A DYNAMICAL APPROACH TO THE LARGE-TIME BEHAVIOR OF SOLUTIONS TO WEAKLY COUPLED SYSTEMS OF HAMILTON–JACOBI EQUATIONS

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Abstract. We investigate the large-time behavior of the value functions of the optimal control problems on the $n$-dimensional torus which appear in the dynamic programming for the system whose states are governed by random changes. From the point of view of the study on partial differential equations, it is equivalent to consider viscosity solutions of quasi-monotone weakly coupled systems of Hamilton–Jacobi equations. The large-time behavior of viscosity solutions of this problem has been recently studied by the authors and Camilli, Ley, Loreti, and Nguyen for some special cases, independently, but the general cases remain widely open. We establish a convergence result to asymptotic solutions as time goes to infinity under rather general assumptions by using dynamical properties of value functions.

1. Introduction and Main Result

In this paper we deal with optimal control problems, or calculus of variations, which appear in the dynamic programming for the system whose states are governed by random changes. More precisely, we consider the minimizing problem:

$$\text{Minimize } \mathbb{E}_i \left[ \int_{-t}^{0} L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds + g_{\nu(-t)}(\gamma(-t)) \right],$$

over all controls $\gamma \in \text{AC}([-t,0])$ with $\gamma(0) = x$ for any fixed $(x,t) \in \mathbb{T}^n \times [0, \infty)$, where the Lagrangians $L_i(x,p) : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ are derived from the Fenchel-Legendre transforms of given Hamiltonians $H_i$ and we denote by AC([-t,0]) the set of absolutely continuous functions on $[-t,0]$ with values in $\mathbb{T}^n$. The functions $g_i$ are given real-valued continuous functions on $\mathbb{T}^n$ for $i = 1, 2$. Here $\mathbb{E}_i$ denotes the expectation of a process with $\nu(0) = i$, where $\nu$ is a $\{1, 2\}$-valued process which is a continuous-time Markov chain on $(-\infty, 0]$ (notice that time is reversed) such that for $s \leq 0$, $\Delta s > 0$,

$$\mathbb{P}(\nu(s - \Delta s) = j \mid \nu(s) = i) = c_i \Delta s + o(\Delta s) \text{ as } \Delta s \to 0 \text{ for } i \neq j,$$

where $c_i$ are given positive constants and $o : [0, \infty) \to [0, \infty)$ is a function satisfying $o(r)/r \to 0$ as $r \to 0$. We call the minimizing costs of (1.1) the value functions of optimal control problems (1.1).
The purpose of this paper is to investigate the large-time behavior of the value functions. From the point of view of partial differential equations it is equivalent to study that of viscosity solutions of quasi-monotone weakly coupled systems of Hamilton–Jacobi equations. From the point of view of partial differential equations it is equivalent to study

\[ (u_1)_t + H_1(x, Du_1) + c_1(u_1 - u_2) = 0 \quad \text{in } \mathbb{T}^n \times (0, \infty), \]
\[ (u_2)_t + H_2(x, Du_2) + c_2(u_2 - u_1) = 0 \quad \text{in } \mathbb{T}^n \times (0, \infty), \]
\[ u_i(x, 0) = g_i(x) \quad \text{on } \mathbb{T}^n, \]

where the Hamiltonians \( H_i(x, p) : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) are given continuous functions for \( i = 1, 2 \), which are assumed throughout the paper to satisfy the followings.

(A1) The functions \( H_i \) are uniformly coercive in the \( x \)-variable, i.e.,

\[ \lim_{r \to \infty} \inf \{ H_i(x, p) \mid x \in \mathbb{R}^n, |p| \geq r \} = \infty. \]

(A2) The functions \( p \mapsto H_i(x, p) \) are strictly convex for any \( x \in \mathbb{T}^n \).

Here \( u_i \) are real-valued unknown functions on \( \mathbb{T}^n \times [0, \infty) \) and \( (u_i)_t = \partial u_i / \partial t, Du_i = (\partial u_i / \partial x_1, \ldots, \partial u_i / \partial x_n) \) for \( i = 1, 2 \), respectively. We are only dealing with viscosity solutions of Hamilton–Jacobi equations here and thus the term “viscosity” will be omitted henceforth.

The existence and uniqueness results for weakly coupled systems \( (C) \) of Hamilton–Jacobi equations have been established by [6, 9]. In recent years, there have been many studies on the properties of viscosity solutions of weakly coupled systems of Hamilton–Jacobi equations. See [3, 11, 12, 4] for instance. In particular, the studies on large-time behaviors were done for some special cases by the authors [11], and Camilli, Ley, Loreti and Nguyen [4], independently. However, the general cases remain widely open and the techniques developed in [11, 4] are not applicable for general cases. The coupling terms cause serious difficulties, which will be explained in details later.

Let us first recall the heuristic derivation of the large-time asymptotics for \( (C) \) discussed by the authors [11] for readers’ convenience. We use the same notations as in [11]. For simplicity, we assume that \( c_1 = c_2 = 1 \) henceforth. Formal asymptotic expansions of the solutions \( u_1, u_2 \) of \( (C) \) are considered to be of the forms

\[ u_1(x, t) = a_{01}(x)t + a_{11}(x) + a_{21}(x)t^{-1} + \ldots, \]
\[ u_2(x, t) = a_{02}(x)t + a_{12}(x) + a_{22}(x)t^{-1} + \ldots. \]

as \( t \to \infty \). Then \( (C) \) becomes

\[ a_{01}(x) - a_{21}(x)t^{-2} + \ldots + H_1(x, Da_{01}(x)t + Da_{11}(x) + Da_{21}(x)t^{-1} + \ldots) \]
\[ + (a_{01}(x) - a_{02}(x))t + (a_{11}(x) - a_{12}(x)) + (a_{21}(x) - a_{22}(x))t^{-1} + \ldots = 0, \] (1.3)

and

\[ a_{02}(x) - a_{22}(x)t^{-2} + \ldots + H_2(x, Da_{02}(x)t + Da_{12}(x) + Da_{22}(x)t^{-1} + \ldots) \]
\[ + (a_{02}(x) - a_{01}(x))t + (a_{12}(x) - a_{11}(x)) + (a_{22}(x) - a_{21}(x))t^{-1} + \ldots = 0. \] (1.4)
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Sum up (1.3) and (1.4) to yield
\[ H_1(x, Da_{01} t + Da_{11} + O(1/t)) + H_2(x, Da_{02} t + Da_{12} + O(1/t)) + O(1) = 0 \]
as \( t \to \infty \). Hence we formally get \( Da_{01} = Da_{02} \equiv 0 \) by the coercivity of \( H_1 \) and \( H_2 \). We
next let \( t \to \infty \) in (1.3), (1.4) to achieve that \( a_{01}(x) = a_{02}(x) \equiv a_0 \) for some constant \( a_0 \in \mathbb{R} \), and
\[
\begin{cases}
H_1(x, Da_{11}(x)) + a_{11}(x) - a_{12}(x) = -a_0, \\
H_2(x, Da_{12}(x)) + a_{12}(x) - a_{11}(x) = -a_0,
\end{cases}
\]
in \( \mathbb{T}^n \). It is then natural to study the ergodic problem
\[
(E) \quad \begin{cases}
H_1(x, Dv_1(x)) + v_1 - v_2 = c & \text{in } \mathbb{T}^n, \\
H_2(x, Dv_2(x)) + v_2 - v_1 = c & \text{in } \mathbb{T}^n.
\end{cases}
\]
We here seek for a triplet \((v_1, v_2, c) \in C(\mathbb{T}^n)^2 \times \mathbb{R}\) such that \((v_1, v_2)\) is a solution of \((E)\). If \((v_1, v_2, c)\) is such a triplet, we call \((v_1, v_2)\) a pair of ergodic functions and \(c\) an ergodic constant. It was proved in [3, 11] that there exists a unique constant \(c\) such that the ergodic problem \((E)\) has continuous solutions \((v_1, v_2)\).

Hence, our goal in this paper is to prove the following large-time asymptotics for \((C)\).

**Theorem 1.1 (Main Result).** Assume that (A1), (A2) hold. For any \((g_1, g_2) \in C(\mathbb{T}^n)^2\) there exists a solution \((v_1, v_2, c) \in C(\mathbb{T}^n)^2 \times \mathbb{R}\) of \((E)\) such that
\[
u_i(x,t) + ct - v_i(x) \to 0 \text{ uniformly on } \mathbb{T}^n \text{ as } t \to \infty \quad (1.5)
\]
for \(i = 1, 2\).

In the last decade, the large time behavior of solutions of single Hamilton–Jacobi equations,
\[
u_t + H(x, Du) = 0 \text{ in } \mathbb{T}^n \times (0, \infty), \quad (1.6)
\]
where \(H\) is coercive, has received much attention and general convergence results for
solutions have been established. The first general result was discovered by Namah and
Roquejoffre in [13] under the following additional assumptions: \(p \mapsto H(x,p)\) is convex,
and
\[
H(x, p) \geq H(x, 0) \text{ for all } (x, p) \in \mathcal{M} \times \mathbb{R}^n \text{ and } \max_{\mathcal{M}} H(x, 0) = 0, \quad (1.7)
\]
where \(\mathcal{M}\) is a smooth compact \(n\)-dimensional manifold without boundary. Then Fathi
used dynamical system approach from weak KAM theory in [7] to establish the same type
of convergence result, which requires uniform convexity (and smoothness) assumptions on
\(H(x, \cdot),\) i.e., \(D_{pp} H(x, p) \geq \alpha I\) for all \((x, p) \in \mathcal{M} \times \mathbb{R}^n\) and \(\alpha > 0\) but does not require
the specific structure \((1.7)\) of Hamiltonians. Afterwards Roquejoffre [14], Davini and
Siconolfi in [5], Ishii in [8] refined and generalized the approach of Fathi and they studied
the asymptotic problem for Hamilton–Jacobi equations on \(\mathcal{M}\) or the whole \(n\)-dimensional
Euclidean space. Besides, Barles and Souganidis [1] also obtained this type of results, for
possibly non-convex Hamiltonians, by using a PDE method in the context of viscosity
solutions.
In the previous paper [11], the authors could establish Theorem 1.1 only in two main specific cases. In the first case, we generalized the approach in [13] and obtain convergence result under additional assumptions similar to (1.7) (see also [4]). The second case is a generalization of [1] under the strong assumption that $H_1 = H_2 = H$, where $H$ satisfies similar assumptions as in [1]. We could not obtain Theorem 1.1 in its full generality because of the appearance of the coupling terms $u_1 - u_2$ and $u_2 - u_1$.

In this paper we develop a dynamical approach to weakly coupled systems of Hamilton–Jacobi equations which is inspired by the works by Davini, Siconolfi [5] and Ishii [8], and establish Theorem 1.1 in its full generality. The results in [7, 14, 5] can be viewed as a particular case of Theorem 1.1 when $H_1 = H_2$, and $g_1 = g_2$. As we consider system (C), we need to take random switchings among the two states in (1.1) into account, which does never appear in the context of single Hamilton-Jacobi equations. The key ingredients in this approach consist of obtaining existence and stability results of extremal curves of (1.1). It is fairly straightforward to prove the existence of extremal curves by using techniques from calculus of variations. However, representation formulas (1.1) are implicit in some sense and prevent us from deriving a stability result (see Theorem 4.1). In order to over come this difficulty, we give more deterministic formulas for the value functions of (1.1) by explicit calculations in Theorem 2.4. By using the new formulas, which are more intuitive, we are able to derive Theorem 4.1, and hence large time behavior results.

Let us call attention to the forthcoming paper [2] by Cagnetti, Gomes and the authors, which provides a completely new approach to the study of large time behaviors of both single and weakly coupled systems of Hamilton–Jacobi equations. A new and different proof of Theorem 1.1 is derived as well.

The paper is organized as follows. In Section 2 we establish new representation formulas, which are more explicit and useful for our study here. We then derive the existence of extremal curves in Section 3, which is pretty standard in the theory of optimal control and calculus of variations. Section 4 concerns the study of stability of extremal curves. This section plays the key roles in this paper and allows us to overcome the technical difficulties coming from the coupling terms. See Remarks 4.4 and 4.5 for details. Section 5 is devoted to the proof of Theorem 1.1. Finally, some lemmata concerning verifications of optimal control formulas for (C) in Section 2 are recorded in Appendix for readers’ convenience.

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2. Preliminaries

In this section, we establish new representation formulas, which give us a clearer intuition about the switching states of the systems. The new formulas allow us to perform deep studies on the extremal curves in Sections 3, and 4.
Lemma 2.1. Let \( \nu \) be a Markov process defined by (1.2) with \( c_1 = c_2 = 1 \) and set \( p_j(t) := \mathbb{P}(\nu(t) = j) \) for \( j \in \{1, 2\} \). Then we have
\[
p_j(t) = 1/2 + e^{2t}(p_j(0) - 1/2) \text{ for all } t < 0.
\]
In particular, \( p_j(t) \rightarrow 1/2 \) as \( t \rightarrow -\infty \) for any \( j \in \{1, 2\} \).

Proof. Set \( \mu(dx) := -(\delta_1(dx) + \delta_2(dx)) \), where \( \delta_j \) are the Dirac measures at the points \( j \), for \( j \in \{1, 2\} \), respectively. For any measurable function \( f \) and \( i = 1, 2 \), by the Itô formula we have
\[
\mathbb{E}_i[f(\nu(t))] = f(i) + \int_0^t \mathbb{E}_i[\int (f(\beta) - f(\nu(s))) \mu(d\beta)] \, ds
\]
\[
= f(i) + \int_0^t -\left\{ \mathbb{P}((\nu(s) = 1) \cdot (f(2) - f(1)) + \mathbb{P}(\nu(s) = 2) \cdot (f(1) - f(2)) \right\} \, ds.
\]
If we take \( f(x) = 1_{\{j\}}(x) \) for \( j \in \{1, 2\} \), where \( 1_{\{j\}} \) is the characteristic function of \( \{j\} \), then we have
\[
p_j(t) = p_j(0) + \int_0^t -\left\{ p_j(s) \cdot (-1) + (1 - p_j(s)) \cdot 1 \right\} \, ds = p_j(0) + \int_0^t (2p_j(s) - 1) \, ds.
\]
Therefore, we get \( p_j(t) = 1/2 + e^{2t}(p_j(0) - 1/2) \) for all \( t < 0 \).

A straightforward result of Lemma 2.1 is

Lemma 2.2. Let \( \phi_i \) be any functions in \( C(\mathbb{T}^n) \) for \( i = 1, 2 \). We have
\[
\mathbb{E}_i[\phi_{\nu(t)}(x)] = \frac{1}{2}(1 + e^{2t})\phi_i(x) + \frac{1}{2}(1 - e^{2t})\phi_j(x)
\]
for all \( x \in \mathbb{T}^n \), \( t < 0 \), and \( i = 1, 2 \), where we take \( j \) so that \( \{i, j\} = \{1, 2\} \).

Remark 2.3. In general if \( c_1, c_2 > 0 \) are arbitrary constants, then we have
\[
\mathbb{E}_i[\phi_{\nu(t)}(x)] = \frac{1}{c_1 + c_2} (c_j + c_i e^{(c_1 + c_2)t}) \phi_i(x) + \frac{c_i}{c_1 + c_2} (1 - e^{(c_1 + c_2)t}) \phi_j(x)
\]
for all \( x \in \mathbb{T}^n \), \( t < 0 \), and \( i = 1, 2 \), where we take \( j \) so that \( \{i, j\} = \{1, 2\} \).

It turns out that the value function of optimal control problems (1.1) can be written in more explicit forms without using continuous Markov chains as follows by using the Fubini theorem.

Theorem 2.4. Let \( u_i \) be the value functions defined by (1.1). Then we can write them as
\[
u(x, t) = \inf \left\{ \int_{-L/2}^{L/2} (1 + e^{2s})L_i(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2} (1 + e^{-2t})g_i(\gamma(-t)) \right.
\]
\[
+ \int_{-L/2}^{L/2} \left( 1 - e^{2s} \right) L_j(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2} (1 - e^{-2t})g_j(\gamma(-t)) \mid \gamma \in AC([-t, 0]), \gamma(0) = x \right\}
\]

Moreover, \( u_i \) are uniformly continuous on \( \mathbb{T}^n \times [0, \infty) \) and the pair \( (u_1, u_2) \) is the unique viscosity solution of (C).
We call $(1/2)(1 + e^{2s})$ and $(1/2)(1 - e^{2s})$ for $s < 0$ the weights corresponding to (C), which comes from the random switchings among the two states in (1.1).

**Proof.** By Fubini’s theorem and Lemma 2.2 we have

\[
\mathbb{E}_i \left[ \int_{-t}^{0} L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) ds + g_{\nu(-t)}(\gamma(-t)) \right] \\
= \int_{-t}^{0} \mathbb{E}_i \left[ L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \right] ds + \mathbb{E}_i \left[ g_{\nu(-t)}(\gamma(-t)) \right] \\
= \int_{-t}^{0} \frac{1}{2}(1 + e^{2s}) L_i(\gamma(s), \dot{\gamma}(s)) ds + \frac{1}{2}(1 + e^{-2s}) g_i(\gamma(-t)) \\
+ \int_{-t}^{0} \frac{1}{2}(1 - e^{2s}) L_j(\gamma(s), \dot{\gamma}(s)) ds + \frac{1}{2}(1 - e^{-2s}) g_j(\gamma(-t))
\]

for any $\gamma \in AC([-t,0])$, which implies the equality (2.1).

In Appendix we prove that $u_i$ are uniformly continuous on $\mathbb{T}^n \times [0, \infty)$ and the pair $(u_1, u_2)$ gives a solution of (C). In the previous paper [12], we showed that the pair $(u_1, u_2)$ defined by (1.1) solves (C) already. But we present it in a different way by using the new formula (2.1) itself to make the paper self-contained. \(\square\)

Let $(v_1, v_2, 0)$ be a solution of (E). Without loss of generality, we may assume that the ergodic constant $c = 0$ henceforth. We notice that $v_i$ satisfies

\[
v_i(x) = \inf \left\{ \mathbb{E}_i \left[ \int_{-t}^{0} L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) ds + v_{\nu(-t)}(\gamma(-t)) \right] \mid \gamma \in AC((-\infty, 0]) \text{ with } \gamma(0) = x \right\}, \tag{2.2}
\]

where $\nu$ is a $\{1, 2\}$-valued process which is a continuous-time Markov chain satisfying (1.2) such that $\nu(0) = i$.

**Proposition 2.5.** Let $(v_1, v_2, 0)$ be a subsolution of (E). Then,

\[
v_i(x) \leq \mathbb{E}_i \left[ \int_{-t}^{0} L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) ds + v_{\nu(-t)}(\gamma(-t)) \right]
\]

for all $t \geq 0$, $\gamma \in AC([-t, 0])$ with $\gamma(0) = x$.

**Lemma 2.6.** Let $t > 0$, $v_i \in W^{1, \infty}(\mathbb{T}^n)$ for $i = 1, 2$ and $\gamma \in AC([-t, 0], \mathbb{T}^n)$ with $\gamma(0) = x$. We have $v_i \circ \gamma \in AC([-t, 0])$ and there exists a function $p_i \in L^{\infty}((-t, 0), \mathbb{R}^n)$ such that

\[
v_i(x) = \mathbb{E}_i \left[ \int_{-t}^{0} p_{\nu(s)}(s) \cdot \dot{\gamma}(s) + \sum_{j=1}^{2} (v_{\nu(s)} - v_j)(\gamma(s)) ds + v_{\nu(-t)}(\gamma(-t)) \right], \tag{2.3}
\]

$p_i(s) \in \partial_c v_i(\gamma(s))$.
for a.e. \( s \in (-t, 0) \) and \( i \in \{1, 2\} \). Here \( \partial_c v_i \) denotes the Clarke differential of \( v_i \) which is defined as

\[
\partial_c v_i(x) = \bigcap_{r > 0} \overline{\{Dv_i(y) \mid y \in B(x, r), \ v_i \text{ is differentiable at } y\}} \quad \text{for } x \in \mathbb{T}^n,
\]

where \( B(x, r) := \{y \in \mathbb{R}^n \mid |x - y| < r\} \), and for \( A \subset \mathbb{R}^n \), \( \overline{A} \) denotes the closed convex hull of \( A \).

**Proof.** Fix any \( i \in \{1, 2\} \). Let \( \rho \in C^\infty(\mathbb{R}^n) \) be a standard mollification kernel, i.e., \( \rho \geq 0 \), \( \text{supp } \rho \subset B(0, 1) \) and \( \int_{\mathbb{R}^n} \rho(x) \, dx = 1 \). Set \( \rho^k(x) := k^n \rho(kx) \) for \( k \in \mathbb{N} \), and

\[
\psi^k(i, t) = \psi^k(t) := (\rho^k \ast v_i)(\gamma(t)) \quad \text{and} \quad p^k(i, t) = p^k(t) := D(\rho^k \ast v_i)(\gamma(t))
\]

for all \( t \in (0, T) \). By the Itô formula for a jump process we have

\[
\mathbb{E}_i \left[ \psi^k(\nu(0), 0) - \psi^k(\nu(-t), -t) \right] = \mathbb{E}_i \left[ \int_{-t}^0 p^k(\nu(s), s) \cdot \dot{\gamma}(s) \, ds + \int_{-t}^0 \sum_{j=1}^2 (\psi^k(j, s) - \psi^k(\nu(s), s)) \, ds \right]
\]

Note that \( \psi^k_i \to v_i(\gamma(\cdot)) \) uniformly on \( [0, t] \) as \( k \to \infty \) and moreover passing to a subsequence if necessary, we may assume that for some \( p_i \in L^\infty((0, T), \mathbb{R}^n) \), \( p^k_i \rightharpoonup p_i \) weakly star in \( L^\infty((-t, 0)) \) as \( k \to \infty \), which implies (2.3).

It remains to show that \( p_i \in \partial_c v_i(\gamma(s)) \) for a.e. \( s \in (-t, 0) \). Since \( \{p^k_i\}_{k \in \mathbb{N}} \) is weakly convergent to \( p_i \) in \( L^2((-t, 0), \mathbb{R}^n) \), by the Mazur theorem, there is a sequence \( \{q^k_i\}_{k \in \mathbb{N}} \subset L^\infty((-t, 0), \mathbb{R}^n) \) such that

\[
q^k_i \rightharpoonup p_i \quad \text{strongly in } L^2((-t, 0), \mathbb{R}^n) \quad \text{as } k \to \infty, \quad q^k_i \in \text{co} \{p^j_i \mid j \geq k\} \tag{2.4}
\]

for all \( j \in \mathbb{N} \). We may thus assume by its subsequence if necessary that

\[
q^k_i(s) \to p_i(s) \quad \text{for a.e. } s \in (-t, 0) \quad \text{as } k \to \infty.
\]

Now, noting that \( D(\rho_k \ast v)(x) = \int_{y \in B(x, 1/k)} \rho_k(x - y) Dv_i(y) \, dy \) for any \( x \in \mathbb{T}^n \) and \( k \in \mathbb{N} \), we find that

\[
p^k_i(s) \in \overline{\cup_{y \in B(\gamma(s), 1/k)} Dv_i(y) \mid v_i \text{ is differentiable at } y}\]

for any \( s \in (-t, 0) \). Therefore,

\[
q^k_i(s) \in \overline{\cup_{y \in B(\gamma(s), 1/k)} Dv_i(y) \mid v_i \text{ is differentiable at } y}\]

for any \( s \in (-t, 0) \). Since \( q^k_i(s) \to p_i(s) \) for a.e. \( s \in (-t, 0) \) as \( k \to \infty \), we get

\[
p_i(s) \in \bigcap_{r > 0} \overline{\cup_{y \in B(\gamma(s), r)} Dv_i(y) \mid v_i \text{ is differentiable at } y}\]

for a.e. \( s \in (-t, 0) \). \( \square \)
Proof of Proposition 2.5. Let $\gamma \in AC([-t, 0])$ with $\gamma(0) = x$ and $p_i$ be the functions given by Lemma 2.6. In view of Lemma 2.6 we have

$$v_i(x) = \mathbb{E}_i \left[ \int_{-t}^{0} p_{\nu(s)}(s) \cdot \gamma(s) + \sum_{j=1}^{2} (v_{\nu(s)} - v_j)(\gamma(s)) \, ds + v_{\nu(-t)}(\gamma(-t)) \right]$$

$$\leq \mathbb{E}_i \left[ \int_{-t}^{0} H_{\nu(s)}(\gamma, p_{\nu(s)}) + L_{\nu(s)}(\gamma, \dot{\gamma}) + \sum_{j=1}^{2} (v_{\nu(s)} - v_j)(\gamma) \, ds + v_{\nu(-t)}(\gamma(-t)) \right]$$

$$\leq \mathbb{E}_i \left[ \int_{-t}^{0} L_{\nu(s)}(\gamma, \dot{\gamma}) \, ds + v_{\nu(-t)}(\gamma(-t)) \right].$$

3. Existence of Extremal Curves

Let $(v_1, v_2, 0)$ be a solution of (E). For any interval $[a, b] \subset (-\infty, 0]$, we denote by $\mathcal{E}([a, b], x, i, (v_1, v_2))$ the set of all curves $\gamma \in AC([a, b])$, which will be called an extremal curve on $[a, b]$ such that $\gamma(b) = x$ and for any $[c, d] \subset [a, b]$,

$$\mathbb{E}_i[v_{\nu(d)}(\gamma(d))] = \mathbb{E}_i \left[ \int_{c}^{d} L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds + v_{\nu(c)}(\gamma(c)) \right]$$

with a continuous-time Markov chain $\nu$ such that $\nu(0) = i$ and satisfies (1.2).

Theorem 3.1. Let $(v_1, v_2)$ be a solution of (E). Then $\mathcal{E}([-\infty, 0], x, i, (v_1, v_2)) \neq \emptyset$.

In order to avoid technical difficulties we make the following additional assumptions in this section which are not necessary to get Theorem 3.1 and Theorem 1.1. We refer the readers to [8, Section 6] for the detail of general settings.

(A3) $H_i \in C^2(\mathbb{T}^n \times \mathbb{R}^n)$ and there exists $\theta > 0$ such that $D_{pp}^2 H_i \geq \theta I$ for $i = 1, 2$, where $I$ is the unit matrix of size $n$.

(A4) There exists a constant $C > 0$ such that

$$\frac{1}{2C} |p|^2 - C \leq H_i(x, p) \leq \frac{C}{2} (|p|^2 + 1) \text{ for } x \in \mathbb{T}^n, \ p \in \mathbb{R}^n, \ i = 1, 2.$$

Note that in this case we can easily see that $L_i \in C^2(\mathbb{T}^n \times \mathbb{R}^n)$ are uniformly convex and satisfy

$$\frac{1}{2C} |p|^2 - C \leq L_i(x, p) \leq \frac{C}{2} (|p|^2 + 1) \text{ for } x \in \mathbb{T}^n, \ p \in \mathbb{R}^n, \ i = 1, 2. \ (3.1)$$

Lemma 3.2. Let $(v_1, v_2, 0)$ be a solution of (E). Then $\mathcal{E}([-1, 0], x, i, (v_1, v_2)) \neq \emptyset$.

Proof. By (2.2) there exists a sequence of curves $\{\gamma_k\} \subset AC([-1, 0])$ with $\gamma_k(0) = x$ such that

$$v_i(x) + \frac{1}{k} > \mathbb{E}_i \left[ \int_{-1}^{0} L_{\nu(s)}(\gamma_k(s), \dot{\gamma}_k(s)) \, ds + v_{\nu(-1)}(\gamma_k(-1)) \right].$$

Since $v_i$ are bounded, we have

$$\mathbb{E}_i \left[ \int_{-1}^{0} L_{\nu(s)}(\gamma_k(s), \dot{\gamma}_k(s)) \, ds \right] \leq C \text{ for some } C > 0. \ (3.2)$$
Combining (3.2) and (3.1), we deduce that $\|\dot{\gamma}_k\|_{L^2([-1,0])} \leq M$ for some $M > 0$. For any $-1 \leq a < b \leq 0$, we have

$$|\gamma(b) - \gamma(a)| \leq \int_a^b |\dot{\gamma}_k(s)| \, ds \leq \left[ \int_a^b |\dot{\gamma}_k(s)|^2 \, ds \right]^{1/2} \left[ \int_a^b 1 \, ds \right]^{1/2} \leq M|b-a|^{1/2}.$$ 

By the Arzela–Ascoli theorem and the weak compactness, by passing to a subsequence if necessary, $\{\gamma_k\}$ converges to $\gamma \in AC([-1,0])$ uniformly, and $\{\dot{\gamma}_k\}$ converges weakly to $\dot{\gamma}$ in $L^2(-1,0)$.

Now we prove that

$$\mathbb{E}_i \left[ \int_{-1}^0 L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds \right] \leq \liminf_{k \to \infty} \mathbb{E}_i \left[ \int_{-1}^0 L_{\nu(s)}(\gamma_k(s), \dot{\gamma}_k(s)) \, ds \right]. \quad (3.3)$$

This is a standard part in the theory of calculus of variations but let us present it here for the sake of clarity. The convexity of $L_i$ gives us that

$$L_i(\gamma_k(s), \dot{\gamma}_k(s)) \geq L_i(\gamma_k(s), \dot{\gamma}(s)) + D_qL_i(\gamma_k(s), \dot{\gamma}(s)) \cdot (\dot{\gamma}_k(s) - \dot{\gamma}(s))$$

$$= L_i(\gamma_k(s), \dot{\gamma}(s)) + [D_qL_i(\gamma_k(s), \dot{\gamma}(s)) - D_qL_i(\gamma_k(s), \dot{\gamma}(s))] \cdot (\dot{\gamma}_k(s) - \dot{\gamma}(s))$$

$$+ D_qL_i(\gamma(s), \dot{\gamma}(s)) \cdot (\dot{\gamma}_k(s) - \dot{\gamma}(s)).$$

Since $\gamma_k$ converges uniformly to $\gamma$, we employ the Lebesgue dominated convergence theorem to get that

$$\lim_{k \to \infty} \mathbb{E}_i \left[ \int_{-1}^0 L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds \right] = \mathbb{E}_i \left[ \int_{-1}^0 L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds \right]. \quad (3.4)$$

We use (3.1) again to yield that

$$|D_qL_i(x, q)| \leq C(|q| + 1) \text{ for } x \in \mathbb{T}^n, \; q \in \mathbb{R}^n, \; i = 1, 2.$$ 

It then straightforward by using the above and the Lebesgue dominated convergence theorem to see that

$$\lim_{k \to \infty} \mathbb{E}_i \left[ \int_{-1}^0 (D_qL_{\nu(s)}(\gamma_k(s), \dot{\gamma}(s)) - D_qL_{\nu(s)}(\gamma(s), \dot{\gamma}(s))) \cdot (\dot{\gamma}_k(s) - \dot{\gamma}(s)) \, ds \right] = 0 \quad (3.5)$$

Besides, the weak convergence of $\{\dot{\gamma}_k\}$ to $\dot{\gamma}$ in $L^2(-1,0)$ implies

$$\lim_{k \to \infty} \mathbb{E}_i \left[ \int_{-1}^0 D_qL_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \cdot (\dot{\gamma}_k(s) - \dot{\gamma}(s)) \, ds \right] = 0.$$ 

We combine (3.4), (3.5), and the above to get (3.3). Thus, $\gamma$ satisfies

$$v_i(x) \geq \mathbb{E}_i \left[ \int_{-1}^0 L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds + v_{\nu(-1)}(\gamma(-1)) \right]. \quad (3.6)$$
On the other hand, for any \(-1 \leq a < b \leq 0,\)

\[ v_i(x) \leq \mathbb{E}_i \left[ \int_b^0 L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds + v_{\nu(b)}(\gamma(b)) \right], \]

\[ \mathbb{E}_i[v_{\nu(b)}(\gamma(b))] \leq \mathbb{E}_i \left[ \int_a^b L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds + v_{\nu(a)}(\gamma(a)) \right], \]

\[ \mathbb{E}_i[v_{\nu(a)}(\gamma(a))] \leq \mathbb{E}_i \left[ \int_{-1}^a L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds + v_{\nu(-1)}(\gamma(-1)) \right]. \]

The above inequalities together with (3.6) yield the conclusion that \(\gamma \in \mathcal{E}([-1,0], x, i, (v_1, v_2)).\)

\[ \square \]

**Proof of Theorem 3.1.** Fix \(x \in \mathbb{T}^n\) and \(i \in \{1, 2\}.\) We define the sequence \(\{\gamma^k\}_{k \in \mathbb{N}} \subset \text{AC}([-k, -k+1])\) recursively as \(\gamma^k \in \mathcal{E}([-k, -k+1], x_{k-1}, i, (v_1, v_2))\), where \(x_k := \gamma^k(-k)\) and \(x_0 = x.\) Define the curve \(\gamma \in \text{AC}((-\infty, 0])\) by \(\gamma(s) = \gamma^k(s)\) for \(s \in [-k, -k+1]\) for \(k \in \mathbb{N}.\) Then it is clear to see that \(\gamma \in \mathcal{E}((-\infty, 0], x, i, (v_1, v_2)).\)

\[ \square \]

4. Stability on the Extremal Curves

In this section, we establish the following stability result, which plays a key role in the proof of Theorem 1.1.

**Theorem 4.1 (Scaling Result).** Let \((v_1, v_2, 0)\) be a solution of (E). For any \(\tau, T \in (0, \infty)\) with \(\tau < T\) such that \(\tau/(T - \tau) < \delta_0,\) where \(\delta_0\) appears in Lemma 4.3, and \(\gamma \in \mathcal{E}((-\infty, 0], x, i, (v_1, v_2)),\) we have

\[ u_i(x, T) - \mathbb{E}_i[u_{\mu(-T)}(\gamma(-T), \tau)] \]

\[ \leq v_i(x) - \mathbb{E}_i[v_{\mu(-T)}(\gamma(-T))] + (1 + \frac{\tau T}{T - \tau}) \omega(\frac{\tau}{T - \tau}) \]

(4.1)

for a function \(\omega : [0, \infty) \to [0, \infty)\) which is continuous and \(\omega(0) = 0.\)

**Lemma 4.2.** For any \(T > 0\) and \(\gamma \in \mathcal{E}((-\infty, 0], x, i, (v_1, v_2)).\) There exists \((p_1, p_2) \in L^\infty((-T, 0], \mathbb{R}^n)^2\) such that

\[ L_i(\gamma(t), \dot{\gamma}(t)) + H_i(\gamma(t), p_i(t)) = p_i(t) \cdot \dot{\gamma}(t), \]

\[ H_i(\gamma(t), p_i(t)) + v_i(\gamma(t)) - v_j(\gamma(t)) = 0, \text{ and } p_i(t) \in \partial_v v_i(\gamma(t)) \]

for a.e. \(t \in (-T, 0).\)

**Proof.** By Lemma 2.6 there exists \((p_1, p_2) \in L^\infty((-T, 0], \mathbb{R}^n)^2\) such that \(p_i(t) \in \partial_v v_i(\gamma(t))\) for a.e. \(t \in (-T, 0)\) and satisfies (2.5) in the proof of Proposition 2.5. Also, note that by the convexity of \(H_i\) and the definition of \(L_i,\) we have \(H_i(\gamma(t), p_i(t)) + v_i(\gamma(t)) - v_j(\gamma(t)) \leq 0\) and \(H_i(\gamma(t), p_i(t)) + L(\gamma(t), \dot{\gamma}(t)) \geq p_i(t) \cdot \dot{\gamma}(t)\) for a.e. \(t \in (-T, 0)\) and \(i = 1, 2.\) Since \(\gamma\) is an extremal curve, all inequalities above must become the equalities, which give the desired conclusion.

\[ \square \]
Lemma 4.3. Let \((v_1, v_2, 0)\) be a solution of \((E)\). There exists \(\delta_0 > 0\) such that for any \(\varepsilon \in [0, \delta_0]\) and \(\gamma \in \mathcal{E}((-\infty, 0], x, i, (v_1, v_2))\) we have

\[
L_i(\gamma(t), (1 + \varepsilon)\dot{\gamma}(t)) \leq (1 + \varepsilon)L_i(\gamma(t), \dot{\gamma}(t)) - \varepsilon(v_i - v_j)(\gamma(t)) + \varepsilon\omega(\varepsilon)
\]

for a function \(\omega : [0, \infty) \to [0, \infty)\) which is continuous and \(\omega(0) = 0\).

Proof. Let \((p_1, p_2)\) be the pair of functions given by Lemma 4.2. We notice that

\[
H_i(\gamma(t), p_i(t)) + H_j(\gamma(t), p_j(t)) = 0 \text{ for a.e. } t \in (-\infty, 0]
\]

by Lemma 4.2. Set

\[
Q := \{(x, p_1, p_2) \in \mathbb{T}^n \times \mathbb{R}^{2n} \mid H_1(x, p_1) + H_2(x, p_2) = 0\},
\]

\[
S := \{(x, q_1, q_2) \mid q_i \in D_p^* H_i(x, p_i) \text{ for some } (x, p_1, p_2) \in Q\}
\]

and then \(Q\) and \(S\) are compact in \(\mathbb{T}^n \times \mathbb{R}^{2n}\) in view of the coercivity of \(H_i\). We notice that \((\gamma(t), \dot{\gamma}(t), \dot{\gamma}(t))\) is in \(S\) for a.e. \(t \in (-\infty, 0)\) and thus \(|\dot{\gamma}(t)| \leq M\) for some \(M > 0\). We choose \(\delta_0 \in (0, 1)\) so that \((x, (1 + \varepsilon)\dot{\gamma}) \in \text{int}(\text{dom } L_1 \cap \text{dom } L_2)\) for all \(\varepsilon \in [0, \delta_0]\), where \(\text{dom } L_i := \{(x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n \mid L_i(x, \xi) < \infty\}\).

By Lemma 4.2,

\[
L_i(\gamma(t), \dot{\gamma}(t)) = p_i(t) \cdot \dot{\gamma}(t) - H_i(\gamma(t), p_i(t)) = D_q L_i(\gamma(t), \dot{\gamma}(t)) \cdot \dot{\gamma}(t) + (v_i - v_j)(\gamma(t)). \tag{4.2}
\]

Note that since \(H_i(x, \cdot)\) are strictly convex, \(D_q L_i(x, \xi)\) exists, and is continuous on \(\text{dom } L_i\).

Due to the mean value theorem and (4.2), there exists \(\theta_\varepsilon \in (0, 1)\) and a function \(\omega : [0, \infty) \to [0, \infty)\) which is continuous and \(\omega(0) = 0\) such that

\[
L_i(\gamma(t), (1 + \varepsilon)\dot{\gamma}(t)) = L_i(\gamma(t), \dot{\gamma}(t)) + \varepsilon D_q L_i(\gamma(t), (1 + \theta_\varepsilon \varepsilon)\dot{\gamma}(t)) \cdot \dot{\gamma}(t)
\]

\[
\leq L_i(\gamma(t), \dot{\gamma}(t)) + \varepsilon D_q L_i(\gamma(t), \dot{\gamma}(t)) \cdot \dot{\gamma}(t) + \varepsilon|\dot{\gamma}(t)|\omega(\varepsilon|\dot{\gamma}(t)|)
\]

\[
\leq (1 + \varepsilon)L_i(\gamma(t), \dot{\gamma}(t)) - \varepsilon(v_i - v_j)(\gamma(t)) + \varepsilon\tilde{\omega}(\varepsilon),
\]

where we set \(\tilde{\omega}(r) := M \max_{s \in [0, M\varepsilon]} \omega(s)\).

Remark 4.4. We notice that the result of Lemma 4.3 is different from the similar one for single equations (see [8, Lemma 7.2] for details). More precisely, the natural appearance of the coupling terms \(-\varepsilon(v_i - v_j)(\gamma(t))\) makes the analysis for weakly coupled systems more difficult. We could not proceed to establish large time behavior results in a crude way. It turns out that the weights \((1/2)(1 + e^{2t})\) and \((1/2)(1 - e^{2t})\) for \(t < 0\) are the key factors helping us overcome this difficulty as in the proof of Theorem 4.1 below.

Proof of Theorem 4.1. Set \(\varepsilon := \tau/(T - \tau)\) and \(T_\varepsilon := T/(1 + \varepsilon)\). Notice that \(T = T_\varepsilon + \varepsilon T_\varepsilon = T_\varepsilon + \tau\). We have

\[
u_i(x, T) = u_i(\gamma(0), T) = u_i(\gamma(0), T_\varepsilon + \tau)
\]

\[
= \inf \left\{ \mathbb{E}_i \left[ \int_{-T_\varepsilon}^{0} L_\nu(s)(\eta(s), \dot{\eta}(s)) \, ds + u_{\nu(-T_\varepsilon)}(\eta(-T_\varepsilon), \tau) \right] \mid \eta \in \text{AC}([-T_\varepsilon, 0]) \text{ with } \eta(0) = x \right\}.
\]
Take $\gamma \in \mathcal{E}((-\infty, 0], x, i, (v_1, v_2))$ and set $\eta(s) := \gamma((1 + \varepsilon)s)$ to derive that

$$u_i(x, T) \leq \mathbb{E}_i \left[ \int_{-T}^{0} L_{\nu(s)}(\gamma((1 + \varepsilon)s), (1 + \varepsilon)\dot{\gamma}((1 + \varepsilon)s)) \, ds + u_{\nu(-T)}(\gamma(-T), \tau) \right].$$

Make the change of variable $t = (1 + \varepsilon)s$ and use Lemma 2.2 to get

$$u_i(x, T) \leq \mathbb{E}_i \left[ \int_{-T}^{0} \frac{1}{1 + \varepsilon} L_{\nu(t/(1+\varepsilon))}(\gamma(t), (1 + \varepsilon)\dot{\gamma}(t)) \, dt + u_{\nu(-T)}(\gamma(-T), \tau) \right]$$

$$= \int_{-T}^{0} \frac{1 + e^{2t/(1+\varepsilon)}}{2(1 + \varepsilon)} L_i(\gamma(t), (1 + \varepsilon)\dot{\gamma}(t)) \, dt + \int_{-T}^{0} \frac{1 - e^{2t/(1+\varepsilon)}}{2(1 + \varepsilon)} L_j(\gamma(t), (1 + \varepsilon)\dot{\gamma}(t)) \, dt$$

$$+ \mathbb{E}_i [u_{\nu(-T)}(\gamma(-T), \tau)].$$

We use Lemma 4.3 in the above inequality to deduce

$$u_i(x, T) - \mathbb{E}_i [u_{\nu(-T)}(\gamma(-T), \tau)]$$

$$\leq \int_{-T}^{0} \frac{1}{2}(1 + e^{2t/(1+\varepsilon)}) L_i(\gamma(t), \dot{\gamma}(t)) + \frac{1}{2}(1 - e^{2t/(1+\varepsilon)}) L_j(\gamma(t), \dot{\gamma}(t)) \, dt$$

$$+ \frac{\varepsilon}{1 + \varepsilon} \int_{-T}^{0} e^{2t/(1+\varepsilon)} (v_j - v_i)(\gamma(t)) \, dt + T\varepsilon \omega(\varepsilon).$$

We use the fact that $v_j - v_i$ is bounded in $\mathbb{T}^n$ to derive that

$$\left| \frac{\varepsilon}{1 + \varepsilon} \int_{-T}^{0} e^{2t/(1+\varepsilon)} (v_j - v_i)(\gamma(t)) \, dt \right| \leq C\varepsilon \int_{-T}^{0} e^{2t/(1+\varepsilon)} \, dt \leq C\varepsilon.$$

Furthermore, for $t < 0$, $|e^{2t/(1+\varepsilon)} - e^{2t}| \leq -2t\varepsilon e^{2t/(1+\varepsilon)}$. This together with the facts that $u_i$ are bounded and $|\dot{\gamma}(t)| \leq M$ imply

$$\mathbb{E}_i [u_{\nu(-T)}(\gamma(-T), \tau)] \leq \mathbb{E}_i [u_{\nu(\cdot)}(\gamma(-T), \tau)] + C\varepsilon,$$

and

$$\left| \int_{-T}^{0} (e^{2t/(1+\varepsilon)} - e^{2t}) L_k(\gamma(t), \dot{\gamma}(t)) \, dt \right| \leq -C_1\varepsilon \int_{-T}^{0} t e^{2t/(1+\varepsilon)} \, dt \leq C_2\varepsilon,$$

for $k = 1, 2$ and $C_1, C_2 > 0$ independent of $\varepsilon$.

Summing up everything, we obtain

$$u_i(x, T) - \mathbb{E}_i [u_{\nu(\cdot)}(\gamma(-T), \tau)]$$

$$\leq \int_{-T}^{0} \left[ \frac{1}{2}(1 + e^{2t}) L_i(\gamma(t), \dot{\gamma}(t)) + \frac{1}{2}(1 - e^{2t}) L_j(\gamma(t), \dot{\gamma}(t)) \right] \, dt + C\varepsilon + T\varepsilon \omega(\varepsilon)$$

$$= v_i(x) - \mathbb{E}_i [v_{\nu(\cdot)}(\gamma(-T))] + C \frac{\tau}{T - \tau} + \frac{\tau T}{T - \tau} \omega\left( \frac{\tau}{T - \tau} \right),$$

which is the desired conclusion. \qed

**Remark 4.5.** The new representation formula (2.1) with the weights $(1/2)(1 + e^{2t})$ and $(1/2)(1 - e^{2t})$ for $t < 0$ appears naturally in both the statement and the proof of Theorem 4.1 pointing out a major difference between single equations and weakly coupled systems. With new representation formula (2.1), we could explicitly calculate (4.4) and (4.5) and thus identify the main obstacle coming from the coupling term, the second last term in
result. As mentioned in Remark 4.4, we could not estimate the coupling term in a crude way. For instance, in (4.3) we can easily see by Lemma 4.3 that

\[
\mathbb{E}_t \left[ \int_{-T}^{0} \frac{1}{1 + \varepsilon} L_{\nu(t/(1+\varepsilon))}(\gamma(t), (1 + \varepsilon)\dot{\gamma}(t)) \, dt \right] \\
\leq \mathbb{E}_t \left[ \int_{-T}^{0} L_{\nu(t/(1+\varepsilon))}(\gamma(t), \dot{\gamma}(t)) - \varepsilon \frac{1}{1 + \varepsilon} (v_{\nu(t/(1+\varepsilon))} - v_{3-\nu(t/(1+\varepsilon))}) \, dt \right] + \varepsilon T \omega(\varepsilon) \\
\leq \mathbb{E}_t \left[ \int_{-T}^{0} L_{\nu(t/(1+\varepsilon))}(\gamma(t), \dot{\gamma}(t)) \, dt \right] + \varepsilon T \omega(\varepsilon) + CT\varepsilon/(1+\varepsilon)
\]

by using the fact that \(\|v_1 - v_2\|_{L^\infty(\mathbb{T}^n)} \leq C\). But the last term in the above, which is of order \(O(\tau)\) and does not vanish as \(\varepsilon \to 0\), is not enough to get the large-time asymptotics as we can see in the proof of Theorem 1.1. It turns out that the weights played an essential role here and helped us in establishing the key estimate (4.6) leading to the large time behavior result.

5. The Proof of Convergence

We define the functions \(u_{g_i}^{-}\) and \(u_{g_i}^{+}\) \((i = 1, 2)\) by

\[
u_{g_i}^{-}(x) := \sup_{\phi_i(x)} \{\phi_i(x) | (\phi_i, \phi_j) \text{ is a subsolution of (E) with } \phi_i \leq g_i, \phi_j \leq g_j\}
\]

\[
u_{g_i}^{+}(x) := \inf_{\psi_i(x)} \{\psi_i(x) | (\psi_i, \psi_j) \text{ is a solution of (E) with } \psi_i \geq u_{g_i}^{-}, \psi_j \geq u_{g_j}^{-}\},
\]

where we take \(j\) so that \(\{i, j\} = \{1, 2\}\). Notice first that the pair \((u_{g_i}^{-}, u_{g_i}^{+})\) is a solution of (E) because of the convexity of the Hamiltonians \(H_i\) for \(i = 1, 2\). Let \((T_t)_{t \geq 0}\) be the semigroup defined by \((T_t g_i)(x) := u_i(x, t)\), where \((u_1, u_2)\) is the unique solution of (C) with a given initial data \((g_1, g_2)\).

Lemma 5.1. We have \((T_t u_{g_i}^{-})(x) = \inf_{s \geq t} u_i(x, s)\), and \(u_{g_i}^{+}(x) = \lim \inf_{s \to \infty} u_i(x, s)\) for \(x \in \mathbb{T}^n\) and \(i = 1, 2\).

Remark 5.2. In view of the uniform continuity of \(u_i\) on \(\mathbb{T}^n \times [0, \infty)\), we have \(\lim \inf_{t \to \infty} u_i(x, t) = \lim \inf_{t \to \infty} u_i(x, t) := \lim_{r \to 0} \inf \{u_i(y, s) | ||x - y|| \leq r, s \geq 1/r\}\) for all \(x \in \mathbb{T}^n\).

Proof. Let \(w_i(x, t) := \inf_{s \geq t} u_i(x, s)\) for \(i = 1, 2\) and \((x, t) \in \mathbb{T}^n \times [0, \infty)\). Noting that \((u_{g_1}^{-}, u_{g_2}^{-})\) is a subsolution of (E) and \(u_{g_i}^{-} \leq g_i\), we get by the comparison principle

\[
u_{g_i}^{-}(x) \leq (T_s g_i^-)(x) \leq (T_s g_i^+)(x) = u_i(x, s).
\]

We use the comparison principle again to deduce that

\[
u_{g_i}^{-}(x) \leq (T_t u_{g_i}^{-})(x, s) = u_i(x, t + s) \text{ for all } s \geq 0.
\]

Hence \((T_t u_{g_i}^{-})(x) \leq \inf_{s \geq t} u_i(x, s) = w_i(x, t)\) for all \((x, t) \in \mathbb{T}^n \times [0, \infty)\).

On the other hand, \((w_1, w_2)\) is a solution of (C) by the convexity of \(H_i\). Moreover, \(w_i\) are increasing in the \(t\)-variable for \(i = 1, 2\) by definition, which give us in addition that \((w_1(\cdot, t), w_2(\cdot, t))\) is a subsolution of (E) for all \(t \geq 0\). In particular, \((w_1(\cdot, 0), w_2(\cdot, 0))\) is a subsolution of (E) with \(w_i \leq g_i\), which implies \(w_i \leq u_{g_i}^{-}\). Thus, the comparison principle yields \(w_i(\cdot, t) \leq (T_t u_{g_i}^{-})(\cdot)\) on \(\mathbb{T}^n\) for all \(t \geq 0\).
Next, note that
\[
\liminf_{s \to \infty} u_i(x, s) = \lim_{t \to \infty} (T_t u_{g_{i}})(x) \leq \lim_{t \to \infty} (T_t u_{i})(x) = u_i(x) \text{ for } i = 1, 2, \ x \in \mathbb{T}^n.
\]
Due to the fact that \((T_t u_{g_{i}})(x) = w_i(x, t)\) are increasing in the \(t\)-variable for \(i = 1, 2\), we see that \((\lim_{t \to \infty} w_1(\cdot, t), \lim_{t \to \infty} w_2(\cdot, t))\) is a solution of (E). Therefore, by the definition of \((u_{1, u_{2}})\) we deduce that \(\lim_{t \to \infty} w_i(\cdot, t) \geq u_i \) on \(\mathbb{T}^n\).

**Proof of Theorem 1.1.** Set \(\overline{u}_i(x) := \limsup_{t \to \infty} u_i(x, t)\). By Lemma (5.1) we have \(\underline{u}_i \leq \overline{u}_i\) on \(\mathbb{T}^n\). We now prove that \(\overline{u}_i = \underline{u}_i\) on \(\mathbb{T}^n\) for \(i = 1, 2\).

Were the above statement false, there would exist a point \(x \in \mathbb{T}^n\) such that
\[
\max \max_{i=1,2} (\overline{u}_i - \underline{u}_i)(z) = (\overline{u}_1 - \underline{u}_1)(x) =: \alpha > 0.
\]
Take \(\gamma \in \mathcal{E}((\infty, 0], x, i, (\underline{u}_1, \underline{u}_2))\). We can choose a sequence \(\{T_m\} \subset (0, \infty)\) converging to \(\infty\) such that \(\lim_{m \to \infty} u_i(x, T_m) = \overline{u}_i(x) > \underline{u}_i(x)\). Without loss of generality, we assume further that \(\gamma(-T_m) \to y \in \mathbb{T}^n\) as \(m \to \infty\).

Fix \(\delta > 0\) and choose \(\tau > 0\) large enough such that \(u_i(y, \tau) < \underline{u}_i(y) + \alpha/2\) and \(u_j(y, \tau) < \overline{u}_i(y) + \alpha/2\). Notice that the first constraint requires \(\tau\) to be specific while the second constraint only requires \(\tau\) to be large enough. We apply Theorem 4.1 to get
\[
u_i(x, T_m) - \left\{ \frac{1}{2}(1 + e^{-2T_m}) u_i(\gamma(-T_m), \tau) + \frac{1}{2}(1 - e^{-2T_m}) u_j(\gamma(-T_m), \tau) \right\}
\]
\[
\leq \underline{u}_i(x) - \left\{ \frac{1}{2}(1 + e^{-2T_m}) \underline{u}_i(\gamma(-T_m)) + \frac{1}{2}(1 - e^{-2T_m}) \underline{u}_j(\gamma(-T_m)) \right\} + \left(1 + \frac{\tau T_m}{T_m - \tau}\right) \omega(\frac{\tau}{T_m - \tau})
\]
for \(m\) large enough. Let \(m \to \infty\) in the above inequality to yield
\[
\overline{u}_i(x) - \frac{1}{2}(u_i(y, \tau) + u_j(y, \tau)) \leq \underline{u}_i(x) - \frac{1}{2}(\underline{u}_i(y) + \underline{u}_j(y)),
\]
which contradicts the choice of \(\tau\). This finishes the proof. \(\square\)

6. **Appendix**

Let \(u_i\) be the value function associated with (1.1), or equivalently the function defined by the right hand side of (2.1).

**Proposition 6.1** (Dynamic Programming Principle). For any \(x \in \mathbb{R}^n\), \(0 \leq h \leq t\) and \(i = 1, 2\), we have
\[
u_i(x, t) = \inf \left\{ \int_{-h}^{0} \frac{1}{2}(1 + e^{2s}) L_i(\gamma(s), \dot{\gamma}(s)) ds + \frac{1}{2}(1 + e^{-2h}) u_i(\gamma(-h), t - h) \right.
\]
\[
+ \int_{-h}^{0} \frac{1}{2}(1 - e^{2s}) L_j(\gamma(s), \dot{\gamma}(s)) ds + \frac{1}{2}(1 - e^{-2h}) u_j(\gamma(-h), t - h) \mid \gamma \in \text{AC}([-h, 0]), \gamma(0) = x \right\},
\]
\[
(6.1)
\]

**Proof.** We denote by \(w_i(x, t, h)\) the right hand side of (6.1). For any \(\gamma \in \text{AC}([-t, 0])\) with \(\gamma(0) = x\), set \(\eta(s) = \gamma(s + h)\) for \(s \in [-t + h, 0]\). Note that for \(s < 0\),
\[
\frac{1}{2}(1 \pm e^{2(s-h)}) = \frac{1}{2}(1 + e^{-2h}) \cdot \frac{1}{2}(1 \pm e^{2s}) + \frac{1}{2}(1 - e^{-2h}) \cdot \frac{1}{2}(1 \mp e^{2s}),
\]
which actually comes from the memoryless property of Markov processes. We have

\[
\int_{-t}^{0} \frac{1}{2}(1 + e^{2s})L_i(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2}(1 + e^{-2t})g_i(\gamma(-t)) \\
+ \int_{-t}^{0} \frac{1}{2}(1 - e^{2s})L_j(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2}(1 - e^{-2t})g_j(\gamma(-t)) \\
= \int_{-h}^{0} \frac{1}{2}(1 + e^{2s})L_i(\gamma(s), \dot{\gamma}(s)) \, ds + \int_{-h}^{0} \frac{1}{2}(1 - e^{2s})L_j(\gamma(s), \dot{\gamma}(s)) \, ds \\
+ \frac{1}{2}(1 + e^{-2h}) \left[ \int_{-t+h}^{0} \frac{1}{2}(1 + e^{2s})L_i(\eta(s), \dot{\eta}(s)) \, ds + \frac{1}{2}(1 + e^{-2(t-h)})g_i(\eta(-t + h)) \right] \\
+ \frac{1}{2}(1 - e^{-2h}) \left[ \int_{-t+h}^{0} \frac{1}{2}(1 - e^{2s})L_j(\eta(s), \dot{\eta}(s)) \, ds + \frac{1}{2}(1 - e^{-2(t-h)})g_j(\eta(-t + h)) \right] \\
\geq \int_{-h}^{0} \frac{1}{2}(1 + e^{2s})L_i(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2}(1 + e^{-2h})u_i(\gamma(-h), t - h) \\
+ \int_{-h}^{0} \frac{1}{2}(1 - e^{2s})L_j(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2}(1 - e^{-2h})u_j(\gamma(-h), t - h),
\]

which implies \( u_i(x, t) \geq w_i(x, t, h) \) for \( i = 1, 2 \).

We also can prove the other inequalities by a similar way. Thus, we omit the details. \( \square \)

**Proposition 6.2.** The functions \( u_i \) are continuous on \( \mathbb{T}^n \times [0, \infty) \).

**Proof.** We first prove that \( u_i \) are Lipschitz continuous on \( \mathbb{T}^n \times [0, \infty) \) under the additional assumption that \( g_i \) are Lipschitz continuous on \( \mathbb{T}^n \). This additional requirement on \( g_i \) will be removed at the end of the proof.

We may choose a constant \( M_1 > 0 \) so that \( H_i(x, Dg_i(x)) + (g_i - g_j)(x) \leq M_1 \) for a.e. \( x \in \mathbb{T}^n \). It is clear that the function \( v_i(x, t) := g_i(x) - M_1t \) on \( \mathbb{T}^n \times [0, \infty) \) is a subsolution of (C).

By a similar argument to the proof of Proposition 2.5 we obtain

\[
v_i(x, t) \leq \mathbb{E}_x \left[ \int_{-t}^{0} L_{\nu(s)}(\gamma(s), \dot{\gamma}(s)) \, ds + v_{\nu(-t)}(\gamma(-t), 0) \right]
\]

for all \( (x, t) \in \mathbb{T}^n \times [0, \infty) \) and \( \gamma \in \text{AC}([-t, 0]) \) with \( \gamma(0) = x \), from which we get \( g_i(x) - M_1t \leq u_i(x, t) \) for all \( (x, t) \in \mathbb{T}^n \times [0, \infty) \).

It follows from (2.1) that

\[
u_i(x, t) \leq \mathbb{E}_x \left[ \int_{-t}^{0} L_{\nu(s)}(x, 0) \, ds + g_{\nu(-t)}(x) \right] \leq g_i(x) + C_1t
\]
for $C_1 := \max\{M_1, \max_{i=1,2;x \in \mathbb{T}^n} |L_i(x, 0)| + \max_{x \in \mathbb{T}^n} |g_1(x) - g_2(x)|\}$, and all $(x, t) \in \mathbb{T}^n \times [0, \infty)$. Therefore we get
\[
|u_i(x, t) - g_i(x)| \leq C_1 t \text{ for all } (x, t) \in \mathbb{T}^n \times [0, \infty). \tag{6.2}
\]

Now, for any $(x, t) \in \mathbb{T}^n \times (0, \infty)$ and $h > 0$, by Dynamic Programming Principle (6.1),
\[
u_i(x, t + h) = \inf \left\{ \int_{-t}^{0} \frac{1}{2}(1 + e^{2s})L_i(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2}(1 + e^{-2t})u_i(\gamma(-t), h)
+ \int_{-t}^{0} \frac{1}{2}(1 - e^{2s})L_j(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2}(1 - e^{-2t})u_j(\gamma(-t), h) \mid \gamma \in AC([-t, 0]), \gamma(0) = x \right\}.
\]

By (6.2), $|u_i(\gamma(-t), h) - g_i(\gamma(-t), h)| \leq C_1 h$ for $i = 1, 2$. Hence, we derive that
\[
|u_i(x, t + h) - u_i(x, t)| \leq C_1 h. \tag{6.3}
\]

We next prove that $u_i$ are Lipschitz continuous in $x$ for $i = 1, 2$. Fix $x, y \in \mathbb{T}^n$ with $x \neq y$ and $t > 0$. In view of the coercivity of $H_i$, there exists a constant $\rho > 0$ such that $L_i(x, \xi) \leq C$ for all $x \in \mathbb{T}^n$ and $\xi \in B(0, \rho)$ (see [8, Proposition 2.1]). Set $\tau := |x - y|/\rho$ and we first consider the case where $\tau < t$. Set $\eta(s) := y - s\rho(x - y)/|x - y|$ for $s \in [-\tau, 0]$. Note that $\eta \in AC([-t, 0]), \eta(0) = y$ and $\eta(-\tau) = x$. By Dynamic Programming Principle (6.1), (6.2) and (6.3),
\[
u_i(y, t) \leq \mathbb{E}_i \left[ \int_{-\tau}^{0} L_{\nu}(\eta(s), \dot{\eta}(s)) \, ds + u_{\nu(-\tau)}(\eta(-\tau), t - \tau) \right]
\leq C\tau + \frac{1}{2}(1 + e^{-2\tau})u_i(x, t - \tau) + \frac{1}{2}(1 - e^{-2\tau})u_j(x, t - \tau)
\leq (C + 2C_1)\tau + u_i(x, t) \leq C|x - y| + u_i(x, t),
\]

By symmetry we conclude $|u_i(x, t) - u_i(y, t)| \leq C|x - y|$, where $C$ depends on $t$ as calculated above. Notice that this is just a fairly crude estimate, but it is good enough for our presentation here.

We consider the case where $t \leq \tau$. By (6.2),
\[
|u_i(x, t) - u_i(y, t)| \leq |u_i(x, t) - g_i(x)| + |g_i(x) - g_i(y)| + |g_i(y) - u_i(y, t)|
\leq 2Ct + M|x - y| \leq C|x - y|,
\]

where $M := \max_{i=1,2} \|Dg_i\|_{L^\infty(\mathbb{T}^n)}$. Thus, we get $|u_i(x, t) - u_i(y, t)| \leq C|x - y|$ for all $x, y \in \mathbb{T}^n, t \geq 0$ and $i = 1, 2$.

We finally remark that we can deduce the continuity of $u_i$ by using an approximation argument. We may choose a sequence $\{g_k^i\}_{k \in \mathbb{N}}$ of Lipschitz continuous functions so that $\|g_k^i - g_i\|_{L^\infty(\mathbb{T}^n)} \leq 1/k$ for all $k \in \mathbb{N}$. Let $u_k^i$ be functions defined by (2.1) with given $g_k^i$. By comparison through the formulas for $u_i$ and $u_k^i$, we see that $|u_i(x, t) - u_k^i(x, t)| \leq \max_{\mathbb{T}^n} |g_i - g_k^i|$. Since $u_k^i \in C(\mathbb{T}^n \times (0, \infty))$ by the above argument for all $k \in \mathbb{N}$ and $u_k^i$ converges uniformly to $u_i$ on $\mathbb{T}^n \times [0, \infty)$, we obtain $u_i \in C(\mathbb{T}^n \times [0, \infty))$.

\[\square\]

_Proof of Theorem 2.4._ It is clear that $(u_1, u_2)(\cdot, 0) = (g_1, g_2)$ on $\mathbb{T}^n$. We now prove that $u_1$ is a subsolution of (C). Take a test function $\phi \in C^1(\mathbb{T}^n \times (0, \infty))$ such that $u_1 - \phi$ has
a maximum at \((x_0, t_0) \in \mathbb{T}^n \times (0, \infty)\) and \((u_1 - \phi)(x_0, t_0) = 0\). Take \(h > 0\) small enough. By Proposition 6.1,

\[
\begin{align*}
u_1(x_0, t_0) &\leq \int_{-h}^{0} \frac{1}{2} (1 + e^{2s})L_1(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2} (1 + e^{-2h})u_1(\gamma(-h), t_0 - h) \\
+ &\int_{-h}^{0} \frac{1}{2} (1 - e^{2s})L_2(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1}{2} (1 - e^{-2h})u_2(\gamma(-h), t_0 - h)
\end{align*}
\]

for any \(\gamma \in AC([-h, 0])\) with \(\gamma(0) = x_0\) and \(\dot{\gamma}(0) = q \in \mathbb{R}^n\). We now use the fact \(u_1(\gamma(-h), t_0 - h) \leq \phi(\gamma(-h), t_0 - h)\) to plug into the above to derive that

\[
\frac{\phi(\gamma(0), t_0) - \phi(\gamma(-h), t_0 - h)}{h} \leq \frac{1}{h} \int_{-h}^{0} \frac{1}{2} (1 + e^{2s})L_1(\gamma(s), \dot{\gamma}(s)) \, ds \\
+ \frac{1}{h} \int_{-h}^{0} \frac{1}{2} (1 - e^{2s})L_2(\gamma(s), \dot{\gamma}(s)) \, ds + \frac{1 - e^{-2h}}{2h}(u_2 - u_1)(\gamma(-h), t_0 - h).
\]

Sending \(h \to 0\), we obtain

\[
\phi_t(x_0, t_0) + D\phi(x_0, t_0) \cdot q \leq L_1(x_0, q) + (u_2 - u_1)(x_0, t_0) \text{ for all } q \in \mathbb{R}^n,
\]

which implies \(\phi_t(x_0, t_0) + H_1(x_0, D\phi(x_0, t_0)) + (u_1 - u_2)(x_0, t_0) \leq 0\).

Next we prove that \(u_1\) is a supersolution of (C). Take a test function \(\phi \in C^1(\mathbb{T}^n \times (0, \infty))\) such that \(u_1 - \phi\) has a minimum at \((x_0, t_0) \in \mathbb{T}^n \times (0, \infty)\) and \((u_1 - \phi)(x_0, t_0) = 0\). Take \(h > 0\) small enough. By Proposition 6.1, there exists \(\gamma_h \in AC([-h, 0])\) with \(\gamma_h(0) = x_0\) such that

\[
u_1(x_0, t_0) + h^2 \geq \int_{-h}^{0} \frac{1}{2} (1 + e^{2s})L_1(\gamma_h(s), \dot{\gamma}_h(s)) \, ds + \frac{1}{2} (1 + e^{-2h})u_1(\gamma_h(-h), t_0 - h) \\
+ \int_{-h}^{0} \frac{1}{2} (1 - e^{2s})L_2(\gamma_h(s), \dot{\gamma}_h(s)) \, ds + \frac{1}{2} (1 - e^{-2h})u_2(\gamma_h(-h), t_0 - h).
\]

We use \(u_1(\gamma_h(-h), t_0 - h) \geq \phi(\gamma_h(-h), t_0 - h)\) in the above to yield

\[
\frac{\phi(\gamma_h(0), t_0) - \phi(\gamma_h(-h), t_0 - h)}{h} + h \geq \frac{1}{h} \int_{-h}^{0} \frac{1}{2} (1 + e^{2s})L_1(\gamma_h(s), \dot{\gamma}_h(s)) \, ds \\
+ \frac{1}{h} \int_{-h}^{0} \frac{1}{2} (1 - e^{2s})L_2(\gamma_h(s), \dot{\gamma}_h(s)) \, ds + \frac{1 - e^{-2h}}{2h}(u_2 - u_1)(\gamma(-h), t_0 - h). \tag{6.4}
\]
On the other hand,

\[
\frac{1}{h} \int_{-h}^{0} \phi_t(\gamma_h(s), t_0 - s) \, ds
\]

\[
\leq \frac{1}{h} \int_{-h}^{0} \phi_t(\gamma_h(s), t_0 - s) \, ds
\]

\[
+ \frac{1}{h} \int_{-h}^{0} \frac{1}{2} (1 + e^{2s}) \left\{ L_1(\gamma_h(s), \dot{\gamma}_h(s)) + H_1(\gamma_h(s), D\phi(\gamma_h(s), t_0 - s)) \right\} \, ds
\]

\[
+ \frac{1}{h} \int_{-h}^{0} \frac{1}{2} (1 - e^{2s}) \left\{ L_2(\gamma_h(s), \dot{\gamma}_h(s)) + H_2(\gamma_h(s), D\phi(\gamma_h(s), t_0 - s)) \right\} \, ds.
\]

(6.5)

Combine (6.4), (6.5) and then send \( h \to 0 \) to get

\[
\phi_t(x_0, t_0) + H_1(x_0, D\phi(x_0, t_0)) + (u_1 - u_2)(x_0, t_0) \geq 0.
\]

It is easy to see the uniform continuity of \( u_i \) due to the coercivity of Hamiltonians. \( \square \)

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LARGE TIME BEHAVIOR OF SOLUTIONS TO WEAKLY COUPLED SYSTEMS

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