Holes of the Leech lattice and the projective models of K3 surfaces

BY ICHIRO SHIMADA
Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 Japan
e-mail: shimada@math.sci.hiroshima-u.ac.jp

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Abstract

Using the theory of holes of the Leech lattice and Borcherds method for the computation of the automorphism group of a K3 surface, we give an effective bound for the set of isomorphism classes of projective models of fixed degree for certain K3 surfaces.

1. Introduction

Let X be a K3 surface defined over an algebraically closed field k, and let d be an even positive integer. Sterk [27] and Lieblich and Maulik [14] showed that, at least when the base field k is not of characteristic 2, there exist only a finite number of projective models of X with degree d up to the action of the automorphism group Aut(X) of X. On the other hand, by means of Borcherds method ([1, 2]), the automorphism groups of several K3 surfaces have been calculated ([5, 6, 8, 10, 11, 23, 25, 28]). Combining this method with the precise description of holes of the Leech lattice due to Borcherds, Conway, Parker, Queen and Sloane ([4, chapters 23–25]), we obtain an effective bound for the set of isomorphism classes of projective models of degree d. This bound is applicable to a wide class of K3 surfaces.

Our result on K3 surfaces is a corollary of Theorem 1.2 on the Conway chamber of the even unimodular hyperbolic lattice \( L := \Pi_{1,25} \) of rank 26.

We fix some terminologies and notation about lattices. A lattice is a free \( \mathbb{Z} \)-module of finite rank with a nondegenerate symmetric bilinear form that takes values in \( \mathbb{Z} \), which we call the intersection form. Let \( M \) be a lattice with the intersection form \( \langle \, , \rangle_M \). We let the orthogonal group \( O(M) \) of \( M \) act on \( M \) from the right. We say that \( M \) is hyperbolic if its rank \( n \) is \( > 1 \) and its signature is \( (1, n - 1) \), whereas \( M \) is negative-definite if its signature is \( (0, n) \).

We say that \( M \) is even if \( \langle v, v \rangle_M \in 2\mathbb{Z} \) holds for all vectors \( v \in M \). Suppose that \( M \) is even. We put

\[ \mathcal{R}_M := \{ r \in M \mid \langle r, r \rangle_M = -2 \}. \]

The dual lattice \( M^\vee \) of \( M \) is the \( \mathbb{Z} \)-module \( \text{Hom}(M, \mathbb{Z}) \), into which \( M \) is embedded by \( \langle \, , \rangle_M \) as a submodule of finite index. We say that \( M \) is unimodular if \( M = M^\vee \) holds.

Suppose that \( M \) is an even hyperbolic lattice. A positive cone of \( M \) is one of the two connected components of \( \{ x \in M \otimes \mathbb{R} \mid \langle x, x \rangle_M > 0 \} \). We denote by \( O^+(M) \) the stabiliser
subgroup of a positive cone of $M$ in $O(M)$. We choose a positive cone $P_M$. Then $O^+(M)$ acts on $P_M$. For each $r \in R_M$, we put

$$(r)^\perp := \{ x \in P_M \mid \langle x, r \rangle_M = 0 \}$$

and denote by $s_r$ the element of $O^+(M)$ given by

$$s_r : x \mapsto x + \langle x, r \rangle_M \cdot r.$$  

Then $s_r$ acts on $P_M$ as the reflection in the real hyperplane $(r)^\perp$. Let $W_M$ denote the subgroup of $O^+(M)$ generated by all the reflections $s_r$, where $r$ ranges through $R_M$. The closure in $P_M$ of a connected component of $P_M \setminus \bigcup_{r \in R_M} (r)^\perp$ is called a standard fundamental domain of the action of $W_M$ on $P_M$.

Let $L$ be an even unimodular hyperbolic lattice of rank 26, and let $\langle \ , \ \rangle_L$ denote the intersection form of $L$. It is well known that $L$ is unique up to isomorphism. We choose a positive cone $P_L$ of $L$. By the negative-definite Leech lattice, we mean an even negative-definite unimodular lattice $\Lambda^-$ of rank 24 with no vectors of square norm $-2$. It is well known that $\Lambda^-$ is unique up to isomorphism. A vector $w \in L$ is called a Weyl vector if $w$ is a nonzero primitive vector of square norm 0 contained in the closure of $P_L$ in $L \otimes \mathbb{R}$ such that the lattice $\langle w \rangle^\perp/\langle w \rangle$ is isomorphic to $\Lambda^-$, where $\langle w \rangle^\perp$ is the orthogonal complement of the submodule $\langle w \rangle := \mathbb{Z}w$ in $L$. A standard fundamental domain of the action of $W_L$ on $P_L$ is called a Conway chamber. For a Weyl vector $w$, we put

$$R_L(w) := \{ r \in R_L \mid \langle r, w \rangle_L = 1 \}$$

and

$$D(w) := \{ x \in P_L \mid \langle r, x \rangle_L \geq 0 \text{ for all } r \in R_L(w) \}.$$  

We have the following theorem.

**Theorem 1.1** (Conway [3]). The mapping $w \mapsto D(w)$ gives rise to a bijection from the set of Weyl vectors to the set of Conway chambers.

Our main result is as follows:

**Theorem 1.2.** Let $w \in L$ be a Weyl vector, and let $d$ be an even positive integer. Then, for any vector $v \in D(w) \cap L$ with $\langle v, v \rangle_L = d$, we have

$$\langle v, w \rangle_L \leq \phi(d) := \frac{\sqrt{1081}}{23} \frac{(529d + 1)}{2} = 756.20698 \ldots d + 1.4295028 \ldots.$$  

We apply Theorem 1.2 to $K3$ surfaces $X$, and obtain an effective bound for the set of nef classes of self-intersection number $d$ modulo the action of $\text{Aut}(X)$ for certain $K3$ surfaces (Corollary 1.8). For this purpose, we give a review of Borcherds method ([1, 2]). See also [23] for the computational aspects of this method.

First we recall the definition of the discriminant forms. Let $M$ be an even lattice. Then the dual lattice $M^\vee$ is equipped with a canonical $\mathbb{Q}$-valued symmetric bilinear form extending $\langle \ , \ \rangle_M$. This $\mathbb{Q}$-valued symmetric bilinear form defines a finite quadratic form

$$q_M : M^\vee/M \longrightarrow \mathbb{Q}/2\mathbb{Z},$$
which is called the discriminant form of $M$. (See Nikulin [15] for the basic properties of discriminant forms.) Let $O(q_M)$ denote the automorphism group of the finite quadratic form $q_M$, and let $\eta_M : O(M) \to O(q_M)$ denote the natural homomorphism.

Let $X$ be a $K3$ surface, and let $S_X$ denote the Néron–Severi lattice of $X$ with the intersection form $\langle \cdot, \cdot \rangle_S$. Suppose that rank $S_X > 1$. Then $S_X$ is an even hyperbolic lattice. Let $\mathcal{P}(X)$ be the positive cone of $S_X$ that contains an ample class. We let $\text{Aut}(X)$ act on $X$ from the left, and on $S_X$ from the right by the pull-back. Hence we have a natural homomorphism

$$\text{Aut}(X) \to O^+(S_X).$$

Suppose that $X$ is defined over $\mathbb{C}$ or is supersingular in characteristic $\neq 2$. Then we can use Torelli theorem (Piatetski–Shapiro and Shafarevich [20], Ogus [18, 19]) for $X$. We put

$$N(X) := \{ x \in \mathcal{P}(X) \mid \langle x, C \rangle_S \geq 0 \quad \text{for all curves } C \text{ on } X \}.$$ 

It is well known that $N(X)$ is a standard fundamental domain of the action of $W_{S_X}$ on $\mathcal{P}(X)$. When $X$ is defined over $\mathbb{C}$, we denote by $H_X$ the unimodular lattice $H^2(X, \mathbb{Z})$ with the cup-product, by $\tilde{G}_X$ the subgroup of $O(H_X)$ consisting of isometries of $H_X$ that preserve the 1-dimensional subspace $H^{2,0}(X)$ of $H_X \otimes \mathbb{C}$, and put

$$G_X := \{ g \in O^+(S_X) \mid g \text{ extends to an isometry } \tilde{g} \in \tilde{G}_X \}.$$ 

When $X$ is supersingular, we put

$$G_X := \{ g \in O^+(S_X) \mid g \text{ preserves the period of } X \}.$$ 

(See Ogus [18, 19] for the definition of the period of a supersingular $K3$ surface.) Note that, in either case, $G_X$ is of finite index in $O^+(S_X)$. By Torelli theorem, the image of the natural homomorphism $\text{Aut}(X) \to O^+(S_X)$ is equal to

$$\{ g \in G_X \mid N(X)^g = N(X) \}.$$ 

Assumption 1.3. We assume that the following conditions hold:

(a) the negative-definite lattice $R$ cannot be embedded into the negative-definite Leech lattice $\Lambda$;
(b) the image $\eta_{S_X}(G_X)$ of $G_X$ by $\eta_{S_X} : O(S_X) \to O(q_{S_X})$ is contained in the image $\eta_R(O(R))$ of $\eta_R : O(R) \to O(q_R)$, where $O(q_{S_X})$ and $O(q_R)$ are identified by the isomorphism $\delta_L$.

Remark 1.4. When $X$ is defined over $\mathbb{C}$, we always have a primitive embedding of $S_X$ into $L$. See [23, proposition 8.1].

A closed subset $D$ of $\mathcal{P}(X)$ is said to be an induced chamber if there exists a Conway chamber $\mathcal{D}(w)$ such that $D = \mathcal{P}(X) \cap \mathcal{D}(w)$ holds and the interior of $D$ in $\mathcal{P}(X)$ is nonempty.
Since $\mathcal{P}_L$ is tessellated by the Conway chambers, $\mathcal{P}(X)$ is tessellated by the induced chambers. Moreover, since $N(X)$ is bounded by hyperplanes of $\mathcal{P}(X)$ perpendicular to vectors in $\mathcal{R}_{S_X}$ and $\mathcal{R}_{S_X}$ is a subset of $\mathcal{R}_L$ by the embedding $S_X \hookrightarrow L$, it follows that $N(X)$ is also tessellated by induced chambers. We say that two induced chambers $D$ and $D'$ are $G_X$-congruent if there exists an element $g \in G_X$ such that $D^g = D'$. Then we have the following theorem.

**Theorem 1.5** ([23]). *Suppose that $S_X$ has a primitive embedding into $L$ satisfying Assumption 1-3 and $\mathcal{P}(X) \subset \mathcal{P}_L$. Then the following statements hold:

1. each induced chamber $D$ is bounded by a finite number of hyperplanes of $\mathcal{P}(X)$, and the group $\text{Aut}_{G_X}(D) := \{g \in G_X \mid D^g = D\}$ is finite;
2. the number of $G_X$-congruence classes of induced chambers is finite.*

In [23], we presented an algorithm to calculate a complete set

$$\{D_0, \ldots, D_{m-1}\}$$

of representatives of $G_X$-congruence classes of induced chambers contained in $N(X)$. We also presented an algorithm to calculate the set of hyperplanes bounding $D_i$ and the finite group $\text{Aut}_{G_X}(D_i)$ for each $D_i$. Then, for any vector $v \in N(X) \cap S_X$, there exist an automorphism $g \in \text{Aut}(X)$ and an index $i$ such that $v^g \in D_i$. Let $\text{pr}_S : L \rightarrow S_X^\vee$ denote the orthogonal projection. Let $w_i \in L$ be a Weyl vector such that

$$D_i = \mathcal{P}_L \cap \mathcal{D}(w_i).$$

We put

$$a_i := \text{pr}_S(w_i).$$

We have $\langle a_i, a_i \rangle_S > 0$. (See Remark 5-4.) Moreover we have $\langle v, w_i \rangle_L = \langle v, a_i \rangle_S$ for any vector $v \in S_X$. Therefore we obtain the following corollary of Theorem 1.2.

**Corollary 1.6.** *Suppose that $S_X$ has a primitive embedding into $L$ satisfying Assumption 1-3 and $\mathcal{P}(X) \subset \mathcal{P}_L$. Then there exist vectors $a_0, \ldots, a_{m-1}$ of $S_X^\vee$ satisfying $\langle a_i, a_i \rangle_S > 0$ such that, for any vector $v \in N(X) \cap S_X$ with $\langle v, v \rangle_S = d > 0$, there exist an automorphism $g \in \text{Aut}(X)$ and an index $i$ satisfying $\langle v^g, a_i \rangle_S \leq \phi(d)$.*

Since $\langle a_i, a_i \rangle_S > 0$, the set of all vectors $v \in S_X$ satisfying $\langle v, v \rangle_S = d$ and $\langle v, a_i \rangle_S \leq \phi(d)$ is finite for each $d > 0$. Therefore, provided that we have obtained, by the algorithm in [23], a set of Weyl vectors $w_0, \ldots, w_{m-1}$ that give the representatives of $G_X$-congruence classes of induced chambers, we get an effective bound for the set of nef vectors of square norm $d$ up to the action of $\text{Aut}(X)$. Unfortunately, we do not yet have a general bound for such a set \{w_0, \ldots, w_{m-1}\}. In some cases, however, the algorithm in [23] terminates very quickly.

**Definition 1.7.** Let $X$ be a $K3$ surface that is defined over $\mathbb{C}$ or is supersingular in characteristic $\neq 2$, and let $h \in S_X \otimes \mathbb{Q}$ be an ample class. We say that $(X, h)$ is a polarized $K3$ surface of simple Borcherds type if $S_X$ admits a primitive embedding $S_X \hookrightarrow L$ satisfying Assumption 1-3, $\mathcal{P}(X) \subset \mathcal{P}_L$, and the following condition; there exists only one $G_X$-congruence classes of induced chambers, and it is represented by $D = \mathcal{P}_L \cap \mathcal{D}(w)$ with $h = \text{pr}_S(w)$. 

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**Corollary 1.8.** Let $(X, h)$ be a polarised $K3$ surface of simple Borcherds type. If $v \in S_X$ is a nef vector with $\langle v, v \rangle_S = d > 0$, then there exists an automorphism $g \in \text{Aut}(X)$ such that $\langle v^g, h \rangle_S \leq \phi(d)$.

**Example 1.9.** The following polarized $K3$ surfaces $(X, h)$ are of simple Borcherds type. For each of them, $\text{Aut}(X)$ was determined by Borcherds method.

- The $K3$ surface $X$ is the complex Kummer surface $\text{Km}(\text{Jac}(C))$ associated with the Jacobian of a generic curve $C$ of genus 2, and $h$ is a polarization of degree 8 that embeds $X$ in $\mathbb{P}^5$ as a complete intersection of multi-degree $(2, 2, 2)$. We have rank $S_X = 17$. See [10].
- The $K3$ surface $X$ is the complex Kummer surface $\text{Km}(E \times F)$, where $E$ and $F$ are generic elliptic curves, and $h$ is a polarisation of degree 28. We have rank $S_X = 18$. See [8].
- The $K3$ surface $X$ is the Fermat quartic surface in characteristic 3, and $h$ is the class of a hyperplane section. We have rank $S_X = 22$. See [11].

See Section 5 for further examples.

The problem to classify all Jacobian fibrations on a given $K3$ surface $X$ up to the action of $\text{Aut}(X)$ has been studied by many authors. For example, this classification was done for the three $K3$ surfaces in Example 1.9. See [12] for $\text{Km}(\text{Jac}(C))$, [13], [16] and [17] for $\text{Km}(E \times F)$ and [22] for the Fermat quartic surface in characteristic 3. This problem is equivalent to the classification modulo $\text{Aut}(X)$ of primitive nef vectors $v$ satisfying $\langle v, v \rangle_S = 0$ and a certain condition corresponding to the existence of a zero section. Our problem can be regarded as an extension of this problem to the case where $\langle v, v \rangle_S > 0$.

The proof of Theorem 1.2 relies on the enumeration [4, table 25-1, chapter 25] of holes of $\Lambda$ carried out by Borcherds, Conway, and Queen. Hence the correctness of their list is crucial for our result. Using the data we computed for the proof of Theorem 1.2, we reconfirmed the correctness of the list. See Remark 2.10. Since the whole computational data are too large to be put in the paper, we present the data only on the most important hole (the deep hole of type $D_{24}$), and the rest is put in the author’s web page [24]. For the computation, we used GAP [7].

The plan of this paper is as follows. In Section 2, we give a review of the theory of holes of the Leech lattice, and describe a method to obtain representatives of equivalence classes of holes. In Section 3, we define several invariants of holes, and relate them to the set of possible values of $\langle v, w \rangle_{\Lambda}$, where $w \in \Lambda$ is a fixed Weyl vector and $v$ ranges through $D(w) \cap \Lambda$. Proposition 3.2 in this section is the principal ingredient of the proof of Theorem 1.2, which is carried out in Section 4. In Section 5, we discuss some examples, and conclude the paper by several remarks.

2. Holes of the Leech lattice

We review the theory of holes of the Leech lattice by Borcherds, Conway, Parker, Queen, and Sloane. See [4, chapters 23–25] for the details.

We denote by $\Lambda$ the positive-definite Leech lattice with the intersection form $\langle \ , \ \rangle_{\Lambda}$. Let $\Lambda_\mathbb{R}$ denote $\Lambda \otimes \mathbb{R}$. We use the basis of $\Lambda$ given in [4, chapter 4, figure 4-12], and write elements of $\Lambda_\mathbb{R}$ as a row vector with respect to this basis. We put $\|x\| := \sqrt{\langle x, x \rangle}_{\Lambda}$ for $x \in \Lambda_\mathbb{R}$, and define the function $d_{\Lambda} : \Lambda_\mathbb{R} \to \mathbb{R}$ by

$$d_{\Lambda}(x) := \min\{ \|x - \lambda\| \mid \lambda \in \Lambda \}.$$
By the main result of [4, chapter 23], we know that the maximum of the function $d_A$ on $\Lambda_\mathbb{R}$ is $\sqrt{2}$.

**Definition 2.1.** A point $c$ of $\Lambda_\mathbb{R}$ is called a hole if $d_A$ attains a local maximum at $c$. The radius $R(c)$ of a hole $c$ is defined to be $d_A(c)$. We say that a hole $c$ is deep if $R(c) = \sqrt{2}$, whereas $c$ is shallow if $R(c) < \sqrt{2}$.

For $\lambda \in \Lambda$, we define the Voronoi cell of $\lambda$ by

$$V(\lambda) := \{ x \in \Lambda_\mathbb{R} \mid \|x - \lambda\| \leq \|x - \lambda'\| \text{ for all } \lambda' \in \Lambda \setminus \{\lambda\} \}.$$ 

Then $V(\lambda)$ is a convex polytope, and $\Lambda_\mathbb{R}$ is tessellated by these Voronoi cells. Moreover, a point $c$ of $\Lambda_\mathbb{R}$ is a hole if and only if $c$ is a vertex of a Voronoi cell $V(\lambda)$ for some $\lambda \in \Lambda$.

Let $c$ be a hole. We put

$$P_c := \{ \lambda \in \Lambda \mid \|\lambda - c\| = R(c) \} = \{ \lambda \in \Lambda \mid c \in V(\lambda) \},$$

and let $\overline{P}_c$ denote the convex hull of $P_c$ in $\Lambda_\mathbb{R}$. The following remark is important in the proof of our main result.

**Remark 2.2.** The affine space $\Lambda_\mathbb{R}$ is tessellated by the convex polytopes $\overline{P}_c$, where $c$ ranges though the set of all holes. This tessellation is dual to the tessellation of $\Lambda_\mathbb{R}$ by the Voronoi cells.

In [4, chapter 23, section 2], it is shown that $\|\lambda_i - \lambda_j\| \in \{2, \sqrt{6}, \sqrt{8}\}$ for any distinct points $\lambda_i, \lambda_j$ of $P_c$. We define $\Delta_c$ to be the graph whose set of nodes is $P_c$ and whose edges are drawn by the following rule:

- $\lambda_i$ and $\lambda_j$ are not connected $\iff \|\lambda_i - \lambda_j\| = 2$,
- $\lambda_i$ and $\lambda_j$ are connected by a single edge $\iff \|\lambda_i - \lambda_j\| = \sqrt{6}$,
- $\lambda_i$ and $\lambda_j$ are connected by a double edge $\iff \|\lambda_i - \lambda_j\| = \sqrt{8}$.

Then each connected component of the graph $\Delta_c$ is an indecomposable Coxeter–Dynkin diagram; that is, the diagram of type $A_k$ or $a_k$ ($k \geq 1$), or $D_k$ or $d_k$ ($k \geq 4$), or $E_k$ or $e_k$ ($k = 6, 7, 8$). See [4, chapter 23, figure 23-1] for these diagram. We say that $A_k, D_k, E_k$ are extended, and $a_k, d_k, e_k$ are ordinary. (The readers are warned that this usage of the symbols $A_k, D_k, E_k$ for extended diagrams and $a_k, d_k, e_k$ for ordinary diagrams is not standard.) Let

$$\Delta_c = \Delta_{c,1} \sqcup \cdots \sqcup \Delta_{c,m}$$

be the decomposition of $\Delta_c$ into the connected components, and let

$$P_c = P_{c,1} \sqcup \cdots \sqcup P_{c,m} \quad (2.1)$$

be the corresponding decomposition of the nodes. Let $\tau_{c,i}$ be the type of the indecomposable Coxeter–Dynkin diagram $\Delta_{c,i}$. We define the hole type $\tau(c)$ of $c$ to be the product

$$\tau(c) := \tau_{c,1} \cdots \tau_{c,m}.$$ 

Note that, if $\tau_{c,i}$ is $A_k, D_k, \text{ or } E_k$, then $|P_{c,i}| = k + 1$, whereas if $\tau_{c,i}$ is $a_k, d_k, \text{ or } e_k$, then $|P_{c,i}| = k$.

For a nonempty subset $S$ of $\Lambda_\mathbb{R}$, we denote by $\langle S \rangle$ the minimal affine subspace of $\Lambda_\mathbb{R}$ containing $S$. For an affine subspace $E$ of $\Lambda_\mathbb{R}$ and a point $x$ of $E$, we denote by $E_x$ the linear space obtained from $E$ by regarding $x$ as the origin. Then $E_x$ is a linear subspace of the linear space $(\Lambda_\mathbb{R})_x$. 
By the classification of the deep holes in [4, chapter 23], we obtain the following:

**Theorem 2.3.** Suppose that \( c \) is deep. Then each \( \tau_{c,i} \) is extended, and the convex hull \( \overline{P}_{c,i} \) of each \( P_{c,i} \) is an \( (n_i - 1) \)-dimensional simplex containing \( c \) in its interior, where \( n_i := |P_{c,i}| \). The linear space \((\Lambda_{\mathbb{R}})_c\) is the orthogonal direct sum of the subspaces \( \langle P_{c,1} \rangle_c, \ldots , \langle P_{c,m} \rangle_c \). In particular, we have \( \sum_i (n_i - 1) = 24 \).

By the classification of the shallow holes in [4, chapter 25], we obtain the following:

**Theorem 2.4.** Suppose that \( c \) is shallow. Then each \( \tau_{c,i} \) is ordinary. Moreover, we have \( |P_c| = 25 \), and \( \overline{P}_c \) is a 24-dimensional simplex containing \( c \) in its interior.

We say that two holes \( c \) and \( c' \) are **equivalent** if there exists an affine isometry \( g \) of \( \Lambda \) such that \( c^g = c' \). For a hole \( c \), we denote by \([c]\) the equivalence class of holes containing \( c \). The equivalence classes of holes are enumerated in [4, table 25-1, chapter 25]. The result is summarised as follows.

**Theorem 2.5.** There exist exactly 23 equivalence classes of deep holes, and 284 equivalence classes of shallow holes. Each equivalence class \([c]\) is determined uniquely by the hole type \( \tau(c) \), except for the following hole types:

\[
a_1^2a_5, \quad d_7a_7a_1, \quad d_7a_1a_3a_2^2, \quad a_3^2a_4a_3.
\]

(2.2)

For each of the hole types in (2.2), there exist exactly two equivalence classes of holes.

**Remark 2.6.** The two equivalence classes of each hole type in (2.2) can be distinguished by another method. See Remark 3.1.

We describe a method to find a representative element \( c \) of each equivalence class \([c]\) of holes and the set \( P_c \) of vertices of \( \overline{P}_c \).

Let \( P \) and \( P' \) be finite sets of \( \Lambda \). A **congruence map** from \( P \) to \( P' \) is a bijection \( \gamma : P \xrightarrow{\sim} P' \) such that

\[
\|v_1 - v_2\| = \|\gamma(v_1) - \gamma(v_2)\|
\]

holds for any \( v_1, v_2 \in P \). Suppose that \( c \) is a hole. Then the congruence class containing \( P_c \) is determined by \( \tau(c) \), and hence is denoted by \([\tau(c)]\). If \( P' \) belongs to \([\tau(c)]\), then the convex hull \( \overline{P}' \) of \( P' \) is circumscribed by a 23-dimensional sphere of radius \( R(c) \) and hence \( \overline{P}' \) has the circumscenter \( c(P') \).

**Proposition 2.7.** Let \( c \) be a hole. Suppose that \( P' \) belongs to \([\tau(c)]\). Then \( c(P') \) is a hole with \( P_{c(P')} = P' \) and \( \tau(c(P')) = \tau(c) \).

**Proof.** For the case where \( c \) is deep, this result follows from [4, chapter 23, theorem 7]. The proof for the case where \( c \) is shallow is almost the same. Let \( c \) be a shallow hole. Then \( \overline{P}' \) is a 24-dimensional simplex whose circumradius \( R' \) is smaller than \( \sqrt{2} \). It is enough to show that there exist no vectors \( z \in \Lambda \) such that \( z \notin P' \) and \( \|z - c(P')\| \leq R' \). Suppose that \( z \in \Lambda \) satisfies \( z \notin P' \) and \( \|z - c(P')\| \leq R' \). Then, for any \( v_i \in P' \), we have

\[
4 \leq \|z - v_i\|^2 = \|z - c(P')\|^2 - 2 \langle z - c(P'), v_i - c(P') \rangle_{\Lambda} + \|v_i - c(P')\|^2,
\]

where the first inequality follows from \( z, v_i \in \Lambda \) and \( z \equiv v_i \). Since \( \|z - c(P')\| \leq R' < \sqrt{2} \) and \( \|v_i - c(P')\| = R' < \sqrt{2} \), we have

\[
\langle z - c(P'), v_i - c(P') \rangle_{\Lambda} < 0.
\]

(2.3)
On the other hand, since \( c(P') \) is the circumcenter of the simplex \( \overline{P'} \) contained in the interior, there exist positive real numbers \( a_i \) such that
\[
\sum_{v_i \in P'} a_i (v_i - c(P')) = 0.
\]
(2.4)

Combining (2.3) and (2.4), we obtain a contradiction. This completes the proof.

Suppose that \( c \) is a hole, and let \( P_1 \) and \( P_2 \) be elements of \([\tau(c)]\). We can determine whether the holes \( c(P_1) \) and \( c(P_2) \) are equivalent or not by the following method. Since \( P_1 \) and \( P_2 \) are finite, we can make the list of all congruence maps \( \gamma \) from \( P_1 \) to \( P_2 \). Since \( \langle P_1 \rangle = \langle P_2 \rangle = \Lambda, \) each congruence map \( \gamma \) induces an affine isometry
\[
\gamma_\Lambda : \Lambda \otimes \mathbb{Q} \xrightarrow{\sim} \Lambda \otimes \mathbb{Q}.
\]
Then \( c(P_1) \) and \( c(P_2) \) are equivalent if and only if there exists a congruence map \( \gamma \) from \( P_1 \) to \( P_2 \) such that \( \gamma_\Lambda \) maps \( \Lambda \subset \Lambda \otimes \mathbb{Q} \) to itself.

\textbf{Remark 2.8.} Let \( c \) be a hole. Let \( \text{Aut}(\overline{P}_c) \) denote the group of all congruence maps from \( P_c \) to \( P_c \), and let \( \text{Aut}(P_c, \Lambda) \) denote the group of all affine isometries of \( \Lambda \) that maps \( P_c \) to \( P_c \). If the order of \( \text{Aut}(\overline{P}_c) \) is not very large, we can calculate \( \text{Aut}(P_c, \Lambda) \) by selecting from \( \text{Aut}(\overline{P}_c) \) all the congruence maps \( g \) such that \( g_\Lambda \) preserves \( \Lambda \).

We describe a method to find a representative \( \mathfrak{c} \) of an equivalence class [\( \mathfrak{c} \)] of hole type \( \tau(\mathfrak{c}) \). The case where \( \tau(\mathfrak{c}) = \mathcal{A}_1^{24} \) is described in [4, chapter 23] in details. Hence we assume that \( \tau(\mathfrak{c}) \neq \mathcal{A}_1^{24} \). Then the graph \( \Delta(\mathfrak{c}) \) contains no double edges. By an affine translation of \( \Lambda \), we can assume that \( P_c \) contains the origin \( O \) of \( \Lambda \). Then \( P_c \) is a subset of the set \( \mathcal{N}_{\leq 6} := \{ O \} \cup \mathcal{N}_4 \cup \mathcal{N}_6 \) of cardinality \( 1 + 196560 + 16773120 \), where
\[
\mathcal{N}_{2d} := \{ \lambda \in \Lambda \mid \langle \lambda, \lambda \rangle_\Lambda = 2d \}.
\]

We make the set \( \mathcal{N}_{\leq 6} \), and search for a subset \( P' \) of \( \mathcal{N}_{\leq 6} \) such that the congruence class of \( P' \) is \([\tau(\mathfrak{c})]\). If \( \tau(\mathfrak{c}) \) is not on the list (2.2), then \( c(P') \) is a representative of \([\mathfrak{c}]\) and \( P_c(P') \) is equal to \( P' \) by Theorem 2.5 and Proposition 2.7. Suppose that \( \tau(\mathfrak{c}) \) is on the list (2.2). We search for subsets \( P'_1, \ldots, P'_K \) of \( \mathcal{N}_{\leq 6} \) contained in the congruence class \([\tau(\mathfrak{c})]\) until \( c(P'_K) \) is not equivalent to \( c(P'_1) \). Then \( c(P'_1) \) and \( c(P'_K) \) are representatives of the two equivalence classes of hole type \( \tau(\mathfrak{c}) \).

\textbf{Remark 2.9.} For the computation, we used the standard backtrack algorithm. See [9] for the definition of this algorithm.

In the author’s web page [24], we present a representative element \( \mathfrak{c} \) of each equivalence class [\( \mathfrak{c} \)] and the set \( P_c \) of vertices of \( \overline{P}_c \) in the vector representation.

\textbf{Remark 2.10.} The computation above relies on the enumeration [4, table 25-1, chapter 25] of equivalence classes of holes of \( \Lambda \). In order to show that this enumeration is complete, Borcherds, Conway and Queen used the volume formula
\[
\sum_{[\mathfrak{c}]} \frac{\text{vol}(\overline{P}_c)}{|\text{Aut}(P_c, \Lambda)|} = \frac{1}{|\text{Co}_0|},
\]
(2.5)
where \( \text{vol}(\overline{P}_c) \) is the volume of \( \overline{P}_c \), \( \text{Aut}(P_c, \Lambda) \) is defined in Remark 2.8, \( \text{Co}_0 \) is the Conway group, and the summation is taken over the set of all equivalence classes of holes. Using...
the sets \( P_c \) that we computed, we have reconfirmed the equality (2.5). The volume \( \text{vol}(\overline{P}_e) \) can be computed easily from \( P_c \). The groups \( \text{Aut}(P_e, \Lambda) \) for deep holes \( c \) are studied in detail in [4, chapters 23 and 24]. For the shallow holes, we can use the method described in Remark 2-8, except for the holes of type
\[
a_{5}d_2^{10}, \ a_{4}d_2^{11}, \ a_{3}d_2^{11}, \ a_{3}a_1^{22}, \ a_1a_2^{12}, \ a_2a_1^{23}, \ a_1^{25}.
\]
For example, for the shallow hole \( c = c_{303} \) of type \( \tau(c_{303}) = a_3d_2^{11} \), the order of \( \text{Aut}(\overline{P}_c) \) is \( 2 \cdot 2^{11} \cdot 11! = 163499212800 \), which is too large to be treated by this naive method. To deal with these holes, we need some consideration involving Golay codes and Mathieu groups.

In particular, a characterization of Golay codes by Pless [21] plays an important role. See a note presented in the web page [24].

3. **Geometry of holes and the integer points in a Conway chamber**

Let \( c \) be a hole of radius \( R(c) \). Suppose that \( c \) is shallow. Then there exists a positive rational number \( s(c) \) that satisfies
\[
R(c) = \sqrt{2 - \frac{1}{s(c)}}.
\]
When \( c \) is deep, we put \( s(c) := \infty \). It is obvious that \( s(c) \) depends only on \([c]\).

Let \( v \) be a point of \( \Lambda \otimes \mathbb{Q} \). We define \( m(v) \) to be the order of \( v \) mod \( \Lambda \) in the torsion group \( (\Lambda \otimes \mathbb{Q})/\Lambda \cong (\mathbb{Q}/\mathbb{Z})^{24} \). It is obvious that \( m(v) \) is invariant under the action of affine isometries of \( \Lambda \).

Note that \( c \) belongs to \( \Lambda \otimes \mathbb{Q} \), because \( c \) is the intersection point of the bisectors of distinct two points of \( P_c \). It is obvious that \( m(c) \) depends only on \([c]\).

**Remark 3.1.** The invariant \( m(v) \) enables us to distinguish the two equivalence classes of each hole type in (2.2).

1. For the two equivalence classes \([c_{42}] \) and \([c_{43}] \) with \( \tau(c_{42}) = \tau(c_{43}) = a_{17}a_8 \), we have \( m(c_{42}) = 33 \) and \( m(c_{43}) = 99 \).
2. For the two equivalence classes \([c_{45}] \) and \([c_{46}] \) with \( \tau(c_{45}) = \tau(c_{46}) = d_7a_{17}a_1 \), we have \( m(c_{45}) = 144 \) and \( m(c_{46}) = 48 \).
3. For the two equivalence classes \([c_{130}] \) and \([c_{131}] \) with \( \tau(c_{130}) = \tau(c_{131}) = d_7a_{11}a_3a_2 \), we have \( m(c_{130}) = m(c_{131}) = 54 \). For \( v = 130 \) and \( 131 \), let \( v_{130}^i \) and \( v_{131}^i \) be the two vertices of \( \overline{P}_c \) that correspond to the two nodes of valency 1 in the Coxeter–Dynkin diagram of type \( a_3 \) in \( d_7a_{11}a_3a_2 \). For \( i = 1 \) and \( 2 \), let \( c_{130}^i \) be the circumcenter of the 23-dimensional face of \( \overline{P}_c \) that does not contain \( v_{130}^i \). Then we have \( \{m(c_{130}^1), m(c_{130}^2)\} = \{120, 240\} \) and \( \{m(c_{131}^1), m(c_{131}^2)\} = \{480\} \). Therefore \( c_{130}^1 \) and \( c_{131}^1 \) are not equivalent.
4. For the two equivalence classes \([c_{181}] \) and \([c_{182}] \) with \( \tau(c_{181}) = \tau(c_{182}) = a_5^0a_4a_3 \), we have \( m(c_{181}) = m(c_{182}) = 60 \). For \( v = 181 \) and \( 182 \), let \( v_{181}^i \) and \( v_{182}^i \) be the two vertices of \( \overline{P}_c \) that correspond to the two nodes of valency 1 in \( a_4 \). For \( i = 1 \) and \( 2 \), let \( c_{181}^i \) be the circumcenter of the 23-dimensional face of \( \overline{P}_c \) that does not contain \( v_{181}^i \). Then we have \( \{m(c_{181}^1), m(c_{181}^2)\} = \{350, 70\} \) and \( \{m(c_{182}^1), m(c_{182}^2)\} = \{350\} \). Therefore \( c_{181}^1 \) and \( c_{182}^1 \) are not equivalent.

We then define the invariant \( N(c) \) of \([c]\) as follows. When \( c \) is deep, we put
\[
N(c) := \begin{cases} 
\frac{m(c)}{2} & \text{if } m(c) \text{ is even,} \\
m(c) & \text{if } m(c) \text{ is odd.}
\end{cases}
\]
When \( e \) is shallow, we define \( N(e) \) to be the least positive integer such that \( N(e)/s(e) \) belongs to \( \mathbb{Z} \).

For a positive real number \( r \), we put

\[
\Xi(r) := \{ x \in \Lambda \mathbb{R} \mid d_\Lambda(x) \geq r \}.
\]

Let \( e \) be a hole. We put

\[
\Xi_e(r) := \{ x \in \overline{P}_e \mid \|x - \lambda\| \geq r \text{ for all } \lambda \in P_e \}.
\]

Then we obviously have

\[
\Xi(r) \cap \overline{P}_e \subset \Xi_e(r).
\]

Note also that, if \( r \leq R(e) \), then we have \( c \in \Xi_e(r) \). Let \( \theta(e) \) be the minimal real number such that, if \( r \) satisfies \( \theta(e) < r \leq R(e) \), then \( \Xi_e(r) \) is contained in the interior of \( \overline{P}_e \). For \( r \) with \( \theta(e) \leq r \leq R(e) \), we put

\[
\sigma(e, r) := \max\{ \|x - e\| \mid x \in \Xi_e(r) \}.
\]

Since \( \theta(e) \) and \( \sigma(e, r) \) depend only on the congruence class of the polytope \( \overline{P}_e \), they depend only on the hole type \( \tau(e) \), and hence only on the equivalence class \([e]\). It is easy to see that \( \sigma(e, r) \) is a decreasing function with respect to \( r \), and that \( \sigma(e, R(e)) = 0 \). For simplicity, we put

\[
\sigma(e, r) := 0 \text{ for } r > R(e).
\]

In fact, the function \( \sigma(e, r) \) can be calculated from the real number \( \theta(e) \) (see Section 4.1).

Using these invariants of holes, we can state our principal result. For each even positive integer \( d \), we put

\[
\rho_d(x) := \sqrt{2 - \frac{d}{x^2},}
\]

which is a function defined for \( x \geq \sqrt{d/2} \).

**Proposition 3.2.** Let \( w \in L \) be a Weyl vector, and let \( d \) be an even positive integer. Let \( v \) be a point of \( D(w) \cap L \) with \( \langle v, v \rangle_L = d \), and suppose that \( b := \langle v, w \rangle_L \) satisfies \( b \geq \sqrt{d/2} \). Then there exists a hole \( e \) for which \( b \) satisfies one of the following conditions:

(I) \( b^2 \) divides \( N(e)^2d \), and \( b^2 \leq s(e)d \);

(II) \( \rho_d(b) \leq \theta(e) \); or

(III) \( \rho_d(b) \geq \theta(e) \) and \( \sigma(e, \rho_d(b)) \geq \frac{2}{m(e)b} \).

**Remark 3.3.** When \( e \) is deep, the second condition in (I) is vacuous.

For the proof of Proposition 3.2, we use the following lemma.

**Lemma 3.4.** For any hole \( e' \in [e] \), we have \( N(e) \langle e', e' \rangle_\Lambda \in \mathbb{Z} \).

**Proof.** Let \( \lambda_0 \in \Lambda \) be an element of \( P_e \), and we put \( e'' := e' - \lambda_0 \). Note that \( e'' \in [e] \) and hence \( m(e) e'' \in \Lambda \). Moreover, we have \( \langle e'', e'' \rangle_\Lambda = R(e)^2 \). Hence we have

\[
\langle e', e' \rangle_\Lambda = R(e)^2 + 2\langle e'', \lambda_0 \rangle_\Lambda + \langle \lambda_0, \lambda_0 \rangle_\Lambda.
\]

Suppose that \( e \) is deep. Then we have \( R(e)^2 = 2 \in \mathbb{Z} \), and \( 2N(e) e'' \in \Lambda \). Therefore \( N(e) \langle e', e' \rangle_\Lambda \in \mathbb{Z} \) holds. Suppose that \( e \) is shallow. Then we have \( N(e)R(e)^2 \in \mathbb{Z} \) by (3.1).
Holes of the Leech lattice and K3 surfaces

By the list [24], we confirm that $m(c)$ divides $2N(c)$, and thus we obtain $2N(c) \langle c'', \lambda_0 \rangle_\Lambda \in \mathbb{Z}$. Therefore $N(c) \langle c', c' \rangle_\Lambda \in \mathbb{Z}$ holds. This completes the proof of Lemma 3.4.

**Proof of Proposition 3.2.** Let $U$ denote the hyperbolic plane; that is, $U$ is the lattice of rank 2 with a basis $e_1, e_2$ with respect to which the Gram matrix is

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
$$

We put

$$
L := U \oplus \Lambda^-,
$$

where $\Lambda^-$ is the negative-definite Leech lattice. Then $L$ is an even unimodular hyperbolic lattice of rank 26. A vector of $L \otimes \mathbb{R}$ is written as $(a, b, v)$, where $(a, b) = a e_1 + b e_2 \in U \otimes \mathbb{R}$ and $v \in \Lambda \otimes \mathbb{R}$. The intersection form $\langle \cdot, \cdot \rangle_L$ of $L$ is given by

$$
\langle (a, b, v), (a', b', v') \rangle_L = ab' + a'b - \langle v, v' \rangle_\Lambda.
$$

We choose the positive cone $P_L$ of $L \otimes \mathbb{R}$ in such a way that the primitive vector $w_0 := (1, 0, 0)$ of square norm 0 is contained in the closure of $P_L$ in $L \otimes \mathbb{R}$. Since $\langle w_0 \rangle_\Lambda / \langle w_0 \rangle_\Lambda$ is isomorphic to $\Lambda^-$, we see that $w_0$ is a Weyl vector. Since the group $O^+(L)$ acts on the set of Weyl vectors transitively, it is enough to prove Proposition 3.2 for the Weyl vector $w_0$.

For $\lambda \in \Lambda$, we put

$$
r_\lambda := \left( \frac{\lambda^2}{2} - 1, 1, \lambda \right) \in R_L, \quad \text{where} \quad \lambda^2 := \langle \lambda, \lambda \rangle_\Lambda.
$$

Then we have $R_L(w_0) = \{ r_\lambda | \lambda \in \Lambda \}$, and hence

$$
D(w_0) = \{ x \in P_L \mid \langle x, r_\lambda \rangle_L \geq 0 \text{ for all } \lambda \in \Lambda \}.
$$

Let $v = (a, b, v)$ be an arbitrary vector of $D(w_0) \cap L$ satisfying $\langle v, v \rangle_L = d$, and suppose that $b = \langle v, w_0 \rangle_L$ satisfies $b \geq \sqrt{d/2}$.

Note that $a, b,$ and $v$ satisfy the following conditions:

(i) $a, b \in \mathbb{Z}$ and $v \in \Lambda$;

(ii) $\langle v, r_\lambda \rangle_L = a + \left( \frac{\lambda^2}{2} - 1 \right) b - \langle v, \lambda \rangle_\Lambda \geq 0$ for all vectors $\lambda \in \Lambda$;

(iii) $\langle v, v \rangle_L = 2ab - \langle v, v \rangle_\Lambda = d$.

By condition (iii), we have

$$
\frac{a}{b} = \frac{1}{2} \left( \frac{d}{b^2} + \frac{\langle v, v \rangle_\Lambda}{b^2} \right).
$$

Combining this with the assumption $b \geq \sqrt{d/2}$, we see that condition (ii) is equivalent to the inequalities

$$
\left\| \frac{v}{b} - \lambda \right\| \geq \sqrt{2 - \frac{d}{b^2}} \text{ for all } \lambda \in \Lambda. \tag{3.3}
$$

In other words, we have

$$
v/b \in \Xi(\rho_d(b)). \tag{3.4}
$$
By Remark 2.2, there exists a hole $c$ such that the convex polytope $\overline{P}_c$ contains the point $v/b$. By (3.2) and (3.4), we have
\[ \frac{v}{b} \in \Xi_c(\rho_d(b)). \] (3.5)

We will show that $b$ satisfies one of conditions (I), (II) or (III) for this hole $c$.

**Lemma 3.5.** Suppose that $v/b$ is equal to the hole $c$, and let $N$ be a positive integer such that $N \langle c, c \rangle / \Lambda_1 \in \mathbb{Z}$. Then $b^2$ divides $N^2d$.

**Proof.** We put $M := N \langle c, c \rangle / \Lambda_1 \in \mathbb{Z}$. By condition (iii) and the assumption $v/b = c$, we have
\[ a = \frac{d}{2b} + \frac{Mb}{2N}. \]

Multiplying $2N$ on both sides, we obtain
\[ L := \frac{Nd}{b} = 2Na - Mb \in \mathbb{Z}. \]

Moreover, we have
\[ a = \frac{d}{2b} + \frac{Md}{2L}. \]

Multiplying $2L$ on both sides, we obtain
\[ \frac{Ld}{b} = \frac{Nd^2}{b^2} = 2La - Md \in \mathbb{Z}. \]

This completes the proof of Lemma 3.5.

**Case 1.** Suppose that $v/b$ is equal to the hole $c$. From the case $\lambda \in P_c$ in (3.3), we obtain $\sqrt{2} - d/b^2 \leq R(c) = \sqrt{2} - 1/s(c)$, and hence $b^2 \leq s(c)d$. By Lemmas 3.4 and 3.5, we also have that $b^2$ divides $N(c)^2d$. Therefore $b$ satisfies condition (I).

**Case 2.** Suppose that $v/b$ is not equal to $c$. Then $m(c)v$ and $b m(c) c$ are distinct points of $\Lambda$ by the definition of $m(c)$ and hence $\|m(c)v - b m(c) c\| \geq 4$ holds. Therefore we have
\[ \frac{\|v - c\|}{b} \geq \frac{2}{m(c)b}. \] (3.6)

We assume that $b$ does not satisfy condition (II). Then $\Xi_c(\rho_d(b))$ is contained in the interior of $\overline{P}_c$. By (3.5) and the definition of $\sigma(c, r)$, we have
\[ \frac{\|v - c\|}{b} \leq \sigma(c, \rho_d(b)). \] (3.7)

Combining (3.6) and (3.7), we see that $b$ satisfies condition (III).

4. **Proof of Theorem 1.2**

4.1. **Computation of the hole invariants**

The values of $s(c)$, $m(c)$, and $N(c)$ can be easily obtained from the set $P_c$ of vertices of $\overline{P}_c$. To calculate the value of $\theta(c)$ and the function $\sigma(c, r)$, we use the following lemma.

**Lemma 4.1.** Let $c$ be a hole. Let $F_1, \ldots, F_M$ be the 23-dimensional faces of $\overline{P}_c$. Then each $F_j$ is a 23-dimensional simplex.
The proof above also indicates a method to make the list of all 23-dimensional faces of the simplex of $F_i$ is a 23-dimensional face of the simplex $F_i$ for $i = 1, \ldots, m$, where $n_i = |P_{c,i}|$. If $F$ is a 23-dimensional face of $\overline{P}_c$, then the intersection $F \cap \langle P_{c,i} \rangle$ is an $(n_i - 2)$-dimensional face of the simplex $\overline{P}_{c,i}$. Conversely, if $F^{(i)}$ is an $(n_i - 2)$-dimensional face of the simplex $\overline{P}_{c,i}$ for $i = 1, \ldots, m$, then the convex hull $F$ of the vertices of $F^{(1)}, \ldots, F^{(m)}$ is a 23-dimensional face of $\overline{P}_c$. By Theorem 2.3, we see that the sum $\sum_i (n_i - 1)$ of the numbers of the vertices of $F^{(1)}, \ldots, F^{(m)}$ is 24. Hence their convex hull $F$ is a 23-dimensional simplex. This completes the proof.

**Proof.** If $c$ is shallow, then the convex polytope $\overline{P}_c$ is a 24-dimensional simplex, and it has exactly 25 faces of dimension 23, each of which is obviously a simplex. Suppose that $c$ is deep. We consider the decomposition (2.1) of $P_c$. Note that $\overline{P}_{c,i}$ is an $(n_i - 1)$-dimensional simplex in the $(n_i - 1)$-dimensional affine space $\langle P_{c,i} \rangle$ containing $P_{c,i}$ for $i = 1, \ldots, m$, where $n_i = |P_{c,i}|$. If $F$ is a 23-dimensional face of $\overline{P}_c$, then the intersection $F \cap \langle P_{c,i} \rangle$ is an $(n_i - 2)$-dimensional face of the simplex $\overline{P}_{c,i}$. Conversely, if $F^{(i)}$ is an $(n_i - 2)$-dimensional face of the simplex $\overline{P}_{c,i}$ for $i = 1, \ldots, m$, then the convex hull $F$ of the vertices of $F^{(1)}, \ldots, F^{(m)}$ is a 23-dimensional face of $\overline{P}_c$. By Theorem 2.3, we see that the sum $\sum_i (n_i - 1)$ of the numbers of the vertices of $F^{(1)}, \ldots, F^{(m)}$ is 24. Hence their convex hull $F$ is a 23-dimensional simplex. This completes the proof.

The proof above also indicates a method to make the list of all 23-dimensional faces $F_1, \ldots, F_M$ of $\overline{P}_c$. Let $h_j$ denote the point on $\langle F_j \rangle$ such that the line passing through $c$
Table 4.2. $||h_j - c_1||^2$

| $j$ | $||h_j - c_1||^2$ |
|-----|------------------|
|     | 1/4324           |
|     | 1/3312           |
|     | 1/2875           |
|     | 1/2484           |
|     | 1/2139           |
|     | 1/1840           |
|     | 1/1587           |

and $h_j$ is perpendicular to $\langle F_j \rangle$. Then $h_j$ lies in the interior of $F_j$, and $F_j$ is circumscribed by a 22-dimensional sphere in the 23-dimensional affine space $\langle F_j \rangle$ with center $h_j$ of radius $R_j := \sqrt{R(c)^2 - ||h_j - c||^2}$.

Therefore we have

$$\theta(c) = \max \{ R_j \mid j = 1, \ldots, M \},$$

$$\sigma(c, r) = \max(0, \sqrt{R(c)^2 - \theta(c)^2} - \sqrt{r^2 - \theta(c)^2}).$$

Example 4.2. Let $c_1 \in \Lambda_R$ be the point such that

$$46c_1 = [15, -2, -1, -2, 5, -1, -2, 4, 0, 0, -6, 12, -1, 0, 0, 0, 5, -4, -2, 0, 3, 12, 2, 14].$$

Then $c_1$ is a deep hole with $\tau(c_1) = D_{24}$. We have $m(c_1) = 46$. The convex polytope $P_{c_1}$ is a 24-dimensional simplex, and its vertices are given in Table 1. The nodes of the graph $\Delta_{c_1}$ correspond to these vertices in the way indicated in the graph in Table 1. Let $F_j$ be the 23-dimensional face of $P_{c_1}$, that does not contain $\lambda_j$. Then $||h_j - c||^2$ is calculated as in Table 2. Note that, by the symmetry of the simplex $P_{c_1}$, we have $||h_j - c_1|| = ||h_{20-j} - c_1||$ and $||h_1 - c_1|| = ||h_2 - c_1||$. Therefore we have

$$\theta(c_1)^2 = 8647/4324.$$
If \( b \notin S_\Pi([e], d) \), then \( \sigma(e, \sqrt{2 - d/b^2}) \) is defined. We put
\[
S_\Pi([e], d) := \left\{ b \in \mathbb{Z}_{>0} \setminus S_\Pi([e], d) \mid \sigma\left( e, \sqrt{2 - \frac{d}{b^2}} \right) \geq \frac{2}{m(e)b} \right\}.
\]

Consider the rational function
\[
\psi_e(t) := \left( \sqrt{R(e)^2 - \theta(e)^2} - \frac{2}{m(e)t} \right)^2 - \left( 2 - \frac{d}{t^2} - \theta(e)^2 \right)
\]
of \( t \). By (4.2), we see that a positive real number \( t_0 \) satisfying \( \sqrt{2 - d/t_0^2} \geq \theta(e) \) satisfies
\[
\sigma\left( e, \sqrt{2 - \frac{d}{t_0^2}} \right) \geq \frac{2}{m(e)t_0}
\]
if and only if \( \psi_e(t_0) \) is non-negative and
\[
\sqrt{R(e)^2 - \theta(e)^2} - \frac{2}{m(e)t_0} \geq 0
\]
holds. We put
\[
\Psi_e(t) := t^2 \psi_e(t) = \left( \frac{4}{m(e)^2} + d \right) - \frac{4\sqrt{R(e)^2 - \theta(e)^2}/m(e) - \sqrt{\theta(e)^2 - \theta(e)^2}}{m(e)} + (R(e)^2 - 2)t^2.
\]
Note that \( \Psi_e \) is a strictly decreasing linear function of \( t \) having a positive root \( \beta(e, d) \) if \( e \) is deep, whereas \( \Psi_e \) is an upward convex quadratic function of \( t \) having a negative root \( \alpha(e, d) \) and a positive root \( \beta(e, d) \) if \( e \) is shallow. Hence we have
\[
S_\Pi([e], d) = \left\{ b \in \mathbb{Z}_{>0} \setminus S_\Pi([e], d) \mid \frac{2}{m(e)\sqrt{R(e)^2 - \theta(e)^2}} \leq b \leq \beta(e, d) \right\}.
\]
In terms of the invariants \( s, m, \) and \( \theta^2 \), the function \( \beta(e, d) \) is given as follows:
\[
\beta(e, d) = \frac{d m(e)^2 + 4}{4m(e)\sqrt{2 - \theta(e)^2}} \quad \text{(4.3)}
\]
when \( e \) is deep, whereas
\[
\beta(e, d) = \frac{\sqrt{4s(e)^2(2 - \theta(e)^2) + ds(e)m(e)^2} - \sqrt{4s(e)^2(2 - \theta(e)^2) - 4s(e)m(e)^2}}{m(e)}
\]
when \( e \) is shallow.

**Example 4.3.** Let \( e_1 \) be the deep hole with \( \tau(e_1) = D_{24} \) given in Example 4-2. Recall that we have \( m(e_1) = 46 \) and \( 2 - \theta(e_1)^2 = 1/4324 \). By (4.3), we see that \( \beta(e_1, d) \) is equal to the function \( \phi(d) \) given in the statement of Theorem 1-2. On the other hand, we have
\[
\frac{2}{m(e_1)\sqrt{R(e_1)^2 - \theta(e_1)^2}} = \frac{2}{23\sqrt{1081}} = 2.859 \ldots
\]
Hence we have
\[
S_\Pi([e_1], d) \cup S_\Pi([e_1], d) = \{ b \in \mathbb{Z}_{>0} \mid b \leq \phi(d) \}.
\]
Finally, we put
\[
S(d) := \bigcup_{[c]} \left( S_1([c], d) \cup S_2([c], d) \cup S_3([c], d) \right),
\]
where \([c]\) ranges through the set of all equivalence classes of holes. Then Proposition 3.2 can be rephrased as follows:

**Proposition 4.4.** Let \(w \in L\) be a Weyl vector, and let \(d\) be an even positive integer. Then, for any vector \(v \in D(w) \cap L\) with \(\langle v, v \rangle_L = d\), we have \(\langle v, w \rangle_L \in S(d)\).

4.3. **Proof of Theorem 1.2**

We compare the sets \(S_1([c], d), S_2([c], d), S_3([c], d)\) and prove Theorem 1.2. After the comparison, it turns out that the set \(S_3([c], d)\) given by the deep hole \(c_1\) of type \(D_{24}\) is the largest.

Theorem 1.2 follows from Proposition 4.4 by the following lemma.

**Lemma 4.5.** The set \(S(d)\) coincides with \(\{ b \in \mathbb{Z}_{>0} \mid b \leq \phi(d) \}\).

**Proof.** The fact that \(S(d)\) includes \(\{ b \in \mathbb{Z}_{>0} \mid b \leq \phi(d) \}\) follows from Example 4.3. In order to show the opposite inclusion, we prove the following claims.

**Claim 4.6.** If \(b \in S_1([c], d)\), then \(b \leq \phi(d)\).

We put
\[
\mu_c := \min(N(c), \sqrt{s(c)}).
\]
Then \(S_1([c], d)\) is included in \(\{ b \in \mathbb{Z}_{>0} \mid b \leq \mu_c \sqrt{d} \}\). Since \(\sqrt{d} < d\) for any even positive integer \(d\) and \(\phi(0) > 0\), Claim 4.6 follows from
\[
\mu_c < \frac{529 \sqrt{1081}}{23} = 756.20 \cdots,
\]
which can be confirmed by numerical computation for each equivalence class \([c]\).

**Claim 4.7.** If \(b \in S_2([c], d)\), then \(b \leq \beta(c, d)\).

This claim follows from
\[
\Psi_c \left( \sqrt{\frac{d}{2 - \theta(c)^2}} \right) = \left( \sqrt{\frac{R(c)^2 - \theta(c)^2}{2 - \theta(c)^2}} \frac{\sqrt{d} - \frac{2}{m}}{2} \right)^2 \geq 0.
\]

**Claim 4.8.** Suppose that \([c] \neq [c_1]\). Then \(\beta(c, d) \leq \phi(d)\) holds for all even positive integers \(d\).

Suppose that \(c\) is deep. Then \(\beta(c, d)\) is a linear function of \(d\), and hence we can write it as \(f(c) d + g(c)\). We have \(f(c) > 0\). Hence the hoped-for inequality \(\beta(c, d) \leq \beta(c_1, d)\) follows from
\[
\frac{529 \sqrt{1081}}{23} = \frac{g(c) - g(c_1)}{f(c) - f(c_1)} < 2,
\]
which we can confirm by numerical computation again. Suppose that \(c\) is shallow. In order to prove \(\beta(c, d) \leq \phi(d)\), it is enough to show that \(\Psi_c(\phi(d)) \leq 0\). Since \(\Psi_c(\phi(d))\) is a quadratic polynomial in \(d\), and its coefficient of \(d^2\) is negative, we can prove \(\Psi_c(\phi(d)) \leq 0\).
for any even positive integer $d$ by showing that the quadratic equation $\Psi_c(\phi(x)) = 0$ in variable $x$ has no roots larger than 2.

Combining these three claims, we complete the proof of Lemma 4.5 and hence that of Theorem 1.2.

5. Examples and remarks

We continue the list of polarized $K3$ surfaces $(X, h)$ of simple Borcherds type in Example 1.9.

A complex $K3$ surface $X$ is said to be singular if $S_X$ is of rank 20. For a singular $K3$ surface $X$, the orthogonal complement of $S_X$ in $H_X = H^2(X, \mathbb{Z})$ is called the transcendental lattice of $X$. By [26], we see that, for each even positive-definite lattice $T_i$ of rank 2 whose Gram matrix

$$
\begin{bmatrix}
  a & b \\
  b & c
\end{bmatrix}
$$

is given in Table 3, there exists a singular $K3$ surface $X_i$, unique up to isomorphism, such that the transcendental lattice of $X_i$ is isomorphic to $T_i$. Then $X_i$ possesses an ample class $h_i$ such that $(X_i, h_i)$ is of simple Borcherds type. The automorphism group $\text{Aut}(X_i)$ of each $X_i$ has been determined in the papers cited in Table 3.

In [6], it was shown that the generic quartic Hessian surface $X$ possesses an ample class $h \in S_X \otimes \mathbb{Q}$ with $h^2 = 20$ such that $(X, h)$ is of simple Borcherds type. In this case, we have rank $S_X = 16$.

In [8], it was shown that the complex Kummer surface $\text{Km}(E \times E)$, where $E$ is a generic elliptic curve, possesses an ample class $h \in S_X \otimes \mathbb{Q}$ with $h^2 = 19$ such that $(X, h)$ is of simple Borcherds type. In this case, we have rank $S_X = 19$.

Remark 5.1. In [5], it was shown that the supersingular $K3$ surface $X$ in characteristic 2 with Artin invariant 1 possesses an ample class $h \in S_X \otimes \mathbb{Q}$ with $h^2 = 14$ such that Corollary 1.8 holds for $(X, h)$.

Remark 5.2. There exists a singular $K3$ surface $X$, unique up to isomorphism, such that its transcendental lattice is of discriminant 11. We showed in [23] that there exists a primitive embedding $S_X \hookrightarrow L$ satisfying Assumption 1.3 and $\mathcal{P}(X) \subseteq \mathcal{P}_L$ such that the number of $G_X$-congruence classes of induced chambers is 1098.
Remark 5.3. In all known examples of polarized $K3$ surfaces $(X, h)$ of simple Borcherds type, the orthogonal complement $R$ of $S_X$ in $L$ contains a sublattice of finite index generated by the set $R_R$ of vectors of $R$ with square norm $-2$. See [1, lemma 5.1] and [23, remark 6.7].

Remark 5.4. Let $S_X \hookrightarrow L$ be a primitive embedding satisfying Assumption 1.3 and $\mathcal{P}(X) \subset \mathcal{P}_L$, and let $a := \text{pr}_S(w)$ be the image of a Weyl vector $w \in L$ by the orthogonal projection $\text{pr}_S : L \rightarrow S_X^\vee$. We show that $\langle a, a \rangle_S > 0$. Since the orthogonal complement $R$ of $S_X$ in $L$ is negative-definite, we have $\langle a, a \rangle_S \geq \langle w, w \rangle_L = 0$, and the equality holds if and only if $a = w$. Therefore, if $\langle a, a \rangle_S = 0$, then we have $w \in S_X$, and hence $\langle w \rangle_{S_X} = \Lambda^-$ contains $R$, which contradicts condition (b) in Assumption 1.3.

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