WHERE IS THE CONFINING STRING IN RANDOM PERCOLATION

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The percolating phase of whatever random percolation process resembles the confining vacuum of a
gauge theory in most respects, with a string tension having a well-behaved continuum limit, a non
trivial glueball spectrum and a deconfinement transition at a well determined temperature $T_c$. Simple
numerical experiments reveal an underlying, strongly fluctuating, confining string, with an internal
vortex structure formed by a core trapping inside a Coulomb-like phase composed by the vacuum at
the percolation threshold. The width of the core almost coincides with $1/T_c$ and it turns out to be
separated from the confining vacuum by a domain wall of definite thickness.

1 Introduction

Recently, it has been pointed out that three-dimensional percolation can be thought as a
gauge theory in disguise[1]. Although percolation is a purely geometrical process in which
no dynamics is involved, it captures all the salient features of a confining vacuum.

Take for instance a bond percolation process. The ensemble of configurations is ob-
tained simply by populating each of the links of a 3D lattice independently with occu-
pation probability $p$. The connected graphs made by occupied links are called clusters.
When $p$ exceeds a threshold value $p_c$ there is a percolating cluster in the infinite lattice.

The key ingredient to extract from this ensemble the relevant information on the un-
derlying gauge theory is the definition of the percolation counterpart of the Wilson oper-
ator $W_\gamma$, that we associate to whatever loop $\gamma$ of the dual lattice. The rule of the game is
very simple. We set $W_\gamma = 1$ if there is no cluster of the configuration topologically linked
to $\gamma$, otherwise we put $W_\gamma = 0$. Therefore, the vacuum expectation value $\langle W_\gamma \rangle$ coincides
with the average probability that there is no path in any cluster linked to $\gamma$. As in usual
gauge theories, evaluating these quantities yields the main physical properties of this sys-

In this way it has been shown that in the percolating phase the theory is confining,
in analogy to what happens in gauge theories with percolating center vortices[2,3,4]. The
string tension $\sigma$ and the other physical observables have the expected scaling behaviour
dictated by the universality class of 3D percolation, therefore such a theory has a well
defined continuum limit. Moreover it has a non-trivial glueball spectrum[5] and a second-
order deconfining transition at finite temperature $T_c$ with a ratio $T_c/\sqrt{\sigma} \simeq 1.5$ which
turns out to be universal, i.e. it does not depend on the kind of lattice utilised nor on the
specific percolation process considered (site or bond percolation)[1].

A particularly interesting feature is the occurrence of universal shape effects in the
Wilson loop[6] which are usually ascribed to the quantum fluctuations of a string. Then
the question arises: where and how are encoded the relevant degrees of freedom of the
underlying confining string? This is the main issue discussed in this paper. The answer
we find, though surprising, confirms once again that percolation is a full-fledged gauge
theory: the confining string reveals a non-trivial microscopic structure which is believed
to be shared by whatever confining gauge theory, according to the picture of the confining
vacuum as a dual superconductor[7,8].
The great advantage of percolation is that its simplicity allows to explore regions that are still inaccessible to the other gauge systems from a computational point of view.

It turns out that the confining string in percolation is made of a core in a Coulomb-like vacuum separated from the confining vacuum by a domain wall. The width $R$ of the core coincides almost exactly with the inverse of the deconfining temperature, while the thickness of the domain wall is about $R/2$. This confining string is not stuck to the minimal surface bounded by $\gamma$, but strongly fluctuates, sweeping a large volume which grows with the size of the Wilson loop.

2 Wilson loop as a source of the flux tube

Each bond configuration of a percolating process in a cubic lattice can be mapped into a plaquette configuration of the dual lattice by setting empty any plaquette orthogonal to an occupied dual link and vice versa. This mapping is one to one, therefore the question whether there is a closed path in a cluster that is linked with a loop $\gamma$ of the dual lattice is equivalent to the existence of a surface $\Sigma$ (at least) of occupied plaquettes having $\gamma$ as boundary, i.e. $\partial \Sigma = \gamma$. The cluster of this kind of surfaces can be considered as the outgoing flux of the "gauge" field generated by a point-like source describing the world-line $\gamma$.

The crucial observation which allows to extract a lot of information form the apparent white noise of the percolation ensemble is that not all the configurations are compatible with the existence of such a source, but only those with $W_\gamma = 1$. In this subset the distribution of occupied plaquettes is not at all flat and may be used to study the spatial distribution of the gauge field. When $p$ is below the percolating threshold the flux is not very constrained and is spread out in the whole space. On the contrary, in presence of a percolating cluster the volume at the disposal of the gauge flux is much more tight, thus
the percolating cluster acts as a sort of superconducting medium, squeezing the gauge flux in a thin tube.

It is useful to associate to each link \( \ell \) of the lattice a projector \( \pi(\ell) \) which is 1 when \( \ell \) is occupied and 0 otherwise. We have trivially \( \langle \pi(\ell) \rangle_{E_p} = p \), where \( E_p \) is the whole ensemble of the percolation process. Much less trivial is the average occupancy probability of the links in the subset of configurations with \( W_\gamma = 1 \), that we denote as

\[
\pi_\ell^\gamma \equiv \langle \pi(\ell) \rangle_{\{W_\gamma = 1\}},
\]

and we use it as a local probe to explore the spatial distribution of the flux. As an example, the quantity \( \pi_\ell^\gamma \) as a function of the links \( \ell \) orthogonal to the plane of a \( 10 \times 20 \) rectangular Wilson loop in the percolating phase at \( p = 0.260 \) is drawn in Fig.1. The data are taken using a simple burning epidemic type of algorithm. The sinking along the perimeter of the rectangle is an effect caused by the finite clusters: since the clusters near \( \gamma \) have a finite probability to be linked to it, the configurations selected by \( W_\gamma = 1 \) have a cluster density smaller with respect the whole ensemble \( E_p \). Therefore an observer near \( \gamma \) experiences an occupation probability \( \pi^\gamma < p \). One has even \( \pi^\gamma < p_c \), as Fig.2 shows. Hence the vacuum around the \( \gamma \) source is in a non-percolating phase. This important fact will be the starting point of the next section.

Inside the rectangular surface encircled by \( \gamma \) the average probability \( \pi_\ell^\gamma \) reaches a plateau \( \tilde{p} < p \) because of the presence of the infinite, percolating cluster. Actually this very condition is sufficient to assure that the vacuum in which the Wilson loop is embedded is confining. Indeed it is easy to prove the following exact inequality for the string tension

\[
\sigma \geq \log \left( \frac{\tilde{p}}{p} \right) \left( \frac{1 - \tilde{p}}{1 - p} \right)^{1 - \tilde{p}}.
\]

\*Also the deepening at the corners of the rectangle is a finite cluster effect, because the linking probability of a cluster is proportional to its overlap with \( \gamma \) and such an overlap is larger near a corner.
To prove it, note that the probability of finding $k$ occupied links among the $N$ links orthogonal to a rectangle $\gamma = r \times t$ is binomial by construction:

$$P_N(k) = \binom{N}{k} p^k (1-p)^{N-k}.$$  \hfill (3)

In our case $N = rt$ and we are assuming that $k \simeq \tilde{p} N$ with $\tilde{p} < p$. For large $k$ and $N$ we have

$$P_N(\tilde{p} N) \simeq \frac{1}{\sqrt{2\pi \tilde{p} (1-\tilde{p}) N}} \left[ \left( \frac{\tilde{p}}{p} \right)^{\tilde{p}} \left( \frac{1 - \tilde{p}}{1 - p} \right)^{1-\tilde{p}} \right]^{-N}. \hfill (4)$$

On the other hand, not all the configurations with $\tilde{p} \simeq \frac{k}{N}$ are compatible with $W_{r,t} = 1$, of course, hence we get the exact inequality

$$\langle W_{r,t} \rangle \leq P_N(\tilde{p} N), \quad N = rt,$$ \hfill (5)

which yields, in turn, eq. (2).

3 **The intrinsic width of the flux tube**

What exactly is the confining string made of? The distribution of the occupied links in close proximity of the source $\gamma$ of the flux tube is much smaller than its asymptotic value and even smaller than the percolation threshold, as Fig. 1 and 2 clearly show. This supports the idea that the confining string may have a microscopic structure similar to that of a (dual) Abrikosov or Nielsen-Olesen vortex, with a core made of a Coulomb-like vacuum as contrasted with the surrounding medium, which is in the confining phase. This core conveys the whole flux in a tube of small but non-zero thickness $R_c$ (see Fig. 3). Some numerical and analytical tests of this scenario were performed long ago on the $Z_2$ gauge model. The semi-classical descriptions of these extended objects neglect an important quantum property of the confining string. In fact, in order to account for the universal shape effects of large Wilson loops $\langle W_\gamma \rangle$ one is forced to assume that the string worldsheet belongs to a rough phase, hence it should resemble much more closely a strongly

\footnote{With a similar reasoning one could “prove” quark confinement out of percolation of center vortices. There is however a flaw in the argument, because center vortices, contrary to the links of random percolation, are correlated, therefore they cannot obey a binomial distribution.}

\footnote{For an updated discussion on this argument and a complete list of references see...}
fluctuating fluid interface rather than a static, smooth structure. As a consequence, we do not expect to find a flux tube stuck to the minimal surface encircled by $\gamma$. Nonetheless, the average occupancy probability in the whole space would provide clues to its internal structure. In this way, enlarging the size of the lattice, one discovers that the confining string sweeps a very large volume surrounding the loop $\gamma$. More precisely, we found that the sum

$$w = \sum_{\ell \in \Lambda} \pi_{\ell}$$

over the $n$ links $\ell$ of the lattice $\Lambda$ is less than the naively expected value $pn$ and their difference $pn - w$ approaches a constant value depending only on the size of $\gamma$. This can be considered as a first hint on the finiteness of the world-volume of flux tube. We start by considering a rather crude approximation of the flux tube, by partitioning the $n$ links of the lattice into two subsets

$$n = n_i + n_e$$

where $n_i$ is the number of links lying inside the Coulomb-like core of the string and $n_e$ the number of those lying outside, in the confining vacuum.

The average number of occupied links in the confining phase is $pn_e$, while the links belonging to the core are occupied with a smaller probability $\hat{p} \leq p_c$ in order to prevent linking. This fact is manifest near the fixed boundary $\gamma$, where the string cannot fluctuate too much, as shown in Fig.2. Thus we can rewrite eq.(6) as

$$w = \hat{p}n_i + pn_e,$$

which tells us that the volume of the core can be expressed simply as

$$n_i = \frac{pn - w}{p - \hat{p}}.$$  \hfill (9)

In order to estimate $n_i$ we need to know the value of $\hat{p}$. A simple scaling argument suggests $\hat{p} = p_c$. When $p \to p_c$, the flux tube is expected to grow because the infinite, percolating cluster is crumbling, then the flux can spread out in the whole space. Therefore, from one hand we expect in this limit $n_i \sim n$. On the other hand in this very limit the confining string disappears, hence we should have $w \sim pn$. According to eq.(9), these two conditions are consistent only if $\hat{p} = p_c$.

Assuming the world-volume of the flux tube to be a slab of size $AR_c$, with $A$ the area of the minimal surface encircled by $\gamma$, we are led to the following effective definition of the intrinsic thickness of the flux tube

$$R_c = \frac{n_i}{A} = \frac{pn - w}{A(p - p_c)}.$$  \hfill (10)

Some numerical estimates of $R_c$ are reported in Tab.1. They seem to approach asymptotically an approximate scaling behaviour, as expected for a physical quantity. Though it gives a length scale of the order of the transverse size of the flux tube, it should not be taken too literally. First, the splitting of the links in two subsets of those inside or outside the Coulombic core is an over-simplification: there should be a domain wall between the two vacua with an average occupancy probability that interpolates between $p$ and $p_c$. This would produce an enlargement of the flux tube which is not taken into account in eq.(10). Similarly, the number of interior links does not coincide exactly with the volume
Figure 4: The slab geometry to probe the internal structure of the flux tube. The transverse direction is periodic and of size $L_\perp$. $\gamma$ denotes a rectangular Wilson loop orthogonal to the compactified dimension.

Figure 5: The intrinsic width of the confining string as a function of the transverse dimension $L_\perp$ of the periodic slab in which is trapped. The data are taken at $p = 0.26$ with a spatial square Wilson loop of size $\ell = 15$. The horizontal arrow denotes the asymptotic value $R_c(\infty)$ of the intrinsic width.
Table 1: The intrinsic width $\mathcal{R}_c$ of the flux tube generated by a square Wilson loop of side $L$. The data are taken at two different values of $p$. The corresponding values of $\sigma$ and of the inverse deconfinement temperature $1/T_c$ are also reported. All the physical quantities are expressed in lattice spacing units.

| $p$   | $\sigma$    | $1/T_c$  | $L$  | $\mathcal{R}_c$ |
|-------|-------------|----------|------|-----------------|
| 0.26000 | 0.00340(5) | 11.5(1)  | 10   | 15.4(3)         |
|       |             |          | 15   | 14.8(3)         |
|       |             |          | 20   | 14.2(3)         |
| 0.26502 | 0.00649(16)| 8.4(1)   | 12   | 12.2(5)         |
|       |             |          | 16   | 10.0(6)         |
|       |             |          | 22   | 10.6(9)         |

Figure 6: Vacuum expectation value of a square Wilson loop of size $\ell = 15$ at $p = 0.26$ as a function of the transverse periodic dimension $L_\perp$ of the lattice. The arrow indicates its asymptotic value.

Long time ago it was observed that in three-dimensional $SU(2)$\textsuperscript{13} and $Z_2$ gauge model\textsuperscript{14,15} the intrinsic width of the flux tube almost coincides with the inverse of the deconfinement temperature (see also\textsuperscript{16}). This result was obtained by studying the response of the vacuum expectation value of a large, planar Wilson loop when one varies the size $L_\perp$ of the dimension orthogonal to the loop in a periodic lattice, like in Fig 4.

We repeated such a numerical experiment in percolation, using the same setting and measuring the intrinsic radius $\mathcal{R}_c$ as defined in eq.\textsuperscript{10}. The surprising results are reported in Fig 5 and 6. Let us follow the behaviour of $\mathcal{R}_c(L_\perp)$ starting from $L_\perp$ large enough, where $\mathcal{R}_c$ coincides with its asymptotic value $\mathcal{R}_c \equiv R_c(\infty)$. As $L_\perp$ decreases, we
observe first an apparent growth of $R_c(L_\perp)$ starting at a value of $L_\perp$ slightly larger than the asymptotic value $R_c$, the reason being perhaps the underestimated contribution of the domain wall. The apparent increasing of $R_c(L_\perp)$ should be ascribed to the fact that the two domain walls separating the core from the confining vacuum are pushed one against the other squeezing a fraction of the core. When $L_\perp$ is of the order of $1/T_c$, the external confining vacuum disappears and the domain walls melt in the Coulomb-like core. At this point the flux tube fills the entire interval $L_\perp$ and loses its own width, which now coincides, for $L_\perp \leq 1/T_c$, with the transverse size of the lattice. This fact is corroborated by value of the slope in this region, which is about 1.

In conclusion, the response of the system to the variation of the size of the transverse dimension seems to confirm the scenario of a confining string made of a core in a Coulomb-like vacuum separated from the confining medium by a domain wall. The width $R$ of the core almost coincides with the inverse of the deconfining temperature, like in other gauge systems, while the thickness of the domain wall seems to be about $R/2$.

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