Generalized Eigenstate Thermalization in 2d CFTs

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Infinite-dimensional conformal symmetry in two dimensions leads to integrability of 2d conformal field theories by giving rise to an infinite tower of local conserved qKdV charges in involution. We discuss how presence of conserved charges constrains equilibration in 2d CFTs. We propose that in the thermodynamic limit large central charge 2d CFTs satisfy generalized eigenstate thermalization, with the values of qKdV charges forming a complete set of thermodynamically relevant quantities, which unambiguously determine expectation values of all local observables from the vacuum family. Equivalence of ensembles further provides that local properties of an eigenstate can be described by the Generalized Gibbs Ensemble that only includes qKdV charges. In the case of a general initial state, upon equilibration, emerging Generalized Gibbs Ensemble will necessarily include negative chemical potentials and holographically will be described by a quasi-classical black hole with quantum soft hair.

The topic of thermalization, and more generally, equilibration of isolated many-body quantum systems has been an active area of research during the past decade. In case of non-integrable systems, i.e. those without an extensive number of local conserved quantities, emergence of the thermal equilibrium has been traced to eigenstate thermalization hypothesis (ETH) which postulates thermal properties of individual energy eigenstates\textsuperscript{[1–3]}. In the simplest form it requires the expectation value of some appropriate (often taken to be local) observable $O$ in a many-body eigenstate $|E_i\rangle$ to be a smooth function of energy,

$$
\langle E_i|O|E_i\rangle = f_O(E_i).
$$

(1)

Qualitatively, eq.\textsuperscript{(1)} postulates that energy is the only thermodynamically relevant quantity, which completely specifies local properties of an eigenstate. The condition\textsuperscript{(1)} may apply to all or most eigenstates, in which case it is referred as strong or weak ETH. The eigenstate thermalization ensures equivalence between the expectation value in the eigen-ensemble, $f_O(E_i)$, and thermal expectation value of $O$ in the Gibbs ensemble, $f_O(E_i) = \text{Tr}(e^{-\beta H}O)/Z$, where the effective temperature $\beta$ is fixed through the energy balance relation, $E_i = \text{Tr}(e^{-\beta H}O)/Z \textsuperscript{[4]}$.

When the system is integrable, with an extensive number of conserved charges $Q_k$, ETH does not apply. Accordingly emerging equilibrium can be different from the Gibbs state. In this case the equilibrium can be described by the Generalized Gibbs Ensemble (GGE), a generalization of grand canonical ensemble that includes an infinite tower of conserved charges\textsuperscript{[5]}. Validity of the GGE has been related to the generalized eigenstate thermalization\textsuperscript{[6–8]}, which generalizes\textsuperscript{(1)} to include an infinite number of conserved quantities,

$$
\langle E_i|O|E_i\rangle = f_O(Q_k(E_i)).
$$

(2)

Here $|E_i\rangle$ is a mutual eigenstate of the Hamiltonian and charges $Q_k$, $Q_k(E_i)$ are the eigenvalues of $Q_k$ associated with $|E_i\rangle$, and function $f_O$ is assumed to be a smooth function of all of its arguments. Similarly to\textsuperscript{(1)}, at the qualitative level,\textsuperscript{(2)} postulates that charges $Q_k$ form a complete set of thermodynamically relevant quantities which fully specify local properties of an eigenstate. Provided\textsuperscript{(2)} applies to most states, it ensures equivalence between the generalized microcanonical ensemble and GGE, establishing validity for the latter to describe emerging equilibrium e.g. following a quantum quench\textsuperscript{[6]}.

In this Letter we discuss thermalization of two-dimensional conformal field theories (CFTs), a rich topic with multiple connections ranging from the cold atom experiments\textsuperscript{[9]} to physics of quantum gravity\textsuperscript{[10]}. It has been shown that following a quantum quench 2d conformal theories equilibrate and reach a steady state, which in many cases can be described in terms of the Gibbs ensemble\textsuperscript{[11, 12]}. At the same time emergence of thermal equilibrium is not universal. Conformal symmetry in two dimensions gives rise to an infinite tower of local mutually commuting conserved qKdV charges $Q_{2k-1}$, the CFT Hamiltonian for the left-movers being $Q_1 \equiv H$, a part of the integrable structure of the 2d CFTs\textsuperscript{[13, 15]}. The question we are concerned with is how presence of these charges affects equilibration. By analogy with the integrable lattice models it is natural to expect that locally equilibrium states can be described in terms of the GGE, which includes all local qKdV charges. Indeed, emergence of exactly such qKdV GGE was analytically
shown for a special family of so-called Cardy-Calabrese initial states [16].

In the context of integrable systems the question which quantities should be included in the GGE is far from being trivial. Early studies in the context of XXZ and Lieb-Liniger models have shown that a full set of extensive local charges does not specify local properties of eigenstates, signaling failure of generalized ETH [17, 18]. These works raised an important question of the validity beyond currently known leading 1/c expansion.

In the regime of quasi-classical gravity [5], c ≫ 1, expectation values of qKdV charges can be calculated explicitly [34].

\[ \ell q_1 = \Delta + \sum_k n_k k, \]
\[ \ell q_3 = \Delta^2 + \sum_k n_k \left( 6\Delta k + \frac{c k^3}{6} \right) + O(c^0), \]
\[ \ell q_{2r-1} = \Delta^r + \sum_k n_k p_{2r-1}(c, \Delta, k) + O(c^{r-2}), \]

where \( p_{2r-1}(c, \Delta, k) \) are some known polynomials of degree 2r − 1 which include only odd powers of k.

Because of translational invariance the expectation value of a full derivative \( \mathcal{O} = \partial \mathcal{O}' \) in energy eigenstate will vanish. Hence it suffices to consider expectation values \( \langle E|\mathcal{O}|E \rangle \) only when \( \mathcal{O} \) is a quasi-primary operator. Below we consider the case when \( \mathcal{O} \) belongs to the vacuum family, i.e. it is a Virasoro descendant of the identity. To streamline the notations we introduce \( \langle \mathcal{O} \rangle \equiv \langle E|\mathcal{O}|E \rangle \). It is convenient to parametrize \( \mathcal{O} \) by its dimension (level). At the levels 2 and 4 there are unique quasi-primaries in the vacuum family,

\[ \mathcal{O}_2 = T, \quad \mathcal{O}_4 = T^2 - \frac{3}{10} T^2 T. \]

Thus expectation values of \( \mathcal{O}_{2,4} \) are identically equal to charge densities \( q_1, q_3 \) [33]. At the level 6 there are two quasi-primaries (we always choose quasi-primaries in the basis which diagonalizes Zamolodchikov metric)

\[ \mathcal{O}_6^{(1)} = T^3 - \frac{9}{10}(T^2 T) + \frac{4}{35} T^4 T + \frac{93}{70c} + 29 O_6^{(2)}, \]
\[ \mathcal{O}_6^{(2)} = (\partial T \partial T) - \frac{4}{5}(T^2 T) + \frac{23}{210} T^4 T. \]

The expectation value of the combination \( \mathcal{O}_6^{(1)} + \frac{5}{9 T^2} \mathcal{O}_6^{(2)} \) is identically equal to \( q_5 \). Similarly to (7-9), at leading order the expectation value of \( \mathcal{O}_6^{(2)} \) has the form of a polynomial in \( \Delta \) and odd powers of k,

\[ \langle \mathcal{O}_6^{(2)} \rangle = \frac{9}{5} \sum_k n_k \left( \frac{c}{6} k^5 + 4\Delta k^3 \right) + O(c^0). \]
It is possible to use (7-9) to express any term of the form \( \sum_{k} n_{k} k^{2r-1} \) via \( q_{2j-1} \), \( j \leq r \), but a priori the result would also depend on \( \Delta \). Thus, at leading order in \( 1/c \), expectation values of \( O(\beta) \) are some functions of \( \Delta \) and \( q_{2r-1} \). Remarkably, because of the non-trivial cancellations the final result is \( \Delta \)-independent, and can be expressed solely in terms of \( q_{2r-1} \). To simplify the answer we introduce dimensionless ratio \( q_{2k-1} = q_{2k-1}/q_{1}^{2} \) such that \( \delta q_{2k-1} \equiv q_{2k-1} - 1 \) is of order \( 1/c \). Then \( O(\beta) \) measured in units of energy density \( q_{1} \) is given by

\[
q_{1}^{-3} O(\beta^{(1)}) = 1 + 3 \delta q_{3} + O(1/c^{2}),
\]

\[
q_{1}^{-3} O(\beta^{(2)}) = \frac{9}{5} c (\delta q_{5} - 3 \delta q_{3}) + O(1/c^{3}).
\]

As we see different quasi-primaries have different scaling with \( c \). Our calculation applies to leading \( 1/c \) behavior of each quasi-primary, except for a special one, which includes maximal power of \( T \) without derivatives. The expectation value of that quasi-primary starts with \( O(\beta^{(1)}) \) and our result applies to the first two terms in \( 1/c \) expansion.

The possibility to express eigenstate expectation value \( \langle O \rangle \) as a polynomial in \( q_{2j-1} \) extends to all higher levels. For an operator of dimension \( 2r \) the answer only depends on \( q_{2j-1} \) for \( j \leq r \). We write down explicit expressions for all operators up to level 10 in terms of \( q_{2j-1} \) in Supplemental Materials. Our results establish generalized eigenstate thermalization for vacuum block observables in large \( c \) CFTs.

That expectation value \( \langle O \rangle \) of an operator of dimension \( 2r \) only includes qKdV charges \( q_{2j-1} \) up to the same dimension \( j \leq r \) can be interpreted as a manifestation of locality. It is analogous to the observation in the context of integrable lattice models that to describe equilibrium state locally, at the length scales not exceeding some distance \( a \), it is only necessary to include local and quasi-local charges in the GGE with the support within a \( \delta \) regime.

Generalized eigenstate thermalization implies validity of the qKdV Generalized Gibbs Ensemble

\[
\rho = \exp \left\{ - \sum_{k} \mu_{2k-1} Q_{2k-1} \right\} / Z, \quad \mu_{1} \equiv \beta,
\]

to describe local properties of individual energy eigenstates, provided chemical potentials \( \mu_{2k-1} \) are tuned to match values of the eigenstate charges

\[
\ell q_{2k-1} = \langle E_{i} Q_{2k-1} E_{i} \rangle = \text{Tr}(\rho Q_{2k-1}).
\]

Provided \( q_{2k-1} \) chosen to represent charge densities of some non-equilibrium initial state \( |\Psi\rangle \), a standard argument would consequently equate the GGE expectation values of local operators with those in the diagonal ensemble of \( |\Psi\rangle \), written in the eigenbasis \( \langle \ell | \). In most cases the latter would be equal to the expectation values in state \( |\Psi\rangle \) upon equilibration. It should be noted though that left and right Hamiltonians \( Q_{1}, \tilde{Q}_{1} \) are highly degenerate, and therefore validity of the diagonal ensemble to describe local physics upon equilibration may be violated.

It remains an open question to establish existence of \( q_{2k-1} \) which would solve (17) for any given set of \( q_{2r-1} \). Using explicit form of the generalized partition function in the large \( c \) limit [34] we can find, up to \( O(1/c^{2}) \) corrections,

\[
\delta q_{2k-1} = \frac{q_{2k-1}}{q_{1}^{2}} - 1 = \frac{24 k}{c} \int_{0}^{\infty} \frac{dkk}{2} \frac{(2k-1)_{2} F_{1}(1,1-k,3/2,-k^{2}) - 1}{\exp^{2\pi k \gamma} - 1},
\]

\[
\gamma = \sum_{j=1}^{\infty} \mu_{2j-1} j(2j-1)\sigma_{j}^{-1/2} F_{1}(1,1-j,3/2,-k^{2}),
\]

where \( \mu_{2k-1} = \sigma_{k} \gamma^{k-1/2} \) and \( \sigma_{j} \) is positive and satisfies

\[
\sum_{k=1}^{\infty} k \mu_{2k-1} \gamma^{k-1/2} = 1.
\]

From here it follows that when all chemical potentials are positive \( q_{2k-1} \) satisfy an infinite series of inequalities (see Supplemental Materials)

\[
\frac{q_{1}}{q_{1}^{2}} - 1 \leq \frac{22}{5c} + O(1/c^{2}), \quad \frac{q_{5}}{q_{1}^{3}} - 1 \leq \frac{302}{21c} + O(1/c^{2}),
\]

\[
\ldots
\]

Thus GGE emerging after equilibration of some general initial state will have to include negative chemical potentials, unless all inequalities (20) are satisfied.

To match GGE to a primary state all qKdV densities should be related to each other via \( q_{2k-1} = q_{1}^{2k} \). This is only possible if the integral in (18) vanishes, which requires \( \gamma \) to be infinite. This is consistent with the observation of [30] that an ensemble with any finite number of non-zero \( \mu_{2k-1} \) cannot describe primary states. This is because in full generality \( q_{2k-1} \geq q_{1}^{2k} \) and hence primary states are at the boundary of the phase space of \( q_{2k-1} \)'s. It is nevertheless possible to describe them in the limit, via a GGE with at least some coefficients approaching infinity. The simplest scenario is to consider \( \mu_{1} > 0 \) and arbitrary \( \beta \equiv \mu_{1} \), while all other chemical potentials are identically zero. Then in the limit \( \tau = \beta/(6/\pi^{2} c \mu_{1})^{1/3} \to \infty \), for all \( k, q_{2k-1}/q_{1}^{k} - 1 \) will vanishes as \( \sim |\tau|^{-3} \), as is shown for \( k = 2, 3 \) in Fig. 1.

With just two chemical potentials \( \beta, \mu_{3} \) being non-zero the values of \( q_{2k-1}/q_{1}^{k} - 1 \) is confined to be between zero and their thermal (Gibbs ensemble) values. This constraint is removed already after turning on one more additional chemical potential. For example by taking \( \beta, \mu_{5} > 0 \) and \( \mu_{3} < 0 \) one can fine-tune function \( \gamma \) to become arbitrarily small for some positive value of \( \kappa \), leading to the divergence of the integral in (18) and violating quasi-classical regime (3).
large central charge 2d CFTs in the thermodynamic limit satisfy generalized eigenstate thermalization with the tower of local qKdV charges forming a complete set of thermodynamically-relevant quantities. Our analysis establishes universal validity of Generalized Gibbs Ensemble that includes all qKdV charges to describe individual energy eigenstates, and hence in most cases, asymptotic equilibrium states in such theories. It would be important to extend the analysis to next order in 1/c, which will likely reveal if the eigenstate thermalization is strong i.e. applies to all finite energy density eigenstates, or weak, i.e. applies to most states.

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In the units of energy density at leading order they are
\[ O_6^{(1)} = T^3 - \frac{9}{10} (T \partial^2 T) + \frac{4}{35} \partial^3 T + \frac{93}{70c + 29} O_6^{(2)}, \] (21)
\[ O_6^{(2)} = (T \partial T) - \frac{4}{5} (T \partial^2 T) + \frac{23}{210} \partial^4 T. \] (22)
In the limit (6) they can be simplified to
\[ O_6^{(1)} = T^3 + O(1/c), \] (23)
\[ O_6^{(2)} = \frac{9}{5} (T \partial T) + O(1/c). \] (24)

In units of the energy density their expectation values are
\[ q_1^{-3} O_6^{(1)} = 1 + 3 \delta q_3 + O(1/c^2), \] (25)
\[ q_1^{-3} O_6^{(2)} = \frac{9}{5} \frac{12}{c} (\delta q_5 - 3 \delta q_3) + O(1/c^3). \] (26)

Level 8
There are three quasi-primaries at level 8,
\[ O_8^{(1)} = T^4 + O(1/c), \] (27)
\[ O_8^{(2)} = \frac{9}{5} (T \partial T) + O(1/c), \] (28)
\[ O_8^{(3)} = \frac{143}{63} (T^2 \partial^2 T) + O(1/c). \] (29)
In the units of energy density at leading order they are
\[ q_1^{-4} O_8^{(1)} = 1 + 6 \delta q_3 + O(1/c^2), \] (30)
\[ q_1^{-4} O_8^{(2)} = \frac{9}{5} \frac{12}{c} (\delta q_5 - 3 \delta q_3) + O(1/c^3), \] (31)
\[ q_1^{-4} O_8^{(3)} = \frac{143}{63} \frac{180}{c^2} (\delta q_7 - 4 \delta q_5 + 6 \delta q_3) + O(1/c^4). \] (32)

Level 9
There are no quasi-primaries of odd dimension smaller than nine. At level nine there is a unique quasi-primary \( O_9 \), which has zero expectation value, as well as all higher odd-dimensional quasi-primaries, due to parity.

There are four quasi-primaries at level 8. In the limit (6) up to some additional factors they are
\[ O_{10}^{(1)} = T^5 + O(1/c), \] (33)
\[ O_{10}^{(2)} = (T(T \partial T)) + O(1/c), \] (34)
\[ O_{10}^{(3)} = (T(T^2 \partial T^2)) + O(1/c), \] (35)
\[ O_{10}^{(4)} = (T^3 \partial^3 T) + O(1/c). \] (36)
In terms of energy density their expectation values are
\[ q_1^{-5} O_{10}^{(1)} = 1 + 10 \delta q_3 + O(1/c^2), \] (37)
\[ q_1^{-5} O_{10}^{(2)} = \frac{1}{c} \frac{1}{\delta q_3} - 3 \delta q_3 + O(1/c^3), \] (38)
\[ q_1^{-5} O_{10}^{(3)} = \frac{180}{c^2} (\delta q_7 - 4 \delta q_5 + 6 \delta q_3) + O(1/c^4), \] (39)
\[ q_1^{-5} O_{10}^{(4)} = \frac{3024}{c^3} (\delta q_9 - 5 \delta q_7 + 10 \delta q_5 - 10 \delta q_3) + O(1/c^5). \] (40)

### GGE with positive chemical potentials
For any positive integer \( j \) hypergeometric function \( _2F_1(1, 1 - j, 3/2, -\kappa^2) \) is polynomial in \( \kappa^2 \) with non-negative coefficients which starts with one,
\[ _2F_1(1, 1 - j, 3/2, -\kappa^2) = 1 + \frac{2}{3} (j - 1) \kappa^2 + \ldots \] (41)
Hence it is a monotonically increasing function of \( \kappa \) which satisfies \( _2F_1(1, 1 - j, 3/2, -\kappa^2) \geq 1 \). From here it follows that when all chemical potentials are non-negative, function \( \gamma \) defined in the equation (18) from the main text satisfies
\[ \gamma \leq \sum_{j=1}^\infty (2k - 1)^{\gamma - 1/2} \sum_{j=1}^\infty \tilde{\mu}_{2j-1} \kappa^{j - 1/2} = 1. \] (42)
Thus at leading order in \( 1/c \), \( q_{2k-1}/q_1^k - 1 \) is bounded from above by its value in the Gibbs ensemble,
\[ \delta q_{2k-1} \leq \frac{24k}{c} \int_0^\infty dk \kappa \left( [2k - 1]_2F_1(1, 1 - k, 3/2, -\kappa^2) - 1 \right) \] (43)
\[ = \frac{k}{c} \kappa \sum_{p=0}^{\infty} \left( \frac{6(k - 1)\Gamma(k)\Gamma(1/2)}{\Gamma(p + 3/2)\Gamma(k - p)} \right) (-1)^{p+1} \zeta(-1 - 2p) - 1 \gamma \] (44)
This yields \( 22/5 \) for \( k = 2 \), \( 302/11 \) for \( k = 3 \), \( 2428/75 \) for \( k = 4 \), and so on.

### GGE with two non-zero chemical potentials
To gain better intuition it is instructive to consider the generalized ensemble which includes only two charges,
the conventional Hamiltonian of CFT $H \equiv Q_1$ and $Q_3$, 
\begin{equation}
\rho = \exp (-\beta H - \mu_3 Q_3) / Z.
\end{equation}

To assure convergence we must require $\mu_3 > 0$ while $\beta$ can be arbitrary. It is convenient to parametrize $\beta, \mu_3$ in terms of 
\begin{equation}
\tau = \beta \left( \frac{6}{\pi^2 c \mu_3} \right)^{1/3},
\end{equation}
and energy density $q_1 = -\ell^{-1} \partial \ln Z / \partial \beta$, such that 
\begin{align*}
\beta &= q_1^{-1/2} \left( \frac{c \pi^2}{6} \right)^{1/2} \frac{2^{3/2} (\tau^3 + 3 \sqrt{6 \tau^3 + 81} + 9) - \tau}{\sqrt{6} \sqrt{\tau^3 + 3 \sqrt{6 \tau^3 + 81} + 9}}, \\
\mu_3 &= q_1^{-3/2} \left( \frac{c \pi^2}{6} \right)^{1/2} \frac{2^{3/2} (\tau^3 + 3 \sqrt{6 \tau^3 + 81} + 9) - \tau}{6 \sqrt{6} \sqrt{\tau^3 + 3 \sqrt{6 \tau^3 + 81} + 9}}.
\end{align*}

Then $\delta q_{2k-1}$ only depends on $\tau$, 
\begin{align*}
\gamma &= 1 + \frac{2^{3/2} (\kappa^2 + 1) \left( \tau^3 + 3 \sqrt{6 \tau^3 + 81} + 9 \right) - \tau}{3^{3/2} \tau^3 + 3 \sqrt{6 \tau^3 + 81} + 9}, \\
\delta q_{2k-1} &= \frac{24k}{e^c} \int_0^\infty d\kappa \kappa \left( (2k - 1) F_1 (1, 1 - k, 3/2, -\kappa^2) - 1 \right).
\end{align*}

When $\tau$ approaches minus infinity while $q_1$ is kept fixed, 
\begin{align*}
\beta &\sim -q_1^{-1/2} \left( \frac{c \pi^2}{6} \right)^{1/2} |\tau|^{3/2} 2^{-1/2}, \\
\mu_3 &\sim q_1^{-3/2} \left( \frac{c \pi^2}{6} \right)^{1/2} |\tau|^{3/2} 2^{-3/2},
\end{align*}
and we find that $c\delta q_{2k-1}$ approaches zero as $1/|\tau|^3$. We plot $\delta q_{2k-1} = q_1^{-k} q_{2k-1} - 1$ in the units of $1/c$ as a function of $\tau$ for $k = 2, 3$ in Fig. [1] in the main text.