Cosmic String Nucleation near the Inflationary Phase Boundary

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Abstract

We investigate the nucleation of circular cosmic strings in models of generalized inflationary universes with an accelerating scale factor. We consider toy cosmological models of a smooth inflationary exit and transition into a flat Minkowski spacetime. Our results establish that an inflationary expanding phase is necessary but not sufficient for quantum nucleation of circular cosmic strings to occur.
1 Introduction and Conclusions

Inflation \cite{1, 2, 3} is a short period of rapid expansion in the early history of the universe whereby its presently observable part originated from a tiny initial region. Topological defects such as cosmic strings, monopoles, and domain walls are extended objects present in the spectrum of grand unified theories that are believed to be typically generated in phase transitions in the early universe. They could have acted as seeds for the generation of density perturbations that resulted in the large scale structure of the universe and the observed anisotropy in the cosmic microwave background radiation. Recently it was realized that such objects can be created spontaneously in a de Sitter spacetime through the process of quantum nucleation \cite{4}. More specifically, using the static parametrization it was found that the classical evolution of a circular string is determined by a simple potential barrier and that strings can nucleate by a quantum mechanical tunnelling through the barrier. In a realistic situation any topological defects formed at the onset of the inflationary period are expected to be inflated away. Strings nucleated towards the inflationary exit will eventually contract upon their entrance into the radiation dominated phase inside the causal horizon \cite{5}. If they are still circular they are expected to form black holes \cite{6}, otherwise they oscillate radiating away their gravitational energy. In fact this was the conclusion in a cosmological model whereby a de Sitter phase is followed "abruptly" by a flat Minkowskian spacetime at which point the Hubble rate of expansion changes discontinuously \cite{5}.

It is the purpose of our present work to explore how a smooth and continuous exit from the inflationary era and the passage of the expanding universe into a radiation dominated one relates to the switch off of the quantum nucleation of circular cosmic strings. This is of some phenomenological importance as the actual size distribution of string loops upon their entrance within our causal horizon must depend on the precise way in which their nucleation ended as the exponential expansion of the universe relaxed into its radiation dominated rate. This problem also hinges on the still unresolved issue of the actual realization of inflation in the cosmic history. A number of models have indeed been proposed which correspond to cosmologies with accelerating scale factors \((d^2a/dt^2 > 0)\) and variable cosmic time dependance \(a(t)\). In a previous work \cite{7} we took a first step towards the important issue of how generic is cosmic string nucleation in inflationary cosmologies. We specifically studied circular string evolution in a time dependent spacetime \((a(t) \simeq t^p; \ p > 0)\) with no static parametrization. In cosmological spacetimes which admit circular string configurations with the property:

\[
\exists \ t_0: \ f(t) \geq f(t_0) > 0, \quad \frac{df}{dt}(t_0) = 0, \quad (1.1)
\]

where \(f\) is the physical radius of the loop and \(t\) is the cosmic time, it is reasonable to assume that string nucleation plays an important role. The presence of \(t_0\) in the time dependent spacetime has of course only a local
significance. It implies that a contracting circular string configuration hits a potential barrier and is energetically prohibited to collapse. The existence of such string loops is a necessary condition for string tunnelling and nucleation in the spacetime under consideration. In fact we found that for power law expanding spacetimes with $p < 2\sqrt{2} + 3$ no such configurations exist. This we interpreted as the absence of string tunnelling. It followed both from an explicit numerical search for such solutions to the string equations of motion and from the analytical observation that eq.(1.1) in fact implies $\dot{f}(t_0) \geq 0$, where a dot denotes differentiation with respect to $t$. In conjunction with the derived string equations of motion it implies that the Hubble rate of expansion $H(t) = \dot{a}(t)/a(t)$ at $t = t_0$ satisfies a master inequality:

$$\dot{H}(t_0) \geq (2\sqrt{2} - 3)H(t_0)^2$$

(1.2)

It provides us with a necessary condition in order that a circular string nucleates at $t = t_0$ with a finite radius $f(t_0)$ by tunnelling through a barrier. Loosely speaking we may say that for time dependent spacetimes a potential barrier that prevents circular strings from collapsing at $t = t_0$ will change in time and in some cases it may only exist for a finite amount of time during the evolution of the universe. This implies that a string nucleated at one moment may collapse at a later one by classically shrinking to a point in the absence of any barrier. This is a very realistic and physical picture as our universe went through a de Sitter like expansion for a short period in its cosmic history before it entered the radiation dominated era where the string evolution equations indicate an unconditional collapse for circular strings. The precise way in which the Hubble expansion relaxed to its radiation dominated era is not known. Under the hypothesis that the scale factor traversed smoothly through all possible time behaviours before it relaxed to its present one, we investigate cosmic string nucleation in models with generalized inflation ($\ddot{a}(t) > 0$). More specifically we study the cases of intermediate and superinflation [8]. Both our analytical and numerical tools suggest that circular string nucleation is almost generic. For a given model of inflationary cosmology we precisely determine the time at which nucleation either sets in or switches off. Interestingly we find that in contrast to the purely de Sitter case, where $H(t)$ is constant, string nucleation does not occur throughout the inflationary phase. Furthermore we proceed and present a toy cosmological model whereby the universe after exponentially expanding for a period of time smoothly enters a flat Minkowskian phase. We find that circular string nucleation stops before the universe exits from the inflationary era.

In conclusion, in cosmological models with generalized inflation cosmic string nucleation appears to be almost generic in the sense that inflation is a necessary but not sufficient condition. As a consequence, for a cosmologically smooth transition from an inflationary era with $\ddot{a}(t) > 0$ and into a radiation dominated one ($\ddot{a}(t) < 0$), string nucleation ceases in a well-defined and determined manner at some $t = t_0$, which nevertheless is model dependent. Circular loops eventually collapse and form black holes upon entering
the causal horizon whereas noncircular ones are likely to become the seeds of density perturbations. Their precise size distribution is of interest for models of large and medium scale structure formation in the late universe.

The paper is organized as follows: In Section 2 we derive the circular string evolution in an arbitrary spatially flat FRW spacetime. In Section 3 we investigate string nucleation for models which exhibit generalized inflation ($\ddot{a}(t) > 0$) and present a toy model for a smooth transition out of a de Sitter phase and into a flat Minkowskian one.

## 2 String Evolution Equations

We want to derive the equations of motion for circular strings in a spatially flat Friedman-Robertson-Walker (FRW) spacetime. These spacetimes are usually parametrized in terms of comoving coordinates:

$$ds^2 = -dt^2 + a(t)^2 \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \quad (2.1)$$

where $a(t)$ is the scale factor. For an observer who employs the comoving coordinate system the circular string will appear to have a physical radius $f$ at the cosmic time $t$:

$$f(t) = r a(t). \quad (2.2)$$

It is convenient for our purposes to use a parametrization for the spacetime directly in terms of these variables. From eqs.(2.1) and (2.2) it is easy to obtain:

$$ds^2 = -(1 - H^2 f^2) dt^2 - 2H f df dt + df^2 + f^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.3)$$

Here we have introduced the Hubble expansion rate:

$$H = \frac{1}{a} \frac{da}{dt}. \quad (2.4)$$

It is generically a function of the cosmic time $t$. Indeed for all generalized inflationary models with $\dot{a}(t) > 0$ this is precisely the case with the exception of de Sitter space ($H = \text{const.} \equiv H_0 \neq 0$). In the latter case we notice that the spacetime metric in eq.(2.3) admits a stationary but non-static parametrization. It is, moreover, related to the well-known static parametrization by the transformation:

$$t_{\text{static}} = t - \log \left| 1 - H_0^2 f^2 \right| \quad (2.5)$$

In what follows, however, we will be mostly interested in spacetimes with time dependent Hubble expansion rate.

A family of circular strings with a time dependent radius is obtained by the ansatz:

$$t = \tau, \quad f = f(t), \quad \theta = \frac{\pi}{2}, \quad \phi = \sigma. \quad (2.6)$$
Here we have identified the cosmic time with the world-sheet time, and the azimuthal angle with the world-sheet spatial and periodic coordinate. The radius $f(t)$ is to be determined by the equations of motion that depend on the specific string model. We hereby consider the Nambu-Goto action:

$$S = \int d\tau d\sigma \sqrt{-\det G_{\alpha\beta}}, \quad (2.7)$$

where $G_{\alpha\beta}$ is the induced metric on the world-sheet:

$$G_{\alpha\beta} = g_{\mu\nu} X^\mu_{\alpha} X^\nu_{\beta} \quad (2.8)$$

with $X^\mu = (t, r, \theta, \phi)$ and $g_{\mu\nu}$ being given by eq.(2.3). By taking also into account the ansatz of eq.(2.6) we obtain:

$$G_{\tau\tau} = \dot{f}^2 - 2Hf\dot{f} + H^2f^2 - 1, \quad G_{\sigma\sigma} = f^2, \quad G_{\tau\sigma} = 0,$$

and we require for a time-like string:

$$(\dot{f} - Hf)^2 < 1. \quad (2.10)$$

It is now straightforward to derive the equation of motion:

$$\ddot{f}f - 2Hf\dot{f}^3 + (6H^2f^2 - 1)\dot{f}^2 + 3Hf(1 - 2H^2f^2)\dot{f} - f^2\ddot{H} + 2H^4f^4 - 3H^2f^2 + 1 = 0. \quad (2.11)$$

It determines the physical string size as a function of the cosmic time in a spatially flat FRW spacetime with a Hubble rate of expansion $H$. The complete analytical solution to this equation is unfortunately not known. It is still though possible to extract some information about the evolution of the circular strings. This was already discussed in our previous work [7]. We nevertheless highlight once more the main points. A contracting string ($\dot{f} < 0$) will continue to contract unless there exists a critical time $t_0$ such that:

$$f(t) \geq f(t_0) > 0, \quad \dot{f}(t_0) = 0. \quad (2.12)$$

This should hold true in a time dependent spacetime locally around some $t = t_0$. The existence of a $t_0$ that satisfies eq.(2.12) means that the contracting string hits a barrier and is therefore energetically forbidden to collapse. Such a bounce is a necessary condition for the existence of a potential barrier and consequently of string tunnelling and nucleation in the spacetime under consideration. Such an interpretation is consistent with the results obtained already [4] for the case of quantum creation of circular strings in de Sitter spacetime. A more detailed discussion for the case of an arbitrary scale factor can be found in [4].
By a Taylor expansion it follows that $\ddot{f}(t_0) \geq 0$ and then eqs.(2.10)-(2.11) lead to [4]:

$$\dot{H}(t_0) \geq (2\sqrt{2} - 3) \ H(t_0)^2.$$  \hspace{1cm} (2.13)

This is our master inequality which provides a necessary condition for a cosmic string to nucleate with a finite radius $f(t_0)$ at a time $t_0$ after tunnelling through the barrier. In the limiting case of equality eq.(2.13) is readily solved by:

$$a(t) \propto t^{2\sqrt{2}+3}.$$ \hspace{1cm} (2.14)

For a power law inflationary universe ($a(t) \propto t^p$) with $p \geq 2\sqrt{2} + 3$ inequality (2.13) is fulfilled and string nucleation is expected for any $t_0 \in [0, \infty[$, whereas for $p < 2\sqrt{2} + 3$ no nucleation is expected to occur. Notice that power law expanding universes have a special property, namely that $\dot{H}$ is proportional to $H^2$. This implies that any dependence on the cosmic time $t_0$ drops out from (2.13) with the power of expansion being the only parameter left.

### 3 String Nucleation and Generalized Inflation

In this section we proceed to employ both our analytical master inequality (2.13) and string evolution equation (2.11) to investigate circular string nucleation in a wide class of inflationary models. Such a diversity of spacetimes was recently shown to arise from the equation of state [5]:

$$p + \rho = \gamma \rho^h.$$ \hspace{1cm} (3.1)

Here $p$ and $\rho$ are the matter pressure and density whereas $\gamma$ and $h$ are arbitrary constants. The resulting spatially flat FRW universes are described by scale factors of the form:

$$a(t) \approx e^{\pm tp}, \ e^{\pm \epsilon t}, \ t^p.$$ \hspace{1cm} (3.2)

The question of circular string tunnelling and nucleation in the power law expanding universes ($a(t) \propto t^p$) has already been dealt with [7]. It was found that a necessary condition for nucleation is that $p \geq 2\sqrt{2} + 3$. This is true for all $t_0 \in [0, \infty[$. In the remaining of this section we show that the latter statement does not hold for the other types of cosmologies given by eq.(3.2). In order to be more specific we will first consider the scale factors of the form:

$$a(t) = e^{\alpha_p t^p}; \ t \geq 0,$$ \hspace{1cm} (3.3)

where $\alpha_p$ is a dimensionful constant. This family of cosmologies includes de Sitter space ($p = 1$) as a special case. In what follows we will be only interested in expanding universes ($\alpha_p > 0, \ p > 0$) and ($\alpha_p < 0, \ p < 0$), respectively.

I. $\alpha_p > 0, \ p > 0$. The necessary condition for string nucleation (2.13) implies:

$$p - 1 \geq (2\sqrt{2} - 3) \ p \ a_p \ t_0^p.$$ \hspace{1cm} (3.4)
We may observe that for \( p \geq 1 \) the above inequality is always fulfilled independently of the nucleation time \( t_0 \). This is to be interpreted that we should expect strings to nucleate at any time \( t_0 \in [0, \infty] \) of the inflationary phase. In the special case of \( p = 1 \) this is consistent with the conclusions of Basu, Guth and Vilenkin [4]. It is not surprising that the de Sitter \((p = 1)\) result generalizes to the superinflationary cosmologies \((p > 1)\). Indeed the master inequality (2.13) expresses the fact that a certain amount of inflation is necessary for nucleation of circular strings. In this case the superinflationary universe undergoes a faster expansion \( \dot{H} > 0 \) than the de Sitter space \((\dot{H} = 0)\).

In the case of \( p \in ]0, 1[\), which corresponds to models of the so-called intermediate inflation type [8], inequality (2.13) leads to:

\[
a_p^{1/p} t_0 \geq (3 + 2\sqrt{2})^{1/p} \left( \frac{1}{p} - 1 \right)^{1/p}.
\]

This is a completely different result from the one obtained for \( p \geq 1 \) and from that of the power law inflationary universes. The inequality (2.13) gives us now a critical time \( t_0^{\text{crit}} \) with the interpretation that nucleation can only take place during the \( t \geq t_0^{\text{crit}} \) period in the evolution of the universe. In the intermediate inflationary models there is actually only inflation \((\ddot{a}(t) > 0)\) for \( t > t^{\text{inf}} \) where:

\[
a_p^{1/p} t^{\text{inf}} = (1 + \frac{1}{|p|} - 1)^{1/p}.
\]

It is interesting to compare the two time scales \( t_0^{\text{crit}} \) and \( t^{\text{inf}} \) for different cosmological models \((p \text{ values})\). This is done in Fig.1. We observe that for \( p \to 1 \) both nucleation and inflation start at \( t \approx 0 \). For \( p \to 0 \) inflation starts long before nucleation. This is certainly of little observational value from the point of view of our presently radiation dominated universe. It demonstrates nevertheless the fact that inflation is necessary but not sufficient for the nucleation phenomenon of circular cosmic strings to occur.

II. \( a_p < 0, \ p < 0 \). The cosmologies represented by this class of models are different from case I. They start at \( t = 0 \) with \( a = 0 \), run through a transient inflationary era and finally enter into a stable static Minkowski space \( a(\infty) = 1 \). The inequality (2.13) now takes the form:

\[
|a_p|^{-1/|p|} t_0 \leq (3 - 2\sqrt{2})^{1/|p|} (1 + \frac{1}{|p|})^{-1/|p|}.
\]

The presence of a critical time \( t_0^{\text{crit}} \) now represents the point at which nucleation ceases to occur. In this sense it is the endpoint of the nucleating phase \((t \leq t_0^{\text{crit}})\). This timescale is to be compared with that of the exit from the inflationary phase \((t < t^{\text{inf}})\) where \( t^{\text{inf}} \) is given by:

\[
|a_p|^{-1/|p|} t^{\text{inf}} = (1 + \frac{1}{|p|})^{-1/|p|}.
\]
In Fig. 2 we give the two timescales for different cosmological models. For $p \to -\infty$ we may observe that circular string nucleation is allowed for the entire inflationary era while for $p \approx -1.6$ the available time for inflationary stretching of the nucleated strings is maximal. This is the physical picture for cosmological models where a de Sitter phase undergoes a smooth transition into the radiation dominated one.

Let us now turn to the second family of scale factors (eq.(3.2)), obtained by Barrow [8], of the form:

$$a(t) = e^{Ae^{Bt}}; \quad t \in ] - \infty, \infty[,$$

where $A$ and $B$ are constants. We are again only interested in expanding universes. We distinguish two cases of interest ($A > 0$, $B > 0$) and ($A < 0$, $B < 0$). The first case is somewhat similar to the superinflationary models already discussed after eq.(3.4). As the expansion rate is now even faster inequality (2.13) is trivially fulfilled and we expect nucleation to occur for any $t_0 \in ] - \infty, \infty[.$

For the second case inequality (2.13) gives:

$$t_0 \leq \frac{1}{B} \log \left[ -\frac{(3 + 2\sqrt{2})}{A} \right],$$

while inflation takes place for $t < t^{inf}$, where:

$$t^{inf} = \frac{1}{B} \log \left( \frac{1}{A} \right)$$

It follows that:

$$t^{inf} - t_0 \geq -\frac{1}{B} \log (3 + 2\sqrt{2})$$

As $B < 0$ we have the string nucleation switched off to predate the inflationary exit. We may consider as a specific example the case $A = B = -1$. The scale factor is shown in Fig. 3. The universe inflates for $t \in ] - \infty, 0[,$ but there can only be nucleation for $t \in ] - \infty, -\log(3 + 2\sqrt{2})].$ Therefore, any nucleated circular string will be stretched by inflation for at least $\Delta t = \log(3 + 2\sqrt{2})$ amount of time in this model. After the inflationary era the universe smoothly undergoes a transition into flat Minkowski space, where the circular strings eventually collapse.

We would like finally to mention another toy model that also illustrates a smooth exit out of a de Sitter inflationary phase and into a flat Minkowski spacetime. Its scale factor is given by:

$$a(t) = \frac{e^{Hot/2}}{e^{Hot/2} + e^{-Hot/2}}$$

It is evident that we can identify a de Sitter phase for $t \to -\infty$ ($a(t) = e^{Hot}$) and a flat Minkowski one for $t \to \infty$ ($a(t) = 1$) with a smooth interpolation.
between them. The relevant quantities for string nucleation are given by:

\begin{align}
H(t) &= \frac{H_0}{2} \left[ 1 - \tanh(H_0 t/2) \right], \\
\dot{H}(t) &= -\frac{H_0^2}{4 \cosh^2(H_0 t/2)}.
\end{align}

(3.14) (3.15)

Inequality (2.13) implies that at \( H_0 t_0^{\text{crit}} = \log(3 - 2 \sqrt{2}) \approx -1.76 \), string nucleation ceases to occur. By evaluating \( \ddot{a}(t) \) we may conclude that the inflationary exit occurs at \( H_0 t_{\text{inf}} = 0 \). Indeed explicit numerical solutions of the string evolution in eq.(2.11), depicted in Fig.4, confirm our analytical expectations. Contracting circular strings can hit a barrier for \( H_0 t \leq H_0 t_0^{\text{crit}} \approx -1.76 \) only, after which they expand. At their entrance into the Minkowski phase \( (t >> 0) \) the strings collapse.
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Figure Captions

Fig.1. Cosmic string nucleation \((t \geq t_0^{\text{crit}})\) near the inflationary boundary \(t > t_{\text{inf}}\) in cosmologies with scale factors \(a(t) = \exp(a_pt^p)\); \((a_p > 0, 0 < p \leq 1)\). Inflation is necessary but not sufficient \((t_0^{\text{crit}} > t_{\text{inf}})\) for \(0 < p < 1\).

Fig.2. Cosmic string nucleation \((t \leq t_0^{\text{crit}})\) near the inflationary boundary \(t < t_{\text{inf}}\) in cosmologies with scale factors \(a(t) = \exp(a_pt^p)\); \((a_p < 0, p < 0)\). Inflation is necessary but not sufficient \((t_0^{\text{crit}} < t_{\text{inf}})\).

Fig.3. A two phase cosmological model of an expanding universe with scale factor \(a(t) = \exp[-\exp(-t)]; \ t \in ]-\infty, \infty[\). The inflationary era \(t \in ]-\infty, 0[\) is followed by a flat Minkowski phase \(t >> 0\). Circular cosmic strings may nucleate for \(t \leq t_0^{\text{crit}} = -\log(3 + 2\sqrt{2})\).

Fig.4. Circular string evolution near the inflationary boundary \(t < t_{\text{inf}} = 0\) in a de Sitter-Minkowski expanding toy model defined by eq.(3.13).
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