The Feasibility of Homotopy Continuation Method for a Nonlinear Matrix Equation

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In this paper, we discuss the feasibility of homotopy continuation method for the nonlinear matrix equations
\[ X + \sum_{i=1}^{s} B_i^* X^{-1} B_i + \sum_{i=s+1}^{m} B_i^* X^i B_i = I \]
for 0 < t_i < 1. This iterative method does not depend on a good initial approximation to the solution of matrix equation.

1. Introduction

In this paper, we consider the Hermitian positive definite (HPD) solutions of the nonlinear matrix equation:
\[ X + \sum_{i=1}^{s} B_i^* X^{-1} B_i + \sum_{i=s+1}^{m} B_i^* X^i B_i = I, \quad 0 < t_i < 1, \quad (1) \]
where \( B_i (i = 1, 2, \ldots, m) \) are \( n \times n \) complex nonsingular matrices, \( I \) is an \( n \times n \) identity matrix, and \( s \geq 1, m \geq 2 \) are positive integers. Here, \( B_i^* \) denotes the conjugate transpose of the matrix \( B_i \).

The nonlinear matrix equations of (1) or some special cases are applicable to many fields such as nanoresearch, ladder networks, dynamic programming, control theory, stochastic filtering, and statistics [1–8].

Equation (1) is recognized as playing an important role in solving a system of linear equations. For example, in many physical calculations, one must solve the system of linear equation \( Mx = f \), where \( x \) and \( f \) are column vectors, and
\[
M = \begin{pmatrix}
I & 0 & \cdots & 0 & A_1 \\
0 & I & \cdots & 0 & A_2 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I & A_m \\
A_1^* & A_2^* & \cdots & A_m^* & I
\end{pmatrix}
\] (2)
arises in a finite difference approximation to an elliptic partial differential equation (for more information, refer to [2]). We can rewrite \( M \) as \( \tilde{M} = M + D \), where
\[
\tilde{M} = \begin{pmatrix}
X & 0 & \cdots & 0 & A_1 \\
0 & X & \cdots & 0 & A_2 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & X & A_m \\
A_1^* & A_2^* & \cdots & A_m^* & I
\end{pmatrix}
\]
and
\[
\tilde{D} = \begin{pmatrix}
I - X & 0 & \cdots & 0 & 0 \\
0 & I - X & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I - X & 0 \\
0 & 0 & \cdots & 0 & I - X
\end{pmatrix}
\] (3)
\[ \tilde{M} \text{ can be factored as } \tilde{M} = \tilde{M}_1 \tilde{M}_2 \text{ if and only if } X \text{ is a solution of equation } X + \sum_{i=1}^{m} A_i^* X^{-1} A_i + \sum_{i=1}^{m} A_i^* X A_i = I, \]

where

\[ \tilde{M}_1 = \begin{pmatrix}
I & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & I & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & I \\
\end{pmatrix} 
\]

\[ \tilde{M}_2 = \begin{pmatrix}
X & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & X^{p-1} & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & X \\
\end{pmatrix} 
\]

(4)

Some special cases of matrix equation (1) and related matrix equations were studied on the solvability, numerical methods, and perturbation analysis by many scholars: \[ X + A^* X^{-1} A = P \] \[ X^t + A^t X^{-t} A = I \] \[ X^t \times A^t X^{-t} A = I \] \[ X^t + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I \] \[ X^t + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I \] \[ X^t \times A^t X^{-t} A = I \]

According to our knowledge, the matrix equation (1) has not been treated explicitly in the literatures. The reason is that (1) does not always have a unique Hermitian positive definite solution. It is hard to find sufficient conditions for the existence of a unique Hermitian positive definite solution because the map \[ F(X) = I - \sum_{i=1}^{m} X^{-1} B_i^* B_i - \sum_{i=1}^{m} B_i^* B_i X \]

is not monotonic. There are two difficulties for discussing the solvability and iterative method for the matrix equation (1). One is how to find a suitable set and some reasonable restrictions on the coefficient matrices ensuring this equation has a unique Hermitian positive definite solution in this set. The other one is how to find a reasonable expression of \[ F(X) - F(Y) \]

which was important for discussing the feasibility of the homotopy continuation iterative method (see [34], for more details), which was not dependent on a good initial approximation to the solution of matrix equation.

In this paper, firstly, we derive necessary and sufficient conditions for the existence of Hermitian positive definite solutions to equation (1) in Section 3. Then, discuss the homotopy continuation methods for obtaining the unique Hermitian positive definite solution in Section 4.

We denote by \( \mathcal{C}_m \), \( \mathcal{H}_m \), and \( \mathcal{W}_m \) the set of all \( n \times n \) complex matrices, Hermitian matrices, and unitary matrices, respectively. For \( A = (a_{ij}) \in \mathcal{C}_m \) and a matrix \( B, A \otimes B = (a_{ij}B) \) is a Kronecker product, and vec \( A \) is a vector defined by \( \text{vec} A = (a_{ij}) \). The symbol \( \| \cdot \| \) stands for the spectral norm. We denote by \( \lambda_i (M) \) the eigenvalues of \( M \), by \( \lambda_j (M) \) and \( \lambda_k (M) \) the maximal and minimal eigenvalues of \( M \), respectively. For \( X, Y \in \mathcal{H}_m \), we write \( X \succeq Y \) (\( Y \preceq X \)), if \( X - Y \) is a Hermitian positive semi-definite (definite) matrix. For \( A, B \in \mathcal{H}_m \), the sets \( [A, B] \) and \( \{A, B\} \) are defined by \( [A, B] = \{X \in \mathcal{H}_m | A \leq X \leq B\} \) and \( \{A, B\} = \{X \in \mathcal{H}_m | A < X \leq B\} \).

2. Preliminaries

In this section, we present some lemmas that will be needed to develop this paper.

**Lemma 1** (see [35]). If \( A \succeq B > 0 \) and \( 0 \leq y \leq 1 \), then \( A^y \succeq B^y \).

**Lemma 2** (see [24], Lemma 3.3). For every Hermitian positive definite matrix \( X \) and \( 0 < t < 1 \), it yields that \( X^t = \sin t \pi/\pi \int_0^{\infty} X (\lambda I + X)^{-\lambda^{-1}} \lambda d\lambda \).

**Lemma 3** (see [36], Theorem 1.9.1). Let \( A \in \mathcal{C}_m \), \( B \in \mathcal{C}_m^p \) or \( C \in \mathcal{C}_m^q \), and \( D \in \mathcal{C}_m^r \). Then,

\[ (A \otimes B) (C \otimes D) = (AC) \otimes (BD), \]

\[ (A \otimes B)^* = A^* \otimes B^*. \]

**Lemma 4** (see [36], Lemma 1.9.1). Let \( A \in \mathcal{C}_m \), \( X \in \mathcal{C}_m^{nm} \), and \( B \in \mathcal{C}_m^{nk} \). Then,

\[ \text{vec}(AXB) = (B^T \otimes A) \cdot \text{vec}X. \]

**Lemma 5** (see [37], Theorem 6.19). Let \( A \in \mathcal{C}_m \) and \( B \in \mathcal{C}_m \) with eigenvalues \( \lambda_j \) and \( \mu_j \), \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \), respectively. Then, the eigenvalues of \( A \otimes B \) are \( \lambda_i \mu_j \), \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \).

**Lemma 6** (see [24], Lemma 3.2). Suppose that \( m \geq 1 \), \( 0 < t < 1 \), and \( (mt/mt+1) < x, y < 1 \). Then,

\[ 0 < f(x, y, t) = \frac{\sqrt{(1 - x)(1 - y)} (x^t - y^t)}{(x - y) x^{(t/2)} y^{(t/2)}} < \frac{1}{m}. \]

3. Hermitian Positive Definite Solutions

In this section, some sufficient and necessary conditions for the existence and uniqueness of HPD solution of (1) are derived.

**Theorem 1.** (1) has a HPD solution if and only if there exist \( Q_i \in \mathcal{H}_m, i = 1, 2, \ldots, m \), \( P \in \mathcal{H}_m \), and diagonal matrices \( \Gamma, \Lambda > 0 \) such that
Proof. Assume $Y$ is an HPD solution of (1). According to the spectral decomposition theorem, we have that there exists $P \in \mathcal{S}^{m \times m}$ and a diagonal matrix $\Gamma > 0$ such that $Y = P^* \Gamma P$.

Then, (1) can be expressed as

$$P^* \Gamma P + \sum_{i=1}^{s} B_i^* P^* \Gamma^{-1} P B_i + \sum_{i=s+1}^{m} B_i^* P^* \Gamma^i P B_i = I.$$  \hspace{1cm} (8)

Multiplying the left side of (8) by $P$ and the right side by $P^*$, we obtain

$$\sum_{i=1}^{s} PB_i^* P^* \Gamma^{-1} PB_i P^* + \sum_{i=s+1}^{m} PB_i^* P^* \Gamma^i PB_i P^* = I - \Gamma.$$  \hspace{1cm} (9)

Recall that $B_i (i = 1, 2, \ldots, m)$ are nonsingular matrices. Then,

$$0 < \Gamma < I.$$  \hspace{1cm} (10)

Therefore, we can rewrite (10) as

$$\sum_{i=1}^{s} (I - \Gamma)^{-(1/2)} PB_i^* P^* \Gamma^{-1} PB_i P^* (I - \Gamma)^{-(1/2)} + \sum_{i=s+1}^{m} (I - \Gamma)^{-(1/2)} PB_i^* P^* \Gamma^i PB_i P^* (I - \Gamma)^{-(1/2)} = I.$$  \hspace{1cm} (11)

Let

$$\Lambda = (I - \Gamma)^{-(1/2)},$$

$$Q_i = \begin{cases} \Gamma^{-(1/2)} PB_i^* P^* \Lambda^{-1} & \text{for } i = 1, 2, \ldots, s, \\ \Gamma^{(1/2)} PB_i^* P^* \Lambda^{-1} & \text{for } i = s+1, s+2, \ldots, m. \end{cases}$$ \hspace{1cm} (12)

It is easy to verify that $\Gamma + \Lambda^2 = I$ and

$$B_i = \begin{cases} P^* \Gamma^{(1/2)} Q_i \Lambda P, & \text{for } i = 1, 2, \ldots, s, \\ P^* \Gamma^{-(1/2)} Q_i \Lambda P, & \text{for } i = s+1, s+2, \ldots, m. \end{cases}$$ \hspace{1cm} (13)

By (11), we obtain $\sum_{i=1}^{m} Q_i^* Q_i = I$.

Conversely, assume there exist $P \in \mathcal{S}^{m \times m}$, $Q_i \in \mathcal{S}^{m \times m}$, $\sum_{i=1}^{m} Q_i^* Q_i = I$, and diagonal matrices $\Gamma, \Lambda > 0, \Lambda^2 + \Gamma = I$ such that

$$B_i = \begin{cases} P^* \Gamma^{(1/2)} Q_i \Lambda P, & \text{for } i = 1, 2, \ldots, s, \\ P^* \Gamma^{-(1/2)} Q_i \Lambda P, & \text{for } i = s+1, s+2, \ldots, m. \end{cases}$$ \hspace{1cm} (14)

Let $Y = P^* \Gamma P$ such that

$$Y + \sum_{i=1}^{s} B_i^* Y^{-1} B_i + \sum_{i=s+1}^{m} B_i^* X^i B_i$$

$$= P^* \Gamma P + \sum_{i=1}^{s} P^* \Lambda^* Q_i \Gamma^{(1/2)} P (P^* \Gamma P)^{-1} P^* \Gamma^{(1/2)} Q_i \Lambda P$$

$$+ \sum_{i=s+1}^{m} P^* \Lambda^* Q_i \Gamma^{-(1/2)} P (P^* \Gamma P)^i P^* \Gamma^{-(1/2)} Q_i \Lambda P$$

$$= P^* \Gamma P + \sum_{i=1}^{m} P^* \Lambda Q_i \Gamma \Lambda P = P^* (\Gamma + \Lambda^2) P = I,$$  \hspace{1cm} (15)

which means $Y$ is an HPD solution of (1). \hfill \Box

Theorem 2. If $\lambda_1 (\sum_{i=1}^{m} B_i^* B_i) < (m/(m+1))^2$, then (1) has a unique HPD solution $X$ on $[(m/(m+1))I, I].$

Proof. \hfill \Box

Step 1. We will prove that (1) has a HPD solution on $[(m/(m+1))I, I]$ under the assumption $\lambda_1 (\sum_{i=1}^{m} B_i^* B_i) < (m/(m+1))^2$.

Let $\Omega = [(m/(m+1))I, I]$. Define

$$F(X) = I - \sum_{i=1}^{s} B_i^* X^{-1} B_i - \sum_{i=s+1}^{m} B_i^* X^i B_i, X \in \Omega.$$  \hspace{1cm} (16)

Obviously, $\Omega$ is a bounded convex closed set and $F$ is continuous on $\Omega$.

Note that $0 < t_i < 1$. For any $X \in \Omega$, it follows from Lemma 1 that $(m/(m+1))^2 I \leq X_i \leq I$, which implies

$$I \geq F(X) \geq I - \sum_{i=1}^{m} \left( \frac{(m+1)}{m} \right) B_i^* B_i - \sum_{i=s+1}^{m} B_i^* B_i$$ \hspace{1cm} (17)

$$\geq I - \left( \frac{(m+1)}{m} \right) \sum_{i=1}^{m} B_i^* B_i.$$  \hspace{1cm} (18)

Recall that $\lambda_1 (\sum_{i=1}^{m} B_i^* B_i) < (m/(m+1))^2$, then $I \geq F(X) \geq (m/(m+1))I$. That is, $F(X) \subseteq \Omega$. By Brouwer’s fixed point theorem, the map $F$ has a fixed point $X \in \Omega$, which is a HPD solution of (1).

Step 2. We will prove that if (1) has a HPD solution on $[(m/(m+1))I, I]$, then the HPD solution is unique.

If $Y_1$ is a HPD solution of (1), according to Lemma 3.1, there exist $P_1 \in \mathcal{S}^{m \times m}$, $Q_i \in \mathcal{S}^{m \times m}$, $\sum_{i=1}^{m} Q_i^* Q_i = I$, and diagonal matrices $\Gamma_1, \Lambda_1 > 0$ such that

$$B_i = \begin{cases} P_1^* \Gamma_1^{(1/2)} Q_i \Lambda_1 P_1, & \text{for } i = 1, 2, \ldots, s, \\ P_1^* \Gamma_1^{-(1/2)} Q_i \Lambda_1 P_1, & \text{for } i = s+1, s+2, \ldots, m. \end{cases}$$ \hspace{1cm} (18)

where

$$\sum_{i=1}^{m} Q_i^* Q_i = I,$$  \hspace{1cm} (19)

$$\Lambda_1^2 + \Gamma_1 = I.$$
In this case, \( Y_1 = \mathcal{P}_1^* \Gamma_1 \mathcal{P}_1 \), where \( \Gamma_1 = \text{diag}(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1n}) \) with \( \{\lambda_{1j}\} \) the eigenvalues of \( Y_1 \). Similarly, if \( Y_2 \) is a HPD solution of (1), then there exist \( P_2 \in \mathcal{R}^{n \times n} \), \( U_j \in \mathcal{G}^{n \times n} \), \( i = 1, 2, \ldots, m \), and diagonal matrices \( \Gamma_2, \Lambda_2 > 0 \) such that

\[
B_i = \begin{cases} 
\mathcal{P}_2^{(1/2)} \Gamma_2 U_i \Lambda_2 \mathcal{P}_2, & i = 1, 2, \ldots, s, \\
\mathcal{P}_2^{(1/2)} \Gamma_2^{-1/2} U_i \Lambda_2 \mathcal{P}_2, & i = s + 1, s + 2, \ldots, m,
\end{cases}
\]

where

\[
\sum_{i=1}^{m} U_i^* U_i = I, \\
\Lambda_2^2 + \Gamma_2 = I.
\] (21)

In this case, \( Y_2 = \mathcal{P}_2^* \Gamma_2 \mathcal{P}_2 \), where \( \Gamma_2 = \text{diag}(\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2n}) \) with \( \{\lambda_{2j}\} \) the eigenvalues of. According to \( Y_2 \) in Lemma 2, we have

\[
Y_1 - Y_2 = \sum_{i=1}^{s} B_i^* (Y_2^{-1} - Y_1^{-1}) B_i + \sum_{i=s+1}^{m} B_i^* (Y_2^{-1} - Y_1^{-1}) B_i
\]

\[
= \sum_{i=1}^{s} B_i^* Y_2^{-1} (Y_1 - Y_2) Y_1^{-1} B_i + \sum_{i=s+1}^{m} B_i^* \sin t_i \pi \int_{0}^{\infty} [Y_2 (\lambda I + Y_2)^{-1} - Y_1 (\lambda I + Y_1)^{-1}] \lambda^{l-1} B_i d\lambda
\]

\[
= \sum_{i=1}^{s} B_i^* Y_2^{-1} (Y_1 - Y_2) Y_1^{-1} B_i - \sum_{i=s+1}^{m} B_i^* \sin t_i \pi \int_{0}^{\infty} (Y_1 - Y_2) (\lambda I + Y_1)^{-1} \lambda^{l-1} B_i d\lambda
\]

\[
+ \sum_{i=s+1}^{m} \frac{B_i^* \sin t_i \pi}{\pi} \int_{0}^{\infty} Y_2 (\lambda I + Y_2)^{-1} (Y_1 - Y_2) (\lambda I + Y_1)^{-1} \lambda^{l-1} B_i d\lambda.
\]

By

\[
(\lambda I + Y_1)^{-1} = (\lambda I + P_i^* \Gamma_i P_i)^{-1} = P_i^{-1} (\lambda I + \Gamma_i)^{-1} P_i, \quad i = 1, 2,
\]

we have

\[
Y_1 - Y_2 = \sum_{i=1}^{s} \left( \mathcal{P}_2^{(1/2)} U_i \Lambda_2 \mathcal{P}_2 \right)^* \mathcal{P}_1^{-1} \mathcal{P}_1 \mathcal{P}_2 (Y_1 - Y_2) \mathcal{P}_1 \mathcal{P}_2 \left( \mathcal{P}_1^{(1/2)} Q_i \Lambda_i \mathcal{P}_1 \right)
\]

\[
- \sum_{i=s+1}^{m} \frac{\sin t_i \pi}{\pi} \int_{0}^{\infty} \mathcal{P}_2 \Lambda_2 U_i^* \Gamma_2 P_i (Y_1 - Y_2) P_i^* \left( \lambda I + \Gamma_i \right)^{-1} \mathcal{P}_1 \Lambda_1 \lambda^{l-1} P_i d\lambda
\]

\[
+ \sum_{i=s+1}^{m} \frac{\sin t_i \pi}{\pi} \int_{0}^{\infty} \mathcal{P}_2 \Lambda_2 U_i^* \Gamma_2^{-l} (\lambda I + \Gamma_2)^{-1} P_i (Y_1 - Y_2) P_i^* \left( \lambda I + \Gamma_i \right)^{-1} \mathcal{P}_1 \Lambda_1 \lambda^{l-1} P_i d\lambda.
\]

Let

\[
W = \mathcal{P}_2 (Y_1 - Y_2) P_i^*.
\]

Then, (24) can be expressed as

\[
W = \sum_{i=1}^{s} A_i U_i^* \Gamma_2^{-l/2} W T_1^{-l/2} Q_i \Lambda_i - \sum_{i=s+1}^{m} \frac{\sin t_i \pi}{\pi} \int_{0}^{\infty} \mathcal{P}_2 \Lambda_2 U_i^* \Gamma_2 P_i (Y_1 - Y_2) P_i^* \left( \lambda I + \Gamma_2 \right)^{-1} \mathcal{P}_1 \Lambda_1 \lambda^{l-1} d\lambda
\]

\[
+ \sum_{i=s+1}^{m} \frac{\sin t_i \pi}{\pi} \int_{0}^{\infty} \mathcal{P}_2 \Lambda_2 U_i^* \Gamma_2^{-l/2} (\lambda I + \Gamma_2)^{-1} W (\lambda I + \Gamma_1)^{-1} \mathcal{P}_1 \Lambda_1 \lambda^{l-1} d\lambda.
\] (26)
By (26), Lemmas 3 and 4, we have that

\[
\vec{W} = \sum_{i=1}^{s} \left[ \Gamma_{1}^{-1/2} Q_i \Lambda_i \right]^T \otimes \left[ \Lambda_2 U_i^* \Gamma_{2}^{-1/2} \right] \cdot \vec{W} \\
- \sum_{i=s+1}^{m} \frac{\sin t_i \pi}{\pi} \int_{0}^{\infty} \left[ (\lambda I + \Gamma_i)^{-1} \Gamma_{1}^{-1/2} Q_i \Lambda_i \right]^T \otimes \left( \Lambda_2 U_i^* \Gamma_{2}^{-1/2} \right) \lambda^{t_i-1} \, d\lambda \vec{W} \\
+ \sum_{i=s+1}^{m} \frac{\sin t_i \pi}{\pi} \int_{0}^{\infty} \left[ (\lambda I + \Gamma_i)^{-1} \Gamma_{1}^{-1/2} Q_i \Lambda_i \right]^T \otimes \left[ \Lambda_2 U_i^* \Gamma_{2}^{-1/2} (\lambda I + \Gamma_2)^{-1} \right] \lambda^{t_i-1} \, d\lambda \vec{W} \\
= \sum_{i=1}^{s} (\Lambda_1 \otimes \Lambda_2) \left( Q_i^T \otimes U_i^* \right) \left( \Gamma_{1}^{-1/2} \otimes \Gamma_{2}^{1/2} \right) \cdot \vec{W} \\
- \sum_{i=s+1}^{m} \frac{\sin t_i \pi}{\pi} (\Lambda_1 \otimes \Lambda_2) \left( Q_i^T \otimes U_i^* \right) \left( \Gamma_{1}^{-1/2} \otimes \Gamma_{2}^{-1/2} \right) \int_{0}^{\infty} \left[ (\lambda I + \Gamma_i)^{-1} \otimes I \right] \lambda^{t_i-1} \, d\lambda \vec{W} \\
+ \sum_{i=s+1}^{m} \frac{\sin t_i \pi}{\pi} (\Lambda_1 \otimes \Lambda_2) \left( Q_i^T \otimes U_i^* \right) \left( \Gamma_{1}^{-1/2} \otimes \Gamma_{2}^{-1/2} \right) \int_{0}^{\infty} \left[ (\lambda I + \Gamma_1)^{-1} \otimes (\lambda I + \Gamma_2)^{-1} \right] \lambda^{t_i-1} \, d\lambda \vec{W}.
\]

Assume that

\[
\Lambda_1 = \text{diag}(\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1m}), \\
\Lambda_2 = \text{diag}(\sigma_{21}, \sigma_{22}, \ldots, \sigma_{2m}).
\]

It follows from (10), (19), and (21) that

\[
0 < \sigma_{ij} = \sqrt{1 - \lambda_{ij}} < 1, \quad 0 < \sigma_{2j} = \sqrt{1 - \lambda_{2j}} < 1, \quad j = 1, 2, \ldots, n.
\]

Let

\[
B = \Lambda_1 \otimes \Lambda_2, J_i = Q_i^T \otimes U_i^*, E = \Gamma_{1}^{(1/2)} \otimes \Gamma_{2}^{(1/2)}, \\
C_i = \left( \Gamma_{1}^{-1/2} \otimes \Gamma_{2}^{-1/2} \right) \frac{\sin t_i \pi}{\pi} \int_{0}^{\infty} \left[ (\lambda I + \Gamma_1)^{-1} \otimes I \right] \lambda^{t_i-1} \, d\lambda, \\
D_i = \left( \Gamma_{1}^{-1/2} \otimes \Gamma_{2}^{-1/2} \right) \frac{\sin t_i \pi}{\pi} \int_{0}^{\infty} \left[ (\lambda I + \Gamma_1)^{-1} \otimes (\lambda I + \Gamma_2)^{-1} \right] \lambda^{t_i-1} \, d\lambda,
\]

Then, (27) can be expressed as

\[
\vec{W} + B \sum_{i=s+1}^{m} J_i (C_i - D_i) - \sum_{i=1}^{s} J_i E \cdot \vec{W} = 0.
\]
By Lemma 5, we have

\[ B = \Lambda_1 \otimes \Lambda_2 = \text{diag}(\sigma_{i1} \cdot \sigma_{i2})_{n^2 \times n^2}, \quad l, j = 1, 2, \ldots, n, \]

\[ C_i = \left( \Gamma_1^{-t_i/2} \otimes \Gamma_2^{-t_i/2} \right) \frac{\sin \frac{t_i \pi}{\nu}}{\pi} \int_0^\infty \left( (\lambda I + \Gamma_1)^{-1} \otimes I \right) \lambda^{t_i-1} d\lambda, \]

\[ = \text{diag} \left( \lambda_1^{t_i/2}, \lambda_2^{t_i/2}, \lambda_1^{-1/2}, \lambda_2^{-1/2} \right)_{n^2 \times n^2} \]

\[ = \text{diag} \left( \lambda_1^{t_i/2}, \lambda_2^{t_i/2}, \lambda_1^{-1/2} \right)_{n^2 \times n^2} \]

\[ = \text{diag} \left( \lambda_1^{t_i/2}, \lambda_2^{t_i/2}, \lambda_1^{-1/2} \right)_{n^2 \times n^2}, \quad i = (s+1), (s+2), \ldots, m, \quad l, j = 1, 2, \ldots, n, \]

\[ D_i = \left( \Gamma_1^{-1/2} \otimes \Gamma_2^{-1/2} \right) \frac{\sin \frac{t_i \pi}{\nu}}{\pi} \int_0^\infty \left( (\lambda I + \Gamma_1)^{-1} \otimes (\lambda I + \Gamma_2)^{-1} \right) \lambda^{-t_i-1} d\lambda \]

\[ = \text{diag} \left( \lambda_1^{-1/2}, \lambda_2^{-1/2} \left( \lambda_1^{t_i-1} - \lambda_2^{t_i-1} \right) \right)_{n^2 \times n^2}, \quad i = (s+1), (s+2), \ldots, m, \quad l, j = 1, 2, \ldots, n, \]

\[ E = \Gamma_1^{-(1/2)} \otimes \Gamma_2^{-(1/2)} = \text{diag}(\lambda_1^{-(1/2)}, \lambda_2^{-(1/2)})_{n^2 \times n^2}, \quad l, j = 1, 2, \ldots, n. \]

It follows that

\[ C_i - D_i = \text{diag} \left( \lambda_1^{t_i/2}, \lambda_2^{t_i/2} - \frac{\lambda_1^{t_i/2}}{\lambda_2^{t_i/2} - \lambda_1^{t_i/2}} \left( \lambda_1^{t_i-1} - \lambda_2^{t_i-1} \right) \lambda_2^{t_i/2} - \lambda_1^{t_i/2} \right)_{n^2 \times n^2} \]

\[ = \text{diag} \left( \lambda_1^{t_i/2}, \lambda_2^{t_i/2} - \frac{\lambda_1^{t_i/2}}{\lambda_2^{t_i/2} - \lambda_1^{t_i/2}} \lambda_2^{t_i/2} \lambda_1^{t_i/2} \right)_{n^2 \times n^2}, \quad i = (s+1), (s+2), \ldots, m. \]

Note that \( B \) is nonsingular. Multiplying the left side of (31) by \( B^{-1} \), we have

\[ B^{-1} \text{vec}W + \left[ \sum_{i=1}^m J_i(C_i - D_i) - \sum_{i=1}^s J_iE \right] \cdot \text{vec}W \]

\[ = \left[ I + \sum_{i=1}^m J_i(C_i - D_i)B - \sum_{i=1}^s J_iEB \right] B^{-1} \cdot \text{vec}W = 0. \]

A combination of (30) and Lemma 3 gives

\[ J_i^* J_i = (Q_i^T \otimes U_i^*)^* (Q_i^T \otimes U_i^*) = (\tilde{Q}_i \otimes U_i)(Q_i^T \otimes U_i^*) \]

\[ = (\tilde{Q}_i Q_i^T) \otimes (U_i U_i^*). \]

\[ \text{It follows } (19), (21), \text{ and Lemma 5 that } 0 < \|J_i\| \leq 1. \]

Therefore,

\[ \sum_{i=1}^m J_i(C_i - D_i)B \leq \sum_{i=1}^m \|C_i - D_i\|B, \quad \sum_{i=1}^s J_iEB \leq \sum_{i=1}^s \|EB\|. \]
By the hypothesis of the theorem, we have \((m/(m + 1)) I < Y_1, Y_2 < I\), which implies that \(m/(m + 1) < \lambda_{ij}, \lambda_{2j} < 1; \) \(i, j = 1, 2, \ldots, m\). Note that \((mt_i/(mt_i + 1)) < \lambda_{i1}, \lambda_{21} < 1; \) \(i = 1, 2, \ldots, m\). Therefore, it follows (32) and (33) that

\[
\|(C_i - D_i)B\| = \max_{i,j} \left\{ \frac{\sqrt{(1 - \lambda_{i1})(1 - \lambda_{21})}}{\lambda_{i1} \lambda_{21}^{1/2}} \left( \frac{1}{\lambda_{i1}^{1/2}} - \frac{1}{\lambda_{21}^{1/2}} \right) \right\}
\]

\[
= \max_{i,j} \left\{ f(\lambda_{i1}, \lambda_{21}, t_i) \right\},
\]

where \(f(x, y, t)\) is defined in Lemma 6. It is easy to verify that

\[
\|EB\| = \max_{i,j} \left\{ \frac{\sqrt{(1 - \lambda_{i1})(1 - \lambda_{21})}}{\lambda_{i1} \lambda_{21}^{1/2}} \left( \frac{1}{\lambda_{i1}^{1/2}} - \frac{1}{\lambda_{21}^{1/2}} \right) \right\} < \frac{1}{m}
\]

(38)

A combination of Lemma 6, (36)–(38) gives that

\[
\left\| \sum_{i=1}^{m} J_i(C_i - D_i)B - \sum_{i=1}^{s} J_iEB \right\| < \frac{s}{m} + \frac{s}{m} = 1,
\]

(39)

which implies \(I + \sum_{i=1}^{m} J_i(C_i - D_i)B - \sum_{i=1}^{s} J_iEB\) is nonsingular. It follows (34) that \(\text{vec}W = 0\). Recall that \(W = P_2(Y_1 - Y_2)P_1^T\). Therefore, \(Y_1 = Y_2\), which means the HPD solution on \([(m/(m + 1))I, I]\) of (1) is unique.

4. The Homotopy Continuation Iterative Method

In this section, by means of the homotopy continuation iterative method (see [34], for more details), we derive a numerical iterative process for solving the matrix equation (1).

Define the nonlinear map \(F: \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}\) by

\[
F(X) = I - \sum_{i=1}^{j} B_i^* X^{-1} B_i - \sum_{i=1}^{m} B_i^* X^i B_i.
\]

(40)

Consider the homotopy \(H: [0, 1] \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}\).

\[
H(t, X) = I - t \sum_{i=1}^{j} B_i^* X^{-1} B_i - t \sum_{i=1}^{m} B_i^* X^i B_i - X.
\]

(41)

Then, at \(t = 0\), the solution of \(H(t, X) = 0\) is a known matrix \(I\), while at \(t = 1\), the solution \(X\) of \(H(t, X) = 0\) also solves \(F(X) = X\). To discuss the numerical method for solving the homotopy equation \(H(t, X) = 0\), we rewrite the homotopy equation \(H(t, X) = 0\) as the following fixed point form.

Assume that \(G: [0, 1] \times \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}\) is a map such that

\[
X(t) = G(t, X(t)), \quad t \in [0, 1],
\]

(42)

where \(X: [0, 1] \rightarrow \mathbb{R}^{m \times n}\) denotes the solution of \(H(t, X) = 0\). Then, for each \(t\), we can consider the iterative process:

\[
X_{n+1} = G(t, X_n).
\]

(43)

Since for a fixed \(t\), this process will converge to \(X(t)\) only for starting values near that point; to overcome the local convergence of iterative process, we consider the following numerical continuation process.

A partition of \(J = [0, 1]\):

\[
0 = t_0 < t_1 < \cdots < t_{N-1} = 1,
\]

and a sequence of integers \(\{j_k\}\), \(k = 1, \ldots, N - 1\), is chosen with the property that the points

\[
X_{k,(j+1)} = \begin{cases} 
X_{k,(j)} = G(t_k, X_{k,j}), & j = 0, \ldots, j_k - 1, \\
X_{k,(j+1)} = X_{k,j}, & j = k, \ldots, N - 1, \\
X_{k,0} = X_{0}, & j = 0,
\end{cases}
\]

(45)

are well-defined and such that

\[
X_{N,(j+1)} = G(1, X_{N,j}),
\]

(46)

converges to \(X(1)\) as \(j \rightarrow \infty\).

The main idea is to choose partition (44) so that \(X(t_k)\) lies in some domain of attraction \(D_{(k+1)}\), for each \(k, 1 \leq k \leq N\). Then, if \(X_{k,0} \in D_{(k+1)}\), the sequence generated by (43) for \(t = t_k\) must produce an iterate \(X_{k,j_k} \in D_{(k+1)}\), which in turn can be taken as the starting point \(X_{k+1,0} = X_{k,j_k}\) for the next iteration involving \(f_{k+1}\). Thus, with \(X_{k,0} = X_{0}\) as initial point, the entire process can be carried out until finally \(t_k = t_N = 1\) is reached. For \(t = 1, X_N = X_{N,(j+1)}\) is then in \(D_1\) which ensures that (45) converges to \(X(1)\) as \(j \rightarrow \infty\).

To discuss the feasibility of the abovementioned numerical continuation process, we will use the following definition and lemmas which can be found in [34].

Definition 1 (see [34]). If a partition (44) exists so that with some sequence of integers \(\{j_k\}\), the entire process (45)–(46) is well defined so that (46) converges to \(X(1)\), and then the numerical continuation process (45)–(46) is called feasible.

Definition 2 (see [34]). Let \(G: \mathbb{R}^n \rightarrow \mathbb{R}^n\) be a given mapping. Then, any nonempty set \(D_0 \subset \mathbb{R}^n\) is a domain of attraction of the iterative process:

\[
x_{n+1} = G(x_n), \quad n = 0, 1, \ldots.
\]

(47)

with respect to the point \(x_0\), if for any \(x_0 \in D_0\), we have \(\{x_n\} \subset D\) and \(\lim_{n \rightarrow \infty} x_n = x_0\).

If \(x_n \in \text{int}(D_0)\) for some domain of attraction \(D_0\), then \(x_n\) is a point of attraction of (47).

Lemma 7 (see [34]). Let \(G: \mathbb{R}^n \rightarrow \mathbb{R}^n\) be Fréchet differentiable at the fixed point \(x_0 \in \text{int}(D)\) of \(G\). If \(\rho(G'(x_0)) < 1\), then \(x_0\) is a point of attraction of (47) and, more precisely, there is an open ball \(S(x_0, r)\) with center \(x_0\) and radius \(r > 0\) which is a domain of attraction of (47) with respect to \(x_0\). Here, \(\rho(\cdot)\) denotes the spectral radius of \(G'(x_0)\).
Lemma 8 (see [34]). Let $G$: $[0, 1] \times D \subset [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $D$ is open and assume that $x$: $[0, 1] \rightarrow D$ is continuous and satisfies $x(t) = G(t, x(t))$. Let $G$ have a Fréchet derivative with respect to $x$ at $(t, x(t))$ for every $t \in [0, 1]$. If $G_x(t, x(t))$ is continuous on $[0, 1] \times D$ and $\rho(G_x(t, x(t))) < 1$ for all $t \in [0, 1]$, then the numerical continuation process (45)--(46) is feasible.

In what follows, we derive a sufficient condition for the existence of a unique HPD solution of the homotopy equation $H(t, x) = 0$ for all $t \in [0, 1]$.

**Theorem 3.** If $\lambda \left( \sum_{i=1}^{m} B_i^* B_i \right) < (m/(m+1))^2$, then for arbitrary $t \in [0, 1]$, the homotopy equation $H(t, x) = 0$ has a unique HPD solution on $[(m/(m+1))I, I]$.

**Proof.** Since $t > 0$, then the homotopy equation $H(t, x) = 0$ can be rewritten as

$$X + \sum_{i=1}^{s} (\sqrt{r} B_i^*)^* X^{-1} (\sqrt{r} B_i) + \sum_{i=(s+1)}^{m} (\sqrt{r} B_i^*)^* X^i (\sqrt{r} B_i) = I.$$  

(48)

By the hypothesis of the theorem, we have

$$\lambda_1 \left( \sum_{i=1}^{m} (\sqrt{r} B_i^*)^* (\sqrt{r} B_i) \right) = \lambda_1 \left( \sum_{i=1}^{m} B_i^* B_i \right) \leq \lambda_1 \left( \sum_{i=1}^{m} B_i^* B_i \right)^{1/2} \lambda \left( \sum_{i=1}^{m} B_i^* B_i \right)^{1/2} \leq \lambda \left( \sum_{i=1}^{m} B_i^* B_i \right) < (m/(m+1))^2.$$  

(49)

It follows from Theorem 2 that the homotopy equation $H(t, x) = 0$ has a unique HPD solution on $[(m/(m+1))I, I]$.

In the next theorem, the local convergence of the iterative process (43) is obtained.

**Theorem 4.** If $F(X) = X$ has a unique HPD solution $X_\ast$ on $((m/(m+1))I, I)$, then there exists an open ball $N(X_\ast, \delta)$ with center $X_\ast$ and radius $\delta > 0$ such that, for any starting value $X_0 \in N(X_\ast, \delta)$, $X_n = F(X_{n-1})$ converges to $X_\ast$ as $n \rightarrow \infty$.

**Proof.** Recall that $F(X) = I - \sum_{i=1}^{s} B_i^* X^{-1} B_i - \sum_{i=(s+1)}^{m} B_i^* X^i B_i$. According to Lemma 2, for any $h \in \mathcal{G}_{n,n}$, we have

$$F(X_\ast + h) - F(X_\ast) = \sum_{i=1}^{s} B_i^* (X_\ast^{-1} - (X_\ast + h)^{-1}) B_i + \sum_{i=(s+1)}^{m} B_i^* (X_\ast^i - (X_\ast + h)^i) B_i$$

$$= \sum_{i=1}^{s} B_i^* (X_\ast + h)^{-1} h X_\ast^{-1} B_i - \sum_{i=(s+1)}^{m} \frac{B_i^* \sin t \pi}{\pi} \int_{0}^{\infty} X_\ast^{-1} h(\lambda I + X_\ast)^{-1} \lambda^{i-1} B_i d\lambda$$

$$+ \sum_{i=(s+1)}^{m} \frac{B_i^* \sin t \pi}{\pi} \int_{0}^{\infty} X_\ast^{-1} h(\lambda I + X_\ast)^{-1} \lambda^{i-1} B_i d\lambda.$$  

(50)

By the definition of Fréchet derivative, we obtain

$$F'(X_\ast)h = \sum_{i=1}^{s} B_i^* X_\ast^{-1} h X_\ast^{-1} B_i - \sum_{i=(s+1)}^{m} \frac{B_i^* \sin t \pi}{\pi} \int_{0}^{\infty} X_\ast^{-1} h(\lambda I + X_\ast)^{-1} \lambda^{i-1} B_i d\lambda$$

$$+ \sum_{i=(s+1)}^{m} \frac{B_i^* \sin t \pi}{\pi} \int_{0}^{\infty} X_\ast^{-1} h(\lambda I + X_\ast)^{-1} \lambda^{i-1} B_i d\lambda.$$  

(51)

Let $\lambda$ be any eigenvalue of $F'(X_\ast)$. Then, there exists a nonzero matrix $h_\ast$ such that

$$F'(X_\ast)h_\ast = \lambda h_\ast.$$  

(52)

Since $X_\ast$ is the unique HPD solution of $F(X) = X$, then by Theorem 1, there exists $P \in \mathcal{G}_{n,n}$, $Q_i \in \mathcal{G}_{m,n}$, $i = 1, 2, \ldots, m$, and diagonal matrices $\Gamma, \Lambda > 0$ such that

$$B_i = \begin{cases} \Gamma^{(1/2)} Q_i \Lambda P, & i = 1, 2, \ldots, s, \\ \Gamma^{-(1/2)} Q_i \Lambda P, & i = s+1, s+2, \ldots, m, \end{cases}$$

(53)

where

$$\sum_{i=1}^{m} Q_i^* Q_i = I,$$

$$\Gamma + \Lambda^2 = I.$$  

In this case, $X_\ast = P^* \Gamma P$, where $\Gamma = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\{\lambda_i\}$ the eigenvalues of $X_\ast$. Therefore, (52) can be rewritten as
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\[ F(X, h) = \sum_{i=1}^{l} P^* \Lambda Q_i^* \Gamma^{-1/2} P h^* \Gamma^{-1/2} Q_i \Lambda P \]

\[ - \sum_{i=(s+1)}^{m} \frac{\sin f_i \pi}{\pi} \int_{0}^{\infty} P^* \Lambda Q_i^* \Gamma^{-1/2} P h^* \Gamma^{-1/2} Q_i \Lambda P \, d\lambda 
\]

\[ + \sum_{i=(s+1)}^{m} \frac{\sin f_i \pi}{\pi} \int_{0}^{\infty} P^* \Lambda Q_i^* \Gamma^{-1/2} \Gamma (\lambda + \Gamma)^{-1} \Gamma^{-1/2} Q_i \Lambda P \, d\lambda \]

\[ = \lambda h^*. \]

Let \( z = Ph^* \). It follows that

\[ \sum_{i=1}^{l} \Lambda Q_i^* \Gamma^{-1/2} \Gamma - (1/2) Q_i \Lambda - \sum_{i=(s+1)}^{m} \frac{\sin f_i \pi}{\pi} \int_{0}^{\infty} \Lambda Q_i^* \Gamma^{-1/2} \Gamma (\lambda + \Gamma)^{-1} \Gamma^{-1/2} Q_i \Lambda \, d\lambda 
\]

\[ + \sum_{i=(s+1)}^{m} \frac{\sin f_i \pi}{\pi} \int_{0}^{\infty} \Lambda Q_i^* \Gamma^{-1/2} \Gamma (\lambda + \Gamma)^{-1} \Gamma^{-1/2} Q_i \Lambda \, d\lambda = \lambda z. \]

Define the operator \( \mathcal{L}_\Gamma : \mathcal{C}^{\text{mon}} \rightarrow \mathcal{C}^{\text{mon}} \) by

\[ \mathcal{L}_\Gamma z = \sum_{i=1}^{l} \Lambda Q_i^* \Gamma^{-1/2} \Gamma - (1/2) Q_i \Lambda - \sum_{i=(s+1)}^{m} \frac{\sin f_i \pi}{\pi} \int_{0}^{\infty} \Lambda Q_i^* \Gamma^{-1/2} \Gamma (\lambda + \Gamma)^{-1} \Gamma^{-1/2} Q_i \Lambda \, d\lambda 
\]

\[ + \sum_{i=(s+1)}^{m} \frac{\sin f_i \pi}{\pi} \int_{0}^{\infty} \Lambda Q_i^* \Gamma^{-1/2} \Gamma (\lambda + \Gamma)^{-1} \Gamma^{-1/2} Q_i \Lambda \, d\lambda \]

Then, using (58) and Lemmas 3 and 4, we can rewrite (57) as

\[ \mathcal{L}_\Gamma z = \lambda z. \]
Let

\[ B = \Lambda \otimes \Lambda, J_i = Q_i^T \otimes Q_i^*, \]

\[ C_i = \left( \Gamma^{-t_i/2} \otimes \Gamma^{-t_i/2} \right) \cdot \frac{\sin t_i \pi}{\pi} \int_0^\infty \left[ (\lambda I + \Gamma)^{-1} \otimes I \right] \lambda^{s_i - 1} d\lambda, \quad i = s + 1, s + 2, \ldots, m, \]

\[ D_i = \left( \Gamma^{-t_i/2} \otimes \Gamma^{-t_i/2} \right) \cdot \frac{\sin t_i \pi}{\pi} \int_0^\infty \left( (\lambda I + \Gamma)^{-1} \otimes (\lambda I + \Gamma)^{-1} \right) \lambda^{s_i - 1} d\lambda, \quad i = s + 1, s + 2, \ldots, m, \]

\[ E = \Gamma^{-1/2} \otimes \Gamma^{-1/2} = \text{diag}(\lambda_i^{-1/2}, \lambda_j^{-1/2}), \quad i, j = 1, 2, \ldots, n. \]

Then, (59) can be rewritten as

\[ \text{vec}(\mathcal{F}z) = \left[ \sum_{i=s+1}^m B_j(D_i - C_i) + \sum_{i=1}^r B_iE \right] \cdot \text{vec} = \lambda \cdot \text{vec}z. \quad (61) \]

According to (36), (55), (57), (59), and (61), we have that

\[ \rho(F'(X_\star)) = \max[|\lambda|] = \rho\left( \sum_{i=s+1}^m B_j(D_i - C_i) + \sum_{i=1}^r B_iE \right) \]

\[ \leq \left\| \sum_{i=s+1}^m B_j(D_i - C_i) + \sum_{i=1}^r B_iE \right\| \leq \sum_{i=s+1}^m \|B(C_i - D_i)\| + \sum_{i=1}^r \|BE\|. \quad (62) \]

Assume that

\[ \Lambda = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n). \quad (63) \]

According to Lemma 5 and (64), we have

\[ 0 < \sigma_j = \sqrt{1 - \lambda_j}, \quad j = 1, 2, \ldots, n. \]

By (54), we have

\[ B = \text{diag}(\sigma_j \cdot \sigma_j)_{n^2 \times n^2} = \text{diag}(\sqrt{1 - \lambda_j} \cdot \sqrt{1 - \lambda_j})_{n^2 \times n^2}, \quad (65) \]

\[ C_i = \left( \Gamma^{-t_i/2} \otimes \Gamma^{-t_i/2} \right) \cdot \frac{\sin t_i \pi}{\pi} \int_0^\infty \left[ (\lambda I + \Gamma)^{-1} \otimes I \right] \lambda^{s_i - 1} d\lambda \]

\[ = \text{diag}(\lambda_i^{-t_i/2} \cdot \lambda_j^{-t_i/2}) \cdot \frac{\sin t_i \pi}{\pi} \int_0^\infty (\lambda + \lambda_i)^{-1} \lambda^{s_i - 1} d\lambda \]

\[ = \text{diag}(\lambda_i^{-t_i/2} \cdot \lambda_j^{-t_i/2})_{n^2 \times n^2}, \quad i = (s + 1), (s + 2), \ldots, m, l, j = 1, 2, \ldots, n, \]

\[ D_i = \left( \Gamma^{-t_i/2} \otimes \Gamma^{-t_i/2} \right) \cdot \frac{\sin t_i \pi}{\pi} \int_0^\infty \left( (\lambda I + \Gamma)^{-1} \otimes (\lambda I + \Gamma)^{-1} \right) \lambda^{s_i - 1} d\lambda \]

\[ = \text{diag}(\lambda_i^{-t_i/2} \cdot \lambda_j^{-t_i/2}) \cdot \frac{\sin t_i \pi}{\pi} \int_0^\infty (\lambda + \lambda_i)^{-1} (\lambda + \lambda_j)^{-1} \lambda^{s_i - 1} d\lambda \]

\[ = \text{diag}(\frac{\lambda_i^{-t_i/2} \cdot \lambda_j^{-t_i/2}}{\lambda_i - \lambda_j})_{n^2 \times n^2}, \quad i = (s + 1), (s + 2), \ldots, m, l, j = 1, 2, \ldots, n. \quad (67) \]
We have that 

\[
\|C_i - D_i\| = \max_{i,j} \left\{ \frac{(1 - \lambda_i)(1 - \lambda_j)}{(\lambda_i - \lambda_j)^{1/2}\lambda_i^{1/2}} \right\} = \max_{i,j} \{ f(\lambda_i, \lambda_j, t_i) \} \tag{68} \]

where \( f(x, y, t) \) is defined in Lemma 6. Combining Lemma 6 \((62)\) with \((68)\) gives that

\[
\rho(F'(X_i)) \leq \sum_{i=1}^{i} \|B(C_i - D_i)\| + \sum_{i=1}^{i} \|BE\| < 1. \tag{69} \]

According to Lemma 7, there exists an open ball \( N(X_*, \delta) \) with center \( X_* \) and radius \( \delta > 0 \) such that, for any starting value \( X_0 \in N(X_*, \delta), X_n = F(X_{n-1}) \) converges to \( X_* \) as \( n \rightarrow \infty \).

In the next theorem, we will prove the numerical continuation process \((45)-(46)\) is feasible.

**Theorem 5.** If \( \lambda_i \left( \sum_{i=1}^{m} B_i^* B_i \right) < m/(m+1)^2 \), then the numerical continuation process \((45)-(46)\) is feasible.

**Proof.** Define the map \( G: [0,1] \times \mathcal{G}^{\text{proj}} \rightarrow \mathcal{G}^{\text{proj}} \) by

\[
G(t, X(t)) = I - t \sum_{i=1}^{m} B_i^*X^iB_i - t \sum_{i=1}^{m} B_i^*X_iB_i. \tag{70} \]

In the following, we will prove the numerical continuation processes \((45)-(46)\) are feasible.

By Lemma 2, for any \( h \in \mathcal{G}^{\text{proj}} \), we have

\[
G(t, X(t) + h) - G(t, X(t)). \tag{71} \]

By the definition of Fréchet derivative, we obtain

\[
G_t(t, X(t))h = \sum_{i=1}^{m} B_i^*X^iX_iB_i - \sum_{i=1}^{m} B_i^* \frac{\sin t_i \pi}{\pi} \int_0^\infty h(\lambda I + X)X(\lambda I + X)^{-1}h(\lambda I + X)^{-1}X_iB_i d\lambda. \tag{73} \]

Using the same technique described in Theorem 4, we have that

\[
\rho(G_t(t, X(t))) < \left( \sum_{i=1}^{m} \max_{i,j} \left\{ \frac{(1 - \lambda_i)(1 - \lambda_j)}{(\lambda_i - \lambda_j)^{1/2}\lambda_i^{1/2}} \right\} + \sum_{i=1}^{m} \max_{i,j} \left\{ \frac{(1 - \lambda_i)(1 - \lambda_j)}{(\lambda_i - \lambda_j)^{1/2}\lambda_i^{1/2}} \right\} \right) < 1. \tag{74} \]
where \((mt_i/(mt_i + 1)) < (m/(m + 1)) < \lambda_i, \lambda_j < 1, \quad l, j = 1, 2, \ldots, n\).

According to Lemma 2, the numerical continuation processes (45)–(46) are feasible. □

5. Conclusions

We have introduced a class of nonlinear matrix equations that is wider than those studied earlier in the literature. We have derived some sufficient and necessary conditions for the existence and uniqueness of HPD solution of (1). In order to apply the homotopy continuation iterative method proposed by Avila [34] to the nonlinear matrix equation (1), we have constructed a related homotopy matrix equation \(H(t, X) = 0\) and have derived a sufficient condition for the existence of a unique HPD solution of this equation. We have cited the definition and the judgment theorem of existence of a unique HPD solution of this equation. We have introduced a class of nonlinear matrix equations (45)–(46), which was proposed in [34]. And then we have derived the condition for feasible of the numerical continuation processes (45)–(46). Furthermore, we obtained the feasible conclusion of the numerical method by verifying the conditions in Theorem 2.5 in [34].

Data Availability

No data were used to support our study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors’ Contributions

All authors read and approved the final manuscript.

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