Quantum correction to thermodynamical entropy of black hole

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Abstract

The entropy of a black hole can differ from a quarter of the area of the horizon because of quantum corrections. The correction is related to the contribution to the Euclidean functional integral from quantum fluctuations but is not simply equal to the correction to the effective action. A (2+1) dimensional rotating black hole is explicitly considered.

I. INTRODUCTION

It has long been known that classical black hole physics has a set of laws parallel to the laws of thermodynamics [1]. By virtue of this parallelism, the area of the horizon of a black hole was interpreted as its entropy [2]. After the discovery of Hawking radiation and the development of a semiclassical concept of temperature for black holes, this analogy became more well-defined and at present a quarter of the area of the horizon is supposed to be a quantitative measure of the entropy at the semiclassical level. While this refers to the thermodynamic entropy, it should be mentioned that recently a statistical interpretation of this expression in terms of a counting of microscopic states has been suggested [3].

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As mentioned above, the standard expressions for the temperature and entropy are semi-classical ones. It would be interesting to see what deviations from the standard expression for the entropy arise because of higher quantum corrections. This will be useful for comparison with statistical computations when they get more refined.

For the simplest black hole, namely the one discovered by Schwarzschild, the Hawking temperature is given by

$$T = \frac{\hbar}{8\pi GM},$$

(1)

where $M$ is the mass of the black hole. Accordingly, the first law of thermodynamics can be written as

$$dM = TdS = \frac{\hbar}{8\pi GM}dS,$$

(2)

which shows that the entropy must be $4\pi GM^2/\hbar$ up to an additive constant, i.e., essentially a quarter of the area. While this result is obtained directly from thermodynamics, the expression used for the temperature in the above argument is a semiclassical one and therefore subject to further quantum corrections. As such, the expression for the entropy too must be approximate. This can be seen more clearly in the Euclidean approach to entropy [4], where the functional integral is semiclassically evaluated. The entropy is given essentially by the effective action, which can be approximated by the on-shell classical action, equal to a quarter of the area. This action naturally acquires quantum corrections.

In this paper we shall consider the correction term. The correction to the entropy can be obtained directly by calculating the functional integral, i.e., the effective action, and interpreting it in terms of the thermodynamic potential related to a grand canonical ensemble, but the value is not equal to the correction to the effective action although in the literature the words entropy and action have often been used synonymously.

As the Schwarzschild black hole does not involve any charge or angular momentum, it is comparatively simple. It is more instructive to consider a rotating black hole in (2+1) dimensions [5]. In this case a quantum correction to the functional integral is known [6],
so that one can determine the corrected entropy. As mentioned above, it turns out to be different from what one would imagine on the basis of the usual relation between the action and the entropy.

**II. ROTATING BLACK HOLE IN 2+1 DIMENSIONS**

A 2+1 dimensional black hole has been studied in [5]. The rotating version is described by the metric

\[ ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 (d\phi - \frac{4GJ}{r^2} dt)^2, \]  

(3)

where

\[ f^2 = -8GM + \frac{r^2}{l^2} + \frac{16G^2 J^2}{r^2}, \]  

(4)

and \( l^{-2} \) is a cosmological constant [5]. The outer horizon is at \( r_+ \) where

\[ r_+^2 = 4GMl^2 \left[ 1 + \sqrt{1 - \frac{J^2}{M^2l^2}} \right]. \]  

(5)

The Hawking temperature is

\[ T = \frac{\hbar (r_+^2 - 4GMl^2)}{\pi r_+ l^2}. \]  

(6)

The first law of black hole physics can be written in the form

\[ T d\left( \frac{\pi r_+}{2\hbar G} \right) = dM - \Omega dJ. \]  

(7)

The area is \( 2\pi r_+ \) because of the 1-dimensional nature of the horizon here. This equation can be directly checked from the expression for \( r_+ \). The analogue of the chemical potential for the angular momentum is

\[ \Omega = \left. \frac{\partial M}{\partial J} \right|_A = \frac{4GJ}{r_+^2}. \]  

(8)

Comparing (7) with the first law of thermodynamics, we can write
As mentioned before, this can also be understood from a functional integral approach. At the semiclassical level, the on-shell euclidean action is given by

\[
\frac{-I_0^E}{\hbar} = \int d\tau \left( \frac{r_+^2}{4\hbar G l^2} - \frac{M}{\hbar} \right)
\]

\[
= \frac{r_+^2}{4Gl^2T} - \frac{M}{T}
\]

\[
= -\frac{M}{T} + \frac{J\Omega}{T} + \frac{\pi r_+}{2hG}.
\] (10)

In the second line, the standard finite-temperature result that the range of the \(\tau\)-integration is from zero to \(\frac{h}{T}\) has been made use of. This on-shell action is to be identified with the logarithm of the grand canonical partition function and hence with \(-\frac{M-OJ-TS}{T}\). This demonstrates the entropy at this level to be \(\frac{\pi r_+}{2hG}\).

Now the partition function is not really given by the exponential of an on-shell action but involves functional integration with appropriate boundary conditions. What is shown above is the leading term in the effective action. Quantum fluctuations about the classical configuration lead to corrections. A one-loop corrected action was given in \([6]\) as

\[
-\frac{I_E}{\hbar} = \int d\tau \left( \frac{r_+^2}{4\hbar G l^2} - \frac{M}{\hbar} \right) + \frac{2\pi r_+}{l}.
\] (11)

We rewrite this as

\[
-\frac{I_E}{\hbar} = \frac{r_+^2}{4Gl^2T} - \frac{M}{T} + \frac{2\pi r_+}{l}
\]

\[
= (1 + \frac{8hG}{l}) \frac{\pi^2 l^2 T^2}{2h^2 G (1 - \Omega^2 l^2)},
\] (12)

where in the first line, we have again used the fact that the range of the \(\tau\)-integration is from zero to \(\frac{h}{T}\) and in the last line we have rewritten the expression in terms of the independent variables \(T\) and \(\Omega\) of the grand canonical ensemble. It is assumed that the semiclassical values of these quantities can continue to be used in this grand canonical description. The above equation implies that the one-loop corrected thermodynamic potential is given by

\[
F = -(1 + \frac{8hG}{l}) \frac{\pi^2 l^2 T^2}{2h^2 G (1 - \Omega^2 l^2)}.
\] (13)
Consequently, the entropy, which is the negative of the temperature derivative of the thermodynamic potential at constant chemical potential $\Omega$ is

$$S = (1 + \frac{8hG}{l}) \frac{\pi^2 l^2 T}{\hbar^2 G(1 - \Omega^2 l^2)}$$

$$= (\frac{\pi r_+}{2hG})(1 + \frac{8hG}{l}). \quad (14)$$

This is the corrected entropy, and it is different from what one would naïvely expect from the action because the correction term is twice the change in the action.

The parameter $r_+$ which occurs in this expression is the original value of $r_+$ and is related to the uncorrected mass $M$ and the uncorrected angular momentum $J$ through (13). It has to be noted that $M$, $J$ are no longer the correct physical mass or angular momentum of the black hole. There are corrected values which are easily calculated from (13) using the formalism of the grand canonical ensemble:

$$\tilde{J} = -\frac{\partial F}{\partial \Omega} \bigg|_T = J(1 + \frac{8hG}{l}), \quad (15)$$

$$\tilde{M} = F + TS + \Omega \tilde{J} = M(1 + \frac{8hG}{l}). \quad (16)$$

It is now possible to express the temperature and the chemical potential in terms of these physical parameters. If a corrected $r_+$ is defined by

$$\tilde{r}_+^2 = 4G\tilde{M}l^2 \left[ 1 + \sqrt{1 - \frac{\tilde{J}^2}{\tilde{M}^2 l^2}} \right], \quad (17)$$

one has

$$T = \frac{\hbar(\tilde{r}_+^2 - 4G\tilde{M}l^2)}{\pi \tilde{r}_+ l^2} (1 - \frac{4hG}{l}), \quad (18)$$

$$\Omega = \frac{4G\tilde{J}}{\tilde{r}_+^2}, \quad (19)$$

and the entropy can be rewritten as

$$S = (\frac{\pi \tilde{r}_+}{2hG})(1 + \frac{4hG}{l}). \quad (20)$$
III. DISCUSSION

The corrections that we are talking about involve extra powers of the Planck constant and are therefore small compared to the semiclassical results for large systems. In the present case, the condition for the corrections to be small is that $l$ has to be large. For large black holes, these $\hbar^0$ contributions are small relative to the area term where Planck’s constant appears in the denominator. On the other hand, for some extremal black hole solutions obtained in string theory the area of the horizon vanishes. Quantum corrections will be especially important in these cases.

To sum up, we have demonstrated that the entropy of a black hole is not simply given by the effective action. However, if corrections to the functional integral are known, it is possible to calculate corrections to the entropy as well as to other thermodynamic variables.
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