Surface critical behavior of random systems at the special transition

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We study the surface critical behavior of semi-infinite quenched random Ising-like systems at the special transition using three dimensional massive field theory up to the two-loop approximation. Besides, we extend up to the next-to leading order, the previous first-order results of the √ε expansion obtained by Ohno and Okabe [Phys. Rev. B 46, 5917 (1992)]. The numerical estimates for surface critical exponents in both cases are computed by means of the Padé analysis. Moreover, in the case of the massive field theory we perform Padé-Borel resummation of the resulting two-loop series expansions for surface critical exponents. The obtained results confirm that in a system with quenched bulk randomness the plane boundary is characterized by a new set of surface critical exponents.

I. INTRODUCTION

The investigation of the critical behavior of the real physical systems is of considerable theoretical and experimental interest. As usual, the real physical systems are characterized by the presence of different kinds of imperfections. In common, the defects and impurities may be localized inside the bulk as well as at the boundary.

Historically the systematic investigation of the quenched disordered systems was initiated in the seminal works by Harris, Lubensky [1,2] and Khmelnitskii [3]. It is shown that investigation of the Ising-like systems is of main concern from the whole class of O(n) symmetric n-vector models in d dimensions, because they satisfy the Harris criterion [4], which states that the presence of randomness is relevant for such pure systems which have a positive specific heat exponent α. The introducing of the bulk dilution into a system shifts the critical temperature of the bulk phase transition and drives the system to another, ‘random’ fixed point in which unconventional scaling behavior is observed. As it is confirmed by the Wilson’s renormalization group and ε expansions [1–3,5–7], the massive field theory in three dimensions [8–10], experiments [12–14], and Monte-Carlo simulations [15,16], the critical behavior of three-dimensional disordered Ising-like systems is characterized by a new set of critical exponents [17]. The case of the Ising model at d = 2 is marginal, because in this case α = 0 and the correspondent logarithmic corrections to the power laws singularities of the pure model take place, as was confirmed in a series of papers [21,22].

The presence of a surface leads to the appearance of additional problems in critical phenomena. The most general review of critical behavior at surfaces and list of related publications are given in [24,25]. It is well known [26–30] that each surface universality class is defined by the bulk universality class and specific properties of a given boundary. At the present time three surface universality classes, called ordinary, special and extraordinary, are known [27,28]. They correspond to the respective surface transitions which occur at the bulk critical point m^2_0 = m^2_0c [31] and are characterized by different fixed points

\begin{align}
\eta^*_0,ord &= +\infty, \\
\eta^*_0,sp &= \eta^*_0,sp = \eta^*_0,extr = -\infty.
\end{align}

Here c_0 is so called ‘bare surface enhancement’, which measures the enhancement of the interactions at the surface, and (m^2_0, c_0) = (m^2_0c, c^*0,sp) is a multicritical point, called special point.

The influence of quenched surface disorder on the surface critical behavior was investigated in a series of theoretical works [32,33] and Monte Carlo calculations [22,4]. General irrelevance-relevance criterion of the Harris type for short-range as well as for long-range correlated surface disorder was derived [32]. In the case of special transition it has been demonstrated [24,33] that the fixed point describing the surface critical behavior of three dimensional pure systems is stable with respect to short-range correlated surface disorder. Thus, the weak short-range surface disorder is irrelevant for three dimensional systems, but long-range correlated random field disorder is relevant in d ≤ 4 dimensions.

In a recent paper [34], we quantitatively confirm the previous results by Ohno and Okabe [14] that the introducing of quenched bulk randomness in semi-infinite systems bounded by a plane surface affects the surface critical behavior of these systems. Thus obtained surface critical exponents of quenched dilute semi-infinite systems at the ordinary transition differ from the surface critical exponents of the pure semi-infinite systems [35]. Besides, we showed that to order ε, the √ε expansion for surface critical exponents η|| and η⊥ has given negative value of the correlation function critical exponent η for the random bulk Ising system according to the scaling relation η = 2η⊥ − η||. It confirms
the well known fact that the second order of the $\sqrt{\epsilon}$ expansion is not enough to give correct positive value of critical exponent $\eta$ \cite{11}. The obtained results \cite{11} have shown that this kind of deficiencies do not appear at the calculations using the massive field-theoretic approach directly in $d = 3$ dimensions \cite{12}.

All these have stimulated us to perform the investigation of the special transition occurring in quenched bulk dilute semi-infinite systems bounded with a plane surface. It should be mentioned that the problem of investigation of the special transition is very important from such point of view that at some conditions it may be reduced to the problem of the adsorption of $\theta$ polymers on a wall \cite{13,14}.

The investigation of the critical behavior of the systems with quenched, i.e., time independent, randomness is possible to split on two directions. One of them is renormalization-group approach introduced by Harris and Lubensky \cite{1}. This approach involves applying the renormalization-group transformation to the random system directly and subsequent averaging over disorder. Ohno and Okabe \cite{15} employed the above mentioned method to analyze the influence of randomness on the surface critical behavior at $d = 4 - \epsilon$ dimensions in the frames of $\sqrt{\epsilon}$ expansion.

Another technique introduced by Grinstein and Luther \cite{5} involves firstly removing the randomness by averaging, and subsequent employing the renormalization group. They considered an $mn$ vector model and showed that analytic continuation of this model to $n = 0$ is equivalent to a model of a random $m$-component spin system. An elegant derivation of this equivalence has been given by Emery \cite{16}. Our main calculations are performed with this technique to treat randomness.

The present paper is dedicated to the investigation of the special surface transition in semi-infinite, quenched dilute Ising-like systems at the bulk ‘random’ critical point directly in $d = 3$ dimensions using the massive field theory up to the two-loop approximation. Besides, we extend up to the next-to leading order of the $\sqrt{\epsilon}$ expansion, the previous first-order results obtained by Ohno and Okabe \cite{15}. The numerical estimates for surface critical exponents of the special transition in both cases are calculated using extensive Padé analyses. Moreover, in the case of the massive field theory we performed Padé-Borel resummations of the resulting two-loop series expansions and obtained quite reasonable and reliable numerical estimates for surface critical exponents. The obtained results confirm that in the case of quenched bulk randomness in semi-infinite systems the new set of the surface critical exponent appears.

II. MODEL

In the previous work \cite{36}, we presented some notations about possibilities to use effective Landau-Ginzburg-Wilson Hamiltonian with cubic anisotropy defined in semi-infinite space

$$H_{LGW}[\phi] = \int_0^\infty dz \int d^{d-1}r \left[ \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} m_0^2 |\phi|^2 \right] + \frac{1}{4} v_0 \sum_{n=1}^{n} \phi_n^4 + \frac{1}{6} u_0 (|\phi|^2)^2 \right], \quad (2.1)$$

description for surface critical behavior of quenched dilute semi-infinite Ising-like systems at the ordinary transition in the replica limit $n \to 0$. It should be mentioned that here $\phi$ is an $n$ - vector field with the components $\phi_i$, $i = 1, ..., n$ defined on a half-space $\mathbb{R}^d_+ \equiv \{ x = (r, z) \in \mathbb{R}^d \mid r \in \mathbb{R}^{d-1}, z \geq 0 \}$ bounded by a plane free surface at $z = 0$.

For the first time the above model was introduced by Grinstein and Luther \cite{5}. The $O(n)$ symmetric term in (2.1) arises in the process of the configurational averaging of the free energy over disorder using the replica trick

$$F = -T \lim_{n \to 0} \frac{1}{n} \frac{1}{n} (\langle Z^n \rangle_{conf} - 1), \quad (2.2)$$

via cumulant expansion and in accordance with this its coupling constant $u_0 \propto -\Delta < 0$. Here $Z$ is the partition function with Boltzmann weight $e^{-H[\phi]}$, where $H[\phi]$ is the effective Landau-Ginzburg-Wilson Hamiltonian of an original Ising system with scalar field $\phi = \phi(x)$

$$H[\phi] = \int_0^\infty dz \int d^{d-1}r \left[ \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} \tau_0 \phi^2 + \frac{1}{6} u_0 \phi^4 \right]. \quad (2.3)$$

Parameter $\tau_0$ involves local random temperature fluctuations $\delta \tau(x)$ via $\tau_0 = m_0^2 + \delta \tau(x)$, where $\langle \delta \tau(x) \rangle_{conf} = 0$ and $\langle \delta \tau(x) \delta \tau(x') \rangle_{conf} = \Delta \delta (x - x')$ with $\Delta > 0$. It should be noticed that $m_0^2$ corresponds to a "bare mass" representing linear temperature deviations from the mean - field critical temperature.

In the present paper we wish to investigate the surface critical behavior of semi-infinite Ising-like systems with quenched bulk randomness at the special transition. The critical behavior of the special transition has own peculiarities. In general case effective Hamiltonian for such systems must involve terms describing surface interactions \cite{20,21,22,23,24,25}. 


\[ U_1(\phi) = \frac{1}{2} c_0 \phi^2 - \tilde{h} \phi. \] (2.4)

Usually these external surface fields \( \tilde{h} \) are position and time dependent, so that means they may be used as sources. But in the present investigation we restrict ourselves to the case \( \tilde{h} = 0 \). Thus, the common form of the effective Hamiltonian describing the surface critical behavior of quenched dilute semi-infinite Ising-like systems in the replica limit \( n \to 0 \) can be written in the form
\[ H(\phi) = \int_0^\infty dz \int d^{d-1} r \left[ \frac{1}{2} | \nabla \phi |^2 + \frac{1}{2} m^2_0 | \phi |^2 \right. \]
\[ \left. + \frac{1}{4!} \sum_{i=1}^{n} \phi_i^4 + \frac{1}{4!} u_0 \left( | \phi |^2 \right)^2 \right] + \frac{1}{2} \int d^{d-1} rc_0 \phi^2. \] (2.5)

It should be mentioned that fields \( \phi(\mathbf{r}, z) \) satisfy the Neumann boundary condition \[29,47\], so we have at \( z = 0 \) that \( \partial_z \phi(\mathbf{r}, z) = 0 \). This Hamiltonian takes into account surface interaction in the form of additional term \( \frac{1}{2} \int d^{d-1} rc_0 \phi^2 \). The model defined in \(2.5\) is restricted to translations parallel to the bounding surface. Thus, only parallel Fourier transformations in \( d-1 \) dimensions take place.

### III. RENORMALIZATION OF THE CORRELATION FUNCTION

The correlation function of the model (see Eq. \(2.5\)), which involves \( N \) fields \( \phi(x_i) \) at distinct points \( x_i (1 \leq i \leq N) \) in the bulk and \( M \) field \( \phi(r_j, z = 0) \equiv \phi_s(r_j) \) at distinct surface points with parallel coordinates \( r_j (1 \leq j \leq M) \), has the form
\[ G^{(N,M)} \{ x_i \} \{ r_j \} = \left\langle \prod_{i=1}^{N} \phi(x_i) \prod_{j=1}^{M} \phi_s(r_j) \right\rangle. \] (3.1)

The corresponding full free propagator in the \( px \) representation is given by
\[ G(p, z, z') = \frac{1}{2k_0} \left[ e^{-\kappa_0 |z-z'|} - \frac{c_0 - \kappa_0}{c_0 + \kappa_0} e^{-\kappa_0 (z+z')} \right], \] (3.2)

with the standard notation \( \kappa_0 = \sqrt{p^2 + m_0^2} \). Here, \( p \) is the value of parallel momentum associated with \( d-1 \) translationally invariant directions in the system. The first term in \(3.2\) corresponds to usual free bulk propagator in coordinate space, between the points \( \mathbf{r} = (p, z) \) and \( \mathbf{r}' = (p, z') \), and the second, so called ‘surface’ term, depends on the distance between the point \( \mathbf{r} \) and its ‘mirror image’ \( \mathbf{r}' = (0, -z') \).

The formulation of the randomness problem in the spirit of approach introduced by Grinstein and Luther indicate that the renormalization process for the random systems is similar to the ‘pure’ case \[27,33\]. As it is known \[27,33\], in the theory of semi-infinite systems the bulk field \( \phi(x) \) and the surface field \( \phi_s(r) \) should be reparameterized by different uv-finite renormalization factors \( Z_\phi(u, v) \) and \( Z_1(u, v) \)
\[ \phi(x) = Z_\phi^{1/2} \phi_R(x) \quad \text{and} \quad \phi_s(r) = Z_\phi^{1/2} Z_1^{1/2} \phi_{s,R}(r). \]

The renormalized correlation function involving \( N \) bulk and \( M \) surface fields \( (N, M) \neq (0, 2) \) can be written as
\[ G_R^{(N,M)}(0; m, u, v, c) = Z_\phi^{-(N+M)/2} Z_1^{M/2} G^{(N,M)}(0; m_0, u_0, v_0, c_0). \] (3.3)

In order to obtain the critical exponent \( \eta_\parallel \) which characterizes surface correlations at special transition, it is sufficient to consider a two-point correlation function of surface fields \( G^{(0,2)}(p) = \langle \phi(p, z = 0) \phi(-p, z' = 0) \rangle \).

It should be mentioned that the renormalized mass \( m \), coupling constants \( u, v \), and the renormalization factor \( Z_\phi \) are fixed via the standard normalization conditions of the infinite-volume theory \[13,18,42\]. In order to remove short-distance singularities of the correlation function \( G^{(0,2)} \) located in the vicinity of the surface and to define the surface-enhancement shift \( \delta c \) and surface renormalization factor \( Z_1 \), new normalization conditions should be introduced. We normalize the renormalized surface two-point correlation function in such a manner that at zero external momentum it should coincide with the lowest order of perturbation expansion of the surface susceptibility \( \chi_\parallel(p) = G^{(0,2)}(p) \).
\[ G^{(0,2)}(p; m_0, u_0, v_0, c_0) = \frac{1}{c_0 + \sqrt{p^2 + m_0^2}} + O(u_0, v_0) \]  
(3.4)

and its first derivatives with respect to \( p^2 \). Thus we obtain necessary surface normalization conditions

\[ G_R^{(0,2)}(0; m, u, v, c) = \frac{1}{m + c} \]  
(3.5)

and

\[ \left. \frac{\partial G_R^{(0,2)}(p; m, u, v, c)}{\partial p^2} \right|_{p=0} = -\frac{1}{2m(m + c)^2}. \]  
(3.6)

Eq. (3.5) defines the required surface-enhancement shift \( \delta c \). From Eq. (3.5) it is easy to see that the surface susceptibility diverge at \( m = c = 0 \). This point corresponds to the multicritical point \((m_0^2, c_0^p)\) at which special transition takes place.

Taking into account the above normalization condition (3.6) and expression for renormalized correlation function (3.3) it is possible to define the renormalization factor \( Z_\parallel = Z_1 Z_\phi \) via

\[
Z_\parallel = 2m \frac{\partial}{\partial p^2} \left[ G^{(0,2)}(p) \right]^{-1} \bigg|_{p^2=0} = \lim_{p^2 \to 0} \frac{m}{p} \frac{\partial}{\partial p} \left[ G^{(0,2)}(p) \right]^{-1}.
\]  
(3.7)

It should be mentioned that the next singular behavior of these renormalization \( Z \) factors in the critical region takes place

\[
Z_\phi \propto m^\eta, \quad Z_1^{sp} \propto m^{\eta_1^{sp}},
\]  
(3.8)

where \( m \) is identified via the inverse bulk correlation length \( \xi^{-1} \propto t^\nu \), \( t = (T - T_c) / T_c \). Here \( \eta \) is the standard bulk correlation exponent and exponent \( \eta_1^{sp} \) is specific for our quenched random semi-infinite system. As it is known [49,35] these exponents \( \eta \) and \( \eta_1^{sp} \) arise as a RG arguments of an inhomogeneous Callan-Symanzik equation for correlation functions \( G_R^{(0,2)} \) (3.3)

\[
\eta_\phi = m \frac{\partial}{\partial m} \ln Z_\phi \big|_{FP}, \quad \eta_1^{sp} = m \frac{\partial}{\partial m} \ln Z_1 \big|_{FP}.
\]  
(3.9)

The simple scaling dimensional analysis of \( G_R^{(0,2)} \) and mass dependence of \( Z \) factors (3.8) defines the surface correlation exponent \( \eta_\parallel^{sp} \) via

\[
\eta_\parallel^{sp} = \eta_1^{sp} + \eta.
\]  
(3.10)

According to (3.7), (3.9) and (3.10) we obtain for surface correlation exponent \( \eta_\parallel^{sp} \)

\[
\eta_\parallel^{sp} = m \frac{\partial}{\partial m} \ln Z_\parallel \big|_{FP} = \beta_u(u,v) \frac{\partial \ln Z_\parallel(u,v)}{\partial u} + \beta_v(u,v) \frac{\partial \ln Z_\parallel(u,v)}{\partial v} \bigg|_{FP}.
\]  
(3.11)

The above value should be calculated at the infrared-stable random fixed point (FP) of the underlying bulk theory. The other surface critical exponents of the special transition can be determined via the set of surface scaling relations [27].

IV. THE PERTURBATION SERIES UP TO TWO-LOOPS.

As was indicated in the previous section, the calculation of the surface critical exponent \( \eta_\parallel^{sp} \) can be performed in accordance with Eq. (3.11), where renormalization factor \( Z_\parallel \) is defined via (3.7). It should be mentioned that here,
by analogy with the infinite-volume theory, we start from inverse surface correlation function \( G_R^{(0,2)} \)^{-1} in order to avoid the external lines depending both on the external momentum \( p \) and the surface enhancement \( c_0 \) in each external propagator

\[
G(p; z, 0) = \frac{e^{-\kappa_0 z}}{\kappa_0 + c_0}.
\]  

(4.1)

In other words such procedure of the correlation function inversion allows to amputate the denominators \( \kappa_0 + c_0 \) from each external propagator. Thus we consider the Feynman diagram expansion of the unrenormalized surface correlation function \( G_R^{(0,2)} \)^{-1} in terms of the free propagator (3.2) up to the two-loop order

\[
[G^{(0,2)}(p; m_0, u_0, v_0)]^{-1} = c_0 + \kappa_0 - \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram1.png}}
\end{array} - \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram2.png}}
\end{array}^2
\]  

(4.2)

Here the full internal lines denote the full free propagator (3.2) and the dashed external lines correspond to the factors \( e^{-\kappa_0 z} \). In accordance with the presence solely \( d - 1 \) translational invariance in (2.5) only a partial cancellation in the external legs takes place and as a result the one-line reducible graphs do not disappear. All these graphs in (4.2) have their own weights arising from the standard symmetry factors of the effective Hamiltonian (2.5)

\[
t_1^{(0)} = \frac{n + 2}{3} u_0 + v_0, \\
t_2^{(0)} = \frac{n + 2}{3} u_0^2 + v_0^2 + 2v_0u_0, \\
t_3^{(0)} = t_4^{(0)} = (t_1^{(0)})^2
\]  

(4.3a, 4.3b, 4.3c)

where \(-t_1^{(0)}\) corresponds to the one-loop diagram, \( t_2^{(0)} \) to the two-loop melon-like diagram, \( t_3^{(0)} \) and \( t_4^{(0)} \) to the reducible and irreducible two-loop diagrams in (4.2), respectively. With a view to avoid the usual bulk uv singularities, which are present in (4.2), we perform the mass renormalization (see [36]). After the mass renormalization we obtain that the divergent parts of the correlation function (4.2) associated with the bulk terms of the free propagator (3.2) cancel and the "melon-like" diagram becomes subtracted. Thus we obtain the next expression for inverse correlation function \( [G^{(0,2)}]^{-1} \)

\[
[G^{(0,2)}(p; m, u_0, v_0)]^{-1} = c_0 + \kappa - \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram3.png}}
\end{array} - \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram4.png}}
\end{array}^2
\]  

(4.4)

The superscript \( G \) means the full free propagator (3.2), its bulk and surface parts are denoted with "-" and ++ signs, respectively. The line labeled by \( D \) represents the Dirichlet propagator

\[
G(p, z, z') = \frac{1}{2\kappa_0} \left[ e^{-\kappa_0 |z-z'|} - e^{-\kappa_0 (z+z')} \right],
\]  

(4.5)

which arises from the difference

\[
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram5.png}}
\end{array} - \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram6.png}}
\end{array} = \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram7.png}}
\end{array} + \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram8.png}}
\end{array}
\]  

(4.6)

The expansion (4.4) still holds the uv divergences which are connected with the presence of the surface in the system. These divergent parts represented by the closed lines with the index ++. The short-distance singularities of
the inverse correlation function \( [G^{(0,2)}(p)]^{-1} \) can be removed via the renormalization of the surface enhancement. The surface enhancement renormalization is defined via the normalization condition \( (3.3) \). This condition can be rewritten for our inverse unrenormalized correlation function in the form
\[
Z\| [G^{(0,2)}(0; m_0, u_0, v_0, c_0)]^{-1} = m + c. \tag{4.7}
\]
Substituting \( [G^{(0,2)}(p)]^{-1} \) (see Eq.\( (1.4) \)) at zero external momentum \( p = 0 \) into the normalization condition \( (4.7) \) gives us the equation for the surface enhancement shift \( \delta c \)
\[
\delta c = (Z\|^{-1} - 1)(m + c) + \sigma_0(0; m, c_0) = c + \delta c. \tag{4.8}
\]
This equation can be resolved using method of sequential iteration to determine the dependence of \( c_0 = c + \delta c \) on \( c \) and \( m \). Here \( \sigma_0(0; m, c_0) \) denotes the sum of all loop diagrams in \( (4.4) \). Among them \( \sigma_1 \) corresponds to the one-loop graph, \( \sigma_2 \) denotes the melon-like two-loop diagrams
\[
\sigma_2(p; m, c_0) = \ldots \frac{\partial}{\partial a} \ldots - \frac{1}{2\kappa} + \frac{m^2}{2\kappa} \frac{\partial}{\partial k^2} \ldots \bigg|_{k^2 = 0}, \tag{4.9}
\]
\( \sigma_3 \) and \( \sigma_4 \) represent the last two terms in Eq.\( (4.4) \), respectively.
Taking into account Eq.\( (4.8) \) the expansion \( (4.4) \) for inverse surface correlation function can be written in the form
\[
[G^{(0,2)}(p; m, c)]^{-1} = \kappa - m + Z\|^{-1}(m + c) - [\sigma_0(p; m, c_0) - \sigma_0(0; m, c_0)]. \tag{4.10}
\]
According to Eq.\( (4.11) \), the surface enhancement renormalization of the inverse surface correlation function reduces to the subtraction from each diagram contribution the same diagram at zero external momentum. Thus, the uv divergences connected with the presence of the surface in the system are mutually canceled.

The iteration process of finding solution of the Eq. \( (4.8) \) implies that at zero-order we put \( \delta c = 0 \) and find the first-order solution
\[
\delta c^{(1)} = (Z\|^{-1} - 1)(m + c) + \sigma_1(0; m, c_0) = c. \tag{4.11}
\]
Taking into account that we have dealt with the special transition where \( c = 0 \), for the renormalization factor \( Z\| \) according to Eq.\( (4.7) \) at one-loop order we obtain
\[
(Z\|^{-1})^{(1)} = -2m \frac{\partial \sigma_1(p; m, 0)}{\partial p^2} \bigg|_{p = 0} - \frac{\bar{\ell}_1}{4}, \tag{4.12}
\]
where \( \bar{\ell}_1 = t_1/(8\pi m) \). At two-loop order, the first-order contribution \( \delta c^{(1)} \) must be taken into account in \( \sigma_1 \) because in common we have
\[
\sigma_1(p; m, c_0) = \sigma_1(p; m, c_0)|_0 + \frac{\partial \sigma_1(p; m, c_0)}{\partial c_0} \bigg|_0 \delta c^{(1)} + ... \tag{4.13}
\]

The uv singular part of the second term of this expansion remove the divergence of the correspondent surface subgraph in \( \sigma_4 \), denoted via “+”. As we can see from Eq.\( (4.4) \), the similar subgraphs arise twice in \( \sigma_3 \). But the contribution \( \sigma_3 \) is uv finite because the Dirichlet propagator \( G_D \) vanishes whenever \( z \) or \( z' \) approach zero. Thus, the perturbation expansion of the inverse surface correlation function of our system up to two-loop order after bulk and surface enhancement renormalization can be written in the form
\[
[G^{(0,2)}]^{-1} = \kappa + (Z\|^{-1})^{(1)} m - (\sigma_1(p; m, 0) - \sigma_1(0; m, 0)) + \frac{\partial \sigma_1(0; m, c_0)}{\partial c_0} \bigg|_0 \delta c^{(1)} \tag{4.14}
\]
\[
- \frac{\partial \sigma_1(p; m, c_0)}{\partial c_0} \bigg|_0 \delta c^{(1)} - \sum_{i=2}^4 \sigma_1(p; m, 0). \tag{4.14}
\]
Following to the definition \( (3.7) \) we obtain the next expansion for the renormalization factor \( Z\| \) up-to to loop-order
\[
Z^{-1} = 1 - \frac{\bar{\ell}_1}{4} + \frac{\bar{\ell}_1^2}{4}(1 - 2 \ln 2) - \lim_{p \to 0} \frac{m}{p} \frac{\partial}{\partial p} \frac{\partial \sigma_1(p; m, c_0)}{\partial c_0} \bigg|_0 \sigma_1(0; m, 0) \lim_{p \to 0} \frac{m}{p} \frac{\partial}{\partial p} \sum_{i=2}^4 \sigma_i(p; m, 0). \tag{4.15}
\]
Performing the integration of Feynman integrals in \((1.17)\) by analogy with \([35]\) we derive the result
\[
Z^{-1}(\bar{u}_0, \bar{v}_0) = 1 - \frac{t_1(0)}{4} + \frac{t_2(0)}{4}(\frac{1}{4} - \ln 2 + \ln 2^2) + t_2(0)A, \tag{4.16}
\]
where the constant \(A\) arise from the two-loop contribution in \((1.15)\),
\[
A \simeq 0.202428. \tag{4.17}
\]

Here the renormalization factor \(Z\) is expressed as a second-order series expansion in powers of bare dimensionless parameters \(u_0 = u_0/(8\pi m)\) and \(v_0 = v_0/(8\pi m)\). The corresponding weighting factors \(t_1(0)\) and \(t_2(0)\) are obtained by replacements \((u_0, v_0) \to (\bar{u}_0, \bar{v}_0)\) in the original combinations \(t_1(0)\) and \(t_2(0)\) from (3.3). As it is usual in superrenormalizable theories the next step is the vertex renormalizations
\[
\bar{u}_0 = \bar{u} \left(1 + \frac{n + 8}{6} \bar{u} + \bar{v}\right),
\]
\[
\bar{v}_0 = \bar{v} \left(1 + \frac{3}{2} \bar{v} + 2 \bar{u}\right). \tag{4.18a}
\]

Again, the vertex renormalization at \(d = 3\) is a finite reparameterization. The result for a modified series expansion of \(Z\) in terms of a new renormalized coupling constants \(\bar{u}\) and \(\bar{v}\), normalized in a standard fashion \((4.18a)\), is
\[
Z^{-1}(\bar{u}, \bar{v}) = 1 - \frac{n+2}{12} \bar{u} - \frac{\bar{v}}{4} + \frac{n+2}{3} \left(A - \frac{1}{4} + \frac{n+2}{12}(\ln 2^2 - \ln 2)\right) \bar{u}^2
\]
\[
+ \left(A - \frac{1}{4} + \frac{1}{4}(\ln 2^2 - \ln 2)\right) \bar{u}^2 + 2 \left(A - \frac{1}{4} + \frac{n+2}{12}(\ln 2^2 - \ln 2)\right) \bar{u} \bar{v}. \tag{4.19}
\]

Combining the renormalization factor \(Z(\bar{u}, \bar{v})\) together with the one-loop pieces of the beta functions
\[
\beta_{\bar{u}}(\bar{u}, \bar{v}) = -\bar{u} \left(1 - \frac{n+8}{6} \bar{u} - \bar{v}\right),
\]
\[
\beta_{\bar{v}}(\bar{u}, \bar{v}) = -\bar{v} \left(1 - \frac{3}{2} \bar{v} - 2 \bar{u}\right), \tag{4.20a}
\]
through \((1.11)\), we obtain the desired series expansion for \(\eta_{sp}^p\)
\[
\eta_{sp}^p(u, v) = -\frac{n+2}{2(n+8)} u - v + \frac{12(n+2)}{(n+8)^2} A(n) u^2 + \frac{4}{9} A(1) v^2 + \frac{8}{n+8} A(n) uv, \tag{4.21}
\]
where \(A(n)\) is a function of the replica number \(n\), defined as
\[
A(n) = 2A + \frac{n-10}{48} + \frac{n+2}{6}(\ln 2^2 - \ln 2), \tag{4.22}
\]
and renormalized coupling constants \(u\) and \(v\), normalized in a standard fashion \(u = \frac{n+8}{6} \bar{u}\) and \(v = \frac{3}{2} \bar{v}\).

In fact, the last expression \((4.21)\) for \(\eta_{sp}^p\) provides a result for the cubic anisotropic model given by the effective Hamiltonian \((2.3)\) with general number \(n\) of order-parameter components. In the case of infinite space, this cubic anisotropic model attracted much attention very recently (see e.g. \([13, 46, 41]\) and references therein).

In the present paper we restrict our discussion to the case of semi-infinite random Ising-like systems by taking the replica limit \(n \to 0\). According to \((4.21)\), we obtain the next two-loop expansion for the surface critical exponent \(\eta_{sp}^p\) at the special transition
\[
\eta_{sp}^p = -\frac{u}{8} - v + \frac{3}{2} A(0) u^2 + \frac{4}{9} A(1) v^2 + A(0) uv. \tag{4.23}
\]

As it is well known, the knowledge of one surface critical exponent gets access via usual scaling relations \([27]\) to the other surface critical exponents. For convenience further below we suppress the superscript \(sp\) at the surface critical exponents.
V. CALCULATION OF THE SURFACE CRITICAL EXPONENTS

The present section is devoted to numerical calculation of the surface critical exponents at the special transition. The individual RG series expansions for other surface critical exponents can be derived from (4.23) through standard scaling relations \[27\] (with \(d = 3\))

\[
\begin{align*}
\eta_\perp &= \eta + \eta_\parallel / 2, \\
\beta_1 &= \nu (d - 2 + \eta_\parallel), \\
\gamma_{11} &= \nu (1 - \eta_\parallel), \\
\gamma_1 &= \nu (2 - \eta_\perp), \\
\Delta_1 &= \nu (d - \eta_\parallel), \\
\delta_1 &= \Delta_1 / \beta_1 = (d + 2 - \eta_\parallel) / (d - 2 + \eta_\parallel), \\
\delta_{11} &= \Delta_1 / \beta_1 = (d - \eta_\parallel) / (d - 2 + \eta_\parallel).
\end{align*}
\]

Each of these critical exponents characterizes certain properties of the system with the surface in the vicinity of the critical point (see \[36\]). The values \(\nu, \eta,\) and \(\Delta = \nu (d + 2 - \eta_\parallel) / 2\) are the standard bulk exponents. The correspondent series expansions for \(\nu\) and \(\eta\) at \(d = 3\) are given by \((8–10)\)

\[
\begin{align*}
\nu &= \nu \left[1 + \frac{v}{6} + \frac{(n + 2)}{2(n + 8)} u\right] \\
&\quad - \frac{1}{324} \left[\frac{11}{9} v^2 - \frac{2}{n + 8} (27n - 38) uv - \frac{3(n + 2)}{(n + 8)^2} (27n - 38) u^2\right], \\
\eta &= \frac{8}{27} \left[v^2 + \frac{2uv}{3(n + 8)} + \frac{(n + 2)}{(n + 8)^2} u^2\right].
\end{align*}
\]

For each of surface critical exponents we obtain from \((5.1)\) and \((4.21)\) at \(d = 3\) a double series expansion in powers of \(u\) and \(v\) truncated at the second order

\[
f(u, v) = \sum_{j,l\geq 0} f_{jl} u^j v^l.
\]

Since perturbation expansions of this kind are generally divergent \[52\], the sufficiently powerful resummation procedure of the series is essential to obtain accurate estimates of the critical exponents. One of the simplest way is to calculate for each quantity we consider a sequence of rational Padé approximants in two variables from the original series expansions. This should work already well when the series behave in lowest orders ‘in a convergent fashion’. Besides, if the series are alternating in sign \[55\], we can use more modern Padé-Borel resummation procedure \[56\] for their analysis.

The results of our Padé and Padé-Borel analysis for the surface critical exponents at the special transition are represented in Table I. We evaluate the exponents at the standard RG random fixed point \[8\]

\[
\begin{align*}
u^* &= -0.60509, \\
v^* &= 2.39631.
\end{align*}
\]

The values \([0/0], [1/0],\) and \([2/0]\) are simply the direct partial sums up to the zeroth, first, and second orders, respectively. Padé approximants \([0/1]\) and \([0/2]\) represent the partial sums of the inverse series expansions up to the first and second order.

Besides, by analogy with \[36\] we performed the calculation of nearly-diagonal two-variable rational approximants of the type

\[
[11/1] = \frac{1 + a_1 u + a_1 v + a_1 u v}{1 + b_1 u + b_1 v}
\]

and
\[ \frac{1}{11} = \frac{1 + a_1 u + a_1 v}{1 + b_1 u + b_1 v + b_{11} uv}, \] (5.6)

which are given in eighth and ninth columns of Table I.

The second column of Table I contains the ratios of magnitudes of first- \( O_1 i \) and second-order \( O_2 i \) perturbative corrections appearing in inverse series expansions of our critical exponents. The larger (absolute) values of this ratios correspond to the better apparent convergence of truncated series. It is easy to see that the series of inverse expansions for all critical exponents, except \( \beta_1 \), are alternating in sign and consequently adapted to the above-mentioned Padé-Borel resummation analysis (see Appendix 1). Among the direct series the situation is more complicated. The ratios of first- \( O_1 \) and second-order \( O_2 \) perturbative corrections of the direct series expansions for the critical exponents \( \delta_1 \) and \( \gamma_1 \) are positive \( [60] \). This means that the signs of the first- and second-order corrections do not alternate and hence the correspondent series are not suitable to the Padé-Borel resummation technique, since the \( [11/1] \) approximant of the Borel transform have a pole in the integration range. But these series are slowly convergent, because the contribution of the second-order are considerably less than contribution of the first-order. For example the ratio \( O_1/O_2 \) for the critical exponent \( \Delta_1 \) will be equal 35.1. Thus the above mentioned series adapted to the Padé analysis. It should be noted, that a very similar situation has been meet in the analysis of the surface critical exponents perturbation series expansions at the ordinary transition in pure \( [12] \) and quenched dilute semi-infinite Ising-like systems \( [11] \).

The results of Padé-Borel analysis of the inverse series expansions are given in the last column of the Table I. These values give the most reliable numerical estimates. In common, we assume that the results obtained from the Padé-Borel analysis \( R^{-1} \) are the best we could achieve from the available knowledge about the series expansions in the frames of the present approximation scheme

\[ \eta_\parallel = -0.238, \quad \Delta_1 = 1.101, \quad \eta_\perp = -0.116, \quad \beta_1 = 0.258, \quad \gamma_{11} = 0.845, \quad \gamma_1 = 1.442, \quad \delta_1 = 6.343, \quad \delta_{11} = 4.172. \] (5.7)

Here the estimate of \( \beta_1 \), for which no Padé-Borel approximant \( R^{-1} \) exists, has been derived from the scaling relation \( \beta_1 = \frac{v}{2}(d - 2 + \eta_1) \), where \( v = 0.678 \) \( [8] \) and the above value of \( \eta_\parallel = -0.238 \).

The deviations of estimates \( (5.7) \) from the other second-order estimates of the table might serve as a rough measure of the achieved numerical accuracy.

**VI. \( \sqrt{\epsilon} \) Expansion**

As was mentioned above, there is an alternative method to analyze the influence of randomness on the critical behavior based on the renormalization-group approach introduced by Harris and Lubensky \( [1] \). This method was employed by Ohno and Okabe \( [11] \) for study critical behavior of semi-infinite systems with a Gaussian randomness in \( 4 - \epsilon \) dimensions in the frames of \( \sqrt{\epsilon} \) expansion for obtaining the two-loop approximation for correlation function and deriving corresponding series expansions for the surface critical exponents \( \eta_\parallel \) and \( \eta_\perp \). Their results in the case of special transition at \( n = 1 \) (see \( [13] \)) with corresponding changes of coupling constants normalizations \( (u \to v/24, \quad w \to -u/3) \) in accordance with our notations are written in the form

\[ \eta_\parallel = -\frac{u}{3} - \frac{v}{2} + \frac{7}{12} u^2 + \frac{11}{12} v^2 + \frac{7}{4} u v + O(\epsilon^{4/3}), \]
\[ \eta_\perp = -\frac{u}{6} - \frac{v}{4} + \frac{11}{36} u^2 + \frac{23}{48} v^2 + \frac{11}{12} u v + O(\epsilon^{4/3}). \] (6.1)

Unfortunately, the surface critical exponents \( \eta_\parallel \) and \( \eta_\perp \) have been obtained only to the first order in \( \sqrt{\epsilon} \) from these equations in \( [11] \). In the present paper we derive the next term in \( \sqrt{\epsilon} \) expansion for above mentioned surface critical exponents using the fixed-point values up to \( O(\epsilon) \) \( [6,7] \)

\[ u^* = -3 \sqrt{\frac{6\epsilon}{53}} + 18 \frac{110 + 63\zeta(3)}{53^2 \epsilon}, \]
\[ v^* = 4 \sqrt{\frac{6\epsilon}{53}} - 72 \frac{19 + 21\zeta(3)}{53^2 \epsilon}, \] (6.2)

where \( \zeta(3) \approx 1.2020569 \) is the Riemann \( \zeta \)-function, and the usual geometric factor \( K_d = 2^{1-d}\pi^{-d/2}/\Gamma(d/2) \) has been absorbed into the redefinitions of the coupling constants. As a result we obtain
\[ \eta_{\parallel} = -\sqrt{\frac{6\epsilon}{53}} + \frac{756\zeta(3) - 641}{2 \cdot 53^2} \epsilon, \]
\[ \eta_{\perp} = -\sqrt{\frac{3\epsilon}{106}} + \frac{378\zeta(3) - 347}{2 \cdot 53^2} \epsilon. \]  

(6.3)

Taking into account scaling relations for surface critical exponents and \( \sqrt{\epsilon} \) expansions for random bulk exponents \( \nu \) and \( \eta \) we obtain respective perturbation series expansions for other surface critical exponents

\[ \Delta_1 = \frac{3}{4} + \frac{5}{8} \sqrt{\frac{6\epsilon}{53}} + \frac{3523 - 3780\zeta(3)}{16 \cdot 53^2} \epsilon, \]
\[ \beta_1 = \frac{1}{4} - \frac{1}{8} \sqrt{\frac{6\epsilon}{53}} + \frac{3 (252\zeta(3) - 461)}{16 \cdot 53^2} \epsilon, \]
\[ \gamma_{11} = \frac{1}{2} + \frac{3}{4} \sqrt{\frac{6\epsilon}{53}} + \frac{2268\zeta(3) - 2453}{8 \cdot 53^2} \epsilon, \]
\[ \gamma_1 = 1 + \frac{3}{4} \sqrt{\frac{6\epsilon}{53}} - \frac{3 (378\zeta(3) - 347)}{4 \cdot 53^2} \epsilon, \]
\[ \delta_1 = 5 + 5 \sqrt{\frac{6\epsilon}{53}} - \frac{3 (630\zeta(3) - 1073)}{53^2} \epsilon, \]
\[ \delta_{11} = 3 + 4 \sqrt{\frac{6\epsilon}{53}} - \frac{2 (756\zeta(3) - 1277)}{53^2} \epsilon. \]  

(6.4)

Similarly as in the case \( d = 3 \) of the previous section, we perform a Padé analysis of our \( \sqrt{\epsilon} \) expansions at \( \epsilon = 1 \). The numerical values of surface critical exponents obtained in this way are represented in Table II. It should be noticed that as was shown in \( [18] \), the \( \sqrt{\epsilon} \) expansion is not Borel summable.

The Padé approximants \([1/0]\) for the exponents \( \eta_{\parallel} \) and \( \eta_{\perp} \) reproduce the first-order results obtained by Ohno and Okabe [45]. On the other hand, the other exponents, \( \beta_1, \gamma_{11} \) and \( \gamma_1 \) slightly differ from the previous results [16].

\[ \beta_1 = 0.17, \quad \gamma_{11} = 0.78, \quad \gamma_1 = 1.26. \]  

(6.5)

The reason is that we calculated our \([1/0]\) estimates directly from each respective \( \sqrt{\epsilon} \) expansion, while in Ref. [45] they were obtained from the scaling relations using the above mentioned numerical values of \( \eta_{\parallel} \) and \( \eta_{\perp} \)[6,7]. In addition we performed the analysis of the correspondent series expansions for the surface magnetic shift exponent \( \Delta_1 \), exponents \( \delta_1 \) and \( \delta_{11} \) which give relations between the surface magnetization and the surface and bulk external magnetic fields, respectively.

It can be easily verified that the above first-order approximants of the critical exponents satisfy the scaling relation

\[ \eta_{\perp} = (\eta + \eta_{\parallel})/2 \]  

(6.6)

with the value of the bulk exponent \( \eta = -\frac{1}{106} + O(\epsilon^2) \) [6,7].

Comparing the results given in Tables I and II we see that the values of the first-order approximants, denoted by \([1/0]\) and \([0/1]\) in both cases, are of comparable magnitudes. But, on the other hand, the values of the second-order approximants are significantly different in both tables. As was shown previously [6,7], \( \sqrt{\epsilon} \) series expansions possess rather irregular structure and are practically unsuitable for subsequent resummation and are ineffective for obtaining reliable numerical estimates. Our results confirm this assumption. If we try to reproduce the numerical value of \( \eta \) (see (6.6)) from our second-order data of Table II, we always obtain negative values. But, this does not agree with the sufficiently precise positive results of massive field-theoretic approach for random bulk systems in three dimensions up to two-loop [5,9], three-loop [7,10,11] and to the four-loop [12] order. Besides, very recently was obtained the value \( \eta = 0.025 \pm 0.01 \) in the frames of five-loop renormalization-group expansions [9] and up-to six-loop order \( \eta = 0.030(3) \) [18]. This discrepancy is not present in our calculation performed directly at \( d = 3 \) (see Table I). From the surface scaling relation (6.6) and the second-order results of Table I we obtain the value \( \eta = 0.031 \), which quite well agree with previous estimates.

VII. SUMMARY

The main aim of the present article was an investigation of the influence of quenched bulk randomness on the surface critical behavior of semi-infinite Ising-like systems at the special transition. To summarize, we have calculated
the surface critical exponents of the special transition for such systems using two alternative technique: the massive field qtheory approach directly at $d = 3$ dimensions up-to two-loop order, and the $\sqrt{\epsilon}$ expansion at $d = 4 - \epsilon$ dimensions to the order of $O((\sqrt{\epsilon})^2)$. In the last case we extend up to the next to leading order, the previous first-order results obtained by Ohno and Okabe [43]. But, the $\sqrt{\epsilon}$ expansions possess rather irregular structure, as was shown in [44,45]. This makes them practically unsuitable for subsequent resummation and ineffective for getting quantitative numerical estimates.

In both cases the resummation of obtained perturbation series expansions for surface critical exponents was performed using Padé analysis. Moreover, in the case of the massive field theory the resulting two-loop series expansions were resummed by means of more precise Padé-Borel resummation technique. In the previous section we have discussed some merits of using the massive field-theoretic approach directly in $d = 3$ dimensions for obtaining most accurate numerical estimates for critical exponents. Thus, the best estimates for surface critical exponents of semi-infinite systems with quenched bulk disorder at the special transition, which we can obtain in the frames of the present approximation scheme, are

\[
\eta_\parallel = -0.238, \quad \Delta_1 = 1.101, \quad \eta_\perp = -0.116, \quad \beta_1 = 0.258, \\
\gamma_{11} = 0.845, \quad \gamma_1 = 1.442, \quad \delta_1 = 6.343, \quad \delta_{11} = 4.172.
\]

(7.1)

The obtained results are evidently different from results obtained for pure semi-infinite Ising-like systems [48,41]

\[
\eta_\parallel = -0.165, \quad \Delta_1 = 0.997, \quad \eta_\perp = -0.067, \quad \beta_1 = 0.263, \\
\gamma_{11} = 0.734, \quad \gamma_1 = 1.302, \quad \delta_1 = 5.951, \quad \delta_{11} = 3.791
\]

(7.2)

and confirm the assumption that the presence of quenched bulk disorder affects the critical behavior of boundaring surface. So, in the case of special transition, similarly as in the case of previously investigated ordinary transition [38], the new set of surface critical exponents appear. It should be mentioned that at the present time the value of the surface crossover exponent $\Phi$ for semi-infinite systems with quenched bulk disorder is still an open question. This question will be the topic of our next publication.

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APPENDIX

As was mentioned above, the power series expansions of surface critical exponents (see Eq. (5.3)) are in general not convergent. In order to obtain meaningful and rather accurate numerical estimates we must apply to them sufficiently powerful "resummation" procedure. In the present paper we employ a two-variable resummation technique which is a simple generalization of the single-variable Padé-Borel method. The starting point of this calculation is construction for truncated power series (5.3) the Borel transform

\[
F(ut, vt) = \sum_{j,l \geq 0} \frac{f_{jl}}{(j + l)!} (ut)^j(vt)^l.
\]

(7.3)

Then we construct the rational approximant $F^B(x, y)$

\[
F^B(u, v) = \frac{1 + a_{10}u + a_{01}v + a_{11}uv}{1 + b_{10}u + b_{01}v},
\]

(7.4)

which is extrapolation of the Borel transform (7.3). It is clear that at $u = 0$ or $v = 0$ we obtain from (7.4) the usual [1/1] Padé approximant. The coefficients $a_{jl}$ and $b_{jl}$ are expressed via the appropriate expansion coefficients of the initial function $f(u, v)$ (5.3)

\[
a_{01} = g_{10} + b_{10}, \quad b_{10} = -g_{20}/g_{10}, \\
a_{01} = g_{01} + b_{01}, \quad b_{01} = -g_{02}/g_{01}, \\
a_{11} = g_{11} + b_{10}g_{01} + b_{01}g_{10},
\]

(7.5)
where $g_{jl} = f_{jl}/(j + l)!$. Hence, for the resummed function by means of the Padé-Borel resummation technique we obtain

$$
\bar{f}(u, v) = \int_0^{\infty} F^B(ut, vt)e^{-t}dt.
$$

(7.6)

This technique has been applied to the direct power series expansions of surface critical exponents to find the corresponding Padé-Borel approximants $R^{-1}$.

TABLE I. Surface critical exponents of the special transition for $d = 3$ up to two-loop order at the random-fixed point $u^* = -0.60509$, $v^* = 2.39631$.

| exp | $\frac{\partial^2}{\partial^2 u^*}$ | $[0/0]$ | $[1/0]$ | $[0/1]$ | $[2/0]$ | $[0/2]$ | $[11/1]$ | $[1/11]$ | $R^{-1}$ |
|-----|----------------------------------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\eta_\parallel$ | -23.82 | 0.00 | -0.324 | -0.245 | -0.205 | -0.237 | -0.244 | -0.238 | -0.238 |
| $\Delta_1$ | -3.39 | 0.75 | 1.074 | 1.229 | 1.083 | 1.046 | 1.083 | 1.090 | 1.101 |
| $\eta_\perp$ | -3.35 | 0.00 | -0.162 | -0.139 | -0.087 | -0.102 | -0.115 | -0.114 | -0.116 |
| $\beta_1$ | 0.00 | 0.25 | 0.25 | 0.25 | 0.263 | 0.263 | — | — | — |
| $\gamma_{11}$ | -3.14 | 0.50 | 0.824 | 0.979 | 0.825 | 0.783 | 0.825 | 0.834 | 0.845 |
| $\gamma_1$ | -2.56 | 1.00 | 1.405 | 1.680 | 1.410 | 1.327 | 1.410 | 1.421 | 1.442 |
| $\delta_1$ | -1.41 | 5.00 | 6.619 | 7.394 | 7.062 | 5.521 | 6.205 | 6.236 | 6.343 |
| $\delta_{11}$ | -1.40 | 3.00 | 4.295 | 5.279 | 3.926 | 3.418 | 4.032 | 4.070 | 4.172 |

TABLE II. Surface critical exponents of the special transition from the $\sqrt{\epsilon}$ expansion.

| exp | $[0/0]$ | $[1/0]$ | $[0/1]$ | $[2/0]$ | $[0/2]$ | $[11/1]$ | $[1/11]$ |
|-----|--------|--------|--------|--------|--------|--------|--------|
| $\eta_\parallel$ | 0.00 | -0.336 | -0.252 | -0.289 | -0.287 | -0.287 | -0.295 |
| $\Delta_1$ | 0.75 | 0.960 | 1.016 | 0.938 | 0.917 | 0.917 | 0.940 |
| $\eta_\perp$ | 0.00 | -0.168 | -0.144 | -0.149 | -0.151 | -0.151 | — | — |
| $\beta_1$ | 0.25 | 0.208 | 0.210 | 0.197 | 0.198 | 0.198 | — | — |
| $\gamma_{11}$ | 0.50 | 0.752 | 0.838 | 0.740 | 0.714 | 0.714 | 0.741 |
| $\gamma_1$ | 1.00 | 1.252 | 1.338 | 1.224 | 1.190 | 1.190 | 1.227 |
| $\delta_1$ | 5.00 | 6.682 | 7.535 | 7.019 | 6.729 | 6.729 | 7.104 |
| $\delta_{11}$ | 3.00 | 4.436 | 5.441 | 4.608 | 4.232 | 4.232 | 4.671 |
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For this kind of series, the expansion coefficients at large orders of perturbation theory grow nearly factorial. In fact, this is an intuitive picture conveyed from the theory of bulk regular systems. Much less is known about the large-order behavior of perturbative expansions pertaining to infinite random systems (see Refs. [53,38,54]), especially at large space dimensionalities. To our knowledge, there are no explicit results on large orders for surface quantities, even in the absence of disorder.

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