Random attractor for the 3D viscous primitive equations driven by fractional noises

Guoli Zhou †
March 5, 2018

Abstract

We develop a new and general method to prove the the existence of the random attractor (strong attractor) for the primitive equations (PEs) of large-scale ocean and atmosphere dynamics under non-periodic boundary conditions and driven by infinite-dimensional additive fractional Wiener processes. In contrast to our new method, the common method, compact Sobolev embedding theorem, is to obtain the uniform a priori estimates in some Sobolev space whose regularity is high enough. But this is very complicated for the 3D stochastic PEs with the non-periodic boundary conditions. Therefore, the existence of universal attractor (weak attractor) was established in previous work. The main idea of our method is that we first derive that \( \mathbb{P} \)-almost surely the solution operator of stochastic PEs is compact. Then we construct a compact absorbing set by virtue of the compact property of the solution operator and the existence of a absorbing set. We should point out that our method has some advantages over the common method of using compact Sobolev embedding theorem, i.e., if the random attractor in some Sobolev space do exist in view of the common method, our method would then further implies the existence of random attractor in this space. The present work provides a general way for proving the existence of random attractor for common classes of dissipative stochastic partial differential equations and improves the existing results concerning random attractor of stochastic PEs. In a forthcoming paper, we use this new method to prove the existence of strong attractor for the stochastic moist primitive equations, improving the results, the existence of weak (universal) attractor of the deterministic model.

Keywords: Primitive equations, Fractional noise, Random attractor

Mathematics Subject Classification (2000): 60H15, 35Q35.

1 Introduction

The paper is concerned with the PEs in a bounded domain with fractional noises. To outline its content in detail, we introduce a smooth bounded domain \( M \subset \mathbb{R}^2 \) and the cylindrical domain \( \bar{M} = \mathbb{R}^2 \times (0,1) \).
$M \times (-h, 0) \subset \mathbb{R}^3$, and consider the following 3D stochastic PEs of Geophysical Fluid Dynamics.

\[ \partial_t v + (v \cdot \nabla) v + \nu \partial_z v + f v^\perp + \nabla p + L_1 v = W^H_1, \]

\[ \partial_z p + T = 0, \]

\[ \nabla \cdot v + \partial_z w = 0, \]

\[ \partial_t T + v \cdot \nabla T + w \partial_z T + L_2 T = Q + W^H_2. \]

The unknowns for the 3D stochastic viscous PEs are the fluid velocity field $(v, w) = (v_1, v_2, w) \in \mathbb{R}^3$ with $v = (v_1, v_2)$ and $v^\perp = (-v_2, v_1)$ being horizontal, the temperature $T$ and the pressure $p$. $f = f_0(\beta + y)$ is the given Coriolis parameter, $Q$ is a given heat source. The viscosity and the heat diffusion operators $L_1$ and $L_2$ are given by

\[ L_i = -\nu_i \Delta - \mu_i \partial_{zz}, \quad i = 1, 2. \]

Here the positive constants $\nu_1, \mu_1$ are the horizontal and vertical Reynolds numbers, respectively, and $\nu_2, \mu_2$ are positive constants which stand for the horizontal and vertical heat diffusivity, respectively. To simplify the nations, we assume $\nu_i = \mu_i = 1, \quad i = 1, 2$. The results in this paper are still valid when we consider the general cases. We set $\nabla = (\partial_x, \partial_y)$ to be the horizontal gradient operator and $\Delta = \partial_x^2 + \partial_y^2$ to be the horizontal Laplacian. Here, we take $W^H_1(t, x, y, z), i = 1, 2$, the informal derivative for the fractional Wiener process $W^H_1$ given below.

The boundary of $\mathcal{U}$ is partitioned into three parts: $\Gamma_u \cup \Gamma_b \cup \Gamma_s$, where

\[ \Gamma_u = \{(x, y, z) \in \mathcal{U} : z = 0\}, \]

\[ \Gamma_b = \{(x, y, z) \in \mathcal{U} : z = -h\}, \]

\[ \Gamma_s = \{(x, y, z) \in \mathcal{U} : (x, y) \in \partial M, -h \leq z \leq 0\}. \]

Here $h$ is a sufficiently smooth function. Without loss generality, we assume $h = 1$. We consider the following boundary conditions of the stochastic 3D viscous PEs.

\[ \partial_z v = n, \quad w = 0, \quad \partial_z T = -\alpha(T - \tau) \quad \text{on} \quad \Gamma_u, \]

\[ \partial_z v = 0, \quad w = 0, \quad \partial_z T = 0 \quad \text{on} \quad \Gamma_b, \]

\[ v \cdot \vec{n} = 0, \quad \partial_n v \times \vec{n} = 0, \quad \partial_n T = 0 \quad \text{on} \quad \Gamma_s, \]

where $n(x, y)$ is the wind stress on the surface of the ocean, $\alpha$ is a positive constant, $\tau$ is the typical temperature distribution on the top surface of the ocean and $\vec{n}$ is the norm vector to $\Gamma_s$. We assume for the sake of simplicity that $Q$ is independent of time and $\eta = \tau = 0$. It is worth pointing that results presented in the paper can still be obtained provided some simple modifications are made.

By elementary calculus, we have that

\[ w(x, y, z, t) = -\int_{-1}^{z} \nabla \cdot v(x, y, \lambda, t)d\lambda, \]

\[ p(x, y, z, t) = p_s(x, y, t) - \int_{-1}^{z} T(x, y, \lambda, t)d\lambda. \]

Using the fact, we obtain the following equivalent 3D stochastic PEs:

\[ \partial_t v + L_1 v + (v \cdot \nabla) v - \left( \int_{-1}^{z} \nabla \cdot v(x, y, \lambda, t)d\lambda \right) \partial_z v + \nabla p_s(x, y, t) - \int_{-1}^{z} \nabla T(x, y, \lambda, t)d\lambda + f v^\perp = W^H_1; \]

\[ \partial_t T + v \cdot \nabla T - \left( \int_{-1}^{z} \nabla \cdot v(x, y, \lambda, t)d\lambda \right) \partial_z T + L_2 T = Q + W^H_2; \]

\[ \partial_z v |_{\Gamma_u} = \partial_z v |_{\Gamma_b} = 0, \quad v \cdot \vec{n} |_{\Gamma_u} = 0, \quad \partial_n v \times \vec{n} |_{\Gamma_s} = 0; \]

\[ \left( \partial_z T + \alpha T \right) |_{\Gamma_u} = \partial_z T |_{\Gamma_b} = 0, \quad \partial_n T |_{\Gamma_s} = 0; \]

\[ v(x, y, z, 0) = v_0(x, y, z), \quad T(x, y, z, 0) = T_0(x, y, z). \]
The Primitive Equations are the basic model used in the study of climate and weather prediction, which describe the motion of the atmosphere when the hydrostatic assumption is enforced \[ \text{[17, 25, 26].} \] As far as we know, their mathematical study was initiated by J.L.Lions, R.Teman and S.Wang (\[ 32 - 35 \]). And this research field has developed and has received considerable attention from the mathematical community over the last two decades. Lions, Temam and Wang \[ 33 \] obtained the existence of global weak solutions for the primitive equations. Guillén-González, et al. \[ 24 \] obtained the global existence of strong solutions to the primitive equations with small initial data. Moreover, they proved the local existence of strong solutions to the equation. The local existence of strong solutions to the primitive equations under the small depth hypothesis was studied by Hu et al. \[ 27 \]. Taking advantage of the fact that the pressure is essentially two-dimensional in the PEs, Cao and Titi \[ 10 \] proved the global results for the existence of strong solutions of the full three-dimensional PEs. Subsequently, I. Kukavica and M. Ziane \[ 29 \] developed a different proof which allows one to treat non-rectangular domains as well as different, physically realistic, boundary conditions. The existence of the global attractor is given by Ju \[ 28 \]. For the PEs with partial dissipation, we refer the reader to the papers \[ 5, 7, 8, 9, 11 \].

Despite the developments in the deterministic case, the theory for the stochastic PEs remains underdeveloped. B. Ewald, M. Petcu, R. Teman \[ 13 \] and N. Glatt-Holtz, M. Ziane \[ 21 \] considered a two-dimensional stochastic PEs. Then N. Glatt-Holtz and R. Temam \[ 22, 23 \] extended the case to the greater generality of physically relevant boundary conditions and nonlinear multiplicative noise. Following the methods closer to \[ 10, 19 \], Boling Guo and Dawen Huang \[ 19 \] studied the global well-posedness of the three-dimensional system with a additive noise in the horizontal momentum equations and obtained some kind of weak type compactness properties of the solutions to the stochastic system. Using methods different from \[ 19 \], A. Debuessche, N. Glatt-Holtz, R. Temam and M. Ziane considered three-dimensional system with multiplicative noise.

Although the PEs express very fundamental laws of physics, the deterministic models are numerically intractable. Studies have shown that resolved states are associated with many possible unresolved states. This calls for stochastic methods for numerical weather and climate prediction which potentially allow a proper representation of the uncertainties, a reduction of systematic biases and improved representation of long-term climate variability. Furthermore, while current deterministic parameterization schemes are inconsistent with the observed power-law scaling of the energy spectrum, new statistical dynamical approaches that are underpinned by exact stochastic model representations have emerged that overcome this limitation. The observed power spectrum structure is caused by cascade processes which can be best represented by a stochastic non-Markovian Ansatz. Non-Markovian terms are necessary to model memory effects due to model reduction. It means that in order to make skillful predictions the model has to take into account also past states and not only the current state (as for a Markov process). For more details, please refer to \[ 6 \] and other references.

Based on the fact, we consider stochastic PEs, where both the horizontal velocity field and the temperature are perturbed by fractional Brownian motion (fBm). We define stochastic integrals through pathwise generalized Stieltjes integrals as is the case in \[ 36, 43, 52 \]. So far, various forms of stochastic integrals with respect to fBm have been developed by several authors (see Chapter 5 in Nualart \[ 41 \] and the references therein). The theory of stochastic partial differential equations of parabolic type driven by an fractional noise have received much attention (see Maslowski and Nualart \[ 36 \], Maslowski and Schmalfuss \[ 37 \], Nualart \[ 39, 40 \], Nualart and Vuillermot \[ 42 \], Tindel, Tudor and Viens \[ 50 \], and the references therein).

In this article, we mainly study the existence of random attractor for the stochastic PEs perturbed by fractional noises. We should point out that the definitions of the attractors between our paper and \[ 19 \] have essential differences. The random attractor obtained in this work is \( \mathbb{P} \)-a.e. \( \omega \) compact in \( (H^1(\mathbb{T}))^3 \) and attracts any orbit starting from \(-\infty\) in the strong topology of \( (H^1(\mathbb{T}))^3 \). While the attractor studied in \[ 19 \] is not necessary a compact subset in \( (H^1(\mathbb{T}))^3 \). In addition, the attractor attracts any orbit in the weak topology of \( (H^1(\mathbb{T}))^3 \).
To consider the dynamical behavior of the strong solution of 3D stochastic PEs, we will encounter the following difficulties.

The first difficulty involved here is to study moment estimates and growth property of the Ornstein-Uhlenbeck (O-U) processes driven by fractional noises. Since fBm is not a semimartingale and has not independent increment, we can not follow the common method to use the Itô isometry and the law of large numbers to study its properties. In addition, the integrals with Hölder continuous integrators are defined pathwise, which is a qualitative difference to the definition of the classical stochastic integral where the integrand is a white noise. We overcome the difficulties by using analysis techniques and taking advantage of stationary increments and polynomial growth property as well as regularity of fBm. The fundamental results can apply to the study of long time behavior of stochastic partial differential equations driven by fractional noises.

Secondly, to prove the existence of the random attractor, by virtue of the common method Sobolev compact theorem we should obtain the uniform a priori estimates with respect to initial data in $(H^2(\Omega))^3$(see [2, 3]). But it is very difficult to achieve under the non-periodic boundary conditions. To overcome the difficulty, we need to establish a new method to obtain a compact absorbing ball in $(H^1(\Omega))^3$ which guarantee the existence of random attractor in $(H^1(\Omega))^3$. The main idea of our new method is that we first try to derive that $P$-almost surely the solution operator of stochastic PEs is compact in $(H^1(\Omega))^3$. Then we construct a compact absorbing ball in the strong solution space $(H^1(\Omega))^3$ by using the the solution operator to act on a absorbing ball.

Thirdly, showing the compact property of solution operator is still difficult. To overcome the difficulty, we take advantage of the regularity of solution operator and Aubin-Lions Lemma to achieve our goal. Specifically, we establish the continuity of the strong solutions to the 3D stochastic PEs in the space $(H^1(\Omega))^3$ with respect to time $t$ and with respect to the initial condition $(\nu_0, T_0)$. Notice that [19] only proved the strong solution is Lipschitz continuous in the space $(L^2(\Omega))^3$ with respect to the initial data but this is not enough to study the asymptotical behavior in $(H^1(\Omega))^3$ considered here. The new difficulty arose here in obtaining the regularities of the strong solution about time $t$ and initial condition is that we have no valid boundedness for the derivatives of the vertical velocity. To overcome the difficulty, the special geometry involved with the vertical velocity is used to obtain delicate a priori estimates.

It is important to point out that our method develops a general way for proving the existence of random attractor for common classes of dissipative stochastic partial differential equations and has some advantages over the common method of using compact Sobolev embedding theorem, i.e., if an absorbing ball for the solutions in space $(H^2(\Omega))^3$ does exist, our method would then further imply the existence of global random attractor in $(H^2(\Omega))^3$. In our forth coming paper [51] using the new method we prove the existence of strong attractor for the stochastic moist primitive equations, which improves the results, the existence of weak (universal) attractor for the deterministic model in [20].

The remaining of this paper is organized as follows. In section 2, we state some preliminaries and recall some results. The main result is presented in section 3, and section 4 is for its proof. Finally, in section 5, the appendix of our work, we obtain the a priori estimates for the global existence of the strong solutions to the stochastic PEs. As usual, constants $C$ may change from one line to the next, unless, we give a special declaration ; we denote by $C(a)$ a constant which depends on some parameter $a$.

## 2 Preliminaries

For $1 \leq p \leq \infty$, let $L^p(\Omega), L^p(M)$ be the usual Lebesgue spaces with the norm $| \cdot |_p$ and $| \cdot |_{L^p(M)}$ respectively. If there is no confusion, we will write $| \cdot |_p$ instead of $| \cdot |_{L^p(M)}$. For a positive integer $m$, we denote by $(H^{m,p}(\Omega), \| \cdot \|_{m,p})$ and $(H^{m,p}(M), \| \cdot \|_{H^{m,p}(M)})$ the usual Sobolev spaces, see [2]. When $p = 2$, we denote by $(H^m(\Omega), \| \cdot \|_m)$ and $(H^m(M), \| \cdot \|_{H^m(M)})$ for short. Without confusion,
we shall sometime abuse notation and denote by \( \| \cdot \|_m \) the norm in \( H^m(M) \). Let

\[
V_1 = \{ v \in (C^\infty(\Omega))^2 : \partial_z v|_{z=0} = 0, \partial_z v|_{z=-h} = 0, \quad v \cdot \overline{n}|_{\Gamma_s} = 0, \partial_n v \times \overline{n}|_{\Gamma_s} = 0, \int_{-1}^{0} \nabla \cdot v dz = 0 \},
\]

\[
V_2 = \{ T \in C^\infty(\Omega) : \partial_z T|_{z=-h} = 0, (\partial_z T + \alpha T)|_{z=0} = 0, \partial_n T|_{\Gamma_s} = 0 \}.
\]

We denote by \( V_1 \) and \( V_2 \) be the closure spaces of \( V_1 \) in \( (H^1(\Omega))^2 \), and \( V_2 \) in \( H^1(\Omega) \) under \( H^1 \)-topology, respectively. Let \( H_1 \) be the closure space of \( V_1 \) with respect to the norm \( | \cdot |_2 \). Define \( H_2 = L^2(\Omega) \).

Set

\[
\mathcal{V} = V_1 \times V_2, \quad \mathcal{H} = H_1 \times H_2.
\]

Let \( U_i := (v_i, T_i) \) be the horizontal velocity and temperature with \( i = 1, 2 \). We equip \( \mathcal{V} \) with the inner product

\[
\langle U_1, U_2 \rangle_{\mathcal{V}} := \langle v_1, v_2 \rangle_{V_1} + \langle T_1, T_2 \rangle_{V_2},
\]

\[
\langle v_1, v_2 \rangle_{V_1} := \int_{\Omega} (\nabla v_1 \cdot \nabla v_2 + \partial_z v_1 \cdot \partial_z v_2) d\delta,
\]

\[
\langle T_1, T_2 \rangle_{V_2} := \int_{\Omega} (\nabla T_1 \cdot \nabla T_2 + \partial_z T_1 \cdot \partial_z T_2) d\delta + \alpha \int_{\Gamma_u} T_1 T_2 d\Gamma_u.
\]

Subsequently, the norm of \( \mathcal{V} \) is defined by \( \| U_i \|_1 = \langle U_i, U_i \rangle_{\mathcal{V}}^{\frac{1}{2}} \). We define the inner product of \( \mathcal{H} \) by

\[
\langle U_1, U_2 \rangle_{\mathcal{H}} := \langle v_1, v_2 \rangle_{V_1} + \langle T_1, T_2 \rangle_{V_2},
\]

\[
\langle v_1, v_2 \rangle_{V_1} := \int_{\Omega} v_1 \cdot v_2 d\delta,
\]

\[
\langle T_1, T_2 \rangle_{V_2} := \int_{\Omega} T_1 \cdot T_2 d\delta.
\]

Denote \( \mathcal{V}_i' \) the dual space of \( \mathcal{V}_i \) with \( i = 1, 2 \). And define the linear operator \( A_i : \mathcal{V}_i \mapsto \mathcal{V}_i', i = 1, 2 \) as follows:

\[
\langle A_1 v_1, v_2 \rangle = \langle v_1, v_2 \rangle_{V_1}, \quad \forall \ v_1, v_2 \in \mathcal{V}_1; \quad \langle A_2 T_1, T_2 \rangle = \langle T_1, T_2 \rangle_{V_2}, \quad \forall \ T_1, T_2 \in \mathcal{V}_2.
\]

Let \( D(A_i) := \{ \eta \in \mathcal{V}_i, A_i \eta \in \mathcal{H}_i \} \). Because \( A_i^{-1} \) is a self-adjoint compact operators in \( \mathcal{H}_i \), thanks to the classic spectral theory, we can define the power \( A_i^s \) for any \( s \in \mathbb{R} \). Then \( D(A_i)^s = D(A_i^{-1}) \) is the dual space of \( D(A_i) \) and \( V_i = D(A_i^\frac{1}{2}), V_i' = D(A_i^{-\frac{1}{2}}) \). Furthermore, we have the compact embedding relationship

\[
D(A_i) \subset V_i \subset H_i \subset V_i' \subset D(A_i)',
\]

and

\[
\langle \cdot, \cdot \rangle_{\mathcal{V}_i} = \langle A_i \cdot, \cdot \rangle = \langle A_i^\frac{1}{2} \cdot, A_i^{-\frac{1}{2}} \cdot \rangle, \quad i = 1, 2.
\]

And set \( \alpha \in (0, 1) \). For a function \( f : [0, T] \rightarrow \mathbb{R} \), we define the Weyl derivative \( D_{0+}^{\alpha} f \) by

\[
D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(t)}{t^\alpha} + \alpha \int_0^t \frac{f(t) - f(u)}{(t-u)^{\alpha+1}} du \right),
\]

provided the singular integral in the right hand side exists for almost all \( t \in (0, T) \), where \( \Gamma \) denotes the Euler function. Similarly, we can define \( D_{0-}^{\alpha} f(t) \) as

\[
D_{0-}^{\alpha} f(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(T-t)^\alpha} + \alpha \int_t^T \frac{f(t) - f(u)}{(u-t)^{\alpha+1}} du \right).
\]
Let $\phi \in L^1([0,T])$, a Lebesgue space with values in $\mathbb{R}$. Then the left and right-sided fractional Riemann-Liouville integrals of $\phi$ of order $\alpha$ are defined for almost all $t \in (0,T)$ by

$$I_{0+}^\alpha \phi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u) du$$

and

$$I_{T-}^\alpha \phi(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^T (u-t)^{\alpha-1} \phi(u) du,$$

respectively. Let $f = I_{0+}^\alpha \phi$. Then the Weyl derivative of $f$ exists and $D_{0+}^\alpha f = \phi$. A similar result holds for the right-sided fractional integral. For the theory of fractional integrals and derivatives we refer to [66].

Let $W^{\alpha,1}([0,T];\mathbb{R})$ be the space of measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$|f|_{\alpha,1} := \int_0^T \left( \frac{|f(s)|}{s^\alpha} + \int_0^s \frac{|f(s) - f(u)|}{(s-u)^{\alpha+1}} du \right) ds < \infty,$$

where $0 < \alpha < \frac{1}{2}$. Similarly, we can define $W^{\alpha,1}(a,b]$ with $a,b \in \mathbb{R}$ and $a < b$. Let

$$C_\alpha(g) := \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \sup_{0<s<t<T} \left( \frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(u) - g(s)|}{(u-s)^{2-\alpha}} du \right).$$

Following [52] we define the generalized Stieltjes integral $\int_0^T f dg$ by

$$\int_0^T f dg = (-1)^{\alpha} \int_0^T D_{0+}^\alpha f(s) D_{T-}^{1-\alpha} g_T(s) ds,$$

where $g_T(s) = g(s) - g(T)$. Under the above hypotheses the integral $\int_0^t f dg$ exists for all $t \in [0,T]$, and (cf. [43])

$$\int_0^t f dg = \int_0^T f 1_{[0,t]} dg.$$

Furthermore, we have

$$|\int_0^t f dg| \leq C_\alpha(g) |f|_{\alpha,1}. \quad (2.7)$$

Let $\left((B^H(t))_{t \in \mathbb{R}^+}\right)_{i \in \mathbb{N}^+}$ be a sequence of one-dimensional, independent, identically distributed fractional Brownian motions with Hurst parameter $H \in (0,1)$, defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and starting at the origin. For $j \in \{1,2\}$, let $G_j$ be a linear, self-adjoint, positive trace-class operator with generating kernel $\eta_j$, and we write $(\epsilon_{i,j})_{i \in \mathbb{N}^+}$, for an orthonormal basis of $H_j$, consisting of eigenfunctions of the operator $G_j$ and $(\lambda_{i,j})_{i \in \mathbb{N}^+}$, for the sequence of the corresponding eigenvalues. We introduce the $H_j$-valued fractional Wiener process $(W^H_j(\cdot,t))_{t \in \mathbb{R}^+}$ with $j = 1,2$, by setting

$$W^H_j(\cdot,t) := \sum_{i=1}^{\infty} \lambda_{i,j}^{\frac{1}{2}} \epsilon_{i,j}(\cdot) B^H_i(t), \quad (2.8)$$

where the series converges a.s. in the strong topology of $H_j$ by virtue of the basic properties of the $B^H_i(t)$’s and the fact that $G_j$ is trace-class. From these basic properties, we can also conclude that $(W^H_j(\cdot,t))_{t \in \mathbb{R}^+, j=1,2}$ is a centered Gaussian process whose covariance is given by

$$
\mathbb{E}((W^H_j(\cdot,s),v)(W^H_j(\cdot,t),\hat{v})) = \frac{1}{2} \left(s^{2H} + t^{2H} - |s-t|^{2H}\right) \langle G_j v, \hat{v} \rangle
= \frac{1}{2} \left(s^{2H} + t^{2H} - |s-t|^{2H}\right) \int_{\Omega \times \Omega} \eta_j v \cdot \hat{v} d\mathbb{P} d\mathbb{P},
$$

for $v, \hat{v} \in \mathcal{S}^\prime$, where $\mathcal{S}^\prime$ is the space of Schwartz functions.
for all \( s, t \in \mathbb{R}^+ \) and all \( \nu, \nu' \in H_j \). Take a parameter \( \alpha \in (1 - H, \frac{1}{2}) \) which will be fixed throughout the paper. Let \( f \in W^{\alpha, 1}([0, T]; \mathbb{R}) \), we define
\[
\int_0^T f(s) dB^H_t(s)
\]
in sense of \((2.6)\) pathwise, since by \([43]\) we have \( C_\alpha(B^H_t) < \infty, \mathbb{P} - \text{a.s. for } t \in \mathbb{R}^+ \) and \( i \in \mathbb{N}_+ \). Denote \( L(V_j) \) the space of linear bounded operators on \( V_j \) with \( j = 1, 2 \). Suppose \( l : \Omega \times [0, T] \to L(V_j) \) be an operator-valued function such that \( le_{i,j} \in W^{\alpha, 1}([0, T]; V_j) \) for \( i \in \mathbb{N}_+, \omega \in \Omega \). We define
\[
\int_0^T \varphi(s) dW^H_j(s) := \sum_{i=1}^{\infty} \int_0^T l(s)G^2_i e_{i,j} dB^H_i(s)
\]
where \( S = \sum_{i,j} (\lambda_{i,j} - P_9) \) being understood as \( \mathbb{P} - \text{a.s. convergence in } V_j \). We consider the equation:
\[
dZ_j(t) = -A_j Z_j dt + dW^H_j(t), \tag{2.10}
\]
with initial condition \( Z_j(0) = 0, \, j = 1, 2 \). The solution of \((2.10)\) can be understood pathwise in the mild sense, that is, the solution \((Z_j(t))_{t \in [0, T]}\) is a \( V_j \)-valued process whose paths are with probability one elements of the space \( W^{\alpha, 1}([0, T]; V_j) \) for \( \alpha \in (1 - H, \frac{1}{2}) \), such that
\[
Z_j(t) = \int_0^t S_j(t-s) dW^H_j(s), \tag{2.11}
\]
where \( S_j(t-s) = e^{-A_j(t-s)}, t \in [0, T] \). Denote by \( 0 < \gamma_{1,j} \leq \gamma_{2,j} \leq \cdots \) the eigenvalues of \( A_j \) with corresponding eigenvectors \( e_{1,j}, e_{2,j}, \cdots \).

**Proposition 2.1** Assume \( \tau > 0 \) and
\[
\sum_{i=1}^{\infty} \lambda^\frac{1}{2}_{i,1} \gamma_{i,1}^\frac{3}{2} < \infty, \tag{2.12}
\]
then \((Z_1(t))_{t \in [0, \tau]}\) exists as a generalized Stieltjes integral in the sense of \([53]\) and we have
\[
(Z_1(t))_{t \in [0, \tau]} \in C([0, \tau]; (H^3(\mathcal{U}))^2) \, \text{ a.s.},
\]
where \( C([0, \tau]; (H^3(\mathcal{U}))^2) \) is the set of continuous functions defined on \([0, \tau]\) with values in \((H^3(\mathcal{U}))^2\).

Proof. According to Lemma 2.2 in \([36]\) and \((2.7)\), there exists a finite, positive random variable \( C_\alpha(B^H_t) \), depending only on \( \alpha, B^H_t \) such that
\[
\|Z_1(t)\| \leq C \sum_{i=1}^{\infty} \lambda^\frac{1}{2}_{i,1} \int_0^t A^\frac{3}{2}_i S_1(t-s)e_{i,1} dB^H_i(s)\|_2
\]
\[
\leq C \sum_{i=1}^{\infty} \lambda^\frac{1}{2}_{i,1} C_\alpha(B^H_t) \|A^\frac{3}{2}_i S_1(t-s)e_{i,1}\|_{\alpha,1}
\]
\[
= C \sum_{i=1}^{\infty} \lambda^\frac{1}{2}_{i,1} C_\alpha(B^H_t) \int_0^t \frac{|A^\frac{3}{2}_i S_1(t-s)e_{i,1}|}{s^\alpha} ds
\]
\[
+ \int_0^t \frac{|A^\frac{3}{2}_i S_1(t-s)e_{i,1} - A^\frac{3}{2}_i S_1(t-u)e_{i,1}|}{(s-u)^{1+\alpha}} du \, ds.
\]

7
Since the orthonormal basis \((e_{i,1})_{i \in \mathbb{N}^+}\) of \(H_1\) consist of eigenfunctions of operator \(A_1\) with corresponding eigenvalues \((\gamma_{i,1})_{i \in \mathbb{N}^+}\), we get

\[
\|Z_1(t)\|_3 \leq C \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 C_{\alpha}(B_i^H) \int_0^t \frac{e^{-\gamma_{i,1}(t-s)}}{s^\alpha} ds + \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 C_{\alpha}(B_i^H) \int_0^t \left( \int_0^s \frac{|e^{-\gamma_{i,1}(t-s)} - e^{-\gamma_{i,1}(t-u)}|}{(s-u)^{1+\alpha}} du \right) ds \leq C \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 C_{\alpha}(B_i^H)(t^{1-\alpha} + t^{\frac{1}{2}-\alpha}),
\]

where \(\alpha \in (0, \frac{1}{2})\). From (2.12), we deduce

\[
\sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 E(C_{\alpha}(B_i^H)) \leq C \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 < \infty
\]

by virtue of the fact that the \(B_i^H\)'s are identically distributed. Therefore, for each \(t \in [0, \tau]\)

\[
\|Z_1(t)\|_3 \leq C \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 C_{\alpha}(B_i^H) < \infty \ a.s. \quad (2.13)
\]

Furthermore, we obtain

\[
\|Z_1(t) - Z_1(s)\|_3 \leq \left\| \int_s^t S_1(t-u)dW_1^H(u) \right\|_3 + \left\| \int_0^s (S_1(t-u) - S_1(s-u))dW_1^H(u) \right\|_3 = I_1 + I_2. 
\]

Analogously to the derivation of (2.13), we get for \(I_1\)

\[
I_1 \leq C \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 C_{\alpha}(B_i^H)((t-s)^{1-\alpha} + (t-s)^{\frac{3}{2}-\alpha}).
\]

As for \(I_2\), from (2.7) and (2.14), we deduce

\[
I_2 \leq \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \| \int_0^s (S_1(t-u) - S_1(s-u))e_{i,1} dB_i^H(u) \|_3 \leq C \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 C_{\alpha}(B_i^H) \left\| (S_1(t-u) - S_1(s-u))e_{i,1} \right\|_{L^1} \leq C \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 C_{\alpha}(B_i^H) \left( \int_0^s \frac{|(S_1(t-u) - S_1(s-u))e_{i,1}|^2}{u^\alpha} du \right) + \int_0^s \int_0^u \frac{|(S_1(t-u) - S_1(s-u))e_{i,1} - (S_1(t-s_1) - S_1(s-s_1))e_{i,1}|^2 ds_1 du}{(u-s_1)^{1+\alpha}} = C \sum_{i=1}^{\infty} \lambda_{i,1}^{\frac{1}{2}} \gamma_{i,1}^2 C_{\alpha}(B_i^H)(I_3 + I_4).
\]
Similarly to the above proof, we can derive that

$$ I_3 \leq C \gamma_{i,1}^2 s^{1-\alpha}(t-s)^{\frac{1}{2}}. $$

Proceeding as in (2.13) we get

$$ I_4 \leq \int_0^s \int_0^u \left( e^{-\gamma_{i,1}(t-u)} - e^{-\gamma_{i,1}(t-s)} \right) \left( e^{-\gamma_{i,1}(t-s)} - e^{-\gamma_{i,1}(s-u)} \right) \frac{ds_1}{(u-s_1)^{1+\alpha}} du. $$

Since

$$ 1 - (e^{-\gamma_{i,1}(s-s_1)} - e^{-\gamma_{i,1}(t-s_1)})(e^{-\gamma_{i,1}(s-u)} - e^{-\gamma_{i,1}(t-u)})^{-1} = 1 - (e^{-\gamma_{i,1}(s)} - e^{-\gamma_{i,1}(t)})(e^{-\gamma_{i,1}(s-u)} - e^{-\gamma_{i,1}(t-u)})^{-1} $$

we get the estimates of $I_4$ that

$$ I_4 \leq \gamma_{i,1}^{\frac{1}{2}}(t-s)^{\frac{1}{2}} \int_0^s \int_0^u \frac{1 - e^{-\gamma_{i,1}(u-s_1)}}{(u-s_1)^{1+\alpha}} ds_1 du $$

$$ \leq C \gamma_{i,1}^{\frac{1}{2}}(t-s)^{\frac{1}{2}} \int_0^s \int_0^u \frac{\gamma_{i,1}^{\frac{1}{2}}(u-s_1)^{\frac{1}{2}}}{(u-s_1)^{1+\alpha}} du $$

$$ \leq C \gamma_{i,1} s^{2-\alpha}(t-s)^{\frac{1}{2}}. $$

By (2.14) and estimates of $I_1 - I_4$ we obtain that

$$ ||Z_1(t) - Z_1(s)||_3 \leq C \sum_{i=1}^\infty \lambda_{i,2}^{\frac{1}{2}} \gamma_{i,1}^{\frac{3}{2}} C_{\alpha}(B_i^H)(t-s)^{\frac{1}{2}}, $$

which complete the proof. \qed

Following the same steps, we can also have the result below.

**Proposition 2.2** Assume $\tau > 0$ and

$$ \sum_{i=1}^\infty \lambda_{i,2}^{\frac{1}{2}} \gamma_{i,2} < \infty, \quad (2.15) $$

then $(Z_2(t))_{t\in[0,\tau]}$ exists as a generalized Stieltjes integral in the sense of [32] and we have

$$ (Z_2(t))_{t\in[0,\tau]} \in C([0,\tau];H^3(\mathbb{R})) \text{ a.s.,} $$

where $C([0,\tau];H^3(\mathbb{R}))$ is the set of continuous functions defined on $[0,\tau]$ with values in $H^3(\mathbb{R})$.

Before proving that the mild solution (2.11) is equivalent to a distribution solution of (2.10), we introduce some notations and definitions from [38].

**Definition 2.1** The nonrandom function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called piecewise Hölder of order $\alpha$ on the interval $[T_1, T_2] \subset \mathbb{R}$ ($f \in C^\alpha_{pw}[T_1, T_2]$), if there exists a finite set of disjoint subintervals $\{[a_i, b_i], 1 \leq i \leq N \} \cup_{i=1}^N [a_i, b_i] \cup T_2 = [T_1, T_2]$ and the function $f \in C^\alpha[a_i, b_i]$ for $1 \leq i \leq N$. As before, we denote

$$ ||f||_{C^\alpha[a_i, b_i]} := \sup_{a_i \leq t < b_i} |f(t)| + \sup_{a_i \leq s < t < b_i} \frac{|f(t) - f(s)|}{|t-s|^\alpha}. $$
For $f \in C^{\alpha}[T_1, T_2]$, let

$$
\|f\|_{C^\alpha_p[T_1, T_2]} := \max_{1 \leq i \leq N} \|f\|_{C^\alpha[a_i, b_i]}.
$$

To prove the equivalence of mild solution and distribution solution of linear stochastic system, we need stochastic Fubini theorem with respect to fractional Brownian motion. For convenience, we cite the theorem here, of which proof can be found in [35].

**Theorem 2.1** Denote by $(B^H(u))_{u \in [T_1, T_2]}$ the scalar fBm with Hurst index $H \in (0, 1)$ and $T_1 < T_2, T_1, T_2 \in \mathbb{R}$. Let $\Omega' \subset \Omega$ such that $P(\Omega') = 1$ and assume for any $\omega \in \Omega'$ the function $\Phi(t, u, \omega)$ satisfy the conditions:

1) $\forall t \in (T_1, T_2)$, $\Phi(t, u, \omega)$ is piecewise Hölder of order $\beta > 1 - H$ in $u \in [T_1, T_2]$, and there exists $C = C(\omega) > 0$ such that $\|\Phi(t, \cdot, \omega)\|_{C^\alpha_p[T_1, T_2]} \leq C$;

2) the function $\int_{T_1}^{T_2} \Phi(t, u, \omega) dB^H(u)$ is Riemann integrable in the interval $[T_1, T_2]$.

Then there exist the repeated integrals

$$I_1 := \int_{T_1}^{T_2} \left( \int_{T_1}^{T_2} \Phi(t, u, \omega) dB^H(u) \right) dt \text{ and } I_2 := \int_{T_1}^{T_2} \left( \int_{T_1}^{T_2} \Phi(t, u, \omega) dt \right) dB^H(u),$$

and $I_1 = I_2, \mathbb{P}$-a.s..

**Proposition 2.3** The mild solution (2.11) to (2.10) is also a distribution solution to (2.10) and vice versa.

Proof. Let $\xi \in D(A^*_j)$. For $t \in [0, T]$, using (2.13) and Proposition 2.2. of [52], we have

$$
\int_0^t \langle Z_j(s), A^*_j \xi \rangle ds = \int_0^t \left( \int_0^s S_j(s - u) dW^H_j(u), A^*_j \xi \right) ds = \sum_{i=1}^{\infty} \lambda_{i,j}^2 \int_0^t \int_0^s \langle S_j(s - u) e_{i,j}, A^*_j \xi \rangle dB^H_i(u) ds = \sum_{i=1}^{\infty} \lambda_{i,j}^2 \int_0^t \int_0^s \mathbb{1}_{[0, s]}(u) \langle S_j(s - u) e_{i,j}, A^*_j \xi \rangle dB^H_i(s) ds,
$$

(2.16)

where $j = 1, 2$. Now we define the function $f_j : [0, t] \times [0, t] \to \mathbb{R}$ by

$$f_j(s, u) := 1_{[0, s]}(u) \langle S_j(s - u) e_{i,j}, A^*_j \xi \rangle.$$

It is easy to check that for all $s \in [0, t], f_j(s, \cdot)$ is piecewise Hölder of order 1 in $u \in [0, t]$ (see Definition 2.1), and there exists $C = C(\omega) > 0$ such that $\|f_j(s, \cdot)\|_{C^\alpha_p[0, t]} \leq C$. Indeed, for $s \in [0, t]$, in order to check the regularity of $f_j(s, \cdot)$ we only need to consider $u \leq s$. For $u_1, u_2 \in [0, s]$, we have

$$|f_j(s, u_1) - f_j(s, u_2)| \leq |\langle (S_j(s - u_1) - S_j(s - u_2)) e_{i,j}, A^*_j \xi \rangle | \leq |A^*_j \xi| e^{-\gamma(s-u_1)\gamma_{i,j}} - e^{-\gamma(s-u_2)\gamma_{i,j}} \leq C \gamma_{i,j} |u_1 - u_2|.$$

Since the integrand in $Z_j$ is infinitely differentiable with respect to time, by Theorem 4.2.1 in [52] we get that the stochastic calculus agrees with the Riemann–Stieltjes integral. Using (2.13) again,
we know $Z_j$ is Riemann integrable. Therefore, applying Theorem 2.1 to (2.16), we obtain
\[
\int_0^t (Z_j(s), A_j^* x) ds = \sum_{i=1}^\infty \lambda^\frac{1}{2}_{i,j} \int_0^t \int_0^t \int_0^t 1_{[0,i]}(u) \langle S_j(s - u) e_{i,j}, A^*_j x \rangle ds dB^H_t(u)
\]
\[
= \sum_{i=1}^\infty \lambda^\frac{1}{2}_{i,j} \int_0^t \int_0^t \langle A_j S_j(s - u) e_{i,j}, x \rangle ds dB^H_t(u)
\]
\[
= \sum_{i=1}^\infty \lambda^\frac{1}{2}_{i,j} \int_0^t \langle (S_j(t - u) - I) e_{i,j}, x \rangle dB^H_t(u)
\]
\[
= \langle Z_j(t), x \rangle - \langle W^H_j(t), x \rangle
\]
which implies that mild solution (2.11) to (2.10) is the distribution solution to (2.10). Conversely, by Theorem 4.2.1 in [52] we get that the stochastic calculus $Z_j$ agrees with the Riemann–Stieltjes integral. Then following the steps in [44] we can show that a distribution solution to (2.10) is also a mild solution to (2.10).

To simplify the notations, we denote
\[
w(x, y, z, t) = -\int_{-1}^z \nabla \cdot v(x, y, \lambda, t) d\lambda := \varphi(v)(x, y, z, t)
\]
and set
\[
\int_\Omega \cdot d\mathcal{O} := \int_\Omega \cdot , \quad \int_M \cdot dM := \int_M \cdot .
\]

**Definition 2.2** Given $T > 0$, we say a continuous $\mathcal{V}$-valued $(\mathcal{F}_t) = (\sigma(W^H_j(s), s \in [0, t]), j = 1, 2)$ adapted random field $(U(. , t))_{t \in [0, T]} = (\nu(., t), T(., t))_{t \in [0, T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a strong solution (weak solution) to problem (1.1) - (1.5) if the following two conditions hold:

1. We have $U \in L^2([0, T]; (H^2(\Omega))^3) \cap C([0, T]; \mathcal{V})$ ($U \in L^2([0, T]; (H^1(\Omega))^3) \cap C([0, T]; \mathcal{H})$) a.s.

2. The integral relation
\[
\int_\Omega v(t) \cdot \phi_1 - \int_0^t dt \int_\Omega \{(v \cdot \nabla) \phi_1 + \varphi(v) \partial_z \phi_1\} - [(f k \times v) \cdot \phi_1 + (\int_{-1}^z T dz') \nabla \cdot \phi_1]
\]
\[
+ \int_\Omega \int_0^t v \cdot L_1 \phi_1 = \int_\Omega v_0 \cdot \phi_1 + \int_\Omega W^H_1(t, w) \cdot \phi_1,
\]
\[
\int_\Omega T(t) \phi_2 - \int_0^t \int_\Omega \{(v \cdot \nabla) \phi_2 + \varphi(v) \partial_z \phi_2\} - TL_2 \phi_2 = \int_\Omega T_0 \phi_2
\]
\[
+ \int_0^t \int_\Omega Q \phi_2 + \int_\Omega W^H_2(t, w) \cdot \phi_2,
\]
hold a.s. for all $t \in [0, T]$ and $\phi = (\phi_1, \phi_2) \in D(A_1) \times D(A_2)$.

Let $u(t) = v(t) - Z_1(t)$ and $\theta(t) = T(t) - Z_2(t), t \in \mathbb{R}^+$. A stochastic process $U(t, w) = (v, T)$ is a strong solution to (1.1) - (1.5) on $[0, T]$, if and only if $(u, \theta)$ is a strong solution to the following
problem on $[0, T]$:

$$\partial_t u - \Delta u - \partial_{zz} u + [(u + Z_1) \cdot \nabla](u + Z_1) + \varphi (u + Z_1) \partial_z (u + Z_1) + f(u + Z_1) + \nabla p_x - \int_1^z \nabla T dz' = 0;$$

$$\partial_t \theta - \Delta \theta - \partial_{zz} \theta + [(u + Z_1) \cdot \nabla](\theta + Z_2) + \varphi (u + Z_1) \partial_z (\theta + Z_2) = Q;$$

$$\int_1^0 \nabla \cdot udz = 0;$$

$$\partial_z u|_{r_u} = \partial_z u|_{r_b} = 0; u \cdot \vec{n}|_{r_u} = 0; \partial_{\vec{n}} u \times \vec{n}|_{r_u} = 0;$$

$$\partial_z \theta + \alpha \theta|_{r_u} = \partial_z \theta|_{r_b} = 0; \partial_{\vec{n}} \theta|_{r_u} = 0;$$

$$\left. (u|_{t=0}, \theta|_{t=0}) \right) = (0, T_0).$$

\textbf{Definition 2.3} Let $Z_j, j = 1, 2,$ are defined above, $v_0 \in V_1, T_0 \in V_2$ and $T$ be a fixed positive time.

For $P - a.e., \omega \in \Omega$, $(u, \theta)$ is called a strong solution of the system (2.17) - (2.22) on the time interval $[0, T]$ if it satisfies (2.17) - (2.18) in the weak sense such that

$$u \in C([0, T]; V_1) \cap L^2([0, T]; (H^2(\Omega))^2),$$

$$\theta \in C([0, T]; V_2) \cap L^2([0, T]; H^2(\Omega)).$$

Let $(X, d)$ be a polish space and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a probability space, where $\tilde{\Omega}$ is the two-sided Wiener space $C_0(\mathbb{R}; X)$ of continuous functions with values in $X$, equal to 0 at $t = 0$. We consider a family of mappings $S(t, s; \omega) : X \to X$, $-\infty < s \leq t < \infty$, parametrized by $\omega \in \tilde{\Omega}$, satisfying for $\tilde{\mathbb{P}}$-a.e. $\omega$ the following properties (i)-(iv):

(i) $S(t, r; \omega)S(r, s; \omega)x = S(t, s; \omega)x$ for all $s \leq r \leq t$ and $x \in X$;

(ii) $S(t, s; \omega)$ is continuous in $X$, for all $s \leq t$;

(iii) for all $s < t$ and $x \in X$, the mapping $\omega \mapsto S(t, s; \omega)x$

is measurable from $(\tilde{\Omega}, \tilde{\mathcal{F}})$ to $(X, B(X))$ where $B(X)$ is the Borel-$\sigma$-algebra of $X$;

(iv) for all $t, x \in X$, the mapping $s \mapsto S(t, s; \omega)$ is right continuous at any point.

A set valued map $K : \tilde{\Omega} \to 2^X$ taking values in the closed subsets of $X$ is said to be measurable if for each $x \in X$ the map $\omega \mapsto d(x, K(\omega))$ is measurable, where $d(A, B) = \sup \{ \inf \{ d(x, y) : y \in B \} : x \in A \}$ for $A, B \in 2^X$, $A, B \neq \emptyset$; and $d(x, B) = d(\{x\}, B)$. Since $d(A, B) = 0$ if and only if $A \subset B$, $d$ is not a metric. A closed set valued measurable map $K : \tilde{\Omega} \to 2^X$ is named a random closed set.

Given $t \in \mathbb{R}$ and $\omega \in \tilde{\Omega}$, $K(t, \omega) \subset X$ is called an attracting set at time $t$ if, for all bounded sets $B \subset X$,

$$d(S(t, s; \omega)B, K(t, \omega)) \to 0, \text{ provided } s \to -\infty.$$ Moreover, if for all bounded sets $B \subset X$, there exists $t_B(\omega)$ such that for all $s \leq t_B(\omega)$

$$S(t, s; \omega)B \subset K(t, \omega),$$

we say $K(t, \omega)$ is an absorbing set at time $t$.

Let $\{\vartheta_t : \tilde{\Omega} \to \tilde{\Omega}\}, t \in T, T = \mathbb{R}$, be a family of measure preserving transformations of the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that for all $s < t$ and $\omega \in \tilde{\Omega}$

(a) $t, \omega \mapsto \vartheta_t(\omega)$ is measurable;

(b) $\vartheta_t(\omega)(s) = \omega(t + s) - \omega(t)$;

(c) $S(t, s; \omega)x = S(t - s, 0; \vartheta_s(\omega)x)$.

Thus $\{\vartheta_t\}_{t \in T}$ is a flow, and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), (\vartheta_t)_{t \in T}$ is a measurable dynamical system.
Definition 2.4 Given a bounded set \( B \subset X \), the set 
\[
A(B, t, \omega) = \bigcap_{T \leq t} \bigcup_{s \leq T} S(t, s, \omega)B
\]
is said to be the \( \Omega \)-limit set of \( B \) at time \( t \). Obviously, if we denote \( A(B, 0, \omega) = A(B, \theta_t \omega) \), we have \( A(B, t, \omega) = A(B, \theta_t \omega) \).

We may identify
\[
A(B, t, \omega) = \{ x \in X : \text{there exists } s_n \to -\infty \text{ and } x_n \in B \text{ such that } \lim_{n \to \infty} S(t, s_n, \omega)x_n = x \}. 
\]

Furthermore, if there exists a compact attracting set \( K(t, \omega) \) at time \( t \), it is not difficult to check that \( A(B, t, \omega) \) is a nonempty compact subset of \( X \) and \( A(B, t, \omega) \subset K(t, \omega) \).

Definition 2.5 If, for all \( t \in \mathbb{R} \) and \( \omega \in \tilde{\Omega} \), the random closed set \( \omega \to A(t, \omega) \) satisfying the following properties:

(1) \( A(t, \omega) \) is a nonempty compact subset of \( X \),

(2) \( A(t, \omega) \) is the minimal closed attracting set, i.e., if \( \bar{A}(t, \omega) \) is another closed attracting set, then \( A(t, \omega) \subset \bar{A}(t, \omega) \),

(3) it is invariant, in the sense that, for all \( s \leq t \),
\[
S(t, s; \omega)A(s, \omega) = A(t, \omega),
\]

\( A(t, \omega) \) is called the random attractor.

Let
\[
A(\omega) = A(0, \omega).
\]

Then the invariance property writes
\[
S(t, s; \omega)A(\theta_s \omega) = A(\theta_t \omega).
\]

We will prove the existence of the random attractor using Theorem 2.2 in [3]. For the convenience of reference, we cite it here.

Theorem 2.2 Let \( (S(t, s; \omega))_{t \geq s, \omega \in \tilde{\Omega}} \) be a stochastic dynamical system satisfying (i), (ii), (iii) and (iv). Assume that there exists a group \( \theta_t, t \in \mathbb{R} \), of measure preserving mappings such that condition (c) holds and that, for \( \tilde{\mathbb{P}} \)-a.e. \( \omega \), there exists a compact attracting set \( K(\omega) \) at time 0. For \( \tilde{\mathbb{P}} \)-a.e. \( \omega \), we set
\[
A(\omega) = \bigcup_{B \subset X} A(B, \omega)
\]
where the union is taken over all the bounded subsets of \( X \). Then we have for \( \tilde{\mathbb{P}} \)-a.e. \( \omega \in \tilde{\Omega} \).

(1) \( A(\omega) \) is a nonempty compact subset of \( X \), and if \( X \) is connected, it is a connected subset of \( K(\omega) \).

(2) The family \( A(\omega), \omega \in \Omega \), is measurable.

(3) \( A(\omega) \) is invariant in the sense that
\[
S(t, s; \omega)A(\theta_s \omega) = A(\theta_t \omega), \quad s \leq t.
\]

(4) It attracts all bounded sets from \( -\infty \): for bounded \( B \subset X \) and \( \omega \in \tilde{\Omega} \)
\[
d(S(t, s; \omega)B, A(\theta_t \omega)) \to 0, \quad \text{when } s \to -\infty.
\]
Moreover, it is the minimal closed set with this property: if \( \tilde{A}(\vartheta_t\omega) \) is a closed attracting set, then \( A(\vartheta_t\omega) \subset \tilde{A}(\vartheta_t\omega) \).

(5) For any bounded set \( B \subset X \) and \( t \), \( d(S(t,s;\omega)B,A(\vartheta_t\omega)) \to 0 \) in probability when \( t \to \infty \). And if the time shift \( \vartheta_t, t \in \mathbb{R} \) is ergodic

(6) there exists a bounded set \( B \subset X \) such that

\[
A(\omega) = A(B,\omega).
\]

(7) \( A(\omega) \) is the largest compact measurable set which is invariant in sense of Definition 2.5.

Before showing the existence of random attractor, we will cite Aubin-Lions Lemma which is vital to prove our main result Theorem 3.1 in this section.

**Lemma 2.1** Let \( B_0, B, B_1 \) be Banach spaces such that \( B_0, B_1 \) are reflexive and \( B_0 \subset B \subset B_1 \). Define, for \( 0 < T < \infty \),

\[
X := \left\{ h \left| h \in L^2([0,T];B_0), \frac{dh}{dt} \in L^2([0,T];B_1) \right. \right\}.
\]

Then \( X \) is a Banach space equipped with the norm \( |h|_{L^2([0,T];B_0)} + |h'|_{L^2([0,T];B_1)} \). Moreover,

\[
X \subset L^2([0,T];B).
\]

The following Lemma, a special case of a general result of Lions and Magenes [31], will help us to show the continuity of the solution to stochastic PEs with respect to time in \( (H^1(\Omega))^3 \). For the proof of the Lemma we can see [18].

**Lemma 2.2** Let \( V, H, V' \) be three Hilbert spaces such that \( V \subset H = H \subset V' \), where \( H' \) and \( V' \) are the dual spaces of \( H \) and \( V \) respectively. Suppose \( u \in L^2(0,T;V) \) and \( u' \in L^2(0,T;V') \). Then \( u \) is almost everywhere equal to a function continuous from \( [0,T] \) into \( H \).

## 3 Main Result

The aim of the paper is to prove:

**Theorem 3.1** Let \( Q \in L^2(\Omega), u_0 \in V_1, T_0 \in V_2, \alpha > \frac{1}{8} \) in (1.4). Assume conditions (2.12) and (2.15) hold. Then the solution operator \( (S(t,s;\omega))_{t \geq s, \omega \in \Omega} \) of 3D stochastic PEs (1.1) – (1.5) : \( S(t,s;\omega)(v_s,T_s) = (v(t),T(t)) \) has properties (i) – (iv) of Theorem 2.2 and possesses a compact absorbing ball \( B(0,\omega) \) in \( V \) at time 0. Furthermore, for \( \tilde{P} \)-a.e. \( \omega \), the set

\[
A(\omega) = \bigcup_{B \subset V} A(B,\omega)
\]

where the union is taken over all the bounded subsets of \( V \), is the random attractor of stochastic PEs (1.1) – (1.5) and possesses the properties (1) – (7) of Theorem 2.2 with space \( X \) replaced by space \( V \).

To prove this theorem, we need the following result concerning global well-posedness of strong solution to stochastic PEs. The regularity of the strong solution is key to prove the Theorem 3.1.

**Theorem 3.2** Let \( Q \in L^2(\Omega), u_0 \in V_1, T_0 \in V_2, T > 0 \). Assume conditions (2.12) and (2.15) hold. Then there exists a unique strong solution \( (v,T) \) of the system (1.1) – (1.5) or equivalently \( (u,\theta) \) of the system (2.17) – (2.22) on the interval \([0,T]\) which is Lipschitz continuous with respect to the initial data and the noises in \( V \) and \( C([0,T];V) \) respectively.
Remark 3.1

(1) Compared with regularity of strong solution in [19], we improve the regularity of the strong solution by proving the continuity of strong solution with respect to initial data in \((H^1(\Omega))^3\). This is key to prove the compact property of the solution operator in \(V\) and construct the compact absorbing ball in \(V\). Notice that [19] only proved the strong solution is Lipschitz continuous in the space \((L^2(\Omega))^3\) with respect to the initial data but this is not enough to study the asymptotical behavior in \((H^1(\Omega))^3\) considered here.

(2) With the help of Lemma 2.2, we prove the the continuity of strong solution with respect to time in \(V\) and obtain a priori estimates to prove the compact property of the solution operator in \(V\).

(3) We reduce the regularity of \(Q\) from \(H^1(\Omega)\) to \(L^2(\Omega)\) which is more natural. There is some gap between \(H^1(\Omega)\) and \(L^2(\Omega)\).

(4) Due to the radiation properties of the clouds, we should consider the effect of the noise on the equations of the heat conduction. Therefore, the stochastic PEs where both horizontal momentum equations and heat conduction equations are disturbed by the fractional noises are considered. Since the authors of [19] didn’t consider the effect of the noises on the heat conduct equation, our model is more complicated than the previous one. There are more challenges for the present model, especially when we consider the existence of random attractor.

(5) We simplify the proof of the the global well-posedness of stochastic PEs in [19]. By making a delicate and careful argument of the a priori estimates in \((L^4(\Omega))^3\), we find the a priori estimates in \((L^3(\Omega))^3\) is not necessary.

(6) Lastly, we establish the regularity of the strong solution with respect to the fractional noises. We know this property is important to the ergodicity of stochastic PEs with Wiener noises.

4 Proof Of Main Result

The aim of this section is to prove the Theorem 3.1. Since we need the regularity of the strong solution and the a priori estimates to prove the compact property of solution operator, we should firstly prove Theorem 3.2. Then we study the growth property and the moment estimates of O-U process driven by fractional noises. The proof of the Theorem 3.1 is completed in the last of this section.

4.1 Proof Of Theorem 3.2. Before giving our proof, we should notify that the global well-posedness of (2.17) – (2.22) is equivalent to (5.86) – (5.91).

In the following, we will complete our proof of the global well-posedness of stochastic PEs by three steps. Firstly, we will prove the global existence of strong solution. Then, we will show that the solution is continuous in the space \(V\) with respect to \(t\). Finally, we will obtain the continuity in \(V\) with respect to the initial data.

Step 1: We prove the global existence of strong solution.

In the appendix, we prove the local existence of the strong solution to stochastic PEs and obtain the a priori estimates in \(V\). As we have indicated in the appendix that \([0, \tau_s]\) is the maximal interval of existence of the solution of (5.86) – (5.91), we infer that \(\tau_s = \infty\), a.s.. Otherwise, if there exists \(A \in \mathcal{F}\) such that \(\mathbb{P}(A) > 0\) and for fixed \(\omega \in A, \tau_s(\omega) < \infty\), it is clear that

\[
\limsup_{t \to \tau_s(\omega)} (\|u(t)\|_1 + \|\theta(t)\|_1) = \infty, \quad \text{for any } \omega \in A,
\]

which contradicts the priori estimates (5.134), (5.138) and (5.140). Therefore \(\tau_s = \infty\), a.s., and the strong solution \((u, \theta)\) exists globally in time a.s..

Step 2: We show the continuity of strong solutions with respect to \(t\).
Multiplying (5.86) by $\eta \in \mathcal{V}_1$, integrating with respect to space variable, yields
\[
\langle \partial_t A^\frac{1}{2}_1 u, \eta \rangle = \langle \partial_t u, A^\frac{1}{2}_1 \eta \rangle = -\langle A_1 u, A^\frac{1}{2}_1 \eta \rangle - \langle (u + Z_1) \cdot \nabla \rangle (u + Z_1), A^\frac{1}{2}_1 \eta \rangle \\
-\langle \varphi (u + Z_1) \partial_z (u + Z_1), A^\frac{1}{2}_1 \eta \rangle - \langle f(u + Z_1)^{\perp}, A^\frac{1}{2}_1 \eta \rangle \\
+ \left\langle \int_{-1}^1 \nabla \theta dz', A^\frac{1}{2}_1 \eta \right\rangle + \langle \beta Z_1, A^\frac{1}{2}_1 \eta \rangle,
\]
where we have used $\langle \nabla p_x, A^\frac{1}{2}_1 \eta \rangle = 0$ which follows by integration by parts formula. Taking a similar argument in (5.136), we get
\[
\langle \varphi (u + Z_1) \partial_z (u + Z_1), A^\frac{1}{2}_1 \eta \rangle \leq \|u + Z_1\|_1 \|u + Z_1\|_2 \|A^\frac{1}{2}_1 \eta\|_2.
\]
By Hölder inequality and Sobolev embedding theorem, we have
\[
\|\partial_t (A^\frac{1}{2}_1 u)\|_{\mathcal{V}'_1} \leq C (\|u\|_2 + \|u + Z_1\|_1 \|u + Z_1\|_2 + |u|_2 + |Z_1|_2 + |\nabla \theta|_2).
\]
In view of Proposition 2.1 and the following results about the regularity of $u$
\[
u \in L^\infty([0, T]; \mathcal{V}_1) \cap L^2([0, T]; (H^2(\Omega))^2), \quad Z_1 \in C([0, T]; (H^3(\Omega))^2) \quad \forall T > 0,
\]
we obtain
\[
A^\frac{1}{2}_1 u \in L^2([0, T]; \mathcal{V}_1), \quad \partial_t (A^\frac{1}{2}_1 u) \in L^2([0, T]; \mathcal{V}'_1), \quad \forall T > 0,
\]
which together with Lemma 2.2 implies
\[
A^\frac{1}{2}_1 u \in C([0, T]; H_1) \text{ or } u \in C([0, T]; \mathcal{V}_1) \text{ a.s.}
\]
To study the regularity of $\theta$, we choose $\xi \in \mathcal{V}_2$. By (5.87) we have
\[
\langle \partial_t A^\frac{1}{2}_2 \theta, \xi \rangle = \langle \partial_t \theta, A^\frac{1}{2}_2 \xi \rangle = \langle A_2 \theta, A^\frac{1}{2}_2 \xi \rangle - \langle (u + Z_1) \cdot \nabla (\theta + Z_2), A^\frac{1}{2}_2 \xi \rangle \\
+ \langle \varphi (u + Z_1) \partial_z (\theta + Z_2), A^\frac{1}{2}_2 \xi \rangle + \langle Q, A^\frac{1}{2}_2 \xi \rangle + \langle \beta Z_2, A^\frac{1}{2}_2 \xi \rangle.
\]
Taking a similar argument as above, we get
\[
\langle \varphi (u + Z_1) \partial_z (\theta + Z_2), A^\frac{1}{2}_2 \xi \rangle \leq C \|u + Z_1\|_1 \|u + Z_1\|_2 \|\theta + Z_2\|_1 \|\theta + Z_2\|_2 \|A^\frac{1}{2}_2 \xi\|_2.
\]
Then by Hölder inequality and Sobolev embedding theorem, we arrive at
\[
|\partial_t A^\frac{1}{2}_2 \theta|_{\mathcal{V}_2} \leq C (\|A_2 \theta\|_2 + \|u + Z_1\|_1 \|\theta + Z_2\|_2 \\
+ \|u + Z_1\|_1 \|u + Z_1\|_2 \|\theta + Z_2\|_1 \|\theta + Z_2\|_2 + |\theta|_2 + |Z_2|_2).
\]
By step one and Proposition 2.2, we have also proved that
\[
\theta \in L^\infty([0, T]; \mathcal{V}_2) \cap L^2([0, T]; H^2(\Omega)), \quad Z_2 \in C([0, T]; H^3(\Omega)), \quad \forall T > 0.
\]
Therefore, by the above argument we have
\[
A^\frac{1}{2}_2 \theta \in L^2([0, T]; \mathcal{V}_2), \quad \partial_t (A^\frac{1}{2}_2 \theta) \in L^2([0, T]; \mathcal{V}'_1), \quad \forall T > 0,
\]
which together with Lemma 2.2 implies
\[
A^\frac{1}{2}_2 \theta \in C([0, T]; H_2) \text{ or } \theta \in C([0, T]; \mathcal{V}_2), \quad \text{a.s.}
\]
Step 3: We obtain the continuity in $\mathcal{V}$ with respect to the initial data. In order to show the uniqueness of the solutions. Let $(v_1, T_1)$ and $(v_2, T_2)$ be two solutions of the system (5.86) - (5.91) with corresponding pressure $p_v'$ and $p_v''$, and initial data $((v_0)_1, (T_0)_1)$ and $((v_0)_2, (T_0)_2)$, respectively. Denote by $v = v_1 - v_2, p_b = p_v' - p_v''$ and $\mathbf{T} = T_1 - T_2$. Then we have

$$
\begin{align*}
\partial_t v - \Delta v - \partial_z v + [(v_1 + Z_1) \cdot \nabla] v + (v \cdot \nabla)(v_2 + Z_1) \\
+ \varphi(v_1 + Z_1)v_z + \varphi(v) \partial_z (v_2 + Z_1) + f k \cdot v + \nabla p_b - \int_{-1}^{z} \nabla \mathbf{T} dz' = 0, \\
\partial_t \mathbf{T} - \Delta \mathbf{T} - \partial_z \mathbf{T} + [(v_1 + Z_1) \cdot \nabla] \mathbf{T} + (v \cdot \nabla)(T_2 + Z_2) \\
+ \varphi(v_1 + Z_1) \mathbf{T}_z + \varphi(v) \partial_z (T_2 + Z_2) = 0,
\end{align*}
$$

(4.23)

$$
\int_{-1}^{0} \nabla \cdot v dz = 0,
$$

(4.24)

$$
v(x, y, z, 0) = (v_0)_1 - (v_0)_2, \quad \mathbf{T}(x, y, z, 0) = (T_0)_1 - (T_0)_2,
$$

(4.26)

$(v, \mathbf{T})$ satisfies the boundary value conditions (5.89) - (5.90).

(4.27)

Recall the notations $L_1$ in the introduction with $i = 1, 2$. Multiplying $L_1 v$ in equation (4.23) and integrating with respect to spatial variable yields,

$$
\begin{align*}
\frac{1}{2} \partial_t (|\nabla v|^2 + |\partial_z v|^2) + |\Delta v|^2 + |\partial_z v|^2 + |\nabla v|^2 \\
= - \int_{0}^{1} \{[(v_1 + Z_1) \cdot \nabla] v \} \cdot L_1 v - \int_{0}^{1} \varphi(v_1 + Z_1) v_z \cdot L_1 v \\
- \int_{0}^{1} [\varphi(v) \partial_z (v_2 + Z_1)] \cdot L_1 v - \int_{0}^{1} [(v \cdot \nabla)(v_2 + Z_1)] \cdot L_1 v \\
- \int_{0}^{1} (f k \cdot v) \cdot L_1 v + \int_{0}^{1} (\int_{-1}^{z} \nabla \mathbf{T} dz' \cdot L_1 v \\
= \Sigma_{i=1}^{6} K_i.
\end{align*}
$$

(4.28)

To estimate $K_1$, by the Agmon inequality and Hölder inequality, we obtain

$$
K_1 \leq |v_1 + Z_1|_{\infty} |\nabla v|_2 |L_1 v|_2 \\
\leq C \|v_1 + Z_1\|_{1}^{\frac{3}{2}} \|v_1 + Z_1\|_{2}^{\frac{1}{2}} \|v\|_1 \|v\|_2 \\
\leq C \|v_1 + Z_1\|_{1} \|v_1 + Z_1\|_{2} \|v\|_1^2 + \varepsilon \|v\|_2^2.
$$

Then, using Hölder inequality, interpolation inequality and Sobolev embedding theorem, we get the estimate of $K_2$,

$$
K_2 \leq \int_{0}^{1} \left| \int_{-1}^{0} (\nabla \cdot v_1 + \nabla \cdot Z_1) dz \right| \cdot |v_2| \cdot |L_1 v| \\
\leq |\nabla \cdot \tilde{v}_1 + \nabla \cdot \tilde{Z}_1|_{L^4(M)} \int_{-1}^{0} \|v_2\|_{L^4(M)} |L_1 v|_{L^2(M)} \\
\leq C \left( \|v_1\|_1^{\frac{1}{4}} + \|Z_1\|_1^{\frac{1}{4}} \right) \left( \|v_1\|_2^{\frac{1}{4}} + \|Z_1\|_2^{\frac{1}{4}} \right) \|v\|_1^{\frac{1}{4}} \|v\|_2^{\frac{1}{4}} \\
\leq \varepsilon \|v_2\|_2^2 + C \left( \|v_1\|_1^{\frac{1}{4}} + \|Z_1\|_1^{\frac{1}{4}} \right) \left( \|v_1\|_2^{\frac{1}{4}} + \|Z_1\|_2^{\frac{1}{4}} \right) \|v\|_1^{2}.
$$
By an analogous argument as above, we infer that
\[
K_3 \leq \int_0^1 \left( \int_{-1}^0 \nabla \cdot v \, dz \right) \cdot |\partial_z (v_2 + Z_1)| \cdot |L_1 v| \\
\leq |\nabla \cdot v|_{L^4(M)} \int_{-1}^0 |\partial_z (v_2 + Z_1)|_{L^4(M)} |L_1 v|_{L^2(M)} \\
\leq C \|v\|_1^2 |v_2|_2^2 \int_{-1}^0 |v_2 + Z_1|^2_{H^1(M)} |v_2 + Z_1|^2_{H^2(M)} |v|_2^2 \, dz \\
\leq \varepsilon \|v\|_2^2 + C \|v\|_1^2 |v_2 + Z_1|^2_{H^1} |v_2 + Z_1|_2^2.
\]

By virtue of Hölder inequality,
\[
K_4 \leq |v|_4 (|\nabla v_2|_4 + |\nabla Z_1|_4) |L_1 v|_2 \leq \varepsilon \|v\|_2^2 + C \|v\|_2^2 (\|v_2\|_2^2 + \|Z_1\|_2^2).
\]

After some basic calculus, we obtain the estimates of \(K_5\) and \(K_6\) as the following,
\[
K_5 + K_6 \leq \varepsilon \|v\|_2^2 + C |v_2|_2^2 + C \|T\|_2^2.
\]

By the boundary conditions, \(|T|_2^2\) is equivalent to \(|T_z|_2^2 + |T(z = 0)|_2^2\), then \(|T|_1^2\) is equivalent to \(|\nabla T|_2^2 + |T_z|_2^2 + |T(z = 0)|_2^2\). Keeping this in mind and taking an inner product of the equation (4.24) with \(L_2 T\), we have
\[
\frac{1}{2} \partial_t (|\nabla T|^2_2 + |T_z|^2_2 + \alpha |T(z = 0)|_2^2) + |\Delta T|^2_2 + |T_{zz}|_2^2 + |\nabla T_z|^2_2 + \alpha |\nabla T(z = 0)|_2^2 = - \int_0^1 [(v_1 + Z_1) \cdot \nabla T] L_2 T - \int_0^1 [v \cdot \nabla (T_2 + Z_2)] L_2 T \\
- \int_0^1 \varphi (v_1 + Z_1) T_z L_2 + \varphi (v) \partial_z (T_2 + Z_2) L_2 T := \sum_{i=1}^4 J_i. \tag{4.29}
\]

By Agmon inequality, we get that
\[
J_1 + J_2 \leq |v_1 + Z_1|_\infty |\nabla T|_2 |L_2 T|_2 + |v_4|_4 |\nabla (T_2 + Z_2)|_4 |L_2 T|_2 \\
\leq C |v_1 + Z_1|_1 \frac{1}{2} |v_1 + Z_1|^\frac{1}{2} |T|_1 |T|_2 + C |v_1|_1 |T_2 + Z_2|_2 |T|_2 \\
\leq \varepsilon \|T\|_2^2 + C \|v\|_2^2 |T_2 + Z_2|_2^2 + C |v_1 + Z_1|_1 |v_1 + Z_1|_2 |T|_2^2.
\]

Taking an similar argument as \(K_2\), we obtain
\[
J_3 \leq \int_0^1 \left( \int_{-1}^0 |\nabla \cdot (v_1 + Z_1)| \, dz \right) \cdot |T_z| \cdot |L_2 T| \\
\leq \left( \int_{-1}^0 |L_2 T|_{L^2(M)} |T_z|_{L^4(M)} \, dz \right) \int_{-1}^0 |\nabla \cdot (v_1 + Z_1)| \, dz |L^4(M) \\
\leq C \int_{-1}^0 \|T|_{H^2(M)} \|T_z|_{H^1(M)} \|v_1 + Z_1|_{L^4(M)} \, dz \\
\leq C \|T|_2^2 \|T_z|_2^2 \int_{-1}^0 \|v_1 + Z_1|_{H^1(M)} \|v_1 + Z_1|_{H^2(M)} \, dz \\
\leq C \|T|_2^2 \|T_z|_2^2 \|v_1 + Z_1|_{H^1} \|v_1 + Z_1|_2 \|
\leq \varepsilon \|T\|_2^2 + C \|T|_2^2 \|v_1 + Z_1|_2^2 \|v_1 + Z_1|_2^2.
\]
Follow the similar steps, we can prove that

\[ J_4 \leq \left( \int_{-1}^{0} |L_z| |\partial_z(T_2 + Z_2)| \right) \int_{-1}^{0} |\nabla \cdot v| dz |L_z| \]

\[ \leq \left( \int_{-1}^{0} |L_z| L_z^2(M) |\partial_z(T_2 + Z_2)| L_4(M)dz \right) \int_{-1}^{0} |\nabla \cdot v| dz |L_z| L_4(M) \]

\[ \leq C \|T\|_2 \|T_2 + Z_2\|_2^{1/2} \|T_2 + Z_2\|_2^{1/2} \|v\|_2^{1/2} \|v\|_2^{1/2} \]

Since \( |\nabla v|^2 + |\partial_z v|^2 \) is equivalent to \( |v|^2 \) and \( |\nabla T|^2 + |T_2|^2 + |T(z = 0)|^2 \) is equivalent to \( |T|^2 \), letting \( \varepsilon \) be small enough, by (4.28) – (4.29) and estimates of \( K_1 - K_6 \) and \( J_1 - J_4 \) we have

\[ \frac{d\eta(t)}{dt} + \|v\|^2 + \|T\|^2 \leq \eta(t)\xi(t). \]  

(4.30)

Since \( (v_i(t), T_i(t)), i = 1, 2, \) is the solution of stochastic PEs in sense of Definition 2.3 which ensure that

\[ \int_0^t \xi(s)ds < \infty, \text{ a.s., for all } t \in (0, \infty), \]

we conclude that

\[ \eta(t) \leq \eta(0)e^{C \int_0^t \xi(s)ds}. \]

Therefore, we proved that for any \( t \in (0, \infty), (u(t), \theta(t)) \) is Lipschitz continuous in \( \mathcal{V}_1 \times \mathcal{V}_2 \) with respect to the initial data \( (u(0), \theta(0)) \), which is equivalent to that strong solution \( (v(t), T(t)) \) of (1.1) – (1.5) is Lipschitz continuous in \( \mathcal{V}_1 \times \mathcal{V}_2 \) with respect to the initial data \( (v_0, T_0) \), for any \( t \in (0, \infty) \). Following an analogous argument, we can also show that \( (v(t), T(t))_{t \in [0, \tau]} \) is Lipschitz continuous with respect to the noises in \( C([0, \tau]; \mathcal{V}) \) with the norm \( \sup_{t \in [0, \tau]} \|v\|_1 \) for arbitrary \( \tau > 0 \).

\[ \square \]

4.2. Growth Property And Moment Estimates Of O – U Processes.

In order to obtain the existence of random attractor for system (1.1) – (1.5), we need the following property of Ornstein-Uhlenbeck process driven by fBm. First of all, we extend the definition of generalized Stieltjes integral (2.6) to infinite interval. For arbitrary \( a, b \in \mathbb{R} \) and \( a < b \), we assume \( f \in W^{a,1}([a, b]; \mathbb{R}) \) (see section 2) and \( g \) satisfies

\[ C_\alpha(g)^b_a = \frac{1}{\Gamma(1 - \alpha)\Gamma(\alpha)} \sup_{a < s_1 < s_2 < b} \left( \int_{s_2}^{s_1} \left| \frac{g(s_1) - g(s_2)}{(s_2 - s_1)^{1-\alpha}} + \int_{s_1}^{s_2} \frac{|g(u) - g(s_1)|}{(u - s_1)^{2-\alpha}} du \right| \right) \]

\[ = \frac{1}{\Gamma(1 - \alpha)\Gamma(\alpha)} \sup_{a < s_1 < s_2 < b} \left| \int_{s_1}^{s_2} D^{1-\alpha}_{s_2} \cdot g_s_{-} - (s_1) \right| < \infty. \]

Then, for \( t \in \mathbb{R} \), we define

\[ \int_{-\infty}^{t} f dg = \lim_{\tau \to \infty} \int_{\tau}^{t} f dg. \]
provided the limit exists and \( \int_0^t f\, dg \) is defined as (2.6) (or see [43] and [52]).

To study the long-term behavior of stochastic PEs, we should consider the moment estimates and growth properties of O-U processes (see [43]) and fBm \( B^H \) and growth properties of O-U processes (provided the limit exists and \( \int_0^t f\, dg \)).

\[
Z_j(t) = \int_{-\infty}^t e^{-(t-s)(A_j+\beta)} dW^H_j(s).
\]

**Lemma 4.1** Assume (2.12) hold, then for any \( \varepsilon > 0 \) and \( m \geq 2 \), there exists positive constant \( \beta \) depending only on \( \varepsilon \) and \( m \) such that

\[
E\| Z_1 (0) \|^m < \varepsilon. \tag{4.31}
\]

**Proof.** Due to (2.12) and Theorem 2.5 in [52], we have for \( t \in [-1,0] \)

\[
\| Z_1 (t) \|_3 \leq \sum_{i=1}^{\infty} \sqrt{\lambda_i \gamma_i} \int_{-\infty}^t e^{-(\gamma_i+\beta)(t-s)} dB^H_i(s) \leq \sum_{i=1}^{\infty} \sqrt{\lambda_i \gamma_i} \int_{t-1}^t e^{-(\gamma_i+\beta)(t-s)} dB^H_i(s) + \sum_{i=1}^{\infty} \sqrt{\lambda_i \gamma_i} \int_{-\infty}^{t-1} e^{-(\gamma_i+\beta)(t-s)} dB^H_i(s) := I_1 + I_2. \tag{4.32}
\]

By the definition of stochastic calculus and inequality (2.7), we infer that

\[
I_1 \leq \sum_{i=1}^{\infty} \sqrt{\lambda_i \gamma_i} \left( C_\alpha (B^H_i) \right) \left[ \int_{t-1}^t e^{-(\gamma_i+\beta)(t-s)} ds + \int_{t-1}^s e^{-(\gamma_i+\beta)(t-s)} \frac{e^{-\gamma_i (t-s)}}{(s-u)^{1+\alpha}} du \right] \leq C \sum_{i=1}^{\infty} \sqrt{\lambda_i \gamma_i} \left( C_\alpha (B^H_i) \right) \left[ \int_{t-1}^t e^{-(\gamma_i+\beta)(t-s)} ds + \int_{t-1}^s (s-u)^{-\frac{1}{2}} du \right] \leq C \beta^{-\frac{1}{2}} \sum_{i=1}^{\infty} \sqrt{\lambda_i \gamma_i} \left( C_\alpha (B^H_i) \right). \tag{4.33}
\]

From (4.33) and Minkovski inequality, we reach

\[
(EI^m)^{\frac{1}{m}} \leq C (EC^m_\alpha (B^H_i))^{\frac{1}{m}} \beta^{-\frac{1}{2}}, \tag{4.34}
\]

where the first two inequalities follow by the facts that \( (B^H_i)_{i \in \mathbb{N}^+} \) is a i.i.d. sequence and each fBm \( B^H \) has stationary increments which satisfy Lemma 7.5 in [43]. To estimate \( I_2 \), by (4.32) we deduce that

\[
I_2 \leq \sum_{i=1}^{\infty} \sqrt{\lambda_i \gamma_i} \sum_{n=1}^{\infty} \int_{n(t-1)}^{n(t-1)} e^{-(\gamma_i+\beta)(t-s)} dB^H_i(s).
\]
Since $e^{-((\gamma_1+\beta)((t-1)-s))}$ has all derivatives of any order in any interval and $B^H_t$ is $\gamma$-Hölder continuous for all $\gamma < H$, by the Theorem 4.2.1 in [22] stochastic calculus $\int_{(n+1)(t-1)}^{(n+1)(t-1)} e^{-((\gamma_1+\beta)((t-1)-s))} dB^H_t(s)$ defined by (2.6) is equal to Riemann-Stieltjes integral. Therefore

$$I_2 \leq \sum_{i=1}^{n}(\sum_{i=1}^{n} e^{-((\gamma_1+\beta)((t-1)-s))} dB^H_t(s)).$$

For $n \geq 2$, by (2.7) and elementary arguments we deduce

$$\left| \int_{(n+1)(t-1)}^{n(t-1)} e^{-((\gamma_1+\beta)((t-1)-s))} dB^H_t(s) \right|$$

$$\leq C_\alpha(B^H_t)^{n(t-1)}(n+1)(t-1) \left( \int_{(n+1)(t-1)}^{n(t-1)} \frac{e^{-((\gamma_1+\beta)((t-1)-s))}}{(s-(n+1)(t-1))^{\alpha}} ds \right)$$

$$+ \int_{(n+1)(t-1)}^{n(t-1)} \int_s^t \frac{|e^{-((\gamma_1+\beta)((t-1)-s))} - e^{-((\gamma_1+\beta)((t-1)-u))}|}{(s-u)^{1+\alpha}} duds$$

$$\leq C_\alpha(B^H_t)^{n(t-1)}(n+1)(t-1) \left( \int_{(n+1)(t-1)}^{n(t-1)} \frac{(\gamma_1+\beta)^{-2}(t-1-s)^{-2}}{(s-(n+1)(t-1))^{\alpha}} ds \right)$$

$$+ \int_{(n+1)(t-1)}^{n(t-1)} e^{-((\gamma_1+\beta)((t-1)-s))} \int_s^t \frac{1-e^{-((\gamma_1+\beta)(s-u))}}{(s-u)^{1+\alpha}} duds$$

$$\leq C_\alpha(B^H_t)^{n(t-1)}(n+1)(t-1) \left( (\gamma_1+\beta)^{-2}(t-1-s)^{-2} \int_{(n+1)(t-1)}^{n(t-1)} \frac{(\gamma_1+\beta)^{\frac{1}{2}}(s-u)^{\frac{1}{2}}}{(s-u)^{1+\alpha}} duds \right)$$

$$+ (\gamma_1+\beta)^{-\frac{3}{2}}(n-1)^{-2}(t-1)^{-2} |t-1|^1 - \alpha$$

$$\leq C\alpha(B^H_t)^{n(t-1)}(\gamma_1, \beta)^{-\frac{3}{2}}(n-1)^{-2}.$$
\[ a, b \in \mathbb{R}, a < b \text{ and } \lambda \in (0, 1) \text{ in the last step. Combining (4.35) - (4.37), we reach} \]
\[
I_2 \leq C \sum_{i=1}^{\infty} \sqrt{\lambda_i} \frac{\gamma_i}{\sqrt{\gamma_i + \beta}} e^{-(\gamma_i + \beta)} C_\alpha(B_i^{H})_{t}^{(t-1)} \\
+ \sum_{i=1}^{\infty} \sqrt{\lambda_i} \gamma_i e^{-(\gamma_i + \beta)} \sum_{n=2}^{\infty} C_\alpha(B_i^{H})_{(n+1)(t-1)}^{(n-1)} (n-1)^{-2}.
\]

Since \((B_i^{H})_{t}^{(t-1)}\) is i.i.d sequence and each fBm \(B_i^{H}\) has stationary increments, by Lemma 7.5 in [43] we obtain
\[
EI_2 \leq CE\left(C_\alpha(B_i^{H})_{t}^{(t-1)}\right) \sum_{i=1}^{\infty} \sqrt{\lambda_i} \gamma_i e^{-(\gamma_i + \beta)} \left(1 + \sum_{n=2}^{\infty} C(n-1)^{-2}\right) \\
\leq CE\left(C_\alpha(B_i^{H})_{t}^{(t-1)}\right) \sum_{i=1}^{\infty} \sqrt{\lambda_i} \gamma_i e^{-(\gamma_i + \beta)} \sum_{n=2}^{\infty} (n-1)^{-2} < \infty,
\]
which implies \(I_2 < \infty, \text{a.s.}\). Therefore, Minkowsky inequality we have
\[
(EI_2)^{\frac{1}{m}} \leq C(E C_\alpha^{m}(B_i^{H})_{t}^{(t-1)})^{\frac{1}{m}} \sum_{i=1}^{\infty} e^{-(\gamma_i + \beta)} \\
+ (E C_\alpha^{m}(B_i^{H})_{t}^{(t-1)})^{\frac{1}{m}} \sum_{i=1}^{\infty} e^{-(\gamma_i + \beta)} (n-1)^{-2} \\
\leq C e^{-\beta} \sum_{i=1}^{\infty} e^{-\gamma_i}. \tag{4.38}
\]

On account of (4.34) and (4.38), we can choose \(\beta\) big enough such that \(EI^m \leq \varepsilon\) which complete the proof. \(\square\)

Taking a similar argument, we have the moment estimates for \(Z_2(0)\) in the following.

**Lemma 4.2** For any \(\varepsilon > 0\) and \(m \geq 2\), under condition (2.15) there exists a positive constant \(\beta\) depending only on \(\varepsilon\) and \(m\) such that
\[
E\|Z_2(0)\|_{3}^{m} < \varepsilon. \tag{4.39}
\]

Concerning the growth properties of \((Z_j(t))_{t \in \mathbb{R}}, j = 1, 2\), we have Lemma 4.3 and Lemma 4.4 below.

**Lemma 4.3** Under the condition of Proposition 2.1, there exists a random variable \(r(\omega)\) taking finite values, i.e., \(r(\omega) < \infty\), for all \(\omega \in \Omega\), such that
\[
\|Z_1(t)\|_{3} < r(\omega),
\]
for all \(t \in (-\infty, 0]\).

Proof.
\[
\|Z_1(t)\|_{3}^{2} = \sum_{i=1}^{\infty} \lambda_i \gamma_i^{3} \lim_{\tau \to -\infty} \left| \int_{\tau}^{t} e^{-(\gamma_i + \beta)(t-s)} dB_i^{H}(s) \right|^{2}. \tag{4.40}
\]

By the concept of stochastic integral driven by fBm, we have
\[
\int_{\tau}^{t} e^{-(\gamma_i + \beta)(t-s)} dB_i^{H}(s) = (-1)^{\alpha} \int_{\tau}^{t} \frac{1}{\Gamma(\alpha)} \left( \frac{e^{-(\gamma_i + \beta)(t-s)}}{(s - \tau)^{\alpha}} \right) + \alpha \int_{\tau}^{s} \frac{e^{-(\gamma_i + \beta)(t-s)} - e^{-(\gamma_i + \beta)(t-u)}}{(s - u)^{\alpha+1}} du \]
\[
\cdot \frac{(-1)^{1-\alpha} B_i^{H}(t)}{(t - s)^{1-\alpha}} + (1 - \alpha) \int_{s}^{t} \frac{B_i^{H}(s) - B_i^{H}(v)}{(v - s)^{2-\alpha}} dv) ds. \tag{4.41}
\]
Substituting (4.41) into (4.40) yields

\[
\|Z_1(t)\|_2^2 \leq C \sum_{i=1}^{\infty} \lambda_{i,1}^{\gamma_{i,1}^3} \lim_{t \to -\infty} \left| \int_{\tau}^{t} e^{-\gamma_{i,1}^3 (t-s)} B_i^H(t) - B_i^H(s) \, ds \right|^2
\]

\[
+ C \sum_{i=1}^{\infty} \lambda_{i,1}^{\gamma_{i,1}^3} \lim_{t \to -\infty} \left| \int_{\tau}^{t} \frac{B_i^H(t) - B_i^H(s)}{(t-s)^{1-\alpha}} \left( \int_{\tau}^{s} e^{-\gamma_{i,1}^3 (t-s)} - e^{-\gamma_{i,1}^3 (t-u)} \frac{du}{(s-u)^{\alpha+1}} \right) \, ds \right|^2
\]

\[
+ C \sum_{i=1}^{\infty} \lambda_{i,1}^{\gamma_{i,1}^3} \lim_{t \to -\infty} \left| \int_{\tau}^{t} \frac{B_i^H(t) - B_i^H(s)}{(t-s)^{1-\alpha}} \left( \int_{s}^{t} \frac{B_i^H(s) - B_i^H(v)}{(v-s)^{2-\alpha}} \, dv \right) \, ds \right|^2
\]

\[
:= I_1 + I_2 + I_3 + I_4. \quad (4.42)
\]

Applying Lemma 2.6 in [37], we deduce for fixed \( t < 0 \) and all \( s \in (-\infty, t] \)

\[
e^{-\gamma_{i,1}^3 (t-s)} |B_i^H(t) - B_i^H(s)|
\]

\[
= e^{-\gamma_{i,1}^3 (t-s)} |B_i^H(t) - B_i^H(s)| \sum_{n=1}^{\infty} I_{[(n+1)t \leq s \leq nt]}(s, t)
\]

\[
\leq \sum_{n=1}^{\infty} e^{-\gamma_{i,1}^3 (t-s)} |B_i^H(t) - B_i^H(s)| I_{[(n+1)t \leq s \leq nt]}(s, t)
\]

\[
\leq \sum_{n=1}^{\infty} e^{(\gamma_{i,1}^3 - 1) t} [(n+1)^2 t^2 + k(\omega)]
\]

\[
\leq C \sum_{n=2}^{\infty} \frac{(n+1)^2 t^2 + k(\omega)}{[(\gamma_{i,1}^3 + \beta)(n+1)t]^4}
\]

\[
= \frac{C}{t^2(\gamma_{i,1}^3 + \beta)^2} \left( \sum_{n=1}^{\infty} \frac{(n+1)^2}{(n-1)(n+1)} + \sum_{n=2}^{\infty} \frac{k(\omega)}{(n-1)^4 t^2} \right)
\]

\[
\leq \frac{C k(\omega)}{t^2(\gamma_{i,1}^3 + \beta)^4} + \frac{C k(\omega)}{t^4(\gamma_{i,1}^3 + \beta)^4}. \quad (4.43)
\]

Since \( Z_1 \) is continuous in \((H^3(0))^2\) with respect to time \( t \), we only need to show the result of this theorem is true when \( t \leq -1 \). Therefore, in the following we assume \( t \leq -1 \). By (4.42) and (4.43) we have

\[
I_1 \leq C \sum_{i=1}^{\infty} \lambda_{i,1}^{\gamma_{i,1}^3} \frac{C + C k(\omega)}{(\gamma_{i,1}^3 + \beta)^8} \frac{1}{t^8} \leq \frac{(C + C k(\omega))}{t^8} < \infty.
\]
To estimate $I_2$, we first consider

$$
| \int_\tau^t \frac{B^H_i(t) - B^H_i(s)}{(t-s)^1-\alpha} \left( \int_s^t e^{-(\gamma_i,1+\beta)(t-s)} - e^{-(\gamma_i,1+\beta)(t-u)} \frac{ds}{(s-u)^{\alpha+1}} \right) du | \leq \int_\tau^t \frac{B^H_i(t) - B^H_i(s)}{(t-s)^1-\alpha} \left( \int_s^{s-1} e^{-(\gamma_i,1+\beta)(s-u)} \frac{du}{(s-u)^{\alpha+1}} \right) ds | + | \int_\tau^t \frac{B^H_i(t) - B^H_i(s)}{(t-s)^1-\alpha} \left( \int_{s-1}^{-\infty} e^{-(\gamma_i,1+\beta)(s-u)} \frac{du}{(s-u)^{\alpha+1}} \right) ds | $$

$$\leq \int_\tau^t \frac{|B^H_i(t) - B^H_i(s)| e^{-(\gamma_i,1+\beta)(t-s)}}{(t-s)^1-\alpha} \left( \int_s^{s-1} \frac{1}{(s-u)^{\alpha+1}} du \right) ds | + | \int_\tau^t \frac{|B^H_i(t) - B^H_i(s)| e^{-(\gamma_i,1+\beta)(t-s)}}{(t-s)^1-\alpha} \left( \int_{s-1}^{-\infty} \frac{1}{(s-u)^{\alpha+1}} du \right) ds | $$

$$= \int_\tau^t \frac{|B^H_i(t) - B^H_i(s)| e^{-(\gamma_i,1+\beta)(t-s)}}{(t-s)^1-\alpha} ds | \left( \frac{\gamma_i,1 + \beta}{\varepsilon} \frac{\alpha+\varepsilon}{\varepsilon} + \frac{1}{\alpha} \right), \quad (4.44)$$

where $\varepsilon$ is small positive constant such that $\alpha + \varepsilon < 1$ and we have used the fact

$$e^{-|x|} - e^{-|y|} \leq C_r |x - y|^r,$$

for $x, y \in \mathbb{R}, r \in [0, 1]$ in the second inequality. Note that from (4.43) and (4.44) it follows that

$$| \int_\tau^t \frac{B^H_i(t) - B^H_i(s)}{(t-s)^1-\alpha} \left( \int_s^t e^{-(\gamma_i,1+\beta)(t-s)} - e^{-(\gamma_i,1+\beta)(t-u)} \frac{ds}{(s-u)^{\alpha+1}} \right) du | \leq \int_\tau^t \frac{B^H_i(t) - B^H_i(s)}{(t-s)^1-\alpha} \left( \int_s^{s-1} \frac{1}{(s-u)^{\alpha+1}} du \right) ds | + \frac{C}{\alpha(\gamma_i,1 + \beta)^{\alpha+\varepsilon}} \left( \frac{\gamma_i,1 + \beta}{\varepsilon} \frac{\alpha+\varepsilon}{\varepsilon} + \frac{1}{\alpha} \right)$$

$$+ \frac{C}{\alpha(\gamma_i,1 + \beta)^{\alpha+\varepsilon}} \left( \frac{\gamma_i,1 + \beta}{\varepsilon} \frac{\alpha+\varepsilon}{\varepsilon} + \frac{1}{\alpha} \right) \int_{-\infty}^0 \left( \frac{\gamma_i,1 + \beta}{2} \right)^{-2\alpha} ds $$

$$\leq \frac{C + Ck_\omega}{\alpha(\gamma_i,1 + \beta)^{\alpha+\varepsilon}} \left( \frac{\gamma_i,1 + \beta}{\varepsilon} \frac{\alpha+\varepsilon}{\varepsilon} + \frac{1}{\alpha} \right) \int_{-\infty}^0 \left( \frac{\gamma_i,1 + \beta}{2} \right)^{-2\alpha} ds $$

which by (4.42) yields that

$$I_2 \leq C \sum_{i=1}^{\infty} \lambda_i,1^{s+2\alpha-2\varepsilon} \frac{C + Ck_\omega}{\alpha(\gamma_i,1 + \beta)^{\alpha+\varepsilon}} < \infty.$$ 

In order to estimate $I_3$, we consider

$$\left| \int_\tau^t \left( \int_s^t e^{-(\gamma_i,1+\beta)(t-s)} - e^{-(\gamma_i,1+\beta)(t-u)} \frac{du}{(s-u)^{\alpha+1}} \right) \frac{B^H_i(s) - B^H_i(v)}{(v-s)^{2-\alpha}} dv ds \right| \leq \left| \int_\tau^t \left( \int_s^t e^{-(\gamma_i,1+\beta)(t-s)} - e^{-(\gamma_i,1+\beta)(t-u)} \frac{du}{(s-u)^{\alpha+1}} \right) \frac{|B^H_i(s) - B^H_i(v)|}{(v-s)^{2-\alpha}} dv ds \right|$$

$$+ \left| \int_\tau^t \left( \int_s^t e^{-(\gamma_i,1+\beta)(t-s)} - e^{-(\gamma_i,1+\beta)(t-u)} \frac{du}{(s-u)^{\alpha+1}} \right) \frac{|B^H_i(s) - B^H_i(v)|}{(v-s)^{2-\alpha}} dv ds \right|$$

$$:= J_1 + J_2. \quad (4.45)$$
By Lemma 2.6 in [37] we can derive the following estimate for $J_1$

$$J_1 \leq \int_0^t \left( \int_0^s e^{-(\gamma_i,1+\beta)(t-s)} - e^{-(\gamma_i,1+\beta)(t-u)} \frac{du}{(s-u)^{\alpha+1}} \right) 2([s]+1)^2 + k(\omega)(t-s) ds$$

$$\leq \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} s^2(t-s) \left( \int_0^s \frac{1 - e^{-(\gamma_i,1+\beta)(s-u)}}{(s-u)^{\alpha+1}} du \right) ds$$

$$\leq \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} s^2(t-s) \left( \int_{s-1}^s \frac{1 - e^{-(\gamma_i,1+\beta)(s-u)}}{(s-u)^{\alpha+1}} du \right) ds$$

$$+ \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} s^2(t-s) \left( \int_{-\infty}^{s-1} \frac{1 - e^{-(\gamma_i,1+\beta)(s-u)}}{(s-u)^{\alpha+1}} du \right) ds$$

$$\leq \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} s^2(t-s) \left( \int_{s-1}^s \frac{(\gamma_i,1+\beta)^{\alpha+\varepsilon}(s-u)^{\alpha+\varepsilon}}{(s-u)^{\alpha+1}} du \right) ds$$

$$+ \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} s^2(t-s) \left( \int_{-\infty}^{s-1} \frac{1}{(s-u)^{\alpha+1}} du \right) ds.$$

After some elementary calculations, we arrive at

$$J_1 \leq C(\gamma_i,1+\beta)^{\alpha+\varepsilon} \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} s^2(t-s) ds$$

$$\leq C(\gamma_i,1+\beta)^{\alpha+\varepsilon} \sum_{n=1}^{\infty} \int_{(n+1)t}^{nt} e^{-(\gamma_i,1+\beta)(t-s)} s^2(t-s) ds$$

$$\leq C(\gamma_i,1+\beta)^{\alpha+\varepsilon} \sum_{n=1}^{\infty} \int_{(n+1)t}^{nt} e^{(\gamma_i,1+\beta)(n-1)t} (nt)^2[t-(n+1)t] ds$$

$$\leq C(\gamma_i,1+\beta)^{\alpha+\varepsilon} \sum_{n=1}^{\infty} \frac{n^3t^4}{(\gamma_i,1+\beta)^{\beta}(n-1)^\beta|t|^5}$$

$$\leq C(\gamma_i,1+\beta)^{\beta-\alpha-\varepsilon}.$$

Choosing a positive constant $\varepsilon$ such that $\alpha + H - \varepsilon > 1$ and using Lemma 7.4 in [33] we obtain

$$J_2 \leq C \left| \int_0^t \left( \int_0^s \frac{e^{-(\gamma_i,1+\beta)(t-s)} - e^{-(\gamma_i,1+\beta)(t-u)}}{(s-u)^{\alpha+1}} du \right) \left( \int_s^{s+1} \frac{v-s}{(v-s)^{2-\alpha}} dv \right) ds \right|$$

$$\leq C(\gamma_i,1+\beta)^{\alpha+\varepsilon} \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} \left( \int_0^s \frac{1 - e^{-(\gamma_i,1+\beta)(s-u)}}{(s-u)^{\alpha+1}} du \right) ds$$

$$\leq C \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} \left( \int_{s-1}^s \frac{(\gamma_i,1+\beta)^{\alpha+\varepsilon}(s-u)^{\alpha+\varepsilon}}{(s-u)^{\alpha+1}} du \right) ds$$

$$+ C \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} \left( \int_{-\infty}^{s-1} \frac{1}{(s-u)^{\alpha+1}} du \right) ds$$

$$\leq C(\gamma_i,1+\beta)^{\alpha+\varepsilon} \int_0^t e^{-(\gamma_i,1+\beta)(t-s)} ds$$

$$\leq C(\gamma_i,1+\beta)^{\alpha+\varepsilon-1}.$$

Therefore, by (4.45) and estimates of $J_1$ and $J_2$ we conclude

$$I_3 \leq \sum_{i=1}^{\infty} \lambda_i(\gamma_i,1)^2 \left( \frac{C}{(\gamma_i,1+\beta)^{\beta-\alpha-\varepsilon}} + C(\gamma_i,1+\beta)^{\alpha+\varepsilon-1} \right)^2 < C \sum_{i=1}^{\infty} \lambda_i(\gamma_i,1)^2 < \infty.$$
To estimate $I_4$, by Lemma 7.4 in [43] we note that

$$
| \int_{\tau}^{t} e^{-\gamma_{i,1} + \beta}(t-s) (s-\tau)^{\alpha} (s) \int_{\tau}^{t} B^H(s) - B^H(v) (v-s)^{2-\alpha} dv) ds |

\leq | \int_{\tau}^{t} e^{-\gamma_{i,1} + \beta}(t-s) (s-\tau)^{\alpha} \int_{\tau}^{t} B^H(s) - B^H(v) (v-s)^{2-\alpha} dv) ds |

+ | \int_{\tau}^{t} e^{-\gamma_{i,1} + \beta}(t-s) (s-\tau)^{\alpha} \int_{\tau}^{t} B^H(s) - B^H(v) (v-s)^{2-\alpha} dv) ds |

\leq C | \int_{\tau}^{t} e^{-\gamma_{i,1} + \beta}(t-s) (s-\tau)^{\alpha} \int_{\tau}^{t} B^H(s) - B^H(v) (v-s)^{2-\alpha} dv) ds |

+ C \int_{\tau}^{t} e^{-\gamma_{i,1} + \beta}(t-s) (s-\tau)^{\alpha} \int_{\tau}^{t} B^H(s) - B^H(v) (v-s)^{2-\alpha} dv) ds |

\leq C \frac{\gamma_{i,1} + \beta}{2} \alpha - 1 \int_{\tau}^{t} (s-\tau)^{\alpha} (t-s)^{\alpha} e^{-\gamma_{i,1} + \beta}(t-s)^{3/2} s^{2}(t-s) ds

+ C \frac{\gamma_{i,1} + \beta}{2} \alpha - 1 \int_{\tau}^{t} (s-\tau)^{\alpha} (t-s)^{\alpha} e^{-\gamma_{i,1} + \beta}(t-s)^{3/2} s^{2}(t-s) ds

\leq C \sum_{n=1}^{\infty} \frac{n^3 |t|^3}{(\gamma_{i,1} + \beta)^n (n-1)! |t|^n} \leq \frac{C}{(\gamma_{i,1} + \beta)^5},

which by (4.46) yields

$$
| \int_{\tau}^{t} e^{-\gamma_{i,1} + \beta}(t-s) (s-\tau)^{\alpha} \int_{\tau}^{t} B^H(s) - B^H(v) (v-s)^{2-\alpha} dv) ds |

\leq C \frac{\gamma_{i,1} + \beta}{2} \alpha - 1 \int_{\tau}^{t} (s-\tau)^{\alpha} (t-s)^{\alpha} e^{-\gamma_{i,1} + \beta}(t-s)^{3/2} s^{2}(t-s) ds

\leq C \sum_{i=1}^{\infty} \lambda_i \gamma_{i,1}^{1+2\alpha} \leq C \sum_{i=1}^{\infty} \lambda_i \gamma_{i,1}^{2} < C \sum_{i=1}^{\infty} \lambda_i \gamma_{i,1}^{2} < \infty.

(4.47)

Since

$$
e^{-\gamma_{i,1} + \beta}(t-s)^{3/2} s^{2}(t-s)
$$

\quad = \sum_{n=1}^{\infty} e^{-\gamma_{i,1} + \beta}(t-s)^{3/2} s^{2}(t-s) I_{(n+1)t \leq s \leq nt}(s)

\quad \leq \sum_{n=1}^{\infty} e^{-\gamma_{i,1} + \beta}(n-1)! |t|^{2} (n+1)^{2} t^{2} |nt|

\quad \leq C \sum_{n=1}^{\infty} \frac{n^3 |t|^3}{(\gamma_{i,1} + \beta)^n (n-1)! |t|^n} \leq \frac{C}{(\gamma_{i,1} + \beta)^5},

which by (4.46) yields

Subsequently, it follows that

$$I_4 \leq C \sum_{i=1}^{\infty} \lambda_i \gamma_{i,1}^{1+2\alpha} \leq C \sum_{i=1}^{\infty} \lambda_i \gamma_{i,1}^{2} < \infty.

(4.47)

Combining the estimates of $I_1 - I_4$ and Proposition 2.1, we complete the proof. □

Analogously, considering the growth properties of $(Z_2(t))_{t \in \mathbb{R}}$ we have Lemma 4.4 whose proof is similar to Lemma 4.3 and is omitted.

**Lemma 4.4** Under the condition of Proposition 2.1, there exists a random variable $r(\omega)$ taking finite values, i.e., $r(\omega) < \infty$, for all $\omega \in \Omega$, such that

$$\|Z_2(t)\|_3 < r(\omega),$$

for all $t \in (-\infty, 0]$.

4.3. **Proof Of Theorem 3.1.** Under conditions of Theorem 3.2, for $j \in \{1, 2\}$, $(W_j^H(., t))_{t \in \mathbb{R}^+}$, is a fractional Wiener process with values in $V_j$. As usual in this context, we extend the fractional Wiener process $(W_j^H(., t))_{t \in \mathbb{R}^+}$ to all $\mathbb{R}$ by setting

$$W_j^H(., t) = V_j^H(., -t), \ t \leq 0,$$

and $W_j^H(., t) = V_j^H(., t), \ t > 0$.
where \((V^H_j(.,t))_{t \in \mathbb{R}^+}\) is another fractional Wiener process with the same covariance operator as \((W^H_j(.,t))_{t \in \mathbb{R}^+}\). We will consider a canonical version \((\omega_j(.,t))_{t \in \mathbb{R}}\) of this process given by the probability space \((C_0(\mathbb{R}, V), \mathcal{B}(C_0(\mathbb{R}, V)), \mathbb{P}_i, \vartheta)\) where \(\mathbb{P}_i\) is the Gaussian-measure generated by \((W^H_j(.,t))_{t \in \mathbb{R}}\). So, if we set \((\omega(.,t))_{t \in \mathbb{R}} := (\omega_1(.,t), \omega_2(.,t))_{t \in \mathbb{R}},\) the process \((\omega(.,t))_{t \in \mathbb{R}}\) is given by the probability space \((C_0(\mathbb{R}, V), \mathcal{B}(C_0(\mathbb{R}, V)), \tilde{\mathbb{P}}, \vartheta)\), where \(\tilde{\mathbb{P}} = \mathbb{P}_1 \times \mathbb{P}_2.\) Now we may define the stochastic dynamical system \((S(t, s; \omega))_{t \geq s, \omega \in \tilde{\Omega}}\) by

\[
S(t, s; \omega)(u_s, T_s) = (u(t, \omega_1) + Z_1(t, \omega_1), \theta(t, \omega_2) + Z_2(t, \omega_2))
\]  

(4.48)

where \((u, T)\) is the strong solution to (1.1) – (1.5) with \((u_s, T_s) = (u_s + Z_1(s, \omega_1), \theta_s + Z_2(s, \omega_2)\) and \((u, \theta)\) is the strong solution to (2.17) – (2.22) or equivalently it is the strong solution to (5.86) – (5.91). It can be checked that assumptions (i)-(iv) and (a)-(c) are satisfied with \(X = V.\)

Further more, by [37], we have the result: \((C_0(\mathbb{R}, V), \mathcal{B}(C_0(\mathbb{R}, V)), \tilde{\mathbb{P}}, \vartheta)\) is an ergodic metric dynamical system. Properties (i), (ii), (iv) of the solution operator \((S(t, s; \omega))_{t \geq s, \omega \in \tilde{\Omega}}\) follows by Theorem 3.2 and property (iii) of the solution operator also holds regarding the fact that the proof of the global existence of strong solution to (1.1) – (1.3) rest upon Faedo-Galerkin method.

Next, we will prove the existence of the random attractor.

For \(j = 1, 2\) and \(t \in \mathbb{R},\) we have

\[
Z_j(t) = \int_t^\infty e^{-(t-s)(A_j + \beta)} dW^H_j(s)
= \sum_{i=1}^\infty \lambda_i^\frac{1}{2} e_{i,j} \int_t^\infty e^{-(t-s)(\gamma_{i,j} + \beta)} d\beta_i^H(s).
\]

Since \(e^{-(t-s)(\gamma_{i,j} + \beta)}\) is infinitely many times continuously differentiable with respect to \(s\) and \(\beta_i^H(s)\) has \(\alpha-\)Hölder continuous paths for all \(\alpha \in (0, H)\) with \(H > \frac{1}{2}\), by Theorem 4.2.1 in [32], we know the stochastic integral on the right hand side of the above equality is equivalent to Riemann–Stieltjes integral. Therefore,

\[
Z_j(t) = \sum_{i=1}^\infty \lambda_i^\frac{1}{2} e_{i,j} \lim_{\tau \to -\infty} \int_\tau^0 e^{s(\gamma_{i,j} + \beta)} d\theta_i^H(s).
\]

By integration by parts and \(\theta_i\beta_i^H(s)\) has polynomial growth with respect to \(s\), we have

\[
Z_j(t) = -\sum_{i=1}^\infty \lambda_i^\frac{1}{2} e_{i,j} \lim_{\tau \to -\infty} \int_\tau^0 (\gamma_{i,j} + \beta) e^{s(\gamma_{i,j} + \beta)} \theta_i \beta_i^H(s) d(s)
= -\int_\infty^0 (A + \beta) e^{s(A+\beta)} \theta_i W^H_j(s) ds.
\]

Since the integral is convergent in \(H^1(\Omega)\), by Fubini theorem, we know \(Z_j(t)\) is adapted with respect to \(\mathcal{F}_t := \sigma(W^H_j(s), s \leq t, j = 1, 2)\). Therefore, for \(s < 0\), applying the properties of ergodic metric dynamical system we have

\[
\lim_{s \to -\infty} \frac{1}{s} \int_s^0 (\|Z_1\|^2 + \|Z_1\|^2 + \|Z_2\|^2) ds = E(\|Z_1(0)\|^2 + \|Z_1(0)\|^2 + \|Z_2(0)\|^2).
\]

From Lemma 4.1 and Lemma 4.2, there exists a \(\beta\) which is big enough such that

\[
E(\|Z_1(0)\|^2 + \|Z_1(0)\|^2 + \|Z_2(0)\|^2) \leq \frac{\gamma_1}{2}
\]

which implies that there exists a random variable \(\tau(\omega)\) such that for \(s < \tau(\omega)\) we have

\[
\int_s^0 (-\gamma_1 + \|Z_1\|^2 + \|Z_1\|^2 + \|Z_2\|^2) ds \leq \frac{\gamma_1}{2}s.
\]

(4.49)
Due to (5.102),
\[
|u(-4)|^2 + |\theta(-4)|^2 \leq (|u(t_0)|^2 + |\theta(t_0)|^2) e^{\int_{-4}^{\infty} \int_{-4}^{\infty} (-\gamma_1 + \|Z_1\|^2 + \|Z_1\|^2 + \|Z_2\|^2) ds dt} + \int_{-4}^{t} e^{\int_{-4}^{\infty} (-\gamma_1 + \|Z_1\|^2 + \|Z_1\|^2 + \|Z_2\|^2) ds} ds. \tag{4.50}
\]

In the following, we denote by \(u(t, \omega; t_0, u_0), \theta(t, \omega; t_0, \theta_0)\) the solution to (5.86) – (5.91) with \((u(t_0) = u_0, \theta(t_0) = \theta_0)\). Then, by (4.49) and (4.50), there exists a random variable \(c_1(\omega) > 0\), depending only on \(\gamma_1, Z_1, Z_2\), such that for arbitrary \(\rho > 0\) there exists \(t(\omega) \leq -4\) such that the following holds P-a.s.: For all \(t \leq t(\omega)\) and \((u_0, \theta_0) \in H\) with \(|u_0|^2 + |\theta_0|^2 \leq \rho\), the solution \((u(t, \omega; t_0, u_0), \theta(t, \omega; t_0, \theta_0))\) over \([t_0, \infty)\), satisfies
\[
|u(-4, \omega; t_0, u_0)|^2 + |\theta(-4, \omega; t_0, \theta_0)|^2 \leq c_1(\omega). \tag{4.51}
\]

Using (5.102) again, for \(t \in [-4, 0]\) we have
\[
|u(t)|^2 + |\theta(t)|^2 \leq (|u(-4)|^2 + |\theta(-4)|^2) e^{\int_{-4}^{t} (-\gamma_1 + \|Z_1\|^2 + \|Z_1\|^2 + \|Z_2\|^2) ds} + \int_{-4}^{t} e^{\int_{-4}^{\infty} (-\gamma_1 + \|Z_1\|^2 + \|Z_1\|^2 + \|Z_2\|^2) ds} ds. \tag{4.52}
\]

Moreover, integrating (5.101) over \([-4, 0]\) we obtain
\[
\int_{-4}^{0} (|u|^2 + |\theta|^2) ds \leq |u(-4)|^2 + |\theta(-4)|^2 + C \int_{-4}^{0} (|Q|^2 + |Z|^2) ds + C \int_{-4}^{0} (|u|^2 + |\theta|^2) (\|Z_1\|^2 + \|Z_1\|^2 + \|Z_2\|^2 + \|Z_2\|^2) ds. \tag{4.53}
\]

Therefore, from (4.51) – (4.53), we conclude that there exists two random variables \(r_1(\omega)\) and \(c_1(\omega)\), depending only on \(\gamma_1, Z_1, Z_2\), such that for arbitrary \(\rho > 0\) there exists \(t(\omega) \leq -4\) such that the following holds P-a.s.: For all \(t \leq t(\omega)\) and \((u_0, \theta_0) \in H\) with \(|u_0|^2 + |\theta_0|^2 \leq \rho\), the solution \((u(t, \omega; t_0, u_0), \theta(t, \omega; t_0, \theta_0))\) over \([t_0, \infty)\), satisfies
\[
|u(t, \omega; t_0, u_0)|^2 + |\theta(t, \omega; t_0, \theta_0)|^2 \leq r_1(\omega) \quad \text{for all } t \in [-4, 0] \tag{4.54}
\]
and
\[
\int_{-4}^{0} (|u(s, \omega; t_0, u_0)|^2 + |\theta(s, \omega; t_0, \theta_0)|^2) ds \leq c_2(\omega). \tag{4.55}
\]

For \(t < -3\), by (5.109) we have
\[
|\theta(-3, \omega; t_0, \theta_0)|^2 \leq |\theta(t, \omega; t_0, \theta_0)|^2 e^{-C(-3-t)} + C \int_{t}^{\infty} e^{-C(-3-s)} (|Q|^2 + \|Z_2(s)|^2 + \|Z_1(s)|^2 + \|Z_2(s)|^2 + \|u(s, \omega; t_0, u_0)|^2) ds dt.
\]

Integrating in \(t\) over the interval \([-4, -3]\) yields
\[
|\theta(-3, \omega; t_0, \theta_0)|^2 \leq \int_{-4}^{-3} |\theta(t, \omega; t_0, \theta_0)|^2 e^{-C(-3-t)} dt + C \int_{-4}^{-3} \int_{t}^{\infty} e^{-C(-3-s)} (|Q|^2 + \|Z_2(s)|^2 + \|Z_1(s)|^2 + \|Z_2(s)|^2 + \|u(s, \omega; t_0, u_0)|^2) ds dt dt
\]
\[
\leq C \int_{-4}^{-3} (|\theta(t, \omega; t_0, \theta_0)|^2 + \|u(t, \omega; t_0, u_0)|^2) dt + C \int_{-4}^{-3} e^{-C(-3-s)} (|Q|^2 + \|Z_2(s)|^2 + \|Z_1(s)|^2) dt. \tag{4.56}
\]
where the second inequality follows by the boundedness of $e^{-C(3-s)}\|Z_2(s)\|^2$ for $s \leq -3$.

By (4.53) and (4.56), there exists random variable $c_3(\omega)$, depending only on $\gamma_1, Z_1$ and $Z_2$, such that for arbitrary $\rho > 0$ there exists $t(\omega) \leq -3$ such that the following holds $P$-a.s.. For all $t_0 \leq t(\omega)$ and $\theta_0 \in L^4(\Omega)$ with $|\theta_0|_4 \leq \rho$, $\theta(t, \omega; t_0, \theta_0)$ satisfies
\begin{equation}
|\theta(-3, \omega; t_0, \theta_0)|^2_4 \leq c_3(\omega). \tag{4.57}
\end{equation}

Repeating the argument as (4.54), by (4.57) we have that there exists random variable $c_4(\omega)$, depending only on $\gamma_1, Z_1$ and $Z_2$, such that for arbitrary $\rho > 0$ there exists $t(\omega) \leq -3$ such that the following holds $P$-a.s.. For all $t_0 \leq t(\omega)$ and $\theta_0 \in L^4(\Omega)$ with $|\theta_0|_4 \leq \rho$, $\theta(t, \omega; t_0, \theta_0)$ satisfies
\begin{equation}
|\theta(t, \omega; t_0, \theta_0)|^2_4 \leq c_4(\omega), \quad \text{for all} \ t \in [-3, 0]. \tag{4.58}
\end{equation}

By (5.118), taking a similar argument as (4.58) we deduce that there exists random variable $c_5(\omega)$, depending only on $\gamma_1, Z_1$ and $Z_2$, such that for arbitrary $\rho > 0$ there exists $t(\omega) \leq -3$ such that the following holds $P$-a.s.. For all $t_0 \leq t(\omega)$ and $\tilde{u}_0 \in (L^4(\Omega))^2$ with $|\tilde{u}_0|_4 \leq \rho$, $\tilde{u}(t, \omega; t_0, \tilde{u}_0)$ satisfies
\begin{equation}
|\tilde{u}(t, \omega; t_0, \tilde{u}_0)|^2_4 \leq c_5(\omega), \quad \text{for all} \ t \in [-3, 0]. \tag{4.59}
\end{equation}

By (5.117), making an analogous argument as (4.55) we infer that that there exists random variable $c_6(\omega)$, depending only on $\gamma_1, Z_1$ and $Z_2$, such that for $\rho > 0$ there exists $t(\omega) \leq -3$ such that the following holds $P$-a.s.. For all $t_0 \leq t(\omega)$ and $\tilde{u}_0 \in (L^4(\Omega))^2$ with $|\tilde{u}_0|_4 \leq \rho$, $\tilde{u}(t, \omega; t_0, \tilde{u}_0)$ satisfies
\begin{equation}
\int_{-3}^0 \|\tilde{u}(s, \omega; t_0, \tilde{u}_0)||\nabla \tilde{u}(s, \omega; t_0, \tilde{u}_0)||^2_2 ds \leq c_6(\omega). \tag{4.60}
\end{equation}

By (5.123) and (4.59) – (4.60), proceeding as (4.59) we find that there exists random variable $c_7(\omega)$, depending only on $\gamma_1, Z_1$ and $Z_2$, such that for arbitrary $\rho > 0$ there exists $t(\omega) \leq -2$ such that the following holds $P$-a.s.. For all $t_0 \leq t(\omega)$ and $\nabla \tilde{u}_0 \in (L^2(\Omega))^2$ with $|\nabla \tilde{u}_0|_2 \leq \rho$, $\nabla \tilde{u}(t, \omega; t_0, \tilde{u}_0)$ satisfies
\begin{equation}
|\nabla \tilde{u}(t, \omega; t_0, \tilde{u}_0)|^2_2 \leq c_7(\omega), \quad \text{for all} \ t \in [-2, 0]. \tag{4.61}
\end{equation}

In view of (5.133), (4.55) and (4.59) – (4.61), following the steps in (4.59) – (4.60) we get that there exists two random variables $r_2(\omega)$ and $c_8(\omega)$, depending only on $\gamma_1, Z_1$ and $Z_2$, such that for arbitrary $\rho > 0$ there exists $t(\omega) \leq -1$ such that the following holds $P$-a.s.. For all $t_0 \leq t(\omega)$ and $\partial_z u_0 \in (L^2(\Omega))^2$ with $|\partial_z u_0|_2 \leq \rho$, $\partial_z u(t, \omega; t_0, u_0)$ satisfies
\begin{equation}
|\partial_z u(t, \omega; t_0, u_0)|^2_2 \leq r_2(\omega), \quad \text{for all} \ t \in [-1, 0] \tag{4.62}
\end{equation}
and
\begin{equation}
\int_{-1}^0 |\nabla u_z(s, \omega; t_0, u_0)|^2_2 ds \leq c_8(\omega). \tag{4.63}
\end{equation}

Regarding (5.137), (4.59) and (4.61) – (4.63), we repeat the procedures of deriving (4.59) – (4.60) to get that there exists two random variables $r_3(\omega)$ and $c_9(\omega)$, depending only on $\gamma_1, Z_1$ and $Z_2$, such that for arbitrary $\rho > 0$ there exists $t(\omega) \leq -1$ such that the following holds $P$-a.s.. For all $t_0 \leq t(\omega)$ and $\nabla u_0 \in (L^2(\Omega))^4$ with $|\nabla u_0|_2 \leq \rho$, $u(t, \omega; t_0, u_0)$ satisfies
\begin{equation}
|\nabla u(t, \omega; t_0, u_0)|^2_2 \leq r_3(\omega), \quad \text{for all} \ t \in [-1, 0] \tag{4.64}
\end{equation}
and
\begin{equation}
\int_{-1}^0 |\Delta u(s, \omega; t_0, u_0)|^2_2 ds \leq c_9(\omega). \tag{4.65}
\end{equation}
By (5.139), (4.59), (4.62) and (4.64) – (4.65), proceeding as above we have that there exists random variable $r_4(\omega)$, depending only on $\gamma_1, Z_1$ and $Z_2$, such that for arbitrary $\rho > 0$ there exists $t(\omega) \leq -1$ such that the following holds $P$-a.s.. For all $t_0 \leq t(\omega)$ and $\theta_0 \in H^1(\Omega)$ with $\|\theta_0\|_1 \leq \rho$, $\theta(t, \omega; t_0, \theta_0)$ satisfies

$$\|\theta(t, \omega; t_0, \theta_0)\|_1^2 \leq r_4(\omega), \quad \text{for all } t \in [-1, 0].$$

Now we are ready to prove the desired compact result. Let $r(\omega) = \Sigma_{i=1}^4 r_i(\omega) + \|Z_1(-1)\|_1^2 + \|Z_2(-1)\|_1^2$, then $B(-1, r(\omega))$, the ball of center 0 in $\mathcal{V}$ and radius $r(\omega)$, is an absorbing set at time $-1$ for $(S(t, s; \omega))_{t \geq s, \omega \in \widetilde{\Omega}}$. Therefore, in order to prove the existence of the global random attractor of the stochastic dynamical system in space $\mathcal{V}$, we need to to construct a compact absorbing set at time 0 in $\mathcal{V}$ according to Theorem 2.2. Denote by $\mathcal{B}$ a bounded subset $\mathcal{V}$ and set $\mathcal{C}_T$ as a subset of the function space:

$$\mathcal{C}_T := \{ (A_T^\frac{1}{2}u, A_T^\frac{1}{2}\theta) \mid (v(-1), T(-1)) \in \mathcal{B}, (v(t), T(t)) = S(t, -1; \omega)(v(-1), T(-1)), t \in [-1, 0] \}. $$

Since $\mathcal{V}_1 \subset H_1$ is compact, $\mathcal{V}_1 \times \mathcal{V}_2 \subset H_1 \times H_2$ is also compact. Let $(v(-1), T(-1)) \in \mathcal{B}$, by the argument of step 2 in the proof of Theorem 3.2 we know

$$(A_T^\frac{1}{2}u, A_T^\frac{1}{2}\theta) \in L^2([-1, 0]; \mathcal{V}_1 \times \mathcal{V}_2), \quad (\partial_t A_T^\frac{1}{2}u, \partial_t A_T^\frac{1}{2}\theta) \in L^2([-1, 0]; \mathcal{V}_1^1 \times \mathcal{V}_2^1).$$

Therefore, by Lemma 2.1 with

$$(\nu_0, \theta_0) \in L^2([-1, 0]; \mathcal{V}),$$

and a subsequence of $\mathcal{C}_T$, there is a function $(\nu_*, \theta_*)$:

$$(\nu_*, \theta_*) \in L^2([-1, 0]; \mathcal{V}),$$

and a subsequence of $\mathcal{C}_T$, still denoted by $\{S(t, -1; \omega)(\nu_0, \tau_0)\}_{n \in \mathbb{N}}$ for simplicity, such that

$$\lim_{n \to \infty} \int_{-1}^0 \|S(t, -1; \omega)(\nu_0, \tau_0) - (\nu_*(t), \theta_*(t))\|_1^2 dt = 0. \quad (4.66)$$

By measure theory, convergence in mean square implies almost sure convergence of a subsequence. Therefore, it follows from (4.66) that there exists a subsequence of $\mathcal{C}_T$, still denoted by $\{S(t, -1; \omega)(\nu_0, \tau_0)\}_{n \in \mathbb{N}}$ for simplicity, such that

$$\lim_{n \to \infty} \|S(t, -1; \omega)(\nu_0, \tau_0) - (\nu_*(t), \theta_*(t))\|_1 = 0, \quad a.e. \ t \in (-1, 0]. \quad (4.67)$$

Fix any $t \in (-1, 0)$. By (4.67), we can select a $t_0 \in (-1, t)$ such that

$$\lim_{n \to \infty} \|S(t_0, -1; \omega)(\nu_0, \tau_0) - (\nu_*(t_0), \theta_*(t_0))\|_1 = 0.$$
Hence for any \( t \in (-1, 0], \{ S(t, -1; \omega)(\nu_{0,n}, \tau_{0,n}) \}_{n \in \mathbb{N}} \) contains a subsequence which is convergent in \( V \), which implies that for any fixed \( t \in (-1, 0], \omega \in \Omega, S(t, -1; \omega) \) is a compact operator in \( V \). Let \( B(0, \omega) = S(0, -1; \omega)B(1, r(\omega)) \) be the closed set of \( S(0, -1; \omega)B(1, r(\omega)) \). Then, by the above arguments, we know \( B(0, \omega) \) is a random compact set in \( V \). More precisely, \( B(0, \omega) \) is a compact absorbing set in \( V \) at time 0. Indeed, for \( (\nu_{0,n}, \tau_{0,n}) \in B \), there exists \( s(B) \in \mathbb{R}^{-} \) such that if \( s \leq s(B) \), we have

\[
S(0, s; \omega)(\nu_{0,n}, \tau_{0,n}) = S(0, -1; \omega)S(-1, s; \omega)(\nu_{0,n}, \tau_{0,n}) \subset S(0, -1; \omega)B(-1, r(\omega)) \subset B(0, \omega).
\]

Therefore, conclusions (1) – (7) of Theorem 3.1 follows by Theorem 2.2.

\[\square\]

5 Appendix: A Priori Estimates

5.1. proof of local – existence of strong solutions. Using a similar argument in [24], we obtain the existence of local strong solutions to stochastic PEs. Assume that \( \eta \) is the solution of the initial boundary value problem

\[
\begin{align*}
\partial_{t}\eta + \nabla p_{t} - \Delta \eta - \partial_{zz} \eta &= 0, \\
\partial_{z} \eta|_{\Gamma_{1}} &= 0, \quad \eta \cdot \overline{n}|_{\Gamma_{0}} = 0, \quad \partial_{\eta} \eta \times \overline{n}|_{\Gamma_{0}} = 0, \\
\int_{-1}^{0} \nabla \cdot \eta dz &= 0, \\
\eta(0, w) &= v_{0}.
\end{align*}
\]

If \( v_{0} \in V_{1} \), then, for any \( T > 0 \) and a.s. \( w \in \Omega \),

\[
\eta \in L^{\infty}(0, T; V_{1}) \cap L^{2}(0, T; (H^{2}(\Omega))^{2}),
\]

see, e.g., [24]. Let \( u(t) = v(t) - Z_{1}(t) - \eta(t) =: v(t) - \tilde{Z}_{1}(t) \) and \( \theta(t) = T(t) - Z_{2}(t), t \in \mathbb{R}^{+} \). Given \( \mathcal{T} = T > 0 \), a stochastic process \( U(t, w) = (v, T) \) is a strong solution to (1.1) – (1.5) on \([0, \mathcal{T}]\), if and only if \( (u, \theta) \) is a strong solution to the following problem on \([0, \mathcal{T}]\):

\[
\begin{align*}
\partial_{t}u - \Delta u - \partial_{zz} u + [(u + \tilde{Z}_{1}) \cdot \nabla](u + \tilde{Z}_{1}) + \varphi(u + \tilde{Z}_{1})\partial_{z}(u + \tilde{Z}_{1}) &= f(u + \tilde{Z}_{1}) + \nabla p_{s} - \int_{-1}^{z} \nabla T dz' = 0; \\
\partial_{t} \theta - \Delta \theta - \partial_{zz} \theta + [(u + \tilde{Z}_{1}) \cdot \nabla](\theta + Z_{2}) + \varphi(u + \tilde{Z}_{1})\partial_{z}(\theta + Z_{2}) &= Q; \\
\int_{-1}^{0} \nabla \cdot udz &= 0; \\
\partial_{z} u|_{\Gamma_{1}} &= \partial_{z} u|_{\Gamma_{0}} = 0; u \cdot n|_{\Gamma_{1}} = 0, \partial_{n} u \times \overline{n}|_{\Gamma_{0}} = 0; \\
(\partial_{z} \theta + \alpha \theta)|_{\Gamma_{1}} &= \partial_{z} \theta|_{\Gamma_{0}} = 0, \quad \partial_{\eta} \theta|_{\Gamma_{0}} = 0; \\
(u|_{t=0}, \theta|_{t=0}) &= (0, T_{0}).
\end{align*}
\] (5.68)

(5.69)

(5.70)

(5.71)

(5.72)

(5.73)

Theorem 5.1 (Existence of local solutions to (5.68)-(5.73)). If \( Q \in L^{2}(\Omega), v_{0} \in V_{1} \) and \( T_{0} \in V_{2}, \) then, for \( P \) – a.e. \( w \in \Omega, \) there exists a stopping time \( \mathcal{T} > 0 \) such that \( (u, \theta) \) is a strong solution of the system (5.68) – (5.73) on the interval \([0, \mathcal{T}]\).

Proof. We use Fuedo-Galerkin method to prove the result. Let \( (u_{m}, T_{m}) \) be an approximate solution for the problem (5.68) – (5.73), where \( (u_{m}, T_{m}) = \sum_{i=1}^{m} c_{i,m}(t) \xi_{i}(x) \) and \( \{ \xi_{i} \}_{i \in \mathbb{N}} \) is a completely
orthonormal basis of $\mathcal{V}$. Then $(u_m, T_m)$ satisfies
\begin{align}
\int_{\Omega} \nu_m \cdot \partial u_m + \int_{\Omega} \nu_m \cdot \{(u_m + \tilde{Z}_1) \cdot \nabla (u_m + \tilde{Z}_1) + \varphi (u_m + \tilde{Z}_1) \partial_z (u_m + \tilde{Z}_1)\}
+ \int_{\Omega} \nu_m \cdot (u_m + \tilde{Z}_1)^{\frac{1}{2}} - \int_{\Omega} \nu_m \cdot \int_{-1}^{z} \nabla T_m d\lambda + \int_{\Omega} \nu_m \cdot L_1 u_m = 0, \quad (5.74)
\end{align}
\begin{align}
\int_{\Omega} \tau_m \cdot \partial T_m + \int_{\Omega} \tau_m \cdot \{(u_m + \tilde{Z}_1) \cdot \nabla (T_m + Z_2) + \varphi (u_m + \tilde{Z}_1) \partial_z (T_m + Z_2)\}
+ \int_{\Omega} \tau_m \cdot L_2 T_m = \int_{\Omega} \tau_m Q, \quad (5.75)
\end{align}
\begin{align}
u_m (0) = 0, T_m (0) = T_0 m \rightarrow T_0, \quad (5.76)
\end{align}
where $\nu_m \in \mathcal{V}_{1 m}$, $\tau_m \in \mathcal{V}_{2 m}$, $T_0 m \in \mathcal{V}_{2 m}$, and $\mathcal{V}_{1 m} \times \mathcal{V}_{2 m} = \text{span}\{\xi_1, ..., \xi_m\}$. We first estimate $u_m$ and $T_m$ in $(L^2 (\Omega))^2$ and $L^2 (\Omega)$ respectively. Let $\tau_m = T_m$. By integration by parts and Hölder inequality as well as Sobolev embedding theorem, we have
\begin{align}
\partial_t |T_m|^2 + C \|T_m\|^2 \leq C |Q|^2 + \int_{\Omega} T_m \{(u_m + \tilde{Z}_1) \cdot \nabla Z_2 + \varphi (u_m + \tilde{Z}_1) \partial_z Z_2\}
\leq C |Q|^2 + |\nabla Z_2|_\infty |T_m| \|u_m + \tilde{Z}_1\|_2 + |\partial_z Z_2|_\infty |T_m|_2 \|u_m + \tilde{Z}_1\|_1 \leq \varepsilon \|u_m\|^2_1 + C \|Z_2\|_3^2 |T_m|_2^2 + C \|Z_2\|_3 |T_m|_2 \|\tilde{Z}_1\|_1 + C |Q|^2. \quad (5.77)
\end{align}
Using Hölder inequality, Sobolev embedding theorem and interpolation inequalities, we have
\begin{align}
\int_{\Omega} u_m [(\tilde{Z}_1 \cdot \nabla) u_m + (u_m \cdot \nabla) \tilde{Z}_1 + (\tilde{Z}_1 \cdot \nabla) \tilde{Z}_1]\n\leq |\tilde{Z}_1|_\infty \|u_m\|_2 \|\nabla u_m\|_2 + |u_m|_3^2 \|\tilde{Z}_1\|_3 + |\tilde{Z}_1|_\infty \|u_m\|_2 \|\nabla \tilde{Z}_1\|_2 \leq \varepsilon \|u_m\|_1^2 + C \|\tilde{Z}_1\|_2^2 (\|u_m\|_2^2 + \|u_m\|_2^2). \quad (5.78)
\end{align}
By virtue of Minkowski inequality, interpolation inequalities as well as Sobolev imbedding theorem, we obtain
\begin{align}
\int_{\Omega} u_m \varphi (u_m) \partial_z \tilde{Z}_1 d\Omega
\leq \int_{M} \left( \int_{-1}^{0} |\text{div} u_m| dz \int_{-1}^{0} \|u_m\|_{\partial_z \tilde{Z}_1} \|d\Omega dz\right) dM
\leq \int_{M} \int_{-1}^{0} |\text{div} u_m| dz \left( \int_{-1}^{0} \|u_m\|_{\partial z \tilde{Z}_1}^2 dz\right)^{\frac{1}{2}} \left( \int_{-1}^{0} \|\partial_z \tilde{Z}_1\|_{L^2 (M)}^2 dz\right)^{\frac{1}{2}} dM
\leq |\text{div} u_m| \left( \int_{-1}^{0} \|u_m\|_{L^2 (M)}^2 dz\right)^{\frac{1}{2}} \left( \int_{-1}^{0} \|\partial_z \tilde{Z}_1\|_{L^2 (M)}^2 dz\right)^{\frac{1}{2}} \left( \int_{-1}^{0} \|\tilde{Z}_1\|_{H^1 (M)}^2 \|\tilde{Z}_1\|_{H^2 (M)}^2 dz\right)^{\frac{1}{2}} \leq C \|u_m\| \left( \int_{-1}^{0} \|u_m\|_{L^2 (M)} \|u_m\|_{H^1 (M)} dz\right)^{\frac{1}{2}} \left( \int_{-1}^{0} \|\tilde{Z}_1\|_{H^1 (M)} \|\tilde{Z}_1\|_{H^2 (M)} dz\right)^{\frac{1}{2}} \leq \varepsilon \|u_m\|_1^2 + C \|u_m\|_2^2 \|\tilde{Z}_1\|_1^2 + \|\tilde{Z}_1\|_2^2.
\end{align}
Analogously, we have
\begin{align}
\int_{\Omega} u_m \varphi (\tilde{Z}_1) \partial_z u_m d\Omega
\leq |\partial_z u_m| \left( \int_{-1}^{0} |\text{div} \tilde{Z}_1|_{L^2 (M)} dz\right) \left( \int_{-1}^{0} \|u_m\|_{L^2 (M)}^2 dz\right)^{\frac{1}{2}} \leq C \|u_m\|_1 \left( \int_{-1}^{0} \|\tilde{Z}_1\|_{H^1 (M)}^2 \|\tilde{Z}_1\|_{H^2 (M)}^2 dz\right)^{\frac{1}{2}} \left( \int_{-1}^{0} \|u_m\|_{L^2 (M)} \|u_m\|_{H^1 (M)} dz\right)^{\frac{1}{2}} \leq \varepsilon \|u_m\|_1^2 + C \|u_m\|_2^2 \|\tilde{Z}_1\|_1^2 + \|\tilde{Z}_1\|_2^2.
\end{align}
and

\[
\int_\Omega u_m\varphi(\tilde{Z}_1)\partial_z \tilde{Z}_1 d\tilde{\Omega}
\leq |\partial_z \tilde{Z}_1|^2 \int_0^1 |\text{div} \tilde{Z}_1|_{L^2(M)} dz \left( \int_{-1}^0 |u_m|_{L^2(M)} dz \right)^{\frac{1}{2}}
\leq \varepsilon |u_m|^2 + C|u_m|^2 + C|\tilde{Z}_1|^2 \|	ilde{Z}_1\|_2.
\]

Combining the above inequalities, we arrive at

\[
\int_\Omega u_m[\varphi(u_m)\partial_z \tilde{Z}_1 + \varphi(\tilde{Z}_1)\partial_z u_m + \varphi(\tilde{Z}_1)\partial_z \tilde{Z}_1] d\tilde{\Omega}
\leq \varepsilon |u_m|^2 + C|u_m|^2 \|	ilde{Z}_1\|^2_2 + C|u_m|^2 + C|\tilde{Z}_1|^2 \|	ilde{Z}_1\|_2.
\]

(5.79)

Let \( v_m = u_m \). Due to integration by parts, we get

\[
\int_\Omega [(u_m \cdot \nabla) u_m + \varphi(u_m) \partial_z u_m] \cdot u_m = 0
\]

and

\[
\int_\Omega u_m \cdot \int_{-1}^z \nabla T_m d\lambda = -\int_\Omega \nabla \cdot u_m \int_{-1}^z T_m d\lambda.
\]

(5.80)

Therefore, from (5.74) and (5.79) – (5.80), we conclude that

\[
\partial_t |u_m|^2 + ||u_m||^2 \leq C\|	ilde{Z}_1\|^2 + C\|	ilde{Z}_1\|^1 \|	ilde{Z}_1\|_2
\]

\[
+ C|u_m|^2 \left( 1 + \|	ilde{Z}_1\|^1 + \|	ilde{Z}_1\|^2 + \|	ilde{Z}_1\|^2 \|	ilde{Z}_1\|^2_2 \right)
\]

which together with (5.77) implies

\[
\partial_t (|T_m|^2 + |u_m|^2) + ||T_m||^2 + ||u_m||^2
\]

\[
\leq C(1 + \|	ilde{Z}_1\|^2 + \|	ilde{Z}_1\|^2 + \|	ilde{Z}_2\|^2)(|T_m|^2 + |u_m|^2)
\]

\[
+ C(|Q|^2 + ||\tilde{Z}_1||^2 + ||\tilde{Z}_1||^2 ||\tilde{Z}_1||^2) + |Z_2||^2 ||\tilde{Z}_1||^2 ||\tilde{Z}_1||^2
\]

From the estimate above we infer that \( T_m \) and \( u_m \) is uniformly with respect to \( m \) bounded in \( C([0, T^*]; (L^2(\tilde{\Omega}))^2) \cap L^2([0, T^*]; (H^1(\tilde{\Omega}))^2) \) for any \( T^* > 0 \). In the following, we will estimate \( u_m \) and \( T_m \) in \((H^1(\tilde{\Omega}))^2\) and \( H^1(\tilde{\Omega}) \) respectively. Using Hölder inequality and interpolation inequality, we have

\[
\int_\Omega L_{1u_m} \cdot [(u_m + \tilde{Z}_1) \cdot \nabla](u_m + \tilde{Z}_1)
\]

\[
\leq |u_m|_2 |\nabla u_m|_2 + |\tilde{Z}_1|_2 ||u_m||_2 |\nabla u_m|_2
\]

\[
+ |\tilde{Z}_1|_2 ||u_m||_2 ||\tilde{Z}_1||_1
\]

\[
\leq C||u_m||^2_2 ||u_m||^2_2 + ||u_m||_2 ||u_m||_2 ||\tilde{Z}_1||^2_2 ||\tilde{Z}_1||^2_2 + C||u_m||_2 ||\tilde{Z}_1||^2_2 ||\tilde{Z}_1||^2_2
\]

\[
\leq \varepsilon |u_m|^2 + C||u_m||^2_2 ||\tilde{Z}_1||^2_2 ||\tilde{Z}_1||^2_2 + C||Z_1||^2_2 ||\tilde{Z}_1||^2_2.
\]

(5.81)

Taking a similar argument in [10] to obtain the estimates of \( \tilde{v} \) in \((L^6(\tilde{\Omega}))^2\), we get

\[
\int_\Omega L_{1u_m} \cdot \varphi(u_m + \tilde{Z}_1) \partial_z (u_m + \tilde{Z}_1)
\]

\[
\leq \varepsilon |u_m|^2 + C||u_m||^2_2 ||u_m||_1 + C(1 + ||u_m||^2_1) ||\tilde{Z}_1||^2_2 ||\tilde{Z}_1||^2_2.
\]

(5.82)
Let $v_m = L_1u_m$ in (5.74). Since by Young inequality we get

$$
\int \Omega L_1u_m \cdot u_m^1 \leq \varepsilon \|u_m\|_1^2 + C\|u_m\|_2^2,
$$

which combined with (5.81) – (5.82) implies

$$
\partial_t \|u_m\|_1^2 + \|u_m\|_2^2 \leq C\|u_m\|_1^2 \|u_m\|_1 + C \left(1 + \|\tilde{Z}_1\|^2_1\|\tilde{Z}_1\|^2_2 + \|T_m\|^2_1\right) + C \left(\|u_m\|_1^2 + \|\tilde{Z}_1\|^2_1\|\tilde{Z}_1\|^2_2 + \|\tilde{Z}_1\|^2_1\|\tilde{Z}_1\|^2_2\right)\|u_m\|_1^2.
$$

(5.83)

As $u_m(0) = 0$ and $u_m$ is continuous in time with values in $H^1(\Omega)$ and progressively measurable, we can choose a stopping time $T_m$ such that

$$
\sup_{0 \leq t \leq T_m} \|u_m(t)\|_1 \leq \frac{1}{2C}.
$$

Next we will show that $T_m$ can be independent of $m$. Integrating (5.83) from 0 to $t$, for $t \in [0, T_m]$, we get

$$
\|u_m(t)\|_1^2 + \int_0^t \|u_m(t)\|_2^2 ds \leq C \int_0^t \left(1 + \|\tilde{Z}_1\|^2_1\|\tilde{Z}_1\|^2_2 + \|T_m\|^2_1\right) ds.
$$

Since $\int_0^t \|T_m(t)\|^2_2 dt$ is uniformly bounded for arbitrary $t > 0$, so we can choose a small $T^*$ independent of $m$ such that

$$
C \int_0^{T^*} \left(1 + \|\tilde{Z}_1\|^2_1\|\tilde{Z}_1\|^2_2 + \|T_m\|^2_1\right) ds \leq \frac{1}{4C^2},
$$

which implies that for all $m$, $T_m$ can be chosen to be equal to $T^*$ such that

$$
\sup_{t \in [0,T^*]} \|u_m(t)\|_1^2 + \int_0^{T^*} \|u_m(t)\|_2^2 ds \leq C(T^*, \tilde{Z}_1, Z_2, Q).
$$

(5.84)

Therefore, $u_m$ uniformly bounded with respect to $m$ in $L^\infty([0, T^*]; (H^1(\Omega))^2) \cap L^2([0, T^*]; (H^2(\Omega))^2)$. To estimate $T_m$ in $H^1(\Omega)$, setting $\tau_m = L_2T_m$ and using the methods to prove (5.81) and (5.82), we arrive at

$$
\int \Omega (L_2T_m)(\varphi(u_m + \tilde{Z}_1) \cdot \nabla)(T_m + Z_2)
\leq \varepsilon \|T_m\|_2^2 + C\|u_m\|_2^2 \|Z_2\|_2^2 + C\|T_m\|_1^2 \|u_m\|_4 + C\|\tilde{Z}_1\|^2_2\|T_m\|^2_2 + C\|\tilde{Z}_1\|^2_1\|Z_2\|^2_2
$$

and

$$
\int \Omega (L_2T_m)\varphi(u_m + \tilde{Z}_1) \partial_z(T_m + Z_2)
\leq \varepsilon \|T_m\|_2^2 + C\|T_m\|_2^2 \left(\|u_m\|_2^2 \|\tilde{Z}_1\|_2 \|\tilde{Z}_1\|_2^2 + \|Z_2\|_2^2 \|u_m\|_1 \|u_m\|_2 + C\|\tilde{Z}_1\|^2_2\|Z_2\|^2_2\right).
$$

Combining the above bounds yields

$$
\partial_t \|T_m\|_1^2 + \|T_m\|_2^2
\leq C\|T_m\|_1^2 \left(\|u_m\|_4^2 + \|u_m\|_2^2 \|\tilde{Z}_1\|_2 \|\tilde{Z}_1\|_2 \|\tilde{Z}_1\|_2^2\right) + C\|u_m\|_2^2 \|Z_2\|_2^2 \|u_m\|_1 \|u_m\|_2 + C\|\tilde{Z}_1\|^2_2\|Z_2\|^2_2 + C|Q|^2_2,
$$

(5.85)

which together with (5.84) implies that $T_m$ is uniformly bounded with respect to $m$ in $L^\infty([0, T^*]; (H^1(\Omega))^2) \cap L^2([0, T^*]; (H^2(\Omega))^2)$. Therefore, by (5.84), (5.85) and a standard argument (see [30] [47]),
that at this point do not require any novelty, one can prove further bounds on the time derivative of \((u_m, T_m)\), and reasoning on weakly and strongly convergent subsequences one gets the existence of a solution \((u, \theta)\) with the regularity specified by the theorem.

5.2. A Priori Estimates For The Global Existence Of Strong Solutions. In the previous subsections we have proven the existence of strong solution for a short interval of time, whose length depends on the initial data and other physical parameters of the system (5.68)–(5.73). Let \((v_0, T_0)\) be a given initial condition. In this section we will consider the strong solution that corresponds to this initial data in its maximal interval of existence \([0, \tau_*]\). Specially, for fixed \(\omega \in \Omega\), we will establish \textit{a priori} upper estimates for various norms of this solution in the interval \([0, \tau_*]\). In particular, we will show that if \(\tau_* < \infty\) then the \((H^1(\Omega))^{3}\) norm of the strong solution is bounded over the interval \([0, \tau_*]\). This key observation plays an important role in the proof of global regularity of strong solution to the system (5.68)–(5.73).

Similarly as in [4], to study the long time behavior of stochastic PEs, we introduce a modified stochastic convolution. For \(j = 1, 2\), \(\beta > 1\) and \(t \in \mathbb{R}\), we define

\[
Z_j(t) := \int_{-\infty}^{t} e^{-(t-s)(A_j+\beta)}dW_j^H(s).
\]

Then \(Z_j\) is the mild solution of the linear equation

\[
dZ_j = (-A_j Z_j - \beta Z_j)dt + dW_j^H,
Z_j(0) = z_0^j,
\]

where \(z_0^j = \int_{-\infty}^{0} e^{s(A_j+\beta)}dW_j^H(s)\), \(j = 1, 2\). It is easy to see that if \((u, \theta)\) is the global solution of the following system (5.86)–(5.91), then equivalently it is also the global solution of (5.68)–(5.73).

\[
\begin{align*}
\partial_t u - \Delta u - \partial_z u & + [(u + Z_1) \cdot \nabla](u + Z_1) + \varphi(u + Z_1)\partial_z(u + Z_1) \\
& + f(u + Z_1) + \nabla p_s - \int_{-1}^{t} \nabla \theta dz' = \beta Z_1; \\
\partial_t \theta - \Delta \theta - \partial_z \theta & + [(u + Z_1) \cdot \nabla](\theta + Z_2) + \varphi(u + Z_1)\partial_z(\theta + Z_2) = Q + \beta Z_2; \\
\int_{-1}^{0} \nabla \cdot udz & = 0; \\
\partial_z u|_{\Gamma_s} & = \partial_z u|_{\Gamma_s} = 0; u \cdot \bar{n}|_{\Gamma_s} = 0, \partial_{\bar{n}} u \times \bar{n}|_{\Gamma_s} = 0; \\
\partial_z \theta + \alpha \theta & \big|_{\Gamma_s} = \partial_z \theta|_{\Gamma_s} = 0, \partial_{\bar{n}} \theta|_{\Gamma_s} = 0; \\
(u|_{t=0}, \theta|_{t=0}) & = (v_0 - z_0^1, T_0 - z_0^2).
\end{align*}
\]

We denote by

\[
\tilde{\phi}(x, y) = \int_{-1}^{0} \phi(x, y, \xi)d\xi, \quad \forall (x, y) \in M.
\]

In particular,

\[
\tilde{u}(x, y) = \int_{-1}^{0} u(x, y, \xi)d\xi, \quad \text{in } M.
\]

Let

\[
\tilde{u} = u - \bar{u}.
\]

Notice that

\[
\tilde{u} = 0
\]
and
\[ \nabla \cdot \tilde{u} = 0, \quad \text{in } M. \]

In the following, we will study the properties of \( \tilde{u} \) and \( \bar{u} \). By taking the average of equations (5.86) in the \( z \) direction, over the interval \( (-1,0) \), and using boundary conditions (5.89), we have
\[
\partial_t \bar{u} + [(u + Z_1) \cdot \nabla] (u + Z_1) + \varphi(u + Z_1) \partial_z (u + Z_1) + (\bar{u} + \bar{Z}_1)^\perp
\]
\[+ \nabla p_s - \int_{-1}^{0} \int_{-1}^{z} \nabla \theta dz' dz - \Delta \bar{u} = \beta \bar{Z}_1. \quad (5.92) \]

Based on the above and integration by parts we have
\[
\int_{-1}^{0} \varphi(u + Z_1) \partial_z (u + Z_1) dz = \int_{-1}^{0} (u + Z_1) \nabla \cdot (u + Z_1) dz = \int_{-1}^{0} \nabla \cdot (\bar{u} + \bar{Z}_1)(\bar{u} + \bar{Z}_1) dz \quad (5.93)
\]
and
\[
\int_{-1}^{0} (u + Z_1) \cdot \nabla (u + Z_1) dz = \int_{-1}^{0} (\bar{u} + \bar{Z}_1) \cdot \nabla (\bar{u} + \bar{Z}_1) dz + (\bar{u} + \bar{Z}_1) \cdot \nabla (\bar{u} + \bar{Z}_1). \quad (5.94)
\]

Therefore, substituting (5.94) and (5.93) into (5.92), we can see that \( \bar{u} \) satisfies the following equations and boundary conditions:
\[
\partial_t \bar{u} - \Delta \bar{u} + (\bar{u} + \bar{Z}_1) \cdot \nabla (\bar{u} + \bar{Z}_1) + (\bar{u} + \bar{Z}_1) \nabla \cdot (\bar{u} + \bar{Z}_1) + (\bar{u} + \bar{Z}_1) \cdot \nabla (\bar{u} + \bar{Z}_1)^\perp
\]
\[+ (\bar{u} + \bar{Z}_1)^\perp + \nabla p_s - \int_{-1}^{0} \int_{-1}^{z} \theta(x, y, \lambda, t) d\lambda dz = \beta \bar{Z}_1, \quad (5.95)\]
\[
\nabla \cdot \bar{u} = 0, \quad \text{in } M, \quad (5.96)
\]
\[
\bar{u} \cdot \bar{n} = 0, \quad \partial_{\Gamma} \bar{u} \times \bar{n} = 0 \quad \text{on } M. \quad (5.97)
\]

By substracting (5.95) from (5.86) and using (5.89), (5.97) we conclude that \( \tilde{u} \) satisfies the following equations and boundary conditions:
\[
\partial_t \tilde{u} - \Delta \tilde{u} - \partial_{zz} \tilde{u} + [(\tilde{u} + \tilde{Z}_1) \cdot \nabla(\tilde{u} + \tilde{Z}_1) + \varphi(\tilde{u} + \tilde{Z}_1) \partial_z (\tilde{u} + \tilde{Z}_1)]
\]
\[+ [(\tilde{u} + \tilde{Z}_1) \cdot \nabla (\tilde{u} + \tilde{Z}_1)] + [(\tilde{u} + \tilde{Z}_1) \cdot \nabla (\tilde{u} + \tilde{Z}_1)] - (\tilde{u} + \tilde{Z}_1) \cdot \nabla (\tilde{u} + \tilde{Z}_1)^\perp
\]
\[= (\tilde{u} + \tilde{Z}_1) \nabla \cdot (\tilde{u} + \tilde{Z}_1) + (\tilde{u} + \tilde{Z}_1)^\perp - \int_{-1}^{z} \nabla \theta dz' + \int_{-1}^{0} \int_{-1}^{z} \nabla \theta dz' dz = 0, \quad (5.98)
\]
\[
\partial_z \tilde{u} |_{z=0} = 0, \quad \partial_z \tilde{u} |_{z=-1} = 0, \quad \tilde{u} \cdot \tilde{n}|_{\Gamma_s} = 0, \quad \partial_{\Gamma} \tilde{u} \times \tilde{n}|_{\Gamma_s} = 0. \quad (5.99)
\]

5.2.1. \( L^2 \) estimates. We take the inner product of equation (5.87) with \( \theta \), in \( L^2(\bar{U}) \), and get
\[
\frac{1}{2} \partial_t |\theta|^2 + |\nabla \theta|^2 + |\partial_z \theta|^2 + \alpha |\theta(z = 0)|^2
\]
\[= \int_{\bar{U}} Q \theta - \int_{\bar{U}} (u + Z_1) \cdot \nabla (\theta + Z_2) + \varphi(u + Z_1) \partial_z (\theta + Z_2) - \beta Z_2 \theta. \]

By integration by parts, we have
\[
\int_{\bar{U}} (u + Z_1) \cdot \nabla \theta + \varphi(u + Z_1) \partial_z \theta \theta = 0.
\]
On account of Hölder inequality and Sobolev imbedding theorem, we obtain
\[ \int_{\Omega} |\partial_t(\varphi(u) + Z_1)\partial_z Z_2| \theta \leq C|\partial_z Z_2|_{\infty} |\nabla \cdot u + \nabla \cdot Z_1|_2|\theta|_2 \]
\[ \leq \varepsilon|\nabla \cdot u|_2^2 + C\|Z_2\|_3^3|\theta|_2^2 + C\|Z_1\|_2^2. \]
Since \( u \cdot \tilde{n} = 0 \) on \( \partial\Omega \), by Exercise II.5.15 in [18] there exists a positive constant \( C = C(\Omega) \) such that \( |u|_2 \leq C|\nabla u|_2 \). Therefore, taking a similar argument as above we have
\[ \int_{\Omega} [(u + Z_1) \cdot \nabla Z_2] \theta \leq \varepsilon|\nabla u|_2^2 + C|Z_1|_2^2 + C\|Z_2\|_3^3|\theta|_2^2. \]
Combining the above bounds, we arrive at
\[ \partial_t|\theta|_2^2 + (2 - \varepsilon)|\nabla \theta|_2^2 + (2 - \varepsilon)|\theta_z|_2^2 + (2\alpha - \varepsilon)|\theta(z = 0)|_2^2 \leq 2\varepsilon|\nabla u|_2^2 + C|Q|_2^2 + C\|Z_1\|_2^2 + C\|Z_2\|_3^3|\theta|_2^2, \]
where we have used \( |\theta|_2^2 \leq C\|\theta\|_1^2 \) and
\[ c\|\theta\|_1^2 \leq |\nabla \theta|_2^2 + |\theta_z|_2^2 + \alpha|\theta(z = 0)|_2^2 \leq C\|\theta\|_1^2, \]
for some \( c, C > 0 \).
To obtain the global well-posedness, we take an analogous argument in section 5.1 and reach
\[ \partial_t|u|_2^2 + 2|\nabla u|_2^2 + 2|u_z|_2^2 \leq C|u|_2^2(\|Z_1\|_2^2 + \|Z_1\|_4^2) + C\|Z_1\|_2^2 + C|\theta_z|_2^2, \]
which together with the bound of \( |\theta|_3^2 \) and Gronwall inequality imply that
\[ \sup_{t \in [0, T_\ast]} (|u(t)|_2^2 + |\theta(t)|_2^2) + \int_{0}^{T_\ast} \|u(s)|_2^2 + \|\theta(s)|_2^2 ds \leq C(v_0, T_0, Z_1, Z_2). \tag{5.100} \]
In order to prove the existence of random attractor of stochastic PEs, we need to get a more delicate and careful \textit{a priori} estimate of \( |u(t)|_2^2 \). From (5.80) and Hölder inequality we infer that there exists \( c \in (\frac{1}{2\alpha} \vee \frac{1}{2}, 4) \) such that
\[ \int_{\Omega} u \cdot \int_{-1}^{z} \nabla \theta dz' \leq |\nabla u|_2|\theta|_2 \leq \frac{c}{2}|\nabla u|_2^2 + \frac{1}{2c} |\theta|_2^2, \]
where we assume \( \alpha > \frac{1}{8} \) and \( \frac{1}{2\alpha} \vee \frac{1}{2} = \max\{\frac{1}{2\alpha}, \frac{1}{2}\} \). Then, by the estimates of \( |u_m|_2 \) in section 5.1 we have
\[ \partial_t|u|_2^2 + 2|\nabla u|_2^2 + 2|u_z|_2^2 \leq C|u|_2^2(\|Z_1\|_2^2 + \|Z_1\|_4^2) + C\|Z_1\|_2^2 + \frac{1}{2c} |\theta|_2^2 + \frac{c}{2} |\nabla u|_2^2. \]
By (48) in [10],
\[ |\theta|_2^2 \leq 2|\theta_z|_2^2 + 2|\theta(z = 0)|_2^2, \]
which implies
\[ \frac{1}{2c} |\theta|_2^2 \leq \frac{1}{c} |\theta_z|_2^2 + \frac{1}{c} |\theta(z = 0)|_2^2 < 2|\theta_z|_2^2 + 2\alpha|\theta(z = 0)|_2^2. \]
In view of the bounds of \( \theta \) and \( u \), we conclude that
\[ \partial_t(|u|_2^2 + |\theta|_2^2) + (2 - \frac{c}{2} - 3\varepsilon)(|\nabla u|_2^2 + |\nabla \theta|_2^2) + (2 - \frac{1}{c} - \varepsilon)|u_z + \theta_z|_2^2 \]
\[ + (2\alpha - \frac{1}{c} - \varepsilon)|\theta(z = 0)|_2^2 \leq C(|u|_2^2 + |\theta|_2^2)(\|Z_1\|_2^2 + \|Z_1\|_4^2 + \|Z_2\|_3^3) \]
\[ + C(|Q|_2^2 + \|Z_1\|_2^2). \tag{5.101} \]
Since there exists $\gamma_1 > 0$ such that
\[
(2 - \frac{c}{2} - 3\varepsilon)(|\nabla u|_2^2 + |\nabla \theta|_2^2) + (2 - \frac{1}{c} - \varepsilon)|u_z + \theta_z|_2^2
\]
\[+(2\alpha - \frac{1}{c} - \varepsilon)|\theta(z = 0)|_2^2 > \gamma_1(|u|_2^2 + |\theta|_2^2).
\]
Then, for each $t \in [0, \tau_*)$, by Gronwall inequality we obtain
\[
|u(t)|_2^2 + |\theta(t)|_2^2 \leq (|u(0)|_2^2 + |\theta(0)|_2^2)e^{\int_0^t -\gamma_1 + C(|Z_1|_3^2 + |Z_2|_3^2)/ds}
\]
\[+ \int_0^t e^{\int_0^s -\gamma_1 + C(|Z_1|_3^2 + |Z_2|_3^2)/ds}(|Q|_2^2 + |Z_1|_3^2)ds,
\]
which also implies (5.100).

5.2.2. $L^4$ estimates about $\theta$ and $\ddot{u}$. Taking the inner product of the equation (5.87) with $\theta^3$ in $L^2(\Omega)$ and obtain
\[
\frac{1}{4} \partial_t |\theta|^4 + \frac{3}{4} \nabla \theta^2|_2^2 + \frac{3}{4} |(\theta^2)_z|_2^2 + \alpha \int_M |\theta(z = 0)|^4
\]
\[= \int_{\Omega} Q \theta^3 - \int_{\Omega} [(u + Z_1) \cdot \nabla (\theta + Z_2) + \varphi(u + Z_1) \partial_z (\theta + Z_2) + \beta Z_2] \theta^3.
\]
By integration by parts, we have
\[
\int_{\Omega} [(u + Z_1) \cdot \nabla \theta + \varphi(u + Z_1) \partial_z \theta] \theta^3 = 0.
\]
Using Hölder inequality,
\[
\int_{\Omega} [\varphi(u + Z_1) \partial_z Z_2] \theta^3 \leq |\partial_z Z_2|_\infty |\nabla \cdot u + \nabla \cdot Z_1|_2 |\theta^3|_2
\]
\[\leq \|Z_2\|_3 |\nabla \cdot u + \nabla \cdot Z_1|_2 |\theta^2|_3^\frac{4}{3}.
\]
Applying interpolation inequality to $|\theta^2|_3$, we obtain
\[
|\theta^2|_3 \leq C|\theta^2|_2^\frac{4}{3} (|\nabla \theta^2|_2^\frac{1}{3} + |\partial_z \theta^2|_2^\frac{1}{3} + \alpha |\theta^2(z = 0)|_2^\frac{1}{3}),
\]
which together with Hölder inequality implies that
\[
\int_{\Omega} [\varphi(u + Z_1) \partial_z Z_2] \theta^3 \leq \varepsilon (|\nabla \theta^2|_2^\frac{4}{3} + |\partial_z \theta^2|_2^\frac{4}{3} + \alpha |\theta^2(z = 0)|_2^\frac{1}{3}) + C\|Z_2\|_3^\frac{8}{3} \|u + Z_1\|_1^\frac{8}{3} |\theta|_4^\frac{12}{4}.
\]
Taking a similar argument, we have
\[
\int_{\Omega} [(u + Z_1) \cdot \nabla Z_2] \theta^3 \leq \varepsilon (|\nabla \theta^2|_2^\frac{4}{3} + |\partial_z \theta^2|_2^\frac{4}{3} + \alpha |\theta^2(z = 0)|_2^\frac{1}{3}) + C\|Z_2\|_3^\frac{8}{3} \|u + Z_1\|_1^\frac{8}{3} |\theta|_4^\frac{12}{4}.
\]
Analogously, we deduce
\[
\int_{\Omega} (Q + Z_2) \theta^3 \leq \varepsilon (|\nabla \theta^2|_2^\frac{4}{3} + |\partial_z \theta^2|_2^\frac{4}{3} + \alpha |\theta^2(z = 0)|_2^\frac{1}{3}) + C\|Q\|_2^\frac{8}{3} + \|Z_2\|_2^\frac{8}{3} |\theta|_4^\frac{12}{4}.
\]
Therefore, combining (5.103) – (5.107), we arrive at
\[
\partial_t |\theta|^4 + |\nabla \theta^2|_2^2 + |(\theta^2)_z|_2^2 + \alpha \int_M |\theta(z = 0)|^4
\]
\[\leq C(\|Q\|_2^\frac{8}{3} + \|Z_2\|_3^\frac{8}{3} + \|Z_1\|_1^\frac{8}{3} + \|Z_2\|_2^\frac{8}{3} \|u\|_1^\frac{8}{3}) |\theta|_4^\frac{12}{4}.
\]
Since by Young’s inequality
\[
|\theta|^4 \leq \int_{\Omega} \theta^4 = -\int_M \int_{z=0}^t \int_{\Omega} (\theta^2 z)^2 + \int_M \int_{z=0}^t \theta^4 (z = 0) \\
\leq 8|\theta^2 z|^2 + \frac{1}{2}|\theta|^4 + \int_M \theta^4 (z = 0),
\]
we have
\[
|\theta|^4 \leq 16|\partial_z \theta|^2 + 2|\theta (z = 0)|^4,
\]
which combined (5.108) implies
\[
\partial_t |\theta|^4 + |\theta|^4 \leq C(|Q|^2 + \|Z_2\|^2 + \|Z_1\|^2 + \|Z_2\|^3 \|u\|^2) |\theta|^1
\]
or
\[
\partial_t |\theta|^2 + |\theta|^2 \leq C(|Q|^2 + \|Z_2\|^2 + \|Z_1\|^2 + \|Z_2\|^3 \|u\|^2) |\theta|^1.
\]
Then, using Gronwall inequality yields,
\[
|\theta(t)|^2 \leq |\theta(t = 0)|^2 e^{Ct} + C \int_0^t e^{C(t-s)} (|Q|^2 + \|Z_2\|^2 + \|Z_1\|^2 + \|Z_2\|^3 \|u\|^2) ds \quad (5.109)
\]
for \( t \in [0, \tau_*] \). Since, by integration by parts and boundary conditions (5.99) we have
\[
\int_{\Omega} \left[(\tilde{u} \cdot \nabla)\tilde{u} - (\int_{z=0}^t \nabla \cdot \tilde{u} d\lambda) \partial_z \tilde{u}\right] |\tilde{u}|^2 \tilde{u} = 0
\]
and
\[
\int_{\Omega} \left[(\tilde{u} + \tilde{Z}_1) \cdot \nabla \tilde{u}\right] |\tilde{u}|^2 \tilde{u} = -\frac{1}{4} \int_{\Omega} |\tilde{u}|^4 \nabla \cdot (\tilde{u} + \tilde{Z}_1) = 0,
\]
as well as
\[
\int_{\Omega} [(\tilde{u} + \tilde{Z}_1) \cdot \nabla)(\tilde{u} + \tilde{Z}_1)] \cdot |\tilde{u}|^2 \tilde{u} \\
= -\int_{\Omega} [(\tilde{u} + \tilde{Z}_1) \cdot \nabla) |\tilde{u}|^2 \tilde{u}] \cdot (\tilde{u} + \tilde{Z}_1) - \int_{\Omega} \left( \nabla \cdot (\tilde{u} + \tilde{Z}_1) \right) |\tilde{u}|^2 \tilde{u} \cdot (\tilde{u} + \tilde{Z}_1)
\]
and
\[
\int_{\Omega} (\tilde{u} + \tilde{Z}_1) \nabla \cdot (\tilde{u} + \tilde{Z}_1) = (\tilde{u} + \tilde{Z}_1) \cdot \nabla (\tilde{u} + \tilde{Z}_1) \cdot |\tilde{u}|^2 \tilde{u} \\
= -\int_{\Omega} (\tilde{u}_k + \tilde{Z}_{1,k})(\tilde{u}_j + \tilde{Z}_{1,j}) \partial_{x_k} (|\tilde{u}|^2 \tilde{u}_j),
\]
where \( \tilde{u}_k \) and \( \tilde{Z}_{1,k} \) are the k'th coordinate of \( \tilde{u} \) and \( \tilde{Z}_1 \) respectively with \( k = 1, 2 \). Taking the inner
product of the equation (5.98) with $|\bar{u}|^2 \bar{u}$ in $(L^2(\Omega))^2$, by the above equalities about $\bar{u}$ we get

$$
\frac{1}{4} \partial_t |\bar{u}|^4 + \frac{1}{2} \int_\Omega \left( |\nabla (|\bar{u}|^2)|^2 + |\partial_z (|\bar{u}|^2)|^2 \right) + \int_\Omega |\bar{u}|^2 (|\nabla \bar{u}|^2 + |\partial_z \bar{u}|^2) \\
= - \int_\Omega [Z_1 \cdot \nabla \bar{u} + \bar{u} \cdot \nabla Z_1 + \nabla Z_1] \cdot |\bar{u}|^2 \bar{u} \\
- \int_\Omega [\varphi(Z_1) \partial_z \bar{u} + \varphi(\bar{u}) \partial_z \bar{Z}_1 + \varphi(\bar{Z}_1) \partial_z \bar{Z}_1] \cdot |\bar{u}|^2 \bar{u} \\
+ \int_\Omega (\bar{u} + Z_1) \cdot [(\bar{u} + Z_1) \cdot \nabla] |\bar{u}|^2 \bar{u} + \int_\Omega [\nabla \cdot (\bar{u} + Z_1)] (\bar{u} + Z_1) \cdot |\bar{u}|^2 \bar{u} \\
- \int_\Omega \{(\bar{u} + Z_1) \cdot \nabla \bar{Z}_1 \} |\bar{u}|^2 \bar{u} - \int_\Omega (\bar{u}_k + Z_{1,k})(\bar{u}_j + Z_{1,j}) \partial_{\bar{u}_j} (|\bar{u}|^2 \bar{u}_j) \\
- \int_\Omega (f \times \bar{Z}_1) \cdot |\bar{u}|^2 \bar{u} \\
- \int_\Omega \left( \int_{-1}^0 \int_{-1}^0 \partial_t \lambda dz - \int_{-1}^0 \int_{-1}^0 \partial_t \lambda dz \right) \nabla \cdot |\bar{u}|^2 \bar{u} := \Sigma_{j=1}^8 I_j. \tag{5.110}
$$

Next, we estimate $I_j$ respectively, for $j = 1, \cdots, 8$. Since by integration by parts, we have

$$
\int_\Omega (\bar{Z}_1 \cdot \nabla \bar{Z}_1) |\bar{u}|^2 \bar{u} + \int_\Omega (\varphi(\bar{Z}_1) \partial_z \bar{u}) |\bar{u}|^2 \bar{u} \\
= \int_\Omega (\bar{Z}_1 \cdot \nabla \bar{u}_i) |\bar{u}|^2 \bar{u}_i + \int_\Omega (\varphi(\bar{Z}_1) \partial_z \bar{u}_i) |\bar{u}|^2 \bar{u}_i \\
= \frac{1}{4} \int_\Omega |\bar{Z}_1 \cdot \nabla |\bar{u}|^4 + \frac{1}{4} \int_\Omega \varphi(\bar{Z}_1) \partial_z |\bar{u}_i|^4 = 0. \tag{5.111}
$$

Therefore, to estimate $I_1$ (or $I_2$), we need only to estimate the other two terms. By interpolation inequalities and Hölder inequality, we get

$$
\int_\Omega (\bar{u} \cdot \nabla \bar{Z}_1) |\bar{u}|^2 \bar{u} \leq |\nabla \bar{Z}_1|_3 \left( \int_\Omega (|\bar{u}|^2)^3 \right)^{\frac{1}{3}} \\
\leq C |\nabla \bar{Z}_1|_3 |(|\bar{u}|^2)_2||(|\bar{u}|^2)_1 \\
\leq C |\nabla \bar{Z}_1|_3 |(|\bar{u}|^2)_2 \left( |\nabla (|\bar{u}|^2)|_2 + |\partial_z (|\bar{u}|^2)|_2 + |(|\bar{u}|^2)|_2 \right) \\
\leq \varepsilon \left( |\nabla (|\bar{u}|^2)|_2 + |\partial_z (|\bar{u}|^2)|_2 \right) + C \| \bar{Z}_1 \|_2^2 |\bar{u}|_4^2.
$$

Analogously, we have

$$
\int_\Omega (\bar{Z}_1 \cdot \nabla \bar{Z}_1) \cdot |\bar{u}|^2 \bar{u} \leq |(|\bar{u}|^2)_2|_3^3 \nabla \bar{Z}_1|_3 \bar{Z}_1|_6 \\
\leq C |(|\bar{u}|^2)|_2^\frac{3}{2} |(|\bar{u}|^2)|_1^\frac{3}{4} \nabla \bar{Z}_1|_3 \bar{Z}_1|_6 \\
\leq \varepsilon \left( |\nabla (|\bar{u}|^2)|_2^2 + |\partial_z (|\bar{u}|^2)|_2^2 \right) + C \| \bar{Z}_1 \|_2^2 |\bar{u}|_4^4 + C \| \bar{Z}_1 \|_2^\frac{2}{3} |\bar{u}|_4^\frac{2}{3}.
$$

Therefore, based on the above bounds we infer that

$$
I_1 \leq 2 \varepsilon \left( |\nabla (|\bar{u}|^2)|_2^2 + |\partial_z (|\bar{u}|^2)|_2^2 \right) + C \| \bar{Z}_1 \|_2^2 |\bar{u}|_4^4 + C \| \bar{Z}_1 \|_2^\frac{2}{3} |\bar{u}|_4^\frac{2}{3}.
$$

Applying integration by parts formula on $M$, we reach

$$
\int_\Omega \varphi(\bar{u}) \partial_z \bar{Z}_1 \cdot |\bar{u}|^2 \bar{u} = \int_{-1}^0 \int_M \varphi(\bar{u}) \partial_z \bar{Z}_1, i |\bar{u}|^2 \bar{u}_i dz \\
= \int_{-1}^0 \left( \int_M (\int_{-1}^z \bar{u} d\bar{z}')(\nabla \partial_z \bar{Z}_1) \cdot |\bar{u}|^2 \bar{u}_i + (\partial_z \bar{Z}_1) \nabla |\bar{u}|^2 \bar{u}_i) dz' \right) dz
$$
which together with Hölder inequality implies
\[
\int_{\mathcal{D}} \varphi(\tilde{u}) \partial_z \tilde{Z}_1 \cdot |\tilde{u}|^2 \tilde{u} \leq |\nabla \partial_z \tilde{Z}_1|_9 (\int_{-1}^{\epsilon} \tilde{u} dz') |\tilde{u}| \leq C \|Z_1\|_2 |\tilde{u}|^4 + C \|Z_1\|_2 |\tilde{u}|^2_6 |\nabla |\tilde{u}|^2_2|.
\]
Since, by interpolation inequality,
\[
|\tilde{u}|^3_8 = |\tilde{u}|^2_4 \leq C |\tilde{u}|^2_2 \left( |\nabla |\tilde{u}|^2_4| + |\partial_z |\tilde{u}|^2_2| \right)
\]
and
\[
|\tilde{u}|^2_6 = |\tilde{u}|^2_3 \leq C |\tilde{u}|^2_2 \left( |\nabla |\tilde{u}|^2_3| + |\partial_z |\tilde{u}|^2_2| \right),
\]
taking into account of Hölder inequality we conclude
\[
\int_{\mathcal{D}} \varphi(\tilde{u}) \partial_z \tilde{Z}_1 \cdot |\tilde{u}|^2 \tilde{u} \leq \epsilon (|\nabla |\tilde{u}|^2_2| + |\partial_z |\tilde{u}|^2_2|) + C \|Z_1\|_2^3 |\tilde{u}|^4_1. \tag{5.112}
\]

Using again integration by parts formula and Hölder inequality,
\[
\int_{\mathcal{D}} \varphi(\tilde{Z}_1) \cdot \partial_z \tilde{Z}_1 \cdot |\tilde{u}|^2 \tilde{u} \leq |(\tilde{u}^3)|_2 |\partial_z \tilde{Z}_1|_3 |\nabla \tilde{Z}_1|_6 \leq C |(\tilde{u}^3)|_2^2 |\tilde{Z}_1|^2_6
\leq C \epsilon (|\nabla |\tilde{u}|^2_2| + |\partial_z |\tilde{u}|^2_2|) + C \|Z_1\|_2^4 |\tilde{u}|^4_4 + C \|Z_1\|_2^3 |\tilde{u}|^3_4. \tag{5.113}
\]
Therefore, from (5.111) – (5.113) we conclude
\[
I_2 \leq \epsilon \left( |\nabla |\tilde{u}|^2_2| + |\partial_z |\tilde{u}|^2_2| \right) + C \|Z_1\|_2^4 |\tilde{u}|^4_4 + C \|Z_1\|_2^3 |\tilde{u}|^3_4 + C \|Z_1\|_2^2 |\tilde{u}|^2_4.
\]
To estimate $I_3$, we first consider
\[
\int_{\mathcal{D}} (\tilde{u} + \tilde{Z}_1) \cdot \{(\tilde{u} + \tilde{Z}_1) \cdot \nabla |\tilde{u}|^2 \tilde{u} \}
= \int_{\mathcal{D}} \tilde{u} \cdot \{(\tilde{u} + \tilde{Z}_1) \cdot \nabla |\tilde{u}|^2 \tilde{u} \} + \int_{\mathcal{D}} \tilde{Z}_1 \cdot \{(\tilde{u} + \tilde{Z}_1) \cdot \nabla |\tilde{u}|^2 \tilde{u} \}. \tag{5.114}
\]
Applying Hölder inequality repeatedly on the first term on the right hand side of (5.114) ,
\[
\int_{\mathcal{D}} \tilde{u} \cdot \{(\tilde{u} + \tilde{Z}_1) \cdot \nabla |\tilde{u}|^2 \tilde{u} \}
\leq C \int_{M} \int_{-1}^{0} \tilde{u} |\nabla \tilde{u}| |\tilde{u}|^2 dz + |\tilde{Z}|_{L^\infty} \int_{M} \int_{-1}^{0} |\nabla \tilde{u}| |\tilde{u}|^2 dz 
\leq \int_{M} |\tilde{u}| (\int_{-1}^{0} |\tilde{u}|^4 dz)^{\frac{1}{2}} (\int_{-1}^{0} |\tilde{u}|^2 dz)^{\frac{1}{2}} 
+ |\tilde{Z}|_{L^\infty} \int_{M} |\tilde{u}| (\int_{-1}^{0} |\tilde{u}|^2 dz)^{\frac{1}{2}} (\int_{-1}^{0} |\tilde{u}|^2 dz)^{\frac{1}{2}} 
\leq |\tilde{u}|_{L^4(M)} |\nabla |(\tilde{u})^2|_2 (\int_{M} (\int_{-1}^{0} |\tilde{u}|^4 dz)^{\frac{1}{2}} )^{\frac{1}{4}} 
+ |\tilde{Z}|_{L^\infty} |\tilde{u}|_{L^4(M)} |\nabla |(\tilde{u})^2|_2 (\int_{M} (\int_{-1}^{0} |\tilde{u}|^2 dz)^{\frac{1}{2}} )^{\frac{1}{4}}.
\]
Then by Minkowski inequality and interpolation inequalities, proceeding from above we have

\[
\int_{\Omega} \tilde{u} \cdot \{[(\tilde{u} + \tilde{Z}) \cdot \nabla] |\tilde{u}|^2 \tilde{u} \} \leq |\tilde{u}|_{L^4(M)} |\nabla (|\tilde{u}|^2)|_2 \left( \int_{-1}^{0} (|\tilde{u}_{\varepsilon}|^2) \right)^{\frac{1}{2}} \\
+ |\tilde{Z}_1|_{\infty} |\tilde{u}|_{L^4(M)} |\nabla (|\tilde{u}|^2)|_2 |\tilde{u}|_4
\]

\[
\leq |\tilde{u}|_{L^4(M)} |\nabla (|\tilde{u}|^2)|_2 \left( \int_{-1}^{0} (|\tilde{u}|^2)_{L^2(M)} |||\tilde{u}|^2||_{H^1(M)} dz \right)^{\frac{1}{2}} \\
+ |\tilde{Z}_1|_{\infty} |\tilde{u}|_{L^4(M)} |\nabla (|\tilde{u}|^2)|_2 |\tilde{u}|_4
\]

\[
\leq |\tilde{u}|_{L^4(M)} |\nabla (|\tilde{u}|^2)|_2 \left( |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( |\tilde{u}|^2 \right)^{\frac{1}{2}} + |\nabla (|\tilde{u}|^2)|_2 |\tilde{u}|_4
\]

\[
+ |\tilde{Z}_1|_{\infty} |\tilde{u}|_{L^4(M)} |\nabla (|\tilde{u}|^2)|_2 |\tilde{u}|_4,
\]

which together with Hölder inequality implies

\[
\int_{\Omega} \tilde{u} \cdot \{[(\tilde{u} + \tilde{Z}_1) \cdot \nabla] |\tilde{u}|^2 \tilde{u} \} \leq \varepsilon (|\nabla (|\tilde{u}|^2)|_2^2 + |\partial_x (|\tilde{u}|^2)|_2^2)
\]

\[
+ C(|\tilde{u}|_{L^4(M)}^4 + |\tilde{u}|_{L^4(M)}^4 |\tilde{u}|_4^4) + C^{\varepsilon}(\tilde{Z}_1|_{\infty}^4 + |\tilde{u}|_{L^4(M)}^4 |\tilde{u}|_4^4).
\]  

By Hölder inequality,

\[
\int_{\Omega} \tilde{Z}_1 \cdot \{[(\tilde{u} + \tilde{Z}_1) \cdot \nabla] |\tilde{u}|^2 \tilde{u} \} \leq \varepsilon \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2 + C[Z_1|_{L^2(x)}^4 |\tilde{u}|_4^4 + C[Z_1|_{L^2(x)}^4 |\tilde{u}|_4^4
\]

\[
\leq \varepsilon \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2 + C[Z_1|_{L^2(x)}^4 |\tilde{u}|_4^4 + C[Z_1|_{L^2(x)}^4 |\tilde{u}|_4^4.
\]

Therefore, from (5.114) – (5.116) we have

\[
I_3 \leq \varepsilon (|\nabla (|\tilde{u}|^2)|_2^2 + |\partial_x (|\tilde{u}|^2)|_2^2 + \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2
\]

\[
+ C(|\tilde{u}|_{L^4(M)}^4 + |\tilde{u}|_{L^4(M)}^4 + |\tilde{u}|_{L^4(M)}^4 + C[Z_1|_{L^2(x)}^4 |\tilde{u}|_4^4 + C[Z_1|_{L^2(x)}^4 |\tilde{u}|_4^4.
\]

Similar to the estimate for $I_2$, we obtain

\[
I_4 = \int_{\Omega} |\nabla \cdot (\tilde{u} + \tilde{Z}_1)(\tilde{u} + \tilde{Z}_1) \cdot |\tilde{u}|^2 \tilde{u}
\]

\[
\leq \varepsilon \left( \int_{\Omega} (|\nabla \tilde{u}|^2 |\tilde{u}|^2 + |\nabla (|\tilde{u}|^2)|_2^2 + |\partial_x (|\tilde{u}|^2)|_2^2) \right)
\]

\[
+ C(|\tilde{u}|_{L^4(M)}^4 + |\tilde{u}|_{L^4(M)}^4 + |\tilde{u}|_{L^4(M)}^4 + C[Z_1|_{L^2(x)}^4 |\tilde{u}|_4^4 + C[Z_1|_{L^2(x)}^4 |\tilde{u}|_4^4.
\]

By Hölder inequality and interpolation inequality,

\[
I_5 = \int_{\Omega} \{[(\tilde{u} + \tilde{Z}_1) \cdot \nabla] \tilde{Z}_1 \} |\tilde{u}|^2 \tilde{u}
\]

\[
\leq |\tilde{u}|_{L^4(M)}^4 |\nabla \tilde{Z}_1|_{L^4(M)} + |\tilde{Z}_1|_{L^4(M)}
\]

\[
\leq |(|\tilde{u}|^2)|_2^4 \left( |\tilde{u}|^2 \right)^{\frac{3}{17}} |\tilde{Z}_1|_{L^2(x)} |\tilde{u}|_{L^4(M)} + |\tilde{Z}_1|_{L^4(M)}
\]

\[
\leq \varepsilon (|\nabla (|\tilde{u}|^2)|_2^2 + |\partial_x (|\tilde{u}|^2)|_2^2)
\]

\[
+ C[Z_1|_{L^2(x)}^4 + C[Z_1|_{L^2(x)}^4 + C[Z_1|_{L^2(x)}^4 |\tilde{u}|_4^4.
\]

\[
42
\]
Using Hölder inequality, we have

\[ I_6 = -\int_{\Omega} (\tilde{u}_k + \tilde{Z}_{1,k})(\tilde{u}_j + \tilde{Z}_{1,j}) \partial_{x_k}(|\tilde{u}|^2 \tilde{u}_j) \]

\[ \leq \int_M \left( \int_{-1}^0 |\tilde{u}|^2 dz \int_{-1}^0 |\nabla \tilde{u}| |\tilde{u}|^2 dz \right) \]

\[ + |Z_1| \int_M \left( \int_{-1}^0 |\tilde{u}| \right) \int_{-1}^0 |\nabla \tilde{u}| |\tilde{u}|^2 dz + |Z_1| \int_M |\nabla \tilde{u}| |\tilde{u}| |\tilde{u}_2| \]

\[ \leq \left( \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} \left( \int_{-1}^0 |\tilde{u}|^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} \]

\[ + |Z_1| \int_{\Omega} \left( \int_{-1}^0 |\nabla \tilde{u}|^2 |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} \left( \int_{-1}^0 |\tilde{u}|^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} + C \| Z_1 \|_2 \| \nabla \tilde{u} \|_2 \| \tilde{u}_2 \|_2, \]

which together with Minkowski inequality and interpolation inequality imply

\[ I_6 \leq \left( \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} \left( \int_{-1}^0 |\tilde{u}|^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} \]

\[ + |Z_1| \int_{\Omega} \left( \int_{-1}^0 |\nabla \tilde{u}|^2 |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} \left( \int_{-1}^0 |\tilde{u}|^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} + C \| Z_1 \|_2 \| \nabla \tilde{u} \|_2 \| \tilde{u}_2 \|_2 \]

\[ \leq C \left( \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( \int_{-1}^0 |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( \int_{-1}^0 |\tilde{u}|^2 \right)^{\frac{3}{2}} \]

\[ + |Z_1| \| \nabla \tilde{u} \|_2 \| \tilde{u}_2 \|_2 \]

\[ \leq C \left( \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( \int_{-1}^0 |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( \int_{-1}^0 |\tilde{u}|^2 \right)^{\frac{3}{2}} \]

\[ + C \| Z_1 \|_2 \| \nabla \tilde{u} \|_2 \| \tilde{u}_2 \|_2, \]

Therefore, using Hölder inequality again we arrive at

\[ I_6 \leq \varepsilon \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2 + C \| Z_1 \|_2 |\tilde{u}|^2 + C (\| Z_1 \|_2 + \| \tilde{u} \|_1) |\tilde{u}|_4. \]

By Hölder inequality and Sobolev imbedding theorem,

\[ I_7 = -\int_{\Omega} (f_k \cdot \tilde{Z}_1) \cdot |\tilde{u}|^2 \tilde{u} \leq C \| Z_1 \|_1 |\tilde{u}|^3. \]

Analogously, we have

\[ I_8 = -\int_{\Omega} \left( \int_{-1}^0 \int_{-1}^0 \partial_\theta \right) \nabla \cdot |\tilde{u}|^2 \tilde{u} \]

\[ \leq \left( \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2 \right)^{\frac{1}{2}} \left( \int_{-1}^0 \theta^4 \right)^{\frac{1}{4}} \left( \int_{-1}^0 |\theta|^4 \right)^{\frac{1}{4}} \]

\[ \leq \varepsilon \int_{\Omega} |\nabla \tilde{u}|^2 |\tilde{u}|^2 + C |\tilde{u}|^2 |\theta|^2 \]

Therefore, by (5.110) and the estimates of \( I_1 - I_8 \), we have

\[ \partial_t |\tilde{u}|^4 + \int_{\Omega} \left( |\nabla(|\tilde{u}|^4)|^2 + \partial_z(|\tilde{u}|^2)^2 \right) + \int_{\Omega} |\tilde{u}|^2 (|\nabla \tilde{u}|^2 + |\partial_z \tilde{u}|^2) \]

\[ \leq C (\| Z_1 \|_2^2 + \| Z_1 \|_2^2 + \| u \|_2^2 + \| u_1 \|_2 \| u \|_1) |\tilde{u}|^4 \]

\[ + C (\| Z_1 \|_1 + \| Z_1 \|_2 \| u \|_1 + \| Z_1 \|_1 \| u \|_1) |\tilde{u}|^4 \]

\[ + C (\| Z_1 \|_2^2 + \| u \|_2 \| u \|_1 + \| u \|_1 \| u \|_1) |\tilde{u}|^4 \]

\[ + C (\| Z_1 \|_2^2 + \| Z_1 \|_2 \| u \|_1 + \| u \|_1 \| u \|_1) |\tilde{u}|^4 \] (5.117)
\[ \partial_t |\bar{u}|^2 \leq C(\|Z_1\|^2_2 + \|Z_1\|^2_2 + \|u\|^2_2 + |u|^2_2 \|u\|^2_2 + \|Z_1\|^2_2 \|u\|^2_2) |\bar{u}|^2_2 + C(\|Z_1\|^2_2 + \|Z_1\|^2_2 \|u\|^2_2 + |\theta|^2_2). \] (5.118)

Subsequently, by Gronwall inequality, (5.110) and (5.117) – (5.118), we conclude that
\[ \sup_{t \in [0, \tau^*]} |\bar{u}(t)|^2_1 + \int_0^{\tau^*} \left( |\nabla (|\bar{u}|^2)|^2 + |\partial_x (|\bar{u}|^2)|^2 \right) ds + \int_0^{\tau^*} |\bar{u}|^2_2 (|\nabla \bar{u}|^2 + |\partial_x \bar{u}|^2) ds \leq C(\tau^*, Q, Z_1, Z_2, v_0, T_0). \] (5.119)

5.2.3. \( H^1 \) estimates about \( \theta \) and \( u \).
By integration by parts and (5.96) – (5.97) (for more detail, see [10]), we have
\[ \int_M f_k \times \bar{u} \cdot \Delta \bar{u} = 0, \quad \int_M \nabla p_\lambda \cdot \Delta \bar{u} = 0 \]
and
\[ \int_M \nabla \int_{-1}^0 \int_{-1}^1 \theta(x, y, \lambda, t) d\lambda dz \cdot \Delta \bar{u} = 0. \]

Then, taking the inner product of equation (5.92) with \(-\Delta \bar{u}\) in \( L^2(M) \), we arrive at
\[ \frac{1}{2} \partial_t |\nabla \bar{u}|^2_2 + |\Delta \bar{u}|^2_2 = \int_M (\bar{u} + \bar{Z}_1) \cdot \nabla (\bar{u} + \bar{Z}_1) \cdot \Delta \bar{u} + \int_M (\bar{u} + \bar{Z}_1) \cdot \nabla (\bar{u} + \bar{Z}_1) \cdot \Delta \bar{u} \]
\[ + \int_M (\bar{u} + \bar{Z}_1) \nabla \cdot (\bar{u} + \bar{Z}_1) \cdot \Delta \bar{u} + \int_M f_k \cdot \bar{Z}_1 \cdot \Delta \bar{u} + \int_M \bar{Z}_1 \cdot \Delta \bar{u}. \] (5.120)

By H"older inequality and interpolation inequalities, we have
\[ \int_M (\bar{u} + \bar{Z}_1) \cdot \nabla (\bar{u} + \bar{Z}_1) \cdot \Delta \bar{u} \]
\[ = \int_M \bar{u} \cdot \nabla (\bar{u} + \bar{Z}_1) \cdot \Delta \bar{u} + \int_M \bar{Z}_1 \cdot \nabla (\bar{u} + \bar{Z}_1) \cdot \Delta \bar{u} \]
\[ \leq C|\bar{u}|^\frac{3}{2} (|\nabla \bar{u}|_2 + |\nabla \bar{Z}_1|_2) |\Delta \bar{u}|^\frac{1}{2} + C|Z_1|_{\infty} (|\nabla \bar{u}|_2 + |\nabla \bar{Z}_1|_2) |\Delta \bar{u}|_2 \]
\[ \leq \varepsilon |\Delta \bar{u}|^3_2 + C|\bar{u}|^\frac{3}{2} (|\nabla \bar{u}|^2_2 + |\nabla \bar{Z}_1|_2^3) + C|Z_1|_{\infty} (|\nabla \bar{u}|^2_2 + |\nabla \bar{Z}_1|_2^3). \] (5.121)

Using H"older inequality, Minkowski inequality and Sobolev imbedding theorem, we obtain
\[ \int_M (\bar{u} + \bar{Z}_1) \nabla \cdot (\bar{u} + \bar{Z}_1) \cdot \Delta \bar{u} + \int_M (\bar{u} + \bar{Z}_1) \cdot \nabla (\bar{u} + \bar{Z}_1) \cdot \Delta \bar{u} \]
\[ \leq 2 \int_M \left( \int_{-1}^0 (|\bar{u}| + |\bar{Z}_1|)(|\nabla \bar{u}| + |\nabla \bar{Z}_1|) \right) |\Delta \bar{u}| \]
\[ \leq 2 |\Delta \bar{u}|_2 \left[ \left( \int_M |\bar{u}|^2 |\nabla \bar{u}|^2 \right)^\frac{1}{2} + \left( \int_M |\bar{u}|^2 |\nabla \bar{Z}_1|^2 \right)^\frac{1}{2} \right] \]
\[ + \left( \int_M |\bar{Z}_1|^2 |\nabla \bar{u}|^2 \right)^\frac{1}{2} \left[ \left( \int_M |\bar{Z}_1|^2 |\nabla \bar{Z}_1|^2 \right)^\frac{1}{2} \right] \]
\[ \leq \varepsilon |\Delta \bar{u}|^3_2 + C \left( \int_M |\bar{u}|^2 |\nabla \bar{u}|^2 + |\bar{u}|^2 |Z_1|_2^2 + |Z_1|_{\infty} |u|_2^2 + |Z_1|_{\infty} |Z_1|_1^2 \right). \] (5.122)

From (5.120) – (5.122), we conclude that
\[ \partial_t |\nabla \bar{u}|^2_2 + |\Delta \bar{u}|^2_2 \leq C(|u|^2_2 |u|_2^2 + |Z_1|^2_2) |\nabla \bar{u}|^2_2 + C(|Z_1|^2_2 + \|Z_1\|^2_2 + |Z_1|^2_2) \]
\[ + \|Z_1\|^2_2 |u|_2^2 + \|Z_1\|^2_2 |\bar{u}|_4^2 + \|Z_1\|_2 |u|_2^2 + \int_M |\bar{u}|^2 |\nabla \bar{u}|^2. \] (5.123)
Therefore, by Proposition 2.1, Proposition 2.2, (5.100), (5.119), (5.123) and Gronwall inequality, we arrive at

$$\sup_{t \in [0, \tau^*]} |\nabla \bar{u}(t)|^2 \leq C(\tau^*, Q, Z_1, Z_2, v_0, T_0).$$  \hspace{1cm} (5.124)

Taking the derivative, with respect to $z$, of equation (5.86), we get

\begin{align*}
\partial_t u_z - \Delta u_z - \partial_{zz} u_z + [(u + Z_1) \cdot \nabla](u_z + \partial_z Z_1) + [(u_z + \partial_z Z_1) \cdot \nabla](u + Z_1) \\
- (u_z + \partial_z Z_1) \nabla \cdot (u + Z_1) + \varphi(u + Z_1)(u_{zz} + \partial_{zz} Z_1) \\
+ f_k \times (u_z + \partial_z Z_1) - \nabla \theta - \beta \partial_z Z_1 = 0. \hspace{1cm} (5.125)
\end{align*}

By integration by parts, we obtain

$$\int_\Omega \left( (u \cdot \nabla) u_z + \varphi(u) u_{zz} \right) \cdot u_z = 0. \hspace{1cm} (5.126)$$

Similarly, using Hölder inequality, interpolation inequality and Sobolev embedding theorem, we have

$$\int_\Omega [(u \cdot \nabla) \partial_z Z_1] \cdot u_z \leq \|Z_1\|_2 |u|_4 |u_z|_4$$

$$\leq \|Z_1\|_2 |u|_4 |u_z|_4 \frac{1}{4} |\nabla u_z|_2^2 + |u_{zz}|_2^2 + |u_z|_4^2$$

$$\leq \varepsilon (|\nabla u_z|_2^2 + |u_{zz}|_2^2) + C\|Z_1\|_2^2 (|\bar{u}|_4^2 + |\nabla \bar{u}|_2^2) + C|u_z|_2^2. \hspace{1cm} (5.127)$$

By virtue of Hölder inequality, we have

$$\int_\Omega (Z_1 \cdot \nabla)(u_z + \partial_z Z_1) \cdot u_z \leq |Z_1|_\infty (|\nabla u_z|_2 + |\nabla \partial_z Z_1|_2) |u_z|_2$$

$$\leq \varepsilon |\nabla u_z|_2^2 + C\|Z_1\|_2^2 (|u_z|_2^2 + 1). \hspace{1cm} (5.128)$$

Thanks to Hölder inequality, interpolation inequality and Sobolev imbedding theorem, we reach

$$\int_\Omega [(u \cdot \nabla) u] \cdot u_z + \int_\Omega [(u \cdot \nabla) Z_1] \cdot u_z$$

$$\leq C \int_\Omega |u||u_z||\nabla u_z| + C \int_\Omega |Z_1||u_z||\nabla u_z|$$

$$\leq C|\nabla u_z|_2 |u|_4 |u_z|_4 + C|Z_1|_\infty |u_z|_2 |\nabla u_z|_2$$

$$\leq C|\nabla u_z|_2 |u|_4 |u_z|_2 \left( |\nabla u_z|_2^2 + |\partial_z u_z|_2^2 + |u_z|_2^2 \right) + C|Z_1|_\infty |u_z|_2 |\nabla u_z|_2$$

$$\leq \varepsilon (|\nabla u_z|_2^2 + |\partial_z u_z|_2^2) + C(|u_z|_2^2 + \|Z_1|_2^2 + 1)|u_z|_2^2. \hspace{1cm} (5.129)$$

By integration by parts, Hölder inequality and interpolation inequality, we have

\begin{align*}
\int_\Omega [(\partial_z Z_1 \cdot \nabla)(u + Z_1)] \cdot u_z \\
= \int_\Omega [\partial_z Z_1 \cdot \nabla(u_j + Z_{1,j})] \partial_z u_j = - \int_\Omega \nabla \cdot [(\partial_z u_j)(\partial_z Z_1)](u_j + Z_{1,j}) \\
\leq |\nabla \partial_z Z_1|_2 |u_z|_4 |u + Z_1|_4 + |\nabla \partial_z u_j|_2 |\partial_z Z_1|_4 |u + Z_1|_4 \\
\leq C\|Z_1\|_2 |u_z|_2 \frac{1}{2} (|\nabla u_z|_2^2 + |u_z|_2^3 + |u_{zz}|_2^3) |u + Z_1|_4 \\
+ C\|\nabla \partial_z u_j\|_2 \|Z_1\|_2 |u + Z_1|_4$$

$$\leq \varepsilon (|\nabla u_z|_2^2 + |u_{zz}|_2^2) + C\|Z_1\|_2^2 |u + Z_1|_2^2 + |u_z|_2^2. \hspace{1cm} (5.130)$$
where \( j = 1, 2, Z_1 = (Z_{1,1}, Z_{1,2}) \) and \( u = (u_1, u_2) \). Since

\[
\int_{\Omega} [(u_z + \partial_z Z_1) \cdot \nabla (u + Z_1)] \cdot u_z
\]

\[
= \int_{\Omega} [(u_z \cdot \nabla) u] \cdot u_z + \int_{\Omega} [(u_z \cdot \nabla) Z_1] \cdot u_z + \int_{\Omega} [(\partial_z Z_1 \cdot \nabla) (u + Z_1)] \cdot u_z,
\]

we conclude from the above two bounds that

\[
\int_{\Omega} [(u_z + \partial_z Z_1) \cdot \nabla (u + Z_1)] \cdot u_z
\]

\[
\leq \varepsilon (|\nabla \partial_z u|^2 + |u_{zz}|^2) + C (|u_1^4 + \|Z_1\|_1^8 + 1) |u_z|^2 + C \|Z_1\|_2^2 |u + Z_1|_4^2
\]

\[
\leq \varepsilon (|\nabla \partial_z u|^2 + |u_{zz}|^2) + C (|\bar{u}|_4^8 + |\nabla \bar{u}|_2^2 + \|Z_1\|_2^8 + 1) |u_z|^2 + C \|Z_1\|_2^2 (|\bar{u} + Z_1|_4^2 + |\nabla \bar{u}|_2^2).
\]

(5.129)

Proceeding as in (5.129), we get

\[
\int_{\Omega} \nabla \cdot (u + Z_1)(u_z + \partial_z Z_1) \cdot u_z
\]

\[
\leq \varepsilon (|\nabla u_z|^2 + |u_{zz}|^2) + C (|\bar{u}|_4^8 + |\nabla \bar{u}|_2^2 + \|Z_1\|_2^8 + 1) |u_z|^2.
\]

(5.130)

Using integration by parts,

\[
\int_{\Omega} \varphi(u)(\partial_{zz} Z_1) u_z = \int_{\Omega} (\nabla \cdot u)(\partial_z Z_1) u_z - \int_{\Omega} \varphi(u)(\partial_z Z_1) u_{zz}.
\]

Similarly, we have

\[
\int_{\Omega} (\nabla \cdot u) \partial_z Z_1 \cdot u_z = - \int_{\Omega} u \cdot (\nabla \partial_z Z_1) \cdot u_z - \int_{\Omega} u \cdot [\partial_z Z_1 \cdot (\nabla u_z)].
\]

Then, due to Hölder inequality, Sobolev imbedding theorem and interpolation inequality,

\[
\int_{\Omega} (\nabla \cdot u) \partial_z Z_1 \cdot u_z \leq |\int_{\Omega} u \cdot [(\nabla \partial_z Z_1) \cdot u_z]| + |\int_{\Omega} u \cdot [\partial_z Z_1 \cdot (\nabla u_z)]| 
\]

\[
\leq |\nabla \partial_z Z_1|_2 |u_z|_4 + |\nabla u_z|_2 |\partial_z Z_1|_4 |u_z|_4
\]

\[
\leq \|Z_1\|_2 (|\bar{u}|_4 + |\nabla \bar{u}|_2) |u_z|_2^2 (|\nabla u_z|_2^2 + |u_{zz}|_2^2 + |u_z|^2)
\]

\[
+ |\nabla u_z|_2 (\|Z_1\|_2(|\bar{u}|_4 + |\nabla \bar{u}|_2)
\]

\[
\leq \varepsilon (|\nabla u_z|^2 + |u_{zz}|^2) + C (|u_z|^2 + \|Z_1\|_2^2 (|\bar{u}|_2^2 + |\nabla \bar{u}|_2^2).
\]

By Hölder inequality, Minkowski inequality and interpolation inequality, we obtain

\[
\int_{\Omega} \varphi(u)(\partial_z Z_1) u_{zz} \leq \|\partial_z Z_1\|_\infty |\nabla u_z|_2 |u_{zz}|_2 \leq \varepsilon |u_{zz}|_2^2 + C \|Z_1\|_2^2 |\nabla u_z|_2^2.
\]

Therefore, by the above argument we reach

\[
\int_{\Omega} \varphi(u)(\partial_{zz} Z_1) u_z \leq \varepsilon (|u_{zz}|_2^2 + |\nabla u_z|_2^2) + C |u_z|^2 + C \|Z_1\|_2^2 (|\bar{u}|_2^2 + |\nabla \bar{u}|_2^2) + C \|Z_1\|_2^2 |\nabla u_z|_2^2.
\]

(5.131)

According to Hölder inequality, Sobolev imbedding theorem and interpolation inequality, we obtain

\[
\int_{\Omega} \varphi(Z_1)(u_{zz} + \partial_{zz} Z_1) u_z \leq |u_{zz}|_2 |u_z|_4 \varphi(Z_1)|_4 + |\partial_{zz} Z_1|_2 \|\nabla Z_1\|_4 |u_z|_4
\]

\[
\leq C(|u_{zz}|_2 \|Z_1\|_2 + \|Z_1\|_2^2) |u_z|_2^2 (|u_{zz}|_2^2 + |\nabla u_z|_2^2)
\]

\[
\leq \varepsilon (|u_{zz}|_2^2 + |\nabla u_z|_2^2) + C \|Z_1\|_2^2 |u_z|_2^2 + C \|Z_1\|_2^2 |u_{zz}|_2^2.
\]

(5.132)
Taking the inner product of the equation (5.125) with $u_z$ in $(L^2(\Omega))^2$ and by (5.126) – (5.132), we obtain
\[
\partial_t|u_z|^2 + |\nabla u_z|^2 + |u_z|^2 \leq C(|\tilde{u}_z|^4 + |\nabla u_z|^2 + \|Z_1\|^2_4 + 1)|u_z|^2 + C(|\tilde{u}_z|^2 + |\nabla u_z|^2 + 1)\|Z_1\|^2_5. 
\] (5.133)

Therefore, from Gronwall inequality, (5.100), (5.119) and (5.124), we reach
\[
\sup_{t \in [0, \tau^*]} |u_z(t)|^2 + \int_0^{\tau^*} \left( |\nabla u_z(s)|^2 + |u_z(s)|^2 \right) ds \leq C(\tau^*, Q, Z_1, Z_2, v_0, T_0). 
\] (5.134)

By Hölder inequality, interpolation inequality and Sobolev inequality,
\[
\int_0^{\tau^*} \{(u + Z_1) \cdot \nabla \}(u + Z_1) \cdot \Delta u 
\leq |\Delta u|_2|\nabla u|_4|u + Z_1|_4 + |\Delta u|_2|\nabla Z_1|_4|u + Z_1|_4 
\leq |\Delta u|_2|\nabla u|_2^3(2|\nabla u|^2_2 + |\nabla u|^3_2)|u + Z_1|_4 
+|\Delta u|_2\|Z_1\|_2|u + Z_1|_4 
\leq \varepsilon(|\Delta u|^2_2 + |\nabla u|^2_2) + C\|Z_1\|^2_2(u + Z_1) 
+C(|u + Z_1|^2_4 + |u + Z_1|^3_8)\|\nabla u|^2_2 
\leq \varepsilon(|\Delta u|^2_2 + |\nabla u|^2_2) + C\|Z_1\|^2_2(|\tilde{u}|^2 + |\nabla \tilde{u}|^2 + \|Z_1\|^2_1) 
+C(|\tilde{u}|^2 + |\nabla \tilde{u}|^2 + \|Z_1\|^2_1)\|\nabla u|^2_2. 
\] (5.135)

Due to Hölder inequality,
\[
\int_0^{\tau^*} \{\varphi(u + Z_1)(u_z + \partial_z Z_1) \cdot \Delta u 
\leq \int_M \int_{-1}^0 \nabla \cdot u + \nabla \cdot Z_1|dz| \int_{-1}^0 |u_z + \partial_z Z_1| \cdot |\Delta u|dz 
\leq |\Delta u|^2_2 \left( \int_M \left( \int_{-1}^0 |\nabla \cdot u + \nabla \cdot Z_1|dz \right)^4 \right)^{\frac{1}{4}} \left( \int_M \left( \int_{-1}^0 |u_z + \partial_z Z_1|^2dz \right)^2 \right)^{\frac{1}{4}}. 
\]

Then by Minkowsky inequality and interpolation inequality, as well as Sobolev embedding theorem and Hölder inequality,
\[
\int_0^{\tau^*} \{\varphi(u + Z_1)(u_z + \partial_z Z_1) \cdot \Delta u 
\leq C|\Delta u|^2_2 \left( \int_{-1}^0 \|\nabla u\|^2_2(dz) + \|\nabla u\|^3_2(dz) \right) 
\cdot \left( \int_{-1}^0 |u_z|_2^2|dz| \|\nabla u_z\|_2^2(dz) + \|u_z||_2^2|dz| \right) 
\leq \varepsilon(|\Delta u|^2_2 + |\nabla u_z|^2|dz|) + C\|Z_1\|^2_2|u_z|^2 + C\|Z_1\|^2_2 
+C|\nabla u|^2_2(|u_z|^2 + \|Z_1\|^2_1 + |u_z||_2^2 + |u_z||_2^2 + \|u_z||_2^2 + \|Z_1\|^2_1). 
\] (5.136)

Since
\[
\int_0^{\tau^*} (f_k \cdot u) \cdot \Delta u = 0 \quad \text{and} \quad \int_0^{\tau^*} \nabla p \cdot \Delta u = 0,
\]
taking the inner product of equation (5.86) with $-\Delta u$ in $(L^2(\Omega))^2$, by (5.135) – (5.136) we reach
\[
\partial_t|\nabla u|^2_2 + |\Delta u|^2_2 
\leq C(\|Z_1\|^2_2 + \|\theta||_2^4 + \|Z_1\|^2_2|\tilde{u}|^2 + \|Z_1\|^2_2|\nabla \tilde{u}|^2 + \|Z_1\|^2_2|u_z|^2 + \|Z_1\|^2_2|\nabla u_z|^2) 
+ C(|Z_1|^2_2 + |\tilde{u}|^2_2 + |\tilde{u}|^2_2 + |u_z|^2 + \|Z_1\|^2_2|\nabla \tilde{u}|^2 + |\nabla u|^2_2 
+ |u_z|^2 + |u_z|^2 + |\nabla u_z|^2 + |u_z|^2|\nabla u_z|^2)|\nabla u|^2_2. 
\] (5.137)
By (5.100), (5.119), (5.124), (5.134) and thanks to Gronwall inequality, we obtain

$$\sup_{t \in [0, \tau^*]} |\nabla u(t)|^2 + \int_0^{\tau^*} |\Delta u(t)|^2 dt \leq C(\tau^*, Q, Z_1, Z_2, v_0, T_0).$$  \hspace{1cm} (5.138)

Taking the inner product of the equation (5.87) with $-\Delta \theta - \theta_{zz}$ in $L^2(\mathcal{U})$, and making an analogous argument in (5.138) we get

$$\frac{1}{2} \partial_t(|\nabla \theta|^2 + |\theta_z|^2 + \alpha|\nabla \theta(z = 0)|^2) + |\Delta \theta|^2 + 2(|\nabla \theta_z|^2 + \alpha|\nabla \theta(z = 0)|^2) + |\theta_{zz}|^2 = \int_{\mathcal{U}} [(u + Z_1) \cdot \nabla (\theta + Z_2) + \varphi(u + Z_1)(\theta_z + \partial_z Z_2) - Q + \beta Z_2](\Delta \theta + \theta_{zz})$$

$$\leq c(|\Delta \theta|^2 + |\theta_{zz}|^2 + |\nabla \theta_z|^2) + C|Q|_2^2 + C\|Z_2\|_3^2(1 + \|u\|_1^2 + \|Z_1\|_1^2) + C(|\nabla u|^2 + \|Z_1\|_2^2)(|\Delta u|^2 + \|Z_1\|_2^2)\|\theta_z\|^2 + C|\overline{u}|_1^2 + \|\nabla \overline{u}\|^2 + \|Z_1\|_1^2 + |\overline{u}|_2^2 + \|Z_1\|_2^2)|\nabla \theta|^2. \hspace{1cm} (5.139)$$

Therefore, by Gronwall inequality, we conclude that

$$\sup_{t \in [0, \tau^*]} \|\theta(t)\|_1^2 + \int_0^{\tau^*} \|\theta(t)\|_2^2 dt \leq C(\tau^*, Q, Z_1, Z_2, T_0). \hspace{1cm} (5.140)$$

**Acknowledgments.** This work was started during the author’s visit to Professor Zhao Dong in July 2014. When the first version of this manuscript was finished, it was reported in Chinese Academy of Science in August 2015. The author is grateful for the kind invitations and the hospitality of Professor Zhao Dong. The author also thanks Professor Zhenqing Chen for interesting discussions and kind suggestions during the visit to University of Washington in 2016. He is also deeply grateful for Professor Boling Guo’s long-time and constant help, encouragement and kind introduction of hydrodynamics into his interest.

**Bibliography**

[1] L. Arnold, Random Dynamical Systems, Springer, New York, 1998.
[2] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[3] H. Crauel, A. Debussche, F. Flandoli, Random attractors, J. Dynam. Differential Equations 9 (1997) 307–341.
[4] H. Crauel, F. Flandoli, Attractors for random dynamical systems, Probab. Theory Relat. Fields. 100(1994), 365–393.
[5] C. Cao, S. Ibrahim, K. Nakanishi, E.S. Titi, Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics, Comm. Math. Phys. 337(2015), 473–482.
[6] CLE. Franzke, T.J. O’Kane, J. Berner, PD. Williams, V. Lucarini, Stochastic climate theory and modeling, [arXiv:1409.0423v1 [physics.ao-ph]] 1 Sep 2014.
[7] C. Cao, J. Li, E.S. Titi, Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity, J. Differential Equations 257 (2014), 4108–4132.
[8] C. Cao, J. Li, E.S. Titi, Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity, Arch. Ration. Mech. Anal. 214 (2014), 35–76.
[9] C. Cao, J. Li, E.S. Titi, Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion, Communications on Pure and Applied Mathematics Vol. LXIX(2016), 1492–1531.

[10] C. Cao, E.S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, Ann. of Math. 166(2007), 245–267.

[11] C. Cao, E.S. Titi, Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusion, Comm. Math. Phys. 310 (2012), 537–568.

[12] G. Da Prato, and J. Zabczyk, Stochastic equations in infinite dimensions, Cambridge University Press, Cambridge, 1992.

[13] B. Ewald, M. Petcu and R. Temam, Stochastic solutions of the two-dimensional primitive equations of the ocean and atmosphere with an additive noise, Anal. Appl. (Singap.) 5(2007), 183–198.

[14] CLE. Franzke, TJ. O’Kane, J. Berner, PD. Williams, V. Lucarini, Stochastic climate theory and modeling, arXiv:1409.0423v1 [physics.ao-ph] 1 Sep 2014.

[15] C. Foias and G. Prodi, Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2, Rend. Sem. Mat. Univ. Padova. 39 (1967), 1–34.

[16] F. Flandoli and B. Schmalfuss, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise, Stochastics Stochastics Rep. 59(1996), 21–45.

[17] A. E. Gill, Atmosphere-ocean dynamics, International Geophysics Series, Vol. 30, Academic Press, San Diego, 1982.

[18] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Second Edition, Springer-Verlag, New York, 2011.

[19] B. Guo and D. Huang, 3d stochastic primitive equations of the large-scale ocean: global well-posedness and attractors, Commun. Math. Phys. 286(2009), 697–723.

[20] B. Guo, D. Huang, Existence of the universal attractor for the 3-D viscous primitive equations of large-scale moist atmosphere, J. Differential Equations 251(2011), 457–491.

[21] N. Glatt-Holtz and M. Ziane, The stochastic primitive equations in two space dimensions with multiplicative noise, Discrete Contin. Dyn. Syst. Ser. B 10(2008), 801–822.

[22] N. Glatt-Holtz and R. Temam, Cauchy convergence schemes for some nonlinear partial differential equations, Appl. Anal. 90(2011), 85–102.

[23] N. Glatt-Holtz and R. Temam, Pathwise solutions of the 2-d stochastic primitive equations, Appl. Math. Optim. 63(2011), 401–433.

[24] F. Guillén-González, N. Masmoudi, M.A. Rodríguez-Bellido, Anisotropic estimates and strong solutions for the primitive equations, Diff. Int. Equ. 14(2001), 1381–1408.

[25] G. J. Haltiner, Numerical weather prediction, J.W. Wiley & Sons, New York, 1971.

[26] G. J. Haltiner and R. T. Williams, Numerical prediction and dynamic meteorology, John Wiley & Sons, New York, 1980.

[27] C. Hu, R. Temam, M. Ziane, The primitive equations of the large scale ocean under the small depth hypothesis, Disc. and Cont. Dyn. Sys. 9(2003), 97–131.
[28] N. Ju, The global attractor for the solutions to the 3D viscous primitive equations, Discret. Contin. Dyn. Syst. 17(2007), 159–179.

[29] I. Kukavica and M. Ziane, On the regularity of the primitive equations of the ocean, Nonlinearity 20(2007), 2739–2753.

[30] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites non LINéaires, Dunod, Paris, 1969.

[31] J. Lions and B. Magenes, Nonhomogeneous Boundary Value Problems and Applications, Springer-Verlag, New York, 1972.

[32] J.L. Lions, R. Temam and S. Wang, New formulations of the primitive equations of atmosphere and applications, Nonlinearity 5(1992), 237–288.

[33] J.L. Lions, R. Temam and S. Wang, On the equations of the large scale ocean, Nonlinearity 5(1992), 1007–1053.

[34] J.L. Lions, R. Temam and S. Wang, Models of the coupled atmosphere and ocean (CAOI), Computational Mechanics Advance 1(1993), 1–54.

[35] J.L. Lions, R. Temam and S. Wang, Mathematical theory for the coupled atmosphere-ocean models (CAOIII), J. Math. Pures Appl. 74(1995), 105–163.

[36] B. Maslowski and D. Nualart, Evolution equations driven by a fractional Brownian motion, Journal of Functional Analysis 202 (2003), 277–305.

[37] B. Maslowski and B. Schmalfuss, Random dynamical systems and stationary solutions of differential equations driven by the fractional Brownian motion, Stochastic Anal. Appl. 22(6) (2004), 1577–1607.

[38] S. Yuliya, Mishura, Stochastic Calculus for Fractional Brownian Motion and Related Processes, Springer-Verlag, Berlin Heidelberg, 2008.

[39] D. Nualart, Stochastic integration with respect to fractional Brownian motion and applications, Stochastic models (2002), 3–39.

[40] D. Nualart, Stochastic calculus with respect to fractional Brownian motion, Ann. Fac. Sci. Toulouse Math. 6(2006), 63–78.

[41] D. Nualart, Malliavin Calculus and Related Topics, Second Edition. Springer, 2006.

[42] D. Nualart, P.A. Vuillermot, Variational solutions for partial differential equations driven by a fractional noise, Journal of Functional Analysis 232(2006), 390–454.

[43] D. Nualart, A. Rascanu, Differential equations driven by fractional Brownian motion, Collectanea Math. 53(2002), 55–81.

[44] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge Univ. Press, Cambridge, 1992.

[45] B. Schmalfuss, Backward cocycles and attractors of stochastic differential equations, in International Seminar on Applied Mathematics–Nonlinear Dynamics: Attractor Approximation and Global Behaviour, V. Reitmann, T. Riedrich, and N. Koksch, editors, pp. 185–192, 1992.

[46] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, London, 1993.
[47] R. Temam, Navier-Stokes Equations, North-Holland, Amsterdam, 1984.

[48] R. Temam, Navier-Stokes equations. Theory and Numerical Analysis, reprint of 3rd edition, AMS 2001.

[49] R. Temam, Some mathematical aspects of geophysical fluid dynamics equations, Milan J. Math. 71 (2003), 175–198.

[50] S. Tindel, C. Tudor, and F. Viens, Stochastic evolution equations with fractional Brownian motion, Probab. Theory Related Fields 127(2003), 186–204.

[51] Lidan Wang, and Guoli Zhou, Random Attractors of 3-D Stochastic Primitive equations of large-scale moist atmosphere with additive noise, In preparation.

[52] M. Zähle, Integration with respect to fractal functions and stochastic calculus, I, Probab. Theory Related Fields 111(1998), 333–374.