Nonlinear free vibration of size-dependent microbeams with nonlinear elasticity under various boundary conditions

F. Lin, J.S. Peng, S.F. Xue, L. Yang and J. Yang

1 College of Pipeline and Civil Engineering, China University of Petroleum (Huadong), Qingdao, China
2 School of Mechanical Engineering, Chengdu University, Chengdu, China
3 School of Engineering, RMIT University, Melbourne, Australia

ABSTRACT

In this study, nonlinear couple stress–strain constitutive relationships in the modified couple stress theory (MCST) are derived on the basis of previous classical stress–strain constitutive relationships of nonlinear elasticity materials. Hamilton’s principle is employed to obtain higher-order nonlinear governing equations within the framework of the updated MCST, von Kármán geometric nonlinearity and Bernoulli–Euler beam theory. These mathematical formulations are solved numerically by the differential quadrature method together with an iterative algorithm to determine the nonlinear dynamic features of microbeams with four groups of boundary conditions. A detailed parametric study is conducted to analyze the influences of nonlinear elasticity properties on the nonlinear free vibration characteristics of the microbeams. Results show that these microbeams exhibiting nonlinear couple constitutive relationships have lower frequencies than their approximately simplified linear couple constitutive relationships. In addition, the frequencies of microbeams with nonlinear elasticity properties decrease as the vibration amplitude increases.

KEYWORDS: material nonlinearity, nonlinear couple stresses, size effect, nonlinear natural frequency

1. INTRODUCTION

Electrostatically actuated micro- and nanoelectromechanical systems (MEMS/NEMS) that integrate electrical and mechanical elements are widely used in many engineering applications, including inkjet printers [1], microwave switches [2], magnetic sensitive transistors [3], MEMS accelerometers [4], pressure sensors [5], micromirrors [6], micropumps [7] and MEMS microgrippers [8]. Thin beams, which are key components of MEMS/NEMS devices, are extensively used to convert electrical energy into mechanical energy.

In recent years, studies have applied the nano- and microbeams on linear and nonlinear free vibrations to predict dynamic features such as natural frequencies and mode shapes. Kong et al. [9] established a model to solve analytically the dynamic problems of Bernoulli–Euler beams on the basis of modified couple stress theory (MCST). They concluded that linear natural frequencies are size dependent and that the difference of natural frequencies between the new and the classical beam model is very significant. On the basis of the sinusoidal beam theory and the modified strain gradient theory, Wang et al. [10] studied size-dependent bending and vibration of microbeams made of porous metal foams. Wang and Wang [11] first presented an exact linear free vibration solution for exponentially tapered cantilever with tip mass and obtained the exact results, i.e. linear frequencies and mode shapes. Thereafter, Wang [12] investigated the linear free vibration of a tapered cantilever of constant thickness and linearly tapered width. Nourbakhsh et al. [13] investigated the problems of nonlinear free and forced vibrations of microscale simply supported beams with initial lateral displacement. For free vibrations, they concluded that the nonlinear frequencies are much higher than the linear ones. On the basis of Eringen’s nonlocal elasticity theory and von Kármán’s geometric nonlinearity, Ke et al. [14] studied the nonlinear free vibration of embedded double-walled carbon nanotubes (DWNTs) and investigated the influences of nonlocal parameter, tube length, spring constant and end supports on the nonlinear free vibration characteristics of DWNTs. They found that both linear frequency and nonlinear frequency ratios decrease as the length of DWNTs increases. Subsequently, Ke et al. [15] investigated the nonlinear free vibration of microbeams made from functionally graded materials (FGMs) on the basis of the MCST and von Kármán geometric nonlinearity. Their findings showed that linear and nonlinear frequencies increase significantly when the thickness of the FGM microbeam is comparable to the material length scale parameter. Asghari et al. [16] introduced a size-dependent nonlinear Timoshenko microbeam model on the basis of strain gradient theory to investigate nonlinear static and free vibration behaviors of hinged–hinged beams. Some methods are applicable for the nonlinear analysis of structures, such as the multiple-scale method [17, 18], the harmonic balance method [19], and the differential quadrature method (DQM) [14, 15, 20, 21].
It is well known that the classical continuum mechanics theories cannot correctly explain the mechanical behavior of microscale structural elements without taking the size effect into account. In order to describe the size effect in the classical continuum mechanics theories, higher-order (nonlocal) continuum theories that contain additional material constants such as Eringen’s nonlocal elasticity theory [22], the couple stress theory [23], the MCST [24], the strain gradient elasticity theory [25] and the modified strain gradient elasticity theory [26] have been introduced on the basis of linear elasticity theory. In recent decades, these theories have been used to study the mechanical behavior of microscale structural elements made of linear elasticity materials (e.g. see [9, 13–16, 27, 28]).

Many materials such as metal alloy, single-crystal silicon, polycrystalline silicon, silicon oxide, silicon nitride and shape memory alloy [28–35] exhibit inherent nonlinear elasticity properties; that is to say, these materials have nonlinear stress–strain constitutive relationships. There is some previous work done on the mechanical behavior of microstructures made of above-mentioned nonlinear elasticity materials. To study the nonlinear forced vibration of microbeams made of nonlinear elasticity materials, Peng et al. [36] utilized a third-order power function to fit the nonlinear classical stress–strain constitutive relationship curve approximately. Initially, they presented a nonlinear dynamic analysis of a microactuator made of nonlinear elasticity material Al–1% Si. The theoretical formulations were based on Bernoulli–Euler beam theory and included the effects of material nonlinearity and von Kármán geometric nonlinearity due to the exact strain of large deformation. Then, the same nonlinear elasticity material was utilized by Peng et al. [37] to investigate the dynamic pull-in instability of the microactuator. Thereafter, nonlinear electrodynamic analysis for a size-dependent microbeam with nonlinear elasticity was conducted by Peng et al. [38] within the framework of the MCST and the Bernoulli–Euler beam model. The above-mentioned studies showed that the influence of nonlinear elasticity is important and cannot be neglected.

To the best of our knowledge, no literature is available to research the nonlinear free vibration of microbeams exhibiting nonlinear elasticity properties in MEMS/NEMS. Moreover, no nonlinear couple stress–strain constitutive relationships in the MCST describe the mechanical behaviors of nonlinear elasticity materials.

This study aims to introduce nonlinear couple stress–strain constitutive relationships of the MCST on the basis of previous classical stress–strain constitutive relationships of nonlinear elasticity materials and then investigate nonlinear free vibration behaviors of microbeams made from nonlinear elasticity materials. Hamilton’s principle is employed to derive higher-order nonlinear governing equations and boundary conditions on the basis of von Kármán geometric nonlinearity, Bernoulli–Euler beam theory and the updated MCST that considers both nonlinear classical stress–strain constitutive relationships and nonlinear couple ones. The nonlinear free vibrations of microbeams with four different groups of boundary conditions are determined by using a numerical method that combines the DQM with an iterative algorithm. The effects of nonlinear elasticity properties on the nonlinear free vibration characteristics of the microbeams are discussed in detail.

2. THE MCST IN NONLINEAR ELASTICITY

2.1 The MCST in linear elasticity

On the basis of the MCST given by Yang et al. in 2002 [24], the strain energy density $\phi(\varepsilon_{ij}, \eta_{ij})$ for an arbitrary volume element is given by

$$
\phi(\varepsilon_{ij}, \eta_{ij}) = \int_0^{\varepsilon_{ij}} \sigma_{ij} \, d\varepsilon_{ij} + \int_0^{\eta_{ij}} m_{ij} \, d\eta_{ij} \quad (i, j = x, y, z),
$$

where $\sigma_{ij}$, $\varepsilon_{ij}$, $m_{ij}$ and $\eta_{ij}$ represent the stress tensor, strain tensor, deviatoric part of the couple stress tensor and symmetric curvature tensor, respectively. They are defined as follows:

$$
\sigma_{ij} = \lambda \varepsilon_{ij} + 2G\varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{1}{2} \left[ \nabla \ddot{u}_i + (\nabla \ddot{u}_i)^T \right],
$$

$$
m_{ij} = 2 \mu \eta_{ij}, \quad \eta_{ij} = \frac{1}{2} \left[ \nabla \gamma_i + (\nabla \gamma_i)^T \right], \quad \gamma_i = \frac{1}{2} \text{curl} (\ddot{u}_i),
$$

$$
G = \frac{E}{2(1 + \mu)}, \quad \lambda = \frac{E \mu}{(1 + \mu)(1 - 2\mu)},
$$

where $\gamma_i$ is the component of the rotation vector; $l$ is a material length scale parameter; $\mu$ is Poisson’s ratio; and $\lambda$, $E$ and $G$ are Lamé constants.

2.2 Shear stress–strain constitutive relationships in nonlinear elasticity

As stated before, Peng et al. [36] utilized the following third-order power function to fit the experimentally determined nonlinear normal stress–strain constitutive relationship curve of isotropic nonlinear elasticity material Al–1% Si [33]:

$$
\sigma_{xx} = E_1 \varepsilon_{xx} - E_2 (\varepsilon_{xx})^3,
$$

where $E_1$ and $E_2$ are two generalized Young’s modulus coefficients. Moreover, Eq. (2a) could be rewritten as

$$
\sigma_{xx} = \sigma_{xx}^l - \sigma_{xx}^N,
$$

where $\sigma_{xx}^l = \sigma_{xx}^l - \sigma_{xx}^N$. 

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where
\[ \sigma^L_{xx} = E_1 \varepsilon_{xx}, \quad \sigma^N_{xx} = E_2 (\varepsilon_{xx})^3 \] (2c, d)
and \( \sigma^\alpha_{xx} (\alpha = L, N) \) represent linear and nonlinear components, respectively.

Taking the first derivative of Eq. (2a) with respect to \( \varepsilon_{xx} \), we could get its differential equation written as
\[ d\sigma_{xx} = E_{xx} d\varepsilon_{xx}, \] (3a)
where \( E_{xx} \) is the generalized nonlinear elastic modulus, i.e. the gradient of Eq. (2a) at arbitrary strain \( \varepsilon_{xx} \), along the \( x \) direction and its expression is given as
\[ E_{xx} = \frac{d\sigma_{xx}}{d\varepsilon_{xx}} = E_1 - 3E_2 (\varepsilon_{xx})^2. \] (3b)

Derived from the nonlinear classical stress–strain constitutive relationship (2a), differential equation (3a) exhibits linear relationship between the differentials of the classical normal stress \( d\sigma_{xx} \) and strain \( d\varepsilon_{xx} \) at arbitrary strain \( \varepsilon_{xx} \) so it follows Hooke’s law.

For the isotropic nonlinear elasticity materials, we could also have the following similar differential equations like Eqs (3a) and (3b) along the other two directions by replacing their double subscript \( xx \) with \( ii \) \((i = y, z)\):
\[ d\sigma_{ii} = E_{ii} d\varepsilon_{ii}, \] (4a)
where \( E_{ii} \) is the generalized nonlinear elastic modulus along the \( i \) direction and is given as
\[ E_{ii} = E_1 - 3E_2 (\varepsilon_{ii})^2. \] (4b)

Now consider a cube element \( abcd'a'b'c'd' \) of isotropic nonlinear elasticity materials and it is only subjected to differential of plane stresses \( d\sigma_{\beta\phi} \) \((\beta, \phi = x, y)\) as shown in Fig. 1a, where the three axes of Cartesian coordinate \( xyz \) are vertical to three planes of the cube element. Referring to the effect of the Poisson’s ratio, the differential of strain \( d\varepsilon_{xx} \) in the \( x \) direction due to the differential of stress \( d\sigma_{xx} \) is equal to \( d\sigma_{xx}/E_{xx} \) and in the \( y \) direction due to the differential of stress \( d\sigma_{yy} \) is \(-\mu d\sigma_{yy}/E_{yy}\). Thus, the differential of the resultant strain in the \( x \) direction is
\[ d\varepsilon_{xx} = \frac{d\sigma_{xx}}{E_{xx}} - \mu \frac{d\sigma_{yy}}{E_{yy}}. \] (5a)

In a similar manner, the differential of the strain \( d\varepsilon_{yy} \) in the \( y \) direction is obtained as
\[ d\varepsilon_{yy} = \frac{d\sigma_{yy}}{E_{yy}} - \mu \frac{d\sigma_{xx}}{E_{xx}}. \] (5b)

However, for the element of isotropic nonlinear elasticity materials subjected to the differential of plane shear stresses \( d\sigma_{xy} \), we assume that the relationship between the differentials of the shear stress \( d\sigma_{xy} \) and strain \( d\varepsilon_{xy} \) follows Hooke’s law and the differential of shear strain \( d\varepsilon_{xy} \) is given as
\[ d\varepsilon_{xy} = \frac{d\sigma_{xy}}{G_{xy}}, \] (5c)
where \( G_{xy} \) is the generalized shear modulus of the shear stress–strain constitutive relationships for the nonlinear elasticity materials and is unknown.
Equations (5a)–(5c) could be rewritten for the differential of the stresses in terms of the differential of the strains as
\[
\begin{align*}
\sigma_{xx} &= \frac{E_{xx}}{1-\mu^2} (d\varepsilon_{xx} + \mu d\varepsilon_{yy}), \\
\sigma_{yy} &= \frac{E_{yy}}{1-\mu^2} (d\varepsilon_{yy} + \mu d\varepsilon_{xx}), \\
\sigma_{xy} &= G_{xy} d\varepsilon_{xy}.
\end{align*}
\]

As we know, general equations of plane-stress transformation and plane-strain transformation are obtained on the basis of linear elasticity. However, Eqs (6a)–(6c) are the linear relationships between differentials of the stresses \(d\sigma_{\alpha\beta}\) and strains \(d\varepsilon_{\alpha\beta}\) \((\alpha,\beta = x, y)\), and they follow Hooke’s law. We cut a triangular subelement \(ebf\) along the \(z\) direction from the element \(abcda'b'c'd'\) and it is shown in Fig. 1b. Referring to the general equations of plane-stress transformation in Mechanics of Materials [39], we could obtain general equations of differential-of-plane-stress transformation as follows:
\[
\begin{align*}
\frac{d\sigma_{xx}}{2} &= \frac{d\sigma_{xx} + d\sigma_{yy}}{2} \cos 2\theta + d\sigma_{xy} \sin 2\theta, \\
\frac{d\sigma_{yy}}{2} &= -\frac{d\sigma_{xx} - d\sigma_{yy}}{2} \sin 2\theta + d\sigma_{xy} \cos 2\theta, \\
\frac{d\sigma_{xy}}{2} &= -\frac{d\sigma_{xx} - d\sigma_{yy}}{2} \cos 2\theta - d\sigma_{xy} \sin 2\theta,
\end{align*}
\]
where \(\theta\) is the angle between the normal direction \(\hat{x}\) of the plane of and the \(x\) direction.

Similarly, referring to general equations of plane-strain transformation, general equations of differential-of-plane-strain transformation of the triangular subelement \(ebf\) could be obtained directly as
\[
\begin{align*}
\frac{d\varepsilon_{xx}}{2} &= \frac{d\varepsilon_{xx} + d\varepsilon_{yy}}{2} \cos 2\theta + d\varepsilon_{xy} \sin 2\theta, \\
\frac{d\varepsilon_{yy}}{2} &= -\frac{d\varepsilon_{xx} - d\varepsilon_{yy}}{2} \sin 2\theta + d\varepsilon_{xy} \cos 2\theta, \\
\frac{d\varepsilon_{xy}}{2} &= -\frac{d\varepsilon_{xx} - d\varepsilon_{yy}}{2} \cos 2\theta - d\varepsilon_{xy} \sin 2\theta.
\end{align*}
\]

On the basis of general equations of differential-of-plane-stress transformation, Eqs (7a)–(7c), the cube element \(abcd\) in Fig. 1c, which is only subjected to differentials of pure shear stresses \(d\varepsilon_{xy}\) and \(d\varepsilon_{yz}\), is equal to the cube element \(a_1b_1c_1d_1\) subjected to differentials of principal stresses \(d\sigma_{xx}^{45}\) and \(d\sigma_{yy}^{45}\) at \(\theta = 45^\circ\). Meanwhile, by setting \(\theta = 45^\circ\) in Eqs (7a)–(7c) and (8a)–(8c), one has the following differential equations of transformation of stress and strain:
\[
\begin{align*}
d\sigma_{xx}^{45} &= d\sigma_{yy}, \\
-d\sigma_{yy} &= =-d\sigma_{xy}, \\
d\sigma_{xy}^{45} &= =0.
\end{align*}
\]

Integrating Eq. (10a) with respect to \(\varepsilon_{xy}\) from zero to \(\varepsilon_{xy}\) under the condition that \(\varepsilon_{xy}^{45} = 0\) at \(\varepsilon_{xy} = 0\), one obtains
\[
\varepsilon_{xy}^{45} = \frac{1}{2} \varepsilon_{xy}.
\]

On the basis of Eq. (6a), differential of principal stress \(d\sigma_{xx}^{45}\) is obtained as
\[
\begin{align*}
d\sigma_{xx}^{45} &= \frac{E_1 - 3E_2 (\varepsilon_{xy}/2)^2}{1 - \mu^2} (d\varepsilon_{xx}^{45} + \mu d\varepsilon_{yy}^{45}).
\end{align*}
\]

By substituting Eqs (9a), (10a), (10b) and (10d) into Eq. (11a), one obtains
\[
\begin{align*}
d\sigma_{xy} &= \frac{E_1 - 3E_2 (\varepsilon_{xy}/2)^2}{2(1-\mu^2)} (d\varepsilon_{xy} - \mu d\varepsilon_{xy}) = \frac{E_1 - 3E_2 (\varepsilon_{xy}/2)^2}{2(1+\mu)} d\varepsilon_{xy}.
\end{align*}
\]

Integrating the above equation with respect to \(\varepsilon_{xy}\) from zero to \(\varepsilon_{xy}\) under the condition that \(\varepsilon_{xy} = 0\) at \(\varepsilon_{xy} = 0\), the nonlinear classical shear stress–strain constitutive relationship is obtained as
\[
\begin{align*}
\sigma_{xy} &= \frac{1}{2} \frac{E_1 - 3E_2 (\varepsilon_{xy}/2)^2}{2(1+\mu)} \int_{0}^{\varepsilon_{xy}} d\varepsilon_{xy} = \frac{E_1 \varepsilon_{xy} - \frac{1}{4}E_2 (\varepsilon_{xy})^3}{2(1+\mu)}.
\end{align*}
\]

For the isotropic nonlinear elasticity materials, the nonlinear classical shear stress–strain constitutive relationships could be obtained by repeating the above process and their expressions are given as
\[
\begin{align*}
\sigma_{ij} &= \frac{E_1 \varepsilon_{ij} - \frac{1}{4}E_2 (\varepsilon_{ij})^3}{2(1+\mu)} (i, j = x, y, z; i \neq j).
\end{align*}
\]
2.3 Couple stress–strain constitutive relationships in nonlinear elasticity

We have obtained the nonlinear classical shear stress–strain constitutive relationships Eq. (12) for the nonlinear elasticity materials. Next, we are required to determine the expressions of the couple stress–strain constitutive relationships for the nonlinear elasticity materials.

First, let us obtain the gradient of rotation angle at an arbitrary position. As we know in Mechanics of Materials, the structure that is subjected to torques or to bending moments can cause a three–dimensional rotation angle $\gamma$ at an arbitrary position ($x, y, z$) of the structure, which is described in terms of the one-dimensional rotation angles $\gamma_i$ around the $i$-axis written as

$$\gamma = \gamma_x e_x + \gamma_y e_y + \gamma_z e_z,$$

in which $e_i$ is a unit vector component around the $i$-axis. The one-dimensional rotation angles $\gamma_i$ are defined as in Eq. (1f).

Considering an arbitrary cube element $abcdABCD$ with three lengths, i.e., $||dr||$ of the line element $AD$, $||dr'||$ of $aA$ and $||dr''||$ of $AB$ (Fig. 2a), we suppose that one of end points position of vertical line element $rs$ between the two parallel planes $abBA$ and $dcCD$ is ($x, y, z$) in plane $abBA$, and the other is $(x + dx, y + dy, z + dz)$ in plane $dcCD$. Therefore, the vertical line element has the length vector $dr$ and the direction cosine vector $n$ respectively equal to

$$dr = dx e_x + dy e_y + dz e_z, \quad n = \frac{dx e_x + dy e_y + dz e_z}{||dr||} = \nu_x e_x + \nu_y e_y + \nu_z e_z,$$

where the operator $||||$ represents the vector norm, and

$$||dr|| = \sqrt{dx^2 + dy^2 + dz^2}.$$  

Similarly, the other two length vectors and the direction cosine vectors are given respectively for two parallel planes $abcd$ and $ABCD$ as

$$dr' = dx' e_x + dy' e_y + dz' e_z, \quad n' = \nu'_x e_x + \nu'_y e_y + \nu'_z e_z,$$

and for $aADd$ and $bBCc$ as

$$dr'' = dx'' e_x + dy'' e_y + dz'' e_z, \quad n'' = \nu''_x e_x + \nu''_y e_y + \nu''_z e_z.$$  

Subjecting the cube element $abcdABCD$ to torques and to bending moments, therefore, we gain rotation angle components $\gamma_x(x, y, z), \gamma_y(x, y, z)$ and $\gamma_z(x, y, z)$ of the plane $abBA$ at point $r$, and $\gamma_x(x + dx, y + dy, z + dz), \gamma_y(x + dx, y + dy, z + dz)$ and $\gamma_z(x + dx, y + dy, z + dz)$ of the plane $cCDd$ at point $s$, where the line element $rs$ is parallel to $AD$. Regarding the rotation angles of the plane $abBA$ at point $r$ as the references and on basis of the first order Taylor formula with the remain of Peano $o(||dr||)$, the components of the relative rotation angle vector between points $r$ and $s$ are
\[ \begin{align*}
\frac{d\gamma_x}{dx} &= (\gamma_x) - (\gamma_s), \\
\frac{d\gamma_y}{dx} &= (\gamma_y) - (\gamma_s), \\
\frac{d\gamma_z}{dx} &= (\gamma_z) - (\gamma_s),
\end{align*} \]

(15a) (15b) (15c)

The relative rotation angle vector \( d\gamma \) is projected onto the direction cosine vector \( n \) of the plane \( abBA \), i.e., \( \delta = d\gamma \cdot n \) is relative twist angle between two parallel planes \( abBA \) and \( c'C'd' \) (Fig. 2b), and then is divided by the vector norm ||\( d\gamma \)|| of the line element \( rs \), and we get the gradient of twist angle \( \zeta_{rs} \) stated as

\[ \zeta_{rs} = \frac{d\gamma \cdot n}{||d\gamma||} = H(n) \cdot n, \quad (16a) \]

where \( H(n) = d\gamma/||d\gamma|| \) called gradient of rotation angle vector is the relative rotation angle vector \( d\gamma \) per unit length along the direction \( n \), and its components are given by

\[ \begin{align*}
H_x &= \frac{d\gamma_x}{||d\gamma||} = \frac{\partial \gamma_x}{\partial x} v_x + \frac{\partial \gamma_y}{\partial y} v_y + \frac{\partial \gamma_z}{\partial z} v_z, \\
H_y &= \frac{d\gamma_y}{||d\gamma||} = \frac{\partial \gamma_x}{\partial y} v_x + \frac{\partial \gamma_y}{\partial y} v_y + \frac{\partial \gamma_z}{\partial z} v_z, \\
H_z &= \frac{d\gamma_z}{||d\gamma||} = \frac{\partial \gamma_x}{\partial z} v_x + \frac{\partial \gamma_y}{\partial y} v_y + \frac{\partial \gamma_z}{\partial z} v_z.
\end{align*} \]

(16b–e)

Substituting Eqs. (16b–e) and (14b) in Eq. (16a), one gives

\[ \zeta_{rs} = \frac{\partial \gamma_x}{\partial x} (v_x) + \frac{\partial \gamma_y}{\partial y} (v_y) + \frac{\partial \gamma_z}{\partial z} (v_z) \]

(16f)

The relative rotation angle vector \( d\gamma \) is projected onto the direction \( n' \), i.e., \( \xi = d\gamma \cdot n' \) is relative bending angle between two parallel planes \( abCD \) and \( b'C'C' \) (Fig. 2c), and then is divided by the vector norm ||\( d\gamma'||\) and we get the gradient of bending angle \( \alpha \) stated as

\[ \alpha \approx \frac{d\gamma \cdot n'}{||d\gamma'||} = H(n') \cdot n', \quad (17a) \]

where \( H(n'') \) is the gradient of rotation angle vector along the direction \( n' \) and could gained by substituting \( n' \) for \( n \) of \( H(n) \) in Eqs. (16b–e). Similarly, we can also obtain the gradient of bending angle \( \beta \) between two parallel planes \( abCD \) and \( ABCD \) stated as

\[ \beta = H(n'') \cdot n''. \quad (17b) \]

Consequently, we can get the total gradient of bending angle \( \zeta_{rs} \) as

\[ \zeta_{rs} = \alpha + \beta = H(n'') \cdot n' + H(n') \cdot n''. \quad (17c) \]

Substituting Eqs. (14d–g), (16b–e) and (17a,b) in Eq. (17c), one gives

\[ \begin{align*}
\zeta_{rs} &= 2 \frac{\partial \gamma_x}{\partial x} u_x u'' + 2 \frac{\partial \gamma_y}{\partial y} u_y u'' + 2 \frac{\partial \gamma_z}{\partial z} u_z u'' + \left( \frac{\partial \gamma_x}{\partial y} + \frac{\partial \gamma_y}{\partial x} \right) u_x u_y + \left( \frac{\partial \gamma_x}{\partial z} + \frac{\partial \gamma_z}{\partial x} \right) u_x u_z + \left( \frac{\partial \gamma_y}{\partial z} + \frac{\partial \gamma_z}{\partial y} \right) u_y u_z \\
&+ \left( \frac{\partial \gamma_x}{\partial z} + \frac{\partial \gamma_z}{\partial y} \right) (u_x u'' + u_y u'') + \left( \frac{\partial \gamma_x}{\partial y} + \frac{\partial \gamma_y}{\partial z} \right) (u_x u'' + u_y u'')
\end{align*} \]

(17d)

Eqs. (16f) and (17d) are respectively rewritten in tensor as

\[ \frac{1}{2} \left( \gamma_x, \gamma_y, \gamma_z \right) u_i u_j = \frac{1}{2} \left( \gamma_x, \gamma_y, \gamma_z \right) u_i u_j, \quad \frac{1}{2} \zeta_{rs} = \frac{1}{2} \left( \gamma_x, \gamma_y, \gamma_z \right) u_i u_j = \frac{1}{2} \left( \gamma_x, \gamma_y, \gamma_z \right) u_i u_j. \quad (18a, b) \]

The two tensor Eqs. above are combined to a following tensor Eq. as

\[ \tilde{\eta}_{ij} = \frac{1}{2} \left( \gamma_x, \gamma_y, \gamma_z \right) u_i u_j, \quad (18c) \]

where \( \tilde{\eta}_{ij} = \tilde{\eta}_{ji} \) are symmetric curvature tensor.

By using the Eqs. (15) without the remain of Peano \( o(||d\gamma||) \), the rotation angle vector \( \gamma \) at point \( s \) can be expressed in the block matrix form in terms of the rotation angle vector \( \gamma \) at point \( r \), symmetric curvature tensor \( \tilde{\eta} \), skew–symmetric curvature tensor \( \tilde{\chi} \) and length vector \( dr \) as

\[ \gamma_s = \gamma_r + \tilde{\eta}_{ij} \cdot dr + \tilde{\chi}_{ij} \cdot dr, \quad (19a) \]

where \( \tilde{\chi}_{ij} = -\tilde{\chi}_{ji} \) are skew–symmetric curvature tensor, which are given by

\[ \frac{1}{2} \left( \frac{\partial \gamma_i}{\partial j} - \frac{\partial \gamma_j}{\partial i} \right), \quad [\gamma_r] = \begin{bmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{bmatrix}, \quad [\tilde{\eta}_{ij}] = \begin{bmatrix} \tilde{\eta}_{xx} & \tilde{\eta}_{xy} & \tilde{\eta}_{xz} \\ \tilde{\eta}_{yx} & \tilde{\eta}_{yy} & \tilde{\eta}_{yz} \\ \tilde{\eta}_{zx} & \tilde{\eta}_{zy} & \tilde{\eta}_{zz} \end{bmatrix}, \quad [\tilde{\chi}_{ij}] = \begin{bmatrix} 0 & \tilde{\chi}_{xy} & \tilde{\chi}_{xz} \\ -\tilde{\chi}_{xy} & 0 & \tilde{\chi}_{yz} \\ -\tilde{\chi}_{xz} & -\tilde{\chi}_{yz} & 0 \end{bmatrix}, \quad [dr] = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}. \quad (19b–f) \]
Let us define a gradient of rotation angle vector \( \mathbf{\omega} = -\chi_{j} e_{j} + \chi_{i} e_{i} - \chi_{i} n_{j} e_{j} \), that is, a skew-symmetric curvature tensor \( \chi_{ij} \). By Eq. (19a), we then can find the orbital angle vector \( \mathbf{\omega} \times \mathbf{dr} \) equals to the third term of right side of Eq. (19a), that is,
\[
\mathbf{\omega} \times \mathbf{dr} = \chi \cdot \mathbf{dr}. \tag{20}
\]

Introducing another Cartesian coordinate \( \hat{x}\hat{y}\hat{z} \) whose \( \hat{z} \) axis is parallel to the direction of rotation vector \( \mathbf{\omega} \) (Fig. 2d), we can rewrite \( \mathbf{\omega} \) and \( \mathbf{dr} \) as
\[
\mathbf{\omega} = \|\mathbf{\omega}\| e_{z}, \quad \mathbf{dr} = d\hat{x} e_{x} + d\hat{y} e_{y} + d\hat{z} e_{z}, \tag{21a, b}
\]
where \( \|\mathbf{\omega}\| \) is the vector norm of the gradient of rotation vector \( \mathbf{\omega} \), and \( e_{i} \) is a unit vector component around the \( i \)-axis.

Substituting Eqs. (21) in \( \mathbf{\omega} \times \mathbf{dr} \), we can obtain
\[
\mathbf{\omega} \times \mathbf{dr} = \|\mathbf{\omega}\| (d\hat{x} e_{y} - d\hat{y} e_{x}), \tag{22a}
\]
where
\[
\|\mathbf{\omega} \times \mathbf{dr}\| = \sqrt{(\mathbf{\omega} \times \mathbf{dr}) \cdot (\mathbf{\omega} \times \mathbf{dr})} = \|\mathbf{\omega}\| \sqrt{(dx)^2 + (dy)^2}. \tag{22b}
\]

For a small gradient of rotation angle \( \mathbf{\omega} \) corresponding to the gradient of rotation angle vector norm \( \|\mathbf{\omega}\| \ll 1 \), the orbital angle vector norm \( \|\mathbf{\omega} \times \mathbf{dr}\| \) approximatively equals to arc length \( bb' \) as
\[
\|\mathbf{\omega} \times \mathbf{dr}\| \approx \hat{b}b', \tag{22c}
\]
where arc length \( \hat{b}b' \) equals to the small gradient of rotation angle \( \mathbf{\omega} \) multiplied by length of segment \( sr \), and \( sr \) is shadow of the line element \( sr \) in \( \hat{x}\hat{y} \) plane, and the following two equations can be obtained respectively
\[
bb' = \mathbf{\omega} \cdot \mathbf{sr}, \quad ss' = \sqrt{(dx)^2 + (dy)^2}. \tag{22d, e}
\]

Substituting Eqs (22b, d, e) in Eq. (22c), we can obtain the gradient of the rotation angle \( \mathbf{\omega} \) as
\[
\mathbf{\omega} = \frac{\|\mathbf{\omega}\| \mathbf{dr}}{\|\mathbf{dr}\|} \approx \|\mathbf{\omega}\|. \tag{23}
\]

When the line element \( sr \) rotates through the gradient of rigid angle \( \|\mathbf{\omega}\| \), norm of line element \( sr \) is given by
\[
\|\mathbf{dr} + \mathbf{\omega} \times \mathbf{dr}\| = \sqrt{\|\mathbf{dr}\|^2 + \|\mathbf{\omega} \times \mathbf{dr}\|^2} = \sqrt{\|\mathbf{dr}\|^2 + \|\mathbf{\omega}\|^2 (dx)^2 + (dy)^2}. \tag{24a}
\]

For the small rotation angle, i.e., \( \|\mathbf{\omega}\| \ll 1 \), the above equation has an approximated relationship as
\[
\|\mathbf{dr} + \mathbf{\omega} \times \mathbf{dr}\| \approx \|\mathbf{dr}\|. \tag{24b}
\]

Therefore, the vector \( \mathbf{\omega} \times \mathbf{dr} \) suggests the end point \( s \) of the line element \( sr \) rotates through the gradient of rigid angle \( \|\mathbf{\omega}\| \) around the direction of rotation vector \( \mathbf{\omega} \), where the rotation axis passes through the other end point \( r \).

Substituting Eq. (20) in Eq. (19a), one gives
\[
\gamma_{i} = \gamma_{r} + \mathbf{\eta} \cdot \mathbf{dr} + \mathbf{\omega} \times \mathbf{dr}. \tag{25}
\]

Above equation suggests that the rotation angle vector \( \gamma_{i} \) of the plane \( \hat{c}\hat{D}\hat{a}d \) at the point \( s \) consists of three parts: the rotation angle vector \( \gamma_{r} \) of the plane \( abBA \) at the point \( r \), rotation angle deformation vector \( \mathbf{\eta} \cdot \mathbf{dr} \) and rigid rotation angle vector \( \mathbf{\omega} \times \mathbf{dr} \). Especially, symmetric curvature tensor \( \eta_{ij} \) are two types of gradient of rotation angles: gradient of bending angle \( \eta_{ii} \) (\( i \neq j \)) and gradient of twist angle \( \eta_{ij} \) (\( i = j \)); skew–symmetric curvature tensor \( \chi_{ij} \) are gradient of rotation angle for rigid body rotation.

In nonlinear elasticity, we also expect that the skew–symmetric curvature tensor \( \chi_{ij} \) will not contribute as a fundamental measure of deformation as the same as in the MCST and obtain the rotation angle vector \( \gamma_{i} \) only in terms of the rotation angle vector \( \mathbf{\omega} \), and the rotation angle deformation vector \( \mathbf{\eta} \) as
\[
\gamma_{i} = \gamma_{r} + \mathbf{\eta} \cdot \mathbf{dr}. \tag{26}
\]

On basis of definition Eq. (16a) for the gradient of twist angle \( \xi_{m} \), we can get the differential of twist angle \( \gamma_{m}^{n} \) between two arbitrary parallel plane elements along whose direction cosine \( \mathbf{n} \) as
\[
d\gamma_{m}^{n} = d\gamma \cdot \mathbf{n} = \xi_{m} \|\mathbf{dr}\|. \tag{27a}
\]

Substituting Eqs (14c) and (18a) in above equation, we can obtain the differential of twist angle \( d\gamma_{m}^{n} \) along an arbitrary direction \( \mathbf{n} \) as
\[
d\gamma_{m}^{n} = \frac{1}{2} (\gamma_{i,j} + \gamma_{j,i}) n_{i} n_{j} \sqrt{dx^2 + dy^2 + dz^2}. \tag{27b}
\]

Thus, based on above equation, we also can obtain the differential of twist angle scalar \( \gamma_{i} \) along \( i \)-axis (\( i = x, y, z \)) as
\[
d\gamma_{i} = \frac{d\gamma_{i}}{dt} dt. \tag{27c}
\]
Supposing the gradient of twist angle scalar $\gamma_i$ constant, Eq. (27c) is rewritten by

\[ d\gamma_i = \varsigma_i \eta_{ii} di, \quad (27d) \]

where $\varsigma_i$ are the weighting coefficients, and

\[ \eta_{ii} = \frac{\partial \gamma_i}{\partial i}. \quad (27e) \]

Integrating Eq. (27d) with respect to $i$ from zero to $L_i$ under the condition that $\gamma_i = 0$ at $i = 0$, one gives

\[ \gamma_i = \varsigma_i \eta_{ii} L_i. \quad (27f) \]

where $L_i$ is structure length along $i$–axis.

In the same way, we also obtain bending angle scalar $\gamma_{ij} (i, j = x, y, z; i \neq j)$

\[ \gamma_{ij} = \varsigma_{ij} \eta_{ij} L_i. \quad (28a) \]

in which the gradient of bending angle scalar

\[ \eta_{ij} = \frac{\partial \gamma_i}{\partial j} + \frac{\partial \gamma_j}{\partial i}. \quad (28b) \]

Considering a uniform circular cross–section shaft of length $L_i$ and of radius $R_i$ that was made of nonlinear elasticity materials and whose left end was fixed along $i$–axis ($i = x, y, z$), we exert a torque named $T_i$ parallel to $i$–axis at its the other end as shown in Fig. 3a. Due to deformation caused by the twisting couples, there is an angle of twist $\gamma_i$ that is a one–dimensional twist angle parallel to $i$–axis.

Based on the assumption in Mechanics of Materials that the one–dimensional deformed circular shaft is made of separate slats parallel to the $i$–axis, its torsional deformation actually consists of shear deformation of these slats. We detach from the solid shaft an element of the hollow shaft of inner radius $\rho_i$ and of the outer $\rho_i + d\rho_i$, and then consider the element of the hollow shaft made of element of separate slat (Fig. 3c), for example, the element of slat (Fig. 3b) that has element of line $ab$ parallel to the axis of the shaft on the surface of shaft of inner radius $\rho_i$ and that has the element of area $dA_i$ on the cross–section of right end of the shaft (Fig. 3d).
As stated in Mechanics of Materials, the element of slot of the shaft will be subjected to shears stresses $\sigma_{yi}$ and $\gamma_{yi}$, we could obtain the engineering shear strain $\varepsilon_{yi}$ (i.e., the angle $bab'$ in Fig. 3b) corresponding to an angle of twist $\gamma_i^i$ at the right end. For small values of shear strain $\varepsilon_{yi}$, we could both express the arc length $bb'$ as $bb' = L_i\varepsilon_{yi}$ and as $bb' = \rho_i\gamma_i^i$. It follows that $L_i\varepsilon_{yi} = \rho_i\gamma_i^i$, or

$$
\varepsilon_{yi} = \frac{\rho_i\gamma_i^i}{L_i}.
$$

Substituting Eq. (27f) in above equation, one gives

$$
\varepsilon_{yi} = \rho_i\delta_i^i\eta_i.
$$

Substituting for $\varepsilon_{yi}$ from Eq. (29b) into Eq. (11c), one gives

$$
\sigma_{yi} = \frac{1}{2(1 + \mu)} \left[ E_1\rho_i\delta_i^i\eta_i^2 - \frac{1}{4} E_2(\rho_i\delta_i^i\eta_i)\right].
$$

The torques $T_i$ at the cross–section of right end of the shaft (Fig. 3d) is equal to the sum of the moments of shear forces $\sigma_{yi}dA_i$, rotating round the axis of the shaft, in other words,

$$
T_i = \int \int_{A_i} \rho_i\sigma_{yi}dA_i,
$$

where $\rho_i$ denotes the vertical distance between the differential of shear force $\sigma_{yi}dA_i$ and the rotation axis of the shaft, and $\sigma_{yi}$ is the shear stress on the element of area $dA_i$ as shown in Fig. 3d that is expressed as

$$
dA_i = \rho_i d\rho_i d\kappa_i,
$$

where $d\kappa_i$ is the element of central angle corresponding to the element of area $dA_i$.

Substituting Eqs (30) and (31b) into Eq. (31a) and integrating with respect to $\kappa_i$ from zero to $2\pi$, and then with respect to $\rho_i$ from zero to $R_i$, the torque $T_i$ is obtained as

$$
T_i = \int \int_{A_i} \rho_i\sigma_{yi}dA_i = \frac{\pi}{1 + \mu} \left[ E_1\rho_i\delta_i^i\eta_i^2 - \frac{1}{4} E_2(\rho_i\delta_i^i\eta_i)\right].
$$

Now, let us consider the elementary work $d\kappa_i$ done by the torques $T_i$ as the shaft twists by one small amount $d\gamma_i^i$. The elementary work is expressed as

$$
d\kappa_i = T_id\gamma_i^i.
$$

Substituting Eqs (32) and (27f) in above equation and integrating with respect to $i$ from zero to $L_i$, the work $\lambda_i$ is obtained as

$$
\lambda_i = \int_0^{\gamma_i^i} T_id\gamma_i^i = \frac{\delta_i^i\eta_i^2\pi}{1 + \mu} \left[ E_1\rho_i\delta_i^i\eta_i^2 - \frac{1}{4} E_2(\rho_i\delta_i^i\eta_i)\right] di
$$

$$
= \frac{L_i\pi}{1 + \mu} \left[ E_1\rho_i\delta_i^i\eta_i^2 (\eta_i^2) - \frac{1}{4} E_2(\rho_i\delta_i^i\eta_i)(\eta_i^2)\right].
$$

Dividing the work $\lambda_i$ by the volume $V = \pi(R_i)^2L_i$ of the undeformed circular shaft in Fig. 3a, we have the average strain–energy density $\bar{h}_i$, i.e., the strain energy per unit volume, written as

$$
\bar{h}_i = \frac{\lambda_i}{V} = \frac{1}{1 + \mu} \left[ \frac{(R_i)^4}{4} E_1(\delta_i^i\eta_i^2) - \frac{(R_i)^4}{24} E_2(\delta_i^i\eta_i)(\eta_i^2)\right].
$$

Taking the first derivative of Eq. (35) with respect to twist angle per unit length $\eta_i$, we could obtain torsion stress scalar $m_i$, written as

$$
m_i = \frac{\partial \bar{h}_i}{\partial \eta_i} = m_i^L + m_i^N \quad (i = x, y, z),
$$

where

$$
m_i^L = \left(\frac{l_i}{\rho_i}\right)^2\frac{E_1}{1 + \mu}, \quad m_i^N = \left(\frac{l_i}{\rho_i}\right)^3\frac{E_2}{1 + \mu}, \quad l_i = \sqrt{\frac{1}{2}R_i\delta_i^i}, \quad l_i^L = \sqrt{\frac{1}{6}R_i\delta_i^i},
$$

and $m_i^N$ ($\alpha = L, N$) represent linear and nonlinear components of the torsion stress scalar, respectively; $l_i^L$ and $l_i^N$ are material length scale parameters as stated in linear couple stress–strain constitutive relationships Eq. (1d).

Similarly, considering a uniform circular cross–section shaft that is subjected to bending moment parallel to $i$–axis, there are a bending angle $\gamma_i^i$ and bending angle per unit length $\eta_i = \partial \gamma_i^i/\partial j + \partial \gamma_i^i/\partial i$ ($i \neq j$) in Eq. (28b). We believe that through the derivation of...
the formulas like above, bending–moment stress scalar called the couple stress scalar $m_{ij}$ has the same expression as the torsion stress scalar $m_{ii}$ and is written as

$$m_{ij} = m_{ij}^t - m_{ij}^N, \quad (i, j = x, y, z; i \neq j),$$

where

$$m_{ij}^t = \left(\frac{r_i^j}{r_i^t}\right)^2 E_1 (1 + \mu) \eta_{ij}, \quad m_{ij}^N = \left(\frac{r_i^j}{r_i^N}\right)^4 E_2 (\eta_{ij})^3.$$

This is because torque consists of the sum of shear stress multiplied by its arm and bending moment consists of the sum of normal stress multiplied by its arm. What's more, for the nonlinear elasticity materials both shear stress and normal stress are nonlinear and can interconvert into each other.

On basis of Eqs (2a), (4b), (12), (27e), (28b) and (36), the strain energy density of an arbitrary volume element $\varphi(\varepsilon_{ij}; \eta_{ij})$ described in scalar is given by

$$\phi = \int_0^{r_x} \sigma_{xx} \, dx + \int_0^{r_y} \sigma_{yy} \, dy + \int_0^{r_z} \sigma_{zz} \, dz + \int_0^{r_{xy}} \sigma_{xy} \, dxy + \int_0^{r_{xz}} \sigma_{xz} \, dxz + \int_0^{r_{yz}} \sigma_{yz} \, dyz + \int_0^{r_{xx}} \eta_{xx} \, \varepsilon_{xx} \, dx + \int_0^{r_{yy}} \eta_{yy} \, \varepsilon_{yy} \, dy + \int_0^{r_{zz}} \eta_{zz} \, \varepsilon_{zz} \, dz + \int_0^{r_{xy}} \eta_{xy} \, \varepsilon_{xy} \, dxy + \int_0^{r_{xz}} \eta_{xz} \, \varepsilon_{xz} \, dxz + \int_0^{r_{yz}} \eta_{yz} \, \varepsilon_{yz} \, dyz$$

Where

$$\sigma_{ii} = E_i \varepsilon_{ii} - E_2 (\eta_{ii})^3, \quad \varepsilon_{ij} = \frac{\partial \varepsilon_{ii}}{\partial j}, \quad \sigma_{ij} = \frac{E_i \varepsilon_{ij} - \frac{1}{2} E_2 (\eta_{ij})^3}{2 (1 + \mu)}, \quad \varepsilon_{ij} = \frac{\partial \varepsilon_{ii}}{\partial j} + \frac{\partial \varepsilon_{ij}}{\partial i}, \quad \eta_{ij} = \frac{(r_i^j)^2 E_1 (1 + \mu) \eta_{ij} - (r_i^j)^4 E_2 (\eta_{ij})^3}{1 + \mu}, \quad \eta_{ii} = \frac{\partial \gamma_i}{\partial i},$$

$$m_{ij} = \left(\frac{r_i^j}{r_i^t}\right)^2 E_1 (1 + \mu) \eta_{ij} - \left(\frac{r_i^j}{r_i^N}\right)^4 E_2 (\eta_{ij})^3, \quad \eta_{ii} = \frac{\partial \gamma_i}{\partial i}, \quad \gamma_i = \frac{1}{2} \text{curl} (\varepsilon_{ij}), \quad (i, j = x, y, z; i \neq j).$$

Setting the generalized Young's modulus coefficient $E_2 = 0$, the strain energy density Eq. (37a) will come back to the linear one.

3. MATHEMATICAL FORMULATION

Figure 4 illustrates the geometry and coordinate system of microbeams with uniform cross section $A$. The microstructures have length $L$, thickness $H$, width $B$ and density $\varrho$. Three kinds of boundary ends are present: hinged (H), clamped (C) and free (F) ends. Four pairs of boundary conditions, i.e. H–H, C–C, C–H and C–F, are loaded on the two ends of the microbeams, which could be obtained with the combination of the three kinds of boundary ends.

According to the Bernoulli–Euler beam theory [40], the displacement field can be written as

$$u(x, y, z, t) = u(x, t) - z \frac{\partial w(x, t)}{\partial x}, \quad u_y(x, y, z, t) = 0, \quad u_z(x, y, z, t) = w(x, t),$$

where $u_x(x, y, z, t)$, $u_y(x, y, z, t)$ and $u_z(x, y, z, t)$ are the displacements of an arbitrary space point $(x, y, z)$ in the microbeams along the $x$, $y$ and $z$ directions, respectively; $u(x, t)$ and $w(x, t)$ represent the displacements of the mid-plane in the longitudinal and transverse directions, respectively; and $t$ is time.

Using Eq. (38) in Eqs (37i) and (37j), the nonzero components of $\varepsilon_{ij}$ can be written as

$$\varepsilon_{xx} = -\frac{1}{2} \frac{\partial^2 w}{\partial x^2}.$$

Figure 4 Schematic of the Cartesian coordinate system for Bernoulli–Euler microbeams: (a) a movable microbeam and (b) three kinds of boundary ends.
Similarly, substituting Eq. (38) into Eqs (37c) and (37e), the nonzero components of $\varepsilon_{ij}$ can be given as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}.$$  \hspace{1cm} (40a)

Taking into account von Kármán geometric nonlinearity due to the exact strain of large deformation, the nonlinear strain–displacement relationship \([41, 42]\) is obtained by

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2.$$  \hspace{1cm} (40b)

By substituting Eqs (37b) and (37h) into Eq. (37a), the strain energy density can be obtained as

$$\phi(\varepsilon_{ij}, \eta_{ij}) = \int_0^L \left[ E_1 \varepsilon_{xx} - E_2 (\varepsilon_{xx})^3 \right] d\varepsilon_{xx} + \int_0^L \left[ \left( \frac{\partial \varepsilon_{xx}}{\partial x} \right)^2 \eta_{xx} - \left( \frac{\partial \varepsilon_{xx}}{\partial x} \right)^2 (\eta_{xx})^3 \right] d\eta_{xx}$$

$$= \frac{1}{2} E_1 (\varepsilon_{xx})^2 - \frac{1}{4} E_2 (\varepsilon_{xx})^4 + \frac{(\frac{\partial \varepsilon_{xx}}{\partial x})^2}{2(1+\mu)} (\eta_{xx})^2 - \frac{(\frac{\partial \varepsilon_{xx}}{\partial x})^2}{4(1+\mu)} (\eta_{xx})^4.$$  \hspace{1cm} (41a)

Using Eqs (2b)–(2d) and (36f)–(36h), Eq. (41a) could be rewritten as follows:

$$\phi(\varepsilon_{ij}, \eta_{ij}) = \frac{1}{2} \left( \sigma_{xx} + \frac{1}{2} \sigma_{NN} \right) \varepsilon_{xx} + \frac{1}{2} \left( m_{xx} + \frac{1}{2} m_{NN} \right) \eta_{xx}.$$  \hspace{1cm} (41b)

Consequently, the strain energy density equation (41b) based on the MCST in nonlinear elasticity has the additional nonlinear normal stress $\sigma_{NN}$ and couple stresses $m_{NN}$ compared with Eq. (1a) based on the MCST in linear elasticity.

Next, by substituting Eqs (39) and (40b) into Eq. (41b), the strain energy $\Phi$ of the isotropic nonlinear elasticity material microbeam occupying region $\Lambda$ can be derived as

$$\Phi = \int_\Lambda \phi(\varepsilon_{ij}, \eta_{ij}) dV$$

$$= \frac{1}{2} \int_0^L \int_\Lambda \left\{ \left( \sigma_{xx} + \frac{1}{2} \sigma_{NN} \right) \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]^2 - z \left( \sigma_{xx} + \frac{1}{2} \sigma_{NN} \right) \frac{\partial^2 w}{\partial x^2} - \frac{1}{2} \left( m_{xx} + \frac{1}{2} m_{NN} \right) \frac{\partial^2 w}{\partial x^2} \right\} dA dx$$

$$= \frac{1}{2} \int_0^L \int_\Lambda \left\{ \left( N_{xx} + \frac{1}{2} N_{NN} \right) \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]^2 - \left( M_{xx} + \frac{1}{2} M_{NN} \right) \frac{\partial^2 w}{\partial x^2} - \frac{1}{2} \left( Y_{xx} + \frac{1}{2} Y_{NN} \right) \frac{\partial^2 w}{\partial x^2} \right\} dA dx,$$  \hspace{1cm} (42a)

where the normal resultant force $N_{xx}$, bending moment $M_{xx}$ and couple moment $Y_{xx}$, and the nonlinear generalized normal resultant force $N_{NN}$, bending moment $M_{NN}$ and couple moment $Y_{NN}$, which are caused by the nonlinear elasticity, can be defined as follows:

$$N_{xx} = \int_\Lambda \sigma_{xx} dA$$

$$= A_{11} E_1 \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] - E_2 \left[ A_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right)^3 + 3A_{22} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \right],$$  \hspace{1cm} (42b)

$$N_{NN} = \int_\Lambda \sigma_{NN} dA = -E_2 \left[ A_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right)^3 + 3A_{22} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \right],$$  \hspace{1cm} (42c)

$$M_{xx} = \int_\Lambda \sigma_{xx} dA = -A_{22} E_1 \left[ \frac{\partial^2 w}{\partial x^2} \right] + E_2 \left[ 3A_{22} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{33} \left( \frac{\partial^2 w}{\partial x^2} \right)^3 \right],$$  \hspace{1cm} (42d)

$$M_{NN} = \int_\Lambda \sigma_{NN} dA = E_2 \left[ 3A_{22} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{33} \left( \frac{\partial^2 w}{\partial x^2} \right)^3 \right],$$  \hspace{1cm} (42e)

$$Y_{xx} = \int_\Lambda \sigma_{xx} dA = -A_{44} E_1 \left[ \frac{\partial^2 w}{\partial x^2} \right] + A_{55} E_2 \left( \frac{\partial^2 w}{\partial x^2} \right)^3, \hspace{1cm} Y_{NN} = \int_\Lambda \sigma_{NN} dA = -\frac{1}{4} A_{55} E_2 \left( \frac{\partial^2 w}{\partial x^2} \right)^3.$$  \hspace{1cm} (42f, g)

The corresponding coefficients in Eqs (42b)–(42g) are given by

$$\{A_{11}, A_{22}, A_{33}\} = B \int_{-H/2}^{H/2} \left[ x^2, x^4 \right] dz, \quad \{A_{44}, A_{55}\} = \frac{B}{2(1+v)} \int_{-H/2}^{H/2} \left[ \left( \frac{\partial \varepsilon_{xx}}{\partial x} \right)^2, \left( \frac{\partial \varepsilon_{xx}}{\partial x} \right)^4 \right] dz.$$  \hspace{1cm} (42h, i)
For the simplification of Eq. (42a), one obtains
\[
\Phi = \frac{1}{2} \int_0^L \left\{ N^* \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + M^* \frac{\partial^2 w}{\partial x^2} + Y^* \frac{\partial^2 w}{\partial x^2} \right\} dx, \tag{43a}
\]
where
\[
N^* = N_{xx} + \frac{1}{2} N_{xx}, \quad M^* = -M_{xx} - \frac{1}{2} M_{xx}, \quad Y^* = -\frac{1}{2} Y_{xx} - \frac{1}{4} Y_{xx}. \tag{43b–d}
\]
On the other hand, the kinetic energy $\Theta$ of the system can be expressed as
\[
\Theta = \frac{1}{2} \int_0^L A_{66} \left[ \left( \frac{\partial w}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \right] dx, \tag{44a}
\]
where
\[
A_{66} = B \int_{-H/2}^{H/2} \vartheta \, dz. \tag{44b}
\]
The governing equations for the free vibration of microbeams can be derived from Hamilton’s principle [9, 43]:
\[
\int_{t_1}^{t_2} \delta F \, dt = 0, \tag{45a}
\]
where $\delta$ is the first variation operator and $F$ is the Lagrange function of the free vibration system without the external forces, which is given by
\[
F = F(w_x, w_{xx}, w_t, u_x, u_t) = \Theta - \Phi, \tag{45b}
\]
where
\[
w_x = \frac{\partial w}{\partial x}, \quad w_{xx} = \frac{\partial^2 w}{\partial x^2}, \quad w_t = \frac{\partial w}{\partial t}, \quad u_x = \frac{\partial u}{\partial x}, \quad u_t = \frac{\partial u}{\partial t}. \tag{45c–g}
\]
On the basis of the derivation rule of compound function, the first variation of Lagrange function can be rewritten as
\[
\int_{t_1}^{t_2} \delta F(w_x, w_{xx}, w_t, u_x, u_t) \, dt
\]
\[
= \int_{t_1}^{t_2} \int_0^L \left\{ \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_x} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_{xx}} \right) + \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial w_t} \right) \right\} \delta w + \left[ \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial u_t} \right) \right] \delta u \right\} dx \, dt
\]
\[
+ \int_{t_1}^{t_2} \left\{ \frac{\partial F}{\partial w_{xx}} \delta w + \frac{\partial F}{\partial w_{xx}} \delta w_x + \frac{\partial F}{\partial u_x} \delta u \right\}_0^L \int_0^L \left[ \frac{\partial F}{\partial w_t} \delta w + \frac{\partial F}{\partial u_t} \delta u \right]_{t_1}^{t_2} dx = 0. \tag{46}
\]
Substituting Eqs (43a), (44a) and (45b) into Eq. (46) and setting the coefficients of $\delta w$, $\delta u$ and $\delta w_x$ to zero yield the nonlinear partial differential governing equations of microbeams: the transverse equation
\[
-\frac{1}{2} \frac{\partial^2 M^*}{\partial x^2} - \frac{1}{2} \frac{\partial^2 Y^*}{\partial x^2} + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial N^*}{\partial w_x} \frac{\partial w}{\partial x} \right) + \frac{1}{4} \frac{\partial}{\partial x} \left( \frac{\partial N^*}{\partial w_{xx}} \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial N^*}{\partial u_x} \frac{\partial u}{\partial x} \right) + \frac{1}{4} \frac{\partial}{\partial x} \left( \frac{\partial N^*}{\partial u_{xx}} \frac{\partial u}{\partial x} \right)^2
\]
\[
- \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial N^*}{\partial w_{xx}} \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial N^*}{\partial u_{xx}} \frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial M^*}{\partial w_{xx}} \frac{\partial^2 w}{\partial x^2} \right) + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial M^*}{\partial u_{xx}} \frac{\partial^2 u}{\partial x^2} \right) + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial Y^*}{\partial w_{xx}} \frac{\partial^2 w}{\partial x^2} \right) = A_{66} \frac{\partial^2 w}{\partial t^2}, \tag{47a}
\]
the longitudinal equation
\[
\frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial N^*}{\partial u_x} \frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial N^*}{\partial u_{xx}} \frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial M^*}{\partial u_{xx}} \frac{\partial^2 u}{\partial x^2} \right) + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial Y^*}{\partial u_{xx}} \frac{\partial^2 u}{\partial x^2} \right) = A_{66} \frac{\partial^2 u}{\partial t^2}, \tag{47b}
\]
and the three kinds of boundary ends: C end
\[
w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad u = 0, \tag{48a–c}
\]
H end
\[
w = 0, \quad u = 0, \quad \frac{1}{2} M^* - \frac{1}{2} Y^* - \frac{1}{2} w_{xx} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{1}{2} w_{xx} \left( \frac{\partial u}{\partial x} \right) - \frac{1}{2} w_{xx} \frac{\partial^2 w}{\partial x^2} - \frac{1}{2} \frac{\partial Y^*}{\partial w_{xx}} \frac{\partial^2 w}{\partial x^2} = 0 \tag{49a–c}
\]
and F end
\[
\begin{align*}
-\frac{1}{2} M^* - \frac{1}{2} Y^* - \frac{1}{4} \frac{\partial N^*}{\partial \omega} (\frac{\partial \omega}{\partial x})^2 - \frac{1}{2} \frac{\partial N^*}{\partial \omega} \frac{\partial u}{\partial x} - \frac{1}{2} \frac{\partial M^*}{\partial \omega} \frac{\partial^2 \omega}{\partial x^2} - \frac{1}{2} \frac{\partial Y^*}{\partial \omega} \frac{\partial^2 \omega}{\partial x^2} &= 0, \\
\frac{1}{2} \frac{\partial M^*}{\partial x} + \frac{1}{2} \frac{\partial Y^*}{\partial x} - \frac{1}{2} N^* \frac{\partial \omega}{\partial x} - \frac{1}{4} \frac{\partial N^*}{\partial \omega} \frac{\partial^2 \omega}{\partial x^2} - \frac{1}{2} \frac{\partial N^*}{\partial \omega} \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial \omega}{\partial \omega} \left( \frac{\partial \omega}{\partial x} \right)^2 \right] \\
+ \frac{1}{2} \frac{\partial \omega}{\partial x} \left( \frac{\partial N^*}{\partial \omega} \frac{\partial u}{\partial x} - \frac{1}{2} \frac{\partial M^*}{\partial \omega} \frac{\partial^2 \omega}{\partial x^2} + \frac{1}{2} \frac{\partial N^*}{\partial \omega} \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial \omega}{\partial \omega} \left( \frac{\partial \omega}{\partial x} \right)^2 \right) = 0.
\end{align*}
\]

We obtain the Lagrange function \(F\), which could be expressed in derivatives of the displacement field with respect to \(x\) and \(t\) (because \(N^*, M^*\) and \(Y^*\) are functions of derivatives of the displacement field and \(F\) is compound function of \(N^*, M^*, Y^*\) and derivatives of the displacement field), and then take the variation of the Lagrange function \(F\) with respect to derivatives of the displacement field. Thus, there are many derivatives of \(N^*, M^*\) and \(Y^*\) with respect to derivatives of the displacement field (such as \(\partial N^*/\partial \omega\)) in the above-mentioned equations. Meanwhile, there are no derivatives of the normal resultant force, bending moment and couple moment with respect to derivatives of the displacement field as obtained by Şimşek et al. [42, 44]. This is because they take the variation of the strain energy density with respect to the strain directly and then take the variation of the Lagrange function \(F\) with respect to derivatives of the displacement field. The two procedures could obtain the same results even though a difference in expression exists.

### 4. SOLUTION TECHNIQUE

To gain the dimensionless equations of motion, the dimensionless parameters are defined as follows:

\[
W = \frac{w}{H}, \quad U = \frac{u}{H}, \quad X = \frac{x}{L}, \quad \Gamma = \frac{H}{L_0} \sqrt{\frac{E_1}{12\rho}},
\]

where \(L_0\) indicates that the length \(L\) of microbeams is constant.

On the basis of the discretization rules of the DQM [27], the dimensionless displacement field components \(W, U\) and their \(n\)th derivatives with respect to \(X\) can be approximated as

\[
\{W, U\} = \sum_{m=1}^{N} \gamma_m(X)\{W_m(\Gamma), U_m(\Gamma)\},
\]

where \(N\) is the total number of nodes distributed along the \(x\) direction, \(\gamma_m(X)\) are the Lagrange interpolation polynomials and \(c_m^{(n)}\) are the weighting coefficients. The sampling points are generated by using the following cosine pattern:

\[
X_i = \frac{1}{2} \left\{ 1 - \cos \left( \frac{\pi (i-1)}{N-1} \right) \right\}, \quad i = 1, 2, \ldots, N.
\]

Substituting Eqs (51) and (52) into Eqs (47)–(50), the discretized ordinary differential equations governing the nonlinear vibration of microbeams are written in the block matrix form as

\[
\begin{bmatrix}
\left[ B_{WL}^{(0)} \right]_{2 \times 2} & \left[ B_{WL}^{(1)} \right]_{2 \times 2} & \left[ B_{WL}^{(2)} \right]_{2 \times 1} & \left[ B_{WL}^{(3)} \right]_{2 \times 1} \\
\left[ B_{WR}^{(0)} \right]_{2 \times 2} & \left[ B_{WR}^{(1)} \right]_{2 \times 1} & \left[ B_{WR}^{(2)} \right]_{2 \times 1} & \left[ B_{WR}^{(3)} \right]_{2 \times 1} \\
\left[ B_{UL}^{(0)} \right]_{2 \times 2} & \left[ B_{UL}^{(1)} \right]_{2 \times 1} & \left[ B_{UL}^{(2)} \right]_{2 \times 1} & \left[ B_{UL}^{(3)} \right]_{2 \times 1} \\
\left[ B_{UR}^{(0)} \right]_{2 \times 2} & \left[ B_{UR}^{(1)} \right]_{2 \times 1} & \left[ B_{UR}^{(2)} \right]_{2 \times 1} & \left[ B_{UR}^{(3)} \right]_{2 \times 1}
\end{bmatrix}\begin{bmatrix}
\{D_1\} \\
\{D_2\} \\
\{D_3\} \\
\{D_4\}
\end{bmatrix} = \{0\},
\]

\[
\begin{bmatrix}
\left[ G_{W}^{(0)} \right]_{(N-4) \times 2} & \left[ G_{W}^{(1)} \right]_{(N-4) \times 2} & \left[ G_{W}^{(2)} \right]_{(N-4) \times 1} & \left[ G_{W}^{(3)} \right]_{(N-4) \times 1} \\
\left[ G_{U}^{(0)} \right]_{(N-2) \times 2} & \left[ G_{U}^{(1)} \right]_{(N-2) \times 2} & \left[ G_{U}^{(2)} \right]_{(N-2) \times 1} & \left[ G_{U}^{(3)} \right]_{(N-2) \times 1}
\end{bmatrix}\begin{bmatrix}
\{D_1\} \\
\{D_2\} \\
\{D_3\} \\
\{D_4\}
\end{bmatrix} = \{0\}.
\]
Table 1 Effect of the number of nodes N on the first three linear and nonlinear natural frequencies $\varphi_{\alpha\alpha\alpha}$ (MHz) of microbeams with four types of boundary conditions ($H/l = 1, L/H = 100, W_{max}^1 = 1.0$).

| N  | $\varphi_{LLL}^{1}$ First step | $\varphi_{LLL}^{1}$ Last step | $\varphi_{LLL}^{2}$ Last step | $\varphi_{LLL}^{3}$ First step | $\varphi_{LLL}^{3}$ Last step | $\varphi_{LLL}^{4}$ First step | $\varphi_{LLL}^{4}$ Last step |
|----|--------------------------------|--------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 7  | 0.1608                         | 0.2463                         | 0.5728                        | 0.6304                        | 1.023                         | 1.015                         |
| 9  | 0.1601                         | 0.2281                         | 0.6486                        | 0.7081                        | 1.626                         | 1.751                         |
| 11 | 0.1601                         | 0.2318                         | 0.6400                        | 0.6886                        | 1.433                         | 1.455                         |
| 13 | 0.1601                         | 0.2312                         | 0.6404                        | 0.6923                        | 1.441                         | 1.480                         |
| 15 | 0.1601                         | 0.2312                         | 0.6404                        | 0.6916                        | 1.441                         | 1.476                         | 12.65                        | 12.64                        |
| 17 | 0.1601                         | 0.2312                         | 0.6404                        | 0.6917                        | 1.441                         | 1.476                         | 12.06                        | 12.00                        |
| 19 | 0.1601                         | 0.2312                         | 0.6404                        | 0.6917                        | 1.441                         | 1.476                         | 12.87                        | 12.81                        |
| 21 | 0.1601                         | 0.2312                         | 0.6404                        | 0.6917                        | 1.441                         | 1.476                         | 12.99                        | 12.95                        |

Equations (54) and (55) are rewritten in more simplified matrix form for obtaining eigenvalue equations easily as

$$\begin{align*}
\{D\} = \{\bar{D}\}
\end{align*}$$

where $[B_{\alpha\beta}^{ij}]_{j \times k}$ represents the size of the submatrices and $[G_{\alpha\beta}^{ij}]_{j \times k}$ are the stiffness submatrices of the boundary conditions and governing equations, respectively; $[M]_{j \times k}$ are the mass submatrices; $\{D\} = \{\bar{D}\}$ are the components of the displacement vectors, which are written in displacement form as $\{d_{1}\} = \{w_1 w_2\}^T, \{d_{2}\} = \{w_2 w_3 w_4\}^T, \{d_{3}\} = \{w_3 w_4 w_5\}^T, \{d_{4}\} = \{w_4 w_5 w_6\}^T, \{d_{5}\} = \{w_5 w_6 w_7\}^T$; and the two dots above $\{d_{2}\}$ and $\{d_{3}\}$ represent the second-order partial derivatives with respect to dimensionless time $\Gamma$. The expressions of the stiffness and mass submatrices are presented in Appendix A.

Equations (54) and (55) are rewritten in more simplified matrix form for obtaining eigenvalue equations easily as

$$
[B_{\alpha\beta}] \{D\} + [G_{\alpha\beta}] \{\bar{D}\} = \{0\},
$$

where $[B_{\alpha\beta}] = [B_{\alpha\beta}]_a + [B_{\alpha\beta}]_b [D]_b$.

Inserting Eq. (56) into Eq. (57), one obtains

$$
[K_{\alpha\beta}] \{D\} + [M] \{\bar{D}\} = \{0\},
$$

where

$$
[K_{\alpha\beta}] = [G_{\alpha\beta}]_b - [G_{\alpha\beta}]_a [B]_a^{-1} [B]_b.
$$
Table 2 First three linear natural frequencies $\varphi_{LLL}^j$ (MHz) of microbeams with four groups of boundary conditions ($H/l = 2$, $L/H = 100$, $W_{\text{max}}^1 = 1.0$).

| Boundary conditions | $l = 0$ | $l = 17.6 \mu \text{m}$ |
|---------------------|---------|-------------------------|
|                     | $\varphi_{LLL}^1$ | $\varphi_{LLL}^2$ | $\varphi_{LLL}^3$ | $\varphi_{LLL}^1$ | $\varphi_{LLL}^2$ | $\varphi_{LLL}^3$ |
| Present             | 0.04436 | 0.1775 | 0.3993 | 0.06471 | 0.2589 | 0.5824 |
| Ref. [9]            | 0.04436 | 0.1775 | 0.3993 | 0.06471 | 0.2589 | 0.5824 |
| C–C                 | 0.1006  | 0.2772 | 0.5435 | 0.1467  | 0.4044 | 0.7928 |
| Ref. [9]            | 0.1006  | 0.2772 | 0.5435 | 0.1467  | 0.4044 | 0.7928 |
| C–H                 | 0.06931 | 0.2246 | 0.4686 | 0.1011  | 0.3276 | 0.6835 |
| Ref. [9]            | 0.06931 | 0.2246 | 0.4686 | 0.1011  | 0.3276 | 0.6836 |
| C–F                 | 0.01580 | 0.09905 | 0.2774 | 0.02305 | 0.1445 | 0.4046 |
| Present             | 0.01580 | 0.09904 | 0.2774 | 0.02305 | 0.1445 | 0.4046 |

Table 3 Nonlinear fundamental frequency ratio $\varphi_{LNN}^j/\varphi_{LLL}^j$ of microbeams with three groups of boundary conditions ($H/l = 2$, $L/H = 100$).

| $w_{\text{max}} / k$ | H–H | C–C | C–H | H–H | C–C | C–H |
|-----------------------|-----|-----|-----|-----|-----|-----|
| Present               |     |     |     |     |     |     |
| Ref. [42]             |     |     |     |     |     |     |
| Relative error (%)    |     |     |     |     |     |     |
| 1.0                   | 1.119 | 1.11920 | 0.01787 | 1.029 | 1.03029 | 0.1252 |
|                      | 1.060 | 1.05923 | -0.07269 | 1.222 | 1.21789 | -0.3375 |
|                      | 1.091 | 1.074 | -1.583 | 1.715 | 1.69576 | -1.135 |
|                      | 1.229 | 1.192 | -3.104 | 2.001 | 1.94717 | -2.765 |

| $w_{\text{max}} / H$ | H–H | C–C | C–H | H–H | C–C | C–H |
|----------------------|-----|-----|-----|-----|-----|-----|
| Present              |     |     |     |     |     |     |
| Ref. [44]            |     |     |     |     |     |     |
| Relative error (%)   |     |     |     |     |     |     |
| 0.2                  | 1.042 | 1.034 | -0.7737 | 1.111 | 1.009 | -0.1982 |
|                      | 1.024 | 1.024 | -0.029 | 1.089 | 1.089 | -0.029 |
|                      | 1.290 | 1.290 | -0.029 | 1.407 | 1.407 | -0.029 |

Table 4 Relative errors between the first three dimensionless nonlinear frequencies $\omega_{LNN}^j$ and the dimensionless linear fundamental frequencies $\omega_{LLL}^j$ for the microbeams with different boundary conditions ($H/l = 1$, $L/H = 100$, $W_{\text{max}}^1 = 1.0$).

| $\omega_{LLL}^j$ | H–H | C–C | C–H | C–F |
|------------------|-----|-----|-----|-----|
| $\omega_{LNN}^1$ | 35.62 | 80.74 | 55.64 | 12.69 |
| $\omega_{LNN}^2$ | 35.31 (−0.86%) | 77.21 (−4.57%) | 54.46 (−2.17%) | 12.68 (−0.92%) |
| $\omega_{LNN}^3$ | 142.46 | 222.56 | 180.31 | 79.52 |
| $\omega_{LNN}^4$ | 142.01 (−0.32%) | 220.11 (−1.11%) | 179.17 (−0.64%) | 79.49 (−0.028%) |
| $\omega_{LNN}^5$ | 320.54 | 436.30 | 376.19 | 222.65 |
| $\omega_{LNN}^6$ | 319.32 (−0.38%) | 430.00 (−1.46%) | 373.20 (−0.80%) | 222.58 (−0.029%) |
Figure 5 Comparisons between the fundamental nonlinear dimensionless frequencies $\omega_{LLN}$ and linear dimensionless frequencies $\omega_{LLL}$ of microbeams with H–H (a), C–C (b), C–H (c) and C–F (d) ends under different dimensionless amplitudes $W_{\text{max}}^i$ ($H/l = 1, L/H = 100$).

Next, the dynamic displacement vector $\{\mathbf{D}\}_b$ can be expanded in the form of

$$\{\mathbf{D}\}_b = \{\mathbf{D}^*\}_b e^{i\omega t},$$

where

$$\{\mathbf{D}^*\}_b = \left\{\{D^*_2\}^T, \{D^*_5\}^T\right\}^T$$

and $\omega = \varphi L_0^2 \sqrt{12\rho/E_1H^2}$ is the dimensionless nonlinear frequency; $\varphi$ is the nonlinear free vibration frequency; $\{\mathbf{D}^*\}_b$ is the vibration mode shape vector; and $\{D^*_2\} = \{W^*_3, W^*_4, \ldots, W^*_N-3\}$ and $\{D^*_5\} = \{U^*_2, U^*_3, \ldots, U^*_N-2\}$ are the vibration mode shapes along the $z$ and $x$ directions, respectively.

Then, substituting Eq. (59a) into Eq. (58a) yields the nonlinear eigenvalue equations as shown below:

$$[\mathbf{K}] \{\mathbf{D}^*\}_b = \omega^2 [\mathbf{M}] \{\mathbf{D}^*\}_b.$$  

(60)

The above nonlinear eigenvalue equation (60) can be solved through a direct iterative process as follows, applied in [14, 15, 20, 21]:

**Step 1:** By neglecting nonlinear terms in the stiffness matrix $[\mathbf{K}]$ of Eq. (60), we could obtain the linear stiffness matrix $[\mathbf{K}]_L$ and the corresponding linear eigenvalue problem is solved.

**Step 2:** The linear eigenvectors obtained in Step 1 are appropriately scaled up such that the maximum transverse displacement is equal to a given vibration amplitude. Then, inserting the scaled normalized linear eigenvectors into the stiffness matrix $[\mathbf{K}]$, we could use the scaled normalized linear eigenvectors to get the nonlinear stiffness matrix $[\mathbf{K}]_{NL}$. The nonlinear eigenvalues and eigenvectors are obtained from the updated eigensystem (60).

**Step 3:** The eigenvectors are scaled up again, and Step 2 is repeated until the frequency values from the two subsequent iterations $i$ and $i+1$ satisfy the prescribed convergence criteria as

$$\frac{|\omega^{i+1} - \omega^i|}{\omega^i} \leq \varepsilon_0,$$

where $\omega^k$ is the frequency at iteration $k (k = i, i+1)$ and $\varepsilon_0$ is a small value number, which is set to be $10^{-4}$ in this paper.

5. NUMERICAL RESULTS

Numerical results of the nonlinear free vibration of microbeams made from nonlinear elasticity material Al–1% Si are presented in Tables 1–4 and Figs 5–13. In previous studies [36, 38], two generalized nonlinear Young’s modulus coefficients of Eq. (3a) of Al–1% Si were given as $E_1 = 65$ GPa and $E_2 = 6 \times 10^5$ GPa, and the linear elastic modulus was $E = 69$ GPa. However, the generalized Young's modulus coefficient $E_1$ of Eq. (2a) should be equal to $E$ of Eq. (1g) when $\varepsilon_{xx} = 0$. To obtain the relationship $E_1 = E$, we use the third-order nonlinear algebraic equation of Eq. (2a) again to refit the nonlinear normal stress–strain constitutive relationship.
Figure 6 Comparisons between the second nonlinear dimensionless frequencies $\omega_{LLN}^2$ and linear dimensionless frequencies $\omega_{LLL}^2$ of microbeams with H–H (a), C–C (b), C–H (c) and C–F (d) ends under different dimensionless amplitudes $W_{\max}^1 (H/l = 1, L/H = 100)$.

Figure 7 Comparisons between the third nonlinear dimensionless frequencies $\omega_{LLN}^3$ and linear dimensionless frequencies $\omega_{LLL}^3$ of microbeams with H–H (a), C–C (b), C–H (c) and C–F (d) ends under different dimensionless amplitudes $W_{\max}^1 (H/l = 1, L/H = 100)$.

curve of Al–1% Si and obtain new values of material properties: $E_1 = 70$ GPa, $E_3 = 5.2 \times 10^6$ GPa and $E = 70$ GPa. Other values of geometric and material properties, such as thickness $H = 35.2 \ \mu m$, constant length $L_0 = 100H$, width $B = 17.6 \ \mu m$, Poisson’s ratio $\nu = 0.33$ and density $\theta = 2330 \ \text{kg/m}^3$, are also present. For the material length scale parameters of nonlinear elasticity materials, there are no available experimental data and $l = 17.6 \ \mu m$ for a homogeneous epoxy microbeam by Lam et al. [45]. In order to observe the size effect in quantitative analysis, the two material length scale parameters are assumed to be equal and $l_{1\alpha} = l_{2\beta} = 17.6 \ \mu m$. These values of geometric and material properties can be regarded as constants until they are redefined as other values again.

Note that $\varphi_{\alpha\beta\varsigma}^j$, $\omega_{\alpha\beta\varsigma}^j$ and $W_{\alpha\beta\varsigma}^j$ ($j = 1, 2, ..., 2N - 5, 2N - 6; \alpha, \beta, \varsigma = L, N$) are the dimensional frequencies, dimensionless frequencies and mode shapes of microbeams, respectively, where the superscript variable $j$ represents the $j$th frequency, the first subscript $\alpha$ represents the von Kármán geometric nonlinearity (N) or linearity (L), the second subscript $\beta$ represents the nonlinear (N)
Figure 8 Comparisons between the fundamental nonlinear dimensionless frequencies $\omega_{LNL}^1$ and linear dimensionless frequencies $\omega_{LLL}^1$ of microbeams with H–H (a), C–C (b), C–H (c) and C–F (d) ends under different dimensionless amplitudes $W_{\text{max}}^1$ ($H/l = 1, L/H = 100$).

Figure 9 Comparisons between the second nonlinear dimensionless frequencies $\omega_{LNL}^2$ and linear dimensionless frequencies $\omega_{LLL}^2$ of microbeams with H–H (a), C–C (b), C–H (c) and C–F (d) ends under different dimensionless amplitudes $W_{\text{max}}^1$ ($H/l = 1, L/H = 100$).

or linear (L) classical stress, and the third subscript $\varsigma$ represents nonlinear (N) or linear (L) couple stresses. For example, $\omega_{LNL}^1$ is the fundamental nonlinear dimensionless frequency within the framework of the geometric linearity, and the linear classical and nonlinear couple stress–strain constitutive relationships; $\omega_{NNN}^1$ is the fundamental nonlinear dimensionless mode shape within the framework of the geometric nonlinearity, and the nonlinear classical and couple stress–strain constitutive relationships.

Table 1 examines the effect of the number of nodes $N$ of Eq. (53) on the first three linear and nonlinear natural frequencies $\varphi_{\text{free}}^j$ ($j = 1, 2, 3; \alpha = L, N$) of microbeams with four couples of boundary conditions, where $\varphi_{LNL}^j$ and $\varphi_{NNN}^j$ are obtained in the first step and the last step of the iterative eigensystem (60), respectively. The maximum relative errors almost decrease with an increasing number of nodes $N$, and excellent convergence results are obtained when $N \geq 15$. In what follows, $N = 15$ is utilized. Running the
Figure 10 Comparisons between the third nonlinear dimensionless frequencies $\omega_{LNL}$ and linear dimensionless frequencies $\omega_{LLL}$ of microbeams with H–H (a), C–C (b), C–H (c) and C–F (d) ends under different dimensionless amplitudes $W_{\text{max}}^1 (H/l = 1, L/H = 100)$.

Figure 11 Comparisons between the fundamental nonlinear dimensionless frequencies $\omega_{\text{NNN}}^1$ and linear dimensionless frequencies $\omega_{LLL}^1$ of microbeams with H–H (a), C–C (b), C–H (c) and C–F (d) ends under different dimensionless amplitudes $W_{\text{max}}^1 (H/l = 1, L/H = 100)$.

computer programming on MATLAB platform to get a nonlinear frequency, for an example $\varphi_{\text{NNN}}^1 = 0.2312 \text{ MHz}$ with H–H ends when $N = 15$, we find that number of iterations is two times, CPU time is about 0.850 s and the used memory size is about 887 MB.

To validate the present study, two direct comparisons with theoretical predictions of the linear free vibration responses in Table 2 and numerical results of the nonlinear free vibration responses in Table 3 are made, respectively. A size effect exists in the MCST when $l \neq 0$. Otherwise, no size effect exists in the classical beam theory. Table 2 lists the first three linear natural frequencies $\varphi_{LLL}^j (j = 1, 2, 3)$ of microbeams with four groups of boundary conditions on the basis of geometric linearity, and Table 3 displays the fundamental nonlinear frequency ratio $\varphi_{NLL}^1/\varphi_{LLL}^1$ of microbeams with three groups of boundary conditions on the basis of geometric
nonlinearity. The vibration amplitude ratio is $w_{\text{max}}/k$ for classical beams and $w_{\text{max}}/H$ for nonclassical beams, where $k = 2\sqrt{3}H$ designates the gyration radius of the beam and relative errors represent (Ref. — Present) $\times 100$%/Ref.. These numerical results are in good agreement with the previous results.

Ke et al. [14, 15] found that employment of von Kármán geometric nonlinearity for nonlinear free vibration exhibits a typical “hard spring” behavior, i.e. taking into account von Kármán geometric nonlinearity increases vibrational frequencies as the vibration amplitude increases. Therefore, to eliminate the disturbance of von Kármán geometric nonlinearity in Eq. (40b), the linear strain-
Euler beam theory. Then the nonlinear partial differential governing equations are discretized into a set of nonlinear ordinary differential equations and boundary conditions in the framework of the updated MCST, von Kármán geometric nonlinearity and Bernoulli–Euler beam theory. The nonlinear partial differential governing equations are discretized into a set of nonlinear ordinary differential equations using the DQM. Then, an iterative algorithm is employed to solve these nonlinear ordinary differential equations for obtaining the nonlinear vibration frequencies of the microbeams with four types of boundary conditions. The influences of nonlinear elasticity properties on the nonlinear free vibration characteristics of the microbeams are discussed. The following conclusions are obtained: (1) the use of nonlinear couple constitutive relationships for nonlinear elasticity materials has a significant softening effect on the stiffness, i.e. employing nonlinear couple stress–strain constitutive relationships leads to lower vibrational frequencies than the previous linear couple stress–strain constitutive relationships; and (2) the nonlinear elasticity materials exhibit a typical “soft spring” behavior, i.e. their stiffness softens as the vibration amplitude increases.

6. CONCLUSIONS

In this study, the nonlinear couple stress–strain constitutive equations are obtained on the basis of previous nonlinear stress–strain constitutive relationships of nonlinear elasticity materials. Hamilton’s principle is utilized to derive higher-order nonlinear governing equations and boundary conditions within the framework of the updated MCST, von Kármán geometric nonlinearity and Bernoulli–Euler beam theory. The nonlinear partial differential governing equations are discretized into a set of nonlinear ordinary differential equations using the DQM. Then, an iterative algorithm is employed to solve these nonlinear ordinary differential equations for obtaining the nonlinear vibration frequencies of the microbeams with four types of boundary conditions. The influences of nonlinear elasticity properties on the nonlinear free vibration characteristics of the microbeams are discussed. The following conclusions are obtained: (1) the use of nonlinear couple constitutive relationships for nonlinear elasticity materials has a significant softening effect on the stiffness, i.e. employing nonlinear couple stress–strain constitutive relationships leads to lower vibrational frequencies than the previous linear couple stress–strain constitutive relationships; and (2) the nonlinear elasticity materials exhibit a typical “soft spring” behavior, i.e. their stiffness softens as the vibration amplitude increases.

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APPENDIX A

Four pairs of boundary conditions (i.e. H–H, C–C, C–H and C–F), which could be obtained with the combination of the three kinds of boundary ends in Fig. 4, are loaded on the two ends of the microbeams, respectively. Expressions of non-zero components of block matrices in Eq. (S6) are written for four pairs of boundary conditions as follows:
(a) H–H:

\[
\begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(1,1)} = 1, \quad \begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(2, n)} = \frac{1}{2} \left( M^* W_{xx}^{-1} C_{(1)}^{(1)} + M^* W_{xx}^{-1} C_{(n)}^{(2)} \right) - \frac{1}{2} Y^* W_{xx}^{-1} C_{(1)}^{(2)} - \frac{1}{2} Y^* W_{xx}^{-1} C_{(n)}^{(2)} - \frac{1}{2} \partial M^* W_{xx}^{-1} C_{(1)}^{(2)} - \frac{1}{2} \partial M^* W_{xx}^{-1} C_{(n)}^{(2)}
\]

\[
\begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(n, 1)} = \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(1)}^{(2)} - \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(n)}^{(2)} - \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(1)}^{(2)} - \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(n)}^{(2)}
\]

\[
\begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(n, 2)} = \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(1)}^{(2)} - \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(n)}^{(2)} - \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(1)}^{(2)} - \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(n)}^{(2)}
\]

\[
\begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(1, n)} = \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(1)}^{(2)} - \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(n)}^{(2)} - \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(1)}^{(2)} - \frac{1}{2} \partial Y^* W_{xx}^{-1} C_{(n)}^{(2)}
\]

\[
\begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(2, 2)} = 1
\]

\[
\begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(2, 1)} = 1
\]

(b) C–C:

\[
\begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(1, 1)} = 1, \quad \begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(2, n)} = C_{(1)}^{(1)} + \begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(1, 1)} = 1
\]

\[
\begin{bmatrix}
B_{3W}^{UL}
\end{bmatrix}_{(2, 2)} = 1, \quad \begin{bmatrix}
B_{3W}^{UL}
\end{bmatrix}_{(1, 1)} = C_{(1)}^{(1)} + \begin{bmatrix}
B_{3W}^{UL}
\end{bmatrix}_{(1, 1)} = 1
\]

(c) C–H:

\[
\begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(1, 1)} = 1, \quad \begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(2, n)} = C_{(1)}^{(1)} + \begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(1, 1)} = 1
\]

\[
\begin{bmatrix}
B_{3W}^{UL}
\end{bmatrix}_{(2, 2)} = 1, \quad \begin{bmatrix}
B_{3W}^{UL}
\end{bmatrix}_{(1, 1)} = C_{(1)}^{(1)} + \begin{bmatrix}
B_{3W}^{UL}
\end{bmatrix}_{(1, 1)} = 1
\]

(d) C–F:

\[
\begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(1, 1)} = 1, \quad \begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(2, n)} = \begin{bmatrix}
B_{1W}^{UL}
\end{bmatrix}_{(1, 1)} = 1
\]

\[
\begin{bmatrix}
B_{3W}^{UL}
\end{bmatrix}_{(2, 2)} = 1, \quad \begin{bmatrix}
B_{3W}^{UL}
\end{bmatrix}_{(1, 1)} = \begin{bmatrix}
B_{3W}^{UL}
\end{bmatrix}_{(1, 1)} = 1
\]

where \([B_{ij}^{ul}]_{(1,1)}\) is a component of submatrices \([B_{ij}^{ul}]_{(1,1)}\), i.e. the two subscripts in parentheses represent values of row and column, respectively; \(n = 1, 2\) when \(i = 1; n = 3, ..., N - 2\) when \(i = 2; n = N - 1, N\) when \(i = 3; n = 1\) when \(j = 4; n = 2, ..., N - 1\) when \(j = 5\) and \(n = N\) when \(j = 6\).
Expressions of block matrices in Eq. (57) are written for governing equations as follows:

\[
\begin{align*}
\left[ G^W_i \right]_{(m-2,n)} &= -\frac{1}{2L^2} \frac{\partial^2 M^*}{\partial X^2} \left[ \begin{array}{c} C^{(1)}_{mn} \frac{C^{(2)}_{mn}}{W_{XX}} + C^{(3)}_{mn} \frac{C^{(4)}_{mn}}{W_{XXX}} \end{array} \right] + \frac{H}{2L^2} \left[ \begin{array}{c} \frac{\partial N^*}{\partial X} \frac{C^{(1)}_{mn}}{W_{XX}} + \frac{\partial N^*}{\partial X} \frac{C^{(2)}_{mn}}{W_{XXX}} \end{array} \right] - \frac{1}{2L^2} \frac{\partial^2 Y^*}{\partial X^2} \left[ \begin{array}{c} C^{(2)}_{mn} \frac{C^{(3)}_{mn}}{W_{XX}} + C^{(4)}_{mn} \frac{C^{(1)}_{mn}}{W_{XXX}} \end{array} \right], \\
\left[ C^W_j \right]_{(q-1,n)} &= \frac{1}{2L} \frac{\partial W}{\partial X} C^{(2)}_{qn} + \frac{\partial^2 W}{\partial X^2} C^{(1)}_{qn} - \frac{1}{2L} \frac{\partial W}{\partial X} C^{(3)}_{qn} - \frac{\partial^2 W}{\partial X^2} C^{(4)}_{qn} + \frac{\partial^2 W}{\partial X^2} C^{(1)}_{qn}, \\
\left[ M^W \right]_{(m-2,n-2)} &= \frac{BH^2}{2L^3} E_i \left[ \begin{array}{c} U_X \end{array} \right] C^{(2)}_{qn} + \frac{1}{2L} \frac{\partial N^*}{\partial X} C^{(2)}_{qn} + \frac{\partial^2 N^*}{\partial X^2} C^{(1)}_{qn}, \\
\left[ M^U \right]_{(q-1,n-1)} &= \frac{BH^2}{2L^3} E_i \left[ \begin{array}{c} U_X \end{array} \right] C^{(2)}_{qn} + \frac{1}{2L} \frac{\partial N^*}{\partial X} C^{(2)}_{qn} + \frac{\partial^2 N^*}{\partial X^2} C^{(1)}_{qn},
\end{align*}
\]

where \( m = 3, ..., N - 2; q = 2, ..., N - 1; n = 1, 2 \) when \( i = 1; n = 3, ..., N - 2 \) when \( i = 2; n = N - 1, N \) when \( i = 3; n = 1 \) when \( j = 4; n = 2, ..., N - 1 \) when \( j = 5 \); and \( n = N \) when \( j = 6 \).

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