Reeb complexes and topological persistence

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Abstract

We introduce Reeb complexes in order to capture how generators of homology flow along sections of a real-valued continuous function. This intuition suggests a close relation of Reeb complexes to established methods in topological data analysis such as levelset zigzags and persistent homology. We make this relation precise and in particular explain how Reeb complexes and levelset zigzags can be extracted from the first pages of respective spectral sequences with the same termination.

1 Introduction

In this paper we study two different structures associated to a real-valued continuous function \( f: X \rightarrow \mathbb{R} \).

1. Covers \( U = \{ U_a = f^{-1}(I_a) \subset X \} \) of a topological space \( X \) pulled back from covers \( \{ I_a \subset \mathbb{R} \} \) of the real line.

2. Sections of \( f \), that is, for real numbers \( a \leq b \) continuous maps \( s: [a, b] \rightarrow X \), such that \( f \circ s = \text{id} \).

In both cases the available information is neatly organized in a simplicial space. For the pulled back cover \( U \), this is the well known Čech complex:

\[
\bigsqcup U_{\alpha_0} \leftrightarrow \bigsqcup U_{\alpha_0 \alpha_1} \leftrightarrow \bigsqcup U_{\alpha_0 \alpha_1 \alpha_2} \cdots
\]

It combines the topology of intersections of cover elements \( U_{\alpha_0 \ldots \alpha_n} = U_{\alpha_0} \cap \ldots \cap U_{\alpha_n} \) with the combinatorics of the various possible inclusions. Combining these two pieces of information, it is possible to recover the homology of the base space \( X \). Indeed, it can be shown that the realization of the Čech complex is homotopy equivalent to \( X \). To compute the homology of \( X \) from the Čech complex it can be helpful to utilize a spectral sequence.

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It turns out that various objects of interest to topological data analysis can be extracted from the first page of this spectral sequence, i.e. as intermediate steps of the corresponding homology computation. In Section 2, we exemplify this with the persistence modules of a filtration [Car09] and the levelset zigzag of a real valued height function like defined in [CdSM09].

The simplicial space associated to the sections between a subset of heights $A \subset \mathbb{R}$ is the section complex.

$$S^A_f: (S^A_f)_0 \leftrightarrow (S^A_f)_1 \leftrightarrow (S^A_f)_2 \ldots$$

In analogy to the Čech complex, the section complex encodes the topology and combinatorics of sections. We discuss it thoroughly at the beginning of Section 3, but in short: the topology is supposed to capture how sections relate via homotopies, while the combinatorics encode the various ways to concatenate sections.

Applying homology levelwise to the section complex yields

$$G^A_q: \quad H_q(S^A_f)_0 \leftrightarrow H_q(S^A_f)_1 \leftrightarrow H_q(S^A_f)_2 \ldots$$

- a simplicial abelian group that we call the Reeb complex. It is an object that captures how generators of homology flow between fibers of $f$ along sections. We may arrange the Reeb complexes as the first page of a spectral sequence, called the section spectral sequence. In [Try21] it is shown how, under mild regularity assumptions on $f$, this sequence collapses on the second page and computes the homology of the base space $X$.

The Reeb complexes can then be understood as an intermediate step in a calculation of the homology of $X$. Thus, it comes at no surprise that they are intimately related to the well known persistence modules, that we can extract from the spectral sequence of the Čech complex. We make these relations precise in Proposition 3.4 and Proposition 3.7.

It is worthwhile to emphasize that if $f$ is a piecewise linear function, the Reeb complexes are amenable for practical computations. Indeed, combining Proposition 3.2 of this paper with Corollary 4.4 of [VHT22] implies that for piecewise linear functions we can compute the Reeb complexes in terms of the simplicial theory developed in the latter paper.

To summarise, this paper relates the theory of sections developed in [Try21] and [VHT22] to topological data analysis. To this end we define Reeb complexes and prove a close correspondence to known objects in topological data analysis like the persistence module of a filtration and levelset zigzag modules. The existence of this correspondence is motivated by the following discussion of the Čech complex.
2 The Čech complex and topological data analysis

We explain how to obtain the persistence module of a filtration as well as the levelset zigzag module of a Morse-type function by applying homology levelwise to the Čech complex of an appropriate cover. Then we exhibit these modules as constituting the first page of a spectral sequence, that computes the homology of the covered space. This perspective on topological data analysis is very related to the sheaf-theoretic ideas explored for example in [DSMP16], [BS14], [CGN16] and [Cur14]. Consequently this section makes no claim to be original. The intention is rather to present well-known methods of topological data analysis in a way that makes their relation to the Reeb complexes of Section 3 plausible.

Given a covering $U = (U_\alpha)_{\alpha \in \Sigma}$ of a topological space $X$, consider the following diagram of continuous maps:

$$
\coprod U_{\alpha_0} \leftrightarrow \coprod U_{\alpha_0 \alpha_1} \leftrightarrow \coprod U_{\alpha_0 \alpha_1 \alpha_2} \cdots
$$

Here $U_{\alpha_0 \ldots \alpha_n}$ denotes the intersection $U_{\alpha_0} \cap \ldots \cap U_{\alpha_n}$ and the arrows are the various ways of omitting indices and then applying inclusions. Interpreting these maps as face maps yields a simplicial space $\check{\mathcal{C}}(U)$, which is also commonly referred to as the Čech complex of the cover $U$.

Consider a continuous real-valued function $f : X \to \mathbb{R}$. We can pull back a cover of $\mathbb{R}$ along $f$ and thereby obtain a cover of the space $X$. The corresponding simplicial vector space of this cover is then often a well-known object in topological data analysis, like we now wish to demonstrate with two examples.

**Levelset zigzag** Let $I_k \subset \mathbb{R}$ be a finite collection of open intervals such that $U_k = f^{-1}I_k$ forms an open cover of $X$ for which $U_k \cap U_l \neq \emptyset$ only if $l = k \pm 1$. Applying homology to the Čech complex of this pullback cover gives

$$
\bigoplus_{k=1}^n H_q(U_k) \leftrightarrow \bigoplus_{k=1}^{n-1} H_q(U_k \cap U_{k+1})
$$

We can then wrap out the direct sums to obtain a zigzag module:

$$
H_q(U_1) \leftrightarrow H_q(U_1 \cap U_2) \leftrightarrow H_q(U_2) \leftrightarrow H_q(U_2 \cap U_3) \leftrightarrow \cdots \leftrightarrow H_q(U_{n-1} \cap U_n) \rightarrow H_q(U_n).
$$

Note that this was possible because of the absence of higher simplices or else the diagram would have been much more complicated. Furthermore, if $f$ is of Morse type with $n-1$ critical values $c_1, \ldots, c_{n-1}$, we may arrange a cover that is pulled back from $k+1$ intervals $I_1, \ldots, I_{k+1} \subset \mathbb{R}$, like above, but with the
further requirement $c_k$ lies in the intersection $I_k \cap I_{k+1}$. Then the above zigzag module is isomorphic to

$$H_q(f^{-1}(-\infty, c_1)) \leftarrow H_q(f^{-1}[c_1, c_2]) \leftarrow H_q(f^{-1}c_2) \rightarrow \cdots \leftarrow H_q(f^{-1}c_n) \rightarrow H_q(f^{-1}[c_n, \infty)).$$

This is the levelset zigzag module of $f$.

The persistence module associated to a filtration can in a certain way be seen as a special case of this construction.

**Persistent homology** Consider a filtration of topological spaces:

$$X : X_{i_0} \leftrightarrow X_{i_1} \leftrightarrow \cdots X_{i_{n-1}} \leftrightarrow X_{i_n}$$

We denote its mapping telescope by $C_X$. It is obtained as the colimit of the following diagram:

$$
\begin{array}{cccc}
X_{i_0} \times [i_0, i_1] & \longrightarrow & X_{i_1} \times [i_1, i_2] & \longrightarrow & \cdots & \longrightarrow & X_{i_{n-1}} \times [i_{n-1}, i_n] & \longrightarrow & X_{i_n} \\
(id, i_1) & & (1, i_1) & & & & (id, i_n) & & (1, i_n)
\end{array}
$$

and we get an induced height function $f_X : C_X \rightarrow \mathbb{R}$. Along this function, pull back a cover of $C_X$ from open intervals in $I_k \subset \mathbb{R}$ as in the construction of the levelset zigzag above, so that $i_k \in I_k \cap I_{k+1}$. We then obtain the zigzag

$$H_q(f_X^{-1}i_0) \overset{\alpha_0}{\leftarrow} H_q(f_X^{-1}[i_0, i_1]) \overset{\beta_1}{\rightarrow} \cdots \overset{\alpha_{n-1}}{\leftarrow} H_q(f_X^{-1}[i_{n-1}, i_n]) \overset{\beta_n}{\rightarrow} H_q(f_X^{-1}i_n),$$

where we observe that all $\alpha_k$ are isomorphisms. We may thus write the persistence module

$$H_q(f_X^{-1}i_0) \overset{\beta_1 \circ \alpha_0^{-1}}{\rightarrow} H_q(f_X^{-1}i_1) \overset{\beta_2 \circ \alpha_1^{-1}}{\rightarrow} \cdots \overset{\beta_n \circ \alpha_{n-1}^{-1}}{\rightarrow} H_q(f_X^{-1}i_n).$$

Compare this to the usual persistence module associated to the filtration

$$H_q X : H_q X_{i_0} \longrightarrow H_q X_{i_1} \longrightarrow \cdots \longrightarrow H_q X_{i_n},$$

and observe that they are isomorphic (0-interleaved).

**The spectral sequence of an open cover** We return to a general cover $U = (U_\alpha)_{\alpha \in \Sigma}$ of $X$. Applying any homology functor levelwise to the Čech complex of this cover gives

$$\bigoplus H_q(U_\alpha_0) \leftrightarrow \bigoplus H_q(U_\alpha_0 \alpha_1) \leftrightarrow \bigoplus H_q(U_\alpha_0 \alpha_1 \alpha_2) \cdots,$$
which we recognize as a simplicial abelian group. We may turn it into a chain complex

\[ \bigoplus H_q(U_{\alpha_0}) \xleftarrow{\partial_{1,q}^1} \bigoplus H_q(U_{\alpha_0\alpha_1}) \xleftarrow{\partial_{2,q}^1} \bigoplus H_q(U_{\alpha_0\alpha_1\alpha_2}) \cdots. \]

The differential \( \partial_{p,q}^1 \) is induced in \( H_q \) from the alternating sum of face maps. For example, for \( p = 1 \) these are \( \partial_{1,q}^1 = H_q d_0 - H_q d_1 \). Organizing all the chain complexes for different \( q \)'s gives us the first page of a spectral sequence.

Due to a result that goes back to Segal [Seg68], which was later generalized in [DI04], we know that the termination of this sequence is always \( H_q X \) - the homology of the covered space. Furthermore, for finite covers the sequence will eventually collapse and we are thus able to recover the homology of \( X \) by combining the topology and combinatorics of the cover \( U \). In practice this is achieved by solving Mayer-Vietoris-like extension problems to compute higher differentials.

Let us look at two cases in which these computations are particularly straightforward. The first is the one of a cover \( U = (U_{\alpha})_{\alpha \in \Sigma} \) of \( X \) for which all intersections are contractible, also commonly referred to as a good cover. Then, only the lowest row of the above first spectral sequence page is non-trivial. Furthermore, all the summands in its terms are either singletons or zero. We recognize it as the chain complex associated to the nerve of the cover \( U \). In this case, the topology of the cover is trivial and we can recover the homology of \( X \) just with the combinatorics of the cover, encoded in its nerve.

As another special case, we consider a cover that has at most \( n \) non-empty pairwise intersections. Then, just the first two columns are non-trivial. Again, the spectral sequence collapses on the second page and we read of the homology of \( X \) as

\[
H_q(X) \cong \begin{cases} 
\text{coker} \partial_{1,0}^1 & q = 0 \\
\ker \partial_{1,q-1}^1 \oplus \text{coker} \partial_{1,q}^1 & q \geq 1 
\end{cases}
\]

Note that in particular we can use this strategy to compute the homology of \( X \) from the levelset zigzag modules described above.
3 Reeb complexes

The section complex Let \( f: X \to \mathbb{R} \) be a continuous function on a topological space. A section of \( f \) between heights \( a \leq b \) is a continuous map \( \rho: [a, b] \to X \), such that the composition \( f \circ \rho \) is the inclusion \( [a, b] \hookrightarrow \mathbb{R} \). These sections assemble into \( \text{Sect}_f[a, b] \) - a subspace of the mapping space \( \text{map}([a, b], X) \) with the compact-open topology. Fix a subset \( A \subset \mathbb{R} \). We define a series of spaces \((S^A_f)_0, (S^A_f)_1, (S^A_f)_2, \ldots\)

- The space \((S^A_f)_0\) is given as \( \coprod_{a \in A} f^{-1}(a) \), i.e. the disjoint union of fibers of the map \( f \).
- To obtain \((S^A_f)_1\) collect all the sections going between heights in \( A \) into one space of sections by taking the disjoint union over ordered pairs in \( A \), that is \( \coprod_{a \leq b} \text{Sect}_f[a, b] \).
- For \( p \geq 2 \), the space \((S^A_f)_p\) has as points all the ways to concatenate \( p \) sections. Let for example \( \sigma \in \text{Sect}_f[a, b] \) and \( \rho \in \text{Sect}_f[b, c] \) be two sections with compatible ending and starting points. These can be concatenated to a section \( \sigma \ast \rho \in \text{Sect}_f[a, c] \). We denote all possible ways to obtain such concatenations by \( \text{Sect}_f[a, b, c] \). Then we induce the structure of a topological space from \( \text{Sect}_f[a, b] \) and \( \text{Sect}_f[b, c] \). Again, taking the disjoint union over all triples of heights \( a \leq b \leq c \), gives us the space \((S^A_f)_2\).

These spaces may be collected as in the following diagram.

\[
(S^A_f)_0 \iff (S^A_f)_1 \iff (S^A_f)_2 \ldots
\]

The arrows denote the various ways to naturally map from \((S^A_f)_{p+1}\) to \((S^A_f)_p\). For example: \( \sigma \in \text{Sect}_f[a, b] \) and \( \rho \in \text{Sect}_f[b, c] \) as above associate to a point in \( \text{Sect}_f[a, b, c] \) and thus in \((S^A_f)_2\). We can map this point to \( \sigma \in \text{Sect}_f[a, b], \rho \in \text{Sect}_f[b, c] \) or \( \sigma \ast \rho \in \text{Sect}_f[a, c] \), all of which lie in \((S^A_f)_1\). In this way we obtain three continuous maps \((S^A_f)_1 \leftarrow (S^A_f)_2\). For \( p \geq 2 \), the \( p + 1 \) maps \((S^A_f)_p \leftarrow (S^A_f)_{p+1}\) are obtained in the same manner. The two arrows \((S^A_f)_1\) to \((S^A_f)_0\) finally correspond to the two ways to evaluate a section at its end-points.

The above diagram defines a simplicial space. We denote it by \( S^A_f \) and call it the section complex. In the same way that the Čech complex \( \check{C}(U) \) encodes how the topology of the cover \( U \) combinatorially fits together, the section complex \( S^A_f \) contains the information how topological information about sections between heights in \( A \) combinatorially fits together.
**Reeb complexes** In Section 2, we applied homology functors to the Čech complex and obtained simplicial abelian groups, that we eventually turned into chain complexes. These chain complexes were then collected on the first page of a spectral sequence. In the previous paragraph we then reviewed the section complex $S^A_f$ that encoded information about sections of the function $f$. We may now apply homology levelwise to the section complex as well. Intuitively, this gives an object that captures how homological information flows between fibers of $f$ along sections.

**Definition 3.1.** The $q$'th Reeb complex associated to a continuous function $f: X \to \mathbb{R}$ and a subset $A \subset \mathbb{R}$ is defined to be the simplicial vector space denoted by $\mathcal{G}_q^A$, given as

$$H_q(S^A_f)_0 \leftrightarrow H_q(S^A_f)_1 \leftrightarrow H_q(S^A_f)_2 \ldots,$$

that is, by applying the $q$'th homology functor levelwise to $S^A_f$.

As for the Čech complexes, we may organize all the Reeb complexes as the rows of the first page of a spectral sequence by considering their corresponding chain complexes.

**Truncated Reeb complexes** We now assume that $f$ is a Reeb function with finitely many critical height levels $A = (c_1 < \ldots < c_n)$. The class of Reeb functions includes Morse functions on smooth manifolds and piecewise linear functions on CW-complexes. See Definition 2.6 of [Try21] for a precise definition. We then take the following truncation of the $q$'th Reeb complex, just considering sections between adjacent critical levels

$$\bigoplus_{i=1}^{n} H_qf^{-1}(c_i) \leftrightarrow \bigoplus_{i=1}^{n-1} H_q \text{Sect}_f[c_i, c_{i+1}].$$

We denote this object as $\mathcal{T}^f_q$. The complex $\mathcal{T}^f_q$ is much smaller than the original Reeb complex $\mathcal{G}_q^A$, which makes the following statement valuable for computations.
**Proposition 3.2.** For \( f : X \to \mathbb{R} \) a Reeb function, let \( A \) be its set of critical values. Then the chain complexes associated to \( \mathcal{G}_q^A \) and \( \mathcal{T}_q^f \) are quasi-isomorphic.

**Proof.** Because \( A \) contains all the critical values of the Reeb function \( f \), we know from Proposition 4.2 of [Try21] that the spectral sequence associated to \( \mathcal{G}_q^A \) converges on the second page and that \( H_p\mathcal{G}_q^A = 0 \) for \( p \geq 2 \). Furthermore, we recognize the differential induced by the facemaps of \( \mathcal{T}_q^f \) as the critical differential

\[
\partial^c_{i,q} : \bigoplus_{c_i} H_q\text{Sect}_f[c_i,c_{i+1}] \to \bigoplus_{c_i} H_qf^{-1}(c_i)
\]

as defined in Section 4.2 of [Try21]. Then, by Proposition 4.9 of [Try21], \( H_p\mathcal{G}_q^A \cong H_p\mathcal{T}_q^f \).

**Zigzag persistence and Reeb complexes** We can wrap out the direct sums of the complex \( \mathcal{T}_q^f \) into a zigzag module

\[
\begin{align*}
H_qf^{-1}a_1 & \leftarrow H_q\text{Sect}_f[a_1,a_2] \to \cdots \leftarrow H_q\text{Sect}_f[a_{n-1},a_n] \to H_qf^{-1}(a_n).
\end{align*}
\]

Comparing this zigzag to the levelset zigzag of \( f \), we notice two differences:

1) \( \text{Sect}_f[a_{i-1},a_i] \) and \( f^{-1}[a_{i-1},a_i] \) are different spaces in general and

2) the arrows in the levelset zigzag are reversed compared to \( \mathcal{T}_q^f \).

The following example illuminates these differences:

**Example 3.3.** Consider a cylinder with pinched boundary circles

\[
\begin{array}{ccc}
\alpha & \beta & \mathbb{R} \\
\gamma & X \\
\end{array}
\]

with a mapping to \( \mathbb{R} \) defined as

\[
S^1 \times [0,1] \xrightarrow{\text{pr}_1} [0,1] \xrightarrow{f} \mathbb{R}.
\]
We compute $\mathcal{T}_0^f$ and $\mathcal{T}_1^f$ in coordinates:

\[
\begin{align*}
&k \quad 1 \\
&-1 \\
\end{align*}
\begin{align*}
&k^2 \\
&-1 \\
\end{align*}
\begin{align*}
&k \\
&1 \\
\end{align*}
\begin{align*}
&k^2 \\
&1 \\
\end{align*}

The pre-image $h^{-1}(0, 1) = X$ deformation retracts onto the two horizontal circles $\alpha, \beta$ and the vertical circle $\gamma$ depicted above. Pick these three circles as generators in $H_1$ to calculate $H_0$ and $H_1$ of the corresponding levelset zigzags in coordinates:

\[
\begin{align*}
&k \\
&1 \\
\end{align*}
\begin{align*}
&k \\
&1 \\
\end{align*}
\begin{align*}
&k \\
&1 \\
\end{align*}
\begin{align*}
&k^3 \\
&1 \\
\end{align*}
\begin{align*}
&k^2 \\
&0 \\
\end{align*}
\begin{align*}
&k^2 \\
&0 \\
\end{align*}
\begin{align*}
&k \\
&1 \\
\end{align*}
\begin{align*}
&k \\
&1 \\
\end{align*}

Note the distinct difference both in zeroth and first homology. However, taking direct sums across the middle rows in the concatenated diamonds results in sequences

\[
\begin{align*}
&k \\
&1 \\
\end{align*}
\begin{align*}
&k \\
&1 \\
\end{align*}
\begin{align*}
&k \\
&1 \\
\end{align*}
\begin{align*}
&k^3 \\
&1 \\
\end{align*}
\begin{align*}
&k^2 \\
&0 \\
\end{align*}
\begin{align*}
&k^2 \\
&0 \\
\end{align*}
\begin{align*}
&k \\
&1 \\
\end{align*}
\begin{align*}
&k \\
&1 \\
\end{align*}

that are exact in the middle term. In this example, we can thus translate between the barcode of $\mathcal{T}_d^f$ and the levelset zigzag modules via the diamond principle in [CdSM09].

The above observation generalises as we now demonstrate.
Proposition 3.4 (Diamond Principle). Let \( f : X \to \mathbb{R} \) be a Reeb function. Then, for every pair of successive critical values \( a < b \), the sequence
\[
H_q \text{Sect}_f[a, b] \to H_q f^{-1} a \oplus H_q f^{-1} b \to H_q f^{-1} [a, b]
\]
is exact at the middle term.

Proof. Evaluation at \( \frac{a+b}{2} \) defines a homotopy equivalence \( \text{Sect}_f[a, b] \to f^{-1}(\frac{a+b}{2}) \) by Proposition 3.10 in [Try21]. The homotopy inverse is given by associating canonical sections (flow lines) to points in the intermediate fiber \( f^{-1}(\frac{a+b}{2}) \), which defines a map \( f^{-1}(\frac{a+b}{2}) \to f^{-1} a \coprod f^{-1} b \). This gives a commutative ladder
\[
\begin{array}{c}
H_q \text{Sect}_f[a, b] \\
\downarrow \\
H_q f^{-1}(\frac{a+b}{2}) \\
\downarrow \\
H_q f^{-1} I_a \cap f^{-1} I_b
\end{array} \quad \begin{array}{c}
\to H_q f^{-1} a \oplus H_q f^{-1} b \\
\to H_q f^{-1} [a, b] \\
\to H_q f^{-1} I_a \cup f^{-1} I_b.
\end{array}
\]

where all the vertical arrows are isomorphisms. Let \( I_a \) and \( I_b \) be open intervals in \( \mathbb{R} \) that contain \( a \) and \( b \), respectively. We can safely assume that \( a \) is the only critical value contained in \( f(I_a) \) and similarly that \( b \) is the only such value contained in \( f(I_b) \). Further we assume that the union \( f^{-1} I_a \cup f^{-1} I_b \) contains \( f^{-1}[a, b] \). By using the available inclusions, we get
\[
\begin{array}{c}
H_q f^{-1}(\frac{a+b}{2}) \\
\downarrow \\
H_q f^{-1} I_a \cap f^{-1} I_b \\
\downarrow \\
H_q f^{-1} I_a \cup f^{-1} I_b
\end{array} \quad \begin{array}{c}
\to H_q f^{-1} a \oplus H_q f^{-1} b \\
\to H_q f^{-1} [a, b] \\
\to H_q f^{-1} I_a \cup f^{-1} I_b.
\end{array}
\]

The vertical arrows are isomorphisms due to Lemma 2.8 in [Try21]. We recognize the final row as part of the well-known Mayer-Vietoris sequence which is exact. \( \square \)

Persistent homology and Reeb complexes In Section 2, we encountered the mapping telescope \( C_X \) associated to a filtration
\[
X : X_{i_0} \leftrightarrow X_{i_1} \leftrightarrow \ldots \leftrightarrow X_{i_{n-1}} \leftrightarrow X_{i_n}.
\]
This came with a height function \( f_X : C_X \to \mathbb{R} \) of which we want to consider the section spaces. It turns out that in this case \( T^f_q \) is isomorphic to the \( q' \)th persistence module of the filtration. This will be a consequence of the following two lemmas.
Lemma 3.5. Let $i : X_{i_0} \hookrightarrow X_{i_1}$ be an inclusion, let $C_i$ be the associated mapping cylinder and let $f_i : C_i \to \mathbb{R}$ be the induced height function that maps $X_{i_0}$ to $i_0$ and $X_{i_1}$ to $i_1$. Then the diagram

\[
\begin{array}{ccc}
\text{Sect}_{f_i}[i_0, i_1] & \xleftarrow{d_1} & f_i^{-1}i_0 \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
& f_i^{-1}i_1 & \xrightarrow{d_0}
\end{array}
\]

where $d_0$ and $d_1$ denote the respective face-maps in the section complex, commutes up to homotopy.

Proof. We construct a homotopy

\[
\eta : \text{Sect}_{f_i}[i_0, i_1] \times [i_0, i_1] \to X_{i_1}
\]

It will be convenient to define it in terms of its adjoint

\[
\tilde{\eta} : \text{Sect}_{f_i}[i_0, i_1] \to \text{map}([i_0, i_1], X_{i_1}).
\]

Postcomposing a section $(\rho : [i_0, i_1] \to C_i) \in \text{Sect}_{f_i}[i_0, i_1]$ with the map $\phi : C_i \to X_{i_1}$ induced from the universal property of the pushout

\[
\begin{array}{ccc}
X_{i_0} & \xrightarrow{\text{id}} & X_{i_1} \\
\downarrow & & \downarrow \\
X_{i_0} \times [i_0, i_1] & \xrightarrow{\phi} & C_i \\
\downarrow & & \downarrow \\
X_{i_1}
\end{array}
\]

yields a continuous map $\phi \circ \rho : [i_0, i_1] \to X_{i_1}$. We define $\tilde{\eta}(\rho) = \phi \circ \rho$. \hfill \Box

Lemma 3.6. In the setting of the previous lemma, the face map

\[d_1 : \text{Sect}_{f_i}[i_0, i_1] \to X_{i_0}\]

is a homotopy equivalence.

Proof. This follows immediately from Proposition 4.10 in [Try21]. \hfill \Box

We return to the section complex associated with the mapping telescope of the filtration $X$. The zigzag obtained from $T^f_q$ may be extended as follows:

\[
\begin{array}{cccccc}
H_q\text{Sect}_{f_X}[i_0, i_1] & \to & H_qX_{i_1} & \leftarrow & \cdots & \leftarrow & H_q\text{Sect}_{f_X}[i_{n-1}, i_n] & \to & H_qX_{i_n} \\
\downarrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
H_qX_{i_0} & \to & H_qX_{i_1} & \to & \cdots & \to & H_qX_{i_{n-1}} & \to & H_qX_{i_n}
\end{array}
\]
where the lower row is just the ordinary $q$’th persistence module of the filtration $\mathbb{X}$, i.e. all the maps are induced by inclusions. We note that all squares in this diagram commute due to Lemma 3.5. Furthermore, all the arrows pointing to the left in the top row can be inverted due to Lemma 3.6. Thus,

**Proposition 3.7.** The commutative ladder

\[
\begin{array}{c}
H_q \text{Sect}_{f_{\mathbb{X}}} [i_0, i_1] \longrightarrow H_q X_{i_1} \longrightarrow \cdots \longrightarrow H_q \text{Sect}_{f_{\mathbb{X}}} [i_{n-1}, i_n] \longrightarrow H_q X_{i_n} \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
H_q X_{i_0} \longrightarrow H_q X_{i_1} \longrightarrow \cdots \longrightarrow H_q X_{i_{n-1}} \longrightarrow H_q X_{i_n}
\end{array}
\]

defines an isomorphism of persistence modules.
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