Finite dimensional global and exponential attractors for a coupled time-dependent Ginzburg-Landau equations for atomic Fermi gases near the BCS-BEC crossover

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Abstract

We study a coupled nonlinear evolution system arising from the Ginzburg-Landau theory for atomic Fermi gases near the BCS-BEC crossover. First, we prove that the initial boundary value problem generates a strongly continuous semigroup on a suitable phase-space which possesses the global attractor. Then we establish the existence of an exponential attractor. As a consequence, we show that the global attractor is of finite fractal dimension.

Keywords: time-dependent Ginzburg-Landau equations, BCS-BEC crossover, global attractor, exponential attractors.

1 Introduction

The superfluidity in the ultra-cold atomic Fermi gases has been paid much attention by many researchers in recent years, since it provides a useful testbed for the study of high-temperature superconductivity in strongly correlated fermionic systems. In the superfluid atomic Fermi gases near the Feshbach resonance the strong attractive interaction is realized between fermion atoms which can cause a crossover from the weak-coupling BCS state to the strong-coupling BEC one [15,18]. The Ginzburg-Landau theory plays an important role in superconductivity research which was applied in the pioneering works [6,17] and later in the single-component fermion system (single channel model) [120]. Recently, Machida & Koyama [15] developed a time-dependent Ginzburg-Landau (TDGL) theory for the superfluid atomic Fermi gases near the Feshbach resonance from the fermion-boson model on the basis of the functional integral formalism. This

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two-component TDGL model describes the dynamics of the superfluid atomic Fermi gases, in which BCS pairs and tightly bound diatomic condensate coexist. The resulting system consists of a nonlinear time-dependent complex Ginzburg-Landau equation coupled with a Schrödinger type equation, which reads as follows (cf. [15])

\[
\begin{aligned}
-i\dot{u}_t &= \left( -\frac{\partial^2 u + a}{U} + a \right) u + g[a + d(2\nu - 2\mu)]\phi + \frac{c}{4m} \Delta u + \frac{b}{4m}(c - d)\Delta \phi \\
-b|u + g\phi|^2(u + g\phi), \\
i\phi_t &= -\frac{b}{4} u + (2\nu - 2\mu)\phi - \frac{1}{4m} \Delta \phi.
\end{aligned}
\]

(1.1)

\(u\) and \(\phi\) are both complex-valued unknown functions, which stand for the fermion-pair field and the condensed boson field, respectively. \(2\nu\) is the threshold energy of the Feshbach resonance while \(g\) is the coupling constant in the Feshbach resonance. \(\mu\) is the chemical potential and \(U > 0\) denotes the BCS coupling constant. The coefficients \(a, b\) and \(c\) correspond to the Ginzburg-Landau coefficients in the TDGL theory. All these seven coefficients are real numbers. The coefficient \(d\) is generally complex, which dominates the dynamics of the superfluid atomic Fermi gases. In the BCS limit, \(d\) can be considered to be purely imaginary while in the BEC region, the imaginary part of \(d\) usually vanishes. In the BCS-BEC crossover region, both the real and imaginary parts of \(d\) have finite values that \(d\) is a complex number (see e.g., (A3) below). For the detailed discussions on these physical coefficients, we refer to [15].

By introducing a new variable

\[v = u + g\phi,\]

we can transform the original system (1.1) into the following form, which is more convenient to be treated from the mathematical point of view (cf. [3][5][11]):

\[
\begin{aligned}
dv_t - \left( a - \frac{1}{U} \right) iv - \frac{i\phi}{4m} \Delta v + ib|v|^2v &= 0, \\
\phi_t - \frac{ig_0}{4} v + \frac{ig^2}{4} \phi + i(2\nu - 2\mu)\phi - \frac{i}{4m} \Delta \phi &= 0.
\end{aligned}
\]

(1.2)

In Chen & Guo [3], the authors proved the existence and uniqueness of weak solutions to (1.2) subject to periodic boundary conditions. Later in [11], the global existence of weak solutions to the periodic boundary value problem of system (1.2) with a general nonlinearity \(ib|v|^p v\) was obtained for certain power \(p\) instead of 2. As far as the classical solution is concerned, Chen & Guo [5] studied the initial boundary problem of (1.2) subject to homogeneous Dirichlet/Neumann boundary conditions for arbitrary spatial dimension. They proved the global existence and uniqueness of classical solutions under some specific restrictions on the complex coefficient \(d\). However, no results on the long-time behavior of the global weak/classical solutions to (1.2) were obtained in the papers [3][5] mentioned above. One possible difficulty is that we do not have enough dissipative mechanism in the equation for \(\phi\). As a first step for the study of the long-time dynamics of the problem, in the present paper, we consider the system (1.2) with a linear weak dissipation in the equation for \(\phi\), that is

\[
\begin{aligned}
dv_t - \left( a - \frac{1}{U} \right) iv - \frac{i\phi}{4m} \Delta v + ib|v|^2v &= f, \quad (x, t) \in \Omega \times \mathbb{R}_+, \\
\phi_t + \gamma \phi - \frac{ig_0}{4} v + \frac{ig^2}{4} \phi + i(2\nu - 2\mu)\phi - \frac{i}{4m} \Delta \phi &= h, \quad (x, t) \in \Omega \times \mathbb{R}_+, 
\end{aligned}
\]

(1.3)
where $\gamma > 0$ is the damping parameter. For the sake of simplicity, we consider the problem in a bounded domain $\Omega \subset \mathbb{R}^3$ whose boundary $\Gamma$ is smooth. $f$ and $h$ are given external forces. System (1.3) is subject to the homogeneous Dirichlet boundary conditions

$$v = \phi = 0, \quad (x, t) \in \Gamma \times \mathbb{R}_+,$$

and the initial conditions

$$v|_{t=0} = v_0(x), \quad \phi|_{t=0} = \phi_0(x), \quad x \in \Omega.$$ (1.4)

To formulate our results, we first introduce some notions on the functional settings. Let $L^2(\Omega)$ (or $L^2(\Omega)$) be the Lebesgue space of complex-valued (real-valued) functions. By $(\cdot, \cdot)$ and $\| \cdot \|$, we denote the scalar product and the norm in $L^2(\Omega)$ (or $L^2(\Omega)$), respectively:

$$(w_1, w_2) = \int_{\Omega} w_1 \overline{w_2} dx, \quad \|w\| = \sqrt{(w, w)}.$$ (1.5)

Let $W^{k, p}(\Omega)$ be the standard Sobolev spaces for real-valued functions and as usual, $H^k(\Omega) = W^{k, 2}(\Omega)$. Correspondingly, Sobolev spaces of complex-valued functions are denoted by $W^k(\Omega)$ and similarly, $H^k(\Omega) = W^{k, 2}(\Omega)$. We note that $L^p(\Omega) = W^{0, p}(\Omega)$, $L^p(\Omega) = W^{0, p}(\Omega)$ and $H^{-1}(\Omega)$ (or $H^{-1}(\Omega)$) is the dual space of $H^1_0(\Omega)$ (or $H^1_0(\Omega)$).

Let $A$ be the unbounded linear operator defined by $A = -\Delta$, whose domain is $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. It is well-known that (cf. e.g., [19]) one can define spaces $D(A^s)$ for $s \in \mathbb{R}$, with inner product $\langle \cdot, \cdot \rangle_s = (A^s \cdot, A^s \cdot)$ and corresponding norm $| \cdot |_s = \sqrt{\langle \cdot, \cdot \rangle_s}$. In particular, $D(A^{1/2}) = H^1_0(\Omega)$, $D(A^0) = L^2(\Omega)$, $D(A^{-1/2}) = H^{-1}(\Omega)$. We note that corresponding results hold for the complex-valued functional spaces.

In this paper, we make the following assumptions on external forces $f$, $h$ and the coefficients of system (1.3):

(A1) $f, h \in H^1_0(\Omega)$ are independent of time,

(A2) $U > 0, b > 0, c > 0, m > 0, aU < 1, \gamma > 0$,

(A3) $d := d_r + id_i$, where $d_r, d_i \in \mathbb{R}$ and $d_i > 0$. $|d| = \sqrt{d_r^2 + d_i^2}$.

Next, we introduce the weak formulation of problem (1.3)–(1.5):

**Definition 1.1.** A pair of complex-valued functions $(v, \phi)$ is called a weak solution to problem (1.3)–(1.5) in $Q_T := \Omega \times [0, T]$ for arbitrary $T > 0$, if

$$v, \phi \in C([0, T], H^1_0(\Omega)), \quad v \in L^2((0, T), L^2(\Omega)), \quad \phi \in L^2((0, T), H^{-1}(\Omega)),$$

and for arbitrary complex-valued functions $\psi \in H^1_0(\Omega)$ and $\xi \in C^1[0, T]$ with $\xi(T) = 0$, it holds

$$\int_0^T \left[ -d(v, \xi v) - i \left( a - \frac{1}{U} \right) (v, \xi v) - \frac{ig}{U} (\phi, \xi \psi) + \frac{ic}{4m} (\nabla v, \xi \nabla \psi) + ib(|v|^2 v, \xi \psi) \right] dt = 0 + (v_0, \xi(0)\psi),$$

$$\int_0^T \left[ -(\phi, \xi \psi) - \frac{ig}{U} (v, \xi \psi) + i \left( \frac{g^2}{U} + 2\nu - 2\mu \right) (\phi, \xi \psi) + \frac{i}{4m} (\nabla \phi, \xi \nabla \psi) + \gamma(\phi, \xi \psi) \right] dt = 0 + (\phi_0, \xi(0)\psi).$$

(1.6)
The main results of this paper are as follows:

(a) Existence and uniqueness of global weak solutions (cf. Theorem 2.1 and Corollary 2.1);

(b) Existence of a global attractor with finite fractal dimension (cf. Theorem 3.1 and Corollary 4.2);

(c) Existence of an exponential attractor (cf. Theorem 4.1).

We note that in the recent paper [11], the authors also considered the long-time behavior of system (1.1) with a linear weak dissipation term in the equation for \( \phi \). In particular, they proved the existence of a weakly compact attractor under some specific restrictions on the coefficients \( \gamma, g, c \) and \( d \) when the spatial dimension is three. However, comparing their results, our present work has some new features. (i) We prove the existence of an absorbing set in \( H_1^0 \times H_1^0 \) for our problem (1.3)-(1.5) under much simpler assumptions on the physical coefficients (cf. (A2), (A3)). In [11], the corresponding result was obtained under some rather specific restrictions on the coefficients. For instance, it was required that \( 0 < g < 2 \) and the positive damping parameter \( \gamma \), denoted by \( \beta \) in [11], was assumed to be bounded from below by a positive constant such that 
\[
\beta > \frac{2\gamma + \frac{\gamma + 1}{2}}{2-g} > 0.
\]
Although the weakly damped system considered in [11] is slightly different from ours in the formulation, by a careful calculation, one can obtain the same a priori estimates without those restrictions therein. (ii) The equation for \( \phi \) is a Schrödinger type equation, which does not enjoy the smoothing property like parabolic equations. To show the precompactness of \( \phi \), we use a suitable decomposition to split the trajectory into two parts: one decays exponentially fast to zero, and the other one satisfies a certain compactness property. We recall that in [11], no results on the compactness of weak solutions were obtained and only the existence of a weakly compact attractor was proved. (iii) We prove the finite dimensionality (in terms of fractal dimension) of the global attractor and the existence of an exponential attractor. Although the global attractor represents the first important step in the understanding of long-time dynamics of a given evolutionary problem, it may also present some severe drawbacks. Indeed, as simple examples show, the rate of convergence to the global attractor may be arbitrarily slow. This fact makes the global attractor very sensitive to perturbations and to numerical approximation. In addition, it is usually very difficult to estimate the rate of convergence to the global attractor and to express it in terms of the physical parameters of the system. The concept of exponential attractor has then been proposed in [7] to possibly overcome these drawbacks. The exponential attractors contain the global attractor, are finite dimensional, and attract the trajectories exponentially fast. Comparing with the global attractor, an exponential attractor turns out to be much more robust to perturbations. Besides, it provides a way of proving that the global attractor has finite fractal dimension. We refer to [16] for a survey. In this paper, we apply a simple method that also works in Banach spaces, due to [8] (see [2, 9, 13] for generalizations) to prove the existence of an exponential attractor. As a byproduct, we obtain the finite fractal dimensionality of the global attractor.

The remaining part of this paper is organized as follows. In Section 2, we prove the existence and uniqueness of global weak solutions to problem (1.3)-(1.5). In Section 3, we show that problem (1.3)-(1.5) possesses a compact global attractor \( \mathcal{A} \) in \( H_1^0 \times H_1^0 \). In the last Section 4, we
prove the existence of an exponential attractor \( \mathcal{E} \), whose basin of attraction is the whole space \( (H^2 \cap H^1_0) \times (H^2 \cap H^1_0) \).

2 Global Existence and Uniqueness of Weak Solutions

In order to prove the existence of weak solutions to problem (1.3)-(1.5), we shall use the Faedo-Galerkin method to find approximate solutions. After deriving some uniform a priori estimates for the approximate solution, we can pass to the limit. We denote by \( C \) and \( C_i \) positive constants that may vary from place to place. Special dependence will be indicated if it is necessary.

**Theorem 2.1.** Suppose that assumptions (A1)-(A3) are satisfied. For any \((v_0, \phi_0) \in H^1_0(\Omega) \times H^1_0(\Omega) \) and \( T > 0 \), the initial boundary value problem (1.3)-(1.5) admits a global weak solution \((v, \phi)\).

**Proof.** Step 1. Galerkin’s approximation

Let \( \{\omega_j\}, j = 1, 2, \ldots \) be a system of eigenfunctions of the operator \( A \), that is,

\[- \Delta \omega_j = \lambda_j \omega_j, \quad \text{in} \ \Omega, \quad \text{and} \quad \omega_j = 0, \quad \text{on} \ \Gamma, \quad (2.1)\]

where \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) are the eigenvalues. It is easy to see that \( \{\omega_j\} \) forms base functions of \( H^1_0(\Omega) \) as well as \( L^2(\Omega) \). Moreover, \( \omega_j \in C^\infty, \ j \in \mathbb{N} \).

Let \( l \) be a given positive integer. We denote the approximate solutions of problem (1.3)-(1.5) by \( v_l(x, t) \) and \( \phi_l(x, t) \) such that \( v_l(x, t) = \sum_{j=1}^l \alpha_j(t) \omega_j(x) \), \( \phi_l(x, t) = \sum_{j=1}^l \beta_j(t) \omega_j(x) \), where \( \alpha_j(t), \ \beta_j(t), \ (j = 1, 2, \ldots, l) \) are complex-valued functions that satisfy the following system of ordinary differential equations of first order: for \( j = 1, 2, \ldots, l \),

\[ d(v_l, \omega_j) - i \left( a - \frac{1}{U} \right) (v_l, \omega_j) - \frac{ig}{U} (\phi_l, \omega_j) - \frac{ic}{4m} (\Delta v_l, \omega_j) + ib \| v_l \|^2 (v_l, \omega_j) = (f, \omega_j), \quad (2.2) \]

\[ (\phi_l, \omega_j) - \frac{ig}{U} (v_l, \omega_j) + i \left( \frac{g^2}{U} + 2 \nu - 2 \mu \right) (\phi_l, \omega_j) - \frac{i}{4m} (\Delta \phi_l, \omega_j) + \gamma (\phi_l, \omega_j) = (h, \omega_j), \quad (2.3) \]

with the initial data

\[ (v_l(0), \omega_j) = \eta_{jl}, \quad (\phi_l(0), \omega_j) = \zeta_{jl}. \quad (2.4) \]

\( \eta_{jl}, \zeta_{jl} \) are constants such that as \( l \to +\infty \)

\[ \sum_{j=1}^l \eta_{jl} \omega_j \to v_0, \quad \sum_{j=1}^l \zeta_{jl} \omega_j \to \phi_0, \quad \text{strongly in} \ H^1_0(\Omega). \]

Existence of such \( \eta_{jl}, \zeta_{jl} \) follows from the fact that \( (v_0, \phi_0) \in H^1_0(\Omega) \times H^1_0(\Omega) \) and the definition of \( \{\omega_j\} \). Actually, we can just take \( \eta_{jl} = \eta_{jl}(t_0), \ \zeta_{jl} = \phi_0, \omega_j \) at \( t_0 \).

The standard theory for nonlinear ordinary differential equations of first order (i.e., the Picard iteration method) ensures that for each \( l \), the initial value problem (2.2), (2.3) admits a unique local solution \((v_l, \phi_l)\) on \([0, t_0]\) where \( t_0 \) depends only on \(|\xi_{jl}|\) and \(|\zeta_{jl}|\). We omit the details here.

**Step 2. a priori estimates**
We now try to obtain some \textit{a priori} estimates for the approximate solutions. Multiplying (2.2) by \(\alpha_j\), \(\frac{d}{dt}\alpha_j\), and \(\lambda_j\), respectively, summing over \(j\) from 1 to \(l\) and taking the imaginary part of the results, we get

\[
\frac{d}{dt}\|v_I\|^2 + \frac{1}{U} - a \right) \|v_I\|^2 + \frac{c}{4m} \|\nabla v_I\|^2 + b\|v_I\|^2  \\
= \text{Im} \int_\Omega f\nu d\Omega + \frac{g}{U} \text{Re} \int_\Omega \phi_I \nu d\Omega - d_r \text{Im} \int_\Omega v_I \nu d\Omega \\
\leq \frac{d}{dt}\|v_I\|^2 + \|\phi_I\|^2 + \|f\|^2 + C\|v_I\|^2. 
\tag{2.5}
\]

\[
\frac{d}{dt}\|v_I\|^2 + \frac{1}{U} - a \right) \|v_I\|^2 + \frac{c}{8m} \|\Delta v_I\|^2 + 2b \int_\Omega |\nabla v_I|^2 v_I^2 dx \\
= - \text{Im} \int_\Omega f \Delta \nu d\Omega - \frac{g}{U} \text{Re} \int_\Omega \phi_I \Delta \nu d\Omega - b \text{Re} \int_\Omega \nabla \nu \cdot \nabla v_I^2 d\Omega + d_r \text{Im} \int_\Omega v_I \Delta \nu d\Omega \\
\leq b \int_\Omega |\nabla v_I|^2 v_I^2 dx + \frac{c}{8m} \|\Delta v_I\|^2 + C(\|f\|^2 + \|\phi_I\|^2) + C\|v_I\|^2. 
\tag{2.7}
\]

On the other hand, multiplying (2.3) by \(\beta_j\), \(\lambda_j\), and \(\lambda_j^{-1}\frac{d}{dt}\beta_j\), respectively, summing over \(j\) from 1 to \(l\), and taking the real part, we obtain

\[
\frac{1}{2} \frac{d}{dt}\|\phi_I\|^2 + \gamma\|\phi_I\|^2 = \text{Re} \int_\Omega h\nu d\Omega - \frac{g}{U} \text{Im} \int_\Omega v_I \nu d\Omega \leq \frac{g}{U} \|\phi_I\|^2 + C(\|\nu\|^2 + \|v_I\|^2). 
\tag{2.8}
\]

\[
\frac{1}{2} \frac{d}{dt}\|\phi_I\|^2 + \gamma \|\phi_I\|^2 = \text{Re} \int_\Omega \phi_I \nabla \nu d\Omega - \frac{g}{U} \text{Im} \int_\Omega \nabla v_I \cdot \nabla \phi_I d\Omega \\
\leq \frac{g}{U} \|\phi_I\|^2 + C(\|\nabla \nu\|^2 + \|\nabla v_I\|^2). 
\tag{2.9}
\]

\[
\frac{g}{2} \frac{d}{dt}\|\phi_I\|^2_{H^{k-1}} + \|\phi_I\|^2_{H^{k-1}} = \left(\frac{g^2}{U} + 2\nu - 2\mu\right) \text{Im} \int_\Omega \phi_I \Delta^{-\frac{1}{2}} \phi_I d\Omega - \frac{1}{4m} \text{Im} \int_\Omega \phi_I \phi_I d\Omega \\
- \frac{g}{U} \text{Im} \int_\Omega v_I \Delta^{-\frac{1}{2}} \phi_I d\Omega + \text{Re} \int_\Omega h \Delta^{-\frac{1}{2}} \phi_I d\Omega \\
\leq \frac{1}{2} \|\phi_I\|^2_{H^{k-1}} + C(\|\phi_I\|^2_{H^{k-1}} + \|\phi_I\|^2_{H^{k-1}} + \|h\|^2_{H^{k-1}}). 
\tag{2.10}
\]

Multiplying (2.7) by a small positive constant \(\kappa \in (0, \frac{d}{8m})\), adding it with (2.5), (2.6), (2.8), (2.9) together, we obtain

\[
\frac{d}{dt} \mathbf{T}_1(t) + \mathbf{T}_2(t) \leq C(\|v_I(t)\|^2 + \|\phi_I(t)\|^2 + \|f\|^2 + \|h\|^2_{H^{k-1}}), 
\tag{2.11}
\]
Finally, we infer from (2.10), (2.15) and assumptions (A1), (A2) that the problem (2.2)-(2.4) can be extended to (2.11). The above uniform estimates imply that the solution \( l \) of the system. Turning back to (2.11) and integrating with respect to time, we can see that
\[
\int_0^T \frac{d}{dt} \Upsilon_1(t) dt \leq C_2 \Upsilon_1(t) + C_3 (\|f\|^2 + \|h\|^2_{\mathcal{H}^1}).
\]

By the Gronwall inequality and assumption (A1), we conclude that for arbitrary \( T > 0 \):
\[
\Upsilon_1(t) \leq e^{C_2 t} \left[ \Upsilon_1(0) + \frac{C_3}{C_2} (\|f\|^2 + \|h\|^2_{\mathcal{H}^1}) \right], \quad \forall t \in [0, T].
\]

As a result,
\[
\|v_l(t)\|^2_{\mathcal{H}^1} + \|\phi_l(t)\|^2_{\mathcal{H}^1} \leq C_T, \quad \forall t \in [0, T],
\]
where \( C_T \) is a constant depending on \( \|v_0\|_{\mathcal{H}^1}, \|\phi_0\|_{\mathcal{H}^1}, \|f\|, \|h\|_{\mathcal{H}^1}, T, \Omega \), and the coefficients of the system. Turning back to (2.11) and integrating with respect to time, we can see that
\[
\int_0^T \Upsilon_2(t) dt \leq \Upsilon_1(0) + C_2 \int_0^T \Upsilon_1(t) dt + C_3 T (\|f\|^2 + \|h\|^2_{\mathcal{H}^1}) \leq C_T,
\]
which implies that
\[
\int_0^T (\|v_l(t)\|^2_{\mathcal{H}^1} + \|v_{lt}(t)\|^2 + \|\phi_l(t)\|^2_{\mathcal{H}^1}) dt \leq C_T.
\]

Finally, we infer from (2.10), (2.15) and assumptions (A1), (A2) that
\[
\int_0^T \|\phi_{lt}(t)\|^2_{\mathcal{H}^{-1}} dt \leq C_T.
\]

The above uniform estimates imply that the solution \((\alpha_l(t), \ldots, \alpha_l(t), \beta_l(t), \ldots, \beta_l(t))\) to ODE problem (2.2)-(2.4) can be extended to \([0, T]\), for any \( T > 0 \). Moreover, on \([0, T]\) we have the following uniform \textit{a priori} estimates:

\[
\begin{align*}
\nu_l, \phi_l & \quad \text{uniformly bounded in } L^\infty((0, T), \mathcal{H}_0^1), \\
v_l & \quad \text{uniformly bounded in } L^2((0, T), \mathcal{H}^2), \\
\phi_l & \quad \text{uniformly bounded in } L^2((0, T), \mathcal{H}_0^1), \\
v_{lt} & \quad \text{uniformly bounded in } L^2((0, T), \mathcal{L}^2), \\
\phi_{lt} & \quad \text{uniformly bounded in } L^2((0, T), \mathcal{H}^{-1}),
\end{align*}
\]

\textit{Step 3. Convergence of the approximate solutions as } \( l \to +\infty \)
The uniform bounds \(\text{(2.19)}\) yield that there exist functions \((v, \phi)\) and subsequences of \(\{v_l\}\) and \(\{\phi_l\}\) (still denoted by \(\{v_l\}\) and \(\{\phi_l\}\) for the sake of simplicity) such that as \(l \to +\infty\),

\[
\begin{align*}
    v_l & \to v, \quad \phi_l \to \phi, \quad \text{weakly-* in } L^\infty((0, T), \mathcal{H}^1_0), \\
    v_l & \to v \quad \text{weakly in } L^2((0, T), \mathcal{H}^2), \\
    \phi_l & \to \phi \quad \text{weakly in } L^2((0, T), \mathcal{H}^1_0), \\
    v_{ll} & \to v_t \quad \text{weakly in } L^2((0, T), \mathcal{L}^2), \\
    \phi_{ll} & \to \phi_t \quad \text{weakly in } L^2((0, T), \mathcal{H}^{-1}).
\end{align*}
\]

(2.20)

From \(\phi \in L^2((0, T), \mathcal{H}^1_0), \phi_l \in L^2((0, T), \mathcal{H}^{-1})\) and \([19\text{ Lemma II.3.2}]\) we know that \(\phi \in C([0, T], \mathcal{L}^2)\). Besides, by the following result (cf. e.g., \([14]\))

**Lemma 2.1.** Let \(X \subset Y\) be two Hilbert spaces, and suppose that the embedding of \(X\) into \(Y\) is compact. The following continuous embedding holds: \(\{f \in L^2((0, T), X), \ f_l \in L^2((0, T), Y)\} \hookrightarrow C([0, T]; [X, Y]_q)\).

and the fact that \(\mathcal{H}^1_0 = [\mathcal{H}^2 \cap \mathcal{H}^1_0, \mathcal{L}^2]_\frac{1}{2}\) (cf. \([19]\)), we have (up to a subsequence)

\[
v_l \to v \quad \text{weakly in } C([0, T], \mathcal{H}^1_0).
\]

(2.21)

We infer from \([19\text{ Lemma II.3.3}]\) that \(\phi\) is weakly continuous with values in \(\mathcal{H}^1_0\). Namely, for any \(\psi \in \mathcal{H}^1_0\), \(t \mapsto \int_0^T \nabla \phi(t) \cdot \nabla \psi dt\) is continuous. Arguing as in \([19]\), we can get an equality similar to \(\text{(2.9)}\) which holds in the distributional sense on \((0, T)\):

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \phi\|^2 + \gamma \|\nabla \phi\|^2 = \text{Re} \int_0^T \nabla h \cdot \nabla \phi dx - \frac{q}{U} \text{Im} \int_0^T \nabla v \cdot \nabla \phi dx.
\]

(2.22)

As a result, \(t \mapsto \|\nabla \phi(t)\|^2\) is also continuous on \([0, T]\). Since \(\|\nabla \cdot \|\) is the equivalent norm on \(\mathcal{H}^1_0\), we conclude that \(\phi \in C([0, T], \mathcal{H}^1_0)\).

The well-known Aubin-Lions lemma implies that there is a subsequence of \(v_l\), still denoted by \(v_l\) such that

\[
v_l \to v \quad \text{strongly in } L^2((0, T), \mathcal{H}^1_0).
\]

(2.23)

Hence, there is a subsequence of \(v_l\), still denoted by \(v_l\) such that \(v_l\) almost everywhere converges to \(v\) in \(Q_T = \Omega \times [0, T]\). It turns out that \(|v_l|^2 v_l\) almost everywhere converges to \(|v|^2 v\) in \(Q_T\).

On the other hand, it follows from \(\text{(2.19)}\) that \(|v_l|^2 v_l\) is uniformly bounded in \(L^\infty((0, T), \mathcal{L}^2)\) and hence in \(L^2((0, T), \mathcal{L}^2)\). Therefore, we infer that the weak limit of \(|v_l|^2 v_l\) in \(L^2([0, T], \mathcal{L}^2(\Omega))\) equals to \(|v|^2 v\):

\[
|v_l|^2 v_l \to |v|^2 v \quad \text{weakly in } L^2((0, T), \mathcal{L}^2).
\]

(2.24)

Passing to the limit \(l \to +\infty\), we can infer from the above convergence properties of \(v_l, \phi_l\) that \([1.6]\) and \([1.7]\) are satisfied. Concerning the initial data, we infer from \(\text{(2.20)}\) that (cf. e.g., \([21\text{ Lemma 3.1.7}]\)) that

\[
\begin{align*}
    v_l(0) & = \sum_{j=1}^l \eta_j \omega_j \to v(0) \quad \text{weakly in } \mathcal{L}^2(\Omega), \\
    \phi_l(0) & = \sum_{j=1}^l \zeta_j \omega_j \to \phi(0) \quad \text{weakly in } \mathcal{H}^{-1}(\Omega).
\end{align*}
\]

(2.25)
On the other hand, we know that \((v_1(0), \phi_1(0))\) strongly converges in \(H^1_0 \times H^1_0\); hence, it also weakly converges to \((v_0, \phi_0)\) in \(L^2 \times H^{-1}\). By the uniqueness of the limit, we have \(v(0) = v_0, \phi(0) = \phi_0\).

Summing up, we have proved the existence of a global weak solution \((v, \phi)\) to problem (1.3)-(1.5). The proof is complete. \(\square\)

Next, we show the continuous dependence result on the initial data that yields the uniqueness of weak solutions to problem (1.3)-(1.5):

**Theorem 2.2.** For any \((v_0, \phi_0), (v_2, \phi_0) \in H^1_0(\Omega) \times H^1_0(\Omega),\) we denote the corresponding global weak solutions to problem (1.3)-(1.5) by \((v_1, \phi_1)\) and \((v_2, \phi_2)\), respectively. For any \(T > 0,\) it holds

\[
\|v_1(t) - v_2(t)\|_{H^1_0}^2 + \|\phi_1(t) - \phi_2(t)\|_{H^1_0}^2 + \int_0^t \|v_1(t) - v_2(t)\|^2 dt \\ \leq L_1 e^{L_2 t} (\|v_0 - v_2\|_{H^1_0}^2 + \|\phi_0 - \phi_2\|_{H^1_0}^2), \quad \forall \ 0 \leq t \leq T,
\]

where \(L_1, L_2\) are positive constants depending on \(\|v_0\|_{H^1_0}, \|\phi_0\|_{H^1_0}, \|v_2\|_{H^1_0}, \|\phi_2\|_{H^1_0}, |\Omega|, f, h\) and coefficients of system (1.3).

**Proof.** We shall just perform formal computations that can be justified within the same Galerkin scheme used above. Let \(v = v_1 - v_2, \phi = \phi_1 - \phi_2, v(0) = v_0 - v_2, \phi(0) = \phi_0 - \phi_2\). Then the differences \((v, \phi)\) satisfy a.e. in \([0, T]\) that

\[
dv_t + i \left( \frac{1}{U} - a \right) v - i \frac{g}{U} \phi - \frac{i c}{4 m} \Delta v + i b (|v_1|^2 - |v_2|^2 v_2) = 0,\]

\[
\phi_t - i \frac{g}{U} v + i \left( \frac{g^2}{U} + 2 \nu - 2 \mu \right) \phi - \frac{i}{4 m} \Delta \phi + \gamma \phi = 0.
\]

Multiplying (2.27) by \(\overline{v_t}\), integrating over \(\Omega\) and taking the imaginary part of the result, we have

\[
\frac{d}{dt} \left[ \frac{1}{2} \left( \frac{1}{U} - a \right) \|v\|^2 + \frac{c}{8 m} \|\nabla v\|^2 \right] + d_t \|v_t\|^2 \\ = \frac{g}{U} \text{Re} \int_\Omega \phi \overline{v_t} dx - b \text{Re} \int_\Omega (|v_1|^2 v_1 - |v_2|^2 v_2) \overline{v_t} dx \\ \leq \frac{d_t}{2} \|v_t\|^2 + C \|\phi\|^2 + C \|v_1|^2 \|v_1 - |v_2|^2 v_2\|^2 \\ \leq \frac{d_t}{2} \|v_t\|^2 + C\|\phi\|^2 + C \left[ \|v_1\|^2 \|v\|_{L^6} + \|v_2\|^2 \|v\|_{L^6} \right] \|v\|_{L^6}^2 \\ \leq \frac{d_t}{2} \|v_t\|^2 + C(\|\phi\|^2 + \|v\|_{H^1}^2).
\]

In above, we have used the uniform-in-time estimate (3.11) instead of (2.13). Multiplying (2.28) by \(\overline{\phi} - \Delta \overline{\phi}\), integrating over \(\Omega\) and taking the real part, we get

\[
\frac{1}{2} \frac{d}{dt} (\|\phi\|^2 + \|\nabla \phi\|^2 + \gamma (\|v\|^2 + \|\nabla v\|^2) = - \frac{g}{U} \text{Im} \int_\Omega (\overline{v \phi} + \nabla v \cdot \nabla \overline{\phi}) dx \\ \leq \frac{\gamma}{2} (\|\phi\|^2 + \|\nabla \phi\|^2) + C (\|v\|^2 + \|\nabla v\|^2).
\]
Adding the above estimates together, we have

\[
\frac{d}{dt} \left[ \frac{1}{2} \left( \frac{1}{U} - a \right) \|v\|^2 + \frac{c}{8m} \|\nabla v\|^2 + \frac{1}{2} \|\phi\|^2_{H^1} \right] + \frac{d}{2} \|v_t\|^2 + \frac{\gamma}{2} \|\phi\|^2_{H^1} \\
\leq C(\|\phi\|^2_{H^1} + \|v\|^2_{H^1}).
\] (2.31)

Then our conclusion (2.26) easily follows from (2.31) and the standard Gronwall lemma. The proof is complete. \qed

**Corollary 2.1.** Under the assumptions of Theorem 2.1, the global weak solution \((v, \phi)\) to problem (1.3)-(1.5) is unique.

The above results imply that the unique global weak solution to problem (1.3)-(1.5) defines a strongly continuous nonlinear semigroup \(S(t)\) acting on \(H^1_0(\Omega) \times H^1_0(\Omega)\), such that \((v(t), \phi(t)) = S(t)(v_0, \phi_0)\).

### 3 Existence of the Global Attractor

In this section, we study the existence of a global attractor to problem (1.3)-(1.5). For this purpose, we will show the existence of an absorbing set and some precompactness of the weak solution \((v, \phi)\). In the remaining part of the paper, we shall exploit some formal \textit{a priori} estimates, which can be justified rigorously by the approximate procedure in the previous section and the standard dense argument.

**Proposition 3.1.** Let assumptions (A1)-(A3) be satisfied. There exists a positive constant \(R_0\) such that the ball

\[ B_0 = \{(v, \phi) \in H^1_0(\Omega) \times H^1_0(\Omega) \mid \|v\|^2_{H^1} + \|\phi\|^2_{H^1} \leq R_0\} \]

is a bounded absorbing set for the dynamical system \(S(t)\) associated with problem (1.3)-(1.5). Namely, for any bounded set \(B \subset H^1_0(\Omega) \times H^1_0(\Omega)\), there is \(t_0 = t_0(B)\) such that \(S(t)B \subset B_0\) for every \(t \geq t_0\).

**Proof.** Within the proof, we denote by \(C_j\) \((j = 1, 2, \ldots)\) positive constants that may depend on the coefficients of the system (1.3), \(\Omega\), but not on the initial data \(v_0, \phi_0\) and time. Multiplying the first equation in (1.3) by \(\overline{v}\) and \(\overline{v}_t\), respectively, integrating over \(\Omega\) and taking the imaginary part of the results, we have

\[
\frac{d}{dt} \frac{d}{dk} \|v\|^2 + \left( \frac{1}{U} - a \right) \|v\|^2 + \frac{c}{4m} \|\nabla v\|^2 + b \int_{\Omega} |v|^4 \, dx \\
= \frac{g}{U} \text{Re} \int_{\Omega} \phi \overline{v} \, dx - d_r \text{Im} \int_{\Omega} v_t \overline{v} \, dx + \text{Im} \int_{\Omega} f \overline{v} \, dx \\
\leq \frac{1}{2} \left( \frac{1}{U} - a \right) \|v\|^2 + C_1(\|\phi\|^2 + \|v_t\|^2 + \|f\|^2),
\] (3.1)
By the Young inequality, we have for some $\kappa$

$$
\frac{d}{dt} \left[ \frac{1}{2} \left( \frac{1}{U} - a \right) \|v\|^2 + \frac{c}{8m} \|\nabla v\|^2 + \frac{b}{4} \int_\Omega |v|^4 dx \right] + d_i \|v_i\|^2
= \frac{g}{U} \text{Re} \int_\Omega \bar{\phi} \tau_i dx + \text{Im} \int_\Omega \phi \tau_i dx
\leq \frac{d_i}{2} \|v_i\|^2 + C_2 (\|\phi\|^2 + \|f\|^2). \tag{3.2}
$$

Multiplying the second equation in (1.3) by $\bar{\phi}$ and $-\Delta \bar{\phi}$, respectively, integrating over $\Omega$ and taking the real part, we get

$$
\frac{1}{2} \frac{d}{dt} \|\phi\|^2 + \gamma \|\phi\|^2 = \text{Re} \int_\Omega \bar{h} \phi dx - \frac{g}{U} \text{Im} \int_\Omega \bar{\phi} dx
\leq \frac{\gamma}{2} \|\phi\|^2 + C_3 (\|v\|^2 + \|h\|^2), \tag{3.3}
$$

$$
\frac{1}{2} \frac{d}{dt} \|\nabla \phi\|^2 + \gamma \|\nabla \phi\|^2 = \text{Re} \int_\Omega \nabla h \cdot \nabla \bar{\phi} dx - \frac{g}{U} \text{Im} \int_\Omega \nabla \phi \cdot \nabla \bar{\phi} dx
\leq \frac{\gamma}{2} \|\nabla \phi\|^2 + C_4 (\|\nabla v\|^2 + \|\nabla h\|^2). \tag{3.4}
$$

Now multiplying (3.1) by $\kappa_1 > 0$, (3.3) by $\kappa_2 > 0$ and (3.4) by $\kappa_3 > 0$, adding together the resulting inequalities with (3.2), we obtain that

$$
\frac{d}{dt} \left[ \frac{1}{2} \left( \kappa_1 \kappa_4 + \frac{1}{U} - a \right) \|v\|^2 + \frac{c}{8m} \|\nabla v\|^2 + \frac{b}{4} \int_\Omega |v|^4 dx + \frac{\kappa_2}{2} \|\phi\|^2 + \frac{\kappa_3}{2} \|\nabla \phi\|^2 \right]
+ \frac{\kappa_1}{2} \left( \frac{1}{U} - a \right) \|v\|^2 + \left( \frac{c \kappa_1}{4m} - C_4 \kappa_3 \right) \|\nabla v\|^2 + \kappa_1 b \int_\Omega |v|^4 dx
+ \left( \frac{\gamma \kappa_2}{2} - C_1 \kappa_1 - C_2 \right) \|\phi\|^2 + \frac{\gamma \kappa_3}{2} \|\nabla \phi\|^2 + \left( \frac{d_i}{2} - C_1 \kappa_1 \right) \|v_i\|^2
\leq C_3 \kappa_2 \|v\|^2 + (C_1 \kappa_1 + C_2) \|f\|^2 + C_3 \kappa_2 \|h\|^2 + C_4 \kappa_3 \|\nabla h\|^2. \tag{3.5}
$$

By the Young inequality, we have for some $\kappa_4 > 0$,

$$
\|v\|^2 \leq \kappa_4 \|v\|^2_{L^4} + \frac{|\Omega|}{4 \kappa_4}. \tag{3.6}
$$

Take

$$
\kappa_1 = \frac{d_i}{4C_1}, \quad \kappa_2 = \frac{4C_2 + d_i}{\gamma}, \quad \kappa_3 = \frac{cd_i}{32mC_1C_4}, \quad \kappa_4 = \frac{\gamma bd_i}{8C_1C_3(4C_2 + d_i)}. \tag{3.7}
$$

We infer from (3.5) that the following inequality holds

$$
\frac{d}{dt} E_1(t) + C_5 E_1(t) + \frac{d_i}{4} \|v_i\|^2 \leq C_6, \tag{3.8}
$$

where

$$
E_1(t) = \frac{1}{2} \left( \kappa_1 \kappa_4 + \frac{1}{U} - a \right) \|v(t)\|^2 + \frac{c}{8m} \|\nabla v(t)\|^2 + \frac{b}{4} \int_\Omega |v(t)|^4 dx + \frac{\kappa_2}{2} \|\phi(t)\|^2 + \frac{\kappa_3}{2} \|\nabla \phi(t)\|^2.
$$

Then (3.8) yields that

$$
E_1(t) \leq e^{-C_5 t} E_1(0) + \frac{C_6}{C_5}, \quad \forall t \geq 0. \tag{3.9}
$$
Finally, we can take $R_0 = \frac{2C_8}{C_6C_7}$. The proof is complete.

**Remark 3.1.** Proposition 3.1 implies that the trajectories $(v(t), \phi(t))$ starting from any bounded set $B$ will eventually enter the ball $B_0$ in $H_1^0 \times H_1^0$ of radius $(R_0)^\frac{1}{2}$ uniformly in time. Noticing that, $B_0 \subset \tilde{B}_0 := \bigcup_{t \geq 0} S(t)B_0$, we can see that $\tilde{B}_0$ also serves as an absorbing set of $S(t)$. Moreover, $\tilde{B}_0$ is invariant under $S(t)$ for $t \geq 0$.

Our next goal is to study the precompactness of the weak solution $(v, \phi)$ of problem (1.3)-(1.5).

**Lemma 3.1.** Under assumptions of Theorem 2.1, the following uniform estimate holds:

$$
\|v(t)\|_{H^2(\Omega)} \leq C \left(1 + \frac{1}{r^2}\right), \quad \forall \ t \geq r > 0,
$$

where $C$ is a constant depending on $\|v_0\|_{H^1}$, $\|\phi_0\|_{H^1}$, $\Omega$, $f$, $h$ and the coefficients of the system (1.3), but independent of $t$.

**Proof.** For any $t \geq 0$ and $r > 0$, integrating (3.8) from $t$ to $t + r$, we infer from (3.11) that

$$
\int_t^{t+r} (\|v(\tau)\|_{H^1}^2 + \|v(\tau)\|_{L^4}^4 + \|\phi(\tau)\|_{H^1}^2 + \|v(\tau)\|^2) d\tau \leq C.
$$

(3.13)

Multiplying the first equation in (1.3) by $-\Delta v$, integrating over $\Omega$ and taking the imaginary part, we have

$$
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \left(\frac{1}{U} - a\right) \|\nabla v\|^2 + \frac{c}{4m} \|\Delta v\|^2 + 2b \int_{\Omega} |v|^2 |\nabla v|^2 dx
$$

$$
= -\frac{g}{U} \text{Re} \int_{\Omega} \phi \Delta v dx - b \text{Re} \int_{\Omega} \nabla v \cdot \nabla \psi^* + d_v \text{Im} \int_{\Omega} v_i \Delta v dx - \text{Im} \int_{\Omega} f \Delta v dx
$$

$$
\leq \frac{c}{8m} \|\Delta v\|^2 + b \int_{\Omega} |v|^2 |\nabla v|^2 dx + C(\|v_0\|^2 + \|\phi\|^2 + \|f\|^2).
$$

(3.14)

Integrating the above inequality from $t$ to $t + r$, we infer from (3.13) that

$$
\int_t^{t+r} \|v(\tau)\|_{H^2}^2 d\tau \leq C.
$$

(3.15)

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Next, multiplying the first equation in (1.3) by $-\Delta v$, integrating over $\Omega$ and taking the imaginary part, we get
\[
\frac{d}{dt} \left[ \frac{1}{2} \left( \frac{1}{U} - a \right) \|\nabla v\|^2 + \frac{c}{8m} \|\Delta v\|^2 \right] + d_i \|\nabla v_i\|^2
\]
\[= \frac{g}{U} \text{Re} \int_\Omega \nabla \phi \nabla v_i dx - b \text{Re} \int_\Omega \nabla (|v|^2 v) \cdot \nabla v_i dx + \text{Im} \int_\Omega \nabla f \cdot \nabla v_i dx
\]
\[\leq \frac{d_i}{2} \|\nabla v_i\|^2 + C \left( \|\nabla \phi\|^2 + \|\nabla f\|^2 + \int_\Omega |\nabla v|^2 dx \right)
\]
\[\leq \frac{d_i}{2} \|\nabla v_i\|^2 + C(\|\nabla \phi\|^2 + \|\nabla f\|^2 + \|v\|_{L^\infty}^2 \|\Delta v\|^2)
\]
\[\leq \frac{d_i}{2} \|\nabla v_i\|^2 + C(\|\nabla \phi\|^2 + \|\nabla f\|^2 + \|v\|_{L^1}^4 \|\Delta v\|^2).
\]
In the last step, we use the three-dimensional Agmon inequality that for any $v \in H^2 \cap H^1_0$, it holds $\|v\|_{L^\infty}^2 \leq c(\Omega) \|\nabla v\| \|\Delta v\|$. Then it easily follows from (3.16) that
\[
\frac{d}{dt} y(t) \leq Ch_1(t) y(t) + Ch_2(t),
\]
where
\[
y(t) = \frac{1}{2} \left( \frac{1}{U} - a \right) \|\nabla v(t)\|^2 + \frac{c}{8m} \|\Delta v(t)\|^2, \quad h_1(t) = \|v(t)\|_{L^1}^4, \quad h_2(t) = \|\phi(t)\|_{L^1}^2 + \|f\|_{L^1}^2.
\]
Applying the well-known uniform Gronwall lemma (cf. e.g., [19, Lemma III.1.1]), we infer from (3.11) and (3.15) that for any $r > 0$
\[
y(t + r) \leq C \left( 1 + \frac{1}{r} \right), \quad \forall \ t \geq 0.
\]
The proof is complete. \[\square\]

Since the continuous embedding $H^2 \hookrightarrow H^1$ is compact, Proposition 3.1 implies that $v(t)$ is precompact in $H^1$ for $t \geq r$.

Next, we prove the precompactness of $\phi(t)$. We note that $\phi$ satisfies a Schrödinger type equation, which does not enjoy the smoothing property like parabolic equations. To overcome this difficulty, we shall decompose the solution $\phi$ into a uniformly stable part and a compact part such that
\[
\phi = \phi^d + \phi^c,
\]
where $\phi^d(t)$ and $\phi^c(t)$ satisfy the following systems
\[
\begin{aligned}
\phi^d_t + i \left( \frac{\Delta}{2} + 2\nu - 2\mu \right) \phi^d - \frac{i}{4m} \Delta \phi^d + \gamma \phi^d &= 0, \\
\phi^d|_{\Gamma} &= 0, \\
\phi^d|_{t=0} &= \phi_0,
\end{aligned}
\]
and
\[
\begin{aligned}
\phi^c_t - \frac{i}{2} v + i \left( \frac{\Delta}{2} + 2\nu - 2\mu \right) \phi^c - \frac{i}{4m} \Delta \phi^c + \gamma \phi^c &= h, \\
\phi^c|_{\Gamma} &= 0, \\
\phi^c|_{t=0} &= 0.
\end{aligned}
\]
Lemma 3.2. Problem \([3.18]\) admits a unique global weak solution \(\phi^d(t) \in C([0, +\infty), \mathcal{H}_0^1)\) and the following estimate holds:

\[
\|\phi^d(t)\|_{\mathcal{H}^1} = \|\phi_0\|_{\mathcal{H}^1} e^{-\gamma t}, \quad \forall t \geq 0.
\]  

(3.20)

Proof. The existence and uniqueness of solution \(\phi^d\) to equation \([3.18]\) can be easily proven as in Section 2. Multiplying \([3.18]\) by \(\phi^d - \Delta \phi^d\), integrating over \(\Omega\) and taking the real part, we obtain

\[
\frac{1}{2} \frac{d}{dt} (\|\phi^d\|^2 + \|\nabla \phi^d\|^2) + \gamma (\|\phi^d\|^2 + \|\nabla \phi^d\|^2) = 0,
\]  

(3.21)

which easily yields \([3.20]\).

Lemma 3.3. For any \(r > 0\), it holds

\[
\|\phi^c(t)\|_{\mathcal{H}^2} \leq C, \quad t \geq r,
\]  

(3.22)

where \(K\) is a constant depending on \(\|v_0\|_{\mathcal{H}^1_0}, \|\phi\|_{\mathcal{H}^1_0}, \Omega, f, h, r\) and the coefficients of system \([1.3]\).

Proof. It follows from \([3.11]\) and \([3.20]\) that

\[
\|\phi^c(t)\|_{\mathcal{H}^1} \leq C, \quad \forall t \geq 0.
\]  

(3.23)

Differentiating \([3.19]\) with respect to \(t\), multiplying the resultant by \(\overline{\phi^c}\), integrating over \(\Omega\) and taking the real part, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\phi^c\|^2 + \gamma \|\phi^c\|^2 = -\frac{g}{U} \text{Im} \int_{\Omega} v_t \overline{\phi^c} dx \leq \frac{\gamma}{2} \|\phi^c\|^2 + \frac{g^2}{2\gamma U^2} \|v_t\|^2.
\]  

(3.24)

Namely,

\[
\frac{d}{dt} \|\phi^c\|^2 + \gamma \|\phi^c\|^2 \leq \frac{g^2}{\gamma U^2} \|v_t\|^2.
\]  

(3.25)

It follows that for \(\forall t \geq r\),

\[
\|\phi^c(t)\|^2 \leq e^{-\gamma t} \|\phi^c(0)\|^2 + \frac{g^2}{\gamma U^2} e^{-\gamma t} \int_{0}^{t} e^{\gamma \tau} \|v_t(\tau)\|^2 d\tau
\]

\[
\leq Ce^{-\gamma \tau} (\|v_0\|^2 + \|h\|^2) + \frac{g^2}{\gamma U^2} e^{\gamma \tau} \int_{0}^{\tau} \|v_t(\tau)\|^2 d\tau + \frac{g^2}{\gamma U^2} (1 - e^{-\gamma \tau}) \sup_{\tau \geq r} \|v_t(\tau)\|^2
\]

\[
\leq C + \frac{g^2}{\gamma U^2} \sup_{\tau \geq r} (\|v(\tau)\|^2_{\mathcal{H}^2} + \|\phi(\tau)\|^2 + \|v(\tau)\|^6_{L^6})
\]

\[
\leq C.
\]  

(3.26)

Thus, we can deduce from the equation \([3.19]\) and Lemma \(3.1\) that

\[
\|\phi^c(t)\|_{\mathcal{H}^2} \leq C (\|v(t)\| + \|\phi^c(t)\| + \|\phi^c(t)\| + \|h\|) \leq C, \quad \forall t \geq r.
\]  

(3.27)

The proof is complete.

Thus, after the previous preparations, we are able to state the main result of this section:
where

Inserting (4.2) into (4.1), we have

The following proposition implies the dissipativity of the dynamical system

4 Existence of Exponential Attractors

Theorem 3.1. Suppose that (A1)-(A3) are satisfied. The semigroup $S(t)$ defined by the global weak solutions to problem (1.3)-(1.5) on $\mathcal{H}_0 \times \mathcal{H}_0$ possesses a compact connected global attractor $\mathcal{A} \subset \mathcal{H}_0 \times \mathcal{H}_0$, which is the $\omega$-limit set of the absorbing set $\mathcal{B}_0$ such that $\mathcal{A} = \omega(\mathcal{B}_0)$.

Proof. Since $\mathcal{B}_0$ is a connected, invariant, bounded absorbing set, our conclusion follows from Lemmas 3.1-3.3 and the classical theory of dynamical systems (cf. e.g., [19, Theorem I.1.1]). $\square$

4 Existence of Exponential Attractors

The following proposition implies the dissipativity of the dynamical system $S(t)$ when it is restricted to the regular space $(\mathcal{H}^2 \cap \mathcal{H}_0^1) \times (\mathcal{H}^2 \cap \mathcal{H}_0^1)$.

Proposition 4.1. There exists $R_1 \geq 0$ such that the ball

$$\mathcal{B}_1 = \{(v, \phi) \in (\mathcal{H}^2 \cap \mathcal{H}_0^1) \times (\mathcal{H}^2 \cap \mathcal{H}_0^1) \mid \|v\|^2_{\mathcal{H}^2} + \|\phi\|^2_{\mathcal{H}^2} \leq R_1\}$$

is a bounded absorbing set for $S(t)$ in $(\mathcal{H}^2 \cap \mathcal{H}_0^1) \times (\mathcal{H}^2 \cap \mathcal{H}_0^1)$.

Proof. Let $\mathcal{B}$ be any bounded set in $(\mathcal{H}^2 \cap \mathcal{H}_0^1) \times (\mathcal{H}^2 \cap \mathcal{H}_0^1)$. In particular, there exist $r_1 \geq r_0 \geq 0$ such that

$$\sup_{(v, \phi) \in \mathcal{B}} \|(v, \phi)\|_{\mathcal{H}^2 \cap \mathcal{H}_0^1} \leq r_0 \quad \text{and} \quad \sup_{(v, \phi) \in \mathcal{B}} \|(v, \phi)\|_{\mathcal{H}_0^1} \leq r_1.$$

Within the proof, we denote by $K_j$ ($j = 1, 2, \ldots$) positive constants that may depend on the coefficients of the system (1.3), $\Omega$, $f$, $h$, but not on the initial data $v_0, \phi_0$ and time.

Differentiating the first equation in (1.3) with respect to time, multiplying the result by $\phi$ and taking the imaginary/real part, respectively, we have

$$\frac{d}{dt} \int_\Omega |\phi_t| dx = b \operatorname{Re} \int_\Omega |v|^2 |\phi_t| dx - \frac{g}{U} \int_\Omega \phi_t \bar{\phi}_t dx + \int_\Omega \phi_t \bar{\phi}_t dx$$

(1.1)

$$\frac{d}{dt} \int_\Omega |v_t|^2 dx = -\frac{g}{U} \int_\Omega \phi_t \bar{\phi}_t dx + \int_\Omega |v_t|^2 dx$$

(1.2)

Inserting (1.2) into (1.1), we have

$$\left(\frac{d}{dt} + \frac{d^2}{dt^2}\right) \int_\Omega |v_t|^2 dx + \frac{c}{4m} \int_\Omega |\nabla v_t|^2 dx + \frac{1-aU}{U} \int_\Omega |v_t|^2 dx$$

$$= \frac{g}{U} \int_\Omega \phi_t \bar{\phi}_t dx - \frac{d}{dt} \int_\Omega \phi_t \bar{\phi}_t dx - \frac{d}{dt} \int_\Omega |v_t|^2 dx$$

$$+ \frac{d}{dt} \int_\Omega |v_t|^2 dx$$

$$:= I_1 + I_2,$$

where

$$I_1 \leq C \phi_t ||v_t||^2 \leq \frac{\gamma}{2} \phi_t ||v_t||^2 + C ||v_t||^2,$$

(4.3)

$$I_2 \leq C ||v||^2 ||v_t||^2 \leq C ||v||^2 ||\nabla v_t||^2 ||v_t|| \leq \frac{c}{8m} ||\nabla v_t||^2 + C ||v||^4 ||v_t||^2.$$

(4.4)
As a result,
\[
\left(\frac{d_t}{2} + \frac{d_t^2}{2d_t}\right) \frac{d}{dt} \|v_t\|^2 + \frac{c}{8m} \|\nabla v_t\|^2 + \frac{1-aU}{U}\|v_t\|^2 \leq \frac{\gamma}{2} \|\phi_t\|^2 + C\|v\|_H^4 \|v_t\|^2.
\] (4.5)

On the other hand, differentiating the \(\phi\)-equation in (4.3) with respect to \(t\), multiplying the resultant by \(-\phi_t\), integrating over \(\Omega\) and taking the real part, in analogy to (3.22) we obtain that
\[
\frac{d}{dt} \|\phi_t\|^2 + \gamma \|\phi_t\|^2 \leq \frac{g^2}{\gamma U^2} \|v_t\|^2.
\] (4.6)

By (4.5), (4.6) and Cauchy-Schwarz inequality, we get
\[
\frac{d}{dt} \|v_t\|^2 + 2\|\phi_t\|^2 \geq \frac{4}{4m} \|\nabla v_t\|^2 + \gamma \|\phi_t\|^2 \leq \left(\frac{2g^2}{\gamma U^2} + C\|v\|_H^4\right) \|v_t\|^2.
\] (4.7)

It follows from (3.11) that there exists \(t_0 = t_0(r_0) > 0\) such that \(\|v(t)\|_{H^1} \leq M\) for \(t \in [0, t_0]\) and \(\|v(t)\|_{H^1} \leq M'\) for all \(t \geq t_0\), with \(M\) being a constant depending on \(r_0\) while \(M'\) being independent of \(r_0\). Thus, on \([0, t_0]\), (4.7) implies that
\[
\frac{d}{dt} \left[\left(d_t + \frac{d_t^2}{d_t}\right) \|v_t\|^2 + 2\|\phi_t\|^2\right] \leq \left(\frac{2g^2}{\gamma U^2} + CM^4\right) \|v_t\|^2,
\] (4.8)

which together with the Gronwall inequality yields
\[
\|v_t(t_0)\|^2 + \|\phi_t(t_0)\|^2 \leq M_1,
\] (4.9)

where \(M_1\) depends on \(r_1, r_0\) and \(t_0\). Let us start from time \(t_0\). We infer from (4.7) that
\[
\frac{d}{dt} \left[\left(d_t + \frac{d_t^2}{d_t}\right) \|v_t\|^2 + 2\|\phi_t\|^2\right] + \frac{c}{4m} \|\nabla v_t\|^2 + \gamma \|\phi_t\|^2 \leq K_1 \|v_t\|^2, \quad \forall t \geq t_0,
\] (4.10)

where \(K_1 = \frac{2g^2}{\gamma U^2} + C(M')^4\). Denote
\[
E_3(t) = \left(d_t + \frac{d_t^2}{d_t}\right) \|v_t(t)\|^2 + 2\|\phi_t(t)\|^2 + \frac{4K_1 + 4}{d_t} E_1(t).
\] (4.11)

Then it follows from (3.38) and (4.10) that
\[
\frac{d}{dt} E_3(t) + K_2 E_3(t) \leq K_3, \quad \forall t \geq t_0,
\] (4.12)

which yields
\[
E_3(t) \leq e^{-K_2 t} e^{K_2 t_0} E_3(t_0) + \frac{K_3}{K_2}, \quad \forall t \geq t_0.
\] (4.13)

From (4.11) and (4.9), we know that \(E_3(t_0)\) can be bounded by a constant depending on \(r_1, r_0\) and \(t_0\). Then it follows from (4.13) that there exists a time \(t_1 \geq t_0\) depending on \(r_1, r_0\) and \(t_0\) such that
\[
\left(d_t + \frac{d_t^2}{d_t}\right) \|v_t(t)\|^2 + 2\|\phi_t(t)\|^2 \leq E_3(t) \leq \frac{2K_3}{K_2}, \quad \forall t \geq t_1.
\] (4.14)

On the other hand, we deduce from (4.3) that
\[
\|v(t)\|_{H^2} \leq C(\|v_t(t)\| + \|\phi(t)\| + \|v(t)\| + \|v(t)\|_2^2 + \|f\|),
\]
\[
\|\phi(t)\|_{H^2} \leq C(\|\phi_t(t)\| + \|\phi(t)\| + \|v(t)\| + \|h_t\|),
\]

where
where $C$ is a constant depending only on the coefficients of system (1.3). Thus, from (4.14) and Proposition 3.1 we can see that there exists a constant $R_1 > 0$ independent of $r_0, r_1$ such that

$$\sup_{(v_0, \phi_0) \in B} \sup_{t \geq t_1} \| S(t)(v_0, \phi_0) \|_{H^2 \times H^2} \leq R_1.$$ 

The proof is complete. □

As a byproduct, the above lemma gives the following integral estimate

**Corollary 4.1.** There holds

$$\sup_{\|(v, \phi)\|_{H^2 \times H^2} \leq R} \sup_{t \geq 0} \int_{t}^{t+1} (\|v_t\|^2_{H^1} + \|\phi_t\|^2) dt \leq C(R). \tag{4.15}$$

Next, we prove the following proposition that enables us to confine the dynamics of system (1.3)-(1.5) to a regular set $\tilde{B}_1 \subset (H^2 \cap H^1_0) \times (H^2 \cap H^1_0)$.

**Proposition 4.2.** There exists a closed ball $\tilde{B}_1 \subset (H^2 \cap H^1_0) \times (H^2 \cap H^1_0)$ such that

(i) there is a positive increasing function $M$ such that for every bounded set $B \subset H^1_0 \times H^1_0$ with $R = \sup_{(v, \phi) \in B} \|(v, \phi)\|_{H^1_0 \times H^1_0}$, the following estimate holds:

$$\text{dist}_{H^1_0 \times H^1_0}(S(t)B, \tilde{B}_1) \leq M(R)e^{-\gamma t}; \tag{4.16}$$

(ii) there is a time $\tilde{t}_1 \geq 0$ depending on $\tilde{R} = \sup_{(v, \phi) \in \tilde{B}_1} \|(v, \phi)\|_{H^2 \times H^2}$ such that

$$S(t)\tilde{B}_1 \subset \tilde{B}_1, \quad \forall t \geq \tilde{t}_1. \tag{4.17}$$

**Proof.** The existence of a bounded exponential attracting ball $G \subset (H^2 \cap H^1_0) \times (H^2 \cap H^1_0)$ in the $H^1_0 \times H^1_0$ metric is given by Lemmas 3.3-3.3. Next, using Proposition 4.1 we can enlarge the ball $G$ properly such that under the action of $S(t)$, after a time $t_1 = t_1(G)$, $G$ is absorbed into itself. Taking $\tilde{B}_1 = G$, we complete the proof. □

We can now state the main result of this section:

**Theorem 4.1.** The semigroup $S(t)$ possesses an exponential attractor $E \subset (H^2 \cap H^1_0) \times (H^2 \cap H^1_0)$. Thus, by definition, we have that

(i) $E$ is a closed compact set in $H^1_0 \times H^1_0$ that is positively invariant for $S(t)$.

(ii) The fractal dimension of $E$ is finite.

(iii) $E$ satisfies the following exponential attraction property: there exist a constant $\omega > 0$ and a positive increasing function $J$ such that, for every bounded set $B \subset (H^2 \cap H^1_0) \times (H^2 \cap H^1_0)$ with $R = \sup_{(v, \phi) \in B} \|(v, \phi)\|_{H^1_0 \times H^1_0}$, it holds

$$\text{dist}_{H^1_0 \times H^1_0}(S(t)B, E) \leq J(R)e^{-\omega t}, \quad \forall t \geq 0. \tag{4.18}$$
Proof. The proof of Theorem 4.1 consists of several steps.

Step 1. First, we confine the dynamics of system (1.3)-(1.5) to the regular set $\tilde{B}_{1} \subset (\mathcal{H}_{2}^{1} \cap \mathcal{H}_{1}^{1})$ obtained in Proposition 4.2. In order to prove the existence of an exponential attractor, we shall use the simple constructive method introduced in [8, Proposition 1] and follow the strategy in [2] (cf. also [12]). For the reader’s convenience, we report the following lemma adapted to our present case (cf. [2, Lemma 5.3]).

Lemma 4.1. Let $\tilde{B}_{1}$ and $\tilde{t}_{1}$ be as in Proposition 4.2 and denote $z = (v, \phi)$. Suppose that there exists $t^{*} \geq \tilde{t}_{1}$ such that the following conditions are satisfied:

(C1) The map $(t, z) \mapsto S(t)z : [t^{*}, 2t^{*}] \times \tilde{B}_{1} \rightarrow \tilde{B}_{1}$ is $\frac{1}{2}$-Hölder continuous in time and Lipschitz continuous in the initial data, when $\tilde{B}_{1}$ is endowed with the $\mathcal{H}_{1}^{1} \times \mathcal{H}_{2}^{1}$-topology.

(C2) Setting $S = S(t^{*})$, there are $\lambda \in (0, \frac{1}{2})$ and $\Lambda \geq 0$ such that, for every $z_{01}, z_{02} \in \tilde{B}_{1}$,

$$\|D(z_{01}, z_{02})\|_{\mathcal{H}_{1}^{1} \times \mathcal{H}_{2}^{1}} \leq \lambda\|z_{01} - z_{02}\|_{\mathcal{H}_{1}^{1} \times \mathcal{H}_{2}^{1}}, \quad \|K(z_{01}, z_{02})\|_{\mathcal{H}_{2}^{1} \times \mathcal{H}_{2}^{1}} \leq \Lambda\|z_{01} - z_{02}\|_{\mathcal{H}_{1}^{1} \times \mathcal{H}_{2}^{1}}.$$

Then there exists a bounded set $E \subset \tilde{B}_{1}$, closed and of finite fractal dimension in $\mathcal{H}_{1}^{1} \times \mathcal{H}_{2}^{1}$, positively invariant for $S(t)$, such that for some $J_{0} > 0$ and $J_{0} \geq 0$, it holds

$$\text{dist}_{H_{1}^{1} \times H_{2}^{1}}(S(t)\tilde{B}_{1}, E) \leq J_{0}e^{-\omega_{0}t}. \quad (4.19)$$

It will be shown in the appendices that the conditions (C1) and (C2) in Lemma 4.1 are satisfied when the dynamics of system (1.3)-(1.5) is confined to the regular set $\tilde{B}_{1}$. Hence, there exists a set $E \subset \tilde{B}_{1}$, closed and of finite fractal dimension in $\mathcal{H}_{1}^{1} \times \mathcal{H}_{2}^{1}$, positively invariant for $S(t)$ and satisfying (4.19).

Step 2. In order to complete the proof, we are left to show that (4.19) actually holds for any bounded subset $B \subset (\mathcal{H}_{2}^{1} \cap \mathcal{H}_{1}^{1}) \times (\mathcal{H}_{2}^{1} \cap \mathcal{H}_{1}^{1})$ instead of $\tilde{B}_{1}$, with possibly different $J_{0}$ and $\omega_{0}$. In other words, we have to prove that the basin of exponential attraction can be the whole space $(\mathcal{H}_{2}^{1} \cap \mathcal{H}_{1}^{1}) \times (\mathcal{H}_{2}^{1} \cap \mathcal{H}_{1}^{1})$ (cf. (4.18)).

For any bounded set $B \subset (\mathcal{H}_{2}^{1} \cap \mathcal{H}_{1}^{1}) \times (\mathcal{H}_{2}^{1} \cap \mathcal{H}_{1}^{1})$ with $R = \sup_{(v, \phi) \in B} \|(v, \phi)\|_{\mathcal{H}_{1}^{1} \times \mathcal{H}_{2}^{1}}$, it follows from Proposition 4.2 that

$$\text{dist}_{H_{1}^{1} \times H_{2}^{1}}(S(t)B, \tilde{B}_{1}) \leq M(R)e^{-\gamma t}. \quad (4.20)$$

On the other hand, for any $z_{01} = (v_{01}^{1}, \phi_{01}^{1})$, $z_{02} = (v_{02}^{2}, \phi_{02}^{2}) \in B$, by Theorem 2.2 (and Poincaré inequality), we have

$$\|S(t)z_{01} - S(t)z_{02}\|_{\mathcal{H}_{1}^{1} \times \mathcal{H}_{2}^{1}} \leq C_{P}L_{1}^{\frac{1}{2}}e^{{\frac{C_{P}}{2}}t}\|z_{01} - z_{02}\|_{\mathcal{H}_{1}^{1} \times \mathcal{H}_{2}^{1}}, \quad (4.21)$$

where $C_{P} > 0$ depends only on $\Omega$. Applying the abstract result on the transitivity of exponential attraction (cf. [10, Theorem 5.1]), we conclude from (4.19), (4.20) and (4.21) that

$$\text{dist}_{H_{1}^{1} \times H_{2}^{1}}(S(t)B, E) \leq Je^{-\omega t}, \quad (4.22)$$

where

$$J = J(R) = C_{P}L_{1}^{\frac{1}{2}}M(R) + J_{0}, \quad \omega = \frac{\gamma\omega_{0}}{L_{2}^{2} + \gamma + \omega_{0}}.$$

The proof is complete. \qed
We note that the exponential attractor $\mathcal{E}$ actually contains the global attractor $\mathcal{A}$ that is obtained in Section 3. As a consequence, we have

**Corollary 4.2.** The global attractor $\mathcal{A}$ has finite fractal dimension.

## 5 Appendices

We verify the conditions (C1) and (C2) in Lemma 4.1 when the dynamics of system (1.3)-(1.5) is confined to the regular set $\tilde{B}_1$.

1. **Verifying condition (C1).**

For any $t, \tau \in [t^*, 2t^*]$ satisfying $t \geq \tau$, we take the difference of the $\phi$-equation:

$$\phi_t(t) - \phi_t(\tau) + \gamma(\phi(t) - \phi(\tau)) - \frac{i g}{U}(v(t) - v(\tau)) + i \left( \frac{g^2}{U} + 2\nu - 2\mu \right)(\phi(t) - \phi(\tau))$$

$$- \frac{i}{4m}(\Delta \phi(t) - \Delta \phi(\tau)) = 0.$$  

Multiplying it by $\overline{\phi}(t) - \overline{\phi}(\tau)$, integrating over $\Omega$ and taking the imaginary part, we obtain

$$\frac{1}{4m}||\nabla \phi(t) - \nabla \phi(\tau)||^2 = - \left( \frac{g^2}{U} + 2\nu - 2\mu \right) ||\phi(t) - \phi(\tau)||^2 + \frac{g}{U} \text{Re} \int_{\Omega} (v(t) - v(\tau))(\overline{\phi}(t) - \overline{\phi}(\tau))dx$$

$$- \text{Im} \int_{\Omega} (\phi_t(t) - \phi_t(\tau))(\overline{\phi}(t) - \overline{\phi}(\tau))dx \leq C(||\phi(t) - \phi(\tau)||^2 + ||v(t) - v(\tau)||^2 + ||\phi_t(t) - \phi_t(\tau)|| ||\phi(t) - \phi(\tau)||).$$  

(5.1)

By (3.11) and (4.14), we know that for $t \geq \tilde{t}_1$, $||\phi(t)||_{\mathcal{H}^1}$, $||v(t)||_{\mathcal{H}^1}$, $||\phi_t(t)||$ and $||v_t(t)||$ can be uniformly bounded by a constant independent of the initial data. Then we infer from (5.1) that

$$||\phi(t) - \phi(\tau)||^2_{\mathcal{H}^1} \leq C(t - \tau).$$  

(5.3)

This and the Poincaré inequality yield that

$$||\phi(t) - \phi(\tau)||^2_{\mathcal{H}^1} \leq C(t - \tau).$$  

(5.2)

On the other hand, it follows from Corollary 4.1 that

$$||v(t) - v(\tau)||_{\mathcal{H}^1} \leq \int_{\tau}^{t} ||v_t(s)||_{\mathcal{H}^1} ds \leq \left( \int_{\tau}^{t} ||v_t(s)||^2_{\mathcal{H}^1} ds \right)^{\frac{1}{2}} \sqrt{t - \tau}.$$  

(5.4)

Denote $z = (v, \phi)$. For any $t^* \geq \tilde{t}_1$, $t, \tau \in [t^*, 2t^*]$ with $t \geq \tau$, and $z_1, z_2 \in \tilde{B}_1$, we infer from (2.26), (5.3) and (5.4) that

$$||S(t)z_1 - S(\tau)z_2||_{\mathcal{H}^1 \times \mathcal{H}^1} \leq ||S(t)z_1 - S(t)z_2||_{\mathcal{H}^1 \times \mathcal{H}^1} + ||S(t)z_2 - S(\tau)z_2||_{\mathcal{H}^1 \times \mathcal{H}^1}$$

$$\leq C(t^*)(||z_1 - z_2||_{\mathcal{H}^1 \times \mathcal{H}^1} + \sqrt{t - \tau}).$$
(2) Verifying condition (C2).

For any initial data $z_{01} = \left( v_{01}^{(1)}, \phi_{01}^{(1)} \right)$, $z_{02} = \left( v_{01}^{(2)}, \phi_{01}^{(2)} \right) \in \mathcal{B}_1$, we set $z_0 = (v_0, \phi_0) : = \left( v_0^{(1)} - v_0^{(2)}, \phi_0^{(1)} - \phi_0^{(2)} \right)$. The difference of the solutions $S(t)z_0 = (v^{(j)}, \phi^{(j)})$, $j = 1, 2$ can be decomposed as

$$\left( v, \phi \right) = (v^{(1)} - v^{(2)}, \phi^{(1)} - \phi^{(2)}) = (v^d, \phi^d) + (v^c, \phi^c),$$

where $(v^d, \phi^d)$ solves the linear problem

$$ \begin{align*}
& dv^d_t - i (\frac{1}{U} - \frac{1}{\nu}) v^d - \frac{i \varphi}{\nu} \phi^d - \frac{k^2}{4m} \Delta v^d = 0, \\
& \phi^d_t + \gamma \phi^d + i \left( \frac{\varphi^2}{\nu} + 2\nu - 2\mu \right) \phi^d - \frac{i}{4m} \Delta \phi^d = 0, \\
& v^d|_\Gamma = \phi^d|_\Gamma = 0, \\
& v^d(0) = v_0, \quad \phi^d(0) = \phi_0,
\end{align*} \tag{5.5}$$

while $(v^c, \phi^c)$ satisfies

$$ \begin{align*}
& dv^c_t - i (\frac{1}{U} - \frac{1}{\nu}) v^c - \frac{i \varphi}{\nu} \phi^c - \frac{k^2}{4m} \Delta v^c + ib|v^{(1)}|^2 v^{(1)} - ib|v^{(2)}|^2 v^{(2)} = 0, \\
& \phi^c_t + \gamma \phi^c + i \left( \frac{\varphi^2}{\nu} + 2\nu - 2\mu \right) \phi^c - \frac{i}{4m} \Delta \phi^c = 0, \\
& v^c|_\Gamma = \phi^c|_\Gamma = 0, \\
& v^c(0) = 0, \quad \phi^c(0) = 0.
\end{align*} \tag{5.6}$$

Similar to (3.21), we have

$$\frac{1}{2} \frac{d}{dt} \| \phi^d \|^2_{H^1} + \gamma \| \phi^d \|^2_{H^1} = 0, \tag{5.7}$$

which implies

$$\| \phi^d(t) \|^2_{H^1} \leq e^{-2\gamma t} \| \phi^d \|^2_{H^1}, \quad t \geq 0. \tag{5.8}$$

Multiplying the first equation in (5.5) by $\overline{v^d} + \alpha v^d_t$ ($\alpha > 0$), integrating over $\Omega$ and taking the imaginary part, we have

$$\frac{d}{dt} \left[ \left( \frac{\alpha}{2} \left( \frac{1}{U} - a \right) + \frac{d_i}{2} \right) \| v^d \|^2 + \frac{\alpha c}{8m} \| \nabla v^d \|^2 \right] + \alpha d_i \| v^d \|^2 + \left( \frac{1}{U} - a \right) \| v^d \|^2 + \frac{c}{4m} \| \nabla v^d \|^2 \\
= \frac{\alpha \varphi}{U} \text{Re} \int_{\Omega} \phi^d \overline{v^d_t} dx + \frac{\varphi}{U} \text{Re} \int_{\Omega} \phi^d \overline{v^c_t} dx - d_i \text{Im} \int_{\Omega} v^d \overline{v^c_t} dx \\
\leq \frac{\alpha d_i}{4} \| v^d_t \|^2 + C_1 \alpha \| \phi^d \|^2 + \frac{1}{2} \left( \frac{1}{U} - a \right) \| v^d \|^2 + C_2 (\| \phi^d \|^2 + \| v^d \|^2). \tag{5.9}$$

Taking $\alpha = \frac{4C_1}{d_i}$ in the above inequality, we arrive at

$$\frac{d}{dt} \left[ \left( \frac{\alpha}{2} \left( \frac{1}{U} - a \right) + \frac{d_i}{2} \right) \| v^d \|^2 + \frac{\alpha c}{8m} \| \nabla v^d \|^2 \right] + \frac{\alpha d_i}{2} \| v^d \|^2 + \frac{1}{2} \left( \frac{1}{U} - a \right) \| v^d \|^2 + \frac{c}{4m} \| \nabla v^d \|^2 \leq (C_1 \alpha + C_2) \| \phi^d \|^2. \tag{5.10}$$

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It easily follows from (5.8) and (5.11) that there exists $\gamma_1 \in (0, \gamma)$

$$\|v^d(t)\|_{\mathcal{H}^l}^2 \leq e^{-2\gamma_1 t} \|v_0\|_{\mathcal{H}^l}^2 + Ce^{-2\gamma_1 t} \int_0^t e^{2\gamma_1 \tau} \|\phi^d(\tau)\|_{\mathcal{H}^l}^2 d\tau$$

$$\leq e^{-2\gamma_1 t} \|v_0\|_{\mathcal{H}^l}^2 + C \frac{1}{(\gamma - \gamma_1)} e^{-2\gamma_1 t} \|\phi_0\|_{\mathcal{H}^l}^2, \quad \forall t \geq 0,$$  \hspace{1cm} (5.11)

and

$$\int_0^t \|v^d(\tau)\|_{\mathcal{H}^l}^2 d\tau \leq C \|v_0\|_{\mathcal{H}^l}^2 + C \int_0^t \|\phi^d(\tau)\|_{\mathcal{H}^l}^2 d\tau \leq C(\|v_0\|_{\mathcal{H}^l}^2 + \|\phi_0\|_{\mathcal{H}^l}^2).$$  \hspace{1cm} (5.12)

Then (5.8), (5.11) and (2.26) yield that

$$\|v^c(t)\|_{\mathcal{H}^l} + \|\phi^c(t)\|_{\mathcal{H}^l} \leq \|v(t)\|_{\mathcal{H}^l} + \|v^d(t)\|_{\mathcal{H}^l} + \|\phi(t)\|_{\mathcal{H}^l} + \|\phi^d(t)\|_{\mathcal{H}^l}$$

$$\leq C(t)(\|v_0\|_{\mathcal{H}^l} + \|\phi_0\|_{\mathcal{H}^l}), \quad \forall t \geq 0.$$  \hspace{1cm} (5.13)

Next, we try to get higher-order estimate of $(v^c, \phi^c)$. For this purpose, we take the time derivative of equations in (5.6):

$$dv^c_t - i\left(a - \frac{1}{U}\right)v^c - i\frac{g}{U}\phi^c - i\frac{c}{4m}\Delta v^c + ib(|v(1)|^2v^c(1) - ib|v(2)|^2v^c(2))_t = 0,$$  \hspace{1cm} (5.14)

$$\phi^c_t + \gamma\phi^c - i\frac{g}{U}v_t + i\left(\frac{g^2}{U} + 2\nu - 2\mu\right)\phi^c - i\frac{c}{4m}\Delta \phi^c = 0.$$  \hspace{1cm} (5.15)

Multiplying (5.14) by $\overline{v^c}$, integrating over $\Omega$ and taking the imaginary part/real part, respectively, we have

$$\frac{d}{dt}\|v^c\|^2 + \frac{c}{4m}\|\nabla v^c\|^2 + \frac{1 - aU}{U}\|v^c\|^2$$

$$= \frac{g}{U}\text{Re} \int_\Omega \phi^c v^c \overline{v} dx - b\text{Re} \int_\Omega (|v(1)|^2v^c(1) - |v(2)|^2v^c(2))v^c v^c dx - \frac{d}{dt} \text{Im} \int_\Omega v^c \overline{v^c} dx,$$  \hspace{1cm} (5.16)

$$\frac{1}{2} \frac{d}{dt}\|v^c\|^2 - \frac{d}{dt} \text{Im} \int_\Omega v^c \overline{v^c} dx = -\frac{g}{U} \text{Im} \int_\Omega \phi^c v^c \overline{v} dx + b \text{Im} \int_\Omega (|v(1)|^2v^c(1) - |v(2)|^2v^c(2))v^c v^c dx.$$  \hspace{1cm} (5.17)

As in the previous section, we can insert (5.17) into (5.16) to cancel the higher-order term $\text{Im} \int_\Omega v^c \overline{v^c} dx$:

$$\frac{d}{dt}\|v^c\|^2 + \frac{c}{4m}\|\nabla v^c\|^2 + \frac{1 - aU}{U}\|v^c\|^2$$

$$= \frac{g}{U} \left(\text{Re} \int_\Omega \phi \overline{v^c} dx - \frac{d}{dt} \text{Im} \int_\Omega \phi \overline{v^c} dx\right)$$

$$- b \left(\text{Re} \int_\Omega (|v(1)|^2v^c(1) - |v(2)|^2v^c(2))v^c \overline{v} dx - \frac{d}{dt} \text{Im} \int_\Omega (|v(1)|^2v^c(1) - |v(2)|^2v^c(2))v^c \overline{v} dx\right)$$

$$: = J_1 + J_2.$$  \hspace{1cm} (5.18)

$$J_1 \leq C\|\phi^c\|\|v^c\| \leq \frac{\gamma}{2} \|\phi^c\|^2 + C\|v^c\|^2.$$  \hspace{1cm} (5.19)

By Proposition 4.1, we know that

$$\|v^{(j)}(t)\|_{\mathcal{H}^2} + \|\phi^{(j)}(t)\|_{\mathcal{H}^2} \leq C, \quad j = 1, 2, \forall t \geq 0.$$  \hspace{1cm} (5.20)
It easily follows from (5.20), the Sobolev embedding theorem and the Poincaré inequality that
\[
J_2 \leq C(\|v^{(1)}\|_{L^\infty} + \|v^{(2)}\|_{L^\infty})\|v\|_{L^6} \|v^{(1)}\|_{L^3} + C\|v^{(2)}\|_{L^\infty} \|v_t\|_{L^2} \|v_t^\perp\|_{L^2}
\]
\[
\leq C\|v\|_{\dot{H}^1} \|\nabla v_t^\perp\|^\frac{3}{2} \|v_t^\perp\|^\frac{1}{2} + C\|v_t\|_{L^2} \|v_t^\perp\|
\]
\[
\leq \frac{c}{8m} \|\nabla v_t^\perp\|^2 + C\|v_t^\perp\|^2 + C\|v_t\|^2 + C\|v\|^2_{\dot{H}^1}.
\]
(5.21)

We can conclude from (5.18), (5.19) and (5.21) that
\[
\left(\frac{d_i}{2} + \frac{d_i^2}{2d_i}\right) \frac{d}{dt} \|v_t^\perp\|^2 + \frac{c}{8m} \|\nabla v_t^\perp\|^2 \leq \frac{\gamma}{2} \|\phi_t^\perp\|^2 + C\|v_t^\perp\|^2 + C\|v_t\|^2 + C\|v\|^2_{\dot{H}^1}.
\]
(5.22)

On the other hand, similar to (4.6), there holds
\[
\frac{d}{dt} \|\phi_t^\perp\|^2 + \gamma \|\phi_t^\perp\|^2 \leq \frac{g^2}{\gamma U^2} \|v_t\|^2.
\]
(5.23)

Then it follows from (5.22) and (5.23) that
\[
\frac{d}{dt} \left[ \left(\frac{d_i}{2} + \frac{d_i^2}{2d_i}\right) \|v_t^\perp\|^2 + \|\phi_t^\perp\|^2 \right] \leq C\|v_t^\perp\|^2 + C\|v_t\|^2 + C\|v\|^2_{\dot{H}^1}.
\]
(5.24)

Integrating with respect to time, using (2.26) and (5.13), we get
\[
\left(\frac{d_i}{2} + \frac{d_i^2}{2d_i}\right) \|v_t^\perp(t)\|^2 + \|\phi_t^\perp(t)\|^2 \leq C \int_0^t (\|v(\tau)\|_{\dot{H}^1}^2 + \|v_t(\tau)\|^2 + \|v_t^\perp(\tau)\|^2) d\tau
\]
\[
\leq C \int_0^t (\|v(\tau)\|_{\dot{H}^1}^2 + \|v_t(\tau)\|^2 + \|v_t^\perp(\tau)\|^2) d\tau \leq C(t)(\|v_0\|_{\dot{H}^1}^2 + \|\phi_0\|_{\dot{H}^1}^2).
\]
(5.25)

By (2.26), (5.13), (5.24) and the Sobolev embedding theorem, we deduce from equation (5.6) and the elliptic regularity theorem that
\[
\|v^\perp(t)\|_{\dot{H}^2} + \|\phi^\perp(t)\|_{\dot{H}^2} \leq C(\|v^\perp(t)\|_{\dot{H}^1} + \|\phi^\perp(t)\|_{\dot{H}^1} + \|v(t)\|_{\dot{H}^1} + \|v^\perp(t)\| + \|\phi^\perp(t)\|)
\]
\[
\leq C(t)(\|v_0\|_{\dot{H}^1} + \|\phi_0\|_{\dot{H}^1}).
\]
(5.26)

Due to (5.8) and (5.11), for any fixed $\lambda \in (0, \frac{1}{2})$, we can choose $t^* \geq \tilde{t}_1$ sufficiently large such that
\[
\|v^d(t^*)\|_{\dot{H}^1}^2 + \|\phi^d(t^*)\|_{\dot{H}^1}^2 \leq \lambda^2(\|v_0\|_{\dot{H}^1}^2 + \|\phi_0\|_{\dot{H}^1}^2).
\]
(5.27)

Set
\[
D(z_{01}, z_{02}) = (v^d(t^*), \phi^d(t^*)), \quad K(z_{01}, z_{02}) = (v^c(t^*), \phi^c(t^*)).
\]
(5.28)

It follows from (5.20) and (5.21) that condition (C2) are satisfied.

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