Girth conditions and Rota’s basis conjecture

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July 2019

Abstract

Rota’s basis conjecture (RBC) states that given \( n \) bases \( B_1, \ldots, B_n \) of a matroid \( M \) of rank \( n \), one can always find \( n \) disjoint transversal bases of \( B_1, \ldots, B_n \). By modifying the arguments of Bucić et al. [3], we show that one can obtain vastly improved lower bounds on the number of transversal bases attainable, assuming a lower bound on the girth \( g \). Specifically, we show that if \( M \) is regular, and \( g \in \omega(\log n) \), then we can attain \((1 - o(1))n\) disjoint transversal bases. Furthermore, we employ a result of Nelson and van Zwam [17] to prove the following: Let \( F \) be a finite field of characteristic \( p \), and let \( \mathcal{M} \) be a minor-closed class of \( F \)-representable matroids, not containing all projective geometries over \( F_p \). If \( M \in \mathcal{M} \) and \( g \geq \delta n \), where \( \delta \in (0, 1) \), then we can attain \((1 - o(1))n\) disjoint transversal bases.

Keywords: base, girth, matroid, transversal.

AMS Subject Classifications (2012): 05B35.

1 Introduction

For basic concepts and notation pertaining to matroids, we follow Oxley [19]. Given a matroid \( M = (E, \mathcal{I}) \), we let \( r(M) \) denote the rank of \( M \), and we let \( \mathcal{E}(M) = |E(M)| \). We denote the collection of bases of \( M \) by \( \mathcal{B}(M) \). The girth of \( M \), denoted \( g(M) \), is the size of a smallest circuit, provided that \( M \) has a circuit; otherwise \( g(M) = \infty \). For a subset \( X \subseteq E \), we denote the closure of \( X \) in \( M \) by \( \text{cl}_M(X) \). Given a graph \( G \), we let \( V(G) \) and \( E(G) \) denote its vertex set and edge set, respectively. Furthermore, we let \( \nu(G) = |V(G)| \) and \( \mathcal{E}(G) = |E(G)| \).

Let \( M \) be a matroid of rank \( n \). A basis sequence of \( M \) is an element \( \mathcal{B} = (B_1, \ldots, B_n) \in \mathcal{B}(M)^n \), where we think of each base \( B_c \) as “coloured” with colour \( c \). A transversal base of \( \mathcal{B} \) is a base that contains exactly one distinguished element of each colour. Transversal bases of \( \mathcal{B} \) are said to be disjoint if for each colour \( c \), the representatives of colour \( c \) are distinct. We let \( t_M(\mathcal{B}) \) denote the cardinality of a largest set of disjoint transversal bases of \( \mathcal{B} \).

In 1989, Rota made the following conjecture, first communicated in [14]:

Conjecture 1.1 (Rota’s Basis Conjecture (RBC)) Let \( M \) be a matroid of rank \( n \), and let \( \mathcal{B} \in \mathcal{B}(M)^n \). Then \( t_M(\mathcal{B}) = n \).

Rota’s original conjecture concerns \( n \) bases of a vector space of dimension \( n \), over a field of characteristic 0. For even dimensions, this version of RBC is implied by the Alon-Tarsi conjecture (ATC), which states that the sum of the signs of the Latin squares of even order \( n \) is non-zero [2]. See [9] for an overview of ATC. The fact that ATC implies RBC for even \( n \) was known to Huang and Rota [13], but the proof was simplified

∗Research supported by NSERC USRA
†Research supported by NSERC discovery grant
by Onn [18]. As a result of the work of Drisko [8] and Glynn [13] on ATC, RBC is known to be true for vector spaces of dimension \( p \pm 1 \), where \( p \) is an odd prime.

Chan [4] and Cheung [6] proved RBC for matroids of rank 3 and 4, respectively. Geelen and Humphries [11] proved RBC for paving matroids, and Wild [20] proved the conjecture for strongly base-orderable matroids.

One approach to RBC is to determine lower bounds on \( t_M(B) \). This approach was taken by Geelen and Webb [12], who proved that \( t_M(B) \in \Omega(\sqrt{n}) \), Dong and Geelen [7], who showed that \( t_M(B) \in \Omega(n/\log n) \), and most recently by Bucić et al. [3], who proved that \( t_M(B) \geq (1/2 - o(1))n \).

In this paper, we adapt the arguments of Bucić et al. [3] to show that their (1-\( \epsilon \))-RBC is true for a much more general class of matroids, with a strong girth condition: that \( n \) can be vastly improved if a girth requirement is imposed. In the following theorems, we shall modify their arguments to obtain bounds involving a girth requirement.

**Theorem 1.2** If \( M \) is graphic, and \( g(M) \geq \psi(n) \), then for every \( \epsilon > 0 \), we have \( t_M(B) \geq (1-\epsilon)n \), provided that \( n \) is sufficiently large.

We generalize the theorem above to regular matroids by applying a theorem of Kashyap [16]:

**Theorem 1.3** If \( M \) is regular, and \( g(M) \geq \psi(n) \), then for every \( \epsilon > 0 \), we have \( t_M(B) \geq (1-\epsilon)n \), provided that \( n \) is sufficiently large.

Finally, using a result of Nelson and van Zwam [17] on asymptotically good classes of matroids, we prove a theorem for a much more general class of matroids, with a stronger girth condition:

**Theorem 1.4** Suppose \( \mathbb{F} \) is a finite field of characteristic \( p \), and \( \mathfrak{M} \) is a minor-closed class of \( \mathbb{F} \)-representable matroids, not containing all projective geometries over \( \mathbb{F}_p \). If \( M \in \mathfrak{M} \), and \( g(M) \geq \delta n \), for some \( \delta \in (0,1) \), then for every \( \epsilon > 0 \), we have \( t_M(B) \geq (1-\epsilon)n \), provided that \( n \) is sufficiently large.

## 2 Swapping numbers and the number of disjoint transversal bases

The concepts and ideas from Bucić et al. [3] are a critical ingredient in the proof of the main theorems. We shall modify their arguments to obtain bounds involving a swapping number, a concept to be defined in this section.

### 2.1 RIS’s, swapping and adding elements, swapping numbers

Throughout this section, we let \( B = (B_1,\ldots,B_n) \) be a basis sequence of a matroid \( M = (E,I) \) of rank \( n \). As in [3], we let \( U = \{(x,c) : x \in B_c\} \) and \( \pi : U \rightarrow E \) be the projection \( \pi : (x,c) \mapsto x \). A subset \( S \subset U \) is called a **rainbow independent set** (RIS) if \( \pi(S) \subset I \), and all elements of \( S \) have a distinct colour. Thus, an RIS corresponds to an independent partial transversal of \( B \), and an RIS of size \( n \) corresponds to a transversal base of \( B \). Let \( \mathcal{B}(B) \) be the set of all collections of disjoint RIS’s. We define the **volume** of \( \mathcal{J} \in \mathcal{B}(B) \) to be \( \text{vol}(\mathcal{J}) = \sum_{S \in \mathcal{J}} |S| \), and we define \( F(\mathcal{J}) = \cup_{S \in \mathcal{J}} S \). One should think of \( F(\mathcal{J}) \) as the set of “used elements”. In [3], it is shown that starting with a collection \( \mathcal{J} \) of \( f < \frac{n}{2} \) RIS’s and any small \( 0 < \epsilon < 1 \), one can construct, provided \( n \) is large enough, \( f - \epsilon \frac{n}{2} \) transversal bases in a step-by-step procedure where at each step one

![Figure 1](image-url)
increases the volume of $\mathcal{S}$ by adding an element of $U$ directly, or one increases the volume by a sequence of ‘swaps’. We shall use the same strategy, with the additional assumption that the matroids involved belong to a class having the asymptotic swapping number property. We show in Section 3 that the classes of matroids with this property include regular matroids and larger classes satisfying a certain girth condition.

We say that $\mathcal{S} \in \mathcal{W}_M(B)$ is maximized if $\text{vol}(\mathcal{S}) \geq \text{vol}(\mathcal{S}')$ for all $\mathcal{S}' \in \mathcal{W}_M(B)$ such that $|\mathcal{S}'| = |\mathcal{S}|$. Given $\mathcal{S} \in \mathcal{W}_M(B)$, we let $t_M(\mathcal{S})$ denote the number of number of transversal bases in $\mathcal{S}$. That is, $t_M(\mathcal{S}) = |\{S \in \mathcal{S} : |S| = n\}|$. Note that $t_M(B) \geq t_M(\mathcal{S})$, for all $\mathcal{S} \in \mathcal{W}_M(B)$.

Let $\mathcal{S} \in \mathcal{W}_M(B)$, and suppose $S \in \mathcal{S}$ is missing a colour $b$. Following the definitions in [3], a colour $c$ is said to be $(S,b)$-swappable if there exists an element $(x',c) \in S$, and an element $(y,b) \in U - F(\mathcal{S})$, such that $S - (x',c) + (y,b)$ is an RIS. The set $S - (x',c) + (y,b)$ is the result of a simple swap on $S$. We call $y \in B_b$ a witness to the $(S,b)$-swappability of $c$. An element $(x,c) \in U$ is said to be $(S,b)$-addable if either

- $S + (x,c)$ is an RIS, or
- there are elements $(x',c) \in S, (y,b) \in U - F(\mathcal{S})$ such that $S - (x',c) + (y,b) + (x,c)$ is an RIS.

In the former case, we say that $(x,c)$ is directly $(S,b)$-addable, and in the latter case, we say that $(x,c)$ is $(S,b)$-addable with a witness $y \in B_b$. Note that addition of an element with a witness is equivalent to a simple swap followed by a direct addition. Observe that if $\mathcal{S}$ is maximized, then all $(S,b)$-addable elements must be contained in $F(\mathcal{S})$. We will make use of the following two lemmas, which appear in [3]:

**Lemma 2.1** Let $S \in \mathcal{S}$ be non-empty and missing a colour $b$. If $c$ is an $(S,b)$-swappable colour with witness $y$, then either $(y,b)$ is $(S,b)$-addable, or $(x,c)$ is $(S,b)$-addable, for every $x \in B_c$ independent of $\pi(S)$.

**Lemma 2.2** Let $S \in \mathcal{S}$. Then for any colour $b$, there exists an injection $\phi_b : S \to B_b$ such that $\phi_b(x,c)$ is independent of $\pi(S - (x,c))$, for all $(x,c) \in S$.

For a colour $b$, let $F_b = \pi(F(\mathcal{S})) \cap B_b$. We will need the following simple but important lemma:

**Lemma 2.3** Suppose $S \in \mathcal{S}$ is missing a colour $b$. Let $S' \subset S$ be the set of elements of $S$ with an $(S,b)$-swappable colour. If there are no $(S,b)$-addable elements, then $B_b - F_b \subset \text{cl}_M(\pi(S'))$.

**Proof:** By contradiction. We may assume that there are no $(S,b)$-addable elements and that there exists an element $y \in B_b - F_b$ such that $y \notin \text{cl}_M(\pi(S'))$. Since there are no $(S,b)$-addable elements, we have $B_b - F_b \subseteq \text{cl}_M(\pi(S))$. Let $C$ be the (unique) circuit in $\pi(S) + y$ (where $y \in C$). Then there exists an $x \in C \cap (\pi(S) - S') \neq \emptyset$ and we see that $\pi(S) - x + y$ is independent. Letting $c$ be the colour such that $(x,c) \in S$, it follows that $c$ is $(S,b)$-swappable, and hence $(x,c) \in S'$, a contradiction.

In addition to the definitions found in [3], we define the so-called swapping number and the asymptotic swapping number property. For $\mathcal{S} \in \mathcal{W}_M(B)$, we define $\lambda(\mathcal{S}) \in \mathbb{Z}_+$ to be the minimum integer $\lambda$ such that for all $S \in \mathcal{S}$ and for any colour $b$ not appearing in $S$, either there is an element $(y,b) \in U - F(\mathcal{S})$ which is $(S,b)$-addable, or there are at least $\lambda$ colours which are $(S,b)$-swappable. We call $\lambda(\mathcal{S})$ the swapping number of $\mathcal{S}$.

**Definition:** We say that a class of matroids $\mathcal{M}$ satisfies the asymptotic swapping number property, or ASN property, if for every $\epsilon > 0$, there exists an $n_0(\epsilon) \in \mathbb{Z}_+$ such that the following holds for all $M \in \mathcal{M}$ of rank $n > n_0(\epsilon)$: given any base sequence $B = (B_1, \ldots, B_n)$ of $M$, and any maximized $\mathcal{S} \in \mathcal{W}_M(B)$ of size $|\mathcal{S}| \leq (1 - \epsilon)n$, we have $\lambda(\mathcal{S}) \geq (1 - \epsilon)n$.

We are now equipped to state the main theorem of this section:

**Theorem 2.4** Let $\mathcal{M}$ be a class of matroids satisfying the ASN property, and let $B = (B_1, \ldots, B_n)$ be a basis sequence of a matroid $M \in \mathcal{M}$ of rank $n$. Then for all $\epsilon > 0$, we have $t_M(B) \geq (1 - \epsilon)n$, provided that $n$ is sufficiently large.

Theorem 2.4 is implied by the following lemma:
**Lemma 2.5** Let \( \mathcal{M} \) be a class of matroids satisfying the ASN property, and let \( \mathcal{B} = (B_1, \ldots, B_n) \) be a basis sequence of a matroid \( M \in \mathcal{M} \) of rank \( n \). Suppose that \( 0 < \epsilon < 1 \), \( 0 < \delta < 1 \) and \( f \in \mathbb{Z}_+ \) with \( f \leq (1 - \epsilon)\delta n \). Then for sufficiently large \( n \), there exists a maximized \( \mathcal{F} \in \mathcal{W}_M(\mathcal{B}) \) where \( |\mathcal{F}| = f \) and \( t_M(\mathcal{F}) \geq f - \epsilon\delta n \).

A proof of this implication is as follows:

**Proof:** Let \( \mathcal{M} \) be a class of matroids satisfying the ASN property, and let \( \mathcal{B} = (B_1, \ldots, B_n) \) be a basis sequence of a matroid \( M \in \mathcal{M} \) of rank \( n \). Suppose that \( 0 < \epsilon < 1 \). Choose \( \epsilon' \) so that \( 0 < \epsilon' < \epsilon/2 \) and choose \( \delta \) so that \( 1 - \epsilon < (1 - 2\epsilon')\delta < 1 \). For large enough \( n \), there exists an integer \( f \) such that \((1 - \epsilon)n + \epsilon'\delta n \leq f \leq (1 - \epsilon')\delta n \). By Lemma 2.5 there exists a maximized \( \mathcal{F} \in \mathcal{W}_M(\mathcal{B}) \) where \( |\mathcal{F}| = f \) and \( t_M(\mathcal{F}) \geq f - \epsilon'\delta n \geq (1 - \epsilon)n \), provided that \( n \) is sufficiently large. This completes the proof, since \( t_M(\mathcal{B}) \geq t_M(\mathcal{F}) \).

### 2.2 Proof of Lemma 2.5

Let \( M \in \mathcal{M} \) and \( \mathcal{B} = (B_1, \ldots, B_n) \) be as described in Lemma 2.5. We may assume that \( \epsilon < \delta^{-1} - 1 \), so that \((1 + \epsilon)\delta < 1 \). By the ASN property, we must assume that \( n \) is large enough so that for all maximized \( \mathcal{F} \in \mathcal{W}_M(\mathcal{B}) \) such that \( |\mathcal{F}| = f \leq (1 - \epsilon)\delta n \), we have \( \lambda(\mathcal{F}) \geq (1 + \epsilon)\delta n \). We may assume that \( n/2 \leq f \); otherwise, the result follows from the proof of the main theorem of [3].

#### 2.2.1 Cascading Swaps

The following definition can be found in [3]. Let \( S_0, \ldots, S_{\ell-1} \in \mathcal{F} \) be a sequence of distinct RIS’s. We say that \( (x_\ell, c_\ell) \in U - \bigcup_{i=0}^{\ell-1} S_i \) is cascade-addable with respect to \( S_0, \ldots, S_{\ell-1} \) if there is a sequence of colours \( c_0, \ldots, c_{\ell-1} \) and a sequence of elements \( (x_1, c_1) \in S_1, \ldots, (x_{\ell-1}, c_{\ell-1}) \in S_{\ell-1} \) such that:

- The colour \( c_0 \) does not appear in \( S_0 \),
- \( (x_1, c_1) \) is \((S_0, c_0)\)-addable with a witness \((y_0, c_0)\),
- for \( 1 \leq i \leq \ell - 1 \), the element \((x_{i+1}, c_{i+1}) \) is \((S_i - (x_i, c_i), c_i)\)-addable with a witness \((y_i, c_i)\).

Since we are assuming that \( \mathcal{F} \) is maximized, it follows that any cascade-addable element \((x_\ell, c_\ell) \) must be contained in \( F(\mathcal{F}) \). Otherwise we can construct new collection \( \mathcal{F}' \) of larger volume from \( \mathcal{F} \) by performing a **cascading swap** (see Figure 2), an operation defined as follows: For \( 0 \leq i \leq \ell - 1 \), let \((x_i', c_{i+1}) \) be the element of \( S_i \) with colour \( c_{i+1} \) (exists since \((x_{i+1}, c_{i+1}) \) is addable with a witness). We define \( S'_0 = S_0 - (x_0', c_1) + (y_0, c_0) + (x_1, c_1) \), and for \( 0 \leq i \leq \ell - 1 \), we define \( S'_i = S_i - (x_i, c_i) - (x_{i+1}', c_{i+1}) + (x_{i+1}, c_{i+1}) + (y_i, c_i) \).
Then the collection \( \mathcal{J}' = \mathcal{J} - \{S_0, \ldots, S_{\ell-1}\} + \{S'_0, \ldots, S'_{\ell-1}\} \), and \( \operatorname{vol}(\mathcal{J}') = \operatorname{vol}(\mathcal{J}) + 1 \). We denote the set of all cascade-addable elements with respect to \( S_0, \ldots, S_{\ell-1} \) by \( Q(S_0, \ldots, S_{\ell-1}) \). By the reasoning above, we have \( Q(S_0, \ldots, S_{\ell-1}) \subseteq \mathcal{F}(\mathcal{J}) \).

### 2.2.2 Extending sequences

The following claim, which we present without proof, can be found in [3, Lemma 2.10]:

**Claim 1** If \((x, c) \in S_\ell\) is cascade-addable with respect to \( S_0, \ldots, S_{\ell-1} \), and \((x, c)\) is \((S_\ell - (x, c), c_\ell)\)-addable with a witness, then either:

- \((x, c) \in \bigcup_{i=0}^f S_i\)
- \(c \in \{c_0, \ldots, c_\ell\}\), or
- \((x, c)\) is cascade-addable with respect to \( S_0, \ldots, S_\ell\).

The next claim, a modification of a lemma [3, Lemma 2.11], tells us that given a sequence of RIS’s \( S_0, \ldots, S_{\ell-1} \), we can always choose a new RIS \( S_\ell \) such that the pool of cascade-addable elements grows larger.

**Claim 2** Let \( S_0, \ldots, S_{\ell-1} \in \mathcal{J} \) be a sequence of distinct RISs, with \( 1 \leq \ell < \epsilon \delta n \). Then we can choose \( S_\ell \neq S_0, \ldots, S_{\ell-1} \) such that

\[
|Q(S_0, \ldots, S_{\ell-1})| \geq |Q(S_0, \ldots, S_{\ell-1})| \cdot \frac{1}{1 - \epsilon} - (\ell + 1)n. \tag{1}
\]

In order to prove Claim 2, we first choose \( S_\ell \in \mathcal{J} - \{S_0, \ldots, S_{\ell-1}\} \) such that \( |S_\ell \cap Q(S_0, \ldots, S_{\ell-1})| \) is a maximum. Let \( Q = S_\ell \cap Q(S_0, \ldots, S_{\ell-1}) \). Then

\[
|Q| \geq \frac{Q(S_0, \ldots, S_{\ell-1})}{f - \ell}.
\]

By Lemma 2.2, we may assume that for every colour \( b \), there exists an injection \( \phi_b : S_\ell \to B_b \) such that \( \phi_b(x, c) \) is independent of \( \pi(S_\ell - (x, c)) \), for all \((x, c) \in S_\ell\).

Fixing \((x, c) \in Q\), let \( \mathcal{J}' \) be the collection formed from \( \mathcal{J} \) by performing a cascading swap along the sequence \( S_0, \ldots, S_{\ell-1} \), adding \((x, c)\) to \( S_{\ell-1} \) (as in Figure 2). Then \( S_\ell - (x, c) \in \mathcal{J}' \). Furthermore, \( \operatorname{vol}(\mathcal{J}') = \operatorname{vol}(\mathcal{J}) \), and hence \( \mathcal{J}' \) is also a maximized collection. By assumption, we have \( \lambda(\mathcal{J}') \geq (1 + \epsilon) \delta n \). In particular, since \( \mathcal{J}' \) is maximized (and hence there are no \((S_\ell - (x, c), c_\ell)\)-addable elements in \( U - \mathcal{F}(\mathcal{J}') \)), there are at least \((1 + \epsilon) \delta n - \ell\) colours that are \((S_\ell - (x, c), c_\ell)\)-swappable and not in the set \( \{c_0, \ldots, c_{\ell-1}\} \). If \( c \) is a such a colour, then \((\phi_b(x, c), c)\) is \((S_\ell - (x, c), c_\ell)\)-addable, by Lemma 2.1. It follows that there are at least \((1 + \epsilon) \delta n - \ell\) colours \( c \not\in \{c_0, \ldots, c_{\ell-1}\} \) such that \((\phi_b(x, c), c)\) is \((S_\ell - (x, c), c_\ell)\)-addable.

We have \(|Q|\) choices of \((x, c_\ell)\). For each choice, there are at least \((1 + \epsilon) \delta n - \ell\) elements \((\phi_b(x, c), c)\) (distinct since \( \phi_b \) is injective) which are \((S_\ell - (x, c), c_\ell)\)-addable. By Claim 1 each such element is in \( Q(S_0, \ldots, S_\ell) \), unless it is in one of the sets \( S_0, \ldots, S_\ell \). But \(|U_{\ell=0}^f S_i| \leq (\ell + 1)n\), so we arrive at the following inequality:

\[
|Q(S_0, \ldots, S_{\ell-1})| \geq |Q(S_0, \ldots, S_{\ell-1})| \cdot \frac{(1 + \epsilon) \delta n - \ell}{f - \ell} - (\ell + 1)n.
\]

Now, noting that

\[
\frac{(1 + \epsilon) \delta n - \ell}{f} \geq \frac{(1 + \epsilon) \delta n - \ell}{(1 - \epsilon) \delta n} \geq \frac{\delta n + \epsilon \delta n - \epsilon \delta n}{(1 - \epsilon) \delta n} = \frac{1}{1 - \epsilon},
\]

we see that Claim 2 follows.

In the next claim, we largely repeat the arguments in [3, Lemma 2.12]. Let \( C = C(\epsilon) \in \mathbb{R} \) be large enough so that

\[
C(1 + \epsilon/2)^{\ell-1} \frac{1}{1 - \epsilon} - \ell - 1 \geq C(1 + \epsilon/2)^\ell, \quad \forall \ell \geq 1. \tag{2}
\]
We iteratively apply Claim 2, building a sequence $S_0, \ldots, S_h$ of distinct RIS's in $\mathcal{J}$ so that for all $\ell \in \{1, \ldots, h\}$, inequality (1) holds. We may assume that $h = \lceil \delta n \rceil$. For $\ell \in \{0, \ldots, h\}$, we let $Q_\ell = Q(S_0, \ldots, S_\ell)$.

Claim 3 If $|Q_0| \geq Cn$ or $|Q_1| \geq Cn$, then for $0 < \ell \leq h$, we have

$$|Q_\ell| \geq C(1 + \epsilon/2)^{\ell-1}n.$$

We prove Claim 3 by induction. Suppose $\ell = 1$. By assumption, we have $|Q_0| \geq Cn$ or $|Q_1| \geq Cn$. Since the base case follows immediately from the latter possibility, so we assume that $|Q_0| \geq Cn$. By (1), we have that

$$|Q_1| \geq |Q_0| \cdot \frac{1}{1-\epsilon} - 2n \geq \left( C \cdot \frac{1}{1-\epsilon} - 2 \right) n$$

$$\geq C(1 + \epsilon/2)n > Cn \text{ (by (2))}.$$

This proves the claim in the case $\ell = 1$.

Now suppose $|Q_\ell| \geq C(1 + \epsilon/2)^{\ell-1}n$ for $\ell < h$. Then by (1)

$$|Q_{\ell+1}| \geq C(1 + \epsilon/2)^{\ell-1}n \cdot \frac{1}{1-\epsilon} - (\ell + 1)n$$

$$\geq \left( C(1 + \epsilon/2)^{\ell-1} \cdot \frac{1}{1-\epsilon} - \ell - 1 \right) n \geq C(1 + \epsilon/2)^\ell n.$$

This proves Claim 3.

Assume that $n$ is large enough so that

$$(1-\epsilon)\delta n \leq C(1 + \epsilon/2)^{h-1}.$$

Suppose for the sake of contradiction that $t_M(\mathcal{J}) < f - h$. By Claim 3, we have $|Q_h| \geq C(1 + \epsilon/2)^{h-1} \cdot n \geq (1-\epsilon)\delta n^2 \geq fn$. Since $\mathcal{J}$ is maximized, we have $Q_h \subseteq F(\mathcal{J})$, and so $|F(\mathcal{J})| = \text{vol}(\mathcal{J}) \geq fn$. It follows that $f - h > t_M(\mathcal{J}) = f$, establishing the contradiction. Thus Lemma 2.5 is proven if we can justify the assumption that $|Q(S_0)| \geq Cn$ or $|Q(S_0, S_1)| \geq Cn$.

2.2.3 The initial addable-elements

The problem now is to justify the assumption that $|Q(S_0)| \geq Cn$ or $|Q(S_0, S_1)| \geq Cn$. Claims 4 and 5 below are modifications to lemmas 3, Lemma 2.15, Lemma 2.16. To a large extent, we use the same arguments, except that we are using $\lambda$.

Claim 4 There exists a maximized $\mathcal{J}' \in \Psi_M(B)$ for which $|\mathcal{J}'| = f$, and there is an injection $b : \mathcal{J}' \to \{1, \ldots, n\}$ such that for all $S \in \mathcal{J}'$, if $S$ is not a transversal base, then $S$ is missing the colour $b(S)$.

Let $\mathcal{J} = \{S_1, \ldots, S_f\}$. We proceed iteratively. Suppose that for all $j < i$, if $S_j$ is not a transversal base, then $S_j$ is missing $b_j := b(S_j)$.

If $S_i$ is a transversal base, then let $b(S_i)$ be an arbitrary colour not in $\{b_1, \ldots, b_{i-1}\}$. Otherwise, $S_i$ is missing a colour, say $b$. Then there are at least $\lambda(\mathcal{J}) \geq (1 + \epsilon)\delta n$ colours that are $(S,b)$-swappable. Since $i - 1 < f \leq \delta n$, at least one such colour $c$ does not appear in $\{b_1, \ldots, b_{i-1}\}$. A simple swap performed on $S_i$ where $(x,c) \in S_i$ is swapped with some $(y,b) \in U - F$ yields a new RIS $S'_i$ and a new maximized collection $\mathcal{J}' \in \Psi_M(B)$, where $b(S'_i) = c$ and continue. This proves Claim 4.

Let $D = 2C + 4$. Note that $D((1 + \epsilon)\delta n - 1) - 2n \geq Cn$ for sufficiently large values of $n$ (since by assumption, $n/2 \leq f \leq (1-\epsilon)\delta n$). Thus the following claim suffices to prove our assumption that $|Q(S_0)| \geq Cn$ or $|Q(S_0, S_1)| \geq Cn$:

Claim 5 Suppose that at least $c\delta n$ of the RIS's contained in $\mathcal{J}$ are not transversal bases. For sufficiently large $n$, we can modify $\mathcal{J}$ so that either:
(i) \( \text{vol}(\mathcal{S}) \) is unchanged, and there exists an \( S_0 \in \mathcal{S} \) missing at least \( D \) colours, or

(ii) \( \text{vol}(\mathcal{S}) \) is unchanged, and there exist distinct \( S_0, S_1 \in \mathcal{S} \) where \( S_1 \) contains at least \( D \) elements that are \((S_0, b)\)-addable for some colour \( b \).

Let \( E \) be the largest integer such that there are at least \( M_E = \left( \frac{\epsilon}{4D^2} \right)^E n \) RIS’s in \( \mathcal{S} \) missing at least \( E \) colours. If \( E \geq D \), then (i) holds, and we are done. On the other hand, we know that \( E \geq 1 \), since at least \( \epsilon \cdot \delta n \) RIS’s in \( \mathcal{S} \) are missing at least one colour. Therefore we may assume that \( 1 \leq E < D \). To complete the proof, it suffices to show that \( \mathcal{S} \) can be modified without changing the volume in such a way that \( E \) is increased.

We define a digraph \( G \) whose vertices are the RIS’s in \( \mathcal{S} \), such that if \( S_0, S_1 \in \mathcal{S} \), then there is an arc from \( S_1 \) to \( S_0 \) if \( S_0 \) is missing at least \( E \) colours, and \( S_1 \) contains at least \( E + 1 \) elements that are \((S_0, b(S_0))\)-addable.

An \((E + 1)\)-out-star is a subgraph of \( G \) consisting of \( E + 1 \) arcs, directed away from a single vertex. Given an \((E + 1)\)-out-star centered at \( S_1 \), for each out-neighbour \( S_0 \), we can transfer a distinct \((S_0, b(S_0))\)-addable element from \( S_1 \) to \( S_0 \). The result of this transfer is that \( S_1 \) is now missing at least \( E + 1 \) colours. Thus it is sufficient to prove the existence of at least \( M_{E+1} \) vertex-disjoint \((E + 1)\)-out-stars.

Suppose \( S_0 \in \mathcal{S} \) is missing at least \( E \) colours. We shall show that the in-degree \( d^-(S_0) \) of \( S_0 \) in \( G \) is at least \( 2E \cdot \delta n / D \). We see this by counting \((S_0, b(S_0))\)-addable elements. By definition of the swapping number, there are at least \( \lambda \) colours which are \((S_0, b(S_0))\)-swappable. For each such colour \( c \), we see by Lemma 2.1 that every element of \( B_c \) independent of \( \pi(S_0) \) is \((S_0, b(S_0))\)-addable. By augmentation, there are at least \( n - |S_0| \geq E \) such elements. Thus, in total there are at least \( E \lambda \) elements which are \((S_0, b(S_0))\)-addable.

Let \( N^-(S_0) \) be the set of in-neighbours of \( S_0 \). Then \( d^-(S_0) = |N^-(S_0)| \). Any vertex not in \( N^-(S_0) \) has, by definition, at most \( E \) elements which are \((S_0, b(S_0))\)-addable. We may assume that any vertex has at most \( D \) elements which are \((S_0, b(S_0))\)-addable, since otherwise (ii) occurs. Thus we have

\[
Dd^-(S_0) + E(f - d^-(S_0)) \geq E\lambda,
\]

which yields

\[
d^-(S_0) \geq \frac{E(\lambda - f)}{D - E} = \frac{\lambda - f}{D} \geq \frac{2E\delta n}{D},
\]

as claimed.

We are finally ready to show that there are at least \( M_{E+1} \) vertex-disjoint \((E + 1)\)-out-stars. Assume that we have already found fewer than \( M_{E+1} \) vertex-disjoint \((E + 1)\)-out-stars. Let \( G' \) be the digraph obtained from \( G \) by deleting the vertices of the \((E + 1)\)-out-stars that we have previously found. We will prove that there exists an \((E + 1)\)-out-star in \( G' \), and thus an additional \((E + 1)\)-out-star in \( G \), vertex-disjoint from those that have already been found. The number of arcs in \( G' \) is greater than

\[
2M_E \frac{\epsilon \delta n}{D} - M_{E+1}(E + 2) \cdot 2f = 2M_E \frac{\epsilon \delta n}{D} - \frac{M_E \cdot \epsilon}{2D^2} (E + 2)f
\]

\[
\geq 2M_E \frac{\epsilon \delta n}{D} - \frac{M_E \cdot \epsilon}{2D^2} (E + 2)(1 - \epsilon)\delta n
\]

\[
\geq M_E \frac{\epsilon^2 \delta n}{D} (1 - (1 - \epsilon)) = M_E \frac{\epsilon^2 \delta n}{D}.
\]

Now, note that

\[
M_E \frac{\epsilon^2 \delta n}{D} \geq (E + 1)f,
\]

as long as

\[
n \geq \left( \frac{(E + 1)(1 - \epsilon)D}{\epsilon^2} \cdot \left( \frac{A}{\epsilon} \right)^E \right).
\]

Thus, since \( G' \) has at most \( f \) vertices, then for sufficiently large \( n \), there must be a vertex in \( G' \) of out-degree at least \( E + 1 \), and therefore there must be an \((E + 1)\)-out-star. This proves Claim 5, completing the proof of Lemma 2.5. \( \blacksquare \)
3 Classes of matroids satisfying the ASN property

3.1 Graphic matroids

We will make use of the following theorem of Alon, Hoory, and Linial [1):

**Theorem 3.1 (Alon, Hoory, and Linial, 2002)** Let $G$ be a graph of girth $g < \infty$ and average vertex degree $d \geq 2$. Then

$$\nu(G) \geq \nu_0(d, g),$$

where $\nu_0(d, g)$ is the so called Moore bound, defined by

$$\nu_0(d, 2r + 1) = 1 + d \sum_{i=0}^{r-1} (d-1)^i, \quad \nu_0(d, 2r) = 2 \sum_{i=0}^{r-1} (d-1)^i.$$

The above is easily seen to imply the following corollary:

**Corollary 3.2** If $G$ is a graph of girth $g < \infty$ and average vertex degree $d > 2$, then

$$\nu(G) \geq 2 \cdot \frac{(d-1)^{g/2} - 1}{d-2}.$$  

We can now prove Theorem 1.2

**Proof:** Let $\Gamma$ denote the class of all graphic matroids, and let $\overline{\Gamma} \subset \Gamma$ denote the following class of matroids:

$$\overline{\Gamma} = \{ M \in \Gamma \mid g(M) \geq \psi(r(M)) \}.$$ 

By Theorem 2.4, it suffices to show that $\overline{\Gamma}$ satisfies the ASN property.

Let $M = M(G) \in \overline{\Gamma}$ have $n$ and let $B = \{ B_1, \ldots, B_n \}$ be a basis sequence of $M$. The proof is trivial in the case that $g(M) = \infty$, so we may assume that $g(M) < \infty$. Let $0 < \epsilon < 1$ and let $n_0$ be the largest integer $x$ such that

$$(1-\epsilon)x + 1 \geq \frac{(1 + 2\epsilon)^{\psi(x)/2} - 1}{\epsilon}.$$ 

Such an integer $n_0$ exists, since $\psi(n) \in \omega(\log n)$. Suppose that $\mathcal{S} \in \mathcal{W}_M(B)$ is maximized, and $|\mathcal{S}| = f \leq (1-\epsilon)n$. We will show that if $n > n_0$, then $\lambda(\mathcal{S}) \geq (1-\epsilon)n$. Assume for the sake of contradiction that $n > n_0$ and $\lambda(\mathcal{S}) < (1-\epsilon)n$. Then there exists $S \in \mathcal{S}$ missing a colour $b$ for which there are no $(S, b)$-addable elements and there are fewer than $(1-\epsilon)n$ colours that are $(S, b)$-swappable.

Let $S' \subseteq S$ be the set of elements of $S$ with a $(S, b)$-swappable colour. Then $|S'| < (1-\epsilon)n$. By Lemma 2.3, we see that $B_b - B_b \subset \text{cl}_M(\pi(S'))$.

Let $G'$ be the subgraph of $G$ induced by the edges in $\text{cl}_M(\pi(S'))$. We may assume that $G'$ is connected; to see this, if $G'$ were not connected, then we can let $K$ be a complete set of representatives of the components of $G'$. By identifying the vertices in $K$, we obtain a connected graph $G''$ with $g(G'') = g(G')$ and $M(G'') \simeq M(G')$, in which case, we may use $G''$ in place of $G'$. Since $G'$ is connected and $r(M(G')) = |S'|$, it follows that $\nu(G') = |S'| + 1$. Furthermore, since $B_b - F_b \subset \text{cl}_M(\pi(S'))$, the number of edges $\mathcal{E}(G')$ is

$$\mathcal{E}(G') \geq |S'| + |B_b - F_b| \geq |S'| + n - f + 1 \geq |S'| + 1 + \epsilon n,$$

so that the average degree $d$ of $G'$ is

$$d = \frac{2\mathcal{E}(G')}{|S'| + 1} \geq 2 \left( \frac{|S'| + 1 + \epsilon n}{|S'| + 1} \right) \geq 2(1 + \epsilon).$$ 

The girth of $G'$ is at least $g(M) \geq \psi(n)$, since $G'$ is a subgraph of $G$. Thus by Corollary 3.2, we have

$$(1-\epsilon)n + 1 > |S'| + 1 \geq 2 \cdot \frac{(d-1)^{\psi(n)/2} - 1}{d-2} \geq \frac{(1 + 2\epsilon)^{\psi(n)/2} - 1}{\epsilon}.$$ 

However, since $n > n_0$, we have a contradiction, establishing the proof. ■
3.2 Regular matroids

In this section, we generalize the result for graphic matroids to the class of regular matroids. The following theorem is equivalent to [16, Lemma C.2], translated into matroidal terms:

**Theorem 3.3 (Adapted from Kashyap, 2008)** Let $M$ be a regular matroid such that $r(M) > k \cdot \mathcal{E}(M)$. Then, if $\mathcal{E}(M)$ is sufficiently large, we have

$$g(M^*) \leq \frac{8\log \mathcal{E}(M)}{\log(1 + k)}.$$  

We now prove Theorem 3.3.

**Proof:** Let $\mathcal{R}$ denote the class of all regular matroids, and let $\overline{\mathcal{R}}$ denote the following subclass of $\mathcal{R}$:

$$\overline{\mathcal{R}} = \{M \in \mathcal{R} \mid g(M) \geq \psi(r(M))\}.$$  

We will show that $\overline{\mathcal{R}}$ satisfies the ASN property, implying Theorem 3.3 by Theorem II.

Suppose $M \in \overline{\mathcal{R}}$ has rank $n$ and $B = (B_1, \ldots, B_n)$ is a basis sequence of $M$. Let $0 < \epsilon < 1$ and let $0 < k < \epsilon$. Let $n_0$ be the largest integer $x$ such that

$$\psi(x) \leq \frac{8 \log 2x}{\log(1 + k)}.$$  

Suppose $\mathcal{S} \in \mathcal{G}(B)$ is maximized, and that $|\mathcal{S}| = f \leq (1 - \epsilon)n$. We will show that if $n > n_0$, then $\lambda(\mathcal{S}) \geq (1 - \epsilon)n$. Suppose for the sake of contradiction that $n > n_0$ and $\lambda(\mathcal{S}) < (1 - \epsilon)n$. Then there exists an RIS $S \in \mathcal{S}$ such that $S$ is missing a colour $b$, and there are fewer than $(1 - \epsilon)n$ colours that are $(S, b)$-swappable. Let $S' \subseteq S$ be the set of elements of $S$ with an $(S, b)$-swappable colour, so that $|S'| < (1 - \epsilon)n$.

By Lemma II, we know that $B_b - F_b \subseteq cl_M(\pi(S'))$. Let $N = M \mid cl_M(\pi(S'))$, so that $r(N) = |S'|$, and

$$\mathcal{E}(N) \geq |S'| + n - f \geq r(N) + \epsilon n > \left(1 + \frac{\epsilon}{1 - \epsilon}\right) r(N) = \left(\frac{1}{1 - \epsilon}\right) r(N).$$  

Let $N'$ be a restriction of $N$ so that $r(N') = r(N)$ and $\mathcal{E}(N') = \left(\frac{1}{1 - \epsilon}\right) r(N') = \left(\frac{1}{1 - \epsilon}\right) r(N') + \epsilon'$, where $0 \leq \epsilon' < 1$. The rank of the dual $(N')^*$ is given by $r((N')^*) = \left(\frac{1}{1 - \epsilon}\right) r(N') + \epsilon' \geq \epsilon \cdot \mathcal{E}((N')^*)$.

Since $N'$ is a restriction of $M$, we have $g(N') \geq g(M) \geq \psi(n)$. For sufficiently large $n$, and since $k < \epsilon$, we have by Theorem III that

$$\psi(n) \leq g(N') \leq \frac{8 \log \mathcal{E}(N')}{\log(1 + k)} < \frac{8 \log 2n}{\log(1 + k)},$$  

since $\mathcal{E}(N') < 2n$. Since $n > n_0$, we have a contradiction.  

3.3 Asymptotically good classes of matroids

Let $\epsilon, \delta \in (0, 1]$. A sequence $\{M_i\}_{i=1}^\infty$ of matroids is said to be $(\epsilon, \delta)$-good if $\lim_{i \to \infty} \mathcal{E}(M_i) = \infty$, and for all $i \geq 1$, we have

$$r(M_i) \geq \epsilon \cdot \mathcal{E}(M_i),$$

$$g(M_i^*) \geq \delta \cdot \mathcal{E}(M_i).$$

A class $\mathcal{M}$ of matroids is said to be asymptotically good if it contains an $(\epsilon, \delta)$-good sequence of matroids, for some $\epsilon, \delta \in (0, 1]$. This definition is intimately related to the notion of an asymptotically good class of linear codes; see [16, 17].

In [17], Nelson and van Zwam characterized asymptotically good minor-closed classes of matroids, using a consequence of the matroid minors project of Geelen, Gerards, and Whittle [11]. The following theorem has been published without proof, and as such we shall refer to it as a hypothesis.
Hypothesis 3.4 (Geelen, Gerards, and Whittle, 2015) Let $\mathbb{F}$ be a finite field having characteristic $p$ and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids not containing all matroids representable over $\mathbb{F}_p$. Then there exist $k, m \in \mathbb{Z}$ such that for every vertically $k$-connected matroid $M$ in $\mathcal{M}$ of rank at least $m$, there exists some $\mathbb{F}$-representable frame matroid $N$ such that

$$\min\{\text{dist}(M, N), \text{dist}(M, N^*)\} \leq k.$$ 

Here, $\text{dist}(M, N)$ is the minimum number of elementary lifts or projections needed to transform $M$ into $N$. Using Hypothesis 3.4, Nelson and van Zwaan were able to prove the following:

Theorem 3.5 (Nelson and van Zwaan, 2015) Let $\mathbb{F}$ be a finite field having characteristic $p$ and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids. Then $\mathcal{M}$ is asymptotically good if and only if $\mathcal{M}$ contains all projective geometries over $\mathbb{F}_p$.

Theorem 3.5 allows us to prove Theorem 1.4.

Proof: We may assume that $\mathcal{M}$ is closed under duals. Otherwise, we could replace $\mathcal{M}$ by $\mathcal{M}' = \mathcal{M} \cup \mathcal{M}^*$, where $\mathcal{M}^* = \{M^* \mid M \in \mathcal{M}\}$, noting that $\mathcal{M}'$ is minor-closed, closed under duals, and does not contain all projective geometries over $\mathbb{F}_p$.

Let $0 < \epsilon < 1$ and $0 < \delta < 1$. Let $n_0$ be the largest integer $x$ such that there exists an $M \in \mathcal{M}$ with $\mathcal{E}(M) = x$, and

$$r(M^*) \geq \epsilon \cdot \mathcal{E}(M)$$
$$g(M) \geq \delta \cdot (\mathcal{E}(M) - 1).$$

If such an $n_0$ did not exist, then there would be a $(\epsilon, \delta')$-good sequence of matroids in $\mathcal{M}$ for every $\delta' < \delta$, which contradicts Theorem 3.5. Let

$$\mathcal{M}_{\epsilon, \delta} = \{M \in \mathcal{M} \mid g(M) \geq \delta \cdot r(M)\}.$$ 

By Theorem 2.4, it suffices to show that $\mathcal{M}_{\epsilon, \delta}$ satisfies the ASN property. Suppose $B = \{B_1, \ldots, B_n\}$ is a basis sequence of $M \in \mathcal{M}_{\epsilon, \delta}$, where $r(M) = n$. We claim that if $\mathcal{T} \in \mathcal{W}_M(B)$ is maximized and $|\mathcal{T}| = f \leq (1-\epsilon)n$, and if $n > n_0/\delta$, then $\lambda(\mathcal{T}) \geq (1-\epsilon)n$. Assume for the sake of contradiction that $n > n_0/\delta$ and $\lambda(\mathcal{T}) < (1-\epsilon)n$. Then there exists an $S \in \mathcal{T}$ missing a colour $b$, such that $S$ has fewer than $(1-\epsilon)n$ colours that are $(S, b)$-swappable. Define $S' \subset S$ to be the set of elements of $S$ with an $(S, b)$-swappable colour. By assumption, $|S'| < (1-\epsilon)n$.

We know by Lemma 1.3 that $B_b - F_b \subset c_{\mathcal{M}}(\pi(S'))$. Let $N = M|c_{\mathcal{M}}(\pi(S'))$. Then $r(N) = |S'|$ and

$$\mathcal{E}(N) \geq |S'| + |B_b - F_b| \geq |S'| + (n - f) \geq |S'| + \epsilon n.$$ 

Thus

$$\mathcal{E}(N) \geq \left(1 + \frac{\epsilon n}{|S'|}\right) |S'| > \left(1 + \frac{\epsilon}{1-\epsilon}\right) r(N) = \left(\frac{1}{1-\epsilon}\right) r(N).$$

Let $N'$ be a restriction of $N$ such that $r(N') = r(N)$ and $\mathcal{E}(N') \geq \left[\left(\frac{1}{1-\epsilon}\right) r(N)\right] = \left(\frac{1}{1-\epsilon}\right) r(N') + \epsilon'',$

where $0 \leq \epsilon' < 1$. Thus we have, $\mathcal{E}(N') = \left(\frac{1}{1-\epsilon}\right) (\mathcal{E}(N') - r((N')^*) + \epsilon'\), from which it follows that $r((N')^*) = \epsilon \cdot \mathcal{E}(N') + \epsilon'(1-\epsilon).$ Consequently, $r((N')^*) \geq \epsilon \cdot \mathcal{E}(N')$. On the other hand,

$$\mathcal{E}(N') = \left(\frac{1}{1-\epsilon}\right) r(N') + \epsilon' = \left(\frac{1}{1-\epsilon}\right) |S'| + \epsilon'$$

$$\leq \left(\frac{1}{1-\epsilon}\right) \cdot (1-\epsilon)n + \epsilon' = n + \epsilon'.$$

Thus $n \geq \mathcal{E}(N') - \epsilon'$. Since $N'$ is a restriction of $M$, it follows that $g(N') \geq g(M) \geq \delta n \geq \delta (\mathcal{E}(N') - \epsilon') \geq \delta (\mathcal{E}(N') - 1)$. Thus $\mathcal{E}(N') \leq n_0$. But $\mathcal{E}(N') > r(N')$, so $N'$ contains a circuit. Thus $\delta n \leq \mathcal{E}(N') \leq n_0$, so that $n \leq n_0 / \delta$, a contradiction.
4 Kahn’s basis conjecture

Let \((B_{i,j})\) be an \(n \times n\) array of bases in a matroid \(M\) of rank \(n\). We let \(t_M(B_{i,j})\) be the size of a largest set \(L \subseteq \{1, \ldots, n\}\) such that for some choice of the representatives \(b_{i,j} \in B_{i,j}\), we have

- \(\{b_{i,j} : i \in L\}\) is independent for every \(j \in \{1, \ldots, n\}\), and
- \(\{b_{i,j} : j = 1, \ldots, n\}\) is a base for every \(i \in L\).

In 1991, Kahn made the following conjecture \[15\], a strengthening of RBC:

**Conjecture 4.1 (Kahn, 1991)** For all \(1 \leq i, j \leq n\), let \(B_{i,j}\) be a base of a matroid \(M\) of rank \(n\). Then \(t_M(B_{i,j}) = n\).

In the companion note \[4\] to their main paper \[3\], Bucić et al. showed that a slight modification of their arguments yields the following theorem:

**Theorem 4.2** For all \(1 \leq i, j \leq n\), let \(B_{i,j}\) be a base of a matroid \(M\) of rank \(n\). Then for every \(\epsilon > 0\), we have \(t_M(B_{i,j}) \geq (1 - \epsilon)n\), provided \(n\) is sufficiently large.

Our arguments here can be modified in a similar way, albeit using our notion of the asymptotic swapping number property (we omit the details). In the following theorems, \(B_{i,j}\) are \(n^2\) bases of a matroid \(M\) of rank \(n\), where \(1 \leq i, j \leq n\), and \(\psi : \mathbb{Z}_+ \to \mathbb{R}\) is a function such that \(\psi(n) \in \omega(\log n)\):

**Theorem 4.3** If \(M\) is regular, and \(g(M) \geq \psi(n)\), then for every \(\epsilon > 0\), we have \(t_M(B_{i,j}) \geq (1 - \epsilon)n\), provided that \(n\) is sufficiently large.

**Theorem 4.4** Suppose \(\mathbb{F}\) is a finite field of characteristic \(p\), and \(\mathcal{M}\) is a minor-closed class of \(\mathbb{F}\)-representable matroids, not containing all projective geometries over \(\mathbb{F}_p\). If \(M \in \mathcal{M}\), and \(g(M) \geq \delta n\), for some \(\delta \in (0,1)\), then for every \(\epsilon > 0\), we have \(t_M(B_{i,j}) \geq (1 - \epsilon)n\), provided that \(n\) is sufficiently large.

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