Quantum Geometry II :
The Mathematics of Loop Quantum Gravity

Three dimensional quantum gravity

J. Manuel García-Islas *

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Abstract

Loop quantum gravity is a physical theory which aims at unifying general relativity and quantum mechanics. It takes general relativity very seriously and modifies it via a quantisation. General relativity describes gravity in terms of geometry. Therefore, quantising such theory must be equivalent to quantising geometry and that is what loop quantum gravity does. This sounds like a mathematical task as well. This is why in this paper we will present the mathematics of loop quantum gravity. We will do it from a mathematician point of view. This paper is intended to be an introduction to loop quantum gravity for postgraduate students of physics and mathematics. In this work we will restrict ourselves to the three dimensional case.

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1 Introduction

It is very well known in physics that classical physical theories have a quantum version. Classical Mechanics and Quantum Mechanics, Classical Electrodynamics and Quantum Electrodynamics, Special Relativity and Quantum Field Theory.

The question then is: what is the quantum version of general relativity? There is no agreement on the answer to this question.

Loop quantum gravity \[1, 2, 3, 4\], is a theory which describes the quantum version of general relativity. Since general relativity is a geometrical theory, quantising such a theory is equivalent to thinking of quantising classical geometry.\footnote{Mathematically speaking, quantum geometry may also refer to a subject known as non-commutative geometry \[5\]. But for us it will refer to loop quantum gravity which is also a quantum geometry.} This is what loop quantum gravity does in its own original way. However, unfortunately mathematicians are not aware of loop quantum gravity and it may be due to the fact that it is a theory mostly invented by physicists. The truth is that loop quantum gravity is such a beautiful theory with very precise, rigorous mathematical background which exploits many different subjects of outstanding level.

Our intention and our project is to write papers introducing loop quantum gravity to postgraduate students in physics and mathematics. We write it from the point of view of a mathematician. We hope that more mathematicians get to know and become interested in loop quantum gravity.

However, the project may also be considered from the loop quantum gravity community itself as writing about the mathematics of loop quantum gravity.

Loop quantum gravity is a very extensive field of study and there are numerous things to deal with. Our project is to write about the mathematics of loop quantum gravity and therefore write many works on this subject by dealing with a particular theme in each work. We have already started this project describing quantum polyhedra in \[6\].

In this work we will deal with three dimensional quantum gravity which has been extensively studied in the literature and there are also many things in this particular theme to write about. We concentrate on the essentials. Our presentation of the mathematics of loop quantum gravity is completely original and described in our own way. In this sense this is an original work and an original project.

In section 2 and 3 we present the mathematical background which is also useful for many other themes in loop quantum gravity. In section 4 and 5 we introduce three dimensional loop quantum gravity.
2 Topological groups and representations

In this section we review some mathematical background related to Lie groups and linear representations which is used in loop quantum gravity. This section is based on [7], [8].

**Definition 1.** A topological group $G$, is a group which is also a topological space such that the product and inverse operations are continuous. It is said to be compact if it is a compact space.

We will only consider compact topological groups.

There exists a unique measure on $G$ with the following properties.

1) $\int_G f(g) \, dg = \int_G f(gh) \, dg$ for every $h \in G$.
2) $\int_G f(g) \, dg = \int_G f(hg) \, dg$ for every $h \in G$.
3) $\int_G f(g) \, dg = 1$.

The measure $dg$ is called invariant measure or the Haar measure of $G$.

**Definition 2.** A linear representation of $G$ in a Hilbert space $H$ is a group homomorphism $\rho : G \rightarrow GL(H)$, such that $\rho(g)x$ is a continuous map for every $x \in H$.

Therefore, we have that for $g$ and $g' \in G$, $\rho(gg') = \rho(g)\rho(g')$, $\rho(g^{-1}) = \rho(g)^{-1}$ and $\rho(e) = Id_H$, where $e$ is the identity of the group and $Id_H$ is the identity map in $H$.

The dimension of $H$ is called the dimension of the representation and a representation will be denoted by the pair $(H, \rho)$.

**Definition 3.** A linear representation of $G$ in a Hilbert space $H$ is a unitary representation if $\rho(g)$ is a unitary operator for every $g \in G$. Equivalently, $\rho$ is unitary if $\langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle$ for every $g \in G$ and every $x, y \in H$.

If $(e_i)$ is a basis of $H$ we have the matrix coefficients of the representation $(H, \rho)$ given by

\[
\rho_{ij}(g) = \langle e_i, \rho(g)e_j \rangle
\]

Since we are considering compact Lie groups only, it is known that right invariant measures are also left invariant and that a normalisation condition such as 3) can be chosen. We will not be concerned with this here.

$\langle \cdot, \cdot \rangle$ is the scalar product in $H$ which we assume to be conjugate linear in the first entry and linear in the second.
**Definition 4.** An intertwining operator between two unitary representations \((H_1, \rho_1)\) and \((H_2, \rho_2)\) is a continuous linear map \(T : H_1 \rightarrow H_2\) such that for every \(g \in G\) the following diagram commutes

\[
\begin{array}{ccc}
H_1 & \xrightarrow{T} & H_2 \\
\downarrow & & \downarrow \\
H_1 & \xrightarrow{T} & H_2
\end{array}
\]

\[
\rho_1(g) \quad \rho_2(g)
\]

**Definition 5.** The representations \((H_1, \rho_1)\) and \((H_2, \rho_2)\) are said to be equivalent if there exists a bijective \(T : H_1 \rightarrow H_2\) intertwining operator.

This implies that we have an equivalence class of a representation.

**Definition 6.** A vector subspace \(W \subset H\) is said to be invariant under \((H, \rho)\), if \(\forall g \in G, \rho(g)W \subset W\).

**Definition 7.** A representation \((H, \rho)\) is said to be irreducible if \(H \neq \{0\}\) and if \(\{0\}\) and \(H\) are the only vector subspaces of \(H\) which are invariant under \(\rho\).

**Definition 8.** Let \((H_1, \rho_1)\) and \((H_2, \rho_2)\) be representations of \(G\). The direct sum is the representation \((H_1 \oplus H_2, \rho_1 \oplus \rho_2)\) so that \(\forall g \in G\) and \(x_1 \in H_1, x_2 \in H_2\).

\[
(\rho_1 \oplus \rho_2)(g)(x_1, x_2) = (\rho_1(g)x_1, \rho_2(g)x_2)
\]

The matrix representation of a direct sum is given by

\[
\begin{pmatrix}
\rho_1(g) & 0 \\
0 & \rho_2(g)
\end{pmatrix}
\]

Analogously the direct sum of an arbitrary finite number of representations is defined.

**Definition 9.** Let \((H_1, \rho_1)\) and \((H_2, \rho_2)\) be representations of \(G\). The tensor product is the representation \((H_1 \otimes H_2, \rho_1 \otimes \rho_2)\) so that \(\forall g \in G\) and \(x_1 \in H_1, x_2 \in H_2\).

\[
(\rho_1 \otimes \rho_2)(g)(x_1 \otimes x_2) = \rho_1(g)x_1 \otimes \rho_2(g)x_2
\]

The matrix representation of a tensor product is given by the Kronecker tensor product of the matrices

\[
\begin{pmatrix}
\rho_1(1)\rho_2(g) & \cdots & \rho_1(n)\rho_2(g) \\
\vdots & \ddots & \ddots \\
\rho_1(1)\rho_2(g) & \cdots & \rho_1(n)\rho_2(g)
\end{pmatrix}
\]
**Definition 10.** A representation \((H, \rho)\) is said to be reducible if it is the direct sum of irreducible representations \((W_1, \rho_1), (W_2, \rho_2), \ldots, (W_k, \rho_k)\); this is expressed \((H, \rho) = (W_1 \oplus W_2 \oplus \cdots \oplus W_k, \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k)\)

or simply

\[(H, \rho) = (\oplus_{i=1}^k W_i, \oplus_{i=1}^k \rho_i)\]

**Theorem 1.** Every unitary representation \((H, \rho)\) of \(G\) is reducible.

*Proof.* If the representation is irreducible, then we are finished. Suppose then that the representation \((H, \rho)\) is unitary and not irreducible; then there must be an invariant subspace \(W \subset H\). Since there is a scalar product we can consider the complement of \(W\) in \(H\) denoted \(W^\perp\) such that \(H = W \oplus W^\perp\). It can be shown that \(W^\perp\) is also an invariant subspace. By induction the theorem follows. \(\square\)

**Lemma 1.** *(Schur)* Let \((H_1, \rho_1)\) and \((H_2, \rho_2)\) be unitary irreducible representations of \(G\). And let \(T : H_1 \to H_2\) be an intertwining operator. Then \(T = 0\) or \(T\) is an isomorphism.

Consider the vector space \(\mathbb{C}^G = \{f : G \to \mathbb{C}\}\) of complex valued functions on \(G\), and define the scalar product

\[\langle f_1 | f_2 \rangle = \int_G \overline{f_1(g)} f_2(g) \, dg\]

where \(dg\) is the Haar measure. By completing this space we obtain the Hilbert space of square integrable functions \(L^2(G)\).

**Theorem 2.** Let \((H_1, \rho_1)\) and \((H_2, \rho_2)\) be irreducible unitary representations of \(G\). Let \(x_1, y_1 \in H_1\) and \(x_2, y_2 \in H_2\). Then

\[\langle \rho_{x_1y_1}(g) | \rho_{x_2y_2}(g) \rangle = 0\]

if the representations are not equivalent, or

\[\langle \rho_{x_1y_1}(g) | \rho_{x_2y_2}(g) \rangle = \frac{1}{n} < x_2, x_1 > < y_1, y_2 >\]

if the representations are equivalent and where \(n\) is the dimension of \(H_1 \sim H_2\).

In particular, if we have orthonormal basis

\[\langle \rho_{ij}(g) | \rho_{kl}(g) \rangle = 0\]

if the representations are not equivalent, or

\[\langle \rho_{ij}(g) | \rho_{kl}(g) \rangle = \frac{1}{n} \delta_{ik} \delta_{jl}\]
if the representations are equivalent, where \( \delta_{ij} \) is the Kronecker delta.

The set of equivalence classes of irreducible representations of \( G \) is denoted by \( \hat{G} \).

**Theorem 3. Peter - Weyl.** Let \( f \in L^2(G) \). Then

\[
f(g) = \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{\dim \rho^\alpha} c^\alpha_{ij} \rho^\alpha_{ij}(g)
\]

where \( \alpha \) runs over the different irreducible unitary representations, \( \rho^\alpha_{ij} \) are the matrix coefficients of the representations in orthonormal basis.

Therefore, the coefficients are given by

\[
c^\alpha_{ij} = \dim \rho^\alpha \int_G \overline{\rho^\alpha_{ij}(g)} f(g) \, dg
\]

2.1 SU(2)

In this part, we will summarise some important aspects of the irreducible representations of the group \( G = SU(2) \) which are used in LQG. \( SU(2) \) consist of the matrices given by

\[
\begin{pmatrix}
    a & b \\
    -\overline{b} & \overline{a}
\end{pmatrix}
\]

where \( a, b \in \mathbb{C} \) and \( |a|^2 + |b|^2 = 1 \). It can be easily seen that \( SU(2) \) is diffeomorphic to the three dimensional sphere \( S^3 \).

\( SU(2) \) acts on the space of complex valued functions on \( \mathbb{C}^2 \) as follows: we have that for \( g \in SU(2) \), \( g^{-1} = \overline{g} ; \) the action is given by

\[
\rho(g) f = f \circ g^{-1}
\]

so that

\[
\rho(g) f(z_1, z_2) = f(\overline{az}_1 - bz_2, \overline{b}z_1 + az_2)
\]

Now we restrict the space of complex valued functions on \( \mathbb{C}^2 \) to homogeneous polynomials and denote by \( H^j \) the vector space of homogeneous polynomials of degree \( 2j \) where \( j = \frac{1}{2} N \). The dimension of \( H^j \) is \( 2j + 1 \). \(^4\)

This space has basis given and denoted by

\[
f^j_m(z_1, z_2) = z_1^{j+m} z_2^{j-m}
\]

\(^4\)This view is common to physicists and we will use it. Mathematicians are more used to think of the space \( H^n \) as the space of complex homogeneous polynomials on \( \mathbb{C}^2 \) of degree \( n = 2j, n \in \mathbb{N} \), which has dimension \( n + 1 \).
where \(-j \leq m \leq j\); \(m\) is integer if \(j\) is integer and half-integer if \(j\) is half-integer. When restricted to the space \(H^j\) the action \(\rho\) gives a representation which we denote by \((H^j, \rho^j)\).

The Lie algebra \(\mathfrak{su}(2)\) of \(SU(2)\) is a vector space of dimension three of traceless anti-Hermitian matrices with basis given by

\[
\begin{align*}
\xi_1 &= \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
\xi_2 &= \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
\xi_3 &= \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\end{align*}
\]

which satisfy the commutation relations \([\xi_1, \xi_2] = \xi_3, [\xi_2, \xi_3] = \xi_1, [\xi_3, \xi_1] = \xi_2\).

Physicists in LQG prefer to use the Hermitian matrices given by

\[
J_k = i\xi_k \quad \text{for} \quad k = 1, 2, 3
\]

where the commutation relations are given by

\[
\begin{align*}
[J_1, J_2] &= iJ_3 \\
[J_2, J_3] &= iJ_1 \\
[J_3, J_1] &= iJ_2
\end{align*}
\]

Moreover, we consider the matrices

\[
J_3, J_+ = J_1 + iJ_2, \quad J_- = J_1 - iJ_2
\]

which have commutation relations \([J_+, J_-] = 2J_3, [J_3, J_+] = J_+, [J_3, J_-] = -J_-\). It is known that the differentials \(D\) of the representations \(\rho^j\) are representations of the Lie algebra and therefore give rise to operators \((D\rho^j)_J, (D\rho^j)_{J+}, (D\rho^j)_{J-}\) that act on the space \(H^j\) as follows. Rename the basis \(f^j_m\) of \(H^j\) with notation used in physics

\[
| j, m > = \frac{1}{\sqrt{(j-m)!(j+m)!}} f^j_m
\]

then

\[
\begin{align*}
(D\rho^j)_{J_3} | j, m > &= m | j, m > \\
(D\rho^j)_{J_+} | j, m > &= \sqrt{(j-m)(j+m+1)} | j, m + 1 > \\
(D\rho^j)_{J_-} | j, m > &= \sqrt{(j+m)(j-m+1)} | j, m - 1 >
\end{align*}
\]

It is a well known proven fact that each representation \((H^j, \rho^j)\) of \(SU(2)\) is unitary and irreducible with the fact that the basis \(| j, m >\) is orthonormal.

The tensor product of representations is used thoroughly in LQG. The tensor product of two irreducible representations is reducible and therefore is the direct sum of irreducible representations which for \(SU(2)\) is known as the Clebsch-Gordan decomposition

\[
(H^j_1 \otimes H^j_2, \rho^{j_1} \otimes \rho^{j_2}) = (H^{j_2-j_1 \oplus H^{j_2-j_1} \oplus H^{j_2-j_1+1} \oplus \ldots \oplus H^{j_1+j_2}, \rho^{j_2-j_1} \oplus \ldots \oplus \rho^{j_1+j_2})
\]

This decomposition of the tensor product of two unitary irreducible representations implies that we can consider two orthonormal basis, one given by

\footnote{The differential of the representation \((E, \rho)\) of a Lie Group \(G\) gives a Lie Algebra representation. Studying Lie Algebra representations is equivalent to studying the representations of the Lie Group. See [7] for details.}
\[ |j_1, m_1 > \otimes | j_2, m_2 > \text{ where } -j_1 \leq m_1 \leq j_1, \quad -j_2 \leq m_2 \leq j_2 \]

And the second one given by

\[ |J, M > \text{ where } \quad |j_1 - j_2| \leq J \leq j_1 + j_2 \quad \text{and} \quad -J \leq M \leq J \]

Since the tensor product and the direct sum decomposition are isomorphic spaces we can change basis and write down one of the basis vectors as a linear combination of the others

\[ |J, M > = \sum_{m_1, m_2} C(J, M, j_1, m_1, j_2, m_2) \quad |j_1, m_1 > \otimes |j_2, m_2 > \]

where the coefficients \( C(J, M, j_1, m_1, j_2, m_2) \) are known as Clebsch-Gordan coefficients. It is more common in physics to use Wigner notation of these coefficients because of the symmetries which are easier to visualise. Let

\[
\begin{pmatrix}
  j_1 & j_2 & J \\
  m_1 & m_2 & -M
\end{pmatrix} = \frac{(-1)^{j_1 - j_2 + M}}{\sqrt{2J + 1}} C(J, M, j_1, m_1, j_2, m_2)
\]

Then we can write the change of basis formula as

\[ |J, M > = \sum_{m_1, m_2} (-1)^{j_2 - j_1 - M} \frac{\sqrt{2J + 1}}{2J + 1} \begin{pmatrix}
  j_1 & j_2 & J \\
  m_1 & m_2 & -M
\end{pmatrix} \quad |j_1, m_1 > \otimes |j_2, m_2 > \]

It is also useful to write down the change of basis the other way around. This is given by

\[ |j_1, m_1 > \otimes | j_2, m_2 > = \sum_{J,M} (-1)^{j_2 - j_1 - M} \frac{\sqrt{2J + 1}}{2J + 1} \begin{pmatrix}
  j_1 & j_2 & J \\
  m_1 & m_2 & -M
\end{pmatrix} \quad |J, M > \]

The symmetry properties of the Wigner symbol are can be found in the literature.

### 3 Graphs and Triangulations

This section is based on [9].

**Definition 11.** A point configuration in Euclidean space \( \mathbb{R}^n \) is a finite collection of points \( \Pi = \{p_1, p_2, ..., p_k\} \)
Definition 12. A convex hull of $\Pi$ is the intersection of all convex sets which contain the points of $\Pi$. It is denoted by $\text{conv}(\Pi)$.

Definition 13. A $d$-simplex is the convex hull of $d + 1$ affinely independent points in $\mathbb{R}^n$. We must have $n \geq d$.

Definition 14. A $j$-face of a $d$-simplex is the convex hull of $j + 1$ of its vertices.

Definition 15. A triangulation of a point configuration $\Pi \in \mathbb{R}^n$ is a collection of $n$-simplices all of whose vertices are points of $\Pi$ such that the union of all these simplices equals $\text{conv}(\Pi)$ and any pair of these simplices intersects in a common face.

It is allowed that the intersection of these pair of simpleces is empty.

Definition 16. A graph is a pair $\Gamma = (V, E)$ of sets, such that $E \subseteq [V]^2$.

$E \subseteq [V]^2$ means that the elements of $E$ are two-element subsets of $V$. It is always assumed that $V \cap E = \emptyset$. The elements of $V$ will be called vertices and the elements of $E$ will be called edges.

The vertex set of the graph $\Gamma$, will be denoted as $V(\Gamma)$ and its vertex set by $E(\Gamma)$.

Definition 17. The number of vertices of the graph $\Gamma$ is its order and is denoted $|\Gamma|$.

Definition 18. A vertex is incident to edge $e$ if $v \in e$. It is also said that $e$ is an edge at $v$. The two vertices incident to edge $e$ are called its endvertices.

The set of all edges at a vertex $v$ is denoted by $E(v)$.

Definition 19. The number of edges $E(v)$ at vertex $v$ is called the degree of vertex $v$ and it is denoted by $d_\Gamma(v)$ or simply by $d(v)$.

Definition 20. Given a triangulation of a point configuration $\Pi \in \mathbb{R}^n$ we associate a graph called the dual graph; this graph is drawn as follows: draw a node for each $n$-simplex and an edge when two $n$-simplices intersect in a $(n-1)$-face.

Consider a triangulation of a point configuration in $\mathbb{R}^3$ and its dual graph. The dual graph is called a 2-complex in LQG. Observe that the dual graph associated to the boundary simplicial triangulation is given by a trivalent graph $\Gamma = (V, E)$. 
4 Three dimensional LQG

We now start with the three dimensional study of quantum gravity as it is understood in LQG. This section is our main section and we try to be as mathematical as possible as in our previous sections. This section is entirely inspired on [2]. A mathematical description of loop quantum gravity can be found in [10], and reference [3] is a very advanced book on loop quantum gravity which is very mathematical. A very nice and useful introduction to the subject which uses more physical notation is [11].

4.1 Spin Networks

In LQG there is a Hilbert space which is interpreted as the quantum space of states. Consider a trivalent directed graph $\Gamma = (V, E)$ with $|V|$ number of vertices and $|E|$ number of edges. The Hilbert space of three dimensional LQG is a subspace of the Hilbert space $L^2(SU(2) \times L^2(SU(2)) \times \cdots \times L^2(SU(2))$ of $|E|$ products which we denote by $L^2(SU(2)^{|E|})$. This subspace is called the invariant space and it is the Hilbert space of the theory. This space is given as follows.

Consider a function $c : E \to SU(2)$ which goes from the set of edges to the set of equivalence classes of irreducible representations of $SU(2)$ with the following properties. This function $c$ is called labelling function. So for each directed edge $e$ we have $c(e) = (\mathcal{H}_j, \rho_j)$. Denote the matrix coefficients of this representation by

$$\rho_{mn}^j(g) = \langle e_m, \rho_j(g)e_n \rangle$$

We can think of this in a graphical way, where the directed edge is labelled $j$ and its vertices are labelled $m$ and $n$ and the edge is directed from vertex labelled $n$ to vertex labelled $m$.

For each vertex $v$ we have three directed edges $e_1, e_2, e_3$ incident to $v$ and suppose that $c(e_1) = (\mathcal{H}^{j_1}, \rho^{j_1})$ and $c(e_2) = (\mathcal{H}^{j_2}, \rho^{j_2})$ and $c(e_3) = (\mathcal{H}^{j_3}, \rho^{j_3})$.

Assign to the vertex $v$ the Wigner coefficients

$$v \mapsto \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} := \tilde{C}(j_1, m_1, j_2, m_2, j_3, m_3)$$

This is equivalent to assigning to the vertex $v$ an intertwining operator given by $T : \mathbb{C} \to \mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2} \otimes \mathcal{H}^{j_3}$. Although the notation of the Wigner coefficients is the $2 \times 3$ matrix, we will use the $\tilde{C}(j_1, m_1, j_2, m_2, j_3, m_3)$ notation of our own.

6The Wigner coefficient $\tilde{C}(j_1, m_1, j_2, m_2, j_3, m_3)$ in a graphical way is thought as a trivalent vertex with its edges directed outwards and labelled $j_1, j_2, j_3$. If we had $\tilde{C}(j_1, m_1, j_2, m_2, j_3, -m_3)$ then it is thought as a trivalent vertex with two of its edges directed outwards labelled $j_1, j_2$ and one inwards labelled $j_3$ Analogously if any of the indices $m_i$ changes sign.
Now we are ready to present an element of the invariant Hilbert space of three dimensional LQG.

By the Peter-Weyl theorem if $f \in L^2(SU(2)^{|E|})$ it must be of the form

$$f(g_1, g_2, \cdots, g_{|E|}) = \sum_{j_1 \cdots j_{|E|}} \sum_{m,n=1}^{dim} \rho^{j_1 j_2 \cdots j_{|E|}} c_{m_1 m_2 \cdots m_{|E|} n_{1 n_2 \cdots n_{|E|}}} \rho^{j_1}_{m_1 n_1}(g_1) \cdots \rho^{j_{|E|}}_{m_{|E|} n_{|E|}}(g_{|E|})$$

According to [2], the invariant Hilbert space is spanned by the functions $\Psi \in L^2(SU(2)^{|E|})$ given by [3]

$$\Psi(g_1, g_2, \cdots, g_{|E|}) = \sum_{m_1 m_2 \cdots m_{|E|-1 n_{|E|}}} \tilde{C}_1(j_1, m_1, j_2, m_2, j_3, m_3) \times \cdots \times \tilde{C}_{|V|}(j_{|E|-2}, n_{|E|-2}, j_{|E|-1}, n_{|E|-1}, j_{|E|}, n_{|E|}) \times \rho^{j_1}_{m_1 n_1}(g_1) \cdots \rho^{j_{|E|}}_{m_{|E|} n_{|E|}}(g_{|E|})$$

These functions are known as the quantum state functions. Therefore the invariant Hilbert space is given by linear combinations of these functions. This space is called Quantum Space.

Let us show with examples how these generator functions $\Psi$ are given.

4.1.1 Example

The theta graph

The theta graph $\Theta = (V, E)$ has two vertices $v_1, v_2$ and three edges $e_1, e_2, e_3$. If $c(e_1) = (H^{j_1}, \rho^{j_1})$, $c(e_2) = (H^{j_2}, \rho^{j_2})$ then recall that $j_3$ must be chosen between $\{|j_2 - j_1|, |j_2 - j_1| + 1, \cdots |j_1 + j_2|\}$. In this case the state function is given by

$$\Psi(g_1, g_2, g_3) = \sum_{m_1 m_2 m_3} \sum_{n_1 n_2 n_3} \tilde{C}(j_1, m_1, j_2, m_2, j_3, m_3) \tilde{C}(j_1, -n_1, j_2, -n_2, j_3, -n_3) \rho^{j_1}_{m_1 n_1}(g_1) \rho^{j_2}_{m_2 n_2}(g_2) \rho^{j_3}_{m_3 n_3}(g_3)$$

where the sum takes the values $-j_1 \leq m_1 \leq j_1$, $-j_1 \leq n_1 \leq j_1$, $-j_2 \leq m_2 \leq j_2$, $-j_2 \leq n_2 \leq j_2$, $-j_3 \leq m_3 \leq j_3$, $-j_3 \leq n_3 \leq j_3$.

---

[3] Recall that we have considered a trivalent graph $\Gamma = (V, E)$ with $|V|$ number of vertices and $|E|$ number of edges.
5 State sum model

As we have discussed it is equivalent to think of a triangulation of a point configuration or to think of the dual graph. Let $M$ be a triangulation of a point configuration in $\mathbb{R}^3$. Let $\partial M = \Sigma$ be its boundary\footnote{This boundary may have one component or may be composed of different components.} Observe that the boundary is a two dimensional triangulation in the sense that it is composed of 0-simplices, 1-simplices and 2-simplices.

**Definition 21.** A colouring of $M$ is a map $c : E(M) \to \widehat{SU}(2)$, where $E(M)$ is the set of 1-simplices of $M$ and $\widehat{SU}(2)$ is the set of equivalent irreducible representations of $SU(2)$.

**Definition 22.** Such colouring $c$ is called admissible if for each 2-simplex there is an intertwining operator $T : H_1 \otimes H_2 \otimes H_3 \to \mathbb{C}$.

Consider now the tensor product of three irreducible representations of $SU(2)$, $(H^{j_{12}} \otimes H^{j_{23}} \otimes H^{j_{34}}, \rho^{j_{12}} \otimes \rho^{j_{23}} \otimes \rho^{j_{34}})$. We know from section 2.1 that the tensor product of two irreducible representations decomposes in a direct sum of irreducible ones. Therefore, now have two possible homomorphisms

\[
F_1 = H^{j_{14}} \longrightarrow (H^{j_{12}} \otimes H^{j_{23}}) \otimes H^{j_{34}}
\]

and

\[
F_2 = H^{j_{14}} \longrightarrow H^{j_{12}} \otimes (H^{j_{23}} \otimes H^{j_{34}})
\]

As

\[
H^{j_{12}} \otimes H^{j_{23}} = \bigoplus_{j_{13}} H^{j_{13}}
\]

where $j_{13}$ runs from $|j_{23} - j_{12}|$ to $j_{23} + j_{12}$. We also have that

\[
H^{j_{23}} \otimes H^{j_{34}} = \bigoplus_{j_{24}} H^{j_{24}}
\]

where $j_{24}$ runs from $|j_{34} - j_{23}|$ to $j_{34} + j_{23}$.

**Definition 23.** The coefficients which appear in the change of basis in the diagram

\[
\begin{array}{ccc}
H^{j_{14}} & \xrightarrow{F_2} & H^{j_{12}} \otimes (H^{j_{23}} \otimes H^{j_{34}}) \\
\downarrow Id & & \downarrow Id \\
H^{j_{14}} & \xleftarrow{F_1} & (H^{j_{12}} \otimes H^{j_{23}}) \otimes H^{j_{34}}
\end{array}
\]
are called the \(6j\)-symbols and are denoted by
\[
\{ j_{12} \ j_{23} \ j_{13} \ \ j_{34} \ j_{14} \ j_{24} \}
\]

Explicitly the \(6j\)-symbols are given by
\[
\{ j_{1} \ j_{2} \ j_{3} \ j_{4} \ j_{5} \ j_{6} \} = \\
\sum_{m_{1},\ldots,m_{6}} (-1)^{\sum_{i=1}^{6}(j_{i}-m_{i})}\tilde{C}(j_{1},-m_{1},j_{2},-m_{2},j_{3},-m_{3})\tilde{C}(j_{1},m_{1},j_{5},-m_{5},j_{6},m_{6}) \\
\times \tilde{C}(j_{4},m_{4},j_{2},m_{2},j_{6},-m_{6})\tilde{C}(j_{3},m_{3},j_{4},-m_{4},j_{5},m_{5})
\]

The \(6j\)-symbol has the following symmetries
\[
\{ j_{1} \ j_{2} \ j_{3} \ j_{4} \ j_{5} \ j_{6} \} = \\
\{ j_{2} \ j_{1} \ j_{3} \ j_{4} \ j_{5} \ j_{6} \} = \\
\{ j_{3} \ j_{2} \ j_{1} \ j_{4} \ j_{5} \ j_{6} \} = \\
\{ j_{4} \ j_{2} \ j_{1} \ j_{5} \ j_{4} \ j_{6} \}
\]

**Definition 24.** Let \(M\) be a triangulation of a point configuration in \(\mathbb{R}^{3}\) and \(\partial M = \Sigma\) be its boundary. Let \(E^{o} = \{E(M) \in M^{o}\}\) be the interior edges of \(M\). The state sum is defined by
\[
Z_{M}[\Sigma] = \sum_{c} \prod_{E^{o}}(-1)^{2j}(2j+1) \prod_{\Delta}(-1)^{\sum_{i=1}^{6}j_{i}} \{ j_{1} \ j_{2} \ j_{3} \ j_{4} \ j_{5} \ j_{6} \}
\]
where the sum is over all functions \(c : E^{o} \to \tilde{SU}(2)\) of admissible colourings of the interior 1-simplices \(E^{o}\), and \(\Delta\) is the set of 3-simplices of \(M\).

The state sum \(Z_{M}[\Sigma]\) is a function of the colouring of the boundary \(\Sigma\) of \(M\) The colouring of the boundary is fixed since the state sum is only over the colouring of the interior 1-simplices. And if our triangulation configuration \(M\) has no boundary then \(Z_{M}[\Sigma] \in \mathbb{C}\).

It is known that this state sum is interpreted geometrically as a sum over metrics on the interior of a manifold which match the fixed boundary metric. Moreover, for most triangulations this state sum diverges and therefore regularisation of the sum is needed. We will not be concerned with the metric problem here but in a future work. Neither will we be concerned with the regularisation issue in this work as it is a different topic which has been extensively studied, see for example [12], [13]. These subjects will be treated in a different one.
Physically the state sum is a Feynman path integral of general relativity. In this case of three dimensional Euclidean general relativity. This state sum is known as the Ponzano-Regge model \cite{14} and its connection to loop quantum gravity was pointed out in \cite{15}. We will deal with the regularisation and the mathematics of this state sum in another work and will describe its properties.

6 Discussion

We have introduced a well known state sum of three dimensional loop quantum gravity. The model is a quantisation of three dimensional Euclidean general relativity without cosmological constant. Originally this model came from a discretisation of a Feynman path integral of three dimensional general relativity. But now, in the loop quantum gravity this discrete model is not something we impose for simplicity problems but it is a real consequence of the theory. Space is really quantised.

This three dimensional model is one of the simplest models of loop quantum gravity since it is a toy model in the physical sense. Moreover, we have introduced mathematical background which is thoroughly used in many topics of loop quantum gravity.

We have described the model in a mathematical way as much as formal as we could. Our purpose is to introduce loop quantum gravity to mathematicians or it could also be thought as a project in which we are introducing an approach to the field of loop quantum gravity from a mathematical side; from the point of view of a mathematician.

We hope that the present work, article \cite{6}, and future writings from a mathematical point of view inspire mathematics and physics students as well as researchers in the field to approach the subject of loop quantum gravity in a more rigorous mathematical way.
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