CONNECTING TORIC MANIFOLDS BY CONICAL KÄHLER-EINSTEIN METRICS

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Abstract. We give criterions for the existence of toric conical Kähler-Einstein and Kähler-Ricci soliton metrics on any toric manifold in relation to the greatest Ricci and Bakry-Emery-Ricci lower bound. We also show that any two toric manifolds with the same dimension can be joined by a continuous path of toric manifolds with conical Kähler-Einstein metrics in the Gromov-Hausdorff topology.

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1. INTRODUCTION

The existence of Kähler-Einstein metrics has been a central problem in Kähler geometry since Yau’s celebrated solution [52] to the Calabi conjecture. Constant scalar curvature metrics with conical singularities have been extensively studied in [34, 48, 31] for Riemann surfaces. In general, one considers a pair \((X, D)\) for an \(n\)-dimensional compact Kähler manifold and a smooth complex hypersurface \(D\) of \(X\). A conical Kähler metric \(g\) on \(X\) with cone angle \(2\pi\beta\) along \(D\) is locally equivalent to the following model edge metric

\[
g = |z_1|^{-2(1-\beta)}dz_1 \otimes d\bar{z}_1 + \sum_{j=2}^{n} dz_j \otimes d\bar{z}_j
\]

if \(D\) is locally defined by \(z_1 = 0\). Applications of conical Kähler metrics are proposed and applied to obtain various Chern number inequalities [43, 38]. Donaldson has developed the linear theory to study the existence of canonical conical Kähler metrics in [17]. It plays an

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essential role in the recent breakthrough of the Yau-Tian-Donaldson conjecture \cite{41, 11, 45, 12, 13, 14}. Brendle \cite{8} solves Yau’s Monge-Ampère equations for conical Kähler metrics with cone angle $2\pi \beta$ for $\beta \in (0, 1/2)$ along a smooth divisor $D$. The general case is settled by Jeffres, Mazzeo and Rubinstein \cite{24} for all $\beta \in (0, 1)$. As an immediate consequence, there always exist conical Kähler-Einstein metrics with negative or zero constant scalar curvature with cone angle $2\pi \beta$ along a smooth divisor $D$ for $\beta \in (0, 1)$. When $X$ is a Fano manifold, Donaldson \cite{17} proposes to study the conical Kähler-Einstein equation

$$\text{Ric}(\omega) = \beta \omega + (1 - \beta)[D],$$

where $D$ is smooth simple divisor in the anticanonical class $[-K_X]$ and $\beta \in (0, 1)$.

The solvability of equation (1.1) is closely related to the following holomorphic invariant for Fano manifolds which is known as the greatest Ricci lower bound first introduced by Tian in \cite{42}.

**Definition 1.1.** Let $X$ be a Fano manifold. The greatest Ricci lower bound $R(X)$ is defined by

$$R(X) = \sup \{ \beta \mid \text{Ric}(\omega) \geq \beta \omega, \text{ for some } \omega \in c_1(X) \cap \mathcal{K}(X) \},$$

where $\mathcal{K}(X)$ is the space of all Kähler metrics on $X$.

It is proved by Szekelyhidi in \cite{39} that $(0, R(X))$ is the maximal interval for the continuity method to solve the Kähler-Einstein equation on a Fano manifold $X$. In particular, it is independent of the choice for the initial Kähler metric when applying the continuity method. The invariant $R(X)$ is explicitly calculated for $\mathbb{P}^2$ blown up at one point by Szekelyhidi \cite{39}, and for all toric Fano manifolds by Li \cite{28}. Recent results \cite{29} show that $R(X) = 1$ if and only if $X$ is K-semistable, and such a Fano manifold satisfies the Chern-Miyaoka inequality \cite{38}. It is shown in \cite{38, 30} that (1.1) cannot be solved for $\beta > R(X)$, answering a question of Donaldson \cite{17} while it can always be solved for $\beta \in (0, R(X))$ if one replace $D$ by a smooth divisor in the pluri-anticanonical system of $X$. In this paper, we will give various generalizations of the greatest Ricci lower bound.

The Bakry-Emery-Ricci curvature on a Riemannian manifold $(M, g)$ is defined by

$$\text{Ric}_f(g) = \text{Ric}(g) + \text{Hess} f$$

for a smooth real valued function $f$ on $M$ \cite{4}. If $(M, g, f)$ satisfies the equaiton $\text{Ric}_f(g) = \lambda g$ for some $\lambda \in \mathbb{R}$, it is called a gradient Ricci soliton with the gradient vector field $V = \nabla f$. We can define the greatest Bakry-Emery-Ricci lower bound on Fano manifolds as an analogue of the greatest Ricci lower bound.

**Definition 1.2.** Let $X$ be a Fano manifold. The greatest Bakry-Emery-Ricci lower bound $R_{BE}(X)$ is defined by

$$R_{BE}(X) = \sup \{ \beta \mid \text{Ric}(\omega) \geq \beta \omega + \mathcal{L}_{\xi} \omega, \text{ for some } \omega \in c_1(X) \cap \mathcal{K}(X) \text{ and } \xi \in H^0(X, TX) \},$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to $\xi$.

Since $X$ is Fano, it is simply connected and $\mathcal{L}_{\text{Re} \xi} \omega = -\sqrt{-1} \partial \overline{\partial} f_\xi$ for some real-valued smooth function $f_\xi$ with $\nabla_z \nabla_{\overline{z}} f_\xi = 0$ in holomorphic coordinates. This implies that $\text{Ric}(\omega) \geq \beta \omega + \mathcal{L}_{\text{Re} \xi} \omega$ is equivalent to

$$R_{ij} + \nabla_i \nabla_j f_\xi \geq \beta g_{ij}$$

in real coordinates. Hence

$$R_{BE}(X) = \sup \{ \beta \mid \text{Ric}(\omega) + \sqrt{-1} \partial \overline{\partial} f \geq \beta \omega, \omega \in c_1(X) \cap \mathcal{K}(X), f \in C^\infty(X), \uparrow \overline{\partial} f \text{ is holomorphic } \}. $$
One can relate $R_{BE}(X)$ to the continuity method for solving the Kähler-Ricci soliton equation on $X$ as introduced in [47] as analogue of $R(X)$ and explicitly calculate the value of $R_{BE}(X)$ for toric Fano manifolds. In fact, we conjecture that $R_{BE}(X) = 1$ for any Fano manifold $X$. However, in this paper, we are more interested in generalizing $R(X)$ and $R_{BE}(X)$ for log Fano manifolds and more specifically, toric conical metrics on toric manifolds.

We start with a few definitions. Let $X$ be an $n$-dimensional toric manifold and $L$ a Kähler class (or equivalently, an ample $\mathbb{R}$ divisor) on $X$. In [16, 38], smooth toric conical Kähler metrics are defined and studied in detail and a brief review is given in section 2. We let $\mathcal{K}_c(X)$ be the set of all smooth toric conical Kähler metrics with each cone angle in $(0, 2\pi]$.

**Definition 1.3.** Let $X$ be a toric manifold. Let $\omega \in \mathcal{K}_c(X)$ be a smooth toric conical Kähler metric on $X$. We say

$$\text{Ric}(\omega) > \alpha \omega$$

if there exists $\eta \in \mathcal{K}_c(X)$ and an effective toric divisor $D$ such that

$$\text{Ric}(\omega) = \alpha \omega + \eta + [D].$$

In fact, $\omega$ and $\eta$ have the same cone angles and the divisor $D$ can be explicitly calculated in terms of the cone angles of $\omega$.

**Definition 1.4.** A smooth toric conical Kähler metric $\omega \in \mathcal{K}_c(X)$ is called a conical Kähler-Ricci soliton metric if it satisfies

$$\text{Ric}(\omega) = \alpha \omega + \mathcal{L}_\xi \omega + [D]$$

for some holomorphic vector field $\xi$ and effective toric divisor $D$. If $\xi = 0$, the metric is a smooth toric conical Kähler-Einstein metric.

Associated to any toric Kähler class, we define the following geometric invariants $\mathcal{R}(X, L)$, $\mathcal{R}_{BE}(X, L)$ and $\mathcal{S}(X, L)$.

**Definition 1.5.** Let $X$ be a toric manifold and $L$ be a Kähler class on $X$. Let $\{D_j\}_{j=1}^N$ be the set of all prime toric divisors on $X$. Then we define

1. $\mathcal{R}(X, L) = \sup\{\alpha \mid \text{Ric}(\omega) > \alpha \omega \text{ for some } \omega \in c_1(L) \cap \mathcal{K}_c(X)\},$
2. $\mathcal{R}_{BE}(X, L) = \sup\{\alpha \mid \text{Ric}(\omega) + \mathcal{L}_\xi \omega > \alpha \omega \text{ for a } \omega \in c_1(L) \cap \mathcal{K}_c(X) \text{ and a toric } \xi \in H^0(X, TX)\},$
3. $\mathcal{S}(X, L) = \sup\{\alpha \mid \text{there exists } D = \sum_{j=1}^N a_j D_j \sim -K_X - \alpha L \text{ with } a_j \in [0, 1)\}.$

$\mathcal{R}(X, L)$ and $\mathcal{R}_{BE}(X, L)$ are natural generalizations of $R(X)$ and $R_{BE}(X)$ for log Fano manifolds with polarization $L$. $\mathcal{S}(X, L)$ characterizes when $(X, D)$ is log Fano as by definition $K_X + D$ is klt and negative. In the special case that $X$ is toric Fano and $L = -K_X$, $\mathcal{R}(X, -K_X)$ is the usual greatest Ricci lower bound studied in [39] and $\mathcal{S}(X, -K_X) = 1$. In fact, for any toric pair $(X, L)$, $\mathcal{R}(X, L)$ and $\mathcal{S}(X, L)$ are both positive. In general, one can define $\mathcal{R}(X, L)$ and $\mathcal{R}_{BE}(X, L)$ for any log Fano pair $(X, L)$ by requiring $\text{Ric}(\omega) - \alpha \omega \geq 0$ and $\text{Ric}(\omega) + \mathcal{L}_\xi \omega - \alpha \omega \geq 0$ in the current sense.

Any toric manifold $X$ is induced by an integral Delzant polytope $P$ and $P$ determines a Kähler class on $X$. Without loss of generality, we let

$$(1.3) \quad P = \{x \in \mathbb{R}^n \mid l_j(x) > 0, \ j = 1, \ldots, N\},$$

where $l_j(x) = v_j \cdot x + \lambda_j$, $v_j$ is a prime integral integral vector in $\mathbb{Z}^n$ and $\lambda_j \in \mathbb{R}$ for all $j = 1, \ldots, N$. As a special case, when $X$ is Fano, one can choose $\lambda_j = 1$ for all $j$ and the
polytope gives the anti-canonical polarization of $X$. The existence of smooth toric Kähler-Einstein and Kähler-Ricci soliton metrics on toric Fano manifolds is completely settled by Wang-Zhu [50]. We generalize their results to toric conical Kähler-Einstein and Kähler-Ricci soliton metrics on any toric manifold.

**Theorem 1.1.** Let $X$ be an $n$-dimensional toric Kähler manifold and $L$ be the Kähler class on $X$ induced by the Delzant polytope $P$. Then

1. $\mathcal{R}_{BE}(X, L) = S(X, L) > 0$ and

$$\mathcal{R}_{BE}(X, L) = \sup \{ \alpha \mid \text{there exists } \tau \in P \text{ with } 1 - \alpha l_j(\tau) > 0, j = 1, \ldots, N \},$$

2. For any $\alpha \in (0, S(X, L))$ and $\tau \in P$ satisfying $1 - \alpha l_j(\tau) \geq 0$ for all $j$, there exists a unique $\omega \in L \cap K_c(X)$ solving the Kähler-Ricci soliton equation

$$\text{Ric}(\omega) = \alpha \omega + L \xi \omega + [D].$$

Moreover the divisor $D$ and the vector field $\xi$ are given by

$$D = \sum_{j=1}^{N} (1 - \alpha l_j(\tau)) D_j, \quad \xi = \sum_{i=1}^{n} c_i z_i \frac{\partial}{\partial z_i},$$

where $z_i$’s are the standard coordinates on $(\mathbb{C}^*)^n$ and $c \in \mathbb{R}^n$ is uniquely given by

$$\tau = \frac{\int_{P} xe^{cx} \, dx}{\int_{P} e^{cx} \, dx}.$$ 

3. There does not exist a toric conical Kähler-Ricci soliton metric $\omega \in L \cap K_c(X)$ solving the soliton equation (1.5) for any $\alpha > \mathcal{R}_{BE}(X, L)$.

The existence of toric conical Kähler-Ricci soliton metrics on log Fano toric varieties is derived in [6] and for general toric manifolds by allowing the cone angle in $(0, \infty)$ [27]. Our result gives a complete classification for the existence of toric conical Kähler-Einstein and Kähler-Ricci soliton metrics using the invariants $\mathcal{R}(X, L)$ and $\mathcal{R}_{BE}(X, L)$ for any Kähler class. We are only interested in the toric conical Kähler metrics with cone angle in $(0, 2\pi)$ since the smooth part is geodesic convex and various Riemannian geometric properties can be applied. In particular, it gives optimal regularity and a complete classification of smooth toric conical Kähler-Ricci soliton metrics for any toric pair $(X, L)$. The proof is based on a family version of the $C^0$ estimate and we will apply it to toric degenerations for smooth toric conical Kähler-Einstein metrics.

As a special case, we obtain an existence result for conical Kähler-Einstein metrics on toric manifolds and apply it to characterize the invariant $\mathcal{R}(X, L)$ in terms of the polytope data.

**Theorem 1.2.** Let $X$ be an $n$-dimensional toric Kähler manifold and $L$ be the Kähler class on $X$ induced by the Delzant polytope $P$. Let $P_C$ be the barycenter of $P$. Then

1. $\mathcal{R}(X, L) > 0$ and

$$\mathcal{R}(X, L) = \sup \{ \alpha \mid 1 - \alpha l_j(P_C) > 0, j = 1, \ldots, N \},$$

2. For all $\alpha \in (0, \mathcal{R}(X, L)]$, there exists a unique toric conical Kähler-Einstein metric $\omega \in L \cap K_c(X)$ solving

$$\text{Ric}(\omega) = \alpha \omega + [D].$$
Moreover the divisor $D$ is given by

\begin{equation}
D = \sum_{j=1}^{N} (1 - \alpha l_j(P_C)) D_j.
\end{equation}

(3) There does not exist a toric conical Kähler-Einstein metric $\omega \in L \cap K_c(X)$ solving the equation (1.9) for any $\alpha > R(X, L)$.

In the special case when $X$ is Fano and $L = -K_X$, $l_j(0) = 1$ and so $1 - \alpha l_j(P_C) = (1 - \alpha)l_j(-\frac{\alpha P_C}{1-\alpha})$. By the theorem, $R(X, L)$ is the maximum of all $\alpha$ such that $-\frac{\alpha P_C}{1-\alpha}$ remains inside the polytope, generalizing the results in the smooth case in [39] and [28]. There is a subtle difference between Theorem 1.1 and Theorem 1.2 that in Theorem 1.2, $\alpha$ can be taken to be $R_{BB}(X, L)$ while in Theorem 1.1, $\alpha$ has to be strictly less than $R_{BE}(X, L)$. Such a phenomena will be explained in Example 5.1.

Smooth Kähler-Einstein and Kähler-Ricci soliton metrics on Fano manifolds are unique, while the space of conical Kähler-Einstein and Kähler-Ricci soliton metrics is much bigger. One would ask under what assumptions the space of conical Kähler-Einstein metrics is connected. In other words, given two log Fano manifolds $X_0$ and $X_1$, we ask when and how one can connect $X_0$ and $X_1$ by a family of conical Kähler-Einstein spaces in Gromov-Hausdorff topology. Now we state our main theorem of the paper to answer such a question in the toric case.

**Theorem 1.3.** Let $X_0$ and $X_1$ be two $n$-dimensional toric manifolds. Suppose $\omega_0 \in K_c(X_0)$ and $\omega_1 \in K_c(X_1)$ are two smooth toric conical Kähler-Einstein metrics on $X_0$ and $X_1$ respectively. Then, there exist a family $\{(X_t, \omega_t)\}_{t \in [0, 1]}$ of $n$-dimensional toric manifolds $X_t$ with smooth toric conical Kähler-Einstein metrics $\omega_t \in K_c(X_t)$ for $t \in [0, 1]$, such that

1. $(X_t, \omega_t)$ is a continuous path in Gromov-Hausdorff topology for $t \in [0, 1]$,
2. $\omega_t$ is piecewise smooth in $t$ on the complex torus $(\mathbb{C}^*)^n$.

Theorem 1.3 can be considered to be an analytic analogue of the weak factorization theorem for toric varieties in algebraic geometry. Combined with Theorem 1.2 it implies that any two toric manifolds of same dimension can be joined by a continuous path of conical Kähler-Einstein spaces in Gromov-Hausdorff topology. It is a natural question to ask if for any two birationally equivalent Fano manifolds, there exists a continuous path connecting them by Fano varieties coupled with conical Kähler-Einstein metrics, in Gromov-Hausdorff topology. This is related to the connectedness of moduli space of log Fano varieties coupled with conical Kähler-Einstein metrics.

**2. Preliminaries**

2.1. **Basics of toric varieties.** In this section, let us recollect some well-known facts of projective toric varieties.

**Definition 2.1.** A convex polytope $P \subset \mathbb{R}^n$ is called a Delzant polytope if a neighborhood of any vertex $p \in P$ is $SL(n, \mathbb{Z})$ equivalent to $\{x_j \geq 0, j = 1, \ldots, n\} \subset \mathbb{R}^n$. $P$ is called an integral Delzant polytope if each vertex $p \in P$ is a lattice point in $\mathbb{Z}^n \subset \mathbb{R}^n$.

Let $P$ be an integral Delzant polytope in $\mathbb{R}^n$ defined by

\begin{equation}
P = \{x \in \mathbb{R}^n \mid l_j(x) > 0, j = 1, \ldots, N\},
\end{equation}
where
\[ l_j(x) = v_j \cdot x + \lambda_j \]
and \( v_i \) is a primitive integral vector in \( \mathbb{Z}^n \) and \( \lambda_j \in \mathbb{Z}_+ \) for all \( j = 1, ..., N \). Let \( \Sigma_P \) be the fan consisting of the cones over the faces of the polar polytope
\[ P = \{ y \in \mathbb{R}^n \mid \langle y, x \rangle \geq -1 \text{ for all } x \in P \}. \]
Then \( \Sigma_P \) defines an \( n \)-dimensional smooth projective toric variety \( X_P \). Its Picard group \( \text{Pic}(X) \) is generated by \( D_i \)'s, the toric divisors corresponding to the generators of edges \( e_i \)'s of the fan \( \Sigma_P \). For any toric divisor \( D = \sum a_i D_i \), it determines a rational convex polyhedron
\[ P_D = \{ \alpha \in \mathbb{R}^n \mid \langle \alpha, e_i \rangle \geq -a_i \text{ for all } i \} \subset \mathbb{R}^n , \]
and the space of global sections of the line bundle \( \mathcal{O}_X(D) \) is given by
\[ (2.12) \quad H^0(X, \mathcal{O}_X(D)) = \bigoplus_{\alpha \in P_D \cap \mathbb{Z}^n} \mathbb{C} \cdot \chi^\alpha, \]
where \( \chi \)'s are the characters \( \text{Hom}(T, \mathbb{C}^*) \). In particular, we have
\[ (2.13) \quad \dim H^0(X, \mathcal{O}(kD)) = k^n \text{Vol}(P_D) + O(k^n-1), \]
where \( \text{Vol}(P_D) \) denote the Euclidean volume of \( P_D \in \mathbb{R}^n \). In particular, for the anti-canonical divisor \( -K_{X_P} = \sum_i D_i \), we have \( P_{-K_X} = \{ \alpha \in \mathbb{R}^n \mid \langle \alpha, e_i \rangle \geq -1 \text{ for all } i \} \), hence \( \text{Vol}(P_{-K_X}) > 0 \). By \[ (2.13) \], we conclude with the following well-known lemma.

**Lemma 2.1.** Let \( X \) be a smooth projective toric variety, then \( -K_X \) is big.

Next we study how far away is a toric variety \( X \) from being Fano. To do that, we need to introduce some notions.

**Definition 2.2.** Let \( X \) be a smooth projective variety and \( D = \sum a_i D_i \geq 0 \) be a \( \mathbb{Q} \)-divisor with \( D_i \)'s being prime divisors, so that \( a_i \)'s are non-negative rational numbers. If \( (X, D) \) is a log pair, then we say that a birational map: \( \pi : (Y, D') \to (X, D) \) is a log resolution if the strict transform of \( D \) union the exceptional locus is log smooth, i.e. normal crossings of smooth divisors. We say that \( (X, D) \) is Kawamata log terminal (klt) if when we write
\[ K_Y + D' = \pi^*(K_X + D) \]
then \( |D'| \leq 0 \), that is, if we write \( D' = \sum a'_i D'_i \) with \( D'_i \)'s being prime divisors then \( a'_i < 1 \) for all \( i \).

Here we list the properties of being klt.

**Lemma 2.2.** \[ (33) \text{Lemma 4.2] Let } X \text{ be a smooth projective variety} \]

1. If \( D \geq 0 \) is an effective \( \mathbb{Q} \)-divisor then there is an \( \epsilon > 0 \) such that \( (X, \epsilon D) \) is klt.
2. If \( (X, D) \) is klt and \( D + D' \geq 0 \) then \( (X, D + \epsilon D') \) is klt for any small \( \epsilon > 0 \).
3. If \( (X, D) \) is klt and \( G \) is semistable then we may find \( G' \equiv G \) such that \( K_X + D + G' \) is klt.

**Remark 2.1.** One notices that, if \( X \) is toric then all the divisors in the statements above can be chosen to be toric divisors.

Now we are ready to introduce the main terminology of this section

**Definition 2.3.** Let \( X \) be a smooth projective variety. We say that \( X \) is a log Fano variety if there is a divisor \( D \) such that \( K_X + D \) is klt and \( -(K_X + D) \) is ample.

In order to characterize the log Fano varieties, we have the following criterion.
**Theorem 2.1.** Let $X$ be a smooth projective variety, then $X$ is log Fano if and only if there is an effective big divisor $D$ such that $K_X + D$ is klt and numerically trivial.

**Proof.** First, we prove the sufficiency. Suppose $(X, G)$ is log Fano then $K_X + G$ is klt and $-(K_X + G)$ is ample. So there is an ample divisor $A$ such that $-(K_X + G) = A/m$ for some $m \in \mathbb{Z}$. Clearly, we have $K_X + G + A/m \equiv 0$. Since $A/m$ is ample, by Lemma 2.2 there is a $A' \equiv A$ such that $K_X + G + A'/m$ is klt. Now we define $D := G + A'/m \geq 0$, which is big (cf. [25, Lemma 2.60]) then $K_X + D \equiv 0$ and it is klt. This finishes the proof of sufficiency.

Conversely, by our assumption, $D$ is big, so $D \sim A + B$ with $A$ being ample and $B \geq 0$ (cf. [25, Lemma 2.60]). Now we define $K_X + G := K_X + (1 - \epsilon)D + \epsilon B = K_X + D + \epsilon(B - D)$. Let $D' = B - D$ then $D + D' = B \geq 0$, by part 2 of Lemma 2.2 we conclude $K_X + G = K_X + D + \epsilon D'$ is klt for sufficient small $\epsilon$. On the other hand, since $K_X + D \equiv 0$, $-(K_X + G) \equiv -\epsilon D' \sim \epsilon A$ is ample, hence $(X, G)$ is log Fano by Definition 2.2. Our proof is completed.

\[\Box\]

As a direct consequence, when $X$ is toric, $-K_X$ is effective and big by Lemma 2.1. Let $D = -K_X$ and apply Lemma 2.3 above, we obtain a toric (cf. Remark 2.1) $G \geq 0$ such that $K_X + G$ is log canonical and ample. Since $D$ is big, we may choose ample divisor $A$ and $\epsilon$ such that $D - \epsilon A$ is still big and $|D - \epsilon A| \leq 0$. This implies $(X, D - \epsilon A)$ is klt. Now $\epsilon A$ is ample, by part 3 Lemma 2.2 there is a $A' \equiv A$ such that for $D' = D - \epsilon A + \epsilon A'$, $(X, D')$ is klt. Now $D' \equiv D$ is also big and $K_X + D' \equiv 0$. By Lemma 2.3 we obtain $X$ is log Fano, which is well-known (cf. [33]). In the following, we give an elementary proof without using Lemma 2.3.

**Theorem 2.1.** Let $X$ be a projective toric variety and $D \subset X$ be any ample divisor, there is a toric divisor $G \sim mD$ for some $m \in \mathbb{Z}$ such that $(X, -K_X - \epsilon G)$ is log Fano for all $0 < \epsilon \ll 1$.

**Proof.** Since $D$ is ample, for $m$ large, there is a $0 \leq G' \in |mD|$, the linear system of $mD$. Take a general one parameter subgroup $\lambda \subset (\mathbb{C}^\ast)^n$ and let $G = \lim_{t \to 0} \lambda(t) \cdot G'$, then $G \geq 0$ is $G$ is toric. So $G = \sum_{i=1}^N a_i D_i$ with all $a_i \geq 0$, where $D_i$’s are the toric divisors generating $\text{Pic}(X)$.

We claim that $-K_X - \epsilon G \geq 0$ and $|-K_X - \epsilon G| \leq 0$ for $0 < \epsilon \ll 1$, from which we obtain $(X, -K_X - \epsilon G)$ being log Fano. But this follows from the fact $-K_X - \epsilon G = \sum_{i=1}^N (1 - \epsilon a_i) D_i \geq 0$ and $[(1 - \epsilon a_i)] = 0$ for any $0 < \epsilon \ll 1$. Hence the proof is completed.

\[\Box\]

The following weak factorization theorem first proved in [51] (cf. also [1]) reduces the proof of our main result to the case of a simple blow-up or blow-down of a smooth toric center.

**Theorem 2.2.** Let $f : X \dashrightarrow Y$ be a toric birational map between two complete nonsingular toric varieties $X$ and $Y$ over $\mathbb{C}$, and let $U \subset X$ be an open set where $f$ is an isomorphism. Then $f$ can be factored into a sequence of blow-ups and blow-downs with nonsingular irreducible toric centers disjoint from $U$, namely, there is a sequence of birational maps between complete nonsingular toric varieties

$$X = X_0 \overset{f_1}{\dashrightarrow} X_1 \overset{f_2}{\dashrightarrow} \cdots \overset{f_i}{\dashrightarrow} X_i \overset{f_{i+1}}{\dashrightarrow} \cdots \overset{f_n}{\dashrightarrow} X_n = Y,$$

where

(1) $f = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$,
that beginning of this section) and a polarization \(X \supset X\) (cf. Definition 2.1), one can always associate a toric variety \(\mathbb{P} \in \mathbb{C}^n\), where as usual we sum over repeated indices. Moreover, the moment map for the Hamiltonian action of \((S^1)^n\) on \(X\) is bounded and \(\nabla\varphi : X \to \mathbb{R}^n\), whose image is a bounded convex polytope \(P \subset \mathbb{R}^n\) by the famous Atiyah and Guillemin-Sternberg convexity theorem. Conversely, a theorem of Delzant says that if a polytope \(P\) is integral Delzant (cf. Definition 2.1), one can always associate a toric variety \(X_P\) (as we introduced in the beginning of this section) and a polarization \(L \to X_P\) with a Kähler metric \(\omega \in c_1(L)\) such that \(\nabla\varphi(\mathbb{R}^n) = P\).

Let \(\varphi = \log(\sum_{\alpha \in \mathbb{Z}^n \cap \mathcal{T}} |z^n|)\). Then \(\omega_P = \sqrt{-1} \partial \bar{\partial} \varphi_P\) is a smooth Kähler metric on \((\mathbb{C}^*)^n\) and it can be smoothly extended to a smooth global toric Kähler metric on \(X_P\) in \(c_1(L)\). Then the space of toric Kähler metrics in \(c_1(L)\) is equivalent to the set of all smooth plurisubharmonic function \(\varphi\) on \((\mathbb{C}^*)^n\) such that \(\varphi - \varphi_P\) is bounded and \(\sqrt{-1} \partial \bar{\partial} \varphi\) extends to a smooth Kähler metric on \(X_P\). Let \(u\) be the Legendre transform of \(\varphi\). The recall that \(u\) and \(\varphi\) are related by

\[\varphi(\rho) = Lu(\rho) = \sup_{x \in P} (x \cdot \rho - u(x)), \quad u(x) = \mathcal{L}_\varphi(x) = \sup_{\rho \in \mathbb{R}^n} (x \cdot \rho - \varphi(\rho))\]

or equivalently

\[\varphi(\rho) = x \cdot \rho - u(x), \quad u(x) = x \cdot \rho - \varphi(\rho), \quad x = \nabla_\rho \varphi(\rho), \quad \rho = \nabla_x u(x)\]

The Guillemin boundary conditions imply that \(\sqrt{-1} \partial \bar{\partial} \varphi\) extends to a global Kähler metric on \(X_P\) if and only if

\[(2.14) \quad u = u(x) = \sum_{j=1}^N l_j(x) \log l_j(x) + f(x)\]

for some \(f \in C^\infty(\mathbb{P})\) and \(u\) is strictly convex on \(P\).

Now we extend the Guillemin condition to toric conical Kähler metrics on \(X_P\), a generalization of orbifold Kähler metrics. On each coordinate chart determined by the pair \((p, \{v_{p,j} \}_{i=1}^n)\) associated to a vertex \(P\), we let \(z = (z_1, ..., z_n)\) be the coordinates on \(\mathbb{C}^n\). The closure of \(\{z_i = 0\} \subset X\) give rise to a smooth toric divisor of \(X_P\). Let \(D\) be a toric divisor of \(X_P\) and suppose \(D\) restricted to this coordinate chart is given by

\[\sum_{i=1}^n a_i \{z_i = 0\}\]
For any function $f(z)$ invariant under the $(S^1)^n$-action, we can lift it to a function
\[ \tilde{f}(w) = f(z) \]
by letting
\[ |w_i| = |z_i|^\beta_i, \quad w = (w_1, ..., w_n) \in \mathbb{C}^n, \]
and clearly $\tilde{f}(w)$ is also $(S^1)^n$-invariant. $w \in \mathbb{C}^n$ can be regarded a $\beta$-covering of $z \in \mathbb{C}^n$.

Now we introduce consider the $(S^1)^n$-invariant function space for $k \in \mathbb{Z}^+$ and $\gamma \in [0, 1]$
\[ C^{k, \gamma}_\beta(X_P), \beta = (\beta_1, ..., \beta_N) \in (\mathbb{R}_+)^N \]
whose restriction on each chart belongs to $C^{k, \gamma}_\beta$ with respect to the weight $\beta$ and $\beta_j$ corresponding to the divisor induced by $l_j(x) = 0$.

**Definition 2.4.** A Kähler current $\omega \in c_1(L)$ is said to be a smooth $\beta$-conical metric if for each vertex $p$ of the polytope $P$,
\[ \omega|_{U_p} = \sqrt{-1} \partial \bar{\partial} \varphi_p \]
for some $\varphi_p \in C^\infty_{\beta, p}$. Such a metric naturally has a cone angle of $2\pi \beta_j$ along the divisor $D_j$.

The local lifting $\tilde{\varphi}(w)$ is a smooth plurisubharmonic function on the lifting space $w \in \mathbb{C}^n$. We can also define the space of weighted toric Kähler potential $\varphi$ on $(\mathbb{C}^*)^n$ such that $\varphi - \varphi_P$ is bounded and $\sqrt{-1} \partial \bar{\partial} \varphi$ extends to a smooth weighted Kähler metric on $X_P$. In particular, we have the following conical extension of the Guillemin condition for toric Kähler metrics.

**Proposition 2.1.** Let $\varphi$ be a toric potential on $(\mathbb{C}^*)^n$, $u$ it’s Legendre transform and $P$ the image of $\mathbb{R}^n$ under $\nabla \varphi$. Then $\omega = \sqrt{-1} \partial \bar{\partial} \varphi$ extends to a global smooth $\beta$-conical metric on $X_P$ if and only if
\[
(2.15) \quad u(x) = \sum_{j=1}^{N} \beta_j^{-1} l_j(x) \log l_j(x) + f(x)
\]
for some $f \in C^\infty(P)$ and $0 \leq \beta_j \leq 1$. Moreover, the angle along the divisor corresponding to $l_j$ is precisely $2\pi \beta_j$.

The main advantage of dealing with conical metrics on toric manifolds is that one has all the curvature bounds.

**Lemma 2.4.**[35] Let $g$ be a smooth toric conical metric on a toric manifold $X$. Let $D$ be a toric divisor consist of all toric prime divisors and let $Rm$ denote the full curvature tensor of $g$. Then for any $k \geq 0$ there exists a constant $C_k$ such that for all $p \in X \setminus D$,
\[
(2.16) \quad |\nabla^k g Rm|^2_g(p) \leq C_k.
\]

2.3. **Comparison theorems for conical Kähler manifolds.** In this subsection, we will state comparison theorems for cone metric with Ricci curvature bounded below, which will be needed to prove the Gromov-Hausdorff convergence in section 4.
2.3.1. Geodesic convexity. Assume $D$ is a smooth divisor, $g$ is a cone metric with cone angle $2\pi \alpha$ ($\alpha \in (0, 1)$) along $D$. Around a point $p \in D$, we choose local coordinates $\{z_1, z_2, \ldots, z_n\}$ with $D = \{z_1 = 0\}$ and $p = (0, \ldots, 0)$. Set $w_1 = z_1^\alpha / \alpha$, which can be viewed as a singular holomorphic change of coordinates. Although $w_1$ is multi-valued, we can work locally in the logarithmic Riemann surface which uniformizes this variable. Written in terms of the coordinates $\{w_1, z_2, \ldots, z_n\}$, the model flat metric $g_0$ can be written as

$$g_0 = |dw_1|^2 + \sum_{k=2}^n |dz_k|^2,$$

which is the standard flat metric on $\mathbb{C}^n$.

It is well-known that $(X\setminus D, g)$ is geodesic convex, i.e., any minimal geodesic $\gamma$ connecting two points $p_1, p_2 \in X\setminus D$ is strictly contained in $X\setminus D$. For reader’s convenience, we present a proof of the geodesic convexity.

**Lemma 2.5.** Let $(X, D)$ be given as above, $g$ a smooth cone metric with cone angle $2\pi \alpha$ along $D$ with $\alpha \in (0, 1)$. Let $\gamma : [0, 1] \to X$ be a minimal geodesic with $\gamma(0) = p \in X\setminus D$,

$$\gamma(1) = q \in X\setminus D.$$

Then $\gamma|_{[0,1]} \subset X\setminus D$.

**Proof.** We only need to show that if $\gamma : (-\delta, \delta) \to X$ is a minimal geodesic with $\gamma(0) = s \in D$, then there exist an $\varepsilon > 0$ such that $\gamma|_{[-\varepsilon, \varepsilon]} \subset D$, from which it follows that $\gamma|_{(-\delta, \delta)} \subset D$ by the closedness of $D$. Recall that in appropriate coordinates $\{z_1, \cdots, z_n\}$ centered at a point in $D$, the conic metric $g$ can be approximated by the model flat metric $g_0$ in the sense that there exists a constant $\varepsilon(r) > 0$ with $\varepsilon(r) \to 0$ as $r \to 0$ such that

$$(1 - \varepsilon(r))g_0(z) \leq g(z) \leq (1 + \varepsilon(r))g_0(z), \quad \text{on } \{|z| \leq r, z_1 \neq 0\}.$$

For two points $p, q$ outside $D$ with $|z(p)|, |z(q)| \leq r$, assume the minimal geodesic $\gamma$ connecting $p, q$ passes through the origin $0 \in D$. Viewed in terms of the coordinates $(w_1, z_2, \cdots, z_n)$ as above, in which $g_a$ is flat and the two parts $\gamma|_{[p,0]}$ and $\gamma|_{[0,q]}$ intersect at 0 with angle $\theta < \pi$. We know that $L_g(\gamma) \geq (1 - \varepsilon(r))L_{g_0}(\gamma) \geq (1 - \varepsilon(r))(|w(p)| + |w(q)|)$, where $|w(p)|^2 = |w_1(p)|^2 + |z_2(p)|^2 + \ldots + |z_n(p)|^2$ and similar for $|w(q)|$. On the other hand, the straight line $\ell_{p,q}$ (in the coordinates $\{w_1, z_2, \cdots, z_n\}$ connecting $p$ and $q$ has length $L_g(\ell_{p,q}) \leq (1 + \varepsilon(r))L_{g_0}(\ell_{p,q}) = (1 + \varepsilon(r))\sqrt{|w(p)|^2 + |w(q)|^2 - 2 \cos \theta_{p,q}|w(p)||w(q)|}$. Since $\gamma$ is a minimal geodesic connecting $p$ and $q$, we have $L_g(\gamma) \leq L_g(\ell_{p,q})$. Hence we get

$$\tag{2.17} (1 - \varepsilon(r))(|w(p)| + |w(q)|) \leq (1 + \varepsilon(r))\sqrt{|w(p)|^2 + |w(q)|^2 - 2 \cos \theta_{p,q}|w(p)||w(q)|}.$$

Note that (2.17) also holds for any points $\tilde{p} \in \gamma|_{[p,0]}$ and $q \in \gamma|_{[0,q]}$, since $\gamma$ is also a minimal geodesic connecting $\tilde{p}$ and $\tilde{q}$. Take $\tilde{p}, \tilde{q}$ sufficiently close to 0 such that $|w(\tilde{p})| = |w(\tilde{q})|$, then

$$\frac{\sqrt{|w(\tilde{p})|^2 + |w(\tilde{q})|^2 - 2 \cos \theta_{\tilde{p},\tilde{q}}|w(\tilde{p})||w(\tilde{q})|}}{|w(\tilde{p})| + |w(\tilde{q})|} = \sqrt{\frac{1 - \cos \theta_{\tilde{p},\tilde{q}}}{2}} < 1,$$

while

$$\frac{1 - \varepsilon(r)}{1 + \varepsilon(r)} \to 1, \quad \text{as } r \to 0.$$

Thus we would get a contradiction to (2.17) if $r << 1$, that is, $\gamma$ cannot intersect with $D$.

$\square$

**Remark 2.2.** If $(X, D)$ is a toric manifold with normal crossing toric divisor $D = \sum_i D_i$, and $g$ is a smooth toric cone metric with angle $2\pi \alpha_i \in (0, 2\pi)$ along each component $D_i$. Since around any point $p \in \text{Supp}D$, $g$ can be extended smoothly to a smooth metric in the
\( \alpha \)-covering (see section 2.2), we don’t need to approximate cone metric by the model metric as in Lemma 2.5, indeed, similar method as in the orbifold case \([9]\) can be applied to give that \((X \setminus \text{Supp} D, g)\) is geodesic convex.

2.3.2. Volume comparison. Assumptions as above, observe that

\[
\text{Vol}_g(D) = 0.
\]

To see this, we can take a tubular neighborhood of \( D \), say, locally \( D \subset \{ z : |z_1| \leq \varepsilon \} =: \Sigma_\varepsilon \). Since the metric \( g \) is equivalent to \( g_\alpha \), then

\[
\text{Vol}(D) \leq \text{Vol}(\Sigma_\varepsilon) = \int_{\Sigma_\varepsilon} \det g \leq C \int_0^\varepsilon r^{2n-2}rdr \leq C\varepsilon^{2n} \to 0, \quad \text{as} \ \varepsilon \to 0.
\]

Hence we can define the volume of a set \( A \subset X \) to be

\[
\text{Vol}(A) := \int_{A \cap X \setminus D} \omega^n,
\]

where \( \omega \) is the Kähler form associated to \( g \).

For a constant \( k \), let \( S^m_k \) denote the simply-connected \( m \)-dimensional space form of constant curvature \( k \), \( V_k(r) \) the volume of geodesic ball in \( S^m_k \) with radius \( r \).

**Theorem 2.3** (Relative Volume Comparison). Let \((X, D)\) be a compact complex manifold with smooth divisor \( D \), \( g \) be a smooth cone metric with cone angle \( 2\pi\alpha \in (0, 2\pi) \) along \( D \). Suppose \( \text{Ric}(g) \geq (2n - 1)k \) in \( X \setminus D \) for some constant \( k \). Let \( p, x \in X \) be fixed points, \( \rho = d(p, x) \), \( (V(p, r) = \text{Vol}_g(B(p, r)) \). Then we have the following:

1. The function \( r \mapsto \frac{V(p, r)}{V_k(r)} \) is non-increasing and if \( p \in D \) then \( \frac{V(p, r)}{V_k(r)} \to \alpha \) as \( r \to 0^+ \).
2. For \( r_1 \leq r_2 \leq r_3 \),

\[
\frac{V(p, r_3) - V(p, r_2)}{V_k(r_3) - V_k(r_2)} \leq \frac{V(p, r_1)}{V_k(r_1)}.
\]

3. For \( r_1 \leq r_2 \),

\[
\frac{V(p, r_2)}{V_k(\rho + r_2)} \leq \frac{V(x, r_1)}{V_k(r_1)}.
\]

4. If \( r_2 + r \leq \rho \), then

\[
\frac{V(p, r_2)V_k(r)}{V_k(\rho + r_2) - V_k(\rho - r_2)} \leq V(x, r).
\]

5. Let \( p \in X \setminus D \), \( A \) be a star-convex set centered at \( p \), then for any \( r_1 \leq r_2 \), we have

\[
\frac{V(B(p, r_2 \cap A)) - V(B(p, r_2) \cap A)}{V_k(r_2) - V_k(r_1)} \leq \frac{V(\partial B(p, r_1))}{V_k(\partial B^\varepsilon_k)}.
\]

**Proof.** For any \( p \in D \), we can choose a sequence \( \{p_i\} \subset X \setminus D \) such that \( d(p, p_i) \to 0 \), and it’s clear that for any \( r > 0 \), \( V(p_i, r) \to V(p, r) \). Hence, we only need to prove the above inequalities for \( p \in X \setminus D \). Note that the cone metric \( g \) is smooth in \( X \setminus D \) (being incomplete), we can define the cut-loci of \( p \) to be the same as in the smooth case, say, a point \( q \notin D \) is at the cut-loci of \( p \), if either \( q \) is a conjugate point of \( p \), or there exist at least two different minimal geodesics connecting \( p \) and \( q \). Moreover, in this cone metric case, we also define \( D \) to be contained in the cut-loci of \( p \) because of Lemma 2.5. Denote \( CL(p) \) to be the cut-loci of \( p \). Clearly \( CL(p) \) has measure 0. For any star-convex set \( A \subset X \) (in the sense that any point \( q \in A \) can be joined to \( p \) with the whole geodesic contained in \( A \), in particular, \( X \setminus CL(p) \) is a star-convex set). In \( A \cap X \setminus CL(p) \), the metric \( g \) is smooth and the distance function
\(d(\cdot) := d(\cdot, p)\) is smooth except at \(p\), and the metric has Ricci curvature bounded below by \((2n - 1)k\). By relative comparison in Riemannian geometry, we can see that the function

\[
\frac{r}{V_k(\partial B(r))} = f(r)
\]

is non-increasing, where \(B_r^k\) is a geodesic ball in \(S_k^n\) with radius \(r\) and \(m = 2n\). Then the proof of the items except the second part in (1) goes similar as that in [10] by taking \(A = X \setminus CL(p)\).

For the second part of (1), viewed in the coordinates \(\{w_1, z_2, \ldots, z_n\}\), the metric \(g\) can be viewed as a metric with Ricci curvature bounded below by \((2n - 1)k\) in \(S \times \mathbb{C}^{n-1}\), where \(S\) is a fan-shape space with angle \(2\pi\alpha\) in \(\mathbb{R}^2 \cong \mathbb{C}\). Precisely, \(\arg w_1 = \alpha \arg z_1\), thus for \(p \in D\)

\[
\frac{V(B(p, r))}{V_k(r)} \to \alpha,
\]
as \(r \to 0\), which can be seen by writing the volume in terms of polar coordinates.

\[\square\]

If the cone metric satisfies \(\text{Ric}(g) \geq (2n - 1)k > 0\), then we have the diameter bound.

**Theorem 2.4** (Myers Theorem). Let \((X, D, g)\) be a cone metric with cone angle \(2\pi\alpha \in (0, 2\pi)\) along \(D\), and \(\text{Ric}(g) \geq (2n - 1)k > 0\), then the diameter of \((X, g)\) is bounded above by \(\pi/\sqrt{k}\).

**Proof.** Note that \(g\) is smooth in \(X \setminus D\) and its Ricci curvature is bounded below by \((2n - 1)k > 0\), and any two points \(p, q \in X \setminus D\) can be joined by a minimal geodesic which has no intersection with \(D\), thus the same proof as that in the classic Myers Theorem applies to this case, and we have \(\text{diam}(X \setminus D, g) \leq \pi/\sqrt{k}\). For any points \(p, q \in D\), there exist \(\{p_i\}, \{q_i\} \subset X \setminus D\) such that \(d(p, p_i) \to 0\) and \(d(q, q_i) \to 0\). Then triangle inequality implies \(d(p, q) \leq \lim d(p_i, q_i) + d(p, p_i) + d(q, q_i) \leq \pi/\sqrt{k}\). Thus \(\text{diam}(X, g) \leq \pi/\sqrt{k}\). \(\square\)

Next we state a lemma due to Gromov [20].

**Lemma 2.6.** Suppose \(g\) is a cone metric on \((X, D)\) with cone angle \(2\pi\alpha \in (0, 2\pi)\), and \(\text{Ric}(g) \geq (2n - 1)k > 0\). Let \(E\) be a small tubular neighborhood of \(D\). Let \(\varepsilon > 0\) and \(p_1, p_2\) be two points in \(X \setminus E\) such that \(B(p_i, \varepsilon) \cap E = \emptyset\) \((i = 1, 2)\). If for any point \(p \in B(p_2, \varepsilon)\), there is a minimal geodesic connecting \(p_1\) and \(p\) and intersecting with \(\partial E\), then there exists a constant \(C = C(n, \varepsilon, k)\) such that

\[
\text{Vol}(B(p_2, \varepsilon)) \leq C\text{Vol}(\partial E).
\]

**Proof.** We proceed the proof following Gromov [20]. For any \(p \in B(p_2, \varepsilon)\), let \(\gamma_p\) be a minimal geodesic connecting \(p_1\) and \(p\), by assumption, \(\gamma_p \cap \partial E \neq \emptyset\), denoted by \(\tilde{p} \in \gamma \cap \partial E\) (if there are more than one intersection points, we take the one closest to \(p\)). Set \(d_1(\gamma_p) := d(p_1, \tilde{p})\) and \(d_2(\gamma_p) := d(p, \tilde{p})\). Then clearly, \(\varepsilon < d_i(\gamma_p) \leq D, i = 1, 2\), and here \(D\) is the diameter of \((X, g)\), which is bounded above by \(\pi/\sqrt{k}\). Applying the relative volume comparison theorem (the 5th item), we have

\[
\frac{\text{Vol}(B(p_2, \varepsilon))}{\text{Vol}(\partial E)} \leq \sup_{\gamma} \frac{V_k(d_1(\gamma) + d_2(\gamma)) - V_k(d_1(\gamma))}{V_k(\partial B_k(d_1(\gamma)))} \leq \frac{V_k(D) - V_k(\varepsilon)}{V_k(\partial B_k(\varepsilon))} =: C(n, k, \varepsilon).
\]

\(\square\)
Remark 2.3. Although we only state the volume comparison theorem, Myers theorem and Gromov’s lemma, for smooth divisor case, they can be easily generalized for smooth toric conical Kähler metrics. We will need the geodesic convexity as in Remark 2.2.

3. Toric Kähler-Ricci solitons with conical singularities

In this section, we will prove Theorem 1.1 and Theorem 1.2. The key ingredient is to prove a family version of $C^0$ estimates in [50] which will help us prove Theorem 1.3 for a possibly degenerate family of toric manifolds.

3.1. Setting up the continuity method.

For the rest of this section, $X$ will be a toric manifold with a fixed polarization by an ample line bundle $L$. We borrow notation from the last section. In particular, recall that the polytope given by $L$ is fixed to be

\[ P = \{ x \in \mathbb{R}^n \mid l_j(x) = v_j \cdot x + \lambda_j > 0 \}. \]

Also, we once and for all fix a reference metric in $c_1(L)$ by setting $\hat{\omega} = \sqrt{-1} \partial \bar{\partial} \hat{\phi}$ on $(\mathbb{C}^*)^n$ where,

\[ \hat{\phi} = N \sum_{j=1}^N \frac{l_j(x) \log(l_j(x))}{\beta_j}, \]

and $\hat{\phi}$ is the Legendre transform of $\hat{u}$. Then, by the discussion in the last section, $\hat{\omega}$ is a global toric smooth conical metric with angles $2\pi \beta_j$ along the divisor $D_j$.

Our aim in this section is to solve the following conical soliton equation

\[ \text{(**)} \quad \text{Ric}(\omega) = \alpha \omega + \mathcal{L}_\xi \omega + [D], \]

where $\alpha > 0$, $\omega$ is a smooth toric conical Kähler metric, $\xi$ is a holomorphic toric vector field on $X$ and $D$ is an effective toric $\mathbb{R}$-divisor. This is a generalization of Wang-Zhu [50] in the case of smooth Fano manifolds. We will prove our estimates in the framework of [50] combined with some techniques of [16]. On the open part $(\mathbb{C}^*)^n$, we can write $\omega = \sqrt{-1} \partial \bar{\partial} \phi$. $\xi$ being holomorphic then implies that $\mathcal{L}_\xi \omega = \partial \bar{\partial} \xi(\phi)$. Since $\xi$ is also toric, it is generated by the standard vector fields $\{ z_i \partial / \partial z_i \}$. Consequently, there exists a vector $\vec{c} \in \mathbb{R}^n$ such that

\[ \xi(\phi) = \sum_{i=1}^n c_i \frac{\partial \phi}{\partial \rho_i}. \]

Since on the open part one does not see the divisor, the soliton equation can be re-written as a real Monge-Ampere equation -

\[ \det(\nabla^2 \phi) = e^{-\alpha \phi - c \cdot \nabla \phi + \alpha \tau \cdot \rho} \]

for some $\tau \in \mathbb{R}^n$. Here the linear part shows up when one gets rid of the $\partial \bar{\partial}$ and in some sense corresponds to the divisor and controls the blow up of the metric as is seen below.

Lemma 3.1. If there exists a solution of (3.19) then $\tau$ and the vector $\vec{c}$ must satisfy

\[ \tau = \frac{\int_P x e^{c \cdot x} dx}{\int_P e^{c \cdot x} dx}. \]

Moreover, the divisor $D$ in (**) is given by
\[ D = \sum_j (1 - \alpha l_j(\tau))D_j \]

and so the cone angle along each \( D_j \) is \( 2\pi \beta_j \) where \( \beta_j = \alpha l_j(\tau) \).

**Proof.** Since \( \det(\nabla^2 \varphi) \to 0 \) as \(|\rho| \to \infty\), for \( \alpha > 0 \),

\[
0 = \int_{\mathbb{R}^n} \nabla(e^{-\alpha \varphi} + \alpha \tau \cdot \rho) d\rho = \alpha \tau \int_P e^{c \cdot x} dx - \alpha \int_P xe^{c \cdot x} dx.
\]

To compute the cone angles we consider the asymptotics at infinity. First, the equation for the symplectic potential \( u \), the Legendre transform of \( \varphi \), is given by

\[
\det(\nabla^2 u) = e^{-\alpha u + \alpha (x-x) \cdot \nabla u - c \cdot x}.
\]

By the conic Guillemin boundary conditions,

\[
u = \hat{u} + f(x) = \sum_{j=1}^N \frac{l_j(x) \log(l_j(x))}{\beta_j} + f(x)
\]

for some \( f \in C^\infty(\bar{P}) \). By direct computation it can be seen that

\[
\det(\nabla^2 u) = \frac{G(x)}{l_1(x) \ldots l_N(x)}
\]

for some non vanishing \( G \in C^\infty(\bar{P}) \). On the other hand, once again using the formula for \( u \), the order of \( l_j \) on the right hand side of equation (3.21) can be seen to be \( \alpha l_j(\tau)/\beta_j \).
Comparing the orders of \( l_j(x) \) on both sides of the equation we conclude that \( \beta_j = \alpha l_j(\tau) \).

\[ \square \]

Hence \( D \) is effective if and only if \( 1 - \alpha l_j(\tau) > 0 \) for all \( j \) and in this case Lemma 3.1 implies that \( \tau \in P \). Conversely, we have the following Lemma due to Wang-Zhu and Donaldson [50, 16]

**Lemma 3.2.** For each \( \tau \in P \), there exists a unique vector \( \vec{c} \in \mathbb{R}^n \) satisfying

\[
(3.22) \quad \tau = \frac{\int_P xe^{c \cdot x} dx}{\int_P e^{c \cdot x} dx}.
\]

**Proof.** By translating the polytope by \( \tau \), we can assume without loss of generality that \( \tau = 0 \). Consider the function

\[
F(\vec{c}) = \int_P e^{c \cdot x} dx.
\]

Clearly this function is strictly convex as can be seen by differentiating it twice. It is also proper. This follows from 0 being an interior point. Hence the function has a unique minimum \( \vec{c} \). But then \( \nabla F(\vec{c}) = 0 \) which is precisely what we need.

\[ \square \]
By the Cartan formula, for any Kähler metric $\omega$, $\mathcal{L}_\xi \omega = d\iota(\xi \omega)$. Since $\xi$ is holomorphic, clearly $\partial \iota \xi \omega = 0$. Now, all toric manifolds are simply connected i.e $H^{0,1}(X, \mathbb{C}) = 0$. So there exists a potential function $\theta_\xi$ such that $\xi = \nabla \theta_\xi$. Of course the function also depends on the metric. The Lie derivative is now given by

$$\mathcal{L}_\xi \omega = \sqrt{-1} \bar{\partial} \bar{\partial} \theta_\xi.$$  

From now on, we fix $\tau \in P$ with $1 - \alpha l_j(\tau) > 0$ for all $j$ and $\xi$ is the unique holomorphic vector field determined by $\tau$ as in Lemma 3.1.

For the continuity method, we need to set up the Monge-Ampere equation. For that we need an analogue of the $\partial \bar{\partial}$-lemma in this conical setting. We also set $\beta(\alpha) = (\alpha l_1(\tau), \ldots, \alpha l_N(\tau))$. By lifting the smooth conical Kähler metric $\hat{\omega}$ to each uniformization covering, we can obtain the following lemma.

**Lemma 3.3.** There exists a unique (up to constants) function $h \in C^\infty_{\beta(\alpha)}(X)$ satisfying

$$\text{Ric}(\hat{\omega}) - \alpha \hat{\omega} - [D] - \mathcal{L}_\xi \hat{\omega} = \sqrt{-1} \bar{\partial} \bar{\partial} h.$$  

We now write the Monge-Ampere equation for the conical soliton. Set $\omega = \hat{\omega} + \sqrt{-1} \bar{\partial} \bar{\partial} \psi$. Then the equation for the conical soliton is

$$\begin{cases}
(\hat{\omega} + \sqrt{-1} \bar{\partial} \bar{\partial} \psi)^n = e^{-n \psi - \xi(\psi)} + h \hat{\omega}^n \\
\hat{\omega} + \sqrt{-1} \bar{\partial} \bar{\partial} \psi > 0 \\
\psi \in C^\infty_{\beta(\alpha)}(X),
\end{cases}$$  

where $h$ is from the above lemma.

To solve this equation, like usual, we introduce a parameter $s \in [0, \alpha]$ and look at the following family of equations.

$$\begin{cases}
(\hat{\omega} + \sqrt{-1} \bar{\partial} \bar{\partial} \psi_s)^n = e^{-n s \psi_s - \xi(\psi_s)} + h \hat{\omega}^n \\
\hat{\omega} + \sqrt{-1} \bar{\partial} \bar{\partial} \psi_s > 0 \\
\psi_s \in C^\infty_{\beta(\alpha)}(X)
\end{cases}$$  

or equivalently

$$\begin{cases}
\text{Ric}(\omega_s) = s \omega_s + (\alpha - s) \hat{\omega} + \mathcal{L}_\xi \omega_s + D.
\end{cases}$$  

The corresponding linearized operator is given by

$$L_s(\psi) = L(\psi) + s \psi = \Delta \psi + \xi(\psi) + s \psi.$$  

Recall that we are only looking at the space of functions invariant under the toric action. One can define an inner product by

$$(\psi_1, \psi_2) = \int_X \psi_1 \bar{\psi}_2 e^{\theta \xi} \omega^n$$

and denote the corresponding Hilbert space of square integrable functions by $L^2(e^{\theta \xi})$. Then $L$ restricted to $C^\infty_\beta(X)$ is self adjoint and hence can be thought of as an operator from $L^2(e^{\theta \xi})$ to itself. Also, by virtue of being self adjoint, $L$ only has real eigenvalues. The linear theory for the spaces $C^{k,\gamma}_\beta(X)$ is summarized below.

**Lemma 3.4.** Let $\omega$ be a $\beta$-conical metric, $\Delta$ be the corresponding Laplacian and $L$ be defined as above. Then

1. For $k \geq 2$, $\Delta : C^{k,\gamma}_\beta(X) \to C^{k-2,\gamma}_\beta(X)$ is an invertible operator, modulo constants
2. The Fredholm alternative holds for $L$.  

(3) All nonzero eigenvalues of $-L$ are positive. Moreover, if $\text{Ric}(\omega) > t\omega + L\xi\omega$ and $-L\psi = \lambda\psi$, then $\lambda > t$.

The lemma that follows is essentially an observation of Zhu \[55\], adapted to the conical setting and is required in all the subsequent estimates. The proof in the toric case is in fact much easier.

**Lemma 3.5.** There exists a uniform constant $C$ depending only on $\hat{\omega}$ and $\xi$ such that, for any function $\psi \in C_\beta^\infty(X) \cap PSH(X,\hat{\omega})$

$$|\xi(\psi)| \leq C.$$ 

**Proof.** In the toric situation, the proof is almost trivial. Locally on $(\mathbb{C}^*)^n$, $\hat{\omega} = \partial\bar{\partial}\hat{\phi}$ and for any $\phi = \hat{\phi} + \psi$, since $\psi$ is globally bounded and plurisubharmonic, it is easy to see that $\nabla\phi(\mathbb{R}^n) = \nabla\hat{\phi}(\mathbb{R}^n) = P$ and $\partial\bar{\partial}\phi$ extends to a global Kähler metric. So there exists a uniform constant $C$ such that

$$|\nabla\psi| \leq C.$$ 

But then, since $\xi$ is given by

$$\xi = \sum_{i=1}^n c_i z_i \frac{\partial}{\partial z_i},$$

we have that

$$\xi(\psi) = c \cdot \nabla\psi.$$ 

This gives us the required bound. \qed

The following proposition shows that there exists a solution to equation (3.25) at $s = 0$

**Proposition 3.1.** For any $h \in C_\beta^\infty(\alpha)(X)$ there exists a unique function $\psi \in C_\beta^\infty(\alpha)(X)$ satisfying

$$\begin{cases}
(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{h-\xi(\psi)}\hat{\omega}^n \\
\sup \psi = 0 \\
\omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi > 0.
\end{cases}$$

**Proof.** We proceed by the continuity method. Consider the family of equations

$$\begin{cases}
(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s)^n = e^{sh-\xi(\psi_s)}\hat{\omega}^n \\
\sup \psi_s = 0 \\
\omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s > 0
\end{cases}$$

and set $S = \{ s \in [0,1] | \text{equation (3.25) has a solution } \psi_s \in C_\beta^\infty(\alpha)(X) \text{ at } s \}$. The set $S$ is clearly nonempty, since $0 \in S$. In what follows we suppress the index $s$ for convienience.

**Openness.** This follows straight from part(a) of Lemma 3.4 and the implicit function theorem on the space $C_\beta^\infty(\alpha)(X)$, since the linearized operator is just $L$.

**$C^0$ estimates.** By Lemma 3.5, the right hand side of the equation is uniformly bounded in $s$ and hence in particular there exists a uniform $L^p$ bound for any $p > 1$. Now, Kolodziej’s results and their generalizations \[26, 18, 54\] give a uniform $C^0$ bound.

**Second order estimates.** Consider the quantity...
where $A$ is some large number to be chosen later. Since both $\hat{\omega}$ and $\omega_s$ have poles of same order, the quantity is bounded. Let $\sup_X H_s = H_s(q)$. We lift all the local calculations to the $\mathbb{S}^1$ invariant $\beta$-covering space. The second order estimates easily follow from [52] and [17].

**C³ and higher order estimates.** Calabi’s method for third order estimates can again be carried out by lifting the calculations to the $\beta$-cover. The reader should refer to [35] for the simplified computations. Higher order derivatives can be obtained by a standard bootstrapping argument. Closedness now follows from Ascoli-Arzela. Hence $1 \in S$.

3.2. Estimates and proofs of Theorems 1.1 and 1.2. For later applications, we need to get the precise dependence of the $C^0$ estimate on the polytope. So we introduce some notation. Recall that

$$P = \{x \in \mathbb{R}^n \mid l_j(x) = v_j \cdot x + \lambda_j > 0\}.$$  

We let $\nu$ and $\sigma$ be two constants such that

$$\nu^{-1} < \text{Vol}(P) < \nu,$$

$$(\sigma)^{-1} < \text{diam}(P) < \sigma.$$  

On $(\mathbb{C}^*)^n$ we write $\hat{\omega} = \sqrt{-1} \partial \bar{\partial} \hat{\varphi}$ and $\omega_s = \sqrt{-1} \partial \bar{\partial} \varphi_s$. Using the standard logarithmic coordinates like before one can rewrite equation $\text{[**]}$ as a real Monge-Ampère on $\mathbb{R}^n$

$$\begin{cases}
\det(\nabla^2 \varphi_s) = e^{-s(\varphi_s - \tau \rho) - (\alpha - s)(\hat{\varphi} - \tau \rho) - e \cdot \nabla \varphi_s}, \\
u_s = \mathcal{L} \varphi_s = \sum_{j=1}^{N} (\beta_j)^{-1} l_j(x) \log l_j(x) + f(x), \quad f \in C^\infty(\mathcal{P}) \\
\nabla^2 \varphi_s > 0.
\end{cases}$$

**Proposition 3.2.** For any $s_0 \in (0, \alpha)$ there exists a constant $C = C(n, s_0, \nu, \sigma, \Lambda, \sup_{j,P} |l_j|)$ such that

$$|\varphi_s - \hat{\varphi}| \leq C$$

for all $s \in [s_0, \alpha]$. Here $\Lambda$ is the constant from Lemma 3.7 below.

We use the arguments in [50] with inputs and simplifications from [16], most notably the last step which helps us avoid the Harnack inequality. We first need two technical lemmas.

**Lemma 3.6.** Suppose $v \geq 0$ is a strictly convex function on $\mathbb{R}^n$ such that $v(0) = 0$ and $\det(\nabla^2 v) \geq \lambda$ on $v \leq 1$. Then there exists $C > 0$ such that

$$\text{Vol}(v \leq 1) \leq C \lambda^{-1/2}.$$  

The proof is a standard barrier function argument and so we skip it.

**Lemma 3.7.** If $\hat{\varphi}$ is the Legendre transform of $\hat{\varphi}$ and we define the function

$$g_j(\rho) = \log(l_j(\nabla \hat{\varphi}(\rho))).$$  

Then, there exists a constant $\Lambda$ such that

$$\sup_{\mathbb{R}^n} |\nabla g_j| \leq \Lambda.$$  

Here, $\Lambda$ depends only on $\beta_j$, $N$, $n$ and the normal vectors $v_j$.  

Proof. Recall that the polytope $P$ is given by faces $l_j(x) = v_j \cdot x + \lambda_j$ and let $\hat{u}$ be the usual conical symplectic potential given by

\begin{equation}
\hat{u} = \sum_{j=1}^{N} \frac{1}{\beta_j} l_j \log l_j.
\end{equation}

We then set,

\[ V = \{v_j\}, \quad A = \{\frac{v_j}{\sqrt{\beta_j l_j}}\} := \{a_{j}\}, \]

where $v_j = \{v_j\}$ is the vector normal to the face $l_j$. For any $J = \{j_1, \ldots, j_n\} \subset \{1, \ldots, N\}$ in some order, we let $M_J$ be the corresponding $n \times n$ minor of $V$ and $C_J = \text{det}(M_J)$. Let $M_J$ and $\tilde{C}_J$ be the corresponding quantities for $A$. Then, in our notation

\begin{equation}
\nabla^2 \hat{u} := \{\hat{u}_{\gamma\mu}\} = (A)^t A.
\end{equation}

**Claim 1**

\[
\text{det}(\nabla^2 \hat{u}) = \sum_{j_1 < \ldots < j_n} \tilde{C}_{j_1 \ldots j_n}^2 = \sum_{j_1 < \ldots < j_n} \frac{C_{j_1 \ldots j_n}^2}{(\beta_{j_1} l_{j_1}) \ldots (\beta_{j_n} l_{j_n})}
\]

This is known as the Cauchy-Binet formula in literature. The proof is of course just a simple exercise in undergraduate linear algebra and so we skip it.

Now, for $J = \{j_1, \ldots, j_{n-1}\}$ let $M_{J,\gamma}$ be the minor obtained by deleting the $\gamma^{th}$ column from the matrix of row vectors $v_{j_i}$. We once again set $C_{J,\gamma} = \text{det}(M_{J,\gamma})$

**Claim 2**

\begin{equation}
\text{det}(\nabla^2 \hat{u}) \sum_{\mu=1}^{n} a_{j\mu} \hat{u}_{\mu\gamma} = (-1)^{\gamma+1} \sum_{j_1 < \ldots < j_{n-1}} C_{j, j_1 \ldots j_{n-1}, \gamma} \tilde{C}_{j, j_1 \ldots j_{n-1}, \gamma}
\end{equation}

\begin{equation}
\text{det}(\nabla^2 \hat{u}) \sum_{\mu=1}^{n} a_{j\mu} \hat{u}_{\mu\gamma} = \frac{(-1)^{\gamma+1}}{\sqrt{\beta_{j} l_{j}}} \sum_{j_1 < \ldots < j_{n-1}} C_{j, j_1 \ldots j_{n-1}, \gamma} C_{j, j_1 \ldots j_{n-1}, \gamma}
\end{equation}

where $\{u^{\mu\gamma}\}$ denotes the inverse matrix of $\nabla^2 \hat{u}$.

Proof. The proof proceeds along the lines of the proof for Cauchy-Binet formula, only it requires more book-keeping and is as follows - Denoting by $\chi_{\gamma\mu}$, the co-factor matrix of $\nabla^2 u$ and employing Cramer’s rule,

\[
\text{det}(\nabla^2 \hat{u}) \sum_{\mu=1}^{n} a_{j\mu} \hat{u}_{\mu\gamma} = \sum_{\mu=1}^{n} a_{j\mu} \chi_{\gamma\mu}
\]

\[
= \sum_{\mu=1}^{n} a_{j\mu} \sum_{\sigma = \{1, \ldots, j, \ldots, n\}}^{\sigma \rightarrow \{1, \ldots, j, \ldots, n\}} (-1)^{\gamma+\mu} \text{sgn}(\sigma) u_{1\sigma(1)} \cdots u_{i-1\sigma(i-1)} u_{i+1\sigma(i+1)} \cdots u_{n\sigma(n)}
\]

\[
= \sum_{\mu=1}^{n} a_{j\mu} \sum_{j_1=1}^{N} \sum_{j_{n-1}=1}^{N} \sum_{\sigma = \{1, \ldots, j, \ldots, n\}}^{\sigma \rightarrow \{1, \ldots, j, \ldots, n\}} (-1)^{\gamma+\mu} \text{sgn}(\sigma) a_{j_1 A j_1 \sigma(1)} \cdots a_{j_{n-1} A j_{n-1} \sigma(n)}
\]
where the product in the above summation includes exactly two entries from all columns except the $\mu^{th}$ and the $\gamma^{th}$ ones which have one entry each. Clearly, the innermost summation, as $\sigma$ runs over all permutations, is some determinant. More precisely,

$$\det(\nabla^2 \tilde{u}) \sum_{\mu=1}^{n} a_{j\mu} \tilde{u}^{\mu\gamma} = \sum_{\mu=1}^{n} a_{j\mu} \sum_{j_1=1}^{N} \cdots \sum_{j_{n-1}=1}^{N} (-1)^{\gamma+\mu} a_{j_11} \cdots a_{j_{n-1}n} \tilde{C}_{j_1,\ldots,j_{n-1};\gamma}.$$ 

Again, like in the proof of the first claim, $\tilde{C}_{j_1,\ldots,j_{n-1};\gamma} = 0$ unless the $j_i$’s are distinct. Also, if $j_1 < \ldots j_{n-1}$ and $\tau$ any permutation of this indices, then $\tilde{C}_{\tau(j_1),\ldots,\tau(j_{n-1});\gamma} = \tilde{\text{sgn}}(\tau)C_{j_1,\ldots,j_{n-1};\gamma}.$ Thus,

$$\det(\nabla^2 \tilde{u}) \sum_{\mu=1}^{n} a_{j\mu} \tilde{u}^{\mu\gamma} = \sum_{\mu=1}^{n} a_{j\mu} \sum_{j_1 < \ldots < j_{n-1}} \sum_{\tau} (-1)^{\gamma+\mu} \tilde{\text{sgn}}(\tau) a_{\tau(j_1)1} \cdots a_{\tau(j_{n-1})n} \tilde{C}_{j_1,\ldots,j_{n-1};\gamma}$$

$$= \sum_{\mu=1}^{n} a_{j\mu} \sum_{j_1 < \ldots < j_{n-1}} \sum_{\tau: \{1,\ldots,\mu,\ldots,n\}} (-1)^{\gamma+\mu} \tilde{\text{sgn}}(\tau) a_{j_1\tau(1)} \cdots a_{j_{n-1}\tau(n)} \tilde{C}_{j_1,\ldots,j_{n-1};\gamma}$$

$$= \sum_{j_1 < \ldots < j_{n-1}} (-1)^{\gamma+1} \tilde{C}_{j_1,\ldots,j_{n-1}} \tilde{C}_{j_1,\ldots,j_{n-1};\gamma}.$$

\[\square\]

**Proof of Lemma 3.7** We compute the derivative of $g_j$ using the correspondence $\nabla^2 \tilde{\phi} = (\nabla^2 \tilde{u})^{-1}$ and $x = \nabla \phi$.

\[
\frac{\partial g_j}{\partial \rho_\gamma} = l_j(\nabla (\varphi_\gamma)) = \frac{1}{l_j(\nabla \varphi)} \sum_{\mu=1}^{n} v_{j\mu} \tilde{u}^{\mu\gamma} = \frac{\sqrt{\beta_j}}{a_{j\mu}} \sum_{\mu=1}^{n} a_{j\mu} \tilde{u}^{\mu\gamma}
\]

$$= \frac{1}{\det(\nabla^2 \tilde{u})} \sum_{j_1 < \ldots < j_{n-1}} \frac{1}{l_j(\beta_{j_1} l_{j_1} \ldots (\beta_{j_{n-1} l_{j_{n-1}}}))} \sum_{j_1 \neq j} \frac{C_{j_1,\ldots,j_{n-1} \gamma} C_{j_1,\ldots,j_{n-1};\gamma}}{C_{j_1,\ldots,j_{n-1}}^{2n-2}}$$

$$\leq \frac{\sum_{j_1 < \ldots < j_{n-1}} \frac{1}{l_j(\beta_{j_1} l_{j_1} \ldots (\beta_{j_{n-1} l_{j_{n-1}}}))} \sum_{j_1 \neq j} \frac{C_{j_1,\ldots,j_{n-1} \gamma} C_{j_1,\ldots,j_{n-1};\gamma}}{C_{j_1,\ldots,j_{n-1}}^{2n-2}}}{\sum_{j_1 < \ldots < j_{n-1}} \frac{1}{l_j(\beta_{j_1} l_{j_1} \ldots (\beta_{j_{n-1} l_{j_{n-1}}}))} \sum_{j_1 \neq j} \frac{C_{j_1,\ldots,j_{n-1} \gamma}}{C_{j_1,\ldots,j_{n-1}}}}.$$ 

The summation can be taken to be only over all $J = \{j_1, \ldots, j_{n-1}\}$ such that $C_{j,J} \neq 0$. For such terms, one has the trivial bound 

$$\left| \frac{C_{j_1,\ldots,j_{n-1};\gamma}}{C_{j_1,\ldots,j_{n-1}}} \right| \leq M',$$

where $M'$ depends only on upper bounds on $|v_j|$ and $\beta_j$ and a positive lower bound on $|C_J|$ as $J \subset \{1, \ldots, N\}$ varies over all subsets with $C_J \neq 0$. Together with the above computation,
we get
\[ \left| \frac{\partial g_j}{\partial \rho_i} \right| \leq \beta_j \Lambda'. \]
This proves the Lemma with \( \Lambda = \beta_j \Lambda' \).

**Proof of Proposition 3.2** There are several steps following [50] and [16] combined with Lemma 3.7 Let \( \phi_s = \varphi_s - \tau \cdot \rho, \hat{\phi} = \hat{\varphi} - \tau \cdot \rho \) and define
\[ w_s = s\phi_s + (\alpha - s)\hat{\phi}. \]
Set
\[ m_s := \inf_{\mathbb{R}^n} w_s = w_s(\rho_s) \]

**Step 1.** We claim that there exist \( C, \zeta > 0 \) independent of \( s \) such that for all \( s \in [s_0, \alpha] \),
\[ (a) \ |m_s| \leq C \]
\[ (b) \ w_s \geq \zeta |\rho - \rho_s| - C. \]

It follows from the definition of \( w_s \) that \( \det(\nabla^2 w_s) \geq s_0^n \det(\nabla^2 \phi_s) \). Set \( K = \{m_s \leq w_s \leq m_s + 1\}, K_\mu = \{m_s \leq w_s \leq m_s + \mu\} \) and \( V_\mu = Vol(K_\mu) \). From the equation, \( \det(\nabla^2 \phi_s) = e^{-w_s-c \nabla \phi_s} \) and so on \( K \),
\[ \det(\nabla^2 w_s) \geq s_0^n \det(\nabla^2 \phi_s) = s_0^n e^{-w_s-c \nabla \phi_s} \geq C e^{-m_s}, \]
where \( C \) only depends on \( s_0 \) and \( \sigma \) which is an upper bound for \( |\nabla \varphi_s| \). So, Lemma 3.6 applied to \( v = w_s - m_s \) implies that \( Vol(K) \leq C e^{m_s/2} \). But \( K_\mu \subseteq \mu K \), where by \( \mu K \), we mean dilation with center \( \rho_s \). So we have the volume estimate
\[ V_\mu \leq C \mu^n e^{m_s/2}. \]

Now,
\[ \nu^{-1} \leq Vol(X) = \int_{\mathbb{R}^n} \det(\nabla^2 \phi_s) d\rho = \int_{\mathbb{R}^n} e^{-w_s-c \nabla \phi_s} d\rho \leq C e^{-m_s} \int_0^\infty e^{-\mu} V_\mu d\mu \leq C e^{-m_s/2} \]
and so \( m_s \leq C(n, s_0, \nu, \sigma) \). For the lower bound, notice that \( \nabla w_s(\mathbb{R}^n) = P - \tau \) and so \( |\nabla w_s| \leq 2\sigma \). This implies that \( K \) contains a ball of radius \( 1/2\sigma \). But the volume of \( K \) is bounded above by \( C e^{m_s/2} \) and so we immediately have a lower bound for \( m_s \). Hence (a) is proved with \( C = C(n, s_0, \nu, \sigma) \).

Suppose now there exists a point \( \rho \in K \) such that \( |\rho - \rho_s| = R \). Because \( B = B(\rho_s, 1/(2\sigma)) \subseteq K \), by convexity, the entire cone \( \kappa \) with vertex at \( \rho \) and base as \( B \) lies inside \( K \). So, \( Vol(K) \geq C R \), where \( C \) depends only on dimension and \( \sigma \). But \( Vol(K) \leq C e^{m_s/2} \) and so is less than some fixed constant \( C \) by part (a). Hence \( R \) is uniformly bounded. That
is, there exists a uniform $R$ such that $K \subseteq B(\rho_s, R)$. But then, convexity implies that $K_\mu \subseteq B(\rho_s, \mu R)$. From this and the lower bound on $m_s$, it easily follows that

$$w_s \geq \frac{1}{R}|\rho - \rho_s| - C.$$  

This proves (b) with $\zeta = 1/R$.

**Step 2.** We now claim that, there exists uniform constant $C$ such that

(3.37) $|\rho_s| \leq C$.

We first observe,

$$0 = \int_{\mathbb{R}^n} \nabla (e^{-w_s}) = -\int_{\mathbb{R}^n} [s(\nabla \varphi_s - \tau) + (\alpha - s)(\nabla \hat{\varphi} - \tau)]e^{-w_s} \, d\rho$$

$$= -\int_{\mathbb{R}^n} (\alpha - s)(\nabla \hat{\varphi} - \tau)]e^{-w_s},$$

where we use the change of coordinates $x = \nabla \varphi_s(\rho)$ along with the equation $e^{-w_s} = \det(\nabla^2 \varphi_s)e^{c\cdot\nabla \varphi_s}$ and the fact that $\vec{c}$ and $\tau$ are compatible to conclude that the first term is zero. This computation gives us the crucial identity

$$\int_{\mathbb{R}^n} (\nabla \hat{\varphi} - \tau)e^{-w_s} \, d\rho = 0,$$

or equivalently,

(3.38) $\frac{1}{V_s} \int_{\mathbb{R}^n} (\nabla \hat{\varphi})e^{-w_s} \, d\rho = \tau$,

where $\bar{V}_s$ is the weighted volume given by

$$\bar{V}_s = \int_{\mathbb{R}^n} e^{-w_s} \, d\rho.$$

Note that when the Futaki invariant vanishes, this is precisely the identity in the paper of Wang and Zhu since in that case $\tau$ is the barycenter which is zero.

Suppose the claim is false i.e for all $M > 0$ there exists a pair $(s, \rho_s)$ with $|\rho_s| > M$. Applying $l_j$ to both sides of the identity (3.38),

(3.39) $\frac{1}{V_s} \int_{\mathbb{R}^n} l_j(\nabla \hat{\varphi})e^{-w_s} \, d\rho = l_j(\tau) > \delta$

for some $j$ and some $\delta > 0$. Fix an $\epsilon > 0$. From the estimates in the previous step there exists an $R_\epsilon >> 1$ such that

(3.40) $\int_{\mathbb{R}^n \setminus B(\rho_s, R_\epsilon)} e^{-w_s} \, d\rho \leq \epsilon.$

Recall that as $\rho$ goes to infinity, the image under $\nabla \hat{\varphi}$ goes to the boundary of $P$. So, by hypothesis, one can choose a big $M >> 1$ such that $|\rho_s| > M$ and $\log(l_j(\nabla \hat{\varphi}(\rho_s))) < -M$ for some $s$ and some face $l_j$. By the gradient estimate in Lemma 3.7 there exists a constant $\Lambda$ (which does not depend on $s$) such that on $B = B(\rho_s, R_\epsilon)$

(3.41) $\log(l_j(\nabla \hat{\varphi}(\rho))) < -M + \Lambda R_\epsilon < \frac{-M}{2} < \log \epsilon$
for $M$ sufficiently big. So combining 3.40 and 3.41 we estimate the integral in 3.39

$$\frac{1}{V} \int_{\mathbb{R}^n} l_j(\nabla \hat{\varphi}) e^{-ws} = \frac{1}{V} \int_B l_j(\nabla \hat{\varphi}) e^{-ws} + \frac{1}{V} \int_{\mathbb{R}^n \setminus B} l_j(\nabla \hat{\varphi}) e^{-ws} \leq \epsilon + C\epsilon,$$

where $C$ only depends on an upper bound for the image of $P$ under $l_j$ and a lower bound for the total volume of $X$. Now choose $\epsilon$ small enough so that $\epsilon + C\epsilon < \delta/2$. But then, this contradicts (3.39), completing Step 2.

**Step 3.** We first observe the elementary identity from convex analysis

(3.42) $\sup_{\mathbb{R}^n} |\varphi_s - \hat{\varphi}| = \sup_P |u_s - \hat{u}|$.

So, to complete the proof, one only needs to control the $C^0$ norm of $u_s$ since from the definition it is easy to see that the bound for $\hat{u}$ only depends on $\beta_j$ and an upper bound on $l_j$. From (3.36) and (3.37),

(3.43) $w_s(\rho) \geq \zeta |\rho| - C$.

Let $u_s$ be the Legendre transform of $\varphi_s$, then for any $p > n$,

$$\int_{\mathbb{R}^n} |\nabla u_s|^p dx = \int_{\mathbb{R}^n} |\rho|^p e^{-w-s-\zeta|\rho|} d\rho \leq C \int_{\mathbb{R}^n} |\rho|^p e^{-\zeta|\rho|} d\rho \leq C(p).$$

By Morrey’s inequality $\text{osc}_P u_s < C$ for some $C$ independent of $s$. Now, if we set $x_s = \nabla \varphi_s(\rho_s)$, then,

$$u_s(x_s) = \rho_s \cdot x_s - \varphi_s(\rho_s).$$

The first term is clearly bounded from Step-2. Moreover by Step-1, $w_s(\rho_s)$ is bounded. Since $\rho_s$ stays bounded, there is a uniform bound on $\varphi_s(\rho_s)$, which in turn gives a uniform bound on $w_s(\rho_s)$. This shows that $|u_s(x_s)|$ is uniformly bounded. Hence the oscillation bound implies

$$|u_s|_{C^0(P)} \leq C.$$

This completes the proof of the proposition. \qed

We are now in a position to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.**

**Step 1.** We first characterize the invariant $S(X, L)$ in terms of the polytope data as follows.

- The polytope for the linear system $| -K_X - \alpha L |$ can be taken to be $P^\alpha = \{ x \in \mathbb{R}^n \mid l^\alpha_j = v_j \cdot x + 1 - \alpha \lambda_j \}$. For any $\alpha > 0$ and any $j$,

$$\tau \in P \text{ with } 1 - \alpha l_j(\tau) \geq 0 \Leftrightarrow 0 \leq 1 - \alpha l_j(\tau) \leq 1 \Leftrightarrow 0 \leq v_j \cdot (-\alpha \tau) + 1 - \alpha \lambda_j \leq 1 \Leftrightarrow 0 \leq l^\alpha_j(-\alpha \tau) \leq 1.$$

But then divisor $D = \sum l^\alpha_j(-\alpha \tau)D_j$ is an effective divisor in $| -K_X - \alpha L |$ with coefficients less than 1.
**Step 2.** We next outline a proof of the existence of solutions to the soliton equation. Let
\[ S = \{ s \in [0, \alpha] \mid \exists a solution \psi \in C^3\beta(\alpha) \text{ to eqn. (**)} \}. \]
By proposition 3.1, 0 \in S and hence S is nonempty. We now need to show that S is both open and closed.

**Openness** - The linearized operator for equation \((**)_s\) is \( L_s = \Delta_s + \xi + sI \). Since \([D] \geq 0, Ric(\omega_s) > s\omega_s + L_\xi \omega_s \). By lemma 3.4 all eigenvalues of \(-L_s\) are strictly positive and hence the Fredholm alternative implies that \(L_s\) is invertible. Implicit function theorem then implies that S is open.

**C^0 estimates** - Since there is a solution at \(s = 0\) by openness there exists an \(s_0\) such that there is a solution on \([0, s_0]\). With this choice of \(s_0\), by proposition 3.2 there exists a constant \(C\) independent of \(s\) such that
\[ |\psi_s| = |\varphi_s - \hat{\varphi}| \leq C. \]

**\(C^2\) and higher order estimates** - Once the uniform bound is obtained, the argument for the second and higher order estimates is the same as that in the proof of proposition 3.1. Hence the upshot is that S is nonempty, open and closed. Hence \(S = [0, \alpha]\) and in particular \(\alpha \in S\). This completes the proof of the second part of the theorem.

**Step 3.** Finally, to complete the proof of Theorem 1.1, we now prove that \(R_{BE}(X, L) = S(X, L)\). From the existence part of the theorem, it is easy to see that \(R_{BE}(X, L) \geq S(X, L)\). In order to prove the reverse inequality, let \(\alpha \in (0, R_{BE})\). Then by definition, there exist smooth toric \(\beta\)-conical metrics \(\omega = \sqrt{-1} \partial \bar{\partial} \varphi\) and \(\eta = \sqrt{-1} \partial \bar{\partial} \psi\), and a holomorphic vector field \(\xi\) \(\tau \in \mathbb{R}^n\), such that
\[ Ric(\omega) = \alpha \omega + L_\xi \omega + \eta + [D] \]
for some smooth conical Kähler metric \(\eta\) and some effective divisor \(D\). Note that the volume form can be expressed as
\[ \omega^n = \frac{\Omega}{\prod_{j=1}^{N} |s_j|^{2(1-\beta_j)}} \]
for some global volume form \(\Omega\) with log \(\Omega\) bounded. From this, it is clear that the divisor is given by
\[ D = \sum_{j=1}^{N} (1 - \beta_j) D_j \]
and consequently one can take the polytope for \(\eta\) to be
\[ P^\eta = \{ x \in \mathbb{R}^n | v_j \cdot x + \beta_j - \alpha \lambda_j > 0 \ j = 1, \ldots, N \}. \]
Locally on \((\mathbb{C}^*)^n, \omega = \sqrt{-1} \partial \bar{\partial} \varphi\) and \(\eta = \sqrt{-1} \partial \bar{\partial} \psi\), and the corresponding real Monge-Ampere equation reads
\[ \det \nabla^2 \varphi = e^{-\alpha \varphi - \psi - c \nabla \varphi - \tau \rho} \]
for some \(\tau \in \mathbb{R}^n\). As before, we take \(\varphi\) so that \(\nabla \varphi(\mathbb{R}^n) = P\). Furthermore we normalize \(\psi\) so that \(\nabla \psi(\mathbb{R}^n) = P^\eta\). With this normalization, we claim that \(\tau = 0\).
Since \(\nabla \varphi\) is bounded, It suffices to prove that
\[ (3.44) \quad | \log \det \nabla^2 \varphi + \alpha \varphi + \psi | \]
is bounded. Let $\varphi = \alpha \varphi + \psi$ be the potential for the smooth $\beta$-conical metric $\varpi = \alpha \omega + \eta$. Then $\varpi^n/\omega^n$ is a global bounded function. This is because both the metrics have the same poles at the divisors. Consequently it is enough to show that

$$|\log \det \nabla^2 \varphi + \varpi|$$

is bounded. But, as in the proof of Lemma 3.1,

$$|\log \det \nabla^2 \varphi + \varpi| \leq \sum_{j=1}^N \left(1 + \frac{x \cdot v_j}{\beta_j}\right) \log l_j| + C$$

$$\leq \sum_{j=1}^N \left(1 - \frac{l_j(0)}{\beta_j}\right) \log l_j| + C$$

$$\leq C,$$

where $l_j(x) = v_j \cdot x + \beta_j$ and we used the fact that $l_j \log l_j$ is a bounded function in the second line. Note that the polytope for $\varphi$ is given precisely by the intersection of $l_j > 0$ for $j = 1, \ldots, N$. This completes the proof of (3.44) and hence proves the claim that $\tau = 0$. But then using the integration by parts trick from the proof of Lemma 3.1

$$0 = \int_{\mathbb{R}^n} \nabla(e^{-\alpha \varphi - \psi}) d\rho = -\alpha \int_P xe^{\varphi} dx - \int_P \nabla \psi e^{\varphi} dx.$$

So, if we set

$$\tilde{\tau} = \int_P xe^{\varphi} dx / \int_P e^{\varphi} dx = -\int_P \nabla \psi e^{\varphi} dx / \alpha \int_P e^{\varphi} dx.$$

Obviously, $\tilde{\tau} \in P$ and applying $l_j$, we have

$$1 - \alpha l_j(\tilde{\tau}) = \frac{\int_P (1 + v_j \cdot \nabla \psi - \alpha \lambda_j) e^{\varphi} dx}{\int_P e^{\varphi} dx} \geq 1 - \beta_j \geq 0,$$

where we used the definition of $P^n$ for the first inequality and $D \geq 0$ for the second inequality. Hence $\alpha < S(X,L)$ and this completes the proof of the theorem. \(\square\)

**Proof of Theorem 1.2.** This theorem follows directly from Theorem 1.1 by taking $\tau = P_C$. For uniqueness we refer to results of Berndtsson \[7\]. The only slightly subtle point is the existence of conical Kähler-Einstein metrics for $\alpha = R(X,L)$. This follows from the fact that barycenter always stays in the interior of the polytope and hence Proposition 3.2 also holds for this choice of $\alpha$ (Contrast this, for instance, with the case when $\alpha = S(X,L)$ as discussed in Example 5.1 below). All the higher order estimates then follow from the $C^0$ bound exactly as in the proof above.

4. Connectedness of the space of toric conical Kähler-Einstein metrics

We will prove Theorem 1.3 in this section.

4.1. **Reducing the proof of Theorem 1.3 to the case of one blow-up.** Let us fix a toric manifold $Y$ with an ample toric line bundle $L$ and corresponding polytope $P$. Let $X$ be the blow-up of $Y$ along a $k$-dimensional smooth toric variety $V$ with $\pi : X \to Y$ as the blow-down map. Set $L_t = \pi^* L + t A$ for some ample toric line bundle $A$ on $X$ and for $t \in [0, 1]$. Recalling the definition of the invariant $R$, we make the following simple observation
Lemma 4.1. Let $(X_t, L_t)$ be a family of toric manifolds with ample $\mathbb{R}$-line bundles $L_t$ for $t \in (0, 1]$. Then as long as the corresponding polytopes $P_t$ stay bounded, we have

\begin{equation}
\inf_t R(X_t, L_t) > 0.
\end{equation}

Proof. By Theorem 1.1

\[ R(X, L) = \sup \{ \alpha |1 - \alpha \alpha_j(P_C) > 0, \ j = 1, \ldots, N \} \]

\[ = \inf \{ \frac{1}{l_j(P_C)} \}, \]

which stays bounded away from zero if the polytopes stay bounded. \hfill \Box

For $t \in [0, t_0]$ we can now choose a continuous path $\alpha_t$ such that

\[ 0 < \alpha_t < \min \{ R(X, L_t), R(Y, L_0) \} \]

and let $\omega_t$ be the unique toric conical Kähler-Einstein metrics in $c_1(L_t) \cap K_c(X)$ with Einstein constant $\alpha_t$. We also let $\omega_Y \in c_1(L) \cap K_c(Y)$ be the the toric conical Kähler-Einstein metric on $Y$ with Einstein constant $\alpha_0$. Denoting the corresponding Riemannian metrics by $g_t$ and $g_Y$, we have the following proposition.

**Proposition 4.1.** $(X, g_t)$ is a continuous path in the Gromov-Hausdorff topology and

\[ (X, g_t) \xrightarrow{t \to 0} (Y, g_Y). \]

By restricting $\alpha_t$ to be less than $R(X, L_t)$ we ensure that $g_t$ are geodesically convex, thus facilitating the application of tools from comparison geometry. In particular, we will make use of lemma 2.7. Taking the above proposition for granted, we now prove Theorem 1.3

**Proof of Theorem 1.3.** We fix $(X_j, \omega_j)$ for $j = 0, 1$ as in the statement of the theorem and we let $L_j$ be the Kähler class of $\omega_j$ and $\alpha_j < R(X, L_j)$, and we call $(L_j, \omega_j, \alpha_j)$ compatible triples on $X_j$. By the factorization theorem 2.2 there exist a sequence $0 = t_0 < t_1 < \ldots < t_k = 1$ and pairs $(X_{t_i}, f_{t_i})$ such that

\[ X = X_0 \xrightarrow{f_{t_1}} X_{t_1} \xrightarrow{f_{t_2}} \ldots \xrightarrow{f_{t_i}} X_{t_i} \xrightarrow{f_{t_i+1}} \ldots \xrightarrow{f_{t_k}} X_{t_k} = X_1. \]

We start from the left and construct the family of metrics inductively. Suppose we are at stage $t_i$ i.e we have already constructed $(X_{t_i}, L_{t_i}, \alpha_{t_i}, \omega_{t_i})$. Then there are two cases

**Case-1** - $f_{t_{i+1}}$ is a blow-down map.

On $X_{t_{i+1}}$, we take an arbitrary choice of compatible triples $(L_{t_{i+1}}, \alpha_{t_{i+1}}, \omega_{t_{i+1}})$. Then we connect this to $(X_{t_i}, L_{t_i}, \alpha_{t_i}, \omega_{t_i})$ in two steps. Fix a $\mu \in (t_i, t_{i+1})$ and ample line bundle $A$ on $X_{t_i}$.

**Step-1** For $t \in [\mu, t_{i+1}]$, set

\[
\begin{align*}
X_t &= X_{t_i} \\
L_t &= f_{t_{i+1}}^* L_{t_{i+1}} + (t_{i+1} - t)A \\
\alpha_t &< \min (R(X_t, L_t), R(X_{t_{i+1}}, L_{t_{i+1}}))
\end{align*}
\]
where we choose $\alpha_t$ to be continuous. We now let $\omega_t$ be the $\alpha_t$-conical Kähler-Einstein metric in $L_t$. Then by Proposition 4.1, $(X_t, \omega_t)$ is continuous for $t \in [\mu, t_{i+1})$ and converges to $(X_{t_{i+1}}, \omega_{t_{i+1}})$ in the Gromov-Hausdorff topology.

Step 2 - For $t \in [t_i, \mu]$ set

\[
X_t = X_{t_i}, \\
L_t = \frac{\mu - t}{\mu - t_i} L_{t_i} + \frac{t - t_i}{\mu - t_i} L_{\mu}, \\
\alpha_t < R(X_t, L_t)
\]

Again, let $\omega_t$ be the corresponding $\alpha_t$-conical Kähler-Einstein metrics. Since in this case, $L_t$ is uniformly Kähler all the estimates of Proposition 4.1 go through and we in fact get that $(X_t, \omega_t)$ is continuous in $t$ in the smooth topology on $X_{t_i}$.

**Case 2.** $f_{t_{i+1}}$ is a blow-up map. This can be treated by the same argument as in Case 1 by moving $t$ backward from $t_{i+2}$ to $t_{i+1}$.

The smoothness of $g_t$ on the complex torus $(\mathbb{C}^*)^n$ follows if we take $\alpha_t$ to be a smooth path in $t$.

\[\square\]

### 4.2. Uniform estimates and proof of proposition 4.1

In this section we prove Proposition 4.1 thereby completing the proof of Theorem 1.3. Throughout the section we fix an $\alpha > 0$, such that $\alpha \in (0, \min(R(X, L_t), R(Y, L)))$.

Without loss of generality, we can assume that the polytope $P$ that induces the toric manifold $Y$ is given by $(N - 1)$ defining functions $l_j(x) = v_j \cdot x + \lambda_j \geq 0$, $j = 1, \ldots, N - 1$. Let $P^A$ be the polytope corresponding to the ample line bundle $A$ on $X$ with $N$ defining functions $l_j^A(x) = v_j \cdot x + \lambda_j^A$ for $j = 1, \ldots, N$. The blow-up process corresponds to the $(n - 1)$-dimensional face given by $l_N$ contracting to a $k$-dimensional face given by the intersection of $(n - k)$ co-dimension one faces, say $l_1, \ldots, l_{n-k}$. We denote the divisor corresponding to $l_j$ on $X$ by $D_j$ with defining section $s_j$, while on $Y$ we denote the divisor corresponding to $l_j$ by $D_j$ and the corresponding section by $\tilde{s}_j$. Then it follows from the definition of blow-ups that

\[
\begin{aligned}
\pi^* D_j &= D_j + D_N, & j &= 1, \ldots, n - k, \\
\pi^* \tilde{D}_j &= D_j, & j &= n - k + 1, \ldots, N - 1,
\end{aligned}
\]

where as before $D_j$ denotes the divisor corresponding to $l_j$ and $\pi^*$ is the total transform. Using this fact, one can explicitly write down the polytope $P^t$ for $L_t$ by defining

\[
\begin{aligned}
l_j^t(x) &= v_j \cdot x + \lambda_j + t \lambda_j^A, & j &= 1, \ldots, N - 1, \\
l_N^t(x) &= v_N \cdot x + (\sum_{j=1}^{n-k} \lambda_j) + t \lambda_N^A,
\end{aligned}
\]

where $v_N = \sum_{j=1}^{n-k} v_j$.

We denote the barycenters of the evolving polytopes by $P^t_C$ and the corresponding angles by $\beta_j^t = \alpha l_j^t(P_C^t)$. We then set $l_j^0, P_C^0$ and $\beta_j^0$ to be the limit of the respective quantities as $t$ goes to zero. We first prove an important identity that will be very useful, among other things, in proving that the limiting Monge-Ampere equation descends to the Einstein equation on $Y$. 
Lemma 4.2.

\[(4.47) \quad (1 - \beta^0_N) - \sum_{j=1}^{n-k} (1 - \beta^0_j) = -(n - k - 1).\]

Proof. Since \(D_N\) is obtained by blowing up the intersection of \(D_1, \ldots, D_{n-k}\), it is well known that

\[v_N = \sum_{j=1}^{n-k} v_j.\]

Now at \(t = 0\), there are \((n - k + 1)\) affine linear functions \(l_1, \ldots, l_{n-k}\) and \(l_N\) vanishing on a \(k\)-dimensional face (see the figure given below). So they must be linearly related i.e there exist real numbers \(a_j\) so that

\[l^0_N = \sum_{j=1}^{n-k} a_j l^0_j.\]

But then, since \(v_j\)'s are linearly independent, the two equations together force the \(a_j\)'s to be one i.e

\[l^0_N = \sum_{j=1}^{n-k} l^0_j.\]

The lemma now follows.

\(\square\)

Example. Let \(X = \mathbb{P}^2 \# \mathbb{P}^2\) and \(Y = \mathbb{P}^2\). On \(Y\) we take \(\mathcal{L}\) to be the anti-canonical bundle and the corresponding \(P \subset \mathbb{P}^2\) to be defined by \(\{x+1 \geq 0, y+1 \geq 0, 0 \leq x-y \leq 0\}\). One can view \(X\) as a projective bundle over \(\mathbb{P}^1\) with a zero section \(D_0\) and a section \(D_\infty\) at infinity. We take \(A\) to be \(2[D_\infty] - [D_0]\), with the polytope \(P^A\) given by \(\{x \geq 0, y \geq 0, 0 \leq x-y \leq 0, -1+x+y \geq 0\}\). It follows from the Nakai criteria that \(A\) is ample. Then the polytope for \(L_t\) is given by the inequalities \(\{x+1 \geq 0, y+1 \geq 0, 1+2t - x-y \geq 0, 2-t+x+y \geq 0\}\). Computing the \(\beta^0_j\) for this example we see that \(\beta^0_1 = 1, \beta^0_2 = 1, \beta^0_3 = 1, \beta^0_4 = 2\). One can now easily verify Lemma 4.2 for this simple example.

Now let \(\omega_t\) and \(\omega_Y\) be the unique toric conical Kähler-Einstein metrics with Einstein constant \(\alpha\) on \(X\) and \(Y\) in the class \(c_1(L_t)\) and \(c_1(\mathcal{L})\) respectively. In section 3, we worked with conical reference metrics coming from the symplectic potential. However, for dealing with convergence issues as the Kähler class degenerates, it is more convenient to work with smooth reference forms. So we pick a Kähler form \(\tilde{\omega}_Y \in c_1(\mathcal{L})\) and a Kähler form \(\chi \in c_1(A)\). More explicitly, by taking the embedding of \(Y\) into a big projective space via the sections coming from the lattice points of \(P\), we can set \(\tilde{\omega}_Y\) to be the pull-back of the Fubini-Study metric. One can make a similar choice for \(\chi\). We then set \(\tilde{\omega}_t = \pi^* \tilde{\omega}_Y + t \chi\). Clearly, there exist locally bounded functions \(\psi_t\) such that \(\omega_t = \tilde{\omega}_t + \sqrt{-1} \partial \bar{\partial} \psi_t\). We similarly have a potential \(\psi_Y\) for \(\omega_Y\).

Lemma 4.3. There exists a uniform constant \(C\) independent of \(t\) such that

\[(4.48) \quad \sup_X \psi_t - \inf_X \psi_t \leq C.\]
Proof. We fix a volume form on $X$ say $\Omega = \chi^n$. Recall from section 3, that $\omega_t$ is given locally on $(\mathbb{C}^*)^n$ by $\omega_t = \sqrt{-1} T \partial \bar{\partial} \varphi_t$, where $\varphi_t$ is a function only of $\rho \in \mathbb{R}^n$ and satisfies the real Monge-Ampere

$$\det(\nabla^2 \varphi_t) = e^{-\alpha(\varphi_t - P^t \cdot \rho)} = e^{-w_t}$$

The volume for $\omega_t^n$ is given by

$$\omega_t^n = (\det \nabla^2 \varphi_t) d\rho_1 \wedge \ldots \wedge d\rho_n \wedge d\theta_1 \ldots \wedge d\theta_n$$

All the estimates in the proof of Proposition 3.2 remain uniform under small perturbations of the polytope. In particular, the estimate (3.43) holds with constants $\zeta$ and $C$ independent of $t$. That is

$$\det(\nabla^2 \varphi_t) < Ce^{-\zeta |\rho|}.$$ 

Similarly on $(\mathbb{C}^*)^n$, $\chi = \sqrt{-1} T \partial \bar{\partial} \phi$. Since $\chi$ is the pull back of the Fubini-Study metric, one can take

$$\phi = \log \left( \sum_{\nu \in \mathcal{A} \cap \mathbb{Z}^n} e^{\nu \cdot \rho} \right).$$

By an elementary calculation, there exist constant $B_1, B_2, B_3, B_4$ depending only on $\mathcal{A}$ such that

$$B_3 e^{-B_1 |\rho|} < \det(\nabla^2 \phi) < B_4 e^{-B_2 |\rho|}.$$ 

Now, consider the trivial identity

$$(4.49) (\tilde{\omega} + \sqrt{-1} T \partial \bar{\partial} \psi_t)^n = \omega_t^n = \frac{\omega_t^n}{\Omega}.$$

We claim that $\omega_t^n / \Omega$ is in $L^{1+\epsilon}(X, \Omega)$ for some $\epsilon > 0$. This is because

$$\int_X \left( \frac{\omega_t^n}{\Omega} \right)^{1+\epsilon} \Omega \leq C \int_{\mathbb{R}^n} \left( \frac{e^{-\zeta |\rho|}}{e^{-B_1 |\rho|}} \right)^\epsilon e^{-B_2 |\rho|} d\rho$$

if $\epsilon$ is small enough. Then in lieu of (4.49), since we have a uniform $L^{1+\epsilon}(X, \Omega)$ bound on $\omega_t^n / \Omega$, applying [26, 18, 54] we directly obtain a uniform bound on the oscillation of $\psi_t$.

We now spend some time in deriving a complex Monge-Ampere equation satisfied by $\psi_t$. Let $\Omega$ and $\Omega_Y$ be two fixed volume forms on $X$ and $Y$ respectively and let $\xi_Y, \xi_A$ be metrics on $\mathcal{J}$ and $A$ such that $\omega_Y = -\sqrt{-1} T \partial \bar{\partial} \log \xi_Y$ and $\chi = -\sqrt{-1} T \partial \bar{\partial} \log \xi_A$. One can also view $\Omega$ and $\Omega_Y$ as metrics on $-K_X$ and $-K_Y$. We recall the adjunction formula

$$K_X = \pi^* K_Y + (n-k-1)[D_N].$$

By the $\partial \bar{\partial}$-lemma there exists a metric $h_N$ on $[D_N]$ such that

$$(4.50) \Omega = \frac{\pi^* \Omega_Y}{|s_N|^{2(n-k-1)}}.$$
Next, since \(-K_Y = \alpha L + \tilde{D}\), one can choose smooth hermitian metrics \(\tilde{h}_1, \ldots, \tilde{h}_{N-1}\) on \(\tilde{D}_1, \ldots, \tilde{D}_{N-1}\) such that

\[
(4.51) \quad \prod_{j=1}^{N-1} \tilde{h}_j^{(1-\beta_j^Y)} = \pi^* \left( \frac{\Omega_Y}{(\xi_Y)^\alpha} \right).
\]

Using (4.46) we then define smooth metrics on \(\tilde{D}_j\) for \(j < N\) by setting

\[
(4.52) \quad \begin{cases} 
  h_j = \pi^* \tilde{h}_j/h_N & j = 1, \ldots, n-k \\
  h_j = \pi^* \tilde{h}_j & j = n-k+1, \ldots, N-1.
\end{cases}
\]

Finally we define a family of metrics on \([D_N]\) by

\[
(4.53) \quad h_N(t) = \left( \frac{\Omega \xi_A^{-\alpha} \pi^* (\xi_Y^{-\alpha})}{\prod_{j=1}^{N-1} h_j^{(1-\beta_j^Y)}} \right)^{1/(1-\beta_N^Y)}.
\]

We claim

**Lemma 4.4.** At all points of \(X\),

\[
(4.54) \quad \lim_{t \to 0} \frac{h_N(t)}{h_N} = 1.
\]

**Proof.** If we consider \(\Omega\) as a metric on \(-K_X\) and \(\pi^* \Omega_Y\) as a pull back metric on \(-\pi^* K_Y\), then by equation (4.50), \(\Omega = \pi^* \Omega_Y h_N^{-(n-k-1)}\).

\[
\begin{align*}
\lim_{t \to 0} \frac{h_N(t)}{h_N} &= \left( \frac{\Omega \pi^* (\xi_Y)^{-\alpha}}{h_N^{(1-\beta_N^Y)} \prod_{j=1}^{N-1} h_j^{(1-\beta_j^Y)}} \right)^{1/(1-\beta_N^Y)} \\
&= \left( \frac{\pi^* (\Omega_Y \xi_Y^{-\alpha}) h_N^{-(n-k-1)}}{h_N^{(1-\beta_N^Y)} \prod_{j=1}^{N-1} h_j^{(1-\beta_j^Y)}} \right)^{1/(1-\beta_N^Y)} \\
&= \left( \frac{\pi^* (\Omega_Y \xi_Y^{-\alpha})}{\prod_{j=1}^{N-1} \pi^* h_j^{(1-\beta_j^Y)}} \right)^{1/(1-\beta_N^Y)} \\
&= 1,
\end{align*}
\]

where we used lemma 4.47, equation (4.52) in line three and equation (4.51) in line four.

\[
\square
\]

By applying logarithm and taking \(\partial \bar{\partial}\) we see that \(h_1, \ldots, h_{N-1}\) and \(h_N(t)\) satisfy

\[-\partial \bar{\partial} \log \Omega = \alpha \tilde{\omega}_t - \sum_{j=1}^{N-1} (1 - \beta_j^Y) \partial \bar{\partial} \log h_j - (1 - \beta_N^Y) \partial \bar{\partial} \log h_N(t).\]

The purpose of the above constructions is that \(\psi_t\) and \(\psi_Y\) now satisfy, possibly after modification by some constant, the following Monge-Ampere equations
Combining this with the above estimate, also, there exists a constant \( C \) such that
\[
\sum_{j=1}^{N-1} |s_j|^{-2(1-\beta_j^t)}.
\]
\[
(\omega_t + \sqrt{-1} \partial \bar{\partial} \psi_t)^n = \frac{e^{-\alpha \psi_t} \Omega}{|s_N|^{2(1-\beta_N^t)} \prod_{j=1}^{N-1} |s_j|^{-2(1-\beta_j^t)}}.
\]

We remark that since modification by a constant doesn’t change the oscillation, the estimate of Lemma 4.48 holds for this modified \( \psi_t \). Immediately, we have the following corollary from Lemma 4.3, because the total volume of \((X, \omega_t)\) is \([L_t]^n\) and is uniformly bounded.

**Corollary 4.1.** There exists a unique constant \( C > 0 \) such that for all \( t \in (0, 1) \),
\[
\|\psi_t\|_{L^\infty(X)} \leq C.
\]

We next prove uniform estimates away from \( D \) on all derivatives of \( \psi_t \).

**Proposition 4.2.** For all \( l \geq 0 \) and \( K \subset \subset X \setminus D \), there exist constants \( C_{K,l} \) independent of \( t \) such that
\[
\|\psi_t\|_{C^l(K)} < C_{K,l}.
\]

Here the norm is with respect to some fixed reference metric.

**Proof.** Since the usual arguments for \( C^{2,\gamma} \) estimates, using the theory of Evans, Krylov and Safanov, are local in nature they can be used in the present context. Hence, to prove the proposition, we only need uniform \( C^2 \) estimates.

We follow the argument in [37] using Tsuji’s trick [49]. By Kodaira’s lemma, \( L_\epsilon = \pi^* \omega - \epsilon [D_N] > 0 \) for small \( \epsilon > 0 \). So, we pick a new smooth hermitian metric \( \xi_N \) on \([D_N]\) such that
\[
\eta = \pi^* \omega_Y + \epsilon \partial \bar{\partial} \log \xi_N > 0
\]
and consider
\[
Q_t = \log \left( |s_N|^{2(1+\epsilon)} \prod_{j=1}^{N-1} |s_j|^{-2} \right) + A \psi_t
\]
for some big constant \( A \) to be chosen later. We note that \( Q \) goes to negative infinity near \( D \). This is because the order of poles of \( \omega_t \) near each \( D_j \) is \( 2(1-\beta_j^t) \) which is strictly less than two. So for each \( t \), the maximum is attained in \( X \setminus D \). Following Yau, we compute \( \Delta_t Q_t \) where \( \Delta_t \) is the Laplacian with respect to \( \omega_t \). Since on \( X \setminus D \), \( Ric(\omega_t) = \alpha \omega_t \), standard calculations show that there exists a constant \( C \) depending only on the dimension and curvature of \( \eta \) such that
\[
\Delta_t Q_t \geq -C \text{tr}_{\omega_t} \eta + (1 + 4\epsilon) \Delta_t \log \xi_N + \sum_{j=1}^{N-1} \Delta_t \log h_j + A \text{tr}_{\omega_t} \omega_t - C.
\]
Also, there exists a constant \( C' \) independent of \( t \) such that
\[
\Delta_t \log \xi_N > -C' \text{tr}_{\omega_t} \eta,
\]
\[
\Delta_t \log h_j > -C' \text{tr}_{\omega_t} \eta.
\]
Combining this with the above estimate
\[
\Delta_t Q_t \geq -C \text{tr}_{\omega_t} \eta + A \text{tr}_{\omega_t} (\omega_t + \epsilon \partial \bar{\partial} \log \xi_N) - C
\]
\[
= -C \text{tr}_{\omega_t} \eta + A \text{tr}_{\omega_t} (\eta + t\chi) - C
\]
\[
> \text{tr}_{\omega_t} \eta - C,
\]
where we choose $A = C + 1$. So, at the maximum point of $Q_t$, $tr_{\omega_t} \eta(p_t) < C$. Now, standard arguments show
\[
\left( s^N_j \xi_N \frac{2}{|N|} \prod_{j=1}^{N-1} |s_j|^2 \right) tr_{\eta} \omega_t < CE^{\sup \psi_t - \inf \psi_t} \left( s^N_j \xi_N \frac{2}{|N|} \prod_{j=1}^{N-1} |s_j|^2 \frac{\omega_t}{\eta} \right)(p_t).
\]
Using the equation and the oscillation bound on $\psi_t$ and the fact that $\beta^j_t < 1$,
\[
tr_{\eta} \omega_t < \frac{C}{\left( s^N_j \xi_N \frac{2}{|N|} \prod_{j=1}^{N-1} |s_j|^2 \right)}.
\]
Hence, we have uniform second order estimates away from $D$ and this completes the proof of the proposition.

With the above uniform local estimates and the uniqueness of $\omega_Y$, we have the following local uniform convergence away from the divisors.

**Proposition 4.3.** For any compact subset $K \subset X / D$, we have the following uniform convergence
\[
\omega_t \xrightarrow{\mathcal{C}^\infty(K)} \omega_Y.
\]

Using Moser iteration, one can in fact show that $\psi_t$ converges to $\pi^* \psi_Y$ globally in $L^\infty$.

**Proof of Proposition 4.1.** We let $t \to 0$. Fix an $\epsilon > 0$. Let $E$ be a tubular neighborhood of $D \subset Y$ such that $A = Y \setminus E$ is $\epsilon$-dense in $Y$ with respect to the metric $g_Y$. Note, that since $X$ and $Y$ are bi-holomorphic away from $D$, $A$ can be identified as a subset of $X$. We also pick $E$ close enough to $D$ so that $Vol_{g_Y}(E) < \epsilon^4 n$ and we set $\tilde{E} = \pi^*(E)$. Finally, we denote the distances with respect to $g_t$ and $g_Y$, by $d_t$ and $d_Y$ respectively.

**Claim 1.** For $t$ small enough, $A = Y \setminus E = X \setminus \tilde{E}$ is $\epsilon$- dense in $(X, g_t)$. If not, then there exists a sequence $t_k \to 0$ and points $x_k \in \tilde{E}$ such that $B_{g_{t_k}}(x_k, \epsilon) \subset \tilde{E}$. Using volume comparison, uniform diameter bounds and the fact that the volumes converge, for small $t_k$
\[
k \epsilon^{2n} < Vol_{g_k}(B_{g_k}(x_k, \epsilon)) < Vol_{g_k}(\tilde{E}) < 2Vol_{g_Y}(E) < 2\epsilon^{4n}
\]
for some constant $k$ if $k$ is sufficiently large. But if $\epsilon$ is small enough, this is a contradiction.

**Claim 2.** There exists a $t(\epsilon)$ such that for all $0 < t < t(\epsilon)$ and for all $p, q \in A$,
\[
d_t(p, q) < d_Y(p, q) + \epsilon.
\]

**Proof.** By the geodesic convexity of $Y \setminus D$, one can choose a small tubular neighborhood, $T \subset E$ of $D$ in $Y$ such that any two points in $A$ can be connected by a $g_Y$- minimal geodesic in $Y \setminus T$. Set $\tilde{T} = \pi^{-1}(T)$. Let $\gamma$ be a $g_Y$- minimal geodesic connecting $p$ and $q$ lying in $Y \setminus T$. Since the metrics converge uniformly on compact sets of $X \setminus \tilde{E}$, for $t$ sufficiently small,
\[
d_t(p, q) < \mathcal{L}_t(\gamma) < \mathcal{L}_Y(\gamma) + \epsilon = d_Y(p, q) + \epsilon,
\]
where $\mathcal{L}$ denotes the length functional.
Claim 3. There exists a \( t(\epsilon) \) such that for all \( 0 < t < t(\epsilon) \) and for all \( p, q \in A \),
\[
d_t(p, q) > d_Y(p, q) - \epsilon.
\]

Proof. The proof of this claim relies on the generalization of Gromov’s lemma to the conical setting (Lemma 2.6). We once again choose a tubular neighborhood \( T \) of \( D \) contained in \( E \) with smooth boundary such that for all \( q \in A \),
\[
B_{g_Y}(q, \epsilon/2) \subset Y \setminus T
\]
\[
\text{Vol}_{g_Y}(\partial T) < \delta/2
\]
and set \( \tilde{T} = \pi^*(T) \). \( D \) being of real co-dimension two, one can choose \( \delta \) to be as small as needed. Since, away from \( D \), the metric converges uniformly, we can assume that \( \text{Vol}_{g_t}(\partial T) < \delta \) by choosing \( t \) sufficiently small. Furthermore, since \( d_{g_Y}(q, \partial T) > \epsilon/2 \), once again by the uniform convergence of the metric on \( X \setminus \tilde{T} \), for small \( t \),
\[
d_{g_t}(q, \partial T) > \epsilon/4,
\]
\[
B_{g_t}(q, \epsilon/4) \subset X \setminus \tilde{T}.
\]
We claim that there exists at least one minimal geodesic from \( p \) to a point \( \tilde{q} \in B_{g_t}(q, \epsilon/4) \) that lies entirely in \( X \setminus \tilde{T} \). If not, then by Gromov’s lemma there exists a constant \( c \) uniform in \( t \) (but depending on \( \epsilon \)) such that
\[
\kappa \epsilon^n < \text{Vol}_{g_t}(B_{g_t}(q, \epsilon/4)) < c \text{Vol}_{g_t}(\partial \tilde{T}) < c \delta.
\]
Letting \( \delta \) go to zero, we get a contradiction.

So there exists at least one \( g_t \)-minimal geodesic \( \gamma_t \) connecting \( p \) to a point \( \tilde{q} \in B_{g_t}(q, \epsilon/4) \). By compactness, there exists a \( \tilde{q} \in B_{g_Y}(q, \epsilon/2) \) such that \( \gamma_t \rightarrow \tilde{q} \) and moreover the geodesics \( \gamma_t \) converge to a curve, denoted by \( \gamma \), joining \( p \) to \( \tilde{q} \).
\[
d_{g_t}(p, q) > L_{g_t}(\gamma) - \epsilon/4
\]
\[
> L_{g_Y}(\gamma) - \epsilon/2
\]
\[
> d_{g_Y}(p, \tilde{q}) - \epsilon/2
\]
\[
> d_{g_Y}(p, q) - \epsilon
\]
and this proves Claim 3.

Now we complete the proof of the proposition. For sufficiently small \( t \),
\[
d_{GH}((X, d_t), (Y, d_Y))
\]
\[
\leq d_{GH}((X, d_t), (A, d_t)) + d_{GH}((A, d_t), (A, d_Y)) + d_{GH}((A, d_Y), (Y, d_Y))
\]
\[
< 3\epsilon,
\]
where we use Claim 1 to bound the first term, Claim 2 and Claim 3 to bound the second term, while the last term is bound by \( \epsilon \) from the choice of \( A \). Now, letting \( \epsilon \) go to zero, we see that \( (X, g_t) \) converges in Gromov-Hausdorff distance to \( (Y, g_Y) \). \( \square \)

5. Discussions

In this section, we propose some questions in relation to our main results. In Theorem 1.1, given the toric pair \( (X, L) \), the smooth toric conical Kähler-Ricci soliton equation can in general be solved for \( \alpha \in (0, \mathcal{S}(X, L)) \), while the smooth toric conical Kähler-Einstein equation can be solved for \( \alpha \in (0, \mathcal{R}(X, L)) \). The following example illustrates when the soliton equation can be solved at \( \mathcal{S}(X, L) \).
Example 5.1. Let $X = \mathbb{P}^2 \# \mathbb{P}^2$, i.e., $\mathbb{P}^2$ blow-up at one point, given by a polytope $P$ defined by $l_0 = x + y + \varepsilon > 0$, $l_1 = y + 1 + \varepsilon > 0$, $l_2 = x + 1 > 0$, and $l_\infty = -x - y + \varepsilon > 0$, where $\varepsilon > 0$ is a small constant. Let $L$ be the Kaehler class induced by $P$. The corresponding ample line bundle $L$ over $X$ is determined by the divisor $D_1 + D_2 + \varepsilon D_\infty + \varepsilon D_0$ (noting that the bundle being ample is equivalent to $\varepsilon \in (0, 2)$). By Theorem 1.1, $\frac{1}{\varepsilon} S(X, L)$ can be characterized as

\begin{equation}
\inf_{(x,y) \in P} \max \{x + 1, y + 1, x + y + \varepsilon, -x - y + \varepsilon\}.
\end{equation}

By the symmetry of $x, y \in P$, the extremal point must be at the line $y = x = 0$, hence the aimed function is reduced to

\begin{equation}
\inf_{x \in (-\varepsilon/2, \varepsilon/2)} \max \{x + 1, 2x + \varepsilon, -2x + \varepsilon\}.
\end{equation}

We have three cases: $\varepsilon \in (0, \frac{2}{3}), (\frac{2}{3}, 1), [1, 2)$.

(1) When $\varepsilon \in (0, \frac{2}{3})$, the unique extremal point of (5.58) is at $x = -\frac{2}{3}$, or, that of (5.57) is at $(-\frac{2}{3}, \frac{2}{3})$, which is at the boundary of $P$, hence the conical Kaehler-Ricci soliton equation cannot be solved at $S(X, L)$ for this case.

(2) When $\varepsilon \in (\frac{2}{3}, 1)$, the unique extremal point of (5.58) is at $x = -\frac{1-\varepsilon}{3}$, and the point $(-\frac{1-\varepsilon}{3}, -\frac{1-\varepsilon}{3})$ is in the interior of $P$, hence by our proof above, the conical Kaehler-Ricci soliton equation can be solved at $S(X, L)$ in this case.

(3) When $\varepsilon \in [1, 2)$, the unique extremal point of (5.58) is at $x = 0$, and the origin $(0, 0)$ is always in $P$, so in this case, the conical Kaehler-Ricci soliton equation can also be solved up to $S(X, L)$.

It is a natural question to ask in the above example, when $\varepsilon \in (0, 2/5]$, if there exists a limiting space for the Kaehler-Ricci solitons as $\alpha$ tends to $S(X, L)$. Since the Futaki invariant blows up as $\alpha$ tends to $S(X, L)$, most likely the limiting space will be a complete shrinking or steady soliton with an complete end at the exceptional divisor after some appropriate scaling.

We would also like to mention that since for $\alpha \in (0, \mathcal{R}_{BE})$, the toric conical Kaehler-Ricci soliton metric in Theorem 1.1 is not unique and in fact, there are infinitely many of them. Let $\mathcal{K}(X, L, \alpha)$ be the space of Kaehler-Ricci soliton metrics $\omega \in L \cap \mathcal{K}_c(X)$ with $\text{Ric}(\omega) = \alpha \omega + L_\xi \omega + |D|$, if $\alpha \in (0, \mathcal{R}_{BE}(X, L))$. Then we define

$$
F_{\text{min}}(X, L, \alpha) = \inf \{Fut(X, \omega) \mid \omega \in \mathcal{K}(X, L, \alpha)\},
$$

where $Fut(X, \omega)$ is the Futaki invariant for the holomorphic vector field $\xi$ and $Fut(X, \omega) = \int_X |\xi|^2 \omega^n$. After some calculations, one can show that $F_{\text{min}}(X, L, \alpha)$ is achieved for some smooth toric Kaehler-Ricci soliton metric in $L \cap \mathcal{K}_c(X)$. We call such a conical Kaehler-Ricci soliton the minimal Kaehler-Ricci soliton with respect to $(X, L, \alpha)$.

Finally we propose the following regression scheme to obtain models for any log Fano variety $(X, D)$ by maximizing the Ricci curvature and Bakry-Emery-Ricci curvature.

We first consider the Fano case.

(1) Let $X$ be a Fano manifold and $D$ be a smooth simple divisor in $| - mK_X|$ for some $m \in \mathbb{Z}^+$. Let $T = \sup \{t \mid \text{Ric}(\omega) = t\omega + (1-t)|D|/m \}$ for a conical Kähler metric $\omega$ and $\omega_t$ be the unique Kähler-Einstein metric solving $\text{Ric}(\omega_t) = t\omega_t + (1-t)|D|/m$ for $t \in (0, T)$. 
(1.2) Then \((X, D/m, \omega_I)\) converges in Gromov-Hausdorff sense to a compact metric space \((X', D', \omega')\) homeomorphic to a Fano variety \(X'\) with log terminal singularities. \(D'\) is an effective \(\mathbb{R}\)-divisor in \(-K_{X'}\) and \(\omega'\) is a Kähler-Einstein metric solving 
\[
Ric(\omega') = T\omega' + (1 - T)[D'],
\]
in particular, \(\omega'\) is smooth outside \(D'\) and the singularities of \(X'\).

(1.3) We replace \((X, D/m)\) by \((X', D')\) and go back to (1) and continue the procedure.

(1.4) The regression terminates in finite steps for finitely many data \(\{(X_i, D_i), T_i\}\), \(i = 0, ..., I\), with \((X_0, D_0) = (X, D/m)\) and \(T_0 = 0\).

(1.5) \((X_I, D_I, \omega_I)\) is a \(\mathbb{Q}\)-Fano conical Kähler-Einstein space with Ricci curvature equal to \(T_I = R(X, D) \leq R(X)\). We cannot increase \(T_I\) by repeating the above process. \(R(X, D) = R(X)\) for a generic smooth divisor \(D\) and \(I = 1\). If \(R(X) = 1\), we stop with \(D_I\) being 0, otherwise we continue with the next step.

(2) Let \((X_{KE}, D_{KE}) = (X_I, D_I)\).

(2.1) Let \(T = \sup\{t \mid Ric(\omega) = t\omega + L_\xi\omega + (1 - t)[D]\}\) for a conical Kähler metric \(\omega\) and \((\omega_I, \xi_I)\) be the minimal Kähler-Ricci soliton metric solving 
\[
Ric(\omega_I) = t\omega_I + L_{\xi_I}\omega_I + (1 - t)[D_I]
\]
for \(t \in [R(X), T]\). The effective divisor \(D_t\) depends on \(t\).

(2.2) Then \((X_{KE}, D_t, \omega_I, \xi_I)\) converges in Gromov-Hausdorff sense to a compact metric space \((X', D', \omega', \xi')\) homeomorphic to a Fano variety \(X'\) with log terminal singularities. \(D'\) is an effective \(\mathbb{R}\)-divisor in \(-K_{X'}\) and \(\omega'\) is a conical Kähler-Ricci soliton metric solving 
\[
Ric(\omega') = T\omega' + L_{\xi'}\omega' + (1 - T)[D'],
\]
in particular, \(\omega'\) is smooth outside \(D'\) and the singularities of \(X'\).

(2.3) We replace \((X_{KE}, D_{KE})\) by \((X', D')\) and go back to (2) and continue the procedure.

(2.4) The regression terminates in finite steps for finitely many data \(\{(X_j, D_j), T_j\}\) for \(j = 1, ..., J\) with \((X_I, D_I) = (X_{KE}, D_{KE})\).

(2.5) \(T_J = R_{BE}(X, D_I) \leq R_{BE}(X) = 1\). For generic \(D_0, D_J = 0\) and \((X_{KR}, \omega_{KR}) = (X_J, \omega_I)\) is a \(\mathbb{Q}\)-Fano Kähler-Ricci soliton space solving \(Ric(\omega_{KR}) = \omega_{KR} + L_{\xi_{KR}}\omega_{KR}\).

(3) We call \((X_{KE}, \omega_{KE})\) when \(R(X) = 1\) and \((X_{KR}, \omega_{KR})\) when \(R(X) < 1\) the maximal model of \(X\) in the sense that it maximizes Bakry-Emery-Ricci curvature. It should be unique and does not depend on the choice of the initial divisor \(D\), while the intermediate model \((X_{KE}, D_{KE})\) is not necessarily unique and it might depend on \(D\), if \(R(X) < 1\).

If the above speculation holds, we also conjecture that the maximal model of a Fano manifold \(X\) should coincide with the limit of the Fano Kähler-Ricci flow on \(X\), generalizing \([36]\). The above scheme can be generalized to any log Fano variety \(\{X, D\}\) with some modifications if the polarization \(-K_X - D\) satisfies log \(K\)-stability for the log pair \((X, -(1 - \epsilon)K_X + \epsilon D)\) with sufficiently small \(\epsilon > 0\). This will give a unique maximal model for any log Fano variety if \(R_{BE}(X, -K_X - D)\) is achieved. In this case, the divisor where the conical singularities occur
changes continuously $t$ increases so that the deforming soliton metrics are always minimizing the Futaki invariant for each $t$. The minimal model program deforms projective varieties with positive Kodaira dimension to their minimal model by birational tranformations, while in the Fano case, our regression scheme looks for a model of Fano manifold by deformations of complex structures which should be also identified as certain algebraic operations in relation to log K-stability. A good model to test the above scheme is the example in [40].

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