A DOUBLY CRITICAL SEMILINEAR HEAT EQUATION IN THE $L^1$ SPACE

YASUHITO MIYAMOTO

ABSTRACT. We study the existence and nonexistence of a Cauchy problem of the semilinear heat equation

$$\begin{cases}
\partial_t u = \Delta u + |u|^{p-1} u & \text{in } \mathbb{R}^N \times (0, T), \\
u(x, 0) = \phi(x) & \text{in } \mathbb{R}^N
\end{cases}$$

in $L^1(\mathbb{R}^N)$. Here, $N \geq 1$, $p = 1 + 2/N$ and $\phi \in L^1(\mathbb{R}^N)$ is a possibly sign-changing initial function. Since $N(p-1)/2 = 1$, the $L^1$ space is scale critical and this problem is known as a doubly critical case. It is known that a solution does not necessarily exist for every $\phi \in L^1(\mathbb{R}^N)$. Let $X_q := \{ \phi \in L^1_{\text{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\phi| |\log(e + |\phi|)|^q \, dx < \infty \} (\subset L^1(\mathbb{R}^N))$. In this paper we construct a local-in-time mild solution in $L^1(\mathbb{R}^N)$ for $\phi \in X_q$ if $q \geq N/2$. We show that, for each $0 \leq q < N/2$, there is a nonnegative initial function $\phi_0 \in X_q$ such that the problem has no nonnegative solution, using a necessary condition given by Baras-Pierre [Ann. Inst. H. Poincaré Anal. Non Linéaire 2 (1985), 185–212]. Since $X_q \subset X_{N/2}$ ($q \geq N/2$), $X_{N/2}$ becomes a sharp integrability condition. We also prove a uniqueness in a certain set of functions which guarantees the uniqueness of the solution constructed by our method.

1. Introduction and main results

We consider the existence and nonexistence of a Cauchy problem of the semilinear heat equation

$$(1.1) \quad \begin{cases}
\partial_t u = \Delta u + |u|^{p-1} u & \text{in } \mathbb{R}^N \times (0, T), \\
u(x, 0) = \phi(x) & \text{in } \mathbb{R}^N,
\end{cases}$$

where $N \geq 1$, $p = 1 + 2/N$ and $\phi$ is a possibly sign-changing initial function. When $\phi \in L^\infty(\mathbb{R}^N)$, one can easily construct a solution by using a fixed point argument. When $\phi \not\in L^\infty(\mathbb{R}^N)$, the solvability depends on the balance between the strength of the singularity of $\phi$ and the growth rate of the nonlinearity. Weissler [13] studied the solvability of (1.1), and obtained the following:

**Proposition 1.1.** Let $q_c := N(p-1)/2$. Then the following (i) and (ii) hold:

(i) (Existence, subcritical and critical cases) Assume either both $q > q_c$ and $q \geq 1$ or $q = q_c > 1$. The problem (1.1) has a local-in-time solution for $\phi \in L^q(\mathbb{R}^N)$.

(ii) (Nonexistence, supercritical case) For each $1 \leq q < q_c$, there is $\phi \in L^q(\mathbb{R}^N)$ such that (1.1) has no local-in-time nonnegative solution.
| ranges of $q$ | $1 \leq q < q_c$ supercritical | $1 = q = q_c$ doubly critical | $1 < q = q_c$ critical | $q > q_c$, $q \geq 1$ subcritical |
|---------------------------------|-------------------------------|-----------------------------|-----------------------------|----------------------------------|
| existence/ nonexistence | not always exist | not always exist | exist | exist |
| Prop. 1.1 (ii) | Prop. 1.1 (i) | Prop. 1.1 (i) | Prop. 1.1 (i) | Prop. 1.1 (i) |
| Thm. 1.3 (i) | not exist: [2, 3, 7], Thm. 1.3 (i) | exist: [14, p.32], Thm. 1.3 (i) | Prop. 1.1 (i) | Prop. 1.1 (i) |

Table 1. Existence and nonexistence of a local-in-time solution of (1.1) in $L^q(\mathbb{R}^N)$.

Let $u(x, t)$ be a function such that $u$ satisfies the equation in (1.1). We consider the scaled function $u_\lambda(x, t) := \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t)$. Then, $u_\lambda$ also satisfies the same equation. We can easily see that $\|u_\lambda(x, 0)\|_p = \|u(x, 0)\|_q$ if and only if $q = q_c$. It is well known that $q_c$ is a threshold as Proposition 1.1 shows. However, the case $q = q_c = 1$, i.e., $p = 1 + 2/N$, is not covered by Proposition 1.1 and it is known that there is a nonnegative initial function $\phi \in L^1(\mathbb{R}^N)$ such that (1.1) with $p = 1 + 2/N$ has no local-in-time nonnegative solution. See Brezis-Cazenave [2, Theorem 11], Celik-Zhou [3, Theorem 4.1] or Laister et. al. [7, Corollary 4.5] for nonexistence results. See [1, 6, 11] and references therein for existence and nonexistence results with measures as initial data. In [2, Section 7.5] the case $p = 1 + 2/N$ is referred to as “doubly critical case”. Several open problems were given in [2]. It was mentioned in [14, p.32] that (1.1) has a local-in-time solution if $\phi \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for some $q > 1$. However, a solvability condition was not well studied. See Table 1. For a detailed history about the existence, nonexistence and uniqueness of (1.1), see [3, Section 1].

In this paper we obtain a sharp integrability condition on $\phi \in L^1(\mathbb{R}^N)$ which determines the existence and nonexistence of a local-in-time solution in the case $p = 1 + 2/N$. We also show that a solution constructed in Theorem 1.3 is unique in a certain set of functions. Throughout the present paper we define $f(u) := |u|^{p-1}u$. Let $L^q(\mathbb{R}^N)$, $1 \leq q \leq \infty$, denote the usual Lebesgue space on $\mathbb{R}^N$ equipped with the norm $\| \cdot \|_q$. For $\phi \in L^1(\mathbb{R}^N)$, we define

$$S(t)[\phi](x) := \int_{\mathbb{R}^N} G_t(x - y)\phi(y)dy,$$

where $G_t(x - y) := (4\pi t)^{-N/2} \exp\left(-\frac{|x-y|^2}{4t}\right)$. The function $S(t)[\phi]$ is a solution of the linear heat equation with initial function $\phi$. We give a definition of a solution of (1.1).

**Definition 1.2.** Let $u$ and $\bar{u}$ be measurable functions on $\mathbb{R}^N \times (0, T)$.

(i)(Integral solution) We call $u$ an integral solution of (1.1) if there is $T > 0$ such that $u$ satisfies the integral equation

$$u(t) = \mathcal{F}[u](t) \quad a.e. \ x \in \mathbb{R}^N, \ 0 < t < T, \ and \ \|u(t)\|_{\infty} < \infty \ for \ 0 < t < T,$$

where

$$\mathcal{F}[u](t) := S(t)[\phi] + \int_0^t S(t - s)f(u(s))ds.$$

(ii)(Mild solution) We call $u$ a mild solution if $u$ is an integral solution and $u(t) \in C([0, T), L^1(\mathbb{R}^N))$.

(iii) We call $\bar{u}$ a supersolution of (1.1) if $\bar{u}$ satisfies the integral inequality $\mathcal{F}[\bar{u}](t) \leq \bar{u}(t) < \infty$ for a.e. $x \in \mathbb{R}^N, 0 < t < T$. 
For $0 \leq q < \infty$, we define a set of functions by

$$X_q := \{ \phi(x) \in L^1_{\text{loc}}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\phi| [\log(e + |\phi|)]^q dx < \infty \}.$$  

It is clear that $X_q \subset L^1(\mathbb{R}^N)$ and that $X_{q_1} \subset X_{q_2}$ if $q_1 \geq q_2$. The main theorem of the paper is the following:

**Theorem 1.3.** Let $N \geq 1$ and $p = 1 + 2/N$. Then the following (i) and (ii) hold:

(i) (Existence) If $\phi \in X_q$ for some $q \geq N/2$, then (1.1) has a local-in-time mild solution $u(t)$, and this mild solution satisfies the following:

$$\text{there is } C > 0 \text{ such that } \|u(t)\|_{\infty} \leq Ct^{-\frac{N}{2}}(-\log t)^{-q} \text{ for small } t > 0.$$  

In particular, (1.1) has a local-in-time mild solution for every $\phi \in X_{N/2}$.

(ii) (Nonexistence) For each $0 \leq q < N/2$, there is a nonnegative initial function $\phi_0 \in X_q$, which is explicitly given by (4.1), such that (1.1) has no local-in-time nonnegative integral solution, and hence (1.1) has no local-in-time nonnegative mild solution.

**Remark 1.4.** (i) The function $\phi$ in Theorem 1.3 (i) is not necessarily nonnegative.

(ii) Theorem 1.3 indicates that $X_{N/2} \subset L^1(\mathbb{R}^N)$ is an optimal set of initial functions for the case $p = 1 + 2/N$, and $X_{N/2}$ is slightly smaller than $L^1(\mathbb{R}^N)$. This situation is different from the case $p > 1 + 2/N$, since (1.1) is always solvable in the scale critical space $L^{N(p-1)/2}$ for $p > 1 + 2/N$ (Proposition 1.1 (i)).

(iii) $L^1(\mathbb{R}^N)$ is larger than the optimal set for $p = 1 + 2/N$. On the other hand, it follows from Proposition 1.1 (i) that if $1 < p < 1 + 2/N$, then (1.1) has a solution for all $\phi \in L^1(\mathbb{R}^N)$.

Therefore, $L^1(\mathbb{R}^N)$ is small enough for the case $1 < p < 1 + 2/N$.

(iv) The function $\phi_0$ given in Theorem 1.3 (ii) is modified from $\psi(x)$ given by (1.4). This function comes from Baras-Pierre [1], and Theorem 1.3 (ii) is a rather easy consequence of Proposition 3.2. However, we include Theorem 1.3 (ii) for a complete description of the borderline property of $X_{N/2}$.

(v) Laister et.al. [7] obtained a necessary and sufficient condition for the existence of a local-in-time nonnegative solution of

$$\left\{ \begin{array}{l}
\partial_t u = \Delta u + h(u) \quad \text{in } \mathbb{R}^N \times (0,T), \\
u(x,0) = \phi(x) \geq 0 \quad \text{in } \mathbb{R}^N.
\end{array} \right.$$  

They showed that when $h(u) = u^{1+2/N}[\log(e + u)]^{-r}$, (1.4) has a local-in-time nonnegative solution for every nonnegative $\phi \in L^1(\mathbb{R}^N)$ if $1 < r < \lambda p$, and (1.4) does not always have if $0 \leq r \leq 1$. Here, $\lambda > 0$ is a certain constant. Therefore, the optimal growth of $h(u)$ for $L^1(\mathbb{R}^N)$ is slightly smaller than $u^{1+2/N}$.

(vi) The exponent $p = 1 + 2/N$, which is called Fujita exponent, also plays a key role in the study of global-in-time solutions. If $1 < p \leq 1 + 2/N$, then every nontrivial nonnegative solution of (1.1) blows up in a finite time. If $p > 1 + 2/N$, then (1.1) has a global-in-time nonnegative solution. See Fujita [4]. In particular, in the case $p = 1 + 2/N$ we cannot expect a global existence of a classical solution for small initial data.

The next theorem is about the uniqueness of the integral solution in a certain class.
Theorem 1.5. Let \( N \geq 1, p = 1 + 2/N \) and \( q > N/2 \). Then an integral solution \( u(t) \) of (1.1) is unique in the set
\[
\left\{ u(t) \in L^1(\mathbb{R}^N) \left| \sup_{0 \leq t \leq T} t^{N/2}(- \log t)^q \|u(t)\|_{\infty} < \infty \right. \right\}.
\]
Therefore, a solution given by Theorem 1.3 is unique.

Remark 1.6. (i) If there were a solution that does not satisfy (1.5), then the uniqueness fails. However, it seems to be an open problem.
(ii) In the case \( q = N/2 \) the uniqueness under (1.3) is left open.
(iii) For general \( p \) and \( q \), the uniqueness of a solution of (1.1) is known in the set
\[
\left\{ u(t) \in L^q(\mathbb{R}^N) \left| \sup_{0 \leq t \leq T} t^{\frac{N}{2}}(\frac{1}{q} - \frac{1}{p}) \|u(t)\|_{pq} < \infty \right. \right\}.
\]
See Haraux-Weissler [5] and [13]. For an unconditional uniqueness with a certain range of \( p \) and \( q \), see [2, Theorem 4].
(iv) The nonuniqueness in \( L^q(\mathbb{R}^N) \) is also known for (1.1). For \( p > 1 + 2/N \) and \( 1 \leq q < N(p - 1)/2 < p + 1 \), see [5]. For \( p = q = N/(N - 2) \), see Ni-Sacks [8] and Terraneo [12].

Let us mention technical details. We assume that \( \phi \in X_q \) for some \( q \geq N/2 \). Using a monotone method, we construct a nonnegative mild solution \( w(t) \) of
\[
\begin{aligned}
\partial_t w &= \Delta w + f(w) \quad \text{in } \mathbb{R}^N \times (0, T), \\
 w(x, 0) &= |\phi(x)| \quad \text{in } \mathbb{R}^N.
\end{aligned}
\]
We define \( g(u) \) by
\[
g(u) := u [\log(\rho + |u|)]^q,
\]
where \( \rho > 1 \) is chosen appropriately. We will see that if \( \rho \geq e \), then \( g(u) \) is convex for \( u \geq 0 \) and \( g \) plays a crucial role in the construction of the solution of (1.6). In order to construct a nonnegative solution we use a method developed by Robinson-Sierżega [10] with the convex function \( g \), which was also used in Hisa-Ishige [6]. We define a sequence of functions \( (u_n)_{n=0}^{\infty} \) by
\[
\begin{aligned}
u_n(t) &= F[u_{n-1}](t) \quad \text{for } 0 \leq t < T \quad \text{if } n \geq 1, \\
u_0(t) &= 0.
\end{aligned}
\]
Then, we show that \( -w(t) \leq u_n(t) \leq w(t) \) for \( 0 \leq t < T \). Since \( |u_n(t)| \leq w(t) \), we can extract a convergent subsequence in \( C_{\text{loc}}(\mathbb{R}^N \times (0, T)) \), using a parabolic regularization, the dominated convergence theorem and a diagonal argument. The limit function becomes a mild solution (1.1).

In the nonexistence part we use a necessary condition for the existence of a nonnegative solution of (1.1) obtained by Baras-Pierre [11], which is stated in Proposition 2.2 in the present paper. Using their result, one can show that there is \( c_0 > 0 \) such that if \( \phi(x) \leq c_0 \psi(x) \) in a neighborhood of the origin, then (1.1) has no nonnegative integral solution. Here,
\[
\psi(x) := |x|^{-N} (- \log |x|)^{-\frac{N}{2} - 1} \quad \text{for } 0 < |x| < 1/e.
\]
See also [6]. For each \( 0 \leq q < N/2 \) we will see that a modified function \( \phi_0 \), which is given by (4.1), belongs to \( X_q \). We show that \( \phi_0 \) does not satisfy the necessary condition for the
existence of an integral solution stated in Proposition 2.2. Hence, (1.1) with \( \phi_0 \) has no nonnegative solution for each \( 0 \leq q < N/2 \).

This paper consists of five sections. In Section 2 we recall known results including a monotone method, a necessary condition on the existence for (1.1) and \( L^p-L^q \)-estimates. In Section 3 we prove Theorem 1.3 (i). In Section 4 we prove Theorem 1.3 (ii). In Section 5 we prove Theorem 1.5.

2. Preliminaries

First we recall the monotonicity method.

Lemma 2.1. Let \( 0 < T \leq \infty \) and let \( f \) be a continuous nondecreasing function such that \( f(0) \geq 0 \). The problem (1.1) has a nonnegative integral solution for \( 0 < t < T \) if and only if (1.1) has a nonnegative supersolution for \( 0 < t < T \). Moreover, if a nonnegative supersolution \( \bar{u}(t) \) exists, then the solution \( u(t) \) obtained in this lemma satisfies \( 0 \leq u(t) \leq \bar{u}(t) \).

Proof. This lemma is well known. See [10, Theorem 2.1] for details. However, we briefly show the proof for readers' convenience.

If (1.1) has an integral solution, then the solution is also a supersolution. Thus, it is enough to show that (1.1) has an integral solution if (1.1) has a supersolution. Let \( \bar{u}(t) \) be a supersolution for \( 0 < t < T \). Let \( u_1 = S(t)\phi \). We define \( u_n = F[u_{n-1}] \).

Then we can show by induction that
\[
0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq \bar{u} < \infty \quad \text{a.e. } x \in \mathbb{R}^N, \quad 0 < t < T.
\]

This indicates that the limit \( \lim_{n \to \infty} u_n(x,t) \) which is denoted by \( u(x,t) \) exists for almost all \( x \in \mathbb{R}^N \) and \( 0 < t < T \). By the monotone convergence theorem we see that
\[
\lim_{n \to \infty} F[u_{n-1}] = F[u],
\]
and hence \( u = F[u] \). Then, \( u \) is an integral solution of (1.1). It is clear that \( 0 \leq u(t) \leq \bar{u}(t) \).

Baras-Pierre [1] studied necessary conditions for the existence of an integral solution in the case \( p > 1 \). See also [6] for details of necessary conditions including Proposition 2.2. The following proposition is a variant of [1] Proposition 3.2.

Proposition 2.2. Let \( N \geq 1 \) and \( p = 1 + 2/N \). If \( u(t) \) is a nonnegative integral solution, i.e., \( u(t) \) satisfies (1.2) with a nonnegative initial function \( \phi \) and some \( T > 0 \), then there exists a constant \( \gamma_0 > 0 \) depending only on \( N \) and \( p \) such that
\[
\int_{B(\tau)} \phi(x)dx \leq \gamma_0 |\log \tau|^{-N/2} \quad \text{for all } 0 < \tau < T,
\]
where \( B(\tau) := \{ x \in \mathbb{R}^N \mid |x| < \tau \} \).

Lemma 2.3. Let \( q \geq 0 \) be fixed, and let
\[
X_{q,\rho} := \left\{ \phi \in L^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\phi| |\log(\rho + |\phi|)|^q dx < \infty \right\}.
\]
Then, \( \phi \in X_{q,\rho} \) for all \( \rho > 1 \) if and only if \( \phi \in X_{q,\sigma} \) for some \( \sigma > 1 \).
Proof. We consider only the case \( q > 0 \). It is enough to show that \( \phi \in X_{q, \rho} \) for all \( \rho > 1 \) if \( \phi \in X_{q, \sigma} \) for some \( \sigma > 1 \). Let \( \rho > 1 \) be fixed, and let \( \xi(s) := \log(\rho + s) / (\log(s + \sigma)) \). By L'Hôpital's rule we see that \( \lim_{s \to \infty} \xi(s) = \lim_{s \to \infty} s^∗(s + \sigma)/(s + \rho) = 1 \). Since \( \xi(s) \) is bounded on each compact interval in \([0, \infty)\), we see that \( \xi(s) \) is bounded in \([0, \infty)\), and hence there is \( C > 0 \) such that \( \log(\rho + s) \leq C \log(\sigma + s) \) for \( s \geq 0 \). This inequality indicates that \( \phi \in X_{q, \rho} \) if \( \phi \in X_{q, \sigma} \).

Because of Lemma 2.1 we do not care about \( \rho > 1 \) in Proposition 2.4. In particular, if \( \phi \in X_q \), then \( \|g(\phi)\|_1 < \infty \) for every \( \rho > 1 \).

Proposition 2.4. (i) Let \( N \geq 1 \) and \( 1 \leq \alpha \leq \beta \leq \infty \). There is \( C > 0 \) such that, for \( \phi \in L^\alpha(\mathbb{R}^N) \),

\[
\|S(t)\phi\|_\beta \leq Ct^{-N\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)} \|\phi\|_\alpha \quad \text{for } t > 0.
\]

(ii) Let \( N \geq 1 \) and \( 1 \leq \alpha < \beta \leq \infty \). Then, for each \( \phi \in L^\alpha(\mathbb{R}^N) \) and \( C_0 > 0 \), there is \( t_0 = t_0(C_0, \phi) \) such that

\[
\|S(t)\phi\|_\beta \leq C_0t^{-N\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)} \quad \text{for } 0 < t < t_0.
\]

For Proposition 2.4 (i) (resp. (ii)), see [9, Proposition 48.4] (resp. [2, Lemma 8]). Note that \( C_0 > 0 \) in (ii) can be chosen arbitrary small.

We collect various properties of \( g \) defined by (1.7).

Lemma 2.5. Let \( q > 0 \) and let \( g_1(s) := s[\log(\rho + s)]^{-q} \). Then the following hold:

(i) If \( \rho > 1 \), then \( g'(s) > 0 \) for \( s > 0 \).

(ii) If \( \rho \geq e \), then \( g''(s) > 0 \) for \( s > 0 \).

(iii) If \( \rho \geq e \), then \( g_1(s) \leq g^{-1}(s) \) for \( s \geq 0 \).

(iv) If \( \rho > 1 \), then there is \( C_1 > 0 \) such that \( g^{-1}(s) \leq g_1(C_1 s) \) for \( s \geq 0 \).

(v) If \( \rho > e^{\nu/(q-1)} \), then \( g^{-1}(s)^p/s \) is nondecreasing for \( s \geq 0 \).

(vi) If \( \rho \geq e \), then, for \( \phi \in L^1(\mathbb{R}^N) \),

\[
S(t)\phi \leq g^{-1}(S(t)g(\phi)) \quad \text{for } t \geq 0.
\]

Proof. By direct calculation we have

\[
g'(s) = \left[\log(\rho + s)\right]^{q-1}\left\{\log(\rho + s) + \frac{qs}{s + \rho}\right\},
\]

\[
g''(s) = \frac{q[\log(\rho + s)]^{q-2}}{(s + \rho)^2}\left[s\left\{\log(\rho + s) + q - 1\right\} + 2\rho \log(\rho + s)\right].
\]

Thus, (i) and (ii) hold.

(iii) Since \( \rho \geq e \), we have

\[
g(g_1(s)) = \frac{s}{[\log(\rho + s)]^q}\left[\log\left(\rho + \frac{s}{[\log(\rho + s)]^q}\right)\right]^q \leq \frac{s}{[\log(\rho + s)]^q}[\log(\rho + s)]^q = s
\]

for \( s \geq 0 \). By (i) we see that \( g^{-1}(s) \) exists and it is increasing. By (2.3) we see that \( g_1(s) \leq g^{-1}(s) \) for \( s \geq 0 \).

(iv) Let \( \xi(s) := (g(g_1(s)))^{1/q} = \log(\rho + s)/[\log(\rho + s)]^q) \). Then, for each compact interval \( I \subset [0, \infty) \), there is \( c > 0 \) such that \( \xi(s) > c \) for \( s \in I \). By L'Hôpital’s rule we have

\[
\lim_{s \to \infty} \xi(s) = \lim_{s \to \infty} \frac{1 + \frac{2}{s}[\log(\rho + s)]^q}{1 + \frac{2}{s}[\log(\rho + s)]^q}\left\{1 - \frac{1}{1 + \frac{2}{s}[\log(\rho + s)]^q}\right\} = 1.
\]
and hence there is \( c_0 > 0 \) such that \( \xi(s) \geq c_0 \) for \( s \geq 0 \). Thus, \( g^{-1}(c_0 s) \leq g_1(s) \) for \( s \geq 0 \). Then, the conclusion holds.

(v) By (i) we see that \( g(\tau) \) is increasing. Let \( s := g(\tau) \). Then, \( g^{-1}(s)^p/s = \tau^{p-1}[\log(\rho + \tau)]^{-q} \). Since \( \rho > e^{q/(p-1)} \), we have

\[
\frac{d}{d\tau} \tau^{p-1}[\log(\rho + \tau)]^q \leq \frac{\tau^{p-2}}{[\log(\rho + \tau)]^{q+1}} \left\{ (p-1) \log(\rho + \tau) - \frac{q\tau}{\rho + \tau} \right\} > 0.
\]

Thus, \( g^{-1}(s)^p/s \) is increasing for \( s \geq 0 \).

(vi) Because of (ii), \( g \) is convex. By Jensen’s inequality we see that \( g(S(t)\phi) \leq S(t)g(\phi) \). Since \( g^{-1} \) exists and \( g^{-1} \) is increasing, the conclusion holds. The proof is complete. \( \square \)

3. Existence

**Lemma 3.1.** Let \( N \geq 1 \) and \( p = 1 + 2/N \). Assume that \( \phi \geq 0 \). If \( \phi \in X_q \) for some \( q \geq N/2 \), then (1.7) has a local-in-time nonnegative mild solution \( u(t) \), and \( \|u(t)\|_\infty \leq Ct^{-N/2}(-\log t)^{-q} \) for small \( t > 0 \).

**Proof.** First, we consider the case \( q = N/2 \). Let \( \rho \geq \max\{e^{q/(p-1)}, e\} \) be fixed. Let \( g \) be defined by (1.7). Here, \( q = N/2 \) and \( g \) satisfies Lemma 2.5. We define

\[
u(t) := 2g^{-1}(S(t)g(\phi)).
\]

We show that \( \nu \) is a supersolution. By Lemma 2.5 (vi) we have

\[
S(t)\phi \leq g^{-1}(S(t)g(\phi)) = \frac{\nu(t)}{2}.
\]

Next, we have

\[
\int_0^t S(t-s)f(\nu(s))ds = 2^p \int_0^t S(t-s) \left[ S(s)g(\phi) \frac{g^{-1}(S(s)g(\phi))^p}{S(s)g(\phi)} \right] ds
\leq 2^p S(t)g(\phi) \int_0^t \left\| \frac{g^{-1}(S(s)g(\phi))^p}{S(s)g(\phi)} \right\|_\infty ds
\leq 2^p g^{-1}(S(t)g(\phi)) \left\| \frac{S(t)g(\phi)}{g^{-1}(S(t)g(\phi))} \right\|_\infty \int_0^t \left\| \frac{g^{-1}(S(s)g(\phi))^p}{S(s)g(\phi)} \right\|_\infty ds.
\]

Since \( g(\phi) \in L^1(\mathbb{R}^N) \), by Proposition 2.4 (ii) we have

\[
\|S(t)g(\phi)\|_\infty \leq C_0 t^{-N/2}.
\]

By Lemma 2.5 (v) we see that \( g^{-1}(u)^p/u \) is nondecreasing for \( u \geq 0 \). Using (3.3) and Lemma 2.5 (iv), we have

\[
\left\| \frac{g^{-1}(S(s)g(\phi))^p}{S(s)g(\phi)} \right\|_\infty \leq \frac{g^{-1}(\|S(s)g(\phi)\|_\infty)^p}{\|S(s)g(\phi)\|_\infty}
\leq \frac{g^{-1}(C_0 s^{-N/2})^p}{C_0 s^{-N/2}} \leq \frac{C_1^p C_0^{2/N}}{s \log (\rho + C_1 s^{-N/2})^{pq}} \leq \frac{C_1^2 C_0^{2/N}}{s (-\log s)^{pq}}.
\]
for \(0 < s < s_0(C_0)\), where \(C'_1\) is a constant independent of \(C_0\). Using Lemma 2.5 (iii) and (3.3), we have

\[
(3.5) \quad \left\| \frac{S(t)g(\phi)}{g^{-1}(S(t)g(\phi))} \right\|_{\infty} \leq \left\| \frac{S(t)g(\phi)}{S(s)g(\phi)} \right\|_{\infty} = \left\| \log(\rho + S(t)g(\phi)) \right\|_{\infty} \leq \left( \log(\rho + t^{-N/2}) \right)^q \leq C'_2(-\log t)^q
\]

for \(0 < t < t_0(C_0)\), where \(g_t\) is defined in Lemma 2.5 and \(C'_2\) is a constant independent of \(C_0\). By (3.4) and (3.5) we have

\[
(3.6) \quad \left\| \frac{S(t)g(\phi)}{g^{-1}(S(t)g(\phi))} \right\|_{\infty} \int_0^t \left\| \frac{S^{-1}(S(s)g(\phi))}{S(s)g(\phi)} \right\|_{\infty} ds \leq C'_0^2 NC'_1 C'_2(-\log t)^q \int_0^t \frac{ds}{s(-\log s)^p} = C'_0^2 NC'_1 C'_2(-\log t)^q \frac{2}{N(-\log t)^q} = C'_0^2 NC'_1 C'_2 \frac{2}{N}
\]

for \(0 < t < \min\{s_0(C_0), t_0(C_0)\}\). By Proposition 2.4 (ii) we can take \(C_0 > 0\) such that \(2^{p+1} C'_0^2 NC'_1 C'_2/N < 1\). By (3.1), (3.2) and (3.6) we have

\[
\mathcal{F}[\bar{u}](t) = S(t)\phi + \int_0^t S(t-s)f(\bar{u}(s))ds \leq \frac{1}{2} \bar{u}(t) + \frac{1}{2} \bar{u}(t) = \bar{u}(t)
\]

for small \(t > 0\). Thus, there is \(T > 0\) such that \(\mathcal{F}[\bar{u}] \leq \bar{u}\) for \(0 < t < T\), and hence \(\bar{u}\) is a supersolution. By Lemma 2.1 we see that there is \(T > 0\) such that (1.1) has a solution for \(0 < t < T\), and \(u(t)\) is clearly nonnegative. Moreover,

\[
(3.7) \quad 0 \leq u(t) \leq \bar{u}(t) = 2g^{-1}(S(t)g(\phi)) \leq Ct^{-\frac{N}{2}}(-\log t)^{-q},
\]

which is the estimate in the assertion. We show that \(u(t) \in C([0, T), L^1(\mathbb{R}^N))\). Since \(\|g^{-1}(u)\|_1 \leq C\|u\|_1\), by (3.6) and Proposition 2.4 (i) we have

\[
(3.8) \quad \|u(t) - S(t)\phi\|_1 \leq \left\| \int_0^t S(t-s)f(\bar{u}(s))ds \right\|_1 \leq C'_0^2 NC'_1 C'_2 \frac{2}{N} \left\| g^{-1}(S(t)g(\phi)) \right\|_1 \leq C'_0^2 NC'_1 C'_2 \frac{2}{N} C\|S(t)g(\phi)\|_1 \leq C'_0^2 NC'_1 C'_2 \frac{2}{N} C'\|g(\phi)\|_1
\]

for small \(t > 0\), where \(C'\) is independent of \(C_0\). By Proposition 2.4 (ii) we can take \(C_0 > 0\) arbitrary small, and hence

\[
\|u(t) - S(t)\phi\|_1 \to 0 \quad \text{as} \quad t \downarrow 0.
\]

Since \(S(t)\) is a strongly continuous semigroup on \(L^1(\mathbb{R}^N)\) (see e.g., [9] Section 48.2)), we have

\[
(3.9) \quad \|u(t) - \phi\|_1 \leq \|u(t) - S(t)\phi\|_1 + \|S(t)\phi - \phi\|_1 \to 0 \quad \text{as} \quad t \downarrow 0.
\]

It follows from (3.2) and (3.6) that \(\left\| \int_0^t S(t-s)f(\bar{u}(s))ds \right\|_1 < \infty \) for \(0 < t < T\). We see that if \(0 < t < T\), then

\[
(3.10) \quad \|u(t + h) - u(t)\|_1 \to 0 \quad \text{as} \quad h \to 0.
\]

By (3.9) and (3.10) we see that \(u(t) \in C([0, T), L^1(\mathbb{R}^N))\). The proof of (i) is complete.
Next, we consider the case $q > N/2$. The argument is the same until (3.6). We have
\begin{equation}
(3.11) \left\| \frac{S(t)g(\phi)}{g^{-1}(S(t)g(\phi))} \right\|_\infty \int_0^t \left\| \frac{g^{-1}(S(s)g(\phi))}{S(s)g(\phi)} \right\|_\infty ds \leq C_0^{2/N} C_1^r C_2' (-\log t)^q \int_0^t \frac{ds}{s(-\log s)^{pq}}
= C_1^{2/N} C_1^r C_2' (-\log t)^{1 - \frac{2q}{pq}}
\end{equation}

instead of (3.6). Since the RHS of (3.11) goes to 0 as $t \downarrow 0$, the rest of the proof is almost the same with obvious modifications. In particular, (3.7) holds even for $q > N/2$. We omit the details. \hfill \square

We consider (1.6), where $\phi$ is given in (1.1). By Lemma 3.1 we see that (1.6) has a local-time solution which is denoted by $w(t)$. We consider the sequence $(u_n)_{n=0}^\infty$ defined by (1.8). Then, the following lemma says that $\|u_n(t)\|_\infty$ can be controlled by $w(t)$.

**Lemma 3.2.** Let $u_n$ be as defined by (1.8), and let $w$ be a solution of (1.6) on $(0,T)$. Then,
\begin{equation}
-w(t) \leq u_n(t) \leq w(t) \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and } 0 < t < T.
\end{equation}

**Proof.** It is clear from the definitions of $u_0$ and $w(t)$ that
\[ u_0(t) \leq w(t) \quad \text{for } 0 < t < T. \]

We assume that $u_{n-1}(t) \leq w(t)$ on $(0,T)$. Then, we have
\[ w(t) = S(t)|\phi| + \int_0^t S(t-s)f(w(s))ds \geq S(t)|\phi| + \int_0^t S(t-s)f(u_{n-1}(s))ds = u_n(t), \]
and hence $u_n(t) \leq w(t)$ for $0 < t < T$. Thus, by induction we see that, for $n \geq 0$,
\begin{equation}
(3.13) \quad u_n(t) \leq w(t) \quad \text{on } 0 < t < T.
\end{equation}

It is clear that $u_0(t) \geq -w(t)$ for $0 < t < T$. We assume that $u_{n-1}(t) \geq -w(t)$ on $(0,T)$. Then, we have
\[ u_n(t) = S(t)|\phi| + \int_0^t S(t-s)f(u_{n-1}(s))ds \geq -S(t)|\phi| + \int_0^t S(t-s)f(-w(s))ds = -w(t), \]
and hence, $u_n(t) \geq -w(t)$ on $(0,T)$. Thus, by induction we see that for $n \geq 0$,
\begin{equation}
(3.14) \quad -w(t) \leq u_n(t) \quad \text{on } 0 < t < T.
\end{equation}

By (3.13) and (3.14) we see that (3.12) holds. \hfill \square

**Proof of Theorem 1.3.** (i) Let $(u_n)_{n=0}^\infty$ be defined by (1.8). Using an induction argument with a parabolic regularity theorem, we can show that, for each $n \geq 1$, $u_n \in C^{2,1}(\mathbb{R}^N \times (0,T))$ and $u_n$ satisfies the equation
\[ \partial_t u_n = \Delta u_n + f(u_{n-1}) \quad \text{in } \mathbb{R}^N \times (0,T) \]
in the classical sense. Let $K$ be an arbitrary compact subset in $\mathbb{R}^N \times (0,T)$, and let $K_1$, $K_2$ be two compact sets such that $K \subset K_1 \subset K_2 \subset \mathbb{R}^N \times (0,T)$. Because of Lemma 3.2, $f(u_{n-1})$ is bounded in $C(K_2)$. By a parabolic regularity theorem we see that $u_n$ is bounded in $C^{2+\gamma/2}(K_1)$. Using a parabolic regularity theorem again, we see that $u_{n+1}$ is bounded in $C^{2+\gamma+1/2+\gamma/2}(K_1)$.

In the following we use a diagonal argument to obtain a convergent subsequence in $\mathbb{R}^N \times (0,T)$. Let $Q_j := \{x \in \mathbb{R}^N | |x| \leq j\} \times \left[ \frac{T}{j+2}, \frac{(j+1)T}{j+2} \right]$. Since $(u_n)_{n=3}^\infty$ is bounded in $C^{2,1}(Q_1)$,
Thus, we take a limit of $u(3.17) \lim_{k \to \infty} F$ arbitrariness of $C(3.15)$ $u \in C(Q_1)$ as $k \to \infty$. Since $(u_{1,k})_{k=1}^\infty$ is bounded in $C^{2,1}(Q_2)$, there is a subsequence $(u_{2,k}) \subset (u_{1,n})$ and $u^*_2 \in C(Q_2)$ such that $u_{2,k} \to u^*_2$ in $C(Q_2)$ as $k \to \infty$. Repeating this argument, we have a double sequence $(u_{j,k})$ and a sequence $(u^*_j)$ such that, for each $j \geq 1$, $u_{j,k} \to u^*_j$ in $C(Q_j)$ as $k \to \infty$. We still denote $u_{n,n}$ by $u_n$, i.e., $u_n := u_{n,n}$. It is clear that $u^*_1 \equiv u^*_2 \in Q_1$ if $j_1 \leq j_2$. Since $\mathbb{R}^N \times (0, T) = \bigcup_{j=1}^\infty Q_j$, there is $u^* \in C(\mathbb{R}^N \times (0, T))$ such that $u_n \to u^*$ in $C(K)$ as $n \to \infty$ for every compact set $K \subset \mathbb{R}^N \times (0, T)$. In particular, $u_n \to u^*$ a.e. in $\mathbb{R}^N \times (0, T)$.

Let $w$ be a solution of (1.6). It follows from Lemma 3.2 that $|u_n(x, t)| \leq w(x, t)$. Since $G_t(x - y)u_n(y, t) | \leq G_t(x - y)w(y, t) |$ for $y \in \mathbb{R}^N$, and

$$G_t(x - y)w(y, t) \in L^1_g(\mathbb{R}^N),$$

by the dominated convergence theorem we see that

$$\lim_{n \to \infty} S(t)u_n = \lim_{n \to \infty} \int_{\mathbb{R}^N} G_t(s - y)u_n(y, t)dy = \int_{\mathbb{R}^N} G_t(s - y)u^*(y, t)dy = S(t)u^*.$$

By (3.2) and (3.6) we see that if $T > 0$ is small, then

$$\int_0^t \int_{\mathbb{R}^N} G_{t-s}(x - y)f(w(y, s))dyds \leq Cg^{-1}(S(t)g(\phi)) < \infty$$

for each $(x, t) \in \mathbb{R}^N \times (0, T)$, and hence $G_{t-s}(x - y)f(w(y, s)) \in L^1_{(y,s)}(\mathbb{R}^N \times (0, T))$. Since $|G_{t-s}(x - y)f(u_{n-1}(y, s)) | \leq |G_{t-s}(x - y)f(w(y, s)) |$ for a.e. $(y, s) \in \mathbb{R}^N \times (0, T)$ and

$$G_{t-s}(x - y)f(w(y, s)) \in L^1_{(y,s)}(\mathbb{R}^N \times (0, T)),$$

by the dominated convergence theorem we see that

$$\lim_{n \to \infty} \int_0^t S(t-s)f(u_{n-1}(s))ds = \lim_{n \to \infty} \int_0^t \int_{\mathbb{R}^N} G_{t-s}(x - y)f(u_{n-1}(y, s))dyds = \int_0^t \int_{\mathbb{R}^N} G_{t-s}(x - y)f(u^*(y, s))dyds = \int_0^t S(t-s)f(u^*(s))ds.$$

Thus, we take a limit of $u_n = F[u_{n-1}]$. By (3.15), (3.16) and (3.17) we see that $u^*(t) = F[u^*](t)$ for $0 < t < T$.

Since $|u_n| \leq w$, we see that $|u^*| \leq w$. Since $|u^*| \leq w$ in $\mathbb{R}^N \times (0, T)$, by (3.8) and the arbitrariness of $C_0 > 0$ we have

$$\|u^*(t) - S(t)\phi\|_1 = \left\| \int_0^t S(t-s)f(u^*(s))ds \right\|_1 \leq \left\| \int_0^t S(t-s)f(w(s))ds \right\|_1 \to 0 \text{ as } t \downarrow 0.$$

Then, $\|u^*(t) - \phi\|_1 \leq \|u^*(t) - S(t)\phi\|_1 + \|S(t)\phi - \phi\|_1 \to 0$ as $t \downarrow 0$. Since $\left\| \int_0^t S(t-s)f(w(s))ds \right\|_1 < \infty$ for $0 < t < T$, we can show by a similar way to the proof of Lemma 3.1 that $u^*(t) \in C((0, T), L^1(\mathbb{R}^N))$. Thus, $u^*(t) \in C((0, T), L^1(\mathbb{R}^N))$, and hence $u^*(t)$ is a mild solution. Since $|u^*(t)| \leq w(t)$, by Lemma 3.1 we have (1.3). The proof of (i) is complete. \qed
4. Nonexistence

Let $0 \leq q < N/2$ be fixed. Then there is $0 < \varepsilon < N/2 - q$. We define $\phi_0$ by

\[
\phi_0(x) := \begin{cases} 
|x|^{-N} (- \log |x|)^{-N/2 - 1 + \varepsilon} & \text{if } |x| < 1/e, \\
0 & \text{if } |x| \geq 1/e.
\end{cases}
\]

Lemma 4.1. Let $0 \leq q < N/2$, and let $\phi_0$ be defined by (4.1). Then the following hold:
(i) $\phi_0 \in X_q(\subset L^1(\mathbb{R}^N))$.
(ii) The function $\phi_0$ does not satisfy (2.1) for any $T > 0$.

Proof. (i) We write $\phi_0(r) = r^{-N} (- \log r)^{-N/2 - 1 + \varepsilon}$ for $0 < r < 1/e$. Since $\log(e + s) \leq 1 + \log s$ for $s \geq 0$, we have

\[
\int B(1/e) |\phi_0| [\log(e + |\phi_0|)]^q dx \leq \omega_{N-1} \int_0^{1/e} \frac{(2N)^q (- \log r)^q r^{N-1} dr}{r^N (- \log r)^{N/2 + 1 - \varepsilon}} \leq (2N)^q \omega_{N-1} \int_0^{1/e} \frac{dr}{r (- \log r)^{N/2 + 1 - q - \varepsilon}} = \frac{(2N)^q \omega_{N-1}}{N - q - \varepsilon} < \infty,
\]

where $\omega_{N-1}$ denotes the area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$. By (4.1) we see that $\phi_0 \in X_q$.
(ii) Suppose the contrary, i.e., there exists $\gamma_0 > 0$ such that (2.1) holds. When $0 < \tau < 1/e$, we have

\[
\int_{B(\tau)} \phi_0(x) dx = \omega_{N-1} \int_0^\tau \frac{dr}{r (- \log r)^{N/2 + 1 - \varepsilon}} = \frac{C}{(- \log \tau)^{N/2 - \varepsilon}},
\]

where $C > 0$ is independent of $\tau$. Then,

\[
\gamma_0 \geq \frac{\int_{B(\tau)} \phi_0(x) dx}{(- \log \tau)^{-N/2}} \geq C(- \log \tau)^\varepsilon \to \infty \text{ as } \tau \downarrow 0,
\]

which is a contradiction. Thus, the conclusion holds.

Proof of Theorem 1.3 (ii). Let $0 \leq q < N/2$. It follows from Lemma 4.1 (i) that $\phi_0 \in X_q$. By Lemma 4.1 (ii) we see that there does not exist $\gamma_0 > 0$ such that (2.1) holds. By Proposition 2.2 the problem (1.1) with $\phi_0$ has no nonnegative integral solution.

5. Uniqueness

Proof of Theorem 1.5. Let $q > N/2$. Suppose that (1.1) has two integral solutions $u(t)$ and $v(t)$. Using Young’s inequality and the inequality $\|u(t)\|_\infty \leq C t^{-N/2}(- \log t)^{-q}$, we have

\[
\|u(t) - v(t)\|_1 \leq \int_0^t \|G_{t-s} \ast \{ (p|u|^{p-1} + p|v|^{p-1})(u - v) \}\|_1 ds \leq p \int_0^t \|G_{t-s}\|_1 \|u\|_\infty^{p-1} \|v\|_\infty^{p-1} ds \sup_{0 \leq s \leq t} \|u(s) - v(s)\|_1 \leq C \int_0^t \frac{ds}{s^{N/2}(- \log s)^{N/2 - q}} \sup_{0 \leq s \leq t} \|u(s) - v(s)\|_1.
\]
Since
\[ \int_0^t s^{-N(p-1)/2}(-\log s)^{-(p-1)q}ds = \frac{N(-\log t)^{1-2q/N}}{2q-N} \]
and \(1 - 2q/N < 0\), we can choose \(T > 0\) such that \(C \int_0^t s^{-N(p-1)/2}(-\log s)^{-(p-1)q}ds < 1/2\) for every \(0 \leq t \leq T\). Then, we have
\[ \sup_{0 \leq t \leq T} \|u(t) - v(t)\|_1 \leq \frac{1}{2} \sup_{0 \leq s \leq T} \|u(s) - v(s)\|_1, \]
which implies the uniqueness. \(\square\)

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