The Chess conjecture

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Abstract We prove that the homotopy class of a Morin mapping $f : P^p \to Q^q$ with $p - q$ odd contains a cusp mapping. This affirmatively solves a strengthened version of the Chess conjecture [5], [3]. Also, in view of the Saeki-Sakuma theorem [10] on the Hopf invariant one problem and Morin mappings, this implies that a manifold $P^p$ with odd Euler characteristic does not admit Morin mappings into $\mathbb{R}^{2k+1}$ for $p \geq 2k + 1 \neq 1, 3, 7$.

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1 Introduction

Let $P$ and $Q$ be two smooth manifolds of dimensions $p$ and $q$ respectively and suppose that $p \geq q$. The singular points of a smooth mapping $f : P \to Q$ are the points of the manifold $P$ at which the rank of the differential $df$ of the mapping $f$ is less than $q$. There is a natural stratification breaking the singular set into finitely many strata. We recall that the kernel rank $kr_x(f)$ of a smooth mapping $f$ at a point $x$ is the rank of the kernel of $df$ at $x$. At the first stage of the stratification every stratum is indexed by a non-negative integer $i_1$ and defined as

$$\Sigma^{i_1}(f) = \{ x \in P \mid kr_x(f) = i_1 \}.$$

The further stratification proceeds by induction. Suppose that the stratum $\Sigma_{n-1}(f) = \Sigma^{i_1, \ldots, i_{n-1}}(f)$ is defined. Under assumption that $\Sigma_{n-1}(f)$ is a submanifold of $P$, we consider the restriction $f_{n-1}$ of the mapping $f$ to $\Sigma_{n-1}(f)$ and define

$$\Sigma^{i_1, \ldots, i_n}(f) = \{ x \in \Sigma_{n-1}(f) \mid kr_x(f_{n-1}) = i_n \}.$$

Boardman [4] proved that every mapping $f$ can be approximated by a mapping for which every stratum $\Sigma_n(f)$ is a manifold.

We abbreviate the sequence $(i_1, \ldots, i_n)$ of $n$ non-negative integers by $I$. We say that a point of the manifold $P$ is an $I$-singular point of a mapping $f$ if
it belongs to a singular submanifold $\Sigma^I(f)$. There is a class of in a sense the simplest singularities, which are called Morin. Let $I_1$ denote the sequence $(p - q + 1, 0)$ and for every integer $k > 1$, the symbol $I_k$ denote the sequence $(p - q + 1, 1, ..., 1, 0)$ with $k$ non-zero entries. Then Morin singularities are singularities with symbols $I_k$. A Morin mapping is an $I_k$-mapping if it has no singularities of type $I_{k+1}$. For $k = 1, 2$ and $3$, points with the symbols $I_k$ are called fold, cusp and swallowtail singular points respectively. In this terminology, for example, a fold mapping is a mapping which has only fold singular points.

Given two manifolds $P$ and $Q$, we are interested in finding a mapping $P \to Q$ that has as simple singularities as possible. Let $f : P \to Q$ be an arbitrary general position mapping. For every symbol $I$, the $\mathbb{Z}_2$-homology class represented by the closure $\overline{\Sigma^I(f)}$ does not change under general position homotopy. Therefore the homology class $[\Sigma^I(f)]$ gives an obstruction to elimination of $I$-singularities by homotopy.

In [5] Chess showed that if $p - q$ is odd and $k \geq 4$, then the homology obstruction corresponding to $I_k$-singularities vanishes. Chess conjectured that in this case every Morin mapping $f$ is homotopic to a mapping without $I_k$-singular points.

We will show that the statement of the Chess conjecture holds. Furthermore we will prove a stronger assertion.

**Theorem 1.1** Let $P$ and $Q$ be two orientable manifolds, $p - q$ odd. Then the homotopy class of an arbitrary Morin mapping $f : P \to Q$ contains a cusp mapping.

**Remark** The standard complex projective plane $\mathbb{C}P^2$ does not admit a fold mapping [9] (see also [1], [12]). This shows that the homotopy class of $f$ may contain no mappings with only $I_1$-singularities.

**Remark** The assumption on the parity of the number $p - q$ is essential since in the case where $p - q$ is even homology obstructions may be nontrivial [5].

**Remark** We refer to an excellent review [11] for further comments. In particular, see Remark 4.6, where the authors indicate that Theorem 1.1 does not hold for non-orientable manifolds.

In [10] (see also [7]) Saeki and Sakuma describe a remarkable relation between the problem of the existence of certain Morin mappings and the Hopf invariant.

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one problem. Using this relation the authors show that if the Euler characteristic of \( P \) is odd, \( Q \) is almost parallelizable, and there exists a cusp mapping \( f: P \to Q \), then the dimension of \( Q \) is 1, 2, 3, 4, 7 or 8.

Note that if the Euler characteristic of \( P \) is odd, then the dimension of \( P \) is even. We obtain the following corollary.

**Corollary 1.2** Suppose the Euler characteristic of \( P \) is odd and the dimension of an almost parallelizable manifold \( Q \) is odd and different from 1, 3, 7. Then there exist no Morin mappings from \( P \) into \( Q \).

### 2 Jet bundles and suspension bundles

Let \( P \) and \( Q \) be two smooth manifolds of dimensions \( p \) and \( q \) respectively. A germ at a point \( x \in P \) is a mapping from some neighborhood about \( x \) in \( P \) into \( Q \). Two germs are equivalent if they coincide on some neighborhood of \( x \). The class of equivalence of germs (or simply the germ) at \( x \) represented by a mapping \( f \) is denoted by \([f]_x\).

Let \( U \) be a neighborhood of \( x \) in \( P \) and \( V \) be a neighborhood of \( y = f(x) \) in \( Q \). Let

\[
\tau_U: (U, x) \to (\mathbb{R}^p, 0) \quad \text{and} \quad \tau_V: (V, y) \to (\mathbb{R}^q, 0)
\]

be coordinate systems. Two germs \([f]_x\) and \([g]_x\) are \( k \)-equivalent if the mappings \( \tau_V \circ f \circ \tau_U^{-1} \) and \( \tau_V \circ g \circ \tau_U^{-1} \), which are defined in a neighborhood of \( 0 \in \mathbb{R}^p \), have the same derivatives at \( 0 \in \mathbb{R}^p \) of order \( \leq k \). The notion of \( k \)-equivalence is well-defined, i.e. it does not depend on choice of representatives of germs and on choice of coordinate systems. A class of \( k \)-equivalent germs at \( x \) is called a \( k \)-jet. The set of all \( k \)-jets constitute a set \( J^k(P,Q) \). The projection \( J^k(P,Q) \to P \times Q \) that takes a germ \([f]_x\) into a point \( x \times f(x) \) turns \( J^k(P,Q) \) into a bundle (for details see [4]), which is called the \( k \)-jet bundle over \( P \times Q \).

Let \( y \) be a point of a manifold and \( V \) a neighborhood of \( y \). We say that two functions on \( V \) lead to the same local function at \( y \), if at the point \( y \) their partial derivatives agree. Thus a local function is an equivalence class of functions defined on a neighborhood of \( y \). The set of all local functions at the point \( y \) constitutes an algebra of jets \( \mathcal{F}(y) \). Every smooth mapping \( f: (U, x) \to (V, y) \) defines a homomorphism of algebras \( f^*: \mathcal{F}(y) \to \mathcal{F}(x) \). The maximal ideal \( m_y \) of \( \mathcal{F}(y) \) maps under the homomorphism \( f^* \) to the maximal ideal \( m_x \subset \mathcal{F}(x) \).
The restriction of $f^*$ to $m_y$ and the projection of $f^*(m_y) \subset m_x$ onto $m_x/m_x^{k+1}$ lead to a homomorphism

$$f_{k,x} : m_y \rightarrow m_x/m_x^{k+1}.$$ 

It is easy to verify that $k$-jets of mappings $(U, x) \rightarrow (V, y)$ are in bijective correspondence with algebra homomorphisms $m_y \rightarrow m_x/m_x^{k+1}$. That is why we will identify a $k$-jet with the corresponding homomorphism.

The projections of $P \times Q$ onto the factors induce from the tangent bundles $TP$ and $TQ$ two vector bundles $\xi$ and $\eta$ over $P \times Q$. The latter bundles determine a bundle $HOM(\xi, \eta)$ over $P \times Q$. The fiber of $HOM(\xi, \eta)$ over a point $x \times y$ is the set of homomorphisms $Hom(\xi_x, \eta_y)$ between the fibers of the bundles $\xi$ and $\eta$. The bundle $\xi$ determines the $k$-th symmetric tensor product bundle $\odot^k \xi$ over $P \times Q$, which together with $\eta$ leads to a bundle $HOM(\odot^k \xi, \eta)$.

**Lemma 2.1** The $k$-jet bundle contains a vector subbundle $C^k$ isomorphic to $HOM(\odot^k \xi, \eta)$.

**Proof** Define $C^k$ as the union of those $k$-jets $f_{k,x}$ which take $m_y$ to $m_x^k$. With each $f_{k,x} \in C^k$ we associate a homomorphism (for details, see [4, Theorem 4.1])

$$\underbrace{\xi_x \circ \ldots \circ \xi_x}_{k} \otimes m_y/m_y^2 \rightarrow \mathbb{R}$$

(1)

which sends $v_1 \circ \ldots \circ v_k \otimes \alpha$ into the value of $v_1 \circ \ldots \circ v_k$ at a function representing $f_{k,x}(\alpha)$. In view of the isomorphism $m_y/m_y^2 \cong Hom(\eta_y, \mathbb{R})$, the homomorphism (1) is an element of $Hom(\odot^k \xi_x, \eta_y)$. It is easy to verify that the obtained correspondence $C^k \rightarrow HOM(\odot^k \xi_x, \eta_y)$ is an isomorphism of vector bundles. \[\square\]

**Corollary 2.2** There is an isomorphism $J^{k-1}(P, Q) \oplus C^k \approx J^k(P, Q)$.

**Proof** Though the sum of two algebra homomorphisms may not be an algebra homomorphism, the sum of a homomorphism $f_{k,x} \in J^k(P, Q)$ and a homomorphism $h \in C^k$ is a well defined homomorphism of algebras $(f_{k,x}+h) \in J^k(P, Q)$. This defines an action of $C^k$ on $J^k(P, Q)$. Two $k$-jets $\alpha$ and $\beta$ map under the canonical projection

$$J^k(P, Q) \longrightarrow J^k(P, Q)/C^k$$

onto one point if and only if $\alpha$ and $\beta$ have the same $(k-1)$-jet. Therefore $J^k(P, Q)/C^k$ is canonically isomorphic to $J^{k-1}(P, Q)$. \[\square\]
Remark The isomorphism $J^{k-1}(P, Q) \oplus C^k \approx J^k(P, Q)$ constructed in Corollary 2.2 is not canonical, since there is no canonical projection of the $k$-jet bundle onto $C^k$.

In [8] Ronga introduced the bundle

$$S^k(\xi, \eta) = \mathcal{HOM}(\xi, \eta) \oplus \mathcal{HOM}(\xi \circ \xi, \eta) \oplus \ldots \oplus \mathcal{HOM}(\circ^k \xi, \eta),$$

which we will call the $k$-suspension bundle over $P \times Q$.

Corollary 2.3 The $k$-jet bundle is isomorphic to the $k$-suspension bundle.

3 Submanifolds of singularities

There are canonical projections $J^{k+1}(P, Q) \to J^k(P, Q)$, which lead to the infinite dimensional jet bundle $J(P, Q) := \lim J^k(P, Q)$. Let $f: P \to Q$ be a smooth mapping. Then at every point $x \times f(x)$ of the manifold $P \times Q$, the mapping $f$ determines a $k$-jet. The $k$-jets defined by $f$ lead to a mapping $j^k f : P \to J^k(P, Q)$, which is called the jet extension of $f$. We will call a subset of $J(P, Q)$ a submanifold of the jet bundle if it is the inverse image of a submanifold of some $k$-jet bundle. A function $\Phi$ on the jet bundle is said to be smooth if locally $\Phi$ is the composition of the projection onto some $k$-jet bundle and a smooth function on $J^k(P, Q)$. In particular, the composition $\Phi \circ jf$ of a smooth function $\Phi$ on $J(P, Q)$ and a jet extension $jf$ is smooth. A tangent to the jet bundle vector is a differential operator. A tangent to $J(P, Q)$ bundle is defined as a union of all vectors tangent to the jet bundle.

Suppose that at a point $x \in P$ the mapping $f$ determines a jet $z$. Then the differential of $jf$ sends differential operators at $x$ to differential operators at $z$, that is $d(jf)$ maps $T_x P$ into some space $D_z$ tangent to the jet bundle. In fact, the space $D_z$ and the isomorphism $T_x P \to D_z$ do not depend on representative $f$ of the jet $z$. Let $\pi$ denote the composition of the jet bundle projection and the projection of $P \times Q$ onto the first factor. Then the tangent bundle of the jet space contains a subbundle $D$, called the total tangent bundle, which can be identified with the induced bundle $\pi^* TP$ by the property: for any vector field $v$ on an open set $U$ of $P$, any jet extension $jf$ and any smooth function $\Phi$ on $J(P, Q)$, the section $V$ of $D$ over $\pi^{-1}(U)$ corresponding to $v$ satisfies the equation

$$V \Phi \circ jf = v(\Phi \circ jf).$$
We recall that the projections $P \times Q$ onto the factors induce two vector bundles $\xi$ and $\eta$ over $P \times Q$ which determine a bundle $\mathcal{HOM}(\xi, \eta)$. There is a canonical isomorphism between the 1-jet bundle and the bundle $\mathcal{HOM}(\xi, \eta)$. Consequently 1-jet component of a $k$-jet $z$ at a point $x \in P$ defines a homomorphism $h: T_x P \to T_y Q$, $y = z(x)$. We denote the kernel of the homomorphism $h$ by $K_{1,z}$. Identifying the space $T_x P$ with the fiber $D_z$ of $D$, we may assume that $K_{1,z}$ is a subspace of $D_z$. Hence at every point $z \in J(P, Q)$ we have a space $K_{1,z}$. Boardman showed that the union $\Sigma^i = \Sigma^i(P, Q)$ of jets $z$ with $\text{dim} \ K_{1,z} = i$ is a submanifold of $J(P, Q)$.

Suppose that we have already defined a submanifold $\Sigma_{n-1} = \Sigma^{i_1, \ldots, i_{n-1}}$ of the jet space. Suppose also that at every point $z \in \Sigma_{n-1}$ we have already defined a space $K_{n,z}$. Then the space $K_{n,z}$ is defined as $K_{n-1,z} \cap T_z \Sigma_{n-1}$ and $\Sigma_n$ is defined as the set of points $z \in \Sigma_{n-1}$ such that $\text{dim} \ K_{n,z} = i_n$. Boardman proved that the sets $\Sigma_n$ are submanifolds of $J(P, Q)$. In particular every submanifold $\Sigma_n$ comes from a submanifold of an appropriate finite dimensional $k$-jet space. In fact the submanifold with symbol $I_n$ is the inverse image of the projection of the jet space onto $n$-jet bundle. To simplify notation, we denote the projections of $\Sigma_n$ to the $k$-jet bundles with $k \geq n$ by the same symbol $\Sigma_n$.

Let us now turn to the $k$-suspension bundle. Following the paper [4], we will define submanifolds $\tilde{\Sigma}^I$ of the $k$-suspension bundle.

A point of the $k$-suspension bundle over a point $x \times y \in P \times Q$ is the set of homomorphisms $h = (h_1, \ldots, h_k)$, where $h_i \in \text{Hom}(\sigma^i \xi_x, \eta_y)$. For every $k$-suspension $h$ we will define a sequence of subspaces $T_x P = K_0 \supset K_1 \supset \ldots \supset K_k$. Then we will define the singular set $\tilde{\Sigma}^{i_1, \ldots, i_n}$ as

$$\tilde{\Sigma}^{i_1, \ldots, i_n} = \{ h \mid \text{dim} \ K_j = i_j \text{ for } j = 1, \ldots, n \}.$$  

We start with definition of a space $K_1 \supset K_0$ and a projection of $P_0 = T_y Q$ onto a factor space $Q_1$. The $h_1$-component of $h$ is a homomorphism of $K_0$ into $P_0$. We define $K_1$ and $Q_1$ as the kernel and the cokernel of $h_1$:

$$0 \to K_1 \to K_0 \xrightarrow{h_1} P_0 \to Q_1 \to 0.$$  

The cokernel homomorphism of this exact sequence gives rise to a homomorphism $\text{Hom}(K_1, P_0) \to \text{Hom}(K_1, Q_1)$, coimage of which is denoted by $P_1$. The sequence of the homomorphisms

$$\text{Hom}(K_1 \circ K_1, P_0) \to \text{Hom}(K_1, \text{Hom}(K_1, P_0)) \to \text{Hom}(K_1, P_1)$$

takes the restriction of $h_2$ on $K_1 \circ K_1$ to a homomorphism $\sigma(h_2): K_1 \to P_1$. Again the spaces $K_2$ and $Q_2$ are respectively defined as the kernel and the cokernel of the homomorphism $\sigma(h_2)$.
The definition continues by induction. In the $n$-th step we are given some spaces $K_i, Q_i$ for $i \leq n$, spaces $P_i$ for $i \leq n - 1$ and projections

$$\text{Hom}(K_i^{n-1}, P_0) \to P_{n-1},$$

$$P_{n-1} \to Q_n,$$

where $K_i^{n-1}$ abbreviates the product $K_{n-1} \circ \ldots \circ K_1$.

First we define $P_n$ as the coimage of the composition

$$\text{Hom}(K_n, P_0) \to \text{Hom}(K_n, \text{Hom}(K_i^{n-1}, P_0)) \to \text{Hom}(K_n, Q_n),$$

where the latter homomorphism is determined by the two given projections. Then we transfer the restriction of the homomorphism $h_{n+1}$ on $K_n \circ K_i^{n-1}$ to a homomorphism $\sigma(h_{n+1})$: $K_n \to P_n$ using the composition

$$\text{Hom}(K_n \circ K_i^{n}, P_0) \to \text{Hom}(K_n, \text{Hom}(K_i^{n}, P_0)) \to \text{Hom}(K_n, P_n).$$

Finally we define $K_{n+1}$ and $Q_{n+1}$ by the exact sequence

$$0 \to K_{n+1} \to K_n \to P_n \to Q_{n+1} \to 0.$$

In the previous section we established a homeomorphism between the fibers of the $k$-jet bundle and $k$-suspension bundle. Suppose that neighborhoods of points $x \in P$ and $y \in Q$ are equipped with coordinate systems. Then every $k$-jet $g$ which takes $x$ to $y$ has the canonical decomposition into the sum of $k$-jets $g_i$, $i = 1, \ldots, k$, such that in the selected coordinates the partial derivatives of the jet $g_i$ at $x$ of order $\neq i$ and $\leq k$ are trivial. In other words the choice of local coordinates determines a homeomorphism

$$J^k(P, Q)|_{x \times y} \to C^1|_{x \times y} \oplus \ldots \oplus C^k|_{x \times y}. \quad (2)$$

Since $C^i|_{x \times y}$ is isomorphic to $\text{Hom}(\sigma^i \xi_x, \eta_y)$, we obtain a homeomorphism between the fibers of the $k$-jet bundle and $k$-suspension bundle.

**Remark** From [4] we deduce that this homeomorphism takes the singular submanifolds $\Sigma^I$ to $\tilde{\Sigma}^I$. Suppose that a $k$-jet $z$ maps onto a $k$-suspension $h = (h_1, \ldots, h_k)$. The homomorphisms $\{h_i\}$ depends not only on $z$ but also on choice of coordinates in $U_i$. However Boardman [4] showed that the spaces $K_i$, $Q_i$, $P_i$ and the homomorphisms $\sigma(h_i)$ defined by $h$ are independent from the choice of coordinates.

**Lemma 3.1** For every integer $k \geq 1$, there is a homeomorphism of bundles $r_k: J^k(P, Q) \to S^k(\xi, \eta)$ which takes the singular sets $\Sigma^I$ to $\tilde{\Sigma}^I$. 

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Choose covers of $P$ and $Q$ by closed discs. Let $U_1, \ldots, U_t$ be the closed discs of the product cover of $P \times Q$. For each disc $U_i$, choose a coordinate system which comes from some coordinate systems of the two disc factors of $U_i$. We will write $J^k$ for the $k$-jet bundle and $J^k|_{U_i}$ for its restriction on $U_i$. We adopt similar notations for the $k$-suspension bundle. The choice of coordinates in $U_i$ leads to a homeomorphism

$$\beta_i: J^k|_{U_i} \to S^k|_{U_i}.$$ 

Let $\{\varphi_i\}$ be a partition of unity for the cover $\{U_i\}$ of $P \times Q$. We define $r_k: J^k \to S^k$ by

$$r_k = \varphi_1 \beta_1 + \varphi_2 \beta_2 + \ldots + \varphi_k \beta_k.$$ 

Suppose that $U_i \cap U_j$ is nonempty and $z$ is a $k$-jet at a point of $U_i \cap U_j$. Suppose

$$\beta_i(z) = (h^i_1, \ldots, h^i_k) \quad \text{and} \quad \beta_j(z) = (h^j_1, \ldots, h^j_k).$$

Then by the remark preceding the lemma, the homomorphisms $\sigma(h^i_s)$ and $\sigma(h^j_s)$ coincide for all $s = 1, \ldots, k$. Consequently, $r_k$ takes $\Sigma^f$ to $\tilde{\Sigma}^f$.

The mapping $r_k$ is continuous and open. Hence to prove that $r_k$ is a homeomorphism it suffices to show that $r_k$ is one-to-one.

For $k = 1$, the mapping $r_k$ is the canonical isomorphism. Suppose that $r_{k-1}$ is one-to-one and for some different $k$-jets $z_1$ and $z_2$, we have $r_k(z_1) = r_k(z_2)$. Since $r_{k-1}$ is one-to-one, the $k$-jets $z_1$ and $z_2$ have the same $(k-1)$-jet components. Hence there is $v \in C^k$ for which $z_1 = z_2 + v$. Here we invoke the fact that $C^k$ has a canonical action on $J^k$.

For every $i$, we have $\beta_i(z_1) = \beta_i(z_2) + \beta_i(v)$. Therefore

$$r_k(z_1) = r_k(z_2) + r_k(v). \quad (3)$$

The restriction of the mapping $r_k$ to $C^k$ is a canonical identification of $C^k$ with $\text{HOM}(\circ^k \xi_k, \eta)$. Hence $r_k(v) \neq 0$. Then (3) implies that $r_k(z_1) \neq r_k(z_2)$. 

**Corollary 3.2** There is an isomorphism of bundles $r: J(P,Q) \to S(\xi, \eta)$ which takes every set $\Sigma_n$ isomorphically onto $\tilde{\Sigma}_n$.

The space $J^k(P,Q)$ may be also viewed as a bundle over $P$ with projection

$$\pi: J^k(P,Q) \to P \times Q \to P.$$ 

Let $f: P \to Q$ be a smooth mapping. Then at every point $p \in P$ the mapping $f$ defines a $k$-jet. Consequently, every mapping $f: P \to Q$ gives rise to a section $j^k f: P \to J^k(P,Q)$, which is called the $k$-extension of $f$ or the $k$-jet
section afforded by } f \). The sections \( \{ j^k f \}_{k} \) determined by a smooth mapping \( f \) commute with the canonical projections \( J^{k+1}(P, Q) \rightarrow J^k(P, Q) \). Therefore every smooth mapping \( f: P \rightarrow Q \) also defines a section \( j f: P \rightarrow J(P, Q) \), which is called the jet extension of \( f \).

A smooth mapping \( f \) is in general position if its jet extension is transversal to every singular submanifold \( \Sigma^I \). By the Thom Theorem every mapping has a general position approximation.

Let \( f \) be a general position mapping. Then the subsets \( (jf)^{-1}(\Sigma^I) \) are submanifolds of \( P \). Every condition \( kr_x(f_{n-1}) = i_n \) in the definition of \( \Sigma^I(f) \) can be substituted by the equivalent condition \( \dim K_{n,x}(f) = i_n \), where the space \( K_{n,x}(f) \) is the intersection of the kernel of \( df \) at \( x \) and the tangent space \( T_{x_n} \Sigma_{n-1}(f) \). Hence the sets \( (jf)^{-1}(\Sigma^I) \) coincide with the sets \( \Sigma^I(f) \). In particular the jet extension of a mapping \( f \) without \( I \)-singularities does not intersect the set \( \Sigma^I \).

Let \( \Omega_r = \Omega_r(P, Q) \subset J(P, Q) \) denote the union of the regular points and the Morin singular points with indexes of length at most \( r \).

**Theorem 3.3** (Ando-Eliashberg, [2], [6]) *Let \( f: P^p \rightarrow Q^q, p \geq q \geq 2 \), be a continuous mapping. The homotopy class of the mapping \( f \) contains an \( I_r \)-mapping, \( r \geq 1 \), if and only if there is a section of the bundle \( \Omega_r \).*

Note that every general position mapping \( f: P^p \rightarrow Q^q, q = 1 \), is a fold mapping. That is why for \( q = 1 \), Theorem 1.1 holds and we will assume that \( q \geq 2 \).

Let \( \tilde{\Omega}_r \) denote the subset of the suspension bundle corresponding to the set \( \Omega_r(P, Q) \subset J(P, Q) \). Every mapping \( f: P \rightarrow Q \) defines a section \( j f \) of \( J(P, Q) \). The composition \( r \circ (jf) \) is a section of \( S(P, Q) \). In view of Lemma 3.1 the Ando-Eliashberg Theorem implies that to prove that the homotopy class of a mapping \( f \) contains a cusp mapping, it suffices to show that the section of the suspension bundle defined by \( f \) is homotopic to a section of the bundle \( \tilde{\Omega}_2 \subset S(\xi, \eta) \).

### 4 Proof of Theorem 1.1

We recall that in a neighborhood of a fold singular point \( x \), the mapping \( f \) has the form

\[
T_i = t_i, \quad i = 1, 2, ..., q - 1, \\
Z = Q(x), \quad Q(x) = \pm k_1^2 \pm ... \pm k_{2q-1}^2.
\]

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If \( x \) is an \( I_r \)-singular point of \( f \) and \( r > 1 \), then in some neighborhood about \( x \) the mapping \( f \) has the form
\[
T_i = t_i, \quad i = 1, 2, ..., q - r, \\
L_i = l_i, \quad i = 2, 3, ..., r, \\
Z = Q(x) + \sum_{t=2}^{r} l_t k^{t-1} + k^{r+1}, \quad Q(x) = \pm k_1^2 \pm ... \pm k_{p-q}^2.
\]

Let \( f: P \to Q \) be a Morin mapping, for which the set \( \Sigma_2(f) \) is nonempty. We define the section \( f_i: P \to \text{Hom}(\circ^i \xi, \eta) \) as the \( i \)-th component of the section \( r \circ (jf) \) of the suspension bundle \( S(\xi, \eta) \to P \). Over \( \Sigma_2(f) \) the components \( f_1 \) and \( f_2 \) defined by the mapping \( f \) determine the bundles \( K_i, Q_i, \ i = 1, 2 \) and the exact sequences
\[
0 \to K_1 \to TP \to TQ \to Q_1 \to 0, \\
0 \to K_2 \to K_1 \to \text{HOM}(K_1, Q_1) \to Q_2 \to 0.
\]

From the latter sequence one can deduce that the bundle \( Q_2 \) is canonically isomorphic to \( \text{HOM}(K_2, Q_1) \) and that the homomorphism
\[
K_1/K_2 \otimes K_1/K_2 \to Q_1,
\]
which is defined by the middle homomorphism of the second exact sequence, is a non-degenerate quadratic form (see Chess, [5]). Since the dimension of \( K_1/K_2 \) is odd, the quadratic form (6) determines a canonical orientation of the bundle \( Q_1 \). In particular the 1-dimensional bundle \( Q_1 \) is trivial. This observation also belongs to Chess [5].

Assume that the bundle \( K_2 \) is trivial. Then the bundle \( Q_2 \) being isomorphic to \( \text{HOM}(K_2, Q_1) \) is trivial as well. Let
\[
h: K_2 \to \text{HOM}(K_2, Q_2) \approx \text{HOM}(K_2 \otimes K_2, Q_1)
\]
be an isomorphism over \( \Sigma_2(f) \) and \( h: P \to \text{HOM}(\circ^3 \xi, \eta) \) an arbitrary section, the restriction of which on \( \circ^3 K_2 \) over \( \Sigma_2(f) \) followed by the projection given by \( \eta \to Q_1 \), induces the homomorphism \( \hat{h} \). Then the section of a suspension bundle whose first three components are \( f_1, f_2 \) and \( h \) is a section of the bundle \( \hat{\Omega}_2 \). Since for \( i > 0 \) the bundle \( \text{HOM}(\circ^i \xi, \eta) \) is a vector bundle, we have that the composition \( r \circ (jf) \) is homotopic to the section \( s \) and therefore the original mapping \( f \) is homotopic to a cusp mapping.

Now let us prove the assumption that \( K_2 \) is trivial over \( \Sigma_2(f) \).

**Lemma 4.1** The submanifold \( \Sigma_2(f) \) is canonically cooriented in the submanifold \( \Sigma_1(f) \).
Proof For non-degenerate quadratic forms of order \( n \), we adopt the convention to identify the index \( \lambda \) with the index \( n - \lambda \). Then the index \( \text{ind} \, Q(x) \) of the quadratic form \( Q(x) \) in (4) and (5) does not depend on choice of coordinates.

With every \( I_k \)-singular point \( x \) by (4) and (5) we associate a quadratic mapping of the form \( Q(x) \). It is easily verified that for every cusp singular point \( y \) and a fold singular point \( x \) of a small neighborhood of \( y \), we have \( Q(x) = Q(y) \pm k^{2p-q+1} \). Moreover, if \( x_1 \) and \( x_2 \) are two fold singular points and there is a path joining \( x_1 \) with \( x_2 \) which intersects \( \Sigma_2(f) \) transversally and at exactly one point, then \( \text{ind} \, Q(x_1) - \text{ind} \, Q(x_2) = \pm 1 \). In particular, the normal bundle of \( \Sigma_2(f) \) in \( \Sigma_1(f) \) has a canonical orientation.

Lemma 4.2 Over every connected component of \( \Sigma_2(f) \) the bundle \( K_2 \) has a canonical orientation.

Proof At every point \( x \in \Sigma_2(f) \) there is an exact sequence
\[
0 \to K_{3,x} \to K_{2,x} \to \mathcal{HOM}(K_{2,x}, Q_{2,x}) \to Q_{3,x} \to 0.
\]
If the point \( x \) is in fact a cusp singular point, then the space \( K_{3,x} \) is trivial and therefore the sequence reduces to
\[
0 \to K_{2,x} \to \mathcal{HOM}(K_{2,x}, Q_{2,x}) \to 0
\]
and gives rise to a quadratic form
\[
K_{2,x} \otimes K_{2,x} \to Q_{2,x} \cong \mathcal{HOM}(K_{2,x}, Q_{1,x}).
\]
This form being non-degenerate orients the space \( \mathcal{HOM}(K_{2,x}, Q_{1,x}) \). Since \( Q_{1,x} \) has a canonical orientation, we obtain a canonical orientation of \( K_{2,x} \).

Let \( \gamma: [-1, 1] \to \Sigma_2(f) \) be a path which intersects the submanifold of non-cusp singular points transversally and at exactly one point.

Lemma 4.3 The canonical orientations of \( K_2 \) at \( \gamma(-1) \) and \( \gamma(1) \) lead to different orientations of the trivial bundle \( \gamma^*K_2 \).

Proof If necessary we slightly modify the path \( \gamma \) so that the unique intersection point of \( \gamma \) and the set \( \Sigma_3(f) \) is a swallowtail singular point. Then the statement of the lemma is easily verified using the formulas (5).

Now we are in position to prove the assumption.
Lemma 4.4 The bundle $K_2$ is trivial over $\Sigma_2(f)$.

Proof Assume that the statement of the lemma is wrong. Then there is a closed path $\gamma : S^1 \to \Sigma_2(f)$ which induces a non-orientable bundle $\gamma^*K_2$ over the circle $S^1$.

We may assume that the path $\gamma$ intersects the submanifold $\Sigma_3(f)$ transversally. Let $t_1, \ldots, t_k, t_{k+1} = t_1$ be the points of the intersection $\gamma \cap \Sigma_3(f)$. Over every interval $(t_i, t_{i+1})$ the normal bundle of $\Sigma_2(f)$ in $\Sigma_1(f)$ has two orientations. One orientation is given by Lemma 4.1 and another is given by the canonical orientation of the bundle $K_2$. By Lemma 4.3 if these orientations coincide over $(t_{i-1}, t_i)$, then they differ over $(t_i, t_{i+1})$. Therefore the number of the intersection points is even and the bundle $\gamma^*K_2$ is trivial. Contradiction.

Remark The statement similar to the assertion of Lemma 4.4 for the jet bundle $J(P, Q)$ is not correct. The vector bundle $K_2$ over $\bar{\Sigma}_2 \subset J(P, Q)$ is non-orientable. This follows for example from the study of topological properties of $\Sigma_{1r}$ in [2, §4].

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