The moment $\langle x \rangle_{u-d}$ of the nucleon from $N_f = 2$ lattice QCD down to nearly physical quark masses

Gunnar S. Bali, Sara Collins, Benjamin Gläble, Meinulf Göckeler, Johannes Najjar, Rudolf H. Rödl, Andreas Schäfer, Rainer W. Schiel, André Sternbeck, and Wolfgang Söldner
Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany
(Dated: October 27, 2014)

We present an update of our analysis [1] which includes additional ensembles at different quark masses, lattice spacings and volumes, all with high statistics. We use $N_f = 2$ mass-degenerate quark flavours, employing the non-perturbatively improved clover action. The lattice matrix elements are converted to the $\overline{\text{MS}}$ scheme via renormalization factors determined non-perturbatively in the RI’MOM scheme. We have systematically investigated excited state contributions, in particular, at the smallest, near physical, pion mass. While our results (with much increased precision) are consistent with Ref. [1], comparing with previous determinations we find that excited state contributions can be significant if the quark smearing is not suitably optimized, in agreement with other recent studies. The difference with respect to the value for $\langle x \rangle_{u-d}$ extracted from experimental data is reduced but not resolved. Using lattice sizes in the range $L \pi \sim 3.4 - 6.7$, no significant finite volume effects were observed. Performing a controlled continuum limit that may remove the discrepancy will require simulations at lattice spacings $a < 0.06$ fm.

PACS numbers: 12.38.Gc,13.85.-t,14.20.Dh

I. INTRODUCTION

The focus of modern hadron structure physics has evolved from ordinary parton distribution functions (PDFs) and form factors to more complex quantities like generalized parton distributions (GPDs), transverse momentum dependent PDFs (TMDs), double distributions (DDs) and distribution amplitudes (DAs), see e.g. Ref. [2]. Many of these quantities cannot easily be accessed by experiment and lattice results often have to be used as a substitute. In this situation, observables where lattice results and experiment can be compared to judge the reliability of lattice predictions (and of phenomenological fits to experimental data) play a key role. In recent years good agreement has been found for such benchmark quantities, e.g., in hadron spectroscopy [3,4] however, for a fundamental and well-known nucleon structure observable, namely the iso-vector quark momentum fraction, $\langle x \rangle_{u-d}$, significant disagreement remains. This needs to be resolved to improve prospects for hadron physics beyond the level of PDFs. For recent reviews concerning lattice hadron structure determinations see e.g. Refs. [5,6].

Past lattice predictions were complicated by the need to extrapolate lattice results determined at larger than physical quark masses using, e.g., parameterizations given by chiral perturbation theory (ChPT). The range of validity of ChPT depends on the quantity studied and can ultimately only be tested by lattice simulations, including the physical point. In addition, any extrapolation is unreliable if the lattice results themselves do not include reasonable estimates of their main systematics. Consequently, we started a dedicated effort to produce high statistics results for $N_f = 2$ fermions at nearly physical quark masses.

This began with a study at $m_\pi \lesssim 160$ MeV on a single small volume, with linear lattice extent $L$ in units of the pion mass $L m_\pi \sim 2.77$, detailed in Ref. [1]. With the aim of investigating the main sources of systematic uncertainty we have since expanded our data set to include a larger volume with $m_\pi \sim 150$ MeV and $L m_\pi \sim 3.49$ and several ensembles at larger quark masses (up to $m_\pi \sim 490$ MeV), a range of volumes ($L m_\pi \sim 3.4 - 6.7$) and a limited range of lattice spacings ($a \sim 0.06 - 0.08$ fm). In addition, we have performed a thorough analysis of excited state contributions, which $\langle x \rangle_{u-d}$ is known to be sensitive to [1,8–12]. Nevertheless we find our results for $\langle x \rangle_{u-d}$ still to deviate from the phenomenological values by roughly 25%.

The structure of this paper is as follows: After detailing our lattice set up in Section II we discuss our analysis of excited states in Section III and present consistency checks performed involving finite momentum data in Section IV. Our final results are compared with other recent determinations in Section V and we discuss remaining systematics in the conclusions, Section VI. We note that a preliminary analysis of some of our ensembles appeared in Ref. [13].

II. LATTICE SET UP

We used configurations generated by the Regensburg QCD collaboration (RQCD) and QCDSF with $N_f = 2$ non-perturbatively improved clover fermions and the Wilson gauge action, see Table I for the simulation parameters. While many lattice simulations now also include dynamical strange quarks, so far strangeness has

---

* sara.collins@ur.de
been found to play a minor role in nucleon structure \cite{14-17} and $N_f = 2$ simulations remain relevant.

The two-point and three-point functions, needed to extract the quark momentum fraction have the form

$$C_{2\text{pt}}(t) = \sum_{\vec{x},\vec{y}} \langle N(\vec{x}, t_1) N(\vec{y}, 0) \rangle$$  \hspace{1cm} (1)

$$C_{3\text{pt}}(t_1, t) = \sum_{\vec{x},\vec{y},\vec{z}} \langle N(\vec{x}, t_1) O(\vec{y}, t) N(\vec{z}, 0) \rangle$$  \hspace{1cm} (2)

for a nucleon, $N$, created at a time $t_1 = 0$, destroyed at a time $t$ and with an operator $O$ inserted at a time $t$. In the limit of large Euclidean time separations, $t_1 \gg t \gg 0$, one obtains

$$C_{2\text{pt}}(t) = |Z_0|^2 e^{-m t} + |Z_1|^2 e^{-m t} \ldots$$ \hspace{1cm} (3)

$$C_{3\text{pt}}(t_1, t) = |Z_0|^2 \langle N_0 | O | N_0 \rangle e^{-m t}$$

$$+ Z_0^* Z_1 \langle N_0 | O | N_1 \rangle e^{-m_1 t}$$

$$+ Z_1^* |N_1 | O | N_1 \rangle e^{-m t}$$

$$+ \ldots,$$ \hspace{1cm} (4)

where $Z_i = \langle N_i | \overline{N} \rangle$ are the overlaps of the state $\overline{N}|0\rangle$, created by a nucleon interpolator $\overline{N}$ with the ground and first excited states $|N_0\rangle$ and $|N_1\rangle$, respectively. We denote the corresponding masses as $m_0$ and $m_1$. The “…” indicate the neglected higher excitations. The target matrix element

$$\langle N_0 | O | N_0 \rangle = -\langle x \rangle_{u-d}^{\text{LAT}} m_0,$$ \hspace{1cm} (5)

where $(y = (\vec{y}, t))$.

$$O = \sum_{\vec{y}} \tilde{a}_y \left( \gamma_4 \overleftrightarrow{D}_4 - \frac{1}{3} \gamma \overleftrightarrow{D} \right) u_y - \overline{d}_y \left( \gamma_4 \overleftrightarrow{D}_4 - \frac{1}{3} \gamma \overleftrightarrow{D} \right) d_y,$$ \hspace{1cm} (6)

can be extracted more easily if the ground state contribution dominates $C_{3\text{pt}}$ (and similarly $C_{2\text{pt}}$). The covariant derivative is defined as $\overleftrightarrow{D}_4 = \frac{1}{2} \left( \overleftrightarrow{D}_\mu - \overleftrightarrow{D}_\mu \right)$. For each ensemble the quark smearing was optimized to minimize the excited state contributions to the nucleon two-point function. We used gauge invariant Wuppertal \cite{18-19} smeared quark sources and sinks with APE smoothed spatial gauge links $U_{x,j}$ \cite{20}. The Wuppertal algorithm involves $N_{\text{sm}}$ iterations of

$$q_x^{(n)} = \frac{1}{1 + \delta s} \left[ q_x^{(n-1)} + \delta \sum_{j=1}^3 \left( U_{x,j} q_{x+j}^{(n-1)} + U_{x-j,j} q_{x-j}^{(n-1)} \right) \right],$$ \hspace{1cm} (7)

where $n$ labels the iteration number and we choose $\delta = 0.25$. $N_{\text{sm}}$ for each ensemble is given in Table I. Naively, at equal pion mass, $N_{\text{sm}} \propto 1/\alpha^2$. In addition, for ensemble IX, we also computed the two and three-point functions using Jacobi smearing \cite{21}, for different numbers of iterations, with and without APE smoothed links, for comparison. The Jacobi algorithm is given by

$$q_x^{(n)} = q_x^{(0)} + \kappa \sum_{j=1}^3 \left( U_{x,j} q_{x+j}^{(n-1)} + U_{x-j,j} q_{x-j}^{(n-1)} \right)$$ \hspace{1cm} (8)

where we used $\kappa = 0.21$.

The three-point functions were generated using the standard sequential propagator method \cite{22} which involves fixing $t_1$. Alternative approaches using stochastic estimates have also been investigated recently, see Refs. 23, 24. The value of $t_1$ was optimized using ensemble IV. From the last term in Eq. (4), one can see that $C_{3\text{pt}}$ may contain contributions proportional to $\langle N_j | O | N_j \rangle$, $j \geq 1$, which can only be resolved by varying $t_1$. As discussed below, by using $t_1/a$ in the range $7 - 17$ we found that for our choice of smearing the last term in Eq. (4) is sufficiently suppressed at $t_1 \approx 1.1 \text{ fm} \approx 15a$ for $\beta = 5.29$ within given statistics. This value was then used for all $\beta = 5.29$ ensembles and rescaled to $t_1 = 13a$ and $17a$ for $\beta = 5.20$ and $\beta = 5.40$, respectively. As an additional check at the lightest mass point we generated $C_{3\text{pt}}$ with $t_1/a = 9, 12$ and 15. Multiple measurements were performed on each configuration for all ensembles and autocorrelations were investigated by binning the data.

The lattice matrix elements are converted to the $\overline{\text{MS}}$ scheme at a scale $\mu = 2 \text{ GeV}$ using renormalization factors determined non-perturbatively in the RI'-MOM scheme in Refs. 25, 26 (and 3-loop continuum perturbative factors relating the RI'-MOM and $\overline{\text{MS}}$ schemes). Applying $O(a)$ Symanzik improvement (as was implemented for the quark action) the relation between the renormalized and lattice operators has the form \cite{27}

$$O^{\overline{\text{MS}}}(\mu) = Z^{\overline{\text{MS}}, \text{LAT}}(a\mu) \left[ (1 + \lambda m a) O^{\text{LAT}} + a c_O O_{\text{LAT}}^{\text{lat}} \right]$$ \hspace{1cm} (9)

with $\lambda m a = \frac{1}{2} \left( \kappa - 1 - \kappa_c^{-1} \right)$, where $\kappa_c = 0.1360546(39), 0.1364281(12)$ and 0.1366792(108) for $\beta = 5.20, 5.29$ and 5.40, respectively. We set $b = 1$ and $c_O = 0$, thus, our values for $\langle x \rangle_{u-d}$ still have $O(a)$ leading discretization errors.

### III. SUPPRESSION OF EXCITED STATES

In the past few years a number of studies have highlighted excited state contamination as one of the main systematic uncertainties in lattice determinations of $\langle x \rangle_{u-d}$ \cite{1, 8-12}. Fig. 1, which shows the ratio

$$C_{3\text{pt}}(t_1, t)/C_{2\text{pt}}(t_1) = \langle N_0 | O | N_0 \rangle + \ldots$$ \hspace{1cm} (10)

for different smearings, illustrates the difficulty in extracting the contribution of the ground state matrix element. The ratio is multiplied by the renormalization factors in Eq. (9) and divided by the ground state mass (cf. Eq. 5) to give $\langle x \rangle_{u-d}^{\overline{\text{MS}}}(2 \text{ GeV})$ in the region of ground state dominance. This ratio is symmetric about $t = t_1/2$.
TABLE I. Overview of ensembles used for this analysis. \(N(n)\) indicates the number of configurations, \(N\), and the number of measurements, \(n\), per configuration of the two- and three-point functions. Statistical noise decreases with decreasing \(t_f\) and for some of the three-point functions we used a smaller number of measurements per configuration as indicated in brackets in the next-to-last column of the table. \(N_{sm}\) refers to the number of iterations used for Wuppertal smearing. Note that the errors of the (finite volume) pion masses combine the statistical uncertainty with an estimate of the variation in the mass arising from the choice of fitting range. For \(\langle x \rangle_{u-d}^{MS}(2\text{GeV})\), the error includes the same sources of uncertainty and in addition the error associated with the renormalization factors.

| ensemble | \(\beta\) | \(a\) [fm] | \(\kappa\) | \(V\) | \(a_{ms}\) [GeV] | \(m_{\pi}\) [GeV] | \(Lm_{\pi}\) | \(N(n)\) | \(N_{sm}\) | \(t_f/a\) | \(\langle x \rangle_{u-d}^{MS}(2\text{GeV})\) |
|----------|----------|------------|----------|-------|----------------|---------------|-------------|-----------|----------|----------|-----------------|
| I        | 5.20     | 0.081      | 0.13596  | 32\(\times\)64 | 0.11516(73)   | 0.2795(18)    | 3.69        | 1986(4)   | 300      | 13       | 0.195(10) |
| II       | 5.29     | 0.071      | 0.13620  | 24\(\times\)48 | 0.15449(74)   | 0.4264(20)    | 3.71        | 1999(2)   | 300      | 15       | 0.230(10) |
| III      | 0.13620  | 32\(\times\)64 | 0.15298(46) | 0.4222(13)   | 4.89        | 1998(2)    | 300      | 15.17     | 15       | 0.215(04) |
| IV       | 0.13632  | 32\(\times\)64 | 0.10675(51) | 0.2946(14)   | 3.42        | 2023(2)    | 400      | 7(1), 9(1), 11(1), 206(09) |
| V        |          |            |          | 40\(\times\)64 | 0.10465(38)   | 0.2888(11)    | 4.19        | 2025(2)   | 400      | 15       | 0.218(08) |
| VI       | 4\(\times\)64 | 0.10487(24) | 0.2895(07) | 6.70        | 1232(2)    | 400      | 15       | 0.196(06) |
| VII      | 0.13640  | 48\(\times\)64 | 0.05786(55) | 0.1597(15)   | 2.77        | 3442(2)    | 400      | 15       | 0.21(00) |
| VIII     | 64\(\times\)64 | 0.05425(49) | 0.1497(13) | 3.49        | 1593(3)    | 400      | 9(1), 12(2), 15 | 0.217(09) |
| IX       | 5.40     | 0.060      | 0.13640  | 32\(\times\)64 | 0.15020(53)   | 0.4897(17)    | 4.81        | 1123(2)   | 400      | 17       | 0.216(07) |
| X        | 0.13647  | 32\(\times\)64 | 0.13073(62) | 0.4262(20)   | 4.18        | 1999(2)    | 450      | 17       | 0.212(06) |
| XI       | 0.13660  | 48\(\times\)64 | 0.07959(27) | 0.2595(09)   | 3.82        | 2177(2)    | 600      | 17       | 0.196(08) |

FIG. 1. Results for the ratio of three- and two-point functions for ensemble IX using different smearing algorithms, with and without APE smoothed links and for different numbers of smearing iterations. The ratio is multiplied by the appropriate renormalization factor to give \(\langle x \rangle_{u-d}^{MS}(2\text{GeV})\) in the limit of ground state dominance. The Wuppertal smeared results have been shifted horizontally for clarity.

and one may mis-identify ground state dominance, e.g., for the results with Jacobi smearing and \(N_{sm} = 75\) if no other data are available. One can see that using APE smoothed links improves the overlap with the ground state and, as might be expected, both Wuppertal and Jacobi smearing give compatible results once the number of smearing iterations is high enough.

If the ground state is not dominant, similar problems arise if only one value of \(t_f\) is available, in particular, since, as mentioned previously, there are terms that cannot be resolved without varying \(t_f\). With the sequential source method the computational expense increases linearly with the number of \(t_f\) values (and similarly if the smearing is varied). However, with the computing resources now generally available, such studies have become possible.

Our strategy was to minimize excited state contributions by optimizing the smearing for the nucleon source/sink operators and to investigate residual contamination using a range of \(t_f\) values. This analysis was performed for ensemble IV \((m_\pi \sim 295\text{MeV})\) with \(t_f/a = 7, 9, 11, 13, 15, 17, \) and ensemble VIII \((m_\pi \sim 150\text{MeV})\) with \(t_f/a = 9, 12, 15, \) where \(15a \sim 1.1\) fm. We perform simultaneous fits to the two- and three-point correlators for all \(t_f\)s using a functional form which includes the first excited state:

\[
C_{2pt}(t_f) = A_0 e^{-m_0 t_f} + A_1 e^{-(m_0+\Delta m) t_f},
\]

\[
C_{3pt}(t, t_f) = A_0 e^{-m_0 t} \left[ B_0 + B_1 \left( e^{-\Delta m(t-t_f)} + e^{-\Delta m t_f} \right) \right],
\]

where \(\Delta m = m_1 - m_0, B_0 = \langle N_0 | O | N_0 \rangle\) and

\[
B_1 = \sqrt{\frac{A_1}{A_0}} \langle N_1 | O | N_0 \rangle, \quad B_2 = \frac{A_1}{A_0} \langle N_1 | O | N_1 \rangle.
\]

FIG. 2 shows a typical fit for ensemble IV. The contributions to \(C_{3pt}\) from excited states are large for \(t_f = 7a\), however, they steadily reduce, so that for \(t_f \geq 11a\) the results are consistent around \(t \sim t_f/2\). For \(t_f \geq 15a\) one can reasonably identify a plateau over several timeslices. The fit reproduces the data well, with reasonable values
of $\chi^2$/d.o.f. < 2, where correlations between timeslices and the different correlators are taken into account. To avoid any possible bias from an ill-determined covariance matrix, all final results are taken from uncorrelated fits. The systematic uncertainty in $\langle x \rangle_{u-d}(2\text{ GeV})$ arising from the choice of fit is estimated by varying the fitting range for both $C_{3pt}(t_i)$ ($t_{\text{min}}$ to $t_{\text{max}}$) and $C_{2pt}(\delta t)$ to $t_i - \delta t$ where $\delta t, t_{\text{min}} \geq 2a$ is allowed. The number of $t_i$s used in the fit was also varied. Note that the fitted value for $\langle x \rangle_{u-d}(2\text{ GeV})$ indicated in Fig. 2 (the green shaded region) only corresponds to a single fit and the errors are purely statistical.

The above approach, which we call “combined” fits, can be compared to the traditional method of fitting the ratio $C_{3pt}(t_i)/C_{2pt}(t_i)$, for fixed $t_i$, to a constant $B_0$, cf. Eq. (10). In Fig. 3 one can see that for $t_i \geq 11a$ the results for the quark momentum fraction are consistent with each other and are also consistent with the results of the combined fits. Figs. 4 and 5 show the corresponding results for ensemble VIII. In particular, Fig. 3 suggests $t_i = 15a$ is sufficient for suppressing excited state contributions and obtaining ground state dominance for $t$ close to $t_i/2$ at the present level of statistical errors. However, this conclusion is only possible through the use of optimized smearing and our extensive analysis.

For completeness we also considered the summation method [22], which has been been advertised in several recent studies [11, 14, 28, 30]. This involves summing the ratio of the three-point and two-point functions over a range of $t$ values:

$$S(t_i) = \sum_{t_i - \delta t} C_{3pt}(t_i)/C_{2pt}(t_i).$$

Using Eqs. (11) and (12) one can show that

$$S(t_i) = B_0 + B_1 e^{-\Delta m (t_i - t)} + e^{-\Delta m t}$$

$$+ B_2 e^{-\Delta m t_i} \left[ 1 + A_1 e^{\Delta m t_i}/A_0 \right]^{-1} + B_3 t_i + C + O(t_i e^{-\Delta m t_i}),$$

where $C$ contains $t_i$-independent terms. Thus, one can extract $B_0$ by performing a linear fit to $S(t_i)$ as a function of $t_i$. A large number of $t_i$ values are required in
order to obtain reliable results with this approach, where the $e^{-\Delta m t_f/2}$ corrections of the traditional ratio method were traded in against $1/t_f$ corrections for the slope. We compare the summation method and the combined fit approach using ensemble IV in Fig. 6. In this case we chose $\delta t = 3a$ (cf. Eq. 14) to minimize the contributions from excited states. Similarly, one can omit the results with the smallest $t_f$ in the fit. However, we found consistent results for $B_0$, with and without the $t_f = 7a$ points. Agreement is also seen with $S(t_f)$ obtained using Eq. 15 and the parameters determined from the combined fit. No advantage was found in using the summation method, in particular, given the need for many $t_f$ values in order to confirm the linear behaviour of $S(t_f)$.

For the other ensembles the three-point functions were computed with only one value $t_f = 15a$, rescaled for different lattice spacings at $\beta = 5.20$ ($t_f = 13a$) and $\beta = 5.40$ ($t_f = 17a$). This is justified by the observation that

$$\frac{|B_2|e^{-\Delta m 1^{-1}fm}}{|B_0|} < 2 \cdot 10^{-3}$$

(17)

on ensembles IV and VIII. We performed combined fits of the $C_{3pt}$ and $C_{2pt}$ using Eqs. 11 and 12 setting $B_2 = 0$. A typical example is shown in Fig. 7. Also included in the figure for comparison is the ratio $C_{3pt}/C_{2pt}$ and the result for a constant fit to this ratio. Reasonable agreement is found between both fitting methods.

As a consistency check we also utilized the value of $B_2$ determined in the fits to ensemble IV:

$$B_2 = \frac{A_1}{A_0} \langle N_1 | O | N_1 \rangle = -m_1 \frac{A_1}{A_0} \tilde{B}_2$$

(18)

where $\tilde{B}_2$ corresponds to $\langle x \rangle_{u-d}$ for the first excited state. Assuming a weak dependence of this on $m_\pi$ (as is the case for the ground state), the $B_2$ term in Eq. 12 can be replaced by the r.h.s. above and $\tilde{B}_2$ can be fixed to the value obtained from ensemble IV. However, $a\Delta m$ typically around 0.5, which means this term is very small at $t_f \geq 13a$, and no noticeable changes occurred in the fit results.
IV. EXTRAPOLATION OF THE ISO-VECTOR GENERALIZED FORM FACTOR $A_{20}$

As a further check we extracted the (iso-vector) generalised form factor, $A_{20}(q^2)$, which in the forward limit is equal to $\langle x \rangle_{u-d}$. The generalised form factor appears in the Lorentz decomposition of the matrix element (in Euclidean notation)

$$\langle N(p_t)|\bar{q}_i \gamma_\mu D_{\mu
u} q|N(p)\rangle = S(\mu, \nu) \left[ \bar{u}(p_t) \left( i\gamma_\mu \slashed{p}_t A_{20}(Q^2) + i\sigma_{\mu\nu} \frac{q_0 \slashed{p}_t}{2m_N} B_{20}(Q^2) \right) + \frac{q_0 q_\nu}{m_N} C_{20}(Q^2) \right] u(p),$$

(19)

where $u(p)$ and $\bar{u}(p_t)$ are fermion and anti-fermion spinors, respectively, and $\slashed{p} = \frac{1}{2} (p_t + p)$, $q = (p_t - p)$ and $Q^2 = -q^2$. The indices $\mu, \nu$ are symmetrized on both sides of the equation (indicated by the curly brackets on the l.h.s. and the symmetrization function $S(\mu, \nu)$ on the r.h.s.) and the traces are subtracted. One constructs combinations of $O_{\mu\nu} = \bar{q}_i \gamma_\mu D_{\mu\nu} q$ which correspond to irreducible representations of the hypercubic group to avoid mixing under renormalization.

The corresponding matrix elements can be extracted using the following ratio of finite momentum three-point and two-point functions in the region of ground state dominance:

$$\frac{C_{3pt}(t, t_i; \vec{p}, \vec{p}_i)}{C_{2pt}(t; \vec{p}_i)} = \frac{C_{2pt}(t; \vec{p}_i) C_{2pt}(t_i; \vec{p}) C_{2pt}(t_i - t; \vec{p})}{C_{2pt}(t; \vec{p}) C_{2pt}(t_i; \vec{p}) C_{2pt}(t_i - t; \vec{p})} \frac{1}{2},$$

(20)

where

$$C_{3pt}(t, t_i; \vec{p}, \vec{p}_i) = \sum_{\vec{x}, \vec{y}} e^{-i\vec{p}\cdot\vec{x}} \langle N(\vec{x}, t_i)|\mathcal{N}(\vec{0}, 0)\rangle$$

(21)

$$C_{3pt}(t, t_i; \vec{p}, \vec{p}_i) = \sum_{\vec{x}, \vec{y}} e^{-i\vec{p}\cdot\vec{x}} e^{i(\vec{p}_i - \vec{p})\cdot\vec{y}} \times \langle N(\vec{x}, t)|O(\vec{y}, t)|\mathcal{N}(\vec{0}, 0)\rangle.$$  

(22)

Further details and results on generalized form factors will be provided in a forthcoming publication, including an analysis of the excited state contributions.

Figure 8 shows $A_{20}^{MS}(Q^2)$ at $\mu = 2$ GeV for ensemble VIII for the three values of $t_f$. In general, excited state contributions depend on the momentum of the correlation functions. We find the dependence on $t_f$ to be similar to that in the forward limit for the lowest three values of $Q^2$. Performing a linear extrapolation in $Q^2$, as suggested, e.g., by leading order chiral perturbation theory, Fig. 8 shows we obtain consistency with our analysis at $Q^2 = 0$. In addition, we found agreement between results obtained using different hypercubic irreducible representations, which are sensitive to different discretization effects, albeit within large statistical uncertainties.

V. RESULTS

Our results for $\langle x \rangle_{u-d}^{MS}$ at the scale $\mu = 2$ GeV for all ensembles are given in Fig. 9 as a function of $m^2$. Strikingly, we see that there is no significant dependence on the quark mass, neither for the range $m_\pi \sim 490 - 289$ MeV and $4.0 \leq L_{m_\pi} \leq 6.0$ (the circles in the figure) nor for $m_\pi \sim 295 - 150$ MeV and $3.4 \leq L_{m_\pi} \leq 4.0$ (crosses). At the three values of $m_\pi$ in the range $422 - 150$ MeV, where we have more than one volume, we also find no sig-
significant finite volume effects. To emphasize this point we show the ratio of the three-point to the two-point functions, Eq. (10), in Figs. 10 and 11 for \( m_\pi \sim 295 \) MeV and \( m_\pi \sim 150 \) MeV, respectively. For the larger pion mass, the three volumes correspond to \( L m_\pi \sim 3.42 \) – 6.70 while at the near physical pion mass \( L m_\pi \) is only varied in the range 2.77 – 3.49. However, at fixed \( L m_\pi \), if the finite volume effects arise from pion exchange, the relative finite volume correction is proportional to \( m_\pi^2 \). Thus, we do not expect the emergence of significant finite volume effects for larger volumes for \( m_\pi \sim 150 \) MeV. In terms of lattice spacing effects, our analysis is not conclusive. Although no significant effects are seen, we have leading \( O(a) \) discretization errors and the lattice spacing is only varied in a very limited range \( a \sim 0.08 \) – 0.06 fm.

\[
\begin{align*}
\text{FIG. 10.} & \quad \text{The ratio of the three-point to two-point functions for ensembles IV, V and VI, which have the same } \beta \text{ and } \kappa \text{ values but have different spatial extents } L. \ m_\pi \sim 290 \text{ MeV for } L = 64a. \\
\end{align*}
\]

\[
\begin{align*}
\text{FIG. 11.} & \quad \text{The same as in Fig. 10 for ensembles VII and VIII. } m_\pi \sim 150 \text{ MeV for } L = 64a. \\
\end{align*}
\]

In Ref. [1] we compared our results for ensemble VII, \( m_\pi \sim 160 \) MeV (with lower statistics) with earlier, much larger, \( N_f = 2 \) results, also shown in Fig. 9 which used similar analyses and (non-perturbative) renormalization but different smearing. At that time, this suggested a strong dependence on \( m_\pi \) as one approaches the chiral limit. However, from our present analysis including larger pion masses, optimized smearing and excited state fits throughout, we conclude the observed difference is probably due to excited state contamination (cf. Fig. 1). Nonetheless there remains a \( \sim 25\% \) discrepancy with the values obtained from phenomenological fits to the experimental data [34,37].

In Fig. 12 a comparison is made with recent determinations employing \( N_f = 2 + 1 \) dynamical fermions (LHPC [11] using tree-level improved clover fermions with 2-HEX link smearing and RBC/UKQCD [39] with domain wall fermions) and with \( N_f = 2 + 1 + 1 \) simulations (ETMC [40,41] using twisted mass fermions). All collaborations use non-perturbative renormalization and the unimproved lattice operator. Overall, within the larger errors of these collaborations, consistency can be seen with our results (one high statistics ETMC point being the only exception). Higher precision is needed to resolve any effects of including strange quarks in the sea or to uncover discretization effects. For instance, LHPC [11] reports agreement with the phenomenological value at almost physical pion masses using, predominantly, coarse \( a \sim 0.12 \) fm ensembles, however, within quite large errors, see Fig. 12.

\[1\] Note that these fits are only performed at NNLO while we converted the lattice results from the RI'MOM to the \( \overline{\text{MS}} \) scheme at three-loop order. We also evolved the scale to this order. However, differences between running the scale to 2 GeV at two-, three- or four-loops [35] are only on the few per mille level.
VI. CONCLUSIONS

In this article we presented high statistics results for the iso-vector quark momentum fraction, $\langle x \rangle_{u-d}$, with pion masses down to 150 MeV and with volumes up to $|ln V| > 6$. This quantity is sensitive to excited state contributions and through the use of optimized smearing, multiple sink time positions and excited state fits we were able to extract ground state signals unambiguously. No significant dependence was observed on the quark mass within the range, 150 MeV < $m_q$ < 490 MeV, nor on the lattice volume, even close to the physical point. The consistency found with other recent determinations suggests that strange sea quarks do not play an important role for this valence quantity. The remaining discrepancy between our result $\langle x \rangle_{MS}^{SM}(2\text{GeV}) = 0.217(9)$ obtained at $m_\pi \sim 150$ MeV and $a \sim 0.071$ fm and phenomenological values $0.15-0.17$ may be due to discretization effects. At present, we do not have these under control with $O(a)$ leading lattice spacing effects and only a small variation in $a$. We remark that all lattice studies have leading order $a$ discretization effects for this quantity and lattice spacings $a \gtrsim 0.06$ fm in common. We are also in the process of revisiting the determination of the non-perturbative renormalization factor since this has been computed using similar methods in all recent lattice investigations. In the future we plan to realize $a < 0.06$ fm, simulating $N_f = 2 + 1$ sea quarks with open boundaries [42].

ACKNOWLEDGMENTS

The ensembles were generated primarily on the QPACE computer [33, 34], which was built as part of the Deutsche Forschungs Gemeinschaft SFB/TRR 55 project. The analyses were performed on the iDataCool cluster in Regensburg and the superMUC system at the Leibniz Supercomputing Center in Munich. Additional support was provided by the EU: ITN STRONGnet and FP7-PEOPLE-2009-IRG, No.256594. The BQCD [15] and CHROMA [16] software packages were used extensively along with the locally deflated domain decomposition solver implementation of openQCD [17].

[1] G. S. Bali, S. Collins, M. Deka, B. Glässlé, M. Göckeler, J. Najjar, A. Nobile, D. Pleiter, A. Schäfer, and A. Sternebeck, Phys. Rev. D86, 054504 (2012) [arXiv:1207.1110 [hep-lat]]
[2] D. Boer et al., (2011), arXiv:1108.1713 [nucl-th]
[3] S. Dürr et al., Science 322, 1224 (2008) [arXiv:0906.3599 [hep-lat]]
[4] Z. Fodor and C. Hoelbling, Rev. Mod. Phys. 84, 449 (2012) [arXiv:1203.4789 [hep-lat]]
[5] S. Syritsyn, Proc. Science LATTICE2013, 009 (2014), arXiv:1403.4666 [hep-lat]
[6] N. Brambilla et al., (2014), arXiv:1404.3723 [hep-ph]
[7] C. Alexandrou, [AIP Conf. Proc. 1560, 3 (2013)]
[8] S. Dinter, C. Alexandrou, M. Constantiniou, V. Drach, K. Jansen, and D. B. Renner, Phys. Lett. B704, 89 (2011) [arXiv:1108.1076 [hep-lat]]
[9] B. J. Owen, J. Dragos, W. Kamleh, D. B. Leinweber, M. S. Mahbub, B. J. Menadue, and J. Zanotti, Phys. Lett. B723, 217 (2013) [arXiv:1212.4666 [hep-lat]]
[10] S. Capitani, M. Della Morte, G. von Hippel, B. Jäger, A. Jüttner, B. Knippischl, H. Meyer, and H. Wittig, Phys. Rev. D86, 074502 (2012) [arXiv:1205.0180 [hep-lat]]
[11] J. Green, M. Engelhardt, J. Krieg, J. Negele, A. Pochinsky, and S. Syritsyn, Phys. Lett. B734, 290 (2014) [arXiv:1209.1687 [hep-lat]]
[12] T. Bhattacharya, S. D. Cohen, R. Gupta, A. Joseph, H.-W. Lin, and B. Yoon, Phys. Rev. D89, 094502 (2014) [arXiv:1306.5435 [hep-lat]]
[13] G. Bali, S. Collins, B. Glässlé, M. Göckeler, J. Najjar, R. Rödl, A. a Schäfer, R. Schiel, A. Sternebeck, and W. Söldner, Proc. Science LATTICE2013, 290 (2013), arXiv:1311.7041 [hep-lat]
[14] T. Doi, M. Deka, S.-J. Dong, T. Draper, K.-F. Liu, D. Mankame, N. Mathur, and T. Streuer, Phys. Rev. D80, 094503 (2009) [arXiv:0903.3232 [hep-ph]]
[15] G. S. Bali et al. (QCDSF Collaboration), Phys. Rev. Lett. 108, 222001 (2012) [arXiv:1112.3354 [hep-lat]]
[16] G. S. Bali et al. (QCDSF Collaboration), Phys. Rev. D85, 054502 (2012) [arXiv:1111.1600 [hep-lat]]
[17] A. Abdel-Rehim, C. Alexandrou, M. Constantiniou, V. Drach, K. Hadjiyiannakou, K. Jansen, G. Koutsou, and A. Vaquero, Phys. Rev. D89, 034501 (2014) [arXiv:1310.6339 [hep-lat]]
[18] S. Gürken, U. Löw, K. Mütter, R. Sommer, A. Patel, and K. Schilling, Phys. Lett. B227, 266 (1989)
[19] S. Gürken, Nucl. Phys. B Proc. Suppl. 17, 361 (1990)
[20] M. Falcioni, M. Paciello, G. Parisi, and B. Taglienti, Nucl. Phys. B251, 624 (1985)
[21] C. Allton et al. (UKQCD Collaboration), Phys. Rev. D47, 5128 (1993) [arXiv:hep-lat/9303009 [hep-lat]]
[22] L. Maianti, G. Martineilli, M. Paciello, and B. Taglienti, Nucl. Phys. B293, 420 (1987)
[23] G. S. Bali, S. Collins, B. Glässlé, M. Göckeler, J. Najjar, R. Rödl, A. Schäfer, A. Sternbeck, and W. Söldner, Proc. Science LATTICE2013, 271 (2013), arXiv:1311.1718 [hep-lat]
[24] C. Alexandrou, S. Dinter, V. Drach, K. Hadjiyiannakou, K. Jansen, and D. B. Renner (ETM collaboration), Eur. Phys. J. C74, 2602 (2014) arXiv:1302.2608 [hep-lat]
[25] M. Göckeler et al., Phys. Rev. D82, 114511 (2010), arXiv:1003.5756 [hep-lat]
[26] M. Constantiniou, M. Costa, M. Göckeler, R. Horsley, H. Panagopoulos, H. Perl, P. Rakow, G. Schierholz, and A. Schiller, Phys. Rev. D87, 096019 (2013) [arXiv:1303.6776 [hep-lat]
[27] S. Capitani, M. Göckeler, R. Horsley, H. Perl, P. E. Rakow, G. Schierholz, and A. Schiller, Nucl. Phys. B593, 183 (2001) [arXiv:hep-lat/0007004 [hep-lat]]
[28] M. Deka et al., (2013), arXiv:1312.4816 [hep-lat]
[29] S. Capitani, B. Knippschild, M. Della Morte, and H. Wittig, Proc. Science LATTICE2010, 147 (2010), arXiv:1011.1358 [hep-lat].

[30] J. Bulava, M. Donnellan, and R. Sommer (ALPHA Collaboration), Proc. Science LATTICE2010, 303 (2010), arXiv:1011.4393 [hep-lat].

[31] D. Pleiter et al. (QCDSF/UKQCD Collaboration), Proc. Science LATTICE2010, 153 (2010), arXiv:1101.2326 [hep-lat].

[32] A. Sternbeck et al., Proc. Science LATTICE2011, 177 (2011), arXiv:1203.6579 [hep-lat].

[33] C. Alexandrou, J. Carbonell, M. Constantinou, P. Harraud, P. Guichon, K. Jansen, C. Kallidonis, T. Korzec, and M. Papinutto, Phys. Rev. D83, 114513 (2011), arXiv:1104.1600 [hep-lat].

[34] J. Gao, M. Guzzi, J. Huston, H.-L. Lai, Z. Li, P. Nadolski, J. Pumpkin, D. Stump, and C.-P. Yuan, Phys. Rev. D89, 054028 (2014), arXiv:1302.6246 [hep-ph].

[35] S. Alekhin, J. Blümlein, and S. Moch, Phys. Rev. D89, 033009 (2014), arXiv:1302.6246 [hep-ph].

[36] R. D. Ball et al., Nucl. Phys. B867, 244 (2013), arXiv:1207.1303 [hep-ph].

[37] A. Martin, W. Stirling, R. Thorne, and G. Watt, Eur. Phys. J. C64, 653 (2009), arXiv:0905.3331 [hep-ph].

[38] V. Velizhanin, Nucl. Phys. B860, 288 (2012), arXiv:1112.3954 [hep-ph].

[39] Y. Aoki, T. Blum, H.-W. Lin, S. Ohta, S. Sasaki, R. Tweedle, T. Yamazaki, and J. Zanotti, Phys. Rev. D82, 014501 (2010), arXiv:1003.3387 [hep-lat].

[40] C. Alexandrou, M. Constantinou, V. Drach, K. Jansen, C. Kallidonis, and G. Koutsou, Proc. Science LATTICE2013, 292 (2013), arXiv:1312.2874 [hep-lat].

[41] C. Alexandrou, M. Constantinou, S. Dinter, V. Drach, K. Jansen, C. Kallidonis, and G. Koutsou, Phys. Rev. D88, 014509 (2013), arXiv:1303.5979 [hep-lat].

[42] M. Lüscher and S. Schaefer, Comput. Phys. Commun. 184, 519 (2013), arXiv:1206.2809 [hep-lat].

[43] H. Baier et al., Proc. Science LAT2009, 001 (2009), arXiv:0911.2174 [hep-lat].

[44] Y. Nakamura, A. Nobile, D. Pleiter, H. Simma, T. Streuer, T. Wettig, and F. Winter, (2011), arXiv:1103.1363 [hep-lat].

[45] Y. Nakamura and H. Stiben, Proc. Science LATTICE2010, 040 (2010), arXiv:1011.0199 [hep-lat].

[46] R. G. Edwards and B. Joó (SciDAC, LHPC, UKQCD), Nucl. Phys. Proc. Suppl. 140, 832 (2005), arXiv:hep-lat/0409003 [hep-lat].

[47] http://luscher.web.cern.ch/luscher/openQCD,