THE KERNEL OF THE MONODROMY OF THE UNIVERSAL FAMILY OF DEGREE d SMOOTH PLANE CURVES

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Abstract. We consider the parameter space $U_d$ of smooth plane curves of degree $d$. The universal smooth plane curve of degree $d$ is a fiber bundle $E_d \rightarrow U_d$ with fiber diffeomorphic to a surface $\Sigma_g$. This bundle gives rise to a monodromy homomorphism $\rho_d : \pi_1(U_d) \rightarrow \text{Mod}(\Sigma_g)$, where $\text{Mod}(\Sigma_g) := \pi_0(\text{Diff}^+(\Sigma_g))$ is the mapping class group of $\Sigma_g$. The main result of this paper is that the kernel of $\rho_4 : \pi_1(U_4) \rightarrow \text{Mod}(\Sigma_3)$ is isomorphic to $F_\infty \times \mathbb{Z}/3\mathbb{Z}$, where $F_\infty$ is a free group of countably infinite rank. In the process of proving this theorem, we show that the complement $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ of the hyperelliptic locus $\mathcal{H}_g$ in Teichmüller space $\text{Teich}(\Sigma_g)$ has the homotopy type of an infinite wedge of spheres. As a corollary, we obtain that the moduli space of plane quartic curves is aspherical. The proofs use results from the Weil-Petersson geometry of Teichmüller space together with results from algebraic geometry.

1. Introduction

Let $\mathbb{P} \left( \text{Sym}^d \left( \mathbb{C}^3 \right) \right) = \mathbb{P}^N$, where $N = \binom{d+2}{2} - 1$, be the parameter space of plane curves of degree $d > 0$. Elements of $\mathbb{P}^N$ are homogeneous degree $d$ polynomials in variables $x, y, z$. Let $U_d$ denote the parameter space of smooth plane curves of degree $d$. More precisely, $U_d = \mathbb{P}^N \setminus \Delta_d$ is the complement of the discriminant locus $\Delta_d \subset \mathbb{P}^N$ which is the set of polynomials $f$ such that the curve $V(f) = \{ p \in \mathbb{P}^2 : f(p) = 0 \}$ is singular.

The universal smooth plane curve of degree $d$ is the fiber bundle $E_d \rightarrow U_d$ defined by

$$E_d := \{(f, p) \in U_d \times \mathbb{P}^2 : f(p) = 0\} \rightarrow U_d$$

$$(f, p) \mapsto f$$

There exists a monodromy homomorphism $\rho_d : \pi_1(U_d) \rightarrow \text{Mod}(\Sigma_g)$,

where $\text{Mod}(\Sigma_g) := \pi_0(\text{Diff}^+(\Sigma_g))$ is the mapping class group. We omit reference to the basepoint in $\pi_1(U_d)$, however, it can be taken to be the Fermat curve $f_F(x, y, z) = x^d + y^d + z^d = 0$. The homomorphism $\rho_d$ is called the geometric monodromy of the universal smooth plane curve of degree $d$. A finite presentation for $\pi_1(U_d)$ has been given by Lönne [L09, Main Theorem].

Two natural questions are to determine the image $\text{Im}(\rho_d)$ and kernel $K_d := \ker(\rho_d)$. Dolgachev and Libgober have given a description of $\pi_1(U_3)$ as an extension

$$0 \rightarrow \text{Heis}_3(\mathbb{Z}/3\mathbb{Z}) \rightarrow \pi_1(U_3) \xrightarrow{\rho_3} \text{Mod}(\Sigma_1) \rightarrow 0$$
[DL81, Exact Sequence 4.8] of $\text{Mod}(\Sigma_1)$ by the $\mathbb{Z}/3\mathbb{Z}$-points of the 3-dimensional Heisenberg group [DL81, Page 12]

$$\text{Heis}_3(\mathbb{Z}/3\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{Z}/3\mathbb{Z} \right\}$$

The action $\text{Mod}(\Sigma_1) \circlearrowleft H_1(\text{Heis}_3(\mathbb{Z}/3\mathbb{Z}); \mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^2$ is the action on the Weierstraß points of the elliptic curve. This action is exactly the composition $\text{Mod}(\Sigma_1) \xrightarrow{\Psi_1} \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/3\mathbb{Z})$, where $\Psi_1 : \text{Mod}(\Sigma_1) \cong \text{SL}_2(\mathbb{Z})$ is the action on $H_1(\Sigma_1; \mathbb{Z})$, see [FM12, Theorem 2.5], and $\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/3\mathbb{Z})$ is the natural projection.

For higher degrees $d \geq 4$, there is an exact sequence

$$0 \to K_d \to \pi_1(\mathcal{U}_d) \xrightarrow{\rho_d} \text{Mod}(\Sigma_g).$$

The map $\rho_d$ is, in general, not surjective. However, Salter [Sal19, Theorem A] has shown that $\text{Im}(\rho_d)$ always has finite index in $\text{Mod}(\Sigma_g)$. For $d = 4$, Kuno has shown that $\text{Im}(\rho_4) = \text{Mod}(\Sigma_3)$ and that $K_4$ is infinite [Kun08, Proposition 6.3]. For $d = 5$, Salter [Sal16, Theorem A] shows that $\text{Im}(\rho_5)$ is the stabilizer $\text{Mod}(\Sigma_6)([\phi])$ of a certain spin structure $\phi$ on $\Sigma_6$, the spin structure $\phi = e^*\mathcal{O}(1)$ induced on $\Sigma_6$ by its embedding $e : \Sigma_6 \to \mathbb{P}^2$ as a plane curve. For odd $d \geq 5$, Salter shows that the monodromy group $\text{Im}(\rho_d)$ is the stabilizer of a spin structure on $\Sigma_g$, for $g = \left(\frac{d+1}{2}\right)$. For even $d \geq 6$, $\text{Im}(\rho_d)$ is only known to be finite index in this stabilizer, hence in $\text{Mod}(\Sigma_g)$ [Sal19, Theorem A].

Another result in this vein $\pi_1(\mathcal{U}_d)$ can be found in [CT99]. Recall that $\text{Mod}(\Sigma_g)$ acts on $H_1(\Sigma_g; \mathbb{Z})$ preserving the intersection form. This gives rise to the symplectic representation $\Psi_g : \text{Mod}(\Sigma_g) \to \text{Sp}_{2g}(\mathbb{Z})$.

Consider the composition

$$\Psi_g \circ \rho_d : \pi_1(\mathcal{U}_d) \to \text{Sp}_{2g}(\mathbb{Z}).$$

This representation is called the algebraic monodromy of the universal smooth plane curve of degree $d$. Carlson and Toledo show that $\tilde{K}_d := \ker(\Psi_g \circ \rho_d)$ is large [CT99, Theorem 1.2], i.e. there is a homomorphism $\tilde{K}_d \to G$ to a noncompact semisimple real algebraic Lie group $G$ with Zariski-dense image.

In this paper we prove the following theorem, which is a refinement of Kuno’s theorem [Kun08, Proposition 6.3] that $K_4$ is infinite. In the statement, $\text{SMod}(\Sigma_g) \subset \text{Mod}(\Sigma_g)$ denotes the centralizer of a fixed hyperelliptic involution, the homotopy class of an order 2 homeomorphism $\tau : \Sigma_g \to \Sigma_g$ which acts on $H_1(\Sigma_g; \mathbb{Z})$ by multiplication by $-1$.

**Theorem 1.1.** The group $K_4$ is isomorphic to $F_\infty \times \mathbb{Z}/3\mathbb{Z}$, where $F_\infty$ is an infinite rank free group. Moreover, $F_\infty$ has a free generating set in bijection with the set of cosets of the hyperelliptic mapping class group $\text{SMod}(\Sigma_3)$, and

$$H_1(K_4; \mathbb{Q}) \cong \mathbb{Q}[\text{Mod}(\Sigma_3)/\text{SMod}(\Sigma_3)]$$

as $\text{Mod}(\Sigma_3)$-modules.

The idea for the proof of Theorem 1.1 is to exhibit the cover $\mathcal{U}_4^{\text{mark}} \to \mathcal{U}_4$ corresponding to $K_4$ as a principal fiber bundle over the complement $\text{Teich}(\Sigma_3) \setminus \mathcal{H}_3$ of the hyperelliptic locus $\mathcal{H}_3$ in Teichmüller space $\text{Teich}(\Sigma_3)$. The following theorem determines the homotopy type of $\text{Teich}(\Sigma_3) \setminus \mathcal{H}_3$. 
**Theorem 1.2.** Let $g \geq 3$. The hyperelliptic complement $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ has the homotopy type of a wedge
\[ \bigvee_{i=1}^{\infty} S^n \] of infinitely many n-spheres, where $n = 2g - 5$.

From Theorem 1.2, we can conclude that $\mathcal{U}_d^{\text{mark}} \to \text{Teich}(\Sigma_3) \setminus \mathcal{H}_3$ is trivial and Theorem 1.1 follows.

We will also show that the structure of the group $K_d$ is closely related to that of the hyperelliptic mapping class group. The failure of our proof method in Theorem 1.1 for degrees $d > 4$ is due to the lack of knowledge of the topology of the locus of planar curves in the moduli space of Riemann surfaces; there are many more obstructions to being planar than being hyperelliptic.

The paper is organized as follows. Section 2 recalls basic facts about the Weil-Petersson metric on Teichmüller space and the hyperelliptic locus. Section 3 introduces the geodesic length functions. These will then be used to prove Theorem 1.2. The proof of Theorem 1.1 is carried out in section 4.

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2. **The Hyperelliptic Locus and the Weil-Petersson Metric**

For the rest of the paper, let $g \geq 2$ unless otherwise stated. In this section we give the necessary background on Teichmüller space and its geometry. We review the Weil-Petersson metric on Teichmüller space and describe the geometric properties of the hyperelliptic locus in terms of this metric, see Proposition 2.1.

2.1. **Teichmüller Space.** We recall the basic theory of Teichmüller space and of the moduli space of Riemann surfaces of genus $g$. For additional background, see e.g. [FM12]. Let $\text{Teich}(\Sigma_g)$ denote the Teichmüller space of genus $g \geq 2$ curves. That is, $\text{Teich}(\Sigma_g)$ is the set of equivalence classes $[X, h]$ of pairs $(X, h)$, where $X$ is a complex curve of genus $g$ and $h$ is a marking, i.e. a homeomorphism $\Sigma_g \to X$. Two pairs $(X, h)$ and $(Y, g)$ are equivalent if $h \circ g^{-1} : Y \to X$ is isotopic to a biholomorphism. We will also denote such an equivalence class $[X, h]$ by $X$. The (complex) dimension of $\text{Teich}(\Sigma_g)$ is $3g - 3$.

The mapping class group $\text{Mod}(\Sigma_g)$ acts on $\text{Teich}(\Sigma_g)$ by
\[ [f] \cdot [X, h] = [X, h \circ f^{-1}] \]
where $[f] \in \text{Mod}(\Sigma_g)$. This action is properly discontinuous [FM12, Theorem 12.2] so that the quotient space $\mathcal{M}_g := \text{Mod}(\Sigma_g) \setminus \text{Teich}(\Sigma_g)$, the moduli space of genus $g$ Riemann surfaces, is an orbifold. Let $\pi : \text{Teich}(\Sigma_g) \to \mathcal{M}_g$ denote the quotient map. The space $\mathcal{M}_g$ can also be defined as the space of all complex curves of genus $g$, up to biholomorphism. Note that the orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{M}_g)$ of $\mathcal{M}_g$ is $\text{Mod}(\Sigma_g)$. 
2.2. Weil-Petersson Metric. In this subsection we recall the Weil-Petersson (WP) metric and some of its properties. The WP metric is a certain Kähler metric on $\text{Teich}(\Sigma_g)$ which gives rise to a Riemannian structure on $\text{Teich}(\Sigma_g)$. For more on the Weil-Petersson metric, see the survey [Wol09].

The cotangent space $T^*_X \text{Teich}(\Sigma_g)$ at a point $X = [X,h] \in \text{Teich}(\Sigma_g)$ can be identified with the space $Q(X)$ of holomorphic quadratic differentials on $X$. Define a (co)metric on $T^*_X \text{Teich}(\Sigma_g)$ by

$$\langle \langle \varphi, \psi \rangle \rangle := \int_X \varphi \overline{\psi} (ds^2)^{-1},$$

where $ds^2$ is the hyperbolic metric on $X$ and $(ds^2)^{-1}$ is its dual. The Weil-Petersson (WP) metric is defined to be the dual of $\langle \langle \cdot, \cdot \rangle \rangle$.

The WP metric is a $\text{Mod}(\Sigma_g)$-invariant, incomplete [Wol75, Section 2], smooth Riemannian metric of negative sectional curvature [Tro86, Theorem 2]. Teichmüller space $\text{Teich}(\Sigma_g)$ equipped with the WP metric is geodesically convex [Wol87, Subsection 5.4], meaning that any two points $X,Y \in \text{Teich}(\Sigma_g)$ are connected by a unique geodesic. When referring to any metric properties of Teichmüller space, we will assume they are with respect to the WP metric unless otherwise stated.

2.3. Hyperelliptic Locus. A hyperelliptic curve $X$ is a complex curve equipped with a biholomorphic involution $\tau : X \to X$ such that $X/\tau$ is isomorphic to $\mathbb{P}^1$. Such a map $\tau$, if it exists, is called a hyperelliptic involution. An element $[\tau] \in \text{Mod}(\Sigma_g)$ is called a hyperelliptic mapping class if $[\tau]^2 = 1$ and $\Sigma_g/\tau$ is homeomorphic to $\mathbb{P}^1$, or equivalently, if $[\tau]$ acts on $H_1(\Sigma_g; \mathbb{Z})$ by multiplication by $-1$.

Let $\mathcal{H}_g \subset \mathcal{M}_g$ denote the locus of hyperelliptic curves and let $\mathcal{H}_g := \pi^{-1}(\mathcal{H}_g)$, where $\pi : \text{Teich}(\Sigma_g) \to \mathcal{M}_g$ is the quotient map. The set $\mathcal{H}_g$ is called the hyperelliptic locus. It has (complex) dimension $2g - 1$. Note that when $g = 3$, the hyperelliptic locus $\mathcal{H}_3$ has complex codimension 1 in $\text{Teich}(\Sigma_g)$.

The following proposition collects some facts that will be useful in later sections.

**Proposition 2.1.** The locus $\mathcal{H}_g$ is a complex-analytic submanifold of $\text{Teich}(\Sigma_g)$. Moreover, $\mathcal{H}_g$ has infinitely many connected components (see Figure 1). If $H$ is any component of $\mathcal{H}_g$ then $H$ is totally geodesic in $\text{Teich}(\Sigma_g)$ and $H$ is biholomorphic to $\text{Teich}(\Sigma_{0,2g+2})$, the Teichmüller space of a sphere with $2g + 2$ punctures. In particular, each component of $\mathcal{H}_g$ is contractible.

**Proof.** Let $[\tau] \in \text{Mod}(\Sigma_g)$ be a hyperelliptic mapping class. Then $[\tau]$ acts on $\text{Teich}(\Sigma_g)$ with fixed set

$$\text{Fix}([\tau]) := \{ [Y,g] \in \text{Teich}(\Sigma_g) : [Y,g] = [Y,g \circ \tau] \}.$$

First, we show that

$$\mathcal{H}_g = \bigcup_{[\tau] \text{ hyperelliptic}} \text{Fix}([\tau]),$$

where the union is taken over all hyperelliptic mapping classes $[\tau] \in \text{Mod}(\Sigma_g)$. If $[X,h] \in \text{Fix}([\tau])$ then $\tau : X \to X$ is isotopic to a biholomorphism $\tau_b$. The map $\tau_b$ must be a hyperelliptic involution, and so
[X, h] ∈ \mathcal{H}_g. Conversely, if [X, h] ∈ \mathcal{H}_g then there is a hyperelliptic involution \( \tau : X \to X \) which is a biholomorphism and so [X, h] ∈ Fix([\tau]).

If [\tau] and [\eta] are two distinct hyperelliptic mapping classes, then Fix([\tau]) ∩ Fix([\eta]) = \emptyset. More explicitly, if [X, h] ∈ Fix([\tau]) ∩ Fix([\eta]) then, [\tau] and [\eta] contain biholomorphic representatives \( \tau_b, \eta_b : X \to X \). By [FK80, Section III.7.9, Corollary 2], we must have \( \tau_b = \eta_b \).

Each set Fix([\tau]) is totally geodesic in Teich(Σ_g). This follows from the uniqueness of geodesics in the WP metric: if \( \gamma \) is any geodesic with endpoints lying in Fix([\tau]), then [\tau] \cdot \gamma must be another geodesic with the same endpoints as \( \gamma \), hence \( \gamma \) must be fixed by \( \tau \).

For a proof that \( \mathcal{H}_g \) is a complex-analytic submanifold of Teich(Σ_g) and that each component is biholomorphic to Teich(Σ_{0,2g+2}), we refer the reader to [Nag88, Section 4.1.5].

![Figure 1. A schematic of the hyperelliptic locus \( \mathcal{H}_g \) in Teich(Σ_g). The submanifold \( \mathcal{H}_g \subset \text{Teich}(\Sigma_g) \) has infinitely many connected components, each of which is totally geodesic with respect to the Weil-Petersson metric.](image)

### 3. Homotopy Type of the Hyperelliptic Complement

In Section 3.1, we prove, Lemma 3.1, the existence of certain Morse functions on Teich(Σ_g). These functions will be used to prove Theorem 1.2 in Section 3.2.

#### 3.1. Geodesic Length Functions

This section is devoted to proving the existence of sufficiently well-behaved functions on Teich(Σ_g).

**Lemma 3.1.** Let \( g \geq 3 \). There exists a function \( f : \text{Teich}(\Sigma_g) \to \mathbb{R}_+ \) which satisfies the following properties.

1. The function \( f \) is proper, strictly convex and has positive-definite Hessian everywhere.
(2) The function \( f \) has a unique critical point in \( \text{Teich}(\Sigma_g) \), denoted \( x_0 \).

(3) For any component \( H \) of \( H_g \), the restriction \( f|_H \) has a unique critical point, denoted \( x_H \).

(4) Any two critical values are distinct. That is, for any component \( H \) of \( H_g \), \( f(x_H) \neq f(x_0) \). Also, if \( H' \) is any other component of \( H_g \), then \( f(x_H) = f(x_{H'}) \) if and only if \( H = H' \).

(5) The set of critical values 
\[ \{ f(x_H) : H \text{ is a component of } H_g \} \cup \{ f(x_0) \} \]

is a discrete subset of \( \mathbb{R}_+ \).

In particular, such a function \( f \) is Morse on \( \text{Teich}(\Sigma_g) \) and for each component \( H \) of \( H_g \), the restriction \( f|_H \) is Morse.

Proof. The function \( f \) is built using \emph{geodesic length functions}. These functions are defined as follows. Let \( \alpha \) be a free homotopy class of simple closed curves on \( \Sigma_g \) and let \( [X, h] \) be a point in \( \text{Teich}(\Sigma_g) \). Then \( h(\alpha) \) is a free homotopy class of simple closed curves in \( X \). Recall that \( h(\alpha) \) contains a unique geodesic \( \gamma \). The \emph{geodesic length function} \( \ell_\alpha : \text{Teich}(\Sigma_g) \to \mathbb{R}_+ \) associated to \( \alpha \) is defined by
\[ \ell_\alpha([X, h]) := \text{length of the unique geodesic in the free homotopy class } h(\alpha) \text{ on } X, \]
where \( X = [X, h] \). Any other choice \( (X', h') \) of representative of \( [X, h] \) would differ from \( (X, h) \) by an isometry, hence \( \ell_\alpha \) is well-defined. Fix a finite collection \( \mathcal{A} \) of (homotopy classes of) simple closed curves which fills \( \Sigma_g \), and let \( c = (c_\alpha) \in \mathbb{R}^+_A \) be a collection of positive real numbers for each \( \alpha \in \mathcal{A} \). For each choice of \( c \in \mathbb{R}^+_A \), there is a function
\[ L_{\mathcal{A}, c} := \sum_{\alpha \in \mathcal{A}} c_\alpha \ell_\alpha : \text{Teich}(\Sigma_g) \to \mathbb{R}_+. \]
The function \( f \) in the statement of the theorem will be defined to be \( L_{\mathcal{A}, c} \) for a specific value of \( c \).

Wolpert [Wol87, Theorem 4.6] states that for any free homotopy class of simple closed curves \( \alpha \) on \( \Sigma_g \), the geodesic length function \( \ell_\alpha \) has positive-definite Hessian everywhere. In particular, \( \ell_\alpha \) is strictly convex along WP geodesics.

Recall that the Hessian operator \( \text{Hess} \) is given in local coordinates by
\[ f \mapsto \left( \frac{\partial^2 f}{\partial x^i \partial x^j} + \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right) dx^i \otimes dx^j, \]
where \( \Gamma^k_{ij} \) are the Christoffel symbols given by \( g \). Thus, \( \text{Hess} \) is \( \mathbb{R} \)-linear. It follows that
\[ \text{Hess} L_{\mathcal{A}, c} = \sum_{\alpha \in \mathcal{A}} c_\alpha \cdot (\text{Hess } \ell_\alpha). \]
For any \( v \in T_X \text{Teich}(\Sigma_g) \),
\[ \text{Hess } L_{\mathcal{A}, c}(v, v) = \sum_{\alpha \in \mathcal{A}} c_\alpha \cdot (\text{Hess } \ell_\alpha)(v, v) > 0 \]
and so \( \text{Hess } L_{\mathcal{A}, c} \) is positive-definite. This also shows that \( L_{\mathcal{A}, c} \) is strictly convex.
Let $\mathbf{1}$ denote the element of $\mathbb{R}^A_+$ such that $c_\alpha = 1$ for all $\alpha \in A$. For $c = (c_\alpha) \in \mathbb{R}^A_+$, let $c_{\text{min}} := \min_{\alpha \in A} c_\alpha$. Then,

$$c_{\text{min}} L_{A,1} \leq L_{A,c}.$$ 

Kerckhoff [Ker83, Lemma 3.1] states that the functions $L_{A,1}$ are proper. If $K = [a, b] \subset \mathbb{R}_+$ is compact, then

$$(L_{A,c})^{-1}(K) \subset (L_{A,1})^{-1} [0, b/c_{\text{min}}],$$

so $(L_{A,c})^{-1}(K)$ is a closed subset of a compact set, hence is compact. Thus, $L_{A,c}$ is proper. This proves (1) in the statement of the theorem.

If $L_{A,c}$ has distinct critical points $x_0$ and $x'_0$ in $\text{Teich}(\Sigma_g)$, then these are local minima of $L_{A,c}$ since $\text{Hess } L_{A,c}$ is positive definite at both $x_0$ and $x'_0$. Without loss of generality, assume $L_{A,c}(x'_0) \leq L_{A,c}(x_0)$. However, by strict convexity, this is impossible. Let $\gamma$ be the unique geodesic with $\gamma(0) = x_0$ and $\gamma(1) = x'_0$. Then

$$L_{A,c}(\gamma(t)) < (1 - t)L_{A,c}(x_0) + tL_{A,c}(x'_0) \leq L_{A,c}(x_0)$$

for all $t \in (0, 1)$, contradicting the fact that $x_0$ must be a local minimum. Hence $x_0 = x'_0$ and $L_{A,c}$ has a unique critical point in $\text{Teich}(\Sigma_g)$, denoted $x_0$. This proves property (2).

Since the components of $\mathcal{H}_g$ are totally geodesic in the WP metric, the same argument shows that the restriction $L_{A,c}|_H$ will have a unique critical point, denoted $x_H$, for each component $H$ of $\mathcal{H}_g$. This proves property (3) of the theorem. Thus, properties (1) through (3) of the theorem above are satisfied by the function $L_{A,c}$ for any value of $c$.

Let $S = \{H : H$ is a component of $\mathcal{H}_g\} \cup \{0\}$. For each pair $i, j \in S$ of distinct elements, there is an open dense subset $U_{i,j}$ of $\mathbb{R}^A_+$ given by

$$U_{i,j} = \{c \in \mathbb{R}^A_+ : L_{A,c}(x_i) \neq L_{A,c}(x_j)\}.$$ 

By the Baire Category Theorem, $\bigcap_{i \neq j} U_{i,j}$ is open and dense in $\mathbb{R}^A_+$. Let $c' \in \bigcap_{i \neq j} U_{i,j}$. We now define $f := L_{A,c'}$. Then, $f$ satisfies property (4).

Lastly, we wish to show that $f(S)$ is discrete. Choose a neighborhood $U_0$ of $x_0$ and $U_H$ of $x_H$, for each component $H$ of $\mathcal{H}_g$ which are mutually disjoint. Properness of $f$ then implies that $f(S)$ is discrete. This shows that $f$ satisfies property (5). $\square$

### 3.2. Relative Morse theory of the pair $(\text{Teich}(\Sigma_g), \mathcal{H}_g)$

The goal of this subsection is to prove Theorem 1.2. The idea is that the Morse function $f$ found in Lemma 3.1 may be used to determine a handle decomposition of both $\mathcal{H}_g$ and $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g$. For a reference on relative Morse theory, see e.g. [Sha88, Section 3].

**Theorem 1.2.** Let $g \geq 3$. The hyperelliptic complement $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ has the homotopy type of a wedge $\bigvee_{i=1}^{\infty} S^n$ of infinitely many $n$-spheres, where $n = 2g - 5$. 

Note that since every curve of genus $g = 2$ is hyperelliptic, $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_2 = \emptyset$. The proof of Theorem 1.2 is similar to Mess’s proof that the image of the period mapping on $\text{Teich}(\Sigma_g)$ has the homotopy type of an infinite wedge of circles [Mes92, Proposition 4]. We now prove Theorem 1.2.

**Proof.** The idea behind relative Morse theory is that such a function as given by Lemma 3.1 can be used to determine a handle decomposition not only of $\mathcal{H}_g$, but of its complement $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g$. Let $f$ be the function that satisfies the conclusion of Lemma 3.1. We let $x_0$ denote the unique minimum point of $f$ in $\text{Teich}(\Sigma_g)$. For each component $H$ of $\mathcal{H}_g$, let $x_H$ denote the unique critical point of $f|_H$. We refer to $x_0$ as a critical point of $f$ of type I and each $x_H$ are referred to as critical points of $f$ of type II. The values $c_0 = f(x_0)$ and $c_H = f(x_H)$ are called critical values of type I and II, respectively.

For $r$ a real number, let $X_r := \{X \in \text{Teich}(\Sigma_g) : f(X) \leq r\}$. If $(c_0, c_0 + \epsilon]$ contains no type II critical values, then $X_{c_0 + \epsilon} \setminus \mathcal{H}_g$ is diffeomorphic to a 0-handle, i.e. a closed ball. Consider an arbitrary interval $[a, b] \subset \mathbb{R}$. If $[a, b]$ contains no critical value of type I or II of $f$, then $X_a \setminus \mathcal{H}_g$ is diffeomorphic to $X_b \setminus \mathcal{H}_g$. To see this, we can construct a vector field $V$ which is equal to $\text{grad}(f)$ outside a neighborhood of $\mathcal{H}_g$ and such that $V|_{\mathcal{H}_g}$ is equal to $\text{grad}(f|_{\mathcal{H}_g})$. The flow along this vector field gives the required diffeomorphism.

Let $x$ be a critical point of type II, and let $c = f(x)$. By Lemma 3.1, the set of critical values of $f$ is discrete, so there exists some $\epsilon > 0$ such that $[c - \epsilon, c + \epsilon]$ contains no other critical values of $f$. We wish to show that $X_{c+\epsilon} \setminus \mathcal{H}_g$ is diffeomorphic to $X_{c-\epsilon} \setminus \mathcal{H}_g$ with an $n$-handle attached, where $n = 2g - 5$ (see Figure 2).

Let $H$ be the component of $\mathcal{H}_g$ containing $x$. There exists a coordinate system $(u, y) \in \mathbb{R}^{2g-4} \times \mathbb{R}^{4g-2}$ in a neighborhood $U$ of $x$ such that [Sha88, 3.3]

1. $U \cap H$ is given by $u = 0$,
2. $f = c + \|y\|^2$ on $U \cap H$.

The coordinates $y$ are “tangent” coordinates to $H$ and the coordinates $u$ are “normal” coordinates to $H$. Note that since $H$ has complex dimension $2g - 1$, it has real dimension $4g - 2$.

![Figure 2](image)

**Figure 2.** Start with $X_{c-\epsilon}$. As $c - \epsilon$ increases to $c + \epsilon$, the level set $X_{c+\epsilon}$ intersects exactly one more component $H$ of $\mathcal{H}_g$, the component containing the critical point $x$. 

Then, \( X_{c+\epsilon} \setminus \mathcal{H}_g \) is diffeomorphic to the union of \( X_{c-\epsilon} \setminus \mathcal{H}_g \) and a tubular neighborhood of
\[
\{(u, 0) : \|u\|^2 = \delta\},
\]
for some small \( \delta > 0 \). This tubular neighborhood deformation retracts to the \((2g-5)\)-sphere \( \{(u, 0) : \|u\|^2 = \delta\} \). Hence, \( \text{Teich}(\Sigma_g) \setminus \mathcal{H}_g \) has a handle decomposition consisting of a 0-handle with infinitely many (one for each component of \( \mathcal{H}_g \)) \( n \)-handles attached, where \( n = 2g - 5 \).

Let \( \mathcal{M}_{g}^{hyp} \) denote the moduli space of hyperelliptic curves of genus \( g \). Since \( \text{Teich}(\Sigma_3) \setminus \mathcal{H}_3 \) is a covering space for \( \mathcal{M}_3^{hyp} \), the moduli space \( \mathcal{M}_3^{hyp} \) has contractible universal cover and \( \mathcal{M}_3^{hyp} \) is aspherical. If \( g \geq 4 \) then \( \pi_n(\mathcal{M}_g^{hyp}) \), where \( n = 2g - 5 > 1 \), is an infinite rank abelian group. In particular, \( \mathcal{M}_g^{hyp} \) is not aspherical for \( g \geq 4 \).

We can be even more precise. The components of the hyperelliptic locus \( \mathcal{H}_g \) are enumerated by the set of cosets of the group \( \text{SMod}(\Sigma_g) \) in \( \text{Mod}(\Sigma_g) \). Recall that \( \text{SMod}(\Sigma_g) \) is the centralizer in \( \text{Mod}(\Sigma_g) \) of a fixed hyperelliptic involution \( \tau \in \text{Mod}(\Sigma_g) \). The group \( \text{SMod}(\Sigma_g) \) is called the hyperelliptic mapping class group of genus \( g \). If \( \eta \) is another hyperelliptic involution, then the centralizers of \( \tau \) and \( \eta \) are conjugate in \( \text{Mod}(\Sigma_g) \).

Corollary 3.2. Let \( g \geq 3 \). There is a homotopy equivalence
\[
\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g \cong \bigvee_{[h] \in \text{Mod}(\Sigma_g)/\text{SMod}(\Sigma_g)} S^{2g-5}.
\]
In particular,
\[
H_{2g-5}(\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g; \mathbb{Z}) \cong \mathbb{Z}[\text{Mod}(\Sigma_g)/\text{SMod}(\Sigma_g)]
\]
as \( \text{Mod}(\Sigma_g) \)-modules.

Proof. The mapping class group \( \text{Mod}(\Sigma_g) \) acts on the set of components of \( \mathcal{H}_g \) by permutations. Then, there is a map
\[
\text{Orb}(H_0) \to \text{Mod}(\Sigma_g)/\text{Stab}(H_0)
\]
\[
h \cdot H_0 \mapsto h\text{Stab}(H_0)
\]
from the orbit \( \text{Orb}(H_0) \) of \( H_0 \) to the left coset space of the stabilizer \( \text{Stab}(H_0) \). It suffices to show that \( \text{Stab}(H_0) = \text{SMod}(\Sigma_g) \) and \( \text{Mod}(\Sigma_g) \) acts transitively on the set of components of \( \mathcal{H}_g \).

First, since \( H_0 = \text{Fix}(\tau) \), the mapping class \( h \in \text{Stab}(H_0) \) if and only if
\[
h \cdot \text{Fix}(\tau) = \text{Fix}(h\tau h^{-1}) = \text{Fix}(\tau).
\]
Since no hyperelliptic curve can have two distinct hyperelliptic involutions, it must follow that \( h\tau h^{-1} = \tau \) so \( h \in \text{SMod}(\Sigma_g) \). Therefore, \( \text{Stab}(H_0) = \text{SMod}(\Sigma_g) \).

Secondly, if \( H \) is any other component of \( \mathcal{H}_g \), then \( H = \text{Fix}(\eta) \) for some hyperelliptic involution \( \eta \in \text{Mod}(\Sigma_g) \). Since hyperelliptic involutions in \( \text{Mod}(\Sigma_g) \) are conjugate, there exists some \( h \in \text{SMod}(\Sigma_g) \) such that
\[
H = \text{Fix}(\eta) = \text{Fix}(h\tau h^{-1}) = h \cdot \text{Fix}(\tau) = h \cdot H_0.
\]
Therefore, Mod(Σ_g) acts transitively on the set of components of H_g.

4. The Parameter Space of Smooth Plane Curves

In this section, we prove Proposition 4.2, showing that the cover of U_d determined by the subgroup K_d of π_1(U_d) carries the structure of a principal fiber bundle. This will be critical in the proof of Theorem 1.1 in Section 4.2.

4.1. Covers of U_d and principal fiber bundles. The main result of this subsection is to prove Proposition 4.2, exhibiting a cover of U_d as a principal fiber bundle over a certain subspace of Teich(Σ_g).

Associating each point of U_d to the curve it determines gives rise to a map ϕ_d : U_d → M_g into the moduli space of Riemann surfaces of genus g(d), where g = g(d) := (d−1)/2 by the degree-genus formula. Let M^pl_g denote the image of this map. For d ≥ 4, the locus M^pl_g ⊆ M_g and for d = 3, M^pl_3 = M_1.

There is a (disconnected) covering U^mark_d of U_d defined as follows. A point (f, [h]) ∈ U^mark_d is an ordered pair consisting of f ∈ U_d and a homotopy class [h] of orientation-preserving homeomorphisms h : Σ_g → V(f) of some fixed Σ_g with the complex curve V(f) given by f(x, y, z) = 0.

Let π_1(U^mark_d) be the fundamental group of a chosen component of U^mark_d. Note that π_1(U^mark_d) ∼= K_d.

Remark 4.1. There is a commutative diagram

\[ \begin{array}{ccc}
U^\text{mark}_d & \xrightarrow{\hat{\varphi}_d} & \text{Teich}(\Sigma_g) \\
\downarrow & & \downarrow \pi \\
U_d & \xrightarrow{\varphi_d} & M_g
\end{array} \]

The map \( \varphi_d : U_d \to M_g \) lifts to a map \( \hat{\varphi}_d : U^\text{mark}_d \to \text{Teich}(\Sigma_g) \) into Teichmüller space defined by

\[ \varphi_d : (f, [h]) \mapsto [V(f), h]. \]

Let Teich(Σ_g)^pl denote the image of \( \varphi_d \).

Recall that a principal G-bundle is a fiber bundle \( P \to X \) with a G-action that acts freely and transitively on the fibers.

Proposition 4.2. For d ≥ 4, the map \( \hat{\varphi}_d : U^\text{mark}_d \to \text{Teich}(\Sigma_g)^\text{pl} \) is a principal PGL_3(ℂ)-bundle.

Proof. First, PGL_3(ℂ) acts on U^mark_d by \( g \cdot (f, [h]) = (g \cdot f, [g \circ h]) \) where \( g \cdot f \) denotes the action of \( g \) on polynomials \( f(x, y, z) \), by acting on the triple of variables \( (x, y, z) \). This induces a map \( g : V(f) \to V(g \cdot f) \) and \( g \circ h \) is the composition of this map with the marking \( h : \Sigma_g \to V(f) \).
This action is free. Indeed, if \( g \cdot (f, [h]) = (f, [h]) \) then \( g \cdot f = f \) and \( [g \circ h] = [h] \). Thus \( g \) induces an automorphism on the curve \( V(f) \). Moreover, this automorphism acts trivially on the marking, hence trivially on \( H_1(V(f); \mathbb{Z}) \). An automorphism of \( V(f) \) acting trivially on homology must be the identity [FM12, Theorem 6.8]. The fixed set of any automorphism of \( \mathbb{P}^2 \) is a linear subspace, so any \( g \in \text{PGL}_3(\mathbb{C}) \) point-wise fixing a smooth quartic curve must be the identity automorphism.

Next, we show that this action is transitive on fibers. It suffices to show that if \( \tilde{\varphi}_d(f_1, [h_1]) = \varphi_d(f_2, [h_2]) \), then the \( (f_1, [h_1]) \) lie in the same \( \text{PGL}_3(\mathbb{C}) \)-orbit. By assumption, \([V(f_1), h_1] = [V(f_2), h_2]\) and there is some biholomorphism \( \psi : V(f_1) \rightarrow V(f_2) \) such that \([\psi \circ h_1] = [h_2]\). Then the pullback of the hyperplane bundle \( H \) along the embeddings \( e_i : V(f_i) \rightarrow \mathbb{P}^2 \) gives line bundles \( L_i := e_i^*(H) \) on \( V(f_i) \) of degree \( d \) with \( h^0(L_i) = 3 \).

A \( g_d \) line bundle is a line bundle \( L \rightarrow C \) such that \( \deg(L) = d \) and \( h^0(L) \geq r + 1 \). Smooth plane curves have a unique \( g_2 \) given by the pullback of the hyperplane bundle [Ser87, Theorem 3.13]. Therefore, \( L_1 \) and \( \psi^*L_2 \) are isomorphic line bundles on \( V(f_1) \).

For any smooth curve \( C \), there is a correspondence between maps \( C \rightarrow \mathbb{P}^r \) up to the action of \( \text{PGL}_{r+1}(\mathbb{C}) \) and pairs \( (L, V) \) where \( L \) is a line bundle over \( C \) and \( V \subset H^0(C; L) \) is an \((r + 1)\)-dimensional subspace. The fact that there is a unique line bundle \( L \) on \( V(f_1) \) with \( h^0(L) \geq 3 \) implies that there is only one such map \( V(f_1) \rightarrow \mathbb{P}^2 \) up to the action of \( \text{PGL}_3(\mathbb{C}) \). Therefore, the two embeddings \( e_1 \) and \( e_2 \circ \psi \) are equivalent up to the action of \( \text{PGL}_2(\mathbb{C}) \), i.e. there is some \( g \in \text{PGL}_2(\mathbb{C}) \) such that \( g \circ e_1 = e_2 \circ \psi \). This implies that \( g \cdot f_1 = f_2 \) and \( g : V(f_1) \rightarrow V(f_2) \) coincides with \( \psi \). Thus, \((f_1, [h_1])\) and \((f_2, [h_2])\) lie in the same \( \text{PGL}_3(\mathbb{C}) \)-orbit.

Finally, it remains to prove local triviality. This is a consequence of a much more general fact that if \( G \) acts on a manifold \( P \) freely such that \( P/G \) is a manifold, then \( q : P \rightarrow P/G \) is locally trivial. Indeed, a local trivialization of \( q : P \rightarrow P/G \) can be built over any contractible subset \( U \) by first taking a section \( \sigma : U \rightarrow P \) and defining \( \varphi : q^{-1}(U) \rightarrow U \times G \) by \( \varphi(x) = (q(x), g(x)) \), where \( g(x) \in G \) is the unique element such that \( x = g(x) \cdot \sigma(q(x)) \).

**Proposition 4.3.** Let \( d \geq 3 \) and \( g = \binom{d-2}{2} \). The space \( \mathcal{U}^\text{mark}_d \) has finitely many components. Consequently, \( \text{Teich}(\Sigma_g)^{\text{mark}} \) has finitely many components.

**Proof.** A single component of \( \mathcal{U}^\text{mark}_d \) is the connected covering space of \( \mathcal{U}_d \) corresponding to \( K_d \). Hence, its deck transformation group is the image of the homomorphism \( \rho_d : \pi_1(\mathcal{U}_d) \rightarrow \text{Mod}(\Sigma_g) \). The components of \( \mathcal{U}^\text{mark}_d \) are enumerated by the cosets of \( \text{Im}(\rho_d) \) in \( \text{Mod}(\Sigma_g) \). It was shown in and [Sal19, Theorem A] that the index \([\text{Mod}(\Sigma_g) : \text{Im}(\rho_d)] \) is finite.

**4.2. The kernel of the geometric monodromy of the universal quartic.** In this subsection, we prove Theorem 1.1.

**Theorem 1.1.** The group \( K_4 \) is isomorphic to \( F_\infty \times \mathbb{Z}/3\mathbb{Z} \), where \( F_\infty \) is an infinite rank free group. Moreover, \( F_\infty \) has a free generating set in bijection with the set of cosets of the hyperelliptic mapping class group \( \text{SMOD}(\Sigma_3) \), and

\[
H_1(K_4; \mathbb{Q}) \cong \mathbb{Q}[\text{Mod}(\Sigma_3)/\text{SMOD}(\Sigma_3)]
\]
as Mod(\(\Sigma_3\))-modules.

**Proof of Theorem 1.1.** Classically, Teich\((\Sigma_3)^{pl}\) is exactly the complement of the hyperelliptic locus \(\mathcal{H}_3\) in Teich\((\Sigma_3)\): the canonical map \(C \to \mathbb{P}^2\) is an embedding precisely when \(C\) is nonhyperelliptic [GH94, Pages 246-7]. Consider the following principal fiber bundle.

\[
PGL_3(\mathbb{C}) \longrightarrow \mathcal{U}_4^{mark} \xrightarrow{\varphi_4} \text{Teich}(\Sigma_3) \setminus \mathcal{H}_3
\]

Because \(\rho_4 : \pi_1(\mathcal{U}_4) \to \text{Mod}(\Sigma_3)\) is surjective [Kun08, Proposition 6.3], \(\mathcal{U}_4^{mark}\) is connected.

By Theorem 1.2, Teich\((\Sigma_3) \setminus \mathcal{H}_3\) is homotopy equivalent to an infinite wedge of circles and, since \(\text{PGL}_3(\mathbb{C})\) is connected, there must exist some continuous section \(\sigma : \text{Teich}(\Sigma_3) \setminus \mathcal{H}_3 \to \mathcal{U}_4^{mark}\). Because \(\varphi_4\) is a principal \(\text{PGL}_3(\mathbb{C})\)-bundle, the existence of such a section implies that \(\mathcal{U}_4^{mark}\) is homeomorphic to \(\text{PGL}_3(\mathbb{C}) \times (\text{Teich}(\Sigma_3) \setminus \mathcal{H}_3)\), and so

\[
\pi_i(\mathcal{U}_4^{mark}) = \begin{cases} 
\mathbb{Z}/3\mathbb{Z} \times F_{\infty}, & \text{for } i = 1 \\
\pi_i(\text{PGL}_3(\mathbb{C})), & \text{for } i > 1.
\end{cases}
\]

This also shows that \(\pi_i(\mathcal{U}_4) \cong \pi_i(\text{PGL}_3(\mathbb{C}))\) for \(i \geq 2\).

We now wish to show that \(H_1(K_4; \mathbb{Q})\) is isomorphic to \(\mathbb{Q}[\text{Mod}(\Sigma_3)/\text{SMod}(\Sigma_3)]\) as Mod\((\Sigma_3)\)-modules. The calculation of \(K_4 \cong \pi_1(\mathcal{U}_4^{mark})\) in equation 4.1 shows that the projection

\[
\mathcal{U}_4^{mark} \xrightarrow{\cong} \text{PGL}_3(\mathbb{C}) \times (\text{Teich}(\Sigma_3) \setminus \mathcal{H}_3) \to \text{Teich}(\Sigma_3) \setminus \mathcal{H}_3
\]

induces an isomorphism

\[
H_1(K_4; \mathbb{Q}) \cong H_1(\text{Teich}(\Sigma_3) \setminus \mathcal{H}_3; \mathbb{Q}).
\]

The action of Mod\((\Sigma_3)\) on \(\mathcal{U}_4^{mark}\) commutes with the projection map

\[
\mathcal{U}_4^{mark} \to \text{Teich}(\Sigma_3) \setminus \mathcal{H}_3,
\]

so that the above isomorphism of \(\mathbb{Q}\)-vector spaces is an isomorphism of Mod\((\Sigma_3)\)-modules.

The group \(H_1(\text{Teich}(\Sigma_3) \setminus \mathcal{H}_3; \mathbb{Z})\) is the free abelian group on the set of cycles in Teich\((\Sigma_3) \setminus \mathcal{H}_3\) represented by meridians around the components of the hyperelliptic locus \(\mathcal{H}_3\); that is, the boundaries of disks transversely intersecting \(\mathcal{H}_3\) in a single point. Such cycles are in bijection with the cosets of Mod\((\Sigma_3)/\text{SMod}(\Sigma_3)\) (see proof of Corollary 3.2). This bijection commutes with the action of Mod\((\Sigma_3)\) and therefore this Mod\((\Sigma_3)\)-module is isomorphic to the permutation representation \(\mathbb{Q}[\text{Mod}(\Sigma_3)/\text{SMod}(\Sigma_3)]\). □

The following table shows \(\pi_i(\mathcal{U}_4) \cong \pi_i(\text{PGL}_3(\mathbb{C}))\) for small values of \(i \geq 2\) (c.f. [MT64, Introduction], where we have used the fact that SL\(_3(\mathbb{C})\) covers PGL\(_3(\mathbb{C})\) and is homotopy equivalent to SU\((3))\).
The kernel of the monodromy of the universal family of degree $d$ smooth plane curves

\[ \pi_i(U_d) \] for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

\[ \mathbb{Z} \] \quad \mathbb{Z}/6\mathbb{Z} \quad \mathbb{Z}/12\mathbb{Z} \quad \mathbb{Z}/3\mathbb{Z} \quad \mathbb{Z}/30\mathbb{Z} \quad \mathbb{Z}/4\mathbb{Z} \quad \mathbb{Z}/60\mathbb{Z} 

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