A class of nonergodic interacting particle systems with unique invariant measure

Benedikt Jahnel * and Christof Külske †

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Abstract

We consider a class of discrete $q$-state spin models defined in terms of a translation-invariant quasilocal specification with discrete clock-rotation invariance which have extremal Gibbs measures $\mu_\varphi^\iota$ labelled by the uncountably many values of $\varphi$ in the one-dimensional sphere (introduced by van Enter, Opoku, Kuelske [8]). In the present paper we construct an associated Markov jump process with quasilocal rates whose semigroup $(S_t)_{t \geq 0}$ acts by a continuous rotation $\mu_\varphi^\iota S_t = \mu_{\varphi+t}^\iota$. The jump rates are given in terms of certain expectations w.r.t. continuous-spin Gibbs measures constrained to take prescribed discretization values, which yields quasilocality in a regime of fine discretizations. We then show that the infinite-volume discretization map is an equivariant bijection for the rotation action on continuous-spin Gibbs measures and discrete Gibbs measures.

As a further consequence our construction provides examples of interacting particle systems with unique invariant measure, which is not long-time limit of all starting measures, answering an old question (compare Liggett [23] question four chapter one.) The construction of this particle system is inspired by recent conjectures of Maes and Shlosman about the intermediate temperature regime of the nearest neighbor clock-model. We define our generator of the interacting particle system as a (non-commuting) sum of the rotation part and a Glauber part.

Technically the paper rests on the control of the spread of weak non-localities and relative entropy-methods, both in equilibrium and dynamically, based on Dobrushin-uniqueness bounds for conditional measures.

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*Ruhr-Universität Bochum, Fakultät für Mathematik, D44801 Bochum, Germany, Benedikt.Jahnel@ruhr-uni-bochum.de, http://http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Kuelske/jahnel.html
†Ruhr-Universität Bochum, Fakultät für Mathematik, D44801 Bochum, Germany, Christof.Kuelske@ruhr-uni-bochum.de, http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Kuelske/kuelske.html /~kuelske/
1 Introduction

Consider an interacting particle system with finite local state space with quasilocal rates. Consider a translation-invariant measure which is invariant under the dynamics. Suppose there is only one such measure. Is it true that the dynamics is necessarily ergodic?

This is an old question which was picked up again in a recent very interesting paper by Maes and Shlosman [24] about dynamics of Clock models (see [14], [15], [2] and [25]), where they conjecture that this may not be the case and suggest a mechanism of rotating states in discrete rotator models with standard scalar product nearest neighbor interactions at intermediate temperatures. While their conjectures seemed plausible, at the same time no simple proof based on their heuristics in their model seemed possible.

For the more degenerated situation of discrete-time, parallel updating PCAs, an example was recently given in [3].

In the present paper we construct a dynamics for a $q$-state particle system ($q$ possibly large but finite) which does the job: It has a unique invariant measure for which the dynamics is not ergodic. Our construction is inspired by the conjectures of Maes and Shlosman which we put to a situation where they can be proved. Technically it builds on earlier works of [8], [22]. However, the present main new idea is the definition of a rotation dynamics and proving that it can be realized as a generator with quasilocal jump rates.

The construction hints at the existence of synchronization phenomena on the lattice, even for discrete local spaces. Such phenomena have been successfully studied in the mean-field framework of the Kuramoto model [1], [4], [17].

1.1 Main result

Consider an $S^1$ rotation-invariant and translation-invariant Gibbsian specification $\gamma^\Phi$ on the lattice $G = \mathbb{Z}^d$, with local state space $S^1$, which is given by an absolutely summable $S^1$-invariant and translation-invariant Gibbsian potential $\Phi = (\Phi_A)_{A \subseteq \mathbb{Z}^d}$, w.r.t. to the Lebesgue measure $\lambda$ on the spheres. A standard example is provided by the nearest neighbor scalar product interaction rotator model.

Assume moreover that the extremal translation-invariant Gibbs measures can be obtained as weak limits with homogeneous boundary conditions

$$\text{ex } \mathcal{G}_0(\gamma^\Phi) = \{ \mu_\varphi | \mu_\varphi = \lim_{\Lambda} \gamma^\Phi_{\Lambda, \varphi} : \varphi \in S^1 \}$$

and that different boundary conditions $\varphi$ yield different measures so that there is a unique labelling of states $\mu_\varphi$ with the angles $\varphi$ in the sphere $S^1$. This is known to be the case in the standard rotator model in $d = 3$ for $\lambda$-a.a. temperatures in the low temperature region as discussed in [24], [13], [27].

Denote by $T$ the local coarse-graining with equal arcs of $S^1$ to $\{1, \ldots, q\}$. Extend this map to infinite-volume configurations by performing it sitewise and extend it also to measures.
Choose \( q \geq q_0(\Phi) \) large enough so that the condition from Theorem 2.5 of [22] is fulfilled (ensuring a uniform Dobrushin regime for the so-called constrained first-layer models.)

Define a Markov process with state space \( \{1, \ldots, q\}^G \) which we will refer to as the coarse-grained layer in terms of the following generator

\[
(L\psi)(\omega') = \sum_{i \in G} c_L(\omega', (\omega')^i)(\psi((\omega')^i) - \psi(\omega'))
\]

with jump rates

\[
c_L(\omega', (\omega')^i) = \frac{\mu_{G\setminus i}[\omega'_{G\setminus i}]}{\mu_{G\setminus i}[\omega'_{G\setminus i}]}(\lambda^i(e^{-H_1(\omega')}))
\]

(\( (\omega')^i \)) being the discrete configuration which coincides with \( \omega' \) except at the site \( i \) where it is increased by the amount of one unit. \((\omega')^r \) is the right endpoint of the interval in continuous single-spin space at \( i \) prescribed by \( \omega'_i \). From the definition it is clear that the corresponding dynamics will be irreversible since jumps are only possible in one direction. Note that the rates depend on the original Hamiltonian in two places namely in the \( H_i \) and in the \( \mu_{G\setminus i} \).

Here \( H_i = \sum_{A \ni i} \Phi_A \) is the Hamiltonian and \( \mu_{G\setminus i}[\omega'_{G\setminus i}] \) is the unique continuous-spin Gibbs measure restricted to the volume \( G\setminus i \) for a conditional specification with open boundary conditions at \( i \) and constrained to take values \( \omega'_{G\setminus i} \) with discretization images \( T\omega_{G\setminus i} = \omega'_{G\setminus i} \). This object is well-defined and well-behaved for sufficiently fine discretization \( q \geq q_0(\Phi) \), see [8], [22], [11] and below [12]. For general background on preservation of Gibbsianness see [6], [11] and [21].

Next we consider another generator \( K \) on the same space \( \{1, \ldots, q\}^G \)

\[
(K\psi)(\omega') = \sum_{i \in A} \left[c_K(\omega', (\omega')^i)((\psi((\omega')^i) - \psi(\omega'))) + c_K(\omega', (\omega')^{i-})(\psi((\omega')^{i-}) - \psi(\omega'))\right]
\]

It will be of Glauber-type in a sense explained below. Here the rates to go up (modulo \( q \)) and down respectively satisfy

\[
c_K(\omega', (\omega')^i) = \frac{\mu_{G\setminus i}[\omega'_{G\setminus i}]}{\mu_{G\setminus i}[\omega'_{G\setminus i}]}(\lambda^i(e^{-H_1(\omega')}))
\]

with \( (\omega')^{i-} \) being the discrete configuration which coincides with \( \omega' \) except at the site \( i \) where it is decreased by the amount of one unit. Then we have as our main result the following theorem.

**Theorem 1.1** Consider a translation-invariant, rotation-invariant and continuously differentiable potential with

\[
\sum_{A \ni i} \sum_{k \in G} \epsilon^{|k|} \delta_k(\Phi_A) < \infty
\]

where \( \epsilon > 0 \). Let \( \alpha > 0 \) and assume fine enough discretization \( q \geq q_0(\Phi) \).
1. Then \( L + \alpha K \) gives rise to a well-defined interacting particle system with quasilocal rates.

2. The class of translation-invariant measures which are invariant under the associated Markov semigroup consists of a single element.

3. There are translation-invariant measures which do not converge under the dynamics to the unique invariant measure. Notice any finite range potential or exponentially-decaying pair-potential satisfies (5). We further note, the requirements on the potential can be relaxed. For example one could replace exponential decay by polynomial decay of sufficiently high order as will become clear from the proof. The conditions will be presented whenever they get used for the first time.

1.2 Idea of Proof

Define the quasilocal specification \( \gamma \) for the discretized model by

\[ \gamma_{A}(\omega_{A} | \omega_{G \setminus A}) = \frac{\mu_{G \setminus A}[\omega'_{G \setminus A}](\lambda^{A}(e^{-H_{A}}1_{\omega_{A}}))}{\mu_{G \setminus A}[\omega'_{G \setminus A}](\lambda^{A}(e^{-H_{A}}))}. \] (6)

Here \( H_{A} = \sum_{A_{ab}} \Phi_{A} \) is the Hamiltonian in \( A \) and \( \mu_{G \setminus A}[\omega'_{G \setminus A}] \) is the unique continuous-spin Gibbs measure restricted to the volume \( G \setminus A \) for a conditional specification constrained to take values with discretization images \( \omega'_{G \setminus i} \), with open boundary conditions at \( A \).

The map \( T \) is injective on the translation-invariant states in the continuum model. More precisely we have the following

**Theorem 1.2** \( T \) is a bijection from \( G_{\theta}(\gamma^{\Phi}) \) to \( G_{\theta}(\gamma') \) with inverse given by the kernel \( \mu_{G}[\omega'](d\omega) \).

Remarkably, when we denote \( T(\mu_{\gamma}) = : \mu_{\gamma}' \) for a discretized measure, we have \( \mu_{\gamma}' \in G_{\theta}(\gamma') \) and we can get back from the discrete to the continuous measure by \( \mu_{\gamma}(d\omega) = \int \mu_{\gamma}'(d\omega') \mu_{G}[\omega'](d\omega) \).

That \( T \mu \) is Gibbs when \( \mu \) is, is already contained in [8], [22] and uses the uniform Dobrushin condition on the coarse-graining. The part that each translation-invariant discrete Gibbs measure has a discretization preimage in the continuous Gibbs measures is new and uses the Gibbs variational principle (see [16]).

Next we show that rotation on the level of discrete states \( \mu_{\gamma}' \) can be realized with the generator \( L \) defined above with local jump rates. This is a main new structure of our paper.

**Theorem 1.3**

1. The semigroup \( (S_{t}^{L})_{t \geq 0} \) is well-defined.

2. \( S_{t}^{L}T \mu_{\varphi} = T \mu_{\varphi+t} \) for all \( t, \varphi \).
In group theoretical language \((t, \mu_\psi) \mapsto \mu_{\psi+t}\) is an \(S^1\)-action on \(\text{ex} \, G_\theta(\gamma^\Phi)\) and \((t, \mu'_\psi) \mapsto \mu'_{\psi+t}\) is an \(S^1\)-action on \(\text{ex} \, G_\theta(\gamma')\). The second statement of the theorem says that \(T\) is an equivariant map (that is a group-action preserving map).

Technically the proof relies on the introduction of weighted triple-semi-norms to control the weak non-localities which are present in the rates and the spreading of these under the action of the dynamics.

Having defined the discretized local specification \(\gamma'\) we note that the generator \(K\) defined above plays the role of a Glauber dynamics. To understand the argument providing us with a unique invariant measure for the joint dynamics and understand better this Glauber part of the dynamics we prove the following intermediate result.

**Proposition 1.4**

1. The semigroup \((S^K_t)_{t \geq 0}\) is well-defined.

2. \(K\) is the generator of a Glauber dynamics for the coarse-grained specification \(\gamma'\) that satisfies local detailed balance.

3. The translation-invariant measures which are invariant under the Glauber dynamics \((S^K_t)_{t \geq 0}\) are precisely the discrete Gibbs measures \(G_\theta(\gamma')\).

To see that invariance under dynamics implies Gibbs we use an adaption of the relative entropy arguments of Liggett ("Holley's argument") [23] from the Ising lattice gas context to our situation. The standard idea here is to consider time derivatives of relative entropies of the time-evolved measure relative to a suitable finite-volume version of a Gibbs measure in \(\Lambda\) which, along with translation-invariance and estimation of boundary terms, produces a single-site DLR equation. This argument will have to be modified with new terms arising from the joint dynamics corresponding to \(L + \alpha K\) which we want to consider now. The result is the following.

**Proposition 1.5** Let \(\alpha > 0\).

1. The semigroup \((S^{L+\alpha K}_t)_{t \geq 0}\) is well-defined.

2. \(S^{L+\alpha K}_t T \mu_\psi = T \mu_{\psi+t}\) for all \(t\).

3. The translation-invariant measures which are invariant under the joint dynamics \((S^{L+\alpha K}_t)_{t \geq 0}\) must necessarily be elements of the discrete Gibbs measures \(G_\theta(\gamma')\).

Obviously it uses that the Glauber part leaves the discrete Gibbs measures invariant. A bit of care however needs to be taken for the second part since the rotation part \(L\) and the Glauber part \(K\) don’t commute. However, it follows by arguments as above controlling also quadratic terms in \(L, K\) which use weighted triple-semi-norms to control the weak non-localities as above. The idea of the third part is this: To see that invariance under joint dynamics implies Gibbs we use again the relative entropy arguments and note that we have to deal with a sum of two terms each corresponding to \(L\) and \(K\). For the part corresponding to
we note that a comparison of the terms with those obtained by open boundary conditions in the generator and in the measure in the second slot of the relative entropy provides us only with a boundary-order correction. The bulk terms have a good sign. Together we arrive at the desired single-site DLR equation.

Combing the second and the third part we conclude

**Corollary 1.6** Let \( \alpha > 0 \). Then the only translation-invariant measure which is invariant under the joint dynamics \((S_t^{L+\alpha K})_{t\geq 0}\) is the measure \( \frac{1}{2\pi} \int d\varphi T \mu_\varphi \).

Together with property 2 which shows that there is no relaxation of the pure measure \( \mu'_\varphi \) under \((S_t^{L+\alpha K})_{t\geq 0}\) we arrive at the proof of Theorem 1.1.

The remainder of the paper contains the following. In Section 2 we prove Theorem 1.2 using the variational principle. For this we need to present generalities and facts on discretizations and recall criteria on the preservation of Gibbsianness. In Section 3 we consider the rotation dynamics and prove Theorem 1.3. In Section 4 we consider the Glauber dynamics and prove Proposition 1.4. In Section 5 we consider the joint dynamics and prove the main Proposition 1.5.

### 1.3 Extensions

Theorem 1.2 stays true also for models where for every angle there are more than one Gibbs measures as could occur for potentials with highly nonconvex shapes \([10]\). The well-definedness of the rotation semigroup is untouched and one has

**Theorem 1.7** The map \( T : \text{ex } G_\theta (\gamma'') \mapsto \text{ex } G_\theta (\gamma') \) is an equivariant bijection for the \( S^1 \)-actions on continuous and discrete-spin Gibbs measures.

The equivariance property says \( S_t^{L+\alpha K} T \mu = T R_t \mu \) for all \( \alpha \geq 0 \), where \( R_t \mu \) is the measure obtained by joint rotation of the realizations of the measure \( \mu \) by an angle \( t \). The same conclusions apply.

[Diagram]

### 1.4 Acknowledgement

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2 Discretizations

Take an underlying site space $G$, a measurable local spin-space $S$ and a configuration space $\Omega = S^G$. The local state space will be often just the sphere $S^1$, but we can also consider a subset of an Euclidean space or finite-dimensional manifold. We will refer to this space as the continuous spin-space and. Consider a Gibbsian potential $\Phi = (\Phi_A)_{A \subset G}$, assumed to be absolutely summable and translation-invariant. Let $\gamma = (\gamma_A)_{A \subset G}$ be the associated Gibbsian specification with a priori measure $\lambda$, write for the Hamiltonian in finite $\Lambda \gamma = \sum_{A \subset \Lambda} \Phi_A$. We denote by $G(\gamma^p)$ the corresponding Dobrushin measures, defined by the DLR equation and by $G_\Theta(\gamma^p)$ the translation-invariant Gibbs measures. Together we call this the first-layer system.

Next a discretization is defined, and as in [8], [22] we start with a decomposition of the local state space $S = \bigcup_{s' \in (1, \ldots, q)} S_{s'}$. Denote $T(s) := s'$ for $S_{s'} \ni s$. This defines a deterministic transformation on $S$, called the discretization map. The space $\Omega' := \{1, \ldots, q\}^G$ will be referred to as the discrete spin-space.

For fixed discrete-spin variable $\omega' \in \Omega'$ define a specification on the continuous spin-space in terms of the local kernels
$$\gamma^\omega_{\Lambda} = \frac{\gamma_{\Lambda}(\varphi | \omega_{\Lambda'})}{\gamma_{\Lambda}(\varphi | \omega_{\Lambda'})}. \tag{7}$$
Notice that here and in many other cases we look at $s' \in \{1, \ldots, q\}$ as a subset of $S$ and write $1_{s'}(s) = 1$ iff $T(s) = s'$.

This specification is non-null on the constrained first-layer local spin-spaces, i.e. it is a specification on $\Omega'' = \times_{i \in G} S_{\omega_i'}$.

One verifies the defining properties of a specification: To begin with, from the compatibility property of $\gamma$ follows that for $\gamma^\omega_{\Lambda}$ for each fixed $\omega'$. This does not use the fact that it is coming from a Gibbsian specification with a potential $\Phi$, but it follows from a computation using the corresponding compatibility property of $\gamma$ that, indeed,
$$\gamma^\omega_{\Lambda} = \gamma_{\Lambda}^{\omega'} \tag{8}$$
for all finite $\Lambda \subset \Delta$. The quasilocality of $\gamma$ implies that of $\gamma^\omega_{\Lambda}$ for all $\omega'$. Since $\gamma$ is proper it is simple that $\gamma^\omega_{\Lambda}$ is proper, where properness means for all finite $\Lambda \subset G$ and $A \subset \Omega$ measurable and dependent only on sites in $\Lambda$ it holds $\gamma_{\Lambda}(A | \omega) = 1_A$ for all $\omega \in \Omega$. We will refer to this as the second-layer system.

This specification has a Dobrushin matrix, which with another supremum over the discrete spins, is
$$C_{ij} := \sup_{\omega'} \sup_{\omega, \omega'} \| \gamma^\omega_{\Lambda} (\cdot | i | \omega) - \gamma^\omega_{\Lambda} (\cdot | i | \omega') \|_i \tag{8}$$
as well as a corresponding Dobrushin constant $\bar{c} := \sup_i \sum_j C_{ij}$ which is defined to incorporate already the supremum over $\omega'$.

Suppose that $\bar{c} := \sup_i \sum_j C_{ij} < 1$ or a slightly stronger exponential decay property that we will present later. Then it follows from Dobrushin theory that for each finite or infinite-volume $V \subset G$ there is a kernel from coarse-grained
configurations $\omega'$ (inside $V$) and boundary conditions of first-layer configurations $\omega$ outside $V$, namely $\gamma_{V}(\cdot | \omega_{G \setminus V})$, which has the infinite-volume compatibility property $\gamma_{V}^G \gamma'_{W} = \gamma_{V}'$, for all (and not only finite) $W \subset V$.

Let us write for the unique first-layer Gibbs measure on discretizations $\omega'$

$$\mu_G[\omega_G'](d\omega) = \gamma_{G}^G(d\omega).$$

Since it is also measurable as a function of the coarse-grained configuration, $\mu[\cdot](d\omega)$ really is a probability kernel from $\Omega'$ to $\Omega$.

We cite the bound on the matrix elements of the Dobrushin matrix from [8]

$$\tilde{C}_{ij} \leq \sup_{s'} \text{diam}_{ij} S_{s'}/4$$

and hence the bound on the Dobrushin constant is

$$\tilde{c} \leq \frac{1}{4} \sum_{j \in G \setminus i} \sup_{s'} \text{diam}_{ij} (S_{s'})$$

with the family of metrics $(d_{ij})_{j \in G \setminus i}$ on the local spin-space at the site $i \in G$ defined by

$$d_{ij}(\sigma_i, \tau_i) := \sup_{\zeta, \tilde{\zeta}} \left| H_i(\sigma_i \zeta^c) - H_i(\sigma_i \tilde{\zeta}^c) - \left( H_i(\tau_i \zeta^c) - H_i(\tau_i \tilde{\zeta}^c) \right) \right|,$$

where for any $i \in G$, $i^c = G \setminus \{i\}$.

In the standard nearest-neighbour models, (the plane rotor or XY-model) with Hamiltonian

$$H_{\Lambda}(\sigma_{\Lambda} \eta_{\Lambda^c}) = -\beta \sum_{<i,j> \in \Lambda} \sigma_i \cdot \sigma_j - \beta \sum_{i \in \Lambda, j \in \Lambda^c} \sigma_i \cdot \eta_j$$

with local spin-spaces $S^1$ we have that $2d\beta(\sin \frac{\pi}{q})^2 < 1$ implies $\tilde{c} < 1$ (see [8], similar criteria are immediate for high-dimensional rotators.)

Suppose we are conditionally uniformly in the Dobrushin regime $\tilde{c} < 1$. We have that $f$ quasilocal implies that $f'(\omega') := \mu_G[\omega_{G}'](f)$ is quasilocal on $\Omega'$. Denoting by $\mathcal{F}'$ the sigma algebra over $\Omega$ generated by the infinite-volume coarse-graining map $T$ we have that $\mu_G[\omega_{G}'](f)$ is a regular version of the conditional expectation $\mu(f|\mathcal{F}')(\omega')$ for every Gibbs measures $\mu \in \mathcal{G}(\gamma^G)$.

Moreover we have for $\mu' = T\mu$

$$\left| \left( \mu(fg) - \mu(f)\mu(g) \right) - \left( \mu'(fg') - \mu'(f')\mu'(g') \right) \right| \leq \frac{1}{4} \sum_{i,j \in S} \delta_i(f)\delta_j(g) \tilde{D}_{ij}.$$  \hspace{1cm} (13)

In particular the following holds. If there are observables $f, g$ such that $\mu(f(g \circ \theta_k)) - \mu(f)\mu(g \circ \theta_k)$ decays slower than the matrix $(\tilde{D}_{ij})_{i,j \in G} := \sum_{n \geq 0} \tilde{C}^n$ then there are coarse-grained observables $f', g'$ such that their correlation decay is also slower than the decay of the matrix $\tilde{D}_{ij}$. 

8
To see that (13) holds, write
\[
\mu(fg) - \mu(f)\mu(g) = \mu(\mu(fg|F')) - \mu(f)\mu(g) \\
= \mu'(\mu(fg|F') - \mu(f|F')\mu(g|F')) + \mu'((\mu(f|F'))\mu(\mu(g|F')) - \mu'(\mu(f|F'))\mu'(\mu(g|F'))).
\]

Further the standard estimate (see Proposition 8.34 in [16]), in the Dobrushin uniqueness regime yields
\[
\sup_\omega |\mu_G[\omega'](fg) - \mu_G[\omega'](f)\mu_G[\omega'](g)| \leq \frac{1}{4} \sum_{i,j \in S} \delta_i(f)\delta_j(g)D_{ij} \tag{14}
\]
which proves (13).

The map from \(\mu\) to \(\mu' := T\mu\) is injective when viewed on the (not necessarily translation-invariant) Gibbs measures of the continuous-spin system. Indeed we can restore an initial Gibbs measure \(\mu\) from its coarse-grained image where we have that \(\mu(\varphi) = \int \mu'(d\omega')\mu_G[\omega'](\varphi)\) where \(\mu_G[\omega'](\varphi)\) does not depend on \(\mu\). Hence different \(\mu\)'s must have different images \(\mu'\).

Next recall the definition of the specification \(\gamma'\) for the coarse-grained system (see also [22]), given in (6). Standard arguments show that, in the uniform Dobrushin regime and we have
\[
\lim_{\Lambda \to \Lambda'} E[\gamma' \omega' | \Lambda \setminus \Lambda'] = \lim_{\Lambda \to \Lambda'} E[\gamma' \omega' \omega'_{\Lambda \setminus \Lambda'} | \Lambda \setminus \Lambda'],
\]
where \(\gamma' \omega' | \Lambda \setminus \Lambda'\) is the specification on \(\Omega_{\Lambda \setminus \Lambda'}\) obtained by putting all potentials \(\Phi_A\) with \(A \cap \Lambda' \neq \emptyset\) equal to zero.

Then, by martingale convergence \(\mu'(\omega'_{\Lambda'}) \omega'_{\Lambda \setminus \Lambda'}\) converges, as \(\Lambda\) tends to \(G\) in the a.s. and \(L^1\)-sense to \(\mu(1_{\omega'_{\Lambda'}} | F'_{\Lambda \setminus \Lambda'})(\omega'_{\Lambda \setminus \Lambda'})\) where \(1_{\omega'_{\Lambda \setminus \Lambda'}}(\omega'_{\Lambda \setminus \Lambda'}) = 1\) for all \(\Lambda \supset \Lambda'\).

On the other hand, for any finite \(\Lambda'\), there is convergence uniformly in the integration variable \(\omega\) under the \(\mu\)-integrals since the conditional specification is in the uniform Dobrushin regime and we have
\[
\gamma'_{\Lambda'}(\omega'_{\Lambda'} | \omega'_{\Lambda \setminus \Lambda'}) = \lim_{\Lambda \to \Lambda'} E[\gamma' \omega' \omega'_{\Lambda \setminus \Lambda'} | \Lambda \setminus \Lambda'] = \lim_{\Lambda \to \Lambda'} E[\gamma' \omega' \omega'_{\Lambda \setminus \Lambda'} | \Lambda \setminus \Lambda']/\omega'_{\Lambda \setminus \Lambda'}
\]
where
\[
\omega'_{\Lambda \setminus \Lambda'}(\omega') = \frac{\mu_{\Lambda \setminus \Lambda'}[\omega'_{\Lambda \setminus \Lambda'}](\lambda^{\Lambda'}(e^{-H_{\Lambda'}(1_{\omega'_{\Lambda'}})))}{\mu_{\Lambda \setminus \Lambda'}[\omega'_{\Lambda \setminus \Lambda'}](\lambda^{\Lambda'}(e^{-H_{\Lambda'}})).
\]
The limiting measure in the last line is the unique Gibbs measure of the specification restricted to \(G' \setminus \Lambda'\) with open boundary conditions and this proves (9). It is easy to see using the standard Dobrushin estimates that the specification \(\gamma'\) built with these kernels is quasilocal.

We also have that, uniformly in the configuration \(\omega'\),

\[
\log \frac{\gamma'_\Lambda(\omega'_\Lambda | \omega'_\Lambda G | \Lambda)}{\gamma_{\Lambda'}^{p}(\omega'_{\Lambda'} | \tilde{\omega}'_{\Lambda'} G \setminus \Lambda)} \leq 4 \sum_{\Lambda' \setminus \Lambda = \emptyset, \Lambda' \setminus \Lambda = \emptyset} \| \Phi_A \|. \tag{17}
\]

Note that for summable potentials and \(\Lambda'\) being cubes on the lattice, the r.h.s. is bounded by a constant times the length of the boundary of \(\Lambda'\), in particular

\[
\log \frac{d\gamma'_\Lambda(\omega'_\Lambda | \omega'_\Lambda G | \Lambda)}{d\gamma_{\Lambda'}^{p}(\omega'_{\Lambda'} | \tilde{\omega}'_{\Lambda'} G \setminus \Lambda)} = O(|\partial \Lambda'|).
\]

Let us now restrict to the lattice case, i.e. \(G = \mathbb{Z}^d\) and discuss relative entropy density. The following lemma should be seen as a generalization of the contractivity of the relative entropy (density) between two measures (see Lemma 3.3 in [18]) under strictly local transforms to transforms which are not strictly but “sufficiently” local. In this context note also the monotonicity property of relative entropy between two measures which are restricted to a sub-sigma-algebra w.r.t. to this sigma-algebra (see Proposition 15.5 [19]).

**Lemma 2.1** Let \(\mu'_1, \mu'_2 \in \mathcal{G}(\gamma')\) for some specification where \(\log \frac{d\gamma'_\Lambda(\omega'_\Lambda | \omega'_\Lambda G | \Lambda)}{d\gamma_{\Lambda'}^{p}(\omega'_{\Lambda'} | \tilde{\omega}'_{\Lambda'} G \setminus \Lambda)}\) is of the order \(O(|\Lambda|)\) for cubes. Take a kernel \(\mu_G[\omega'_G](d\omega')\) where \(\log \frac{d\mu_G[\omega'_G]| | \Lambda \rangle}{d\mu_G[\omega_G]| | \Lambda \rangle}\) is also of the order \(O(|\Lambda|)\) uniformly in all configurations \(\omega'\) and \(\tilde{\omega}'\) which coincide on \(\Lambda\). Then the relative entropy density between the mapped measures equals zero, i.e.

\[
\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} H \left( \int \mu'_1(d\tilde{\omega}') \mu_G[\tilde{\omega}']| | \Lambda \rangle \int \mu'_2(d\tilde{\omega}') \mu_G[\tilde{\omega}']| | \Lambda \rangle \right) = 0 \tag{18}
\]

along cubes.

**Proof.** We need to estimate the relative entropy \(H\) in a volume \(\Lambda\) where \(\Lambda \subset \mathbb{Z}^d\) is a finite cube appearing in the formula above, which is

\[
\int \mu'_1(d\tilde{\omega}') \mu_G[\tilde{\omega}']| | \Lambda \rangle \left( \log \frac{d\int \mu'_1(d\tilde{\omega}') \mu_G[\tilde{\omega}']| | \Lambda \rangle}{d\int \mu'_2(d\tilde{\omega}') \mu_G[\tilde{\omega}']| | \Lambda \rangle} \right). \tag{19}
\]

Using the DLR equation for the integrand as well as the conditions on the Radon Nikodým derivatives we find

\[
\log \frac{d\int \mu'_1(d\tilde{\omega}') \mu_G[\tilde{\omega}']| | \Lambda \rangle}{d\int \mu'_2(d\tilde{\omega}') \mu_G[\tilde{\omega}']| | \Lambda \rangle} = \log \frac{\int \mu'_1(d\tilde{\omega}') \frac{d\mu_G[\tilde{\omega}']| | \Lambda \rangle}{d\lambda}}{\int \mu'_2(d\tilde{\omega}') \frac{d\mu_G[\tilde{\omega}']| | \Lambda \rangle}{d\lambda}} \leq \sup_{\lambda_1, \lambda_2} \log \frac{\int (\gamma_\Lambda) \langle \lambda | d\tilde{\omega}' \rangle \frac{d\mu_G[\tilde{\omega}']| | \Lambda \rangle}{d\lambda}}{\int (\gamma_\Lambda) \langle \lambda | d\tilde{\omega}' \rangle \frac{d\mu_G[\tilde{\omega}']| | \Lambda \rangle}{d\lambda}} = o(|\Lambda|) \tag{20}
\]
where the estimate in the last line uses the two assumptions in the hypothesis. Hence the relative entropy density as the limit of the relative entropy devided by the volumes of cofinal sequence of cubes is equal to zero.

Applying the lemma and using now the Gibbs variational principle in the form of Theorem 15.37 of [16] it follows that every discrete Gibbs measure has a continuous preimage:

*Proposition 2.2* Let $\mu' \in \mathcal{G}_0(\gamma')$, then $\mu(d\omega) := \int \mu'(d\omega')\mu_G[\omega'](d\omega) \in \mathcal{G}_0(\gamma^\Phi)$.

**Proof.** Let $\mu_0 \in \mathcal{G}_0(\gamma^\Phi)$ be a Gibbs measure for the original system and $\mu'_0 := T\mu_0$ its coarse-grained image. We want to use the preceding lemma, i.e. justify the conditions and therefore conclude that the relative entropy density between the two translation-invariant measures is zero. Hence, by the variational principle applied to the original system, also $\mu \in \mathcal{G}_0(\gamma^\Phi)$.

Indeed, (17) asserts the condition of Lemma 2.1 for the coarse-grained specification $\gamma'$. Also we have for $\omega', \tilde{\omega}'$ coinciding on $\Lambda$

$$\log \frac{d\mu_G[\omega']|\Lambda}{d\mu_G[\tilde{\omega}']|\Lambda} = \log \frac{\int \mu_G[\omega'](d\tilde{\omega}_1) \frac{d(\gamma^\Phi)}{d\lambda}(\cdot |\tilde{\omega}_1)}{\int \mu_G[\tilde{\omega}'](d\tilde{\omega}_2) \frac{d(\gamma^\Phi)}{d\lambda}(\cdot |\tilde{\omega}_2)} \leq \sup_{\tilde{\omega}_1, \tilde{\omega}_2} \log \frac{d\gamma^\Phi}{d\gamma^\Phi}(\cdot |\tilde{\omega}_1, \tilde{\omega}_2) \leq 4 \sum_{\Lambda \cap \Lambda' \neq \emptyset, \Lambda \cap \Lambda'' \neq \emptyset} ||\Phi_A|| = o(|\Lambda|) \quad (21)$$

This means the map from the translation-invariant Gibbs measures of the original system $\mathcal{G}_0(\gamma^\Phi)$ to the translation-invariant measures for the coarse-grained configuration $\mathcal{G}_0(\gamma')$ is one-to-one.

If $\mu$ is tail-trivial, then so is $T\mu$ since the tail sigma-algebra of discrete events is contained in the tail sigma-algebra of all events, $\mathcal{T}' \subset \mathcal{T}$. In particular $T(\text{ex} \ \mathcal{G}(\gamma^\Phi)) \subset \text{ex} \ \mathcal{G}(\gamma')$. To see that also $\mu \in \text{ex} \ \mathcal{G}(\gamma^\Phi)$ for $T\mu \in \text{ex} \ \mathcal{G}(\gamma')$ one can use the fact that the mapping $T$ is affine: Let’s assume $T\mu \in \text{ex} \ \mathcal{G}(\gamma')$ and $\mu = s\mu_1 + (1-s)\mu_2$ for $s \in [0,1]$ and $\mu_1, \mu_2 \in \mathcal{G}(\gamma^\Phi)$. Then we have $T\mu = sT\mu_1 + (1-s)T\mu_2$ and hence $T\mu = T\mu_1 = T\mu_2$ since $T\mu$ is extremal. But that means $\mu = \mu_1 = \mu_2$ and thus $\mu \in \text{ex} \ \mathcal{G}(\gamma^\Phi)$.

The proof of the preceding remark also follows from the fact that tail-triviality is preserved under the kernel (even not assuming initial Gibbs measures). Let us give some detail here since it provides useful background and explains the “essentially local” nature of the transformation from the perspective of the tail events.

*Proposition 2.3* Assume that $\mu'$ is a probability measure on $\Omega'$ which is trivial on $\mathcal{T}'$. Then $\mu(d\omega) := \int \mu'(d\omega')\mu_G[\omega'](d\omega)$ is trivial on the tail-sigma algebra $\mathcal{T}$.  

11
Proof. We assume that also sup \( \sum_i \tilde{C}_{ij} < 1 \) which is guaranteed in the fine-discretization regime ensured by our criteria.

If \( A \in \mathcal{T} \) then \( \mu_G'[(A) \cap V] \leq \mathcal{T}' \)-measurable. To see this, suppose that \( W \) is a finite subset of \( G \), that \( V \) contains \( W \) and that \( A \) is in \( T_V \), the sigma-algebra of events not depending on spins inside \( V \).

Assuming that \( A \) is a cylinder at first we have

\[
\sup_{\omega', \omega' : \omega'_{W,c} = \omega'_{W,c}} \mu_G[\omega'](A) - \mu_G[\omega'](A) \leq \sum_{i \in \text{supp}(A), j \in W} \tilde{D}_{ij} \leq \sum_{i \in V, j \in W} \tilde{D}_{ij}.
\]

Next we note that this inequality also holds by approximation of probabilities of general events by cylinders, (by a semiring-approximation argument) for all \( A \in \mathcal{T}_V \). Since \( A \in \mathcal{T} \) is in any \( \mathcal{T}_V \) we may let \( V \not\supset G \) and obtain that

\[
\sup_{\omega', \omega' : \omega'_{W,c} = \omega'_{W,c}} \mu_G[\omega'](A) - \mu_G[\omega'](A) \leq 0.
\]

Since \( W \) was arbitrary this is the tail-measurability.

Further we note that \( \mu_G[\omega'](A) \in \{0,1\} \) for each fixed \( \omega' \) and \( A \in \mathcal{T} \) since the original measure constrained to coarse-grained configurations is in the Dobrushin-uniqueness regime, hence tail-trivial. So \( \mu_G[\omega'](A) = 1_{A'}(\omega') \) for some \( A' \in \mathcal{T}' \) and this implies \( \mu(A) = \int \mu'(d\omega') \mu_G[\omega'](A) = \mu'(A') \in \{0,1\} \) by tail-triviality of \( \mu' \).

\[\qed\]

3 Continuous rotations for discrete-spin models

Let us specialize to a translation-invariant \( S^1 \)-model and define a Markov process in the coarse-grained layer given by the following generator

\[
(L\psi)(\omega') = \sum_{i \in G} c_L(\omega', (\omega')^i)(\psi((\omega')^i) - \psi(\omega'))
\]

with jump rates

\[
c_L(\omega', (\omega')^i) = \frac{\mu_G'_{ij}[(\omega'_{G_{ij}}^c)(e^{-H_i((\omega')^r_i, e)})]}{\mu_G'_{ij}[(\omega'_{G_{ij}}^c)(e^{-H_i 1_{\omega_i^c})])}
\]

with notation as explained in the introduction.

Intuitively, if we think about the coarse-grained version of the \( d \geq 3 \) XY-Model, this process should deterministically and homogeneously in time rotate any initial second-layer extremal Gibbs measure \( \mu'_i \) in an anti-clockwise direction. The \( \mu'_i \)'s are characterized by the order-parameter \( \mu'_i(\hat{m}_i) = m_\beta e_i \) with the quasi-local observable \( \hat{m}_i(\omega') = \mu_G[\omega'_G](\sigma_i) \) taking discrete variables to the unit disc, where \( m_\beta \) is the modulus of the magnetisation of the underlying continuous model.

In other words if \( (S^L_t)_{t \geq 0} \) is the associated semigroup of \( L \), the rotation should be performed as \( \mu'_i((t+s) \mod 2\pi) = S^L_t \mu'_i \). In order to achieve this, for a given configuration \( \omega' \), the rate to jump up one unit at site \( i \) should satisfy

\[
c_L(\omega', (\omega')^i)dt = \mu_G[\omega'_G]((\omega'_i)^r - dt, (\omega'_i)^r)]
\]
where \([(\omega'_i)^r - dt, (\omega'_i)^r)\] is the first-layer part of \(\omega'_i\) that after an infinitesimal shift \(dt\) would be deterministically under \(T\) be mapped to \((\omega'_i)^i\).

Taking the limit \(dt\) to zero we arrive at the probability density

\[
c_L(\omega', (\omega')^i) = \frac{d\mu_G[\omega'_G]_G}{d\xi}(\omega'_i)^r). \tag{26}
\]

In terms of the constrained Gibbs measure restricted to \(G'\setminus i\) we may write this as \(\text{Proposition 3.2} \tag{27}\).

**Definition 3.1** Let us fix the following notations. We write

1. \(\mathcal{L}' := \{ f : \Omega' \to \mathbb{R} : f \text{ is local} \}\) for the local functions.

2. \(C(\Omega') = \mathcal{L}'\) equivalently for the space of continuous functions on the compact configuration-space \(\Omega'\) which, since \(\varrho\) is finite, coincides with the space of bounded quasilocal functions which is just the \(\| \cdot \|\)-completion of the local functions. Here \(\| \cdot \|\) denotes the uniform norm.

3. \(D(\Omega') := \{ f \in C(\Omega') : \| f \| := \sum_{i \in G} \delta_i(f) < \infty \}\) for the core functions.

4. \(\mathcal{L}'_{\| \cdot \|}\) for the triple-semi-norm completion of the local functions.

5. \(D_{p(\varrho)}(\Omega') := \{ g \in C(\Omega') : \| g \|_{p(\varrho)} := \sum_{i \geq 0} \varrho(i,0) \delta_i(g) < \infty \}\) for the space of weighted triple-semi-normed functions, where \(\varrho\) is an increasing, translation-invariant semi-metric on the site space and \(p : \mathbb{R}_0^+ \to \mathbb{R}_0^+\) any weight-function.

Let us clarify the relations between those spaces and specialize to \(p\) being either an exponential or monomial function with power \(m \in \mathbb{N}\) and the semi-metric as being just the euclidean metric \(| \cdot |\) on \(G\) with some factor \(\varepsilon > 0\). We have

\[
\mathcal{L}' \subset D_{\varepsilon 1}(\Omega') \subset D_{11}(\Omega') \subset D_{1}(\Omega') \subset L_{\| \cdot \|} \subset D(\Omega') \subset C(\Omega'). \tag{28}
\]

Notice, all those spaces are dense in \(C(\Omega')\) with respect to the \(\| \cdot \|\)-norm. All inclusions should be clear except \(D_{1}(\Omega') \subset L_{\| \cdot \|}\).

**Proposition 3.2** Let \(f \in C(\Omega')\). If there exists a sequence of finite volumes \(\Lambda_n \not\ni G\) such that \(\sum_{i \in \Lambda_n} |\Lambda_n| \delta_i(f) \to 0\) for \(n \to \infty\) then there exists a sequence of local functions \(f_n\) with \(\| f - f_n \| \to 0\) for \(n \to \infty\). Thus \(f \in L_{\| \cdot \|}\).

**Proof.** First note that \(f \in D(\Omega')\). In particular there exists finite \(\Lambda\) such that \(\sum_{i \in \Lambda} \delta_i(f) < \frac{\varepsilon}{2}\). Let \(\eta \in \Omega'\) and define \(f_n(\omega) := f(\omega_{\Lambda_n}, \eta)\) then we have

\[
|\Lambda_n| \| f - f_n \| \leq |\Lambda_n| \sup_{\eta = \eta_{\Lambda_n}} |f(\eta) - f(\tilde{\eta})| \leq |\Lambda_n| \sum_{i \in \Lambda_n} \delta_i(f) \to 0\text{ for }n \to \infty. \tag{29}
\]

Hence we have with \(n\) such that \(|\Lambda_n| \| f - f_n \| \leq \frac{\varepsilon}{2}\) and \(\Lambda \subset \Lambda_n\)

\[
\| f - f_n \| = \sum_{i \in \Lambda_n} \delta_i(f - f_n) + \sum_{i \in \Lambda_n^c} \delta_i(f) \leq 2|\Lambda_n|\| f - f_n \| + \frac{\varepsilon}{2} \leq \varepsilon. \tag{30}
\]
Remark 3.3 If \( n : G \to \mathbb{N} \) is an ordering of \( G \) then \( \Lambda_i := \{ j \in G : n(j) \leq n(i) \} \) is an exhausting sequence of finite volumes such that \( \sum_{i \in G} |\Lambda_i| \delta_i(f) = \sum_{i \geq 0} i \delta_{n^{-1}(i)}(f) \).

Let \( f \) be a function such that \( \sum_{i \geq 0} i \delta_{n^{-1}(i)}(f) < \infty \) then we have

\[
\sum_{i \in \Lambda_k} |\Lambda_k| \delta_i(f) = \sum_{i > n(k)} n(k) \delta_{n^{-1}(i)}(f) \leq \sum_{i > n(k)} i \delta_{n^{-1}(i)}(f) \to 0 \quad \text{for} \quad k \to \infty
\]

(30)

thus \( f \in \mathcal{L}' \) if \( \sum_{i \geq 0} i \delta_{n^{-1}(i)}(f) < \infty \). In other words \( D_{\omega}^1(\Omega') \subset \mathcal{L}' \).

We will drop the notation \( n^{-1}(i) \) and just write \( \sum_{i \geq 0} i \delta_i(f) \).

3.1 Well-definedness of the rotation generator

Here we prove Theorem 1.3 part 1. The jump rates are uniformly bounded since we assumed the potential to be absolutely summable and translation-invariant and the coarse-graining to be finite. In order to ensure well-definedness of the dynamics we have to check

\[
\sup_{i \in G} \sum_{j \neq i} \delta_j(c_L(\cdot, i)) < \infty.
\]

(31)

But this follows from the Dobrushin comparison theorem (see [16] Theorem 8.20.), indeed

\[
\delta_j(c_L(\cdot, i)) \leq C e^{H_i} \sup_{\omega' = \omega'|_{i \leftarrow j}} |\mu_{G^i}|(e^{-H_{\lambda^i}((\omega')_r, \cdot, \omega')_l}) - |\mu_{G^i}|(e^{-H_{\lambda^i}((\omega')_r, \cdot, \omega')_l})| + C e^{3H_i} \sup_{\omega' = \omega'|_{i \leftarrow j}} |\mu_{G^i}|(e^{-H_{\lambda^i}1_{\omega'}}) - |\mu_{G^i}|(e^{-H_{\lambda^i}1_{\omega'}})|
\]

therefore it suffices to look at the Gibbs measures \( |\mu_{G^i}|(\omega'_{G^i})(\cdot) \) and \( |\mu_{G^i}|(\omega^{\lambda^i}_{G^i})(\cdot) \) on \((S^1)^{G^i}\) applied to the quasilocal functions \( \psi_1(\cdot) := e^{-H_{\lambda^i}((\omega')_r, \cdot, \omega')_l} \) and \( \psi_2(\cdot) := \lambda^i(e^{-H_{\lambda^i}((\omega')_r, \cdot, \omega')_l}) \). For any fixed first-layer boundary condition \( \omega \in \Omega \) the measure \( |\mu_{G^i}|(\omega'_{G^i})(\cdot) \) is uniquely specified by the specification

\[
\gamma_{\omega^{\lambda^i}_{G^i}} := \left( (|\omega^{\lambda^i}_{G^i}|_{G^i})_{G^i} \right)_{\lambda \subset G^i}
\]

(32)

A being finite subsets of \( G^i \). We have for \( \omega'_{G^i} = \omega^{\lambda^i}_{G^i} \)

\[
\| (|\omega_{G^i}|_{G^i})_{G^i} (\cdot | \omega_{G^i}) - (|\omega^{\lambda^i}_{G^i}|_{G^i})_{G^i} (\cdot | \omega_{G^i}) \|_1 \leq 1_{1-j}.
\]

(33)

Hence for \( \psi''_{G^i} = \omega''_{G^i} \) and \( \psi'' \in \{ \psi_1'', \psi_2'' \} \) the comparison theorem gives us

\[
|\mu_{G^i}(\omega'_{G^i})(\psi''_{G^i}) - |\mu_{G^i}(\omega^{\lambda^i}_{G^i})(\psi''_{G^i})| \leq \sum_{k \neq i} \delta_k(\psi''_{G^i}) D_{kj}(\gamma_{\omega^{\lambda^i}_{G^i}}) \leq \sum_{k \neq i} \delta_k(\psi''_{G^i}) \tilde{D}_{kj}
\]
where we used the fact that the specifications $\gamma_{\omega}^{c;\iota}$ are in the Dobrushin region uniformly in the constraint $\omega'$. Since $\hat{c} := \sup_{\iota} \sum_{j} \tilde{C}_{ij} < 1$ we have $\sum_{j\in G} \tilde{D}_{kj} < \infty$ for all $k \in G$ and can therefore conclude
\[
\sup_{i \in G} \sum_{j \neq i} \sum_{k \neq i} \delta_k(\psi_{i}^{\omega}) \tilde{D}_{kj} \leq C \sum_{k \in G} \sup_{i \neq i} \sup_{j \neq j} \delta_k(\psi_{i}^{\omega}) \leq C \sup_{i \in G} \sum_{k \in G} \delta_k(\psi_{i}^{\omega})
\]
with $\psi_{i} \in \{\psi_{1}^{\omega}; \psi_{2}^{\omega}\}$ and $\psi_{1}^{\omega}(\cdot) := e^{-H_{L}(\cdot;\cdot)}$ and $\psi_{2}^{\omega}(\cdot) := \lambda^{i}(e^{-H_{L}(\cdot;\cdot)})$. In case $\psi_{i}^{\omega}$ is a local function, uniformly bounded in $i$ (for instance in the XY-Model) the sum is finite and thus less than infinity. In the general case where the $\psi_{i}^{\omega}$ are coming from an uniformly bounded Hamiltonian which is only quasilocal the summability is not guaranteed. But if we stipulate
\[
\sup_{\iota \in G} \sum_{A: \iota} \|\Phi_{A}\| < \infty
\]
we have for $\psi_{1}^{\omega}$ and $\psi_{2}^{\omega}$
\[
\sum_{k \in G} \delta_k(\psi_{2}^{\omega}) \leq C \sum_{k \in G} \delta_k(\psi_{1}^{\omega}) \leq C e^{\|H_{0}\|} \sum_{k \in G} \sum_{i \neq i} \delta_k(\Phi_{A}) = Ce^{\|H_{0}\|} \sum_{A: \iota} \|\Phi_{A}\| < \infty
\]
where we used $|x - y| \leq |x - y| e^{\max(|x|,|y|)}$. Hence we proved (31).

If we impose the following exponential decay condition on the Dobrushin matrix
\[
\hat{c}_{\omega} := \sup_{i \neq i} \sum_{j \neq j} e^{\frac{\omega(i,j)}{\omega}} \tilde{C}_{ij} < 1
\]
for a translation-invariant semi-metric $\omega$ on $G$, we have $\sup_{i \in G} \sum_{i \neq i} e^{\omega(i,j)} \tilde{D}_{ij} \leq \frac{1}{1 - \hat{c}_{\omega}}$ and by the triangle inequality
\[
\sup_{i \neq i} \sum_{j \neq j} e^{\omega(i,j)} \delta_{j}(c_{\omega}(\cdot, \cdot)) \leq \sup_{i \neq i} \sum_{j \in G, i} \sum_{k \in G, j} e^{\omega(i,j)} \delta_{k}(\psi_{i}^{\omega}) \tilde{D}_{kj}
\]
\[
\leq \frac{1}{1 - \hat{c}_{\omega}} \sup_{i \neq i} \sum_{k \in G, i} e^{\omega(i,k)} \delta_{k}(\psi_{i}^{\omega})
\]
which is again finite for finite-range first-layer potentials as for example the XY-Model. In the general case were the $\psi_{i}^{\omega}$ are coming from an uniformly bounded non-local Hamiltonian we can stipulate
\[
\sum_{A: \iota} \sum_{k \in G} e^{\omega(i,k)} \delta_{k}(\Phi_{A}) < \infty
\]
uniformly in $i \in G$ and find for $\psi_{1}^{\omega}$ and $\psi_{2}^{\omega}$ using the same arguments as above
\[
\sum_{k \in G} e^{\omega(i,k)} \delta_{k}(\psi_{2}^{\omega}) \leq C \sum_{k \in G} e^{\omega(i,k)} \delta_{k}(\psi_{1}^{\omega}) \leq Ce^{2K} \sum_{A: \iota} \sum_{k \in G} e^{\omega(i,k)} \delta_{k}(\Phi_{A}) < \infty.
\]
Hence we have $\sum_{j \neq i} \delta_j(c_L(\cdot, i)) < Ce^{-\theta(i,j)}$ for all $i \in G$. We will use that in the sequel. Note that in particular $c_L(\cdot, i) \in D(\Omega') \subset L'_1$ for all $i \in G$ thus the rates are quasilocal.

Instead of imposing an exponential decay property of the Dobrushin matrix one could just consider polynomial weights $p(\theta(i,j))$ which would admit Hamiltonians with polynomial dependence. In fact for our purposes that would be sufficient.

Theorem 3.9. of [23] now asserts the following: 1. The closure $\bar{L}$ of $L$ is a Markov generator of a Markov semigroup $(S^L_t)_{t \geq 0}$ connected to the generator via the Hille-Yosida Theorem. $D(\Omega')$ is a core for $\bar{L}$. 2. For observables $f \in D(\Omega')$ we can control the oscillation of $S_t f$ at any site $i \in G$ via

$$\delta_i(S_t f) \leq [e^{t \Gamma} \delta (f)](i)$$

where $\Gamma : L_1 \to L_1$, $[\Gamma \delta (f)](i) := \sum_{j \neq i} \delta_j(c_L(\cdot, j))\delta_j(f)$ is a bounded operator with $\|\Gamma\| := M$. In particular for $f \in D(\Omega')$ we have $\|S_t f\| \leq e^{tM}\|f\|$ and thus $S_t f \in D(\Omega')$.

### 3.2 Rotation property of the generator

The goal of this subsection is to verify Theorem [13] part 2. We use the following strategy:

1. We verify the rotation property for infinitesimal times by comparing the generator to the derivative on the level of the probability density. We do this directly on local observables.

2. In order to get from infinitesimal to finite time, we consider the associated semigroup $(S^L_t)_{t \geq 0}$ and use Taylor expansion. To match the first-order terms it is necessary to verify the infinitesimal rotation for local functions propagated by $S^L_t$. Those functions are no longer strictly local but lie in a larger space, namely $L'_1$. Since later we need (and will verify) the stronger result $S^L_t f \in D(p(x))(\Omega')$ for local $f$ and weight-function $p(x) = x^2$, at this point we just assume $S^L_t f \in L'_1$.

3. The two second-order error terms need to be estimated. As for the first one we can use the contraction property of the semigroup. For the other one we compute the second derivative of the measure again on the level of the probability density and local observables. It turns out the desired upper bound exists as long as the observable lies in a space of weighted triple-semi-normed functions.

4. By assuming exponential decay of the Dobrushin matrix the rates of the generator are elements of this space even for arbitrary polynomial weights. One can think of these spaces as containing functions with a certain degree of locality. The amount of non-locality the semigroup injects into a local function is controlled by the degree of locality of the rates. This can simplest
be captured by looking at the operator $\Gamma$ mentioned above. We can show under these assumptions that local observables propagated by the semigroup stay in the space of weighted triple-semi-normed functions.

Let us start with an infinitesimal rotation and show $\mu'_t((t+s)_{mod \ 2\pi})(f) = \mu'_t(S^t f)$ for all $t \in [0, 2\pi)$, $s > 0$ and local observables $f$ on $\Omega'$. Since the coarse-graining is finite it suffices to use $f = 1_{sA}$ for finite $A$. Write $\rho_A = \frac{d\lambda}{d\rho}$ for the Lebesgue-density of the local specification in $A$. We have

$$\frac{d}{d\varepsilon} \mu'_{t+\varepsilon}(1_{sA}) = \int \mu_t(d\omega) \frac{d}{d\varepsilon} \left( \prod_{j \in A} \int_{s^j_l - \varepsilon}^{s^j_l} d\varphi \rho_A(\varphi, \omega) \right)$$

$$= \sum_{j \in A} \int \mu_t(d\omega) \left( \prod_{i \in A \cup j} \int_{s^j_i}^{s^j_i + \varepsilon} d\varphi \right) \left( \rho_A(s^j_i, \varphi), \omega) - \rho_A(s^j_i, \varphi, \omega) \right)$$

since $\mu_t$ admits $\gamma_A$ for all $t \in [0, 2\pi)$. On the other hand

$$\frac{d}{d\varepsilon} \mu'_t S^t_{\varepsilon}(1_{sA}) = \mu'_t(L1_{sA})$$

$$= \sum_{j \in A} \left( \sum_{\omega' : \omega'_{\Lambda} = sA} c_L(\omega', (\omega')^j) \mu'_t(\omega') - \sum_{\omega' : \omega'_{\Lambda} = sA} c_L(\omega', (\omega')^j) \mu'_t(\omega') \right).$$

Looking at the individual summands we find

$$\sum_{\omega' : \omega'_{\Lambda} = sA} c_L(\omega', (\omega')^j) \mu'_t(\omega') = \int \mu'_t(d\omega') 1_{sA}(\omega') \frac{\mu_{G\Lambda}(\omega'_{\Lambda})}{\mu_{G\Lambda}(\omega'_{\Lambda})} \left( e^{-H_A(s^j_i, \varphi, \omega)} \right)$$

where we used the DLR equation in the second last line and the fact that

$$\frac{\mu_{G\Lambda}(\omega'_{\Lambda})}{\mu_{G\Lambda}(\omega'_{\Lambda})} = \mu_G(\varphi(\Lambda')). \quad (37)$$

Similary for the other summand. Thus we have $\frac{d}{d\varepsilon} \mu'_{t+\varepsilon}(f) = \mu'_t(Lf)$ for all local observables. Later we want to apply $S^t_t f$ and will show $S^t_t f \in \mathcal{L}^r_{\Lambda}$ if $f$ is local. So let us prove the following
Proposition 3.4 If $f \in \mathcal{L}_{\| \cdot \|}$ then $\frac{d}{dt} \mu'_{t+\varepsilon}(f) = \mu_t'(Lf)$.

Proof. Assume $(f_n)_{n \in \mathbb{N}}$ to be a sequence of local functions such that $\| f - f_n \| \to 0$ for $n \to \infty$ then we have according to Proposition 3.2 of [23]

$$|\mu_t'(Lf) - \mu_t'(Lf_n)| \leq \|Lf - Lf_n\| \leq C\|f - f_n\| \xrightarrow{n \to \infty} 0. \quad (38)$$

On the other hand with $g := f - f_n$ and $n : G \to \mathbb{N}$ an ordering of $G$

$$\mu'_{t+\varepsilon}(g) - \mu'_t(g) = \int \mu_t(d\bar{\omega})(g(T(\bar{\omega} - \varepsilon 1_G)) - g(T(\bar{\omega})))$$

$$= \int \mu_t(d\bar{\omega}) \sum_{j \in G}(g(T(\bar{\omega} - \varepsilon 1_{\{0, \ldots, n(j)\}})) - g(T(\bar{\omega} - \varepsilon 1_{\{0, \ldots, (n(j)-1)\}})))$$

$$\leq \sum_{j \in G} \delta_j(g) \mu_t(\{\bar{\omega} : T(\bar{\omega}_j - \varepsilon) = T(\bar{\omega}_j) - 1\})$$

where we use a telescopic sum in the second line. Further we have with $A_j := \{\bar{\omega} : T(\bar{\omega}_j - \varepsilon) = T(\bar{\omega}_j) - 1\} = \{\bar{\omega} : \bar{\omega}_j \in [s^t, s^t + \varepsilon]\}$ for some $s \in S'$

$$\mu_t(A_j) \leq \sup_{\omega \in \Omega} \gamma_j(A_j|\omega) \leq \frac{\varepsilon |S'| e^{\|H_j\|}}{2\pi e^{-\|H_j\|}} \leq K\varepsilon \quad (39)$$

uniformly in $t$ and $j$, hence $\frac{1}{\varepsilon} |\mu'_{t+\varepsilon}(f - f_n) - \mu'_t(f - f_n)| \leq K\|f - f_n\| \xrightarrow{n \to \infty} 0$ and we can conclude

$$\left|\frac{d}{dt} \mu'_{t+\varepsilon}(f) - \mu'_t(Lf)\right| \leq \left|\frac{d}{dt} \mu'_{t+\varepsilon}(f - f_n)\right| + \left|\mu'_t(L(f - f_n))\right| \xrightarrow{n \to \infty} 0. \quad (40)$$

Assume for the moment $S^L_t f \in \mathcal{L}_{\| \cdot \|}$ for local $f$. In order to verify the rotation we use the following iteration procedure. Let $f$ be local, $k \in \mathbb{N}$, $t \in [0, 2\pi)$, $s > 0$ and $\varepsilon := \frac{\pi}{k}$. On the one hand

$$\mu'_t(S^L_s f) = \mu'_t(S^L_{\varepsilon} S^L_{s-\varepsilon} f) = \mu'_t((1 + \varepsilon L + S^L_{\varepsilon} - (1 + \varepsilon L))S^L_{s-\varepsilon} f)$$

$$= \mu'_t(g) + \varepsilon \mu'_t(Lg) + \mu'_t((S^L_{\varepsilon} - (1 + \varepsilon L))g) \quad (41)$$

where we set $g := S^L_{s-\varepsilon} f$. On the other hand one can use Taylor expansion

$$\mu'_{t+\varepsilon}(g) = \mu'_t(g) + \varepsilon \frac{d}{d\varepsilon} \mu'_t(g) + \varepsilon^2 \frac{d^2}{d\varepsilon^2} \mu'_t(g)|_{\varepsilon=[t,t+\varepsilon]}$$

$$= \mu'_t(g) + \varepsilon \mu'_t(Lg) + \varepsilon^2 \frac{d^2}{d\varepsilon^2} \mu'_t(g)|_{\varepsilon=[t,t+\varepsilon]} \quad (42)$$

By iteration

$$\mu'_t(S^L_s f) = \mu'_{t+s}(f) - \varepsilon^2 \sum_{l=0}^{k-1} \frac{d^2}{d\varepsilon^2} \mu'_{t+l\varepsilon}(S^L_{s-(l+1)\varepsilon} f)|_{\varepsilon=[t+l\varepsilon,t+(l+1)\varepsilon]}$$

$$+ \sum_{l=0}^{k-1} \mu'_{t+l\varepsilon}((S^L_{\varepsilon} - (1 + \varepsilon L))S^L_{s-(l+1)\varepsilon} f)$$

18
where the error terms should go to zero as $k$ tends to infinity. Let us look at the second error term first and use the uniform continuity of the Markov semigroup

$$
\sum_{l=0}^{k-1} \mu_{l+\varepsilon}((S_{\varepsilon}^L - (1 + \varepsilon L))S_{s-(l+1)\varepsilon}^L) \leq \varepsilon \sum_{l=0}^{k-1} \left\| \frac{S_{\varepsilon}^L S_{s-(l+1)\varepsilon}^L f - S_{s-(l+1)\varepsilon}^L f}{\varepsilon} - L S_{s-(l+1)\varepsilon}^L f \right\| 
$$

where the r.h.s goes to zero as $\varepsilon$ goes to zero since the semigroup is generated by $L$ and $f$ in the domain of $L$. In particular this is true for core observables of $L$.

Let us check the first error term. Set $t' \in [t \varepsilon, t+(l+1)\varepsilon]$ and $t'' := s-(l+1)\varepsilon$. It suffices to find a constant $C(s, f)$ such that

$$
\frac{d^2}{d\varepsilon^2} \mu_{t'\varepsilon}(S_{t''}^L f) \leq C(s, f)
$$

since then we have $\varepsilon^2 \sum_{l=0}^{k-1} \frac{d^2}{d\varepsilon^2} \mu_{t'\varepsilon}(S_{t''}^L f) \leq \varepsilon^2 k C(s, f) \xrightarrow{k \to \infty} 0$.

Consider the second derivative when we apply a local indicator function $1_{s^L}$.

$$
\frac{d^2}{d\varepsilon^2} \mu_{t'\varepsilon}(1_{s^L}) = \sum_{j \in \Lambda} \mu_t(d\omega) \left[ \left( \prod_{i \in \Lambda} \int_{s_i^0}^{s_i^t} \right) d\varphi_{\Lambda,t} \left( \rho_\Lambda(s_j^t, \varphi_{\Lambda,t}, \omega) \frac{d}{ds_j} H_\Lambda(s_j^t, \varphi_{\Lambda,t}, \omega) 

- \rho_\Lambda(s_j^t, \varphi_{\Lambda,t}, \omega) \frac{d}{ds_j} H_\Lambda(s_j^t, \varphi_{\Lambda,t}, \omega) 

- \rho_\Lambda(s_j^t, s_k^t, \varphi_{\Lambda,t}, \omega) - \rho_\Lambda(s_j^t, s_k^t, \varphi_{\Lambda,t}, \omega) 

+ \rho_\Lambda(s_j^t, s_k^t, \varphi_{\Lambda,t}, \omega) + \rho_\Lambda(s_j^t, s_k^t, \varphi_{\Lambda,t}, \omega) \right] 

= \sum_{j \in \Lambda} \mu_t(d\omega) \left[ A(j, s_J, \omega) + \sum_{k \in \Lambda} B(j, k, s_J, \omega) \right]
$$

where, as we see from the formalism, the Hamiltonian of the first-layer system needs to be differentiable as a function on $S^1$. Let us assume these partial derivatives are also uniformly bounded with $K' := \sup_{\omega \in \mathbb{S}^1} \sup_{\omega \in \mathbb{S}^1} \| \frac{d}{d\omega} H_\Lambda(\cdot, \omega, \omega) \| < \infty$, then we have $A(j, s_J, \omega) =$

$$
\left( \prod_{i \in \Lambda} \int_{s_i^0}^{s_i^t} \right) d\varphi_{\Lambda,t} \left( e^{-H_\Lambda(s_j^t, \varphi_{\Lambda,t}, \omega)} \frac{d}{ds_j} H_\Lambda(s_j^t, \varphi_{\Lambda,t}, \omega) - \frac{d}{ds_j} H_\Lambda(s_j^t, \varphi_{\Lambda,t}, \omega) 

+ \frac{d}{ds_j} H_\Lambda(s_j^t, \varphi_{\Lambda,t}, \omega) e^{-H_\Lambda(s_j^t, \varphi_{\Lambda,t}, \omega)} \right).
$$
where
\[
A_1(j, s_\Lambda, \omega) := \frac{d}{ds_j} H_{\Lambda}(s_j^r, \varphi_{\Lambda \setminus j}, \omega) - \frac{d}{ds_j} H_{\Lambda}(s_j^l, \varphi_{\Lambda \setminus j}, \omega)
\]
\[
= \frac{d}{ds_j} H_{\Lambda}(s_j^r, \varphi_{\Lambda \setminus j}, \omega) - \frac{d}{ds_j} H_{\Lambda}(s_j^l, \varphi_{\Lambda \setminus j}, \omega) \leq \delta_j \left( \frac{d}{dj} H_j \right) \leq 2K',
\]
\[
A_2(j, s_\Lambda, \omega) := e^{-H_{\Lambda}(s_j^r, \varphi_{\Lambda \setminus j}, \omega_A)} - e^{-H_{\Lambda}(s_j^l, \varphi_{\Lambda \setminus j}, \omega_A)} \leq \sum_{j \notin A} \Phi_A(\varphi_{\Lambda \setminus j}, \omega_A)
\]
and thus
\[
A(j, s_\Lambda, \omega) \leq 2K'e^{2K} 2\pi \left( \frac{\Pi_{i \in \Lambda \setminus j} \sum_{t_i} \delta_i}{d \varphi_{\Lambda \setminus j} e^{-\sum_{j \notin A} \Phi_A(\varphi_{\Lambda \setminus j}, \omega_A)}} \right)
\]
\[+ 2e^{2K} (K' + (|\Lambda| - 1)K) 2\pi \left( \frac{\Pi_{i \in \Lambda \setminus j} \sum_{t_i} \delta_i}{d \varphi_{\Lambda \setminus j} e^{-\sum_{j \notin A} \Phi_A(\varphi_{\Lambda \setminus j}, \omega_A)}} \right).
\]

Let us look at the second term. We can write
\[
B(j, k, s_\Lambda, \omega) \leq 4e^{4K} 4\pi^2 \left( \frac{\Pi_{i \in \Lambda \setminus j} \sum_{t_i} \delta_i}{d \varphi_{\Lambda \setminus j} e^{-\sum_{j \notin A} \Phi_A(\varphi_{\Lambda \setminus j}, \omega_A)}} \right)
\]
\[+ 2e^{2K} (K' + (|\Lambda| - 1)K) 2\pi \left( \frac{\Pi_{i \in \Lambda \setminus j} \sum_{t_i} \delta_i}{d \varphi_{\Lambda \setminus j} e^{-\sum_{j \notin A} \Phi_A(\varphi_{\Lambda \setminus j}, \omega_A)}} \right).
\]

For convenience set \(\tilde{K} := \max\{K, K'\}\) and \(\bar{K} := \max\{4\pi \tilde{K} e^{2\tilde{K}}, 8\pi^2 e^{4\tilde{K}}\}\). Also we want to adopt a notation we introduced earlier
\[
\gamma_{\Lambda \setminus j} e^{-\sum_{j \notin A} \Phi_A(\varphi_{\Lambda \setminus j}, \omega_A)}
\]
Before we combine these estimates, let us apply a local functions \(h\) on the coarse-grained space with support \(\Lambda\). \(h\) can be written as \(h(\omega') = \sum_{s_\Lambda \in \{1, \ldots, \kappa\}} \kappa_{s_\Lambda} 1_{s_\Lambda} (\omega')\) with \(\|h\| = \sup_{s_\Lambda} |\kappa_{s_\Lambda}|\). Hence
\[
\frac{d^2}{dz^2} \mu_{t+\epsilon}(h) \leq \|h\| \sum_{j \in \Lambda} \int \mu_t (d\omega) \left[ q\tilde{K}(|\Lambda| + 1) \sum_{s_\Lambda \in \{1, \ldots, \kappa\}} \gamma_{\Lambda \setminus j} e^{-\sum_{j \notin A} \Phi_A(\varphi_{\Lambda \setminus j}, \omega_A)} \right]
\[+ \sum_{j \in \Lambda} q^2 \tilde{K} \sum_{s_\Lambda \in \{1, \ldots, \kappa\}} \gamma_{\Lambda \setminus j k} \left( \gamma_{\Lambda \setminus j} e^{-\sum_{j \notin A} \Phi_A(\varphi_{\Lambda \setminus j}, \omega_A)} \right) \right]
\leq \|h\| \|\Lambda\| (q\tilde{K}(|\Lambda| + 1) + q^2 \tilde{K} (|\Lambda| - 1)) \leq \tilde{K}|\Lambda|^2 \|h\|.
\]
For a general quasilocal function \(f\) one can write again a telescopic sum using an ordering of \(G\) and a generic configuration \(\eta'\)
\[
f(\omega') = f(\omega', \eta'_{[1]} e) + (f(\omega', \omega'_{[1]} e, \eta'_{[1]} e) - f(\omega', \eta'_{[1]} e))
\[+ \sum_{n \geq 3} (f(\omega'_{[1, \ldots, n]}, \eta'_{[1, \ldots, n]} e) - f(\omega'_{[1, \ldots, n-1]}, \eta'_{[1, \ldots, n-1]} e)) \right)
\]
\[= 43]
Let us define \( g_n(\omega') := (f(\omega'_{[1,n]}, \eta'_{[1,n-1]}), f(\omega'_{[1,n-1]}, \eta'_{[1,n-1]})) \in \mathcal{F}_{[1,n]} \).
In particular \( \|g_n\| \leq \delta_n(f) \). Hence we can write
\[
\frac{d^2}{d\varepsilon^2} \mu_{t+\varepsilon}^f(f) \leq \|f\| \hat{K} + \sum_{n \geq 2} \|g_n\| \hat{K} n^2 \leq 2\hat{K} \sum_{n \geq 1} n^2 \delta_n(f).
\] (44)

Thus it suffices to show \( S_t^L f \in D_{\nu,t|\Omega} \) for local \( f \). We use the exponential decay property of the Dobrushin matrix introduced in (34) and the exponentially decaying Hamiltonian, i.e. assume the model to satisfy \( \sup_{v \in G} \sum_{j \neq i} e^{e(\gamma,j)\delta_j(c_L(\cdot,\cdot))} =: \tilde{M} \leq 0 \) for some translation-invariant increasing semi-metric \( \rho \) in \( G \). Then we can prove the following

**Proposition 3.5** Let \( f \) be a local observable on \( \Omega' \) and \( (S_t^L)_{t \geq 0} \) associated to the rotation generator \( L \). For all polynomials \( p \) on \( \mathbb{R}^+ \) we have \( S_t^L f \in D_{\rho,\gamma}(\Omega') \).

**Proof.** Let us consider the monomials \( x^m \). It suffices to look at \( n = 2^m \) for some \( m \in \mathbb{N} \). We know from Theorem 3.9. in [23]
\[
\sum_{i \geq 0} \theta(i,0)^m \delta_i(\gamma_i^L f) \leq \sum_{i \geq 0} \theta(i,0)^m [e^{\gamma T} \delta_i(f)](i)
\]
where \( [\gamma \delta_i(f)](i) := \sum_{j \neq i} \delta_i(c_L(\cdot,\gamma)) \delta_j(f) \). There exists a constant \( K_{m,\rho} \) such that for fixed \( j, m \in \mathbb{N} \) we have \( \theta(i,j)^m \leq K_{m,\rho} e^{\theta(i,j)} \). Of course local \( f \in D_{\rho,\gamma}(\Omega') \) for all \( m \in \mathbb{N} \) and also for exponential weight. Under the above condition on the jump-rates, the operator \( \Gamma \) is bounded as well in the exponential weighted triple-semi-norm with norm \( \tilde{M} \), indeed
\[
\|\Gamma\|_{ee} = \sup_{\|a\|_{ee} \leq 1} \frac{\|\Gamma a\|_{ee}}{\|a\|_{ee}} = \sup_{\|a\|_{ee} \leq 1} \frac{\sum_{j \geq 0} \sum_{j \neq i} e^{\theta(i,j)0} \delta_i(c_L(\cdot,\gamma)) a_j}{\sum_{j \geq 0} e^{\theta(j,0)} a_j} \leq \frac{\tilde{M} \sum_{j \geq 0} e^{\theta(j,0)} a_j}{\sum_{j \geq 0} e^{\theta(j,0)} a_j} = \tilde{M}.
\] (45)

Then we can write
\[
\sum_{i \geq 0} \theta(i,0)^m [e^{\gamma T} \delta_i(f)](i) \leq K_{m,\rho} \sum_{i \geq 0} e^{\theta(i,j)0} [e^{\gamma T} \delta_i(f)](i) = K_{m,\rho} \|e^{\gamma T} \delta_i(f)\|_{ee} \leq K_{m,\rho} \|e^{\gamma T} \delta(f)\|_{ee} \leq K_{m,\rho} e^{\gamma T} \|\Gamma\|_{ee} \|\delta_i(f)\|_{ee}.
\]

In particular for local \( f \), we have \( S_{t,\varepsilon}^L f \in D_{\rho,\gamma}(\Omega') \subset L^\prime_{\rho,\gamma} \) for all polynomial and even exponential weights \( \rho \). In other words, we can control the diffusion of the semi-group applied to a local function by looking at the decay property of the conditional Dobrushin matrix as well as of the first-layer Hamiltonian. In particular if those are well behaved (which is the case for the XY-model with some slightly refined coarse-graining) no second-order blow-ups can appear and we can conclude \( \mu_{t+\varepsilon}^f = S_{t,\varepsilon}^L \mu_t^f \) for all extremal Gibbs measures labelled by \( t \in S^1 \).
4 Reversible dynamics for discrete-spin models

A reversible dynamics $K$ in the infinite volume with finite local state space and given specification $\gamma$ can be defined by

$$(K\psi)(\omega') = \sum_{i \in \Lambda} [c_K(\omega', (\omega')^i)(\psi(\omega')^i - \psi(\omega')) + c_K(\omega', (\omega')^{i_2})((\psi(\omega')^{i_2}) - \psi(\omega'))]$$

with $(\omega')^{i_2}$ being the discrete configuration which coincides with $\omega'$ except at the site $i$ where it is decreased by the amount of one unit. With rates satisfying

$$\frac{c_K(\omega', (\omega')^i)}{c_K((\omega')^i, \omega')} = \frac{\mu_G[i(\omega')^i]}{\mu_G[i]} \frac{\mu_G[i]}{\mu_G[i]} \frac{\lambda^i(e^{-H_i(\omega')^i})}{\lambda^i(e^{-H_i(\omega')^i})}.$$ \hspace{1cm} (46)

Thus the corresponding infinite-volume generator $K$ has detailed balance with respect to the specification, which was stated in Proposition (1.4) part 2.

These rates are bounded (by boundedness of $H_j$), translation-invariant (by the translation-covariance of the $\mu_G[j]$, in the conditional Dobrushin regime) and of exponentially decaying influence (however not strictly local).

The rates are uniformly bounded and bounded in the triple-norm by the same arguments as used for the rotation dynamics, so Proposition (1.4) part 1 is true.

In the next subsection we adapt a line of arguments presented for $q = 2$ in [26] for general finite $q$.

4.1 Translation-invariant invariant measures are Gibbs measures

Let us put ourselves in dimension $d \geq 3$. In the right temperature region there are multiple Gibbs measures, ferromagnetically ordered on $S^1$, in the initial continuous system.

Since in the following subsections we will only deal with second-layer configurations it is convenient to suppress the primes and write $c_K(\omega, \omega^i)$ for the up-flip at site $i \in G$ and $c_K(\omega, \omega^{i_2})$ for the down-flip. Assume the rates to be defined as in (46), in particular for the corresponding process we have $\mathcal{G}(\gamma') \subseteq \mathcal{I}_K$.

We show $\mathcal{I}_K \cap \mathcal{S} \subseteq \mathcal{G}(\gamma')$, i.e invariant measures w.r.t. $K$ that are also translation-invariant are Gibbs measures. This is precisely part 3 of Proposition (1.4). We use Holleys argument [19]. Recall the definition of the second-layer specification and define the local relative entropy

$$H_\Lambda(\nu|\gamma_\Lambda(\cdot|\zeta)) := \sum_{\omega \in \{1, \ldots, q\}^\Lambda} \nu(1_\omega) \log \frac{\nu(1_\omega)}{\gamma_\Lambda(\cdot|\zeta)}.$$ \hspace{1cm} (47)

where $\Lambda \subseteq \mathbb{Z}^d$ is finite, $\nu \in \mathcal{P}(\Omega')$ and $\zeta \in \Omega'$ an arbitrary but fixed boundary condition. Let $(S^K_t)_{t \geq 0}$ be the semigroup for the generator $K$ and define $\nu_t := \nu S^K_t$. 

22
Let us compute \( \frac{d}{dt} H_{\Lambda}(\nu|\gamma_{\Lambda}^t(\cdot|\zeta))_{t=0} \) in two parts

\[
\frac{d}{dt}_{|t=0} \sum_\omega \nu_t(1_\omega) \log \nu_t(1_\omega) = \sum_\omega [1 + \log \nu(1_\omega)] \int K1_\omega d\nu \\
= \sum_{\omega, i \in \Lambda} \log \nu(1_\omega) \int \nu(d\eta) [c_K(\eta, \eta^i)(1_\omega(\eta^i) - 1_\omega(\eta)) + c_K(\eta, \eta^{i-})(1_\omega(\eta^{i-}) - 1_\omega(\eta))] \\
= \sum_{\omega, i \in \Lambda} \left[ \Gamma(\omega, i^+) \log \frac{\nu(1_\omega)}{\nu(1_\omega)} + \Gamma(\omega, i^-) \log \frac{\nu(1_\omega^-)}{\nu(1_\omega)} \right]
\]

where we wrote \( \Gamma(\omega, i^+) := \int \nu(d\eta) c_K(\eta, \eta^{i^+}) 1_\omega(\eta) \) for the outflows of \( 1_\omega \) in the direction \( i^+ \).

Note if \( \nu \in \mathcal{I}_K \) then \( \nu(1_\omega) > 0 \) for all \( \omega \in \{1, ..., q\}^\Lambda \), indeed

\[
0 = \int K1_\omega d\nu \\
= \sum_{i \in \Lambda} \int d\nu(d\eta) [c_K(\eta, \eta^i)(1_\omega^i(\eta) - 1_\omega^1(\eta)) + (\eta, \eta^-)(1_\omega^1(\eta) - 1_\omega^1(\eta))].
\]  

(48)

Since all flip-rates are positive \( \nu(1_\omega^-) = 0 \) would imply \( \nu(1_\omega^1) = 0 = \nu(1_\omega^1^-) \) for all \( i \in \Lambda \) and thus by iteration \( \nu(1_\omega^1) = 0 \) for all \( \eta \in \{1, ..., q\}^\Lambda \) which is a contradiction to \( \nu \) being a probability measure.

Let us look at the second part of \( \frac{d}{dt} H_{\Lambda}(\nu|\gamma_{\Lambda}^t(\cdot|\zeta))_{t=0} \). Since the normalizing constant in the specification is independent of \( \omega \) and \( \sum_{\omega} \frac{d}{dt} \sum_{\omega} \nu_t(d\omega) = 0 \), we can directly compute

\[
\frac{d}{dt} \int \nu_t(d\omega) \log \mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}})) \\
= \sum_{\omega, i \in \Lambda} \int \nu(d\eta_{\Lambda^c}) \nu(\omega | \eta_{\Lambda^c}) K \log \mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}})) (\omega \eta_{\Lambda^c}) \\
= \sum_{\omega, i \in \Lambda} \left[ \log \mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}})) \int \nu(d\eta) 1_{\omega^1}(\eta) c_K(\eta, \eta^i) \\
+ \log \frac{\mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}}))}{\mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}}))} \int \nu(d\eta) 1_{\omega^1}(\eta) c_K(\eta, \eta^{i-}) \right] \\
= \sum_{\omega, i \in \Lambda} \left[ V(\omega, i^+) \Gamma(\omega, i^+) + V(\omega, i^-) \Gamma(\omega, i^-) \right]
\]

where we defined \( V(\omega, i^+) := \log \frac{\mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}}))}{\mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}}))}. Notice we have

\[
\frac{\mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}}))}{\mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}}))} = \frac{\mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}}))}{\mu_{G^i_{\Lambda}}[\zeta_{G^i_{\Lambda}}](\lambda^A(e^{-H^i_{\Lambda} 1_{\omega^1}}))} = \frac{c(\omega^1 \zeta_{\Lambda^c}, \omega^1 \zeta_{\Lambda_c})}{c(\omega^1 \zeta_{\Lambda^c}, \omega^1 \zeta_{\Lambda_c})}.
\]
Combining the two parts we have \( \frac{d}{dt}H_\Lambda(\nu | \gamma'_\Lambda(\cdot | \zeta))_{t = 0} = \)

\[
\sum_{\omega, i \in \Lambda} \left[ \Gamma(\omega, i^+) \left( \log \frac{\nu(1, \omega)}{\nu(1, \omega)} - V(\omega, i^+) \right) + \Gamma(\omega, i^-) \left( \log \frac{\nu(1, \omega^-)}{\nu(1, \omega)} - V(\omega, i^-) \right) \right].
\] (50)

Since \( \log \frac{\nu(1, \omega)}{\nu(1, \omega)} - V(\omega, i^+) = - (\log \frac{\nu(1, \omega)}{\nu(1, \omega)} - V(\omega, i^-)) \) we can write

\[
2 \frac{d}{dt}H_\Lambda(\nu | \gamma'_\Lambda(\cdot | \zeta))_{t = 0} = \sum_{\omega, i \in \Lambda} \left[ (\Gamma(\omega, i^+) - \Gamma(\omega, i^-))(\log \frac{\nu(1, \omega)}{\nu(1, \omega)}) - V(\omega, i^+) \right)
+ (\Gamma(\omega, i^-) - \Gamma(\omega, i^+))(\log \frac{\nu(1, \omega^-)}{\nu(1, \omega)} - V(\omega, i^-))].
\] (51)

Adding zeros we have \( 2 \frac{d}{dt}H_\Lambda(\nu | \gamma'_\Lambda(\cdot | \zeta))_{t = 0} = \)

\[
- \sum_{\omega, i \in \Lambda} \left[ (\Gamma(\omega, i^+) - \Gamma(\omega, i^-))(\log \frac{\Gamma(\omega, i^+)}{\Gamma(\omega, i^-)}) + (\Gamma(\omega, i^-) - \Gamma(\omega, i^+))(\log \frac{\Gamma(\omega, i^-)}{\Gamma(\omega, i^+)}) \right]
+ \sum_{\omega, i \in \Lambda} \left[ (\Gamma(\omega, i^+) - \Gamma(\omega, i^-))(\log \frac{\Gamma(\omega, i^+)}{\Gamma(\omega, i^-)}) - (\log \frac{\nu(1, \omega^-)}{\nu(1, \omega)} - V(\omega, i^+)) \right]
+ (\Gamma(\omega, i^-) - \Gamma(\omega, i^+))(\log \frac{\Gamma(\omega, i^-)}{\Gamma(\omega, i^+)} - \log \frac{\nu(1, \omega^+)}{\nu(1, \omega^-)} - V(\omega, i^-)).
\]

If \( \nu \in \mathcal{I}_K \) it follows

\[
\sum_{\omega, i \in \Lambda} \left[ (\Gamma(\omega, i^+) - \Gamma(\omega, i^-))(\log \frac{\Gamma(\omega, i^+)}{\Gamma(\omega, i^-)}) + (\Gamma(\omega, i^-) - \Gamma(\omega, i^+))(\log \frac{\Gamma(\omega, i^-)}{\Gamma(\omega, i^+)}) \right]
= \sum_{\omega, i \in \Lambda} \left[ (\Gamma(\omega, i^+) - \Gamma(\omega, i^-))(\log \frac{\Gamma(\omega, i^+)}{\Gamma(\omega, i^-)}) - (\log \frac{\nu(1, \omega^-)}{\nu(1, \omega)} - V(\omega, i^+)) \right]
+ (\Gamma(\omega, i^-) - \Gamma(\omega, i^+))(\log \frac{\Gamma(\omega, i^-)}{\Gamma(\omega, i^+)} - \log \frac{\nu(1, \omega^+)}{\nu(1, \omega^-)} - V(\omega, i^-)).
\]

where the left hand side is non-negative. We want to exploit properties of the \( d \)-dimensional lattice in order to show the r.h.s. of the last equation goes to zero for \( \Lambda \not\ni G \). Let us define

\[
\kappa_\Lambda(i^\pm) := \sum_\omega (\Gamma(\omega, i^\pm) - \Gamma(\omega^\pm, i^\mp)) \log \frac{\Gamma(\omega, i^\pm)}{\Gamma(\omega^\pm, i^\mp)}
\]
\[
\beta_\Lambda(i^\pm) := \sum_\omega \Gamma(\omega, i^\pm) - \Gamma(\omega^\pm, i^\mp)
\]
\[
\vartheta_\Lambda(i) := \sum_{j \not\in \Lambda} \sup_{\eta_j = \tilde{\eta}_j \neq i_j} \left| c(\eta, i^\pm) - c(\tilde{\eta}, i^\mp) \right| + \sum_{j \not\in \Lambda} \sup_{\eta_j = \tilde{\eta}_j \neq i_j} \left| c(\eta, i^-) - c(\tilde{\eta}, i^+ \mp) \right|.
\] (52)
We estimate $-V(\omega, i^+) + \log \frac{\Gamma(\omega, i^+)}{\nu(\omega)} - \log \frac{\Gamma(\omega, i^-)}{\nu(\omega)} =$

$$\sum \log \frac{c_K(\omega, i^+)}{\nu(\omega)} - \sum \log \frac{c_K(\omega, i^-)}{\nu(\omega)} \leq \sup \left\{ \log \frac{c_K(\eta_1, \eta_1^+)}{c_K(\eta_2, \eta_2^+)} : \eta_1 = \eta_2 \text{ on } \Lambda \right\} + \sup \left\{ \log \frac{c_K(\eta_1, \eta_1^-)}{c_K(\eta_2, \eta_2^-)} : \eta_1 = \eta_2 \text{ on } \Lambda \right\} \leq \vartheta_\Lambda(i) \text{.}$$

Using $\log a \leq a - 1$ and expressing the oscillation on $\Lambda^c$ via single-point oscillations we arrive at $\vartheta_\Lambda(i)$. Similarily for the second summand:

$$- V(\omega, i^-) + \log \frac{\Gamma(\omega, i^-)}{\nu(\omega)} - \log \frac{\Gamma(\omega, i^+)}{\nu(\omega)} =$$

$$\leq \sup \left\{ \log \frac{c_K(\eta_1, \eta_1^-)}{c_K(\eta_2, \eta_2^-)} : \eta_1 = \eta_2 \text{ on } \Lambda \right\} + \sup \left\{ \log \frac{c_K(\eta_1, \eta_1^+)}{c_K(\eta_2, \eta_2^+)} : \eta_1 = \eta_2 \text{ on } \Lambda \right\} \leq \vartheta_\Lambda(i) \text{.}$$

Hence

$$\sum_{i \in \Lambda} (\kappa_\Lambda(i^+) + \kappa_\Lambda(i^-)) \leq \sum_{i \in \Lambda} [(\beta_\Lambda(i^+) + \beta_\Lambda(i^-)) \vartheta_\Lambda(i)] \text{.}$$

Notice $\vartheta_\Lambda(i) \to 0$ for all $i \in G$ as $\Lambda \nearrow G$ since our flip-rates are quasilocal and summable, indeed by the well-definedness we have for all $i \in G$

$$\sum_{j \neq i} \sup_{\eta_c = \tilde{\eta}_c} \left| \frac{c_K(\eta, \eta^i) - c_K(\tilde{\eta}, \tilde{\eta}^i)}{c_K(\eta, \eta^i)} \right| \leq \frac{2\pi e^{|H_0|}}{\min_{i \in \Lambda} \lambda(l)} \sum_{j \in G} \sup_{\eta_c = \tilde{\eta}_c} \left| c_K(\eta, \eta^i) - c_K(\tilde{\eta}, \tilde{\eta}^i) \right| =: A \times B^+ < \infty \text{.}$$

Notice also that $\kappa_\Lambda(i^+) \leq \kappa_\Lambda(i^\pm)$ if $i \in \Lambda_1 \subset \Lambda_2$. Indeed if we look at the subadditive function $\varphi(x, y) = (x - y) \log \frac{x}{y}$ for $x, y > 0$ and use

$$\Gamma_{\Lambda_1}(\omega_1, i^\pm) = \sum_{\omega_2 = \omega_1 \text{ on } \Lambda_1} \Gamma_{\Lambda_2}(\omega_2, i^\pm)$$

we have

$$\kappa_{\Lambda_1}(i^\pm) = \sum_{\omega_1 \in \Lambda_1} \varphi[\Gamma_{\Lambda_1}(\omega_1, i^\pm), \Gamma_{\Lambda_1}(\omega_1^\pm, i^\mp)] \leq \sum_{\omega_2 \in \Lambda_2} \varphi[\Gamma_{\Lambda_2}(\omega_2, i^\pm), \Gamma_{\Lambda_2}(\omega_2^\pm, i^\mp)] = \kappa_{\Lambda_2}(i^\mp) \text{.}$$

\textbf{Theorem 4.1} Suppose that $G = \mathbb{Z}^d$ and the Glauber dynamics flip-rates

$$\frac{c_K(\omega', (\omega')^i)}{c_K((\omega')^i, \omega')} = \frac{\mu_G(\omega')}{\mu_G(\omega')} \frac{\lambda^i(e^{-H_1(\omega')^i})}{\lambda^i(e^{-H_1(\omega')^i})} \text{.}$$

are defined for a translation-invariant first-layer potential $H$. Then $\mathcal{I}_K \cap \mathcal{S} \subset \mathcal{G}(\gamma')$. 

\text{25}
Proof. Let \( \nu \in \mathcal{I}_K \cap \mathcal{S} \) and \( \Lambda_n \) cubes in \( \mathbb{Z}^d \) of side length \( n \) then we have

\[
\frac{1}{(kn)^d} \sum_{i \in \Lambda_n} \left[ \kappa_{\Lambda_n}(i^+) + \kappa_{\Lambda_n}(i^-) \right] \geq \frac{1}{n^d} \sum_{i \in \Lambda_n} \left[ \kappa_{\Lambda_n}(i^+) + \kappa_{\Lambda_n}(i^-) \right].
\]

(55)

On the other hand \( \beta_\Lambda(i^+) \) and \( \beta_\Lambda(i^-) \) are uniformly bounded and

\[
\frac{1}{n^d} \sum_{i \in \Lambda_n} \partial_{\Lambda_n}(i) \leq A \sum_{i \in \Lambda_n} \sum_{j \notin \Lambda_n} \left[ \delta_j(c_K(\cdot, i^+)) + \delta_j(c_K(\cdot, i^-)) \right]
\]

\[
= A \sum_{i \in \Lambda_n} \sum_{j \notin \Lambda_n} \left[ \delta_{j-1}(c_K(\cdot, 0^+)) + \delta_{j-1}(c_K(\cdot, 0^-)) \right]
\]

\[
= A \sum_{i \in \mathbb{Z}^d} [\delta_i(c_K(\cdot, 0^0)) + \delta_i(c_K(\cdot, 0^-))] \frac{\# \{ i \in \Lambda_n : i + l \notin \Lambda_n \}}{n^d}.
\]

(56)

This tends to zero since the oscillations are bounded by \( B^\pm \) and the fact that an increasing strip of boundary of cubes goes to infinity slower than the volume. Together we have

\[
\frac{1}{n^d} \sum_{i \in \Lambda_n} \left[ \kappa_{\Lambda_n}(i^+) + \kappa_{\Lambda_n}(i^-) \right] \leq \frac{1}{n^d} \sum_{i \in \Lambda_n} \left[ (\beta_{\Lambda_n}(i^+) + \beta_{\Lambda_n}(i^-)) \partial_{\Lambda_n}(i) \right]
\]

\[
\leq C \frac{1}{n^d} \sum_{i \in \Lambda_n} \partial_{\Lambda_n}(i) \to 0 \text{ for } n \to \infty
\]

(57)

and hence by non-negativity of \( \kappa_\Lambda(i^\pm) \) we have \( \kappa_{\Lambda_n}(i^+) = \kappa_{\Lambda_n}(i^-) = 0 \) for all \( i \in \Lambda_n \), and by the subadditivity argument \( \kappa_\Lambda(i^+) = \kappa_\Lambda(i^-) = 0 \) for all \( \Lambda \) and \( i \in \Lambda \). Hence we have for all finite \( \Lambda, \omega \in \{1, \ldots, q\}^\Lambda \) and \( i \in \Lambda \)

\[
0 = \Gamma(\omega, i^+) - \Gamma(\omega^i, i^-)
\]

\[
= \int \nu(d\eta)[\nu(\omega_{i\Lambda}; \eta_{\Lambda^c})c_K(\omega_{i\Lambda}^{\Lambda^c}, \omega_{\Lambda}^{\Lambda^c}) - \nu(\omega^i_{i\Lambda}; \eta_{\Lambda^c})c_K(\omega_{\Lambda^c}^{\Lambda^c}, \omega_{\Lambda}^{\Lambda^c})].
\]

So \( \nu \)-a.s. we can write

\[
\frac{\nu(\omega^i_{i\Lambda}; \eta_{\Lambda^c})}{\nu(\omega_{i\Lambda}; \eta_{\Lambda^c})} = \frac{c_K(\omega_{i\Lambda}^{\Lambda^c}, \omega_{\Lambda}^{\Lambda^c})}{c_K(\omega_{\Lambda^c}^{\Lambda^c}, \omega_{\Lambda}^{\Lambda^c})} = \frac{\mu_{G,i}[\eta_{\Lambda^c}; \omega_{\Lambda^c}]}{\mu_{G,i}[\eta_{\Lambda}; \omega_{\Lambda}]}(\lambda(e^{-H_1 \omega^i})).
\]

Since we compare discrete measures on sites \( i \in G \) it follows by the remark below, \( \nu \) almost everywhere \( \nu(\omega_{i}; \omega_{i^\prime}) = \gamma'(\omega_{i}; \omega_{i^\prime}) \) and thus \( \nu \in G(\gamma') \).

Remark 4.2 Let \( (a_1, \ldots, a_q) \) and \( (b_1, \ldots, b_q) \) be probability vectors with \( \frac{a_k}{a_{k+1}} = \frac{b_k}{b_{k+1}} \) for all \( k \in \{1, \ldots, q\} \), then we have \( \frac{a_k}{a_{k+1}} = \frac{b_k}{b_{k+1}} \) for all \( k, l \in \{1, \ldots, q\} \) and thus

\[
a_l = \frac{a_l}{\sum_{k=1}^q a_k} = \frac{1}{1 + \sum_{k \neq l} \frac{a_k}{a_l}} = \frac{1}{1 + \sum_{k \neq l} \frac{b_k}{b_l}} = b_l.
\]

(58)
5 Joint dynamics

Let us now consider the joint dynamics $L + \alpha K$ for $\alpha > 0$. Of course well-definedness (Proposition (1.5) part 1) follows directly from the fact, that the individual rates of $L$ and $K$ are well-defined.

As a warning we note, the generators $L$ and $K$ do not commute (except in the limit $q \to \infty$). To see this we apply $LK - KL$ to the local observable $\psi := 1_{\eta_A}$ for a finite $\Lambda \subset G$ and evaluated for instance at $\omega_\Lambda = \eta_\Lambda$, we find

$$
\sum_{i \in \Lambda} \left( c_L(\omega, \omega^i)c_K(\omega^i, \omega) - c_K(\omega, \omega^{i-})c_L(\omega^{i-}, \omega) \right)
= \sum_{i \in \Lambda} \left( \mu_{G, i}[\omega_{G', i}] (e^{-H_i((\omega_i)^+, \cdot)} - \mu_{G, i}[\omega_{G', i}] (e^{-H_i((\omega_i^i)^-, \cdot)}) \right) \tag{59}
$$

Notice the direction of rotation is such that $(\omega_i)^i = (\omega_i^{i-})^r$. One can isolate and evaluate the other summands in a similar fashion by looking at $\omega_\Lambda = \eta_\Lambda$, $\omega_\Lambda^i = \eta_\Lambda$, $\omega_\Lambda^{ij} = \eta_\Lambda$ for $i, j \in \Lambda$.

In general the commutator is thus not zero, but if we consider the limit of the coarse-graining, i.e. letting $q$ the number of discrete states go to infinity, we reach a commutative setting. This result reflects the continuum situation in the Maes-Shlosman program [24].

As a consequence $S_{L^a}^{L+\alpha K} \neq S_{L^a}^L S_{L^a}^K$ and it is not immediate that the joint dynamics also rotates the discrete Gibbs measures in the sense of Proposition (1.5) part 2. To see that this is nevertheless true one has to follow the same arguments as in section 3.2 and notice $\|\Gamma^{\text{joint}}\|_{\epsilon^*} < \infty$.

5.1 The invariant measure for the joint dynamics

In this subsection we show Proposition (1.5) part 3 and Corollary (1.6). First let us verify that indeed the symmetrically mixed measure is invariant and in the set of Gibbs measures this is the only one. Finally we prove $\mathcal{I}_{L+\alpha K} \subset \mathcal{G}({\gamma'})$. The strategy for this is the following: We use the relative entropy argument again. The bulk contribution of the rotation dynamics is strictly decreasing, this we show via an open boundary version of $L$. In other words the rotation in the bulk only ‘helps’ $K$. The error we make by using the approximation has the right order.

The mixture of all translation-invariant extremal Gibbs measures $\mu'_t$

$$
\mu'_t := \frac{1}{2\pi} \int_0^{2\pi} \mu'_s dt
$$

is invariant for the rotation dynamics and hence for the joint dynamics $L + \alpha K$. Indeed, let $(S'_t)_{t \geq 0}$ be the semigroup for $L$ and $f$ a quasilocal observable we have

$$
\int S'_t f(\eta) \mu'_s(\eta) d\eta = \frac{1}{2\pi} \int_0^{2\pi} \int S'_t f(\eta) \mu'_s(\eta) d\eta ds = \int f(\eta) \frac{1}{2\pi} \int_0^{2\pi} \mu'_{s+t}(d\eta) ds = \int f(\eta) \mu'_s(\eta).
\tag{60}
$$
Proposition 5.1 There are no translation-invariant invariant Gibbs measures for the rotation dynamics other than $\mu_s'$.

Proof. We know every Gibbs measure $\mu' \in \mathcal{G}_\Theta(\gamma')$ has a unique representation

$$\mu' = \int_{ex\mathcal{G}(\gamma')} \bar{\mu} w_{\mu'}(d\bar{\mu})$$

where $w_{\mu'} \in \mathcal{P}(ex\mathcal{G}(\gamma'), \sigma(ex\mathcal{G}(\gamma']))$ and $\sigma(\mathcal{P})$ is the so-called evaluation $\sigma$-algebra. Since the Gibbs measures can be ordered as described above, there is a bijection

$$b : ex\mathcal{G}_\Theta(\gamma') \to [0, 2\pi], \quad \mu' \mapsto -i \log \left( \frac{e_{\tilde{m}'}(\mu')}{m_\beta} \right)$$

where $b$ is $(\sigma(ex\mathcal{G}_\Theta(\gamma')), \mathcal{B}([0, 2\pi]))$ measurable. Indeed since $\tilde{m}'_0$ is bounded and measurable, so is $e_{\tilde{m}'} : \mu' \mapsto \mu'(\tilde{m}'_0)$ and thus $b(\mu') = -i \log \left( \frac{e_{\tilde{m}'}(\mu')}{m_\beta} \right)$ is a composition of measurable functions. Hence we can consider image measures $\nu_{\mu'}$ of $w_{\mu'}$ under $b$.

On the other hand for all local coarse-grained sets $A' \in \mathcal{F}'$ the mapping

$$c_{A'} : [0, 2\pi, \omega^t] \to [0, 1], \quad t \mapsto \mu'_t(A') = \mu_t(A) = \lim_{\Lambda_i \to \omega^t} \gamma_{\Lambda}(A|\omega^t)$$

where $A := T^{-1}(A')$ and $\omega^t$ the homogeneous first-layer configuration in the direction $e^t$ is borel-measurable as a composition of measurable maps, where we also used the measurability of $t \mapsto \omega^t$. Hence this is true for all $A' \in \mathcal{F}'$.

By the transformation theorem for measurable maps we have for all $A' \in \mathcal{F}'$

$$\mu'(A') = \int_{ex\mathcal{G}(\gamma')} \bar{\mu}(A') w_{\mu'}(d\bar{\mu}) = \int_{ex\mathcal{G}(\gamma')} c_{A'}(b(\bar{\mu})) w_{\mu'}(d\bar{\mu})$$

$$= \int_{0}^{2\pi} c_{A'}(t) w_{\mu'}(b^{-1}(dt)) = \int_{0}^{2\pi} c_{A'}(t) v_{\mu'}(dt) = \int_{0}^{2\pi} \mu'_t(A') v_{\mu'}(dt). \tag{61}$$

By looking at tail measurable interval sets

$$A_{[0,u)} := \{ \omega \in \Omega : \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{j \in \Lambda_n} -i \log \frac{\omega_j}{m_\beta} = [0, u) \}$$

and $\varphi_{[0,u)}(\omega') := \mu_G[\omega'](A_{[0,u)})$ we see, $\nu_{\mu'}$ has to be a translation-invariant Borel-measure, indeed

$$v_{\mu'}([0, u)) = \int_{0}^{2\pi} \mu'_t(\varphi_{[0,u)}) v_{\mu'}(dt) = \mu'(\varphi_{[0,u)}) = S^L_s \mu'(\varphi_{[0,u)})$$

$$= \int_{0}^{2\pi} \mu'_t(\varphi_{[-s,u-s)}) v_{\mu'}(dt) = v_{\mu'}([s, u-s)) \tag{62}$$

for all $s \in [0, 2\pi)$. Since $\{[0, u) : u \in [0, 2\pi)\}$ is a generator for the Borel-$\sigma$-algebra and $v_{\mu'}$ is a probability measure we have $v_{\mu'}(dt) = \frac{1}{2\pi} \lambda(dt)$.

Since $S^L_{s+dK} \mu'_t = \mu'_{t+s} = S^L_s \mu'_t$ we know $\mathcal{T}_L \cap \mathcal{G}_\Theta(\gamma') = \{ \mu'_s \}$ as well. The next proposition proves Proposition (1.5) part 3.
**Proposition 5.2** Every translation-invariant measure that is invariant for the joint dynamics \(L + \alpha K\) with \(\alpha > 0\) is a Gibbs measure.

**Proof.** Let \(\Lambda \subset \mathbb{Z}^d\) be a finite set and \(\zeta \in \Omega'\) an arbitrary but fixed boundary condition for the second-layer specification, i.e. consider the coarse-grained measure \(\gamma^\Lambda_\zeta(\omega) = \frac{\mu_{\Lambda}\llbracket \zeta_{\Lambda}^\Lambda \rrbracket(\lambda^\Lambda(e^{-H_1}1_\omega))}{\mu_{\Lambda}\llbracket \zeta_{\Lambda}^\Lambda \rrbracket(\lambda^\Lambda(e^{-H_1}1_\Lambda))}\) on \(\{1, \ldots, q\}^\Lambda\). Our strategy for the proof is to look at the derivative of the local relative entropy \(H_\Lambda(\nu|\gamma^\Lambda_\zeta) := \sum_{\omega \in \{1, \ldots, q\}^\Lambda} \nu(1_\omega) \log \frac{\nu(1_\omega)}{\gamma^\Lambda_\zeta(1_\omega)}\) again for \(\nu\) translation-invariant and invariant with respect to the joint dynamics. We have seen in case of the Glauber dynamics how to verify Gibbsianness for invariant measures by estimating certain terms in the derivative of the local relative entropy. Those terms are only of the order of the boundary \(\partial \Lambda\) and thus increase slower than the volume \(\Lambda\). This allows us to prove the DLR equality for the invariant measure. A crucial ingredient of course is the translation-invariance of both, the model as well as the invariant measure.

Essentially we follow the same line of arguments here, taking special care of the contribution of the rotation. We look at an approximating local open boundary rotation dynamics and show its relative entropy is decreasing. This means the approximating rotation only "helps" the Glauber dynamics argument. The error we make by using the approximation instead of the infinite-volume rotation dynamics is only of boundary order and thus again increases slower than the volume.

Since the time-derivative of the local relative entropy is additive as a sum of the two terms corresponding to the two generators \(K\) and \(L\), we can calculate separately for the Glauber- and for the rotation dynamics. We write \(\nu_{t,K}\) (resp. \(\nu_{t,L}\), \(\nu_{t,L+\alpha K}\)) for the measure \(\nu\) propagated only by the rotation (resp. by the Glauber dynamics, by the joint dynamics).

Let us compute for the rotation \(\frac{d}{dt} H_\Lambda(\nu_{t,L}|\gamma^\Lambda_\zeta)|_{t=0}\). Again we do this in two parts. Similarly to (4.11) we find

\[
\frac{d}{dt}_{|t=0} \sum_{\omega} \nu_{t,L}(1_\omega) \log \nu_{t,L}(1_\omega) = \sum_{\omega, i \in \Lambda} \Gamma_L(\omega, i^+) \log \frac{\nu(1_\omega)}{\nu(1_\omega)}
\]  

(63)

where we again wrote \(\Gamma_L(\omega, i^+) := \int \nu(d\eta) c_L(\eta, \eta^+ 1_\omega)\) for the outflows of \(1_\omega\) in the direction \(i^+\). For the second part of the summand of \(\frac{d}{dt} H_\Lambda(\nu_{t,L}|\gamma^\Lambda_\zeta)|_{t=0}\) we find similarly to (49)

\[
\frac{d}{dt} \int \nu_{t,L}(d\omega) \log \mu_{\Lambda}\llbracket \zeta_{\Lambda}^\Lambda \rrbracket(\lambda^\Lambda(e^{-H_1}1_\omega)) = \sum_{\omega, i \in \Lambda} V^\zeta(\omega, i^+) \Gamma_L(\omega, i^+)
\]  

(64)

where we again defined \(V^\zeta(\omega, i^+) := \log \frac{\mu_{\Lambda}\llbracket \zeta_{\Lambda}^\Lambda \rrbracket(\lambda^\Lambda(e^{-H_1}1_\omega))}{\mu_{\Lambda}\llbracket \zeta_{\Lambda}^\Lambda \rrbracket(\lambda^\Lambda(e^{-H_1}1_\Lambda))}\). Together we have

\[
\frac{d}{dt} H_\Lambda(\nu_{t,L}|\gamma^\Lambda_\zeta)|_{t=0} = \sum_{\omega, i \in \Lambda} \Gamma_L(\omega, i^+) \left( \log \frac{\nu(1_\omega^+)}{\nu(1_\omega)} - V^\zeta(\omega, i^+) \right).
\]  

(65)
We define the approximating local generator $\tilde{L}_\Lambda$ via the following open boundary rates
\[
c_{\tilde{L}_\Lambda}(\eta, \eta^i) := \begin{cases} 
\frac{\lambda^\Lambda(e^{-\tilde{H}_\Lambda(\omega^i)})_1}{\lambda^\Lambda(e^{-\tilde{H}_\Lambda(\eta)})}, & \text{if } i \in \Lambda \\
0, & \text{if } i \in \Lambda^c
\end{cases}
\]
where $\Lambda$ is a fixed finite volume and $\tilde{H}_\Lambda := \sum_{i \in \Lambda} \Phi_i$ is the open boundary Hamiltonian for $\Lambda$ in the first-layer model. Let $(S^\Lambda_t)_{t \geq 0}$ be the corresponding semigroup. Since we assume the underlying first-layer potential to be rotation-invariant, the open boundary measure $\tilde{\gamma}_\Lambda(\omega) := \frac{\lambda^\Lambda(e^{-\tilde{H}_\Lambda(\omega)})}{\lambda^\Lambda(e^{-\tilde{H}_\Lambda})}$ on $\{1, ..., q\}^\Lambda$ is invariant for $\tilde{L}_\Lambda$.

Indeed for all $\omega \in \{1, ..., q\}^\Lambda$ we have
\[
\tilde{\gamma}_\Lambda(\tilde{L}_\Lambda(\omega)) = \sum_{i \in \Lambda} [\lambda^\Lambda, \omega_{\Lambda \setminus i}] - [\lambda^\Lambda, \omega_{\Lambda \setminus i}]
\]
\[
= \frac{d}{de} \int \tilde{\gamma}_\Lambda(1_{\omega + e}) = \frac{d}{de} \int \tilde{\gamma}_\Lambda(1_{\omega}) = 0.
\]

We can employ a standard argument for the decrease of relative entropy in finite volume in order to determine the sign of $\frac{d}{dt} H_\Lambda(\nu, \tilde{L}_\Lambda | \tilde{\gamma}_\Lambda)_{t = 0}$. Indeed if we use the convex function $\psi(x) = x \log x + x - 1$ the relative entropy reads
\[
H_\Lambda(\nu, \tilde{L}_\Lambda | \tilde{\gamma}_\Lambda) = \sum_\omega \tilde{\gamma}_\Lambda(1_{\omega}) \psi\left( \frac{\nu(1_{\omega})}{\tilde{\gamma}_\Lambda(1_{\omega})} \right)
\]
\[
= \sum_\omega \tilde{\gamma}_\Lambda(1_{\omega}) \psi\left( \frac{1}{\tilde{\gamma}_\Lambda(1_{\omega})} \sum_\eta S^\Lambda_t(1_{\omega})(\eta) \frac{\nu(\eta)}{\tilde{\gamma}_\Lambda(\eta)} \tilde{\gamma}_\Lambda(\eta) \right)
\]
where $\frac{S^\Lambda_t(1_{\omega})}{\tilde{\gamma}_\Lambda(1_{\omega})} \tilde{\gamma}(d\eta) = \frac{1_{\omega}}{\tilde{\gamma}_\Lambda(1_{\omega})} \tilde{\gamma}(d\eta)$ is a probability measure. Hence we can use Jensen's inequality and obtain
\[
H_\Lambda(\nu, \tilde{L}_\Lambda | \tilde{\gamma}_\Lambda) \leq \sum_\omega \tilde{\gamma}(1_{\omega}) \frac{1}{\tilde{\gamma}(1_{\omega})} \sum_\eta S^\Lambda_t(1_{\omega})(\eta) \psi\left( \frac{\nu(\eta)}{\tilde{\gamma}_\Lambda(\eta)} \right) \tilde{\gamma}_\Lambda(\eta)
\]
\[
= \sum_\omega \psi\left( \frac{\nu(\omega)}{\tilde{\gamma}_\Lambda(\omega)} \right) \tilde{\gamma}_\Lambda(\omega) = H_\Lambda(\nu | \tilde{\gamma}_\Lambda)
\]
with equality iff $\nu, \tilde{L}_\Lambda = \tilde{\gamma}_\Lambda$. Thus the derivative must be non-positive
\[
0 \geq \frac{d}{dt} H_\Lambda(\nu, \tilde{L}_\Lambda | \tilde{\gamma}_\Lambda)_{t = 0} = \sum_{\omega, i \in \Lambda} \nu(1_{\omega}) c_{\tilde{L}_\Lambda}(\omega, \omega^i)(\log \nu(1_{\omega}) - \log \tilde{\gamma}_\Lambda(\omega^i))
\]
\[
=: \sum_{\omega, i \in \Lambda} \Gamma_{\tilde{L}_\Lambda}(\omega, i^+) (\log \nu(1_{\omega}) - \nu(1_{\omega}) - V_{\tilde{L}_\Lambda}(\omega, i^+)).
\]

We are going to show $\frac{d}{dt} H_\Lambda(\nu, \tilde{L}_\Lambda | \tilde{\gamma}_\Lambda)_{t = 0} = o(|\Lambda|)$. Let us
Let us start with the following estimate
\[
\frac{d}{dt} H_\Lambda(\nu_t,L | \gamma^\Lambda)_{|t=0} - \frac{d}{dt} H_\Lambda(\nu_t, L | \gamma^\Lambda)_{|t=0} = \sum_{\omega, i \in \Lambda} \left[ \Gamma_L(\omega, i^+) - \Gamma_{L_A}(\omega, i^+) \right] \log \frac{\nu(1_\omega)}{\nu(1_\omega)} \\
+ \sum_{\omega, i \in \Lambda} \left[ (V_{L_A}(\omega, i^+) - V(\omega, i^+)) \Gamma_L(\omega, i^+) - [\Gamma_L(\omega, i^+) - \Gamma_{L_A}(\omega, i^+)] V_{L_A}(\omega, i^+) \right] \\
\leq \sum_{\omega, i \in \Lambda} A(\omega, i^+) \log \frac{\nu(1_\omega)}{\nu(1_\omega)} \left[ \sum_{\omega, i \in \Lambda} B(\omega, i^+) \Gamma_L(\omega, i^+) - A(\omega, i^+) \nu(1_\omega) V_{L_A}(\omega, i^+) \right].
\]

(70)

where we defined \( B(\omega, i^+) := |V_{L_A}(\omega, i^+) - V(\omega, i^+)| \) and used the following estimate and definition
\[
\Gamma_L(\omega, i^+) - \Gamma_{L_A}(\omega, i^+) = \int \nu(d\eta) c_L(\eta, \eta') \omega^1(\eta) - \nu(1_\omega) c_{L_A}(\omega, \omega') \\
= \int \nu(d\eta) \nu(1_\omega | \eta) \left[ c_L(\omega, \eta \nu, \omega', \eta) - c_{L_A}(\omega, \omega') \right] \\
\leq \sup_{\eta \nu} \left[ \frac{\mu_G \nu(\eta \nu | \omega_{L_A}) \left( e^{-H_\Lambda(\omega, \eta \nu | \omega_{L_A})} \right)}{\mu_G \nu(\eta \nu | \omega_{L_A}) \left( \lambda(e^{-H_\Lambda(\omega, \eta \nu | \omega_{L_A})}) \right)} - \frac{\lambda^{\Lambda} \nu(1_\omega | \eta \nu)}{\lambda^{\Lambda} \nu(1_\omega | \eta \nu)} \right] \nu(1_\omega) \\
=: A(\omega, i^+) \nu(1_\omega).
\]

We first verify \( \sup_\omega \sum_{i \in \Lambda} A(\omega, i^+) = o(\left|\Lambda\right|) \) and \( \sup_\omega \sum_{i \in \Lambda} B(\omega, i^+) = o(\left|\Lambda\right|) \).

Let us start with \( A(\omega, i^+) \). In order to see similar terms we define for a given second-layer boundary condition inside \( \Lambda \), namely \( \omega_{L_A} \), and open boundary conditions outside \( \Lambda \), the conditional first-layer probability measures on \((S^1)^{G_{\Lambda}}\)
\[
\tilde{\mu}_{G_{\Lambda}}(\omega_{L_A} | \nu) := \frac{\lambda^{\nu} \left( \nu e^{-\sum_{\omega \in \Lambda} A(\omega_{L_A}) 1_\omega} \right)}{\lambda^{\nu} \left( e^{-\sum_{\omega \in \Lambda} A(\omega_{L_A}) 1_\omega} \right)}.
\]

(71)

In particular \( \tilde{\mu}_{G_{\Lambda}}(\omega_{L_A} | \lambda(e^{-H_{\Lambda}(\omega_{L_A})}) \nu) = \frac{\lambda^{\nu} \left( \nu e^{-\sum_{\omega \in \Lambda} A(\omega_{L_A}) 1_\omega} \right)}{\lambda^{\nu} \left( e^{-\sum_{\omega \in \Lambda} A(\omega_{L_A}) 1_\omega} \right)} \). These fractions themselves constitute again a specification \( \tilde{\gamma} \) on the second layer.

In essence we want to exploit the Dobrushin comparison theorem. Since we can bound every term by some constant times \( e^{\pm |H_\Lambda|} = e^{\pm |H_\Lambda|} : = e^K \) it suffices to estimate the distance of the conditional first-layer Gibbs measures \( \mu_{G_{\Lambda}}(\omega_{L_A} | \nu) \) and \( \tilde{\mu}_{G_{\Lambda}}(\omega_{L_A} | \nu) \) applied to the quasilocal functions
\[
\psi_{\nu}^{(\mu)}(\omega) := e^{-H_\Lambda(\omega | \nu, r, \nu)} \quad \text{and} \quad \tilde{\psi}_{\nu}^{(\mu)}(\omega) := \lambda(e^{-H_\Lambda(\omega | \nu, r, \nu)} 1_\omega),
\]

(72)

Notice that we have done similar computations in the section about well-definedness of the rotation dynamics. For any fixed first-layer boundary condition \( w \in \Omega \) the measure \( \mu_{G_{\Lambda}}(\omega_{L_A} | \nu) \) is uniquely admitted by the specification
\[
\gamma^{\nu \omega_{L_A} | \nu} := \left( \gamma^{\omega_{L_A} | \nu} \Delta(\cdot | w_{L_A}) \right)_{\Delta \in \nu}.
\]

(73)
\( \tilde{\mu}_{G,i}[\omega_{\Lambda,i}] \) is admitted by \( \tilde{\gamma}^{\omega_{\Lambda,i}}|_{e^i} := \left( (\tilde{\gamma}^{\omega_{\Lambda,i}}|_{e^i})_{\Delta \in e^i} \right) \), \( \Delta \) being finite subsets of \( e^i \). The total variational distance between the two specifications on the site \( l \neq i \) can be estimated by

\[
\begin{align*}
\sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} & \left\{ \left( (\gamma^{\eta_{\Lambda \times \omega_{\Lambda,i}}}|_{e^i})_{l}(\cdot|w_{\Delta^l,i}) - (\tilde{\gamma}^{\omega_{\Lambda,i}}|_{e^i})_{l}(\cdot|w_{\Delta^l,i}) \right) \right\} \\
= & \sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} \left\{ \lambda(1_{B | \beta_{\Lambda \times \omega_{\Lambda,i}}} e^{-\sum_{i \notin A \Delta \Lambda} \Phi_A(\cdot,|w_{\Delta^l,i})}) - \lambda(1_{B | \beta_{\Lambda \times \omega_{\Lambda,i}}} e^{-\sum_{i \notin A \Delta \Lambda} \Phi_A(\cdot,|w_{\Delta^l,i})}) \right\} \\
\leq & \begin{cases} 
1, & \text{if } l \in \Lambda^c, \\
K \sum_{i \in A \Phi: \Lambda} \| \Phi_A \|, & \text{if } l \in \Lambda_\Lambda \setminus i, 
\end{cases}
\end{align*}
\]

where we set \( a := \min_{k \in \{1, \ldots, q\}} \lambda(k) \) and again used \( |e^x - e^y| \leq |x - y| e^{\max(|x|,|y|)} \).

Notice, for any fixed \( l \) when \( \Lambda \) tends to \( \mathbb{Z}^d \), because of the absolute summability of the Hamiltonian, this goes to zero. We want to estimate

\[
\begin{align*}
\sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} & \left\{ \mu_{G,i}[\eta_{\Lambda \times \omega_{\Lambda,i}}](\tilde{\psi}^{\omega_{\Lambda,i}}_1) - \tilde{\mu}_{G,i}[\omega_{\Lambda,i}](\tilde{\psi}^{\omega_{\Lambda,i}}_1) \right\} \quad \text{and} \\
\sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} & \left\{ \mu_{G,i}[\eta_{\Lambda \times \omega_{\Lambda,i}}](\tilde{\psi}^{\omega_{\Lambda,i}}_2) - \tilde{\mu}_{G,i}[\omega_{\Lambda,i}](\tilde{\psi}^{\omega_{\Lambda,i}}_2) \right\}.
\end{align*}
\]

We do this for both terms simultaneously by just writing \( \psi \) instead of \( \tilde{\psi}_1, \tilde{\psi}_2 \).

\[
\begin{align*}
\sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} & \left\{ \mu_{G,i}[\eta_{\Lambda \times \omega_{\Lambda,i}}](\psi^{\omega_{\Lambda,i}}) - \tilde{\mu}_{G,i}[\omega_{\Lambda,i}](\psi^{\omega_{\Lambda,i}}) \right\} \\
\leq & \sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} \left\{ \mu_{G,i}[\eta_{\Lambda \times \omega_{\Lambda,i}}](\tilde{\psi}^{\omega_{\Lambda,i}}) - \tilde{\mu}_{G,i}[\omega_{\Lambda,i}](\tilde{\psi}^{\omega_{\Lambda,i}}) \right\} \\
& + \sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} \left\{ \mu_{G,i}[\eta_{\Lambda \times \omega_{\Lambda,i}}](\psi^{\omega_{\Lambda,i}}) - \mu_{G,i}[\eta_{\Lambda \times \omega_{\Lambda,i}}](\tilde{\psi}^{\omega_{\Lambda,i}}) \right\}.
\end{align*}
\]

For the second part we have

\[
\sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} \left\{ \mu_{G,i}[\eta_{\Lambda \times \omega_{\Lambda,i}}](\tilde{\psi}^{\omega_{\Lambda,i}}) - \tilde{\mu}_{G,i}[\omega_{\Lambda,i}](\tilde{\psi}^{\omega_{\Lambda,i}}) \right\} \leq \sup_{\omega_i} \| \psi^{\omega_{\Lambda,i}} - \tilde{\psi}^{\omega_{\Lambda,i}} \| \leq K \sum_{i \in A \Phi: \Lambda} \| \Phi_A \|
\]

which tends to zero as \( \Lambda \not\subset \mathbb{Z}^d \) by the absolute summability of the potential. In particular there exists a radius \( r \in \mathbb{N} \) such that \( \sup_{i \in \Lambda_n \setminus r} \sum_{i \in A \Phi: \Lambda_n} \| \Phi_A \| < \varepsilon \) for all centered cubes \( \Lambda_n \) such that \( n - r \geq 0 \). Hence

\[
\frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} \sum_{i \in A \Phi: \Lambda_n} \| \Phi_A \| < \varepsilon + \| H_0 \| \frac{|\Lambda_n \setminus \Lambda_{n-r}|}{|\Lambda_n|}
\]

where the r.h.s becomes arbitrarily small as \( n \to \infty \).

Let us look at the first part and use the Dobrushin comparison theorem, which states

\[
\begin{align*}
\sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} & \left\{ \mu_{G,i}[\eta_{\Lambda \times \omega_{\Lambda,i}}](\tilde{\psi}^{\omega_{\Lambda,i}}) - \tilde{\mu}_{G,i}[\omega_{\Lambda,i}](\tilde{\psi}^{\omega_{\Lambda,i}}) \right\} \leq \sup_{\eta_{\Lambda \times \omega_{\Lambda,i}}} \sum_{k \neq i, l \neq i} \delta_k(\tilde{\psi}^{\omega_{\Lambda,i}}) D_{kl}(\eta_{\Lambda \times \omega_{\Lambda,i}}) b_l \\
& \leq \sup_{\omega_i} \sum_{k \in \Lambda \setminus i} \sum_{l \in \Lambda \setminus i} \delta_k(\tilde{\psi}^{\omega_{\Lambda,i}}) D_{kl} + \sup_{\omega_i} \sum_{k \in \Lambda \setminus i} \sum_{l \in \Lambda \setminus i} \delta_k(\tilde{\psi}^{\omega_{\Lambda,i}}) D_{kl} b_l.
\end{align*}
\]
As for the second term we have
\[
\sum_{i \in \Lambda} \sum_{k \in \Lambda \setminus \{i\}} \sum_{l \in \Lambda \setminus \{i\}} \sup_{\omega_i} \delta_k(\tilde{p}^{(i)}) \mathcal{D}_{kl} b_l \leq \left( \sum_{i \in \Lambda} b_i \right) \left( \sup_k \sum_{i \in \Lambda} \mathcal{D}_{kl} \right) \left( \sup_{i \in \Lambda} \sup_{\omega_i} \delta_k(\tilde{p}^{(i)}) \right)
\leq K \sum_{i \in \Lambda} b_i = o(|\Lambda|).
\]

Indeed we have for all \(k\)
\[
\sum_{i \in \Lambda} \sup_{\omega_i} \delta_k(\tilde{p}^{(i)}) \leq e^C \sum_{i \in \Lambda} \sum_{k \in \Lambda} \delta_k(\Phi_A) \leq e^C \sum_{0 \in \Lambda} \|\Phi_A\| < \infty \tag{78}
\]
And also \(\sum_{k \in \Lambda \setminus \{i\}} \mathcal{D}_{kl} \leq \sum_k \mathcal{D}_{kl} = \sum_k \mathcal{D}_{0,j-k} = \sum_j \mathcal{D}_{0,j} < \infty\) for all \(l\). Finally
\[
\frac{1}{|\Lambda|} \sum_{i \in \Lambda} b_l \leq \frac{K}{|\Lambda|} \sum_{i \in \Lambda} \sum_{l \in \Lambda} \|\Phi_A\| \to 0 \text{ as } \Lambda \not\rightarrow \mathbb{Z}^d
\]
by the Cesàro argument as in (77). Let us consider for the first term of (77)
\[
\sum_{i \in \Lambda} \sum_{k \in \Lambda \setminus \{i\}} \sum_{l \in \Lambda} \sup_{\omega_i} \delta_k(\tilde{p}^{(i)}) \mathcal{D}_{kl} = \sum_{i \in \Lambda} \sum_{k \in \Lambda \setminus \{i\}} \sum_{l \in \Lambda} \sup_{\omega_i} \delta_k(\tilde{p}^{(i)}) \mathcal{D}_{kl}. \tag{79}
\]
Notice we assume the model to have the exponential decay property (31) with increasing translation invariant semi-metric \(\rho\) on \(G\) and again summability of the potential in the triple-semi-norm. Thus for all \(i\) and \(l\) by the triangle inequality
\[
\sum_{k \in \Lambda \setminus \{i\}} e^{\rho(i,l)} \sup_{\omega_i} \delta_k(\tilde{p}^{(i)}) \mathcal{D}_{kl} \leq \left( \sup_{i \in \Lambda} \sum_{k} e^{\rho(i,k)} \sup_{\omega_i} \delta_k(\tilde{p}^{(i)}) \right) \left( \sup_{l,k} e^{\rho(k,l)} \mathcal{D}_{kl} \right) \leq \tilde{C}. \tag{80}
\]
Hence we can write
\[
\sum_{i \in \Lambda} \sum_{k \in \Lambda \setminus \{i\}} \sum_{l \in \Lambda} \sup_{\omega_i} \delta_k(\tilde{p}^{(i)}) \mathcal{D}_{kl} = \tilde{C} \sum_{j \in \mathbb{Z}^d} e^{-\rho(0,j)} \# \{i \in \Lambda : i + j \not\in \Lambda\} \tag{81}
\]
which again tends to infinity slower than \(|\Lambda|\).

Let us look at the next error term in (70)
\[
B(\omega, i^+) = | \log \frac{\tilde{\mu}_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i}))}{\tilde{\mu}_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i}))} - \log \frac{\mu_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i}))}{\mu_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i}))} | \\
\leq \frac{| \tilde{\mu}_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i})) - \mu_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i})) |}{\mu_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i}))} \\
+ \frac{| \mu_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i})) - \mu_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i})) |}{\mu_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i}))} \tag{82}
\]
where we assumed \(\tilde{\mu}_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i})) \geq \mu_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i}))\) and \(\tilde{\mu}_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i})) \geq \mu_{G,\gamma[i][\omega_{\Lambda \setminus i}]}(\lambda(e^{-H_1 \omega_i}))\). In that case as well as in all other cases we can follow the exact same arguments as before and get
\[
\frac{1}{|\Lambda|} \sup_{\omega} \sum_{i \in \Lambda} B(\omega, i^+) \to 0 \text{ as } \Lambda \not\rightarrow \mathbb{Z}^d. \tag{83}
\]
Let us consider the remaining parts of (70). For the last term we have

\[
\sup_{\omega} \sum_{i \in \Lambda} \nu(1,\omega)|V_{\frac{d}{\Lambda}}(\omega, i^+)| \leq K \sup_{\omega} \sum_{i \in \Lambda} \nu(1,\omega) = K. \tag{84}
\]

The first term of (70) requires some extra care. We verify

\[
\sum_{\omega, i \in \Lambda} A(\omega, i^+) \nu(1,\omega) \frac{\nu(1,\omega)}{\nu(1,\omega)} \leq \sum_{\omega} A(\omega, i^+) \sum_{\omega} \nu(1,\omega) \log \frac{\nu(1,\omega)}{\nu(1,\omega)} = o(|\Lambda|). \tag{85}
\]

Since the rates are bounded from below away from zero and bounded from above, i.e.\( e^{-2|H_0|} \leq c_L(\omega, \omega^i) \leq \tilde{k} e^{2|H_0|}, e^{-|H_0|} \leq c_K(\omega, \omega^i) \leq \tilde{k} e^{|H_0|}, e^{-|H_0|} \leq c_K(\omega, \omega^i) \leq \tilde{k} e^{|H_0|}\) and \(\nu\) is invariant, i.e. \(0 = \int (L + \alpha K) 1_{\omega_A} d\nu\) we have for all \(\omega \in \{1, \ldots, q\}^\Lambda\)

\[
\hat{K} \geq \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \frac{\nu(1,\omega^i)}{\nu(1,\omega)} \geq \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \nu(1,\omega^i) = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \nu(1,\omega^i) \tag{86}
\]

**Lemma 5.3** Let \(\nu \in \mathcal{P}(\Omega')\) be translation-invariant and invariant for the joint dynamics. There exists a constant \(\hat{K}\) such that for all finite sets \(\Delta\) we have

\[
\frac{1}{\nu(\omega^i_0|\omega^i_{\Delta, 0})} < \hat{K}.
\]

**Proof.** By the Jensen inequality, it suffices to show this for centered cubes \(\Delta\). Let us consider the \(\nu\)-expectation of (86) and apply Jensen’s inequality

\[
\hat{K} \geq \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \nu(d\omega) \frac{\nu(1,\omega^{|\omega^i_{\Lambda, i}|})}{\nu(1,\omega^i)} \geq \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sum_{l=1}^q \nu(d\omega) \frac{\nu(k,\omega^{|\omega^i_{\Lambda, i}|})}{\nu(k+1,\omega^{|\omega^i_{\Lambda, i}|})}^2
\]

\[
\geq \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \min_l \nu(d\omega) \frac{\nu(l,\omega^{|\omega^i_{\Lambda, i}|})}{\nu(1+1,\omega^{|\omega^i_{\Lambda, i}|})} \sum_{k=1}^q \nu(d\omega) \frac{\nu(k,\omega^{|\omega^i_{\Lambda, i}|})}{\nu(k+1,\omega^{|\omega^i_{\Lambda, i}|})}
\]

\[
\geq \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \min_l \nu(d\omega) \frac{\nu(1,\omega^{|\omega^i_{\Lambda, i}|})}{\nu(1,\omega^{|\omega^i_{\Lambda, i}|})} \sum_{k=1}^q \nu(d\omega) \frac{1}{\nu(1,\omega^{|\omega^i_{\Lambda, i}|})}
\]

\[
\geq \frac{1}{|\Lambda|} \sum_{l \in \Lambda} \min \nu(l) \sum_{k=1}^q \nu(d\omega) \frac{1}{\nu(1,\omega^{|\omega^i_{\Lambda, i}|})} \geq \frac{\varepsilon_0}{|\Lambda|} \sum_{i \in \Lambda} \nu(d\omega) \frac{1}{\nu(1,\omega^{|\omega^i_{\Lambda, i}|})}
\]

where we used \(\min_l \nu(l) \geq \varepsilon_0 > 0\).

**Remark 5.4** In fact take \(\Lambda = \{0\}\) in (86) then we have \(\frac{\nu(k)}{\nu(k+1)} \leq \hat{K}\) and hence

\[
\nu(k) = \frac{\nu(k)}{\nu(k+1)} \geq \frac{1}{1 + \varepsilon_0 K^q} \geq \frac{1}{\sum_{l=0}^{q-1} K^l} = \varepsilon_0.
\]
Consider \( \Lambda_n := [-n, n]^d \) and \( m := n - \lfloor n^k \rfloor \) with \( k \in (0, 1) \). Then the above inequality can be further estimated by

\[
\frac{\hat{K}}{\varepsilon_0} \geq \frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_m} \int \nu(d\omega) \frac{1}{\sqrt{\nu(\omega_i^*|\omega_{\Lambda_n \setminus i})}} \geq \left( \frac{1}{|\Lambda_n|} \right) \int \nu(d\omega) \frac{1}{\sqrt{\nu(\omega_i^*|\omega(\Delta + i)\setminus i))}}
\]

where \( \Delta \) is the largest centred cube such that for all \( i \in \Lambda_m \) we have \( \Delta + i \subset \Lambda \). We used the conditional Jensen inequality in the last line. Because of translation-invariance we have \( \frac{\hat{K}}{\varepsilon_0} \geq \frac{1}{|\Lambda_m|} \int \nu(d\omega) \frac{1}{\sqrt{\nu(\omega_i^*|\omega(\Delta + i)\setminus i))}} \). Since \( \frac{|\Lambda_m|}{|\Lambda_n|} \to 1 \) for \( n \to \infty \) and \( \Lambda_n \setminus \Lambda_m \) allows \( \Delta \) to become arbitrarily large, the result follows.

Consider centered cubes \( \Lambda_n \) of side-length \( 2n + 1 \) and write \( \partial_r(\Lambda_n) := \{ i \in \Lambda : d(i, \Lambda^c) = r \} \) where \( d(\cdot, \cdot) \) is the uniform norm. We show for \( i \in \partial_r(\Lambda_n) \)

\[
\sup_{\omega} A(\omega, i^+) \leq f(r)
\]

with \( \lim_{r \to \infty} f(r) = 0 \). Indeed, let us look at (85) again. We can estimate the second part by

\[
\sup_{\omega} \sup_{i \in \partial_r(\Lambda_n)} \| \psi^{\omega_i} - \tilde{\psi}^{\omega_i} \| \leq \sup_{i \in \partial_r(\Lambda_n)} K \sup_{i \in \partial_r(\Lambda_n)} \| \Phi_A \| \leq K \sum_{0 \in \partial_r(\Lambda_n)} \| \Phi_A \| \quad (88)
\]

which goes to zero as \( r \) tends to infinity. For the first part of (85) we have

\[
\sup_{\eta \in \Lambda_n} \| \mu_{G_i}[\eta_{\Lambda_n \setminus i}|\omega_{\Lambda_n \setminus i}] (\psi^{\omega_i}) \| \leq \sup_{\omega_i} \frac{1}{|\Lambda_n|} \sum_{k \neq i \neq l} \delta_k (\tilde{\psi}^{\omega_i}) \overline{D}_{kl} = \sum_{k \neq i \neq l} \delta_k (\tilde{\psi}^{\omega_i}) \overline{D}_{kl} + \sum_{i \neq k \neq l} \delta_k (\psi^{\omega_i}) \overline{D}_{kl} \quad (89)
\]

By looking at (80) we notice for \( i \in \partial_r(\Lambda_n) \)

\[
\sup_{\omega_i} \sum_{k \in \Lambda_n \setminus i} \sum_{l \in \Lambda_n \setminus i} \delta_k (\tilde{\psi}^{\omega_i}) \overline{D}_{kl} \leq C \sum_{i \in \Lambda_n \setminus i} e^{-g(0, l)} \leq \tilde{C} \sum_{i \in (\Lambda_n \setminus i)} e^{-g(0, l)} \leq \tilde{C} \sum_{i \in \Lambda_n^c} e^{-g(0, l)}
\]

which goes to zero as \( r \) tends to infinity. Let us define \( n_l := d(l, \Lambda_n^c) \) for the distance between the site \( l \) and \( \Lambda_n^c \). For the other summand in (89) we have

\[
\sup_{\omega_i} \sum_{k \in \Lambda_n \setminus i} \sum_{l \in \Lambda_n \setminus i} \delta_k (\tilde{\psi}^{\omega_i}) \overline{D}_{kl} b_l \leq \sum_{\omega_0} \sup_{l \in \Lambda_n \setminus i} \overline{D}_{0,l} \sum_{k \in \Lambda_n \setminus i} \| \Phi_A \| \quad (90)
\]

Notice if \( n_i \to \infty \) then \( n_{i+l+k} \to \infty \) for every fixed \( l, k \). In particular since \( \sum_l \overline{D}_{0,l} < \infty \) we have for \( n_i \to \infty \)

\[
\sum_l \overline{D}_{0,l} \sum_{k \in \Lambda_n \setminus i} \| \Phi_A \| \to 0
\]

by the dominated convergence theorem. Similarly to (78) we have

\[
\sum_k \| \tilde{\psi}^{\omega_0} \| \leq e^C \sum_k \sum_{i \in A^c} \delta_k (\Phi_A) \leq e^C \sum_{0 \in \partial_r(\Lambda_n)} \| \Phi_A \| < \infty
\]
and thus for \( n \to \infty \) we can conclude again with the dominated convergence theorem
\[
\sum_k \sup_{\omega_0} \delta_k (\bar{\nu}^{00}) \sum_l D_{0,l} \sum_{0 \in A \pm \Lambda_{n+1+k}} \| \Phi_A \| \to 0.
\]

Since \(- \log(x) < \frac{1}{\sqrt{x}} < \frac{1}{x}\) on \((0, 1]\), we can estimate \(\| \Phi_A \|\) in the following way
\[
\sum_{r=1}^n \sum_{i \in \partial_r (A_n)} \sup_{\omega} A(\omega, i^+) \int \nu(d\omega) |\log \frac{\nu(\omega_1^{j} [\omega_{A_n} \backslash i])}{\nu(\omega_1^{j} [\omega_{A_n} \backslash i])}| \\
\leq \sum_{r=1}^n f(r) \sum_{i \in \partial_r (A_n)} \left[ \int - \log \nu(\omega_1^{j} [\omega_{A_n} \backslash i]) \nu(d\omega) + \int - \log \nu(\omega_1^{j} [\omega_{A_n} \backslash i]) \nu(d\omega) \right] \\
\leq \sum_{r=1}^n f(r) \sum_{i \in \partial_r (A_n)} \left[ \int \nu(d\omega) \frac{1}{\nu(\omega_1^{j} [\omega_{A_n} \backslash i])} + \int \nu(d\omega) \frac{1}{\nu(\omega_1^{j} [\omega_{A_n} \backslash i])} \right] \\
= \sum_{r=1}^n f(r) \sum_{i \in \partial_r (A_n)} \left[ \sum_{k=1}^q \int \nu(d\omega) \frac{\nu(k|\omega_{A_n} \backslash i)}{\nu(k|\omega_{A_n} \backslash i)} + \int \nu(d\omega) \frac{1}{\nu(k|\omega_{A_n} \backslash i)} \right] \\
\leq \sum_{r=1}^n f(r) \sum_{i \in \partial_r (A_n)} [q + \hat{K}] = K \sum_{r=1}^n f(r) |\partial_r (A_n)| \\
\]

where we used the last lemma in the last line. Since \( f(r) \to 0 \) for \( r \to \infty \) there exists a \( R \in \mathbb{N} \) such that for all \( r \geq R \) we have \( f(r) < \varepsilon \), hence for large \( n \)
\[
\frac{1}{|A_n|} \sum_{r=1}^n f(r) |\partial_r (A_n)| = \frac{1}{|A_n|} \left( \sum_{r=1}^n f(r) |\partial_r (A_n)| + \sum_{r=R+1}^n f(r) |\partial_r (A_n)| \right) \leq \varepsilon + K \frac{\hat{\sigma}(A_n)}{|A_n|}
\]

where the second summand goes to zero as \( n \) tends to infinity. We proved \(\| \Phi_A \|\).

Together we see the combined error caused by the approximation vanishes from the point of view of difference between time derivatives of relative entropy densities, i.e. \(\frac{d}{dt} H_A(\nu_{t,L} [\gamma_{A}])_{t=0} - \frac{d}{dt} H_A(\nu_{t,L} [\gamma_{A}])_{t=0} = o(\| \Lambda \|)\). Together we have
\[
0 = \frac{d}{dt} H_A(\nu_{t,L+\alpha K} [\gamma_{A}])_{t=0} - \frac{d}{dt} H_A(\nu_{t,L} [\gamma_{A}])_{t=0} + \alpha \frac{d}{dt} H_A(\nu_{t,K} [\gamma_{A}])_{t=0}
\]
and hence with the notation of \(\| \Lambda \|\) we can combine the result for the Glauber dynamics together with the negative sign of the local rotation approximation and the above estimates on the approximation error. We get
\[
\alpha \sum_{i \in A} [\kappa_{\Lambda,K}(i^+) + \kappa_{\Lambda,K}(i^-)] \leq \alpha \sum_{i \in A} [\beta_{\Lambda,K}(i^+) + \beta_{\Lambda,K}(i^-)] \delta_{\Lambda,K}(i) + 2 \frac{d}{dt} H_A(\nu_{t,L} [\gamma_{A}])_{t=0} \\
\leq \alpha C \sum_{i \in A} \delta_j (c_G(j,i)) + 2C \left| \frac{d}{dt} H_A(\nu_{t,L} [\gamma_{A}])_{t=0} - \frac{d}{dt} H_A(\nu_{t,L} [\gamma_{A}])_{t=0} \right| \\
\leq \alpha C o(\| \Lambda \|) + 2C \hat{K} o(\| \Lambda \|) = o(\| \Lambda \|)
\]
but this implies \( \nu \) is Gibbs. \(\square\)
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