Integrable modules for Lie Torus.

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Dedicated to my teachers at Andhra University, Waltair.

Abstract

In the last two decades the structure of Extended Affine Lie Algebra (EALA) is extensively studied. In explicitly constructing an EALA, the centerless Lie Torus play an important role. In this paper we consider the Universal central extension of a centerless Lie Torus and classify the irreducible integrable modules for them when the center acts non-trivially. They turn out to be “highest weight modules” for direct sum of finitely many affine Lie algebras.

Key words: Lie Torus, Universal Central extension, Finite order automorphism and Integrable modules.

MSC: Primary 17B67, Secondary 17B65, 17B70

Introduction

An extended affine Lie algebra (EALA) is a Lie algebra together with a nondegenerate invariant symmetric bilinear form, a nonzero finite dimensional ad-diagonalizable subalgebra such that a list of natural axioms is imposed. As the term (EALA) suggest, the defining axioms for an EALA are modeled after the properties of affine Kac-Moody Lie algebras. The
EALA’s are natural generalization of affine Lie algebras in several variables. See \( [AABGP, ABF, ABFP1, ABFP2, AF, AZ, BA, NK, EN1, EN2] \) and the references there in.

The classical procedure of realizing affine Lie algebra using loop algebra in one variable proceeds in two steps [K, chaps. 7 and 8]. In the first step, the derived algebra module its center of the affine Lie algebra is constructed as the loop algebra of a diagram automorphism of a finite dimensional simple Lie algebra. In the second step, the affine Lie algebra itself, together with a Cartan subalgebra and a nondegenerate invariant bilinear form for the affine Lie algebra, is built from the graded loop algebra by forming a central extension (with one dimensional center) and adding a (one dimensional) graded algebra of derivations.

The replacement for the derived algebra modulo its center in EALA theory is the centerless core of the EALA, and the centerless core has been characterized axiomatically as centerless Lie Torus \([Y, EN3]\). Starting with a centerless Lie Torus, Erahard Neher \([EN3]\) has shown how to carry out the second step of the classical realization procedure that is to construct an EALA.

In this paper we consider Lie Torus which are multiloop algebras (see 1.7). These algebras cover allmost all Lie Torus except one class of Lie Torus which appear in type A and coming from quantum torus \([ABFP1]\). We start with a centerless Lie Torus and consider its universal central extension denoted by \(\mathbf{LT}\). We add a finite set derivations of zero degree which measures the natural gradation on \(\mathbf{LT}\), and denote it by \(\widetilde{\mathbf{LT}}\).

The purpose of the paper is to classify irreducible integrable modules for \(\widetilde{\mathbf{LT}}\) with finite dimensional weight spaces when the center acts non-trivially. They turn out to be highest weight modules for direct sum of finitely may affine Kac-Moody Lie algebras. Some very special cases are done in \([E3],[EB],[EZ]\) and \([XT]\). These highest weight modules are different from the standard highest weight modules of affine Lie algebra. The ad-diagonalizable subalgebra we consider is much smaller than the Cartan
subalgebra of the affine Lie algebra. Consequently the zero root space need not be abelian. They need further investigation.

The contents of the paper are the following. Let $LT$ be the centerless Lie Torus (see (1.7) for the definition). Let $\overline{LT}$ be the universal central extension of $LT$ (see Proposition (2.2)). Both $LT$ and $\overline{LT}$ are naturally $\mathbb{Z}^{n+1}$ graded for some $n$. Let $D$ be the spaces spanned by zero degree derivations $d_0, d_1, \cdots, d_n$. Let $\widetilde{LT} = \overline{LT} \oplus D$. The role of the $D$ is to keep track of the gradation for $\overline{LT}$.

The Lie Torus is defined as multiloop algebra using finitely many finite order automorphisms (see 1.4). In the case where all automorphisms are trivial we get the standard non-twisted multiloop algebra. In this case $\widetilde{LT}$ is nothing but toroidal Lie algebra (see 1.3). The irreducible integrable modules for toroidal case has been classified in $[E3]$. The classification of irreducible integrable modules (where the zero degree center acts non trivially) for the universal central extension of Lie Torus runs parallel to the toroidal case. But there are several places where we need completely different arguments. For example, in the toroidal case, highest weight exists at a graded level. But in the case of $\widetilde{LT}$, we could only able to prove that the highest weight exists at a non-graded level. Because of the presence of automorphisms, the proof becomes more complex.

We start with a subquotient of $\widetilde{LT}$ denoted by $\mathfrak{L}$ which is obtained by removeing certain derivations and going modulo a part of the center. We establish a one-one correspondence between irreducible modules of $\widetilde{LT}$ and $\mathfrak{L}$. The former are called graded modules and the later are called non-graded modules. (see Section 7 and 8).

We defined highest weight irreducible modules for $\mathfrak{L}$ which need not have finite dimensional weight spaces. We give necessary and sufficient condition for an irreducible highest weight module to have finite dimensional weight spaces (Theorem 3.10). We prove that an irreducible integrable module, where the center acts by a positive integer is actually an highest weight module (Theorem 6.2). We next prove that an irreducible integrable highest
weight module is actually a module for the direct sum of finitely many affine Lie algebras (Proposition 4.3 and Lemma 5.5).

1 Notation and Preliminaries

Throughout this work will use the following notation.

(1.1) All vector spaces, algebras and tensor products are over complex numbers \( \mathbb{C} \). Let \( \mathbb{Z}, \mathbb{N} \) and \( \mathbb{Z}_+ \) denote integers, non-negative integers and positive integers respectively.

(1.2) Let \( g \) be simple finite dimensional Lie algebra and let \( (, ) \) be a non-degenerate symmetric bilinear form on \( g \). Fix a positive integer \( n \) and let \( \sigma_0, \sigma_1, \ldots, \sigma_n \) be commuting finite order automorphisms of \( g \) of order \( m_0, m_1, \ldots, m_n \) respectively. Let \( m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \). Let \( \Gamma = m_1 \mathbb{Z} \oplus \cdots \oplus m_n \mathbb{Z} \) and \( \Gamma_0 = m_0 \mathbb{Z} \). Let \( \Lambda = \mathbb{Z}^n / \Gamma \) and \( \Lambda_0 = \mathbb{Z} / \Gamma_0 \). Let \( \overline{k}, \overline{l} \) denote the images in \( \Lambda \). For any integers \( k_0 \) and \( l_0 \), let \( \overline{k}_0 \) and \( \overline{l}_0 \) denote images in \( \Lambda_0 \).

Let
\[
A = \mathbb{C}[t_0^{\pm 1}, \ldots, t_n^{\pm 1}],
\]
\[
A_n = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}],
\]
\[
A(m) = \mathbb{C}[t_1^{\pm m_1}, \ldots, t_n^{\pm m_n}], \quad \text{and}
\]
\[
A(m_0, m) = \mathbb{C}[t_0^{\pm m_0}, \ldots, t_n^{\pm m_n}]
\]
be Laurent polynomial algebras with respective variables.

(1.3) For \( k \in \mathbb{Z}^n \), let \( t^k = t_1^{k_1} \cdots t_n^{k_n} \in A_n \). Let \( \Omega_A \) be the vector space spanned by symbols \( t_0^{k_0} k_i t^k i, 0 \leq i \leq n, k_0 \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \). Let \( dA \) be the subspace of \( \Omega_A \) spanned by \( \sum_{i=0}^n k_i t_0^{k_0} k_i t^k i \). Similar spaces can be defined for all other Laurent polynomial algebras.

Let \( Z = \Omega_A / dA, Z(n) = \Omega_{A_n} / dA_n, Z(m) = \Omega_{A(m)} / dA(m) \) and \( Z(m_0, m) = \Omega_{A(m_0, m)} / dA(m_0, m) \). Notice that \( Z(n), Z(m), Z(m_0, m) \) are all subspaces of \( Z \). Notice that \( g \otimes A \) has a natural structure of a Lie algebra. We will now define toroidal Lie algebra. See[EM] and [EMY].

\[
\tau = g \otimes A \oplus \Omega_A / dA.
\]
Let $X(k_0, k) = X \otimes t_0^{k_0} t^k$, $X \in \mathfrak{g}$, $k_0 \in \mathbb{Z}$, $k \in \mathbb{Z}^n$

Define

(a) $[X(k_0, k), Y(l_0, l)] = [X, Y](k_0 + l_0, k + l) + (X, Y) \sum_{i=0}^{n} k_it_0^{k_0+l_0} t^k [k_i]$ \[\Omega_A/dA \text{ is central.}\]

It is well known that $\tau$ is the universal central extension of $\mathfrak{g} \otimes A$. See [EMY] and [Ka].

(1.4) We will now define multiloop algebra as a subalgebra of $\mathfrak{g} \otimes A$. For $0 \leq i \leq n$, let $\xi_i$ denote a $m_i$th primitive root of unity.

Let

$$\mathfrak{g}(k_0, k) = \{X \in \mathfrak{g} | \sigma_i X = \xi_i^{m_i} X, 0 \leq i \leq n\}$$

Then $\bigoplus_{(k_0, k) \in \mathbb{Z}^{n+1}} \mathfrak{g}(k_0, k)t_0^{k_0} t^k$ is called multiloop algebra. The finite dimensional irreducible modules are classified by Michael Lau [ML].

(1.5) Suppose $\mathfrak{h}_1$ is a finite dimensional ad-diagonalizable subalgebra of a Lie algebra $\mathfrak{g}_1$. We set, for $\alpha \in \mathfrak{h}_1^*$

$$\mathfrak{g}_{1, \alpha} = \{x \in \mathfrak{g}_1 | [h, x] = \alpha(h)x, h \in \mathfrak{h}_1^*\}$$

Then we have

$$\mathfrak{g}_1 = \bigoplus_{\alpha \in \mathfrak{h}_1^*} \mathfrak{g}_{1, \alpha}$$

Let $\Delta(\mathfrak{g}_1, \mathfrak{h}_1) = \{\alpha \in \mathfrak{h}_1^*, |\mathfrak{g}_{1, \alpha} \neq 0\}$ which includes 0.

Let $\Delta^\times(\mathfrak{g}_1, \mathfrak{h}_1) = \Delta(\mathfrak{g}_1, \mathfrak{h}_1) \setminus 0$

(1.6) Throughout this paper we assume that $\mathfrak{g}(\mathfrak{h}, \mathfrak{h})$ is a simple Lie algebra. We can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}(o) \subseteq \mathfrak{h}$ where $\mathfrak{h}(o)$ is a Cartan subalgebra of $\mathfrak{g}(\mathfrak{h}, \mathfrak{h})$. We always fix such a choice $\mathfrak{h}(o)$ and $\mathfrak{h}$ throughout this paper. It is well known that $\Delta^\times_0 = \Delta^\times_0(\mathfrak{g}(\mathfrak{h}, \mathfrak{h}), \mathfrak{h}(o))$ is irreducible reduced finite root system and has at most two root lengths. Let $\Delta^\times_{0, sh}$ be the set of non-zero short roots.

Define

$$\Delta^\times_{0, en} = \begin{cases} \Delta^\times_0 \cup 2\Delta^\times_{0, sh} & \text{if } \Delta^\times_0 \text{ of type } B_l. \\ \Delta^\times_0 & \text{is otherwise} \end{cases}$$
\[ \Delta_{0,\text{en}} = \Delta_{0,\text{en}}^\times \cup \{0\} \]

An irreducible finite dimensional highest weight module for \( \mathfrak{g}(\overline{\sigma}, \overline{\tau}) \) is said to have property (\( M \)) if the highest weight is one of the following:

(a) highest root

(b) highest short root if \( \Delta_{0}^\times \) is not simplylaced.

(c) twice highest short root if \( \Delta_{0}^\times \) is of type \( B_l \).

In all these three cases the weight system of the module is contained in \( \Delta_{0,\text{en}} \). Future the multiplicity of non-zero weight is one.

(1.7) We will now define Lie Torus (centerless) which is the main object of our study. A multi-loop algebra \( \bigoplus_{(k_0, k) \in \mathbb{Z}^{n+1}} \mathfrak{g}(\overline{k_0}, \overline{k}) t_0^{k_0} t^k \) is called a Lie Torus and denoted by \( LT \) if

(a) \( \mathfrak{g}(\overline{\sigma}, \overline{\tau}) \) is a simple Lie algebra

(b) As \( \mathfrak{g}(\overline{\sigma}, \overline{\tau}) \) module, each \( \mathfrak{g}(\overline{k_0}, \overline{k}) = U(\overline{k_0}, \overline{k}) \oplus V(\overline{k_0}, \overline{k}) \) where \( U(\overline{k_0}, \overline{k}) \) is trivial module and either \( V(\overline{k_0}, \overline{k}) \) is zero or satisfy the property (\( M \)).

(1.8) Properties of Lie Torus \( LT \).

(a) \( \Delta(\mathfrak{g}, \mathfrak{h}(o)) = \Delta_{0,\text{en}} \) if \( \Delta_0 \) is of type \( B_l \) and \( V(\overline{k_0}, \overline{k}) \) is isomorphic to highest weight irreducible finite dimensional module with highest weight \( 2\beta_s \) for some \( (k_0, \overline{k}) \neq (o, \overline{o}) \). Here \( \beta_s \) is the highest short root in \( \Delta_{0}^\times \).

(b) \( \Delta(\mathfrak{g}, \mathfrak{h}(o)) = \Delta_0 \) in all other cases.

(c) For \( \alpha \in \mathfrak{h}(o)^* \), let \( \mathfrak{g}(\overline{k_0}, \overline{k}, \alpha) = \{ g \in \mathfrak{g}(\overline{k_0}, \overline{k}) \mid [h, g] = \alpha(h) g, h \in \mathfrak{h}(o) \} \)

Then \( \mathfrak{g}(\overline{k_0}, \overline{k}) = \bigoplus_{\alpha \in \mathfrak{h}(o)^*} \mathfrak{g}(\overline{k_0}, \overline{k}, \alpha) \). The dimension of \( \mathfrak{g}(\overline{k_0}, \overline{k}, \alpha) \leq 1 \) for \( \alpha \neq 0 \).

(d) For \( \alpha \neq 0, \mathfrak{g}(\overline{k_0}, \overline{k}, \alpha), \mathfrak{g}(\overline{-k_0}, \overline{-k}, -\alpha) \) and \( \left[ \mathfrak{g}(\overline{k_0}, \overline{k}, \alpha), \mathfrak{g}(\overline{-k_0}, \overline{-k}, -\alpha) \right] \)
forms an \( sl_2 \) - copy.
For details see Proposition 3.2.5 of [ABFP1].

(1.9) Change of co-ordinates. We work with $n$ variables for notational convenience. Let $G = GL(n, \mathbb{Z})$ be the group of $n \times n$ matrices with entries in $\mathbb{Z}$ and determinant $\pm 1$. Then the group $G$ acts naturally on $\mathbb{Z}^n$. Denote the action by $B.k$ for $B \in G$ and $k \in \mathbb{Z}^n$. Let $s_i = t^{B.e_i}$ where $\{e_i\}$ is the standard $\mathbb{Z}$ basis of $\mathbb{Z}^n$. Then clearly $A_n = C[s_1^{\pm 1}, \ldots, s_n^{\pm 1}]$. Consider the map $B : g \otimes A_n \oplus \mathbb{Z}(n) \rightarrow g \otimes A_n \oplus \mathbb{Z}(n)$ given by

$$B.X \otimes t^k = X \otimes t^{B.k}$$

$$B.d(t^k)t^s = d(t^{B.k})t^{B.s}$$

for $X \in g$ and $k, s, \in \mathbb{Z}^n$. Where $d(t^k)t^s = \sum k_it^{k+s}K_i$. Then it is easy to see that $B$ defines an automorphism.

Suppose $B = (b_{ij})$ then one can verify that $B(t^kK_i) = \sum a_{ij}t^{B.k}K_j$.

In particular $B(K_i) = \sum a_{ij}K_j$.

Now define a new Lie Torus $\overline{LT}(B)$ by replacing variables $t_i$ by $s_i$ and note that $B$ takes the Lie Torus $\overline{LT}$ to $\overline{LT}(B)$.

This is what we call change of co-ordinates. We will use this change of co-ordinates in the paper without any mention and just say that “upto a choice of co-ordinates.”

2 Universal central extension of Lie Torus.

Let $LT = \bigoplus_{(k_0,k) \in \mathbb{Z}^{n+1}} g(\overline{k_0}, \overline{k})t_0^{k_0}t^k$ be a Lie Torus as defined in Section 1.

(2.1) Let $\overline{LT} = LT \oplus \mathbb{Z}(m_0, m)$

Let $X(k_0, k) = X \otimes t_0^{k_0}t^k, X \in g(\overline{k_0}, \overline{k})$.

Define a Lie algebra structure on $\overline{LT}$ by

(a) $[X(k_0, k), Y(l_0, l)] = [X, Y](k_0 + l_0, k + l) + (X, Y)\sum_{i=0}^{n} k_it_0^{k_0}t^kK_i$

(b) $\mathbb{Z}(m_0, m)$ is central in $\overline{LT}$.

Notice that $(X, Y) \neq 0 \Rightarrow k + l \in \Gamma$ and $k_0 + l_0 \in \Gamma_0$. This follows from the standard fact that $(,)$ is invariant under $\sigma_i, 0 \leq i \leq n$. This proves that the
above Lie bracket is closed. Notice also the above Lie bracket is restriction defined in (1.3).

(2.2) Proposition: $\mathcal{LT}$ is the universal central extension of $\mathcal{L}T$.

See Corollary (3.27) of [JS]

Both $\mathcal{L}T$ and $\mathcal{LT}$ are naturally $\mathbb{Z}^{n+1}$ graded. To reflect this fact we add derivations. Let $\mathcal{D}$ be the space spanned by $d_0, d_1, \cdots, d_n$.

(2.3) Let $\widetilde{\mathcal{L}}T = \mathcal{LT} \oplus \mathcal{D}$. Extended the Lie bracket in the following way.

\[
\begin{align*}
[d_i, X(k_0, k)] &= k_i X(k_0, k) \\
[d_i, t^{k_0}t^k K_j] &= k_i t^{k_0}t^k K_j \\
[d_i, d_j] &= 0
\end{align*}
\]

Notice that $Z(m_0, m)$ is no more central in $\widetilde{\mathcal{L}}T$ but only an abelian ideal. In fact any graded subspace of $Z(m_0, m)$ is an ideal.

(2.4) Let $\widetilde{\mathfrak{h}} = \mathfrak{h}(o) \oplus \sum_{0 \leq i \leq n} \mathbb{C}K_i \oplus \mathcal{D}$ which is an abelian subalgebra of $\widetilde{\mathcal{L}}T$.

Let $\delta_i \in \widetilde{\mathfrak{h}}^*$ such that $\delta_i(\mathfrak{h}(o)) = \delta_i(K_j) = 0$ and $\delta_i(d_j) = \delta_{ij}, 0 \leq i, j \leq n$. For $(k_0, k) \in \mathbb{Z}^{n+1}, \delta(k) = \sum k_i \delta_i$ and $\delta(k_0, k) = k_0 \delta_o + \delta(k)$

Let

\[
\begin{align*}
\widetilde{\mathcal{L}}T_{\alpha + \delta(k_0, k)} &= \begin{cases} 
\mathfrak{g}(k_0, k, \alpha) \otimes t^{k_0}t^k, \alpha \neq 0 \\
\mathfrak{g}(k_0, k, o) \otimes t^{k_0}t^k \oplus \sum_{0 \leq i \leq n} t^{k_0}t^k K_i, \alpha = 0 \text{ and } (k_0, k) \neq (0, 0)
\end{cases}
\end{align*}
\]

Then $\widetilde{\mathcal{L}}T = \bigoplus_{\alpha \in \widetilde{\mathfrak{h}}^*(o) \atop (k_0, k) \in \mathbb{Z}^{n+1}} \widetilde{\mathcal{L}}T_{\alpha + \delta(k_0, k)}$ is a root space decomposition with respect to $\widetilde{\mathfrak{h}}$ and each root space is finite dimensional.

(2.5) Definition. A module $V$ of $\widetilde{\mathcal{L}}T$ is called weight module if

(a) $V = \bigoplus_{\lambda \in \widetilde{\mathfrak{h}}^*} V_\lambda$, $V_\lambda = \{v \in V | hv = \lambda(h)v, h \in \widetilde{\mathfrak{h}}\}$

(b) $\dim V_\lambda < \infty$

(2.6) Definition. A root $\alpha + \delta(k_0, k)$ of $\widetilde{\mathcal{L}}T$ is called real root if $\alpha \neq o$ and null root if $\alpha = 0$.

(2.7) Definition. A weight module $V$ of $\widetilde{\mathcal{L}}T$ is called integrable if every real
root vector acts locally nilpotent on $V$. That is for $X \in \tilde{L}T_{\alpha+\delta(k_o,k)}$, $\alpha \neq 0$ and $v \in V$, there exists $b = b(v, \alpha + \delta(k_o,k))$ such that $X^b.v = 0$.

The purpose of this paper is to classify irreducible integrable modules for $\tilde{L}T$. We will first reduce the problem to a subquotient of $\tilde{L}T$ and then classify those modules.

### 3 Subquotient of $\tilde{L}T$ and highest weight module

(3.1) We will first define a Lie-algebra $\mathfrak{L} = LT \oplus K \otimes A(m) \oplus \mathbb{C}d_o$. Let $X(k_o,k) \in \mathfrak{g}(k_o,k)$ and $Y(l_o,l) \in \mathfrak{g}(l_o,l)$ and define.

(a) $[X(k_o,k), Y(l_o,l)] = [X, Y](k_o+l_o, k+l) + (X, Y)\delta_{l_o+k_o} K \otimes t^{l+k}$

(b) $K \otimes A(m)$ is central.

(c) $[d_o, X(k_o,k)] = k_o X(k_o,k)$

(3.2) Let $\tilde{\mathfrak{L}} = \mathfrak{L} \oplus D$ where $D$ is the space spanned by derivations $d_1, d_2, \cdots, d_n$. Extend the Lie bracket to $\tilde{\mathfrak{L}}$ by defining $D$ action on $\mathfrak{L}$ as in (2.3). We will first give a Lie algebra surjective homomorphism from $\Phi : \tilde{L}T \rightarrow \tilde{\mathfrak{L}}$ by

$$
\begin{align*}
\Phi X(k_o,k) &= X(k_o,k), X \in \mathfrak{g}(k_o,k) \\
\Phi t_i^{k_o} &\quad 0 \text{ if } i \neq 0 \text{ or } k_o \neq 0 \\
\Phi K_o &\quad K \otimes t^k \\
\Phi d_i &\quad d_i, 0 \leq i \leq n.
\end{align*}
$$

It is easy to see that $\Phi$ is a Lie algebra homomorphism. In fact ker $\Phi$ is a graded subspace of $\mathbb{Z}(m_o,m)$ hence an ideal in $\tilde{L}T$ as noted in (2.3).

(3.3) We will now indicate the plan of the rest of the paper. The first step is to prove that any irreducible weight module $V$ of $\tilde{L}T$ is actually an irreducible module for $\tilde{\mathfrak{L}}$ upto a suitable choice of co-ordinates. See (1.9). The second step is to prove that any irreducible weight module for $\tilde{\mathfrak{L}}$ give rise to an irreducible module for $\mathfrak{L}$ with finite dimensional weight spaces with respect to $\mathfrak{h}(o) = \mathfrak{h}(o) \oplus \mathbb{C}K \oplus \mathbb{C}d_o$. We will also indicate how to recover
the original module for $\tilde{LT}$ from the irreducible module of $\mathfrak{L}$. The third step is to prove that an irreducible integrable module for $\mathfrak{L}$ where $m_oK$ acts as positive integer is an highest weight module (definition given below). The last step is to classify such highest weight integrable irreducible module to some extent.

In this section we will first take of the last step.

(3.4) Let $\delta \in \tilde{\mathfrak{h}}(o)^*$ such that $\delta(\tilde{\mathfrak{h}}(o)) = 0, \delta(K) = 0$ and $\delta(d_o) = 1$.

Let
\[ \mathfrak{L}_{\alpha+k_o\delta} = \begin{cases} \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(k_o, k, \alpha)t^{k_o}t^k & \text{if } \alpha + k_o\delta \neq 0 \\ \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(o, K, o)t^k & \text{if } \alpha + k_o\delta = 0 \end{cases} \]

Then $\mathfrak{L} = \bigoplus_{\alpha \in \tilde{\mathfrak{h}}*, k_o \in \mathbb{Z}} \mathfrak{L}_{\alpha+k_o\delta}$ is a root space decomposition with respect to $\mathfrak{h}(o)$.

But the dimensions of the root spaces are infinite dimensional.

(3.5) Recall $\Delta_o = \Delta(\tilde{\mathfrak{g}}(\mathfrak{h}, \mathfrak{o}), \tilde{\mathfrak{h}}(o))$ and $\Delta(\mathfrak{g}, \mathfrak{h}(o))$ from (1.8). Let $\alpha_1, \cdots, \alpha_d$ be a set of simple roots in $\Delta_o$. Let $\beta$ be a maximal root in $\Delta(\mathfrak{g}, \mathfrak{h}(o))$. Let $\alpha_o = -\beta + \delta$ which may not be root of $\mathfrak{L}$. Let $Q = \bigoplus_{i=0}^{d} \mathbb{Z}\alpha_i$ and define $\leq$ on $\tilde{\mathfrak{h}}^*$ for $\lambda, \mu, \in \tilde{\mathfrak{h}}(o)^*, \lambda \leq \mu$ if $\mu - \lambda \in Q^+ = \bigoplus_{i=0}^{d} \mathbb{N}\alpha_i$. Clearly this ordering is consistant with the standard order on $\Delta(\mathfrak{g}, \mathfrak{h}(o))$. Let $\Delta$ be the set of all roots of $\mathfrak{L}$.

\[ \Delta_+ = \{ \alpha + k_o\delta \in \Delta | k_o > 0 \text{ or } k_o = 0, \alpha > 0 \} \]
\[ \Delta_- = \{ \alpha + k_o\delta \in \Delta | k_o < 0 \text{ or } k_o = 0, \alpha < 0 \} \]

Clearly $\Delta_+$ is the set of positive root with the above ordering.

(3.6) Let
\[ \mathfrak{L}^+ = \bigoplus_{\alpha+k_o\delta > o} \mathfrak{L}_{\alpha+k_o\delta} \text{ and } \]
\[ \mathfrak{L}^- = \bigoplus_{\alpha+k_o\delta < o} \mathfrak{L}_{\alpha+k_o\delta} \]
\[ \mathfrak{L}^o = \mathfrak{L}_o \]

Then clearly $\mathfrak{L} = \mathfrak{L}^+ \oplus \mathfrak{L}^o \oplus \mathfrak{L}^-$ and note that $\mathfrak{L}^o$ is a subalgebra but need not be ablian. The bracket in $\mathfrak{L}^o$ does not produce the center $K \otimes A(m)$. In otherword $K \otimes A(m)$ is a direct summand of $\mathfrak{L}^o$ as Lie algebras.
Construction of highest weight module for $\mathfrak{L}$.

Let $N$ be an irreducible finite dimensional module for $\mathfrak{L}$. Since $\bar{\mathfrak{h}}(\alpha) + K \otimes A(m)$ is central, it is easy to see they act by scalars on $N$. Thus $\bar{\mathfrak{h}}(\alpha)$ acts on $N$ by a single linear function. Let $U(\mathfrak{L})$ denote the universal enveloping algebra of $\mathfrak{L}$. Define Verma module for $\mathfrak{L}$.

$$M(N) = U(\mathfrak{L}) \otimes N$$

where $\mathfrak{L}^+$ acts trivially on $N$. By standard arguments, one can see that $M(N)$ admits a unique irreducible quotient say $V(N)$. It is easy to see that $M(N)$ and $V(N)$ are weight module with respect to $\bar{\mathfrak{h}}(\alpha)$. But they may not have finite dimensional weight spaces.

We will now give a necessary and sufficient condition for $V(N)$ to have finite dimensional weight spaces.

Let $I$ be an ideal in $A(m)$ and let

$$\mathfrak{L}^\alpha(I) = \bigoplus_{\bar{k} \in \Lambda} \mathfrak{g}(\bar{\alpha}, \bar{\alpha}) t^k I \oplus K \otimes I$$

$$\mathfrak{L}(I) = \bigoplus_{\bar{k} \in \Lambda} \mathfrak{g}(\bar{k}, \bar{\alpha}) t^k I \oplus K \otimes I$$

Clearly $\mathfrak{L}^\alpha(I)$ is an ideal in $\mathfrak{L}^\alpha$ and $\mathfrak{L}(I)$ is an ideal in $\mathfrak{L}$.

Define $\mathfrak{L}_{k_\alpha} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(\bar{k}, \bar{\alpha}) t^k I$, $k_\alpha \in \mathbb{Z}$

and $\mathfrak{L}_{-k_\alpha, -\alpha} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(\bar{k}, \bar{\alpha}) t^k I$

Define $\mathfrak{L}_{k_\alpha}(I)$ and $\mathfrak{L}_{-k_\alpha, -\alpha}(I)$ by replacing $t^k$ by $t^k I$ in $\mathfrak{L}_{k_\alpha}$ and $\mathfrak{L}_{-k_\alpha, -\alpha}$ respectively.

**Proposition (3.9)** Suppose $N$ is finite dimensional irreducible module for $\mathfrak{L}^\alpha$ such that $\mathfrak{L}^\alpha(I).N = 0$ for some ideal $I$ of $A(m)$. Then $\mathfrak{L}(I).V(N) = 0$.

**Proof** Suppose $\alpha + k_\alpha \delta > 0$ then $\alpha + k_\alpha \delta = \sum_{i=0}^{n} s_i \alpha_i$, $s_i \in \mathbb{N}$.

Define $ht(\alpha + k_\alpha \delta) = \sum_{i=0}^{n} s_i > 0$.

**Claim** $\mathfrak{L}_{\bar{k}_\alpha} \mathfrak{L}_{-k_\alpha, -\alpha}(I) \subseteq N$ for all roots $\bar{\alpha} + l_\alpha \delta > 0$.

We will prove the claim by induction on the $ht(\alpha + k_\alpha \delta)$. The claim is trivial if $ht(\alpha + k_\alpha \delta) = 0$ as $\alpha + k_\alpha \delta = 0$ and $\mathfrak{L}_{-k_\alpha, -\alpha}(I) \subseteq N$. Assume the claim for all roots $\alpha + k_\alpha \delta$ such that $ht(\alpha + k_\alpha \delta) \leq s$ for a fixed $s > 0$. 
Consider the root $\tilde{\alpha} + l_0 \delta > 0$ and

$$\mathfrak{L}_{l_0, \tilde{\alpha}}, \mathfrak{L}_{-k_0, -\alpha}(I).N$$

$$= \mathfrak{L}_{-k_0, -\alpha}(I).\mathfrak{L}_{l_0, \tilde{\alpha}}N + [\mathfrak{L}_{l_0, \tilde{\alpha}}, \mathfrak{L}_{-k_0, -\alpha}(I)]N$$

The first term is zero as $N$ is the highest weight space. There are four cases $\tilde{\alpha} - \alpha + (l_0 - k_0)\delta > 0, < 0, = 0$ or may not be a root for the second term. The second term is zero in the first case as $N$ is highest weight space. The second term is zero in the second case by induction. In the third case, $[\mathfrak{L}_{l_0, \tilde{\alpha}}, \mathfrak{L}_{-k_0, -\alpha}(I)] \subseteq \mathfrak{L}^0(I)$ and hence zero by assumption. In the fourth case the bracket itself is zero. This proves the claim.

From the claim it follows $\mathfrak{L}_{-k_0, -\alpha}(I).N$ generates a proper submodule inside an irreducible module $V(N)$ for $\alpha + k_0 \delta < 0$. Hence $\mathfrak{L}_{-k_0, -\alpha}(I).N = 0$ for all $\alpha + k_0 \delta < 0$. The same is true for $\alpha + k_0 \delta > 0$ as $N$ is highest weight space. It is also true for the zero part by assumption.

This proves $\mathfrak{L}(I).N = 0$.

Consider $S = \{v \in V(N) | \mathfrak{L}(I).v = 0\}$ which contains $N$. Since $\mathfrak{L}(I)$ is an ideal in $\mathfrak{L}$, one can check that $S$ is $\mathfrak{L}$-module. This proves $S = V(N)$ and hence $\mathfrak{L}(I).V(N) = 0$. This completes the proof of the proposition.

**Theorem (3.10)** $V(N)$ has finite dimensional weight spaces with respect to $\tilde{\mathfrak{h}}(o)$ if and only if there exists a co-finite ideal $I$ of $A(m)$ such that $\mathfrak{L}^0(I).N = 0$.

**Proof** Suppose there exists a co-finite ideal $I$ of $A(m)$ such that $\mathfrak{L}^0(I).N = 0$. Then by Proposition (3.9), $\mathfrak{L}(I)V(N) = 0$. Thus $V(N)$ is a module for $\mathfrak{L}(A/I) \oplus \mathbb{C}d_o$. Here $\mathfrak{L}(A/I)$ is a Lie algebra in an obvious sense and admits root space decomposition with respect to $\tilde{\mathfrak{h}}(o)$. Since $A/I$ is finite dimensional, each root space of $\mathfrak{L}(A/I)$ is finite dimensional. By stands arguments (using PBW Theorem) it follows that any highest weight module for $\mathfrak{L}(A/I)$ has finite dimensional weight spaces. In particular $V(N)$ has finite dimensional weight spaces.

Conversely, suppose $V(N)$ has finite dimensional weight spaces with respect to $\tilde{\mathfrak{h}}(o)$. Since $\mathfrak{g}$ is simple and hence perfect, it follows that
\((3.11)\) \( g = \sum_{k_o,l_o \in \Lambda \atop \bar{k}, \bar{l} \in \Lambda} [g(\bar{k}_o, \bar{k}), g(\bar{l}_o, \bar{l})]. \)

It is now easy to see that \( \mathcal{L}^o(I) \) is spanned by \([g(\bar{k}_o, \bar{k}), g(\bar{l}_o, \bar{l})]t^{k+l}I \) where \( k_o + l_o = 0 \) and \( \bar{k}, \bar{l} \in \Lambda \) and \( K \otimes I \).

**Claim** There exists a polynomial \( Q_i \) in \( t_i^{m_i}, (1 \leq i \leq n) \) such that \( K \otimes t^k Q_i = 0 \) on \( V(N) \) for all \( k \in \Gamma \).

To see the proof of the claim, note that the non-degenerate form \((,\) on \( g \) remains non-degenerate on \( g(o, o) \). Future it remains non-degenerate on \( h(o) \). This is a general fact for any simple Lie algebra. Thus there exists \( h_1, h_2 \in h(o) \) such that \( (h_1, h_2) \neq 0 \). Fix a vector \( v \) in \( N \) and \( k_o > 0 \).

Consider \( \{h_2t^m q t^{-m_o}v, q \in \mathbb{N}\} \)

Which belongs to a single weight space of \( V(N) \). Thus there exists a polynomial \( Q_i \) in variable \( t_i^{m_i} \) such that

\[
  h_2t^{-m_o}Q_iv = 0
\]

Consider, for any \( k \in \Gamma, h_1t_o^{m_o}t^k h_2t_o^{-m_o}Q_i v \\
= h_2t_o^{-m_o}Q_i h_1t_o^{m_o}t^k v + (h_1, h_2)m_o K t^k Q_i v = 0. \) Here and below we omit \( \otimes \) for convenience.

The first term is zero as \( v \) belongs to \( N \) and \( m_o \delta \) is a positive root. Thus \( K t^k Q_i v = 0 \) for all \( k \in \Gamma \). This proves the claim.

Let \( B(N) = \{v_1, v_2, \ldots, v_d\} \) be a basis of \( N \). Fix \( (k_o, k) \in \mathbb{Z}^{n+1} \) such that \( o < k_i \leq m_i \) for all \( i \). Note that \( k_o > 0 \) but \( \bar{k}_o \) could be zero.

Let

\[
  B\left(g(\bar{k}_o, \bar{k})\right) = \{X_1, X_2, \ldots, X_s\}
\]

be a basis of \( g(\bar{k}_o, \bar{k}) \) in such a way that they are all \( h(o) \) weight vectors.

Consider, for a fixed \( v_j, X_p \) and \( 1 \leq i \leq n \)

\[
  \{X_p t^{-k_o} t^{k_i m_q} v_j, q \in \mathbb{N}\}
\]

Which belongs to a single weight space of \( V(N) \), which is finite dimensional. Thus there exists a polynomial \( S_i := S_i(X_p, v_j, k, k_o) \) in variable \( t_i^{m_i} \) such that \( X_p t^{-k_o} t^{k_i} S_i v_j = 0. \)

Let \( P_i = \Pi S_i(X_p, v_j, k_o, k) Q_i \), where the product runs over all \( p, j, k_o \) and \( k \).
such that $1 \leq p \leq s$, $1 \leq j \leq d$, $0 < k_0 \leq m_o$ and $0 < k_i \leq m_i$. Here $Q_i$ is a polynomial given in the above claim. Since the product is finite, $P_i$ is a polynomial in $t_i^{m_i}$. Consider, for any $l \in \mathbb{Z}^n$ and $v_j \in N$
\[ g(\kappa_o, l)t_0^l t^l X_p t_0^{-k_0 t^k} P_i v_j = X_p t_0^{-k_0 t^k} P_i g(\kappa_o, l)t_0^l t^l v_j + [g(\kappa_o, l), X_p] t_l^{l+k} P_i v_j \]
The central term does not appear as it is zero by the claim. For this note that $Q_i$ is a factor of $P_i$. The first term is zero as $v_j$ belongs to $N$ and $k_0 \delta > 0$. Thus we have proved that
\[ [g(\kappa_o, l), g(-\kappa_o, k)] t_l^{l+k} P_i v_j = 0 \]
for $l \in \mathbb{Z}^n$ and $(k_o, k) \in \mathbb{Z}^n$ such that $0 < k_i \leq m_i$ for all $i$.

We already know that $K t_k P_i = 0$ for all $k \in \Gamma$ by the claim. Let $I$ be the ideal generated by $P_1, P_2, \cdots, P_n$ inside $A(m)$. It is easy to see that $I$ is a co-finite ideal.

Now from (3.11) it is easily follows that $\mathfrak{L}^o(I).N = 0$. This completes the proof of Theorem.

\section{Integrable modules for $\mathfrak{L}$.}

\textbf{Definition (4.1)} A module $V$ of $\mathfrak{L}$ is called a weight module if
\[ V = \bigoplus_{\lambda \in \mathfrak{h}(o)^*} V_{\lambda}, V_{\lambda} = \{ v \in V | hv = \lambda(h)v, \forall h \in \mathfrak{h}(o)^* \} \text{ and } \dim V_{\lambda} < \infty \]
Recall that the roots of $\mathfrak{L}$ are of the form $\alpha + k_0 \delta$ and is called real root if $\alpha \neq 0$.

\textbf{Definition (4.2)} A weight module $V$ of $\mathfrak{L}$ is called integrable if all real root vectors are locally nilpotent on $V$. See Definition 2.7 for more details.

In this section we will classify irreducible integrable highest weight module for $\mathfrak{L}$.

Let $V(N)$ be an irreducible highest weight module for $\mathfrak{L}$. Throughout this section we assume $V(N)$ is integrable. In particular $V(N)$ has finite dimensional weight spaces with respect to $\tilde{\mathfrak{h}}(o)$.

Thus by Theorem 3.10, there exists a co-finite ideal $I$ of $A(m)$ such that
We can assume that the ideal $I$ generated by polynomials $P_i$ in variable $t_i^{m_i}, 1 \leq i \leq n$. We can further assume that the constant term is one.

Write $P_i(t_i^{m_i}) = \prod_{j=1}^{q_i} (t_i^{m_i} - a_{ij}^{m_i})^{b_{ij}}$ for some positive integers $b_{ij}$ and $q_i$.

Further $a_{ij}^{m_i} \neq a_{ij'}^{m_i}$ for $j \neq j'$.

Let $P'_i(l_i^{m_i}) = \prod_{j=1}^{q_i} (t_i^{m_i} - a_{ij}^{m_i})$.

Let $I'$ be the ideal of $A(m)$ generated by $P'_i, 1 \leq i \leq n$. Then clearly $I \subseteq I'$.

**Proposition (4.3)** $\mathfrak{L}(I')V(N) = 0$.

We need some Lemmas. In view of proposition it is sufficient to prove that $\mathfrak{L}(I')N = 0$. Fix a positive root $\alpha + k_o \delta$ such that $\alpha \neq 0$. Let $J$ be a subquotient of $A(m)$ of the form $J_1/J_2$ where $J_1$ is an ideal of $A(m)$ and $J_2$ is an ideal of $J_1$.

Recall $\mathfrak{L} = \bigoplus_{\alpha \in \mathfrak{h}^+, k_o \in \mathbb{Z}} \mathfrak{L}_{\alpha + k_o \delta}$

Fix a root $\alpha + k_o \delta > 0$. Let $\mathfrak{g}(J)$ be the linear space spanned by

(a) $\bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(\bar{k}_o, k, \alpha)t_o^{k_o}t^k J$

(b) $\bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(-\bar{k}_o, k, -\alpha)t_o^{-k_o}t^k J$

(c) $\bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(2\bar{k}_o, k, 2\alpha)t_o^{2k_o}t^k J$

(d) $\bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(-2\bar{k}_o, k, -2\alpha)t_o^{-2k_o}t^k J$

(e) $\left[ \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(\bar{k}_o, k, \alpha)t_o^{k_o}t^k J, \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(-\bar{k}_o, k, -\alpha)t_o^{-k_o}t^k \right]$

(f) $\left[ \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(2\bar{k}_o, k, 2\alpha)t_o^{2k_o}t^k J, \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(-2\bar{k}_o, k, -2\alpha)t_o^{-2k_o}t^k \right]$

Note that the second term in (e) and (f) do not have $J$. In case $2\alpha + 2k_o \delta$ is not a root, then the terms (c), (d) and (f) donot occur. Now it is easy to check that $\mathfrak{g}(J)$ is a Lie algebra using Jacobi identity.

Let $\tilde{\mathfrak{g}}(J) = \mathfrak{g}(J) + \sum \mathfrak{g}(\alpha, k, o)t^k J + K \otimes J \oplus C \mathfrak{d}$

Note that $\mathfrak{L}^o(J) \subseteq \tilde{\mathfrak{g}}(J)$ and the terms at (e) and (f) are contained in $\mathfrak{L}^o(J)$. 

It is direct checking that \( \mathfrak{g}(J) \) is an ideal in \( \tilde{\mathfrak{g}}(J) \).

Let \( \tilde{N} \) be the \( \tilde{\mathfrak{g}}(J) \)-module generated by \( N \).

Note the following.

(1) \( \tilde{N} \) is actually module gor \( \tilde{\mathfrak{g}}(A(m)/I) \) which is finite dimensional.

(2) The first two terms in the definition of \( \mathfrak{g}(A(m)) \) acts trivially on \( N \) as they correspond to positive root spaces.

(3) \( \mathfrak{g}^o(A_m) \) leaves \( N \) invariant.

(4) The 3rd and 4th terms of \( \mathfrak{g}(A(m)) \) acts locally nilpotently on \( N \) as they correspond to real root spaces.

Thus by \( PBW \) Theorem we conclude that \( \tilde{N} \) is finite dimensional. \( \tilde{N} \) is also \( d_o \)-invariant.

Let \( \tilde{\mathfrak{g}}(A(m)) = \tilde{\mathfrak{g}}(A(m))_- \oplus \mathfrak{g}^o(A(m)) \oplus \tilde{\mathfrak{g}}(A(m))_+ \) be a natural decomposition. The positive part corresponds to \( \alpha + k_o \delta \) and \( 2\alpha + 2k_o \delta \). Similarly the negative part. So \( \tilde{N} \) is an highest weight module for \( \tilde{\mathfrak{g}}(A(m)) \) and hence has a unique irreducible quotient \( N_1 \).

Let \( \pi : \tilde{N} \to N_1 \) be the natural map.

(4.4) Note that \( \ker \pi \cap N = \{0\} \)

**Lemma (4.5)** Fik \( k \) and \( l \in \mathbb{Z}^n \).

\[
Y \in \mathfrak{g}(-k_o, I, -\alpha)t_o^{-k_o}t^l \\
X \in \mathfrak{g}(k_o, \alpha)t_o^k t^k I' \\
H = [X, Y], \quad Y_1 = [H, Y].
\]

Suppose there exists a vector \( v \) in \( N_1 \) such that \( Xv = Y_1, v = [Y_1, Y]v = 0 \) and \( Hv = \lambda v \) for some \( \lambda \in \mathbb{C} \).

Then

(1) \( HY^q v = \lambda Y^q v, q \geq 0 \)

(2) \( XY^q v = q\lambda Y^{q-1}v, q > 0 \).

**Proof** We have the following formula from \([K]\)

\[
Y^q Y_1 = \sum_{s=0}^{q} \binom{q}{s} ((adY)^s Y_1) Y^{q-s}
\]
We first note that \((adY)^s Y_1 = 0\) for \(s \geq 2\) as \((s + 1)\alpha\) is not a root.

We also have \(Y^q Y_1 v = 0\).

Thus
\[
Y_1 Y^q v + q[Y, Y_1] Y^{q-1} v = 0
\]

But \([Y, Y_1] Y^{q-1} v\) is zero as \([Y, Y_1]\) commutes with \(Y\) and kills \(v\).

This proves \([H, Y]^q v = 0\) for \(q \geq 0\). We will now prove (1) by induction on \(q\). By assumption in the Lemma, it is true for \(q = 0\). Assume for \(q \geq 1\) and consider
\[
HY^{q+1} v = HY Y^q v = HY Y^q v + [H, Y] Y^q v = \lambda Y^{q+1} v \text{ by (4.6)}.
\]

This completes the induction step. We will now prove (2) again by induction on \(q\). (2) is true for \(q = 1\) as \(XY v = Y X v + [X, Y] v = Hv = \lambda v\)

Assume for \(q > 1\) and consider
\[
XY^{q+1} v = Y XY^q v + HY^q v = q\lambda Y^q v + \lambda Y^q v \text{ (by 1 and induction)} = (q + 1)\lambda Y^q v
\]

This completes the induction step. Thus the proof of the Lemma is completed.

**Lemma (4.7)** \(\tilde{\mathfrak{g}} (I'/I)\) is a solvable Lie-algebra.

**Proof** It is easy checking. Just note that \(I'^P \subseteq I\) for large \(P\).

**Lemma (4.8)** \(\mathfrak{g}(I'/I) N_1 = 0\).

**Proof** Since \(\mathfrak{g}(I'/I)\) is solvable and \(N_1\) is finite dimensional, there exists a vector \(v\) in \(V\) such that \(\mathfrak{g}(I'/I)\) acts as scalars.

**Claim** \(\mathfrak{g}(I'/I)v = 0\)

The positive and negative part act as locally nilpotent operators and also as scalars. Hence they acts trivially. It remains to prove the zero part of \(\tilde{\mathfrak{g}} (I'/I)\) acts trivially on \(v\). Fix \(k, l \in \mathbb{Z}^n\) and \(X, Y, Y_1\) and \(H\) be as in Lemma 4.5. We have already seen \(X v = 0, Y_1 v = [Y_1, Y] v = 0\) as \(Y_1\) and \([Y_1, Y]\) are in the negative in part of \(\mathfrak{g}(I'/I)\). We have \(H v = \lambda v\).

Thus we have Lemma (4.5).
We will prove that $\lambda = 0$. Let $P$ be least positive integer such that $Y^P v = 0$ and $Y^{P-1} v \neq 0$. Consider

$$\lambda Y^{P-1} v = H Y^{P-1} v = [X,Y] Y^{P-1} v = (XY - YX) Y^{P-1} v = -(P-1) \lambda Y^{P-1} v$$

This means $P \lambda = 0 \Rightarrow \lambda = 0$ as $P \neq 0$. Recall the definition of $g(I'/I)$ and we have proved that the term from (e) acts trivially on $v$. Similar argument proves that the term (f) acts trivially. This completes the proof of claim.

**Lemma (4.9)** $g(I'/I) N = 0$

**Proof** Let $M = \{ v \in N_1 | g(I'/I) v = 0 \}$. Since $g(I'/I)$ is in ideal in $\tilde{g}$ $(I'/I)$, $M$ will be $\tilde{g}$ $(I'/I)$-module. From Lemma (4.8), we know that $M \neq 0$. Since $N_1$ is $\tilde{g}$ $(I'/I)$ irreducible it follows that $N_1 = M$. But by (4.4) $N \subseteq N_1$ and the Lemma follows.

**Proof of proposition (4.3)**

We only need to prove that $L(I') N = 0$. In view of the above Lemma, what remains to be shown is that $g(0, \overline{k}, 0) \in t^k I' \oplus K \otimes I'$ is zero on $N$ for all $k \in \mathbb{Z}^n$. From the proof of Lemma (4.9) we know the following.

$$\left( g(\overline{k}_o, \overline{k}, \alpha) t^k I', g(-\overline{k}_o, \overline{l}, -\alpha) t^{-k} I \right) = g(\overline{k}_o, \overline{k}, \alpha), g(-\overline{k}_o, \overline{l}, -\alpha) t^k I'$$

is zero on $N$.

The above statement is true for any positive root $\alpha + k_o \delta$. Now replacing $k_o$ by $m_o + k_o$ we see that first term in (4.10) does not change as $m_o + k_o = \overline{k}_o$.

By substracting we see that each term in (4.10) is zero on $N$. Now noting that $(g(\overline{k}_o, \overline{k}, \alpha), g(-\overline{k}_o, \overline{l}, -\alpha))$ is non-zero, we see that $K \otimes t^k N = 0$ for all $k \in \Lambda$. To complete the proof of the proposition, it is sufficent to see that $g(o, \overline{k}, o)$ is spanned by $g(\overline{k}_o, \overline{k}, \alpha), g(-\overline{k}_o, \overline{l}, -\alpha)$. Now this follows from (3.11) and the observation that $g = \bigoplus_{\alpha \neq 0} (g_\alpha + [g_\alpha, g_{-\alpha}])$. 
5 Affine Lie algebras.

(5.1) In the last section we have seen that the irreducible integrable module \(V(N)\) is actually a module for \(\mathfrak{L}(A(m)/I')\). In this section we will describe \(\mathfrak{L}(A(m)/I')\) and prove that it is isomorphic to direct sum of finitely many affine Lie algebras.

Recall that \(\sigma_o\) is an automorphism of order \(m_o\) and \(\xi_o\) is \(m_o\)th primitive root of unity.

Let \(g_{k_o} = \{ x \in g | \sigma_o x = \xi_o^{k_o} x \} \)

Write

\[ \mathfrak{L}(g, \sigma_o) = \bigoplus_{k_o \in \mathbb{Z}} g_{k_o} \otimes t^{k_o} \oplus \mathbb{C}K \]

which is known to be an affine Lie algebra.

(5.2) Some notation.

For \(1 \leq i \leq n\), let \(N_i\) be a positive integer and let \(N = N_1 \cdots N_n\) be the product.

Let \(a_i = (a_{i1}, \cdots, a_{iN_i})\) be non-zero complex numbers such that \(a_{ij}^m \neq a_{ij}^{m'}\) for \(j \neq j'\). Let \(I = (i_1, \cdots, i_n)\) such that \(1 \leq i_j \leq N_j\). There are \(N\) of them and let \(I_1, \cdots, I_N\) be some order.

Write

\[ a_i^q = a_{i1}^{q_1} \cdots a_{in}^{q_n}, q = (q_1, \cdots, q_n) \in \mathbb{Z}^n. \]

Fix \(k = (k_1, \cdots, k_n) \in \mathbb{Z}^n\) such that \(0 \leq k_i < m_i\).

(5.3) Lemma Consider the \(N \times N\) matrix

\[ B = (a_{i1}^{l_{1m_1+k_1}}, \cdots, a_{in}^{l_{nm_n+k_n})}_{1 \leq i, j \leq N_i, 1 \leq i, j \leq N_j}. \]

Then \(B\) is invertible.

Proof Let \(B_i = (a_{ij}^{m_{ij}})_{1 \leq j \leq N_i}\) and \(1 \leq i \leq N_i\)

\[ B_i = \begin{pmatrix}
  a_{i1}^{k_i} & 0 \\
  & \ddots \\
  0 & a_{iN_i}^{k_i}
\end{pmatrix}. \]

Let \(D_i = \begin{pmatrix}
  a_{i1}^{k_i} & 0 \\
  & \ddots \\
  0 & a_{iN_i}^{k_i}
\end{pmatrix}. \]

Since \(B_i\) is Vandermonde matrix, it is invertible. \(D_i\) is also invertible.
Then it is not too difficult to see $B$ is a tensor product of $B_i D_i$. It is well known that the tensor product of invertible matrices is invertible. Hence $B$ is invertible. See $[B]$. □

\[(5.4)\] Let $\mathcal{L}' = \bigoplus_{(k_o,k) \in \mathbb{Z}^n} g(\overline{k_o}, \overline{k}) t_o^{k_o}t^k \oplus K \otimes A(m)$

$\mathcal{L}'(I)$ is similarly defined.

Define a Lie algebra homomorphism

$$\varphi : \mathcal{L}' \to \bigoplus_{N\text{-copies}} \mathcal{L}(g, \sigma_o)$$

$$\varphi(X t_o^{k_o}t^k) = (X t_o^{k_o} a^k_{t_1}, \ldots, X t_o^{k_o} a^k_{t_N})$$

Let $P'_i(t^m) = \prod_{j=1}^{N_i} (t^m_i - a^m_{ij})$ and let $I'$ be an ideal generated by $P'_1, \ldots, P'_n$ inside $A(m)$.

\[(5.5)\] Lemma

(1) $\varphi$ is surjective

(2) $\mathcal{L}'/\mathcal{L}'(I') \cong \bigoplus_{N\text{-copies}} \mathcal{L}(g, \sigma_o)$

(3) $\varphi$ is $d_o$ grade preserving map.

Proof Fix $k \in \mathbb{Z}^n$ such that $o \leq k_i < m_i$. Fix any $k_o \in \mathbb{Z}$.

Let $T(\overline{k_o}, \overline{k})$ be the span of $g(\overline{k_o}, \overline{k}) t_o^{k_o}t^k Q$ where $Q = (q_1 m_1, \ldots, q_n m_n) \in \Gamma$ and $1 \leq q_i \leq N_i$. Then by Lemma (5.3), the restriction of $\varphi$ to

$$T(\overline{k_o}, \overline{k}) \to \oplus g(\overline{k_o}, \overline{k}) t_o^{k_o}$$

is both injective and surjective as the corresponding matrix is invertible.

Similar argument holds good for $\varphi$ restricted to $K \otimes A(m)$. This proves (1).

To see (2), first note that $\mathcal{L}'(I') \subseteq \ker \varphi$. It is easy to see that $T(\overline{k_o}, \overline{k})$ is a spanning set for $g(\overline{k_o}, \overline{k}) t_o^{k_o} t^k A(m)/I'$. In fact it is a basis as the map $\varphi$ is injective on $T(\overline{k_o}, \overline{k})$. It proves $\mathcal{L}'(I') = \ker \varphi$.

(3) is obvious.

\[(5.6)\] Remarks.
By a result of Kac[K] we can assume that $\mathcal{L}(\mathfrak{g}, \sigma_0)$ is an affine Lie algebra. The highest weight modules that appears here is this paper are different from the standard highest weight modules in [K]. The ad-dionalizable subalgebra $\tilde{\mathfrak{h}}(o)$ is much smaller that the Cartan subalgebra of $\mathcal{L}(\mathfrak{g}, \sigma_0)$. Consequently the zero root space could be non-abelian.

6 Integrable modules revisited

(6.1) Recall the definition of integrable modules for $\mathfrak{L}$ from (4.2). In this section we will prove that an irreducible integrable module for $\mathfrak{L}$ where $m_oK$ acts as positive integer is an highest weight module.

Let $\mathfrak{g}_{aff} = \mathfrak{g}(\mathfrak{g}(o, o)) \otimes \mathbb{C}[t^m_0, t^{-m}_0] \oplus \mathbb{C}K \oplus \mathbb{C}d_o$ which is a subalgebra of $\mathcal{L}$. Recall that $\tilde{\mathfrak{h}}(o) = \mathfrak{h}(o) \oplus \mathbb{C}K \oplus \mathbb{C}d_o \subseteq \mathfrak{g}_{aff}$.

(6.2) Theorem. Suppose $V$ is an irreducible module for $\mathfrak{L}$ with finite dimensional weight spaces with respect to $\tilde{\mathfrak{h}}$. Further suppose $V$ is integrable for $\mathfrak{g}_{aff}$ where the canonical central element $m_oK$ acts as positive integer. Then $V$ is an highest weight module for $\mathcal{L}$.

Before proving the above theorem, we need some notation and prove some Lemmas.

(6.3) Recall that $\delta \in \tilde{\mathfrak{h}}^*$ such that $\delta(\mathfrak{h}(o)) = 0, \delta(K) = o$ and $\delta(d_o) = 1$. Now define $w \in \tilde{\mathfrak{h}}(o)^*$ such that $w(\mathfrak{h}(o)) = 0, w(K) = 1$ and $w(d_o) = 1$. Given $\lambda \in \tilde{\mathfrak{h}}(o)^*$, let $\overline{\lambda}$ be the restriction to $\mathfrak{h}(o)$. Conversely given any $\overline{\lambda} \in \mathfrak{h}(o)^*$ we can extend $\overline{\lambda}$ to $\tilde{\mathfrak{h}}$ such that $\overline{\lambda}(K) = \overline{\lambda}(d_o) = 0$. Given any $\lambda \in \tilde{\mathfrak{h}}^*$ we can write in a unique way

$$\lambda = \overline{\lambda} + \lambda(d_o)\delta + \lambda(K)w.$$ 

Recall that $\Delta_o = \Delta(\mathfrak{g}(\mathfrak{g}(o, o)), \mathfrak{h}(o))$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}(o))$. Let $\beta$ be maximal root in $\Delta_o$ and let $\beta_{o}$ be maximal root in $\Delta$. We have already note that $\beta = \beta_{o}$ except possibly in the case $B_t$. In that case $\beta_{o} = 2\beta_s$ where $\beta_s$ is a highest short in $\Delta_o$. In any case $\beta \leq_o \beta_{o}$ where $\leq_o$ denote the standard ordering in $\Delta_o$.

(6.4) Let $\alpha_o = -\beta_{o} + \delta$ and $\alpha^1 = -\beta + m_o\delta$. Note that $\alpha_o$ may not be a
root of \(L\).

Recall that \(\alpha_1, \alpha_2, \ldots, \alpha_d\) is a set of simple root in \(\Delta_o\). Let \(\tilde{Q} = \bigoplus_{i=1}^{d} \mathbb{Z}\alpha_i\),

\[Q_{aff} = \tilde{Q} + \mathbb{Z}\alpha^1\text{ and } Q = \tilde{Q} + \mathbb{Z}\alpha_o.
\]

Let \(\hat{Q} = \bigoplus_{i=1}^{d} \mathbb{N}\alpha_i\), \(Q^+_{aff} = \hat{Q} + \mathbb{N}\alpha^1\) and \(Q^+ = \hat{Q} + \mathbb{N}\alpha_o\).

Consider \(\alpha^1 = -\beta + m_o\delta = -\beta_o + \delta + (\beta_o - \beta) + (m_o - 1)(\beta_o + \alpha_o)\)

\[\hat{Q} \subseteq Q_{aff} \subseteq Q \text{ and } Q^+ \subseteq Q^+_{aff} \subseteq Q^+
\]

Define three orderings on \(\tilde{h}(o)^*\) in the following way. Let \(\lambda, \mu \in \tilde{h}(o)^*\)

\[\lambda \leq_o \mu \text{ if } \mu - \lambda \in \hat{Q}^+
\]

\[\lambda \leq \mu \text{ if } \mu - \lambda \in Q^+_{aff}
\]

\[\lambda \leq_1 \mu \text{ if } \mu - \lambda \in Q^+
\]

From above we have \(\lambda \leq_o \mu \Rightarrow \lambda \leq \mu \text{ and } \lambda \leq \mu \Rightarrow \lambda \leq_1 \mu\).

\[\text{(6.4) We note that } \lambda \geq_1 0 \text{ and } \lambda(d_o) = 0 \text{ then } \lambda \geq_0 0\]

\[\text{(6.5) Let } \Delta^+_{aff} = \{\alpha + k_o m_o \delta | k_o \in \mathbb{Z}, \alpha \in \Delta_o\}\]

Let

\[\Delta^+_{aff} = \{\alpha + k_o m_o \delta | k_o > 0 \text{ or } k_o = 0, \alpha >_o 0\} = \{\alpha + k_o m_o \delta \in \Delta_{aff} | \alpha + k_o m_o \delta > 0\}\]

It is obvious that \(g_{aff}\) has root space decomposition

\[g_{aff} = \bigoplus_{\alpha \in \Delta^+_{aff}} g_{aff,\alpha} \text{ and } g_{aff,0} = \tilde{h}(o).
\]

Let \(N^+_{aff} = \bigoplus_{\alpha > 0} g_{aff,\alpha}\)

\[\text{(6.6) Denote } V(\lambda) \text{ the irreducible integrable highest weight module for } g_{aff}.
\]

The following are well known.

1. An highest weight integrable module for \(g_{aff}\) is necessarily irreducible.

2. Suppose \(\mu\) and \(\lambda\) are dominate integral weights for the simple roots \(\alpha^1, \alpha_1, \ldots, \alpha_d\) such that \(\lambda \leq \mu\). Then \(\lambda\) is a weight of \(V(\mu)\).

\[\text{(6.7) Recall that } \tilde{Q} \text{ is a root lattice of } g(\vec{\mathfrak{g}}, \vec{\mathfrak{g}}). \text{ let } \tilde{\Lambda} \text{ be the weight latice of } g(\vec{\mathfrak{g}}, \vec{\mathfrak{g}}) \text{ and let } \tilde{\Lambda}^+ \text{ be the set of dominant integral weights.}
\]

\[\text{Definition : } \lambda \in \tilde{\Lambda}^+ \text{ is called minimal if } \mu \leq_o \lambda, \mu \in \tilde{\Lambda}^+ \text{ then } \mu = \lambda. \text{ It is}
\]
known that every coset $\hat{Q} / \hat{\Lambda}$ has a unique minimal weight.

Let $P(V(\lambda))$ denote the set of weight of $V(\lambda)$ and let

$$\overline{P}(\lambda) = \{\overline{\mu} | \mu \in P(V(\lambda))\}.$$ Clearly $\overline{P}(\lambda)$ determines a unique coset in $\hat{Q} / \hat{\Lambda}$.

Let $\overline{\mu}_o$ be the minimal weight and note that $\overline{\mu}_o \leq \overline{\lambda}$. Let $s$ be any complex number such that $\lambda(d) - s$ is a non-negative integer divisible by $m_o$.

(6.8) $\mu_o = \overline{\mu}_o + s\delta + \lambda(K)w \in P(L(\lambda))$. See [E3] for details.

We fix an irreducible integrable module $V$ for $\mathfrak{L}$. We assume that $m_oK$ acts as positive integer. Let $P(V)$ be the set of weights.

Let $P_+(V) = \{\lambda \in P(V)|\lambda + \eta \notin P(V) \text{ and } \forall \eta > o\}$

(6.9) Lemma Given $\lambda \in P(V)$, there exist $\eta \geq o$ such that $\lambda + \eta \in P_+(V)$. In particular $P_+(V) \neq 0$.

The proof proceeds as in [E3].

Let $T = \{\mu \in P(V)|\mu(d) = \lambda(d)\}$

Consider $W = \bigoplus_{\mu \in T} V_\mu$ which is $\mathfrak{g}(\mathfrak{g}, \mathfrak{g})$ module and integrable. As in the proof of Lemma 2.6 of [E3] one can prove $W$ is finite dimensional. Consider

$T_\lambda = \{\mu \in T|\lambda \leq_o \mu\}$ which is finite and hence has a maximal weight say $\gamma$. Now $\gamma - \lambda \geq 0$ and $(\gamma - \lambda)(d) = 0$. By (6.4) we have $\gamma - \lambda \geq o$. Take $\eta = \gamma - \lambda$. We need to prove $\lambda + \eta \in P_+(V)$. Suppose $\lambda + \eta + \eta_1 \in P(V)$ for $\eta_1 > o$. Then $\eta_1(d_o) = 0$

Then $\lambda + \eta + \eta_1 \in T$ and that contradict the fact that $\lambda + \eta$ is maximal. This proves $\lambda + \eta_1 \in P_+(V)$. Hence Lemma is proved.

(6.10) Lemma: Given $\lambda \in P_+(V)$, there exists $\lambda_1 = \lambda + P_1m_o\delta + \eta_1 \in P(V)$ for $\eta_1 \geq 0$ and a $\lambda_1$ weight vector $\mu_1$ such that $N_{aff}^+ u_1 = 0$.

The Lemma follows from the proof of Theorem (2.4) of [C]. The assumption of irreducibility is not needed for this part.

(6.11) Proposition: Suppose for any $\lambda \in P(V)$ there exists $\eta > 1, 0$ such that $\lambda + \eta \in P(V)$. Then there exists infinitely many $\lambda_i \in P(V), i \in \mathbb{Z}_+$ such that

1. $\lambda_i <_1 \lambda_{i+1}, \lambda_{i+1}(d_o) - \lambda_i(d_o) \in \mathbb{Z}_+$ for $i \in \mathbb{Z}_+$.

2. There exists $u_i \in V_{\lambda_i}$ such that $N_{aff}^+ u_i = 0$. The corresponding irre-
ducible module $V(\lambda_i) \subseteq V$.

3. There exists a common weight for all $V(\lambda_i)$.

**Proof** Choose $\lambda \in P_+(V)$. Then by Lemma (6.10) there exists $\lambda_1 = \lambda + P_1 m_\alpha \delta + \eta_1 \in P(V)$ for $P_1 \geq 0$ and $\eta_1 \geq 0$ and a weight vector $u_1$ of weight $\lambda_1$ such that $N^+_{aff} u_1 = 0$. Now by assumption there exists $\tilde{\eta}_1 > 0$ such that $\lambda_1 + \tilde{\eta}_1 \in P(V)$. Now by Lemma (6.9) there exists $\alpha \geq 0$ such that $\lambda_1 + \tilde{\eta}_1 + \alpha \in P(V)$. Now by Lemma (6.10) there exists $\lambda_2 = \lambda_1 + \tilde{\eta}_1 + \alpha + P_2 m_\alpha \delta + \eta_2 \in P(V)$. Where $P_2 \geq 0$ and $\eta_2 \geq 0$. Further there exists $\lambda_2$ weight vector $\mu_2$ such that $N^+_{aff} \mu_2 = 0$. Now by construction it is clear $\lambda_1 < \lambda_2$. We claim that $\lambda_2(d_o) - \lambda_1(d_o) > 0$. In any case $\lambda_2(d_o) - \lambda_1(d_o) \geq 0$ as $A = \tilde{\eta}_1(d_o) + \alpha(d_o) + P_2 m_\alpha + \eta_2(d_o) \geq 0$.

Suppose $A = 0$. By using 6.4 we see that $B = \tilde{\eta}_1 + \alpha + P_2 m_\alpha \delta + \eta_2 \in Q^+$. This contradict the fact that $\lambda_1 \in P_+(V)$ unless $B = 0$. But $B \neq 0$ as $\tilde{\eta}_1 > 0$. This proves our claim. Now by repeating infinitely many times we see (1) and (2).

Now we have $\lambda_i(d_o) = \lambda_1(d_o) + S_i$ where $\{S_i\}_i \geq 2$ is strictly increasing sequence of positive integer. It is easy to see that there exists $j \geq 0 \leq j < m_\alpha$ such that $S_i \equiv j(m_\alpha)$ for infinitely many indices. Denote such sequence by $S_{i_g}, g = 1, 2, \ldots$.

Write $S_{i_g} = f_{i_g} m_\alpha + j$ and can assume $f_{i_g} > 0$.

Let $\mu^j_o = \overline{\mu}_o + (\lambda_1(d_o) + j)\delta + \lambda_1(K)w$

Consider

$$\lambda_{i_g} - \mu^j_o = \overline{\lambda}_{i_g} - \overline{\mu}_o + f_{i_g} m_\alpha \delta > 0$$

This proves $\mu^j_o \leq \lambda_{i_g}$ and hence $\mu^j_o \in P(V(\lambda_{i_g}))$ by (6.8). For this subsequence (1), (2) and (3) are true.

**Proof of Theorem (6.2)**. Suppose $V$ is not a highest weight module. Then the assumption of Proposition 6.11 are true. Hence by the proposition 6.11, there exists infinitely many non-isomorphic irreducible modules with common weight. Then the dimension of the common weight is infinite dimensional. Contradicting the definition of weight module. See Definition 4.1.
Therefore $V$ is highest weight module. Let $\lambda$ be the top weight and put $N = V_\lambda$. Since $V$ is irreducible, by weight argument we see that $N$ is irreducible $L^\omega$- module. Thus $V \cong V(N)$.

7 Reduction to non-graded module

(7.1) Recall the Universal central extension of a Lie Tours $\tilde LT$ from Section 2 and $\mathcal{L}$ from section 3.

$\tilde LT = LT + D$ where $D$ is spanned by derivations $d_o, d_1, \cdots, d_n$. Now let $D$ be the span of $d_1, \cdots, d_n$. In this section, for an irreducible weight module $V$ of $\tilde LT$, we associate an irreducible weight module $V_1$ for $L$ (need not be unique). In the next section we will show how to recover the original module $V$ from the $\mathcal{L}$ module $V_1$. There by proving that the study of irreducible weight modules for $\tilde LT$ is reduced to the irreducible weight modules for $L$.

We generally refer $L$ modules as non-graded because the grade measuring $D$ is removed.

We fix an irreducible weight module $V$ of $\tilde LT$ for the rest of the section.

(7.2) Definition A linear map $z$ from $V \to V$ is called a central operator of degree $(k_o, k) \in \mathbb{Z}^{n+1}$ if $z$ commutes with $LT$ and $d_i z - zd_i = k_i z$ for all $i$.

For example $t_o^s t^r K_j$ is a central operator of degree $(r_o, r)$ and $t_o^s t^s K_i t_o^s t^r K_j$ is a central operator of degree $(r_o + s_o, r + s)$.

(7.3) Lemma: (a) Suppose $z$ is a central operator of degree $(k_o, k)$ and suppose $zu \neq 0$ for some $u$. Then $zw \neq 0$ for all $w$.

(b) Suppose $z$ is a non-zero central operator of degree $(k_o, k)$. Then there exists a non-zero central operators $T$ of degree $(-k_o, -k)$ such that $zT = Tz = Id$ on $V$.

(c) Suppose $z_1$ and $z_2$ are non-zero central operator of same degree. Then there exists a scalar $\lambda$ such that $z_1 = \lambda z_2$.

The proof are exactly as given in Lemma 1.7 and Lemma 1.8 of $[E4]$.

(7.4) Let $S = \{(k_o, k) \in \mathbb{Z}^{n+1} | t_o^{k_o} t^k K_i \neq 0 \text{ for some } i \} \subseteq \Gamma \oplus \Gamma_o$.

Let $\langle S \rangle$ be the subgroup of $\Gamma \oplus \Gamma_o$ generated by $S$. Clearly rank
\( < S > = P \leq n + 1. \) Note that for any \((k, o, k) \in < S >\) there exists a nonzero central operator of degree \((k, o, k)\). Let \(e_i = (0, \cdots, 1, \cdots 0), 0 \leq i \leq n\) be the standard \(\mathbb{Z}\)-basis of \(\mathbb{Z}^{n+1}\). Now it is standard fact that up to a choice of co-ordinates, there exists positive integer \(l_1 e_i, \cdots l_n, m_i | l_i\) such that \(\{l_i e_i, i \geq n - P + 1\}\) is a \(\mathbb{Z}\)-basis for \(< S >\).

(7.5) Proposition  Notation as in (7.2) and (7.4).

1. There exists non-zero central operators \(z_i, i \geq n - P + 1\) of degree \(l_i e_i\).
2. \(P < n + 1\)
3. Suppose \(t^k i t^k I_i \neq 0\) on \(V\). Then \(k = k = \cdots k_{n-p} = 0\) and \(i \leq n - P\).
4. There exists \(\mathcal{L}T \oplus D_p\) proper submodule \(W\) of \(V\) such that \(V/W\) is a weight module for \(\mathcal{L}T + D_p\) with respect to \(h(o) + \sum \mathbb{C} k_i \oplus D_p\). Here \(D_p\) is space spanned by \(d_o, d_1, \cdots d_{n-p}\).
5. We can choose \(W\) such that \(\{z_i v - v | v \in V\} \subseteq W\).

The proof is exactly as given in Theorem (4.5) of [E3].

(7.6) Proposition:  Notation as in (7.2) and suppose \(t^k i t^k I_j \) acts non-trivially on \(V\) for some \(j\) and for some \((k, o, k) \in \mathbb{Z}^{n+1}\) then \(P = n\).

The proof is similar to Theorem 1.10 of [E4]. We record the following which is of independent interest.

(7.7) Proposition: Suppose \(\tilde{V}\) is any weight module for \(\tilde{\mathcal{L}}T\) such that \(t^k i t^k I_i \) is zero on \(\tilde{V}\) for \((k, o, k) \neq (0, 0)\) and any \(i\). Then each \(I_i\) acts trivially on \(V\). This follows from Corollary 2.14 [E4]. Just note that \(\tilde{V}\) is a module to the toroidal Lie algebra. \(g(\mathfrak{g}, \mathfrak{g}) \otimes \mathbb{C}[t^\pm m_n, \cdots t^\pm m_o] \oplus z(m, m) \oplus D\)

Also note that we are not assuming \(\tilde{V}\) is irreducible.

(7.8) We have started with an irreducible weight module \(V\) for \(\tilde{\mathcal{L}}T\). Then we have proved it is actualy a module for \(\mathcal{L} \oplus D\). See (3.2) and observe \(\ker \varphi\) is trivial on \(V\) up to a choice of co-ordinates.

Note that \(D_n = \mathbb{C}d_o\) and \(V\) contains \(\mathcal{L}\) submodule \(W\). We can assume that \(\{z_i v - v | v \in v, i \geq 1\}\) \(\subseteq W\)

(7.9) Proposition: Notation as above. \(V\) contains maximal \(\mathcal{L}\) submodule
8 Passage from non-graded to graded

The main ideas come from \([E4]\) but arguments differ.

(8.1) We will first record some general results on weight modules. Let \(G\) be any Lie algebra and let \(H\) be ad-dionalizable sub algebra of \(G\).

Let \(G = \bigoplus_{\alpha \in H^*} G_{\alpha}\) where \(G_{\alpha} = \{x \in G|[h, x] = \alpha(h)x, \forall h \in H\}\) Let \(U\) be the Universal enveloping algebra of \(G\).

1. Write \(U = \bigoplus_{\eta \in H^*} U_{\eta}, U_{\eta} = \{x \in U|[h, x] = \eta(h)x\forall h \in H\}\)

   Clearly \(U_o\) is a subalgebra of \(U\) and contains \(H\).

2. Let \(J\) be an irreducible weight module for \(G\).

   Write \(J = \bigoplus_{\lambda \in H^*} J_{\lambda}, J_{\lambda} = \{v \in J|h v = \lambda(h) v, \forall h \in H\}\)

   Then it is standard fact that each \(J_{\lambda}\) is an irreducible module for \(U_o\).

Fix \(\lambda \in H^*\) such that \(J_{\lambda} \neq 0\) and consider the induced module

\[ M(\lambda) = U \bigotimes_{U_o} J_{\lambda} \]

from the universal properties of induced modules, the following can be seen.

(8.2) Proposition

1. \(M(\lambda)_{\lambda} \cong J_{\lambda}\) as \(U_o\)- modules.

2. \(M(\lambda)\) has unique irreducible quotient \(V(\lambda)\) and \(V(\lambda)_{\lambda} \cong J_{\lambda}\) as \(U_o\)- modules.

3. For any \(G\) weight modules \(W\) such that \(W_{\lambda} \cong J_{\lambda}\) as \(U_o\)- modules, then \(W\) is a quotient of \(M(\lambda)\).

(8.3) Corollary: Suppose \(J\) and \(W\) are irreducible. \(G\) weight modules such that \(J_{\lambda} \cong W_{\lambda}\) as \(U_o\) modules. Then \(J \cong W\) as \(G\)- modules.
We will now continue with Section 7. We have an irreducible weight module \( V \) for \( \tilde{L}T \). We have non-zero central operators \( z_1, \ldots, z_n \) of degree \( l_i e_i, m_i | l_i \). We also have a maximal \( \mathfrak{L} \) submodule \( W \) of \( V \) and \( V_1 = V/W \) is irreducible weight module. We assume \( W \) contains the space spanned by \( \{ z_i v - v | v \in v, 1 \leq i \leq n \} \). Also we have the group \( < S > = \mathbb{Z}l_1 \oplus \cdots \oplus \mathbb{Z}l_n \subseteq \Gamma \). For every \( s \in < S > \), there is a non-zero central operator of degree \( s \).

Let \([S]\) be the space of such operators. As operators on \( V, K \otimes A(m) \subseteq [S] \).

Equality may not hold. Let \( \overline{G} = \overline{L}T + [S] \oplus \mathbb{C}d_o \) which is a Lie algebra in an obvious sense and contains \( \mathfrak{L} \).

Let \( G = \overline{G} \oplus D \) and contains \( \tilde{L}T \).

Let \( H = \mathfrak{h}(o) \oplus \sum \mathbb{C}K_i + \mathbb{C}d_o + D \). We will note that \( V \) is irreducible \( G \) module and \( V_1 = V/W \) is irreducible \( \overline{G} \) module. Write \( U = U(\overline{G}) \) and \( U = U(\overline{G}) \) which are Universal enveloping algebra of \( \overline{G} \) and \( G \).

Note that by our assumption, each \( z_i \) acts as 1 on \( V_1 \). It is easy to see that \( z_i^{-1} \) also acts as 1 on \( V_1 \). Suppose \( z \) is any non-zero Central operator of degree \( k \in < S > \). Then \( z = \lambda \Pi z_i^{P_i}, P_i \in \mathbb{Z}, \lambda \neq 0 \) and \( k = (P_1, \ldots, P_n) \). This can be seen from 7.3(c).

Let \( \mathfrak{L}(V_1) = V_1 \otimes A(m) \) and define \( G \) module action on \( (\mathfrak{L}(V_1), \Pi(\alpha)) \) where \( \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n \). Recall that \( \overline{G} \) is \( \mathbb{Z}^n \)- graded via \( D \).

Write \( \overline{G} = \bigoplus_{k \in \mathbb{Z}^n} \overline{G}_k \). Recall that \( V_1 \) is \( \overline{G} \) module.

Define
\[
X.v(r) = (Xv)(r + k), X \in \overline{G}_k, \\
d_i v(r) = (r_i + \alpha_i)v(r), r \in \mathbb{Z}^n.
\]

It is easy to see that \( \mathfrak{L}(V_1) \) is a \( G \) module and each central operator from \( [S] \) as invertible operators. Any \( \lambda \in \tilde{\mathfrak{h}}(o)^* \) can be extended to \( \tilde{\mathfrak{h}} \) as \( \lambda(D) = 0 \). Recall that \( \delta_r \in \tilde{\mathfrak{h}} \) from (2.4).

Clearly \( \mathfrak{L}(V_1)_{\lambda + \delta_r} = (V_1)_{\lambda} \otimes t^r \)

which is clearly finite dimensional. Thus \( \mathfrak{L}(V) \) is a \( G \) weight module.

In this section our aim is to prove that \( \mathfrak{L}(V_1) \) is direct sum of finitely many irreducible \( G \) modules. All components are isomorphic upto grade shift. Further we will prove that one of the components is isomorphic to \( V \) as \( G \)-
modules for suitable $\alpha$. There by recovering the original module $V$ from $V_1$.

(8.6) Consider $\rho : \mathfrak{L}(V_1) \to V_1$ defined by $\rho(v \otimes t^r) = v$ which is a surjective $G$-module map. Let $v(r) = v \otimes t^r$.

(8.7) Lemma Suppose $W_1$ is a non-zero submodule of $\mathfrak{L}(V_1)$. Then
1. $\rho(W_1) = V_1$
2. Suppose $\lambda \in P(V_1)$ then $\lambda + \delta_k \in P(W_1)$ for some $k \in \mathbb{Z}^n$.

Proof Since $V_1$ is irreducible, to see (1) it is sufficient to prove that $\rho(W_1) \neq 0$. But $W_1$ is a $D$-module and hence contains vectors of the form $v(k)$. Then $P(v(k)) = v \neq 0$.

To see (2), let $v \in V_1$ of weight $\lambda \in \tilde{h}(o)^*$. Then there exists $w \in W_1$ such that $\varphi(w) = v$ by (1). Write $w = \sum w_i(\gamma^i)$ for some $w_i \in V_1$ and $\gamma^i \in \mathbb{Z}^n$. Each $w_i$ is weight $\lambda$. As $w_i$ is $D$-module, $w_i(\gamma^i) \in W_1$ for all $i$. Thus $\lambda + \delta_{\gamma^i} \in P(W_1)$.

(8.8) Let $W^1$ be a non-zero submodule of $\mathfrak{L}(V_1)$. Then for any $z \in [S]$ of degree $k$, we have $zW^1_\lambda = W^1_{\lambda + \delta_k}$. This is because $W^1$ is $[S]$ invariant and $z$ is invertible.

(8.9) Fix a $\lambda \in P(V_1)$ and define $K(W^1) = \sum_{0 \leq s_i < l_i} \dim(W^1_{\lambda + \delta_s})$ for any submodule $W^1$ of $\mathfrak{L}(V_1)$. We claim that $K(W^1) > 0$ for non-zero submodule $W^1$. From Lemma 8.7(a), $\lambda + \delta_k \in P(W^1)$ for some $k \in \mathbb{Z}^n$. Now using (8.8) we can assume $0 \leq k_i < l_i$. This proves $K(W^1) > 0$. Suppose $W^1 \subseteq W^2$ are submodules then clearly $K(W^1) \leq K(W^2)$.

(8.10) Lemma Let $W^1$ be a non-zero $G$ submodule of $\mathfrak{L}(V_1)$. Then $W^1$ contains an irreducible $G$-submodule.

Proof All submodules consider here are non-zero $G$-modules. Choose a submodule $W^2$ of $W^1$ such that $K(W^2)$ is minimal.

Let $Q = \bigoplus_{0 \leq k_i < l_i} W^2_{\lambda + \delta_k}$ and let $\tilde{W}$ is a submodule generated by $Q$.

Clearly $K(\tilde{W}) = K(W^2)$. We claim that $\tilde{W}$ is irreducible. For that let $W^3$ a submodule of $\tilde{W}$. We have $\bigoplus_{0 \leq k_i < l_i} W^3_{\lambda + \delta_k} \subseteq \bigoplus_{0 \leq k_i < l_i} W^2_{\lambda + \delta_k}$. By minimality we know $K(W^3) = K(W^2)$ and hence equality holds above. Thus $W^3$ contains $Q$ and hence contains $\tilde{W}$. Thus $W^3 = \tilde{W}$. This proves that $\tilde{W}$ is irreducible.

For $v \in V_1$ and $k \in \mathbb{Z}^n$ let $Uv(k)$ be the $G$ submodule of $\mathfrak{L}(V_1)$ generated by
(8.11) Proposition} There exists a weight vector $v$ in $V_1$ such that

1. $Uv(k)$ is irreducible $G$-module for all $k \in \mathbb{Z}^n$

2. $\sum_{k \in \mathbb{Z}^n} UV(k) = L(V_1)$

3. $\sum_{0 \leq k < l_i} UV(k) = L(V_1)$.

Before we prove the proposition we note the following.

(8.12) Remark} Let $v$ be a weight vector of $V_1$ and let $r$ and $s \in \mathbb{Z}^n$.

Consider the map

$$\varphi : Uv(r) \to Uv(s)$$

by $\varphi(w(k)) = w(k + s - r)$ for $w(k) \in Uv(r)$ where $w \in V_1$ and $k \in \mathbb{Z}^n$. It is easy to check that $\varphi$ is $G$-module isomorphism. But need not be $G$-module map. Suppose $D$ acts on $v(r)$ by $\alpha \in \mathbb{C}^n$. That means $d_i v(r) = \alpha_i v(r)$. Suppose $D$ acts on $v(s)$ by $\beta$. Then it is easy to see that $Uv(r) \cong Uv(s) \otimes \mathbb{C}$ as $G$-modules where $\mathbb{C}$ is a one dimensional module for $G$. $\overline{G}$ acts trivially on $\mathbb{C}$ and $D$ acts as $\beta - \alpha$.

This is what we call $Uv(r)$ isomorphic to $Uv(s)$ as $G$-modules upto a grade shift. Now it is easy to see that $Uv(r)$ is irreducible if $Uv(s)$ is irreducible as $G$-modules.

**Proof of the Proposition (8.11).** Note that (3) follows from (2) as $Uv(k) = Uv(k + r)$ for any $r \in < S >$. One can use central operators of degree $r$. See (8.8).

1. Let $W^1$ be an irreducible $G$-module of $\mathcal{L}(V_1)$ (Lemma 8.10). From the proof of Lemma (8.7)(1) we see that $W^1$ contains a vector $v(k)$ for some $k \in \mathbb{Z}^n$ and some weight vector $v$ in $V_1$. Now $Uv(k) \subseteq W^1$ and hence $Uv(k)$ is irreducible as it is contained in an irreducible module. Now by Remark (8.12) it follows that $Uv(s)$ is irreducible for any $s \in \mathbb{Z}^n$.

2. Let $w(s) \in \mathcal{L}(V_1)$ for $w \in V_1$ and $s \in \mathbb{Z}^n$. Since $V_1$ is $\overline{G}$ irreducible and $v \in V_1$ there exists $X \in \overline{U}$ such that $Xv = w$. Write $X = \sum X_{k_i}$ where $D$ acts on $X_{k_i}$ by $k^i$.

Consider $\sum \pi(\alpha)X_{k_i}v(s-k_i) = \sum(X_{k_i}v)(s) = w(s)$ (recall the definition
of $G$ action on $L(V_1)$. $L(V_1) \subseteq \sum_{s \in \mathbb{Z}^n} Uv(s)$ and hence we have (2). This completes the proof of the Proposition (8.11).

(8.13) Theorem

1. $(L(V_1), \pi(\alpha))$ is completely reducible as $G$-modules and the number of components are finite. Further all components are $G$-isomorphic upto grade shift.

2. For a suitable $\alpha$, a component of $(L(V_1), \pi(\alpha))$ is isomorphic to $V$ as $G$-modules.

Proof We have already seen that $\sum_{0 \leq s_i < l_i} Uv(s) = L(V_1)$ for some weight vector $v$ in $V_1$. Let $T = \{s \in \mathbb{Z}^n | 0 \leq s_i < l_i\}$

Suppose for some $\gamma$, $Uv(\gamma) \cap \sum_{s \in T, s \neq \gamma} Uv(s) \neq 0$. Since $Uv(\gamma)$ is irreducible it follows that $Uv(\gamma) \subseteq \sum_{s \in T, s \neq \gamma} Uv(s) \neq 0$. Thus we have reduced one components from the sum. By repeating this finitely many times we see that $L(V_1)$ is direct sum of fewer terms. This proves 1.

2. Recall that $U$ and $\overline{U} = U_\sigma$ are Universal enveloping algebra of $G$ and $\overline{G}$ respectively. Recall the notation from (8.1). We have $U_\sigma = \overline{U}_\sigma.U(D)$. Consider the weight space $V_{\lambda+\delta,} \subseteq V$ which is irreducible $G$-module. As noted earlier $V_{\lambda+\delta,}$ is an irreducible $U_\sigma$-module and remains irreducible as $\overline{U}_{\sigma}$ as $U(D)$ acts as scalars. Recall the map $\rho : L(V_1) \rightarrow V_1$ from (8.6). Now $\rho$ restricted to $V_{\lambda+\delta,}$ is injective as ker $\rho$ cannot contain $h$ weight vectors.

Now $\rho(V_{\lambda+\delta,}) \subseteq (V_1)_{\lambda}$ is an irreducible $\overline{U}_\sigma$ module and $\rho$ is a $\overline{U}_\sigma$ module map.

Therefore $\rho(V_{\lambda+\delta,}) = (V_1)_{\lambda}$ and hence $\rho(V_{\lambda+\delta,}) \cong (V_1)_{\lambda}$ as $\overline{U}_{\sigma}$-modules.

As $V_1 \subseteq L(V_1)$ we see that $(V_1)_{\lambda} = L(V_1)_{\lambda} = \oplus (Uv(\gamma))_{\lambda}$ as $\overline{U}_{\sigma}$-modules. (See (1)).

Each $(Uv(\gamma))_{\lambda}$ is $U_\sigma$ irreducible and remains irreducible as $\overline{U}_{\sigma}$-modules. This proves $V_{\lambda+\delta,} \cong (Uv(s))_{\lambda}$ as $\overline{U}_{\sigma}$-modules for some $s$.

We know that $D$ acts as scalars on $V_{\lambda+\delta,}$ and $(Uv(s))_{\lambda}$. We can choose $\alpha$ in such a way, $D$ action is same on $V_{\lambda+\delta,}$ and $(Uv(s))_{\lambda}$. Thus for the special
choice of $\alpha$ we see that $V_{\lambda+\delta_s} \cong (Uv(s))_\lambda$ as $U_\alpha$-modules. Now by corollary (8.3) it follows that $V \cong Uv(s)$ as $G$-modules. This completes the (2) of the Theorem.

References

[AABGP] Allison,B., S. Azam, S. Berman, Y. Gao and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc., 126(603),1997.

[ABF] Allison, B., S. Berman and A. Pianzola, *Multiloop Algebras, Iterated loop algebras and Extended Affine Lie algebras of Nullity 2.*, arxiv: 1002.2674, [Math RA.]

[ABFP1] Allison, B., S. Berman, J. Faulkner and A. Pianzola, *Multiloop realization of Extended Affine Lie algebras and the Lie Tori*, Trans. Amer. Math. Soc., 361(2009), 4807-4842

[ABFP2] Allison, B., S. Berman, J. Faulkner and A. Pianzola, *Realization of graded simple Lie algebras*, Forum Math. 20(2008), 395-432.

[AB] Allison, B., *Some isomorphism invariants for Lie Tori*, Journal of Lie Theory 22(2012), No.1, 163-204.

[AF] Allison, B. and J. Faulkner, *Isotopy for Extended Affine Lie algebras and Lie Tori*, Developments and trends in infinite dimensional Lie theory, 3-43, Progress. Math. 288, Birkhavser Boston, Inc. Boston, M.A, 2011.

[AZ] Azam, S., *Generalized reductive Lie algebras: Connection with Extended Affine Lie algebras and Lie Tori*, Canad. J. Math. 58(2006), 225-248.

[B] Batra, P., *Representations of twisted multiloop algebras*, Journal of algebra, 272(2004), No. 1, 404-416.
[C] Chari, V., *Integrable representations of Affine Lie algebras*, Invent. Math., *85*(1986), 317-335

[E1] Eswara Rao, S., *Iterated loop modules and a filtration for Vertex representation of Toroidal Lie algebras*, Pacific Journal of Mathematics, *171*, No.2 (1995), 511-528.

[E2] Eswara Rao, S., *Classification of loop modules with finite dimensional weight spaces*, Math. Ann., *305*(1996), 651-663.

[E3] Eswara Rao, S., *Classification of irreducible integrable modules for Toroidal Lie algebras with finite dimensional weight spaces*, Journal of Algebra, *277*(2004), 318-348.

[E4] Eswara Rao, S., *Irreducible representation for Toroidal Lie algebras*, Journal of Pure and Applied Algebra, *202*(2005), 102-117.

[EB] Eswara Rao, S. and Batra, P. *Classification of irreducible integrable modules for twisted Toroidal Lie algebras with finite dimensional weight spaces*, Pacific Journal of Mathematics, *237*(2008), No.1, 151-181.

[EM] Eswara Rao, S. and Moody, R.V., *Vertex representations for n-toroidal Lie algebras and a generalization of Virasoro algebra*, Commun. Math. Phys., *159-(1994)*, 239-264.

[EMY] Eswara Rao, S., Moody, R.V. and Yokonuma, T., *Toroidal Lie algebras and Vertex representations*, Geom. Dedicata, *35*(1990), 283-307.

[EZ] Eswara Rao and K. Zhao, *Integrable representations of toroidal Lie algebras co-ordinated by rational quantum tori*, Journal of Algebra, *361*(2012), 225-247.

[EN1] Erhard Neher, *Extended affine Lie algebras- an Introduction to their structure theory*, Geometric representation theory and extended affine Lie algebras, 107-167, Fields Inst. commun., *59*, Amer. Math. Soc., Providence, RI, 2011.
[EN2] Erhard Neher, *Lie Tori*, C.R. Math. Acad. Sci. Soc. R. Can., **26**(2004), 84-89.

[EN3] Erhard Neher, *Extended Affine Lie Algebras*, C.R. Math. Acad. Sci. Soc. R. Can., **26**(2004), No.3, 90-96.

[H] Humphreys, J.E., *Introduction to Lie algebras and Representation theory*, Berlin, Heidelberg, New York, 1972.

[JS] Jie Sun, *Universal central extensions of twisted forms of split simple Lie algebras over rings*, Journal of Algebra, **322**(2009), No.5, 1819-1829.

[K] Kac, V.G., *Infinite dimensional Lie algebras*, 3rd ed. Cambridge University Press, 1990.

[Ka] Kassel, C., *Kahler differentials and coverings extended over a commutative algebra*, Journal of Pure and Applied Algebra, **34**(1984), 265-275.

[ML] Michael Lau, *Representation of Multiloop algebras*, Pacific Journal of Mathematics, **245**, No.1, (2010), 167-184.

[NK] Naoi, K., *Multiloop Lie algebras and the construction of Extended Affine Lie algebras*, Journal of Algebra, **323**, No. 8 (2012), 2103-2129.

[XT] Chang, Xuewu and Shaobin Tan, *A class of irreducible integrable modules for the extended baby TKK algebra*, Pacific Journal of Mathematics, **252**, No. 2(2011), 293-312.

[Y] Yoshii, Y., *Lie Tori- a simple characterization of Extended Affine Lie algebras*, Publ. Res. Inst. Math. Sci. **42**(2006), 739-762.