \hat{Y}_{D2} \).

From (16) and (40) we have

$$MSE_{min} \left( \hat{\bar{Y}}_{USV} \right) - MSE_{min} (t)$$

$$= \frac{\bar{Y}^2}{\Delta \Delta_1} \left[ \frac{A_7(srs) \left( A_0(srs) A_1(srs) - A_3(srs) \right)}{A_0(srs) A_1(srs) - 2A_3(srs)} \right]^2 \geq 0$$

(50)

It shows that the proposed class of estimators \( t \) is more efficient than the Upadhyaya et al. (1985) estimator \( \hat{Y}_{USV} \).

Hence from (42), (43), (44), (45), (46), (48) and (49) it is observed that the suggested generalized class of estimators \( t \) is better than the estimators \( \hat{y}, \hat{Y}_{Rao} \), \( \hat{Y}_{G}, \hat{Y}_{P}, \hat{y}_{\alpha}, \hat{Y}_{D1}, \hat{Y}^* \) and \( \hat{Y}_{USV} \).

From (24) and (40) we have that \( MSE_{min}(t) < MSE_{min}(\hat{Y}_{Rao}) \) if

$$\left[ 1 - \frac{(\Delta_0 + A_6(srs) \Delta_1 + A_7(srs) \Delta_2)}{\Delta} \right] < \frac{(1 - f_n) \left( C_y^2 - K_{yx}^2 \rho_{xz}^2 \right)}{1 + (1 - f_n) \left( C_y^2 - K_{yx}^2 \rho_{xz}^2 \right)}$$

(51)

From (28) and (40) it is observed that \( MSE_{min}(t) < MSE_{min}(\hat{Y}_{GS}) \) if

$$\frac{a_2 a_4^2 - 2a_3 a_4 a_5 + a_1 a_5^2}{(a_1 a_2 - a_3^2)} < \frac{(\Delta_0 + A_6(srs) \Delta_1 + A_7(srs) \Delta_2)}{\Delta}$$

(52)
Further from (31) and (40) we note that $MSE_{\min}(t) < MSE_{\min}(\hat{Y}_{D2})$ if

$$1 - \left( \frac{A_{6(srs)}\Delta_1 + A_{7(srs)}\Delta_2}{\Delta} \right) < \left( \frac{1 - f}{n} \right) C_y^2(1 - R_{y,xz}^2)$$

(53)

Thus from (51), (52) and (53) it is observed that the proposed generalized class of estimators $t$ is more efficient than $\hat{Y}_{Rao}, \hat{Y}_{GS}$ and $\hat{Y}_{D2}$ as long as the conditions (51), (52) and (53) are satisfied respectively.

5 Empirical Study

For numerical comparisons of different estimators, we use the following data sets.

- **Data I:** [Source: Singh and Chaudhary (1986), page 177]
- **Data II:** [Source: Abu-Dayyeh et al. (2003)]
- **Data III:** [Source: Steel and Torrie (1960)]
- **Data IV:** [Source: Cochran (1977)]
- **Data V:** [Source: Ahmed (1997)]
- **Data VI:** [Source: PCR (1998)]

| Data | I  | II | III | IV  | V   | VI  |
|------|----|----|-----|-----|-----|-----|
| $N$  | 34 | 332| 30  | 34  | 376 | 424 |
| $N'$ | 20 | 80 | 6   | 15  | 159 | 169 |
| $\bar{Y}$ | 856.41 | 1093.1 | 0.6860 | 4.92 | 316.65 | 646.215 |
| $\bar{X}$ | 208.88 | 181.57 | 4.6537 | 2.59 | 141.13 | 4533.981 |
| $\bar{Z}$ | 199.44 | 143.37 | 0.8077 | 2.91 | 1075.31 | 325.0325 |
| $C_y$ | 0.86 | 0.7626 | 0.4803 | 1.01232 | 0.7721 | 1.509 |
| $C_x$ | 0.72 | 0.7684 | 0.2295 | 1.23187 | 0.845 | 1.342 |
| $C_z$ | 0.75 | 0.7616 | 0.7493 | 1.05351 | 0.7746 | 1.335 |
| $\rho_{yx}$ | 0.45 | 0.973 | 0.7194 | 0.7326 | 0.9106 | 0.623 |
| $\rho_{yz}$ | 0.45 | 0.862 | 0.04996 | 0.643 | 0.9094 | 0.907 |
| $\rho_{xz}$ | 0.98 | 0.842 | 0.4074 | 0.6837 | 0.8614 | 0.682 |

Table 1 gives the $PRE$’s of $\hat{Y}_R, \hat{Y}_P, \hat{Y}_{D1}, \hat{Y}_{Rao}, \hat{Y}_{GS}$ and $\hat{Y}_{D2}$ estimators with respect to $\bar{y}$ for six data sets respectively.

Table 2 gives the $PRE$ of $\hat{Y}_{USV}$ with respect to $\bar{y}$ for $\alpha_1 = (-1,1)$, for six data sets.
Table 1  PRE’s of different estimators of population mean $\bar{Y}$ with respect to $\bar{y}$

| Estimator | Data I | Data II | Data III | Data IV | Data V | Data VI |
|-----------|--------|---------|----------|---------|--------|---------|
| $\hat{Y}_R$ | 105.55 | 1835.92 | 94.62    | 143.30  | 488.77 | 146.46  |
| $\hat{Y}_P$ | 40.74  | 25.15   | 71.44    | 23.45   | 23.86  | 34.49   |
| $\hat{Y}_{D1}$ | 125.39 | 1877.19 | 103.33   | 215.84  | 585.45 | 163.43  |
| $\hat{Y}_{Rao}$ | 126.92 | 1877.75 | 106.40   | 219.66  | 585.67 | 164.24  |
| $\hat{Y}_{Gs}$ | 126.93 | 1877.75 | 106.42   | 219.89  | 585.67 | 164.25  |
| $\hat{Y}_{D2}$ | 125.71 | 2127.83 | 174.04   | 235.09  | 907.16 | 563.97  |

Table 2  PRE’s of $\hat{Y}_{USV}$ with respect to $\bar{y}$

| Data | $\alpha_1$ | PRE | $\alpha_2$ | PRE | $\alpha_1$ | PRE | $\alpha_2$ | PRE | $\alpha_1$ | PRE | $\alpha_2$ | PRE | $\alpha_1$ | PRE | $\alpha_2$ | PRE |
|------|------------|-----|------------|-----|------------|-----|------------|-----|------------|-----|------------|-----|------------|-----|------------|-----|-------|
| I    | −1         | 127.66 | −1         | 1878.66 | −1         | 106.76 | −1         | 227.99 | −1         | 586.30 | −1         | 164.74 |
|      | 1          | 126.24 | 1          | 2049.28 | 1          | 106.20 | 1          | 215.95 | 1          | 588.70 | 1          | 163.54 |

Table 3(a) depicts the PREs of proposed class of estimators $t$ with respect to $\bar{y}$ at different values of $(\alpha_1, \alpha_2)$ for data sets I, II, III.

Table 3(a)  PRE’s of proposed class of estimators $t$ for population mean $\bar{Y}$ with respect to $\bar{y}$ (for data sets I, II, III)

| $\alpha_1$ | $\alpha_2$ | PRE | $\alpha_1$ | PRE | $\alpha_1$ | PRE | $\alpha_2$ | PRE | $\alpha_1$ | PRE | $\alpha_2$ | PRE | $\alpha_1$ | PRE | $\alpha_2$ | PRE |
|------------|------------|-----|------------|-----|------------|-----|------------|-----|------------|-----|------------|-----|------------|-----|------------|-----|-------|
| −16.56     | −16.56     | 17654.39 | −6.5 | −6.9 | 772384 | −4 | −4 | 1286.97 |
| −16.55     | −16.55     | 11434.21 | −6.5 | −6.8 | 234504.2 | −3 | −3 | 240.98 |
| −16.54     | −16.54     | 8469.43 | −6.5 | −6.7 | 139357.5 | −2 | −2 | 194.41 |
| −16.53     | −16.53     | 6734.46 | −6.5 | −6.6 | 99699.33 | −1 | −1 | 180.01 |
| −16.52     | −16.52     | 5595.56 | −6.5 | −6.5 | 77951.88 | 1 | 1 | 174.25 |
| −16.51     | −16.51     | 4790.57 | −6.4 | −6.4 | 34176.73 | 2 | 2 | 177.14 |
| −16.5      | −16.5      | 4191.43 | −6.3 | −6.3 | 22046.63 | 3 | 3 | 183.83 |
| −16        | −16        | 657.28  | −6.2 | −6.2 | 16359.33 | 4 | 4 | 196.20 |
| −10        | −10        | 153.84  | −6.1 | −6.1 | 13060.19 | 5 | 5 | 219.66 |
| −5         | −5         | 133.88  | −6 | −6 | 10906.9 | 6 | 6 | 273.01 |
| −4         | −4         | 131.95  | −5 | −5 | 4435.19 | 7 | 7 | 484.92 |
| −3         | −3         | 130.37  | −4 | −4 | 3039.74 | −4 | 6 | 9201.75 |
| −2         | −2         | 129.08  | −3 | −3 | 2477.50 | −3 | 5 | 341.57 |
| −1         | −1         | 128.03  | −2 | −2 | 2220.59 | −2 | 4 | 230.89 |
| 1          | 1          | 126.56  | −1 | −1 | 2130.16 | −1 | 3 | 195.37 |
| 2          | 2          | 126.10  | 1 | 1 | 2358.74 | 1 | 2 | 178.42 |
| 3          | 3          | 125.82  | 2 | 2 | 2796.94 | 2 | 1 | 174.06 |
| 4          | 4          | 125.72  | 3 | 3 | 3848.96 | −4 | 5 | 386.31 |
| 5          | 5          | 125.81  | 4 | 4 | 7795.28 | −4 | 4 | 253.76 |
| 8          | 8          | 127.58  | 4.5 | 4.5 | 20478.73 | −4 | 3 | 209.95 |

(Continued)
Table 3(a) Continued

| Data I | Data II | Data III |
|--------|---------|----------|
| \(\alpha_1\) | \(\alpha_2\) | PRE | \(\alpha_1\) | \(\alpha_2\) | PRE | \(\alpha_1\) | \(\alpha_2\) | PRE |
| 10 | 10 | 130.8 | 4.6 | 4.6 | 31462.35 | −4 | 2 | 189.32 |
| 12 | 12 | 137.87 | 4.7 | 4.7 | 69729.23 | −4 | 1 | 178.73 |
| 15 | 15 | 187.23 | 1 | 2 | 2404.66 | −3 | 6 | 1354.17 |
| 16 | 16 | 328.47 | 2 | 3 | 2902.67 | −3 | 5 | 341.57 |
| 16.1 | 16.1 | 378.07 | 4 | 5 | 9706.82 | −3 | 4 | 241.09 |
| 16.2 | 16.2 | 456.93 | −1 | −2 | 2135.10 | −3 | 3 | 204.27 |
| 16.3 | 16.3 | 601.84 | −3 | −4 | 2539.23 | −3 | 2 | 186.40 |
| 16.4 | 16.4 | 955.32 | −4 | −5 | 3182.99 | −2 | 6 | 788.06 |
| 16.5 | 16.5 | 3122.77 | −5 | −6 | 4877.23 | −2 | 5 | 310.28 |

Table 3(b) indicates the \(PRE\)s of proposed class of estimators \(t\) with respect to \(\bar{y}\) at different values of \((\alpha_1, \alpha_2)\) for data sets IV, V, VI.

Table 3(b) \(PRE\)'s of proposed class of estimators \(t\) with respect to \(\bar{y}\) (for data sets IV, V, VI)

| Data IV | Data V | Data VI |
|--------|--------|---------|
| \(\alpha_1\) | \(\alpha_2\) | PRE | \(\alpha_1\) | \(\alpha_2\) | PRE | \(\alpha_1\) | \(\alpha_2\) | PRE |
| −5.5 | −5.6 | 18686.57 | −13.8 | −13.8 | 146480.3 | −11.3 | −11.3 | 56544.35 |
| −5.5 | −5.5 | 9127.82 | −13.5 | −13.5 | 16447.9 | −11 | −11 | 7475.86 |
| −5 | −5 | 1011.91 | −13 | −13 | 6808.17 | −10 | −10 | 2084.82 |
| −4 | −4 | 443.93 | −12 | −12 | 3275.97 | −8 | −8 | 986.33 |
| −3 | −3 | 323.39 | −10 | −10 | 1749.71 | −5 | −5 | 659.03 |
| −2 | −2 | 273.50 | −8 | −8 | 1285.83 | −4 | −4 | 616.64 |
| −1 | −1 | 248.82 | −5 | −5 | 1012.38 | −3 | −3 | 588.91 |
| 1 | 1 | 235.59 | −4 | −4 | 967.68 | −2 | −2 | 572.19 |
| 2 | 2 | 244.97 | −3 | −3 | 936.87 | −1 | −1 | 564.63 |
| 3 | 3 | 274.84 | −2 | −2 | 917.52 | 1 | 1 | 575.12 |
| 4 | 4 | 372.70 | −1 | −1 | 908.24 | 2 | 2 | 594.85 |
| 5 | 5 | 2084.42 | 1 | 1 | 918.17 | 3 | 3 | 627.61 |
| 5.1 | 5.1 | 8806.16 | 2 | 2 | 938.25 | 4 | 4 | 679.08 |
| −5.5 | −5.4 | 6136.23 | 3 | 3 | 970.32 | 5 | 5 | 760.58 |
| −5.5 | −5.2 | 3812.23 | 4 | 4 | 1017.20 | 6 | 6 | 897.22 |
| −5.5 | −5.1 | 3242.17 | 5 | 5 | 1083.64 | 7 | 7 | 1156.67 |
| −5.5 | −5 | 2838.5 | 6 | 6 | 1177.62 | 8 | 8 | 1802.22 |
| −5.5 | −4.5 | 1851.49 | 7 | 7 | 1313.39 | 9 | 9 | 5816.34 |
| −5.5 | −4 | 1470.75 | 8 | 8 | 1518.41 | 9.3 | 9.3 | 24933.39 |
| −5.5 | −3 | 1210.17 | 10 | 10 | 2479.08 | 9 | 9.4 | 100422 |
| −5.5 | −2 | 1286.14 | 12 | 12 | 1384.51 | * | * | * |
| −5.5 | −1 | 2344.68 | 12.1 | 12.1 | 18724.25 | * | * | * |
| −5.4 | −5.6 | 5179.15 | 12.2 | 12.2 | 29101.79 | * | * | * |
| −5.3 | −5.6 | 3067.52 | 12.3 | 12.3 | 66293.65 | * | * | * |
| −5.2 | −5.6 | 2208.87 | * | * | * | * | * | * |
| −5.1 | −5.6 | 1743.17 | * | * | * | * | * | * |
We measured Percent Relative Efficiencies (PREs) of various estimators along with our proposed generalized class of estimators \( t \) with respect to \( \bar{y} \). It is observed that from the entries of the Tables 1, 2, 3(a) and 3(b) that the suggested generalized class of estimators \( t \) gives the largest PRE (17654.39\%, 772384.00\%, 9201.75\%, 18686.57\%, 146480.30\%, and 100422.00\%) for data set I to IV respectively. Using the proposed generalized class of estimators \( t \) over other existing estimators, there is considerable gain in efficiency. Thus there is ample room to pick up the scalars \((\alpha_1, \alpha_2)\) in order to obtain estimators better than the existing estimators. Finally our recommendation is in favor of the proposed generalized class of estimators \( t \) for its use in practice.

6 Estimation of Population Mean Under Stratified Random Sampling

We consider a finite population \( \Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_N\} \) of \( N \) units divided into \( L \) strata with the \( h \)th stratum \((h = 1, 2, \ldots, L)\) having \( N_h \) units such that \( \sum_{i=1}^{L} N_h = N \). Let \( y_{hi}, (x_{hi}, z_{hi})(i = 1, 2, \ldots, N_h) \) respectively be the observations of study variable \( y \) and auxiliary variables \((x, z)\) for the \( i \)th population unit in the \( h \)th stratum. A simple random sample of size \( n_h \) is drawn without replacement from the \( h \)th stratum such that \( \sum_{i=1}^{L} n_h = n \).

Let \( \bar{y}_{(st)} = \sum_{h=1}^{L} W_h \bar{y}_h, \bar{x}_{(st)} = \sum_{h=1}^{L} W_h \bar{x}_h \) and \( \bar{z}_{(st)} = \sum_{h=1}^{L} W_h \bar{z}_h \) be the sample means corresponding to the population means \( \bar{Y} = \sum_{h=1}^{L} W_h Y_h, \bar{X} = \sum_{h=1}^{L} W_h X_h \) and \( \bar{Z} = \sum_{h=1}^{L} W_h Z_h \) of the variables \( y, x \) and \( z \) respectively, where \( \bar{y}_h = \frac{1}{n_h} \sum_{i=1}^{N_h} y_{hi}, \bar{x}_h = \frac{1}{n_h} \sum_{i=1}^{N_h} x_{hi} \) and \( \bar{z}_h = \frac{1}{n_h} \sum_{i=1}^{N_h} z_{hi} \) be the sample means corresponding to the population means \( Y_h = \sum_{i=1}^{N_h} y_{hi}, X_h = \sum_{i=1}^{N_h} x_{hi} \) and \( Z_h = \sum_{i=1}^{N_h} z_{hi} \) in the \( h \)th stratum respectively with known stratum weight \( W_h = \frac{N_h}{N} \).

Further we denote

\[
C_{yh} = \frac{S_{yh}}{Y_h}, \quad C_{xh} = \frac{S_{xh}}{X_h}, \quad C_{zh} = \frac{S_{zh}}{Z_h}, \quad \rho_{yxh} = \frac{S_{yxh}}{S_{yh}S_{xh}},
\]

\[
\rho_{yzh} = \frac{S_{yzh}}{S_{yh}S_{zh}}, \quad \rho_{xzh} = \frac{S_{xzh}}{S_{xh}S_{zh}},
\]

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In the following section we have presented review of some existing estimators with their properties.

7 Reviewing Some Existing Estimators in Stratified Random Sampling

The conventional stratified sample mean estimator for population mean $\bar{Y}$ of $y$ is defined by

$$\hat{Y}_{0(st)} = \bar{y}_{(st)} = \sum_{h=1}^{L} W_h \bar{y}_h$$

(54)
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whose variance/MSE is given by

\[ \text{Var} \left( \hat{Y}_{0(st)} \right) = \text{MSE} \left( \hat{Y}_{0(st)} \right) = V_{200} = \sum_{h=1}^{L} \gamma_h W_h^2 S_{gh}^2 \] (55)

The combined ratio estimator for \( \bar{Y} \) is given by

\[ \hat{Y}_{R(st)} = \bar{y}(st) \left( \bar{X} \right) \] (56)

To the \( fda \), the MSE of \( \hat{Y}_{R(st)} \) is given by

\[ \text{MSE} \left( \hat{Y}_{R(st)} \right) = \left( V_{200} + R_1^2 V_{020} - 2R_1 V_{110} \right) \] (57)

The combined product estimator for \( \bar{Y} \) is defined by

\[ \hat{Y}_{P(st)} = \bar{y}(st) \left( \bar{x}(st) \right) \] (58)

The MSE of \( \hat{Y}_{P(st)} \) to the \( fda \) is given by

\[ \text{MSE} \left( \hat{Y}_{P(st)} \right) = \left( V_{200} + R_1^2 V_{020} + 2R_1 V_{110} \right) \] (59)

Following the approach adopted by Srivastava (1967), we define a class of estimators for population mean \( \bar{Y} \) as

\[ \hat{Y}_{S(st)} = \bar{y}(st) \left( \frac{x(st)}{\bar{X}} \right)^{\alpha_1} \] (60)

We mention that for \( \alpha_1 = -1 \), \( \hat{Y}_{S(st)} \) reduces to \( \hat{Y}_{R(st)} \) while for \( \alpha_1 = 1 \) it reduces to the product estimator \( \hat{Y}_{P(st)} \). If we set \( \alpha_1 = 0 \), then \( \hat{Y}_{S(st)} \) reduces to usual unbiased estimator \( \bar{y}(st) \).

The MSE of \( \hat{Y}_{S(st)} \) to the \( fda \) is given by

\[ \text{MSE} \left( \hat{Y}_{S(st)} \right) = \left( V_{200} + \alpha_1^2 R_1^2 V_{020} + 2\alpha_1 R_1 V_{110} \right) \] (61)

which is minimum when

\[ \alpha_1 = -\frac{V_{110}}{R_1 V_{020}} = \alpha_{1(\text{opt})}, \text{ say} \] (62)
Thus the corresponding minimum $MSE$ of $\hat{Y}_{S(st)}$ is given by

$$MSE_{\min} \left( \hat{Y}_{S(st)} \right) = \left( V_{200} - \frac{V_{110}^2}{V_{020}} \right)$$  \hspace{1cm} (63)

which is same as the minimum $MSE$ of the difference estimator

$$\hat{Y}_{D1(st)} = \bar{y}(st) + d \left( \bar{X} - \bar{x}_{(st)} \right)$$  \hspace{1cm} (64)

i.e.

$$MSE_{\min} \left( \hat{Y}_{S(st)} \right) = MSE_{\min} \left( \hat{Y}_{D1(st)} \right) = \left( V_{200} - \frac{V_{110}^2}{V_{020}} \right)$$  \hspace{1cm} (65)

The stratified version of Upadhyaya et al. (1985) estimator is given by

$$\hat{Y}_{USV(st)} = w_0 \bar{y}(st) + w_1 \bar{y}(st) \left( \frac{\bar{x}_{(st)}}{X} \right)^{\alpha_1}$$  \hspace{1cm} (66)

The $MSE$ of $\hat{Y}_{USV(st)}$ to the $fda$ is given by

$$MSE \left( \hat{Y}_{USV(st)} \right) = \left[ \bar{Y}^2 + w_0^2 A_0(st) + w_1^2 A_1(st) \right.$$  
$$\left. + 2w_0w_1 A_3(st) - 2w_0 \bar{Y}^2 - 2w_1 A_6(st) \right]$$  \hspace{1cm} (67)

where

$$A_0(st) = (\bar{Y}^2 + V_{200})$$
$$A_1(st) = \left[ \bar{Y}^2 + V_{200} + 4\alpha_1 R_1 V_{110} + \alpha_1 (2\alpha_1 - 1) R_1^2 V_{020} \right]$$
$$A_3(st) = \left[ \bar{Y}^2 + V_{200} + 2\alpha_1 R_1 V_{110} + \alpha_1 \frac{(\alpha_1 - 1)}{2} R_1^2 V_{020} \right]$$
$$A_6(st) = \left[ \bar{Y}^2 + \alpha_1 R_1 V_{110} + \alpha_1 \frac{(\alpha_1 - 1)}{2} R_1^2 V_{020} \right]$$

The $MSE(\hat{Y}_{USV(st)})$ is minimized for

$$w_0 = \frac{\Delta_0^*(st)}{\Delta_{(st)}^*}, w_1 = \frac{\Delta_1^*(st)}{\Delta_{(st)}^*}$$  \hspace{1cm} (68)
Thus the corresponding minimum $MSE$ of $\hat{\bar{Y}}_{USV(st)}$ is given by

$$MSE_{\min} \left( \hat{\bar{Y}}_{USV(st)} \right)$$

$$= \left[ \frac{\left( A_{1(st)} \bar{Y}^{2} - 2A_{3(st)}A_{6(st)} \bar{Y}^{2} + A_{0(st)}A_{6(st)}^{2} \right)}{A_{0(st)}A_{1(st)} - A_{3(st)}^{2}} \right]$$

(69)

where

$$\Delta_{(st)} = \left( A_{0(st)}A_{1(st)} - A_{3(st)}^{2} \right)$$

$$\Delta_{0(st)} = \left( A_{1(st)} - A_{3(st)}A_{6(st)} \right)$$

$$\Delta_{1(st)} = \left( A_{0(st)}A_{6(st)} - A_{3(st)} \right)$$

For $w_{0} + w_{1} = 1 \Rightarrow w_{1} = (1 - w_{0})$ in (66), the class of estimators $\hat{Y}_{USV(st)}$ reduces to the estimator

$$\hat{\bar{Y}}_{USV(st)} = w_{0} \bar{y}_{st} + (1 - w_{0}) \bar{y}_{st} \left( \frac{\bar{x}_{st}}{X} \right)^{a_{1}}$$

(70)

To the $fda$, the $MSE$ of $\hat{\bar{Y}}_{USV(st)}$ is given by

$$MSE \left( \hat{\bar{Y}}_{USV(st)} \right) = \left[ V_{100} + A_{1(st)} - 2A_{6(st)} + w_{0}^{2} \left( A_{0(st)} + A_{1(st)} - 2A_{3(st)} \right) \right.$$

$$\left. - 2w_{0} \left( \bar{Y}^{2} + A_{1(st)} - A_{3(st)} - A_{6(st)} \right) \right]$$

(71)

which is minimized for

$$w_{0(\text{opt})} = \frac{\left( \bar{Y}^{2} + A_{1(st)} - A_{3(st)} - A_{6(st)} \right)}{\left( A_{0(st)} + A_{1(st)} - 2A_{3(st)} \right)} = \frac{V_{110} + \alpha_{1} R_{1} V_{020}}{\alpha_{1} R_{1} V_{020}}$$

(72)

Thus the corresponding minimum $MSE$ of $\hat{\bar{Y}}_{USV(st)}$ is given by

$$MSE_{\min} \left( \hat{\bar{Y}}_{USV(st)} \right)$$

$$= \left[ \frac{\left( \bar{Y}^{2} + A_{1(st)} - 2A_{6(st)} - \left( \bar{Y}^{2} + A_{1(st)} - A_{3(st)} - A_{6(st)} \right)^{2} \right)}{\left( A_{0(st)} + A_{1(st)} - 2A_{3(st)} \right)} \right]$$

$$= \left( V_{200} - \frac{V_{110}^{2}}{V_{020}} \right) = MSE_{\min} \left( \hat{\bar{Y}}_{D1(st)} \right)$$

(73)
Stratified version of Rao (1991) estimator for population mean $\bar{Y}$ is given by

$$\hat{Y}_{Rao(st)} = \alpha_1 \bar{y}_{st} + \alpha_2 (\bar{X} - \bar{x}_{st})$$  \hspace{1cm} (74)

where $(\alpha_1, \alpha_2)$ are suitably chosen constants such that $MSE(\hat{Y}_{Rao(st)})$ is minimum.

The optimum values of $(\alpha_1, \alpha_2)$ along with minimum $MSE$ of $\hat{Y}_{Rao(st)}$ are respectively given by

$$\alpha_{1(opt)} = \left\{ \frac{\bar{Y}^2 V_{020}}{V_{020} (\bar{Y}^2 + V_{200}) - V_{110}^2} \right\}$$  \hspace{1cm} (75)

and

$$\alpha_{2(opt)} = -\left\{ \frac{\bar{Y}^2 V_{110}}{V_{020} (\bar{Y}^2 + V_{200}) - V_{110}^2} \right\}$$

and

$$MSE_{\min}(\hat{Y}_{Rao(st)}) = \frac{\bar{Y}^2 \{ V_{020} V_{200} - V_{110}^2 \}}{V_{020} (\bar{Y}^2 + V_{200}) - V_{110}^2}$$  \hspace{1cm} (76)

Gupta and Shabbir (2008) suggested the following estimator for $\bar{Y}$ as,

$$\hat{Y}_{GS(st)} = \left\{ \alpha_3 \bar{y}_{st} + \alpha_4 (\bar{X} - \bar{x}_{st}) \right\} \left( \frac{\bar{X}}{\bar{x}_{st}} \right)$$  \hspace{1cm} (77)

The $MSE$ of $\hat{Y}_{GS(st)}$ to the $fda$ is given by

$$MSE(\hat{Y}_{GS(st)}) = [\bar{Y}^2 + \alpha_3^2 a_1(st) + \alpha_4^2 a_2(st) + 2\alpha_3 \alpha_4 a_3(st)$$

$$- 2\alpha_3 a_4(st) - 2\alpha_4 a_5(st)]$$  \hspace{1cm} (78)

where

$$a_{1(st)} = [\bar{Y}^2 + V_{200} + 3R_1^2 V_{020} - 4R_1 V_{110}]$$

$$a_{2(st)} = V_{020}$$

$$a_{3(st)} = (2R_1 V_{020} - V_{110})$$

$$a_{4(st)} = (R_1^2 V_{020} - R_1 V_{110} + \bar{Y}^2)$$

$$a_{5(st)} = R_1 V_{020}$$
The $MSE$ of $\hat{\bar{Y}}_{GS(st)}$ is minimum when

$$\alpha_3(\text{opt}) = \frac{(a_2(st)a_4(st) - a_3(st)a_5(st))}{(a_1(st)a_2(st) - a_2^2(st))}$$

$$\alpha_4(\text{opt}) = \frac{(a_1(st)a_5(st) - a_3(st)a_4(st))}{(a_1(st)a_2(st) - a_3^2(st))}$$

(79)

Thus the minimum $MSE$ of $\hat{\bar{Y}}_{GS(st)}$ is given by

$$MSE_{\text{min}} (\hat{\bar{Y}}_{GS(st)}) = \left[ \bar{Y}^2 - \frac{(a_2(st)a_3^2(st) - 2a_3(st)a_4(st)a_5(st) + a_4(st)a_5^2(st))}{(a_1(st)a_2(st) - a_3^2(st))} \right]$$

(80)

The usual difference estimator using two auxiliary variables in stratified random sampling is defined by

$$\hat{Y}_{D2(st)} = \bar{y}_{st} + k_1 (\bar{X} - \bar{x}_{st}) + k_2 (\bar{Z} - \bar{z}_{st})$$

(81)

where $k_1$ and $k_2$ are constants whose values are to be obtained. The $MSE$ of $\hat{Y}_{D2(st)}$ is given by

$$MSE (\hat{Y}_{D2(st)}) = \left[ V_{200} + k_1^2 V_{002} + k_2^2 V_{002} + 2k_1k_2 V_{011} - 2k_1 V_{110} - 2k_2 V_{101} \right]$$

(82)

which is minimized for

$$k_1(\text{opt}) = \frac{(V_{002}V_{110} - V_{011}V_{101})}{(V_{020}V_{002} - V_{011}^2)}$$

$$k_2(\text{opt}) = \frac{(V_{020}V_{101} - V_{011}V_{110})}{(V_{020}V_{002} - V_{110}^2)}$$

(83)

Thus the corresponding minimum $MSE$ of $\hat{Y}_{D2(st)}$ is given by

$$MSE_{\text{min}} (\hat{Y}_{D2(st)}) = \left[ V_{200} - \frac{(V_{110}^2 V_{002} - 2V_{011}V_{101}V_{110} + V_{020}V_{101}^2)}{(V_{020}V_{002} - V_{011}^2)} \right]$$

(84)
Suggested Class of Estimators for Population Mean in Stratified Random Sampling

Motivated by Upadhyaya et al. (1985), we propose a generalized class of estimators based on two auxiliary variables \((x, z)\) for population mean \(\bar{Y}\) in stratified random sampling as

\[
t_{(st)} = w_0 \bar{y}_{st} + w_1 \bar{y}_{st} \left( \frac{\bar{x}_{st}}{X} \right)^{\alpha_1} + w_2 \bar{y}_{st} \left( \frac{\bar{z}_{st}}{Z} \right)^{\alpha_2}
\]

(85)

where \((w_0, w_1, w_2)\) are appropriately elected weights whose sum need not be unity and \((\alpha_1, \alpha_2)\) are design parameters. The constants \((\alpha_1, \alpha_2)\) may take positive \((+, +)\) or negative \((-,-)\) or positive-negative \((+, -)\) or negative-positive \((-,+)\) values to form product-type or ratio-type or product-cum-ratio-type or ratio-cum-product-type estimator.

To obtain the bias and \(MSE\) of the proposed estimator \(t_{(st)}\), we write,

\[
\bar{y}_{st} = \bar{Y}(1 + e_0(st)), \quad \bar{x}_{st} = \bar{X}(1 + e_1(st)) \quad \text{and} \quad \bar{z}_{st} = \bar{Z}(1 + e_2(st))
\]

such that

\[
E(e_0^{2}(st)) = \frac{1}{Y^2} \sum_{h=1}^{L} W_h^2 \gamma_h S_{yh}^2, \quad E\left(e_1^{2}(st)\right) = \frac{1}{X^2} \sum_{h=1}^{L} W_h^2 \gamma_h S_{xh}^2,
\]

\[
E\left(e_2^{2}(st)\right) = \frac{1}{Z^2} \sum_{h=1}^{L} W_h^2 \gamma_h S_{zh}^2,
\]

\[
E(e_0(st)e_1(st)) = \frac{1}{XY} \sum_{h=1}^{L} W_h^2 \gamma_h S_{yxh},
\]

\[
E(e_0(st)e_2(st)) = \frac{1}{YZ} \sum_{h=1}^{L} W_h^2 \gamma_h S_{yzh} \quad \text{and}
\]

\[
E(e_1(st)e_2(st)) = \frac{1}{XZ} \sum_{h=1}^{L} W_h^2 \gamma_h S_{xzh}.
\]

Expressing (85) in terms of \(e\’s\) we have

\[
t_{(st)} = \bar{Y} \left[ w_0 \left(1 + e_0(st)\right) + w_1 \left(1 + e_0(st)\right) \left(1 + e_1(st)\right)^{\alpha_1} + w_2 \left(1 + e_0(st)\right) \left(1 + e_1(st)\right)^{\alpha_2} \right]
\]

(86)
We assume that $|e_i| \leq 1$, $i = 1, 2$ so that $(1 + e_i)^\alpha_i$, $i = 1, 2$ are expandable. Expanding the right hand side of (85), multiplying out and ignoring terms of $e$'s having power greater than two, we have

$$t_{(st)} \cong \bar{Y} \begin{bmatrix} w_0 (1 + e_{0(st)}) \\
+ w_1 \left\{ 1 + e_{0(st)} + \alpha_1 e_{1(st)} + \alpha_1 e_{0(st)} e_{1(st)} \right\} \\
+ \frac{\alpha_1 (\alpha_1 - 1)}{2} e_{1(st)} \\
+ w_2 \left\{ 1 + e_{0(st)} + \alpha_2 e_{2(st)} + \alpha_2 e_{0(st)} e_{2(st)} \right\} \\
+ \frac{\alpha_2 (\alpha_2 - 1)}{2} e_{2(st)} \end{bmatrix}$$

or

$$t_{(st)} - \bar{Y} \cong \bar{Y} \begin{bmatrix} w_0 (1 + e_{0(st)}) \\
+ w_1 \left\{ 1 + e_{0(st)} + \alpha_1 e_{1(st)} + \alpha_1 e_{0(st)} e_{1(st)} \right\} \\
+ \frac{\alpha_1 (\alpha_1 - 1)}{2} e_{1(st)} \\
+ w_2 \left\{ 1 + e_{0(st)} + \alpha_2 e_{2(st)} + \alpha_2 e_{0(st)} e_{2(st)} \right\} \\
+ \frac{\alpha_2 (\alpha_2 - 1)}{2} e_{2(st)} \end{bmatrix} - 1$$

(87)

Taking expectation of both sides of (87), we get the bias of $t_{(st)}$ to the $fda$ as

$$B(t_{(st)}) = \bar{Y} (w_0 - 1) + w_1 \left\{ \bar{Y} + \frac{\alpha_1 (\alpha_1 - 1)}{2} R_1 V_{020} X + \alpha_1 \frac{V_{110}}{X} \right\}$$

$$+ w_2 \left\{ \bar{Y} + \frac{\alpha_2 (\alpha_2 - 1)}{2} R_2 V_{002} Z + \alpha_2 \frac{V_{101}}{Z} \right\}$$

(88)

Squaring both sides of (87), ignoring terms of $e$'s having power greater than two and then taking expectation of both sides we get the $MSE$ of $t_{(st)}$ to
the fda as

\[
MSE \left( t_{(st)} \right) = \begin{bmatrix}
\hat{Y}^2 + w_0^2 A_{0(st)} + w_1^2 A_{1(st)} + w_2^2 A_{2(st)} + 2w_0 w_1 A_{3(st)} \\
+2w_0 w_2 A_{4(st)} + 2w_1 w_2 A_{5(st)} - 2w_0 \hat{Y}^2 \\
-2w_1 A_{6(st)} - 2w_2 A_{7(st)}
\end{bmatrix}
\]

(89)

where

\[
A_{2(st)} = \left[ \hat{Y}^2 + V_{200} + 4\alpha_2 R_2 V_{101} + \alpha_2 \left( 2\alpha_2 - 1 \right) R_2^2 V_{002} \right]
\]

\[
A_{4(st)} = \left[ \hat{Y}^2 + V_{200} + 2\alpha_2 R_2 V_{101} + \alpha_2 \frac{\left( \alpha_2 - 1 \right)}{2} R_2^2 V_{002} \right]
\]

\[
A_{5(st)} = \left[ \hat{Y}^2 + V_{200} + 2\alpha_1 R_1 V_{110} + 2\alpha_2 R_2 V_{101} + \alpha_1 \alpha_2 R_1 R_2 V_{011} \\
+ \alpha_1 \frac{\left( \alpha_1 - 1 \right)}{2} R_1^2 V_{020} + \alpha_2 \frac{\left( \alpha_2 - 1 \right)}{2} R_2^2 V_{002} \right]
\]

\[
A_{7(st)} = \left[ \hat{Y}^2 + \alpha_2 R_2 V_{101} + \alpha_2 \frac{\left( \alpha_2 - 1 \right)}{2} R_2^2 V_{002} \right]
\]

\[
(A_{0(st)}, A_{1(st)}, A_{3(st)} \text{ and } A_{6(st)}) \text{ are same as defined earlier.}
\]

Minimization of \(MSE(t_{(st)})\) at (89) with respect to \((w_0, w_1, w_2)\) yields

\[
\begin{bmatrix}
A_{0(st)} & A_{3(st)} & A_{4(st)} \\
A_{3(st)} & A_{1(st)} & A_{5(st)} \\
A_{4(st)} & A_{5(st)} & A_{2(st)}
\end{bmatrix}
\begin{bmatrix}
w_0 \\
w_1 \\
w_2
\end{bmatrix}
= \begin{bmatrix}
\hat{Y}^2 \\
A_{6(st)} \\
A_{7(st)}
\end{bmatrix}
\]

(90)

Solving (90), we get the optimum values of \((w_0, w_1, w_2)\) respectively as

\[
w_{00} = \frac{\triangle 0(st)}{\triangle (st)}, \quad w_{10} = \frac{\triangle 1(st)}{\triangle (st)}, \quad w_{20} = \frac{\triangle 2(st)}{\triangle (st)}.
\]

(91)

where

\[
\triangle (st) = \begin{vmatrix}
A_{0(st)} & A_{3(st)} & A_{4(st)} \\
A_{3(st)} & A_{1(st)} & A_{5(st)} \\
A_{4(st)} & A_{5(st)} & A_{2(st)}
\end{vmatrix}
\]

\[
= A_{0(st)} \left( A_{1(st)} A_{2(st)} - A_{5(st)}^2 \right)
\]
A Generalized Class of Estimators for Finite Population Mean

\[ A_{3(st)} \left( A_{2(st)} A_{3(st)} - A_{4(st)} A_{5(st)} \right) \]
\[ + A_{4(st)} \left( A_{3(st)} A_{5(st)} - A_{1(st)} A_{4(st)} \right) \]

\[ \Delta_{0(st)} = \begin{vmatrix} \bar{Y}^2 & A_{4(st)} \\ A_{3(st)} & A_{5(st)} \\ A_{7(st)} & A_{2(st)} \end{vmatrix} \]
\[ = \bar{Y}^2 \left( A_{1(st)} A_{2(st)} - A_{5(st)}^2 \right) \]
\[- A_{3(st)} \left( A_{2(st)} A_{6(st)} - A_{5(st)} A_{7(st)} \right) \]
\[ + A_{4(st)} \left( A_{5(st)} A_{6(st)} - A_{1(st)} A_{7(st)} \right) \]

\[ \Delta_{1(st)} = \begin{vmatrix} A_{0(st)} & \bar{Y}^2 & A_{4(st)} \\ A_{3(st)} & A_{6(st)} & A_{5(st)} \\ A_{4(st)} & A_{7(st)} & A_{2(st)} \end{vmatrix} \]
\[ = A_{0(st)} \left( A_{2(st)} A_{6(st)} - A_{5(st)} A_{7(st)} \right) \]
\[- A_{3(st)} \left( A_{2(st)} A_{5(st)} - A_{4(st)} A_{5(st)} \right) \]
\[ + A_{4(st)} \left( A_{3(st)} A_{7(st)} - A_{4(st)} A_{6(st)} \right) \]

\[ \Delta_{2(st)} = \begin{vmatrix} A_{0(st)} & A_{3(st)} & \bar{Y}^2 \\ A_{3(st)} & A_{1(st)} & A_{6(st)} \\ A_{4(st)} & A_{5(st)} & A_{7(st)} \end{vmatrix} \]
\[ = A_{0(st)} \left( A_{1(st)} A_{7(st)} - A_{5(st)} A_{6(st)} \right) \]
\[- A_{3(st)} \left( A_{3(st)} A_{7(st)} - A_{4(st)} A_{6(st)} \right) \]
\[ + \bar{Y}^2 \left( A_{3(st)} A_{5(st)} - A_{1(st)} A_{4(st)} \right) \]

Thus the corresponding minimum MSE of \( t_{(st)} \) is given by

\[ MSE_{\text{min}}(t_{(st)}) = \left[ \bar{Y}^2 - \frac{\Delta_{0(st)} \bar{Y}^2}{\Delta_{(st)}} - \frac{A_{6(st)} \Delta_{1(st)}}{\Delta_{(st)}} - \frac{A_{7(st)} \Delta_{2(st)}}{\Delta_{(st)}} \right] \]

Thus we state the following theorem.
Theorem-8.1 – The \( \text{MSE} \) of \( t_{st} \) is always greater than equal to the minimum \( \text{MSE} \) of \( t_{(st)} \) i.e.

\[
\text{MSE}(t_{(st)}) \geq \text{MSE}_{\text{min}}(t_{(st)})
\]

\[
= \left[ Y^2 - \frac{\Delta_{0(st)} Y^2}{\Delta_{(st)}} - \frac{A_{6(st)} \Delta_{1(st)}}{\Delta_{(st)}} - \frac{A_{7(st)} \Delta_{2(st)}}{\Delta_{(st)}} \right]
\]

with equality holding if

\[
w_{0i} = \frac{\Delta_i}{\Delta}, \quad i = 0, 1, 2.
\]

*A remark similar to Remark 3.1 follows here.

9 Comparison of the Proposed Class of Estimator with Some Existing Estimators in Stratified Random Sampling

From (55), (57), (59) and (65), we have

\[
\text{MSE}\left(\hat{\bar{y}}_{(st)}\right) - \left[ \text{MSE}_{\text{min}}\left(\hat{\bar{y}}_{D1(st)}\right) = \text{MSE}_{\text{min}}\left(\hat{\bar{y}}_{S(st)}\right) \right]
\]

\[
= \frac{V_{110}^{2}}{V_{020}} \geq 0
\]  

(93)

\[
\text{MSE}\left(\bar{y}_{R(st)}\right) - \left[ \text{MSE}_{\text{min}}\left(\hat{\bar{y}}_{D1(st)}\right) = \text{MSE}_{\text{min}}\left(\hat{\bar{y}}_{S(st)}\right) \right]
\]

\[
= \frac{(R_{1} V_{020} - V_{110})^2}{V_{020}} \geq 0
\]  

(94)

\[
\text{MSE}\left(\bar{y}_{P(st)}\right) - \left[ \text{MSE}_{\text{min}}\left(\hat{\bar{y}}_{D1(st)}\right) = \text{MSE}_{\text{min}}\left(\hat{\bar{y}}_{S(st)}\right) \right]
\]

\[
= \frac{(R_{1} V_{020} + V_{110})^2}{V_{020}} \geq 0
\]  

(95)

Expressions (93), (94) and (95) clearly indicates that the usual difference estimator \( \hat{\bar{y}}_{D1(st)} \) and Srivastava (1967) estimator \( \hat{\bar{y}}_{S(st)} \) are better than the estimators \( \bar{y}_{st} \), \( \hat{\bar{y}}_{R(st)} \) and \( \hat{\bar{y}}_{P(st)} \).
From (65) and (76), we have
\[ \text{MSE}_{\text{min}} \left( \hat{Y}_{D1(st)} \right) = \text{MSE}_{\text{min}} \left( \hat{Y}_{S(st)} \right) - \text{MSE}_{\text{min}} \left( \hat{Y}_{\text{Rao}(st)} \right) \]
\[ = \frac{\left( V_{200} V_{020} - V_{110}^2 \right)^2}{(V^2 V_{020} + V_{020} V_{200} - V_{110}^2)} \geq 0 \quad (96) \]

From (93)–(96), we have the following inequalities:
\[ \text{MSE}_{\text{min}} \left( \hat{Y}_{\text{Rao}(st)} \right) \leq \left[ \text{MSE}_{\text{min}} \left( \hat{Y}_{D1(st)} \right) = \text{MSE}_{\text{min}} \left( \hat{Y}_{S(st)} \right) \right] \]
\[ \leq \text{MSE} \left( \bar{y}_{st} \right) \quad (97) \]
\[ \text{MSE}_{\text{min}} \left( \hat{Y}_{\text{Rao}(st)} \right) \leq \left[ \text{MSE}_{\text{min}} \left( \hat{Y}_{D1(st)} \right) = \text{MSE}_{\text{min}} \left( \hat{Y}_{S(st)} \right) \right] \]
\[ \leq \text{MSE} \left( \hat{Y}_{R(st)} \right) \quad (98) \]
\[ \text{MSE}_{\text{min}} \left( \hat{Y}_{\text{Rao}(st)} \right) \leq \left[ \text{MSE}_{\text{min}} \left( \hat{Y}_{D1(st)} \right) = \text{MSE}_{\text{min}} \left( \hat{Y}_{S(st)} \right) \right] \]
\[ \leq \text{MSE} \left( \hat{Y}_{P(st)} \right) \quad (99) \]

It follows from (97), (98) and (99) that the Rao (1991) estimator \( \hat{Y}_{\text{Rao}(st)} \) is more precise than \( \bar{y}_{st} \), \( \hat{Y}_{R(st)} \), \( \hat{Y}_{P(st)} \), \( \hat{Y}_{D1(st)} \) and Srivastava’s (1967) estimator \( \hat{Y}_{S(st)} \).

From (65) and (69), we have
\[ \left[ \text{MSE}_{\text{min}} \left( \hat{Y}_{D1(st)} \right) = \text{MSE}_{\text{min}} \left( \hat{Y}_{S(st)} \right) = \text{MSE}_{\text{min}} \left( \hat{Y}_{USV(st)} \right) \right] \]
\[ - \text{MSE}_{\text{min}} \left( \hat{Y}_{USV(st)} \right) \]
\[ = \bar{Y}^2 \left[ A_{1(st)} \left( \bar{Y}^2 - A_0(st) \right) + A_{3(st)} \left( A_{3(st)} - \bar{Y}^2 \right) \right]^2 + A_6(st) \left( A_0(st) - A_{3(st)} \right) \left( A_0(st) + A_{1(st)} - 2 A_{3(st)} \right) \]
\[ \geq 0 \quad (100) \]

It follows that the Upadhyaya et al. (1985) estimator \( \hat{Y}_{USV(st)} \) is more efficient than \( \hat{Y}_{D1(st)} \), \( \hat{Y}_{S(st)} \) and \( \hat{Y}_{USV(st)} \).
From (69) and (92), we have
\[
MSE_{\text{min}} \left( \hat{\bar{y}}_{USV(st)} \right) - MSE_{\text{min}} \left( t_{(st)} \right)
= \frac{\bar{y}^2}{\Delta_{(st)} \Delta_1(st)} \left[ A_{7(st)} \left( A_{0(st)} A_{1(st)} - A_{3(st)}^2 \right) + (A_{3(st)} A_{5(st)} - A_{1(st)} A_{4(st)}) \right. \\
\left. + A_{6(st)} \left( A_{3(st)} A_{4(st)} - A_{0(st)} A_{5(st)} \right) \right] ^2 \geq 0 \tag{101}
\]
which shows that the proposed generalized class of estimators \( t_{(st)} \) is more efficient than Upadhyaya et al. (1985) estimator \( \hat{y}_{USV(st)} \). Hence the estimator \( t_{(st)} \) is more precise than the estimators \( \bar{y}_{st}, \hat{y}_{R(st)}, \hat{y}_{P(st)}, \hat{y}_{D1(st)}, \hat{y}_{S(st)} \) and \( \hat{y}^*_{USV(st)} \).

From (73) and (84) we have
\[
\left[ MSE_{\text{min}} \left( \hat{\bar{y}}_{D1(st)} \right) = MSE_{\text{min}} \left( \hat{\bar{y}}_{S(st)} \right) = MSE_{\text{min}} \left( \hat{\bar{y}}_{USV(st)} \right) \right] \\
- MSE_{\text{min}} \left( \hat{\bar{y}}_{D2(st)} \right) = \frac{(V_{020} V_{011} - V_{110} V_{011})}{V_{020} (V_{020} V_{002} - V_{011})} \geq 0 \tag{102}
\]
which shows that the difference estimator \( \hat{\bar{y}}_{D2(st)} \) is more efficient than the estimator \( \hat{\bar{y}}_{D1(st)} \).

From (76), (80), (84) and (92), we have
\[
\begin{align*}
& \text{• } MSE_{\text{min}}(t_{(st)}) < MSE_{\text{min}}(\hat{\bar{y}}_{Rao(st)}) \text{ if } \\
& \bar{y}^4 V_{020} \left[ \frac{V_{020} (Y^2 + V_{200}) - V_{110}^2}{V_{020} (V_{020} V_{002} - V_{011})} \right] < \left[ \frac{(\Delta_{0(st)} \bar{y}^2 + A_{6(st)} \Delta_1(st) + A_{7(st)} \Delta_2(st))}{\Delta_{(st)}} \right] ^2 \tag{103} \\
& \text{• } MSE_{\text{min}}(t_{(st)}) < MSE_{\text{min}}(\hat{\bar{y}}_{GS(st)}) \text{ if } \\
& \left[ \frac{(a_{2(st)} a_{5(st)}^2 - 2a_{3(st)} a_{4(st)} a_{5(st)} + a_{4(st)} a_{5(st)}^2)}{(a_{1(st)} a_{2(st)} - a_{3(st)}^2)} \right] ^2 < \left[ \frac{(\Delta_{0(st)} \bar{y}^2 + A_{6(st)} \Delta_1(st) + A_{7(st)} \Delta_2(st))}{\Delta_{(st)}} \right] ^2 \tag{104}
\end{align*}
\]
To examine the performance of the proposed generalized class of estimators \( t_{(st)} \) over existing estimators, we use the data sets given below

**Data I**: Source: [Murthy (1967), P. 228]

\[
N = 80, \ n = 22
\]

| \( N_1 \) | \( N_2 \) | \( N_3 \) | \( n_1 \) | \( n_2 \) | \( n_3 \) |
|---|---|---|---|---|---|
| 19 | 32 | 29 | 5 | 9 | 8 |

\[
\begin{align*}
Y_1 &= 2967.95 \quad Y_2 = 4657.63 \quad Y_3 = 7212.97 \quad X_1 = 65.16 \quad X_2 = 139.97 \quad X_3 = 589.41 \\
Z_1 &= 349.68 \quad Z_2 = 706.59 \quad Z_3 = 2098.69 \quad C_{y_1} = 0.25509 \quad C_{y_2} = 0.14366 \quad C_{y_3} = 0.11848 \\
C_{x_1} &= 0.17168 \quad C_{x_2} = 0.31693 \quad C_{x_3} = 0.8415 \quad C_{x_1} = 0.3130 \quad C_{x_2} = 0.15477 \quad C_{x_3} = 0.30386 \\
\rho_{yx_1} &= 0.81 \quad \rho_{yx_2} = 0.89 \quad \rho_{yx_3} = 0.98 \quad \rho_{zx_1} = 0.94 \quad \rho_{zx_2} = 0.93 \quad \rho_{zx_3} = 0.98 \\
\rho_{xz_1} &= 0.90 \quad \rho_{xz_2} = 0.85 \quad \rho_{xz_3} = 0.97
\end{align*}
\]

**Data II**: [Source: Koyuncu and Kadilar (2009)]

\[
N = 923, \ n = 180
\]

| \( N_1 \) | \( N_2 \) | \( N_3 \) | \( N_4 \) | \( N_5 \) | \( N_6 \) |
|---|---|---|---|---|---|
| 127 | 117 | 103 | 170 | 205 | 201 |

| \( n_1 \) | \( n_2 \) | \( n_3 \) | \( n_4 \) | \( n_5 \) | \( n_6 \) |
|---|---|---|---|---|---|
| 31 | 21 | 29 | 38 | 22 | 39 |

\[
\begin{align*}
Y_1 &= 703.74 \quad Y_2 = 413.0 \quad Y_3 = 513.17 \quad Y_4 = 424.66 \quad Y_5 = 267.03 \quad Y_6 = 393.84 \\
X_1 &= 20804.59 \quad X_2 = 9211.79 \quad X_3 = 14309.0 \quad X_4 = 9478.85 \quad X_5 = 5569.95 \quad X_6 = 12997.59 \\
Z_1 &= 498.28 \quad Z_2 = 318.33 \quad Z_3 = 413.36 \quad Z_4 = 311.32 \quad Z_5 = 227.20 \quad Z_6 = 313.71 \\
S_{y_1} &= 881.84 \quad S_{y_2} = 644.92 \quad S_{y_3} = 1033.46 \quad S_{y_4} = 810.58 \quad S_{y_5} = 403.65 \quad S_{y_6} = 711.72 \\
S_{x_1} &= 30486.7 \quad S_{x_2} = 15180.77 \quad S_{x_3} = 27549.78 \quad S_{x_4} = 18218.93 \quad S_{x_5} = 8497.77 \quad S_{x_6} = 2394.14 \\
S_{x_1} &= 555.58 \quad S_{x_2} = 365.46 \quad S_{x_3} = 612.95 \quad S_{x_4} = 458.03 \quad S_{x_5} = 260.85 \quad S_{x_6} = 397.05 \\
\rho_{yx_1} &= 0.936 \quad \rho_{yx_2} = 0.996 \quad \rho_{yx_3} = 0.994 \quad \rho_{yx_4} = 0.983 \quad \rho_{yx_5} = 0.989 \quad \rho_{yx_6} = 0.965 \\
\rho_{yx_1} &= 0.979 \quad \rho_{yx_2} = 0.976 \quad \rho_{yx_3} = 0.984 \quad \rho_{yx_4} = 0.983 \quad \rho_{yx_5} = 0.964 \quad \rho_{yx_6} = 0.983 \\
\rho_{zx_1} &= 0.9396 \quad \rho_{zx_2} = 0.9696 \quad \rho_{zx_3} = 0.977 \quad \rho_{zx_4} = 0.964 \quad \rho_{zx_5} = 0.9676 \quad \rho_{zx_6} = 0.996
\end{align*}
\]
Table 4  PRE’s of different estimators of population mean $\bar{Y}$ with respect to $\bar{y}_{st}$

| Estimator          | Data I  | Data II |
|--------------------|---------|---------|
| $\hat{Y}_{R(st)}$  | 14.42   | 1025.10 |
| $\hat{Y}_{P(st)}$  | 5.89    |  24.22 |
| $\hat{Y}_{D1(st)}$| 235.83  | 1141.85 |
| $\hat{Y}_{Rao(st)}$| 235.91  | 1143.02 |
| $\hat{Y}_{GS(st)}$| 183.44  | 1109.11 |
| $\hat{Y}_{D2(st)}$| 273.99  | 2621.61 |

Table 5  PRE’s of the estimator $\hat{Y}_{USV(st)}$ with respect to $\bar{y}_{st}$

| Data I | Data II |
|--------|---------|
| $\alpha_1$ | PRE | $\alpha_1$ | PRE |
| −1     | 238.07 | −1     | 1146.96 |
| 1      | 235.84 | 1      | 1260.08 |

Table 4 presents the PRE’s of $\hat{Y}_{R(st)}, \hat{Y}_{P(st)}, \hat{Y}_{D1(st)}, \hat{Y}_{Rao(st)}, \hat{Y}_{GS(st)}$ and $\hat{Y}_{D2(st)}$ estimators with respect to $\bar{y}_{(st)}$ for two data sets respectively.

Table 5 shows the PRE of $\hat{Y}_{USV(st)}$ with respect to $\bar{y}_{(st)}$ for $\alpha_1 = −1, 1$, for two data sets.

Table 6 depicts the PRE of proposed estimator $t_{(st)}$ wrt $\bar{y}_{(st)}$ at different values of $\alpha_1$ and $\alpha_2$, for two data sets.

Table 6  PRE’s of the proposed estimator $t_{(st)}$ with respect to $\bar{y}_{st}$ for different values of $(\alpha_1, \alpha_2)$

| Data I | Data II |
|--------|---------|
| $\alpha_1$ | $\alpha_2$ | PRE | $\alpha_1$ | $\alpha_2$ | PRE |
| −1     | −1     | 275.29 | −1 | −1 | 2626.83 |
| −2     | −2     | 277.33 | −2 | −2 | 2720.91 |
| −3     | −3     | 280.47 | −3 | −3 | 3216.10 |
| −4     | −4     | 284.92 | −4 | −4 | 4826.13 |
| −5     | −5     | 291.07 | −5 | −5 | 20279.5 |
| −8     | −8     | 327.57 | −5.2 | −5.2 | 96957.81 |
| −10    | −10    | 391.89 | 1  | 1  | 3590.02 |
| −12    | −12    | 642.36 | 2  | 2  | 6380.74 |
| −13    | −13    | 1645.27 | 2.1 | 2.1 | 7060.90 |
| −13.1  | −13.1  | 2078.85 | 2.2 | 2.2 | 7938.96 |
| −13.2  | −13.2  | 2897.43 | 2.5 | 2.5 | 13252.63 |
| −13.3  | −13.3  | 4928.01 | 2.8 | 2.8 | 51458.14 |
| −13.4  | −13.4  | 20017.07 | −1 | 1  | 3067.82 |

(Continued)
It is observed from Tables 4, 5 and 6 that for various values of \((\alpha_1, \alpha_2)\) the proposed generalized class of estimators \(t_{(st)}\) is more efficient than the estimators \(\bar{y}_{(st)}\), \(\hat{y}_{R_{(st)}}\), \(\hat{y}_{P_{(st)}}\), \(\hat{y}_{D1_{(st)}}\), \(\hat{y}_{Rao_{(st)}}\), \(\hat{y}_{GS_{(st)}}\), \(\hat{y}_{D2_{(st)}}\) and \(\hat{y}_{USV_{(st)}}\), with considerable gain in efficiency. The proposed generalized class of estimators \(t_{(st)}\) yields the largest percent relative efficiency 22036.60% for data set I while it is 96957.81% for data set II. It is further observed from Table 6 that there is enough scope of selecting the scalars \((\alpha_1, \alpha_2)\) in acquiring efficient estimators (from the suggested generalized class of estimators \(t_{(st)}\)) than the existing estimators. Thus we conclude that the proposed generalized class of estimators \(t_{(st)}\) can be used in practice just by selecting the appropriate values of \((\alpha_1, \alpha_2)\).

### 11 Discussion and Conclusion

This article considers the problem of estimating the population mean \(\bar{Y}\) of the study variable \(y\) using information on two auxiliary variables \(x\) and \(z\). We have

| Data I |  | Data II |  |
|--------|--------|----------|--------|
| \(\alpha_1\) | \(\alpha_2\) | PRE | \(\alpha_1\) | \(\alpha_2\) | PRE |
| 1      | 1      | 274.03   | 2      | -1     | 2923.05 |
| 2      | 2      | 274.74   | -1     | 2      | 1002.942 |
| 3      | 3      | 276.39   | 3      | -1     | 2734.047 |
| 4      | 4      | 279.07   | 4      | -1     | 2871.735 |
| 5      | 5      | 282.99   |        | *      | *       |
| 8      | 8      | 306.25   | *      | *      | *       |
| 10     | 10     | 342.15   | *      | *      | *       |
| 12     | 12     | 434.69   | *      | *      | *       |
| 14     | 14     | 1047.46  | *      | *      | *       |
| 14.1   | 14.1   | 1180.86  | *      | *      | *       |
| 14.3   | 14.3   | 1634.81  | *      | *      | *       |
| 14.5   | 14.5   | 2885.63  | *      | *      | *       |
| 14.6   | 14.6   | 4977.54  | *      | *      | *       |
| 14.7   | 14.7   | 22036.36 | *      | *      | *       |
|       | -1     | 274.16   | *      | *      | *       |
|       | -1     | 275.62   | *      | *      | *       |
|       | 3      | 277.21   | *      | *      | *       |
|       | 4      | 279.03   | *      | *      | *       |
|       | -5     | 259.76   | *      | *      | *       |
proposed a generalized class of estimators for the population mean $\bar{Y}$ using information on two supplementary variables $x$ and $z$. Expressions of bias and mean square error up to the $fda$ have been obtained in $SRSWOR$ as well as in stratified random sampling. It is interesting to mention that the envisaged class of estimators includes several existing estimators. Thus the properties of the proposed generalized class of estimators unify results at one place. We have proved theoretically that the proposed generalized class of estimators is more efficient than the several existing estimators in both sampling designs $SRSWOR$ and stratified random sampling.

Empirical studies are carried out to throw light on the merits of the envisaged generalized class of estimators over some existing competitors. Larger gain in efficiency is observed by using the proposed generalized class of estimators over some existing estimators in both the sampling designs: $SRSWOR$ and stratified random sampling. Results incorporated in this study are very sound and quite illuminating. Thus it is recommended that the proposed study is useful in practice.

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**References**

Abu-Dayyeh, W. A., Ahmed, M. S., Ahmed, R. A. and Mutlak, H. A. (2003). Some estimators of a finite population mean using auxiliary information, *Applied Mathematics and Computation*, 139, 287–298.

Ahmed, M. S. (1997). The general class of chain estimators for ratio of two means using double sampling, *Communications in Statistics-Theory and Methods*, 26(9), 2249–2254.

Cochran, W. G. (1977). Sampling Techniques, *Jhon Wiley and Sons, New York*.

Das, A.K. and Tripathi, T. P. (1978). Use of auxiliary information in estimating the finite population variance, *Sankhya*, C, 40, 139–148.

Diana, G. (1993). A class of estimators of the population mean in stratified random sampling, *Statistica*, 53(1), 59–66.

Gleser, L. J. and Healy, J. D. (1976). Estimating the mean of a normal distribution with known coefficient of variation, *Journal of American Statistical Association*, 71, 977–981.
Gupta, S. and Shabbir, J. (2008). On estimating in estimating the population mean in simple random sampling, *Journal of Applied Statistics*, 35(5), 559–566.

Kadilar, C. and Cingi, H. (2003). Ratio estimator in stratified sampling, *Biometrical Journal*, 45, 218–225.

Koyuncu, N. (2013). Families of estimators for population mean using information on auxiliary attribute in stratified random sampling, *Gazi University Journal of Sciences*, 26(2), 181–193.

Koyuncu, N. (2016). Improved exponential type estimators for finite population mean in stratified random sampling, *Pakistan Journal of Statistics and Operation Research*, 12(3), 429–441.

Koyuncu, N. and Kadilar C. (2009). Family of estimators of population mean using two auxiliary variables in stratified sampling, *Communications in Statistics-Theory and Methods*, 38, 2398–2417.

Koyuncu, N. and Kadilar, C. (2010). On the family of estimators of population mean in stratified random sampling, *Communications in Statistics-Theory and Methods*, 26(2), 427–443.

Malik, S. and Singh, R. (2017). A new estimator for population mean using two auxiliary variables in stratified random sampling, *Journal of Information and Optimization Sciences*, 38(8), 1243–1252.

Mishra, M., Singh, B. P. and Singh, R. (2017). Estimation of population mean using two auxiliary variables in the stratified random sampling, *Journal of Reliability and Statistical Studies*, 10(1), 59–68.

Munner, S., Shabbir, J. and Khalil, A. (2016). Estimation of finite population mean in simple random sampling and stratified random sampling using two auxiliary variables, *Communications in Statistics-Theory and Methods*, DOI:10.1080/03610926.2015.1035394.

Murthy, M. N. (1967). *Sampling Theory and Methods*, Statistical Publishing Society, India.

Olkin, I. (1958). Multivariate Ratio Estimation for Finite Population *Biometrika*, 45, 154–165.

Olufadi, Y. (2013). Dual to ratio-cum-product estimator in simple and stratified random sampling, *Pakistan Journal of Statistics and Operation Research*, 9(3), 305–319.

PCR (1998). Population Census Report of Multan District, *Punjab Bureau of Statistics, Lahore, Pakistan*.

Raj, D. (1965). On a method of using multiauxiliary information sample surveys, *Journal of American Statistical Association*, 60, 270–277.
Rao, P. S. R. S. and Mudholkar, G. D. (1967). Generalized multivariate estimator for the mean of finite populations, *Journal of American Statistical Association*, 62, 1009–1012.

Rao, T. J. (1991). On certain method of improving rations and regression estimators, *Communications in Statistics, Theory and Methods*, 20(10), 3325–3340.

Ray, S. K., Sahai, A. and Sahai, A. (1979). A note on ratio and product-type estimators, *Annals of the Institute of Statistical Mathematics*, 31, 141–144.

Reddy, V. N. (1978). A study on the use of prior knowledge on certain population parameters in estimation, *Sankhya*, C, 40, 29–34.

Searls, D. T. (1964). The utilization of a known coefficient of variation in the estimation procedure, *Journal of American Statistical Association*, 59, 1225–226.

Shabbir, J. (2018). Efficient utilisation of two auxiliary variables in stratified double sampling, *Communications in Statistics-Theory and Methods*, 47(1), 92–101.

Shabbir, J. and Gupta, S. (2005). Improved ratio estimators in stratified sampling, *American Journal of Mathematics and Management Sciences*, 25(3–4), 293–311.

Shabbir, J. and Gupta, S. (2015). Estimation of finite population mean using two auxiliary variables in stratified two-phase sampling, *Communications in Statistics-Simulation and Computation*, DOI:10.1080/03610918.2014.995817.

Shabbir, J. and Gupta, S. (2016). Estimation of population coefficient of variation in simple and stratified random sampling under two-phase sampling scheme when using two auxiliary variables, *Communications in Statistics-Theory and Methods*, DOI:10.1080/03610926.2016.1175627.

Sharma, P. and Singh, R. (2014). Improved ratio-type estimators using two auxiliary variables under second order approximation, *Mathematical Journal of Interdisciplinary Sciences*, 2(2), 193–204.

Sharma, P. and Singh, R. (2015). A class of exponential ratio estimators of finite population mean using two auxiliary variables, *Pakistan Journal of Statistics and Operation Research*, 11(2), 221–229.

Shukla, G. K. (1966). An alternative multivariate ratio estimator for finite population, *Calcutta Statistical Association Bulletin*, 15, 127–134.

Singh, D. and Chaudhary, F. S. (1986). Theory and Analysis of Sample Survey Designs, *John Wiley and Sons, New York*.
Singh, H. P. (1986). A Generalized class of estimators of ratio, product and mean using supplementary information on an auxiliary character in PPSWR sampling scheme, *Gujarat Statistical Review*, 13(2), 1–30.

Singh, H. P. and Tailor, R. (2005). Estimation of finite population mean using known correlation coefficient between auxiliary characters, *Statistica*, anno LXV, 4, 407–418.

Singh, H. P. and Vishwakarma, G. K. (2008). A family of estimators of population mean using auxiliary formation in stratified sampling, *Communications in Statistics-Theory and Methods*, 37(7), 1038–1050.

Singh, H. P., Upadhyaya, L. N. and Tailor, R. (2009). Ratio-cum-product type exponential estimator, *Statistica*, anno LXIX, 4, 300–310.

Singh, M. P. (1967). Ratio-cum-product method of estimation, *Metrika*, 12(1), 34–62.

Singh, M. P. (1969). Comparison of some ratio-cum-product estimators, *Sankhya, series B*, 31, 375–378.

Singh, R. and Kumar, M. (2012). Improved estimators of population mean using two auxiliary variables in stratified random sampling, *Pakistan Journal of Statistics and Operation Research*, 8, 65–72.

Singh, R., Chouhan, P. and Sawan, N. (2008). On linear combination of ratio-product-type exponential estimator for estimating finite population mean, *Statistics in Transition*, 9(1), 105–115.

Singh, S. (2003). Advanced sampling theory with applications- How michel “selected” Amy, Kluwer Academic Publishers, I edition volume I and II.

Srivastava, S. K. (1965). An estimation of the mean of a finite population using several auxiliary variables, *Journal of the Indian Statistical Association* 3, 189–194.

Srivastava, S. K. (1966). A note on Olkin’s multivariate ratio estimator, *Journal of Indian Statistical Association*, 4, 202–208.

Srivastava, S. K. (1967). An estimator using auxiliary information in sample surveys, *Calcutta Statistical Association*, 16, 121–132.

Srivastava, S. K. (1971). A generalized estimator for the mean of a finite population using multiauxiliary information, *Journal of American Statistical Association*, 66, 404–407.

Srinivataramana, T. and Tracy, D. S. (1984). Positive and negative valued auxiliary variates in surveys, *Metron*, XLII, (3–4), 3–14.

Steel, R. G. D. and Torrie, J. H. (1960). Principles and Procedures of Statistics, McGraw Hill Book Co.

Swain, A. K. P. C. (2013). On some modified ratio and product-type estimators, Revisited, *Investigacion operational*, 34(1), 35–57.
Tailor, R. and Chouhan, S. (2014). Ratio-cum-product-type exponential estimator of finite population mean in stratified random sampling, *Communications in Statistics-Theory and Methods*, 43(2), 343–354.
Tailor, R., Chouhan, S., Tailor, R. and Garg, N. (2012). A ratio-cum-product estimator of population mean in stratified random sampling using two auxiliary variables, *Statistica*, anno LXXII, n.3, 287–297.
Tripathi, T. P., Maiti, P. and Sharma, S. D. (1983). Use of prior information on some parameter in estimating population mean, *Sankhya*, A, 45(3), 372–376.
Upadhyaya, L. N., Singh, H. P., and Vos, J. W. E. (1985). On the estimation of population means and ratios using supplementary information, *Statistica Neerlandica*, 39, 309–318.
Verma, H. K., Sharma, P. and Singh, R. (2015). Some families of estimators using two auxiliary variables in a stratified random sampling, *Revista Investigacion Operacional*, 36(2), 140–150.
Vos, J. W. E. (1980). Mixing of direct, ratio and product method estimators, *Statistica Neerlandica*, 34, 209–218.
Yadav, S. K., Mishra, S., Kadilar, C. and Shukla, A. K. (2015a). Improved dual to ratio cum dual to product estimator in the stratified random sampling, *American Journal of Operational Research*, 5(3), 57–63.
Yadav, S. K., Mishra, S., Mishra, S. S. and Shukla, A. K. (2015b). Searching efficient estimator of population mean in stratified random sampling, *American Journal of Operational Research*, 5(4), 75–81.

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