REMARKS ON DIMENSION OF UNIONS OF CURVES

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Abstract. We study an analogue of Marstrand’s circle packing problem for curves in higher dimensions. We consider collections of curves which are generated by translation and dilation of a curve $\gamma$ in $\mathbb{R}^d$, i.e., $x + t\gamma, (x, t) \in \mathbb{R}^d \times (0, \infty)$. For a Borel set $F \subset \mathbb{R}^d \times (0, \infty)$, we show the unions of curves

$$\bigcup_{(x,t) \in F} (x + t\gamma(I))$$

has Hausdorff dimension at least $\alpha + 1$ whenever $F$ has Hausdorff dimension bigger than $\alpha$, $\alpha \in (0, d-1)$. We also obtain results for unions of curves generated by multi-parameter dilation of $\gamma$. One of the main ingredients is a local smoothing type estimate (for averages over curves) relative to fractal measures.

1. Introduction

Let $\gamma$ be a smooth curve from $I_\circ = [0, 1]$ to $\mathbb{R}^d$, $d \geq 2$. For a given $\gamma$, we consider the curves given by translation and dilation of $\gamma(I) := \{\gamma(s) : s \in I \subset \subset I_\circ\}$. For a Borel set $F \subset \mathbb{R}^d \times (0, \infty)$, denote

$$\Gamma(F) := \bigcup_{(x,t) \in F} (x + t\gamma(I)),$$

where $(x, t) \in \mathbb{R}^d \times (0, \infty)$. In this paper, we are concerned with the problem of determining how small the dimension of $\Gamma(F)$ can be depending on that of $F$.

This problem has a rich history when $\gamma$ is the unit circle in $\mathbb{R}^2$. Besicovitch–Rado [4] and Kinney [15] showed that there exists a set (so-called BRK set) of Lebesgue measure zero which contains a circle of every diameter, i.e., $\Gamma(F)$ with $F = \{(X(t), t) : t \in (0, \infty)\}$ for some function $X : \mathbb{R}^+ \to \mathbb{R}^2$ (see also [8, 18] for further generalization). In the same vein, it is natural to ask whether or not there exists a plane set of measure zero containing circles with centers at every point, i.e., $\Gamma(F)$ with $F = \{(x, T(x)) : x \in \mathbb{R}^2\}$ for a function $T : \mathbb{R}^2 \to \mathbb{R}^+$ (see [19] or [9, p.105]). Marstrand [20] showed that such a set cannot be of Lebesgue measure zero.

Those problems for the circle in $\mathbb{R}^2$ generalize naturally with subsets $F \subset \mathbb{R}^2 \times (0, \infty)$. Let $\text{Proj}_x F$ be the orthogonal projection of $F$ onto $x$-space. It was proved in [20] that if $\text{Proj}_x F$ has positive Lebesgue measure, then so does $\Gamma(F)$ (see also [4]). Later, Mitsis [22] showed that it is sufficient for $\Gamma(F)$ being of positive measure that $\dim_H(\text{Proj}_x F) > 3/2$. Here, $\dim_H$ denotes the Hausdorff dimension. It was also conjectured that $\Gamma(F)$ has positive Lebesgue measure if $\dim_H(\text{Proj}_x F) > 1$. This condition is optimal in view of Talagrand’s construction [31] of a measure zero set which contains a circle centered at every point on a straight line. The conjecture was later proved by Wolff [33]. In fact, a stronger result was obtained: For any set
$F \subset \mathbb{R}^2 \times (0, \infty)$ satisfying $\dim_H F > 1$ (not necessarily $\dim_H (\text{Proj}_x F) > 1$), any set containing $\Gamma(F)$ can not be of Lebesgue measure zero.

The main object of this paper is to study an extension of the aforementioned problem to space curves in $\mathbb{R}^d$, $d \geq 3$. Similar problems for non-planar curves are less well understood (see \cite{25,17} for partial results). We assume that $\gamma$ is nondegenerate, namely,

\begin{equation}
\det (\gamma'(s) \cdots \gamma^{(d)}(s)) \neq 0, \quad \forall s \in I_0.
\end{equation}

The next theorem is our first main result, which generalizes the circle packing problem to nondegenerate curves in $\mathbb{R}^d$, $d \geq 3$.

**Theorem 1.1.** Let $d \geq 3$ and $\gamma$ be a nondegenerate curve in $\mathbb{R}^d$. Suppose that a Borel set $F$ satisfies $\dim_H F > \alpha$ for some $0 < \alpha \leq d - 1$. Then,

\begin{equation}
\dim_H E \geq \alpha + 1
\end{equation}

holds whenever a Borel set $E \subset \mathbb{R}^d$ contains $\Gamma(F)$.

In $\mathbb{R}^2$, K"aenm"aki–Orponen–Venieri \cite{14} provided a geometric proof of \cite{163} when $F$ is an analytic set.

Theorem 1.1 is optimal in that $\dim_H (\Gamma(F)) \leq \dim_H F + 1$ for a Borel family $\Gamma(F)$, as can be easily seen by the standard covering argument. The case that $\Gamma(F)$ has positive Lebesgue measure was of special interest. In \cite{17} Corollary 1.6], it was shown that $\Gamma(F)$ has positive Lebesgue measure if $\dim_H F > d - 1$. Conversely, modifying Kinney’s example \cite{13}, one can show that the condition $\dim_H F > d - 1$ is necessary, in general, for $\Gamma(F)$ to have positive Lebesgue measure. See Proposition 4.1.

Our result also extends to curves of different type. We say that $\gamma$ is of finite type if for each $s \in I_0$, there exists a $d$-tuple of positive integers $(a_1, \ldots, a_d)$ such that

\[ \det (\gamma^{(a_1)}(s) \cdots \gamma^{(a_d)}(s)) \neq 0. \]

Theorem 1.1 also continues to hold for finite type curves, since a finite type curve contains a nondegenerate sub-curve. See \cite{25,17} for a previous result for finite type curves. One can easily see that Theorem 1.1 does not hold in general if the finite type condition is not satisfied. For example, consider the curve $\gamma$ contained in a $k$-dimensional linear subspace $V_k \subset \mathbb{R}^d$ for $2 \leq k \leq d - 1$ such that $\gamma'(s), \ldots, \gamma^{(k)}(s)$ spans $V_k$ for all $s \in I_0$. For $\varepsilon \in (0, 1/2)$, let $F_k \subset V_k \times (0, \infty)$ such that $\dim_H F_k = k - 1 + 2\varepsilon$. Then $\dim_H \Gamma(F_k) \leq k$ since $x_k + t\gamma(I) \subset V_k$ for any $(x_k, t) \in F_k$. Let $G_k \subset V_k^s$ such that $\dim_H G_k = \dim_B G_k = 1 - \varepsilon$. Here, $\dim_B$ denotes the upper box counting dimension. Clearly, $\dim_H (F_k + G_k) > k$. If we take $F = F_k + G_k$, it follows that $\dim_H \Gamma(F) \leq \dim_H \Gamma(F_k) + \dim_H G_k < k + 1$ even if $\dim_H F > k$.

**Averages over curves.** Our proof of Theorem 1.1 relies on local smoothing estimates relative to fractal measures for averaging operators rather than geometric approach used in some of the previous works (see \cite{32,22,33,14}).

We consider an averaging operator given by

\[ A_{\psi} f(x, t) = \chi(t) \int f(x + t\gamma(s))\psi(s) \, ds, \]

where $\chi \in C_0^\infty((1/2, 4))$ and $\psi$ is a nonnegative smooth function satisfying $\psi = 1$ on $I$ and $\text{supp} \, \psi \subset I_0$. 

Definition 1.2. Let $B_d(z, \rho)$ denote the ball of radius $\rho$ centered at $z$ in $\mathbb{R}^d$. For a non-negative Borel measure $\mu$ defined on $\mathbb{R}^d$ we say $\mu$ is $\alpha$-dimensional if there is a constant $C_\mu$ such that

$$
\mu(B_d(z, \rho)) \leq C_\mu \rho^\alpha, \quad \forall (z, \rho) \in \mathbb{R}^d \times \mathbb{R}^+
$$

for some $\alpha \in (0, d]$. We denote by $\mathcal{C}^d(\alpha)$ the set of all $\alpha$-dimensional measures in $\mathbb{R}^d$. For $\mu \in \mathcal{C}^d(\alpha)$, we set

$$
\langle \mu \rangle_\alpha := \sup_{z \in \mathbb{R}^d, \rho > 0} \rho^{-\alpha} \mu(B_d(z, \rho)).
$$

It is well-known that Marstrand’s results [20] can be deduced from $L^p$, $p \neq \infty$, estimate for the circular maximal function. In [12], some of the authors extended the $L^p$ circular maximal estimate to that relative to $\alpha$-dimensional measures, which recovers the aforementioned Wolff’s result in [33]. Moreover, $L^p$ maximal bound was extended to space curves (see [28, 16, 1, 17]). Various forms of local smoothing estimates played important roles in proving those results. Similarly, to prove Theorem [14], we make use of local smoothing estimate relative to $\alpha$-dimensional fractal measures on $\mathbb{R}^{d+1}$:

$$
(1.4) \quad \|A_\gamma f\|_{L^p(\mathbb{R}^d)} \leq C\langle \mu \rangle_{\frac{\alpha}{d}}^\frac{1}{p} \|f\|_{L^p(\mathbb{R}^d)}.
$$

Here, $\|f\|_{L^p(\mathbb{R}^d)} := \|(1 - \Delta)^{\frac{\alpha}{p}} f\|_{L^p(\mathbb{R}^d)}$. Let us set $p(d) = 4$, if $d = 2$, and $p(d) = 4d - 2$, if $d \geq 3$.

Theorem 1.3. Let $d \geq 2$, $0 < \alpha \leq d + 1$, and $\mu \in \mathcal{C}^{d+1}(\alpha)$. Suppose that $\gamma$ is a smooth curve satisfying (1.2). If $p > p(d)$, then the estimate (1.4) holds for

$$
\sigma > (d - 1 - \alpha)/p.
$$

Considering a specific function and $\alpha$-dimensional measure, one can show that $\sigma \geq (d - 1 - \alpha)/p$ is optimal for (1.4) to hold. See Proposition [4.2] with $m = 1$.

Multi-parameter dilation. The approach for curves can be further generalized to other submanifolds via the corresponding local smoothing estimates for the associated averaging operators. In particular, we can extend Theorem [1.1] to families of curves generated by multi-parameter dilations.

Let $1 \leq m \leq d$ and denote $t = (t_1, \ldots, t_m)$, $t_i \in (1, 2)$. For an onto map $\omega$ from $\{1, \ldots, d\}$ to $\{1, \ldots, m\}$, we define a family of nondegenerate curves $\gamma^\omega_t$ generated by $m$ parameters by

$$
(1.5) \quad \gamma^\omega_t(s) = (t_{\omega(1)} \gamma_1(s), \ldots, t_{\omega(d)} \gamma_d(s)).
$$

For $F' \subset \mathbb{R}^d \times (1, 2)^m$, we obtain a lower bound of Hausdorff dimension of

$$
\Gamma^\omega(F') = \bigcup_{(x, t) \in F'} (x + \gamma^\omega_t(I)).
$$

Theorem 1.4. Let $d \geq 2$, $2 \leq m \leq d$, and $0 < \alpha < d + m - 2$. Suppose $F' \subset \mathbb{R}^d \times (1, 2)^m$ is a Borel set with $\dim_H F' > \alpha$. Then,

$$
(1.6) \quad \dim_H E \geq \max\{\alpha + 2 - m, 0\}
$$

holds whenever a Borel set $E$ contains $\Gamma^\omega(F')$. 
When \( d = 3 \) and \( m = 2 \), a special case was considered by Pramanik and Seeger [28]. It was shown \([28, \text{Proposition 6.1}]\) that the union of a two-parameter family of helices \( \gamma_{t_1,t_2}(s) = (t_1 \cos(2\pi s), t_1 \sin(2\pi s), t_2 s) \) has Hausdorff dimension at least \( 8/3 \). Theorem 1.4, in particular, proves the optimal result that the union of \( \gamma_{t_1,t_2} \) has Hausdorff dimension 3. More generally, note that \( \dim_H E = d \) if \( \dim_H F' > d + m - 2 \). Furthermore, from Theorem 2.1, one can show \( E \) is of positive Lebesgue measure if \( \dim_H F' > d + m - 2 \) (see Remark 4 in p. 8). This is sharp in that, for \( 1 \leq m \leq d \), there exists a set \( E \) of Lebesgue measure zero containing \( \Gamma^\omega(F') \) generated by a nondegenerate curve \( \gamma \) while \( \dim_H F' = d + m - 2 \) (see Proposition 4.1).

**Relation to local smoothing estimates for averages.** For \( 2 \leq k \leq d - 1 \), one may consider a similar packing problem with \( k \)-dimensional submanifolds under a suitable curvature condition. The problem for the case \( k = d - 1 \) which includes spheres and hyperplanes is relatively well known (see \([18, 22, 25, 26, 27]\)).

Let \( \mathcal{S} \subset \mathbb{R}^d \) be a smooth compact \( k \)-dimensional submanifold. Let \( dm \) denote the surface measure on \( \mathcal{S} \) and consider

\[
A f(x,t) = \chi(t) \int_{\mathcal{S}} f(x + ty) \, dm(y).
\]

It is known that the averaging operator has smoothing properties under a suitable curvature condition on \( \mathcal{S} \). There are various model cases for which \( f \to A f(\cdot,t) \) with a fixed \( t \neq 0 \) is bounded from \( L^p(\mathbb{R}^d) \) to \( L^p_\sigma(\mathbb{R}^d) \) for some \( \sigma < 0 \) and the best possible regularity exponent is \( \sigma = -k/p \) (e.g., see \([23, 29, 2, 17]\)). Taking additional integration over \( t \) allows an extra regularity gain on a certain range of \( p \). This phenomenon is known as local smoothing. It is not difficult to see the smoothing order can not exceed \((k+1)/p\) (e.g., see Section 4.2). We refer to as sharp local smoothing estimate the following:

\[
(1.7) \quad \|A f\|_{L^p_\sigma(\mathbb{R}^{d+1})} \lesssim \|f\|_{L^p_\sigma(\mathbb{R}^d)}, \quad \sigma > -(k+1)/p.
\]

If \( k = 1 \), \((1.7)\) has been established in \([28, 11, 17]\) under the assumption that the curve satisfies \((1.2)\). For \( k = d - 1 \geq 2 \), the estimate \((1.7)\) is equivalent to Sogge’s local smoothing conjecture when \( \mathcal{S} \) is strictly convex \((16)\) and similar results are also known when \( \mathcal{S} \) has non-vanishing gaussian \((12, 3)\) with principal curvatures of different signs.

We now intend to draw a connection between the sharp local estimate \((1.7)\) and the problem determining dimension of union of \((x+t\mathcal{S})\). The following gives optimal lower bounds on dimension of union of \((x+t\mathcal{S})\). It can be shown by adapting the proof of Theorem 1.1 so we state it without proof.

**Proposition 1.5.** Let \( 1 \leq k \leq d - 1 \). Suppose that \((1.7)\) holds for some \( p \neq \infty \). Suppose that a Borel set \( F \subset \mathbb{R}^d \times (0,\infty) \) satisfies \( \dim_H F > \alpha \) for some \( 0 < \alpha \leq d - k \). Then, if a Borel set \( E \subset \mathbb{R}^d \) contains

\[
\bigcup_{(x,t) \in F} (x + t\mathcal{S}),
\]

then \( \dim_H E \geq k + \alpha \).
Generalized averaging operators. Theorem 1.1 can also be extended to an analogous result for unions of variable curves in $\mathbb{R}^2$, and more generally variable hypersurfaces in $\mathbb{R}^d$, $d \geq 2$.

Let $U_0, V_0 \subset \mathbb{R}^d$ be small open neighborhoods of $x_0$ and $y_0$ in $\mathbb{R}^d$, respectively. Also, let $\Phi_t \in C^\infty(U_0 \times V_0)$ vary smoothly in the parameter $t \in J_0 := [1, 2]$. We set $U \subset U_0$, $V \subset V_0$, and $J \subset J_0$. For $(x, t) \in U \times J$, we consider a family of variable hypersurfaces given by

$$G_{x,t} = \{ y \in V : \Phi_t(x, y) = 0 \}.$$

We assume that $\Phi_t$ satisfies the rotational curvature condition:

$$\det \begin{pmatrix} \Phi_t & \partial_y \Phi_t \\ \partial_x \Phi_t & \partial^2_{xy} \Phi_t \end{pmatrix} \neq 0 \quad \text{where} \quad \Phi_t(x, y) = 0. \tag{1.8}$$

Then, $\partial_y \Phi_t \neq 0$ for any $(x, y) \in U_0 \times V_0$, and $G_{x,t}$ is a smooth embedded hypersurface in $\mathbb{R}^d$. Furthermore, there exists a homogeneous function $q$ of degree 1 in $\xi$ such that $q$ is smooth if $\xi \neq 0$ and satisfies $q(x, t, \partial_x \Phi_t(x, y)) = \partial_t \Phi_t(x, y)$ whenever $\Phi_t(x, y) = 0$ (see [30] for detailed construction of $q$). We assume that

$$\text{rank} \partial^2_{\xi \xi} q(x, t, \xi) = d - 1 \quad \text{for} \quad (x, t, \xi) \in U_0 \times J_0 \times (\mathbb{R}^d \setminus \{0\}). \tag{1.9}$$

We say that $\Phi_t$ satisfies cinematic curvature condition if (1.8) and (1.9) hold.

Proving a sharp estimate for a maximal average over a neighborhood of $G_{x,t}$ in $\mathbb{R}^2$, Zahl [34] showed that a Borel set containing curves $G_{x,t}$ for every $0 < t < 1$ has Hausdorff dimension 2 (see also [18]). By utilizing the local smoothing estimates in [10, 3], we obtain the following.

**Theorem 1.6.** Let $d \geq 2$ and $0 < \alpha < 1$. Suppose $F \subset U \times J$ is a Borel set satisfying $\dim_H F > \alpha$ and $\Phi_t$ satisfies the cinematic curvature condition. Then a Borel set containing $G_{x,t}$ for $(x, t) \in F$ has Hausdorff dimension at least $d - 1 + \alpha$.

**Organization of the paper.** In Section 2, we prove Theorem 1.1 and 1.4 by using local smoothing type estimates and Bessel capacity. In Section 3, we provide a proof of Theorem 1.6. Finally, we discuss sharpness of Theorem 1.4 and 2.1 in Section 4.

## 2. Proof of Theorem 1.1 and 1.4

In this section we prove Theorem 1.1 and 1.4 by applying Theorem 1.3 combined with Proposition 2.5 which shows that Hausdorff dimension of unions of curves can be determined in terms of the regularity of the smoothing estimate 1.3.

### 2.1. Local smoothing estimates relative to fractal measure.

In this section, we prove Theorem 1.3. In fact, we prove a slightly general result considering multi-parameter dilation. Let $1 \leq m \leq d$ and $t = (t_1, \ldots, t_m)$. Recall that $\gamma_t^\omega$ is given by (1.3). We consider

$$\mathcal{A}_t^\omega f(x, t) = \prod_{j=1}^m \chi(t_j) \int f(x + \gamma_t^\omega(s)) \psi(s) \, ds$$

where $\chi \in C^\infty_0((1/2, 4))$ such that $\chi = 1$ on $[1, 2]$ and $\psi$ is a real valued smooth function supported in $I_0$. Theorem 1.3 follows from the next theorem with $m = 1$. 

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**Proof of Theorem 1.3.**
Theorem 2.1. Let $d \geq 2$ and $1 \leq m \leq d$. Also let $0 < \alpha \leq n := d + m$ and $\mu \in C^n(\alpha)$. Suppose that $\gamma$ is a smooth curve satisfying (1.2). If $p > p(d)$, then
\begin{equation}
\|A_\gamma^\omega f\|_{L^p(\mathbb{R}^n, d\mu)} \leq C(\|f\|_{L^p})^{\frac{1}{p}}
\end{equation}
for $\sigma > (n - \alpha - 2)/p$.

If $\mu$ is the Lebesgue measure, i.e., $\alpha = n$, the smoothing order $\sigma > -2/p$ does not depend on $m$. We deduce Theorem 2.1 from the following sharp local smoothing estimates for $A_\gamma$.

Theorem 2.2 (cf. Lemma 2.7 in [12]). For $d \geq 2$, let $\gamma$ be a smooth curve in $\mathbb{R}^d$ satisfying (1.2). Then, for $p > p(d)$ and any $r > -2/p$, we have
\begin{equation}
\|A_\gamma f\|_{L^p(\mathbb{R}^{d+1})} \leq C\|f\|_{L^p}.
\end{equation}
We also use the next lemma which is commonly used to obtain estimates relative to fractal measures.

Lemma 2.3 (cf. Lemma 2.7 in [12]). Let $\mu \in C^n(\alpha)$ for $0 < \alpha \leq n$. Suppose $\hat{F}$ is supported on $B^n(0, \lambda)$. Then $\|F\|_{L^p(d\mu)} \lesssim \langle \mu \rangle^{\frac{1}{p}} \lambda^\sigma \|F\|_{L^p(\mathbb{R}^n)}$ for $p \geq 1$.

Proof of Theorem 2.1. Let $\beta \in C^\infty_c((2^{-1}, 2))$ such that $\sum_\beta = 1$ for $s \neq 0$ and set $\beta_0 = \sum_{|j| \leq 0} \beta(2^{-j})$. Let $P_\lambda$ be a standard Littlewood-Paley projection operator given by $P_\lambda f(\xi) = \beta(\lambda^{-1} |\xi|) \hat{f}(\xi)$. Then, we have
\begin{equation}
A_\gamma^\omega P_0 f(x, t) = A_\gamma^\omega P_0 f(x, t) + \sum_{j \geq 1} A_\gamma^\omega P_0 f(x, t),
\end{equation}
where $\hat{P_0 f} = \beta_0(|\xi|) \hat{f}(\xi)$. Since the kernel of $A_\gamma^\omega P_0 f(x, t)$ decays rapidly outside of the unit ball, we have $\|A_\gamma^\omega P_0 f\|_{L^p(d\mu)} \leq C\|f\|_{L^p}$ by Schur’s test. Thus it suffices to show that, for $\lambda \geq 1$,
\begin{equation}
\|A_\gamma^\omega P_0 f\|_{L^p(d\mu)} \leq C(\|f\|_{L^p})^{\frac{1}{p}}, \quad \sigma > (n - \alpha - 2)/p.
\end{equation}
Since $\chi$ is compactly supported, the support of the space-time Fourier transform of $A_\gamma^\omega P_0 f$ is unbounded. To apply Lemma 2.3 we decompose the frequency support of $A_\gamma^\omega P_0 f$ into a bounded set and its complement.

For $\tau = (\tau_1, \ldots, \tau_m)$, let us set
\begin{equation}
m_\lambda(\xi, \tau) = \beta(\lambda^{-1} |\xi|) \int e^{2\pi i \xi \cdot \tau - t \cdot \tau} \prod_{j=1}^m \chi(t_j) \psi(s) ds dt,
\end{equation}
so we have $F_{x,t}(A_\gamma^\omega P_0 f)(\xi, \tau) = m_\lambda(\xi, \tau) \hat{f}(\xi)$. We define a frequency localized operator $A_\gamma^\omega$ by
\begin{equation}
F_{x,t}(A_\gamma^\omega f)(\xi, \tau) = \prod_{j=1}^m \beta_0((C \lambda)^{-1} \tau_j) m_\lambda(\xi, \tau) \hat{f}(\xi)
\end{equation}
for a sufficiently large constant $C \geq 1$ to be chosen later. We set
\begin{equation}
A_\gamma^\omega = A_\gamma^\omega - A_\gamma^\omega.
\end{equation}

First, we show
\begin{equation}
\|A_\gamma^\omega P_0 f\|_{L^p(d\mu)} \lesssim (\mu)^{\frac{1}{p}} \lambda^{-N} \|f\|_{L^p}.
\end{equation}
for any $N \geq 1$. We may write $A^p_\gamma P_\lambda f(x,t) = \mathcal{K}_\lambda(\cdot,t) * f(x)$, where

$$
\mathcal{K}_\lambda(x,t) = \int_{\mathbb{R}} \left(1 - \sum_{j=1}^{m} \beta_0((C\lambda)^{-1} \tau_j)\right) m_\lambda(\xi,\tau) e^{2\pi i(x,\xi) \cdot t} d\xi d\tau.
$$

By integration by parts in $t$, we have $|\partial^\kappa_{t,r} m_\lambda(\xi, \tau)| \lesssim \lambda^{-N} (1 + |\tau|)^{-N}$ for any multi-indices $\kappa$ and $N \geq 1$, provided that $|\tau| \geq C|\xi|$ for a sufficiently large $C$ satisfying $C \geq 1 + 10 \sup_{\xi \in \mathbb{R}} |\gamma(s)|$. Then, integration by parts yields $|\mathcal{K}_\lambda(x,t)| \lesssim \lambda^{-N} (1 + |x|)^{-N} (1 + |t|)^{-N}$ for any $N \geq 1$. By Schur's test, we get the estimate (2.4) for any $\mu \in \mathcal{C}^\alpha(\alpha)$.

Now, we show

$$
\|A^p_\gamma P_\lambda f\|_{L^p(d\mu)} \lesssim \langle \mu \rangle^{\frac{1}{p} - \frac{n}{p} - r} \|f\|_{L^p(\mathbb{R}^d)}, \quad r > -\frac{2}{p}.
$$

Since $\mathcal{F}_{x,t}(A^p_\gamma P_\lambda f)$ is supported in $\mathbb{B}^n(0, C(d,m)\lambda)$, we can apply Lemma 2.2 to get

$$
\|A^p_\gamma P_\lambda f\|_{L^p(d\mu)} \lesssim \langle \mu \rangle^{\frac{1}{p} - \frac{n}{p} - r} \|A^p_\gamma P_\lambda f\|_{L^p(\mathbb{R}^n)}.
$$

Using $|A^p_\gamma P_\lambda f| \leq |A^p_\gamma P_\lambda f| + |A^p_\gamma P_\lambda f|$ and (2.4), we see

$$
\|A^p_\gamma P_\lambda f\|_{L^p(d\mu)} \lesssim \langle \mu \rangle^{\frac{1}{p} - \frac{n}{p} - r} \|A^p_\gamma P_\lambda f\|_{L^p(\mathbb{R}^n)} + \langle \mu \rangle^{\frac{1}{p} - \frac{n}{p} - r} \|f\|_{L^p}.
$$

for any $N \geq 1$. In order to prove (2.5), it suffices to show

$$
\|A^p_\gamma P_\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^r \|f\|_{L^p}, \quad r > -\frac{2}{p}.
$$

Using Fubini's theorem, we make change of variables $(t_1, t_2, \ldots, t_m) \mapsto (t_1, t_1 t_2, \ldots, t_1 t_m)$ to get

$$
\|A^p_\gamma P_\lambda f\|_{L^p}^p = \int \cdots \int \left( \int \left| \left. A^p_\gamma P_\lambda f(x,t) \right| \right|^p \prod_{j=2}^{m} \chi(t_j) dxdt_1 dt_2 \cdots dt_m,
$$

where $t' = (t_1, t_2, \ldots, t_m)$. For fixed $t_1, \ldots, t_m$, it is obvious that $\gamma_{t'}$ is nondegenerate. Applying Theorem 2.2 to the inner integral, we obtain (2.6), and thus (2.5) follows.

Combining (2.4) and (2.5), we get the desired estimate (2.3) for $\sigma = (n - \alpha)/p + r > (n - \alpha - 2)/p$.

2.2. Bessel capacity. To prove Theorem 1.1 and 1.4, we begin with Bessel capacity which is closely related to Hausdorff dimension. For a Borel set $E \subset \mathbb{R}^d$, let us define the Bessel capacity $B^\sigma_p(E)$ by

$$
B^\sigma_p(E) = \inf \{ \|f\|_{L^p}^p : f \in C_0^\infty(\mathbb{R}^d), f \geq \chi_E \}
$$

for $\sigma > 0$ and $p > 1$. By $\mathcal{H}^\sigma$ we denote $\sigma$-dimensional Hausdorff measure. The following describes a relation between the Bessel capacity and the Hausdorff measure.

**Theorem 2.4** (Theorem 2.6.16 in [35]). Let $p > 1$ and $0 < \sigma p < d$. Also let $E \subset \mathbb{R}^d$ be a Borel set. If $\mathcal{H}^{d-\sigma p}(E) < \infty$, then $B^\sigma_p(E) = 0$. Conversely, if $B^\sigma_p(E) = 0$, then $\mathcal{H}^{d-\sigma p + \epsilon}(E) = 0$ for every $\epsilon > 0$.

Using (2.3) and Theorem 2.4, we can obtain lower bounds of Hausdorff dimension of unions of curves.
Proposition 2.5. Let $0 < \alpha \leq n := d + m$ and $\mu \in C^\alpha(\Omega)$. Suppose that the estimate \((2.1)\) holds with $\sigma > \sigma_0$ for some $\sigma_0 = \sigma_0(\alpha, n, p) > 0$ and $p > 1$. If $F' \subset \mathbb{R}^d$ is a Borel set with $\dim F' > \alpha$, then a Borel set $E \subset \mathbb{R}^d$ containing $\Gamma^{\sigma}(F')$ has Hausdorff dimension at least $d - \sigma_0 p$.

Proof. Suppose that there exists a Borel set $E$ containing $\Gamma^{\sigma}(F')$ such that

$$\dim_H E < d - \sigma_0 p.$$ 

Taking a number $\sigma > \sigma_0$ such that $\dim_H E < d - \sigma p < d - \sigma_0 p$, we see $\mathcal{H}^{d-\sigma p}(E) < \infty$. By Theorem 2.4, it follows

\[(2.7)\]

$$B^p_\sigma(E) = 0.$$ 

For $\alpha_1$ satisfying $0 < \alpha_1 < \dim_H F'$, there exists a compact set $F_1 \subset F'$ such that $\mathcal{H}^{{\alpha_1}}(F_1) > 0$. By Frostman’s lemma (see \cite{21} for example), there exists $\mu \in C^\alpha(\alpha_1)$ with $\mu \subset F_1$. We consider a function $f \in C^\infty(\mathbb{R}^d)$ such that $f \geq \chi_E$. Since $\supp f \supset E \supset \cup_{x \in F_1} (x + \gamma I)$, we have $\mathcal{A}_E^\sigma f(x, t) \gtrsim 1$ for all $(x, t) \in F_1$. Since we are assuming \((2.1)\) for $\sigma > \sigma_0$, it follows that

\[(2.8)\] 

$$\mu(F_1)^{\frac{1}{p}} \lesssim \|\mathcal{A}_E^\sigma f\|_{L^p(\mu)} \leq C(\mu)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$ 

for $\sigma, p$ satisfying \((2.7)\). Note that \((2.8)\) holds for any function $f \in C^\infty(\mathbb{R}^d)$ such that $f \geq \chi_E$. So, $\mu(F_1)^{\frac{1}{p}} = 0$ from \((2.7)\), which contradicts to $\supp \mu \subset F_1$. Hence $\dim_H E \geq d - \sigma_0 p$ for $E$ containing $\Gamma^{\sigma}(F')$. 

Now we prove Theorem 1.1 and Theorem 1.4.

Proof of Theorem 1.1. Since $\dim_H F > \alpha$, there exists $\mu \in C^{d+1}(\alpha)$ which is supported in $F$ by Frostman’s lemma. For such $\mu$, there exists $\sigma_0 > 0$ such that the estimate \((1.3)\) holds with $\sigma > \sigma_0 > (d - 1 - \alpha)/p$ by Theorem 1.3. Since $\alpha \leq d - 1$ and $p > 1$, by Proposition 2.5 with $m = 1$ we see that a set $E \subset \mathbb{R}^d$ containing $\Gamma(F)$ has Hausdorff dimension at least $d - \sigma_0 p$. Taking $\sigma_0$ arbitrarily close to $(d - 1 - \alpha)/p$, we get \((1.3)\). 

Proof of Theorem 1.4. Theorem 1.4 can be obtained in the same manner. The difference is that we have \((2.1)\) with $\sigma > \sigma_0 > (d + m - \alpha - 2)/p$. If $\alpha \leq d + m - 2$, we can apply Proposition 2.5 to obtain $\dim_H E \geq d - \sigma_0 p$. We omit the details. 

Remark 1. By using Theorem 2.1, one can show that $E$ is of positive Lebesgue measure if $\dim_H F' > d + m - 2$. For $\alpha_1$ satisfying $\dim_H F' > \alpha_1 > d + m - 2$, let $F_1$ be a compact subset of $F$ with $\mathcal{H}^{{\alpha_1}}(F_1) > 0$. Then there exists $\mu \in C^{d+m}(\alpha_1)$ with $\supp \mu \subset F_1$. Applying Theorem 2.1 with $\mu \in C^{d+m}(\alpha_1)$, we have

$$\mu(F_1)^{\frac{1}{p}} \lesssim \|\mathcal{A}_E^\sigma \chi_E\|_{L^p(\mu)} \leq C(\mu)^{\frac{1}{p}} \|\chi_E\|_{L^p}$$ 

for $\sigma > (d + m - \alpha_1 - 2)/p$. Hence, $E$ can not be of Lebesgue measure zero.

3. Dimension of Unions of Variable Hypersurfaces

In this section, we prove Theorem 1.6 by making use of the sharp local smoothing estimates for variable coefficient averaging operators. We consider the Fourier integral operator given by

$$\mathcal{A} f(x, t) = \int a(x, t, y) \delta(\Phi_t(x, y)) f(y) \, dy$$
where \( a \in C^\infty_c(U \times J \times V) \) is a positive bump function such that \( a = 1 \) on \( U \times J \times V \).

Sharp local smoothing estimates for \( \mathcal{A}f \) were shown by Gao–Liu–Miao–Xi [10] for \( d = 2 \), and Beltran–Hickman–Sogge [3] for higher dimensions.

**Theorem 3.1** ([3] [10]). Suppose \( \Phi_t \) satisfies the conditions (1.8) and (1.9) on the support of \( a \). Then

\[
\|\mathcal{A}f\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \quad r > -d/p
\]

holds for \( 4 \leq p < \infty \) if \( d = 2 \), and for \( 2(d+1)/(d-1) \leq p < \infty \) if \( d \geq 3 \).

The range of \( p \) can not be generally extended when \( d = 2 \) and odd \( d \geq 3 \). In the same manner as in the proof of Theorem 2.1, we obtain the following from Theorem 3.1.

**Corollary 3.2.** Let \( 0 < \alpha \leq d + 1 \) and \( \mu \in \mathcal{C}^{d+1}(\alpha) \). Then, we have

\[
\|\mathcal{A}f\|_{L^p(\mathbb{R}^{d+1}, d\mu)} \leq C(\mu)^{\frac{1}{\alpha}} \|f\|_{L^p(\mathbb{R}^d)}, \quad \sigma > (1 - \alpha)/p
\]

for \( 4 \leq p < \infty \) when \( d = 2 \), and for \( 2(d+1)/(d-1) \leq p < \infty \) when \( d \geq 3 \).

**Proof.** In local coordinates, modulo a smoothing operator, \( \mathcal{A}f \) can be written as a finite sum of the operators

\[
\mathcal{A}f(x, t) = \int e^{2\pi i \phi(x, t, \xi)} a(x, t, \xi) \hat{f}(\xi) \, d\xi,
\]

where \( a \in C^\infty(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d) \) is compactly supported in \( (x, t) \) and satisfies

\[|\partial^\alpha_x \partial^\beta_t a(x, t, \xi)| \lesssim (1 + |\xi|)^{-\frac{d+1}{2} - |\kappa|}\]

for any \( \kappa_1, \kappa_2 \). Also, \( \phi \) is a smooth function on \( \mathbb{R}^d \times \mathbb{R} \times (\mathbb{R}^d \setminus \{0\}) \) of homogeneous of degree 1 in \( \xi \) satisfying the cinematic curvature condition on \( \text{supp } a \) (see [13, 24]).

So, it is enough to show that

\[
\|\mathcal{A}_{\lambda} f\|_{L^p(\mathbb{R}^{d+1}, d\mu)} \leq C(\mu)^{\frac{1}{\alpha}} \lambda^\sigma \|f\|_p
\]

holds for \( \lambda \geq 1 \) if \( \sigma > (1 - \alpha)/p \).

By (3.1), we have

\[
\|\mathcal{A}_{\lambda} f\|_{L^p(\mathbb{R}^{d+1}, d\mu)} \lesssim \lambda^\sigma \|f\|_{L^p(\mathbb{R}^d)}, \quad r > -d/p.
\]

As in the proof of Theorem 2.1, we write

\[
\mathcal{A}_{\lambda} f = \tilde{\mathcal{A}}_{\lambda} f + \mathcal{P}_{\lambda} f,
\]

where

\[
F_{\lambda, t}(\tilde{\mathcal{A}}_{\lambda} P_{\lambda} f)(\zeta, \tau) = \beta_0((C\lambda)^{-1}(|\zeta|, \tau)))F_{\lambda, t}(\tilde{\mathcal{A}}_{\lambda} P_{\lambda} f)(\zeta, \tau),
\]

\[
F_{\lambda, t}(\tilde{\mathcal{A}}_{\lambda} P_{\lambda} f)(\zeta, \tau) = (1 - \beta_0((C\lambda)^{-1}(|\zeta|, \tau)))F_{\lambda, t}(\tilde{\mathcal{A}}_{\lambda} P_{\lambda} f)(\zeta, \tau)
\]

for a sufficiently large \( C \geq 1 \).

By rapid decay of the kernel, one can show

\[
\|\tilde{\mathcal{A}}_{\lambda} P_{\lambda} f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_p
\]

for any \( N \geq 1 \). Indeed, note that \( F_{\lambda, t}(\tilde{\mathcal{A}}_{\lambda} P_{\lambda} f)(\zeta, \tau) = \int w_{\lambda}(\zeta, \tau, y) f(y) \, dy \)

where

\[
w_{\lambda}(\zeta, \tau, y) = \int e^{-2\pi i(x' \cdot \zeta + t' \cdot \tau)} e^{2\pi i \phi(x', t', \xi)} e^{-2\pi i y \cdot \xi} a(x', t', \xi) \beta(\lambda^{-1} |\xi|) \, dx' \, dt' \, d\xi.
\]
as long as a projection of $\gamma$ is clear from the proof of Proposition 4.1 below, Proposition 4.1 remains valid.

Let Proposition 4.1.

Proof of Theorem 1.6. The proof is similar to that of Theorem 1.1. By following the proof of Proposition 2.5 one can easily see Proposition 2.5 with $m = 1$ holds for $\cup(x,t) \in F$, which replaces $\Gamma^\omega(F')$. Indeed, taking $f \geq \chi_E$ such that $E \supset \cup(x,t) \in F G_{x,t}$, since $a = 1$ on $U \times J \times V$, we see $\mathfrak{A}f(x,t) \gtrsim 1$ holds whenever $(x,t) \in F$. So, by applying (3.2) in place of (2.1), we obtain

$$\parallel \mathfrak{A} f \parallel_{L^p(\mu)} \lesssim (\mu)_{\frac{1}{d} + r} \lambda^{-\frac{1}{d} + r} \parallel f \parallel_{L^p(\mu)} + (\mu)_{\frac{1}{d}} \lambda^{-\frac{1}{d}} \parallel f \parallel_{L^p(\mu)}$$

for any $N \geq 1$. By this and (3.6), we get (3.4) for $\sigma > (1 - \alpha)/p$. □

Combining Corollary 3.2 and Proposition 2.5, we prove Theorem 1.6.

4. Sharpness of Theorem 1.4 and 2.1

In this section, we discuss optimality of the exponent $\alpha = d + m - 2$ in Theorem 1.4 and 2.1, and sharpness of the regularity exponent $s$ in Theorem 2.1 (and Theorem 1.3).

4.1. Construction of a type of BRK set for space curves. We show that the condition $\dim_H F' > d + m - 2$ in Theorem 1.4 (or $\dim_H F' > d - 1$ in Theorem 1.1) is necessary, in general, for $\Gamma(F')$ to have positive Lebesgue measure. Here, we may assume that $F'$ is a subset of $\mathbb{R}^d \times (0,1)^m$.

Proposition 4.1. Let $1 \leq m \leq d$, and let $\gamma$ be a curve defined on $I_\gamma$ by $\gamma(s) = (s^{a_1}, \ldots, s^{a_d})$ for integers $1 \leq a_1 < \cdots < a_d$. Then, there exists a Borel set $F' \subset \mathbb{R}^d \times (0,1)^m$ with $\dim_H F' = d + m - 2$ such that $\Gamma^\omega(F')$ is of Lebesgue measure zero.

To prove Proposition 4.1, we use Kinney’s construction of Lebesgue measure zero set which consists of similitudes of a planar convex curve \((15)\). When $m \leq d - 1$, as is clear from the proof of Proposition 4.1 below, Proposition 4.1 remains valid as long as a projection of $\gamma$ to 2-plane is convex. Thus, by a suitable change of
coordinates and Taylor expansion, one can see that Proposition 4.1 continues to hold if $\gamma$ is of finite type. However, when $m = d$, we need an additional assumption that at least two components of $\gamma$ are monomial.

**Proof of Proposition 4.1.** Let $\gamma_{a,b}(s) = (s^a, s^b)$ for $s \in I_\omega$ and $1 \leq a < b$. Following the argument in [15], we first construct a compact set $K^{a,b} \subset \mathbb{R}^2$ of Lebesgue measure zero such that $K$ contains $t\gamma_{a,b}(I)$ for every $0 < t \leq 1$. Let $C$ be the standard 1/3-Cantor set on $[0, 1]$. Note that $[0, 1]$ is the distance set of $C$. So, for each $t \in (0, 1]$, there exist the smallest numbers $c_1(t), c_2(t) \in C$ such that $c_2(t) - c_1(t) = t$. Then the curve $(c_1(t), c_1(t)) + t\gamma_{a,b}(I)$ connects two points $(c_1(t), c_1(t))$ and $(c_2(t), c_2(t))$. Since $\gamma_{a,b}$ is a convex curve, adapting the argument in [15], one can show that the union of curves

$$K^{a,b} := \bigcup_{0 < t \leq 1} \left( (c_1(t), c_1(t)) + t\gamma_{a,b}(I) \right)$$

is of Lebesgue measure zero in $\mathbb{R}^2$ (see [15, p.1080]).

We treat the cases $m \leq d - 1$ and $m = d$, separately. We consider the case $m \leq d - 1$ first. Since $m \leq d - 1$, $\omega(i) = \omega(j)$ for some $i \neq j$. There exists a dilation parameter, which we denote by $t_1$, appearing at least twice in $\gamma_{i'} = (t_{\omega(1)}\gamma_1(s), \ldots, t_{\omega(d)}\gamma_d(s))$. We may assume $i = 1$, $j = 2$, that is to say, $t_{\omega(1)} = t_{\omega(2)} = t_1$. We set

$$F' = \{(x, t) \in [0, 1]^d \times (0, 1)^m : (x_1, x_2) = (c_1(t_1), c_1(t_1))\}.$$ 

Obviously, $\dim H F' = d + m - 2$ and

$$\Gamma^\omega(F') = \bigcup_{(x, t) \in F'} (x + \gamma_{i'}(I)) \subset K^{a,b} \times [-C, C]^{d-2}$$

for some constant $C > 0$ where $(a, b) = (a_i, a_j)$ for some $i < j$. Therefore, it is clear that $\Gamma^\omega(F')$ is of Lebesgue measure zero in $\mathbb{R}^d$.

Now we consider the case $m = d$, for which we make use of homogeneity of monomials. We set

$$g_k(s) = \gamma_{i'}((t_1/t_2)^{(a_2-a_1)}s).$$

The first two components of $g_k(s)$ are given by $b(t)s^{a_1}, b(t)s^{a_2}$, where $b(t) = (t_1^{a_2}/t_2^{a_1})^{1/(a_2-a_1)}$. Set $J_\omega = \{(t_1, t_2) \in [0, 1]^2 : t_1 \geq t_2, t_1^{a_2} \leq t_2^{a_1}\}$ and consider

$$F' = \{(x, t) \in [0, 1]^d \times J_\omega \times (0, 1)^{m-2} : (x_1, x_2) = (c_1(b(t)), c_1(b(t)))\}.$$ 

Then, we have

$$\bigcup_{(x, t) \in F'} (x + g_k(I)) \subset \bigcup_{(t_1, t_2) \in J_\omega} \left( (c_1(b(t)), c_1(b(t))) + b(t)\gamma_{a_1,a_2}(I) \right) \times [-C, C]^{d-2}$$

for some $C > 0$. Note that $0 < b(t) < 1$ and $\gamma_{i'}(I) \subset g_k(I)$ if $t \in J_\omega \times (0, 1)^{m-2}$. Thus, it follows that $\Gamma^\omega(F') \subset K^{a_1,a_2} \times [-C, C]^{d-2}$. So, $\Gamma^\omega(F')$ is of Lebesgue measure zero.

4.2. Optimality of the regularity exponent in Theorem 2.1. We prove that the regularity exponent in Theorem 2.1 is sharp for some range of $\alpha$. Recall that $\gamma_{i'}(s) = (t_{\omega(1)}\gamma_1(s), \ldots, t_{\omega(d)}\gamma_d(s))$ for an onto map $\omega : \{1, \ldots, d\} \to \{1, \ldots, m\}$. We set

$$k(\omega) = \max_{1 \leq j \leq m} \{ i \in [1, d] : \omega(i) = j \}$$
which is the maximum number of repetition in \{\omega(1), \ldots, \omega(d)\}. For example, let 
\gamma(s) = (\cos(2\pi s), \sin(2\pi s), s). If \omega(1) = \omega(2) = 1 and \omega(3) = 2, then \gamma_{t_1,t_2}(s) = (t_1 \cos(2\pi s), t_1 \sin(2\pi s), t_2 s) is a two parameter family of helices with \nu = 2.

**Proposition 4.2.** Let \(d \geq 2, 1 \leq m \leq d\), and let \(\gamma\) be a finite type curve in \(\mathbb{R}^d\). Set \(n = d + m\). For \(0 < \alpha \leq n\), suppose \((2.1)\) holds for all \(\mu \in \mathcal{C}^n(\alpha)\). Then, if \(m \leq d - 1\), we have

\[
\sigma \begin{cases} 
(n - \alpha - 2)/p, & \text{if } n - k(\omega) - 1 < \alpha \leq n, \\
(k(\omega) - 1)/p, & \text{if } 0 < \alpha \leq n - k(\omega) - 1.
\end{cases}
\]

If \(m = d\) and \(\gamma(s) = (s^{a_1}, \ldots, s^{a_d})\) for \(1 \leq a_1 < \cdots < a_d\) and \(s \in I_0\), then

\[
\sigma \begin{cases} 
(n - \alpha - 2)/p, & \text{if } n - 3 \alpha \leq n, \\
1/p, & \text{if } 0 < \alpha \leq n - 3.
\end{cases}
\]

**Proof.** Let \(d, m\), and \(k = \nu\) be fixed, and denote \(\vec{x} = (x_1, \ldots, x_k)\) and \(\vec{\gamma} = (\gamma_1, \ldots, \gamma_d)\).

We consider the case \(1 \leq m \leq d - 1\) first. Note that \(k \geq 2\). Without loss of generality, we may assume that \(t_1\) is the parameter repeated \(k\)-times. Also, by a permutation, we may assume that \((t_1, \ldots, t_{d}) = (t_1, \ldots, t_1, t_2, \ldots, t_m)\). So, let us write

\[
x + \gamma_1^\nu(s) = (\vec{x} + t_1 \gamma_1(s), \vec{\gamma} + \gamma_1^\nu(s) \in \mathbb{R}^k \times \mathbb{R}^{d - k}.
\]

Since \(\vec{\gamma}\) is of finite type in \(\mathbb{R}^k\), by Taylor’s expansion \(\gamma(s) = \sum_{j=1}^k \gamma^{(j)}(0)s^j + O(s^{k+1})\) such that \(\gamma^{(j)}(0) \neq 0\) for some integers \(1 \leq j_1 < \cdots < j_k\) if \(s \leq \delta_1\) for a sufficiently small \(\delta_1\). By a change of variables, we may assume that there is an interval \(I_0 = [\delta_0, 2\delta_0] \subset [0, 1]\) such that \(\gamma(s) \geq c\) on \(I_0\) for a constant \(c > 0\).

Let \(\eta \in \mathcal{C}_c^\infty([-2, 2]^k)\) such that \(\eta \geq 0\) and \(\eta = 1\) on \([-1, 1]^k\). We also choose a positive function \(h \in \mathcal{S}(\mathbb{R}^{d-k})\) such that \(\text{supp} h \subset [-1, 1]^{d-k}\) and \(h \geq 1\) on \([-C, C]^{d-k}\) for \(C \geq 1 + 10\sup_{\xi \in I_0} |\gamma(s)|\). For a fixed \(\lambda > \delta_0^{-1}\), we set

\[
g(\vec{x}) = \sum_{\nu \in \lambda^{-1} \mathbb{Z} \cap I_0} \eta(\lambda(\vec{x} - \overline{\gamma}(\nu)))
\]

and \(f(x) = g(\vec{x})h(\vec{\gamma})\). Let

\[
Q = \{(x, t) \in [-\epsilon_0, \epsilon_0]^d \times [2^{-1}, 2]^m : |\vec{x}| \leq \epsilon_0 \lambda^{-1}, |t_1 - 1| \leq \epsilon_0 \lambda^{-1}\}.
\]

Then, we have

\[
\mathcal{A}_1^\nu f(x, t) = \prod_{j=1}^m \chi(t_j) \int g(\vec{x} + t_1 \gamma(s)) h(\vec{x} + \gamma_1^\nu(s)) \psi(s) ds \geq 1,
\]

whenever \((x, t) \in Q\) for a sufficiently small \(\epsilon_0 > 0\). To see this, note that

\[
g(\vec{x} + t_1 \gamma(s)) = \sum_{\nu \in \lambda^{-1} \mathbb{Z} \cap I_0} \eta(\lambda(\vec{x} + (t_1 - 1) \gamma(s) + \gamma(s) - \overline{\gamma}(\nu))).
\]

So, \(g(\vec{x} + t_1 \gamma(s)) \geq 1\) for \(s \in I_0\) if \((x, t) \in Q\). Also, note \(g(\vec{x} + t_1 \gamma(s)) \geq 0\), for \(s \in [-1, 1] \setminus I_0\), and \(h(\vec{\gamma} + \gamma_1^\nu(s)) \geq 1\), for \(s \in [-1, 1]\) and \((x, t) \in Q\). Hence, we get \((4.3)\).

Now, we show

\[
\|f\|_{L^p_{x}(\mathbb{R}^d)} \lesssim \lambda^{\sigma - (k-1)/p}.
\]
We observe that $\|f\|_{L^r_\xi(\mathbb{R}^d)} \lesssim \|h\|_{L^r_\xi(\mathbb{R}^d)} + \|g\|_{L^r_\xi(\mathbb{R}^d)}$ for $\sigma \geq 0$, and similarly $\|f\|_{L^r_\xi(\mathbb{R}^d)} \lesssim \|h\|_{L^r_\xi(\mathbb{R}^d)}$ for $\sigma < 0$. Those can be shown by using the Mikhlin multiplier theorem. Since $\|h\|_{L^r_\xi(\mathbb{R}^d)} \lesssim 1$ for any $\sigma \in \mathbb{R}$, it suffices to show that

$$
(4.5) \quad \|g\|_{L^r_\xi(\mathbb{R}^d)} \lesssim \lambda^{\sigma-(k-1)/p}.
$$

Recall $\beta, \beta_0$ which are defined in the proof of Theorem 2.21. We decompose $g = \sum_{j\geq 0} g_j$ such that $\hat{g}_j(\xi) = \hat{g}(\xi)\beta_j(\lambda^{-1}|\xi|)$ with $\beta_j = \beta(2^{-j} \cdot)$ for $j \geq 1$. Note that

$$
((1 + |\cdot|/2)^{\nu/2} \hat{g}_j) \hat{v}(\xi) = \sum_{\nu \in \lambda^{-1}\mathbb{Z}} \Phi_{j,\nu}(\xi),
$$

where

$$
\Phi_{j,\nu}(\xi) = \int e^{2\pi i (\xi - \hat{\nu}(\nu)) \cdot \xi - \lambda^{-k}\hat{\beta}(\lambda^{-1}\xi)\beta_j(\lambda^{-1}|\xi|)(1 + |\xi|^2)^{\nu/2}}d\xi.
$$

By rescaling $\xi \rightarrow \lambda \xi$, it is easy to show that $|\Phi_{j,\nu}(\xi)| \lesssim 2^{-jN}\lambda^\nu (1 + \lambda|\xi - \hat{\nu}(\nu)|)^{-N}$ for any $N \geq 1$. Since $|\hat{\nu}(\nu)| \geq c$ on $I$, for a constant $c > 0$, we have $|\hat{\nu}(\nu)| \geq c|\nu - \nu'|$ for $\nu, \nu' \in I$. Therefore, we see $\sum_{\nu} |\Phi_{j,\nu}|_p \lesssim 2^{-jN}\lambda^{\nu-(k-1)/p}$ for any $N \geq 1$. This gives $\sum_{j\geq 0} \|g_j\|_{L^p_\lambda} \lesssim \lambda^{\sigma-(k-1)/p}$ and hence (4.5).

We now take

$$
d\mu(x,t) = \begin{cases} 
\lambda^{n-\alpha} \chi_\delta(x,t) dx dt, & \text{if } n - k - 1 < \alpha \leq n, \\
\lambda^{k+1} \chi_\delta(x,t) dx dt, & \text{if } 0 < \alpha \leq n - k - 1.
\end{cases}
$$

It is easy to see that $\mu \in \mathcal{C}_n(\alpha)$ and $\langle \mu \rangle_\alpha \lesssim 1$. Indeed, when $n - k - 1 < \alpha \leq n$, we note that $\mu(B^n(z, r)) \lesssim \lambda^{n-\alpha}r^n$ if $r \leq \lambda^{-1}$, and $\mu(B^n(z, r)) \lesssim \lambda^{n-\alpha-k-1}r^{n-k-1}$ if $r \geq \lambda^{-1}$. Similarly, when $0 < \alpha \leq n - k - 1$, we have $\mu(B^n(z, r)) \lesssim \lambda^{k+1}r^n$ if $r \leq \lambda^{-1}$ and $\mu(B^n(z, r)) \lesssim r^{n-k-1}$ if $r \geq \lambda^{-1}$. Combining the two cases, we see $\mu(B^n(z, r)) \lesssim r^\alpha$ as desired.

Note that $|Q| \sim \epsilon_{0}^{d+1} \lambda^{-k-1}$ and $\mu(Q) \gtrsim \epsilon_{0}^{d+1} \min \{ \lambda^{n-\alpha-k-1}, 1 \}$. Therefore,

$$
(4.6) \quad \min \{ \lambda^{(n-\alpha-k-1)/p}, 1 \} \lesssim \|A_{\gamma}^w f\|_{L^p_\lambda(\mu)} \lesssim \|f\|_{L^{\infty}_\xi(\mathbb{R}^d)} \lesssim \lambda^{-\sigma-(k-1)/p}.
$$

Letting $\lambda \rightarrow \infty$, we get (4.1).

We now consider the case $m = d$ and $\gamma(s) = (s^{a_1}, \ldots, s^{a_d})$. As before, we exploit homogeneity of monomials. We denote $\bar{x}$ and $\bar{r}$ same as above with $k = 2$, i.e., $\bar{x} = (x_1, x_2)$ and $\bar{r} = (x_3, \ldots, x_d)$. Set $c(t) = (t_1/t_2)^{1/(a_2-a_1)}$. By the change of variable $s \mapsto c(t)s$, we have

$$
A_{\gamma}^w f(x,t) = \prod_{j=1}^{d} \chi(t_j) \int f(x - \gamma^w(t)(c(t)s)) \psi(c(t)s)c(t) ds.
$$

Note that $\gamma^w(t)(c(t)s) = (t_1^{a_2}/t_2^{a_1})^{1/(a_2-a_1)}(s^{a_1}, s^{a_2})$. Then, we can repeat the same argument as above. Let $f$ and $\mu$ be given in the same manner as before with $k = 2$. We also set

$$
Q' = \{(x, t) \in [-\epsilon_0, \epsilon_0]^d \times [2^{-2}, 2]^d : |\bar{x}| \leq \epsilon_0\lambda^{-1}, \|(t_1^{a_2}/t_2^{a_1})^{1/(a_2-a_1)} - 1\| \leq \epsilon_0\lambda^{-1}\}.
$$

Then $A_{\gamma}^w f(x,t) \gtrsim 1$ whenever $(x, t) \in Q'$ for a sufficiently small $\epsilon_0 > 0$. Note that for each fixed $t_2$, we have $|t_1 - t_2^{a_2}/a_1| \lesssim \epsilon_0\lambda^{-1}$ whenever $(x, t) \in Q'$. Thus $|Q'| \sim \epsilon_0^{d+1} \lambda^{-3}$ and hence $\mu(Q') \gtrsim \epsilon_0^{d+1} \min \{ \lambda^{n-a-3}, 1 \}$. This gives (4.2). \qed
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References

[1] D. Beltran, S. Guo, J. Hickman, A. Seeger, Sharp $L^p$ bounds for the helical maximal function, arXiv:2102.08272.
[2] ______., Sobolev improving for averages over curves in $\mathbb{R}^4$, Adv. Math., 393 (2021) 108089.
[3] D. Beltran, J. Hickman, C. D. Sogge, Variable coefficient Wolff-type inequalities and sharp local smoothing estimates for wave equations on manifolds, Anal. PDE, 13 (2020), 403–433.
[4] A. Besicovitch, R. Rado, A plane set of measure zero containing circumferences of every radius, J. London Math. Soc., 43 (1968), 717–719.
[5] J. Bourgain, Averages in the plane over convex curves and maximal operators, J. Analyse. Math. 47 (1986), 69–85.
[6] J. Bourgain, C. Demeter, The proof of the $l^2$ decoupling conjecture, Ann. of Math. 182 (2015), 351–389.
[7] ______., Decouplings for curves and hypersurfaces with nonzero Gaussian curvature, J. Anal. Math. 133 (2017), 279–311.
[8] R. O. Davies, Another thin set of circles, J. London Math. Soc. 5 (1972) 191–192.
[9] K. J. Falconer, The geometry of fractal sets, Cambridge University Press 1985.
[10] C. Gao, B. Liu, C. Miao, Y. Xi, Square function estimates and Local smoothing for Fourier integral operators, arXiv:2010.14390.
[11] L. Guth, H. Wang, R. Zhang, A sharp square function estimate for the cone in $\mathbb{R}^3$, Ann. of Math. 192 (2020), 551–581.
[12] S. Ham, H. Ko, S. Lee, Circular average relative to fractal measures, To appear in Commun. Pure Appl. Anal., (2022), doi: 10.3934/cpaa.2022100.
[13] L. Hörmander, Fourier integral operators. I, Acta Math. 127 (1971), 79–183.
[14] A. Käenmäki, T. Orponen, L. Venieri, A Marstrand-type restricted projection theorem in $\mathbb{R}^3$, arXiv:1708.04859v2.
[15] J. Kinney, A thin set of circles, Amer. Math. Monthly, 75 (1968), 1077–1081.
[16] H. Ko, S. Lee, S. Oh, Maximal estimate for average over space curve, Invent. Math., 228 (2022), 991–1035.
[17] ______., Sharp smoothing properties of averages over curves, arXiv:2105.01628v4.
[18] L. Kolasa, T. Wolff, On some variants of the Kakeya problem, Pacific J. Math. 190 (1999), 111–154.
[19] B. Lepson, On a problem of Peter Fenton and the distance set of the Cantor set, Notices Amer. Math. Soc. 23 (1976) A-507.
[20] J. Marstrand, Packing circles in the plane, Proc. London Math. Soc. 55 (1987), 37–58.
[21] P. Mattila, Fourier analysis and Hausdorff dimension, Cambridge University Press, Cambridge, United Kingdom 2015.
[22] T. Mitsis, On a problem related to sphere and circle packing, J. London Math. Soc. 60 (1999), 501–516.
[23] A. Miyachi, On some estimates for the wave equation in $L^p$ and $H^p$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), 331–354.
[24] G. Mockenhaupt, A. Seeger, C. D. Sogge, Local smoothing of Fourier integral operators and Carleson-Sjölin estimates, J. Amer. Math. Soc. 6 (1993), 65–130.
[25] D. Oberlin, Packing spheres and fractal Strichartz estimates in $\mathbb{R}^d$ for $d \geq 3$, Proc. Amer. Math. Soc., 134 (2006), 3201–3209.
[26] ______., Restricted Radon transforms and unions of hyperplanes, Rev. Mat. Iberoamericana, 22 (2006), 977–992.
[27] ______., Unions of hyperplanes, unions of spheres, and some related estimates, Illinois J. of Math. 51 (2007), 1265–1274.
[28] M. Pramanik, A. Seeger, $L^p$ regularity of averages over curves and bounds for associated maximal operators, Amer. J. Math. 129 (2007), 61–103.
[29] A. Seeger, C.D. Sogge, E.M. Stein, Regularity properties of Fourier integral operators, Ann. of Math. 134 (1991), 251–251.
[30] ______, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, Cambridge University Press, 2017.

[31] M. Talagrand, *Sur la mesure de la projection d’un compact et certaines familles de cercles*, Bull. Sci. Math. **104** (1980), 225–231.

[32] T. Wolff, *A Kakeya type problem for circles*, Amer. J. Math. **119** (1997), 985–1026.

[33] ______, *Local smoothing estimates on L^p for large p*, Geom. Funct. Anal. **10** (2000), 1237–1288.

[34] J. Zahl, *On the Wolff circular maximal function*, Illinois J. Math. **56** (2012), 1281–1295.

[35] W. P. Ziemer, *Weakly Differentiable Functions*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1989.

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