Effect of short-range interaction for collision of ultracold dipoles

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We consider the scattering of two ultracold polarized dipoles with both a short-range interaction (SRI) and a weak dipole-dipole interaction (DDI) which is far away from a shape-resonance. In previous works the scattering amplitude is usually calculated via 1st-order Born approximation. Our results show that significant derivation from this approximation can arise in some cases. In these cases the SRI can significantly modify the dipole-dipole scattering amplitude, even if the scattering amplitude for the SRI alone is negligibly smaller than the dipolar length of the DDI. We further obtain approximate analytical expressions for the inter-dipole scattering amplitude.

I. INTRODUCTION

During the past a few decade, fast progresses have been made in the experimental study of complicated magnetic atoms\textsuperscript{[1–12]} and polar molecules\textsuperscript{[13–15]}. In an ultracold gas of these atoms or molecules, the interactions between particles include both a long-range dipole-dipole interaction (DDI) and a short-range interaction (SRI) (e.g., van der Waals interaction). The interplay between these two parts of interactions can usually lead to intriguing quantum phenomena in both two-body and many-body level. A well known example is that, when the DDI is near a shape resonance\textsuperscript{[16]}, a small hard-core SRI can significantly shift the location of the shape resonance\textsuperscript{[17, 18]}. In this paper we consider the opposite case where the DDI is away from a shape resonance, and furthermore the scattering amplitude for the SRI is much smaller than the characteristic length of the DDI (“dipole length”). In many previous works in this subject, the scattering amplitude contributed by the DDI is usually calculated via 1st-order Born approximation. In our investigation we calculate the scattering amplitude given by distorted wave Born approximation\textsuperscript{[27–29]} (DWBA), which is the accurate 1st-order approximation of the DDI. In the expression of scattering amplitude given by DWBA, there is one term which describes the DDI-SRI interplay (i.e., the SRI-induced effect for the inter-dipole scattering). We find that for the scattering of two ultracold bosonic dipoles, such term is negligible and the 1st-order Born approximation works very well, when two ultracold fermionic dipoles, such term is negligible and the SRI-DDI interplay is very important even if the scattering amplitude for SRI itself is negligibly smaller than the dipolar length of the DDI.

II. SCATTERING AMPLITUDE GIVEN BY DWBA

A. scattering amplitude

We consider the scattering of two dipoles polarized along $z$-direction. The Hamiltonian for the relative motion is given by $H = \frac{p^2}{2\mu} + V_{s\tau}(r) + V_d(r)$, where $p$ is inter-dipole relative momentum, $r$ is the relative coordinate, $\mu$ is the reduced mass, while $V_{s\tau}$ and $V_d$ are potentials for SRI and DDI, respectively. For simplicity, we assume $V_{s\tau}$ is isotropic. The DDI potential $V_d(r)$ is given by ($2\mu = \hbar = 1$)

$$V_d(r) = 2a_d \left(1 - 3 \cos^2 \theta_r \right), \quad (1)$$

where $\theta_r$ is the angle between $r$ and the $z$-axis (polar angle of $r$), and $a_d = d_1d_2/2$ is the dipolar moment, with $d_j$ the dipole moment of the $j$-th dipole.

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In such system, the scattering amplitude is defined as
\[ f(k_i, k_f) = -2\pi^2 |\langle V_d | k_f \rangle| |\langle k_i | V_d \rangle| \langle s_{k_f}^- | s_{k_i}^+ \rangle, \]
where \( k_i \) and \( k_f \) are incident and outgoing momentum, respectively. Due to energy conservation, they satisfy \( |k_i| = |k_f| = k \). Here \( |k| \) is the plane-wave eigenstate of the relative momentum operator with eigen-value \( k \), and \( \langle s_{k_i}^+ \rangle \) is the two-body scattering state. In this paper we consider the system where the dipole moment is weak enough, so that in the expression of \( f(k_i, k_f) \) we can only keep the terms up to the 1st order of \( V_d \). On the other hand, in reality the intensity of SRI is strong (for instance, the depth of the SRI between ultracold atoms can be as large as \( 10^{14} \text{Hz} \)), and cannot be treated as a perturbation. Thus, in our calculation we treat \( V_d \) non-perturbatively. Such treatment is the so called DWBA. As shown in appendix A, the scattering amplitude given by DWBA can be expressed as
\[ f(k_i, k_f) = f_{sr}(k_i, k_f) - 2\pi^2 (s_{k_f}^- | V_d | s_{k_i}^+) \tag{2} \]
where \( f_{sr}(k_i, k_f) \) is the exact scattering amplitude for \( V_{sr} \), while \( s_{k_f}^+ \) and \( s_{k_i}^- \) are the outgoing and incoming scattering states for \( V_{sr} \), respectively.

To investigate the influence of SRI on dipole-dipole scattering, i.e., the SRI-DDI interplay, we re-express the scattering amplitude as
\[ f(k_i, k_f) = f_{sr}(k_i, k_f) - 2\pi^2 (k_f | V_d | k_i) + g(k_i, k_f) \tag{3} \]
where \(-2\pi^2 (k_f | V_d | k_i)\) is contributed by the matrix element of the dipole-dipole interaction in the plane-wave basis, and the remaining term
\[ g(k_i, k_f) = -2\pi^2 \left( (s_{k_f}^- | V_d | s_{k_i}^+) - (k_f | V_d | k_i) \right) \tag{4} \]
is given by the differences between the scattering states \( s_{k_f}^+ \) for \( V_{sr} \) and the plane wave \( |k| \). The amplitude \( g(k_i, k_f) \) describes SRI-induced effect for the scattering amplitude.

In the 1st-order Born approximation which is widely used in the research for ultracold gases with weak dipolar interactions, the SRI-induced amplitude \( g(k_i, k_f) \) is neglected. As a result, the scattering amplitude is approximated as \( f(k_i, k_f) \approx f_{sr}(k_i, k_f) - 2\pi^2 (k_f | V_d | k_i) \). In this paper, we carefully investigate the importance and behavior of the amplitude \( g(k_i, k_f) \), and the applicability of the 1st-order Born approximation.

### B. Partial-wave expansion

To understand the problem more clearly, we make partial-wave expansion for the total scattering amplitude \( f(k_i, k_f) \) in Eq. (3):
\[ f(k_i, k_f) = 4\pi \sum_{l,l',m} F_{l,l'}^{(m)}(k) Y_l^m(\hat{k}_f) Y_l^m(\hat{k}_i)^* \tag{5} \]
where \( \hat{k} = k/|k| \) is the unit vector along the direction of \( k \), and \( Y_l^m \) is the spherical harmonic of degree \( l \) and order \( m \). Here \( F_{l,l'}^{(m)}(k) \) is the generalized partial-wave scattering amplitude, and can be expressed as
\[ F_{l,l'}^{(m)}(k) = F_{l,l'}^{(sr)}(k) \delta_{l,l'} + \mathcal{I}_{l,l'}^{(m)} + G_{l,l'}^{(m)}(k) \tag{6} \]
with \( F_{l,l'}^{(sr)}(k) \) the \( l \)-th partial-wave scattering amplitude for \( V_{sr} \). In Eq. (6) the \( k \)-independent parameter \( F_{l,l'}^{(sr)} \) and the function \( G_{l,l'}^{(m)}(k) \) are the partial-wave components of \(-2\pi^2 (k_f | V_d | k_i) \) and \( g(k_i, k_f) \), respectively, and satisfy the relations
\[ -2\pi^2 (k_f | V_d | k_i) = 4\pi \sum_{l,l',m} \mathcal{I}_{l,l'}^{(m)}(k) Y_l^m(\hat{k}_f) Y_l^m(\hat{k}_i)^* \tag{7} \]
and
\[ g(k_i, k_f) = 4\pi \sum_{l,l',m} G_{l,l'}^{(m)}(k) Y_l^m(\hat{k}_f) Y_l^m(\hat{k}_i)^* \tag{8} \]

It is clear that, the generalized partial-wave scattering amplitude given by 1st-order Born approximation is
\[ F_{\text{Born}(l,l')}^{(m)}(k) = F_{l,l'}^{(sr)}(k) \delta_{l,l'} + \mathcal{I}_{l,l'}^{(m)}, \tag{9} \]
and the SRI-induced effect or SRI-DDI interplay for the partial-wave scattering is described by the amplitude \( G_{l,l'}^{(m)}(k) \).

In the following sections, we investigate the magnitude of \( G_{l,l'}^{(m)}(k) \) for ultracold gases of polarized bosonic and fermionic dipoles. To this end, we first re-express this amplitude as
\[ G_{l,l'}^{(m)}(k) = a_d D_{l,l'}^{(m)} \times \int_0^\infty \frac{1}{r} \phi_{l}^{(+)}(k, r) \phi_{l}^{(+)}(k, r) - j_l(kr) j_l(kr) dr \tag{10} \]
with
\[ D_{l,l'}^{(m)} = 2i^{(l-l')} \int d\mathbf{r} Y_l^m(\hat{r})^* Y_{l'}^m(\hat{r})(3\cos^2 \theta_r - 1). \tag{11} \]

Here \( j_l(z) = j_l(a) \) is the \( l \)-th order regular spherical Bessel function, \( \phi_{l}^{(+)}(k, r) \) is the short-range radial scattering wave function in the \( l \)-th partial-wave channel, and satisfies \( \langle r | \phi_{l}^{(+)}(k, r) \rangle = \sqrt{\frac{2}{\pi \sqrt{\pi l(l+1)}}} \sum_{m} i^l \phi_{l}^{(m)}(|k|, r) Y_l^m(\hat{r}) Y_l^m(\hat{k})^* \), with \( |r| \) the eigen-state of inter-dipole relative coordinate operator with eigen-value \( r \). In the derivation of Eq. (10), we have used the fact \( \langle r | s_{k_i}^{(-)} \rangle = \langle r | s_{k_f}^{(+)} \rangle^* \), which is due to the boundary conditions satisfied by \( s_{k_i}^{(+)} \).

Eq. (10) shows that the magnitude of \( G_{l,l'}^{(m)}(k) \) is determined by the difference between the short-range scattering wave functions \( \phi_{l}^{(+)}(k, r) \) and the free wave function \( j_l(kr)(kr) \) in the \( l \)-th\((l'\text{-th}) \) partial wave channels. In the following two sections, for simplicity, we assume
SRI effects in high-partial-wave channels of our systems are very weak, so that in the calculation of $G^{(m)}_{1,l,l'}(k)$ for dipolar bosons (fermions), the scattering wave function $\phi_l(k,r)$ with $l > 0$ ($l > 1$) can be approximated as $j_l(kr)$. Namely, we only consider the effects induced by the difference between $\phi_0(k,r)$ and $j_0(kr)$ for dipolar bosons, and the effects from the difference between $\phi_l(k,r)$ and $j_l(kr)$ for dipolar fermions. In Sec. V we consider a system the difference between $\phi_2^+(k,r)$ and $j_2(kr)$ cannot be neglected.

III. SRI-INDUCED EFFECTS FOR BOSONIC DIPOLES

In this section we consider ultracold gas of polarized bosonic dipoles. As shown above, we only consider the effect given by the difference between $\phi_0(k,r)$ and $j_0(kr)$. Therefore, the SRI-induced terms $G^{(m)}_{1,l,l'}(k)$ is non-zero only for $(l, l') = (0, 2)$ or $(2, 0)$. It is also easy to prove that $G^{(0)}_{0,2}(k) = G^{(0)}_{2,0}(k)$. Thus, the SRI-induced effect for inter-dipole scattering is completely described by the amplitude $G^{(0)}_{0,2}(k)$.

With a straightforward calculation based on Eq. (10), we obtain (appendix B)

$$G^{(0)}_{0,2}(k) = \frac{a_d}{3\sqrt{5}} \frac{ka}{(ika + 1)},$$

where $a$ is scattering length of the SRI. Furthermore, the amplitudes $P^{(0)}_{0,2}$ contributed by 1st-order Born approximation is $P^{(0)}_{0,2} = -a_d/(3\sqrt{5})$. Therefore, $G^{(0)}_{0,2}(k)$ is comparable with $P^{(0)}_{0,2}$ only when $ka \to \infty$. In such a limit the SRI-induced effect is significant. Nevertheless, in the systems where the s-wave scattering length $a$ is comparable or smaller than the dipolar length $a_d$, and the inter-atomic momentum $k$ is much smaller than $1/a_d$, the SRI-induced amplitude $G^{(0)}_{0,2}(k)$ is much smaller than the terms from 1st-order Born approximation, and can be safely neglected. In these systems the 1st-order Born approximation is a very good approximation. This conclusion is also supported by Fig. 1 where the behaviors of $P^{(0)}_{0,2}$ and $G^{(0)}_{0,2}(k)$ for systems with different scattering length $a$ are illustrated.

IV. SRI-INDUCED EFFECTS FOR FERMIonic DIPOLES

A. “fast-decay” type SRI

Now we investigate the SRI-induced effects for the scattering of two fermionic dipoles. We first assume the SRI potential $V_{sr}(r)$ is a “fast-decay” type potential, which decreases faster than every power of $1/r$ as $r \to \infty$ and becomes negligible when $r$ is larger than a characteristic distance $b$. In our system the SRI-induced term $G^{(m)}_{1,l,l'}(k)$ can be obtained with numerical calculation based on Eq. (10). Furthermore, in the low-energy limit where the short-range $p$-wave scattering amplitude can be approximated as $F^{(sr)}_{1}(k) = -vk^2$ with $v$ the scattering volume of the SRI [31], with the help of effective-range theory we can obtain the approximated analytical expression for $G^{(m)}_{1,l,l'}(k)$ (appendix C):

$$G^{(m)}_{1,l,l'}(k) \approx a_d \frac{v^2k^2}{r_a^4} \lambda_m,$$

$$G^{(m)}_{1,l,l'}(k) \approx 0, \quad \text{for} \quad (l, l') \neq (1, 1).$$

Here $\lambda_0 = 2/5$ and $\lambda_{l+1} = -1/5$. $r_a$ is a $k$-independent parameter determined by the short-range detail of $V_{sr}(r)$. It is defined as $r_a = [b^2 + 4 + \frac{6}{b}\beta(r)r^2]^{-1/4}$, where $\beta(r)$ is the $k$-independent real function which satisfies $\phi_1(k,r) \propto F^{(sr)}_{1}(k)\beta(r)/k$ for $r \ll 1/k$ (appendix C). Eq. (13) yields that, the generalized partial-wave scattering amplitude modified by the SRI-induced amplitude $G^{(m)}_{1,l,l'}(k)$ is

$$F^{(m)}_{1}(k) \approx -vk^2 + a_d \left(1 + \frac{v^2k^2}{r_a^4}\right) \lambda_m.$$

Here we have used the fact that $P^{(m)}_{1,1} = a_d\lambda_m$.

Eq. (13) and (15) show that, when the scattering energy is exactly zero, i.e., $k = 0$, we have $G^{(m)}_{1,1}(k) = 0$, and thus the total scattering amplitude $F^{(m)}_{1,1}(k)$ equals to $F^{(m)}_{1,1}(k)$ from 1st-order Born approximation. However, for the cases with finite incident momentum $k$, $F^{(m)}_{1,1}(k)$ is modified by SRI-induced scattering amplitude $G^{(m)}_{1,1}(k)$. In particular, when $|v| \gg r_a^4/a_d$, $G^{(m)}_{1,1}(k)$ is much larger than the short-range scattering amplitude $-vk^2$, and
dominates the variation of $F_{1,1}^{(m)}(k)$ with $k$. In that case the increasing of $F_{1,1}^{(m)}(k)$ with $k$ is seriously enhanced by such SRI-induced term.

Such SRI-induced effect is important in the system where $|v|$ is much larger than both $r_a^2/b$ and $r_b^2/a_d$. In such system, when $k$ has the same order of magnitude as $r_a^2/|v|$ [32], we have $v^2k^2/r_a^2 \sim 1$ and thus $G_{1,1}^{(m)}(k) \sim P_{1,1}^{(m)}$. Namely, the SRI-induced scattering term $G_{1,1}^{(m)}(k)$ becomes comparable to the one given by 1st-order Born approximation, and cannot be neglected in the expression of the total scattering amplitude $F_{1,1}^{(m)}$. Accordingly, the 1st-order Born approximation $F_{1,1}^{(m)} \approx F_{\text{Born}(1,1)}^{(m)}$ is no longer applicable. Furthermore, in that case we also have $|v|k^2 \sim r_a^2/|v| \ll a_d$, which implies $|vk^2| \ll |F_{1,1}^{(m)}| \sim |G_{1,1}^{(m)}(k)|$. Therefore, the SRI-induced effect is strong although the short-range scattering amplitude $-vk^2$ is still negligibly small.

In Fig. 2 we illustrate such SRI-induced effect with a simple square-well model of $V_{sr}$, i.e., $V_{sr}(r < b) = -V_0$ and $V_{sr}(r > b) = 0$. In this model the scattering volume $v$ is determined by the depth $V_0$ of the square well. We calculate the total scattering amplitude $F_{1,1}^{(m)}$ with the exact numerical calculation based on Eqs. (6) (10), and the approximated expression (15). For comparison, we also calculate the scattering amplitude $F_{\text{Born}(1,1)}^{(m)}$ from 1st-order Born approximation, and the short-range scattering amplitude $F_{1}^{(sr)}$. In our calculation we take $a_d = 0.05b$. As shown in Fig. 2, when the scattering volume $v$ is small, 1st-order Born approximation works very well. Nevertheless, when $v$ is large, $F_{1,1}^{(m)}$ is modified by the SRI-induced amplitude $G_{1,1}^{(m)}$, and significantly differs from the result $F_{\text{Born}(1,1)}^{(m)}$ given by 1st-order Born approximation, although the short-range scattering amplitude $F_{1}^{(sr)}$ is still negligible. Furthermore, it is also shown that our analytical expression (15) is always a good approximation for $F_{1,1}^{(m)}$. For such a system we also calculate the SRI-induced amplitude $G_{1,3}^{(m)}(k)$ and $G_{3,1}^{(m)}(k)$. The calculation shows that, as predicted in Eq. (14), when $F_{1}^{(sr)} \ll a_d$ these amplitudes are 3 order of magnitude smaller than the amplitudes $P_{1,3}^{(m)}$ and $P_{3,1}^{(m)}$ from 1st-order Born approximation, and thus can be safely neglected.

The reason of the significant SRI-induced effect we derived above can be understood with the following rough analysis. According to the scattering theory [29], in the region $r > b$ we have $|\phi_1(k, r) - j_1(kr)| \propto |F_{1}^{(sr)}(k)|$. As a result, in the SRI-induced scattering amplitude $G_{1,1}^{(m)}(k)$ given by Eq. (10), the leading term is proportional to $|F_{1}^{(sr)}(k)|^2$. Therefore, in some cases the term $G_{1,1}^{(m)}(k)$ can become significantly large, although the short-range scattering amplitude $F_{1}^{(sr)}(k)$ is still very small.

Here we also point out that, Eq. (15) can be re-written as

$$F_{1,1}^{(m)}(k) \approx -v_{\text{eff}}^{(m)}k^2 + P_{1,1}^{(m)},$$

where

$$v_{\text{eff}}^{(m)} = v - \frac{a_d\lambda_m}{r_a^4}v^2.$$  

With this result, the effect from the amplitude $G_{1,1}^{(m)}(k)$ can also be understood as “DDI-induced modification of scattering volume”. Namely, due to the DDI-SRI interplay, the effective scattering volume of the two dipoles is shifted from $v$ to $v_{\text{eff}}^{(m)}$, which depends on the quantum number $m$ for the relative angular momentum along $z$-direction. In previous works people have studied the variation of scattering volume by DDI in the close channel of a $p$-wave Feshbach resonance [33]. Since our calculation is based on a single-channel model, the shift of scattering volume we study here is due to the DDI in the open channel.

### B. “Van-der-Waals-ttype” SRI.

Now we consider another case where the SRI is a “van der Waals type potential”, which has the behavior $-\beta_6/r^6$ when $r$ is larger than a distance $c$. Here we assume both $\beta_6$ and $c$ are much smaller than $1/k$. In this case the effective-range theory cannot be used to study the scattering problem for $V_{sr}$ [34]. As a result, we cannot use analytical techniques to derive approximated expression for the scattering amplitudes. Thus, we study the SRI-induced effect via direct numerical calculation for $F_{1,1}^{(m)}(k)$ with Eq. (6). In our calculation we consider the limit $c \rightarrow 0$. In such a limit the scattering-state wave function $(r|s_k^\pm)$ is the superposition of the incident plane wave and the exact solutions of the $p$-wave Schroedinger equation with van der Waals potential, which is analytically obtained by Bo Gao in Ref. [35]. The superposition coefficient is determined by the scattering volume of the system [34].

In Fig. 3 we show our numerical results for the exact scattering amplitude $F_{1,1}^{(m)}(k)$, the scattering amplitude $F_{\text{Born}(1,1)}^{(m)}(k)$ from 1st-order Born approximation, and the short-range scattering amplitude $F_{1}^{(sr)}$. In our calculation we take $a_d = 0.05\beta_6$. It is shown that, in our system strong SRI-induced effect can also appear when the $p$-wave scattering volume is large enough. In these cases $F_{1,1}^{(m)}(k)$ cannot be approximated as $F_{\text{Born}(1,1)}^{(m)}(k)$, although $F_{1}^{(sr)}$ is much smaller than $a_d$.

Furthermore, we are surprised to find that our numerical also agrees very well with expression (15) with $r_a = 0.4845\beta_6$ (Fig. 3). Such fact implies that, although Eq. (15) is derived for the system with fast-decay type SRI, it can still be used to describe the scattering amplitude for the system with van der Waals type
SRI. Nevertheless, in the current system $r_a$ becomes a $k$-independent fitting parameter which is, in principle, a function of $v$.

The SRI-induced effect we discussed above is possible to be experimentally observed via the measurement of scattering crosssection of ultracold fermionic dipoles. In Fig. 4 we illustrate the total crosssection $\sigma = \int d\mathbf{k}_i d\mathbf{k}_f |f_{\ell}(\mathbf{k}_i, \mathbf{k}_f)|^2 /4\pi$ with $f_{\ell}(\mathbf{k}_i, \mathbf{k}_f) = [f(\mathbf{k}_i, \mathbf{k}_f) - f(\mathbf{k}_i, -\mathbf{k}_f)]/\sqrt{2}$ for the system with van der Waals type SRI with $a_0 = 0.05\beta_0$. It is shown that when the scattering volume of the SRI is large enough, the scattering crosssection is significantly enhanced by the SRI-induced effect.

V. SRI-INDUCED EFFECTS FOR BOSONIC DIPOLES WITH STRONG D-WAVE SRI

In above two sections, we neglect the difference between the short-range scattering wave function $\phi_1^{(+))(k,r)}$ and the free wave function $j_1(kr)$ in high partial-wave channels. Such approximation is applicable when the low-energy short-range scattering effects in these channels are very weak. In this section we consider the ultracold bosonic dipoles with a “fast-decay”-type SRI which is strong in $d$-wave channel. For such a system, both the difference between $\phi_0^{(+))(k,r)}$ and $j_0(kr)$ and the difference between $\phi_2^{(+))(k,r)}$ and $j_2(kr)$ should be taken into account. We find that under particular conditions, the generalized partial-wave scattering amplitudes $F_{2,2}^{(m)}(k)$ and $F_{2,0}^{(m)}(k)$ can differ from the result from Born approximation even when $k = 0$.

According to the effective range theory, for “fast-decay” type SRI the low-energy $d$-wave scattering amplitude $F_2^{(sr)}(k)$ is given by

$$F_2^{(sr)}(k) = \frac{-1}{ik + \frac{w}{\alpha} + \frac{1}{\beta} + \frac{1}{\alpha'}},$$

with parameters $w$, $s_*$ and $u$ determined by the detail of the SRI. In this section we consider the systems where $s_*$ and $u$ have the same order of magnitude as $b$. For such
a system, with direct calculations (appendix D) we find that the SRI-induced amplitudes can be approximately expressed as

\[
G_{2,2}^{(m)}(k) \approx a_d \left[ \frac{3}{2} \frac{F_2^{(sr)}(k)^2}{r_b^2 k^4} + \frac{2}{5} \frac{F_2^{(sr)}(k)}{b} \right] D_{2,2}^{(m)}; \quad (19)
\]

\[
G_{2,0}^{(m)}(k) \approx a_d \left[ \frac{F_2^{(sr)}(k)}{r_c(k)^3 k^2} + \frac{F_0^{(sr)}(k)}{kb^3} \right] D_{2,0}^{(m)}, \quad (20)
\]

\(G_{0,2}^{(m)}(k) = G_{2,0}^{(m)}(k),\) and \(G_{1,1}^{(m)}(k, \Delta) \approx 0\) for \((l, l') \neq (2, 2), (2, 0)\) and \((0, 2)\). Here the parameters \(D_{2,2}^{(m)}\) and \(D_{2,0}^{(m)}\) are defined in Eq. (11). The ranges \(r_b\) and \(r_c(k)\) are defined as \(r_b = (b^{-6} + \frac{2}{3} \int_0^b \beta_2(r)^2/rdr)^{-1/6}\) and \(r_c(k) = (b^{-3} + \int_0^b \phi_0^{(+)}(k, r) \beta_2(r)/rdr)^{-1/3}\), respectively, with function \(\beta_2(r)\) satisfying \(\phi_2^{(+)}(k, r) \leq b = F_2^{(sr)}(k)/2k^2\).

Using Eq. (18) and Eqs. (19) and (20), we find that when \(w = \infty\) the SRI-induced amplitudes \(G_{2,2}^{(m)}(k)\) and \(G_{2,0}^{(m)}(k)\) are non-zero in the zero-energy limit \(k = 0\), i.e., we have

\[
G_{2,2}^{(m)}(k = 0) = \frac{3a_d s}{2r_b^2} D_{2,2}^{(m)}; \quad (21)
\]

\[
G_{2,0}^{(m)}(k = 0) = \frac{a_d s}{k r_c(0)^3} D_{2,0}^{(m)}. \quad (22)
\]

Therefore, when \(w = \infty\) the SRI can significantly modify the generalized parital-wave scattering amplitudes \(F_{2,2}^{(m)}(k = 0) = 0\) and \(F_{2,2}^{(m)}(k = 0) = 0\) in the zero-energy limit, although in such limit the \(d\)-wave scattering amplitude \(F_2^{(sr)}(k)\) of the SRI is zero.

When the parameter \(w\) is finite, we have \(G_{2,2}^{(m)}(k = 0) = G_{2,0}^{(m)}(k = 0) = 0\). Thus, when \(k = 0\) the total scattering amplitude equals to the one from Born approximation. Nevertheless, when \(k\) is finite and \(w\) is much larger than \(b^5\), the SRI-induced amplitudes \(G_{2,2}^{(m)}(k)\) and \(G_{2,0}^{(m)}(k)\) can still be significant, even if \(F_2^{(sr)}(k)\) is much smaller than the dipolar length \(a_d\). This SRI-induced
FIG. 5. (color online) Scattering amplitudes (SFs) for system with square-well-type SRI. Here we consider the cases with $w = \infty$ (a, b) and $w = 2.2 \times 10^4 b^5$ (c, d). For each case we plot $F_{22}^{(0)}$ and $F_{20}^{(0)}$ given by numerical calculation based on Eqs. (6, 10) (red circles) and the approximated expressions (19, 20) (black solid line below the red circles), as well as the results from Born approximation (blue dashed lines). We also illustrate the $d$-wave short-range scattering amplitude $F_{2}^{(sr)}$ (black dashed-dotted line). Here we also neglect the imaginary parts of the scattering amplitudes, which are at least 6 order of magnitude smaller than the real parts.

VI. CONCLUSIONS AND DISCUSSIONS

In this work we investigate the interplay of SRI and DDI in the scattering of two polarized weak dipoles. We show that such interplay can be safely neglected for ultracold bosonic dipoles when the scattering length of the SRI is much smaller than the dipolar length $a_d$, and the effects of SRI in high partial-wave channels are negligible. Nevertheless, for ultracold fermionic dipoles with large scattering volume, or ultracold bosonic dipoles with a strong “fast-decay” type SRI in $d$-wave channel, such interplay is important even when the scattering amplitude for the SRI is much smaller than $a_d$. In these systems the 1st-order Born approximation is not applicable. We find analytical expressions for the scattering amplitudes of these systems. With these expression one can construct appropriate effective interaction potential in the many-body effective Hamiltonian.

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Appendix A: DWBA of DDI

In this appendix we derive the scattering amplitude given by DWBA, i.e., Eq. (2) of our main text. As shown in Sec. II, the scattering amplitude is defined as

$$f(k_f, k_i) = -2\pi^2 \langle k_f | (V_{sr} + V_d) | \psi_{k_i}^{(+)} \rangle.$$  \hspace{1cm} (A1)

Here the 2-dipolar scattering state $| \psi_{k_i}^{(+)} \rangle$ is given by the Lippman-Schwinger equation

$$| \psi_{k_i}^{(+)} \rangle = | k_i \rangle + G[V_{sr} + V_d]|k_i \rangle,$$  \hspace{1cm} (A2)

where the Green’s operator $G$ is defined as $G = [k^2 + i0^+ - (p^2 + V_{sr} + V_d)]^{-1}$, with $p$ is the relative-momentum operator. It is clear that $G$ satisfies

$$G = G_{sr} + G_{sr}V_dG,$$  \hspace{1cm} (A3)

with $G_{sr} = [k^2 + i0^+ - (p^2 + V_{sr})]^{-1}$. Substituting Eq. (A3) into Eq. (A2), we obtain

$$| \psi_{k_i}^{(+)} \rangle = | k_i \rangle + G_{sr}V_d|\psi_{k_i}^{(+)} \rangle = | k_i \rangle + G_{sr}V_d|s_{k_i}^{(+)} \rangle + \mathcal{O}(V_d^2),$$  \hspace{1cm} (A4)

where $|s_{k_i}^{(+)} \rangle = | k_i \rangle + G_{sr}V_{sr}|k_i \rangle$ is the outgoing scattering state for $V_{sr}$ itself. Substituting Eq. (A4) into Eq. (A1), and keeping the terms up to the 1st order of $V_d$, we have

$$f(k_f, k_i) = -2\pi^2 \left[ \langle k_f | V_{sr} | s_{k_i}^{(+)} \rangle + \langle k_f | V_{sr}G_{sr}V_d | s_{k_i}^{(+)} \rangle \right]$$

$$-2\pi^2 \langle k_f | V_d | s_{k_i}^{(+)} \rangle.$$  \hspace{1cm} (A5)

On the other hand, the scattering amplitude $f_{sr}(k_f, k_i)$ and the incoming scattering state $|s_{k_i}^{(-)} \rangle$ for $V_{sr}$ are given by $f_{sr}(k_f, k_i) = -2\pi^2 \langle k_f | V_{sr} | s_{k_i}^{(+)} \rangle$ and $|s_{k_i}^{(-)} \rangle = | k_i \rangle + G_{sr}^+V_{sr}|k \rangle$, respectively. With these results, we can re-write Eq. (A5) as

$$f(k_f, k_i) = f_{sr}(k_f, k_i) - 2\pi^2 \langle s_{k_f}^{(-)} | V_{sr} | s_{k_i}^{(+)} \rangle.$$  \hspace{1cm} (A6)

That is Eq. (2) in our main text.

Appendix B: SRI-induced amplitudes for bosonic dipoles

In this appendix we prove Eq. (12) in Sec. III. According to Eq. (10), the amplitude $G_{0,2}^{(0)}(k)$ is given by $G_{0,2}^{(0)}(k) = I_1 + I_2$ where

$$I_1 = -\frac{4\alpha_d}{\sqrt{3}} \int_0^b \frac{1}{r} \left[ \phi_0^{(+)}(k, r)\phi_2^{(+)}(k, r) - j_0(kr)j_2(kr) \right] dr;$$  \hspace{1cm} (B1)

$$I_2 = -\frac{4\alpha_d}{\sqrt{3}} \int_b^\infty \frac{1}{r} \left[ \phi_0^{(+)}(k, r)\phi_2^{(+)}(k, r) - j_0(kr)j_2(kr) \right] dr.$$  \hspace{1cm} (B2)

with $b$ the range of the SRI. Here we have used the fact $D_{0}^{(0)} = -4/\sqrt{3}$.

As shown in Sec. II, we assume that in our system the difference between $\phi_1^{(+)}(k, r)$ and $j_1(kr)$ with $l > 0$ can be neglected. Therefore, in above equations we can replace $\phi_2^{(+)}(k, r)$ with $j_2(kr)$. Furthermore, in the realistic systems of ultracold dipolar gases the integration $I_1$ for the wave functions in the region $r < b$ can be neglected. Thus, the function $G_{0,2}^{(0)}(k)$ can be approximated as $G_{0,2}^{(0)}(k) \approx I_2$. In addition, in the region $r > b$ the $s$-wave scattering wave function $\phi_0^{(+)}(k, r)$ can be expressed as $\phi_0^{(+)}(k, r > b) = j_0(kr) + kF_0(k)e^{ikr}$, with $F_0(k)$ the $s$-wave scattering amplitude for the SRI. Therefore, we have

$$G_{0,2}^{(0)}(k) \approx I_2 = -\frac{2d^2}{\sqrt{3}} \int_b^\infty \frac{1}{r} \left[ kF_0(k)e^{ikr}j_2(kr) \right] dr.$$  \hspace{1cm} (B3)

For ultracold gases with $k \ll 1/b$, we can approximate the lower limit $b$ of the integration in above equation as zero. Under this approximation and the relation $F_0(k) = -1/[ik + 1/a]$ with $a$ the scattering length, we get Eq. (12) in our main text.

Appendix C: SRI-induced amplitudes for fermionic dipoles

In this appendix we calculate the SRI-induced amplitude $G_{1,1,1}^{(m)}(k)$ for fermionic dipoles with “fast-decay” type SRI, and prove Eq. (13) in our main text. According to the effective-range theory [21], we have

$$F_1^{(sr)}(k) = -\frac{1}{ik + \frac{1}{\pi \tau} + \frac{r_{\text{eff}}}{\tau}},$$  \hspace{1cm} (C1)

where $\tau$ is the scattering volume and $r_{\text{eff}}$ is the effective range. Here we assume $r_{\text{eff}}$ has the same order of magnitude as $b$. In our calculation we consider the low-energy limit where $k$ is much smaller than both $1/b$ and $\sqrt{r_{\text{eff}}/\tau}$. In such limit the $p$-wave short-range scattering amplitude given by Eq. (C1) can be approximated as $F_1^{(sr)}(kr) = -vk^2$.

As shown in Sec. II, for ultracold dipolar fermions we only consider the amplitudes induced by the difference between $\phi_1(k, r)$ and $j_1(kr)$, i.e., the amplitudes $G_{1,1,1}^{(m)}(k)$, $G_{3,1,1}^{(m)}(k)$, and $G_{1,3,1}^{(m)}(k)$. Since $V_{sr}$ is negligible in the region $r > b$, we have

$$\phi_1^{(s)}(k, r) \equiv \phi_1^{(+)}(k, r) - j_1(kr)$$

$$= -iF_1^{(sr)}(k) \left( 1 + \frac{i}{kr} \right)e^{ikr},$$  \hspace{1cm} (C2)

According to Eq. (10), we have

$$G_{1,1,1}^{(m)}(k) = a_dD_{1,1}^{(m)}(\alpha_1 + \alpha_2 + \alpha_3),$$  \hspace{1cm} (C3)
where

\[ \alpha_1 = \int_b^\infty \phi_1^{(s)}(k, r) \frac{1}{r} dr + 2 \int_b^\infty \phi_1^{(s)}(k, r) j_1(kr) \frac{1}{r} dr; \]  
\[ \alpha_2 = - \int_0^b j_1(kr) \frac{1}{r} dr; \]  
\[ \alpha_3 = \int_0^b \phi_1^{(+)}(k, r) \frac{1}{r} dr. \]  
\[ \text{(C4)} \]
\[ \text{(C5)} \]
\[ \text{(C6)} \]

Using Eq. (C2), we directly obtain

\[ \alpha_1 = 8\pi a_d \left[ \frac{F_{1}^{(sr)}(k)^2}{4b^3 k^2} + \frac{2 F_{1}^{(sr)}(k)}{3} \right]. \]  
\[ \text{(C7)} \]

In addition, due to the fact \( kb \ll 1 \), we have \( |\alpha_2| \ll 1 \). Therefore, in \( G_{1,1}^{(m)}(k) \) the contribution given by \( \alpha_2 \) are much smaller than \( P_{1,1}^{(m)} = a_d D_{1,1}^{(m)}/4 \), and thus can be neglected.

The factor \( \alpha_3 \) can be calculated as following. It is clear that \( \phi_1^{(+)}(k, r) \) satisfies the radial Schroedinger equation

\[ \left( -\frac{1}{r^2} \frac{d}{dr} \right)^2 + V_{sr} \phi_1^{(+)}(k, r) = k^2 \phi_1^{(+)}(k, r) \]  
\[ \text{(C8)} \]

with boundary condition \( \phi_1^{(+)}(k, 0) = 0 \). In the region \( r \ll 1/k \), one can neglect the term \( k^2 \phi_1^{(+)}(k, r) \) and thus obtain \( \phi_1^{(+)}(k, r) = \alpha(k) \beta(r) \), where \( \beta(r) \) is a \( k \)-independent real function. On the other hand, using the facts that \( b \) and \( r_{\text{eff}} \) are on the same order of magnitude and \( kb \ll 1 \), we find that when \( r \) is both larger than \( b \) and on the same order of magnitude as \( b \) (e.g., \( r = 2b \)) we have

\[ \phi_1^{(+)}(k, r) = j_1(kr) - \frac{F_{1}^{(sr)}(k)}{k} \left( 1 + \frac{i k}{r} \right) e^{ikr} \approx \frac{F_{1}^{(sr)}(k)}{k} \left( \frac{1}{r^2} + \frac{k^2 r^2}{3} + \frac{k^2 r^2}{3 F_{1}^{(sr)}(k)} \right). \]  
\[ \text{(C9)} \]

The fact \( k \ll 1/b \) yields \( k^2/2 \ll 1/r^2 \) and \( k^3 r \ll 1/r^2 \). These results and \( F_{1}^{(sr)}(k) = -vk^2 \) gives

\[ \phi_1^{(+)}(k, r) \approx \frac{F_{1}^{(sr)}(k)}{k} \left( \frac{1}{r^2} - \frac{k^2 r}{3} \right). \]  
\[ \text{(C10)} \]

Therefore, we have \( \alpha(k) = F_{1}^{(sr)}(k)/(2k) \), and thus

\[ \alpha_3 = 8\pi a_d \left[ \frac{F_{1}^{(sr)}(k)}{2k} \right]^2 \int_0^b \frac{\beta(r)^2}{r} dr. \]  
\[ \text{(C11)} \]

Using the above results for \( \alpha_{1,2,3} \), we obtain

\[ G_{1,1}^{(m)}(k) = a_d D_{1,1}^{(m)} \left[ \frac{v^2 k^2}{4r_a^2} - \frac{2 v k^2}{3b} \right]. \]  
\[ \text{(C12)} \]

where the length \( r_a \) is defined as

\[ r_a = \left[ \frac{1}{b^4} + \int_0^b \frac{\beta(r)^2}{r} dr \right]^{-1/4}. \]  
\[ \text{(C13)} \]

It is clear that we have \( r_a \ll b \). Here we have used the fact that \( F_{1}^{(sr)}(k) = -vk^2 \).

We can calculate \( G_{1,1}^{(m)}(k) \) and \( G_{1,3}^{(m)}(k) \) with the similar approach as above. The calculation gives

\[ G_{1,1}^{(m)}(k) = G_{1,3}^{(m)}(k) = a_d D_{1,3}^{(m)} (\beta_1 + \beta_2), \]  
\[ \text{(C14)} \]

where

\[ \beta_1 = \frac{i}{36} k F_{1}^{(sr)}(k), \]  
\[ \text{(C15)} \]
\[ \beta_2 = \frac{1}{105} \left( F_{1}^{(sr)}(k) k^2 \right) \int_0^b \beta(r) r^2 dr. \]  
\[ \text{(C16)} \]

Here we have neglected the terms which is much smaller than \( P_{1,3}^{(m)} = P_{3,1}^{(m)} = a_d D_{1,3}^{(m)}/36 \).

Furthermore, in the low-energy limit \( k \ll \sqrt{|r_{\text{eff}}|} \) we have

\[ \frac{|v k^2|}{b} \ll \frac{|r_{\text{eff}}|}{b} \sim 1 \]  
\[ \text{(17)} \]

and

\[ |k F_{1}^{(sr)}(k)| \sim |v k^3| \ll |r_{\text{eff}} k| \sim |kb| \ll 1, \]  
\[ \text{(18)} \]

which yields

\[ a_d D_{1,3}^{(m)}/36 \ll P_{1,1}^{(m)}, a_d D_{1,3}^{(m)} \beta_1 \ll P_{1,3}^{(m)} \]  
\[ \text{(19)} \]

In addition, in the practical cases, the short-range interaction potentials \( V_{sr} \) between ultracold dipolar atoms or molecules are deep potential wells. As a result, in the region \( r < b \) the low-energy wave function \( \beta(r) \) is a rapid oscillating function with small amplitude. Here we estimate the upper limit of \( |\beta_2| \) as \( \frac{1}{105} \int_0^b \beta(r)^2 dr \cdot |\beta(r \sim b)|. \) Using the facts \( |\beta(r \sim b)| \ll |\sqrt{1} + \frac{|b|}{3r_{\text{eff}}} \) and \( kb \ll 1 \), we find that in the low-energy limit we have

\[ |a_d D_{1,3}^{(m)} \beta_2| \ll P_{1,3}^{(m)} \]  
\[ \text{(20)} \]

Using Eqs. (C10) and (C20), we find that in the low-energy limit Eqs. (C12) and (C14) can be simplified as

\[ G_{1,1}^{(m)}(k) \approx a_d \frac{\alpha^2 k^2}{r_a^2} \lambda_m, \quad G_{1,3}^{(m)}(k) = G_{3,1}^{(m)}(k) \approx 0, \]  
\[ \text{(21)} \]

with \( \lambda_m = 8/5 \) for \( m = 0 \) and \( \lambda_m = -4/5 \) for \( m = \pm 1 \). That is Eq. (13) in Sec. IV.
Appendix D: SRI-induced amplitudes for bosonic dipoles with strong d-wave SRI

In this appendix we prove Eqs. (11) and (20) in our main text. According to the effective-range theory [29], the d-wave scattering wave function for the “fast-decay” type SRI satisfies

$$\phi^+_2(k, r > b) = j_2(kr) + F_2^{(sr)}(k)kh_2^+(k),$$  \hspace{1cm} (D1)

with the scattering amplitude $F_2^{(sr)}(k)$ given by Eq. (18). Here we assume the parameters $s$, and $u$ have the same order of magnitude as $b$. In addition, it is clear that $\phi^+_2(k, r)$ satisfies the radial Schroedinger equation

$$\left(-\frac{1}{r} \frac{d^2}{dr^2} r + \frac{6}{r^2} + V_{sr}\right) \phi^+_2(k, r) = k^2 \phi^+_2(k, r)$$  \hspace{1cm} (D2)

with boundary condition $\phi^+_2(k, 0) = 0$. In the region $r \ll 1/k$, one can neglect the term $k^2 \phi^+_2(k, r)$ and thus obtain $\phi^+_2(k, r) = \alpha_2(k) \beta_2(r)$, where $\beta_2(r)$ is a $k$-independent real function. On the other hand, using the facts that $b$ and $s$, and $u$ are on the same order of magnitude and $kb \ll 1$, we find that when $r$ is both larger than $b$ and on the same order of magnitude as $b$ (e.g., $r = 2b$) we have

$$\phi^+_2(k, r) \approx F_2^{(sr)}(k) \left(\frac{3}{r^3} - \frac{r^2}{15w}\right).$$  \hspace{1cm} (D3)

Therefore, the function $\alpha_2(k)$ can be chosen as $\alpha_2(k) = F_2^{(sr)}(k)/k^2$. Namely, we have

$$\phi^+_2(k, r < b) = \frac{F_2^{(sr)}(k)}{k^2} \beta_2(r).$$  \hspace{1cm} (D4)

Substituting Eqs. (D1) and (D4) into Eq. (10), we can straightforwardly prove Eq. (11) and (20). In addition, we can also obtain $G_{2,4}^{(m)}(k) = G_{4,2}^{(m)}(k) = -ia_k F_2^{(sr)}(k) D_{2,4}^{(m)}(k)/72$. Since $(s, u) \sim b$ and $kb \ll 1$, we have $F_2^{(sr)}(k) << b$ and thus $|kF_2^{(sr)}(k)| << 1$. Therefore, the amplitudes $G_{2,4}^{(m)}(k)$ and $G_{4,2}^{(m)}(k)$ can be neglected.

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Due to the facts $r_a < b$ and $r_{\text{eff}} \sim b$, the low-energy condition $k \ll \sqrt{r_{\text{eff}}/v}$ is still satisfied when $k \sim r_a^2/|v|$. 

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