Domain Wall Junctions are 1/4-BPS States

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Abstract

We study $N = 1$ SUSY theories in four dimensions with multiple discrete vacua, which admit solitonic solutions describing segments of domain walls meeting at one-dimensional junctions. We show that there exist solutions preserving one quarter of the underlying supersymmetry – a single Hermitian supercharge. We derive a BPS bound for the masses of these solutions and construct a solution explicitly in a special case. The relevance to the confining phase of $N = 1$ SUSY Yang-Mills and the M-theory/SYM relationship is discussed.

\[\text{CWRU-P21-99}\]  \hspace{1cm} \textbf{hep-th/9905217}

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I. INTRODUCTION

Duality has played a fundamental role in recent progress in understanding quantum field theories at strong coupling. One of the most indispensable tools in formulating and providing evidence for duality conjectures for supersymmetric theories has been the existence of states preserving some but not all of the underlying supersymmetry of the Hamiltonian. These states, referred to as BPS states, are useful because they lie in shortened multiplets of the supersymmetry algebra, and therefore they cannot disappear or appear as parameters of the theory, such as the coupling constant, are varied continuously. The spectrum of BPS states is thus one of the few characteristics of a quantum field theory that can be predicted easily at strong coupling.

A celebrated application of this tool has been to gauge theories in four dimensions with extended ($N = 2, 4$) supersymmetry. These theories feature particle-like solitons — magnetic monopoles — that preserve half the supersymmetry of the Hamiltonian, lie in short multiplets, and therefore can be followed to strong coupling, where they become the fundamental excitations of a dual, weakly coupled theory.

In $N = 1$ theories in four dimensions, the situation is different. The SUSY algebra forbids a rotationally invariant central charge and thus massive zero-dimensional objects, such as magnetic monopoles, cannot lie in short representations of the algebra. However, it has been noted that the $N = 1$ SUSY algebra in 4D can admit a central extension if the central charges transform nontrivially under the rotation group [1,2]. Consequently, in such theories there can exist states which are extended objects preserving half the supersymmetry — two Hermitian supercharges. These objects are domain walls separating two of a set of disconnected vacua.

In this paper, we show that, if an $N = 1$ theory has three or more mutually disconnected vacua, there exist states preserving a quarter of the underlying supersymmetry — a single Hermitian supercharge. These states are junctions of BPS domain walls. Networks of intersecting walls have been studied in a cosmological context [3], and theories supporting topological defects ending on other defects of various dimensions (including models of walls ending on walls) have also been constructed [4]. In addition, reference [5] contains a general discussion of domain wall intersections in $N = 1$ theories.

The organization of the paper is as follows. In section II, we give a brief review of BPS domain walls in $N = 1$ theories. In section III, we find the BPS equations satisfied by a soliton preserving a quarter of the underlying supersymmetry. In sections IV and V we use...
the BPS equations to derive some constraints on the kinematics of these solutions, and in section [VI] we use what we’ve learned to construct solutions. In section [VII] we then apply these considerations to the particular case of $N = 1$ supersymmetric Yang-Mills theory, before concluding and discussing open questions in section [VIII].

As this work was being completed, we became aware of a related paper by Gibbons and Townsend [6], in which they also argue on general grounds for the existence of BPS wall junctions preserving 1/4 of the $N = 1$ supersymmetry.

II. SUPERSYMMETRIC DOMAIN WALLS

To understand the origin of central charges in $N = 1$ theories with only chiral superfields, consider the theory with superpotential

$$W = \Lambda^2 \Phi - \Lambda^{2-n} \frac{\Phi^{n+1}}{n+1},$$

(2.1)

where $n$ is an integer $\geq 2$. The anticommutator $\{Q_\alpha, Q_\beta\}$ of two supercharges in this theory doesn’t automatically vanish, but, for a static configuration, is proportional to the integral of the total derivative

$$\epsilon_{\alpha \gamma} (\sigma_0 \bar{\sigma}^a)_{\beta}^\gamma \partial_a W(\phi)^*,$$

where $a$ runs over spacelike indices and $\phi$ is the scalar component of the superfield $\Phi$. The supersymmetry algebra therefore closes on a non-scalar central charge $Z_a$, which is proportional to the change in the value of the superpotential between spatial infinities in different directions. This extension of the SUSY algebra [1][2] is relevant to the confining phase of pure supersymmetric gluodynamics in 4 dimensions, where it has been shown [3] that, for the gauge group $SU(n)$, the theory has $n$ distinct vacua corresponding to distinct values of the superpotential. It was shown in [4] that this central charge allowed the existence of 1/2-supersymmetric domain walls, where the order parameter which changes across the wall is essentially the expectation value of the gluino condensate.

More concretely, consider the full SUSY algebra

$$\{Q_\alpha, \bar{Q}_\bar{\beta}\} = 2 \sigma^{\mu}_{\alpha \beta} P_\mu,$$

(2.2)

$$\{Q_\alpha, Q_\beta\} = -2Z^a \epsilon_{\alpha \gamma} (\sigma_0 \sigma^a)_{\beta}^\gamma,$$

(2.3)

where $P_\mu$ is the energy-momentum vector of the system. We see that a state with energy $H = (Z^a Z^{*a})^{1/2}$ preserves the supercharges.
\[ Q_\alpha - \epsilon_{\alpha\gamma} \tilde{\sigma}^{a\beta\gamma} \frac{Z^a}{(Z^b Z^{\ast b})^{1/2}} \tilde{Q}_{\beta} \]  

(2.4)

and their Hermitian conjugates. Notice that, although this appears to give four Hermitian supercharges, there are really only two linearly independent Hermitian supercharges unbroken.

### III. BPS BOUNDS FOR JUNCTIONS

We now turn to the general BPS properties of the wall junctions we have just described, and those we will introduce later. The \( N = 1 \) chiral theory of the previous section has supercurrent

\[ j_{\mu\alpha} = i \sqrt{2} (\sigma_{\mu} \tilde{\sigma}^{\nu})^{\alpha\beta} \partial_\nu \phi^{*} \cdot \psi_\beta + i \sqrt{2} W'(\phi)^* \sigma_{\mu\alpha\gamma} \tilde{\psi}_\gamma, \]  

(3.1)

with associated supercharge

\[ Q_\alpha = i \sqrt{2} \int d^3 x \left[ (\sigma^0 \tilde{\sigma}^{\nu})^{\alpha\beta} \partial_\nu \phi^{*} \cdot \psi_\beta (\bar{x}) + W'(\phi(\bar{x}))^* \sigma^0_{\alpha\gamma} \tilde{\psi}_\gamma (\bar{x}) \right]. \]  

(3.2)

A careful calculation of the anticommutator \( \{ Q_\alpha, \tilde{Q}_{\beta} \} \) recovers not only the usual momentum term, but also a total derivative given by \( (\sigma^{a}_{\alpha\beta} \cdot Y^a) \), where

\[ Y^a \propto \epsilon^{abc} \int d^3 x \left[ \partial_b \phi(\bar{x}) \partial_c \phi^{*}(\bar{x}) + \text{h.c.} \right]. \]  

(3.3)

A similar central term arises in theories which admit supersymmetric string solutions, including many supergravity theories \[ [6], \] and \( N = 1 \) SUSY gauge theories with abelian gauge group factors and non-vanishing F-I parameters \[ [7]. \] Static string (or multi-string) solutions have a tension determined by the value of the central charge \( Y^a \), and preserve half the supersymmetry of the Hamiltonian.

If analogous partially supersymmetric states exist in the theory we are considering, then they must look very different, since clearly the Lagrangian cannot admit string solutions with finite tension. We would like to know whether there are any BPS states in chiral \( N = 1 \) theories with a nonzero value of the central term \( Y^a \).

As we shall see in the rest of this paper, the answer is yes. We find that in theories with only chiral superfields, the central term \( Y^a \) admits an interpretation not as string charge but rather as junction charge. In any 4D field theory with more than two disconnected vacua, the domain walls may meet in one-dimensional junctions. If the theory is supersymmetric,
we demonstrate that junctions of the 1/2-supersymmetric domain walls may have stable junction solutions preserving 1/4 of the supersymmetry of the Hamiltonian.

From this point onwards, we will assume the junction state to be static and translationally invariant in the direction $x_3$. For such a configuration, the magnitude of the central term $Y^a$ plays the role of an additional contribution to the mass of a junction, above and beyond that contributed by the half-walls themselves. That is, not only do the “spokes” of the junction have a tension associated with them, but the “hub” has its own non-vanishing contribution to the total energy, in contrast to the string junctions of [10].

As in the more familiar examples of central charges in SUSY algebras, the central term here arises at the semiclassical level as a topological term entering a classical BPS bound. To see this, note that the Hamiltonian for static configurations,

$$H = \int d^3x \left[ (\partial_{x_1} \phi)(\partial_{x_1} \phi^*) + (\partial_{x_2} \phi)(\partial_{x_2} \phi^*) + (\partial_{x_3} \phi)(\partial_{x_3} \phi^*) + W'(\phi)W'(\phi) \right] ,$$

(3.4)
can be rewritten, for any phase $\Omega$, as the sum of positive definite terms and a total derivative term:

$$H = \int d^3x \left[ (\partial_{x_3} \phi)(\partial_{x_3} \phi^*) + (\partial_{x_1} \phi - i\partial_{x_2} \phi - \Omega W'(\phi^*)) (\partial_{x_1} \phi^* + i\partial_{x_2} \phi^* - \Omega^* W'(\phi)) ight. \\
+ (\partial_{x_1} - i\partial_{x_2})(\Omega^* W) + (\partial_{x_1} + i\partial_{x_2})(\Omega W^*) + i\partial_{x_1} \phi^* \partial_{x_2} \phi - i\partial_{x_1} \phi \partial_{x_2} \phi^* \\
= \int d^3x \left[ (\partial_{x_3} \phi)(\partial_{x_3} \phi^*) + 4(\partial_{y} \phi - \frac{1}{2} \Omega W'(\phi^*)) (\partial_{y} \phi^* - \frac{1}{2} \Omega^* W'(\phi)) ight. \\
+ 2 \partial_{y} (\Omega^* W) + 2\partial_{y} (\Omega W^*) + \partial_{y} (\phi^* \partial_{y} \phi - \phi \partial_{y} \phi^*) + \partial_{y} (\phi \partial_{y} \phi^* - \phi^* \partial_{y} \phi) \right] (3.5)

Since the mass in any given region is equal to a positive definite term plus a surface term, this imposes a classical BPS lower bound on the mass in a region in terms of the values of the fields on its boundary. If the positive definite terms are set to zero (which will turn out precisely to impose the BPS equations for a static configuration) then the total mass of the state becomes a surface term. Upon doing the integral we find that the first pair of surface terms (involving the superpotential) gives the contribution to the mass corresponding to the central charge $Z^a$, and the second pair gives the contribution corresponding to the central charge $Y^a$.

For later convenience, note that this calculation is easily generalized to the case of a nontrivial Kähler metric $K_{\phi \phi^*}$. In that case, the relevant surface terms in the energy are

$$2\partial_{x} (\Omega^* W) + 2\partial_{x} (\Omega W^*) + \partial_{x} (K_{\phi \phi} \partial_{x} \phi - K_{\phi^* \phi} \partial_{x} \phi^*) + \partial_{x} (K_{\phi \phi^*} \partial_{x} \phi^* - K_{\phi^* \phi} \partial_{x} \phi) .$$

(3.6)
Therefore, the mass contributed by the junction charge \( Y^a \) is equal to a line integral of the pullback of the one-form \( iK_{\phi}d\phi - iK_{\phi'}d\phi^* \), or, equivalently:

*The junction mass is proportional to the area in field space spanned by the fields of the solution, as measured by the Kähler metric.*

Although, unlike the masses of the domain walls themselves, the mass of the junction is not protected by supersymmetry from perturbative quantum corrections, this is still a compact and useful result, and we shall use it later to derive an interesting quantitative prediction about the behavior of certain states in M-theory.

**IV. DOMAIN WALL JUNCTIONS: THE \( Z_3 \) CASE**

In order understand the subtleties involved in constructing our 1/4-BPS states, let us specialize to the simplest nontrivial example, the case \( n = 3 \). The superpotential (2.1) becomes

\[
W = \Lambda^2 \Phi - \frac{\Phi^4}{4\Lambda},
\]

so that the theory has three supersymmetric vacua \( I, II, \) and \( III \), in which the scalar component takes on vacuum expectation values

\[
\phi_I = \Lambda, \quad \phi_{II} = e^{2\pi i/3} \Lambda, \quad \phi_{III} = e^{-2\pi i/3} \Lambda.
\]

We wish to consider an initial field configuration in which \( \phi \) tends to each of these values in three different directions, as in figure \( [4] \). If we allow the field to radiate away energy, it will settle down to a stable configuration whose topology is that of a junction of three half-walls meeting at 120° angles.

We now analyze how much supersymmetry may be preserved by such a configuration. First, consider the supercharges left unbroken by each individual half-wall. The changes \( \tilde{\Delta}W = \int d^3x \tilde{\nabla}W \) in the superpotential across the walls between vacua \( I \) and \( II \), between vacua \( II \) and \( III \), and between vacua \( III \) and \( I \), are

\[
\tilde{\Delta}W_{I,II} = \left( -\frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \right) \cdot \left( \exp \frac{2\pi i}{3} - 1 \right) \cdot \frac{3\Lambda^3}{4}
\]

\[
\tilde{\Delta}W_{II,III} = (0, -1, 0) \cdot \left( \exp \frac{4\pi i}{3} - \exp \frac{2\pi i}{3} \right) \cdot \frac{3\Lambda^3}{4}
\]

\[
\tilde{\Delta}W_{III,I} = \left( -\frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \right) \cdot \left( 1 - \exp \frac{4\pi i}{3} \right) \cdot \frac{3\Lambda^3}{4}
\]
FIG. 1. A field configuration that interpolates among the three supersymmetric vacua. The energy density is concentrated inside the contour.
respectively. Therefore, the supercharges preserved by each half-wall alone are

\[ Q_{I,II}^{(1)} = Q_\uparrow - Q_\uparrow^\dagger \]  
\[ Q_{I,II}^{(2)} = Q_\downarrow + Q_\downarrow^\dagger \]  
\[ Q_{III,I}^{(1)} = Q_\uparrow - Q_\uparrow^\dagger \]  
\[ Q_{III,I}^{(2)} = Q_\downarrow + \exp\left\{-\frac{2\pi i}{3}\right\} Q_\downarrow^\dagger \]  
\[ Q_{II,III}^{(1)} = Q_\uparrow - Q_\uparrow^\dagger \]  
\[ Q_{II,III}^{(2)} = Q_\downarrow + \exp\left\{+\frac{2\pi i}{3}\right\} Q_\downarrow^\dagger . \]

Here we have written the conjugates of \( Q \) in terms of \( Q^\dagger \) instead of \( \bar{Q} \) as we are dealing with static states, and wish to emphasize Hermiticity rather than Lorentz covariance. The subscripts \( \uparrow, \downarrow \) represent the supercharges with spin \( \pm \frac{1}{2} \) in the \( z \)-direction, respectively. Thus the entire configuration can preserve at most the Hermitian supercharge

\[ Q_{\text{junc}} = i(Q_\uparrow - Q_\uparrow^\dagger) \]

The equations satisfied by a semiclassical solitonic junction state \( |J\rangle \) preserving this supercharge can be found by taking the expectation value of the SUSY variations of the fermions:

\[ 0 = -i\langle J|\{Q_{\text{junc}}, \psi_\downarrow\} |J\rangle \]
\[ = \langle J|\left(\partial_x - i\partial_y\right)\phi - W'(\phi^*) |J\rangle \]
\[ \simeq \left(\partial_x - i\partial_y\right)\langle J|\phi|J\rangle - W'(\langle J|\phi^*|J\rangle) , \]

\[ 0 = \langle J|\{Q_{\text{junc}}, \psi_\uparrow\} |J\rangle \]
\[ = \langle J|\dot{\phi} - \partial_z \phi|J\rangle , \]

in the \( \hbar \to 0 \) limit. Notice that these are exactly the equations obtained by assuming saturation of the classical BPS bound (3.5). The existence of 1/4-BPS domain wall junctions depends on the existence of a solution to these equations. However, in general it is quite difficult to solve the equations directly.

In order to address this, we pursue two different strategies. First, in section we present a picture of wall junctions and junction networks viewed on scales large compared to the thickness of the walls, with emphasis on the \( Z_3 \) case. We derive a new form of the BPS
equations appropriate to this regime, and use it to show that the BPS equations translate into a condition on the kinematics of the junctions.

In section VI we take a different tack, and outline an explicit construction for junction and network solutions to the BPS equations. We apply this approach to a particular intersecting domain wall solution, in which two BPS walls intersect, rather than one terminating on the other. While we do not resolve the existence question for domain wall junctions in general, the construction demonstrates that 1/4-BPS solutions do indeed exist in many cases.

V. LONG DISTANCE LIMIT AND INTEGRAL BPS EQUATIONS

We will begin with the long-distance limit. Our strategy is to derive an integral, global form of the BPS equations and use this to examine domain wall junctions on scales much larger than the thickness of the walls. We will see that in this long-distance regime, the BPS condition is indeed satisfied for various configurations.

We consider field configurations which tend to some vacuum everywhere at spatial infinity, except perhaps along codimension-one defects separating different vacua. Such configurations have approximately step-function behavior across domain walls, when viewed on scales much larger than any length scale appearing in the Lagrangian.
How do we check to see whether or not the BPS equations are satisfied for a given configuration in this limit? Since we want to work in an approximation in which the fields vary discontinuously, the differential form of the BPS equations is clearly unsatisfactory. However we can write down an integral form of the equations, analogous to the integral form of the first-order Maxwell equations.

We begin by noting that the BPS equations

\[ 2K_{\phi\phi} \partial_z \phi = \Omega' W'(\phi)^* \]  

and

\[ 2K_{\phi\phi} \partial_{\bar{z}} \phi^* = \Omega^* W'(\phi) \]  

imply the relations

\[ \partial_z W(\phi) = W'(\phi) \partial_z \phi = \frac{\Omega}{2} K_{\phi\phi} W'(\phi) W'(\phi^*) = \frac{\Omega}{2} V(\phi, \phi^*) \]  

and

\[ \partial_{\bar{z}} W(\phi)^* = \frac{\Omega^*}{2} V(\phi, \phi^*) , \]

where \( V \) denotes the potential energy. We then integrate these equations over any large region \( R \) in the \( z-\bar{z} \) plane; in particular, integrate over an elongated rectangle containing a segment of one of the walls, whose long side has length \( L >> \Lambda^{-1} \).

Using Stokes’s theorem,

\[ \int_R dz \wedge d\bar{z} (\partial_z v_\bar{z} - \partial_{\bar{z}} v_z) = \oint_{C=\partial R} (dz v_\bar{z} + d\bar{z} v_z) , \]

and setting

\[ v_\bar{z} \equiv -i \Omega^* W \]

\[ v_z = i \Omega W^* \]

we obtain

\[ -i \int_R dz \wedge d\bar{z} V = i \oint_C (\Omega W^* dz - \Omega^* W d\bar{z}) . \]

It is straightforward to check that for an isolated BPS wall, the kinetic and potential energies of the solution are equal, and so this must hold as well for the junction solution to order \( L^1 \). Since \( dz \wedge d\bar{z} = -2i dx \wedge dy \), the equation above then says that the total mass
enclosed by the contour $C$, for a $1/4$-supersymmetric junction state, is given to order $L^1$ by the contour integral

$$M_{\text{enclosed}} = -i \oint_C (\Omega W^* dz - \Omega^* W d\bar{z}) .$$

(5.8)

This reformulation of the BPS equations immediately yields a useful set of restrictions on the statics of wall junctions. Assuming the walls themselves to saturate the BPS bound

$$T = 2|\Delta W|,$$

(5.9)
taking $L \to \infty$ and matching terms of order $L^1$, we find that each wall must be oriented such that $\Omega \Delta W \omega$ is real and positive, where $\omega$ is the phase characterizing the orientation of the wall in the complex plane.

Thus, since $\Delta W_{I,II}, \Delta W_{II,III},$ and $\Delta W_{III,I}$ differ from each other in phase by $120^0$, this means that the orientations of the walls in the $z-\bar{z}$ plane must also differ by the same amount. Therefore, the integral forms of the BPS equations confirm reasonable physical expectations for the angles at which the walls must meet, based on the balance of forces on the junction. Although the $Z_3$ symmetry of this particular case makes the statics particularly simple, it is straightforward to check that the integral form of the BPS equations yields the same consistent picture for an arbitrary superpotential with multiple vacua: the relative angles of walls at a junction must be arranged so as to cancel the total force on the junction point. Therefore, for theories with four or more disconnected supersymmetric vacua, we argue that unless there may to exist full moduli spaces of domain wall networks, as in figure [3]. We caution, however, that the moduli space may not exist beyond this limit; the low-frequency dynamics on this space are governed by a $1 + 1$-dimensional field theory with only a single supercharge, which allows for the existence of potentials. The question of the existence or nonexistence of such potentials lies beyond the scope of this paper.

It remains an open question whether there exists a topological index to count the number of moduli directly, as has been done for moduli spaces of self-dual instantons, monopoles, and other BPS states in SUSY theories.

Last we note that the junctions described in this paper bear some rough resemblance to string junctions of [10], and at first one might guess that string junctions might be described, perhaps after a series of duality transformations, by the field theory wall junctions described here. Even leaving aside the different amounts of supersymmetry preserved by the two types of configurations – eight supercharges versus one – the analysis of sections [V] and [V]
In the $Z_4$ theory, the long-distance limit suggests the existence of a one-dimensional moduli space of BPS wall junction networks with four external walls. The two branches of moduli space, which meet at a $Z_4$-symmetric branch point, resemble t- and s-channel Feynman diagrams, respectively. Makes it clear that the relation between the two, if any, cannot be too straightforward. It is known that type $IIB$ string junctions can be arranged to form infinite network lattices, and a simple argument shows that this cannot be the case for domain wall junctions in our theory with $Z_3$-symmetric superpotential. While one can arrange a hexagonally symmetric configuration as described in [6], such a configuration cannot be BPS, since if one chooses to preserve a fixed supercharge, the orientation of a domain wall segment is completely determined via the long-distance limit of the integral BPS equations in terms of the phase of the superpotential difference across the segment, something which does not hold for the hexagonal lattice. The same considerations apply to rectangular lattices in theories with the analogous $Z_4$-symmetric superpotential.

One can imagine, of course, evading this no-go principle for BPS lattices by choosing the superpotential to be of the form $W(\phi) = p(x) + k \cdot \phi$, where $p(\phi)$ is some holomorphic, doubly periodic function in the complex plane. One could then construct formal long-distance limits of junction lattices which preserve supersymmetry. However since doubly periodic holomorphic functions of one variable are necessarily singular, one would have to resolve the singularities with new degrees of freedom in order to give any physical meaning to such configurations, and we do not pursue this problem here.

VI. GEOMETRIC REVERSE-ENGINEERING: AN EXPLICIT CONSTRUCTION

Despite the consistent picture we have presented of 1/4-BPS domain wall junctions, a skeptic might still suspect that there could be a subtle obstruction to the existence of such a
state. In order to allay any such anxieties, we will demonstrate that it is straightforward to construct 1/4-supersymmetric configurations describing wall junctions and networks thereof.

We begin with the observation that the statement \( \Omega^* \partial z W = \Omega \partial \bar{z} W^* \) for some phase \( \Omega \) always holds for 1/4-BPS states, independent of the form of the Kähler metric. In other words, the condition for a function \( W(z, \bar{z}) \) describing the behavior of the superpotential as a function of space is simply the condition that \( (W^*, W) \) be equal to \( (\Omega^* \partial A, \Omega \partial \bar{z} A) \) for some real function \( A \).

Furthermore, if we know more or less what the energy density of the state should look like, we can simply construct the function \( A \) by solving the linear Poisson equation \( \partial z \partial \bar{z} A = \frac{1}{4} V(z, \bar{z}) \). Having solved for the superpotential as a function of space, the value of the field \( \phi \) is then implicitly defined.

There are two catches to this procedure. The first is that one cannot start with a given Kähler metric and use this procedure to solve the BPS equations for that metric. Once one has a profile for the field \( \phi \) one can, of course, reconstruct the corresponding metric, if not necessarily in closed form, although one must then check that the resulting metric is nonsingular and positive definite. We shall give an example below. The second catch is that the process of solving for \( \phi(z, \bar{z}) \) in terms of the superpotential \( W(z, \bar{z}) \) can break down if the function \( W(z, \bar{z}) \) ever attains a value for which \( W'(\phi) \) vanishes. These caveats, however, are really blessings in disguise. If not for such obstructions, one could clearly start with energy density distributions with no reasonable interpretation as a junction state or a network thereof, and use them to generate 1/4-BPS states.

There is another, more subtle, property that the function \( W(z, \bar{z}) \) must satisfy for a BPS junction or network. The BPS equation for a single static wall,

\[
K_{\phi \phi^*} \bar{n} \cdot \nabla \phi = \omega W'(\phi)^*,
\]

where \( \bar{n} \) is the direction normal to the wall and \( \omega \) is a phase, means that the imaginary part of \( \omega W^* \) remains constant over a BPS wall trajectory – that is, the trajectory in the \( W \)-plane of a single domain wall solution is always a straight line segment connecting two vacua \([11]\). This implies a restriction on \( W(z, \bar{z}) \) for a BPS junction or network state: that the set of values assumed by \( W \) over the \( z - \bar{z} \) plane must exactly fill out the convex hull in the \( W \)-plane of the \( k \) vacua among which the junction interpolates. That is, the image in the \( W \)-plane of the junction with \( k \) legs is simply a \( k \)-sided polygon, which we will refer to henceforth as the BPS polygon in the \( W \)-plane.
To make this less abstract, it would be nice to construct explicitly the functions $W(z, \bar{z})$ and $A(z, \bar{z})$ for the basic three-wall junction in the $Z_3$ theory. However, we have not been able to find an energy density profile which allowed us to solve the Poisson equation in closed form for such a configuration. Instead, we will turn to a case in which a simple closed-form solution does exist, and verifiably has the correct properties to describe a well-behaved junction state: the $Z_4$-symmetric four-wall junction in the $Z_4$ theory which lies at the intersection of the $s$-channel and $t$-channel branches of the moduli space of four-wall networks in that theory [5].

Given the superpotential

$$W(\Phi) = \Lambda^2 \Phi - \frac{\Phi^5}{5\Lambda^2},$$

which has vacua at $\pm \Lambda, \pm i\Lambda$, there is a natural guess at a profile function $W(z, \bar{z})$ for a $Z_4$-symmetric four-wall junction:

$$W = \frac{(2 - 2i)\Lambda^3}{5} \left( \tanh\{\Lambda(z + \bar{z})\} - i \tanh\{i\Lambda(z - \bar{z})\} \right)$$

Note that our initial profile does indeed map to itself under a combined discrete spatial rotation and $R$-symmetry transformation, as we expect the actual solution to do. We now verify that this function has the correct behavior to be a junction state.

First consider the behavior of $W$ as $x \to +\infty$, and $y = mx$, with $m > 0$. In this limit

$$W \to \frac{(2 - 2i)\Lambda^3}{5} (1 + i) = \frac{4\Lambda^3}{5},$$

which is the correct value of the superpotential in the vacuum at $\phi = \Lambda$. Since we know how $W$ transforms under the discrete $Z_4$ rotation subgroup, this then implies that $W$ has the correct limiting behavior in all four quadrants of the $z - \bar{z}$ plane.

Second, it is clear that the set of values assumed by $W$ is precisely the convex hull in the $W$ plane of the four vacuum values of the superpotential (see figure [4]) so the solution’s $W$-profile does indeed fill out the BPS polygon whose edges are the trajectories of the superpotential along the four individual BPS domain (half-) wall solutions. Since $W(z, \bar{z})$ stays within the BPS polygon for all $z, \bar{z}$, the function $\phi(z, \bar{z})$ can be recovered from equation (6.2), since $W'(\phi)$ is non-vanishing everywhere in this region.

Until now we have let the Kähler potential be an arbitrary function $K(\Phi, \Phi^\dagger)$. Then the condition for a junction state to preserve the supercharge $Q_+^\dagger - \Omega Q_+^\dagger$ is
FIG. 4. BPS polygons in the $W$-plane for the two examples discussed in this section. Vertices (vacua) preserve four supercharges, edges (half-walls) preserve two, and the interior (the junction) preserves one.

\[ 2K_{\phi\phi^*} \partial_z \phi = \Omega \left( \Lambda^2 - \frac{\phi^*}{\Lambda^2} \right) . \] (6.5)

Therefore, the Kähler metric can be expressed as

\[ K_{\phi\phi^*}(z, \bar{z}) = \Omega \left[ \frac{\Lambda^2 - \phi^*}{2\partial_z \phi} \right] . \] (6.6)

For this to be a sensible metric, the function $K_{\phi\phi^*}$ must be real and non-degenerate. To check this, we point out that the requirement that $K_{\phi\phi^*}$ be real is equivalent to the requirement that

\[ \Omega^*(\partial_z \phi)W'(\phi) = \Omega(\partial_z \phi^*)W'(\phi^*) \] (6.7)

However, the superpotential can be written as $W = \Omega \partial_z A(z, \bar{z})$ where $\Omega = \exp(-\frac{\pi i}{4})$ and $A$ is the real function

\[ A = \frac{2\sqrt{2}\Lambda^2}{5} \left[ \ln \cosh(\Lambda(z + \bar{z})) + \ln \cosh(i\Lambda(z - \bar{z})) \right] , \] (6.8)

and we may therefore infer that $K_{\phi\phi^*}(z, \bar{z})$ is indeed everywhere real. Moreover, as the gradient of $W$ is everywhere non-vanishing in the interior of the BPS polygon, this means that $K_{\phi\phi^*}$ is real and nonzero, and thus positive definite for all $z, \bar{z}$. Therefore, since our theory contains only one chiral superfield, the metric defined in this way satisfies the Kähler condition trivially.
Hence, we have demonstrated that, for some choice of reasonable metric and the superpotential \(6.2\), there exists a domain wall junction preserving a single supercharge. It remains to note that the map from the \((z, \bar{z})\) plane to the BPS polygon is nonsingular and one-to-one. Thus we could formally re-express the metric as a positive definite function of \(\phi\) and \(\phi^*\), which could be derived from a Kähler potential \(K_{\text{initial}}\).

Having developed this existence proof for nontrivial Kähler metric, we may wonder again about the case with trivial metric. At present we have a partial answer to this question. Let

\[
K(t; \phi, \phi^*) \equiv (1 - t)K_{\text{initial}}(\phi, \phi^*) + t\phi\phi^* .
\]  

(6.9)

Then the corresponding metric is positive definite \(\forall t \in [0, 1]\). We expect the number of supersymmetric configurations in a given topological class to remain the same as we vary the parameters of the Lagrangian in a sufficiently smooth way. And so as we vary the metric, we can vary the solution to the BPS equations, knowing that a solution must exist for all values of \(t\), including \(t = 1\). Therefore it seems that a solution exists for trivial Kähler metric, and our argument is complete.

While it is still possible that there is some obstruction to continuing the solution to \(t = 1\), other than an ill-defined metric, at very least, we have shown that a metric can be constructed which allows BPS junctions.

**VII. RELEVANCE TO \(N = 1\) SYM**

The models we wrote down were quite simple, based on a single chiral superfield and a polynomial superpotential. However they are closely related to a particular physical system of great interest. The simplest four-dimensional supersymmetric gauge theories, pure super-Yang-Mills theories with no matter multiplets, are believed to have a finite number of equivalent supersymmetric vacua (\(n\) of them for the gauge group \(SU(n) \[7\]) related by a spontaneously broken discrete \(R\)-symmetry, as does our model.

The picture is as follows. The \(SU(n)\) theory has, at the classical level, a \(U(1)\) \(R\)-symmetry under which the gauge fields transform trivially and the gluinos \(\lambda^a\) have charge +1. The Dynkin index of the adjoint representation is equal to \(2n\), so the \(R\)-symmetry is broken by anomalies down to a \(Z_{2n}\) subgroup. Since bosonic operators transform under a \(Z_n\) subgroup, strong coupling effects can at most spontaneously break \(Z_{2n}\) to \(Z_2\), so the maximum number of supersymmetric vacua the theory may have is \(n\). An index calculation \[8\] shows that this upper bound is attained.
The existence of disconnected vacua means that $SU(n)$ SYM has domain walls interpolating between them, and the anomaly term (2.2) makes it possible for the domain walls to preserve half the supersymmetry. Indeed, this is the context in which BPS-saturated domain walls were originally discovered. Can they form BPS junctions? And if so, can we calculate their tensions explicitly?

The description of the effective theory of SYM is quite complicated, and at present still poorly understood. But the considerations of sections (V) and (VI) suggest that within the category of SUSY theories with multiple vacua related by a spontaneously broken cyclic $R$-symmetry, domain wall junction and network states are a fairly robust feature whose existence depends little on the details of the metric. Indeed, the only essential input to our calculation was that the theory contain a single chiral superfield.

The authors of [12], [13], building on earlier work [14], derive an effective action for large-$n$ SUSY gluodynamics in its confining phase. Their model describes spontaneous $Z_n$ breaking and domain walls in terms of a single superfield $X$, which is closely related to the gluino condensate $\langle \lambda \lambda \rangle$. The superpotential for this model,

$$W_{\text{eff}} = n \Lambda^2 X - \frac{C N X^{n+1}}{n+1},$$

(7.1)

(where $C$ is a constant), clearly gives the correct global behavior for the system: the superpotential transforms nontrivially under a spontaneously broken $Z_n$ $R$-symmetry, and there are BPS domain walls interpolating between distinct vacua. Indeed, this superpotential is trivially related to ours: by rescaling $X$, one can make the two superpotentials the same. The explicit dependence of $W_{\text{eff}}$ on $C$ and $n$ is then absorbed into the normalization of the metric.

We therefore expect that:

$SU(n)$ SYM contains in its Hilbert space wall junction states preserving a single Hermitian supercharge, at least for sufficiently large $n$.

The tension of the walls was calculated in [15]. In order to calculate the additional energy contributed by the junction itself, one would need to use the Kähler metric for this model, which, unfortunately, is not known. So we can at present make a qualitative prediction about the existence of 1/4-supersymmetric junctions in $N = 1$ SYM, but not yet a quantitative one about their tensions.
VIII. DISCUSSION

SYM with gauge group $SU(n)$ is believed to have a dual formulation in terms of $M$-theory. In [13], Witten describes a configuration of M-theory fivebranes with a three-dimensional intersection, the effective field theory on which is argued to be in the same universality class as pure $N = 1$ supersymmetric Yang-Mills theory. Many of the strong-coupling phenomena present in SYM, such as color flux tubes and domain walls, then have simple descriptions in terms of intersecting branes in M-theory. Indeed, explicit M-theory states have been written down which correspond to BPS domain walls in this effective theory [16], [17].

We expect that there ought to exist solutions of 11-dimensional supergravity preserving $1/32$ of the supersymmetry of the theory, corresponding to the full spectrum of BPS wall junctions described in this paper. The question of the existence of such solutions therefore may provide a stringent and quantitative test of the duality between $M$-theory and $N = 1$ SYM.

ACKNOWLEDGMENTS

The authors would like to thank Oren Bergman, Zurab Kakushadze, Andrew Sornborger and Cyrus Taylor for valuable discussions. This work was supported in part by the National Science Foundation under grants PHY/94-07194 and PHY/97-22022, and by the U.S. Department of Energy (D.O.E.).
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