The Soliton-Ricci Flow with variable volume forms

NEFTON PALI

June 9, 2014

Abstract

We introduce a flow of Riemannian metrics and positive volume forms over compact oriented manifolds whose formal limit is a shrinking Ricci soliton. The case of a fixed volume form has been considered in our previous work. We still call this new flow the Soliton-Ricci flow. It corresponds to a forward Ricci type flow up to a gauge transformation generated by the gradient of the density of the volumes. The new Soliton-Ricci flow exist for all times and represents the gradient flow of Perelman’s $W$ functional with respect to a pseudo-Riemannian structure over the space of metrics and normalized positive volume forms. We obtain an expression of the Hessian of the $W$ functional with respect to such structure. Our expression shows the elliptic nature of this operator in directions orthogonal to the orbits obtained by the action of the group of diffeomorphism. In the case the initial data is Kähler then the Soliton-Ricci flow preserves the Kähler structure and the symplectic form. The space of tamed complex structures embeds naturally to the space of metrics and normalized positive volume forms via the Chern-Ricci map. Over such space the pseudo-Riemannian structure restricts to a Riemannian one. We perform a study of the sign of the restriction of the Hessian of the $W$ functional over such space. This allows us to obtain a finite dimensional reduction, and thus the solution, of the well known problem of the stability of Kähler-Ricci solitons.

1 Introduction and statement of the main result

This is the first of a series of papers whose purpose is the study the following problem.

Let $(X, J_0)$ be a Fano manifold. We remind that the first Chern class $c_1(X, [J_0]) \in H^2_\mathbb{Z}(X, \mathbb{R})$ depends only on $X$ and the coboundary class $[J_0]$ of the complex structure $J$.

Let also $\omega \in 2\pi c_1(X, [J_0])$ be an arbitrary $J_0$-invariant Kähler form over $X$. We want to find under which conditions on $J_0$ and $\omega$ there exists a smooth
complex structure $J \in [J_0]$ and a smooth volume form $\Omega > 0$ over $X$ such that
\[
\begin{cases}
\omega = \text{Ric}_J(\Omega), \\
\partial_{T_{X,J}} (\omega^{-1} d \log \frac{\omega^n}{\Omega}) = 0,
\end{cases}
\]
i.e. the Riemannian metric $g := -\omega J$, is a $J$-invariant Kähler-Ricci soliton.

This set up represents a particular case of the Hamilton-Tian conjecture with a stronger conclusion, namely we avoid the singularities in the solution of the Kähler-Ricci soliton equation.

Proofs of the Hamilton-Tian conjecture have been posted on the arXiv server in (2013) by Tian-Zhang [Ti-Zha] in complex dimension 3 and quite recently by Chen-Wang [Ch-Wa] in arbitrary dimensions.

Our starting point of view is Perelman’s twice contracted second Bianchi type identity introduced in [Per].

We remind first what this is about. Let $\Omega > 0$ be a smooth volume form over an oriented compact and connected Riemannian manifold $(X,g)$. We remind that the $\Omega$-Bakry-Emery-Ricci tensor of $g$ is defined by the formula
\[
\text{Ric}_g(\Omega) := \text{Ric}(g) + \nabla_g d \log dV_g \Omega.
\]
A Riemannian metric $g$ is called a $\Omega$-Shrinking Ricci soliton if $g = \text{Ric}_g(\Omega)$. We define following fundamental objects
\[
h \equiv h_{g,\Omega} := \text{Ric}_g(\Omega) - g,
\]
\[
2H \equiv 2H_{g,\Omega} := -\Delta_g \Omega + \text{Tr}_g h + 2f,
\]
\[
f \equiv \log \frac{dV_g}{\Omega}.
\]
We equip the set smooth Riemannian metrics $\mathcal{M}$ with the scalar product
\[
(u,v) \mapsto \int_X (u,v)_g \Omega, \tag{1.1}
\]
for all $u,v \in \mathcal{H} := L^2(X, S^2_{\mathbb{R}} T^*_X)$. Let $P_g^*$ be the formal adjoint of some operator $P$ with respect to a metric $g$. We observe that the operator
\[
P_g^{*\alpha} := e^f P_g^* (e^{-f} \bullet),
\]
with $f := \log \frac{dV_g}{\Omega}$, is the formal adjoint of $P$ with respect to the scalar product (1.1). We define also the $\Omega$-Laplacian operator
\[
\Delta^\Omega_g := \nabla^g \nabla_g = \Delta_g + \nabla_g f - \nabla_g.
\]
It is also useful to introduce the $\Omega$-divergence operator acting on vector fields as follows;
\[
\text{div}^\Omega \xi := \frac{d((\xi - \Omega))}{\Omega} = e^f \text{div}_g (e^{-f} \xi) = \text{div}_g \xi - g(\xi, \nabla_g f).
\]
We infer in particular the identity $\text{div}^\Omega \nabla_g u = -\Delta_g^\Omega u$, for all functions $u$. We observe also the integration by parts formula

$$-\int_X u \text{div}^\Omega \xi \Omega = \int_X g(\nabla_g u, \xi)\Omega.$$ 

Then an elementary computation made by Perelman [Per] (see also [Pal2]) shows that the functions $h$ and $H$ satisfy Perelman’s twice contracted second Bianchi type identity

$$\nabla^{*\Omega} h^*_{g,\Omega} + \nabla_g H_{g,\Omega} = 0,$$ \hspace{1cm} (1.2)

where $h^*_{g,\Omega} := g^{-1}h_{g,\Omega}$ is the endomorphism associated to $h_{g,\Omega}$. We remind now that for any symmetric 2-tensor $u$ the tensor $R_g^u$, defined by the formula

$$(R_g^u)(\xi,\eta) := -\text{Tr}_g [u (R_g (\xi, \cdot) \eta, \cdot)],$$

is also symmetric (see section 3). For any smooth symmetric 2-tensor $u$ we define the $\Omega$-Lichnerowicz Laplacian $\Delta^\Omega_{L,g}$ as

$$\Delta^\Omega_{L,g} u := \Delta_g^\Omega u - 2R_g * u + u \text{Ric}^*_g(\Omega) + \text{Ric}_g(\Omega)u^*_g.$$ 

This operator is self-adjoint with respect to the scalar product (1.1) thanks to the identity

$$\langle R_g * u, v \rangle_g = \langle u, R_g * v \rangle_g,$$ \hspace{1cm} (1.3)

for all symmetric 2-tensors $u$ and $v$ (see section 3). We define also the set of normalised volume forms $V_1 := \{\Omega > 0 \mid \int_X \Omega = 1\}$. From now on we consider that the functions $h$ and $H$ are defined over $M \times V_1$. Notice that the tangent space of $M \times V_1$ is $T_{M \times V_1} = C^\infty(X, S^2T_X^\ast) \oplus C^\infty(X, \Lambda^m T_X^\ast)_{\Omega}$, where

$$C^\infty(X, \Lambda^m T_X^\ast)_0 := \left\{ V \in C^\infty(X, \Lambda^m T_X^\ast) \mid \int_X V = 0 \right\}.$$ 

We denote by $\text{End}_g(T_X)$ the bundle of $g$-symmetric endomorphisms of $T_X$ and by $C_{\Omega}^\infty(X, \mathbb{R})_0$ the space of smooth functions with zero integral with respect to $\Omega$. We will systematically use the fact that for any $(g, \Omega) \in M \times V_1$ the tangent space $T_{M \times V_1, (g,\Omega)}$ identifies with $C^\infty(X, \text{End}_g(T_X)) \oplus C_{\Omega}^\infty(X, \mathbb{R})_0$ via the isomorphism

$$(v, V) \mapsto (v^*_g, V^*_\Omega) := \left( g^{-1}v, V/\Omega \right).$$

With this notations hold the fundamental variation formulas

$$2D_{g,\Omega} h (v, V) = \Delta_{L,g}^\Omega v - L_{\nabla^\Omega_{g,\Omega} v^*_g + \nabla_g V^*_\Omega} g - 2v,$$ \hspace{1cm} (1.4)

and

$$2D_{g,\Omega} H (v, V) = \Delta_{g,\Omega}^V V^*_\Omega - \left( L_{\nabla^\Omega_{g,\Omega} v^*_g + \nabla_g V^*_\Omega} \right)^*_\Omega - 2V^*_\Omega - \langle v, h_{g,\Omega} \rangle_g.$$ \hspace{1cm} (1.5)
We will give a detailed proof in section 3.) We infer that the variations of the non-linear operators $h$ and $H$ are elliptic in restriction to the space

\[ \mathcal{F}_{g,\Omega} := \{(v,V) \in T_{\mathcal{M}} \times \mathcal{V}_1 \mid \nabla_g^* v_g^* + \nabla_g V_\Omega^* = 0\} . \]

This fact strongly suggests that the following flow represents a strictly parabolic system.

**Definition 1** The Soliton-Ricci flow is the smooth curve $(g_t,\Omega_t)_{t \geq 0} \subset \mathcal{M} \times \mathcal{V}_1$ solution of the evolution system

\[
\begin{cases}
\dot{g}_t = -h_{g_t,\Omega_t}, \\
\dot{\Omega}_t = -H_{g_t,\Omega_t}, \\
\Omega_t,
\end{cases}
\]

with

\[ H_{g,\Omega} := H_{g,\Omega} - \int_X H_{g,\Omega} \Omega. \]

Indeed this is the case as shown in the proof of the following basic fact.

**Lemma 1** For every $(g_0,\Omega_0) \in \mathcal{M} \times \mathcal{V}_1$ there exists a unique smooth solution $(g_t,\Omega_t)_{t \geq 0} \subset \mathcal{M} \times \mathcal{V}_1$ of the Soliton-Ricci flow equation with initial data $(g_0/\lambda,\Omega_0)$, for some $\lambda > 0$. In the case $(X,J_0)$ is a Fano variety and $g_0J_0 \in 2\pi c_1(X)$ we can choose $\lambda = 1$. In this case the Soliton-Ricci flow represents a smooth family of Kähler structures and normalized positive volumes $(J_t,g_t,\Omega_t)_{t \geq 0}$ uniquely determined by the evolution system

\[
\begin{cases}
\dot{g}_t = -h_{g_t,\Omega_t}, \\
\dot{\Omega}_t = -\frac{H_{g_t,\Omega_t}}{2}, \\
2\dot{J}_t = [J_t,\dot{g}_t],
\end{cases}
\]

that we call the **Soliton-Kähler-Ricci flow**.

We will show in section 3 that if the initial data $(J_0,g_0,\Omega_0)$ satisfies

\[ \omega := g_0J_0 = \text{Ric}_{J_0}(\Omega_0), \int_X \Omega_0 = 1, \]

then the Soliton-Kähler-Ricci flow equation is equivalent with the evolution system

\[
\begin{cases}
\omega = \text{Ric}_{J_t}(\Omega_t), \int_X \Omega_t = 1, \\
\dot{J}_t = \mathfrak{T}_{TX,J_t} \left( \omega^{-1} d\log \frac{\omega^n}{\Omega_t} \right).
\end{cases}
\]
Thus the Soliton-Kähler-Ricci flow preserves the initial symplectic structure \( \omega \).

Over a \( m \)-dimensional compact Riemannian manifold \((X, g)\) we consider Perelman’s \( W \)-functional
\[
W(g, f) := \int_X \left[ |\nabla g f|^2 + \text{Scal}(g) + 2f - m \right] e^{-f} dV_g
\]
\[
= \int_X \left[ -\Delta_g f + \text{Scal}(g) + 2f - m \right] e^{-f} dV_g.
\]
(We can use here the identity \( \Delta_g e^{-f} = -(|\nabla g f|^2 + \Delta_g f) e^{-f} \).)
If we use the identifications \( f \leftrightarrow \Omega := e^{-f} dV_g \) and \( W(g, f) \equiv W(g, \Omega) \) then
\[
W(g, \Omega) = \int_X \left[ \text{Tr}_g h_{g, \Omega} + 2 \log \frac{dV_g}{\Omega} \right] \Omega = 2 \int_X H_{g, \Omega} \Omega.
\]
With these notations Perelman’s first variation formula for the functional \( W : \mathcal{M} \times \mathcal{V}_1 \to \mathbb{R} \) writes as
\[
D_{g, \Omega} W(v, V) = -\int_X \left[ \langle v, h_{g, \Omega} \rangle_g - 2V^*_\Omega H_{g, \Omega} \right] \Omega.
\]
Thus if we consider the pseudo-Riemannian structure over the space \( \mathcal{M} \times \mathcal{V}_1 \) given by the formula \((g, \Omega) \in \mathcal{M} \times \mathcal{V}_1 \mapsto G_{g, \Omega} \), with
\[
G_{g, \Omega}(u, U; v, V) = \int_X \left[ \langle u, v \rangle_g - 2U^*_\Omega V^*_\Omega \right] \Omega,
\]
for all \((u, U), (v, V) \in T_{\mathcal{M} \times \mathcal{V}_1} \), then we infer the identity
\[
\nabla_G W(g, \Omega) = - \left( h_{g, \Omega}, H_{g, \Omega} \right).
\]
This shows that the Soliton-Ricci flow is the gradient flow of the \( W \) functional with respect to the pseudo-Riemannian structure \( G \). Perelman’s twice contracted second Bianchi identity \((1.2)\) implies the equality
\[
\{(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1 \mid D_{g, \Omega} W = 0\} = \{(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1 \mid h_{g, \Omega} = 0\},
\]
i.e the critical points of \( W \) are precisely the shrinking Ricci solitons. We can provide at this point a geometric interpretation of the space \( \mathbb{F}_{g, \Omega} \). It represents the orthogonal space, with respect to \( G \), to the tangent space at the point \((g, \Omega) \in \mathcal{M} \times \mathcal{V}_1 \) of the orbit
\[
[g, \Omega] = \text{Diff}_0(X) \cdot (g, \Omega),
\]
of the point \((g, \Omega)\) under the action of the identity component of the group of smooth diffeomorphisms \( \text{Diff}_0(X) \) of \( X \). In formal terms hold the equality
\[
T^0_{(g, \Omega)} = \mathbb{F}_{g, \Omega}.
\]
We define the anomaly space of the pseudo-Riemannian structure \( G \) at an arbitrary point \((g, \Omega)\) as the vector space
\[
\mathcal{A}_g^\Omega := \mathcal{F}_{g, \Omega} \cap T_{[g, \Omega], g, \Omega}.
\]
In the case \((g, \Omega)\) is a shrinking Ricci-Soliton then the map
\[
\ker(\Delta_g^\Omega - 2\Pi) \rightarrow \mathcal{A}_g^\Omega
\]
\[
u \mapsto 2(\nabla_g d\nu, -\nu \Omega),
\]
is an isomorphism (see section \[\text{8}\]). In the case \((J, g, \Omega)\) is a Kähler-Ricci soliton then \(\mathcal{A}_g^\Omega\) is canonically isomorphic with the space of Killing vector fields of \(g\). This is a consequence of a non trivial result (see corollary \[\text{5}\]).

We denote by \(\nabla^2_G \mathcal{W}(g, \Omega)\) the Hessian endomorphism of the \(\mathcal{W}\) functional with respect to the pseudo-Riemannian structure \(G\) at the point \((g, \Omega) \in \mathcal{M} \times \mathcal{V}_1\). We show in lemma \[\text{7}\] that its restriction to the space \(\mathcal{F}_{g, \Omega}\) is a strictly elliptic operator for any point \((g, \Omega)\). A simple consequence of Perelman’s twice contracted second Bianchi type identity \[\text{1.2}\] is that the map
\[
\nabla^2_G \mathcal{W}(g, \Omega) : \mathcal{F}_{g, \Omega} \rightarrow \mathcal{F}_{g, \Omega},
\]
is well defined in the case \((g, \Omega)\) is a shrinking Ricci-Soliton (see section \[\text{10}\]). In this case holds also the inclusion
\[
\mathcal{A}_g^\Omega \subseteq \mathcal{F}_{g, \Omega} \cap \ker \nabla^2_G \mathcal{W}(g, \Omega).
\]
(See lemma \[\text{8}\]) In general (see section \[\text{10}\]) for any point \((g, \Omega)\) holds the fundamental and deep property
\[
\nabla^2_G \mathcal{W}(g, \Omega)(h_{g, \Omega}, H_{g, \Omega}, \Omega) \in \mathcal{F}_{g, \Omega}.
\] (1.8)
This is quite crucial for the stability of the Soliton-Kähler-Ricci flow (see \[\text{Pal7}\]).

The following basic fact is a meaningful geometric reformulation of the monotony statement for Perelman’s \(\mathcal{W}\) functional discovered by the author in 2006 \[\text{Pal1}\] and published in 2008.

**Lemma 2** Let \((X, J)\) be a Fano manifold, let \(g\) be a \(J\)-invariant Kähler metric with symplectic form \(\omega := gJ \in 2\pi c_1(X, [J])\) and let \(\Omega > 0\) be the unique smooth volume form with \(\int_X \Omega = 1\) such that \(\omega = \text{Ric}(J)\). Then Perelman’s \(\mathcal{W}\) functional is monotone increasing along the Soliton-Kähler-Ricci flow with initial data \((J_0, g_0, \Omega_0) = (J, g, \Omega)\). The monotony is strict unless \((J, g)\) is a Kähler-Ricci soliton.

From now on we will refer to the Soliton-Kähler-Ricci flow only if the initial data are as in the previous lemma. The fact that the Soliton-Kähler-Ricci flow preserves the symplectic form \(\omega\) strongly suggests the study of the restriction of
Perelman’s $W$ functional over the image $\mathcal{S}_\omega \subset \mathcal{M} \times \mathcal{V}_1$ of the natural embedding of the space of $\omega$-compatible complex structures

$$J_\omega := \{ J \in J_{\text{int}} \mid \omega = J^* \omega J, \omega J < 0 \},$$

(where $J_{\text{int}}$ denotes the space of integrable complex structures). Precisely let $\mathcal{M}_\omega := -\omega \cdot J_\omega \subset \mathcal{M}$. Then

$$S_\omega := \{ (g, \Omega) \in \mathcal{M}_\omega \times \mathcal{V}_1 \mid \omega = \text{Ric}_J(\Omega), J = g^{-1} \omega \}.$$

The fact that the space $J_\omega$ may be singular in general implies that also the space $S_\omega$ may be singular. We denote by $\text{TC}_{S_\omega,g}$ the tangent cone of $S_\omega$ at an arbitrary point $(g, \Omega)$ of $\mathcal{S}_\omega$. This is by definition the union of all tangent vectors of $S_\omega$ at the point $(g, \Omega)$. We notice that (see for example [Pal3]) the tangent cone $\text{TC}_{\mathcal{M}_\omega,g}$ of $\mathcal{M}_\omega$ at an arbitrary point $g \in \mathcal{M}_\omega$ satisfies the inclusion

$$\text{TC}_{\mathcal{M}_\omega,g} \subseteq \frac{\partial}{\partial W} J_g, [0],$$

with

$$\mathcal{D}_{g,[0]} := \left\{ v \in C^\infty(X, S^2 \mathcal{T}_X^*) \mid v^J = 0, \partial_T_{X,J}(v^J)^*_g = 0 \right\},$$

where $v^J$ and $v^J_J$ denote respectively the $J$-invariant and $J$-anti-invariant parts of $v$. The first variation of the Chern-Ricci form (see lemma [18]) shows that for any $(g, \Omega) \in S_\omega$ hold the inclusion

$$\text{TC}_{S_\omega,(g, \Omega)} \subseteq \mathcal{T}_{g,\Omega},$$

with

$$\mathcal{T}_{g,\Omega} := \left\{ (v, V) \in \mathcal{D}_{g,[0]} \times \mathcal{T}_{\mathcal{V}_1} \mid d \left( \nabla^*_g v^*_g + \nabla_g V^*_\Omega - \omega \right) = 0 \right\}.$$

We consider also its sub-space

$$\mathcal{F}_{g,\Omega}[0] := \left\{ (v, V) \in \mathcal{F}_{g,\Omega} \mid v \in \mathcal{D}_{g,[0]} \right\}.$$

In the case $(X, J, g)$ is a compact Kähler-Ricci soliton then the map

$$\nabla^2 \mathcal{W}(g, \Omega) : \mathcal{F}_{g,\Omega}[0] \longrightarrow \mathcal{F}_{g,\Omega}[0],$$

is well defined. Furthermore for any $(g, \Omega) \in S_\omega$ the fundamental property [18] implies

$$\nabla^2 \mathcal{W}(g, \Omega) (h_{g,\Omega}, \mathcal{H}_{g,\Omega}) \in \mathcal{F}_{g,\Omega}[0].$$

This is precisely the key statement needed for the study of the stability of the Soliton-Kähler-Ricci flow in [Pal7]. For any $(g, \Omega) \in S_\omega$ we denote by

$$[g, \Omega]_\omega := \text{Symp}^0(X, \omega) \cdot (g, \Omega) \subset S_\omega,$$
the orbit of the point \( (g, \Omega) \) under the action of the identity component of the group of smooth symplectomorphisms \( \text{Symp}^0(X, \omega) \) of \( X \). With these notations hold the property
\[
T_{[\omega, \Omega],(g,\Omega)}^{1,\omega} \cap T_{T,\Omega}^J = F_{g,\Omega}^J [0].
\]
(1.13)
This combined with (1.10) implies directly the more geometric identity
\[
T_{[\omega, \Omega],(g,\Omega)}^{1,\omega} \cap TC_{S_{\omega},(g,\Omega)} = F_{g,\Omega}^J [0] \cap TC_{S_{\omega},(g,\Omega)},
\]
(1.14)
for any \( (g, \Omega) \in S_{\omega} \). An other remarkable fact is that for any \( (g, \Omega) \in S_{\omega} \) the restriction of the symmetric form \( G_{g,\Omega} \) to the vector space \( T_{g,\Omega}^J \), with \( J := g^{-1} \omega \), is positive definite. This implies the \( G \)-orthogonal decomposition
\[
L^2 T_{g,\Omega}^J = L^2 T_{[\omega, \Omega],(g,\Omega)} \oplus_G L^2 F_{g,\Omega}^J [0],
\]
where for any subset \( S \) of smooth sections over \( X \) we denote by \( L^2 S \) its closure with respect to the \( L^2 \)-topology. The finite dimensional vector space of \( \Omega \)-harmonic \( T_{X,J} \)-valued \( (0,1) \)-forms \( g_{1,\Omega} \) embeds naturally inside \( F_{g,\Omega}^J [0] \) via the map \( A \in g_{1,\Omega} (T_{X,J}) \mapsto (gA,0) \). By abuse of notations we still denote by \( G_{g,\Omega} \) the image of this embedding. There exists an infinite dimensional vector space \( E_{g,\Omega}^J [0] \subset F_{g,\Omega}^J [0] \) (see the sub-section 18.2 for its definition) such that the \( G \)-orthogonal decomposition hold true
\[
F_{g,\Omega}^J [0] = E_{g,\Omega}^J [0] \oplus_G H_{g,\Omega}^{0,1} (T_{X,J}).
\]
We can explain now a more precise property of the tangent cone \( TC_{S_{\omega},(g,\Omega)} \). For this purpose we consider the Kuranishi space \( K_{g,J} \subset H_{g,\Omega}^{0,1} (T_{X,J}) \), \( 0 \in K_{g,J} \) of \( X \). (See theorem 3 in the sub-section 21.4.4 of appendix B for its definition and properties.) In the sub-section 21.4.5 we define also the Kuranishi space of \( \omega \)-polarized complex deformations \( K_{\omega} \subset K_{g,J} \) of the Fano manifold \( (X, J, \omega) \). (See the definition 2.) Then hold the inclusions
\[
L^2 T_{[\omega, \Omega],(g,\Omega)} \oplus_G L^2 E_{g,\Omega}^J [0] \oplus_G TC_{K_{\omega}}[0]
\]
(1.15)
\[
\subseteq L^2 TC_{S_{\omega},(g,\Omega)}
\]
\[
\subseteq L^2 T_{[\omega, \Omega],(g,\Omega)} \oplus_G L^2 E_{g,\Omega}^J [0] \oplus_G TC_{K_{J,0}}[0].
\]
(1.16)
We define now the non-negative cone of \( \Omega \)-harmonic variations
\[
H_{g,\Omega}^{0,1} (T_{X,J})_{\geq 0} := \left\{ A \in H_{g,\Omega}^{0,1} (T_{X,J}) \mid \int_X |A|^2 F \Omega \geq 0 \right\},
\]
and the sub-cone
\[
H_{g,\Omega}^{0,1} (T_{X,J})_{0} := \left\{ A \in H_{g,\Omega}^{0,1} (T_{X,J}) \mid \int_X |A|^2 F \Omega = 0 \right\}.
\]
In the Kähler-Einstein case holds the obvious identities
\[ H_{g,\Omega}^{0,1} (T_{X,J}) \geq 0 = H_{g,\Omega}^{0,1} (T_{X,J})_0. \]

In the Dancer-Wang Kähler-Ricci soliton case \( H_{g,\Omega}^{0,1} (T_{X,J})_0 \neq \{0\} \) thanks to a result in Hall-Murphy [Ha-Mu2]. Let \( H_{T_{X,J}} \) be the \( L^2_\Omega \)-projector over the space \( H_{g,\Omega}^{0,1} (T_{X,J}) \). We define now the non-negative cone
\[ T_{g,\Omega}^J \geq 0 := \left\{ (v, V) \in T_{g,\Omega}^J \mid H_{T_{X,J}} v_g \in H_{g,\Omega}^{0,1} (T_{X,J}) \geq 0 \right\}, \]
and in a similar way \( T_{g,\Omega}^J \). The meaningful non-negative cone from the geometric view point is clearly
\[ T_{C_{\omega,\Omega}^0} := T_{C_{S_{\omega,\Omega}}} \cap T_{g,\Omega}^J \geq 0. \]

We observe also that that Perelman’s twice contracted second Bianchi type identity implies that the set of all Kähler-Ricci solitons inside \( S_{\omega} \) is given by
\[ KRS_{\omega} := \left\{ (g, \Omega) \in S_{\omega} \mid H_{g,\Omega} = 0 \right\}. \]

Notice that for any \( (g, \Omega) \in KRS_{\omega} \) hold the obvious inclusions \([g, \Omega]_\omega \subseteq KRS_{\omega}\) and
\[ T_{[g,\Omega]_\omega} \subseteq TC_{KRS_{\omega}} \subseteq Ker D_{g,\Omega}H \cap TC_{S_{\omega}}. \]

The following statement provides a finite dimensional reduction which represents the solution of the stability of Kähler-Ricci solitons problem.

**Theorem 1 (Main result. The stability of Kähler-Ricci solitons)**

Let \((X, J, g)\) be compact Kähler-Ricci soliton and let \( \Omega > 0 \) be the unique smooth volume form with \( \int_X \Omega = 1 \) such that \( \omega = \text{Ric}_J(\Omega) \). Then for all \((v, V) \in T_{g,\Omega}^J \) the Hessian form of Perelman’s \( W \) functional with respect to the pseudo-Riemannian structure \( G \) at the point \( (g, \Omega) \), in the direction \((v, V)\) satisfies the inequality
\[ \nabla_G DW (g, \Omega) (v, V ; v, V) \leq 0, \tag{1.17} \]
with equality if and only if
\[ (v, V) \in Ker D_{g,\Omega}H \cap T_{g,\Omega}^J \] 
\[ = T_{[g,\Omega]_\omega} \oplus G H_{g,\Omega}^{0,1} (T_{X,J})_0 \] 
\[ \supseteq TC_{KRS_{\omega}}. \]
In more explicit/classic terms, the previous statement shows that for any smooth curve \((g_t, \Omega_t)_{t \in \mathbb{R}} \subset \mathcal{M} \times \mathcal{V}_1\) (not necessarily in \(\mathcal{S}_\omega\!\)!) with \((g_0, \Omega_0) = (g, \Omega)\) a Kähler-Ricci soliton and with \((\dot{g}_0, \dot{\Omega}_0) = (v, V) \in T^J_{\Omega_0}\) hold the inequality
\[
\left. \frac{d^2}{dt^2} W(g_t, \Omega_t) \right|_{t = 0} \leq 0,
\]
with equality if and only if \((v, V) \in \text{Ker } D_{g, \Omega} H \cap T^J_{g_0, \Omega_0}\). The identity (1.19) and the inclusion (1.20) are part of the statement the main theorem.

In section 17 we obtain also quite general and sharp second variation formulas for Perelman’s \(W\) functional with respect to variations \((v, V) \in B_{g, \Omega}\) over a Kähler-Ricci soliton point which arise from variations of Kähler structures preserving the first Chern class of \(X\).

This formulas provide a precise control of the sign of the second variation of Perelman’s \(W\) functional over a Kähler-Ricci soliton point. This can be of independent interest for experts. (In particular we will see below some general consequences for the classical stability of Kähler-Einstein metrics.) For our geometric applications the most striking particular case is the one corresponding to the main theorem.

The highly geometric nature of the Soliton-Kähler-Ricci flow combined with the main theorem suggest to the author the following version of the Hamilton-Tian conjecture (compare with the statements made in [Ti-Zha] and [Ch-Wa])

**Conjecture 1** Let \((X, J_0)\) be a Fano manifold and let \(\omega \in 2\pi c_1(X, [J_0])\) be an arbitrary \(J_0\)-invariant Kähler form. Then there exists a complex analytic subset \(\Sigma\) of complex codimension greater or equal to 2 (which may be empty!), a smooth complex structure \(J \in [J_0]\) outside \(\Sigma\) and a smooth volume form \(\Omega > 0\) outside \(\Sigma\) such that:

\[
\begin{align*}
\omega &= \text{Ric}_J(\Omega), \\
\partial_{T_X, J} \left( \omega^{-1} d \log \frac{\omega^n}{n!} \right) &= 0,
\end{align*}
\]

outside \(\Sigma\), i.e., the Riemannian metric \(g := -\omega J\), is a smooth \(J\)-invariant Kähler-Ricci soliton outside \(\Sigma\). The triple \((J, g, \Omega)\) is obtained as the limit in the smooth topology of \(X \setminus \Sigma\) as \(t \to +\infty\) of the Soliton-Kähler-Ricci flow with initial data \((J_0, g_0, \Omega_0)\) where \(g_0 := -\omega J_0\) and \(\omega = \text{Ric}_{J_0}(\Omega_0)\), with \(\int_X \Omega_0 = 1\).

We explain now that a very particular consequence of our study of the stability problem provides a result on the stability in the classical sense of Kähler-Einstein manifolds. We introduce first a few basic notations.

Let \((X, J)\) be a compact Kähler manifold and let \(c_1 \equiv c_1(X, [J]) \in H^2_c(X, \mathbb{R})\). We denote by \(K\mathcal{S}\) the space of Kähler structures over \(X\) and we set
\[
K\mathcal{S}_{2\pi c_1} := \{ (J, g) \in K\mathcal{S} \mid gJ \in 2\pi c_1 \}.
\]

We define also the set \(K\mathcal{V}'_J(2\pi c_1)\) of symmetric variations of Kähler structures preserving the first Chern class of \(X\) as the set of elements \(v \in C^\infty \left( X, S^2 \mathbb{R} T_X^* \right)\).
such that there exists a smooth curve $(J_t, g_t) \subset \mathcal{KS}_{2\pi c_1}$ with $(J_0, g_0) = (J, g)$, $g_0 = v$ and $J_0 = (J_0)_t^g$. In section 14 we show the inclusion
\[ \mathbb{K}_{g_0}^J(2\pi c_1) \subseteq \mathbb{D}_{g_0}^J, \] with
\[ \mathbb{D}_{g_0}^J := \left\{ v \in C^\infty(X, S^2 T^*_X) \mid \partial_{T_{X,J}}^g \langle v'_J \rangle^*_g = 0, \overline{\partial}_{T_{X,J}} (v''_J)_g = 0, \{v'_J J\}_d = 0 \right\}, \]
where $\{\alpha\}_d$ denotes the De Rham cohomology class of any $d$-closed form $\alpha$. We introduce also the classical stability operator (see [Bes])
\[ \mathcal{L}_g := \Delta_g - 2R_g^*, \]
acting on smooth symmetric 2-tensors. With these notations we can state the following stability, (in the classical sense) result.

**Theorem 2** Let $(X, J, g)$ be a Fano Kähler-Einstein manifold. Then for any $v \in \text{Ker} \nabla^*_g \cap \mathbb{D}_{g_0}^J$, hold the inequality
\[ \int_X \langle \mathcal{L}_g v, v \rangle_g \, dV_g \geq 0, \]
with equality if and only if $v^*_g \in H_{g_0}^{0,1}(T_{X,J})$.

(See sub-section 17.1 for the proof). A similar result in the case of negative or vanishing first Chern class has been proved in the remarkable paper [D-W-W2] (see also [D-W-W1]). The statement about the equality case hold also under more general assumptions (see lemma 29 in the appendix B).

In the next section we enlighte the results obtained by other authors in the long standing problem of the stability of Kähler-Ricci solitons and on the Hamilton-Tian conjecture.

### 2 Other works on the subject

A question of central importance in complex differential geometry is the Hamilton-Tian conjecture.

Solutions of this conjecture have been posted on the arxiv server in (2013) by Tian-Zhang [T-Z] in complex dimension 3 and quite recently by Chen-Wang [C-W] in general.

Since we have learned about this conjecture in 2004 we asked ourself immediately which one is the precise notion of gauge needed for the convergence.

(The Kähler-Ricci flow $(J_t, \hat{g}_t)_{t \geq 0}$ needs to be modified since its formal limit $(J_0, \hat{g}_\infty)$ as $t \to +\infty$ is a is a Kähler-Einstein metric, but Fano manifolds do not always admit such ones!)

It turns out that the Soliton-Kähler-Ricci flow introduced in this paper corresponds to a modification of the Kähler-Ricci flow via the gauge provided by the gradient of the Ricci potentials.

11
To the very best of our knowledge the Soliton-Kähler-Ricci flow with variable volume forms introduced in this paper does not appear nowhere in the literature.

In our previous works [Pal4] and [Pal5], we introduced also the notion of Soliton-Kähler-Ricci flow with fixed volume form. This leads to a complete different approach which leads naturally to the study of the existence of ancient solutions of the Kähler-Ricci flow and their (modified, according to [Pal4] and [Pal5]) convergence as \( t \to -\infty \). This approach requires some particular geometric conditions (which imply some strong regularity) on the initial data. The key point in [Pal4] and [Pal5] is that these conditions represent a conservative law along the Soliton-Kähler-Ricci flow with fixed volume form. These conditions imply good convexity properties for the convergence of this flow.

We review now the modifications of the Kähler-Ricci flow made by other authors. We can find two frequent approaches in the literature. One is based on the gauge transformation generated by a holomorphic vector fields with imaginary part generating an \( S^1 \)-action on the manifold (see [Ti-Zhu1] and [P-S-S-W2] for a very elegant construction). A Kähler-Ricci-soliton vector field provides such example.

The second approach, which has been used quite intensively in the last years is based on the gauge modification constructed via the minimizers of Perelman’s \( W \) functional (see [Ti-Zhu3] and [Su-Wa]). As far as known the minimizers are unique only in a small neighborhood of the Kähler-Ricci soliton. Therefore the ”modified Kähler-Ricci flow” in [Ti-Zhu3] and [Su-Wa] exists only in such small neighborhood.

For historical reasons it is important to remind that Hamilton [Ham] pointed out first that to any flow of Kähler structures with fixed complex structure corresponds another flow of Kähler structures which preserves the symplectic form (see also Donaldson [Don] for the same remark). He suggested this approach for the study of the Kähler-Ricci flow. As far as we know he did not pursue on this idea.

As explained in the introduction our definition of the Soliton-Ricci flow with variable volume forms was inspired to us from Perelman’s twice contracted second Bianchi type identity and the strict ellipticity of the first variation of the functions \( h \) and \( H \) in the directions \( \mathbb{F} \).

It was surprising for us to discover that the corresponding Soliton-Kähler-Ricci flow with variable volume forms (from now on we will refer only to this flow) preserves the symplectic structure.

We realized quickly the power of this fact since it allows us to apply Futaki’s weighted complex Bochner identity with uniform lower bound on the first eigenvalue of the corresponding weighted Laplacian [Fu1]. The main feature of the Soliton-Kähler-Ricci flow in this paper is that it presents the jumping of the complex structure at the limit when \( t \to +\infty \). This phenomenon is necessary for the existence of Kähler-Ricci solitons in general. We learned for the first time about this key phenomenon in the Pioneer work of [P-S-S-W1]. In this fundamental work the authors introduce a condition on stability (is the condition (B) in [P-S-S-W1]) which is the key phenomenon occurring in the convergence of the Kähler-Ricci flow. We refer also to [P-S-S-W3] for further developments.
We remind now that by definition, the stability of a critical point of a functional corresponds to determine a sign of its second variation in determinate directions.

The stability of critical metrics for natural geometric functionals was naturally born with differential geometry (see [Bes]). The main classic example is the Einstein metric. In the case of this metric the corresponding functional is the integral of the scalar curvature.

In 2003 Grigory Perelman astonished the mathematical community with his spectacular proof of the Poincaré conjecture. In this celebrated paper [Per] he introduced various entropy functionals for Ricci-solitons. Shrinking Ricci-solitons corresponds to critical points of his $W$ functional or to his entropy functional $\nu$.

Since then the second variation of Perelman’s functional $W$ and $\nu$ has been studied quite intensively. It started in 2004 with the works of Cao-Hamilton-Imlanen [C-H-I], [Ca-Zhu] and Tian-Zhu [Ti-Zhu2] independently. It continued with [Ca-He] and [Ha-Mu1], [Ha-Mu2].

We wish to point out that the results in this paper and in [Pal3] are of completely different nature with respect to the previous works. The reason is that in our work we compute the second variation of Perelman’s $W$ functional with respect to the pseudo-Riemannian structure $G$. (The work [Pal3] is a particular case.)

An important fact about Kähler-Ricci solitons is that once they exist, one can obtain the Einstein condition by proving the vanishing of the Futaki invariant [Fut]. From our point of view they provide a natural and necessary generalization in order to control the Einstein condition.

The stability of Kähler-Ricci solitons is important in order to understand the convergence of the Kähler-Ricci flow. The first work on the subject is due to Tian-Zhu, see [Ti-Zhu2].

In 2009 Sun-Wang [Su-Wa] posted on the arxiv server a stability result for the Kähler-Ricci flow basing on the Lojasiewicz inequality (see [Co-Mi]). In this paper the authors use the modified flow in [Ti-Zhu3]. The same method was used in Ache [Ach], where a uniform bound on the curvature is made. We report finally a quite recent work on the same subject by Kröncke [Kro] which combines the technical details in [Su-Wa], [Ach] and [Co-Mi] in the Riemannian set up.

The statements made in this section are based on the very best of our knowledge and understanding of the subject. We sincerely apologize to other authors in case of inaccuracies or omissions in the claims of this section.
3 Proof of the first variation formulas for the maps $h$ and $H$

3.1 The first variation of the Bakry-Emery-Ricci tensor

We remind (see [Pal3]) that the first variation of the Bakry-Emery-Ricci tensor with fixed volume form $\Omega > 0$ is given by the formula

$$2 \frac{d}{dt} \text{Ric}_{g_t}(\Omega) = -\nabla^*_{g_t} \mathcal{D}_{g_t} \dot{g_t},$$

(3.1)

where $\mathcal{D}_g := \tilde{\nabla}_g - 2\nabla_g$, with $\tilde{\nabla}_g$ being the symmetrization of $\nabla_g$ acting on symmetric 2-tensors.

Explicitly

$$\tilde{\nabla}_g \alpha(\xi_0, \ldots, \xi_p) := \sum_{j=0}^p \nabla\alpha(\xi_j, \xi_0, \ldots, \tilde{\nabla}_g \xi_j, \ldots, \xi_p),$$

for all $p$-tensors $\alpha$. Fixing an arbitrary time $\tau$ and time deriving at $t = \tau$ the decomposition

$$\text{Ric}_{g_t}(\Omega_t) = \text{Ric}_{g_{\tau}}(\Omega_{\tau}) - \nabla_{g_t} d \log \frac{\Omega_t}{\Omega_{\tau}},$$

we deduce, thanks to (3.1), the general variation formula

$$2 \frac{d}{dt} \text{Ric}_{g_t}(\Omega_t) = -\nabla^*_{g_t} \mathcal{D}_{g_t} \dot{g_t} - 2\nabla_{g_t} d \frac{\dot{\Omega}_t}{\Omega_t}.$$  

(3.2)

This formula implies directly Perelman’s general first variation formula for the $W$ functional (see appendix A). We define the Hodge Laplacian (resp. the $\Omega$-Hodge Laplacian) operators acting on $q$-forms as

$$\Delta^\Omega_{T X, g} := \nabla_{T X, g} \nabla^*_{g} + \nabla^*_{g} \nabla_{T X, g},$$

$$\Delta^\Omega_{T X, g} := \nabla_{T X, g} \nabla^*_{g} + \nabla^*_{g} \nabla_{T X, g}.$$

We remind also the following Weitzenböck type formula proved in [Pal4]

**Lemma 3** Let $(X, g)$ be a orientable Riemannian manifold, let $\Omega > 0$ be a smooth volume form and let $A \in C^\infty(X, \text{End}(TX))$. Then

$$\Delta^\Omega_{T X, g} A = \Delta^\Omega_{T X, g} A - \mathcal{R}_g \ast A + A \text{Ric}^*_{g}(\Omega),$$

where $(\mathcal{R}_g \ast A) \xi := \text{Tr}_{\xi} (\xi \ast \mathcal{R}_g A)$ for all $\xi \in T_X$.

In analogy to the $\Omega$-Hodge Laplacian we can define the Laplace type operator

$$\tilde{\Delta}^\Omega_{g} := \nabla^*_{g} \nabla_{g} - \nabla_{g} \nabla^*_{g} : C^\infty(X, S^p T^*_X) \rightarrow C^\infty(X, S^2 T^*_X).$$
Using this notation we observe that for any \( u \in C^\infty(X, S^2 T_X) \) hold the identities

\[-\nabla^*_g \mathcal{D}_g u = \left( 2\Delta^g - \hat{\Delta}^g \right) u - \hat{\nabla}^*_g \nabla^*_g u\]

The last one follows from the equalities \( \nabla^*_g u = g \nabla^*_g u^* \) and \( \hat{\nabla}^g (g\xi) = L\xi g \), \( \xi \in C^\infty(X, T_X) \). We observe now that for any symmetric 2-tensor \( u \) the tensor \( R^*_g u \) is also symmetric. In fact let \( (e_k)_k \) be a \( g \)-orthonormal base of \( T_{x,X} \). Then

\[-(R_g * u)(\xi, \eta) = R_g(\xi, e_k, u^*_g e_k, \eta) = R_g(\eta, u^*_g e_k, e_k, \xi).\]

Furthermore if we choose the \( g(x) \)-orthonormal base \( (e_k)_k \) such that \( u \) is diagonal with respect to this one, then

\[R_g(\eta, u^*_g e_k, e_k, \xi) = R_g(\eta, e_k, u^*_g e_k, \xi) = -(R_g * u)(\eta, \xi).\]

We observe also that the previous computation shows the identity

\[(R_g * u)(\xi, \eta) = R_g(\xi, e_k, \eta, u^*_g e_k)\]

\[= g(R_g(\xi, e_k)u^*_g e_k, \eta)\]

\[= g((R_g * u^*_g) \xi, \eta),\]

i.e

\[(R_g * u)^*_g = R_g * u^*_g. \tag{3.3}\]

We deduce in particular the equality

\[R_g * u^*_g = (R_g * u^*_g)^T_g. \tag{3.4}\]

We remind that the \( \Omega \)-Lichnerowicz Laplacian \( \Delta^g_{\Omega,g} \) is self-adjoint with respect to the scalar product \( \langle \cdot, \cdot \rangle \) thanks to the identity \([1,3]\) that we show now.

We pick a \( g(x) \)-orthonormal base \( (e_k)_k \subset T_{x,X} \) such that \( v \) is diagonal with respect to this one at the point \( x \). Using \([3,3]\) we infer

\[\langle R_g * u, v \rangle_g = \text{Tr} \left[ (R_g * u^*_g) v^*_g \right]\]

\[= R_g(u^*_g e_l, e_k, e_l, u^*_g e_k)\]

\[= R_g(e_l, e_k, v^*_g e_l, u^*_g e_k)\]

\[= R_g(e_k, e_l, u^*_g e_k, v^*_g e_l)\]

\[= \langle R_g * v, u \rangle_g,\]
since these identities are independent of the choice of the $g(x)$-orthonormal base $(e_k)_k \subset T_x X$.

**Lemma 4** For any $g \in \mathcal{M}$ and $u \in C^\infty(X, S^2 T^*_X)$ hold the Weitzenböck type formula

$$-\nabla^* g \nabla u = \Delta^\Omega_{L,g} u - L_{\nabla^* g} u.$$ 

**Proof** The required formula follows from the identity

$$\Delta^\Omega_{L,g} u = \left(2\Delta^\Omega_g - \hat{\Delta}^\Omega_g\right) u. \quad (3.5)$$

In order to show this identity we expand $\hat{\Delta}^\Omega_g u = \nabla^* g \hat{\nabla} g u - \hat{\nabla} g \nabla^* g u$. We observe first

$$\nabla^* g \hat{\nabla} g u(\xi, \eta) = \nabla^* g \hat{\nabla} g u(\xi, \eta) + \hat{\nabla} g u(\nabla f, \xi, \eta).$$

We fix an arbitrary point $x_0$ and we choose the vector fields $\xi$ and $\eta$ such that $0 = \nabla g \xi(x_0) = \nabla g \eta(x_0)$. Let $(e_k)_k$ be a $g$-orthonormal local frame such that $\nabla g e_k(x_0) = 0$. Then at the point $x_0$ hold the identities

$$\hat{\nabla} g u(\xi, \eta) = -\cdot \hat{\nabla} g u(e_k, \xi, \eta)$$

$$= -\nabla g \nabla g e_k \hat{\nabla} g u(e_k, \xi, \eta)$$

$$= -\nabla g \nabla g e_k \left[\nabla g u(e_k, \xi, \eta) + \nabla g u(\xi, e_k, \eta) + \nabla g u(\eta, e_k, \xi)\right]$$

$$= -\nabla g \nabla g e_k \nabla g u(e_k, \xi, \eta) - \nabla g \nabla g e_k \nabla g u(\eta, e_k, \xi),$$

and

$$\hat{\nabla} g u(\nabla f, \xi, \eta) = \nabla g u(\nabla g f, \xi, \eta) + \nabla g u(\xi, \nabla g f, \eta) + \nabla g u(\eta, \nabla g f, \xi).$$

Moreover

$$\hat{\nabla} g \nabla^* g u(\xi, \eta) = \hat{\nabla} g \nabla^* g u(\xi, \eta) + \hat{\nabla} g (\nabla g f - u)(\xi, \eta),$$

and at the point $x_0$ hold the identities

$$\hat{\nabla} g \nabla^* g u(\xi, \eta) = \nabla_{\xi} g \nabla^* g u \cdot \eta + \nabla_{\eta} g \nabla^* g u \cdot \xi$$

$$= \nabla_{\xi} g \left[\nabla^* g u \cdot \eta\right] + \nabla_{\eta} g \left[\nabla^* g u \cdot \xi\right]$$

$$= -\nabla_{\xi} g \left[\nabla g e_k u(e_k, \eta)\right] - \nabla_{\eta} g \left[\nabla g e_k u(e_k, \xi)\right]$$

$$= -\nabla_{\xi} g \nabla g e_k u(e_k, \eta) - \nabla_{\eta} g \nabla g e_k u(e_k, \xi),$$
and
\[
\nabla g (\nabla g f \nabla u) (\xi, \eta) = \nabla g,\xi (\nabla g f \nabla u) \cdot \eta + \nabla g,\eta (\nabla g f \nabla u) \cdot \xi
\]
\[
= \nabla g,\xi [u(\nabla g f, \eta)] + \nabla g,\eta [u(\nabla g f, \xi)]
\]
\[
= \nabla g u(\xi, \nabla g f, \eta) + (u \nabla^2 g f) (\xi, \eta)
\]
\[
+ \nabla g u(\eta, \nabla g f, \xi) + (\nabla g df u^* g) (\xi, \eta).
\]

Let now \( A \in C^\infty(X, \text{End}(T_X)) \). We denote by \( A \nabla u \) the 2-tensor defined by the formula
\[
(A \nabla u)(\xi, \eta) := u(A\xi, \eta) + u(A\eta).
\]
We observe that if \( \mu, \zeta \) are two germs of vector fields near \( x_0 \) such that \([\mu, \zeta](x_0) = 0\) then hold the identity at the point \( x_0 \)
\[
\nabla g,\mu \nabla g,\zeta u - \nabla g,\zeta \nabla g,\mu u = -R_g(\mu, \zeta) \nabla u.
\]

Using this identity we infer the equalities at the point \( x_0 \)
\[
(\nabla g,\xi \nabla g, e_k u - \nabla g, e_k \nabla g,\xi u) (e_k, \eta) = -u(\mathcal{R}_g(\xi, e_k) e_k, \eta) - u(e_k, \mathcal{R}_g(\xi, e_k) \eta)
\]
\[
= -(u \text{Ric}^*(g))(\xi, \eta) + (\mathcal{R}_g \ast u)(\xi, \eta),
\]
\[
(\nabla g,\eta \nabla g, e_k u - \nabla g, e_k \nabla g,\eta u) (e_k, \xi) = -(\text{Ric}(g) u^* g)(\xi, \eta) + (\mathcal{R}_g \ast u)(\xi, \eta),
\]
by obvious symmetries. Combining the identities obtained so far and simplifying we obtain the identity
\[
\Delta^\Omega g u = \Delta^\Omega g u + 2\mathcal{R}_g \ast u - u \text{Ric}^*_g(\Omega) - \text{Ric}_g(\Omega) u^*_g,
\]
which in its turn implies the required identity (3.3).

The Weitzenböck type identity in lemma 2 combined with the variation formula (3.2) implies directly the variation formula (1.4).

### 3.2 Proof of the first variation formula for Perelman H-function

We show now the variation formula (1.4). For this purpose let \( 0 < (g_t, \Omega_t) \) be a smooth family and set as usual \( f_t := \log \frac{dV}{dt} \). We start time deriving the identity
\[
-\Delta^\Omega g_t f_t = \text{div}^\Omega g_t \nabla g_t f_t.
\]
We compute first the variation of the $\Omega$-divergence operator. Set $u_t := \dot{\Omega}_t^*$ and time derive the definition identity
\[
d (\xi-\Omega_t) = (\text{div}^{\Omega_t} \xi) \Omega_t,
\]
we infer
\[
d (\xi-u_t \Omega_t) = \left( \frac{d}{dt} \text{div}^{\Omega_t} \xi \right) \Omega_t + u_t (\text{div}^{\Omega_t} \xi) \Omega_t.
\]
Moreover expanding the left hand side we obtain
\[
d (\xi-u_t \Omega_t) = (\xi,u_t) \Omega_t + u_t d (\xi-\Omega_t),
\]
which implies the formula
\[
\left( \frac{d}{dt} \text{div}^{\Omega_t} \right) \xi = g \left( \nabla_g \dot{\Omega}_t^*, \xi \right).
\]
We observe also the variation formulas
\[
\frac{d}{dt} (\nabla_g f_t) = \nabla_g \dot{f}_t - \dot{g}_t \nabla_g f_t, \tag{3.6}
\]
and
\[
\dot{f}_t = \frac{1}{2} \text{Tr}_g \dot{g}_t - \dot{\Omega}_t^*. \tag{3.7}
\]
Combining all this formulas we obtain
\[
-\frac{d}{dt} \Delta_g f_t = \frac{d}{dt} (\nabla_g f_t) + \text{div}^{\Omega_t} \frac{d}{dt} (\nabla_g f_t)
\]
\[
= g_t \left( \nabla_g \Omega_t^*, \nabla_g f_t \right) + \text{div}^{\Omega_t} \left( \dot{\Omega}_t^* - \frac{1}{2} \text{Tr}_g \dot{g}_t \right)
\]
\[
- \text{div}^{\Omega_t} (\dot{g}_t^* \nabla_g f_t).
\]
We expand last term using the identity
\[
\text{div}^{\Omega} \xi = \text{Tr}_\mathbb{R} (\nabla_g \xi) - g (\xi, \nabla g f).
\]
We obtain with respect to a $g_t(x)$-orthonormal basis $(e_k)_k \subset T_{X,x}$ at an arbitrary space-time point $(x,t)$
\[
\text{div}^{\Omega_t} (\dot{g}_t^* \nabla_g f_t)
\]
\[
= g_t (\nabla_g e_k (\dot{g}_t^* \nabla_g f_t), e_k) - g_t (\dot{g}_t^* \nabla_g f_t, \nabla_g f_t)
\]
\[
= g_t (\nabla_g e_k \dot{g}_t^* \cdot \nabla_g f_t + \dot{g}_t^* \nabla_g e_k \nabla_g f_t, e_k) - g_t (\dot{g}_t^* \nabla_g f_t, \nabla_g f_t)
\]
\[
= g_t (\nabla_g f_t, \nabla_g e_k \dot{g}_t^* e_k) + g_t (\nabla^2 g_{e_k} f_t, \dot{g}_t^* e_k) - g_t (\nabla_g f_t, \dot{g}_t^* \nabla_g f_t)
\]
\[
= -g_t (\nabla^* g_t \dot{g}_t^* \nabla_g f_t) + (\nabla_g d f_t, \dot{g}_t)_{g_t}.
\]
We infer the variation formula

\[- \frac{d}{dt} \Delta_{g^t} f_t = \Delta_{g^t}^2 \left( \dot{\Omega}_{t}^* - \frac{1}{2} \text{Tr}_{g^t} \dot{g}_t \right) + g_t \left( \nabla_{g^t}^* \dot{g}_t^* + \nabla_{g^t} \dot{\Omega}_t^* , \nabla_{g^t} f_t \right) \]

\[- \langle \dot{g}_t , \nabla_{g^t} df_{t \Omega} \rangle_{g^t} . \]  

(3.8)

We observe next the identity

\[ 2 \frac{d}{dt} h_t^* = 2 \dot{h}_t^* - 2 \dot{g}_t^* h_t^* \]

\[ = \Delta_{g^t} \dot{g}_t^* - 2 (\mathcal{R}_{g^t} * \dot{g}_t^*)_{t} + \dot{g}_t^* \text{Ric}_{g^t}^* (\Omega_t) + \text{Ric}_{g^t}^* (\Omega_t) \dot{g}_t^* \]

\[- \left( L_{\nabla_{g^t} \dot{\Omega}_t^* + \nabla_{g^t} \dot{\Omega}_t^*} \dot{g}_t \right)^* - 2 \dot{g}_t^* - 2 \dot{g}_t^* h_t^* , \]

thanks to the variation formula (1.4). We deduce the formula

\[ \frac{d}{dt} \text{Tr}_{g^t} h_t = \frac{1}{2} \Delta_{g^t} \text{Tr}_{g^t} \dot{g}_t - \langle \dot{g}_t , \text{Ric}(g_t) \rangle_{g_t} - \text{div}_{g_t} \left( \nabla_{g^t}^* \dot{g}_t^* + \nabla_{g^t} \dot{\Omega}_t^* \right) , \]

(3.9)

thanks to the identities

\[ \text{Tr}_{g} (\mathcal{R}_g * v) = \langle v, \text{Ric}(g) \rangle_g , \]

(3.10)

\[ \text{Tr}_{g} (L_{\xi} g) = 2 \text{Tr}_{\mathcal{R}} (\nabla_{g} \xi) = 2 \text{div}_g \xi . \]

(3.11)

In order to show the identity (3.11) we expand with respect to a $g(x)$-orthonormal basis $(e_k)_k \subset T_{X,x}$ the therm

\[ \text{Tr}_{g} (\mathcal{R}_g * v) = (\mathcal{R}_g * v) (e_k, e_k) \]

\[ = - v (\mathcal{R}_g (e_k, e_l) e_k , e_l) \]

\[ = g (v^* e_l , \text{Ric}^* (g) e_l) \]

\[ = \langle v , \text{Ric}(g) \rangle_g . \]

The first equality in (3.11) follows from the elementary identity

\[ (L_{\xi} g)_g^* = \nabla_{g} \xi + (\nabla_{g} \xi)_g^T , \]

where $A_g^T$ denotes the transpose of an endomorphism $A$ of $T_X$ with respect to $g$.

In conclusion combining the variation formulas (3.8), (3.9) and (3.7) we infer the variation identity

\[ 2 D_{g^t, \Omega} H (v, V) = \Delta_{g^t}^2 V_{\Omega}^* - \text{div}_{g^t} \left( \nabla_{g^t}^* v^* + \nabla_{g^t} V_{\Omega}^* \right) - 2 V_{\Omega}^* - \langle v , h_{g, \Omega} \rangle_g , \]

and thus the required variation formula (1.5).
4 The Soliton-Kähler-Ricci Flow with variable volume forms

4.1 Existence of the Soliton-Kähler-Ricci flow

We prove in this sub-section lemma 1.

Proof From now on we will set for notation simplicity $h_t \equiv h_{g_t, \Omega_t}$, $H_t \equiv H_{g_t, \Omega_t}$, and $H_t \equiv H_{g_t, \Omega_t}$. We observe that for any smooth curve $(g_t, \Omega_t)_{t \geq 0} \subset M \times \mathcal{V}_1$ the identity

\[
f_t = \log \frac{dV_{g_t}}{\Omega_t},
\]

is equivalent to the evolution equation

\[
2 \dot{f}_t = \text{Tr}_{g_t} \dot{g}_t - 2 \dot{\Omega}_t^*.
\] (4.1)

with initial data $f_0 := \log \frac{dV_{g_0}}{\Omega_0}$. Along the Soliton-Ricci flow, the equation (4.1) rewrites as

\[
2 \dot{f}_t = - \text{Tr}_{g_t} h_t + 2 H_t,
\]

\[
= - \Delta_{g_t} f_t + 2 f_t - 2 \int_X H_t \Omega_t.
\]

We infer that the Soliton-Ricci flow equation is equivalent to the evolution system

\[
\begin{aligned}
\dot{g}_t &= g_t - \text{Ric}(g_t) - \nabla_{g_t} df_t, \\
2 \dot{f}_t &= - \Delta_{g_t} f_t - |\nabla_{g_t} f_t|^2_{g_t} + 2 f_t - \mathcal{W}(g_t, f_t),
\end{aligned}
\] (4.2)

with $f_0 := \log \frac{dV_{g_0}}{\Omega_0}$. We consider now the flow of diffeomorphisms $(\varphi_t)_{t \geq 0}$ solution of

\[
2 \dot{\varphi}_t = (\nabla_{g_t} f_t) \circ \varphi_t,
\]

with $\varphi_0 = \text{Id}_X$ and we define $(\dot{g}_t, \dot{f}_t) := \varphi_t^*(g_t, f_t)$. We observe the evolution formulas

\[
\frac{d}{dt} \dot{g}_t = \varphi_t^* \left( \dot{g}_t + \frac{1}{2} L_{\nabla_{g_t} f_t} g_t \right)
\]

\[
= \varphi_t^* \left[ g_t - \text{Ric}(g_t) \right]
\]

\[
= \dot{g}_t - \text{Ric}(\dot{g}_t),
\]

20
and
\[
2 \frac{d}{dt} \hat{f}_t = 2 \hat{f}_t \circ \varphi_t + 2 d \varphi_t f_t \cdot \hat{\varphi}_t
\]
\[
= 2 \hat{f}_t \circ \varphi_t + d \varphi_t f_t \cdot [\nabla g_t f_t] \circ \varphi_t
\]
\[
= \left(2 \hat{f}_t + df_t \cdot \nabla g_t f_t\right) \circ \varphi_t
\]
\[
= \left(2 \hat{f}_t + |\nabla g_t f_t|^2\right) \circ \varphi_t,
\]

We deduce thanks to the diffeomorphism invariance of the $W$ functional that the evolution system (4.2) is equivalent to

\[
\begin{aligned}
\frac{d}{dt} \hat{g}_t &= \hat{g}_t - \text{Ric}(\hat{g}_t), \\
2 \frac{d}{dt} \hat{f}_t &= -\Delta_{\hat{g}_t} \hat{f}_t + 2 \hat{f}_t - W(\hat{g}_t, \hat{f}_t),
\end{aligned}
\]

with initial data $(\hat{g}_0, \hat{f}_0) := (g_0, f_0)$. Notice indeed that we can obtain (4.2) from (4.3) by performing the inverse gauge transformation $(g_t, f_t) := \psi_t^*(\hat{g}_t, \hat{f}_t)$ with $\psi_t = \phi_t^{-1}$ being characterized by the evolution equation

\[
2 \dot{\psi}_t = -\left(\nabla_{\hat{g}_t} \hat{f}_t\right) \circ \psi_t,
\]

$\psi_0 = \text{Id}_X$. In order to show all time existence and uniqueness of the solutions of the evolution system (4.3) we consider a solution of the Ricci flow $(\hat{g}_t)_{t \in [0,T]}$,

\[
\frac{d}{dt} \hat{g}_t = -2 \text{Ric}(\hat{g}_t),
\]

with initial data $\hat{g}_0$ and $0 < T < +\infty$. Then $(\hat{g}_t)_{t \geq 0}$ defined by

\[
\hat{g}_t := e^{\lambda t} \hat{g}_T (1-e^{-t}),
\]

satisfies the evolution equation relative to the metrics in (4.3). Then we set $\lambda := 2T$. In the case $(X, J_0)$ is a Fano variety and $\hat{g}_0, J_0 \in 2\pi c_1(X)$ we can choose $\lambda = 1$ since the the evolution equation of $\hat{g}_t$ in (4.3) represents a solution of the Kähler-Ricci flow equation.

The existence and uniqueness of the solutions of the evolution equation for $\hat{f}_t$ in (4.3) follows directly from standard parabolic theory with respect to H"older spaces. Notice indeed that the presence of the integral therm $W(\hat{g}_t, \hat{f}_t)$ (we consider the expression involving the $H^1(X)$ norm of $\hat{f}$) does not produce any issue in this theory.

In the Fano set up we define the complex structures $J_t := \psi_t^* J_0$. Then the family $(J_t, g_t)_{t \geq 0}$ represents a flow of Kähler structures since $(J_0, g_t)_{t \geq 0}$ is also
a flow of Kähler structures. The identity \( \varphi^*_t J_t \equiv J_0 \) is equivalent to the equality

\[
0 = \frac{d}{dt} (\varphi^*_t J_t)
= \varphi^*_t \left( J_t + \frac{1}{2} L_{\nabla_{g_t} f_t} J_t \right)
= \varphi^*_t \left( J_t + J_t \partial_{TX,J_t} \nabla_{g_t} f_t \right),
\]

i.e to the equation

\[
\dot{J}_t = -J_t \partial_{TX,J_t} \nabla_{g_t} f_t.
\]

This combined with the \( J_t \)-linearity of the first two terms in the right hand side of the complex decomposition

\[
\text{Ric}^*_t(\Omega_t) = \text{Ric}^*(g_t) + \partial_{TX,J_t} \nabla_{g_t} f_t + \partial_{TX,J_t} \nabla_{g_t} f_t,
\]

implies the required characterization \( 2\dot{J}_t = [J_t, \dot{g}_t^*] \) of the evolution of the complex structures \( J_t \).

\[
\square
\]

4.2 Monotony of Perelman’s \( \mathcal{W} \)-functional along the soliton Soliton-Kähler-Ricci flow

We observe first the following elementary fact.

**Lemma 5** Let \((X, J)\) be a Fano manifold and let \( g \) be a \( J \)-invariant Kähler metric with symplectic form \( \omega := gJ \in 2\pi\mathcal{C}_1(X, [J]) \). Then \((J, g)\) is a Kähler-Ricci soliton if and only if there exists a smooth volume form \( \Omega > 0 \) with \( \int_X \Omega = 1 \) such that

\[
(S) \begin{cases}
\omega = \text{Ric}_J(\Omega), \\
\Delta^\Omega f - 2f + 2\int_X f\Omega = 0, f := \log \frac{\omega}{\omega_\Omega}.
\end{cases}
\]

**Proof** We assume first that \((J, g)\) is a Kähler-Ricci soliton. Then Perelman’s twice contracted Bianchi type identity (1.2) implies \( H_{g,\Omega} = 0 \). This last is equivalent to the second equation of the system \( S \) thanks to the identity \( \text{Tr}_g h_{g,\Omega} = 0 \). We show now that the solution of the system \( S \) implies that \((J, g)\) is a Kähler-Ricci soliton. Indeed multiplying by \( \nabla_g f \) both sides of the identity (1.2) and integrating by parts we obtain the general formula

\[
\int_X \langle h_{g,\Omega}, \nabla^2_{g} f \rangle g \Omega = -\int_X H_{g,\Omega} \Delta^\Omega_{g} f g\Omega.
\]

(4.4)

In our case this rewrites as

\[
2\int_X \left| \partial_{TX,J} \nabla_{g} f \right|^2 g \Omega = \int_X (\Delta^\Omega_{g} f - 2f) \Delta^\Omega_{g} f \Omega,
\]

(4.5)
thanks to the condition \( \omega = \text{Ric}_J(\Omega) \) and the complex decomposition of the Bakry-Emery-Ricci tensor. We infer the required conclusion.

We provide now a proof of the monotony statement in lemma 2.

**Proof** STEP I. Let \((J, \hat{g}_t)_{t \geq 0}\) be a solution of the Kähler-Ricci flow and observe that this equation rewrites in the equivalent form

\[
\begin{cases}
\frac{d}{dt} \hat{\omega}_t = i \partial_J \overline{\partial_J} \log \frac{\hat{\omega}_t^n}{\hat{\Omega}_t^n}, \\
\hat{\omega}_t = \text{Ric}_J(\hat{\Omega}_t), \int_X \hat{\Omega}_t = 1,
\end{cases}
\]  

with \( \hat{\omega}_t := \hat{g}_t J, \) and \( \hat{\omega}_0 := \omega. \) We define the function

\[
\hat{f}_t := \log \frac{\hat{\omega}_t^n}{\hat{\Omega}_t^n},
\]

and we observe the analogue of (4.1)

\[
2 \frac{d}{dt} \hat{f}_t = \text{Tr} \hat{\omega}_t \frac{d}{dt} \hat{\omega}_t - 2 \left( \frac{d}{dt} \hat{\Omega}_t \right)\ast.
\]

This combined with the first equation in (4.6) implies

\[
2 \frac{d}{dt} \hat{f}_t = -\Delta_{\hat{g}_t} \hat{f}_t - 2 \left( \frac{d}{dt} \hat{\Omega}_t \right)\ast.
\]  

On the other hand time differentiating the identity \( \hat{\omega}_t = \text{Ric}_J(\hat{\Omega}_t) \) in (4.6) we obtain

\[
2 \frac{d}{dt} \hat{\omega}_t = 2 \frac{d}{dt} \text{Ric}_J(\hat{\Omega}_t) = -2i \partial_J \overline{\partial_J} \left( \frac{d}{dt} \hat{\Omega}_t \right)\ast,
\]

which combined with (4.7) implies

\[
2i \partial_J \overline{\partial_J} \hat{f}_t = i \partial_J \overline{\partial_J} \left( 2 \frac{d}{dt} \hat{f}_t + \Delta_{\hat{g}_t} \hat{f}_t \right),
\]

i.e.

\[
2 \frac{d}{dt} \hat{f}_t = -\Delta_{\hat{g}_t} \hat{f}_t + 2 \hat{f}_t + C_t,
\]

for some time dependent constant \( C_t \) which can be determined time deriving the integral condition \( \int_X \hat{\Omega}_t = 1. \) Indeed using (4.7) we obtain

\[
0 = -2 \int_X \left( \frac{d}{dt} \hat{\Omega}_t \right)\ast \hat{\Omega}_t = \int_X \left[ 2 \frac{d}{dt} \hat{f}_t + \Delta_{\hat{g}_t} \hat{f}_t \right] \hat{\Omega}_t = C_t + 2 \int_X \hat{f}_t \hat{\Omega}_t.
\]
We infer the evolution formula
\[ \frac{2}{dt} \hat{f}_t = -\Delta_g \hat{f}_t + 2 \hat{f}_t - 2 \int_X \hat{f}_t e^{-\hat{f}_t} dV_{g_t}, \quad (4.8) \]
with initial data
\[ \hat{f}_0 := \log \frac{\omega^n}{\Omega^n}. \]
We observe now that the identity \( \hat{\omega}_t = \text{Ric}_J(\hat{\Omega}_t) \) in (4.6) implies
\[ \hat{g}_t = -\text{Ric}_J(\hat{\Omega}_t) J = \text{Ric}_{\hat{g}_t}(\hat{\Omega}_t) - \hat{g}_t \partial_{TX,J} \nabla_{\hat{g}_t} \hat{f}_t, \quad (4.9) \]
and thus \( \text{Tr}_{\hat{g}_t} h_{\hat{g}_t,\hat{\Omega}_t} = 0 \). We deduce the equality
\[ \mathcal{W}(\hat{g}_t, \hat{f}_t) = 2 \int_X \hat{f}_t e^{-\hat{f}_t} dV_{g_t}. \quad (4.10) \]
We infer by Cauchy’s uniqueness that the evolution equation (4.8) is equivalent with the second evolution equation in (4.3). We obtain, as in the proof of lemma 1, a Soliton-Kähler-Ricci flow \((J_t, \omega_t, \Omega_t)_{t \geq 0}\) with initial data \((J_0, \omega_0, \Omega_0) = (J, \omega, \Omega)\). We observe that thanks to (4.9) and (4.10) the Soliton-Ricci flow evolution system (4.2) writes in our case as
\[ \begin{cases} \dot{g}_t = -g_t \bar{\partial}_{TX,J_t} \nabla_{g_t} \hat{f}_t, \\ 2 \dot{f}_t = -\Delta_{g_t} \hat{f}_t + 2 \hat{f}_t - 2 \int_X \hat{f}_t e^{-\hat{f}_t} dV_{g_t}. \end{cases} \quad (4.11) \]
Time deriving the identity \( \omega_t = g_t J_t \) and using the evolution formula for the complex structure \( 2 \dot{J}_t = [J_t, \dot{g}_t^*] \) in the Soliton-Kähler-Ricci flow equation we infer
\[ \begin{align*}
\dot{\omega}_t &= \dot{g}_t J_t + g_t \dot{J}_t \\
&= \frac{1}{2} g_t (\dot{g}_t^* J_t + J_t \dot{g}_t^*) \\
&= \frac{1}{2} \omega_t (\dot{g}_t^* - J_t \dot{g}_t^* J_t) \\
&= \omega_t (\dot{g}_t^*)_{J_t}^{1,0} \\
&= 0,
\end{align*} \]
thanks to the first equation in (4.11). We deduce \( \omega_t \equiv \omega \) and thus the identity in time
\[ \omega = \text{Ric}_J(\Omega_t). \quad (4.12) \]
STEP IIa. We provide now a first proof of the monotony statement for the Soliton-Kähler-Ricci flow. The equality (4.10) rewrites as

\[ \mathcal{W}(g_t, \Omega_t) = 2 \int_X f_t e^{-f_t} \frac{\omega^n}{n!} \equiv 2 \int_X f_t \Omega_t, \quad (4.13) \]

thanks to the invariance by diffeomorphisms of \( \mathcal{W} \). Let

\[ F_t := f_t - \int_X f_t \Omega_t, \]

and observe that the second evolution equation in (4.11) rewrites as

\[ 2 \dot{f}_t = -\Delta^\Omega_{g_t} F_t + 2 F_t. \quad (4.14) \]

Time deriving the expression (4.13) and using the evolution equation (4.14) we infer

\[ \frac{d}{dt} \mathcal{W}(g_t, \Omega_t) = 2 \int_X (\dot{f}_t - f_t \dot{f}_t) \Omega_t \]

\[ = -2 \int_X f_t \dot{f}_t \Omega_t \]

\[ = \int_X (\Delta^\Omega_{g_t} F_t - 2 F_t) F_t \Omega_t \geq 0, \]

thanks to the estimate \( \lambda_1(\Delta^\Omega_g) \geq 2 \) for the first eigenvalue \( \lambda_1(\Delta^\Omega_g) \) of the weighted Laplacian \( \Delta^\Omega_g \) in the case \( gJ = \text{Ric}_J(\Omega) \). (See the estimate (13.15) in the section 13.) Indeed by the variational characterization of the first eigenvalue hold the estimate

\[ 2 \leq \lambda_1(\Delta^\Omega_g) = \inf \left\{ \int_X \Delta^\Omega_g uu \Omega \mid u \in C^\infty_0(X, \mathbb{R})_0 : \int_X u^2 \Omega = 1 \right\}, \quad (4.15) \]

which implies

\[ 0 \leq \int_X (\Delta^\Omega_g F - 2F) F \Omega, \quad (4.16) \]

with

\[ F := f - \int_X f \Omega, f := \log \frac{dV_2}{\Omega}. \]

We assume now equality in (4.10). We assume also \( F \neq 0 \) otherwise \( g \) will be a \( J \)-invariant Kähler-Einstein metric. Equality in (4.10) implies \( 2 = \lambda_1(\Delta^\Omega_g) \) and

\[ u_0 := F \left[ \int_X F^2 \Omega \right]^{-1/2}, \]
attains the infinitum in (4.15). Thus we can apply the method of Lagrange multipliers to the functionals

\[ \Phi(u) := \int_X \Delta g^\Omega uu \Omega, \]
\[ \Psi(u) := \int_X u^2 \Omega, \]
over the space \( C^\infty_\Omega(X, \mathbb{R}) \). We have the equalities

\[ 2 = \min_{\Psi=1} \Phi = \Phi(u_0), \]

which imply \( D_{u_0} \Phi = \mu D_{u_0} \Psi \), i.e. \( \Delta g^\Omega u_0 = \mu u_0 \), with \( \mu = 2 \). This last is equivalent to the equation \( \Delta g^\Omega F = 2F \). Then the required conclusion follows from lemma 5.

**STEP IIb.** We give here a different proof of the monotony statement. We remind first that Perelman’s first variation formula for the \( W \) functional [Per] writes as

\[ D_{g_t,\Omega} W(v, V) = -\int_X \left[ \langle v, h_{g_t,\Omega} \rangle_{g_t} - 2V_{g_t}^* H_{g_t,\Omega} \right] \Omega. \]

Thus along the Soliton-Ricci flow hold the identity

\[ \frac{d}{dt} W(g_t, \Omega_t) = \int_X \left[ |h_{g_t,\Omega}|_{g_t}^2 - 2H_{g_t,\Omega}^2 \right] \Omega_t. \]

Then the conclusion follows from the identity (4.12) combined with the elementary lemma below.

**Lemma 6** Let \((X, J)\) be a Fano manifold, let \( g \) be a \( J \)-invariant Kähler metric with symplectic form \( \omega := gJ \in 2\pi c_1(X, [J]) \) and let \( \Omega > 0 \) be a smooth volume form with \( \int_X \Omega = 1 \) such that \( \omega = \mathrm{Ric}_J(\Omega) \). Then

\[ \int_X |h_{g,\Omega}|_g^2 \Omega \geq 2 \int_X H_{g,\Omega}^2 \Omega, \]  

with equality if and only if \((J, g)\) is a Kähler-Ricci soliton.

**Proof** The condition \( \omega = \mathrm{Ric}_J(\Omega) \) and the complex decomposition of the BER tensor imply

\[ h_{g,\Omega} = g^{\overline{\beta}} \partial_{X,\beta} \nabla_g f, \]

and thus \( \Tr_g h_{g,\Omega} = 0 \). We deduce

\[ 2H_{g,\Omega} = -(\Delta g^\Omega - 2\overline{\Omega}) F. \]  

26
Then
\[\int_X \left| \partial_{T,X,J} g \right|^2 \Omega = \int_X \left| \partial_{T,X,J} \nabla_g f \right|^2 \Omega = \int_X (\Delta_g^2 - 2\Pi) F \cdot F \Omega,\]
thanks to the integral identity (4.5). The conclusion follows from the variational argument at the end of step IIa.

Remark 1 We observe that the elementary identities
\[\nabla_g f = J \omega^{-1} df = 2 \omega^{-1} d_q f,\]
with \(2\omega^{-1} df := -df \cdot J,\) allow to rewrite the Soliton-Ricci flow evolution system (4.11) as
\[
\begin{aligned}
\dot{J}_t &= \partial_{T,X,J_t} (\omega^{-1} df_t), \\
2 \dot{f}_t &= \text{Tr} (dd_q f_t - df_t \wedge d_q f_t) + 2 f_t - 2 \int_X f_t e^{-f_t} \omega^n.
\end{aligned}
\]
(4.19)

We notice also that the Soliton-Kähler-Ricci flow evolution system with initial data \((J_0, g_0, \Omega_0) = (J, g, \Omega)\) such that \(\omega := gJ = \text{Ric}_J(\Omega)\) is equivalent to the system (1.6). Indeed the argument in step I of the proof of lemma 2 shows that our Soliton-Kähler-Ricci flow is equivalent to the Kähler-Ricci flow equation (4.6) via the gauge transformation given by the diffeomorphisms \(\varphi_t\). But (1.6) is also equivalent to (4.6) via the same gauge transformation. Notice in fact the identities
\[
\frac{d}{dt} \hat{\omega}_t = \frac{1}{2} \varphi_t^* (L_{\nabla_{g_t} f_t} \omega) = \varphi_t^* (i \partial_{J_t} \partial_{J_t} f_t) = i \partial_J \partial_J \hat{f}_t.
\]
The corresponding identities for the transformation of the complex structure have been considered at the end of the proof of lemma 1. We infer the equivalence of our Soliton-Kähler-Ricci flow with (1.6).

Remark 2 Let \((g_t, \Omega_t)_{t \geq 0}\) be the Soliton-Ricci flow and set for notation simplicity \(V_t := V(g_t, \Omega_t), h_t := h_{g_t, \Omega_t}, H_t := H_{g_t, \Omega_t}\). Perelman’s twice contracted differential Bianchi identity (1.2) implies
\[\nabla^{\ast}_{g_t} g_t^* + \nabla g_t \Omega_t^* = 0.\]
(4.20)

Then the fundamental variation formula (1.5) implies the evolution equation along the Soliton-Ricci flow
\[2 \frac{d}{dt} H_t = -(\Delta_{g_t} - 2\Pi) H_t + |h_t|_{g_t}^2 - V_t.\]
(4.21)

This combined with the monotony statement in lemma 2 or in [11] implies the inequality
\[2 \frac{d}{dt} H_t \leq -(\Delta_{g_t} - 2\Pi) H_t + |h_t|_{g_t}^2,\]
(4.22)
along the Soliton-Kähler-Ricci flow.
5 The second variation of the $\mathcal{W}$ functional along the Soliton-Kähler-Ricci flow

Let $(J_t, g_t, \Omega_t)_{t \geq 0}$ be the Soliton-Kähler-Ricci flow. In the proof of step I of lemma 2 we obtained the identity

$$\mathcal{W}_t = -2 \int_X f_t \dot{f}_t e^{-f_t} \frac{\omega^n}{n!}.$$ 

Time deriving this we obtain

$$\ddot{\mathcal{W}}_t = -2 \int_X \dot{f}_t^2 \Omega_t - 2 \int_X f_t \left( \dddot{f}_t - \dddot{f}_t^2 \right) \Omega_t.$$ 

Time deriving the identity

$$0 = \int_X \dot{f}_t \Omega_t \equiv \int_X \dot{f}_t e^{-f_t} \frac{\omega^n}{n!},$$

we deduce

$$0 = \int_X \left( \dddot{f}_t - \dddot{f}_t^2 \right) \Omega_t,$$

and thus the evolution formula

$$\ddot{\mathcal{W}}_t = -2 \int_X \dot{f}_t^2 \Omega_t - 2 \int_X F_t \left( \dddot{f}_t - \dddot{f}_t^2 \right) \Omega_t. \quad (5.1)$$

We observe now that the second evolution equation in the system (4.11) rewrites as

$$2 \dddot{f}_t = -\Delta g_t f_t + 2 f_t - \mathcal{W}_t,$$

thanks to (4.13). Time deriving this we infer

$$-2 \dddot{f}_t = \frac{d}{dt} \Delta_{g_0} f_t - 2 \dot{f}_t + \dot{\mathcal{W}}_t. \quad (5.2)$$

Plunging the identity (4.20) in the variation formula (3.8) and using the first equation in the system (4.11) we obtain

$$\frac{d}{dt} \Delta_{g_t} f_t = \Delta_{g_t} \frac{\partial}{\partial g_t} \left( \mathcal{W}_t \right) - \left| \nabla_{g_t} f_t \right|^2_{g_t}$$

$$= \Delta_{g_t} H - \left| h_t \right|_{g_t}^2,$$

with $2H_t = -(\Delta_{g_t} - 2\parallel) F_t$. Thus $\dot{f}_t = H_t$ thanks to (4.14). Using (5.2) we infer

$$-2 \dddot{f}_t = \left( \Delta_{g_t} - 2\parallel \right) H_t - \left| h_t \right|_{g_t}^2 + \dot{\mathcal{W}}_t.$$
(This last follows also from the general evolution formula \(4.21\).) Integrating by parts we obtain the identity

\[-2 \int_X F_t \dot{f}_t \Omega_t = - \int_X \left[ 2H_t^2 + F_t |h_t|^2 \right] \Omega_t,
\]

(since \( \int_X F_t \Omega_t = 0 \)). Plunging this identity in the evolution formula \(5.1\) we deduce the simple second variation formula

\[\dot{W}_t = - \int_X \left[ 4H_t^2 + \left( |h_t|^2 - 2H_t^2 \right) F_t \right] \Omega_t\]

\[= - \int_X \left[ |(\Delta_{g_t} - 2\mathbb{I}) F_t|^2 + \left( |h_t|^2 - 2H_t^2 \right) F_t \right] \Omega_t.\]

6 The Levi-Civita connection of the pseudo-Riemannian structure \(G\)

In this section we compute the Levi-Civita connection of the pseudo-Riemannian structure \(G\). This is needed for the computation of the second variation of the \(W\) functional with respect to such structure. In the computations that will follow we set for notations simplicity \(T := T_{M \times V_1}\) and we compute the first variation of \(G\) at an arbitrary point \((g, \Omega)\),

\[D_{g, \Omega} G : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}^*.\]

In a direction \((\theta, \Theta) \in \mathcal{T}\) this is given by the identity

\[D_{g, \Omega} G(\theta, \Theta; u, V)(v, V) = \frac{d}{dt}_{|t=0} G_{g_t, \Omega_t}(u, U_t; v, V),\]

where \((g_t, \Omega_t)_{t \in (-\varepsilon, \varepsilon)} \subset M \times V_1\) is a smooth curve with \((g_0, \Omega_0) = (g, \Omega)\) and \((\dot{g}_0, \dot{\Omega}_0) = (\theta, \Theta)\). For notation simplicity let denote \(u_t^* := g_t^{-1} u\) and \(U_t^* := U_t/\Omega_t\). Then hold the equality

\[D_{g, \Omega} G(\theta, \Theta; u, V)(v, V) = \frac{d}{dt}_{|t=0} \left[ \int_X \text{Tr}_R(u_t^* v_t^*) \Omega_t - 2 \int_X U_t^* V \right]\]

\[= \int_X \left[ \frac{d}{dt}_{|t=0} \text{Tr}_R(u_t^* v_t^*) \right] \Omega_t + \int_X \text{Tr}_R(u_t^* v_t^*) \Theta\]

\[= 2 \int_X \frac{d}{dt}_{|t=0} U_t^* V.\]

Using the identity \(\frac{d}{dt}_{|t=0} U_t^* = -\dot{g}_t^* u_t^*\), which follows from the formula

\[\frac{d}{dt}_{|t=0} g_t^{-1} = -g_t^{-1} \dot{g}_t g_t^{-1},\]

29
we obtain
\[
\frac{d}{dt} \text{Tr}_R (u^*_i v^*_i) = \text{Tr}_R \left( \frac{d}{dt} u^*_i v^*_i + u^*_i \frac{d}{dt} v^*_i \right) \\
= - \text{Tr}_R (\dot{g}^*_i u^*_i v^*_i + u^*_i \dot{g}^*_i v^*_i) \\
= - 2 \text{Tr}_R (\dot{g}^*_i u^*_i v^*_i),
\]
since \(\dot{g}_t\) is also symmetric. Indeed we observe the elementary identities
\[
\text{Tr}_R [(u^*_i \dot{g}^*_i) v^*_i] = \text{Tr}_R [v^*_i (u^*_i \dot{g}^*_i)] \\
= \text{Tr}_R [v^*_i (u^*_i \dot{g}^*_i)]^T_i \\
= \text{Tr}_R [(u^*_i \dot{g}^*_i)^T_i v^*_i] \\
= \text{Tr}_R (\dot{g}^*_i u^*_i v^*_i),
\]
where \(A^T_i\) denotes the transpose of \(A\) with respect to \(g_t\). Time deriving the identity \(U = U^*_t \Omega_t\) we infer
\[
0 = \frac{dU^*_i}{dt} \Omega_t + U^*_i \dot{\Omega}_t,
\]
and thus
\[
\frac{dU^*_i}{dt} = - U^*_i \dot{\Omega}_t^*.
\]
Summing up we infer the expression of the variation of \(G\) at the point \((g, \Omega)\) in the direction \((\theta, \Theta)\)
\[
D_{g,\Omega} G(\theta, \Theta; u, U)(v, V) = \int_X \left\{ \text{Tr}_R [(\Theta_{ti}^* - 2\theta_t^*)u^*_g v^*_g] + 2\Theta_{ti}^* U^*_t V^*_t \right\} \Omega.
\]
We can compute now the Levi-Civita connection \(\nabla_G = D + \Gamma_G\) of the pseudo-Riemannian structure \(G\). At a point \((g, \Omega)\) the symmetric bilinear form
\[
\Gamma_G(g, \Omega) : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T},
\]
is identified by the expression
\[
2G_{g,\Omega} (\Gamma_G(g, \Omega)(u, U; v, V); \theta, \Theta) \\
= [D_{g,\Omega} G(u, U; v, V) + D_{g,\Omega} G(v, V; u, U)](\theta, \Theta) \\
- D_{g,\Omega} G(\theta, \Theta; u, U)(v, V).
\]
Expanding and arranging the terms of the right hand side we obtain

\[ 2G_{g,\Omega}(\Gamma_G(g,\Omega)(u,U;v,V);\theta,\Theta) \]

\[ = \int_X \{ \text{Tr}_R[(U^*_\Omega - 2u^*_g)v^*_g \theta^*_g] + 2U^*_\Omega V^*_\Omega \theta^*_\Omega \} \Omega \]

\[ + \int_X \{ \text{Tr}_R[(V^*_\Omega - 2v^*_g)u^*_g \theta^*_g] + 2V^*_\Omega U^*_\Omega \theta^*_\Omega \} \Omega \]

\[ - \int_X \{ \text{Tr}_R[(\Theta^*_\Omega - 2\theta^*_g)u^*_g v^*_g] + 2\Theta^*_\Omega U^*_\Omega V^*_\Omega \} \Omega \]

\[ = \int_X \text{Tr}_R[(u^*_g(V^*_\Omega - v^*_g) + v^*_g(U^*_\Omega - u^*_g)) \theta^*_g] \Omega \]

\[ - \int_X \text{Tr}_R[u^*_g v^*_g] - 2U^*_\Omega V^*_\Omega \theta^*_\Omega \Omega \]

\[ = \int_X \langle u(V^*_\Omega - v^*_g) + v(U^*_\Omega - u^*_g), \theta \rangle_g \Omega \]

\[ - 2 \int_X \frac{1}{2} \langle u,v \rangle_g - U^*_\Omega V^*_\Omega - \frac{1}{2} G_{g,\Omega}(u,U;v,V) |\Theta^*_\Omega \Omega, \]

since \( \int_X \Theta = 0 \). We infer the expression

\[ (\psi, \Psi) = \Gamma_G(g,\Omega)(u,U;v,V), \]

\[ \psi = \frac{1}{2} [u(V^*_\Omega - v^*_g) + v(U^*_\Omega - u^*_g)], \]

\[ \Psi = \frac{1}{4} [\langle u,v \rangle_g - 2U^*_\Omega V^*_\Omega - G_{g,\Omega}(u,U;v,V)] \Omega. \]

This concludes the computation of the Levi-Civita connection \( \nabla_G \).

7 The second variation of the \( \mathcal{W} \) functional with respect to the pseudo-Riemannian structure \( G \)

We justify first the geometric interpretation of \( \mathbb{F}_{g,\Omega} \) provided by the identity \[117\]. We observe indeed that \( (v,V) \in T_{[g,\Omega]}^{\perp G}(g,\Omega) \) if and only if

\[ G_{g,\Omega}(L_{\xi}g, L_{\xi}\Omega; v,V) = 0, \]
for all $\xi \in C^\infty(X, T_X)$, i.e

$$0 = \int_X \left[ (L_\xi g, v)_g - 2(L_\xi \Omega)_\Omega V^*_g \right] \Omega$$

$$= 2 \int_X \left[ (\nabla_g \xi, v^*_g) - (\text{div}^\Omega \xi) V^*_g \right] \Omega$$

$$= 2 \int_X \langle \xi, \nabla^\omega v^*_g + \nabla_g V^*_\Omega \rangle \Omega,$$

which shows the required conclusion. We introduce now the operator

$$L^\Omega_g : C^\infty(X, \text{End}(T_X)) \rightarrow C^\infty(X, \text{End}(T_X)),$$

defined by the formula

$$L^\Omega_g A := \Delta^\Omega_g A - 2R_g * A.$$

By abuse of notations we define also

$$L^\Omega_g : C^\infty(X, S^2 T^*_X) \rightarrow C^\infty(X, S^2 T^*_X),$$

defined by the same formula

$$L^\Omega_g v := \Delta^\Omega_g v - 2R_g * v.$$

We observe that (3.3) implies the identity $(L^\Omega_g v)_g = L^\Omega_g v^*_g$. We show now the second variation formula for the $W$ functional.

**Lemma 7** The Hessian endomorphism $\nabla^2 W(g, \Omega)$ of the $W$ functional with respect to the pseudo-Riemannian structure $G$ at the point $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$ in the directions $(v, V) \in \mathcal{F}_{g, \Omega}$ is given by the expressions

$$(u, U) = \nabla^2 W(g, \Omega)(v, V),$$

$$u := -\frac{1}{2} (L^\Omega_g + H_{g, \Omega}) v - \frac{1}{2} V^*_\Omega h_{g, \Omega},$$

$$U^*_\Omega := -\frac{1}{2} (\Delta^\Omega_g + H_{g, \Omega} - 2\Pi) V^*_\Omega + \frac{1}{4} \langle h_{g, \Omega}, v \rangle_g + \frac{1}{4} D_{g, \Omega} W(v, V).$$

In particular if $h_{g, \Omega} = 0$ then

$$u = -\frac{1}{2} L^\Omega_g v,$$

$$U^*_\Omega = -\frac{1}{2} (\Delta^\Omega_g - 2\Pi) V^*_\Omega.$$
Proof We consider a smooth curve \((g_t, \Omega_t)_{t \in \mathbb{R}} \subset \mathcal{M} \times \mathcal{V}_1 \) with \((g_0, \Omega_0) = (g, \Omega)\) and with arbitrary speed \((\dot{g}_0, \dot{\Omega}_0) = (v, V)\). We observe that the \(G\)-covariant derivative of its speed is given by the expressions

\[
(\theta_t, \Theta_t) = \nabla_G(\dot{g}_t, \dot{\Omega}_t)(\dot{g}_t, \dot{\Omega}_t) = (\ddot{g}_t, \ddot{\Omega}_t) + \Gamma(g_t, \Omega_t)(\dot{g}_t, \dot{\Omega}_t),
\]

\[
\theta_t := \ddot{g}_t + \dot{g}_t \left( \dot{\Omega}_t^* - \dot{g}_t^* \right),
\]

\[
\Theta_t := \ddot{\Omega}_t + \frac{1}{4} \left[ |\dot{g}_t|^2 - 2(\dot{\Omega}_t^*)^2 - G_{g_t, \Omega_t}(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t) \right] \Omega_t.
\]

We infer

\[
\theta_t^* = \frac{d}{dt} \dot{g}_t^* + \dot{\Omega}_t^* \dot{g}_t^*,
\]

\[
\Theta_t^* = \frac{d}{dt} \dot{\Omega}_t^* + \frac{1}{2} (\dot{\Omega}_t^*)^2 + \frac{1}{4} |\dot{g}_t|^2 - \frac{1}{4} G_{g_t, \Omega_t}(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t).
\]

Using this expressions and Perelman's first variation formula we expand the Hessian form

\[
\nabla_G DW(g_t, \Omega_t)(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t)
\]

\[
= \frac{d^2}{dt^2} W(g_t, \Omega_t) - D_{g_t, \Omega_t} W(\theta_t, \Theta_t)
\]

\[
= - \frac{d}{dt} \int_X \left[ \text{Tr}_R (\dot{g}_t^* h_t^*) - 2 \dot{\Omega}_t^* H_t \right] \Omega_t
\]

\[
+ \int_X \left[ \text{Tr}_R (\theta_t^* h_t^*) - 2 \Theta_t^* H_t \right] \Omega_t
\]

\[
= - \int_X \left[ \text{Tr}_R \left( \frac{d}{dt} \dot{g}_t^* h_t^* + \dot{g}_t^* \frac{d}{dt} h_t^* \right) - 2 \frac{d}{dt} \dot{\Omega}_t^* H_t - 2 \dot{\Omega}_t^* H_t \right] \Omega_t
\]

\[
- \int_X \left[ \text{Tr}_R (\dot{g}_t^* h_t^*) - 2 \dot{\Omega}_t^* H_t \right] \dot{\Omega}_t
\]

\[
+ \int_X \left[ \text{Tr}_R (\theta_t^* h_t^*) - 2 \Theta_t^* H_t \right] \Omega_t
\]

\[
= - \int_X \left\{ \text{Tr}_R \left[ \dot{g}_t^* \left( \dot{h}_t^* - \dot{g}_t^* h_t^* \right) \right] - 2 \dot{\Omega}_t^* H_t \right\} \Omega_t
\]

\[
- \frac{1}{2} \int_X \left[ |\dot{g}_t|^2 - 2(\dot{\Omega}_t^*)^2 - G_{g_t, \Omega_t}(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t) \right] H_t \Omega_t.
\]
Using the variation formulas (1.4) and (1.5) and evaluating the previous identity at time $t = 0$ we obtain the expression

$$\nabla_G DW(g, \Omega)(v; v, V) = \frac{1}{2} \int_X \left[ \left( L^g v - L_{\nabla^g v} g \right) \right] \Omega$$

$$- \frac{1}{2} \int_X \left\{ -2V^\alpha \left( (\Delta^\Omega \Omega - 2\Omega) V^\beta \right) - \text{div}^\Omega (\nabla^g v^\alpha + \nabla_g V^\alpha) - (v, h_g, \Omega) \right\} \Omega$$

$$- \frac{1}{2} \int_X \left[ \left| v \right|^2 - 2\left( V^\alpha \right)^2 \right] \Omega,$$

since $\int_X H_{g,\Omega} \Omega = 0$. Arranging symmetrically the integrand terms via the identity

$$\nabla_G DW(g, \Omega)(v; v, V) = G_{g,\Omega}(u; v, V),$$

$$\left( u, U \right) = \nabla^2 G W(g, \Omega)(v, V),$$

we infer the general expressions

$$u = -\frac{1}{2} \left( L^g + H_{g,\Omega} \right) v + \frac{1}{2} L_{\nabla^g v} g - \frac{1}{2} V^\alpha h_g, \Omega,$$

$$U^\alpha = -\frac{1}{2} \left( \Delta^\Omega - H_{g,\Omega} - 2\Omega \right) V^\beta$$

$$+ \frac{1}{2} \left( L_{\nabla^g v} + \nabla_g V \right)^\beta \Omega + \frac{1}{4} \Omega (h_g, \Omega, v) g + \frac{1}{4} D_g, \Omega W(v, V).$$

Then the required expression of the Hessian of $W$ follows from the assumption $(v, V) \in \mathbb{F}_{g,\Omega}$. If $h_g, \Omega = 0$ then the required conclusion follows from Perelman’s twice contracted second Bianchi identity (1.2) which implies $H_{g,\Omega} = 0$. □

8 The anomaly space of the pseudo-Riemannian structure $G$

Let $\text{Isom}^0_{g,\Omega}$ be the identity component of the group

$$\text{Isom}_{g,\Omega} := \{ \varphi \in \text{Diff}(X) \mid \varphi^* g = g, \varphi^* \Omega = \Omega \},$$

and let

$$\text{Kill}_{g,\Omega} := \text{Lie} (\text{Isom}^0_{g,\Omega}) \equiv \{ \xi \in C^\infty(X, TX) \mid L_\xi g = 0, L_\xi \Omega = 0 \}.$$
We define the anomaly space of the pseudo-Riemannian structure $G$ at an arbitrary point $(g, \Omega)$ as the vector space

$$\mathcal{A}_g^\Omega := \mathbb{F}_g \cap T_g \Omega, \Omega.$$ 

We will study some properties of this space. It is clear by definition that this space is generated by the vector fields $\xi \in C^\infty \left( X, T^* X \right)$ such that

$$0 = \left[ \nabla^*_g \Omega \right] \left( g\xi \right) = \hat{\Delta}_g^\Omega \left( g\xi \right).$$

More precisely there exists the exact sequence of finite dimensional vector spaces

$$0 \rightarrow \text{Kill}_g \Omega \rightarrow \text{Ker} \hat{\Delta}_g^\Omega \rightarrow \mathcal{A}_g^\Omega \rightarrow 0$$

$$\xi \mapsto \ g\xi = \alpha \mapsto \left( \nabla^*_g \alpha, (\text{div}^\Omega g \alpha) \Omega \right).$$

We observe that if $\alpha = du \in \text{Ker} \hat{\Delta}_g^\Omega$ then the function $u$ satisfies the equation

$$2\Delta_g^\Omega \nabla_g u - \nabla_g \Delta_g^\Omega u = 0,$$

which is equivalent to the equation

$$\left[ \Delta_g^\Omega - \text{Ric}_g^* (\Omega) \right] \nabla_g u = 0,$$  \hspace{1cm} (8.1)

thanks to the general identity

$$\nabla_g \Delta_g^\Omega u = \Delta_g^\Omega \nabla_g u + \text{Ric}_g^* (\Omega) \nabla_g u.$$  \hspace{1cm} (8.2)

Let $C^\infty_0 \left( X, \mathbb{R} \right)_0$ be the space of smooth functions with zero integral with respect to $\Omega$. We set

$$\mathbb{V}_{g, \Omega} := \left\{ \alpha \in \text{Ker} \hat{\Delta}_g^\Omega \mid \alpha = du \right\}$$

$$\cong \left\{ u \in C^\infty_0 \left( X, T^*_X \right)_0 \mid \left[ \Delta_g^\Omega - \text{Ric}_g^* (\Omega) \right] \nabla_g u = 0 \right\}.$$

We observe that in the soliton case $h_{g, \Omega} = 0$ we have

$$\mathbb{V}_{g, \Omega} \cong \text{Ker} (\Delta_g^\Omega - 2\mathbb{I}) \subset C^\infty_0 \left( X, \mathbb{R} \right)_0,$$  \hspace{1cm} (8.3)

thanks to the identity \hspace{1cm} (8.2). By duality we can consider $\text{Kill}_{g, \Omega} \subset \text{Ker} \hat{\Delta}_g^\Omega$ and we observe the inclusion

$$\mathbb{V}_{g, \Omega} \subset \text{Kill}_{g, \Omega}^\perp,$$  \hspace{1cm} (8.4)

where the symbol $\perp$ indicates the orthogonal space inside $\text{Ker} \hat{\Delta}_g^\Omega$ with respect to the scalar product at the level of 1-forms. The previous inclusion hold true for any $(g, \Omega)$ since

$$\int_X \left( du, \beta \right)_g \Omega = - \int_X \left( u, \text{div}_g^\Omega \beta \right)_g \Omega = 0.$$

35
for any $\beta \in \text{Kill}_g,\Omega$. We infer that in the soliton case the previous exact sequence can be reduced to the sequence

$$0 \longrightarrow \text{Ker}(\Delta^\Omega - 2\mathbb{I}) \longrightarrow A^\Omega_g$$

$$u \mapsto 2(\nabla_g du, -u\Omega).$$

In order to show that the previous map is also surjective we need to show a few differential identities. We show first the Weitzenböck type formula

$$\hat{\Delta}^\Omega_g \alpha = \Delta^\Omega_g \alpha - \alpha \text{Ric}_g^*(\Omega). \quad (8.5)$$

(This implies in particular the identification of $C^\Omega_g,\Omega$ in terms of functions). We decompose the expression

$$\hat{\Delta}^\Omega_g \alpha = \left[\nabla^*_g \nabla_g - d\nabla^*_g\right] \alpha. \quad (8.6)$$

We decompose first the therm

$$\nabla^*_g \nabla_g \alpha \cdot \xi = \nabla^*_g \nabla_g \alpha \cdot \xi + \nabla_g f(\nabla_g f, \xi).$$

We fix an arbitrary point $p$ and we choose the vector fields $\xi$ and $\eta$ such that $0 = \nabla_g \xi(p) = \nabla_g \eta(p)$. Let $(e_k)_k$ be a $g$-orthonormal local frame such that $\nabla_g e_k(p) = 0$. Then at the point $p$ hold the identities

$$\nabla_g \nabla_g \alpha \cdot \xi = -\nabla_{g,ek} \nabla_g \alpha(e_k, \xi)$$

$$= -\nabla_{g,ek} \left[\nabla_g \alpha(e_k, \xi)\right]$$

$$= -\nabla_{g,ek} \left[\nabla_{g,ek} \alpha \cdot \xi + \nabla_{g,ek} \alpha \cdot \xi\right]$$

$$= -\nabla_{g,ek} \nabla_{g,ek} \alpha \cdot \xi - \nabla_{g,ek} \nabla_{g,ek} \alpha \cdot \xi.$$

We infer the expression

$$\nabla^*_g \nabla_g \alpha \cdot \xi = \Delta^\Omega_g \alpha - \nabla_{g,ek} \nabla_{g,ek} \alpha \cdot e_k + \nabla_g f(\nabla_g f, \xi).$$

Moreover

$$d\nabla^*_g \alpha(\xi) = -\nabla_{g,ek} \nabla_{g,ek} \alpha \cdot e_k + \nabla_{g,ek} \alpha \cdot \nabla_g f + \alpha \cdot \nabla_{g,ek} \alpha \cdot \nabla_g f.$$

Summing up we deduce

$$\hat{\Delta}^\Omega_g \alpha \cdot \xi = \Delta^\Omega_g \alpha \cdot \xi + \left(\nabla_{g,ek} \nabla_{g,ek} \alpha - \nabla_{g,ek} \nabla_{g,ek} \alpha\right) \cdot e_k - \alpha \cdot \nabla_{g,ek} \alpha \cdot \nabla_g f$$

$$= \Delta^\Omega_g \alpha \cdot \xi - \alpha \cdot R_g(\xi, e_k) e_k - \alpha \cdot \nabla_{g,ek} \alpha \cdot \nabla_g f,$$
thanks to the dual identity
\[ \nabla_{g,\xi} \nabla_{g,\eta} \alpha - \nabla_{g,\eta} \nabla_{g,\xi} \alpha = \nabla_{g,[\xi,\eta]} \alpha - \alpha \cdot R_g(\xi, \eta), \] (8.7)
and to the fact that \([\xi, e_k](p) = 0\). We infer the required formula (8.5). We deduce that in the soliton case \(h_{g,\Omega} = 0\) hold the equality
\[ \text{Ker} \hat{\Delta}^\Omega_g = \text{Ker}(\Delta^\Omega_g - \mathbb{1}) \subset C^\infty(X, T^*_X). \] (8.8)

We define now the \(\Omega\)-Hodge Laplacian acting on scalar valued differential forms as the operator
\[ \Delta^\Omega_{d,g} := d\nabla^\omega + \nabla^\omega d. \]
At the level of scalar valued 1-forms we observe the identities
\[ \left( \Delta^\Omega_{d,g} + \hat{\Delta}^\Omega_g \right) \alpha = \nabla^\omega \left( d + \hat{\nabla}_g \right) \alpha = 2\nabla^\omega \nabla_g \alpha = 2\Delta^\Omega_g \alpha. \]
We infer thanks to the identity (8.5) that for any scalar valued 1-form \(\alpha\) hold the Weitzenbőck type formula
\[ \Delta^\Omega_g \alpha = \Delta^\Omega_{d,g} \alpha - \alpha \mathcal{Ric}^*_g(\Omega). \] (8.9)

Applying the \(\nabla^\omega\)-operator to both sides of this identity and using the fact that \((\nabla^\omega)^2 = 0\) at the level of scalar valued differential forms we obtain
\[ \nabla^\omega \Delta^\Omega_g \alpha = \Delta^\Omega_g \nabla^\omega \alpha - \nabla^\omega \left[ \alpha \mathcal{Ric}^*_g(\Omega) \right]. \]

In the soliton case \(h_{g,\Omega} = 0\) this implies the formula
\[ \nabla^\omega \Delta^\Omega_g \alpha = \Delta^\Omega_g \nabla^\omega \alpha - \nabla^\omega \alpha. \] (8.10)

Then the identity (8.8) implies that the map
\[ \text{Ker} \hat{\Delta}^\Omega_g \rightarrow \text{Ker}(\Delta^\Omega_g - 2\mathbb{1}) \subset C^\infty(X, \mathbb{R})_0 \]
\[ \alpha \mapsto \text{div}^\omega \alpha, \]
is well defined. More precisely there exists the exact sequence of finite dimensional vector spaces
\[ 0 \rightarrow \text{Kill}_{g,\Omega} \rightarrow \text{Ker} \hat{\Delta}^\Omega_g \rightarrow \text{Ker}(\Delta^\Omega_g - 2\mathbb{1}) \rightarrow 0 \]
\[ \xi \mapsto g\xi = \alpha \mapsto \text{div}^\omega \alpha. \]

Indeed the surjectivity follows from the isomorphism (8.3). The injectivity follows from the fact that
\[ \text{Kill}_{g,\Omega} \cong \left\{ \alpha \in \text{Ker} \hat{\Delta}^\Omega_g \mid \text{div}^\omega \alpha = 0 \right\}. \]
This holds true thanks to the identity
\[ \int_X \left| \text{div}_g \alpha \right|^2 \Omega = \frac{1}{2} \int_X \left| \nabla_g \alpha \right|^2 \Omega, \]
which follows from the expression
\[ \Delta_g^\Omega \alpha = \left[ \frac{1}{2} \nabla_g^* \nabla_g - d \nabla_g^* \right] \alpha. \]
For dimensional reasons we conclude the existence of the required exact sequence
\[ 0 \to \text{Ker}(\Delta_g^\Omega - 2\mathbb{I}) \to \mathcal{A}_g^\Omega \to 0 \]
\[ u \mapsto 2(\nabla_g du, -u\Omega). \]
(We observe also that for dimensional reasons (8.4) is an equality.)

9 Properties of the kernel of the Hessian of \( \mathcal{W} \)

Lemma 8 In the soliton case \( h_g,\Omega = 0 \) hold the inclusion
\[ \mathcal{A}_g^\Omega \subseteq F_{g,\Omega} \cap \text{Ker} \nabla_g^2 \mathcal{W}(g,\Omega). \]

We start with a few notations. For any tensor \( A \in C^\infty(X, (T^*_X)^{\otimes p+1} \otimes T_X) \) we define the divergence type operations
\[ \text{div}_g A(u_1, \ldots, u_p) := \text{Tr}_g \left[ \nabla_g A(\cdot, u_1, \ldots, u_p, \cdot) \right], \]
\[ \text{div}_g^\Omega A(u_1, \ldots, u_p) := \text{div}_g A(u_1, \ldots, u_p) - A(u_1, \ldots, u_p, \nabla g f). \]
The once contracted differential Bianchi identity writes often as \( \text{div}_g^\Omega R_g = -\nabla_{T_X,g} \text{Ric}^*_g. \) This combined with the identity \( \nabla_{T_X,g} \nabla^2 g f = R_g \cdot \nabla g f \) implies
\[ \text{div}_g^\Omega R_g = -\nabla_{T_X,g} \text{Ric}^*_g(\Omega). \] (9.1)

We define the \( \Omega \)-Lichnerowicz Laplacian \( \Delta_L^\Omega \) acting on \( g \)-symmetric endomorphisms \( A \) as
\[ \Delta_L^\Omega A := \mathcal{L}_g^\Omega A + \text{Ric}^*_g(\Omega) A + A \text{Ric}^*_g(\Omega). \]
Let now \( (x_1, \ldots, x_m) \) be \( g \)-geodesic coordinates centered at an arbitrary point \( p \) and set \( e_k := \frac{\partial}{\partial x_k}. \) Then the local frame \( (e_k)_k \) is \( g(p) \)-orthonormal at the point \( p \) and satisfies \( \nabla g e_k(p) = 0 \) for all \( k \). We take now an arbitrary vector field \( \xi \) with constant coefficients with respect to the \( g \)-geodesic coordinates \( (x_1, \ldots, x_m) \).

Therefore \( \nabla g \xi(p) = 0 \). We expand the identity at the point \( p \)
\[ (\Delta_L^\Omega \nabla^2 g u) \xi = -\nabla_{g,e_k} \nabla g^3 u(e_k, \xi) + \nabla g^3 u(\nabla g f, \xi). \]
Commuting derivatives at the point \( p \) we obtain
\[
\nabla_{g,e_k} \nabla^3_u(e_k;\xi) = \nabla_{g,e_k} \left[ \nabla^3_u(e_k;\xi) \right]
\]
\[
= \nabla_{g,e_k} \left[ \nabla_{g,e_k} \nabla_{g,e_k} \nabla u - \nabla^2_g u \cdot \nabla_{g,e_k} \xi \right]
\]
\[
= \nabla_{g,e_k} \left[ \nabla_{g,e_k} \nabla_{g,e_k} \nabla u + R_g(e_k,\xi) \nabla_g u - \nabla^2_g u \cdot \nabla_{g,e_k} \xi \right]
\]
\[
= \nabla_{g,e_k} \nabla_{g,e_k} \nabla_g u + 2 R_g(e_k,\xi) \nabla_{g,e_k} \nabla_g u
\]
\[
+ \nabla_{g,e_k} R_g(e_k,\xi) \nabla_g u - \nabla^2_g u \cdot \nabla_{g,e_k} \nabla_{g,e_k} \xi,
\]
since \([e_k,\xi] \equiv 0\). The choice of \( g\)-geodesic coordinates centered at the point \( p \) implies the expansion
\[
\Delta_g \nabla u = -\nabla_{g,e_k} \nabla_{g,e_k} \nabla_g u + \nabla^2_g u \cdot \nabla_{g,e_k} e_k + O(|x|^2).
\]
Taking a covariant derivative of this identity we infer
\[
\nabla_{g,\xi} \Delta_g \nabla u = -\nabla_{g,\xi} \nabla_{g,e_k} \nabla_{g,e_k} \nabla_g u + \nabla^2_g u \cdot \nabla_{g,\xi} \nabla_{g,e_k} e_k,
\]
at the point \( p \). Combining with the previous expression we obtain
\[
\nabla_{g,e_k} \nabla^3_u(e_k;\xi) = -2 (R_g * \nabla^2_g u) \xi - \nabla_{g,\xi} \Delta_g \nabla_g u
\]
\[
+ (\text{Ric}^*(g)\xi) - \nabla^2_g u + \nabla_{g,e_k} R_g(e_k,\xi) \nabla_g u.
\]
On the other hand deriving the identity
\[
\Delta^\Omega_g \nabla u = \Delta_g \nabla u + \nabla^2_g u \cdot \nabla f,
\]
we infer
\[
\nabla_{g,\xi} \Delta^\Omega_g \nabla u = \nabla_{g,\xi} \Delta_g \nabla u + \nabla_{g,\xi} \nabla^2_g u \cdot \nabla f + \nabla^2_g u \cdot \nabla_{g,\xi} f,
\]
and thus
\[
\nabla_{g,e_k} \nabla^3_u(e_k;\xi) = -2 (R_g * \nabla^2_g u) \xi - \nabla_{g,\xi} \Delta^\Omega_g \nabla_g u + \nabla_{g,\xi} \nabla^2_g u \cdot \nabla f
\]
\[
+ \nabla^2_g u \cdot \text{Ric}^*_g(\Omega) \xi - \text{div}_g R_g(\xi, \nabla_g u) - (\nabla_g u - \nabla^*_g R_g) \xi,
\]
thanks to the algebraic Bianchi identity. We obtain
\[
(\Delta^\Omega_g \nabla^2_g u) \xi = 2 (R_g * \nabla^2_g u) \xi - \nabla_{TX,g} \nabla^2_g u(\xi, \nabla f)
\]
\[
+ \text{div}_g R_g(\xi, \nabla_g u) + (\nabla_g u - \nabla^*_g R_g) \xi
\]
\[
+ \nabla_{g,\xi} \Delta^\Omega_g \nabla_g u - \nabla^2_g u \cdot \text{Ric}^*_g(\Omega) \xi.
\]
The identity \( \nabla_{TX,g} \nabla_g^2 u = \mathcal{R}_g \cdot \nabla_g u \) implies

\[
- \nabla_{TX,g} \nabla_g^2 u(\xi, \nabla_g f) = \mathcal{R}_g(\nabla_g f, \xi) \nabla_g u
\]

\[
= -\mathcal{R}_g(\xi, \nabla_g u) \nabla_g f + \mathcal{R}_g(\nabla_g f, \nabla_g u) \xi,
\]

thanks again to the algebraic Bianchi identity. We infer

\[
(L^\Omega g \nabla_g^2 u) \xi = \left[ \nabla_g u - \left( \nabla_g^{*\Omega} \mathcal{R}_g - \text{div}_g \nabla_g \mathcal{R}_g \right) \right] \xi
\]

\[
+ \nabla_g \xi \Delta^\Omega g \nabla_g u - \nabla_g^2 u \cdot \text{Ric}_g^*(\Omega) \xi
\]

\[
= \left[ \nabla_g u - \left( \nabla_g^{*\Omega} \mathcal{R}_g + \nabla_{TX,g} \text{Ric}_g^*(\Omega) \right) \right] \xi
\]

\[
+ (\nabla_g^2 \Delta^\Omega g) \xi - \nabla_g \xi [\text{Ric}_g^*(\Omega) \nabla_g u] - \nabla_g^2 u \cdot \text{Ric}_g^*(\Omega) \xi,
\]

thanks to (9.1) and (8.2). Thus

\[
\Delta^\Omega_{L,g} \nabla_g^2 u = \nabla_g^2 \Delta^\Omega g u + \nabla_g u - \left[ \nabla_g^{*\Omega} \mathcal{R}_g + \nabla_{TX,g} \text{Ric}_g^*(\Omega) \right] - 2\nabla_g \text{Ric}_g^*(\Omega) \nabla_g u. \quad (9.2)
\]

We observe now that the endomorphism section \( \nabla_g u \nabla^*_g \nabla_g^{*\Omega} \mathcal{R}_g \) is \( g \)-anti-symmetric thanks to the identity

\[
\mathcal{R}_g(\xi, \eta) = - (\mathcal{R}_g(\xi, \eta))^T_g,
\]

which is a consequence of the alternating property of the \( (4,0) \)-Riemann curvature operator. Notice indeed that the previous identity implies

\[
\nabla_g \mu \mathcal{R}_g(\xi, \eta) = - (\nabla_g \mu \mathcal{R}_g(\xi, \eta))^T_g,
\]

for all vector fields \( \xi, \eta, \mu \). Combining the \( g \)-symmetric and \( g \)-anti-symmetric parts in the identity (9.2) we infer the formulas

\[
\Delta^\Omega_{L,g} \nabla_g^2 u = \nabla_g^2 \Delta^\Omega g u + \nabla_g u - \nabla_{TX,g} \text{Ric}_g^*(\Omega) - [\nabla_g \text{Ric}_g^*(\Omega) \nabla_g u]^T_g,
\]

\[
\xi - \nabla_g^{*\Omega} \mathcal{R}_g = \nabla_g \text{Ric}_g^*(\Omega) \xi - [\nabla_g \text{Ric}_g^*(\Omega) \xi]^T_g,
\]

for all \( \xi \in TX \) since the function \( u \) is arbitrary. In the case \( \nabla_g \text{Ric}_g^*(\Omega) = 0 \) we deduce the identities \( \Delta^\Omega_{L,g} \nabla_g^2 u = \nabla_g^2 \Delta^\Omega g u \) and \( \nabla_g^{*\Omega} \mathcal{R}_g = 0 \). More in particular in the soliton case \( h_{\mu,u} = 0 \) the first formula reduces to the differential identity

\[
L^\Omega g \nabla_g^2 u = \nabla_g^2 (\Delta^\Omega g - 2\mu) u. \quad (9.3)
\]

We infer the conclusion of lemma 8. This formula will be also quite crucial for the study of the sign of the second variation of the \( \mathcal{W} \) functional at a Kähler-Ricci soliton point.
10 \hspace{1cm} \textbf{Invariance of} \ \mathbb{F} \ \textbf{under the action of the Hessian endomorphism of} \ \mathcal{W}

We observe that Perelman's twice contracted second Bianchi type identity (1.2) rewrites as:

\begin{equation}
\nabla^*_g h_g, \Omega + dH_g, \Omega = 0.
\end{equation}

If we differentiate this over the space \( \mathcal{M} \times \mathcal{V}_1 \) we obtain

\[ [(D_g, h) \nabla^*_g v, V)] h_g, \Omega + \nabla^*_g [(D_g, h) v, V] + d[D_g, h] (v, V) = 0. \]

We deduce using the fundamental variation formulas (1.4) and (1.5)

\[ \nabla^*_g \left( L_g^* v + \nabla^*_g h_g, \Omega v^* + \nabla^*_g \Omega \right) \]

\[ = -2 [D_g, h] \nabla^*_g (v, V)] h_g, \Omega \]

in the directions \((v, V) \in \mathbb{F}_g, \Omega\). We infer that in the soliton case \( h_g, \Omega = 0 \) the map

\[ \nabla^2_G \mathbb{W}(g, \Omega) : \mathbb{F}_g, \Omega \to \mathbb{F}_g, \Omega, \]

is well defined. In order to investigate the general case we use a different method which has the advantage to involve less computations. Let \((e_k)_k\) be a \(g\)-orthonormal local frame of \(T_X\). For any \(u, v \in C^\infty (X, S^2 T^* X)\) we define the real valued 1-form

\[ M_g(u, v)(\xi) := 2 \nabla_g v(e_k, u^*_g e_k, \xi) + \nabla_g u(\xi, v^*_g e_k, e_k) , \]

for all \(\xi \in T_X\). One can show that the operator

\[ T_g(u, v) := M_g(u, v) - M_g(v, u), \]

is related with the torsion of the distribution \(\mathbb{F}\). We observe now that by lemma \(\exists\) hold the identity

\[ \Delta^*_g v^* - \nabla_g \nabla^*_g v^* = \nabla^*_g \nabla^* T_X, g v^* + \mathcal{R}_g * v^* - v^* \text{Ric}_g^*(\Omega). \]

Applying the \(\nabla^*_g\)-operator to both sides of this identity we deduce the commutation formula

\[ [\nabla^*_g, \Delta_g] v^* = \nabla^*_g \left[ \nabla^*_g \nabla^* T_X, g v^* + \mathcal{R}_g * v^* - v^* h_g, \Omega - v^* \right]. \]
We observe now that for any $\psi \in C^\infty(X, \Lambda^2 T_X \otimes \mathbb{R} T_X)$ and $\xi \in C^\infty(X, T_X)$ hold the equalities
\[
\int_X \langle (\nabla^* g)^2 \psi, \xi \rangle_g \Omega = \int_X \langle \nabla^* g \psi, \nabla g \xi \rangle_g \Omega
\]
\[
= \frac{1}{2} \int_X \langle \nabla_{T_X,g}^2 \psi, \nabla g \xi \rangle_g \Omega
\]
\[
= \frac{1}{2} \int_X \langle \nabla^* g \psi, \nabla g \xi \rangle_g \Omega
\]
\[
= \frac{1}{2} \int_X \langle \psi, \nabla g \xi \rangle_g \Omega
\]
and
\[
\langle \psi, \mathcal{R}_g \cdot \xi \rangle_g = \langle \psi(e_k, e_l), \mathcal{R}_g(e_k, e_l) \xi \rangle_g = -\langle \mathcal{R}_g(e_k, e_l) \psi(e_k, e_l), \xi \rangle_g.
\]
We infer
\[
(\nabla^* g)^2 \nabla_{T_X,g}^2 v_g^* = -\frac{1}{2} \mathcal{R}_g(e_k, e_l) \left[ \nabla_g v_g^*(e_k, e_l) - \nabla_g v_g^*(e_l, e_k) \right]
\]
\[
= \mathcal{R}_g(e_k, e_l) \nabla_g v_g^*(e_k, e_l).
\]
This combined with the expression
\[
\nabla^* g (\mathcal{R}_g * v_g^*) = \nabla^* g \mathcal{R}_g(e_k) v_g^* e_k + \mathcal{R}_g(e_l, e_k) \nabla_g v_g^*(e_k, e_l),
\]
implies the identity
\[
\nabla^* g \mathcal{L}_g \nabla^* g = (\mathcal{D}_g^\Omega - \Box) \nabla^* g v_g^* + \nabla_g v_g^*(e_k, h_g^* e_k)
\]
\[
- \nabla^* g \mathcal{R}_g(e_k) v_g^* e_k,
\]
which rewrites also under the form
\[
\nabla^* g \mathcal{L}_g v_g^* = (\mathcal{D}_g^\Omega - \Box) \nabla^* g v + \nabla_g v(e_k, h_g^* e_k, \bullet)
\]
\[
- v \nabla^* g h_g^* e_k + v(e_k, \nabla^* g \mathcal{R}_g(e_k) \bullet),
\]
thanks to (3.3) and the anti-symmetry property
\[
e_k - \nabla^* g \mathcal{R}_g = -\left(e_k - \nabla^* g \mathcal{R}_g \right)^T.
\]
On the other hand the once contracted differential Bianchi type identity (9.1) rewrites as
\[
-\nabla_{T_X,g} \mathcal{R}_g^*(\Omega) = \text{Alt} \left( \nabla^* g \mathcal{R}_g \right),
\]
thanks to the algebraic Bianchi identity. Therefore for any \( \xi \in C^\infty(X, T_X) \) hold the identities

\[
\nabla^\alpha g R_g(e_k, \nabla^* g \Omega, g R_g(e_k, \xi)) = \nabla^\alpha g R_g(e_k, \xi) + \nabla^* g(R_g^{\star}(\Omega) \xi, e_k) = \text{Tr}_g \left[ v(\xi \nabla^* g \Omega, g R_g(e_k, \xi)) \right] = \text{Tr}_g \left[ v(\xi \nabla^* g \Omega, h^g_{\star,\Omega}) \right],
\]

since the endomorphism section \( \nabla^\alpha g R_g(\xi) \) is \( g \)-anti-symmetric. Notice indeed that if \( A, B \in C^\infty(X, \text{End}(T_X)) \) satisfy \( A = A^T g \) and \( B = -B^T g \) then

\[
\text{Tr}_g(AB) = \text{Tr}_g(BA) = \text{Tr}_g(A^T g B^T g) = -\text{Tr}_g(AB),
\]

i.e \( \text{Tr}_g(AB) = 0 \). We deduce in conclusion the formula

\[
\nabla^\alpha g L^g_{\Omega}(\varphi) = (\Delta^g_{\Omega} - \nabla^* g \varphi - \nabla^* g h^g_{\star,\Omega}) + \nabla_g v(e_k, h^g_{\star,\Omega}), \quad \text{Tr}_g \left[ v(\xi \nabla^* g \Omega, h^g_{\star,\Omega}) \right] = \nabla_g v(e_k, h^g_{\star,\Omega}), \quad \text{Tr}_g \left[ v(\xi \nabla^* g \Omega, h^g_{\star,\Omega}) \right].
\]

Using the general formula

\[
\nabla^\alpha g (\varphi v) = -\nabla_g \varphi + \varphi \nabla^\alpha g v,
\]

with \( \varphi \in C^\infty(X, \mathbb{R}) \) we infer

\[
\nabla^\alpha g [H_{\star,\Omega} v + V^g_{\star,\Omega} h^g_{\star,\Omega}] = -\nabla_g H_{\star,\Omega} + H_{\star,\Omega} \nabla^\alpha g v - h^g_{\star,\Omega} \nabla_g V^g_{\star,\Omega} + V^g_{\star,\Omega} \nabla^\alpha g h^g_{\star,\Omega}
\]

thanks to Perelman’s twice contracted differential Bianchi type identity (1.2). Using the identity (15.1) we expand the term

\[
\nabla_g \left[ (\Delta^g_{\Omega} + H_{\star,\Omega} - 2 \nabla^* g) V^g_{\star,\Omega} - \frac{1}{2} \langle v, h^g_{\star,\Omega} \rangle_{g} \right]
\]

\[
= \Delta^g_{\Omega} V^g_{\star,\Omega} + dV^g_{\star,\Omega} \cdot \text{Ric}^g_{\star}(\Omega) + V^g_{\star,\Omega} dH_{\star,\Omega} + H_{\star,\Omega} dV^g_{\star,\Omega} - 2dV^g_{\star,\Omega} - \frac{1}{2} d \langle v, h^g_{\star,\Omega} \rangle_{g}
\]

\[
= (\Delta^g_{\Omega} + H_{\star,\Omega} - 4) dV^g_{\star,\Omega} + dV^g_{\star,\Omega} \cdot h^g_{\star,\Omega} + V^g_{\star,\Omega} dH_{\star,\Omega} - \frac{1}{2} d \langle v, h^g_{\star,\Omega} \rangle_{g}.
\]

43
Summing up we infer
\[ \nabla^* g \left[ (\mathcal{L}_g^\Omega + H_g^\Omega) v + V^*_\Omega h_g^\Omega \right] \]
\[ + d \left[ (\Delta_g^\Omega + H_g^\Omega - 2\Pi) V^*_\Omega - \frac{1}{2} \langle v, h_g^\Omega \rangle_g \right] \]
\[ = (\Delta_g^\Omega + H_g^\Omega - \Pi) (\nabla^*_g v + dV^*_\Omega) \]
\[ + \nabla_g v(e_k, h_g^* e_k, \xi) + Tr_g \left[ v (\bullet - \nabla_{T_X} h_g^* \Omega) \right] - \frac{1}{2} d \langle v, h_g^\Omega \rangle_g. \]

We observe now the identity
\[ \nabla_g v(e_k, h_g^* e_k, \xi) + Tr_g \left[ v (\bullet - \nabla_{T_X} h_g^* \Omega) \right] - \frac{1}{2} d \langle v, h_g^\Omega \rangle_g = \frac{1}{2} T_g(h_g^\Omega, v). \]

We deduce the formula
\[ \nabla^*_g \left[ (\mathcal{L}_g^\Omega + H_g^\Omega) v + V^*_\Omega h_g^\Omega \right] \]
\[ + d \left[ (\Delta_g^\Omega + H_g^\Omega - 2\Pi) V^*_\Omega - \frac{1}{2} \langle v, h_g^\Omega \rangle_g \right] \]
\[ = (\Delta_g^\Omega + H_g^\Omega - \Pi) (\nabla^*_g v + dV^*_\Omega) + \frac{1}{2} T_g(h_g^\Omega, v). \]

Setting \((v, V) = (h_g^\Omega, H_g^\Omega \Omega) \in \mathbb{F}_{g, \Omega}\) in the previous identity we infer
\[ 0 = \nabla^*_g \left[ (\mathcal{L}_g^\Omega + 2 H_g^\Omega) h_g^\Omega \right] \]
\[ + d \left[ (\Delta_g^\Omega + H_g^\Omega - 2\Pi) H_g^\Omega - \frac{1}{2} |h_g^\Omega|^2 \right]. \]
This shows the fundamental property (1.8) of the Soliton-Ricci flow.

## 11 The Kähler set up

In this sections we introduce a few basic notations needed in sequel. Let \((X, J, g)\) be a compact connected Kähler manifold with symplectic form \(\omega := gJ\). Let \(h := g - igJ = 2g^{1,0} \) be the hermitian metric over \(T_{X,J}\) induced by \(g\). We remind that in the Kähler case the Chern connection
\[ D^g_{T_{X,J}} = \partial^g_{T_{X,J}} + \partial^*_{T_{X,J}} : \quad C^\infty(T_{X,J}) \rightarrow C^\infty(T_X^* \otimes_{\mathbb{R}} T_{X,J}), \]
of the hermitian vector bundle \((T_{X,J}, h)\) coincides with the Levi-Civita connection \(\nabla_g\). We set \(\mathcal{C}T_X := T_X \otimes_{\mathbb{C}} \mathcal{C}\) and \(\mathcal{C}T_X^* := T_X^* \otimes_{\mathbb{C}} \mathcal{C}\). We observe further that the sesquiliner extension of \(g\)
\[ g_{\mathbb{C}} \in C^\infty(X, \mathcal{C}T_X^* \otimes_{\mathbb{C}} \mathcal{C}T_X), \quad g_{\mathbb{C}}(\xi, \eta) := g(\xi, \bar{\eta}), \forall \xi, \eta \in \mathcal{C}T_X. \]
is a hermitian metric over $\mathcal{C}T_X$ and the $\mathbb{C}$-linear extension of the Levi-Civita connection $\nabla_g$

$$\nabla^\mathbb{C}_g : C^\infty(\mathcal{C}T_X) \rightarrow C^\infty(\mathcal{C}T_X^* \otimes_\mathbb{C} \mathcal{C}T_X),$$

is a $g_\mathbb{C}$-hermitian connection over the vector bundle $T_X \otimes_\mathbb{R} \mathbb{C}$. We will focus our interest on the sections of the hermitian vector bundle

$$\left( (\mathcal{C}T_X^*)^{\otimes p} \otimes_\mathbb{C} T_{X,J}, g_\mathbb{C} \otimes h \right),$$

and we will denote by abuse of notations $\nabla_g = \nabla^\mathbb{C}_g \otimes D g_{T_X,J}$ the $g_\mathbb{C} \otimes h$-hermitian connection over this vector bundle. Still by abuse of notations we will use the identification $(\cdot, \cdot)_\omega : = g_\mathbb{C} \otimes h$. With these notations we define the operators

$$\nabla_{g,1,0} : C^\infty \left( (\mathcal{C}T_X^*)^{\otimes p} \otimes_\mathbb{C} T_{X,J} \right) \rightarrow C^\infty \left( A^{1,0}_J T_X^* \otimes_\mathbb{C} (\mathcal{C}T_X^*)^{\otimes p} \otimes_\mathbb{C} T_{X,J} \right),$$

$$\nabla_{g,0,1} : C^\infty \left( (\mathcal{C}T_X^*)^{\otimes p} \otimes_\mathbb{C} T_{X,J} \right) \rightarrow C^\infty \left( A^{0,1}_J T_X^* \otimes_\mathbb{C} (\mathcal{C}T_X^*)^{\otimes p} \otimes_\mathbb{C} T_{X,J} \right),$$

by the formulas

$$2\nabla_{g,1,0} : = \nabla_g - J \nabla_{g,J},$$

$$2\nabla_{g,0,1} : = \nabla_g + J \nabla_{g,J}.$$)

Then the formal adjoints of the operators $\partial^2_{T_{X,J}}$ and $\overline{\partial}_{T_{X,J}}$ acting on $T_{X,J}$-valued differential forms satisfy the identities (see [Pal6])

$$\partial_{T_{X,J}}^* \alpha = -q \text{ Tr}_g \nabla_{g,J}^{0,1} \alpha,$$

$$\overline{\partial}_{T_{X,J}}^* \alpha = -q \text{ Tr}_g \nabla_{g,J}^{1,0} \alpha,$$

for any $\alpha \in C^\infty(X, \Lambda^q T_X^* \otimes_\mathbb{C} T_{X,J})$. We remind now that with our conventions (see [Pal3]) the Hodge Laplacian operator acting on $T_X$-valued $q$-forms satisfies the identity

$$\Delta_{T_X} = \frac{1}{q} \nabla_{T_X} \nabla_{T_X}^* + \frac{1}{q+1} \nabla_{T_X}^* \nabla_{T_X},$$

We define also the holomorphic and antiholomorphic Hodge Laplacian operators acting on $T_X$-valued $q$-forms as

$$\Delta^f_{T_X} := \frac{1}{q} \partial^f_{T_{X,J}} \partial^*_{T_{X,J}} + \frac{1}{q+1} \partial^*_{T_{X,J}} \partial^f_{T_{X,J}},$$

$$\Delta^{f^*}_{T_X} := \frac{1}{q} \overline{\partial}^f_{T_{X,J}} \overline{\partial}^*_{T_{X,J}} + \frac{1}{q+1} \overline{\partial}^*_{T_{X,J}} \overline{\partial}^f_{T_{X,J}}.$$
with the usual convention $\infty \cdot 0 = 0$. This Hodge Laplacian operators coincide with the standard ones used in the literature. We remind that in the Kähler case hold the decomposition identity

$$\Delta_{T_{X,g}} = \Delta^{J}_{T_{X,g}} + \Delta^{-J}_{T_{X,g}}.$$ 

We observe now that the formal adjoint of the $\partial^{g}_{T_{X,J}}$ operator with respect to the hermitian product

$$\langle \cdot, \cdot \rangle_{\omega, \Omega} := \int_{X} \langle \cdot, \cdot \rangle_{\omega} \Omega,$$

is the operator

$$\partial^{g,\alpha}_{T_{X,J}} := e^{f} \partial^{g}_{T_{X,J}} (e^{-f} \bullet).$$

In a similar way the formal adjoint of the $\overline{\partial}^{g}_{T_{X,J}}$ operator with respect to the hermitian product (11.1) is the operator

$$\overline{\partial}^{g,\alpha}_{T_{X,J}} := e^{f} \overline{\partial}^{g}_{T_{X,J}} (e^{-f} \bullet).$$

With these notations we define the holomorphic and anti-holomorphic $\Omega$-Hodge Laplacian operators acting on $T_{X}$-valued $q$-forms as

$$\Delta^{\Omega,J}_{T_{X,g}} := \frac{1}{q} \partial^{g}_{T_{X,J}} \partial^{* g,\alpha}_{T_{X,J}} + \frac{1}{q+1} \partial^{* g,\alpha}_{T_{X,J}} \partial^{g}_{T_{X,J}},$$

$$\Delta^{\Omega,-J}_{T_{X,g}} := \frac{1}{q} \overline{\partial}^{g}_{T_{X,J}} \overline{\partial}^{* g,\alpha}_{T_{X,J}} + \frac{1}{q+1} \overline{\partial}^{* g,\alpha}_{T_{X,J}} \overline{\partial}^{g}_{T_{X,J}}.$$ 

### 12 The decomposition of the operator $L_{g}^{\Omega}$ in the Kähler case

For any $A \in \text{End}(T_{X})$ we denote by $A^{J}_{t}$ and by $A^{J}_{a}$ the $J$-linear, respectively the $J$-anti-linear parts of $A$. We observe that the operator

$$L_{g}^{\Omega} : C^{\infty}(X, \text{End}(T_{X})) \rightarrow C^{\infty}(X, \text{End}(T_{X})), $$

defined by the formula

$$L_{g}^{\Omega} A := \Delta_{g}^{\Omega} A - 2 \mathcal{R}_{g} \ast A,$$

restricts as:

$$L_{g}^{\Omega} : C^{\infty}(X, T^{*}_{X,J} \otimes T_{X,J}) \rightarrow C^{\infty}(X, T^{*}_{X,J} \otimes T_{X,J}), \quad (12.1)$$

$$L_{g}^{\Omega} : C^{\infty}(X, T^{*}_{X,-J} \otimes T_{X,J}) \rightarrow C^{\infty}(X, T^{*}_{X,-J} \otimes T_{X,J}), \quad (12.2)$$
Indeed this properties follow from the identities
\[(R_g * A)'_J = R_g * A'_J, \quad (12.3)\]
\[(R_g * A)''_J = R_g * A''_J, \quad (12.4)\]
for any \(A \in \text{End}(T_X)\). In their turn they are direct consequence of the identities
\[J(R_g * A) = R_g * (JA), \quad (12.5)\]
\[(R_g * A)J = R_g * (AJ), \quad (12.6)\]

In order to see (12.5) and (12.6) let \((e_k)\) be a \(g\)-orthonormal real basis. Using the \(J\)-invariant properties of the curvature operator we infer
\[J(R_g * A)\xi = R_g(J\xi, e_k)Ae_k = R_g(Je_k, A)\xi = [R_g * (JA)]\xi, \quad (12.7)\]
\[(R_g * A)J\xi = R_g(J\xi, e_k)Ae_k = -R_g(\xi, Je_k)Ae_k = R_g(\xi, \eta_k)AJ\eta_k, \quad (12.8)\]
where \(\eta_k := Je_k\). The fact that \((\eta_k)_k\) is also a \(g\)-orthonormal real frame implies (12.6). By (12.1) and (12.2) we conclude the decomposition formula
\[
\int_X \langle L^\Omega_{g} A, A \rangle_g \Omega = \int_X \langle L^\Omega_{g} A', A'_J \rangle_g \Omega + \int_X \langle L^\Omega_{g} A'', A''_J \rangle_g \Omega. \quad (12.7)\]

We observe that the properties (12.1) and (12.2) imply also that \(A \in \text{Ker} L^\Omega_{g}\) if and only if \(A'_J \in \text{Ker} L^\Omega_{g}\) and \(A''_J \in \text{Ker} L^\Omega_{g}\). We observe further that the identity (9.3) combined with the properties (12.1) and (12.2) implies the formulas
\[
L^\Omega_{g} \partial^\Omega_{T_X,J} \nabla_g u = \partial^\Omega_{T_X,J} \nabla_g (\Delta^\Omega_{g} - 2\mathbb{I}) u, \quad (12.8)\]
\[
L^\Omega_{g} \overline{\overline{\partial}}^\Omega_{T_X,J} \nabla_g u = \overline{\overline{\partial}}^\Omega_{T_X,J} \nabla_g (\Delta^\Omega_{g} - 2\mathbb{I}) u, \quad (12.9)\]
in the Kähler-Ricci soliton case. The properties (12.1) and (12.2) imply (11.11) and (11.12).

### 13 Basic complex Bochner type formulas

We need to review in detail now some facts from [Fut], (see also [Pal1]). Let \((X, J, g)\) be a compact connected Kähler manifold with symplectic \(\omega := gJ\). We remind that the hermitian product induced by \(\omega\) over the bundle \(\Lambda^1,0_{T^*_X}\) satisfies the identity
\[2 \langle \alpha, \beta \rangle_\omega = \text{Tr}_\omega (i\alpha \wedge \beta). \quad (13.1)\]
Let $\Omega > 0$ be a smooth volume form and set as usual $f := \log \frac{dV_g}{d\lambda}$. We define the $\Omega$-weighted complex Laplace type operator acting on functions $u \in C^\infty(X, \C)$ as

$$\Delta^\Omega_{g,J} u := e^f \Tr \omega [i\overline{\partial}_J (e^{-f} \partial_J u)]$$

$$= \Delta_g u + 2 \langle \partial_J u, \partial_J f \rangle_\omega$$

$$= \Delta_g u + 2 \partial_J u \cdot \nabla_g f.$$

We notice the identities $\Delta^\Omega_{g,J} f = \Delta^\Omega_g f$ and $2\Delta^\Omega_g = \Re(\Delta^\Omega_{g,J})$. The complex operator $\Delta^\Omega_{g,J}$ is self-adjoint with respect to the the $L^2_\Omega$-hermitian product

$$\langle u, v \rangle_\Omega := \int_X u \overline{v} \Omega.$$

(13.1)

Indeed integrating by parts we obtain

$$\int_X i\overline{\partial}_J (e^{-f} \partial_J u) \overline{\omega} \wedge \omega^{n-1} = \int_X \partial_J u \wedge i e^{-f} \overline{\partial}_J \overline{\omega} \wedge \omega^{n-1}$$

$$= -\int_X u i \partial_J (e^{-f} \overline{\partial}_J \overline{\omega}) \wedge \omega^{n-1}.$$

(Notice the equality $\Omega = e^{-f} \omega^n/n!$.) We observe in particular the identity

$$\int_X \Delta^\Omega_{g,J} u \cdot \overline{\Omega} = \int_X 2 \langle \partial_J u, \partial_J v \rangle_\omega \Omega,$$

which implies that all the eigenvalues satisfy $\lambda_j(\Delta^\Omega_{g,J}) \geq 0$. For any function $u \in C^\infty(X, \C)$ we define the $J$-complex $g$-gradient as the real vector field;

$$\nabla_{g,J} u := \nabla_g \Re u + J \nabla_g \Im u \in C^\infty(X, T_X).$$

With these notations hold the complex decomposition formula

$$\nabla_{g,J} u-g = \partial_j \overline{\omega} + \overline{\partial}_J u.$$

(13.2)

We consider now the linear operator

$$B^\Omega_{g,J} : C^\infty(X, \R) \longrightarrow C^\infty(X, \R)_0,$$

$$B^\Omega_{g,J} u := \div^\Omega(\nabla_g u).$$

This is a first order differential operator. Indeed

$$B^\Omega_{g,J} u = \Tr_{\mathbb{R}} (J \nabla_g^2 u) - df \cdot J \nabla_g u$$

$$= g(\nabla_g u, J \nabla_g f),$$

48
since $J$ is $g$-anti-symmetric. We extend $B^\Omega_{g,J}$ over $C^\infty(X, \mathbb{C})$ by complex linearity. Let also

$$2d_c^J u := i(\overline{\partial} J - \partial_J)u = -du \cdot J.$$  

Then the identity $2\partial_J = d + 2i d_c^J$ implies the decomposition

$$\Delta^\Omega_{g,J} = \Delta^\Omega_g + 2i\nabla_g f - d_c^J.$$

In other terms

$$\Delta^\Omega_{g,J} = \Delta^\Omega_g - iB^\Omega_{J,g}.$$  

The following lemma is needed for the study of the operator $\Delta^\Omega_{g,J}$. (Compare also with [Fut].)

**Lemma 9** Let $(X, J, g)$ be a Kähler manifold with symplectic form $\omega := gJ$ and let $\Omega > 0$ be a smooth volume form. Then for all $u \in C^\infty(X, \mathbb{R})$ and $v \in C^\infty(X, \mathbb{C})$ hold the complex Bochner type formulas

$$
2\partial^*_{T_X,J} \partial^g_{T_X,J} \nabla_g u = \nabla_g J \Delta^\Omega_{g,J} u - 2\overline{\partial}_{T_X,J} \nabla_g J \nabla_g u,
$$

(13.3)

$$
2\overline{\partial}^*_{T_X,J} \partial^g_{T_X,J} \nabla_g,J \nabla_g,J \nabla_g,J \nabla_g u = \nabla_g,J \Delta^\Omega_{g,J} u - 2\text{Ric}^*_J(\Omega)\omega \nabla_g,J \nabla_g,J \nabla_g u.
$$

(13.4)

**Proof** Let $\xi \in C^\infty(X, T_X)$ and observe that for bi-degree reasons hold the identity

$$
2\partial^*_{T_X,J} \partial^g_{T_X,J} \xi = 2\nabla^*_g \partial^g_{T_X,J} \xi
$$

$$
= \Delta^\Omega_g \xi - \nabla^*_g (J \nabla_g,J \xi)
$$

$$
= \Delta^\Omega_g \xi - \nabla^*_g (J \nabla_g,J \xi) - J \nabla_g,J \nabla_g,J \xi.
$$

Let $(e_k)^{2n}_{k=1}$ be a local $g$-orthonormal frame over a neighborhood of an arbitrary point $p$ such that $\nabla_g e_k(p) = 0$. Then at the point $p$ hold the equalities

$$
-\nabla^*_g (J \nabla_g,J \xi) = J \nabla_{g,e_k} \nabla_g,J e_k \xi
$$

$$
= \frac{1}{2} (J \nabla_{g,e_k} \nabla_g,J e_k \xi - J \nabla_g,J e_k \nabla_{g,e_k} \xi),
$$

since $(J e_k)^{2n}_{k=1}$ is also a local $g$-orthonormal frame. Then the fact that $[e_k, J e_k] (p) = 0$ implies

$$
-\nabla^*_g (J \nabla_g,J \xi) = \frac{1}{2} J \text{Ric}_g(e_k, J e_k) \xi = \text{Ric}^*(g) \xi.
$$

We infer the complex Bochner type formula

$$
2\partial^*_{T_X,J} \partial^g_{T_X,J} \xi = \Delta^\Omega_g \xi + \text{Ric}^*(g) \xi - J \nabla_g,J \nabla_g,J \xi.
$$

(13.5)
In a similar way we obtain

$$2\partial^{\ast}_X, J \nabla g = \Delta^\Omega g \xi - \operatorname{Ric}^\ast (g) \xi + J \nabla g, J \nabla g \xi. \quad (13.6)$$

Using formulas (13.5) and (8.2) we deduce the expressions

$$2\partial^{\ast}_X, J \nabla g = \nabla g \Delta^\Omega g \xi - \nabla g^2 f \nabla g \xi - J \nabla g^2 u J \nabla g f
\begin{align*}
\quad &= \nabla g \Delta^\Omega g \xi - (\nabla^2 g f + J \nabla^2 g f J) \nabla g u \\
\quad &= J (\nabla^2 g u J \nabla g f - \nabla^2 g f J \nabla g u)
\end{align*}$$

Using the first order expression of $B^\Omega_{g,j}$ we obtain

$$2\partial^{\ast}_X, J \nabla g = [\nabla g \Delta^\Omega g \xi - \nabla g^2 u J \nabla g f] - 2 \partial T_X, J \nabla g f \nabla g u.$$

We infer the complex differential Bochner type formula (13.3). In a similar way using formulas (13.6) and (8.2) we deduce

$$2\partial^{\ast}_X, J \nabla g = \nabla g \Delta^\Omega g \xi - \nabla g^2 f \nabla g \xi - J \nabla g^2 u J \nabla g f
\begin{align*}
\quad &= \nabla g \Delta^\Omega g \xi - (\nabla^2 g f + J \nabla^2 g f J) \nabla g u \\
\quad &= J (\nabla^2 g u J \nabla g f - \nabla^2 g f J \nabla g u)
\end{align*}$$

Using the first order expression of $B^\Omega_{g,j}$ we obtain

$$2\partial^{\ast}_X, J \nabla g = [\nabla g \Delta^\Omega g \xi - \nabla g^2 u J \nabla g f] - 2 \partial T_X, J \nabla g f \nabla g u.$$

We infer the complex differential Bochner type formula (13.7). More in general for all $v \in C^\infty (X, \mathcal{C})$ this writes as (13.4). □

Notice that for bi-degree reasons the identity (8.2) decomposes as

$$2\partial^{\ast}_X, J \nabla g = \nabla g, J \Delta^\Omega g \xi - 2 \operatorname{Ric}^\ast (g) \omega \nabla g u.$$
Then we can obtain (13.7) from (13.3) and vice versa. We observe also that the complex Bochner identities (13.3), (13.4) write in the KS case as

\[ 2\partial^*g,\Omega T X,J \partial g T X,J \nabla g u = \nabla g,J \Delta g,J u, \tag{13.8} \]

\[ 2\partial^*g,\Omega T X,J \partial g T X,J \nabla g,J v = \nabla g,J (\Delta g,J - \frac{2}{C_1}) v, \tag{13.9} \]

for all \( u \in C^\infty(X,\mathbb{R}) \) and \( v \in C^\infty(X,\mathbb{C}) \). Obviously the identity (13.9) still hold in the more general case \( \text{Ric}_J(\Omega) = \omega \). We observe now an other integration by parts formula.

Let \( \xi \in C^\infty(X,T_X), A \in C^\infty(X,T^*_X,\mathbb{R}) \) and observe that the comparison between Riemannian and hermitian norms of \( T_X \)-valued 1-forms (see the appendix in [Pal2]) implies

\[
\int_X \langle \partial_{T X,J} \xi, A \rangle_g \Omega = \frac{1}{2} \int_X \left( \langle \partial_{T X,J} \xi, A \rangle_\omega + \langle A, \partial_{T X,J} \xi \rangle_\omega \right) \Omega
\]

\[
= \frac{1}{2} \int_X \left( \langle \xi, \partial_{T X,J}^* A \rangle_\omega + \langle \partial_{T X,J}^* A, \xi \rangle_\omega \right) \Omega
\]

\[
= \int_X \langle \xi, \partial_{T X,J}^* A \rangle_g \Omega.
\]

Using this and multiplying both sides of (13.9) by \( \nabla g,J v \) we obtain the identity

\[
2 \int_X \left\| \partial_{T X,J} \nabla g,J v \right\|^2_g \Omega = \int_X \langle \nabla g,J (\Delta g,J - \frac{2}{C_1}) v, \nabla g,J v \rangle_g \Omega,
\]

in the case \( \text{Ric}_J(\Omega) = \omega \). We consider now the \( J \)-anti-linear component of the complex Hessian map:

\[
H_{g,J}^{0,1} : C^\infty(\Omega,\mathbb{C})_0 \rightarrow C^\infty(X,\Lambda^0_{g,J} T_X \otimes_{\mathbb{C}} T_{X,J})
\]

\[
u \mapsto \overline{\partial}_{T X,J} \nabla g,J u.
\]

We observe that \( H_{g,J}^{1,0}(X,\mathbb{C}) = 0 \) in the case of Fano manifolds and we remind the following well known fact

**Lemma 10** Let \((X,J,g)\) be a compact connected Kähler manifold such that \( H_{g,J}^{1,0}(X,\mathbb{C}) = 0 \). Then the map

\[
\text{Ker} H_{g,J}^{0,1} \rightarrow H^0(X,T_{X,J})
\]

\[
u \mapsto \nabla g,J u,
\]

is an isomorphism of complex vector spaces.
Proof. We observe first the injectivity. Using the complex decomposition (13.2) we infer the formula
\[ d(\nabla_{g,J}u - g) = \partial_J \overline{\partial}_J (u - \nu), \]
which in the case \( \nabla_{g,J}u = 0 \) implies \( \text{Im} u = 0 \) and thus \( \text{Re} u = 0 \). In order to show the surjectivity we consider an arbitrary \( \xi \in H^0(X, T_{X,J}) \). Then the identity (13.11) below implies
\[ \overline{\partial}_J (\xi^{1,0}_J - \omega) = 0. \]
By Hodge decomposition hold the identity \( H^{0,1}_\Omega(X, \mathbb{C}) = 0 \). We deduce the existence of a unique function \( u \in C^\infty(\Omega_{X,J}) \) such that
\[ i \partial_J u = \xi^{1,0}_J - \omega = i \xi^{1,0}_J - g. \]
Thus \( \xi = \nabla_{g,J}u \) thanks to the complex decomposition (13.2). □

Lemma 11 Let \((X, J)\) be a complex manifold and let \( \omega \in C^\infty(X, \Lambda^{1,1}_{T^*X}) \), \( \xi \in C^\infty(X, T^{1,0}_X) \). Then hold the identity
\[ \overline{\partial}_J (\xi - \omega) = \overline{\partial}_{T^{1,0}} \xi - \xi - \overline{\partial}_J \omega. \]  
(13.11)

Proof. Let \( \eta, \mu \in C^\infty(T^{0,1}) \) and observe the identities (see [Pal])
\[ \overline{\partial}_J (\xi - \omega)(\eta, \mu) = \eta \omega(\xi, \mu) - \mu \omega(\xi, \eta) - \omega(\xi, [\eta, \mu]), \]
\[ \overline{\partial}_J \omega(\eta, \xi, \mu) = \eta \omega(\xi, \mu) + \mu \omega(\eta, \xi) \]
\[ - \omega(\eta, \xi^{1,0}, \mu) + \omega([\eta, \mu], \xi) - \omega([\xi, \mu^{1,0}], \eta) \]
\[ = \overline{\partial}_J (\xi - \omega)(\eta, \mu) - \omega(\overline{\partial}_{T^{1,0}} \xi(\eta), \mu) + \omega(\overline{\partial}_{T^{1,0}} \xi(\mu), \eta) \]
\[ = \overline{\partial}_J (\xi - \omega)(\eta, \mu) - \omega(\overline{\partial}_{T^{1,0}} \xi(\eta), \mu) - \omega(\eta, \overline{\partial}_{T^{1,0}} \xi(\mu)) \]
\[ = \overline{\partial}_J (\xi - \omega) - \overline{\partial}_{T^{1,0}} (\xi - \omega) \in \ker H^{0,1}, \]
which implies the required identity. □

On the other hand the identities (13.9) and (13.10) show that in the case \( \text{Ric}_J(\Omega) = \omega \) hold the identity
\[ \ker(\Delta_{g,J} - 2\Pi) = \ker H^{0,1}_{g,J}. \]  
(13.12)

We infer the following well known result due to Futaki [Fut]. (See also [Gau] and the sub-section 21.2 in appendix B for a more complete statement.)
Corollary 1 Let \((X, J)\) be a Fano manifold and let \(g\) be a \(J\)-invariant Kähler metric such that \(\omega := gJ \in 2\pi c_1(X, [J])\). Let also \(\Omega > 0\) be the unique smooth volume form with \(\int_X \Omega = 1\) such that \(\text{Ric}_J(\Omega) = \omega\). Then the map
\[
\overline{\text{Ker}}(\Delta^\Omega_{g,J} - 2\Pi) \longrightarrow H^0(X, T_{X,J})
\]
\[
u \mapsto \nabla_{g,J} u,
\]
is well defined and it represents an isomorphism of complex vector spaces. The first eigenvalue \(\lambda_1(\Delta^\Omega_{g,J})\) of the operator \(\Delta^\Omega_{g,J}\) satisfies the estimate \(\lambda_1(\Delta^\Omega_{g,J}) \geq 2\), with equality in the case \(H^0(X, T_{X,J}) \neq 0\). Moreover if we set \(\text{Kill}_g := \text{Lie}(\text{Isom}_g^0)\) and
\[
\text{Ker}_R(\Delta^\Omega_{g,J} - 2\Pi) := \text{Ker}(\Delta^\Omega_{g,J} - 2\Pi) \cap C^\infty_\Omega(X, \mathbb{R})_0,
\]
then the map
\[
J\nabla_g : \text{Ker}_R(\Delta^\Omega_{g,J} - 2\Pi) \longrightarrow \text{Kill}_g,
\] (13.13)
is well defined and it represents an isomorphism of real vector spaces.

Proof We only need to show the statement concerning the map (13.13). Let \(\xi \in \text{Kill}_g\) and let \((\varphi_t)_{t \in \mathbb{R}} \subset \text{Isom}_g^0\) be the corresponding 1-parameter sub-group. The Kähler condition \(\nabla_g J = 0\) implies \(\Delta_d g \omega = 0\) and thus \(\Delta_d g (\varphi_t^* \omega) = 0\). Time deriving this last at \(t = 0\) we infer
\[
\Delta_d g L \xi \omega = 0.
\] (13.14)

But \(L \xi \omega = d(\xi - \omega)\) and (13.14) rewrites as \(d^* d(\xi - \omega) = 0\). We infer
\[
0 = L \xi \omega = gL \xi J = 2\omega \partial_{T_{X,J}} \xi,
\]
and thus
\[
\text{Kill}_g = \{ \xi \in H^0(X, T_{X,J}) \mid L \xi \omega = 0 \} = \{ \xi \in H^0(X, T_{X,J}) \mid d(\xi - \omega) = 0 \} = \{ \xi \in H^0(X, T_{X,J}) \mid \exists u \in C^\infty_\Omega(X, \mathbb{R})_0 : \xi - \omega = du \},
\]
thanks to the fact that \(H^1_d(X, \mathbb{R}) = 0\). But this last identity rewrites as
\[
\text{Kill}_g = \{ \xi \in H^0(X, T_{X,J}) \mid \exists u \in C^\infty_\Omega(X, \mathbb{R})_0 : \xi = J \nabla_g u \},
\]
which shows that the map (13.13) is well defined thanks to the first statement of corollary\[1\]. The surjectivity of the map (13.13) follows from the identity (13.9) applied to the function \(v := -iu\), with \(u \in C^\infty_\Omega(X, \mathbb{R})_0\) such that \(J \nabla_g u \in \text{Kill}_g\). The injectivity of the map (13.13) is obvious. \(\square\)
Using the variational characterization of the first eigenvalue we observe:

\[
\lambda_1(\Delta_{g,j}) = \inf \left\{ \frac{\int_X \Delta_{g,j} uu}{\int_X |u|^2} \mid u \in C^\infty_0(X,\mathbb{C}) \setminus \{0\} \right\} \\
\leq \inf \left\{ \frac{\int_X 2|\partial_j u|^2}{\int_X u^2} \mid u \in C^\infty_0(X,\mathbb{R}) \setminus \{0\} \right\} \\
= \inf \left\{ \frac{\int_X \Delta_{g,j} uu}{\int_X u^2} \mid u \in C^\infty_0(X,\mathbb{R}) \setminus \{0\} \right\} \\
= \lambda_1(\Delta_g),
\]

thanks to the identity \(2|\partial_j u|^2 = |\nabla_g u|^2\). We deduce that in the set-up of corollary \(1\) hold the estimate

\[
\lambda_1(\Delta_g^\Omega) \geq 2. \quad (13.15)
\]

### 14 Symmetric variations of Kähler structures

We show a few fundamental facts about the space of symmetric variations of Kähler structures \(\mathbb{K}\mathbb{V}_g^J\) given by the elements \(v \in C^\infty(X,S^2_R T_X^*)\) such that there exists a smooth family \((J_t, g_t) \subset KS\) with \((J_0, g_0) = (J, g)\), \(g_0 = v\) and \(J_0 = (J_0)_g^T\). One can observe (see \[Pal3\]) that \(\mathbb{K}\mathbb{V}_g^J \subseteq \mathbb{D}_g^J\) with

\[
\mathbb{D}_g^J := \left\{ v \in C^\infty(X,S^2_R T_X^*) \mid \partial^2_{X,J}(v^*_g)_j^{1,0} = 0, \partial^2_{X,J}(v^*_g)_j^{0,1} = 0 \right\}, \quad (14.1)
\]

where \((v^*_g)_j^{1,0}\) and \((v^*_g)_j^{0,1}\) denote respectively the \(J\)-linear and \(J\)-anti-linear parts of the endomorphism \(v^*_g\). We remind here some lines of this basic fact. We define

\[
2\nabla_{g,1,J}^{1,0} := \nabla_g - i\nabla_g J^* \\
2\nabla_{g,1,J}^{0,1} := \nabla_g + i\nabla_g J^*
\]

Let \(v'_j\) and \(v''_j\) be respectively the \(J\)-invariant and \(J\)-anti-invariant parts of \(v\) and set for notation simplicity

\[
A' := (v'^*_g)_j^{1,0} = (v'_j)_g, \\
A'' := (v'^*_g)_j^{0,1} = (v''_j)_g.
\]

The identity \(A'' = -J\dot{J}_0\) (see \[Pal3\]) implies directly \(\dot{\omega}_0 = v'_j J = \omega A'\). We infer

\[
0 = d\dot{\omega}_0 = d(v'_j J).
\]
The fact that the \((1,1)\)-form \(v'_JJ\) is real implies that the identity \(d(v'_JJ) = 0\) is equivalent to the identity \(\partial_J(v'_JJ) = 0\). In its turn this is equivalent to the identity \(\partial^2_J A = 0\). We observe indeed that for all \(\xi, \eta, \mu \in C^\infty(X, TX \otimes \mathbb{R})\) hold the equalities

\[
\partial_J (\omega A')(\xi', \eta', \mu') = \nabla^{1,0}_{g,i,J}(\omega A')(\xi', \eta', \mu') - \nabla^{1,0}_{g,i,J}(\omega A')(\eta', \xi', \mu')
\]

\[+
\nabla^{1,0}_{g,i,J}(\omega A')(\mu', \xi', \eta')
\]

\[=
\omega \left( \left[ \nabla^{1,0}_{g,i,J}(\xi', \eta') - \nabla^{1,0}_{g,i,J}(\eta', \xi') \right] , \mu' \right)
\]

\[-
\omega \left( \nabla^{1,0}_{g,i,J}(\mu', \xi'), \eta' \right)
\]

\[=
\omega \left( \left[ \nabla^{1,0}_{g,i,J}(\xi', \eta') - \nabla^{1,0}_{g,i,J}(\eta', \xi') \right] , \mu' \right)
\]

\[=
\omega \left( \partial^2_J A'(\xi', \eta'), \mu'' \right).
\]

In order to continue the study of the space \(D_J^g\) we need to show a few general and fundamental facts. We start with the following weighted complex Weitzenböck type formula.

\textbf{Lemma 12} Let \((X, J, g)\) be a Kähler manifold, let \(\Omega > 0\) be a smooth volume form and let \(A \in C^\infty(X, T^*_X \otimes TX)\). Then hold the identity

\[
\Delta^{0,-J}_X A = \nabla^*_g \nabla^{0,1}_{g,J} A - R_g * A + A \text{Ric}_J^g(\Omega) \omega
\]

\textbf{Proof} We observe that for bi-degree reasons hold the identities

\[
\Delta^{0,-J}_X A := \nabla^*_g \nabla^{0,1}_{g,J} A + \frac{1}{2} \nabla^*_g \nabla^{0,1}_{g,J} A
\]

\[
= \nabla^*_g \nabla^{0,1}_{g,J} A + \frac{1}{2} \nabla^*_g \nabla^{0,1}_{g,J} A
\]

Let

\[
\nabla^{0,1}_{g,J} A(\xi, \eta) := \nabla^{0,1}_{g,J} A(\eta, \xi).
\]

Then

\[
2\Delta^{0,-J}_X A = \nabla^*_g \nabla^{0,1}_{g,J} A + J \nabla^*_g \nabla^{0,1}_{g,J} A
\]

\[+
2\nabla^*_g \nabla^{0,1}_{g,J} A - 2\nabla^*_g \nabla^{0,1}_{g,J} A.
\]
We fix an arbitrary point $p$ and we choose an arbitrary vector field $\xi$ such that $\nabla_g \xi(p) = 0$. Let $(e_k)_k$ be a $g$-orthonormal local frame such that $\nabla_g e_k(p) = 0$. We observe the local expression

$$\nabla^* \Omega^g \nabla^0_{g,\xi} \nabla^0_{g,J} A \cdot \xi = -\nabla_{g,e_k} \nabla^0_{g,J} A(e_k, \xi) + \nabla^0_{g,1} A(\xi, \nabla g f).$$

At the point $p$ hold the identities

$$2\nabla_{g,e_k} \nabla^0_{g,J} A(e_k, \xi) = 2\nabla_{g,e_k} \nabla^0_{g,J} A(\xi, e_k)$$

and thus

$$2\nabla^* \Omega^g \nabla^0_{g,J} A \cdot \xi = -\nabla_{g,e_k} \nabla_{g,\xi} A \cdot e_k - J\nabla_{g,e_k} \nabla_{g,J} \xi \cdot e_k + \nabla_{g,\xi} A \cdot \nabla g f + J\nabla_{g,\xi} A \cdot \nabla g f.$$

We obtain the identity at the point $p$,

$$2\Delta^*_g X, g A \cdot \xi = -\nabla_{g,\xi} \nabla_{g,e_k} A \cdot e_k + \nabla_{g,\xi} (A \cdot \nabla g f)$$

$$- J\nabla_{g,J} \xi \nabla_{g,e_k} A \cdot e_k + J\nabla_{g,J} \xi (A \cdot \nabla g f)$$

$$+ 2\nabla^* \Omega^g \nabla^0_{g,J} A \cdot \xi$$

$$+ \nabla_{g,e_k} \nabla_{g,\xi} A \cdot e_k + J\nabla_{g,e_k} \nabla_{g,J} \xi A \cdot e_k$$

$$- \nabla_{g,\xi} A \cdot \nabla g f - J\nabla_{g,J} \xi A \cdot \nabla g f.$$

We remind that for any $A \in C^\infty(X, \text{End}(T_X))$ and $\xi, \eta \in C^\infty(X, T_X)$ hold the general formula

$$\nabla_{g,\xi} \nabla_{g,\eta} A - \nabla_{g,\eta} \nabla_{g,\xi} A = [\nabla g(\xi, \eta), A] + \nabla g([\xi, \eta]) A. \quad (14.3)$$
Using (14.3) and the fact that in our case \[ e_k, \xi \] (p) = \[ e_k, J\xi \] (p) = 0 we obtain
\[
2 \Delta_{T_{X,g}} \cdot e_k + J [R_g(e_k, J\xi)A \cdot e_k - AR_g(e_k, J\xi) \cdot e_k] \\
+ 2 \nabla^{*g} \nabla^{0,1}_{g,J} A \cdot \xi + A \nabla^2_{g,\xi} f + JA \nabla_{g,J} \xi \nabla_{g} f \\
= - (R_g * A) \cdot \xi + A \text{Ric}^*(g) \cdot \xi \\
- J (R_g * A) \cdot J\xi + JA \text{Ric}^*(g) \cdot J\xi \\
+ 2 \nabla^{*g} \nabla^{0,1}_{g,J} A \cdot \xi + A \partial^g_{T_{X,J}} \nabla_{g} f \cdot \xi \\
= 2 \left[ \nabla^{*g} \nabla^{0,1}_{g,J} A - R_g * A + A \text{Ric}^*(\Omega) \right] \cdot \xi,
\]
thanks to (12.4).

We observe also the following corollary.

**Corollary 2** Let \((X, J, g)\) be a Kähler manifold, let \(\Omega > 0\) be a smooth volume form and let \(A \in C^\infty (X, T_{X,J}^* \otimes T_{X,J})\). Then hold the identities
\[
\mathcal{L}^\Omega_g A = 2 \Delta_{T_{X,g}}^{\Omega,J} A + \text{div}^\Omega_g \nabla_{g,J}(J_A) - 2A \text{Ric}^*(\Omega) \omega, \quad (14.5)
\]
\[
\text{div}^g \nabla_{g,J}(J_A) = \text{Ric}^*(g)A + A \text{Ric}^*(g) - (J \nabla_{g} f - (J \nabla_{g} A)). \quad (14.6)
\]

**Proof** It is obvious that the identity (14.2) rewrites as (14.5). In order to show (14.6) let \((\eta_k)_{k=1}^n\) be a local complex frame of \(T_{X,J}\) in a neighborhood of a point \(p\) with \(\nabla_{g} \eta_k(p) = 0\) such that the real frame \((e_l)_{l=1}^{2n}, e_l = \eta_l, l = 1, \ldots, n\)
and $c_{n+k} = J\eta_k$, $k = 1, \ldots, n$ is $g$-orthonormal. Then at the point $p$ hold the equalities

$$\text{div}_g \nabla g, J^* (JA''_f) = \sum_{l=1}^{2n} \nabla_{g,e_l} \nabla_{g,Je_l} (JA''_f)$$

$$= \sum_{k=1}^{n} [\nabla_{g,\eta_k} \nabla_{g,J\eta_k} (JA''_f) - \nabla_{g,J\eta_k} \nabla_{g,\eta_k} (JA''_f)]$$

$$= \sum_{k=1}^{n} [R_g (\eta_k, J\eta_k), JA''_f],$$

thanks to the general formula (14.3) and thanks to the fact that $[\eta_k, J\eta_k] (p) = 0$. Using the $J$-linear and $J$-anti-linear properties of the tensors involved in the previous equality we obtain

$$\text{div}_g \nabla g, J^* (JA''_f) = \sum_{k=1}^{n} [JR_g (\eta_k, J\eta_k)A''_f + A''_f JR_g (\eta_k, J\eta_k)]$$

$$= \text{Ric}^*(g) A''_f + A''_f \text{Ric}^*(g).$$

Notice indeed the identities

$$2 \text{Ric}^*(g) = \sum_{l=1}^{2n} JR_g (e_l, J e_l)$$

$$= \sum_{k=1}^{n} [JR_g (\eta_k, J\eta_k) - JR_g (J\eta_k, \eta_k)]$$

$$= 2 \sum_{k=1}^{n} JR_g (\eta_k, J\eta_k).$$

We conclude the required formula (14.6). \qed

We define now the vector spaces

$$\mathcal{H}_{g,\Omega}^{0,1} (T_X, J) := \ker \Delta_{T_X, g}^{\Omega, -J} \cap C^\infty (X, T^*_X, -J \otimes T_X, J),$$

$$\mathcal{H}_{g,\Omega}^{0,1} (T_X, J)_{\text{sm}} := \left\{ A \in \mathcal{H}_{g,\Omega}^{0,1} (T_X, J) \mid A = A^T_g \right\}.$$

**Lemma 13** Let $(X, J)$ be a Fano manifold, let $g$ be a $J$-invariant Kähler metric with symplectic form $\omega := gJ \in 2\pi c_1 (X, [J])$ and let $\Omega > 0$ be the unique smooth volume form with $\int_X \Omega = 1$ such that $\omega = \text{Ric}_J (\Omega)$. Then hold the identity

$$\mathcal{H}_{g,\Omega}^{0,1} (T_X, J) = \mathcal{H}_{g,\Omega}^{0,1} (T_X, J)_{\text{sm}}.$$
Proof We consider the decomposition $A = A_{\text{sm}} + A_{\text{as}}$, where $A_{\text{as}}$ and $A_{\text{sm}}$ are respectively the $g$-symmetric and $g$-anti-symmetric parts of $A$. We observe the symmetries

\[ R_g * A_{\text{sm}} = (R_g * A_{\text{sm}})^T_g, \]
\[ R_g * A_{\text{as}} = -(R_g * A_{\text{as}})^T_g. \]

The fact that $A \in C^\infty(X, T_{X,J} \otimes T_{X,J})$ implies $A_{\text{sm}}, A_{\text{as}} \in C^\infty(X, T_{X,J} \otimes T_{X,J})$ and thus

\[ \nabla^*_g \nabla^0_{g,J} A_{\text{sm}} = \left( \nabla^*_g \nabla^0_{g,J} A_{\text{sm}} \right)^T_g, \]
\[ \nabla^*_g \nabla^0_{g,J} A_{\text{as}} = -\left( \nabla^*_g \nabla^0_{g,J} A_{\text{as}} \right)^T_g. \]

Then the identity (14.2) implies the equalities

\[ \left( \Delta^\Omega_{T_{X,J}} A_{\text{sm}} \right)^T_g - \Delta^\Omega_{T_{X,J}} A_{\text{sm}} = [\text{Ric}^*_J(\Omega), A_{\text{sm}}], \tag{14.7} \]
\[ \left( \Delta^\Omega_{T_{X,J}} A_{\text{as}} \right)^T_g + \Delta^\Omega_{T_{X,J}} A_{\text{as}} = [A_{\text{as}}, \text{Ric}^*_J(\Omega)]. \tag{14.8} \]

We deduce that in the case $\text{Ric}_{J}(\Omega) = \lambda \omega$, with $\lambda = \pm 1, 0$, the condition $A \in \mathcal{H}^{0,1}_g(T_{X,J})$ is equivalent to the conditions $A_{\text{sm}}, A_{\text{as}} \in \mathcal{H}^{0,1}_g(T_{X,J})$. We focus now on the Fano case $\lambda = 1$. We remind the identity $R_g * A_{\text{as}} = 0$. (See (20.9) in the appendix.) Thus if $A \in \mathcal{H}^{0,1}_g(T_{X,J})$ and $\text{Ric}_{J}(\Omega) = \omega$ then the integral formula (14.4) reduces to

\[ 0 = \int_X \left[ |\nabla_{g,J} A_{\text{as}}|^2_g + |A_{\text{as}}|^2_g \right] \Omega, \]

which shows $A_{\text{as}} = 0$ and thus the required conclusion of the lemma.

\[ \square \]

We obtain also the following statement (the case $c_1 < 0$ has been proved in [D-W-W2]):

**Lemma 14** Let $(X, J, g)$ be a compact non Ricci flat Kähler-Einstein manifold. Then hold the identity

\[ \mathcal{H}^{0,1}_g(T_{X,J}) = \mathcal{H}^{0,1}_g(T_{X,J})_{\text{sm}}. \]

**Proof** Using the identities (14.7) and (14.8) with $\Omega = CdV_g$ we deduce that in the Kähler-Einstein case $\text{Ric}(g) = \lambda g$, with $\lambda = \pm 1, 0$, the condition $A \in \mathcal{H}^{0,1}_g(T_{X,J})$ is equivalent to the conditions $A_{\text{sm}}, A_{\text{as}} \in \mathcal{H}^{0,1}_g(T_{X,J})$. On the other hand the identities (14.5) and (14.6) imply in the case $\Omega = CdV_g$ the formula

\[ \mathcal{L}_g A = 2\Delta^J_{T_{X,J}} A + [\text{Ric}^*(g), A], \tag{14.9} \]
for any $A \in C^\infty (X, T_{X,-} \otimes T_{X,J})$. The fact that $\mathcal{R}_g * A_{as} = 0$ implies the formula

$$\Delta_g A_{as} = 2 \Delta_{T_g,g} A_{as} + [\text{Ric}^*(g), A_{as}].$$

We conclude that in the Kähler-Einstein case $\text{Ric}(g) = \lambda g$, with $\lambda = \pm 1, 0$, any $A \in H^{0,1}_g (T_{X,J})$ satisfies $\nabla_g A_{as} = 0$. Then the formula [11.14] implies

$$0 = \text{div}_g \nabla_{g,J} (A_{as}) = \text{Ric}^*(g) A_{as} + A_{as} \text{Ric}^*(g) = 2 \lambda A_{as}.$$

We deduce $A_{as} = 0$ in the case $\lambda = \pm 1$. This shows the required conclusion. □

We denote by

$$\Lambda g,J \Omega := \text{Ker}(\Delta g,J - 2 1/ C_1 g,J) \subset C^\infty (X, \Lambda \Omega^0 (X, \Lambda 1)),$$

and by

$$\Lambda g,J \Omega,⊥ := \left[\text{Ker}(\Delta g,J - 2 1/ C_1 g,J)\right] \Omega \subseteq C^\infty (X, \Lambda \Omega^0 (X, \Lambda 1)).$$

its $L^2$-orthogonal inside $C^\infty (X, \Lambda \Omega^0 (X, \Lambda 1))$. We obtain as corollary of lemma (13) the following fundamental fact.

**Corollary 3 (Decomposition of the variation of the complex structure)**

Let $(X,J)$ be a Fano manifold, let $g$ be a $J$-invariant Kähler metric with symplectic form $\omega := gJ \in 2 \pi c_1(X,J)$ and let $\Omega > 0$ be the unique smooth volume form with $\int_X \Omega = 1$ such that $\omega = \text{Ric}_J(\Omega)$. Then for all $v \in D^g_J$ there exists a unique $\psi \in \Lambda g,J \Omega,⊥$ and a unique $A \in H^{0,1}_g (T_{X,J})$ such that

$$(v_g^*)^{0,1} = \overline{\partial}_{T_{X,J}} \nabla_{g,J} \psi + A.$$

**Proof** We observe that the identity

$$\overline{\partial}_{T_{X,J}} (v_g^*)^{0,1} = 0,$$

combined with the $\Omega$-Hodge isomorphism

$$H^{0,1}_g (T_{X,J}) \simeq H^{0,1} (X, T_{X,J})$$

implies the decomposition

$$(v_g^*)^{0,1} = \overline{\partial}_{T_{X,J}} \xi + A,$$

with $\xi \in C^\infty (X, T_X)$ and unique $A \in H^{0,1}_g (T_{X,J})$. Then the fact that the endomorphism $(v_g^*)^{0,1}$ is $g$-symmetric combined with lemma [13] implies that
\( \bar{\partial}_{TX,\xi} \) is also \( g \)-symmetric. Then formula (13.11) implies that for all \( \eta, \mu \in C^\infty(X, T^0_1 X) \) hold the identity
\[
\bar{\partial}_J (\xi^{1,0}_J - \omega)(\eta, \mu) = \omega (\bar{\partial}_{TX,\xi} \eta, \mu) + \omega \left( \eta, \bar{\partial}_{TX,\xi} \xi \cdot \mu \right)
\]
\[
= g \left( J\bar{\partial}_{TX,\xi} \eta, \mu \right) + g \left( J\eta, \bar{\partial}_{TX,\xi} \xi \cdot \mu \right)
\]
\[
= g \left( \left[ (\bar{\partial}_{TX,\xi})^T_g - \bar{\partial}_{TX,\xi} \right] : J\eta, \mu \right)
\]
\[
= 0.
\]
Then the argument showing the surjectivity of the map in lemma 10 in the section 13 implies the existence of a function \( \Psi \in C^\infty(X, \mathbb{C})_0 \) such that \( \xi = \nabla_g, J, \Psi \).

This combined with the identity (13.12) implies the existence and uniqueness of \( \psi \in \Lambda^1_{g, J} \) such that
\[
\bar{\partial}_{TX,\xi} = \bar{\partial}_{TX,\xi} \nabla_g, J, \psi.
\]

We infer the required conclusion. \( \square \)

We show now the inclusion (1.21). Time deriving the condition \( \omega_t := g_t J_t \in 2\pi C_1 \) we infer \( \{ \dot{\omega}_t \}_{t} = 0 \). Then (1.21) follows from the complex decomposition identity
\[
v^*_g = g^{-1} \dot{y}_0 = \omega^{-1} \dot{\omega}_0 - J \dot{J}_0 = (v'_J)^*_g + (v''_J)^*_g.
\]

15 The decomposition of the space \( \mathbb{F}_{g, \Omega} \) in the soliton case

Lemma 15 Let \((X, g, \Omega)\) be a compact shrinking Ricci soliton. Then the linear map
\[
T_{g,\Omega} : C^\infty(X, \mathbb{R})_0 \oplus \left[ \text{Ker } \nabla^*_g \cap C^\infty(X, S^2 T^*_X) \right] \longrightarrow \mathbb{F}_{g, \Omega}
\]
\[
(\varphi, \theta) \longmapsto (\nabla_g \varphi + \theta, (\varphi - \Delta_g^1 \varphi) \Omega),
\]
is an isomorphism of vector spaces.

Proof STEP I. We observe first that in the compact shrinking Ricci soliton case the first eigenvalue \( \lambda_1(\Delta^1_g) \) of \( \Delta^1_g \) satisfies the inequality \( \lambda_1(\Delta^1_g) > 1 \). Indeed multiplying both sides of the identity (5.2) with \( \nabla_g u \) and integrating we
Let now $u \in C_0^\infty(X,\mathbb{R})$ be an eigen-function corresponding to $\lambda_1(\Delta^g) > 0$. By definition $u \not\equiv 0$. Thus by the previous integral identity we deduce

$$\lambda_1(\Delta^g) \int_X |\nabla^g u|^2 \Omega = \int_X \langle \nabla^g \Delta^g u, \nabla^g u \rangle \Omega$$

$$= \int_X \left[ |\nabla^g u|^2 + \text{Ric}_g(\nabla^g u, \nabla^g u) \right] \Omega,$$

which implies the required estimate.

**STEP II.** Multiplying both sides of the the identity (8.2) with $g$ we obtain

$$d\Delta^g u = \Delta^g du + du \cdot \text{Ric}^*_g(\Omega).$$

Let now $(v, V) := T_{g,\Omega}(\varphi, \theta)$ and observe the equalities

$$\nabla^*_{\varphi} v = \nabla^*_{\varphi} \nabla^g d\varphi = \Delta^g d\varphi = d(\Delta^g \varphi - \varphi).$$

The last one follows from (15.1). We infer that the linear map $T_{g,\Omega}$ is well defined. The fact that in the soliton case $h_{g,\Omega} = 0$ the differential operator $\Delta^g - \Pi$ is invertible over $C_0^\infty(X,\mathbb{R})$ implies the injectivity of the map $T_{g,\Omega}$.

In order to show the surjectivity of the map $T_{g,\Omega}$ let $(v, V) \in \mathbb{F}_{g,\Omega}$ and define the function

$$\varphi := (\Pi - \Delta^g)^{-1} V_{\Omega} \in \mathbb{F}_0^\infty(X,\mathbb{R}).$$

Then the identity

$$\nabla^*_{\varphi} \nabla^g d\varphi = d(\Delta^g \varphi - \varphi),$$

implies that the tensor $\theta := v - \nabla^g d\varphi$ satisfies $\nabla^*_{\varphi} \theta = 0$. We deduce the orthogonal decomposition with respect to the scalar product (1.1)

$$v = \nabla^g d\varphi + \theta,$$

with $\nabla^*_{\varphi} \theta = 0$. We deduce the required surjectivity statement. \hfill \square

We need to introduce a few notations. From now on we assume $H^1_\delta(X,\mathbb{R}) = 0$ (this is the case of any Fano manifold) and we observe that the first projection map

$$p_1 : \mathbb{F}_{g,\Omega} \rightarrow \mathbb{S}_{g,\Omega} := \{ v \in C_0^\infty(X, S^2 T_X) \mid d\nabla^*_{\varphi} v = 0 \}.$$
is an isomorphism. Over a compact Kähler manifold we define the real vector spaces

\[ S^J_{g,\Omega} := S_{g,\Omega} \cap D^J_g, \]
\[ S^J_{g,\Omega}(0) := S_{g,\Omega} \cap D^J_{g,0}, \]
\[ S^J_{g,\Omega}[0] := S_{g,\Omega} \cap D^J_{g,[0]}, \]

and

\[ E^\Omega_{g,J} := \left\{ \psi \in \Lambda^1_g \mid \Delta^\Omega_{g,J}(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi = \Delta^\Omega_{g,J}(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi \right\}. \]

With the notations introduced so far we can state the following decomposition result.

**Lemma 16** Let \((J, g)\) be a KRS and let \(\Omega > 0\) be the unique smooth volume form such that \(g_J = \text{Ric}_J(\Omega)\) and \(\int_X \Omega = 1\). Then the linear map

\[ C^\infty_\Omega(X, \mathbb{R})_0 \oplus E^\Omega_{g,J} \oplus H^{0,1}_{g,\Omega}(T_X, J) \rightarrow S^J_{g,\Omega}(0) \]

\[ (\varphi, \psi, A) \mapsto v, \]
\[ (v^+)^{1,0} := \partial^g_{T_X,J} \nabla g(\varphi + \tau), \]
\[ (v^+)^{0,1} := \overline{\partial}_{T_X,J} \nabla g_J(\varphi + \overline{\psi}) + A, \]

with \(\tau \in C^\infty_\Omega(X, \mathbb{R})_0\) the unique solution of the equation

\[ -\Delta^g_{g,J}\tau = (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi, \]

is an isomorphism of real vector spaces. In particular the linear map

\[ E^\Omega_{g,J} \oplus H^{0,1}_{g,\Omega}(T_X, J) \rightarrow S^J_{g,\Omega}[0] \]

\[ (\psi, A) \mapsto v, \]
\[ (v^+)^{0,1} := \overline{\partial}_{T_X,J} \nabla g_J(\varphi + \overline{\psi}) + A, \]

with \(\varphi \in C^\infty_\Omega(X, \mathbb{R})_0\) the unique solution of the equation

\[ \overline{\Delta^\Omega_{g,J}}\varphi = (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi, \]

is also an isomorphism of real vector spaces.
Let first \( v \in \mathbb{S}^{J} \) and observe that the decomposition formula (15.2)
rewrites as
\[
v^* = \partial^g_{T_X,J} \nabla g \varphi + \partial g T X,J \nabla g \phi + \theta^*_g.
\]
This implies that \( v \in \mathbb{D}^J_g \) if and only if \( \theta \in \mathbb{D}^J_g \), and also \( v \in \mathbb{D}^J_{g,0} \) if and only if \( \theta \in \mathbb{D}^J_{g,0} \).

Let now \( v \in \mathbb{S}^{J} \). Then the decomposition of the variation of the complex structure in corollary 3 implies the existence of unique \( \tau \in \Lambda^\Omega_{g,J} \) and \( \psi \in \Lambda^\Omega_{g,J} \) such that
\[
\theta^*_g = \partial^g_{T_X,J} \nabla g \tau + \partial g T X,J \nabla g \psi + A.
\] (15.5)

For bi-degree reasons the condition \( \nabla^{*g} \theta = 0 \) is equivalent to the identity
\[
0 = 2 \partial^g_{T_X,J} \partial^g_{T_X,J} \nabla g \tau + 2 \partial g T X,J \nabla g \psi.
\]
This last is equivalent to the equation
\[
0 = \nabla g \Delta^\Omega_{g,J} \tau + \Delta^\Omega_{g,J} \nabla g \psi,
\]
thanks to the complex Bochner identities (13.8) and (13.9). We remind that if \( u \in C^\infty (X, \mathbb{R}) \) satisfies \( \nabla g u = 0 \) then \( u = 0 \). (See the proof of the injectivity statement in lemma 10 in the section 13.) We conclude that the condition \( \nabla^{*g} \theta = 0 \) is equivalent to the equation (15.3) via the decomposition (15.5) of \( \theta \).

Then the required decomposition statement concerning the space \( \mathbb{S}^{J} \) follows from the fact that the condition \( \tau \) real valued is equivalent to the equation defining \( \psi \in \mathbb{E}^\Omega_{g,J} \). In order to see this we show first the commutation identity
\[
[\Delta^\Omega_{g,J}, B^\Omega_{g,J}] = 0.
\] (15.6)

Indeed using an arbitrary \( g \)-orthonormal local frame \( (e_k)_k \) we obtain
\[
\Delta^\Omega_{g,J} B^\Omega_{g,J} u = \Delta^\Omega_{g,J} [g(\nabla g u, J \nabla g f)]
\]
\[
= g(\Delta^\Omega_{g} \nabla g u, J \nabla g f) - 2 g(\nabla^2 g u \cdot e_k, J \nabla^2 g f \cdot e_k) + g(\nabla g u, J \Delta^\Omega_{g} \nabla g f)
\]
\[
= g(\Delta^\Omega_{g} \nabla g u + \nabla g u, J \nabla g f) - 2 \text{Tr}_{\mathbb{R}} (\nabla^2 g u J \nabla^2 g f)
\]
thanks to formula (8.2) applied to \( f \) and thanks to the fact that \( (\Delta^\Omega_{g} - 2 \mathbb{I}) f = 0 \).

Moreover the endomorphism \( J \nabla^2 g f \) is \( g \)-anti-symmetric since in the soliton case \([J, \nabla^2 g f] = 0\). We deduce
\[
\Delta^\Omega_{g,J} B^\Omega_{g,J} u = g(\nabla g \Delta^\Omega_{g} u, J \nabla g f) = B^\Omega_{g,J} \Delta^\Omega_{g} u.
\]

thanks to formula (8.2) applied to \( u \). We infer the identity (15.6) which implies
\[
[\Delta^\Omega_{g,J}, B^\Omega_{g,J}] = 2i [B^\Omega_{g,J}, \Delta^\Omega_{g,J}] = 0.
\] (15.7)
Multiplying both sides of (15.3) with $\Delta_{g,J}^{\Omega}$ we obtain

$$-\left(\Delta_{g,J}^{\Omega} \Delta_{g,J}^{\Omega}\right) \tau = \Delta_{g,J}^{\Omega}(\Delta_{g,J}^{\Omega} - 2\Pi)\psi.$$ \hspace{1cm} (15.8)

The invertible operator

$$\Delta_{g,J}^{\Omega} : C^\infty_{\overline{\Omega}}(X, \mathbb{C})_0 \rightarrow C^\infty_{\overline{\Omega}}(X, \mathbb{C})_0,$$

is real thanks to (15.7). We deduce that the condition $\tau$ real valued is equivalent to the left hand side of (15.8) being real valued, thus equivalent to the equation defining $\psi \in \mathbb{E}_{g,J}^{\Omega}$.

We observe finally that a variation $v \in \mathbb{S}_{g,\Omega}[0] \subset \mathbb{S}_{g,\Omega}(0)$ corresponds to $\varphi = -\tau$, i.e. to $(\varphi, \psi)$ solution of the equation (15.4). \hspace{1cm} $\square$

**Remark 1.** If we write $\psi = \psi_1 + i\psi_2$, with $\psi_1, \psi_2 \in C^\infty_{\overline{\Omega}}(X, \mathbb{R})_0$, then (15.3) is equivalent to the system

$$\begin{cases}
-\Delta_{g,J}^{\Omega} \tau = (\Delta_{g,J}^{\Omega} - 2\Pi)\psi_1 + B_{g,J}^{\Omega}\psi_2, \\
-B_{g,J}^{\Omega} \tau = (\Delta_{g,J}^{\Omega} - 2\Pi)\psi_2 - B_{g,J}^{\Omega}\psi_1.
\end{cases} \hspace{1cm} (S)$$

Moreover separating real and imaginary parts in the equation defining $\psi \in \mathbb{E}_{g,J}^{\Omega}$ and using the commutation identity (15.6) we obtain

$$\mathbb{E}_{g,J}^{\Omega} = \left\{ \psi \in \Lambda_{g,J}^{\Omega,+} \mid [\Delta_{g,J}^{\Omega}(\Delta_{g,J}^{\Omega} - 2\Pi) - (B_{g,J}^{\Omega})^2] \psi_2 = 2(\Delta_{g,J}^{\Omega} - \Pi)B_{g,J}^{\Omega}\psi_1 \right\},$$

Using (15.9) and the complex Bochner formula (13.9) we obtain also the identity

$$\mathbb{E}_{g,J}^{\Omega} = \left\{ \psi \in \Lambda_{g,J}^{\Omega,+} \mid -\text{div} \sum_{\partial T_{X,J}} \partial T_{X,J} \nabla g \psi_2 = (\Delta_{g,J}^{\Omega} - \Pi)B_{g,J}^{\Omega}\psi_1 \right\}.$$ 

**Remark 2.** We observe that the linear map

$$\Delta_{g,J}^{\Omega} : \Lambda_{g,J}^{\Omega,+} \rightarrow \Lambda_{g,J}^{\Omega,+},$$

is well defined and it represents an isomorphisms of complex vector spaces. In fact this follows from the identity

$$2 \int_X u \tau = \int_X \Delta_{g,J}^{\Omega} u \tau,$$

for all $v \in \Lambda_{g,J}^{\Omega}$. Thus the linear map

$$\Delta_{g,J}^{\Omega} \rightarrow \Pi : \Lambda_{g,J}^{\Omega,+} \rightarrow \Lambda_{g,J}^{\Omega,+},$$

is also well defined and represents an isomorphisms of complex vector spaces. The surjectivity of the latter follows from the finiteness theorem for elliptic operators. By definition of $\mathbb{E}_{g,J}^{\Omega}$ we deduce the existence of the isomorphism of real vector spaces

$$\Delta_{g,J}^{\Omega}(\Delta_{g,J}^{\Omega} - 2\Pi) : \mathbb{E}_{g,J}^{\Omega} \rightarrow \Lambda_{g,J}^{\Omega,+} \cap C^\infty_{\overline{\Omega}}(X, \mathbb{R})_0.$$

We notice also the inclusion

$$\Lambda_{g,J}^{\Omega,+} \cap C^\infty_{\overline{\Omega}}(X, \mathbb{R})_0 \supset (\Delta_{g,J}^{\Omega} - 2\Pi)(\Delta_{g,J}^{\Omega} - 2\Pi)C^\infty_{\overline{\Omega}}(X, \mathbb{R})_0.$$
16 The geometric meaning of the space $F^J_{g,\Omega}[0]$

We define the subspaces

\[ F^J_{g,\Omega}(0) := \{ (v,V) \in F_{g,\Omega} | v \in S^J_{g,\Omega}(0) \}, \]

\[ F^J_{g,\Omega}[0] := \{ (v,V) \in F_{g,\Omega} | v \in S^J_{g,\Omega}[0] \}. \]

In the previous section we gave a parametrization of the space $S^J_{g,\Omega}(0)$, and thus of $F^J_{g,\Omega}(0)$, which is fundamental for the computation of a general second variation formula for the $\mathcal{W}$ functional at a Kähler-Ricci soliton point. In this section we give a simpler parametrization of the sub-space $F^J_{g,\Omega}[0]$ and a useful geometric interpretation of it. Indeed let

\[ S_\omega := \{ (g,\Omega) \in M_\omega \times V_1 | \omega = \mathrm{Ric}_J(\Omega), J = g^{-1}\omega \}, \]

\[ [g,\Omega]_\omega := \text{Sym}^0(X,\omega) \cdot (g,\Omega) \subset S_\omega, (g,\Omega) \in S_\omega. \]

We denote by $TC_{S_\omega,(g,\Omega)}$ the tangent cone of $S_\omega$ at an arbitrary point $(g,\Omega) \in S_\omega$. With these notations we show the following useful fact.

**Lemma 17** For any point $(g,\Omega) \in S_\omega$ hold the identities (1.13) and (1.14).

We show first a quite general variation formula for the Ricci-Chern form.

**Lemma 18** Let $(g_t,J_t) \subset K\mathcal{S}_1$, $(\Omega_t) \subset V_1$ be two smooth families such that $J_t = (J_t)^{\mathcal{K}}_{g_t}$. Then hold the first variation formula

\[ 2 \frac{d}{dt} \mathrm{Ric}_J(\Omega_t) = -d \left( g_t \nabla^{\ast}_{g_t} J_t \nabla_{g_t} J_t + 2d_{J_t} \dot{\Omega}_t \right). \]  

(16.1)

**Proof** In the case of a fixed volume form $\Omega > 0$ we have the variation formula (see [Palö])

\[ 2 \frac{d}{dt} \mathrm{Ric}_J(\Omega) = -d \left( g \nabla^{\ast}_{g} J \nabla_{g} J \right). \]

For an arbitrary family $(\Omega_t) \subset V_1$ we fix an arbitrary time $\tau$ and we time derive at $t = \tau$ the decomposition

\[ \mathrm{Ric}_J(\Omega_t) = \mathrm{Ric}_J(\Omega_\tau) - dd^c_{J_t} \log \frac{\Omega_t}{\Omega_\tau}. \]

We obtain the required variation formula.

We show now that for any point $(g,\Omega) \in S_\omega$ hold the inclusion (1.10). Indeed for any smooth curve $(g_t,\Omega_t) \subset S_\omega$, with $(g_0,\Omega_0) = (g,\Omega)$ we have $\dot{g}_t = -J_t \dot{J}_t$ and thus

\[ 0 = 2 \frac{d}{dt} \mathrm{Ric}_J(\Omega_t) = -d \left[ \left( \nabla^{\ast}_{g_t} \dot{g}_t + \nabla_{g_t} \dot{\Omega}_t \right) - \omega \right], \]

66
thanks to the variation formula \[16.1\]. Then the inclusion \[1.10\] follows from \[1.9\]. We can provide at this point the proof of lemma 17.

**Proof** We remind that by the orthogonal decomposition in corollary 3 any element \(v \in \mathbb{D}_{g,[0]}^J\) decomposes as

\[v_g^* = \overline{\partial}_{TX,J} \nabla_g J \psi_v + A_v,\]

with unique \(\psi_v \in \Lambda^{0,J}_{g,J}\) and \(A_v \in \mathcal{H}^0_{g,\Omega}(TX,J)\). Moreover the weighted complex Bochner identity \[13.9\] implies the equality

\[
\overline{\partial}_{TX,J}^\alpha v_g^* + \nabla_g V_\Omega^* = \frac{1}{2} \nabla_{g,J} \left[ (\Delta_{g,J} - 2\mathbb{I})\psi_v + 2V_\Omega^* \right],
\]

(16.2) for any \((v,V) \in \mathbb{D}_{g,[0]}^J \times T_{V_1}\). Thus

\[
\mathbb{T}_{g,\Omega}^J [0] = \left\{ (v,V) \in \mathbb{D}_{g,[0]}^J \times T_{V_1} \mid (\Delta_{g,J} - 2\mathbb{I})\psi_v = -2V_\Omega^* \right\}.
\]

Let

\[
R_\psi := \text{Re} \left[ (\Delta_{g,J} - 2\mathbb{I})\psi \right],
\]

(16.4)

\[
I_\psi := \text{Im} \left[ (\Delta_{g,J} - 2\mathbb{I})\psi \right],
\]

(16.5)

(for any \(z \in \mathbb{C}\) we write \(z = \text{Re} z + i \text{Im} z\) and observe that \[16.2\] implies the identity

\[
(\nabla^\alpha_g v_g^* + \nabla_g V_\Omega^*) \cdot \omega = \frac{1}{2} dI_\psi + d_J^* (R_\psi + 2V_\Omega^*),
\]

for any \((v,V) \in \mathbb{D}_{g,[0]}^J \times T_{V_1}\). Thus

\[
\mathbb{T}_{g,\Omega}^J = \left\{ (v,V) \in \mathbb{D}_{g,[0]}^J \times T_{V_1} \mid R_\psi = -2V_\Omega^* \right\}.
\]

(16.6)

We notice now the equalities

\[
T_{[g,\Omega],(g,\Omega)} = \{ (L_\xi g, L_\xi \Omega) \mid \xi \in C^\infty(X,T_X) : L_\xi \omega = 0 \}
\]

\[
= \{ (L_J\nabla_g u, L_J\nabla_g \Omega) \mid u \in C^\infty_\Omega(X,\mathbb{R})_0 \}
\]

\[
= \{ (2gJ \overline{\partial}_{TX,J} \nabla_g u, \text{div}^\Omega(J\nabla_g u)\Omega) \mid u \in C^\infty_\Omega(X,\mathbb{R})_0 \},
\]

indeed

\[
(L_J\nabla_g u)_{g}^* = J\nabla^2_g u - \nabla^2_g u J = 2J \overline{\partial}_{TX,J} \nabla_g u.
\]

67
We deduce that \((v, V) \in T_{\varphi, \Omega}^{\perp, \omega} (g, \Omega)\) if and only if for all \(u \in C_0^\infty (X, \mathbb{R})\) hold the equalities

\[
0 = 2 \int_X \left[ \langle J\overline{\partial}_{X, J} \nabla_g u, u^* \rangle_g - \text{div}^\Omega (J \nabla_g u) \cdot V^*_g \right] \Omega
\]

\[
- 2 \int_X \left( \nabla_g u \cdot J \left( \overline{\partial}_{X, J} u^* + \nabla_g V^*_g \right) \right) \Omega
\]

\[
= 2 \int_X u \cdot \text{div}^\Omega \left[ J \left( \overline{\partial}_{X, J} u^* + \nabla_g V^*_g \right) \right] \Omega.
\]

If we assume \((v, V) \in \mathbb{T}_{g, \Omega}^J\), then

\[
\overline{\partial}_{X, J} v^* + \nabla_g V_g^* = -\frac{1}{2} J \nabla_g I_{\psi_v},
\]

thanks to (16.2) and (16.6). Thus if \((v, V) \in T_{\varphi, \Omega}^{\perp, \omega} (g, \Omega) \cap \mathbb{T}_{g, \Omega}^J\) then

\[
0 = - \int_X u \cdot \Delta_g I_{\psi_v} \Omega,
\]

for all \(u \in C_0^\infty (X, \mathbb{R}), \) i.e. \(\Delta_g I_{\psi_v} = 0\), which is equivalent to the condition \(I_{\psi_v} = 0\). We infer

\[
T_{\varphi, \Omega}^{\perp, \omega} (g, \Omega) \cap \mathbb{T}_{g, \Omega}^J \subseteq \mathbb{F}_{g, \Omega}^J [0].
\]

The reverse inclusion is obvious. We deduce the identity (1.13). Then the identity (1.14) follows from the inclusion (1.10).

17 The sign of the second variation of the \(\mathcal{W}\) functional at a Kähler-Ricci soliton point

**Proposition 1** Let \((J, g)\) be a KRS and let \(\Omega > 0\) be the unique smooth volume form such that \(gJ = \text{Ric}_J(\Omega)\) and \(\int_X \Omega = 1\). Let also \((g_t, \Omega_t)_{t \in \mathbb{R}} \subset \mathcal{M} \times \mathcal{V}_1\) be a smooth curve with \((g_0, \Omega_0) = (g, \Omega)\) and with \((g_0, \Omega_0) = (v, V) \in \mathbb{F}_{g, \Omega}^J (0)\). Then with the notations of lemma (16) hold the second variation formula

\[
\frac{d^2}{dt^2} |_{t=0} \mathcal{W}(g_t, \Omega_t) = \nabla_G D \mathcal{W}(g, \Omega)(v, V; v, V)
\]

\[
= \frac{1}{2} \int_X \left( \Delta_g^\Omega - \mathbb{I} \right) \left( \Delta_g^\Omega - 2 \mathbb{I} \right) \phi \cdot \left( \Delta_g^\Omega - 2 \mathbb{I} \right) \phi \Omega
\]

\[
- \frac{1}{2} \int_X \left[ (\Delta_g^\Omega - \mathbb{I}) \Delta_g^\Omega \tau \cdot \Delta_g^\Omega \tau + P^\Omega_{g, J} \text{Im} \psi \cdot \text{Im} \psi + |A|_g^2 \right] \Omega.
\]
where
\[ P_{g,J}^\Omega := \frac{1}{C_1} (\Delta_{g,J}^\Omega - 2\mathbb{I})(\Delta_{g,J}^\Omega - 2\mathbb{I}), \]
is a non-negative self-adjoint real elliptic operator with respect to the \( L^2_\Omega \)-hermitian product. In particular if \( (v,V) \in F_{g,\Omega} \) then
\[
\frac{d^2}{dt^2} W(g_t, \Omega_t) = \nabla G D W(g_t, \Omega_t)(v; v, V)
\]
\[
= -\frac{1}{2} \int_X \left[ 4 |(\Delta_g^\Omega - \mathbb{I})\varphi|^2 + P_{g,J}^\Omega \text{Im} \psi \cdot \text{Im} \psi + |A|^2 F \right] \Omega.
\]

**Proof** **STEP I.** Let \((g, \Omega)\) be a shrinking Ricci soliton point and let \((g_t, \Omega_t)_{t \in \mathbb{R}} \subset \mathcal{M} \times \mathcal{V}_1\) be a smooth curve with \((g_0, \Omega_0) = (g, \Omega)\) and with arbitrary speed \((\dot{g}, \dot{\Omega}) = (v,V) \in F_{g,\Omega}.\) We know from lemma \([7]\)
\[
\frac{d^2}{dt^2} W(g_t, \Omega_t) = \nabla G D W(g, \Omega)(v; v, V)
\]
\[
= -\frac{1}{2} \int_X \left[ (\nabla_g^\Omega v, v)_g + 2(\Delta_g^\Omega - 2\mathbb{I}) V_\Omega^* \cdot V_\Omega^* \right] \Omega.
\]
By the considerations in the beginning of section \([10]\) we deduce that in the soliton case \(h_{g,\Omega} = 0\) hold the identity
\[
(\nabla_g^\Omega)^2 L_g^\Omega v + 2(\Delta_g^\Omega - 2\mathbb{I}) V_\Omega^* V_\Omega = 0,
\]
for all \((v,V) \in F_{g,\Omega}.\) Applying the operator \(\nabla_g^\Omega\) to both sides of this identity we infer
\[
(\nabla_g^\Omega)^2 L_g^\Omega v + \Delta_g^\Omega (\Delta_g^\Omega - 2\mathbb{I}) V_\Omega^* V_\Omega = 0. \tag{17.2}
\]
For any function \(\varphi \in C_0^\infty(X, \mathbb{R})\) let \((v,V) := T_{g,\Omega}(\varphi, 0).\) Integrating by parts and using the identity \(\tag{17.2}\) we infer the equalities
\[
\nabla G D W(g, \Omega)(v; v, V)
\]
\[
= -\frac{1}{2} \int_X \left[ (\nabla_g^\Omega)^2 L_g^\Omega v \cdot \varphi + 2(\Delta_g^\Omega - 2\mathbb{I})(\Delta_g^\Omega - \mathbb{I}) \varphi \cdot (\Delta_g^\Omega - 2\mathbb{I}) \varphi \right] \Omega
\]
\[
= -\frac{1}{2} \int_X \Delta_g^\Omega (\Delta_g^\Omega - 2\mathbb{I}) \varphi \cdot \varphi \Omega
\]
\[
- \frac{1}{2} \int_X 2(\Delta_g^\Omega - 2\mathbb{I})(\Delta_g^\Omega - \mathbb{I}) \varphi \cdot (\Delta_g^\Omega - 2\mathbb{I}) \varphi \Omega
\]
\[
= \frac{1}{2} \int_X (\Delta_g^\Omega - \mathbb{I})(\Delta_g^\Omega - 2\mathbb{I}) \varphi \cdot (\Delta_g^\Omega - 2\mathbb{I}) \varphi \Omega.
\]
Remark 1. We can also compute the integral
\[ \int_X \langle L_g \nabla^2_x \varphi, \nabla^2_x \varphi \rangle_g \Omega, \]
in the previous expansion via the formula (9.3). Indeed
\[ \int_X \langle L_g \nabla^2_x \varphi, \nabla^2_x \varphi \rangle_g \Omega = \int_X \langle \nabla_g^2 (\Delta_g^\Omega - 2\Pi) \varphi, \nabla_g^2 \varphi \rangle_g \Omega \]
thanks to the identity (8.2). We conclude integrating by parts
\[ \int_X \langle L_g \nabla^2_x \varphi, \nabla^2_x \varphi \rangle_g \Omega = \int_X \Delta_g^\Omega (\nabla_g^2 (\nabla_g^2 (\Delta_g^\Omega - 2\Pi) \varphi, \nabla_g^2 (\nabla_g^2 \varphi \rangle_g \Omega, \]
Remark 2. We set \( \Phi := (\Delta_g^\Omega - 2\Pi) \varphi \in C^\infty_0(X, \mathbb{R})_0. \) Then the previous variation formula rewrites also as
\[ \frac{d^2}{dt^2} \bigg|_{t=0} W(g_t, \Omega) = \nabla_G D W(g, \Omega)(v, V; v, V) = \frac{1}{2} \int_X \left[ |\nabla_g \Phi|^2_g - \Phi^2 \right] \Omega \geq 0. \]
the last inequality follows from the variational characterization of the first eigenvalue of \( \Delta_g^\Omega, \]
\[ \lambda_1(\Delta_g^\Omega) = \inf \left\{ \frac{\int_X |\nabla_g u|^2_g \Omega}{\int_X u^2_g \Omega} \mid u \in C^\infty_0(X, \mathbb{R})_0 \setminus \{0\} \right\}, \]
which satisfies the inequality \( \lambda_1(\Delta_g^\Omega) > 1. \)

STEP II. Let \( (v, V) \in \mathcal{F}_g, \Omega. \) Using the \( L^2 \)-orthogonal decomposition (15.2) in the proof of lemma (15) we expand the integral term
\[ \int_X \langle L_g^\Omega v, v \rangle_g \Omega = \int_X \left[ \langle L_g^\Omega \nabla_g d\varphi, \nabla_g d\varphi \rangle_g + \langle L_g^\Omega \nabla_g d\varphi + L_g^\Omega \theta, \nabla_g d\varphi \rangle_g \right] \Omega. \]
We observe that
\[ \int_X \langle L_g^\Omega \theta, \nabla_g d\varphi \rangle_g = \int_X \langle \nabla_g^{\star \Omega} L_g^\Omega \theta, d\varphi \rangle_g \Omega = 0, \]
since \( \nabla_g^{\star \Omega} L_g^\Omega \theta = 0 \) thanks to the identity (17.1) applied to \( (\theta, 0) \in \mathcal{F}_g, \Omega. \) On the other hand formula (17.3) implies
\[ \int_X \langle L_g^\Omega \nabla_g d\varphi, \theta \rangle_g \Omega = \int_X \langle d(\Delta_g^\Omega - 2\Pi) \varphi, \nabla_g^{\star \Omega} \theta \rangle_g \Omega = 0. \]
We conclude the decomposition identity
\[
\int_X \langle L^\Omega_g v, v \rangle_g \Omega = \int_X \left[ \langle L^\Omega_g \nabla_g d\varphi, \nabla_g d\varphi \rangle_g + \langle L^\Omega_g \theta, \theta \rangle_g \right] \Omega.
\]

Then step I implies
\[
\frac{d^2}{dt^2} |_{t=0} \mathcal{W}(g_t, \Omega_t) = \nabla_G D \mathcal{W}(g, \Omega)(v, V; v, V)
= \frac{1}{2} \int_X (\Delta^\Omega_g - \Pi)(\Delta^\Omega_g - 2\Pi) \varphi \cdot (\Delta^\Omega_g - 2\Pi) \varphi \Omega
\]
\[
- \frac{1}{2} \int_X \langle L^\Omega_g \theta, \theta \rangle_g \Omega.
\]

On the other hand using the decomposition (15.5) of \( \theta \) and the decomposition formula (12.7) we infer
\[
\int_X \langle L^\Omega_g \theta, \theta \rangle_g \Omega = \int_X \left[ \langle L^\Omega_g \partial_{T,X,J} \nabla_g \tau, \partial_{T,X,J} \nabla_g \tau \rangle_g \Omega
+ \int_X \langle L^\Omega_g \partial_{T,X,J} \nabla_g J \psi, \partial_{T,X,J} \nabla_g J \psi + A \rangle_g \Omega
+ \int_X \langle L^\Omega_g A, \partial_{T,X,J} \nabla_g J \psi + A \rangle_g \Omega.
\]

Using the identities (12.8), (12.9) and the property (12.5) we deduce
\[
\int_X \langle L^\Omega_g \theta, \theta \rangle_g \Omega = \int_X \left[ \langle \partial_{T,X,J} \nabla_g (\Delta^\Omega_g - 2\Pi) \tau, \partial_{T,X,J} \nabla_g \tau \rangle_g \Omega
+ \int_X \langle \partial_{T,X,J} \nabla_g J (\Delta^\Omega_g - 2\Pi) \psi, \partial_{T,X,J} \nabla_g J \psi + A \rangle_g \Omega
+ \int_X \left[ \langle \partial_{T,X,J} L^\Omega_g A, \nabla_g J \psi \rangle_g + \langle L^\Omega_g A, A \rangle_g \right] \Omega.
\]

By bi-degree reasons \( \nabla^\omega g A = 0 \), which means \( (gA, 0) \in \mathbb{F}_{g, \Omega} \). We infer \( \nabla^\omega g L^\Omega_g A = 0 \) thanks to the identity (17.1). Then the property (12.2) implies
\[
\nabla_{T,X,J}^\omega L^\Omega_g A = 0,
\]
(17.3)

71
by bi-degree reasons. Integrating by parts further and using the weighted complex Bochner identities (13.8), (13.9) we obtain

$$
\int_X \langle \mathcal{L}^\Omega \theta, \theta \rangle_g \Omega = \int_X \left\langle \nabla_g (\Delta^\Omega_g - 2\underline{\Pi}) \tau, \partial^\sharp_{X,T_X} \partial^\flat_{X,T_X} \nabla_g \tau \right\rangle_g \Omega
$$

$$
+ \int_X \left\langle \nabla_g J (\Delta^\Omega_g - 2\underline{\Pi}) \bar{\psi}, \partial^\sharp_{X,T_X} \partial^\flat_{X,T_X} \nabla_g J \bar{\psi} \right\rangle_g \Omega
$$

$$
+ \int_X \left\langle \mathcal{L}^\Omega g A, A \right\rangle_g \Omega
$$

$$
= \frac{1}{2} \int_X \left\langle \nabla_g (\Delta^\Omega_g - 2\underline{\Pi}) \tau, \nabla_g J \Delta^\Omega_{g,J} \tau \right\rangle_g \Omega
$$

$$
+ \frac{1}{2} \int_X \left\langle \nabla_g J (\Delta^\Omega_g - 2\underline{\Pi}) \bar{\psi}, \nabla_g J (\Delta^\Omega_{g,J} - 2\underline{\Pi}) \bar{\psi} \right\rangle_g \Omega
$$

$$
+ \int_X \left\langle \mathcal{L}^\Omega g A, A \right\rangle_g \Omega.
$$

Using the integration by parts formulas (20.4) and (20.3) in the subsection 20.2 of the appendix, we deduce

$$
\int_X \langle \mathcal{L}^\Omega g \theta, \theta \rangle_g \Omega = \frac{1}{4} \int_X \Delta^\Omega_{g,J} (\Delta^\Omega_g - 2\underline{\Pi}) \tau \cdot \Delta^\Omega_{g,J} \tau \Omega
$$

$$
+ \frac{1}{4} \int_X \Delta^\Omega_{g,J} (\Delta^\Omega_g - 2\underline{\Pi}) \psi \cdot \Delta^\Omega_{g,J} \psi \Omega
$$

$$
+ \frac{1}{4} \int_X \Delta^\Omega_{g,J} (\Delta^\Omega_g - 2\underline{\Pi}) \bar{\psi} \cdot (\Delta^\Omega_{g,J} - 2\underline{\Pi}) \bar{\psi} \Omega
$$

$$
+ \frac{1}{4} \int_X \Delta^\Omega_{g,J} (\Delta^\Omega_g - 2\underline{\Pi}) \bar{\psi} \cdot (\Delta^\Omega_{g,J} - 2\underline{\Pi}) \bar{\psi} \Omega
$$

$$
+ \int_X \langle \mathcal{L}^\Omega g A, A \rangle_g \Omega.
$$
We observe now that the commutation identity (15.6) implies
\[
\int_X \langle L^\Omega_g \theta, \theta \rangle_g \Omega = \frac{1}{4} \int_X (\Delta^\Omega_g - 2\Pi) \Delta^\Omega_{g,j,\tau} \cdot \Delta^\Omega_{g,j,\tau} \Omega
\]
\[+ \frac{1}{4} \int_X (\Delta^\Omega_g - 2\Pi) \frac{\Delta^\Omega_{g,j,\tau}}{\sqrt{\Delta^\Omega_{g,j,\tau} \Omega}} \cdot \frac{\Delta^\Omega_{g,j,\tau}}{\sqrt{\Delta^\Omega_{g,j,\tau} \Omega}} \]
\[+ \frac{1}{4} \int_X (\Delta^\Omega_g - 2\Pi) \Delta^\Omega_{g,j,\psi} \cdot \frac{\Delta^\Omega_{g,j,\psi}}{\sqrt{\Delta^\Omega_{g,j,\psi} \psi \Omega}} \]
\[+ \frac{1}{4} \int_X (\Delta^\Omega_g - 2\Pi) \frac{\Delta^\Omega_{g,j,\psi}}{\sqrt{\Delta^\Omega_{g,j,\psi} \psi \Omega}} \cdot \frac{\Delta^\Omega_{g,j,\psi}}{\sqrt{\Delta^\Omega_{g,j,\psi} \psi \Omega}} \]
\[+ \int_X \langle L^\Omega_g A, A \rangle_g \Omega.
\]
Completing the square we obtain
\[
\int_X \langle L^\Omega_g \theta, \theta \rangle_g \Omega = \frac{1}{4} \int_X (\Delta^\Omega_g - 2\Pi) \Delta^\Omega_{g,j,\tau} \cdot \Delta^\Omega_{g,j,\tau} \Omega
\]
\[+ \frac{1}{4} \int_X (\Delta^\Omega_g - 2\Pi) \frac{\Delta^\Omega_{g,j,\tau}}{\sqrt{\Delta^\Omega_{g,j,\tau} \Omega}} \cdot \frac{\Delta^\Omega_{g,j,\tau}}{\sqrt{\Delta^\Omega_{g,j,\tau} \Omega}} \]
\[+ \frac{1}{4} \int_X (\Delta^\Omega_g - 2\Pi) (\Delta^\Omega_{g,j,\psi} \cdot \frac{\Delta^\Omega_{g,j,\psi}}{\sqrt{\Delta^\Omega_{g,j,\psi} \psi \Omega}}) \]
\[+ \frac{1}{4} \int_X (\Delta^\Omega_g - 2\Pi) \frac{\Delta^\Omega_{g,j,\psi}}{\sqrt{\Delta^\Omega_{g,j,\psi} \psi \Omega}} \cdot \frac{\Delta^\Omega_{g,j,\psi}}{\sqrt{\Delta^\Omega_{g,j,\psi} \psi \Omega}} \]
\[+ \frac{1}{2} \int_X (\Delta^\Omega_g - 2\Pi) \psi \cdot \frac{\Delta^\Omega_{g,j,\psi}}{\sqrt{\Delta^\Omega_{g,j,\psi} \psi \Omega}} \]
\[+ \frac{1}{2} \int_X (\Delta^\Omega_g - 2\Pi) \frac{\Delta^\Omega_{g,j,\psi}}{\sqrt{\Delta^\Omega_{g,j,\psi} \psi \Omega}} \cdot \frac{\Delta^\Omega_{g,j,\psi}}{\sqrt{\Delta^\Omega_{g,j,\psi} \psi \Omega}} \]
\[+ \int_X \langle L^\Omega_g A, A \rangle_g \Omega.
\]
Using the equation (15.3) we infer
\[
\int_X \langle L^\Omega \theta, \theta \rangle_g \Omega = \frac{1}{4} \int_X (\Delta^\Omega_g - 2\Pi) \left( \Delta^\Omega_{g,\tau} + \Delta^\Omega_{g,\bar{\tau}} \right) \cdot \left( \Delta^\Omega_{g,\tau} + \Delta^\Omega_{g,\bar{\tau}} \right) \Omega
\]
\[+ \int_X \left| (\Delta^\Omega_{g,\tau} - 2\Pi) \psi \right|^2 \Omega \]
\[+ \frac{i}{2} \int_X \left[ B^\Omega_{g,j} \psi \cdot \left( \Delta^\Omega_{g,\tau} - 2\Pi \right) \psi - B^\Omega_{g,j} \bar{\psi} \cdot \left( \Delta^\Omega_{g,\tau} - 2\Pi \right) \psi \right] \Omega \]
\[+ \int_X \langle L^\Omega A, A \rangle_g \Omega \]
\[= \int_X \left[ (\Delta^\Omega_g - 2\Pi) \Delta^\Omega_{g,\tau} \cdot \Delta^\Omega_{g,\bar{\tau}} + \left| \Delta^\Omega_{g,\tau} \right|^2 + \langle L^\Omega A, A \rangle_g \right] \Omega \]
\[- \frac{i}{2} \int_X \left[ B^\Omega_{g,j} \psi \cdot \Delta^\Omega_{g,\tau} - B^\Omega_{g,j} \bar{\psi} \cdot \Delta^\Omega_{g,\tau} \right] \Omega.
\]
We observe now that the operator \( B^\Omega_{g,j} \) is \( L^2_\Omega \)-anti-adjoint. This implies in particular
\[
\int_X \Delta^\Omega_{g,\tau} \cdot B^\Omega_{g,j} \tau \Omega = 0,
\]
and
\[
- \frac{i}{2} \int_X \left[ B^\Omega_{g,j} \psi \cdot \Delta^\Omega_{g,\tau} - B^\Omega_{g,j} \bar{\psi} \cdot \Delta^\Omega_{g,\tau} \right] \Omega
\]
\[= \frac{i}{2} \int_X \left[ \psi \cdot \Delta^\Omega_{g,j} B^\Omega_{g,j} \tau - \bar{\psi} \cdot \Delta^\Omega_{g,j} B^\Omega_{g,j} \bar{\tau} \right] \Omega
\]
\[= \frac{i}{2} \int_X \left[ \Delta^\Omega_{g,j} \psi - \Delta^\Omega_{g,j} \bar{\psi} \right] B^\Omega_{g,j} \tau \Omega
\]
\[= - \int_X \left( \Delta^\Omega_{g,\psi_1} + B^\Omega_{g,j} \psi_1 \right) B^\Omega_{g,j} \tau \Omega.
\]
thanks to the commutation identity (15.6). Thus
\[
\int_X \langle L^\Omega_\theta, \theta \rangle_g \Omega = \int_X \left[ (\Delta^\Omega_g - \Pi) \Delta^\Omega_{g,\tau} \cdot \Delta^\Omega_{g,\bar{\tau}} + \left| B^\Omega_{g,j} \tau \right|^2 + \langle L^\Omega_\theta A, A \rangle_g \right] \Omega
\]
\[- \int_X \left( \Delta^\Omega_\psi + B^\Omega_{g,j} \psi_1 \right) B^\Omega_{g,j} \tau \Omega
\]
\[= \int_X \left[ (\Delta^\Omega_g - \Pi) \Delta^\Omega_{g,\tau} \cdot \Delta^\Omega_{g,\bar{\tau}} + \langle L^\Omega_\theta A, A \rangle_g \right] \Omega
\]
\[- 2 \int_X \left( \Delta^\Omega_{\psi_2} + B^\Omega_{g,j} \psi_2 \right) B^\Omega_{g,j} \tau \Omega.
\]
thanks to the second equation in (S). Using again the second equation in (S) we expand the term

\[-2 \int_X (\Delta_g - \Delta_g \Omega) \psi_2 \cdot B_{g,j} \Omega = 2 \int_X (\Delta_g - \Delta_g \Omega) \psi_2 \cdot (\Delta_g - 2\Pi) \psi_2 \Omega\]

\[-2 \int_X (\Delta_g - \Delta_g \Omega) \psi_2 \cdot B_{g,j} \psi_1 \Omega\]

\[= \int_X \left[ (\Delta_g - \Delta_g \Omega) \psi_2 \right]^2 + \Delta_g \psi_2 \cdot (\Delta_g - 2\Pi) \psi_2 \Omega\]

\[-2 \int_X \psi_2 \cdot (\Delta_g - \Delta_g \Omega) \psi_1 \psi_2 \Omega\]

\[= \int_X \left[ (\Delta_g - \Delta_g \Omega) \psi_2 \right]^2 + (B_{g,j})^2 \psi_2 \psi_2 \Omega,\]

thanks to the expression (15.3). We observe further that the formula

\[P_{g,j} = (\Delta_g - \Delta_g \Omega)^2 + (B_{g,j})^2, \quad (17.4)\]

hold thanks to the commutation identity (15.6). We conclude

\[\int_X \langle L_g \theta, \theta \rangle_g \Omega = \int_X \left[ (\Delta_g - \Delta_g \Omega) \Delta_g \tau + P_{g,j} \psi_2 \psi_2 + \langle L_g A, A \rangle_g \right] \Omega,\]

which implies the required formula for the variations \((v, V) \in \mathbb{F}_{g, \Omega}(0)\).

**STEP III.** We compute now the stability integral involving \(A\). The trivial identity

\[L_g A = L_g A + \nabla_g f \nabla_g A,\]

combined with the formula (14.5) implies

\[L_g A = 2\bar{\partial}_{TX,j} \bar{\partial}_{TX,j} A + [\text{Ric}^*(g), A] + \nabla_g f \nabla_g A,\]

since \(\bar{\partial}_{TX,j} A = 0\). Integrating by parts we deduce

\[\int_X \langle L_g A, A \rangle_g \Omega = \int_X \left[ 2 \langle \bar{\partial}_{TX,j} \bar{\partial}_{TX,j} A, A \rangle_g + \langle \nabla_g f \nabla_g A, A \rangle_g \right] \Omega\]

\[= \int_X \left[ 2 \langle \bar{\partial}_{TX,j} A, \bar{\partial}_{TX,j} A \rangle_g + \frac{1}{2} \nabla_g f \cdot |A|_g^2 \right] \Omega\]

\[= \frac{1}{2} \int_X \Delta_g f \cdot |A|_g^2 \Omega\]

\[= \int_X F |A|_g^2 \Omega,\]
since \( \partial^\infty_{X,g,J} A = 0 \) and \((J,g)\) is a KS. (This last identity has been obtained by Hall-Murphy \[Ha-Mu2\] using a different integration by parts method.)

We show now the second variation formula corresponding to the particular case \((v,V) \in F^j_{g,\Omega}[0]\). With this assumption hold the relation \( \varphi = -\tau \). Thus we rearrange the expression

\[
E := \frac{1}{2} \int_X (\Delta^\Omega_g - \|)(\Delta^\Omega_g - 2 \|)\varphi \cdot (\Delta^\Omega_g - 2 \|)\varphi \Omega
\]

\[
- \frac{1}{2} \int_X (\Delta^\Omega_g - \|)\Delta^\Omega_g \varphi \cdot \Delta^\Omega_g \varphi \Omega
\]

\[
= \frac{1}{2} \int_X \left[ -4(\Delta^\Omega_g - \|)\Delta^\Omega_g \varphi \cdot \varphi + 4(\Delta^\Omega_g - \|)\varphi \cdot \varphi \right] \Omega
\]

\[
= -2 \int_X (\Delta^\Omega_g - \|)^2 \varphi \cdot \varphi \Omega
\]

\[
= -2 \int_X (\Delta^\Omega_g - \|)\varphi \Omega
\]

which implies the required formula in the particular case \((v,V) \in F^j_{g,\Omega}[0]\).

Let \( A := \Delta^\Omega_{g,J} - 2 \| \) and observe that \([A, \bar{A}] = -2i[\Delta^\Omega_{g,J}, B^\Omega_{g,J}] = 0\), thanks to \[15.3\]. Then the statement concerning the operator \( p^\Omega_{g,J} \) follows from the elementary lemma below. \(\square\)

**Lemma 19** Let \( H := L^2(X, \mathbb{C})_0 \) and \( A, B : D \subset H \rightarrow H \) be closed densely defined linear operators such that \( 0 \leq A = A^*, 0 \leq B = B^*, [A, B] = 0 \). If \( A \) and \( B \) are differential operators of same order with \( A \) elliptic then \( AB \geq 0 \). In particular if \([A, A] = 0\) then \( AA \geq 0 \).

**Proof** Let \( E_{\lambda_k}(A) \subset H \) be the eigenspace of \( A \) corresponding to an eigenvalue \( \lambda_k \in \mathbb{R}_{\geq 0} \). Then the identity \([A, B] = 0\) implies that the restriction \( B : E_{\lambda_k}(A) \rightarrow E_{\lambda_k}(A) \) is well defined and represents a non-negative self-adjoint operator. We deduce by the spectral theorem in finite dimensions the existence of an orthonormal basis \((e_{k,l})_{l \in I_k} \subset E_{\lambda_k}(A)\) such that \( Be_{k,l} = \mu_{k,l} e_{k,l} \), with \( \mu_{k,l} \in \mathbb{R}_{\geq 0} \). Moreover \( A e_{k,l} = \lambda_k e_{k,l} \). We consider a strictly monotone increasing parametrization \((\lambda_k)_k\). Then any \( u \in H \) writes as

\[
u = \sum_{k \geq 0} \sum_{l \in I_k} c_{k,l} e_{k,l},
\]

\( c_{k,l} \in \mathbb{C} \). In particular for \( u \in C^\infty(X, \mathbb{C})_0 \) hold the expressions

\[
Au = \sum_{k \geq 0} \sum_{l \in I_k} \lambda_k c_{k,l} e_{k,l},
\]

\[
Bu = \sum_{k \geq 0} \sum_{l \in I_k} \mu_{k,l} c_{k,l} e_{k,l}.
\]
The inequality in the general case \( u \in D \) follows from the density of the smooth functions in the graph topology of \( A \). In order to see that \( \bar{A} \geq 0 \) we observe the trivial equalities
\[
0 \leq \int_X A u \cdot \bar{u} \Omega = \int_X u \cdot \overline{A u} \Omega = \int_X \bar{A} v \cdot u \Omega = \int_X u \cdot \overline{A v} \Omega.
\]
with \( v := \overline{u} \). In order to show its self-adjointness we observe also the trivial equalities
\[
\int_X \bar{A} u \cdot \bar{v} \Omega = \int_X \bar{v} \cdot \overline{A u} \Omega = \int_X A v \cdot u \Omega = \int_X u \cdot \overline{A v} \Omega.
\]
□

We deduce the following corollary of proposition 1.

**Corollary 4** In the setting of proposition 1 assume \((v, V) \in \mathbb{F}^J_{gJ \Omega}[0]\) with \(A_v = 0\). Then
\[
\left. \frac{d^2}{dt^2} \right|_{t=0} W(g_t, \Omega_t) \leq 0,
\]
with equality if and only if \((v, V) = (0, 0)\).

We notice indeed that the equality hold if and only if \(\varphi = 0\).

### 17.1 The Kähler-Einstein case

In the Kähler-Einstein case the complex operator \(\Delta^{\Omega, J}_{gJ}\) reduces to the real operator \(\Delta^\Omega_g\). Let
\[
\Lambda_g := \text{Ker}_\mathbb{R}(\Delta_g - 2\mathbb{I}) \subset C^\infty(X, \mathbb{R})_0,
\]
and let \(\Lambda_g^\perp \subset C^\infty(X, \mathbb{R})_0\) be its \(L^2\)-orthogonal with respect to the measure \(dV_g\).

We observe the decomposition \(\Lambda^\Omega_{gJ} = \Lambda_g \oplus i\Lambda_g\), which implies the decomposition
\[
\Lambda^\Omega_{gJ}^\perp = \Lambda_g^\perp \oplus i\Lambda_g^\perp,
\]
and thus the identity \(\mathbb{E}^{\Omega}_{gJ} = \Lambda_g^\perp\). With the notations of lemma 16 let \(\Phi := (\Delta_g - 2\mathbb{I})\varphi\), and \(\Psi := (\Delta_g - 2\mathbb{I})\psi\). Then the second variation formulas in proposition 1 reduce to
\[
\left. \frac{d^2}{dt^2} \right|_{t=0} W(g_t, \Omega_t) = \nabla_G DW(g, \Omega)(v, V; v, V) = \frac{1}{2 \text{Vol}_g(X)} \int_X [(\Delta_g - \mathbb{I}) \Phi \cdot \Phi - (\Delta_g - \mathbb{I}) \Psi \cdot \Psi] dV_g,
\]
77
in the case \((v, V) \in \mathbb{F}_g'(0)\) and to
\[
\frac{d^2}{dt^2} \mathcal{W}(g_t, \Omega_t) = \nabla g \mathcal{D} \mathcal{W}(g, \Omega)(v, V; V, V) \\
= -\frac{2}{\text{Vol}_g(X)} \int_X |\nabla g^{-1}(\nabla g - \mathbb{I})\Psi|^2 \, dV_g \geq 0,
\]
in the case \((v, V) \in \mathbb{F}_g'(0)\], with equality if and only if \(v^*_g \in \mathcal{H}_g^{0,1}(T_X, J)\).

**Proof of step II in the Kähler-Einstein case.**

The most difficult part in the proof of proposition \([4]\) is the computation of the stability integral
\[
\int_X \langle L^\Omega g, \theta \rangle_g \Omega,
\]
in step II of the proof. In the Kähler-Einstein case the argument is much more simple. We include the details for readers convenience.

We remind first the isomorphism \(g^{-1} : S^2 g T_X^* \cong \text{End}_g(T_X)\). We have the \(g\)-orthogonal spiting
\[
\text{End}_g(T_X) = E'_{g,J} \oplus g E''_{g,J},
\]
where
\[
E'_{g,J} := \text{End}_g(T_X) \cap \mathcal{C}^\infty(X, T_{X,J}^* \otimes T_{X,J}),
\]
\[
E''_{g,J} := \text{End}_g(T_X) \cap \mathcal{C}^\infty(X, T_{X,-J} \otimes T_{X,J}).
\]

We observe that if \(\alpha \in \Lambda^2 g T_X^*\) then hold the identity \(\alpha^*_\omega = -g\alpha^*_J\), where \(\alpha^*_\omega := \omega^{-1}\alpha\). We define also the vector bundle
\[
\Lambda^{1,1}_{J,R} := \Lambda^{1,1}_{J} T_X^* \cap \Lambda^2 g T_X^* ,
\]
and we notice the isomorphism \(\omega^{-1} : \Lambda^{1,1}_{J,R} \cong E'_{g,J}\). Moreover the identity (3.4) combined with the properties (12.1) and (12.2) implies that the maps
\[
L^\Omega g : \mathcal{C}^\infty(X, E'_{g,J}) \longrightarrow \mathcal{C}^\infty(X, E'_{g,J}),
\]
(17.5)
\[
L^\Omega g : \mathcal{C}^\infty(X, E''_{g,J}) \longrightarrow \mathcal{C}^\infty(X, E''_{g,J}),
\]
(17.6)
are well defined. We observe also that by (12.3) and (20.7) we deduce the formula
\[
\omega L^\Omega g \alpha^*_\omega = \Delta^\Omega g \alpha + 2 R_g * \alpha,
\]
(17.7)
for all \(\alpha \in \mathcal{C}^\infty(X, \Lambda^{1,1}_{J,R})\). Notice indeed that the endomorphism \(J \alpha^*_J\) is \(g\)-anti-symmetric thanks to the \(J\)-linearity of \(\alpha^*_J\). We deduce using (17.7) and the identity \(\text{Tr}_\omega \alpha = \text{Tr}_R \alpha^*_g\),
\[
\text{Tr}_\omega \left( \Delta^\Omega g \alpha + 2 R_g * \alpha \right) = \text{Tr}_R \left( L^\Omega g \alpha^*_\omega \right).
\]
This combined with the identity (3.10), which in our case rewrites as

\[ \text{Tr}_R (R_g * \alpha^*_\omega) = \text{Tr}_R [\text{Ric}^*_\omega (g) \alpha^*_\omega], \]

implies that in the Einstein case hold the trace formula

\[ \text{Tr}_\omega (\Delta_g \alpha + 2R_g * \alpha) = (\Delta_g - 2\Pi) \text{Tr}_\omega \alpha. \quad (17.8) \]

We observe also the identity

\[ \langle \alpha, \beta \rangle_g = \langle \alpha^*_\omega, \beta^*_\omega \rangle_g, \quad (17.9) \]

for all \( \alpha, \beta \in \Lambda^{1,1}_{J,R}. \) Indeed we consider the equalities

\[
\begin{align*}
\langle \alpha, \beta \rangle_g &= \langle \alpha^*_\omega, \beta^*_\omega \rangle_g \\
&= \text{Tr}_R [\alpha^*_\omega (\beta^*_\omega)^T_g] \\
&= - \text{Tr}_R [\alpha^*_g \beta^*_g] \\
&= \text{Tr}_R [J \alpha^*_g \beta^*_g] \\
&= \text{Tr}_R [\alpha^*_\omega \beta^*_\omega] \\
&= \langle \alpha^*_\omega, \beta^*_\omega \rangle_g.
\end{align*}
\]

We deduce by the identity (20.6) in the appendix and by the Stokes theorem that over a compact Kähler manifold if \( \alpha, \beta \in C^\infty(X, \Lambda^{1,1}_{J,R}), d\alpha = d\beta = 0 \) satisfy \( \{\alpha\}_d = 0, \) or \( \{\beta\}_d = 0 \) then

\[ 2 \int_X \langle \alpha, \beta \rangle_g \, dV_g = \int_X \text{Tr}_\omega \alpha \, \text{Tr}_\omega \beta \, dV_g. \quad (17.10) \]

(Notice indeed the identity \( \langle \alpha, \beta \rangle_g = \langle \alpha, \beta \rangle_\omega \) for all \( \alpha, \beta \in \Lambda^{1,1}_{J,R}. \)) We decompose now the endomorphism section \( \theta^*_g = A'_g + A''_g \) and we estimate the integral

\[
\int_X \langle \mathcal{L}^\Omega_g \theta, \theta \rangle_g \, \Omega = \int_X \langle \mathcal{L}^\Omega_g \theta^*_g, \theta^*_g \rangle_g \, \Omega
\]

\[ = \int_X \left[ \langle \mathcal{L}^\Omega_g A'_g, A'_g \rangle_g + \langle \mathcal{L}^\Omega_g A''_g, A''_g \rangle_g \right] \Omega. \]

The last equality hold thanks to the identity (12.7). Let \( \alpha := \omega A'_g \) and assume \( \{\alpha\}_d = 0. \) Using the identity (20.11) we obtain

\[ \Delta^g \alpha + 2R_g * \alpha = (\Delta^g_{d,g} - 2\Pi) \alpha = d\nabla^*_g \alpha - 2\alpha, \]

79
and thus $\{\Delta g^\alpha + 2 R_g \ast \alpha\}_d = 0$. Then the identities (17.9), (17.7), (17.10) and (17.8) imply
\[
\int_X \langle \mathcal{L}_g A'_J, A'_J \rangle_g dV_g = \int_X \langle \Delta g \alpha + 2 R_g \ast \alpha, \alpha \rangle_g dV_g = \frac{1}{2} \int_X (\Delta g - 2 \mathbb{I}) \text{Tr}_\omega \alpha \cdot \text{Tr}_\omega \alpha dV_g \geq 0,
\]

since $\lambda_1(\Delta_g) \geq 2$ in the Kähler-Einstein case. (Notice that the condition $\int_X \text{Tr}_\omega \alpha dV_g = 0$ hold thanks to the assumption $\{\alpha\}_d = 0$.) In the set up of lemma 16 we have $A = i \partial J \partial J \tau$ and
\[
A''_J = \partial T_{X,J} \nabla_g \psi + A,
\]
with $\psi \in \Lambda^\perp$ and $A \in H^{0,1}(T_{X,J})$. Thus by the previous computation
\[
\int_X \langle \mathcal{L}_g A''_J, A''_J \rangle_g dV_g = \frac{1}{2} \int_X (\Delta g - 2 \mathbb{I}) \Delta g \tau \cdot \Delta g \tau dV_g.
\]
On the other hand formula (14.9) implies in the Kähler-Einstein case
\[
\mathcal{L}_g A''_J = 2 \Delta_{T_{X,g}} A''_J = 2 \overline{\partial}_{T_{X,J}} \overline{\partial}_{T_{X,J}} A''_J,
\]
since $\overline{\partial}_{T_{X,J}} A''_J = 0$. Integrating by parts we deduce
\[
\int_X \langle \mathcal{L}_g A''_J, A''_J \rangle_g dV_g = 2 \int_X \left| \overline{\partial}_{T_{X,J}} A''_J \right|^2_g dV_g = 2 \int_X \left| \overline{\partial}_{T_{X,J}} \nabla_g \psi \right|^2_g dV_g.
\]
In the Kähler-Einstein case the complex Bochner type formula (13.9) combined with the equation (15.3) implies
\[
2 \overline{\partial}_{T_{X,J}} \nabla_g \psi = \nabla_g (\Delta g - 2 \mathbb{I}) \psi = - \nabla_g \Delta g \tau.
\]
We obtain
\[
\int_X \langle \mathcal{L}_g A''_J, A''_J \rangle_g dV_g = \frac{1}{2} \int_X \left| \nabla_g \Delta g \tau \right|^2_g dV_g = \frac{1}{2} \int_X \Delta^2 g \tau \cdot \Delta g \tau dV_g,
\]
and thus the required formula
\[
\int_X \langle \mathcal{L}_g \theta, \theta \rangle_g dV_g = \int_X (\Delta g - \mathbb{I}) \Delta g \tau \cdot \Delta g \tau dV_g \geq 0.
\]
We notice also that this last implies the statement of theorem 2. Indeed in the equality case hold $\Delta g \tau = 0$ since $\lambda_1(\Delta_g) \geq 2$. Then the equation (15.3) implies $\psi = 0$. The conclusion follows from the decomposition identity (15.5).
Remark 3 We consider the particular case of a smooth curve \((g_t, \Omega_t)_t \subset S_\omega\) with \(g_0\) Kähler-Einstein metric. Time deriving twice the expression
\[
W(g_t, \Omega_t) = 2 \int_X \log \left( \frac{\omega^n}{\Omega_t} \right) \Omega_t - 2 \log n!,
\]
we infer
\[
\frac{d^2}{dt^2} W(g_t, \Omega_t) = -2 \int_X \left| \dot{\Omega}_t \right|^2 \Omega_0 - 2 \int_X \log \left( \frac{\Omega_0}{\omega^n} \right) \ddot{\Omega}_0
\]
thanks to the Kähler-Einstein condition. Then a trivial change of variables allows to deduce our previous second variation formula in the particular case \((\dot{g}_0, \dot{\Omega}_0) \in F_{g_0, \Omega_0}^J[0]\).

17.2 The case of variations in the direction \(T_{g_0}^J\Omega\)

Proposition 2 Let \((J, g)\) be a KRS and let \(\Omega > 0\) be the unique smooth volume form such that \(gJ = \text{Ric}_J(\Omega)\) and \(\int_X \Omega = 1\). Let also \((g_t, \Omega_t)_{t \in \mathbb{R}} \subset M \times V_1\) be a smooth curve with \((g_0, \Omega_0) = (g, \Omega)\) and with \((\dot{g}_0, \dot{\Omega}_0) = (v, V) \in T_{g_0, \Omega}^J\). Then with respect to the decomposition
\[
v_g^* = \overline{\partial}_{T_{X,J}} \nabla_{g,J} \psi + A,
\]
with unique \(\psi \in \Lambda^{1,0}_{g,J} \) and \(A \in H^1_{g,J}(T_{X,J})\), hold the second variation formula
\[
\frac{d^2}{dt^2} W(g_t, \Omega_t) = \nabla G DW(g, \Omega)(v, V; v, V)
\]
\[
= -\frac{1}{2} \int_X \left[ P_{g,J}^{\Omega} \text{Re} \psi \cdot \text{Re} \psi + |A|^2 \right] \Omega,
\]
where
\[
P_{g,J}^{\Omega} := (\Delta_{g,J}^{\Omega} - 2 \mathbb{I})(\Delta_{g,J}^{\Omega} - 2 \mathbb{I}),
\]
is a non-negative self-adjoint real elliptic operator with respect to the \(L_\Omega^2\)-hermitian product. Moreover if \((v, V) \in F_{g_0, \Omega}^J[0]\) then the previous formula writes as
\[
\frac{d^2}{dt^2} W(g_t, \Omega_t) = \nabla G DW(g, \Omega)(v, V; v, V)
\]
\[
= -\frac{1}{2} \int_X \left[ 4 |\psi|_\Omega^2 + P_{g,J}^{\Omega} \text{Im} \psi \cdot \text{Im} \psi + |A|^2 \right] \Omega.
\]
**Proof Step I.** Reconsidering a computation in the proof of step II of the proposition 1 we have that for all variations \( v \in \mathbb{D}^J_{g,[0]} \) hold the identity

\[
\int_X \langle L_g^0 v, v \rangle_g \Omega = \int_X \langle L_g^0 \overline{\partial}_{TX,J} \nabla_{g,J} \overline{\psi}, \overline{\partial}_{TX,J} \nabla_{g,J} \overline{\psi} + A \rangle_g \Omega
\]

\[
+ \int_X \langle L_g^0 A, \overline{\partial}_{TX,J} \nabla_{g,J} \overline{\psi} \rangle_g \Omega.
\]

Using the identity (12.9) and the property (12.5) we deduce

\[
\int_X \langle L_g^0 v, v \rangle_g \Omega = \int_X \langle \overline{\partial}_{TX,J} \nabla_{g,J} (\Delta_g^0 - 2\mathbb{I}) \overline{\psi}, \overline{\partial}_{TX,J} \nabla_{g,J} \overline{\psi} + A \rangle_g \Omega
\]

\[
+ \int_X \left[ \langle \overline{\partial}_{TX,J} L_g^0 A, \nabla_{g,J} \overline{\psi} \rangle_g + \langle L_g^0 A, A \rangle_g \right] \Omega.
\]

Using the identity (17.3), integrating by parts further and using the weighted complex Bochner identity (13.9) we obtain

\[
\int_X \langle L_g^0 v, v \rangle_g \Omega = \int_X \langle \nabla_{g,J} (\Delta_g^0 - 2\mathbb{I}) \overline{\psi}, \overline{\partial}_{TX,J} \nabla_{g,J} \overline{\psi} \rangle_g \Omega
\]

\[
+ \int_X \langle L_g^0 A, A \rangle_g \Omega
\]

\[
= \frac{1}{2} \int_X \langle \nabla_{g,J} (\Delta_g^0 - 2\mathbb{I}) \overline{\psi}, \nabla_{g,J} (\Delta_g^0 - 2\mathbb{I}) \overline{\psi} \rangle_g \Omega
\]

\[
+ \int_X \langle L_g^0 A, A \rangle_g \Omega.
\]

Using the integration by parts formula (20.3) in the subsection 20.2 of the appendix we infer

\[
\int_X \langle L_g^0 v, v \rangle_g \Omega = \frac{1}{4} \int_X (\Delta_g^0 - 2\mathbb{I}) \psi \cdot (\Delta_g^0 - 2\mathbb{I}) \overline{\psi} \Omega
\]

\[
+ \frac{1}{4} \int_X (\Delta_g^0 - 2\mathbb{I}) \psi \cdot (\Delta_g^0 - 2\mathbb{I}) \overline{\psi} \Omega
\]

\[
+ \int_X \langle L_g^0 A, A \rangle_g \Omega
\]

for all \( v \in \mathbb{D}^J_{g,[0]} \).

**STEP II.** We show first the variation formula in the case \( (v,V) \in \mathbb{E}^J_{g,\Omega,[0]} \) since the proof is simpler than in the general one. Using the expression (16.3)
for the space \( \mathbb{F}_{g,\Omega}(0) \) inside the identity (17.11) we deduce the equalities

\[
\int_X \langle L_g^\Omega v, v \rangle_g \Omega = - \int_X (\Delta_g^\Omega - 2\mathbb{I}) \text{Re}(\Delta_{g,J}^\Omega \psi) \cdot V_\Omega^* \Omega \\
+ \int_X \langle L_g^\Omega A, A \rangle_g \Omega \\
= - \int_X (\Delta_g^\Omega - 2\mathbb{I})(\Delta_g^\Omega \psi_1 + B_{g,J}^\Omega \psi_2) \cdot V_\Omega^* \Omega \\
+ \int_X \langle L_g^\Omega A, A \rangle_g \Omega.
\]

Let write \( \psi = \psi_1 + i\psi_2 \), with \( \psi_j \) real valued functions. Then the condition in the expression (16.3) rewrites as

\[
(\Delta_g^\Omega - 2\mathbb{I}) \psi_1 + B_{g,J}^\Omega \psi_2 = -2V_\Omega^*, \\
(\Delta_g^\Omega - 2\mathbb{I}) \psi_2 - B_{g,J}^\Omega \psi_1 = 0.
\]

We use now the condition (17.12) in the formula

\[
-2 \frac{d^2}{dt^2} |_{t=0} \mathcal{W}(g_t, \Omega_t) = -2\nabla_G D\mathcal{W}(g, \Omega)(v; v, V) \\
= \int_X \left[ \langle L_g^\Omega v, v \rangle_g - 2(\Delta_g^\Omega - 2\mathbb{I})V_\Omega^* \cdot V_\Omega^* \right] \Omega \\
= -2 \int_X (\Delta_g^\Omega - 2\mathbb{I})\psi_1 \cdot V_\Omega^* \Omega \\
+ \int_X \langle L_g^\Omega A, A \rangle_g \Omega.
\]

Using again the condition (17.12), we expand the integral

\[
-2 \int_X (\Delta_g^\Omega - 2\mathbb{I})\psi_1 \cdot V_\Omega^* \Omega \\
= \int_X (\Delta_g^\Omega - 2\mathbb{I})\psi_1 \cdot \left[ (\Delta_g^\Omega - 2\mathbb{I}) \psi_1 + B_{g,J}^\Omega \psi_2 \right] \Omega \\
= \int_X \psi_1 \left[ (\Delta_g^\Omega - 2\mathbb{I})^2 \psi_1 + B_{g,J}^\Omega (\Delta_g^\Omega - 2\mathbb{I}) \psi_2 \right] \Omega \\
= \int_X \psi_1 \left[ (\Delta_g^\Omega - 2\mathbb{I})^2 \psi_1 + (B_{g,J}^\Omega)^2 \psi_1 \right] \Omega,
\]

thanks to the identities (15.6) and (17.13). We infer using the formula (17.4)

\[
\frac{d^2}{dt^2} |_{t=0} \mathcal{W}(g_t, \Omega_t) = \nabla_G D\mathcal{W}(g, \Omega)(v; v, V) \\
= - \frac{1}{2} \int_X \left[ P_{g,J}^\Omega \psi_1 \cdot \psi_1 + \langle L_g^\Omega A, A \rangle_g \right] \Omega.
\]
Using again the condition (17.12) and the commutation identity (15.6) we expand the integral

\[ \int X |V^*| V^2 \Omega \]

\[ = \int X \left[ (\Delta^g g - 2\Pi)^2 \psi_1 \cdot \psi_1 + (B^\Omega g,J)^2 \psi_2 \cdot \psi_2 + 2\psi_1 \cdot B^\Omega g,J (\Delta^g g - 2\Pi) \psi_2 \right] \Omega, \]

thanks to the fact that the operator \( B^\Omega g,J \) is \( L^2 \)-anti-adjoint. Using again this fact and the condition (17.13) we deduce

\[ \int X 2\psi_1 \cdot B^\Omega g,J (\Delta^g g - 2\Pi) \psi_2 \Omega \]

\[ = \int X \psi_1 \cdot (B^\Omega g,J)^2 \psi_1 \Omega \]

\[ - \int X B^\Omega g,J \psi_1 \cdot (\Delta^g g - 2\Pi) \psi_2 \Omega \]

\[ = \int X \left[ (B^\Omega g,J)^2 \psi_1 \cdot \psi_1 - (B^\Omega g,J)^2 \psi_2 \cdot \psi_2 \right] \Omega, \]

and thus

\[ \int X |V^*| V^2 \Omega = \int X \left[ P^\Omega g,J \psi_1 \cdot \psi_1 - P^\Omega g,J \psi_2 \cdot \psi_2 \right] \Omega. \]

We infer the second variation formula

\[ \frac{d^2}{dt^2} |W(g_t, \Omega_t) = \nabla^2 G \nabla W(g, \Omega)(v, V; v, V) \]

\[ = \frac{1}{2} \int X \left[ 4|V^*| V^2 + P^\Omega g,J \psi_2 \cdot \psi_2 + \langle L^\Omega g A, A \rangle \right] \Omega. \]

The conclusion follows from the computation in the beginning of step III in the proof of the proposition [1].

**STEP III.** We show now the second variation formula in the more general case of variations \((v, V) \in \mathbb{T} g, \Omega \). We observe first that the general expression of \( \nabla^2 G W(g, \Omega) \) obtained at the end of the proof of lemma [7] implies that over a
shrinking-Ricci-Soliton point hold the variation formula

\[
-2 \frac{d^2}{dt^2} W(g_t, \Omega_t)_{t=0} = -2 \nabla_G D W(g, \Omega)(v, V; v, V)
\]

\[
= \int_X \left( \mathcal{L}_g^\Omega v - L v^g e + \nabla_g V^* \nabla_g g, v \right)_g \Omega
\]

\[
- 2 \int_X \left[ (\Delta_g^\Omega - 2 \mathbb{II}) V^*_\Omega - \text{div}^\Omega (\nabla_g^\Omega v^* + \nabla_g V^*\nabla_g) \right] V^*_\Omega \Omega
\]

\[
= \int_X \left[ \langle \mathcal{L}_g^\Omega v, v \rangle_g - 2(\Delta_g^\Omega - 2 \mathbb{II}) V^*_\Omega \cdot V^*_\Omega \right] \Omega
\]

\[
- 2 \int_X \left( \nabla_g (\nabla_g^\Omega v^* + \nabla_g V^*_\nabla_g^\Omega), v^* \right)_g \Omega
\]

\[
- 2 \int_X \left( \nabla_g^\Omega v^* + \nabla_g V^*_\nabla_g^\Omega \right)_g \Omega
\]

\[
= \int_X \left[ \langle \mathcal{L}_g^\Omega v, v \rangle_g - 2(\Delta_g^\Omega - 2 \mathbb{II}) V^*_\Omega \cdot V^*_\Omega - 2 \left| \nabla_g^\Omega v^* + \nabla_g V^*_\nabla_g \right|^2 \right] \Omega.
\]

for arbitrary directions \((v, V) \in T_{M^\Omega} V_\Omega\). Using now the fact that in the case \((v, V) \in T_{g^\Omega} V_\Omega\) hold the expressions \(R_\psi = -2V^*_\Omega\), (we use here the definitions \((16.4), (16.5)\) and \(16.7\)) we obtain

\[
-2 \nabla_G D W(g, \Omega)(v, V; v, V)
\]

\[
= \int_X \left[ \langle \mathcal{L}_g^\Omega v, v \rangle_g - \frac{1}{2} (\Delta_g^\Omega - 2 \mathbb{II}) R_\psi \cdot R_\psi - \frac{1}{2} |\nabla_g I_\psi|^2 \right] \Omega,
\]

for all \((v, V) \in T_{g^\Omega} V_\Omega\). Thanks to the commutation identity \((15.5)\) we can rewrite the identity \((17.11)\) as

\[
\int_X \left[ \langle \mathcal{L}_g^\Omega v, v \rangle_g - \langle \mathcal{L}_g^\Omega A, A \rangle_g \right] \Omega
\]

\[
= \frac{1}{4} \int_X (\Delta_g^\Omega - 2 \mathbb{II}) \Delta_g^\Omega, \psi \cdot (\Delta_g^\Omega, - 2 \mathbb{II}) \psi \Omega
\]

\[
+ \frac{1}{4} \int_X (\Delta_g^\Omega - 2 \mathbb{II}) \Delta_g^\Omega, \psi \cdot (\Delta_g^\Omega, - 2 \mathbb{II}) \psi \Omega.
\]

Adding and subtracting \(2\psi\) to the factor \(\Delta_g^\Omega \psi\) and respectively \(2\psi\) to the factor
\[ \Delta^\Omega_{g,J} \psi, \text{ we infer} \]
\[ \int_X \left[ \langle L_g^\Omega v, v \rangle_g - \langle L_g^\Omega A, A \rangle_g \right] \Omega \]
\[ = \frac{1}{4} \int_X (\Delta^\Omega_g - 2 \Pi) (\Delta^\Omega_{g,J} - 2 \Pi) \psi \cdot (\Delta^\Omega_{g,J} - 2 \Pi) \psi \Omega \]
\[ + \frac{1}{4} \int_X (\Delta^\Omega_g - 2 \Pi) (\Delta^\Omega_{g,J} - 2 \Pi) \psi \cdot (\Delta^\Omega_{g,J} - 2 \Pi) \psi \Omega \]
\[ + \frac{1}{2} \int_X (\Delta^\Omega_g - 2 \Pi) \psi \cdot (\Delta^\Omega_{g,J} - 2 \Pi) \psi \Omega \]
\[ + \frac{1}{2} \int_X (\Delta^\Omega_g - 2 \Pi) \psi \cdot (\Delta^\Omega_{g,J} - 2 \Pi) \psi \Omega. \]

We deduce the equalities

\[ -2 \nabla_G DW(g, \Omega)(v, V ; v, V) \]
\[ = \int_X \left[ (\Delta^\Omega_g - 2 \Pi) (\Delta^\Omega_{g,J} - 2 \Pi) \psi \cdot \bar{\psi} + \frac{1}{2} (\Delta^\Omega_g - 2 \Pi) I_\psi \cdot I_\psi - \frac{1}{2} |\nabla_g I_\psi|^2 \right] \Omega \]
\[ + \int_X \langle L_g^\Omega A, A \rangle_g \Omega \]
\[ = \int_X \left\{ \left[ P_\Omega^{g,J} - i B_\Omega^{g,J} (\Delta^\Omega_g - 2 \Pi) \right] \psi \cdot \bar{\psi} - I_\psi \cdot I_\psi + \langle L_g^\Omega A, A \rangle_g \right\} \Omega. \]

Using the expression

\[ I_\psi = (\Delta^\Omega_g - 2 \Pi) \psi_2 - B_\Omega^{g,J} \psi_1, \]

we find the formula

\[ -2 \nabla_G DW(g, \Omega)(v, V ; v, V) \]
\[ = \int_X \left[ P_\Omega^{g,J} - i B_\Omega^{g,J} (\Delta^\Omega_g - 2 \Pi) - (B_\Omega^{g,J})^2 \right] \psi \cdot \bar{\psi} \Omega \]
\[ - \int_X \left[ (\Delta^\Omega_g - 2 \Pi) \psi_2 \right]^2 + \left| B_\Omega^{g,J} \psi_1 \right|^2 - 2 (\Delta^\Omega_g - 2 \Pi) \psi_2 \cdot B_\Omega^{g,J} \psi_1 \right] \Omega \]
\[ + \int_X \langle L_g^\Omega A, A \rangle_g \Omega. \]

The fact that the operator \( B_\Omega^{g,J} \) is \( L_0^2 \)-anti-adjoint combined with the commutation identity \((15.\text{II})\) implies that \( B_\Omega^{g,J} (\Delta^\Omega_g - 2 \Pi) \) is also \( L_0^2 \)-anti-adjoint. We
deduce in particular the identity
\[ \int_X B_{g,J}^\Omega (\Delta_g^\Omega - 2\mathbb{I})\psi_j \cdot \psi_j \, \Omega = 0, \]
and thus the equality
\[ -2\nabla G D\mathcal{W}(g,\Omega)(v,V;v,V) = \int_X \left[ P_{g,J}^\Omega \psi_1 \cdot \psi_1 + P_{g,J}^\Omega \psi_2 \cdot \psi_2 \right] \Omega \]
\[ + \int_X \left[ B_{g,J}^\Omega (\Delta_g^\Omega - 2\mathbb{I}) \psi_2 \cdot \psi_1 - B_{g,J}^\Omega (\Delta_g^\Omega - 2\mathbb{I}) \psi_1 \cdot \psi_2 \right] \Omega \]
\[ - \int_X \left[ (B_{g,J}^\Omega)^2 \psi_1 \cdot \psi_1 + (B_{g,J}^\Omega)^2 \psi_2 \cdot \psi_2 \right] \Omega \]
\[ - \int_X \left[ \left| (\Delta_g^\Omega - 2\mathbb{I})\psi_2 \right|^2 + \left| B_{g,J}^\Omega \psi_1 \right|^2 - 2(\Delta_g^\Omega - 2\mathbb{I}) \psi_2 \cdot B_{g,J}^\Omega \psi_1 \right] \Omega \]
\[ + \int_X \langle L_g^\Omega A, A \rangle_g \Omega \]
Using the fact that the operator \( B_{g,J}^\Omega \) is \( L^2_\Omega \)-anti-adjoint and the commutation identity (15.6) we can simplify in order to obtain the required variation formula.

\[ \square \]

18 \quad \textbf{Positivity of the metric } \textit{G}_{g,\Omega} \textit{ over the space } \mathbb{T}_{g,\Omega}^J

\textbf{Lemma 20} \quad \text{For any } (g,\Omega) \in \mathcal{S}_\omega \text{ the restriction of the symmetric form } \textit{G}_{g,\Omega} \text{ to the vector space } \mathbb{T}_{g,\Omega}^J, \text{ with } J := g^{-1}\omega, \text{ is positive definite.}

\textbf{Proof} \quad \text{Let } (u,U),(v,V) \in \mathbb{T}_{g,\Omega}^J. \text{ Using the expression (16.6) for the space } \mathbb{T}_{g,\Omega}^J, \text{ we have}
\[ u_g^* = \mathcal{J}_{T_X,J} \nabla_{g,J} f + A, \]
\[ -2U_{\Omega}^* = \text{Re} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) f \right], \]
and
\[ v_g^* = \mathcal{J}_{T_X,J} \nabla_{g,J} \bar{f} + B, \]
\[ -2V_{\Omega}^* = \text{Re} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) f \right], \]
with unique \( \varphi, \psi \in \Lambda^\Omega_{g,J} \) and \( A, B \in H^0_{g,\Omega} (T_X,J) \). We decompose now the term

\[
G_{g,\Omega}(u,U;v,V) = \int_X \left[ \langle u,v \rangle_g - 2U_1^g \cdot V_1^g \right] \Omega
\]

\[
= \int_X \left[ \langle \partial^g_{T_X,J} \nabla g_j \varphi, \partial^g_{T_X,J} \nabla g_j \psi \rangle_g + \langle A, B \rangle_g \right] \Omega
\]

\[
- \frac{1}{2} \int_X \text{Re} \left[ (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \right] \text{Re} \left[ (\Delta^\Omega_{g,J} - 2\mathbb{I}) \psi \right] \Omega.
\]

Integrating by parts and using the weighted complex Bochner formula (13.9) we transform the integral

\[
I_1 := \int_X \langle \partial^g_{T_X,J} \nabla g_j \varphi, \partial^g_{T_X,J} \nabla g_j \psi \rangle_g \Omega
\]

\[
= \int_X \langle \partial^g_{T_X,J} \nabla g_j \varphi, \partial^g_{T_X,J} \nabla g_j \psi \rangle_g \Omega
\]

\[
= \frac{1}{2} \int_X \left[ \nabla g_j (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi, \nabla g_j \psi \right]_g \Omega.
\]

Using the integration by parts formula (20.4) in the subsection 20.2 of the appendix we deduce

\[
I_1 = \frac{1}{4} \int_X \left[ \Delta^\Omega_{g,J} (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot \psi + \Delta^\Omega_{g,J} (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot \psi \right] \Omega
\]

\[
= \frac{1}{4} \int_X \left[ (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot \Delta^\Omega_{g,J} \psi + (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot \Delta^\Omega_{g,J} \psi \right] \Omega.
\]

Adding and subtracting \( 2\psi \) to the factor \( \Delta^\Omega_{g,J} \psi \) and respectively \( 2\varphi \) to the factor \( \Delta^\Omega_{g,J} \varphi \), we infer

\[
I_1 = \frac{1}{2} \int_X \left[ (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot \psi + (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot \psi \right] \Omega
\]

\[
+ \frac{1}{4} \int_X \left[ (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot \Delta^\Omega_{g,J} \psi + (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot (\Delta^\Omega_{g,J} - 2\mathbb{I}) \psi \right] \Omega
\]

\[
= \frac{1}{2} \int_X \left[ (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot \psi + \varphi \cdot (\Delta^\Omega_{g,J} - 2\mathbb{I}) \psi \right] \Omega
\]

\[
+ \frac{1}{4} \int_X \left[ (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot (\Delta^\Omega_{g,J} - 2\mathbb{I}) \psi + (\Delta^\Omega_{g,J} - 2\mathbb{I}) \varphi \cdot (\Delta^\Omega_{g,J} - 2\mathbb{I}) \psi \right] \Omega.
\]
We deduce the general formula

\[ G_{g,\Omega}(u,U;v,V) = \int_X \left\{ \frac{1}{2} \left[ (\Delta_{g,J}^\Omega - 2\Pi) \varphi \cdot \overline{\psi} + (\Delta_{g,J}^\Omega - 2\Pi) \psi \cdot \overline{\varphi} \right] + (A,B)_g \right\} \Omega + \frac{1}{2} \int_X \text{Im} \left[ (\Delta_{g,J}^\Omega - 2\Pi) \varphi \right] \text{Im} \left[ (\Delta_{g,J}^\Omega - 2\Pi) \psi \right] \Omega. \]

In particular

\[ G_{g,\Omega}(u,U;u,U) = \int_X \left[ (\Delta_{g,J}^\Omega - 2\Pi) \varphi \cdot \overline{\psi} + |A|_g^2 \right] \Omega + \frac{1}{2} \int_X \left\{ \text{Im} \left[ (\Delta_{g,J}^\Omega - 2\Pi) \varphi \right] \right\}^2 \Omega \geq 0, \]

with equality if and only if \( \varphi = 0 \) and \( A = 0 \), i.e. \( (u,U) = (0,0) \), thanks to the variational characterization of the first eigenvalue \( \lambda_1(\Delta_{g,J}^\Omega) \geq 2 \) of the elliptic operator \( \Delta_{g,J}^\Omega \).

\[ \square \]

**Corollary 5** For any \((g,\Omega) \in S_\omega\) hold the identity

\[ \text{Ker}_R(\Delta_{g}^\Omega - 2\Pi) = \text{Ker}_R(\Delta_{g,J}^\Omega - 2\Pi), \]  

(18.1)

with \( J := g^{-1}\omega \).

**Proof** Let \( u \in C^\infty_X(X,\mathbb{R})_0 \) and \((\varphi_t)_{t \in \mathbb{R}} \subset \text{Symp}^0(X,\omega)\) the 1-parameter sub-group generated by the symplectic vector field \( \xi := (du)_u^* \). We set \( J_t := \varphi_t^* J, \ g_t := \varphi_t^* g, \Omega_t := \varphi_t^* \Omega \) and we compute \( \dot{g}_0 = L_\xi g \) and \( \dot{\Omega}_0 = L_\xi \Omega \). The expression of the tangent space to the symplectic orbit \([g,\Omega]_\omega\) in the proof of lemma 17 implies

\[ \dot{g}_0^* = -2J\overline{\nabla_{g,J} u}, \]

\[ \dot{\Omega}_0^* = -B_{g,J}^\Omega u. \]

Then the weighted complex Bochner formula (13.9) implies

\[ \overline{\nabla_{g,J}^\alpha} \dot{g}_0^* + \nabla_g \dot{\Omega}_0^* = -2J\overline{\nabla_{g,J}^\alpha} \overline{\nabla_{g,J} u} + \nabla_g \dot{\Omega}_0^* \]

\[ = -J\nabla_g (\Delta_{g}^\Omega - 2\Pi) u + \nabla_g B_{g,J}^\Omega u + \nabla_g \dot{\Omega}_0^* \]

\[ = -J\nabla_g (\Delta_{g}^\Omega - 2\Pi) u. \]

89
We deduce that \((\dot{g}_0, \dot{\Omega}_0) \in \mathbb{F}_{\Omega}^J [0]\) if and only if \((\Delta_{g,J}^{\Omega} - 2I)u = 0\). On the other hand the (strict) positivity of the metric \(G_{g,\Omega}\) over \(\mathbb{T}_{g,\Omega}^J \supset T_{[g,\Omega]_{\omega}}(g,\Omega)\) implies

\[
T_{[g,\Omega]_{\omega}}(g,\Omega) \cap T_{[\dot{g},\dot{\Omega}]_{\omega}}(g,\Omega) = 0.
\]

Then lemma \([17]\) implies

\[
T_{[g,\Omega]_{\omega}}(g,\Omega) \cap T_{[\dot{g},\dot{\Omega}]_{\omega}}(g,\Omega) = T_{[g,\Omega]_{\omega}}(g,\Omega) \cap \mathbb{F}_{\Omega}^J [0] = \{0\},
\]

So if \((\dot{g}_0, \dot{\Omega}_0) \in \mathbb{F}_{\Omega}^J [0]\) then \((\dot{g}_0, \dot{\Omega}_0) = (0,0)\). We infer the inclusion

\[
\text{Ker}_R (\Delta_{g,J}^{\Omega} - 2I) \subseteq \text{Ker}_R B_{g,J},
\]

and thus the required identity (18.1).

**18.1 Double splitting of the space \(\mathbb{T}_{g,\Omega}^J\)**

Let \(H^k\), with \(H^0 = L^2\), be a Sobolev space of sections over \(X\). For any subset \(S\) of smooth sections over \(X\) we denote with \(H^k_S\) its closure with respect to the \(H^k\)-topology. The pseudo-Riemannian metric \(G_{g,\Omega}\) is obviously continuous with respect to the \(L^2\)-topology. At the moment we are unable to say if the topology induced by \(G_{g,\Omega}\) over \(L^2 \mathbb{T}_{g,\Omega}^J\) is equivalent with the \(L^2\)-topology. Nevertheless we can show the following basic decomposition result

**Corollary 6** For any \((g, \Omega) \in \mathcal{S}_{\omega}\) hold the decomposition identity

\[
L^2 \mathbb{T}_{g,\Omega}^J = L^2 T_{[g,\Omega]_{\omega}}(g,\Omega) \oplus L^2 \mathbb{F}_{\Omega}^J [0],
\]

with \(J := \frac{1}{g} \omega\).

**Proof** We set

\[
\Lambda_{g,R}^{\Omega,\perp} := \text{Ker}_R (\Delta_{g,J}^{\Omega} - 2I),
\]

and let \(\Lambda_{g,R}^{\Omega,\perp} \subset L^2(X, \mathbb{R})_0\) be its \(L^2\)-orthogonal with respect to the measure \(\Omega\). Then corollary \([6]\) and its proof show that the map

\[
\chi : \Lambda_{g,R}^{\Omega,\perp} \cap C_0^\infty (X, \mathbb{R})_0 \longrightarrow T_{[g,\Omega]_{\omega}}(g,\Omega),
\]

\[
\varphi \longmapsto \left(2\omega \overrightarrow{\partial_{X,J}} \nabla_g \varphi, (B_{g,J}^{\Omega} \varphi) \right),
\]

is an isomorphism. We notice also that the expression of the metric \(G_{g,\Omega}\) obtained at the end of the proof of lemma \([20]\) hold true for arbitrary functions \(\Phi\) and \(\Psi\). So we put \((u, U) := \chi (\varphi)\) and \(\Phi = -2\varphi\) in this formula. Using the fact that the operator \(B_{g,J}^{\Omega}\) is \(L^2\)-anti-adjoint and the expression

\[
\text{Im} \left[ (\Delta_{g,J}^{\Omega} - 2I) \Phi \right] = -2(\Delta_{g,J}^{\Omega} - 2I) \varphi,
\]

90
we infer

\[ G_{g, \Omega}(u, U; u, U) = 2 \int_X \left[ \left| (\Delta_\Omega^g - 2\Pi) \varphi \right|^2 + 2(\Delta_\Omega^g - 2\Pi) \varphi \cdot \varphi \right] \Omega \]

\[ =: \gamma_{g, \Omega}(\varphi, \varphi) \geq 0, \]

(with equality if and only if \( \varphi = 0 \)). We remark now that the proof of the weighted Bochner formula (13.9) shows the identity

\[ -2 \text{div} \Omega \overline{\delta}_{T_{X,J}} \delta_{T_{X,J}} \nabla_g = \Delta_\Omega^g (\Delta_\Omega^g - 2\Pi) - (B_\Omega^g)^2. \]

Thus the operator

\[ \left( \overline{\delta}_{T_{X,J}} \nabla_g \right)^{\ast g, \Omega} \delta_{T_{X,J}} \nabla_g = -\text{div} \Omega \overline{\delta}_{T_{X,J}}^{\ast g, \Omega} \delta_{T_{X,J}} \nabla_g, \]

is elliptic. This implies (see for example [Ebi]) that the image

\[ \overline{\delta}_{T_{X,J}} \nabla_g \left[ H^2(X, \mathbb{R}) \right] \subset L^2, \]

is closed in the \( L^2 \)-topology. We infer that the map

\[ \overline{\delta}_{T_{X,J}} \nabla_g : \Lambda_{g, \mathbb{R}}^{\Omega, \perp} \cap H^2(X, \mathbb{R}) \longrightarrow \overline{\delta}_{T_{X,J}} \nabla_g \left[ H^2(X, \mathbb{R}) \right] \subset L^2, \]

is a topological isomorphism. We deduce that the extension in the sense of distributions

\[ \chi : \Lambda_{g, \mathbb{R}}^{\Omega, \perp} \cap H^2(X, \mathbb{R}) \longrightarrow L^2 T_{[g, \Omega]_{\omega, (g, \Omega)}}, \]

of the map \( \chi \) is also a topological isomorphism and a \((\gamma_{g, \Omega}, G_{g, \Omega})\)-isometry. The fact that the map

\[ \Delta_\Omega^g - 2\Pi : \Lambda_{g, \mathbb{R}}^{\Omega, \perp} \cap H^2(X, \mathbb{R}) \longrightarrow \Lambda_{g, \mathbb{R}}^{\Omega, \perp}, \]

is a topological isomorphism provides the estimate

\[ \gamma_{g, \Omega}(\varphi, \varphi) \geq 2 \int_X \left| (\Delta_\Omega^g - 2\Pi) \varphi \right|^2 \Omega \]

\[ \geq 2 \| (\Delta_\Omega^g - 2\Pi)^{-1} \| \| \varphi \|^2_{H^2}. \]

Then the Lax-Milgram theorem implies that the map

\[ \gamma_{g, \Omega} : \Lambda_{g, \mathbb{R}}^{\Omega, \perp} \cap H^2(X, \mathbb{R}) \longrightarrow \left[ \Lambda_{g, \mathbb{R}}^{\Omega, \perp} \cap H^2(X, \mathbb{R}) \right]^{\ast}, \]

is a topological isomorphism. (The sign * here denotes the topological dual). We infer that the restricted map

\[ G_{g, \Omega} : L^2 T_{[g, \Omega]_{\omega, (g, \Omega)}} \longrightarrow \left[ L^2 T_{[g, \Omega]_{\omega, (g, \Omega)}} \right]^{\ast}, \]

is also a topological isomorphism. (The sign * here denotes the topological dual).
is also a topological isomorphism thanks to the fact that the extended map \( \chi \) is a \((\gamma, \Omega, G, \Omega)\)-isometry. Applying the elementary lemma \([\text{21}]\) below to the spaces \( E := L^2(X, S^2T_X^\gamma) \oplus L^2 \Omega(X, \mathbb{R})_0 \) and \( V := L^2T_{[g, \Omega], (g, \Omega)} \) we deduce the \( G \)-orthogonal decomposition

\[
L^2(X, S^2T_X^\gamma) \oplus L^2 \Omega(X, \mathbb{R})_0 = L^2T_{[g, \Omega], (g, \Omega)} \oplus L^2T_{[g, \Omega], (g, \Omega)},
\]

and thus

\[
L^2T_{g, \Omega}^J = L^2T_{[g, \Omega], (g, \Omega)} \oplus \left[ L^2T_{[g, \Omega], (g, \Omega)} \cap L^2T_{g, \Omega}^J \right] = L^2T_{[g, \Omega], (g, \Omega)} \oplus L^2 \left[ T_{[g, \Omega], (g, \Omega)} \cap T_{g, \Omega}^J \right].
\]

Then the conclusion follows from the identity \([\text{1.13}]\). \( \Box \)

**Lemma 21** Let \( E \) be a real Banach space, \( E^* \) its topological dual and \( G : E \times E \to \mathbb{R} \) be a topologically non degenerate bilinear form, i.e. \( G : E \to E^* \) is an isomorphism. If there exists a closed subspace \( V \subset E \) such that the restriction \( G : V \times V \to \mathbb{R} \) is also topologically non degenerate then \( E = V \oplus V^\perp \).

**Proof** Let \( j : V \to E \) be the canonical inclusion and notice the trivial identity

\[
V^\perp := \{ \alpha \in E^* \mid \alpha \cdot v = 0, \forall v \in V \} = \text{Ker } j^*.
\]

By assumption for any element \( e \in E \) there exists a unique \( v \in V \) such that \( j^*(e) = j^*(v) \). Thus \( (e - v) - G \in V^\perp \). By definition the restriction \( G : V^\perp \to V^\perp \) provides an isomorphism. We conclude \( e - v \in V^\perp \). \( \Box \)

We notice that the condition \( V \cap V^\perp = \{0\} \) is equivalent to \( \text{Ker}(G : V \to V^*) = \{0\} \) but in general not sufficient to insure the surjectivity of \( G : V \to V^* \).

### 18.2 Triple splitting of the space \( T_{g, \Omega}^J \)

By abuse of notations we will denote by \( G_{g, \Omega} \) the scalar product over \( \Lambda_{g, J}^{\Omega, \perp} \subset C^\infty \) induced by the isomorphism

\[
\eta : \Lambda_{g, J}^{\Omega, \perp} \oplus H_{g, \Omega}^{\emptyset, 1}(T_X, J) \to T_{g, \Omega}^J,
\]

\[
(\psi, A) \mapsto \left( g \left( \nabla_{T_X, J} \nabla_{g, J} \overline{\psi} + A \right), -\frac{1}{2} \text{Re } \left[ (\Delta^\Omega_{g, J} - 2\Pi) \psi \right] \Omega \right).
\]

Explicitly

\[
G_{g, \Omega}(\varphi, \psi) = \frac{1}{2} \int_X \left[ (\Delta^\Omega_{g, J} - 2\Pi) \varphi \cdot \overline{\psi} + (\Delta^\Omega_{g, J} - 2\Pi) \psi \cdot \overline{\varphi} \right] \Omega
\]

\[
+ \frac{1}{2} \int_X \text{Im } \left[ (\Delta^\Omega_{g, J} - 2\Pi) \varphi \right] \text{Im } \left[ (\Delta^\Omega_{g, J} - 2\Pi) \psi \right] \Omega.
\]

92
By abuse of notations also we will consider from now on \( \Lambda^{\Omega,\perp}_{g,J} \subset C^\infty \). We introduce the vector space

\[
\mathbb{E}^J_{g,\Omega} := \left\{ u \in \Lambda^{\Omega,\perp}_{g,J} \mid (\Delta^\Omega_{g,J} - 2\Pi) u \in \Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega (X, \mathbb{R}) \right\},
\]

and we observe that the expression (16.3) for the space \( \mathbb{E}^J_{g,\Omega}[0] \) shows that the map \( \eta \) restricts to the isomorphism

\[
\eta : \mathbb{E}^J_{g,\Omega} \oplus \mathcal{H}^{0,1}_{g,\Omega} (T_X,J) \longrightarrow \mathbb{E}^J_{g,\Omega}[0].
\]

The subspaces \( \mathbb{E}^J_{g,\Omega}[0] := \eta \mathbb{E}^J_{g,\Omega} \) and \( \mathcal{H}^{0,1}_{g,\Omega} (T_X,J) \subset \mathbb{E}^J_{g,\Omega}[0] \) embedded via the previous isomorphism) are \( G \)-orthogonal thanks to the expression of the restriction of \( G \) over \( \mathbb{E}^J_{g,\Omega}[0] \) computed in the proof of lemma 20. We deduce the \( G \)-orthogonal decomposition

\[
\mathbb{E}^J_{g,\Omega}[0] = \mathbb{E}^J_{g,\Omega}[0] \oplus \mathcal{H}^{0,1}_{g,\Omega} (T_X,J).
\]

Let now

\[
\mathcal{D}^J_{g,\Omega} := (\mathbb{E}^J_{g,\Omega})^{\perp_G} \cap \Lambda^{\Omega,\perp}_{g,J},
\]

and observe that the decomposition in corollary 4 implies

\[
L^2 T_{[g,\Omega],(g,\Omega)} = \left[ L^2 \mathbb{E}^J_{g,\Omega}[0] \right]^{\perp_G} \cap L^2 \mathbb{T}^J_{g,\Omega} = \mathbb{E}^J_{g,\Omega}[0]^{\perp_G} \cap L^2 \mathbb{T}^J_{g,\Omega},
\]

and thus

\[
T_{[g,\Omega],(g,\Omega)} = \mathbb{E}^J_{g,\Omega}[0]^{\perp_G} \cap \mathbb{T}^J_{g,\Omega}.
\]

We deduce that the map \( \eta \) restricts to a \( G \)-isometry

\[
\eta : \mathcal{D}^J_{g,\Omega} \longrightarrow T_{[g,\Omega],(g,\Omega)}.
\]

Furthermore the decomposition in corollary 4 implies also the \( G \)-orthogonal decomposition

\[
H^2 \Lambda^{\Omega,\perp}_{g,J} = H^2 \mathcal{D}^J_{g,\Omega} \oplus_G H^2 \mathbb{E}^J_{g,\Omega},
\]

since \( \eta \) extends to an isomorphism

\[
\eta : H^2 \Lambda^{\Omega,\perp}_{g,J} \oplus \mathcal{H}^{0,1}_{g,\Omega} (T_X,J) \longrightarrow L^2 \mathbb{T}^J_{g,\Omega}.
\]

We observe now the following elementary lemma.

**Lemma 22** Let \( T : D \subset L^2 (X, \mathbb{C}) \longrightarrow L^2 (X, \mathbb{C}) \) be a closed densely defined \( L^2_{\Omega} \)-self-adjoint operator such that \( [T, \overline{T}] = 0 \). Then

\[
\ker(T\overline{T}) \cap L^2 (X, \mathbb{R}) = \{ \text{Re} u \mid u \in \ker T \}.
\]
Proof The assumption \([T, T] = 0\) implies that the restriction \(\overline{T} : \text{Ker} T \rightarrow \text{Ker} T\) is well defined. This combined with the fact that \(\overline{T}\) is also \(L^2_\Omega\)-self-adjoint implies that the restriction
\[
\overline{T} : D \cap (\text{Ker} T)^\perp \rightarrow (\text{Ker} T)^\perp,
\]
is also well defined. The inclusion \(\text{Ker}(\overline{T}T) \supset \text{Ker} T + \text{Ker} \overline{T}\) is obvious. In order to show the reverse inclusion let \(u \in \text{Ker}(\overline{T}T)\), i.e. \(\overline{T}u \in \text{Ker} T\), and consider the decomposition \(u = u_1 + u_2\) with \(u_1 \in \text{Ker} T\) and \(u_2 \in (\text{Ker} T)^\perp\). Then \(\overline{T}u \in \text{Ker} T\) if and only if \(\overline{T}u_2 \in \text{Ker} T\) since \(\overline{T}u_1 \in (\text{Ker} T)^\perp\). We infer the reverse inclusion. Thus
\[
\text{Ker}(\overline{T}T) = \{ u + v \mid u, v \in \text{Ker} T \},
\]
which implies the required conclusion. \(\square\)

We remind that if \((g, \Omega) \in S_\omega\) is a KS with \(J := g^{-1}\omega\) then
\[
\left[ \Delta^\Omega_{g,j} - 2\Pi, \Delta^\Omega_{g,j} - 2\Pi \right] = 0,
\]
which allows to apply the previous lemma to the \(L^2_\Omega\)-self-adjoint operator \(P^\Omega_{g,j}\). Thus
\[
\text{Ker} P^\Omega_{g,j} \cap C^\infty_\Omega (X, \mathbb{R})_0 = \{ \text{Re} u \mid u \in \Lambda^\Omega_{g,j}\} =: \text{Re} \Lambda^\Omega_{g,j}.
\]
The finiteness theorem for elliptic operators implies
\[
P^\Omega_{g,j} C^\infty_\Omega (X, \mathbb{R})_0 = \left( \text{Re} \Lambda^\Omega_{g,j} \right)^\perp \cap C^\infty_\Omega (X, \mathbb{R})_0 \supseteq \Lambda^\Omega_{g,j} \cap C^\infty_\Omega (X, \mathbb{R})_0.
\]
The last inclusion is obvious. The inclusion \(P^\Omega_{g,j} C^\infty_\Omega (X, \mathbb{R})_0 \subseteq \Lambda^\Omega_{g,j} \cap C^\infty_\Omega (X, \mathbb{R})_0\) is also obvious. We conclude
\[
P^\Omega_{g,j} C^\infty_\Omega (X, \mathbb{R})_0 = \Lambda^\Omega_{g,j} \cap C^\infty_\Omega (X, \mathbb{R})_0 = \left( \text{Re} \Lambda^\Omega_{g,j} \right)^\perp \cap C^\infty_\Omega (X, \mathbb{R})_0. \quad (18.3)
\]

Lemma 23 If \((g, \Omega) \in S_\omega\) is a KS then hold the identity
\[
\mathcal{D}^j_{g,\Omega} = \left\{ \psi \in \Lambda^\Omega_{g,j} \cap \left( P^\Omega_{g,j} \text{Re} \psi = 0 \right) \right\},
\]
with \(J := g^{-1}\omega\).

Proof We notice that for any \(\varphi \in \mathcal{E}^j_{g,\Omega}\) and \(\psi \in \mathcal{D}^j_{g,\Omega}\) hold the identity
\[
0 = G_{g,\Omega}(\varphi, \psi) = \int_X (\Delta^\Omega_{g,j} - 2\Pi) \varphi \cdot \text{Re} \psi \Omega.
\]

94
We infer
\[ D_{g,\Omega}^J = \left\{ \psi \in L_{g,J}^{\Omega,\perp} \mid \Re \psi \in \left[ L_{g,J}^{\Omega,\perp} \cap C_{\Omega}^\infty (X, \mathbb{R})_0 \right] \right\} \]
\[ = \left\{ \psi \in L_{g,J}^{\Omega,\perp} \mid \Re \psi \in \Re L_{g,J}^{\Omega} \right\}, \]
thanks to (18.3).
\[ \square \]

19 Infinitesimal properties of the function \( H \)

We observe that lemma 5 implies; \((g, \Omega) \in \mathcal{S}_\omega\) is a KS if and only if \( H_{g,\Omega} = 0 \). Furthermore the identity (4.18) rewrites as
\[ 2H_{g,\Omega} = - (\Delta_{g,J}^{\Omega} - 2\Pi)F \in L_{g,J}^{\Omega,\perp} \cap C_{\Omega}^\infty (X, \mathbb{R})_0, \]
for all \((g, \Omega) \in \mathcal{S}_\omega\). We show now the following fact

**Lemma 24** If \((g, \Omega) \in \mathcal{S}_\omega\) is a KS then the linear map
\[ D_{g,\Omega} H : \mathbb{F}_{g,\Omega}^J [0] \rightarrow L_{g,J}^{\Omega,\perp} \cap C_{\Omega}^\infty (X, \mathbb{R})_0, \] (19.1)
with \( J := g^{-1}\omega \), is well defined and represents an isomorphism of real vector spaces.

**Proof** The identity \( 2H_{g,\Omega} = 2H_{g,\Omega} - \mathcal{W}(g, \Omega) \) combined with the basic variation formula (1.5) implies
\[ 2D_{g,\Omega}H (v, V) = (\Delta_g^{\Omega} - 2\Pi)V^\ast, \]
for all \((v, V) \in \mathbb{F}_{g,\Omega}\) over a shrinking Ricci soliton point \((g, \Omega)\). In our KS set up this last rewrites as
\[ 2D_{g,\Omega}H (v, V) = - \frac{1}{2} (\Delta_g^{\Omega} - 2\Pi) (\Delta_g^{\Omega} - 2\Pi) \psi_v, \] (19.2)
for all \((v, V) \in \mathbb{F}_{g,\Omega}^J [0]\). The commutation identity
\[ \left[ \Delta_g^{\Omega} - 2\Pi, \Delta_g^{\Omega} - 2\Pi \right] = 0, \]
implies the inclusion
\[ (\Delta_g^{\Omega} - 2\Pi) L_{g,J}^{\Omega,\perp} \subseteq L_{g,J}^{\Omega,\perp}, \] (19.3)
and thus
\[ (\Delta_g^{\Omega} - 2\Pi) L_{g,J}^{\Omega,\perp} \subseteq L_{g,J}^{\Omega,\perp}. \] (19.4)
Then the identity (19.2) shows that the map (19.1) is well defined. We will deduce that it is an isomorphism if we show that the map
\[ \Delta_g^{\Omega} - 2\Pi : L_{g,J}^{\Omega,\perp} \cap C_{\Omega}^\infty (X, \mathbb{R})_0 \rightarrow L_{g,J}^{\Omega,\perp} \cap C_{\Omega}^\infty (X, \mathbb{R})_0, \] (19.5)
is an isomorphism. Indeed this is the case. The injectivity of (19.5) follows from the inclusion

$$\mathcal{C} \Lambda_{g,R}^{\Omega} \subseteq \Lambda_{g,J}^{\Omega},$$

which hold thanks to the identity (18.1). This inclusion implies also

$$\mathcal{C} \Lambda_{g,R}^{\Omega,\perp} = (\mathcal{C} \Lambda_{g,R}^{\Omega})^\perp \supseteq \Lambda_{g,J}^{\Omega,\perp}, \quad (19.6)$$

and thus

$$\Lambda_{g,R}^{\Omega,\perp} \supseteq \Lambda_{g,J}^{\Omega,\perp} \cap C_0^\infty (X, \mathbb{R})_0.$$

We use now the obvious fact that

$$\Delta_{g,J}^{\Omega} - 2\mathbb{I} : \Lambda_{g,R}^{\Omega,\perp} \longrightarrow \Lambda_{g,R}^{\Omega,\perp},$$

is an isomorphism. Thus for any \( f \in \Lambda_{g,J}^{\Omega,\perp} \cap C_0^\infty (X, \mathbb{R})_0 \) there exists a unique \( u \in \Lambda_{g,R}^{\Omega,\perp} \) such that

$$\Delta_{g,J}^{\Omega} u = f.$$

We decompose \( u = u_1 + u_2 \), with \( u_1 \in \Lambda_{g,J}^{\Omega,\perp} \) and \( u_2 \in \Lambda_{g,R}^{\Omega,\perp} \). Then the inclusions (19.3) and (19.4) imply the \( L_2^{\Omega,J} \)-orthogonal decomposition

$$\Delta_{g,J}^{\Omega} u_1 + (\Delta_{g,J}^{\Omega} - 2\mathbb{I}) u_2 = f.$$

We deduce \( u_1 \in \mathcal{C} \Lambda_{g,R}^{\Omega,\perp} \). But \( u_2 \in \mathcal{C} \Lambda_{g,R}^{\Omega,\perp} \) thanks to the inclusion (19.6). We infer \( u_1 = 0 \) since \( u \in \Lambda_{g,R}^{\Omega,\perp} \). Thus

$$u = u_2 \in \Lambda_{g,J}^{\Omega,\perp} \cap C_0^\infty (X, \mathbb{R})_0.$$

We obtain the surjectivity of the map (19.5) and thus the required conclusion. \( \square \)

**Lemma 25** If \((g, \Omega) \in \mathcal{S}_\omega\) is a KS then hold the identity

$$\ker D_{g,0} : \mathbb{H} \cap \mathcal{T}_{g,\Omega}^{J} = T_{(g,\Omega)} \cap \mathbb{H}^{0,1} (T_{X,J}) \quad (19.7)$$

with \( J := g^{-1}\omega \).

**Proof** With the notations in the proof of lemma (17) the basic variation formula (16.5) combined with the identities (16.6) and (16.7) implies that for all \((v, V) \in \mathbb{H} \cap \mathcal{T}_{g,\Omega}^{J}\)
over a shrinking Ricci soliton point \((g, \Omega)\) hold the equalities

\[
2D_{g,\Omega} H(v, V) = -\frac{1}{2}(\Delta g - 2\Pi)R\psi + \frac{1}{2}(L_J v \cdot I_\psi \cdot \Omega)^*_{\Omega} \\
= \frac{1}{2}(\Delta g - 2\Pi)R\psi + \frac{1}{2}B_{g,\Omega}I_\psi \\
= \frac{1}{2}\Re \left[ (\Delta g - 2\Pi)(\Delta g, J - 2\Pi)\psi \right] \\
= \frac{1}{2}\Re \left[ P_{g,\Omega}^J(\Omega) \psi \right] \\
= \frac{1}{2}P_{g,\Omega}^J \Re \psi,
\]

since \(P_{g,\Omega}^J\) is a real operator in the KS case. Then lemma 23 implies

\[
\text{Ker} \ D_{g,\Omega} H \cap \mathbb{T}^J_{g,\Omega} \simeq \mathcal{D}_{g,\Omega}^J \oplus \mathcal{H}_{g,\Omega}^{0,1}(T_{g,\Omega}),
\]

i.e. the required conclusion.

\(\square\)

**Proof of the main theorem**

**Proof** The inequality in the statement follows immediately from proposition 2. If equality hold then obviously \(A \in \mathcal{H}_{g,\Omega}^{0,1}(T_{g,\Omega})_0\) and

\[
\int_X P_{g,\Omega}^J \Re \psi \cdot \Re \psi \Omega = 0.
\]

Then the spectral theorem applied to the non-negative \(L^2_{\Omega}\)-self-adjoint real elliptic operator \(P_{g,\Omega}^J\) implies \(P_{g,\Omega}^J \Re \psi = 0\). Thus the conclusion

\[
(v, V) \in T_{g,\Omega} \cap \mathbb{T}^J_{g,\Omega} = \text{Ker} \ D_{g,\Omega} H \cap \mathbb{T}^J_{g,\Omega},
\]

follows from lemma 23 and the identity 19.7. In order to show the inclusion 17.20 we observe that if \((g_t, \Omega_t) \in \mathbb{R} \subset \text{KRS}_{\omega}\) is a smooth curve with \((g_0, \Omega_0) = (g, \Omega)\) and with \((\dot{g}_0, \dot{\Omega}_0) = (v, V)\) then hold the identity \(H_{g_0,\Omega_0} \equiv 0\) and thus

\[
(v, V) \in \text{Ker} \ D_{g,\Omega} H \cap \mathbb{T}^J_{g,\Omega} = T_{g,\Omega} \oplus \mathcal{H}_{g,\Omega}^{0,1}(T_{g,\Omega})_0,
\]

thanks to the identity 19.7. On the other side if we set \(W_t := W(g_t, \Omega_t)\) then \(W_t \equiv 0\) and thus

\[
0 = \tilde{\nabla} \tilde{V}_0 = \int_X |A|^2 \Omega,
\]

thanks to proposition 2 and lemma 23. We conclude the required inclusion. \(\square\)
20 Appendix A

20.1 The first variation of Perelman’s $W$ functional

We give a short proof of Perelman’s first variation formula [Per] for the $W$ functional based on the identity (3.2). Let $(g_t, \Omega_t) \subset M \times \mathcal{V}_1$ be a smooth family and set $f_t := \log \frac{\text{Vol}}{t^\frac{n}{2}}$. Then

$$\frac{d}{dt} W(g_t, \Omega_t) = \frac{d}{dt} \int_X \left[ \text{Tr}_R \left( g_t^{-1} h_t \right) + 2 f_t \right] \Omega_t$$

$$= \int_X \left[ \text{Tr}_R \left( - \hat{g}_t^* h_t + \hat{h}_t \right) + \text{Tr}_{g_t} \hat{g}_t - 2 \hat{\Omega}_t \right] \Omega_t$$

$$+ \int_X \left[ \text{Tr}_{g_t} h_t + 2 f_t \right] \hat{\Omega}_t$$

$$= \int_X \left[ - \langle \hat{g}_t, h_t \rangle_{g_t} + \left< g_t, \frac{d}{dt} \text{Ric}_{g_t}(\Omega_t) \right>_{g_t} - 2 \hat{\Omega}_t \right] \Omega_t$$

$$+ \int_X \left[ \text{Tr}_{g_t} h_t + 2 f_t \right] \hat{\Omega}_t.$$

Using the variation formula (3.2) and integrating by parts we infer

$$\int_X \left< g_t, \frac{d}{dt} \text{Ric}_{g_t}(\Omega_t) \right>_{g_t} \Omega_t = \int_X \left[ - \frac{1}{2} \left< g_t, \nabla_{\hat{g}_t} \nabla_{\hat{g}_t} \hat{g}_t \right>_{g_t} + \Delta_{g_t} \hat{\Omega}_t \right] \Omega_t$$

$$= - \int_X \left[ \frac{1}{2} \left< \nabla_{\hat{g}_t} \nabla_{\hat{g}_t} \hat{g}_t \right>_{g_t} + \left< \nabla_{\hat{g}_t} \hat{\Omega}_t, \nabla_{\hat{g}_t} f_t \right>_{g_t} \right] \Omega_t$$

$$= - \int_X \hat{\Omega}_t \Delta_{g_t} f_t \Omega_t,$$

which implies Perelman’s first variation formula

$$\frac{d}{dt} W(g_t, \Omega_t) = - \int_X \left[ \langle \hat{g}_t, h_t \rangle_{g_t} - 2 \hat{\Omega}_t (H_t - 1) \right] \Omega_t$$

$$= - \int_X \left[ \langle \hat{g}_t, h_t \rangle_{g_t} - 2 \hat{\Omega}_t H_t \right] \Omega_t,$$

since $\int_X \hat{\Omega}_t = 0.$

20.2 Basic complex identities

We provide a useful expression of the hermitian product $\langle \cdot, \cdot \rangle_{\omega}$ on $T^*_X \otimes \mathbb{C}$, which is the sesquilinear extension of the dual of $g$. We observe first that for any $\xi \in T^*_X \otimes \mathbb{C}$ and any $\alpha \in T^*_X \otimes \mathbb{C}$ hold the elementary equalities

$$0 = \xi \langle \alpha \wedge \omega^n \rangle = (\alpha \cdot \xi) \omega^n - \alpha \wedge (\xi \wedge \omega^n).$$

98
We obtain the formula
\[(\alpha \cdot \xi)\omega^n = n\alpha \wedge (\xi \omega) \wedge \omega^{n-1},\] (20.1)
and thus
\[2\alpha \cdot \xi = \text{Tr}_\omega [\alpha \wedge (\xi \omega)].\] (20.2)
Using (20.2) we deduce that for all \(\alpha, \beta \in T_X^* \otimes \mathbb{C}\) hold the equalities
\[2 \langle \alpha, \beta \rangle \omega = 2\alpha \cdot \bar{\beta}^* = \text{Tr}_\omega [\alpha \wedge (\bar{\beta}^* \omega)],\]
where \(\bar{\beta}^*: g^{-1}\beta\). Then the identity \(\bar{\beta} J = -\bar{\beta}^* \omega\) implies the formula
\[2 \langle \alpha, \beta \rangle \omega = -\text{Tr}_\omega [\alpha \wedge (\bar{\beta} J)].\]
Thus in the case \(\alpha, \beta \in \Lambda^1 T_X^*\) we deduce the identities
\[2 \langle \alpha, \beta \rangle \omega = \text{Tr}_\omega (i\alpha \wedge \bar{\beta}),\]
\[\langle \alpha, \beta \rangle \omega = \langle \alpha, \bar{\beta} \rangle \omega.\]

We show now the following integration by parts formulas

**Lemma 26** For any \(u, v \in C^\infty(X, \mathbb{C})\) hold the integration by parts identity
\[\int_X \left[ \Delta^\Omega_{g, J} u \cdot \overline{\nabla g, J} v + \Delta^\overline{\Omega}_{g, J} u \cdot v \right] \Omega = 2 \int_X g(\nabla g, J u, \nabla g, J v) \Omega.\] (20.3)
If \(u \in C^\infty(X, \mathbb{R})\) then hold also the integration by parts identity
\[\int_X \left[ \Delta^\Omega_{g, J} u \cdot v + \Delta^\overline{\Omega}_{g, J} u \cdot \overline{v} \right] \Omega = 2 \int_X g(\nabla g u, \nabla g, J v) \Omega.\] (20.4)

**Proof** Using the complex decomposition (13.2) and the fact that hermitian product \(\langle \cdot, \cdot \rangle_\omega\) on \(T_X^* \otimes \mathbb{C}\) is the sesquilinear extension of the dual of \(g\), we deduce
\[g(\nabla g, J u, \nabla g, J v) = \langle \partial_f \overline{\nabla g, J u}, \partial_f \overline{\nabla g, J v} \rangle_g\]
\[= \langle \partial_f \overline{\nabla g, J u}, \partial_f \overline{\nabla g, J v} \rangle_\omega\]
\[= \langle \partial_f \overline{\nabla g, J u}, \partial_f \overline{\nabla g, J v} \rangle_\omega\]
\[= \langle \partial_f \overline{\nabla g, J u}, \partial_f \overline{\nabla g, J v} \rangle_\omega\]
Integrating by parts and taking the conjugate we infer the identity
\[2 \int_X g(\nabla g, J u, \nabla g, J v) \Omega = \int_X \left[ \Delta^\Omega_{g, J} \overline{\nabla g, J u} + \Delta^\overline{\Omega}_{g, J} \overline{\nabla g, J v} \right] \Omega.\] (20.5)
Replacing \( u \) with \( \tau \), \( v \) with \( \tau \) in (20.5) we obtain (20.3). In the case \( u \in C^{\infty}(X, \mathbb{R}) \) formula (20.5) implies directly (20.4).

We show now that for all \( \alpha, \beta \in \Lambda^{1,1}T_{X}^{*} \cap \Lambda^{2}T_{X}^{*} \) hold the identity

\[
4n(n-1) \frac{\alpha \wedge \beta \wedge \omega^{n-2}}{\omega^{n}} = \text{Tr}_{\omega} \alpha \text{Tr}_{\omega} \beta - 2 \langle \alpha, \beta \rangle_{\omega}.
\]

Indeed we consider the local expressions

\[
\omega = \frac{1}{2} \sum_{k} \zeta_{k}^* \wedge \bar{\zeta}_{k}^*, \quad \alpha = i \sum_{k,l} \alpha_{kl} \zeta_{k}^* \wedge \bar{\zeta}_{l}^*, \quad \beta = i \sum_{k,l} \beta_{kl} \zeta_{k}^* \wedge \bar{\zeta}_{l}^*,
\]

and we set

\[
\Psi := \alpha \wedge \beta = \sum_{k_1,k_2,l_1,l_2} \alpha_{k_1,l_1} \beta_{k_2,l_2} \zeta_{k_1}^* \wedge \bar{\zeta}_{l_1}^* \wedge \zeta_{k_2}^* \wedge \bar{\zeta}_{l_2}^*,
\]

where \( K = (k_1, k_2), 1 \leq k_1 < k_2 \leq n \) and the same hold for \( L \). Explicitly the coefficients \( \Psi_{K,L} \) are given by the expression

\[
\Psi_{K,L} = \alpha_{k_1,l_1} \beta_{k_2,l_2} + \alpha_{k_2,l_2} \beta_{k_1,l_1} - \alpha_{k_1,l_2} \beta_{k_2,l_1} - \alpha_{k_2,l_1} \beta_{k_1,l_2}.
\]

We conclude the identity

\[
4n(n-1) \frac{\Psi \wedge \omega^{n-2}}{\omega^{n}} = 16 \sum_{|K|=|L|=2} \Psi_{L,L} = 16 \sum_{k,l} \alpha_{kl} \beta_{kl} - 16 \sum_{k,l} \alpha_{kl} \beta_{kl} = \text{Tr}_{\omega} \alpha \text{Tr}_{\omega} \beta - 2 \langle \alpha, \beta \rangle_{\omega}.
\]

### 20.3 Action of the curvature on alternating 2-forms

We observe that as in the symmetric case we can define an action of the curvature operator over alternating 2-forms as follows

\[
(R_{g} \ast \alpha)(\xi, \eta) := - \text{Tr}_{g} [\alpha (R_{g}(\xi, \cdot) \eta, \cdot)],
\]

for any \( \alpha \in \Lambda^{2}T_{X}^{*} \). The tensor \( R_{g} \ast \alpha \) is anti-symmetric. In fact let \( (e_{k})_{k} \) be a \( g(x) \)-orthonormal base of \( T_{X,x} \) and consider the local expression \( \alpha_{x}^{*} = A_{k,l} e_{k}^{*} \otimes e_{l} \),
with $A_{l,k} = -A_{k,l}$. Then
\[
(R_g * \alpha)(\xi, \eta) = -g(\alpha_g^* R_g(\xi, e_k)\eta, e_k)
\]
\[
= g(R_g(\xi, e_k)\eta, \alpha_g^* e_k)
\]
\[
= R_g(\xi, e_k, \alpha_g^* e_k, \eta)
\]
\[
= R_g(\xi, e_k, A_{l,k} e_l, \eta)
\]
\[
= -R_g(\xi, A_{k,l} e_k, e_l, \eta)
\]
\[
= -R_g(\xi, \alpha_g^* e_l, e_l, \eta)
\]
\[
= -R_g(\eta, e_l, \alpha_g^* e_l, \xi)
\]
\[
= -(R_g * \alpha)(\eta, \xi),
\]
thanks to the symmetry properties of the curvature form. We observe also that the previous computation shows the identity
\[
(R_g * \alpha)(\xi, \eta) = -R_g(\xi, e_k, \eta, \alpha_g^* e_k)
\]
\[
= -g(R_g(\xi, e_k)\alpha_g^* e_k, \eta)
\]
\[
= -g((R_g * \alpha_g^*)(\xi, \eta)),
\]
i.e.
\[
(R_g * \alpha)_g^* = -R_g * \alpha_g^*.
\]  
(20.7)

On the other hand using the algebraic Bianchi identity we obtain the equalities
\[
(R_g * \alpha)(\xi, \eta) = -g(\alpha_g^* R_g(\xi, e_k)\eta, e_k)
\]
\[
= g(\alpha_g^* \mathcal{R}_g(e_k, \eta)\xi, e_k) + g(\alpha_g^* \mathcal{R}_g(\eta, \xi)e_k, e_k)
\]
\[
= -g(\alpha_g^* \mathcal{R}_g(\eta, e_k)\xi, e_k) - g(\mathcal{R}_g(\eta, \xi)e_k, \alpha_g^* e_k)
\]
\[
= (R_g * \alpha)(\eta, \xi) + g(\mathcal{R}_g(\eta, \xi)e_k, \alpha_g^* e_k)
\]
\[
= -(R_g * \alpha)(\xi, \eta) - \text{Tr}_R[\mathcal{R}_g(\xi, \eta)\alpha_g^*],
\]
101
and thus the formula
\[(R_g \ast \alpha)(\xi, \eta) = -\frac{1}{2} \text{Tr}_R [R_g(\xi, \eta)\alpha^*_g] . \tag{20.8}\]

We assume further that \((X, J, g)\) is Kähler and \(\alpha\) is \(J\)-anti-invariant. In this case \(\alpha^*_g = A\) is \(J\)-anti-linear and so is the endomorphism \(R_g(\xi, \eta)\alpha^*_g\). We deduce
\[R_g \ast \alpha = 0, \quad \text{i.e.} \quad R_g \ast A = 0. \tag{20.9}\]

thanks to the identity \(20.7\).

20.4 Weighted Weitzenböck formula for alternating 2-forms

We show the weighted Weitzenböck type formula
\[\Delta^\Omega_{g, g} \alpha = \Delta^\Omega g \alpha + 2R_g \ast \alpha + \alpha \text{Ric}^g(\Omega) + \text{Ric}_g(\Omega)\alpha^*_g, \tag{20.10}\]

for any alternating 2-form \(\alpha\) over a Riemannian manifold. For this purpose we fix an arbitrary point \(x_0\) and we choose the vector fields \(\xi\) and \(\eta\) such that \(0 = \nabla_g \xi(x_0) = \nabla_g \eta(x_0)\). Let \((e_k)_k\) be a \(g\)-orthonormal local frame such that \(\nabla_g e_k(x_0) = 0\). Then at the point \(x_0\) hold the identities
\[
d \nabla^*_g \alpha(\xi, \eta) = \nabla_g \xi \nabla^*_g \alpha \cdot \eta - \nabla_g \eta \nabla^*_g \alpha \cdot \xi
\]
\[
= \nabla_g \xi [\nabla^*_g \alpha \cdot \eta] - \nabla_g \eta [\nabla^*_g \alpha \cdot \xi]
\]
\[
= -\nabla_g \xi [\nabla g, e_k \alpha(e_k, \eta)] + \nabla_g \eta [\nabla g, e_k \alpha(e_k, \xi)]
\]
\[
= -\nabla_g \xi \nabla g, e_k \alpha(e_k, \eta) + \nabla_g \eta \nabla g, e_k \alpha(e_k, \xi),
\]

and
\[
\nabla^*_g d \alpha(\xi, \eta) = -\nabla g, e_k d \alpha(e_k, \xi, \eta)
\]
\[
= -\nabla g, e_k [d \alpha(e_k, \xi, \eta)]
\]
\[
= -\nabla g, e_k [\nabla g, e_k \alpha(e_k, \xi, \eta) - \nabla g, \xi \alpha(e_k, \eta) + \nabla g, \eta \alpha(e_k, \xi)]
\]
\[
= -\nabla g, e_k \nabla g, e_k \alpha(e_k, \eta) + \nabla g, e_k \nabla g, \xi \alpha(e_k, \eta) - \nabla g, e_k \nabla g, \eta \alpha(e_k, \xi).
\]

We remind now that for any vector fields \(\mu, \zeta\) such that \([\mu, \zeta](x_0) = 0\) hold the identity at the point \(x_0\)
\[
\nabla g, \mu \nabla g, \zeta \alpha - \nabla g, \zeta \nabla g, \mu \alpha = R_g(\zeta, \mu) \alpha,
\]
where the contraction operation $T \dashv : \Lambda^2 T^*_X \to \Lambda^2 T^*_X$ associated to an endomorphism $T \in \text{End}(T_X)$ is defined by the formula

$$(T \circ \alpha)(\xi, \eta) := \alpha(T\xi, \eta) + \alpha >= \xi, T\eta >=. $$

We deduce

$$(\nabla_{g,e_k} \nabla_{g,\xi} \alpha - \nabla_{g,\xi} \nabla_{g,e_k})(e_k, \eta) = \alpha (\mathcal{R}_g(\xi, e_k)e_k, \eta) + \alpha (e_k, \mathcal{R}_g(\xi, e_k)\eta)$$

and also

$$(\nabla_{g,\eta} \nabla_{g,e_k} \alpha - \nabla_{g,e_k} \nabla_{g,\eta}\alpha)(e_k, \xi) = -[\alpha \text{Ric}^*(g) + (\mathcal{R}_g * \alpha)](\eta, \xi)$$

Summing up the terms $d\nabla_g \circ \alpha(\xi, \eta)$ and $\nabla_g \circ \alpha(\xi, \eta)$ and using this last identities we infer the formula (20.10) in the case $\Omega = C\text{dV}_g$. In order to obtain the general case we observe the decompositions

$$d\nabla_g \circ \alpha = d\nabla_g \circ \alpha + d(\nabla_g \circ \alpha - \alpha),$$

$$\nabla_g \circ \alpha = \nabla_g \circ \alpha + \nabla_g \circ \alpha - \alpha,$$

and the identities at the point $x_0$.

$$d(\nabla_g \circ \alpha - \alpha)(\xi, \eta) = \nabla_{g,\xi}(\nabla_g \circ \alpha - \alpha) \cdot \eta - \nabla_{g,\eta}(\nabla_g \circ \alpha - \alpha) \cdot \xi$$

$$(\nabla_g \circ \alpha - \alpha)(\xi, \eta) = \nabla_{g,\xi} \alpha(\nabla_g \circ \alpha - \alpha) \cdot \eta - \nabla_{g,\eta} \alpha(\nabla_g \circ \alpha - \alpha) \cdot \xi$$

$$(\nabla_g \circ \alpha - \alpha)(\xi, \eta) = \nabla_{g,\xi} \alpha(\nabla_g \circ \alpha - \alpha) \cdot \eta - \nabla_{g,\eta} \alpha(\nabla_g \circ \alpha - \alpha) \cdot \xi$$

Summing up we infer the required formula (20.10).
21 Appendix B

21.1 Reformulation of the weighted complex Bochner identity \((13.9)\)

We define the complex operator

\[
\Delta^\Omega_{g,-J} := \overline{\Delta^\Omega_{g,J}}.
\]

With this notation the weighted complex Bochner type identity \((13.9)\) rewrites also as

\[
2\Delta^\Omega_{T_X,g} \nabla_{g,J} u = \nabla_{g,J}(\Delta^\Omega_{g,-J} - 2\Pi)u,
\]

for all \(u \in C^\infty(X,\mathbb{C})\). We show now that the fundamental identity \((13.9)\) implies an other important formula. We need a few preliminaries.

**Lemma 27** For any \(u, v \in C^\infty(X,\mathbb{C})\) hold the integration by parts identity

\[
\int_X \Delta^\Omega_{g,-J} u \cdot \nabla_{g,J} v \omega = 2 \int_X \langle \nabla_{g,J} u, \nabla_{g,J} v \rangle \omega \Omega.
\]

**Proof** We define the complex components of the \(g\)-gradient as

\[
\nabla^{1,0}_{g,J} u := (\nabla_g u)^{1,0} \in C^\infty(X, T_{X,J}^{1,0}),
\]

\[
\nabla^{0,1}_{g,J} u := (\nabla_g u)^{0,1} \in C^\infty(X, T_{X,J}^{0,1}).
\]

With these notations hold the decomposition formula

\[
\nabla_{g,J} u = \nabla^{1,0}_{g,J} u + \nabla^{0,1}_{g,J} \overline{\nabla}. \tag{21.1}
\]

We observe that for all \(\xi, \eta \in T_X\) hold the identity

\[
\langle \xi, \eta \rangle_\omega \equiv h(\xi, \eta) = 2i\omega(\eta^{0,1}_{J}, \xi^{1,0}_{J}).
\]

This combined with \((21.1)\) implies

\[
\langle \nabla_{g,J} u, \nabla_{g,J} v \rangle_\omega = 2i\omega(\nabla^{0,1}_{g,J} \overline{\nabla}, \nabla^{1,0}_{g,J} u).
\]

We observe now that the complex spiting of the \(g\)-gradient

\[
\nabla u = \nabla^{1,0}_{g,J} u + \nabla^{0,1}_{g,J} u
\]

implies the identities

\[
\nabla^{1,0}_{g,J} u \cdot \omega = i\overline{\partial}_J u,
\]

\[
\nabla^{0,1}_{g,J} u \cdot \omega = -i\partial_J u.
\]
Using this and the identity (20.2) we deduce
\[ \langle \nabla g, J \rangle u, \nabla g, J \rangle v \rangle_\omega = 2 \partial_J v \cdot \nabla 1_{\omega} \rho_g, J \rangle u \]
\[ = \text{Tr}_\omega \left[ \partial_J J \wedge (\nabla 1_{\omega} u - \omega) \right] \]
\[ = \text{Tr}_\omega \left[ i \partial_J \nabla J u \right] \]
\[ = 2 \langle \partial_J \nabla J u, \partial_J \nabla J v \rangle_\omega . \]

We infer the equalities
\[ \int_X \langle \nabla g, J \rangle u, \nabla g, J \rangle v \rangle_\omega \Omega = 2 \int_X \langle \partial_J \nabla J u, \partial_J \nabla J v \rangle_\omega \Omega \]
\[ = \int_X \Delta_{\omega} \rho_g, J \rangle u \cdot \nabla \nabla J v \Omega \]
\[ = \int_X \Delta_{\omega} \rho_g, J \rangle u \cdot \nabla \nabla J v \Omega . \]

We equip \( C^\infty_\Omega(X, \mathbb{C}) \) with the \( L^2_\Omega \)-product (13.1) and the space \( C^\infty(X, \Lambda^{0,1}_T X \otimes_{\mathbb{C}} T_{X,J}) \) with the \( L^2_{\omega,\Omega} \)-hermitian product (11.1). Then the formal adjoint of \( H^{0,1}_{g,J} \) with respect to such products \( (H^{0,1}_{g,J})^{*_{\omega,\Omega}} : C^\infty(X, \Lambda^{0,1}_T X \otimes_{\mathbb{C}} T_{X,J}) \rightarrow C^\infty_\Omega(X, \mathbb{C}) \),
\[ \int_X \langle H^{0,1}_{g,J} u, A \rangle_\omega \Omega = \int_X u \cdot (H^{0,1}_{g,J})^{*_{\omega,\Omega}} A \Omega , \]

satisfies the identity
\[ (H^{0,1}_{g,J})^{*_{\omega,\Omega}} = \nabla_{g,J} \nabla^{*_{\omega,\Omega}}. \]

Moreover lemma implies the identity
\[ \Delta_{\omega} = \nabla_{g,J} \nabla_{g,J} \nabla^{*_{\omega,\Omega}}. \]

Then the complex Bochner type identity (13.9) implies
\[ 2(H^{0,1}_{g,J})^{*_{\omega,\Omega}} H^{0,1}_{g,J} \pi = 2 \nabla_{g,J} \nabla^{*_{\omega,\Omega}} \nabla_{T_{X,J}} \nabla_{g,J} \pi \]
\[ = \Delta_{\omega} \Delta_{g,J} - 2 \pi \]

or in other terms
\[ 2(H^{0,1}_{g,J})^{*_{\omega,\Omega}} H^{0,1}_{g,J} u = \Delta_{\omega} \Delta_{g,J} - 2 \pi u , \]
for all \( u \in C^\infty(X, \mathbb{C}) \).
A Poisson structure on the first eigenspace of $\Delta^\Omega_{g,J}$

For any complex valued function $u$ and symplectic form $\omega$ we define the complex vector field
\[
(du)^*_{\omega} := \omega^{-1}du = -J\nabla_g u,
\]
and the Poisson bracket
\[
\{u,v\}_{\omega} := dv \cdot (du)^*_{\omega} = -\omega((du)^*_{\omega}, (dv)^*_{\omega}).
\]

We define the Poisson bracket over the space $C^\infty(X, \mathbb{C})$ as
\[
\{u,v\}_{\omega,\Omega} := \{u,v\}_{\omega} - \int_X \{u,v\}_{\omega} \Omega.
\]

With these notations hold the following lemma (see also [Fut], [Gau]).

Lemma 28 Let $(X,J)$ be a Fano manifold and let $g$ be a $J$-invariant Kähler metric such that $\omega := gJ \in 2\pi c_1(X, [J])$. Let also $\Omega > 0$ be the unique smooth volume form with $\int_X \Omega = 1$ such that $\text{Ric}_J(\Omega) = \omega$.

A) Then the map
\[
\chi : \left( \text{Ker}(\Delta^\Omega_{g,J} - 2\Pi), i \{\cdot, \cdot\}_{\omega,\Omega} \right) \rightarrow (H^0(X, T_{X,J}), [\cdot, \cdot])
\]
\[u \mapsto -\nabla_{g,J} u,
\]
is well defined and it represents an isomorphism of complex lie algebras.

B) The first eigenvalue $\lambda_1(\Delta^\Omega_{g,J})$ of the operator $\Delta^\Omega_{g,J}$ satisfies the estimate $\lambda_1(\Delta^\Omega_{g,J}) \geq 2$, with equality in the case $H^0(X, T_{X,J}) \neq 0$.

C) If we set $\text{Kill}_g := \text{Lie} (\text{Isom}_g^0)$ then the map
\[
J\nabla_g : \text{Ker}_R(\Delta^\Omega_{g,J} - 2\Pi) \rightarrow \text{Kill}_g,
\]
is well defined and it represents an isomorphism of real vector spaces.

D) The hermitian form
\[
(u,v) \mapsto \int_X i \{u,\overline{v}\}_{\omega} \Omega,
\]
over $\text{Ker}(\Delta^\Omega_{g,J} - 2\Pi)$ is non-negative and let $(\mu_j)_{j=0}^N \subset \mathbb{R}_{\geq 0}$, $\mu_0 = 0$, be its spectrum with respect to the $L^2_\Omega$-product. If $g$ is a $J$-invariant Kähler-Ricci soliton then hold the decomposition
\[
H^0(X, T_{X,J}) = \bigoplus_{j=0}^N V_{\mu_j},
\]
\[
V_{\mu_j} := \{\xi \in H^0(X, T_{X,J}) \mid [\nabla_g \xi, \xi] = \mu_j \xi\},
\]
\[
V_0 = \text{Kill}_g \oplus J \text{Kill}_g.
\]
Proof Step A. In this step we show the statement A. The fact that $\chi$ is an isomorphism follows from corollary \[\square\] We show now that $\chi$ is also a morphism of complex Lie algebras. Let

$$K_\pm := \text{Ker} (\Delta_{g,J} - 2\Pi).$$

For any $\xi \in H^0(X,T_{X,J})$ we denote $u^\xi := \chi^{-1}(\xi) \in K_-$ and we decompose $u^\xi = u_1^\xi + iu_2^\xi$, with $u_2^\xi \in C_\infty(X,\mathbb{R})$. For any $u,v \in K_- = \overline{K_+}$ we set $\xi := \nabla_{g,J}u$, $\eta := \nabla_{g,J}v$ and as in \[\text{[Gau]}\] we observe the identities

$$L_{[\xi,\eta]} \omega = L_{\xi}L_{\eta} \omega - L_{\eta}L_{\xi} \omega$$

$$= 2L_{\xi}i\partial_j \overline{J} J u_1^\eta - 2L_{\eta}i\partial_j \overline{J} J u_1^\xi$$

$$= 2i\partial_j \overline{J} J \left( \xi, u_1^\eta - \eta, u_1^\xi \right),$$

since $\xi, \eta$ are holomorphic. We infer that for some constant $C_1 \in \mathbb{R}$ hold the identities

$$u_1^{[\xi,\eta]} + C_1 = \xi, u_1^\eta - \eta, u_1^\xi$$

$$= \xi, v_1 - \eta, u_1$$

$$= g(\nabla_g v_1, \nabla_g u_1 + J \nabla_g u_2)$$

$$- g(\nabla_g u_1, \nabla_g v_1 + J \nabla_g v_2)$$

$$= g(\nabla_g v_1, J \nabla_g u_2) - g(\nabla_g u_1, J \nabla_g v_2)$$

$$= \omega(\nabla_g u_2, \nabla_g v_1) - \omega(\nabla_g v_2, \nabla_g u_1)$$

$$= \omega(J \nabla_g u_2, J \nabla_g v_1) + \omega(J \nabla_g u_1, J \nabla_g v_2)$$

$$= \omega((du_2)^\omega, (dv_1)^\omega) + \omega((du_1)^\omega, (dv_2)^\omega)$$

$$= - \{u_1, v_2\}_\omega - \{u_2, v_1\}_\omega.$$

On the other hand

$$u_2^{[\xi,\eta]} = -u_1^{[\xi,\eta]} = -u_1^{[\xi,J\eta]},$$

107
since \( \xi \) is holomorphic. We infer that for some constant \( C_2 \in \mathbb{R} \) hold the identities

\[
\begin{align*}
\left. u_2 \right|^{\xi, \eta}_2 + C_2 &= -\xi, u_1^\eta + J\eta, u_1^\xi \\
&= \xi, u_2^\eta + J\eta, u_1^\xi \\
&= \xi, v_2 + J\eta, u_1 \\
&= g(\nabla_g v_2, \nabla_g u_1 + J\nabla_g u_2) \\
&+ g(\nabla_g u_1, J\nabla_g v_1 - \nabla_g v_2) \\
&= g(\nabla_g v_2, J\nabla_g u_2) + g \left( \nabla_g u_1, J\nabla_g v_1 \right) \\
&= \omega(\nabla_g v_2, \nabla_g v_1 + \nabla_g u_1) \\
&= \omega(J\nabla_g u_2, J\nabla_g v_2) - \omega(J\nabla_g u_1, J\nabla_g v_1) \\
&= \omega((du_2)_\omega, (dv_2)_\omega) - \omega((du_1)_\omega, (dv_1)_\omega) \\
&= \{u_1, v_1\}_\omega - \{u_2, v_2\}_\omega.
\end{align*}
\]

We conclude that for all \( u, v \in \mathbb{K}_- \) hold the identity

\[
\nabla_g, i \{u, v\}_\omega, \Omega = [\nabla_g, i u, \nabla_g, J v],
\]

which shows that \( i \{\cdot, \cdot\}_\omega, \Omega \) is a complex Lie algebra product over \( \mathbb{K}_- \) and that the map \( \chi \) is a morphism of complex Lie algebras.

**Step B, C.** The statements B and C follow from corollary \[1\] and the remarkable identity \[18.1\].

**Step D.** We show now the statement D. We observe first that for all \( u, v \in C^\infty(X, \mathbb{C}) \) hold the identity

\[
\int_X i \{u, v\}_\omega, \Omega = - \int_X iB_{\nabla_g, J u} \cdot v, \Omega.
\]
Indeed thanks to the computations in step A we deduce

\[
\int_X i\{u,v\}_\omega \Omega = -\int_X [\{u_1,v_2\}_\omega + \{u_2,v_1\}_\omega] \Omega \\
+ i \int_X \{u_1,v_1\}_\omega - \{u_2,v_2\}_\omega \Omega \\
= \int_X [(\nabla_g v_1, J \nabla_g u_2) + (\nabla_g u_1, J \nabla_g v_2)] \Omega \\
+ i \int_X [\nabla_g v_2, J \nabla_g u_2] + [\nabla_g u_1, J \nabla_g v_1] \Omega.
\]

Integrating by parts we infer

\[
\int_X i\{u,v\}_\omega \Omega = \int_X [v_1 \cdot B_{g,J}^\Omega u_2 - u_1 \cdot B_{g,J}^\Omega v_2] \Omega \\
+ i \int_X [v_2 \cdot B_{g,J}^\Omega u_2 + u_1 \cdot B_{g,J}^\Omega v_1] \Omega \\
= \int_X [B_{g,J}^\Omega u_1 \cdot v_2 + B_{g,J}^\Omega u_2 \cdot v_1] \Omega \\
- i \int_X [B_{g,J}^\Omega u_1 \cdot v_1 - B_{g,J}^\Omega u_2 \cdot v_2] \Omega \\
= - \int_X iB_{g,J}^\Omega u \cdot \bar{v} \Omega,
\]

thanks to the fact that \(B_{g,J}^\Omega\) is \(L^2_\Omega\)-anti-adjoint. Thus if \(u \in \mathbb{K}_-\)

\[
\int_X i\{u,\bar{u}\}_\omega \Omega = \int_X (\Delta_g^\Omega - 2\bar{\Omega}) u \cdot \bar{v} \Omega \geq 0.
\]

The KS assumption implies the commutation identity

\[
[\Delta_{g,J}^\Omega, \Delta_{g,-J}^\Omega] = 0.
\]

We infer that

\[
\Delta_{g,J}^\Omega - 2\bar{\Omega} : \mathbb{K}_- \rightarrow \mathbb{K}_-,
\]

is a well defined non-negative \(L^2_\Omega\)-self-adjoint operator and let \((\lambda_j)_{j=0}^{N} \subset \mathbb{R}_{\geq 0}, \lambda_0 = 0\) be it’s spectrum. Notice also that by definition of \(\mathbb{K}_-\) this operator coincides with the operator

\[
-2iB_{g,J}^\Omega : \mathbb{K}_- \rightarrow \mathbb{K}_-.
\]

Thus \(u \in \mathbb{K}_-\) is an eigen-vector corresponding to the eigenvalue \(\lambda_j\) if and only if \(u \in \mathbb{K}_-\) satisfies

\[
(J \nabla_g f)_u = \frac{\lambda_j}{2} iu.
\]

109
This rewrites as
\[ i \{ f, u \}_\omega, \Omega = \frac{\lambda_j}{2} u, \]
and is equivalent to the equation
\[ [\nabla_g f, \nabla_g J u] = \frac{\lambda_j}{2} \nabla_g J u. \]

Notice also that the kernel of (21.3) is given by the identity
\[ \mathbb{K}_+ \cap \mathbb{K}_- = \mathbb{K}_R \oplus J \mathbb{K}_R, \]
\[ \mathbb{K}_R := \text{Ker}_R(\Delta_{g, J}^{\Omega} - 2 \Pi). \]

We deduce the required conclusion with \( \mu_j = \lambda_j/2. \)

21.3 Consequences of the Bochner-Kodaira-Nakano formula

The holomorphic and antiholomorphic Hodge Laplacian operators are related by the Bochner-Kodaira-Nakano identity. At the level of \( T_X \)-valued 1-forms it reduces to the identity
\[ \Delta_{T_X, g}^J A = \Delta_{T_X, g} A + \frac{1}{6} (JR_g \wedge A) (\omega^* \wedge \bullet), \tag{21.4} \]
where \( \omega^* \equiv \omega^{-1} \in C^\infty(X, \Lambda^{1,1}_J T_X \cap \Lambda^2_R T_X) \) is the dual element associated to \( \omega \).

If in holomorphic coordinates \( \omega \) writes as
\[ \omega = \frac{i}{2} \omega_{k\bar{l}} dz_k \wedge d\bar{z}_l, \]
then
\[ \omega^* = 2i \omega_{k\bar{l}} \partial \partial z_k \wedge \partial \partial\bar{z}_l. \]

The factor 1/6 in front of the last term on the right hand side of (21.4) is due to the convention
\[ v_1 \wedge \ldots \wedge v_p := \sum_{\sigma \in S_p} \varepsilon\sigma v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_p}. \]

We explicit this last term. For this purpose we observe first that for any \( \alpha \in \Lambda^{1,1}_J T_X^* \otimes \mathbb{C} E \) hold the identity
\[ \text{Tr}_\omega \alpha = - \text{Tr}_g [\alpha(J\cdot, \cdot)]. \]
We infer the expressions
\[
\frac{1}{6} (J\mathcal{R}_g \wedge A)(\omega^* \wedge \xi) = (J\mathcal{R}_g \wedge A) \left( 2i\omega^{l,k} \frac{\partial}{\partial z^k} \cdot \frac{\partial}{\partial \bar{z}^l} \xi \right)
\]
\[
= -\frac{1}{2} \text{Tr}_\omega ([J\mathcal{R}_g \wedge A](\cdot, \cdot, \xi)]
\]
\[
= \frac{1}{2} (J\mathcal{R}_g \wedge A)(Je_k, e_k, \xi),
\]
for an arbitrary $g$-orthonormal real frame $(e_k)_k$. We explicit the exterior product using the $J$-invariant properties of the curvature operator. We obtain
\[
(J\mathcal{R}_g \wedge A)(Je_k, e_k, \xi) = -\frac{1}{2} \text{Tr}_\omega (J\mathcal{R}_g \wedge A)(\cdot, \cdot, \xi)]
\]
\[
= 12 \text{Tr}_\omega (J\mathcal{R}_g \wedge A)(Je_k, e_k, \xi),
\]
where $\eta_k := Je_k$. But $(\eta_k)_k$ is also a $g$-orthonormal real frame. We infer
\[
(J\mathcal{R}_g \wedge A)(Je_k, e_k, \xi) = -2 \text{Ric}^*(g)A\xi - 2 [\mathcal{R}_g * (JAJ)] \xi.
\]
We deduce that the Bochner-Kodaira-Nakano identity rewrites at the level of $T_X$-valued 1-forms as
\[
\Delta_{TX,g}^J A = \Delta_{TX,g} A - \text{Ric}^*(g)A - \mathcal{R}_g * (A''_J - A'_J),
\] (21.5)
where $A'_J$ and $A''_J$ are respectively the $J$-linear and $J$-anti-linear parts of $A$. Using the Weitzenböck type formula in lemma 3 with $\Omega = CdV_g$ we infer
\[
\mathcal{L}_g^\Omega A = \Delta_g A + \nabla_g f - \nabla_g A - 2 \mathcal{R}_g * A
\]
\[
= \Delta_{TX,g} A - \mathcal{R}_g * A - A \text{Ric}^*(g) + \nabla_g f - \nabla_g A
\]
\[
= \left( \Delta_{TX,g}^J + \Delta_{TX,g}^J \right) A - \mathcal{R}_g * A - A \text{Ric}^*(g) + \nabla_g f - \nabla_g A.
\]
Using the Bochner-Kodaira-Nakano identity (21.5) we deduce the formulas
\[
\mathcal{L}_g^\Omega A = 2\Delta_{TX,g} A - \text{Ric}^*(g)A - A \text{Ric}^*(g) - 2 \mathcal{R}_g * A'_J + \nabla_g f - \nabla_g A,
\]
\[
\mathcal{L}_g^\Omega A = 2\Delta_{TX,g} A + \text{Ric}^*(g)A - A \text{Ric}^*(g) - 2 \mathcal{R}_g * A'_J + \nabla_g f - \nabla_g A,
\]
and thus the identities

\[ \mathcal{L}_g A'_f = 2 \Delta^J_{T_{X,g}} A'_f - \text{Ric}^*(g) A'_f - A'_f \text{Ric}^*(g) + \nabla_g f \nabla_g A'_f, \quad (21.6) \]

\[ \mathcal{L}_g A''_f = 2 \Delta^J_{T_{X,g}} A''_f + \text{Ric}^*(g) A''_f - A''_f \text{Ric}^*(g) + \nabla_g f \nabla_g A''_f. \quad (21.7) \]

We point out that one can obtain directly this formulas by using the methods in the proof of formulas (14.2), (14.5) and (14.6). We remind now that the properties (12.1) and (12.2) imply that \( A \in \text{Ker} \mathcal{L}_g \) if and only if \( A'_f \in \text{Ker} \mathcal{L}_g \) and \( A''_f \in \text{Ker} \mathcal{L}_g \). Thus if \( A \in \text{Ker} \mathcal{L}_g \) we infer thanks to the identity (21.6) with \( \Omega = CdV_g \),

\[ 0 = \int_X \langle \mathcal{L}_g A'_f, A'_f \rangle_g \, dV_g \]

\[ = 2 \int_X \left[ \langle \Delta^J_{T_{X,g}} A'_f, A'_f \rangle_g - \langle \text{Ric}^*(g) A'_f, A'_f \rangle_g \right] \, dV_g. \]

Using the identity between Riemannian and hermitian norms of \( T_X \)-valued forms we obtain

\[ 2 \int_X \langle \Delta^J_{T_{X,g}} A'_f, A'_f \rangle_g \, dV_g = 2 \int_X \langle \Delta^J_{T_{X,g}} A'_f, A'_f \rangle_\omega \, dV_g \]

\[ = \int_X \left[ 2 \left| \partial^{\nu*}_{T_{X,g}} A'_f \right|^2_\omega + \left| \partial^g_{T_{X,g}} A'_f \right|^2_\omega \right] \, dV_g \]

\[ = \int_X \left[ 2 \left| \partial^{\nu*}_{T_{X,g}} A'_f \right|^2_g + \left| \partial^g_{T_{X,g}} A'_f \right|^2_g \right] \, dV_g. \]

We deduce

\[ \int_X \left[ 2 \left| \partial^{\nu*}_{T_{X,g}} A'_f \right|^2_g + \left| \partial^g_{T_{X,g}} A'_f \right|^2_g - 2 \langle \text{Ric}^*(g) A'_f, A'_f \rangle_g \right] \, dV_g = 0. \quad (21.8) \]

Assume from now on the Kähler-Einstein condition \( \text{Ric}(g) = \lambda g \), \( \lambda = \pm 1, 0 \). The identity (21.7) with \( \Omega = CdV_g \) implies in this case

\[ \mathcal{L}_g A''_f = 2 \Delta^J_{T_{X,g}} A''_f, \]

and thus

\[ \text{Ker} \mathcal{L}_g \cap C^\infty (X, T^*_{X,-} \otimes T_{X,+}) = \mathcal{H}^{0,1}_g (T_{X,+}). \]

Let now \( A \in \text{Ker} \nabla^*_g \) and observe that for be-degree reasons hold the decomposition

\[ 0 = \nabla^*_g A = \nabla^*_T_{X,g} A'_f + \nabla^*_T_{X,g} A''_f \]

\[ = \partial^{\nu*}_{T_{X,g}} A'_f + \partial^g_{T_{X,g}} A''_f. \]

112
Thus if \( A \in \text{Ker} \nabla^*_g \cap \text{Ker} \mathcal{L}_g \) then \( \overline{\partial}^{\ast}_{T_X,J} A'_j = 0 \) and thus \( \partial^{\ast}_{T_X,J} A'_j = 0 \) which implies

\[
\int_X \left[ \left| \partial^{\ast}_{T_X,J} A'_j \right|_g^2 - 2 \langle \text{Ric}^*(g) A'_j, A'_j \rangle_g \right] dV_g = 0,
\]

thanks to (21.8). We will still denote by \( \mathcal{L}_g \) the analogue operator over \( C^\infty (X, S^2_R T^*_X) \).

We infer that if \( \lambda \neq 0 \) then hold the identity

\[
\text{Ker} \nabla^*_g \cap \text{Ker} \mathcal{L}_g \cap \mathcal{D}_g^J = \{ v \in C^\infty (X, S^2_R T^*_X) \mid v = v'_J, v^*_g \in \mathcal{H}^{0,1}_g (T_X,J) \},
\]

i.e. there exists an isomorphism

\[
\text{Ker} \nabla^*_g \cap \text{Ker} \mathcal{L}_g \cap \mathcal{D}_g^J \longrightarrow \mathcal{H}^{0,1}_g (T_X,J)_{\text{sm}}
\]

\[
v \mapsto v^*_g.
\]

But \( \mathcal{H}^{0,1}_g (T_X,J)_{\text{sm}} = \mathcal{H}^{0,1}_g (T_X,J) \), thanks to lemma 14. We conclude the following fact.

**Lemma 29** Over any compact non Ricci flat Kähler-Einstein manifold \( (X, J, g) \) there exists the canonical isomorphism

\[
\text{Ker} \nabla^*_g \cap \text{Ker} \mathcal{L}_g \cap \mathcal{D}_g^J \longrightarrow H^{0,1}(X, T_X,J) \simeq H^1(X, \mathcal{O}(T_X,J))
\]

\[
v \mapsto \{ v^*_g \}.
\]

This result was proved by [D-W-W2] in the negative Kähler-Einstein case \( \text{Ric}(g) = -g \).

### 21.4 Polarised deformations of Fano manifolds

In this subsection we review a few basic facts on deformation theory which clarifies the Fano set up in the paper.

#### 21.4.1 The Maurer-Cartan equation

Let \( (V, J_0) \) be a complex vector space of dimension \( n \). We remind that the data of a complex structure \( J \) over \( V \) is equivalent with a \( n \)-dimensional complex subspace data \( \Gamma \subset \mathbb{C}V := V \otimes \mathbb{C} \) such that \( \Gamma \cap \overline{\Gamma} = \{0\} \). A complex structure \( J \) over \( V \) is called \( J_0 \)-compatible if the projection map

\[
\pi^{0,1}_{J_0} : V^{0,1}_J \longrightarrow V^{0,1}_{J_0},
\]

is surjective, i.e. a \( \mathbb{C} \)-isomorphism. This is equivalent to the condition \( V^{0,1}_J \cap V^{1,0}_{J_0} = \{ 0 \} \), which in its turn is equivalent to the existence of a \( \mathbb{C} \)-linear map \( \theta : V^{0,1}_{J_0} \longrightarrow V^{1,0}_{J_0} \) such that

\[
V^{0,1}_J = (I + \theta)V^{0,1}_{J_0}.
\]
If we set $\mu := (\theta + \bar{\theta})_V \in \text{End}_{-J_0}(V)$ then the condition $V^0_1 \cap V^1_1 = \{0\}$ is equivalent to say $\mathbb{I} + \mu \in \text{GL}_R(V)$. Notice that we can obtain $\theta$ by the formula $\theta = \mu_C \cdot J_{-}^{0,1}$, with $\mu_C \in \text{End}_C(\mathbb{C}V)$ the natural complexification of $\mu$. If we denote by $\mathcal{J}(V, J_0)$ the set of $J_0$-compatible complex structures over $V$ and if we set

$$C(V, J_0) := \{ \mu \in \text{End}_{-J_0}(V) \mid \mathbb{I} + \mu \in \text{GL}_R(V) \},$$

we infer the existence of a bijection, called the Cayley transform (see [Gau]),

$$\chi : C(V, J_0) \longrightarrow \mathcal{J}(V, J_0)$$

$$\mu \longmapsto J := (\mathbb{I} + \mu)J_0(\mathbb{I} + \mu)^{-1}$$

$$\mu := (J_0 + J)^{-1}(J_0 - J) \longmapsto J.$$  

Notice indeed that $\mathcal{J}(V, J_0)$ is the sub-set of the complex structures such that $J_0 + J \in \text{GL}_R(V)$. We observe that for any $\mu \in C(V, J_0)$ as above $\mathbb{I} - \mu \in \text{GL}_R(V)$. Indeed

$$-J_0J = (\mathbb{I} - \mu)(\mathbb{I} + \mu)^{-1}.$$  

Thus $\mu \in \text{End}_{-J_0}(V)$ satisfies $\mathbb{I} + \mu \in \text{GL}_R(V)$ if and only if $(\mathbb{I} - \mu^2) = (\mathbb{I} - \mu)(\mathbb{I} + \mu) \in \text{GL}_{J_0}(V)$. This last condition is equivalent with

$$\pi^1_{J_0} \cdot (\mathbb{I} - \mu^2) = (\mathbb{I}V^1_{J_0} - \theta \bar{\theta}) \in \text{GL}_C(V^1_{J_0}).$$  

We assume from now on that $(X, J_0)$ is a compact complex manifold and let $\mathcal{J}(X, J_0)$ be the set of $J_0$-compatible smooth almost complex structures. For any $J \in \mathcal{J}(X, J_0)$ let

$$\theta \equiv \theta_J \in C^\infty(X, \Lambda^{0,1}_{J_0}T_X^* \otimes_{\mathbb{C}} T^{1,0}_{X,J_0}), (\mathbb{I}V^1_{J_0} - \theta \bar{\theta}) \in \text{GL}_C(T^{1,0}_{X,J_0}),$$

be the corresponding Cayley transform. We show that the subset $\mathcal{J}_{\text{int}}(X, J_0)$ of integrable almost complex structures is given by the Maurer-Cartan equation

$$\overline{\partial}_{T^{1,0}_{X,J_0}} \theta + \frac{1}{2}[\theta, \theta] = 0,$$  

(21.9)

where for any $\alpha, \beta \in C^\infty(X, \Lambda^{0,\bullet}_{J_0}T_X^* \otimes_{\mathbb{C}} T^{1,0}_{X,J_0})$ we define the exterior differential Lie product

$$[\alpha, \beta] \in C^\infty(X, \Lambda^{0,\bullet}_{J_0}T_X^* \otimes_{\mathbb{C}} T^{1,0}_{X,J_0}),$$

of degree $d = \deg \alpha + \deg \beta$ by the formula

$$[\alpha, \beta](\xi) := \sum_{|I| = \deg \alpha} \varepsilon_I [\alpha(\xi_I), \beta(\xi_{\bar{I}})],$$

114
for all \( \xi \in \tilde{\mathcal{O}}(T^{0,1}_{X,J_0})^{x,d} \). Notice that this formula defines a priori only an element
\[
[\alpha, \beta] \in \text{Alt}^d_\mathcal{O}(\tilde{\mathcal{O}}(T^{0,1}_{X,J_0}); C^\infty(T^{1,0}_{X,J_0})).
\]

However we can define pointwise the section \([\alpha, \beta]\) as follows. For any \( v \in (T^{0,1}_{X,J_0,x})^{x,d} \)
\[
[\alpha, \beta](v) := [\alpha, \beta](\xi)|_x,
\]
with \( \xi \in \tilde{\mathcal{O}}(T^{0,1}_{X,J_0})^{x,d} \) such that \( \xi_x = v \). This is well defined by the \( \tilde{\mathcal{O}} \)-linearity of \([\alpha, \beta]\). Indeed the coefficients of \( \xi \) with respect to the local frame \( (\tilde{\zeta}_k)_{k=1}^n \subset \tilde{\mathcal{O}}(U, T^{0,1}_{X,J_0}) \), with \( \zeta_j := \frac{\partial}{\partial z_j} \), and \( J_0 \)-holomorphic coordinates \((z_1, \ldots, z_n)\), are \( J_0 \)-anti-holomorphic functions which value at the point \( x \) is uniquely determined by \( v \). The section \([\alpha, \beta]\) is smooth since its coefficients with respect to the frame \((\tilde{\zeta}_k)_{k=1}^n \) are smooth functions.

Notice now that \( (\Pi + \theta)(\tilde{\zeta}_k), (\Pi + \theta)(\tilde{\zeta}_l) \) holds true. On the other hand if \((21.9)\) is satisfied then the previous identity is satisfied and (21.10) hold true. We infer that if \((21.10)\)
\[
[(\Pi + \theta)(\tilde{\zeta}_k), (\Pi + \theta)(\tilde{\zeta}_l)] \in C^\infty(U, T^{1,0}_{X,J_0}),
\]

since the torsion form \( \tau_J \in C^\infty(X, \Lambda^0,2 T^*_X \otimes_{\mathbb{C}} T^{1,0}_{X,J_0}) \) of \( J \) satisfies
\[
\tau_J ((\Pi + \theta)(\tilde{\zeta}_k), (\Pi + \theta)(\tilde{\zeta}_l)) = [(\Pi + \theta)(\tilde{\zeta}_k), (\Pi + \theta)(\tilde{\zeta}_l)]_{J_0}.
\]

We observe also the identities
\[
[(\Pi + \theta)(\tilde{\zeta}_k), (\Pi + \theta)(\tilde{\zeta}_l)] = [\zeta_k, \theta(\tilde{\zeta}_l)] + [\theta(\tilde{\zeta}_k), \tilde{\zeta}_l] + [\theta(\tilde{\zeta}_k), \theta(\tilde{\zeta}_l)]
\]
\[
= [\zeta_k, \theta(\tilde{\zeta}_l)]_{J_0} - [\zeta_l, \theta(\tilde{\zeta}_k)]_{J_0} + \frac{1}{2} [\theta, \theta] (\tilde{\zeta}_k, \tilde{\zeta}_l)
\]
\[
= \left( \tilde{\mathcal{T}}_{T^{1,0}_{X,J_0}} \theta + \frac{1}{2} [\theta, \theta] \right) (\tilde{\zeta}_k, \tilde{\zeta}_l) \in C^\infty(U, T^{1,0}_{X,J_0}).
\]

We have \( T^{0,1}_{X,J_0} \cap T^{1,0}_{X,J_0} = 0_X \) by the \( J_0 \)-compatibility of \( J \). We infer that if \((21.10)\)
hold then also \((21.9)\) hold true and
\[
[(\Pi + \theta)(\tilde{\zeta}_k), (\Pi + \theta)(\tilde{\zeta}_l)] = 0.
\]

On the other hand if \((21.9)\) is satisfied then the previous identity is satisfied and thus \((21.10)\) hold true.

**Remark 4** For any \( \alpha, \beta \in C^\infty(X, \Lambda^0,0 T^*_X \otimes_{\mathbb{C}} T_{X,J_0}) \) we define the exterior differential Lie product
\[
[\alpha, \beta] \in C^\infty(X, \Lambda^0,0 T^*_X \otimes_{\mathbb{C}} T_{X,J_0}),
\]

115
of degree \( d = \deg \alpha + \deg \beta \) by the formula
\[
[\alpha, \beta] := [\pi^{1,0}_J \cdot \alpha_c, \pi^{1,0}_J \cdot \beta_c] + [\pi^{1,0}_J \cdot \alpha_c, \pi^{1,0}_J \cdot \beta_c].
\]
Then the Maurer-Cartan equation (21.9) can be rewritten in the equivalent form
\[
\partial T_{X,J_0} \mu + \frac{1}{2} [\mu, \mu] = 0,
\]
since
\[
\partial T_{X,J_0} \mu = \partial T_{X,J_0} \theta + \partial T_{X,J_0}^\theta,
\]
\[
[\mu, \mu] = [\theta, \theta] + [\theta, \theta].
\]

Let now \( B \subset \mathbb{C}^p \) be the unitary open ball and observe that, by a refinement of Ehresmann theorem for any proper holomorphic submersion \( \pi : \mathcal{X} \to B \) of a complex manifold \( \mathcal{X} \) onto \( B \) with central fibre \( (X, J_0) = \pi^{-1}(0) \) there exists a smooth map \( \varphi : \mathcal{X} \to X \) such that the map
\[
(\varphi, \pi) : \mathcal{X} \to X \times B,
\]
is a diffeomorphism with \( \varphi|_X = \mathbb{I}_X \) and with \( \varphi^{-1}(x) \subset \mathcal{X} \) complex sub-variety for all \( x \in X \).

Let now \( \theta := (\theta_t)_{t \in B} \subset C^\infty(X, \Lambda^{0,1}_X T^*_X \otimes_{\mathbb{C}} T^{1,0}_{X,J_0}) \) with \( \theta_0 = 0 \) and
\[
\det(\mathbb{I}_{T^{1,0}_{X,J_0}} - \theta_t \mathbb{I}_t) \neq 0,
\]
be a smooth family of \( J_0 \)-compatible complex structures. We observe that the almost complex manifold
\[
\mathcal{X} \equiv (\mathcal{X}, \theta) := \bigsqcup_{t \in B} (X, \theta_t),
\]
is a complex one if and only if \( \theta_t \) satisfies the Maurer-Cartan equation (21.9) for all \( t \in B \) and the map
\[
t \in B \mapsto \theta_t(x) \in \Lambda^{0,1}_{J_0} T^*_{X,x} \otimes_{\mathbb{C}} T^{1,0}_{X,J_0,x},
\]
is holomorphic for all \( x \in X \). Indeed the distribution \( T^{0,1}_{X,\theta} \) is integrable if and only if its local generators \( \bar{\tau}_r := \frac{\partial}{\partial \bar{z}_r}, \ r = 1, \ldots, p, (\mathbb{I} + \theta_t)(\zeta_k), \ k = 1, \ldots, n, \) satisfy the conditions
\[
[\bar{\tau}_r, (\mathbb{I} + \theta_t)(\zeta_k)] \in C^\infty(U, T^{0,1}_{X,\theta_t}), \quad (21.11)
\]
and
\[
[(\mathbb{I} + \theta_t)(\zeta_k), (\mathbb{I} + \theta_t)(\zeta_l)] \in C^\infty(U, T^{0,1}_{X,\theta_t}). \quad (21.12)
\]

116
The latter is equivalent with the Maurer-Cartan equation (21.9). Let \( \theta_t = \theta_t^k,l \zeta^k \otimes \zeta^l \) be the local expression of \( \theta_t \). Then the identity
\[
[\bar{\tau}_r,(\Pi + \theta_t)(\zeta_k)] = \bar{\tau}_r,\theta_t^k,l \zeta^l \in C^\infty(U,T_{X,J_0}^0),
\]
combined with the property \( T_{X,J_0}^0 \cap T_{X,J_0}^1 = 0_X \), shows that (21.11) hold true if and only if the map \( t \mapsto \theta_t \) is holomorphic.

For any \( p \in X \) a coordinate chart of \( X \) in a open neighborhood \( U_p \times B \) of \((p,0)\) is given by a smooth function \( f : U_p \times B \to \mathbb{C}^n \times \mathbb{C}^p \) such that
\[
\begin{align*}
\bar{\partial}_B f &= 0, \\
\partial B f &= 0, \\
\det (df) &\neq 0.
\end{align*}
\]

In order to produce such family \( \theta \) we need to remind a few basic facts about Hodge theory.

### 21.4.2 Basic facts about Hodge theory and \( \bar{\partial} \)-equations

Let \( \omega \) be a hermitian metric over \( X \) and let \((E,\bar{\partial}_E,h)\) be a hermitian holomorphic vector bundle over it. We define the anti-holomorphic Hodge Laplacian
\[
\Delta''_E := \bar{\partial}_E \bar{\partial}_E + \bar{\partial}_E \bar{\partial}_E,
\]
acting on the sections of \( \Lambda^p,q T^*_X \otimes_E E \). Let \( \mathcal{E}^{p,q}(E) := C^\infty(X,\Lambda^p,q T^*_X \otimes_E E) \) and set
\[
\mathcal{H}^{p,q}(E) := \text{Ker} \Delta''_E \cap \mathcal{E}^{p,q}(E).
\]

We remind the \( L^2 \)-Hodge decomposition
\[
\mathcal{E}^{p,q}(E) = \mathcal{H}^{p,q}(E) \oplus \bar{\partial}_E \mathcal{E}^{p,q-1}(E) \oplus \bar{\partial}_E^* \mathcal{E}^{p,q+1}(E).
\]

We observe that if there exists two subspaces \( L,V \subset C^\infty(X,E) \) such that the \( L^2 \)-decomposition
\[
C^\infty(X,E) = L \oplus V,
\]
hold then \( L \) and \( V \) are closed subspaces of \( C^\infty(X,E) \). Indeed \( L = V^\perp \) and \( V = L^\perp \) by the \( L^2 \)-decomposition. The same consideration hold for the Sobolev spaces \( W^k(X,E) \). Thus the \( L^2 \)-Hodge decomposition implies that the spaces \( \bar{\partial}_E \mathcal{E}^{p,q-1}(E) \) and \( \bar{\partial}_E^* \mathcal{E}^{p,q+1}(E) \) are closed in the smooth topology. We infer the \( L^2 \)-decomposition
\[
\mathcal{E}^{p,q}(E) = [\text{Ker} \bar{\partial}_E \cap \mathcal{E}^{p,q}(E)] \oplus \bar{\partial}_E \mathcal{E}^{p,q+1}(E),
\]

117
and thus
\[ \text{Ker } \overline{\partial}_E \cap \mathcal{E}^{p,q}(E) = \mathcal{H}^{p,q}(E) \oplus \overline{\partial}_E \mathcal{E}^{p,q-1}(E). \]

An other way to see this decomposition is the following. Let
\[ H_E : \mathcal{E}^{p,q}(E) \rightarrow \mathcal{H}^{p,q}(E), \]
be the $L^2$-projection operator over $\mathcal{H}^{p,q}(E)$. For any $\alpha \in \mathcal{E}^{p,q}(E)$ there exists $\beta \in \mathcal{E}^{p,q-1}(E)$ and $\gamma \in \mathcal{E}^{p,q+1}(E)$ such that
\[ \alpha = H_E \alpha + \overline{\partial}_E \beta + \overline{\partial}_E \gamma. \]

Now if $\overline{\partial}_E \alpha = 0$ then $\overline{\partial}_E \overline{\partial}_E \gamma = 0$, i.e. $\overline{\partial}_E \gamma = 0$. Let $W_k^{p,q}(E) := W^k(X, \Lambda^p_q T^*_X \otimes \mathcal{E})$. We remind that the Green operator
\[ G_E : W^{p,q}_k(E) \rightarrow \Delta''_E W^{p,q}_{k+4}(E), \]
is defined by the identity $\mathbb{1} = H_E + \Delta''_E G_E$. The latter implies that $\text{Ker } G_E = \text{Ker } \Delta''_E$ and the $L^2$-orthogonal decomposition
\[ \alpha = H_E \alpha + \overline{\partial}_E \overline{\partial}_E G_E \alpha + \overline{\partial}_E G_E \alpha. \]

We show now the identity $\overline{\partial}_E G_E = G_E \overline{\partial}_E$. Indeed $\overline{\partial}_E$-differentiating the identity defining $G_E$ we infer
\[ \overline{\partial}_E \alpha = \overline{\partial}_E \Delta'_k G_E \alpha = \Delta''_E \overline{\partial}_E G_E \alpha. \]

Applying the same identity to $\overline{\partial}_E \alpha$ we obtain $\overline{\partial}_E \alpha = \Delta'_k G_E \overline{\partial}_E \alpha$, since $H_E \overline{\partial}_E = 0$ by orthogonality. Thus
\[ \Delta''_E (\overline{\partial}_E G_E \alpha - G_E \overline{\partial}_E \alpha) = 0. \]

The fact that by definition $G_E W^{p,q}_k(E) = \Delta''_E W^{p,q}_{k+4}(E)$ implies the existence of $\beta \in W^{p,q}_{k+4}(E)$ and $\gamma \in W^{p,q+1}_{K+3}(E)$ such that
\[ G_E \alpha = \Delta''_E \beta, \]
\[ G_E \overline{\partial}_E \alpha = \Delta''_E \gamma. \]

We deduce
\[ \overline{\partial}_E G_E \alpha - G_E \overline{\partial}_E \alpha = \Delta''_E (\overline{\partial}_E \beta - \gamma) = 0, \]
thanks to the orthogonality of the Kernel and image of $\Delta''_E$.

We observe finally that the equation $\overline{\partial}_E \alpha = \beta$ admits a solution if and only if $\overline{\partial}_E \beta = 0$ and $H_E \beta = 0$. In this case the unique solution of minimal $L^2$-norm is given by $\alpha = \overline{\partial}_E G_E \beta$. 

118
21.4.3 The equation of holomorphic maps

For any smooth map \( f : (X, J_X) \longrightarrow (Y, J_Y) \) we define the operators
\[
2\partial_{J_X,J_Y} f := df - (J_Y \circ f) \cdot df \cdot J_X \in C^\infty \left( X, \Lambda^{1,0}_X T_X^e \otimes c f^* T_Y, J_Y \right),
\]
\[
2\bar{\partial}_{J_X,J_Y} f := df + (J_Y \circ f) \cdot df \cdot J_X \in C^\infty \left( X, \Lambda^{0,1}_X T_X^e \otimes c f^* T_Y, J_Y \right),
\]
and we notice the elementary identities
\[
\partial_{J_X,J_Y} f = \pi^{1,0}_Y \cdot df \cdot \pi^{0,1}_X + \pi^{0,1}_Y \cdot df \cdot \pi^{1,0}_X,
\]
\[
\bar{\partial}_{J_X,J_Y} f = \pi^{1,0}_Y \cdot df \cdot \pi^{0,1}_X + \pi^{0,1}_Y \cdot df \cdot \pi^{1,0}_X.
\]
The map \( f \) is called holomorphic if \((J_Y \circ f) \cdot df = df \cdot J_X\). We deduce that the map \( f \) is holomorphic if and only if \( \bar{\partial}_{J_X,J_Y} f = 0 \), thus if and only if
\[
\pi^{1,0}_Y \cdot df \cdot \pi^{0,1}_X = 0.
\]
We infer that a map \( f : (X, J_\varphi) \longrightarrow (Y, J_Y) \) is holomorphic if and only if
\[
\pi^{1,0}_Y \cdot df|_{T_X^{0,1}_{X,J_\varphi}} = 0.
\]
The identity
\[
T^{0,1}_{X,J_\varphi} = \left( \frac{\pi^{0,1}_{j_0} + \varphi}{\pi^{1,0}_{j_0}} \right) \otimes c T_X,
\]
implies that \( f : (X, J_\varphi) \longrightarrow (Y, J_Y) \) is holomorphic if and only if
\[
\pi^{1,0}_Y \cdot df \cdot \left( \frac{\pi^{0,1}_{j_0} + \varphi}{\pi^{1,0}_{j_0}} \right) = 0.
\]
This last condition rewrites as
\[
0 = \bar{\partial}_{j_0,J_Y} f \cdot \pi^{0,1}_{j_0} + \pi^{1,0}_Y \cdot df \cdot \pi^{1,0}_{j_0} \cdot \varphi
\]
\[
= \bar{\partial}_{j_0,J_Y} f \cdot \pi^{0,1}_{j_0} + \partial_{j_0,J_Y} f \cdot \varphi.
\]
We explicit this last condition in the case of a smooth map $f : (X, J_\varphi) \rightarrow (X, J_\theta)$. Indeed

\[
0 = 2\partial J_{j_0, j_0} f \cdot \pi_{j_0}^{0,1} + 2 \partial J_{j_0, j_0} f \cdot \varphi \\
= [\Pi - i (J_\theta \circ f)] \cdot (df \cdot \pi_{j_0}^{0,1} + df \cdot \varphi) \\
= [\Pi - i (J_\theta \circ f)] \cdot \pi_{j_0}^{1,0} \cdot (df \cdot \pi_{j_0}^{0,1} + df \cdot \varphi) \\
+ [\Pi - i (J_\theta \circ f)] \cdot \pi_{j_0}^{0,1} \cdot (df \cdot \partial J_{j_0} f \cdot \varphi) \\
+ [\Pi - i (J_\theta \circ f)] \cdot \pi_{j_0}^{0,1} \cdot (\partial J_{j_0} f \cdot \varphi). 
\]

We explicit at this point the expression of $J_\theta$. For this purpose let $\mu := \theta + \bar{\theta}$ and decompose the identity

\[
J_\theta := J_0 (\Pi - \mu) (\Pi + \mu)^{-1} \\
= J_0 (\Pi - \mu)^2 (\Pi - \mu^2)^{-1} \\
= J_0 (\Pi - 2\mu + \mu^2) (\Pi - \mu^2)^{-1} \\
= J_0 (\Pi - 2\theta - 2\bar{\theta} + \theta \bar{\theta} + \bar{\theta} \theta) (\Pi - \theta \bar{\theta} - \bar{\theta} \theta)^{-1}. 
\]

Decomposing in types we infer

\[
J_\theta = i (\Pi_{1,0} + \theta \bar{\theta}) (\Pi_{1,0} - \theta \bar{\theta})^{-1} + 2i\bar{\theta}(\Pi_{1,0} - \theta \bar{\theta})^{-1} \\
- 2i\theta(\Pi_{0,1} - \theta \bar{\theta})^{-1} - i(\Pi_{0,1} + \theta \bar{\theta})(\Pi_{0,1} - \theta \bar{\theta})^{-1}. 
\]

Let $A := \theta \bar{\theta}$. Using the trivial identity $(\Pi + A) (\Pi - A)^{-1} = \Pi + 2A (\Pi - A)^{-1}$ we conclude the expression

\[
J_\theta = \frac{i\Pi_{1,0} + 2i\theta\bar{\theta}(\Pi_{1,0} - \theta \bar{\theta})^{-1} + 2i\bar{\theta}(\Pi_{1,0} - \theta \bar{\theta})^{-1}}{\epsilon \in C^{1,0} (T_{X, J_{j_0}})} \\
- \frac{i\Pi_{0,1} - 2i\theta\bar{\theta}(\Pi_{0,1} - \theta \bar{\theta})^{-1}}{\epsilon \in C^{0,1} (T_{X, J_{j_0}})}. 
\]
For notation simplicity we identify \( J_\theta \equiv J_\theta \circ f \) and thus \( \theta \equiv \theta \circ f \). Using the previous expression we infer the equalities

\[
\frac{1}{2} [\Pi - i (J_\theta \circ f)] \cdot \pi_{J_\theta}^{1,0} = \Pi_{1,0} + \theta \theta (\Pi_{1,0} - \theta \theta)^{-1} + \overline{\theta} (\Pi_{1,0} - \overline{\theta \theta})^{-1} \\
= (\Pi_{1,0} - \theta \theta)^{-1} + \overline{\theta} (\Pi_{1,0} - \theta \theta)^{-1},
\]

\[
\frac{1}{2} [\Pi - i (J_\theta \circ f)] \cdot \pi_{J_\theta}^{0,1} = -\theta (\Pi_{0,1} - \theta \theta)^{-1} - \overline{\theta \theta} (\Pi_{0,1} - \overline{\theta \theta})^{-1}.
\]

The second equality follows from the trivial identity \( \Pi + A (\Pi - A)^{-1} = (\Pi - A)^{-1} \). We deduce that the holomorphy condition for \( f \) writes in the form

\[
0 = \overline{\theta} J_{J_\theta} f \cdot \pi_{J_\theta}^{0,1} + \overline{\partial} J_{J_\theta} f \cdot \varphi
\]

\[
= (\Pi_{1,0} - \theta \theta)^{-1} \cdot (\overline{\partial} J_{J_\theta} f + \partial J_{J_\theta} f \cdot \varphi) - \theta (\Pi_{0,1} - \theta \theta)^{-1} \cdot (\partial J_{J_\theta} f + \overline{\theta} J_{J_\theta} f \cdot \varphi) + \overline{\theta} (\Pi_{1,0} - \theta \theta)^{-1} \cdot (\overline{\theta} J_{J_\theta} f + \partial J_{J_\theta} f \cdot \varphi) - \theta (\Pi_{0,1} - \theta \theta)^{-1} \cdot (\partial J_{J_\theta} f + \overline{\theta} J_{J_\theta} f \cdot \varphi).
\]

The fact that the second line is composed by elements in \( \mathcal{E}^{0,1} \left( T_{X,J_{J_\theta}} \right) \) and the third by elements in \( \mathcal{E}^{0,1} \left( T_{X,J_{J_\theta}} \right) \) implies that the holomorphy condition for \( f \) is equivalent to the equations

\[
(\Pi_{1,0} - \theta \theta)^{-1} \cdot (\overline{\partial} J_{J_\theta} f + \partial J_{J_\theta} f \cdot \varphi) = \theta (\Pi_{0,1} - \theta \theta)^{-1} \cdot (\partial J_{J_\theta} f + \overline{\theta} J_{J_\theta} f \cdot \varphi),
\]

\[
\overline{\theta} (\Pi_{1,0} - \theta \theta)^{-1} \cdot (\overline{\partial} J_{J_\theta} f + \partial J_{J_\theta} f \cdot \varphi) = \overline{\theta} \theta (\Pi_{0,1} - \overline{\theta \theta})^{-1} \cdot (\partial J_{J_\theta} f + \overline{\theta} J_{J_\theta} f \cdot \varphi).
\]

But the last one is obtained multiplying both sides of the first with \( \overline{\theta} \). We infer that the holomorphy condition for \( f \) writes as

\[
\pi_{J_\theta}^{1,0} \cdot \overline{\partial} J_{J_\theta} f + \partial J_{J_\theta} f \cdot \varphi = (\Pi_{1,0} - \theta \theta) \theta (\Pi_{0,1} - \overline{\theta \theta})^{-1} \cdot (\partial J_{J_\theta} f + \overline{\theta} J_{J_\theta} f \cdot \varphi).
\]

We notice now the identity

\[
\theta = (\Pi_{1,0} - \theta \theta) \theta (\Pi_{0,1} - \overline{\theta \theta})^{-1}.
\]

This last follows decomposing the trivial identity

\[
\mu = (\Pi - \mu^2) \mu (\Pi - \mu^2)^{-1}.
\]

We conclude finally that the map \( f : (X,J_\varphi) \rightarrow (X,J_\theta) \) is holomorphic if and only if

\[
\pi_{J_\theta}^{1,0} \cdot \overline{\partial} J_{J_\theta} f + \partial J_{J_\theta} f \cdot \varphi = (\theta \circ f) \cdot (\partial J_{J_\theta} f + \overline{\theta} J_{J_\theta} f \cdot \varphi).
\]
For any \( f \in \text{Diff}(X) \) sufficiently close to the identity in \( C^1 \)-norm, the almost complex structure \( f^* J_0 \) is \( J_0 \)-compatible, i.e. \( \det (J_0 + f^* J_0) \neq 0 \). Thus there exists a unique form \( \theta_f \) such that \( f^* J_0 = J_{\theta_f} \). By definition the map \( f : (X, J_{\theta_f}) \rightarrow (X, J_0) \) is holomorphic. We conclude that \( \theta_f \) is given by the formula

\[
\pi_{j_0}^{1,0} \cdot \overline{\partial}_{j_0} f - (\theta \circ f) \cdot \partial_{j_0} f = -[\partial_{j_0} f - (\theta \circ f) \cdot \overline{\partial}_{j_0} f] \cdot \theta_f, \quad (21.13)
\]

and thus

\[
\theta_f = -[\partial_{j_0} f - (\theta \circ f) \cdot \overline{\partial}_{j_0} f]^{-1} \left[ \pi_{j_0}^{1,0} \cdot \overline{\partial}_{j_0} f - (\theta \circ f) \cdot \partial_{j_0} f \right],
\]

as long as

\[
[\partial_{j_0} f - (\theta \circ f) \cdot \overline{\partial}_{j_0} f]_{T_X^{1,0}} \in \text{GL}_c \left( T_X^{1,0} \right).
\]

Adding the complex conjugate we infer

\[
\overline{\partial}_{j_0} f - (\mu \circ f) \cdot \partial_{j_0} f = -[\partial_{j_0} f - (\mu \circ f) \cdot \overline{\partial}_{j_0} f] \cdot \mu_f, \quad (21.14)
\]

and thus

\[
\mu_f = -[\partial_{j_0} f - (\mu \circ f) \cdot \overline{\partial}_{j_0} f]^{-1} \left[ \overline{\partial}_{j_0} f - (\mu \circ f) \cdot \partial_{j_0} f \right],
\]

as long as

\[
[\partial_{j_0} f - (\mu \circ f) \cdot \overline{\partial}_{j_0} f]_{T_X^{1,0}} \in \text{GL}(T_X).
\]

### 21.4.4 The Kuranishi space of a compact complex manifold

Let \((X, J)\) be a complex manifold and consider

\[
E''_{g,J} := T_X^{1,0} \otimes T_X,J,
\]

\[
E''_{g',J} := \text{End}_g(T_X) \cap E''_{g,J},
\]

\[
\mathcal{C}_J := \left\{ \mu \in \mathcal{E}(E''_{g,J}) \mid (1 + \mu) \in \text{GL}_\mathbb{R}(T_X), \overline{\partial}_{T_X,J} \mu + \frac{1}{2} [\mu, \mu] = 0 \right\}.
\]

Then the Cayley transform (see \[\text{Gau}\]) provides a bijection

\[
\text{Cal}_J : \mathcal{C}_J \rightarrow J_{\text{int}}
\]

\[
\mu \leftrightarrow J := (I + \mu)J_0(I + \mu)^{-1}
\]

\[
\mu := (J_0 + J)^{-1}(J_0 - J) \leftrightarrow J.
\]
For notations convenience we will restrict our considerations to the Fano case even if the result that will follow and its argument hold for a general compact complex manifold. For any polarized Fano manifold \((X, J, \omega)\) we define also the sub-set of \(\Omega\)-divergence free tensors in \(\mathcal{C}_J\)

\[
\mathcal{C}^{\text{div}}_{J,g} := \left\{ \mu \in \mathcal{C}_J \mid \overline{\partial}_{T_{X,J}}^{g,\Omega} \mu = 0 \right\}.
\]

We denote by \(H^0(T_{X,J})^\perp \cap W_k(T_{X,J})\) the \(L^2_{g,\Omega}\)-orthogonal space to the space of holomorphic vector fields of type \((1,0)\) inside \(W_k(T_{X,J})\). For any \(\xi \in \mathcal{E}(T_{X,J})\) of sufficiently small norm the map \(e(\xi) : X \rightarrow X\) defined by

\[
e(\xi)_x := \exp_{g,x}(\xi_x),
\]

is a smooth diffeomorphism. For readers convenience we provide a proof (in the Fano case) of the following fundamental result due to Kuranishi [Kur].

**Theorem 3 (The Kuranishi space \(K_{J,g}\))** For any polarized Fano manifold \((X, J, \omega)\) and any integer \(k > n + 1\) with \(n := \dim_X X\) there exists:

(A) \(\varepsilon, \delta \in \mathbb{R}_{>0},\) a complex analytic subset \(K_{J,g} \subset H^{0,1}_{g,\Omega}(T_{X,J}) \cap B^{g}_\delta(0),\)

with \(\mu_0 = 0\), which restricts to a bijection

\[
\mu : K_{J,g} \rightarrow \mathcal{C}^{\text{div}}_{J,g} \cap B^{W_k}_{\varepsilon}(0),
\]

with the property \(d_0 \mu(v) = v\), for all \(v \in TC_{K_{J,g},0} := \text{the tangent cone of } K_{J,g}\) at the origin.

(B) \(\varepsilon_0 \in \mathbb{R}_{>0}, \varepsilon_0 < \varepsilon,\) and a smooth map

\[
B^{W_k^{0,1}(T_{X,J})}_{\varepsilon_0}(0) \rightarrow H^0(T_{X,J})^\perp \cap W_k(T_{X,J})
\]

\(\varphi \mapsto \xi_\varphi,\)

with \(\xi_0 = 0\), such that \(\overline{\partial}_{T_{X,J}}^{g,\Omega} \varphi_{\varepsilon(\xi_\varphi)} = 0\) which restricts to an application

\[
B^{W_k^{0,1}(T_{X,J})}_{\varepsilon_0}(0) \cap C^{0,1}(T_{X,J}) \rightarrow H^0(T_{X,J})^\perp \cap \mathcal{E}(T_{X,J}),
\]

and such that the map

\[
\mathcal{C}_J \cap B^{W_k^{0,1}(T_{X,J})}_{\varepsilon_0}(0) \rightarrow \mathcal{C}^{\text{div}}_{J,g} \cap B^{W_k^{0,1}(T_{X,J})}_{\varepsilon}(0)
\]

\(\varphi \mapsto \mu(\varphi) := \varphi_{\varepsilon(\xi_\varphi)},\)

is well defined.
Proof. We divide Kuranishi’s proof in a few steps.

STEP A1. We show first that the system

\begin{equation}
\left\{ \begin{array}{l}
\overline{\partial}_{TX,J} \mu + \frac{1}{2} [\mu, \mu] = 0, \\
\overline{\partial}_{TX,J} \mu = 0,
\end{array} \right.
\end{equation}

is equivalent to the system

\begin{equation}
\left\{ \begin{array}{l}
\mu + \frac{1}{2} \overline{\partial}_{TX,J} G_{TX,J} [\mu, \mu] = H_{TX,J} \mu, \\
H_{TX,J} [\mu, \mu] = 0,
\end{array} \right.
\end{equation}

provided that \( \mu \) is sufficiently close to 0. Indeed let \( \mu \) be a solution of (S1). Then the considerations about the resolution of the \( \overline{\partial} \)-equation imply the second equation in (S2). Moreover if we set

\[ \varphi := -\frac{1}{2} \overline{\partial}_{TX,J} G_{TX,J} [\mu, \mu], \]

then \( \alpha := \mu - \varphi \) satisfies \( \overline{\partial}_{TX,J} \alpha = 0 \) and \( \overline{\partial}_{TX,J} \alpha = 0 \). Thus \( \alpha \in H^{0,1}_{g,\Omega}(T_{X,J}) \) and \( H_{TX,J} \mu = \alpha \) since

\[ H_{TX,J} \overline{\partial}_{TX,J} = 0, \]

by orthogonality. This shows that also the first equation in (S2) hold. Assume now that \( \mu \) is a solution of (S2). It is clear that the second equation in (S1) hold true. We set

\[ \psi := \overline{\partial}_{TX,J} \mu + \frac{1}{2} [\mu, \mu], \]

and we observe the equalities

\[ \psi = -\frac{1}{2} \overline{\partial}_{TX,J} \overline{\partial}_{TX,J} G_{TX,J} [\mu, \mu] + \frac{1}{2} [\mu, \mu] \]

\[ = \frac{1}{2} \overline{\partial}_{TX,J} \overline{\partial}_{TX,J} G_{TX,J} [\mu, \mu] \]

\[ = \frac{1}{2} \overline{\partial}_{TX,J} G_{TX,J} \overline{\partial}_{TX,J} [\mu, \mu] \]

\[ = \overline{\partial}_{TX,J} G_{TX,J} [\overline{\partial} \mu, \mu]. \]

We deduce the identity

\[ \psi = \overline{\partial}_{TX,J} G_{TX,J} [\psi, \mu]. \]
The assumption \( k > n + 1 \) implies that the Sobolev embedding \( W^k \subset C^{k-n} \) hold true. Using the standard estimates on the Sobolev norms of \( W_k^\cdot \cdot \)

\[
\| G_{T_X,J} \varphi \|_{k+2} \leq C_0 \| \varphi \|_k,
\]

\[
\| [\varphi, \psi] \|_{k-1} \leq C_2 \| \varphi \|_k \| \psi \|_k,
\]

we obtain

\[
\| \psi \|_k \leq C_1 \| [\psi, \mu] \|_{k-1} \leq C_1 C_2 \| \psi \|_k \| \mu \|_k.
\]

Thus if \( \| \mu \|_k \leq \varepsilon/(C_1 C_2) \) for some \( \varepsilon \in (0,1) \) then \( (1-\varepsilon) \| \psi \|_k \leq 0 \), which hold true if and only if \( \psi = 0 \).

**STEP A2.** We remind that the previous discussion shows that the first equation in (S2) is equivalent to the condition

\[
F(\mu) := \mu + \frac{1}{2} \overline{\Omega}_{T_X,J}^\cdot [\mu, \mu] \in \mathcal{H}^{0,1}_{g,\Omega}(T_X,J).
\]

Let \( \Xi_k \subset W^{0,1}_k(T_X,J) \) be the subset of the elements satisfying this condition. We notice that the map

\[
F : W^{0,1}_k(T_X,J) \rightarrow W^{0,1}_k(T_X,J),
\]

is well defined and continuous thanks to the estimate

\[
\| \overline{\Omega}_{T_X,J}^\cdot [\mu, \mu] \|_k \leq C_1 \| [\mu, \mu] \|_{k-1} \leq C_1 C_2 \| \mu \|_k^2.
\]

We infer that \( F \) is also holomorphic since \( F - \mathbb{I} \) is a continuous quadratic form. The fact that the differential of \( F \) at the origin is the identity implies the existence of an inverse holomorphic map \( F^{-1} \) in a neighborhood \( B_{B_k}^0(0) \) of the origin. Restricting this to \( \mathcal{H}^{0,1}_{g,\Omega}(T_X,J) \cap B_{B_k}^0(0) \) we deduce the existence of a holomorphic map

\[
\alpha \in \mathcal{H}^{0,1}_{g,\Omega}(T_X,J) \cap B_{B_k}^0(0) \rightarrow \mu_\alpha \in W^{0,1}_k(T_X,J),
\]

such that

\[
\mu_\alpha + \frac{1}{2} \overline{\Omega}_{T_X,J}^\cdot [\mu_\alpha, \mu_\alpha] = \alpha.
\]

By construction \( \text{Im}(\alpha \mapsto \mu_\alpha) \) represents a neighborhood of the origin inside \( \Xi_k \). It is clear that \( \mu_\alpha \) is of class \( C^{k-n} \) by the Sobolev embedding. We show further that \( \mu_\alpha \) is smooth for a sufficiently small choice of \( \varepsilon \). Indeed applying the Hodge Laplacian \( \Delta_{T_X,g}^{\cdot -}\) to both sides of the previous identity and using the equalities

\[
\Delta_{T_X,g}^{\cdot -} \overline{\Omega}_{T_X,J}^\cdot [\mu, \mu] = \overline{\Omega}_{T_X,J}^\cdot \Delta_{T_X,g}^{\cdot -} G_T = \overline{\Omega}_{T_X,J}^\cdot G_T,
\]

125
(notice that $\overline{\partial}_{T_{X,J}}^{\alpha}\tilde{H}_{T_{X,J}} = 0$) we obtain the equation

$$\Delta_{T_{X,g}}^{\Omega,-J} \mu_{\alpha} + \frac{1}{2} \overline{\partial}_{T_{X,J}}^{\alpha} [\mu_{\alpha}, \mu_{\alpha}] = 0,$$

which rewrites also as

$$\Delta_{T_{X,g}}^{\Omega,-J} \mu_{\alpha} + \frac{1}{2} \mu_{\alpha} * \nabla_{g}^{2} \mu_{\alpha} = \frac{1}{2} \nabla_{g} \mu_{\alpha} * \nabla_{g} \mu_{\alpha} + \frac{1}{2} \mu_{\alpha} * \nabla_{g} \mu_{\alpha} * \nabla_{g} f,$$

where $*$ denotes adequate contraction operators. The fact that the $C^0$-norm of $\mu_{\alpha}$ can be made arbitrary small for sufficiently small $\varepsilon$ implies that the operator

$$\Delta_{T_{X,g}}^{\Omega,-J} + \frac{1}{2} \mu_{\alpha} * \nabla_{g}^{2},$$

is elliptic. Then the smoothness of $\mu_{\alpha}$ follows by standard elliptic bootstrapping.

We denote by $K_{J,g}$ the zero set of the holomorphic map

$$\chi : \mathcal{H}_{g,\Omega}^{0,1}(T_{X,J}) \cap B_{\varepsilon}^{W_k}(0) \rightarrow \mathcal{H}_{g,\Omega}^{0,2}(T_{X,J})$$

$$\alpha \mapsto H_{T_{X,J}}^{0,1}[\mu_{\alpha}, \mu_{\alpha}].$$

Then the set $\{\mu_{\alpha} | \alpha \in K_{J,g}\}$ covers the set of the solutions of the system $(S_2)$ in a neighborhood of the origin.

**STEP B.** We observe first that $\overline{\partial}_{T_{X,J}}^{\alpha} \varphi = 0$ if and only if $G_{T_{X,J}} \overline{\partial}_{T_{X,J}}^{\alpha} \varphi = 0$. Indeed

$$\text{Im} \overline{\partial}_{T_{X,J}}^{\alpha} \perp \text{Ker} G_{T_{X,J}},$$

since $\text{Ker} G_{T_{X,J}} = \text{Ker} \Delta_{T_{X,g}}^{\Omega,-J}$. Thus in order to construct the application $\varphi \mapsto \xi_{\varphi}$ we need to find the zeros of the map

$$R : W_{k}^{0,1}(T_{X,J}) \times \left[H^{0}(T_{X,J}) \perp \cap B_{\varepsilon}^{W_k}(T_{X,J})\right] \rightarrow H^{0}(T_{X,J}) \perp \cap W_k(T_{X,J})$$

$$(\varphi, \xi) \mapsto G_{T_{X,J}} T_{X,J}^{\alpha} \varphi(\xi),$$

$$(0, 0) \mapsto 0.$$

For notations simplicity we denote $\Psi : (\mu, f) \mapsto \mu_f$. With these notations the formula \cite{2014} writes as

$$\overline{\partial}_{J} e(t\xi) = -\partial_{J} e(t\xi) \cdot \Psi(0, e(t\xi)).$$

Time deriving this identity at $t = 0$ and using the fact that $\frac{d}{dt}|_{t=0} e(t\xi) = \xi$, $\Psi(0, \text{Id}_X) = 0$ and $e(0) = \text{Id}_X$ we obtain

$$\overline{\partial}_{T_{X,J}} \xi = -D_{J} \Psi(0, \text{Id}_X) \cdot \xi,$$

126
where $D_f \Psi$ denotes the partial Frechet derivative of $\Psi$ in the variable $f$. We observe now that for any $\xi \in W_k(T_{X,J})$ hold the decomposition formula

$$\xi = H_{T_{X,J}} \xi + G_{T_{X,J}} \overline{\partial}_{T_{X,J}}^* \overline{\partial}_{T_{X,J}} \xi.$$ 

Thus if $\xi \in H^0(T_{X,J}) \cap W_k(T_{X,J})$ then hold the identity

$$\xi = G_{T_{X,J}} \overline{\partial}_{T_{X,J}}^* \partial_{T_{X,J}} \xi.$$ 

We conclude the identity $D_{\xi} R(0,0) = C_1$ and the existence of the map $\varphi \mapsto \xi_\varphi$ by the implicit function theorem. In local coordinates we can consider the expansion

$$e(\xi) = \text{Id}_X + \xi + O(|\xi|^2).$$

Then the formula (21.14) implies the local identity

$$\varphi_{e(\xi)} = \overline{\partial}_{T_{X,J}} \xi + \varphi + Q(\varphi, \xi),$$

with $Q$ an analytic function (depending on the local coordinates). Then the condition $\overline{\partial}_{T_{X,J}}^* e(\varphi_{e(\xi)}) = 0$ implies

$$\Delta_{-J}^{\Omega} \xi + \overline{\partial}_{T_{X,J}} \varphi + \overline{\partial}_{T_{X,J}}^* Q(\varphi, \xi) = 0.$$ 

Thus $\xi_\varphi$ is smooth if $\varphi$ is smooth by elliptic regularity. \hfill $\square$

### 21.4.5 Parametrization of a sub-space of the $\omega$-compatible complex structures

Let $(X, J, \omega)$ be a polarized Fano manifold and consider the set

$$\mathcal{C}_{\omega,J} := \left\{ \mu \in \mathcal{E}(X, E_g) \mid g(1 \pm \mu) > 0, \overline{\partial}_{T_{X,J}} \mu + \frac{1}{2} [\mu, \mu] = 0 \right\},$$

with $g := -\omega J$. Then the Cauchy transform restricts to a bijection (see Gaud)

$$\text{Cald}_{J} : \mathcal{C}_{\omega,J} \longrightarrow \mathcal{J}_{\omega}.$$ 

We define also the sub-set of $\Omega$-divergence free tensors in $\mathcal{C}_{\omega,J}$

$$\mathcal{C}_{\omega,J}^{\text{div}} := \left\{ \mu \in \mathcal{C}_{\omega,J} \mid \overline{\partial}_{T_{X,J}}^* \mu = 0 \right\}.$$ 

**Definition 2 (The Kuranishi space of polarised deformations)** For any polarized Fano manifold $(X, J, \omega)$ we define the Kuranishi space of $\omega$-polarized complex deformations as the complex analytic subset

$$\mathcal{K}_{J}^{\omega} := \left\{ \alpha \in \mathcal{K}_{J,g} \mid \mu_\alpha - (\mu_\alpha)_g^T \right\}.$$
With these notations the map $\mu$ in theorem 3 restricts to a bijection

$$
\mu : K_{\omega} \longrightarrow C_{\omega,i}^{\text{div}} \cap B_{k}^{W_{k}^{0,1}(T_{X,J})}(0).
$$

Let now $U_{\omega} \subset C_{\omega}^{\infty}(X,\mathbb{C})_{0}$ be an open neighborhood of the origin such that $\omega + dd_{c}^{J}u_{1} > 0$ for all $\alpha \in K_{\omega}$ and $u = u_{1} + iu_{2} \in U_{\omega}$, with $u_{j}$ real valued. We define the real vector field

$$
\xi_{t}^{\alpha,u} := - \left( \omega + dd_{c}^{J}u_{1} \right)^{-1} \left( d_{c}^{J}u_{1} + \frac{1}{2} du_{2} \right),
$$

for all $t \in (-\varepsilon, 1 + \varepsilon)$, for some small $\varepsilon > 0$. We define also the family of diffeomorphisms $(\Phi_{t}^{\alpha,u})_{t \in (-\varepsilon,1+\varepsilon)}$ over $X$ given by $\partial_{t} \Phi_{t}^{\alpha,u} = \xi_{t}^{\alpha,u} \circ \Phi_{t}^{\alpha,u}$, with $\Phi_{0}^{\alpha,u} = \text{Id}_{X}$. We set finally

$$
J_{\alpha,u} := (\Phi_{1}^{\alpha,u})^{*} J_{\alpha}.
$$

With these notations hold the following lemma.

**Lemma 30** The map

$$
K_{\omega} \times U_{\omega} \longrightarrow J_{\omega},
$$

$$(\alpha, u) \longmapsto J_{\alpha,u},
$$

is well defined and its differential at the origin is given by the fiberwise injection

$$
TC_{\omega} \longrightarrow TC_{J_{\omega}},
$$

$$(A, v) \longmapsto -J \left[ \bar{T}_{X,J} \nabla_{g,J} v + 2A \right].
$$

**Proof** Let denote for simplicity $\omega_{t} := \omega + dd_{c}^{J}u_{1}$ and we observe the elementary identities

$$
\dot{\omega}_{t} = dd_{c}^{J}u_{1} = -d(\xi_{t}^{\alpha,u} \omega_{t}) = -L_{\xi_{t}^{\alpha,u}} \omega_{t}.
$$

We infer

$$
\frac{d}{dt} \left[ (\Phi_{t}^{\alpha,u})^{*} \omega_{t} \right] = (\Phi_{t}^{\alpha,u})^{*} \left( \dot{\omega}_{t} + L_{\xi_{t}^{\alpha,u}} \omega_{t} \right) = 0,
$$

and thus $(\Phi_{1}^{\alpha,u})^{*} \omega_{1} = (\Phi_{0}^{\alpha,u})^{*} \omega_{0} = \omega$, i.e.

$$
(\Phi_{1}^{\alpha,u})^{*} \left( \omega + dd_{c}^{J}u_{1} \right) = \omega.
$$

The fact that the complex structure $J_{\alpha}$ is integrable implies that the form $\omega_{1}$ is $J_{\alpha}$-invariant. (This is no longer true in the non-integrable case!) We conclude $J_{\alpha,u} \in J_{\omega}$. 128
We compute now the differential at the origin. We consider for this purpose a smooth family \((u(s))_s \subset \mathcal{U}_\omega\) such that \(u(0) = 0\) and \(\dot{u}(0) = v\). We denote for simplicity \(\xi_{t,s} := \xi_t^{0,u(s)}\) and \(\Phi_{t,s} := \Phi_t^{0,u(s)}\). Then deriving with respect to \(s\) at \(s = 0\) the identity
\[
\frac{\partial}{\partial t} \Phi_{t,s} = \xi_{t,s} \circ \Phi_{t,s},
\]
and using the fact that \(\xi_{t,0} = 0\), (which implies in particular \(\Phi_{t,0} = \text{Id}_X\)) we obtain
\[
\frac{\partial}{\partial s} \bigg|_{s=0} \frac{\partial}{\partial t} \Phi_{t,s} = \frac{\partial}{\partial s} \bigg|_{s=0} \xi_{t,s} + d_x \xi_{t,0} \cdot \frac{\partial}{\partial s} \bigg|_{s=0} \Phi_{t,s} = \frac{\partial}{\partial s} \bigg|_{s=0} \xi_{t,s}.
\]
On the other hand deriving with respect to \(s\) at \(s = 0\) the identity
\[
\xi_{t,s} \circ \left( \omega + tdd_{J} u_1 (s) \right) = - \left( d_{J} v_1 (s) + \frac{1}{2} du_2 (s) \right),
\]
we obtain
\[
\left( \frac{\partial}{\partial s} \bigg|_{s=0} \xi_{t,s} \right) \circ \omega = - \left( d_{J} v_1 + \frac{1}{2} dv_2 \right),
\]
and thus
\[
\frac{\partial}{\partial s} \bigg|_{s=0} \xi_{t,s} = - \frac{1}{2} \nabla_{g,J} v.
\]
Commuting the derivatives in \(s\) and \(t\) we infer the identity
\[
\frac{\partial}{\partial t} \frac{\partial}{\partial s} \bigg|_{s=0} \Phi_{t,s} = - \frac{1}{2} \nabla_{g,J} v.
\]
Integrating in \(t\) from 0 to 1 we deduce
\[
\eta := \frac{\partial}{\partial s} \bigg|_{s=0} \Phi_{1,s} = - \frac{1}{2} \nabla_{g,J} v,
\]
since \(\Phi_{0,s} = \text{Id}_X\). We infer
\[
\frac{d}{ds} \bigg|_{s=0} J_{u(s)} = L_0 J = - J \overline{\partial}_{X,J} \nabla_{g,J} v.
\]
Assume now \((\alpha)_{s} \subset \mathcal{K}_{J,\omega}\) is a smooth curve with \(\alpha(0) = 0\) and \(\dot{\alpha}(0) = A\). Then
\[
\frac{d}{ds} \bigg|_{s=0} J_{\alpha(s),u(s)} = \frac{d}{ds} \bigg|_{s=0} J_{\alpha(s)} + \frac{d}{ds} \bigg|_{s=0} J_{0,u(s)},
\]

with
\[
\frac{d}{ds}_{|s=0} J_{\alpha(s)} = -2JA,
\]
thanks to the properties of the differential of the Caley transform. □

**Lemma 31** For any point \( J \in J_\omega \) hold the inclusions
\[
\left[ \partial_{T_X,J} \nabla_{g,J} C^\infty (X, \mathcal{C}) \right] \oplus \Omega TC_{K_\gamma,0} 
\subseteq \ TC_{J_\omega,J} 
\subseteq \left[ \partial_{T_X,J} \nabla_{g,J} C^\infty (X, \mathcal{C}) \right] \oplus \Omega TC_{K_{J,g},0}.
\]

**Proof** The first inclusion is a direct consequence of lemma 30. In order to show the second one let \((\varphi_t)_t \subset C_J \cap B^{W,1}_{CJ}(T_X,J)(0)\) with \(\varphi_0 = 0\) and set for notation simplicity \(e_t := e(\xi, \varphi_t)\). With this notation the identity (21.14) writes as
\[
\partial_{T_X,J} [D \xi (0) \dot{\varphi}_0] - \dot{\varphi}_0 = - \frac{d}{dt}_{|t=0} \mu (\varphi_t).
\]
Time deriving this at \(t = 0\) and using the obvious equality \(\dot{\varphi}_0 = D\xi (0) \dot{\varphi}_0\) we deduce the equality
\[
\partial_{T_X,J} [D \xi (0) \dot{\varphi}_0] - \dot{\varphi}_0 = - \frac{d}{dt}_{|t=0} \mu (\varphi_t).
\]
This combined with the identity \(\partial_{T_X,J}^* \mu (\varphi_t) \equiv 0\) implies
\[
\partial_{T_X,J}^* \partial_{T_X,J} [D \xi (0) \dot{\varphi}_0] - \partial_{T_X,J}^* \dot{\varphi}_0 = 0.
\]
Thus if \(\partial_{T_X,J}^* \dot{\varphi}_0 = 0\) then \(D \xi (0) \dot{\varphi}_0 = 0\) and
\[
\dot{\varphi}_0 = \frac{d}{dt}_{|t=0} \mu (\varphi_t).
\]
We infer the equality
\[
\left\{ A \in H^{0,1}_{g,\Omega}(T_X,J) \mid \exists (J_t)_t \subset J_\omega : J_0 = A \right\} 
= \left\{ A \in H^{0,1}_{g,\Omega}(T_X,J) \mid \exists (\varphi_t)_t \subset C^{\text{div}}_J : \dot{\varphi}_0 = A \right\} = TC_{K_{J,g},0}.
\]
By gauge transformation we deduce
\[
\left\{ A \in H^{0,1}_{g,\Omega}(T_X,J) \mid \exists (J_t)_t \subset J_\omega : H_{T_X,J}J_0 = A \right\} 
\subseteq \left\{ A \in H^{0,1}_{g,\Omega}(T_X,J) \mid \exists (J_t)_t \subset J_\omega : J_0 = A \right\}.
\]
and thus the required inclusion.

This result combined with the decomposition statement in corollary \[6\] implies the inclusions (1.15) and (1.16) in the introduction of the paper.

Acknowledgments. We warmly thank Jean-Michel Bismut for giving us the opportunity to explain our Soliton-Kähler-Ricci flow at his weekly seminar in Orsay.

We warmly thank also Duong H. Phong for sharing with us a huge number of quite delicate points in contemporary Kähler geometry.

We are also very grateful to Robert Bryant for explaining us in detail the key Spin$^c$ computation in [D-W-W2] and to Xianzhe Dai for providing us more technical details about their beautiful work.

We thank Paul Gauduchon for sending us a copy of his monumental book [Gau] and for sharing his great knowledge in Kähler geometry with us.

We thank also Stuart Hall for explaining to us in great detail the construction of the Dancer-Wang Kähler-Ricci-Soliton and the vanishing of the stability integral in [Ha-Mu2] in some particular harmonic directions over this soliton.

References

[Ach] Ache, A.G., On the uniqueness of asymptotic limits of the Ricci flow, arXiv:1211.3387v2, (2013).

[Bes] Besse, A.L., Einstein Manifolds, Springer-Verlag, 2007.

[C-H-I] Cao, H.D, Hamilton, R.S., Ilmanen, T., Gaussian densities and stability for some Ricci solitons, arXiv:math.DG/0404169

[Ca-Zhu] Cao, H.D, Zhu, M., On second variation of Perelman’s Ricci shrinker entropy, arXiv:1008.0842v5 (2011), Math. Ann., 353 (2012), no. 3, 747-763.

[Ca-He] Cao, H.D, He, C., Linear stability of Perelman’s $\nu$-entropy on symmetric spaces of compact type, arXiv:1304.2697v1, (2013).

[Ch-Wa] Chen, X.X, Wang, B., Space of Ricci flows (II), arXiv:1405.6797v1, (2014).

[Co-Mi] Colding, T.H., Minicozzi, W.P., On uniqueness of tangent cones for Einstein manifolds, arXiv:1206.4929, (2012), Invent. Math. 196 (2014), no. 3, 515588.

[D-W-W1] Dai, X., Wang, X., Wei, G., On the stability of Riemannian manifold with parallel spinors, Invent.Math. 161, (2005), no. 1, 151-176.
[D-W-W2] Dai, X., Wang, X., Wei, G., On the variational stability of Kähler-Einstein metrics, Commun. Anal. Geom. 15 (2007), no. 4, 669-693.

[Don] Donaldson, S.K., Remarks on gauge theory, complex geometry and 4-manifold topology. In Fields Medallists’ lectures, volume 5 of World Sci. Ser. 20th Century Math., pages 384-403. World Sci. Publ., River Edge, NJ, 1997.

[Ebi] Ebin, D.G. The manifolds of Riemannian metrics, in Proceedings of the AMS Symposia on Pure Mathematics, XV, (1970)

[Fut] Futaki, A. An obstruction to the existence of Einstein Kähler metrics, Invent. Math. 73, 1983, 437-443.

[Fu1] Futaki, A. Kähler-Einstein metrics and Integral Invariants, Lecture Notes in Mathematics, 1314. Springer-Verlag, Berlin, 1988, 437-443.

[Gau] Gauduchon, P. Calabi extremal metrics: An elementary introduction, book in preparation.

[Ha-Mu1] Hall, S., Murphy, T., On the linear stability of Kähler-Ricci solitons, arXiv:1008.1023, (2010), Proc. Amer. Math. Soc. 139, No. 9, (2011) 3327-3337.

[Ha-Mu2] Hall, S., Murphy, T., Variation of complex structures and the stability of Kähler-Ricci Solitons, arXiv:1206.4922, (2012), Pacific J. Math. 265 (2013), no. 2, 441-454.

[Ham] Hamilton, R.S., The formation of singularities in the Ricci flow, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7136, Int. Press, Cambridge, MA, 1995.

[Kur] Kuranishi, M., On the locally complete families of complex analytic structures, Ann. of Math. (2) 75 1962 536577.

[Kro] Kröncke, K., Stability and instability of Ricci solitons, arXiv:1403.3721v1, (2014).

[Pal] Pali, N., Plurisubharmonic functions and positive (1,1)-currents over almost complex manifolds, Manuscripta Mathematica, volume 118, (2005) issue 3, pp. 311 - 337.

[Pal1] Pali, N., Characterization of Einstein-Fano manifolds via the Kähler-Ricci flow, arXiv:math/0607581v2, (2006), Indiana Univ. Math. J. 57, (2008), no. 7, 3241-3274.

[Pal2] Pali, N., A second variation formula for Perelman’s W-functional along the modified Kähler-Ricci flow, arXiv:1201.0970v1, (2012), Mathematische Zeitschrift. 276 (2014), no. 1-2, 173-189.
[Pal3] Pali, N., The total second variation of Perelman's W-functional, arXiv:1201.0969v1. (2012), Calc. Var. Partial Differential Equations 50 (2014), no. 1-2, 115144.

[Pal4] Pali, N., The Soliton-Ricci Flow over Compact Manifolds, arXiv:1203.3682

[Pal5] Pali, N., The Soliton-Kähler-Ricci Flow over Fano Manifolds, arXiv:1203.3684

[Pal6] Pali, N., Variation formulas for the complex components of the Bakry-Emery-Ricci endomorphism, arXiv:math

[Pal7] Pali, N., The stability of the Soliton-Kähler-Ricci flow, in preparation.

[Per] Perelman, G., The entropy formula for the Ricci flow and its geometric applications, arXiv/math/0211159

[P-S-S-W1] Phong, D.H., Sturm, J., On stability and the convergence of the Kähler-Ricci flow, arxiv:0412185v1, (2004), J. Differential Geom. 72 (2006), no. 1, 149168.

[P-S-S-W2] Phong, D.H., Song, J., Sturm, J., Weinkove, B., On the convergence of the modified Kähler-Ricci flow and solitons, arXiv:0809.0941v1 (2008), Comment. Math. Helv. 86 (2011), no. 1, 91112.

[P-S-S-W3] Phong, D.H., Song, J., Sturm, J., Weinkove, B., The Kähler-Ricci flow and the ∂-operator on vector fields, arXiv:0705.4048v2 (2008), J. Differential Geom. 81 (2009), no. 3, 631647.

[Su-Wa] Sun, S., Wang, Y.Q, On the Kähler Ricci flow near a Kähler Einstein metric, arXiv:1004.2018v3 (2013). To appear in J. Reine Angew. Math.

[Ti-Zhu1] Tian, G., Zhu, X.H, Convergence of Kähler-Ricci flow, J. Amer. Math. Soc. 20 (2007), no. 3, 675699.

[Ti-Zhu2] Tian, G., Zhu, X.H, Perelman’s W-functional and stability of Kähler-Ricci flow, arXiv:0801.3504v1. (2008).

[Ti-Zhu3] Tian, G., Zhu, X.H, Convergence of Kähler-Ricci flow on Fano Manifolds II, arXiv:1102.4798v1. (2011). J. Reine Angew. Math. 678 (2013), 223245.

[T-Z-Z-Z] Tian, G., Zhang, S., Zhang, Z.L., Zhu, X.H, Perelman’s Entropy and Kähler-Ricci Flow on a Fano Manifold, arXiv:1107.4018 (2011), Trans. Amer. Math. Soc. 365 (2013), no. 12, 66696695.

[Ti-Zha] Tian, G., Zhang, Z.L., Regularity of Kähler-Ricci flows on Fano manifolds, arXiv:1310.5897v1, (2013).
