Casimir effect on nontrivial topology spaces in Krein space quantization

M. Naseri\textsuperscript{1,2}, S. Rouhani\textsuperscript{3}, M.V. Takook\textsuperscript{2}\n
February 1, 2008

\textsuperscript{1} Islamic Azad University, Kermanshah Branch, Kermanshah, IRAN
\textsuperscript{2} Department of Physics, Razi University, Kermanshah, IRAN
\textsuperscript{3} Plasma Physics Research Centre, Islamic Azad University, P.O.BOX 14835-157, Tehran, IRAN

Abstract

Casimir effect of a topologically nontrivial two-dimensional space-time, through Krein space quantization \cite{1, 2}, has been calculated. In other words, auxiliary negative norm states have been utilized here. Presence of negative norm states play the role of an automatic renormalization device for the theory. The negative norm states (which do not interact with the physical world) could be chosen in two perspective. In the first case our method results in zero or vanishing values for energy. In the second case, however, the result are the same as the renormalization procedure.

\textit{Proposed PACS numbers:} 04.62.+v, 03.70+k, 11.10.Cd, 98.80.H

*e-mail: takook@razi.ac.ir*
1 Introduction

Consideration of the negative norm states was proposed by Dirac in 1942. In 1950, Gupta applied this idea in QED. The presence of higher derivatives in the lagrangian also led to ghosts states with negative norms. In order to preserve the covariance principle in the gauge theory, the auxiliary negative norms states were utilized. In previous paper [1], it was shown that consideration of the negative norm states is necessary for a fully covariant quantization of the minimally coupled scalar field in de Sitter space (Krein QFT). We have shown that for physical states (positive norm states) the energy is positive, whereas, for the negative norm states (so called un-physical states) the energy is negative. It was also shown that the effect of these un-physical states merely appears in the physics of the problem as a tool for an automatic renormalization of the theory in one-loop approximation [1, 2, 3, 4, 5, 6, 7].

In a previous paper [8], the Casimir effect in Krein QFT has been studied as well. Once again it is found that the theory is automatically renormalized. This method is once again reexamined here by analysis of Casimir effect in space-time with nontrivial topology. The paper is organized as follows. The next section presents a brief review of ordinary Casimir effect in two-dimensional space-time with nontrivial topology. Section 3 is devoted to study of the vacuum energy density of scalar field in two-dimensional space-time with $R \times S^1$ topology. Finally, the results are discussed and analyzed in Section 5.

2 Casimir energy: a review

Consider a real scalar field $\varphi(t, x)$ defined on an interval $0 < x < a$ in an one-dimensional space with $S^1$ topology. In this case, the boundary conditions can be written as

$$\varphi(t, 0) = \varphi(t, a), \quad \partial_x \varphi(t, 0) = \partial_x \varphi(t, a).$$  \hfill (1)

The scalar field equation is

$$(\Box + m^2)\varphi(t, x) = 0.$$  \hfill (2)

The scalar product associated with this equation is

$$(f, g) = i \int_{t=\text{const.}} dx (f^* \partial_0 g - g \partial_0 f^*),$$  \hfill (3)

where $f$ and $g$ are solutions of the eq. (2). It can be seen that the positive- and negative-frequency solutions of eq. (2) are

$$u_p(k, t, x) = \frac{e^{ikx - iwt}}{\sqrt{(2\pi)2w}}, \quad u_n(k, t, x) = \frac{e^{-ikx + iwt}}{\sqrt{(2\pi)2w}}.$$  \hfill (4)

These modes are orthonormalized by the following relations:

$$(u_p(k, x, t), u_p(k', x, t)) = \delta(k - k'),$$

$$(u_n(k, x, t), u_n(k', x, t)) = -\delta(k - k'),$$

$$(u_p(k, x, t), u_n(k', x, t)) = 0.$$  \hfill (5)
\( u_\rho \) modes are positive norm states and the \( u_\sigma \)'s are negative norm states. By imposing the boundary conditions (1) into (4) the positive- and negative frequency solutions can be obtained as follows:
\[
\varphi_{N}^{\pm}(t, x) = \frac{1}{\sqrt{(2a \omega_N)}} \exp[\pm i(\omega_N t - k_N x)],
\]
where
\[
\omega_N = (m^2 + k_N^2)^{1/2}, \quad k_N = \frac{2\pi N}{a}, \quad N = 0, \pm 1, \pm 2, \ldots.
\]
Now the standard quantization of the field is performed by means of the expansion
\[
\phi(t, x) = \sum_{N=\infty}^{\infty} \left[ \varphi_{N}^{(+)}(t, x)a_N + \varphi_{N}^{(-)}(t, x)a_{N}^\dagger \right].
\]
The energy density operator is given by the 00-component of the energy-momentum tensor
\[
T_{00} = \frac{1}{2} \{ (\partial_t \phi(t, x))^2 + (\partial_x \phi(t, x))^2 \}.
\]
The vacuum energy density of a scalar field on \( S^1 \) can be calculated as follow [15]
\[
\langle 0 | T_{00} | 0 \rangle = \frac{1}{2a} \sum_{N=-\infty}^{\infty} \omega_N.
\]
The total vacuum energy is
\[
E_0(a, m) = \int_0^a \langle 0 | T_{00} | 0 \rangle \, dx = \frac{1}{2} \sum_{N=-\infty}^{\infty} \omega_N = \sum_{N=0}^{\infty} \omega_N - \frac{m}{2}.
\]
The renormalization of this infinite quantity is performed by subtracting the contribution of the Minkowski space
\[
E_0(a, m) = \left[ \sum_{N=0}^{\infty} \omega_N - \frac{a}{2\pi} \int_0^{\infty} \omega(k)dk \right] - \frac{m}{2}.
\]
Substituting \( A = \frac{am}{2\pi} \) and \( t = \frac{ak}{2\pi} \), one can obtain
\[
E_0(a, m) = \frac{2\pi}{a} \left[ \sum_{N=0}^{\infty} \sqrt{A^2 + N^2} - \frac{a}{2\pi} \int_0^{\infty} \sqrt{A^2 + t^2}dt \right] - \frac{m}{2}.
\]
Using the Abel-Plana formula [?]
\[
\sum F(N) - \int_0^{\infty} F(t)dt = \frac{1}{2} F(0) + i \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1}[F(it) - F(-it)],
\]
where \( F(t) = \sqrt{A^2 + N^2} \), we finally obtain,
\[
E^{ren}_0(a, m) = -\frac{4\pi}{a} \int_0^{\infty} \sqrt{\frac{t^2 - A^2}{e^{2\pi t} - 1}}dt = -\frac{1}{\pi a} \int_{\mu}^{\infty} \frac{\sqrt{\xi^2 - \mu^2}}{e^\xi - 1}d\xi.
\]
where $\xi = 2\pi t$, $\mu = 2\pi A$. In the massless case ($\mu = 0$) we have

$$E_{0}^{\text{ren}}(a, 0) = -\frac{1}{\pi a} \int_{0}^{\infty} \frac{\xi}{\exp(\xi) - 1} d\xi = -\frac{\pi}{6a}.$$ (15)

For $\mu \ll 1$ it follows from (13)

$$E_{0}^{\text{ren}}(a, m) \approx -\frac{\sqrt{\mu}}{\sqrt{2\pi a}} e^{-\mu},$$ (16)

i.e., the vacuum energy of the massive field is exponentially small.

### 3 Vacuum energy in Krein space quantization

In the previous paper [2], we present the free field operator in the Krein space quantization

$$\phi(t, x) = \phi_{p}(t, x) + \phi_{n}(t, x),$$ (17)

where

$$\phi_{p}(t, x) = \int dk[a(k)\varphi_{p}(k, x, t) + a^\dagger(k)\varphi^{*}_{p}(k, x, t)],$$

$$\phi_{n}(t, x) = \int dk[b(k)\varphi_{n}(k, x, t) + b^\dagger(k)\varphi^{*}_{n}(k, x, t)],$$

and $a(k)$ and $b(k)$ are two independent operators. Creation and annihilation operators are constrained to obey the following commutation rules

$$[a(k), a^\dagger(k')] = 0, \quad [a^\dagger(k), a^\dagger(k')] = 0, \quad [a(k), a(k')] = \delta(k - k'),$$ (18)

$$[b(k), b(k')] = 0, \quad [b^\dagger(k), b^\dagger(k')] = 0, \quad [b(k), b^\dagger(k')] = -\delta(k - k'),$$ (19)

$$[a(k), b(k')] = 0, \quad [a^\dagger(k), b^\dagger(k')] = 0, \quad [a(k), b(k')] = 0, \quad [a^\dagger(k), b^\dagger(k')] = 0.$$ (20)

The vacuum state $|\Omega\rangle$ is then defined by

$$a^\dagger(k) |\Omega\rangle = 1_{k} >; \quad a(k) |\Omega\rangle = 0,$$ (21)

$$b^\dagger(k) |\Omega\rangle = \bar{1}_{k} >; \quad b(k) |\Omega\rangle = 0,$$ (22)

$$b(k) |_{1_{k}} = 0; \quad a(k) |_{\bar{1}_{k}} = 0,$$ (23)

where $|_{1_{k}}$ is called a one particle state and $|_{\bar{1}_{k}}$ is called a one “unparticle state”.

We are now in a position to calculate the vacuum energy of two-dimensional space-time with nontrivial topology in Krein space. The field operator in Krein space is build by joining two possible solutions of field equation, positive and negative norms. The negative norm states, which do not interact with the physical world could be constructed with two perspective. These perspectives lead us to build two possible field operators.
3.1 First perspective

In this case the boundary conditions (1) are intrinsic properties of the space-time. Consequently both positive and negative energy basis are affected by these conditions. So the scalar field operator in such this perspective (through Krein quantization method) can be written as:

\[ \phi(t, x) = \sum_{N=-\infty}^{\infty} [\varphi_N^{(+)}(t, x)a_N + \varphi_N^{(-)}(t, x)a_N^+] + \sum_{N=-\infty}^{\infty} [\varphi_N^{(-)}(t, x)b_N + \varphi_N^{(+)}(t, x)b_N^+] , \]

where \(\varphi_N^{(\pm)}(t, x)\) is defined in (5). The vacuum energy density of a scalar field on \(S^1\) can be calculated as follows

\[ \langle \Omega | T_{00}^{Kre} | \Omega \rangle = \frac{1}{2a} \sum_{N=-\infty}^{\infty} \omega_N - \frac{1}{2a} \sum_{N=-\infty}^{\infty} \omega_N = 0. \]

(25)

Therefore the vacuum energy is automatically renormalized and it is equal to zero. In this perspective the structure of space-time is not affected by the vacuum energy.

3.2 Second perspective

In this perspective, the gravitational field appears as an external phenomena imposed on the structure of space-time. This is due to the fact that the boundary conditions (1) are only imposed on positive norm states.

By imposing the above boundary conditions, the field operator in Krein QFT can be written as follows:

\[ \phi(t, x) = \sum_{N=-\infty}^{\infty} [\varphi_N^{(+)}(t, x)a_N + \varphi_N^{(-)}(t, x)a_N^+] + \int dk [b(k)u_n(k, x, t) + b^\dagger(k)u_n^*(k, x, t)] \]

(26)

where \(\varphi_N^{(\pm)}(t, x)\) and \(u_n(k, x, t)\) are defined in (6) and (4) respectively. By Substituting the above field operator in (7) and using Eqs (24),(25) and (26), one can easily obtain:

\[ \langle \Omega | T_{00}^{Kre} | \Omega \rangle = \frac{1}{2a} \sum_{N=-\infty}^{\infty} \omega_N - \frac{1}{2a} \int_0^\infty \omega(k)dk. \]

(27)

The total vacuum energy in Krein QFT is

\[ E_0^{Kre}(a) = \int_0^a \langle \Omega | T_{00}^{Kre} | \Omega \rangle dx = \left[ \sum_{N=-\infty}^{\infty} \omega_N - \frac{a}{2\pi} \int_0^\infty \omega(k)dk \right] - \frac{m}{2}. \]

(28)

It is clearly seen that once again the previous result (10) has attained. It should be noted that this energy can not be detected locally. In the semiclassical treatment of gravitional field, however, it affects the curvature globally through

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \]

where \( T_{\mu\nu} \) is the energy-momentum tensor. (29)

i.e. it changes the structure of the space-time. In this perspective we have the quantum instability of the space-time topology.
4 Conclusion

The Casimir effect is the simplest example of interaction field. The zero point energy of quantum fields in this case, i.e. Casimir energy, can be alternatively calculated as an interaction lagrangian without reference to zero point energies [9]. The goal of the presentation of Krein space quantization is to eliminate the singularity which appears in the interaction field. In the present paper, Casimir energy for two-dimensional space-time with nontrivial topology, has been calculated through the Krein space quantization. Once again it is found that the theory is automatically renormalized. The zero point energy of vacuum is found to be zero if we suppose that the structure of space-time is dependent on the matter it contains. If we take the perspective that the structure of space-time is not independent of matter and only boundary conditions are imposed as the external interaction, we obtain the regular results found by other authors but we have the quantum instability of the space-time.

References

[1] J.P. Gazeau, J. Renaud, M.V. Takook, Class. Quan. Grav, 17(2000)1415, gr-qc/9904023
[2] M.V. Takook, Int. J. Mod. Phys. E, 11(2002)509, gr-qc/0006019
[3] M.V. Takook, Int. J. Mod. Phys. E, 14 (2005) 219, gr-qc/0006052.
[4] M.V. Takook, Thèse de l’université Paris VI, 1997 Théorie quantique des champs pour des systèmes élémentaires “massifs” et de “masse nulle” sur l’espace-temps de de Sitter.
[5] J.P. Gazeau, S. Rouhani, M.V Takook, Linear covariant quantum gravity in de Sitter space, in preparation
[6] M.V. Takook, Mod. Phy. lett. A, 16(2001)1691, gr-qc/0005020
[7] Takook M.V., Proceeding of the Wigsym6, 16-22 August, 1999, Istanbul, Turkey, gr-qc/0001052
[8] H. Khosravi, M. Naseri, S. Rouhani and M.V. Takook, Phys. Lett. B 640(2006)48, gr-qc/0604036
[9] R.L. Jaffe, Phys. Rev. D, 72(2005)021301, hep-th/0503158