Maximal Concurrent Limited Cost Flow Problems on Extended Multi-commodity Multi-cost Network

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Abstract: Graphs are excellent mathematical tools applied in many fields such as transportation, communication, informatics, economy, .... A network and a flow network is a useful device to solve many problems in many fields in reality. However, most of the network applications in traditional graphs have only considered the weights of edges and vertexes independently, in which the length of a path is the sum of weights of the edges and vertexes on the path. However, in many practical problems, weights at a vertex are not the same for all paths passing the vertex, but depend on the edges coming to and leaving the vertex. For example, the transit time on the transport network depends on the direction of transportation: turn right, turn left or go straight, even some directions are forbidden. Furthermore, on a network, there are many types of commodities, each of which are at different costs. Types of commodities share the capacity of edges and vertexes. Therefore, it is necessary to study a network with multiple commodities at multiple costs. The article builds a model of extended multi-commodity multi-cost network in order to modelise practical problems more exactly and effectively. The maximal concurrent multi-commodity multi-cost flow limited cost problems, that are modelized by implicit linear programming problems. On the basis of duality theory in linear programming, an effective polynomial approximation algorithm is developed.

Keywords: Network, Graph, Multi-cost Multi-commodity Flow, Linear Optimization, Approximation

1. Introduction

Flows on networks are excellent mathematical means used in many applications as communication, transportation, economy, informatics, .... So far, most of the network applications in traditional graphs have only considered the weights of edges and vertexes independently, in which the length of a path is the sum of weights of the edges and the vertexes on the path. However, in many practical problems, the weight at one node is not the same for all paths passing through that node, but also depends on coming and leaving edges. For example, the transit time on the transport network depends on the direction of transportation: turn right, turn left or go straight, even some directions are forbidden. The idea of using duality theory of linear programming to solve these problems is motivated by the work [1]. Paper [2] proposes switching cost only for directed graphs. Therefore, it is necessary to build an extended mixed network model in order to apply more accurate and effective modeling of practical problems. Multi-commodity flow in traditional network problems have been studied in the papers [3-11]. Multi-commodity singlecost flow problems in extended transport networks are studied in the papers [12-22]. The papers [23, 24] study maximal multi-commodity multi-cost flow problems. The papers [25, 26] study maximal multi-commodity multi-cost flow limited cost problems. The papers [27] and [28] study maximal concurrent flow problems on extended multi-commodity multi-cost networks.

2. Multi-commodity Flows in Extended Multi-commodity Multi-cost Network

Given mixed graph $G = (V, E)$ with node set $V$ and edge set $E$. The edges may be directed or undirected. The symbol $E$, is
the set of edges incident the node \( v \in V \). There are many types of commodities circulating on the network. Commodities share the capacities of the edges, but have different costs. The undirected edges represent the two-way edge, in which the commodities on the same edge, but reverse directions, share the capacity of the edge.

The symbol \( r \) is the number of commodity types, \( q_i > 0 \) is the coefficient of conversion of commodity type \( i, i=1, 2, \ldots, r \).

We define the following functions:

**Edge circulating capacity function** \( ce:E \rightarrow R^r \), where \( ce(e) \) is the circulating capacity of the edge \( e \in E \).

**Edge service coefficient function** \( ze:E \rightarrow R^r \), where \( ze(e) \) is the circulating ratio of the edge \( e \in E \) (the real capacity of the edge \( e \) is \( ze(e) \)).

**Node circulating capacity function** \( cv:V \rightarrow R^r \), where \( cv(u) \) is the circulating capacity of the vertex \( u \in V \).

**Node service coefficient function** \( zv:V \rightarrow R^r \), where \( zv(u) \) is the circulating ratio of the vertex \( v \in V \) (the real capacity of the vertex \( v \) is \( zv(v) \)).

The tuple \((V, E, ce, zv, cv, zv)v\) is called an extended network.

**Edge cost function** \( q = 1, 2, \ldots, r \), \( be:E \times E \times E \rightarrow R^r \), where \( be(e) \) is the cost of circulating the edge \( e \in E \) a converted unit of commodity of type \( i \). Note that with 2-way paths, the cost of each direction may be different.

**Node switch cost function** \( i, j, i = 1, 2, \ldots, r \), \( bv_i:V \times E \times E \rightarrow R^r \), where \( bv_i(v,e,e') \) is the cost of transferring a converted unit of commodity of type \( j \) from edge \( e \in E \) to edge \( e' \in E \).

The set \((V, E, ce, zv, cv, {be_i,bv_i,q_i}|i=1, r)\) is called an extended multi-cost multi-commodity network.

Note: If \( be(e) = \infty \), commodity of type \( i \) is banned from passing on path \( e \). If \( bv_i(v,e,e') = \infty \), commodity of type \( i \) is prohibited from circulating edge \( e \) through \( v \) to edge \( e' \).

Let \( p \) be the path from vertex \( u \) to vertex \( v \) through edges \( e_p \), \( j=1,2,\ldots, (h+1) \), and vertices \( u_p \), \( j=1,2,\ldots, h \) as follows:

\[
p = (u, e_1, u_1, e_2, u_2, \ldots, e_h, u_h, e_{h+1})
\]

The cost of passing a converted unit of commodity of kind \( i, i=1, 2, \ldots, r \), through the path \( p \), is denoted by the symbol \( b(p) \), and calculated by the following formula:

\[
b(p) = b(e_1) + \sum_{j=1}^{h} b(e_j) + \sum_{j=1}^{h} b(v_j, e_j, e_{j+1})
\]

Given a multi-cost multi-commodity network \( G=(V, E, ce, zv, cv, {be_i,bv_i,q_i}|i=1, r) \). Assume, for each commodity type \( i, i=1, 2, \ldots, r \), there are \( k_i \) source-target pairs \((s_{ij}, t_{ij})\), \( j=1, 2, \ldots, k_i \), each pair assigned a quantity of commodity of type \( i \), that is necessary to move from source node \( s_{ij} \) to target node \( t_{ij} \).

Denote \( P_i \) the set of paths from node \( s_{ij} \) to node \( t_{ij} \) in \( G \), which commodity of type \( i \) can be passed through, \( i=1, 2, \ldots, r \), \( j=1, 2, \ldots, k_i \). Set

\[
P_i = \bigcup_{j=1}^{k_i} P_{i,j}, \quad \forall i=1, 2, \ldots, r
\]

For each path \( p \in P_{i,j}, i=1, 2, \ldots, r \), \( j=1, 2, \ldots, k_i \), denote \( x_{ij}(p) \) the flow of converted commodity of type \( i \) from the source node \( s_{ij} \) to the target node \( t_{ij} \) along the path \( p \).

Let \( P_{i,e} \) denote the set of paths in \( P_i \) passing through the edge \( e \), \( \forall e \in E \).

Let \( P_{i,j} \) denote the set of paths in \( P_j \) passing through the node \( v \), \( \forall v \in V \).

A set

\[
F = \{ x_{ij}(p) \mid p \in P_{i,j}, i=1, 2, \ldots, r, j=1, 2, \ldots, k_i \}
\]

is called a multi-commodity flow on the extended multi-cost multi-commodity network, if it satisfies the edge capacity constraints:

\[
\sum_{i=1}^{r} \sum_{j=1}^{k_i} x_{ij}(p) \leq ce(e)ze(e), \forall e \in E
\]

and the vertex capacity constraints:

\[
\sum_{i=1}^{r} \sum_{j=1}^{k_i} x_{ij}(p) \leq cv(v)zv(v), \forall v \in V
\]

The expressions

\[
f_{ij} = \sum_{p \in P_{i,j}} x_{ij}(p), i=1, 2, \ldots, r, j=1, 2, \ldots, k_i
\]

are called the flow value of commodity kind \( i \) of the source-target pair \((s_{ij}, t_{ij})\) of the multi-commodity flow \( F \).

The expressions

\[
f_i = \sum_{j=1}^{k_i} f_{ij}, i=1, r
\]

are called the flow value of commodity kind \( i \) of the multi-commodity flow \( F \).

The expression

\[
f_i = \sum_{i=1}^{r} f_i
\]

is called the flow value of the multi-commodity flow \( F \).

### 3. Maximal Concurrent Limited Cost Multi-commodity Multi-cost Flow Problems

Given an extended linear multi-commodity multi-cost network \( G=(V, E, ce, zv, cv, {be_i, bv_i, q_i}|i=1, r) \). Assume, for each commodity type \( i, i=1, 2, \ldots, r \), there are \( k_i \)
source-target pairs \((s_{ij}, t_{ij}), j = 1, 2, \ldots, k_o\), each pair assigned a quantity \(D_{ij}\) of commodity of kind \(i\), that is required to transferred from source vertex \(s_{ij}\) to target vertex \(t_{ij}\). Given a limited cost \(B\).

The mission of the problem is to find a maximum number \(\lambda\) such that there exists a flow converting \(\lambda D_{ij}\) unit of commodity kind \(i, i = 1, 2, \ldots, r, j = 1, 2, \ldots, k_o\), from source vertex \(s_{ij}\) to target vertex \(t_{ij}\), \(\forall j = 1, 2, \ldots, \), and the total cost does not exceed the limited cost \(B\).

Set
\[
\sum_{j=1}^{r} \sum_{p \in P_{ij}} x_{i,j}(p) \leq ce(e).ze(e), \forall e \in E
\]
\[
\sum_{j=1}^{r} \sum_{p \in P_{ij}} x_{i,j}(p) \leq cv(v).zv(v), \forall v \in V
\]
\[
\sum_{p \in P_{ij}} x_{i,j}(p) \geq 0, \forall j = 1, 2, \ldots, k_i
\]
\[
\sum_{i=1}^{r} \sum_{j=1}^{k} x_{i,j}(p).b_i(p) \leq B, \lambda \geq 0, x_{i,j}(p) \geq 0, \forall j = 1, 2, \ldots, r, \forall j = 1, 2, \ldots, k_i, \forall p \in P_{ij}
\]

The dual linear programming problem of \((P)\), called \((D)\), is constructed as follows: each edge \(ce(e).ze(e)\) is assigned a dual variable \(le(e)\), each vertex \(v \in V\) is assigned a dual variable \(lv(v)\), each requirement \(d_{ij}\) is assigned a dual variable \(z_{ij} \), \(\forall i = 1, 2, \ldots, r, \forall j = 1, 2, \ldots, k_o\), and the cost constraint is assigned a dual variable \(\varphi\). The problem \((D)\) states as following:

\[
D(le, lv) = \sum_{e \in E} ce(e).ze(e).le(e) + \sum_{v \in V} cv(v).zv(v).lv(v) + B.\varphi \rightarrow \min
\]

\[
\sum_{e \in P} le(e) + \sum_{v \in P} lv(v) \geq z_{i,j}, \forall i = 1, 2, \ldots, r, \forall j = 1, 2, \ldots, k_i, \forall i \in P_{ij}
\]

\[
\sum_{i=1}^{r} \sum_{j=1}^{k} d_{i,j}.z_{i,j} \geq 1, le(e) \geq 0, \forall e \in E, lv(v) \geq 0, \forall v \in V, z_{i,j} \geq 0, \forall i = 1, 2, \ldots, r, \forall j = 1, 2, \ldots, k_i, \varphi \geq 0
\]

Now, given \(p \in P\) a path from vertex \(u\) to vertex \(v\) through edges \(e\), \(i = 1, 2, \ldots, (h+1)\), and vertex \(u_0\), \(i = 1, 2, \ldots, h\), as follows

\[
p = [u, e_1, u_1, e_2, u_2, \ldots, e_h, u_h, e_{h+1}, v]
\]

We define the path length of \(p\), denoted by \(length(p)\), depending on the variables \(le(e)\), \(lv(v)\) by the following formula:

\[
length(p) = \sum_{j=1}^{h+1} le(e_j) + \sum_{j=1}^{h} lv(u_j) + b_i(p) \varphi
\] (9)

Denote \(dist_{i,j}(le, lv, \varphi)\) the shortest path length from \(s_{ij}\) to \(t_{ij}\) calculated by function \(length(p)\), \(\forall i = 1, 2, \ldots, r, \forall j = 1, 2, \ldots, k_i\).

Set

\[
\alpha(le, lv, \varphi) = \sum_{i=1}^{r} \sum_{j=1}^{k} d_{i,j}.dist_{i,j}(le, lv, \varphi)
\] (10)

Consider the problem \((D_\alpha)\):

\[
d_{ij} = q_{ij}, D_{ij}, \forall i = 1, 2, \ldots, r, \forall j = 1, 2, \ldots, k_i
\]

The problem is expressed by an implicit linear programming model \((P)\) as follows:

\[
\lambda \rightarrow \max
\]

Satisfies

\[
\sum_{i=1}^{r} \sum_{j=1}^{k} d_{i,j}.z_{i,j} \geq 1, le(e) \geq 0, \forall e \in E, lv(v) \geq 0, \forall v \in V, z_{i,j} \geq 0, \forall i = 1, 2, \ldots, r, \forall j = 1, 2, \ldots, k_i, \varphi \geq 0
\]

\[
\alpha = \min \left\{ \frac{D(le, lv, \varphi)}{\alpha(le, lv, \varphi)} \right\}
\]

\[
\beta = \min \left\{ \frac{D(le, lv, \varphi)}{\alpha(le, lv, \varphi)} \right\}
\]

\[
le' = le(e) / \alpha(le, lv, \varphi), lv'(v) = lv(v) / \alpha(le, lv, \varphi) \forall e \in E, \forall v \in V
\]

\[
\lambda' = \varphi / \alpha(le, lv, \varphi)
\]

\[
d_{ij} = dist_{i,j}(le', lv', \lambda')
\]

We have

\[
\alpha(le, lv, \varphi) = \sum_{i=1}^{r} \sum_{j=1}^{k} d_{i,j}.dist_{i,j}(le, lv, \varphi)
\]
\[ \sum_{e \in p} le'(e) + \sum_{v \in p} lv'(v) + b(p) \varphi' \geq z'_{ij}, \]
\[
\forall i=1, 2, \ldots, r, \forall j=1, 2, \ldots, k_i \forall p \in P_{ij}
\]
and
\[ \sum_{i=1}^{r} \sum_{j=1}^{k_i} d_{i,j}^{l} - \sum_{j=1}^{k_i} d_{i,j}^{r} = \frac{1}{\alpha(le, lv, \varphi)} \]
\[ \sum_{i=1}^{r} \sum_{j=1}^{k_i} d_{i,j}^{l} \text{dist}_{i,j}(le, lv, \varphi) = 1 \]

So \((le', lv', \varphi', z'_{ij})\) is an accepted solution of \((D)\) and
\[ D(le', lv', \varphi') = \frac{D(le, lv, \varphi)}{\alpha(le, lv, \varphi)} \]
Hence,
\[ \min(D) \leq \min(D_\alpha) \quad (11) \]

On the contrary, let \((le, lv, \varphi, z_{ij})\) be an admitted solution of \((D)\). Then, we have:
\[ z_{ij} \leq \text{dist}_{i,j}(le, lv, \varphi), \forall i=1, 2, \ldots, r, \forall j=1, 2, \ldots, k_i \]
\[ \Rightarrow \alpha(le, lv, \varphi) = \sum_{i=1}^{r} \sum_{j=1}^{k_i} d_{i,j}^{l} \text{dist}_{i,j}(le, lv, \varphi) \geq \sum_{i=1}^{r} \sum_{j=1}^{k_i} d_{i,j}^{l} z_{i,j} \geq 1 \]

It follows
\[ D(le, lv, \varphi) \leq D(le, lv, \varphi). \]
Hence,
\[ \min(D) \geq \min(D_\alpha) \quad (12) \]

From \((11)\) and \((12)\) it follows \(\min(D) = \min(D_\alpha)\).

Next, if \((le, lv, \varphi, z_{ij})\) is an optimal solution of the problem \((D_\alpha)\), then \((le', lv', \varphi', z'_{ij})\) where
\[ le'(e) = le(e) \alpha(le, lv, \varphi) \forall e \in E, \]
\[ lv'(v) = lv(v) \alpha(le, lv, \varphi) \forall v \in V, \]
\[ \varphi' = \varphi \alpha(le, lv, \varphi) \quad \text{and} \quad z'_{ij} = \text{dist}_{i,j}(le', lv', \varphi') \]

is an optimal solution of \((D)\). Conversely, if \((le, lv, \varphi, z_{ij})\) is an optimal solution of the problem \((D_\alpha)\), then \((le', lv', \varphi', z'_{ij})\) is an optimal solution of the problem \((D)\).

\section{4. Algorithm}

Ideas

Algorithm is implemented through several phases. Each phase consists of \(k\) loops, \(k=k_1+k_2+\ldots+k_r\). At the loop \([i,j]\), \(\forall i=1, 2, \ldots, r, \forall j=1, 2, \ldots, k_i\), of a phase \(t\) we move \(d_{i,j}\) converted units of commodity of kind \(i\) from source vertex \(s_{i,j}\) to target vertex \(t_{i,j}\). This move is implemented in several steps. Algorithm

\[ \text{Input: Extended multi-cost multi-commodity network} \]
\[ G=(V, E, ce, cv, zv, \{be_{ij}, bv_{ij}, q_{ij}\}_{i=1, 2, \ldots, r}) \]
Assume, for each commodity of kind \(i\), \(i=1, 2, \ldots, r\), there are \(k_i\) source-target pairs \((s_{i,j}, t_{i,j})\), \(j=1, 2, \ldots, k_i\), each pair assigned a quantity of commodity \(D_{i,j}\) of kind \(i\), that is necessary to move from source vertex \(s_{i,j}\) to target vertex \(t_{i,j}\). Given a limited cost \(B\) and an approximation ratio \(\alpha\). Let denote \(n=|V|, m=|E|\).

\[ F' = \{f_{i,j}'(e) | e \in E, i=1, 2, \ldots, r, j=1, 2, \ldots, k_i\} \]

Maximal concurrent multi-commodity flow \(F'\) represents a set of real commodity flows at edges
\[ rF' = \{r_{i,j}'(e) | e \in E, i=1, 2, \ldots, r, j=1, 2, \ldots, k_i\} \]

Total cost \(B_j \leq B\), maximal concurrent coefficient \(\lambda\).

\[ \text{//Procedure} \]
\[ \text{//Initialization: Calculate } \varepsilon \text{ and } \delta \]
\[ \varepsilon = 1 - \frac{1}{1+\omega}; \delta = \left(\frac{m+n+1}{1-\varepsilon}\right)^{\frac{1}{\varepsilon}}; \]
\[ d_{i,j} = D_{i,j} q_{i,j}, \forall i=1, 2, \ldots, r, j=1, 2, \ldots, k_i; \]
\[ le(e) = \delta (ce(e) ze(e)), \forall e \in E; \]
\[ lv(v) = \delta (cv(v) zv(v)), \forall v \in V; \]
\[ x_{i,j} = 0, \forall i=1, 2, \ldots, r, j=1, 2, \ldots, k_i, \forall v \in E; \]
\[ \varphi = \delta / B; \]
\[ D(le, lv, \varphi) = \sum_{e \in E} ce(e) ze(e) le(e) + \sum_{v \in V} cv(v) zv(v) lv(v) = B, \varphi = (m+n+1) \delta \]
\[ t = 0; \text{// phase counts in iteration……..while (D(t) < 1) } \]
\[ \text{//Denote } le_t, lv_t, D_t, \alpha_t \text{ the corresponding quantities after phase } t. \]
\[ D(0) = (m+n+1) \delta; le_0(e) = le(e); \forall e \in E; lv_0(v) = lv(v); \forall v \in V, \quad \varphi_0 = \varphi; \]
\[ \text{do } \text{// phases } \}
\[ \text{for } (i=1; i<n; i++) \]
\[ \text{for } (j=1; j<k_i; j++) \text{// loops } \}
\[ \text{do } \text{// steps } \}
\[ \text{Find the shortest path } p \text{ from } s_{i,j} \text{ to } t_{i,j} \text{ calculated by function length}_t(). \text{ Note that the path } p \text{ must be valid for commodity of type } i, \text{ i.e., not containing the edge with edge cost } \infty \text{ or the node with the switch cost } \infty, \text{ dist} \text{ is the shortest path } p \text{ from } s_{i,j} \text{ to } t_{i,j} \text{ calculated by function length}(p), \text{ be}(p) \text{ is the cost of a converted unit of commodity type } i \text{ on the path } p. \text{ Set } c = \min \{\min_{e \in p} \{ce(e) ze(e)\}, \min_{v \in p} \{cv(v) zv(v)\}\}, d_{i,j} = c, b(p); \]
\[ f(B') = B; \]
algorithm, $\forall i=1,2,\ldots,r,j=1,2,\ldots,k_i$, we have transferred $(r-1)d_{ij}$ units of converting commodity type $i$ from $s_{ij}$ to $t_{ij}$. However, the transferred flow may exceed the throughput capacity of the edges.

The following lemma resolves the above problem

\begin{equation}
\lambda > \frac{1}{\log_{1+\epsilon} \frac{1}{\delta}}.
\end{equation}

**Proof.** Consider any edge $e$. Initially,

\[ le(e) = \delta((ce)(ze)(e)), \forall e \in E, lv(v) = \delta((cv)(cz)(v)), \forall v \in V, \]

\[ \varphi = \delta/B. \]

After $(t-1)$ phases implemented, we have $D(t-1) < 1$, it means

\[ \sum_{e \in E} ce(e)ze(e)Le_{t-1}(e) + \sum_{e \in E} cv(e)ze(e)Lv_{t-1}(e) + B. \varphi_{t-1} < 1, \]

that follows

\[ le_{t-1}(e) < 1/(ce)(ze)(e)), \forall e \in E, lv_{t-1}(v) < 1/(cv)(cz)(v)), \forall v \in V. \]

Let $fe(e)$ be the sum of the converted units of commodities passing $eeE$ and $fs(v)$ is the sum of the converted units of commodities passing $vE$ in $(t-1)$ phases.

Consider $eeE$. Suppose in the process of constructing $fe(e)$ there is $ce(e)ze(e)$ units of the flow passing $e$ through $q$ steps, each step transfers $g_{e}$ converted units of commodity, i.e.

\[ \sum_{t} g_{e} = ce(e)ze(e). \]

Through each step, $le(e)$ is increased by the factor $(1+\epsilon,g_{e}/(ce)(ze)(e)))$. So, through $q$ steps, $le(e)$ is increased by the factor

\[ \prod_{t} (1+\epsilon,g_{e}/(ce)(ze)(e))) > (1+\epsilon) \sum_{t} g_{e}/(ce)(ze)(e)) = (1+\epsilon) \]

We see, for every edge $eeE$, for each transfer of $ce(e)ze(e)$ converted units of commodities through $e$, $le(e)$ increases by at least one factor $(1+\epsilon)$. Similarly, for every node $vE$, for every $cv(v)cz(v)$ converted units of commodity passed $v$, $lv(v)$ increases by at least one factor $(1+\epsilon)$.

On the other hand, the number of times to send $ce(e)ze(e)$ converted unit of commodity over each edge $eeE$ is at least $fe(e)/(ce(e)ze(e))$ and the number of times to send $cv(v)cz(v)$ converted unit of commodity through each node $vE$ is at least $fs(v)/(cv(v)cz(v))$.

At this point, the edge and node functions will satisfy the following inequality:

\[ le_{t-1}(e) \geq le_{0}(e)(1+\epsilon)^{(ce)(ze)(e)}), \forall e \in E \]

and

\[ lv_{t-1}(v) \geq lv_{0}(v)(1+\epsilon)^{(cv)(cz)(v)}), \forall v \in V \]

\[ \sum_{t} g_{e} = ce(e)ze(e). \]
from (17), we have

$$\frac{v(e)}{v_0(e)} <$$

and

$$\frac{v(e)}{v_0(e)} = \frac{1}{\delta}$$

Thus, divide \( f(e) \) by \( \log_{1+\varepsilon} \delta \), we receive accepted flows.

The above analysis shows that after \((t-1)\) of phases, \(i=1, 2, \ldots, r; j=1, 2, \ldots, k \), we have moved \( (t-1)d_{ij} \) converted units of commodities from \( s_{ij} \) to \( t_{ij} \). However, in order for the flow to be accepted, we must divide the flow by \( \log_{1+\varepsilon} \delta \). So, \( i=1, 2, \ldots, r; j=1, 2, \ldots, k \), we move \( \frac{t-1}{\log_{1+\varepsilon} \delta} \cdot d_{ij} \) converted units of commodities from \( s_{ij} \) to \( t_{ij} \). Then, flow through edge \( e \) is no greater than \( ce(e), zv(e) \), \( \forall e \in E \), and flow through \( v \) is not greater than \( cv(v), zv(v) \), \( \forall v \in V \). So, we have

$$\lambda > \frac{t-1}{\log_{1+\varepsilon} \delta} \cdot$$

Lemma 4.2

Assume \( \beta \geq 1 \). The algorithm’s found flow, after being divided by \( \log_{1+\varepsilon} \delta \), is the concurrent maximal flow, where

$$\frac{t-1}{\log_{1+\varepsilon} \delta}$$

is the maximal coefficient with the approximation ratio \((1+\varepsilon)\).

Proof.

Since \( le \) and \( lv \) functions are incremental after each update,

$$D(q) \leq D(q-1) + \varepsilon a(q), \forall q = 1, 2, \ldots, t. \quad (15)$$

Next, we have

$$\frac{D(q)}{a(q)} \geq \beta \Rightarrow a(q) \leq \frac{D(q)}{\beta}, \forall q = 1, 2, \ldots, t. \quad (16)$$

From (15) and (16) we receive:

$$D(q) \leq D(q-1) + \varepsilon D(q)/\beta, \forall q = 1, 2, \ldots, t,$$

$$\Rightarrow D(q) \leq \frac{D(q-1)}{1-\varepsilon/\beta} \leq \frac{D(q-2)}{(1-\varepsilon/\beta)^2} \leq \cdots \leq \frac{D(0)}{(1-\varepsilon/\beta)^t} = \frac{(m+n+1)\delta}{(1-\varepsilon/\beta)^t}, \forall q = 1, 2, \ldots, t.$$
Lemma 4.3
Assume $\beta < 1$. Let $l > 1$ such that $l^{\beta} > 1$. Apply the algorithm to the requirements

$$d_{ij}^* = \frac{1}{l} d_{ij}, \forall i=1, 2, \ldots, r, j=1, 2, \ldots, k_i.$$

The algorithm's found flow, after divide $t \frac{t-1}{\log_{\epsilon} \frac{1}{\delta}}$, is the maximal concurrent flow of the original problem, where the maximal coefficient is

$$1 \frac{t-1}{\log_{\epsilon} \frac{1}{\delta}}$$

with the approximation ratio $(1+\alpha \delta)$. 

Proof

According to Lemma 4.2, the algorithm's found flow, after dividing by $\log_{\epsilon} \frac{1}{\delta}$, is the maximal concurrent flow of the problem to the requirement $d_{ij}^* = \frac{1}{l} d_{ij}, \forall i=1..r, j=1..k_i$, and

$$\lambda = \frac{t-1}{t \log_{\epsilon} \frac{1}{\delta}}$$

is the maximal coefficient with the approximation ratio $(1+\alpha \delta)$. 

Hence, $1 \frac{t-1}{l \log_{\epsilon} \frac{1}{\delta}}$ is the maximal coefficient with the approximation ratio $(1+\alpha \delta)$ of the original problem.

Lemma 4.4. The total cost after $(t-1)$ phases does not exceed $B. \log_{\epsilon} \frac{1}{\delta}$. That means, that after dividing the flows by $\log_{\epsilon} \frac{1}{\delta}$, the total cost after does not exceed $B$.

Proof

We have $\phi_l = \delta / B$. After $(t-1)$ phases we get $D(t-1) < 1$, t.e.

$$\sum_{e \in E} ce_l e(z_l) = ce_l e(z_l) + \sum_{e \in E} cv_l e(z_l) \leq B \phi_{t-1} < 1.$$

It follows $\phi_{t-1} < 1/B$. Furthermore, for each transfer of flow such that the total cost is augmented by an amount $B$, $\varphi$ increases by at least one factor $(1+\epsilon)$. 

Therefore, denoting $x$ the times of increasing the total cost by $B$, after $(t-1)$ phases, we have $\varphi_0 (1+\epsilon)^x \leq \phi_{t-1} \leq 1/B$, suy ra $x \leq \log_{\epsilon} \frac{1}{\delta}$.

So, the total cost is $B. \log_{\epsilon} \frac{1}{\delta}$.

5. Algorithm Complexity

Theorem 5.1.

The algorithm’s complexity is

$$O(\omega^2.(cemin/dmax).(\chi+k).m.n^2.\log_{\epsilon}(m+n+1)),$$

where $m$ is the number of edges, $n$ is the number of vertices of the network, $k = k_1 + \ldots + k_i$, $\chi = \max\{ce_l e(z_l) | e \in E\}$, $dmax = \max\{d_{ij} | i=1, 2, \ldots, r, j=1, 2, \ldots, k_i\}$, and $\chi = \sum_{i=1}^{r} \sum_{j=1}^{k_i} d_{ij} cemin$.

Lemma 4.3 with $cemin = \min\{ce_l e(z_l) | e \in E\}$ and $cemin = \min\{cv_l e(z_l) | e \in E\}$.

Proof

First, we find the number of phases the algorithm has taken. According to the proof of Lemma 4.2 above and for $B = \lambda$, we have

$$1 \leq \chi = \beta \frac{t-1}{\log_{\epsilon} \frac{1}{\delta}}.$$

$$\Rightarrow t \leq \frac{1+\beta}{\log_{\epsilon} \frac{1}{\delta}} \Rightarrow t = \frac{1}{\log_{\epsilon} \frac{1}{\delta}}.O(\beta),$$

where $\epsilon$ and $\delta$ depend on $\alpha$. Besides, $t$ depends on $\beta$. Further, denote $imax, jmax$ indexes satisfying

$$dmax = \max\{d_{ij} | i=1, 2, \ldots, r, j=1, 2, \ldots, k_i\} = d_{imax,jmax}.$$

From the constraint of the problem (P)

$$\sum_{p=i_j} x_{i_j} = \lambda d_{ij}, \forall i=1,2,\ldots,r, j=1,2,\ldots,k_i$$

we have

$$\lambda \leq \sum_{p=i_{j_{max}}} \chi_{i_{j_{max}}} / d_{imax,jmax} \leq \sum_{e \in E} \chi_{i_{j_{max}}} / d_{imax,jmax}.$$

That implies

$$\beta = \lambda \leq \frac{cemin/dmax}{m.\text{cemin/dmax}}$$

Replacing $\delta = \left(\frac{m+n+1}{1-\epsilon}\right)^{-\frac{1}{2}}$ to the above expression, we have

$$t = \frac{1}{\epsilon} \log_{\epsilon} \frac{m+n+1}{1-\epsilon} \cdot O(m.\text{cemin/dmax}).$$

On the other hand, each phase implements $k$ loop, so the loop number is $k.t$. Consider the loop transferring $d_{ij}$ converted units of commodities from $s_{ij}$ to $t_{ij}, i=1, 2, \ldots, r, j=1, 2, \ldots, k_i$. Since $cemin$ is the minimal capacity of edges and nodes, the
number of steps required to execute the loop is not exceeded \((d_{ij}/c_{\min}+1)\). The main procedure in each step, finding the shortest path from \(s_{ij}\) to \(t_{ij}\), has a complexity of \(n^3\) \cite{8}. So the complexity of the loop is \((d_{ij}/c_{\min}+1)n^3\). Hence the complexity of each phase is:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} (d_{ij}/c_{\min}+1)n^3 = (\sum_{i=1}^{n} d_{ij}/c_{\min}+1)n^3 = (\chi+k)n^3.
\]

So the algorithm’s complexity is

\[
t. (\chi+k)n^3 = \frac{1}{\varepsilon} \log_{1+\varepsilon} m + n + 1 \cdot O(m. c_{\max}/d_{\max}). (\chi+k)n^3
\]

\[
= \frac{1}{\varepsilon} \log_{1+\varepsilon} m + n + 1 \cdot m.(c_{\max}/d_{\max}).(\chi+k)n^3
\]

\[
= O(\omega^2 \cdot (c_{\max}/d_{\max}).(\chi+k)n^3 \cdot \log (m+n+1)),
\]

for \(\varepsilon = 1 - \frac{1}{1+\omega} = O(\omega^2)\) và \(\log((1+\varepsilon)^{-\varepsilon}) = \varepsilon\).

6. Conclusions

The contribution defines the maximal concurrent limited cost flow problems on extended multi-commodity multi-cost networks, that can be more exactly and effectively applied to model many practical problems. The maximal concurrent limited cost flow problems are modeled as implicit linear optimization problems. On the base of dual theory in linear optimization, an effective polynomial approximate algorithm is developed. Correctness and algorithm complexity are proved.

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