Abstract

For the spacelike momenta $k$ of the virtual photon $\gamma^*$, the $\pi^0(p)\gamma^*(k)\gamma(k')$ transition form factor is considered in the coupled Schwinger-Dyson and Bethe-Salpeter approach in conjunction with the generalized impulse approximation using the dressed quark–photon–quark vertices of the Ball-Chiu and Curtis-Pennington type. These form factors are compared with the ones predicted by the vector meson dominance, operator product expansion, QCD sum rules, and the perturbative QCD for the large spacelike transferred momenta $k$. The most important qualitative feature of the asymptotic behavior, namely the $1/k^2$ dependence, is in our approach obtained in the model-independent way. Again model-independently, our approach reproduces also the Adler-Bell-Jackiw anomaly result for the limit of both photons being real. For the case of one highly virtual photon, we find in the closed form the asymptotic expression which can be easily generalized both to the case of other unflavored pseudoscalar mesons $P^0 = \pi^0, \eta_8, \eta_0, \eta_c, \eta_b$, and to the case of arbitrary virtuality of the other photon. Implications thereof for certain important theoretical and experimental applications are pointed out.

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1. The interest in the form factor $T_{\pi^0(-Q^2, 0)}$ for the transition $\gamma^*(k)\gamma(k') \rightarrow \pi^0(p)$ (where $k^2 = -Q^2 \neq 0$ is the momentum-squared of the spacelike off-shell photon $\gamma^*$), has again been growing lately for both experimental (the new CLEO data [1] and plans for new TJNAF measurements [2]) and theoretical reasons – such as its relevance for the hadronic light-by-light scattering contribution to muon $g - 2$ [3], which is in turn again relevant for the experiment E821 at BNL [4]. Nevertheless, this transition form factor still represents a more demanding theoretical challenge than usually thought, as the perturbative QCD (pQCD) may be not sufficient even at the highest of the presently accessible momenta ($Q^2 \lesssim 10$ GeV$^2$) – e.g., see Refs. [5]. These references therefore indicate desirability of having direct calculations of $T_{\pi^0(-Q^2, 0)}$ without any a priori assumptions about the shape of the pion distribution amplitude $\varphi_\pi$. Even more important for the present work is that the pQCD approaches, oriented mostly at reproducing the large $Q^2$ behavior, have problems at low $Q^2$. This was most recently stressed in Ref. [6] discussing the status of QCD–based theoretical predictions for this process. See especially its Sec. IV for clarifications how such approaches [7] can fail to reproduce the $Q^2 = 0$ value [8] corresponding to the Abelian axial alias Adler-Bell-Jackiw (ABJ) anomaly which explains the $\pi^0$ decay into two real photons ($k^2 = k'^2 = 0$).

On the other hand, the approaches (such as the present one) which rely on the chirally well–behaved Schwinger–Dyson (SD) and Bethe–Salpeter (BS) equations\footnote{See, for example, Ref. [8], and the recent reviews [9], or [10].} for the light pseudoscalar mesons ($\pi, K, \eta, ...$) while taking care that the Ward-Takahashi identities (WTI) of QED are respected, satisfy in this respect even what Ref. [6] calls the “maximalist” requirement – namely that the fundamental axial anomaly result

$$T_{\pi^0}(0, 0) = \frac{1}{4\pi^2 f_\pi}$$

should be satisfied automatically in the chiral (and soft) limit. (This is clearly superior to imposing it as an external condition, which can by analogy be termed the “minimalist” attitude.) In the present approach [11–13], the Abelian axial anomaly amplitude [4] is obtained without imposing any requirements or constraints on the solutions for the $\pi^0$ wave function or the solutions for the quark propagators (e.g., see [14,15]). Since the anomaly – and thus also $T_{\pi^0}(0, 0)$ of Eq. (1) – must not depend on the internal structure of the pion, this is exactly as it should be. What must then be explored in the coupled SD-BS approach is if the large $Q^2$ behavior can be satisfactorily understood in that approach. In particular, the comparison with the new CLEO data [1] – at $Q^2$ up to 8 GeV$^2$ – must be made, since the results of the SD-BS approach [11] (as well as the results of the earlier and closely related Ansatz approach of [13]) have so far been compared only with the older CELLO data [17] at $Q^2$ not exceeding some 2.5 GeV$^2$. In this letter we will show how the SD-BS approach to modeling QCD provides a good description for both low and high values of $Q^2$. In other words, it shows how one can in fact obtain something similar to the Brodsky–Lepage (BL) interpolation formula $T_{\pi^0(-Q^2, 0)} = (1/4\pi^2 f_\pi) / (1 + Q^2/8\pi^2 f_\pi^2)$ proposed [18] as a desirable behavior for the transition form factor because it reduces to the ABJ anomaly amplitude [4] at $Q^2 = 0$, while agreeing with the following type of leading behavior for large $Q^2$:
\[ T_{\pi^0}(-Q^2, 0) = \mathcal{J} \frac{f_\pi}{Q^2} \quad (\mathcal{J} = \text{constant} \text{ for large } Q^2), \]

which is favored both experimentally \[1\] and theoretically (e.g., see the QCD-based predictions in Refs. \[18\ \text{[2,14,8]}, \text{and references therein.} \]

2. In the coupled SD-BS approach, the BS equation for the pion bound-state \( q\bar{q} \) vertex \( \Gamma_{\pi^0}(q, p) \) employs the dressed quark propagator \( S(k) = [A(k^2)\not{k} - B(k^2)]^{-1} \), obtained by solving its SD equation. Solving the SD and BS equations in a consistent approximation is crucial (e.g., see Refs. \[21,22\]) for obtaining \( q\bar{q} \) bound states which are, in the case of light pseudoscalar mesons, simultaneously also the (pseudo-)Goldstone bosons of dynamical chiral symmetry breaking (D\( \chi \)SB).

Following Jain and Munczek \[21,22\], we adopt the ladder-type approximation sometimes called the improved \[8\] or generalized \[15\] ladder approximation (employing bare quark–gluon–quark vertices but dressed propagators). For the gluon propagator we use an effective, \( \text{IR} \) part of the gluon propagator, we adopt from Ref. \[22\] \[A\], i.e., the functions \( \Gamma_{\pi^0}(\not{q}, \not{p}) \equiv 1/2 \mu \) is an improved\( \text{UV} \) effective propagator function \( \text{Ansatz} \) is often called the “Abelian approximation” \[24\]. What is essential is that the perturbative part of the propagator \( \mu \), which is well-known from perturbative QCD, so that \( G_{\mu \nu} = G_{\mu \nu}(Q^2 + G_{\mu \nu}(Q^2), \ (Q^2 = -l^2) \). The perturbative part \( G_{\mu \nu} \) is required to reproduce correctly the ultraviolet (UV) asymptotic behavior that unambiguously follows from QCD in its high–energy regime. Therefore, this part must be given -- up to the factor \( 1/Q^2 \) -- by the running coupling constant \( \alpha_s(Q^2) \) which is well-known from perturbative QCD, so that \( G_{\mu \nu} \) is \textit{not} modeled. As in Refs. \[11,13\], we follow Refs. \[21,22\] and employ the two–loop asymptotic expression for \( \alpha_s(Q^2) \).

For the modeled, IR part of the gluon propagator, we adopt from Ref. \[22\] \( G_{1R}(Q^2) = (16\pi^2/3)\alpha_s^2 e^{-\mu Q^2} \), with their parameters \( \alpha_s = (0.387 \text{GeV})^{-1} \text{ and } \mu = (0.510 \text{GeV})^{-2} \).

More details on our calculational procedures can be found in our Refs. \[11,13\]. We essentially reproduce Jain and Munczek’s \[22\] solutions for the dressed propagators \( S(q) \), i.e., the functions \( A(q^2) \text{ and } B(q^2) \), as well as the solutions for the four functions comprising the pion bound-state vertex \( \Gamma_{\pi^0} \). Actually, Ref. \[22\] employs the BS amplitude \( \chi_{\pi^0}(q, p) \equiv S(q + p/2)\Gamma_{\pi^0}(q, p)S(q - p/2) \), which is completely equivalent.

3. We assume that the \( n^0\gamma^*\gamma \) transition proceeds through the triangle graph (Fig. 1), and that we calculate the pertinent amplitude \( T_{\pi^0}^{\mu\nu}(k, k') = \varepsilon^{\alpha\beta\nu\mu} k_\alpha k'_\beta T_{\pi^0}(k^2, k'^2) \) as in Refs. \[11,13\], using the framework advocated by (for example) \[14,16,24,26\] in the context of electromagnetic interactions of BS bound states, and often called the generalized impulse approximation (GIA) - e.g., by \[16,27\]. To evaluate the triangle graph, we therefore use the \textit{dressed} quark propagator \( S(q) \) and the pseudoscalar BS bound–state vertex \( \Gamma_{P}(q, p) \). Another ingredient, crucial for GIA’s ability to reproduce the correct Abelian anomaly result, is employing an appropriately dressed \textit{electromagnetic} vertex \( \Gamma^\mu(q', q) \), which satisfies the vector Ward–Takahashi identity (WTI) \( (q' - q)_{\mu} \Gamma_{\mu}(q', q) = S^{-1}(q') - S^{-1}(q) \). Namely, assuming that photons couple to quarks through the bare vertex \( \gamma^\mu \) would be inconsistent with our quark propagator \( S(q) \), which, dynamically dressed through its SD-equation, contains the momentum-dependent functions \( A(q^2) \text{ and } B(q^2) \). The bare vertex \( \gamma^\mu \) obviously violates the vector WTI, implying the nonconservation of the electromagnetic vector current and charge. Solving the pertinent SD equation for the dressed quark–photon–quark \( (q\gamma\gamma) \) vertex \( \Gamma^\mu \) is a difficult and still unsolved problem, and using the realistic Ansätze for \( \Gamma^\mu \) still
remains the only practical way to satisfy the WTI. The simplest solution of the vector WTI is the Ball–Chiu (BC) \[27\] vertex

\[
\Gamma_{\mu BC}(q', q) = A_+(q'^2, q^2) \frac{\gamma^\mu}{2} + \frac{(q' + q)^\mu}{(q'^2 - q^2)} \{ A_-(q'^2, q^2) \frac{q' + \not{q}}{2} - B_-(q'^2, q^2) \} ,
\]

where \( H_\pm(q'^2, q^2) \equiv [H(q'^2) \pm H(q^2)] \), for \( H = A \) or \( B \). This particular solution of the vector WTI reduces to the bare vertex in the free-field limit as must be in perturbation theory, has the same transformation properties under Lorentz transformations and charge conjugation as the bare vertex, and has no kinematic singularities. Note that it does not introduce any new parameters as it is completely determined by the dressed quark propagator \( S(q) \). In phenomenological calculations in the SD-BS approach, this minimal WTI-satisfying Ansatz \([3]\) is still the most widely used \( qq\gamma \) vertex (e.g., Refs. \([12,23,13,23]\)). A general WTI-satisfying vertex can be written \([27]\) as \( \Gamma^\mu = \Gamma_{\mu BC} + \Delta \Gamma^\mu \), where the addition \( \Delta \Gamma^\mu \) does not contribute to the WTI, since it is transverse, \((q' - q)_\mu \Delta \Gamma^\mu(q', q) = 0\). That is, \( \Delta \Gamma^\mu(q', q) \) entirely lies in the hyperplane spanned by the eight vectors \( T_i^\mu(q', q) \) transverse to the photon momentum \( k = q' - q \). Curtis and Pennington (CP) \([28]\) advocated a transverse Ansatz for \( \Delta \Gamma^\mu(q', q) \) exclusively along the basis vector usually denoted \( T_6^\mu(q', q) \):

\[
\Delta \Gamma^\mu(q', q) = T_6^\mu(q', q) \frac{A_-(q'^2, q^2)}{2d(q', q)}; \quad T_6^\mu(q', q) \equiv \gamma^\mu(q'^2 - q^2) - (q' + q)^\mu(q' - q). \tag{4}
\]

Then, the coefficient multiplying \( T_6^\mu(q', q) \) can be suitably chosen to ensure multiplicative renormalizability in the context of solving fermion SD equations beyond the ladder approximation in QED\(_4\) \([23]\). To this end, \( d(q', q) \) should be a symmetric, singularity free function of \( q' \) and \( q \), with the limiting behavior \( \lim_{q^2 \to q'^2} d(q', q) = q^2 \); for example,

\[
d_\pm(q', q) = \frac{1}{q'^2 + q^2} \left\{ (q'^2 \pm q^2)^2 + \left[M^2(q'^2) + M^2(q^2)\right]^2 \right\} , \tag{5}
\]

where \( M(q^2) \equiv B(q^2)/A(q^2) \) is the \( \chi \)SB-generated dynamical mass function, which in our case has the large-\( q^2 \) dependence \([13]\) in agreement with pQCD.

The choice \( d = d_- \) corresponds to the CP vertex Ansatz \( \Gamma^\mu_{CP} \) suggested in Ref. \([28]\). We will use it in analytic calculations of \( T(-Q^2, 0) \), which are possible for \( Q^2 = 0 \) and \( Q^2 \to \infty \). However, in the numerical calculations, which are necessary for finite values of \( Q^2 \neq 0 \), we prefer to use the modified CP (mCP) vertex, \( \Gamma^\mu_{mCP} \), resulting from the choice \( d = d_+ \) in Eq. \((4)\). Namely, although the original CP denominator function \( d_-(q', q) \) never vanishes strictly, the dynamical masses it contains become negligible with growing momenta, so that \( d_-(q', q) \) can become arbitrarily small for the large spacelike quark loop momenta \( q, q' \) if simultaneously \( q'^2 - q^2 \) is small, causing numerical problems. On the other hand, the numerical calculations of \( T(-Q^2, 0) \) employing our standard computational methods \([12,13]\) and using the mCP vertices have no such problems and they are as reliable as those using the minimal, BC vertices. In contrast to the BC one, the mCP vertex is consistent with multiplicative renormalizability by the same token as the CP one. It should also be noted
that $\Gamma_{mCP}^\mu$ is essentially equal to the high-$q^2$ or $q^2$ leading part of the vertex of Cudell et al. [23] [see their Eq. (3.14), and comments below it].

In the present context, the important qualitative difference between the BC vertex on one side, and the CP as well as the modified, mCP vertex on the other side, will be that $\Gamma_{BC}(q', q) \to \gamma^\mu$ when both $q'^2, q^2 \to \pm \infty$, whereas $\Gamma_{CP}^\mu(q', q) \to \gamma^\mu$ and $\Gamma_{mCP}^\mu(q', q) \to \gamma^\mu$ as soon as one of the squared momenta tends to infinity.

The correct ABJ anomaly result (1) cannot be obtained analytically in the chiral limit unless such a $qq\gamma$ vertex satisfying the WTI is used [14,15]. We have checked by the explicit GIA calculation that the $\pi^0 \to \gamma\gamma$ amplitude [11] is analytically reproduced employing the CP and mCP vertices, in the same way as when the BC vertices were employed in earlier applications, e.g. [14,16,11,12].

In the case of $\pi^0$, GIA yields (e.g., see Eq. (24) in Ref. [12]) the amplitude $T_{\pi^0}^{\mu\nu}(k, k')$:

$$T_{\pi^0}^{\mu\nu}(k, k') = -N_c \frac{1}{3\sqrt{2}} \int \frac{d^4q}{(2\pi)^4} \text{tr}\{\Gamma^\mu(q - \frac{p}{2}, k + q - \frac{p}{2}) S(k + q - \frac{p}{2}) \chi(q, p)\} + (k \leftrightarrow k', \mu \leftrightarrow \nu). \tag{6}$$

Here, $\chi$ is the BS amplitude of both $u\bar{u}$ and $d\bar{d}$ pseudoscalar bound states: $\chi \equiv \chi_{u\bar{u}} = \chi_{d\bar{d}}$ thanks to the isospin symmetry assumed here. This symmetry likewise enables us to continue suppressing flavor labels also on the quark propagators $S$ and $qq\gamma$ vertices $\Gamma^\mu$. We follow the conventions of Ref. [12], including those for the flavor factors and flavor matrices $\lambda^a$.

Then, $\chi_{\pi^0}(q, p) \equiv \chi(q, p) \lambda^3/\sqrt{2}$, so that the prefactor $1/3\sqrt{2}$ in Eq. (6) is just the flavor trace $\text{tr}(Q^2\lambda^3/\sqrt{2})$ where $Q$ is the matrix of the fractional quark charges.

In Ref. [11] (where only the BC vertex was employed), the transition form factor $T_{\pi^0}(-Q^2, 0)$ was numerically evaluated (for $0 < Q^2 < 2.8$ GeV$^2$ only) employing the soft and chiral limit. That is, in Ref. [11] we approximated (for the pion only!) the BS vertex by its leading, $O(p^0)$ piece: $\Gamma_{\pi^0}(q, p) \approx \Gamma_{\pi^0}(q, 0) = \gamma_5 \lambda^3 B(q^2)/f_\pi$. For the sake of comparison, we again give $T_{\pi^0}(-Q^2, 0)$ evaluated in that approximation (denoted by the dash-dotted line in Fig. 2), but now up to $Q^2 \approx 8$ GeV$^2$. However, in the present work we go beyond this approximation, using our complete solution for the BS vertex $\Gamma_{\pi^0}(q, p)$, viz. the BS amplitude $\chi_{\pi^0}(q, p)$, given by the decomposition into 4 scalar functions multiplying independent spinor structures. The approximation we still keep, is discarding the second and higher derivatives in the momentum expansions.

4. $T_{\pi^0}(-Q^2, 0)$ obtained in this way, and using the BC vertices, is depicted (after multiplication by $Q^2$) in Fig. 2 by the solid line. It tends to be on the high side of the data, but the general agreement of our $T_{\pi^0}(-Q^2, 0)$ with the experimental points is rather good, except for the intermediate momenta around the interval from 1 to 2 GeV$^2$ interval. The dashed line denotes $Q^2T_{\pi^0}(-Q^2, 0)$ obtained in the same way, but employing the mCP vertices. For both curves, the agreement with experiment would be improved by lowering them somewhat

\[\text{Up to the inclusion of the mass functions } M(q^2), \text{ and for the simplest choice of their } \eta \text{-function: } \eta(q', q) \equiv 1, \text{ i.e., their choice } n = 0 \text{ for the exponent in } \eta.\]
(at least in the momentum region $Q^2 \lesssim 4 \text{ GeV}^2$), which could be achieved by modifying the model [22] and/or its parameters so that such a new solution for $A(q^2)$ is somewhat lowered towards its asymptotic value $A(q^2 \to -\infty) \to 1$. (Of course, in order to be significant, this must not be a specialized refitting aimed only at $A(q^2)$). Lowering of $A(q^2)$ must be a result of a broad fit to many meson properties, comparable to the original fit [22]. This, however, is beyond the scope of this letter.)

While the empirically successful anomaly result (1) at $Q^2 = 0$ is model–independent in any consistently coupled SD-BS approach (being the consequence only of the correct chiral behavior due to the incorporation of $D\chi$SB), the transition form factor for $Q^2 \neq 0$ of course depends not only on the WTI-preserving $qq\gamma$-vertex Ansatz, but also on the chosen bound-state model. However, it must be stressed that we we did not do any parameter fitting whatsoever, as we have used the parameters obtained from Jain and Munczek’s [22] broad fit to the meson spectrum (except the $\eta$–$\eta'$ complex) and the pseudoscalar decay constants. Our model choice [22] has subsequently given us predictions for $\eta_c, \eta_b \to \gamma\gamma$ [11], the description of $\eta$–$\eta'$ complex and $\eta, \eta' \to \gamma\gamma$ [12], and now, with the same set of parameters, the transition form factors which are empirically successful for the presently accessible high values of $Q^2$, and at the same time in agreement (in the chiral limit even exactly and analytically [14–16,11,12]) with the anomalous amplitude (1) at $Q^2 = 0$.

Our transition form factors thus also agree rather well with other successful theoretical approaches. The vector–meson dominance (VMD) model (which is still the most successful one from the purely phenomenological point of view [8]) and the QCD sum rule approach [4] give the transition form factor which is some 10% below our “BC” $T_{\pi^0}(-Q^2, 0)$ for large values of $Q^2$. Therefore, in the large $Q^2$ region, our “BC” values are halfway between the uppermost line in Fig. 2 denoting the asymptotic pQCD [18] version ($J = 2$) of the result (4), and VMD as well as the QCD sum rule results [4] of Radyushkin and Ruskov [amounting to Eq. (3)] with $J \approx 1.6$. By the highest presently accessible momenta, our “mCP” version of $T_{\pi^0}(-Q^2, 0)$ has crossed to the low side of both other theoretical predictions and the data, but is still within the error bars.

The large-$Q^2$ leading power-law behavior (2) was first derived from the parton picture in the infinite momentum frame – e.g., see Ref. [18]. In this and other similar pQCD approaches, the precise value of the coefficient of the leading $1/Q^2$ term depends on the pion distribution amplitude $\varphi_\pi(x)$ which should contain the necessary nonperturbative information about the probability that a partonic quark carries the fraction $x$ of the total longitudinal momentum. E.g., the well-known example of a “broad” distribution $\varphi^{\text{CZ}}_\pi(x) = \frac{5}{3\sqrt{3}}(1 - x)(1 - 2x)^2$ (proposed by Chernyak and Zhitnitsky [19] motivated by sum-rule considerations) leads to $J = 10/3$, but it is too large in the light of the latest CLEO data [1]. In contrast, the asymptotic $\varphi^A_\pi(x) = \sqrt{3}(1 - x)$ favored by Lepage and Brodsky [18] yields $J = 2$, resulting in the line touching the error bars of the presently highest $Q^2$ data from above in Fig. 2. In fact, in the strict $\ln Q^2 \to \infty$ limit, every distribution amplitude must evolve into the asymptotic one, $\varphi_\pi(x) \to \varphi^A_\pi(x)$, if the effects of the pQCD evolution are taken into account. However, even at $Q^2$-values considerably larger than the presently accessible ones, other effects may still be more important than the effects of the pQCD evolution. This is the reason why other approaches and other forms of $\varphi_\pi(x)$ should be considered even when they do not incorporate the pQCD evolution. The form
\( \varphi_\pi(x) = \frac{f_\pi \Gamma(2\zeta + 2)}{2\sqrt{3} [\Gamma(\zeta + 1)]^2} x^\zeta (1 - x)^\zeta, \quad \zeta > 0, \)  

(7)

is suitable for representing various distribution amplitudes because it is relatively general [18]: \( \zeta > 1 \) yields the (empirically favored) distributions that are “peaked” or “narrowed” with respect to the asymptotic one (\( \zeta = 1 \)), whereas \( \zeta < 1 \) gives the “broadened” distributions which, however, now seem to be ruled out for the same reason as \( \varphi_\pi^{CZ}(x) \) quoted above, since \( J > 2 \) is ruled out empirically by CLEO [1]. Namely, it is easy to see that Eq. (7) implies \( \zeta = 2/3(J - 4) \).

Indeed, at the highest presently accessible momenta, the asymptotic prediction (\( J = 2 \)) is lowered by some 20\% by the lowest order QCD radiative corrections [30], amounting to \( J \approx 1.62 \), which fits the CLEO data well. Of course, these \( O(\alpha_s) \) corrections mean that \( Q^2 T_{\pi^0}(-Q^2, 0) \) is not strictly constant, but according to Ref. [30] it rises towards \( 2f_\pi \) so slowly (by just 4\% from \( Q^2 = 9 \text{ GeV}^2 \) to \( Q^2 = 36 \text{ GeV}^2 \)) that we can take it constant in practice. The situation is similar with the sum-rule approach of Radyshtin and Ruskov [7]. Their \( Q^2 T_{\pi^0}(-Q^2, 0) \) starts actually falling after \( Q^2 \approx 7 \text{ GeV}^2 \), but so slowly that Eq. (4) with the constant \( J \approx 1.6 \) represents it accurately at the presently accessible values of \( Q^2 \). This corresponds to a rather narrow distribution (4) with \( \zeta = 2.5 \).

The same type of the leading large-\( Q^2 \) behavior as in Eq. (4), was obtained by Manohar [20] using the operator product expansion (OPE). According to his OPE calculation, the coefficient in Eq. (4) giving the leading term is \( J = 4/3 \), which is below our large-\( Q^2 \) \( T_{\pi^0}(-Q^2, 0) \) by \( \sim 20\% \) when we use the BC vertex, but as we will see below – exactly coincides with the \( Q^2 \to \infty \) limit obtained using the mCP and CP vertices. The coefficient \( J = 4/3 \) is the lowest one still consistent with the form (4) because it corresponds to \( \zeta = \infty \). The pion distribution amplitude (4) then becomes infinitely peaked delta function: \( \varphi_\pi(x) = (f_\pi/2\sqrt{3}) \delta(x - 1/2) \).

For \( Q^2 > 4 \text{ GeV}^2 \), our “BC” \( T_{\pi^0}(-Q^2, 0) \) also behaves in excellent approximation as Eq. (2), with \( J \approx 1.78 \). This \( J \) would, in the pQCD factorization approach, correspond to \( \varphi_\pi(x) \) (3) with \( \zeta = 1.5 \). On the other hand, our “mCP” \( T_{\pi^0}(-Q^2, 0) \) falls off faster than \( 1/Q^2 \) for even the largest of the \( Q^2 \) values depicted in Fig. 2. However, it does not fall off much faster, as our “mCP” \( Q^2 T_{\pi^0}(-Q^2, 0) \) at \( Q^2 = 18 \text{ GeV}^2 \) is only 6\%, and at the huge \( Q^2 = 36 \text{ GeV}^2 \) is only 10 \% smaller than at \( Q^2 = 9 \text{ GeV}^2 \) (i.e., at the highest presently accessible momenta). Moreover, we can show analytically that, generally, the \( 1/Q^2 \)-behavior of Eq. (3) must at some point be reached in our approach, although Fig. 2 shows that for the mCP \( qq\gamma \) vertices this can happen only at significantly higher \( Q^2 \) than it happens for the BC \( qq\gamma \) vertices.

5. It is very pleasing that in the present approach (the coupled SD-BS approach in conjunction with GIA), besides numerical results, one can get also some analytical insights in the regime of asymptotically large \( Q^2 \). This way we can make more illuminative comparisons with the asymptotic behaviors predicted by the various approaches – notably pQCD on the light cone [18,19] and OPE [24].

Since the pion is light, \( k \cdot k' \approx Q^2/2 \) for large negative \( k^2 = -Q^2 \) and \( k'^2 = 0 \). Taking into account the behavior of the propagator functions \( A(q^2) \) and \( B(q^2) \) (e.g., see [13]) we can then in Eq. (3) approximate those quark propagators that depend on the photon momenta \( k \) and \( k' \), by their asymptotic forms:
\[ S(q - \frac{p}{2} + k) \approx S(k - \frac{p}{2}) \approx -\frac{2}{Q^2}(k - \frac{1}{2}p) , \]  

(8)

and analogously for the propagator where \( k \) is replaced by \( k' \). Although the relative loop momentum \( q \) can be large in the course of integration, its neglecting is justified because the BS amplitude \( \chi(q, p) \) decays quickly and thus strongly damps the integrand for the large \( q \)’s.

For the moment let us make an additional approximation by replacing the dressed electromagnetic vertices in Eq. (6) by the bare, free ones: \( \Gamma^{\mu}(q, q') \to \gamma^{\mu} \). The tensor amplitude (9) then becomes

\[ T_{\pi^0}(k, k') = \frac{\sqrt{2}}{Q^2} \int \frac{d^4q}{(2\pi)^4} \text{tr} \left( \chi(q, p) \left[ (k - \frac{p}{2})_\gamma \gamma^{\mu} \gamma^{\lambda} \gamma^{\gamma} + (k' - \frac{p}{2})_\lambda \gamma^{\mu} \gamma^{\gamma} \gamma^{\sigma} \right] \right) , \]

(9)

which is readily rewritten as

\[ T_{\pi^0}(k, k') = -\frac{i\sqrt{2}}{Q^2} \varepsilon^{\mu\lambda\sigma}(k - k')_\lambda \int \frac{d^4q}{(2\pi)^4} \text{tr}[\chi(q, p)\gamma_\sigma\gamma_5] = -\frac{2}{N_c Q^2} \varepsilon^{\mu\lambda\sigma}(k - k')_\lambda p_\sigma f_\pi . \]

(10)

The second equality holds since in the Mandelstam formalism, the integral in Eq. (10) is equal to \(-ip_\sigma f_\pi \sqrt{2}/N_c - \text{ e.g.}, see [2] [12]. Using the definition of the scalar amplitude \( T_{\pi^0}(k^2, k'^2) \), we finally recover Eq. (4) with the coefficient having the special value \( J = 4/3 \). We have thus found that the asymptotic behavior predicted by the present approach with bare \( qq\gamma \) vertices (and, as seen below, also with dressed mCP and CP vertices), is in exact agreement with the leading term predicted by OPE [24].

The asymptotic behavior \( \propto 1/Q^2 \) obtained in the present approach is especially satisfying when compared with that resulting from the calculation of the triangular quark loop carried out in the simple constituent quark model (with the constant light–quark mass parameter \( m_u \)), where \( T_{\pi^0}(-Q^2, 0) \propto (m_u^2/Q^2) \ln^2(Q^2/m_u^2) \) as \( Q^2 \to \infty \), which overshoots the data considerably [3] because of the additional \( \ln^2(Q^2) \)-dependence.

As seen from Fig. 2, the asymptotic behavior \( Q^2T_{\pi^0}(-Q^2, 0) = (4/3)f_\pi \) barely touches the experimental error bars from below. However, Manohar [20] pointed out that his OPE approach also indicates the existence of potentially large corrections to his leading term. (Note that he also pointed out that, for the same reason, significant corrections should similarly affect the light-cone prediction \( (J = 2) \) of Lepage and Brodsky [15].) It is easy to see that in our case when the BC vertex is used, the main origin of the corrections raising \( J \) from \( J = 4/3 \) resulting from the usage of the bare vertices, to \( J \approx 1.78 \) of our full numerical calculation, is the usage of these dressed \( qq\gamma \) vertices. It is nevertheless instructive to formulate another successive approximation to illustrate the gradual transition to the \( T_{\pi^0}(-Q^2, 0) \) resulting from the full calculation. The idea is that at high \( Q^2 \), only the \( H_+ \)-term contributes significantly in the BC-vertex (3), as the \( H_- \)-terms are suppressed by their \( Q^2 \)-dominated denominators in the both BC-vertices: \( p_1^2 - p_2^2 = \pm(q \mp p/2)^2 \mp (k + q - p/2)^2 \approx \pm Q^2/2 \). The detailed derivation shows that, as \( Q^2 \) grows, this indeed happens, and only \( \frac{1}{2}\gamma_\mu[A(p_1^2) + A(p_2^2)] \) contributes significantly to both of the BC vertices. Since \( p_1^2 \) in one of the vertices, and \( p_2^2 \) in the other, also contain \( Q^2/2 \), the BC vertices reduce to \( \gamma_\mu[1 + A([q \pm p/2]^2)]/2 \) since \( A(-Q^2/2) \to 1 \) as \( Q^2 \to \infty \). The adequacy of the chiral limit
approach, provides from the modern version of the constituent quark model which is given by the coupled SD-BS QCD predictions (2) for the leading large-\( Q^2 \) independently of model details – consistent with the Abelian ABJ anomaly and with the Euclidean bound-state solutions) in this and earlier papers [11–13].

The most important result of the present paper is that we have demonstrated that our confidence in the accuracy of our numerical methods and procedures (employing the asymptotics of our full calculation and the empirical fit.

However, all this happens for the BC vertex, where the “soft” leg adjacent to the pion BS-amplitude always contributes \( A(q^2) \), but not for the mCP vertex, nor for the CP one. Since they tend to the bare vertex as soon as the high momentum flows through one of its legs, \( f_\pi \) does not get replaced by \( \tilde{f}_\pi \), and the “bare” result, Eq. (3) with \( J = 4/3 \), continues to hold for \( Q^2 \to \infty \) when the mCP or CP vertex is used. Of course, the usage of the mCP vertex or the CP vertex instead of \( \gamma^\mu \) causes considerable differences for the finite \( Q^2 \).

6. The most important result of the present paper is that we have demonstrated that the modern version of the constituent quark model which is given by the coupled SD-BS approach, provides from \( Q^2 = 0 \) to \( Q^2 \to \infty \) the description for \( \gamma^\mu \to \pi^0 \) which is – independently of model details – consistent with the Abelian ABJ anomaly and with the QCD predictions (2) for the leading large-\( Q^2 \) behavior. Since \( f_\pi \) is a calculable quantity in the SD-BS approach, the model dependence is present in the (successfully reproduced [22, 14, 13]) value of \( f_\pi \), but our derivation of the asymptotic forms (4) and (11) is model-independent. Of special importance is also that this derivation [Eqs. (3), (11)] of the large-\( Q^2 \) behavior applies to both Minkowski and Euclidean space. The same holds for the considerations involving \( \tilde{f}_\pi \) and the \( [1 + A(q^2)]^2/4 \) correction factor, which have enabled us to make smooth and accurate contact between our large-\( Q^2 \) numerical results calculated with the Euclidean solutions of the chosen model [22], and analytical results on the large-\( Q^2 \) asymptotics. The agreement between the analytical and numerical results enhances our confidence in the accuracy of our numerical methods and procedures (employing the Euclidean bound-state solutions) in this and earlier papers [11, 13].

To mention one possible application, the relevance of the present work for the light-by-light scattering contribution to the muon \( g - 2 \) is obvious. Nevertheless, the relevance of its extension to the case when both photons are off-shell, i.e., to the \( \pi^0 \gamma^* \gamma^* \) form factor, is even larger (for \( g - 2 \)), since there are no experimental data on \( T_{\pi^0}(-Q^2, -Q^2) \) at present. In the light of its consistency with OPE and axial anomaly, but also the (at least) qualitative
consistency with the QCD sum rules, light-cone pQCD and VMD dominance, we propose that the present approach be used in a re-calculation of some \( \pi^0 \gamma^* \gamma^* \)-dependent predictions of Ref. [3] instead of the double-VMD Ansatz (or instead of the Extended Nambu-Jona-Lasinio model which does not give the correct asymptotic behavior even for the \( \pi^0 \gamma^* \gamma^* \) vertex [31]). The reason is that this VMD-motivated Ansatz for the \( \pi^0 \gamma^* \gamma^* \) form factor (Eqs. (3.5) and (4.1) in Ref. [3]) disagrees with what it should be according to the simple and unambiguous extension of \( T_{\pi^0}(-Q^2,0) \) presented in this work. In particular, the VMD Ansatz for \( T_{\pi^0}(-Q^2,-Q'^2) \) behaves as \( 1/(Q^2 Q'^2) \) for asymptotically large \( Q^2, Q'^2 \), whereas it should be only as the inverse of the sum of these squared momenta. Our full treatment of the \( \pi^0 \gamma^* \gamma^* \) form factor will be given in another, more detailed paper [32], but already here we can point out that it is almost trivial to generalize the derivation of the asymptotic behaviors (2) and (11) to the case when both photons are off-shell, \( k^2 = -Q^2 \ll 0 \) and \( k'^2 = -Q'^2 \leq 0 \). Not much changes, except that \( k \cdot k' \approx (Q^2 + Q'^2)/2 \). Our prediction with the bare (or mCP, or CP) \( qq\gamma \)-vertices for this generalization of Eq. (2) turns out to be

\[
T_{\pi^0}(-Q^2, -Q'^2) = \frac{4}{3} \frac{f_\pi}{Q^2 + Q'^2},
\]

which is in agreement with the leading term of the OPE result derived by Novikov et al. [33] for the special case \( Q^2 = Q'^2 \). The distribution-amplitude-dependence of the pQCD factorization approach cancels out for that symmetric case, so that \( T_{\pi^0}(-Q^2, -Q'^2) \) in this approach (e.g., see [34]), in the limit \( Q^2 = Q'^2 \to \infty \), exactly agrees with both our Eq. (12) and Ref. [13]. The inclusion of the dressed, WTI-preserving vertices does not require any modifications of the asymptotic forms in the case of \( qq\gamma \) vertices (such as mCP and CP ones) which reduce to the bare one as soon as one of their quark momenta squared tends to infinity. In the case of the BC vertex (3), the generalization of the better-approximated expression (11) requires nothing except the substitution \( f_\pi \to \tilde{f}_\pi \) in Eq. (12). For our particular \( A(q^2) \), \( \tilde{f}_\pi \) can, in a good approximation, be factored as explained above.

We note that the VMD-motivated Ansatz for the \( P^0 \gamma^* \gamma^* \) \( (P^0 = \pi^0, \eta, \eta') \) transition form factor is also used as a theoretical input in the CLEO data analysis – see Eq. (8) in Ref. [1]. A re-analysis of the data in the spirit of the present insights from the SD-BS approach, is therefore desirable. At least, one should examine the effects on the data analysis when one makes a change in Eq. (8) of Ref. [1] like \( 1/[(1 + Q^2 /\Lambda^2)(1 + Q'^2 /\Lambda^2)] \to 1/(1 + Q^2 /\Lambda^2 + Q'^2 /\Lambda^2) \).

Actually, \( \eta \) and \( \eta' \) appear not only in the CLEO data [1], but are also relevant for the \( g-2 \) analysis [3] discussed above. This underscores the importance of extending the present work to include the \( \eta \gamma^* \gamma \) and \( \eta' \gamma^* \gamma \) transition form factors. While the full calculation thereof obviously must be relegated to another paper [32], our predictions for the \( Q^2 \to \infty \) behavior of the SU(3)_f states \( \eta_8 \) and \( \eta_0 \), and also \( \eta_c \) and \( \eta_b \), can be easily obtained by redoing the derivation of the asymptotic expression so that the track of their flavor content is kept. But, one then realizes that the generalization to both photons being virtual can be easily done also for \( \eta_8, \eta_0, \eta_c \), and \( \eta_b \) in the way discussed two passages above for \( \pi^0 \). Therefore, in the approximation of the bare \( qq\gamma \) vertices, in the asymptotic regime with one of the photon momenta large (say \( Q^2 \to \infty \)), and the other (say \( Q'^2 \)) anywhere between 0 and \( \infty \), the \( \eta_8 \) and \( \eta_0 \) transition form factors are given by the expressions analogous to Eq. (12), except that its factor \( 4f_\pi/3 \equiv C_{\pi^0} \), appropriate for \( P^0 = \pi^0 \), is replaced by the respective factors.
$C_{\eta_8} = \left(4/9\sqrt{3}\right)(5f_\pi - 2f_{s\bar{s}})$ and $C_{\eta_0} = \left(4\sqrt{2}/9\sqrt{3}\right)(5f_\pi + f_{s\bar{s}})$. The auxiliary quantity $f_{s\bar{s}}$ is the decay constant of the unphysical $s\bar{s}$ pseudoscalar bound state, evaluated in [12] to be $f_{s\bar{s}} = 1.47f_\pi$ for the chosen bound-state model [22], yielding the model-dependent values $C_{\eta_8} = 0.53f_\pi = 49.3$ MeV and $C_{\eta_0} = 2.35f_\pi = 219$ MeV.

When the $\eta_8, \eta_0$ asymptotic expressions are improved by using the dressed, WTI-preserving vertices instead of the bare vertex $\gamma^\mu$, the asymptotic expressions do not change at all in the case of the mCP and CP vertex, and in the case of the BC-vertex the only changes are again just the substitutions $f_\pi \to \tilde{f}_\pi$ and $f_{s\bar{s}} \to \tilde{f}_{s\bar{s}}$.

Similarly, for $\eta_c$ and $\eta_b$, the analogous expressions when at least one of $Q^2$ or $Q^2$ is large, are obtained from Eq. (12) by replacing $4f_\pi/3 \equiv C_{\eta_0}$ with $C_{\eta_c} = (16\sqrt{2}/9)f_{\eta_c}$ and $C_{\eta_b} = (4\sqrt{2}/9)f_{\eta_b}$, respectively. (Of course, if $Q^2 < M^2_{P0}$, neglecting the meson mass $M_{P0}$ is for $Q^2 < M^2_{P0}$ a rougher approximation than setting $Q^2 = 0$. $M_{\eta_c}$ and especially $M_{\eta_b}$ are already so large that keeping $k \cdot k' = (Q^2 + Q'^2 + M^2_{P0})/2$ in Eq. (12) can be important for momenta $Q^2$ accessible in practice.) In our chosen bound-state model [22], we have obtained $f_{\eta_c} = 213$ MeV and $f_{\eta_b} = 284$ MeV, leading to the model-dependent values $C_{\eta_c} = 536$ MeV and $C_{\eta_b} = 179$ MeV. This is of immediate importance for experiment, as the $\eta_c$ transition form factor $T_{\eta_c}(-Q^2, 0)$ can be measured at L3 [35].

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FIGURE CAPTIONS

Fig. 1: The diagram for $P^0 \rightarrow \gamma\gamma$ decays ($P^0 = \pi^0, \eta, \eta', \eta_c, \eta_b$). Within the scheme of generalized impulse approximation, the propagators and vertices are dressed.

Fig. 2: The comparison of our results for $Q^2 T_{\pi^0}(-Q^2, 0)$, the pion transition form factor, with the CELLO (circles) and CLEO (triangles) data and with the Brodsky-Lepage interpolation formula (the dotted line, which also denotes the pQCD limiting value $2f_\pi$). The dash-dotted line represents our $Q^2 T_{\pi^0}(-Q^2, 0)$ evaluated as in our earlier paper [11] (but now to higher squared momenta), i.e., exclusively with the BC $q q \gamma$ vertices, and in the chiral and soft limit approximation for the pion $[\Gamma_{\pi^0}(q, p) \approx \Gamma_{\pi^0}(q, 0) = \gamma_5 \lambda^3 B(q^2)/f_\pi]$. Our results for $Q^2 T_{\pi^0}(-Q^2, 0)$ without that approximation are depicted by the solid line for the case of the BC vertices, and by the dashed line for the mCP vertices. The one obtained with the BC vertex practically saturates beyond $Q^2 \sim 6$ GeV$^2$ at higher values [$Q^2 T_{\pi^0}(-Q^2, 0) \approx 164$ MeV] than that obtained with the mCP vertices. This latter one gets lower beyond $Q^2 \sim 2.5$ GeV$^2$ and does not yet saturate at the presently accessible momenta although approaches asymptotically $Q^2 T_{\pi^0}(-Q^2, 0) \rightarrow 4f_\pi/3$ for much higher $Q^2$. (This limit, which is also the leading term resulting from OPE [20], is denoted by the dashed straight line.)
$Q^2 T(-Q^2,0)$ [GeV]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Graph showing the distribution of $Q^2 T(-Q^2,0)$ with data points from CELLO and CLEO, plotted against $Q^2$ [GeV$^2$].}
\end{figure}