Reggeon interactions in perturbative QCD

R. Kirschner
DESY - Institut für Hochenergiephysik Zeuthen
Platanenallee 6, D-15735 Zeuthen, Germany

Abstract

We study the pairwise interaction of reggeized gluons and quarks in the Regge limit of perturbative QCD. The interactions are represented as integral kernels in the transverse momentum space and as operators in the impact parameter space. We observe conformal symmetry and holomorphic factorization in all cases.
1 Introduction

The data on deep-inelastic scattering at small $x$ show the features expected from the perturbative Regge asymptotics in QCD [1]. This provides the motivation to study the perturbative Regge region of QCD in more detail and to apply more effort to go essentially beyond the leading logarithmic approximation.

It is known that in the leading logarithmic approximation of perturbative QCD the Regge asymptotics with vacuum quantum number exchange is determined by the exchange of two reggeized gluons [2]. In the case of meson quantum number exchange we have instead the exchange of reggeized quark and anti-quark (of different flavours) [3] [4]. It is also known that the leading logarithmic contribution does not give the true asymptotics. Unitarity corrections become important with increasing energy. A minimal set of unitarity corrections consists of the contributions from the exchange of an arbitrary number of reggeized gluons with the same overall t-channel quantum numbers. There the trajectories of the reggeized gluons and quarks are to be taken in the leading logarithmic approximation. The exchanged reggeons interact only pairwise and the effective vertices, which determine this interaction, are just the same as in the two-reggeon exchange of the leading logarithmic approximation. This minimal improvement obeying unitarity in all sub-energy channels can be called generalized leading logarithmic approximation.

Beyond the leading logarithmic approximation there are other contributions not improving s-channel unitarity. They will not be considered here. In the Regge asymptotics they are to be treated as corrections to the reggeon interactions and to the reggeon-particle vertices [4].

In the case where the number of exchanged gluons and quarks does not change, the equations for the partial waves, the reggeon Green function, can written as a straightforward generalization of the two-reggeon case Fig. 1 [6] [7] [8].

$$\omega \ f^{(i)}_{\{i_1\}} = f^{(0)}_{\{i_1\}} + \sum_{i<j} \mathcal{H}^{(i,j)}_{\{i_1\};\{j_1\}} f^{(i')}_{\{i_1\}} \quad (1.1)$$

The set $\{i_1\} = (i_1, ..., i_r)$ labels the reggeons, where $i = G$ stands for a reggeized gluon and $i = F$ or $i = F$ stands for reggeized fermions in dependence on the helicity. Corresponding to each reggeon $i$ the partial wave $f$ carries an index $\alpha_i$, labelling the state in the corresponding gauge group representation and the argument $\kappa_i$, the transverse momentum, or $x_i$, the impact parameter of the reggeon. The term with $i$ and $j$ in the sum in (1.1) is the contribution from the interaction between the corresponding two reggeons.

The $r$-reggeon contribution to the partial-wave of the scattering amplitude is obtained from the $r$-reggeon Green function by convolution with impact factors $\Phi$, depending on the scattering particles. The angular momentum $j$ is related to $\omega$ by

$$j = 1 + \omega - \frac{r_F}{2} \quad (1.2)$$

$r_F$ is the number of exchanged fermions.
The present paper is devoted to the analysis of all two-reggeon interactions $\mathcal{H}^{(i,j)}$ in QCD. The cases $i = j = G$ and $i = F, j = \overline{F}$ are known from the studies of the two-reggeon exchange with vacuum \[2\] \[10\] and meson quantum numbers \[4\]. Besides of these there is the case $i = j = F$ of two exchanged fermions carrying the same helicity and the case $i = F, j = G$ of an exchanged fermion interacting with an exchanged gluon. The latter case appears with two contributions, corresponding to the interaction via $s$-channel gluon or fermion.

The graphical rules of the multi-Regge effective action \[11\] provide the framework for a simple derivation of $\mathcal{H}^{(i,j)}$. We show that in all cases the eigenvalues have a form compatible with conformal symmetry and holomorphic factorization. We consider the representation both in transverse momentum and in impact parameter spaces. In the first case $\mathcal{H}^{(i,j)}$ are written as integral kernels and in the second case as operators. The operator representation allows to write the interaction explicitly in a conform-symmetric and factorizable form.

The ideas of conformal symmetry in the Regge asymptotics and the operator approach appeared first in the study of gluon exchange \[10\] \[9\] \[12\] and have been developed in application to the fermion exchange ($i = F, j = \overline{F}$) \[4\]. Here we show that the methods allow an unified treatment of all reggeon interactions.

2 The interaction kernels

Consider the equation for the reggeon Green function $f$ (1.1) represented graphically in Fig. 1. The arguments referring to the reggeons $\overrightarrow{i}$ are not affected by the operator in the equation and will be suppressed. In the transverse momentum representation we have

$$\mathcal{H}^{(i,j)} f_{i_{1} \ldots i_{r}} = \frac{g^{2}}{(2\pi)^{3}} \int d\kappa_{1} d\kappa_{2} \delta(\kappa_{i} + \kappa_{j} - \kappa_{1}' - \kappa_{2}')$$

$$[(T_{i} \otimes T_{j})_{\mathcal{H}} \mathcal{H}^{(i,j)}_{i_{1} \ldots i_{r}} + (T_{i} \otimes T_{j})_{\mathcal{G}} \mathcal{G}^{(i,j)}_{i_{1} \ldots i_{r}}]$$

(2.1)

$T_{a}^{i}, a = 1, \ldots, N,$ are the generators of the $SU(N)$ representation of the reggeon $i$. The first contribution in (2.1) corresponds to the interaction via an $s$-channel gluon. Therefore the gauge group matrix is obtained from the generators by summing over the gluon colour states $a$.

$$(T_{i} \otimes T_{j})_{\mathcal{H}}^{(a_{i}, a'_{j})} = \prod_{\ell \neq i, j} \delta_{a_{i}, a'_{j}} \cdot (T_{a}^{i})_{a_{i}, a'_{j}} (T_{a}^{j})_{a_{j}, a'_{j}}$$

(2.2)

The second contribution in (2.1) corresponds to the interaction via an $s$-channel fermion.

In this case the gauge group matrix os obtained by summing over the fermion colour state $a$, e.g. if $i = G, j = F$,

$$(T_{i} \otimes T_{j})_{\mathcal{G}}^{(a_{i}, a'_{j})} = \prod_{\ell \neq i, j} \delta_{a_{i}, a'_{j}} \cdot (T_{a}^{i})_{a, a'_{j}} (T_{a}^{j})_{a, a'_{j}}$$

(2.3)

The overall gauge group state in the $t$ channel is the singlet state. (1.1) is valid only in this case.
There are important special cases, when the interaction in gauge-group representation space reduces to multiplication by a number. This simplification happens in the two- and three-gluon exchange and becomes an approximation at large N.

The kernels are functions of $\kappa_i, \kappa_j, \kappa'_i, \kappa'_j$ and as arguments of $f$ we have $\kappa', \kappa'_j$ and $\kappa_\ell$ for $\ell \neq i,j$. In the momentum representation we prefer to modify the definition of the reggeon Green function $f$ in such a way that it includes from the propagator of the $r$-reggeon state,

$$\left[ \omega - \sum_{\ell=1}^r \alpha_\ell(\kappa_\ell) \right]^{-1} \prod_{\ell=1}^r d_\ell(\kappa_\ell),$$

only the first factor but not the $t$-channel propagators $d_\ell(\kappa_\ell), \ell = 1, ..., r$,

$$d_G(\kappa) = |\kappa|^{-2}, \quad d_F(\kappa) = \kappa^{*-1}, d_{\overline{F}} = \kappa^{-1}.$$  \hfill (2.5)

We use the complex notation for the two-dimensional transverse momenta, $\kappa = \kappa_1 + i\kappa_2$. $\alpha_i(\kappa)$ determine the trajectories of the reggeized gluons ($i = G$) or quarks ($i = F, \overline{F}$). The gluon trajectory is

$$1 + N \alpha_G(\kappa), \quad \alpha_G(\kappa) = \frac{g^2}{2(2\pi)^3} \int \frac{d^2\kappa'}{|\kappa - \kappa'|^2|\kappa'|^2}$$

and the fermion trajectories are

$$\frac{1}{2} + C_2 \alpha_F(\kappa), \quad \alpha_F(\kappa) = \frac{g^2}{(2\pi)^3} \int \frac{d^2\kappa'|\kappa'|^2}{|\kappa - \kappa'|^2|\kappa'|^2}.$$

The interaction kernels can be derived easily by applying the effective graphical rules \[11\]. We use the effective vertices (Fig. 2a) $V_{ij}^\lambda$, where $\lambda$ denotes the helicity of the produced gluon,

$$V_{GG}^+ = \kappa\kappa'^*, \quad V_{GG}^- = \kappa'^*\kappa, \quad V_{FF}^- = \kappa'^*\kappa, \quad V_{FF}^- = \kappa^*.$$ \hfill (2.8)

By complex conjugation one obtains the corresponding vertices for the opposite helicities of the fermions and of the produced gluon.

We need also the transverse momentum factors $D_{ij}(\kappa - \kappa')$ in the propagators of the s-channel gluon or quark. For the gluon we have

$$D_{GG} = D_{FF} = D_{\overline{F}F} = |\kappa - \kappa'|^{-2}$$

and for fermions

$$D_{FG} = -D_{GF} = (\kappa - \kappa')^{*-1}, \quad D_{\overline{F}G} = -D_{GF} = (\kappa - \kappa')^{-1}.$$ \hfill (2.10)
Working with partial waves the longitudinal momentum integral of the loop in the last graph in Fig. 1 can be calculated. We are left with the transverse momentum integral of the form (2.1) with the kernel obtained from the graph Fig. 2b with the vertices (2.8) and the propagators (2.5) and (2.9), (2.10).

\[ \mathcal{H}^{(0)}_{ij,i'j'}(\kappa_i, \kappa_j; \kappa_{i'}, \kappa_{j'}) = \sum_{\lambda=\pm} V_{ii'}^-\lambda(\kappa_i, \kappa_{i'})D_{jj'}^\lambda(\kappa_j, \kappa_{j'})d_{i'}(\kappa_{i'})d_{j'}(\kappa_{j'}) + (i \leftrightarrow j) \]
\[ \mathcal{H}^{(0)}_{ij,i'j'} = \delta_{i'i} \delta_{jj'} \mathcal{H}^{(0)} + \delta_{ij'} \delta_{j'i} \mathcal{G}^{(0)} \] (2.11)

In the case of interaction via s-channel gluon there is a sum over its helicity states.

In the case \( i = i' = F, j = j' = \bar{F} \) the double logarithmic divergencies require a more careful treatment of the longitudinal momentum integration, which leads to additional \( \omega \)-dependent factors in the final expression. We shall not repeat the details given in [4] and quote just the result.

The sum over all loop contributions enters the equation Fig. 1 originally multiplied by the angular momentum factor of the \( r \)-reggeon propagator (2.4). We multiply both sides by the inverse of this factor, \( [\omega - \sum \alpha_\ell] \), and add \( \sum \alpha_\ell f \). Then the latter contribution can be included into the kernel \( \mathcal{H}_{ij,i'j'} \) (2.11) with the same gauge group operator in front (2.1). This is a non-trivial step, which can be done only because the overall gauge group state in the t-channel is the singlet one.

The resulting kernels as they enter the equation (1.1) in the transverse momentum representation (2.1) are obtained in the following form. We replace \( \kappa_i \rightarrow \kappa_1, \kappa_{i'} \rightarrow \kappa'_1, \kappa_j \rightarrow \kappa_2, \kappa_{j'} \rightarrow \kappa'_2 \).

\[ \mathcal{H}_{GG} = |\kappa_1 - \kappa'_1|^{-2} \left( \frac{\kappa_1 \kappa_2^*}{\kappa_1^2 \kappa_2} + \frac{\kappa_1^* \kappa_2}{\kappa_1 \kappa_2^2} - (\alpha_G(\kappa_1) + \alpha_G(\kappa_2)) \delta(\kappa_1 - \kappa'_1) \right), \]
\[ \mathcal{H}_{TF}^{(0)} = |\kappa_1 - \kappa'_1|^{-2} \left( \frac{\kappa_1^* \kappa_2}{\kappa_1^* \kappa_2} + \frac{\kappa_1 \kappa_2^*}{\kappa_1 \kappa_2^*} - (\alpha_T(\kappa_1) + \alpha_T(\kappa_2)) \delta(\kappa_1 - \kappa'_1) \right), \]
\[ \mathcal{H}_{FG} = |\kappa_1 - \kappa'_1|^{-2} \left( \frac{\kappa_1 \kappa_2^*}{\kappa_1^* \kappa_2} + \frac{\kappa_1^* \kappa_2}{\kappa_1 \kappa_2} - (\alpha_F(\kappa_1) + \alpha_F(\kappa_2)) \delta(\kappa_1 - \kappa'_1) \right), \]
\[ \mathcal{H}_{FF} = |\kappa_1 - \kappa'_1|^{-2} \left( \frac{\kappa_1^* \kappa_2}{\kappa_1 \kappa_2^*} + \frac{\kappa_1 \kappa_2^*}{\kappa_1^* \kappa_2} - (\alpha_F(\kappa_1) + \alpha_F(\kappa_2)) \delta(\kappa_1 - \kappa'_1) \right), \]
\[ \mathcal{G}_{FG} = |\kappa_1 - \kappa'_1|^{-2} \left( \frac{\kappa_1}{\kappa_1^*} - 1 \right). \] (2.12)

The cases with the replacement \( F \leftrightarrow \bar{F} \) are obtained by complex conjugation.

3 The eigenvalues of the interaction kernels

We study the special case if the momentum transfer in the t-subshannel (i,j) vanishes, \( \kappa_i = -\kappa_j = \kappa, \kappa'_i = -\kappa'_j = \kappa' \). The second term in (2.1) can be understood as \( P_{ij}\mathcal{G}_{ij} \),
where $P_{ij}$ is the operator of the permutation of the reggeon $i$ and $j$. Therefore it is reasonable to look at the eigenvalue problem for $G_{ij}$ as well as for $H_{ij}$.

\[
\int d^2\kappa' H_{ij}(\kappa,\kappa') f(\kappa') = \pi \Omega_{ij} f(\kappa), \\
\int d^2\kappa' G_{ij}(\kappa,\kappa') f(\kappa') = \pi \Omega_{ij} f(\kappa).
\]  

(3.1)

In the subchannel with vanishing momentum transfer we have rotation symmetry in the plane of transverse momenta. The appropriate complete orthogonal set of functions is

\[
\phi_{n,\nu}(\kappa) = |\kappa|^{-1+2\nu} \left( \frac{\kappa^*}{|\kappa|} \right)^n,
\]

(3.2)

parametrized by $\nu$, running over the real axis, and by $n$, taking all integer or all half-integer values in dependence of whether the fermion number in the sub-channel $(ij)$ is even or odd.

We choose the eigenfunctions $f^{n,\nu}$ in such a way that their orthogonality relation is written with the propagators of the particles $i$ and $j$ as a weight. Therefore we define

\[
f^{n,\nu} = (d_i(\kappa)d_j(\kappa))^{-1/2} \phi_{n,\nu}. \]

(3.3)

In the calculation it is convenient to extend the dimension to $2 + 2\epsilon$. The pole terms in $\epsilon$ cancel, since the kernels are infrared finite, and we obtain

\[
\Omega_{F F} = 4\psi(1) - \psi(m) - \psi(1 - m) - \psi(\tilde{m}) - \psi(1 - \tilde{m}) - \psi(1 + \frac{1}{2}) - \psi(1 - \frac{1}{2}) - \psi(1 + \frac{1}{2}) - \psi(1 - \frac{1}{2}), \\
\Omega_{GG} = 4\psi(1) - \psi(m) - \psi(1 - m) - \psi(\tilde{m}) - \psi(1 - \tilde{m}), \\
\Omega_{F G} = \Omega_{G G}, \\
\Omega_{F F} = \Omega_{G G}, \\
\Omega_{F G} = (\tilde{m} - \frac{1}{2})^{-1}.
\]

(3.4)

We have used the digamma function,

\[
\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \psi(1) - \sum_{\ell=0}^{\infty} \left( \frac{1}{\ell + z} - \frac{1}{\ell + 1} \right).
\]

(3.5)

The notations

\[
m = \frac{1}{2} + i\nu + \frac{n}{2}, \tilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}
\]

(3.6)

appear in the following as conformal weights. All eigenvalue functions (3.4) can be written as sums of two functions, one depending on the eigenvalue $m(1 - m)$ of the holomorphic and the other depending on the eigenvalue $\tilde{m}(1 - \tilde{m})$ of the anti-holomorphic Casimir operator of the linear conformal transformations. This is the necessary condition for conformal symmetry and holomorphic factorization of the equation.
We define
\[ \chi_\Delta(z) = \sum_{\ell=0}^{\infty} \left( \frac{2(\ell + \Delta) + 1}{(\ell + \Delta)(\ell + \Delta + 1) + z} - \frac{2}{\ell + 1} \right) \] (3.7)
and observe from (3.5)
\[ \chi_\Delta(m(1 - m)) = 2\psi(1) - \psi(m + \Delta) - \psi(1 - m + \Delta) \] (3.8)
to obtain the desired result for the eigenvalue functions,
\[ \Omega_{FG} = \frac{1}{2} \left( \chi_{\frac{1}{2}}(m(1 - m)) + \chi_{\frac{-1}{2}}(m(1 - m)) + \chi_{\frac{1}{2}}(\bar{m}(1 - \bar{m})) + \chi_{\frac{-1}{2}}(\bar{m}(1 - \bar{m})) \right), \]
\[ \Omega_{GG} = \chi_0(m(1 - m)) + \chi_0(\bar{m}(1 - \bar{m})), \]
\[ \Omega_{FG} = \chi_0(m(1 - m)) + \chi_0(\bar{m}(1 - \bar{m})), \]
\[ \Omega_{F/G} = \left( -\bar{m}(1 - \bar{m}) + \frac{1}{4} \right)^{-1/2}. \] (3.9)

We find that \( \Omega_{FG} \) consists of the holomorphic part of \( \Omega_{GG} \) and the anti-holomorphic part of \( \Omega_{FF} \), where in the latter \( \omega \) is put to zero. \( \Omega_{FF} \) coincides with \( \Omega_{GG} \). The holomorphic part of \( \Omega_{F/G} \) vanishes.

4 Operators in the impact parameter space

We study the reggeon Green function in the impact parameter representation, \( f(\omega; x_1, ..., x_r) \), keeping the symbol \( f \) also for the Fourier transformed function. We use an operator representation for the equation (1.1).

\[ \mathcal{H}^{(ij)} f_{i_1i_2...j_1j_2} = \frac{g^2}{8\pi^2} \left( (T_i \otimes T_j) \mathcal{H} \mathcal{H}_{ij} f_{i_1i_2...j_1j_2} + (T_i \otimes T_j) \mathcal{G} \mathcal{G}_{ij} f_{i_1i_2...j_1j_2} \right) \] (4.1)

The operators \( \mathcal{H}_{ij} \) and \( \mathcal{G}_{ij} \) are functions of the elementary operators of multiplication with \( x_i \) and \( x_j \) and differentiations \( \partial_i \) and \( \partial_j \). They can be read off from the kernels (2.12) by the following simple substitution rules. The momenta are replaced by derivatives, \( \kappa_1 \rightarrow \partial_1^*, \kappa_1^* \rightarrow \partial_1 \), and \( \delta(\kappa_1 - \kappa_2) \) by 1. The propagators of the s-channel particles are replaced as

\[ D_G(\kappa_1 - \kappa_1') = \frac{1}{|\kappa_1 - \kappa_1'|^2} \rightarrow -\ln |x_{12}^2| + \psi(1), \]
\[ D_F(\kappa_1 - \kappa_1') = \frac{1}{(\kappa_1 - \kappa_1')^2} \rightarrow (x_{12}^*)^{-1}, \] (4.2)

where \( x_{12} = x_1 - x_2 \). The trajectories are replaced as

\[ \frac{1}{2} \alpha_G(\kappa_1) = \alpha_F(\kappa_1) \rightarrow -\ln(\partial_1 \partial_1^*). \] (4.3)

(4.2) and (4.3) can be understood in dimensional regularization. The poles in \( \epsilon \) from the s-channel gluon propagator (4.2) and from the trajectories cancel in the operators.
The resulting operators decompose into sums of holomorphic (acting only on $x_1, x_2$) and anti-holomorphic parts.

\[
\begin{align*}
\hat{H}_{GG} &= H_G + H^*_G, \\
\hat{H}_{FF}^{(\omega)} &= H_F^{(\omega)} + P_{12}H_F^{(\omega)*}P_{12}, \\
\hat{H}_{FF} &= H_G + \hat{H}_F, \\
\hat{H}_{FG} &= H_G + P_{12}H_F^{(0)*}P_{12}, \\
\hat{G}_{FG} &= (x_{12}\partial_2^*)^{-1} = -P_{12}D_1^{-1}P_{12}.
\end{align*}
\]

$P_{12}$ denotes the operator of permutation of $x_1$ and $x_2$. We use the notation $D_1 = x_{12}\partial_1$ and $D_2 = x_{12}\partial_2$.

The operators $H_G$ and $H_F^{(\omega)}$ have been encountered before [9] [12] [4].

\[
\begin{align*}
H_G &= 2\psi(1) - \partial_1^{-1} \ln x_{12} \partial_1 - \partial_2^{-1} \ln x_{12} \partial_2 - \ln \partial_1 - \ln \partial_2, \\
H_F^{(\omega)} &= 2\psi(1) - \partial_1^{1+\omega/2} \ln x_{12} \partial_1^{1-\omega/2} - \ln \partial_1 - \partial_2^{-\omega/2} \ln x_{12} \partial_2^{\omega/2} - \ln \partial_2.
\end{align*}
\]

The new operators are $\hat{H}_F$, being the complex conjugate of

\[
\hat{H}_F = 2\psi(1) - 2\ln x_{12} - \ln \partial_1 - \ln \partial_2,
\]
and the simple operator $\hat{G}_{FG}$ given explicitly in (4.5). We express $H_G, H_F$ and $\hat{H}_F$ as $\psi$-functions of $D_1 = x_{12}\partial_1$ and $D_2 = x_{12}\partial_2$. This representation is obtained by observing that $x^2\partial$ and \( \partial \) can be written as similarity transformations of $x$ and $x^{-1}$, respectively.

\[
x^2\partial = \Gamma(x\partial)x(\Gamma(x\partial))^{-1}, \\
\partial = (\Gamma(x\partial + 1))^{-1}x^{-1}\Gamma(x\partial + 1).
\]

The operators as functions of $x\partial$ are determined actually only up to periodic functions with the period 1 of the same argument. This ambiguity matters if we apply (4.7) to logarithms. The relations imply the identity

\[
\ln(x^2\partial) - \ln x = \partial^{-1} \ln x\partial - \ln \partial
\]
and lead to the following representation,

\[
\begin{align*}
H_G &= 2\psi(1) - \frac{1}{2} (\psi(D_1) + \psi(1 - D_1) + \psi(-D_2) + \psi(1 + D_2)), \\
H_F^{(\omega)} &= 2\psi(1) - \frac{1}{2} \left( \psi(-D_1 + \frac{\omega}{2}) + \psi(1 - D_1 - \frac{\omega}{2}) + \frac{\omega}{2} \left( \psi(-D_2 + \frac{\omega}{2}) + \psi(1 + D_2 - \frac{\omega}{2}) \right) \right) + \\
&\quad \frac{1}{2} \left( \psi(-D_2 + \frac{\omega}{2}) + \psi(1 + D_2 - \frac{\omega}{2}) \right), \\
\hat{H}_F &= 2\psi(1) - \frac{1}{2} (\psi(-D_1) + \psi(D_2) + \psi(1 + D_1) + \psi(1 - D_2)).
\end{align*}
\]

5 Conformal symmetry

The result (3.9) about the eigenvalues tells us that the operators can be written in a conform-symmetric form. We shall express the holomorphic operators in terms of the holomorphic Casimir operator of the linear conformal group.
Starting with the form (4.5), (4.6) of the operators and applying the relation (4.8) we observe the following behaviour under conformal inversions $I$.

\[
\mathcal{I} H G \mathcal{I} = H G, \quad \mathcal{I} H_F^{(0)} \mathcal{I} = x_2 H_F^{(0)} x_2^{-1}, \quad \mathcal{I} P_{12} D_1 \mathcal{I} = x_2 (P_{12} D_1) x_2^{-1}.
\] (5.1)

The equation with these operators is conform-symmetric if the Green function $f$ transforms correspondingly, i.e. as a correlator with operators in the points $x_1, x_2$ of the conformal weights $(\Delta_1, \tilde{\Delta}_1), (\Delta_2, \tilde{\Delta}_2)$.

\[
\Delta_\ell = \frac{1}{2} (\delta_\ell + s_\ell), \quad \tilde{\Delta}_\ell = \frac{1}{2} (\delta_\ell - s_\ell).
\] (5.2)

$\delta_\ell, \ell = 1, 2$ are the scaling dimensions and $s_\ell$ are the conformal spins. From (5.1) and (4.5) we see that the conformal operator corresponding to a reggeized gluon has

\[
\delta_G = s_G = 0
\] (5.3)

and the conformal operators representing fermionic reggeon have in dependence on the helicity have

\[
\delta_F = -s_F = \frac{1}{2}, \quad \tilde{\delta}_F = \tilde{s}_F = \frac{1}{2}
\] (5.4)

The eigenvalue problem can also be studied in the operator representation. Because of the conformal symmetry there is a simple solution without restrictions on the momentum transfer. The eigenfunctions are the conformal 3-point functions,

\[
E_{\delta_1, s_1, \delta_2, s_2}^{(n, \nu)} = \langle \phi_{\delta_1, s_1}(x_1) \phi_{\delta_2, s_2}(x_2) O^{(n, \nu)}(x_0) \rangle,
\] (5.5)

which are determined up to a factor by the scaling dimensions $\delta_1, \delta_2, \frac{1}{2} + i\nu$ and the conformal spins $s_1, s_2, n$. For given $\delta_1, \delta_2, s_1, s_2$, characterizing the interacting reggeons (compare (5.3), (5.4)), they form a twofold overcomplete basis of functions of two impact parameters $x_1, x_2$. The set is parametrized by the scaling dimension $\nu$, running over the real axis, the conformal spin $n$, taking all integer values for $s_1 + s_2$ integer or all half-integer values for $s_1 + s_2$ half-integer, and by the position $x_0$, running over the impact parameter space. The eigenvalues coincide with the above results (3.4), (3.9).

The explicite form of the 6 generators $M_{12}^{\pm}, M_{12}^{(0)}, \tilde{M}_{12}^{\pm}, \tilde{M}_{12}^{(0)}$ of the coformal transformations acting on functions of $x_1$ and $x_2$ depends on the conformal weights $(\Delta_1, \tilde{\Delta}_1), (\Delta_2, \tilde{\Delta}_2)$.

\[
M_{12}^{+} = x_1^2 \partial_1 + 2\Delta_1 x_1 + x_2^2 \partial_2 + 2\Delta_2 x_2,
\]
\[
M_{12}^{-} = \partial_1 + \partial_2,
\]
\[
M_{12}^{(0)} = x_1 \partial_1 + \Delta_1 + x_2 \partial_2 + \Delta_2.
\] (5.6)

The anti-holomorphic generators are obtained by complex conjugation and by replacing $\Delta_1, \Delta_2$ by $\tilde{\Delta}_1, \tilde{\Delta}_2$. The explicite form of the holomorphic Casimir operator depends on the weights $\Delta_1, \Delta_2$ as follows.

\[
C_{\Delta_1, \Delta_2} = -M_{12}^{(0)2} + \frac{1}{2} (M_{12}^{+} M_{12}^{-} + M_{12}^{+} M_{12}^{+})
\]
\[ x_{12}^2 \partial_1 \partial_2 + 2x_{12}(\Delta_1 \partial_2 - \Delta_2 \partial_1) + (\Delta_1 + \Delta_2)(1 - \Delta_1 - \Delta_2). \]  

For values of \( \Delta_1, \Delta_2 \) differing by \( \frac{1}{2} \) the Casimir operator can be expressed as a square of a simpler operator plus a constant, a property related to supersymmetry. We need the cases \( \delta_1 = 0, \delta_2 = \frac{1}{2} \) and \( \delta_1 = \frac{1}{2}, \delta_2 = 0 \).

\[ C_{0\frac{1}{2}} = -A_{F2}^2 + \frac{1}{4}, \quad A_{F2} = P_{12}x_{12}\partial_1, \]
\[ C_{\frac{1}{2}0} = -A_{F1}^2 + \frac{1}{4}, \quad A_{F1} = P_{12}x_{12}\partial_2. \]  

We find immediately from (4.5) that the interaction by fermion exchange is expressed in terms of these operators,

\[ P_{12}G_{FG} = \left(A_{F1}^*\right)^{-1}, \]  

in agreement with the eigenvalues (3.4), (3.9).

We look for the expressions of the remaining operators in terms of the Casimir operators starting from (4.9). We are aware of the ambiguity in this representation because of the non-uniqueness of the similarity transformation (4.7). The transformations, which we are going to perform, are valid in general only approximately on a restricted class of functions. In the \( \rho q \) representation these functions should be small outside the region \( \rho q \ll 1 \). The \( \rho q \) representation is obtained from the impact parameter representation \( f(x_1, x_2) \) by identifying \( \rho \) with \( x_{12} = x_1 - x_2 \) and by Fourier transformation with respect to \( R = (x_{10} + x_{20})/2 \). The Fourier conjugate variable to \( R \) is the momentum transfer \( q \). Acting on this class of functions we have for the operator

\[ D_1 + D_2 \ll 1. \]  

Within this approximation we obtain from (4.9) the expression in terms of the Casimir operators using (3.5). Despite of the approximation the result is valid in general, without restrictions on the functions, because of the conformal symmetry.

\[ H_{G} = \chi_0(C_{00}), \]
\[ H_{F}^{(\omega)} = \frac{1}{2}(\chi_{\frac{1}{2}0}(C_{0\frac{1}{2}}) + \chi_{\frac{1}{2}1}(C_{0\frac{1}{2}})), \]
\[ \tilde{H}_{F} = \chi_0(C_{\frac{1}{2}0}). \]  

These results confirm the results (3.9) about the eigenvalues.

6 Discussions

The Regge asymptotics in perturbative QCD can be represented in terms of reggeized gluons and quarks in the exchange channel which interact by emitting and absorbing s-channel gluons and quarks. This representation is quite useful although those reggeons have no meaning independent of a regularization because their trajectories are infrared
divergent. However in gauge group singlet channels these divergencies cancel against divergencies in the reggeon interactions. Therefore in these channels the reggeon interaction can be represented in terms of infrared finite operators.

In the generalized leading logarithmic approximation we impose the conditions of multi-Regge kinematics on all s-channel intermediate states. In this case the reggeon interact only pairwise.

We have studied all cases of these two-reggeon interactions that occur in QCD. We have represented them as intergral kernels in transverse momenta and as operators in the impact parameter space. The eigenvalues can be obtained in the momentum representation in the limit of vanishing momentum transfer. The form of the resulting eigenvalue functions is compatible with the properties of holomorphic factorization and conformal symmetry. The operator approach in the impact parameter space allows to represent the two-reggeon interaction operators as sums of holomorphic and anti-holomorphic parts. Moreover these parts are functions of the Casimirir operator of the holomorphic or anti-holomorphic linear conformal transformations, respectively. More details about this approach will be published elsewhere [14].

It is remarkable that the treatment invented for the gluon exchange extends to the case involving fermions. Whereas the reggeized gluons are represented by operators with vanishing conformal dimension and spin, the reggeized fermions correspond to conformal dimension $\frac{1}{2}$ and spin $\pm \frac{1}{2}$. This result provides a further piece of information about the simplicity and the symmetry of perturbative QCD in the Regge limit.

Acknowledgements

The author is grateful to L.N. Lipatov and L. Szymanowski for discussions.

References

[1] I. Abt et al., Nucl. Phys. B407 (1993), 515;
M. Derrick et al., Phys. Lett. B316 (1993), 412

[2] V.S. Fadin, E.A. Kuraev and L.N. Lipatov, Phys. Lett. 60B(1975)50; ZhETF 71(1976)840; ibid 72(1977)377;
Y.Y. Balitski and L.N. Lipatov, Sov. J. Nucl. Phys. 28(1978) 882

[3] R. Kirschner, Zeitschr. f. Phys. C31 (1986) 135
[4] R. Kirschner, "Regge asymptotics of scattering with flavour exchange in QCD", preprint DESY 94-090

[5] V.S. Fadin and L.N. Lipatov, Pisma ZhETF 46 (1989) 311; Yad. Fiz., 50 (1989) 1141; V.S. Fadin and R. Fiore, Phys. Lett. B294 (1992) 286

[6] J. Kwiecinski and M. Praszalowicz, Phys. Lett. B94(1980) 413

[7] J. Bartels, Nucl. Phys. B175 (1980) 365

[8] L.N. Lipatov in ”Perturbative QCD” A.H. Mueller ed., World Scientific 1989

[9] L.N. Lipatov, Phys. Lett 251B (1990) 284

[10] L.N. Lipatov, ZhETF 90 (1986), 536

[11] R. Kirschner, L.N. Lipatov and L. Szymanowski, ”Effective action for multi-Regge processes in QCD”, hep-th 9402010. Siegen Univ. preprint (1994), to be published in Nucl. Phys. B ; ”Symmetry properties of the eff. action ...”, hep-th 9403082, preprint DESY 94-064, to be published in Phys. Rev. D

[12] L.N. Lipatov, Phys. Lett. 309B (1993) 393

[13] A.M. Polyakov, ZhETF 66 (1974) 23

[14] R. Kirschner and L.N. Lipatov, work in progress

**Figure captions**

Fig. 1 The equation for the r-reggeon Green function.
   The horizontal lines represent reggeized gluons or quarks.
   The vertical line is a gluon (H_{ij}) or a quark (G_{FG}).
   In the last term a sum over i, j is understood.

Fig. 2 Graphical rules for the interaction kernels.
   a) Effective vertex.
   b) Two-reggeon interaction.
\[ f = \cdots \] + \[ f \]}

Fig. 1

\[ a \]

\[ b \]

Fig. 2