EINSTEIN-MAXWELL FIELDS GENERATED FROM THE $\gamma$-METRIC AND THEIR LIMITS

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Abstract

Two solutions of the coupled Einstein-Maxwell field equations are found by means of the Horský-Mitskievitch generating conjecture. The vacuum limit of those obtained classes of spacetimes is the seed $\gamma$-metric and each of the generated solutions is connected with one Killing vector of the seed spacetime. Some of the limiting cases of our solutions are identified with already known metrics, the relations among various limits are illustrated through a limiting diagram. We also verify our calculation through the Ernst potentials. The existence of circular geodesics is briefly discussed in the Appendix.

1 Introduction

Several constructing techniques has been proposed that suggest how to generate Einstein-Maxwell (E-M) fields from pure gravitational ones (see e.g. [1]). Hereafter in this paper we use an alternative method based on the Horský-Mitskievitch (H-M) conjecture [2] which prescribes quite close connection between isometries of vacuum spacetimes (seed metrics) and an electromagnetic four-potential of generated E-M fields. Taking the $\gamma$-metric as a seed vacuum spacetime, we obtain two classes of
E-M fields, each of which corresponds to one Killing vector of the seed metric. Main properties of those solutions are summarized too.

The paper is organized in the following way: we start resuming the basic characteristic of the $\gamma$-metric, then we gradually come to new classes of E-M fields that further generalize the results presented in [3]. We demonstrate that some already known E-M fields can be regarded as special cases of these solutions. The most important limits and special cases of the generated E-M fields are listed in section 4, the relations among them are illustrated through a limiting diagram. At the same time we touch on problems connected with the physical interpretation of obtained solution, with the existence of curvature singularities and circular geodesics.

2 The seed $\gamma$-metric

The $\gamma$-metric is a vacuum solution of Einstein equations discovered by Darmois in 1927 and reinvestigated by various authors many times since (see references cited in [4] for more details). Known also as Darmois-Voorhees-Zipoy metric, it represents an interesting class of static axially symmetric spacetimes. Therefore it can be expressed in the Weyl-Lewis-Papapetrou cylindrical coordinates in the form (see e.g. [4])

$$\text{d}s^2 = -e^{2\mu}\text{d}t^2 + e^{-2\nu}[e^{2\nu}(\text{d}r^2 + \text{d}z^2) + r^2d\varphi^2], \tag{1}$$

where $\mu = \mu(r, z)$, $\nu = \nu(r, z)$ are functions of $r$ and $z$ only. Particularly, the $\gamma$-metric is the line-element (1) with

$$e^{2\mu} = \left(\frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m}\right)^{\gamma} = f_1(r, z),$$

$$e^{2\nu} = \left[\frac{(R_1 + R_2 - 2m)(R_1 + R_2 + 2m)}{4R_1R_2}\right]^{\gamma^2} = f_2(r, z), \tag{2}$$

$$R_1 = \sqrt{r^2 + (z - m)^2}, \quad R_2 = \sqrt{r^2 + (z + m)^2}.$$ 

It is well known that the function $\mu(r, z)$ satisfies flat-space Laplace’s equation $\Delta \mu = 0$ for any Weyl metric [4] and that it is possible to construct an infinite number axially symmetric solutions of Einstein’s equations using various realistic Newtonian potentials identified with $\mu(r, z)$ [3]. Particularly, the $\gamma$-metric can be generated by the Newtonian potential of a linear mass source (“rod”) with linear density $\gamma/2$ and length $2m$ [3]

$$\phi = \mu(r, z) = \frac{1}{2}\ln|g_{tt}| = \frac{\gamma}{2}\ln\left(\frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m}\right).$$

This fact is the main supporting argument for the “standard” interpretation of the $\gamma$-metric as an exterior field of a finite linear source located along the z-axis symmetrically with respect to the origin $z = 0$. However, though the function
µ(r, z) may give us a guide to the physical meaning of the exact solution, this correspondence has to be used with caution; the Darmois-Voorhees-Zipoy solution can be also explained as a gravitational field of counter-rotating relativistic discs or a field of an oblate (γ > 1) or prolate (γ < 1) spheroid. Geodesic motion in Darmois-Voorhees-Zipoy spacetime and its difference from the case of spherical symmetry has been analyzed by Herrera, Paiva and Santos.

The γ-metric has an interesting singularity structure: it has a directional singularity for γ > 2, but not for γ < 2. For a distant observer at infinity such gravitational field behaves as an isolated body with monopole and higher mass moments which can be – at least principally – measured by gyroscope experiments. A superposition of two or more γ-solutions was studied by Letelier and Oliveira.

Moreover, if

\[ \lim_{r \to 0} \nu(r, z) \neq 0, \]

then any Weyl metric includes a conical singularity representing stresses on the z-axis (see e.g. [4, 6]).

The case γ = 1 corresponds to the exterior Schwarzschild metric outside the horizon. It can be more easily demonstrated when one uses so-called Erez-Rosen spherical coordinates θ, φ introduced by the transformation

\[ r^2 = (\rho^2 - 2m\rho) \sin^2 \theta, \quad z = (\rho - m) \cos \theta. \]

Naturally, our results concerning the γ-metric must be consistent with this thoroughly studied Schwarzschild limit.

Recently, Herrera et al. showed that the limit m → ∞ locally gives the Levi-Civita (L-C) spacetime

\[ ds^2 = -r^{4\sigma} dt^2 + r^{4\sigma}(2\sigma - 1) (dr^2 + dz^2) + C^{-2} r^{-4\sigma} d\phi^2, \]

i.e. another Weyl metric with constant parameters σ, C (for the relation between γ and σ in this limit see section 3). We have already shown that every Killing vector of the L-C spacetime is connected with a particular class of E-M fields and that those fields can be generated through the H-M conjecture. The existence of the solutions described in section 3 proves that for two Killing vectors the answer is positive.

The H-M generating conjecture, which we apply to the γ-metric, seems to outline an efficient and fruitful way, how to obtain solutions of E-M equations as a generalization of some already known vacuum seed metrics, though no general proof has been given so far and that we do not know exact limits of its applicability (see [12] and references therein). In this paper we do not intend to tackle general theoretical problems, we would like to concentrate on the application of the conjecture to our particular case instead.

In [3] we have proposed a simplified ad hoc empirical algorithmic scheme, by means of which six classes of E-M fields were obtained; each of those classes corresponds to another Killing vector of the seed L-C solution. The fact that the
L-C solution represents a limiting case of the $\gamma$-metric provides us with the motivation to apply this algorithm also in the case of Darmois-Voorhees-Zipoy vacuum solution.

In the following section the H-M conjecture is applied to the seed $\gamma$-metric in the same way we have applied it to the seed L-C metric [3]. Therefore we do not concentrate on the reexplanation of the generation procedure, we rather describe basic characteristics of the obtained solutions and then we study their limiting cases (section 4). For our further calculations it is more convenient to rewrite the line-element (1) in the form

$$d s^2 = -f_1(r, z) \, dt^2 + f_1(r, z)^{-1} \left[ f_2(r, z) \left( dr^2 + dz^2 \right) + r^2 \, d\varphi^2 \right],$$

where the functions $f_1(r, z), f_2(r, z)$ are defined by (2).

### 3 The $\gamma$-metric with an electromagnetic field

#### 3.1 The $\gamma$-metric with an electric field

Let us first employ the timelike Killing vector $\partial_t$. Following [3], we modify the line-element (3) into the form

$$d s^2 = -f_1(r, z) \frac{f(r, z)}{f_1(r, z)} \, dt^2 + \frac{f(r, z)^2}{f_1(r, z)} \left[ f_2(r, z) \left( dr^2 + dz^2 \right) + r^2 \, d\varphi^2 \right],$$

and set the four-potential

$$A = q \frac{f_1(r, z)}{f(r, z)} \, dt.$$  

It can be verified through standard calculations that sourceless E-M equations are fulfilled if

$$f(r, z) = 1 - q^2 f_1(r, z).$$

Generated E-M field is of an electric type, because of the non-positive electromagnetic invariant

$$F_{\alpha\beta} F^{\alpha\beta} =$$

$$= - \frac{32 q^2 m^2 \gamma^2 f_1(r, z)^2 \left\{ r^2 (R_1 + R_2)^2 + [z (R_1 + R_2) + m (R_1 - R_2)]^2 \right\}}{f(r, z)^4 f_2(r, z) R_1^2 R_2^2 (R_1 + R_2 - 2m)^2 (R_1 + R_2 + 2m)^2} \leq 0.$$

This can be also checked when we calculate the non-zero components of the electric field strength $E^1 = F^{01}$ and $E^3 = F^{03}$. The analytic expressions for the Kretschmann scalar as well as for the Weyl scalars are too lengthy to reproduce them here in full extend. The spacetime (3) generally belongs to the Petrov type $I$, with the trivial exception $\gamma = 0$ when it belongs to the Petrov type $0$ and the electromagnetic field disappears.
Figure 1: Newtonian potential $\phi = \phi(r, z)$ for the “charged” $\gamma$-metric solutions with $m = 2, \gamma = 0.5$: a) the solution with an electric field, $q = 1$; b) the solution with an electric field, $q = 2$; c) the solution with a magnetic field, $q = 1$.

The spacetime (7) has two curvature singularities. The attractive one is inherited from the seed metric (1) and its location coincides with the Newtonian source—a massive rod situated along the $z$-axis symmetrically with respect to the origin. The Kretschmann scalar also blows up to infinity at “points”, the coordinates of which satisfy the transcendent equation

$$f(r, z) = 1 - q^2 f_1(r, z) = 1 - q^2 \left( \frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m} \right)^\gamma = 0. \quad (9)$$

The presence and the “shape” of this latter repulsive curvature singularity can be also illustrated by means of the corresponding generating Newtonian potential extracted from the $g_{tt}$ component of the metric tensor
\begin{equation}
\phi = \frac{1}{2} \ln |g_{tt}| = \frac{1}{2} \ln f_1 (r, z) - \frac{1}{2} \ln \left[ 1 - q^2 f_1 (r, z) \right]^2,
\end{equation}

the dependence of which on the coordinates \( r, z \) together with the equipotential curves are illustrated in figures 1(a), (b). While for some set of the parameters the condition (9) cannot hold and the spacetime has only one rod-like singularity at \( z \)-axis as in figure 1(a), in other cases the spacetime is endowed with a singularity surrounding the linear Newtonian source and acting as a repulsing potential barrier in figure 1(b). At least part of the electric field could be intuitively connected with the charge distribution along the finite rod, but the physical character of the second singularity present in this solution is not clear. It might seem that the existence of such repulsive singularity excludes any “reasonable” physical interpretation of our solution. On the other hand, as it is shown in section 3, the metric (4) includes the Reissner-Nordström solution as its limit. Therefore, the Weyl-Lewis-Papapetrou coordinates are probably not the best suitable coordinates for the study of this “strange” singularity. The existence of circular geodesics is studied in the Appendix.

### 3.2 The \( \gamma \)-metric with a magnetic field

Next, turn our attention to the Killing vector \( \partial_\phi \). In accordance with (3) we assume the line-element

\begin{equation}
\begin{split}
ds^2 &= - f (r, z)^2 f_1 (r, z) \, dt^2 + \\
&\quad + \frac{1}{f_1 (r, z)} \left[ f (r, z)^2 f_2 (r, z) \left( dr^2 + dz^2 \right) + \frac{r^2}{f (r, z)^2} \, d\phi^2 \right]
\end{split}
\end{equation}

and four-potential

\begin{equation}
A = q \frac{r^2}{f (r, z) f_1 (r, z)} \, d\phi.
\end{equation}

One can again verify the validity of sourceless E-M equations and the condition of traceless Einstein tensor. Gradually, we get the result

\begin{equation}
f (r, z) = 1 + q^2 r^2 / f_1 (r, z).
\end{equation}

The obtained electromagnetic field is of magnetic type because

\begin{equation}
F_{\alpha\beta} F^{\alpha\beta} = \frac{32 q^2}{f (r, z)^4 f_2 (r, z) R_1^2 R_2^2 (R_1 + R_2 - 2m)^2 (R_1 + R_2 + 2m)^2} \times \\
\times \left\{ \left[ R_1 R_2 (r^2 + z^2 - m^2) - m^2 r^2 (R_1 + R_2) \right]^2 + \\
+ r^2 m^2 \gamma^2 \left[ R_2 (z - m) + R_1 (z + m) \right]^2 \right\} \geq 0;
\end{equation}

both longitudinal and radial components of the magnetic field strength \( B^3 = F^{12}, \quad B^1 = F^{23} \) are nonzero. The Kretschmann and Weyl scalars are again too lengthy.
to reproduce them here. Generated E-M field generally belongs to the Petrov type I besides the trivial exception $\gamma = 0$ in which case we get the Petrov class $D$.

The solution (10) can be evidently written in the Weyl form (1) and thus it can be considered as a spacetime generated by the Newtonian gravitational potential

$$\phi = \frac{1}{2} \ln |g_{tt}| = \frac{1}{2} \ln f_1(r, z) + \ln \left[ 1 + \frac{q^2 r^2}{f_1(r, z)} \right];$$

the dependence of $\phi$ on Weyl coordinates $r$, $z$ is drawn in figure 1(c), from which the presence of rod-like curvature singularity along $z$-axis can be deduced. This curvature singularity is, of course, inherited from the seed $\gamma$-metric.

### 3.3 Ernst potentials and dual solutions

The coupled E-M filed equations for any axially symmetric stationary metric

$$ds^2 = -F(r, z) \left( dt - \omega_j dx^j \right)^2 + F(r, z)^{-1} \left[ e^{2\nu} (dr^2 + dz^2) + r^2 d\varphi^2 \right],$$

are often formulated in terms of two complex functions, so called Ernst potentials $\Phi$, $\mathcal{E}$ (an index $j = 1, 2, 3$ runs over space-like coordinates). As both solutions (7) and (10) are static (i.e. $\omega_j = 0 \forall j$), in their case the Ernst potentials take a simplified form

$$\Phi = A_t + i A_\varphi'$$

$$\mathcal{E} = F(r, z) - |\Phi|^2$$

where $A_\varphi'$ depends on the vector potential components $A_t, A_\varphi$ via equation

$$\frac{1}{r} F(r, z) A_\varphi = \mathbf{e}_\varphi \times A_\varphi'. \tag{13}$$

Hereafter in this section the differential operators are to be understood as three dimensional operators expressed in cylindrical coordinates in the Euclidean space, $\mathbf{e}_\varphi$ is a unit azimuthal vector, $i = \sqrt{-1}$. The E-M equations then turn into a pair of complex equations $\tag{14}$

$$\left( \text{Re} \, \mathcal{E} + |\Phi|^2 \right) \Delta \mathcal{E} = (\nabla \mathcal{E} + 2 \Phi^* \nabla \Phi) \cdot \nabla \mathcal{E},$$

$$\left( \text{Re} \, \mathcal{E} + |\Phi|^2 \right) \Delta \Phi = (\nabla \mathcal{E} + 2 \Phi^* \nabla \Phi) \cdot \nabla \Phi. \tag{14}$$

It is not difficult to verify, that the Ernst equations (14) are fulfilled for the “charged” $\gamma$-metrics (7) and (10) when one sets

$$\Phi = A_t = \frac{q f_1(r, z)}{1 - q^2 f_1(r, z)},$$

$$\mathcal{E} = \frac{f_1(r, z)}{|1 - q^2 f_1(r, z)|^2} - |\Phi|^2 = \frac{1}{q} \Phi. \tag{15}$$
for the $\gamma$-metric with an electric field (7) and

$$\Phi = iA'_\varphi, \quad A'_\varphi = 2qz + q\gamma (R_1 - R_2),$$

$$E = f_1 (r, z) \left[ 1 + \frac{q^2 r^2}{f_1 (r, z)} \right]^2 - |\Phi|^2.$$  \hfill (16)

for the $\gamma$-metric with a magnetic field (10). While in the former case $\Phi$ is straightforwardly determined by the vector potential (8), in the latter case one has to solve a pair of coupled partial differential equations to get $A'_\varphi$; substituting (11) into (13) these equations read as

$$\frac{\partial A'_\varphi}{\partial r} = qr \frac{\partial}{\partial z} [\ln f_1 (r, z)], \quad \frac{\partial A'_\varphi}{\partial z} = 2q - qr \frac{\partial}{\partial r} [\ln f_1 (r, z)].$$

Formulating our problem through Ernst potentials $E$ and $\Phi$ we can easily found all dual solutions, as the duality rotation is represented by the transformation $\Phi \rightarrow \Phi \exp(i\alpha)$, where $\alpha$ is a real constant [14]. Thus, according to (16) our “magnetic” solution (10) has a dual electric counterpart the vector potential of which has a non-zero time-like component $A_t = A'_\varphi = 2qz + q\gamma (R_1 - R_2)$. Using (15), a magnetic solution dual to the metric (7) then requires $A'_\varphi = A_t = qf_1 (r, z) / \left[ 1 - q^2 f_1 (r, z) \right]$. Solving a set of equations (13) for $A_\varphi$ we obtain a vector potential with the only non-zero azimuthal component $A_\varphi = q\gamma (R_1 - R_2)$. In this sense, our distinguishing of “electric” and “magnetic” solutions of E-M equations is certainly rather conventional: it always depends on the choice of the vector potential.

4 Limiting cases

4.1 Vacuum solutions

In the preceding section we have found two classes of E-M fields. It is well known that some vacuum Weyl metrics (e.g. the Curzon metric) represent limiting cases of the Darmois-Voorhees-Zipoy solution (2). Consequently, we can explore limiting cases of the spacetimes (7) and (10). In their inspiring paper [5] Herrera et al. presented a fruitful point of view that enables us to systematize the particular cases efficiently. The unifying idea is based on the comparison of the corresponding generating Newtonian potentials of considered Weyl metrics.

The relations among the seed vacuum spacetimes are illustrated in figure 2. All these vacuum spacetimes represent axisymmetric solutions of Einstein’s equations or they can be brought into the Weyl form (1) via a suitable transformation. In the limiting diagram 2 there are two qualitatively different types of limits. The first one represents a usual limit, when we set a particular value to some metric parameter (or parameters); as an example we can take the Minkowski limit of the $\gamma$-metric (6) for $\gamma = 0$. The second type of limits (denoted by the abbreviation “loc.” in the diagram) is achieved through the Cartan scalars [5]. Thus, such limit does not treat
global properties such as topological defects. To be concrete, setting $\sigma = 0$ in the line-element of the L-C metric, we come to the Minkowski metric in cylindrical coordinates, but the conical singularity described by the parameter $C$ survives. Therefore, the obtained limit globally differs from the Minkowski spacetime.

The choice of a coordinates in the limiting diagram follows Herrera et al. [5], i.e. the Weyl-Lewis-Papapetrou cylindrical coordinates of the L-C solution are introduced. The coordinates in which the $\gamma$-metric takes the form are just rescaled L-C coordinates with scaling ratios explicitly given in [5]. Working with the L-C cylindrical coordinates, it seems most convenient to express the parameters of other spacetimes through the parameter $\sigma$ of the L-C solution: the relations given in the lower part of figure are also derived in [5].
While relativistic limits sometimes require rather tedious calculations, the relations among considered spacetimes become quite transparent when we think about their generating Newtonian potentials. Thus, the relations among the limits of the $\gamma$-metric provide a supporting argument for the interpretation of the studied spacetimes according to the corresponding Newtonian gravitational fields though e.g. in the case of L-C solution (2), such interpretation is acceptable only for a limited range of the linear density parameter $\sigma$ [15].

In that sense, starting from the $\gamma$-metric (6), the Newtonian image of which is a field of a finite rod laid along the $z$-axis, and prolonging the rod at both ends to $-\infty$ and $+\infty$, i.e. mathematically in the limit $m \to \infty$, we get a Newtonian field of an infinite line-mass, the Newtonian analogy of the L-C solution (5). If, on the contrary, one contracts the length of the rod to zero at the same time keeping its mass $M = 2\sigma m = \gamma m$ finite we obtain the Curzon metric, another Weyl metric (1) with functions $\mu$ and $\nu$ in the form [4]

$$\mu(r, z) = -\frac{M}{R}, \quad \nu(r, z) = -\frac{M^2 r^2}{2 R^4}, \quad R = \sqrt{r^2 + z^2}. \quad (17)$$

Its generating Newtonian potential $\phi \equiv \mu(r, z)$ describes the gravitational field of a spherical particle. The Curzon metric is again endowed with a directional singularity and for a distant observer it looks like a gravitational fields of a point particle with multipoles on it [4] (let us remind that this vacuum solution can also arise as a gravitational field of counter-rotating relativistic discs [6, 7]). Another limit of the $\gamma$-metric for $\gamma = 1$ is the well-known Schwarzschild spacetime, as it has been already mentioned in section 2. The $\gamma$-metric, the Curzon metric as well as the Schwarzschild solution reduce to the Minkowski spacetime when the mass of their source falls down to zero.

Let us concentrate on the upper part of the diagram which includes the seed vacuum solutions studied thoroughly by Bonnor who has also found most of the corresponding coordinate transformations [4, 15, 16]. An infinite linear source, the Newtonian image of the vacuum L-C solution (5), can be considered as a “composition” of two semi-infinite linear sources, each of which generates the Weyl metric

$$ds^2 = -X^{2\sigma} dt^2 + X^{-2\sigma} \left[ \frac{X}{2R} (dr^2 + dz^2) + r^2 d\varphi^2 \right], \quad (18)$$

The source of the generating Newtonian potential, a semi-infinite line-mass with linear density $\sigma$ is again located along the $z$-axis either from $z_1$ to $\infty$ ($\epsilon = -1$) or from $-\infty$ to $z_1$ ($\epsilon = 1$). As it has been pointed by Bonnor [1, 3], the solution (18) is isometric with a solution for a non-uniform infinite plane

$$ds^2 = -Z^{4\sigma} dt^2 + \rho^2 Z^{2-4\sigma} d\varphi^2 + Z^{8\sigma-4}\left( Z^2 + \rho^2 \right)^{1-4\sigma^2} (d\rho^2 + dZ^2); \quad (19)$$

(10)
the corresponding coordinate transformation \([4, 15]\) maps \((18)\) into the half-space \(Z > 0\). In the particular case \(\sigma = -1/2\) all the solutions \((5), (18), (19)\) are isometric with the Taub metric

\[
ds^2 = \frac{1}{\sqrt{\xi}} \left( -d\tau^2 + d\xi^2 \right) + \xi \left( dx^2 + dy^2 \right),
\]

the general plane symmetric solution \([4, 16]\). Moreover, the metric \((20)\) can be further transformed into \([16]\).

\[
ds^2 = -\frac{2\alpha}{p} d\eta^2 - 2 dp d\eta + p^2 \left( dx^2 + dy^2 \right), \quad \alpha = \text{const.},
\]

a particular case of radiative vacuum solutions discovered by Robinson and Trautman in 1962. The metric \((21)\) is usually interpreted as a gravitational field of a particle on a null line (properties of Robinson-Trautman’s solutions are summarized e.g. in \([3]\) with rather skeptic conclusion about the cosmological and astrophysical relevance of these solutions). This ambiguity illustrates the difficulty in physical interpretation of the seed spacetimes listed in figure 2 and consequently, the problems with the interpretation of our generated E-M fields.

The lower part of the figure 2 has been described by Herrera at al. \([5]\). We do not intend to concentrate on the exploration of these limits and coordinate transformations. From our point of view all the vacuum spacetimes in the limiting diagram are interesting from another reason: we have managed to apply the generalized H-M conjecture to all these seed metrics successfully and so we have obtained several classes of E-M fields. The existence of the limits and coordinate transformations \textit{ex post} justifies the application of the H-M conjecture to all these vacuum spacetimes according to the algorithmic scheme formulated in \([3]\). Some of the obtained limiting E-M fields have been found before by other authors while the others are new at least in Weyl cylindrical coordinates. The limiting diagram enables us to systematize our earlier results because some known E-M fields may be considered as special cases of the metrics \((7)\) or \((10)\).

The rest of this section is devoted to a brief survey of E-M fields generated from the seed metrics in the limiting diagram. As the method through which the E-M solutions were found was described in \([3]\) and demonstrated in section 3, we just list the line-elements \(ds^2\), vector potentials \(A\), Petrov types and – in case of already known solutions – relevant references. Validity of both Einstein and sourceless Maxwell’s equations can be easily checked by means of a suitable computer algebra program.

### 4.2 E-M fields of Curzon’s type

Evidently, one obtains these limits from equations \((7), (10)\) when redefining the functions \(f_1(r, z)\) and \(f_2(r, z)\) according to relations \((17)\).
a) The Curzon solution with an electric field
It represents a limit of the solution (7) and its line element has the form
\[ f_1(r, z) = \exp \left( -\frac{2M}{R} \right), \quad f_2(r, z) = \exp \left( -\frac{M^2r^2}{R^3} \right), \quad f(r, z) = 1 - q^2 \exp \left( -\frac{2M}{R} \right). \]
and the vector potential reduces into
\[ \mathbf{A} = q \frac{1}{\exp (2M/R) - q^2} \, dt. \]

b) The Curzon solution with a magnetic field
Similarly, this limit of the solution (10) again represents the Weyl metric with the choice
\[ f_1(r, z) = \exp \left( -\frac{2M}{R} \right), \quad f_2(r, z) = \exp \left( -\frac{M^2r^2}{R^4} \right), \quad f(r, z) = 1 + q^2 r^2 \exp \left( \frac{2M}{R} \right), \quad \mathbf{A} = q \frac{r^2}{\exp (-2M/R) + r^2 q^2} \, d\varphi. \]

4.3 E-M fields of Levi-Civita’s type
Next two solutions has been found in [3]. In the context of this paper and especially of the limiting diagram in figure 3 they can be considered as limits of the metrics (7) when one extends \( m \to \infty \) keeping \( \sigma = \gamma/2 \) constant.

a) The L-C solution with a radial electric field
This E-M field represents a limit of (1), its line element and fourpotential read as
\[ ds^2 = -r^{4\sigma} \frac{dt^2}{f(r)^2} + f(r)^2 r^{4\sigma (2\sigma - 1)} [dr^2 + dz^2] + f(r)^2 C^{-2} r^{2 - 4\sigma} d\varphi^2, \]
\[ f(r) = 1 - q^2 r^{4\sigma}, \quad \mathbf{A} = -\frac{q r^{4\sigma}}{f(r)} \, dt. \]

The metric (22) represents a special case of the general class of cylindrically symmetric solution with radial electric field given in [1], §20.2, equation 20.9c. The line-element of this general class has the form
\[ ds^2 = \varrho^{2m^2} G^2 (d\varrho^2 + dz^2) + \varrho^2 G^2 d\varphi^2 - G^{-2} dt^2, \]
where \( G = C_1 \varrho^m + C_2 \varrho^{-m} \) and \( C_1, C_2, m \) are real constants. The solution (22) belongs generally to the Petrov type I, for \( \sigma = \pm 1/2 \) to the Petrov type D, for \( \sigma = 0 \) the spacetime becomes flat (Petrov type 0).
b) The L-C solution with a longitudinal magnetic field

Similarly, the limit of the metric (10) takes the form

\[ ds^2 = -f(r)^2 r^{2\sigma} dt^2 + f(r)^2 r^{4\sigma(2\sigma-1)} [dr^2 + dz^2] + \frac{r^{2-4\sigma}}{f(r)^2} C^2 \, d\varphi^2. \]

\[ f(r) = 1 + \frac{q^2}{C^2} r^{2(1-2\sigma)}, \quad A = \frac{q r^{2(1-2\sigma)}}{C^2 f(r)} \, d\varphi. \]  

(23)

Submitting the metric (23) to the coordinate transformation

\[ \rho = r^{(2\sigma-1)^2}, \quad m = \frac{1}{2\sigma - 1}, \]  

(24)

one can show that the solution (23) belongs to the class of the general static cylindrically symmetric solution with longitudinal magnetic field [1], §20.2, equation 20.9b

\[ ds^2 = \rho^{3m^2} G^2 (d\rho^2 - dt^2) + G^{-2} d\varphi^2 + \rho^2 G^2 dz^2, \]

where \( G = C_1 \rho^m + C_2 \rho^{-m} \) and \( C_1, C_2, m \) are real constants. The spacetime (23) belongs to the Petrov type I, for \( \sigma = 0 \) to the type D and for \( \sigma = 1/2 \) it becomes flat. In the case \( \sigma = 0 \) the metric is equivalent to the Bonnor-Melvin universe (35) (see also subsection 4.9).

4.4 E-M fields of Taub’s type

As it is apparent from the limiting diagram, these solutions are just the space-times (22) and (23) in different coordinates for a particular case \( \sigma = -1/2 \). To simplify analytic expressions a new parameter \( Q \) characterizing the strength of E-M field is introduced below. It differs from the parameter \( q \) used above (namely in (22) and (23)) only by some multiplicative constant, the coordinate transformation can be found in [16].

a) The Taub solution with an electric field.

The L-C solution with radial electric field (22) with \( \sigma = -1/2 \) transforms into the one-parameter class of solutions

\[ ds^2 = -\frac{dr^2}{\sqrt{\xi} \left( 1 - \frac{Q^2}{\sqrt{\xi}} \right)^2} + \left( 1 - \frac{Q^2}{\sqrt{\xi}} \right)^2 \left[ d\xi^2 + \xi (dx^2 + dy^2) \right] \]

\[ A = -\frac{Q}{\sqrt{\xi} \left( 1 - \frac{Q^2}{\sqrt{\xi}} \right)} \, dr, \quad Q = 2^{-2/3} q; \]  

(25)

belonging to the Petrov type D.
b) The Taub solution with a magnetic field.

Analogously, the L-C solution with longitudinal magnetic field \(23\) with \(\sigma = -1/2\) transforms into the one parameter class

\[
ds^2 = (1 + Q^2 \xi^2)\left[\frac{1}{\sqrt{\xi}}(-d\tau^2 + d\xi^2) + \xi \, dx^2\right] + \frac{\xi \, dx^2}{(1 + Q^2 \xi^2)^2}
\]

\[A = \frac{Q \xi}{1 + Q^2 \xi} \, dx, \quad Q = 2^{4/3} \, q;\]

belonging to the Petrov type \(I\).

4.5 E-M fields of Robinson-Trautman’s type

Next two one-parameter classes again do not represent new solution as they can be obtained from the metrics of the Levi-Civita’s type \(22\), \(23\) setting \(\sigma = -1/2\). For the sake of simplicity a new rescaled constant parameter \(Q\) for the strength of the E-M field is again introduced as in the preceding subsection.

a) The Robinson-Trautman subclass with an electric field.

Submitting the L-C solution with a radial electric field \(22\) to the Bonnor’s transformation \(16\), one obtains the metric

\[
ds^2 = -\frac{2\alpha}{pf(p)^2} \, d\eta^2 - \frac{2}{f(p)^2} \, dp \, d\eta + p^2 f(p)^2 \, (dx^2 + dy^2) +
\]

\[
+ \frac{dp^2}{f(p)^2} \left[Q^2 \left(\frac{2}{\alpha}\right)^{2/3} + Q^4 \left(\frac{2}{\alpha}\right)^{1/3} + \frac{Q^6}{p^2} + \frac{Q^8}{8p^4} (2\alpha)^{1/3}\right]
\]

\[f(p) = 1 + Q^2 \left(\frac{\alpha}{2}\right)^{1/3} \, \frac{1}{p}, \quad A = -\frac{Q}{(4\alpha)^{1/3} f(p)} (2\alpha \, d\eta + p \, dp), \quad Q = q\]

belonging to the Petrov type \(D\).

b) The Robinson-Trautman subclass with a magnetic field.

Similarly, the L-C solution with a longitudinal magnetic field \(23\) is equivalent to the Petrov class \(f\) metric

\[
ds^2 = -\frac{2\alpha}{p} f(p)^2 \, d\eta^2 - 2 f(p)^2 \, dp \, d\eta + p^2 \left[\frac{dx^2}{f(p)^2} + f(p)^2 \, dy^2\right]
\]

\[f(p) = 1 + Q^2 p^2, \quad A = \frac{Qp^2}{f(p)} \, dx, \quad Q = \left(\frac{2}{\alpha}\right)^{1/3} \, \frac{q}{C}.\]

4.6 General plane solution with an electromagnetic field

The vacuum solution \(13\) representing a non-uniform infinite plane \(\gamma\) has not been proved to be a limiting case either of the \(\gamma\)-metric \(1\) nor the L-C solution \(3\). Nevertheless, the H-M conjecture can be successfully applied to \(13\) in the simplified way described in \(3\) and provides following E-M fields.
a) An infinite plane with an electric field.

Employing the timelike Killing vector $\partial_t$ of the seed metric (19), the application of the H-M conjecture leads to the metric

\[
\begin{align*}
ds^2 &= f(Z)^2 \left[ \varrho^2 Z^{2-4\sigma} \, \dd{\varphi}^2 + Z^{8\sigma^2-4\sigma} \left( Z^2 + \varrho^2 \right)^{1-4\sigma^2} \left( \dd{\varrho}^2 + \dd{Z}^2 \right) \right] - \\
&\quad - \frac{Z^{4\sigma}}{f(Z)^2} \dd{t}^2, \quad f(Z) = 1 - q^2 Z^{4\sigma}, \quad A = \frac{q Z^{4\sigma}}{f(Z)} \dd{t}.
\end{align*}
\]

(29)

This solution belongs generally to the Petrov type I, for $\sigma = \pm 1/2$ it reduces to the type D and for $\sigma = 0$ we get the Petrov type 0.

b) An infinite plane with a magnetic field.

Starting from the second Killing vector of the seed vacuum solution (19) $\partial_\varphi$, we come to the spacetime

\[
\begin{align*}
ds^2 &= - Z^{4\sigma} f(Z, \varrho)^2 \dd{t}^2 + f(Z, \varrho)^2 \dd{\varphi}^2 + \\
&\quad + f(Z, \varrho)^2 Z^{8\sigma^2-4\sigma} \left( Z^2 + \varrho^2 \right)^{1-4\sigma^2} \left( \dd{\varrho}^2 + \dd{Z}^2 \right), \\
f(Z, \varrho) &= 1 + q^2 \varrho^2 Z^{2-4\sigma}, \quad A = \frac{q \varrho^2 Z^{2-4\sigma}}{f(Z, \varrho)} \dd{\varphi}.
\end{align*}
\]

(30)

which generally belongs to the Petrov type I, for particular values $\sigma = 0, 1/2, 1$ to the Petrov type D.

4.7 Semi-infinite line-mass with an electromagnetic field

The vacuum solution (18) is isometric with the metric (19). Thus, transforming the metrics (29), (30) we straightforwardly obtain the following E-M fields.

a) Semi-infinite linear source with an electric field.

The class of E-M fields (29) transforms into

\[
\begin{align*}
ds^2 &= - X^{2\sigma} \frac{1}{f(z, r)^2} \dd{t}^2 + f(z, r)^2 X^{-2\sigma} \left[ \frac{X}{2R} \right]^{4\sigma^2} \left( \dd{r}^2 + \dd{z}^2 \right), \\
f(z, r) &= 1 - q^2 X^{2\sigma}, \quad A = \frac{q X^{2\sigma}}{f(z, r)} \dd{t}.
\end{align*}
\]

(31)

b) Semi-infinite linear source with a magnetic field

Analogously, transforming the metric (30), we come to the magneto-vacuum E-M
field

\[ ds^2 = -f(z,r)^2 \left[ X^{2\sigma} dt^2 + X^{-2\sigma} \left( \frac{X}{2R} \right)^{4\sigma^2} (dr^2 + dz^2) \right] + \frac{r^2}{X^{2\sigma} f(z,r)^2} d\varphi^2, \]

\[ f(z,r) = 1 + q^2 r^2 X^{-2\sigma}, \quad A = \frac{qr^2 X^{-2\sigma}}{f(z,r)} d\varphi. \]

The Petrov types are, of course, the same as in subsection 4.6.

4.8 E-M fields of Schwarzschild’s type

We obtain these limits of both the solutions (7) and (10) straightforwardly setting \( \gamma = 1 \) (i.e. \( \sigma = 1/2 \)) and introducing Erez-Rosen coordinates (4).

\( a) \) The Schwarzschild metric with a radial electric field.

Under above specified conditions the metric (7) reduces to

\[ ds^2 = -\frac{(1 - 2m/\varrho)}{(1 - q^2 + 2q^2 m/\varrho)^2} dt^2 + (1 - q^2 + 2q^2 m/\varrho)^2 \left[ \frac{d\varrho^2}{1 - 2m/\varrho} + \varrho^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right] \]

and the vector potential (8) to

\[ A = \frac{(1 - 2m/\varrho)}{1 - q^2 + 2q^2 m/\varrho} dt. \]

Introducing a new radial coordinate \( R = \varrho \left( 1 - q^2 + 2q^2 m/\varrho \right) \) and rescaling the time coordinate \( \tau = t / (1 - q^2) \), we come to a familiar Reissner-Nordström solution, which has been explored in connection with the H-M conjecture in [2]. The mass of the Reissner-Nordström source \( M \) as well as its electric charge \( Q \) depend both on \( m \) and \( q \)

\[ M = m \left( 1 + q^2 \right), \quad Q = 2mq; \]

the vector potential then takes the form

\[ A = q \left( 1 - \frac{2m}{R} \right) dR. \]

Evidently, in the vacuum case \( q \to 0 \) one gets the Schwarzschild solution as \( M \to m \) and \( Q \to 0 \) and \( R \to \varrho \).

\( b) \) The Schwarzschild metric with a magnetic field.

Analogously, the metric (10) reduces to

\[ ds^2 = (1 + q^2 \varrho^2 \sin^2 \theta)^2 \left[ -\left( 1 - \frac{2m}{\varrho} \right) dt^2 + \frac{dr^2}{1 - 2m/\varrho} + \varrho^2 d\theta^2 \right] + \frac{\varrho^2 \sin^2 \theta}{(1 + q^2 \varrho^2 \sin^2 \theta)^2} d\varphi^2, \]

\[ (32) \]
and the vector potential (11) to
\[
A = \frac{q \varrho \sin^2 \vartheta}{1 + q^2 \varrho^2 \sin^2 \vartheta} \, d\varphi.
\]

4.9 Minkowski’s limit

The last two limits represent undoubtedly the simplest cases of all the E-M fields listed in this section. They can be obtained either from the fields of the Schwarzschild type (33), (34) when putting \( m = 0 \) and returning to the cylindrical coordinates \( z = \varrho \cos \theta, \, r = \varrho \sin \theta \), or from the E-M fields of the Levi-Civita type (22), (23) setting \( \sigma = 0 \). In the latter case the solutions might inherit a conical singularity of the Levi-Civita solution determined by the constant \( C \). Hereafter we assume \( C = 1 \).

While the “electric” case is trivial as it is equivalent to the Minkowski spacetime itself without any electromagnetic field, the solution with a magnetic field has a well known form
\[
ds^2 = \left( 1 + q^2 r^2 \right)^2 \left\{ -dt^2 + dr^2 + dz^2 \right\} + \frac{r^2}{\left( 1 + q^2 r^2 \right)^2} \, d\varphi^2.
\]
(35)

Evidently, this metric with \( q = B_0/2 \) represents so called Bonnor-Melvin magnetic universe (see e.g. [1], §20.2, equation 20.10). As it was mentioned in subsection 3.3, the spacetime (35) need not include a longitudinal magnetic field only – the background electromagnetic field might be also electric (a particular example of this metric with a longitudinal electric field was found in [3]).

5 Conclusion

We have again proved explicitly that H-M conjecture outlined serves as a useful, efficient tool for the generating of E-M fields. Let us briefly summarize obtained results.

(i) We present a successful application of the H-M conjecture in its generalized formulation to the whole class of seed vacuum metrics. In each case two Killing vectors of the corresponding seed vacuum solution were used to generate new E-M fields. Thus, an algorithmic scheme for the H-M conjecture formulated in [3] can be evidently used in more various situations, namely, in case of all vacuum solutions listed in figure 2.

(ii) We have obtained several E-M fields generated from the seed vacuum spacetimes of Weyl’s type. Most of them represent a special case or a limit of the general classes of E-M fields (7) and (10), some solutions can be derived through an appropriate coordinate transformations. Components of basic tensors are explicitly given and general Petrov classes are determined for all metrics.
It is typical for the solutions generated through the H-M conjecture that
their interpretation is determined by the physical features of their seed vac-
uum metrics. Unfortunately, for most vacuum metrics discussed above, sev-
eral possible, often qualitatively different interpretations can be proposed.
However, Bonnor’s interpretation \[4, 15\] seems to be most convenient for the
discussion of the limits in section \[4\]. Therefore, we prefer to interpret the seed
Weyl metrics according to their generating Newtonian potentials. E-M fields
obtained via the H-M conjecture then might represent a gravitational field of
linear sources with some electric charge distribution or the linear sources in
a background electromagnetic field of the Bonnor-Melvin’s type.

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Appendix: Circular geodesics in the equatorial plane

The study of circular (azimuthal) geodesic can impose additional constraints on
the spacetime parameters. Hereafter we use Rindler-Perlick method \[17\] designed
for axially symmetric stationary metrics. The angular velocity \(\omega = \frac{d\varphi}{dt}\) along
azimuthal geodesics in the equatorial plane \(z = 0\) for any Weyl solution \[9\] was
calculated by Herrera and Pastora \[9\] and in the Weyl-Lewis-Papapetrou spherical
coordinates it reads as

\[
\omega = \frac{|g_{tt}| \left( \frac{\partial |g_{tt}|}{\partial R} \right)^{1/2}}{\left( 2R |g_{tt}| - R^2 \frac{\partial |g_{tt}|}{\partial R} \right)^{1/2}}. \tag{36}
\]

Applying \eqref{36} to our “electric” solution \eqref{7} we obtain

\[
\omega = \frac{\sqrt{m\gamma} (R_e^2 - m^2)^\gamma}{r \left( (R_e + m)^\gamma - q^2 (R_e - m)^\gamma \right)^2} \times
\]

\[
\times \sqrt{(R_e + m)^\gamma + q^2 (R_e - m)^\gamma}, \tag{37}
\]
analogously for our “magnetic” solution we get

$$\omega = \frac{q^2 r^2 (R_e + m)^{\gamma} + (R_e - m)^{\gamma}}{r (R_e^2 - m^2)^{\gamma}} \times \sqrt{\frac{q^2 r^2 (2R_e - m_{\gamma}) (R_e + m)^{\gamma} + \gamma m (R_e - m)^{\gamma}}{(R_e - \gamma m)^{(R_e - m)^{\gamma} - q^2 r^2 (R_e + m)^{\gamma}}}}$$

(38)

where \( R_e = R_1 |_{z=0} = R_2 |_{z=0} = \sqrt{r^2 + m^2} \); in the equatorial plane both spherical and cylindrical radial coordinates \( R, r \) coincide. It is worth checking that in the vacuum Schwarzschild limit \( (q = 0 \text{ and } \gamma = 1) \) we come to the relation

$$\omega^2 = \frac{m (\sqrt{r^2 + m^2} - m)}{r^2 (\sqrt{r^2 + m^2} + m)^2},$$

which expressed in Erez-Rosen coordinates (4) leads to the “exact Kepler law” \( \omega^2 = m q^{-3} \) for the Schwarzschild solution (it is derived e.g. in [17], \( q = \sqrt{(r^2 + m^2) + m} \)). The expressions (36)–(38) would be much more complicated for non-equatorial planes \( z \neq 0 \).

The dependence \( \omega = \omega (r, q) \) for the (7) with \( \gamma = 1 \) (the “charged” Schwarzschild spacetime) is illustrated in figure 3a. The circular geodesics are allowed only for \( q \in (-1, 1) \) and \( \omega \) increases towards the symmetry axis \( r = 0 \). According to the figure 3b, this conclusion is valid for \( \gamma \leq 1 \), but for \( \gamma > 1 \) the angular velocity \( \omega \) diverges at “points” \( r \neq 0 \) and no azimuthal geodesics can be found in the region with sufficiently small \( r \). In case of the “magnetic” spacetime (10) the existence of circular geodesics is limited not only at the side of small \( r \), but also at the side of large \( r \), which is demonstrated in figure 3.

Figure 3: The angular velocity for the azimuthal geodesics in the equatorial plane of the solution (7): a) \( \gamma = 1 \); b) \( q = 0.5 \).
Figure 4: The angular velocity for the circular geodesics in the equatorial plane of the solution (10): a) $\gamma = 1$; b) $q = 0.1$.

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