Antiferromagnetic spin ladders effectively coupled by one-dimensional electron liquids

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We study a model of the stripe state in strongly correlated systems consisting of an array of antiferromagnetic spin ladders, each with \( n_{\text{leg}} \) legs, coupled to each other through the spin-exchange interaction to charged stripes in between each pair of ladders. The charged stripes are assumed to be Luttinger liquids in a spin-gap regime (Luther-Emery). An effective interaction for a pair of neighboring ladders is calculated by integrating out the gapped spin degree of freedom in the charged stripe. The low energy effective theory of each ladder is the usual nonlinear \( \sigma \)-model with additional cross couplings of neighboring ladders. These interactions are found to favor either in-phase or anti-phase short range spin orderings depending on whether the charge stripe is site-centered or bond-centered as well as on its filling factor and other physical parameters of the charged stripe.

Experiments in high \( T_c \) cuprates and other doped antiferromagnets have established that electronic stripe ordering exists in these materials [1]. In this phase, the doped holes are confined into quasi-one-dimensional structures separating locally antiferromagnetic (AF) regions. In addition, there is by now extensive evidence suggesting that dynamical or static stripe order persist in both the normal and superconducting states of the underdoped cuprates such as La\(_{2-x}\)Sr\(_x\)CuO\(_4\). [2] These experimental findings have led support to the picture that stripe order may be intimately related to the mechanism of high \( T_c \) superconductivity. For this, it is important to understand the mechanisms of stripe formation.

The interplay between the charge stripe and the intervening Mott-insulating regions has received some attention. The spin-gap proximity effect was found to be induced by pair hopping between the stripe and its environment in the theory of Emery, Kivelson and Zachar. [3] A Landau theory was also proposed by Zachar, Kivelson, and Emery [4] to describe the intermediate relation of the charge stripe order and the incommensurate spin modulation. They argued that the stripe phase is charge driven through continuous or first-order phase transitions: charge order sets in first and then anti-phase spin domains set later. Their interpretation qualitatively agrees with the experimental observations of La\(_{1.6-x}\)Nd\(_{0.4}\)Sr\(_x\)CuO\(_4\) [5] and La\(_2\)NiO\(_{4+\delta}\) [6]. Granath and Johannesson [7] studied a Hubbard chain (representing the stripe) coupled to an AF spin ladder and found that the magnetic correlations on the ladder generate a spin gap on the stripe. Very recently, Zachar [8] proposed a domain wall (stripe) model of electron dynamics in the AF environment to investigate the energetics of anti-phase and in-phase spin modulation.

In this paper, we attempt to find the effective interaction between two neighboring spin regions mediated by the charge stripe. Such a calculation is important in understanding the magnetic properties of the (planar) doped Mott insulator in the stripe phase, and may serve as a microscopic basis to various spin models studied recently [3,4,8]. Although the model used by Zachar [8] differs in many ways from the picture that we will present here, our conclusions are in full agreement with his.

It has been suggested both theoretically [4,11] and experimentally [1,2] that the charge ordering into hole rich stripes takes place before anti-phase spin domain modulations set in. In this spirit we will consider a model of the stripe phase of a doped AF Mott insulator in two dimensional square lattice based on the following assumptions: 1) the doped holes form an array of metallic stripes (domain walls) lining up, say in \( x \)-direction; 2) between the charge stripes are AF insulating regions, each described by a Heisenberg spin ladder with \( n_{\text{leg}} \) legs of length \( N \):

\[
H_S = J \sum_j \sum_{r} n_{\text{leg}} \vec{S}(j,r) \cdot \left[ \vec{S}(j+1,r) + \vec{S}(j,r+1) \right].
\]

3) the metallic stripes are Luttinger liquids in the Luther-Emery regime which has a spin gap. Thus, low energy hopping process between the stripe and the ladders is suppressed. [3] Our model system qualitatively corresponds to the electronic smectic phase of a doped Mott insulator first suggested by Kivelson, Fradkin and Emery [12], but we shall not include the effects of transverse fluctuations in the present paper.

Now we apply the standard approach to a nonlinear \( \sigma \)-model (see, e.g., Ref. [13]) for the spin ladder. The partition function can be expressed as a path integral by means of the spin coherent states. For antiferromagnetic interaction (\( J > 0 \)), the ladder is expected to be of short-ranged Néel order in the ground state. Therefore, we split the (staggered) spin fields into a slowly varying piece \( \vec{M} \), the order parameter field, and a small rapidly varying part \( \vec{l} \):

\[
\vec{S}(j,r) \approx s \left[ (-)^{j+r} \vec{M}(j) + a\vec{l}(j,r) \right],
\]

where \( \vec{M} \cdot \vec{l} = 0 \) and \( \vec{M}^2 = 1 \). \( s \) is the spin (equal to 1/2
for the physical case). We have assumed that the AF correlation length along the legs is much greater than the width of the ladder, which allows us to treat $\tilde{M}$ constant along the rungs [14]. The low energy theory of the spin ladder is then found to be the familiar nonlinear $\sigma$-model with the (Euclidean) action [13]

$$S_{\text{int}} = \int dx d\tau \left( \frac{1}{2g_\sigma} \left[ \frac{1}{4\pi} \langle \partial_x \tilde{M} \rangle^2 + c_\sigma \langle \partial_x \tilde{M} \rangle^2 \right] + 2\pi i \sigma \left( \frac{1}{4\pi} \langle \partial_x \tilde{M} \rangle \times \langle \partial_y \tilde{M} \rangle \right) \right) \tag{3}$$

where the coupling constant and velocity are $g_\sigma = \frac{1}{2} \left( n_{\text{leg}} / \sum_{r^r} L^{-1}_{r^r} \right)^{-1/2}$, $c_\sigma = a J \left( n_{\text{leg}} / \sum_{r^r} L^{-1}_{r^r} \right)^{1/2}$, and the matrix $L_{r^r} \equiv 4\delta_{rr'} + \delta_{r, r'\pm 1}$. The quantity $\sigma \left( n_{\text{leg}} \right) = 0.1$ for $n_{\text{leg}}$ even and odd respectively. Obviously, the topological term appearing as the last in (3) vanishes for even-leg ladders which are in a Haldane gap phase [15].

The charge stripe can be thought of a quasi-one-dimensional electron gas (1DEG). In the intermediate temperature regime where the system is spin gapped but not superconducting yet, it becomes the Luther-Emery liquid [16] with a (Euclidean) Lagrangian in the bosonized form

$$\mathcal{L}_{\text{1DEG}} = \mathcal{L}_c + \mathcal{L}_s \ . \tag{4}$$

The charge and spin fields are each of the sine-Gordon variety:

$$\mathcal{L}_c = \frac{1}{2\pi K_s} \left[ \frac{1}{v_\alpha} (\partial_x \phi_\alpha)^2 + v_\alpha (\partial_x \phi_\alpha)^2 \right] + V_\alpha \cos(\sqrt{8\pi} \phi_\alpha) \ , \tag{5}$$

where $\alpha = c, s$ for the charge and spin fields, respectively, and $V_\alpha = 0$ (thus for cases of no Umklapp scattering). We consider only the Luther-Emery regime [17] of intermediate temperature where $V_\alpha$ is relevant and a spin gap, $\Delta_s$, is dynamically generated. In this regime, the sine-Gordon theory is known to scale to a strong fixed point, $K_s = 1/2$, sometimes called the Luther-Emery point. In the following, we shall restrict ourselves in performing calculations to this point, where some exact solution is possible. Our analysis can be perturbatively extended to around this fixed point.

Now we are in a position to introduce the coupling between the charge stripe and the spin ladders on both sides ($A, B$):

$$H_{\text{int}} = \frac{g_{J}}{2} \sum_j \left( \mathcal{S}^a_{A}(j, n_{\text{leg}}) + \mathcal{S}^b_{B}(j, 1) \right) \cdot c^\dagger_\sigma(j) \cdot c_\sigma(j) \ , \tag{6}$$

where $\mathcal{S}^a_{A}$ and $\mathcal{S}^b_{B}$ for the charge and spin fields, respectively, are $\mathcal{S}^a_{A}(j, n_{\text{leg}}) + \mathcal{S}^b_{B}(j, 1) \cdot c^\dagger_\sigma(j) \cdot c_\sigma(j)$. For the high $T_c$ cuprates. Seeking for a low energy theory, we write $c_\sigma(j) = \sqrt{2a} \left[ e^{-i K_F x} \psi_{L}(x, j) + e^{-i K_F x} \psi_{R}(x, j) \right]$, where $\psi_{L, R}$ are slowly varying left/right moving fermion fields with respect to the Fermi points $\mp K_F$, and bosonize them [17]. The value of $K_F$ measures the filling factor of the stripe. We split each spin on the ladders into the slowly varying order parameter field $\tilde{M}$ and the rapidly varying field $\tilde{I}$ as in (2). $H_{\text{int}}$ splits into two separate parts, involving either $\tilde{M}$ or $\tilde{I}$. We shall consider the part of $\tilde{M}$ first and come back to another afterwards.

The part of interaction involving the order parameter fields amounts to an extra term to the Lagrangian:

$$\mathcal{L}_{\text{int,M}} = g_{JS} (-)^{n_{\text{leg}}} \phi^a_{M}(x) \cdot \left[ e^{-i(G_N-2K_F) x} e^{-i \sqrt{2\pi} \phi_\alpha \mathcal{J}_{2K_F}(x) + \text{h.c.}} \right] \ , \tag{7}$$

where $\phi^a_{M} \equiv \tilde{M}_A + \eta_s \tilde{M}_B$ for short-hand notation, and $\mathcal{J}_{2K_F}(x)$ are the spin piece of the usual $2K_F$ spin density wave order parameter (see, e.g., [18]): $\mathcal{J}_{2K_F}^a = \frac{1}{\sqrt{\pi} \alpha} \cos(\sqrt{2\pi} \phi_s),$ and $\mathcal{J}_{2K_F}^b = \frac{1}{\sqrt{\pi} \alpha} \sin(\sqrt{2\pi} \phi_s)$ with $\alpha$ the short distance cutoff and $\phi_s$ the dual field to $\phi_\alpha$. $G_N = \pi/\alpha$ is the Néel ordering wave vector. The relative phase factor, $\eta_s$, is $+1 (-1)$ for site-centered (bond-centered) stripes, arising from staggering the spins of two neighbor ladders. The overall factor $(-)^{n_{\text{leg}}}$ appears simply because we have arbitrarily chosen the spin $\tilde{S}^a_{A}(1, 1)$ as the unique reference vector when staggering spins for both ladders. As we shall see, this factor plays no physical role in the following calculation.

Since the spin excitations in the stripe are gapped, the spin degree of freedom can be integrated out to obtain a correction term to the effective action of $\tilde{M}$:

$$e^{-S'_{\text{eff}}} = \left[ \prod_{x, \tau} d\phi_\alpha(x, \tau) \right] e^{-\int dx d\tau \left[ \mathcal{L}_c + \mathcal{L}_{\text{int,M}} \right]} \ . \tag{8}$$

To the lowest order in $g_J$, the corrected effective action is

$$S'_{\text{eff}} = -\frac{\left( g_{JS} \right)^2}{2} \int dx d\tau d\tau' \phi^a_M(x, \tau) \phi^b_M(x', \tau') \times (T_r \mathcal{J}^a_{2K_F}(x, \tau) \mathcal{J}^b_{2K_F}(x', \tau') )_s \times \left\{ e^{-i(G_N-2K_F) x} e^{-i \sqrt{2\pi} \phi_\alpha(x, \tau) + \text{h.c.}} \right\} \ , \tag{9}$$

Here, the indices $a, b = x, y, z$. At the Luther-Emery point $K_s = 1/2$, the sine-Gordon theory [14] for the spin sector is equivalent to a theory of free massive Dirac fermions [19]. The correlation function in (9) can then be calculated exactly. At zero temperature, the result is

$$\langle \mathcal{S}^a_{2K_F}(x, \tau) \mathcal{S}^b_{2K_F}(0) \rangle_s = \frac{2\delta_{ab}}{2(2\pi \xi_s)^2} \left[ K_1 \left( \sqrt{x^2 + v_F^2 \tau^2} / \xi_s \right)^2 - K_0 \left( \sqrt{x^2 + v_F^2 \tau^2} / \xi_s \right)^2 \right] \ . \tag{10}$$
where $K_v(z)$ are Bessel functions. Here $\xi_s$ is the spin correlation length related to the spin gap of the stripe, $\xi_s = v_s/\Delta_s$, as introduced in Carlson et. al. [10]. (10) is manifestly spin rotationally invariant.

Since the correlation function falls off at long distance ($>\xi_s$) exponentially fast (due to the spin gap), we can use the operator product expansion to find an effective local interaction. The product of the (normal ordered) vertex operators $e^{\pm i\sqrt{2}\pi \phi_s}$ in (9) at short distances yields a contribution proportional to the identity operator of the form

$$e^{\pm i\sqrt{2}\pi \phi_s(x,\tau)} \sim \left[(x-x')^2 + v_s^2(\tau-\tau')^2\right]^{\alpha_s/2} + \cdots$$

where we have dropped derivative operators.

Inserting expressions (10) and (11) to (9), one finds $S'_\sigma$ quadratic in $\phi_M$ (and so $\tilde{M}'s$), in which the interaction falls off exponentially fast. Hence, the long wave length limit of (9) becomes (apart from an additive constant)

$$S'_\sigma = 2V\eta_s \int dx d\tau \tilde{M}_A \cdot \tilde{M}_B + \cdots \text{(derivatives)}$$

The effective coupling constant $V$ is given by

$$V = -\alpha K_c \left(\frac{g_J s}{2\pi \xi_s} \right)^2 \int dx d\tau \frac{\cos[(G_N - 2k_F)x]}{[x^2 + v_s^2 \tau^2]^{\alpha_s/2}} \times$$

$$\left[K_1 \left(\frac{\sqrt{x^2 + v_s^2 \tau^2}}{\xi_s} \right)^2 - K_0 \left(\frac{\sqrt{x^2 + v_s^2 \tau^2}}{\xi_s} \right)^2 \right]$$

(13)

The effective interaction term $S'_\sigma$ is a correction to the nonlinear $\sigma$-model described by the action $S_{\sigma\sigma}$ in (3). We should comment that we have used an effective long wavelength theory to determine the effective coupling (12) and that our expressions therefore depend strongly on the short distance cutoff. Also, a small parameter of $G_N - 2k_F$ cannot make our calculation more controllable for the same reason. This is natural since we are computing a non-universal quantity.

So far we have left out the coupling between the 1DEG and the rapidly fluctuating fields $\tilde{I}$, which is part of the $H_{int}$ in (4). Analogous to (4), which comes from the interaction part of $\tilde{M}$, we find an interaction term $\mathcal{L}_{int,I}$ of the fields $\tilde{I}$ coupled to both the spin density and the $2k_F$ spin density waves. Nevertheless, the coupling to the latter becomes negligible for the rapidly oscillations in comparison with the former, so we ignore it. We then follow the same route as above to integrate out the spin degree of freedom in 1DEG, $\phi_s$, and end up with effective couplings of $\tilde{I}_A$ and $\tilde{I}_B$. Adding these to the terms obtained of each spin ladder itself, we find the total effective Lagrangian involving the fields $\tilde{I}$ as follows

$$\mathcal{L}_I = \left[-is(\tilde{M}_A \times \partial_r \tilde{M}_A) \cdot \sum_{r=1}^{n_{neg}} \tilde{I}_A(r) + \frac{aJ s^2}{2} \sum_r \left(4I_A^2(r) + 2\tilde{I}_A(r) \cdot \tilde{I}_A(r + 1)\right) \right]$$

$$+ [A \rightarrow B] + V' a \left[\tilde{I}_A(m_{eg}) + \tilde{I}_B(1)\right]^2$$

(14)

where the coupling constant $V' = (g_J s)^2 a/(4\pi v_s)$ is calculated exactly at the point $K_c = 1/2$ from integrating out the spin fields $\phi_s$ of 1DEG. Since we generally expect

$$\frac{V'}{J} \sim \frac{g_J}{J} \frac{g_J}{v_s(1/a)} \ll 1,$$

we have reduced the problem of an infinite number of weekly coupled parallel spin ladders to a problem of two coupled ladders in (14).

\[ \text{FIG. 1. The (static) effective interaction of } \eta_s \tilde{M}_A \cdot \tilde{M}_B \text{ given by the integral expression (14). The coupling constant } V \text{ is scaled with } V_0 = \frac{2}{a^2} \left(\frac{a^2 - K_v_a K_c}{l^2}\right). \text{ The lattice spacing } a \text{ is taken as the short distance cutoff such that integration in the area of } \sqrt{x^2 + v_s^2 \tau^2} < a \text{ is excluded. Parameters are } v_s/v_a = 3, \text{ and (a) } K_c = 1/4; (b) } K_c = 3/4. \]

The effective Lagrangian $\mathcal{L}_I$ is quadratic in the fluctuating fields $\tilde{I}$, so they can be exactly integrated out, leading to an effective coupling between the order parameter fields of two ladders, which is of the form

$$\mathcal{L}_\sigma'' \sim C''(\tilde{M}_A \times \partial_r \tilde{M}_A) \cdot (\tilde{M}_B \times \partial_r \tilde{M}_B)$$

(15)

with $C''$ a constant coefficient. Comparing with (12), the above effective interaction is a higher order correction to the low energy effective theory.

The static interaction of two adjacent spin ladders is solely given by (12). The integral in (13) has a short distance divergence, which will be dealt with by imposing
a short distance cutoff. A natural cutoff is the lattice spacing \(a\). Thus, our results will depend on the choice of cutoff, which is natural since this effective coupling is controlled by local physics. Obviously, additional operator will yield corrections to the effective coupling but their effects should be considerably weaker.

Fig. 1 shows that the effective interaction \(V\) changes sign when varying \(k_F\), namely the filling factor of the stripe, which for a free 1DEG are related by: \(k_F = (1 - \rho_h)\pi/(2a)\) with \(\rho_h\) the (linear) density of holes in each charge stripe.

Table I summarizes what spin configurations are favored by this effective interaction for different regions of \(k_F\). Our results suggest that the effective interaction could favor either in-phase or anti-phase spin modulation depending on whether the stripe is site-centered or bond-centered and on its physical parameters (such as line density). We would like to compare our results with others in two special cases. Firstly, we observe from Table I that for the bond-centered stripe the in-phase spin modulation is favored for small \(k_F\), i.e., high \(\rho_h\). Recently, Han, Wang and Lee \(^{20}\) studied the mean-field phases of the \(t-J+\)Coulomb model numerically. They found that for a doping range \(0.02 < x < 0.14\) the ground states are the bond-centered in-phase stripes, in which the line density of holes in each stripe varies from \(0.38\) at \(x = 0.095\) to \(0.56\) at \(x = 0.14\), corresponding to a \(k_F\) from \(0.31\pi/a\) to \(0.22\pi/a\) if assuming a free 1DEG. Since such a range of \(k_F\) can be well considered below some \(k_F^*\), our result is consistent with that of Han, Wang and Lee \(^{20}\) despite that their model and approach are completely different from ours. Secondly, we find for a site-centered stripe configuration that a transition from anti-phase to in-phase should occur at a critical \(k_F^*\) (and thus a critical line density). That agrees with what Zachar \(^{8}\) found in a different domain wall model, in which he estimated the critical (electron) filling factor to be \(0.28 < \delta_e < 0.30\).

The consistency of our result can also be checked by thinking of a special (but unphysical) case: half-filled (electron) charge stripe with \(k_F = \pi/2a\). In this case, the \(2k_F\) spin density wave of the stripe is commensurate with the short-ranged Néel order (with \(G_N = \pi/2\)) of the neighboring AF spin ladders. Therefore, one expects that effective coupling mediated by the stripe should become largest in this limit. Fig. 1 shows that the coupling constant \(V\) is indeed largest in magnitude for \(k_F = \pi/2a\).

TABLE I. Possible spin stripe modulations favored by the effective interaction \(^{(12)}\). Note that \(k_F^*\) denotes qualitatively some phase transition point that can be read off from Fig. 1.

| Stripe configuration | site-centered \((\eta_s = +1)\) | bond-centered \((\eta_s = -1)\) |
|----------------------|-------------------------------|-------------------------------|
| \(0 \leq k_F < k_F^*\) | anti-phase | in-phase |
| \(k_F^* < k_F \leq \pi/2a\) | in-phase | anti-phase |

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\[\begin{align*}
1 & \text{For a recent review of stripe in the high temperature superconductors, see, for instance, V. J. Emery, S. A. Kivelson and J. M. Tranquada, Proc. Natl. Acad. Sci., 96, 8814 (1999); and references therein.} \\
2 & \text{T. Suzuki et. al., Phys. Rev. B 57, R3229 (1998). H. Kimura et. al., Phys. Rev. B 59, 6517 (1999); Phys. Rev. B 61, 14366 (2000) and references therein.} \\
3 & \text{V. J. Emery, S. A. Kivelson, and O. Zachar, Phys. Rev. B 56, 6120 (1997).} \\
4 & \text{O. Zachar, S. A. Kivelson, and V. J. Emery, Phys. Rev. B 57, 1422 (1998).} \\
5 & \text{J. Tranquada et al, Nature 375, 561 (1995); Phys. Rev. B 54, 7489 (1996); J. M. Tranquada, J. D. Axe, N. Ichikawa, A. R. Moodenaugh, Y. Nakamura, and S. Uchida, Phys. Rev. Lett. 78, 338 (1997).} \\
6 & \text{C. H. Chen, S.-W. Cheong, and A. S. Cooper, Phys. Rev. Lett. 71, 2461 (1993); J. M. Tranquada, D. J. Buttrey, V. Sachan, and J. E. Lorenzo, Phys. Rev. Lett. 73, 1003 (1994); J. M. Tranquada, J. E. Lorenzo, D. J. Buttrey, and V. Sachan, Phys. Rev. B 52, 3581 (1995); V. Sachan et al., Phys. Rev. B 51, 12742 (1995).} \\
7 & \text{M. Granath and H. Johannesson, Phys. Rev. Lett. 83, 199 (1999).} \\
8 & \text{O. Zachar, cond-mat/0001217.} \\
9 & \text{A. H. Castro Neto and D. Hone, Phys. Rev. Lett. 76, 2165 (1996).} \\
10 & \text{J. Tworzydlo, O. Y. Osman, C. N. A. van Duin, and J. Zaanen, Phys. Rev. B 59, 115 (1999).} \\
11 & \text{G. Seibold, C. Castellani, C. Di Castro, and M. Grilli, Phys. Rev. B 58, 13506 (1998).} \\
12 & \text{S. A. Kivelson, E. Fradkin, and V. J. Emery, Nature 393, 550 (1998).} \\
13 & \text{E. Fradkin, Field Theories of Condensed Matter Systems (Addison-Wesley, Redwood City, Calif., 1991).} \\
14 & \text{S. Dell’Aringa et al., Phys. Rev. Lett. 78, 2457 (1997).} \\
15 & \text{F. D. M. Haldane, Phys. Rev. Lett. 50, 1153 (1983).} \\
16 & \text{E. W. Carlson, D. Orgad, S. A. Kivelson, and V. J. Emery, Phys. Rev. B 62, 3422 (2000).} \\
17 & \text{For a recent review, see, for example, H. J. Schulz, G. Cuniberti, and P. Pieri, cond-mat/9807363 and references therein.} \\
18 & \text{J. Voit, Rep. Prog. Phys. 57, 977 (1995).} \\
19 & \text{This equivalence was originally shown by S. Coleman, Phys. Rev. D 11, 2088 (1975). It was first applied to the present context by Carlson et al. \(^{10}\).} \\
20 & \text{J. H. Han, Q.-H. Wang, and D.-H. Lee, cond-mat/0006046 (unpublished).} \\
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