Submanifolds associated to Toda theories

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Abstract

A set of two-dimensional semi-riemannian submanifolds of flat semi-riemannian manifolds is associated to each Toda theory. The method and an example are given to Toda theories associated to real finite dimensional Lie algebras.

1 Introduction

The subject of this work is concerned with the theory of classical integrability. Toda theories are two-dimensional relativistic integrable theories. Their field equations are expressed in terms of a zero curvature condition. Toda theories associated to finite dimensional Lie algebras are called conformal Toda theories (see [1] for a review). In this work we consider only Toda theories associated to finite dimensional real Lie algebras.

The purpose of this work is to translate the analytical information, that is, the field equations, into geometrical information. More precisely, we are going to apply a method by which a system, such that its field equations are given by a set of zero curvature conditions, associated to a real Lie algebra, in circumstances to be explained, can be associated to a set of semi-riemannian submanifolds of a given semi-riemannian manifold (see [2] for a review).

This work is organized as follows: In the sections 2, 3 and 4 we review some basic facts about Toda theories, semi-riemannian geometry and semi-riemannian submanifolds, respectively. In the section 5 we review the above cited method. In the section 6 we apply the method of section 5 to Toda theories associated to real Lie algebras and give an example. In the appendix a typical calculation of this work is shown.
2 Toda theories

In this section we review some basic facts about Toda theories.

Suppose a Lie algebra $\mathcal{G}$ can be decomposed as a direct sum of vector spaces $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$, $i \in \mathbb{Z}$, where the subspaces $\mathcal{G}_i$ are defined by, $Q \in \mathcal{G}$, $[Q, T] = iT$, $\forall T \in \mathcal{G}_i$. The element $Q$ is called a grading operator. As a consequence of the Jacobi identity, $\forall T \in \mathcal{G}_i$, $\forall V \in \mathcal{G}_j$, $[T, V] \in \mathcal{G}_{i+j}$. Note that if $i \neq j$, then $\mathcal{G}_i$ and $\mathcal{G}_j$ have only the zero element (0 $\in \mathcal{G}$) in common.

Toda theories are defined on a flat manifold, the two-dimensional Minkowski space $Mk^2$. In a natural coordinate system, globally defined, $x^1 = x$, $x^2 = t$, the metric tensor has constant components given by $\eta_{11} = -1$, $\eta_{22} = 1$, $\eta_{12} = \eta_{21} = 0$. We introduce new coordinates, $z = t + x$, $\bar{z} = t - x$. Then $\bar{\partial}_t = \partial + \bar{\partial}$ and $\partial_x = \partial - \bar{\partial}$, where $\bar{\partial} \equiv \partial / \bar{z}$ and $\partial \equiv \partial / z$.

Toda theories can be obtained as constrained Wess-Zumino-Witten models (WZW). Given a Lie algebra $\mathcal{G}$ and a grading operator $Q$, define $\mathcal{G}_+ \equiv \bigoplus_{i>0} \mathcal{G}_i$ and $\mathcal{G}_- \equiv \bigoplus_{i<0} \mathcal{G}_i$. If the dynamical variable of a WZW model is a group element $g$, use the Gauss decomposition $\mathcal{G}$ to write $g = NBM$, where $N = \exp(\mathcal{G}_-)$, $M = \exp(\mathcal{G}_+)$ and $B = \exp(\mathcal{G}_0)$. Then the dynamical variable in the Toda theories is the group element $B$ and the field equations are given by

$$\bar{\partial}(B^{-1} \partial B) + [\varepsilon^-, B^{-1} \varepsilon^+ B] = 0,$$

where $\varepsilon^+$, $\varepsilon^-$ are constants such that $\varepsilon^+ \in \mathcal{G}_1$ and $\varepsilon^- \in \mathcal{G}_{-1}$. As

$$B(\bar{\partial}(B^{-1} \partial B) + [\varepsilon^-, B^{-1} \varepsilon^+ B])B^{-1} = \partial(\bar{\partial}B^{-1}B) - [\varepsilon^+, B \varepsilon^- B^{-1}],$$

the field equations can be expressed in an equivalent way by:

$$\bar{\partial}(\bar{\partial}B^{-1}B) - [\varepsilon^+, B \varepsilon^- B^{-1}] = 0.$$

Note that $Q \in \mathcal{G}_0$ because $[Q, Q] = 0$ and $\mathcal{G}_0$ is a subalgebra as a consequence of the Jacobi identity. If the $\mathcal{G}_0$ subalgebra is abelian, the model is called abelian. There is a procedure to get the action of the Toda theories, given the action of the WZW model $\mathcal{I}$.

Let $A_t(t, x)$ and $A_x(t, x)$ be gauge potentials. Define $A \equiv A_x = 1/2(A_t + A_x)$ and $\bar{A} \equiv A_x = 1/2(A_t - A_x)$. The curvature is given by $F_{\bar{z}z} = \partial \bar{A} - \bar{\partial}A + [A, \bar{A}]$. The Toda theories field equations can be seen as a zero curvature condition, $F_{\bar{z}z} = 0$, where

$$A = B \varepsilon^- B^{-1} \quad \text{and} \quad \bar{A} = -\varepsilon^+ - \bar{\partial}B^{-1}B.$$

Given a group element $g$ and a gauge transformation

$$A^g = gAg^{-1} - \partial gg^{-1} \quad \text{and} \quad \bar{A}^g = g\bar{A}g^{-1} - \bar{\partial}g^{-1},$$

(2. 1)
the curvature is transformed to \( F^g_{z\bar{z}} \equiv F_{ar{z}z}(A^g) = gF_{z\bar{z}}g^{-1} \). Then \((F^g_{z\bar{z}} = 0) \iff (F_{z\bar{z}} = 0)\) and the field equations are invariant by a gauge transformation.

Let \( \lambda \in \mathbb{R}, \partial_z \lambda = \partial_{\bar{z}} \lambda = 0 \). Define

\[
a(z, \bar{z}, \lambda) \equiv e^{-\lambda}B\varepsilon^{-B^{-1}} \quad \text{and} \quad \bar{a}(z, \bar{z}, \lambda) \equiv -e^\lambda \varepsilon^+ - \bar{\partial}BB^{-1}.
\]

The zero curvature condition

\[
\partial \bar{a} - \bar{\partial} a + [a, \bar{a}] = 0
\]

is equivalent to the Toda theories field equations. Note that:

- (a)

\[
a(z, \bar{z}, \lambda = 0) = A(z, \bar{z}) \quad \text{and} \quad \bar{a}(z, \bar{z}, \lambda = 0) = \bar{A}(z, \bar{z}).
\]

- (b) \( a \) and \( \bar{a} \) are obtained applying a gauge transformation to \( A \) and \( \bar{A} \) (equations (2.1) and (2.2)), where \( g = e^{\lambda Q} \).

## 3 Semi-Riemannian geometry

In this section we review some basic facts about semi-riemannian geometry [4]. We consider \( C^\infty \) differentiable manifolds \( M \) (Hausdorff and second countable) with a complete atlas. \( \mathcal{F}(M) \) is the set of \( C^\infty \) differentiable functions \( f : M \to \mathbb{R} \). Given \( p \in M \), \( T_p(M) \) is the set of tangent vectors to \( M \) at \( p \). Given \( p \in M \), an open set \( V \subset M \), \( p \in V \) and a coordinate system \( \varphi : V \to \mathbb{R}^m, m = \text{dim}(M) \), \( \varphi(p) = (x^1(p), \ldots, x^m(p)) \), then \( \frac{\partial}{\partial x^i} |_p, \ldots, \frac{\partial}{\partial x^m} |_p \) form a basis for the tangent space \( T_p(M) \), that is, \( v = v^i \frac{\partial}{\partial x^i} |_p, \forall v \in T_p(M) \). Let \( \phi : M \to N \) be a \( C^\infty \) map, where \( M \) and \( N \) are manifolds. The differential map of \( \phi \) at \( p \in M \) is denoted by \( d\phi_p : T_p(M) \to T_{\phi(p)}(N) \). \( \mathfrak{X}(M) \) is the set of \( C^\infty \) vector fields on \( M \). Given \( V, W \in \mathfrak{X}(M) \), the Lie bracket is denoted by \([V, W]\).

A metric tensor \( g \) on \( M \) is a symmetric nondegenerate \( \mathcal{F}(M) \)-bilinear map \( g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M) \) of constant index. The term nondegenerate means that, \( \forall p \in M \), given \( v \in T_p(M), \) if \( g(v, w) = 0, \forall w \in T_p(M), \) then \( v = 0 \). Given a basis to \( T_p(M) \) and the matrix \( (g_{ij}(p)) \) which corresponds to \( g \) evaluated on this basis, it follows that \( (g_{ij}(p)) \) is invertible. We denote the inverse by \( (g^{ij}(p)) \). Given \( p \in M \), one can always find an orthonormal basis for \( T_p(M) \). On this basis \( (g_{ij}(p)) \) is diagonal, \( g_{ij}(p) = \delta_{ij} \epsilon_j, \) where \( \epsilon_j = \pm 1, \forall j \in \{1, \ldots, m\} \). The signature is \( (\epsilon_1, \ldots, \epsilon_m) \) and the index is defined as the number of negative signs in the signature.

The name semi-riemannian corresponds to a general signature. The riemannian and
lorentzian cases correspond, respectively, to the indexes 0 and 1. Let \( M \) and \( N \) be semi-riemannian manifolds with metric tensors \( g_M \) and \( g_N \). An isometry is a diffeomorphism \( \phi : M \to N \) such that \( \phi^*(g_N) = g_M \), where \( \phi^*(g_N) \) is the pullback of \( g_N \).

A connection \( D \) is a map \( D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \) such that

(a) \( D_V W \) is \( \mathcal{F}(M) \)-linear in \( V \),

(b) \( D_V W \) is \( \mathbb{R} \)-linear in \( W \) and

(c) \( D_V(fW) = (Vf)W + fD_V W \),

\( \forall f \in \mathcal{F}(M), \forall V, W \in \mathfrak{X}(M) \). \( D_V W \) is called the covariant derivative of \( W \) with respect to \( V \).

There is a unique connection \( D \) such that

(a) \( [V, W] = D_V W - D_W V \) and

(b) \( X g(V, W) = g(D_X V, W) + g(V, D_X W) \),

\( \forall X, V, W \in \mathfrak{X}(M) \). \( D \) is called the Levi-Civita connection of \( M \). These two properties are known as torsion free condition and metric compatibility.

Let \( x^1, \ldots, x^m \) be a coordinate system on an open set \( V \subset M \). The Christoffel symbols are defined by \( D_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k \), \( (1 \leq i, j \leq m) \). In the case of a Levi-Civita connection \( \Gamma^k_{ij} = \Gamma^k_{ji} \) and

\[
\Gamma^k_{ij} = \frac{1}{2} g^{kr} \left( \frac{\partial g_{jr}}{\partial x^i} + \frac{\partial g_{ir}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^r} \right), \quad (3.1)
\]

\( \forall i, j, k \in \{1, \ldots, m\} \).

The Riemann tensor \( R : [\mathfrak{X}(M)]^3 \to \mathfrak{X}(M) \) is defined by \( R(X, Y)Z = D_{[X,Y]}Z - [D_X, D_Y]Z \), \( \forall X, Y, Z \in \mathfrak{X}(M) \) and is \( \mathcal{F}(M) \)-linear in \( X, Y \) and \( Z \). Given a coordinate system, \( R(\partial_i, \partial_l) \partial_j = R^k_{ijkl} \partial_i \), where

\[
R^i_{jkl} \equiv \frac{\partial \Gamma^i_{kj}}{\partial x^l} - \frac{\partial \Gamma^i_{lj}}{\partial x^k} + \Gamma^i_{lr} \Gamma^r_{kj} - \Gamma^i_{kr} \Gamma^r_{lj} \quad (3.2)
\]

and we define \( R_{ijkl} \equiv g_{ir} R^r_{jkl} \), \( \forall i, j, k, l \in \{1, \ldots, m\} \). Different definitions of the Riemann tensor (concerning its sign) and its components \( R^i_{jkl} \) can be found.

Given \( p \in M \), a two-dimensional subspace \( P \) of \( T_p(M) \) is called a tangent plane to \( M \) at \( p \). If \( g \) restricted to \( P \) is nondegenerate, \( P \) is said to be a nondegenerate tangent plane to \( M \) at \( p \). In this case take a basis \( v, w \) for \( P \) and define

\[
K(P) \equiv \frac{g(R(v, w)v, w)}{Q(v, w)},
\]

where \( Q(v, w) \equiv g(v, v)g(w, w) - [g(v, w)]^2 \). Then \( K(P) \) is independent of the choice \( v, w \) for \( P \) and is called the sectional curvature of \( P \). One can verify that \( K(P) = 0 \).
for every nondegenerate plane $P$ in $T_p(M)$ if and only if the Riemann tensor satisfies $R(p) = 0$, that is, $R(x, y)z = 0$, $\forall x, y, z \in T_p(M)$. $M$ is said to be flat if $R(p) = 0$, $\forall p \in M$. $M$ is said to have constant curvature if $K(P)$ is constant for every $P$ in $T_p(M)$ and for every $p \in M$.

The Ricci tensor is a $\mathcal{F}(\mathcal{M})$-bilinear symmetric map $\text{Ric} : [\mathcal{F}(\mathcal{M})]^2 \to \mathcal{F}(\mathcal{M})$ whose components are given by $R_{ij} = R^k_{ijk}$, $\forall i, j \in \{1, \ldots, m\}$. $M$ is said to be Ricci flat if $\text{Ric}(p) = 0$, $\forall p \in M$. The scalar curvature is given by $S = g^{ij}R_{ij}$.

Let $\mathbb{R}^m$ be the $C^\infty$ manifold with its usual differentiable structure. A natural coordinate system on $\mathbb{R}^m$ is one (globally defined) that associates to $(x^1, \ldots, x^m) \in \mathbb{R}^m$ the coordinates $(x^1, \ldots, x^m)$. We define a metric tensor on $\mathbb{R}^m$ in such a way that in a natural coordinate system the components of $g$ are given by $g_{ij}(p) = \delta_{ij} \epsilon_j$, $\forall p \in \mathbb{R}^m$, $\forall i, j \in \{1, \ldots, m\}$, where $\epsilon_j = -1$ for $1 \leq j \leq \nu$ and $\epsilon_j = 1$, for $\nu + 1 \leq j \leq m$. The number $\nu$ is the index. This manifold with a Levi-Civita connection is denoted by $\mathbb{R}^m_\nu$. In natural coordinates $\Gamma^k_{ij}(p) = 0$, $\forall p \in \mathbb{R}^m_\nu$, $\forall i, j, k \in \{1, \ldots, m\}$. The Riemann tensor satisfies $R(p) = 0$, $\forall p \in \mathbb{R}^m_\nu$. These flat manifolds are called semi-euclidean spaces. $\mathbb{R}^m_0$ is the $m$-dimensional Euclidean space $E^m$ and, if $m \geq 2$, $\mathbb{R}^m_1$ is the $m$-dimensional Minkowski space $M^{k,m}$.

Let $M$ be a two-dimensional manifold. Then $T_p(M)$ is the only tangent plane at $p \in M$. The sectional curvature in this case is denoted by $K$ and is called the gaussian curvature. Then $R(X, Y)Z = K[g(Z, X)Y - g(Y, X)Z], \forall X, Y, Z \in \mathcal{F}(M)$, $\text{Ric} = Kg$, $S = 2K$ and $K = -R_{1212}$.

4 Semi-Riemannian submanifolds

In this section we review some basic facts about semi-riemannian submanifolds [4]. At first place, submanifolds, immersions and imbeddings are defined. Then it is shown that, given an immersion, there is always a submanifold locally defined. The next concept introduced is that of semi-riemannian submanifolds. By last, the Gauss-Weingarten equations and the Gauss-Codazzi-Ricci equations are discussed.

Let $\bar{M}$ be a topological space and $M \subset \bar{M}$. The induced topology is the one such that a subset $V$ of $M$ is open if and only if there is an open set $\bar{V}$ of $\bar{M}$ such that $\bar{V} \cap M = V$. $M$ is a topological subspace of $\bar{M}$ if it has the induced topology. A manifold $M$ is a submanifold of a manifold $\bar{M}$ provided:

(a) $M$ is a topological subspace of $\bar{M}$.

(b) The inclusion map $j : M \subset \bar{M}$ is $C^\infty$ and at each point $p \in M$ its differential map $dj$ is injective.

Let $\bar{M}$ and $N$ be $C^\infty$ manifolds. An immersion $\phi : N \to \bar{M}$ is a $C^\infty$ map such that $d\phi_p$ is injective, $\forall p \in N$. An imbedding of a manifold $N$ into $\bar{M}$ is an injective immersion $\phi : N \to \bar{M}$ such that the induced map $N \to \phi(N)$ is a homeomorphism onto the subspace $\phi(N)$ of $\bar{M}$, where $\phi(N) \subset \bar{M}$ has the induced topology. An open set $V$ of a manifold $N$ is a manifold, considering the restriction of the complete
atlas of $N$ to $V$. Let $\phi : N \to \bar{M}$ be an immersion. $\forall p \in N$, there is an open set $V \subseteq N$, $p \in V$, such that $\phi : V \to \bar{M}$ is an imbedding \cite{10}. If $\phi : V \to \bar{M}$ is an imbedding, make its image $\phi(V)$ a manifold so that the induced map $\hat{\phi} : V \to \phi(V)$ is a diffeomorphism. Then $\phi(V)$ is a topological subspace of $\bar{M}$ and the inclusion map $j : \phi(V) \subseteq \bar{M}$ is $\hat{\phi} \circ [\hat{\phi}]^{-1}$, which by the chain rule is an immersion. Thus $\phi(V)$ is a submanifold of $\bar{M}$ \cite{4}.

Let $M$ be a submanifold of the semi-riemannian manifold $\bar{M}$ with the metric tensor $\bar{g}$. If the pullback $j^*(\bar{g})$ is a metric tensor on $M$, then $M$ is called a semi-riemannian submanifold of $\bar{M}$. In the particular case of a riemannian manifold $\bar{M}$, the previous condition is always true.

Let us explain better this last topic. If $p \in M$ and $g$ is the metric tensor on the semi-riemannian submanifold $M$, then $g(v, w) = \bar{g}(dj_p(v), dj_p(w)) \equiv \bar{g}(v, w)$, $\forall v, w \in T_p(M)$, where we have simplified our notation, as it is usually done, by the identification of $T_pM$ and $dj_p(T_p(M))$, which turns out to be equivalent to say $T_p(M) \subset T_p(\bar{M})$. The previous condition has to be verified in order to define a semi-riemannian submanifold, because the restriction of $\bar{g}$ to $T_p(M) \times T_p(M), \forall p \in M$, in some cases does not define a metric tensor on $M$. The metric tensor $g$ is called the first fundamental form of the submanifold $M$. There is another usual simplification in the notation. We write $\langle , \rangle$ for the scalar product defined by $\bar{g}$ and understand that, when it is restricted to $T_p(M) \times T_p(M), \forall p \in M$, it is the scalar product defined by $g$.

Suppose $M$ is a semi-riemannian submanifold of $\bar{M}$. Then define, $\forall p \in M$,

$$T_p(M) = \{ v \in T_p(\bar{M}) | \bar{g}(v, w) = 0, \forall w \in T_p(M) \}.$$

As $\bar{g}$ restricted to $T_p(M) \times T_p(M)$ is nondegenerate, $T_p(M) = T_p(M) \oplus T_p(M)^\perp$ and $\bar{g}$ restricted to $T_p(M)^\perp \times T_p(M)^\perp$ is also nondegenerate. The subspaces in the direct sum are called tangent and normal subspaces. The projections are denoted by $Tan : T_p(M) \to T_p(M)$ and $Nor : T_p(\bar{M}) \to T_p(M)^\perp, \forall p \in M$.

We denote by $\mathfrak{X}(\bar{M})$ the set of $C^\infty$ vector fields defined on $M$ such that if $X \in \mathfrak{X}(\bar{M})$, then $X(p) \in T_p(\bar{M}), \forall p \in M$. Similarly, the two sets of $C^\infty$ vector fields defined on $M$ denoted by $\mathfrak{X}(M)$ and $\mathfrak{X}(M)^\perp$ are such that, if $Y \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}(M)^\perp$, then $Y(p) \in T_p(M)$ and $Z(p) \in T_p(M)^\perp, \forall p \in M$.

We write $\bar{D} : \mathfrak{X}(\bar{M}) \times \mathfrak{X}(\bar{M}) \to \mathfrak{X}(\bar{M})$ for the Levi-Civita connection on $\bar{M}$, where $\mathfrak{X}(\bar{M})$ is the set of $C^\infty$ vector fields on $\bar{M}$. There is a natural defined induced connection $\bar{D} : \mathfrak{X}(M) \times \mathfrak{X}(\bar{M}) \to \mathfrak{X}(M)$, obtained from the Levi-Civita connection $\bar{D}$ on $\bar{M}$ by taking appropriate $C^\infty$ local extensions of the vector fields in the sets $\mathfrak{X}(M)$ and $\mathfrak{X}(\bar{M})$ to $\mathfrak{X}(M)$. The induced connection has analogous properties to the Levi-Civita connection $\bar{D}$ on $\bar{M}$.

The Gauss-Weingarten equations are:

\begin{align}
\bar{D}_V W &= Tan(\bar{D}_V W) + Nor(\bar{D}_V W) & (4.1) \\
\bar{D}_V Z &= Tan(\bar{D}_V Z) + Nor(\bar{D}_V Z), & (4.2)
\end{align}

$\forall V, W \in \mathfrak{X}(M), Z \in \mathfrak{X}(M)^\perp$, with the following identifications:
(a) \(\tan(\bar{D}_V W) = D_V W\), where \(D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)\) is the Levi-Civita connection on \(M\).

(b) \(\text{Nor}(\bar{D}_V W) = \Pi(V, W)\), where \(\Pi : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^\perp\) is \(\mathcal{F}(M)\)-bilinear and symmetric. \(\Pi\) is denominated the second fundamental form of the submanifold \(M\).

(c) \(\tan(\bar{D}_V Z) = \tilde{\Pi}(V, Z)\), where \(\tilde{\Pi} : \mathfrak{X}(M) \times \mathfrak{X}(M)^\perp \to \mathfrak{X}(M)\) is \(\mathcal{F}(M)\)-bilinear. \(\tilde{\Pi}\) is given in terms of \(\Pi\), because \(\langle \tilde{\Pi}(V, Z), W \rangle = -\langle \Pi(V, W), Z \rangle\), \(\forall V, V \in \mathfrak{X}(M), Z \in \mathfrak{X}(M)^\perp\).

(d) \(\text{Nor}(\bar{D}_V Z) = D^\perp_V Z\), where \(D^\perp_V Z\) is called the normal covariant derivative of \(Z\) with respect to \(V\) and \(D^\perp\) is called the normal connection on \(M\).

One can verify that

(a) \(D^\perp_V Z\) is \(\mathcal{F}(M)\)-linear in \(V\) and \(\mathbb{R}\)-linear in \(Z\),

(b) \(D^\perp_V (f Z) = f D^\perp_V Z + (V f) Z\) and

(c) \(V(Y, Z) = \langle D^\perp_V Y, Z \rangle + \langle Y, D^\perp_V Z \rangle\),

\(\forall V \in \mathfrak{X}(M), Y, Z \in \mathfrak{X}(M)^\perp\) and \(f \in \mathcal{F}(M)\). The normal connection plays on the normal bundle \(\mathcal{N}M\) the role played by the Levi-Civita connection \(D\) on the tangent bundle \(TM\).

The maps \(g, \Pi\) and \(D^\perp\) are called the fundamental forms of the submanifold \(M\). As a consequence of the Gauss-Weingarten equations \([4]\), we have:

(a) The Gauss equation:

\[
\langle R(V, W)X, Y \rangle = \langle \bar{R}(V, W)X, Y \rangle + \langle \Pi(V, X), \Pi(W, Y) \rangle - \langle \Pi(V, Y), \Pi(W, X) \rangle,
\]

\(\forall V, W, X, Y \in \mathfrak{X}(M)\), where \(\bar{R}\), \(R\) are the Riemann tensors on \(\bar{M}\) and \(M\), respectively.

(b) The Codazzi equation:

\[
\text{Nor} \bar{R}(V, W)X = -(\nabla_V \Pi)(W, X) + (\nabla_W \Pi)(V, X),
\]

where

\[
(\nabla_V \Pi)(X, Y) = D^\perp_V (\Pi(X, Y)) - \Pi(D_V X, Y) - \Pi(X, D_V Y),
\]

\(\forall V, W, X, Y \in \mathfrak{X}(M)\).
(c) The Ricci equation:

$$\langle R^\perp(V,W)X,Y \rangle = \langle \bar{R}(V,W)X,Y \rangle + \langle \bar{\Pi}(V,X),\bar{\Pi}(W,Y) \rangle - \langle \bar{\Pi}(V,Y),\bar{\Pi}(W,X) \rangle$$

for all $V,W \in \mathcal{X}(M), X,Y \in \mathcal{X}(M)^\perp$, where

$$R^\perp: \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M)^\perp \to \mathcal{X}(M)^\perp$$

is given by

$$R^\perp(V,W)X = D^\perp_{[V,W]}X - [D^\perp_V, D^\perp_W]X$$

and is $\mathcal{F}(M)$-multilinear.

One can get a component representation of the Gauss-Weingarten (GW) equations and the Gauss-Codazzi-Ricci (GCR) equations. Let $\dim(M) = \bar{m}$, $\dim(M) = m$ and adopt the following convention about indexes: latin lower case letters $(a,b,\ldots)$ take values on the set $\{1,\ldots,\bar{m}\}$, latin upper case letters $(A,B,\ldots)$ take values on the set $\{1,\ldots,m - m\}$ and greek letters $(\alpha,\beta,\ldots)$ take values on the set $\{1,\ldots,m\}$. The summation convention over indexes appearing twice, once as a superscript and once as a subscript, is employed.

Let $\tilde{V}$ be a coordinate neighborhood of $p \in \tilde{M}$ and $V$ a coordinate neighborhood of $p \in M, p \in V \subset \tilde{V}$, such that there is a set of $(\bar{m} - m)$ locally defined $C^\infty$ vector fields $\{N_A\}, N_A(q) \in T_q(M), \forall q \in V, \langle N_A, N_B \rangle = \eta_{AB}$, where $\eta_{AB}(q) = \delta_{AB}\epsilon^B$ and $\epsilon^B = \pm 1$. The index corresponding to the signature $(\epsilon^1,\ldots,\epsilon^m)$ is denoted by $\nu^\perp$. We denote the inverse of the matrix $(\eta_{AB})$ by $(\eta^{AB})$. Similarly, $(g^{ij})$ and $(g^{\mu\nu})$ are the inverses of $(g_{ij})$ and $(g_{\mu\nu})$. The normal frame field $\{N_A\}$ is typically employed in order to construct bundle charts on the normal bundle $\nu$. We denote by $\{\Gamma^B_{ij}\}$ and $\{\Gamma^A_{\alpha\beta}\}$ the Christoffel symbols of $\tilde{M}$ and $M$ respectively.

Given local coordinates $(y^1,\ldots,y^m)$ on $\tilde{V}$, $(z^1,\ldots,z^m)$ on $V$, then $\partial_{\alpha} = y^k_{\alpha} \partial_k$, where $\partial_{\alpha} \equiv \frac{\partial}{\partial y^\alpha}$, $\partial_k \equiv \frac{\partial}{\partial z^k}$ and $y^k_{\alpha} \equiv \frac{\partial y^k}{\partial y^\alpha}$. Define $\Pi(\partial_{\alpha}, \partial_{\beta}) \equiv \eta^{AB}b_{\alpha\beta}N_A$ and $D^\perp_{\partial_{\alpha}}N_A = \mu^B_{\alpha A}N_B$. Then $\Pi(\partial_{\alpha}, N_A) = -b_{\alpha\beta}g^{\beta\gamma}\partial_{\gamma}$. The set of functions $\{\{g_{\alpha\beta}\}, \{b_{\alpha\beta}\}, \{\mu^A_{\alpha\beta}\}\}$ gives the component representation of the fundamental forms $(g,\Pi, D^\perp)$. Note that $b_{\alpha\beta} = b_{\beta\alpha}$.

Taking $V = \partial_{\alpha}$, $W = \partial_{\beta}$ and $Z = N_A$ in the equations (4.1) and (4.2), the GW equations can be written as:

$$y^k_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}y^k_{\gamma} - \Gamma^k_{pq}y^p_{\alpha}y^q_{\beta} + \eta^{AB}b_{\alpha\beta}N^k_B$$

and

$$N^i_{A,\alpha} = \Gamma^i_{pq}y^p_{\alpha}N^q_A + \mu^C_{A\alpha}N^i_C - b_{\alpha\gamma}g^{\gamma\rho}y^i_{\rho},$$

where $N_A \equiv N^i_{A,\partial_i}$ and $y^k_{\alpha\beta} \equiv \partial_{\alpha}\partial_{\beta}y^k$.

We define $r^C_{A\alpha\beta}$ by $R^\perp(\partial_{\alpha}, \partial_{\beta})N_A \equiv r^C_{A\alpha\beta}N_C$. Thus

$$r^C_{A\alpha\beta} = \partial_{\beta}\mu^C_{A\alpha} + \mu^B_{A\alpha}\mu^C_{B\beta} - \partial_{\alpha}\mu^C_{A\beta} - \mu^B_{A\beta}\mu^C_{B\alpha}.$$  

The GCR equations can be written as:

(a) The Gauss equation:

$$R^i_{\delta\gamma\alpha\beta} = y^i_{\alpha\gamma}y^j_{\beta\delta}y^k_{\delta\gamma}R^j_{ikij} + \eta^{CD}(b_{C\alpha\gamma}b_{D\beta\delta} - b_{C\alpha\delta}b_{D\beta\gamma}).$$
(b) The Codazzi equation:

\[ R_{\mkij y_i^\alpha y_j^\beta y_k^\gamma N^m_D} = b_{B\beta\gamma} \mu_{D\alpha}^B - b_{B\alpha\gamma} \mu_{D\beta}^B - b_{D\beta\gamma;\alpha} + b_{D\alpha;\beta}, \]

where

\[ b_{D\beta\gamma;\alpha} = \partial_\alpha b_{D\beta\gamma} - \Gamma^\theta_{\alpha\gamma} b_{D\beta\theta} - \Gamma^\theta_{\alpha\beta} b_{D\theta\gamma}. \]

(c) The Ricci equation:

\[ \eta_{CB} r^C_{A\alpha\beta} = y^i_{\alpha} y^j_{\beta} N_{N^i_k N^j_l} \bar{\tilde{R}}_{\mkij} + b_{A\alpha\gamma} b_{B\beta\theta} g^{\theta\gamma} - b_{B\alpha\gamma} b_{A\beta\theta} g^{\theta\gamma}. \]

We raise and lower latin uppercase / latin lowercase / greek indexes with \((\eta^{AB})\), \((\eta_{AB})\) / \((g^{ij})\), \((g_{ij})\) / \((\bar{g}^{\alpha\beta})\), \((\bar{g}_{\alpha\beta})\). These six matrices are symmetric ones.

As \((\eta_{AC})\) is constant in \(V\),

\[ 0 = \partial_\alpha \langle N_A, N_C \rangle = \langle D_{\alpha}^i N_A, N_C \rangle + \langle N_A, D_{\alpha}^i N_C \rangle = \mu_{A\alpha}^B \eta_{BC} + \mu_{C\alpha}^B \eta_{AB} \equiv \mu_{CA\alpha} + \mu_{AC\alpha}. \]

Then \(\mu_{CA\alpha} = -\mu_{AC\alpha}\). Thus \(\mu_{1\alpha} = \mu_{2\alpha} = \ldots = \mu_{(m-n)(m-n)\alpha} = 0\).

Let \(M\) be a hypersurface, that is, \(m = m + 1\). In this case, latin uppercase indexes take just one value \((1)\). Then, as \(\mu_{A\alpha}^C = \eta^{AB} \mu_{BC\alpha}\), \(\mu_{1\alpha}^1 = 0\) and \(r_{1\alpha\beta}^1 = 0\). \(\bar{R}\) satisfies \(\bar{R}_{\mkij} = -\bar{\bar{R}}_{\mkij}\). Thus both sides of the Ricci equation are equal to zero and the equation is trivially satisfied in this case.

The mean curvature vector field \(\vec{\mathcal{H}}\) is defined by:

\[ \vec{\mathcal{H}}(p) = \frac{1}{m} \sum_{\alpha=1}^{m} \epsilon_\alpha \Pi(e_\alpha, e_\alpha), \quad (4.5) \]

where \(\{e_\alpha\}\) is any orthonormal basis for \(T_p(M)\), \(g(e_\alpha, e_\beta) = \delta_{\alpha\beta}\epsilon_\beta\) and \(\epsilon_\beta = \pm 1\).

In the classical theory of surfaces in the three-dimensional euclidean space \(E^3\), the gaussian curvature \(K\) and the mean curvature \(\mathcal{H}\) are defined in terms of the differential of the normal map of Gauss in such a way that \(\mathcal{H}(p) = \frac{k_1 + k_2}{2}\) and \(K(p) = k_1 k_2\), where \(k_1, k_2\) are the principal curvatures of the surface \(S\) at \(p \in S\).

The result obtained by Gauss that \(K\) is invariant by local isometries is known as the Egregium Theorem [5].

One can ask if given functions \(\{g_{\alpha\beta}\}, \{b_{A\alpha\beta}\}, \{\mu_{A\alpha}^B\}\) satisfying the GCR equations is there a submanifold such that these functions are the corresponding components of the fundamental forms. The answer to this question in the classical theory of surfaces in \(E^3\) is known as the Bonnet theorem and, basically, it says that locally, up to rigid motions, there is one such surface. See the reference [5] for the precise statement of this theorem. About the generalized Bonnet theorem on manifolds \(\bar{M}\) with constant sectional curvature, also called the fundamental theorem of submanifolds, see [7].
5 Submanifolds defined by zero curvature conditions

In this section we review a method by which a system, such that its field equations are given by a set of zero curvature conditions, associated to a real Lie algebra, in circumstances to be explained, can be associated to a set of semi-riemannian submanifolds of a given semi-riemannian manifold (see [2] for a review).

The first step is to associate to a given real Lie algebra $G$ a manifold with a metric tensor and a corresponding Levi-Civita connection. One of the basic ideas is that all $\bar{m}$-dimensional vector spaces $V$ over $\mathbb{R}$ are isomorphic to the vector space $\mathbb{R}^m$ by the choice of a basis in $V$. Corresponding to a $\bar{m}$-dimensional Lie algebra $G$ there is a $\bar{m}$-dimensional vector space. We can associate to $\mathbb{R}^m$ a manifold with its usual differentiable structure. In this way we can associate to a $\bar{m}$-dimensional Lie algebra $G$ a differentiable manifold which is diffeomorphic to the differentiable manifold $\mathbb{R}^m$.

The Killing form $[8]$ on a real Lie algebra $G$ is a symmetric $\mathbb{R}$-bilinear map $k : G \times G \rightarrow \mathbb{R}$. This map is nondegenerate if and only if the Lie algebra is semisimple. From now on we will suppose the Lie algebra is semisimple. As $k$ is nondegenerate it is possible to find an orthonormal basis $\{T_i\}$, $T_i \in G$, that is, $k(T_i, T_j) = \delta_{ij}\bar{\epsilon}_j$, $\bar{\epsilon}_j = \pm 1$. We denote by $\bar{\nu}$ the index of $k$, that is, the number of negative signs in the signature $(\bar{\epsilon}_1, \ldots, \bar{\epsilon}_\bar{m})$. The Killing form is invariant by automorphisms and satisfy

$$k([T_i, T_j], T_r) = k(T_i, [T_j, T_r]). \quad (5.1)$$

Let us denote for a moment the manifold associated to the Lie algebra $G$ by $\mathbb{R}^m_G$. There is a diffeomorphism relating $\mathbb{R}^m$ and $\mathbb{R}^m_G$. Then there is an isomorphism given by the differential of this diffeomorphism relating the tangent spaces of these two manifolds at corresponding points. In this way a natural coordinate system on $\mathbb{R}^m$ induces a globally defined coordinate system $\{y^i\}$ on $\mathbb{R}^m_G$, which we also call natural, and corresponding coordinate fields $\{\partial/\partial y^i\}$.

We introduce a set of bijective (injective and surjective) linear maps, $\forall p \in \mathbb{R}_{G}^m$, $L_p : T_p(\mathbb{R}^m_G) \rightarrow G$, which are defined in terms of the basis $\{\partial/\partial y^i\}$ as $L_p(\partial/\partial y^i(p)) \equiv T_i$, where $\{T_i\}$ is the orthonormal basis defined by the Killing form. That is, each one of the $L_p$ in the set $\{L_p\}$ is an isomorphism. We denote the inverse by $L_p^{-1} : G \rightarrow T_p(\mathbb{R}^m_G)$, which is also bijective and linear. Thus $L_p^{-1}(T_i) = \partial/\partial y^i(p)$.

Let $c \in \mathbb{R}$, $c \neq 0$. Then we introduce a $F(\mathbb{R}^m_G)$-bilinear map $\bar{g} : F(\mathbb{R}^m_G) \times F(\mathbb{R}^m_G) \rightarrow F(\mathbb{R}^m_G)$ which is defined in terms of the basis $\{\partial_i\}$ as

$$\bar{g}(\partial_i(p), \partial_j(p)) \equiv ck(L_p(\partial_i(p)), L_p(\partial_j(p))) = ck(T_i, T_j) = c\delta_{ij}\bar{\epsilon}_j, \quad (5.2)$$

$\forall p \in \mathbb{R}^m_G$. One can see the map $\bar{g}$ is a metric tensor on $\mathbb{R}^m_G$. Note that

$$\bar{g} \left( \frac{\partial_i(p)}{\sqrt{|c|}}, \frac{\partial_j(p)}{\sqrt{|c|}} \right) = \frac{c}{|c|} \delta_{ij}\bar{\epsilon}_j.$$
That is, \( \{ \partial_i(p) / \sqrt{|c|} \} \) is an orthonormal basis at \( T_p(\mathbb{R}^n_\nu) \).

Define \( \bar{\nu}(c) \) by \( \bar{\nu}(c) \equiv \bar{\nu} \) if \( c > 0 \) and \( \bar{\nu}(c) \equiv \bar{m} - \bar{\nu} \) if \( c < 0 \). Then \( \mathbb{R}^n_\nu \) with the metric tensor \( \bar{g} \) and a corresponding Levi-Civita connection is isometric to the semi-euclidean space \( \mathbb{R}^m_{\bar{\nu}(c)} \). To simplify our notation we will just denote this manifold by \( \mathbb{R}^m_{\bar{\nu}(c)} \).

We consider field theories such that their field equations are given in terms of a set of semisimple real Lie algebra valued gauge potentials \( a_1(z^1, \ldots, z^m, \lambda), \ldots, a_m(z^1, \ldots, z^m, \lambda) \) defined on \( \mathbb{R}^m \times \mathbb{R} \) satisfying a set of zero curvature conditions

\[
\partial_\alpha a_\beta - \partial_\beta a_\alpha + [a_\alpha, a_\beta] = 0,
\]

\( \forall \alpha, \beta \in \{1, \ldots, m\} \).

Let \( U \) be defined by:

\[
(\partial_\mu + a_\mu)U = 0,
\]

\( \forall \mu \in \{1, \ldots, m\} \).

We now introduce an expression which is supposed to parametrize a submanifold \( M \). After we have derived expressions for the corresponding components of the fundamental forms, we are going to indicate the conditions that must be satisfied in order to have a consistent construction and a well defined semi-riemannian submanifold. The maps \( L_p \) and \( L^{-1}_p \) are implicitly employed in order to identify \( T_p(\mathbb{R}^m_{\bar{\nu}(c)}) \) and \( G \), \( \forall p \in \mathbb{R}^m_{\bar{\nu}(c)} \).

Let the position vector corresponding to points in the submanifold \( M \) be given by

\[
r = y^1(z^1, \ldots, z^m, \lambda)\partial_1 + \cdots + y^m(z^1, \ldots, z^m, \lambda)\partial_m \equiv U^{-1}U_\lambda,
\]

(5. 5)

where \( U_\lambda \equiv u_\lambda \), \( \{z^\alpha\} \) are local coordinates on \( M \) and \( \{y^i\} \) are natural coordinates on \( \mathbb{R}^m_{\bar{\nu}(c)} \).

Then, using (5. 4),

\[
r_\mu = -U^{-1}a_{\mu,\lambda}U \quad \text{and} \quad r_{\mu\nu} = U^{-1}([a_{\mu,\lambda}, a_\nu] - a_{\mu,\nu}\lambda)U,
\]

(5. 6)

(5. 7)

where \( a_{\mu,\lambda} \equiv \frac{\partial a_\mu}{\partial \lambda} \) and \( a_{\mu,\nu}\lambda \equiv \frac{\partial}{\partial \lambda} \frac{\partial}{\partial z^n} a_\mu \).

If \( r \) corresponds to a point \( p \in M \), described in terms of the local coordinates \( \{z^\alpha\} \), then \( r_{\mu}(p) \in T_pM \), \( r_{\mu}(p) = (\partial y^i / \partial z^n) \partial_i(p) = \partial_\mu(p) \) and

\[
g_{\mu\nu}(p) \equiv \bar{g}(r_{\mu}, r_{\nu})(p) = ck(-U^{-1}a_{\mu,\lambda}U, -U^{-1}a_{\nu,\lambda}U)(p) = ck(a_{\mu,\lambda}, a_{\nu,\lambda})(p),
\]

(5. 8)

because \( U^{-1}GU \) is an automorphism of \( G \) and the Killing form is invariant by automorphisms. As we see, one of the conditions we need to have a well defined semi-riemannian submanifold is that the matrix defined by \( k(a_{\mu,\lambda}, a_{\nu,\lambda}) \) has determinant
different of zero. In this case, one can find $N^0_1, N^0_2, \ldots, N^0_{m-m} \in G$ such that

$$k(a_{\mu, \lambda}, N^0_A) = 0 \quad \text{and} \quad (5. \ 9)$$

$$ck(N^0_A, N^0_B) = \eta_{AB} = \delta_{AB} \epsilon^B_\perp, \quad (5. \ 10)$$

where $\epsilon^B_\perp = \pm 1$ and the index is denoted by $\nu^\perp$. Note that $\{\epsilon^B_\perp\}$ and $\nu^\perp$ depend on the sign of $c$. That is, if a set $\{N^0_A\}$ satisfy (5. 9) and (5. 10) when $c > 0$, then the same set will satisfy (5. 9) and (5. 10) when $c < 0$, where now $\{\eta_{AB}\} \rightarrow \{-\eta_{AB}\}, \{\epsilon^B_\perp\} \rightarrow \{-\epsilon^B_\perp\}$ and $\nu^\perp \rightarrow (\bar{m} - m - \nu^\perp)$. For instance, in the example given in the next section ($\bar{m} - m = 1$ and $\eta_{11} = c/(|c|)$.

Define

$$N_A = U^{-1}N^0_A U. \quad (5. \ 11)$$

Then $ck(N_A, N_B) = \eta_{AB}$ and, using (5. 4), $N_{A, \mu} = U^{-1}(N^0_{A, \mu} + [a_{\mu, \lambda}])U$. Note that $\bar{g}(r_{\mu}, N_A) = 0$.

Let us employ a natural coordinate system on $\mathbb{R}^m_\mu$. Then $\Gamma^\mu_{ij} = 0$ and the GW equations (4. 3) and (4. 4) can be written as:

$$r_{,\alpha\beta} = \Gamma^\gamma_{\alpha\beta} r_{,\gamma} + \eta^{AB} b_{A\alpha\beta} N_B \quad \text{and} \quad (5. \ 12)$$

$$N_{A, \alpha} = \mu C_{A\alpha} N_C - b_{A\alpha\gamma} g^{\gamma\rho} r_{,\rho}. \quad (5. \ 13)$$

From (5. 12), $\bar{g}(r_{,\alpha\beta}, r_{,\rho}) = \Gamma^\gamma_{\alpha\beta} g_{\gamma\rho}$. Then $\Gamma^\mu_{\alpha\beta} = g^{\mu\rho} \bar{g}(r_{,\alpha\beta}, r_{,\rho})$. Thus

$$\Gamma^\mu_{\alpha\beta} = cg^{\mu\rho} k(a_{\rho, \lambda}, a_{\alpha, \beta\lambda}) \quad (5. \ 14)$$

where $(g^{\mu\rho})$ is the inverse of the matrix defined by (5. 8).

Similarly, from (5. 12),

$$b_{C\alpha\beta} = ck(N^0_C, [a_{\alpha, \lambda}, a_\beta] - a_{\alpha, \beta\lambda}) \quad (5. \ 15)$$

and, from (5. 13),

$$b_{C\alpha\beta} = ck(a_{\beta, \lambda}, N^0_{C, \alpha} + [a_\alpha, N^0_C]). \quad (5. \ 16)$$

One can verify, using (5. 1), (5. 3) and (5. 9), that the expressions (5. 15) and (5. 16) coincide.

From (5. 13),

$$\mu^D_{A\alpha} = c \eta^{BD} k(N^0_B, N^0_{A, \alpha} + [a_\alpha, N^0_A]). \quad (5. \ 17)$$

It follows that

$$\mu_{B A\alpha} = ck(N^0_B, N^0_{A, \alpha} + [a_\alpha, N^0_A]). \quad (5. \ 18)$$

It is possible verify, using (5. 3), that in the expressions (5. 15), (5. 16) and (5. 18) $b_{C\alpha\beta} = b_{C\beta\alpha}$ and $\mu_{D, A\alpha} = -\mu_{A, D\alpha}$. We show in the appendix that $\Gamma^\mu_{\alpha\beta}$ from the expression (5. 11) and $\Gamma^\mu_{\alpha\beta}$ obtained from $g_{\mu\nu}$ by (3. 1) are identical by the use of (5. 3). As in the expression (3. 1) $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$, the same holds in the expression (5. 11). This can be seen also directly in (5. 13), because

$$a_{\alpha, \beta\lambda} - [a_{\alpha, \lambda}, a_\beta] = (1/2)(a_{\alpha, \beta\lambda} - [a_{\alpha, \lambda}, a_\beta]) + (1/2)(a_{\beta, \alpha\lambda} - [a_{\beta, \lambda}, a_\alpha]).$$
We are now going to discuss the conditions that must be imposed in order such construction defines a semi-riemannian submanifold. Let \( \phi : \mathbb{R}^m_{\nu} \to \mathbb{R}^m_{\nu(c)} \) be defined by \( \phi(z^1, \ldots, z^m) = (y^1, \ldots, y^m) \), where \( \{y^i\} \) are the functions defined by \((5.3)\), evaluated at a fixed value of \( \lambda \). The conditions are: There is an open set \( V \subseteq \mathbb{R}^m_{\nu} \) such that

1. \( \phi \) is a \( C^\infty \) differentiable function.
2. \( a_{1,\lambda}(z^1, \ldots, z^m, \lambda), \ldots, a_{m,\lambda}(z^1, \ldots, z^m, \lambda) \) are linearly independent elements of \( G \).
3. \( \det(k(a_{\mu,\lambda}, a_{\nu,\lambda})) \neq 0 \).

To understand the meaning of the second condition, suppose there is \( p \in V \) such that \( r_{1}(p), \ldots, r_{m}(p) \) are linearly dependent. Then there are \( c_{1}, \ldots, c_{m} \in \mathbb{R} \) such that there is at least of them that satisfies \( c_{j} \neq 0 \) and, by \((5.6)\),

\[
U^{-1}(c_{1}a_{1,\lambda} + \cdots + c_{m}a_{m,\lambda})U(p) = 0.
\]

Thus \( (c_{1}a_{1,\lambda} + \cdots + c_{m}a_{m,\lambda})(p) = 0 \), which would correspond to linearly dependent \( a_{1,\lambda}(p), \ldots, a_{m,\lambda}(p) \). Then \( r_{1}(p), \ldots, r_{m}(p) \) are linearly independent elements of \( G \), \( \forall p \in V \). It follows that the \( m \times m \) matrix \( \partial(y^1, \ldots, y^m)/\partial(z^1, \ldots, z^m) \) corresponding to the differential of \( \phi \) has the maximal rank \( m \). Thus \( d\phi \) is injective. We have the conclusion by the first and second conditions that \( \phi : V \to \mathbb{R}^m_{\nu(c)} \) is an immersion. As explained in the previous section, \( \forall p \in V \subseteq \mathbb{R}^m_{\nu} \) there is an open set \( W \), \( p \in W \subseteq V \), such that \( \phi : W \to \mathbb{R}^m_{\nu(c)} \) is an imbedding and \( \phi(W) \) is a submanifold of \( \mathbb{R}^m_{\nu(c)} \).

The last condition implies that the map given by \((5.8)\) is nondegenerate. We can take \( W \) as a conex open set. Then, as \( \phi : W \to \phi(W) \) is a homeomorphism, \( \phi(W) \) is conex. It is known [3] that if a map \( g : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(\mathcal{M}) \) on a conex manifold \( M \) is \( \mathcal{F}(\mathcal{M}) \)-bilinear, symmetric and nondegenerate, then it has constant index. Thus the last condition implies \( \phi(W) \) is a semi-riemannian submanifold.

Note that, as we are taking derivatives with respect to the parameter \( \lambda \) in the expressions written in this section, we are implicitly supposing that differentiability requirements about this parameter are satisfied.

The field equations are invariant by a gauge transformation, as explained in the section 2. It is possible to show that the fundamental forms given by \((5.8)\), \((5.15)\) and \((5.17)\) are invariant by a gauge transformation given by a \( \lambda \)-independent group element \( g \). That is, in any matrix representation, \( \partial_{\lambda}g = 0 \). To do that, as in the equations \((5.9)\) and \((5.10)\), we need to find \( \{N^0_A\} \) such that

\[
k(a_{\mu,\lambda}^g, N^0_A) = 0 \quad \text{and} \quad c\kappa(N^0_A, N^0_B) = \eta_{AB},
\]

where

\[
a_{\mu}^g = ga_{\mu}g^{-1} - \partial_{\mu}gg^{-1} \quad \text{and} \quad (5.19)
a_{\mu,\lambda}^g = ga_{\mu,\lambda}g^{-1}. \quad (5.20)
\]
We take
\[ N_{A}^{0'} \equiv gN_{A}^{0}g^{-1}. \] (5.21)

Then one can verify that (5.8), (5.15) and (5.17) calculated at \( \{a_{\mu}\}, \{N_{A}^{0}\} \) and \( \{a_{\mu}'\}, \{N_{A}^{0'}\} \) are identical.

By last, we emphasize that we are associating to a field theory not just one but a set of semi-riemannian submanifolds. Different solutions of the field equations correspond, in general, to different explicit expressions of the gauge potentials and these last ones correspond, in general, to different submanifolds.

6 Submanifolds associated to Toda theories

In this section we apply the method of the previous section to Toda theories associated to semisimple real Lie algebras of finite dimension. In this case, \( \mathbb{R}_{g}^{m} \) is \( \mathbb{R}_{g}^{1} \simeq Mk^{2} \) and we denote \( z^{1} \equiv z, z^{2} \equiv \bar{z} \). The gauge potentials are given by (2.3) and (2.4). Then
\[ a_{1,\lambda} = -e^{-\lambda}B\varepsilon^{-}B^{-1} \quad \text{and} \quad a_{2,\lambda} = -e^{\lambda}\varepsilon^{+}. \]

The condition (b) of section 5 is that \( a_{1,\lambda} \) and \( a_{2,\lambda} \) are linearly independent. Note that \( a_{1,\lambda} \in G_{-1} \) and \( a_{2,\lambda} \in G_{1} \). Then, if \( \varepsilon^{+} \neq 0 \) and \( \varepsilon^{-} \neq 0 \), that condition is satisfied.

Given a semisimple real Lie algebra \( G \), suppose there is a grading operator \( Q \) and let \( T_{n} \in G_{n}, T_{m} \in G_{m} \). Then, using (5.1), \( k([Q, T_{n}], T_{m}) = -k(T_{n},[Q, T_{m}]) \). We see that \( k([Q, T_{n}], T_{m}) = nk(T_{n}, T_{m}) \) and \( k(T_{n}, [Q, T_{m}]) = mk(T_{n}, T_{m}) \). Then \( (n + m)k(T_{n}, T_{m}) = 0 \). Thus, if \( (n + m) \neq 0 \), then \( k(T_{n}, T_{m}) = 0 \).

Let us analyze the condition (c) of section 5. We have
\[ k(a_{1,\lambda}, a_{1,\lambda}) = k(a_{2,\lambda}, a_{2,\lambda}) = 0, \]
\[ k(a_{1,\lambda}, a_{2,\lambda}) = k(a_{2,\lambda}, a_{1,\lambda}) = k(B\varepsilon^{-}B^{-1}, \varepsilon^{+}). \]

As the Killing form is invariant by authomorphisms, \( k(B\varepsilon^{-}B^{-1}, \varepsilon^{+}) = k(\varepsilon^{-}, B^{-1}\varepsilon^{+}B) \). Thus the condition (c) of section 5 is
\[ k(B\varepsilon^{-}B^{-1}, \varepsilon^{+}) = k(\varepsilon^{-}, B^{-1}\varepsilon^{+}B) \neq 0. \] (6.1)

The metric tensor has components given by:
\[ g_{11} = g_{22} = 0 \quad \text{and} \quad g_{12} = g_{21} = ck(B\varepsilon^{-}B^{-1}, \varepsilon^{+}) = ck(\varepsilon^{-}, B^{-1}\varepsilon^{+}B), \]
where \( c \neq 0 \).
Given \( p \in M \), where \( M \) denotes the submanifold, by \( \{5, 6\} \),

\[
c_1[U^{-1}(e^{-\lambda}B\varepsilon^{-B^{-1}})U](p) + c_2[U^{-1}(e^{\lambda}\varepsilon^+)^2](p) \in T_p(M),
\]

\( \forall c_1, c_2 \in \mathbb{R} \). Let

\[
V_1 = e^{-\lambda}U^{-1}B\varepsilon^{-B^{-1}}U + \frac{e^{\lambda}U^{-1}\varepsilon^+U}{2ck(\varepsilon^+, B\varepsilon^{-B^{-1}})} \quad \text{and} \quad (6. 2)
\]

\[
V_2 = e^{-\lambda}U^{-1}B\varepsilon^{-B^{-1}}U - \frac{e^{\lambda}U^{-1}\varepsilon^+U}{2ck(\varepsilon^+, B\varepsilon^{-B^{-1}})}. \quad (6. 3)
\]

Then \( V_1(p) \) and \( V_2(p) \) \( \in T_p(M) \), \( \forall p \in M \) and \( g(V_1, V_1) = ck(V_1, V_1) = 1 \),

\( g(V_2, V_2) = -1 \) and \( g(V_1, V_2) = 0 \) (Note that \((V_1, V_2) \mid_{c>0} = (V_2, V_1) \mid_{c<0}\)). Thus the index associated to the submanifold is \( \nu_{\text{sub}} = 1 \) and does not depend on the sign of \( c \). We have \( \nu_{\text{sub}} \mid_{c>0} + \nu_{\text{sub}} \mid_{c<0} = m = 2 \). Then \((\nu_{\text{sub}} \mid_{c>0} = 1) \iff (\nu_{\text{sub}} \mid_{c<0} = 1)\).

Note that \( \nu_{\text{sub}} = \nu \), where \( \nu = 1 \) is the index associated to \( \mathbb{R}^2 \simeq M^2 \).

The inverse of \( g \) has components given by:

\[
g_{11} = g_{22} = 0 \quad \text{and} \quad g_{12} = g_{21} = \frac{1}{ck(B\varepsilon^{-B^{-1}}, \varepsilon^+)} = \frac{1}{ck(\varepsilon^-, B^{-1}\varepsilon+B)}.
\]

The Christoffel symbols can be obtained using \( \{2, 3\} \), \( \{2, 4\} \), \( \{5, 14\} \) and the fact that if \((n + m) \neq 0 \) then \( k(G_n, G_m) = 0 \):

\[
\Gamma^1_{11} = k(\varepsilon^+, \partial_1(B\varepsilon^{-B^{-1}})) \quad \text{and} \quad \Gamma^1_{21} = k(\varepsilon^+, \partial_2(B\varepsilon^{-B^{-1}})),
\]

\[
\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = \Gamma^2_{22} = 0 \quad \text{and} \quad \Gamma^2_{22} = \frac{k([B\varepsilon^{-B^{-1}}, \varepsilon^+], \partial_2BB^{-1})}{k(\varepsilon^+, B\varepsilon^{-B^{-1}})}.
\]

The gaussian curvature \( K = -R_{1212} = -g_{1\mu}R^{\mu}_{212} \), can be obtained using \( \{3, 2\} \) and the metric tensor components and Christoffel symbols given in this section:

\[
K = ck(B\varepsilon^{-B^{-1}}, \varepsilon^+)\partial_1 \left[ \frac{k([B\varepsilon^{-B^{-1}}, \varepsilon^+], \partial_2BB^{-1})}{k(\varepsilon^+, B\varepsilon^{-B^{-1}})} \right].
\]

As we explained in the section 3, the Riemann tensor, the Ricci tensor and the scalar curvature of the submanifold are given in terms of the gaussian curvature.

We now give an example of an abelian Toda theory. Let \( G \) be the simple real Lie algebra \( sl(2, \mathbb{R}) \), which is a non-compact real form of the simple complex Lie algebra \( A_1 \). The commutation relations are: \([H, E_{\pm\alpha}] = \pm \alpha E_{\pm\alpha} \) and \([E_{\alpha}, E_{-\alpha}] = 2\alpha H/\alpha^2 \).

The Killing form is \( k(H, H) = 1 \), \( k(E_{\alpha}, E_{-\alpha}) = 2/\alpha^2 \) and zero in the other cases.
In the Chevalley basis \( \{ h, E_{\alpha}, E_{-\alpha} \} \), we have: \( h \equiv 2\alpha H/\alpha^2 \), \([h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}\), \([E_{\alpha}, E_{-\alpha}]=h \) and \( k(h, h) = 4/\alpha^2 \).

Note that

\[
k \left( \frac{\alpha}{2}(E_{\alpha} + E_{-\alpha}), \frac{\alpha}{2}(E_{\alpha} + E_{-\alpha}) \right) = 1,
k \left( \frac{\alpha}{2}(E_{\alpha} - E_{-\alpha}), \frac{\alpha}{2}(E_{\alpha} - E_{-\alpha}) \right) = -1 \quad \text{and}
k \left( \frac{\alpha}{2}(E_{\alpha} + E_{-\alpha}), \frac{\alpha}{2}(E_{\alpha} - E_{-\alpha}) \right) = 0.
\]

One can verify that \( \{ H, \frac{\alpha}{2}(E_{\alpha} + E_{-\alpha}), \frac{\alpha}{2}(E_{\alpha} - E_{-\alpha}) \} \) is an orthonormal basis for \( \mathfrak{sl}(2, \mathbb{R}) \) with respect to the Killing form. Thus \( \bar{m} = \dim(\mathfrak{sl}(2, \mathbb{R})) = 3 \) and \( \nu = 1 \).

The fundamental weight \( \Lambda \) is defined by \( 2\alpha \Lambda/\alpha^2 = 1 \). We define \( Q \equiv 2\Lambda H/\alpha^2 \).

Then \( h \in G_0, E_{\alpha} \in G_1 \) and \( E_{-\alpha} \in G_{-1} \). We define \( \varepsilon^+ = \mu^+ E_{\alpha} \) and \( \varepsilon^- = \mu^- E_{-\alpha} \), where \( \mu^+, \mu^- \in \mathbb{R} \), \( \mu^+ \neq 0 \) and \( \mu^- \neq 0 \). The group element \( B \) is parametrized as \( B \equiv \exp(\phi h) \), where \( \phi \in \mathcal{F}(Mk^2) \), that is, \( \phi \) is a \( C^\infty \) differentiable function \( \phi: Mk^2 \to \mathbb{R} \).

The field equation is

\[
\partial_1 \partial_2 \phi = \mu^+ e^{-2\phi} E_{-\alpha} \quad \text{and} \quad \partial_2 \phi = -\mu^- e^\phi E_{\alpha} - \partial_2 \phi h. \quad (6.5)
\]

The metric tensor components are given by:

\[
g_{11} = g_{22} = 0 \quad \text{and} \quad g_{12} = g_{21} = \frac{2c}{\alpha^2} \mu^+ \mu^- e^{-2\phi}.
\]

The inverse of \( g \) has components given by:

\[
g^{11} = g^{22} = 0 \quad \text{and} \quad g^{12} = g^{21} = \frac{\alpha^2 e^{2\phi}}{2c \mu^+ \mu^-}.
\]

The Christoffel symbols are given by:

\[
\Gamma^1_{11} = -2\partial_1 \phi, \quad \Gamma^2_{12} = \Gamma^1_{21} = \Gamma^2_{21} = \Gamma^1_{22} = 0 \quad \text{and} \quad \Gamma^2_{22} = -2\partial_2 \phi.
\]

The gaussian curvature is given by:

\[
K = -\frac{4c}{\alpha^2} (\mu^+ \mu^-)^2 e^{-4\phi},
\]

where \( c \neq 0 \). Note that:
(a) \( \varphi = \text{constant} \) is not a solution of the field equation. Thus \( K \) is not constant.

(b) If \( c > 0 \), then \( K(p) < 0, \forall p \in M \). If \( c < 0 \), then \( K(p) > 0, \forall p \in M \).

As shown in the section 5, we need a set of \( (\bar{m} - m) \) elements \( \{ N_A^0 \} \) satisfying (5. 9) and (5. 10). In our example, \( (\bar{m} - m) = 1 \). Then \( N_1^0 = \bar{H}/\sqrt{|c|} \). As \( k(\bar{H}, H) = 1 \), \( \eta_{11} = \eta_{11} = c/|c| \).

From (5. 15), (5. 16), (6. 4) and (6. 5),

\[
\begin{align*}
    b_{111} &= b_{122} = 0 \\
    b_{112} &= b_{121} = -\frac{2c\mu^+ e^{-2\varphi}}{\alpha \sqrt{|c|}}.
\end{align*}
\]

In our example the submanifold is a hypersurface \( (\bar{m} = m + 1) \). Then, as shown in the section 4, we have that \( \mu^1_{11} = \mu^1_{12} = 0 \).

Given \( p \in M \), then \((V_1(p), V_2(p))\) (equations (6. 2) and (6. 3)) is an orthonormal basis at \( T_p(M) (\epsilon_1 = 1 \text{ and } \epsilon_2 = -1) \). As \( (r_1 = \partial_1, r_2 = \partial_2) \), they can be written as:

\[
\begin{align*}
    V_1 &= \partial_1 + \frac{1}{2ck(\varepsilon^+, B\varepsilon^- B^{-1})} \partial_2 \\
    V_2 &= \partial_1 - \frac{1}{2ck(\varepsilon^+, B\varepsilon^- B^{-1})} \partial_2.
\end{align*}
\]

The mean curvature vector field \( \tilde{H} \) defined in the equation (4. 5) is given by:

\[
\tilde{H} = \frac{1}{2} [\Pi(V_1, V_1) - \Pi(V_2, V_2)].
\]

Using that \( \Pi \) is \( \mathcal{F}(\mathcal{M}) \)-bilinear, \( \Pi(\partial_\mu, \partial_\nu) = \eta^{AB} b_{B\mu} N_A \) and \( N_A = U^{-1} N_A^0 U \):

\[
\tilde{H} = -\frac{\alpha}{c} U^{-1} H U = -\frac{\alpha^2}{2c} U^{-1} h U.
\]

Note that:

(a) \( \bar{g}(\tilde{H}, \tilde{H})(p) = ck(\tilde{H}, \tilde{H})(p) = \alpha^2/c = \text{constant}, \forall p \in M \). The vector field \( \tilde{H} \) has the same causal character at all the points of the submanifold. That is, if \( c > 0 \) \((c < 0)\), then \( \tilde{H}(p) \) is a spacelike (timelike) vector, \( \forall p \in M \).

(b) \( \tilde{H} = -\alpha \sqrt{|c|} N_1/c \). Then

\[
D_V^\perp \tilde{H} = -V^\beta \partial_\beta \left( \frac{\alpha \sqrt{|c|}}{c} \right) N_1 - \alpha \frac{\sqrt{|c|}}{c} D_V^\perp N_1 = 0,
\]

\( \forall V \in \mathcal{X}(M) \), because \( (\alpha \sqrt{|c|}/c) \) is constant, \( \mu^1_{11} = \mu^1_{12} = 0 \) and \( D_V^\perp N_1 = V^\beta \mu^1_{1,\beta} N_1 = 0 \).
By last, we want to analyze the structure of the set \( \{ N_0^A \} \) in the case of an abelian Toda theory associated to a higher rank algebra. Let \( G \) be the simple real Lie algebra \( sl(3, \mathbb{R}) \), which is a non-compact real form of the simple complex Lie algebra \( A_2 \).

In an analogous way to that described in the case of \( sl(2, \mathbb{R}) \), the commutation relations of \( sl(3, \mathbb{R}) \) associated to the basis \( \{ \{H_i\}, \{E_{\alpha}\} \} \) or \( \{ \{h_i\}, \{E_{\alpha}\} \} \) can be obtained from the commutation relations of \( A_2 \) [8]. We define

\[
Q \equiv 2 \frac{\Lambda_1 \cdot H}{\alpha_1^2} + 2 \frac{\Lambda_2 \cdot H}{\alpha_2^2},
\]

where \( \Lambda_1, \Lambda_2 \) are the fundamental weights. Then \( \{h_1, h_2\} \in G_0, \{E_{\alpha_1}, E_{\alpha_2}\} \in G_1, \{E_{-\alpha_1}, E_{-\alpha_2}\} \in G_{-1}, \{E_{\alpha_1+\alpha_2}\} \in G_2 \) and \( \{E_{-\alpha_1-\alpha_2}\} \in G_{-2} \), where \( h_i = 2\alpha_i \cdot H/\alpha_i^2 \).

We define \( \varepsilon^+ = \mu^+(E_{\alpha_1} + E_{\alpha_2}) \) and \( \varepsilon^- = \mu^-(E_{-\alpha_1} + E_{-\alpha_2}) \), where \( \mu^+, \mu^- \in \mathbb{R}, \mu^+ \neq 0 \) and \( \mu^- \neq 0 \). The group element is parametrized as \( B = \exp(\varphi_1 h_1 + \varphi_2 h_2) \), where \( \varphi_1, \varphi_2 \in \mathcal{F}(Mk^2), \varphi_1 : Mk^2 \to \mathbb{R}, \varphi_2 : Mk^2 \to \mathbb{R} \). Note that \( k(B\varepsilon^-B^{-1}, \varepsilon^+) = \mu^+\mu^-(e^{-2\varphi_1+\varphi_2} + e^{\varphi_1-2\varphi_2}) \neq 0 \), by the normalization \( \alpha_1^2 = 2 \) for all the roots. Thus the condition \( (6.1) \) is satisfied.

The gauge potentials are given by:

\[
a_1 = e^{-\lambda}\mu^- (e^{-2\varphi_1+\varphi_2}E_{-\alpha_1} + e^{\varphi_1-2\varphi_2}E_{-\alpha_2}) \quad \text{and} \quad a_2 = -e^{-\lambda}\mu^+(E_{\alpha_1} + E_{\alpha_2}) - (\partial_2\varphi_1 h_1 + \partial_2\varphi_2 h_2).
\]

We have \( (\bar{m} - m) = 8 - 2 = 6 \) elements in the set \( \{ N_0^A \} \):

\[
N_1^0 = \frac{H_1}{\sqrt{|c|}}, \quad N_2^0 = \frac{H_2}{\sqrt{|c|}}, \quad N_3^0 = \frac{1}{\sqrt{2 |c|}}(c_1 E_{\alpha_1} - c_2 E_{\alpha_2} - E_{-\alpha_1} + E_{-\alpha_2}),
\]
\[
N_4^0 = \frac{1}{\sqrt{2 |c|}}(c_1 E_{\alpha_1} - c_2 E_{\alpha_2} + E_{-\alpha_1} - E_{-\alpha_2}),
\]
\[
N_5^0 = \frac{1}{\sqrt{2 |c|}}(E_{\alpha_1+\alpha_2} + E_{-\alpha_1-\alpha_2}), \quad N_6^0 = \frac{1}{\sqrt{2 |c|}}(E_{\alpha_1+\alpha_2} - E_{-\alpha_1-\alpha_2}),
\]

where

\[
c_1 = \frac{\exp[(3/2)(\varphi_1 - \varphi_2)]}{2 \cosh[(3/2)(\varphi_1 - \varphi_2)]} \quad \text{and} \quad c_2 = \frac{\exp[-(3/2)(\varphi_1 - \varphi_2)]}{2 \cosh[(3/2)(\varphi_1 - \varphi_2)]}.
\]

Note that:

(a) Each one of the elements in the set \( \{ N_0^A \} \) belongs to one of the subspaces: \( G_0, G_1 \oplus G_{-1}, G_2 \oplus G_{-2} \).

(b) The elements associated to the subspaces \( G_0 \) and \( G_2 \oplus G_{-2} \) do not depend on the variables \( z_1, z_2 \). That is, they do not depend on the field variables.

In a similar way, one can construct the set \( \{ N_3^A \} \) in the case of a general Toda theory.
Appendix

We want to show that

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}cg^{\rho\mu}[\partial_\alpha k(a_{\beta,\lambda}, a_{\rho,\lambda}) + \partial_\beta k(a_{\alpha,\lambda}, a_{\rho,\lambda}) - \partial_\rho k(a_{\alpha,\lambda}, a_{\beta,\lambda})]$$

is identical to $\Gamma^\mu_{\alpha\beta}$ given by the equation (5.14). Taking the derivatives in the order they appear, we have six terms: $\Gamma^\mu_{\alpha\beta} = (\Gamma^\mu_{\alpha\beta})^1 + \cdots + (\Gamma^\mu_{\alpha\beta})^6$. For instance, using (5.1) and (5.3),

$$(\Gamma^\mu_{\alpha\beta})^2 + (\Gamma^\mu_{\alpha\beta})^5 = \frac{1}{2}cg^{\rho\mu}[k(a_{\beta,\lambda}, a_{\rho,\alpha\lambda}) - k(a_{\alpha,\rho\lambda}, a_{\beta,\lambda})]$$

$$= \frac{1}{2}cg^{\rho\mu}[k(a_{\beta,\lambda}, a_{\rho,\alpha\lambda}) - a_{\rho,\alpha\lambda} - [a_{\alpha,\lambda}, a_{\rho}] - [a_{\alpha}, a_{\rho,\lambda}])$$

$$= \frac{1}{2}cg^{\rho\mu}[k(a_{\rho,\lambda}, [a_{\alpha}, a_{\beta,\lambda}]) - k(a_{\rho}, [a_{\beta,\lambda}, a_{\alpha,\lambda}])].$$

Similarly, $(\Gamma^\mu_{\alpha\beta})^3 = (1/2)cg^{\rho\mu}k(a_{\rho,\lambda}, 2a_{\alpha,\beta\lambda} - [a_{\alpha,\lambda}, a_{\beta}] - [a_{\alpha}, a_{\beta,\lambda}])$

and $(\Gamma^\mu_{\alpha\beta})^4 + (\Gamma^\mu_{\alpha\beta})^6 = (1/2)cg^{\rho\mu}[k(a_{\rho,\lambda}, [a_{\beta}, a_{\alpha,\lambda}]) - k(a_{\rho}, [a_{\alpha,\lambda}, a_{\beta,\lambda}])].$

Thus $\Gamma^\mu_{\alpha\beta} = cg^{\rho\mu}k(a_{\rho,\lambda}, [a_{\alpha,\beta,\lambda}] - [a_{\alpha,\lambda}, a_{\beta}]).$

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