ERGODIC PROPERTIES OF A MODEL FOR TURBULENT DISPERSION OF INERTIAL PARTICLES

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Abstract. We study a simple stochastic differential equation that models the dispersion of close heavy particles moving in a turbulent flow. In one and two dimensions, the model is closely related to the one-dimensional stationary Schrödinger equation in a random \( \delta \)-correlated potential. The ergodic properties of the dispersion process are investigated by proving that its generator is hypoelliptic and using control theory.

1. INTRODUCTION

Transport by turbulent flows belongs to phenomena whose understanding is both important for practical applications and abounds in intellectual challenges. Unlike the reputedly difficult problem of turbulence per se, turbulent transport allows simple modeling that accounts, at least qualitatively, for many of its observable features. The simplest of such models study transport properties of synthetic random velocity fields with presupposed distributions that only vaguely render the statistics of realistic turbulent velocities. The advection by velocity fields of quantities like temperature or tracer density may be derived from the dynamics of the Lagrangian trajectories of fluid elements. In synthetic velocity ensembles, such dynamics is described by a random dynamical system. One of the best studied schemes of this type is the so called Kraichnan model based on a Gaussian ensemble of velocities decorrelated in time but with long-range spatial correlations \([8, 18]\). In this case, the random dynamical system that describes the Lagrangian flow is given by stochastic differential equations (SDE’s). It was successfully studied with the standard tools of the theory of random dynamical systems, but it also led to non-trivial extensions of that theory \([12, 19, 20]\).

The problem of turbulent transport of matter composed of small but heavy particles (like water droplets in turbulent atmosphere) may be also studied by modeling turbulent velocities by a random synthetic ensemble, but it requires a modification of the previous approach. The reason is that heavy particles do not follow Lagrangian trajectories due to their inertia. On the other hand, the assumptions of the time decorrelation of random velocities may be more realistic for inertial particles on scales where the typical relaxation time of particle trajectories (called the Stokes time) is much longer than the typical correlation time of fluid velocities. There have been a number of papers that pursued the study of dynamics of inertial particles with various simplifying assumptions, see e.g. \([1–7, 9, 17, 22, 23, 29, 33]\). The primary focus of those studies, combining analytical and numerical approaches, was the phenomenon of intermittent clustering of inertial particles transported by turbulent flow. A good understanding of that phenomenon is of crucial importance for practical applications.

The aim of the present article is to show that the simplest among the models of inertial particles dynamics are amenable to rigorous mathematical analysis. More concretely, we study the SDE’s that describe the pair dispersion of close inertial particles in shortly correlated moderately turbulent homogeneous and isotropic \(d\)-dimensional velocity fields (not necessarily compressible). Such models were discussed in some detail in \([2, 17, 23, 24, 29, 33]\). In particular, it was noted in \([33]\) that the \(d=1\) version of the model is closely related to the one-dimensional stationary Schrödinger equation with \(\delta\)-correlated potential studied already in the sixties of the last century \([14]\) as a model for Anderson localization. As was stressed in \([17]\), the \(d=2\) model for the inertial particle dispersion is also related to the one-dimensional stationary Schrödinger equation, but this time with \(\delta\)-correlated complex potential. The models for dispersion were used to extract information about the (top) Lyapunov exponent for the inertial particles which is a rough measure of the tendency of particles to separate or to cluster \([2, 33]\). The numerical calculations of the Lyapunov exponents in two or more dimensional models of particle dispersion presumed certain ergodic properties that seemed consistent with results of the simulations. Here, we shall establish those properties rigorously by showing the hypoellipticity of the generator of the Markov process solving the corresponding SDE and by proving the irreducibility of the process with the help of control theory. For a quick introduction to such, by now standard, methods, we refer the reader to \([11, 30]\). More information about the ergodic theory of Markov processes may be found in the treatise \([25]\).
The paper is organized as follows. In Sec. 2 we present the SDE modeling the inertial particle dispersion. In Sec. 3 we recall its relation to models of one-dimensional Anderson localization. Sec. 4 establishes the hypoelliptic properties of the generator of the dispersion process. In Sec. 5 we introduce the (real-)projective version of the dispersion process whose compact space of states may be identified with the \((2d-1)\)-dimensional sphere \(S^{2d-1}\). Sec. 6 is devoted to proving that the dispersion process is controllable. Together with the hypoelliptic properties of the generator, this implies that the projectivized version of the process has a unique invariant probability measure with a smooth strictly positive density. The analytic expression for such a measure may be written down explicitly in \(d = 1\) but not in higher dimensions. The smoothness and strict positivity of its density provides, however, in conjunction with the isotropy assumption, valuable information about the equal-time statistics of the projectivized dispersion. The isotropy permits to project further the projectivized dispersion to the quotient space \(S^{2d-1}/SO(d)\). For \(d = 2\), this space may be identified with the complex projective space \(\mathbb{CP}^1 = \mathbb{C} \cup \infty\) and the projected process with the complex-projectivized dispersion. In fact, the point at infinity may be dropped from \(\mathbb{CP}^1\) since the complex-projectivized dispersion process stays in \(\mathbb{C}\) with probability one. For \(d \geq 3\), the quotient space \(S^{2d-1}/SO(d)\) is not smooth but has an open dense subset that may be identified with the complex upper-half-plane that the projected process never leaves. These non-explosive behaviors are established in Sec. 8 by constructing a Lyapunov function with appropriate properties. Finally, in Sec. 9 and Appendix B we demonstrate how the established ergodic properties of the projectivized dispersion process lead to the formulae for the top Lyapunov exponent for inertial particles that were used in the physical literature. Appendix A derives a formula, used in the main text, that expresses the \(SO(2d)\)-invariant measure on \(S^{2d-1}\) in terms of \(SO(d)\) invariants.

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2. BASIC EQUATIONS

The motion in a turbulent flow of a small body of large density, called below an **inertial particle**, is well described by the equation \([1, 2, 22, 23, 33]\)

\[
\dot{r} = -\frac{1}{\tau} (\dot{r} - u(t, r)),
\]

where \(r(t)\) is the position of the particle at time \(t\) and \(u(t, r)\) is the fluid velocity field. Relation (2.1) is the Newton equation with the particle acceleration determined by a viscous friction force proportional to the relative velocity of the particle with respect to the fluid. Constant \(\tau\) is the Stokes time. Much of the characteristic features of the distribution of non-interacting inertial particles moving in the flow according to Eqs. (2.1) is determined by the dynamics of the separation \(\delta r(t) \equiv \rho(t)\), called **particle dispersion**, of very close trajectories. In a moderately turbulent flow, the particle dispersion evolves according to the linearized equation:

\[
\dot{\rho} = -\frac{1}{\tau} (\dot{\rho} - (\rho \cdot \nabla)u(t, r(t)))
\]

(2.2)

or, in the first-order form:

\[
\dot{\rho} = \frac{1}{\tau} \chi, \quad \dot{\chi} = -\frac{1}{\tau} \chi + (\rho \cdot \nabla)u(t, r(t)).
\]

(2.3)

For sufficiently heavy particles, the correlation time of \((\nabla u)(t, r(t))\) is short with respect to the Stokes time \(\tau\) and one may set in good approximation \([2]\)

\[
\nabla_j u^j(t, r(t)) \, dt = dS_j(t)
\]

(2.4)

where \(dS(t)\) is a matrix-valued white noise with the isotropic covariance

\[
\langle dS_j^k(t) dS_l^j(t') \rangle = D_{jl}^{ik} \delta(t - t') \, dt \, dt', \quad D_{jl}^{ik} = A \delta_{jl} \delta_{ik} + B (\delta_{ij} \delta_{lk} + \delta_{il} \delta_{jk}).
\]

(2.5)

Positivity of the covariance requires that

\[
A \geq |B|, \quad A + (d + 1)B \geq 0.
\]

(2.6)

Incompressibility implies that \(A + (d + 1)B = 0\), but we shall not impose it, in general. We shall only assume that \(A + 2B > 0\) for \(d = 1\) and that \(A > 0\) for \(d \geq 2\). After the substitution of (2.4), Eq. (2.2) becomes the linear SDE

\[
\dot{\rho} = -\frac{1}{\tau} \dot{\rho} + \frac{1}{\tau} \frac{dS(t)}{dt} \rho
\]

(2.7)
that may be written in the first order form in a more standard notation employing differentials as

\[
d\begin{pmatrix} \rho \\ \chi \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \frac{dS(t)}{dt} \\ \frac{1}{2} \frac{dS(t)}{dt} & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \chi \end{pmatrix}. \tag{2.8}
\]

We shall interpret the latter SDE using the Itô convention, but the Stratonovich convention would lead to the same process. The solution of Eq. (2.8) exists with probability 1 for all times and has the form

\[
\begin{pmatrix} \rho(t) \\ \chi(t) \end{pmatrix} = \mathcal{T} \exp \left[ \int_0^t \begin{pmatrix} 0 & \frac{1}{4} \frac{dS(s)}{ds} \\ \frac{1}{2} \frac{dS(s)}{ds} & 0 \end{pmatrix} ds \right] \begin{pmatrix} \rho(0) \\ \chi(0) \end{pmatrix}, \tag{2.9}
\]

where the time ordered exponential may be defined as the sum of its Wiener chaos decomposition

\[
\mathcal{T} \exp \left[ \int_0^t \begin{pmatrix} 0 & \frac{1}{4} \frac{dS(s)}{ds} \\ \frac{1}{2} \frac{dS(s)}{ds} & 0 \end{pmatrix} ds \right] = \sum_{n=0}^{\infty} \int_{0<s_1<\cdots<s_n<t} e^{(t-s_n)\begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{2} & 0 \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \end{pmatrix} e^{(s_n-s_{n-1})\begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{2} & 0 \end{pmatrix}} \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix} e^{(s_1-0)\begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{2} & 0 \end{pmatrix}} ds_1 \cdots ds_n \tag{2.10}
\]

that converges in the \( L^2 \)-norm for functionals of the white noise \( dS(t) \). The resulting stochastic process \( (\rho(t), \chi(t)) \equiv \rho(t) \) is Markov and has generator

\[
L = \frac{1}{\tau} (\chi \cdot \nabla \rho - \rho \cdot \nabla \chi) + \frac{1}{2} \sum_{i,j,k,l} \rho^i \rho^j D^{ik}_{jl} \nabla \chi^i \nabla \chi^j. \tag{2.11}
\]

In other words, for smooth functions \( f \),

\[
\frac{d}{dt} \langle f(\rho(t)) \rangle = \langle (Lf)(\rho(t)) \rangle \tag{2.12}
\]

for \( \langle \cdots \rangle \) denoting the expectation.

For the process \( \rho(t) \) given by Eq. (2.9), \( \rho(t) = 0 \) for all \( t \geq 0 \) if \( \rho(0) = 0 \). On the other hand, if \( \rho(0) \neq 0 \) then \( \rho(t) \neq 0 \) with probability 1 for all \( t \geq 0 \) so that we may restrict the space of states of the Markov process \( \rho(t) \) to \( \mathbb{R}^{2d} \setminus \{0\} = \mathbb{R}^{2d}_0 \).

3. RELATION TO ONE-DIMENSIONAL LOCALIZATION

In the lowest dimensions, there is a simple relation between the stochastic process \( \rho(t) = (\rho(t), \chi(t)) \) constructed above and simple models of Anderson localization in one space dimension. Let us set

\[
\psi(t) = e^{\frac{i}{\tau} t} \rho(t) \tag{3.1}
\]

exponentially blowing up the long-time values of \( \rho(t) \). Eq. (2.9) implies that

\[
-\frac{d^2}{dt^2} \psi + \frac{1}{\tau} \frac{dS(t)}{dt} \psi = -\frac{1}{4\tau^2} \psi, \tag{3.2}
\]

or, in the first order form,

\[
d\begin{pmatrix} \psi \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \frac{dS(t)}{dt} \\ \frac{1}{2} \frac{dS(t)}{dt} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \xi \end{pmatrix}. \tag{3.3}
\]

Similarly as before, the above SDE defines a Markov process. Clearly,

\[
(\psi(t), \xi(t)) \equiv e^{\frac{i}{\tau} t} (\rho(t), \chi(t) + \frac{1}{\tau} \rho(t)). \tag{3.4}
\]

Viewing \( t \) as the one-dimensional spatial coordinate, Eq. (3.4) takes the form of the vector-like stationary Schrödinger equation

\[
-\frac{d^2}{dx^2} \psi + V(t) \psi = E \psi, \tag{3.5}
\]

where \( V(t) = \frac{1}{\tau} \frac{dS(t)}{dt} \) plays the role of the random matrix-valued white-noise potential and \( E = -\frac{1}{4\tau^2} \) of the (negative) energy. In particular, in \( d = 1 \), \( \psi(t) \) is a real scalar function and so is the \( \delta \)-correlated potential

\[
V(t) = \frac{1}{\tau} \sqrt{A + 2B} \frac{dS(t)}{dt}, \tag{3.6}
\]
where $\beta(t)$ is the Brownian motion. The scalar version of Eq. (4.5) was studied in [14] as a model of one-dimensional Anderson localization, see also [21]. In $d = 2$, interpreting $\psi$ as a complex number $\psi^1 + i\psi^2$, one may replace the matrix valued $\delta$-correlated potential $V(t)$ in the SDE (3.5) with the complex valued one

$$V(t) = \frac{1}{2} \left( \sqrt{A + 2B} \frac{d\beta^1(t)}{dt} + i\sqrt{A} \frac{d\beta^2(t)}{dt} \right)$$

(3.7)

where $\beta^1(t)$, $\beta^2(t)$ are two independent Brownian motions (the two realizations of $V(t)$ lead to the Markov processes with the same generator and, consequently, with the same law). Consequently, as stressed in [17], Eq. (4.5) in $d = 2$ may be viewed as a model of localization for a one-dimensional non-hermitian random Schrödinger operator of the type not studied before.

4. HYPOELECTRIC PROPERTIES OF THE GENERATOR

The generator (2.11) of the particle dispersion process $p(t) = (p(t), \chi(t))$ has certain non-degeneracy properties which imply smoothness of the transition probabilities. Let us start with the following fact about the covariance (2.5) of the matrix-valued white noise $dS(t)$:

**Lemma 4.1.** We have

$$D_{ij}^{kl} = \sum_{m,n=1}^d (E\delta^m_i \delta^m_j + F\delta^m_i \delta^m_j + G\delta^m_i \delta^m_j)(E\delta^m_i \delta^m_j + F\delta^m_i \delta^m_j + G\delta^m_i \delta^m_j)$$

(4.1)

for

$$E = d^{-1} \left( -\sqrt{A + B} + \sqrt{A + (d + 1)B} \right),$$

$$F = \frac{1}{2} \left( \sqrt{A + B} + \sqrt{A - B} \right),$$

$$G = \frac{1}{2} \left( \sqrt{A + B} - \sqrt{A - B} \right).$$

(4.2)

**Proof.** The right hand side of Eq. (4.1) is:

$$(E^2 d + 2EF + 2EG)\delta^i_j \delta^k_l + (F^2 + G^2)\delta^i_j \delta^i_j + 2FG\delta^i_j \delta^i_j.$$  

(4.3)

Hence, in order to satisfy Eq. (4.1), we must have

$$A = F^2 + G^2, \quad B = E^2 d + 2E(F + G) = 2FG.$$  

(4.4)

The assumed values of $E$, $F$, $G$ solve these equations.

Define the vector fields

$$X_0 = \frac{1}{\tau} (\chi \cdot \nabla \rho - \chi \cdot \nabla \chi),$$

$$X_n^m = \sum_{i,j} \rho^j (E\delta^m_i \delta^m_j + F\delta^m_i \delta^m_j + G\delta^m_i \delta^m_j) \nabla \chi^i,$$

$$Y_n^m = -\sum_{i,j} \rho^j (E\delta^m_i \delta^m_j + F\delta^m_i \delta^m_j + G\delta^m_i \delta^m_j) \nabla \rho^i,$$

$$+ \sum_{i,j} (\chi^i + \rho^j) (E\delta^m_i \delta^m_j + F\delta^m_i \delta^m_j + G\delta^m_i \delta^m_j) \nabla \chi^i = \tau [X_0, X_n^m],$$

$$Z_n^m = -\sum_{i,j} (2\chi^i + \rho^j) (E\delta^m_i \delta^m_j + F\delta^m_i \delta^m_j + G\delta^m_i \delta^m_j) \nabla \rho^i,$$

$$+ \sum_{i,j} (\chi^i + \rho^j) (E\delta^m_i \delta^m_j + F\delta^m_i \delta^m_j + G\delta^m_i \delta^m_j) \nabla \chi^i = \tau [X_0, Y_n^m].$$

(4.5)

Using Eq. (4.1), one infers that Eq. (2.11) giving the generator $L$ may be rewritten in the form:

$$L = X_0 + \frac{1}{2} \sum_{m,n} (X_n^m)^2.$$  

(4.6)

**Remark 4.2.** The process $p(t)$ may be equivalently obtained from the SDE

$$dp = X_0(p) \, dt + \sum_{m,n=1}^d X_n^m(p) \, d\beta^m_n(t),$$

(4.7)

where $\beta^m_n(t)$ are independent Brownian motions. Here, we adopt the Stratonovich convention and hence the generator corresponding to that equation has the form (4.6) so that the latter SDE leads to a process with
the same law as \( p(t) \). The convention of the stochastic integral we choose, however, is insignificant as the process \( \rho(t) \) is of bounded variation.

In order to establish hypoelliptic properties of \( L \), we shall use the following non-degeneracy relation satisfied by the vector fields \( X^m_n, Y^m_n \) and \( Z^m_n \):

**Proposition 4.3.** Suppose that \( p = (\rho, \chi) \neq 0 \). Then the vectors

\[
X^m_n(p), \ Y^m_n(p), \ Z^m_n(p) \quad \text{with} \quad m, n = 1, \ldots, d
\]

span the 2d-dimensional space.

**Proof.** First suppose that \( \rho = 0 \) so that \( \chi \neq 0 \). We have

\[
Y^m_n(0, \chi) = \sum_{i,j} \chi^j (E\delta^i_j \delta^m_n + F\delta^i_m \delta^j_n + G\delta^i_n \delta^j_m) \nabla \chi^i.
\]

Let \( \phi \in \mathbb{R}^d \). Then

\[
\sum_{m,n} (\alpha \chi^m \chi^m + \beta \chi^m \phi^m) Y^m_n(0, \chi) = [E(\alpha \chi^2 + \beta \chi \cdot \phi) + F \alpha \chi^2 + G(\alpha \chi^2 + \beta \chi \cdot \phi)] \chi \cdot \nabla \chi
\]

\[
+ F \beta \chi^2 \phi \cdot \nabla \chi.
\]

Setting

\[
\alpha = -\frac{E + G}{(E + F + G)F} \frac{\chi \cdot \phi}{(\chi^2)^2}, \quad \beta = \frac{1}{F \chi^2}
\]

(note that \( F > 0 \) and \( E + F + G > 0 \)), we obtain

\[
\sum_{m,n} (\alpha \chi^m \chi^m + \beta \chi^m \phi^m) Y^m_n(0, \chi) = \phi \cdot \nabla \chi.
\]

Hence the vector \( \phi \cdot \nabla \chi \) is in the span of (4.8) for arbitrary \( \phi \). We have still to show that an arbitrary vector \( \sigma \cdot \nabla \rho \) is in that span. To this aim note that

\[
Z^m_n(0, \chi) = -2 \sum_{i,j} \chi^j (E\delta^i_j \delta^m_n + F\delta^i_m \delta^j_n + G\delta^i_n \delta^j_m) \nabla \rho^i + \sum_i (...) \nabla \chi^i.
\]

Proceeding as before, we show that an appropriate combination of \( Z^m_n(0, \chi) \) gives the vector

\[
\sigma \cdot \nabla \rho + \sum_i (...) \nabla \chi^i
\]

from which the term \( \sum_i (...) \nabla \chi^i \) may be removed by subtracting an appropriate combination of \( Y^m_n(0, \chi) \). That ends the proof of the claim of Proposition 4.3 for \( \rho = 0 \).

Suppose now that \( \rho \neq 0 \). Proceeding as before, we see that arbitrary vector \( \phi \cdot \nabla \chi \) may be obtained by taking an appropriate combination of the vectors \( X^m_n(\rho, \chi) \). Similarly, arbitrary vector \( \sigma \cdot \nabla \rho \) may be obtained as an appropriate combination of the vectors \( Y^m_n(\rho, \chi) \) and \( X^m_n(\rho, \chi) \). This completes the proof of Proposition 4.3. □

The representation (4.6) and Proposition 4.3 imply, in virtue of Hörmander’s theory \cite{11,16,27,28}, the following result:

**Corollary 4.4.** The operators \( L, L^\dagger, \partial_t - L, \partial_t - L^\dagger \) and \( 2\partial_t - L \otimes 1 - 1 \otimes L^\dagger \) are hypoelliptic\(^1\) on \( \mathbb{R}^d_\neq 0 \), \( \mathbb{R}_+ \times \mathbb{R}^d_\neq 0 \) and \( \mathbb{R}_+ \times \mathbb{R}^d_\neq 0 \times \mathbb{R}^d_\neq 0 \), respectively.

In particular, the hypoellipticity of \( 2\partial_t - L \otimes 1 - 1 \otimes L^\dagger \) implies that the transition probabilities of the dispersion process \( p(t) \),

\[
P_t(p_0, dp) = P_t(p_0, p) dp,
\]

have densities (annihilated by \( 2\partial_t - L \otimes 1 - 1 \otimes L^\dagger \)) that are smooth functions of \( (t, p_0, p) \) for \( t > 0 \) and away from the origin in \( \mathbb{R}^d_\neq 0 \).

\(^1\)A differential operator \( D \) on a domain \( \Omega \) is hypoelliptic if for all distributions \( f, g \) such that \( Df = g \), smoothness of \( g \) on an open subset \( U \subset \Omega \) implies smoothness of \( f \) on \( U \).
5. CONTROL THEORY AND IRREDUCIBILITY

The additional important property of the process \( p(t) \) restricted to \( \mathbb{R}^{2d}_{\neq 0} \) is its irreducibility assured by the strict positivity the smooth transition probability densities \( P_t(p_0, p) \) for all \( t > 0 \) and \( p_0 \neq 0 \neq p \).

The latter property results, according to Stroock-Varadhan’s Support Theorem [32], see also [30], from the controllability of the process \( p(t) \) on \( \mathbb{R}^{2d} \) that is established in the following

**Proposition 5.1.** For every \( T > 0 \) and \( p_0 \neq 0 \neq p_1 \) there exists a piecewise smooth curve \( [0, T] \ni t \mapsto (u_{n}^{m}(t)) \in \mathbb{R}^{d} \) such that the solution of the ODE

\[
\dot{p} = X_0(p) + \sum_{m,n} u_{m}^{n}(t) X_{m}^{n}(p)
\]

(5.1)

with the initial condition \( p(0) = p_0 \) satisfies \( p(T) = p_1 \).

**Proof.** First suppose that \( p_0 \neq 0 \neq p_1 \). Let \( [0, T] \ni t \mapsto \rho(t) \) be any curve such that

\[
\begin{align*}
\rho(0) &= \rho_0, \\
\rho(T) &= \rho_1,
\end{align*}
\]

(5.2)

and such that \( \rho(t) \neq 0 \) for all \( t \in [0, T] \). Set \( \chi(t) = \tau \dot{\rho}(t) \). Let

\[
\phi(t) = \tau \dot{\chi}(t) + \chi(t).
\]

(5.3)

Then the formula

\[
u_n^m = \frac{1}{\tau} (\alpha \rho^n \rho^m + \beta \rho^n \phi^m),
\]

(5.4)

where now

\[
\alpha = -\frac{E + G}{(E + F + G) F} \frac{\rho \cdot \dot{\phi}}{\rho^2}, \quad \beta = \frac{1}{F} \frac{\rho \cdot \phi}{\rho^2},
\]

(5.5)

defines smooth control functions \( [0, T] \ni t \mapsto (u_{m}^{n}(t)) \) such that Eq. (5.1) holds.

Now suppose that \( \rho_0 = 0 \neq \rho_1 \). Choose \( 0 < \epsilon < T \) and for \( 0 \leq t \leq \epsilon \), set

\[
\rho(t) = (1 - e^{-\frac{t}{\epsilon}}) \chi_0
\]

(5.6)

and \( \chi(t) = \tau \dot{\rho}(t) \). Then \( p(t) = (\rho(t), \chi(t)) \) satisfies Eq. (5.1) with \( u_{m}^{n}(t) \equiv 0 \) for \( 0 \leq t \leq \epsilon \), with the correct initial condition at \( t = 0 \). Note that

\[
(\rho(\epsilon), \chi(\epsilon)) = (1 - e^{-\frac{\epsilon}{\tau}}) \chi_0, e^{-\frac{\epsilon}{\tau}} \chi_0.
\]

(5.7)

Since, by the assumptions, \( \chi_0 \neq 0 \), we infer that \( \rho(\epsilon) \neq 0 \) and the solution of Eq. (5.1) for \( \epsilon \leq t \leq T \) may be constructed as in the previous point but taking \( \chi(t) \) as the initial conditions at \( t = \epsilon \).

Similarly, if \( \rho_0 \neq 0 = \rho_1 \) then set for \( T - \epsilon \leq t \leq T \)

\[
\rho(t) = (1 - e^{\frac{\epsilon}{\tau}}) \chi_1.
\]

(5.8)

and \( \chi(t) = \tau \dot{\rho}(t) \). Then \( p(t) = (\rho(t), \chi(t)) \) satisfies Eq. (5.1) with \( u_{m}^{n}(t) \equiv 0 \) for \( T - \epsilon \leq t \leq T \), with the correct final condition at \( t = T \). One has

\[
(\rho(T - \epsilon), \chi(T - \epsilon)) = (1 - e^{\frac{\epsilon}{\tau}}) \chi_1, e^{\frac{\epsilon}{\tau}} \chi_1.
\]

(5.9)

Since, by the assumptions, \( \chi_1 \neq 0 \) now, we infer that \( \rho(T - \epsilon) \neq 0 \) and the solution of Eq. (5.1) for \( 0 \leq t \leq T - \epsilon \) with \( \rho(t) \neq 0 \) may be constructed as in the first point but taking \( \chi(t) \) as the final condition at \( t = T - \epsilon \).

Finally, if \( \rho_0 = 0 = \rho_1 \), we combine the above solutions for \( 0 \leq t \leq \epsilon \) and \( T - \epsilon \leq t \leq T \) with vanishing \( u_{m}^{n} \) with the a solution with \( \rho(t) \neq 0 \) and appropriate \( u_{m}^{n}(t) \) for \( \epsilon \leq t \leq T - \epsilon \).

\[\square\]

**Remark 5.2.** Note that the solution \( p(t) \) of the ODE (5.1) satisfying \( p(0) = p_0 \neq 0 \) and \( p(T) = p_1 \neq 0 \) is everywhere nonzero.
6. PROJECTION OF THE DISPERSION TO $S^{2d-1}$

The generator $L$ of the process commutes with the multiplicative action of $\mathbb{R}_+$ on $\mathbb{R}^{2d}$ given by

$$ p \xrightarrow{\Theta_{\sigma}} \sigma p $$

for $\sigma > 0$. It follows that if $p(0) \neq 0$ then the projection

$$ [p(t)] \equiv \pi(t) $$

of the process $p(t)$ on the quotient space $\mathbb{R}^{2d}/\mathbb{R}_+$ is also a Markov process whose generator may be identified with $L$ acting on functions on $\mathbb{R}^{2d}_{\neq 0}$ that are homogeneous of degree zero. The quotient space $\mathbb{R}^{2d}_{\neq 0}/\mathbb{R}_+$ may be naturally identified with the sphere

$$ S^{2d-1} = \{ (\rho, \chi) \mid \rho^2 + \chi^2 = R^2 \} $$

for a fixed $R$ and we shall often use this identification below. The transition probabilities $P_t(\pi_0; d\pi)$ of the process $\pi(t)$ are obtained by projecting the original transition probabilities from $\mathbb{R}^{2d}_{\neq 0}$ to the quotient space. Note that the vector fields $X_0, X^m_n, Y^m_n, Z^m_n$ also commute with the action $\mathbb{R}_+$ so may be identified with vector fields on $\mathbb{R}^{2d}_{\neq 0}/\mathbb{R}_+$ and Eq. (6.3) still holds. Viewed as vector fields on $S^{2d-1}$, $X^m_n, Y^m_n$ and $Z^m_n$ still span at each point the tangent space to $S^{2d-1}$. It follows that the operators $L, L^1, \partial_t - L, \partial_t - L^1, 2\partial_t - L \otimes 1 - 1 \otimes L^1$ (with the adjoints defined now with respect to an arbitrary measure with smooth positive density on $S^{2d-1}$, e.g. the normalized standard $SO(2d)$-invariant one $\mu_0(d\pi)$) are still hypoelliptic and the transition probabilities of the projected process have smooth densities $P_t(\pi_0; \pi)$ with respect to $\mu_0(d\pi)$ for $t > 0$. Consequently, the process $\pi(t)$ is strongly Feller: for bounded measurable functions $f$ on $S^{2d-1}$, the functions

$$ (T_t f)(\pi_0) = \int_{S^{2d-1}} P_t(\pi_0; d\pi) f(\pi) = \int_{S^{2d-1}} P_t(\pi_0; \pi) f(\pi) \mu_0(d\pi) $$

are continuous (and even smooth) for $t > 0$. Besides, the projected process is still irreducible since $P_t(\pi_0; \pi) > 0$ for all $t > 0$ and $\pi_0, \pi \in S^{2d-1}$. The latter property follows from the relation between $P_t(\pi_0; \pi)$ and $P_t(\pi_0; \pi)$ and from the strict positivity of the latter away from the origin of $\mathbb{R}^{2d}$.

The gain from projecting the process $p(t)$ to the compact space $S^{2d-1}$ is that the projected process $\pi(t)$ has necessarily invariant probability measures $\mu(d\pi)$. In particular, each weak-topology accumulation point for $T \to \infty$ of the Cesaro means

$$ T^{-1} \int_0^T P_t(\pi_0; d\pi) dt $$

provides such a measure. Since the (a priori distributional) density $n(\pi)$ of an invariant measure is annihilated by $L^1$, the hypoellipticity of the latter operator assures that $n(\pi)$ is a smooth function. The invariance relation

$$ \int_{S^{2d-1}} P_t(\pi_0, \pi) n(\pi_0) \mu_0(d\pi_0) = n(\pi) $$

together with the strict positivity of $P_t(\pi_0, \pi)$ implies then the strict positivity of the density $n(\pi)$ of the invariant measure and, in turn, the uniqueness of the latter (different ergodic invariant measures have to have disjoint supports, so that there may be only one such measure), see e.g. [30] for more details. One obtains this way

**Theorem 6.1.** The projected process $\pi(t)$ has a unique invariant probability measure $\mu(d\pi)$ with a smooth strictly positive density $n(\pi)$.

The smoothness of the densities $P_t(\pi_0; \pi)$ implies by the Arzelà-Ascoli Theorem that the operators of the semigroup $T_t$ on the space $C(S^{2d-1})$ of continuous function on $S^{2d-1}$ with the sup-norm, defined by Eq. (6.4), are compact for $t > 0$. The uniqueness of the invariant measure implies then that the spectrum of $T_t$ is strictly inside the unit disk except for the geometrically simple eigenvalue 1 corresponding to the constant eigenfunctions, see [30]. It follows that the process $\pi(t)$ is exponentially mixing:

\[ \text{Probability measures on a compact space form a compact set in weak topology.} \]
Theorem 6.2.

\[ \left\langle f_1(\pi(t_1)) f_2(\pi(t_2)) \right\rangle \underset{t_1 \to \pm \infty}{\to} \int f_1(\pi) \mu(d\pi) \int f_2(\pi) \mu(d\pi) \]  

exponentially fast for continuous functions \( f_1, f_2 \).

7. PROPERTIES OF THE INVARIANT MEASURE

Due to the isotropy of the covariance \((2.5)\), the generator \( L \) of the process \( \pi(t) \) commutes with the action of the rotation group \( SO(d) \) induced on \( S^{d-1} \) by the mappings

\[ (\rho, \chi) \mapsto (O\rho, O\chi) \]  

for \( O \in SO(d) \). As a consequence, the process \( \pi(t) \) stays Markov when projected to the quotient space \( P_d = S^{d-1}/SO(d) \). The unique invariant measure \( \mu(d\pi) \) of the process \( \pi(t) \) has to be also invariant under \( SO(d) \) and its projection to \( P_d \) provides the unique invariant probability measure of the projected process\(^3\). The projected invariant measure may be expressed in terms of invariants of the \( SO(d) \)-action. Such invariants will be chosen as the following dimensionless combinations:

- for \( d = 1 \) where \( P_1 = S^1 \)
  \[ x = \frac{\chi}{\rho} \]  
  (7.2)

- for \( d = 2 \) where \( P_2 = \mathbb{CP}^1 \),
  \[ x = \frac{\rho \cdot \chi}{\rho^2} \quad \text{and} \quad y = \frac{\rho^1 \chi^2 - \rho^2 \chi^1}{\rho^2} \]  
  (7.3)

with \( z = x + iy \) providing the inhomogeneous complex coordinate of \( \mathbb{CP}^1 \),

- for \( d \geq 3 \),
  \[ x = \frac{\rho \cdot \chi}{\rho^2} \quad \text{and} \quad y = \frac{\sqrt{\rho^1 \chi^2 - (\rho \cdot \chi)^2}}{\rho^2} \]  
  (7.4)

Note that the right hand side of the \( d \geq 3 \) expression for \( y \) would give in \( d = 2 \) the absolute value of \( y \). The quotient spaces \( P_d \) are not smooth for \( d \geq 3 \).

7.1. \( d = 1 \) case. In one dimension, Eq. \((2.8)\) implies that

\[ dx = -\frac{1}{x}(x + x^2)dt + dS(t) . \]  
(7.5)

The invariant probability measure on \( S^1 \) is easily found \(^{14, 33} \) to have the form \( d\mu = \eta(x)dx \) with

\[ \eta(x) = Z^{-1} \left( e^{-\frac{1}{2}x + x^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x + x^2} dx' \right) dx , \]  
(7.6)

where \( Z \) is the normalization constant. Since the normalized rotationally invariant measure on \( S^1 = \{ (\rho, \chi) \mid \rho^2 + \chi^2 = R^2 \} \) has the form \( d\mu_0 = \frac{dx}{\pi(1 + x^2)} \), it follows from our general result that the density \( n(x) = \pi(1 + x^2)\eta(x) \) of the invariant measure relative to \( d\mu_0 \) must be smooth and positive at \( x = \infty \), i.e. at the origin when expressed in the variable \( x^{-1} \). In particular,

\[ \eta(x) = O(|x|^{-2}) \quad \text{for} \quad |x| \to \infty , \]  
(7.7)

which may also be easily checked directly.

In one dimension, the generator \( L \) given by Eq. \((2.11)\) acts on a function \( f(x) \) according to the formula:

\[ (Lf)(x) = -\frac{1}{x}(x^2 + x) \partial_x f(x) + \frac{1}{x}(A + 2B) \partial_x^2 f(x) . \]  
(7.8)

It coincides with the generator of the process satisfying the SDE \((7.5)\). The trajectories of the latter process with probability one explode to \( -\infty \) in finite time but, in the version of the process that describes the projectivized dispersion of the one-dimensional inertial particle, they re-enter immediately from \( +\infty \).

\(^3\)To see the uniqueness, note that averaging over the action of \( SO(d) \) maps \( C(S^{2d-1}) \) to \( C(P_d) \) and that dual map sends invariant measures for the projected process to invariant measures of \( \pi(t) \).
7.2. $d = 2$ case.

In two dimensions, the invariant measure on $S^3$ has to have the form
\[
d\mu = \frac{1}{2\pi} \eta(z, \bar{z}) d^2 z d\arg(\rho).
\] (7.9)

On the other hand, the $SO(4)$-invariant normalized measure on $S^3$ is
\[
d\mu_0 = \frac{1}{2\pi} \eta_0(z, \bar{z}) d^2 z d\arg(\rho)
\] (7.10)
with
\[
\eta_0(z, \bar{z}) = \frac{1}{\pi(1 + |z|^2)^2}.
\] (7.11)

It follows from the general result obtained above that the density of $d\mu$ relative to $d\mu_0$
\[
n(z, \bar{z}) = \frac{\eta(z, \bar{z})}{\eta_0(z, \bar{z})}
\] (7.12)

has to extend to a smooth positive function on $\mathbb{PC}^1$, i.e. to be smooth and positive at zero when expressed in the variables $(z^{-1}, \bar{z}^{-1})$. In particular,
\[
\eta(z, \bar{z}) = \mathcal{O}(|z|^{-4}) \quad \text{for} \quad |z| \to \infty.
\] (7.13)

The unique invariant probability measure of the Markov process obtained by projecting $\pi(t)$ from $S^3$ to $S^3/SO(2) = \mathbb{PC}^1$ has the form (7.9) with $\frac{d}{dz}d\arg(\rho)$ on the right hand side dropped. Note that the relation (7.13) implies that
\[
\int_{-\infty}^{\infty} \eta(x, y) dy = \mathcal{O}(|x|^{-3}) \quad \text{for} \quad |x| \to \infty.
\] (7.14)

by changing variables $y \mapsto \sqrt{1 + x^2}y$ in the integral. Such behavior was heuristically argued for and numerically checked in [2].

In two dimensions, the generator $L$ of Eq. (2.11) acts on $SO(d)$ invariant functions $f(x, y)$ according to the formula:
\[
(Lf)(x, y) = -\frac{1}{\tau}(x^2 - y^2 + x) \partial_x f(x, y) - \frac{1}{\tau}(2xy + y) \partial_y f(x, y) + \frac{1}{\tau}(A + 2B) \partial_x^2 f(x, y) + \frac{1}{\tau}A \partial_y^2 f(x, y).
\] (7.15)

It coincides with with the generator of the process $z(t) = (x + iy)(t)$ in the complex plane given by the SDE [29]
\[
dz = -\frac{1}{\tau}(z + z^2) dt + \sqrt{A + 2B} d\beta^1(t) + i\sqrt{A} d\beta^2(t),
\] (7.16)
where $\beta^1(t)$ and $\beta^2(t)$ are two independent Brownian motions.

7.3. $d \geq 3$ case.

Finally, in three or more dimensions, the invariant measure on $S^{2d-1}$ has to have the form
\[
d\mu = \eta(x, y) dx dy d[O],
\] (7.17)
where $O \in SO(d)$ is the rotation matrix such that $O^{-1} \rho$ is along the first positive half-axis in $\mathbb{R}^d$ and $O^{-1} \chi$ lies in the half-plane spanned by the first axis and the second positive half-axis. Note that, generically, $O$ is determined modulo rotations in $(d - 2)$ remaining directions. $d[O]$ stands for the normalized $SO(d)$-invariant measure on $SO(d)/SO(d-2)$. In the same notation, the $SO(2d)$-invariant normalized measure on $S^{2d-1}$ takes the form
\[
d\mu_0 = \eta_0(x, y) dx dy d[O].
\] (7.18)

for
\[
\eta_0(x, y) = \frac{(d - 1)2^{d-1}y^{d-2}}{\pi(1 + x^2 + y^2)^d},
\] (7.19)
as is shown in Appendix [A] As before, it follows from the general analysis that the function
\[
n(x, y) = \frac{\eta(x, y)}{\eta_0(x, y)}
\] (7.20)
Definition 8.1. We say that the solution of the form 
\[ \eta(x, y) = O(y^{-d}) \quad \text{for} \quad y \searrow 0 \] (7.21)
i.e. for \( \rho \) and \( \chi \) becoming parallel or \( \chi^2 \) becoming small and 
\[ \eta(x, y) = O(|x|^{-d}) \quad \text{for} \quad |x| \to \infty \] (7.22)
when \( \rho^2 \to 0 \) but the angle between \( \rho \) and \( \chi \) stays away from a multiple of \( \frac{\pi}{2} \). The smoothness and positivity of \( n(x, y) \) on \( S^{2d-1} \) imply (again by changing variables \( y \mapsto \sqrt{1+x^2}y \) in the integral) that now 
\[ \int_0^\infty \eta(x, y) \, dy = O(|x|^{-d-1}) \quad \text{for} \quad |x| \to \infty. \] (7.23)

A straightforward calculation shows that, in three or more dimensions, the action on \( L \) on \( SO(d) \)-invariant functions \( f(x, y) \) is given by a generalization of Eq. (7.15):
\[ (Lf)(x, y) = -\frac{1}{\tau}(x^2 - y^2 + x) \partial_x f(x, y) - \frac{1}{\tau}(2xy + y - \frac{\tau A(d-2)}{2y}) \partial_y f(x, y) + \frac{1}{\tau}(A + 2B) \partial^2_y f(x, y) + \frac{1}{\tau}A \partial^2_x f(x, y). \] (7.24)

It coincides with the generator of the process \( z(t) = (x + iy)(t) \) in the complex plane given by the SDE
\[ dz = -\frac{1}{\tau}(z + z^2 - i\frac{\tau A(d-2)}{2 \text{Im}(z)}) \, dt + \sqrt{A + 2B} \, d\beta^1(t) + i\sqrt{A} \, d\beta^2(t) \] (7.25)
which upon setting \( d = 2 \) reduces to the SDE (7.10). 8. ABSENCE OF EXPLOSION IN THE COMPLEX (HALF-)PLANE

Let us set
\[ Q_d = \begin{cases} \mathbb{R}^2 & \text{if} \quad d = 2, \\ \mathbb{H}_+ & \text{if} \quad d \geq 3, \end{cases} \] (8.1)
where
\[ \mathbb{H}_+ = \{(x, y) \mid y > 0\} \]
is the upper-half plane. Note that \( Q_d \) may be identified with an open dense subset of the quotient space \( P_d = S^{2d-1}/SO(d) \) using the \( SO(d) \)-invariants (7.3) or (7.4) on \( S^{2d-1} \). We shall often use the complex combination \( x + iy \) as a coordinate on \( Q_d \).

In the present section, we shall show that for \( d \geq 2 \) the unique solution of the SDE (7.25) starting from \( z \in Q_d \) remains in \( Q_d \) for all times \( t \geq 0 \) with probability one. This will also have to be the property of the projection of the process \( \pi(t) \) to the quotient space \( P_d = S^{2d-1}/SO(d) \) when described in the complex coordinate \( z = x + iy \). Indeed, the coincidence of the generators of the two processes will assure that they have the same law. Let us start by generalizing and simplifying (7.25).

Let \( w(t) = z(t) + 1/2 \), where \( z(t) \) solves (7.25) with \( z(0) = x + iy \in Q_d \). Clearly, \( w(t) \) satisfies an SDE of the form
\[ dw = \frac{1}{\tau} \left( -w^2 + \alpha + i\frac{\tau b(d-2)}{2 \text{Im}(w)} \right) \, dt + \sqrt{2\alpha_1} \, d\beta^1(t) + i\sqrt{2\alpha_2} \, d\beta^2(t), \] (8.2)
where \( \alpha = \alpha_1 + i\alpha_2 \in \mathbb{C} \), \( b > 0 \), \( \kappa_1 \geq 0 \), \( \kappa_2 > 0 \), and \( \beta^1(t) \) and \( \beta^2(t) \) are two independent Brownian motions. When \( d = 2 \), the term proportional to \( b(d-2) \) is absent from (8.2). When \( d \geq 3 \), we suppose that \( \tau b(d-2) \geq \kappa_2 \). Clearly in (7.25), all of these assumptions are met under the given substitution. Since \( w(t) \) is a horizontal shift of \( z(t) \), \( w(t) \) stays in \( Q_d \) with probability one for all times if and only if \( z(t) \) does. Employing methods of refs. (15,26,30,39), we shall estimate the time at which the process \( w(t) \) leaves \( Q_d \). To this end, it is easy to see that there exists a sequence of precompact open subsets \( \{O_n \mid n \in \mathbb{N}\} \) of \( Q_d \) such that
\[ O_n \uparrow Q_d \quad \text{as} \quad n \to \infty. \] Thus we may define stopping times:
\[ \tau_n = \inf\{s > 0 \mid w(s) \in O_n^c\}, \] (8.3)
for \( n \in \mathbb{N} \). Let \( \tau_\infty \) be the finite or infinite limit of \( \tau_n \) as \( n \to \infty \).

**Definition 8.1.** We say that the solution \( w(t) \) is non-explosive if
\[ P[\tau_\infty = \infty] = 1. \] (8.4)
Naturally, in order to show that \( w(t) \) remains in \( Q_d \) for \( t \geq 0 \) with probability one, it is enough to prove that \( w(t) \) is non-explosive.

Let \( M \) be the generator of the process \( w(t) = x(t) + iy_t \). We see that for \( f \in C^\infty(Q_d) \):
\[
(M_d f)(x, y) = -\frac{1}{c}(x^2 - y^2 - a_1 \partial_x f(x, y) - \frac{1}{c}(2xy - a_2 - \tau b(d - 2)y^{-1}) \partial_y f(x, y) + \frac{a_{12}}{c} \partial_{xy}^2 f(x, y),
\]
where the term \( \tau b(d - 2)y^{-1} \) is absent for \( d = 2 \). Let us define
\[
\partial Q_d = \begin{cases} 
\infty & \text{if } d = 2, \\
\{(x, y) \in \mathbb{R}^2 \ | \ y = 0 \} \cup \infty & \text{if } d \geq 3,
\end{cases}
\]
with \( \infty \) denoting the point compactifying \( \mathbb{R}^2 \). To ensure condition (8.4), it suffices to construct a (Lyapunov) function \( \Phi_d \in C^\infty(Q_d) \) that satisfies:

(I) \( \Phi_d(x, y) \geq 0 \) for all \( (x, y) \in Q_d \),

(II) \( \Phi_d(x, y) \to \infty \) as \( (x, y) \to \partial Q_d, \ (x, y) \in Q_d \),

(III) \( M_d \Phi_d(x, y) \leq C \Phi_d(x, y) \) for all \( (x, y) \in Q_d \), where \( C > 0 \) is a positive constant.

See, for example, [20]. We will show:

**Theorem 8.2.** If \( \kappa_1 \geq 0 \) and \( \tau b(d - 2) \geq \kappa_2 > 0 \) then there exists \( \Phi_d \in C^\infty(Q_d) \) that satisfies (I), (II), and
\[
M_d \Phi_d(x, y) \to -\infty \text{ as } (x, y) \to \partial Q_d, \ (x, y) \in Q_d.
\]

Given such \( \Phi_d \), clearly \( \Phi_d + 1 \) will satisfy (I), (III) and (IV). We will then have:

**Theorem 8.3.** Under the assumptions of Theorem 8.2, the solution \( w(t) \) of the SDE (8.2) stays in \( Q_d \) for all times \( t > 0 \) with probability one if \( w(0) = x + iy \in Q_d \).

**Corollary 8.4.** This implies the same result about the solution \( z(t) \) of the SDE (7.25) with \( A > 0 \) and \( A + 2B \geq 0 \).

The existence of the Lyapunov function with the properties asserted in Theorem 8.2 has another consequence. It allows to show that
\[
\lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{T} \int_0^T P_t(w, O_n) dt = 0
\]
for the SDE (8.2) and \( O_n \uparrow Q_d \) as before, implying the existence of an invariant measure on \( Q_d \), see Theorems 4.1 and 5.1 in Chapter III of [13]. If the generator of the process is elliptic, then the same tools that we used for the projectivized dispersion (i.e. hypoellipticity and control theory [30]) show that the invariant measure must have a smooth strictly positive density and be unique. This gives:

**Theorem 8.5.** Under the assumptions of Theorem 8.2, the system (8.2) on \( Q_d \) has an invariant measure which is unique and has a smooth strictly positive density if \( \kappa_1 > 0 \).

**Remark 8.6.** Theorem 8.5 allows to reaffirm and strengthen what has already been proven earlier since it implies the existence of an invariant measure for the system (7.25) if \( A > 0 \) and \( A + 2B \geq 0 \) and its uniqueness if \( A + 2B > 0 \). Given the non-explosivity result of Corollary 8.3, the approach taken earlier implied the existence and the uniqueness of an invariant measure for the system (7.25) under more stringent conditions: \( A > 0 \), \( A \geq |B| \) and \( A + (d + 1)B \geq 0 \).

The construction of the Lyapunov function \( \Phi_d \) is split up into two cases: \( d = 2 \) and \( d \geq 3 \). The existence of \( \Phi_d \) for \( d \geq 3 \) will be easy, given \( \Phi_2 \). Thus we shall first construct \( \Phi_2 \).

**8.1. \( d = 2 \) case.**

It is not easy to write down a globally defined function \( \Phi_2 \) that satisfies (I), (III), and (IV) in all of \( Q_2 = \mathbb{R}^2 \). This is because the signs of the coefficients of the vector fields in \( M_2 \) vary over different regions in \( \mathbb{R}^2 \). We shall thus construct functions that satisfy these properties in different regions, the union of which is \( \mathbb{R}^2 \). We shall then glue together these functions to form one single globally defined \( \Phi_2 \). One should note that this idea is similar in spirit to that of M. Scheutzow in [31]. Let \( r = \sqrt{x^2 + y^2} \). For the rest of Subsection 8.1 we will drop the use of the subscript 2 in \( M_2 \) and \( \Phi_2 \). We first need the following:

**Definition 8.7.** Let \( X \subset \mathbb{R}^2 \) be unbounded. We say that a function \( f(x, y) \to \pm \infty \) as \( r \to \infty \) in \( X \) if \( f(x, y) \to \pm \infty \) as \( (x, y) \to \infty \), \( (x, y) \in X \).
Proof. Let $X \subset \mathbb{R}^2$ be unbounded and let $\varphi \in C^\infty(X)$ satisfy

(i) $\varphi \geq 0$ for all $(x, y) \in X$,
(ii) $\varphi \to \infty$ as $r \to \infty$ in $X$,
(iii) $M\varphi \to -\infty$ as $r \to \infty$ in $X$.

We call $\varphi$ a Lyapunov function in $X$ corresponding to $M$ and denote

$$\mathcal{N}(\alpha, \kappa_1, \kappa_2, X) = \{ \text{Lyapunov functions in } X \text{ corresponding to } M \}.$$ 

We shall abbreviate “Lyapunov function” by LF.

Definition 8.9. Let $X \subset \mathbb{R}^2$ be unbounded and $f, g : X \to \mathbb{R}$. We shall say that $f$ is asymptotically equivalent to $g$ in $X$ and write $f \simeq_X g$ if

$$\lim_{r \to \infty} \frac{f(x, y)}{g(x, y)} = 1,$$

where the limit is taken only over points $(x, y) \in X$.

It is clearly sufficient to construct LFs in regions that cover $\mathbb{R}^2$, except, possibly, a large ball about the origin. The constructions will be done in a series of propositions. The possibly daunting multitude of parameters is designed to make the gluing possible. There is a total of five LFs in five different regions and the details that follow are not difficult to verify. The crucial LF is the fifth one, $\varphi_5$, defined in a region where explosion occurs in a nonrandom equation, i.e., when $\alpha = \kappa_1 = \kappa_2 = 0$ in (8.2).

Proposition 8.10. Let $X_1 = \{ x \geq 1 \} \subset \mathbb{R}^2$, $C_1 > 0$, and $\delta \in (0, 1/2)$. Define

$$\varphi_1(x, y) = C_1(x^2 + y^2)^{\delta/4}. \quad \text{(LF1)}$$

We claim that $\varphi_1 \in \mathcal{N}(\alpha, \kappa_1, \kappa_2, X_1)$ for all $\alpha \in \mathbb{C}$, $\kappa_1 \geq 0$, $\kappa_2 > 0$.

Proof. $\varphi_1$ is nonnegative everywhere in $\mathbb{R}^2$, hence everywhere in $X_1$. $\varphi \to \infty$ as $r \to \infty$ in all of $\mathbb{R}^2$, hence in all of $X_1$. It is easy to check that $\partial_{xx} \varphi_1$ and $\partial_{yy} \varphi_1$ both go to zero as $r \to \infty$. Thus dropping second order terms in the expression for $M\varphi_1$, we have

$$\tau M\varphi_1 \simeq_{X_1} -C_1\delta x(x^2 + y^2)^{\delta/4} + \frac{C_1\delta}{2} \frac{a_1 x + a_2 y}{(x^2 + y^2)^{1-\delta/4}}$$

$$\simeq_{X_1} -\frac{C_1\delta}{2} x(x^2 + y^2)^{\delta/4} \to -\infty \quad \text{(8.8)}$$

as $r \to \infty$ in $X_1$, since $x \geq 1$ in $X_1$.

We need a remark before we move onto the next region. Let $\mathbb{R} \subset \mathbb{R}^2$ be the real axis.

Remark 8.11. Let $f(x, y) = u(x, |y|)$ be a twice differentiable function in $X \setminus \mathbb{R}$. Then

$$(\tau M f)(x, y) = \kappa_1 u_{xx}(x, |y|) + \kappa_2 u_{y||y|}(x, |y|)$$

$$+ (y^2 - x^2 + a_1) u_x(x, |y|) + (-2x|y| + \text{sgn}(y)a_2) u_{y||y|}(x, |y|).$$

Proof. Apply the chain rule to the operator $\partial_{|y|}$. In the following arguments, often the function will be of the form $f(x, y) = u(x, |y|)$. The above remark will allow for simplifications in the argument for property (iii) in Definition 8.8.

Proposition 8.12. Let $C_2 > 0$, $\delta \in (0, 1/2)$ and

$$\varphi_2 = C_2(-x + |y|^{\delta/2}). \quad \text{(LF2)}$$

Then $\varphi_2 \in \mathcal{N}(\alpha, \kappa_1, \kappa_2, X_2)$ for all $\alpha \in \mathbb{C}$ and all $\kappa_1 \geq 0$, $\kappa_2 > 0$, where

$$X_2 = \{-2 \leq x \leq 2\} \cap \{|y| \geq 2^{3/\delta}\}.$$ 

Proof. $\varphi_2$ is indeed smooth in $X_2$ since $X_2$ is bounded away from $\mathbb{R}$. Note that the region was chosen so that $\varphi_2 \geq 0$ in $X_2$. Moreover, since $x$ is bounded in this region, $r \to \infty$ in $X_2$ if and only if $|y| \to \infty$. Hence, $\varphi_2 \to \infty$ in $X_2$. By Remark 8.11 and noting that $\partial_{xx} \varphi_2 = 0$ and that $\partial_{|y||y|}\varphi_2 \to 0$ as $|y| \to \infty$, we have

$$\tau M\varphi_2(x, y) \simeq_{X_2} C_2(x^2 - y^2 - a_1) + C_2 \frac{\delta}{2} (-2x|y| + \text{sgn}(y)a_2)|y|^{\delta/2 - 1}$$

$$\simeq_{X_2} -C_2 y^2 \to -\infty$$

as $r \to \infty$ in $X_2$. 

□
Proposition 8.13. Let $C_3 > 0$ and $\delta \in (0,1/2)$. Define
\[
\varphi_3 = C_3 \left( \frac{x^2 + y^2}{|y|^{3/2}} \right)^{\delta} \quad \text{(LF3)}
\]
on $X_3 = \{ x \leq -1 \} \cap \{ |y| \geq 1 \}$. Then $\varphi_3 \in \mathcal{N}(\alpha, \kappa_1, \kappa_2, X_3)$ for all $\alpha \in \mathbb{C}$ and all $\kappa_1 \geq 0, \kappa_2 > 0$.

Proof. Smoothness of $\varphi_3$ is not a problem in this region as we are bounded away from $\mathbb{R}$ in $X_3$. Clearly, $\varphi_3 \geq 0$ and note that $\varphi_3 \to \infty$ as $r \to \infty$ in $X_3$. After dropping the $\delta(\delta - 1)$-terms which are negative, we obtain:
\[
\tau M \varphi_3(x, y) \leq C_3 \delta \left( \frac{x^2 + y^2}{|y|^{3/2}} \right)^{\delta-1} \left[ |\varphi_3| - \frac{x^3}{|y|^{3/2}} \right] = \delta x \varphi_3 \to -\infty
\]
as $r \to \infty$ in $X_3$ since $x \leq -1$ in $X_3$. \hfill \Box

Proposition 8.14. Let $C_4 > 0$, $\eta > 1$ and $\delta \in (0,1/2)$. Define
\[
\varphi_4(x, y) = C_4 \left( \frac{|x|^{2\delta} + |y|^{2\delta}}{|y|^{\delta}} \right) \quad \text{(LF4)}
on X_4 = \{ x \leq -1 \} \cap \left\{ \eta \sqrt{\frac{3}{2} \delta + 1} \frac{1}{\sqrt{|x|}} \leq |y| \leq 2 \right\}.
\]
Then $\varphi_4 \in \mathcal{N}(\alpha, \kappa_1, \kappa_2, X_4)$ for all $\alpha \in \mathbb{C}$ and all $\kappa_1 \geq 0, \kappa_2 > 0$.

Proof. Note that $\varphi_4$ is smooth in $X_4$ since this region excludes both $x$ and $y$ axes. Moreover, $\varphi_4 \to \infty$ as $r \to \infty$ in $X_4$ since then $x$ must approach $\infty$ as $r \to \infty$, and $y$ is bounded above. Dropping insignificant terms in the expression for $M \varphi_4$, we see that in $X_4$:
\[
\tau M \varphi_4(x, y) \leq C_4 \left( -\delta \frac{|x|^{2\delta+1}}{|y|^{\delta}} + \kappa_2 \frac{3}{2} \delta \left( \frac{3}{2} \delta + 1 \right) \frac{|x|^{2\delta}}{|y|^{\delta+2}} + \delta |x||y|^{\delta/2} \right)
\]
\[
= C_4 \left( \kappa_2 \frac{3}{2} \delta \left( \frac{3}{2} \delta + 1 \right) \frac{|x|^{2\delta}}{|y|^{\delta+2}} \right) \leq -C_4 \delta (1-\eta^2) \frac{|x|^{2\delta+1}}{|y|^{\delta}} \to -\infty
\]
in $X_4$ as $r \to \infty$. \hfill \Box

Proposition 8.15. Let $C_5, \beta > 0$ and $E > 0$ such that $2\kappa_2 > E\beta$, let $\xi > 1$, and let
\[
\varphi_5(x, y) = C_5 (E|x|^\beta - y^2 |x|^{\beta+1}) \quad \text{(LF5)}
\]
be defined on
\[
X_5 = \{ x \leq -1 \} \cap \left\{ |y| \leq \frac{1}{\xi} \frac{E}{|x|} \right\}.
\]
Then $\varphi_5 \in \mathcal{N}(\alpha, \kappa_1, \kappa_2, X_5)$ for all $\alpha \in \mathbb{C}$ and all $\kappa_1 \geq 0$. 
Proof. The fact that \( \varphi_5 \) is smooth in \( X_5 \) is clear as \( x \leq -1 \) in \( X_5 \). Again by the choice of \( X_5 \), \( \varphi_5 \geq 0 \) and \( \varphi_5 \to \infty \) as \( r \to \infty \) in \( X_5 \). Dropping irrelevant terms in the expression for \( M \varphi_5 \), we see that in \( X_5 \):

\[
\tau M \varphi_5(x, y) \leq C_5 \left( k_5 E \beta (1 - 1)|x|^\beta - 2k_2 |x|^\beta + 1 + E \beta^2 |x|^\beta + 1 + (\beta + 1) y^1 |x|\right) + C_5 \left( \alpha_1 - E \beta |x|^\beta + 1 + (\beta + 1) y^2 |x| - 2a_2 y^1 |x|\right) \\
\approx_{X_5} C_5 \left( E \beta - 2k_2 \right) |x|^{\beta + 1} \to -\infty
\]

as \( r \to \infty \) in \( X_5 \), as \( |x| \) must approach \( \infty \) when \( r \to \infty \) in \( X_5 \). \( \square \)

We now have our desired LFs. It is not obvious, however, that the regions \( X_1, X_2, \ldots, X_5 \) cover \( \mathbb{R}^2 \) except, possibly, a bounded region about the origin. To assure that one has to show that \( X_4 \) and \( X_5 \) overlap. In order to make this more tangible, we will choose some of the parameters given in the previous propositions. With the choices that follow, however, we first need a lemma that says that varying the diffusion coefficients \((k_1, k_2)\) is permitted. This lemma will also be of crucial use later when we glue the LFs to form a globally defined \( \Phi \).

**Lemma 8.16.** Fix \( k_2 > 0 \) and suppose that \( \Phi \in \mathcal{N}(\alpha, k_1, k_2, \mathbb{R}^2) \) for all \( \alpha \in \mathbb{C} \) and all \( k_1 \geq 0 \). Then for every \( \nu_2 > 0 \), we can find \( \Psi \in \mathcal{N}(\alpha, \nu_1, \nu_2, \mathbb{R}^2) \) for all \( \alpha \in \mathbb{C} \) and all \( \nu_1 \geq 0 \).

For the proof of Lemma 8.16 we temporarily use the notation \( M^{\alpha}_{(\nu_1, \nu_2)} \) for the generator \( M \) given by (8.5).

Proof. Let \( \eta > 0 \) be such that \( \eta^3 \nu_2 = k_2 \). Define \( \Psi(x, y) = \Phi(\eta x, \eta y) \). For the function \( \Psi \), smoothness and properties (I) and (II) are immediate. Let \( s = \eta x \) and \( t = \eta y \). Then by the chain rule

\[
\tau M^{\alpha}_{(\nu_1, \nu_2)} \Psi(s, t) = \frac{\nu_1 M^{\alpha}_{(\nu_1, \nu_2)} \Phi(s, t)}{\nu_1 M^{\alpha}_{(\nu_1, \nu_2)} \Phi(s, t)} = \frac{\nu_1 M^{\alpha}_{(\nu_1, \nu_2)} \Phi(s, t)}{-\infty,}
\]

as \( r = \sqrt{x^2 + y^2} \to \infty \). \( \square \)

By Lemma 8.16 it is enough to find a function \( \Phi \in \mathcal{N}(\alpha, k_1, k_2, \mathbb{R}^2) \) for some fixed \( k_2 > 0 \) for all \( \alpha \in \mathbb{C} \) and all \( k_1 \geq 0 \). All of the \( \varphi_i \) satisfy these criteria. In fact, \( \varphi_i \) for \( i = 1, 2, 3, 4 \) work more generally. The reason that \( \varphi_5 \) only works for \( 2k_2 > E \beta \) is due to the fact that when \( \alpha = \kappa_1 = \kappa_2 = 0 \), the solution to (8.2) has an explosive trajectory along the negative real axis.

Now we choose some parameters. Let \( E = 5 \) and \( \xi = \sqrt{5}/2 \), so as to make \( X_5 = \{ x \leq -1 \cap \{ |y| |x|^{1/2} \leq 2 \} \} \).

Let \( \beta = \frac{1}{18} \), \( \delta = \frac{1}{2} k_2 \), and \( \kappa_2 \in (0, 1) \). Then \( E \beta = \frac{18}{5} k_2 < 2k_2 \) and \( \delta \in (0, \frac{1}{2}) \). Hence for all \( i = 1, 2, \ldots, 5 \), \( \varphi_i \) is a LF in the region \( X_i \). Decrease \( \kappa_2 > 0 \) so that

\[
X_4 \supset \{ x \leq -1 \} \cap \{ 1 \leq |y| |x|^{1/2} \leq 2 \}.
\]

One can easily check that \( X_4 \) and \( X_5 \) overlap in such a way that we have covered all of \( \mathbb{R}^2 \) with \( X_1, X_2, \ldots, X_5 \) except a bounded region about the origin.

We fix \( \kappa_2 > 0 \) sufficiently small (this will be made precise later). We will construct a function \( \Phi \in \mathcal{N}(\alpha, k_1, k_2, \mathbb{R}^2) \) for all \( \alpha \in \mathbb{C} \) and all \( k_1 \geq 0 \). The idea is as follows. Note that \( \varphi_1 \) and \( \varphi_2 \) are LFs in the region \( X_1 \cap X_2 \). We shall define a nonnegative smooth auxiliary function \( \zeta(x) \in \mathcal{C}^\infty(\mathbb{R}) \) such that \( \zeta(x) = 0 \) for \( x \geq 2 \) and \( \zeta(x) = 1 \) for \( x \leq 1 \), satisfying some additional properties. We will then show that the combination

\[
(1 - \zeta) \varphi_1 + \zeta \varphi_2
\]

is a LF in the larger region \( X_1 \cup X_2 \). Proceeding inductively this way, we shall construct a LF in all of \( \mathbb{R}^2 \).

Let us first define some auxiliary functions needed to construct such a \( \Phi \). Let \( \zeta : \mathbb{R} \to \mathbb{R}_+ \) be a \( C^\infty \) function such that

\[
\zeta(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases}
\]

and \( \zeta'(x) < 0 \) for all \( x \in (1, 2) \). We define the smooth function \( \mu : \mathbb{R} \to \mathbb{R}_+ \) as the horizontal shift of \( \zeta \), three units to the left, i.e.,

\[
\mu(x) = \zeta(x + 3) \text{ for } x \in \mathbb{R}.
\]
Let
\[ \nu(x, y) = \begin{cases} \zeta(|y|) & \text{for } x \leq -2, \\ 0 & \text{for } |y| \geq 2, \\ 0 & \text{for } x > -1, \end{cases} \]
and assume that \( \nu \) is \( C^\infty \) outside of the ball \( B_4 \). Let \( q : (-\infty, -1] \times \mathbb{R} \to \mathbb{R} \) be defined by
\[ q(x, y) = \begin{cases} 1 & \text{if } |x|^{1/2}|y| \geq 2, \\ |x|^{1/2}|y| - 1 & \text{if } 1 < |x|^{1/2}|y| < 2, \\ 0 & \text{if } |x|^{1/2}|y| \leq 1, \end{cases} \]
and
\[ r(t) = \begin{cases} \exp \left(-\frac{1}{t(2t-1)^2}\right) & \text{if } 0 < t < 1, \\ 0 & \text{otherwise}. \end{cases} \]
Let
\[ s(x) = \frac{1}{N} \int_{-\infty}^{x} r(t) \, dt, \]
where \( N = \int_{\mathbb{R}} r(t) \, dt \). Now define a function on \( \mathbb{R}^2 \) by
\[ \rho(x, y) = \begin{cases} (s(q(x, y))) & \text{if } x \leq -1, \\ 1 & \text{if } |y| \geq 3, \\ 1 & \text{if } x \geq -1/2. \end{cases} \]
Clearly, \( \rho \) is \( C^\infty(\mathbb{R}^2) \) outside of \( B_4 \).

Let \( r(\delta) = \max \left(4, \sqrt{2^{4/3} + 4}\right) \). Define \( \varphi \) for \( x^2 + y^2 \geq r^2(\delta) \) by
\[ \varphi(x, y) = \begin{cases} \varphi_1 & \text{if } x \geq 2, \\ \zeta \varphi_2 + (1 - \zeta) \varphi_1 & \text{if } 1 < x < 2, \\ \varphi_2 & \text{if } -1 \leq x \leq 1, \\ \mu \varphi_3 + (1 - \mu) \varphi_2 & \text{if } -2 < x < -1, \\ \varphi_3 & \text{if } x \leq -2, |y| \geq 2, \\ \nu \varphi_4 + (1 - \nu) \varphi_3 & \text{if } x \leq -2, 1 < |y| < 2, \\ \varphi_4 & \text{if } x \leq -2, |y| \leq 1, |x|^{1/2}|y| \geq 2, \\ \rho \varphi_4 + (1 - \rho) \varphi_5 & \text{if } x \leq -2, 1 < |x|^{1/2}|y| < 2, \\ \varphi_5 & \text{if } x \leq -2, |x|^{1/2}|y| \leq 1, \end{cases} \]
and now
\[ \Phi(x, y) = \begin{cases} \varphi(x, y) & \text{if } x^2 + y^2 \geq B(r(\delta)), \\ \text{arbitrary positive and smooth} & \text{if } x^2 + y^2 < B(r(\delta)). \end{cases} \]
It is easy to see \( \Phi \) can be chosen to be nonnegative and \( C^\infty(\mathbb{R}^2) \). With the aid of Lemma \( 8.16 \) the following lemma implies Theorem \( 8.2 \) in the \( d = 2 \) case.

**Lemma 8.17.** For \( \kappa_2 \) sufficiently small, \( \Phi \in \mathcal{N}(\alpha, \kappa_1, \kappa_2, \mathbb{R}^2) \) for all \( \alpha \in \mathbb{C} \) and all \( \kappa_1 \geq 0 \).

**Proof.** Clearly, \( \Phi \) is smooth and satisfies properties I and II. Since \( M \varphi_i \to -\infty \) as \( r \to \infty \) in \( X_i \) for each \( i \), all we must verify is that \( M \Phi \to -\infty \) as \( r \to \infty \) in the overlapping regions. Let us recall the choices that have already been made:
\[ E = 5, \quad \xi = \frac{\sqrt{5}}{2}, \quad \beta = \frac{11}{4} \delta, \quad \delta = \frac{\kappa_2}{r}, \]
and note that \( \kappa_2 \in (0, 1) \) was chosen such that
\[ X_4 \supset \{ x \leq -1 \} \cap \{ 1 \leq |y||x|^{1/2} \leq 2 \}. \]

Pick \( C_5 > C_4 = C_3 > C_2 > C_1 \). Consider first
\[ \psi_1 := \zeta \varphi_2 + (1 - \zeta) \varphi_1 \]
defined in the region
\[ Y_1 = \{ 1 < x < 2 \} \cap B_{r(\delta)}. \]
We have
\[
\tau M \psi_1 = \zeta \tau M \varphi_2 + (1 - \zeta) \tau M \varphi_1 + (y^2 - x^2 + a_1) \zeta' (\varphi_2 - \varphi_1) \\
+ \kappa_1 (\zeta'' (\varphi_2 - \varphi_1) + 2 \zeta' (\partial_x \varphi_2 - \partial_x \varphi_1)) \\
\simeq_{Y_1} - \zeta C_2 g y^2 - (1 - \zeta) \frac{C_2^2}{2} x (x^2 + y^2)^{3/4} + (y^2 - x^2 + a_1) \zeta' (C_2 - C_1) |y|^{\delta/2} \\
+ \kappa_1 (\zeta'' (C_2 - C_1) |y|^{\delta/2} - 2 \zeta' C_2).
\] (8.10)

Note that if \( x \) is bounded away from 1 and 2 in (1, 2), the dominant term above is \( y^2 \zeta' (x) (C_2 - C_1) |y|^{\delta/2} \to -\infty \). Note also that as \( x \to 1 \) or \( x \to 2, \zeta', \zeta'' \to 0 \). But, the first two terms in the expression above decay to \(-\infty\) at least as fast as \(-C |y|^{\delta/2}\), where \( C > 0 \) is a constant independent of \( \zeta \) and \( x \). Thus we may choose \( \epsilon > 0 \) so that whenever \( x \in (1, 2) \setminus (1 + \epsilon, 2 - \epsilon) \), \( M \psi_1 \) decays at least as fast as \(-D |y|^{\delta/2}\), where \( D \) is some positive constant.

We now consider
\[
\psi_2 := \mu \varphi_3 + (1 - \mu) \varphi_2
\]
in the region
\[
Y_2 = \{-2 < x < -1\} \cap B^c_{r(\delta)}.
\]
We have
\[
\tau M \psi_2 = \mu \tau M \varphi_3 + (1 - \mu) \tau M \varphi_2 + (y^2 - x^2 + a_1) \mu' (\varphi_3 - \varphi_2) \\
+ \kappa_1 (\mu'' (\varphi_3 - \varphi_2) + 2 \mu' (\partial_x \varphi_3 - \partial_x \varphi_2)) \\
\simeq_{Y_2} \mu \delta x \varphi_3 - (1 - \mu) C_2 g y^2 + (y^2 - x^2 + a_1) \mu' (C_3 - C_2) |y|^{\delta/2} \\
+ \kappa_1 (\mu'' (C_3 - C_2) |y|^{\delta/2} + 2 \mu' C_2) \\
\simeq_{Y_2} \mu C_2 x |y|^{\delta/2} - (1 - \mu) C_2 g y^2 + (y^2 - x^2 + a_1) \mu' (C_3 - C_2) |y|^{\delta/2} \\
+ \kappa_1 (\mu'' (C_3 - C_2) |y|^{\delta/2} + 2 \mu' C_2).
\]

Note that, for the very same reasons as in the case of \( \psi_1 \), \( M \psi_2 \to -\infty \) as \( r \to \infty \) in \( Y_2 \).

Let
\[
\psi_3 := \nu \varphi_4 + (1 - \nu) \varphi_3
\]
in the region
\[
Y_3 = \{x \leq -2\} \cap \{1 < |y| < 2\} \cap B^c_{r(\delta)}.
\]
Thus
\[
\tau M \psi_3 = \nu \tau M \varphi_4 + (1 - \nu) \tau M \varphi_3 + (-2x |y| + \text{sgn}(y) a_2) (\partial_{|y|} \nu) (\varphi_4 - \varphi_3) \\
+ \kappa_2 (\partial^2_{|y|} \nu) (\varphi_4 - \varphi_3) + 2 \kappa_2 (\partial_{|y|} \nu) (\partial_{|y|} (\varphi_4 - \varphi_3)) \\
\simeq_{Y_3} \nu \tau M \varphi_4 + (1 - \nu) \tau M \varphi_3 \to -\infty.
\]

This is true since we chose \( C_3 = C_4 \). Hence the first order term
\[
(-2x |y| + \text{sgn}(y) a_2) (\partial_{|y|} \nu) (\varphi_4 - \varphi_3)
\]
approaches infinity at worst as fast as \( C |x| \), where \( C \) is a positive constant. The second order term \( \kappa_2 (\partial^2_{|y|} \nu) (\varphi_4 - \varphi_3) \) is a bounded function in this region. Moreover the term \( 2 \kappa_2 (\partial_{|y|} \nu) (\partial_{|y|} (\varphi_4 - \varphi_3)) \) at worst approaches infinity as fast as \( D |x|^{2\delta} \), where \( D \) is some positive constant. But, both \( M \varphi_3 \) and \( M \varphi_4 \) approach negative infinity at least as fast as \( -D |x|^{2\delta + 1} \), where \( D \) is another positive constant. This gives the desired result.

Let
\[
\psi_4 := \rho \varphi_4 + (1 - \rho) \varphi_5,
\]
in the region
\[
Y_4 = \{1 < |x|^{1/2} |y| < 2\} \cap \{x \leq -2\} \cap B^c_{r(\delta)}.
\]
Note that
\[
\tau M \psi_4 = s(q) \tau M \varphi_4 + (1 - s(q)) \tau M \varphi_5 + \partial_x (s(q))(\varphi_4 - \varphi_5)(y^2 - x^2 + a_1) \\
+ \partial_{|y|} (s(q))(\varphi_4 - \varphi_5) (-2x |y| + \text{sgn}(y) a_2) + \kappa_1 (\partial^2_x (s(q))(\varphi_4 - \varphi_5) \\
+ 2 \kappa_1 \partial_x (s(q)) \partial_x (\varphi_4 - \varphi_5) + \kappa_2 (\partial^2_{|y|} (s(q))(\varphi_4 - \varphi_5) \\
+ 2 \kappa_2 \partial_{|y|} (s(q)) \partial_{|y|} (\varphi_4 - \varphi_5) .
\]
Note that in the expression above, we may drop the $\kappa_1 \partial_x^2(s(q))(\varphi_4 - \varphi_5)$ and $2\kappa_1 \partial_x(s(q))\partial_x(\varphi_4 - \varphi_5)$ terms, as they are asymptotically less than other terms. Dropping other obviously insignificant terms, we obtain:

$$
\tau M \psi_4 \geq 2 \kappa_1 \partial_x(s(q))\partial_x(\varphi_4 - \varphi_5) + \kappa_2 \partial_y(s(q))(\varphi_4 - \varphi_5)x^2
$$

By the choice of $C_5 > C_4$, it is easy to see that for large $|x|$ in this region, there are constants $C, D > 0$ such that

$$
\kappa_2 \partial_y^2(s(q))(\varphi_4 - \varphi_5) + 2\kappa_2 \partial_y(s(q))\partial_y(\varphi_4 - \varphi_5) \leq C \kappa_2 r(q) \frac{|x|^{1/2} + 1}{g(x, y)}
$$

and

$$
-\partial_y(s(q))(\varphi_4 - \varphi_5)x^2 - \partial_y(s(q))(\varphi_4 - \varphi_5)2x|y| \leq -Dr(q)|x|^{1/2 + 1},
$$

where the constants $C, D$ and the function $r(q)$ are independent of $\kappa_2$. $r(q)$ is a function that goes to zero faster than any power of the function $g(x, y) := |(|x|^{1/2} - 1)(|x|^{1/2} - 2)|$ as $|x|^{1/2}|y| \to 1$ or 2. But note that, for all $\epsilon > 0$, we may choose $\kappa_2$ so small so that

$$
D = \frac{C \kappa_2}{g(x, y)^2}
$$

for $1 + \epsilon \leq |x|^{1/2}|y| \leq 2 - \epsilon$. From the first two terms in (8.11), we obtain at least $-C' \delta x^{1/2} + 1$ decay for some $C' > 0$ independent of $\kappa_2$ for all $1 \leq |x|^{1/2}|y| \leq 2$. But since every other term in the expression goes to zero faster than every power of $g$ as $|x|^{1/2}|y| \to 1$ or 2, we can choose an $\epsilon > 0$ so small as above so that $M \psi_4 \to -\infty$ for all $1 < |x|^{1/2}|y| < 2$. This completes the proof. 

8.2. $d \geq 3$ case.

Here we shall complete the proof of Theorems 8.2 for $d \geq 3$. Recall that for $d = 2$ and $\epsilon > 0$ sufficiently small, we defined $\Phi_2(x, y) := \Phi(x, y) \in C^\infty(Q_d)$ that satisfies (I), (II), and (IV) for all $\alpha \in \alpha, \kappa_1 \geq 0$, and $\kappa_2 \in (0, \epsilon)$. Lemma 8.15 implied then that for $\kappa_2 > 0$ arbitrary, the function $\Phi_{2,\eta}(x, y) := \Phi_2(\eta x, \eta y)$ is in $C^\infty(Q_d)$ satisfied (I), (II), and (IV) for all $\alpha$, provided that $c\eta^{-3} = 2\kappa_2$. Let is fix $\kappa_2 > 0$. For $d \geq 3$, we shall define

$$
\Phi_{d,\eta} := \Phi_{2,\eta} + (1 + \log^2(\eta y/2)).
$$

Lemma 8.18. For fixed $\kappa_2 > 0$, $\Phi_{d,\eta}$ is a smooth function on $Q_d$ that satisfies (I), (II), and (IV).

Proof. By definition, $\Phi_{d,\eta}$ is smooth and nonnegative in $Q_d = \mathbb{R}^d$. Clearly, $\Phi_{d,\eta} \to \infty$ as $(x, y) \to \partial Q_d = \{(x, y) \in \mathbb{R}^d : y = 0\} \cup \{\infty\}$, $(x, y) \in Q_d$. Thus we must verify property (IV). To this end, note that:

$$
\tau M \Phi_{d,\eta} = \tau M \Phi_{2,\eta} + \tau M \psi_4 |

$$

$$
= \tau M \Phi_{2,\eta} - \frac{b(d-2)}{y} \partial_y \Phi_{2,\eta} + \frac{2 \log(\eta y/2)}{y^2(1 + \log^2(\eta y/2))}
$$

$$
+ \frac{2 \kappa_2 (1 - \log^2(\eta y/2))}{y^2(1 + \log^2(\eta y/2))}.
$$

Case 1. Suppose first that $y \geq 2\eta^{-1}$. It is easy to check that there exist positive constants $K_1, K_2 > 0$ such that

$$
\tau \frac{b(d-2)}{y} \partial_y \Phi_{2,\eta} \leq K_1, \quad (a_2 y + \tau b(d-2) - \kappa_2) \frac{2 \log(\eta y/2)}{y^2(1 + \log^2(\eta y/2))} + \frac{2 \kappa_2 (1 - \log^2(\eta y/2))}{y^2(1 + \log^2(\eta y/2))} \leq K_2.
$$
If also \( x > -2\eta^{-1} \) then

\[
-2x \frac{\log(\eta y/2)}{1 + \log^2(\eta y/2)} \leq K_3
\]  

for a positive constant \( K_3 \), whereas for \( x \leq -2\eta^{-1} \),

\[
-2x \frac{\log(\eta y/2)}{1 + \log^2(\eta y/2)} \leq -K_4 x.
\]  

(8.13)

(8.14)

Since \( M_2\Phi_{2,\eta} \to -\infty \) as \( (x,y) \to \partial Q_d \) and \( y \geq 2\eta^{-1} \), and, besides, if \( x \leq -2\eta^{-1} \) and \( y \geq 2\eta^{-1} \) then \( \Phi_{2,\eta}(x,y) \) is equal to the rescaled function \( \varphi_3 \) so that, by (8.13), \( M_2\Phi_{2,\eta} \leq K_5 x^2 + y^2 \delta/4 \) for some \( K_5 > 0 \), we infer that \( M_d\Phi_{d,\eta} \to -\infty \) as \( (x,y) \to \partial Q_d \) and \( y \geq 2\eta^{-1} \).

**Case 2.** Suppose now that \( 0 < y < 2\eta^{-1} \). If \( |x| < 2\eta^{-1} \), then \( (x,y) \to \partial Q_d \) if and only if \( y \downarrow 0 \). Since \( \Phi_{2,\eta} \) is smooth on \( \mathbb{R}^2 \), there exists a constant \( K_6 > 0 \) such that

\[
\tau M_2\Phi_{2,\eta} + \frac{\tau(b(d-2))}{y} \frac{\partial\Phi_{2,\eta}}{\partial y} \leq K_6 \frac{1}{y}
\]

for \( (x,y) \in (-2\eta^{-1}, 2\eta^{-1}) \times (0, 2\eta^{-1}) \). Hence, recalling the assumption \( \tau(b(d-2)) \geq \kappa_2 \), we have on this rectangle:

\[
\tau M_d\Phi_{d,\eta} \leq \frac{K_6}{y} + \left(-2xy^2 + a_2y + \tau(b(d-2) - \kappa_2)\right) \frac{2\log(\eta y/2)}{y^2(1 + \log^2(\eta y/2))} + \frac{2\kappa_2(1 - \log^2(\eta y/2))}{y^2(1 + \log^2(\eta y/2))^2}
\]

\[
\leq \frac{K_7}{y} + \frac{2\kappa_2(1 - \log^2(\eta y/2))}{y^2(1 + \log^2(\eta y/2))^2} \to -\infty
\]

as \( y \downarrow 0 \).

If \( x \geq 2\eta^{-1} \), then \( \Phi_{2,\eta} \) is equal to the rescaled function \( \varphi_1 \). We see that by (8.8) there exist constants \( K_8, K_9, K_{10}, K_{11} > 0 \) such that

\[
\tau M_d\Phi_{d,\eta} \leq -K_8 x^2 + y^2 \delta/4 + K_9(x^2 + y^2) \delta/4 - 1
\]

\[
+ (-2xy^2 + a_2y + \tau(b(d-2) - \kappa_2)) \frac{2\log(\eta y/2)}{y^2(1 + \log^2(\eta y/2))} + \frac{2\kappa_2(1 - \log^2(\eta y/2))}{y^2(1 + \log^2(\eta y/2))^2}
\]

\[
\leq -K_8 x^2 + y^2 \delta/4 + K_{10} x + K_{11} \frac{1}{y} + \frac{2\kappa_2(1 - \log^2(\eta y/2))}{y^2(1 + \log^2(\eta y/2))^2}.
\]

Note that as \( (x,y) \to \partial Q_d \) in this region then \( x \to \infty \) or \( y \downarrow 0 \). It is thus easy to see that \( \tau M_d\Phi_{d,\eta} \to -\infty \).

If \( x < -2\eta^{-1} \), then it is easy to check that \( \partial_{\eta}\Phi_{2,\eta} \) is bounded above by the choice of \( C_3 = C_4 \) and \( C_5 > C_4 \). Then, for some \( K_{12} > 0 \),

\[
\tau M_d\Phi_{d,\eta} \leq \tau M_2\Phi_{2,\eta} + K_{12} \frac{1}{y} + \frac{2\kappa_2(1 - \log^2(\eta y/2))}{y^2(1 + \log^2(\eta y/2))^2}
\]

so that \( \tau M_d\Phi_{d,\eta} \to -\infty \) as \( (x,y) \to \partial Q_d \) in this region.

\[
\Box
\]

9. **TOP LYAPUNOV EXPONENT**

The Lyapunov exponent \( \lambda \) for the dispersion process \( p(t) = (\rho(t), \chi(t)) \) is the asymptotic rate of growth in time of the logarithm of the length \( \sqrt{\rho^2 + \chi^2} \). Suppose that the process starts at \( t = 0 \) from \( p_0 = (\rho_0, \chi_0) \neq 0 \). Anticipating the existence of the limit of the below, we shall define:

\[
\lambda = \lim_{T \to \infty} \frac{1}{T} \left\langle \ln \left( \sqrt{\rho^2(T) + \chi^2(T)} - \ln \sqrt{\rho_0^2 + \chi_0^2} \right) \right\rangle
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{d}{dt} \left\langle \ln \sqrt{\rho^2(t) + \chi^2(t)} \right\rangle dt
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\langle L \ln \sqrt{\rho^2(t) + \chi^2(t)} \right\rangle dt
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \int \left( L \ln \sqrt{\rho^2 + \chi^2} \right) P_t(p_0, dp) \right) dt,
\]  

(9.1)
where \( L \) is the generator of the process \( p(t) \) given by Eq. (2.11). Note that the function \( L \ln \sqrt{\rho^2 + \chi^2} \) is smooth on \( \mathbb{R}^{2d} \setminus \{0\} \) and homogeneous of degree zero. It may be viewed as a function \( f_0(\pi) \) on \( S^{2d-1} \) that, besides, is \( SO(d) \)-invariant. We may then rewrite the definition of \( \lambda \) as

\[
\lambda = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \int f_0(\pi) \, P_t(\pi_0; d\pi) \right) dt.
\]  (9.2)

Now, the existence of the limit follows from the fact that the Cesaro means of the transition probabilities \( P_t(\pi_0; d\pi) \) tend in weak topology to the unique invariant probability measure \( \mu(d\pi) \). Hence

\[
\lambda = \int f_0(\pi) \, \mu(d\pi)
\]  (9.3)

and is independent of \( p_0 \). The crucial input that allows to make the latter formula more explicit is the formula

\[
L \ln \sqrt{\rho^2} = \frac{1}{\tau} \frac{\rho \cdot \chi}{\rho^2} = \frac{x}{\tau}.
\]  (9.4)

It implies that

\[
f_0(\pi) = L \ln \sqrt{\rho^2 + \chi^2} = \frac{x}{\tau} + \frac{1}{2} L \ln \left(1 + \frac{\chi^2}{\rho^2}\right) = \frac{x}{\tau} + \frac{1}{2} L \left\{ \ln(1 + x^2) \right\} \text{ for } d = 1
\]

\[
= \frac{x}{\tau} + \frac{1}{2} L \left\{ \ln(1 + x^2 + y^2) \right\} \text{ for } d \geq 2
\]  (9.5)

in terms of the \( SO(d) \) invariants with \( L \) given by explicit formulae (7.8), (7.15) or (7.24). If the functions \( \ln(1 + x^2) \) in \( d = 1 \) and \( \ln(1 + x^2 + y^2) \) in \( d = 2 \) that are homogeneous of degree zero on \( \mathbb{R}^{2d} \setminus \{0\} \) were smooth, then their contributions to the expectation with respect to the invariant measure on the right hand side of (9.2) would drop out by the integration by parts. The problem is, however, the lack of smoothness of those functions at \( \rho = 0 \) and a more subtle argument is required.

9.1. \( d = 1 \) case. In one dimension, Eq. (9.3) reduces to the identity

\[
\lambda = \int_{-\infty}^{\infty} \left( \frac{x}{\tau} + \frac{1}{2} L \ln(1 + x^2) \right) \eta(x) \, dx
\]  (9.6)

with \( \eta(x) \) given by Eq. (7.6). Since the latter integral represents the integration of a smooth function against a smooth measure on \( S^1 \), it converges absolutely. Consequently, the formula for \( \lambda \) may be rewritten in the form:

\[
\lambda = \lim_{n \to \infty} \int_{-n}^{n} \left( \frac{x}{\tau} + \frac{1}{2} L \ln(1 + x^2) \right) \eta(x) \, dx.
\]  (9.7)

Now the integration by parts and the formula \( L^1 \eta = 0 \) for the formal adjoint \( L^1 \) defined with respect to the Lebesgue measure \( dx \) show that the term with \( L \ln \sqrt{1 + x^2} \) drops out (for the cancellation of the boundary terms it is crucial that the integral is over a symmetric finite interval \([−n,n] \)). We obtain this way the identity

\[
\lambda = \frac{1}{\tau} \lim_{n \to \infty} \int_{-n}^{n} x \eta(x) \, dx \equiv \frac{1}{\tau} \text{ p.v.} \int_{-\infty}^{\infty} x \eta(x) \, dx
\]  (9.8)

where “p.v.” stands for “principal value”. The result may be expressed [21] by the Airy functions [13]:

\[
\lambda = -\frac{1}{2\tau} + \frac{1}{4\pi \tau} \frac{d}{dc} \ln \left( \text{Ai}^2(c) + \text{Bi}^2(c) \right) \text{ for } c = \frac{1}{(4\tau(A + 2B))^2}.
\]  (9.9)

The number \( \lambda + \frac{1}{\tau} \) is the Lyapunov exponent for the one-dimensional Anderson problem [33], recall relation (8.1). It is always positive reflecting the permanent localization in one dimension. On the other hand, \( \lambda \) itself changes sign as a function of \( \tau \) and \( A + 2B \) signaling a phase transition in the one-dimensional advection of inertial particles [33].
9.2. \(d \geq 2\) case. In two or more dimensions, Eq. \((9.3)\) becomes
\[
\lambda = \int \left( \frac{x}{\tau} + \frac{1}{2} L \ln (1 + x^2 + y^2) \right) \eta(x, y) \, dx \, dy,
\] (9.10)
where \(\eta(x, y)\) is the density of the invariant measure from Eqs. \((7.9)\) or \((7.1)\). The asymptotic behavior of \(\eta(x, y)\) was established in Sec. 7.2 and Sec. 7.3. We show in Appendix B that it guarantees that the term with \(L \ln (1 + x^2 + y^2)\) may, indeed, be dropped from the expectation on the right hand side of Eq. \((9.2)\) so that
\[
\lambda = \frac{1}{\tau} \int x \eta(x, y) \, dx \, dy,
\] (9.11)
where the integral converges absolutely as follows from the estimates \((7.14)\) and \((7.23)\). In general, there is no closed analytic expressions for the right hand side, unlike in the one-dimensional case. The results of numerical simulations for \(\lambda\), indicating its qualitative dependence on the parameters of the model, together with analytic arguments about its behavior when \(A \tau \to \infty\) with \(A/B = \text{const.}\) or when \(\tau \to 0\) with \(A/\tau = \text{const.}\) and \(B/\tau = \text{const.}\) may be found in [2, 4, 17, 23, 24].

10. CONCLUSIONS

We have studied rigorously a simple stochastic differential equation (SDE) used to model the pair dispersion of close inertial particles moving in a moderately turbulent flow. We have established the smoothness of the transition probabilities and the irreducibility of the dispersion process using Hörmander criteria for hypoellipticity and control theory. For the projectivized version of the dispersion process, these results implied the existence of the unique invariant probability measure with smooth positive density as well as exponential mixing. The latter properties permitted to substantiate the formulae for the top Lyapunov exponents for the inertial particles used in the physical literature. In two space dimensions, we also showed that the complex-projectivized version of the dispersion process is non-explosive when described in the inhomogeneous variable of the complex projective space, unlike the real-projective version of the dispersion in one space dimension. A similar result was established in \(d \geq 3\) for the complex-valued process built from the \(SO(d)\) invariants of the projectivized dispersion that was shown to stay for all times in the upper half-plane. These non-explosive behaviors are the reason why the numerical simulation of the processes in the complex (half-)plane could lead to reliable numerical results [2]. There are other questions about the models studied here that may be amenable to rigorous analysis. Let us list some of them: What about the expressions for the other \(2d - 1\) Lyapunov exponents (the \(2d\) exponents have to sum to \(-d/\tau [10]\))? Can one establish the existence of the large deviation regime for the finite-time Lyapunov exponents (the corresponding rate function for the top exponent was numerically studied in \(d = 2\) model in [2, 4]); it gives access to more subtle information about the clustering of inertial particles than the top Lyapunov exponent itself? Is the SDE modeling the inertial particle dispersion in fully developed turbulence, that was introduced and studied numerically in [34], amenable to rigorous analysis? All those open problems are left for a future study.

APPENDIX A.

We establish here the expressions \((7.18)\) and \((7.19)\) for the \(SO(2d)\)-invariant normalized volume measure on \(S^{2d-1}\) for \(d \geq 3\) (the same proof works also in \(d = 2\), although there, the corresponding formulae are straightforward and well known). Note that for \(S^{2d-1}\) identified with the set of \((\rho, \chi) \in \mathbb{R}^{2d}\) such that \(\rho^2 + \chi^2 = R^2\) for fixed \(R\), we may write
\[
d\mu_0 = \text{const.} \, \delta(R - \sqrt{\rho^2 + \chi^2}) \, d\rho \, d\chi,
\] (A.1)
provided that we identify functions on \(S^{2d-1}\) with homogeneous function of degree zero on \(\mathbb{R}^{2d} \setminus \{0\}\). Let us parametrize:
\[
\rho = O(\rho, 0, \ldots, 0), \quad \chi = O(\rho x, \rho y, 0, \ldots, 0)
\] (A.2)
where \(\rho = |\rho|, \chi = |\chi|, x\) and \(y\) are the \(SO(d)\)-invariants of Eq. \((7.1)\), and \(O \in SO(d)\). Note that \(O^{-1}\) is the rotation that aligns \(\rho\) with first positive half-axis of \(\mathbb{R}^d\) and brings \(\chi\) into the half-plane spanned by the first axis and the second positive half-axis, as required in Sec. 7.3. \(O\) and \(OO'\), for \(O'\) rotating in the
subspace orthogonal to the first two axes, give the same \((\rho, \chi)\). Let \(\Lambda\) be a \(d \times d\) antisymmetric matrix, \(\Lambda_{ij} = -\Lambda_{ji}\). Setting \(O = e^{\Lambda}\) and differentiating Eqs. (A.2) at \(\Lambda = 0\), we obtain:

\[
d\rho = (d\rho, \rho d\Lambda_{21}, \ldots, \rho d\Lambda_{1d}),
\]

\[
d\chi = (-pyd\Lambda_{21}, \rho xd\Lambda_{21}, \rho xd\Lambda_{31} + pyd\Lambda_{32}, \ldots, \rho xd\Lambda_{d1} + pyd\Lambda_{d2})
\]

+ \((xd\rho + \rho dx, yd\rho + \rho dy, 0, \ldots, 0)\).

\[(A.4)\]

Hence for the volume element,

\[
d\rho d\chi = \rho^{2d-1}y^{d-2}d\rho dxdy d\Lambda_{21} \cdots d\Lambda_{d1}d\Lambda_{d2} \cdots d\Lambda_{d3}.
\]

\[(A.5)\]

The product of \(d\Lambda_{ij}\) gives, modulo normalization, the \(SO(d)\)-invariant volume element \(d[O]\) of the homogeneous space \(SO(d)/SO(d-2)\) at point \([1]\). Using the \(SO(d)\)-invariance, we infer that

\[
d\rho d\chi = \text{const.} \rho^{2d-1}y^{d-2}d\rho dxdy d[O].
\]

\[(A.6)\]

Substituting the last expression to Eq. (A.1) and performing the integral

\[
\int \delta(R - \sqrt{\rho^2 + \chi^2}) \rho^{2d-1}d\rho = \int \delta(R - \rho \sqrt{1 + x^2 + y^2}) \rho^{2d-1}d\rho
\]

= \(R^{2d-1}/(1 + x^2 + y^2)^d\)

\[(A.7)\]

that collects the entire \(\rho\)-dependence in the integration against homogeneous function of zero degree, we obtain Eq. (7.19), modulo a constant factor that is fixed by normalizing of the resulting measure.

**Appendix B.**

We show here that

\[
\int (Lg)(x, y) \eta(x, y) dx dy = 0
\]

\[(B.1)\]

in two or more dimensions, where \(g(x, y) = \ln(1 + x^2 + y^2)\) and \(\eta(x, y)\) is the density of the invariant measure as defined by Eqs. (7.9) and (7.17). As mentioned in Sec. 5, the identity (B.1) does not follow immediately by integration by parts since the function \(g(x, y)\), is not smooth on \(S^{2d-1}\). We shall then replace \(g(x, y)\) by the functions

\[
g_\epsilon(x, y) = \ln \left(1 + \frac{x^2 + y^2}{1 + \epsilon(x^2 + y^2)}\right) = \ln \left(\frac{\rho^2 + \chi^2}{\rho^2 + \epsilon^2}\right)
\]

\[(B.2)\]

that are smooth on \(S^{2d-1}\) for \(\epsilon > 0\). The identity (B.1) will follow if we show that

\[
\int (Lg_\epsilon)(x, y) \eta(x, y) dx dy = \lim_{\epsilon \to 0} \int (Lg_\epsilon)(x, y) \eta(x, y) dx dy.
\]

\[(B.3)\]

Note that

\[
\partial_x g_\epsilon(x, y) = \frac{2x}{(1 + (1 + \epsilon)(x^2 + y^2))(1 + \epsilon(x^2 + y^2))},
\]

\[
\partial_y g_\epsilon(x, y) = \frac{2y}{(1 + (1 + \epsilon)(x^2 + y^2))(1 + \epsilon(x^2 + y^2))},
\]

\[
\partial_x^2 g_\epsilon(x, y) = \frac{2(1 + (1 + \epsilon)(x^2 + y^2))(1 + (1 + \epsilon)(x^2 + y^2))}{(1 + (1 + \epsilon)(x^2 + y^2))^2(1 + \epsilon(x^2 + y^2))^2} - \frac{2}{1 + (1 + \epsilon)(x^2 + y^2)},
\]

\[
\partial_y^2 g_\epsilon(x, y) = \frac{2(1 + (1 + \epsilon)(x^2 + y^2))(1 + (1 + \epsilon)(x^2 + y^2))}{(1 + (1 + \epsilon)(x^2 + y^2))^2(1 + \epsilon(x^2 + y^2))^2},
\]

\[
\partial^2 g_\epsilon(x, y) = \frac{2(1 + (1 + \epsilon)(x^2 + y^2))(1 + (1 + \epsilon)(x^2 + y^2))}{(1 + (1 + \epsilon)(x^2 + y^2))^2(1 + \epsilon(x^2 + y^2))^2},
\]

\[(B.4)\]

so that

\[
|\partial_x g_\epsilon(x, y)| \leq \frac{2|x|}{1 + x^2 + y^2}, \quad |\partial_y g_\epsilon(x, y)| \leq \frac{2|y|}{1 + x^2 + y^2},
\]

\[
|\partial_x^2 g_\epsilon(x, y)| \leq 10, \quad |\partial_y^2 g_\epsilon(x, y)| \leq 10.
\]

\[(B.5)\]

Using the explicit forms (7.15) and (7.24) of the generator \(L\), we infer that

\[
|(Lg_\epsilon)(x, y)| \leq C(1 + |x|)
\]

\[(B.6)\]
with an $\epsilon$-independent constant $C$. Since the integral
\[ \int (1 + |x|) \eta(x, y) \, dx \, dy \] converges due to the estimates (7.14) and (7.23), and point-wise
\[ \lim_{\epsilon \searrow 0} (L\eta)(x, y) = (L\eta)(x, y), \] relation (B.3) follows from the Dominated Convergence Theorem.

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