Thin accretion disk signatures in dynamical Chern–Simons-modified gravity

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Abstract
A promising extension of general relativity is Chern–Simons (CS)-modified gravity, in which the Einstein–Hilbert action is modified by adding a parity-violating CS term, which couples to gravity via a scalar field. In this work, we consider the interesting, yet relatively unexplored, dynamical formulation of CS-modified gravity, where the CS coupling field is treated as a dynamical field, endowed with its own stress–energy tensor and evolution equation. We consider the possibility of observationally testing dynamical CS-modified gravity by using the accretion disk properties around slowly rotating black holes. The energy flux, temperature distribution, the emission spectrum as well as the energy conversion efficiency are obtained, and compared to the standard general relativistic Kerr solution. It is shown that the Kerr black hole provides a more efficient engine for the transformation of the energy of the accreting mass into radiation than their slowly rotating counterparts in CS-modified gravity. Specific signatures appear in the electromagnetic spectrum, thus leading to the possibility of directly testing CS-modified gravity by using astrophysical observations of the emission spectra from accretion disks.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Recently, modified theories of gravity have received a considerable amount of attention mainly motivated by the problems of dark energy (see [1] for reviews) and dark matter [2], and from quantum gravity. A promising extension of general relativity is Chern–Simons (CS)-modified gravity [3–5], in which the Einstein–Hilbert action is modified by adding a parity-violating CS term, which couples to gravity via a scalar field. It is interesting to note that the CS
correction introduces a means to enhance parity violation through a pure curvature term, as opposed to through the matter term, as is usually considered in general relativity. In fact, CS-modified gravity can be obtained explicitly from superstring theory, where the CS term in the Lagrangian density is essential due to the Green–Schwarz anomaly canceling mechanism, upon four-dimensional compactification [6]. Two formulations of CS-modified gravity exist as independent theories, namely the nondynamical formulation and the dynamical formulation (see [5] for an excellent recent review). In the former, the CS scalar is an a priori prescribed function, where its effective evolution equation reduces to a differential constraint on the space of allowed solutions; in the latter, the CS is treated as a dynamical field, possessing an effective stress–energy tensor and an evolution equation. The majority of the work, up to date, has considered the nondynamical formulation [7–10], whereas the dynamical formulation remains the mostly unexplored territory.

Relative to rotating black hole spacetimes, several solutions in the nondynamical formulation were found in CS-modified gravity [8–10]. The first solutions were found by Alexander and Yunes [8, 9], using a far-field approximation (where the field point distance is considered to be much larger than the black hole mass). The second rotating black hole solution was found by Konno et al [10], using a small slow-rotation approximation, where the spin angular momentum is assumed to be much smaller than the black hole mass. However, it is interesting to note that recently, using the dynamical formulation of CS-modified gravity, spinning black hole solutions in the slow-rotation approximation have been obtained [11, 12].

An interesting feature of CS-modified gravity is that it has a characteristic observational signature, which could allow us to discriminate an effect of this theory from other phenomena. However, most of the tests of CS-modified gravity to date have been performed with astrophysical observations and concern the nondynamical framework. In particular, it was found that the CS-modified theory predicts an anomalous precession effect [14], which was tested [15] with LAGEOS [16]. Another constraint on the nondynamical theory was proposed in [17], where it was considered that the CS correction could be used to explain the flat rotation curves of galaxies. However, in [18] a bound was placed on the nondynamical model with a canonical CS scalar that is 11 orders of magnitude stronger than the Solar System one, using double binary pulsar data. Recently, using the dynamical formulation of CS-modified gravity, a stringent constraint was placed on the coupling parameter associated with the dynamical coupling of the scalar field [19].

In this work, we further extend the constraints placed on the dynamical formulation of CS gravity by using the observational signatures of thin disk properties around rotating black holes. In the context of stationary axisymmetric spacetimes, the mass accretion around rotating black holes was studied in general relativity for the first time in [20], by extending the theory of non-relativistic accretion [21]. The radiation emitted by the disk surface was also studied under the assumption that black body radiation would emerge from the disk in thermodynamical equilibrium [22, 23]. More recently, the emissivity properties of the accretion disks were investigated for exotic central objects, such as wormholes [24], and non-rotating or rotating quark, boson or fermion stars, brane-world black holes or gravastars [25–34].

Thus, it is the purpose of the present paper to study the thin accretion disk models for slowly rotating black holes in the dynamical formulation of CS-modified theories of gravity, and carry out an analysis of the properties of the radiation emerging from the surface of the disk. As compared to the standard general relativistic case, significant differences appear in the energy flux and electromagnetic spectrum for CS slowly rotating black holes, thus leading to the possibility of directly testing CS-modified gravity by using astrophysical observations of the emission spectra from accretion disks.
The present paper is organized as follows. In section 2, we review the dynamical formulation of CS-modified gravity, and present the Yunes–Pretorius (YP) slowly rotating solution found in [12]. In section 3, we review the formalism and the physical properties of the thin disk accretion onto compact objects, for stationary axisymmetric spacetimes. In section 4, we analyze the basic properties of matter forming a thin accretion disk in slowly rotating black hole spacetimes in CS-modified gravity. We discuss and conclude our results in section 5. Throughout this work, we use a system of units so that $c = G = \hbar = k_B = 1$, where $k_B$ is Boltzmann’s constant.

2. Dynamical Chern–Simons-modified gravity

In this section, we write down the field equations of the Chern–Simons gravity, and present the Yunes–Pretorius (YP) slowly rotating solution found in [12].

2.1. Field equations of Chern–Simons theory

Consider the dynamical CS-modified gravity theory provided by the action in the form

$$ S = S_{EH} + S_{CS} + S_\vartheta + S_{mat}. \tag{1} $$

The first term is the standard Einstein–Hilbert action

$$ S_{EH} = \kappa \int d^4x \sqrt{-g} R, \tag{2} $$

where $\kappa^{-1} = 16\pi G$ and $R$ is the Ricci scalar. The second term defined as

$$ S_{CS} = \frac{\alpha}{4} \int d^4x \sqrt{-g} \vartheta \ast RR \tag{3} $$

is the CS correction; the third term

$$ S_\vartheta = -\frac{\beta}{2} \int d^4x \sqrt{-g} g^{\mu\nu} (\nabla_\mu \vartheta)(\nabla_\nu \vartheta) + 2V(\vartheta) \tag{4} $$

is the scalar field term. The matter action is given by

$$ S_{mat} = \int d^4x \sqrt{-g} L_{mat}, \tag{5} $$

where $L_{mat}$ is the matter Lagrangian.

The parameters $\alpha$ and $\beta$ are the dimensional coupling constants; the CS coupling field, $\vartheta$, is a function of spacetime that parameterizes deformations from GR [12]; $\nabla_\mu$ is the covariant derivative associated with the metric tensor $g_{\mu\nu}$ and the quantity $\ast RR$ is the Pontryagin density defined as

$$ \ast RR = \ast R^\tau_{\sigma \mu \nu} R^\sigma_{\tau \mu \nu}, \tag{6} $$

where the dual Riemann tensor is given by $\ast R^\tau_{\sigma \mu \nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R^\tau_{\sigma \alpha \beta}$, with $\epsilon^{\mu\nu\alpha\beta}$ the four-dimensional Levi-Civita tensor.

Varying the action $S$ with respect to the metric $g_{\mu\nu}$, one obtains the gravitational field equation given by

$$ G_{\mu\nu} + \frac{\alpha}{\kappa} C_{\mu\nu} = \frac{1}{2\kappa} (T_{\mu\nu}^{\text{mat}} + T_{\mu\nu}^{\vartheta}), \tag{7} $$

where $G_{\mu\nu}$ is the Einstein tensor, and $C_{\mu\nu}$ is the cotton tensor defined as

$$ C^{\mu\nu} = \nabla_\alpha \vartheta \epsilon^{\sigma\rho\beta\delta} (\nabla_\sigma R^\rho_{\beta\delta})_\alpha + \nabla_\sigma \nabla_\alpha \vartheta \ast R^{(\mu\nu)\sigma}. \tag{8} $$


The total stress–energy tensor is split into the matter term $T_{\mu\nu}^{\text{mat}}$, and the scalar field contribution $T_{\mu\nu}^{\vartheta}$, which is provided by the following relationship:

$$T_{\mu\nu}^{\vartheta} = \beta \left[ (\nabla_{\mu} \vartheta)(\nabla_{\nu} \vartheta) - \frac{1}{2} g_{\mu\nu} (\nabla^{\rho} \vartheta)(\nabla_{\rho} \vartheta) - g_{\mu\nu} V(\vartheta) \right].$$  

(9)

Varying the action with respect to the scalar field $\vartheta$, one obtains the equation of motion for the CS coupling term given by

$$\beta \nabla_{\mu} \nabla_{\nu} \vartheta = \beta \frac{dV}{d\vartheta} - \frac{\alpha}{4} \ast RR.$$

(10)

Note that the evolution of the CS coupling is not only governed by its stress–energy tensor, but also by the curvature of spacetime. In the nondynamical formulation of CS-modified gravity the constraint $\beta = 0$ is considered, while in the dynamical framework, $\beta$ is allowed to be arbitrary, so that equation (10) is now the evolution equation for the CS coupling field.

Considering the diffeomorphism invariance of the matter part of the action, we have $\nabla_{\mu} T_{\mu\nu}^{\text{mat}} = 0$, and taking into account the Bianchi identities, i.e.

$$\nabla_{\mu} G_{\mu\nu} = 0,$$

provides the following conservation law:

$$\nabla_{\mu} C_{\mu\nu} = -\frac{1}{8} \left( \nabla^{\nu} \vartheta \right)^* RR.$$

(11)

### 2.2. Rotating black hole solutions in the Chern–Simons model

In this paper, we consider the Yunes–Pretorius (YP) slowly rotating solution found in [12], where the CS correction provides an effective reduction of the frame dragging around a black hole, in comparison with that of the Kerr solution. We will not analyze the Konno, Matsuyama and Tanda (KMT) approximation [11], since the YP solution is taken to second order in the rotation parameter, and therefore gives more accurate results.

The YP approximation method employs two schemes [12], namely a small-coupling approximation and a slow-rotation approximation. In particular, the small-coupling scheme treats the CS modification as a small deformation of general relativity. The slow-rotation scheme expands the background perturbations in powers of the Kerr rotation parameter $a$, and the background metric is formalized via the Hartle–Thorne approximation [13]. We refer the reader to [12] for details, and present the final metric given by

$$ds^2 = - \left[ \frac{1}{1 - \frac{2M}{r}} + \frac{2a^2 M}{r^3} \cos^2 \theta \right] dr^2 + \left[ \frac{1}{1 - \frac{2M}{r}} \right]^{-1} \left[ 1 + \frac{a^2}{r^2} \left( \cos^2 \theta - \left( 1 - \frac{2M}{r} \right)^{-1} \right) \right] d\eta^2$$

$$- \left[ 1 + \frac{2M}{7r} + \frac{27M^2}{10r^2} \right] \sin^2 \theta d\phi$$

$$+ (r^2 + a^2 \cos^2 \theta) d\theta^2$$

$$+ \left[ r^2 \sin^2 \theta + a^2 \sin^2 \theta \left( 1 + \frac{2M}{r} \sin^2 \theta \right) \right] d\phi^2,$$

(12)

with $\xi = a^2/(\kappa \beta)$.

In the following, we compare the properties of the metric given by equation (12) with the standard general relativistic Kerr metric, respectively, which in the equatorial approximation can be written as [22]

$$ds^2 = -DA^{-1} dr^2 + r^2 A (d\phi - \omega dr)^2 + D^{-1} dr^2 + dz^2,$$

(13)

where the coordinate $z = r \cos \theta$ was used to replace $\theta$, and $A = 1 + a^2 / r^4 + 2a^2 x^{-6}$ and $D = 1 - 2 x^{-2} + a^2 / x^{-4}$, respectively. The dimensionless coordinate $x$ is defined as $x = \sqrt{r/M}$, and the spin parameter $a_*$ is defined as $a_* = J/M^2 = a/M$. 


3. Thermal equilibrium radiation properties of thin accretion disks in stationary axisymmetric spacetimes

3.1. Stationary and axially symmetric spacetimes

The physical properties and the electromagnetic radiation characteristics of particles moving in circular orbits around general relativistic bodies are determined by the geometry of the spacetime around the compact object. For a stationary and axially symmetric geometry the metric is given in a general form by

$$ds^2 = g_{tt} dt^2 + 2 g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2.$$ (14)

In the equatorial approximation, which is the case of interest for our analysis, the metric functions $g_{tt}$, $g_{t\phi}$, $g_{rr}$, $g_{\theta\theta}$ and $g_{\phi\phi}$ only depend on the radial coordinate $r$, i.e. $|\theta - \pi/2| \ll 1$.

To compute the relevant physical quantities of thin accretion disks, we determine first the radial dependence of the angular velocity $\Omega_1$, of the specific energy $\tilde{E}$ and of the specific angular momentum $\tilde{L}$ of particles moving in circular orbits in a stationary and axially symmetric geometry through the geodesic equations. The latter take the following form [24]:

$$\frac{dr}{d\tau} = \frac{\tilde{E} \delta_{\phi\phi} + \tilde{L} \delta_{t\phi}}{g_{t\phi} - g_{tt} g_{\phi\phi}},$$ (15)

$$\frac{d\phi}{d\tau} = -\frac{\tilde{E} \delta_{t\phi} + \tilde{L} \delta_{t\phi}}{g_{t\phi} - g_{tt} g_{\phi\phi}}.$$ (16)

$$g_{rr} \left( \frac{dr}{d\tau} \right)^2 = -1 + \frac{\tilde{E}^2 \delta_{\phi\phi} + 2 \tilde{E} \tilde{L} \delta_{t\phi} + \tilde{L}^2 \delta_{tt}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}.$$ (17)

From equation (17) one can introduce an effective potential term as

$$V_{\text{eff}}(r) = -1 + \frac{\tilde{E}^2 \delta_{\phi\phi} + 2 \tilde{E} \tilde{L} \delta_{t\phi} + \tilde{L}^2 \delta_{tt}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}.$$ (18)

For stable circular orbits in the equatorial plane the following conditions must hold: $V_{\text{eff}}(r) = 0$ and $V_{\text{eff}}^{,rr}(r) = 0$, where the comma in the subscript denotes a derivative with respect to the radial coordinate $r$. These conditions provide the specific energy, the specific angular momentum and the angular velocity of particles moving in circular orbits for the case of spinning general relativistic compact spheres given by

$$\tilde{E} = \frac{g_{tt} + g_{t\phi} \Omega}{\sqrt{-g_{tt} - 2 g_{t\phi} \Omega - g_{\phi\phi} \Omega^2}}.$$ (19)

$$\tilde{L} = \frac{g_{t\phi} + g_{\phi\phi} \Omega}{\sqrt{-g_{tt} - 2 g_{t\phi} \Omega - g_{\phi\phi} \Omega^2}}.$$ (20)

$$\Omega = \frac{d\phi}{dr} = \frac{-g_{t\phi,r} + \sqrt{(g_{t\phi,r})^2 - g_{tt,r} g_{\phi\phi,r}}}{g_{t\phi,r}}.$$ (21)

The marginally stable orbit around the central object can be determined from the further condition $V_{\text{eff}}^{,rr}(r) = 0$, which provides the following important relationship:

$$0 = (g_{t\phi}^2 - g_{tt} g_{\phi\phi}) V_{\text{eff}}^{,rr} = \tilde{E}^2 g_{\phi\phi,rr} + 2 \tilde{E} \tilde{L} g_{t\phi,rr} + \tilde{L}^2 g_{tt,rr} - (g_{t\phi}^2 - g_{tt} g_{\phi\phi})_{,rr}.$$ (22)
where \( g_{1\phi}^2 - g_{tt} g_{\phi\phi} \) (appearing as a cofactor in the metric determinant) never vanishes. By inserting equations (19)–(21) into equation (22) and solving this equation for \( r \), we obtain the radii of the marginally stable orbits, once the metric coefficients \( g_{tt}, g_{t\phi} \) and \( g_{\phi\phi} \) are explicitly given.

### 3.2. Physical properties of thin accretion disks

For the thin accretion disk, it is assumed that its vertical size is negligible, as compared to its horizontal extension, i.e. the disk height \( H \), defined by the maximum half thickness of the disk, is always much smaller than the characteristic radius \( r \) of the disk, \( H \ll r \). The thin disk has an inner edge at the marginally stable orbit of the compact object potential, and the accreting plasma has a Keplerian motion in higher orbits.

In steady-state accretion disk models, the mass accretion rate \( \dot{M}_0 \) is assumed to be a constant that does not change with time. The radiation flux \( F \) emitted by the surface of the accretion disk can be derived from the conservation equations for the mass, energy, and angular momentum, respectively. Then the radiant energy \( F(r) \) over the disk is expressed in terms of the specific energy of the angular momentum, and of the angular velocity of the particles orbiting in the disk [20, 22]:

\[
F(r) = -\frac{\dot{M}_0}{4\pi\sqrt{-g}} \frac{\Omega}{(\tilde{E} - \tilde{\Omega}L)} \int_{r_m}^{r} (\tilde{E} - \tilde{\Omega}L) L_r \, dr, \tag{23}
\]

where \( \dot{M}_0 \) is the mass accretion rate measuring the rate at which the rest mass of the particles flows inward through the disk with respect to the coordinate time \( t \) and \( r_m \) is the marginally stable orbit obtained from equation (22).

Another important characteristics of the mass accretion process is the efficiency with which the central object converts rest mass into outgoing radiation. This quantity is defined as the ratio of the rate of the radiation energy of photons, escaping from the disk surface to infinity, and the rate at which mass energy is transported to the central compact general relativistic object, both measured at infinity [20, 22]. If all the emitted photons can escape to infinity, the efficiency is given in terms of the specific energy measured at the marginally stable orbit \( r_m \)

\[
\epsilon = 1 - \tilde{E}_{ms}. \tag{24}
\]

For Schwarzschild black holes the efficiency \( \epsilon \) is about 6%, whether the photon capture by the black hole is considered, or not. Ignoring the capture of radiation by the black hole, \( \epsilon \) is found to be 42% for rapidly rotating black holes, whereas the efficiency is 40% with photon capture in the Kerr potential [23].

The accreting matter in the steady-state thin disk model is supposed to be in thermodynamical equilibrium. Therefore, the radiation emitted by the disk surface can be considered as a perfect black body radiation, where the energy flux is given by \( F(r) = \alpha T^4(r) \) (\( \alpha \) is the Stefan–Boltzmann constant), and the observed luminosity \( L(\nu) \) has a redshifted black body spectrum [28]:

\[
L(\nu) = 4\pi d^2 I(\nu) = \frac{8}{\pi} \cos \gamma \int_{r_i}^{r_f} \int_0^{2\pi} \frac{v_r^3 r \, d\phi \, dr}{\exp(v_e/T) - 1}. \tag{25}
\]

Here \( d \) is the distance to the source, \( I(\nu) \) is the thermal energy flux radiated by the disk, \( \gamma \) is the disk inclination angle and \( r_i \) and \( r_f \) indicate the position of the inner and outer edge of the disk, respectively. We take \( r_i = r_{ms} \) and \( r_f \to \infty \), since we expect that the flux over the
disk surface vanishes at $r \to \infty$ for any kind of general relativistic compact object geometry. The emitted frequency is given by $\nu_e = \nu(1 + z)$, and the redshift factor can be written as

$$1 + z = \frac{1 + \Omega r \sin \phi \sin \gamma}{\sqrt{-g_{tt} - 2 \Omega g_{t\phi} - \Omega^2 g_{\phi\phi}}},$$

where we have neglected the light bending [35, 36].

4. Electromagnetic signatures of accretion disks around slowly rotating black holes in dynamical Chern–Simons gravity

Close to the equatorial plane of the slowly rotating black holes, one can introduce the coordinate $z = r \cos \theta$ describing ‘the height above the equatorial plane’ and write the metrics given by equation (12) in the form

$$ds^2 = -f(r) \ dt^2 + \frac{1}{r} \left( 1 + \frac{h_2(r)}{f(r)} \right) dr^2 + r^2 \left[ 1 + \frac{h_3(r)}{f(r)} \right] d\phi^2 + dz^2,$$

where $f(r) = 1 - 2M/r$ is the Schwarzschild form factor, and

$$h_1(r) = -\frac{5}{16} \frac{\xi}{M r^3} \left( 1 + \frac{12 M}{r} + \frac{27 M^2}{10 r^2} \right),$$

$$h_2(r) = -\frac{a^2}{r^2 f(r)},$$

$$h_3(r) = \frac{a^2}{r^2} \left( 1 + \frac{2M}{r} \right)$$

for the YP metric.

4.1. Constants of motion

If we insert the metric components of equation (27) into expressions (19)–(21) of the specific energy, of the specific angular momentum, and of the angular velocity, we obtain

$$\tilde{E} = \frac{f + 2Ma^{-1} \Omega(1 + h_1)}{\sqrt{f + 4Ma^{-1} \Omega(1 + h_1) - r^2 \Omega^2(1 + h_3)}},$$

$$\tilde{L} = \frac{-2Ma^{-1}(1 + h_1) + r^2 \Omega(1 + h_3)}{\sqrt{f + 4Ma^{-1} \Omega(1 + h_1) - r^2 \Omega^2(1 + h_3)}},$$

$$\Omega = \frac{2Ma^2}{r^3} \left[ H_1(r) \pm \sqrt{H_1(r) + \frac{r^3}{Ma^2 H_2(r)}} \right],$$

with

$$H_1(r) = \frac{1 + h_1 - rh_1, r}{H_2(r)}, \quad H_2(r) = \frac{1}{2} (1 + h_3 + rh_3, r).$$

As equations (31)–(33) show, the constants of motion for the particles orbiting in the equatorial plane depend only on the metric functions $f(r)$, $h_1(r)$ and $h_3(r)$, respectively. The coupling constant $\xi$ of the CS gravity appears only in $h_1(r)$. Since $h_3(r)$ is non-zero, it reduces the value of $g_{\phi\phi}$ for any $r$ outside the marginally stable orbit. However, the decrease in $g_{\phi\phi, r}$ is
proportional to \( h_3, r \sim a^2 r^{-3} \), and it causes only a small variation in the radial distribution of the angular velocity. The specific energy and angular momentum, depending on \( g_{\phi \phi} \), decreases only slightly in amplitude as well. The non-vanishing functions \( h_1(r), h_2(r) \) and \( h_3(r) \) give a negligible contribution to the volume element

\[
\sqrt{-g_{CS}} = \sqrt{(1 + h_2) \left[ \frac{4M^2a^2(1 + h_1)^2}{r^2 - 2Mr} + r^2(1 + h_3) \right]},
\]
as compared to the case of the equatorial approximation for slowly rotating general relativistic black holes:

\[
\sqrt{-g_{GR}} = \sqrt{\frac{4M^2a^2}{r^2 - 2Mr} + r^2}.
\]

As a result, the properties of particles orbiting the equatorial plane of slowly rotating black holes in the standard general relativistic theory and in CS-modified gravity are essentially the same. The only difference is in the location of the marginally stable orbits, which are strongly affected by the coupling. Since the inner edge of a thin accretion disk is supposed to be at the radius \( r_{ms} \), the radial profile of the energy flux radiated over the disk surface can indicate the differences in the mass accretion processes in the general relativistic theory, and in its CS-type modification, respectively.

### 4.2. Flux and temperature distribution

In figure 1 we present the flux distribution for the slowly rotating Kerr black holes, and for the slowly rotating YP solution, respectively. We consider the mass accretion driven by black holes with a total mass of \( M = 10^6 M_\odot \), and with a mass accretion rate of \( \dot{M}_0 = 10^{-12} M_\odot/\text{year} \). In units of the Eddington accretion rate \( \dot{M}_{\text{Edd}} = 1.5 \times 10^{17} (M/M_\odot) \text{ g s}^{-1} \) we have \( \dot{M}_0 = 4.22 \times 10^{-10} \dot{M}_{\text{Edd}} \). The spin parameter \( a_\ast \) runs from 0.1 to 0.4, whereas the coupling constant of the CS gravity is set to \( \xi = 28 M^4, 56 M^4, 112 M^4 \) and \( 168 M^4 \), respectively.

The plots show that the energy flux profiles of the disks in the CS-modified gravity models deviate from the slowly rotating general relativistic Kerr black hole case. For the smallest values of \( \xi \), the inner edge of the accretion disk is located at somewhat higher radius than the inner edge of the disk around the Kerr black hole (the location of the marginally stable orbits can be found in table 1, respectively, considered below). As the quantities \( E, L \) and \( \Omega \) in the flux integral (23) are still close for the CS model of gravity to those for the general relativistic case, for the lower boundary \( r_{ms,GR} < r_{ms,CS} \), the integral gives lower flux values. Thus, the maximum of the integrated flux is smaller in the CS-modified theory of gravity, and it decreases further as we increase the coupling constant. The effect of the coupling becomes more important as the black holes are rotating faster: for higher values of the spin parameter the same increase in the coupling constants produces considerably lower flux values, and shifts the marginally stable orbit to somewhat higher radii, as compared to the case of the very slow rotation (like, for example, in the cases with \( a_\ast = 0.1 \) and \( a_\ast = 0.4 \)).

Similar features can be found in figure 2, where we plot the temperature profiles of the disk. Although the differences here are not so large, since the temperature is proportional to \( F^{1/4} \), the CS approximations of the slowly rotating black hole metrics can still be discriminated from the standard general relativistic case.
4.3. Disk spectra and conversion efficiency

In figure 3, we present the spectral energy distribution of the disk radiation around the slowly rotating black holes for the general relativistic case, and for the CS-modified gravity. The plots show that with the increase of the coupling constants of the CS gravity, the cut-off frequency of the spectra decreases, from its value corresponding to the Kerr black hole, to lower frequencies of the order of $10^{14}$ Hz. Similarly to the case of the flux profiles, the effects of the CS coupling on the spectral cut-off are stronger for black holes rotating faster than for very slowly rotating black holes. For the radiation of the accretion disks around black holes this means that the CS theory produces rather similar disk spectra as in standard general relativity, even that with increasing coupling constants the radial distributions of the fluxes differ to some extent for these two theories (see the top left-hand plot in figures 1 and 3, respectively).

In table 1 we also present the conversion efficiency $\epsilon$ of the accreting mass into radiation for the case when the photon capture by the rotating black hole is ignored. The variation of $\epsilon$ as a function of $\xi / M^4$ is presented in figure 4.

The value of $\epsilon$ measures the efficiency of energy-generating mechanism by mass accretion. The amount of energy released by matter leaving the accretion disk and falling down the black hole is the binding energy $E(r)|_{r=r_m}$ of the black hole potential.
The inner edge of the accretion disk and the efficiency for slowly rotating black holes in general relativity and in the CS-modified theory of gravity with the YP approximation. The lines where the value of $\xi$ is not defined correspond to the general relativistic case.

| $\alpha_*$ | $\xi/M^4$ | $r_{\text{in}}/M$ | $\epsilon$ |
|-----------|-----------|-------------------|-------------|
| 0.1       | 5.6739    | 0.0606            |
| 28        | 5.7936    | 0.0599            |
| 56        | 5.8952    | 0.0592            |
| 112       | 6.0762    | 0.0581            |
| 168       | 6.2250    | 0.0571            |
| 0.2       | 5.3315    | 0.0646            |
| 28        | 5.6122    | 0.0626            |
| 56        | 5.8226    | 0.0611            |
| 112       | 6.1405    | 0.0588            |
| 168       | 6.3807    | 0.0572            |
| 0.3       | 4.9818    | 0.0694            |
| 28        | 5.4662    | 0.0653            |
| 56        | 5.7744    | 0.0628            |
| 112       | 6.1951    | 0.0595            |
| 168       | 6.5025    | 0.0573            |
| 0.4       | 4.6182    | 0.0751            |
| 28        | 5.3546    | 0.0679            |
| 56        | 5.7456    | 0.0643            |
| 112       | 6.2550    | 0.0601            |
| 168       | 6.6100    | 0.0574            |

Table 1 shows that $\epsilon$ is always higher for rotating general relativistic black holes than for their counterparts in CS-modified gravity. As the Kerr black holes spin up, the accreted mass-radiation conversion efficiency raises from about 6%, the characteristic value of the mass accretion of the static black holes, to 7.5%. This feature is much more moderate for the rotating black holes in the CS theory: with increasing rotational velocity, $\epsilon$ also increases; however, the rate of this increase becomes smaller for stronger CS coupling. However, these values show that the Kerr black holes provide a more efficient engine for the transformation of the energy of the accreting mass into radiation than their slowly rotating counterparts in the modified CS theory of gravity, no matter what approximation is used.

5. Discussions and final remarks

In the present paper we have considered the basic physical properties of matter forming a thin accretion disk in slowly rotating black hole spacetimes in the context of the dynamical formulation of CS-modified theories of gravity. The physical parameters of the disk—energy flux, temperature distribution and emission spectrum profiles—have been explicitly obtained for several values of the coupling constant for the YP solution. Due to the differences in the spacetime structure, the CS black holes present some very important differences with respect to the disk properties, as compared to the standard general relativistic Kerr case. We have also shown that the Kerr black holes also provide a more efficient engine for the transformation of
Figure 2. The temperature distribution over the accretion disks around slowly rotating black holes for different spin parameters in general relativity and in the modified CS theory of gravity with the YP approximation. The values of $M$, $a_*$ and $M_0$ are the same as in figure 1.

the energy of the accreting mass into radiation than their slowly rotating counterparts in the modified CS theory of gravity.

It is generally expected that most of the astrophysical objects grow substantially in mass via accretion. Recent observations suggest that around most of the active galactic nuclei (AGN’s) or black hole candidates there exist gas clouds surrounding the central far object, and an associated accretion disk, on a variety of scales from a tenth of a parsec to a few hundred parsecs [37]. These clouds are assumed to form a geometrically and optically thick torus (or warped disk), which absorbs most of the ultraviolet radiation and the soft x-rays. The most powerful evidence for the existence of super massive black holes comes from the very long baseline interferometry (VLBI) imaging of molecular H$_2$O masers in the active galaxy NGC 4258 [38]. This imaging, produced by Doppler shift measurements assuming Keplerian motion of the masering source, has allowed a quite accurate estimation of the central mass, which has been found to be a $3.6 \times 10^7 M_\odot$ super massive dark object, within 0.13 parsec. Hence, important astrophysical information can be obtained from the observation of the motion of the gas streams in the gravitational field of compact objects.

The flux and the emission spectrum of the accretion disks around compact objects satisfy some simple scaling relations, with respect to the simple scaling transformation of the accretion rate and mass. In order to analyze the scaling properties of the physical parameters of the accretion disks we introduce the scale-invariant dimensionless coordinate $x = r / M$. Then the functions $h_1(x)$ and $h_3(x)$ given by equations (28) and (30) depend only on the scale-invariant
The accretion disks spectra for slowly rotating black holes with different spin parameters in general relativity and in the modified CS theory of gravity with the YP approximation. The values of $M$, $a_*$ and $\dot{M}_0$ are the same as in figure 1.

The conversion efficiency $\epsilon$ as a function of the CS coupling parameter $\xi/M^4$ for different values of the spin parameter.

dimensionless spin parameter $a_*=a/M$, and they have no explicit dependence on the mass $M$. As a consequence, in equations (19)–(21), only the specific angular momentum and the rotational frequency have an explicit dependence on $M$, in the form $\tilde{L} \propto M$ and $\Omega \propto M^{-1}$.
whereas $\bar{E}$ does depend only on $a_*$. For any rescaling $M_2 = a M_1$ of the mass of the star we obtain $x_2 = a^{-1} x_1$, and the relations
\begin{equation}
\bar{E} (M_2; x_2) = \bar{E}(M_1; x_1), \quad L (M_2; x_2) = a L (M_1; x_1),
\end{equation}
and
\begin{equation}
\Omega (M_2; x_2) = a^{-1} \Omega (M_1; x_1),
\end{equation}
respectively. For any rescaling $M_0^{(2)} = \beta M_0^{(1)}$ of the accretion rate, these relations give
\begin{equation}
F(M_2, M_0^{(2)})(x_2) = \frac{\beta}{a^2} F(M_1, M_0^{(1)})(x_1),
\end{equation}
as the scaling relation of the flux integral given by equation (23). Then the temperature scales like
\begin{equation}
T(M_2, M_0^{(2)})(x_2) = \left( \frac{\beta}{\alpha^2} \right)^{1/4} T(M_1, M_0^{(1)})(x_1). \tag{37}
\end{equation}
For the maximum of the luminosity $L$ we have $v_{\text{max}} \propto T$, which gives $L(v_{\text{max}}) \propto v_{\text{max}}^3$. As the frequency scales like the temperature, we obtain that
\begin{equation}
L(M_2, M_0^{(2)})(v_2) = \left( \frac{\beta}{\alpha^2} \right)^{3/4} L(M_1, M_0^{(1)})(v_1), \tag{38}
\end{equation}
with
\begin{equation}
v_2(M_2, M_0^{(2)}) = \left( \frac{\beta}{\alpha^2} \right)^{1/4} v_1(M_1, M_0^{(1)}). \tag{39}
\end{equation}

On the other hand, the flux is proportional to the accretion rate $M_0$, and therefore an increase in the accretion rate leads to a linear increase in the radiation emission flux from the disk. For a simultaneous scaling of both the accretion rate $M_0$ and of the mass of the black hole $M$, the maximum of the flux scales as $F \to (M_0/M^2) F$, with all the other characteristics of the flux unchanged. Thus, if for a black hole mass with mass $M_1 = 10^6 M_\odot$ and by considering a mass accretion rate of $M_0^{(1)} = 10^{-12} M_\odot$/year = $4.22 \times 10^{-10} M_{\text{Edd}}$, in the case $a_* = 0.1$ and $\xi = 28 M_\odot^4$, the maximum of the flux is $F_{\text{max}}^{(1)} = 4 \times 10^7$ erg s$^{-1}$ cm$^{-2}$. In the case of a black hole with mass $M_2 = 10 M_\odot$ and with an accretion rate $M_0^{(2)} = 10^{-14} M_\odot$/year = $4.22 \times 10^{-2} M_{\text{Edd}}$, the maximum position of the flux is $F_{\text{max}}^{(2)} = \left( \frac{M_2}{M_1} \right)^{2} \left( \frac{M_0^{(2)}}{M_0^{(1)}} \right) F_{\text{max}}^{(1)} = 4 \times 10^5$ erg s$^{-1}$ cm$^{-2}$. The scaling law of the temperature is $T \to (M_0^{1/4}/\sqrt{M}) T$. Due to the temperature scaling, the maximum value of the spectrum increases, but the relative positions of the different spectral curves does not change.

The determination of the accretion rate for an astrophysical object can give a strong evidence for the existence of a surface of the object. A model in which Sgr A* the $3.7 \times 10^6 M_\odot$ super massive black hole candidate at the Galactic center, may be a compact object with a thermally emitting surface was considered in [39]. Given the very low quiescent luminosity of Sgr A* in the near-infrared, the existence of a hard surface, even in the limit in which the radius approaches the horizon, places a severe constraint on the steady mass accretion rate onto the source, $M \leq 10^{-12} M_\odot$ yr$^{-1}$. This limit is well below the minimum accretion rate needed to power the observed submillimeter luminosity of Sgr A*, $M \geq 10^{-10} M_\odot$ yr. Thus, from the determination of the accretion rate it follows that Sgr A* does not have a surface, that is, it must have an event horizon. Therefore, the study of the accretion processes by compact objects is a powerful indicator of their physical nature. However, up to now, the observational results have confirmed the predictions of general relativity mainly in a qualitative way. With
the present observational precision one cannot distinguish between the different classes of compact/exotic objects that appear in the theoretical framework of general relativity [29].

However, important technological developments may allow one to image black holes and other compact objects directly [39]. In principle, detailed measurements of the size and shape of the silhouette could yield information about the mass and spin of the central object, and provide invaluable information on the nature of the accretion flows in low luminosity galactic nuclei.

The spectrum in black hole systems can be dominated by the disk emission [40]. Recently, the RXTE satellite has provided a large number of data of the x-ray observations of the accretion flows in galactic binary systems. There is also a huge increase of the radio data for these systems. The behavior of the spectrum in such systems is consistent with the existence of a last stable orbit, and such data can be used to estimate the black hole spin. At high luminosities these systems can also show very different spectra [40]. Changes in the spectra of the disks are driven by a changing geometry. Presently there exists an enormous amount of data from the x-ray binary systems which can be used to test this assumption. Therefore, the study of accretion processes in low mass x-ray binaries with well-constrained thermal spectra could also lead to the possibility of discriminating between the various extensions of standard general relativity.

Hence the study of the accretion processes by compact objects is a powerful indicator of their physical nature. Since the energy flux, the temperature distribution of the disk, the spectrum of the emitted black body radiation, as well as the conversion efficiency show, in the case of the CS theory vacuum solutions, significant differences as compared to the general relativistic case, the determination of these observational quantities could discriminate, at least in principle, between standard general relativity and CS gravity, and constrain the parameters of the model.

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