Excitation of plasma wakefields by intense ultra-relativistic proton beam

Mithun Karmakar1 | Bhavesh Patel1 | Nikhil Chakrabarti2,3 | Sudip Sengupta1,2

1Basic Theory and Simulation Division, Institute for Plasma Research, Gandhinagar, India
2Homi Bhabha National Institute, Mumbai, India
3Plasma Physics Division, Saha Institute of Nuclear Physics, Kolkata, India

Correspondence
Mithun Karmakar, Basic Theory and Simulation Division, Institute for Plasma Research, Bhat, Gandhinagar 382428, India.
Email: mithun.karmakar1@gmail.com

Abstract
We report here an exact analytical travelling wave solution for a non-linear electron plasma wave excited by an intense ultra-relativistic proton beam. It demonstrates the underlying physics of longitudinal electric field characteristics of the excited wake wave formed behind the drive beam. The results are further supplemented by a fully relativistic particle-in-cell code OSIRIS in two-dimensional geometry. The investigation is further extended by providing an analytical description of the wake wave excited by an equi-spaced train of small proton bunches with the inclusion of the non-relativistic plasma ion dynamics. Our results show that the amplitude of the wake field does not grow indefinitely with the increase in the number of proton bunches. On the contrary, it saturates to a definitive limit.

KEYWORDS
charge particle acceleration, nonlinear plasma wave, plasma wake field, proton beam

1 | INTRODUCTION

The supremacy of plasma-based particle accelerators over many conventional linear accelerators is now accepted undeniably.[1–6] This novel acceleration scheme not only miniaturizes the size of an accelerator but also makes it possible to produce electrons even with TeV energies.[7–15] Over the last few decades, research on plasma-based acceleration processes was mainly focused on the physics of wake wave excitation in a plasma driven by a highly relativistic electron beam or an intense laser pulse.[1,2,6,16–22] In electron- or laser-driven plasma wake field accelerators, energies up to several GeV can be achieved.[17,18,21,22] However, to reach the energy frontier of the present-day high-energy physics research, there is a requirement to accelerate a charged particle up to an energy that is in the range of several TeV. It takes multiple stages of acceleration to reach a TeV order of energy in laser pulse- or electron bunch-driven schemes, which results in various technological difficulties. Moreover, the energy gain is limited by the energy carried by the electron driver, which is very small (~100 J). An alternative approach is to use a proton beam instead of an electron or laser beam to excite the plasma wake wave. Because of their higher energy (~kJ) and mass, protons can drive wake fields over much longer plasma lengths. A proton bunch carrying energy of the order of ~kJ is capable of accelerating electrons in the TeV energy range in a single plasma stage.[8,23] Such proton beams are now routinely produced at various proton synchrotron facilities like the Large Hadron Collider (6.5 TeV, 1.2 × 1011 protons, ~125 kJ) and the CERN Super Proton Synchrotron (SPS; 450 GeV, 3 × 1011 protons, ~20 kJ).[9] Therefore, because of high energy content, easy availability of proton beam, and technological viability to use it as drive beam to reach a TeV energy regime, this new acceleration scheme becomes superior to others.[9,10,15,24]

In case of a negatively charged driver, background plasma electrons are expelled to form a bubble surrounded by a thin electron sheath. Proton beams, on the other hand, suck in the plasma electrons towards the propagation axis and create
oscillations that propagate at nearly the speed of light along with the proton drive beam. The energetically favourable bubble solution obtained using a negatively charged drive beam, however, is very difficult to achieve by using the proton drive beam. Nevertheless, after the formation of wake wave, a trailing witness bunch of electrons injected externally at the proper phase of the wake with sufficient energy will be trapped and accelerated by the longitudinal electric field of the wake to relativistic energies.

The concept of wake field excitation using a proton beam was first realized in 2009 by Caldwell et al.\cite{7} They performed simulations to study the plasma wave excitation using the particle-in-cell (PIC) code VLPL, as well as LCODE. A proton beam of longitudinal size $100 \mu m$ and transverse size of $0.43 mm$ was used to excite the wake field in a plasma of density $6 \times 10^{14} \text{ cm}^{-3}$. With this plasma density, the plasma wavelength of the wake wave is of the order of the transverse size of the beam. It was shown that the proton beam with initial energy of $\sim$ TeV can produce a longitudinal wake field of amplitude $\sim 3 \text{ GV/m}$, which accelerates the $\sim 10 \text{ GeV}$ externally injected witness beam of electrons up to an average energy of $\sim 0.62 \text{ TeV}$ after traversing a distance of $\sim 450 \text{ m}$. A strong transverse field was also observed, which focussed the witness electron beam and thus reduced the transverse spreading of the beam. After this initial study, several plasma physics groups became involved in exploring the physics of the proton beam-driven plasma wakefield accelerator (PDPWFA) either through simulation, theory, or by modelling experiments on it.\cite{7,8,10–14} The Advanced Wakefield Experiment (AWAKE) project at CERN is the first experimental project worldwide that aims to accelerate electrons in the TeV energy range by exciting plasma wake field using a modulated proton beam of TeV order of energy.\cite{10,15,25,26} A 400 GeV/c proton beam extracted from CERN SPS is used to excite a wake field in a 10 m-long plasma cell that is capable of producing a longitudinal electric field of amplitude of up to several $\sim \text{GV/m}$. AWAKE uses plasma densities in the range of $10^{14} – 10^{15} \text{ cm}^{-3}$, which correspond to a plasma wavelength of the wake wave of the order of $\sim$ few mm. The proton bunches available today are much longer in size compared to the plasma wavelength.\cite{9} The proton beam that is available in the CERN SPS with TeV energy is 12 cm long. So, they are not resonant, and excitation of strong wake field is not possible. However, a process called self-modulational instability can cause a long proton bunch to split into a large number of micro-bunches, which then efficiently excite the plasma wake wave.\cite{12–14,27–29} In the recent past, the excitation mechanism of wake field by such trains of equidistant particle bunches has been discussed.\cite{30} The AWAKE experiment uses such a modulated proton beam in the wake field excitation process.

Admittedly, it is really a challenging job to develop a multidimensional theory for PDPWFA in the non-linear regime. In this investigation, we first present analytical results obtained in one dimension for the wake field excited by a single bunch of protons. These analytical results are further verified by two-dimensional (2D) OSIRIS simulation.\cite{31,32} Then, the field excited using a train of proton bunches is discussed.

The paper is organized as follows. In Section 2, an exact analytical solution is presented for the wake wave excited by a single proton drive bunch in an unmagnetized plasma system. In Section 3, 2D PIC simulation results are shown. In Section 4, we discuss the wake wave excited by a train of proton micro-bunches. The main conclusion of the paper is provided in the Section 5.

## 2 Analytical Solution for the Wake Wave Excited by an Ultra-Relativistic Proton Beam

We adopt a simple fluid model to describe the wake field excitation in a two-component unmagnetized electron–ion plasma. The excited wake wave is a longitudinal one-dimensional (1D) electrostatic electron plasma wave propagating along the direction of propagation of a highly relativistic proton beam. An exact analytical solution of the problem can be obtained from the following fluid equations coupled with Maxwell’s equations, viz. continuity equation, the electron fluid momentum, and Poisson’s equation:

\begin{align}
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e v_e) &= 0, \\
\frac{\partial p_e}{\partial t} + v_e \frac{\partial p_e}{\partial x} &= -eE_x, \\
\frac{\partial E_x}{\partial x} &= 4\pi e(n_0 - n_e + n_b).
\end{align}

The proton beam is characterized by its density $n_b$ and velocity $v_b$. The electric field is denoted by $E_x$, which is directed along the positive $x$ axis. All other variables have their usual meanings. The heavy plasma ions are considered to be
immobile, maintaining an overall charged neutrality in the equilibrium plasma system. Disregarding the ion motion is not an issue for some experiments on PDPWFA, such as AWAKE, where the plasma is created by a laser ionization of rubidium vapour. Rubidium ions are heavy and do not move appreciably on the relevant timescales.\textsuperscript{[10,15,25,26]} Nevertheless, the effect of ion motion has been studied extensively, both analytically as well as in simulations in the past.\textsuperscript{[33,34]} In Section 4, however, we consider the non-relativistic ion motion in our extended analysis. For light ions like hydrogen, inclusion of ion motion in the excitation process becomes relevant.

We construct a stationary wave solution of Equations (1)–(3) considering the propagation of a longitudinal electrostatic wave along $x$ axis and assuming all the dynamical variables to be a function of $\xi = k_p(v_{ph}t - x)$, a special combination of space and time. Here, $k_p = \omega_p/v_{ph}$ and $\omega_p = \sqrt{4\pi n_0 e^2/m_e}$, with $n_0$ and $v_{ph}$ being the equilibrium plasma density and the phase velocity of the longitudinal plasma wave, respectively. We rescale the variables by introducing $n = n_e/n_0$, $a = n_b/n_0$, $\beta = v_e/c$, $\beta_{ph} = v_{ph}/c$, $E = eE_x/m_0v_{ph}$, and $p = p_e/mc$, with $c$ being the velocity of light in free space. We assume a rectangular proton beam profile with the longitudinal extension given by

$$\alpha = a_0 \text{ for } 0 \leq \xi \leq l_b,$$

$$= 0, \text{ otherwise}.$$

where $l_b$ is the beam length. Now, it is important to note that the phase velocity of the wake wave is determined by the velocity of the drive beam. In addition, as we are primarily interested in investigating the physics of wake wave excited by an ultra-relativistic proton beam, we work in the limit $\beta_b \to 1$, and hence, $\beta_{ph} \to 1$. In the transformed coordinate system, the electron continuity equation (Equation 1) is directly integrated to give

$$n = \frac{1}{1 - \beta}.$$  \hspace{1cm} (4)

Hence, the wake wave excitation is now determined by the following equations

$$(1 - \beta) \frac{dE}{d\xi} = (1 + \alpha)\beta - \alpha;$$  \hspace{1cm} (5)

$$(1 - \beta) \frac{dp}{d\xi} = -E.$$  \hspace{1cm} (6)

where we have used (Equation 4) in Poisson’s equation to arrive at (Equation 5). Now, we proceed to find the solution of the above equations inside and outside the beam separately.

### 2.1 Solution inside the beam

Combining Equations (5) and (6) and by transforming another variable:

$$ (1 - \beta) \frac{dE}{d\xi} = \frac{d}{d\varphi}; $$  \hspace{1cm} (7)

we obtain

$$ \frac{d^2p}{d\varphi^2} = -\left[\frac{(1 + a_0)p}{\sqrt{1 + p^2}} - a_0\right]. $$  \hspace{1cm} (8)

This equation is integrated once to obtain

$$ \frac{dp}{d\varphi} = \pm \sqrt{2[C + a_0p - (1 + a_0)\sqrt{1 + p^2}]}^{1/2}, $$  \hspace{1cm} (9)

with $C$ being an integration constant.
In order to obtain an exact analytical solution of the above non-linear first-order differential equation, we use the following transformation relations

\[
\sqrt{1 + p^2} = X^2 - p,
\]

\[
a^2 = C + [C^2 - (1 + 2a_0)]^{1/2},
\]

\[
b^2 = C - [C^2 - (1 + 2a_0)]^{1/2}.
\]

We assume that at \( \xi = 0, p = 0 \) and \( \frac{dp}{d\phi} = 0 \). Therefore, the constant of integration \( C \) becomes \( 1 + a_0 \). By using all the above transformations, we obtain the following from Equation (9)

\[
\varphi = \int_1^X \frac{X^2dX}{\sqrt{(a^2 - X^2)(X^2 - b^2)}} + \int_1^X \frac{dX}{X^2\sqrt{(a^2 - X^2)(X^2 - b^2)}}.
\]

We integrate the above two integrals with the substitution

\[
\sin^2\theta = \frac{a^2 - X^2}{a^2 - b^2}.
\]

After performing some simple algebra, we obtain

\[
\varphi = - \left\{ a + \frac{1}{a^3(1 - k^2)} \right\} [E(\theta, k)]_{\theta_0} + \left[ \frac{k^2 \sin \theta \cos \theta}{a^3(1 - k^2)} \right]_{\theta_0}^\theta.
\]

Now, \( \varphi \) is related to \( \xi \) in the following fashion,

\[
\varphi = \xi + \int \beta d\phi.
\]

Eventually, we can write the exact travelling wave solution for the proton beam-driven plasma wake field as

\[
\xi = - \frac{2}{a^3(1 - k^2)} [E(\theta, k)]_{\theta_0} + \left[ \frac{k^2 \sin 2\theta}{a^3(1 - k^2)} \right]_{\theta_0}^\theta,
\]

where \( E(\theta, k) \) is the elliptic integral of second kind and

\[
k^2 = \frac{a^2 - b^2}{a^2},
\]

\[
\theta = \sin^{-1} \sqrt{\frac{a^2 - X^2}{a^2 - b^2}},
\]

\[
\theta_0 = \sin^{-1} \sqrt{\frac{a^2 - 1}{a^2 - b^2}}.
\]

### 2.2 Solution outside the beam

The solution behind the beam can be obtained by setting \( \alpha = 0 \) in Equation (5). The differential equation that we use to solve in this case is

\[
\frac{d^2}{d\xi^2} \frac{1 - \beta}{\sqrt{1 - \beta^2}} = \frac{\beta}{1 - \beta}.
\]
This equation is integrated with the substitution \( y = \sqrt{\frac{1-\beta}{1+\beta}} \) to obtain

\[
\frac{1}{2} \left( \frac{dy}{d\xi} \right)^2 + \frac{1}{2} \left( y + \frac{1}{y} \right) = \gamma_m, \tag{12}
\]

where \( \gamma_m \) is the constant of integration. We can determine the value of \( \gamma_m \) by writing down the differential equation describing field characteristics inside the beam (Equations 5 and 6) in the following form \[35\]

\[
\left( \frac{dy}{d\xi} \right)^2 = 2(1 + a_0) - \frac{1}{y} - (1 + 2a_0)y, \tag{13}
\]

and by using the continuity condition of \( y \) and \( \frac{dy}{d\xi} \) at \( \xi = l_b \). Therefore, from Equations (12) and (13), we obtain the value of \( \gamma_m \) as

\[
\gamma_m = (1 + a_0) + a_0 y_b, \tag{14}
\]

where \( y_b \) is the value of \( y \) at \( \xi = l_b \). From the solution (inside the beam) presented in the earlier subsection, we can calculate \( y_b = \sqrt{\frac{1-\beta_b}{1+\beta_b}} \), where \( \beta_b \) is the value of \( \beta \) at \( \xi = l_b \). Then, we proceed to obtain the solution behind the beam in the same way as obtained by Bera et al. for the case of an electron beam driver. The solution for the wake wave electric field behind the beam can be expressed as

\[
E(y) = \pm \sqrt{2(1 + a_0) - 2a_0 y} - (y + 1/y). \tag{15}
\]

In order to find the solution for the electric field and electron density as a function of \( \xi \), we integrate Equation (12) to obtain

\[
\xi = l_b + 2\sqrt{b}[E(\psi_b, m) - E(\psi, m)], \tag{16}
\]

where \( E(\psi, m) \) is incomplete elliptic integral of the second kind. Here, \( b = \gamma_m + \sqrt{\gamma_m^2 - 1} \); \( m = 2\sqrt{\gamma_m^2 - 1}/b \), and \( y(y_b) \) is related to \( \psi(\psi_b) \) as

\[
y = \gamma_m + \sqrt{\gamma_m^2 - 1} \cos(2\psi). \tag{17}
\]

Equations (15) and (16), together with the above relation, will give the solution for the electric field behind the proton beam.

Figure 1 shows the wake wave electric field and the perturbed electron density profiles as obtained from the solutions inside and outside the beam. As mentioned before, the length of a proton beam available today is much longer in size than the plasma wavelength. However, due to self-modulation, the beam splits into a train of small proton bunches. Their sizes are comparable or even shorter than the typical plasma wavelength. Keeping this in mind, we have discussed the wake wave excitation by a single proton beam whose size is about three times the plasma wavelength (~\(2.8\lambda_p\)). In a later section, we discuss the wake field excited by multiple proton bunches. From the figure, it is seen that a maximum electric field amplitude of several MV/m can be achieved for plasma densities of \( n_0 = 10^{14} - 10^{15} \text{ cm}^{-3} \) with a beam density \( a_0 = 0.5 \) as considered here.

### 3 2D OSIRIS SIMULATION ON PDPWFA

We have carried out our simulation using 2D PIC code OSIRIS. The simulation shows that a rectangular rigid proton beam with energy \(~15\text{ GeV}\) interacts with a plasma and forms wake wave. Plasma is uniform in a rectangular box of dimensions 53 cm x 10.62 cm. Preformed plasma with uniform density \( n_0 = 1.0 \times 10^{14} \text{ cm}^{-3} \) has been considered. The beam is considered to be rigid, which maintains constant normalized density \( a_0 = 0.5 \) throughout the whole simulation run. It is extended by 9.3 mm and 5.31 cm in the longitudinal and transverse directions, respectively. Simulation has been
carried out using a moving window algorithm with a window size of 5.31 cm × 8 cm and a resolution of 11 and 80 μm in the longitudinal and transverse directions, respectively. The number of macro-particles per cell was 16, with a total number of $5 \times 10^6$ cells in the simulation box.

Figure 2 shows the longitudinal electric field profile for the excited wake wave. The beam is propagating along the positive $x$-direction, and behind the driving beam, the wake wave is excited. The perturbed electron density profile is shown in Figure 3. One can easily observe exact matching of the longitudinal field and density profile of our 2D simulations with the 1D exact analytical solution presented in the previous section (Figure 1). This matching is not quite surprising because the transverse extension of the beam is much larger compared to its longitudinal size. However, in 1D theory, it was not possible to extract any information of the transverse electric field produced in the beam plasma interaction process. The transverse field profile obtained from our 2D simulation is shown in Figure 4. In the case of side injection of witness electrons, the electrons’ bunch propagates, making a small angle with the driver beam, and are gradually sucked in at the proper phase by this transverse electric field.\textsuperscript{[23]}

4 | INVESTIGATION OF PDPWFA USING PSEUDOPOTENTIAL APPROACH

Here, we present an alternative approach to arrive at the wake wave solution in the proton driver scheme. Unlike the solution obtained in the previous section, here, we relax the assumption of immobility of plasma ions and also consider
the velocity of the beam to be arbitrary. In the process of generation of plasma waves using an ultra-relativistic charged particle beam, plasma ions carry the main part of the momentum of the source (proton/electron beam). In the strong field excited behind the beam, the plasma ions can reach a velocity that is sufficient to contribute to the process of charge separation, and therefore, this can influence the excited wake field structures.\cite{33,34} Thus, in our investigations, we have included the plasma ion motion as well. We have considered the motion of the plasma ions to follow non-relativistic dynamics.

We rewrite the basic set of equations describing proton beam-driven non-linear 1D plasma waves by incorporating the plasma ion motion as follows:

\( \partial_t n_j + \partial_x (n_j v_j) = 0, \) \hfill (18)

\( (\partial_t + v_j \partial_x) (\gamma_j v_j) = q_j E / m_j, \) \hfill (19)

\( \partial_x E = 4\pi \left[ \sum_j q_j n_j + e n_b \right]. \) \hfill (20)
FIGURE 5  Normalized wake wave electric field in a proton beam-driven plasma [$\beta_{ph} = 0.995$, $\beta_b = 0.995$, and $\mu = 1/1836$], with beam density [$\alpha = 0.5$ for $0 \leq \xi \leq 5.6\pi$ and 0 otherwise]

FIGURE 6  Normalized perturbed electron density in a beam-driven plasma [$\beta_{ph} = 0.995$, $\beta_b = 0.995$, and $\mu = 1/1836$], with beam density [$\alpha = 0.5$ for $0 \leq \xi \leq 5.6\pi$ and 0 otherwise]

Here, $j$ stands for electrons or ions with $q_j = -e$ for electrons and $q_j = e$ for ions. The relativistic Lorentz factors associated with an electron or ion are denoted by $\gamma_j$. The proton beam density and the velocity are $n_p$ and $v_b$, respectively, with $v_b = v_{ph}$. All the other variables used here have their usual meanings.

A stationary wave solution is obtained with the same variable transformation as demonstrated in the earlier section. In the transformed coordinate system [$\zeta = k_p(v_{ph}t - x)$], Equations (18)–(20) take the form as follows:

\[
\frac{d}{d\zeta} [N_j(\beta_{ph} - \beta_j)] = 0, \tag{21}
\]

\[
(\beta_{ph} - \beta_j) \frac{d}{d\zeta} (\beta_j \gamma_j) = \epsilon_j \beta_{ph}^2 E, \tag{22}
\]

\[
\frac{dE}{d\zeta} = N_e - N_i - \alpha. \tag{23}
\]

where $\beta_{ph} = v_{ph}/c$, and $\epsilon_j$ is a constant: $\epsilon_e = 1$ for electron, and $\epsilon_i = -\mu$ for ion. We have also defined $\beta_j = v_j/c$ and $\gamma_j = (1 - \beta_j^2)^{-1/2}$. Electron and ion densities are both normalized by equilibrium plasma density ($N_j = n_j/n_0$).

Now, $E(\zeta) = -\frac{1}{\beta_{ph}^2} \frac{d\varphi}{d\zeta}$, where $\varphi = e\phi/m_e c^2$. Here, $\phi$ is the electrostatic potential. After carrying out some simple algebra, we obtain normalized velocities and densities of plasma electrons and ions as follows:

\[
\beta_e = \frac{\beta_{ph} - \varphi_e (\varphi_e^2 - \gamma_e^{-2})^{1/2}}{\beta_{ph}^2 + \varphi_e^2}, \tag{24}
\]
FIGURE 7  Variation of normalized wake wave electric field in absence of magnetic field driven by a train of equidistant particle bunches [$\beta_{ph} = 0.995$, $\beta_b = 0.995$, and $\mu = 1/1836$]

Here, $\varphi_i = -\mu \varphi$ and $\varphi_e = 1 + \varphi$, with $m_e/m_i = \mu$ being the electron-to-ion mass ratio. Using these expressions for species densities, from Poisson’s equation (Equation 23), we obtain a second-order differential equation for $\varphi$,

$$\frac{d^2 \varphi}{dz^2} = -\frac{dU(\varphi)}{d\varphi}$$

where

$$\frac{dU(\varphi)}{d\varphi} = -\frac{\beta_{ph}^3}{\sqrt{\beta_{ph}^2 + 2\varphi_i}} + \frac{\beta_{ph}^3 \varphi_e}{\sqrt{\varphi_e^2 - \gamma^{-2}}} - \beta_{ph}^4 \gamma^{-2} - \alpha \beta_{ph}^2$$

which may be integrated to obtain

$$U(\varphi) = \frac{\beta_{ph}^3}{\mu} \left[ \beta_{ph}^2 + 2\varphi_i \right]^{1/2} + \beta_{ph}^3 \varphi_e \left[ \varphi_e^2 - \gamma^{-2} \right]^{1/2} - \beta_{ph}^4 \left( \gamma^2 + \frac{1}{\mu} \right)$$

where we set $U(\varphi) = 0$ at $\varphi = 0$. Here, $\alpha = n_b/n_0$ is the normalized proton beam density. This second-order differential equation describes the motion of a fictitious particle of unit mass in a pseudopotential $U(\varphi)$. We solve this equation numerically and obtain the solution for the wake wave electric field excited inside and outside a rectangular proton beam, as well as corresponding perturbed plasma electron density, as shown in Figures 5 and 6, respectively. Corresponding to a beam density $n_b = n_0/2$ and beam velocity $v_b = 0.995c$, the transformer ratio (the ratio of the maximum accelerating field behind the beam to the maximum decelerating field inside the beam) that determines the overall energy efficiency of the accelerated particles can be evaluated easily from this field profile.

So far, we have discussed the wake wave excited by a single long proton drive beam. However, due to self-modulation, the long proton bunch splits into a long chain of equi-spaced micro-bunches because of the strong focusing and defocusing forces of the field. So, it is of fundamental interest to see how a strong wake field can be excited behind such
multi-beams. Figures 7 and 8 represent the electric field and perturbed beam density, respectively, for the wake wave excited by equi-spaced multi-proton bunches (length of the each single beam is \(\sim 0.5\lambda_p\) with separation distance \(\sim 0.16\lambda_p\)) as obtained by the numerical solution of Equation (30). From Figure 7, we observe that electric field amplitude cannot grow indefinitely with the increase of the number of beams; rather, the field saturates to a particular limit.

5 | CONCLUSION

We have obtained an exact analytical solution for the relativistic wake wave excited by a proton beam using a fluid approach. The solution, specifically the longitudinal electric field and the density profile, match with the results obtained from 2D PIC OSIRIS simulation. PIC simulation also reveals the characteristics of an excited transverse electric field. This field plays a crucial role in trapping the side-injected electrons at a proper phase to the excited longitudinal wake wave. Furthermore, we employed a pseudopotential method as an alternative way to deduce the solution for the wake driven by a long chain of equi-spaced micro-bunches of charge. The results conclusively show that the amplitude of the electric field does not grow indefinitely with increase in the number of beams. On the contrary, it saturates to a definitive limit.

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DATA AVAILABILITY STATEMENT

Research data are not shared.

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