Some static properties of the Slinky

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Abstract
In this paper we use a simple discrete model of a Slinky to explore some of its static properties. We derive some relations for vertically and U-shaped suspended Slinkies, based on which some demonstrations are proposed that can be simply done in freshmen physics classes.

Keywords: Slinky, discrete model, vertical suspension, U-shaped suspension

(Some figures may appear in colour only in the online journal)

1. Introduction

A flexible long spring, known as a Slinky, is a very popular toy. There are many videos on the web demonstrating interesting properties of falling Slinkies. In addition to its entertaining aspects, a Slinky can be used for simple but interesting demonstrations for freshmen physics students, supported with simple theoretical analysis.

There are many papers investigating the different properties of a Slinky. In some of these papers, a suspended Slinky is studied [1–6], while in others dynamical properties such as wave propagation and free fall are considered [4, 7–13]. In these studies, mostly a continuous model is considered; however, some authors use a discrete model [3, 5]. For example, in [3] a simple discrete model is considered and some of the properties of a suspended Slinky are explored.

A more sophisticated discrete model can be found in [14], where they consider instead of point-like masses, bars that are connected with springs. In [15], which is an extension of [14], the bars are connected with three axial, rotational and shear springs.

In this paper we consider a simple discrete model for a Slinky, like the one used in [3] and [5], and after restating some of their results for a vertically suspended Slinky, consider a U-shaped suspension. Based on our theoretical results, we propose some demonstrations that can be done in freshmen physics classes. In a real class at Shahid Beheshti University, we actually performed these demonstrations, which seemed to be very attractive and inspiring for the students.
The paper is organized as follows. In section 2, we consider a vertically suspended Slinky and meanwhile introduce the discrete model. In section 3, U-shaped suspension from equal-height points is described, while section 4 considers suspension from two points with different heights. Finally, section 5 concludes our work.

2. Vertical suspension

In order to study a vertically suspended Slinky, we model it with a collection of \((N + 1)\) small point-like objects with mass \(m\), connected with \(N\) ideal springs with spring constant \(k\), as shown in figure 1. At the limit of large \(N\), this model describes a spring with mass \(M_s = (N + 1)m\) and spring constant \(K_s = k/N\). When a Slinky is suspended from one end, because of its mass the upper parts are stretched more than the lower parts. The change of length of the \(n\)-th spring is shown by \(\delta_n\). Using the cancellation of the forces on each object, considering the appropriate boundary condition and doing some straightforward calculations, one can easily show that the length of the suspended spring is given by

\[
L = L_0 + \sum_{n=1}^{N} \delta_n = L_0 + \frac{M_g}{2K_s},
\]

where \(L_0\) is the equilibrium length of the Slinky. One of the properties that makes a Slinky so special is that its equilibrium length is negligible, i.e. \(L_0 \ll \frac{M_g}{K_s}\). Therefore, the length of a suspended spring is approximately \(L = \frac{M_g}{2K_s}\). Based on this relation, we can do the following
demonstration. First we suspend a Slinky from a fixed point and measure its length. Then, we use half of the Slinky and see that, since the mass is halved and the spring constant is doubled, the length is decreased by a factor of four.

Now consider a downward vertical $y$ axis with its origin at the suspension point. The position of the $n$-th object is $y_n = \frac{L_0}{2} + \sum_{i=1}^{n-1} \delta_i$. For the limit of large $N$, in every small element of length we would have many objects and springs. This means that the discrete model can be considered as a continuous one. For this continuous model, neglecting $L_0$, one arrives at the following formula for the mass distribution over length:

$$\sigma(y) = \frac{K_s}{g} \sqrt{\frac{L}{L - y}}.$$  \hfill (2)

This equation satisfies the relation $M_i = \int_0^L \sigma(y) dy$. Moreover, one can easily show that for negligible $L_0$, the center of mass is located at

$$y_m = \frac{1}{M_i} \int_0^L y \sigma(y) dy = \frac{2L}{3}. \hfill (3)$$

As a classroom demonstration, one can drop a Slinky at the same time with an object located at $y_m$. It is expected that the object and Slinky reach the floor at the same time. This can be seen more clearly by filming the experiment and watching the slow motion replay.

### 3. U-Shaped suspension from points with equal heights

In this section, we consider a Slinky suspended from both ends, with two suspension points at equal heights. For this case, our model is shown in figure 2. In this case, we have $(2N + 1)$ objects with mass $m$ and $2N + 2$ springs with spring constant $k$, therefore, $M_i = (2N + 1)m$ and $K_s = \frac{k}{2N + 2}$. We label the objects and springs from $-N$ to $N$ and define the $x - y$ coordinates as shown in figure 2.

In figure 3 we depict the forces exerted on the $n$-th object ($0 < n < N$). Cancellation of the forces yields the following two relations:

$$f_n \sin \theta_n = mg + f_{n-1} \sin \theta_{n-1}, \hfill (4)$$
and

\[ f_n \cos \theta_n = f_{n-1} \cos \theta_{n-1}, \tag{5} \]

where \( f_n \) is the force exerted by the \( n \)-th spring. We consider a case in which \( L_0 \) is very small, that is, the equilibrium lengths of the springs in the model are approximately zero. As a consequence, \( \sin \theta_n = \frac{h_n}{k_n} \) and \( \cos \theta_n = \frac{I_n}{k_n} \). Combining these with the boundary condition \( 2h_0 \sin \theta_0 = mg \) (which is obtained for the object at origin), one can easily show that equations (4) and (5) result in

\[ h_n = \frac{mg}{k} \left( n + \frac{1}{2} \right), \tag{6} \]

and

\[ l_n = \frac{X}{N + 1}, \tag{7} \]

where \( X \) is half the distance between the suspension points, as indicated in figure 2. The coordinates of the \( n \)-th object at the positive half are \( x_n = \sum_{i=1}^{n} l_i \) and \( y_n = \sum_{i=1}^{n} h_i \). Using equations (6) and (7), and taking the limit of large \( N \), one can show that the shape of the Slinky is a parabola with the following equation:

\[ y(x) = \frac{Mg}{8Ks}x^2. \tag{8} \]

Let us define the depth of this parabola to be \( D = y(X) - y(0) \). From equation (8) we see that \( D = \frac{Mg}{8Ks} \), and it is independent of \( X \). As a demonstration one can hold a Slinky with their two hands and show that by changing the distance of the hands the depth of the formed parabola will not change (obviously, this distance between the suspension points can not get so huge that Hook’s law for the springs breaks down). This behavior is shown in figure 4. It is also noticeable that \( D \) is equal to the length of a halved Slinky which is vertically suspended (compare with equation (1)).

This result is not intuitive because of our daily experience with strings, for which the depth obviously depends on the distance between the suspension points. The difference can be understood by noting that the origin of this result in a Slinky is the fact that we can ignore

Figure 3. Forces exerted on the \( n \)-th object.
the equilibrium length of the small springs in our model, which means \( L_0 \ll L \). However, the usual string lies on the other extreme, in which the spring stiffness is huge and \( L \approx L_0 \).

4. U-Shaped suspension from points with different heights

The more general U-shaped Slinky in which the heights of the suspension points are different is considered in this section. The corresponding discrete model is depicted in figure 5. Following similar steps as the former case, one can obtain the \( y \) coordinate as a function of \( x \). Alternatively, we can use the results of section 3 as follows. Equation (5) in combination with \( \cos \theta = \frac{x}{L} \) prove that the horizontal distance between the objects are equal. As a result, the part of the Slinky which is at the right side of the origin represents a spring with mass \( M_1 = \frac{2k}{X_1 + X_2} M \) and spring constant \( K_1 = \frac{k_1 + k_2}{2k} K_0 \), where \( X_1 \) and \( X_2 \) are shown in figure 5. From equation (8), the \( y(x) \) equation for this part is given by

\[
y(x) = \frac{M_1 g}{8K_1 X_1^2} x^2 = \frac{M_1 g}{2K_1 (X_1 + X_2)^2} x^2.
\]

The same reasoning applies to the part on the left side of the origin and the same \( y(x) \) equation would be obtained.

Figure 4. The depth of the U-shaped Slinky is independent of the distance between equal-height suspension points.

Figure 5. Discrete model for a Slinky in U-shaped suspension with different height suspension points.
Equation (9) says that the functional form of $y(x)$ just depends on the horizontal distance between the suspension points and not on the vertical distance. Therefore, identical Slinkies which are suspended from points with equal horizontal distances are all different parts of the same parabola. It should be mentioned that in our analysis we considered that the difference of the heights of the suspension points is not more than $\frac{Mg}{2K}$, so that the valley lies between them; however, using the discrete model it can be easily shown that the result is not restricted to this condition.

5. Conclusions

In conclusion, we considered a simple discrete model for a Slinky and applied it to cases of vertically and U-shaped suspended Slinkies.

In the first case, we obtained the length of a Slinky and the location of its center of mass. We proposed two demonstrations for this part. In one of the demonstrations we halve the Slinky and show that its suspended length is divided by four. In the second demonstration we drop an object simultaneously with a Slinky at the same height as its center of mass, and students can see that the center of mass motion is a simple free fall.

In the case of a U-shaped Slinky with equal-height suspension points, we obtained its shape and showed that it was a parabola. On the other hand, we showed that the depth of this parabola is independent of the horizontal distance of the suspension point (as long as the distance between the suspension points is not too much, so that Hook’s law holds for our spring). This non-intuitive result can be shown in a simple demonstration.

In a more general case, we considered a U-shaped Slinky with different height suspension points and showed that the parabolic equation for its shape is just determined by the horizontal distance between the suspension points. Therefore, identical Slinkies which are suspended from points with equal horizontal distances are all different parts of the same parabola.

Based on simple and beautiful physics contained in the Slinky, we propose it to be used as a pedagogical device in freshmen basic physics.

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