Small \( f \)-vectors of 3-spheres and of 4-polytopes

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Abstract

We present a new algorithmic approach that can be used to determine whether a given quadruple \((f_0, f_1, f_2, f_3)\) is the \( f \)-vector of any convex 4-dimensional polytope. By implementing this approach, we classify the \( f \)-vectors of 4-polytopes in the range \( f_0 + f_3 \leq 22 \).

In particular, we thus prove that there are \( f \)-vectors of cellular 3-spheres with the intersection property that are not \( f \)-vectors of any convex 4-polytopes, thus answering a question that may be traced back to the works of Steinitz (1906/1922). In the range \( f_0 + f_3 \leq 22 \), there are exactly three such \( f \)-vectors with \( f_0 \leq f_3 \), namely \((10, 32, 33, 11)\), \((10, 33, 35, 12)\), and \((11, 35, 35, 11)\).

1 Introduction

In 1906, Ernst Steinitz [15] proved a remarkably simple and complete result: The set of all \( f \)-vectors of 3-polytopes is given by all the integer points in a 2-dimensional polyhedral cone, whose boundary is given by the extremal cases of (\( f \)-vectors of) simple and of simplicial polytopes:

\[
F(P^3) = \{ (f_0, f_1, f_2) \in \mathbb{Z}^3 : f_0 - f_1 + f_2 = 2, \ f_2 \leq 2f_0 - 4, \ f_0 \leq 2f_2 - 4 \}.
\]

Steinitz’s later work [33, 34] from 1922/1934 implies that the same characterization is valid also for the \( f \)-vectors of more general objects such as regular cellular 2-spheres with the intersection property or of interval-connected Eulerian lattices of length 4 (as described below).

The \( f \)-vectors of 4-polytopes, however, provide a much greater challenge. Grünbaum wrote in his 1967 book:

“\textit{It would be rather interesting to find a characterization of those lattice points in } \mathbb{R}^4 \textit{ which are the } f \textit{-vectors of 4-polytopes. This goal seems rather distant, however, in view of our inability to solve even such a small part of the problem as the lower bound conjecture for 4-polytopes.}” (Grünbaum [17] p. 191)]

The lower bound conjecture was solved by Barnette in 1971/73 [3, 5], but the problem to characterize \( F(P^4) \) remains wide open. Grünbaum himself initiated and started in [17 Sect. 10.4] a study of the 2-dimensional coordinate projections of the 3-dimensional set \( F(P^4) \subset \mathbb{R}^4 \), which was eventually completed by Barnette and Reay [7] and Barnette [6]. A typical result in the

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These are extremal problems on or cone( also identified the two key problems that prevent us up to now from determining fatness corresponds to a linear inequality that is tight at the simplex. In [37] the second author though fatness is not defined for the (• upper bound inequalities, and if it is not one of finitely-many “small” exceptions. For example, according to [17, Thm.10.4.1] a pair (f_0, f_3) occurs for a 4-polytope if and only if the upper bound inequalities f_3 \leq \frac{1}{2}f_0(f_0 - 3) and f_0 \leq \frac{1}{2}f_3(f_3 - 3) are satisfied, with no exceptions in this case.

Any characterization of (a projection of) the set of f-vectors \( \mathcal{F}(P^4) \) contains a characterization of the extremal cases and a solution of the corresponding extremal problems. Some of these are visible in 2-dimensional coordinate projections. For example, the \((f_0, f_3)\)-classification quoted above contains the upper bound theorem for 4-polytopes.

As a complete determination of \( \mathcal{F}(P^4) \) seems out of reach, a natural approximation to the problem asks for a characterization of the closed convex cones with apex at the f-vector \( f(\Delta_4) = (5, 10, 10, 5) \) of the 4-simplex that are generated by the f-vectors of 4-polytopes resp. of 3-spheres,

\[
\text{cone}(\mathcal{F}(P^4)) \subseteq \text{cone}(\mathcal{F}(S^3)).
\]

Equivalently, one asks for the linear inequalities that are valid for all f-vectors and tight at \( f(\Delta_4) = (5, 10, 10, 5) \). For example, the inequalities \( f_1 \geq 2f_0 \) and \( f_2 \geq 2f_3 \) are of this form, satisfied with equality by simple resp. simplicial 4-polytopes. Thus, in particular the f-vectors of simple and simplicial 4-polytopes are extremal in the coordinate projections to \((f_0, f_1)\) resp. \((f_2, f_3)\).

It was noted in Ziegler [37] that a key parameter of an f-vector is the fatness

\[
F(f_0, f_1, f_2, f_3) := \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}.
\]

Though fatness is not defined for the (f-vector of a) simplex, every lower or upper bound on fatness corresponds to a linear inequality that is tight at the simplex. In [37] the second author also identified the two key problems that prevent us up to now from determining \text{cone}(\mathcal{F}(P^4)) \) or \text{cone}(\mathcal{F}(S^3)):

- Does fatness have an upper bound for 4-polytopes?
  (It does not for 3-spheres, as proved by Eppstein, Kuperberg & Ziegler [44].)
- Is the fatness lower bound \( F \geq 2.5 \) valid for all 3-spheres?
  (For 4-polytopes it follows from \( g_{\text{sm}}^0 \geq 0 \), see Kalai [21].)

These are extremal problems on \( \mathcal{F}(P^4) \) resp. \( \mathcal{F}(S^3) \) that cannot be solved by looking at the projections to only two coordinates. However, below we will suggest a different projection which displays fatness very clearly.

In this paper we are not directly dealing with the asymptotic questions. Rather we classify the f-vectors of “small” polytopes, and from this derive new insights into what happens asymptotically. For this, we redefine “small” by measuring the size of an f-vector by

\[
\text{size}(f_0, f_1, f_2, f_3) := f_0 + f_3 - 10.
\]

This is a linear quantity that is size(5, 10, 10, 5) = 0 for the f-vector of the 4-simplex.

For the classification we have developed a new algorithmic approach, in order to determine for any given reasonably small \((f_0, f_1, f_2, f_3)\), whether there is a 4-polytope with this f-vector.

We have implemented the algorithm and achieved a complete classification of the f-vectors of size up to 12. That is, for every vector \((f_0, f_1, f_2, f_3)\) with \( f_0 + f_2 \leq 22 \) that satisfies the known necessary conditions on f-vectors of 4-polytopes, we have either constructed a 3-sphere or 4-polytope with this f-vector, or proved that none exists. The results are detailed in Sections 4 and 5. As a main consequence of the enumeration, we obtain that the difference between spheres and polytopes is so substantial that it appears even at the level of f-vectors:
Theorem 1.1. The set of $f$-vectors of 4-polytopes is a strict subset of the set of $f$-vectors of strongly regular 3-spheres:

$$\mathcal{F}(P^4) \subsetneq \mathcal{F}(S^3).$$

Indeed, the sets differ in exactly five such $f$-vectors of size($P^4$) $= f_0 + f_3 - 10 \leq 12$, namely

- of size 11: (10, 32, 33, 11), (11, 33, 32, 10), and
- of size 12: (10, 33, 35, 12), (12, 35, 33, 10), (11, 35, 35, 11).

For simplicial spheres, the question whether all $f$-vectors of $(d-1)$-spheres also occur for $d$-polytopes—in view of the $g$-Theorem for polytopes—is equivalent to the $g$-conjecture for spheres. The answer is known to be “yes” for $d \leq 5$, but the $g$-conjecture for sphere remains open for larger $d$. However, already in 1971, at the end of the paper in which he introduced the $g$-conjecture, McMullen voiced strong doubts:

“in every case in which the $[g]$-conjecture is known to be true, it also holds for the corresponding triangulated spheres. (…) However, there are fundamental differences between triangulated $(d-1)$-spheres and boundary complexes of simplicial $d$-polytopes. (…) We should therefore, perhaps, be wary of extending the conjecture to triangulated spheres.”

(McMullen [27, p. 569])

Our algorithm works in three steps, proceeding from combinatorial models via topological models to polytopes. It starts with an enumeration of the graphs that could be compatible with the given $f$-vector. It then looks at the possible combinatorial types of facets, and enumerates their combinations into an entirely combinatorial model of polytopes, namely interval-connected Eulerian lattices of length 5. This new model will be described in Section 2, where we also prove that every such object corresponds to a regular cell-decomposition of a closed 3-manifold with the intersection property (Proposition 2.2). Thus the combinatorial types of regular cell-decompositions of the 3-sphere with the intersection property (which we simply refer to as “3-spheres,”) form a subset of these Eulerian lattices. The class of combinatorial types of convex 4-polytopes is still more restrictive, as became clear, for example, in the revision and correction of Brückner’s [12] work by Grünbaum & Sreedharan [18]: Not every diagram, and thus not every sphere, does correspond to a convex polytope.

In our search range of size($f$) $\leq 12$, all $f$-vectors of Eulerian lattices also appear as $f$-vectors of spheres. That is, while

$$\# \{f \in \mathcal{F}(S^3) \setminus \mathcal{F}(P^4) : \text{size}(f) \leq 12\} = 5$$

we have

$$\# \{f \in \mathcal{F}(L^5) \setminus \mathcal{F}(S^3) : \text{size}(f) \leq 12\} = 0.$$  

So it may be that $\mathcal{F}(L^5) = \mathcal{F}(S^3)$, but the computations for size($f$) $\leq 12$ should not be counted as strong evidence, as indeed we did not encounter any manifolds that are not spheres in this range. Also, very natural higher-dimensional versions of $\mathcal{F}(L^5) = \mathcal{F}(S^3)$ turn out to be false. For example, simplicial 5-manifolds with negative $g_3$ appear in the enumerations of Lutz [25, pp. 56-58].

In Figure 1 we evaluate our classification results by looking at the $f$-vector set $\mathcal{F}(P^4)$ in a particular projection, which is not a coordinate projection, and which has the virtue to show size (as first coordinate) and fatness (as “slope + 2”) directly.

Let us note two more intriguing aspects of our enumeration results, which can also be seen in Figure 1.
This figure presents a particular 2-dimensional projection of \( \mathcal{F}(\mathcal{P}^4) \subset \mathcal{F}(\mathcal{S}^3) \subset \mathbb{Z}^4 \): The x-axis represents size = \( f_0 + f_3 - 10 \) of a 4-polytope or 3-sphere, while the y-axis represents \( f_1 + f_2 - 20 - 2 \cdot \text{size} \), so the slope of a line through the origin is “fatness – 2.”

Black dots \( \bullet \) mark data points for which Höppner [19] had found polytopes.

Grey crossed dots \( \otimes \) mark coordinates for which 2-simple 2-simplicial polytopes were found by Paffenholz & Werner [30], and Werner [35].

Grey dots \( \bullet \) give additional data points where we now found polytopes.

Red dots \( \circ \) represent coordinates of points for which there are \( f \)-vectors of 3-spheres, but where we found no \( f \)-vectors of 4-polytopes; left of the dotted line this means that these do not exist.

The graph shown here is complete up to size 12, that is, to the left of the dotted vertical line.

White dots \( \bigcirc \) appear only to the right of the dotted line: They mark locations where the existence of spheres or of polytopes has not been decided.

Figure 1: The size/fatness projection of the \( f \)-vector sets \( \mathcal{F}(\mathcal{P}^4) \subset \mathcal{F}(\mathcal{S}^3) \)
Observations 1.2.

(i) The sets of “small” $f$-vectors $f = (f_0, f_1, f_2, f_3)$ of 3-spheres and of 4-polytopes differ in an essential way, which is detected by fatness:
- For $\text{size}(f) \leq 10$, the $f$-vectors of 3-spheres and of 4-polytopes agree.
- For $\text{size}(f) \leq 11$, the maximal fatness for 3-spheres is $4\frac{1}{11}$, for 4-polytopes it is 4.
- For $\text{size}(f) \leq 12$, the maximal fatness for 3-spheres is $4\frac{1}{8}$, for 4-polytopes it is still 4.

(ii) In the range of “small” $f$-vectors of size $(f) \leq 12$, the particularly “fat” 4-polytopes are 2-simple and 2-simplicial in the sense of Grünbaum [17 Sect. 4.5]. The exceptionally fat 3-spheres are not 2-simple and 2-simplicial, but they still have $f$-vectors that are approximately symmetric, with $|f_0 - f_3| \leq 4$.

For this we recall from Grünbaum [17 Sect. 4.5] that a 4-polytope $P$ is 2-simple and 2-simplicial ("2s2s") if all 2-faces are triangles both for $P$ and for its dual. The definition extends to 3-spheres and even to Eulerian lattices of length 5. Any such 2s2s object has a symmetric $f$-vector, with $f_0 = f_3$ and $f_1 = f_2$. The 2s2s property is detected by the flag vector, but not by the $f$-vector alone. Observation 1.2(ii) refers to all the polytopes of fatness at least 4, which in the range size $(f) \leq 12$, according to the classification of 2s2s 4-polytopes and 3-spheres of size at most 14 in Brinkmann & Ziegler [11 Thm. 2.1], are
- Werner’s example $W_9$ with 9 vertices [35 Thm. 4.2.2],
- $W_{10}$ as well as the hypersimplex $\Delta_3(2)$ and its dual with 10 vertices, and
- $P_{11}$ by Paffenholz & Werner [30 Sect. 4.1].

The pattern continues beyond the range size $(f) \leq 12$ of our enumeration, where we find
- the 2s2s polytope $W_{12}^{10}$ of Werner and Miyata [35 Tbl. 7.1 right] [28 Sect. 4.2] and the 2s2s sphere $W_{12}^{40}$ with $f$-vector $(12, 40, 40, 12)$ constructed by Werner [34 Tbl. 7.1 left].

For this last example we had shown in [11] that it is non-polytopal and that it is the only 2s2s 3-sphere with such a flag vector. As a consequence, we established that the sets of flag vectors of 4-polytopes and 3-spheres differ [11 Theorem 1.1], but did not achieve a similar statement for sets of $f$-vectors. This is provided by Theorem 1.1. However, with our new algorithm presented here (plus massive computation) we also achieved a complete classification result for the $f$-vector $(12, 40, 40, 12)$.

Theorem 1.3. There are 4 strongly regular 3-spheres (all of them self-dual, one of them 2-simple 2-simplicial), but no 4-polytopes at all, with the $f$-vector $(12, 40, 40, 12)$.

So altogether this paper provides six examples of $f$-vectors of 3-spheres that are not $f$-vectors of 4-polytopes, namely the five smallest ones listed in Theorem 1.1 and one more in Theorem 1.3. Of course one would now want to provide infinitely many examples, to show that the cones $\text{cone}(F(P^4)) \subseteq \text{cone}(F(S^3))$ do not coincide, and similar results for flag vectors and for their cones in higher dimensions. Our present methods do not seem to provide this.

2 Objects: Polytopes, Spheres, and Eulerian Lattices

There have been numerous substantial attempts to classify all 4-dimensional polytopes with some given parameters (e.g., $f$-vectors), or to classify the parameters that actually occur. They all depend on a hierarchy of combinatorial/topological/geometric models for convex polytopes of decreasing generality, which we use systematically in our algorithmic approach. For basics on convex polytopes, including diagrams/Schlegel diagrams, we refer to Grünbaum [17] and Ziegler [30]. For regular cell complexes, see Cooke & Finney [13] or Munkres [29]. Eulerian posets/lattices as combinatorial models arose from the work of Klee [22], see Stanley [32 Chap. 3]. The less common objects we work with can be summarized as follows.
Definition 2.1.

- A finite graded lattice is Eulerian if any non-trivial interval has the same number of elements of odd and of even rank; it is interval-connected if the proper part of any interval of length at least 3 is connected.
- A CW-sphere is regular if the attaching maps of the cells are homeomorphisms also on the boundary. The sphere has the intersection property if the intersection of any two cells is a single cell (which may be empty).

In this paper we concentrate entirely on the case of 4-dimensional polytopes, and correspondingly 3-spheres and Eulerian lattices of length 5. The interval connectivity for Eulerian lattices and the regularity and intersection property for cellular spheres are always assumed. We write
- $\mathcal{P}^4$ for the set of combinatorial types of 4-polytopes;
- $\mathcal{S}^3$ for the set of combinatorial types of 3-spheres;
- $\mathcal{E}^5$ for the isomorphism types of length-5 Eulerian lattices.

The boundary complex of any 4-polytope is a 3-sphere (regular, cellular, with the intersection property); the face lattice of any such 3-sphere is an interval-connected Eulerian lattice of length 5.

Polytope theory has produced lots of examples to show that there are strict inclusions

$$\mathcal{P}^4 \subset \mathcal{S}^3 \subset \mathcal{E}^5$$

while there is no difference “one dimension lower,” by Steinitz’s theorems. His theorems also yield that interval-connected Eulerian lattices form an excellent entirely combinatorial model for the topological/geometric structures we are studying.

Proposition 2.2. Every interval-connected Eulerian lattice of length $d + 1 \leq 4$ is the face lattice of a $d$-polytope. In particular, $\mathcal{P}^3 = \mathcal{S}^2 = \mathcal{E}^4$.

Every interval-connected Eulerian lattice of length $d + 1 = 5$ is the face lattice of a (connected, closed) regular CW 3-manifold with the intersection property.

Proof sketch. For $d + 1 \leq 3$ there is little to prove.

For $d + 1 = 4$ the Eulerian lattice is the face lattice of a connected 2-manifold of Euler characteristic 2, so we have a sphere. The lattice property corresponds to what Steinitz calls “Bedingung des Nichtübergreifens” [34, S. 179], which is exactly the intersection property for a cellular 2-sphere. Steinitz’s Theorem [33, 34] yields that every such 2-sphere can be realized as a convex polytope.

For $d + 1 = 5$ the Eulerian lattice is the face poset of a closed connected 3-manifold, whose cells and vertex links are polytopal by Steinitz’s theorem. However, the fact that this manifold has Euler characteristic 0 yields no additional information about its type, by Poincaré duality. ☐

3 Enumeration Algorithm

We here propose a new algorithm, which constructs, for a given vector $(f_0, f_1, f_2, f_3)$, first the graphs and then the face lattices of all 3-manifolds with this $f$-vector, by using 0/1 integer programming in order to enumerate all families of facets that fit to this graph and all other constraints. The algorithm has the following outline:

Algorithm 3.1. find_lattices(f)
INPUT: A vector $(f_0, f_1, f_2, f_3) \in \mathbb{Z}^4$
OUTPUT: All Eulerian lattices of length 5 with this $f$-vector
(i) enumerate all graphs $G$ on $f_0$ vertices and $f_1$ edges that are 4-connected;
(ii) for every graph $G$ find all induced subgraphs that are planar and 3-connected;
(iii) construct for every graph $G$ an integer program (IP) with binary variables corresponding to the possible facets and ridges (faces of the facets), and with constraints given by the $f$-vector, proper intersection, the Euler relation, and the graph;
(iv) enumerate all feasible solutions of this IP;
(v) check for every feasible solution whether it gives an Eulerian lattice.

Proposition 3.2. Algorithm 3.1 enumerates all interval-connected Eulerian lattices of length 5 with $f$-vector $(f_0, f_1, f_2, f_3)$.

Proof. We rely on the interpretation of interval-connected length 5 Eulerian lattices as face lattices of cellular regular 3-manifolds with intersection property in Proposition 2.2. Since the graph of any such manifold is 4-connected, Step (i) will not exclude any graph of some 3-manifold with $f$-vector $(f_0, f_1, f_2, f_3)$.

Also by Proposition 2.2 the graphs of interval-connected Eulerian lattices of length 4 (and thus of facets of cellular 4-manifolds) are exactly the planar and 3-connected graphs. Thus, with Step (ii) we find a list $F_G$ of all potential facets for a manifold with the given graph $G$.

From the list $F_G$, we also get the list $R_G$ of the potential ridges, simply from the faces of the facets. We now construct a 0/1-IP whose variables $x_i$ represent the facets $F_i$, and the variables $y_j$ the ridges $R_j$, such that all solutions correspond to pseudomanifolds formed by a subset of the facets in $F_G$ and such that all face lattices of 3-manifolds with graph $G$ and $f$-vector $(f_0, f_1, f_2, f_3)$ are feasible solutions, with the constraints

\[
\begin{align*}
\sum_i x_i &= f_3 \quad (1) \\
\sum_j y_j &= f_2 \quad (2) \\
2y_j - \sum_{F_i \supset R_j} x_i &= 0 \quad \text{for all ridges } R_j \quad (3) \\
x_i, y_j &\in \{0, 1\}. \quad (4)
\end{align*}
\]

Condition (4) says that all variables are binary, which means that if a variable in the solution is 1 the corresponding face will be selected. Equations (1) and (2) enforce that the total number of facets and ridges selected is $f_3$, resp. $f_2$. Equation (3) ensures that ridge $R_j$ is used if and only if precisely two facets containing it are selected. Similarly, we get constraints from the Euler relation for the intervals above the vertices and edges, such that all feasible solutions correspond to Eulerian posets. Moreover, for every edge we get an inequality forcing the number of faces containing it to be larger than zero. Finally, we get inequalities $x_i + x_j \leq 1$ for pairs of facets $F_i, F_j$ if their intersection is not proper (i.e. that not both can appear in a 3-manifold simultaneously). Since the face lattice of any 3-manifold with the given $f$-vector and graph $G$ satisfies the constraints of the IP, it will be in the set of feasible solutions of this IP. Therefore, with the last step we can complete the enumeration of all interval-connected Eulerian lattices with the given $f$-vector.

We implemented Algorithm 3.1 in sage [31], using the geng-function of nauty [26] (which is a built-in function of sage) to enumerate all graphs on $f_0$ vertices, with $f_1$ edges, with minimal vertex degree at least 4, and being 2-connected (nauty cannot enumerate 4-connected graphs, so we had to relax to 2-connectedness, but this did not include too many extra graphs), and the MILP-library of sage to check the IPs for feasibility and to enumerate all their solutions. We
enumerated all feasible solutions iteratively: Given a feasible solution, we store it and set the sum of the $f_3$ variables corresponding to the facets of this solution to be at most $f_3 - 1$. Thus, we excluded with an additional constraint precisely the solution we just found and optimized again. By iterating this until no feasible solution remained, we enumerated all feasible solutions of the original IP.

Finally we had to check every solution to represent an interval-connected Eulerian lattice of length 5: By construction, we were looking at Eulerian posets. For each of these interval-connectivity was easy to check, as was the intersection property: Both these properties were not completely built into our IP. Then we triangulated the corresponding manifold and used sage to calculate the Betti numbers, and thus verified that in all cases considered we were dealing with homology spheres. Then we used BISTELLAR by Lutz [24] to show that each of them was flip-equivalent to the boundary of the simplex, and thus a genuine sphere.

4 Enumeration and Classification Results

For the proof of Theorem 1.1 we started with the generation of all potential flag-vectors bounded by $f_0 + f_3 \leq 22$, that is, all integer vectors $(f_0, f_1, f_2, f_3; f_03) \in \mathbb{Z}^5$ that satisfy all the linear and non-linear conditions on $f$-vectors and of flag vectors that were known to be valid for Eulerian lattices with the intersection property of length 5, as given by Barnette [4], Bayer [8], and Ling [23]. (See Bayer & Lee [9] and Höppner & Ziegler [20] for surveys.) Moreover, as we in the algorithmic approach started to add to the $f$-vector information specific data about the combinatorial types of facets used, we could make use of constraints such as

$$f_{02} - 4f_2 + 3f_1 - 2f_0 \leq \left(\frac{f_0}{2}\right) - \frac{1}{2} \sum_{F \text{ facet, } f_0(F) \geq 7} \left(m_i(F) + f_{02}(F) - 3f_2(F)\right) - \# \text{ facets with 6 vertices} + \frac{1}{2} \# \text{ pyramids over pentagon},$$

where $m_i(F)$ denotes the number of interior edges of a face $F$, proved in Brinkmann [10, Sect. 2.2.1], which sharpens an inequality by Bayer [8].

Moreover, we could (and did) assume that $f_0, f_3 \geq 9$, as the objects with up to 8 vertices and facets have been enumerated and analyzed in detail by Altshuler & Steinberg [1].

Furthermore, we ticked off on our candidate list all those vectors that are known to occur as $f$-vectors of 4-polytopes, for example from the study of Höppner & Ziegler [20] or the enumeration of 2s2s-polytopes in Brinkmann & Ziegler [11].

For all remaining candidate vectors we enumerated all compatible Eulerian lattices by Algorithm find_lattices(f), and then used the methods detailed in Brinkmann & Ziegler [11] in order to

- either use first numerical non-linear optimization techniques and then exact arithmetic sharpenings in order to find rational coordinates for at least one polytope with the given $f$-vector,
- or use biquadratic final polynomials for partial oriented matroids in order to prove that all spheres for the given $f$-vector are non-realizable.

The results are shown in Table 2. It lists, for each potential $f$-vector considered, the number of graphs to be checked (graphs on $f_0$ vertices, with $f_1$ edges, with minimal vertex degree at least 4, and being 2-connected), and the numbers

- $\# \mathcal{E}^5$ of Eulerian lattices of length 5,
- $\# \mathcal{S}^3$ of cellular 3-spheres,
- $\# \text{np}$ of non-polytopal 3-spheres among them, and
- $\# \mathcal{P}^4$ of convex 4-polytopes.
with the given \( f \)-vector. In some instances for the last two quantitites we just give lower bounds, if we did not decide all cases. An asterisk * marks objects where we have exact coordinates for at least one polytope and approximate (floating point) coordinates for the others. Blank spaces represent missing data (e.g. not enumerated/calculated). In particular for \( f_0 = 11 \) we did not enumerate all \( f \)-vectors, but restricted ourselves to constructing polytopes.

Table 1

| \( f \)-vector | \# graphs | \#E5 | \#S5 | \#np | \#P4 | \( m \) |
|----------------|-----------|------|------|------|------|-------|
| \((9, m, m, 9)\) | 170 | 0 | 0 | 0 | 0 | \( m \leq 19 \) |
| \((9, 20, 20, 9)\) | 713 | 1 | 1 | 0 | 1 | \( m \leq 19 \) |
| \((9, 21, 21, 9)\) | 1754 | 0 | 0 | 0 | 0 | \( m \leq 19 \) |
| \((9, 22, 22, 9)\) | 2770 | 129 | 129 | \( \geq 54 \) | \( m \leq 20 \) |
| \((9, 23, 23, 9)\) | 3129 | 211 | 211 | \( \geq 2 \) | \( \geq 113^* \) | \( m \leq 20 \) |
| \((9, 24, 24, 9)\) | 2723 | 118 | 118 | \( \geq 2 \) | \( \geq 81^* \) | \( m \leq 20 \) |
| \((9, 25, 25, 9)\) | 1917 | 7 | 7 | 0 | \( 7^* \) | \( m \leq 20 \) |
| \((9, 26, 26, 9)\) | 1154 | 1 | 1 | 0 | 1 | \( W_9 \ [35 \text{ Thm. 4.2.2}] \) |
| \((9, m, m, 9)\) | 1132 | 0 | 0 | 0 | 0 | \( m \leq 21 \) |
| \((9, m, m + 1, 10)\) | 2673 | 0 | 0 | 0 | 0 | \( m \geq 27 \) |
| \((9, 22, 23, 10)\) | 2770 | 12 | 12 | \( \geq 9^* \) | \( m \leq 19 \) |
| \((9, 23, 24, 10)\) | 3129 | 398 | 398 | \( \geq 1 \) | \( \geq 78^* \) | \( m \leq 20 \) |
| \((9, 24, 25, 10)\) | 2723 | 904 | 904 | \( \geq 7 \) | \( \geq 27^* \) | \( m \leq 20 \) |
| \((9, 25, 26, 10)\) | 1917 | 524 | 524 | \( \geq 15 \) | \( \geq 80^* \) | \( m \leq 20 \) |
| \((9, 26, 27, 10)\) | 1154 | 67 | 67 | \( \geq 2 \) | \( \geq 62^* \) | \( m \leq 20 \) |
| \((9, 27, 28, 10)\) | 610 | 0 | 0 | 0 | 0 | \( m \leq 20 \) |
| \((9, 28, 29, 10)\) | 294 | 0 | 0 | 0 | 0 | \( m \leq 20 \) |
| \((9, 29, 30, 10)\) | 133 | 0 | 0 | 0 | 0 | \( m \leq 20 \) |
| \((9, m, m + 1, 10)\) | 95 | 0 | 0 | 0 | 0 | \( m \geq 30 \) |
| \((9, m, m + 2, 11)\) | 5443 | 0 | 0 | 0 | 0 | \( m \leq 22 \) |
| \((9, 23, 25, 11)\) | 3129 | 66 | 66 | \( \geq 34 \) | \( m \geq 28 \) |
| \((9, 24, 26, 11)\) | 2723 | 1188 | 1188 | \( \geq 105 \) | \( m \geq 30 \) |
| \((9, 25, 27, 11)\) | 1917 | 2650 | 2650 | \( \geq 52 \) | \( m \geq 30 \) |
| \((9, 26, 28, 11)\) | 1154 | 1344 | 1344 | \( \geq 1 \) | \( m \geq 30 \) |
| \((9, 27, 29, 11)\) | 610 | 125 | 125 | \( \geq 60 \) | \( m \geq 30 \) |
| \((9, 28, 30, 11)\) | 294 | 3 | 3 | 1 | 2 | \( m \geq 30 \) |
| \((9, 29, 31, 11)\) | 133 | 0 | 0 | 0 | 0 | \( m \geq 30 \) |
| \((9, m, m + 2, 11)\) | 103 | 0 | 0 | 0 | 0 | \( m \geq 30 \) |
| \((9, m, m + 3, 12)\) | 5443 | 0 | 0 | 0 | 0 | \( m \leq 22 \) |
| \((9, 23, 26, 12)\) | 3129 | 3 | 3 | 0 | 3 | \( m \geq 31 \) |
| \((9, 24, 27, 12)\) | 2723 | 335 | 335 | \( \geq 129 \) | \( m \geq 31 \) |
| \((9, 25, 28, 12)\) | 1917 | 3275 | 3275 | \( \geq 171 \) | \( m \geq 31 \) |
| \((9, 26, 29, 12)\) | 1154 | 5928 | 5928 | \( \geq 276 \) | \( m \geq 31 \) |
| \((9, 27, 30, 12)\) | 610 | 2171 | 2171 | \( \geq 516 \) | \( m \geq 31 \) |
| \((9, 28, 31, 12)\) | 294 | 113 | 113 | \( \geq 33 \) | \( m \geq 31 \) |
| \((9, 29, 32, 12)\) | 133 | 0 | 0 | 0 | 0 | \( m \geq 31 \) |
| \((9, 30, 33, 12)\) | 39 | 0 | 0 | 0 | 0 | \( m \geq 31 \) |
| \((9, m, m + 3, 12)\) | 44 | 0 | 0 | 0 | 0 | \( m \geq 31 \) |
| \((9, m, m + 4, 13)\) | 8536 | 0 | 0 | 0 | 0 | \( m \leq 23 \) |
| \((9, 24, 28, 13)\) | 2723 | 33 | 33 | \( \geq 32^* \) | \( m \geq 31 \) |
| \((9, 25, 29, 13)\) | 1917 | 1223 | 1223 | \( \geq 1 \) | \( \geq 387^* \) | \( m \leq 23 \) |
Table 1 – continued from previous page

| $f$-vector | # graphs | #$E^s$ | #$S^t$ | #np | #$F^s$ |
|------------|----------|--------|--------|-----|--------|
| (9, 26, 30, 13) | 1154 | 7677 | 7677 | $\geq 3$ | $\geq 309$ |
| (9, 27, 31, 13) | 610 | 9773 | 9773 | $\geq 32$ | $\geq 13$ |
| (9, 28, 32, 13) | 294 | 2136 | 2136 | $\geq 439$ |
| (9, 29, 33, 13) | 133 | 27 | 27 | $\geq 1$ | $\geq 9^*$ |
| (9, 30, 34, 13) | 59 | 0 | 0 | 0 | 0 |
| (9, $m$, $m + 4$, 13) | 44 | 0 | 0 | 0 | 0 |
| (10, $m$, $m$, 10) | 10247 | 0 | 0 | 0 | 0 |
| (10, $m$, $m$, 10) | 35219 | 4 | 4 | 0 | 0 |
| (10, $m$, $m$, 10) | 87014 | 16 | 16 | $\geq 2^*$ |
| (10, $m$, $m$, 10) | 152369 | | | | $\geq 296$ |
| (10, $m$, $m$, 10) | 203469 | 5550 | 5550 | $\geq 69$ | $\geq 2^*$ |
| (10, $m$, $m$, 10) | 217596 | 5561 | 5561 | $\geq 204$ | $\geq 90^*$ |
| (10, $m$, $m$, 10) | 192964 | 1662 | 1662 | $\geq 143$ | $\geq 13^*$ |
| (10, $m$, $m$, 10) | 145773 | 128 | 128 | $\geq 2$ | $\geq 21^*$ |
| (10, $m$, $m$, 10) | 95827 | 3 | 3 | 0 | 3 |
| (10, $m$, $m$, 10) | 55762 | 0 | 0 | 0 | 0 |
| (10, $m$, $m$, 10) | 53718 | 0 | 0 | 0 | 0 |
| (10, $m$, $m$, 10) | 45469 | 0 | 0 | 0 | 0 |
| (10, $m$, $m + 1$, 11) | 87014 | 6 | 6 | $\geq 2^*$ |
| (10, $m$, $m + 1$, 11) | 152369 | 136 | 136 | $\geq 10^*$ |
| (10, $m$, $m + 1$, 11) | 203469 | 6794 | 6794 | $\geq 11$ | $\geq 633$ |
| (10, $m$, $m + 1$, 11) | 217596 | 24915 | 24915 | $\geq 22$ |
| (10, $m$, $m + 1$, 11) | 192964 | 30355 | 30355 | $\geq 1$ | $\geq 159$ |
| (10, $m$, $m + 1$, 11) | 145773 | 11916 | 11916 | $\geq 28$ |
| (10, $m$, $m + 1$, 11) | 95827 | 1441 | 1441 | $\geq 61$ | $\geq 1$ |
| (10, $m$, $m + 1$, 11) | 55762 | 35 | 35 | $\geq 9$ | $\geq 20^*$ |
| (10, $m$, $m + 1$, 11) | 29199 | 2 | 2 | 2 | 0 |
| (10, $m$, $m + 1$, 11) | 13981 | 0 | 0 | 0 | 0 |
| (10, $m$, $m + 1$, 11) | 6202 | 0 | 0 | 0 | 0 |
| (10, $m$, $m + 1$, 11) | 2600 | 0 | 0 | 0 | 0 |
| (10, $m$, $m + 1$, 11) | 1736 | 0 | 0 | 0 | 0 |
| (10, $m$, $m + 1$, 11) | 45469 | 0 | 0 | 0 | 0 |
| (10, $m$, $m + 1$, 11) | 87014 | 2 | 2 | $\geq 1$ |
| (10, $m$, $m + 1$, 11) | 152369 | 2 | 2 | $\geq 1$ |
| (10, $m$, $m + 1$, 11) | 203469 | 1051 | 1051 | $\geq 178$ |
| (10, $m$, $m + 1$, 11) | 217596 | 23884 | 23884 | $\geq 768$ |
| (10, $m$, $m + 1$, 11) | 192964 | 91727 | 91727 | $\geq 455$ |
| (10, $m$, $m + 1$, 11) | 145773 | 112266 | 112266 | $\geq 256$ |
| (10, $m$, $m + 1$, 11) | 95827 | 47141 | 47141 | $\geq 13$ | $\geq 1$ |
| (10, $m$, $m + 1$, 11) | 55762 | 5943 | 5943 | $\geq 521$ | $\geq 368$ |
| (10, $m$, $m + 1$, 11) | 29199 | 225 | 225 | $\geq 7$ |
| (10, $m$, $m + 1$, 11) | 13981 | 1 | 1 | 1 | 0 |
| (10, $m$, $m + 1$, 11) | 6202 | 0 | 0 | 0 | 0 |
| (10, $m$, $m + 1$, 11) | 2600 | 0 | 0 | 0 | 0 |
| (10, $m$, $m + 1$, 11) | 1736 | 0 | 0 | 0 | 0 |
| (11, 22, 22, 11) | 265 | 0 | 0 | 0 | 0 |

Table 1 – continued from previous page

$\Delta_4(2), \Delta_4(2)^*, W_{10}$

$m \geq 31$

$\Delta_4(2), \Delta_4(2)^*, W_{10}$

$m \geq 31$

$\Delta_4(2), \Delta_4(2)^*, W_{10}$

$m \geq 31$

$\Delta_4(2), \Delta_4(2)^*, W_{10}$

$m \geq 31$

$\Delta_4(2), \Delta_4(2)^*, W_{10}$

$m \geq 31$

$\Delta_4(2), \Delta_4(2)^*, W_{10}$

$m \geq 31$
Table 1 – continued from previous page

| f-vector       | # graphs | #E³ | #S³ | #np | #P⁴ |
|----------------|----------|-----|-----|-----|-----|
| (11, 23, 23, 11) | 10 391   | 0   | 0   | 0   | 0   |
| (11, 24, 24, 11) | 120 985  | 0   | 0   | 0   | 0   |
| (11, 25, 25, 11) | 696 184  | 0   | 0   | 0   | 0   |
| (11, 26, 26, 11) | 2 504 998| 21  | 21  | ≥ 1 |     |
| (11, 27, 27, 11) | 6 383 318| 322 | 322 | ≥ 1 |     |
| (11, 28, 28, 11) | 12 417 723| ≥ 2635 |     |     |
| (11, 29, 29, 11) | 19 379 000|     |     | ≥ 1 |     |
| (11, 30, 30, 11) | 25 121 426|     |     | ≥ 1 |     |
| (11, 31, 31, 11) | 27 749 332|     |     | ≥ 1 |     |
| (11, 32, 32, 11) | 26 626 961|     |     | ≥ 104|     |
| (11, 33, 33, 11) | 22 528 512|     |     | ≥ 1 |     |
| (11, 34, 34, 11) | 17 005 570| 100 | 100 | ≥ 15| ≥ 1 |
| (11, 35, 35, 11) | 11 561 155| 2   | 2   | 2   | 0   |
| (11, 36, 36, 11) | 7 134 337| 0   | 0   | 0   | 0   |

Table 1: All potential f-vectors with $f_0, f_3 ≥ 9$ and $f_0 + f_3 ≤ 22$.

The spheres with the particular f-vectors (10, 32, 33, 11), (10, 33, 35, 12), and (11, 35, 35, 11) of Theorem 1.1 will be presented and discussed in Section 5.

The proof of Theorem 1.3 follows the same pattern, with considerably higher computation times. Table 2 shows the results of the computation for the potential f-vectors (12, $m$, $m$, 12) for large $m$: The numbers of graphs to check (graphs on $f_0$ vertices, with $f_1$ edges, with minimal vertex degree at least 4, 2-connected) and the numbers of strongly regular 3-manifolds, strongly regular 3-spheres, non-polytopal spheres, and 4-polytopes. Blank spaces represent missing data (e.g. not enumerated or calculated). For time reasons, and since there is a polytope, we did not enumerate the manifolds with f-vector (12, 39, 39, 12). The results for larger $m$ follow as any manifold with such an f-vector would be 2s2s, as verified in Brinkmann [10, Prop. 2.2.19], and these we have enumerated, see Brinkmann & Ziegler [11, Thm. 2.1].

| f-vector       | # graphs | #E³ | #S³ | #np | #P⁴ |
|----------------|----------|-----|-----|-----|-----|
| (12, 39, 39, 12) | 4 078 410 035| ≥ 1 | ≥ 1 | ≥ 1 | $W_{12}^{39}$ |
| (12, 40, 40, 12) | 2 997 683 218| 4   | 4   | 4   | 0   |
| (12, 41, 41, 12) | 2 037 876 411| 0   | 0   | 0   | 0   |
| (12, $m$, $m$, 12) | 4 880 253 668| 0   | 0   | 0   | $m ≥ 42$ |

Table 2: Results for the potential f-vectors (12, 40, 40, 12) and (12, 41, 41, 12)

5 Examples

According to Theorem 1.1 there are five f-vectors for which there is at least one 3-sphere but no 4-polytope. In this section we will present these 3-spheres. For each of these f-vectors,

- the fact that there are no other 3-spheres than those we present in the following depends on massive computation and does not seem to have a reasonably short or “compact” proof,
- the fact that the objects that we present are, indeed, spheres, can be verified in a variety
of ways; in the following we present coordinates and images for a diagram (in the sense of polytope theory, see Ziegler [36, Lect. 5]),

- the fact that the spheres are not polytopal was verified on the computer with oriented matroid techniques; in principle, one can extract human-verifiable short proofs from the computation results; for this we give one example below.

**Examples 5.1.** There are two 3-spheres with \( f \)-vector \((10, 32, 33, 11)\):

- The sphere \((10^0_{32,33})\) is given by the facet list
  
  \[
  F_0 = \{ v_0, v_2, v_4, v_5, v_9 \} \\
  F_1 = \{ v_0, v_2, v_4, v_6, v_8 \} \\
  F_2 = \{ v_1, v_3, v_6, v_7, v_9 \} \\
  F_3 = \{ v_2, v_3, v_4, v_6, v_8 \} \\
  F_4 = \{ v_0, v_2, v_5, v_7, v_8 \} \\
  F_5 = \{ v_1, v_3, v_5, v_7, v_8 \}
  \]

  It is non-polytopal, but it has diagrams based on each of the facets \( F_0, F_1, F_2, F_3, F_4, F_5, F_6, \) and \( F_7 \), but not based on one of \( F_8, F_9, \) or \( F_{10} \). A diagram based on facet \( F_2 \) is given in Figure 2. This sphere cannot be realized by a fan, and thus it is not star-shaped in the sense of Ewald [16, Sect. III.5].

- The sphere \((10^1_{32,33})\) is given by the facet list
  
  \[
  F_0 = \{ v_0, v_3, v_5, v_6, v_8 \} \\
  F_1 = \{ v_0, v_4, v_5, v_7, v_8 \} \\
  F_2 = \{ v_0, v_3, v_6, v_7, v_9 \} \\
  F_3 = \{ v_0, v_1, v_3, v_5, v_7 \} \\
  F_4 = \{ v_1, v_3, v_5, v_8, v_9 \} \\
  F_5 = \{ v_1, v_3, v_6, v_7, v_9 \}
  \]

  It is non-polytopal, but it has a diagram based on every facet and it can be represented by a fan. A diagram based on facet \( F_2 \) is given in Figure 3.

![Figure 2: A diagram based on facet \( F_2 \) for the sphere \((10^0_{32,33})\) with \( f \)-vector \((10, 32, 33, 11)\).](image)

We did not manage to decide whether the second sphere \((10^1_{32,33})\) has a star-shaped embedding. An oriented matroid that would support such an embedding exists. (Clearly every...
Proposition 5.2. The sphere \((10^{1}_{32,33})\) is non-polytopal.

Proof. We will use a similar oriented matroid approach as in \cite{11}. The following arguments may be verified with reference to the list of labeled facets displayed in Figure 4.

With reference to facet \(F_0\), we may choose \(\chi(v_0, v_2, v_4, v_9, v_i) = 1\), for all \(v_i \notin F_0\). With this we can derive:

\[
\begin{align*}
\chi(v_3, v_5, v_6, v_8, v_9) & \overset{F_{10}}{=} 0, \\
\chi(v_0, v_1, v_2, v_4, v_9) & \overset{-1}{\Rightarrow} \chi(v_1, v_2, v_4, v_5, v_9) = 1 \overset{F_5}{\Rightarrow} \chi(v_2, v_4, v_5, v_8, v_9) = -1, \\
\chi(v_0, v_2, v_4, v_8, v_9) & \overset{-1}{\Rightarrow} \chi(v_0, v_1, v_2, v_4, v_8) = 1 \overset{F_5}{\Rightarrow} \chi(v_1, v_2, v_4, v_6, v_8) = -1, \\
\chi(v_0, v_2, v_4, v_7, v_9) & \overset{-1}{\Rightarrow} \chi(v_0, v_2, v_4, v_7, v_8) = 1 \overset{F_5}{\Rightarrow} \chi(v_2, v_4, v_7, v_8, v_9) = 1, \\
\chi(v_0, v_2, v_4, v_5, v_9) & \overset{-1}{\Rightarrow} \chi(v_0, v_2, v_4, v_5, v_8) = 1 \overset{F_5}{\Rightarrow} \chi(v_0, v_2, v_5, v_6, v_8) = -1, \\
\chi(v_0, v_2, v_3, v_6, v_9) & \overset{-1}{\Rightarrow} \chi(v_0, v_2, v_3, v_6, v_8) = -1 \overset{F_6}{\Rightarrow} \chi(v_0, v_1, v_3, v_6, v_8) = -1, \\
\chi(v_0, v_2, v_4, v_6, v_9) & \overset{-1}{\Rightarrow} \chi(v_0, v_2, v_4, v_6, v_8) = -1 \overset{F_6}{\Rightarrow} \chi(v_1, v_3, v_6, v_8, v_9) = -1, \\
\chi(v_0, v_2, v_4, v_5, v_6) & \overset{-1}{\Rightarrow} \chi(v_0, v_1, v_2, v_4, v_6) = 1 \overset{F_5}{\Rightarrow} \chi(v_0, v_1, v_4, v_6, v_8) = 1, \\
\chi(v_0, v_2, v_4, v_5, v_8) & \overset{-1}{\Rightarrow} \chi(v_0, v_2, v_3, v_5, v_8) = 1 \overset{F_5}{\Rightarrow} \chi(v_0, v_3, v_5, v_7, v_8) = 1, \\
\chi(v_0, v_2, v_4, v_7, v_9) & \overset{-1}{\Rightarrow} \chi(v_0, v_1, v_2, v_4, v_7) = 1 \overset{F_5}{\Rightarrow} \chi(v_1, v_2, v_4, v_7, v_8) = 1,
\end{align*}
\]

Figure 3: A diagram based on facet \(F_0\) for the sphere \((10^{1}_{32,33})\) with \(f\)-vector \((10, 32, 33, 11)\).
Figure 4: These are the facets of the sphere $(10^6_{32,33})$ from $F_0$ (top left) to $F_{10}$ (bottom right).

\[
\chi(v_0, v_1, v_2, v_4, v_6) \quad 1 \quad F_8 \quad \chi(v_0, v_1, v_3, v_4, v_6) = 1 \quad F_3 \quad \chi(v_1, v_3, v_4, v_6, v_9) = 1
\]

\[
F_8 \Rightarrow \chi(v_0, v_1, v_3, v_6, v_9) = 1 \quad F_3 \Rightarrow \chi(v_0, v_3, v_6, v_7, v_9) = 1
\]

\[
\chi(v_0, v_1, v_3, v_6, v_9) = 1 \quad F_8 \Rightarrow \chi(v_0, v_1, v_6, v_7, v_9) = -1 \quad F_3 \Rightarrow \chi(v_1, v_6, v_7, v_8, v_9) = 1
\]

\[
\chi(v_0, v_1, v_2, v_3, v_6, v_8) = -1 \quad F_8 \Rightarrow \chi(v_0, v_3, v_4, v_6, v_8) = 1 \quad F_3 \Rightarrow \chi(v_3, v_4, v_6, v_8, v_9) = 1
\]

\[
\chi(v_0, v_2, v_3, v_6, v_8) = -1 \quad F_8 \Rightarrow \chi(v_0, v_3, v_4, v_6, v_8) = 1 \quad F_3 \Rightarrow \chi(v_3, v_4, v_6, v_8, v_9) = 1
\]

\[
\chi(v_0, v_2, v_4, v_7, v_8) = -1 \quad F_8 \Rightarrow \chi(v_0, v_1, v_2, v_7, v_8) = -1 \quad F_3 \Rightarrow \chi(v_1, v_2, v_5, v_7, v_8) = -1
\]

\[
F_8 \Rightarrow \chi(v_2, v_5, v_7, v_8, v_9) = -1
\]

\[
\chi(v_0, v_1, v_2, v_7, v_8) = -1 \quad F_8 \Rightarrow \chi(v_1, v_2, v_3, v_7, v_8) = -1 \quad F_3 \Rightarrow \chi(v_1, v_3, v_7, v_8, v_9) = 1
\]

\[
\chi(v_0, v_3, v_5, v_7, v_8) = -1 \quad F_8 \Rightarrow \chi(v_3, v_5, v_7, v_8, v_9) = 1
\]

\[
\chi(v_0, v_3, v_5, v_7, v_8) = -1 \quad F_8 \Rightarrow \chi(v_3, v_5, v_7, v_8, v_9) = 1
\]
With these values for the partial chirotope, we can find some new values of $\chi$ using the Grassmann–Plücker relations:

\[
\{\chi(v_7, v_8, v_9, v_1, v_3)\chi(v_7, v_8, v_9, v_5, v_6), \chi(v_7, v_8, v_9, v_1, v_5)\chi(v_7, v_8, v_9, v_3, v_6), \chi(v_7, v_8, v_9, v_1, v_6)\chi(v_7, v_8, v_9, v_3, v_5)\}
\]

\[
\{1 \cdot \chi(v_7, v_8, v_9, v_5, v_6), -1 \cdot (-1), 1 \cdot 1\},
\]

\[
\Rightarrow \chi(v_7, v_8, v_9, v_5, v_6) = -1,
\]

\[
\{\chi(v_6, v_8, v_9, v_2, v_3)\chi(v_6, v_8, v_9, v_5, v_7), \chi(v_6, v_8, v_9, v_2, v_5)\chi(v_6, v_8, v_9, v_3, v_7), \chi(v_6, v_8, v_9, v_2, v_7)\chi(v_6, v_8, v_9, v_3, v_5)\}
\]

\[
\{(-1) \cdot 1, -\chi(v_6, v_8, v_9, v_2, v_5) \cdot 1, 0\},
\]

\[
\Rightarrow \chi(v_6, v_8, v_9, v_2, v_5) = -1,
\]

\[
\{\chi(v_6, v_8, v_9, v_1, v_3)\chi(v_6, v_8, v_9, v_4, v_7), \chi(v_6, v_8, v_9, v_1, v_4)\chi(v_6, v_8, v_9, v_3, v_7), \chi(v_6, v_8, v_9, v_1, v_7)\chi(v_6, v_8, v_9, v_3, v_4)\}
\]

\[
\{(-1) \cdot \chi(v_6, v_8, v_9, v_4, v_7), -1 \cdot (-1), (-1) \cdot 1\},
\]

\[
\Rightarrow \chi(v_6, v_8, v_9, v_4, v_7) = -1,
\]

\[
\{\chi(v_6, v_8, v_9, v_3, v_4)\chi(v_6v_8, v_9, v_5, v_7), \chi(v_6, v_8, v_9, v_3, v_5)\chi(v_6, v_8, v_9, v_4, v_7), \chi(v_6, v_8, v_9, v_3, v_7)\chi(v_6, v_8, v_9, v_4, v_5)\}
\]

\[
\{1 \cdot 1, 0, 1 \cdot \chi(v_6, v_8, v_9, v_4, v_5)\},
\]

\[
\Rightarrow \chi(v_6, v_8, v_9, v_4, v_5) = -1,
\]

\[
\{\chi(v_5, v_8, v_9, v_2, v_4)\chi(v_5v_8, v_9, v_6, v_7), \chi(v_5, v_8, v_9, v_2, v_6)\chi(v_5, v_8, v_9, v_4, v_7), \chi(v_5, v_8, v_9, v_2, v_7)\chi(v_5, v_8, v_9, v_4, v_6)\}
\]

\[
\{(-1) \cdot (-1), -1 \cdot \chi(v_5, v_8, v_9, v_4, v_7), (-1) \cdot (-1)\},
\]

\[
\Rightarrow \chi(v_5, v_8, v_9, v_4, v_7) = 1,
\]

Finally, we get the Grassmann–Plücker relation

\[
\{\chi(v_4, v_8, v_9, v_2, v_5)\chi(v_4v_8, v_9, v_6, v_7), \chi(v_4, v_8, v_9, v_2, v_6)\chi(v_4, v_8, v_9, v_5, v_7), \chi(v_4, v_8, v_9, v_2, v_7)\chi(v_4, v_8, v_9, v_5, v_6)\}
\]

\[
\{1 \cdot 1, -1 \cdot (-1), (-1) \cdot (-1)\},
\]

which is neither $\{0\}$, nor contains $\{-1, 1\}$. Thus, the Grassmann–Plücker relations cannot be satisfied, so the sphere $\ell_{0,32,33}^6$ does not support an oriented matroid. In particular, it is not polytopal.
is given by the facet list and Figure 5. The sphere cannot be represented by a fan.

to each other. These spheres are based on each of 

\[ F(11_{35}) \]

\[ F_0 = \{ v_1, v_2, v_4, v_6, v_9 \} \]

\[ F_1 = \{ v_2, v_3, v_5, v_7, v_9 \} \]

\[ F_2 = \{ v_0, v_4, v_6, v_8, v_{10} \} \]

\[ F_3 = \{ v_2, v_3, v_4, v_8, v_9 \} \]

\[ F_4 = \{ v_0, v_1, v_2, v_5, v_6 \} \]

\[ F_5 = \{ v_1, v_3, v_4, v_7, v_9 \} \]

\[ F_6 = \{ v_0, v_5, v_6, v_7, v_9 \} \]

\[ F_7 = \{ v_0, v_1, v_2, v_4, v_8, v_{10} \} \]

\[ F_8 = \{ v_1, v_3, v_5, v_7, v_{10} \} \]

\[ F_9 = \{ v_0, v_1, v_3, v_6, v_9 \} \]

\[ F_{10} = \{ v_1, v_4, v_5, v_7, v_{10} \} \]

Figure 5: A diagram based on facet \( F_2 \) for the sphere \((10_{33,35})\) with f-vector \((10, 33, 35, 12)\).

Example 5.3. There is exactly one 3-sphere with f-vector \((10, 33, 35, 12)\). This sphere \((10_{33,35})\) is given by the facet list

\[
\begin{align*}
F_0 &= \{ v_0, v_1, v_7, v_9 \} \\
F_1 &= \{ v_2, v_3, v_4, v_6 \} \\
F_2 &= \{ v_0, v_5, v_6, v_8 \} \\
F_3 &= \{ v_2, v_3, v_4, v_5 \} \\
F_4 &= \{ v_1, v_2, v_3, v_4 \} \\
F_5 &= \{ v_0, v_1, v_2, v_4 \} \\
F_6 &= \{ v_0, v_1, v_2, v_3 \} \\
F_7 &= \{ v_0, v_1, v_2, v_4 \} \\
F_8 &= \{ v_0, v_1, v_2, v_5 \} \\
F_9 &= \{ v_0, v_1, v_2, v_6 \} \\
F_{10} &= \{ v_0, v_1, v_2, v_7 \} \\
\end{align*}
\]

It is not polytopal. It has a diagram based on each of the facets \( F_2, F_3, F_4, F_5, F_6, F_7, F_8, \) and \( F_{10} \), but not based on one of \( F_0, F_1, F_9, \) or \( F_{11} \). A diagram based on facet \( F_2 \) is given in Figure 5. The sphere cannot be represented by a fan.

Examples 5.4. There are exactly two 3-spheres with f-vector \((11, 35, 35, 11)\). They are dual to each other. These spheres \((11_0_{35})\) and \((11_1_{35})\) are given by facet lists

\[
\begin{align*}
(11_0^{35}) & \\
F_0 &= \{ v_2, v_4, v_7, v_9 \} \\
F_1 &= \{ v_0, v_4, v_6, v_7, v_{10} \} \\
F_2 &= \{ v_0, v_3, v_4, v_7, v_9 \} \\
F_3 &= \{ v_2, v_3, v_4, v_7, v_{10} \} \\
F_4 &= \{ v_0, v_1, v_2, v_5, v_6, v_8 \} \\
F_5 &= \{ v_0, v_1, v_2, v_3, v_5 \} \\
F_6 &= \{ v_0, v_1, v_2, v_3, v_4 \} \\
F_7 &= \{ v_0, v_1, v_2, v_4, v_6 \} \\
F_8 &= \{ v_0, v_1, v_2, v_5 \} \\
F_9 &= \{ v_0, v_1, v_2, v_6 \} \\
F_{10} &= \{ v_0, v_1, v_2, v_7 \} \\
\end{align*}
\]

Both spheres are not fan-like, hence they have no star-shaped embedding. Furthermore, the sphere \((11_0^{35})\) does not have a diagram with base \( F_6, F_9, \) or \( F_{10} \); the sphere \((11_1^{35})\) has a diagram based on each of \( F_4 \) and \( F_6 \), but does not have a diagram with base \( F_0, F_1, F_3, F_5, F_9, \) or \( F_{10} \). A diagram for \((11_1^{35})\) with base \( F_6 \) is given in Figure 6.
$F_6$
$v_0 = (0, 0, 0)$
$v_1 = (1797, 1585, 512)$
$v_2 = (2009, 2395, 1622)$
$v_3 = (460, 1113, 648)$
$v_4 = (0, 0, 1000)$
$v_5 = (8565805/4137, 2055, 1316)$
$v_6 = (2850, 426, 139)$
$v_7 = (521, 1238, 853)$
$v_8 = (2946124555/1064794, 1020, 770)$
$v_9 = (423, 2580, 139)$
$v_{10} = (1161, 1055, 677)$

Figure 6: A diagram based on facet $F_6$ for the sphere $(111_{35})$ with $f$-vector $(11, 35, 35, 11)$.

Similar details can be found in Brinkmann [10, Sect. 3.2.4] for the four self-dual 3-spheres of Theorem 1.3.

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