PION–PAIR FORMATION AND THE PION DISPERSION
RELATION IN A HOT PION GAS

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The possibility of pion–pair formation in a hot pion gas, based on the bosonic gap equation, is
pointed out and discussed in detail. The critical temperature for condensation of pion pairs (Evans–
Rashid transition) is determined as a function of the pion density. As for fermions, this phase
transition is signaled by the appearance of a pole in the two–particle propagator. In bose systems
there exists a second, lower critical temperature, associated with the appearance of the single–
particle condensate. Between the two critical temperatures the pion dispersion relation changes
from the usual quasiparticle dispersion to a Bogoliubov–like dispersion relation at low momenta.
This generalizes the non-relativistic result for an attractive bose gas by Evans et al. Possible
consequences for the inclusive pion spectra measured in heavy–ion collisions at ultra–relativistic
energies are discussed.

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I. INTRODUCTION

In heavy ion collisions at very high energies, the matter created differs qualitatively from what is traditionally studied in nuclear and elementary particle physics. In the initial stages of the collision, copious production of gluons and quarks in a large volume leads to a rapid increase in the entropy, and the distinct possibility of a new phase of matter characterized by deconfined degrees of freedom. One therefore hopes that relativistic heavy ion experiments can provide insight into the structure of the QCD vacuum, deconfinement, and chiral symmetry restoration.

The hot transient system eventually evolves into a gas of hadrons at high energy densities, whose properties may be studied theoretically using, for example, hadronic cascades [1–3]. In principle, these models provide information on the early, dense phase by tracing the evolution of the system from hadronization to freeze–out. Of course, in ultrarelativistic heavy ion collisions, most of the produced secondaries are pions. For example, in central Au+Au collisions at center–of–mass energies of 200 A GeV estimates from the FRITIOF event generator suggest that \( \sim 4000 \) pions per isospin state might be produced. Further, recent measurements [4] at lower energies and comparison to simulations [5] show that freeze–out source sizes probably deviate quite drastically from a simple multiplicity scaling law: present calculations indicate 10–20 fm Au+Au source radii at \( \sqrt{s} = 200 \) A · GeV. In any event, these high energy collisions might well create highly degenerate bose systems, and even possibly Bose–Einstein condensates (BEC). Since practical conclusions from dynamical simulations [6] depend qualitatively on the effect of the medium on particle interactions [7,8], one needs to better understand the properties of such degenerate systems of pions within the environment of a relativistic heavy ion collision.

Non–relativistically, the problem of interacting, degenerate bose systems has been discussed extensively by several authors. Evans and Imry [9] established the pairing theory of a bose superfluid in analogy to the BCS theory of superconductivity. For an attractive interaction, the resulting gap equation may have a non–trivial solution. Further, though, there appears the possibility of having a macroscopic occupation of the \( k = 0 \) particle state when the corresponding BCS quasiparticle energy vanishes. In turn, this leads to a spectrum which is linear and gapless in the long wavelength limit [9]. In a second paper, Evans and Rashid [10] redrew the equations of Ref. [9] using the Hartree–Fock–Gorkov decoupling method, and solved them for the case of superfluid helium. This boson pairing theory has been generalized by Dörre et al. [11], who carried out a thermodynamic variational calculation with a trial Hamiltonian containing a c–number part. An extensive discussion on the boson pairing problem is also given by Nozières and Saint James [12].

It has further been shown by Stoof [13] and, independently, Chanfray et al. [14] that the critical temperature \( T_c \) for the transition from the normal phase to the phase with a non–vanishing gap (the Evans–Rashid transition) is given by a “Thouless criterion” [15,16] for the bosonic \( T–matrix \) in the quasiparticle approximation, in analogy to the fermion case. Moreover, it has been demonstrated that there exists a second critical temperature \( T_{BEC} < T_c \), where the condition for the macroscopic occupation of the zero momentum mode of Ref. [9] is fulfilled [13,14]. The mechanism is the same as for Bose–Einstein condensation in the ideal bose gas [13].

Here we wish to consider \( \pi–\pi \) interactions in the presence of a dense and hot pion gas along the lines of a previous approach [6,4]. We address the question of pion pair formation and the pion dispersion relation in a thermal medium, first in a qualitative way (section II), then in a more detailed numerical calculation with a realistic two pion interaction (section III). As we shall see in section IV, the in–medium \( 2\pi \) propagator exhibits a pole above a certain
critical temperature, signaling a possible instability with respect to p ion pair formation. We conclude in section V with a discussion of the effect in high energy heavy ion collisions.

The effects we present here require rather large phase space densities for the pions, but are independent of whether full thermal equilibration has been reached. Nonetheless, we choose to couch the discussion in terms of thermal language, because it is convenient, but also because the actual situation is probably not too far removed from it. Dynamical calculations show that a high degree of thermal equilibration is quite reasonable. Chemical equilibration, on the other hand, may well cease at later stages of the system’s evolution and lead to a condensation of pions in the adiabatic limit. Of course, the system actually expands rather rapidly, but nonetheless large chemical potentials ($\mu \sim 130$ MeV) may be built up by freezeout ($T \sim 100$ MeV). One might thus expect large phase space occupation numbers at low momenta, which drive the pion pair formation that we discuss here.

II. THE EVANS–RASHID TRANSITION IN A HOT PION GAS

In order to treat the gas of interacting pions we will use the boson pairing theory of Evans et al. In analogy to the fermion (BCS) case, they obtain a system of coupled equations for the gap energy and the density by linearizing the equations of motion. The usual Thouless criterion for fermions can be established analogously for the bose system, and yields the critical temperature below which the gap equation begins to exhibit non–trivial solutions. However, in contrast to the fermion case, a second, lower, critical temperature appears at which the quasiparticle energy vanishes at zero momentum. This temperature is associated with the Bose–Einstein condensation (BEC) of single bosons, in analogy to the ideal bose gas, as discussed in Ref. and in detail for atomic cesium in Ref.. An interesting feature of the formalism developed by Evans et al. is that below the second critical temperature the dispersion relation for the single bosons is of the Bogoliubov form, i.e., linear, or phonon–like, for small momenta.

In this section, we illustrate these remarks concerning the Evans–Rashid transition for a pion gas in a qualitative way, returning to a more detailed numerical calculation in section III. While relativistic kinematics is taken into account, corrections from backward diagrams are ignored. We shall see in section III that such an approximation is justified for the physical regions in which a solution to the gap equation exists. For clarity in this preliminary discussion, we shall also generally neglect the $k$-dependence of the self–energy, $\Sigma(k)$, and further condense it into the chemical potential.

The gap equation for the pion pairs is derived in the appendix, using Gorkov decoupling:

$$\Delta(k) = -\frac{1}{2} \sum_{k'} V(k, k', E = 2\mu) \frac{\Delta(k')}{2E(k')} \coth \frac{E(k')}{2T} , \quad (1)$$

where the quasiparticle energy is given by

$$E(k) = \sqrt{\epsilon(k)^2 - |\Delta(k)|^2} , \quad (2)$$

with $\epsilon(k) = \omega(k) - \mu$, and where $\omega(k)$ is the free pion dispersion. The coth–factor represents the medium effect for a thermalized pion gas at temperature $T$ and chemical potential $\mu$, and $V(k, k', E)$ is the as yet unspecified bare two–particle interaction. The corresponding pion density is

$$n = \sum_{k'} \left[ \frac{\epsilon(k')}{2E(k')} \coth \frac{E(k')}{2T} - \frac{1}{2} \right] . \quad (3)$$
In spite of the formal similarities of Eq. (1) with the corresponding fermionic gap equation, there are important differences: For bosons, the $\Delta^2$ is subtracted in $E(k)$ (fermions: added), and the temperature factor is a hyperbolic cotangent (fermions: tanh).

We discuss the solution to the gap equation for decreasing temperature, at a fixed value of the chemical potential $\mu$. The possibility of a finite chemical potential in a pion gas has been pointed out in the introduction. At very high temperatures, the gap equation (1) has only the trivial solution $\Delta = 0$, and Eq. (3) is the usual quasiparticle density. The dispersion relation is also of the usual form

$$\lim_{\Delta \to 0} E(k) = \epsilon(k) = \omega(k) - \mu = \sqrt{k^2 + m_\pi^2} - \mu . \quad (4)$$

With decreasing temperature, however, a critical temperature $T_c^u$ is reached, at which the gap equation (1) first exhibits a non–trivial solution $\Delta \neq 0$. The value of $T_c^u$ may be found by linearizing the gap equation, i.e., setting $E(k) \approx \epsilon(k)$ in Eq. (1). We return to this point in section III, showing that the resulting equation for $T_c^u$ is identical to the condition for a pole in the two–pion $T$–matrix at the particular energy $E = 2\mu$ (for total momentum $\vec{K} = 0$). Thus we have a bosonic version of the well–known Thouless criterion for the onset of superfluidity in fermion systems with attractive interactions.

Below the critical temperature $T_c^u$ the order parameter $\Delta$ becomes finite, and the corresponding dispersion relation is now given by Eq. (2). As the temperature drops further, $|\Delta|$ increases to a point where the condition $|\Delta(k = 0)| = |m_\pi - \mu|$ is reached. This is the maximum possible value of $|\Delta|$, since otherwise imaginary quasiparticle energies result. It defines a second critical temperature $T_c^l$, below which the occupation $n_\ell$ of the zero momentum state becomes macroscopically large because $E(k) \to 0$ for $k \to 0$ (3). The possibility of a macroscopic occupation of the $k = 0$ mode below $T_c^l$ follows from the pion density Eq. (3); for $E(k = 0) = 0$, the $k = 0$ contribution to the density must be treated separately, as in the case of the ideal bose gas. A similar comment applies to Eq. (2) for the gap, so that we obtain the two inhomogeneous equations

$$\Delta(k) = \frac{1}{2} \sum_{k' \neq 0} V(k, k', 2\mu) \Delta(k') \coth \frac{E(k')}{2T} , \quad (5)$$

$$n = n_0 + \sum_{k' \neq 0} \left[ \frac{\epsilon(k')}{2E(k')} \coth \frac{E(k')}{2T} - \frac{1}{2} \right] . \quad (6)$$

In contrast to the ideal bose case, the condensation of quasiparticles happens at $\mu < m_\pi$, because of the finite value of the gap. Below $T_c^l$ the dispersion relation is given by

$$E(k) = \sqrt{\omega(k)^2 - 2\omega(k)\mu + 2\mu m_\pi - m_\pi^2} ,$$

$$\approx \sqrt{2(m_\pi - \mu) \frac{k^2}{2m_\pi} + \frac{\mu}{m_\pi} \frac{k^2}{(2m_\pi)^2}} . \quad (7)$$

in the small $k$, non–relativistic approximation. Thus, instead of the usual $k^2$–behavior, the pion dispersion is linear in the long wavelength limit. Eq. (4) may be rewritten in the more usual form of the well–known Bogoliubov dispersion relation (3) for a weakly interacting bose gas:

$$E(k) = \sqrt{|V(0, 0, 2\mu)| \frac{k^2}{2m_\pi} + O(k^4)} . \quad (8)$$

Here, we have used $m_\pi - \mu = -V(0, 0, 2\mu) n_0 / 2$, which follows from Eq. (3) for sufficiently low temperatures.
III. NUMERICAL RESULTS FOR THE GAP EQUATION

We now consider the qualitative discussion of the previous section in more detail, by numerically solving our system of equations for a realistic pion–pion interaction in the \( \ell = I = 0 \) channel. We choose a rank–2 separable \( \pi–\pi \) interaction inspired by the linear \( \sigma–\)model (see appendix) which possesses all the desired low energy chiral properties, as is explicitly discussed in Ref. [17]. For vanishing incoming total momentum, \( \vec{k} = 0 \), it reads (see Eq. (A.23))

\[
\langle \vec{k}, -\vec{k} | V_{I=0}(E) | \vec{k}', -\vec{k}' \rangle = \frac{v(\vec{k})}{2\omega(k)} \frac{M_\sigma^2 - m_\pi^2}{f_\pi^2} \left[ 3 \frac{E^2 - m_\pi^2}{E^2 - M_\sigma^2} + \frac{4\omega(k)\omega(k') - 2m_\pi^2}{M_\sigma^2} \right] \frac{v(\vec{k}')}{2\omega(k')},
\]

where, for later convenience, we have introduced the bare invariant matrix

\[
\langle k | M_B(E) | k' \rangle = \lambda_1(E) v_1(k)v_1(k') + \lambda_2 v_2(k)v_2(k'),
\]

with notation \( v_1(k) \equiv v(k) = (1 + (k/8m_\pi)^2)^{-1} \), \( v_2(k) = (\omega(k)/m_\pi)v(k) \), and

\[
\lambda_1(E) \equiv \frac{M_\sigma^2 - m_\pi^2}{f_\pi^2} \left[ 3 \frac{E^2 - m_\pi^2}{E^2 - M_\sigma^2} - \frac{2m_\pi^2}{M_\sigma^2} \right], \quad \lambda_2 \equiv \frac{M_\sigma^2 - m_\pi^2}{f_\pi^2} + \frac{4m_\pi^2}{M_\sigma^2}.
\]

The form factor \( v(k) \) and \( \sigma–\)mass \( M_\sigma = 1 \text{ GeV} \) are fit to experimental phase shifts, as in Ref. [17]. For free \( \pi^+–\pi^- \) scattering this force yields, when used in the \( T–\)matrix (see below), a scattering length which vanishes in the chiral limit, as it should. This feature induces off–shell repulsion below the \( 2\pi–\)threshold in spite of the fact that the positive \( \delta_0^0 \) phase shifts indicate attraction. It is remarkable that the gap equation still shows a non–trivial solution, signaling pion pair formation, as we will show later. It is evident that bound pair formation, or even larger clusters of pions, can deeply influence the dynamics of the pion gas.

In the sigma channel \(( \ell = 0, I = 0 )\) we rewrite the gap equation (10) as

\[
\Delta(k) = \frac{1}{2} \int \frac{d^3k'}{(2\pi)^3} \langle \vec{k}, -\vec{k} | V_{I=0}(E = 2\mu) | \vec{k}', -\vec{k}' \rangle \frac{\Delta(k')}{2E(k')} \coth \frac{E(k')}{2T},
\]

With the form of our interaction solutions of this equation may be written as

\[
\Delta(k) = \frac{m_\pi}{\omega(k)} \left[ \delta_1 v_1(k) + \delta_2 v_2(k) \right],
\]

and Eq. (12) reduces to two coupled non–linear equations for the “gap strengths” \( \delta_1 \) and \( \delta_2 \). For a non–trivial solution, one can show that \( \delta_1 > -\delta_2 > 0 \). We also note that while \( \lambda_2 \) is always repulsive, \( \lambda_1(E) \) is attractive at \( E = 2\mu \) only if \( |\mu| > M_\sigma m_\pi/2\sqrt{3M_\sigma^2 - 2m_\pi^2} \sim 40 \text{ MeV} \). This inequality is also the formal condition for a solution to the gap equation to exist at some temperature. Intuitively, we require at least some attraction because, as we shall see, a solution to the gap equation is connected to the existence of a pole in the \( T–\)matrix. We note that the repulsive part of the \( \pi–\pi \) interaction Eq. (10) helps to avoid collapse. This is different from our previous calculation [14], which was performed with an entirely attractive interaction. The presence of this repulsion is a consequence of chiral symmetry and PCAC [17].

In the previous section, we introduced the critical temperatures \( T_c^\mu \), at which the gap vanishes, and \( T_c^\ell \), where the gap has reached its maximum value and quasiparticle condensation occurs. Fig. 1 shows the numerical results for these temperatures in the \( \mu–T \) plane. The \( T_c^\mu \) (solid line) are obtained by linearizing Eq. (12):
\[
\Delta(k) = -\frac{1}{2} \int \frac{d^3k'}{(2\pi)^3} \langle \vec{k}, -\vec{k} | V_I=0(E=2\mu) | \vec{k}', -\vec{k}' \rangle \frac{\Delta(k')}{2\epsilon(k')} \coth \frac{\epsilon(k')}{2T_c^u},
\]
while the \( T_c^u \) (dashes) result when the gap strength increases to a point where \( E(k=0) = 0 \), i.e., \( m_\pi - \mu = \delta_1 + \delta_2 \).

At high temperatures \( T > T_c^u \) (region III), the system is in the normal state with no gap, while below the dashed line, \( T < T_c^u \) (region I), there is macroscopic occupation of the \( k = 0 \) mode. For \( T_c^u < T < T_c^d \) (region II), non–trivial gap solutions exist. Notice that for physically realistic solutions (\( T < 200 \) MeV, say) we have \( \mu \lesssim m_\pi \), and \( \omega - m_\pi \ll m_\pi \), and, in hindsight, are justified in neglecting relativistic corrections to the gap equation (see appendix).

Fig. 2 shows the gap strengths \( \delta_1 \) (solid line) and \(-\delta_2 \) (dashes) versus temperature for a fixed chemical potential \( \mu = 135 \) MeV. Again, we see that at high temperatures, in region III, only the trivial solution \( \delta_1 = \delta_2 = 0 \) exists. As the temperature drops to \( T_c^u \sim 123 \) MeV, the order parameter \( \Delta \) switches on, and we have a transition to a paired state in region II (see discussion below). Finally, at \( T = T_c^d \sim 77 \) MeV, the gap has reached its maximum value \( \delta_1 + \delta_2 = m_\pi - \mu \sim 3 \) MeV and quasiparticles condense in the lowest energy mode in region I.

The change in the pion dispersion relation \( E(k) \) is investigated in Fig. 3 in the temperature range \( T_c^d \leq T \leq T_c^u \), for a fixed chemical potential of \( \mu = 135 \) MeV. At \( T = T_c^u \sim 123 \) MeV (solid line), and above, we simply have the normal–state pion dispersion relation \( \epsilon(k) = \omega(k) - \mu \). With decreasing temperature the influence of the finite gap becomes visible at long wavelengths: The dot–dashed line shows \( E(k) \) for \( T = 115 \) MeV. A further drop in the temperature to \( T = T_c^d \sim 77 \) MeV qualitatively changes the character of the pion dispersion relation to a linear, phonon–like dispersion at small \( k \).

**IV. THE IN–MEDIUM \( \pi\pi \) SCATTERING MATRIX**

We turn now to a discussion of the \( T \)–matrix \( M_{I=0}(E, K) \) for a pion pair with total momentum \( K = |\vec{K}| \) with respect to a thermal medium. Writing the on–shell \( T \)–matrix (c.f. Eq. (A.24)) as

\[
\langle k^* | M_{I=0}(E, K) | k^* \rangle = \sum_{i=1}^{2} \lambda_i(s) \, v_i(k^*) \, \tau_i(k^*; s, K),
\]

where \( k^* = s/4 - m_\pi^2 \) and \( s = E^2 - K^2 \) is the square of the total c.m. energy, the Lippmann–Schwinger equation (A.27) becomes a set of two linear equations for the functions \( \tau_i \):

\[
\sum_{j=1}^{2} \left[ \delta_{ij} - \lambda_j(s) \, g_{ij}(s, K) \right] \, \tau_j(k^*; s, K) = v_i(k^*), \quad i = 1, 2
\]

with

\[
g_{ij}(s, K) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \, v_i(k) \, \frac{1}{\omega(k)} \, \frac{(1 + f_+ + f_-)}{s - 4\omega_k^2 + i\eta} \, v_j(k).
\]

Here, \( \langle 1 + f_+ + f_- \rangle \) denotes an average over the angles of the c.m. relative momentum of the pair. For thermal occupation numbers it is given by Eq. (A.28), and reduces to unity in free space and \( \coth(\omega(k) - \mu)/2T \) for vanishing total momentum \( \vec{K} \). We note that Eq. (11) does not incorporate the non–linear effect of the gap.

The solid line in Fig. 4 shows \( |M_{I=0}|^2 \) for free space scattering. Compared to our previous calculation [], the \( T \)–matrix is relatively flat above the resonance, this being due to the repulsion in the interaction at high energies. The
short dashes give \( |\mathcal{M}_{I=0}|^2 \) in a thermal bath of \( T = 100 \text{ MeV} \) and \( \mu = 135 \text{ MeV} \), for \( K = 0 \). The medium strongly suppresses the cross section, an effect that also occurs in the \((I = 1, \ell = 1)\) \( \rho \)-channel [3,4]. At high c.m. energies, the phase space occupation becomes negligible, and the cross section returns to its free space value. The three remaining curves show results in the same thermal bath, but for \( K = 200 \text{ MeV}/c \) (long dashes), 1 \( \text{ GeV}/c \) (dot–dashed), and 3 \( \text{ GeV}/c \) (dotted). As \( K \) increases, the pair is boosted more and more out of the occupied phase space of the medium, and the cross section again returns to its free space value. We also see a threshold behavior in Fig. 4: as \( K \) becomes larger, a resonance peak emerges from below the threshold which continues to shift up in energy and strengthen until it coincides with the free scattering peak. We shall see below that this is the continuation of an upward shift of the Cooper pole in the \( T \)-matrix with decreasing phase space occupation [4].

We consider now the existence of poles in the \( T \)-matrix, first for the special case of zero total momentum, \( K = 0 \) [4], and define the determinant function

\[
F_{\mu,T}(E) \equiv -\det \left[ \delta_{ij} - \lambda_j(E) g_{ij}(E) \right].
\]

This function is shown in Fig. 5 for five different temperatures (solid lines) at a fixed pion chemical potential of 135 MeV. The intersection of these curves with zero (horizontal dashes) below \( 2m_\pi \) (the bound state domain) gives the pole position. We see that a pole always occurs provided the temperature lies above some critical value \( T_0^\mu \approx 47 \text{ MeV} \), for which the pole is at threshold and ceases to exist. This \( T_0^\mu \) is close to the lower critical temperature for the gap, \( T_c^\mu \), where the excitation spectrum vanishes at \( k = 0 \) and quasiparticles begin to condense as singles. Thus, the bound state and gap solution disappear at a similar critical temperature; differences are ascribable to the fact that we use free quasiparticle energies in the \( T \)-matrix.

There is a second special temperature \( T_0^\mu \), for which a pole exists at \( E = 2\mu \) (see Fig. 5). It is identical to the upper critical temperature \( T_c^\mu \) at which the gap vanishes, as may easily be seen by rewriting the \( T \)-matrix for \( E \) near \( 2\mu \),

\[
\langle \vec{k}, -\vec{k} | T_{I=0}(E) | \vec{k}', -\vec{k}' \rangle = Z(k) \frac{1}{E - 2\mu} Z(k')
\]

In the non–relativistic limit, \( Z(k) \) follows as (see appendix)

\[
Z(k) = -\frac{1}{2} \int \frac{d^3k'}{(2\pi)^3} \langle \vec{k}, -\vec{k} | V_{I=0}(E = 2\mu) | \vec{k}', -\vec{k}' \rangle \frac{Z(k')}{2(\omega(k') - \mu)} \coth \frac{\omega(k') - \mu}{2T_0^\mu},
\]

which is precisely the same condition as for \( T_0^\mu \), Eq. [14]. The gap equation [12] thus reduces to the \( T \)-matrix pole condition at the particular energy \( E = 2\mu \). In fermion systems, this is the well–known Thouless criterion [14] for the onset of a phase transition to a pair condensate. We note that the Thouless criterion is only approximately valid if relativistic corrections are included.

Several observations can be made. Firstly, one always obtains a pole in the \( T \)-matrix if the temperature lies above \( T_0^\mu(\mu) \). Thus, at fixed \( \mu \), no matter how weak the interaction strength is (provided it is attractive in the neighborhood of the \( 2\pi \) threshold), one always obtains a pole for sufficiently high temperatures (for fermions at a sufficiently low temperature). In practice, \( T_0^\mu(\mu) \) and \( T_0^\mu(\mu) \) will exceed sensible values for pions as soon as \( \mu \) drops below \( \sim 130 \text{ MeV} \), since they are increasing functions of \( \mu \). Secondly, for a fixed interaction strength, the pole position shifts downward with increasing temperature (for fermions: pole position moves up with increasing temperature). As a function of temperature, we therefore see a behavior for bosons opposite to that for fermions.
The fact that increasing temperature reinforces the binding is somewhat counterintuitive, but is an immediate consequence of the coth–factor associated with bose statistics in Eq. (A.27). Indeed, one realizes that the coth–factor increases with increasing temperature and thus effectively enhances the two–body interaction. We can therefore always find a bound state for arbitrarily small attraction: it suffices to increase the temperature or, equivalently, the density accordingly. This is opposite to the fermion case where the corresponding tanh–factor suppresses the interaction with increasing temperature. Therefore, in the fermion case, even at the $T$–matrix level there exists a critical temperature where the Cooper pole ceases to exist. For bosons, on the other hand, once one has reached $T = T^\ell_0$, a bound state (here $E^{2\pi}_B < 2m_\pi$) exists and the bound state energy simply continues to decrease as the temperature increases. Of course, this becomes unphysical as soon as the density of pairs becomes so strong that the bound states start to obstruct each other, and finally dissolve at an upper critical temperature (Mott effect). Precisely this non–linear effect is very efficiently taken care of in the gap equation. In spite of the fact that we still have a coth–factor in the gap equation, there is now a crucial difference: the argument of the coth–factor is the quasiparticle energy, Eq. (2), (over $T$) and thus, due to the presence of $-\Delta^2(k)$ in $E(k)$, the origin of the coth is shifted to the right with respect to the $T$–matrix case. Now, as $T$ increases, the only way to keep the equality of the gap equation is for $\Delta(k)$ to decrease – this pushes the origin of the coth back to the left, counterbalancing its increase due to the increasing temperature. Of course, this only works until $\Delta = 0$, i.e., until the temperature has reached $T^u_0$. This is precisely the temperature $T^u_0$ for which the bound state in the $T$–matrix reaches an energy $E^{2\pi}_B = 2(m_\pi - \mu)$. We therefore see that in spite of the fact that the bosons prefer high phase space density, the formation of bound states ceases to exist beyond a critical temperature – just as for fermions.

Lastly, we return to the behavior of the pole for varying total momentum $K$, and the threshold effect seen in Fig. 4. Since $F_{\mu,T}(s,K)$ becomes complex above threshold, we show in Fig. 6 its magnitude for fixed $T = 100$ MeV and $\mu = 135$ MeV, and various values of $K$. As expected, for increasing $K$ (i.e., decreasing phase space occupation felt by the two pions in question) the pole (zero of $F$) moves up in energy until it disappears at some critical momentum $100$ MeV/$c < K_c < 250$ MeV/$c$. For $K > K_c$, the now non–zero minimum of the determinant function continues to shift to higher energies, corresponding roughly to the similar shift in the resonance peak in Fig. 4.

V. DISCUSSION AND CONCLUSIONS

In the previous section, we investigated the effect of a thermal medium on the pion dispersion relation at low momenta $k$. In particular, one finds a critical temperature $T^\ell_c$ at which the pion dispersion relation is linear (phonon-like) in $k$ for small $k$. This result is independent of the details of the interaction and characteristic of any bose system (see Ref. [10] for the case of $^4$He).

Such a change in the pion dispersion relation at low temperatures would influence the pion spectra at low momentum. For this to occur, rather large medium phase space occupation numbers are required. In particular, for a physically reasonable system with, say, $T < 200$ MeV, this means that we require large chemical potentials. In fact, dynamical calculations [3] show that a buildup of $\mu$ can indeed occur, provided that the scattering rate is sufficiently large compared to expansion rate and the inelastic collisions have ceased to be a factor.

To demonstrate the possible effect in a qualitative way, consider the pion transverse momentum spectrum for longitudinally boost invariant expansion
\[
\frac{dN}{m_t dm_t dy} = (\pi R^2 \tau) \frac{m_\pi}{(2\pi)^2} \sum_{n=1}^{\infty} \exp\left(\frac{n\mu}{T}\right) K_1\left(\frac{nm_t}{T}\right),
\]  

(21)

where \( K_1 \) is a Mc–Donald function, \( m_t \) is the transverse mass, and the normalization volume \( \pi R^2 \tau \) is of the order 200 to 300 fm\(^3\) [18]. At mid-rapidity, the transverse mass coincides with the full energy of the pion, and we follow Ref. [19] in replacing \( m_t \) by the in–medium pion dispersion relation \( E(k) + \mu \) derived in the previous section. Of course, as remarked in Ref. [18], the use of this procedure is rather tenuous since the system is by definition still far from freeze–out. In a dynamical calculation, hard collisions would re–thermalize the system at ever decreasing temperatures.

In Fig. 7, the thermal transverse momentum spectrum for pions with \( \mu = 135 \text{ MeV} \) and \( T = 100 \text{ MeV} > T_{\ell_c}(\mu) \) is shown with (solid line) and without (dashes) the effect of the gap energy. Essentially, the presence of a large chemical potential gives the spectrum the appearance of one for small–mass particles, and the gap energy, which causes \( E(k) \sim k \) for long wavelengths, strengthens this effect. We would like to mention here again that the use of our force, Eq. (9), which respects chiral symmetry constraints [17], considerably reduces the effect of binding with respect to a purely phenomenological interaction, fitted to the phase shifts (see, for example, Ref. [17]). This stems from the fact that the expression (9) becomes repulsive sufficiently below the \( 2m_\pi \) threshold. This is not the case for commonly employed phenomenological forces [17]. As a consequence, the effect we see in Fig. 7 is relatively weak, but one should remember that the force (9) is by no means a definitive expression. It is well known that in a many body system screening effects should be taken into account. Whereas in a fermi system this tends to weaken the force, it is likely that screening strengthens it in bose systems. In this sense our investigation can only be considered schematic. A quantitative answer to the question of bound state formation in a hot pion gas is certainly very difficult to give. Qualitatively, the curves in Fig. 7 agree with the trend in the pion data at SPS [20] to be “concave–up,” but this is mainly an effect from the finite value of the chemical potential [3,18]. While the gap changes the spectrum by a factor of \( \sim 3 \) at \( m_t - m_\pi \sim 0 \), this region is not part of detector acceptances.

In summary, we have shown that finite temperature induces real poles in the \( 2\pi \) propagator below the \( 2m_\pi \) threshold, even for situations where there is no \( 2\pi \) bound state in free space [14]. The situation is analogous to the Cooper pole of fermion systems, and we therefore studied the corresponding bosonic “gap" equation. This equation has non–trivial solutions in a certain domain of the \( \mu–T \) plane. Such a region always exists, even in the limit of infinitesimally weak attraction. This is different from the \( T = 0 \) case discussed by Nozières and Saint James [12], where a nontrivial solution to the gap equation only exists when there is a two boson bound state in free space. Our study has to be considered preliminary. The final aim will be to obtain an equation of state for a hot pion gas within a Br"uckner–Hartree–Fock–Bogoliubov approach. Also, the subtle question of single boson versus pair condensation must be addressed (see Ref. [12] and references therein). Furthermore, the fact that we obtain two pion bound states in a pionic environment leads to the speculation that higher clusters, such as four–pion bound states, etc., may also occur, and perhaps even be favored over pair states. Such considerations, though interesting, are very difficult to treat on a quantitative basis. However, substantial progress towards the solution of four body equations has recently been made [21], and one may hope that investigations for this case will be possible in the near future.

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VI. APPENDIX: DERIVATION OF T–MATRIX AND GAP EQUATION

This appendix is devoted to a derivation of the gap equation for a bosonic system governed by a field–theoretic Hamiltonian. The basic problem one has to deal with is the formal introduction of a chemical potential for bosons, since the total bosons number operator \( (i.e., \pi^+ + \pi^- + \pi^0) \) does not commute with the Hamiltonian. Hence, if we consider a pion gas at a typical temperature of 200 MeV, it will correspond to zero chemical potential. However, for a system lifetime on the order of tens of fermi, the inelastic collision rate is negligible. Therefore, provided the elastic collision rate is sufficiently large, a thermal equilibrium with a finite chemical potential may well be reached.

Let us consider a pion system at temperature \( T \). Inspired by the linear \( \sigma \)–model, with form factors fitted to the \( \pi–\pi \) phase shifts, we take the Hamiltonian

\[
H = H_0 + H_{\text{int}} ,
\]

(A.1)

where \( H_0 \) is the kinetic Hamiltonian for the \( \pi \) and “\( \sigma \)” mesons

\[
H_0 = \sum \omega \sigma_1^\dagger \sigma_1 + \sum \Omega_\alpha \sigma_\alpha^\dagger \sigma_\alpha .
\]

(A.2)

The index “1” refers to the momentum and isospin of the pion, and “\( \alpha \)” to the momentum and identity of the heavy meson carrying the interaction. The interaction Hamiltonian has the form

\[
H_{\text{int}} = \frac{1}{2} \sum \left[ (\sigma_\alpha + \sigma_\alpha^\dagger) (\sigma^\dagger_1 \sigma_2 + b_1 b_2) \right] \langle 12 | W | \alpha \rangle + \frac{1}{4} \sum \left[ b_1^\dagger b_2^\dagger b_3 b_4 + \frac{1}{2} (b_1 b_2 b_3 b_4 + b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger) \right] \langle 12 | V | 34 \rangle
\]

(A.3)

In the linear \( \sigma \)–model one has \( (L^3 \text{ is a normalization volume}) \)

\[
\langle 12 | W | \alpha \rangle = \frac{1}{2 \omega L^3} \left( \frac{2 \omega L^3}{2 \omega L^3} \right)^{1/2} v(k_{12}) (2\pi)^3 \delta(k_1 \cdot k_2 - \delta_{12}) \frac{M^2 - m^2}{f_\pi} \delta_{12} \]  \hspace{1cm} (A.4)

\[
\langle 12 | V | 34 \rangle = \left[ \frac{1}{2 \omega L^3} \right]^{1/2} v(k_{12}) v(k_{34}) (2\pi)^3 \delta(k_1 \cdot k_2 - \delta_{12}) \frac{M^2 - m^2}{f_\pi} \delta_{12} \hspace{1cm} (A.5)
\]

The form factor taken at c.m. momentum \( k^* \) of the pion pair is fitted to the experimental phase shifts, and \( M_\sigma \) is the \( \sigma \)–mass. The static quartic interaction contains the \( \pi^2 \pi^2 \) interaction of the \( \sigma \)–model, and the \( t \) and \( u \) channel \( \sigma \)–exchange terms. We neglect the \( t \) and \( u \) dependence of the denominator (see Ref. [17]), since their effect is extremely small. Further, terms like \( \sigma b^\dagger b \) and \( b^\dagger b b b \) have been dropped, since they are not essential for our purpose.

The Dyson equation for the pion propagator

We now derive the equation of motion for the pion propagator

\[
G_{11}(t, t') = \left\langle \exp(-iHt) b^\dagger_1(t) b_1(t') \exp(iHt) \right\rangle
\]

(A.6)

where the \( b_1 \) are normal Heisenberg operators \( b_1(t) = \exp(iHt) b_1(0) \exp(-iHt) \). In principle, the extension to finite temperature requires a matrix formulation (real time formulation) or Matsubara Green’s function. However, for
simplicity we consider here the normal zero temperature $G$, and replace it by a thermal propagator at the end. We have checked that the final result is not modified.

After standard manipulation, using the Hamiltonian (A.1), we obtain

$$
\left( i \frac{\partial}{\partial t} - \omega_1 \right) G_{11}(t, t') = \delta(t - t') + \int dt'' \Sigma_1(t, t'') G_{11}(t'', t') ,
$$

(A.7)

with $\Sigma_1(t, t') = \Sigma^S_1(t, t') + \Sigma^D_1(t, t')$. The static part of the mass operator is

$$
\Sigma^S_1(t, t') = \sum_{ij} \langle b_i b_j \rangle \langle 12 | V | 12 \rangle \delta(t - t') ,
$$

(A.8)

while the dynamical part is given by

$$
\Sigma^D_1(t, t') = \left\langle -iT \left( \langle 12 | W | \alpha \rangle \sigma_{\alpha} + \frac{1}{2} \langle 12 | V | 34 \rangle \langle b_3 b_4 + b_{-3}^* b_{-4}^* \rangle(t) \right) \right\rangle .
$$

(A.9)

Making a standard factorization approximation, we obtain

$$
\Sigma^D_1(t, t') = i \sum_{ij} G_{22}(t, t')
\times \left\langle -iT \left( \langle 12 | W | \alpha \rangle \sigma_{\alpha} + \frac{1}{2} \langle 12 | V | 34 \rangle \langle b_3 b_4 + b_{-3}^* b_{-4}^* \rangle(t) \right) \right\rangle ,
$$

(A.10)

with $G_{22}(t, t') = \langle -iT(b_2(t), b_2(t')) \rangle$, and there is an implicit summation over repeated indices.

**Extraction of the condensates**

In the above expression the operator $b_3^\dagger b_4^\dagger$ connects states with $N$ particles to states with $N + 2$ particles. Among these states those with excitation energy $2\mu$ play a prominent role (Cooper poles). To separate the influence of these states, we split the fluctuating part of the operator from the condensate

$$
b_3^\dagger b_4^\dagger(t) = \langle b_3^\dagger b_4^\dagger(t) \rangle + : b_3^\dagger b_4^\dagger(t) : .
$$

(A.11)

The time evolution is

$$
\langle b_3^\dagger b_4^\dagger(t) \rangle = \langle b_3^\dagger b_4^\dagger_{-3} \rangle e^{i2\mu t} \delta_{3,-4} ,
$$

(A.12)

where $\langle b_3^\dagger b_4^\dagger_{-3} \rangle$ is the usual time independent pion density. Similarly, we obtain

$$
b_3 b_4(t) = \langle b_3 b_{-3} \rangle e^{-i2\mu t} \delta_{3,-4} + : b_3 b_4(t) : .
$$

(A.13)

We now extract the condensate part of the $\sigma$–field operator from the fluctuating part:

$$
\sigma_\alpha(t) = \langle \sigma_\alpha(t) \rangle + s_\alpha(t) .
$$

(A.14)

The equation of motion gives
\[ i \frac{\partial}{\partial t} \langle \sigma_\alpha(t) \rangle = \Omega_\alpha \langle \sigma_\alpha(t) \rangle + \frac{1}{2} (b_1 b_2 + b_{-1}^\dagger b_{-2}^\dagger) \langle \alpha | W | 12 \rangle. \] (A.15)

We look for a solution of the form
\[ \langle \sigma_\alpha(t) \rangle = (A e^{-i 2 \mu t} + B e^{i 2 \mu t}) \delta_{\alpha 0}. \] (A.16)

\( A \) and \( B \) are straightforwardly obtained from the equation of motion:
\[ \langle \sigma_0(t) \rangle = -\frac{1}{2} \langle b_1 b_{-1} \rangle \langle 0 \mid W \mid 1 - 1 \rangle e^{-i 2 \mu t} - \frac{1}{2} \langle b_{-1}^\dagger b_{-2}^\dagger \rangle \langle 0 \mid W \mid 1 - 1 \rangle e^{i 2 \mu t}. \] (A.17)

In the expression of the dynamical mass operator one can extract a Cooper pole part, where only the condensates occur. The remaining part involves only the fluctuating pieces. Grouping the latter with the static mass operator, we can write
\[ \Sigma_1(t, t') = \Sigma_{1C}(t, t') + \Sigma_{1H}(t, t') \] (A.18)

where \( \Sigma_{1H}(t, t') \) is the normal Hartree mass operator which depends on the full in–medium \( T \)-matrix:
\[ \Sigma_{1H}(t, t') = i \sum_2 G_{22}(t, t') \langle 12 \mid T(t, t') \mid 12 \rangle. \] (A.19)

with
\[
\langle 12 \mid T(t, t') \mid 34 \rangle = \langle 12 \mid V \mid 34 \rangle \delta(t - t') \\
+ \langle -i T \left( \left[ \langle 12 \mid W \mid \alpha \rangle (s_\alpha + s_{-\alpha}^\dagger)(t) + \frac{1}{2} \langle 12 \mid V \mid 56 \rangle (: b_5 b_6 + b_{-5}^\dagger b_{-6}^\dagger :) \langle t \rangle \right] , \right) \] \\
\left[ (s_\alpha^\dagger + s_{-\alpha}^\dagger)(t') \langle \alpha' \mid W \mid 34 \rangle + \frac{1}{2} (: b_{5'} b_{6'}^\dagger + b_{-5'} b_{-6'}^\dagger :) \langle t' \rangle \langle 5'6' \mid V \mid 34 \rangle \right) \right) \] (A.20)

Using the Dyson equation for the \( b \) and \( s \) operators, it is a purely technical matter to show that this scattering amplitude satisfies a Lippmann–Schwinger equation. In energy space, and in the \( I = 0 \) channel, it reads:
\[ \langle 12 \mid T_{I=0}(E) \mid 34 \rangle = \langle 12 \mid V_{I=0}(E) \mid 34 \rangle + \frac{1}{2} \langle 12 \mid V_{I=0}(E) \mid 56 \rangle G_{2\pi}^{56}(E) \langle 56 \mid T_{I=0}(E) \mid 34 \rangle, \] (A.21)

where \( G_{2\pi}(E) \) is the in–medium \( 2\pi \) propagator
\[ G_{2\pi}^{56}(E) = \left[ \frac{1}{E - (\omega_5 + \omega_6) + i\eta} - \frac{1}{E + (\omega_5 + \omega_6) + i\eta} \right] (1 + f_5 + f_6). \] (A.22)

with thermal occupation numbers \( f(k) = \left[ \exp(\omega(k) - \mu) / T - 1 \right]^{-1} \). As mentioned above, we have checked that using the correct matrix form of the two pion propagators instead of \[ A.22 \] yields the same final result. In Eq. \[ A.21 \], \( V_{I=0}(E) \) is the effective \( \pi - \pi \) potential in the \( I = 0 \) channel which incorporates all the tree level diagrams. For total incoming momentum \( \vec{K} = \vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4 \), it reads
\[ \langle \vec{k}_1, \vec{k}_2 \mid V_{I=0}(E) \mid \vec{k}_3, \vec{k}_4 \rangle = \left( \frac{1}{2 \omega_1 2 \omega_2 2 \omega_3 2 \omega_4} \right)^{1/2} \langle \vec{k}_1, \vec{k}_2 \mid \mathcal{M}_B(E) \mid \vec{k}_3, \vec{k}_4 \rangle, \] (A.23)

where the bare invariant interaction \( \mathcal{M}_B \) is

12
\[ \langle \vec{k}_1, \vec{k}_2 | \mathcal{M}_B(E) | \vec{k}_3, \vec{k}_4 \rangle \equiv \langle k_{12}^* | \mathcal{M}_B(s) | k_{34}^* \rangle = \sum_{i=1}^2 \lambda_i(s) v_i(k_{12}^*) v_i(k_{34}^*) , \quad (A.24) \]

with

\[ v_1(k) = v(k) \equiv [1 + (k/8m_\pi)^2]^{-1} , \quad v_2(k) = \frac{\omega(k)}{m_\pi} v(k) , \quad (A.25) \]

\[ \lambda_1(s) = \frac{M^2 - m_\pi^2}{f_\pi^2} \left[ 3 \frac{s - m_\pi^2}{s - M^2} - \frac{2m_\pi^2}{M^2} \right] , \quad \lambda_2(s) = \frac{M^2 - m_\pi^2 - 4m_\pi^2}{f_\pi^2} \frac{4m_\pi^2}{M^2} . \quad (A.26) \]

In these equations \( s = E^2 - \vec{K}^2 \) is the square of the total c.m. energy, and the \( k_{ij}^* \) are the magnitudes of the relative 3-momenta in the c.m. frame

\[ \omega_{ij}^2 = m_\pi^2 + \vec{k}_{ij}^2 = \frac{1}{4} \left( (\omega_i + \omega_j)^2 - \vec{K}^2 \right) , \quad i, j = 1, 2 \text{ or } 3, 4. \]

The form factor \( v(k) \), Eq. (A.25), and \( \sigma \)-mass \( M_\sigma = 1 \) GeV have been fitted to the experimental phase shifts.

The Lippmann–Schwinger equation for the invariant T-matrix, \( \mathcal{M}_{f=0} \), may finally be rewritten in a form suitable for practical purposes:

\[ \langle k_{12}^* | \mathcal{M}_{f=0}(E, K) | k_{34}^* \rangle = \langle k_{12}^* | \mathcal{M}_B(s) | k_{34}^* \rangle + \frac{1}{2} \int \frac{d^3k_{56}^*}{(2\pi)^3} \langle k_{12}^* | \mathcal{M}_B(s) | k_{56}^* \rangle \frac{1}{\omega_{56}} \frac{1 + f_+ + f_-}{s - 4\omega_{56}^2 + i\eta} \langle k_{56}^* | \mathcal{M}_{f=0}(E, K) | k_{34}^* \rangle . \quad (A.27) \]

In the special case of a single fireball of temperature \( T \) and chemical potential \( \mu \), the angle average factor is given by

\[ (1 + f_+ + f_-) = \frac{T}{\gamma \beta k_{56}^*} \ln \frac{\sinh \{ \gamma (\omega_{56}^2 + \beta k_{56}^2) - \mu \} / 2T \}}{\sinh \{ \gamma (\omega_{56}^2 - \beta k_{56}^2) - \mu \} / 2T} , \quad (A.28) \]

where \( \beta \) and \( \gamma \) are the velocity and gamma–factor of the pair with respect to the bath. This factor reduces to \( \coth(\omega_{56} - \mu)/2T \) for vanishing incoming total momentum \( \vec{K} \).

Eq. (A.27) is solved by a separable ansatz

\[ \langle k_{12}^* | \mathcal{M}_{f=0}(E, K) | k_{34}^* \rangle = \sum_{i=1}^2 \lambda_i(s) v_i(k_{12}^*) \tau_i(k_{34}^*; s, \vec{K}) . \quad (A.29) \]

where the functions \( \tau_i \) obey the coupled set of equations

\[ \sum_{j=1}^2 \left[ \delta_{ij} - \lambda_j(s) g_{ij}(s, \vec{K}) \right] \tau_j(k; s, K) = v_i(k) , \quad i = 1, 2 \quad (A.30) \]

with

\[ g_{ij}(s, K) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} v_i(k) \frac{1}{\omega(k)} \frac{1 + f_+ + f_-}{s - 4\omega_k^2 + i\eta} v_j(k) . \quad (A.31) \]

The gap equation

To obtain the Cooper piece of the mass operator we must simply replace \( \sigma \) and \( bb \) by \( \langle \sigma \rangle \) and \( \langle bb \rangle \). According to the previous result, we find after some straightforward algebra, and noting that the index 2 is necessarily \(-1\),
\[\Sigma_{1C}(t, t') = -i G_{-1, -1}(t, t') F_1^2 \left( e^{-i\Delta_1(t-t')} + e^{i\Delta_1(t-t')} \right).\]  

(A.32)

The important point is that \(F_1\) involves the \(I = 0, \ell = 0\) energy dependent \(\pi-\pi\) potential at \(E = 2\mu\):

\[F_1 = -\frac{1}{2} \int \frac{d^3k_2}{(2\pi)^3} \langle \tilde{k}_1, -\tilde{k}_1 | V_{I=0}(E = 2\mu) | \tilde{k}_2, -\tilde{k}_2 \rangle \langle b_2 b_{-2} \rangle.\]  

(A.33)

In energy space, \(\Sigma_{1C}(\omega)\) is

\[\Sigma_{1C}(\omega) = F_1^2 \int d\tau e^{i\omega \tau} \{ \theta(\tau) \langle b_1^\dagger(\tau), b_1(0) \rangle + \theta(-\tau) \langle b_1(0), b_1^\dagger(\tau) \rangle \} \times (e^{-i\Delta_1 \tau} + e^{i\Delta_1 \tau}).\]  

(A.34)

Taking for \(b_1^\dagger(\tau)\) the bare time evolution \(b_1^\dagger(\tau) = b_1 e^{i\omega \tau}\), and keeping only the real part, we finally find

\[\Sigma_{1C}(\omega) = -F_1^2 \left( \frac{1}{\omega + \omega_1 - 2\mu} + \frac{1}{\omega + \omega_1 + 2\mu} \right).\]  

(A.35)

The first term is the usual non-relativistic result, and the second one corresponds to a relativistic correction.

Reinserting the result (A.35) into the Dyson equation for the pion propagator, and ignoring the Hartree correction, we find that the pole of the pion propagator is the solution of

\[(\omega - \mu)^2 = (\omega_1 - \mu)^2 - F_1^2 \left[ 1 + \frac{(\omega - \mu) + (\omega_1 - \mu)}{(\omega - \mu) + (\omega_1 - \mu) + 4\mu} \right].\]  

(A.36)

The second term in the square brackets represents a relativistic correction to the standard dispersion relation, since for typical non-relativistic situation one has

\[\mu \lesssim m_\pi, \quad \omega - m_\pi \ll m_\pi, \quad \text{and} \quad \omega_1 - m_\pi \ll m_\pi.\]  

(A.37)

Calling

\[\Delta_1^2 = F_1^2 \left[ 1 + \frac{E_1 + (\omega_1 - \mu)}{E_1 + (\omega_1 - \mu) + 4\mu} \right],\]  

(A.38)

the quartic equation can be approximated by a quadratic one in terms of \(\omega - \mu\):

\[E_1^2 = (\omega - \mu)^2 - \Delta_1^2,\]  

(A.39)

with a gap equation following from Eq. (A.34) and (A.33)

\[\Delta_1 = -\frac{1}{2} \int \frac{d^3k_2}{(2\pi)^3} \langle \tilde{k}_1, -\tilde{k}_1 | V_{I=0}(E = 2\mu) | \tilde{k}_2, -\tilde{k}_2 \rangle \frac{\Delta_2}{2E_2} \coth \frac{E_2}{2T} \left[ 1 + \frac{E_1 + (\omega_1 - \mu)}{E_1 + (\omega_1 - \mu) + 4\mu} \right]^{1/2},\]  

(A.40)

which is the standard gap equation with a relativistic correction. The presence of the factor 1/2 is somewhat unconventional, but is simply related to the fact that the matrix element of the interaction incorporates the exchange term. Note that the factor 1/4 in front of the quartic term of the interaction Hamiltonian Eq. (A.3) has the same origin.

To calculate the occupation number, we note, using the explicit form of \(G_{11}\), that the bare vacuum is the vacuum of quasi-particle operators \(B_1\), such that
\[ b_1 = \left[ \frac{\omega_1 - \mu}{2E_1} + \frac{1}{2} \right]^{1/2} B_1 + \left[ \frac{\omega_1 - \mu}{2E_1} - \frac{1}{2} \right]^{1/2} B_{-1}^\dagger . \] (A.41)

Using \( \langle B^\dagger B \rangle = [\exp(E_1/T) - 1]^{-1} \), it follows that
\[
\langle b_1^\dagger b_1 \rangle = \frac{\omega_1 - \mu}{2E_1} \coth \frac{E_1}{2T} - \frac{1}{2}
\]
\[
\langle b_1 b_{-1} \rangle = \frac{\Delta_1}{2E_1} \coth \frac{E_1}{2T} .
\] (A.42)

**The Thouless Criterion**

We may also obtain the condition for having a pole in the \( T \)-matrix at \( E = 2\mu \). For \( E \) near \( 2\mu \), we write
\[
\langle \vec{k}_1, -\vec{k}_1 \mid T_{I=0}(E) \mid \vec{k}_2, -\vec{k}_2 \rangle \equiv Z \frac{1}{E - 2\mu} Z_2 .
\] (A.43)

Multiplying (A.27) by \( E - 2\mu \) and taking the limit \( E \to 2\mu \), one obtains an equation for \( Z_1 \)
\[
Z_1 = -\frac{1}{2} \int \frac{d^3k_2}{(2\pi)^3} \langle \vec{k}_1, -\vec{k}_1 \mid V_{I=0}(E = 2\mu) \mid \vec{k}_2, -\vec{k}_2 \rangle \frac{Z_2}{2(\omega_2 - \mu)} \coth \frac{\omega_2 - \mu}{2T}
\]
\[
\times \left[ 1 + \frac{E_2 + (\omega_2 - \mu)}{E_2 + (\omega_2 - \mu) + 4\mu} \right] ,
\] (A.44)

In the non–relativistic limit, \( i.e., \) neglecting the last term in the square brackets, this equation coincides with the linearized form of the (non–relativistic) gap equation. This is just the Thouless criterion: the gap equation begins to exhibit non–trivial solutions at the point where the \( T \)-matrix has a pole at zero energy, \( \langle H - \mu N \rangle = E - 2\mu = 0 \). We see that the Thouless criterion is only approximately valid if relativistic corrections are included.
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**FIGURE CAPTIONS**

Fig. 1 The critical temperatures \( T^u_c \) (solid line) and \( T^\ell_c \) (dashes).

Fig. 2 The gap strengths \( \delta_1 \) (solid) and \( -\delta_2 \) (dashes) vs temperature \( T \), for \( \mu = 135 \) MeV. See text.

Fig. 3 The quasiparticle dispersion relation at fixed \( \mu = 135 \) MeV, for three temperatures: \( T^\ell_c \) (dashes), \( T = 115 \) MeV (dot–dash), and \( T^u_c \) (solid).

Fig. 4 The square of the on–shell invariant \( T \)–matrix for \( I = 0, \ell = 0 \) in free space (solid line). Also shown are the results in a thermal bath with \( T = 100 \) MeV, \( \mu = 135 \) MeV for total momentum \( K = 0 \) (short dashes), 200 MeV/c (long dashes), 1 GeV/c (dot–dashes), and 3 GeV/c (dotted).

Fig. 5 The \( T \)–matrix pole function \( F_{\mu,T}(E) \) at various temperatures for \( \mu = 135 \) MeV and \( K = 0 \). Here, \( T^u_0 = T^u_c \approx 123 \) MeV, \( T^\ell_c \approx 77 \) MeV, and \( T^\ell_0 \approx 47 \) MeV.

Fig. 6 The magnitude of \( F_{\mu,T}(s,K) \), Eq. (18), for a thermal medium with \( T = 100 \) MeV and \( \mu = 135 \) MeV. The solid, short dashed, long dashed, and dot–dashed lines correspond to \( K = 0, 100 \) MeV/c, 250 MeV/c, and 500 MeV/c, respectively.

Fig. 7 The thermal pion transverse mass spectrum at midrapidity, for \( T = 100 \) MeV and \( \mu = 135 \) MeV, without the effect of a gap (dashes) and including the effect of a gap (solid line).
Alm et al.: Figure 1
(MeV)

Alm et al.: Figure 2
Alm et al.: Figure 3
$|F_{\mu, T}(S)|$
$\frac{dN}{m_t \, dm_t \, dy}$ (arb)