A spectral least-squares-type method for heavy-tailed corrupted regression with unknown covariance & heterogeneous noise

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Abstract

We revisit heavy-tailed corrupted least-squares linear regression assuming to have a corrupted \( n \)-sized label-feature sample of at most \( \epsilon n \) arbitrary outliers. We wish to estimate \( \mathbf{b}^* \in \mathbb{R}^p \) given such sample of a label-feature pair \((y, x) \in \mathbb{R} \times \mathbb{R}^p\) satisfying \( y = \langle x, b^* \rangle + \xi \), with heavy-tailed \((x, \xi)\). We only assume \( x \) is \( L^4 - L^2 \) hypercontractive with constant \( L > 0 \) and has covariance matrix \( \Sigma \) with minimum eigenvalue \( 1/\mu^2(\mathbb{B}_2) > 0 \) and bounded condition number \( \kappa > 0 \). The noise \( \xi \in \mathbb{R} \) can be arbitrarily dependent on \( x \) and nonsymmetric as long as \( \xi x \) has finite covariance matrix \( \Xi \). We propose a near-optimal computationally tractable estimator, based on the power method, assuming no knowledge on \((\Sigma, \Xi)\) nor the operator norm of \( \Xi \). With probability at least \( 1 - \delta \), our proposed estimator attains the statistical rate \( \mu^2(\mathbb{B}_2)\|\Xi\|^{1/2}(\frac{p}{n} + \frac{\log(1/\delta)}{n} + \epsilon)^{1/2} \) and breakdown-point \( \epsilon \lesssim \frac{1}{L^4}\kappa^2 \), both optimal in the \( \ell_2 \)-norm, assuming the near-optimal minimum sample size \( L^4\kappa^2(p \log p + \log(1/\delta)) \lesssim n \), up to a log factor. To the best of our knowledge, this is the first computationally tractable algorithm satisfying simultaneously all the mentioned properties. Our estimator is based on a two-stage Multiplicative Weight Update algorithm. The first stage estimates a descent direction \( \hat{v} \) with respect to the (unknown) pre-conditioned inner product \( \langle \Sigma(\cdot), \cdot \rangle \). The second stage estimate the descent direction \( \Sigma \hat{v} \) with respect to the (known) inner product \( \langle \cdot, \cdot \rangle \), without knowing nor estimating \( \Sigma \).
1 Introduction

Least-squares regression is a fundamental problem in statistics and machine learning, either from a practical or theoretical standpoint. However, classical methodologies for this problem assume the collected data is clean and light-tailed. Robust Statistics [23, 19, 32, 24] aim in addressing robust estimation when either the sample is corrupted or the data generating distribution is too heavy-tailed.

In recent work, the minimax optimality of several robust estimation problems have been attained [3, 4] and [18]. The construction of these estimators, however, is based on Tukey’s depth, a hard computational problem in higher dimensions. Fundamental recent works [13, 27] have proposed alternative estimators that are both computationally tractable and statistically (near) optimal. For instance, near optimal robust estimators for the mean of a high-dimensional vector can be computed in nearly-linear time [5, 17, 12, 20, 9]. We refer to [14, 30] for extensive surveys.

In this work, we revisit the problem of heavy-tailed least-squares regression assuming to have an adversarially corrupted $n$-sized sample. Here, “adversarial” means that the sample, corrupted in both labels and features, has at most $\epsilon n$ arbitrary outliers for some contamination fraction $\epsilon \in (0, 1/2)$. In particular, the adversary mechanism can depend on the (unobserved) clean iid sample. The goal of this paper is to establish near-optimal statistical rates for this problem with a computationally tractable estimator and minimal assumptions. Precisely, our main result can be resumed as follows:

a) Optimality in high-probability. We assume that the feature vector $x$ has finite covariance matrix $\Sigma$, with minimum eigenvalue $1/\mu^2(B^2) > 0$, maximum eigenvalue $\|\Sigma\|$ and condition number $\kappa < \infty$, and satisfy the $L^4 - L^2$ hypercontractive property with constant $L > 0$. Moreover, the noise-feature multiplier vector $\xi x$ is assumed to have finite covariance matrix $\Xi$, with maximum eigenvalue $\|\Xi\| < \infty$. Under these standard assumptions, with probability at least $1 - \delta$, our proposed estimator achieves the $\ell_2$-norm estimation rate $\mu^2(B^2)\|\Xi\|^{1/2}(\sqrt{p/n} + \sqrt{\log(1/\delta)/n} + \sqrt{\epsilon})$ with sample size of at least $n \geq C L^4 \kappa^2 (p \log p + \log(1/\delta))$ and contamination fraction of at most $\epsilon \leq \frac{1}{C L^4 \kappa^2}$. Here, $C > 0$ is a absolute constant. The mentioned rate and cut-offs in $(n, \epsilon)$ are all optimal in $(n, p, \delta, \epsilon)$, including the constants $(\mu(B^2), \|\Sigma\|, \|\Xi\|)$, up to a log factor. Using a “least-squares methodology”, the statistical and optimization rates of our algorithm do not depend on the $\ell_2$-norm of the ground truth parameter, only on the condition numbers $(\mu^2(B^2)\|\Xi\|^{1/2}, \kappa)$.

b) Heterogeneous noise. We are mainly concerned with the statistical learning framework over the linear class in mean least-square sense. In this set-up, the noise $\xi$ can be arbitrarily dependent on $x$ and does not need to be symmetric.
c) **Tractability via spectral methods.** The seminal works \[13, 27\] were the first to suggest that, to construct computationally tractable robust mean estimators, one must exploit the eigenstructure of the sample covariance matrix. Various approaches have been developed since then for robust linear regression. Some current approaches make use of significantly more time consuming approaches such as semi-definite programming (SDP) or sum-of-squares algorithms. Our estimator is computationally tractable by means of faster spectral methods \[28, 11\]. The main computational bottleneck is to run a logarithmic number of iterations of a Multiplicative Weight Update algorithm (MWU) \[26, 1\] in which every iteration requires to approximately solve a maximum eigenvalue problem. This can be done e.g. via a randomized power method.

d) **Unknown covariances and noise level.** With a light-tailed iid clean sample, the least squares estimator is known to be optimal without knowing the covariance matrices \((\Sigma, \Xi)\) nor the noise variance \(\sigma^2\). Likewise, our estimator satisfy (a)-(c) without the need to know \((\Sigma, \Xi)\) nor \(\|\Xi\|\).

To the best of our knowledge, as discussed next, we believe this is the first analysis with a computationally tractable algorithm, based on a least-squares methodology instead of a gradient estimation methodology, satisfying simultaneously all the mentioned properties.

### 1.1 Related work

Outlier-robust linear regression has already been subject to a lot of research since the seminal work of Huber \[23\]. In the particular model of label contamination, estimators based on Huber-type losses are optimal. Unlike the label-feature contamination model, optimal estimators for the label-contamination model can be tuned adaptively to \((\epsilon, \delta)\) and have \(\kappa\)-free breakdown points. Also, optimal estimators are asymptotically consistent in case the model is oblivious. See e.g. \[8, 39\] for an extensive review.

The more general problem of label-feature corrupted linear regression has been previously considered in the works \[15, 16, 38, 40, 2, 11, 6, 37, 25\]. \[15, 16, 38\] focused on the sub-gaussian corrupted model. \[40, 2\] considered assumptions and algorithmic approaches that are statistically optimal with polynomial time complexity. Still, they require more restrictively sample complexity and distribution assumptions. For instance, \[2\] is based on sum-of-squares methodology which is more time consuming than spectral methods. \[11, 37\], as this work, are based on a least-squares methodology, but they require full knowledge of the feature covariance matrix \(\Sigma\). \[6, 25\] do not require knowledge of \(\Sigma\) but their estimators, like \[16, 38\], follow a different approach, based on robust gradient
estimation. [6] is based on SDP, a more time consuming approach. Also, their optimization complexity depends, unlike this work, on the $\ell_2$-norm of the ground truth parameter and they assume independence between noise and feature vector. [25], like this work, are constructed with spectral methods. Still, they assume independence between noise and the feature vector, require knowledge of the noise variance $\sigma^2$ and their complexity depend on the $\ell_2$-distance between the ground truth and the initial estimate. [6, 25] are not concerned with optimality with respect to $\delta$. Also, their minimal requirement on the sample size is of order $\tilde{O}(p)/\epsilon \lesssim n$; we require $\tilde{O}(p) + \log(1/\delta) \lesssim n$, independently of $\epsilon$.

It is instructive to conclude this section with a discussion between two methodologies used in heavy-tailed corrupted estimation. In a nutshell, computationally tractable estimators for this problem are based on two frameworks. The first solve (approximately) the semi-definite programming given a set of points $\{z_i\}_{i=1}^n$ and $k \in \{1, \ldots, n\}$:

$$\min_{w \in \Delta_{n,k}} \sup_{v \in \mathbb{B}_2} \sum_{i=1}^n w_i z_i z_i^\top, \quad (1)$$

where $\Delta_{n,k} := \{w \in \mathbb{R}^n : w_i \geq 0, \sum_{i=1}^n w_i = 1, w_i \leq \frac{1}{n-k}\}$ and $\mathbb{B}_2$ is the Euclidean unit ball. This is the approach followed by [17, 12, 20, 6, 25]. A complementary approach, initiated for robust mean estimation in [21, 28], is to consider tractable relaxations of the combinatorial problem

$$\max_{(\theta, \nu, q) \in \mathbb{R} \times \mathbb{B}_2 \times \mathbb{R}^K} \theta$$

subject to

$$q_i |\langle z_i, \nu \rangle| \geq q_i \theta, \quad i = 0, \ldots, K,$$

$$\sum_{i=1}^K q_i > K - k,$$

$$q_i \in \{0, 1\}, \quad i = 0, \ldots, K. \quad (2)$$

Here, $\{z_i\}_{i=1}^K$ are initially pre-processed from data using a Median-of-Means framework. See Section 5.1 and [21, 28] for further discussion on the motivation for why studying this problem. Most closely to our work are [28, 11]. [28] is focused on robust mean estimation. One important difference between robust mean estimation and linear regression is that, unlike approaches based on (1), methods based on (2) do not require high-probability concentration bounds for the 4th order tensor. See [6] for further discussion on this issue. To the best of our knowledge, [11] is the first work aiming in generalizing the approach of [21, 28] to heavy-tailed corrupted linear regression. Still, one important limitation is that [11] requires full knowledge of the covariance matrix $\Sigma$; this explains why their rate is independent of the condition number $\kappa$. [11] requires the number of
buckets $K$ to be depend on the dimension $p$ while our estimator uses $K$ independent of $p$. One key development in our analysis is to show that a two-stage algorithm based on the MWU algorithm is enough to avoid the need of the knowledge of $\Sigma$. The first stage estimates a descent direction $\hat{v}$ with respect to the (unknown) pre-conditioned inner product $\langle \Sigma(\cdot), \cdot \rangle$. The second stage estimate the descent direction $\Sigma\hat{v}$ with respect to the (known) inner product $\langle \cdot, \cdot \rangle$, without knowing nor estimating $\Sigma$.

Finally, one important difference between robust mean estimation and linear regression concerns the initialization. For instance, the easy to compute coordinate-wise median turns out to be sufficient for the initialization of robust mean estimators [12, 9]. One the other hand, there is no coordinate-wise median counterpart for robust linear regression. To the best of our knowledge, the properties needed for the initialization in [11], the most close to our work, are assumed a priori without formal derivation. We formalize guarantees for the initialization of robust linear regression based on Median-of-Least-Squares (MLS) estimators. In that regard, unlike assuming a priori invertability assumptions for the bucket design matrices as in [34] or Srivastava-Vershynin condition, a stronger assumption than hypercontractivity, as in [22], we derive sufficient lower bounds in high-probability for MLS estimators with tight dependence on $(K, \delta)$ assuming only hypercontractivity. See Proposition 2 in Section 3.2. This is a analog for MLS estimators of the PAC-Bayesian tool developed in [36]. We also remark that our initialization is adaptive to any of the parameters $(\mu^2(B_2), \| \Sigma \|, \sigma^2, \| \Xi \|)$.

We conclude with a minor observation regarding randomized rounding, a needed tool in most of the literature. Our rounding scheme is based on the spherical distribution instead of the Gaussian distribution. This somewhat simplifies the rounding analysis in [12, 28] and it also seems to imply a larger confidence interval. See Proposition 3.

1.2 Framework

Let $(y, x) \in \mathbb{R} \times \mathbb{R}^p$ be a label-feature pair with centered feature $x$. Within a statistical learning framework, we wish to explain $y$ through $x$ via the linear class $F_1(\mathbb{R}^p) := \{ \langle \cdot, b \rangle : b \in \mathbb{R}^p \}$. Precisely, giving a sample of $(y, x)$, we wish to estimate

$$b^* \in \arg\min_{b \in \mathbb{R}^p} \mathbb{E} (y - \langle x, b \rangle)^2.$$ 

In particular, one has $y = \langle x, b^* \rangle + \xi$ with $\xi \in \mathbb{R}$ having zero mean and satisfying $\mathbb{E}[\xi|x] = 0$. This is our only assumption on $\xi$: we do not assume $\xi$ and $x$ are independent nor that $\xi$ is symmetric.

**Assumption 1** (Heavy-tails). Assume:
• The feature vector \( x \) has distribution \( \Pi \) and unknown finite non-singular covariance matrix \( \Sigma := \mathbb{E}[xx^\top] \) with known maximum eigenvalue \( \|\Sigma\| < \infty \) and minimum eigenvalue \( \frac{1}{\mu^2(\mathbb{B}_2)} > 0 \). Moreover, \( x \) satisfies the \( L^4 - L^2 \) norm equivalence condition with unknown constant \( L > 0 \): for all \( v \in \mathbb{B}_2 \),

\[
\left\{ \mathbb{E}|\langle z, v \rangle|^4 \right\}^{1/4} \leq L \left\{ \mathbb{E}|\langle z, v \rangle|^2 \right\}^{1/2}.
\]

• The centered noise \( \xi \) satisfy \( \mathbb{E}[\xi x] = 0 \), has finite variance and the multiplier vector \( \xi x \) has unknown finite population covariance matrix \( \Xi := \mathbb{E}[\xi^2 xx^\top] \) with unknown maximum eigenvalue \( \|\Xi\| \).

\( L^4 - L^2 \) norm equivalence is also known as bounded 4th moment, bounded kurtosis or hypercontractivity conditions [33].

We consider available a label-feature sample with adversarial contamination.

**Assumption 2** (Label-feature adversarial contamination). The contamination fraction will be denoted by \( \epsilon := \frac{\omega}{n} \). This means that it is available a label-feature sample \( \{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^{2n} \) having an arbitrary subset of exactly \( o \) data points differing from the label-feature sample \( \{(y_\ell, x_\ell)\}_{\ell=1}^{n} \) which is an independent iid copy of \((y, x) \in \mathbb{R} \times \mathbb{R}^p \). We use the notations \( \xi_\ell := y_\ell - \langle x_\ell, b^* \rangle \) and \( \tilde{\xi}_\ell := \tilde{y}_\ell - \langle \tilde{x}_\ell, b^* \rangle \) for all \( \ell \in [n] \).

We remark that assuming knowledge of \((\|\Sigma\|, \mu^{-2}(\mathbb{B}_2))\) is not restrictive in the heavy-tailed corrupted model of Assumption 2. Using a separate batch of the corrupted sample, there exist tractable robust estimators \((\hat{\lambda}_{\max}, \hat{\lambda}_{\min})\) satisfying, with high-probability, \( a_1 \|\Sigma\| \leq \hat{\lambda}_{\max} \leq a_2 \|\Sigma\| \) and \( a_3 \mu^{-2}(\mathbb{B}_2) \leq \hat{\lambda}_{\min} \leq a_4 \mu^{-2}(\mathbb{B}_2) \) for positive constants \((a_1, a_2, a_3, a_4)\). See for instance [33]. Using those estimates in our algorithms entail the same rate of Theorem 1.1 up to changes in absolute constants.

Next, we formally state our main result. Its full derivation requires several intermediate results developed in the next sections.

**Theorem 1.1.** Grant Assumptions 1 and 2. Assume that \( \max\{1, \|\Xi\|\} < \gamma \zeta_0 \) for some known \( \gamma \in (0, 1) \) and \( \zeta_0 > 0 \); let \( M := \lceil \log_{\gamma^{-1}}(\zeta_0) \rceil \).

Then, given any desired probability of failure \( \delta_0 \in (0, 1) \), if the sample size and contamination fraction satisfy

\[
n \geq (CL^4 \kappa^2 p \log p) \sqrt{(CL^4 \kappa^2 \log(CM/\delta_0))},
\]

\[
\epsilon \leq \frac{1}{CL^4 \kappa^2},
\]
there exists a computationally efficient algorithm (namely Algorithm 6 in Section 6.1) with inputs \( \{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^n \cup \{(y_\ell, x_\ell)\}_{\ell=n+1}^2, \mu(B_2), \|\Sigma\|, M \) and \( \delta_0 \) and \( \epsilon \) satisfying, with probability at least \( 1 - \delta_0 \),

\[
\|\tilde{b} - b^*\|_2 \lesssim \mu^2(B_2)\|\Xi\|^{1/2} \left( \frac{p}{n} + \frac{1 + \log(M) + \log(1/\delta_0) + \epsilon}{n} \right)^{1/2}.
\]

If one knows \( \|\Xi\| \), then we can take \( M = 1 \) above.

Throughout the paper we will denote the \( \ell \)th residual at \( b \in \mathbb{R}^p \) by \( \tilde{\xi}_\ell(b) := \tilde{y}_\ell - \langle \tilde{x}_\ell, b \rangle \). Recall the bilinear form \( \langle v, w \rangle_H := \mathbb{E}[\langle x, v \rangle \langle x, w \rangle] \), and the \( L^2(H) \) pseudo-norm \( \|v\|_H := \sqrt{\langle v, v \rangle_H} \). Given a cone \( C \subset \mathbb{R}^p \), we define the restricted eigenvalue constant \( \mu(C) := \sup_{v \in C} \frac{\|v\|_2}{\|v\|_H} \).

2 Notation

2.1 Basic notation

Let \( [n] := \{1, \ldots, n\} \). Denote by \( \Delta_n \) the \( n \)-dimensional simplex and, for given \( k \in [n - 1] \),

\[
\Delta_{n,k} := \left\{ w \in \Delta_n : w_i \leq \frac{1}{n-k} \right\}.
\]

We denote by \( KL(p\|q) \) the Kullback-Leibler divergence between two distributions in \( \Delta_n \).

We’ll write \( a \lesssim b \) if \( a \leq Cb \) for an absolute constant and say \( a \asymp b \) if \( a \lesssim b \) and \( b \lesssim a \). We use the usual notations \( a_+ := \max\{0, a\} \), \( a_- := \max\{0, -a\} \), \( a \lor b := \max\{a, b\} \) and \( a \land b := \min\{a, b\} \). Given sequence \( \{\sigma_i\}_{i=1}^n \) of numbers, \( \sigma^*_1 \leq \cdots \leq \sigma^*_m \) denotes its non-decreasing order while \( \sigma^*_1 \geq \cdots \geq \sigma^*_m \) denotes its non-increasing order. We denote the inner product by \( \langle \cdot, \cdot \rangle \), the \( \ell^2 \)-norm by \( \|\cdot\|_2 \), the unit balls \( B_2 := \{v \in \mathbb{R}^p : \|v\|_2 \leq 1\} \), unit sphere \( S_2 := \{v \in \mathbb{R}^p : \|v\|_2 = 1\} \), the \( \ell^2 \)-norm ball with center \( a \) and radius \( r \) by \( \mathbb{B}(a, r) \).

The canonical basis in \( \mathbb{R}^p \) will be denoted by \( \{e_1, \ldots, e_p\} \) and \( I_p \) denotes the identity matrix. Given non-zero matrix \( A \), we denote its trace by \( \text{tr}(A) \), its operator norm by \( \|A\| \) and its Frobenius norm by \( \|A\|_F \). We also define \( \rho^2_A = \|\text{diag}(A)\|_\infty \). The standard inner product on its \( \mathbb{R}^{p \times p} \) will be denoted by \( \langle A, B \rangle := \text{tr}(A^\top B) \), where \( \top \) is the transpose operation. Given two vectors \( v, u \in \mathbb{R}^p \), we let \( v \otimes u := vu^\top \). We use the notation \( M \succeq 0 \) for a semi-positive definite symmetric matrix \( M \in \mathbb{R}^{p \times p} \). Also, its associated bilinear form and pseudo-norm will be denoted respectively by \( \langle u, v \rangle_M := \langle M u, v \rangle \) and \( \|u\|_M := \|M^{1/2}u\| \). We define the set of matrices \( \mathcal{M}(\mathcal{B}) \) for some compact convex set \( \mathcal{B} \subset \mathbb{R}^p \) as the convex hull of the set \( \{vv^\top : v \in \mathcal{B}\} \).
2.2 Some probabilistic notions

Let $X$ be a random variable with distribution $\mathbf{P}$ taking values on a measurable set $\mathbb{B}$. We denote by $\{X_i\}_{i=1}^n$ an iid copy of $X$. We reserve the notation of $\{\epsilon_i\}_{i \in [n]}$ to represent an iid sequence of Rademacher random variables. Given a class $F$ of integrable functions $F : \mathbb{B} \to \mathbb{R}$ with respect to $\mathbf{P}$, the Rademacher complexity of $F$ associated to $\{X_i\}_{i=1}^n$ is the quantity

$$\mathcal{R}_{X,n}(F) := \mathbb{E} \left[ \sup_{f \in F} \sum_{i \in [n]} \epsilon_i f(X_i) \right],$$

where $\{\epsilon_i\}_{i \in [n]}$ is independent of $\{X_i\}_{i=1}^n$. A related quantity is

$$\mathcal{D}_{X,n}(F) := \mathbb{E} \left[ \sup_{f \in F} \left| \sum_{i \in [n]} (f(X_i) - \mathbb{E}[f(X_i)]) \right| \right],$$

noting that, by symmetrization, $\mathcal{D}_{X,n}(F) \asymp \mathcal{R}_{X,n}(F)$. We sometimes use the notation $\mathbf{P} f := \mathbb{E}[f(X)]$. The wimpy variance of the class $F$ associated to $X$ is the quantity

$$\sigma_X^2(F) := \sup_{f \in F} \mathbb{E}[(f(X) - \mathbf{P} f)^2].$$

Letting $u, v \in \mathbb{R}^p$ be centered random vectors with distribution $\mathbf{P}$, we define the bilinear form

$$\langle u, v \rangle _\mathbf{P} := \mathbb{E}[(z, u)(z, v)],$$

and the $L^2(\mathbf{P})$ pseudo-norm $\|v\|_\mathbf{P} := \sqrt{\mathbb{E}[(z, v)^2]}$. We will also define the unit ellipsoid $\mathbb{B}_\mathbf{P} := \{v \in \mathbb{R}^p : \|v\|_\mathbf{P} \leq 1\}$ with border $S_\mathbf{P} := \{v \in \mathbb{R}^p : \|v\|_\mathbf{P} = 1\}$. Let $\mathbb{B}$ be a compact subset of $\mathbb{R}^p$. The wimpy variance of $\mathbb{B}$ associated to the distribution of $z$ is the quantity $\sigma_z^2(\mathbb{B}) := \sigma_z^2(F_1(\mathbb{B}))$. 

8
3 Concentration bounds

In all this section, $X$ is a random variable taking values on some set $\mathcal{B}$ with distribution $P$ and $\{X_i\}_{i=1}^n$ an iid copy of $X$. We also split the sample $\{X_i\}_{i=1}^n$ into $K$ blocks of equal size $B := n/K$ indexed by the partition $\bigcup_{k \in [K]} B_k = [n]$. When $\mathcal{B} = \mathbb{R}$, $\mu$ and $\sigma^2_X < \infty$ will denote the mean and variance of $X$ respectively and let $\overline{X} := X - \mu$. When $\mathcal{B} = \mathbb{R}^p$, $X = z$ will be a centered $p$-dimensional random vector with covariance matrix $\Sigma$ and $\{X_i = z_i\}_{i=1}^n$ is an iid copy of $z$.

3.1 Some general bounds

We define, for any $\eta \in [0, 1]$, its $\eta$-quantile by $Q_{X,\eta}$, that is,

$$Q_{X,\eta} := \sup \left\{ x \in \mathbb{R} : \mathbb{P}(X \geq x) \geq 1 - \eta \right\}.$$

We will assume without loss on generality that $X$ is continuous. In particular, $\mathbb{P}(X \geq Q_{X,\eta}) = 1 - \eta$.

The following result follows from the proofs in [31]. We give a proof for completeness in the Appendix.

**Lemma 3.1.** Let $\eta \in (0, 1/2]$. Then, setting $Q := Q_{X,1-\eta/2}$, with probability at least $1 - \exp(-\eta n/1.8)$,

$$\sum_{i=1}^n 1\{X_i \leq Q\} \geq (1 - 0.75\eta)n.$$

Let $X$ be real valued and $\hat{\mu}_k := \frac{1}{B} \sum_{\ell \in B_k} X_\ell$. We restate the following well-known bound for MOM of random variables [29]. We give a proof in the Appendix for completeness.

**Lemma 3.2 (Random variable).** Let $\alpha \in (0, 1)$ and any constant $C_\alpha > 0$ satisfying

$$\frac{1}{3C_\alpha} + \frac{1}{C^2_\alpha} + \frac{\sqrt{2}}{C^{2/3}_\alpha} \leq \alpha.$$

Then, setting $r := \sigma_X \sqrt{\frac{K}{n}}$, with probability at least $1 - e^{-K/C_\alpha}$,

$$\sum_{k=1}^K 1\{|\hat{\mu}_k - \mathbb{E}[X]| > C_\alpha r\} \leq \alpha K.$$
By now it is well-known that the previous lemma can be generalized for the Empirical Process over a general class $F$ of integrable functions $F : \mathbb{B} \to \mathbb{R}$ with respect to $P$. Define, for each $k \in [K]$ and $f \in F$, the block empirical mean $\hat{P}_{B_k} f := \frac{1}{B} \sum_{\ell \in B_k} f(X_\ell)$. For ease, we use the following notation.

**Definition 3.3.** Let

$$r_{X,n,K}(F) := \frac{\sigma_X(F)}{n} \sqrt{\frac{K}{n}}.$$  

The following result is Lemma 1 in [12].

**Lemma 3.4 (Empirical Process).** Let $\alpha \in (0, 1)$ and any constant $C_\alpha > 0$ satisfying

$$\frac{28}{4C_\alpha} + \frac{4}{C_\alpha^2} + \frac{\sqrt{2}}{C_\alpha^{2/3}} \leq \alpha.$$  \hspace{1cm} (3)

Then, setting $r := r_{X,n,K}(F)$, with probability at least $1 - e^{-K/C_\alpha}$,

$$\sup_{f \in F} \sum_{k=1}^{K} \mathbb{1}_{\{|\hat{P}_{B_k} f - P f| > C_\alpha \cdot r\}} \leq \alpha K.$$  

For instance, when $F = F_1(\mathbb{B}_2)$ one has $r_{z,n,k} = 2 \sqrt{\frac{\text{tr} (\Sigma)}{n}} \vee \sqrt{\frac{\| \Sigma \| K}{n}}$. When $F = F_1(\mathbb{B}_p)$ one has $r_{z,n,k} = 2 \sqrt{\frac{p}{n}} \vee \sqrt{\frac{K}{n}}$.

### 3.2 Concentration bounds for linear regression

For linear regression, we shall need the following bound for the Quadratic Process over the linear class.

**Proposition 1 (Quadratic Process).** Suppose $z \in \mathbb{R}^p$ satisfies the $L^4 - L^2$ norm equivalence property with constant $L > 0$. Let $\alpha \in (0, 1)$ and constant $C_\alpha > 0$ satisfying (3). Let $C > 0$ be an absolute constant in Lemma 7.4 in the Appendix. Set

$$r_{n,K} := CL^2 \sqrt{\frac{p \log p}{n}} \vee L^2 \sqrt{\frac{K}{n}}.$$  

Then,
(i) **Upper bound**: given \( \rho \in (0, 1/2] \) and setting \( C'_\rho := 1 + \sqrt{2/\rho} \), on an event of probability at least \( 1 - e^{-\rho n / 1.8} - e^{-K/C_\rho} \), for all \( u \in \mathbb{S}_p \), for at least \( (1 - (\alpha + 0.75\rho))K \) of the blocks,

\[
\frac{1}{B} \sum_{\ell \in B_k} \left( \langle z_\ell, u \rangle^2 - 1 \right) \leq C_\alpha \left[ r_{n,K} \sqrt{CC'_\rho p \log p} \right].
\]

(ii) **Lower bound**: for any \( \theta > 0 \), on an event of probability at least \( 1 - e^{-K/C_\rho} \), for all \( u \in \mathbb{S}_p \), for at least \( (1 - \alpha)K \) of the blocks,

\[
\frac{1}{B} \sum_{\ell \in B_k} \left( 1 - \frac{L^4}{\theta} - \langle z_\ell, u \rangle^2 \right) \leq C_\alpha \left[ r_{n,K} \sqrt{C\theta p \log p} \right].
\]

**Corollary 1** (Product Process). Given \( \rho \in (0, 1/2] \) and \( \alpha \in (0, 1) \), grant assumptions and definitions in Proposition 1. Set

\[
r_{n,\rho,K} := 2r_{n,K} + CC_\alpha p \log p \frac{1}{2n} + 2L^2 \sqrt{CC_\alpha p \log p \frac{1}{2n}}.
\]

Then, on a event of probability at least \( 1 - e^{-\rho n / 1.8} - 2e^{-K/C_\rho} \), for all \( [u, v] \in \mathbb{B}_p \times \mathbb{B}_p \), for at least \( (1 - (2\alpha + 0.75\rho))K \) of the blocks,

\[
\frac{1}{B} \sum_{\ell \in B_k} \left( \langle z_\ell, u \rangle \langle z_\ell, v \rangle - \langle u, v \rangle \right)_p \leq r_{n,\rho,K}.
\]

We next present some bounds that are suboptimal with respect to the confidence level. Nevertheless, they are important to the pre-processing step of linear regression; we make a remark in this regard in the following.

**Proposition 2.** Suppose that \( z \) satisfies the \( L^4 - L^2 \) norm equivalence condition with constant \( L > 0 \).

Then, for all \( k \in [K] \) and all \( t \geq 0 \), setting

\[
r_t := L^2 \left( 7 \sqrt{Kt \frac{n}{p}} + 4 \sqrt{p \frac{n}{p}} \right),
\]

on a event of probability at least \( 1 - 2e^{-t} \), for all \( v \in \mathbb{R}_p \),

\[
\|v\|_p^2 - \frac{1}{B} \sum_{\ell \in B_k} \langle z_\ell, v \rangle^2 \leq r_t \|v\|_p^2.
\]
Remark 1. The above lower bound can be seen as a MOM-type lower bound for quadratic processes \[36\]. It has two important features. First, a direct application of Theorem 3.1 in \[36\] leads to a rate of the form \(r_t \asymp \|z\|^2 \sqrt{K(\log(n)/\delta)}\); this is not useful in the optimal regime when \(K \geq o \log(1/\delta)\) and \(p \vee o \log(1/\delta) \lesssim n\) we are interested. Indeed, we will only use Proposition 2 with fixed confidence \(t \asymp 1\). Second, the above bound holds for every block \(k \in [K]\) uniformly over \(v \in \mathbb{S}_p\). While the rate in Proposition 2 is worse than the one in Proposition 1 with respect to the confidence parameter \(t > 0\), the uniformity on \((k, v)\) does not hold in item (ii) of Proposition 1. Indeed the blocks for which the lower bound in item (ii) of Proposition 1 holds depend on \(v \in \mathbb{S}_p\). The uniformity property will be fundamental in order to show that the initialization of our algorithm with the Median-of-Least-Squares is bounded in the mentioned regime for \((n, p, o, \delta)\).

The following lemma is immediate from Markov’s inequality and the parallelogram law satisfied by the \(\ell^2\)-norm.

Lemma 3.5. For all \(k \in [K]\) and all \(\delta \in (0, 1)\), with probability at least \(1 - \delta\),

\[
\left\| \frac{1}{B} \sum_{\ell \in B_k} z_\ell \right\|_2^2 \leq \left( \frac{1}{B} \mathbb{E} \|z\|_2^2 \right) \left( \frac{1}{\delta} \right).
\]

### 3.3 Random spherical rounding

Only within this section we assume that \(\{z_i\}_{i=1}^m\) is a fixed (nonrandom) sequence of vectors in \(\mathbb{R}^p\). The following proof is inspired by Proposition 1 in \[12\]. Still, we simplify the proof and enhance the probability level significantly by using a spherical distribution instead of the Gaussian distribution.

**Proposition 3** (Spherical rounding of \(\mathcal{M}(S_2)\) to \(S_2\)). Suppose that there exist \(M \in \mathcal{M}\) and \(D, b > 0\) such that

\[
\sum_{i=1}^m \mathbbm{1}_{\{\langle z_i, z_i^\top \rangle, M\rangle > D\}} > bm.
\]

Let \(\theta \sim \mathcal{U}(S_2)\) be the uniform distribution over \(S_2\). Define the random vector \(v_\theta := M^{1/2} \theta\) where \(M^{1/2}\) is the square root of \(M\).

Then for any \(\varphi \in (0, \pi/2)\) and \(a \in (0, 1)\) satisfying \(\frac{2ab}{\pi a} > 1\), it holds with probability (on the randomness of \(\theta\)) of at least \(\left( \frac{2ab}{\pi a} - a \right)^2\),

\[
\sum_{i=1}^m \mathbbm{1}_{\{|\langle z_i, v_\theta \rangle| > (\cos(\varphi)\sqrt{D})\}} > am.
\]
Remark 2. We remark that the argument above is invariant to scaling. In particular, if one has the same assumption of the proposition for some \( M \in \mathcal{M}(R S_2) \) and \( R > 0 \), then for \( \theta \sim \mathcal{U}(S_2) \), the statement of Proposition 3 still holds for \( \nu_\theta = M^{1/2} \theta \in R \mathbb{B}_2 \).

4 Pre-processing & probabilistic arguments

Set-up 1. We observe the corrupted data set \( \{(y_\ell, \bar{x}_\ell)\}_{\ell=1}^n \cup \{(y_\ell, \bar{x}_\ell)\}_{\ell=n+1}^{2n} \) and denote the unobserved clean data set by \( \{(y_\ell, x_\ell)\}_{\ell=1}^n \cup \{(y_\ell, x_\ell)\}_{\ell=n+1}^{2n} \). Given \( K \in [n] \), we further split the observed first bucket \( \{(y_\ell, \bar{x}_\ell)\}_{\ell=1}^n \) into \( K \) buckets of same size \( B := n/K \) indexed by the partition \( \bigcup_{k \in [K]} \bar{B}_k^{(1)} = [n] \). Similarly, the unobserved first bucket \( \{(y_\ell, x_\ell)\}_{\ell=1}^n \) is split into \( K \) buckets of same size \( B := n/K \) indexed by the partition \( \bigcup_{k \in [K]} B_k^{(1)} = [n] \). Here we assume \( n \) is divisible by \( K \) without loss on generality.

We will use the multivariate notion of median considered by Hsu-Sabato [22]. For that purpose, we introduce the following notation.

Definition 4.1. Given \( \mathcal{W} := \{w_1, \ldots, w_K\} \subset \mathbb{R}^p \) and \( z \in \mathbb{R}^p \), define

\[
\Delta_\mathcal{W}(z) := \min \left\{ r \geq 0 : |\mathbb{B}_2(z, r) \cap \mathcal{W}| > \frac{K}{2} \right\}.
\]

Algorithm 1 Pruning(\( \mathcal{D}, K, \eta \))

Input: sample \( \{(y_\ell, \bar{x}_\ell)\}_{\ell=1}^n \cup \{(y_\ell, \bar{x}_\ell)\}_{\ell=n+1}^{2n} \), number of buckets \( K \) & quantile probability \( \eta \in (0, 1) \).

1: Split the first batch \( \{(y_\ell, \bar{x}_\ell)\}_{\ell=1}^n \) into \( K \) buckets/batches of equal size \( B := n/K \) indexed by the partition \( \bigcup_{k \in [K]} \bar{B}_k^{(1)} = [n] \).
2: For each \( k \in [K] \), compute the bucket least-squares estimator \( \hat{b}_k := \arg\min_{b \in \mathbb{R}^p} \frac{1}{B} \sum_{\ell \in \bar{B}_k^{(1)}} (y_\ell - (\bar{x}_\ell, b))^2 \). 
3: Compute the Hsu-Sabato’s multivariate median \( \bar{b}^{(0)} \) of \( \mathcal{W} := \{\hat{b}_1, \ldots, \hat{b}_K\} \), that is, set \( \bar{k} := \arg\min_{k \in [K]} \Delta_\mathcal{W}(\hat{b}_k) \), and \( \bar{b}^{(0)} := \bar{b}_{\bar{k}} \).
4: Using the second batch \( \{(y_\ell, \bar{x}_\ell)\}_{\ell=n+1}^{2n} \), compute the order statistics \( \hat{R}_1^* \leq \ldots \leq \hat{R}_n^* \) of the sequence \( \{\hat{R}_\ell := \|\hat{\ell}(\bar{b}^{(0)})\|_2 \}_{\ell=n+1}^{2n} \).
5: Set \( m := (1 - \eta)n \).
6: Set \( \{(y_{\ell}, \bar{x}_{\ell})\}_{\ell=1}^m \) (with some abuse of notation) as the subsample obtained by removing from \( \{(y_\ell, \bar{x}_\ell)\}_{\ell=n+1}^{2n} \) the \( n - m = \eta m \) “top data points”, that is, points such that \( \hat{R}_\ell > \hat{R}_m^* \).
7: return \( \bar{b}^{(0)}, \hat{R}_m^*, \{(y_{\ell}, \bar{x}_{\ell})\}_{\ell=1}^m \).
Next we state a lemma ensuring that, with high probability, the initialization $\tilde{b}^{(0)}$ lies at a constant distance to the ground truth $b^*$ and the pruned data set is bounded.

**Lemma 4.2** (Boundedness of initialization & pruned sample). *Grant Assumption 1. Define the quantity $r := 2\mu^2(\mathbb{B}_2)\sqrt{\frac{12K}{n}\text{tr}(\Xi)}$. Let $K \in [n]$ and $\eta \in (0, 1/2]$. Let $m := (1 - \eta)n$. Suppose that$
 o < K/4, \quad \epsilon \leq \eta/4, \quad \mathbb{L}^2 \left( 7\sqrt{\frac{K \log(24)}{n}} + 4\sqrt{\frac{p}{n}} \right) \leq \frac{1}{2}.
$

Let $Q_\eta := Q[\xi \vee \|x\|_2 - 1 - \eta/2]$.

Then on an $\{(y_\ell, x_\ell)\}_{\ell=1}^n \cup \{(y_\ell, x_\ell)\}_{\ell=n+1}^{2n}$-measurable event $E_0$ of probability at least $1 - e^{-K/5.4} - \exp(-\eta n/1.8)$, the output $(\tilde{b}^{(0)}, \tilde{R}^*_m, \{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^m)$ of Algorithm 1 satisfies:

$\|\tilde{b}^{(0)} - b^*\|_2 \leq 3r \quad \text{and} \quad \tilde{R}^*_m = \max_{\ell \in [m]} |\tilde{\xi}_\ell(\tilde{b}^{(0)})| \vee \|\tilde{x}_\ell\|_2 \leq Q_\eta(1 + 3r). \quad (\text{BD}(\eta, r))$

**Proof.** **STEP 1:** Let us define, for each $k \in [K]$, the bucket least-squares estimator $b_k := \arg\min_{b \in \mathbb{R}^p} \frac{1}{B} \sum_{\ell \in B(k)} (y_\ell - \langle x_\ell, b \rangle)^2$, correspondent to the (unobserved) clean sample. We will first prove that on an $\{(y_\ell, x_\ell)\}_{\ell=1}^n$-measurable event of probability at least $1 - \exp(-K/5.4)$,

$\sum_{k=1}^K 1\{\|b_k - b^*\|_2 \leq r\} \geq 3K/4. \quad (4)$

Assume first the above claim is correct. If that is the case, let $K$ be the number of buckets of $\{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^n$ without outliers. Since $o < K/4$, $|K^c| < K/4$. We thus conclude that on the same event

$\sum_{k=1}^K 1\{\|b_k - b^*\|_2 > r\} \leq \sum_{k \in K} 1\{\|b_k - b^*\|_2 > r\} + |K^c|$

$\leq \sum_{k \in [K]} 1\{\|b_k - b^*\|_2 > r\} + K/4$

$< K/2.$

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implying that, for \( \mathcal{W} := \{ \vec{b}_1, \ldots, \vec{b}_K \} \), \( \Delta_{\mathcal{W}}(b^*) \leq r \). This fact and a well-known property of Hsu-Sabato’s multivariate median [22] imply that the estimator \( \vec{b}^{(0)} := \vec{b}_k \) satisfies \( \| \vec{b}^{(0)} - b^* \|_2 \leq 3r \).

We now prove the claim \( \text{BD}(\eta, r) \). Set \( Q_k := \mathbb{P}(\| \vec{b}_k - b^* \|_2 > r) \). If we show that, for any \( k \in [K] \), \( Q_k \leq \eta/2 \) for \( \eta := 1/3 \), then a standard argument based on Bernstein’s inequality (as in the proof of Lemma 3.1 in the Appendix) entails the claim (4).

We next prove that \( Q_k \leq \eta/2 \) for any \( k \in [K] \). Let \( \Delta_k := \vec{b}_k - b^* \). By optimality,

\[
\frac{1}{B} \sum_{\ell \in B_k^{(1)}} \langle \vec{x}_\ell, \Delta_k \rangle^2 = \frac{1}{B} \sum_{\ell \in B_k^{(1)}} \xi_\ell \langle \vec{x}_\ell, \Delta_k \rangle.
\]

Let \( \delta \in (0, 1) \) to be determined later and assume that, for \( r_i \) as defined in Proposition 2, \( r_{\log(4/\delta)} \leq 1/2 \). Given any \( k \in [K] \), by Proposition 2, Lemma 3.5 and an union bound, on an event of probability \( 1 - \delta \), we have

\[
\frac{1}{2} \| \Delta_k \|_\Pi \leq \frac{1}{B} \sum_{\ell \in B_k^{(1)}} \langle \vec{x}_\ell, \Delta_k \rangle^2 \quad \text{and} \quad \frac{1}{B} \sum_{\ell \in B_k^{(1)}} \xi_\ell \langle \vec{x}_\ell, \Delta_k \rangle \leq \sqrt{\frac{K}{n} \mathbb{E}\xi^2 \| \vec{x} \|_2^2 \| \Delta_k \|_2}.
\]

This and the optimality condition imply

\[
\| \Delta_k \|_\Pi \leq 2\mu(B_2) \sqrt{\frac{K}{n} \mathbb{E}\xi^2 \| \vec{x} \|_2^2 \| \Delta_k \|_2}.
\]

We now take \( \eta := 2\delta = 1/3 \) and verify that conditions of the proposition imply \( r_{\log(4/\delta)} \leq 1/2 \).

**STEP 2:** We’ll now make use of the order statistics of the unobserved sequences \( Y_\ell := |\xi_\ell| \sqrt{\| \vec{x}_\ell \|_2} \) and \( \bar{Y}_\ell := |\xi_\ell| \sqrt{\| \vec{x}_\ell \|_2} \). Let \( \eta \in (0, 1/2] \) and define \( Q_\eta := Q_{\xi \sqrt{\| \vec{x} \|_2}, 1-\eta/2} \). By Lemma 3.1, on a \( \{ (y_\ell, \vec{x}_\ell) \}_{\ell=n+1}^{2n} \)-measurable event \( \mathcal{E}_3 \) of probability at most \( 1 - \exp(-\eta n/1.8) \),

\[
\sum_{\ell=n+1}^{2n} 1_{\{ |\xi_\ell| \sqrt{\| \vec{x}_\ell \|_2} \leq Q_\eta \}} > (1 - 0.75\eta)n.
\]

We now claim that on the event \( \mathcal{E}_3 \),

\[
\bar{Y}_{(1-\eta)n}^* \leq Q_\eta.
\]

Indeed, there are at least \( (1 - 0.75\eta)n \) points from the \( n \)-sized clean sample \( \{ Y_\ell \}_{\ell=n+1}^{2n} \) satisfying \( Y_\ell \leq Q_\eta + \eta \). Since \( \epsilon \leq 0.25\eta \) and there at most \( \epsilon n \) arbitrary outliers, \( \{ Y_\ell \}_{\ell=n+1}^{2n} \)
has at least \((1 - 0.75\eta - \epsilon)n \geq (1 - \eta)n\) data points satisfying \(\tilde{Y}_\ell \leq Q_\eta\). This implies the claim.

**STEP 3:** When pruning the second batch in Algorithm 1, the first batch is used only to compute \(\tilde{b}^{(0)}\) (the selected number of samples \(m = (1 - \eta)n\) is fixed). Using Steps 1-2, independence between \(\{(y_\ell, x_\ell)\}_{\ell=1}^n\) and \(\{(y_\ell, x_\ell)\}_{\ell=n+1}^{2n}\) and conditioning imply that on an event \(E\) of probability at least \(1 - e^{-K/5.4} - \exp(-\eta n/1.8)\), we have \(\tilde{Y}_m \leq Q_\eta\) and \(\|\tilde{b}^{(0)} - b^*\|_2 \leq 3r\).

In Step 3 we work on the event \(E\). For all \(\ell = n + 1, \ldots, 2n\),

\[
|\tilde{\xi}_\ell(\tilde{b}^{(0)})| \leq |\tilde{\xi}_\ell| + \|\tilde{x}_\ell\|_2\|\tilde{b}^{(0)} - b^*\|_2.
\]

Hence \(\tilde{R}_m = (|\tilde{\xi}_\ell(\tilde{b}^{(0)})| \vee \|\tilde{x}_\ell\|_2\|\tilde{b}^{(0)} - b^*\|_2) \leq Q_\eta + Q_\eta(3r)\).

We conclude the proof by summarizing the conclusions of Steps 1-3 as: on an event \(E_0\) of probability at least \(1 - e^{-K/5.4} - \exp(-\eta n/1.8)\), property \(\text{BD}(\eta, r)\) holds.

---

**Set-up 2.** We now divide the pruned data set \(\{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^m\) outputted in Algorithm 1 into \(K\) buckets of same size \(m/K = B\) indexed by the partition \(\bigcup_{i \in [K]} \tilde{B}_i = [m]\). Similarly, the unobserved data set \(\{(y_\ell, x_\ell)\}_{\ell=1}^m\) is split into \(K\) buckets of same size \(B\) indexed by the partition \(\bigcup_{i \in [K]} B_i = [m]\). For \(\eta \in (0, 1/2]\), we assume \(m = (1 - \eta)n\) is divisible by \(K\) without loss on generality.

Let \(b \in \mathbb{R}^p\). With some abuse of notation, the corresponding residuals will be denoted by \(\tilde{\xi}_\ell(b) := \tilde{y}_\ell - (\tilde{x}_\ell, b)\) and \(\xi_\ell(b) := y_\ell - (x_\ell, b)\). We also define

\[
\tilde{z}_i(b) := \frac{1}{B} \sum_{\ell \in \tilde{B}_i} \tilde{\xi}_\ell(b)\tilde{x}_\ell, \quad \text{and} \quad z_i(b) := \frac{1}{B} \sum_{\ell \in B_i} \xi_\ell(b)x_\ell.
\]

We now establish high-probability bounds satisfied by the pruned data set outputted by Algorithm 1.

**Lemma 4.3** (Pruned sample: Multiplier Process at \(b^*\)). Grant Set-up 2. Let \((\alpha_1, C_{\alpha_1})\) and \((\alpha_2, C_{\alpha_2})\) satisfying (3) respectively. Suppose that

\[
o \leq (\alpha_1 \vee \alpha_2)K.
\]
Then, setting \( r_1 \geq r_{\xi, n, K}(F_1(B_2)) \), on a \( \{(y_{\ell}, x_{\ell})\}_{\ell=1}^{m} \cup \{(y_{\ell}, x_{\ell})\}_{\ell=n+1}^{2n} \)-measurable event \( \mathcal{E}_1 \) of probability at least \( 1 - e^{-K/C_{\alpha_1}} - e^{-K/C_{\alpha_2}} \),

\[
\sup_{v \in B_2} \sum_{i \in [K]} 1\{|(z_i(b^*), v)| \geq C_{\alpha_1} r_1\} \leq 4 \alpha_1 K,
\]

(\( \text{MP1}(\alpha_1, r_1) \))

\[
\sup_{v \in B_2} \sum_{i \in [K]} 1\{|(z_i(b^*), v)| \geq C_{\alpha_2} r_1\} \leq 4 \alpha_2 K.
\]

(\( \text{MP2}(\alpha_2, r_1) \))

**Proof.** In the proof we will only use that Algorithm 1 removes \( \eta n \) data points from the second batch. In the following, let \( S \) be the index set of buckets without outliers in the pruned sample \( \{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^{m} \).

From Lemma 3.4, on an event \( \mathcal{E}_1' \) of probability at least \( 1 - e^{-K/C_{\alpha_1}} \),

\[
\sup_{v \in B_2} \sum_{i \in [K]} 1\{|(z_i(b^*), v)| \geq C_{\alpha_1} r_1\} \leq \alpha_1 K.
\]

Similarly, on an event \( \mathcal{E}_2' \) of probability at least \( 1 - e^{-K/C_{\alpha_2}} \),

\[
\sup_{v \in B_2} \sum_{i \in [K]} 1\{|(z_i(b^*), v)| \geq C_{\alpha_2} r_1\} \leq \alpha_2 K.
\]

We now work on the event \( \mathcal{E}_1' \cap \mathcal{E}_2' \) of probability at least \( 1 - e^{-K/C_{\alpha_1}} - e^{-K/C_{\alpha_2}} \).

By assumption \( m = (1 - \eta)n, |[m] \setminus S| \leq o \) and \( o \leq \alpha_1 K \). Thus,

\[
\sup_{v \in B_2} \sum_{i \in [K]} 1\{|(z_i(b^*), v)| \geq C_{\alpha_1} r_1\} \leq \sup_{v \in B_2} \sum_{i \in S} 1\{|(z_i(b^*), v)| \geq C_{\alpha_1} r_1\} + o
\]

\[
\leq \sup_{v \in B_2} \sum_{i \in [K]} 1\{|(z_i(b^*), v)| \geq C_{\alpha_1} r_1\} + o
\]

\[
\leq 2 \alpha_1 K = 2 \alpha_1 \frac{K}{(1 - \eta)} \leq 4 \alpha_1 K,
\]

implying \( \text{MP1}(\alpha_1, r_1) \). Using that \( o \leq \alpha_2 K \), very similar arguments as above imply \( \text{MP2}(\alpha_2, r_1) \).

The next lemma follows from very similar arguments of Lemma 4.3 but using the lower bound of Proposition 1 optimized at \( \theta \). We omit the proof.

**Lemma 4.4** (Pruned sample: Quadratic Process Lower bound). **Grant Assumption 1 and Set-up 2.** Let \( \alpha_3 \in (0, 1) \) and \( C_{\alpha_3} > 0 \) satisfying (3) and \( C > 0 \) be an absolute constant as in Lemma 7.4. Let \( r_{n, K} \) as in Proposition 1. Suppose that

\[
o \leq \alpha_3 K,
\]

\[
C_{\alpha_3} r_{n, K} + 2L^2 \sqrt{\frac{CC_{\alpha_3} p \log p}{n}} < \frac{1}{2}.
\]

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Then on a \( \{ (y_\ell, x_\ell) \}_{\ell=1}^n \cup \{ (y_\ell, x_\ell) \}_{\ell=n+1}^{2n} \) measurable event \( \mathcal{E}_2 \) of probability at least \( 1 - e^{-K/C_{\alpha_3}} \), given \( v \in \mathbb{B}_\Pi \), for at least \( (1 - 4\alpha_3)K \) of the buckets,

\[
\frac{1}{B} \sum_{\ell \in \tilde{B}_i} \langle \tilde{x}_\ell, v \rangle^2 \geq \frac{1}{2} \quad (\text{QPl}(\alpha_3))
\]

Finally, the next lemma follows from very similar arguments to Lemma 4.3, but using Corollary 1. We also omit the proof.

**Lemma 4.5** (Pruned sample: Product Process Upper bound). Grant Assumption 1 and Set-up 2. Let \( \rho \in (0, 1/2], \alpha_4 \in (0, 1) \) and constant \( C_{\alpha_4} > 0 \) satisfying (3). Suppose that

\[ o \leq \alpha_4 K. \]

Let \( C'_\rho := 1 + \sqrt{2/\rho} \) and \( C > 0 \) be an absolute constant in Lemma 7.4. Let \( r_{n,K} \) as in Proposition 1.

Then, setting

\[ r_2 := 2C_{\alpha_4}r_{n,K} + CC_{\alpha_4}C_{\rho} \frac{p \log p}{2n} + 2L^2 \sqrt{\frac{CC_{\alpha_4}p \log p}{2n}}, \]

on a \( \{ (y_\ell, x_\ell) \}_{\ell=1}^n \cup \{ (y_\ell, x_\ell) \}_{\ell=n+1}^{2n} \) measurable event \( \mathcal{E}_3 \) of probability at least \( 1 - e^{-\rho n/1.8} - e^{-K/C_{\alpha_4}} \), given \( [u, v] \in \mathbb{B}_\Pi \times \mathbb{B}_\Pi \), for at least \( 1 - (3\alpha_4 + 0.75\rho)K \) buckets,

\[
\frac{1}{B} \sum_{\ell \in \tilde{B}_i} \langle \tilde{x}_\ell, u \rangle \langle \tilde{x}_\ell, v \rangle - \langle u, v \rangle_{\Pi} \leq r_2. \quad (\text{PPu}(\alpha_4, \rho, r_2))
\]

### 5 Pre-algorithms & deterministic arguments

In all this section, we work within the Set-up 2 and on the event \( \mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \) where, for the pruned sample \( \{ (\tilde{y}_\ell, \tilde{x}_\ell) \}_{\ell \in [m]} \) and initialization \( \tilde{b}^{(0)} \), the boundedness property \( \text{BD}(\eta, r) \) and the uniform properties \( \text{MP1}(\alpha_1, r_1), \text{MP2}(\alpha_2, r_1), \text{QP}(\alpha_3) \) and \( \text{PP}(\alpha_4, \rho, r_2) \) all hold. The arguments in this section are purely deterministic.
5.1 The benchmark combinatorial problem

The negation of property $\text{MP1}(\alpha_1, r_1)$ leads to the following benchmark problem. Given data set $\tilde{Z} := \{\tilde{z}_i\}_{i=1}^{\mathcal{K}}$, $k \in [\mathcal{K}]$ and $R > 0$, denote by $P(\tilde{Z}, k, R)$ the problem

$$\begin{align*}
\text{maximize} \quad & \theta \\
\text{subject to} \quad & q_i |\langle \tilde{z}_i, v \rangle| \geq q_i \theta, \quad i = 0, \ldots \mathcal{K}, \\
& \sum_{i=1}^{\mathcal{K}} q_i > \mathcal{K} - k, \\
& q_i \in \{0, 1\}, \quad i = 0, \ldots \mathcal{K}.
\end{align*}$$

It turns out that the above problem can be solved approximately using the Multiplicative Update Algorithm together with a rounding algorithm [26, 28]. We highlight the following immediate fact.

**Fact.** $P(\tilde{Z}, k, R)$ is feasible iff there exists $(\theta, v) \in \mathbb{R} \times \mathbb{R} \mathbb{B}_2$ satisfying the property $\neg \text{P1}(\tilde{Z}, \theta, k)$ defined as:

$$\sum_{i=1}^{\mathcal{K}} 1_{\{\langle \tilde{z}_i, v \rangle > \theta\}} > \mathcal{K} - k. \quad (-\text{P1})$$

The geometrical interpretation of problem $P(\tilde{Z}, k, R)$ is of finding two symmetrical hyperplanes with orthogonal direction $v$ from the origin maximizing the margin across at least $\mathcal{K} - k$ data points. If we can approximately solve it, using a simple order statistics argument, one can obtain for some $k' > k$, a relaxed solution of the one-sided problem:

$$\begin{align*}
\text{maximize} \quad & \theta \\
\text{subject to} \quad & q_i \langle \tilde{z}_i, v \rangle \geq q_i \theta, \quad i = 0, \ldots \mathcal{K}, \\
& \sum_{i=1}^{\mathcal{K}} q_i > \mathcal{K} - k', \\
& q_i \in \{0, 1\}, \quad i = 0, \ldots \mathcal{K}.
\end{align*}$$

This problem identifies the hyperplane with positive angle across most data points. The above benchmark problem is the negation of property $\text{MP2}(\alpha_2, r_1)$ (for normalized length $R = 1$).
5.2 Multiplicative Weights Update & Spherical Rounding

The main purpose of this section is to state and analyze Algorithm 2 constructed to obtain an relaxed solution of $P(\tilde{Z}, k, R)$ given data set $\tilde{Z} := \{\tilde{z}_i\}_{i=1}^K, k \in [K]$ and $R > 0$. All we need to assume are that $P(\tilde{Z}, k, R)$ is feasible and we know a lower bound $\tilde{r}_1 > 0$ on the optimal margin.\(^1\) Precisely, we assume the optimal solution $(\theta_{\tilde{Z}, k, R}, v_{\tilde{Z}, k, R})$ satisfies property $\neg P1(\tilde{Z}, \theta_{\tilde{Z}, k, R}, v_{\tilde{Z}, k, R})$ and we know $\tilde{r}_1$ such that $\theta_{\tilde{Z}, k, R} \geq \tilde{r}_1$. Algorithm 2 then outputs a margin-direction pair $(\hat{\theta}, \hat{v}) \in \mathbb{R} \times R\mathbb{B}_2$ satisfying $\neg P1(\tilde{Z}, \hat{\theta}, \hat{v}, k')$ for some $k' > k$ and $\hat{\theta} \geq a\theta_{\tilde{Z}, k, R}$ for some $a \in (0, 1)$.

Different building blocks are needed to analyze Algorithm 2. We run MWU with cost associated with the spectrum of the data matrix with points $\{\tilde{z}_i\}_{i=1}^K$. Its output $M \in \mathcal{M}(R\mathbb{S}_2)$ is rounded into a direction $v \in R\mathbb{B}_2$ using a spherical distribution via Algorithm 3. In the same algorithm, a simple order statistics is used to obtain the margin-direction pair $(\hat{\theta}, \hat{v}) \in \mathbb{R} \times R\mathbb{B}_2$ where $\hat{v} \in \{v, -v\}$. For specific applications, either linear regression or mean estimation, we show in later sections that $\hat{v}$ satisfies “good descent” properties.

\(^1\) Later we show that we can adapt to this parameter.
Algorithm 2 \( \text{MW}(\mathcal{D}, U, S, k, k', R, \tilde{r}_1) \)

**Input:** data points \( \mathcal{D} := \{ \tilde{z}_i \}_{i=1}^{K} \), upper bound \( U > 0 \), simulation sample size \( S \in \mathbb{N} \), lower bound \( \tilde{r}_1 > 0 \), \( k, k' \in [K] \) and \( R > 0 \).

**Output:** margin-direction pair \( (\hat{\theta}, \hat{v}) \in \mathbb{R} \times R_{\mathbb{S}}^2 \).

\[ \begin{align*}
1: & \quad \text{Set } T := \left\lfloor \frac{40U(1-2k/K)}{\tilde{r}_1^2} \right\rfloor . \\
2: & \quad \text{Set } w^{(1)} := \frac{1}{K} 1_K . \\
3: & \quad \text{for } t \in [T] \text{ do} \\
4: & \quad \quad \text{Set matrix } Z^{(t)} \in \mathbb{R}^{[K] \times p} \text{ with } i\text{th row } Z^{(t)}_i := \sqrt{w^{(t)}_i} \tilde{z}_i^\top . \text{ Set } M^{(t)} := (Z^{(t)})^\top Z^{(t)}. \\
5: & \quad \quad \text{Solve } v^{(t)} := \arg\max_{v \in R_{\mathbb{S}}^2} v^\top M^{(t)} v . \\
6: & \quad \quad \text{For each } i \in [K], \text{ set } \tau^{(t)}_i := \langle \tilde{z}_i, v^{(t)} \rangle^2 . \\
7: & \quad \quad \text{Set } \hat{w}^{(t+1)}_i := w^{(t)}_i \left( 1 - \frac{1}{2R} \tau^{(t)}_i \right) \text{ for each } i \in [K] . \\
8: & \quad \quad \text{Normalize by setting } \hat{w}^{(t+1)}_i := \hat{w}^{(t+1)}_i / \sum_{i=1}^{K} \hat{w}^{(t+1)}_i \text{ for each } i \in [K] . \\
9: & \quad \quad \text{Compute the Kullback-Leibler projection } w^{(t+1)} := \arg\min_{w \in \Delta_{K, K-2k}} \text{KL}(w \| \hat{w}^{(t+1)}). \\
10: & \quad \quad \text{end for} \\
11: & \quad \text{Set } M := \frac{1}{T} \sum_{t=1}^{T} v^{(t)} (v^{(t)})^\top . \\
12: & \quad (\hat{\theta}, \hat{v}) \leftarrow \text{ROUND}(\mathcal{D}, M, S, k, k'). \\
13: & \quad \text{return } (\hat{\theta}, \hat{v}).
\]
Algorithm 3 \textsc{Round}(\mathcal{D}, M, S, k, k')

\textbf{Input:} data points \(\mathcal{D} := \{\tilde{z}_i\}_{i=1}^{K}\), symmetric matrix \(M \in \mathbb{R}^{p \times p}\), simulation sample size \(S \in \mathbb{N}\), \(k, k' \in [K]\).

\textbf{Output:} margin-direction pair \((\hat{\theta}, \hat{v})\).

1: Compute square-root \(M^{1/2}\).
2: Sample independently \(\{\theta_\ell\}_{\ell=1}^{S}\) from the uniform distribution over the unit sphere \(S_2\) on \(\mathbb{R}^p\).
3: Set \(v_\ell := M^{1/2} \theta_\ell\) and \(Z_i(\ell) := |\langle \tilde{z}_i, v_\ell \rangle|\) for all \(\ell \in [S]\) and \(i \in [K]\).
4: For each \(\ell \in [S]\) compute the order statistics: \(Z_1^{\ell}(\ell) \geq \ldots \geq Z_K^{\ell}(\ell)\).
5: Set \(\ell_* \in \arg\max_{\ell \in [S]} Z_{K-k'}^{\ell}(\ell), v := v_{\ell_*}\) and \(\hat{\theta} := Z_{K-k'}^{\ell_*}(\ell_*)\).
6: Compute the order statistics \(W_1^z \geq \ldots \geq W_K^z\) of the sequence \(W_i := \langle \tilde{z}_i, v \rangle\) for \(i \in [K]\).
7: \textbf{if} \(W_{K-k-k'}^z > \hat{\theta}\) \textbf{then}
8: \hspace{1em} Set \(\hat{v} := v\).
9: \textbf{else}
10: \hspace{1em} Set \(\hat{v} := -v\).
11: \textbf{end if}
12: \textbf{return} \((\hat{\theta}, \hat{v})\).

We recall a online regret bound for the multiplicative weight algorithm with restricted distributions [1, 26]. We stated for the particular case of Algorithm 2.

\textbf{Lemma 5.1 (Theorem 2.4 in [1]).} Set \(w^{(\ell)} := [w_1^{(\ell)} \cdots w_K^{(\ell)}]^\top\) and \(\tau^{(\ell)} := [\tau_1^{(\ell)} \cdots \tau_K^{(\ell)}]^\top\). Suppose that

\[
\sup_{t \geq 1} \sup_{\ell \in [t]} \sup_{i \in [K]} \tau_i^{(\ell)} \leq U.
\]

Then, for all \(t \geq 1\) and for all \(w \in \Delta_{K, K-2k}\),

\[
\frac{1}{t} \sum_{\ell = 1}^{t} \langle w^{(\ell)}, \tau^{(\ell)} \rangle \leq \frac{1.5}{t} \sum_{\ell = 1}^{t} \langle w, \tau^{(\ell)} \rangle + \frac{2U}{t} KL(w \| w^{(1)}) + 2U.
\]

\textbf{Lemma 5.2.} Suppose that

(i) \(\max_{i \in [K]} \|\tilde{z}_i\|_2 \leq U\) for some \(U > 0\),
(ii) \(P(\tilde{Z}, k, R)\) is feasible with optimal solution \((\theta_{\tilde{Z},k,R}, v_{\tilde{Z},k,R})\).
(iii) \(\theta_{\tilde{Z},k,R} \geq \tilde{r}_1\), for some \(\tilde{r}_1 > 0\).
Instantiate Algorithm 2 with inputs $\mathcal{D} = \tilde{\mathcal{Z}}$ and $U = \max_{i \in [\mathcal{K}]} \|\tilde{z}_i\|_2^2$.

Then, for $D := \theta_{\tilde{\mathcal{Z}},k,R}^2/6$, Algorithm 2 produces matrix $M \in \mathcal{M}(R\mathcal{S}_2)$ satisfying the quantile property $-P1'(\tilde{\mathcal{Z}}, D, M, 2k)$ defined by

$$\sum_{i=1}^{\mathcal{K}} 1\{\langle \tilde{z}_i^\top, M \rangle > D \} > \mathcal{K} - 2k.$$  

($-P1'$)

Proof. By (i) $\tau^{(t)}_i \leq U < \infty$ for all $i, \ell$. Thus, we may apply recursion (5) in Lemma 5.1. Let us denote locally $(\theta, v) := (\theta_{\tilde{\mathcal{Z}},k,R}, v_{\tilde{\mathcal{Z}},k,R})$. By item (iii), $\theta \geq \tilde{\tau}_1 > 0$.

**LOWER BOUND:** Define the index set $I := \{i \in [\mathcal{K}] : |\langle \tilde{z}_i, v \rangle | > \theta \}$. Item (ii) ensures that $|I| \geq \mathcal{K} - k$ and, since $w^{(t)} \in \Delta_{\mathcal{K},\mathcal{K}-2k}$ for all $\ell \in [t]$, we have $\sum_{i \in I} w^{(t)}_i \geq 1 - \frac{k}{2\mathcal{K}} = 0.5$. We conclude that, for any $t \geq 1$,

$$\frac{1}{t} \sum_{\ell=1}^{t} \langle w^{(t)}, \tau^{(t)} \rangle = \frac{1}{t} \sum_{\ell=1}^{t} \sum_{i=1}^{\mathcal{K}} w^{(t)}_i \langle \tilde{z}_i, v^{(t)} \rangle^2 \geq \frac{1}{t} \sum_{\ell=1}^{t} \sum_{i=1}^{\mathcal{K}} w^{(t)}_i \langle \tilde{z}_i, v \rangle^2 \geq \frac{1}{t} \sum_{\ell=1}^{t} \sum_{i \in I} w^{(t)}_i \langle \tilde{z}_i, v \rangle^2 \geq 0.5\theta^2,$$  

(6)

where the first inequality uses that $v^{(t)} \in \arg\max_{u \in \mathcal{R}_{[t]}} u^\top M^{(t)} u$.

**UPPER BOUND:** We show the following claim: for any $t \geq 1$, there exists $\mathcal{I}_t \subset [\mathcal{K}]$ of size $|\mathcal{I}_t| \geq \mathcal{K} - 2k$ such that for any $i \in \mathcal{I}_t$, there exists $w := w[t, i] \in \Delta_{\mathcal{K},\mathcal{K}-2k}$ such that

$$\sum_{\ell=1}^{t} \langle w, \tau^{(t)} \rangle \leq \sum_{\ell=1}^{t} \tau^{(t)}_i = \sum_{\ell=1}^{t} \langle \tilde{z}_i, v^{(t)} \rangle^2.$$  

(7)

Indeed, fix $t \geq 1$ and set $\alpha_j^{(t)} := \sum_{\ell=1}^{t} \tau^{(t)}_j$ for all $j \in [\mathcal{K}]$. Let $\mathcal{I}_t := \{i \in [\mathcal{K}] : \alpha_i^{(t)} \geq (\alpha_{\mathcal{K}-2k}^{(t)})^2 \}$. By construction, $|\mathcal{I}_t| = \mathcal{K} - 2k$. Fix $i \in \mathcal{I}_t$ and let $w$ be the uniform distribution over $J_i := \{j \in [\mathcal{K}] : \alpha_j^{(t)} \leq \alpha_i^{(t)} \}$. Since $i \in \mathcal{I}_t$ we have that $|J_i| \geq 2k$ and hence $w \in \Delta_{\mathcal{K},\mathcal{K}-2k}$. Finally, by construction,

$$\sum_{\ell=1}^{t} \langle w, \tau^{(t)} \rangle = \sum_{j=1}^{\mathcal{K}} w_j \alpha_j^{(t)} = \sum_{j=1}^{\mathcal{K}} w_j (\alpha_j^{(t)})^2 \leq \sum_{j \in J_i} w_j \alpha_j^{(t)} = \alpha_i^{(t)} = \sum_{\ell=1}^{t} \tau^{(t)}_i,$$

implying claim (7).
Given \( t \geq 1 \), let \( \mathcal{I}_t \) as given by claim (7). From Lemma 7.1 in the Appendix, for any \( i \in \mathcal{I}_t \) and some \( w[t, i] \in \Delta_{\mathcal{K}, \mathcal{K}-2k} \) as in claim (7) we have
\[
\frac{2U}{t} \text{KL}(w[t, i] \| w^{(1)}) \leq \frac{10U(1-2k/\mathcal{K})}{t}.
\] (8)

RAPPING UP: Joining the bounds in (6), (7), (8) with the regret bound (5), we conclude that: for all \( t \geq 1 \) and all \( i \in \mathcal{I}_t \),
\[
\frac{\theta^2}{3} - \frac{20U(1-2k/\mathcal{K})}{3t} \leq \frac{1}{t} \sum_{\ell=1}^{t} (\tilde{z}_i, \mathbf{v}(\ell))^2 = \frac{\alpha_i(t)}{t}.
\]

Recall that \( \theta \geq \tilde{r}_1 \). Thus, the LHS of the previous displayed inequality is at least \( \theta^2/6 \) after \( T := \left\lceil \frac{40U(1-2k/\mathcal{K})}{\tilde{r}_1^2} \right\rceil \) iterations. Hence, for all \( i \in \mathcal{I}_T \),
\[
\frac{\theta^2}{6} \leq \frac{\alpha_i(T)}{T} = \frac{1}{T} \sum_{\ell=1}^{T} (\tilde{z}_i, \mathbf{v}(\ell))^2 = \langle \tilde{z}_i \tilde{z}_i^\top, M \rangle,
\]

where \( M := \frac{1}{T} \sum_{\ell=1}^{T} \mathbf{v}(\ell)(\mathbf{v}(\ell))^\top \). As \( |\mathcal{I}_T| = \mathcal{K} - 2k \) and \( M \in \mathcal{M}(\mathbb{R}\mathbb{S}_2) \), the claim is proved.
\[\square\]

Lemma 5.3 (Random spherical rounding: boosted confidence). Grant assumptions in Lemma 5.2.

Let \( \varphi \in (0, \pi/2) \) and \( k' \in [\mathcal{K}] \) such that \( p \in (0, 1) \) where
\[
p := \frac{2\varphi}{\pi} (1 - 2(k/\mathcal{K})) - (1 - (k'/\mathcal{K})).
\]

Then with probability (on the randomness of \( \{\theta_\ell\}_{\ell \in [\mathcal{S}]} \)) of at least \( 1 - e^{-\frac{K^2}{4p}} \), Algorithm 3 (inputted in Algorithm 2) produces margin \( \hat{\theta} \geq (\cos \varphi)\theta_{Z_k, R}/\sqrt{6} \) and direction \( \mathbf{v} \in \mathbb{R}\mathbb{B}_2 \) satisfying property \( -\text{P1}(\tilde{Z}, \hat{\theta}, \mathbf{v}, k') \), that is,
\[
\sum_{i=1}^{\mathcal{K}} \mathbf{1}_{\{|(\tilde{z}_i, \mathbf{v})| > \hat{\theta}\}} > \mathcal{K} - k'.
\]

Proof. We use the local notation \( \theta := (\cos \varphi)\theta_{Z_k, R}/\sqrt{6} \). Let \( \theta \) be an random variable with the uniform distribution over \( \mathbb{S}_2 \) and \( \mathbf{v}_\theta := M^{1/2} \theta \). Define \( Z_i := |\langle \tilde{z}_i, \mathbf{v}_\theta \rangle| \) and \( Z := Z_{k-k'}^2 \). Recall the notations \( Z_i(\ell) := |\langle \tilde{z}_i, \mathbf{v}(\ell) \rangle| \) and \( Z(\ell) := Z_{K-k'}^2(\ell) \) for \( \ell \in [\mathcal{S}] \) in Algorithm 3.
By the one-sided Bernstein’s inequality, we have that, for all \( t \geq 0 \), with probability at least \( 1 - e^{-t} \),
\[
\frac{1}{S} \sum_{\ell=1}^{S} 1_{\{Z(\ell) > \theta\}} \geq \mathbb{P}(Z > \theta) - \sigma \sqrt{\frac{2t}{S}} - \frac{t}{3S},
\]
where \( \sigma^2 := \mathbb{E}(\mathbf{1}_{\{Z > \theta\}} - \mathbb{P}(Z > \theta))^2 \leq \mathbb{P}(Z > \theta) \).
As \( Z = Z^k_{k'-k'} \),
\[
\mathbb{P}(Z > \theta) = \mathbb{P} \left( \sum_{i=1}^{K} \mathbf{1}_{\{|\langle \tilde{z}_i, v_{\theta}\rangle| > \theta\}} > K - k' \right) \geq \frac{p^2}{2},
\]
In the last inequality, we used Proposition 3 and property \( -\mathbf{P}1'(\tilde{Z}, D, M, 2k) \) with \( D := \theta_2^2_{\tilde{Z}, k,R}/6 \) ensured by Lemma 5.2.

Setting \( t := c^2 p^2 S \) we thus conclude that with probability at least \( 1 - e^{-c^2 p^2 S} \),
\[
\max_{\ell \in [S]} 1_{\{Z(\ell) > \theta\}} \geq p^2 \left( 1 - c\sqrt{2p^2 - \frac{c^2}{3}} \right) \geq p^2 \left( 1 - c\sqrt{2} - \frac{c^2}{3} \right) \geq p^2/2,
\]
for small enough \( c \in (0, 1) \). The choice \( c = 0.36 \) suffices. The rest of the proof will happen in this event.

By the previous display, one has \( \hat{\theta} := Z^{\alpha}_{k' - k'}(\ell_*) = Z(\ell_*) > \theta \) where \( \ell_* \in \text{argmax}_{\ell \in [S]} Z(\ell) \).
This shows that \((\hat{\theta}, v) = (\hat{\theta}, v_{\ell_*})\), as returned by Algorithm 3 (inputted in Algorithm 2), satisfies the claim of the lemma. \( \square \)

5.3 Solving the outer loop combinatorial problem

We next set \( \tilde{Z} := \{ \tilde{z}_i(b) \}_{i=1}^{K} \) for some \( b \neq b^* \). For reasons to be made clearer later, we will tune our algorithm with \( R := \mu^2(\mathbb{E}_2) \) and omit the dependence on \( R \) for convenience.

The next lemma formalizes the fact that under the structural conditions \( \text{MP1}(\alpha_1, r_1) \) and \( \text{MP2}(\alpha_2, r_1) \), problem \( P(\{ \tilde{z}_i(b) \}_{i=1}^{K}, k) \) is feasible when \( b \neq b^* \) for small enough \( k \).

**Lemma 5.4** (Two-sided feasibility & margin-distance lower bound). Suppose that

(i) \( \text{MP1}(\alpha_1, r_1) \) holds for some \( \alpha_1 \in (0, 1) \). Let \( k := 4\alpha_1 K \).

(ii) \( \text{QPI}(\alpha_3) \) holds for some \( \alpha_3 \in (0, 1) \).

Let \( b \neq b^* \) satisfying:

Let \( b \neq b^* \) satisfying:
(iv) For some \((\theta, v, k') \in \mathbb{R} \times \mu^2(\mathbb{B}) \mathbb{S} \times \mathbb{K}\), property \(\neg P_1(\{\tilde{z}_i(b)\}_{i=1}^\mathbb{K}, \theta, v, k')\) holds, i.e.,

\[
\sum_{i=1}^{\mathbb{K}} 1\{||\tilde{z}_i(b)|| > \theta\} > \mathbb{K} - k'.
\]

In particular, \(P(\{\tilde{z}_i(b)\}_{i=1}^m, k')\) is feasible with optimal value, say, \(\bar{\theta}\).

(v) \(4(\alpha_1 + \alpha_3)\mathbb{K} < \mathbb{K}\).

(vi) For some \(a \in (0, 1]\), \(\theta \geq a\bar{\theta}\).

Then

(a) \(a [(1/2)||b - b^*||_2 - C_{\alpha_1} \mu^2(\mathbb{B}) r_1] \leq \theta\).

In particular, for any \(b \neq b^*\), \(P(\{\tilde{z}_i(b)\}_{i=1}^{\mathbb{K}}, k)\) is feasible and, its optimal solution, denoted as \((\theta_{b,k}, v_{b,k})\), satisfies \(\neg P_1(\{\tilde{z}_i(b)\}_{i=1}^{\mathbb{K}}, \theta_{b,k}, v_{b,k}, k)\) with margin satisfying

\[
\theta_{b,k} \geq -C_{\alpha_1} \mu^2(\mathbb{B}) r_1 + (1/2)||b^* - b||_2.
\]

Proof. For simplicity we give a proof for \((\theta_{b,k}, v_{b,k})\). The proof is the same for any \((\theta, v, k')\) satisfying conditions (iv)-(vi).

STEP 1: An upper bound on the optimal value is trivial: for any \((\theta, v, q)\) satisfying the constraints of \(P(\{\tilde{z}_i(b)\}_{i=1}^{\mathbb{K}}, k)\) it follows from Cauchy-Schwarz that \(\theta_{b,k} \leq \max_{i \in \mathbb{K}} ||\tilde{z}_i(b)||_2 < \infty\).

STEP 2: setting \(r_1 := r_{\xi,n,K}(F_1(\mathbb{B}))\), we now prove the lower bound \(\theta_{b,k} \geq (1/2)||b - b^*||_2 - C_{\alpha_1} \mu^2(\mathbb{B}) r_1\). MP1(\(\alpha_1, r_1\)) applied to the vector \(v := \mu^2(\mathbb{B}) (b^* - b)/||b^* - b||_2\) implies that \(\langle \tilde{z}_i(b^*), v \rangle \geq -C_{\alpha_1} \mu^2(\mathbb{B}) r_1\) for more than \(\mathbb{K} - k\) buckets \(i\)'s, for which

\[
||\tilde{z}_i(b), v\rangle\| \geq \langle \tilde{z}_i(b), v \rangle = \langle \tilde{z}_i(b^*), v \rangle + \frac{1}{B} \sum_{\ell \in B_i} \langle \tilde{x}_\ell, b^* - b \rangle \langle \tilde{x}_\ell, v \rangle
\]

\[
\geq -C_{\alpha_1} \mu^2(\mathbb{B}) r_1 + \frac{1}{\mu^2(\mathbb{B})} ||b^* - b||_2 \frac{1}{B} \sum_{\ell \in B_i} \langle \tilde{x}_\ell, v \rangle^2.
\]

QPI(\(\alpha_3\)) implies that

\[
\frac{1}{B} \sum_{\ell \in B_i} \langle \tilde{x}_\ell, v \rangle^2 \geq \frac{1}{2} \cdot \frac{\mu^4(\mathbb{B}) ||b^* - b||^2_1}{||b^* - b||_2^2}.
\]

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for more than \((1 - 4\alpha_3)K\) buckets \(i\)'s. We thus conclude that for more than \((1 - \alpha_3)K - k \geq 1\) buckets,

\[
|\langle \tilde{z}_i(b), v \rangle| \geq -C_{\alpha_1}\mu^2(\mathbb{B}_2)r_1 + (\mu^2(\mathbb{B}_2)/2)\frac{\|b^* - b\|^2_1}{\|b^* - b\|_2} \geq -C_{\alpha_1}\mu^2(\mathbb{B}_2)r_1 + (1/2)\|b^* - b\|_2.
\]

In other words, the feasible set of \(P(\{\tilde{z}_i(b)\}_{i=1}^K, k)\) contains the point \((\theta, v, q) \in \mathbb{R} \times \mathbb{R} \times \{0, 1\}^K\) with \(\theta := -C_{\alpha_1}\mu^2(\mathbb{B}_2)r_1 + (1/2)\|b^* - b\|_2\) for some \(q \in \{0, 1\}^K\). By maximality, one must have \(\theta_{b,k} \geq -C_{\alpha_1}\mu^2(\mathbb{B}_2)r_1 + (1/2)\|b^* - b\|_2\). \(\square\)

### 5.4 Computing the outer loop descent direction w.r.t. \(\langle \cdot, \cdot \rangle_\Pi\)

**Lemma 5.5** (One-sided feasibility & margin-angle upper bound). Grant assumptions of Lemma 5.4 and additionally assume:

(iii) \(PPu(\alpha_4, \rho, r_2)\) holds for some \(\alpha_4 \in (0, 1)\) and \(\rho \in (0, 1/2]\).

(vii) \(r_2 \leq \frac{a}{4C_{\alpha_4}\mu^2(\mathbb{B}_2)}\|\Sigma\|\).

(viii) \(\|b - b^*\|_2 \geq 4 \left(\frac{a+1}{a}\right) C_{\alpha_1}\mu^2(\mathbb{B}_2)r_1\).

Then

(b) Let \(k'' := (2\alpha_4 + 0.75\rho)K + k + k'\). There exists \(v' \in \{-v, v\}\) such that property 
\(-MP2(\{\tilde{z}_i(b)\}_{i=1}^K, \theta, v', k'')\), defined below, holds:

\[
\sum_{i=1}^K 1\{\langle z_i(b), v' \rangle > \theta\} > K - k''.
\]

Additionally to the assumptions of Lemma 5.4 and (iii),(vii)-(viii), assume:

(ix) \(MP2(\alpha_2, r_1)\) holds for some \(\alpha_2 \in (0, 1)\). Let \(k_0 := 4\alpha_2K\).

Then, for any \(b \neq b^*\) satisfying (iv)-(viii) and

(x) \(k'' \leq K/3\) and \(k_0 \leq K/3\),

one also has

(c) \(\theta \leq C_{\alpha_2}\mu^2(\mathbb{B}_2)r_1 + \langle b^* - b, v' \rangle_\Pi + C_{\alpha_1}\mu^2(\mathbb{B}_2)\|\Sigma\|_2r_2\|b^* - b\|_2\).

Suppose additionally that, instead of (viii), one has
(viii') $\|b - b^*\|_2 \geq A\mu^2(\mathbb{B}_2)r_1$ where

$$A := \left[4 \left(\frac{a+1}{a}\right)C_{\alpha_1}\right] \sqrt{\left[(\gamma/a)(aC_{\alpha_1} + C_{\alpha_2})\right]}.$$ 

Then one also has

(d) $\langle v', b - b^* \rangle_{\Pi} \leq -\frac{a}{8}\|b - b^*\|_2.$

**Remark 3** (The need for a margin-angle upper bound). The concept of distance-estimate was shown to be the sufficient property in robust mean estimation when using the benchmark combinatorial problem in Section 5.1 [7, 28]. As it will be clearer in the following, we emphasize that, in our analysis based on least-squares methodology with unknown $\Sigma$, a margin-distance upper bound is not enough to obtain the optimal rate for robust linear regression. In order to obtain the optimal rate and breakdown point with respect to the condition number $\kappa = \mu^2(\mathbb{B}_2)\|\Sigma\|$, the margin-angle upper bound in item (c) of Lemma 5.5 is crucially needed. Notice that such margin-angle upper bound follows from the one-sided benchmark problem (item (b) above). Differently, the margin-distance lower bound in item (a) follows from the two-sided benchmark problem.

**Proof.** Proof of (b): by $PPu(\alpha, \rho, r_2)$,

$$\frac{1}{B} \sum_{\ell \in \tilde{B}_i} \langle \tilde{x}_\ell, b - b^* \rangle \langle \tilde{x}_\ell, v \rangle - \langle b - b^*, v \rangle_{\Pi} \leq C_{\alpha_4} r_2 \|b - b^*\|_{\Pi} \|v\|_{\Pi},$$

for more than $(1 - (2\alpha_4 + 0.75\rho))\mathcal{K}$ buckets $i$'s. Let $S$ denote such index set and define:

- $G_{b^*} := \{i \in [\mathcal{K}] : |\langle \tilde{z}_i(b^*), v \rangle| \leq C_{\alpha_1}\mu^2(\mathbb{B}_2)r_1\}$,
- $B_{b}^+ := \{i \in [\mathcal{K}] : \langle \tilde{z}_i(b), v \rangle > \theta\},$
- $B_{b}^- := \{i \in [\mathcal{K}] : \langle \tilde{z}_i(b), -v \rangle > \theta\}.$

We consider two cases.

**Case 1:** $\langle b - b^*, v \rangle_{\Pi} \leq 0$. Given $i \in B_{b}^- \cap S$,

$$\langle \tilde{z}_i(b^*), v \rangle = \langle \tilde{z}_i(b), v \rangle + \frac{1}{B} \sum_{\ell \in \tilde{B}_i} \langle \tilde{x}_\ell, b - b^* \rangle \langle \tilde{x}_\ell, v \rangle$$

$$\leq -\theta + C_{\alpha_4} r_2 \|b - b^*\|_{\Pi} \|v\|_{\Pi}$$

$$\overset{(a)}{<} aC_{\alpha_1}\mu^2(\mathbb{B}_2)r_1 + \left(\frac{-a}{2} + C_{\alpha_4}\|\Sigma\|_2\mu^2(\mathbb{B}_2)r_2\right) \|b - b^*\|_2$$

$$\overset{(vii),(viii)}{<} -C_{\alpha_1}\mu^2(\mathbb{B}_2)r_1,$$
where we used from (viii) that $\|b - b^*\|_2 \geq 4^{(s+1/3)}C_\alpha\mu^2(\mathbb{B}_2)r_1$.

We thus concluded that $B_b^+ \cap S \subseteq G_b^*$. This and the facts

- $|S| \geq [1 - (2\alpha_4 + 0.75\rho)]\mathcal{K}$ by $\text{PPu}(\alpha_4, \rho, r_2)$,
- $|G_b^*| > \mathcal{K} - k$ by $\text{MP1}(\alpha_1, r_1)$,
- $B_b^+$ and $B_b^-$ are disjoint because $\theta > 0$, by (a) and (viii). Also, $|B_b^+| + |B_b^-| > \mathcal{K} - k'$ by (iv),

imply that $|B_b^+| > \mathcal{K} - (2\alpha_4 + 0.75\rho)\mathcal{K} - k - k'$.

**Case 2:** $\langle b - b^*, v \rangle_{\Pi} > 0$. By exchanging $v$ with $-v$ and $B_b^-$ by $B_b^+$ a similar argument shows that $|B_b^-| > \mathcal{K} - (2\alpha_4 + 0.75\rho)\mathcal{K} - k - k'$.

**Proof of (c):** By (b) and (x), one has $\langle \tilde{z}_i(b), v' \rangle \leq \theta$ for less $\mathcal{K}/3$ of buckets $i$'s. By $\text{MP2}(\alpha_2, r_1)$ and (x), one has $\langle \tilde{z}_i(b^*), v' \rangle > C_{\alpha_2}\mu^2(\mathbb{B}_2)r_1$ for less $\mathcal{K}/3$ of buckets $i$'s. Finally, by $\text{PPu}(\alpha_4, \rho, r_2)$ and (x), for less than $\mathcal{K}/3$ of buckets $i$'s one has

$$\frac{1}{B} \sum_{\ell \in B_i} \langle \tilde{x}_\ell, b^* - b \rangle \langle \tilde{x}_\ell, v' \rangle \leq \langle b^* - b, v' \rangle_{\Pi} + C_{\alpha_4}r_2\|b - b^*\|_{\Pi}\|v'\|_{\Pi}.$$ 

By the pigeonhole principle, there is at least one bucket $i$ for which $\langle \tilde{z}_i(b^*), v' \rangle > \theta$, $\langle \tilde{z}_i(b^*), v' \rangle \leq C_{\alpha_2}\mu^2(\mathbb{B}_2)r_1$ and the previous display all hold. Thus

$$\theta < \langle \tilde{z}_i(b), v' \rangle$$

$$= \langle \tilde{z}_i(b^*), v' \rangle + \frac{1}{B} \sum_{\ell \in B_i} \langle \tilde{x}_\ell, b^* - b \rangle \langle \tilde{x}_\ell, v' \rangle$$

$$\leq C_{\alpha_2}\mu^2(\mathbb{B}_2)r_1 + \langle b^* - b, v' \rangle_{\Pi} + C_{\alpha_4}\mu^2(\mathbb{B}_2)\|\Sigma\|r_2\|b^* - b\|_2,$$

entailing the claim.

**Proof of (d):** we join the upper bound (c) and the lower bound (a). Using (vii) and (viii'), so that $\|b - b^*\|_2 \geq (8/a)(aC_{\alpha_1} + C_{\alpha_2})\mu^2(\mathbb{B}_2)r_1$, and rearranging the displayed inequality finishes the proof.

From now on fix the parameters $\alpha_1 = 1/96$, $\alpha_2 = 0.08$, $\alpha_3 = 0.239$, $\alpha_4 = 1/144$, $c_{\alpha_1} := \frac{1}{3}$, $\rho = 1/36$, $a := 0.0128$ and $\varphi := 0.49\pi$. In order to satisfy (3), it suffices to take $C_{\alpha_1} = 2525.26$, $C_{\alpha_2} = 192.4$, $C_{\alpha_3} = 51.9$ and $C_{\alpha_4} = 4330$.

**Corollary 2** (Good descent properties). Let $\eta \in (0, 1/2]$ and suppose that:

(i) $\text{BD}(\eta, r)$, $\text{MP1}(\alpha_1, r_1)$, $\text{QP}(\alpha_3)$, $\text{PPu}(\alpha_4, \rho, r_2)$ and $\text{MP2}(\alpha_2, r_1)$ all hold.
(ii) Assume \( r_2 \leq \frac{1}{312.5 C_{\alpha_1} \mu^2(\mathbb{B}_2)\|\Sigma\|} \).

(iii) Given \( b \in \mathbb{R}^p \), assume \( \|b - b^*\|_2 \geq 8 \cdot 10^5 \mu^2(\mathbb{B}_2) r_1 \).

Let \((\hat{\theta}, \hat{v})\) be the output of \( \mathcal{A}(\mathcal{D}, U, S_1, k, k', R, \tilde{r}_1) \), namely, Algorithm 2 with inputs \( \mathcal{D} = \{\tilde{z}_i(b)\}_{i \in [K]} \) and \( U = \max_{i \in [K]} \|\tilde{z}_i(b)\|_2^2 \), \( k = 4\alpha_1 K \), \( k' = c_\alpha K \) (with \( c_\alpha = 1/4 \)), \( \tilde{r}_1 = C_{\alpha_1} \mu^2(\mathbb{B}_2) r_1 \) and \( R = \mu^2(\mathbb{B}_2) \).

Then on an event of probability (on the randomness of \( \{\theta_i\}_{i \in [S_1]} \)) of at least \( 1 - e^{-\frac{s_1}{353}} \), one has

\[
\sum_{i=1}^{K} \mathbb{1}_{\{\langle \tilde{z}_i(b), v' \rangle > \hat{\theta} \}} > K - k' \quad \text{and} \quad \hat{\theta} > a \theta_{b,k},
\]

where \( k = 4\alpha_1 K \) and \( k' = c_\alpha K \). Moreover,

\[
\begin{align*}
\langle b - b^* \rangle \leq 2525.26 \mu^2(\mathbb{B}_2) r_1 \leq \hat{\theta},
\end{align*}
\]

\[
\hat{\theta} \leq \langle b^* - b, \hat{v} \rangle + 4330 \mu^2(\mathbb{B}_2) \|\Sigma\| r_2 \|b - b^*\|_2 + 192.4 \mu^2(\mathbb{B}_2) r_1,
\]

\[
\langle \hat{v}, b - b^* \rangle \leq -\frac{1}{625} \|b - b^*\|_2.
\]

**Proof.** Setting \( k = 4\alpha_1 K = K/24 \) and \( k' := c_\alpha K = K/4 \) with the parameters displayed before the corollary, one checks that \( a \leq \cos(\phi)/\sqrt{6} \), \( p \geq 0.148 \) and all conditions of Lemmas 5.2, 5.3, 5.4 and 5.5 hold. In particular, \( \|b - b^*\|_2 \geq 4\mu^2(\mathbb{B}_2) C_{\alpha_1} r_1 \) implying condition (iii) of Lemma 5.2 with \( \tilde{r}_1 = C_{\alpha_1} \mu^2(\mathbb{B}_2) r_1 \).

We now work on the event of probability \( 1 - e^{-\frac{s_1}{353}} \) for which the claim of Lemma 5.3 is true. By Lemma 5.3, (9) is satisfied; these are assumptions (iv) and (vi) of Lemma 5.4 for \((\hat{\theta}, v)\). All other assumptions of such lemma hold, yielding (10).

All conditions of Lemma 5.4 and conditions (iii), (vii)-(viii), (viii’) of Lemma 5.5 hold so there must exist \( v' \in \{-v, v\} \) satisfying \( \sum_{i=1}^{K} \mathbb{1}_{\{\langle \tilde{z}_i(b), v' \rangle > \hat{\theta} \}} > K - k'' \), that is, (b) of such lemma. By this property, the order statistics in Algorithm 3 implies \( \hat{v} = v' \). All additional conditions of Lemma 5.5 hold, yielding (11)-(12).

We finalize this section showing we have a sufficiently small stepsize and descent direction, assuming one has an sufficiently good estimate of \( \Sigma \hat{v} \). We will show in the next section how to construct it (without knowing or estimating the covariance matrix \( \Sigma \)).

We first complement Lemma 5.5 and Corollary 2 with additional results. Like Lemma 5.4 and unlike Lemma 5.5 and Corollary 2, the next two results do not to assume that \( \mu^2(\mathbb{B}_2) r_1 \lesssim \|b - b^*\|_2 \) nor \( \hat{\theta} > 0 \). Lemma 5.7 does assume, however, (d”) which is
stronger than (d). Also, the stepsize in Lemma 5.5 is \( \hat{\theta} + C_{\alpha_1} \mu^2(\mathbb{B}_2)r_1 \) instead of \( \hat{\theta} \). These slightly more general results are only used in case one of the iterates follows within the statistical error before the final iteration. We need them to avoid Cauchy-Schwarz when upper bounding \( \langle \hat{\theta}, \mathbf{b}^* - \mathbf{b} \rangle_{\Pi} \). Hence, we can attain the optimal rate and breakdown point with respect to the condition number \( \kappa \). See proof of Theorem 6.1 in Section 6. We omit the proof of Lemma 5.6 as it is very similar to the proof of (c) in Lemma 5.5.

**Lemma 5.6 (Looser margin-angle upper bound).** Grant items (i) and (iv) of Lemma 5.4, item (iii) of Lemma 5.5 and additionally assume:

\( (x') \quad k'' \leq K/3 \) for \( k'' := (2\alpha_4 + 0.75\rho)K + k + k' \).

Then

\( (c') \quad \theta \leq C_{\alpha_2} \mu^2(\mathbb{B}_2)r_1 + \|\langle \mathbf{b}^* - \mathbf{b}, \mathbf{v} \rangle_{\Pi} + C_{\alpha_4} \mu^2(\mathbb{B}_2)\|\mathbf{\Sigma}\|r_2\|\mathbf{b}^* - \mathbf{b}\|_2 \). \]

**Lemma 5.7 (Descent direction).** Let \( \mathbf{b} \neq \mathbf{b}^* \) and \( (\hat{\theta}, \hat{\mathbf{v}}) \) be the output of Algorithm 2 with inputs \( D = \{\hat{\mathbf{z}}_i(\mathbf{b})\}_{i \in [K]} \) and \( U = \max_{i \in [K]} \|\hat{\mathbf{z}}_i(\mathbf{b})\|_2^2 \). Assume there exist positive constants \( (a, a_1, a_2, a_3, a_4) \) such that

\[ \begin{align*}
(a') & \quad (\alpha/2)\|\mathbf{b} - \mathbf{b}^*\|_2 \leq \hat{\theta} + a_1 \mu^2(\mathbb{B}_2)r_1. \\
(b') & \quad \hat{\theta} \leq \langle \mathbf{b}^* - \mathbf{b}, \hat{\mathbf{v}} \rangle_{\Pi} + a_2 \mu^2(\mathbb{B}_2)\|\mathbf{\Sigma}\|r_2\|\mathbf{b}^* - \mathbf{b}\|_2 + a_3 \mu^2(\mathbb{B}_2)r_1.
\end{align*} \]

Then

\[ \begin{align*}
(d') & \quad (a_4\|\mathbf{b}^* - \mathbf{b}\|_2) V((a_1 + a_3)\mu^2(\mathbb{B}_2)r_1) \leq \langle \mathbf{b}^* - \mathbf{b}, \hat{\mathbf{v}} \rangle_{\Pi}.
\end{align*} \]

Suppose further:

- We know an estimate \( \hat{\mathbf{\mu}} \) of \( \mathbf{\Sigma}\hat{\mathbf{v}} \) satisfying \( \|\hat{\mathbf{\mu}} - \mathbf{\Sigma}\hat{\mathbf{v}}\|_2 \leq \Delta \) for some \( \Delta \in (0, 1) \).
- \( \Delta < \frac{a}{16} \).
- \( a_2(\kappa r_2) \leq \frac{1}{4} \).

Let \( c_* := \frac{a}{8(2 + a_2)\kappa^2(\kappa^2 + \Delta^2)} \), and \( \mathbf{b}^+ := \mathbf{b} + c_* (\hat{\theta} + a_1 \mu^2(\mathbb{B}_2)r_1)\hat{\mathbf{\mu}} \).

Then

\[ \|\mathbf{b}^+ - \mathbf{b}^*\|_2^2 \leq \left( 1 - \frac{a^2}{32(2 + a_2)\kappa^2(\kappa^2 + \Delta^2)} \right) \|\mathbf{b} - \mathbf{b}^*\|_2^2. \]

**Proof.** Let us denote \( \theta := \hat{\theta} + a_1 \mu^2(\mathbb{B}_2)r_1 \). We first note that

\[ \|\mathbf{b}^+ - \mathbf{b}^*\|_2^2 = \|\mathbf{b} - \mathbf{b}^*\|_2^2 + 2c_* \theta \langle \hat{\mathbf{\mu}}, \mathbf{b} - \mathbf{b}^* \rangle + c_*^2 \theta^2 \|\hat{\mathbf{\mu}}\|_2^2 \]

\[ \leq \|\mathbf{b} - \mathbf{b}^*\|_2^2 + 2c_* \theta \langle \mathbf{\Sigma}\hat{\mathbf{v}}, \mathbf{b} - \mathbf{b}^* \rangle + 2c_*^2 \theta^2 \|\mathbf{\Sigma}\hat{\mathbf{v}}\|_2^2 + 2c_* \theta \langle \hat{\mathbf{\mu}} - \mathbf{\Sigma}\hat{\mathbf{v}}, \mathbf{b} - \mathbf{b}^* \rangle + 2c_*^2 \theta^2 \|\hat{\mathbf{\mu}} - \mathbf{\Sigma}\hat{\mathbf{v}}\|_2^2. \]
For ease of notation, we let $A_b := \langle \hat{v}, b - b^* \rangle$ and $D_b := \|b^* - b\|_2$.

We next bound the first-order terms in $c_s$. We have

$$T_1 := 2c_s \theta (\Sigma \hat{v}, b - b^*) + 2c_s \theta (\mu - \Sigma \hat{v}, b - b^*)$$

$$(b^*)$$

$$\begin{align*}
\leq & \ 2c_s \theta A_b + 2c_s (-A_b + a_2 \kappa r_2 D_b + (a_1 + a_3) \mu^2 (B_2) r_1) \Delta D_b \\
= & \ 2c_s A_b (\theta - \Delta D_b) + 2c_s ((a_1 + a_3) \mu^2 (B_2) r_1 + a_2 \kappa r_2 D_b) \Delta D_b.
\end{align*}$$

Since $A_b < 0$, $\theta \geq \frac{a}{2} D_b$ and $\Delta < \frac{a}{2}$, the first term above satisfies

$$2c_s A_b (\theta - \Delta D_b) \leq 2c_s ((\frac{a}{2}) - \Delta) D_b A_b.$$

As for the second term, using (d$^*$), it is upper bounded by

$$2c_s (-A_b + a_2 \kappa r_2 D_b) \Delta D_b = 2c_s (-A_b) \Delta D_b + 2c_s a_2 (\kappa r_2) \Delta D_b^2.$$

Using $\Delta < \frac{a}{2}$ and $A_b \leq -a_1 D_b$, we thus conclude that

$$T_1 \leq -2c_s ((\frac{a}{2}) - 2\Delta) a_1 - a_2 (\kappa r_2) \Delta D_b^2 \leq -\frac{a}{2} c_s D_b^2,$$

since $2\Delta a_1 + a_2 (\kappa r_2) \Delta \leq \frac{a}{2}$.

We next bound the second-order terms in $c_s$. From (b$^*$) and (d$^*$), Cauchy-Schwarz and $\|\hat{v}\|_2 \leq \mu^2 (B_2)$, we have $\theta \leq 2(-A_b) + a_2 \kappa D_b \leq (2\kappa + a_2) D_b$. We thus have

$$T_2 := 2c_s^2 \theta^2 \|\Sigma \hat{v}\|_2^2 + 2c_s^2 \theta^2 \|\mu - \Sigma \hat{v}\|_2^2 \leq 2c_s^2 (2\kappa + a_2 \kappa)^2 D_b^2 \left(\kappa^2 + \Delta^2\right).$$

We thus conclude that

$$\|b^+ - b^*\|_2^2 \leq \left[1 - (\frac{a}{2}) c_s + 2c_s^2 (2\kappa + a_2 \kappa)^2 (\kappa^2 + \Delta^2)\right] \|b - b^*\|_2^2.$$ Optimizing on $c_s$ entails the claim. \qed

### 5.5 Estimating the outer loop descent direction

In Sections 5.3 and 5.4, the estimated direction $\hat{v}$ in Algorithm 2 is a descent direction with respect to the conditioned inner product $\langle \cdot, \cdot \rangle_{\Pi} = \langle \Sigma \cdot, \cdot \rangle$. Still, we cannot use it as is without knowing $\Sigma$. We note however that, if an upper estimate of $\|\Sigma\|$ is available, all we need is an estimate of $\Sigma \hat{v}$. In this section, we show that the property $\Pi \mu (a_4, \rho, r_2)$, already shown to be satisfied by the pruned data set $\{x_\ell\}_{\ell=1}^m$, is enough to estimate $\Sigma \hat{v}$ by means of robust mean estimation.
For ease of reference, we make some definitions.

**Set-up 3.** Given \( \mu \in \mathbb{R}^p \), we define
\[
\bar{z}_i(\hat{v}, \mu) := \frac{1}{B} \sum_{\ell \in B_i} (\langle \bar{x}_\ell, \hat{v} \rangle x_\ell - \mu), \quad \text{and} \quad z_i(\hat{v}, \mu) := \frac{1}{B} \sum_{\ell \in B_i} (\langle \bar{x}_\ell, \hat{v} \rangle x_\ell - \mu).
\]

The following statement is immediate from Lemma 4.5 evaluated at \( \hat{v} \). We state it for ease of reference.

**Corollary 3** (Pruned sample: Noise Process at \( \hat{v} \)). Grant Assumption 1 and Set-ups 2 and 3. Let \( \bar{\rho} \in (0, 1/2], \bar{\alpha}_4 \in (0, 1) \) and constant \( C_{\bar{\alpha}_4} > 0 \) satisfying (3). Suppose that
\[
o \leq \bar{\alpha}_4 K.
\]

Let \( C_{\bar{\rho}}' := 1 + \sqrt{2/\bar{\rho}} \) and \( C > 0 \) be an absolute constant in Lemma 7.4. Let \( r_{n,K} \) as in Proposition 1. Let \( \bar{\alpha}_1 \geq (3\bar{\alpha}_4 + 0.75\bar{\rho})/4 \) and \( \bar{\alpha}_2 \geq (3\bar{\alpha}_4 + 0.75\bar{\rho})/2 \) and let \( (C_{\bar{\alpha}_1}, C_{\bar{\alpha}_2}) \) satisfy (3) with respect to \( (\bar{\alpha}_1, \bar{\alpha}_2) \). Set \( \bar{r}_1 := \|\Sigma\|\bar{r}_2 \) where
\[
\bar{r}_2 := 2C_{\bar{\alpha}_4} r_{n,K} + C C_{\bar{\alpha}_4} C_{\bar{\rho}} p \log p \frac{C C_{\bar{\alpha}_4} p \log p}{2n} + 2 L^2 C C_{\bar{\alpha}_4} p \log p \frac{C C_{\bar{\alpha}_4} p \log p}{2n}.
\]

Then on a \( \{ (y_\ell, x_\ell) \}_{\ell=1}^n \cup \{ (y_\ell, x_\ell) \}_{\ell=n+1}^{2n} \)-measurable event \( \tilde{E}_3 \) of probability at least
\[
n = e^{-\bar{\rho}n/1.8} - e^{-K/C_{\bar{\alpha}_4}},
\]

\[
\sup_{v \in B_2} \sum_{i \in [K]} 1_{\{ (\bar{z}_i(\bar{v}, \Sigma v), v) \geq C_{\bar{\alpha}_1} \bar{r}_1 \}} \leq 4 \bar{\alpha}_1 K, \tag{NP1(\bar{\alpha}_1, \bar{r}_1)}
\]

\[
\sup_{v \in B_2} \sum_{i \in [K]} 1_{\{ (\bar{z}_i(\bar{v}, \Sigma v), v) \geq C_{\bar{\alpha}_2} \bar{r}_1 \}} \leq 4 \bar{\alpha}_2 K. \tag{NP2(\bar{\alpha}_2, \bar{r}_1)}
\]

**Set-up 4.** In all this section, we work within the Set-up 2 and 3 and on the event
\( \mathcal{E} := \mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \cap \tilde{E}_3 \) where, for the pruned sample \( \{ (\tilde{y}_\ell, \tilde{x}_\ell) \}_{\ell \in [m]} \) and initialization \( \tilde{b}^{(0)} \), the boundedness property B(\( \eta, r \) and the uniform properties MP1(\( \alpha_1, r_1 \), MP2(\( \alpha_2, r_1 \), QP(\( \alpha_3 \), PP(\( \alpha_4, \rho, r_2 \), NP1(\( \bar{\alpha}_1, \bar{r}_1 \) and NP2(\( \bar{\alpha}_2, \bar{r}_1 \) all hold. The arguments in this section are purely deterministic.
5.5.1 Solving the inner loop combinatorial problem

When \( Z := \{ \hat{z}_i(\hat{v}, \mu) \}_{i=1}^{\mathcal{K}} \) for some \( \mu \neq \Sigma \hat{v} \), problem \( P(Z, k, 1) \) for some \( k \in \mathcal{K} \) becomes an parametrized instance of the Furthest Hyperplane Problem used in prior work for robust mean estimation \([26]\). For simplicity we will omit the length \( R = 1 \) in the following. The next lemma states that under the structural condition \( \text{NP1}(\tilde{\alpha}_1, \tilde{r}_1) \), problem \( P(\{ \hat{z}_i(\hat{v}, \mu) \}_{i=1}^{\mathcal{K}}, k) \) is feasible.

**Lemma 5.8** (Two-sided feasibility & margin-distance lower bound). Suppose that \( \text{NP1}(\tilde{\alpha}_1, \tilde{r}_1) \) holds and, for \( k := 4\tilde{\alpha}_1\mathcal{K} \),

\[ k < \mathcal{K}. \]

Then, for any \( \mu \neq \Sigma \hat{v} \), \( P(\{ \hat{z}_i(\hat{v}, \mu) \}_{i=1}^{\mathcal{K}}, k) \) is feasible; in particular, its optimal solution \((\theta_{\hat{v}, \mu, k}, v_{\hat{v}, \mu, k})\) satisfies \(-\text{NP1}(\{ \hat{z}_i(\hat{v}, \mu) \}_{i=1}^{\mathcal{K}}, \theta_{\hat{v}, \mu, k}, v_{\hat{v}, \mu, k}, k)\) with margin

\[ \theta_{\hat{v}, \mu, k} \geq -C_{\tilde{\alpha}_1}\tilde{r}_1 + \| \Sigma \hat{v} - \mu \|_2. \]

**Proof.** **STEP 1:** An upper bound on the optimal value is trivial: for any \((\theta, v, q)\) satisfying the constraints of \( P(\{ \hat{z}_i(\hat{v}, \mu) \}_{i=1}^{\mathcal{K}}, k) \) it follows from Cauchy-Schwarz that \( \theta_{\hat{v}, \mu, k} \leq \max_{i \in \mathcal{K}} \| \hat{z}_i(\hat{v}, \mu) \|_2 < \infty. \)

**STEP 2:** we now prove the lower bound \( \theta_{\hat{v}, \mu, k} \geq \| \mu - \Sigma \hat{v} \|_2 - C_{\tilde{\alpha}_1}\tilde{r}_1. \) \( \text{NP1}(\tilde{\alpha}_1, \tilde{r}_1) \) applied to the unit vector \( v := \Sigma \hat{v} - \mu \) implies that \( \langle \hat{z}_i(\hat{v}, \Sigma \hat{v}), v \rangle \geq -C_{\tilde{\alpha}_1}\tilde{r}_1 \) for more than \( \mathcal{K} - k \) buckets \( i \)'s, for which

\[
|\langle \hat{z}_i(\hat{v}, \Sigma \hat{v}), v \rangle| \geq \langle \hat{z}_i(\hat{v}, \mu), v \rangle = \langle \hat{z}_i(\hat{v}, \Sigma \hat{v}), v \rangle + \langle \Sigma \hat{v} - \mu, v \rangle \\
\geq -C_{\tilde{\alpha}_1}\tilde{r}_1 + \| \Sigma \hat{v} - \mu \|_2.
\]

In other words, the feasible set of \( P(\{ \hat{z}_i(\hat{v}, \mu) \}_{i=1}^{\mathcal{K}}, k) \) contains the point \((\theta, v, q)\) with \( \theta := -C_{\tilde{\alpha}_1}\tilde{r}_1 + \| \Sigma \hat{v} - \mu \|_2 \) for some \( q \in \{0, 1\}^{\mathcal{K}} \). By maximality, one must have \( \theta_{\hat{v}, \mu, k} \geq -C_{\tilde{\alpha}_1}\tilde{r}_1 + \| \Sigma \hat{v} - \mu \|_2. \)

\( \square \)

5.5.2 Computing the inner loop descent direction

Let \( \mu \in \mathbb{R}^p \) be a current point that is far from \( \Sigma \hat{v} \).

**Lemma 5.9.** Suppose that

(i) \( \text{NP1}(\tilde{\alpha}_1, \tilde{r}_1) \) holds for some \( \tilde{\alpha}_1 \in (0, 1) \). Let \( k := 4\tilde{\alpha}_1\mathcal{K} \).

Let \( \mu \neq \Sigma \hat{v} \) satisfying:
(ii) For some \((\bar{\theta}, \bar{v}, \bar{k}') \in \mathbb{R} \times S_2 \times [K]\), property \(\neg P1\) \((\{\bar{z}_i(\hat{v}, \mu)\}_{i \in [K]}, \bar{\theta}, \bar{v}, \bar{k}')\) holds, i.e.,
\[
\sum_{i=1}^{\kappa} 1_{\{\langle \bar{z}_i(\hat{v}, \mu), \bar{v} \rangle > \bar{\theta} \}} > \kappa - \bar{k}'.
\]

In particular, \(P((\{\bar{z}_i(\hat{v}, \mu)\}_{i \in [K]}, \bar{k}')\) is feasible with optimal value, say, \(\bar{\theta}\).

(iii) \(\bar{k} < \kappa\).

(iv) For some \(\bar{a} \in (0, 1], \bar{\theta} \geq \bar{a}\bar{\theta}\).

Then
\[
(a) \quad \bar{a} (\|\mu - \Sigma \hat{v}\|_2 - C_{\alpha_1} \bar{r}_1) \leq \bar{\theta} \leq \|\mu - \Sigma \hat{v}\|_2 + C_{\alpha_1} \bar{r}_1.
\]

Proof. From (ii), maximality and (iv), \(\bar{a}\bar{\theta} \leq \bar{\theta} \leq \bar{\theta}\). We skip the proof of the lower bound \(\bar{\theta} \geq \|\Sigma \hat{v} - \mu\|_2 - C_{\alpha_1} \bar{r}_1\) as it is proved in the same way as in STEP 2 of the proof of Lemma 5.8 using (i) and condition \(\bar{k} < \kappa\) in (iii).

Next, we prove the upper bound \(\bar{\theta} \leq \|\Sigma \hat{v} - \mu\|_2 + C_{\alpha_1} \bar{r}_1\). Assume by contradiction that \(\bar{\theta} > \|\Sigma \hat{v} - \mu\|_2 + C_{\alpha_1} \bar{r}_1\). By maximality, this implies that there must exist \(\bar{\theta}' \in \mathbb{R}\), \(\bar{v}' \in S_2\) and \(\bar{q}' \in \{0, 1\}^\kappa\) satisfying the constraints of \(P((\{\bar{z}_i(\hat{v}, \mu)\}_{i \in [K]}, \bar{k}')\) such that \(\bar{\theta}' > \|\Sigma \hat{v} - \mu\|_2 + C_{\alpha_1} \bar{r}_1\). In particular, for more than \(\kappa - \bar{k}'\) buckets \(i\)'s, \(\langle \bar{z}_i(\hat{v}, \mu), \bar{v}' \rangle \geq \bar{\theta}'\), implying
\[
|\langle \bar{z}_i(\hat{v}, \Sigma \hat{v}), \bar{v}' \rangle| \geq |\langle \bar{z}_i(\hat{v}, \mu), \bar{v}' \rangle| - |\langle \Sigma \hat{v} - \mu, \bar{v}' \rangle| \geq \bar{\theta}' - \|\mu - \Sigma \hat{v}\|_2 \geq C_{\alpha_1} \bar{r}_1.
\]

This contradicts \(\neg P1(\bar{\alpha}_1, \bar{r}_1)\) in (i), finishing the proof of (a).

Lemma 5.10 (One-sided feasibility & margin-distance upper bound). Grant assumptions of Lemma 5.9 and additionally assume:

(v) \(\|\mu - \Sigma \hat{v}\|_2 \geq A\bar{r}_1\) with
\[
A := \left(\left(\frac{\bar{a} + 1}{\bar{a}}\right) C_{\alpha_1}\right) V \left(\left(\bar{a} - 1\right) (\bar{a} + 1) C_{\alpha_1} + C_{\alpha_2}\right).
\]

Then

(b) Let \(\bar{k}'' := \bar{k} + \bar{k}'\). There exists \(\bar{v}' \in \{-\bar{v}, \bar{v}\}\) such that property \(\neg P2(\hat{v}, \mu, \bar{\theta}, \bar{v}', \bar{k}'')\), defined below, holds:
\[
\sum_{i=1}^{\kappa} 1_{\{\langle \bar{z}_i(\hat{v}, \mu), \bar{v}' \rangle > \bar{\theta} \}} > \kappa - \bar{k}''.
\]
Additionally to the assumptions of Lemma 5.9 and (v), assume:

(vi) \( \text{NP2}(\bar{\alpha}_2, \bar{r}_1) \) holds for some \( \bar{\alpha}_2 \in (0, 1) \). Let \( \bar{k}_0 := 4\bar{\alpha}_2K \).

Then, for any \( \mu \neq \Sigma \hat{v} \) satisfying (ii)-(v) and

(vii) \( \bar{k}'' \leq \mathcal{K}/2 \) and \( \bar{k}_0 \leq \mathcal{K}/2 \),

one also has

(c) \( \langle \bar{v}', \mu - \Sigma \hat{v} \rangle \leq -\frac{3}{2}\|\mu - \Sigma \hat{v}\|_2 \).

Proof. Proof of (b): define the sets

\[
\begin{align*}
G_{\hat{v}} &:= \{i \in [\mathcal{K}] : |\langle \tilde{z}_i(\hat{v}, \Sigma \hat{v}), \bar{v} \rangle| \leq C_{\bar{\alpha}_1} \bar{r}_1 \}, \\
B^+_\mu &:= \{i \in [\mathcal{K}] : \langle \tilde{z}_i(\hat{v}, \mu), \bar{v} \rangle > \bar{\theta} \}, \\
B^-_\mu &:= \{i \in [\mathcal{K}] : \langle \tilde{z}_i(\hat{v}, \mu), \bar{v} \rangle > \bar{\theta} \}.
\end{align*}
\]

We consider two cases.

Case 1: \( \langle \mu - \Sigma \hat{v}, \bar{v} \rangle \leq 0 \). Given \( i \in B^-_\mu \),

\[
\begin{align*}
\langle \tilde{z}_i(\hat{v}, \Sigma \hat{v}), \bar{v} \rangle &= \langle \tilde{z}_i(\hat{v}, \mu), \bar{v} \rangle + \langle \mu - \Sigma \hat{v}, \bar{v} \rangle \\
&\leq -\bar{\theta} \\
&\overset{(a)}{<} a C_{\bar{\alpha}_1} \bar{r}_1 - \bar{a} \|\mu - \Sigma \hat{v}\|_2 \\
&\overset{(v)}{<} -C_{\bar{\alpha}_1} \bar{r}_1,
\end{align*}
\]

where we used from (v) that \( \|\mu - \Sigma \hat{v}\|_2 \geq (1+\bar{a}/a)C_{\bar{\alpha}_1} \bar{r}_1 \).

We thus concluded that \( B^-_\mu \subset G_{\hat{v}} \). This and the facts

- \( |G_{\hat{v}}| > \mathcal{K} - \bar{k} \) by \( \text{NP1}(\tilde{\alpha}_1, \bar{r}_1) \),
- \( B^+_\mu \) and \( B^-_\mu \) are disjoint because \( \bar{\theta} > 0 \), by (a) and (v). Also, \( |B^+_\mu| + |B^-_\mu| > \mathcal{K} - \bar{k}' \) by (ii),

imply that \( |B^+_\mu| > \mathcal{K} - \bar{k} - \bar{k}' \).

Case 2: \( \langle \mu - \Sigma \hat{v}, \bar{v} \rangle \rangle > 0 \). By exchanging \( \bar{v} \) with \( -\bar{v} \) and \( B^-_\mu \) by \( B^+_\mu \) a similar argument shows that \( |B^-_\mu| > \mathcal{K} - \bar{k} - \bar{k}' \).
Proof of (c): By (b) and (vii), one has \(\langle \tilde{z}_i(\hat{v}, \mu), \tilde{v}' \rangle \leq \bar{\theta} \) for less \(K/2\) of buckets \(i\)'s. By NP2(\(\bar{\alpha}_2, \bar{r}_1\)) and (vii), one has \(\langle \tilde{z}_i(\hat{v}, \Sigma \hat{v}), \tilde{v}' \rangle > C_{\bar{\alpha}_2} \bar{r}_1\) for less \(K/2\) of buckets \(i\)'s. By the pigeonhole principle, there is at least one bucket \(i\) for which \(\langle \tilde{z}_i(\hat{v}, \mu), \tilde{v}' \rangle > \bar{\theta} \) and \(\langle \tilde{z}_i(\hat{v}, \Sigma \hat{v}), \tilde{v}' \rangle \leq C_{\bar{\alpha}_2} \bar{r}_1\) hold. Thus

\[
\bar{a} \left[ \| \mu - \Sigma \hat{v} \|_2 - C_{\bar{\alpha}_1} \bar{r}_1 \right]^{(a)} \leq \bar{\theta} \\
< \langle \tilde{z}_i(\hat{v}, \mu), \tilde{v}' \rangle \\
= \langle \tilde{z}_i(\hat{v}, \Sigma \hat{v}), \tilde{v}' \rangle + \langle \Sigma \hat{v} - \mu, \tilde{v}' \rangle \\
\leq C_{\bar{\alpha}_2} \bar{r}_1 + \langle \Sigma \hat{v} - \mu, \tilde{v}' \rangle.
\]

Using (v), so that \(\| \mu - \Sigma \hat{v} \|_2 \geq (2/a)(\bar{a} C_{\bar{\alpha}_1} + C_{\bar{\alpha}_2}) \bar{r}_1\), and rearranging the displayed inequality finishes the proof. \(\square\)

We conclude this section with the following corollary. From now on, we fix the parameters \(\bar{\alpha}_4 = 1/96, \bar{p} = 1/24\) so that \(\bar{\alpha}_1 = 1/64\) and \(\bar{\alpha}_2 = 1/8\) satisfy the conditions of Corollary 3. We also set \(c_{\bar{\alpha}_1} = \frac{1}{4}, \bar{\alpha} := 0.0128\) and \(\bar{\varphi} := 0.49\pi\). In order to satisfy (3), it suffices to take \(C_{\bar{\alpha}_1} = 1666.68\), and \(C_{\bar{\alpha}_2} = 110\).

**Corollary 4.** Let \(\eta \in (0, 1/2]\) and suppose that:

(i) BD(\(\eta, r\), NP1(\(\bar{\alpha}_1, \bar{r}_1\)) and NP2(\(\bar{\alpha}_2, \bar{r}_1\)) all hold.

(ii) Given \(\mu \in \mathbb{R}^p\), assume \(\| \mu - \Sigma \hat{v} \|_2 \geq 1.32 \cdot 10^5 \bar{r}_1\).

Let \((\bar{\theta}, \bar{v})\) be the output of MW(\(D, U, S_2, \bar{k}, \bar{k}', 1, \bar{r}_1\)), that is, Algorithm 2 with inputs \(D = \{\tilde{z}_i(\hat{v}, \mu)\}_{i \in [K]}\) and \(U = \max_{i \in [K]} \| \tilde{z}_i(\hat{v}, \mu) \|_2^2, \bar{k} = 4\bar{\alpha}_1 K\) and \(\bar{k}' = c_{\bar{\alpha}_1} K, R = 1\) and \(\bar{r}_1 = C_{\bar{\alpha}_1} \bar{r}_1\).

Then on an event of probability (on the randomness of \(\{\theta_t\}_{t \in [S_2]}\)) of at least \(1 - e^{-\frac{S_2}{669}}\), one has

\[
\sum_{i=1}^K 1 \{ \langle \tilde{z}_i(\hat{v}, \mu), \tilde{v} \rangle > \bar{k}' \} > K - \bar{k}' \quad \text{and} \quad \bar{\theta} > \bar{a} \theta_{\hat{v}, \mu, \bar{k}}.
\]

Moreover,

\[
(1/78.125) \| \mu - \Sigma \hat{v} \|_2 - 1666.68 \bar{r}_1 \leq \bar{\theta},
\]

\[
\bar{\theta} \leq \| \mu - \Sigma \hat{v} \|_2 + 1666.68 \bar{r}_1,
\]

\[
\langle \tilde{v}, \mu - \Sigma \hat{v} \rangle_2 \leq -\frac{1}{156.25} \| \mu - \Sigma \hat{v} \|_2.
\]
Proof. Setting \( \tilde{k} = 4\tilde{a}_1 \) and \( \tilde{k}' := c_{\tilde{a}_1}K \) with the parameters displayed before the corollary, one checks that \( \tilde{a} \leq \cos(\tilde{\varphi})/\sqrt{6} \) and all conditions of Lemmas 5.2, 5.3, 5.9 and 5.10 hold. In particular, \( \|\mu - \Sigma \hat{v}\|_2 \geq 2C_{\tilde{a}_1}\tilde{r}_1 \) implying condition (iii) of Lemma 5.2 with \( \tilde{r}_1 := C_{\tilde{a}_1}\tilde{r}_1 \).

We now work on the event of probability \( 1 - e^{-\frac{0.35\tilde{a}_1}{\tilde{k}''}} \) for which the claim of Lemma 5.3 is true. By Lemma such lemma, (13) is satisfied; these are assumptions (ii) and (iv) of Lemma 5.9 for \((\hat{\theta}, \hat{v})\). All other assumptions of such lemma hold, yielding (14)-(15).

All conditions of Lemma 5.9 and conditions (v) of Lemma 5.10 holds so there must exist \( \hat{v}' \in \{-\hat{v}, \hat{v}\} \) satisfying \( \sum_{i=1}^{K} c_{\hat{\varphi}}(\hat{v}, \mu, \hat{v}') > K - \tilde{k}'' \), that is, (b) of such lemma. By this property, the order statistics in Algorithm 3 implies \( \hat{v} = \hat{v}' \). All additional conditions of Lemma 5.10 hold, yielding (16).

**Lemma 5.11.** Let \( \tilde{c}_s := 1.045752 \cdot 10^{-06} \) and \( \Delta_0 := 1.093597 \cdot 10^{-12} \).

Let \( \mu \in \mathbb{R}^p \) and grant assumptions (i)-(ii) of Corollary 4 which guarantees the outputted margin-direction pair \((\hat{\theta}, \hat{v})\) produced by Algorithm 2.

Let \( \mu^+ := \mu + \tilde{c}_s \hat{\theta} \hat{v} \).

Then, on the same event of Corollary 4,

\[
\|\mu^+ - \Sigma \hat{v}\|_2^2 \leq \left(1 - \Delta_0\right) \|\mu - \Sigma \hat{v}\|_2^2.
\]

**Proof.** One has

\[
\|\mu^+ - \Sigma \hat{v}\|_2^2 \leq \|\mu - \Sigma \hat{v}\|_2^2 + 2\tilde{c}_s \hat{\theta}(\hat{v}, \mu^+ - \Sigma \hat{v}) - \|\mu^+ - \mu\|_2^2
\]

\[
= \|\mu - \Sigma \hat{v}\|_2^2 + 2\tilde{c}_s \hat{\theta}(\hat{v}, \mu - \Sigma \hat{v}) + E,
\]

where by, Young’s inequality,

\[
E := 2\tilde{c}_s \hat{\theta}(\hat{v}, \mu^+ - \mu) - \|\mu^+ - \mu\|_2^2 \leq c_s^2 \hat{\theta}^2.
\]

Item (ii) and (14) of Corollary 4 imply \( \hat{\theta} \geq 0.00017\|\mu - \Sigma \hat{v}\|_2 \). Hence, from (16), one gets \( 2\hat{\theta}(\hat{v}, \mu - \Sigma \hat{v}) \leq -2.176 \cdot 10^{-6}\|\mu - \Sigma \hat{v}\|_2^2 \). Items (ii) and (15) imply \( \hat{\theta} \leq 1.02\|\mu - \Sigma \hat{v}\|_2 \).

We thus conclude that

\[
\|\mu^+ - \Sigma \hat{v}\|_2^2 \leq \left[1 - 2.176 \cdot 10^{-6}\tilde{c}_s + 1.0404c_s^2\right] \|\mu - \Sigma \hat{v}\|_2^2.
\]

Minimizing over \( \tilde{c}_s \) yields the claim. \( \square \)
6 Master algorithm

We now present two master algorithms for robust regression. The first one, Algorithm 4, assumes knowledge of

\[ r_1 := 2 \sqrt{\frac{\text{tr}(\Xi)}{n}} \sqrt{\frac{\|\Xi\|}{K}}. \]

Note that \((\text{tr}(\Xi), \|\Xi\|)\) requires information of the noise level, a difficult quantity to robustly estimate in practice. We present the alternative Algorithm 6 in the next section, assuming no knowledge of \((\text{tr}(\Xi), \|\Xi\|)\). Both algorithms assume knowledge of the minimum and maximal eigenvalues \((\mu^{-2}(\mathbb{B}_2), \|\Sigma\|)\) of the covariance matrix \(\Sigma\). As mentioned in the introduction, these values can be effectively replaced by their robust estimates (up to absolute constants).

For ease of reference, we recall previously defined constants and rates. We recall the parameters \(\alpha_1 = 1/96, \alpha_2 = 0.08, \alpha_3 = 0.239, \alpha_4 = 1/144, c_{\alpha_1} := \frac{1}{4}, \rho = 1/36, a := 0.0128, C_{\alpha_1} = 2525.26, C_{\alpha_2} = 192.4, C_{\alpha_3} = 51.9\) and \(C_{\alpha_4} = 4330\). Also, we recall \(\bar{\rho} = 1/24, \bar{\alpha}_1 = 1/64, \bar{\alpha}_2 = 1/8, \bar{\alpha}_4 = 1/96, c_{\bar{\alpha}_1} := \frac{1}{4}, \bar{a} := a, C_{\bar{\alpha}_1} = 1666.68, \) and \(C_{\bar{\alpha}_2} = 110\). Finally, we recall the rates

\[ r := 2\mu^2(\mathbb{B}_2)\sqrt{\frac{K}{n} \text{tr}(\Xi)}, \]

\[ r_{n,K} := CL^2 \sqrt{\frac{p \log p}{n}} \sqrt{\frac{K}{n}}, \]

\[ r_2 := 2C_{\alpha_4}r_{n,K} + CC_{\alpha_4}C'_{\rho} \frac{p \log p}{2n} + 2L^2 \sqrt{\frac{CC_{\alpha_4}p \log p}{2n}}, \]

\[ \bar{r}_2 := 2C_{\bar{\alpha}_4}r_{n,K} + CC_{\bar{\alpha}_4}C'_{\rho} \frac{p \log p}{2n} + 2L^2 \sqrt{\frac{CC_{\bar{\alpha}_4}p \log p}{2n}}, \]

\[ \bar{r}_1 := \|\Sigma\| \bar{r}_2, \]

where \(C > 0\) is an absolute constant in Lemma 7.4 in the Appendix and \(C'_{\alpha} := 1 + \sqrt{2/\alpha}\) for any \(\alpha \in (0, 1)\). Recall the constants \(\tilde{c}_* := 1.045752 \cdot 10^{-06}\) and \(\Delta_0 := 1.093597 \cdot 10^{-12}\). Define the constants \(c_* := \frac{a}{8(2+C_{\alpha_4})\kappa^2(\kappa^2+\Delta_0)}\) and

\[ \Delta := 263876.1\bar{r}_1 \sqrt{(1 + \kappa)^2 e^{-T_2 \Delta_0}}, \]

where \(T_2 \in \mathbb{N}\) is to be defined in the following.
Algorithm 4 Robust-Regression($D, T_1, T_2, K, \eta, S_1, S_2, r_1$)

**Input:** sample $D := \{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^n \cup \{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=n+1}^{2n}$, outer and inner number of iterations ($T_1, T_2$), number of buckets $K$, quantile probability $\eta \in (0, 1/2)$, outer and inner simulation sample sizes ($S_1, S_2$), optimal rate $r_1 > 0$.

**Output:** $\hat{b}(r_1)$.

1: Set $(\tilde{b}^{(0)}, \tilde{R}_m, \{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^m) \leftarrow \text{Pruning}(D, K, \eta)$.
2: Split $\{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^m$ into $K$ buckets of same size $m/K = B$ indexed by the partition $\bigcup_{i \in [K]} \tilde{B}_i = [m]$.
3: Set $b^1 \leftarrow \tilde{b}^{(0)}$.
4: for $t = 1 : T_1$ do
5:     For $i \in [K]$, compute $\tilde{z}_i(b^t) := \frac{1}{B} \sum_{\ell \in \tilde{B}_i} (\tilde{y}_\ell - \langle \tilde{x}_\ell, b^t \rangle) \tilde{x}_\ell$.
6:     Set $(\theta_t, v^t) \leftarrow \text{MW}\left(\{\tilde{z}_i(b^t)\}_{i=1}^K, \max_{i \in [K]} \|\tilde{z}_i(b^t)\|_2^2, S_1, 4\alpha_1 K, c_{\alpha_1} K, \mu^2(\mathbb{B}_2), C_{\alpha_1} \mu^2(\mathbb{B}_2) r_1\right)$.
7:     $\mu^t \leftarrow \text{Robust-Direction}(\{\tilde{x}_\ell\}_{\ell=1}^m, v^t, T_2, S_2)$.
8:     if $\theta^t < \theta^{t-1}/\kappa$ then
9:         Set $b^{t+1} := b^t + c_{\epsilon}(\theta_t + C_{\alpha_1} \mu^2(\mathbb{B}_2) r_1) \mu^t$.
10:    else
11:        $b^{t+1} \leftarrow b^t$.
12:    end if
13: end for
14: return $b^{T_1}$.
Algorithm 5 Robust-Direction($D$, $\hat{\nu}$, $T_2$, $S_2$)

**Input:** pruned feature sample $D := \{\tilde{x}_\ell\}_{\ell=1}^m$, direction $\hat{\nu}$, inner number of iterations $T_2$, inner simulation sample size $S_2$.

**Output:** estimate $\hat{\mu}(\hat{\nu})$ of $\Sigma\hat{\nu}$.

1: Set $\mu^1 \leftarrow (1, 0, \ldots, 0) \in \mathbb{R}^p$.
2: for $\tau = 1 : T_2$ do
3:   For $i \in [K]$, compute $\tilde{z}_i(\hat{\nu}, \mu^\tau) := \frac{1}{B} \sum_{\ell \in \tilde{B}_i} \langle \tilde{x}_\ell, \hat{\nu} \rangle \tilde{x}_\ell - \mu^\tau$.
4:   Set $(\tilde{\theta}^\tau, \tilde{\nu}^\tau) \leftarrow \text{MW} \left( \{ \tilde{z}_i(\hat{\nu}, \mu^\tau) \}_{i=1}^K, \max_{i \in [K]} \| \tilde{z}_i(\hat{\nu}, \mu^\tau) \|_2^2, S_2, 4\tilde{\alpha}_1 K, c\tilde{\alpha}_1 K, 1, C\tilde{\alpha}_4 \bar{r}_1 \right)$.
5:   if $\tilde{\theta}^\tau < \tilde{\theta}^{\tau-1}$ then
6:      Set $\mu^{\tau+1} := \mu^\tau + \tilde{c}_s \tilde{\theta}^\tau \tilde{\nu}^\tau$.
7:   else
8:      $\mu^{\tau+1} \leftarrow \mu^\tau$.
9:   end if
10: end for
11: return $\hat{\mu}^{T_2}$.

We introduce the assumptions in the next two main results. Let $K \in [n]$, $\eta \in (0, 1/2)$ and $m = (1 - \eta)n$. Grant Assumptions 1 and 2 and assume that

\begin{align}
K &> \max \{ 4, (\alpha_1 \lor \alpha_2)^{-1}, \alpha_3^{-1}, \alpha_4^{-1}, \tilde{\alpha}_3^{-1} \}, \\
\eta &\geq 4\epsilon,
\end{align}

and

\begin{align}
L^2 \left( 7 \sqrt{\frac{K \log(24)}{n}} + 4 \sqrt{\frac{p}{n}} \right) &\leq \frac{1}{2}, \\
C_{\alpha_3} r_{n,K} + 2L^2 \sqrt{\frac{C_{\alpha_3}^2 p \log p}{n}} &\leq \frac{1}{2}, \\
\mu^2(\mathbb{B}_2) \| \Sigma \| r_2 &\leq \frac{1}{312.5C_{\alpha_4}}.
\end{align}

**Proposition 4** (Inner loop convergence). Grant Grant Assumptions 1 and 2 and (17), (18), (19), (20) and (21). Instantiate Algorithm 4 with inputs $D := \{(\tilde{y}_\ell, \tilde{x}_\ell)\}_{\ell=1}^n \cup \{(\check{y}_\ell, \check{x}_\ell)\}_{\ell=n+1}^{2n}$, $K$, $\eta$, integers $(T_1, T_2, S_1, S_2)$ and $r_1 > 0$. Recall the event $E$ in Set-up 4.
Then, for any \( t \in [T_1] \), after \( T_2 \) iterations after Robust-Direction(\( \{ \hat{\mathbf{x}}_{l} \}_{l=1}^{n}, \mathbf{v}^{t}, T_2, S_2 \) is queried, there is an event \( S(t, T_2) \) of probability at least \( 1 - T_2 e^{-\frac{T_2}{669}} \), such that on the event \( \mathcal{E} \cap S(t, T_2) \),

\[
\| \hat{\mathbf{r}}^{t} - \mathbf{v}^{t} \|_2 \leq 263876.1\tilde{r}_1 \sqrt{(1 + \kappa)^2 e^{-T_2 \bar{\Delta}_0}}.
\]

**Proof.** On \( \mathcal{E} \), conditions \( \text{BD}(\eta, \tau), \text{NP1}(\bar{\alpha}_1, \tilde{r}_1) \) and \( \text{NP2}(\bar{\alpha}_2, \tilde{	au}_1) \) all hold.

Fix outer iteration \( t \in [T_1] \). Let \( \{ \hat{\theta}_{t, \tau} \}_{t \in [S_2], \tau \in [T_2]} \) be the simulated iid sequence from the uniform distribution over \( S_2 \) (independent of the label-feature data set) during the inner query of Algorithm 5 fir given \( t \in [T_1] \). In the following, our arguments are over the event \( \mathcal{E} \cap S(t, T_2) \) where \( S(t, T_2) \) is an event of probability at least \( 1 - T_2 e^{-S/669} \) over the randomness of \( \{ \hat{\theta}_{t, \tau} \}_{t \in [S_2], \tau \in [T_2]} \) conditioned on \( \{ \mathbf{x}_{t} \}_{t=1}^{n} \cup \{ \mathbf{x}_{t} \}_{t=n+1}^{2n} \). We denote by \( \{ \hat{\mathbf{r}}^{t, \tau} \}_{\tau \in [T_2]} \) the iterates during the call of Algorithm 5 for \( t \in [T_1] \).

Let \( \mathcal{G} := \{ \mathbf{r} \in \mathbb{R}^p : \| \mathbf{r} - \mathbf{v}^{t} \|_2 < 1.32 \cdot 10^5 \tilde{r}_1 \} \). We consider two cases.

**Case 1:** There is \( \tau \in [T_2] \) such that \( \hat{\mathbf{r}}^{t, \tau} \notin \mathcal{G} \). If \( \hat{\mathbf{r}}^{t, T_2} \notin \mathcal{G} \) we are done, so suppose \( \hat{\mathbf{r}}^{t, T_2} \in \mathcal{G} \). By Corollary 4,

\[
(1/78.125)\| \hat{\mathbf{r}}^{t, T_2} - \mathbf{v}^{t} \|_2 - 1666.68\tilde{r}_1 \leq \tilde{\theta}_{T_2},
\]

\[
\tilde{\theta}_t \leq \| \hat{\mathbf{r}}^{t, \tau} - \mathbf{v}^{t} \|_2 + 1666.68\tilde{r}_1.
\]

As \( \tilde{\theta}_{T_2} \leq \tilde{\theta}_t \) by construction,

\[
\| \hat{\mathbf{r}}^{t, T_2} - \mathbf{v}^{t} \|_2 \leq 78.125\| \hat{\mathbf{r}}^{t, \tau} - \mathbf{v}^{t} \|_2 + 79.125\tilde{r}_1 \leq 263876.1\tilde{r}_1.
\]

**Case 2:** For all \( \tau \in [T_2] \), \( \hat{\mathbf{r}}^{t, \tau} \notin \mathcal{G} \) so Lemma 5.11 applies across all iterations after an union bound. Thus

\[
\| \hat{\mathbf{r}}^{t, T_2} - \mathbf{v}^{t} \|_2 \leq e^{-T_2 \bar{\Delta}_0}\| \hat{\mathbf{r}}^{t, 1} - \mathbf{v}^{t} \|_2 \leq (1 + \kappa)^2 e^{-T_2 \bar{\Delta}_0}.
\]

Independence between \( \{ \hat{\theta}_{t, \tau} \}_{t \in [S], \tau \in [T]} \) and \( \{ \mathbf{x}_{t} \}_{t=1}^{n} \cup \{ \mathbf{x}_{t} \}_{t=n+1}^{2n} \) and an union bound finishes the proof. \( \square \)

Next, we additionally assume that the sample size and inner loop number of iterations \( T_2 \) are large enough so that

\[
\Delta := 263876.1\tilde{r}_1 \sqrt{(1 + \kappa)^2 e^{-T_2 \bar{\Delta}_0}} \leq \frac{a}{16}.
\]

(22)
For ease of reference, we define the constants $C_s := 8 \cdot 10^5$, $D_s := 1/625$ and $E_s := \max \{C_s, (C_{\alpha_1} + C_{\alpha_2})/D_s \} = (C_{\alpha_1} + C_{\alpha_2})/D_s \geq 17 \cdot 10^5$. For ease of reference, we also define, given input parameters $(K, \eta, T_1, T_2, S_1, S_2)$, the failure probability

\[
\delta := e^{-\frac{K}{a_4}} + e^{-\frac{\eta_n}{10}} + e^{-\frac{K}{a_1}} + e^{-\frac{K}{a_2}} + e^{-\frac{K}{a_3}} + e^{-\frac{\eta_n}{10}} + e^{-\frac{K}{a_4}} + T_1 T_2 e^{-\frac{S_2}{100}} + T_1 e^{-\frac{S_1}{100}}.
\]

(23)

**Theorem 6.1** (Outer loop convergence). Grant Grant Assumptions 1 and 2 and (17), (18), (19), (20), (21) and (22). Instantiate Algorithm 4 with inputs $D := \{(\tilde{y}_{\ell}, \tilde{x}_{\ell})\}_{\ell=1}^n \cup \{(\tilde{y}_{\ell}, \tilde{x}_{\ell})\}_{\ell=n+1}^2$, $K$, $\eta$, integers $(T_1, T_2, S_1, S_2)$ and $r_1 > 0$. Recall the event $E$ in Set-up 4.

Then after $T_1$ iterations of Algorithm 4, on the event $E \cap \cap \{t \in [T_1] \} S(t, T_2)$ of probability at least $1 - \delta$,

\[
\|b_t^{T_1} - b^*\|_2 \leq \left( \frac{2C_{a_2}}{a\kappa} + \frac{2E_s}{a} \left( 1 + \frac{1}{312.5\kappa} \right) + 5050.52 \right) \mu^2(\mathbb{B}) r_1 \left\lVert 3 e^{-\frac{S_1}{100}}. \right.
\]

**Proof.** On $E \cap \cap \{t \in [T_1] \} S(t, T_2)$, conditions BD$(\eta, r)$, MP1$(\alpha_1, r_1)$, MP2$(\alpha_2, r_1)$, QPI$(\alpha_3)$, PPU$(\alpha_4, \rho, r_2)$, NP1$(\alpha_1, \bar{r}_1)$ and NP2$(\alpha_2, \bar{r}_1)$ all hold. By an union bound and independence, we have that $E \cap \cap \{t \in [T_1] \} S(t, T_2)$ has probability at least as given in the statement of the theorem. We next state our arguments on the event $E \cap \cap \{t \in [T_1] \} S(t, T_2)$.

Let $\{\theta_{t, \ell} \} t \in [S_1], t \in [T_1]$ be the simulated iid sequence from the uniform distribution over $S_2$ (independent of the label-feature data set) during queries of the outer loop of Algorithm 4. Let $I \subset [T_1]$ be the set of iterations for which $\|b^t - b^*\|_2 < E_s \mu^2(\mathbb{B}) r_1$.

We consider two cases.

**Case 1:** There is $t \in I$. If $T_1 \in I$ we are done, so suppose $T_1 \notin I$. By construction $\theta_{T_1} \leq \theta_{t}/\kappa$. From Corollary 2 applied to $b_t^{T_1}$, Lemma 5.6 applied to $(b^t, \theta_t, \nu^t)$, $C_{a_4}\mu^2(\mathbb{B}) \|\Sigma\| r_2 \leq 1/312.5$ and $\|\nu\|_2 \leq \mu^2(\mathbb{B})$, we get

\[
\|b_t^{T_1} - b^*\|_2 \leq \frac{2}{a} \theta_{T_1} + 2 \cdot 2525.26 \mu^2(\mathbb{B}) r_1
\]

\[
\leq \frac{2}{a\kappa} \left( C_{a_2}\mu^2(\mathbb{B}) r_1 + |\langle b^* - b^t, \nu^t \rangle_\Pi + C_{a_4}\mu^2(\mathbb{B}) \|\Sigma\| r_2 \|b^* - b^t\|_2 \right)
\]

\[
+ 5050.52 \mu^2(\mathbb{B}) r_1
\]

\[
\leq \left( \frac{2C_{a_2}}{a\kappa} + 5050.52 \right) \mu^2(\mathbb{B}) r_1 + \frac{2}{a\kappa} \cdot \kappa \|b^t - b^*\|_2 + \frac{2}{a\kappa} \cdot \|b^t - b^*\|_2
\]

\[
< \left( \frac{2C_{a_2}}{a\kappa} + \frac{2E_s}{a} \left( 1 + \frac{1}{312.5\kappa} \right) + 5050.52 \right) \mu^2(\mathbb{B}) r_1.
\]
Case 2: \( I = \emptyset \). In particular, \( \| b^t - b^* \|_2 \geq C_* \mu^2(B_2) r_1 \) for all \( t \in [T_1] \). Corollary 2 applies across all iterations after an union bound. In particular, for all \( t \in [T_1] \), \( \langle v^t, b^* - b^t \rangle \geq D_* \| b^* - b^t \|_2 \geq D_* E_* \mu^2(B_2) r_1 \geq (C_{\alpha_1} + C_{\alpha_2}) \mu^2(B_2) r_1 \). Additionally, \( a_2(\kappa r_2) \leq \frac{1}{2} \) and, by Proposition 4, for all \( t \in [T_1] \),

\[
\| \mu^{t,T_2} - \Sigma v^t \|_2 \leq 263876.1 \sqrt{1 + \kappa} e^{-T_2 \Delta_0} =: \Delta \leq \frac{a}{16}.
\]

Hence, can invoke Lemma 5.7 across all iterations (after an union bound) with

\[
a_1 := 2525.26 = C_{\alpha_1}, \quad a_2 := 4330 = C_{\alpha_2}, \quad a_3 := 192.4 = C_{\alpha_2} \quad \text{and} \quad a_4 := D_*.
\]

We get

\[
\| b^{T_1} - b^* \|_2^2 \leq e^{-\frac{a_1}{16}} \| b^{(0)} - b^* \|_2^2 \leq 9 r_2 \eta e^{-\frac{a_1}{16}}.
\]

Independence between \( \{ \theta_{t,t} \}_{t \in [S_1], \tau \in [T_1]} \), \( \{ \bar{\theta}_{t,\tau} \}_{t \in [S_2], \tau \in [T_2]} \) and \( \{(y_{t, \ell}, x_{\ell})\}_{\ell=1}^n \cup \{(y_{t, \ell}, x_{\ell})\}_{\ell=n+1}^{2n} \), an union bound finishes the proof.

### 6.1 Adaptation to \( r_1 \)

We now present algorithm Algorithm 6 which is adaptive to the noise level. Here we assume to know a (loose) upper bound of \( \| \Xi \| \). In this setting, larger values for \( r_1 \) and \( r \).

Defining, for every \( \zeta > 0 \),

\[
r_1(\zeta) := \left( 2 \sqrt{\frac{p}{n}} \sqrt{\frac{K}{n}} \right) \sqrt{\zeta} \quad \text{and} \quad r(\zeta) := 2 \mu^2(B_2) \sqrt{12 \frac{pK}{n} \sqrt{\zeta}},
\]

we let, only in this section, \( r_1 := r_1(\| \Xi \|) \) and \( r := r(\| \Xi \|) \). Since \( \text{tr}(\Xi) \leq p \| \Xi \| \), these values of \( (r_1, r) \) satisfy the conditions of Section 6. We only assume a loose upper bound for \( \| \Xi \| \) and assume, without loss on generality, that \( \| \Xi \| \geq 1 \).

We will need the following rate definition: given fixed \( T_1 \in \mathbb{N} \), for every \( \zeta > 0 \), let

\[
R(\zeta) := \left[ \left( \frac{2C_{\alpha_2}}{a \kappa} + \frac{2E_*}{a} \left( 1 + \frac{1}{312.5K} \right) + 5050.52 \right) \mu^2(B_2) r_1(\zeta) \right] \sqrt{3r(\zeta) e^{-\frac{a_1}{16}}}.\]
Algorithm 6 Adaptive-Robust-Regression\((D, T_1, T_2, K, \eta, S_1, S_2, \zeta_0, \gamma)\)

**Input:** sample \(D := \{(\tilde{y}_t, \tilde{x}_t)\}_{t=1}^n \cup \{(\tilde{y}_t, \tilde{x}_t)\}_{t=n+1}^{2n},\) outer and inner number of iterations \((T_1, T_2),\) number of buckets \(K,\) quantile probability \(\eta \in (0, 1/2),\) outer and inner simulation sample sizes \((S_1, S_2),\) \(\gamma \in (0, 1)\) and \(\zeta_0 > 0\) satisfying \(\gamma \zeta_0 \geq \|\Xi\|\).

**Output:** \(\hat{b}.\)

1. Set \(M := \lceil \log_{\gamma^{-1}}(\zeta_0) \rceil.\)
2. for \(\ell \in [M]\)
   3. Set \(\zeta_\ell := \gamma^\ell \zeta_0.\)
   4. Set \(\hat{b}(\zeta_\ell) \leftarrow \text{Robust-Regression}(D, T_1, T_2, K, \eta, S_1, S_2, r_1(\zeta_\ell)).\)
5. end for
6. Set \(\hat{\ell} \leftarrow \max\{\ell \in [M] : \bigcap_{j \in [\ell]} \mathbb{B}_2(\hat{b}(\zeta_\ell), R(\zeta_\ell)) \neq \emptyset\}.\)
7. Set \(\hat{\mu} \leftarrow \hat{b}(\zeta_{\hat{\ell}}).\)
8. return \(\hat{\mu}.\)

We conclude with the following result. The arguments are standard and based on Lepski’s method. See for instance [9].

**Theorem 6.2** (Noise level adaptive estimation). Grant Assumptions 1 and 2 and (17), (18), (19), (20), (21) and (22). Let \(\delta \in (0, 1)\) as in (23). Suppose that \(\|\Xi\| \geq 1.\)

Then the output \(\hat{b}\) of Algorithm 6 with inputs \(D := \{(\tilde{y}_t, \tilde{x}_t)\}_{t=1}^n \cup \{(\tilde{y}_t, \tilde{x}_t)\}_{t=n+1}^{2n},\)
\((T_1, T_2, K, \eta, S_1, S_2),\) \(\gamma \in (0, 1)\) and \(\zeta_0 > 0\) such that \(\|\Xi\| < \zeta_0 \gamma\) satisfies with probability at least \(1 - M \delta,\)

\[
\|\hat{b} - b^*\|_2 \leq 3R(\|\Xi\|/\gamma).
\]

**Proof.** Let \(\ell^* := \max\{\ell \in \mathbb{N} : \zeta_\ell \geq \|\Xi\|\}.\) Since \(1 \leq \|\Xi\| < \gamma \zeta_0,\) we have \(\ell^* \in [M]\) and \(\zeta_0 > \zeta_{\ell^*} \geq \|\Xi\| \geq \zeta_{\ell^*} \gamma.\) Define the event \(\Omega_\ell := \{b^* \in \mathbb{B}_2(\hat{b}(\zeta_\ell), R(\zeta_\ell))\}\) for all \(\ell \in [M].\)

For all \(\ell \leq \ell^*, \zeta_\ell \geq \zeta_{\ell^*} \geq \|\Xi\| \geq \zeta_{\ell^*} \gamma,\) in particular, \(r_1(\zeta_\ell) \geq r_1(\|\Xi\|)\) and \(r(\zeta_\ell) \geq r(\|\Xi\|)\) so all the conditions of Theorem 6.1 apply for such \(\ell.\) Precisely, we infer from such theorem that \(\mathbb{P}(\Omega_\ell) \geq 1 - \delta\) for all \(\ell \leq \ell^*.\) By an union bound, with probability at least \(1 - M \delta,\) we must have \(b^* \in \bigcap_{\ell=1}^{\ell^*} \mathbb{B}_2(\hat{b}(\zeta_\ell), R(\zeta_\ell)).\) The argument as follows will occur on this event.

By maximality of \(\hat{\ell}, \hat{\ell} \geq \ell^*.\) In particular, \(\zeta_{\hat{\ell}} \leq \zeta_{\ell^*};\) therefore there must exist \(b\) such that

\[
b \in \mathbb{B}_2(\hat{b}(\zeta_{\hat{\ell}}), R(\zeta_{\hat{\ell}})) \cap \mathbb{B}_2(\hat{\mu}(\zeta_{\ell^*}), R(\zeta_{\ell^*})).
\]

This and triangle inequality implies that

\[
\|\hat{b} - \hat{b}(\zeta_{\ell^*})\|_2 \leq \|\hat{b} - b\|_2 + \|\hat{b}(\zeta_{\ell^*}) - b\|_2 \leq R(\zeta_{\hat{\ell}}) + R(\zeta_{\ell^*}) \leq 2R(\zeta_{\ell^*}).
\]
Using that $b^* \in B_2(\hat{b}(\zeta_{\ell^*}), R(\zeta_{\ell^*}))$, $\zeta_{\ell^*} \leq \|\Xi\|/\gamma$ and that $\zeta \mapsto R(\zeta)$ is non-decreasing, we finally obtain that
\[
\|\hat{b} - b^*\|_2 \leq \|\hat{b} - \hat{b}(\zeta_{\ell^*})\|_2 + \|\hat{b}^* - \hat{b}(\zeta_{\ell^*})\|_2 \leq 3R(\zeta_{\ell^*}) \leq 3(\|\Xi\|/\gamma).
\]
This finishes the proof. \(\square\)

Let $\gamma \in (0, 1)$ and $\zeta_0 > 0$ such that $\|\Xi\| < \gamma\zeta_0$ and $M := \lceil \log_{1-\delta_0}(\zeta_0) \rceil$. Let $\delta_0 \in (0, 1)$ be the desired probability of failure. In the following, $C > 1$ is a constant that may change from line to line. From the conditions of Theorem 6.2, it is straightforward to check that if the sample size and contamination fraction satisfy
\[
n \geq (CL^4\kappa^2 p \log p) \sqrt{(CL^4\kappa^2 \log(CM/\delta_0))},
\]
\[
\epsilon \leq \frac{1}{CL^4\kappa^2},
\]
then, for any $T_1 \in \mathbb{N}$, tuning Algorithm 6 such that
\[
K \geq \log \left(\frac{CM/\delta_0}{\epsilon} \right) \sqrt{(C)},
\]
\[
\eta \geq \frac{C \log \left(\frac{CM/\delta_0}{\epsilon} \right)}{n} \sqrt{(4\epsilon)},
\]
\[
T_2 \geq C \log \kappa,
\]
\[
S_1 \geq C \log \left(\frac{CMT_1/\delta_0}{\epsilon} \right),
\]
\[
S_2 \geq C \log \left(\frac{CMT_1T_2/\delta_0}{\epsilon} \right),
\]
the following estimate holds with probability at least $1 - \delta_0$:
\[
\|\hat{b} - b^*\|_2^2 \lesssim \mu^4(B_2) \|\Xi\| \left(\frac{p + K}{n} \right) \sqrt{\left(\frac{pK}{n} \epsilon^{-T_1/(C\epsilon^*)}\right)}
\]
\[
\lesssim \mu^4(B_2) \|\Xi\| \left(\frac{p}{n} + \frac{\log \left(\frac{CM/\delta_0}{\epsilon} \right)}{n} + \epsilon \right) \sqrt{\left(p \left(\frac{\log \left(\frac{CM/\delta_0}{\epsilon} \right)}{n} + \epsilon \right) \epsilon^{-T_1/(C\epsilon^*)}}.
\]
Note that we can tune $T_1$ (without knowledge of $\|\Xi\|$) as
\[
T_1 \geq C\kappa^4 \log \left(\frac{pK}{p + K}\right)
\]
to obtain the optimal statistical rate. In particular, the optimal $T_1$ is independent of $\|\Xi\|$. 46
7 Appendix

Lemma 7.1 (Lemma A.3 in [20]). Let $p \in \Delta_{K,k}$ and $q$ be the uniform distribution on $[K]$. Then $\text{KL}(p\|q) \leq \frac{5k}{K}$.

7.1 Proof of Lemma 3.1

Let $Q := Q_{X,1-n/2}$. The one-sided Bernstein’s inequality applied to $1_{X > Q}$ implies: for any $t \geq 0$, with probability at least $1 - \exp(-t)$,

$$\frac{1}{n} \sum_{i=1}^{n} 1_{X_i > Q} \leq \mathbb{P}(X > Q) + \sigma \sqrt{\frac{2t}{n} + \frac{t}{3n}}, \quad (24)$$

where

$$\sigma^2 := \mathbb{E} \left[ 1_{X > Q} \right] - \mathbb{P}(X > Q) \leq \mathbb{P}(X > Q).$$

By definition of quantile and that $X$ is absolute continuous,

$$\mathbb{P}(X > Q) = \frac{n}{2}.$$

Take $t = c^2 \eta n$ for $c > 0$ satisfying $1/2 + c + c^2/3 \leq 3/4$, e.g., $c^2 = 0.563$. Then the RHS of (24) is at most $3\eta/4$.

7.2 Proof of Lemma 3.2

We only prove the first statement as the second is similar. Let $C_\alpha > 0$ to be determined. The one-sided Bernstein’s inequality applied to $1_{\hat{\mu}_k - \mathbb{E}[X] > C_\alpha r}$ implies: for any $t \geq 0$, with probability at least $1 - \exp(-t)$,

$$\frac{1}{K} \sum_{k=1}^{K} 1_{\hat{\mu}_k - \mathbb{E}[X] > C_\alpha r} \leq \mathbb{P}(\hat{\mu}_1 - \mathbb{E}[X] > C_\alpha r) + \sigma \sqrt{\frac{2t}{n} + \frac{t}{3n}}, \quad (25)$$

where

$$\sigma^2 := \mathbb{E} \left[ 1_{\hat{\mu}_1 - \mathbb{E}[X] > C_\alpha r} \right] - \mathbb{P}(\hat{\mu}_1 - \mathbb{E}[X] > C_\alpha r) \leq \mathbb{P}(\hat{\mu}_1 - \mathbb{E}[X] > C_\alpha r).$$

By definition Chebyshev’s inequality and rotation invariance,

$$\mathbb{P}(\hat{\mu}_1 - \mathbb{E}[X] > C_\alpha r) \leq \frac{K}{n} \cdot \frac{\sigma^2}{C^2_\alpha r^2} \leq \frac{1}{C^2_\alpha},$$

where we used definition of $r$.

Taking $t = K/C_\alpha$ in (25) the claim is satisfied for $C_\alpha$ satisfying

$$\frac{1}{3C_\alpha} + \frac{1}{C^2_\alpha} + \frac{\sqrt{2}}{C^{2/3}_\alpha} \leq \alpha.$$
7.3 Proof of Lemma 3.4

We only prove the first statement. Next we will take \( r := r_{X,n,K}(F) \) and a numerical constant \( C_\alpha > 0 \) to be determined. The uniform Bernstein-type concentration inequality due to Bousquet applied to the empirical process \( f \mapsto \sum_{k=1}^K 1_{\{ \hat{P}_{B_k} f - P f \geq C_\alpha r \}} \) implies in particular that, for all \( t \geq 0 \), with probability at least \( 1 - e^{-t} \),

\[
\sup_{f \in F} \frac{1}{K} \sum_{k \in [K]} 1_{P_{B_k} f - P f \geq C_\alpha r} \leq \frac{2E}{K} + \sigma \sqrt{\frac{2t}{K} + \frac{4t}{3K}},
\]

where

\[
E := \mathbb{E} \left[ \sup_{f \in F} \sum_{k \in [K]} 1_{\{ \hat{P}_{B_k} f - P f \geq C_\alpha r \}} \right],
\]

\[
\sigma^2 := \sup_{f \in F} \mathbb{E} \left( 1_{\{ \hat{P}_{B_k} f - P f \geq C_\alpha r \}} - \mathbb{E} 1_{\{ \hat{P}_{B_k} f - P f \geq C_\alpha r \}} \right)^2 \leq \sup_{f \in F} \mathbb{P}(\hat{P}_{B_k} f - P f \geq C_\alpha r).
\]

Let \( \varphi \) be a \( 2/(C_\alpha r) \)-Lipschitz function such that \( 1_{t \geq C_\alpha r} \leq \varphi(t) \leq 1_{t \geq C_\alpha r/2} \). Typical symmetrization-contraction arguments lead to

\[
\frac{E}{K} \leq \mathbb{E} \left[ \sup_{f \in F} \frac{1}{K} \sum_{k \in [K]} \varphi(\hat{P}_{B_k} f - P f) \right]
\]

\[
= \mathbb{E} \left[ \sup_{f \in F} \frac{1}{K} \sum_{k \in [K]} \varphi(\hat{P}_{B_k} f - P f) - \mathbb{E}[\varphi(\hat{P}_{B_k} f - P f)] \right] + \sup_{f \in F} \mathbb{E}[\varphi(\hat{P}_{B_k} f - P f)]
\]

\[
\leq 2\mathbb{E} \left[ \sup_{f \in F} \frac{1}{K} \sum_{k \in [K]} \epsilon_k \varphi(\hat{P}_{B_k} f - P f) \right] + \sup_{f \in F} \mathbb{E}[\varphi(\hat{P}_{B_k} f - P f)]
\]

\[
\leq \frac{4}{rK} \mathbb{E} \left[ \sup_{f \in F} \sum_{k \in [K]} \epsilon_k (\hat{P}_{B_k} f - P f) \right] + \sup_{f \in F} \mathbb{E}[\varphi(\hat{P}_{B_k} f - P f)],
\]

where, by reverse symmetrization, the Rademacher complexity of the iid sequence \( \{ \hat{P}_{B_1} f, \ldots, \hat{P}_{B_K} f \} \) of block empirical averages may bounded by

\[
\hat{\mathcal{R}} := \mathbb{E} \left[ \sup_{f \in F} \sum_{k \in [K]} \epsilon_k (\hat{P}_{B_k} f - P f) \right] \leq 2\mathbb{E} \left[ \sup_{f \in F} \left| \sum_{k \in [K]} \hat{P}_{B_k} f - P f \right| \right]
\]

\[
= \frac{2K}{n} \mathbb{E} \left[ \sup_{f \in F} \left| \sum_{i \in [n]} f(X_i) - P f \right| \right] = \frac{2K}{n} \mathcal{D}_{X,n}(F).
\]
Note that, by definition of \( r_{X,n,K}(F) \),
\[
\frac{4}{C_\alpha r} \cdot \hat{R} \leq \frac{8}{C_\alpha r} \cdot \varPhi_{X,n}(F) \leq \frac{8}{C_\alpha}.
\]

It remains to bound \( \sup_{f \in F} \mathbb{E}[\varphi(\hat{P}_{B_k}f - Pf)] \) and \( \sigma \) and gather the bounds in (26). In that regard, by Chebyshev’s inequality and rotation invariance of variance
\[
\mathbb{E}[\varphi(\hat{P}_{B_k}f - Pf)] \leq \sup_{f \in F} \mathbb{P}(\hat{P}_{B_k}f - Pf \geq C_\alpha r/2) \leq \frac{4}{n} \cdot \frac{\sigma^2_\chi(F)}{C_\alpha^2 r^2},
\]
\[
\sigma^2 \leq \sup_{f \in F} \mathbb{P}(\hat{P}_{B_k}f - Pf \geq C_\alpha r) \leq \frac{K}{n} \cdot \frac{\sigma^2_\chi(F)}{C_\alpha^2 r^2}.
\]

Again, by definition of \( r_{X,n,K}(F) \),
\[
\frac{K}{n} \cdot \frac{\sigma^2_\chi(F)}{C_\alpha^2 r^2} \leq \frac{1}{C_\alpha^2}.
\]

The statement of the lemma then follows by setting \( t = K/C_\alpha \) in (26) for a sufficiently large \( C_\alpha \) satisfying
\[
\frac{8}{C_\alpha} + \frac{4}{C_\alpha^2} + \frac{\sqrt{2}}{C_\alpha^{2/3}} + \frac{4}{3C_\alpha} \leq \alpha.
\]

### 7.4 Proof of Proposition 1

We prepare the ground to prove Proposition 1 and assume that \( z \) satisfies the \( L^4 - L^2 \) norm equivalence condition for some \( L > 0 \). Without loss on generality, we present a proof assuming \( \Sigma = I_p \) as the general case can be reduce to this one. Given \( R > 0 \), define the map \( \mathbb{R}^p \ni v \mapsto \pi^R(v) := v^R := (1 \wedge \frac{R}{\|v\|_2})v \). In particular, for all \( v, u \in \mathbb{R}^p \),
\[
\|v^R\|_2 = \phi_R(\|v\|_2) \quad \text{and} \quad \langle v^R, u \rangle \leq \langle v, u \rangle.
\]
Next, define the “truncated quadratic class”
\[
F^R := \{ f = \langle \pi^R(\cdot), u \rangle^2 : u \in \mathbb{B}_2 \}.
\]

Of course, for any \( k \in [K] \) and \( f \in F^R \),
\[
\hat{P}_{B_k}f := \frac{1}{B} \sum_{\ell \in B_k} \langle z^R_\ell, u \rangle^2 \quad \text{and} \quad Pf := \langle z^R, u \rangle^2.
\]

**Definition 7.2.** Let
\[
\mathcal{R}^R_{z,n,K}(\mathbb{B}_2) := \frac{\mathcal{R}_{z,n}(F^R)}{n} \sqrt{L^2 \sqrt{\frac{K}{n}}},
\]
Corollary 5 (Truncated Quadratic Process). Let $\alpha \in (0, 1)$ and any constant $C_\alpha > 0$ satisfying (3).

Then letting $r := \Re R_{z,n,K}(\mathbb{B}_2)$, with probability at least $1 - e^{-K/C_\alpha}$,

$$\sup_{f \in F^R} \sum_{k=1}^K 1_{\{|P_{B_k}(f-Pf)| > C_\alpha r\}} \leq \alpha K.$$  

Proof. By $L^4 - L^2$ norm equivalence,

$$\sigma_z(F^R) \leq L^2 \sigma_z(\mathbb{B}_2) = L^2,$$

implying that $r_{z,n,K}(F^R) \leq \Re R_{z,n,K}(\mathbb{B}_2)$. The claim is then immediate applying Lemma 3.4 to the class $F^R$.

We now aim in bounding

$$\mathcal{R}_{z,n}(F^R) = \mathbb{E} \left\| \sum_{i \in [n]} \epsilon_i z_i^R \otimes z_i^R \right\|.$$  

We use a standard approach via the matrix Bernstein’s inequality due to Minsker [35]. We will need the following lemma whose proof we omit.

Lemma 7.3. For all $v \in \mathbb{R}^p$,

$$\mathbb{E} \langle z^R, v \rangle^4 \leq L^4 \langle v, v \rangle^2,$$

$$\mathbb{E} \| z^R \|_2^4 \leq L^4 p^2,$$

$$\mathbb{E} \langle z^R, u \rangle^2 \geq \left(1 - \frac{L^4 p}{R^2}\right) \mathbb{E} \langle z, u \rangle^2.$$  

Next we set $S_i := \epsilon_i z_i^R \otimes z_i^R$ and $S := \sum_{i=1}^n S_i$. For all $i$, $\mathbb{E}[S_i] = 0$ and $\|S_i\| \leq R^2$. Define the “matrix variance”

$$\mathbb{V}(S) := \sum_{i=1}^n \mathbb{E}[S_i S_i^T] = n \mathbb{E}[\|z^R\|_2^2 z^R \otimes z^R].$$

We claim that $\mathbb{V}(S) \leq n L^4 p \cdot I_p$. Indeed by Lemma 7.3, for any $v \in \mathbb{R}^p$,

$$\langle v, \mathbb{V}(S) v \rangle = n \mathbb{E}[\|z^R\|_2^4 \langle v, z^R \rangle^2] \leq n \sqrt{\mathbb{E} \|z^R\|_2^4 \sqrt{\mathbb{E} \langle v, z^R \rangle^4} \leq n L^4 p \|v\|_2^2.}$$

In particular, $\|\mathbb{V}(S)\| \leq n L^4 p$. The effective rank of $n L^4 p \cdot I_p$ is $p$. The bound by Minsker [35] then yields

$$\mathcal{R}_{z,n}(F^R) = \mathbb{E}\|S\| \leq \sqrt{n L^4 p \log p + R^2 \log p},$$

implying the lemma:
Lemma 7.4. For an absolute constant $C > 0$,

$$\mathcal{R}_{z,n}(F^R) \leq C \left( L^2 \sqrt{\frac{p \log p}{n}} \sqrt{\frac{R^2}{n} \log p} \right).$$

We finalize with the proof of Proposition 1.

Proof of Proposition 1. **Upper bound:** We prove the first inequality. Lemma 3.1 applied to $X := \|z\|_2 - \mathbb{E}\|z\|_2$ together with $Q_{1-\rho/2}(X) \leq \sqrt{2p/\rho}$ and $\mathbb{E}[\|z\|_2] \leq \sqrt{p}$ imply that on an event $E_1$ of probability at least $1 - e^{-\rho n/1.8}$, for at least a fraction of $1 - 0.75\rho$ of the $n$ data points,

$$\|z\|_2 \leq \left( 1 + \sqrt{2/\rho} \right) \sqrt{p}.$$

Using Lemma 7.4 with $R := \left( 1 + \sqrt{2/\rho} \right) \sqrt{p}$ and Corollary 5, we have on an event $E_2$ of probability at least $1 - e^{-K/C_\alpha}$, for all $u \in B_2$, for at least a fraction of $1 - \alpha$ of the $K$ blocks (and corresponding data points),

$$\frac{1}{B} \sum_{\ell \in B_k} \left( \langle z^R, u \rangle^2 - \|u\|_2^2 \right) \leq C_\alpha \left[ r_{n,K} \sqrt{CC_p \frac{p \log p}{n}} \right],$$

where we used that $\mathbb{E}[\langle z^R, u \rangle^2] \leq \|u\|_2^2$.

On the event $E_1 \cap E_2$ of probability $1 - e^{-m/1.8} - e^{-K/C_\alpha}$, we invoke the pigeonhole principle so that for a fraction of $1 - (\alpha + 0.75\rho)$ of the blocks (and corresponding data points), both displayed inequalities hold. For these data points, $z^R_{\ell} = z_{\ell}$ so the claim of the lemma holds.

**Lower bound:** we now prove the second inequality. By analogous argument, Lemma 7.4 Corollary 5, we have on an event of probability at least $1 - e^{-K/C_\alpha}$, for all $u \in B_2$, for at least a fraction of $1 - \alpha$ of the $K$ blocks,

$$\frac{1}{B} \sum_{\ell \in B_k} \left( \mathbb{E}[\langle z^R, u \rangle^2] - \langle z^R_{\ell}, u \rangle^2 \right) \leq C_\alpha \left[ r_{n,K} \sqrt{C \frac{R^2 \log p}{n}} \right].$$

We now use the facts that $\langle z^R_{\ell}, u \rangle^2 \leq \langle z_{\ell}, u \rangle^2$ and, by Lemma 7.3,

$$\mathbb{E}[\langle z^R, u \rangle^2] \geq \left( 1 - \frac{L^4 p}{R^2} \right) \mathbb{E}[\langle z, u \rangle^2].$$

The proof is finished taking $R := \sqrt{\theta p}$ for given $\theta > 0$. \qed
7.5 Proof of Corollary 1

By the parallelogram law and Proposition 1, on an event of probability at least $1 - e^{-\rho n/1.8} - 2e^{-K/C_{\alpha}}$, for all $[u, v] \in \mathbb{B}_p \times \mathbb{B}_p$, for at least $(1 - (2\alpha + 0.75\rho))K$ of the blocks,

$$\frac{1}{B} \sum_{\ell \in B_k} (\langle z_\ell, u \rangle \langle z_\ell, v \rangle - \langle u, v \rangle) = \frac{1}{4B} \sum_{\ell \in B_k} (\langle z_\ell, u + v \rangle^2 - \|u + v\|^2_p)$$

$$- \frac{1}{4B} \sum_{\ell \in B_k} (\langle z_\ell, u - v \rangle^2 - \|u - v\|^2_p)$$

$$\leq \frac{|u + v|^2_p}{4} C_{\alpha} \left[ r_{n,K} \sqrt{CC_p \rho \log p} \right]$$

$$+ \frac{|u - v|^2_p}{4} \left( \frac{L^4}{\theta} + C_{\alpha} \left[ r_{n,K} \sqrt{C \theta \rho \log p} \right] \right).$$

Optimizing over $\theta$, one gets

$$\frac{L^4}{\theta} + C_{\alpha} \rho \log p \leq 2L^2 \sqrt{CC_{\alpha} \rho \log p}.$$

Using that $\|u + v\|^2_p \leq 4$ and $\|u - v\|^2_p \leq 4$ finishes the proof.

7.6 Proof sketch of Proposition 2

The proof follows similar lines as Theorem 3.1 in [36]. It suffices to prove for the case $\Sigma$ is the identity. Given $R > 0$, define $z_i^R := \left(1 \wedge \frac{R}{\|z_i\|_2}\right) z_i$. Fix $s > 0$ and $k \in [K]$. We apply the PAC-Bayesian inequality in Proposition 3.1 in [36] with covariance matrix $C := I / (Kp)$ and process

$$Z_{\theta,k} := s \mathbb{E}[\langle \theta, z_i^R \rangle^2] - s \sum_{\ell \in B_k} \frac{\langle \theta, z_\ell^R \rangle^2}{B} - \frac{s^2}{2B} \mathbb{E}[\langle \theta, z_1^R \rangle^4].$$

By Lemma B.2 in [36], one concludes that $\mathbb{E}[e^{Z_{\theta,k}}] \leq 1$ for all $\theta$. By Proposition 3.1 in [36], we deduce that, with probability at least $1 - e^{-t}$, for all $v \in S_2$,

$$\sum_{\ell \in B_k} \Gamma_{v,C} \frac{\langle \theta, z_\ell^R \rangle^2}{B} \geq \Gamma_{v,C} \mathbb{E}[\langle \theta, z_1^R \rangle^2] - \left( \frac{s}{2B} \mathbb{E}[\Gamma_{v,C} \langle \theta, z_1^R \rangle^4] + \frac{Kp + 2t}{2s} \right).$$
As in [36], one deduces from Lemma 3.1 in that paper the estimates
\[
\sum_{\ell \in B_k} \Gamma_{v_1, C} \frac{\langle \theta, z_1^R \rangle^2}{B} \leq \sum_{\ell \in B_k} \frac{\langle v, z_1 \rangle^2}{B} + \sum_{\ell \in B_k} \frac{\|z_1^R\|^2}{KpB},
\]
\[
\Gamma_{v_1, C} \mathbb{E}[\langle \theta, z_1^R \rangle^2] \geq 1 - \frac{L^4p}{R^2} + \frac{\mathbb{E}[\|z_1^R\|^2]}{Kp},
\]
and
\[
\mathbb{E} \left[ \Gamma_{v_1, C} \langle \theta, z_1^R \rangle^4 \right] \leq 4\mathbb{E} [\langle v, z_1^R \rangle^4] + 12\mathbb{E} \left[ \frac{\|z_1^R\|^4}{(Kp)^2} \right] + 8\mathbb{E} \left[ \langle v, z_1^R \rangle^2 \frac{\|z_1^R\|^2}{Kp} \right]
\]
\[
\leq 8\mathbb{E} [\langle v, z_1^R \rangle^4] + 16\mathbb{E} \left[ \frac{\|z_1^R\|^4}{(Kp)^2} \right]
\]
\[
\leq 8L^4 + \frac{16L^4}{K^2}.
\]

We also have, by Bernstein’s inequality, with probability at least \(1 - e^{-t}\),
\[
\sum_{\ell \in B_k} \frac{\|z_1^R\|^2 - \mathbb{E}[\|z_1^R\|^2]}{KpB} \leq L^2 \sqrt{\frac{2t}{K^2B}} + \frac{2R^2t}{3KpB}.
\]

Invoking an union bound and using the previous bounds, we conclude that, with probability at least \(1 - 2e^{-t}\), for all \(v \in S_2\),
\[
1 - \sum_{\ell \in B_k} \frac{\langle v, z_1 \rangle^2}{B} \leq L^2 \sqrt{\frac{2t}{K^2B}} \left( \frac{L^4p}{R^2} + \frac{2R^2t}{3KpB} \right) + \left( \frac{4L^4s}{B} \frac{t}{s} + \frac{8L^4s}{2s^2} \right) .
\]

The first term is \(L^2 \sqrt{\frac{2t}{K^2n}}\). Optimizing \(R > 0\), the second term is less than \(2L^2 \sqrt{\frac{2t}{3n}}\). We now choose \(s = s_1s_2\) with arbitrary \(s_1 > 0, i = 1, 2\). For any \(s_2\), the third term minimized at \(s_1^*\) has value \(2\sqrt{\frac{4L^4s_1s_2}{B}} = 4L^2 \sqrt{\frac{K}{n}}\). Evaluated at \(s_1^*\), the forth term minimized at \(s_2^*\) has value \(2\sqrt{\frac{8L^4s_1^*s_2Kp}{BK^2}} = 4L^2 \sqrt{\frac{p}{n}}\). Using the overestimates \(K \geq 1, 2/3 \leq 2\) and \(1 \leq 2\), finishes the proof.

### 7.7 Proof of Proposition 3

For simplicity we set \(f_i(M) := \langle z_i z_i^T, M \rangle\). The assumption implies in particular that the set
\[
\mathcal{A}(M) := \{i \in [m] : \|M^{1/2}z_i\|_2 > \sqrt{D}\}
\]

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has cardinality $|\mathcal{A}(M)| > bm$.

Let $u := \frac{M^{1/2} z_i}{\|M^{1/2} z_i\|_2}$. The angle $\angle(\theta, u)$ between $\theta$ and $u \in S_2$ is uniformly distributed over $[0, 2\pi]$. Define the constants $C := (\cos \varphi)^{-1}$ and $B := \sqrt{D}/C$. Define also the random variable

$$Z_\theta := \sum_{i=1}^m 1\{\|\langle M^{1/2} z_i, \theta \rangle\| > B\} = \sum_{i=1}^m 1\{\|z_i, M^{1/2} \theta \| > B\}.$$ 

For any $i \in \mathcal{A}(M)$,

$$\Pr\left(\|\langle M^{1/2} z_i, \theta \rangle\| > B\right) \geq \Pr\left(\|\langle M^{1/2} z_i, \theta \rangle\| > \frac{1}{C}\|M^{1/2} z_i\|_2\right)$$

$$= \Pr(\|\langle u, \theta \rangle\| > 1/C)$$

$$= \Pr(\|\cos \angle(\theta, u)\| > 1/C)$$

$$= 4\Pr(0 \leq \angle(\theta, u) < \varphi)$$

$$= \frac{4\varphi}{2\pi}.$$ 

It follows that $\mathbb{E}Z_\theta \geq |\mathcal{A}(M)| \frac{2\varphi}{\pi} \geq \frac{2\varphi}{\pi} bm$.

Note that almost surely $Z_\theta \leq m$. From Paley-Zygmund’s inequality (Proposition 3.3.1 in [10]), for all $a \in (0, 1]$,

$$\Pr(Z_\theta > am) \geq \left(\frac{\mathbb{E}Z_\theta - am}{\mathbb{E}Z_\theta^2}\right)^2 \geq \left(\frac{2\varphi b}{\pi} - a\right)^2 > 0,$$

if we assume that $\frac{2\varphi b}{\pi a} > 1$. In other words, with probability at least $\left(\frac{2\varphi b}{\pi} - a\right)^2$ the vector $v_\theta := M^{1/2} \theta \in \mathbb{B}_2$ satisfies $\sum_{i=1}^m 1\{\|z_i, v_\theta\| > B\} > am$.

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