Cooperative Multi-Sensor Detection under Variable-Length Coding

Mustapha Hamad  
LTCI, Telecom Paris, IP Paris  
9120 Palaiseau, France  
mustapha.hamad@telecom-paris.fr

Michèle Wigger  
LTCI, Telecom Paris, IP Paris  
9120 Palaiseau, France  
michele.wigger@telecom-paris.fr

Mireille Sarkiss  
SAMOVAR, Telecom SudParis, IP Paris  
91011 Evry, France  
mireille.sarkiss@telecom-sudparis.eu

Abstract—We investigate the testing-against-independence problem over a cooperative MAC with two sensors and a single detector under an average rate constraint on the sensors-detector links. For this setup, we design a variable-length coding scheme that maximizes the achievable type-II error exponent when the type-I error probability is limited to $\epsilon$. Similarly to the single-link result, we show here that the optimal error exponent depends on $\epsilon$ and that variable-length coding allows to increase the rates over the optimal fixed-length coding scheme by the factor $(1-\epsilon)^{-1}$.

Index Terms—Distributed Hypothesis Testing, Cooperative MAC, Variable-Length Coding, Error Exponent

I. INTRODUCTION

Motivated by the broadly emerging Internet of Things (IoT) applications, distributed hypothesis testing problems gained increasing attention recently. In such problems, sensors send information about their observations to one or multiple decision centers. Then, the decision centers attempt to detect the joint distributions underlying the data observed at all the terminals including their own observations.

Our focus is on binary hypothesis testing with a null hypothesis and an alternative hypothesis. We are interested in maximizing the exponential decay (in the number of observed samples) of the probability of error under the alternative hypothesis, given a constraint on the probability of error under the null hypothesis. The study of such a Stein setup has a long history in the information theoretic literature, see e.g., [1]–[8] which study point-to-point, interactive, cascaded, and multi-sensor and/or multi-detector systems. All these works constrain the maximum rate of communication between terminals, and a fixed-length communication scheme is obviously optimal. Recently, the authors of [9] proposed to only constrain the average rate of communication, and they presented a variable-length coding scheme that under this weaker constraint improves the maximum achievable error exponent. The present work is the first extension of the point-to-point average-rate scenario in [9] and the corresponding variable-length coding scheme to systems with multiple sensors.

Specifically, we consider the two-sensors single-detector system in Fig. 1, where the first sensor communicates over a shared link to the second sensor and the detector, and after receiving this message, also the second sensor communicates with the detector. The two sensors observe the sequences $X_1^n$ and $X_2^n$, respectively, and the detector observes $Y^n$, where we assume that the following Markov chain holds both under the null hypothesis $\mathcal{H} = 0$ as well as under the alternative hypothesis $\mathcal{H} = 1$:

$$X_1^n \leftrightarrow X_2^n \leftrightarrow Y^n$$

(1)

We consider the testing-against-independence scenario where under the alternative hypothesis $\mathcal{H} = 1$ the observations at the two sensors are independent of the observations at the detector. We further assume that the sensors’ observations $X_1^n, X_2^n$ follow the same joint distribution and the decision center’s observation $Y^n$ follows the same marginal distribution under both hypotheses. A more general version of our problem (without Markov chain (1)) was studied in [10], but under a maximum rate constraint.

In this paper, we characterize the maximum achievable error exponent $\theta^*_\epsilon(R_1, R_2)$ under the alternative hypothesis when the error probability under the null hypothesis is not allowed to exceed $\epsilon$, and where here $R_1$ and $R_2$ denote the rates of communication from the first and the second sensors, respectively. As we show in this paper, and in contrast to the optimal error exponent under a maximum rate constraint $\theta^*_\epsilon_{\text{Fix}}(R_1, R_2)$ [10],1 the optimal exponent $\theta^*_\epsilon(R_1, R_2)$ depends on $\epsilon$. In fact, as a main result, we obtain

$$\theta^*_\epsilon(R_1, R_2) = \theta^*_\epsilon_{\text{Fix}}(R_1/(1-\epsilon), R_2/(1-\epsilon)).$$

(2)

Thus, through variable-length coding we can increase all available rates in the network by the factor $(1-\epsilon)^{-1}$. A similar observation was already made for the point-to-point setup studied in [9]. In this sense, the current paper extends the conclusion to multiple links, and it shows in particular that the rate-increase can be attained on all links simultaneously.

Notation: We follow the notation in [11] and [9]. In particular, we use sans serif font for bit-strings: e.g., $m$ for a deterministic and $M$ for a random bit-string. We let $\text{string}(m)$ denote the shortest bit-string representation of a positive integer $m$, and for any bit-string $m$ we let $\text{len}(m)$ and $\text{dec}(m)$ denote its length and its corresponding positive integer. We use $h_\epsilon(\cdot)$ for the binary entropy function.

1In the converse proof of [10, Theorem 2], the second step used to upper bound the rate $R_2$ relies on the Markov chain $X_1^n \leftrightarrow (M_1, X_2^{n-1}; X_2^n) \leftrightarrow X_2^n$, which does not necessarily hold. The result of [10] remains however valid under the Markov chain (1), see Remark 2 ahead.
II. SYSTEM MODEL

Consider the distributed hypothesis testing problem in Fig. 1 in the special case of testing against independence where

\[ \mathcal{H} = 0 : (X_1^n, X_2^n, Y^n) \sim i.d. P_{X_1} P_{X_2} P_Y \]  
\[ \mathcal{H} = 1 : (X_1^n, X_2^n, Y^n) \sim i.d. P_{X_1} P_{X_2} P_Y \] (3) (4)

Specifically, the system consists of two transmitters (Tx$_1$ and Tx$_2$) and a receiver (Rx$_Y$). Tx$_1$ observes the source sequence $X_1^n$ and sends its bit-string message $M_1 = \phi_1^{(n)}(X_1^n)$ to both Tx$_2$ and Rx$_Y$, where the encoding function is of the form $\phi_1^{(n)} : X_1^n \to \{0,1\}^*$ and satisfies the rate constraint

\[ E[\text{len}(M_1)] \leq nR_1. \] (5)

Tx$_2$ observes the source sequence $X_2^n$ and sends the message $M_2 = \phi_2^{(n)}(X_2^n, M_1)$ using some encoding function $\phi_2^{(n)} : X_2^n \times \{0,1\}^* \to \{0,1\}^*$ satisfying the rate constraint

\[ E[\text{len}(M_2)] \leq nR_2. \] (6)

We examine the gain provided by variable-length coding on the cooperative MAC at hand of an example. Let $X_1, X_2, Y$ be independent Bernoulli random variables of parameters $a, p, q \in [0, 1]$ and set $X_2 = X_1 \oplus T$ and $Y = X_2 \oplus S$. For this example, Fig. 1 plots the optimal error exponents of variable-length and fixed-length coding under a sum-rate constraint:

\[ \theta^*_{\Sigma}(R) := \max_{R_1, R_2 \geq 0} R_1 + R_2 \leq R \] (15)

\[ \theta^*_{\text{Fix}, \Sigma}(R) := \max_{R_1, R_2 \geq 0} R_1 + R_2 \leq R \] (16)

Remark 1: The present setup differs from the one considered by Zhao and Lai [10] only in that [10] imposes the more stringent constraints

\[ \text{len}(M_i) \leq nR_i, \quad i \in \{1, 2\}, \] (12)

instead of the expected rate constraints (5) and (6). Under the rate-constraints (12), without loss of optimality, the two transmitters can send messages $M_1$ and $M_2$ of fixed lengths.

III. MAIN RESULTS

Theorem 1: There exist auxiliary random variables $U_1$ and $U_2$ such that the optimal error exponent is given by:

\[ \theta^*_\epsilon(R_1, R_2) = \max_{P_{U_1|X_1}, P_{U_2|X_2}} I(U_1,U_2; Y) \] (13)

where mutual information quantities are calculated according to the joint pmf $P_{U_1,U_2,X_1,Y} \overset{\Delta}{=} P_{U_1|X_1} P_{U_2|X_2} P_{X_1,X_2} P_Y(X_1, X_2)$.

Proof: Achievability is proved in Section IV and the converse in Section V.

Lemma 1: In Theorem 1, it suffices to choose $U_1$ and $U_2$ over alphabets of sizes $|U_1| \leq |X_1| + 2$ and $|U_2| \leq |U_1||X_2| + 1$.

Proof: Omitted. It follows by standard applications of Carathéodory's theorem, see [11, Appendix C].

A. Comparing Variable-Length with Fixed-Length Coding

For comparison, we also present the optimal error exponent under fixed-length coding.

Remark 2: Under fixed-length coding, i.e., under rate constraints (12), the optimal error exponent $\theta^*_{\text{Fix}}(R_1, R_2)$ is [10]:

\[ \theta^*_{\text{Fix}}(R_1, R_2) = \max_{P_{U_1|X_1}, P_{U_2|X_2}} I(U_1,U_2; Y) \] (14)

where mutual information quantities are calculated according to the joint pmf $P_{U_1,U_2,X_1,Y} \overset{\Delta}{=} P_{U_1|X_1} P_{U_2|X_2} P_{X_1,X_2} P_Y(X_1, X_2)$.

Proof: Achievability can be proved as described in Section IV when the set $S_n$ is replaced by an empty set. The converse can be shown as in Section V if inequality (76), i.e., $H(M_i) \leq \frac{nR_i}{\alpha_n} \left( 1 + b_0 \left( \frac{\alpha_n}{nR_i} \right) \right)$, is replaced by the trivial inequality $H(M_i) \leq nR_i$. A more direct proof is also possible, similar to the one in [10]; the converse proof in [10] relies however on a wrong Markov chain, see our footnote 1.

We examine the gain provided by variable-length coding on the cooperative MAC at hand of an example. Let $X_1, X_2, Y$ be independent Bernoulli random variables of parameters $a, p, q \in [0, 1]$ and set $X_2 = X_1 \oplus T$ and $Y = X_2 \oplus S$. For this example, Fig. 2 plots the optimal error exponents of variable-length and fixed-length coding under a sum-rate constraint:

\[ \theta^*_{\Sigma}(R) := \max_{R_1, R_2 \geq 0} R_1 + R_2 \leq R \] (15)

\[ \theta^*_{\text{Fix}, \Sigma}(R) := \max_{R_1, R_2 \geq 0} R_1 + R_2 \leq R \] (16)
for $\epsilon = 0.07$ and in function of the sum-rate $R$.

Note that the optimal type-II error exponent under an expected rate constraint $R$ coincides with the optimal type-II error exponent under a maximum rate constraint $(1 - \epsilon)R$. Moreover, as $R$ increases, both error exponents $\theta_{\text{opt},\Sigma}^i$ and $\theta_{\text{opt},\Sigma}^{\text{Fix},3}$ tend to the optimal exponent $I(X_1; X_2; Y)$ that can be obtained in a central hypothesis testing problem where the detector directly observes all theses sequences $X_1^n, X_2^n, Y^n$. In particular, both simulated optimal error exponents reach a value of 0.7011 at $R = 1.1$ which is almost 98.25% of $I(X_1; X_2; Y) = 0.7136$.

Fig. 2. Optimal exponents under variable-length and fixed-length coding under a sum-rate constraint for above example with $a = 0.5, p = 0.75, q = 0.95$, and $\epsilon = 0.07$.

**IV. Achievability Proof**

Fix a large blocklength $n$, a small number $\mu \in (0, \epsilon)$, and conditional pmfs $P_{U_1|X_1}$ and $P_{U_2|U_1, X_1}$ such that:

$$R_1 = (1 - \epsilon + \mu)(I(U_1; X_1) + 2\mu)$$

$$R_2 = (1 - \epsilon + \mu)(I(U_2; X_1|U_1) + 2\mu)$$

(17)

(18)

where mutual information quantities are calculated according to the joint pmf

$$P_{U_1, U_2|X_1} \triangleq P_{U_1|X_1} \cdot P_{U_2|U_1, X_1} \cdot P_{X_1} \cdot P_{Y|x_2}.$$  

Randomly generate a codebook

$$C_{U_1} \triangleq \left\{ u_1^n(m_1) : m_1 \in \left\{ 1, \ldots, 2^{n(I(U_1; X_1) + \mu)} \right\} \right\}$$

(20)

by drawing all entries i.i.d. according to the marginal pmf $P_{U_1}$. For each codeword $u_1^n(m_1)$, generate a codebook

$$C_{U_2}(m_1) \triangleq \left\{ u_2^n(m_2|m_1) : m_2 \in \left\{ 1, \ldots, 2^{n(I(U_2; X_1|U_1) + \mu)} \right\} \right\},$$

(21)

by drawing the $j$-th entry of each codeword according to the marginal pmf $P_{U_2|U_1}$. Also, choose a subset $S_n$ of the typical set $T_n^{(\mu)}(P_{X_1})$ with probability slightly less than $\epsilon$:

$$S_n \subseteq T_n^{(\mu)}(P_{X_1}) : \Pr[X_1^n \in S_n] = \epsilon - \mu.$$  

(22)

Transmitter 1: Assume it observes the sequence $X_1^n = x_1^n$. If $x_1^n \notin S_n$, it looks for indices $m_1 \geq 1$ satisfying $(u_1^n(m_1), x_1^n) \in T_n^{(\mu)}(P_{U_1, X_1})$, randomly picks one of these indices, and sends its corresponding bit-string $M_1 = \text{string}(m_1)$ both to Transmitter 2 and the Receiver. Otherwise, it sends the single-bit string $M_1 = [0]$.

Transmitter 2: Assume it observes the sequence $X_2^n = x_2^n$ and receives the bit-string message $M_1 = m_1$ from Transmitter 1. If $m_1 = [0]$, then it sends the bit-string message $M_2 = [0]$. Else, if $m_1 = \text{dec}(m_1) \geq 1$, it looks for an index $m_2 \geq 1$ satisfying $(u_2^n(m_2), x_2^n(m_2|m_1), y^n) \in T_n^{(\mu)}(P_{U_2|X_1})$. It randomly picks one of these indices and sends its corresponding bit-string $M_2 = \text{string}(m_2)$ to the Receiver. Otherwise, it sends $M_2 = [0]$.

Receiver: Assume it observes the sequence $Y^n = y^n$ and receives messages $M_1 = m_1$ and $M_2 = m_2$. If any of the bit-strings $m_1$ or $m_2$ equals $[0]$, it declares $\mathcal{H} = 1$. Else, it sets $m_i = \text{dec}(m_i)$, for $i = 1, 2$, and checks if $(X_1^n(m_1), u_2^n(m_2|m_1), y^n) \in T_n^{(\mu)}(P_{U_1, U_2})$. It declares $\mathcal{H} = 0$ if the condition is verified, and $\mathcal{H} = 1$ otherwise.

**A. Analysis**

Notice first that when $X_1^n \notin S_n$, our variable-length scheme acts like the fixed-length one in [10]. We denote by $\mathcal{H}^{\text{VL}}$ the hypothesis guessed by the scheme in [10].

The type-I error probability of our scheme satisfies:

$$\alpha_n = \Pr[\mathcal{H} = 1|\mathcal{H} = 0]$$

$$= \Pr[\mathcal{H} = 1, X_1^n \in S_n|\mathcal{H} = 0]$$

$$+ \Pr[\mathcal{H} = 1, X_1^n \notin S_n|\mathcal{H} = 0]$$

$$= \Pr[X_1^n \notin S_n|\mathcal{H} = 0]$$

$$+ \Pr[\mathcal{H}^{\text{VL}} = 1, X_1^n \notin S_n|\mathcal{H} = 0]$$

$$\leq \epsilon - \mu + \Pr[\mathcal{H}^{\text{VL}} = 1|\mathcal{H} = 0].$$

(23)

(24)

(25)

(26)

Since by [10], $\Pr[\mathcal{H}^{\text{VL}} = 1|\mathcal{H} = 0] \to 0$ as $n \to \infty$, we conclude that for the proposed scheme: $\lim_{n \to \infty} \alpha_n \leq \epsilon$.

The type-II error probability satisfies:

$$\beta_n = \Pr[\mathcal{H} = 0|\mathcal{H} = 1]$$

$$= \Pr[\mathcal{H} = 0, X_1^n \in S_n|\mathcal{H} = 1]$$

$$+ \Pr[\mathcal{H} = 0, X_1^n \notin S_n|\mathcal{H} = 1]$$

$$= \Pr[\mathcal{H}^{\text{VL}} = 0, X_1^n \notin S_n|\mathcal{H} = 1]$$

$$\leq \Pr[\mathcal{H}^{\text{VL}} = 0|\mathcal{H} = 1]$$

$$\leq 2^{-n(I(U_1, U_2; Y) + \delta(\mu))}.$$  

(27)

(28)

(29)

(30)

(31)

where (31) uses the achievability result in [10] and $\delta(\mu) \to 0$ as $\mu \to 0$. Therefore, our scheme achieves the type-II error exponent

$$\theta \geq I(U_1 U_2; Y) + \delta(\mu).$$

(32)

Define $L_1 \triangleq \text{len}(M_1)$ and $L_2 \triangleq \text{len}(M_2)$. Notice that for sufficiently large blocklengths $n$ and $\mu > 0$:

$$\mathbb{E}[L_1] = \mathbb{E}[L_1|X_1^n \in S_n] \Pr[X_1^n \in S_n]$$

$$+ \mathbb{E}[L_1|X_1^n \notin S_n] \Pr[X_1^n \notin S_n]$$

$$\leq (\epsilon - \mu) + n(I(U_1; X_1) + \mu) \cdot (1 - \epsilon + \mu)$$

(33)

(34)
Similarly, for sufficiently large blocklengths $n$ and $\mu > 0$:

$$ E[L_2] = E[L_2 | X^n_1 \in S_n] \Pr[X^n_1 \in S_n] + E[L_2 | X^n_1 \notin S_n] \Pr[X^n_1 \notin S_n] \leq (\epsilon - \mu) + n(I(U_2;X_2|U_1) + \mu) \cdot (1 - \epsilon + \mu) = n(I(U_2;X_2|U_1) + 2\mu) \cdot (1 - \epsilon + \mu) \leq nR_2. $$ (39) (40) (41)

Letting $n \to \infty$ and $\mu \to 0$ concludes our achievability proof.

V. CONVERSE PROOF TO THEOREM 1

Notice first that it suffices to show

$$ \theta_{\epsilon}^*(R_1, R_2) \leq \max_{p(u_1|x_1)p(u_2|u_1, x_2): \text{RHS constraints on the LHS}} I(U_1; U_2; Y), $$ (42)

i.e., the Markov chain $U_2 \leftrightarrow (U_1, X_2) \leftrightarrow (X_1, Y)$ in Theorem 1 can be replaced by the weaker Markov chain $U_2 \leftrightarrow (U_1, X_2) \leftrightarrow Y$, because the right-hand side of (42) does not depend on the joint pmf of $U_2$ and $X_1$. More formally, we can prove the equivalence

$$ \bigcup_{U_1, U_2} \{ (I(U_1; U_2; Y), I(U_1; X_1), I(U_2; X_2|U_1)) \} = \bigcup_{U_1, U_2, U_2 \leftrightarrow (U_1, X_2) \leftrightarrow Y} \{ (I(U_1; U_2; Y), I(U_1; X_1), I(U_2; X_2|U_1)) \}. $$ (43)

Since the two objective functions coincide and the constraints on the left-hand side (LHS) are more stringent, it suffices to show that the right-hand side (RHS) is included in the LHS. To this end, fix $U_1, U_2$ satisfying the constraints on the LHS, i.e., the Markov chains $U_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y)$ and $U_2 \leftrightarrow (U_1, X_2) \leftrightarrow Y$. Then, construct $\bar{U}_1, \bar{U}_2$ so that

$$ P_{U_1}|X_1, x_2,y(u_1|x_1, x_2, y) = P_{U_1}|X_1(u_1|x_1) $$

$$ P_{U_2}|U_1, X_1, x_2, y(u_2|u_1, x_1, x_2, y) = P_{U_2}|u_1, x_2 (u_2|u_1, x_2), $$

and thus satisfying the Markov chains on the RHS:

$$ \bar{U}_1 \leftrightarrow X_1 \leftrightarrow (X_2, Y) \quad \text{and} \quad \bar{U}_2 \leftrightarrow (\bar{U}_1, X_2) \leftrightarrow (Y, X_1). $$

The proof is concluded by noting that

$$ I(\bar{U}_1; X_1) = I(U_1; X_1), $$ (46)

$$ I(\bar{U}_2; X_2|\bar{U}_1) = I(U_2; X_2|U_1), $$ (47)

$$ I(\bar{U}_1, \bar{U}_2; Y) = I(U_1; U_2; Y). $$ (48)

Equalities (46) and (47) hold trivially by construction. Equality (48) holds because $P_{\bar{U}_1, X_2} = P_{U_1, X_2}$ and $P_{\bar{U}_2}\bar{Y}|\bar{U}_1, x_2 = P_{U_2|U_1, x_2} \cdot P_{\bar{Y}|x_2} = P_{U_2|U_1, x_2} \cdot P_{\bar{Y}|x_2}$.

We proceed to show that (42) holds. Fix $\theta < \theta_{\epsilon}^*(R_1, R_2)$, a sequence of encoding and decision functions satisfying the type-I and type-II error constraints, a blocklength $n$, and a small number $\eta \geq 0$. Define:

$$ B_n(\eta) \triangleq \{(x^n_1, x^n_2) : \Pr[\hat{\mathcal{X}} = 0|X^n_1 = x^n_1, X^n_2 = x^n_2, \mathcal{H} = 0] \geq \eta \}, $$ (49)

$$ \mu_n \triangleq n^{-\frac{1}{2}}, $$ (50)

$$ D_n(\eta) \triangleq T_n^{\hat{\mathcal{x}}}(P_{X_1, X_2}) \cap B_n(\eta). $$ (51)

By constraint (10) on the type-I error probability:

$$ 1 - \epsilon \leq \sum_{x^n_1, x^n_2} \Pr[\hat{\mathcal{X}} = 0|X^n_1 = x^n_1, X^n_2 = x^n_2, \mathcal{H} = 0] \cdot P_{X^n_1, X^n_2}(x^n_1, x^n_2) $$

$$ \leq \sum_{(x^n_1, x^n_2) \in B_n(\eta)} P_{X^n_1, X^n_2}(x^n_1, x^n_2) + \sum_{(x^n_1, x^n_2) \notin B_n(\eta)} \Pr[\hat{\mathcal{X}} = 0|X^n_1 = x^n_1, X^n_2 = x^n_2, \mathcal{H} = 0] \cdot P_{X^n_1, X^n_2}(x^n_1, x^n_2) $$

$$ \leq P_{X^n_1, X^n_2}(B_n(\eta)) + \eta(1 - P_{X^n_1, X^n_2}(B_n(\eta))). $$ (52) (53) (54)

Thus we have:

$$ P_{X^n_1, X^n_2}(B_n(\eta)) \geq \frac{1 - \epsilon - \eta}{1 - \eta}. $$ (55)

Moreover, by [12, Lemma 2.12], the probability that $(X^n_1, X^n_2)$ lie in the jointly strongly typical set $T_n^{\hat{\mathcal{x}}}(P_{X_1, X_2})$ satisfies

$$ P_{X_1, X_2}(T_n^{\hat{\mathcal{x}}}(P_{X_1, X_2})) \geq 1 - \frac{|X_1| |X_2|}{2\mu_n n}, $$ (56)

and since for any events $\mathcal{A}$ and $\mathcal{B}$,

$$ \Pr(\mathcal{A} \cap \mathcal{B}) \geq \Pr(\mathcal{A}) + \Pr(\mathcal{B}) - 1, $$ (57)

then by (51), (55) and (56), we obtain

$$ P_{X^n_1, X^n_2}(D_n(\eta)) \geq \frac{1 - \epsilon - \eta}{1 - \eta} - \frac{|X_1| |X_2|}{2\mu_n n} \triangleq \Delta_n. $$ (58)

We define the random variables $(\tilde{M}_1, M_2, \tilde{X}^n_1, \tilde{X}^n_2, \tilde{Y}^n)$ as the restriction of the random variables $(M_1, M_2, X^n_1, X^n_2, Y^n)$ to $(X^n_1, X^n_2) \in D_n(\eta)$. The probability distribution of the former tuple is given by:

$$ P_{\tilde{M}_1, \tilde{M}_2, \tilde{X}^n_1, \tilde{X}^n_2, \tilde{Y}^n}(m_1, m_2, x^n_1, x^n_2, y^n) \triangleq P_{X^n_1, X^n_2}(y^n) \cdot \frac{\mathbb{I}\{x^n_1, x^n_2 \in D_n(\eta)\}}{P_{X^n_1, X^n_2}(D_n(\eta))} \cdot \mathbb{I}\{\phi(x^n_1) = m_1\} \cdot \mathbb{I}\{\phi(x^n_2, \phi(x^n_1)) = m_2\}, $$ (59)

leading to the following inequalities:

$$ P_{\tilde{X}_1, \tilde{X}_2}(x^n_1, x^n_2) \leq P_{X_1, X_2}(x^n_1, x^n_2) \Delta_n^{-1}, $$ (60)

$$ P_{M_1, M_2}(m_1, m_2) \leq P_{\tilde{M}_1, \tilde{M}_2}(m_1, m_2) \Delta_n^{-1}, $$ (61)

$$ P_{\tilde{Y}^n}(y^n) \leq P_{Y^n}(y^n) \Delta_n^{-1}, $$ (62)

$$ D(P_{X^n_1, X^n_2}||P_{\tilde{X}_1, \tilde{X}_2}) \leq \log \Delta_n^{-1}. $$ (63)
A. Single-Letter Characterization of Rate Constraints

Define the following random variables:

\[ \tilde{L}_i \triangleq \text{Leu}(M_i), \quad i = 1, 2. \]  

(64)

By the rate constraints (5) and (6), we have for \( i = 1, 2 \):

\[ nR_i \geq E[L_i] \]

(65)

\[ \geq E[L_i](X_i^1, X_i^2) \in D_n(\eta)]P_{X_i^1X_i^2}(D_n(\eta)) \]

(66)

\[ = E[L_i]P_{X_i^1X_i^2}(D_n(\eta)) \]

(67)

\[ \geq E[L_i]\Delta_n, \]

(68)

where the last inequality follows by (58). Moreover, by definition, \( L_i \) is a function of \( M_i \), for \( i = 1, 2 \), so we can upper bound the entropy of \( M_i \) as follows:

\[ H(M_i) = H(M_i, L_i) \]

(69)

\[ = H(M_i|L_i) + H(L_i) \]

(70)

\[ = \sum_{l_i} \text{Pr}[L_i = l_i]H(M_i|L_i = l_i) + H(L_i) \]

(71)

\[ \leq \sum_{l_i} \text{Pr}[L_i = l_i]|l_i| + H(L_i) \]

(72)

\[ = E[L_i] + H(L_i) \]

(73)

\[ \leq nR_i + H(L_i) \]

(74)

\[ \leq \frac{nR_i}{\Delta_n} + \frac{nR_i}{\Delta_n}h_b\left(\frac{\Delta_n}{nR_i}\right) \]

(75)

\[ = \frac{nR_i}{\Delta_n}\left(1 + h_b\left(\frac{\Delta_n}{nR_i}\right)\right), \]

(76)

where (74) holds by (68), and (75) holds since the maximum possible entropy of \( L_i \) is obtained by a geometric distribution of mean \( E[L_i] \), which is further bounded by \( \frac{nR_i}{\Delta_n} \) [13, Theorem 12.1.1].

On the other hand, we lower bound the entropy of \( M_1 \) as:

\[ H(M_1) \geq I(M_1; X_1^n X_2^2) + D(P_{X_1^n X_2^2}||P_{X_1^n X_2^2}) + \log \Delta_n \]

(77)

\[ = H(X_1^n X_2^2) + D(P_{X_1^n X_2^2}||P_{X_1^n X_2^2}) \]

(78)

\[ = \sum_{i=1}^{n} \text{H}(X_1^1, X_2^i | M_1 X_1^{i-1} X_2^{i-1} + \log \Delta_n \]

(79)

\[ = \sum_{i=1}^{n} \text{H}(X_1^1, X_2^i | X_1^{i-1} X_2^{i-1}) \]

(80)

\[ = \sum_{i=1}^{n} \text{H}(X_1^1, X_2^i | U_1, T) \]

(81)

\[ = \sum_{i=1}^{n} \text{H}(X_1^1, X_2^i | U_1, T) \]

(82)

\[ \geq n \left[I(U_1; X_1^1 X_2^2) + D(P_{X_1^n X_2^2}||P_{X_1^n X_2^2}) + \log \Delta_n \right]. \]

(83)

Here, (77) holds by (63); (79) holds by the super-additivity property in [14, Proposition 1] and by the chain rule; (80) holds by defining \( U_1 \triangleq (M_1, X_1^{i-1}, X_2^{i-1}); \) (81) holds by defining \( T \) uniform over \( \{1, \ldots, n\} \) independent of all other random variables; and (82) holds by defining \( U_1 \triangleq (U_1 T), X_1 \triangleq X_1 T, \) \( X_2 \triangleq X_2 T. \)

Similarly,

\[ H(M_2) \geq I(M_2; X_1^n X_2^2 | M_1) \]

(84)

\[ = \sum_{i=1}^{n} I(M_2; X_1^i X_2^i | M_1 X_1^{i-1} X_2^{i-1}) \]

(85)

\[ = \sum_{i=1}^{n} I(U_2, T; X_1^i X_2^i | U_1, T) \]

(86)

\[ = nI(U_2, T; X_1^i X_2^i | U_1, T) \]

(87)

\[ = nI(U_2, T; X_1^i X_2^i | U_1, T) \]

(88)

\[ = nI(U_2 T; X_1^i X_2^i | U_1 T) \]

(89)

\[ \geq nI(U_2 T; X_1^i X_2^i | U_1 T) \]

(90)

\[ \geq nI(U_2 T; X_1^i X_2^i | U_1 T) \]

(91)

Here, (85) holds since \( M_2 \) is a function of \( X_2^2 \) and \( M_1 \); (86) holds by the chain rule; (87) holds by the definition of \( U_1 T \), and by defining \( U_2 T \triangleq M_2 \); and (91) holds by defining \( U_2 \triangleq (U_2 T) \).

Combining (76) with (84) and (91) yields:

\[ R_1 \geq \frac{I(U_1; X_1^1) + \frac{1}{n}\log \Delta_n}{(1 + h_b\left(\frac{\Delta_n}{nR_1}\right))} \cdot \Delta_n \]

(92)

\[ R_2 \geq \frac{I(U_2; X_2^i | U_1^1) + \frac{1}{n}\log \Delta_n}{(1 + h_b\left(\frac{\Delta_n}{nR_2}\right))} \cdot \Delta_n. \]

(93)

B. Upper Bounding the Type-II Error Exponent

Define for each \((m_1, m_2)\) the set

\[ \mathcal{A}_n(m_1, m_2) \triangleq \{\mathcal{Y}_n : (m_1, m_2, \mathcal{Y}_n) \in \mathcal{A}_n\}, \]

(94)

and its Hamming neighborhood:

\[ \mathcal{A}_n^\ell(m_1, m_2) \triangleq \{\mathcal{Y}_n : \exists \mathcal{Y}_n \in \mathcal{A}_n(m_1, m_2) \]

(95)

\[ \text{s.t. } d_H(\mathcal{Y}_n, \mathcal{Y}_n^\ell) \leq \ell_n \}

for some real number \( \ell_n \) satisfying \( \lim_{n \to \infty} \ell_n/n = 0 \) and \( \lim_{n \to \infty} \ell_n/\sqrt{n} = \infty \).

Since by definitions (49) and (51), for all \((x_1^n, x_2^n) \in D_n:\)

\[ P_{\mathcal{Y}_n|x_1^n x_2^n}(\mathcal{A}_n(m_1, m_2)|x_1^n, x_2^n) \geq \gamma, \]

(96)

then by the blowing-up lemma [15]:

\[ P_{\mathcal{Y}_n|x_1^n x_2^n}(\mathcal{A}_n^\ell(m_1, m_2)|x_1^n, x_2^n) \geq 1 - \zeta, \]

(97)

for a real number \( \zeta > 0 \) such that \( \lim_{n \to \infty} \zeta = 0 \). Moreover, by taking the expectation over (97):

\[ P_{M_1 M_2 \mathcal{Y}_n}(A_n^\ell) = \sum_{(x_1^n, x_2^n) \in D_n} \sum_{(m_1, m_2) \in \mathcal{M}_1 \times \mathcal{M}_2} P_{\mathcal{Y}_n|x_1^n x_2^n}(\mathcal{A}_n^\ell(m_1, m_2)|x_1^n, x_2^n) \]

(98)
\[ P_{\tilde{X}_2,\tilde{T}}\tilde{Y}_T|\tilde{X}_1,\tilde{T} + D(P_{\tilde{X}_1,\tilde{T}}\tilde{X}_2,\tilde{Y}_T|P_{\tilde{X}_1,\tilde{T}}) \]

\[ + \log \Delta_n - \sum_{t=1}^{n} H(\tilde{X}_t,\tilde{T}t|\tilde{X}_t^{-1}\tilde{T}^{t-1}\tilde{M}_1) = n[H(\tilde{X}_2,\tilde{T}|\tilde{X}_1,\tilde{T}) + D(P_{\tilde{X}_1,\tilde{T}}\tilde{X}_2,\tilde{Y}_T|P_{\tilde{X}_1,\tilde{T}})] \]

\[ + \log \Delta_n - \sum_{t=1}^{n} H(\tilde{X}_t,\tilde{T}t|\tilde{X}_t^{-1}\tilde{T}^{t-1}\tilde{M}_1) = n[H(\tilde{X}_2,\tilde{T}|\tilde{X}_1,\tilde{T}) + D(P_{\tilde{X}_1,\tilde{T}}\tilde{X}_2,\tilde{Y}_T|P_{\tilde{X}_1,\tilde{T}})] \]

\[ ≥ n(H(\tilde{X}_2,\tilde{T}|\tilde{X}_1,\tilde{T}) + D(P_{\tilde{X}_1,\tilde{T}}\tilde{X}_2,\tilde{Y}_T|P_{\tilde{X}_1,\tilde{T}})) \]

\[ + \log \Delta_n - \sum_{t=1}^{n} H(\tilde{X}_t,\tilde{T}t|\tilde{X}_t^{-1}\tilde{T}^{t-1}\tilde{M}_1) = n[H(\tilde{X}_2,\tilde{T}|\tilde{X}_1,\tilde{T}) + D(P_{\tilde{X}_1,\tilde{T}}\tilde{X}_2,\tilde{Y}_T|P_{\tilde{X}_1,\tilde{T}})] \]

\[ + \log \Delta_n - \sum_{t=1}^{n} H(\tilde{X}_t,\tilde{T}t|\tilde{X}_t^{-1}\tilde{T}^{t-1}\tilde{M}_1) = n[H(\tilde{X}_2,\tilde{T}|\tilde{X}_1,\tilde{T}) + D(P_{\tilde{X}_1,\tilde{T}}\tilde{X}_2,\tilde{Y}_T|P_{\tilde{X}_1,\tilde{T}})] \]
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