Phase Thermalization: from Heavy Fermi Liquid to Incoherent Metal

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Thermalization is the ubiquitous phenomenon of a system attaining thermal equilibrium. When a small system (probe) is connected to a larger one, thermalization entails induction of the temperature of the larger system onto the smaller one. We introduce the notion of phase thermalization, a phenomenon where the quantum phase of a large system is induced onto a smaller one. We provide an explicit example where the smaller system is a heavy Fermi liquid and the larger system is an incoherent metal.

I. INTRODUCTION

One of the most familiar physical quantities in nature is temperature. Despite the familiarity, temperature is a rather mysterious quantity from microscopic point of view, since the laws governing individual particles, be it Newton’s laws of motion or Schrödinger equation, do not feature temperature. One of the great achievements of physics is the microscopic understanding of temperature and thermodynamics, through (semi-classical) statistical mechanics (see e.g. [1]). Chaotic dynamics of physical systems is key to this understanding. A fully quantum treatment of thermalization however calls for a new framework, namely Eigenstate Thermalization Hypothesis [2, 3].

One common feature associated with temperature that we experience in day to day life is, it can be induced. Especially a larger system (the bath) can induce its temperature onto a smaller system (the probe). Under more specific circumstances, a bath might be able to induce finer quantities onto the probe [4], such as Lyapunov exponent, the measure of semi-classical/early time chaos. This inspires us to take one step further and ask: can a quantum many body system induce its phase onto another quantum many body system? We show that the answer is in affirmative, by providing an explicit example. For the lack of a better name, we shall refer to this phenomenon as phase thermalization. In our particular setup the bath and the probe will be an incoherent metal (IM) and a heavy Fermi liquid (HFL) respectively.

IM-s [5] are particular kind of strange metals [6] which characteristically have high (linear with temperature) resistivity and don’t have quasiparticles and thus are not described by usual extremely successful framework of Fermi-liquid (FL) theory. Moreover in IM even the momentum is not conserved (i.e. no overlap between momentum and conserved current) and is degraded very fast. Due to disorders/impurities there is no sound mode but only diffusion modes [5]. They can have very large resistivity (violate Mott-Ioffe-Regel (MIR) bound) [7] and hence are examples of bad metals. Whereas usual metals (i.e. free fermionic QFT) in presence of disorder go to insulating phase at some critical density, most holographic metals at strong disorder (at fixed temperature) become IM-s. Although IM-s are easy to get in holographic models (see e.g. [8–13]), these are hard to find in explicit quantum matter realisation – higher dimensional SYK lattice models [14–17] are systems where one can realize them. Higher dimensional SYK like models show diffusion of energy and number density even at $T = 0$ [18, 19].

At low temperatures HFL can be characterized as Fermi liquid, but they show extremely enhanced Landau parameters [20, 21]. Due to the extremely large effective mass they are called Heavy Fermions (HF). It is remarkable that as a phenomenological theory FL description still works. In the particular fermionic model we are considering, HFL phase is realised at low temperature whose scale is set by the relative strength of the couplings [17]. A single universal coherence scale $E_{c} = \frac{\alpha}{V}$ appears for $s_{0} \ll V$. For $T \ll E_{c}$ it’s a FL with quasi-particles (since there exists a peak in the spectral function) but it’s heavily renormalized i.e. the quasiparticles are very heavy with weight $\sim s_{0}/V$ and effective mass, $\gamma = m_{s}/m = \frac{V}{m}$.

Our goal in this paper is to explore how the phase of the probe, which is a HFL, is affected when we let it interact with the bath that is an IM. As we will see, in what follows, via solving the interacting bath-probe system (in some ‘thermodynamic limit’) that the bath induces its phase on the probe.

The rest of the paper is organised as follows. In section II we introduce the bath, a 1D chain of SYK dots, each dot containing $N \gg 1$ Majorana fermions. The quantum dots interact with each other through arbitrarily long range interaction, parameterised by a parameter $\alpha$. At low energies, this system develops a reparameterization symmetry along time direction and is amenable to analytic treatment. Transport properties are analyzed and the system is found to behave as an IM. The Lyapunov spectrum and butterfly velocity are computed. The spatially constant mode saturates the chaos bound. For $\alpha < 1$, the system develops another phase. In section III we introduce the probe, which is also a 1D chain of SYK
dots, each dot containing \( n < N \) Majorana fermions, interacting in a similar fashion, except there is a hopping term as well. The hopping term turns the probe into a HFL. Next we connect the probe and the bath through a random four Fermi interaction. We find consequently the probe turns into an IM, with same diffusion coefficient as the bath to leading order in \( n/N \). Lastly in section IV, we summarize main results and discuss some future directions.

II. INCOHERENT METALLIC BATH

The bath consists of a one dimensional chain of SYK-like dots. SYK model, acronym for Sachdev-Ye-Kitaev model, is a model of \( N \gg 1 \) number of disordered Majorana Fermions with all to all interactions [22–24]. This model describes a non-Fermi liquid state that saturates the chaos bound [25] and exhibits emergent renormalization symmetry, which is further broken spontaneously as well as explicitly. These non-trivial features are expected of a near-extremal black holes, making SYK model a model for near extremal black holes.

SYK model is intriguing from quantum matter perspective as well, since it is a solvable model of a novel non-Fermi liquid. In recent years variants of SYK and SYK-lattice models [16, 26–28] have been studied extensively. A distinguishing feature of the model considered here is arbitrarily long range interaction. Each pair of dots interacts through the Hamiltonian

\[
H_{\text{bath}} = \sum_{x,y=-\infty \atop y>x}^{\infty} \sum_{1 \leq j < k < N} \sum_{1 \leq l < m \leq N} V_{jklm}^{(xy)} (y-x)^{\alpha} \chi_j^x \chi_k^y \chi_l^y \chi_m^y, \tag{1}
\]

where \( V_{jklm}^{(xy)} \)s are random couplings obeying \( V_{jklm}^{(xy)} = 0 \), \( V_{jklm}^{(xy)} = V_{jklm}^{(y'x')} = \sqrt{2/N} \delta_{xx'} \delta_{yy'} \delta_{jklm} \delta_{j'k'l'm'} \), where the overline denotes disorder average. For SYK-dots it is natural to have an on site random quartic term [14, 17]. This term has the effect of shifting \( V_0^2 \) by an additive constant. Hence we have dropped this term for the sake of simplicity.

In the limit \( \alpha \to \infty \), only nearest neighbor interactions survive in (1). This system was studied in detail in [14] and describes an IM. At finite \( \alpha \), the system continues to be an IM for \( \alpha > 1/2 \), where it undergoes a phase transition [29]. We shall work with large enough \( \alpha \), so as to stay away from such any such pathology.

We can also include a random hopping term with strength say \( s_0 \). Fermions have dimension 1/4 in the IM phase described by (1) and thus hopping term is a relevant one. In fact turning on hopping steers the system to a HFL phase [17]. However this change occurs below the critical temperature \( T_c \sim \frac{\pi}{V_0 \sqrt{\zeta(2\alpha)}} \). We shall choose to work above \( T_c \). Since in this regime the hopping term does not affect the phase of the system, we will exclude this term for the sake of simplicity.

Lastly, we have also ignored any interaction with phonons. In similar models with complex fermions, weak enough electron-phonon interaction has been seen not to alter the IM behaviour of the system [30].

Upon performing the disorder average, one finds the effective action to depend on the fermions only through the bilocal bilinear combinations \( \sum_{i=1}^{N} \chi_i^x(\tau) \chi_i^y(\tau') \). It is thus useful to define these fermion bilocals as new fields

\[
\bar{G}_x(\tau, \tau') = \frac{1}{N} \sum_{i=1}^{N} \chi_i^x(\tau) \chi_i^y(\tau'). \tag{2}
\]

A bilocal Lagrange multiplier field \( \bar{\Sigma} \) is needed in order to impose (2). Disorder averaged theory can be expressed as a path integral over the bilocal fields \( \bar{G}, \bar{\Sigma} \) with the action

\[
S_{\text{eff}}[\bar{G}, \bar{\Sigma}] = N \sum_{x=-\infty \atop y>x}^{\infty} \left[ -\log \text{Pf} (\partial_\tau - \bar{\Sigma}^x) + \frac{1}{4} \int_{-\infty}^{\beta} d\tau_1 d\tau_2 \left[ \bar{\Sigma}^x(\tau_1, \tau_2) \bar{G}_x(\tau_1, \tau_2) - \frac{V_0^2}{4} \sum_{w=1}^\infty w^{-2\alpha} \bar{G}_x(\tau_1, \tau_2)^2 \bar{G}_x^*(\tau_1, \tau_2)^2 \right] \right] . \tag{3}
\]

Solving for two point functions of \( \chi \) fields to leading order in \( 1/N \), amounts to computing the saddle point values of \( \bar{G} \) fields. In deep infrared region, the saddle point equations of \( \bar{G} \) and \( \bar{\Sigma} \) together imply

\[
\frac{V_0^2}{2} \bar{G}_x \circ \left( \sum_{y \neq x} \frac{1}{|y-x|^{2\alpha}} \bar{G}_x \left( \begin{array}{c} y \\ 0 \end{array} \right) \right)^2 = -1. \tag{4}
\]

Here \( \circ \) denotes convolution product, e.g. \( \bar{G}^y(\tau_1, \tau_2) := \int d\tau' \bar{G}^y(\tau_1, \tau') \bar{G}^y(\tau', \tau_2) \). The same equation can be obtained using diagrammatic techniques as Schwinger Dyson equation for two point functions. We shall not pursue this equivalent approach here.

Remarkably, (4) is invariant under an emergent reparameterization symmetry

\[
\tau \to f(\tau), \bar{G}_x(\tau, \tau') \to |f'(\tau) f'(\tau')|^{-1/4} \bar{G}_x(\tau, \tau'). \tag{5}
\]

Assuming translational invariance, the solution to (4) is given by

\[
G_\alpha(\tau) = \left( \frac{1}{4\pi V_0^2 \zeta(2\alpha)} \right)^{1/4} \left[ \frac{\pi}{\beta \sin \frac{\pi}{\sqrt{4\alpha}}} \right]^{1/2} \text{sgn} \tau, \tag{6}
\]

\[
\Sigma_\alpha(\tau, \tau') = V_0^2 \zeta(2\alpha) G_\alpha(\tau, \tau')^3, \tag{6}
\]

where \( \beta \) is the inverse temperature and \( \zeta(2\alpha) = \sum_{k \in \mathbb{Z}} \frac{1}{k^{2\alpha}} \) is the Riemann Zeta function. This implies that the fermions have developed a scaling dimension 1/4, markedly different from that of a free fermion. This immediately indicates that the system is in a strongly
function, that we set out to compute, is given by
\[
\frac{1}{N} F_{xy}(\tau_1, \tau_2; \tau_3, \tau_4) := |G_s(\tau_1, \tau_2)|^{-1} |G_s(\tau_3, \tau_4)|^{-1} \langle g^x(\tau_1, \tau_2) g^y(\tau_3, \tau_4) \rangle .
\] (10)

It is easier to work in momentum space, where (8) implies \( \langle g(p)g(-p) \rangle \sim 1/R(p) \). However, we face a problem for the constant mode, for which \( \langle g(0)g(0) \rangle \sim (K_c^{-1} - 1)^{-1} \).

Since \( K_c \) admits a unit eigenvalue, \( (K_c^{-1} - 1)^{-1} \) is divergent. This can be traced back to vanishing action of Goldstone modes, which correspond to \( K_c = 1 \) eigenspace.

The cure to this problem is to endow these modes with small but non-vanishing action. This can be achieved by moving slightly away from the regime we are working in, i.e. deep infrared, or equivalently large \( \beta \nu \), where \( \nu = V_0 \sqrt{2\xi(2\alpha)} \). Moving slightly away from this regime amounts to taking \( O(\frac{1}{\beta \nu}) \) corrections to \( K_c \), which leads to a small \( O(\frac{1}{\beta \nu}) \) action for Goldstones. Note, “Goldstone” is really a misnomer at this point since a true Goldstone is supposed to have vanishing action. We shall call these modes pseudo-Goldstones or simply soft modes, in what follows.

This small action for soft modes implies that the reparameterization symmetry is not only spontaneously but also explicitly broken. This pattern of symmetry breaking resembles that of near-AdS \( S_3 \) spaces \([31]\), which in turn arise as near horizon geometries of near-extremal black holes. This is one of the reasons for SYK model being proposed as a toy model for such black holes. These would not concern us directly though.

Soft modes are change in \( G_s \) under reparameterizations. Any function on the Euclidean time circle defines such a reparameterization. They can conveniently be expanded in Fourier modes. This introduces new quantum numbers for the soft modes. Any soft \( g(p) \) admits an expansion

\[ g(p) = |G_s| \delta_c(p) \tilde{G}(p) = \sum_{|m| \geq 2} \varepsilon_m(p)g_m , \]

\[ g_m = \frac{i\beta^2}{4} \left( \frac{2\pi}{\beta} \right)^2 \frac{e^{-2\pi i m y/\beta}}{\sin \frac{\pi x}{\beta}} \left\{ -m \cos \frac{\pi m x}{\beta} + \frac{\sin \frac{\pi m x}{\beta}}{\tan \frac{\pi x}{\beta}} \right\} , \] (11)

where \( x = \tau_1 - \tau_2, y = \frac{\tau_1 + \tau_2}{2} \) and \( b = (4\pi V_0^2 \xi(2\alpha))^{1/4} \).

These modes satisfy normalization relations

\[ g_m \circ g_{m'} = \delta_{m+m'} \frac{b^4}{16} \left( \frac{2\pi}{\beta} \right)^4 \beta^2 |m(m^2 - 1)|^3 . \] (12)

Note \( g_0, g_{k \pm 1} \) has vanishing norm. These modes correspond to global reparameterizations and are not soft modes. This is why they have been excluded in the expansion (11).

\( \varepsilon_{m-s} \) are the new fields of the theory. Using the fact...
that up to $\mathcal{O}(1/\beta V)$, eigenvalues of $K_c$ are corrected as
$\left[2\alpha K/m\right] = 1 - \frac{\alpha K}{3\beta V} + \ldots$, with $\alpha K \approx 2.85$ a numer-
ical constant, up to $\mathcal{O}(p^2)$ the soft mode action is found to be

$$S_{\text{soft}}^{bath} \sim \frac{N\alpha_K}{256\pi V} \left(\frac{2\pi}{\beta}\right)^2 \int_0^{2\pi} dp \sum_{|m| \geq 2} \varepsilon_m(p)\varepsilon_{-m}(-p)|m|(m^2 - 1) \left[\frac{2\pi|m|}{\beta} + \frac{2\pi V\zeta(2\alpha - 1)}{3\alpha_K \zeta(2\alpha)}p^2\right] . \quad (13)$$

These modes give the $\mathcal{O}(V_0)$ contribution to the four point function (10), which dominates over other $\mathcal{O}(1)$
contributions. In long wavelength or small $p$ regime, this contribution is

$$\frac{F^{\text{big}}(p; \tau_1, \tau_2; \tau_3, \tau_4)}{G^2(x_{12})G^2(x_{34})} := \frac{32\pi V_0 \sqrt{2\zeta(2\alpha)}}{\alpha_K} \sum_{|m| \geq 2} \frac{1}{|m|(m^2 - 1)} \frac{e^{-2\pi(y_{12} - y_{34})/\beta}}{\omega_m + D_{\alpha} p^2} f_m(x_{12}) f_m(x_{34}) , \quad (14)$$

where $f_n(x) = -n \cos \frac{\pi n x}{\beta} + \sin \frac{\pi n x}{\beta} x_{ij} = \tau_i - \tau_j$ and $y_{ij} = \tau_i + \tau_j$ and $\omega_m = \frac{2\pi m}{\beta}$ are the Matsubara frequencies. The diffusion coefficient $D_{\alpha}$ is given by

$$D_{\alpha} = \frac{2\sqrt{2\pi V_0}}{3\alpha_K \sqrt{\zeta(2\alpha)}} \zeta(2\alpha - 1) . \quad (15)$$

This saturates the bound conjectured in [32]. Similar be-

haviour was found in certain holographic theories [33, 34]. Note $D_{\alpha}$ diverges as $\alpha \to 1$. In fact around $\alpha = 1, \zeta(2\alpha) - \frac{\zeta(2\alpha + 1)}{\tan \beta} \neq 0$ no more admits a Taylor ex-
pansion around $p = 0$. More specifically one encounters a $p^2 \ln |p|$ term. The apparent divergence of $D_{\alpha}$ at $\alpha = 1$ is same as the divergence of $\ln p^2$ as $p \to 0$. Note $p^2 \ln |p|$ however is well defined as $p \to 0$. To summarize, in this regime diffusion coefficient is no more a meaningful con-
cept. For $\alpha < 1$, the $p^2$ term is replaced by $|p|^{2\alpha - 1}$. The system seems to undergo a phase transition at $\alpha = 1$. Finally at $\alpha = 1/2$, the theory becomes ill defined due to divergence of $\zeta(2\alpha)$. We shall restrict to large $\alpha$ regime.

Being bereft of quasi-particles, IM-s lack important scales such as quasiparticle mean free time and Fermi velocity. Lyapunov time $\tau_L = 1/\lambda_L$, and butter velocity has been suggested to play these roles respectively [33–35]. Thus chaos is of central importance for transport in IM-s. In classical physics, chaos is characterised by exponential divergence of initially nearby trajectories. Upon quantizing the system, obvious generalization seems to be divergence of nearby wave packets. However this is rendered meaningless over long time scales due to spread of wave packets themselves. The timescale over which this transition happens is Heisenberg time. Nonetheless in “semi-classically” early times, this point of view is meaningful. Upon some further refinement [36] chaos

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$$\lambda_L = \frac{2\pi Li_2 e^{ip} + Li_2 e^{-ip}}{2\zeta(2\alpha)} . \quad (18)$$

Lyapunov spectrum depends on $\alpha$, but the constant mode $\lambda_{bath}(0)$ always saturates the chaos bound. This behaviour continues till $\alpha \to 1/2$. For $\alpha < 1/2$, $Li_2(p)$ diverges in $p \to 0$ limit, hence seemingly violates the chaos bound [25]. However the mean field analysis employed to deduce this also fails in this regime, so we are not too worried.

Another key quantity in characterizing chaos is the butterfly velocity, the velocity with which a chaos spreads spatially. It is related to the diffusion coefficient as [37]

$$v_B^2 = \frac{2\pi D_{\alpha}}{\beta} . \quad (19)$$
This completes our discussion of the IM bath.

III. HEAVY FERMI LIQUID PROBING THE INCOHERENT METALLIC BATH

In this section we analyze a HFL probe connected to the IM bath discussed above.

A. Free heavy Fermi liquid probe

We take the probe to be another chain of Majorana fermions, which we denote by \( \kappa \). But there are two key distinctions. Firstly each site of the probe hosts \( n < N \) number of Majorana fermions and secondly they interact through random hopping, with hopping amplitude decaying with distance. The following Hamiltonian captures the situation.

\[
H_{\text{probe}} = \sum_{x, y=-\infty, \ y>x}^{\infty} \frac{t_{ab}^{xy}}{(y-x)\rho} \kappa_a \kappa_b^\dagger, \tag{20}
\]

where \( t_{ab}^{xy} \)'s are random hoppings, satisfying \( t_{xy}^{ab} = 0 \), \( t_{xy}^{ab} = t_{xy}^{ba} = t_{-y-x}^{xy} \). The overline represents disorder average as usual. In the limit \( \rho \to \infty \) we only have nearest neighbour hopping. In this limit the system is in a HFL phase [17]. We shall take \( \rho \) to be sufficiently large so that this phase persists.

B. Probe-bath interaction

We connect this probe with the bath via a random two-body or equivalently four fermion interaction

\[
H_{\text{int}} = \sum_{x, y=-\infty, \ y>x}^{\infty} \sum_{j,k \leq N, x \leq \kappa, \kappa \leq N} J_{jk, ab}^{(xy)} [x-y]^{-\gamma} \tilde{\chi}_{jk}^x \tilde{\chi}_{ab}^y \kappa_a \kappa_b^\dagger, \tag{21}
\]

which with \( J_{jk}^{(xy)} = 0 \), \( J_{j,k}^{(xy)} = J_{j,k}^{(xy)} = J_{j,k}^{(xy)} \). One can add an onsite interaction term as well, without much effect.

In order to solve for the coupled bath-probe system, we first define the bilocal field

\[
\tilde{G}_x^a(\tau_1, \tau_2) = \frac{1}{n} \sum_{q=1}^n \kappa_q^a(\tau_1) \kappa_q^a(\tau_2), \tag{22}
\]

and the corresponding Lagrange multiplier bilocal field \( \tilde{\Sigma}_x(\tau_1, \tau_2) \). We shall call the bilocal fields \( \tilde{G}, \tilde{\Sigma} \) introduced in previous section as \( G_x^\kappa, \kappa \) respectively. In terms of these fields the effective action describing the disorder averaged combined bath-probe system reads

\[
S_{\text{eff}}[G_x^\kappa, \tilde{G}_x^\kappa, \tilde{\Sigma}_x] = N \sum_{x=-\infty}^{\infty} \left[ -\log \text{Pf}(\partial_x - \tilde{\Sigma}_x^\kappa) - \epsilon \log \text{Pf}(\partial_x - \tilde{\Sigma}_x^\kappa) + \frac{1}{2} \int_0^\beta \int_0^\beta d\tau_1 d\tau_2 \left\{ \tilde{\Sigma}_x^\kappa(\tau_1, \tau_2) \tilde{G}_x^\kappa(\tau_1, \tau_2) + \epsilon \tilde{\Sigma}_x^\kappa(\tau_1, \tau_2) \tilde{G}_x^\kappa(\tau_1, \tau_2) \right\} \right],
\]

where \( \epsilon = n/N \). To begin with let us consider the \( t_0 = 0 \) case. For a saddle point analysis, we assume translational invariance \( \tilde{G}_x^\kappa = G_x^\kappa, \tilde{\Sigma}_x^\kappa = \tilde{\Sigma}_x^\kappa \). With this assumption, in deep infrared saddle point equations for \( \tilde{G}_x^\kappa, \tilde{\Sigma}_x^\kappa \) read

\[
\left(V^2 \tilde{G}_x^\kappa + \frac{n J^2}{N} \tilde{G}_x^\kappa \tilde{G}_x^\kappa \right) \tilde{G}_x^\kappa = -1, \tag{24}
\]

where we have substituted saddle point values of \( \tilde{\Sigma}_x^\kappa, \tilde{\Sigma}_x^\kappa \) and

\[
V^2 := V_0^2 \zeta(2\alpha), \quad J^2 := J_0^2 \zeta(2\gamma). \tag{25}
\]

The remarkable thing about \( (24) \) is that it is invariant under the reparameterization symmetry \( (5) \), with both \( \tilde{G}_x^\kappa \) and \( \tilde{\Sigma}_x^\kappa \) transforming in same manner. In order to solve for \( (24) \), we first note the homogeneous appearance of \( \tilde{G}_x^\kappa \) and \( \tilde{\Sigma}_x^\kappa \), which suggests the following ansatz

\[
G_x^\kappa = AG_x^\kappa. \tag{26}
\]

Consistency demands

\[
A = \frac{V}{J(1-\epsilon)^{1/2}}. \tag{27}
\]

Therefore one has following two point functions at finite
temperature
\[ G_\chi(\tau) = \left( \frac{(1-\epsilon)}{4\pi V^2} \right)^{1/4} \left[ \frac{\pi}{\beta \sin \frac{\tau}{2}} \right]^{1/2} \text{sgn} \tau, \]
\[ G_\kappa(\tau) = \frac{V^{1/2}}{J(1-\epsilon)^{1/4}(4\pi)^{1/4}} \left[ \frac{\pi}{\beta \sin \frac{\tau}{2}} \right]^{1/2} \text{sgn} \tau, \quad (28) \]

which spontaneously break the reparameterization symmetry down to $SL(2, \mathbb{R})$. So, connecting the probe with the bath does not affect the symmetry of the bath and its spontaneous breaking.

Around the saddle, we define the fluctuation fields as
\[ \widetilde{G}_x = G_\chi^{\text{saddle}} + |G_\chi^{\text{saddle}}|^{-1} g_x, \]
\[ \widetilde{G}_\kappa = G_\kappa^{\text{saddle}} + |G_\kappa^{\text{saddle}}|^{-1} g_\kappa, \]
\[ \widetilde{\Sigma}_x = \Sigma_\chi^{\text{saddle}} + |\Sigma_\chi^{\text{saddle}}|^{-1} \sigma_x, \]
\[ \widetilde{\Sigma}_\kappa = \Sigma_\kappa^{\text{saddle}} + |\Sigma_\kappa^{\text{saddle}}|^{-1} \sigma_\kappa. \quad (29) \]

Integrating $\widetilde{\sigma}_\chi$ and $\widetilde{\sigma}_\kappa$ out, we get the following effective action
\[
S_{eff}^{\text{probe}} = \frac{N}{2\pi} \int dp \, g_\kappa(p) \circ R'_\kappa \circ g_\kappa(-p),
\]
\[
R'_\kappa = -3\epsilon \xi(2\gamma)(K_\epsilon - 1)(3 - K_\epsilon + 2K_\epsilon)\frac{4K_\epsilon (3(1-K_\epsilon) + 2\epsilon)}{4K_\epsilon (3(1-K_\epsilon) + 2\epsilon)}
+ J_0^p e^{2K_\epsilon} \frac{K_\epsilon [\xi(2\alpha - 1)\xi(2\gamma)(1-\epsilon) + 3\xi(2\alpha)\xi(2\gamma - 1)(1-K_\epsilon) + 2K_\epsilon \xi(2\alpha)\xi(2\gamma - 1)]}{\xi(2\alpha)(3(1-K_\epsilon) + 2\epsilon K_\epsilon)^2} p^2. \quad (32)
\]

The notable feature of this rather clumsy looking expression is that it vanishes for $p = 0, K_\epsilon = 1$, just like the effective action of the bath (8). From the discussion following (8), we know this vanishing corresponds to Goldstones of spontaneously broken reparameterization symmetry. Subsequent analysis closely follows that of the bath and the dynamics is dominated by the soft modes, which can be endowed with a small action. Using the expansion of $g_\kappa$

\[ g_\kappa(p) = |G_\kappa^*| \delta_\kappa(p) \tilde{G}_\kappa(p) = \frac{V}{f} \sum_m \varepsilon_m(p) g_m, \quad (33) \]
in the soft mode sector, to leading order in $\epsilon$, the soft mode action $S_{eff}^{\text{soft}}$ is found to be exactly the same as $S_{eff}^{\text{bath}}$(13)!

### Section 4.5: The Fate of the Probe

We study the fate of the probe $\chi$ for some interaction strength $U_0$. When $t_0 < U_0$, the IM phase due to random four fermion interactions of strength $U_0$ continues to persist above a transition temperature $T_c \sim t_0^2/U_0$, when a random hopping of strength $t_0$ is turned on. In particular by choosing $t_0^2 \ll U_0$ this transition temperature can be made arbitrarily low. Adapted to present context, this implies even when the hopping is present, as long as it is small compared to the four fermion couplings $V_0, J_0$, the IM phase...
persists until some low temperature.

To sum it up, just as a large bath induces its temperature on a small probe, the IM bath has induced its phase on the HFL probe.

IV. DISCUSSION

In this paper, we have introduced the notion of phase thermalization- the phenomenon of a larger quantum many body system inducing its phase onto a smaller one. We have presented an explicit example of this phenomenon, with the bath being an incoherent metal and the probe being a heavy Fermi liquid. Microscopic description of the bath consists of a one dimensional chain of quantum dots, each dot comprising $N \gg 1$ Majorana fermions. The dots interact pairwise though an arbitrarily long range random interaction, which is quartic in the fermions. This system is then solved in a certain large $N$ limit and found to be in incoherent metallic state. The probe is a one dimensional chain of quantum dots, each dot comprising $n \gg 1$ Majorana fermions, interacting through arbitrarily long range random hopping. This is known to describe a heavy Fermi liquid. Afterwards a random bi-quadratic interaction is turned on between every probe quantum dot and every bath quantum dot, with interaction strength falling with distance as a power law. The combined system is solved in a certain large $N$ as well as large $n$ limit, with $n < N$ and the probe is found to have turned into an incoherent metal itself, with exactly the same diffusion coefficient, and Lyapunov spectrum as the bath, if probe backreaction is ignored.

An immediate question is - do these conclusions survive when various other interactions are turned on? Let us explore various random interactions first. A bilinear term can consistently be ignored, since it violates both $\kappa \rightarrow -\kappa$ and $\chi \rightarrow -\chi$ symmetry individually (but respects their combined action). For remaining random interactions we shall argue that for a wide enough temperature window our conclusions remain intact. To do so, first let us note a quick way to estimate the critical temperature based solely on dimensional analysis. Consider fermions interacting through random hopping as well as random quartic interactions with couplings $t_0$ and $U_0$ respectively. This system is known to go through phase transition from heavy Fermi liquid to incoherent metal phase at a critical temperature $T_c \sim t_0^2/U_0$ [17]. We note $t_0^2/U_0$ is the only combination of these couplings, that has dimension of temperature/energy, both in the HFL phase as well as in IM phase. The same principle predicts a critical temperature $T_c \sim (U_0^3/c_q^4)^{1/(q-4)}$, where $c_q$ is the coupling for q-fermion interaction. The upshot is that for small $c_q$, the critical temperature is very high and therefore our conclusions hold till some fairly large temperature. More realistic analysis must include interaction with phonons, although we shall not attempt it.

A more serious concern is the phase change of the probe being a consequence of the particulars of the bath-probe interaction. This is just a concern since the Hamiltonian (1) describing the isolated bath and the Hamiltonian (21) describing the probe-bath interaction have similar form. However, it is not the case. This is perhaps best reflected in the diffusion coefficient of the probe (which is same as that of the bath (15) in leading order in $\epsilon$) being independent of the bath-probe interaction strength $J_0$.

Another question the reader may have in mind is whether one can think of the probe-bath system as a larger SYK chain with $N + n$ fermions per site and we are simply studying that bipartite. This would have required $\alpha = \gamma$ and also a term quartic in $\kappa$ fermions, analogous to (1) and (21). But for the case at hand, $\alpha$ and $\gamma$ can be very different and there is no quartic term in $\kappa$ fermions either, hence the system combined probe-bath system can not be thought as a larger SYK chain.

One of the key results of our analysis is – the bath-probe system is solvable for any values of $\epsilon = n/N$ as long as $n \rightarrow \infty$, $N \rightarrow \infty$ and $\epsilon$ is kept fixed to a value smaller than 1. Although physically we should restrict ourselves to the regime $n \ll N$ but our analysis remains unaltered as long as $n < N$. For $n = N$, the ansatz (26) no longer makes sense as the prefactor $A$ blows up. This limit of taking ‘the probe’ and ‘the bath’ of similar size is intriguing in itself and deserves further explorations.

There are several directions to be explored. Here we mention few of them.

The diffusion coefficient $D_\alpha$ diverges as we take the exponent $\alpha$, that determines the strength of intersite interaction, to one and it seems to undergo a phase transition at $\alpha = 1$. Throughout the paper we restricted ourselves to the regime $\alpha > 1$. It will be interesting to explore the opposite (i.e. $\alpha < 1$) regime.

Taking cue from the analysis of the bath, we have used a particular ansatz (see (26)) to solve the saddle point equations. Since the solution is obtained using an ansatz, we can not rule out the existence of other saddles (such as solitonic saddles).

Finally it will be fascinating to look for an experimental realization of this phenomenon of phase thermalization in some solid-state systems (see e.g. [38]).

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Above the critical temperature $T_c$, the high temperature superconductors (cupreets, ruthenium oxides, pnictides etc.) show strange metallic phase [39–42]. Therefore $T_c$ would be determined by the balance of free energies between the superconducting phase and the strange metallic phase. This in itself gives huge interest to understand the phase better.