EXTREMES OF $\alpha(t)$-LOCALLY STATIONARY GAUSSIAN PROCESSES WITH NON-CONSTANT VARIANCES

LONG BAI

Abstract: With motivation from [11], in this paper we derive the exact tail asymptotics of $\alpha(t)$-locally stationary Gaussian processes with non-constant variance functions. We show that some certain variance functions lead to qualitatively new results.

Key Words: Fractional Brownian motion; $\alpha(t)$-locally stationary; Pickands constants; Gaussian process.

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1. Introduction and Main Result

For $X(t),\ t \in [0,T],\ T > 0$ a centered stationary Gaussian process with unit variance and continuous sample paths Pickands derived in [20] that

$$
P \left\{ \sup_{t \in [0,T]} X(t) > u \right\} \sim T H_\alpha a^{1/\alpha} u^{2/\alpha}\P \{ X(0) > u \},\ u \to \infty,$$

provided that the correlation function $r$ satisfies

$$
1 - r(t) \sim a |t|^{\alpha},\ t \downarrow 0,\ a > 0,\ \text{and} r(t) < 1, \forall t \neq 0,$$

with $\alpha \in (0, 2]$ ($\sim$ means asymptotic equivalence when the argument tends to 0 or $\infty$). Here the classical Pickands constant $H_\alpha$ is defined by

$$
H_\alpha = \lim_{T \to \infty} T^{-1} \mathbb{E} \left\{ \sup_{t \in [0,T]} e^{\sqrt{2} B_{\alpha}(t) - t^{\alpha}} \right\},
$$

where $B_{\alpha}(t),\ t \geq 0$ is a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0, 1]$, see [20, 21, 8, 13, 14, 9, 23, 10, 12, 5, 15] for various properties of $H_\alpha$.

The deep contribution [3] introduced the class of locally stationary Gaussian processes with index $\alpha$, i.e., a centered Gaussian process $X(t), t \in [0, T]$ with a constant variance function, say equal to 1, and correlation function satisfying

$$
r(t, t + h) = 1 - a(t)|h|^{\alpha} + o(|h|^{\alpha}),\ h \to 0,
$$

uniformly with respect to $t \in [0, T]$, where $\alpha \in (0, 2]$ and $a(t)$ is a bounded, strictly positive and continuous function.

Clearly, the class of locally stationary Gaussian processes includes the stationary ones. It allows for some minor fluctuations of dependence at $t$ and at the same time keeps stationary structure at the local scale. See [3, 4, 18] for studies on the locally stationary Gaussian processes with index $\alpha$.

In [11] the tail asymptotics of the supremum of $\alpha(t)$-locally stationary Gaussian processes are investigated. Such processes and random fields are of interest in various applications, see [11] and the recent contributions [2, 16, 17].

Following the definition in [11], a centered Gaussian process $X(t), t \in [0, T]$ with continuous sample paths and unit variance is $\alpha(t)$-locally stationary if the correlation function $r(\cdot, \cdot)$ satisfies the following conditions:

(i) $\alpha(t) \in C([0, T])$ and $\alpha(t) \in (0, 2]$ for all $t \in [0, T]$;

(ii) $\alpha(t) \in C([0, T])$ and $0 < \inf \{ \alpha(t) : t \in [0, T] \} \leq \sup \{ \alpha(t) : t \in [0, T] \} < \infty$;
A crucial assumption in our result is that similar to the variance function, the function $\alpha$ point of $t$ behaviour around the extreme point $t_0$. Specifically, as in [11] we shall assume:

(v) there exist $\beta, \delta, b > 0$ such that

$$\alpha(t + t_0) = \alpha(t_0) + b|t|^\beta + o(|t|^\beta + \delta), \quad t \to 0.$$ 

Remark 1.1. We remark that $t_0$ does not need to be the unique point such that $\alpha(t)$ is minimal on $[0, T]$, which is different from [11]. For instance, $[0, T] = [0, 2\pi], t_0 = 0$ and $\alpha(t) = 1 + \frac{1}{2}\sin(t)$, then $0$ is not the minimum point of $\alpha(t)$ over $[0, 2\pi]$ which means assumptions about $\alpha(t)$ in [11] are not satisfied but assumption (v) here is satisfied with

$$\alpha(t) = 1 + \frac{1}{2}|t| + o(|t|^\frac{1}{2}), \quad t \to 0.$$ 

Below we set $\alpha := \alpha(t_0), a := a(t_0)$ and write $\Psi$ for the survival function of an $N(0, 1)$ random variable. Further, define $0^a = \infty$ for $a < 0$. Our main result is stated in the next theorem.

**Theorem 1.2.** If a centered Gaussian process $X(t), t \in [0, T]$ with continuous sample paths is such that the assumptions (i)-(v) are valid, then we have as $u \to \infty$

$$\mathbb{P}\left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \hat{I} a^{1/\alpha} H_\alpha e^{2/\alpha (\ln u)^{1/\beta}} \Psi(u) \begin{cases} 2^{-1/\gamma}, & \text{if } \gamma < \beta, \\ \int_0^{2^{-1/\gamma}} e^{\frac{2x}{\alpha}} dx, & \text{if } \gamma = \beta, \\ \int_0^{\infty} e^{-\frac{2x}{\alpha}} dx, & \text{if } \gamma > \beta, \end{cases}$$

where $\gamma \wedge \beta = \min(\gamma, \beta)$ and

$$\hat{I} = \begin{cases} 1, & \text{if } t_0 = 0 \text{ or } t_0 = T, \\ 2, & \text{if } t_0 \in (0, T). \end{cases}$$

Remark 1.3. i) If $\alpha(t) \equiv \alpha$ for all $t$ in a small neighborhood of $t_0$, the asymptotic of $\mathbb{P}\left\{ \sup_{t \in [0, T]} X(t) > u \right\}$ is the same as in the case of $\gamma < \beta$ in Theorem 1.2.

ii) The result of case $\gamma > \beta$ in Theorem 1.2 is the same as the $\alpha(t)$-locally stationary scenario in [11], which means that $\sigma(t)$ varies so slow in a small neighborhood of $t_0$ that $X(t)$ can be considered as $\alpha(t)$-locally stationary in this small neighborhood.

The following example is a straightforward application of Theorem 1.2.

**Example 1.4.** Here we consider a multifractional Brownian motion $B_{H(t)}(t), t \geq 0$, i.e., a centered Gaussian process with covariance function

$$E\left\{ B_{H(t)}(t)B_{H(s)}(s) \right\} = \frac{1}{2} D(H(s) + H(t)) \left[ |s|^{H(s)+H(t)} + |t|^{H(s)+H(t)} - |t - s|^{H(s)+H(t)} \right],$$
where $D(x) = \frac{2\pi}{1 + (x+1)\sin(\frac{\pi}{2})}$ and $H(t)$ is a Hölder function of exponent $\lambda$ such that $0 < H(t) < \min(1, \lambda)$ for $t \in [0, \infty)$. For constants $T_1, T_2$ with $0 < T_1 < T_2$, define

$$B_{H(t)}(t) := \frac{B_H(t)}{\sqrt{\text{Var}(B_{H(t)}(t))}}, \quad t \in [T_1, T_2],$$

and

$$\sigma(t) := 1 - e^{-|t-t_0|^{-\gamma}}, \quad t \in [T_1, T_2],$$

with some $t_0 \in (T_1, T_2)$ and $\gamma > 0$.

By [11], $B_{H(t)}(t)$, $t \in [T_1, T_2]$, is a $2H(t)$-locally stationary Gaussian process with correlation function

$$r(t, t+h) = 1 - \frac{1}{2}t^{-2H(t)}|h|^{2H(t)} + o(|h|^{2H(t)}), \quad h \to 0.$$ 

Further, we assume that there exist $\beta, \delta, b > 0$ such that $H(t+\delta) = H(t) + bt^\beta + o(t^{\beta+\delta})$, as $t \to 0$. Then

$$\mathbb{P}\left\{ \sup_{t \in [T_1, T_2]} \sigma(t)B_{H(t)}(t) > u \right\} \sim 2^{1-1/2H} \frac{\mathcal{H}_{2H}}{t_0} u^{1/H}(\ln u)^{-\frac{1}{2-H}} \Psi(u) \begin{cases} 2^{-1/\gamma}, & \text{if } \gamma < \beta, \\ \int_0^2 e^{\frac{b(x)}{\gamma}} dx, & \text{if } \gamma = \beta, \quad u \to 0. \\ \int_0^\infty e^{\frac{b(x)}{\gamma}} dx, & \text{if } \gamma > \beta, \end{cases}$$

with $H := H(t_0)$.

2. PROOFS

In the rest of the paper, we focus on the case when $t_0 = 0$. The complementary scenario when $t_0 \in (0, T]$ follows by analogous argumentation. Recall that

$$\mathcal{H}_\alpha = \lim_{T \to \infty} \frac{1}{T} \mathcal{H}_\alpha[0, T], \quad \text{with } \mathcal{H}_\alpha[-S_1, S_2] = \mathbb{E}\left\{ \sup_{t \in [-S_1, S_2]} e^{\sqrt{2}B_{\alpha}(t)-|t|^\alpha} \right\} \in (0, \infty),$$

where $S_1, S_2 \in [0, \infty)$ with $\max(S_1, S_2) > 0$ are some constants.

Lemma 2.1. Under the assumptions of Theorem 1.2 we have

$$\mathbb{P}\left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \mathbb{P}\left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\}, \quad u \to \infty. \tag{3}$$

Moreover, there exists a constant $C > 0$ such that for all sufficiently large $u$

$$\mathbb{P}\left\{ \sup_{t \in [\delta_2(u), T]} X(t) > u \right\} \leq C T u^{2/\alpha} (\ln u)^{-4/3\beta} \Psi(u), \tag{4}$$

where for some constant $q > 1$

$$\delta_1(u) = \left( \frac{1}{2 \ln u - q \ln \ln u} \right)^{1/\gamma} \quad \text{and} \quad \delta_2(u) = \left( \frac{\alpha^2(\ln(\ln u))}{\beta(\ln u)} \right)^{1/\beta}. \tag{5}$$

By (4), in the proof of Theorem 1.2, we derive that, as $u \to \infty$,

$$\mathbb{P}\left\{ \sup_{t \in [\delta_2(u), T]} X(t) > u \right\} = o \left( \mathbb{P}\left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\} \right). \tag{6}$$

Since $\delta_1(u) \to 0, \delta_2(u) \to 0$ as $u \to \infty$ and $a(t)$ is continuous, without loss of generality, we may assume that $a(t) \equiv a(0) = a$ for $t \in ([0, \delta_1(u)] \cup [0, \delta_2(u)])$. Moreover, by assumption (iv), we know that $\sigma(t) > 0$ for $t \in ([0, \delta_1(u)] \cup [0, \delta_2(u)])$. Below we use notation $\overline{X}(t) = \frac{X(t)}{\sigma(t)}$ for all $t$ such that $\sigma(t)$ is positive.

**Proof of Theorem 1.2:** First we derive the asymptotic of

$$\pi(u) := \mathbb{P}\left\{ \sup_{t \in \Delta(u)} X(t) > u \right\},$$

where $\Delta(u) = (\omega(u) [1 + o(1)])$. Further, we assume that there exist $\beta, \delta, b > 0$ such that $H(t+\delta) = H(t) + bt^\beta + o(t^{\beta+\delta})$, as $t \to 0$. Then

$$\mathbb{P}\left\{ \sup_{t \in \Delta(u)} \sigma(t)\overline{X}(t) > u \right\} \sim 2^{1-1/2H} \frac{\mathcal{H}_{2H}}{t_0} u^{1/H}(\ln u)^{-\frac{1}{2-H}} \Psi(u) \begin{cases} 2^{-1/\gamma}, & \text{if } \gamma < \beta, \\ \int_0^2 e^{\frac{b(x)}{\gamma}} dx, & \text{if } \gamma = \beta, \quad u \to 0. \\ \int_0^\infty e^{\frac{b(x)}{\gamma}} dx, & \text{if } \gamma > \beta, \end{cases}$$

with $H := H(t_0)$.
as \( u \to \infty \), where \( \Delta(u) = [0, \delta(u)] \) and

\[
\delta(u) = \begin{cases} \delta_1(u), & \text{if } \gamma \leq \beta, \\ \delta_2(u), & \text{if } \gamma > \beta,
\end{cases}
\]

with \( \delta_1(u) \) and \( \delta_2(u) \) in (5), which combined with Lemma 2.1 finally shows that

\[
\mathbb{P} \left\{ \sup_{t \in [0,T]} X(t) > u \right\} \sim \pi(u).
\]

In the following \( Q_i, \ i \in \mathbb{N} \), are some positive constants. For some \( S > 0 \), let \( Y_{\nu,u}(t), t \in [0,S] \) be a family of centered stationary Gaussian processes with

\[
\text{Cov} (Y_{\nu,u}(s), Y_{\nu,u}(t)) = 1 - (1 - \nu)au^{-2}|s - t|^{\alpha + 2b\delta^2(u)},
\]

for \( \nu \in (0, 1), u > 0 \) such that \( \alpha + 2b\delta^2(u) \leq 2 \) and \( s, t \in [0, S] \). Further, let \( Z_{\nu,u}(t), t \in [0, S] \) be another family of centered stationary Gaussian processes with

\[
\text{Cov} (Z_{\nu,u}(s), Z_{\nu,u}(t)) = 1 - (1 + \nu)au^{-2}|s - t|^{\alpha},
\]

for \( \nu \in (0, 1), u > 0 \) and \( s, t \in [0, S] \). Due to assumptions (i) and (v), \( \alpha \) is strictly smaller than 2, which guarantees that covariance function of \( Y_{\nu,u}(t), t \in [0, S] \) and \( Z_{\nu,u}(t), t \in [0, S] \) are positive-definite. Hence the introduced families of Gaussian processes exist.

By assumption (iv), for any small \( \varepsilon \in (0, 1) \)

\[
1 + (1 - \varepsilon)e^{-|t|^{-\gamma}} \leq \frac{1}{\sigma(t)} \leq 1 + (1 + \varepsilon)e^{-|t|^{-\gamma}},
\]

holds for \( t \in [0, \delta(u)] \).

**Case 1**: \( \gamma < \beta \). Set for any \( \epsilon \in (0,1) \) and all \( u \) large

\[
N(0) = N(u, 0) := \left[ \frac{\delta_1(u)u^{2/\alpha}}{S} \right], \quad N_\epsilon(u) = \left[ (1 - \epsilon)\frac{\delta_1(u)u^{2/\alpha}}{S} \right] = \left[ \frac{(1 - \epsilon)u^{2/\alpha}}{2\ln u - q\ln\ln u}^{1/\gamma} \right],
\]

\[
B_j(u) = B_{j,0}(u) = \left[ j \frac{S}{u^{2/\alpha}} \right], \quad j \in \mathbb{N}, \quad G_{\alpha}^{\pm \varepsilon} = u \left( 1 + \pm \varepsilon \right)e^{-((1-\epsilon)\delta_1(u))^{-\gamma}}.
\]

We notice the fact that

\[
\Psi(G_{\alpha}^{\pm \varepsilon}) \sim \Psi(u), \ u \to \infty,
\]

and

\[
I_1(u) \leq \pi(u) \leq I_1(u) + I_2(u),
\]

where

\[
I_1(u) = \mathbb{P}\left\{ \sup_{t \in [0,(1-\epsilon)\delta_1(u)]} X(t) > u \right\}, \quad I_2(u) = \mathbb{P}\left\{ \sup_{t \in [(1-\epsilon)\delta_1(u), \delta_1(u)]} X(t) > u \right\}.
\]

Then by Bonferroni’s inequality, (8), Lemma 3.1 with \( k = 0 \) and Lemma 3.2

\[
I_1(u) \leq \sum_{j=0}^{N_\epsilon(u)} \mathbb{P}\left\{ \sup_{t \in B_{j}(u)} X(t) > u \right\}
\]

\[
\leq \sum_{j=0}^{N_\epsilon(u)} \mathbb{P}\left\{ \sup_{t \in B_{j}(u)} X(t) > G_{\alpha}^{\varepsilon} \right\}
\]

\[
\leq \sum_{j=0}^{N_\epsilon(u)} \mathbb{P}\left\{ \sup_{t \in [jS,(j+1)S]} X(tu^{-2/\alpha}) > G_{\alpha}^{\varepsilon} \right\}.
\]
\[ \left\{ \begin{array}{l}
\lesssim \sum_{j=0}^{N_s(u)} \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{u, \nu}(t) > G_u^{\epsilon} \right\} \\
\sim \sum_{j=0}^{N_s(u)} H_\alpha \left[ 0, S((1 + \nu)a)^{1/\alpha} \right] \Psi \left( G_u^{\epsilon} \right) \\
\sim \sum_{j=0}^{N_s(u)} H_\alpha \left[ 0, S((1 + \nu)a)^{1/\alpha} \right] \Psi(u) \\
\sim (1 - \epsilon) u^{2/\alpha} \delta_1(u) H_\alpha \left[ 0, S((1 + \nu)a)^{1/\alpha} \right] S \Psi(u), \ u \to \infty, \ S \to \infty.
\end{array} \right. \tag{10} \]

Similarly,
\[ \left\{ \begin{array}{l}
\sum_{j=0}^{N_s(u)-1} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u \right\} \geq \sum_{j=0}^{N_s(u)-1} \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{u, \nu}(t) > G_u^{\epsilon} \right\} \\
\sim (1 - \epsilon) \left( (1 + \nu)a \right)^{1/\alpha} H_\alpha u^{2/\alpha} \delta_1(u) \Psi(u), \ u \to \infty, \ S \to \infty.
\end{array} \right. \tag{11} \]

Since
\[ I_1(u) \geq \sum_{j=0}^{N_s(u)-1} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u \right\} - \sum_{0 \leq j < k \leq N_s(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u, \sup_{t \in B_k(u)} X(t) > u \right\}, \tag{12} \]

and by [11][Lemma 4.5]
\[ \sum_{0 \leq j < k \leq N_s(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u, \sup_{t \in B_j(u)} X(t) > u \right\} \leq \sum_{0 \leq j < k \leq N_s(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u, \sup_{t \in B_k(u)} \overline{X}(t) > u \right\} \]
\[ = o \left( u^{2/\alpha} \delta_1(u) \Psi(u) \right), \ u \to \infty, \ S \to \infty, \ \epsilon \to 0. \tag{13} \]

Thus inserting (11) and (13) into (12), we have
\[ \lim_{u \to \infty} \frac{I_1(u)(2 \ln u - q \ln \ln u)^{1/\gamma}}{u^{2/\alpha} \Psi(u)} \geq (1 - \epsilon) \left( (1 + \nu)a \right)^{1/\alpha} H_\alpha, \]
which combined with (10) gives that
\[ I_1(u) \sim \frac{a^{1/\alpha} H_\alpha u^{2/\alpha}}{(2 \ln u - q \ln \ln u)^{1/\gamma}} \Psi(u), \ u \to \infty, \ \nu \to 0, \ \epsilon \to 0. \tag{14} \]

By (iii) and (v), we have for all u large
\[ \mathbb{E} \left\{ (\overline{X}(t) - \overline{X}(s))^2 \right\} = 2 - 2r(s, t) \leq Q_1 |s - t|^\alpha, \]
uniformly holds for \( s, t \in [(1 - \epsilon) \delta_1(u), \delta_1(u)] \). By Piterbarg inequality for u large enough, see e.g., [22][Theorem 8.1] or an extension in [6][Lemma 5.1]
\[ I_2(u) \leq \mathbb{P} \left\{ \sup_{t \in [(1 - \epsilon) \delta_1(u), \delta_1(u)]} \overline{X}(t) > u \right\} \leq Q_2 \epsilon \delta_1(u) u^{2/\alpha} \Psi(u), \tag{15} \]
which implies
\[ \lim_{\epsilon \to 0} \lim_{u \to \infty} \frac{I_2(u)(2 \ln u - q \ln \ln u)^{1/\gamma}}{u^{2/\alpha} \Psi(u)} = 0. \]
Combining this equation with (9) and (14), we get
\[ \pi(u) \sim \frac{a^{1/\alpha} H_\alpha u^{2/\alpha}}{(2 \ln u - q \ln \ln u)^{1/\gamma}} \Psi(u) \sim a^{1/\alpha} H_\alpha u^{2/\alpha} (2 \ln u)^{-1/\gamma} \Psi(u), \ u \to \infty. \]
Case 2: $\gamma = \beta$. Set
\[ d_k = d_k(u) := \left( \frac{k}{\ln(u)(\ln \ln(u))^{1/\beta}} \right)^{1/\beta}, \quad A_k = A_k(u) := [d_k, d_{k+1}]. \]

Further let $M_r(u) = \max(k \in \mathbb{N} : d_k \leq (1 - \epsilon) \delta_1(u))$ for some $\epsilon \in (0, 1)$, then $M_r(u) \to \infty$, $u \to \infty$. Clearly
\[ \bigcup_{k=0}^{M_r(u)-1} A_k \subset [0, (1 - \epsilon) \delta_1(u)] \subset \bigcup_{k=0}^{M_r(u)} A_k. \]

We divide each interval $A_k$ into subintervals of length $S/u^{2/\alpha(d_k)}$, i.e.,
\[ B_{j,k} = B_{j,k}(u) := \left[ d_k + j \frac{S}{u^{2/\alpha(d_k)}}, d_k + (j + 1) \frac{S}{u^{2/\alpha(d_k)}} \right] \]
for $j = 0, 1, \ldots, N(k)$, where $N(k) = N(k, u) := \left\lfloor \frac{d_{k+1} - d_k}{S} u^{2/\alpha(d_k)} \right\rfloor$. Notice that
\[ \bigcup_{k=0}^{N(k)-1} B_{j,k} \subset A_k \subset \bigcup_{k=0}^{N(k)} B_{j,k}. \]

We have
\[ I_1(u) \leq \pi(u) \leq I_1(u) + I_2(u), \]
where
\[ I_1(u) = \mathbb{P} \left\{ \sup_{t \in [0, (1 - \epsilon) \delta_1(u)]} X(t) > u \right\}, \quad I_2(u) = \mathbb{P} \left\{ \sup_{t \in [(1 - \epsilon) \delta_1(u), \delta_1(u)]} X(t) > u \right\}. \]

Then by Bonferroni’s inequality
\[ I_1(u) \geq \sum_{k=0}^{M_r(u)-1} \sum_{j=0}^{N(k)-1} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u \right\} - \sum_{(j,k), (j',k') \in \mathcal{L}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \]
\[ =: J_1(u) - J_2(u), \]
where $\mathcal{L} = \{(j,k) : 0 \leq k \leq M_r(u) - 1, 0 \leq j \leq N(k) - 1\}$ and
\[ (j,k) < (j',k') \text{ iff } (k < k') \lor (k = k' \land j < j'), \]
and by (8), Lemma 3.1 and Lemma 3.2
\[ I_1(u) \leq \sum_{k=0}^{M_r(u)} \sum_{j=0}^{N(k)} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u \right\} \]
\[ \leq \sum_{k=0}^{M_r(u)} \sum_{j=0}^{N(k)} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > \mathcal{G}_u^{-\epsilon} \right\} \]
\[ \leq \sum_{k=0}^{M_r(u)} \sum_{j=0}^{N(k)} \mathbb{P} \left\{ \sup_{t \in [0,S]} Z_{\nu,u}(t) > \mathcal{G}_u^{-\epsilon} \right\} \]
\[ \sim \sum_{k=0}^{M_r(u)} \sum_{j=0}^{N(k)} \mathcal{H}_\alpha \left[ 0, S((1 + \nu)a)^{1/\alpha} \right] \Psi \left( \mathcal{G}_u^{-\epsilon} \right) \]
\[ \sim \sum_{k=0}^{M_r(u)} \frac{d_{k+1} - d_k}{S} u^{2/\alpha(d_k)} \mathcal{H}_\alpha \left[ 0, S((1 + \nu)a)^{1/\alpha} \right] \Psi(u) \]
\[ = \frac{\mathcal{H}_\alpha \left[ 0, S((1 + \nu)a)^{1/\alpha} \right]}{S} \frac{u^{2/\alpha}}{\ln(u)^{1/\beta}} \Psi(u) \sum_{k=0}^{M_r(u)} \left( \ln(u)^{1/\beta} (d_{k+1} - d_k) e^{\ln(u) \left( \frac{2(1 - \epsilon a(d_k))}{\alpha(d_k)} \right)} \right) \]
and

Thus

(18)

\[ \lim_{u \to \infty} I_1(u)(\ln u)^{1/\beta} \leq \frac{\mathcal{H}_0}{S} \int_0^{(1-\varepsilon)(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\varepsilon)\ln u - \beta d_k}{\alpha^2}} dx, \]

and letting \( S \to \infty, \varepsilon, \nu \to 0, \) and \( \epsilon \to 0, \) we get the upper bound. Similarly, we derive that

(19)

\[ \lim_{\epsilon \to 0} \lim_{S \to \infty} \lim_{u \to \infty} \frac{J_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_0 \int_0^{(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\varepsilon)\ln u - \beta d_k}{\alpha^2}} dx. \]

By [11] [Lemma 4.5]

\[ J_2(u) = \sum_{(j,k),(j',k') \in \mathcal{C}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \]

\[ \leq \sum_{(j,k),(j',k') \in \mathcal{C}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \overline{X}(t) > u, \sup_{t \in B_{j',k'}} \overline{X}(t) > u \right\} \]

(20)

\[ = o \left( u^{2/\alpha} (\ln u)^{-1/3} \Psi(u) \right), \ u \to \infty, \ S \to \infty, \ \epsilon \to 0. \]

Thus inserting (19) and (20) into (17), we get

(21)

\[ \lim_{\epsilon \to 0} \lim_{S \to \infty} \lim_{u \to \infty} \frac{I_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_0 \int_0^{(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\varepsilon)\ln u - \beta d_k}{\alpha^2}} dx. \]

By (15)

(22)

\[ \lim_{\epsilon \to 0} \lim_{u \to \infty} \frac{J_2(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} = 0. \]

Hence according to (16), (18), (21), and (22), we have

\[ \pi(u) \sim a^{1/\alpha} \mathcal{H}_0 u^{2/\alpha} (\ln u)^{-1/3} \Psi(u) \int_0^{(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\varepsilon)\ln u - \beta d_k}{\alpha^2}} dx, \ u \to \infty. \]
**Case 3:** $\gamma > \beta$. We consider $\pi(u) = \mathbb{P}\left\{ \sup_{t \in [0, \delta_2(u)]} X(t) > u \right\}$ with

$$\delta_2(u) = \left( \frac{\alpha^2 (\ln(\ln u))}{\beta (\ln u)} \right)^{1/\beta}.$$

Set for some $\varepsilon > 0$

$$\mathcal{F}^{\pm \varepsilon}_u = u \left( 1 + (1 \pm \varepsilon) c e^{-(\delta_2(u))^{-1}} \right), \quad \mathcal{K} = \{ t \in [0, T] : \sigma(t) \neq 0 \},$$

and we observe that

$$\Psi(\mathcal{F}^{\pm \varepsilon}_u) \sim \Psi(u), \quad u \to \infty.$$

By [11][Theorem 2.1]

$$\pi(u) \leq \mathbb{P}\left\{ \sup_{t \in [0, \delta_2(u)]} X(t) > u \right\} \leq \mathbb{P}\left\{ \sup_{t \in \mathcal{K}} X(t) > u \right\} \sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-1} \int_{0}^{\infty} e^{-\frac{2k\beta}{\alpha}} dx \Psi(u), \quad u \to \infty.$$

Let $d_k, A_k, B_{j,k}, N(k)$ be the same as in **Case 2** and $M(u) = \max(k \in \mathbb{N} : d_k \leq \delta_2(u))$. Clearly

$$\bigcup_{k=0}^{M(u)-1} A_k \subset [0, \delta_2(u)] \subset \bigcup_{k=0}^{M(u)} A_k, \quad \bigcup_{k=0}^{N(k)-1} B_{j,k} \subset A_k \subset \bigcup_{k=0}^{N(k)} B_{j,k},$$

and by Bonferroni’s inequality

$$\pi(u) \geq \sum_{k=0}^{M(u)-1} \sum_{j=0}^{N(k)-1} \mathbb{P}\left\{ \sup_{t \in B_{j,k}} X(t) > u \right\} - \sum_{(j,k),(j',k') \in \mathcal{L}', (j,k) \sim (j',k')} \mathbb{P}\left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\}$$

(24) $$=: J'_1(u) - J'_2(u),$$

where $\mathcal{L}' = \{(j, k) : 0 \leq k \leq M(u) - 1, 0 \leq j \leq N(k) - 1\}$. By (8), Lemma 3.1, Lemma 3.2 and similar argumentation as (19) with $G^{\pm \varepsilon}_u$ replaced by $\mathcal{F}^{\pm \varepsilon}_u$ and the fact that $(\ln u)^{1/\beta} d_{M(u)+1} \to \infty, u \to \infty$, we get

$$\lim_{S \to \infty} \lim_{u \to \infty} \frac{J'_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_{0}^{\infty} e^{-\frac{2k\beta}{\alpha}} dx.$$

By[11][Lemma 4.5]

$$J'_2(u) = \sum_{(j,k),(j',k') \in \mathcal{L}', (j,k) \sim (j',k')} \mathbb{P}\left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \leq \sum_{(j,k),(j',k') \in \mathcal{L}', (j,k) \sim (j',k')} \mathbb{P}\left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\}$$

(26) $$= o \left( u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u) \right), \quad u \to \infty.$$

Hence inserting (25) and (26) into (24), we have

$$\lim_{u \to \infty} \frac{\pi(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_{0}^{\infty} e^{-\frac{2k\beta}{\alpha}} dx,$$

which combined with (23) gives that

$$\pi(u) \sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u) \int_{0}^{\infty} e^{-\frac{2k\beta}{\alpha}} dx, \quad u \to \infty.$$
Consequently, according to Lemma 2.1 and
\[ \pi(u) \leq \mathbb{P}\left\{ \sup_{t \in [0,T]} X(t) > u \right\} \leq \pi(u) + \mathbb{P}\left\{ \sup_{t \in [\delta(u),T]} X(t) > u \right\}, \]
(7) is proved and all claims follow.

\[ 3. \text{ Appendix} \]

In this section we present the proofs of the lemmas used in the proof of Theorem 1.2.

**Proof of Lemma 2.1:** Below $Q_k$, $k = 0, 1, 2, \ldots$, are some positive constants.

**Step 1:** First we prove (3). By the continuity of $\sigma(t)$ in $[0,T]$, for any small enough constant $0 < \theta < 1$

\[ \sup_{t \in [\theta,T]} \sigma(t) =: \rho(\theta) < \sigma(t_0) = \sigma(0) = 1. \]

Then by Borell inequality in [1]
\[
\mathbb{P}\left\{ \sup_{t \in [\sigma,T]} X(t) > u \right\} \leq \exp\left(-\frac{(u - Q_0)^2}{2\rho^2(\theta)}\right) = o(\Psi(u)),
\]
as $u \to \infty$, where $Q_0 = \mathbb{E}\{\sup_{t \in [0,T]} X(t)\} < \infty$.

By assumption (iv), for any small $\varepsilon \in (0,1)$, when $\theta$ small enough
\[ 1 + (1 - \varepsilon)ce^{-|t|^{-\gamma}} \leq \frac{1}{\sigma(t)} \leq 1 + (1 - \varepsilon)ce^{-|t|^{-\gamma}}, \]
holds for $t \in [0, \theta]$. Then
\[ \frac{1}{\sigma(t)} \geq 1 + (1 - \varepsilon)ce^{-|t|^{-\gamma}} \geq 1 + (1 - \varepsilon)cu^{-2}(\ln u)\]
uniformly holds for $t \in [\delta_1(u), \theta]$.

Moreover by assumption (i) and (iii), when $\theta$ small enough
\[ \mathbb{E}\{(X(t) - X(s))^2\} = \mathbb{E}\{X^2(t)\} + \mathbb{E}\{X^2(s)\} - 2\mathbb{E}\{X(t)X(s)\} \leq 2 - 2(1 - 2\alpha(t)|t - s|^{\alpha(t)}) \leq Q_1|t - s|^\zeta, \]
holds uniformly for $s, t \in [0, \theta]$, where $Q_1 = \sup_{t \in [0, \theta]} 4\alpha(t)$ and $\zeta = \inf_{t \in [0, \theta]} \alpha(t) > 0$.

Then by Piterbarg inequality
\[ \mathbb{P}\left\{ \sup_{t \in [\delta_1(u), \theta]} X(t) > u \right\} \leq \mathbb{Q}_2u^{2/\zeta}\Psi(u[1 + (1 - \varepsilon)cu^{-2}(\ln u)\]) = o(\Psi(u)), \quad u \to \infty. \]

Further, since
\[ \mathbb{P}\left\{ \sup_{t \in [0,T]} X(t) > u \right\} = \mathbb{P}\left\{ \sup_{t \in [0,\delta_1(u)]} X(t) > u \right\} + \mathbb{P}\left\{ \sup_{t \in [\delta_1(u),\theta]} X(t) > u \right\} + \mathbb{P}\left\{ \sup_{t \in [\theta,T]} X(t) > u \right\}, \]
and
\[ \mathbb{P}\left\{ \sup_{t \in [0,T]} X(t) > u \right\} \geq \mathbb{P}\left\{ \sup_{t \in [0,\delta_1(u)]} X(t) > u \right\} \geq \mathbb{P}\{X(0) > u\} = \Psi(u), \]
we get
\[ \mathbb{P}\left\{ \sup_{t \in [0,T]} X(t) > u \right\} \sim \mathbb{P}\left\{ \sup_{t \in [0,\delta_1(u)]} X(t) > u \right\}, \quad u \to \infty. \]

**Step 2:** Next we prove (4). When $\gamma \leq \beta$, since $\delta_1(u) = o(\delta_2(u))$, as $u \to \infty$ and by **Step 1**
\[ \mathbb{P}\left\{ \sup_{t \in [\delta_1(u),T]} X(t) > u \right\} = o(\Psi(u)), \quad u \to \infty. \]
Then for \( u \) large enough, (4) is obvious.

When \( \gamma > \beta \), for \( u \) large enough, we have \( \delta_2(u) < \delta_1(u) \) and

\[
\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), T]} X(t) > u \right\} \leq \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} + \mathbb{P} \left\{ \sup_{t \in [\delta_1(u), T]} X(t) > u \right\}.
\]

By **Step 1**, we know for all \( u \) large

\[
\mathbb{P} \left\{ \sup_{t \in [\delta_1(u), T]} X(t) > u \right\} \leq \Psi(u),
\]

and then we just need to deal with \( \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} \).

Since \( \delta_1(u) \to 0 \), \( u \to \infty \), then by assumption (v)

\[
\alpha(t) > \alpha + \frac{3}{4} b(\delta_2(u))^{\frac{3}{2}}
\]

holds for all \( t \in [\delta_2(u), \delta_1(u)] \) when \( u \) large enough.

Let \( \eta_u = u^{-2/(\alpha + \frac{3}{2} b(\delta_2(u))^{\frac{3}{2}})} \). For sufficiently large \( u \) and \( s, t \in [\delta_2(u), \delta_1(u)] \), there exists a constant \( Q_3 > 0 \) such that

\[
1 - r(s, t) \leq 1 - e^{-Q_3(s-t)^{\alpha + \frac{3}{2} b(\delta_2(u))^{\frac{3}{2}}}}.
\]

Let \( Y_u(t), t \geq 0 \) be a family of centered stationary Gaussian processes with correlation functions

\[
r_Y(s, t) = e^{Q_3(s-t)^{\alpha + \frac{3}{2} b(\delta_2(u))^{\frac{3}{2}}}}.
\]

Then from Slepian’s inequality we get for any constant \( S > 0 \)

\[
\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} \leq \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} \frac{X(t)}{\sigma(t)} > u \right\}
\]

\[
\leq \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} Y_u(t) > u \right\}
\]

\[
\leq \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_u(t) > u \right\}
\]

\[
\leq \sum_{i=0}^{\lfloor S_{\eta_u^{-1}} \rfloor + 1} \mathbb{P} \left\{ \sup_{t \in [i\eta_u, (i+1)\eta_u]} Y_u(t) > u \right\}
\]

\[
\leq (\lfloor S_{\eta_u^{-1}} \rfloor + 1) \mathbb{P} \left\{ \sup_{t \in [0, \eta_u]} Y_u(t) > u \right\},
\]

for sufficiently large \( u \). Notice that for each \( s, t \in [0, 1] \)

\[
1 - r_Y(\eta_u, \eta_u s) = Q_3 u^{-2}|s - t|^{\alpha + \frac{3}{2} b(\delta_2(u))^{\frac{3}{2}}}(1 + o(1)) = Q_3 u^{-2}|s - t|^{\alpha (1 + o(1))}, u \to \infty.
\]

Hence, from [22][Lemma D.1]

\[
\mathbb{P} \left\{ \sup_{t \in [0, \eta_u]} Y_u(t) > u \right\} \sim \mathcal{H}_\alpha[1]\Psi(u),
\]

as \( u \to \infty \). Combining this with the fact that

\[
\eta_u^{-1} = u^{2/(\alpha + \frac{3}{2} b(\delta_2(u))^{\frac{3}{2}})} = u^{2/\alpha} u^{2/(\alpha + \frac{3}{2} b(\delta_2(u))^{\frac{3}{2}}} - 2/\alpha = u^{2/\alpha} u^{-\frac{3}{2} b(\delta_2(u))^{\frac{3}{2}}/\beta(\delta_2(u))^{\frac{3}{2}}} = u^{2/\alpha} u^{-\frac{3}{2} (\ln u)/(\delta_2(u))^{\frac{3}{2}}} \leq u^{2/\alpha} u^{-\frac{3}{2} (\ln u)/(\delta_2(u))^{\frac{3}{2}}} \leq u^{2/\alpha} u^{-\frac{3}{2} (\ln u)/(\delta_2(u))^{\frac{3}{2}}} = u^{2/\alpha} (\ln u)^{-4/3\beta} = u^{2/\alpha} (\ln u)^{-4/3\beta},
\]

we get for some constant \( Q_4 \) and all \( u \) large enough

\[
\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} \leq Q_4 S u^{2/\alpha} (\ln u)^{-4/3\beta} \Psi(u).
\]
Lemma 3.1. Under the notation in the proof of Theorem 1.2, for \((j, k) \in U = \{(j, k) : 0 \leq k \leq M^*(u), 0 \leq j \leq N(k)\}\) and \(\lim_{u \to \infty} f(u) = 1\), there exists \(u_0\) such that for each \(u \geq u_0\)

1) \(\mathbb{P} \left\{ \sup_{t \in [B, k]} X(t) > f(u) \right\} \geq \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_{\nu, u}(t) > f(u) \right\};\)

2) \(\mathbb{P} \left\{ \sup_{t \in [B, k]} X(t) > f(u) \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{\nu, u}(t) > f(u) \right\},\)

where

\[ M^*(u) = \begin{cases} 
0, & \text{if } \gamma < \beta, \\
M_\epsilon(u), & \text{if } \gamma = \beta, \\
M(u), & \text{if } \gamma > \beta.
\]

Proof of Lemma 3.1: Since the proofs of scenarios \(\gamma < \beta, \gamma = \beta,\) and \(\gamma > \beta\) are similar, we only present the proof of \(\gamma = \beta\). Set \(X_{j, k, u}(t) = \mathbb{X} \left( d_k + \frac{jS + t}{u^{2/\alpha(d_k)}}, \mathbb{X} \left( d_k + \frac{jS + t}{u^{2/\alpha(d_k)}} \right) \right)\), then \(\sup_{t \in [B, k]} X(t) \equiv \sup_{t \in [0, S]} X_{j, k, u}(t)\). It is enough to analyze the supremum of \(X_{j, k, u}(t)\).

1) For sufficiently large \(u\) and \(s, t \in [0, T]\)

\[ 1 - \text{Cov} (X_{j, k, u}(s), X_{j, k, u}(t)) = 1 - \text{Cov} \left( \mathbb{X} \left( d_k + \frac{jS + s}{u^{2/\alpha(d_k)}}, \mathbb{X} \left( d_k + \frac{jS + t}{u^{2/\alpha(d_k)}} \right) \right) \right) \geq \left( 1 - \frac{\nu}{2} \right)^{1/3} a \left| u^{-2/\alpha(d_k)} (s - t) \right|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS + t))} \]

\[ = \left( 1 - \frac{\nu}{2} \right)^{1/3} a u^{-2\alpha(d_k) - \alpha(d_k + u^{-2/\alpha(d_k)}(jS + t))} \]

\[ = \left( 1 - \frac{\nu}{2} \right)^{1/3} a u^{-2\alpha(d_k) - \alpha(d_k + u^{-2/\alpha(d_k)}(jS + t))} \]

\[ \geq u^{-2(1 - \nu/2)^{1/3}}, \]

where the last inequality follows from the fact that

\[ (\ln u) \left| \alpha(d_k) - \alpha \left( d_k + u^{-2/\alpha(d_k)}(jS + t) \right) \right| \leq (\ln u) \left| b(d_k)^{\beta} - b \left( d_k + u^{-2/\alpha(d_k)}(jS + t) \right)^{\beta} \right| + 2\delta_1^{\beta + \delta}(u) \leq (\ln u) \left( \frac{b}{(\ln u)(\ln \ln u)^{1/\beta}} + 2\delta_1^{\beta + \delta}(u) \right) \leq \frac{b}{(\ln u)^{1/\beta}} + 2(\ln u) \left( \frac{1}{2\ln u - q \ln \ln u} \right)^{\beta + \delta} \to 0,\quad u \to \infty. \]

For \(I_2\), we need to prove that

\[ I_2 \geq (1 - \nu/2)^{1/3} |s - t|^{\alpha + 2\delta_1^{\beta}(u)}. \]

Assumption (v) implies that

\[ \alpha \left( d_k + u^{-2/\alpha(d_k)}(jS + t) \right) < \alpha + 2\delta_1^{\beta}(u) \]

for each \((j, k) \in U\). Thus if \(|s - t| < 1\), then (29) holds immediately. If \(1 \leq |s - t| \leq S\), then by (30)

\[ I_2 = \left| (s - t) \right|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS + t))} \geq T^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS + t)) - \alpha - 2\delta_1^{\beta}(u)} |s - t|^{\alpha + 2\delta_1^{\beta}(u)} \geq T^{-2\delta_1^{\beta}(u)} \left| s - t \right|^{\alpha + 2\delta_1^{\beta}(u)}. \]
for sufficiently large \( u \). The above combined with (27), (28) and (29) gives that for sufficiently large \( u \), uniformly with respect to \((j, k) \in U\),

\[
1 - \text{Cov} \left( X_{j,k,u}(s), X_{j,k,u}(t) \right) \geq (1 - \nu/2)a u^{-2} \left| s - t \right|^\beta \left| u^{-2/\alpha(d_k)}(s - t) \right|^\alpha \geq 1 - \text{Cov} \left( Y_{\nu,u}(s), Y_{\nu,u}(t) \right) .
\]

Thus by Slepian’s inequality 1) is proved.

2) For all \( u \) large

\[
1 - \text{Cov} \left( X_{j,k,u}(s), X_{j,k,u}(t) \right) = 1 - \text{Cov} \left( \mathbf{X} \left( d_k + \frac{js + s}{u^{2/\alpha(d_k)}} \right), \mathbf{X} \left( d_k + \frac{js + t}{u^{2/\alpha(d_k)}} \right) \right) 
\leq \left( 1 + \nu \right)^{1/3} a \left| u^{-2/\alpha(d_k)}(s - t) \right|^\alpha \left| d_k + u^{-2/\alpha(d_k)}(js + t) \right| .
\]

Following the argument analogous to that for the proof of 1), we obtain that for sufficiently large \( u \), uniformly with respect to \( k \), and \( s, t \in [0, S] \)

\[
1 - \text{Cov} \left( X_{j,k,u}(s), X_{j,k,u}(t) \right) \leq 1 - \text{Cov} \left( Z_{\nu,u}(s), Z_{\nu,u}(t) \right) .
\]

Again the application of Slepian’s inequality completes the proof.

\[\Box\]

**Lemma 3.2.** For \( S > 1, \nu \in (0, 1), \) and \( \lim_{u \to \infty} \frac{f(u)}{u} = 1, \) as \( u \to \infty, \) we have

1) \( \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_{\nu,u}(t) > f(u) \right\} = \mathcal{H}_a \left[ 0, S((1 - \nu)a)^{1/\alpha} \right] \Psi \left( f(u) \right) \left( 1 + o(1) \right); \)

2) \( \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{\nu,u}(t) > f(u) \right\} = \mathcal{H}_a \left[ 0, S((1 + \nu)a)^{1/\alpha} \right] \Psi \left( f(u) \right) \left( 1 + o(1) \right). \)

**Proof of Lemma 3.2:** We present the proof of 1) and omit the proof of 2) since it follows with similar arguments. Following the definition of \( Y_{\nu,u}(t) \), for each \( s, t \in [0, S] \)

\[
\lim_{u \to \infty} f^2(u) \left[ 1 - \text{Cov} \left( Y_{\nu,u} \left( t(a(1 - \nu))^{-1/\alpha} \right), Y_{\nu,u} \left( s(a(1 - \nu))^{-1/\alpha} \right) \right) \right] = \lim_{u \to \infty} \left( a(1 - \nu) \right) - \left( \alpha + 2\beta^3(u) \right) / \alpha \left| s - t \right|^\beta \left| a^{-1/\alpha(d_k)}(js + t) \right|^\alpha = \left| s - t \right|^\alpha.
\]

Moreover, for all \( s, t \in [0, S] \), sufficiently large \( u \) and some constant \( C > 0 \)

\[
f^2(u) \left[ 1 - \text{Cov} \left( Y_{\nu,u} \left( t(a(1 - \nu))^{-1/\alpha} \right), Y_{\nu,u} \left( s(a(1 - \nu))^{-1/\alpha} \right) \right) \right] \leq \left( a(1 - \nu) \right) - \left( \alpha + 2\beta^3(u) \right) / \alpha \left| s - t \right|^\beta \left| a^{-1/\alpha(d_k)}(js + t) \right|^\alpha \leq C T^2 \left| s - t \right|^\alpha,
\]

where the last inequality follows from the fact that

\[
\left| s - t \right|^\beta \left| a^{-1/\alpha(d_k)}(js + t) \right|^\alpha \leq \left| s - t \right|^\alpha, \text{ if } \left| s - t \right| < 1,
\]

and

\[
\left| s - t \right|^\beta \left| a^{-1/\alpha(d_k)}(js + t) \right|^\alpha \leq T^2 \leq T^2 \left| s - t \right|^\alpha, \text{ if } 1 \leq \left| s - t \right| \leq T.
\]

Hence, by [19][Lemma 7], we conclude that

\[
\mathbb{P} \left\{ \sup_{t \in [0, S]} Y_{\nu,u}(t) > f(u) \right\} = \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_{\nu,u}((a(1 - \nu))^{-1/\alpha} t) > f(u) \right\} = \mathcal{H}_a \left[ 0, ((1 - \nu)a)^{1/\alpha} S \right] \Psi \left( f(u) \right) \left( 1 + o(1) \right),
\]

as \( u \to \infty \). This completes the proof. \(\Box\)

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