Purity-bounded uncertainty relations in multidimensional space — generalized purity

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Abstract. Uncertainty relations for mixed quantum states (precisely, purity-bounded position-momentum relations, developed by Bastiaans and then by Man’ko and Dodonov) are studied in general multi-dimensional case. An expression for family of mixed states at the lower bound of uncertainty relation is obtained. It is shown, that in case of entropy-bounded uncertainty relations, lower-bound state is thermal, and a transition from one-dimensional problem to multi-dimensional one is trivial. Results of numerical calculation of the relation lower bound for different types of generalized purity are presented. Analytical expressions for general purity-bounded relations for highly mixed states are obtained. PACS number: 03.65.Ca.

1. Introduction and review

Well-known position-momentum uncertainty relation for standard deviations of \( \hat{x} \) and \( \hat{p} \) operators,
\[
\Delta x \Delta p \geq \frac{\hbar}{2},
\]
(1)
is valid for any state (described either by a wavefunction or by a density matrix [1,2]) and plays an quite an important role in quantum physics. In particular, uncertainty relation sets the precision limits of measurement process for non-commuting observables [3, 4]. Another important example is that generalized coherent states (and squeezed states) could be defined as a set of states which minimize an uncertainty relation, see [5]. Uncertainty principle and properties of its minimum is also of special interest in theory of operators in Hilbert space, see further references in a recent work by Goh and Miccelli [6].

The inequality (1) have been generalized to include extra dependence on degree of purity [7] of a quantum state
\[
\mu = \text{Tr}(\rho^2)
\]
(\( \rho \) is a density operator), the parameter \( 0 \leq \mu \leq 1 \) and equality \( \mu = 1 \) is achieved only for pure states. An asymptotic inequality for one-dimensional highly mixed states with \( \mu \ll 1 \) has a form [8–14]
\[
\Delta x \Delta p \geq \frac{8}{9 \mu} \frac{\hbar}{2}.
\]
(3)
In addition to the trace of squared density operator, there are other measures of overall purity (see above cited papers for details, especially a recent comprehensive review on purity-bounded relations [14]).

Another approach for treatment of uncertainty relation for mixed states are developed, by Wolf, Ponomarenko, Agarwal [15, 16], and also by Vourdas and his co-authors [17, 18]. In the cited works, the uncertainty relation is expressed in terms of correlations of respective observables. On the other hand, the inequality of the type relates uncertainties in conjugated variables and a measure of overall purity of state.

A generalization of the uncertainty relation to multidimensional space (vector observables, which can appear e. g. for multimode states, or multi-particle situations) was investigated in early days of quantum mechanics [19] (see also a review in [13]) and is still drawing attention of researches [20–22]. In its most simple form, uncertainty relation for \( n \)-dimensional position and momentum operators \( \hat{X} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \), \( \hat{P} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n) \) could be written as

\[
(\Delta X \Delta P)^n \geq \left( \frac{\hbar}{2} \right)^n, \tag{4}
\]

with definitions

\[
(\Delta X)^2 = \frac{1}{n} \prod_{i=1}^{n} \Delta x_i, \quad (\Delta P)^2 = \frac{1}{n} \prod_{i=1}^{n} \Delta p_i.
\]

In fact, due to equality between different coordinates in minimum of uncertainty relation, the inequality has the same meaning as \( n \)-th degree of standard one-dimensional relation, see also discussion in [23].

The problem of generalization of the inequality for multidimensional case was treated in papers by Karelin and Lazaruk [23, 24]. In particular, in our first paper on this topic [23], it has been shown with the help of the Wigner function formalism, that there is a non-trivial dependence of the purity-bounded uncertainty relation limit on the number of dimensions. For highly mixed states with \( \mu \ll 1 \),

\[
(\Delta X \Delta P)^n \geq \frac{C(n)}{\mu} \left( \frac{\hbar}{2} \right)^n, \quad C(n) = \frac{2^{n+1}(n+1)!}{(n+2)^{n+1}}, \tag{5}
\]

where the parameter \( C(n) \) characterizes distance from a minimum \( (\hbar/2)^n \) for pure states. In deriving (5), we assumed, that the Wigner function of the minimum-uncertainty state is nonnegative.

In another paper [24], the structure of the density matrix near the lower bound of uncertainty relation was also found, using decomposition of the density matrix in terms of Fock states (which form an orthogonal basis with a minimal uncertainty)

\[
\hat{\rho} = \sum_{m,m'} a_{m,m'} |m_1\rangle |m_2\rangle \ldots |m_n\rangle \langle m'_1| \langle m'_2| \ldots \langle m'_n|,
\]

with the additional condition

\[
\mu = \sum_{m,m'} |a_{m,m'}|^2 = \text{const},
\]
where \( \mathbf{m} = (m_1, \ldots, m_s) \) is a “vector” index with integer nonnegative components.

At the lower bound of uncertainty relation, density matrix in Fock representation is diagonal, \( a_{\mathbf{m}, \mathbf{m}'} = a_{\mathbf{m}, \mathbf{m}'} \delta_{\mathbf{m}, \mathbf{m}'} \); coefficients \( a_{\mathbf{m}, \mathbf{m}} \) depend linearly on the ‘norm’ of vector index \( \| \mathbf{m} \| = \sum_{i=1}^{n} m_i \), and number of Fock states in representation of \( \hat{\rho} \) is finite. Coefficients \( a_{\mathbf{m}, \mathbf{m}} \) of this decomposition are degenerate, and their multiplicity is determined by norm \( \| \mathbf{m} \| \) and dimensionality \( n \):

\[
g^{(n)}_{\| \mathbf{m} \|} = \frac{\left( \| \mathbf{m} \| + n - 1 \right)!}{\| \mathbf{m} \|! (n-1)!}. \tag{7}
\]

The inequality obtained in [24] correctly describes the whole range of \( \mu \), including perfectly pure case \( \mu = 1 \). In particular, in the interpolating form it becomes

\[
\Delta X \Delta P \geq \frac{\hbar}{2} \frac{n + 2 L(\mu)}{n + 2} \tag{8}
\]

with the auxiliary real parameter \( L(\mu) \) being a root of the transcendental equation

\[
\mu = \frac{(n + 2 L)(n + 1)! \Gamma(L)}{(n + 2) \Gamma(L + n + 1)}, \tag{9}
\]

where \( \Gamma(y) \) is Euler’s gamma-function.

It is also necessary to note that the inequality, mathematically practically the same as uncertainty relation, but with another physical meaning, is often used for classical wave fields, e. g. in optics [8–12,23]. Results of the present article, as well as of preceding papers [23,24] could be used, with appropriate change of notations, for classical partially coherent fields and sources (in 1-, 2- and 3-dimensional space [25]).

The uncertainty relations \( \text{(5), (8)} \) could be further generalized in order to take into account the dependence of inequality minimum on eigenvalues of density operator. Preliminary report on this topic, with stress on partially coherent classical fields, was published in [26]. Obtaining such a relation, together with the study of its asymptotics, is the main aim of the present paper.

2. Uncertainty relation for the diagonal representation of the density matrix

Any density matrix \( \hat{\rho} \) has a spectral decomposition [27]

\[
\hat{\rho} = \sum_{\mathbf{m}} \rho_{\mathbf{m}} |\psi_{\mathbf{m}}\rangle \langle \psi_{\mathbf{m}}|, \tag{10}
\]

where \( \rho_{\mathbf{m}} \) are the eigenvalues, and \( |\psi_{\mathbf{m}}\rangle \) are eigenvectors of the density operator, then, each of vectors \( |\psi_{\mathbf{m}}\rangle \) could be represented via outer products of one-dimensional Fock states \( |k\rangle \)

\[
|\psi_{\mathbf{m}}\rangle = \sum_{k_1, k_2, \ldots, k_n} A^{(m)}_{k_1, k_2, \ldots, k_n} |k_1\rangle |k_2\rangle \ldots |k_n\rangle, \tag{11}
\]

where \( k_i, i = 1, 2, \ldots, n \) corresponds to \( i \)-th one-dimensional subspace.
The right-hand side of the uncertainty relation (11) is calculated using the method of papers [8, 13]. Core idea of calculation is the introduction of auxiliary observable

$$E(\vartheta) = \frac{1}{2} \left[ (\Delta P)^2 / \vartheta + \vartheta (\Delta X)^2 \right],$$

which could be regarded as energy of some oscillator with unit frequency and mass $\vartheta$. The minimum of $E(\vartheta)$ with respect to $\vartheta$ (for $\vartheta = \Delta X / \Delta P$) is exactly left-hand side of uncertainty relation

$$\min_{\vartheta} E(\vartheta) = \Delta X \Delta P.$$

As far as Fock states are eigenstates of the harmonic oscillator with eigenvalue $2k + 1$ (in the one-dimensional case), then substitution of (10) and (11) into (12) leads to

$$\Delta X \Delta P \geq \frac{\hbar}{2n} \sum_{m} \rho_m \sum_{k_1, k_2, \ldots, k_n} \left[ 2(k_1 + k_2 + \ldots + k_n) + n \right] \left| A_{k_1, k_2, \ldots, k_n}^{(m)} \right|^2 .$$

Now, due to isomorphism between the set of all positive integers and the set of combinations of $n$ positive integers, it is possible to consider the coefficients $A_{k_1, k_2, \ldots, k_n}^{(m)}$ as elements of some unitary matrix $\{ \tilde{A}_{k m} \}$. Then, (in)equality (13) can be treated with lemma‡ from [10] to give

$$\Delta X \Delta P \geq \frac{\hbar}{2n} \sum_{k_1, k_2, \ldots, k_n} \left[ 2k_1 + k_2 + \ldots + k_n \right] + n \rho_{m, k_1, k_2, \ldots, k_n},$$

where eigenvalues of the density matrix are ordered in a non-increasing sequence.

Dependence of the expression $2(k_1 + k_2 + \ldots + k_n) + n$ on indices $k_1, k_2, \ldots, k_n$ is degenerate: this expression takes the same values for several combinations of indices. Therefore it is possible to rewrite the (in)equality (14) as

$$\Delta X \Delta P \geq \frac{\hbar}{2n} \sum_{k} \left( 2k + n \right) \left| \rho_{m, k} \right|^2$$

where the values $g_m^{(n)}$ (degeneration multiplicity) are defined by the formula (7), and the eigenvalues of the density matrix are collected in groups of $g_m^{(n)}$ terms. Expression (15) is the main result of the paper, and it is the most general form of the uncertainty relation for mixed states (partially coherent fields) in a multidimensional space. This inequality relates a minimal uncertainty volume of a state to the spectrum of the density operator corresponding to this state.

3. Multidimensional purity-bounded relations

Using the method from papers [10, 11], it is possible to find a dependence of uncertainty relation limit on some characteristics of purity of quantum system. Usually, a family of

‡ See also [14]: to be self-contained, the lemma is reproduced, together with necessary changes for multidimensional case, in Appendix A of this article.
“generalized purities” (Shatten $p$-norms or “generalized entropies” [22]), is used, which is defined as

$$\mu^{(r)} = \left[ \text{Tr} \left( \hat{\rho}^{r/(r-1)} \right) \right]^{r-1},$$  \hspace{1cm} (16)

where $r$ is an arbitrary (not necessary integer) real number with $r > 1$. Important special cases of $\mu^{(r)}$ include $\mu^{(2)} = \mu$ (“usual” purity, see above), “superpurity”

$$\mu^{(1)} = \lim_{r \to 1} \mu^{(r)},$$  \hspace{1cm} (17)

when only the largest eigenvalue of the density matrix is taken into account, and also “entropy-based” purity degree

$$\mu_S = \exp(-S), \quad S = -\text{Tr}(\hat{\rho} \ln \hat{\rho}),$$  \hspace{1cm} (18)

which is defined in terms of Shannon-von Neumann entropy $S$, and so leads to “entropy-bounded” uncertainty relation. As it is shown in [12], $\mu_S$ can be treated as a limiting case of definition (16) for $r \to \infty$: $\mu^{(\infty)} = \mu_S$.

“Superpurity” and entropy-bounded uncertainty relations play a special role for one dimensional case: owing to continuous non-increasing dependence of $\mu^{(r)}$ on $r$ for $r \geq 1$ [12], they are limiting cases of family of characteristics [16]. It is also possible to show, that the non-increasing dependence of $\mu^{(r)}$ on $r$ remains valid in a general multidimensional case, see details in Appendix B.

As it can be easily shown by Lagrange method, at the minimum of uncertainty relation, the definition (16) reduces to

$$\mu = \left[ \sum_k g^{(n)}_k \left( \xi_k/g^{(n)}_k \right)^{r/(r-1)} \right]^{r-1},$$  \hspace{1cm} (19)

where

$$\xi_k = \sum_{m=0}^{g^{(n)}_k-1} \rho_{m(k)}$$

and

$$\sum_k \xi_k = 1.$$  \hspace{1cm} (20)

Then the (in)equality (15) may be rewritten to

$$\Delta X \Delta P \geq \frac{\hbar}{2} \frac{1}{n} \sum_k (2k + n) \xi_k$$  \hspace{1cm} (21)

and, in order to obtain the relation of the type (5) for given structure of the density matrix, (i.e. eigenvalues $\rho_m$) it is necessary to find a minimum with respect to variables $\xi_s$.

As the degeneration multiplicity (7) has a rather complex form, the task of detailed study of uncertainty relation minimum has, in general, no analytical solution (the same as in one-dimensional case [14]). Besides interpolated and asymptotic inequalities, which
will be studied later in the article, it is possible to obtain analytical solution for the case of entropy-bounded relations.

Using Lagrange method, it is easy to show, that a minimum of uncertainty product $\Delta X \Delta P$ for given entropy $S$ is attained if the coefficients $\xi_m$ are given by

$$\xi_m = A g_m^{(n)} \exp(-\beta m), \quad (22)$$

where $A$ is a normalization constant and parameter $\beta$ depends on the entropy. Taking into account structure of the density operator at the minimum of general uncertainty relation (15) [compare representation (11)], in the case of entropy-bounded relation the density matrix is taking the form

$$\hat{\rho}_S^{(n)} = \hat{\rho}_S^{(1)} \hat{\rho}_S^{(1)} \cdots \hat{\rho}_S^{(1)}. \quad (23)$$

Here

$$\hat{\rho}_S^{(1)} = \sum_{k=0}^{\infty} e^{-\beta k} |k\rangle \langle k| \quad (24)$$

is a density matrix corresponding to minimum of one dimensional entropy-bounded relation. Parameter $\beta$ can be found from solution of transcendental equation

$$S = \beta n \frac{\exp(\beta)}{1 - \exp(\beta)} - n \ln (1 - \exp(-\beta)), \quad (25)$$

which is in accordance with appropriate equation for one-dimensional case [13].

Uncertainty relation then could be written as

$$(\Delta X \Delta P)^n \geq \beta(S) \left(\frac{\hbar}{2}\right)^n, \quad (26)$$

or, for highly-mixed states with $S \gg 1$

$$(\Delta X \Delta P)^n \geq \exp(S) \left(\frac{2}{e}\right)^n \left(\frac{\hbar}{2}\right)^n. \quad (27)$$

Obtained structure of eigenstate decomposition is factorized on solutions of one dimensional problem (see [11, 13]), which leads to thermal state (24). In other words, the entropy-bounded uncertainty relation of position-momentum type has no additional effects for multidimensional cases.

In order to study general low-purity case, it is possible to utilize the approach from Bastiaans’ paper [10], which is based on generalization of the Hölder inequality. The mathematical details are presented in Appendix C.

Let’s define an “uncertainty function” $C(\mu^{(r)}, n)$ (see also (5)) as

$$C(\mu^{(r)}, n) = \mu^{(r)} \left(\frac{1}{n} \left\{ 2M + n - 2 \left[ \mu^{(r)} B(M, n, r) \right]^{1/r} \right\} \right)^n, \quad (28)$$

where

$$B(M, n, r) = \sum_{0 \leq m \leq M} \frac{(m + n - 1)!}{(n - 1)! m! (M - n)^r}. \quad (29)$$
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Figure 1. Uncertainty relation minimum for \(\mu^{(r)} \ll 1, n = 1, \ldots, 6\) and different variants of the degree of purity: o’s — \(r \to 1\), x’s — \(r = 2\), *’s — \(r = 3\), +’s — \(r \to \infty\) (entropy-bounded relation).

and additional real and positive minimization parameter \(M\) is introduced.

1-d problem has known asymptotical solution [10]

\[ C(\mu^{(r)}, 1) = 2 \left[ \frac{r}{r + 1} \right]^r, \quad \mu^{(r)} \ll 1. \]

In the same way, highly mixed states could be treated analytically for arbitrary multidimensional case: as far as limit of small \(\mu^{(r)}\) requires \(M\) to be sufficiently large [14,24], it is possible to replace summation in formula (29) by integration together with approximation of degeneration multiplicity by \(m^{n-1}/(n-1)!\) with

\[ B(M, n, r) \approx \frac{1}{(n-1)!} \int_0^M m^{n-1}(M-m)^r \, dm. \quad (30) \]

The last relation could be calculated analytically,

\[ B(M, n, r) \approx M^{n+r+1} \left( \prod_{k=1}^{n+1} (r+k) \right)^{-1} \quad (31) \]

(see Appendix D for details). Further minimization of relation (28) with respect to \(M\), after some tedious but quite elementary algebra gives an asymptotic variant of general purity bounded uncertainty relation

\[ (\Delta X \Delta P)^n \geq \left( \frac{\hbar}{2} \right)^n \frac{C(n, r)}{\mu^{(r)}}, \quad C(n, r) = \frac{2^n r^r}{(n + r)^{n+r}} \prod_{k=1}^n (r+k), \quad (32) \]

which describes the whole range of \(n\) and \(r\) for \(\mu^{(r)} \ll 1\). Resulting dependencies of \(C(n, t)\) on \(r\) for \(n = 2, 3\) together with one-dimensional case are presented in figure 1 and figure 2. It is seen, that the obtained expressions correctly describe whole range of parameter \(r\), demonstrating decrease of uncertainty minimum with increase of \(r\) and leading to entropy-bounded relations at \(r \to \infty\).
4. Concluding remarks

To summarize, it is worth to note that two main forms of uncertainty principle (of position-momentum type) for mixed states in multidimensional space are obtained in the present paper. The first one \((15)\) relates minimal uncertainty product with eigenspectrum of the density matrix and the second \((32)\) is a general (asymptotic) purity-bounded uncertainty relation. In both cases, minimum of uncertainty product is obtained when eigenstates of the density operator are Fock states. In the case of purity-bounded relation, eigenspectrum of the density operator is defined by

\[
\xi_m \propto g_m^{(n)}(M-m)^{r-1}, \quad 0 \leq m \leq M
\]

see Appendix C. In other words, spectral representation of the density matrix is a finite sum of Fock states. Such state is definitely non-classical, see discussion in the Dodonov’s paper \([14]\). Upon transition to \(r \to \infty\) (entropy-bounded relations), the minimum-uncertainty state becomes thermal \((24)\), i.e. classical. More detailed study of minimum-uncertainty states structure will be subject of another publication.

It is also necessary to note, that the results of the paper is applicable for analysis and characterization of entangled quantum states. Indeed, the spectral decomposition of density matrix is closely connected to the Schmidt decomposition of non-separable states, see, e.g. \([28]\). Approach to uncertainty principle for entangled states can be based on mathematically analogous case of uncertainty (reciprocity) relations for pulsed partially coherent classical beam \([25]\).

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Appendix A.

The lemma is reproduced here mainly for completeness of material, and in order to make the above analysis clearer. Initially it was presented in Appendix A of paper [10], it also could be found in [14]. According to [10], idea of this proof was initially proposed by M. L. J. Hautus.

Let the sequence of numbers $b_m$ be defined by

$$b_m = \sum_{k=0}^{\infty} |a_{mk}|^2 \gamma_k, \quad (A.1)$$

where $\gamma_0 \leq \gamma_1 \leq \ldots \leq \gamma_k \leq \ldots$ and coefficients $a_{mk}$ satisfy the orthonormality condition

$$\sum_{k=0}^{\infty} a_{mk} a_{lk}^* = \delta_{ml}, \quad m, l = 0, 1, \ldots \quad (A.2)$$

One may consider the numbers $b_m$ for $m = 0, 1, \ldots, M$ as the diagonal entries of an $(M+1)$-square Hermitian matrix $H = \|h_{ij}\|$ with

$$h_{ij} = \sum_{k=0}^{\infty} a_{ikl}^* \gamma_k, \quad i, j = 0, 1, \ldots, M. \quad (A.3)$$

Let the eigenvalues $\beta$ of $H$ be ordered according to

$$\beta_0 \leq \beta_1 \leq \ldots \leq \beta_k \leq \ldots \leq \beta_M. \quad (A.4)$$

From Cauchy’s inequalities for eigenvalues of a submatrix of a Hermitian matrix [29], we can conclude, that $\beta_m \geq \gamma_m$ ($m = 0, 1, \ldots, M$) and hence

$$\sum_{m=0}^{M} b_m = \sum_{m=0}^{M} h_{mm} = \sum_{m=0}^{M} \beta_m \geq \sum_{m=0}^{M} \gamma_m. \quad (A.5)$$

Furthermore, with the numbers $\lambda_m$ (or $\rho_m$, in this article) satisfying the property $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_m \geq \ldots$, we can formulate the chain of relations

$$\sum_{m=0}^{M} \lambda_m b_m = \lambda_0 b_0 + \sum_{m=1}^{M} \lambda_m b_m = \lambda_0 b_0 + \sum_{m=1}^{M} \lambda_m \left[ \sum_{l=0}^{m-1} b_l - \sum_{l=0}^{m-1} b_l \right]$$

$$= \lambda_0 b_0 + \sum_{m=1}^{M} \lambda_m \sum_{l=0}^{m-1} b_l - \sum_{m=0}^{M-1} \lambda_{m+1} \sum_{l=0}^{m} b_l$$

$$= \sum_{m=0}^{M-1} \lambda_m \sum_{l=0}^{m} b_l + \lambda_M \sum_{l=0}^{M} b_l - \sum_{m=0}^{M-1} \lambda_{m+1} \sum_{l=0}^{m} b_l$$

$$\geq \lambda_M \sum_{l=0}^{M} \gamma_l + \sum_{m=0}^{M-1} (\lambda_m - \lambda_{m+1}) \sum_{l=0}^{m} \gamma_l = \sum_{m=0}^{M} \lambda_m \gamma_m. \quad (A.6)$$
On choosing $\gamma_n = 2n + 1$ and taking the limit $M \to \infty$, we arrive at the inequality
\[
\sum_{m=0}^{\infty} \lambda_m \sum_{n=0}^{\infty} |a_{mn}|^2 (2n + 1) \geq \sum_{m=0}^{\infty} \lambda_m (2m + 1),
\]
which becomes an equality if $|a_{mn}| = \delta_{mn}$.

In order to modify this proof to multidimensional case, it is necessary to choose $\gamma_n$ as $\gamma_n = 2(m_1 + \ldots + m_s) + n$ (here $m = (m_1, \ldots, m_s)$ is a “vectorial” summation index), and then to take into account degeneracy of coefficients $\gamma_n$.

**Appendix B.**

By analogy with Bastiaans’ paper [12] for any $r, q$, with $1 < r < q$, holds
\[
\mu^{(r)} = \left[ \sum_{m} g_m^{(n)} \xi_m^r \right]^{1/r} = \left[ \sum_{m} g_m^{(n)} \left( \theta_m (r-1) \right)^{(r-1)(q-1)/(q-1)} \theta_m \right]^{1/r} \leq \mu^{(q)},
\]
(B.1)

Here $\theta_m = \xi_m / g_m^{(n)}$, and the Hölder inequality for weighted sum [30] is used, see formula (C.3). Therefore, for a family of purities (16), it is possible to conclude, that “superpurity” and entropy-based purity lead to limiting cases of all multidimensional uncertainty relations.

**Appendix C.**

Starting from the equation (20), with $\xi_m$ a sequence of nonnegative numbers and $M$ an arbitrary real nonnegative constant, we write
\[
\sum_{m=0}^{\infty} \xi_m = \frac{1}{2M+n} \left[ 2 \sum_{m=0}^{\infty} \xi_m (M - m) + \sum_{m=0}^{\infty} \xi_m (2m + n) \right] = 1.
\]
(C.1)

The following (in)equalities hold:
\[
\sum_{m=0}^{\infty} \xi_m (M - m) \leq \sum_{0 \leq m \leq M} \xi_m (M - m),
\]
(C.2)

\[
\sum_{0 \leq m \leq M} \xi_m (M - m) = \sum_{0 \leq m \leq M} g_m^{(n)} (M - m) \xi_m / g_m^{(n)}
\]
\[
\leq \left[ \sum_{0 \leq m \leq M} g_m^{(n)} (M - m)^r \right]^{1/r} \left[ \sum_{0 \leq m \leq M} g_m^{(n)} \left( \xi_m / g_m^{(n)} \right)^p \right]^{1/p},
\]
(C.3)

\[
\sum_{0 \leq m \leq M} g_m^{(n)} \left( \xi_m / g_m^{(n)} \right)^p \leq \sum_{m=0}^{\infty} g_m^{(n)} \left( \xi_m / g_m^{(n)} \right)^p
\]
(C.4)
with two real parameters $p, r \geq 1$, $1/p + 1/r = 1$. The equality sign in relations (C.2) and (C.4) holds if $\xi_m = 0$ for $m > M$. Relation (C.3) changes to equality, when $\xi_m \propto g_m^{(n)} (M - m)^{r-1}$ in the interval $0 \leq m \leq M$. (In)equality (C.3) is a general form of the Hölder inequality for the weighted sum [30], see also [31].

Combining the (in)equalities (C.1) – (C.4) gives a relation

$$\frac{1}{2M + n} \left[ 2B(M, n, r)^{1/r} \mu_p + \sum_{m=0}^{\infty} \xi_m (2m + n) \right] \geq 1,$$

where

$$B(M, n, r) = \sum_{0 \leq m \leq M} g_m^{(n)} (M - m)^{r},$$

$$\mu_p = \left[ \sum_{m=0}^{\infty} g_m^{(n)} \left( \frac{\xi_m}{g_m^{(n)}} \right)^p \right]^{1/rp}.$$  \hfill (C.5)

From the condition $1/p + 1/r = 1$ it follows that $p = r/(r - 1)$ and then the (in)equality (28) results.

Appendix D.

In order to find an integral in approximation (30), we start from introduction of a new variable $x = M - m$

$$B(M, n, r) \approx \frac{1}{(n-1)!} \int_0^M \mathrm{dx} \ (M - x)^{n-1} x^r.$$  \hfill (D.1)

Application of binomial formula to $(M - x)^{n-1}$ and interchanging the order of integration and summation leads to

$$B(M, n, r) \approx M^{n+r+1} \sum_{k=0}^{n} \frac{(-1)^k}{k! (n - k)! (k + r + 1)}.$$  \hfill (D.2)

The last sum can be calculated by use of formula (5.41) from Graham, Knuth and Patashnik book [32], leading at last to (31).

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