RELATIVELY UNIFORMLY CONTINUOUS SEMIGROUPS ON VECTOR LATTICES

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ABSTRACT. In this paper we study continuous semigroups of positive operators on general vector lattices equipped with the relative uniform topology \( \tau_{ru} \). We introduce the notions of strong continuity with respect to \( \tau_{ru} \) and relative uniform continuity for semigroups. These notions allow us to study semigroups on non-locally convex spaces such as \( L^p(\mathbb{R}) \) for \( 0 < p < 1 \) and spaces such as \( \text{Lip}(\mathbb{R}) \) and \( C(\mathbb{R}) \). We provide the general framework and standard constructions for such semigroups. We also introduce a version of a Banach-Steinhaus result for relative uniform convergence which holds true for important classes of vector lattices. By using this we prove relative uniform continuity of the (left) translation semigroup on \( C(\mathbb{R}) \).

1. INTRODUCTION AND PRELIMINARIES

In the 1940s, E. Hille [Hil42, Hil48] and K. Yosida [Yos48] introduced the theory of strongly continuous semigroups on Banach spaces in order to treat evolution equations. By now, their theory is well established, and its applications reach well beyond the classical field of partial differential equations. However, from the very beginning many situations occurred in which the underlying space is not a Banach space. In order to deal with such phenomena, already I. Miyadera [Miy59], H. Komatsu [Kom64], K. Yosida [Yos65], K. Singbal-Vedak [SV65], T. Komura [K68], S. Ouchi [O73] and others generalized the theory to strongly continuous semigroups on locally convex spaces. Also, strongly continuous semigroups of positive operators on Banach lattices have been discussed in [BR84], [AGG+86], [BKR17] and others.
Our purpose is to provide a general framework for the theory of strongly continuous semigroups on vector lattices. Although vector lattices themselves are initially order and algebraic theoretical construct, they admit topologies which arise purely from order. The natural question that appears is whether one can study dynamical systems on general vector lattices. Since we want that our notion of strong continuity of semigroups on general vector lattices agrees with strong continuity for semigroups on Banach lattices, relative uniform topology \( \tau_{ru} \) seems to be the correct choice. This allows us to consider semigroups on non-Banach spaces, such as \( C_c(\mathbb{R}) \), Lip(\( \mathbb{R} \)), UC(\( \mathbb{R} \)), C(\( \mathbb{R} \)), and even on non locally convex spaces, such as \( L^p(\mathbb{R}) \) for \( 0 < p < 1 \). We discuss two types of continuity of semigroups on vector lattices: the strong continuity with respect to \( \tau_{ru} \) and the relative uniform continuity. The former notion is defined by \( \tau_{ru} \)-convergence and the latter by relative uniform convergence.

This paper is structured as follows. In Section 2 we consider some general properties of \( \tau_{ru} \) and we provide examples of vector lattices together with corresponding relative uniform topologies. In Section 3 we introduce notions of \( \tau_{ru} \)-strongly continuous semigroups and relatively uniformly continuous semigroups. While relative uniform continuity implies \( \tau_{ru} \)-strong continuity, in general, these notions do not coincide as it is shown by the (left) translation semigroup on \( L^p(\mathbb{R}) \) \( (0 < p < \infty) \). In the rest of Section 3 we present how one can lift strong continuity with respect to \( \tau_{ru} \) from a \( \tau_{ru} \)-dense set to the whole space. Parallel theory for relatively uniformly continuous semigroups is considered in Section 5. A comparison between vector lattice case and Banach space case reveals that general vector lattices lack some property related to the principle of uniform boundedness. We introduce such property and call it “relative uniform Banach-Steinhaus property”. We prove that many important vector lattices posses it. This property enables us to provide the extension theorem for relatively uniformly continuous semigroups. In Section 4 we study standard constructions of new semigroups from a given one.

Let us now recall some preliminary facts and notation that are needed throughout the text. A family \((T(t))_{t \geq 0}\) of linear operators on a vector space \( Y \) is a semigroup if it satisfies the functional equation

\[
T(s + t) = T(t)T(s)
\]
for all $t, s \geq 0$ and $T(0) = I_Y$. If $Y$ is a Banach space, then we call a semigroup $(T(t))_{t \geq 0}$ a $C_0$-semigroup or strongly continuous on $Y$ when the operator $T(t)$ is bounded for each $t \geq 0$ and for each $y \in Y$ the orbit map

$$
\zeta_y : t \mapsto \zeta_y(t) = T(t)y
$$

is continuous with respect to the Euclidean topology $\tau_e$ on $\mathbb{R}_+$ and the norm topology on $Y$. If $Y$ is a vector lattice and $T(t)$ is a positive operator on $Y$ for each $t \geq 0$, then the semigroup $(T(t))_{t \geq 0}$ is called a positive semigroup.

If $\tau$ is a linear topology on $Y$, then a net of linear operators $(T_\alpha)_{\alpha}$ is $\tau$-equicontinuous when for each $\tau$-neighborhood of zero $V \subset Y$ there exists another $\tau$-neighborhood of zero $U \subset Y$ such that $T_\alpha U \subset V$ for all $\alpha$. If for each $s > 0$ the family of operators $\{T(t) : 0 \leq t \leq s\}$ is $\tau$-equicontinuous, then $(T(t))_{t \geq 0}$ is locally $\tau$-equicontinuous. If for $x \in X$, $s > 0$ and a topology $\tau$ on $X$ we have $T(s + t)x \xrightarrow{\tau} T(s)x$ as $t \to 0$ and $T(t)x \xrightarrow{\tau} x$ as $t \searrow 0$, then we write “$T(s + t)x \xrightarrow{\tau} T(s)x$ as $t \to 0$ for $s \geq 0$”.

A net $(x_\alpha)_{\alpha} \subset X$ is relatively uniformly convergent to $x \in X$ if there exists some $u \in X$ such that for each $\varepsilon > 0$ there exists $\alpha_0$ such that $|x_\alpha - x| \leq \varepsilon \cdot u$ holds for all $\alpha \geq \alpha_0$. We call such an element $u \in X$ a regulator of $(x_\alpha)_{\alpha}$ and we write $x_\alpha \overset{ru}{\rightarrow} x$. It is well-known that limits of relatively uniformly convergent sequences in $X$ are unique if and only if $X$ is Archimedean. Throughout this paper, $X$ stands for an Archimedean vector lattice unless specified otherwise.

For the unexplained terminology about vector lattices and semigroups we refer the reader to [LZ71] and [EN00], respectively.

2. RELATIVE UNIFORM TOPOLOGY

A subset $S$ of $X$ is called relatively uniformly closed whenever $(x_n)_{n \in \mathbb{N}} \subset S$ and $x_n \overset{ru}{\rightarrow} x$ imply $x \in S$. By [LM67, Section 3], the relatively uniformly closed sets are exactly the closed sets of a certain topology in $X$, the relative uniform topology which we denote by $\tau_{ru}$. This topology has been first studied by W.A.J. Luxemburg and L.C. Moore in [LM67]; see also [Moo68]. When a net $(x_\alpha)_{\alpha} \subset X$ converges to $x$ in $\tau_{ru}$, we write $x_\alpha \overset{\tau_{ru}}{\rightarrow} x$. Since $X$ is Archimedean, the topological space $(X, \tau_{ru})$ satisfies the $T_1$-separation axiom.
The following proposition yields that if one starts by defining closed sets through nets, one ends up with the same topology.

**Proposition 2.1.** A subset $S$ of $X$ is relatively uniformly closed if and only if for each net $(x_\alpha)_\alpha \subset S$ and $x \in X$ with $x_\alpha \xrightarrow{ru} x$ we have $x \in S$.

**Proof.** It suffices to prove the “only if” statement. Fix a relatively uniformly closed set $S \subset X$, $x \in X$ and a net $(x_\alpha)_\alpha \subset S$ satisfying $x_\alpha \xrightarrow{ru} x$ with respect to some regulator $u$. We show that $x \in S$. For each $n \in \mathbb{N}$ pick any index $\alpha_n$ such that $|x_{\alpha_n} - x| \leq \frac{1}{n} \cdot u$. Then $x_{\alpha_n} \xrightarrow{ru} x$, and since $S$ is relatively uniformly closed, we conclude $x \in S$. \hfill \Box

We proceed with various examples of important vector lattices together with their relative uniform topologies and convergences which will be needed throughout the paper.

**Example 2.2.**

(a) On a vector lattice $X$ with an order unit $u \in X$ the relative uniform topology $\tau_{ru}$ is generated by the norm

$$\|x\|_u := \inf\{\lambda > 0 : |x| \leq \lambda \cdot u\},$$

since $x_\alpha \xrightarrow{ru} x$ if and only if $x_\alpha \xrightarrow{\|\|_u} x$. Such vector lattices are $\tau_{ru}$-complete if and only if they are uniformly complete.

(b) It is well-known that in a completely metrizable locally solid vector lattice $(X, \tau)$ every convergent sequence has a subsequence which converges relatively uniformly to the same limit, see \cite{AT07} Lemma 2.30]. This immediately yields that a subset of $X$ is relatively uniformly closed if and only if it is $\tau$-closed, so that topologies $\tau_{ru}$ and $\tau$ agree. In particular, if $X$ is a Banach lattice, then $\tau_{ru}$ agrees with norm topology.

(c) For $0 < p < 1$ the vector lattice $L^p(\mathbb{R})$ equipped with the topology $\tau$ induced by the metric

$$d_p(f_1, f_2) := \int_{\mathbb{R}} |f_1(x) - f_2(x)|^p \, dx$$

is a completely metrizable locally solid vector lattice which is not locally convex.

(d) The vector lattice $C(\mathbb{R})$ equipped with the topology of uniform convergence on compact sets is a completely metrizable locally convex solid vector lattice.
In the following proposition we characterize relative uniform convergence in $C_c(\mathbb{R})$.

**Proposition 2.3.** A net $(f_\alpha)_\alpha \subset C_c(\mathbb{R})$ converges relatively uniformly to $f \in C_c(\mathbb{R})$ if and only if $f_\alpha \xrightarrow{\|\cdot\|_{\infty}} f$ and there exists a compact set $K \subset \mathbb{R}$ and $\alpha_0$ such that $f_\alpha|_{K^c} = 0$ for all $\alpha \geq \alpha_0$.

**Proof.** $(\Rightarrow)$ Fix $\varepsilon > 0$. There exist $u \in C_c(\mathbb{R})$, independent of $\varepsilon$, and $\alpha_0$ such that

$$|f_\alpha - f| \leq \varepsilon \cdot u$$

for all $\alpha \geq \alpha_0$. This immediately implies that

$$\|f_\alpha - f\|_{\infty} \leq \varepsilon \cdot \|u\|_{\infty} \quad \text{and} \quad |f_\alpha| \leq |f_\alpha - f| + |f| \leq \varepsilon \cdot u + |f|$$

for all $\alpha \geq \alpha_0$ and hence, $f_\alpha \xrightarrow{\|\cdot\|_{\infty}} f$ and $f_\alpha|_{K^c} = 0$ for all $\alpha \geq \alpha_0$ where $K$ is the compact support of the function $\varepsilon \cdot u + |f|$.

$(\Leftarrow)$ To construct a regulator $u \in C_c(\mathbb{R})$, pick compact sets $K_1, K_2 \subset \mathbb{R}$ and $\alpha_0$ such that $f|_{K_1^c} = 0$ and $f_\alpha|_{K_2^c} = 0$ hold for all $\alpha \geq \alpha_0$ and set $K := K_1 \cup K_2$. Then $(|f_\alpha| + |f|)|_{K^c} = 0$ and since $|f_\alpha - f| \leq |f_\alpha| + |f|$, we also have $(|f_\alpha - f|)|_{K^c} = 0$ for all $\alpha \geq \alpha_0$. By assumption, for each $\varepsilon > 0$ there exists $\alpha_1$ such that $\|f_\alpha - f\|_{\infty} \leq \varepsilon$ holds for all $\alpha \geq \alpha_1$. Hence, for any $\alpha \geq \alpha_1, \alpha_0$ we have

$$|f_\alpha(x) - f(x)| \leq \begin{cases} \varepsilon, & x \in K, \\ 0, & x \in K^c. \end{cases}$$

Now it is easy to see that any positive function $u \in C_c(\mathbb{R})$ with $u(x) = 1$ for all $x \in K$ regulates the convergence $f_\alpha \xrightarrow{\tau_u} f$. \qed

By Proposition 2.3, a set $S \subset C_c(\mathbb{R})$ is relatively uniformly closed if and only if for $(f_n)_{n \in \mathbb{N}} \subset S$ and $f \in C_c(\mathbb{R})$ the existence of a compact set $K \subset \mathbb{R}$ such that $f_n|_{K^c} = 0$ for all $n \in \mathbb{N}$ and $f_n \xrightarrow{\|\cdot\|_{\infty}} f$ imply $f \in S$.

If a vector lattice $X$ has an order unit $u$, Example 2.2(a) yields that $\tau_u$ on $X$ agrees with the norm topology induced by the norm $\|\cdot\|_u$. The following proposition shows that vector lattices of Lipschitz continuous functions $\text{Lip}(\mathbb{R})$ and uniformly continuous functions $\text{UC}(\mathbb{R})$ on the real line posses order units.

**Proposition 2.4.** The function $u : x \mapsto 1 + |x|$ is an order unit of vector lattices $\text{Lip}(\mathbb{R})$ and $\text{UC}(\mathbb{R})$. 
Proof. Since $\text{Lip}(\mathbb{R}) \subseteq \text{UC}(\mathbb{R})$, it suffices to prove that $u \in \text{Lip}(\mathbb{R})$ is an order unit of $\text{UC}(\mathbb{R})$. To prove this, fix $f \in \text{UC}(\mathbb{R})$ and find $\delta > 0$ such that $|f(x) - f(y)| \leq 1$ whenever $|x - y| \leq \delta$.

Pick any $x \geq 0$. There exist $N \in \mathbb{N}_0$ and $0 \leq r < \delta$ such that $x = N\delta + r$. Then

$$|f(x)| \leq \sum_{n=1}^{N} |f(n\delta + r) - f((n-1)\delta + r)| + |f(r) - f(0)| + |f(0)|$$

$$\leq N + 1 + |f(0)| \leq \delta^{-1}|N\delta + r| + 1 + |f(0)|$$

$$\leq \lambda(1 + |x|),$$

where $\lambda := \max(\delta^{-1}, 1 + |f(0)|)$. The case when $x < 0$ can be treated similarly. □

It is well-known that $x_n \xrightarrow{ru} x$ implies $x_n \xrightarrow{ru^*} x$, see [LM67, Section 3]. While the backward implication is not true in general, for sequences $\tau_{ru}$ convergence is equivalent to the following. A sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is relatively uniformly $*$-convergent to $x$ if every subsequence of $(x_n)_{n \in \mathbb{N}}$ contains a further subsequence that is relatively uniformly convergent to $x$. Similarly, a net $(x_\alpha)_{\alpha} \subset X$ is relatively uniformly $*$-convergent to $x \in X$ if every subnet of $(x_\alpha)_{\alpha}$ contains a further subnet that is relatively uniformly convergent to $x$. We write $x_\alpha \xrightarrow{ru^*} x$ if a net or a sequence $(x_\alpha)$ relatively uniformly $*$-converges to $x$.

It is clear that relative uniform convergence implies relative uniform $*$-convergence in case of sequences and nets. Moreover, relative uniform $*$-convergence is tightly connected to $\tau_{ru}$-convergence. By [LM67, Theorem 3.5], $x_n \xrightarrow{ru^*} x$ is equivalent to $x_n \xrightarrow{ru} x$. The natural question that appears here is what happens when one replaces sequences by nets.

The following proposition shows that relative uniform $*$-convergence always implies $\tau_{ru}$-convergence. On the other hand, Example 2.6 will show that the converse implication, in general, is not true.

**Proposition 2.5.** If $x_\alpha \xrightarrow{ru^*} x$, then $x_\alpha \xrightarrow{ru} x$.

**Proof.** We first consider the special case when $x_\alpha \xrightarrow{ru} x$. Fix an open $\tau_{ru}$-neighborhood $U \subset X$ for $x$ and $(x_\alpha)_{\alpha} \subset X$ with $x_\alpha \xrightarrow{ru} x$ with respect to a regulator $u \in X$.

We claim that there exists $n \in \mathbb{N}$ such that $|x_\alpha - x| \leq \frac{1}{n} \cdot u$ implies $x_\alpha \in U$. Assume otherwise. Then for each $n \in \mathbb{N}$ there exists $\alpha_n$ such that $x_{\alpha_n} \notin U$ and $|x_{\alpha_n} - x| \leq \frac{1}{n} \cdot u$. From $x_{\alpha_n} \xrightarrow{ru} x$ we conclude
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\[ x_{\alpha n} \xrightarrow{\tau u} x \] which is a contradiction to \( x_{\alpha n} \not\in U \) for all \( n \in \mathbb{N} \). Hence, there exists \( n \in \mathbb{N} \) such that \(|x_{\alpha n} - x| \leq \frac{1}{n} \cdot u\) implies \( x_{\alpha} \in U \). Since \( x_{\alpha} \xrightarrow{\tau u} x \), there exists \( \alpha_0 \) such that \(|x_{\alpha} - x| \leq \frac{1}{n} \cdot u\) holds for all \( \alpha \geq \alpha_0 \) and hence, we have \( x_{\alpha} \in U \) for all \( \alpha \geq \alpha_0 \).

For the general case, assume that \( x_{\alpha} \xrightarrow{\tau u^*} x \) while \( x_{\alpha} \not\xrightarrow{\tau u} x \). Then there exists an open \( \tau u \)-neighborhood \( V \subset X \) of \( x \) such that for each \( \alpha \) there exists \( \beta_{\alpha} \geq \alpha \) with \( x_{\beta_{\alpha}} \not\in V \). We claim that \( (x_{\beta_{\alpha}})_{\alpha} \) is a subnet of \( (x_{\alpha})_{\alpha} \). For each \( \alpha_1 \) and \( \alpha_2 \) find \( \alpha \) such that \( \alpha \geq \beta_{\alpha_1}, \beta_{\alpha_2} \) and take \( \beta_{\alpha} \geq \alpha \). Hence \( (x_{\beta_{\alpha}})_{\alpha} \) is a net and by construction of \( (\beta_{\alpha})_{\alpha} \) it is a subnet of \( (x_{\alpha})_{\alpha} \). By assumption, there exists a subnet of \( (x_{\beta_{\alpha}})_{\alpha} \) which converges relatively uniformly to \( x \). This subnet necessarily \( \tau u \)-converges to \( x \). This is a contradiction to \( x_{\beta_{\alpha}} \not\in V \) for all \( \alpha \). □

**Example 2.6.** Consider the first uncountable ordinal \( \omega_1 \). It is well-known that \( \omega_1 \) is an uncountable well-ordered set and all countable subsets of \( \omega_1 \) have suprema. This immediately yields that no cofinal subset of \( \omega_1 \) is countable.

Let \( X \) be the vector lattice of all real functions on \( \omega_1 \) with countable support. By [Moo68, Example 2.2], the relative uniform topology on \( X \) is the topology of pointwise convergence. Consider the net \((\chi_\alpha)_{\alpha \in \omega} \in X \) where \( \chi_\alpha \) is the characteristic function of \( \{\alpha\} \). It is clear that \((\chi_\alpha)_{\alpha \in \omega} \) converges pointwise to \( 0 \).

Assume that there exists a subnet \((\chi_\beta) \) of \((\chi_\alpha) \) such that \( \chi_\beta \xrightarrow{\tau u} 0 \). Then there exists \( u \in X \) and \( \beta_0 \) such that \(|\chi_\beta| \leq u \) for all \( \beta \geq \beta_0 \). Hence, for all \( \beta \geq \beta_0 \) we have \( u(\beta) \neq 0 \). Since \( \omega_1 \) has no countable cofinal subsets, the set \( \{\beta : \beta \geq \beta_0 \} \) is uncountable, so that the support of \( u \) is uncountable. This is absurd.

3. SEMIGROUPS ON \((X, \tau u)\)

In this section we introduce two notions of continuity for semigroups on general vector lattices. By providing an example we show that these notions truly differ. We show that these notions expand semigroup theory to spaces which are not locally convex or complete. Furthermore, we provide conditions under which it is enough to check continuity of a semigroup on a \( \tau u \)-dense set to obtain continuity on the whole vector lattice.

A semigroup \((T(t))_{t \geq 0} \) on \( X \) is strongly continuous with respect to \( \tau u \) or \( \tau u \)-strongly continuous if for each \( x \in X \) the orbit map \( \zeta_x : (\mathbb{R}_+, \tau_u) \to \)
\[(X, \tau_{ru})\] is continuous, i.e.
\[\zeta_x(t + s) \xrightarrow{\tau_{ru}} \zeta_x(s)\]
for each \(s \geq 0\) as \(t \to 0\). If, in addition, we have
\[\zeta_x(t + s) \xrightarrow{ru} \zeta_x(s)\]
for each \(x \in X\) and \(s \geq 0\) as \(t \to 0\), then \((T(t))_{t \geq 0}\) is \textit{relatively uniformly continuous}. Since relative uniform convergence implies \(\tau_{ru}\)-convergence, every relatively uniformly continuous semigroup is \(\tau_{ru}\)-strongly continuous. In the special case when \(X\) is a Banach lattice, a positive semigroup \((T(t))_{t \geq 0}\) is \(\tau_{ru}\)-strongly continuous on \(X\) if and only if it is a positive \(C_0\)-semigroup on \(X\).

For a function \(f: \mathbb{R} \to \mathbb{R}\) and \(t \geq 0\), we consider the (left) translation operator
\[(T_l(t)f)(x) = f(t + x), \quad x \in \mathbb{R}\]
of \(f\) by \(t\). It is evident that by fixing a translation invariant space \(Y\) of functions on \(\mathbb{R}\) one obtains a semigroup \((T_l(t))_{t \geq 0}\) on \(Y\) which we call the (left) translation semigroup on \(Y\).

**Proposition 3.1.** The (left) translation semigroup \((T_l(t))_{t \geq 0}\) is relatively uniformly continuous on \(C_c(\mathbb{R})\), \(\text{Lip}(\mathbb{R})\) and \(\text{UC}(\mathbb{R})\).

**Proof.** We first prove that the translation semigroup is relatively uniformly continuous on \(C_c(\mathbb{R})\) by applying Proposition 2.3. Fix \(f \in C_c(\mathbb{R})\), \(\varepsilon > 0\) and \(s \geq 0\). Since \(f\) is uniformly continuous on \(\mathbb{R}\), there exists \(\delta > 0\) such that \(\|T_l(s + h)f - T_l(s)f\|_{\infty} < \varepsilon\) for all \(|h| < \delta\). This proves \(T_l(s + h)f \xrightarrow{\|\cdot\|_{\infty}} T_l(s)f\) in \(C_c(\mathbb{R})\) as \(h \to 0\).

Since \(T_l(s)f \in C_c(\mathbb{R})\), there exists \(n \in \mathbb{N}\) such that \(\text{supp} T_l(s)f \subseteq [-n, n]\). Choose any \(h_0 > 0\). If \(|h| < h_0\), then a direct computation shows that \(\text{supp} T_l(s + h)f \subseteq [-n - h_0, n + h_0]\).

We consider the remaining two cases simultaneously. Fix \(f \in \text{UC}(\mathbb{R})\), \(s \geq 0\) and \(\varepsilon > 0\). Since \(f\) is uniformly continuous, there exists \(\delta > 0\) such that \(|f(x) - f(y)| \leq \varepsilon\) whenever \(|x - y| < \delta\). This implies that for each \(x \in \mathbb{R}\) and \(h\) with \(|h| < \delta\) we have \(|(T_l(s + h)f)(x) - (T_l(s)f)(x)| \leq \varepsilon\). To finish the proof note that constant functions are Lipschitz continuous. \(\square\)

In the remaining part of this section we will weaken the hypothesis under which the semigroup is still \(\tau_{ru}\)-strongly continuous or relatively
uniformly continuous. The motivation comes from the general theory of $C_0$-semigroups on Banach spaces. As it is shown in [EN00, Proposition I.5.3], a semigroup on a Banach space $Y$ is a $C_0$-semigroup if and only if it is norm bounded on bounded time intervals and the orbit maps $\zeta_y$ are continuous on a norm dense set of elements of $Y$. This result heavily relies on the principle of uniform boundedness which is unavailable in general vector lattices. The following theorem is a vector lattice version of the above result for strong continuity with respect to $\tau_{ru}$ in the case when $\tau_{ru}$ is a linear topology. By [Moo68, Theorem 2.1], the topology $\tau_{ru}$ is linear whenever it is first countable.

**Theorem 3.2.** If $\tau_{ru}$ is a linear topology on $X$, then a semigroup $(T(t))_{t \geq 0}$ on $X$ is $\tau_{ru}$-strongly continuous if and only if for each $s \geq 0$ the following two assertions hold.

(i) There exists a $\tau_{ru}$-dense subset $D$ of $X$ such that $T(s+t)y \xrightarrow{\tau_{ru}} T(s)y$ as $t \to 0$ for each $y \in D$.

(ii) For each net $(x_\alpha)_\alpha \subset X$ with $x_\alpha \xrightarrow{\tau_{ru}} 0$ and each open $\tau_{ru}$-neighborhood of zero $V_0 \subset X$ there exists $\alpha_0$ and $\delta > 0$ such that

$$T(s+t)x_\alpha - T(s)x_\alpha_0 \in V_0$$

holds for all $t \in [-\delta, \delta]$ when $s > 0$ and all $t \in [0, \delta]$ when $s = 0$.

**Proof.** Since the forward implication is clear we only prove the backward implication.

Fix an open $\tau_{ru}$-neighborhood of zero $V_0 \subset X$ and take any open $\tau_{ru}$-neighborhood of zero $V_1 \subset X$ such that $V_1 + V_1 \subset V_0$. Fix $s \geq 0$ and $x \in X$. By (i) there exists a net $(x_\alpha) \subset X$ such that $x_\alpha \xrightarrow{\tau_{ru}} x$ and $T(s+t)x_\alpha \xrightarrow{\tau_{ru}} T(s)x_\alpha$ as $t \to 0$ for each $\alpha$. Hence, by (ii) there exist $\alpha_0$ and $\delta > 0$ such that

$$T(s+t)(x-x_\alpha_0) - T(s)(x-x_\alpha_0) \in V_1 \quad \text{and} \quad T(s+t)x_{\alpha_0} - T(s)x_{\alpha_0} \in V_1$$

hold for all $t \in [-\delta, \delta]$ when $s > 0$ and all $t \in [0, \delta]$ when $s = 0$. Therefore,

$$T(s+t)x - T(s)x = [T(s+t)(x-x_{\alpha_0}) - T(s)(x-x_{\alpha_0})]$$

$$+ [T(s+t)x_{\alpha_0} - T(s)x_{\alpha_0}] \in V_1 + V_1 \subset V_0$$

holds for all $t \in [-\delta, \delta]$ when $s > 0$ and all $t \in [0, \delta]$ when $s = 0$. This proves that $(T(t))_{t \geq 0}$ on $X$ is $\tau_{ru}$-strongly continuous. $\square$
In Section 5 we will establish an analogous version of Theorem 3.2 for relatively uniformly continuous semigroups on a particular class of vector lattices which allow a version of the principle of uniform boundedness.

The importance of the following corollary lies in its applicability. For locally $\tau_{ru}$-equicontinuous semigroups, $\tau_{ru}$-strong continuity is equivalent to $\tau_{ru}$-strong continuity at zero.

**Corollary 3.3.** Let $\tau_{ru}$ be a linear topology on $X$ and $(T(t))_{t \geq 0}$ a locally $\tau_{ru}$-equicontinuous semigroup on $X$. Then $(T(t))_{t \geq 0}$ is $\tau_{ru}$-strongly continuous if and only if there exists a $\tau_{ru}$-dense subset $D$ of $X$ such that $T(t)y \xrightarrow{\tau_{ru}} y$ as $t \searrow 0$ for each $y \in D$.

**Proof.** It suffices to prove the “only if” statement. We will check (i) and (ii) from Theorem 3.2. Fix $s > 0$ and an open $\tau_{ru}$-neighborhood of zero $V_0 \subset X$. There exists an open $\tau_{ru}$-neighborhood of zero $V_1 \subset X$ such that $V_1 + V_1 \subset V_0$. Since $(T(t))_{t \geq 0}$ is a locally $\tau_{ru}$-equicontinuous semigroup on $X$, there exists a symmetric open $\tau_{ru}$-neighborhood of zero $U \subset X$ such that $T(t)U \subset V_1$ for all $t \in [0, s + 1]$.

(i) Assume that there exists a $\tau_{ru}$-dense subset $D$ of $X$ such that for each $y \in D$ we have $T(t)y \xrightarrow{\tau_{ru}} y$ as $t \searrow 0$. Fix $y \in D$. There exists $\delta > 0$ such that $T(h)y - y \in U$ for all $h \in [0, \delta]$. Then

$$T(s + h)y - T(s)y = T(s)(T(h)y - y) \in V_1 \subset V_0$$

holds for all for all $h \in [0, \delta]$ and

$$T(s - h)y - T(s)y = T(s - h)(y - T(h)y) \in V_1 \subset V_0$$

holds for all $h \in [0, \min(\delta, s)]$ when $s > 0$. This proves (i).

(ii) Pick a net $(x_\alpha)_\alpha \subset X$ with $x_\alpha \xrightarrow{\tau_{ru}} 0$ and find $\alpha_0$ such that $x_{\alpha_0} \in U$. If $0 < h < 1$, then $T(s + h)x_{\alpha_0} \in V_1$. If $0 < h < s$, then again $T(s - h)x_{\alpha_0} \in V_1$. Hence, if $h$ satisfies $|h| < \min\{1, s\}$, we have

$$T(s + h)x_{\alpha_0} - T(s)x_{\alpha_0} \in V_1 + V_1 \subset V_0$$

which completes the proof. \qed

The following proposition shows that the notion of relative uniform continuity is, in general, stronger than the notion of strong continuity with respect to $\tau_{ru}$. Furthermore, it also provides an example of a completely metrizable locally solid vector lattice $(X, \tau_{ru})$ that is not
locally convex and a $\tau_{ru}$-strongly continuous semigroup on $X$ which is not relatively uniformly continuous.

Having in mind that $\tau_{ru}$ on Banach lattices agrees with norm topology, in the case $1 \leq p < \infty$ the following proposition recovers [EN00, Example I.5.4].

**Proposition 3.4.** For each $0 < p < \infty$ the (left) translation semigroup $(T_l(t))_{t \geq 0}$ on $L^p(\mathbb{R})$ is $\tau_{ru}$-strongly continuous but not relatively uniformly continuous.

**Proof.** Pick $0 < p < 1$ and denote by $\tau$ the topology induced by the metric $d_p$ on $L^p(\mathbb{R})$. Since $(L^p(\mathbb{R}), \tau)$ is a completely metrizable locally solid vector lattice, $\tau$ and $\tau_{ru}$ agree on $L^p(\mathbb{R})$ by Example 2.2(c). The same arguments as in the classical sense show that $C_c(\mathbb{R})$ is dense in $(L^p(\mathbb{R}), \tau_{ru})$.

Since $(T(t))_{t \geq 0}$ is relatively uniformly continuous on $C_c(\mathbb{R})$ by Proposition 3.1, it is also $\tau_{ru}$-strongly continuous. Furthermore, for each $t \geq 0$ the operator $T_l(t)$ preserves every open $d_p$-ball with center at zero, from where it follows that the semigroup $(T_l(t))_{t \geq 0}$ is locally $\tau_{ru}$-equicontinuous on $L^p(\mathbb{R})$. By Corollary 3.3, we conclude that $(T_l(t))_{t \geq 0}$ is $\tau_{ru}$-strongly continuous on $L^p(\mathbb{R})$.

To show that $(T_l(t))_{t \geq 0}$ is not relatively uniformly continuous on $L^p(\mathbb{R})$, consider the function

$$f: x \mapsto \frac{1}{|x - \frac{1}{2}|^{\frac{1}{p}}}$$

in $L^p(\mathbb{R})$. Assume that there exist a function $u \in L^p(\mathbb{R})$ and $0 < \delta < \frac{1}{2}$ such that $|T_l(t)f - f| \leq u$ holds in $L^p(\mathbb{R})$ for all $t \in [0, \delta]$, i.e.,

$$\left|\frac{1}{|t + x - \frac{1}{2}|^{\frac{1}{p}}} - \frac{1}{|x - \frac{1}{2}|^{\frac{1}{p}}}\right| \leq u(x)$$

holds for all $t \in [0, \delta]$ and almost every $x \in \mathbb{R}$. Hence, the family $\mathcal{F} = \{T_l(t)f : 0 \leq t \leq \delta\}$ is bounded above and since $L^p(\mathbb{R})$ is Dedekind complete, there exists $g := \sup\{T_l(t)f : 0 \leq t \leq \delta\}$ in $L^p(\mathbb{R})$. This is impossible since $g$ attains infinity on a set of positive measure. \(\square\)

The remaining part of this section is devoted to relatively uniformly continuous semigroups of positive operators. Our goal is to obtain a version (see Proposition 3.6) of Corollary 3.3 for relatively uniformly
continuous semigroups. In the case of $\tau_{ru}$-strongly continuous semigroups, we were able to provide such a result only for locally $\tau_{ru}$-equicontinuous semigroups on $X$. The reason behind is not so surprising. Consider a semigroup of bounded operators on a Banach lattice. Since $\tau_{ru}$ agrees with norm topology, local $\tau_{ru}$-equicontinuity agrees with local equicontinuity which is equivalent to uniform boundedness of the semigroup on bounded time intervals. Furthermore, applying the principle of uniform boundedness, the latter is equivalent to the fact that the semigroup is pointwise bounded on bounded time intervals. For more details see [EN00, Proposition I.5.3]. In the case of relatively uniformly continuous semigroups we conclude that relatively uniformly continuous positive semigroups are pointwise order bounded on bounded time intervals.

**Proposition 3.5.** Suppose $(T(t))_{t \geq 0}$ is a positive semigroup on $X$ such that for each $x \in X$ we have $T(t)x \xrightarrow{ru} x$ as $t \searrow 0$. Then for each $s \geq 0$ and $x \in X$ the set

$$\{|T(t)x|: 0 \leq t \leq s\}$$

is order bounded in $X$.

**Proof.** Fix $s \geq 0$ and $x \in X$. There exist $u \in X_+$ and $\delta > 0$ such that for all $0 \leq h \leq \delta$ we have $|T(h)x - x| \leq u$. Pick $t \in [0, s]$ and find $n \in \mathbb{N}_0$ and $0 \leq h < \delta$ such that $t = n\delta + h$. Then

$$|T(t)x| = |T(\delta)^n T(h)x| \leq T(\delta)^n (|x| + u).$$

Let $n_0$ be the smallest positive integer such that $n_0\delta \geq s$. If we define

$$v := \sqrt[n_0]{\sum_{k=0}^{n_0} T(\delta)^k (|x| + u)} \in X_+,$$

we have $\{|T(t)x|: 0 \leq t \leq s\} \subset [0, v]$. \hfill $\square$

As was already announced, the following result is a version of [EN00, Proposition I.5.3] for relatively uniformly continuous semigroups of positive operators. It says that a semigroup is relatively uniformly continuous if and only if it is relatively uniformly continuous at $t = 0$ and positive $x$.

**Proposition 3.6.** Let $(T(t))_{t \geq 0}$ be a positive semigroup on $X$. Then $(T(t))_{t \geq 0}$ is relatively uniformly continuous on $X$ if and only if $T(t)x \xrightarrow{ru} x$ as $t \searrow 0$ for positive vectors $x \in X_+$. 
Proof. Only the “if statement” requires a proof. Fix $s > 0$ and $x \in X$. Let $u$ be one of the regulators for $T(t)x_+ \overset{ru}{\rightarrow} x_+$ and $T(t)x_- \overset{ru}{\rightarrow} x_-$ as $t \searrow 0$. Pick $\varepsilon > 0$ and find $0 < \delta < s$ such that for all $h \in [0, \delta]$ we have $|T(h)x_+ - x_+| \leq \frac{\varepsilon}{2} \cdot u$ and $|T(h)x_- - x_-| \leq \frac{\varepsilon}{2} \cdot u$. By Proposition 3.5 we can also find $v \in X_+$ such that $T(h)u \leq v$ for all $h \in [0, s]$. Then

$$|T(s + h)x - T(s)x| \leq T(s)(|T(h)x_+ - x_+| + |T(h)x_- - x_-|) \leq \varepsilon \cdot T(s)u \leq \varepsilon \cdot v$$

and, similarly,

$$|T(s - h)x - T(s)x| \leq T(s - h)(|T(h)x_+ - x_+| + |T(h)x_- - x_-|) \leq \varepsilon \cdot T(s - h)u \leq \varepsilon \cdot v$$

hold for all $h \in [0, \delta]$. This proves that $(T(t))_{t \geq 0}$ is relatively uniformly continuous on $X$. \qed

4. Standard constructions

In this section we construct different relatively uniformly continuous semigroups from a given one. All the constructions are motivated by [EN00, Chapter I.5.b]. To prove that a given semigroup is relatively uniformly continuous we will tacitly use Proposition 3.6. For the sake of clarity, in this section $(T(t))_{t \geq 0}$ always denotes a given relatively uniformly continuous positive semigroup on a vector lattice $X$.

**Similar Semigroups.** Let $V : Y \to X$ be a lattice isomorphism between vector lattices $X$ and $Y$. Then $(V^{-1}T(t)V)_{t \geq 0}$ is a relatively uniformly continuous positive semigroup on $Y$.

**Proof.** It is easy to see that that $(S(T))_{t \geq 0}$ where $S(t) := V^{-1}T(t)V$ is a positive semigroup on $Y$. To prove that $(S(T))_{t \geq 0}$ is relatively uniformly continuous on $Y$, pick $y \in Y$ and $\varepsilon > 0$. Due to relative uniform continuity of $(T(t))_{t \geq 0}$ there exist $u \in X_+$ and $\delta > 0$ such that

$$|T(h)Vy -Vy| \leq \varepsilon \cdot u$$

holds for all $h \in [0, \delta]$. Since $V^{-1} : X \to Y$ is a lattice homomorphism, we obtain

$$|S(h)y - y| = V^{-1}|T(h)Vy -Vy| \leq \varepsilon \cdot (V^{-1}u)$$

for all $h \in [0, \delta]$. \qed
Next we consider semigroups on quotient vector lattices. Let $J$ be an ideal in $X$ and let $\pi: X \to X/J$ be the quotient projection between vector lattices $X$ and $X/J$. In order to guarantee that $X/J$ is Archimedean, we require our ideal $J$ to be relatively uniformly closed (see [LM67, Theorem 5.1]).

**Quotient Semigroups.** Suppose $J$ is a relatively uniformly closed ideal which is invariant under operator $T(t)$ for each $t \geq 0$. Then the family of operators $(\widetilde{T}(t))_{t \geq 0}$ defined for each $x \in X$ and $t \geq 0$ by

$$\widetilde{T}(t)\pi(x) = \pi(T(t)x)$$

is a relatively uniformly continuous positive semigroup on $X/J$.

**Proof.** It is easy to check that $(\widetilde{T}(t))_{t \geq 0}$ is a positive semigroup on $X/J$. To prove that $(\widetilde{T}(t))_{t \geq 0}$ is relatively uniformly continuous on $X/J$, pick $x \in X$ and $u \in X_+$ which regulates $T(t)x \xrightarrow{\text{ru}} x$ as $t \downarrow 0$. Choose $\varepsilon > 0$ and find $\delta > 0$ such that for all $h \in [0, \delta]$ we have $|T(h)x - x| \leq \varepsilon \cdot u$.

Since $\pi$ is a lattice homomorphism, we obtain

$$|\widetilde{T}(h)\pi(x) - \pi(x)| = |\pi(T(h)x) - \pi(x)| = \pi(|(T(h)x - x)|) \leq \varepsilon \cdot \pi(u)$$

for all $h \in [0, \delta]$. □

The next standard construction on our list are rescaled semigroups.

**Rescaled Semigroups.** For any numbers $\mu \in \mathbb{R}$ and $\alpha > 0$, the rescaled semigroup $(S(t))_{t \geq 0}$ defined by

$$S(t) := e^{\mu t}T(\alpha t)$$

is relatively uniformly continuous.

**Proof.** A direct computation shows that $(S(t))_{t \geq 0}$ is a positive semigroup. To prove that $(S(t))_{t \geq 0}$ is relatively uniformly continuous on $X$, pick $x \in X$ and find $u \in X_+$ which regulates $T(t)x \xrightarrow{\text{ru}} x$ as $t \downarrow 0$. Given any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for all $h \in [0, \delta_1]$ we have $|T(h)x - x| \leq \varepsilon \cdot u$. Since the function $h \mapsto e^{\mu h}$ is continuous, there exists $\delta_2 > 0$ such that for all $h \in [0, \delta_2]$ we have $|e^{\mu h} - 1| < \varepsilon$. If $\delta := \min\{\frac{\varepsilon}{\alpha}, \delta_2, 1\}$, then we have

$$|S(h)x - x| \leq e^{\mu h} \cdot |T(\alpha h)x - x| + (e^{\mu h} - 1)|x| \leq \varepsilon \cdot (\mu|u| + u)$$

for each $0 \leq h \leq \delta$. □
Next we deal with product semigroups. It is worth pointing out that the proof in our case is more complicated than the proof in the case of \(C_0\)-semigroups on Banach spaces.

**Product Semigroups.** Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be relatively uniformly continuous positive semigroups such that

\[
T(t)S(t) = S(t)T(t)
\]

holds for each \(t \geq 0\). Then \((T(t)S(t))_{t \geq 0}\) is a relatively uniformly continuous positive semigroup.

**Proof.** We prove first that \((T(t)S(t))_{t \geq 0}\) is a semigroup. As in [EN00, I.5.15] one can show that \(T(q_1)S(q_2) = S(q_2)T(q_1)\) holds for all \(q_1, q_2 \in \mathbb{Q}_+\). Fix \(t, s > 0\) and \(x \in X\). Find \(u\) which regulates \(T(t')x \xrightarrow{ra} T(t)x, T(s')x \xrightarrow{ra} T(s)x, T(t')S(s)x \xrightarrow{ra} T(t)S(s)x\) and \(S(s')T(t)x \xrightarrow{ra} S(s)T(t)x\) as \(t' \to t\) and \(s' \to s\). Pick \(\varepsilon > 0\) and find \(0 < \delta < 1\) such that for all \(t' \in [t, t + \delta]\) and \(s' \in [s, s + \delta]\) we have

\[
|T(t')x - T(t)x| \leq \frac{\varepsilon}{2} \cdot u, \quad |S(s')x - S(s)x| \leq \frac{\varepsilon}{2} \cdot u,
\]

\[
|T(t')S(s)x - T(t)S(s)x| \leq \frac{\varepsilon}{2} \cdot u, \quad |S(s')T(t)x - S(s)T(t)x| \leq \frac{\varepsilon}{2} \cdot u.
\]

By Proposition 5.4 we can find \(v \in X_+\) such that

\[
T(t')u \leq v \quad \text{and} \quad S(s')u \leq v
\]

hold for all \(t' \in [t, t + 1]\) and \(s' \in [s, s + 1]\). Pick \(t' \in [t, t + \delta] \cap \mathbb{Q}\) and \(s' \in [s, s + \delta] \cap \mathbb{Q}\). Since \(X\) is Archimedean and for each \(\varepsilon > 0\) we have

\[
|T(t)S(s)x - S(s)T(t)x|
\]

\[
\leq |T(t)S(s)x - T(t')S(s)x| + |T(t')S(s)x - T(t')S(s')x|
\]

\[
+ |T(t')S(s')x - S(s')T(t')x| + |S(s')T(t')x - S(s')T(t)x|
\]

\[
\leq \frac{\varepsilon}{2} \cdot u + T(t')|S(s)x - S(s')x| + S(s')|T(t')x - T(t)x| + \frac{\varepsilon}{2} \cdot u
\]

\[
\leq \varepsilon \cdot u + \frac{\varepsilon}{2} \cdot (T(t')u + S(s')u) \leq \varepsilon \cdot (u + v),
\]

we conclude \(T(t)S(s)x = S(s)T(t)x\). Since this holds for each \(x \in X\), we obtain \(T(t)S(s) = S(s)T(t)\). Now it is easy to deduce that \((T(t)S(t))_{t \geq 0}\) is a positive semigroup.

In order to prove that \((T(t)S(t))_{t \geq 0}\) is relatively uniformly continuous on \(X\), we first find \(u \in X_+\) such that for each \(\varepsilon > 0\) there is \(0 < \delta < 1\) with

\[
|T(h)x - x| \leq \varepsilon \cdot u \quad \text{and} \quad |S(h)x - x| \leq \varepsilon \cdot u
\]
for all $h \in [0, \delta]$. By Proposition 3.5 there exists $v \in X_+$ such that $T(h)u \leq v$ holds for all $h \in [0, 1]$. By combining all of the above we see that

$$|T(h)S(h)x - x| \leq T(h)|S(h)x - x| + |T(h)x - x| \leq \varepsilon \cdot (v + u)$$

holds for all $h \in [0, \delta]$. □

We finish this section with short comments on the subspace and adjoint semigroups.

By [EN00, Chapter I.5.12], if $J$ is a subspace of a Banach space $Y$ and $(T(t))_{t \geq 0}$ is a $C_0$-semigroup on $Y$ such that $T(t)J \subset J$ for each $t \geq 0$, then the restrictions $\tilde{T}(t) := T(t)|_J$ form a $C_0$-semigroup $(\tilde{T}(t))_{t \geq 0}$ on $J$. In general, such construction does not apply to relatively uniformly continuous semigroups as the following example shows.

**Example 4.1.** Consider the (left) translation semigroup on $C(\mathbb{R})$. It is obvious that every operator from the semigroup leaves the ideal $C_b(\mathbb{R})$ in $C(\mathbb{R})$ invariant. Since $C_b(\mathbb{R})$ has an order unit, relative uniform convergence agrees with norm convergence, so that in this case the (left) translation semigroup is relatively uniformly continuous if and only if it is a $C_0$-semigroup. It is well-known, however, that the (left) translation semigroup on $C_b(\mathbb{R})$ is not a $C_0$-semigroup. On the other hand, in Corollary 5.9 we will prove that the (left) translation semigroup is relatively uniformly continuous on $C(\mathbb{R})$.

When comparing to Banach space case, the distinction from the previous example appears when one wants to consider $\tau_{ru}$ on a sublattice $Y$ of a given lattice $X$. In the normed case, norm topology on the subspace agrees with the relative topology induced by the norm of the space. The relative topology on $Y$ induced by $\tau_{ru}$ from $X$ can be different than the relative uniform topology that $Y$ can induce on itself.

The situation with the adjoint semigroup is even subtler. If $X$ is a general vector lattice, probably the most natural candidate for the dual space is the order dual $X^\sim$ which, in fact, can be trivial (see [Zaa97, Theorem 25.1]). If $X$ is a Banach lattice, then $X^\sim = X^*$ and so one can consider the adjoint semigroup on $X^*$. In the Banach space case, the most common example of a positive $C_0$-semigroup whose adjoint semigroup is not a $C_0$-semigroup is the (left) translation semigroup on $L^1(\mathbb{R})$. However, neither the (left) translation semigroup on $L^1(\mathbb{R})$
nor its adjoint semigroup which is the (right) translation semigroup on $L^\infty(\mathbb{R})$ is relatively uniformly continuous.

5. The ru-Banach-Steinhaus property

In this section we present an analogous version of Theorem 3.2 and Corollary 3.3 for relatively uniformly continuous semigroups of positive operators. If one translates Theorem 3.2 to the setting of $C_0$-semigroups on Banach spaces, one instantly realizes that without the principle of uniform boundedness, in general, there is no hope of having such a result. Therefore, we need to restrict ourselves to a particular case of vector lattices. We will say that such vector lattices have the ru-Banach-Steinhaus property (see Definition 5.1). Before we formally introduce this property and elaborate more on its connection to the principle of uniform boundedness in the classical setting of Banach spaces, we need to recall some basic notions and facts on relative uniform convergence and topology.

A set $D \subset X$ is ru-dense if for each $x \in X$ there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset D$ such that $x_n \overset{ru}{\to} x$. A similar argument as in the proof of Proposition 2.1 shows that a set $D \subseteq X$ is ru-dense in $X$ if and only if for each $x \in X$ there is a net $(x_\alpha)$ in $D$ such that $x_\alpha \overset{ru}{\to} x$. Since $x_\alpha \overset{ru}{\to} x$ implies $x_\alpha \overset{\tau_{ru}}{\to} x$, a set $D \subseteq X$ is $\tau_{ru}$-dense in $X$ whenever it is ru-dense in $X$. The converse implication holds for vector lattices whose $\tau_{ru}$-topologies are completely sequential. Recall that $\tau_{ru}$-topology is said to be completely sequential if for any $S \subset X$ and any vector $x$ in the $\tau_{ru}$-closure of $S$ there exists a sequence in $S$ which converges relatively uniformly to $x$. Since in case of sequences $\tau_{ru}$ agrees with relative uniform $*$-convergence, it should be clear that $\tau_{ru}$-density in the completely sequential case implies ru-density. Before we close this short discussion on ru-density we would like to mention that $\tau_{ru}$ is completely sequential whenever $\tau_{ru}$ is first countable (see [Moo68, Theorem 2.1]).

Now we are finally able to introduce the appropriate class of vector lattices which enables us to develop the remaining part of [EN00, Proposition I.5.3] for relatively uniformly continuous semigroups.

**Definition 5.1.** A vector lattice $X$ has the ru-Banach-Steinhaus property if for each net of linear operators $(T_\alpha)_\alpha$ on $X$ the following two assertions imply $T_\alpha x \overset{ru}{\to} 0$ for each $x \in X$. 

\[ \text{Definition 5.1. A vector lattice } X \text{ has the ru-Banach-Steinhaus property if for each net of linear operators } (T_\alpha)_\alpha \text{ on } X \text{ the following two assertions imply } T_\alpha x \overset{ru}{\to} 0 \text{ for each } x \in X. \]
(i) There exists a $ru$-dense subset $D \subset X$ such that $T_\alpha y \overset{ru}{\to} 0$ for each $y \in D$.

(ii) For each sequence $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \overset{ru}{\to} 0$ there exists $u \in X_+$ such that for each $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and $\alpha_\varepsilon$ such that

$$|T_\alpha x_n| \leq \varepsilon \cdot u$$

holds for all $n \geq N_\varepsilon$ and $\alpha \geq \alpha_\varepsilon$.

Although Definition 5.1 at the first glance seems odd, it properly and nicely substitutes the principle of uniform boundedness from Banach spaces. Indeed, it can be easily shown that a sequence $(T_n)_{n \in \mathbb{N}}$ of operators on a Banach space $X$ converges pointwise to zero on $X$ if and only if it is uniformly bounded and it converges pointwise to zero on a dense subset of $X$. This is not true for nets since there are easy examples of norm unbounded nets of operators which converges pointwise to the zero operator. If, however, in Definition 5.1 we replace our net $(T_\alpha)_\alpha$ with a $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Banach space, then [EN00, Proposition I.5.3] yields that the semigroup $(T(t))_{t \geq 0}$ is norm bounded on time bounded intervals which relates to (ii) from Definition 5.1.

The main result of this section is Theorem 5.7 which is an analogous version of Corollary 3.3 for relatively uniformly continuous semigroups. Before we prove it, we will discuss two different classes of vector lattices which satisfy the ru-Banach-Steinhaus property. One of them is the class of completely metrizable locally solid vector lattices and the other one is the class of vector lattices which satisfies condition (R). Following [Vul67, Definition VI.5.1], a vector lattice $X$ satisfies condition (R) whenever for each sequence $(u_n)_{n \in \mathbb{N}} \subset X_+$ there exists a sequence of positive scalars $(\lambda_n)_{n \in \mathbb{N}}$ such that $(\lambda_n u_n)_{n \in \mathbb{N}}$ is order bounded. Vulikh introduced condition (R) in order to study order convergence. From the proof of [Vul67, Theorem VI.5.2] Swartz [Swa88] extracted the following property. A vector lattice $X$ is said to have property (C) whenever each countable set of relatively uniformly convergent sequences has a common regulator. Due to our best knowledge, it seems that it remained unnoticed in the literature that property (C) and condition (R) are equivalent.

**Proposition 5.2.** A vector lattice $X$ has property (C) if and only if $X$ satisfies condition (R).
Proof. ($\Rightarrow$) Fix a sequence $(u_n)_{n \in \mathbb{N}} \subset X_+$ and for each $n \in \mathbb{N}$ define the relatively uniformly converging sequence $(x_m^{(n)})_{m \in \mathbb{N}}$ by $x_m^{(n)} := \frac{1}{m} u_n$. Since $X$ has property (C), there exists a positive vector $u \in X_+$ such that for each $n \in \mathbb{N}$ the vector $u$ regulates the convergence $x_m^{(n)} \rightharpoonup u$ as $m \to \infty$. Hence, for each $n \in \mathbb{N}$ there exists $M_n \in \mathbb{N}$ with

$$\frac{1}{M_n} u_n \leq u.$$ 

This proves that $X$ satisfies condition (R).

($\Leftarrow$) Here we will follow the proof of [Vul67, Theorem VI.5.2]. Fix sequences $(x_n)_{n \in \mathbb{N}}, (u_n)_{n \in \mathbb{N}} \subset X$ and a double sequence $(x_{n,m})_{n,m \in \mathbb{N}} \subset X$ such that for each $n \in \mathbb{N}$ we have $x_{n,m} \rightharpoonup x_n$ as $m \to \infty$ with respect to some regulator $u_n$. Since $X$ satisfies condition (R), there exists a sequence of positive scalars $(\lambda_n)_{n \in \mathbb{N}}$ and $u \in X_+$ such that $\lambda_n u_n \leq u$ for each $n \in \mathbb{N}$. Since $x_{n,m} \rightharpoonup x_n$ as $m \to \infty$ is regulated by $\lambda_n u_n$, it is also regulated by $u$. \hfill $\Box$

Property (C) also holds for nets. The following corollary follows directly from Proposition 5.2.

Corollary 5.3. A vector lattice $X$ has property (C) if and only if any countable set of relatively uniformly convergent nets in $X$ has a common regulator.

It is easy to see that every vector lattice with an order unit has property (C). Theorem 5.4 shows that the class of vector lattice which have the ru-Banach-Steinhaus property is quite big. Apart to vector lattices with order units, it also contains the class of completely metrizable locally solid vector lattices.

Theorem 5.4. For a vector lattice $X$ consider the following assertions.

(a) There exists a topology $\tau$ on $X$ such that $(X, \tau)$ is completely metrizable locally solid vector lattice.

(b) $X$ has property (C).

(c) $X$ has the ru-Banach-Steinhaus property.

Then

$$(a) \Rightarrow (b) \Rightarrow (c)$$

Proof. (a)$\Rightarrow$(b) By Proposition 5.2 it is enough to show that $X$ satisfies condition (R). Since $(X, \tau)$ is completely metrizable there exists a countable neighborhood basis $\{V_n\}_{n \in \mathbb{N}}$ of zero in $(X, \tau)$ consisting of
solid sets such that for each \( n \in \mathbb{N} \) we have \( V_{n+1} + V_{n+1} \subset V_n \). Fix a sequence \((u_n)_{n \in \mathbb{N}} \subset X^+\) and for each \( n \in \mathbb{N} \) pick \( \lambda_n \) such that \( \lambda_n u_n \in V_n \). We claim that the series \( \sum_{i=1}^{\infty} \lambda_i u_i \) \( \tau \)-converges in \( X \). Define \( s_n = \sum_{i=1}^{n} \lambda_i u_i \) for each \( n \in \mathbb{N} \) and pick a solid neighborhood \( V_0 \) of zero in \((X, \tau)\). Find \( n_0 \in \mathbb{N} \) such that \( V_{n_0} \subset V_0 \). Then for \( m > n \geq n_0 \) we have
\[
s_m - s_n = \lambda_{n+1} u_{n+1} + \cdots + \lambda_m u_m \in V_n \subset V_{n_0} \subset V_0,
\]
and hence, the partial sums \((s_n)_{n \in \mathbb{N}}\) of the series \( \sum_{i=1}^{\infty} \lambda_i u_i \) form a Cauchy sequence in \((X, \tau)\). Since \((X, \tau)\) is complete and Hausdorff, by [AB03, Theorem 2.21] the series \( \sum_{i=1}^{\infty} \lambda_i u_i \) converges in \( X \) to some positive vector \( u \). Now it is clear that for each \( n \in \mathbb{N} \) we have \( \lambda_n u_n \leq u \) and so \( X \) satisfies condition \((R)\).

(b)\(\Rightarrow\)(c) Suppose that (i) and (ii) from Definition \ref{def:ru-dense} are satisfied for some ru-dense subset \( D \subset X \) and a net \((T_\alpha)_\alpha\) of linear operators on \( X \). We need to prove that \( T_\alpha x \xrightarrow{ru} 0 \) for each \( x \in X \).

Pick \( x \in X \) and find \((x_n)_{n \in \mathbb{N}} \subset D\) such that \( x_n \xrightarrow{ru} x \). By (ii) there exists \( u \in X^+ \) such that for each \( \varepsilon > 0 \) there exists \( N_\varepsilon \in \mathbb{N} \) and \( \alpha_\varepsilon \) such that
\[
|T_\alpha (x_n - x)| \leq \varepsilon \cdot u
\]
holds for all \( n \geq N_\varepsilon \) and \( \alpha \geq \alpha_\varepsilon \). Since for each \( n \in \mathbb{N} \) the net \((T_\alpha x_n)_\alpha\) converges relatively uniformly to 0 and since \( X \) has property \((C)\), by Corollary \ref{cor:relatively_uniform_convergence} there exists a positive vector \( \tilde{u} \in X \) which regulates the convergence \( T_\alpha x_n \xrightarrow{ru} 0 \) for each \( n \in \mathbb{N} \). Now, find \( \alpha_1 \) such that for all \( \alpha \geq \alpha_1 \) we have \( |T_\alpha x_{N_\varepsilon}| \leq \varepsilon \cdot \tilde{u} \). Find any \( \alpha_0 \) which is greater than \( \alpha_\varepsilon \) and \( \alpha_1 \). If \( \alpha \geq \alpha_0 \), then
\[
|T_\alpha x| \leq |T_\alpha (x - x_{N_\varepsilon})| + |T_\alpha x_{N_\varepsilon}| \leq \varepsilon \cdot (u + \tilde{u}). \tag*{\Box}
\]

At first glance it may seem that we do not require property \((C)\) in its entirety. The problem is that by changing the value \( \varepsilon \), also the integer \( N_\varepsilon \) changes which unfortunately forces the vector \( \tilde{u} = \tilde{u}(\varepsilon) \) to change itself. In this case, we cannot conclude that the vector \( u + \tilde{u} \) is a regulator of our convergence \( T_\alpha x \xrightarrow{ru} 0 \).

The following two examples show that the implications from Theorem \ref{thm:main_result} in general, cannot be reversed.

**Example 5.5.** By Proposition \ref{prop:order_unit} the function \( u : x \mapsto 1 + |x| \) is an order unit of \( \text{Lip} (\mathbb{R}) \) and hence, \( \text{Lip} (\mathbb{R}) \) has property \((C)\). If there would exist a complete metrizable locally solid topology \( \tau \) on \( \text{Lip} (\mathbb{R}) \), then \( \tau = \tau_{ru} \) by Example \ref{ex:order_unit}. Since \( u \) is an order unit for \( \text{Lip} (\mathbb{R}) \),
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τru agrees with norm topology induced by ∥ · ∥u. In order to reach a contradiction, we will show that the normed space (Lip(ℝ), ∥ · ∥u) is not complete.

It is clear that each function fn : ℝ → ℝ defined as fn(x) = \sqrt{|x| + \frac{1}{n}} is in Lip(ℝ). A direct calculation shows that the sequence (fn)n∈ℕ is Cauchy in (Lip(ℝ), ∥ · ∥u) and fn \rightharpoonup u f where f(x) = \sqrt{|x|}. Since f ∉ Lip(ℝ), we conclude (Lip(ℝ), ∥ · ∥u) is not complete.

Example 5.6. The vector lattice Cc(ℝ) has the ru-Banach-Steinhaus property, yet it does not have property (C).

To show that Cc(ℝ) does not have property (C), it suffices to check that Cc(ℝ) does not satisfy the equivalent condition (R). Pick any sequence (fn)n∈ℕ ⊂ Cc(ℝ) with fn = 1 on [−n, n]. It is easy to check that for any choice of positive scalars (λn)n∈ℕ ⊂ ℝ the sequence (λnf)n∈ℕ is not order bounded in Cc(ℝ).

To show that Cc(ℝ) has the ru-Banach-Steinhaus property, suppose that (i) and (ii) from Definition 5.1 are satisfied for some ru-dense subset D ⊂ Cc(ℝ) and some net (Tα)α of linear operators on Cc(ℝ).

We need to prove that Tαf \rightharpoonup 0 for each f ∈ Cc(ℝ).

Pick f ∈ Cc(ℝ) and find (fn)n∈ℕ ⊂ D such that fn \rightharpoonup f. Pick arbitrary ε > 0. First, by applying (ii), we find Nε ∈ ℕ and αε such that |Tα(f − fn)| ≤ \frac{ε}{2\|u\|∞} \cdot u holds for all α ≥ αε and n ≥ Nε. Next, by (i) there exist \tilde{u} ∈ Cc(ℝ) and α1 such that |TαfNε| ≤ \frac{ε}{2\|\tilde{u}\|∞} \cdot \tilde{u}

holds for all α ≥ α1. It is tempting to proceed with the same argument as in the proof of implication (b)⇒(c) of Theorem 5.4, however since Cc(ℝ) does not have property (C) each choice of ε provides a possibly different Nε, and therefore a possibly different \tilde{u}.

By Proposition 2.3, it suffices to show that Tαf \rightharpoonup 0 and that there exists a compact set K and an index β such that for each α ≥ β the function Tαf vanishes outside K. To see this, pick any α0 ≥ αε, α1 and observe that

|Tαf| ≤ |Tα(f − fnε)| + |Tαfnε| ≤ \frac{ε}{2\|u\|∞} \cdot u + \frac{ε}{2\|\tilde{u}\|∞} \cdot \tilde{u}
holds for all $\alpha \geq \alpha_0$. This yields $T_\alpha f \xrightarrow{\|\cdot\|_\infty} 0$ and for each $\alpha \geq \alpha_0$ the function $T_\alpha f$ vanishes outside the union of supports of $u$ and $\tilde{u}$. This finally proves the claim.

In conclusion, the class of vector lattices which have the ru-Banach-Steinhaus property is at least as big that it contains vector lattices $L^p(\mathbb{R})$ ($0 < p < \infty$), $C(\mathbb{R})$, $C_c(\mathbb{R})$, and $\text{Lip}(\mathbb{R})$ which are examples of very important spaces where one wants to solve (partial) differential equations.

The following theorem is the main result of this section. It is a version of Corollary 3.3 for relatively uniformly continuous semigroups.

**Theorem 5.7.** Let $X$ have the ru-Banach-Steinhaus property and $(T(t))_{t \geq 0}$ be a positive semigroup on $X$. Then $(T(t))_{t \geq 0}$ is relatively uniformly continuous on $X$ if and only if the following two assertions hold.

(i) There exists an ru-dense subset $D \subset X$ such that $T(t)y \xrightarrow{\text{ru}} y$ as $t \searrow 0$ for each $y \in D$.

(ii) For each $s \geq 0$ and $x \in X$ the set

$$\{|T(t)x| : 0 \leq t \leq s\}$$

is order bounded in $X$.

**Proof.** $(\Rightarrow)$ If $(T(t))_{t \geq 0}$ is a positive relatively uniformly continuous semigroup on $X$, then (i) obviously holds for $D = X$ and (ii) follows from Proposition 3.5.

$(\Leftarrow)$ Fix $x \in X$ and define a net of linear operators $(T_h)_{h \in [0,1]}$ on $X$ by $T_h := T(h) - I$. By Proposition 3.6, it suffices to prove that $T_h x \xrightarrow{\text{ru}} 0$ as $h \searrow 0$. Since $X$ has the ru-Banach-Steinhaus property, it suffices to check (i) and (ii) of Definition 5.1.

Clearly (i) holds by our assumption. To check (ii), fix a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \xrightarrow{\text{ru}} 0$ with respect to some regulator $u \in X_+$. By Assumption (ii), we can find $v \in X_+$ such that $T(h)u \leq v$ for all $h \in [0,1]$. Hence, for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$ and $h \in [0,1]$ we have

$$|T_h x_n| \leq T(h)|x_n| + |x_n| \leq \varepsilon \cdot T(h)u + \varepsilon \cdot u \leq \varepsilon \cdot (v + u).$$

As an application of Theorem 5.7 we will show in Corollary 5.9 that the (left) translation semigroup is relatively uniformly continuous on $C(\mathbb{R})$. In order to do this, we consider the space $\mathcal{LPA}(\mathbb{R})$ of locally
piecewise affine functions on $\mathbb{R}$ from [AT17]. Although the space of locally piecewise affine functions was defined in [AT17] on $\mathbb{R}^n$, in the case when $n = 1$ it is easy to see that $f \in \mathcal{LPA}(\mathbb{R})$ if and only if there exist sequences $(a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}}, (j_n)_{n \in \mathbb{Z}} \subset \mathbb{R}$ such that $\bigcup_{n \in \mathbb{Z}} [j_n, j_n+1] = \mathbb{R}$ and

1. \(j_n < j_{n+1}\),
2. \(f(x) = a_n \cdot x + b_n\) holds for all $x \in [j_n, j_{n+1}]$, and
3. $b_{n-1} - b_n = (a_n - a_{n-1}) j_n$

hold for all $n \in \mathbb{Z}$. [AT17, Theorem 4.1] yields that $\mathcal{LPA}(\mathbb{R})$ is ru-dense in $C(\mathbb{R})$.

**Proposition 5.8.** The (left) translation semigroup $(T_t(t))_{t \geq 0}$ is relatively uniformly continuous on $\mathcal{LPA}(\mathbb{R})$.

**Proof.** Fix $f \in \mathcal{LPA}(\mathbb{R})$ and pick sequences $(a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}}, (j_n)_{n \in \mathbb{Z}} \subset \mathbb{R}$ from the characterization of functions in $\mathcal{LPA}(\mathbb{R})$. For each $n \in \mathbb{Z}$ we define

$$
\delta_n := \min \left\{ \frac{j_{n+1} - j_n}{2}, \frac{j_n + 2 - j_{n+1}}{2} \right\} \\
c_n := \sup_{y \in [j_n, j_{n+1}]} \frac{|f(\delta_n + y) - f(y)|}{\delta_n} \\
d_n := \max_{n-1 \leq i \leq n+1} \{|a_i|, c_i| \}.
$$

Now we define a function $u: \mathbb{R} \to \mathbb{R}$ on each interval $[j_n, j_{n+1}]$ separately; if $x \in [j_n, j_{n+1}]$ we define

$$
u(x) = \begin{cases} 
  d_n, & x \in [j_n, j_n+1 - \delta_n], \\
  (d_{n+1} - d_n) \frac{x-(j_{n+1} - \delta_n)}{\delta_n} + d_n, & x \in [j_{n+1} - \delta_n, j_{n+1}].
\end{cases}
$$

A direct verification shows that $u \in \mathcal{LPA}(\mathbb{R})$. We claim that the function $u$ regulates $T_t(h)f \xrightarrow{\tau_t} f$ as $h \searrow 0$. To this end, choose $0 < \varepsilon < 1$, set $\delta = \frac{\varepsilon}{4}$ and pick $h \in (0, \delta]$. We will prove that $|T_t(h)f - f| \leq \varepsilon \cdot u$ holds pointwise on each interval $[j_n, j_{n+1}]$ where $n \in \mathbb{Z}$.

**Case 1:** Assume that $0 < h < \delta_n$. If $x \in [j_n, j_n+1 - h]$, then $x + h \in [j_n, j_{n+1}]$, so that

$$
|f(h + x) - f(x)| = |a_n(h + x) + b_n - (a_nx + b_n)| = h \cdot |a_n| \leq h \cdot u(x)
$$
holds since \(|a_n| \leq \min\{d_n, d_{n+1}\} \leq u(x)\). If \(x \in [j_{n+1} - h, j_{n+1}]\), then \(j_{n+1} \leq h + x < \delta_n + j_{n+1} < j_{n+2}\) yields
\[
|f(h + x) - f(x)| = |a_{n+1}(h + x) + b_{n+1} - (a_n \cdot x + b_n)|
\leq |(a_{n+1} - a_n)x + b_{n+1} - b_n| + h \cdot |a_{n+1}|
= |a_{n+1} - a_n(j_{n+1} - x) + h \cdot |a_{n+1}|.
\]
On the other hand, since \(x \in [j_{n+1} - h, j_{n+1}]\), then \(0 \leq j_{n+1} - x \leq h\), from where we further conclude
\[
|f(h + x) - f(x)| \leq h \cdot (|a_{n+1} - a_n| + |a_{n+1}|) \leq h \cdot (2|a_{n+1}| + |a_n|)
\leq 3h \cdot \min(d_n, d_{n+1}).
\]
We finally obtain \(|f(h + x) - f(x)| \leq 3h \cdot u(x) \leq \varepsilon \cdot u(x)\) for each \(x \in [j_n, j_{n+1}]\).

**Case 2:** Assume that \(h \geq \delta_n\). Then there exist \(N_n \in \mathbb{N}\) and \(0 \leq r_n < \delta_n\) such that \(h = N_n\delta_n + r_n\). Fix \(x \in [j_n, j_{n+1}]\). By Case 1, we have \(|f(r_n + x) - f(x)| \leq 3r_n \cdot u(x)\). By an easy application of the triangle inequality, we can estimate
\[
|f(h + x) - f(x)| \leq \sum_{m=1}^{N_n} |f(m\delta_n + r_n + x) - f((m-1)\delta_n + r_n + x)|
+ |f(r_n + x) - f(x)|.
\]
For each \(1 \leq m \leq N_n\) we denote \(y_m := (m-1)\delta_n + r_n + x\). Since \(\varepsilon < 1\), also \(h < 1\) and hence, \(j_n \leq y_m \leq 1 + j_{n+1}\). By definition of the number \(c_n\) and the estimate \(c_n \leq \min(d_n, d_{n+1}) \leq u(x)\), we obtain
\[
|f(h + x) - f(x)| \leq \sum_{m=1}^{N_n} |f(\delta_n + y_m) - f(y_m)| + |f(r_n + x) - f(x)|
\leq N_n \delta_n \cdot c_n + 3r_n \cdot u(x) \leq 4h \cdot u(x) \leq \varepsilon \cdot u(x)
\]
which concludes the proof. \(\square\)

**Corollary 5.9.** The (left) translation semigroup \((T_t(t))_{t \geq 0}\) is relatively uniformly continuous on \(C(\mathbb{R})\).

**Proof.** The vector lattice \(C(\mathbb{R})\) equipped with the topology of uniform convergence on compact sets is a completely metrizable locally solid vector lattice and hence, by Theorem 5.4, it has the ru-Banach-Steinhaus property. By [AT17, Theorem 4.1], the space \(LPA(\mathbb{R})\) is ru-dense in \(C(\mathbb{R})\) and, by Proposition 5.8 for each \(g \in LPA(\mathbb{R})\) we
have $T_t g \xrightarrow{ru} g$ as $t \searrow 0$. Furthermore, for each $f \in C(\mathbb{R})$ and $s \geq 0$ the function $u \in C(\mathbb{R})$ defined by $u(x) := \max_{0 \leq t \leq s} |f(t + x)|$ for all $x \in \mathbb{R}$ satisfies $|T_t f| \leq u$ for all $t \in [0, s]$. Hence, by Theorem 5.7, the semigroup $(T_t)_{t \geq 0}$ is relatively uniformly continuous on $C(\mathbb{R})$. □

Acknowledgments and further remarks

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