Abstract

Carlitz and Scoville in 1973 considered a four variable polynomial that enumerates permutations in $S_n$ with respect to the parity of its descents and ascents. In recent work, Pan and Zeng proved a $q$-analogue of Carlitz-Scoville’s generating function by enumerating permutations with the above four statistics along with the inversion number. Further, they also proved a type B analogue by enumerating signed permutations with respect to the parity of descents and ascents. In this work we prove a $q$-analogue of the type B result of Pan and Zeng by enumerating permutations in $B_n$ with the above four statistics and the type B inversion number. We also obtain a $q$-analogue of the generating function for the type B bivariate alternating descent polynomials. We consider a similar five-variable polynomial in the type D Coxeter groups as well and give their egf. Alternating descents for the type D groups were previously also defined by Remmel, but our definition is slightly different. As a by-product of our proofs, we get bivariate $q$-analogues of Hyatt’s recurrences for the type B and type D Eulerian polynomials. Further corollaries of our results are some symmetry relations for these polynomials and $q$-analogues of generating functions for snakes of types B and D.

1 Introduction

For a positive integer $n$, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ be the set of permutations of $[n]$. For a permutation $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{S}_n$, an index $i \in [n-1]$ is said to be a descent of $\pi$ if $\pi_i > \pi_{i+1}$. Define $\text{DES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$ to be the set of descents of $\pi$ and let $\text{des}(\pi) = |\text{DES}(\pi)|$. The classical Eulerian polynomial is defined as the generating function of the descent statistic over $\mathfrak{S}_n$, that is,

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)}.$$

These polynomials are very well-studied. The books by Foata and Schutzenberger [4] and by Petersen [9] contain many interesting results on these polynomials. An index $i \in [n]$ is called an ascent
of \( \pi \in S_n \) if \( \pi_i < \pi_{i+1} \). Taking parity of the position of the descents, one can define odd ascents, odd descents, even ascents and even descents. Formally, let \( \text{EvenDES}(\pi) = \{ i \in [n-1] : \pi_i > \pi_{i+1}, i \text{ is even} \} \), \( \text{EvenASC}(\pi) = \{ i \in [n-1] : \pi_i < \pi_{i+1}, i \text{ is even} \} \), \( \text{OddDES}(\pi) = \{ i \in [n-1] : \pi_i > \pi_{i+1}, i \text{ is odd} \} \) and \( \text{OddASC}(\pi) = \{ i \in [n-1] : \pi_i < \pi_{i+1}, i \text{ is odd} \} \). Carlitz and Scoville in \([2]\) considered the polynomial

\[
A_n(s_0, s_1, t_0, t_1) = \sum_{\pi \in S_n} s_0^{\text{even} (\pi)} s_1^{\text{odd} (\pi)} t_0^{\text{oddes} (\pi)} t_1^{\text{edes} (\pi)}. \tag{1}
\]

They gave the exponential generating function (egf henceforth) for the above polynomial (see Theorem \([2, \text{Theorem 3.1}]\)). Pan and Zeng considered a \( q \)-analogue of the above polynomial by adding the inversion number as well. For \( \pi \in S_n \) define \( \text{inv}(\pi) = |\{ 1 \leq i < j \leq n : \pi_i > \pi_j \}| \). They considered

\[
A_n(s_0, s_1, t_0, t_1, q) = \sum_{\pi \in S_n} s_0^{\text{odd} (\pi)} s_1^{\text{even} (\pi)} t_0^{\text{edes} (\pi)} t_1^{\text{oddes} (\pi)} q^{\text{inv}(\pi)}. \tag{2}
\]

Pan and Zeng gave the following egf for \( A_n(s_0, s_1, t_0, t_1, q) \). For integers \( i \geq 0 \), define \( [i]_q = (1 + q + \cdots + q^{i-1}) \) and define \( n!_q = \prod_{i=1}^n [i]_q \). Recall that \( e_q(u) = \sum_{n \geq 0} \frac{u^n}{[n]_q!} \). Separating the odd and even terms, let

\[
\cosh_q(u) = \frac{e_q(u) + e_q(-u)}{2} \quad \text{and} \quad \sinh_q(u) = \frac{e_q(u) - e_q(-u)}{2}.
\]

Pan and Zeng in \([7, \text{Theorem 1.2}]\) showed the following (they use the variables \( x, y \) for what we denote \( t, s \) respectively.)

**Theorem 1** (Pan and Zeng). Let \( \alpha = \sqrt{(t_0 - s_0)(t_1 - s_1)} \). Then,

\[
\sum_{n \geq 1} A_n(s_0, s_1, t_0, t_1, q) u^n / n!_q = \frac{(s_1 + t_1) \cosh_q(\alpha u) + \alpha \sinh_q(\alpha u) - t_1 (\cosh_q^2(\alpha u) - \sinh_q^2(\alpha u)) - s_1}{s_0 s_1 - (s_0 t_1 + s_1 t_0) \cosh_q(\alpha u) + t_0 t_1 (\cosh_q^2(\alpha u) - \sinh_q^2(\alpha u))}. \tag{3}
\]

Using the same notation, Theorem \([1]\) gives rise to an identity for the bivariate Eulerian polynomial and the bivariate alternating Eulerian polynomial. It is easy to see (and noted by Pan and Zeng \([7]\)) among the four statistics that involve descents and ascents in Theorem \([1]\) there are choices of two of which determine the other two statistics. Indeed, using our result, we get type B and type D counterparts of Theorem \([1]\) These are presented as Theorem \([23]\) and Theorem \([33]\) respectively.

Pan and Zeng in \([7]\) also gave a type B counterpart of these identities without the variable \( q \) (that is, without taking type \( B \) inversions into account). For a positive integer \( n \), let \( [n] = \{ \pm 1, \pm 2, \ldots, \pm n \} \). \( \mathcal{B}_n \) is the set of permutations \( \pi \) of \( [n] \) that satisfy \( \pi(-i) = -\pi(i) \). Let \( \pi_0 = 0 \) for all \( \pi \in \mathcal{B}_n \) and let \( [n]_0 = \{ 0, 1, 2, \ldots, n \} \). Define \( \text{EvenDES}_B(\pi) = \{ i \in [n-1]_0 : \pi_i > \pi_{i+1}, i \text{ is even} \} \), \( \text{EvenASC}_B(\pi) = \{ i \in [n-1]_0 : \pi_i < \pi_{i+1}, i \text{ is even} \} \), \( \text{OddDES}_B(\pi) = \{ i \in [n-1]_0 : \pi_i > \pi_{i+1}, i \text{ is odd} \} \) and \( \text{OddASC}_B(\pi) = \{ i \in [n-1]_0 : \pi_i < \pi_{i+1}, i \text{ is odd} \} \). Define \( \text{odes}_B(\pi) = |\text{OddDES}_B(\pi)| \), \( \text{edes}_B(\pi) = |\text{EvenDES}_B(\pi)| \), \( \text{oasc}_B(\pi) = |\text{OddASC}_B(\pi)| \) and lastly \( \text{easc}_B(\pi) = |\text{EvenASC}_B(\pi)| \). Further, define

\[
B_n(s, t) = \sum_{\pi \in \mathcal{B}_n} s_{\text{odes}_B(\pi)} t_{\text{edes}_B(\pi)} \quad \text{and} \quad \hat{B}_n(s, t) = \sum_{\pi \in \mathcal{B}_n} s_{\text{easc}_B(\pi)} t_{\text{odes}_B(\pi)}. \tag{4}
\]

Setting \( s = t \) in the polynomial \( \hat{B}_n(s, t) \) gives \( \hat{B}_n(t) \), the type B alternating Eulerian polynomial which has been studied for example by Ma, Fang, Mansour and Yeh \([3]\).
Theorem 2 (Pan and Zeng). Let $\alpha = (1-s)(1-t)$. Then, we have
\[
\sum_{n \geq 1} B_{2n}(s, t) \frac{u^{2n}}{(2n)!} = \frac{(s + t) \sum_{n \geq 0} \frac{\alpha^n (2u)^{2n}}{(2n)!} + \sum_{n \geq 0} \frac{\alpha^{n+1} u^{2n}}{(2n)!}}{(1 + st) - (s + t) \sum_{n \geq 0} \frac{\alpha^n (2u)^{2n}}{(2n)!}} - (1 + st),
\]
(5)
\[
\sum_{n \geq 0} B_{2n+1}(s, t) \frac{u^{2n+1}}{(2n+1)!} = \frac{(s^2 - 1)(t - 1) \sum_{n \geq 0} \frac{\alpha^n u^{2n+1}}{(2n+1)!}}{(1 + st) - (s + t) \sum_{n \geq 0} \frac{\alpha^n (2u)^{2n}}{(2n)!}}.
\]
(6)
They also gave similar results about the type B alternating descent polynomials. Their result is as follows.

Theorem 3 (Pan and Zeng). Let $\alpha = (1-s)(1-t)$. Then, we have
\[
\sum_{n \geq 1} \hat{B}_{2n}(s, t) \frac{u^{2n}}{(2n)!} = \frac{(1 + st) \sum_{n \geq 0} \frac{(-\alpha)^n (2u)^{2n}}{(2n)!} + \sum_{n \geq 0} \frac{(-\alpha)^{n+1} u^{2n}}{(2n)!}}{(s + t) - (1 + st) \sum_{n \geq 0} \frac{(-\alpha)^n (2u)^{2n}}{(2n)!}} - (s + t),
\]
(7)
\[
\sum_{n \geq 0} \hat{B}_{2n+1}(s, t) \frac{u^{2n+1}}{(2n+1)!} = \frac{(1 + s) \sum_{n \geq 0} \frac{(-\alpha)^n u^{2n+1}}{(2n+1)!}}{(s + t) - (1 + st) \sum_{n \geq 0} \frac{\alpha^n (2u)^{2n}}{(2n)!}}.
\]
(8)
Let $H_0(s, t, u) = \sum_{n \geq 0} B_{2n}(s, t) \frac{u^{2n}}{(2n)!}$ and $H_1(s, t, u) = \sum_{n \geq 0} B_{2n+1}(s, t) \frac{u^{2n+1}}{(2n+1)!}$.

Recall that $\cosh(x) = \frac{1}{2}(\exp(x) + \exp(-x))$ and $\sinh(x) = \frac{1}{2}(\exp(x) - \exp(-x))$.

Define $M^2 = \alpha$.

It is easy to see that the following alternate form can be used to state Theorem 2.

Theorem 4 (Pan and Zeng). With the above notation,
\[
H_0(s, t, u) = \frac{M^2 \cosh(uM)}{M^2 \cosh^2(uM) - (s + 1)(t + 1) \sinh^2(uM)},
\]
(10)
\[
H_1(s, t, u) = \frac{M(s + 1) \sinh(uM)}{M^2 \cosh^2(uM) - (s + 1)(t + 1) \sinh^2(uM)}.
\]
(11)
Recall that length in Type B Coxeter groups is defined as follows (see [9] Page 294). For $\pi \in \mathfrak{B}_n$,
\[
\text{inv}_B(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}| + |\{1 \leq i < j \leq n : -\pi_i > \pi_j\}| + |\text{Negs}(\pi)|,
\]
(12)
where $\text{Negs}(\pi) = \{\pi_i : i > 0, \pi_i < 0\}$. Further, recall the definition of $\text{odes}_B(\pi)$ and $\text{edes}_B(\pi)$ from earlier. Define
\[
B_n(t, q) = \sum_{\pi \in \mathfrak{B}_n} t^{\text{odes}_B(\pi)} q^{\text{inv}_B(\pi)} \text{ and } B_n(s, t, q) = \sum_{\pi \in \mathfrak{B}_n} s^{\text{odes}_B(\pi)} \alpha^{\text{edes}_B(\pi)} q^{\text{inv}_B(\pi)},
\]
(13)
\[
H_0(s, t, q, u) = \sum_{k \geq 0} B_{2n}(s, t, q) \frac{u^{2n}}{B_{2n}(1, q)} \text{ and } H_1(s, t, q, u) = \sum_{k \geq 0} B_{2n+1}(s, t, q) \frac{u^{2n+1}}{B_{2n+1}(1, q)}.
\]
(14)
Let $\exp_B(u; q) = \sum_{n \geq 0} \frac{u^n}{B_n(1, q)}$. As before, we separate terms with odd and even exponents and define
\[
\cosh_B(u; q) = \frac{\exp_B(u; q) + \exp_B(-u; q)}{2} \text{ and } \sinh_B(u; q) = \frac{\exp_B(u; q) - \exp_B(-u; q)}{2}.
\]
With this notation, our first main result is the following $q$-analogue of Theorem 4.
Theorem 5. We have

\[ H_0(s, t, q, u) = \frac{(1 - s)(1 - t \cosh_q(Mu)) \cosh_B(Mu; q) + t \sinh_q(Mu) \sinh_B(Mu; q)}{1 - (s + t) \cosh_q(Mu) + ste_q(Mu)e_q(-Mu)} \]  

(15)

\[ H_1(s, t, q, u) = \frac{M \left( (1 - s \cosh_q(Mu)) \sinh_B(Mu; q) + s \sinh_q(Mu) \cosh_B(Mu; q) \right)}{1 - (s + t) \cosh_q(Mu) + ste_q(Mu)e_q(-Mu)} \]  

(16)

Theorem 5 is proved in Subsection 2.1. Recalling (4), define

\[ \hat{H}_0(s, t, u) = \sum_{n \geq 0} \hat{B}_{2n}(s, t) \frac{u^{2n}}{(2n)!} \quad \text{and} \quad \hat{H}_1(s, t, u) = \sum_{n \geq 0} \hat{B}_{2n+1}(s, t) \frac{u^{2n+1}}{(2n+1)!}. \]

We have rewritten Theorem 2 as Theorem 4 and stated our generalization as Theorem 5. Similarly, it is easy to see that Theorem 3 can be rewritten as follows.

Theorem 6 (Pan and Zeng). With the above notation,

\[ \hat{H}_0(s, t, u) = \frac{-(s - 1)(t - 1) \cos(Mu)}{s + t - (ts + 1) \cos(2Mu)}; \]

(17)

\[ \hat{H}_1(s, t, u) = \frac{-M(s + 1) \sin(Mu)}{s + t - (ts + 1) \cos(2Mu)}. \]

(18)

Define \( \hat{B}_n(s, t, q) = \sum_{\pi \in \mathcal{B}_n} \hat{\text{odes}}_{\pi} s^{\text{asc}}_{\pi} q^{\text{inv}}_{\pi} \) and let

\[ \hat{H}_1(s, t, q, u) = \sum_{n \geq 0} \hat{B}_{2n+1}(s, t, q) \frac{u^{2n+1}}{B_{2n+1}(1, q)}; \quad \text{and} \quad \hat{H}_0(s, t, q, u) = \sum_{n \geq 0} \hat{B}_{2n}(s, t, q) \frac{u^{2n}}{B_{2n}(1, q)}. \]

Moreover, let

\[ \cos_B(u; q) = \frac{\exp_B(iu; q) + \exp_B(-iu; q)}{2} \quad \text{and} \quad \sin_B(u; q) = \frac{\exp_B(iu; q) - \exp_B(-iu; q)}{2}. \]

Another of our main results is the following \( q \)-analogue of Theorem 6.

Theorem 7. We have

\[ \hat{H}_0(s, t, q, u) = \frac{(s - 1) \left( (1 - t \cos_q(Mu)) \cos_B(Mu; q) - t \sin_q(Mu) \sin_B(Mu; q) \right)}{s + te_q(iMu)e_q(-iMu) - (ts + 1) \cos_q(Mu)}, \]

(19)

\[ \hat{H}_1(s, t, q, u) = \frac{-M \left( (s - \cos_q(Mu)) \sin_B(Mu; q) + \sin_q(Mu) \cos_B(Mu; q) \right)}{s + te_q(iMu)e_q(-iMu) - (ts + 1) \cos_q(Mu)}. \]

(20)

The proof of Theorem 7 is also given in Subsection 2.1. We move to our counterpart of this result to type D Coxeter groups \( \mathcal{D}_n \). Recall that \( \mathcal{D}_n \) is the subgroup of \( \mathcal{B}_n \) consisting of the signed permutations which have an even number of negative signs. We denote \( -1 \) as \( \overline{T} \) and for \( \pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathcal{D}_n \), define \( \pi_{\overline{T}} = -\pi_1 \) and let \( \text{DES}_D(\pi) = \{ i \in \{-1, 1, \ldots, n - 1 \} : \pi_i > \pi_{i+1} \} \) be its set of descents. Let \( \text{des}_D(\pi) = |\text{DES}_D(\pi)| \). Moreover, let \( \text{OddDES}_D(\pi) = \{ i \in [-1, n - 1] \setminus \{0\} : \pi_i > \pi_{i+1} \text{ and } i \text{ is odd} \} \) be the set of odd indices where descents occur in \( \pi \) and similarly let \( \text{EvenDES}_D(\pi) = \{ i \in [-1, n - 1] \setminus \{0\} : \pi_i > \pi_{i+1} \text{ and } i \text{ is even} \} \). Let \( \text{odes}_D(\pi) = |\text{OddDES}_D(\pi)| \)
Theorem 8. We have the egfs

\[ \text{inv}_D(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}| + |\{1 \leq i < j \leq n : -\pi_i > \pi_j\}|. \]  

(21)

Remmel in [12] has given a definition of alternating descent for type D Coxeter groups based on a total order on the elements of \([\pm n]\). The polynomial that Remmel gets is different from the one we have. Remmel’s main result is a joint distribution of alternating descents and alternating major index in type B and D Coxeter groups. Below, we consider a slightly different polynomial enumerating alternating descents and type D inversion number in \(D_n\). Our definition uses the parity of the position of descents as before. Formally, define

\[
D_n(t, q) = \sum_{\pi \in D_n} e^\text{des}_D(\pi) t^{\text{inv}_D(\pi)} \quad \text{and} \quad D_n(s, t, q) = \sum_{\pi \in D_n} s^{\text{des}_D(\pi)} t^{\text{odes}_D(\pi)} q^{\text{inv}_D(\pi)}.
\]

Define

\[
\hat{D}_0(s, t, q, u) = \sum_{k \geq 0} D_{2k}(s, t, q) \frac{u^{2k}}{D_{2k}(1, q)}, \quad \hat{D}_1(s, t, q, u) = \sum_{k \geq 0} D_{2k+1}(s, t, q) \frac{u^{2k+1}}{D_{2k+1}(1, q)}.
\]

Moreover, let

\[
\exp_D(u; q) = \frac{u}{2} \quad \text{and} \quad \sinh_D(u; q) = \frac{\exp_D(u; q) - \exp_D(-u; q)}{2}.
\]

Recalling \(M\) from [9], let

\[
\text{OD} = ut^2(\cosh_q(Mu) - 1) + \frac{(1 - t)M}{(1 - s)}(\sinh_q(Mu; q) - Mu) + \frac{2t(1 - t)}{M}(\sinh_q(Mu) - Mu),
\]

\[
\text{ED} = 2t(\cosh_q(Mu) - 1) + (1 - t)(\cosh_q(Mu; q) - 1) + \frac{ut^2(1 - s)}{M} \sinh_q(Mu).
\]

For type D Coxeter groups, our main results are the following.

**Theorem 8.** We have the egfs

\[
\hat{D}_0(s, t, q, u) = \frac{\text{ED}(1 - t \cosh_q(Mu)) + \text{OD}(\frac{(1 - s)}{M} \sinh_q(Mu))}{1 - (s + t) \cosh_q(Mu) + \text{se}_q(Mu) e_q(-Mu)}, \tag{23}
\]

\[
\hat{D}_1(s, t, q, u) = \frac{\text{OD}(1 - s \cosh_q(Mu)) + \text{ED}(\frac{(1 - t)}{M} \sinh_q(Mu))}{1 - (s + t) \cosh_q(Mu) + \text{se}_q(Mu) e_q(-Mu)}. \tag{24}
\]

**Theorem 9.** We have the egfs

\[
\hat{D}_0(s, t, q, u) = \frac{T'(\text{ED})(1 - t \cosh_q(Mu)) - T'(\text{OD})(\frac{(1 - s)}{M} \sinh_q(Mu))}{s - (st + 1) \cosh_q(Mu) + \text{te}_q(iMu) e_q(-iMu)}; \tag{25}
\]

\[
\hat{D}_1(s, t, q, u) = \frac{T'(\text{OD})(s - \cos_q(Mu)) - T'(\text{ED})(\frac{(1 - t)}{M} \sinh_q(Mu))}{s - (st + 1) \cos_q(Mu) + \text{te}_q(iMu) e_q(-iMu)}. \tag{26}
\]
where

\[
\begin{align*}
T'(\text{OD}) &= \sqrt{su^2} \cos_q(Mu) - 1 - \frac{(1 - t)^M}{M} (\sin_D(Mu; q) - Mu) \\
&\quad + \frac{2t(1 - t)\sqrt{s}}{M} (\sin_q(Mu) - Mu), \\
T'(\text{ED}) &= 2t(\cos_q(Mu) - 1) + \frac{u^2(s - 1)\sqrt{s}}{sM} \sin_q(Mu) + \frac{(1 - t)}{t} (\cosh_D(Mu; q) - 1).
\end{align*}
\]

The proof of Theorem 8 and Theorem 9 appear in Subsection 3.1. It can be checked that Theorem 8 refines a result of Reiner [10, Corollary 4.5] for type D Euler-Mahonian polynomials. Our proofs in both the type B and type D cases use an inclusion-exclusion based argument.

1.1 Refining Hyatts recurrences for the Type B and Type D Eulerian polynomial

As an outcome of our proofs, we get a refinement of Hyatt’s recurrence for the type B and type D Eulerian polynomials. Hyatt in [5] gave the following recurrences for Eulerian polynomials of types B. We partition \( B_n \) based on the sign of the last element. Define \( B_n^+ = \{ \pi \in B_n : \pi_n > 0 \} \) contain the elements of \( B_n \) with last element being positive and let \( B_n^- = B_n - B_n^+ \). Define \( B_n^+(t) = \sum_{\pi \in B_n^+} t^\text{des}_B(\pi) \).

\[ B_n^+(t) = \sum_{k=0}^{n-1} \binom{n}{k} B_k(t)(t-1)^{n-k-1}. \]

Our extension of Theorem 10 involves the following polynomial. Define

\[ B_n^+(s, t, q) = \sum_{\pi \in B_n^+} s^\text{des}_B(\pi) t^\text{des}_B(\pi) q^{\text{inv}_B(\pi)}. \] (27)

Our type B generalization is the following.

**Theorem 11.** For even positive integers \( n \), we have

\[
B_n^+(s, t, q) = \sum_{r=0}^{\frac{n-1}{2}} q^{(2r+1)} \binom{n}{2r+1} q^{(s-1)^r}(t-1)^r B_{n-2r-1}(s, t, q)
+ \sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{(2r)} \binom{n}{2r} q B_{n-2r}(s, t, q)(s-1)^{r-1}(t-1)^r. \] (28)

For odd positive integers \( n \), we have

\[
B_n^+(s, t, q) = \sum_{r=0}^{\frac{n-1}{2}} q^{(2r+1)} \binom{n}{2r+1} q^{(s-1)^r}(t-1)^r B_{n-2r-1}(s, t, q)
+ \sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{(2r)} \binom{n}{2r} q B_{n-2r}(s, t, q)(s-1)^{r-1}(t-1)^{r-1}. \] (29)

It is clear that setting \( q = 1 \) and \( s = t \) in Theorem 11 gives us Theorem 10. The proof of Theorem 11 appears in Subsection 2.2. For Type D Coxeter groups, our analogous result is Theorem 34.
1.2 More consequences

Another outcome of our results are some symmetry relations. For the type B case, our results are Theorem 25 and Lemma 26. For the type D case, our symmetry results are Theorem 36 and Corollary 37.

From the $q$-analogue of our generating function, we naturally get a $q$-analogue of the enumeration of type B and D snakes. These results are presented in Section 4. Enumeration of type B and D snakes with respect to some statistics and thus $q$-analogues have been obtained, see for example, Verges [13]. However, to the best of our knowledge, we have not seen $q$-analogues involving the appropriate length function in these groups.

2 Type B results

Recall that $\mathfrak{B}_n$ is the set of permutations of $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying $\pi(-i) = -\pi(i)$. We think of $\pi$ as a word $\pi = \pi_0, \pi_1, \pi_2, \ldots, \pi_n$, where $\pi_i = \pi(i)$ and $\pi_0 = 0$.

For positive integers $n$ and an integer $i$ with $0 \leq i \leq n$, let $[n]^i = \{A \subseteq [n] : |A| = i\}$ be the set of subsets of $[n]$ with cardinality $i$. We define a signed subset $(A, \epsilon)$ to be a subset $A \subseteq [n]$ and $\epsilon$ is a string of signs $\pm$ of length $|A|$. Here, each element $a_i \in A$ has either a positive or a negative sign, encoded by $\epsilon_i$, attached to it. When $a \in A$, we denote a positive signed $a$ just by $a$ and a negative signed $a$ by $\overline{a}$. The set of all signed subsets of size $i$ of $[n]$ will be denoted as $\text{sgn}[n]^i$. Clearly, $|\text{sgn}[n]^i| = 2^i \cdot n!$.

Let $G_{n,i}$ be the set of signed permutations $\pi \in \mathfrak{B}_n$ such that the last $n - i$ elements of $\pi$ are increasing, that is we have $\pi_i < \pi_{i+1} < \pi_{i+2} < \cdots < \pi_{n-1} < \pi_n$. It is easy to see that $|G_{n,i}| = 2^n \binom{n}{i}!$.

Define $G_{n,-1}$ to be the signed permutation $\pi = 0, 1, 2, \cdots, n$, the signed permutation whose $n + 1$ elements are increasing.

Let $\sigma = 0, \sigma_1, \cdots, \sigma_{n-i} \in \mathfrak{B}_n$ and $(A, \epsilon) \in \text{sgn}[n]^i$ be a signed subset. Moreover, let $|n| - A = \{c_1, c_2, \ldots, c_{n-i}\}$ be written in ascending order, that is with $c_1 < c_2 < \cdots < c_{n-i}$. We define a map $h : \mathfrak{B}_{n-i} \rightarrow \mathfrak{B}_{\{c_1, c_2, \ldots, c_{n-i}\}}$ which for $1 \leq k \leq n-i$, maps $k$ to $c_k$ and preserves the sign. Formally,

$$h(\sigma) = 0, \pi_1, \pi_2, \ldots, \pi_{n-i},$$

where for $1 \leq i \leq n-i$, if $|\sigma_i| = k$ then $|\pi_i| = c_k$ and $\pi_i$ has the same sign as $\sigma_i$. This map $h$ is clearly a bijection and is hence invertible.

By inverting the map $h$ on the elements of $[0,n] - A$ and appending the elements of $(A, \epsilon)$ in ascending order, we get a signed permutation in $G_{n,i}$. This map is also invertible, and thus we have a bijection $f : \mathfrak{B}_{n-i} \times \text{sgn}[n]^i \rightarrow G_{n,i}$ defined below. Let $\sigma \in \mathfrak{B}_{n-i}$ and $(A, \epsilon) \in \text{sgn}[n]^i$. For a set $S$ (resp. a signed set $(S, \epsilon)$), by $[S]$ (respectively by $[(S, \epsilon)]$), we denote the string obtained by writing the elements of $S$ (respectively $(S, \epsilon)$) in ascending order in the usual linear order of $\mathbb{Z}$. Define $f(\sigma, (A, \epsilon)) = h(\sigma)[(A, \epsilon)]$ where $h(\sigma)[(A, \epsilon)]$ denotes the juxtaposition of $h(\sigma)$ and $[(A, \epsilon)]$.

Example 12. Let $n = 7, i = 4, \sigma = 0, 2, 1, 3 \in \mathfrak{B}_3$ and $(A, \epsilon) = \{1, 4, 5, 6\}$ be a signed subset of $\text{sgn}[\binom{n}{4}]$. Then, $[0,n] - A = \{0, 2, 3, 7\}$ and thus $h(\sigma) = 0, 2, 3, 7$. Moreover, we have $[(0,n] - A] = 0, 2, 3, 7$ and $[(A, \epsilon)] = 6, 4, 1, 5$. Therefore, $f(\sigma, (A, \epsilon)) = 0, 3, 2, 7, 5, 4, 1, 5$. We also have $f([0,7] - A), (A, \epsilon)) = 0, 2, 3, 7, 6, 4, 1, 5$.

Lemma 13. For positive integers $n$, we have

$$\sum_{(A,\epsilon)\in \text{sgn}[\binom{n}{r}]} q^{\text{inv}(f([0,n]-A),(A,\epsilon))} = \binom{n}{r}_q \frac{(1+q^n)(1+q^{n-1}) \cdots (1+q^{n-r+1})}{q!}.$$
Proof. We proceed by induction on \( n \). The base case when \( n = 1 \) is easy to verify. We assume the result is true for \( n \) and want to show it holds for \( n + 1 \). Thus, we want to show that

\[
\sum_{(A, \epsilon) \in \text{sgn}(\binom{n+1}{r+1})} q^{\text{inv}_B(f([0, n+1] - A], (A, \epsilon)))} = \binom{n+1}{r+1} q^{(1 + q^{n+1})(1 + q^n) \cdots (1 + q^{n-r+1})}. \tag{32}
\]

Let \( \eta(n, r) = (1 + q^n) \cdots (1 + q^{n-r+1}) \). We partition \( \text{sgn}(\binom{n+1}{r+1}) \) into the disjoint union of the following three subsets and determine the contribution of each of these three sets.

1. \( A_1 = \{(A, \epsilon) \in \text{sgn}(\binom{n+1}{r+1}) ; n + 1 \in (A, \epsilon)\} \),

2. \( A_2 = \{(A, \epsilon) \in \text{sgn}(\binom{n+1}{r+1}) ; n + 1 \in (A, \epsilon)\} \),

3. \( A_3 = \{(A, \epsilon) \in \text{sgn}(\binom{n+1}{r+1}) ; n + 1 \notin (A, \epsilon)\} \).

If \( n + 1 \in (A, \epsilon) \), as \( [(A, \epsilon)] \) is in ascending order, it will be the rightmost element of \( f([0, n+1] - A], [(A, \epsilon)] \) and thus it will contribute no extra inversions. Thus

\[
\sum_{(A, \epsilon) \in A_1} q^{\text{inv}_B(f([0, n+1] - A], (A, \epsilon)))} = \eta(n, r) \binom{n}{r}_q. \tag{33}
\]

If \( n + 1 \in (A, \epsilon) \), then \( n + 1 \) has to be in the \('n - r + 1'\)th position in \( f([0, n+1] - A], (A, \epsilon)) \). Every element of \([0, n + 1] - A\) will be to its left and will thus contribute 2 inversions. Further, every element to its right will contribute 1 inversion. Thus, we get \( 2n - r + 1 \) new inversions. Therefore,

\[
\sum_{(A, \epsilon) \in A_2} q^{\text{inv}_B(f([0, n+1] - A], (A, \epsilon)))} = \eta(n, r)q^{2n-r+1} \binom{n}{r}_q. \tag{34}
\]

Lastly, when \( n + 1 \in [0, n + 1] - A \), then it has to be the rightmost element in \([0, n + 1] - A\). Every element of \((A, \epsilon)\) will contribute one inversion and thus we get \('r + 1'\) extra inversions. Hence,

\[
\sum_{(A, \epsilon) \in A_3} q^{\text{inv}_B(f([0, n+1] - A], (A, \epsilon)))} = q^{r+1} \eta(n, r + 1) \binom{n}{r+1}_q = q^{r+1}(1 + q^{n-r}) \eta(n, r) \binom{n}{r+1}_q. \tag{35}
\]

Summing up (33), (34) and (35), we get

\[
\sum_{(A, \epsilon) \in \text{sgn}(\binom{n+1}{r+1})} q^{\text{inv}_B(f([0, n+1] - A], (A, \epsilon)))} = \eta(n, r) \binom{n}{r}_q + q^{2n-r+1} \binom{n}{r}_q + q^{r+1}(1 + q^{n-r}) \binom{n}{r+1}_q
\]

The last equation follows from the \( q \)-Pascal recurrence for the Gaussian binomial coefficients (see [9, Chapter 6]). The proof of (32) and hence of Lemma [13] is complete.

**Corollary 14.** Let \( \sigma \in \mathfrak{S}_{n-r} \) be a signed permutation and \( (A, \epsilon) \in \text{sgn}(\binom{n}{r}) \) be a signed subset. Then

\[
\sum_{(A, \epsilon) \in \text{sgn}(\binom{n}{r})} q^{\text{inv}_B(f(\sigma(A, \epsilon)))} = q^{\text{inv}_B(\sigma)} \binom{n}{r}_q (1 + q^n)(1 + q^{n-1}) \cdots (1 + q^{n-r+1}). \tag{36}
\]
Theorem 16. For positive integers \( i \), we have

\[
\text{inv}_B(f((A, \epsilon))) = \text{inv}_B(h(\sigma), [(A, \epsilon)][0, n] - A, (A, \epsilon))) = \text{inv}_B(f([0, n] - A, (A, \epsilon))) + \text{inv}_B(\sigma).
\]

The proof follows as it takes exactly \( \text{inv}_B(\sigma) \) inversions to get \( h(\sigma) \) from the identity permutation in \( \mathfrak{S}_{n-r} \) (recall \( h(\sigma) \) is defined in (30)).

Adding (30) over all \( \pi \in \mathfrak{S}_{n-r} \) gives us the following.

Corollary 15. For positive integers \( n \), we have

\[
\sum_{\sigma \in \mathfrak{S}_{n-r}} \sum_{(A, \epsilon) \in \text{sgn}(\binom{n}{r})} 1^{\text{odes}_B(\sigma)} s^{\text{odes}_B(\sigma)} q^{\text{inv}_B(f((A, \epsilon)))} = B_{n-r}(s, t, q) \binom{n}{r} (1 + q^n) \cdots (1 + q^{n-r+1}).
\]

Reiner in [11] gave the following egf for the polynomial enumerating descents and length in \( \mathfrak{S}_n \).

Theorem 16 (Reiner). We have the following.

\[
\sum_{n \geq 0} B_n(t, q) \frac{u^n}{B_n(1, q)} = \frac{(1 - t) \exp_B(u(1 - t); q)}{1 - t \exp(u(1 - t); q)}
\]

It can be seen that Theorem 16 is equivalent to the following.

Theorem 17 (Reiner). For positive integers \( n \), the polynomials \( B_n(q, t) \) satisfy the following.

\[
\frac{B_n(t, q)}{B_n(1, q)} = t \sum_{k=0}^{n} \frac{B_{n-k}(t, q)(1 - t)^k}{B_{n-k}(1, q)[k]_q!} + \frac{(1 - t)^{n+1}}{B_n(1, q)}.
\]

We are now interested in proving a trivariate analogue of Theorem 17. Towards that, we start with the following lemma.

Lemma 18. Let \( n \) be a positive integer and let \( 0 \leq i \leq n \). When \( i \) is odd, we have

\[
\sum_{\pi' \in G_{n,i}} 1^{\text{odes}_B(\pi')} s^{\text{odes}_B(\pi')} q^{\text{inv}_B(\pi')} = \frac{B_i(s, t, q) B_n(1, q)}{B_i(1, q)[n - i]_q!} + (1 - t) \left\{ \sum_{\pi' \in G_{n,i+1}} 1^{\text{odes}_B(\pi')} s^{\text{odes}_B(\pi')} q^{\text{inv}_B(\pi')} \right\}.
\]

When \( i \) is even, we have

\[
\sum_{\pi' \in G_{n,i}} 1^{\text{odes}_B(\pi')} s^{\text{odes}_B(\pi')} q^{\text{inv}_B(\pi')} = \frac{s B_i(s, t, q) B_n(1, q)}{B_i(1, q)[n - i]_q!} + (1 - s) \left\{ \sum_{\pi' \in G_{n,i+1}} 1^{\text{odes}_B(\pi')} s^{\text{odes}_B(\pi')} q^{\text{inv}_B(\pi')} \right\}.
\]
Proof. We prove (39) first and therefore take \( i \) to be odd. Let \( F_{n,i} = G_{n,i} - G_{n,i-1} \). We have

\[
\sum_{(\pi,(A,\epsilon)) \in \mathfrak{B}_i \times \text{sgn} \left( \binom{n}{i} \right)} \nu_{\text{odes}}(\pi) s_{\text{odes}}(\pi) q_{\text{inv}} B(\pi, (A,\epsilon))
\]

\[
= \sum_{(\pi,(A,\epsilon)) \in f^{-1}(G_{n,i})} \nu_{\text{odes}}(\pi) s_{\text{odes}}(\pi) q_{\text{inv}} B(\pi, (A,\epsilon))
\]

\[
= \sum_{(\pi,(A,\epsilon)) \in f^{-1}(G_{n,i-1})} \nu_{\text{odes}}(\pi) s_{\text{odes}}(\pi) q_{\text{inv}} B(\pi, (A,\epsilon))
\]

\[
+ \sum_{(\pi,(A,\epsilon)) \in f^{-1}(F_{n,i})} \nu_{\text{odes}}(\pi) s_{\text{odes}}(\pi) q_{\text{inv}} B(\pi, (A,\epsilon))
\]

\[
= \sum_{(\pi,(A,\epsilon)) \in G_{n,i-1}} \nu_{\text{odes}}(\pi) s_{\text{odes}}(\pi) q_{\text{inv}} B(\pi)
\]

\[
+ \frac{1}{t} \left\{ \sum_{(\pi,(A,\epsilon)) \in F_{n,i}} \nu_{\text{odes}}(\pi) s_{\text{odes}}(\pi) q_{\text{inv}} B(\pi, (A,\epsilon)) \right\}
\]

\[
= \sum_{\pi' \in G_{n,i-1}} \nu_{\text{odes}}(\pi') s_{\text{odes}}(\pi') q_{\text{inv}} B(\pi')
\]

\[
+ \frac{1}{t} \left\{ \sum_{\pi' \in G_{n,i}} \nu_{\text{odes}}(\pi') s_{\text{odes}}(\pi') q_{\text{inv}} B(\pi') - \sum_{\pi' \in G_{n,i-1}} \nu_{\text{odes}}(\pi') s_{\text{odes}}(\pi') q_{\text{inv}} B(\pi') \right\}.
\]

The second equality follows because \( f \) is a bijection between \( \mathfrak{B}_i \times \text{sgn} \left( \binom{n}{i} \right) \) and \( G_{n,i} \). For the fourth equality, we have used that \( i \) is odd. In the fifth equality, we are again using that \( f \) is a bijection and \( F_{n,i} = G_{n,i} - G_{n,i-1} \).

From Corollary [15] with \( i = n - r \), we have

\[
B_i(s, t, q) \left( \begin{array}{c} n \\ n - i \end{array} \right) q (1 + q^n) \cdots (1 + q^{i+1})
\]

\[
= (t - 1) \sum_{\pi' \in G_{n,i-1}} \nu_{\text{odes}}(\pi') s_{\text{odes}}(\pi') q_{\text{inv}} B(\pi') + \sum_{\pi' \in G_{n,i}} \nu_{\text{odes}}(\pi') s_{\text{odes}}(\pi') q_{\text{inv}} B(\pi').
\]

(41)

The following result is easy to see

\[
\left( \begin{array}{c} n \\ n - i \end{array} \right) q (1 + q^n) \cdots (1 + q^{i+1}) = \frac{B_n(1, q)}{B_i(1, q) [n - i]_q}.
\]

(42)

Combining (41) and (42) completes the proof of (39). The proof when \( i \) is even is similar and hence is omitted.

We are now in a position to give a refinement of Theorem [17].

**Theorem 19.** Let \( B_0(s, t, q) = 1 \). When \( n \geq 1 \), the polynomials \( B_n(s, t, q) \) satisfy the following
This completes the proof of (43). We now consider the case when $n$ is even. Let $n$ be even. By repeatedly applying (40) and (39) we have

\[
\begin{align*}
B_n(s, t, q) & = \sum_{\pi \in G_{n-1}} \text{odes}_B(\pi) s^{\text{des}_B(\pi)} q^{\text{inv}_B(\pi)} \\
& = \frac{B_{n-1}(s, t, q) B_n(1, q)}{B_{n-1}(1, q) [1]_q!} + (1-t) \sum_{\pi \in G_{n-2}} \text{odes}_B(\pi) s^{\text{des}_B(\pi)} q^{\text{inv}_B(\pi)} \\
& = \frac{B_{n-1}(s, t, q) B_n(1, q)}{B_{n-1}(1, q) [1]_q!} + (1-t) s \frac{B_{n-2}(s, t, q) B_n(1, q)}{B_{n-2}(1, q) [2]_q!} \\
& \quad + (1-t)(1-s) \sum_{\pi \in G_{n-3}} \text{odes}_B(\pi) s^{\text{des}_B(\pi)} q^{\text{inv}_B(\pi)} \\
& = (1-t)^k (1-s)^{k+1} + \sum_{r=0}^{k-1} t(1-t)^r (1-s)^{r+1} \frac{B_{n-2r-1}(s, t, q) B_n(1, q)}{B_{n-2r-1}(1, q) [2r+1]_q!} \\
& \quad + \sum_{r=0}^{k} s(1-t)^r (1-s)^{r+1} \frac{B_{n-2r}(s, t, q) B_n(1, q)}{B_{n-2r}(1, q) [2r]_q!}. 
\end{align*}
\]

This completes the proof of (43). We now consider the case when $n$ is odd. Here, we will get

\[
\begin{align*}
B_n(s, t, q) & = \sum_{\pi \in G_{n-1}} \text{odes}_B(\pi) s^{\text{des}_B(\pi)} q^{\text{inv}_B(\pi)} \\
& = s \frac{B_{n-1}(s, t, q) B_n(1, q)}{B_{n-1}(1, q) [1]_q!} + (1-s) \sum_{\pi \in G_{n-2}} \text{odes}_B(\pi) s^{\text{des}_B(\pi)} q^{\text{inv}_B(\pi)} \\
& = s \frac{B_{n-1}(s, t, q) B_n(1, q)}{B_{n-1}(1, q) [1]_q!} + (1-s) t \frac{B_{n-2}(s, t, q) B_n(1, q)}{B_{n-2}(1, q) [2]_q!} \\
& \quad + (1-s)(1-t) \sum_{\pi \in G_{n-3}} \text{odes}_B(\pi) s^{\text{des}_B(\pi)} q^{\text{inv}_B(\pi)}.
\end{align*}
\]

Continuing as in the case when $n$ was even, completes the proof of (44) and hence completes the proof of Theorem [19].

2.1 Type B Generating Functions

We recast Theorem [19] in the language of egfs to prove Theorem [5]. Recall our definitions from (14).
Proof of Theorem 5. For positive integers $n = 2k$, we have

$$\frac{B_n(s, t, q)u^{2k}}{B_n(1, q)} = \frac{(1-t)^k(1-s)^{k+1}u^{2k}}{B_n(1, q)} + \sum_{r=0}^{k} \left( \frac{(1-t)^r(1-s)^{r+1}u^{2r+1}}{[2r]_q!} \right) \left( \frac{tB_{n-2r}(s, t, q)u^{n-2r}}{B_{n-2r}(1, q)} \right) + \sum_{r=0}^{k-1} \left( \frac{(1-t)^r(1-s)^{r+1}u^{2r+1}}{[2r+1]_q!} \right) \left( \frac{sB_{n-2r-1}(s, t, q)u^{n-2r-1}}{B_{n-2r-1}(1, q)} \right).$$

(45)

When $n = 2k + 1$, we have

$$\frac{B_n(s, t, q)u^{2k+1}}{B_n(1, q)} = \frac{(1-t)^{k+1}(1-s)^{k+1}u^{2k+1}}{B_n(1, q)} + \sum_{r=0}^{k} \left( \frac{(1-t)^r(1-s)^{r+1}u^{2r+1}}{[2r]_q!} \right) \left( \frac{tB_{n-2r}(s, t, q)u^{n-2r}}{B_{n-2r}(1, q)} \right) + \sum_{r=0}^{k} \left( \frac{(1-t)^r(1-s)^{r+1}u^{2r+1}}{[2r+1]_q!} \right) \left( \frac{sB_{n-2r-1}(s, t, q)u^{n-2r-1}}{B_{n-2r-1}(1, q)} \right).$$

(46)

Summing (45), (46) over $k \geq 0$ yields

$$(1-s)\cosh_B(Mu; q) + M\sinh_B(Mu; q) = B_0 \left( 1 - s \cosh_q(Mu) - \frac{s \sinh_q(Mu)}{L} \right) + B_1(1 - t \cosh_q(Mu) - L \sinh_q(Mu)).$$

(47)

where $L = \sqrt{(1-s)/(1-t)}$, $B_0 = H_0(s, t, q, u)$ and $B_1 = H_1(s, t, q, u)$. Changing $u$ to $-u$ gives us

$$(1-s)\cosh_B(Mu; q) - M\sinh_B(Mu; q) = B_0 \left( 1 - s \cosh_q(Mu) + \frac{s \sinh_q(Mu)}{L} \right) - B_1(1 - t \cosh_q(Mu) + tL \sinh_q(Mu)).$$

(48)

Solving (47) and (48), completes the proof. \hfill \Box

Remark 20. We show that setting $q = 1$ in Theorem 5 gives Theorem 7. We claim that $H_0(s, t, 1, u) = H_0(s, t, \frac{u}{2})$ and likewise $H_1(s, t, 1, u) = H_1(s, t, \frac{u}{2})$. As $B_n(1, 1) = 2^n n!$, $\cosh_B(u; 1) = \cosh(u)$, $\sinh_B(u; 1) = \sinh(u)$, and $e_q(u) = e_q(-u)|_{q=1} = 1$, setting $q = 1$ on the right hand side of (15) gives the right hand side of (16). Similarly, setting $q = 1$ on the right hand side of (16), we get the right hand side of (11).

We are now in a position to prove Theorem 7.

Proof of Theorem 7. If $B_{2k}(s, t, q)$ is the polynomial defined in (13), it is easy to see that $\hat{B}_{2k}(s, t, q) = s^k B_{2k}(1/s, t, q)$. Therefore,

$$\hat{H}_0(s, t, q, u) = \hat{H}_0 \left( \frac{1}{s}, t, q, \sqrt{s}u \right) = \frac{(s-1) \left( (1-t \cos_q(Mu)) \cos_B(Mu; q) - t \sin_q(Mu) \sin_B(Mu; q) \right)}{s + te_q(iMu) e_q(-iMu) - (ts + 1) \cos_q(Mu)}. $$

(49)
As the proof is complete, we move on to the case when \( n = 2k + 1 \) is odd. Clearly, in this case, we have
\[
\hat{B}_{2k+1}(s, t, q) = s^{k+1} B_{2k+1}\left(\frac{4}{s}, t, q\right).
\]
Therefore,
\[
\hat{H}_1(s, t, q, v) = \sqrt{s} H_1\left(\frac{1}{s}, t, q, \sqrt{s} u\right)
= -M\left(\frac{(s - \cos_q(Mu) \sin_B(Mu; q) + \sin_q(Mu) \cos_B(Mu; q))}{s + t e_q(iMu) e_q(-iMu) - (ts + 1) \cos_q(Mu)}\right).
\]
(50)
This completes the proof. \(\square\)

**Corollary 21.** We have the following egf for the type \( B \) bivariate alternating descent polynomials:
\[
\sum_{n \geq 0} \hat{B}_n(s, t) \frac{u^n}{n!} = \frac{-(s - 1)(t - 1) \cos(Mu) - M(s + 1) \sin(Mu)}{s + t - (ts + 1) \cos(2Mu)}.
\]
(51)

**Corollary 22.** We get an alternate proof of the following egf for the type \( B \) alternating descent polynomials (see also [6] and [8]):
\[
\sum_{n \geq 0} \hat{B}_n(t) \frac{u^n}{n!} = \frac{-(t - 1)^2 \cos(1 - t) u + (t^2 - 1) \sin(1 - t) u}{2s - (t^2 + 1) \cos(2(1 - t) u)}.
\]
(52)

As mentioned in Section [1], though we consider a two variable enumerator, we can get a four variable version and hence a type \( B \) counterpart of Theorem [11]. Define variables \( s_0, t_0, s_1 \) and \( t_1 \) to keep track of even ascents, even descents, odd ascents and odd descents respectively. Let \( m = \sqrt{(s_0 - t_0)(s_1 - t_1)} \).

Define the five variable distribution
\[
B_n(s_0, s_1, t_0, t_1, q) = \sum_{w \in \mathbb{B}_n} e_{asc_B(w)} t_{s_0} e_{asc_B(w)} t_{s_1} e_{des_B(w)} t_{t_0} e_{des_B(w)} t_{t_1} q^{inv_B(w)}.
\]

Further, define the generating functions
\[
H_0(s_0, s_1, t_0, t_1, q, u) = \sum_{k \geq 0} B_{2k}(s_0, s_1, t_0, t_1, q) \frac{u^{2k}}{B_{2k}(1, q)},
\]
(53)
\[
H_1(s_0, s_1, t_0, t_1, q, u) = \sum_{k \geq 0} B_{2k+1}(s_0, s_1, t_0, t_1, q) \frac{u^{2k+1}}{B_{2k+1}(1, q)}.
\]
(54)

**Theorem 23.** We have the egfs
\[
H_0(s_0, s_1, t_0, t_1, q, u) = \frac{\left(s_1 - t_1 \cosh_q(Mu) \cosh_B(Mu; q) + t_1 \sinh_q(Mu) \sinh_B(Mu; q)\right)}{s_0 s_1 - (t_0 s_1 + s_0 t_1) \cosh_q(Mu) + t_0 t_1 e_q(Mu) e_q(-mu)},
\]
\[
H_1(s_0, s_1, t_0, t_1, q, u) = \frac{m \left(s_0 - t_0 \cosh_q(Mu) \sinh_B(Mu; q) + t_0 \sinh_q(Mu) \cosh_B(Mu; q)\right)}{s_0 s_1 - (s_1 t_0 + t_1 s_0) \cosh_q(Mu) + t_0 t_1 e_q(Mu) e_q(-mu)}.
\]

Proof. Recalling [14], it is easy to see that
\[
H_0(s_0, s_1, t_0, t_1, q, u) = H_0\left(\frac{t_0}{s_0}, \frac{t_1}{s_1}, q, \sqrt{s_0 s_1 u}\right),
\]
\[
H_1(s_0, s_1, t_0, t_1, q, u) = \sqrt{s_0 \sqrt{s_1}} H_1\left(\frac{t_0}{s_0}, \frac{t_1}{s_1}, q, \sqrt{s_0 s_1 u}\right).
\]
The proof is complete. \(\square\)
2.2 $q$-analogaues of Hyatt’s recurrences and symmetries

Hyatt gives a proof of Theorem 10 for the polynomials $B_n^+(t)$ by considering a statistic $\text{maxdrop}_B$. We give an inclusion exclusion argument.

**Proof of Theorem 11**. We prove the two recurrences separately. Let $\hat{A}_k$ be the set of signed permutations in $\mathfrak{B}_n$ such that the last $k+1$ elements are positive and are arranged in descending order. Thus, the set $\hat{A}_{k-1} - \hat{A}_k$ is the set of signed permutations with no descent in the $(n-k)$-th position and with their last $k$ elements being positive and descending. Define

$$A_k(s,t,q) = \sum_{w \in \hat{A}_k} t^{\text{odes}_B(w)} s^{\text{odes}_B(w)} q^{\text{inv}_B(w)}.$$

We abbreviate $A_k(s,t,q)$ as $A_k$ for the rest of this proof for better readability. When $n$ is even, we claim that

$$q^2 \binom{n}{2r}_q B_{n-2r}(s,t,q) s^r t^r = sA_{2r-1} - (s-1)A_{2r}, \quad (55)$$

$$q^{2r+1} \binom{n}{2r+1}_q B_{n-2r-1}(s,t,q) s^{r+1} t^{r+1} = tA_{2r} - (t-1)A_{2r+1}. \quad (56)$$

When $n$ is odd, we claim that

$$q^2 \binom{n}{2r}_q B_{n-2r}(s,t,q) s^r t^r = tA_{2r-1} - (t-1)A_{2r}, \quad (57)$$

$$q^{2r+1} \binom{n}{2r+1}_q B_{n-2r-1}(s,t,q) s^{r+1} t^{r+1} = sA_{2r} - (s-1)A_{2r+1}. \quad (58)$$

We only prove (55). The proofs of (56), (57) and (58) follow from a very similar argument. Recall that $\binom{n}{n-i}$ is the set of all $(n-i)$-sized subsets of $[n]$. Given $A \in \binom{n}{n-i}$, we arrange its elements in descending order and list them as $a_1, a_2, \cdots, a_{n-i}$ with $a_1 > a_2 > \cdots > a_{n-i} > 0$. Define a new juxtaposition map $f' : \mathfrak{B}_i \times \binom{n}{n-i} \to \hat{A}_{n-i-1}$ that takes $(\psi, A)$ to the signed permutation $\psi_{[n]-A, a_1, a_2, \cdots, a_{n-i}}$, i.e.

$$f'(\psi, A) = \psi_{[n]-A, a_1, a_2, \cdots, a_{n-i}}.$$

It is easy to see that $f'$ is a bijection from $\mathfrak{B}_i \times \binom{n}{n-i}$ to $\hat{A}_{n-i-1}$. We define $\text{inv}_B([X],[Y])$ to be the number of inversions that occur between the $X$ and $Y$. The LHS of (55) is clearly obtained as follows

$$\sum_{(\psi, A) \in \mathfrak{B}_{n-2r} \times \binom{n-i}{2r}} t^{\text{odes}_B(\psi)+\text{odes}_B(A)} s^{\text{odes}_B(\psi)+\text{odes}_B(A)} q^{\text{inv}_B(\psi)+\text{inv}_B(A)+\text{inv}_B([\psi],[A])}$$

$$= q^2 \binom{n}{2r}_q s^r t^r \sum_{(\psi, A) \in \mathfrak{B}_{n-2r} \times \binom{n-i}{2r}} t^{\text{odes}_B(\psi)} s^{\text{odes}_B(\psi)} q^{\text{inv}_B(\psi)+\text{inv}_B([\psi],[A])}$$

$$= q^2 \binom{n}{2r}_q s^r t^r \sum_{\psi \in \mathfrak{B}_{n-2r}} \sum_{A \in \binom{n-i}{2r}} t^{\text{odes}_B(\psi)} s^{\text{odes}_B(\psi)} q^{\text{inv}_B(\psi)+\text{inv}_B([\psi],[A])}$$

$$= q^2 \binom{n}{2r}_q s^r t^r \binom{n}{2r}_q B_{n-2r}(s,t,q).$$

The expression above does not account for the descent occurring at the $(n-2r)$-th position. Thus, it is off by a factor of $\frac{1}{s}$ on the set $A_{2r}$. Further, it counts correctly on the set $A_{2r-1} - A_{2r}$. This gives us

$$q^2 \binom{n}{2r}_q s^r t^r \binom{n}{2r}_q B_{n-2r}(s,t,q) = A_{2r-1} - A_{2r} + \frac{1}{s}A_{2r},$$

$$= A_{2r-1} - A_{2r},$$
Lemma 24. Let \( f : \mathcal{B}_n \to \mathcal{B}_n \) be the involution that sends \( w = w_1, \ldots, w_n \) to \( \overline{w} = \overline{w_1}, \ldots, \overline{w_n} \). Then, we have the following.

1. When \( n = 2k+1 \), we have \( \text{odes}_B(w) + \text{odes}_B(f(w)) = k \) and when \( n = 2k \), we have \( \text{odes}_B(w) + \text{odes}_B(f(w)) = k \).

2. When \( n = 2k+1 \), we have \( \text{odes}_B(w) + \text{odes}_B(f(w)) = k + 1 \) and when \( n = 2k \), we have \( \text{odes}_B(w) + \text{odes}_B(f(w)) = k \).

3. The sum \( \text{inv}_B(w) + \text{inv}_B(f(w)) = n^2 \).

Proof. The proof of the first two assertions are straightforward and hence omitted. For the third part, we recall that \( \text{inv}_B(w) = \text{inv}(w) + \text{NegSum}(w) \) where \( \text{inv}(w) = |\{(i, j) : 1 \leq i < j \leq n : w_i > w_j\}| \) and \( \text{NegSum}(w) = \sum_{i \in \text{Neg}(w)} i \). Thus, we have

\[
\text{inv}_B(w) + \text{inv}_B(f(w)) = \text{inv}(w) + \text{NegSum}(w) + \text{inv}(\overline{w}) + \text{NegSum}(\overline{w})
\]

\[
= \text{inv}(w) + \text{inv}(\overline{w}) + \text{NegSum}(w) + \text{NegSum}(\overline{w})
\]

\[
= \left( \frac{n+1}{2} \right) + \left( \frac{n}{2} \right) = n^2.
\]

The proof is complete.

---

which is equivalent to \([55]\). Similarly for \(2r + 1\), we get \([56]\)

\[
q^{(2r+1)} \frac{n}{2r+1} \binom{n}{2r+1} B_{n-2r-1}(s, t, q) = t A_{2r} - (t - 1) A_{2r+1}.
\]

Equations \([55]\) and \([56]\) gives

\[
q^{(2r+1)} \frac{n}{2r+1} B_{n-2r-1}(s, t, q)(s-1)^{r-1}(t-1)^r
\]

\[
= \left( \frac{s-1}{s} \right)^{r-1} \left( \frac{t-1}{t} \right)^r A_{2r} - \left( \frac{s-1}{s} \right)^r \left( \frac{t-1}{t} \right)^r A_{2r}, \quad (59)
\]

\[
q^{(2r+1)} \frac{n}{2r+1} B_{n-2r}(s, t, q)(s-1)^r(t-1)^r
\]

\[
= \left( \frac{s-1}{s} \right)^{(r)} \left( \frac{t-1}{t} \right)^r A_{2r} - \left( \frac{s-1}{s} \right)^r \left( \frac{t-1}{t} \right)^r A_{2r+1}. \quad (60)
\]

Summing \([59]\), over the indices \(1 \leq r \leq \frac{n}{2} \) and \([60]\) over the indices \(0 \leq r \leq \frac{n-2}{2} \), we get

\[
A_0 = \sum_{r=0}^{\frac{n-2}{2}} q^{(2r+1)} \frac{n}{2r+1} B_{n-2r-1}(s, t, q)(s-1)^{r-1}(t-1)^r
\]

\[
+ \sum_{r=1}^{\frac{n}{2}} q^{(2r)} \frac{n}{2r} B_{n-2r}(s, t, q)(s-1)^r(t-1)^r.
\]

As \(\hat{A}_0\) is the set of signed permutations with elements having positive last element (ie \(\mathcal{B}_n^+\)), this completes our proof. \(\square\)

We recall the polynomials \(B^+_n(s, t, q)\) and \(B^-_n(s, t, q)\) from \([27]\). We consider the map that flips the sign of all elements below and give a few properties.

Lemma 24. Let \( f : \mathcal{B}_n \to \mathcal{B}_n \) be the involution that sends \( w = w_1, \ldots, w_n \) to \( \overline{w} = \overline{w_1}, \ldots, \overline{w_n} \). Then, we have the following.

1. When \( n = 2k+1 \), we have \( \text{odes}_B(w) + \text{odes}_B(f(w)) = k \) and when \( n = 2k \), we have \( \text{odes}_B(w) + \text{odes}_B(f(w)) = k \).

2. When \( n = 2k+1 \), we have \( \text{odes}_B(w) + \text{odes}_B(f(w)) = k + 1 \) and when \( n = 2k \), we have \( \text{odes}_B(w) + \text{odes}_B(f(w)) = k \).

3. The sum \( \text{inv}_B(w) + \text{inv}_B(f(w)) = n^2 \).

Proof. The proof of the first two assertions are straightforward and hence omitted. For the third part, we recall that \( \text{inv}_B(w) = \text{inv}(w) + \text{NegSum}(w) \) where \( \text{inv}(w) = |\{(i, j) : 1 \leq i < j \leq n : w_i > w_j\}| \) and \( \text{NegSum}(w) = \sum_{i \in \text{Neg}(w)} i \). Thus, we have

\[
\text{inv}_B(w) + \text{inv}_B(f(w)) = \text{inv}(w) + \text{NegSum}(w) + \text{inv}(\overline{w}) + \text{NegSum}(\overline{w})
\]

\[
= \text{inv}(w) + \text{inv}(\overline{w}) + \text{NegSum}(w) + \text{NegSum}(\overline{w})
\]

\[
= \left( \frac{n+1}{2} \right) + \left( \frac{n}{2} \right) = n^2.
\]

The proof is complete. \(\square\)
2.3 Symmetry results

Theorem 25. For positive integers \( n \), we have
\[
B_n^-(s, t, q) = q^{n^2} s^{k+1} t^k B_n^+(s^{-1}, t^{-1}, q^{-1}) \quad \text{when } n = 2k + 1,
\]
\[
B_n^-(s, t, q) = q^{n^2} s^k t^k B_n^+(s^{-1}, t^{-1}, q^{-1}) \quad \text{when } n = 2k.
\]

Therefore, we have
\[
B_n(s, t, q) = B_n^+(s, t, q) + q^{n^2} s^{k+1} t^k B_n^+(s^{-1}, t^{-1}, q^{-1}) \quad \text{when } n = 2k + 1,
\]
\[
B_n(s, t, q) = B_n^+(s, t, q) + q^{n^2} s^k t^k B_n^+(s^{-1}, t^{-1}, q^{-1}) \quad \text{when } n = 2k.
\]

Proof. Let \( f : \mathfrak{B}_n^+ \to \mathfrak{B}_n^- \) be the map that sends \( w = w_1, \ldots, w_n \) to \( \overline{w} = \overline{w_1}, \ldots, \overline{w_n} \). By Lemma 24 when \( n = 2k \), we have
\[
\sum_{w \in \mathfrak{B}_n^-} t^{odes_B(w)} s^{edes_B(w)} q^{inv_B(w)} = \sum_{w \in \mathfrak{B}_n^+} t^{odes_B(f(w))} s^{edes_B(f(w))} q^{inv_B(f(w))} = \sum_{w \in \mathfrak{B}_n^+} t^{k-odes_B(w)} s^{k-edes_B(w)} q^{n^2- inv_B(w)} = q^{n^2} s^k t^k \sum_{w \in \mathfrak{B}_n^+} t^{-odes_B(w)} s^{-edes_B(w)} q^{- inv_B(w)}.
\]

When \( n = 2k + 1 \), we have
\[
\sum_{w \in \mathfrak{B}_n^-} t^{odes_B(w)} s^{edes_B(w)} q^{inv_B(w)} = \sum_{w \in \mathfrak{B}_n^+} t^{odes_B(f(w))} s^{edes_B(f(w))} q^{inv_B(f(w))} = \sum_{w \in \mathfrak{B}_n^+} t^{k+1-odes_B(w)} s^{k+1-edes_B(w)} q^{n^2- inv_B(w)} = q^{n^2} s^{k+1} t^k \sum_{w \in \mathfrak{B}_n^+} t^{-odes_B(w)} s^{-edes_B(w)} q^{- inv_B(w)}.
\]

completing the proof. \( \Box \)

In a similar manner, the following result also follows.

Lemma 26. We have
\[
B_{2k}(s, t, q) = q^{n^2} s^k t^k B_{2k}(\frac{1}{s}, \frac{1}{t}, \frac{1}{q}) \quad \text{when } n = 2k,
\]
\[
B_{2k+1}(s, t, q) = q^{n^2} s^{k+1} t^k B_{2k+1}(\frac{1}{s}, \frac{1}{t}, \frac{1}{q}) \quad \text{when } n = 2k + 1.
\]

3 Type D analogues

Let \( H_{n,i} \) be the set of signed permutations \( \pi \in \mathfrak{D}_n \) such that the last \( n - i \) elements of \( \pi \) are increasing, that is we have \( \pi_{i+1} < \pi_{i+2} < \cdots < \pi_{n-1} < \pi_n \). Clearly, \( |H_{n,i}| = 2^{n-1} \binom{n}{i} i! \).

Let \( \sigma = \sigma_1, \ldots, \sigma_{n-1} \in \mathfrak{D}_{n-i} \) and \( (A, \epsilon) \in \text{sgn} \left( \binom{n}{i} \right) \) be a signed subset. Moreover, let \( \text{sgn} \left( \binom{n}{i} \right) \) be written in ascending order. Thus, \( c_1 < c_2 < \cdots < c_{n-i} \). We define two maps \( h : \mathfrak{D}_{n-i} \to \mathfrak{D}_{\{c_1, c_2, \ldots, c_{n-i}\}} \) and \( h_D : \mathfrak{D}_{n-i} \to \mathfrak{B}_{c_1, c_2, \ldots, c_{n-i}} - \mathfrak{D}_{\{c_1, c_2, \ldots, c_{n-i}\}} \) as follows:
\[
h(\sigma) = \pi_1, \pi_2, \ldots, \pi_{n-i} \quad \text{and} \quad h_D(\sigma) = \overline{\pi_1}, \overline{\pi_2}, \ldots, \overline{\pi_{n-i}},
\]
where for $1 \leq i \leq n-i$, if $|\sigma_i| = k$ then $|\pi_i| = c_k$ and $\pi_i$ has the same sign as $\sigma_i$. Both maps $h, h_D$ are clearly bijections and hence invertible.

If $(A, \epsilon)$ has an even number of negative elements, then by inverting the map $h$ on the elements of $[0, n] - A$ and appending the elements of $(A, \epsilon)$ in ascending order, we get a signed permutation in $H_{n,n-i}$. Similarly, if $(A, \epsilon)$ has odd number of negative elements, then by inverting the map $h_D$ on the elements of $[0, n] - A$ and appending the elements of $(A, \epsilon)$ in ascending order, we get a signed permutation in $H_{n,n-i}$.

These maps are also invertible, so we have a bijection $f_D : D_{n-i} \times \text{sgn}([n]) \mapsto H_{n,n-i}$ defined as follows. Let $\sigma \in D_{n-i}$ and $(A, \epsilon) \in \text{sgn}([n])$.

Define

$$f_D(\sigma, (A, \epsilon)) = \begin{cases} h(\sigma)[(A, \epsilon)] & \text{if } (A, \epsilon) \text{ has even no. of negatives} \\ h_D(\sigma)[(A, \epsilon)] & \text{if } (A, \epsilon) \text{ has odd no. of negatives} \end{cases}$$

where $(A, \epsilon)$ is juxtaposed at the end of the $h(\sigma)$ or $h_D(\sigma)$.

We start with the following type D counterpart of Lemma 13.

**Lemma 27.** Let $(A, \epsilon) \in \text{sgn}([n])$ be a signed subset of $[n]$. Then,

$$\sum_{(A, \epsilon) \in \text{sgn}([n])} q^{\text{inv}_D(f_D([n] - A], [(A, \epsilon)])) = \binom{n}{r} q \left(1 + q^{n-1}\right) \left(1 + q^{n-2}\right) \cdots \left(1 + q^{n-r}\right). \quad \text{(61)}$$

**Proof.** We proceed by induction on $n$. The base case when $n = 1$ is easy. We assume that our Lemma is true for $n$ and show that it holds for $n + 1$. Thus, we need to show the following:

$$\sum_{(A, \epsilon) \in \text{sgn}([n+1])} q^{\text{inv}_D(f_D([n+1] - A], [(A, \epsilon)])) = \binom{n+1}{r+1} q \left(1 + q^{n}\right) \left(1 + q^{n-1}\right) \cdots \left(1 + q^{n-r}\right). \quad \text{(62)}$$

Let $\eta(n, r) = \left(1 + q^{n-1}\right) \cdots \left(1 + q^{n-r}\right)$. We partition $\text{sgn}([n+1])$ into the disjoint union of the following three subsets.

1. $A_1 = \{(A, \epsilon) \in \text{sgn}([n+1]) : n + 1 \in (A, \epsilon)\}$,
2. $A_2 = \{(A, \epsilon) \in \text{sgn}([n+1]) : \overline{n+1} \in (A, \epsilon)\}$,
3. $A_3 = \{(A, \epsilon) \in \text{sgn}([n+1]) : n + 1 \notin (A, \epsilon)\}$.

We next determine the contribution to $\sum_{(A, \epsilon) \in \text{sgn}([n+1])} q^{\text{inv}_D(f_D([n+1] - A], [(A, \epsilon)]))$ from each of the above sets. If $n + 1 \in (A, \epsilon)$, as $[(A, \epsilon)]$ is in ascending order, it will be the rightmost element of $f([n+1] - A], [(A, \epsilon)]$ and thus it will contribute no extra inversions. Thus

$$\sum_{(A, \epsilon) \in A_1} q^{\text{inv}_D(f_D([n+1] - A], [(A, \epsilon)])) = \eta(n, r) \binom{n}{r}. \quad \text{(63)}$$

If $\overline{n+1} \in (A, \epsilon)$, then $\overline{n+1}$ has to be in the $(n - r + 1)$-th position in $f([n+1] - A], (A, \epsilon))$. Every element of $[n+1] - A]$ will be to its left and will thus contribute 2 inversions. Further, every element to its right will contribute 1 inversion. Thus, we get $2n - r$ new inversions. Therefore,

$$\sum_{(A, \epsilon) \in A_2} q^{\text{inv}_D(f_D([n+1] - A], [(A, \epsilon)])) = \eta(n, r)q^{2n-r} \binom{n}{r}. \quad \text{(64)}$$

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Lastly, when \( n + 1 \in [n + 1] - A \), then it has to be the rightmost element in \([n + 1] - A\). Every element of \((A, \epsilon)\) will contribute one inversion and thus we get ‘\(r + 1\)’ extra inversions. Hence,

\[
\sum_{(A,\epsilon)\in A_3} q^{\text{inv}_D(f_D([[n+1]-A],[A,\epsilon]))} = q^{r+1} \eta(n, r+1) \binom{n}{r+1}_q = q^{r+1}(1 + q^{n-r-1}) \eta(n, r+1) \binom{n}{r+1}_q.
\]

(65)

Summing up (63), (64) and (65), we get

\[
\sum_{(A,\epsilon)\in \text{sgn}(\binom{n+1}{r+1})} q^{\text{inv}_D(f_D([[n+1]-A],[A,\epsilon]))} = \eta(n, r+1) \binom{n}{r+1}_q + q^{2n-r} \binom{n}{r}_q + q^{r+1}(1 + q^{n-r-1}) \binom{n}{r+1}_q = \eta(n+1, r+1) \binom{n+1}{r+1}_q.
\]

The last equation follows from the \(q\)-Pascal recurrence for the Gaussian binomial coefficients. This completes the proof.

\[\square\]

**Corollary 28.** Let \( \sigma \in \mathcal{D}_{n-r} \) be a signed permutation, \((A, \epsilon) \in \text{sgn}(\binom{n}{r})\) be a signed subset.

\[
\sum_{(A,\epsilon)\in \text{sgn}(\binom{n}{r})} q^{\text{inv}_D(f_D(\sigma,[A,\epsilon])))} = q^{\text{inv}_D(\sigma)} \binom{n}{r}_q (1 + q^{n-1})(1 + q^{n-2}) \cdots (1 + q^{n-r}).
\]

(66)

**Proof.** The proof follows exactly as the proof of Corollary[14]. The result follows by noting that changing the sign of the first element does not affect the type D inversion statistic.

\[\square\]

**Corollary 29.** We have

\[
\sum_{\sigma \in \mathcal{D}_{n-r}} \sum_{(A,\epsilon)\in \text{sgn}(\binom{n}{r})} \text{odes}_D(\sigma) \text{odes}_D(\sigma) q^{\text{inv}_D(f_D(\sigma,[A,\epsilon])))} = D_{n-r}(s, t, q) \binom{n}{r}_q (1 + q^{n-1})(1 + q^{n-2}) \cdots (1 + q^{n-r}).
\]

**Proof.** For a particular \( \sigma \in \mathcal{D}_{n-r} \), we have

\[
\text{odes}_D(\sigma) = \sum_{(A,\epsilon)\in \text{sgn}(\binom{n}{r})} q^{\text{inv}_D(f_D(\sigma,[A,\epsilon])))}
\]

\[
\binom{n}{r}_q (1 + q^{n-1})(1 + q^{n-2}) \cdots (1 + q^{n-r})
\]

Summing over all possible \( \sigma \in \mathcal{D}_{n-r} \) finishes the proof.

\[\square\]

**Lemma 30.** Let \( X_{\{1,\bar{1}\}} \) be the set of signed permutations in \( \mathcal{D}_n \) such that the descent set is a subset of \( \{1, \bar{1}\} \). Then,

\[
\sum_{w \in X_{\{1,\bar{1}\}}} \text{odes}_D(w) \text{odes}_D(w) q^{\text{inv}_D(w)} = t^2 \frac{D_n(1, q)}{[n-1]_q^t} + t(1-t) \left( \frac{2D_n(1, q)}{[n]_q^t} - 1 \right) + (1-t).
\]

(67)
Proof. Let $Y_1 = X_{(1,\pi)}$ be the set of signed permutations of $D_n$ with the last $n-1$ elements in ascending order and $Y_0$ be the set of signed permutations of $D_n$ such that the descent set is a subset of \{1\} or \{T\}. Then, by inclusion-exclusion, we can say that

$$
\sum_{w \in X_{(1,\pi)}} t^{odes_D(w)} s^{edes_D(w)} q^{inv_D(w)} = t^2 \left( \sum_{w \in Y_1} q^{inv_D(w)} \right) + t(1-t) \left( \sum_{w \in Y_0} q^{inv_D(w)} \right) + (1-t).
$$

(68)

The equation

$$
\sum_{w \in Y_1} q^{inv_D(w)} = \frac{D_n(1,q)}{[n-1]q!}
$$

(69)

comes from (27).

We just need to show that

$$
\sum_{w \in Y_0} q^{inv_D(w)} = \frac{2D_n(1,q)}{[n]q!} - 1
$$

(70)

If we want a descent at \{1\} but not at \{T\} or vice versa, then we need $|\pi(1)| > |\pi(2)|$. This can be done in the following way. We assign signs to the elements of $[n]$ and arrange them in ascending order. Then, choose the sign of the first element accordingly to make it an element of $D_n$ (i.e. to make the total number of negative signs even). An element $i$ will either contribute 1 if it is positive or $q^{i-1}$ if it is negative, giving the term $(1 + q^{i-1})$. Therefore, the total contribution would be $(1 + q^0)(1 + q) \cdots (1 + q^{n-1})$. However, this procedure also produces $1, 2, \ldots, n$ and $\overline{1}, \overline{2}, \ldots, \overline{n}$, out of which we only need the former. The latter has a length of 1 which we subtract to complete the proof. \qed

Lemma 31. With the above notations, when $i$ is odd, we have

$$
\sum_{\pi' \in H_{n,i}} t^{odes_D(\pi')} s^{edes_D(\pi')} q^{inv_D(\pi')} = \frac{D_i(s,t,q)D_n(1,q)}{D_i(1,q)[n-i]q!} + (1-t) \left\{ \sum_{\pi' \in H_{n,i-1}} t^{odes_D(\pi')} s^{edes_D(\pi')} q^{inv_D(\pi')} \right\}
$$

(71)

When $i$ is even, we have

$$
\sum_{\pi' \in H_{n,i}} t^{odes_D(\pi')} s^{edes_D(\pi')} q^{inv_D(\pi')} = \frac{sD_i(s,t,q)D_n(1,q)}{D_i(1,q)[n-i]q!} + (1-s) \left\{ \sum_{\pi' \in H_{n,i-1}} t^{odes_D(\pi')} s^{edes_D(\pi')} q^{inv_D(\pi')} \right\}.
$$

(72)

Proof. We at first prove (71) and therefore take $i$ to be odd. We evaluate
Define Theorem 32. This completes the proof of (71). The proof when \(i\) is odd, we have used that \(f\) is a bijection between \(\mathcal{D}_i \times \text{sgn}(\frac{n}{n-i})\) to \(H_{n,i}\). For the fourth equality, we have again used that \(f\) is a bijection and \(H'_{n,i} = H_{n,i} - H_{n,i-1}\). We determine the contribution of each of these three sets. From (29) and (73), we have

\[
\sum_{(\pi, (A, e)) \in \mathcal{D}_i \times \text{sgn}(\frac{n}{n-i})} \text{odes}_D(\pi) s \text{odes}_D(\pi) q^{\text{inv}_D(f_D(\pi, (A, e)))} = (t - 1) \left\{ \sum_{\pi' \in H_{n,i-1}} \text{odes}_D(\pi') s \text{odes}_D(\pi') q^{\text{inv}_D(\pi')} \right\} + \sum_{\pi' \in H_{n,i}} \text{odes}_D(\pi') s \text{odes}_D(\pi') q^{\text{inv}_D(\pi')}.
\]

This completes the proof of (71). The proof when \(i\) is even is similar and hence is omitted.

Our next result is a type D counterpart of the recurrence given in Theorem 19.

**Theorem 32.** Define \(D_2(s, t, q) = (1 + tq)^2\). When \(n \geq 3\), the polynomials \(D_n(s, t, q)\) satisfy the
repeatedly applying (71) and (72), we have

\[
D_n(s, t, q) \over D_n(1, q) = \left(1 - t\right)^{k+1}(1 - s)^k \over D_n(1, q) + 2t(1 - t)^{k+1}(1 - s)^k \over [n]_q! + \frac{t^2(1 - t)^{k-1}(1 - s)^k}{[n - 1]_q!} + \sum_{r=0}^{k-1} t(1 - t)^r(1 - s)^{r+1} \frac{D_{n-2r-1}(s, t, q)}{D_{n-2r-1}(1, q)[2r + 1]_q!} + \sum_{r=0}^{k-1} s(1 - t)^r(1 - s)^r \frac{D_{n-2r}(s, t, q)}{D_{n-2r}(1, q)[2r]_q!} \quad \text{if } n = 2k \text{ is even,}
\]

\[
D_n(s, t, q) \over D_n(1, q) = \left(1 - t\right)^{k+2}(1 - s)^k \over D_n(1, q) + 2t(1 - t)^{k+1}(1 - s)^k \over [n]_q! + \frac{t^2(1 - t)^{k-1}(1 - s)^k}{[n - 1]_q!} + \sum_{r=0}^{k-1} t(1 - t)^r(1 - s)^{r+1} \frac{D_{n-2r-1}(s, t, q)}{D_{n-2r-1}(1, q)[2r + 1]_q!} + \sum_{r=0}^{k-1} s(1 - t)^r(1 - s)^r \frac{D_{n-2r}(s, t, q)}{D_{n-2r}(1, q)[2r]_q!} \quad \text{if } n = 2k + 1 \text{ is odd.}
\]

Proof. As \(H_{n,i}\) is the set of signed permutations in \(\mathcal{D}_n\) whose rightmost \((n-i)\) entries form an increasing run, we see that \(H_{n,n-1}\) must be the whole of \(\mathcal{D}_n\). We first consider the case when \(n\) is even. By repeatedly applying (71) and (72), we have

\[
D_n(s, t, q) = \sum_{\pi \in H_{n,n-1}} t^{odesD(\pi)} s^{edesD(\pi)} q^{invD(\pi)} = t \frac{D_{n-1}(s, t, q)D_n(1, q)}{D_{n-1}(1, q)[1]_q!} + \left(1 - t\right) \left( \sum_{\pi \in H_{n,n-2}} t^{odesD(\pi)} s^{edesD(\pi)} q^{invD(\pi)} \right) + \frac{D_{n-1}(s, t, q)D_n(1, q)}{D_{n-1}(1, q)[1]_q!} + (1 - s)(1 - t) \sum_{\pi \in H_{n,n-3}} t^{odesD(\pi)} s^{edesD(\pi)} q^{invD(\pi)} + \frac{D_{n-1}(s, t, q)D_n(1, q)}{D_{n-1}(1, q)[1]_q!} + \frac{D_{n-2}(s, t, q)D_n(1, q)}{D_{n-2}(1, q)[2]_q!} + \frac{D_{n-3}(s, t, q)D_n(1, q)}{D_{n-3}(1, q)[3]_q!} + \cdots + (1 - s)\left(1 - t\right)^2 \frac{D_2(s, t, q)D_n(1, q)}{D_2(1, q)[n - 2]_q!} + \frac{D_{n-1}(s, t, q)D_n(1, q)}{D_{n-1}(1, q)[1]_q!} = \sum_{\pi \in H_{n,n-1}} t^{odesD(\pi)} s^{edesD(\pi)} q^{invD(\pi)}.
\]

This completes the proof of (74). We now consider the case when \(n\) is odd. Here, we have
\( D_n(s, t, q) = \sum_{\pi \in H_{n-1}} t^{\operatorname{des}_D(\pi)} s^{\operatorname{des}_D(\pi)} q^{\operatorname{inv}_D(\pi)} \)

\[
= s \frac{D_{n-1}(s, t, q)D_n(1, q)}{D_{n-1}(1, q)[1]_q!} + (1 - s) \left( \sum_{\pi \in H_{n-2}} t^{\operatorname{des}_D(\pi)} s^{\operatorname{des}_D(\pi)} q^{\operatorname{inv}_D(\pi)} \right) 
\]

\[
= s \frac{D_{n-1}(s, t, q)D_n(1, q)}{D_{n-1}(1, q)[1]_q!} + (1 - s) \left( \sum_{\pi \in H_{n-3}} t^{\operatorname{des}_D(\pi)} s^{\operatorname{des}_D(\pi)} q^{\operatorname{inv}_D(\pi)} \right) 
\]

and we can continue as in the case for \( n \) being even, to complete the proof of (75). This completes the proof.

3.1 Type D generating functions

We again cast the recurrences in egf language to get generating functions. We begin with our proof of Theorem 8.

\textbf{Proof of Theorem 8} Recurrences (74) and (75) give rise to this following.

\[
D_0 \left( 1 - s \cosh_q(Mu) - \frac{s(1 - t) \sinh_q(Mu)}{M} \right) + D_1 \left( 1 - t \cosh_q(Mu) - \frac{t(1 - s) \sinh_q(Mu)}{M} \right) = OD + ED.
\]

Changing \( u \) to \( -u \) gives us

\[
D_0 \left( 1 - s \cosh_q(Mu) + \frac{s(1 - t) \sinh_q(Mu)}{M} \right) - D_1 \left( 1 - t \cosh_q(Mu) + \frac{t(1 - s) \sinh_q(Mu)}{M} \right) = -OD + ED.
\]

Solving the above two equations for \( D_0 \) and \( D_1 \) completes the proof.

We can now prove Theorem 9.

\textbf{Proof.} (Of Theorem 9) As done in the proof of Theorem 7, one can check when \( n = 2k \) is even, that \( \hat{D}_{2k}(s, t, q) = s^k D_{2k}(1/s, t, q) \) and when \( n = 2k + 1 \) is odd, that \( \hat{D}_{2k+1}(s, t, q) = s^{k+1} D_{2k+1} \left( \frac{1}{s}, t, q \right) \).

The other details follow as in the proof of Theorem 7, completing the proof.

Using Theorem 8 we get a type D counterpart of Theorem 1 Define

\[
D_n(s_0, s_1, t_0, t_1, q) = \sum_{w \in \mathcal{D}_n} s_0^{\scriptscriptstyle\operatorname{asc}_D(w)} s_1^{\scriptscriptstyle\operatorname{asc}_D(w)} t_0^{\scriptscriptstyle\operatorname{des}_D(w)} t_1^{\scriptscriptstyle\operatorname{des}_D(w)} q^{\scriptscriptstyle\operatorname{inv}_D(w)}.
\]

Further, define the generating functions

\[
D_0(s_0, s_1, t_0, t_1, q, u) = \sum_{k \geq 1} D_{2k}(s_0, s_1, t_0, t_1, q) \frac{u^{2k}}{D_{2k}(1, q)},
\]

\[
D_1(s_0, s_1, t_0, t_1, q, u) = \sum_{k \geq 1} D_{2k+1}(s_0, s_1, t_0, t_1, q) \frac{u^{2k+1}}{D_{2k+1}(1, q)}.
\]

We move to our type D counterpart of Theorem 1. Recall \( D_0(s, t, q, u) \) and \( D_1(s, t, q, u) \) from (22).
Theorem 33. We have the egf

\[
\mathcal{D}_0(s_0, s_1, t_0, t_1, q, u) = \frac{1}{s_0}[T(ED)(s_1 - t_1 \cosh_q (mu))] + T(OD)\left(\frac{(t_1(s_0-t_0))}{m} \sqrt{s_0 s_1} \sinh_q (mu)\right), \quad (80)
\]

\[
\mathcal{D}_1(s_0, s_1, t_0, t_1, q, u) = \frac{1}{s_0} T(ED)\left(\frac{t_1(s_0-t_0)}{m} \sqrt{s_0 s_1} \sinh_q (mu)\right) - T(OD)\left(\frac{t_1(s_0-t_0)}{m} \sqrt{s_0 s_1} \sinh_q (mu)\right).
\]

where

\[
T(OD) = \frac{\sqrt{s_0 s_1} u t_1^2}{s_1^2} (\cosh_q (mu) - 1) + \frac{(s_1 - t_1) m}{(s_0 - t_0) \sqrt{s_0 s_1}} (\sinh_D (mu; q) - mu) + \frac{2 t_1 (s_1 - t_1) \sqrt{s_0 s_1}}{s_1^2 m} (\sinh_q (mu) - mu),
\]

\[
T(ED) = \frac{2 t_1}{s_1} (\cosh_q (mu) - 1) + \frac{u t_1^2 (s_0 - t_0) \sqrt{s_0 s_1}}{s_1^2 s_0 m} \sinh_q (mu) + \frac{(s_1 - t_1)}{t_1} (\cosh_D (mu; q) - 1).
\]

Proof. We proceed as we did in the proof of Theorem 23. It is easy to see that

\[
\mathcal{D}_0(s_0, s_1, t_0, t_1, q, u) = \frac{s_1}{s_0} \mathcal{D}_0(t_0, t_1, q, \sqrt{s_0 s_1} u),
\]

\[
\mathcal{D}_1(s_0, s_1, t_0, t_1, q, u) = \frac{\sqrt{s_1}}{s_0} \mathcal{D}_1(t_0, t_1, q, \sqrt{s_0 s_1} u).
\]

We denote by \( T \) the transformation that sends \( s \) to \( t_0 \), \( t \) to \( t_1 \) and \( u \) to \( \sqrt{s_0 s_1} u \). It is easy to see that \( T(OD) \) and \( T(ED) \) are as given above, completing the proof.

3.2 Type D \( q \)-analogue of Hyatt’s recurrences

We give our \( q \)-analogue of Hyatt-type recurrences in this subsection.

Theorem 34. For even \( n \),

\[
\sum_{w \in \mathcal{D}_+^n} t^{\text{des}_D(w)} s^{\text{des}_D(w)} q^{\text{inv}_D(w)} = \sum_{r=0}^{\frac{n}{2}-1} q^{\binom{2r+1}{2}} n \binom{n}{2r+1} q D_{n-2r-1}(s, t, q)(s-1)^r (t-1)^r + \sum_{r=1}^{\frac{n}{2}} q^{\binom{2r}{2}} n \binom{n}{2r} q D_{n-2r}(s, t, q)(s-1)^{r-1} (t-1)^r.
\]

For odd \( n \),

\[
\sum_{w \in \mathcal{D}_+^n} t^{\text{des}_D(w)} s^{\text{des}_D(w)} q^{\text{inv}_D(w)} = \sum_{r=0}^{\frac{n}{2}} q^{\binom{2r+1}{2}} n \binom{n}{2r+1} q D_{n-2r-1}(s, t, q)(s-1)^r (t-1)^r + \sum_{r=1}^{\frac{n}{2}} q^{\binom{2r}{2}} n \binom{n}{2r} q D_{n-2r}(s, t, q)(s-1)^{r-1} (t-1)^r.
\]

Proof. Define \( \mathcal{D}_k \) to be the signed permutations in \( \mathcal{D}_n^+ \) that have their rightmost \( k + 1 \) elements being positive and arranged in descending order. Thus, the first \( n - k - 1 \) elements must have an even number
of negative signs. With this observation note that the map $f'' : D_k \times \binom{[n]}{n-k} \rightarrow \hat{D}_{n-k-1}$ that carries $(\psi, A)$ to the signed permutation $\psi_{[n]-A} a_1 a_2 \cdots a_{n-k} (a_1 > a_2 > \cdots a_{n-k} > 0)$, that is,

$$f''(\psi, A) = \psi_{[n]-A}, a_1, a_2, \cdots, a_{n-k}.$$

is a bijection from $D_k \times \binom{[n]}{n-k}$ onto $\hat{D}_{n-k-1}$.

Write $\hat{D}_k(s, t, q) = \sum_{w \in \hat{D}_k} \epsilon^\text{odes}_D(w) s^\text{odes}_D(w) q^{\text{inv}_D(w)}$. We will abbreviate the LHS as $\hat{D}_k$ for brevity. The following recurrences are then easy to prove. For even $n$, we have

$$q \binom{n}{2r} q \frac{n}{2r} \frac{D_{n-2r}(s, t, q) s^r t^r}{q} = s_{\hat{D}} A_{2r-1} - (s-1)_{\hat{D}} A_{2r}. \tag{82}$$

$$q \binom{n}{2r+1} q \frac{n}{2r+1} \frac{D_{n-2r-1}(s, t) s^r t^{r+1}}{q} = t_{\hat{D}} A_{2r} - (t-1)_{\hat{D}} A_{2r+1}. \tag{83}$$

For odd $n$, we have

$$q \binom{n}{2r} q \frac{n}{2r} \frac{D_{n-2r}(s, t) s^r t^r}{q} = t_{\hat{D}} A_{2r-1} - (t-1)_{\hat{D}} A_{2r}. \tag{84}$$

$$q \binom{n}{2r+1} q \frac{n}{2r+1} \frac{D_{n-2r-1}(s, t) s^r t^{r+1}}{q} = s_{\hat{D}} A_{2r} - (s-1)_{\hat{D}} A_{2r+1}. \tag{85}$$

The proofs of these recurrences are along the same lines as the proofs of (55), (56), (57) and (58). The only ambiguity might be when $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$, but this is easily resolved as in $\hat{D}_{2r}$, the rightmost $n-1$ elements are positive and descending for even $n$ or $\hat{D}_{2r+1}$ when $n$ is odd, the first element has to be positive due to the constraint that there are an even number of negative signs. Therefore, there is no possibility of $w_1 + w_2$ being less than 0.

To preserve elements being in $D_n$, we consider the map that flips the sign of all elements when $n$ is even and the map that flips the sign of all elements except the first when $n$ is odd.

**Lemma 35.** Let $f_D : D_n \rightarrow D_n$ be the involution that sends $w = w_1, \cdots, w_n$ to $\overline{w} = \overline{w_1}, \cdots, \overline{w_n}$ if $n$ is even and $w = w_1, \cdots, w_n$ to $\overline{w} = \overline{w_1}, \overline{w_2}, \cdots, \overline{w_n}$ if $n$ is odd. Then, we have the following:

1. When $n = 2k + 1$, we have $\text{odes}_D(w) + \text{odes}_D(f_D(w)) = k + 1$ and when $n = 2k$, we have $\text{odes}_D(w) + \text{odes}_D(f_D(w)) = k + 1$.

2. When $n = 2k + 1$, we have $\text{odes}_D(w) + \text{odes}_D(f_D(w)) = k$ and when $n = 2k$, we have $\text{odes}_D(w) + \text{odes}_D(f_D(w)) = k - 1$.

3. $\text{inv}_D(w) + \text{inv}_D(f_D(w)) = n(n-1)$.

**Proof.** The proof of the first two assertions are straightforward and hence omitted. For the third part, recall that $\text{inv}_D(w) = \text{inv}_B(w) - |\text{Negs}(w)|$. Thus, we have, for even $n$,

$$\text{inv}_D(w) + \text{inv}_D(\overline{w}) = \text{inv}_B(w) + \text{inv}_B(\overline{w}) - |\text{Negs}(w)| - |\text{Negs}(\overline{w})| = n^2 - n = n(n-1).$$

When $n$ is odd, it is easy to see that $\text{inv}_D(w_1, \overline{w_2}, \cdots, \overline{w_n}) = \text{inv}_D(\overline{w_1}, \cdots, \overline{w_n})$. The rest follows from the previous argument. The proof is complete. \qed
3.3 Symmetry results

In this Subsection, we give our type D counterparts of our symmetry results.

**Theorem 36.** We have

\[
D_n^{-}(s,t,q) = \begin{cases} 
q^n(n-1)s^k t^{k+1} D_n^{-}(s^{-1}, t^{-1}, q^{-1}) & \text{when } n = 2k + 1, \\
q^n(n-1)s^{k-1} t^{k+1} D_n^{-}(s^{-1}, t^{-1}, q^{-1}) & \text{when } n = 2k.
\end{cases}
\]

Therefore, we have

\[
D_n(s,t,q) = \begin{cases} 
D_n^{+}(s,t,q) + q^n(n-1)s^k t^{k+1} D_n^{+}(s^{-1}, t^{-1}, q^{-1}) & \text{when } n = 2k + 1, \\
D_n^{+}(s,t,q) + q^n(n-1)s^{k-1} t^{k+1} D_n^{+}(s^{-1}, t^{-1}, q^{-1}) & \text{when } n = 2k.
\end{cases}
\]

**Proof.** Let \( f_D : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^- \) be the map described earlier. By Lemma \[5\] when \( n = 2k \), we have

\[
\sum_{w \in \mathcal{D}_n^-} t^{odes_D(w)} s^{edes_D(w)} q^{inv_D(w)} = \sum_{w \in \mathcal{D}_n^+} t^{odes_D(f_D(w))} s^{edes_D(f_D(w))} q^{inv_D(f_D(w))}
\]

\[
= \sum_{w \in \mathcal{D}_n^+} t^{k+1-odes_D(w)} s^{k-1-edes_D(w)} q^{n(n-1)-inv_D(w)}
\]

\[
= q^n(n-1)s^{k-1} t^{k+1} \sum_{w \in \mathcal{D}_n^+} t^{-odes_D(w)} s^{-edes_D(w)} q^{-inv_D(w)}.
\]

When \( n = 2k + 1 \), we have

\[
\sum_{w \in \mathcal{D}_n^-} t^{odes_D(w)} s^{edes_D(w)} q^{inv_D(w)} = \sum_{w \in \mathcal{D}_n^+} t^{odes_D(f_D(w))} s^{edes_D(f_D(w))} q^{inv_D(f_D(w))}
\]

\[
= \sum_{w \in \mathcal{D}_n^+} t^{k+1-odes_D(w)} s^{k-1-edes_D(w)} q^{n(n-1)-inv_D(w)}
\]

\[
= q^n(n-1)s^k t^{k+1} \sum_{w \in \mathcal{D}_n^+} t^{-odes_D(w)} s^{-edes_D(w)} q^{-inv_D(w)}.
\]

completing the proof. \( \square \)

Since the following corollary is straightforward, we only state it and omit its proof.

**Corollary 37** (Type-D Symmetry). We have

\[
D_n(s,t,q) = \begin{cases} 
q^n(n-1)s^k t^{k+1} D_n(s^{-1}, t^{-1}, q^{-1}) & \text{when } n = 2k + 1, \\
q^n(n-1)s^{k-1} t^{k+1} D_n(s^{-1}, t^{-1}, q^{-1}) & \text{when } n = 2k.
\end{cases}
\]

4 Snakes

A snake in \( \mathfrak{S}_n \) is a signed permutation \( w \in \mathfrak{S}_n \) satisfying \( 0 < w_1 > w_2 < \cdots \). Let \( \text{Snake}^B_n \) be the set of snakes in \( \mathfrak{S}_n \) and denote \( |\text{Snake}^B_n| \) by \( S^B_n \). The paper by Arnol’d \[11\] is a good reference for this topic. Let \( S^B_n(q) = \sum_{w \in \text{Snake}^B_n} q^{inv_B(w)} \). Springer in \[13\] showed the following.

**Theorem 38** (Springer). The following is the egf for the numbers \( S^B_n \):

\[
\sum_{n\geq 0} S^B_n \frac{u^n}{n!} = \frac{1}{\cos(u) - \sin(u)}.
\]
The following corollary of Theorem 23 is now easy.

**Corollary 39.** We have the following egf of the $S_n^B(q)$ polynomials:

$$
\sum_{n \geq 0} S_n^B(q) \frac{u^n}{B_n(1, q)} = \frac{\cos_q(u) \cos(u; q) + (\sin_q(u) - 1) \sin_B(u; q)}{\cos_q(u)}
$$

**Proof.** Setting $s_1 = t_0 = 0$ and $t_1 = s_0 = 1$ in both $H_0(s_0, s_1, t_0, t_1, q, u)$ and $H_1(s_0, s_1, t_0, t_1, q, u)$ from Theorem 23 and adding completes the proof. \qed

It is easy to see that setting $q = 1$ in Corollary 39 gives us Theorem 38.

### 4.1 D-snakes

A d-snake in $S_n$ is a signed permutation $w$ in $S_n$ that satisfies $-w_2 > w_1 > w_2 < w_3 > \ldots w_n$. Let $\text{Snake}_D^D(n)$ be the set of all d-snakes in $S_n$. Denote $|\text{Snake}_D^D(n)|$ by $S_D(n)$. Let $S_D(q) = \sum_{w \in \text{Snake}_D^D(q)} q^{\text{inv}(w)}$.

Define $S_D(0, q, u) = \sum_{n \geq 1} S_{2n}^D(q) \frac{u^{2n}}{D_{2n}(1, q)}$ and $S_D(1, q, u) = \sum_{n \geq 1} S_{2n+1}^D(q) \frac{u^{2n+1}}{D_{2n+1}(1, q)}$.

**Corollary 40.** We have the following egf of the $S_n^D(q)$ polynomials:

$$
S_D(0, q, u) = \frac{-2 \cos_q^2(u) + \cos_q(u)(\cos_D(u; q) - 1) - 2 \sin_q^2(u) + \sin_q(u) \sin_D(u; q)}{-\cos_q(u)}, \quad (86)
$$

$$
S_D(1, q, u) = \frac{-2 \sin_q(u) + u \cos_q(u) + \sin_D(u; q)}{-\cos_q(u)}. \quad (87)
$$

**Proof.** Set $t = 1/t$, $u = u\sqrt{t}$, multiply by $t$ and setting $s = t = 0$ in Theorem 8 gives us (86). Set $t = 1/t$, $u = u\sqrt{t}$, multiply by $\sqrt{t}$ and setting $s = t = 0$ in Theorem 8 gives us (87). \qed

Setting $q = 1$ in Corollary 40 gives us the following egf which is given by Springer.

**Corollary 41 (Springer).** The egf for the $S_n^D$ is:

$$
\sum_{n \geq 1} S_{2n}^D \frac{u^{2n}}{(2n)!} = \frac{\cos(u) - \cos(2u) - 1}{-\cos(2u)},
$$

$$
\sum_{n \geq 1} S_{2n+1}^D \frac{u^{2n+1}}{(2n+1)!} = \frac{-\sin(2u) + u \cos(2u) + \sin(u)}{-\cos(2u)}.
$$

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