Glueball Superfield and Argyres Douglas Points

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Abstract

In this paper, we study $\mathcal{N} = 1$ super-symmetric $SO(N)$ gauge theory in Argyres-Douglas points by using the factorization equation of the $\mathcal{N} = 2$ theory. We suppose that all monopoles become massive in the system and obtain a tree level superpotential. Then, we obtain general Picard-fuchs equations for glueball superfields which are hypergeometric equations having regular singular points corresponding to Argyres-Douglas points. Furthermore, we study the solutions of these differential equations and calculate the effective superpotential. Finally, we study scaling behavior of the chiral operators and coupling constants around the AD points.
1 Introduction

Recently, Dijkraaf and Vafa \cite{1, 2, 3} have understood the quantum dynamics of a wide class of $\mathcal{N} = 1$ supersymmetric gauge theory by studying an auxiliary matrix model where they conjectured that the effective superpotentials in $\mathcal{N} = 1$ SQCD’s theories have a correspondence with the free energy of matrix model. The nonperturbative effects of gauge theory can be also obtained from the planar diagrams of matrix model. At first, these correspondence had been proved by considering the geometric transition and topological string theory \cite{4, 5, 6} and Cachazo, Douglas, Seiberg and Witten’s efforts were caused them to be able to prove this correspondence by the field theory concepts without using string theory \cite{7, 8, 9, 10}. In fact, they applied a technology which was based on the anomalous Ward identity of generalized Konishi anomaly \cite{11, 12}. Moreover, they used the factorization equation \cite{5, 6} which relates the $\mathcal{N} = 2$ Seiberg-Witten curves \cite{13, 14} to reduced $\mathcal{N} = 1$ curve in the presence of massless monopoles. Given the tree-level superpotential $W'(x)$ the factorization equation completely determines all the parameters of polynomials which exist in the factorization equation and so the vacuum structure of theory can be identified. Furthermore, one may calculate the monopoles condensate which show at generic points in parameter space this quantity can be non zero and generates mass gap and confinement in the system. However, Using this geometric picture, when all monopoles become massive in the system, Physics is the same as $\mathcal{N} = 1$ theory and One may derive a system of ordinary differential equations, Picard-fuchs equations, for glueball superfields and calculates the exact effective superpotential.

One can combine this method with matrix model and study the non-perturbative effects of $\mathcal{N} = 1$ theory, as well.

Furthermore, these methods give us ability to study other interesting phenomena such as conformal behavior of supersymmetric gauge theory in points called Argyres – Douglas points \cite{19, 13, 20, 21}. In fact, one can break the $\mathcal{N} = 2$ supersymmetry with a perturbed superpotential but in points where monopoles become massless, one can reproduce $\mathcal{N} = 1$ supersymmetry. Following this method, one can find where reduced $\mathcal{N} = 1$ theory has conformal invariance is this point in \cite{19, 22, 23, 24}.

Even more interestingly it had been noted in \cite{15, 16} that for an $\mathcal{N} = 1$ gauge theory with cubic superpotential when the gauge group is unbroken, there are critical values of superpotential couplings where the effective superpotential is non-analytic and so the large $N$ expansion of such an effective superpotential is singular too. But these singularities can be removed by the double scaling limit. Furthermore, these double scaling limits define an $\mathcal{N} = 1$ four dimensional non-critical string theory \cite{17, 18}.

These proposal along with the machinery that was introduced in \cite{7, 8, 9, 10}, was performed in \cite{25} and \cite{26} for $U(N)$ gauge group. Now, we would like to have a survey of this proposal for other classical gauge groups $SO(2N)$ and $SO(2N + 1)$. Therefore, at absence of massless monopoles, we try to find the general Picard-Fuchs
equations for glueball superfield without flavors then, we’ll solve these equations. These differential equations are the Picard-Fuchs equations for the periods of the *memomorphobic* one-form on the spectral curve. The superpotential is also obtained and the conformal behavior of theory will be studied by taking the IR limit respectively.

This paper is organized as follows. In section 2 we review the CDSW’s method for calculating the chiral operators and study the $\mathcal{N} = 1$ U(N) gauge theory around the AD points. In section 3 we study the factorization equation of $SO(N)$ gauge group and then we focus on AD points and obtain $\mathcal{N} = 1$ effective superpotential in terms of chiral superfield $\Phi$.

In section 4 we find the Picard-Fuchs equations for glueball superfields for gauge groups U(N) and SO(N) which are in general hypergeometric equations. Then we find the solutions of these equations for $SO(2N)$ and $SO(2N + 1)$ respectively. In the last section we obtain the superpotential of $SO(N)$ theory and study the scaling behavior of chiral fields around the AD points.

## 2 Review of U(N) theory

The dynamic of $\mathcal{N} = 1$ $U(N)$ gauge theories with superpotential $\mathcal{W}(\Phi)$ can be studied as a perturbation of the $\mathcal{N} = 2$ strongly coupled gauge theory with $\mathcal{W} = 0$. The low energy group is $U(1)^n$ and $N - n$ monopoles of the $\mathcal{N} = 2$ theory are massless. The Seiberg-Witten curve has the following factorization at this points 

$$y^2 = P_N^2(x) - 4\Lambda^{2N} = F_{2n}H_{N-n}(x),$$

where the polynomyal in the r.h.s has simple roots. In [5] Cachazo, Intriligator and Vafa showed that

$$F_{2n}(x) = \frac{1}{g_n^2} W'(x)^2 + f_{n-1}(x),$$

where $W(x)$ is the superpotential for the reduced $\mathcal{N} = 1$ theory and is a polynomial of degree $n$. From this factorization, the gauge group $U(N)$ breaks to $U(1)^n$ and so $N - n$ monopoles become massless. The CDSW’s method for the calculation of chiral operators is as below [7, 8, 9, 10].

$$T(x) = \langle Tr \frac{1}{x - \Phi} \rangle,$$

$$R(x) = -\frac{1}{32\pi^2} \langle Tr \frac{W}{x - \Phi} \rangle,$$

where $W^\alpha$ is the field strength chiral field. In terms of SW curve

$$T(x) = \frac{P_N}{y_{N-2}(x)},$$

$$R(x) = \frac{1}{2}(W'(x) - y_{N-1}(x)).$$

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The expectation value of chiral fields are also given by $U_r$ and $S_r$ where

$$U_r \equiv \langle Tr \Phi^r \rangle = \oint x^r T(x) dx,$$

$$S_r \equiv \langle Tr \Phi^r W_\alpha W^\alpha \rangle = \oint x^r T(x) dx. \quad (7)$$

Finally, when $U(N)$ theory has been broken to $\Pi^u_{k=1} U(N_k)$, the effective superpotential is

$$W_{eff}(S) = \sum_{k=1}^{n} N_k \frac{\partial F}{\partial S_k} + 2\pi i \tau_0 \sum_{k=1}^{n} S_k + 2\pi i \sum_{k=2}^{n} b_k S_k,$$

where $F$ is prepotential. Glueball superfields and prepotential are generally given by

$$S_k = \frac{1}{2\pi i} \oint_{A_k} y d x, \quad \frac{\partial F}{\partial S_k} = \oint_{B_k} y dx, \quad (10)$$

and

$$N_i = \frac{1}{2\pi i} \oint_{A_k} T(x) dx,$$

$$\tau_0 = \frac{1}{2\pi i} \oint_{B_1} T, \quad b_k = -\frac{1}{2\pi i} \oint_{B_k} T - \tau_0 l h 8, \quad (11)$$

where the $A_k$’s are the closed circles around the branch cuts of the spectral curve and the $B_k$’s are the non-compact cycles connecting the points at infinity on the two sheets of the spectral curve passing through the $A_k$—th branch cut. The intersection pairs of these cycles are

$$A_i \cap A_j = B_i \cap B_j = 0, \quad A_i \cap B_j = \delta_{ij}. \quad (13)$$

Let us introduce $C_k$ cycles for the future use

$$C_k = B_{k+1} - B_k, \quad k = 1, \ldots, n - 1, \quad (14)$$

We also introduce a small cycle $A_0$ around the origin $x = 0$.

Now, we recall that Argyres-Douglas points [19, 13, 20, 21] occur where the $\mathcal{N} = 2$ gauge theory exhibits the $\mathcal{N} = 2$ superconformal symmetry [19, 22, 23, 24]. In fact, the vanishing cycles which have non-trivial intersections, imply that the low-energy $\mathcal{N} = 2$ theory has massless solitons with both electrical and magnetical charge under the same $U(1)$ factor [19]. These points in moduli space correspond to higher order singularities and are simply obtained by adjusting the moduli parameters of the characteristic polynomial $P_N(x)$. For example consider the following Seiberg-Witten curve for $SU(N)$

$$y^2 = P_N(x)^2 - 4\Lambda^{2N} = (x^N - u)^2 - 4\Lambda^{2N},$$

3
\[
P_N(x) = \text{Det}(x.I - \Phi(x)) = x^N - \sum_{i=0}^{N-1} u_i x^i.
\]  
(15)

Then AD points are the zeros of discriminants of this curve which are
\[
s_i = 0, \quad (i = 1, \ldots, N - 1), \quad s_0 = \pm 2\Lambda^N.
\]  
(16)

This curve has a \(Z_N\) symmetry where
\[
(x, y) \rightarrow (e^{\frac{2\pi i}{N}} x, y).
\]  
(17)

Being broken of gauge group from \(U(N)\) into \(U(1)^N\), dose in the presence of this superpotential
\[
\mathcal{W}(\Phi) = g_N(\frac{1}{N+1}\phi^{N+1} - u\phi), \quad N \geq 3.
\]  
(18)

Now, one can find the Picard-Fuchs equation as \([26]\)
\[
\left[\partial_u^2 + \left(\frac{N-2}{N}\right) \frac{1}{u^2 - 4\Lambda^2 N} \partial_u - \left(\frac{N^2 - 1}{N^2}\right) \frac{1}{u^2 - 4\Lambda^2 N}\right] S_k = 0,
\]  
(19)

that are a hypergeometric equations. According to these consequences
\[
\mathcal{W}_{\text{eff}}(S) = 2\pi i \sum_{k=1}^{N} b_k S_k;
\]  
(20)

and the effective superpotential of the \(U(N)\) theory is \([26]\)
\[
\mathcal{W}_{\text{eff}}(s) = -2\pi i \frac{N}{e^{2\pi i /N} - 1} S_1(u, \Lambda^{2N}),
\]  
(21)

where
\[
S_1(u, \Lambda^{2N}) = -\frac{2\Lambda^{2N}}{N} e^{2\pi i /N} u^{-1+\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2N}, 1 - \frac{1}{2N}, 2, \frac{4\Lambda^{2N}}{u^2}\right).
\]  
(22)

and the function \(F\) will be defined in section 4. As it would be shown in \([26]\), this effective superpotential has a non trivial behavior in large \(N\) limit but it can be removed by using Double-scaling limit.

Besides, in \([25]\), the scaling behavior of this theory in the IR limit where \(\Lambda \to \infty\) was studied and the scaling dimensions of chiral operators were obtained
\[
U_r = 0, \quad S_r = 0, \quad \text{for all } r.
\]  
(23)

These are consistent with scaling invariance. In addition, if we consider small perturbation around the AD points as
\[
\mathcal{W}'(x) = P_N(x) = x^N - 2\Lambda^N - \sum_{m=0}^{N-1} g_m x^m \Lambda^{N-m},
\]  
(24)
then the scaling dimension for chiral operators and coupling constants are
\[
\Delta(g_m) = \frac{2(N - m)}{N + 2}, \quad \Delta(U_r) = \frac{N + 2r}{N + 2}, \quad \Delta(S_r) = \frac{N + 2(r + 1)}{N + 2}. \tag{25}
\]
It is noticeable that
\[
\Delta(g_m) \leq 1 \iff m \geq \frac{N}{2} - 1 \text{ (even } N) \text{ or } m \geq \left\lceil \frac{N}{2} \right\rceil \text{ (odd } N), \tag{26}
\]
that corresponds to a coupling constant in the $N = 2$ superconformal field theory in 4 dimension \[19, 22, 23, 24\].

### 3 The factorization and spectral curve of $SO(N)$

In this section, we use the factorization of Seiberg-Witten curve to obtain $N = 1$ spectral curve for gauge group $SO(N)$.

At first, we demonstrate for this gauge group
\[
y^2 = P^2_{2N}(x) - \Lambda^2 \tilde{h} x^{2l} = x^2 [(T_{2N-1}(x))^2 - \Lambda^2 \tilde{h} x^{2l-2}], \tag{27}
\]
where
\[
SO(2N) : \tilde{h} = 2N - 2, \quad l = 2, \\
SO(2N + 1) : \tilde{h} = 2N - 1, \quad l = 1. \tag{28}
\]
The factorization equation for gauge group $SO(N)$ is \[27, 28, 29\]
\[
y^2 = x^2 (H_{2N-2n-2}(x))^2 F_{4n+2} = x^2 (H_{2N-2n-2}(x))^2 (\mathcal{W}^2 + f_{2n}), \tag{29}
\]
so, the reduced $N = 1$ curve is
\[
y_{N=1} = F_{4n+2} = \mathcal{W}^2 + f_{2n}. \tag{30}
\]
Then, we follow
\[
y^2 = x^2 [(T_{2N-1}(x)) - \Lambda \tilde{h} x^{l-1}][(T_{2N-1}(x)) + \Lambda \tilde{h} x^{l-1}] \\
= x^2 (H_{2N-2n-2}(x))^2 F_{4n+2}. \tag{31}
\]

Just as $2N-2n-2$ double roots occur in $(H_{2N-2n-2}(x))^2$, then $2N-2n-2$ monopoles become massless. Since $(T_{2N-1}(x) - \Lambda \tilde{h} x^{l-1})$ and $(T_{2N-1}(x) + \Lambda \tilde{h} x^{l-1})$ can not share any zeroes, we can classify the solutions of (31) based on how to divide the zeroes of $(H_{2N-2n-2}(x))^2$ into these two factors where

1. $s_+$: All the zeroes of $(H_{2N-2n-2}(x))^2$ are those of $T_{2N-1}(x) + \Lambda \tilde{h} x^{l-1}$
2. $s_-$: All the zeroes of $(H_{2N-2n-2}(x))^2$ are those of $T_{2N-1}(x) - \Lambda^\hbar x^{l-1}$

3. $s_+s_- \neq 0$: The case other than (i) and (ii).

Also

\[ s_+ + s_- = 2N - 2n - 2. \]  \hfill (32)

Let us consider the case (1). One might obtain

\[
\begin{align*}
T_{2N-1}(x) + \Lambda^\hbar x^{l-1} &= (H_{2N-2n-2}(x))^2 K_{4n-2N+3} \\
T_{2N-1}(x) - \Lambda^\hbar x^{l-1} &= (H_{2N-2n-2}(x))^2 K_{4n-2N+3} - 2\Lambda^\hbar x^{l-1},
\end{align*}
\]  \hfill (33)

then

\[
\begin{align*}
\mathcal{W}'(x)^2 &= [H_{2N-2n-2}(x)K_{4n-2N+3}]^2 \\
f_{2n}(x) &= -2\Lambda^\hbar x^l K_{4n-2N+2}.
\end{align*}
\]  \hfill (34), (35)

Therefore, because the functions $\mathcal{W}$ and $H$ and $K$ are functions of $x^2$, we can write the complete set of the their zeroes as

\[
\begin{align*}
\mathcal{W}'(x) &= \prod_{i=1}^{m} (x^2 - a_i^2), \\
H_{2N-2n-2}(x) &= \prod_{i=1}^{N-n-1} (x^2 - p_i^2), \\
K_{4n-2N+2}(x) &= \prod_{j=1}^{2n-N+1} (x^2 - q_j^2),
\end{align*}
\]  \hfill (36), (37), (38)

where

\[
\{p_1, \ldots, p_{N-n-1}\} \cup \{q_1, \ldots, q_{2n-N+1}\} = \{a_1, \ldots, a_m\}. \hfill (39)
\]

Following the case of $U(N)$ gauge theory in [30, 25], and according to [27], one might obtain this expression for monopole condensation for $SO(N)$

\[
\langle \vec{M}_i M_i \rangle = \text{const} \times \prod_{i=1}^{2n-N+1} p_i^{2l-2} (p_i^2 - q_j^2)^{\frac{1}{2}}.
\]  \hfill (40)
The case (3) is more complicated than the others and it is not necessary to consider it for this paper.

Now, by considering
\[ y^2 = (x^{2N} - ux^2)^2 - \Lambda^2 h x^{2l}, \quad (44) \]
and writing the discriminant of this curve, one finds that the above curve has higher singularities and one can calculate the Argyres-Douglas point. In particular, for gauge group $SO(2N)$
\[ y^2 = x^4(x^{2N-2} - u + \Lambda^2 x^{2N-2})(x^{2N-2} - u - \Lambda^2 x^{2N-2}), \quad (45) \]
and for gauge group $SO(2N + 1)$, for calculating the discriminant, one must solve this algebraic equation
\[ x^2[(x^{2N-1} - ux)^2 - \Lambda^{4N-2}] = 0. \quad (46) \]
As an illustration, let us consider the $SO(5)$ case. The Argyres-douglas points of this curve are
\[ u_1 = \frac{3}{\sqrt{4}} e^{-\frac{2\pi is}{3}} \Lambda^{\frac{4N-2}{3}}, \quad s = 1, 2, 3, \quad (47) \]
\[ u_2 = 3e^{-\frac{2\pi is}{3}} \Lambda^{\frac{4N-2}{3}}, \quad s = 1, 2, 3. \quad (48) \]
In these points, AD points, the effective superpotential has non-analytic behavior and has conformal invariance.

The curve (44) corresponds to the the case where non of the monopoles are massless. On the other hand, if we choose $n = N - 1$ where non of the monopoles become massless, $H_{2N-2n-2} = H_0 = 1$ and
\[ W'(x) = K_{2N-1}, \quad (49) \]
then
\[ W'(x) = x^{2N-1} - ux, \quad (50) \]
\[ f_{2n} = -\Lambda^{2h} x^{2l}, \quad (51) \]
and finally
\[ W(x) = \frac{1}{2N} x^{2N} - \frac{u}{2} x^2. \quad (52) \]
4 Picard-Fuchs equations for glueball superfield

As it was mentioned, to find the glueball superfield in the geometric picture, one must calculate the following integral

\[ S_k = \frac{1}{2\pi i} \oint_{A_k} ydx. \]  

(53)

Now, we follow the method that was introduced in [31], for obtaining the Picard-Fuchs equations for classical gauge group \( U(n) \) and \( SO(n) \). If we consider the case where all the monopoles are massive in the theory then the \( N = 1 \) curve is the same as \( N = 2 \) curve and one may use the following form of Seiberg-Witten curve to calculate the glueball superfield,

\[ y^2 = P^2(x) - \Lambda^{2\hat{h}}x^{2l}, \]  

(54)

where \( \hat{h} \) is the dual coexter number of the Lie gauge group and

\[ P(x) = x^n - \sum_{i=2}^{n} u_i x^{n-i}, \]  

(55)

with \( n = r + 1, i = 2, 3, \ldots, r + 1 \) for \( A_r \) series and \( n = 2r, i = 2, 4, \ldots, 2r \) for \( B_r, D_r \) series, and \( u_i \)'s, are the casimirs of the gauge groups. Also \( l = n - \hat{h} \).

Base on the explicit form of \( S_k \) and using the fact that \( S_k \) is linearly independent of the casimirs, setting \( \frac{\partial}{\partial u_i} = \partial_i \) then

\[ \partial_i S_k = -\frac{1}{2\pi i} \oint \frac{x^{n-i}P}{y} \]  

(56)

\[ \partial_i \partial_j S_k = -\frac{1}{2\pi i} \oint \frac{x^{2n+2l-i-j} \Lambda^{2\hat{h}}}{y^3} lh 32, \]  

(57)

and using a direct calculation, one obtains

\[ \frac{d}{dx}\left(\frac{x^m P}{y}\right) = m\frac{x^{m-1} P}{y} - \frac{\Lambda^{2\hat{h}}}{y^3} (n-l)x^{m+n+2l-1} + \frac{\Lambda^{2\hat{h}}}{y^3} \sum_t (n-t-l)u_t x^{m+n+2l-t-1}. \]  

(58)

Comparing equations (56) and (58) with (58) we can find the second order differential equations for the \( S_k \)'s as follow

\[ L_m = -m\partial_{n-m+1} - (n-l)\partial_2 \partial_{n-m+1} + \sum_i (n-i-l)u_i \partial_i \partial_{n-m+1}, \]  

(59)

where \( m = t-1 \) for \( A_r \) series and \( m = 2lt-1 \) for \( B_r \) and \( D_r \) series and \( t = 1, \ldots, r-1 \). Note that in \( A_r \) series for \( t = 1 \) the above expression does not give the correct equation. By looking at the final step of (59), one might obtain the following identity

\[ L_m^{A_r} = -(n-l)\partial_2 \partial_r + \sum_i (r+1-i)u_i \partial_{i+1} \partial_{r+1}. \]  

(60)
For \( t > r - 1 \) equation (59) does not give the second order differential equation for \( u_i \). Notice that according to equation (59) we have
\[ \mathcal{L}_{i,j;p,q} = \partial_i \partial_j - \partial_p \partial_q, \quad i + j = p + q. \] (61)
The \( r \)-th equation, the exceptional equation, can be obtained according to this linear independent identity
\[ -n(l) \oint A_k d(x^{n+1}y^{-1}) + \sum_i (n-l-i) u_i \oint A_k d(x^{n-1-i}y^{-1}) = 0. \] (62)
In the expansion of the above identity there is a term as
\[ \Upsilon = (n-l)^2 \oint A_k \Lambda h x^{4l} y^3. \] (63)
We must be careful if we want to reproduce this term by partial derivative of \( S_k \). If \( u_n \) does not be zero for \( A_r \) or \( D_r \) series this term equals with
\[ \Upsilon = -(n-l)^2 \Lambda^2 h \partial^2_{h+1} S_k, \] (64)
and if \( u_2n \) and \( u_{2n-2} \) do not be zero for \( B_r \) series this term should be changed as
\[ \Upsilon = -(n-l)^2 \Lambda - 1 \partial^2_{h+1} S_k, \] (65)
Then
\[ \left[ -(n+1)(n-l) - (l+1) \sum_i i u_i \partial_i + \sum_i i^2 u_i \partial_i + \sum_{i,j} i j u_i u_j \partial_i \partial_j + (l+1) \Lambda \partial \Lambda + \Upsilon \right] S_k = 0. \] (66)
From the fact that the \( S_k \)'s are homogeneous functions of degree \( n+1 \), one can see that
\[ \left[ \Lambda \partial \Lambda + \sum_i i u_i \partial_i \right] S_k = (n+1) S_k, \] (67)
therefore, the exceptional equation reads as
\[ \left[ (n+1)(2l-n+1) + \sum_i i(i-2(l+1)) u_i \partial_i \right. \]
\[ \left. + \sum_{i,j} i j u_i u_j \partial_i \partial_j + \Upsilon \right] S_k = 0. \] (68)
These exceptional equations are in general hypergeometric equations. But, if the above conditions do not be satisfied in an special form of geometric curve then one may use the following identity
\[ \Upsilon = (n-l)^2 \oint A_k \Lambda h x^4 \frac{y^3}{y^4} = [2h \Lambda \partial \Lambda - \Lambda \partial \Lambda (\Lambda \partial \Lambda)]. \] (69)
Nevertheless, by (67), one can show that this method does not give a generic second order differential equation for gleuball superfield. This occurs in our paper for gauge group $SO(2N+1)$ hence we focus on semi classical region where $\Upsilon$ goes to zero. We’ll study the solutions of these equations for gauge groups $SO(2N)$ and $SO(2N+1)$ with details in the following sections.

### 4.1 Solution of the Picard-Fuchs equation

In order to solve the Picard-Fuchs equations, at first, we note that for a $U(N)$ gauge theory in which all $u_i$’s $= 0$ unless $u_N = u$ (higher AD point), the exceptional equation (68) reduces to equation (19) and the solution of this equation is

$$S_k(u, \Lambda^{2N}) = -\frac{2\Lambda^{2N}}{N} e^{2\pi ik/N} u^{-1+\frac{1}{2}} F\left(\frac{1}{2} - \frac{1}{2N}, 1 - \frac{1}{2N}, 2, \frac{4\Lambda^{2N}}{u^2}\right).$$  \hspace{1cm} (70)

Now we derive solutions of differential equations for other classical gauge groups.

**The case $SO(2N)$**

For this gauge group we have

$$n = 2N, \quad \hhat = 2N - 2, \quad l = 2,$$ \hspace{1cm} (71)

and because of the maximal confining phase of the system we have

$$u_{2N-2} = u, \quad \text{other} = 0.$$ \hspace{1cm} (72)

Therefore, we only need the exceptional equation that is

$$[(u^2 - \Lambda^{2\hhat}) \partial_u^2 + \frac{N - 4}{N - 1} u \partial_u + \frac{(2N + 1)(5 - 2N)}{4(N - 1)^2}] S_k = 0.$$ \hspace{1cm} (73)

By following change of variable

$$z = \frac{u^2}{\Lambda^{2\hhat}},$$ \hspace{1cm} (74)

this equation becomes a hypergeometric equation as

$$[z(1 - z) \frac{\partial^2}{\partial z^2} + \left(1 - \frac{2N - 5}{2N - 2}\right) z \frac{\partial}{\partial z} - \frac{(2N + 1)(5 - 2N)}{16(N - 1)^2}] S_k = 0,$$ \hspace{1cm} (75)

which can be solved as [32]

$$S_k(z) = C_{1,k} F(a, b, c; z) + C_{2,k} z^{1-c} F(a - c + 1, b - c + 1; 2 - c; z),$$ \hspace{1cm} (76)
where
\[ a = -\frac{5 - 2N}{4(N - 1)}, \quad b = -\frac{2N + 1}{4(N - 1)}, \quad c = \frac{1}{2}, \quad (77) \]
and
\[ F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} = \frac{(a+n-1)!}{(a-1)!}. \quad (78) \]

Now to fix the coefficients \( C_{1,k} \) and \( C_{2,k} \) one can evaluate the \( S_k \) in the semi-classical limit \( \Lambda \to 0 \), where the glueball superfield vanishes. This can be done by performing the analytic continuation of the hypergeometric function, which are defined as power series in the disk \(|z| \leq 1\).

First of all, one may show that
\[ C_{1,k} = (-1)^{c} \frac{\Gamma(b)\Gamma(c-a)\Gamma(2-c)}{\Gamma(c)\Gamma(1-a)\Gamma(b-c+1)} C_{2,k}. \quad (79) \]

After that, we rewrite the above hypergeometric function in terms of a new variable which is dual with \( \Lambda \to 0 \) and we are able to expand the solutions around the zero. Furthermore, in \( \Lambda \to 0 \) the solutions of (73) are asymptotic to
\[ u^{\alpha_{\pm}} f(u), \quad \alpha_{+} = \frac{5 - 2N}{2(N - 1)}, \quad \alpha_{-} = \frac{2N + 1}{2(N - 1)}, \quad (80) \]
then, by considering the \( z = \frac{A_{2k}}{u^{2}} \) and \( S_k = u^{\alpha_{+}} f_k(u) \), the hypergeometric equation reads as
\[ [z(1 - z) \frac{\partial^2}{\partial z^2} + (2 - \left( \frac{5N - 8}{2(N - 1)} \right) z) \frac{\partial}{\partial z} - \left( \frac{7 - 4N}(5 - 2N) \right) \frac{16(N - 1)^2}{16(N - 1)^2}] f_k = 0, \quad (81) \]
and the solution becomes
\[ S_k(z) = C_{3,k} u^{\alpha_{+}} F(a', b'; c'; z), \quad (82) \]
where
\[ a' = a = \frac{2N - 5}{4(N - 1)}, \quad b' = \frac{4N - 7}{4(N - 1)}, \quad c' = 2, \quad (83) \]
The value of \( C_{3,k} \) can be fixed by expanding the \( S_k \) in the semiclassical limit in which
\[ S_k = \frac{1}{2\pi i} \oint_{\gamma_k} dx \sqrt{(P_{2N}(x))^2 - \Lambda^{4N-4} x^4} = -\frac{\Lambda^{4N-4}}{2\pi i} \oint_{\gamma_k} dx \frac{x^4}{2P(x)} + ..., \quad (84) \]
where \( \gamma_k \) is a counterclockwise around the \( k \)-th root of \( P_{2N}(x) = x^{2N} - u x^2 \)
\[ x_k = e^{\frac{\pi i k}{2(N-1)}} u^{rac{1}{2(N-1)}}, \quad k = 1, ..., 2(N - 1), \quad (85) \]
so
\[- \frac{\Lambda^{4N-4}}{2\pi i} \oint_{\gamma_k} dx \frac{x^4}{2P(x)} = - \frac{\Lambda^{4N-4}}{2} \frac{x_k^4}{P'(x_k)} = - \frac{\Lambda^{4N-4}}{4(N-1)} e^{\frac{2N}{(N-1)} \pi i k} u^{\alpha}. \tag{86}\]

Comparing (82) and (86), one might find
\[C_{3,k} = - \frac{\Lambda^{4N-4}}{4(N-1)} e^{\frac{2N}{(N-1)} \pi i k}, \tag{87}\]
and finally
\[S_k = - \frac{\Lambda^{4N-4}}{4(N-1)} e^{\frac{2N}{(N-1)} \pi i k} u^{\alpha} + F(a', b'; c'; \Lambda^{2h}/u^2). \tag{88}\]

Again, one can obtain a closed relation between \(C_{1,k}, C_{2,k}\) and \(C_{3,k}\) and evaluates the solutions in the regions where \(|u_2^{2N-2}| < 1\).

It is easy to check that the above expression is consistent with the fact that in the semiclassical limit
\[\sum_{k=1}^{2N-2} S_k = 0. \tag{89}\]

In fact, this relation is a reflection of \(Z_2\) symmetry of SW curve for gauge group \(SO(2N)\). We’ll get back to this later.

**The case \(SO(2N + 1)\)**

In this case
\[n = 2N, \quad \hat{h} = 2N - 1, \quad l = 1, \tag{90}\]
and we again consider that
\[u_{2N-2} = u, \quad \text{other} = 0. \tag{91}\]

Now, as we mentioned before since \(u_{2N}\) is zero, we can not derive a generic differential equation for glueball superfield. So, we focus on the semi classical region where \(\Upsilon \to 0\) and by changing of variable as \(z = \frac{1}{u}\), the exceptional equation will be changed to
\[\left[ u^2 \frac{\partial^2}{\partial u^2} + \frac{(N+1)}{(N-1)} u \frac{\partial}{\partial u} + \frac{(2N+1)(3-2N)}{4(N-1)^2} \right] S_k = 0, \tag{92}\]
upon using the series method or changing of variable as \(t = lnz\), one can show that the solutions are
\[S_k = A_k(\Lambda) u^{\frac{3-2N}{2(N-1)}} + B_k(\Lambda) u^{\frac{2N+1}{2(N-1)}} . \tag{93}\]
In the region where $\Lambda \to 0$, then $S \to 0$ and it implies that the solution of the above equation is $A(\Lambda)u^{\frac{3-2N}{4}}$, and in this limit
\[
-\frac{\Lambda^{4N-2}}{2\pi i} \int_{x_k} dx \frac{x^2}{2P(x)} = -\frac{\Lambda^{4N-2}}{2} \frac{x_k^2}{P'(x_k)},
\]
where $x_k = e^{\frac{\pi i k}{N-1}} u^{\frac{1}{2(N-2)}}$ are the poles of $P_{2N}(x)$. Then it is easy to see that
\[
S_k = -\frac{\Lambda^{4N-2}}{4(N-1)} \frac{3-2N}{3-2N} e^{\frac{\pi i k}{N-1}} u^{\frac{3-2N}{2(N-1)}}.
\]
Moreover, One can show that in this case too
\[
\sum_{k=1}^{2N-2} S_k = 0.
\]

**The period $S_0$:**

In order to calculate the period $S_0$, we use the Picard-fuchs equations and obtain the solutions that we described in the previous sections. But, by a direct calculation, we show that in the semi classical regions the period $S_0 = 0$. For this, considering the case $SO(2N)$ where
\[
S_0 = \frac{1}{2\pi i} \int_{A_0} y dx = \frac{1}{2\pi i} \int_{A_0} \sqrt{(x^{2N} - ux^2)^2 - \Lambda^{4N-4}x^4} dx,
\]
and expanding the above in the region $|\frac{u^2}{\Lambda^{4N-4}}| > 1$ (where the solution (88) is valid.), one can easily show that
\[
S_0 = 0.
\]

Similar arguments can be given for the gauge group $SO(2N + 1)$ and one obtains $S_0 = 0$ in the semi classical limit. In fact, these results together with (89) are in agreement with the fact that in the semi classical limit glueball superfield vanishes.

### 5 The effective superpotential:

Now, we are going to evaluate the expression for effective superpotential. After the condensation of monopoles, the group structure breaks as
\[
SO(N) \to SO(\tilde{N}_0) \times \prod_{k=1}^{n} U(N_k).
\]
as before, we consider the case that none of the monopoles become massless and so

$$SO(2N)/SO(2N + 1) \to SO(2N_0)/SO(2N_0 + 1) \times \prod_{k=1}^{2N-2} U(1).$$

(100)

The general matrix model formula for the effective superpotential of a $SO(N)$ theory breaking to $SO(\hat{N}_0) \prod_{k=1}^n SO(N_k)$ is [29, 34]

$$-\frac{1}{2\pi i} W_{\text{eff}}(S) = \frac{1}{2\pi i} 2\hat{N}_0 \frac{\partial F}{\partial S_0} + \frac{1}{2\pi i} \sum_{k=1}^n N_k \frac{\partial F}{\partial S_k} + \tau_0 \sum_{k=0}^n S_k + \sum_{k=0}^n b_k S_k,$$

(101)

where the constants $b_k$ are integers and are related to periods as in (??), though in the $SO(N)$ case the definition of memomorphic one form $T dx$ is different from the $U(N)$ case (other variables in (101) are the same as in the section 2). In this case, if we define the new curve [34]

$$z - \frac{2P_{2N}(x, u)}{x^k} + \frac{\Lambda \hat{h}}{z} = 0,$$

(102)

then

$$T dx = -\frac{dz}{z}.$$ 

(103)

Especially, one can show

$$\frac{1}{2\pi i} \int_{A_0} T dx = 2\hat{N}_0 = 2N_0 - l.$$ 

(104)

Now, for the curve

$$y^2 = (x^{2N} - ux^2)^2 - \Lambda \hat{h} x^{2k},$$

(105)

we calculate these constants.

The constant $N_0$ can be determined by noticing the fact that we suppose there are’t any massless monopoles and so the gauge group breaks to $SO(2)/SO(3) \times U(1)^{2N-2}$. Moreover, we can rewrite the $P_{2N}(x) = x^{2N_0} \times \prod_{k=1}^{2N-2}(x^2 + x^2_k)$ [29] in this case. So, for the above curve $N_0 = 1$

Furthermore, one may determine the $N_0$ by direct calculation using the (104) and obtains $N_0 = 1$. We also obtained that because of $Z_2$ symmetry

$$\sum_{k=0}^{2N-2} S_k = 0.$$ 

(106)

Again, using the above symmetry, we can obtain a simple expression for second term as following

$$\frac{\partial F}{\partial S_k} = \int_{B_k} y dx = \lim_{\Lambda_0 \to \infty} [2 \int_{x_k}^{\Lambda_0} y dx - 2 \int_{x_k}^{\Lambda_0} W dx - 2\Lambda_0 W(\Lambda_0)],$$

(107)
where
\[ x_{k,\pm} = e^{\frac{\pi i k}{2}} (u \pm \Lambda^\hat{h}) \frac{1}{2N-2}. \] (108)

Here \( \Lambda_0 \) is the point at infinity on the upper sheet. Considering \( x = e^{i\pi \tilde{x}} \), then
\[
\int_{\Lambda_0}^{\Lambda_0} ydx - \int_{\Lambda_0}^{\Lambda_0} Wdx = \int_{\Lambda_0}^{\Lambda_0} \sqrt{(x^{2N} - u^2x^2) - \Lambda^{2\hat{h}}x^2}dx - \int_{\Lambda_0}^{\Lambda_0} W(\Lambda_0)dx
\]
\[ = (e^{i\pi}) \int_{\Lambda_0}^{\Lambda_0} \sqrt{(\tilde{x}^{2N} - u\tilde{x}^2) - \Lambda^{2\hat{h}}\tilde{x}^2}d\tilde{x} - \int_{\Lambda_0}^{\Lambda_0} W(\Lambda_0)d\tilde{x}, \]

where \( \tilde{\Lambda}_0 = e^{-i\pi} \Lambda_0 \) and we use
\[ W(x) = \frac{1}{2N}x^{2N} - \frac{u}{2}x^2. \] (110)

Then
\[ \frac{\partial F}{\partial S_k} = e^{i\pi} \frac{\partial F}{\partial S_{k-(N-1)}}. \] (111)

Because the gauge symmetry breaks into \( U(1)^{2N-2} \) and by a direct calculation, we have \( N_k = 1, k = 1, ..., 2N - 2 \). Hence
\[ \sum_{k=1}^{2N-2} N_k \frac{\partial F}{\partial S_k} = 0. \] (112)

Now, it remains to determine the \( b_k \)'s which
\[ b_k = -\sum_{j=1}^{k-1} c_j, (b_1 = 0), \quad c_j = \frac{1}{2\pi i} \oint_{C_j} T, \] (113)

Note that, since \( T_N \) is a logarithmic derivative, its period integrals are integers. For calculating \( c_k \)'s, as it has been showed in [26], when \( u \) goes to AD points in \( SO(N) \) gauge theory we have
\[ c_k = 1, \quad k = 1, ..., 2N - 1. \] (114)

In fact in this limit it can be shown that
\[ c_k = \frac{1}{2\pi i} \oint_{C_k} T_N = \frac{1}{2\pi i} \oint_{A_k} T_N = 1. \] (115)

In order to calculate the variables we must determine the first term of \( W_{eff} \). It is obvious that for \( SO(2N) \) gauge group \( 2\hat{N}_0 = 0 \) but for \( SO(2N+1) \) case \( 2\hat{N}_0 = 1 \). So, we should calculate \( \frac{\partial F}{\partial S_0} \) for \( SO(2N+1) \) gauge group. For this, one should compute this integral
\[ \frac{\partial F}{\partial S_0} = \oint_{E_0} ydx = 2 \int_{0^+}^{\Lambda_0} \sqrt{P_{2N}^2(x) - \Lambda^{2\hat{h}}x^2}. \] (116)
Again, we go to regions where \( \Lambda \to 0 \) (as we solved the picard-fuchs equation in these regions) and rewrite the above integral around the \( \Lambda \to 0 \). Setting \( 0^+ = \Delta \) then

\[
\frac{\partial F}{\partial S_0} = 2 \int_{\Delta}^{\Lambda_0} \left[ P_{2N} - \sum_{m=1}^{\infty} \frac{(2m-3)!!}{2m!!} \Lambda^{2m} x^{2m} (x^{2N-2} - u)^{2m-1} \right] dx,
\]

and by integrating by part one can show

\[
G(x) = \int_{x}^{x^2} \frac{x^{2m} (x^{2N-2} - u)^{1-2m} dx'}{(2N-2)(2m-2)}
\]

\[
- \frac{A(2m-1)}{(2N-2)^{2m-1}(2m-2)!} x^{2m(1-2N)+2N+1} \ln (x^{2N-2} - u)
\]

\[
- \sum_{h=1}^{2m-2} \frac{A(h)(2m-2-h)!}{(2N-2)^h(2m-2)!} (x)^{2h(1-N)-2m+3} (x^{2N-2} - u)^{h-2m+1}
\]

\[
+ \frac{A(2m-1)}{(2N-2)^{2m-1}(2m-2)!} \int_{x}^{x^2} x^{2m(2m-1)} \ln (x^{2N-2} - u) dx',
\]

where we suppose that \( (x^{2N-2} > u) \) and \( A(k) = \prod_{i=2}^{k} [2i(1-N) - 2m + 3] \). The last integral can be performed by using the change of variables as \( t = \ln(x^{2N-2} - u) \) and defining a function \( H(x) \) as

\[
H(x) = (x^{2N-2} - u)(x)^{2m(1-2N)+3},
\]

then

\[
\int_{x}^{x^2} x^{2m(2m-1)} \ln (x^{2N-2} - u) dx' = \frac{1}{2N-2} \sum_{j=0}^{\infty} \frac{c_j}{(j+1)(j)!} (\ln(x^{2N-2} - u))^{j+1}
\]

where \( c_j \)'s are the coefficients of expansion of \( H(x) \) around the \( (u+1)^{\frac{1}{N-1}} \). So

\[
G(x) = - \frac{x^{5-2N-2m} (x^{2N-2} - u)^{2-2m}}{(2N-2)(2m-2)}
\]

\[
- \frac{A(2m-1)}{(2N-2)^{2m-1}(2m-2)!} x^{2m(1-2N)+2N+1} \ln (x^{2N-2} - u)
\]

\[
+ \frac{A(2m-1)}{(2N-2)^{2m(2m-1)}(2m-2)!} \sum_{j=0}^{\infty} \frac{c_j}{(j+1)(j)!} (\ln(x^{2N-2} - u))^{j+1}
\]

\[
- \sum_{h=1}^{2m-2} \frac{A(h)(2m-2-h)!}{(2N-2)^h(2m-2)!} (x)^{2h(1-N)-2m+3} (x^{2N-2} - u)^{h-2m+1},
\]

and

\[
\frac{\partial F}{\partial S_0} = - \sum_{m=1}^{\infty} \frac{2(2m-3)!!}{2m!!} \Lambda^{(4N-2)m} [G(\Lambda_0) - G(\Delta)]
\]

\[
+ \frac{2}{2N+1} \Lambda^{2N+1}_0 - \frac{2}{2N+1} \Delta^{2N+1} - \frac{2u}{3} \Lambda^{3}_0 + \frac{2u}{3} \Delta^{3}.
\]
After all of these, the effective superpotential is
\[
W_{\text{eff}}(S) = -2N_0 \frac{\partial F}{\partial S_0} - 2\pi i \sum_{k=1}^{2N-2} b_k S_k \\
= -(2N_0 - 2) \frac{\partial F}{\partial S_0} - 2\pi i(N - 1) \sum_{k=1}^{N-1} S_k, \quad (123)
\]
and so, the effective superpotential becomes
\[
W_{\text{eff}}(S) = \pi i \frac{\Lambda^{4N-4}}{(1 - e^{\frac{2\pi i}{N}})} u^{\frac{3-2N}{2(N-1)}} F\left(\frac{2N - 5}{4(N - 1)}, \frac{4N - 7}{4(N - 1)}, \frac{2}{u^2}, \frac{\Lambda^{4N-4}}{u^2}\right) \quad (124)
\]
for $SO(2N)$, and for $SO(2N+1)$ in the semi classical limit
\[
W_{\text{eff}}(S) = \pi i \frac{\Lambda^{4N-2}}{(1 - e^{\frac{2\pi i}{N}})} u^{\frac{3-2N}{2(N-1)}} \\
- \frac{2}{2N + 1} (\Lambda_0^{2N+1} - \Delta^{2N+1}) + \frac{2u}{3} (\Lambda_0^3 - \Delta^3) \\
+ \sum_{m=1}^{\infty} \frac{2(2m - 3)!!}{2m!!} \Lambda^{4N-2}[G(\Lambda_0) - G(\Delta)] \quad . \quad (125)
\]

### 5.1 Scaling behavior around the AD points

In this section, we consider the IR limit $\Lambda \to \infty$, where the cycles $A_i$ become large and move out to $\infty$, and do not effect the IR physics. So, it is enough to study the scaling behavior of chiral operators around the origin with cycle $A_0$ $[25]$. It is easy to see that at AD points
\[
U_r = 0, \quad S_r = 0, \quad \text{forall} \; r. \quad (126)
\]

Perturbing the effective superpotential as
\[
W'(x) = x^{2N-1} - ux + \sum_{m=0}^{2N-2} g_{2m} x^{2m-1} \Lambda^{2(N-m)}, \quad (127)
\]
and by computing the expectation value of chiral fields we obtain the scaling behavior of them. Now, we apply a scale transformation as
\[
x \to \rho^n x, \quad g_{2m} \to \rho^{2(N-m)} g_{2m}, \quad (m = 0, \ldots, 2N-2), \quad (128)
\]
then
\[
U_r \simeq \int_{A_0} \rho^{2N} \left(2N x^{2N-1} - \sum_{2m} x^{2m-1} \Lambda^{2N-2m}\right) dx, \quad (129)
\]
and
\[ U_r \to \rho^{\gamma(r+N)} U_r. \] (130)

So, the chiral field \( U(r) \) has the scaling dimension \( \Delta = \gamma(r+N) \). If we assume that the field \( \text{Tr}\Phi \) has no anomalous dimension \[25\], then \( \gamma = \frac{1}{N+1} \) and
\[ \Delta(g_{2m}) = \frac{2(N-m)}{N+1}, \quad \Delta(U_r) = \frac{N+r}{N+1}, \quad \Delta(S_r) = \frac{N+r+1}{N+1}. \] (131)

Dimensions of the coupling constants are in agreement with the known results \[23\, 24\]. Also
\[ \Delta(g_{2m}) \leq 1 \iff 2m \geq N - 1. \] (132)

The case \( \Delta(g_{2m}) \leq 1 \) is well-known and should correspond to a coupling constant in the \( \mathcal{N} = 2 \) superconformal field theory in 4− dimensions \[22\].

General chiral perturbation of an \( \mathcal{N} = 2 \) SCFT\(_4\) action should have the form
\[ \Delta S = \int d^4x d^2\theta^+ d^2\theta^- \sum_{m: \Delta(g_{2m}) \leq 1} g_{2m} \mathcal{O}_{2m} \] (133)
\[ \Delta(g_{2m}) + \Delta(\mathcal{O}_{2m}) = 2, \] (134)
so, the chiral operator \( \mathcal{O}_{2m} \) is
\[ \mathcal{O}_{2m} = \text{Tr}\Phi^{m+2-N}, \] (135)
thus, the coupling \( g_{2m} \) is paired with \( \text{Tr}\Phi^{m+2-N} \).

6 conclusion

Using the factorization equation of gauge group \( SO(N) \) we find the spectral curve of \( \mathcal{N} = 1 \) supersymmetric gauge theory and we obtain a relation between this curve with the vev of monopoles. Then, using the fact that all monopoles are massive in the system we obtain general Picard-fuchs equations for classical \( U(N) \) and \( SO(N) \) gauge groups respect to parameters of curves that are casimirs of gauge groups. These differential equations are hypergeometric equations and have regular singular points which are dual to Argyres-Douglas points.

Then, we focus on AD points and give the solutions of these equations and calculate the effective superpotential by using the geometric picture of \( SO(N) \) gauge theory. Scaling behavior in the IR limits, using the CDSW’s machinery, gives us the dimensions of coupling constants and chiral operators that the results are consistent with \[23\].
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