MEDIUM-INDUCED RADIATIVE ENERGY LOSS; EQUVALENCE BETWEEN THE BDMPS AND ZAKHAROV FORMALISMS

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Abstract

We extend the BDMPS formalism for calculating radiative energy loss to the case when the radiated gluon carries a finite fraction of the quark momentum. Some virtual terms, previously overlooked, are now included. The equivalence between the formalism of BDMPS and that of B. Zakharov is explicitly demonstrated.

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1 Introduction

Over the past few years there has been much work done\cite{1–6} studying the radiative energy loss of high energy partons passing through hot and cold matter. These studies are extensions to QCD of the analogous QED problem considered long ago by Landau, Pomeranchuk and Migdal \cite{7, 8}. In one version of this work initiated in Refs. \cite{1} and \cite{2} and continued in Refs. \cite{3} and \cite{4} one follows the multiple scattering \cite{9, 10} of the high energy parton in the QCD matter and the radiative gluon spectrum induced by the multiple scattering is evaluated. A number of interesting, and perhaps surprising results were found. The energy loss of a high energy jet in a hot QCD plasma appears to be much larger than that in cold nuclear matter even at moderate, say 200 MeV, temperatures of the plasma. When a very high energy parton passes through a length $L$ of hot or cold matter the induced radiative energy loss is proportional to $L^2$. A curious relation, $-dE/dz = \text{const} \cdot \alpha_s N_c p_{\perp}^2 W$, was found between the energy loss and the width of the Gaussian transverse momentum broadening of a high energy parton in QCD matter.

A different and very elegant approach to the energy loss problem has been developed by Bronislav Zakharov \cite{5, 6}. In his approach the gluon radiation probability is given by the difference of the probabilities of a virtual fluctuation of a high energy quark into a quark and a gluon occurring in the vacuum and in the medium. The formalism uses a clever device of treating the quark in the complex conjugate amplitude as an antiquark in the amplitude. What is finally calculated then is the amplitude for a quark-antiquark-gluon system to pass through a QCD medium without inducing inelastic reactions in the medium. For a medium having many scatterers such an amplitude is not small only if the quark-antiquark-gluon system is compact in transverse coordinate-space, and this compactness makes the process perturbatively calculable in QCD.

One of the major purposes of the present work is to show that the BDMPS \cite{3, 4} and Z \cite{5, 6} formalisms are equivalent. However, before showing this equivalence it is necessary to extend the BDMPS formalism beyond the soft gluon approximation where it was originally formulated. It is also necessary to include some virtual graphs which were omitted in our original formulation and which led to some numerical discrepancies when our results were compared to those in Refs. \cite{5} and \cite{6}.

In Secs. 2 and 3 we outline a derivation of the BDMPS formulas for the radiative gluon spectrum for a high energy quark passing through either a hot or cold QCD medium, which formulas are valid even when the radiated gluon carries a finite fraction of the quark’s momentum. Special care is taken to show which factors have changed from our previous results when all terms corresponding to virtual corrections (elastic scattering in the medium) are included.

In Sec. 4 we evaluate our formulas for the radiative gluon energy spectrum. We consider both the case when the quark approaches the medium from outside and when the quark is produced by a hard scattering in a medium.

In Sec. 5 we show the equivalence between our approach and that of B. Zakharov. We show that our formula \cite{19} for the radiative spectrum leads to \cite{29} which is equivalent to (4) of Ref. \cite{6}. We then explain more qualitatively why the two formalisms, seemingly very different, are in fact completely equivalent.
2 The Born term for the amplitude

In this section we calculate the lowest order terms for the emission of a gluon from a quark which may enter the QCD matter from the vacuum or which may be produced in the matter through a hard interaction. We begin by giving the basic vertex for emission of a gluon of momentum \( k \) from a quark of momentum \( p \).

We shall continue to call the fermion from which the gluon is emitted a quark. However, in our final formulas results for spin \( 1/2 \) fermions of any colour representation, \( R \), can be obtained by multiplying the gluon emission spectrum (42b) by the ratio of the “colour charges”, \( C_R/C_F \). The generalisation to spin–1 (gluon) and spin–0 projectiles will also be given in Sec. 5.2.

2.1 The basic vertex

The basic vertex for gluon emission is given by

\[
\bar{u}_{r'}(p-k) \gamma \cdot \epsilon u_r(p) = \frac{2\delta_{rr'}}{x\sqrt{1-x}} \epsilon \cdot \{k - xp\}.
\]

(1)

We use a notation \( \{ \} \) such that for any two-dimensional vector \( v_i, i = 1,2 \)

\[
\{v_i\} = \left(1 - \frac{x}{2}\right) v_i - i \frac{x}{2} \epsilon_{ij} v_j,
\]

(2)

where \( \epsilon_{ij} \) is the antisymmetric tensor; \( \epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0 \). The gluon polarisation vector is

\[
\epsilon_{\mu} = (\epsilon_0, \epsilon_z, \epsilon) = \left(\frac{\epsilon \cdot k}{2k}, -\frac{\epsilon \cdot k}{2k}, \epsilon\right).
\]

(3)

In (1–3) we assume that

\[
|k| \ll |\vec{k}| = k,
\]

(4a)

\[
|p| \ll |\vec{p}| = p,
\]

(4b)

but we do not assume that \( x = k/p \) is small. The vertex is diagonal in the quark spin indices \( r \) and \( r' \) because of the vector nature of the coupling and because of our assumption that the fermion is massless.

2.2 The Born terms for the amplitude; scattering in the medium

In our procedure one explicitly integrates the emission time, \( t \), of the gluon between the times the quark, or the quark-gluon system, scatters in the medium. Consider, for example, the graphs in Fig 2a where the quark scatters inelastically in the medium at time \( t_1 \) and then later emits the gluon at time \( t \). The \( t \)-dependent phase factor associated with this graph is

\[
\exp\left[ it \left(\frac{k^2}{2k} + \frac{(p-k)^2}{2(p-k)} - \frac{p^2}{2p}\right)\right] = \exp\left[ it \frac{(k-xp)^2}{2x(1-x)p}\right].
\]

(5)
The lower limit of the \( t \)-integration is \( t_1 \) while the upper limit will be the time of the next interaction or \( t = \infty \) in case the quark-gluon system leaves the medium without further interactions. Keeping the lower limit of the integral of the expression (5) and multiplying by the gluon emission vertex gives the factor

\[
4ip\sqrt{1-x} \exp \left[ it_1 \frac{(k-xp)^2}{2x(1-x)p} \right] T_a \cdot \xi, \tag{6}
\]

where

\[
T_a = \frac{(k-xp)}{(k-xp)^2}. \tag{7a}
\]

Similarly one can evaluate the emission term and energy denominators in the remaining graphs b–g of Fig. 1 to obtain

\[
T_b = -\frac{(k-xp - (1-x)q)}{(k-xp - (1-x)q)^2}, \tag{7b}
\]
\[
T_c = -\frac{(k-xp + xq)}{(k-xp + xq)^2}, \tag{7c}
\]
\[
T_d = -T_e = -T_f = T_a, \tag{7d}
\]
\[
T_g = -\frac{(k-xp - q)}{(k-xp - q)^2}. \tag{7e}
\]

The \((-1)\) factors in some terms in (7) occur when \( t_1 \) is the upper limit of the \( t \) integration, as happens for the graphs b, c, e, f and g of Fig. 1.

Graphs a–c of Fig. 1 correspond to inelastic reactions with the medium while graphs d–g correspond to forward elastic scatterings in the medium. For terms a–c there are corresponding inelastic reactions in the complex conjugate gluon emission amplitude. In the approximation that the forward elastic amplitude for quark scattering off particles in the medium is purely imaginary, the elastic and inelastic terms are proportional to the same function \( V \) to be defined more precisely below.

It is remarkable that the gluon emission vertices are proportional to exactly the same combinations of transverse momenta whose squares enter in the energy denominators for each of the terms in (7). This is a consequence of gauge invariance. The fact that the emission amplitude is inversely proportional to the first power of the relevant transverse momentum follows from the Gribov bremsstrahlung theorem [11], the generalisation of the Low-Barnett-Kroll theorem [12] to hard collinear radiation, \( x \sim 1, k_\perp \ll p \). This will prove crucial in obtaining a simple form for the gluon emission spectrum when \( x \) is not small.

It is our convention that the time of emission of the gluon in the complex conjugate amplitude, \( t' \), is later than the emission time in the amplitude. The opposite sequence of times will be accounted for by a factor of 2 in the gluon spectrum (31). This convention means that for the terms in the complex conjugate amplitude corresponding to a–c of Fig. 1 it is only the quark and not the quark-gluon system which scatters inelastically off the medium at time \( t_1 \). Thus the colour factors associated with the scattering in the medium for the graph in Fig. 1a is obtained
by considering the graph of Fig. 2a where the part of the graph to the right of the vertical line (cut) is the complex conjugate amplitude. The initial and final $p$-lines have the same colour. The graph in Fig. 2b corresponds to the graph in Fig. 1b along with the corresponding complex conjugate amplitude. Using the formulas for colour factors given in Appendix B of Ref. [3] it is straightforward to evaluate the “colour factors” associated with the hooking of the $q$-lines in Fig. 1 (and Fig. 2) with the quark or quark-gluon passing through the medium, namely

$$F_a = -2 F_d = -2 F_f = C_F,$$

$$F_b = -F_c = F_g = \frac{N_c}{2},$$

$$F_c = \frac{-1}{2N_c}.$$  \hfill (8a, 8b, 8c)

In addition to pure colour factors the $F$’s also include a factor of $(-1)$ for virtual terms and a factor of $1/2$ for virtual terms where the $q$ and $-q$ lines attach to the same quark or gluon line. These factors naturally appear in a Feynman diagram description of Glauber multiple scattering. For quark propagating through a QCD medium these factors guarantee probability conservation. Indeed, the virtual corrections to the amplitude and the complex conjugate amplitude, $2 \ast (\frac{-1}{2})$, cancel the $+1$ coming from the scattering in the medium.

It is convenient to introduce scaled momentum variables

$$\frac{k}{\mu} = U, \quad \frac{q}{\mu} = Q, \quad \frac{p}{\mu} = V,$$

where $\mu$ is an appropriate scale for the problem. In case the high energy fermion moves through a hot QCD plasma $\mu$ may be taken to be the inverse Debye screening length, while for cold nuclear matter $\mu$ is naturally taken to be a typical transverse momentum exchanged in a quark-nucleon scattering. In addition to the factors $T_i$ and $F_i$ the graphs in Fig. 1 naturally include a quark-“particle” cross section when corresponding complex conjugate amplitudes are included. Thus, for example, the graph of Fig. 1a when multiplied by a similar complex conjugate amplitude, leads to the graph illustrated in Fig. 2a, and this graph is clearly proportional to the quark-particle scattering cross section in a two-gluon-exchange approximation. We define a normalised quark-particle cross section $\sigma$ by

$$V(Q^2) = \frac{1}{\pi} \frac{d\sigma}{dQ^2}$$

with

$$\sigma = \int \frac{d\sigma}{d^2Q} d^2Q.$$  \hfill (10, 11)

In case the medium is cold nuclear matter the “particle” referred to above is a nucleon while in case the medium is a hot QCD plasma the particle can be taken to be a quark or gluon.

In this paper we do not consider collisional energy loss. To ensure the validity of the independent scattering picture and to guarantee that $d\sigma/dQ^2$ depend only on transverse momentum, it suffices to assume that the energy transfer from the quark to a particle in the medium be small compared to the incident energy $E_0$. 

5
It is useful to combine factors to define a Born-term gluon emission amplitude $f_0(U,V)$ as

$$
\frac{N_c}{2C_F} f_0(U,V) = \frac{1}{C_F} \sum_{i=1}^g \int d^2Q V(Q^2) T_i(U,V,Q) F_i.
$$

(12)

Here $f_0$ is an amplitude for gluon emission although it also embodies colour factors and quark-particle scattering factors from the complex conjugate amplitude. The factor $N_c/2C_F$ is included to agree with our earlier choice of normalisation of $f_0(U,V)$.

We note that the Born amplitude $f_0$ and the full amplitude $f$ to be introduced later in Sec. 3, as well as their impact-parameter images, $\tilde{f}_0(B)$ and $\tilde{f}(B)$, are two-dimensional vectors. It is implied hereafter though we chose not to underscore $f$’s as transverse vectors.

It is convenient to change from momentum space to impact parameter space. Since $f_0$ depends on the two momenta, $U$ and $V$, it might be expected that two impact parameters would be necessary. However, because $k$ and $p$ enter in (7) only in the combination $k - xp$, the amplitude $f_0$ can depend on $U$ and $V$ only in the combination $U - xV$. Thus it is possible to express $f_0$ in terms of a single “impact parameter” as

$$
f_0(U,V) = \int \frac{d^2B}{(2\pi)^2} e^{iB(U - xV)} \tilde{f}_0(B)
$$

(13)

with

$$
\tilde{f}_0(B) = \int d^2(U - xV) e^{-iB(U - xV)} f_0(U,V).
$$

(14)

Using

$$
\frac{U}{U^2} = -\frac{i}{2\pi} \int d^2B e^{iB\cdot U} \frac{B}{B^2},
$$

(15)

it is straightforward to find

$$
\tilde{f}_0(B) = -2\pi i \frac{\{B\}}{B^2} \left( \left[ 1 - \tilde{V}(B(1-x)) \right] + \left[ 1 - \tilde{V}(B) \right] - \frac{1}{N_c} \left[ 1 - \tilde{V}(-Bx) \right] \right),
$$

(16)

with

$$
\tilde{V}(B) = \int d^2Q e^{-iQB} V(Q^2); \quad \tilde{V}(0) = 1.
$$

The $\{ \}$ symbol in (16) is defined in (2).

Now let us compare (16) with our previous small-$x$ results in (4.5) and (4.25) of Ref. [3]. Of the three terms on the right hand side of (16) the first and third terms come from the production terms in the medium, graphs a–c of Fig. 1, while the second term comes from the elastic scattering terms in the medium, graphs d–g of Fig. 1. In the small-$x$ limit the first and second terms on the right hand side of (16) become equal while the third term vanishes. Comparing with (4.25) of Ref. [3] we note that our present result is larger than what we previously found by a factor of 2 in the small-$x$ limit. This factor of 2 is due to an incomplete treatment of virtual contributions in our earlier work.
2.3 The Born amplitude for the hard scattering case

In case the high energy quark is produced in the medium through a hard scattering one must also include the contribution coming when the endpoint of the integration of the gluon emission coincides with the time of the hard scattering. It is not important to know the details of the hard scattering since we are here only interested in the radiative gluon spectrum accompanying the hard scattering and subsequent rescatterings in the medium. Thus we may imagine that the gluon is produced in a collision of a highly virtual photon with a quark in the medium, which collision transfers a large energy to the struck quark which then rescatters in the medium and radiates a gluon.

\[
\gamma^* \to p-k \quad k
\]

The gluon emission amplitude is as given by \( T_a \) in (7a). Since the gluon is emitted after the hard scattering there is no overall colour factor so that the basic Born term here is

\[
\frac{\{U - xV\}}{(U - xV)^2},
\]

(17a)

or

\[
-2\pi i \frac{\{B\}}{B^2}
\]

(17b)

in impact parameter space. The expression given in (17b) must be added to \( \frac{N_c}{2\pi p} f_0(B) \), the Fourier transform of the left hand side of (12).

3 The time evolution of gluons in the medium

After the gluon is emitted from the high energy quark, the quark-gluon system moves through the medium and carries out multiple scatterings \([9, 10]\) with the particles of the medium. It is not certain that the gluon will be produced as a physical gluon until there is gluon emission in the complex conjugate amplitude. Thus, we must follow the time evolution of the quark-gluon system in the amplitude up to the time of emission in the complex conjugate amplitude. This is what will be done in this section.

Let \( f(U, V, t) \) be the quark-gluon amplitude at time \( t \) starting from \( f_0(U, V) \) at \( t = 0 \). \( U \) is the scaled gluon momentum while \( V - U \) is the scaled quark momentum as given in (3) for the graphs of Fig. 1. The amplitude \( f(U, V, t) \) is illustrated below.
We recall that \( t \) is less than the time \( t' \) at which the gluon is emitted in the complex conjugate amplitude. Then the time-dependence of \( f \) comes partly from the free evolution of the quark-gluon system and partly from interactions in the medium. Suppose \( t_1 \) is the time of the last interaction with the medium before time \( t \). Although \( f \) represents the gluon amplitude it also includes interactions of the quark with the medium in the complex conjugate amplitude as already noted for \( f_0 \). Then the possible interactions at \( t_1 \) are shown in Fig. 3. For example the graph shown in Fig. 3a illustrates an inelastic interaction of the gluon with a particle in the medium in the amplitude and an interaction of the quark with the same particle in the complex conjugate amplitude. Perhaps the only unusual graph in Fig. 3 is graph e where the only interaction is a forward elastic scattering of the quark with a particle of the medium in the complex conjugate amplitude.

The amplitude \( f(U, V, t) \) obeys the integral evolution equation

\[
f(U, V, t) = e^{i\phi(t-t_1)} f_0(U, V) + \int_{t_1}^{t} dt' e^{i\phi(t-t')} \frac{\rho \sigma}{C_F} \int d^2 Q V(Q^2) \left[ \frac{N_c}{2} f(U-Q, V-Q, t') - \frac{1}{2N_c} f(U, V-Q, t') - \frac{C_F}{2} f(U, V, t') + \frac{N_c}{2} f(U-Q, V, t') \right].
\]

(18)

The first term in the right hand side of (18) gives the free evolution of the quark-gluon system between \( t_1 \) and \( t \) in the amplitude and of the quark in the complex conjugate amplitude according to the phase factor

\[
e^{i\phi(t-t_1)} \quad \text{with} \quad \phi = \frac{k^2}{2k} + \frac{(p-k)^2}{2(p-k)} - \frac{p^2}{2p} = \frac{(U-xV)^2}{2(x(1-x)p}}.
\]

(19)

Differentiating over \( t \) we obtain the equation

\[
\frac{\partial}{\partial t} f(U, V, t) = \frac{i(U-xV)^2}{2x(1-x)p} f(U, V, t) + \frac{\rho \sigma}{C_F} \int d^2 Q V(Q^2) \left[ \frac{N_c}{2} f(U-Q, V-Q, t) - \frac{1}{2N_c} f(U, V-Q, t) - \frac{C_F}{2} f(U, V, t) + \frac{N_c}{2} f(U-Q, V, t) \right].
\]

(20)

The first term in the right hand side of (20) comes from free propagation and the rest of the terms come from the upper limit of the integration over \( t_1 \) at \( t_1 = t \). In (18), (20) \( \rho \) is the density of scatterers in the medium while \( \sigma \) is the cross section of a quark with a particle in the medium.
The factor $V(Q^2)\sigma$ corresponds to the differential cross section for scattering the high energy quark with momentum transfer $Q\mu$ while the $1/C_F$ factor takes out the quark colour factor for quark scattering with a particle of the medium. The correct colour factors are then inserted in the various six terms in the integrand in (20). The six terms in the integrand in (20) correspond, in order, to the six graphs of Fig. 3. The colour factors are

$$
F_a = -F_c = F_f = \frac{N_c}{2}, \tag{21a}
$$
$$
F_d = F_e = -\frac{C_F}{2}, \tag{21b}
$$
$$
F_b = -\frac{1}{2N_c}. \tag{21c}
$$

As in (8) we include $(-1)$ factors in $F$ for all virtual terms while factors $1/2$ are included for the terms with both gluon lines, $q$ and $-q$, hooking to the same high energy quark or gluon line.

It is easy to see that $f(U-V,t) = f(U-xV,t)$ since the explicit factors of $U$ and $V$ in (20) occur in the combination $U - xV$ while $f(U,V,0) = f_0(U-xV)$. Introducing

$$
\tau = \frac{t}{\lambda} \frac{N_c}{2C_F}, \quad \tilde{\kappa} = \frac{\lambda \mu^2}{2x(1-x)p} \frac{2C_F}{N_c} = \frac{\lambda \mu^2}{2\omega(1-x)} \frac{2C_F}{N_c}, \tag{22}
$$

with $\rho \sigma = 1/\lambda$, where $\lambda$ is the mean free path of the high energy quark in the medium, one can write (20) as

$$
\frac{\partial}{\partial \tau} f(U-xV,\tau) = i\tilde{\kappa}(U-xV)^2 f(U-xV,\tau)
$$

$$
- \int d^2Q V(Q^2) \left( [f(U-xV,\tau) - f(U-xV-(1-x)Q,\tau)]
$$

$$
+ [f(U-xV,\tau) - f(U-xV-Q,\tau)] - \frac{1}{N_c^2} [f(U-xV,\tau) - f(U-xV+xQ,\tau)] \right). \tag{23}
$$

Now it is a simple matter to go to impact parameter space,

$$
\bar{f}(B,\tau) = \int d^2(U-xV) e^{-iB\cdot(U-xV)} f(U-xV,\tau), \tag{24}
$$

to find

$$
\frac{\partial}{\partial \tau} \bar{f}(B,\tau) = -i\tilde{\kappa} \nabla_B^2 \bar{f}(B,\tau)
$$

$$
- \left( [1 - \tilde{V}(B(1-x))] + [1 - \tilde{V}(B)] - \frac{1}{N_c^2} [1 - \tilde{V}(-Bx)] \right) \bar{f}(B,\tau). \tag{25}
$$

This is the basic Schrödinger-type evolution equation for the propagation of the quark-gluon system in a QCD medium. It should be solved with the initial condition

$$
\bar{f}(B,0) = \bar{f}_0(B), \tag{26}
$$

with $\bar{f}_0(B)$ given in (16).

Comparing (23) to (4.23) of Ref. [3] we again find a factor 2 discrepancy in the small-$x$ limit of the second term on the right hand side of (23) as compared to (4.23). Our previous calculation
effectively amounted to keeping graphs a, b, d and e of Fig. 3. (Although graphs d and e were not explicitly considered, they were effectively included through the use of a mean free path term, the \((-1)\) in the first term on the right-hand side of (4.16) of Ref. [3].) That calculation was in error because the virtual terms, graphs c, d, e, f of Fig. 3 are not completely taken into account in the mean free path treatment of Ref. [3]. We note that the coefficient of \(\tilde{f}(\vec{P}, \tau)\), in the second line of (25), the potential term, is of the same form as the three-body cross section used by Zakharov in eq. 23 of Ref. [3].

4 The Born term for the complex conjugate amplitude

Now that we have calculated gluon emission in the amplitude and followed its evolution in time, it becomes necessary to calculate gluon emission in the complex conjugate amplitude. Suppose the gluon is emitted at time \(t\) in the complex conjugate amplitude. As usual we integrate this emission time between elastic or inelastic interactions with the medium. Except for colour factors the calculation proceeds exactly as in Sec. 2. Suppose \(t_1\), the time of interaction in the complex conjugate amplitude with the particle in the medium, serves as the endpoint of the \(t\)-integration. At \(t_1\) the amplitude consists of a quark-gluon system described by \(f(t_1)\). The graphs describing the emission are shown in Fig. 4 where the vertical line indicates that the gluon is put on-shell. Terms to the left of the vertical line belong to the amplitude while to the right belong to the complex conjugate amplitude. We have rearranged the momenta so that the gluon emission amplitude \(f(U - xV, t_1)\) appears uniformly in all the graphs.

The “colour factors” for gluon emission in the complex conjugate amplitude are

\[
\begin{align*}
\frac{1}{2} F_a &= -F_b = F_c = -F_d = F_h = F_i = \frac{N_c}{2} , \\
F_e &= -2 F_g = -2 F_j = C_F , \\
F_f &= - \frac{1}{2N_c} ,
\end{align*}
\]

(27a) (27b) (27c)

where again a factor of \((-1)\) for virtual terms and a factor of 1/2 for identical particles have been included in (27). The gluon emission terms, analogous to (25), for the complex conjugate amplitude are

\[
\begin{align*}
T^*_a &= T^*_a = T^*_g = T^*_h = -T^*_j = - \frac{\{k - xp\}^*}{(k - xp)^2} , \\
T^*_b &= T^*_d = T^*_i = - \frac{\{k - xp - q\}^*}{(k - xp - q)^2} , \\
T^*_c &= \frac{\{k - xp - (1 - x)q\}^*}{(k - xp - (1 - x)q)^2} , \\
T^*_f &= \frac{\{k - xp + xq\}^*}{(k - xp + xq)^2} .
\end{align*}
\]

(28a) (28b) (28c) (28d)

It is now straightforward to check that

\[
\frac{1}{C_F} \sum_{i=a}^{j} \int d^2Q V(Q^2) T^*_i(U, V, Q) F_i = - \frac{N_c}{2C_F} f_0^*(U - xV) ,
\]

(29)
which, except for a minus sign, is similar to \( (12) \). Taking the Fourier transform gives
\[
\int d^2(U - xV) \ e^{-i \cdot 
abla(U - xV)} \ \frac{1}{C_F} \sum_{i=a}^{j} \int d^2 Q V(Q^2) T^i(\gamma, U, V, Q) F_i = \tilde{f}_0(B) \ \frac{N_c}{2C_F},
\]
with \( \tilde{f}_0 \) exactly the same function as given in \( (16) \).

Comparing with what was found in Ref. [3], it is easy to verify that graphs \( a+b+c \) of Fig. 4 generate what was called the “Y” term and that this contribution is equal to the first and second of the three terms on the right hand side of \( (16) \). Graphs \( d+e+f \) generate what was called the “H” term which is equal to the second and third terms on the right hand side of \( (16) \). The virtual graphs, graphs \( g+h+i+j \) of Fig. 4 were not included in Ref. [3] and it is easy to check using \( (27) \) and \( (28) \) that they give \((-1)\) times the second term on the right hand side of \( (16) \). Thus in the small-\( x \) limit where the third term on the right hand side of \( (16) \) is small we now find a factor 2, instead of the 3 coming from the sum of the \( Y + H \) graphs in Ref. [3], due to inclusion of the virtual terms.

5 The spectrum of radiated gluons and energy loss

In this section we remind the reader of the formula for the induced spectrum of radiated gluons \( [3] \). We then solve for the spectrum in the soft gluon limit for a volume of matter large enough that a high energy quark carries out many scatterings. Finally we integrate the spectrum of radiated gluons to find the radiative energy loss in circumstances where the energy loss problem is dominated by soft gluons, that is when a gluon having coherent length on the order of the dimensions of the medium has a longitudinal momentum much less that that of the high energy quark.

5.1 The formula for the induced gluon radiation

We give the formula for the induced gluon spectrum coming from a high energy quark produced in a hard collision in the medium. Then the Born term in the amplitude will be associated either with a scattering in the medium, as in \( (12) \), or with the hard vertex, as in \( (17a) \). In case the gluon is radiated from a high energy “on-shell” quark entering the medium one simply drops the term associated with the hard vertex. The induced gluon spectrum is
\[
\frac{\omega dI}{d\omega dz} = \frac{\alpha_s C_F}{\pi^2 L} 2 \text{Re} \int d^2 U \left\{ \int_0^L \int_0^{t_2} dt_2 \int_0^{t_1} dt_1 \ \rho \sigma N_C \frac{f(U - xV, t_2 - t_1)}{2C_F} \cdot \rho \sigma N_C \frac{f^*(U - xV)}{2C_F} \right\}_{\tilde{\kappa} = 0}
\]
\[
+ \int_0^L dt \ \rho \sigma N_C \frac{f(U - xV, t)}{(U - xV)^2} \cdot \{U - xV\} \frac{f^*(U - xV)}{(U - xV)^2} \right\}_{\tilde{\kappa} = 0}.
\]

The various terms in \( (31) \) have simple interpretations. The \( \alpha_s C_F / \pi^2 \) is the coupling of a gluon to a quark. The \( 1/L \) comes because we calculate the spectrum per unit length of the medium. The factor
\[
\frac{N_c}{2C_F} f(U - xV, t_2 - t_1) \ \rho \sigma dt_1
\]
gives the number of scatterers in the medium, $\rho \sigma \, dt_1$, times the gluon emission amplitude at $t_1$, then evolved to $t_2$. The factor

$$\frac{-N_c}{2C_F} f_0^*(U - xV) \rho \sigma \, dt_2$$

gives the number of scatterers times gluon emission in the complex conjugate amplitude. The overall normalisation in the small-$x$ limit is fixed by comparing with Ref. [3]. We note that this normalisation is correct even when $x$ is not small because the $\sqrt{1-x}$ present in the basic amplitude (3) but not included in our definition of $f_0$ cancels with a $1/(1-x)$ quark phase space factor.

The second term on the right hand side of (31) gives the contribution of gluon emission due to the hard scattering in the medium. If there is no hard scattering in the medium this term should not be included. We assume the hard scattering happens at $t = 0$. This means that in the case the quark is produced by a hard scattering in a medium $L$ is the length of matter measured from the production point.

The subtraction of the value of the integrals at $\tilde{\kappa} = 0$ in (31) eliminates the medium-independent factorisation contribution [1, 2].

It is straightforward to simplify (31). Using $\rho \sigma = 1/\lambda$ and (22) to write $dt = \frac{2C_F}{N_c} \lambda d\tau$, and defining $\tau_0 = N_c L/2C_F \lambda$, one finds

$$\frac{\omega dI}{d\omega dz} = \frac{\alpha_s N_c}{\pi^2 \lambda} \text{Re} \left\{ \int_0^{\tau_0} d\tau \left( 1 - \frac{\tau}{\tau_0} \right) f(U - xV, \tau) \cdot f_0^*(U - xV) \right. + \frac{1}{\tau_0} \int_0^{\tau_0} d\tau f(U - xV, \tau) \cdot \frac{U - xV}{(U - xV)^2} \} \tilde{\kappa} = 0.$$

(32)

It is convenient to express the integrals in impact parameter space. Using (13) and (17b) one finds

$$\frac{\omega dI}{d\omega dz} = \frac{\alpha_s N_c}{2\pi^3 \lambda} \text{Re} \left\{ \int_0^{\tau_0} d\tau \left( 1 - \frac{\tau}{\tau_0} \right) \tilde{f}(B, \tau) \cdot \tilde{f}_0^*(B) \frac{d^2B}{2\pi} + \frac{2\pi i}{\tau_0} \int_0^{\tau_0} d\tau \tilde{f}(B, \tau) \cdot \frac{(B^2)}{B^2} \frac{d^2B}{2\pi} \} \tilde{\kappa} = 0.$$

(33)

In the small-$x$ limit (33) reduces to

$$\frac{\omega dI}{d\omega dz} = \frac{2\alpha_s N_c}{\pi^2 \lambda} \text{Re} \left\{ i \int_0^{\tau_0} d\tau \left( 1 - \frac{\tau}{\tau_0} \right) \frac{1 - \bar{V}(B^2)}{B^2} B \cdot \tilde{f}(B, \tau) \frac{d^2B}{2\pi} + \frac{1}{2\tau_0} \int_0^{\tau_0} d\tau \frac{B}{B^2} \cdot \tilde{f}(B, \tau) \frac{d^2B}{2\pi} \} \tilde{\kappa} = 0.$$

(34)

Comparing with (4.24) of Ref. [3] we see that the first term on the right hand side of (34) is $2/3$ times (4.24). This $2/3$ is exactly the $2/3$ factor we discussed at the end of Sec. 4.

The second term on the right hand side of (34), due to the hard scattering in the medium, was missed in Ref. [3]. At first glance it might seem that this term is small compared to the first term because of the $1/\tau_0$ in front of the integral. However, the second term is enhanced by a $1/B^2$ compared to the first term and $1/B^2$ is of the order of $\tau_0$ in the dominant part of the integral [3].
5.2 The induced spectrum

It is not difficult to solve (25), in a logarithmic approximation for small \( B^2 \), and to use that solution in (33) to obtain the induced gluon spectrum. The details of the procedure are given in Ref. [3], and here we emphasise the differences which occur when \( x \) is not necessarily small. When \( B^2 \) is small, and this will be the case for matter long enough so that many scatterings occur, it is convenient to write the “potential” in (25) as

\[
\left[ 1 - \tilde{V}(B(1-x)) \right] + \left[ 1 - \tilde{V}(B) \right] - \frac{1}{N_c} \left[ 1 - \tilde{V}(-Bx) \right] = \frac{1}{4} B^2 \tilde{u}(B^2, x). \tag{35}
\]

In terms of \( \tilde{v}(B^2) \) used in Ref. [3] one has

\[
\tilde{u}(B^2, x) = 2 \left( 1 - x + \frac{C_F}{N_c} x^2 \right) \tilde{v}(B^2), \tag{36}
\]

where \( \tilde{v} \), and \( \tilde{u} \), have only logarithmic dependence on \( B^2 \) for small \( B^2 \). In this small-\( B \) limit \( \tilde{f}_0(B) \), from (16), becomes

\[
\tilde{f}_0(B) = -\frac{\pi i}{2} \tilde{u}(B^2, x) \{ B \}. \tag{37}
\]

The only change from Ref. [3] is \( B \rightarrow \{ B \} \) and \( \tilde{v} \rightarrow \tilde{u} \). Eqs. (5.12) and (5.13) now become

\[
\tilde{f}(B, \tau) = -\frac{i \pi \{ B \}}{2 \cos^2 \omega \tau} \exp \left( -\frac{i}{2} m \omega_0 B^2 \tan \omega_0 \tau \right) \tag{38}
\]

and

\[
\tilde{f}(B, \tau) = \frac{2 \pi i \{ B \}}{B^2} \frac{\partial}{\partial \tau} \exp \left( -\frac{i}{2} m \omega_0 B^2 \tan \omega_0 \tau \right) \tag{39}
\]

respectively. Parameters \( m \) and \( \omega_0 \) are defined as before but with \( \tilde{u} \) replacing \( \tilde{v} \). That is

\[
m = -\frac{1}{2 \tilde{k}}, \quad \omega_0 = \sqrt{i \tilde{k} \tilde{u}}, \quad \tilde{k} = \frac{2 C_F}{N_c} \frac{\lambda \mu^2}{2 \omega(1-x)}. \tag{40}
\]

Eqs. (38) and (39) are two useful forms of the solution to (25) in the approximation of treating \( \tilde{u} \) as a constant, an approximation which should be good for small \( B^2 \).

If one substitutes (39) for \( \tilde{f}(B, \tau) \) in the first term on the right hand side of (33), and (38) for the second term on the right hand side of (33), then

\[
\frac{\omega dI}{d\omega d\tau} = \frac{\alpha_s N_c}{\pi \lambda \tau_0} \left( 1 - x + \frac{x^2}{2} \right) \text{Re} \left\{ \int_0^{\tau_0} d\tau \frac{\omega_0 \tau}{\tan \omega_0 \tau} \left[ \left( \frac{\omega_0 \tau}{\tan \omega_0 \tau} - 1 \right) - \left( \frac{\omega_0 \tau}{\sin \omega_0 \tau \cos \omega_0 \tau} - 1 \right) \right] \right\} \tag{41}
\]

emerges. The two terms in the integrand in (41) correspond exactly to the terms on the right hand side of (33). In the small-\( x \) limit the first term on the right hand side of (41) is smaller by a factor 1/3 than (5.15) of Ref. [3], this factor of 1/3 being part of the 2/3 found in Sec. 4, with the 2 in the 2/3 going into changing a \( \tilde{v} \) to a \( \tilde{u} \).
The $\tau$-integral in (41) is easily done to give
\[
\frac{\omega dI}{d\omega dz} = \frac{2\alpha_s C_F}{\pi L} \left[ 1 - x + \frac{x^2}{2} \right] \left( \ln \left| \frac{\sin \omega_0 \tau_0}{\omega_0 \tau_0} \right| - \ln \left| \tan \omega_0 \tau_0 \right| \right),
\]
or
\[
\frac{\omega dI}{d\omega dz} = \frac{2\alpha_s C_F}{\pi L} \left[ 1 - x + \frac{x^2}{2} \right] \ln \left| \cos \omega_0 \tau_0 \right|.
\]
We remind the reader that (42b) corresponds to the quark being produced in the medium. For quark approaching the medium from outside the spectrum is given by the first term in the right hand side of (42a).

Now we are in a position to generalise the gluon energy spectra (42) for the case when the projectile is not a spin-1/2 fermion but a vector or a scalar object. To this end we note that the basic gluon emission vertex (1) becomes diagonal in the gluon helicity basis, $s = \pm 1$,
\[
\{v\} \cdot \epsilon^{(s)\ast} = \left( 1 - \frac{x}{2} \right) + \frac{x}{2} = \left\{ \begin{array}{ll}
1 & \text{for } s = +1, \\
1 - x & \text{for } s = -1.
\end{array} \right.
\]
In general, the structure of the gluon radiation vertex is
\[
\frac{2}{x(1 - x)} \cdot \left[ (1 - x)^{|s - r|J_P} \cdot x^{|r - r'|J_P} \right],
\]
where $J_P$ is spin, and $r, r' = \pm 1$ helicity states of the projectile before and after gluon emission. For the quark case we have $J_P = \frac{1}{2}$ while helicity conserves, $r = r'$. Taken together with (43), this brings us back to (1):
\[
\bar{u}_{r'}(p - k) \gamma \cdot \epsilon^{(s)\ast} u_r(p) = \frac{2\delta_{r'r}}{x(1 - x)} \cdot \left[ (1 - x)^{1/2} \cdot \delta_{s,r} + (1 - x)^{3/2} \cdot \delta_{s,-r} \right].
\]
Applying (44) we obtain the expressions in the square brackets in (44) for the amplitude of gluon emission off a scalar particle, $J_P = 0$,
\[
\left[ (1 - x) \cdot (\delta_{s,+1} + \delta_{s,-1}) \right],
\]
and for the vector projectile (gluon), $J_P = 1$,
\[
\left[ (1 \cdot \delta_{r,s} + (1 - x)^2 \cdot \delta_{r,-s}) \cdot \delta_{r,r'} + x^2 \cdot \delta_{r,s} \delta_{r,-r'} \right]
\]
The last term in (45b) corresponds to helicity flip of the incoming hard gluon. (The full answer is symmetric with respect to two outgoing gluons, $x \leftrightarrow (1 - x)$, $s \leftrightarrow r'$, as it should be.)

Adding together the squared helicity amplitudes we finally obtain the $x$-dependent factors $X_{J_P}$ in the gluon energy spectrum (42) for different projectiles:
\[
X_2(x) = \left[ \frac{1 + (1 - x)^2}{2} \right] = \frac{x}{2} \cdot \frac{1 + (1 - x)^2}{x},
\]
\[
X_0(x) = [1 - x] = \frac{x}{2} \cdot 2(1 - x),
\]
\[
X_1(x) = \left[ \frac{1 + (1 - x)^4 + x^4}{2(1 - x)} \right] = \frac{x}{2} \cdot \left\{ \frac{x}{1 - x} + \frac{1 - x}{x} + x(1 - x) \right\}.
\]
Factors \((46)\) are identical in the soft limit, \(X_{J_F}(0) = 1\). They are proportional to the corresponding parton splitting functions.

If the projectile corresponds to colour representation, \(R\), which is different from the fundamental representation, the colour factor \(C_F\) in the gluon energy spectrum \((42)\) should be replaced by the appropriate quadratic Casimir operator, \(C_R\). The colour factor \(1/N_c^2\) in the third term in the “potential” \((35)\) should be replaced by

\[- \frac{1}{N_c^2} \Rightarrow \frac{2C_R - N_c}{N_c},\]

which leads to the substitution \(C_F \rightarrow C_R\) in the expression \((36)\) for \(\tilde{u}\).

### 5.3 The energy loss

The energy loss per unit length,

\[- \frac{dE}{dz} = \int_0^\infty \frac{\omega dI}{d\omega d\tau} d\omega,\]

can be evaluated easily if the dominant values of \(\omega \sim \mu^2 L^2/\lambda\) are such that the small-\(x\) approximation can be used when doing the \(\omega\)-integral of \((42)\). In this case

\[- \frac{dE}{dz} = \frac{\alpha_s N_c \mu^2 L}{4} \frac{\bar{v}(\tau_0^{-1})}{\lambda},\]

where we have used \(\bar{u} \simeq 2\bar{v}\) in the small-\(x\) limit. We remind the reader that in \((47)\) \(L\) is the length of material traversed by the quark beyond its production point.

The relative contributions of the first and second terms on the right hand side of \((41)\) to \((47)\) are \(1/3\) and \(2/3\) respectively. Thus for a quark approaching the medium from outside \((47)\) is replaced by

\[- \frac{dE}{dz} = \frac{\alpha_s N_c \mu^2 L}{12} \frac{\bar{v}(\tau_0^{-1})}{\lambda},\]

Comparing with Ref. [3] in the small-\(x\) limit we note that the first term on the right hand side of \((42a)\) is a factor \(1/3\) times the expression given in (5.16) of Ref. [3] with, of course, \(\bar{v}\) replaced by \(\bar{u}\) in the definition of \(\omega_0\) in the present result.

Comparing energy loss with jet broadening [4], for example for a jet produced in matter, we find

\[- \frac{dE}{dz} = \frac{\alpha_s N_c}{4} p_{\perp W},\]

the coefficient being a factor of 2 larger than that quoted in Ref. [4].

### 6 Comparing to the method of Zakharov

Recently B. Zakharov [5, 6] has proposed a concise and elegant formulation for describing and calculating the energy loss of high energy partons in hot and cold matter. At first sight Zakharov’s
formalism appears very different from that of BDMPS. In Zakharov’s picture radiative energy loss of a high energy quark is described by the interaction of a high energy colour-neutral quark-antiquark-gluon system with the QCD medium it passes through. No trace of a quark radiating a gluon as it passes through a medium and carries out multiple scatterings with that medium remains visible. Nevertheless, as we shall see below the two formalisms are in fact exactly equivalent. We begin by casting (53) in a form given by Zakharov. Then we shall attempt to explain how one can intuitively see the equivalence of the two formalisms.

6.1 Quantitative equivalence of the two formalisms

In this section we shall show that (53) can be expressed in a form identical to Eq. 4 of Ref. [3]. It is convenient to go back to the unscaled variables used in (31), and also to use

\[ \tilde{f}(B_2, t_2) = \int d^2B_1 \, G(B_2, t_2; B_1, t_1) \, \tilde{f}_0(B_1) \, , \]

(50)

where \( G \) is exactly as given in (5.6) of Ref. [3] with, of course, the replacement of \( \tilde{u} \) by \( \tilde{u} \) being understood. Then (53), or (31), takes the form

\[
\frac{\omega dI}{d\omega dz} = \frac{\alpha_s}{4L\pi^3} \left( \frac{N_c}{2C_F} \right) \Re \int d^2B_2 \int d^2B_1 \left\{ \frac{B_1}{B_1^2} \right\} \left\{ \int_0^\infty dz_2 \rho(z_2) \sigma \int_0^\infty dz_1 \tilde{f}_0^*(B_2) G(B_2, z_2; B_1, z_1) \right\}_{\tilde{\kappa}=0} \]

\[
G(B_2, z_2; B_1, z_1) \left( \frac{B_2^2 \tilde{u} (B_1, x)}{4} \right) \left( \frac{N_c}{2C_F} \right) \rho(z_1) \sigma + \int_0^\infty dz_2 \rho(z_2) \sigma \tilde{f}_0^*(B_2) G(B_2, z_2; B_1, 0) \right\}_{\tilde{\kappa}} \]

(51)

where we have allowed the integrations over \( z_1 \) and \( z_2 \) to go to \( \infty \). It is understood, however, that \( \rho(z) \) vanishes outside the interval \( 0 \leq z \leq L \). We have used \( z \) rather than \( t \) as a variable to emphasise the spatial dependence of the density \( \rho \). In what follows it is not important that \( \rho \) be uniform in the region \( 0 < z < L \), as we are only going to the equations for \( G \) and not the explicit solution. The Green function \( G \) obeys the same differential equation as given for \( \tilde{f} \) in (25). Using (22) and (33) one finds

\[
\frac{\partial}{\partial z_2} G = -i \left( \frac{\tilde{\kappa} N_c}{2\lambda C_F} \right) \nabla_{B_2}^2 G - \frac{1}{4} B_2^2 \tilde{u} \left( \frac{N_c}{2C_F} \right) \rho(z_2) \sigma G \, ,
\]

(52a)

and

\[
\frac{\partial}{\partial z_1} G = i \left( \frac{\tilde{\kappa} N_c}{2\lambda C_F} \right) \nabla_{B_1}^2 G + \frac{1}{4} B_1^2 \tilde{u} \left( \frac{N_c}{2C_F} \right) \rho(z_1) \sigma G \, .
\]

(52b)

Using (52) we substitute the combination of \( \partial G/\partial z_2 \) and \( \nabla_{B_2}^2 G \) for the \( \frac{1}{4} B_2^2 \tilde{u} \left( \frac{N_c}{2C_F} \right) \rho(z_1) \sigma G \) in the first term on the right hand side of (51). The lower limit of the \( z_1 \)-integral of \( \partial G/\partial z_1 \) exactly cancels the second term on the right hand side of (51) while its upper limit, at \( z_2 \), contains

\[
G(B_2, z_2; B_1, z_2) = \delta^2(B_2 - B_1) \, ,
\]

comes out \( \tilde{\kappa} \)-independent and therefore cancels due to the \( \tilde{\kappa} \) versus \( \tilde{\kappa} = 0 \) subtraction indicated in (51).
Thus, we arrive at

$$\frac{\omega dI}{d\omega dz} = \frac{\alpha_s C_F}{L \pi^3} \left( \frac{\bar{k} N_c}{2\lambda C_F} \right) \Re \left\{ \int d^2 B_2 \int d^2 B_1 \frac{\{B_1\}}{B_1^2} \right. 
\left. \int_0^\infty dz_1 \int_{z_1}^\infty dz_2 \rho(z_2) \sigma \left( \frac{N_c}{2C_F} \right) \cdot \tilde{f}_0(B_2) \nabla_{B_1}^2 G(B_2, z_2; B_1, z_1) \right\}_{\hat{k}=0}.$$  \hspace{2cm} (53)

Now, using (57) and (52a) gives

$$\frac{\omega dI}{d\omega dz} = -\frac{2\alpha_s C_F}{L \pi^2} \left( \frac{\bar{k} N_c}{2\lambda C_F} \right) \Re i \int d^2 B_1 d^2 B_2 \frac{\{B_1\}}{B_1^2} \frac{\{B_2\}}{B_2^2} \nabla_{B_1}^2 \int_0^\infty dz_1 \int_{z_1}^\infty dz_2 \left\{ \frac{\partial}{\partial z_2} G(B_2, z_2; B_1, z_1) + i \left( \frac{\bar{k} N_c}{2\lambda C_F} \right) \nabla_{B_2}^2 G(B_2, z_2; B_1, z_1) \right\}_{\hat{k}=0}.$$  \hspace{2cm} (54)

The \(z_2\)-integral of the \(\partial G/\partial z_2\) term vanishes: at the upper limit, \(z_2 = \infty\) because \(G(z_2 = \infty) = 0\), and at the lower limit due to \(\bar{k}\)-subtraction, as before. The remaining term in (54) can be simplified by writing

$$\frac{\{B\}}{B^2} = \frac{1}{2} \{\nabla_B \ln B^2\}. \hspace{2cm} (55)$$

After integrating the \(\{\nabla_B\}\) and \(\nabla_B^2\) terms by parts one finds

$$\frac{\omega dI}{d\omega dz} = \frac{2\alpha_s C_F}{L \pi^2} \left( \frac{\bar{k} N_c}{2\lambda C_F} \right)^2 \Re \int d^2 B_1 d^2 B_2 \int_0^\infty dz_2 \int_0^{z_2} dz_1 \left[ \nabla_{B_1}^2 \frac{1}{2} \ln B_1^2 \right] \left[ \nabla_{B_2}^2 \frac{1}{2} \ln B_2^2 \right] \{\nabla_{B_1}\} \cdot \{\nabla_{B_2}\} G(B_2, z_2; B_1, z_1) \left|_{\hat{k}=0} \right. \hspace{2cm} (56)$$

Using

$$\{\nabla_{B_1}\} \cdot \{\nabla_{B_2}\}^* = \left( 1 - x + \frac{x^2}{2} \right) \nabla_{B_1} \cdot \nabla_{B_2} \hspace{2cm} (57)$$

and

$$\nabla_B^2 \frac{1}{2} \ln B^2 = 2\pi \delta^2(B), \hspace{2cm} (58)$$

one finds

$$\frac{\omega dI}{d\omega dz} = \frac{2\alpha_s C_F}{L} \left[ 4 - 4x + 2x^2 \right] \left( \frac{\bar{k} N_c}{2\lambda C_F} \right)^2 \Re \int_0^{z_2} dz_2 \int_0^{z_2} dz_1 \left[ \nabla_{B_1} \nabla_{B_2} G(B_2, z_2; B_1, z_1) \right]_{B_1 = B_2 = 0} \left|_{\hat{k}=0} \right. \hspace{2cm} (59)$$

Noting that

$$\frac{\bar{k} N_c}{2\lambda C_F} = \frac{\mu^2}{2x(1-x)p},$$

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and that $\mu$ normalises our “impact parameter” $B$ to physical transverse coordinate, $x$, so that

$$\mu \nabla B_1 = \nabla x, \quad \mu^2 G(B_2, 0; B_1, 0) = \delta^2(x_2 - x_1),$$

we see that (59) takes exactly the same form as Eq. 4 in Ref. [6].

It is not immediately clear that the subtraction terms, done at $\kappa = 0$ in our case and at zero matter density in Ref. [3], are identical. There is, however, a physical argument that shows that they should be the same. The $\kappa \to 0$ limit is equivalent to the $\omega \to \infty$ limit. However, at large $\omega$ the gluon is surely radiated outside the medium and since the high energy quark is produced in the medium, the gluon has no knowledge of the medium whatsoever. Thus, subtracting out the zero-density calculation is the same as subtracting out the large-$\omega$ gluons.

In closing we note [6] that the two terms on the right hand side of (41) correspond to integrations when $z_1$ and $z_2$ are in the medium and when $z_1$ is inside the medium while $z_2$ is outside, respectively. Thus a formula like (59) but with $z_1$ and $z_2$ restricted to lie in the medium reproduces the induced radiation off a high energy quark approaching the medium from outside. In this case the subtraction at $\kappa = 0$ subtracts out the so-called factorisation contribution [1, 2]. It does not appear that this subtraction can be done in terms of a zero density limit as the subtraction term has a (weak) matter dependence.

6.2 Why the two formalisms are equivalent

In this section we describe, very qualitatively, how the two formalisms are related. We do this in the context of a hard scattering producing a high energy quark jet in a finite-size medium. The quark then radiates a gluon either in the medium of after it has left the medium. In the BDMPS approach one calculates the gluon emission amplitude and evolves it in time up to the time that the gluon is also emitted in the complex conjugate amplitude, at which time it is certain that the gluon will be produced and is not just a virtual fluctuation. After the gluon is emitted in the complex conjugate amplitude the gluon emission spectrum is determined and it is not necessary to follow the system any further in time.

Suppose the gluon is emitted at $t = t_1$ in the amplitude and at $t_2$ in the complex conjugate amplitude. In evolving the amplitude, and the complex conjugate amplitude, between $t_1$ and $t_2$ the quantum mechanical phase depends on the energy of the quark and of the gluon in the amplitude and on the quark in the complex conjugate amplitude as indicated in (19) where the phase contribution from the complex conjugate amplitude comes with a sign opposite to that of the amplitude. Thus, formally, one may insert the phase from the complex conjugate amplitude into the amplitude by introducing a negative kinetic term in the effective 2-dimensional Lagrangian, and that is what is done in Refs. [3] [4] and [14].

As the quark-gluon system in the amplitude, and the quark in the complex conjugate amplitude, evolve between $t_1$ and $t_2$ there may be inelastic collisions with the medium involving both the amplitude and the complex conjugate amplitude. If the elastic scattering amplitude of a quark with a particle in the medium is purely imaginary then the total inelastic scattering cross section is given by the forward elastic amplitude. Then the inelastic contribution can be taken into account, solely in the amplitude, if one brings the quark from the complex conjugate amplitude to the amplitude as an antiquark [3] [4].

Thus one can consider the evolution of a quark-antiquark-gluon system in the amplitude, and with the antiquark having a negative kinetic energy term, as equivalent to the evolution of a
quark-gluon system in the amplitude and a quark in the complex conjugate amplitude. Forward elastic scatterings of the \( q\bar{q}g \) system with particles in the medium are equivalent to the various elastic and inelastic reactions of the original problem. These forward elastic scatterings may be viewed as a two-body imaginary potential between the various pairs of the three-body system. This is also apparent in (23) and (25).

Thus, formally, one can represent gluon emission in terms of the evolution of the amplitude for the three-body quark-antiquark-gluon system in a medium. There is a bit of a mystery in both approaches, and that concerns the number of independent transverse variables required to describe the process. One might expect there to be two independent impact parameters necessary to describe the three-body evolution. In the BDMPS approach one also would, in general, expect two impact parameters to be necessary to describe the amplitude \( f(U, V, \tau) \) since there would seem to be two independent momenta, \( U \) and \( V \). In the BDMPS approach we have seen that \( U \) and \( V \) only appear in the combination \( U - xV \), as is apparent from (7), (23) and (24), so that only one coordinate is required.

The structure of the potential in (25) can be visualised as pairwise interactions between a quark in the amplitude put at transverse coordinate \( 0 \), the gluon at coordinate \( B \) and the quark in the complex conjugate amplitude at \( xB \). This same result was earlier found by Zakharov in Ref. [5]. If \( x_q, x_{\bar{q}} \) and \( x_g \) are the transverse coordinates of the quark, the “antiquark” and the gluon then the two-body interactions only depend on relative distances, so that the total momentum \( P \) is conserved:

\[
(1 - x) \cdot \dot{x}_q + x \cdot \dot{x}_{\bar{q}} - \dot{x}_g = P = \text{const}.
\]

The factors \( 1 - x \), \( x \) and \( -1 \) correspond to the relative “masses” of the quark, gluon and antiquark, respectively. With the boundary conditions \( x_q - x_{\bar{q}} = 0 = x_g - x_{\bar{q}} \) at \( t = t_1, t_2 \) one gets

\[
(1 - x) \left( x_q - x_{\bar{q}} \right) + x \left( x_g - x_{\bar{q}} \right) = 0.
\]

Choosing \( x_q \equiv 0 \) one finds \( x_{\bar{q}} = x x_g \) just as in the BDMPS approach.

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Figure 2.

Figure 3.
Figure 4.