Learning Trajectory in Mathematical Proof

I Minggi¹*, U Mulabar¹ and S F Assagaf¹

¹Department of Mathematics Education, Faculty of Mathematics and Natural Science. Universitas Negeri Makassar

*e-mail: ilham.minggi@unm.ac.id

Abstract. This study aims to obtain a learning trajectory framework for mathematical proofing and to develop teaching materials for the real number system topic. The method is Design research. The results showed the five developed trajectories: (a) experience and fundamentals, (b) simple proof (c) proof of existence, (d) complex proof, and (e) proofing through construction. In proofing, the students still need scaffolding from the lecturer. The difficulties were still existed are writing symbols and writing mathematical reasoning especially in linking a series of statements in the proof.

Keywords: Learning trajectory; mathematical proof; scaffolding, real number system

1. Introduction

Our fundamental research have been done regarding the mathematical proofing [1]. We have found some difficulties of students in proving mathematics. Some students did not understand the meaning of mathematical symbols correctly. When they constructed a proof, they lose the meaning since some symbols did not use in a proper way. The meaning of symbols in mathematics is very specific and unique. It will greatly affect the meaning of the sentences or statements written. In addition, some students also showed the difficulty in developing a proof strategy. Some students experienced obstacles in executing the evidence. They hardly determined which strategy is more appropriate to use to prove the mathematical sentences. Many studies working on proof [2, 3, 4, 5, 6].

It also found two reasons why students found difficulties in proving statements in mathematics i.e. (1) understanding of mathematical proofs and (2) understanding of mathematical concepts and principles. Students turned out to have difficulty in understanding what exactly the proof in mathematics. Even though proving using computer software has now begun to be accepted in mathematics, it is still in the debate about validity when viewed from various rules - logic with an axiomatic deductive system dominates the discussion.

The difficulty in proving mathematics indicates that it needs to develop a learning sequence in mathematical proofs. The sequence should be associated with difficulty level of proof and characteristics of proved statement. The sequence as a learning trajectory consisted of the activities and concepts related to the topic in mathematics such as Calculus, Geometry, and Trigonometry. This developed learning trajectory also needs to be proved by implementing it in the class. This developed trajectory is hypothetical, so it is called a Hypothetical Learning Trajectory (HLT). HLT is a conjecture of a series of activities that children did in solving a problem or understanding a concept. The HLT firstly formulated and then implemented in the class. The implementation brings some changes or improvements. In this study, the HLT refers to series of activities carried out by students in solving mathematical proof problems.
2. Method
This is a design research proposed by Gravemeijer and Cobb [7]. It is characterized by a cyclical process from preparing the experiment, conducting the experiment, and retrospective analysis. It also characterized as development study by Plomp [8]. It is a systematic study in designing, developing, and evaluating the intervention as a solution to solve a complex problem in education field.

The design research focused on developing the order in presenting the material in a class. It starts with a thought experiment, thinking about the path of learning that students will go through. The results of the thought experiment were implemented in a class. By reflecting the results of the experiment in class, the next though experiment is carried out. In long term process, these two activities can be seen as a cumulative cyclical process.

The following describes the activities in each phase of the design research:

2.1 Identification.
Based on results of the teaching experiments and the retrospective analysis on the first cycle research [9], we formulated subsection of learning trajectory in each mathematical proofing activity. The sub-learning trajectories refers to: (a) understand the statement to be proven, (b) choose the type of proof and design a flow diagram or proof sketch, (c) write the details of the proof, and (d) check the validity of the proof.

2.2 Conducting the Experiment
After conducting a pilot experimental in the first cycle in Minggi & Mulbar [9], we conducted a learning experiment by adding the sub-trajectory at each stage of the learning trajectory. The success and failure of the learning trajectory in the pilot experiment are used as the trajectory in the next learning experiment. The learning trajectory leads us to observe the learning process of mathematical proof and to provide a picture of proof of problems. In this study, the learning phase in class is carried out in small groups of five students. The subjects were university’s students in the major of mathematics education. Besides that, a revision of teaching material in the subject of Real Numbers Sequence in the real analysis course has been made.

2.3 The Retrospective Analysis
This phase aims to evaluate the developed HLT through the pilot experiment. The learning trajectory used in a retrospective analysis as a guideline and a basic reference in revising the trajectory. The main objective at this stage is to contribute to the development of hypothetical learning trajectories in supporting students’ understanding in proofing. The analysis process is not only on the factors that support the trajectory but also on unpredictable students’ response.

3. Results and Discussions
The results of the study are based on the phases of proving activities as HLT in the real analysis course. Data is collected through observation and question-and-answer session when students performed their tasks. The following describes the findings in each sequence in the HLT of mathematical proof. This learning trajectory consists of several phases in one cycle of the learning scenario. One learning cycle consists of several meetings until all phases in one scenario cycle have been carried out.

3.1. Result in Preparatory Phase
The formulation of HLT started from the difficulties level in proofing. The researchers together with several mathematical proof learning experts proposed several sequences to learn mathematical proof. Based on the results of a pilot experiment, literature reviews, and the experiences of the researchers in teaching proof, it is developed a sequence in learning mathematical proof based on the level of difficulties. They are (1) presenting concepts in a fundamentally and giving space to experience the
concept rigorously, (2) proving in a simple way, (3) proving the existence, (4) proving a problem with complex proof, and (5) proving through construction. Other revisions were adding some problems and adding some hints for a problem which is difficult to find the idea of proving.

At each proving activity, sub-trajectories are provided. This is the recommendation in our first cycle in pilot experiment (2018). The sub-trajectories refer to the steps of proving activity as follows: (1) understand the statement that will be proven, (2) choose the type of proof and design a flow chart or proof sketch, (3) write the details of the proof, and (4) check the validity of each step of proof.

3.2. Research Results Implementation and Retrospective Phases
The learning trajectory of mathematical proofs implemented and evaluated whether the trajectory in a pilot experiment run as what we expect and observed how the students deal with the problems. The implementation and retrospective phases described as follows:

3.2.1 Activity 1. Experience and Fundamental
The experience and fundamental activities are the initial phase in learning mathematical proof. The results of observations and questions-answers activities during the teaching experiment shows the need for students to understand the proof through a strict definition, the negation of definition, and visual illustrations. The order of the problems must really concerns to the order of the concepts, definitions and theorems have been presented beforehand. In this activity, the students were given experience of daily problems in the form of statements. They were asked to prove the statement. Besides, this activity focused also on definitions of conjunctions and quantifier statements.

In this activity, students were asked to explore (1) basic understanding of mathematical concepts that will be used to solve proof problems, (2) use of conjunctive words and quantifier statements, which are used in the construction of mathematical proofs. The words are "and", "or", "if ... then ...", "... if and only if ...", "it is not true that ...", "for each ...", and "there are ...". The statements using these words such as "for every number \( \theta \) > 0, there existed real numbers x and y, \( |x - y| < 1 \), such that \( |f(x) - f(y)| < \theta \). Another example, "There is a number x, for every number y > x satisfies \( f(y) > f(x) \). Furthermore, the students also were asked to evaluate the truth of the statement where the context is valid, or asked to write the negation of statement, to check the truth of the statement, or the negation in a particular context.

As the result, most of students explained statements, notations, and particular mathematical terms in the statements that will be proven. Just a few students cannot explain them. However, they were asked to review again the notations and the particular terms.

3.2.2 Activity 2. Simple proof
In this activity, the students were asked to prove mathematical statements that can be solved with a single thinking. Most students grasped the idea of the proof and also solve the proof problem, either by using direct or indirect proof. For example, they were asked to prove a statement in the form "if P then Q". The students proof is that knowing P is true, derive the consequence of P to obtain Q. Some also showed that knowing P is true and knowing Q is wrong, deriving the consequences of true P and false Q until finally finding a contradiction.

Before proving a mathematical statement, students were asked to design a proof sketch or proof flow chart. This is expected to be able to guide intuitively in the minds of students about the flow of proof that will be written next. The efforts made in this activity were quite successful. The mistakes that usually appear in this activity are mistakes in writing notation, selecting sentences to write proof. These mistakes can be corrected through question-answer section. The results showed that this activity was considered as successful.

3.2.3 Activity 3. Proof of existence
In this phase, the students are asked to prove the existence and non-existence of a fact or concept in mathematics. One example used is to prove the non-existence of \( \sqrt{2} \) in a set of rational number and prove the existence of numbers \( \sqrt{2} \) in a set of real numbers.
Proof began with asking them illustrate the flowchart of proof, and then make appropriate sentences based on the chart. The description of the flowchart is very helpful for students to guide the direction of proof. However, difficulties generally arise when writing the proof based on the flowchart. This difficulty was discussed again with the lecturer. After that, the way students write the proof was getting better. The results indicate that the students performed a better proof writing with the scaffolding from the lecturer.

### 3.2.4 Activity 4. Complex Proof

There are many proofs with a complex flow of proof. However, this research investigates how students prove the statement of "If \( P \) then \( Q \)" , but it is difficult to do directly, except using other theorems. The proof used by the students such as, let \( P \) and \( P' \) are equivalent statements expressed by a theorem. Likewise, \( Q \) and \( Q' \) also are equivalent statements. So, to prove the statement "If \( P \) then \( Q \)", simply show "If \( P' \) then \( Q' \)".

To guide the proof, the students were asked to re-design the flowchart of proof and to explain the parts of the proof that are difficult to complete. The lecturer gave scaffolding for students to attack the difficulties. Lastly, the students were asked to explain the complete flowchart of proof. This flowchart helps students grasp the big ideas of a proof.

The most common mistakes were errors in writing notations, not explaining the meaning of notation in advanced and logical flaws – draw a proof through invalid logic. Some parts in the proof writing were related to imprecise logic, unrelated arguments.

As the result, the students’ performances indicated that the students shows an understanding in designing the flow of proof. However, most of them cannot pursue the process of writing the proof. The still struggled in writing symbol and providing a valid argument mathematically. In this case, a scaffolding is needed from the lecturer to help students complete the proof.

### 3.2.5 Activity 5. Proof through construction

In this activity, the students were asked to prove statements by constructing a set direction a proof statement. The example that students did was prove that there exists a number \( \sqrt{2} \) in the real number system. Proof begins by defining a set of positive numbers that the quadrat of the number equals to or less than 2. The existence of \( \sqrt{2} \) is the smallest upper bound of a constructed set.

In this activity, the students showed a good flowchart of proof. However, they showed difficulties in providing mathematical reasons for a series of statements in the proof. The students must have a strict understanding of the concept definitions associated with the statements being proven. As the result, it was still needed a scaffolding to help students complete the proof.

### 4. Conclusions

The developed trajectories in mathematical proof: (1) Experience and fundamental, giving examples of proof and strengthening the principles of proof, (2) simple proof, proving a statement with a single line of proof, (3) proof of existence - proof of the existence or non-existence of facts or concepts, (4) complex proof – proving a statement with elaborating more concepts and reasoning, and (e) proof through construction. Besides, The flowchart proof or sketching the proof helps students to construct their idea in writing the proof. However, the proofing activity still needs scaffolding in helping students complete the mathematical proofs. Some difficulties in proof are writing symbols and writing mathematical reasoning especially in linking a series of statements in the proof.

### References

[1] Sabri, S., & Minggi, I. (2014). Students' Difficulties in Mathematics Proofs. In Proceedings of the 1st International Conference on Science (ICOS-1) (pp. 272-280). Faculty of Mathematics and Natural Sciences UNHAS

[2] Harel, G.; Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In AH
Schoenfeld, J. Kaput & E. Dubinsky (Eds.), Research in collegiate mathematics education, III (pp. 234-283). Providence, USA: American Mathematical Society.

[3] Hanna G. (1989) More than formal proof. For the Learning of Mathematics, 9 (1), 20-25.

[4] Recio, AM and Godino, JD (2001). Institutional and personal meanings of proof. Educational Studies in Mathematics, 48 (1), 83-99.

[5] Selden, Annie and Selden, John (2003), "Validations of Proofs Considered as Texts: Can Undergraduates Tell If an Argument Proves a Theorem?", Journal for Research in Mathematics Education, V34 N1, pp. 4-36

[6] Senk, S. (1985). How well do students write geometry proofs? Mathematics Teacher, 78 (6), 448–456.

[7] Gravemeijer, Koen and Cobb, Paul. (2013). Design research from the Learning Design Perspective. In Jan Van Den Akker, et. al. An Introduction to Educational Design Research. London: Routledge.

[8] Plomp, T. (2013). Educational design research: An introduction. Educational design research, 11-50.

[9] Minggi, I., & Mulbar, U. (2019, April). The Design Framework of Mathematical Proof Learning of Mathematics Education Department Students. In the 1st International Conference on Advanced Multidisciplinary Research (ICAMR 2018). Atlantis Press.