Measurement-Induced Non locality in an \( n \)-partite quantum state.

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(Dated: February 21, 2022)

We generalize the concept of measurement-induced non-locality (MiN) to \( n \)-partite quantum states. We get exact analytical expressions for MiN in an \( n \)-partite pure and \( n \)-qubit mixed state. We obtain the conditions under which MiN equals geometric quantum discord in an \( n \)-partite pure state and an \( n \)-qubit mixed state.  
PACS numbers: 03.65.Ud;75.10.Pq;05.30.-d

Measurement induced non-locality (MiN) is a measure of quantum correlations as manifested in the non-local effects of local (on a single part) quantum operations\(^1, 2\). These local quantum operations leave invariant the reduced density operators of the parts on which they act, while changing the global quantum state. MiN concerns the von-Neumann measurement on a part of a quantum system. MiN being an inherently quantum phenomenon, is expected to be useful as a tool for quantitative specification of quantum correlation. Such a quantitative specification of quantum correlations in terms of MiN was given in \(^2\) for bipartite quantum systems. Here we generalize this measure to \( n \)-partite quantum systems. MiN is a manifestation of the quantum verses classical paradigm of quantum correlations and naturally compares with quantum discord \(^3, 4\) which is also a manifestation of such a paradigm. In fact, it is quite relevant to inquire about the conditions on quantum states under which MiN and geometric discord are equal (or, rather are different) and the different kinds of information they give about the quantum correlations in a quantum state. Here we establish such general conditions in \( n \)-partite pure and \( n \)-qubit mixed states.

To understand the non-local effects involved, consider a bipartite quantum system. The state of a bipartite system may be changed by locally invariant operation applied to one of the subsystems. This change in the bipartite state is a non-local effect and can be detected only by measuring the two parts jointly. By employing a Hilbert-Schmidt metric, for example, we can quantify such non-local effects by measuring the distance between initial and final bipartite states. These ideas are further clarified by considering an application like quantum dense coding. In this process two parties share an entangled pair of qubits (in the Bell state) one of which is subjected to a local unitary operation which does not change its reduced density operator. In other words, the marginal statistics of measurements on the particle does not change by the local operation applied to it. Thus the reduced density operators of both the qubits do not change in the process. However, the state of the whole system (the bipartite state) changes after the local unitary operation is applied to one of the qubits. Thus the change in the state of the whole system due to a local operation on a part is a non-local effect and and can be observed only by measuring the two qubits jointly. There is no way to detect this change locally, that is, there is no way for any eavesdropper to succeed by dealing with only one of the two qubits. Further, this is essentially quantum non-locality as it necessarily involves a pair of entangled qubits in a bipartite pure state. The relation of such a nonlocality with other measures of quantum correlations is a naturally interesting question. In this paper, we address this question by exploring the relation of MiN with discord and entanglement in an \( n \)-partite quantum system.

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We may note here that the processes defining discord and MiN naturally divide the \( n \)-partite system into two parts, one subjected to measurement and the remaining part. However, MiN measures the change in the \( n \)-partite state brought out by such a local measurement (see Eq. (1)) and is related to the multipartite correlations implied by it (see theorems 1,2,3 below). The same statement applies to discord as well [6]. Thus both MiN and discord are amenable to genuine multipartite generalization.

**Multipartite generalization of MiN** : Multipartite generalization of MiN can be obtained in a manner analogous to that of geometric quantum discord [6]. For an \( n \)-partite system in a state \( \rho \) we define, for (normalized) MiN [7]

\[
N_l(\rho) = \frac{d_l}{d_l - 1} \max(\|\rho - \Pi_l(\rho)\|^2), l = 1, 2, \ldots, n
\]

where \( \Pi_l = \{\Pi_l^k\} \) stands for the set of von-Neumann measurements on the \( l \)-th part such that \( \Pi_l^k(\rho_l^0) = \sum_k \Pi_l^k \rho_l^0 \Pi_l^k = \rho_l^0, \rho_l^0 \) being the reduced density operator obtained by tracing out all parts other than the \( l \)-th part from the \( n \)-partite state acting on \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \) with \( \text{dim}(\mathcal{H}_m) = d_m, m = 1, 2, \ldots, n \). Such a measurement \( \Pi_l^k \) is defined by the projectors corresponding to the eigenstates of \( \rho_l^0 \). When all the eigenvalues of \( \rho_l^0 \) are non-degenerate, there is only one von-Neumann measurement \( \Pi_l^k \) satisfying \( \Pi_l^k(\rho_l^0) = \sum_k \Pi_l^k \rho_l^0 \Pi_l^k = \rho_l^0 \) and the maximization requirement in Eq. (1) drops out. If one or more eigenvalues of \( \rho_l^0 \) are degenerate, the right hand side of Eq. (1) has to be maximized over the eigenspaces of degenerate eigenvalues, which is, in general, a difficult task.

Throughout this paper the superscript \( t \) denotes the transpose of a vector or a matrix.

Comparing the definitions of MiN \( N_l(\rho) \) and the geometric discord \( D_l(\rho) \) [6] it follows that, for any \( n \)-partite state, \( N_l(\rho) \geq D_l(\rho) \). We are interested in finding the criteria for their equality.

The multipartite non-locality can be evaluated for an \( n \)-partite pure state via the following

**Theorem 1**: Let \( |\psi\rangle = \sum_{i_1 i_2 \cdots i_n} a_{i_1 i_2 \cdots i_n} |i_1 i_2 \cdots i_n\rangle \) be a \( n \)-partite pure state. Then

\[
N_l(|\psi\rangle\langle \psi|) = \frac{d_l}{d_l - 1}(1 - tr(\rho^{l_0})^2),
\]

where \( \rho^{l_0} \) is the reduced density matrix of the \( l \)-th part and \( d_l = \text{dim}(H^l) \).

**Proof**: In order to get \( N_l(|\psi\rangle\langle \psi|) \) we can directly calculate the terms which define it (Eq. (1)). We have

\[
\rho = |\psi\rangle\langle \psi| = \sum_{i_1 i_2 \cdots i_n} \sum_{j_1 j_2 \cdots j_n} a_{i_1 i_2 \cdots i_n}^{*} a_{j_1 j_2 \cdots j_n}^{*} |i_1 i_2 \cdots i_n\rangle\langle j_1 j_2 \cdots j_n|.
\]

Here \( |i_1 i_2 \cdots i_n\rangle \) is the orthonormal product basis in the \( n \)-partite Hilbert space. The set of von-Neumann measurements on the \( l \)-th part is given by

\[
\Pi_l^k = \{\Pi_l^k = U|k_l\rangle\langle k_l|U^\dagger\}
\]

where \( \{|k_l\rangle\}, k_l = 1, \ldots, d_l = \text{dim}(H^l) \) is an orthonormal basis in \( H^l \) and \( U \) is a unitary operator acting on \( H^l \). We can span all orthonormal bases in \( H^l \) by varying \( U \). The post measurement state (after measurement on the \( l \)-th part) is

\[
\Pi_l^k(\rho) = \sum_{k_l} \Pi_l^k(\rho) \Pi_l^k
\]
where \( \Pi_{k_l}^{(l)} = I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes I_{d_{l-1}} \otimes \Pi_{k_l} \otimes I_{d_{l+1}} \otimes \cdots \otimes I_{d_n} \). We need \( \text{tr}(\rho \Pi^{(l)}(\rho)) \). A direct calculation of \( \text{tr}(\rho \Pi^{(l)}(\rho)) \) and comparison with \( \rho^{(l)} = \text{tr}_l(\rho) \) gives, assuming that \( \{U|k_l\} \) is the eigenbasis of \( \rho^{(l)} \),

\[
\text{tr}(\rho \Pi^{(l)}(\rho)) = \sum_{k_l} (|k_l\rangle U^\dagger \rho^{(l)} U |k_l\rangle)^2 = \sum_{k_l} \lambda_{k_l}^2 = \text{tr}(\rho^{(l)})^2,
\]

where \( \{\lambda_{k_l}\} \) are the eigenvalues of \( \rho^{(l)} \). This calculation is done in the appendix.

From the definition of \( N_l(\rho) \) (Eq. (11)) we get

\[
N_l(\rho) = \frac{d_l}{d_l - 1}(||\rho||^2 - \min_{\Pi^{(l)}}(2\text{tr}(\rho \Pi^{(l)}(\rho)) - ||\Pi^{(l)}(\rho)||^2)) \).
\]

For a pure state \( ||\rho||^2 = 1 \) and \( ||\Pi^{(l)}(\rho)||^2 = \text{tr}(\rho \Pi^{(l)}(\rho)) \) so that

\[
N_l(\rho) = \frac{d_l}{d_l - 1}(1 - \min_{\Pi^{(l)}}\text{tr}(\rho \Pi^{(l)}(\rho))).
\]

The minimum is over the von-Neumann measurements leaving the marginal state \( \rho^{(l)} \) invariant, that is \( \sum_k \Pi_k^{(l)} \rho^{(l)} \Pi_k^{(l)} = \rho^{(l)} \), or,

\[
\sum_{k_l} (|k_l\rangle U^\dagger \rho^{(l)} U |k_l\rangle \langle k_l| U^\dagger = \rho^{(l)}).
\]

This is the spectral decomposition of \( \rho^{(l)} \) which is consistent with our choice of \( \{U|k_l\} \) to be the eigenbasis of \( \rho^{(l)} \). Since \( \text{tr}(\rho \Pi^{(l)}(\rho)) \) is simply the trace of \( (\rho^{(l)})^2 \), the minimization in the definition of \( N_l \) Eq. (11) drops out and we get

\[
N_l(|\psi\rangle \langle \psi|) = \frac{d_l}{d_l - 1}(1 - \text{tr}(\rho^{(l)})^2).
\]

Corollary: For an \( n \)-partite pure state \( \rho = |\psi\rangle \langle \psi| \)

\[
D_l(\rho) = N_l(\rho) \quad \text{(6)}
\]

where \( D_l(\rho) \) is the geometric discord of \( \rho \) with von-Neumann measurement on the \( l \)th part [6]. This important result follows trivially, because \( D_l(\rho) \) requires maximization over all von-Neumann measurements on the \( l \)th part which is obtained if the \( \{U|k_l\} \) forms the eigenbasis of \( \rho^{(l)} \). To make it more explicit, note that, for a pure state,

\[
D_l(\rho) = \frac{d_l}{2(d_l - 1)}(1 - \max_{\Pi^{(l)}}\text{tr}(\rho \Pi^{(l)}(\rho))) = \frac{d_l}{2(d_l - 1)}(1 - \max_{U|k_l}\text{tr}(\rho \Pi^{(l)}(\rho))),
\]

where the maximization is over all von Neuman measurements on the \( l \)th part. We get the the maximization in the second term only when \( \{U|k_l\} \) form the eigenbasis of \( \rho^{(l)} \), so

\[
D_l(\rho) = \frac{d_l}{d_l - 1}(1 - \sum_{k_l} \lambda_{k_l}^2) = \frac{d_l}{d_l - 1}(1 - \text{tr}(\rho^{(l)})^2) = N_l(\rho).
\]

It is interesting to compare \( N_l(\rho) \) (Eq. (11)) with measures of entanglement of pure multipartite states. For a bipartite pure state \( \rho_{AB} \) we have, for the concurrence,

\[
C(\rho_{AB}) = \sqrt{2(1 - \text{tr}(\rho_{AB}^2))}
\]
which is related to $N_l(\rho_{AB})$ by
\begin{equation}
N_l(\rho_{AB}) = \frac{d_l}{2(d_l - 1)} C^2(\rho_{AB}).
\end{equation}

Thus, for pure bipartite states, non-locality is simply related to concurrence.

The Meyer-Wallach measure of entanglement of multipartite pure states is
\begin{equation}
Q(|\psi\rangle) = \frac{1}{n} \sum_{k=1}^{n} 2(1 - tr(\rho_k^2))
\end{equation}

where $\rho_k$ is the reduced density operator for the $k$th part. Thus,
\begin{equation}
Q(|\psi\rangle) = \frac{2}{n} \sum_{l=1}^{n} \left( \frac{d_l - 1}{d_l} \right) N_l(|\psi\rangle\langle\psi|).
\end{equation}

Thus the Meyer-Wallach measure of pure state multipartite entanglement is the average of non-locality over the parts of the system.

Non-locality in the multipartite mixed states: To get $N_l(\rho)$ in this case, we start with the Bloch representation of a multipartite state $\rho$ [8]. Bloch representation [3] of a $n$-partite density operator is
\begin{equation}
\rho = \frac{1}{\Pi_k d_k} \{ \otimes_{k} I_{d_k} + \sum_{k \in \mathcal{N}} \sum_{\alpha_k} s_{\alpha_k} \lambda^{(k)}_{\alpha_k} + \sum_{2 \leq M \leq n} \sum_{\{k_1, k_2, \ldots, k_M\}} \sum_{\alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_M}} \tilde{t}_{\alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_M}} \lambda^{(k_1)}_{\alpha_{k_1}} \lambda^{(k_2)}_{\alpha_{k_2}} \cdots \lambda^{(k_M)}_{\alpha_{k_M}} \} \text{ (9)}
\end{equation}

where $\mathcal{N} = \{1, 2, \ldots, n\}$ and
\begin{alignat}{2}
\lambda^{(k_1)}_{\alpha_{k_1}} &= (I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes I_{d_{k_1}} \otimes I_{d_{k_1+1}} \otimes \cdots \otimes I_{d_n}) \\
\lambda^{(k_2)}_{\alpha_{k_2}} &= (I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes I_{d_{k_2}} \otimes I_{d_{k_2+1}} \otimes \cdots \otimes I_{d_n}) \\
\lambda^{(k_1)}_{\alpha_{k_1}} \lambda^{(k_2)}_{\alpha_{k_2}} &= (I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes I_{d_{k_1}} \otimes I_{d_{k_1+1}} \otimes \cdots \otimes I_{d_n}) \text{ (10)}
\end{alignat}

$s^{(k)}$ is a Bloch vector corresponding to $k$th subsystem, $s^{(k)} = [s_{\alpha_k}]_{\alpha_k=1}^{d_k^2-1}$ and
\begin{equation}
\tilde{t}_{\alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_M}} = \frac{d_{k_1} d_{k_2} \cdots d_{k_M}}{2M} tr[\rho \lambda^{(k_1)}_{\alpha_{k_1}} \lambda^{(k_2)}_{\alpha_{k_2}} \cdots \lambda^{(k_M)}_{\alpha_{k_M}}]. \text{ (11)}
\end{equation}

For more details see ref. [8–10].

Recently, for a bipartite system $ab$ ($n = 2$) with states in $\mathcal{H}^a \otimes \mathcal{H}^b$, dim($\mathcal{H}^a$) = $d_a$, dim($\mathcal{H}^b$) = $d_b$, S. Luo and S. Fu introduced the following generic expression for MiN [2]
\begin{equation}
N_a(\rho) = tr(TT^t) - \min_A tr(ATT^tA^t), \text{ (12)}
\end{equation}

where $T = [t_{ij}]$ is an $d_a^2 \times d_b^2$ matrix and the minimum is taken over all $(d_a \times d_a^2 - 1)$-dimensional isometric matrices $A = [a_{ji}]$ such that $a_{ji} = tr(|j\rangle\langle j|X_i) = \langle j|X_i|j\rangle$, $j = 1, 2, \ldots, d_a$ ; $\{X_i\}, i = 1, 2, \ldots, d_a^2 - 1$ forms an orthonormal basis in the space of operators acting on $\mathcal{H}^a$ and $\{|j\rangle\}$ is any orthonormal basis in $\mathcal{H}^a$. we generalize this result to $n$-partite quantum states, in theorem 2 and 3.
Theorem 2. Let $\rho_{12\cdots n}$ be a $n$-partite state defined by Eq. (2), then

$$N_l(\rho) = \frac{d_l}{(d_l - 1)\Pi_k d_k} \left\{ \sum_{1 \leq M \leq n-1} \sum_{\{k_1, k_2, \ldots, k_M\} \subseteq N - \{l\}} \frac{d_l d_{k_1} d_{k_2} \cdots d_{k_M}}{2M+1} ||T^{(l,k_1,k_2,\cdots,k_M)}||^2 - \min_{A^{(l)}} tr(A^{(l)}K^{(l)}(A^{(l)})^t) \right\},$$

where the $(d_l^2 - 1) \times (d_l^2 - 1)$ symmetric matrix $K^{(l)}$ is defined as

$$K^{(l)}_{\alpha_i\beta_l} = \sum_{1 \leq M \leq n-1} \sum_{\{k_1, k_2, \ldots, k_M\} \subseteq N - \{l\}} \alpha_k \alpha_k \cdots \alpha_k M \sum_{t=1}^{d_l-1} d_{k_1} d_{k_2} \cdots d_{k_M} t_{\alpha_l \alpha_l \alpha_k \cdots \alpha_k M} t_{\beta_l \alpha_l \alpha_k \cdots \alpha_k M}.$$

The proofs of theorems 2 and 3 is a straightforward generalization of those of theorems 2 and 3 respectively in ref. [2] to the multipartite case, so that we skip these proofs.

Theorem 3. If the $l$th part of a $n$-partite quantum system is a qubit ($d_l = 2$), then

$$N_l(\rho) = \frac{d_l}{(d_l - 1)\Pi_k d_k} \left[ \sum_{1 \leq M \leq n-1} \sum_{\{k_1, k_2, \ldots, k_M\} \subseteq N - \{l\}} \frac{d_{k_1} d_{k_2} \cdots d_{k_M}}{2M+1} ||T^{(l,k_1,k_2,\cdots,k_M)}||^2 - \frac{\min_{s^{(l)}} tr(s^{(l)}K^{(l)}s^{(l)})}{||s^{(l)}||^2} \right],$$

where $s^{(l)}$ is the coherent vector of $\rho^{(l)}$ and $\eta_{\text{min}}$ is the smallest eigenvalue of the matrix $K^{(l)}$ which is a $3 \times 3$ real symmetric matrix, defined as

$$K^{(l)}_{\alpha_i\beta_l} = \sum_{1 \leq M \leq n-1} \sum_{\{k_1, k_2, \ldots, k_M\} \subseteq N - \{l\}} \alpha_k \alpha_k \cdots \alpha_k M \sum_{t=1}^{d_l-1} d_{k_1} d_{k_2} \cdots d_{k_M} t_{\alpha_l \alpha_l \alpha_k \cdots \alpha_k M} t_{\beta_l \alpha_l \alpha_k \cdots \alpha_k M}.$$

For $n$-qubit ($d_i = 2, i = 1, 2, \cdots, n$),

$$N_l(\rho) = \frac{1}{2(n-1)} \left[ \sum_{1 \leq M \leq n-1} \sum_{\{k_1, k_2, \ldots, k_M\} \subseteq N - \{l\}} ||T^{(l,k_1,\ldots,k_M)}||^2 - \frac{\min_{s^{(l)}} tr(s^{(l)}K^{(l)}s^{(l)})}{||s^{(l)}||^2} \right],$$

and

$$K^{(l)}_{\alpha_i\beta_l} = \sum_{1 \leq M \leq n-1} \sum_{\{k_1, k_2, \ldots, k_M\} \subseteq N - \{l\}} \alpha_k \alpha_k \cdots \alpha_k M \sum_{t=1}^{d_l-1} t_{\alpha_l \alpha_l \alpha_k \cdots \alpha_k M} t_{\beta_l \alpha_l \alpha_k \cdots \alpha_k M}.$$
Relation between the non-locality and geometric quantum discord for arbitrary n-qubit states: We saw (see Eq. (22)) that the non-locality and geometric discord are equal for arbitrary n-partite pure states. In this section we find a class of general n-qubit states for which these quantities coincide. Consider a n-qubit state \( \rho \). The geometric discord for such a state corresponding to the von-Neumann measurement on \( l \)th qubit is given by

\[
D_{l}(\rho) = \frac{1}{2^{(n-1)}} \left[ ||s^{(l)}||^2 + \sum_{1 \leq M \leq n-1} \sum_{\{k_1, \ldots, k_M\} \subseteq N-\{l\}} \left| \left| \mathcal{T}^{(l,k_1,\ldots,k_M)} \right| \right|^2 - \lambda_{\text{max}}, \right]
\]  

(18)

where \( s^{(l)} \) is the coherent vector of \( \rho^{(l)} \) (reduced density operator for the \( l \)th part), \( \mathcal{T} = [t_{\alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_M}}] = [tr(\rho \lambda_{k_1}^{(k_1)} \lambda_{k_2}^{(k_2)} \ldots \lambda_{k_M}^{(k_M)})] \), and \( \lambda_{\text{max}} \) is the largest eigenvalue of the \( 3 \times 3 \) real symmetric matrix

\[
G^{(l)} = s^{(l)}(s^{(l)})^t + K^{(l)}
\]

(19)

where \( K^{(l)} \) is given by Eq. (17) for \( n \) qubits. The non-locality for the \( n \)-qubit state \( \rho \) is given by Eq. (11a). We now consider two cases.

Case I: \( s^{(l)} \neq 0 \). By Eq. (19) we get

\[
\hat{e}^t G^{(l)} \hat{e} = \hat{e}^t s^{(l)}(s^{(l)})^t \hat{e} + \hat{e}^t K^{(l)} \hat{e}
\]

where \( \hat{e} \in \mathbb{R}^3 \) is an arbitrary unit vector, choosing \( \hat{e} = \frac{s^{(l)}}{||s^{(l)}||} \), we get

\[
\frac{(s^{(l)})^t K^{(l)} s^{(l)}}{||s^{(l)}||^2} = \frac{(s^{(l)})^t G^{(l)} s^{(l)}}{||s^{(l)}||^2} - ||s^{(l)}||^2
\]

Substituting in Eq. (11a) we get

\[
N_{l}(\rho) = \frac{1}{2^{(n-1)}} \left[ ||s^{(l)}||^2 + \sum_{1 \leq M \leq n-1} \sum_{\{k_1, \ldots, k_M\} \subseteq N-\{l\}} \left| \left| \mathcal{T}^{(l,k_1,\ldots,k_M)} \right| \right|^2 - \frac{s^{(l)^t G^{(l)} s^{(l)}}}{||s^{(l)}||^2} \right]
\]

(20)

If \( \frac{s^{(l)}}{||s^{(l)}||} \) is the eigenvector of \( G^{(l)} \) with the largest eigenvalue then the right hand side of Eq. (19) gives the geometric discord \( D_l(\rho) \) so that under this condition \( N_{l}(\rho) = D_l(\rho) \). The above condition can be equivalently stated as

\[
[s^{(l)}(s^{(l)})^t, K^{(l)}] = 0
\]

and

\[
||s^{(l)}||^2 + \eta_l \geq \eta_{l \neq l}
\]

where \( \{\eta_l\} \) are the eigenvalues of \( K^{(l)} \) and \( \eta_l \) is the eigenvalue corresponding to the eigenvector \( \frac{s^{(l)}}{||s^{(l)}||} \).

Case II: \( s^{(l)} = 0 \). In this case \( \rho \) has one doubly degenerate eigenvalue. With \( s^{(l)} = 0 \) we get from Eq. (19)

\[
\hat{e}^t G^{(l)} \hat{e} = \hat{e}^t K^{(l)} \hat{e}
\]

(21)

To get non-locality we have to minimize the right hand side while the geometric discord requires maximization of the left hand side. Under these conditions, the equality in Eq. (20) is preserved.
The variation of \( \bar{N}_l(\rho(p)) \) (dashed curve) and the geometric discord \( D_l(\rho(p)) \) (continuous curve) for the state as in (a) Eq(22) (b) Eq(23) (c) Eq(24) (d) Eq(25) with parameter \( p \).

if \( G^{(l)} = K^{(l)} \) has a single three-fold degenerate eigenvalue, \( (\eta_1 = \eta_2 = \eta_3) \). Thus when \( s^{(l)} = 0 \), \( N_l(\rho) = D_l(\rho) \) provided the matrix \( K^{(l)} \) has a single three fold degenerate eigenvalue.

**Examples**: As our first example we consider the set of three qubit states comprising the convex combination of

\[
    |GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad \text{and} \quad |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),
\]

\[
    \rho(p) = p|GHZ\rangle\langle GHZ| + (1 - p)|W\rangle\langle W|.
\]  

(22)

The \( K^{(1)} \) matrix of this state is

\[
    K^{(1)} = \text{diag}[2p^2 + \frac{16}{9}(1-p)^2, 2p^2 + \frac{16}{9}(1-p)^2, 2p^2 + \frac{19}{9}(1-p)^2 - \frac{4}{3}p(1-p)]
\]

with the coherent vector for the first qubit

\[
    s^{(1)} = [0, 0, \frac{1}{3}(1-p)]^t \neq 0
\]

so that case I applies. We find that \( [s^{(1)}(s^{(1)})^t, K^{(1)}] = 0 \) and the condition \( ||s^{(1)}||^2 + \eta_1 \geq \eta_i, i \neq 1, (\eta_1 \text{ is the eigenvalue of } K^{(1)} \text{ matrix corresponding to eigenvector } \frac{s^{(1)}}{||s^{(1)}||}) \), is satisfied when \( p \leq \frac{1}{4} \) and \( p = 1 \).

This is depicted in fig.(1a).

The second example consists of

\[
    \rho(p) = p|\widetilde{W}\rangle\langle \widetilde{W}| + (1 - p)|W\rangle\langle W|
\]  

(23)

where \( |\widetilde{W}\rangle \) is the flipped \( |W\rangle \) state, \( \sigma_x \otimes \sigma_x \otimes \sigma_x |W\rangle \). The \( K^{(1)} \) matrix of this state is

\[
    K^{(1)} = \text{diag}[\frac{16}{9}p^2 + \frac{16}{9}(1-p)^2, \frac{16}{9}p^2 + \frac{16}{9}(1-p)^2, \frac{19}{9}p^2 + \frac{19}{9}(1-p)^2 - \frac{10}{3}p(1-p)]
\]
with the coherent vector for the first qubit
\[ s^{(1)} = [0, 0, \frac{1}{3}(1 - 2p)]^t \neq 0, \]
so that case I applies. We find that \[ \langle s^{(1)} | s^{(1)} \rangle, K^{(1)} \rangle = 0 \] and the condition \[ ||s^{(1)}||^2 + \eta_1 \geq \eta_{i \neq 1}, \] is satisfied when \( p \leq 0.1127 \) and \( p \geq 0.8873 \). The results are shown in fig.(1b).

The third example consists of
\[ \rho(p) = p|GHZ\rangle\langle GHZ| + (1 - p)|GHZ_\rangle\langle GHZ_\mid \]
where \( |GHZ_\rangle = \frac{1}{\sqrt{2}}(|000⟩ - |111⟩) \). The \( K^{(1)} \) matrix of this state is
\[ K^{(1)} = \text{diag}[2(2p^2 - 1)^2, 2(2p^2 - 1)^2, 2(2p^2 + 1)^2] \]
and the coherent vector for the first qubit
\[ s^{(1)} = 0, \]
so that case II applies. \( K^{(1)} \) does not have a single triply degenerate eigenvalue, for all \( p \), except \( p = 0 \) and \( p = 1 \). Therefore \( N_i(\rho) \neq D_i(\rho) \) for all \( p \) between 0 and 1. The results are shown in fig.(1c).

The last example consists of the states
\[ \rho(p) = p|GHZ\rangle\langle GHZ| + (1 - p)|GHZ_1\rangle\langle GHZ_1| \]
where \( |GHZ_1\rangle = \frac{1}{\sqrt{2}}(|001⟩ + |110⟩) \). The \( K^{(1)} \) matrix of this state is
\[ K^{(1)} = \text{diag}[2(p^2 - (1 - p)^2), 2(p^2 - (1 - p)^2), 2(p^2 - (1 - p)^2)] \]
and the coherent vector for the first qubit
\[ s^{(1)} = 0, \]
so that case II applies. \( K^{(1)} \) does have a single triply degenerate eigenvalue, for all \( p \). Therefore \( N_i(\rho) = D_i(\rho) \) for all \( p \) as shown in fig.(1d).

Summary and comments: In this paper, we have given exact analytical expressions for MiN in an \( n \)-partite pure and \( n \)-qubit mixed state. Apart from this we obtain two results which we think are useful in further understanding of quantum correlations in multipartite quantum systems. First we have shown that the geometric discord and MiN are equal for a multipartite pure quantum state. This indicates that, in the classical verses quantum scenario, quantum correlations and non-locality imply each other. This supports the well known result that for a bipartite pure state, entanglement and non-locality are equivalent in the sense that an entangled bipartite pure state breaks bell inequality and vice versa. Further, we have shown that, for a bipartite pure state, concurrence can be obtained from MiN (Eq.(7)), which, in turn, equals geometric discord. Thus for the bipartite pure states geometric discord and MiN do not give any new information on quantum correlations as compared to entanglement and quantum correlations seem to be essentially dominated by entanglement. Interestingly, for multipartite pure states, the Meyer-Wallach measure of entanglement is just the average over MiN with measurement on the \( l \)th part (Eq.(5)). This equation points to a new relation between the entanglement and non-locality in a multipartite pure state and projects entanglement as a kind of average non-local effect. To the best of our knowledge, a quantitative relation between a measure of multipartite entanglement and a measure of non-locality has not appeared in the literature before.
In the light of our result relating MiN, discord and entanglement for the multipartite pure states (Eq.s[21]), we can see that quantum non-locality and hence discord is complementary to entanglement for a two qubit pure state by comparing the teleportation and dense coding protocols. In the dense coding protocol, quantum non-locality is operating, because a local unitary operation encodes global information in the two qubit joint state which can be deciphered only by the joint measurement of the two entangled qubits. In the teleportation process we need to communicate two classical bits to translate the state of the particle from one party to another far away party provided two parties are sharing maximally entangled state, while the dense coding process is the reverse of teleportation process, as we transfer one particle so as to communicate two classical bits provided the parties are sharing maximally entangled state.

Whereas MiN and geometric discord are equal for all multipartite pure states, these coincide only for a class of multiqubit mixed states. Thus, in general, discord and MiN show different characters in multiqubit mixed states. We have obtained exact analytical conditions necessary for the equality of MiN and quantum discord in a mixed multiqubit state. These conditions obtained in case I and case II above identify a class of states for which MiN and the geometric quantum discord coincide. Recently quantum discord is shown to measure the quantumness of the state rather than genuine quantum correlation [13]. Thus for the class of states with equal \( N_{i}(\rho) \) and \( D_{l}(\rho) \), MiN seems to be identical or simply related to quantumness of the state. An understanding of relation between quantumness and non-locality by other routes will then be interesting. At any rate, it is interesting to explore the relation between \( N_{i}(\rho) \) and \( D_{l}(\rho) \) in the states for which they do not coincide, because these will improve our understanding of quantum correlations. In such situations MiN is distinct from quantumness and may even be independent of it [14]. Finally, the results of this paper may be useful for a unified classification of correlations in a multipartite quantum state [15].

Acknowledgments: This work was supported by the BCUD grant RG-13. ASMH acknowledges University of Pune for hospitality during his visit when this work was carried out.

Appendix:

We obtain Eq.(5). We use various symbols defined in the proof of theorem 1. Using Eq.s(3,4) and the orthonormality of the product basis \( |i_{1}i_{2}\cdots i_{n}\rangle \), a bit lengthy but straightforward calculation gives

\[
tr(\rho^{(l)}(\rho)) = \sum_{k_{l}} \left[ \sum_{q_{l}} \sum_{i_{1}\cdots i_{m}} a_{i_{1}\cdots i_{m}}^{*} a_{i_{1}\cdots i_{m}} \langle k_{l}|U^{\dagger}|i_{l}\rangle \langle q_{l}|U|k_{l}\rangle \right] \times \left[ \sum_{p_{l}} \sum_{j_{1}\cdots j_{m}} a_{j_{1}\cdots j_{m}}^{*} a_{j_{1}\cdots j_{m}} \langle k_{l}|U^{\dagger}|p_{l}\rangle \langle |j_{l}|U|k_{l}\rangle \right].
\] (A1)

Now, we get for \( \rho^{(l)} \)

\[
\rho^{(l)} = \sum_{j_{l}} \sum_{i_{1}\cdots i_{m}} a_{i_{1}\cdots i_{m}}^{*} a_{i_{1}\cdots i_{m}} |i_{l}\rangle \langle j_{l}|.
\] (A2)

If \( \{|\psi_{q}\rangle\} \) are the eigenvectors of \( \rho^{(l)} \), then by spectral theorem we can write

\[
\rho^{(l)} = \sum_{q} \langle \psi_{q}|\rho^{(l)}|\psi_{q}\rangle |\psi_{q}\rangle \langle \psi_{q}|.
\] (A3)

We now put Eq.(A2) in Eq.(A3) and find \( (\langle \psi_{q}|\rho^{(l)}|\psi_{q}\rangle)^{2} \), take \( |\psi_{k_{l}}\rangle = U|k_{l}\rangle \) and compare with
Eq. (A1) to get

\[ tr(\rho \Pi_\ell^{(l)}(\rho)) = \sum_{k_l} (\langle k_l | U^\dagger \rho^{(l)} U | k_l \rangle)^2 = \sum_{k_l} \lambda^2_{k_l} = tr(\rho^{(l)})^2. \] (A4)