Blow Up Dynamics for Equivariant Critical Schrödinger Maps

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Abstract: For the Schrödinger map equation \( u_t = u \times \Delta u \) in \( \mathbb{R}^{2+1} \), with values in \( S^2 \), we prove for any \( \nu > 1 \) the existence of equivariant finite time blow up solutions of the form \( u(x, t) = \phi(\lambda(t)x) + \zeta(x, t) \), where \( \phi \) is a lowest energy steady state, \( \lambda(t) = t^{-1/2-\nu} \) and \( \zeta(t) \) is arbitrary small in \( H^1 \cap H^2 \).

1. Introduction

1.1. Setting of the problem and statement of the result. In this paper we consider the Schrödinger flow for maps from \( \mathbb{R}^2 \) to \( S^2 \):

\[
\begin{align*}
  u_t &= u \times \Delta u, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \\
  u|_{t=0} &= u_0,
\end{align*}
\]

(1.1)

where \( u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in S^2 \subset \mathbb{R}^3 \).

Equation (1.1) conserves the energy

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^2} dx |\nabla u|^2.
\]

(1.2)

The problem is critical in the sense that both (1.1) and (1.2) are invariant with respect to the scaling \( u(x, t) \rightarrow u(\lambda x, \lambda^2 t), \lambda \in \mathbb{R}_+ \).

To a finite energy map \( u : \mathbb{R}^2 \rightarrow S^2 \) one can associate the degree:

\[
\deg(u) = \frac{1}{4\pi} \int_{\mathbb{R}^2} dx u_{x_1} \cdot J_u u_{x_2},
\]

where \( J_u \) is defined by

\[
J_u v = u \times v, \quad v \in \mathbb{R}^3.
\]
It follows from (1.2) that

$$E(u) \geq 4\pi |\deg(u)|. \quad (1.3)$$

This inequality is saturated by the harmonic maps $\phi_m, m \in \mathbb{Z}^+$:

$$\phi_m(x) = e^{m\theta R} Q^m(r), \quad Q^m = (h_1^m, 0, h_3^m) \in S^2,$$

$$h_1^m(r) = \frac{2r^m}{r^{2m} + 1}, \quad h_3^m(r) = \frac{r^{2m} - 1}{r^{2m} + 1}. \quad (1.4)$$

Here $(r, \theta)$ are polar coordinates in $\mathbb{R}^2$: $x_1 + i x_2 = e^{i\theta} r$, and $R$ is the generator of the horizontal rotations:

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

or equivalently

$$Ru = k \times u, \quad k = (0, 0, 1).$$

One has

$$\deg \phi_m = m, \quad E(\phi_m) = 4\pi m.$$

Up to the symmetries $\phi_m$ are the only energy minimizers in their homotopy class.

Since $\phi_1$ will play a central role in the analysis developed in this paper, we set $\phi = \phi_1, Q = Q_1, h_1 = h_1^1, h_3 = h_3^1$.

The local/global well-posedness of (1.1) has been extensively studied in past years. Local existence for smooth initial data goes back to [18], see also [14]. The case of small data of low regularity was studied in several works, the definite result being obtained by Bejenaru et al. in [3], where the global existence and scattering was proved for general $H^1$ small initial data. Global existence for equivariant small energy initial data was proved earlier in [6] (by $m$-equivariant map $u : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3, m \in \mathbb{Z}^+$ one means a map of the form $u(x) = e^{m\theta R} v(r)$, where $v : \mathbb{R}^+ \to S^2 \subset \mathbb{R}^3, m$-equivariance being preserved by the Schrödinger flow (1.1)). In the radial case $m = 0$, the global existence for $H^2$ data was established by Gustafson and Koo [11]. Very recently, Bejenaru et al. [4] proved global existence and scattering for equivariant data with energy less than $4\pi$. The dynamics of $m$-equivariant Schrödinger maps with initial data close to $\phi_m$ was studied by Gustafson et al. [9,10,12] and later by Bejenaru and Tataru [5] in the case $m = 1$. The stability/instability results of these works strongly suggest a possibility of regularity breakdown in solutions of (1.1) via concentration of the lowest energy harmonic map $\phi$. For a closely related model of wave maps this type of regularity breakdown was proved by Kriger et al. [13] and by Raphael and Rodnianski [17]. These authors showed the existence of $1$- equivariant blow up solutions close to $\phi(\lambda(t)x)$ with $\lambda(t) \sim e^{\sqrt{\ln(T^* - t)}}$ as $t \to T^*$ [17], and with $\lambda(t) \sim \frac{1}{(T^* - t)^{1/\nu}}$ as $t \to T^*$ where $\nu > 1/2$ can be chosen arbitrarily [13] (here $T^*$ is the blow up time). While the blow up dynamics exhibited in [17] is stable (in some strong topology), the continuum of blow up solutions constructed by Kriger et al. is believed to be non-generic. Recently, the results of [17] were generalized to the case of Schrödinger map equation (1.1) by Merle et al. in [15] where they proved the existence of $1$-equivariant blow up solutions of (1.1) close to $\phi(\lambda(t)x)$ with $\lambda(t) \sim \frac{(\ln(T^* - t)^3}{T^* - t}$. 


Our objective in this paper is to show that (1.1) also admits 1-equivariant Kriger–Schlag–Tataru type blow up solutions that correspond to certain initial data of the form

\[ u_0 = \phi + \xi_0, \]

where \( \xi_0 \) is 1-equivariant and can be chosen arbitrarily small in \( \dot{H}^1 \cap \dot{H}^3 \). Let us recall (see [5,9,10,12]) that such initial data result in unique local solutions of the same regularity, and as long as the solution exists it stays \( \dot{H}^1 \) close to a two parameter family of 1-equivariant harmonic maps \( \phi^{\alpha, \lambda} \), \( \alpha \in \mathbb{R}/2\pi \mathbb{Z} \), \( \lambda \in \mathbb{R}_+ \) generated from \( \phi \) by rotations and scaling:

\[ \phi^{\alpha, \lambda}(r, \theta) = e^{\alpha R} \phi(\lambda r, \theta). \]

The following theorem is the main result of this paper.

**Theorem 1.1.** For any \( \nu > 1 \), \( \alpha_0 \in \mathbb{R} \), and any \( \delta > 0 \) sufficiently small there exist \( t_0 > 0 \) and a 1-equivariant solution \( u \in C((0, t_0], \dot{H}^1 \cap \dot{H}^3) \) of (1.1) of the form:

\[ u(x, t) = e^{\alpha(t) R} \phi(\lambda(t)x) + \xi(x, t), \tag{1.5} \]

where

\[ \lambda(t) = t^{-1/2-\nu}, \quad \alpha(t) = \alpha_0 \ln t, \tag{1.6} \]

\[ \|\xi(t)\|_{\dot{H}^1 \cap \dot{H}^2} \leq \delta, \quad \|\xi(t)\|_{\dot{H}^3} \leq C_{\nu, \alpha_0} t^{-1}, \quad \forall t \in (0, t_0]. \tag{1.7} \]

**Furthermore, as** \( t \to 0 \), \( \xi(t) \to \xi^* \) in \( \dot{H}^1 \cap \dot{H}^2 \) with \( \xi^* \in H^{1+2s} \).

**Remark 1.2.** In fact, using the arguments developed in this paper one can show that the same result remains valid with \( \dot{H}^3 \) replaced by \( \dot{H}^{1+2s} \) for any \( 1 \leq s < \nu \).

1.2. **Strategy of the proof.** The proof of Theorem 1.1 contains two main steps. The first step is a construction of approximate solutions \( u^{(N)} \) that have the form (1.5), (1.6), (1.7), and solve (1.1) up to an arbitrarily high order error \( O(t^N) \), very much in the spirit of the work of Kriger et al. [13].

The second step is to build the exact solution by solving the problem for the small remainder forward in time with zero initial data at \( t = 0 \). The control of remainder is achieved by means of suitable energy type estimates, see Sect. 3 for the details. The assumption \( \nu > 1 \) ensures that the approximate solutions that we have constructed, belong to \( \dot{H}^1 \cap \dot{H}^3 \), which allows us to work on the level of the \( H^3 \) well-posedness theory.

2. **Approximate Solutions**

2.1. **Preliminaries.** We consider (1.1) under the 1-equivariance assumption

\[ u(x, t) = e^{\theta R} v(r, t), \quad v = (v_1, v_2, v_3) \in S^2 \subset \mathbb{R}^3. \tag{2.1} \]

Restricted to the 1-equivariant functions (1.1) takes the form

\[ v_t = v \times (\nabla v + \frac{R^2}{r^2} v), \tag{2.2} \]

the energy being given by

\[ E(u) = \pi \int_0^\infty dr r (|v_r|^2 + \frac{v_1^2 + v_2^2}{r^2}). \]
\[ Q = (h_1, 0, h_3) \] is a stationary solution of (2.2) and one has the relations
\[
\begin{align*}
\partial_r h_1 &= -\frac{h_1 h_3}{r}, & \partial_r h_3 &= \frac{h_1^2}{r}, \\
\Delta Q + \frac{R^2}{r^2} Q &= \kappa(r) Q, & \kappa(r) &= -\frac{2h_1^2}{r^2}. 
\end{align*}
\] (2.3)

(2.4)

The goal of the present section is to prove the following result.

**Proposition 2.1.** For any \( \delta > 0 \) sufficiently small and any \( N \) sufficiently large there exists an approximate solution \( u^{(N)} : \mathbb{R}^2 \times \mathbb{R}_+^* \rightarrow S^2 \) of (1.1) such that the following holds.

(i) \( u^{(N)} \) is a \( C^\infty \) \( 1 \)-equivariant profile of the form: \( u^{(N)} = e^{\alpha(t)R}(\phi(\lambda(t)x) + \chi^{(N)}(\lambda(t)x, t)) \), where \( \chi^{(N)}(y, t) = e^{\phi R}Z^{(N)}(\rho, t), \rho = |y| \), verifies
\[
\begin{align*}
\|\partial_\rho Z^{(N)}(t)\|_{L^2(\rho \rho p)} &+ \|\rho^{-1}Z^{(N)}(t)\|_{L^2(\rho \rho p)} + \|\rho \partial_\rho Z^{(N)}(t)\|_\infty \leq C\delta^{2\nu}, \\
\|\rho^{-1}\partial_\rho^k Z^{(N)}(t)\|_{L^2(\rho \rho p)} &\leq C\delta^{2\nu-1}t^{1/2+\nu}, \quad k + l = 2, \\
\|\rho^{-1}\partial_\rho^k Z^{(N)}(t)\|_{L^2(\rho \rho p)} &\leq Ct^{2\nu}, \quad k + l = 3, \\
\|\partial_\rho Z^{(N)}(t)\|_\infty, \|\rho^{-1}Z^{(N)}(t)\|_\infty &\leq C\delta^{2\nu-1}t^{\nu}, \\
\|\rho^{-1}\partial_\rho^k Z^{(N)}(t)\|_\infty &\leq Ct^{2\nu}, \quad 2 \leq l + k \leq 3, 
\end{align*}
\] (2.5)

(2.6)

(2.7)

(2.8)

(2.9)

for any \( 0 < t \leq T(N, \delta) \) with some \( T(N, \delta) > 0 \). The constants \( C \) here and below are independent of \( N \) and \( \delta \).

In addition, one has
\[
\|\chi^{(N)}(t)\|_{W^{4,\infty}} + \|\chi^{(N)}(t)\|_{W^{5,\infty}} \leq Ct^{2\nu},
\] (2.10)

and \( (x)^{2(\nu-1)}\nabla^4 u^{(N)}(t), (x)^{2(\nu-1)}\nabla^2 u^{(N)}(t) \in L^\infty(\mathbb{R}^2) \).

Furthermore, there exists \( \zeta^*_N \in \hat{H}^1 \cap \hat{H}^{1+2\nu} \) such that as \( t \to 0 \),
\[
e^{\alpha(t)R} \chi^{(N)}(\lambda(t)t) \to \zeta^*_N \text{ in } \hat{H}^1 \cap \hat{H}^2.
\]

(ii) The corresponding error \( r^{(N)} = -u^{(N)} + u^{(N)} \times \Delta u^{(N)} \) verifies
\[
\|r^{(N)}(t)\|_{H^3} + \|\partial_t r^{(N)}(t)\|_{H^1} + \|\chi r^{(N)}(t)\|_{L^2} \leq t^N, \quad 0 < t \leq T(\delta, N).
\] (2.11)

**Remarks.** 1. Note that estimates (2.5), (2.6) imply:
\[
||u^{(N)}(t) - e^{\alpha(t)R} \phi(\lambda(t)t)||_{\hat{H}^{1+2\nu} \cap \hat{H}^2} \leq \delta^{2\nu-1}, \quad \forall t \in (0, T(\delta, N)].
\] (2.12)

2. It follows from our construction that \( \chi^{(N)}(t) \in \hat{H}^{1+2\nu} \) for any \( s < \nu \) with the estimate
\[
||\chi^{(N)}(t)||_{\hat{H}^{1+2\nu}(\mathbb{R}^2)} \leq C(t^{2\nu} + t^{(1+2\nu)}\delta^{2\nu-2\nu}).
\]

3. The remainder \( r^{(N)} \) verifies in fact, for any \( m, l, k, \)
\[
||\chi^{(N)}(t)\|_{H^k} \leq C_{l,m,k} t^{N-C_{l,m,k}},
\]
provided \( N \geq C_{l,m,k} \).
We will give the proof of Proposition 2.1 in the case of \( \nu \) irrational only, which allows us to slightly simplify the presentation. The extension to \( \nu \) rational is straightforward.

To construct an arbitrarily good approximate solution we analyze separately the three regions that correspond to three different space scales: the inner region with the scale \( r < 1 \), the self-similar region where \( r = O(t^{1/2}) \), and finally the remote region where \( r = O(1) \). The inner region is the region where the blowup concentrates. In this region the solution will be constructed as a perturbation of the profile \( e^{\alpha(t)R} Q(\lambda(t)r) \).

The self-similar and remote regions are the regions where the solution is close to \( k \) and is described essentially by the corresponding linearized equation. In the self-similar region the profile of the solution will be determined uniquely by the matching conditions coming out of the inner region, while in the remote region the profile remains essentially a free parameter of the construction, only the limiting behavior at the origin is prescribed by the matching process, see Sects. 2.3 and 2.4 for the details, see also [1, 2] for some closely related considerations in the context of the critical harmonic map heat flow.

2.2. Inner region \( r \lambda(t) \lesssim 1 \). We start by considering the inner region \( 0 \leq r \lambda(t) \leq 10r^{-\nu+\varepsilon_1} \), where \( 0 < \varepsilon_1 < \nu \) to be fixed later. Writing \( v(r, t) \) as

\[
v(r, t) = e^{\alpha(t)R} V(\lambda(t)r, t), \quad V = (V_1, V_2, V_3),
\]

we get from (2.2)

\[
t^{1+2\nu} V_t + \alpha_0 t^{2\nu} RV - t^{2\nu}(v + \frac{1}{2})\rho V_\rho = V \times (\Delta V + \frac{R^2}{\rho^2} V), \quad \rho = \lambda(t) r. \tag{2.13}
\]

We look for a solution of (2.13) as a perturbation of the harmonic map profile \( Q(\rho) \). Write

\[
V = Q + Z,
\]

and further decompose \( Z \) as

\[
Z(\rho, t) = z_1(\rho, t) f_1(\rho) + z_2(\rho, t) f_2(\rho) + \gamma(\rho, t) Q(\rho),
\]

where \( f_1, f_2 \) is the orthonormal frame on \( T_Q S^2 \) given by

\[
f_1(\rho) = \begin{pmatrix} h_3(\rho) \\ 0 \\ -h_1(\rho) \end{pmatrix}, \quad f_2(\rho) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

One has

\[
\gamma = \sqrt{1 - |z|^2} - 1 = O(|z|^2), \quad z = z_1 + iz_2.
\]

Note also the relations

\[
\partial_\rho Q = -\frac{h_1}{\rho} f_1, \quad \partial_\rho f_1 = \frac{h_1}{\rho} Q, \quad f_2 = Q \times f_1,
\]

\[
\Delta f_1 + \frac{R^2}{\rho^2} f_1 = -\frac{1}{\rho^2} f_1 - \frac{2h_3 h_1}{\rho^2} Q.
\]
We now rewrite (2.13) in terms of $z$. One has
\[ RV = -h_3 z_2 f_1 + (h_3 z_1 + h_1 (1 + \gamma)) f_2 - h_1 z_2 Q, \]
\[ \rho \partial_\rho V = (\rho \partial_\rho z_1 - h_1 (1 + \gamma)) f_1 + \rho \partial_\rho z_2 f_2 + (h_1 z_1 + \rho \partial_\rho \gamma) Q. \]  
(2.14)

We next compute the nonlinear term $V \times (\Delta V + \frac{R^2}{\rho^2} V)$. In the basis $\{f_1, f_2, Q\}$, the expression $\Delta V + \frac{R^2}{\rho^2} V$ can be written as follows:
\[ \Delta V + \frac{R^2}{\rho^2} V = [\Delta z_1 - \frac{z_1}{\rho^2} - 2 \frac{h_1}{\rho} \gamma] f_1 + [\Delta z_2 - \frac{z_2}{\rho^2}] f_2 + [\Delta_V + \kappa(\rho)(1 + \gamma) + 2 \frac{h_1}{\rho} \partial_\rho z_1 - 2 \frac{h_1 h_3 z_1}{\rho^2}] Q, \]
which gives
\[ V \times (\Delta V + \frac{R^2}{\rho^2} V) = [(1 + \gamma) L z_2 + F_1(z)] f_1 - [(1 + \gamma) L z_1 + F_2(z)] f_2 + F_3(z) Q, \]  
(2.15)

where
\[ L = -\Delta + \frac{1 - 2 h_1^2}{\rho^2}, \]
\[ F_1(z) = z_2 (\Delta_V + \gamma) - 2 \frac{h_1 h_3}{\rho^2} z_1 + 2 \frac{h_1}{\rho} \partial_\rho z_1, \]
\[ F_2(z) = z_1 (\Delta_V + \gamma) - 2 \frac{h_1 h_3}{\rho^2} z_1 + 2 \frac{h_1}{\rho} \partial_\rho z_1 + \frac{2 h_1}{\rho} (1 + \gamma) \gamma, \]
\[ F_3(z) = z_1 \Delta z_2 - z_2 \Delta z_1 + \frac{2 h_1}{\rho} \Delta z_2 \gamma. \]  
(2.16)

Projecting (2.13) onto span$\{f_1, f_2\}$ and taking into account (2.14), (2.15), (2.16), we get the following reformulation of (2.13):
\[ i t^{1+2v} z_t - \alpha_0 t^{2v} h_3 z - i (\frac{1}{2} + v) t^{2v} \rho z_\rho = L z + F(z) + d t^{2v} h_1, \]
\[ d = \alpha_0 - i (\frac{1}{2} + v), \]  
(2.17)
\[ F(z) = \gamma L z + z (\Delta_V + \gamma) - 2 \frac{h_1 h_3 z_1}{\rho^2} \partial_\rho z_1 + \frac{2 h_1}{\rho} (1 + \gamma) \gamma + d t^{2v} \gamma h_1. \]

Note that $F$ is at least quadratic in $z$.

We look for a solution of (2.17) as a power expansion in $t^{2v}$:
\[ z(\rho, t) = \sum_{k \geq 1} t^{2vk} z^k(\rho). \]  
(2.18)

Substituting (2.18) into (2.17) we get the following recurrent system for $z^k, k \geq 1$:
\[ L z^1 = -d h_1, \]
\[ L z^k = F_k, \quad k \geq 2, \]  
(2.19) (2.20)
where $\mathcal{F}_k$ depends on $z^j$, $j = 1, \ldots, k - 1$ only. We subject (2.19), (2.20) to zero initial conditions at $\rho = 0$:

$$z^k(0) = \partial_\rho z^k(0) = 0.$$  \hspace{1cm} (2.21)

**Lemma 2.2.** System (2.19), (2.20), (2.21) has a unique solution $(z^k)_{k \geq 1}$, with $z^k \in C^\infty(\mathbb{R}_+)$ for all $k \geq 1$. In addition, one has:

(i) $z^k$ has an odd Taylor expansion at 0 that starts at order $2k + 1$;

(ii) as $\rho \to \infty$, $z^k$ has the following asymptotic expansion

$$z^k(\rho) = \sum_{l=0}^{2k} \sum_{j \leq k-(l-1)/2} c^k_{j,l} \rho^{2j-l}(\ln \rho)^l,$$  \hspace{1cm} (2.22)

with some constants $c^k_{j,l}$. The asymptotic expansion (2.22) can be differentiated any number of times with respect to $\rho$.

**Proof.** First note that the equation $Lf = 0$ has two explicit solutions: $h_1(\rho)$ and $h_2(\rho) = \frac{\rho^4 + 4\rho^2 \ln \rho - 1}{\rho(\rho^2 + 1)}$.

Consider the case $k = 1$:

$$Lz^1 = -dh_1,$$

$$z^1(0) = \partial_\rho z^1(0) = 0.$$

One has

$$z^1(\rho) = -\frac{d}{4} \int_0^{\rho} dss(h_1(s)h_2(s) - h_1(s)h_2(\rho))h_1(s)$$

$$= -\frac{d\rho}{(1 + \rho^2)} \int_0^{\rho} ds \left(\frac{s^4 + 4s^2 \ln s - 1}{(1 + s^2)^2}\right) + \frac{d(\rho^4 + 4\rho^2 \ln \rho - 1)}{\rho(\rho^2 + 1)} \int_0^{\rho} ds \frac{s^3}{(1 + s^2)^2}.$$  \hspace{1cm} (2.23)

Since $h_1$ is a $C^\infty$ function that has an odd Taylor expansion at $\rho = 0$ with a linear leading term, one can easily write an odd Taylor series for $z^1$ with a cubic leading term, which proves (i) for $k = 1$.

The asymptotic behavior of $z^1$ at infinity can be obtained directly from the representation (2.23). As claimed, one has

$$z^1(\rho) = c^1_{1,0} \rho + c^1_{1,1} \rho \ln \rho + \sum_{j \leq 0} \sum_{l=0,1,2} c^1_{j,l} \rho^{2j-l}(\ln \rho)^l,$$

with $c^1_{1,0} = -c^1_{1,1} = -d$.

Consider $k > 1$. Assume that $z^j$, $j \leq k - 1$, verify (i) and (ii). Then, using (2.17), one can easily check that $\mathcal{F}_k$ is an odd $C^\infty$ function vanishing at $\rho = 0$ at order $2k - 1$, with the following asymptotic expansion as $\rho \to \infty$:

$$\mathcal{F}_k = \sum_{j=1}^{k-1} \sum_{l=0}^{2k-2j-1} \alpha^k_{j,l} \rho^{2j-l}(\ln \rho)^l + \sum_{l=0}^{2k-2} \alpha^k_{0,l} \rho^{-1}(\ln \rho)^l$$

$$+ \sum_{l=0}^{2k-1} \alpha^k_{-1,l} \rho^{-3}(\ln \rho)^l + \sum_{j \leq -2} \sum_{l=0}^{2k} \alpha^k_{j,l} \rho^{2j-l}(\ln \rho)^l.$$
As a consequence, \( z^k(\rho) = \frac{1}{z} \int_0^\rho ds (h_1(s)h_2(s) - h_1(s)h_2(\rho))F_k(s) \) is a \( C^\infty \) function with an odd Taylor series at zero starting at order \( 2k + 1 \) and as \( \rho \to \infty \),

\[
z^k(\rho) = \sum_{l=0}^{2k} \sum_{j \leq k-(l-1)/2} c^{k}_{j,l}(\ln \rho)^l \rho^{2j-1},
\]
as required. This concludes the proof of Lemma 2.2. \( \Box \)

Returning to \( v \) we get a formal solution of (2.2) of the form

\[
v(r, t) = e^{\alpha(t)R}V(\lambda(t)r, t), \quad V(\rho, t) = Q + \sum_{k \geq 1} t^{2vk} z^k(\rho), \quad (2.24)
\]

\( Z^k = (Z^k_1, Z^k_2, Z^k_3) \), where \( Z^k_i, i = 1, 2 \), are smooth odd functions of \( \rho \) vanishing at 0 at order \( 2k + 1 \), and \( Z^k_3 \) is an even function vanishing at zero at order \( 2k + 2 \). As \( \rho \to \infty \), one has

\[
\begin{align*}
Z^k_1(\rho) &= \sum_{l=0}^{2k} \sum_{j \leq k-(l-1)/2} c^{k,1}_{j,l}(\ln \rho)^l \rho^{2j-1}, \quad i = 1, 2, \\
Z^k_3(\rho) &= \sum_{l=0}^{2k} \sum_{j \leq k+1-l/2} c^{k,3}_{j,l}(\ln \rho)^l \rho^{2j-2},
\end{align*}
\]

with some real coefficients \( c^{k,i}_{j,l} \) verifying

\[
c^{k,3}_{k+1,0} = 0, \quad \forall k \geq 1.
\]

The asymptotic expansions (2.25) can be differentiated any number of times with respect to \( \rho \).

Note that in the limit \( \rho \to \infty \), \( y \equiv rt^{-1/2} \to 0 \), expansion (2.24), (2.25), rewritten in terms of \( y \), give at least formally

\[
\begin{align*}
V_i(\lambda(t)r, t) &= \sum_{j \geq 0} t^{(2j+1)} \sum_{l=0}^{2j+1} (\ln y - v \ln t)^l V_i^{j,l}(y), \quad i = 1, 2, \\
V_3(\lambda(t)r, t) &= 1 + \sum_{j \geq 1} t^{2vj} \sum_{l=0}^{2j} (\ln y - v \ln t)^l V_3^{j,l}(y), \\
V_i^{j,l}(y) &= \sum_{k \geq -j+l/2} c^{k+j,i}_{k,l} y^{2k-1}, \quad i = 1, 2, \\
V_3^{j,l}(y) &= \sum_{k \geq -j+l/2} c^{k+j,3}_{k+1,l} y^{2k},
\end{align*}
\]

where the coefficients \( c^{k,i}_{j,l} \) with \( k \neq 0 \) are defined by (2.25) and \( c^{0,i}_{j,0} \) come from the expansion of \( Q \) as \( \rho \to \infty \):

\[
\begin{align*}
h_1(\rho) &= \sum_{j \geq 0} c^{0,1}_{j,0} \rho^{2j-1}, \quad h_3(\rho) = 1 + \sum_{j \geq 0} c^{0,3}_{j,0} \rho^{2j-2}, \quad c^{0,2}_{j,0} = 0.
\end{align*}
\]
The role of $\gamma$-expansion (2.26) will become clear in the next subsection where we will use it to perform the transition from the inner region to the self-similar region.

For $N \geq 2$ define

$$z_{\text{in}}^{(N)}(t) = \sum_{k=1}^{N} t^{2k} z_{\text{in}}^{(N)} z_{\text{in}}^{(N)} = z_{\text{in},1}^{(N)} + iz_{\text{in},2}^{(N)}.$$ 

Then $z_{\text{in}}^{(N)}$ solves (2.17) up to the error $X_N = -it^{1+2\nu} \partial_t z_{\text{in}}^{(N)} + \alpha_0 t^{2\nu} \partial_z z_{\text{in}}^{(N)} + i(\frac{1}{\nu} + \nu)^{2\nu} \rho \partial_\rho z_{\text{in}}^{(N)} + dt^{2\nu} h_1 + L z_{\text{in}}^{(N)} + F(z_{\text{in}}^{(N)}).$ Using the fact that $z^k$ are defined recursively it is not difficult to check that the error $X_N$ verifies

$$|\rho^{-l} \partial_\rho z_{\text{in}}^{(N)} X_N| \leq C_{k,l,m} t^{2\nu N-m} (\rho)^{2N-1-l-k} \ln(2+\rho),$$

for any $k, m \in \mathbb{N}$, $0 \leq l \leq (2N+1-k)_+$, $0 \leq \rho \leq 10t^{-v+1}$, $0 < t \leq T(N)$, with some $T(N) > 0$.

Set

$$y_{\text{in}}^{(N)} = \sqrt{1 - |z_{\text{in}}^{(N)}|^2} - 1,$$

$$Z_{\text{in}}^{(N)} = z_{\text{in},1}^{(N)} f_1 + z_{\text{in},2}^{(N)} f_2 + y_{\text{in}}^{(N)} Q,$$

$$V_{\text{in}}^{(N)} = Q + Z_{\text{in}}^{(N)} \in S^2.$$ 

Then $V_{\text{in}}^{(N)}$ solves

$$t^{1+2\nu} \partial_t V_{\text{in}}^{(N)} + \alpha_0 t^{2\nu} R V_{\text{in}}^{(N)} - t^{2\nu} (\nu + 1) \rho \partial_\rho V_{\text{in}}^{(N)} = V_{\text{in}}^{(N)} \times (\Delta \rho V_{\text{in}}^{(N)} + \frac{R^2}{\rho^2} V_{\text{in}}^{(N)}) + R_{\text{in}}^{(N)},$$

(2.28)

with $R_{\text{in}}^{(N)} = \text{im} X_N f_1 - \text{re} X_N f_2 + \text{im} (\langle \bar{X}_N z_{\text{in}}^{(N)} \rangle) Q$ admitting the same estimate as $X_N$.

Note also that it follows from our analysis that for $0 \leq \rho \leq 10t^{-v+1}$, $0 < t \leq T(N)$,

$$|\rho^{-l} \partial_\rho Z_{\text{in}}^{(N)}| \leq C_{k,l} t^{2\nu} (\rho)^{1-l-k} \ln(2+\rho), \quad k \in \mathbb{N}, \quad l \leq (3-k)_+.$$ 

(2.29)

As a consequence, we obtain the following result.

**Lemma 2.3.** There exists $T(N) > 0$ such that for any $0 < t \leq T(N)$ the following holds.

(i) The profile $Z_{\text{in}}^{(N)}(\rho, t)$ verifies

$$\|\partial_\rho Z_{\text{in}}^{(N)}(t)\|_{L^2(\rho \partial_\rho, 0 \leq \rho \leq 10t^{-v+1})} \leq C t^{\nu},$$

(2.30)

$$\|\rho^{-l} Z_{\text{in}}^{(N)}(t)\|_{L^2(\rho \partial_\rho, 0 \leq \rho \leq 10t^{-v+1})} \leq C t^{\nu},$$

(2.31)

$$\|Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0 \leq \rho \leq 10t^{-v+1})} + \|\rho \partial_\rho Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0 \leq \rho \leq 10t^{-v+1})} \leq C t^{\nu},$$

(2.32)

$$\|\rho^{-l} k Z_{\text{in}}^{(N)}(t)\|_{L^2(\rho \partial_\rho, 0 \leq \rho \leq 10t^{-v+1})} \leq C t^{2\nu} (1 + |\ln t|), \quad k + l = 2,$$

(2.33)

$$\|\rho^{-l} k Z_{\text{in}}^{(N)}(t)\|_{L^2(\rho \partial_\rho, 0 \leq \rho \leq 10t^{-v+1})} \leq C t^{2\nu}, \quad k + l \geq 3, \quad l \leq (3-k)_+, \quad k \leq 2,$$

(2.34)

$$\|\partial_\rho Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0 \leq \rho \leq 10t^{-v+1})} + \|\rho^{-l} Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0 \leq \rho \leq 10t^{-v+1})} \leq C t^{2\nu}(1 + |\ln t|),$$

(2.35)

$$\|\rho^{-l} k Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0 \leq \rho \leq 10t^{-v+1})} \leq C t^{2\nu}, \quad 2 \leq l + k, \quad l \leq (3-k)_+, \quad k \leq 2.$$ 

(2.36)
ii) The error $\mathcal{R}^{(N)}_{in}$ admits the estimates
\[
\|\rho^{-l} \partial^k \mathcal{R}^{(N)}_{in}(t)\|_{L^2(\rho d\rho, 0 \leq \rho \leq 10t^{-\nu+\epsilon_1})} \leq t^{N\epsilon_1}, \quad 0 \leq l + k \leq 3,
\]
\[
\|\rho^{-l} \partial^{k} \partial_t \mathcal{R}^{(N)}_{in}(t)\|_{L^2(\rho d\rho, 0 \leq \rho \leq 10t^{-\nu+\epsilon_1})} \leq t^{N\epsilon_1}, \quad 0 \leq k + l \leq 1,
\]
provided $N > \epsilon_1^{-1}$.

2.3. Self-similar region $rt^{-1/2} \lesssim 1$. We next consider the self-similar region $\frac{1}{10} t^{\epsilon_1} \leq rt^{-1/2} \leq 10t^{-\epsilon_2}$, where $0 < \epsilon_2 < 1/2$ to be fixed later. In this region we expect the solution to be close to $k$. In this regime it will be convenient to use the stereographic representation of (2.2):
\[(v_1, v_2, v_3) = v \to w = \frac{v_1 + i v_2}{1 + v_3} \in \mathbb{C} \cup \{\infty\}.
\]
Equation (2.2) is equivalent to
\[iw_t = -\Delta w + r^{-2} w + G(w, \bar{w}, w_r), \quad G(w, \bar{w}, w_r) = \frac{2\bar{w}}{1 + |w|^2} (w_r^2 - r^{-2} w^2).
\]
Slightly more generally, if $w(r, t)$ is a solution of
\[iw_t = -\Delta w + r^{-2} w + G(w, \bar{w}, w_r) + A,
\]
then $v = \left( \frac{2 \text{Re} w}{1 + |w|^2}, \frac{2 \text{Im} w}{1 + |w|^2}, \frac{1 - |w|^2}{1 + |w|^2} \right) \in S^2$ solves
\[v_t = v \times (\Delta v + \frac{R^2}{r^2} v) + A,
\]
with $A = (A_1, A_2, A_3)$ given by
\[A_1 + i A_2 = -2i \frac{A + w^2 \bar{A}}{(1 + |w|^2)^2}, \quad A_3 = \frac{4 \text{Im}(w \bar{A})}{(1 + |w|^2)^2}.
\]
Consider (2.38). Write $w$ as
\[w(r, t) = e^{i\alpha(t)} W(y, t), \quad y = rt^{-1/2}.
\]
Then (2.38) becomes
\[it W_t - \alpha_0 W = \mathcal{L} W + G(W, \bar{W}, W_y),
\]
where
\[\mathcal{L} = -\Delta + y^{-2} + i \frac{1}{2} v \partial_y.
\]
Note that as $y \to 0$, (2.26) gives the following expansion:
\[W(y, t) = \sum_{j \geq 0} \sum_{l=0}^{2j+1} \sum_{i \geq -j+l/2} \alpha(j, i, l) t^{(2j+1)} (\ln y - v \ln t)^l y^{2i-1},
\]
where the coefficients $\alpha(j, i, l)$ can be expressed explicitly in terms of $e_{j, l}', 1 \leq k \leq j+i, j' \leq i, 0 \leq l' \leq l$. This suggests the following ansatz for $W$:

$$W(y, t) = \sum_{j \geq 0} \sum_{l=0}^{2j+1} t^{(2j+1)}(\ln y - y \ln t)^l W_{j,l}(y). \quad (2.43)$$

Substituting (2.43) into (2.41) one gets the following recurrent system for $W_{j,l}, 0 \leq l \leq 2j+1$, $j \geq 0$:

$$\begin{cases}
(\mathcal{L} - \mu_0) W_{0,0} = 0, \\
(\mathcal{L} - \mu_0) W_{0,l} = -i(1/2 + \nu) W_{0,1} + 2y^{-1}\partial_y W_{0,1}, \\
(\mathcal{L} - \mu_j) W_{j,2j+1} = G_{j,2j+1}, \\
(\mathcal{L} - \mu_j) W_{j,2j} = G_{j,2j} - i(2j+1)(1/2 + \nu) W_{j,2j+1} + 2(2j+1)y^{-1}\partial_y W_{j,2j+1}, \\
(\mathcal{L} - \mu_j) W_{j,l} = G_{j,l} - i(l+1)(1/2 + \nu) W_{j,l+1} + 2(l+1)y^{-1}\partial_y W_{j,l+1} + (l+1)(l+2)y^{-2} W_{j,l+2}, \quad 0 \leq l \leq 2j-1.
\end{cases} \quad (2.44)$$

Here $\mu_j = -\alpha_0 + iv(2j+1)$, and $G_{j,l}$ is the contribution of the nonlinear term $G(W, \bar{W}, W_\nu)$, that depends only on $W_{i,n}, i \leq j-1$:

$$G(W, \bar{W}, W_\nu) = -\sum_{j \geq 1} \sum_{l=0}^{2j+1} t^{(2j+1)}(\ln y - y \ln t)^l G_{j,l}(y),$$

$$G_{j,l}(y) = G_{j,l}(y; W_{i,n}, 0 \leq n \leq 2i + 1, 0 \leq i \leq j-1).$$

One has

**Lemma 2.4.** Given coefficients $a_j, b_j, j \geq 0$, there exists a unique solution of (2.44), (2.45), $W_{j,l} \in C^\infty(\mathbb{R}^*_+), 0 \leq l \leq 2j+1, j \geq 0$, such that as $y \to 0$, $W_{j,l}$ has the following asymptotic expansion

$$W_{j,l}(y) = \sum_{i \geq -j+l/2} d^{j,l}_{i} y^{2i-1}, \quad (2.46)$$

with

$$d^{j,1}_{1} = a_j, \quad d^{j,0}_{1} = b_j. \quad (2.47)$$

The asymptotic expansion (2.46) can be differentiated any number of times with respect to $y$.

**Proof.** First note that equation $(\mathcal{L} - \mu_j)f = 0$ has a basis of solutions $\{e^1_j, e^2_j\}$ such that

(i) $e^1_j$ is a $C^\infty$ odd function, $e^1_j(y) = y + O(y^3)$ as $y \to 0$;

(ii) $e^2_j \in C^\infty(\mathbb{R}_+)$ and admits the representation:

$$e^2_j(y) = y^{-1} + \kappa_j e^1_j(y) \ln y + \hat{e}^2_j(y), \quad \kappa_j = -\frac{i}{4} - \frac{\mu_j}{2},$$

where $\hat{e}^2_j$ is a $C^\infty$ odd function, $\hat{e}^2_j(y) = O(y^3)$ as $y \to 0$. 


Consider (2.44). From \((L - \mu_0)W_{0,1} = 0\) and (2.46), (2.47), we get
\[
W_{0,1} = a_0 e_0^1.
\]
Consider the equation for \(W_{0,0}\):
\[
(L - \mu_0)W_{0,0} = -i(1/2 + \nu)W_{0,1} + 2y^{-1}\partial_y W_{0,1}.
\]
The right hand side has the form: \(2a_0 y^{-1} + a C^\infty\) odd function. Therefore, the equation has a unique solution \(W_{0,0}\) of the form
\[
W_{0,0}(y) = d_0 y^{-1} + \tilde{W}_{0,0}(y),
\]
where \(d_0 = \frac{a_0}{k_j}\) and \(\tilde{W}_{0,0}\) is a \(C^\infty\) odd function, \(\tilde{W}_{0,0}(y) = O(y^3)\) as \(y \to 0\). Together with (2.46), (2.47), this gives:
\[
W_{0,0} = W_{0,0}^0 + b_0 e_0^1.
\]
Consider the case \(j \geq 1\). We have
\[
(L - \mu_j)W_{j,l} = F_{j,l}, \quad 0 \leq l \leq 2j + 1,
\]
where
\[
F_{j,2j+1} = G_{j,2j+1},
\]
\[
F_{j,2j} = G_{j,2j} - i(2j + 1)(1/2 + \nu)W_{j,2j+1} + 2(2j + 1)y^{-1}\partial_y W_{j,2j+1},
\]
\[
F_{j,l} = G_{j,l} - i(l + 1)(1/2 + \nu)W_{j,l+1} + 2(l + 1)y^{-1}\partial_y W_{j,l+1} + (l + 1)(l + 2)y^{-2}W_{j,l+2}, \quad 0 \leq l \leq 2j - 1.
\]
The resolution of (2.48) is based on the following ODE lemma whose proof is left to the reader.

**Lemma 2.5.** Let \(F\) be a \(C^\infty(\mathbb{R}^*_+\) function of the form
\[
F(y) = \sum_{j=k}^{0} F_j y^{2j-1} + \tilde{F}(y),
\]
where \(\tilde{F}\) is a \(C^\infty\) odd function and \(k \leq -1\). Then there exists a unique constant \(A\) such that the equation \((L - \mu_j)u = F + Ay^{-3}\) has a solution \(u \in C^\infty(\mathbb{R}^*_+\) with the following behavior as \(y \to 0\):
\[
u_j y^{2j-1}, \quad u_1 = 0.
\]
More precisely, we proceed as follows. Assume that $W_{j,n}, 0 \leq n \leq 2i + 1, i \leq j - 1$ has the prescribed behavior (2.46), (2.47). Then it is not difficult to check that $G_{j,l}$ admit the following expansion as $y \to 0$:

$$G_{j,2j+1}(y) = \sum_{i \geq 1} g_{j,2j+1}^i y^{2i-1},$$

$$G_{j,2j}(y) = \sum_{i \geq 0} g_{j,2j}^i y^{2i-1},$$

$$G_{j,l}(y) = \sum_{i \geq -l+2-1} g_{j,l}^i y^{2i-1}, \quad l \leq 2j - 1. \tag{2.50}$$

Consider $W_{j,2j+1}$. From $(\mathcal{L} - \mu_j) W_{j,2j+1} = G_{j,2j+1}$ we get

$$W_{j,2j+1} = W_{j,2j+1}^0 + c_0 e_j^1, \tag{2.51}$$

where $W_{j,2j+1}^0$ is a unique $C^\infty$ odd solution of $(\mathcal{L} - \mu_j) f = G_{j,2j+1}$ that satisfies $W_{j,2j+1}^0(y) = O(y^3)$ as $y \to 0$. The constant $c_0$ remains undetermined at this stage.

Consider $F_{j,2j}$. It has the form: $(g_{j,2j}^0 + 2(2j + 1)c_0)y^{-1} + a C^\infty$ odd function. Therefore, for $W_{j,2j}$ we obtain

$$W_{j,2j} = W_{j,2j}^0 + c_1 e_j^1, \tag{2.52}$$

where $W_{j,2j}^0$ is a unique solution of $(\mathcal{L} - \mu_j) f = F_{j,2j}$, that satisfies as $y \to 0$,

$$W_{j,2j}^0 = d_1 y^{-1} + O(y^3), \quad d_1 = \frac{g_{j,2j}^0 + 2(2j + 1)c_0}{2k_j}. \tag{2.53}$$

Similarly to $c_0$, the constant $c_1$ is arbitrary here.

Consider $F_{j,2j-1}$. It follows from (2.49), (2.50), (2.51), (2.52), (2.53) that

$$F_{j,2j-1} = (g_{j,2j-1}^{-1} - 4jd_1)y^{-3} + \text{const} \ y^{-1} + \text{an \ } C^\infty \ \text{odd function.}$$

The equation $(\mathcal{L} - \mu_j) W_{j,2j-1} = F_{j,2j-1}$ has a solution of form (2.46) iff

$$g_{j,2j-1}^{-1} - 4jd_1 = 0,$$

which gives

$$c_0 = \frac{k_j g_{j,2j-1} - 2jg_{j,2j}^0}{4j(2j + 1)}.$$

With this choice of $c_0$ one gets

$$W_{j,2j-1} = W_{j,2j-1}^0 + c_2 e_j^1,$$

where $W_{j,2j-1}^0$ is a unique solution of $(\mathcal{L} - \mu_j) f = F_{j,2j-1}$, that satisfies as $y \to 0$,

$$W_{j,2j-1}^0 = \text{const} \ y^{-1} + O(y^3).$$
Continuing the procedure one successively finds $W_{j,2j-2}, \ldots, W_{j,0}$ in the form $W_{j,2j+1-k} = W_{j,2j+1-k}^0 + c_k e_{j}^1, k \leq 2j + 1$, where $W_{j,2j+1-k}^0$ is an unique solution of $(\mathcal{L} - \mu_j)f = F_{j,2j+1-k}$, that as $y \to 0$ has an asymptotic expansion of the form (2.46) with vanishing coefficients $d_1^{l,l}$. The constant $c_k, k \leq 2j - 1$, is determined uniquely by the solvability condition of the equation for $W_{j,2j-k-1}$ (see lemma 2.5). Finally, $c_{2j+1}, c_{2j+2}$ are given by (2.47):

$$c_{2j+1} = a_j, \quad c_{2j+2} = b_j.$$  

\[ \square \]

We denote by $W^{ss}_{j,l}(y), 0 \leq l \leq 2j + 1, j \geq 0$, the solution of (2.44), (2.45) given by Lemma 2.4 with $a_j = \alpha(j, 1, 1), b_j = \alpha(j, 1, 0)$, see (2.42). Since expansion (2.42) is a solution of (2.41), the uniqueness part of Lemma 2.4 ensures that

$$W^{ss}_{j,l}(y) = \sum_{i \geq -j+1/2} \alpha(j, i, l)y^{2i-1}, \quad \text{as } y \to 0. \quad (2.54)$$

We next study the behavior of $W^{ss}_{j,l}, 0 \leq l \leq 2j + 1, j \geq 0$, at infinity. One has

**Lemma 2.6.** Given coefficients $a_{j,i}, b_{j,i}, 0 \leq l \leq 2j + 1, j \geq 0$, there exists a unique solution of (2.44), (2.45) of the following form.

$$W_{0,l} = W_{0,l}^0 + W_{0,l}^1, \quad l = 0, 1, \quad (2.55)$$

$$W_{j,l} = W_{j,l}^0 + W_{j,l}^1 + W_{j,l}^2, \quad 0 \leq l \leq 2j + 1, \quad j \geq 1, \quad (2.56)$$

where $(W_{j,l}^i)_{0 \leq i \leq 2j+1, j \geq 1}$ are two solutions of (2.44), (2.45) that, as $y \to \infty$, have the following asymptotic expansion

$$\sum_{l=0}^{2j+1} (\ln y - v \ln t)^l W^{i}_{j,l}(y) = \sum_{l=0}^{2j+1} (\ln y + (-1)^l \ln t/2)^l \hat{W}^{i}_{j,l}(y), \quad i = 0, 1,$$

$$\hat{W}^{0}_{j,l}(y) = y^{2i\alpha_0+2v(2j+1)} \sum_{k \geq 0} \hat{w}^{i,l,0}_k y^{-2k}, \quad (2.57)$$

$$\hat{W}^{1}_{j,l}(y) = e^{iy^{2/4}} y^{-2i\alpha_0-2-2v(2j+1)} \sum_{k \geq 0} \hat{w}^{i,l,-1}_k y^{-2k},$$

with

$$\hat{w}^{i,l,0}_0 = a_{j,l}, \quad \hat{w}^{i,l,-1}_0 = b_{j,l}. \quad (2.58)$$

Finally, the interaction part $W^{2}_{j,l}(y)$ can be written as

$$W^{2}_{j,l}(y) = \sum_{-j-1 \leq m \leq j} e^{-imy^{2/4}} y^{2i\alpha_0(2m+1)} W_{j,l,m}(y), \quad (2.59)$$
where $W_{j,l,m}$ have the following asymptotic expansion as $y \to \infty$:

$$
W_{j,l,m}(y) = \sum_{k \geq m+2} \sum_{m-j \leq i \leq m-1} \sum_{s=0}^{2j+1-l} w_{k,i,s}^{j,l,m} y^{2v(2j+1)-2k} (\ln y)^s, \quad m \geq 1,
$$

$$
W_{j,l,m}(y) = \sum_{k \geq -m} \sum_{-m-2 \leq i \leq j+m} \sum_{s=0}^{2j+1-l} w_{k,i,s}^{j,l,m} y^{2v(2j+1)-2k} (\ln y)^s, \quad m \leq -2,
$$

$$
W_{j,l,0}(y) = \sum_{k \geq 1} \sum_{-j \leq i \leq j-2} \sum_{s=0}^{2j+1-l} w_{k,i,s}^{j,l,0} y^{2v(2j+1)-2k} (\ln y)^s,
$$

$$
W_{j,l,-1}(y) = \sum_{k \geq 1} \sum_{-j-1 \leq i \leq j-1} \sum_{s=0}^{2j+1-l} w_{k,i,s}^{j,l,-1} y^{2v(2j+1)-2k} (\ln y)^s.
$$

The asymptotic expansion (2.57), (2.60) can be differentiated any number of times with respect to $y$.

Any solution of (2.44), (2.45) has form (2.55), (2.56), (2.57), (2.59), (2.60).

**Proof.** First note that equation $(\mathcal{L} - \mu_j)f = 0$ has a basis of solutions $\{f^1_j, f^2_j\}$ with the following behavior at infinity:

$$
f^1_j(y) = y^{2i\alpha_0+2v(2j+1)} \sum_{k \geq 0} f^k_{j,1} y^{-2k}, \quad f^2_j(y) = e^{iy^2/4} y^{-2i\alpha_0-2v(2j+1)-2} \sum_{k \geq 0} f^k_{j,2} y^{-2k},
$$

$f^0_{j,1} = f^0_{j,2} = 1$. As a consequence, the homogeneous system

$$(\mathcal{L} - \mu_j)g_{2j+1} = 0,$$

$$(\mathcal{L} - \mu_j)g_{2j} = -i(2j + 1)(1/2 + \nu)g_{2j+1} + 2(2j + 1)y^{-1} \partial_y g_{2j+1},$$

$$(\mathcal{L} - \mu_j)g_l = -i(l + 1)(1/2 + \nu)g_{l+1} + 2(l + 1)y^{-1} \partial_y g_{l+1} + (l + 1)(l + 2)y^{-2} g_{l+2}, \quad 0 \leq l \leq 2j - 1.\tag{2.61}
$$

has a basis of solutions $\{g_{i,m}^{j} \}_{i=1,2; m=0,\ldots,2j+1}$,

$$
g_{i,m}^{j} = (g_{j,0}^{i,m}, \ldots, g_{j,2j+1}^{i,m}), \quad 0 \leq m \leq 2j+1, \quad i = 1, 2,
$$

defined by

$$
\sum_{l=0}^{2j+1} (\ln y - \nu \ln t)^l g_{i,m}^{j}(y) = \sum_{l=0}^{2j+1} (\ln y + (-1)^{i-1} \ln t/2)^l \tilde{g}_{j,l}^{i,m}(y),\tag{2.62}
$$

where $(\tilde{g}_{j,l}^{i,m})_{l=0,\ldots,2j+1}$ is the unique solution of

$$(\mathcal{L} - \mu_j)\tilde{g}_{2j+1} = 0,$$

$$(\mathcal{L} - \mu_j)\tilde{g}_{2j} = -i(2j + 1)(i - 1)\tilde{g}_{2j+1} + 2(2j + 1)y^{-1} \partial_y \tilde{g}_{2j+1},$$

$$(\mathcal{L} - \mu_j)\tilde{g}_l = -i(l + 1)(i - 1)\tilde{g}_{l+1} + 2(l + 1)y^{-1} \partial_y \tilde{g}_{l+1} + (l + 1)(l + 2)y^{-2} \tilde{g}_{l+2}, \quad 0 \leq l \leq 2j - 1.\tag{2.63}
$$
verifying
\[ \xi_{j,m}^{i,m}(y) = 0, \quad l > 2j + 1 - m, \]
\[ \xi_{j,2j+1-m}^{i,m}(y) = f_j^i(y), \]
\[ \xi_{j,l}^{1,m}(y) = y^{2\alpha_0 + 2v(2j+1)} \sum_{k \geq 2j+1-l-m} \xi_{l,k}^1 y^{-2k}, \quad y \to +\infty, \tag{2.64} \]
\[ \xi_{j,l}^{2,m}(y) = e^{iy^2/4} y^{-2\alpha_0 - 2v(2j+1) - 2} \sum_{k \geq 2j+1-l-m} \xi_{l,k}^2 y^{-2k} \quad y \to +\infty. \]

Consider \( W_{0,l}, l = 0, 1 \). We have

\[ (\mathcal{L} - \mu_0) W_{0,1} = 0, \]
\[ (\mathcal{L} - \mu_0) W_{0,0} = -i (1/2 + v) W_{0,1} + 2y^{-1} \partial_y W_{0,1}, \]

which gives

\[ W_{0,l}(y) = \sum_{i=1,2, m=0,1} A_{i,m} g_{0,l}^{i,m}(y), \quad l = 0, 1, \]

with some constants \( A_{i,m}, i = 1, 2, m = 0, 1 \). It follows from (2.62), (2.64) that \( W_{0,l}, l = 0, 1 \) have the form (2.55), (2.57) with \( \hat{\omega}_0^{j,l,0} = A_{1,1-l}, \hat{\omega}_0^{j,l,-1} = A_{2,1-l}, l = 0, 1 \), which together with (2.58) gives \( A_{1,m} = a_{0,1-m}, A_{2,m} = b_{0,1-m}, m = 0, 1 \).

We next consider \( j \geq 1 \). Assume that \( W_{i,n}, 0 \leq n \leq 2i + 1, i \leq j - 1 \) has the prescribed behavior (2.56), (2.57), (2.59), (2.60). Then it is not difficult to check that \( G_{j,l} \)

has the form

\[ G_{j,l}(y) = \sum_{-j-1 \leq m \leq j} e^{-imy^2/4} y^{2i\alpha_0(2m+1)} G_{j,l}^m(y), \tag{2.65} \]

where \( G_{j,l}^m, m = 0, -1, \) are given by

\[ G_{j,l}^m(y) = G_{j,l}^{m,0}(y) + G_{j,l}^{m,1}(y), \quad m = 0, -1, \]
\[ G_{j,l}^{0,0}(y) = G_{j,l}(y; W_{i,n}^0, 0 \leq n \leq 2i + 1, 0 \leq i \leq j - 1), \]
\[ e^{iy^2/4} G_{j,l}^{0,1}(y) = G_{j,l}(y; W_{i,n}^1, 0 \leq n \leq 2i + 1, 0 \leq i \leq j - 1), \tag{2.66} \]

and admit the following asymptotic expansions and as \( y \to \infty \):

\[ G_{j,l}^{0,0}(y) = \sum_{k \geq 1} \sum_{s=0}^{2j+1-l} T_{k,j,s}^{j,l,0} y^{2v(2j+1) - 2k} (\ln y)^s, \]
\[ G_{j,l}^{0,1}(y) = \sum_{k \geq 2} \sum_{-j-s \leq j-2} \sum_{s=0}^{2j+1-l} T_{k,j,s}^{j,l,0} y^{2v(2j+1) - 2k} (\ln y)^s, \tag{2.67} \]
\[ G_{j,l}^{-1,0}(y) = \sum_{k \geq 2} \sum_{s=0}^{2j+1-l} T_{k, -j-1, s} y^{-2v(2j+1)-2k}(\ln y)^s, \]
\[ G_{j,l}^{-1,1}(y) = \sum_{k \geq 1} \sum_{-j+1 \leq j-1} \sum_{j-1+i \in \mathbb{Z}} \sum_{s=0}^{2j+1-l} T_{k, i, s} y^{2v(2i+1)-2k}(\ln y)^s. \]

Finally, \( G_{j,l}^m, m \neq 0, -1 \), have the following behavior as \( y \to \infty \):
\[ G_{j,l}^m(y) = \sum_{k \geq m+1} \sum_{m-j \leq j-m} \sum_{j-m-i \in \mathbb{Z}} \sum_{s=0}^{2j+1-l} T_{k, i, s} y^{2v(2i+1)-2k}(\ln y)^s, \quad m \geq 1, \]
\[ G_{j,l}^m(y) = \sum_{k \geq |m|-1} \sum_{-j-m \leq j+m} \sum_{j+m-i \in \mathbb{Z}} \sum_{s=0}^{2j+1-l} T_{k, i, s} y^{2v(2i+1)-2k}(\ln y)^s, \quad m \leq -2. \]

Therefore, integrating (2.45), one gets
\[ W_{j,l} = \tilde{W}_{j,l} + \sum_{i=1, 2} \sum_{m=0, \ldots, 2j+1} A_{i, m} G_{j,l}^{i, m}, \]
\[ \tilde{W}_{j,l}(y) = \sum_{-j-1 \leq m \leq j} e^{-imy^2/4} y^{2i\omega_0(2m+1)} \tilde{W}_{j,l}^m(y), \quad \] (2.70)

where \( e^{-imy^2/4} y^{2i\omega_0(2m+1)} \tilde{W}_{j,l}^m(y) \) is a unique solution of (2.45) with \( G_{j,l} \) replaced by \( e^{-imy^2/4} y^{2i\omega_0(2m+1)} G_{j,l}^m(y) \), that has the following behavior as \( y \to +\infty \):
\[ \tilde{W}_{j,l}^m(y) = \sum_{k \geq m+2} \sum_{m-j \leq j-m} \sum_{j-m-i \in \mathbb{Z}} \sum_{s=0}^{2j+1-l} \tilde{w}_{k, i, s} y^{2v(2i+1)-2k}(\ln y)^s, \quad m \geq 1, \]
\[ \tilde{W}_{j,l}^m(y) = \sum_{k \geq -m} \sum_{m-j \leq j+m} \sum_{j+m-i \in \mathbb{Z}} \sum_{s=0}^{2j+1-l} \tilde{w}_{k, i, s} y^{2v(2i+1)-2k}(\ln y)^s, \quad m \leq -2. \]

Finally for \( m = 0, -1 \) one has:
\[ \tilde{W}_{j,l}^0(y) = \tilde{W}_{j,l}^{0,0}(y) + \tilde{W}_{j,l}^{0,1}(y), \]
\[ \tilde{W}_{j,l}^{-1}(y) = \tilde{W}_{j,l}^{-1,0}(y) + \tilde{W}_{j,l}^{-1,1}(y), \quad \] (2.72)
where $\tilde{W}_{j,l}^{0,i}$ and $e^{iy^2/4}y^{-2i\alpha_0} \tilde{W}_{j,l}^{-1,i}$ are solutions of (2.45) with $G_{j,l}$ replaced by $G_{j,l}^0$ and $y^{-2i\alpha_0}e^{iy^2/4}G_{j,l}^{-1,i}$ respectively, with the following asymptotics as $y \to \infty$:

$$\tilde{W}_{j,l}^{0,0}(y) = \sum_{k \geq 1} \sum_{s=0}^{2j+1-l} \tilde{w}_{k,j,s}^{j,l,0} y^{2v(2j+1) - 2k} (\ln y)^s,$$

$$\tilde{W}_{j,l}^{0,1}(y) = \sum_{k \geq 1} \sum_{s=0}^{2j+1-l} \tilde{w}_{k,j,s}^{j,l,0} y^{2v(2j+1) - 2k} (\ln y)^s,$$

$$\tilde{W}_{j,l}^{-1,0}(y) = \sum_{k \geq 2} \sum_{s=0}^{2j+1-l} \tilde{w}_{k,j-s}^{j,l,-1} y^{-2v(2j+1) - 2k} (\ln y)^s,$$

$$\tilde{W}_{j,l}^{-1,1}(y) = \sum_{k \geq 1} \sum_{s=0}^{2j+1-l} \tilde{w}_{k,j,s}^{j,l,-1} y^{2v(2j+1) - 2k} (\ln y)^s.$$  

Clearly, $W_{j,l}^0 = \tilde{W}_{j,l}^{0,0} + \sum_{m=0}^{2j+1} A_{1,m} G_{j,l}^{1,m}$, and $W_{j,l}^1 = e^{-imy^2/4} \tilde{W}_{j,l}^{-1,0} + \sum_{m=0}^{2j+1} A_{2,m} G_{j,l}^{2,m}$ are solutions of (2.45) with $G_{j,l}$ replaced by $G_{j,l}^0 = G_{j,l}(W_{i,n}^0$, $i \leq j - 1)$ and $e^{imy^2/4}G_{j,l}^{-1,0} = G_{j,l}(\tilde{W}_{i,n}^1$, $i \leq j - 1$) respectively. As a consequence, $W_{j,l}^i$, $i = 0, 1$, $0 \leq l \leq 2j + 1$, have the form (2.57) with $\tilde{w}_{0}^{j,l,i} = A_{1-i}^{1-i}, i = 0, -1, l = 0, \ldots, 2j + 1$, which together with (2.58) gives $A_{1,m} = a_{j,2j+1-m}$, $A_{2,m} = b_{j,2j+1-m}$, $m = 0, \ldots, 2j + 1$. □

Let $W^{(N)}(\nu, t)$ be the the stereographic representation of $V^{(N)}(t^{-v}y, t) = (V_{in,1}^{(N)}(t^{-v}y, t), V_{in,2}^{(N)}(t^{-v}y, t), V_{in,3}^{(N)}(t^{-v}y, t))$:

$$W^{(N)}(\nu, t) = \frac{V^{(N)}_{in,1}(t^{-v}y, t) + i V^{(N)}_{in,2}(t^{-v}y, t)}{1 + V^{(N)}_{in,3}(t^{-v}y, t)}.$$  

For $N \geq 2$ define

$$W^{(N)}_{ss}(\nu, t) = \sum_{j=0}^{N} \sum_{l=0}^{2j+1} \nu^{v(2j+1)} \ln(\rho) W_{j,l}^{s,s}(y),$$

$$A_{ss}^{(N)} = -i t \partial_t W_{ss}^{(N)} + \alpha_0 W_{ss}^{(N)} + \mathcal{L} W_{ss}^{(N)} + G(W_{ss}^{(N)}, \tilde{W}_{ss}^{(N)}, \partial_y W_{ss}^{(N)}),$$

$$V_{ss}^{(N)}(\rho, t) = \left( \frac{2 \text{re } W_{ss}^{(N)}}{1 + |W_{ss}^{(N)}|^2}, \frac{2 \text{im } W_{ss}^{(N)}}{1 + |W_{ss}^{(N)}|^2}, \frac{1 - |W_{ss}^{(N)}|^2}{1 + |W_{ss}^{(N)}|^2} \right),$$

$$Z_{ss}^{(N)}(\rho, t) = V_{ss}^{(N)}(\rho, t) - Q(\rho).$$  

Fix $\varepsilon_1 = \frac{\nu}{2}$. Then, as a direct consequence of the previous analysis, we obtain the following result.
Lemma 2.7. For $0 < t \leq T(N)$ the following holds.

(i) For any $k$, $l$, and $\frac{1}{10} t^{\varepsilon_1} \leq y \leq 10 t^{\varepsilon_1}$, one has

$$|y^{-l} \partial_y^k \partial_t^l (W_{ss}^{(N)} - W_{in}^{(N)})| \leq C_{k,l,i} t^{v(N + 1 - \frac{1}{2} k) - i}, \quad i = 0, 1.$$  \tag{2.74}

(ii) The profile $Z_{ss}^{(N)}$ verifies

$$\|\partial_y Z_{ss}^{(N)}(t)\|_{L^2(\rho d\rho, \frac{1}{10} t^{\varepsilon_1} \leq \rho \leq 10 t^{\varepsilon_2})} \leq C t^{\eta},$$  \tag{2.75}

$$\|\rho^{-1} Z_{ss}^{(N)}(t)\|_{L^2(\rho d\rho, \frac{1}{10} t^{\varepsilon_1} \leq \rho \leq 10 t^{\varepsilon_2})} \leq C t^{\eta},$$  \tag{2.76}

$$\|Z_{ss}^{(N)}(t)\|_{L^\infty(\frac{1}{10} t^{\varepsilon_1} \leq \rho \leq 10 t^{\varepsilon_2})} \leq C t^{\eta},$$  \tag{2.77}

$$\|\rho \partial_y Z_{ss}^{(N)}(t)\|_{L^\infty(\frac{1}{10} t^{\varepsilon_1} \leq \rho \leq 10 t^{\varepsilon_2})} \leq C t^{\eta},$$  \tag{2.78}

$$\|\rho^{-1} \partial_y Z_{ss}^{(N)}(t)\|_{L^2(\rho d\rho, \frac{1}{10} t^{\varepsilon_1} \leq \rho \leq 10 t^{\varepsilon_2})} \leq C t^{v + \frac{1}{2} + \eta}, \quad k + l = 2,$$  \tag{2.79}

$$\|\rho^{-1} \partial_y Z_{ss}^{(N)}(t)\|_{L^2(\rho d\rho, \frac{1}{10} t^{\varepsilon_1} \leq \rho \leq 10 t^{\varepsilon_2})} \leq C t^{2 v}, \quad k + l \geq 3,$$  \tag{2.80}

$$\|\rho^{-1} \partial_y Z_{ss}^{(N)}(t)\|_{L^\infty(\frac{1}{10} t^{\varepsilon_1} \leq \rho \leq 10 t^{\varepsilon_2})} \leq C t^{v + \eta}, \quad k + l = 1,$$  \tag{2.81}

$$\|\rho^{-1} \partial_y Z_{ss}^{(N)}(t)\|_{L^\infty(\frac{1}{10} t^{\varepsilon_1} \leq \rho \leq 10 t^{\varepsilon_2})} \leq C t^{2 v}, \quad 2 \leq l + k.$$  \tag{2.82}

Here and below $\eta$ stands for small positive constants depending on $v$ and $\varepsilon_2$, that may change from line to line.

(iii) The error $A_{ss}^{(N)}$ admits the estimate

$$\|y^{-l} \partial_y^k \partial_t^l A_{ss}^{(N)}(t)\|_{L^2(dy, \frac{1}{10} t^{\varepsilon_1} \leq y \leq 10 t^{\varepsilon_2})} \leq t^{v N(1 - 2 \varepsilon_2) - i}, \quad 0 \leq l + k \leq 4, \quad i = 0, 1.$$  \tag{2.83}

2.4. Remote region $r \sim 1$. We next consider the remote region $r^{-\varepsilon_2} \leq r^{-1/2}$. Consider the formal solution $\sum_{j \geq 0} \sum_{l=0}^{2j+1} t^{v(2j+1)} (\ln y - v \ln t)^l W_{ss}^{j,l}(y)$ constructed in the previous subsection. By Lemma 2.6, it has form (2.55), (2.56), (2.57), (2.59), (2.60), with some coefficients $\hat{w}_{k,j,l}^{j,l,i}, w_{k,j,l,i}^{j,l,m}$. Note that in the limit $y \to \infty$, $r \to 0$, the main order terms of the expansion $\sum_{j \geq 0} \sum_{l=0}^{2j+1} t^{v(2j+1)+i \alpha_0} (\ln y - v \ln t)^l W_{ss}^{j,l}(r^{-1/2} r)$ are given by

$$\sum_{j \geq 0} \sum_{l=0}^{2j+1} t^{v(2j+1)+i \alpha_0} (\ln y - v \ln t)^l W_{ss}^{j,l}(r^{-1/2} r)$$

$$\sim \sum_{k \geq 0} \frac{t^k}{r^{2k}} \sum_{j \geq 0} \sum_{l=0}^{2j+1} \hat{w}_{k,j,l,0}^{j,l,0} (\ln r)^l r^{2i \alpha_0 + 2v(2j+1)}$$

$$+ \frac{e^{\frac{i \alpha^2}{4r^2}}}{t} \sum_{k \geq 0} \frac{t^k}{r^{2k}} \sum_{j \geq 0} \sum_{l=0}^{2j+1} \hat{w}_{k,j,l,-1}^{j,l,-1} \left( \frac{r}{t} \right)^{-2i \alpha_0 - 2v(2j+1) - 2} \left( \ln \left( \frac{r}{t} \right) \right)^l,$$  \tag{2.84}
which means that in the region $t^{-\varepsilon_2} \leq r t^{-1/2}$ we have to look for the solution of (2.38) as a perturbation of the time independent profile

$$
\sum_{j \geq 0} \sum_{l=0}^{2j+1} \beta_0(j, l)(\ln r)^l r^{2\nu(2j+1)},
$$
with $\beta_0(j, l) = \hat{\omega}_0^{j, l, 0}$.

Let $\theta \in C_0^\infty(\mathbb{R})$, $\theta(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2. \end{cases}$ For $N \geq 2$, and $\delta > 0$ we define

$$
f_0(r) \equiv f_0^{(N)}(r) = \theta(\delta^{-1}r) \sum_{j=0}^{N} \sum_{l=0}^{2j+1} \beta_0(j, l)(\ln r)^l r^{2\nu 0 + 2\nu(2j+1)}.
$$

Note that $e^{i\theta} f_0 \in H^{1+2\nu}$ and

$$
\|e^{i\theta} f_0\|_{H^s} \leq C\delta^{1+2\nu-s}, \quad \forall \, 0 \leq s < 1 + 2\nu. \tag{2.85}
$$

Write $w(r, t) = f_0(r) + \chi(r, t)$. Then $\chi$ solves

$$
i \chi_t = -\Delta \chi + r^{-2} \chi + \nu_0 \partial_r \chi + \nu_1 \chi + \nu_2 \bar{\chi} + \mathcal{N} + \mathcal{D},
$$

$$
\nu_0 = \frac{4\tilde{f}_0 \partial_r \tilde{f}_0}{1 + |\tilde{f}_0|^2}, \quad \nu_1 = -\frac{2|f_0|^2(2 + |f_0|^2)}{r^2(1 + |f_0|^2)^2} - \frac{2\tilde{f}_0^2(\partial_r \tilde{f}_0)^2}{(1 + |f_0|^2)^2},
$$

$$
\nu_2 = \frac{2(r^2(\partial_r \tilde{f}_0)^2 - \tilde{f}_0^2)}{r^2(1 + |f_0|^2)^2},
$$

$$
\mathcal{D}_0 = (-\Delta + r^{-2}) f_0 + f_0, \quad \mathcal{D}_0, \partial_r f_0).
$$

Finally, $\mathcal{N}$ contains the terms that are at least quadratic in $\chi$ and it has the form

$$
\mathcal{N} = N_0(\chi, \bar{\chi}) + \chi \cdot \bar{\chi} N_1(\chi, \bar{\chi}) + \chi^2 N_2(\chi, \bar{\chi}),
$$

$$
N_0(\chi, \bar{\chi}) = G(f_0 + \chi, \tilde{f}_0 + \bar{\chi}, \partial_r f_0), \quad N_1(\chi, \bar{\chi}) = G(f_0, \tilde{f}_0, \bar{\chi}, \partial_r f_0) - G(f_0, \tilde{f}_0, \partial_r f_0) - \nu_1 \chi - \nu_2 \bar{\chi},
$$

$$
N_2(\chi, \bar{\chi}) = \frac{4\partial_r f_0(\tilde{f}_0 + \bar{\chi})}{1 + |\tilde{f}_0 + \bar{\chi}|^2} - \nu_0, \tag{2.87}
$$

Accordingly to (2.55), (2.56), (2.75), (2.59), (2.60), we look for $\chi$ as

$$
\chi(r, t) = \sum_{q \geq 0} r^{2\nu q + k} \sum_{k \geq 1} q e^{-im\Phi}(\ln r - \ln t)^s g_{k, q, m, s}(r), \tag{2.88}
$$

where

$$
\Phi = \frac{r^2}{4t} + 2\alpha_0 \ln t + \varphi(r),
$$

with $\varphi$ to be chosen later.
Substituting this ansatz to the expressions $-i\chi_t - \Delta \chi + r^{-2} \chi + V_0 \partial_r \chi + V_1 \chi + V_2 \tilde{\chi}$, $N$, we get

$$
-i\chi_t + \Delta \chi - r^{-2} \chi + V_0 \partial_r \chi + V_1 \chi + V_2 \tilde{\chi} = \sum_{q \geq 0} t^{2q+k-2} \sum_{k \geq 2} -\min[k,q] \leq m \leq \min[(k-2)+q] \sum_{s=0}^q e^{-im\Phi}(\ln r - \ln t)^s \Psi^{lin}_{k,q,m,s},
$$

$$
N_0(\chi, \tilde{\chi}) = \sum_{q \geq 0} t^{2q+k-2} \sum_{k \geq 4} -\min[k,q] \leq m \leq \min[(k-2)+q] \sum_{s=0}^q e^{-im\Phi}(\ln r - \ln t)^s \Psi^{nl,0}_{k,q,m,s},
$$

$$
\chi, N_1(\chi, \tilde{\chi}) = \sum_{q \geq 0} t^{2q+k-2} \sum_{k \geq 3} -\min[k,q] \leq m \leq \min[(k-2)+q] \sum_{s=0}^q e^{-im\Phi}(\ln r - \ln t)^s \Psi^{nl,1}_{k,q,m,s},
$$

$$
(\chi_t)^2 N_2(\chi, \tilde{\chi}) = \sum_{q \geq 0} t^{2q+k-2} \sum_{k \geq 2} -\min[k,q] \leq m \leq \min[(k-2)+q] \sum_{s=0}^q e^{-im\Phi}(\ln r - \ln t)^s \Psi^{nl,2}_{k,q,m,s},
$$

Here

$$
\Psi^{lin}_{k,q,m,s} = \frac{m(m+1)^2}{4} g_{k,q,m,s} + \Psi^{lin,1}_{k,q,m,s} + \Psi^{lin,2}_{k,q,m,s},
$$

with $\Psi^{lin,1}_{k,q,m,s}$ and $\Psi^{lin,2}_{k,q,m,s}$ depending respectively on $g_{k-1,q,m,s}', s' = s, s+1$ and $g_{k-2,q,m,s}, s' = s, s+1, s+2$ only:

$$
\Psi^{lin,1}_{k,q,m,s} = -i(2q+k-1-m-2im\alpha_0)g_{k-1,q,m,s} + i(m+1)(s+1)g_{k-1,q,m,s+1}
$$

$$
+ imr(\partial_r - im\varphi'(r) - \frac{1}{2} V_0(r))g_{k-1,q,m,s},
$$

$$
\Psi^{lin,2}_{k,q,m,s} = -e^{im\varphi} \Delta (e^{-im\varphi} g_{k-2,q,m,s}) - \frac{(s+1)(s+2)}{r^2} e^{im\varphi} \partial_r (e^{-im\varphi} g_{k-2,q,m,s+1})
$$

$$
- \frac{2(s+1)}{r} e^{im\varphi} \partial_r (e^{-im\varphi} g_{k-2,q,m,s+1})
$$

$$
+(r^{-2} + V_1)g_{k-2,q,m,s} + V_2 \tilde{g}_{k-2,q,m,s}.
$$

Here and below we use the convention $g_{k,q,m,s} = 0$ if $(k, q, m, s) \notin \Omega$, where

$$
\Omega = \{ k \geq 1, q \geq 0, 0 \leq s \leq q, q - m \in 2\mathbb{Z}, -\min[k,q] \leq m \leq \min[k-1,q] \}.
$$

The nonlinear terms $\Psi^{nl,i}_{k,q,m,s}, i = 0, 1$, depend only on $g_{k',q',m',s'}$ with $k' \leq k-2$. More precisely,

$$
\Psi^{nl,0}_{k,q,m,s} = \Psi^{nl,0}_{k,q,m,s}(r; g_{k',q',m',s'}, k' \leq k-3),
$$

$$
\Psi^{nl,1}_{k,q,m,s} = \Psi^{nl,1}_{k,q,m,s}(r; g_{k',q',m',s'}, k' \leq k-2).
$$
Finally, $\Psi_{k,q,m,s}^{nl}$ has the following structure

$$
\Psi_{k,q,m,s}^{nl} = -\delta_{m,-2} g_{1,q_1,-1,s_1} g_{1,q_2,-1,s_2} + \frac{r^2 f_0}{2(1 + |f_0|^2)} \sum_{q_{1+q_2}=q} g_{1,q_1,-1,s_1} g_{1,q_2,-1,s_2},
$$

$$
\Psi_{k,q,m,s}^{nl,2} = \Psi_{k,q,m,s}^{nl,2,0} + \Psi_{k,q,m,s}^{nl,2,1},
$$

$$
\Psi_{k,q,m,s}^{nl,2} = \frac{(m+1) r^2 f_0}{1 + |f_0|^2} \sum_{q_{1+q_2}=q} g_{1,q_1,-1,s_1} g_{k-1,q_2,m+1,s_2},
$$

with $\hat{\Psi}_{k,q,m,s}^{nl,2}$ depending on $g_{k',q',m',s'}$, $k' \leq k - 2$ only:

$$
\hat{\Psi}_{k,q,m,s}^{nl,2} = (r; g_{k',q',m',s'}, k' \leq k - 2).
$$

Note that

$$
\Psi_{k,q,-1,s}^{nl,0} = 0, \forall k, q, s.
$$

Equation (2.86) is equivalent to

$$
\begin{cases}
\psi_{2,0,0,0}^{lin} + D_0 = 0, \\
\psi_{k,q,m,s}^{lin} \psi_{k,q,m,s}^{nl} = 0, \quad (k, q, m, s) \in \Omega, \quad (k, q, m, s) \neq (2, 0, 0, 0),
\end{cases}
$$

Here $\psi_{k,q,m,s}^{nl} = \psi_{k,q,m,s}^{nl,0} + \psi_{k,q,m,s}^{nl,1} + \psi_{k,q,m,s}^{nl,2}$.

We view (2.93) as a recurrent system with respect to $k \geq 1$ of the form

$$
\begin{cases}
\psi_{2,0,0,0}^{lin} + D_0 = 0, \\
\psi_{2,2j,0,s}^{lin} = 0, \quad (j, s) \neq (0, 0), \\
\psi_{2,2j+1,1,s}^{lin} = 0,
\end{cases}
$$

and

$$
\begin{cases}
\psi_{k+1,q,m,s}^{lin} \psi_{k+1,q,m,s}^{nl} = 0, \quad m = 0, 1, \\
\psi_{k,q,m,s}^{lin} \psi_{k,q,m,s}^{nl} = 0, \quad m \neq 0, 1, \quad k \geq 2.
\end{cases}
$$

Consider (2.94). Choosing $\varphi$ as

$$
\varphi(r) = -i \int_0^r ds \frac{\tilde{f}_0(s) \partial_s f_0(s) - f_0(s) \partial_s \tilde{f}_0(s)}{1 + |f_0(s)|^2},
$$

we can rewrite (2.94) in the following form

$$
\begin{cases}
(4v j + 1) g_{1,2j,0,s} - (s + 1) g_{1,2j,0,s+1} = 0, \quad (j, s) \neq (0, 0), \\
g_{1,0,0,0} = -i D_0, \\
r \partial_r g_{1,2j+1,1,s} + (2v(2j + 1) + 2 + 2i \alpha_0 - r (\ln(1 + |f_0|^2))') g_{1,2j+1,1,s} = 0.
\end{cases}
$$
Accordingly to (2.84), we solve this system as follows:

\[
\begin{align*}
g_{1.2,j,0,s} &= 0, \quad (j, s) \neq (0, 0), \\
g_{1.0,0,0} &= -iD_0, \\
g_{1.2,j+1,-1,s} &= \beta_1(j, s)(1 + |f_0|^2)r^{-2i\omega_0-2\nu(2j+1)-2}, \quad 0 \leq s \leq 2j + 1, \quad 0 \leq j \leq N, \\
g_{1.2,j+1,-1,s} &= 0, \quad j > N.
\end{align*}
\]  

(2.98)

where \(\beta_1(j, s) = \tilde{w}_0^{ij,l,-1}\).

Consider (2.95). We will solve it with the “zero boundary conditions” at zero. To formulate the result we need to introduce some notations. For \(m \in \mathbb{Z}\), we denote by \(A_m\) the space of continuous functions \(a : \mathbb{R}_+ \to \mathbb{C}\) such that

(i) \(a \in C^\infty(\mathbb{R}_+), \text{ supp } a \subset \{r \leq 2\delta\}\);

(ii) for \(0 \leq r < \delta\), \(a\) has an absolutely convergent expansion of the form

\[
a(r) = \sum_{n \geq K(m)} \sum_{n-m-1 \in \mathbb{Z}} \alpha_n,j (\ln r)^j r^{2\nu n},
\]

where \(K(m) = m + 1\) if \(m \geq 0\), and \(K(m) = |m| - 1\) if \(m \leq -1\). For \(k \geq 1\) we define \(B_k\) as the space of continuous functions \(b : \mathbb{R}_+ \to \mathbb{C}\) such that

(i) \(b \in C^\infty(\mathbb{R}_+);\)

(ii) for \(0 \leq r < \delta\), \(b\) has an absolutely convergent expansion of the form

\[
b(r) = \sum_{n=0}^{\infty} \sum_{l=0}^{2n} \beta_n,l,r^{4\nu n} (\ln r)^l,
\]

(iii) for \(r \geq 2\delta\), \(b\) is a polynome of degree \(k - 1\).

Finally, we set \(B_k^0 = \{b \in B_k, b(0) = 0\}\).

Clearly, for any \(m, k, \) one has \(r \partial_r A_m \subset A_m, r \partial_r B_k \subset B_k, B_k A_m \subset A_m\). Note also that

\[
f_0 \in r^{2i\omega_0} A_0, \quad \varphi \in B_0^0, \quad g_{1.0,0,0} \in r^{2i\omega_0-2} A_0,
\]

\[
g_{1.2,j+1,-1,s} \in r^{-2i\omega_0-2\nu(2j+1)-2} B_1, \quad 0 \leq s \leq 2j + 1.
\]

Furthermore, one checks easily that if for all \((k, q, m, s) \in \Omega, g_{k,q,m,s} \in r^{2i\omega_0(1+2m)-2\nu-2k} A_m\) if \(m \neq -1\) and \(g_{k,q,-1,s} \in r^{-2i\omega_0-2\nu-2k} B_k\), then

\[
\begin{align*}
\psi_{lin,1}^{k,q,m,s}, \quad \psi_{lin,2}^{k,q,m,s}, \quad \tilde{\psi}_{lin,2}^{k,q,m,s} &\in r^{2i\omega_0(1+2m)-2\nu-2(k-1)} A_m, \quad m \neq -1, \\
\psi_{lin,1}^{k,q,-1,s}, \quad \psi_{lin,2}^{k,q,-1,s}, \quad \tilde{\psi}_{lin,2}^{k,q,-1,s} &\in r^{-2i\omega_0-2\nu-2(k-1)} B_{k-2}, \quad i = 1, 2, \quad j = 0, 1, 2.
\end{align*}
\]

Consider (2.95). Using (2.89), (2.90), (2.91), (2.92), (2.96), one can rewrite it as

\[
\begin{cases}
\frac{1}{4}m(m+1)r^2 g_{k,q,m,s} = B_{k,q,m,s}, \quad m \neq 0, -1, \\
r \partial_r g_{k,q,m,s} + (2\nu q + k + 1 + 2i\omega_0 - \frac{r(f_0^2 + f_0 \partial_r f_0)}{1 + |f_0|^2}) g_{k,q,-1,s} = C_{k,q,-1,s}, \\
(2\nu q + k)g_{k,q,0,s} - (s + 1)g_{k,q,0,s+1} = C_{k,q,0,s} + D_{k,q,s}.
\end{cases}
\]

(2.100)
where $B_{k,q,m,s}, C_{k,q,m,s}$ depend on $g_{k',q',m',s'}, k' \leq k - 1$ only:

\[
B_{k,q,m,s} = B_{k,q,m,s}(r, g_{k',q',m',s'}, k' \leq k - 1), \ m \neq 0, -1, \\
C_{k,q,m,s} = C_{k,q,m,s}(r, g_{k',q',m',s'}, k' \leq k - 1), \ m = 0, -1,
\]

and have the following form

\[
B_{k,q,m,s} = -\psi_{k,q,m,s}^{lin,1} - \psi_{k,q,m,s}^{lin,2} - \psi_{k,q,m,s}^{nl}, \ m \neq 0, -1
\]

\[
C_{k,q,m,s} = -i\psi_{k+1,q,m,s}^{lin,2} - i\psi_{k+1,q,m,s}^{nl}, \ m = 0, -1.
\]

Finally $D_{k,q,s}$ depend only on $g_{k.q,1.s}$ and is given by

\[
D_{k,q,s} = -i\psi_{k+1,q,0,s}^{nl,2,0} = -i \left( \frac{r^2 \tilde{f}_0}{1 + |f_0|^2} \right) \sum_{q_1 + q_2 = q, s_1 + s_2 = s} g_{1,q_1,-1,s_1} g_{k,q_2,1,s_2}.
\]

Note that $D_{2,q,s} = 0$.

**Remark 2.8.** It is not difficult to check that if

\[
g_{k,q,m,s} = 0, \ \forall q > (2N + 1)(2k - 2), \ m \neq 0, 1, \\
g_{k,q,m,s} = 0, \ \forall q > (2N + 1)(2k - 1), \ m = 0, 1,
\]

then

\[
B_{k,q,m,s} = 0, \ \forall q > (2N + 1)(2k - 2), \ m \neq 0, 1, \\
C_{k,q,m,s} = 0, \ \forall q > (2N + 1)(2k - 1), \ m = 0, 1, \\
D_{k,q,s} = 0, \ \forall q > (2N + 1)(2k - 1).
\]

We are now in position to prove the following result.

**Lemma 2.9.** There exists a unique solution $(g_{k,q,m,s})_{k \geq 2} \in \Omega$ of (2.100) verifying

\[
g_{k,q,m,s} \in r^{2\imath \alpha_0(2m+1) - 2\nu q - 2k} A_m, \ m \neq -1, \ (2.103)
\]

\[
g_{k,q,-1,s} \in r^{-2\imath \alpha_0 - 2\nu q - 2k} B_k.
\]

In addition, one has

\[
g_{k,q,m,s} = 0, \ \forall q > (2N + 1)(2k - 2), \ m \neq 0, 1, \\
g_{k,q,m,s} = 0, \ \forall q > (2N + 1)(2k - 1), \ m = 0, 1,
\]

\[
(2.104)
\]

**Proof.** For $k = 2$ (2.100), (2.101), (2.92) give

\[
\frac{1}{2} r^2 g_{2,2j,-2,s} = B_{2,2j,-2,s}, \ 0 \leq s \leq 2j, \ 1 \leq j, \ (2.105)
\]

\[
r \partial_r g_{2,2j+1,-1,s} + \left( 2\nu (2j + 1) + 3 + 2\imath \alpha_0 - \frac{r (\tilde{f}_0 \partial_r f_0 + f_0 \partial_r \tilde{f}_0)}{1 + |f_0|^2} \right) g_{2,2j+1,-1,s} \\
= C_{2,2j+1,-1,s}, \ 0 \leq s \leq 2j + 1, \ 0 \leq j, \ (2.106)
\]

\[
(4\nu j + 2)g_{2,2j,0,s} - (s + 1)g_{2,2j,0,s+1} = C_{2,2j,0,s}, \ 0 \leq s \leq 2j, \ 0 \leq j. \ (2.107)
\]
Recall that $B_{2,q,m,s}, C_{2,q,m,s}$ depend only on $g_{1,q',m',s'}$ and therefore, are known by now. By (2.99), (2.101) and Remark 2.8 they verify
\[
\begin{align*}
B_{2,q,-2,s} &\in r^{-6i\alpha_0-2\nu q-2}\mathcal{A}_{-2}, \quad m \neq 0, -1 \\
C_{2,q,0,s} &\in r^{2i\alpha_0-2\nu q-4}\mathcal{A}_0, \quad C_{2,q,-1,s} \in r^{-2i\alpha_0-2\nu q-4}\mathcal{B}_1, \\
B_{2,q,-2,s} & = 0, \quad q > 2(2N + 1), \\
C_{2,q,m,s} & = 0, \quad q > 3(2N + 1), \quad m = 0, 1.
\end{align*}
\]
Therefore, we get from (2.105), (2.106),
\[
\begin{align*}
g_{2,2j,-2,s} &= \frac{2}{r^2}B_{2,2j,-2,s} \in r^{-6i\alpha_0-4\nu j-4}\mathcal{A}_{-2}, \quad 0 \leq s \leq 2j, \quad 1 \leq j, \\
g_{2,2j,0,2j} &= \frac{1}{4j\nu + 2}C_{2,2j,0,2j} \in r^{2i\alpha_0-4\nu j-4}\mathcal{A}_0, \quad 0 \leq j, \\
g_{2,2j,0,s} &= \frac{1}{4j\nu + 2}C_{2,2j,0,s} + \frac{s + 1}{4j\nu + 2}g_{2,2j,0,s+1} \in r^{2i\alpha_0-4\nu j-4}\mathcal{A}_0, \quad 0 \leq s \leq 2j, \\
g_{2,2j,-2,s} & = 0, \quad j > 2N + 1, \\
g_{2,2j,0,s} & = 0, \quad j \geq 3N + 2.
\end{align*}
\]
Consider (2.107). Write
\[
g_{2,2j+1,-1,s} = r^{-2i\alpha_0-3-2\nu(2j+1)}(1 + |f_0|^2)\hat{g}_{2,2j+1,-1,s}.
\]
Then $\hat{g}_{2,2j+1,-1,s}$ solves
\[
\partial_r \hat{g}_{2,2j+1,-1,s} = r^{-2}\hat{C}_{2,2j+1,-1,s}, \tag{2.109}
\]
where
\[
\hat{C}_{2,2j+1,-1,s} = r^{2i\alpha_0+4+2\nu(2j+1)}(1 + |f_0|^2)^{-1}C_{2,2j+1,-1,s}.
\]
Since $C_{2,2j+1,-1,s} \in r^{-2i\alpha_0-2\nu(2j+1)-4}\mathcal{B}_1$, we have:

(i) for $0 \leq r < \delta$, $\hat{C}_{2,2j+1,-1,s}$ admits an absolutely convergent expansion of the form
\[
\hat{C}_{2,2j+1,-1,s} = \sum_{n=0}^{\infty} \sum_{l=0}^{2n} \beta_{n,l} r^{4\nu(n)}(\ln r)^l.
\]

(ii) for $r \geq 2\delta$, $\hat{C}_{2,2j+1,-1,s}$ is a constant.

Clearly, there exists a unique solution $\hat{g}_{2,2j+1,-1,s}$ of (2.109) such that $\hat{g}_{2,2j+1,-1,s} \in r^{-1}\mathcal{B}_2$. It is given by
\[
\hat{g}_{2,2j+1,-1,s}(r) = \int_0^r d\rho \rho^{-2}(\hat{C}_{2,2j+1,-1,s}(\rho) - \beta_{0,0}) - \beta_{0,0} r^{-1}, \quad 0 \leq s \leq 2j + 1, \quad 0 \leq j.
\]

Finally, since $C_{2,q,-1,s} = 0$ for $q > 3(2N + 1)$, one has
\[
g_{2,2j+1,-1,s} = 0, \quad j > 3N + 1.
We next proceed by induction. Suppose we have solved (2.100) with \( k = 2, \ldots, l-1, \ l \geq 3, \) and have found \((g_{k,q,m,s})_{(k,q,m,s) \in \Omega}^{|2 < k < l-1|}\) verifying (2.103) and (2.104). Consider \( k = l.\) From the first line in (2.100) we have:

\[
\frac{1}{4} m(m+1)r^2 g_{l,q,m,s} = B_{l,q,m,s}, \ m \neq 0, -1,
\]

where \( B_{l,q,m,s} \) are known by now and, by (2.99), (2.101) and Remark 2.8, satisfy

\[
B_{l,q,m,s} \in r^{2ia_0(2m+1)-2vq-2(l-1)} A_m, \\
B_{l,q,m,s} = 0, \ q > 2(2N+1)(2l-2).
\]

As a consequence, one obtains for \( m \neq 0, -1:\)

\[
g_{l,q,m,s} = \frac{4}{m(m+1)r^2} B_{l,q,m,s} \in r^{2ia_0(2m+1)-2vq-2l} A_m, \\
g_{l,q,m,s} = 0, \ q > 2(2N+1)(2l-2).
\]

We next consider the equations for \( g_{l,2j,0,s}:\)

\[
(4vj+l)g_{l,2j,0,s} - (s+1)g_{l,2j,0,s+1} = C_{l,2j,0,s} + D_{l,2j,s}, \ 0 \leq s \leq 2j, \ 0 \leq j. \quad (2.111)
\]

The right hand side \( C_{l,2j,0,s} + D_{l,2j,s} \) depends only on \( g_{l,q_1,1,s_1} \) and \( g_{k,q_2,m_2,s_2}, k \leq l-1, \) and by (2.99), (2.101), (2.110) and Remark 2.8, satisfies

\[
C_{l,2j,0,s} + D_{l,2j,s} \in r^{2ia_0-4vj-2l} A_0, \\
C_{l,2j,0,s} + D_{l,2j,s} = 0, \ j > (2N+1)(2l-1).
\]

Therefore, the solution of (2.111) verifies

\[
g_{l,2j,0,s} \in r^{2ia_0-4vj-2l} A_0, \ 0 \leq s \leq 2j, \ 0 \leq j, \\
g_{l,2j,0,s} = 0, \ j > (2N+1)(2l-1).
\]

Finally for \( g_{l,2j+1,1,s}, 0 \leq s \leq 2j+1, 0 \leq j \) we have

\[
r \partial_r g_{l,2j+1,1,m,s} + \left( 2v(2j+1) + l + 1 + 2ia_0 - \frac{r( \tilde{f}_0 \partial_r \tilde{f}_0 + \tilde{f}_0 \partial_r \tilde{f}_0)}{1 + |\tilde{f}_0|^2} \right) g_{l,2j+1,1,s} = C_{l,2j+1,1,s},
\]

with \( C_{l,2j+1,1,s} \in r^{-2ia_0-2v(2j+1)-2l} B_{l-1} \) such that

\[
C_{l,2j+1,1,s} = 0, \ 2j+1 > (2N+1)(2l-1). \quad (2.113)
\]

Equation (2.112) has a unique solution \( g_{l,2j+1,1,s} \) verifying \( g_{l,2j+1,1,s} \in r^{-2ia_0-2v(2j+1)-2l} B_l, \) which is given by

\[
g_{l,2j+1,1,s} = r^{-2ia_0-2v(2j+1)-l-1}(1 + |\tilde{f}_0|^2) \hat{g}_{l,2j+1,1,s}, \]

\[
\hat{g}_{l,2j+1,1,s} = \int_0^r d\rho \rho^{-l} \left( \hat{C}_{l,2j+1,1,s} - \sum_{0 \leq n \leq l-1} \sum_{p=0}^{2n} \beta_{n,p} \rho^{4vn} (\ln \rho)^p \right)
\]

\[- \int_r^\infty d\rho \rho^{-l} \sum_{0 \leq n \leq l-1} \sum_{p=0}^{2n} \beta_{n,p} \rho^{4vn} (\ln \rho)^p,
\]
where
\[
\hat{C}_{l,2j+1,-1,s} = r^{2ia \nu + 2v(2j+1)+2l}(1 + |f_0|^2)^{-1}C_{l,2j+1,-1,s},
\]
\[
\hat{C}_{l,2j+1,-1,s} = \sum_{n=0}^{\infty} \sum_{p=0}^{2n} \beta_n p r^n (\ln r)^p, \quad r < \delta.
\]

By (2.113),
\[
g_{l,2j+1,-1,s} = 0, \quad 2j + 1 > (2N + 1)(2l - 1).
\]

Let us define
\[
w_{\text{rem}}^{(N)}(r, t) = f_0(r) + \sum_{(k,q,m,s) \in \Omega, k \leq N} t^{k+2q} e^{-im\Phi} (\ln r - \ln t)^r g_{k,q,m,s}(r),
\]
\[
A_{\text{rem}}^{(N)} = -i \partial_r u_{\text{rem}}^{(N)} - \Delta u_{\text{rem}}^{(N)} + r^{-2} w_{\text{rem}}^{(N)} + G(u_{\text{rem}}^{(N)}, w_{\text{rem}}^{(N)}, \partial_r w_{\text{rem}}^{(N)})
\]
\[
W_{\text{rem}}^{(N)}(y, t) = e^{-ia\Phi} w_{\text{rem}}^{(N)}(rt^{-1/2}, t).
\]

As a direct consequence of the previous analysis we get:

**Lemma 2.10.** There exists \(T(N, \delta) > 0\) such that for \(0 < t \leq T(N, \delta)\) the following holds.

(i) For any \(0 \leq l, k \leq 4, i = 0, 1\) and \(\frac{1}{10} t^{-\varepsilon_2} \leq y \leq 10 t^{-\varepsilon_2}\), one has
\[
|y^{-l} \partial_y^k \partial_t^i (W_{s\alpha}^{(N)} - W_{\text{rem}}^{(N)})| \leq t^{v(1-2\varepsilon_2)} + t^\varepsilon_2 N, \tag{2.114}
\]
provided \(N\) is sufficiently large (depending on \(\varepsilon_2\)).

(ii) The profile \(w_{\text{rem}}^{(N)}(r, t)\) verifies
\[
\|r^{-l} \partial_y^k w_{\text{rem}}^{(N)}(t) - f_0\|_{L^2(rdr, r \geq \frac{1}{10} t^{1/2-\varepsilon_2})} \leq C t^\eta, \quad 0 \leq k + l \leq 3, \tag{2.115}
\]
\[
\|r \partial_r w_{\text{rem}}^{(N)}(t)\|_{L^\infty(r \geq \frac{1}{10} t^{1/2-\varepsilon_2})} \leq C \delta^{2\nu}, \tag{2.116}
\]
\[
\|r^{-l} \partial_y^k w_{\text{rem}}^{(N)}(t)\|_{L^\infty(r \geq \frac{1}{10} t^{1/2-\varepsilon_2})} \leq C (\delta^{2\nu-k-l} + t^{v-(k+l)/2+\eta}), \quad 0 \leq k + l \leq 4, \tag{2.117}
\]
\[
\|r^{-l-1} \partial_y^k \partial_t^i w_{\text{rem}}^{(N)}(t)\|_{L^\infty(r \geq \frac{1}{10} t^{1/2-\varepsilon_2})} \leq C (\delta^{2\nu-k} + t^{v-3+\eta}), \quad k + l = 5 \tag{2.118}
\]

(iii) The error \(A_{\text{rem}}^{(N)}(r, t)\) admits the estimate
\[
\|r^{-l} \partial_y^k \partial_t^i A_{\text{rem}}^{(N)}(t)\|_{L^2(rdr, r \geq \frac{1}{10} t^{1/2-\varepsilon_2})} \leq t^{\varepsilon_2 N}, \quad 0 \leq l + k \leq 3, \quad i = 0, 1, \tag{2.119}
\]
provided \(N\) is sufficiently large.
2.4.1. Proof of Proposition 2.1. We are now in position to finish the proof of Proposition 2.1. Fix \( \varepsilon_2 \) verifying \( 0 < \varepsilon_2 < \frac{1}{2} \). For \( N \geq 2 \), define

\[
\hat{W}_{\text{ex}}^{(N)}(\rho, t) = \theta(t^{v-\varepsilon_1} \rho) W_{\text{in}}^{(N)}(t^{v} \rho, t) + (1 - \theta(t^{v-\varepsilon_1} \rho)) \theta(t^{v+\varepsilon_2} \rho) W_{\text{s}}^{(N)}(t^{v} \rho, t) + (1 - \theta(t^{v+\varepsilon_2} \rho)) e^{-i\alpha(t)} W_{\text{rem}}^{(N)}(t^{v+1/2} \rho, t),
\]

\[
V_{\text{ex}}^{(N)}(\rho, t) = \begin{pmatrix} \text{Re} \hat{W}_{\text{ex}}^{(N)}(\rho, t) / |\hat{W}_{\text{ex}}^{(N)}(\rho, t)|^2, & 2 \text{Im} \hat{W}_{\text{ex}}^{(N)}(\rho, t) / |\hat{W}_{\text{ex}}^{(N)}(\rho, t)|^2, & 1 - |\hat{W}_{\text{ex}}^{(N)}(\rho, t)|^2 \\
1 + |\hat{W}_{\text{ex}}^{(N)}(\rho, t)|^2, & 1 + |\hat{W}_{\text{ex}}^{(N)}(\rho, t)|^2, & 1 + |\hat{W}_{\text{ex}}^{(N)}(\rho, t)|^2\end{pmatrix}.
\]

Clearly, \( V_{\text{ex}}^{(N)}(\rho, t) \) is well defined for \( \rho \) is sufficiently large, and for \( \rho < t^{v-\varepsilon_1} \), \( V_{\text{ex}}^{(N)}(\rho, t) \) coincides with \( V_{\text{in}}^{(N)}(\rho, t) \). Therefore, setting

\[
V^{(N)}(\rho, t) = \begin{pmatrix} V_{\text{in}}^{(N)}(\rho, t), & \rho \leq \frac{1}{2} t^{-v+\varepsilon_1}, \\
V_{\text{ex}}^{(N)}(\rho, t), & \rho \geq \frac{1}{2} t^{-v+\varepsilon_1}.
\end{pmatrix}
\]

we get a \( C^\infty \) 1-equivariant profile \( u^{(N)} : \mathbb{R}^2 \times \mathbb{R}_+^* \rightarrow S^2 \) that, by Lemmas 2.3 (i), 2.7 (ii), 2.10 (ii), for any \( N \geq 2 \) verifies part (i) of Proposition 2.1, \( \zeta_N^* \) being given by

\[
\zeta_N^*(x) = e^{\theta R} \bar{\zeta}_N^*(|x|), \quad \bar{\zeta}_N^* = \begin{pmatrix} \frac{2 \text{Re} f_0}{1 + |f_0|^2}, & \frac{2 \text{Im} f_0}{1 + |f_0|^2}, & 1 - |f_0|^2 \end{pmatrix}.
\]

By Lemmas 2.3 (ii), 2.7 (i), (iii) and 2.10 (i), (iii), for \( N \) sufficiently large the error \( r^{(N)} = -u^{(N)} + u^{(N)} \times \Delta u^{(N)} \) satisfies

\[
\|r^{(N)}(t)\|_{H^3} + \|\partial_t r^{(N)}(t)\|_{H^1} + \|\langle x \rangle r^{(N)}(t)\|_{L^2} \leq t^{\eta N}, \quad t \leq T(N, \delta),
\]

with some \( \eta = \eta(\nu, \varepsilon_2) > 0 \). Re-denoting \( N = \frac{N}{\eta} \) we obtain a family of approximate solutions \( u^{(N)}(t) \) verifying Proposition 2.1.

3. Proof of the Theorem

3.1. Main proposition. The proof of Theorem 1.1 will be achieved by compactness arguments that rely on the following result. Let \( u^{(N)} \), \( T = T(N, \delta) \) be as in Proposition 2.1. Consider the Cauchy problem

\[
u_t = u \times \Delta u, \quad t \geq t_1, 
\]

\[
u|_{t=t_1} = u^{(N)}(t_1),
\]

with \( 0 < t_1 < T \).

One has

Proposition 3.1. For \( N \) sufficiently large there exists \( 0 < t_0 < T \) such that for any \( t_1 \in (0, t_0) \) the solution \( u(t) \) of (3.1) verifies:

(i) \( u - u^{(N)} \) is in \( C([t_1, t_0], H^3) \) and

\[
\|u(t) - u^{(N)}(t)\|_{H^3} \leq t^{N/2}, \quad \forall t_1 \leq t \leq t_0.
\]
(ii) Furthermore, \( \langle x \rangle (u(t) - u^{(N)}(t)) \in L^2 \) and
\[
\| \langle x \rangle (u(t) - u^{(N)}(t)) \|_{L^2} \leq t^{N/2}, \quad \forall t_1 \leq t \leq t_0.
\] (3.3)

**Proof.** The proof is by a bootstrap argument. Write
\[
u^{(N)}(x, t) = e^{\alpha(t) R} U^{(N)}(\lambda(t)x, t),
\]
\[r^{(N)}(x, t) = \lambda^2(t) e^{\alpha(t) R} R^{(N)}(\lambda(t)x, t),
\]
\[u(x, t) = e^{\alpha(t) R} U(\lambda(t)x, t),
\]
\[U^{(N)}(y, t) = \phi(y) + \chi^{(N)}(y, t).
\]
Then \( S(t) \) solves
\[
t^{1+2\nu} S_t + \alpha_0 t^{2\nu} RS - (v + \frac{1}{2}) y \cdot \nabla S = S \times \Delta U^{(N)} + U^{(N)} \times \Delta S + S \times \Delta S + R^{(N)}(t).
\] (3.4)

Assume that
\[
\|S\|_{L^\infty(\mathbb{R}^2)} \leq \delta_1,
\] (3.5)
with \( \delta_1 \) sufficiently small. Note that since \( S \) is 1-equivariant and
\[
(\phi, S) + (\chi^{(N)}, S) + |S|^2 = 0
\] (3.6)
where \( \|\chi^{(N)}\|_{L^\infty(\mathbb{R}^2)} \leq C \delta^{2\nu} \) (see (2.5)), the bootstrap assumption (3.5) implies
\[
\|S\|_{L^\infty(\mathbb{R}^2)} \leq C \|\nabla S\|_{L^2(\mathbb{R}^2)}.
\] (3.7)

3.1.1. Energy control. We will first derive a bootstrap control of the energy norm:
\[
J_1(t) = \int_{\mathbb{R}^2} dy (|\nabla S|^2 + \kappa(\rho)|S|^2), \quad \rho = |y|.
\]

It follows from (3.4) that
\[
t^{1+2\nu} \frac{d}{dt} \int_{\mathbb{R}^2} dy |\nabla S|^2 = -2 \int_{\mathbb{R}^2} dy (S \times \Delta U^{(N)} + \Delta S) + 2 \int_{\mathbb{R}^2} dy (\nabla R^{(N)}, \nabla S),
\] (3.8)
\[
+ 2 \int_{\mathbb{R}^2} dy \kappa(U^{(N)} \times \Delta S, S) + 2 \int_{\mathbb{R}^2} dy \kappa(R^{(N)}, S).
\] (3.9)

Recall that \( U^{(N)} = \phi + \chi^{(N)} \), with \( \phi \) solving \( \Delta \phi = \kappa \phi \), which means that
\[
(S \times \Delta \phi, \Delta S) - \kappa (\phi \times \Delta S, S) = 0.
\]

Therefore, combining (3.8), (3.9), we get
\[
t^{1+2\nu} \frac{d}{dt} J_1(t) = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4,
\]
where
\[ \mathcal{E}_1 = -2 \int dy(S \times \Delta \chi^{(N)} \Delta S), \]
\[ \mathcal{E}_2 = 2 \int dy \kappa(\chi^{(N)} \times \Delta S, S), \]
\[ \mathcal{E}_3 = -\left(\frac{1}{2} + \nu\right)t^{2\nu} \int dy(2\kappa + \rho \kappa')(S, S), \]
\[ \mathcal{E}_4 = 2 \int dy \left[ (\nabla R^{(N)}, \nabla S) + \kappa(R^{(N)}, S) \right]. \]

From Proposition 2.1 we have
\[ |\mathcal{E}_j| \leq Ct^{2\nu} \|S\|^2_{H^1}, \quad j = 1, \ldots, 3, \]
\[ |\mathcal{E}_4| \leq Ct^{N+\nu+1/2} \|\nabla S\|_{L^2}. \]

Combining these inequalities we obtain
\[ \left| \frac{d}{dt} J_1(t) \right| \leq Ct^{-1} \|S\|^2_{H^1} + Ct^{2N-2\nu}. \] (3.10)

3.1.2. Control of the $L^2$ norm. Consider $J_0(t) = \int_{\mathbb{R}^2} dy |S|^2$. We have
\[ t^{1+2\nu} \frac{d}{dt} J_0(t) = \mathcal{E}_5 + \mathcal{E}_6 + \mathcal{E}_7, \]
\[ \mathcal{E}_5 = 2 \int dy (U^{(N)} \times \Delta S, S), \]
\[ \mathcal{E}_6 = -2(1 + 2\nu)t^{2\nu} J_0(t), \]
\[ \mathcal{E}_7 = 2 \int dy (R^{(N)}, S). \]

Consider $\mathcal{E}_5$. Decomposing $U^{(N)}$ and $S$ in the basis $f_1, f_2, Q$:
\[ U^{(N)}(y, t) = e^{\theta R}((1 + z_3^{(N)}(\rho, t)) Q(\rho) + z_1^{(N)}(\rho, t) f_1(\rho) + z_2^{(N)}(\rho, t) f_2(\rho)), \]
\[ S(y, t) = e^{\theta R}(\xi_1(\rho, t) f_1(\rho) + \xi_2(\rho, t) f_2(\rho) + \xi_3(\rho, t) Q(\rho)), \]
one can rewrite $\mathcal{E}_5$ as follows.
\[ \mathcal{E}_5 = \mathcal{E}_8 + \mathcal{E}_9 + \mathcal{E}_{10}, \]
\[ \mathcal{E}_8 = -4 \int_{\mathbb{R}^+} d\rho \rho \frac{h_1}{\rho} \xi_2 \partial_\rho \xi_3, \]
\[ \mathcal{E}_9 = -2 \int_{\mathbb{R}^+} d\rho \rho (\partial_\rho z^{(N)} \times \partial_\rho \xi, \xi), \quad z^{(N)} = (z_1^{(N)}, z_2^{(N)}, z_3^{(N)}), \quad \xi = (\xi_1, \xi_2, \xi_3), \]
\[ \mathcal{E}_{10} = 2 \int_{\mathbb{R}^+} d\rho \rho (z^{(N)} \times l, \xi), \]
where
\[ l = \left( -\frac{1}{\rho^2} \xi_1 - \frac{2h_1}{\rho} \partial_\rho \xi_3, -\frac{1}{\rho^2} \xi_2, \kappa(\rho) \xi_3 + \frac{2h_1}{\rho} \partial_\rho \xi_1 - \frac{2h_1 h_3}{\rho^2} \partial_\rho \xi_1 \right). \]
Clearly,

$$|l| \leq C \rho^{-2}(|\zeta| + |\partial_\rho \zeta|).$$

Therefore,

$$|E_{10}| \leq C t^{2\nu} \|S\|_{H^1}^2.$$

(3.11)

Consider $E_8$. It follows from

$$2(\zeta, k + z^{(N)}) + |\zeta|^2 = 0,$$

(3.12)

that

$$|\partial_\rho \zeta| \leq C (|\partial_\rho z^{(N)}| |\zeta| + |z^{(N)}| |\partial_\rho \zeta| + |\partial_\rho \zeta| |\zeta|).$$

As a consequence,

$$|E_8| \leq C \left[ t^{2\nu} \|S\|_{H^1}^2 + \|\nabla S\|_{L^2}^3 \right].$$

(3.13)

Consider $E_9$. Denote $e_0 = k + z^{(N)}$ and write $\zeta = \zeta^{\perp} + \mu e_0$, $\mu = (\zeta, e_0)$. It follows from (3.12) that

$$|\mu| \leq C |\zeta|^2,$$

$$|\mu_\rho| \leq C |\zeta| |\partial_\rho \zeta|.$$

Therefore, $E_9$ can be written as

$$E_9 = -2 \int_{\mathbb{R}^+} d\rho \rho (\partial_\rho \zeta^{\perp} \times \zeta^{\perp}, \partial_\rho e_0) + O(\|S\|_{H^1}^2, \|\nabla S\|_{L^2}).$$

(3.14)

Let $e_1$, $e_2$ be a smooth orthonormal basis of the tangent space $T_{e_0}S^2$ that verifies $e_2 = e_0 \times e_1$. Then the expression $(\partial_\rho \zeta^{\perp} \times \zeta^{\perp}, \partial_\rho e_0)$ can be written as follows:

$$(\partial_\rho \zeta^{\perp} \times \zeta^{\perp}, \partial_\rho e_0) = (\zeta^{\perp}, \partial_\rho e_0) \left[ (\zeta^{\perp}, e_2)(\partial_\rho e_0, e_1) - (\zeta^{\perp}, e_1)(\partial_\rho e_0, e_2) \right],$$

which leads to the estimate

$$\left| \int_{\mathbb{R}^+} d\rho \rho (\partial_\rho \zeta^{\perp} \times \zeta^{\perp}, \partial_\rho e_0) \right| \leq C \|\partial_\rho z^{(N)}\|_{L^\infty} J_0(t) \leq C t^{2\nu} J_0(t).$$

(3.15)

Combining (3.11), (3.13), (3.14), (3.15) we obtain

$$\left| \frac{d}{dt} J_0(t) \right| \leq C \left[ t^{-1} \|S\|_{H^1}^2 + t^{-1-2\nu} \|\nabla S\|_{L^2}^2 + t^{2N-2\nu} \right].$$

(3.16)
3.1.3. Control of the weighted $L^2$ norm. Using (3.4) to compute the derivative $\frac{d}{dt} \| yS(t) \|_{L^2}$, we obtain

$$t^{1+2\nu} \frac{d}{dt} \| y|S(t)\|_{L^2}^2 = -4 \int dy_i \left( U^{(N)} \times \partial_i S, S \right)$$

$$- 2 \int dy |y|^2 (\partial_i U^{(N)} \times \partial_i S, S)$$

$$- 2(1 + 2\nu)t^{2\nu} \| yS(t) \|_{L^2}^2 + 2 \int dy |y|^2 (R^{(N)}, S).$$

Here and below $\partial_j$ stands for $\partial y_j$, the summation over the repeated indexes being assumed.

As a consequence, we get

$$\left| \frac{d}{dt} \| y|S(t)\|_{L^2}^2 \right| \leq \frac{C}{t} \left[ \| y|S(t)\|_{L^2}^2 + t^{-4\nu} \| S \|_{H^1}^2 + t^{2N-4\nu} \right].$$

(3.17)

3.1.4. Control of the higher regularity. In addition to (3.5), assume that

$$\| S(t) \|_{H^3} + \| y|S(t)\|_{L^2} \leq t^{2N/5}.$$

(3.18)

We will control $\dot{H}^3$ norm of the solution by means of $\| \nabla S_t \|_{L^2}$. More precisely, consider the functional

$$J_3(t) = t^{2+4\nu} \int dx |\nabla s_t(x, t)|^2 + t^{1+2\nu} \int dx \kappa (t^{-1/2-\nu} x) |s_t(x, t)|^2,$$

where $s(x, t)$ is defined by

$$s(x, t) = e^{\alpha(t)R} S(\lambda(t) x, t).$$

Write $s_t(x, t) = e^{\alpha(t)R} \lambda^2(t) g(\lambda(t) x, t)$. In terms of $g$, $J_3$ can be written as $J_3(t) = \int dy |\nabla g(y, t)|^2 + \int dy \kappa(\rho) |g(y, t)|^2$. Let us compute the derivative $\frac{d}{dt} J_3(t)$. Clearly, $g(y, t)$ solves

$$t^{1+2\nu} g_t + \alpha_0 t^{2\nu} R g - (v + \frac{1}{2})t^{2\nu} (2 + y \cdot \nabla) g$$

$$= (S + U^{(N)}) \times \Delta g + g \times (\Delta U^{(N)} + \Delta S)$$

$$+ (U^{(N)} \times \Delta U^{(N)} - R^{(N)}) \times \Delta S$$

$$+ S \times \Delta(U^{(N)} \times \Delta U^{(N)} - R^{(N)}) + t^{2+4\nu} r^{(N)}.$$

(3.19)

Therefore, we get

$$t^{1+2\nu} \frac{d}{dt} J_3(t) = (2 + 4\nu)t^{2\nu} \| \nabla g \|_{L^2}^2 + (\frac{1}{2} + \nu)t^{2\nu} \int (2\kappa - \rho \kappa') |g|^2 dy$$

$$+ E_1 + E_2 + E_3 + E_4 + E_5,$$

(3.20)
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where

\[
E_1 = -2 \int dy (g \times \Delta \chi^{(N)} \Delta g) + 2 \int dy \kappa (\chi^{(N)} \times \Delta g, g),
\]

\[
E_2 = -2 \int dy ((U^{(N)} \times U^{(N)} - R^{(N)}) \times \Delta S, \Delta g)
+ 2 \int dy (\Delta (U^{(N)} \times U^{(N)} - R^{(N)}) \times S, \Delta g)
+ 2 \int dy \kappa ((U^{(N)} \times U^{(N)} - R^{(N)}) \times S, g)
- 2 \int dy \kappa (\Delta (U^{(N)} \times U^{(N)} - R^{(N)}) \times S, g),
\]

\[
E_3 = -2 \int dy (g \times \Delta S, \Delta g),
\]

\[
E_4 = 2 \int dy \kappa (S \times \Delta g, g),
\]

\[
E_5 = -2 t^{2+4v} \int dy (r_t, \Delta g) + 2 t^{2+4v} \int dy \kappa (r_t, g).
\]

The terms \(E_j, j = 1, 4, 5\) can be estimated as follows.

\[
|E_1| \leq C t^{2v} \|g\|_{H^1}^2,
\]

\[
|E_4| \leq C \|g\|_{H^3}^2 \|S\|_{H^3} \leq C t^{2v} \|g\|_{H^1}^2,
\]

\[
|E_5| \leq C(t^{2v} \|g\|_{H^1}^2 + t^{2N+4+4v}_0),
\]

provided \(N\) is sufficiently large and \(t \leq t_0\) with some \(t_0 = t_0(N) > 0\).

For \(E_2\) we have

\[
|E_2| \leq C(\|\Delta \chi^{(N)}\|_{W^{2,\infty}} + \|R^{(N)}\|_{H^3}) \|g\|_{H^1} \|S\|_{H^3}
+ C \|\langle y \rangle^{-1} \nabla \Delta ^2 \chi^{(N)}\|_{L^\infty} \|\nabla g\|_{L^2} \|\langle y \rangle S\|_{L^2}.
\]

As a consequence,

\[
|E_2| \leq C t^{2v} (\|g\|_{H^1} \|S\|_{H^3} + \|\nabla g\|_{L^2} \|\langle y \rangle S\|_{L^2}).
\]

Note that since

\[
g = (U^{(N)} + S) \times \Delta S + S \times \Delta U^{(N)} + R^{(N)},
\]

the bootstrap assumption (3.18) implies

\[
\|g\|_{L^2} \leq C(\|S\|_{H^2} + \|R^{(N)}\|_{L^2}),
\]

\[
\|\nabla g\|_{L^2} \leq C(\|S\|_{H^3} + \|\nabla R^{(N)}\|_{L^2}).
\]

Therefore, (3.21), (3.22) can be rewritten as

\[
|E_1| + |E_2| + |E_4| + |E_5| \leq C t^{2v} \|\|S\|_{H^3}^2 + (\|S\|_{H^3} + t^{N+1+2v}) \|\langle y \rangle S\|_{L^2}
+ C t^{2N+1+4v}.\]
Consider $E_3$. One has
\[
g \times \Delta S = (U^{(N)} + S, \Delta S) \Delta S - |\Delta S|^2 (U^{(N)} + S) \\
+ (S \times \Delta U^{(N)} + R^{(N)}) \times \Delta S, \\
\Delta g = (U^{(N)} + S) \times \Delta^2 S + Y,
\]
\[
Y = 2(\partial_j U^{(N)} + \partial_j S) \times \Delta \partial_j S + S \times \Delta^2 U^{(N)} \\
+ 2 \partial_j S \times \Delta \partial_j U^{(N)} + \Delta R^{(N)}.
\]
(3.26)

Therefore, one can write $E_3$ as $E_3 = E_6 + E_7 + E_8$, where
\[
E_6 = -2 \int dy(U^{(N)} + S, \Delta S)(\Delta S, \Delta g), \\
E_7 = 2 \int dy|\Delta S|^2(U^{(N)} + S, \Delta g) = 2 \int dy|\Delta S|^2(U^{(N)} + S, Y), \\
E_8 = -2 \int dy((S \times \Delta U^{(N)}) + R^{(N)}) \times \Delta S, \Delta g).
\]

For $E_6$ we have:
\[
E_6 = 2 \int dy[(\Delta U^{(N)}, S) + 2(\partial_j U^{(N)}, \partial_j S) + (\partial_j S, \partial_j S)](\Delta S, \Delta g) \\
= -2 \int dy[(\Delta U^{(N)}, S) + 2(\partial_j U^{(N)}, \partial_j S) + (\partial_j S, \partial_j S)](\Delta \partial_k S, \partial_k g) \\
- 2 \int dy(\Delta S, \partial_k g)\partial_k[(\Delta U^{(N)}, S) + 2(\partial_j U^{(N)}, \partial_j S) + (\partial_j S, \partial_j S)].
\]

As a consequence, one obtains:
\[
|E_6| \leq C \|S\|^2_{H^3} \|g\|_{H^1} \leq Ct^{2\nu} \|S\|^2_{H^3}. \tag{3.27}
\]

Consider $E_7$. From (3.26) we have
\[
\|Y\|_{L^2} \leq C(\|S\|_{H^3} + t^N).
\]
Therefore, we obtain:
\[
|E_7| \leq Ct^{2\nu} \|S\|^2_{H^3}. \tag{3.28}
\]

Finally, $E_8$ can be estimated as follows
\[
|E_8| \leq C \|g\|_{H^1} (\|S\|^2_{H^3} + t^N \|S\|_{H^3}) \leq Ct^{2\nu} \|S\|^2_{H^3} + C t^{3N}. \tag{3.29}
\]

Combining (3.27), (3.29), (3.28) we get
\[
|E_3| \leq C(t^{2\nu} \|S\|^2_{H^3} + t^{3N}), \tag{3.30}
\]

which together with (3.25) gives
\[
\left| \frac{d}{dt} J_3(t) \right| \leq \frac{C}{t} \left[ \|S\|^2_{H^3} + (\|S\|_{H^3} + t^{N+1+2\nu}) \|y|S\|_{L^2} \right] + Ct^{2N+2\nu}. \tag{3.31}
\]
3.1.5. **Proof of Proposition 3.1.** To prove the proposition it is sufficient to show that (3.5), (3.18) implies (3.2), (3.3).

Under the bootstrap assumption (3.18), (3.10), (3.16) become
\[ \left| \frac{d}{dt} J_1(t) \right| + \left| \frac{d}{dt} J_0(t) \right| \leq C t^{-1} \| S \|_{H^1}^2 + C t^{2N-2\nu}, \quad \forall t \leq t_0, \tag{3.32} \]
provided \( N \) is sufficiently large, \( t_0 \) sufficiently small.

Note that for \( c_0 > 0 \) sufficiently large one has \( \| S \|_{H^1}^2 \leq J_1 + c_0 J_0 \). Therefore, denoting \( J(t) = J_1(t) + c_0 J_0(t) \) one can rewrite (3.32) as
\[ \left| \frac{d}{dt} J(t) \right| \leq C t^{-1} J(t) + C t^{2N-2\nu}. \tag{3.33} \]

Integrating this inequality with zero initial condition at \( t_1 \) one gets
\[ J(t) \leq C N t^{2N+1-2\nu}, \quad \forall t \in [t_1, t_0], \tag{3.34} \]
provided \( N \) is sufficiently large. As a consequence, we obtain
\[ \| S \|_{H^1}^2 \leq C N t^{2N+1-2\nu}, \quad \forall t \in [t_1, t_0]. \tag{3.35} \]

Consider \( \| |y| S(t) \|_{L^2} \). From (3.17), (3.35) we have
\[ \left| \frac{d}{dt} \| |y| S(t) \|_{L^2}^2 \right| \leq C t \left[ \| |y| S(t) \|_{L^2}^2 + t^{2N+1-6\nu} \right]. \tag{3.36} \]

Integrating this inequality and assuming that \( N \) is sufficiently large, we get
\[ \| |y| S(t) \|_{L^2}^2 \leq \frac{C}{N} t^{2N+1-6\nu}, \quad \forall t \in [t_1, t_0], \tag{3.37} \]
which gives in particular,
\[ \| x |S(t) \|_{L^2}^2 \leq t^{N/2}, \quad \forall t \in [t_1, t_0]. \tag{3.38} \]

We next consider \( \| \nabla \Delta S(t) \|_{L^2(\mathbb{R}^2)} \). It follows from (3.23), (3.18) that for any \( j = 1, 2 \)
\[ \| \partial_j g - (U^{(N)} + S) \times \Delta \partial_j S \|_{L^2} \leq C \| S \|_{H^2(\mathbb{R}^2)} + t^{N+1+2\nu}. \tag{3.39} \]

Note also that since \( |U^{(N)} + S| = 1 \), we have
\[ |(U^{(N)} + S) \times \Delta \partial_j S|^2 = |\Delta \partial_j S|^2 - (U^{(N)} + S, \Delta \partial_j S)^2, \]

\[ (U^{(N)} + S, \Delta \partial_j S) = - (\Delta U^{(N)} + \Delta S, \partial_j S) - \Delta (\partial_j U^{(N)}, S) \]
\[ - 2(\partial_k U^{(N)} + \partial_k S, \partial^2_{jk} S), \]

which together with (3.18) gives
\[ \| \Delta \partial_j S \|^2_{L^2} - \| (U^{(N)} + S) \times \Delta \partial_j S \|^2_{L^2} \leq C \| S \|^2_{H^2}. \tag{3.40} \]
Consider the functional $\tilde{J}_3(t) = J_3(t) + c_1 J_0(t)$. It follows from (3.24), (3.39), (3.40) that for $c_1 > 0$ sufficiently large we have

$$c_2 \|S\|_{H^3}^2 - C t^{2N+1+2v} \leq \tilde{J}_3(t) \leq C (\|S\|_{H^3}^2 + t^{2N+1+2v}),$$

with some $c_2 > 0$.

From (3.31), (3.32), (3.37) one gets

$$\left| \frac{d}{dt} \tilde{J}_3(t) \right| \leq C \left[ t^{-1} (\|S\|_{H^3(\mathbb{R}^2)}^2 + \|y|S\|_{L^2(\mathbb{R}^2)}^2) + t^{2N-2v} \right]$$

(3.42)

Integrating this inequality between $t_1$ and $t$ and observing that $\tilde{J}_3(t_1) = t_1^{2+4v} \int dx |\nabla r(N)(x, t_1)|^2 + t_1^{1+2v} \int dx \kappa (t^{-1/2+v} x) |r(N)(x, t_1)|^2$, and therefore, $|\tilde{J}_3(t)| \leq C t_1^{2N+1+2v}$, we obtain

$$\tilde{J}_3(t) \leq C t^{2N+1-6v}, \quad \forall t \in [t_1, t_0].$$

Combining this inequality with (3.41), one gets

$$\|S\|_{H^3(\mathbb{R}^2)}^2 \leq C t^{2N+1-6v}, \quad \forall t \in [t_1, t_0],$$

which implies that

$$\|s\|_{H^3(\mathbb{R}^2)}^2 \leq t^{N/2}, \quad \forall t \in [t_1, t_0].$$

This concludes the proof of Proposition 3.1.

3.2. Proof of the theorem. The proof of the theorem is now straightforward. Fix $N$ such that Proposition 3.1 holds. Take a sequence $(t^j)$, $0 < t^j < t_0$, $t^j \to 0$ as $j \to \infty$. Let $u_j(x, t)$ be the solution of

$$\partial_t u_j = u_j \times \Delta u_j, \quad t \geq t^j,$n

$$u_j|_{t=t^j} = u(N)(t^j).$$

(3.43)

By Proposition 3.1, for any $j$, $u_j - u(N) \in C([t^j, t_0], H^3)$ and satisfies

$$\|u_j(t) - u(N)(t)\|_{H^3} + \|x(u_j(t) - u(N)(t))\|_{L^2} \leq 2 t^{N/2}, \quad \forall t \in [t^j, t_0].$$

(3.44)

This implies in particular, that the sequence $u_j(t_0) - u(N)(t_0)$ is compact in $H^2$ and therefore after passing to a subsequence we can assume that $u_j(t_0) - u(N)(t_0)$ converges in $H^2$ to some 1-equivariant function $w \in H^3$, with $\|w\|_{H^3} \leq \delta^{2v}$, $|u(N)(t_0) + w| = 1$.

Consider the Cauchy problem

$$u_t = u \times \Delta u, \quad t \leq t_0,$n

$$u|_{t=t_0} = u(N)(t_0) + w.$$n

(3.45)

By the local well-posedness, (3.45) admits a unique solution $u \in C((t^*, t_0], H^1 \cap H^3)$ with some $0 \leq t^* < t_0$. By $H^1$ continuity of the flow (see [10]), $u_j \to u$ in $C((t^*, t_0], H^1)$, which together with (3.44) gives

$$\|u(t) - u(N)(t)\|_{H^3} \leq 2 t^{N/2}, \quad \forall t \in (t^*, t_0).$$

(3.46)

This implies that $t^* = 0$ and combined with Proposition 2.1 gives the result stated in Theorem 1.1.
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