UNIFORM BOUNDARY OBSERVABILITY WITH LEGENDRE-GALERKIN FORMULATIONS OF THE 1-D WAVE EQUATION

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Abstract. We study the boundary observability of the 1-D homogeneous wave equation when using a Legendre-Galerkin semi-discretization method. It is already known that spurious high frequencies are responsible for its lack of uniformity with respect to the discretization parameter [4] which may prevent convergence in the approximation of the associated controllability problem. A classical remedy is to filter out the highest frequency components but this comes with a high computational cost in several space dimensions. We present here three remedies: a spectral filtering method, a mixed formulation (already used in the context of finite element method [14]) and a Nitsche’s method. Our numerical results show that the uniform boundary observability inequalities are recovered. On the other hand, surprisingly, none of them seem to provide the trace (or direct) inequality uniformly, a property used to prove the convergence of the numerical controls [11]. However, our numerical tests suggest that convergence of the numerical controls is ensured when the uniform observability inequality holds.

1. Introduction.

1.1. General framework and motivation. Let us consider the n-dimensional wave equation with homogeneous Dirichlet boundary conditions,

\[
\begin{align*}
    u_{tt} - \Delta u &= 0, & \text{in } \Omega \times (0,T), \\
    u &= 0, & \text{on } \partial \Omega \times (0,T), \\
    u(.,0) &= u_0, u_t(.,0) = u_1, & \text{in } \Omega.
\end{align*}
\]

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Here $\Omega$ is a smooth bounded domain of $\mathbb{R}^n$ and $\partial \Omega$ is its boundary. With initial data $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$, the energy of the solution,
\[
E(u(t)) := \frac{1}{2} \int_{\Omega} \left[ |u_t(x,t)|^2 + |\nabla u(x,t)|^2 \right] \, dx,
\]
is conserved along time: $E(u(t)) = E(u(0))$, $t \geq 0$.

We are interested in the boundary observability of the wave equation on a part $\Gamma$ of the boundary, $\Gamma \subset \partial \Omega$. We say that it is observable in time $T > 0$ if there exists $c_T > 0$ such that
\[
c_T E(u(0)) \leq \int_{\Gamma \times (0,T)} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma dt, \quad \forall (u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega).
\]
This is often referred as the observability inequality with $c_T$ the observability constant.

It is well-known since the work of Bardos, Lebeau and Rauch [3] (see also [5]) that the observability inequality holds for $T > T^*$ under the geometric control condition, that is if every geodesic travelling at speed 1 and obeying the law of the geometric optic hits $\Gamma$ transversally in time $T^*$. The observability inequality plays a fundamental role in the Dirichlet boundary controllability of the wave equation [26], the reconstruction of initial data [32, chap. 6.1] and feedback stabilization problems like the infinite time horizon linear quadratic regulatory (LQR) problem [25].

From a numerical point of view, the convergence of the numerical approximations of the solution to the wave equation resulting from a space semi-discretization is not sufficient to prove the convergence of the controls. A key ingredient for proving the convergence of the exact control approximations is that a discrete version of the observability inequality holds uniformly with respect to the discretization parameter [36, 11, 12]. This is also essential for approximations for related optimal control problems like the infinite time horizon LQR problem [25].

Unfortunately, classical approximation methods do not ensure the observability inequality to be satisfied uniformly with respect to the discretization parameter. This pathology is caused by spurious high frequencies of the numerical approximations which are not suitable approximations of the corresponding eigenmodes of the continuous problem. For all these standard space discretization methods, a remedy to recover the uniformity of the observability is to filter out the high frequency components of the solution. This method is often referred as Fourier filtering. It was shown in [19] (finite element method in 1D), [35] (finite element method in 2D) and [4] (Legendre-Galerkin method in 1D) that this method leads to the uniformity of the observability inequality. This method, however, has a major drawback: it requires the computation of the eigenmodes (eigenvalues and associated eigenfunctions), which can be costly for spatial domains in several space dimensions.

Based on numerical experiments with finite-element or finite-difference semi-discretizations, other remedies were suggested by Glowinski et al., like a Tychonoff regularization in [15], a multi-grid method [14] and a mixed formulation of (1) [13]. It was later proved in [28] that a bi-grid method applied to a finite-difference semi-discretization of the 1D wave equation allows one to obtain the uniformity of the observability inequality and the convergence of the numerical controls. A mixed finite element approximation was also proved to be successful in 1D [7] and 2D [8].

To our knowledge, no remedies other than the Fourier filtering have been devised for a semi-discretization of spectral type, even in 1D. In this paper we propose three
remedies to recover the uniformity of the boundary observability inequality in the case of the Legendre-Galerkin semi-discretization, with approximation vector spaces made of polynomials of degree \( N \). These remedies are:

1. a spectral filtering, a standard procedure in spectral methods when dealing with undesirable or spurious high frequency components;
2. a mixed method, similar to the one considered in [7] and [8] with finite elements;
3. Nitsche’s method, a method to append Dirichlet-type boundary conditions in a weak form.

The mixed method put aside, the other remedies are new in the context of the numerical observability problem, and the first one (spectral filtering) is specific to spectral methods. Although these remedies are not specific to 1D problems, we present them in 1D and show numerical evidences of the uniformity of the observability constant for all these three methods.

Moreover, we show that these remedies are not fully satisfactory since they do not seem to guarantee the uniformity of the discrete analogue of the direct inequality

\[
\int_{\Gamma \times (0,T)} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma \, dt \leq C_T E(u(0)), \quad \forall (u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega).
\]

which holds for every \( T > 0 \) with \( C_T > 0 \) [22, 21]. The situation here is very different from what happens with classical finite element or finite difference methods, for which the uniform direct inequality holds even without filtering out the highest frequencies [19]. With the Legendre-Galerkin method, not only \( C_{N,T} \) cannot be bounded from above as \( N \) grows without filtering, but the same still happens with each of the three remedies to the non-uniform observability. These negative results, if confirmed theoretically, would prevent us from using the abstract convergence results for the control approximations in [11, 12] whose proofs use both the uniform observability and the uniform direct inequalities.

Nevertheless, our numerical investigations show that convergence of the control approximations does not hold with the classical Legendre-Galerkin method (without uniform observability), whereas convergence seems to hold if we compute the approximate control using any of the three remedies (giving uniform observability inequalities) proposed in this paper.

1.2. Present framework and outline. Let us now present the precise context of this paper. Consider the one-dimensional wave equation with homogeneous Dirichlet boundary conditions,

\[
\begin{cases}
  u_{tt}(x,t) - u_{xx}(x,t) = 0, & \text{for } (x,t) \in (-1,1) \times (0,T) \\
  u(-1,t) = u(1,t) = 0, & \text{for } t \in (0,T) \\
  u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{for } x \in (-1,1)
\end{cases}
\]

with initial data \((u_0, u_1) \in H^1_0(-1,1) \times L^2(-1,1)\).

The 1D wave equation is said to be boundary observable at \( x = 1 \) in time \( T > 0 \) if there exists \( c_T > 0 \) such that

\[
c_T E(u(0)) \leq \int_0^T |u_x(1,t)|^2 \, dt, \quad \forall (u_0, u_1) \in H^1_0(-1,1) \times L^2(-1,1).
\]
where
\[ E(u(t)) := \frac{1}{2} \int_{-1}^{1} \left[ |u_t(x,t)|^2 + |u_x(x,t)|^2 \right] \, dx, \]

The direct inequality writes
\[ \int_0^T |u_x(1,t)|^2 \, dt \leq C_T E(u(0)), \quad \forall (u_0, u_1) \in H^1_0(-1,1) \times L^2(-1,1). \tag{3} \]

We recall that for this 1D wave equation, the observability inequality holds only if \( T \geq 4 \) (see for instance [20]). Time \( T = 4 \) is the minimal time for all waves travelling within the spatial domain \((-1, 1)\) at propagation speed 1 to reach the region of observation which is \( x = 1 \). In fact, if \( T = 4 \), inequalities (2) and (3) become the same equalities, with \( c_T = C_T = 4 \).

In the present work, we are interested in a discrete version of the observability inequality (2) which may take the form:
\[ c_{N,T} E(u^N(0)) \leq \int_0^T |u^N_x(1,t)|^2 \, dt. \tag{4} \]

Here \( u^N(x, t) \) denotes the approximation of \( u \) resulting from a space semi-discretization of the wave equation (1), and \( N \) denoting the discretization parameter, typically the size of the finite dimensional approximation spaces for the initial data.

More precisely, we look for methods to ensure the uniformity with respect to \( N \) of inequality (4), that is with
\[ c_{N,T} \geq c_T, \]
for some \( c_T > 0 \).

The discrete analogue of the direct inequality (3), reads
\[ \int_0^T |u^N_x(1,t)|^2 \, dt \leq C_{N,T} E(u^N(0)), \tag{5} \]
and it is uniform with respect to \( N \) if
\[ C_{N,T} < C_T, \]
for some \( C_T > 0 \).

We finally note that, depending on the remedy, modified versions of the discrete observability and direct inequalities (4) and (5) must be considered. Indeed, not only some remedies change the discrete formulation of the wave equation (like with the mixed method and Nitsche’s method), some of them (like the spectral filtering method or Nitsche’s method) also need to change the observation (here \( u^N_x(1,t) \)).

This is more understandable in the light of the dual controllability problem and of how the control and its approximations are obtained. This is the reason why we shall begin (section 2) with a review of the associated controllability problem and its link with the observability and direct inequalities. In Section 3 we introduce the Legendre-Galerkin approximation of (1), we recall some results on the behaviour of \( c_{N,T} \) and \( C_{N,T} \) obtained in [4] and we show numerically the unboundedness of these constants as well as the boundedness of their counterparts when Fourier filtering is applied. In Sections 4, 5 and 6 we consider a spectral filtering technique, a mixed formulation and Nitsche’s method, respectively. Finally, in Section 7, we present numerical results for an exact controllability problem, showing the convergence of the control approximations for all the remedies proposed in this paper and showing divergence without these remedies.
2. The dual controllability problem. In the sequel, \( H^1_0 \) stands for \( H^1_0(-1,1) \), and the same notational simplification is adopted for (Sobolev) spaces like \( L^2(-1,1) \) and \( H^{-1}(-1,1) \), the dual space of \( H^1_0 \) with \( L^2 \) as the pivot space. Moreover, \( <.,.> \) stands for the duality product between \( H^{-1} \times L^2 \) and \( H^1_0 \times L^2 \);

\[
< (y_1, y_0), (\phi_0, \phi_1) > = \int_{-1}^{1} y_1 \phi_0 dx + \int_{-1}^{1} y_0 \phi_1 dx,
\]

where \( \int_{-1}^{1} y_1 \phi_0 dx \) has a sense if \( y_1 \in L^2 \) but otherwise still denotes the duality product between \( H^{-1} \) and \( H^1_0 \) if \( y_1 \notin L^2 \).

Let us consider the wave equation with Dirichlet boundary condition and final null state,

\[
\begin{align*}
y_{tt}(x,t) - y_{xx}(x,t) &= 0, & x \in (-1,1), \ t \in (0,T), \\
y(-1,t) &= 0, & t \in (0,T), \\
y(y,t) &= v(t), & t \in (0,T), \\
y(x,T) &= y_1(x,T) = 0, & x \in (-1,1).
\end{align*}
\]

(6)

We say that (6) is null controllable in time \( T > 0 \) if, for all \((y_0,y_1) \in L^2 \times H^{-1}\), there exists a control \( v \in L^2(0,T) \) such that the solution \( y \) of (6) (backwards in time) satisfies

\[
y(.,0) = y_0, \quad y_t(.,0) = y_1.
\]

Because of the linearity and the time-reversibility of the wave equation, null controllability is equivalent to controllability, which is the property to steer the solution of (6) from any initial state \((at t=0)\) to any final state \((at t=T)\) in \( L^2 \times H^{-1} \).

In order to showcase the duality between observation and control as well as the construction of a control of minimal \( L^2(0,T) \)-norm in the case of the wave equation with Dirichlet control we follow [31] (see also [24, 23] for other but similar type of equations or control/observations as well as [10] for a general theory in Banach spaces) where this is exposed for the wave equation in any space dimension.

The control to state map

\[
\mathcal{L}_T : L^2(0,T) \longrightarrow H^{-1} \times L^2
\]

v \rightarrow (y(.,0), y_t(.,0))

is linear and continuous. Its adjoint \( \mathcal{L}_T^* : H^1_0 \times L^2 \longrightarrow L^2(0,T) \), defined by

\[
(\mathcal{L}_T^*(u_0, u_1), w)_{L^2(0,T)} = < \mathcal{L}_T(w), (u_0, u_1) >,
\]

is the data to trace map associated with our initial system (1),

\[
\mathcal{L}_T^*(u_0, u_1) = u_x(1,t).
\]

Controllability of (1) is equivalent to the surjectivity of \( \mathcal{L}_T \), which in turn is equivalent to the existence of a constant \( c > 0 \) such that

\[
|| \mathcal{L}_T^* (u^0, u^1) ||_{L^2} \geq c || (u^0, u^1) ||_{H^1_0 \times L^2},
\]

(7)

\( \forall (u_0, u_1) \in H^1_0 \times L^2 \). This inequality is nothing else than the observability inequality (3) with \( c = \sqrt{T} \).

Moreover, when controllability holds, for initial data \((y_0, y_1)\) to be steered to zero the control minimizing the \( L^2(0,T) \)-norm among all control candidates is

\[
v = \mathcal{L}_T^* (\mathcal{L}_T \mathcal{L}_T^*)^{-1} (-y_1, y_0),
\]
where $\mathcal{L}_T^*(\mathcal{L}_T\mathcal{L}_T^*)^{-1}$ is the (Moore-Penrose) pseudoinverse of $\mathcal{L}_T$. If we now define the bounded linear map $\Lambda_T = \mathcal{L}_T\mathcal{L}_T^* : H^1_0 \times L^2 \rightarrow H^{-1} \times L^2$ then we can decompose the computation of $v$ in two steps:

1. Solve $\Lambda_T(u_0, u_1) = (-y_1, y_0)$; \hfill (8)
2. Set $v = \mathcal{L}_T^*(u_0, u_1)$. \hfill (9)

Note that $\Lambda_T$ is the controllability grammian, well known in the control litterature. This is also known as the HUM operator in the approach of [26] to explain the duality between observability and controllability and the construction of this control of minimal norm. The equation (8) to solve in step 1 can also be rewritten in the form:

$$\langle \mathcal{L}_T^*(u_0, u_1), \mathcal{L}_T^*(\phi_0, \phi_1) \rangle_{L^2(0,T)} = \langle (-y_1, y_0), (\phi_0, \phi_1) \rangle.$$ \hfill (10)

This equation can be made more explicit by expliciting $\mathcal{L}_T^*$:

$$\int_0^T u_x(1,t), \phi_x(1,t) \, dt = -\int_{-1}^1 y_1 \phi_0 \, dx + \int_{-1}^1 y_0 \phi_1 \, dx,$$

where $\phi$ is the solution of the homogeneous wave equation (1) with initial state $(\phi_0, \phi_1)$. And the control with minimal $L^2(0,T)$-norm (see (9)) is given by $v(t) = u_x(1,t), \quad 0 < t < T$.

The left hand side (resp. right hand side) of (11) is a symmetric continuous bilinear (resp. linear) form on $H^1_0 \times L^2$. When observability holds, the coercivity of the bilinear form is ensured by the observability inequality (3) with coercivity constant $c_T$. Note also that, due to its symmetry, the solution $(u_0, u_1)$ of (10) can also be viewed as the minimizer over $H^1_0 \times L^2$ of the quadratic functional $J(u_0, u_1) = \frac{1}{2} \| \mathcal{L}_T^*(u_0, u_1) \|^2 - \langle (-y_1, y_0), (u_0, u_1) \rangle$, that is

$$J(u_0, u_1) = \frac{1}{2} \int_0^T |u_x(1,t)|^2 \, dt + \int_{-1}^1 y_1 u_0 \, dx - \int_{-1}^1 y_0 u_1 \, dx.$$ \hfill (12)

Moreover, (10) (or equivalently (11)) provides an upper bound on the control of minimal $L^2(0,T)$-norm, namely

$$\|v\|_{L^2}^2 \leq \frac{C}{c_T} \| (y_0, y_1) \|_{L^2 \times H^{-1}},$$

where $C$ does not depend on $T$ and $c_T$ is the observability constant. Note that this bound does not depend on $C_T$.

We finally note that an approximation $v_N$ of the control is obtained by minimizing the discrete analogue of the functional $J,$

$$J_N(u_0^N, u_1^N) = \frac{1}{2} \int_0^T |u_x^N(1,t)|^2 \, dt + \int_{-1}^1 y_1 u_0^N \, dx - \int_{-1}^1 y_0 u_1^N \, dx,$$ \hfill (13)

over the approximation spaces for the initial data $(u_0^N, u_1^N)$ and where $u^N$ is the approximation of $u$ with initial data $(u_0^N, u_1^N)$. A control approximation is then $v_N(t) = u_x^N(1,t)$, where $u^N$ is generated by the optimal initial data. A uniform observability thus ensures that $J_N$ is uniformly coercive and provides a uniform upper bound on $\|v_N\|_{L^2(0,T)}$. 

3. Legendre-Galerkin approximation and discrete observability properties.

3.1. Legendre-Galerkin approximation. First of all, let us recall that the solution of the wave equation admits the Fourier expansion

\[ u(x,t) = \sum_{k=1}^{\infty} \left( \alpha_k \cos(\sqrt{\lambda_k} t) + \frac{\beta_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \right) \phi_k(x), \]

where

\[ \alpha_k = (u_0, \phi_k)_{L^2}, \quad \beta_k = (u_1, \phi_k)_{L^2}, \]

and \((\lambda_k, \phi_k)_{k \geq 1}\) are the solutions of the eigenproblem: find \(\lambda \in \mathbb{R}, \phi \in H^1_0\) such that

\[ (\phi, \psi)_{H^1_0} = \lambda (\phi, \psi)_{L^2}, \quad \forall \psi \in H^1_0, \tag{14} \]

with \((\phi_k, \phi_k)_{L^2} = 1\), which are

\[ \lambda_k = \left(\frac{k\pi}{2}\right)^2, \quad \phi_k(x) = \sin(\sqrt{\lambda_k} x), \quad k = 1, 2, \ldots \]

Let us now briefly describe the semi-discretization of (1) studied in [4]. Let \(L_k\) denote the \(k\)-th order Legendre polynomial, given by the recurrence relation,

\[ \begin{array}{c}
L_0(x) = 1, \\
L_1(x) = x, \\
(k + 1)L_{k+1}(x) = (2k + 1)xL_k(x) - kL_{k-1}(x), \quad k \geq 1.
\end{array} \]

Let \(P_N\) denote the space of polynomials defined on \([-1, 1]\) of degree \(N\) or less, and let \(P_0^N\) denote the space of polynomials of degree \(N\) or less vanishing at \(x = -1\) and \(x = 1\). We have (see [30]) :

\[ \begin{align*}
P_N &= \text{span}\{L_0(x), \ldots, L_N(x)\}, \\
P_0^N &= \text{span}\{\tilde{L}_1(x), \ldots, \tilde{L}_{N-1}(x)\},
\end{align*} \]

where

\[ \tilde{L}_k(x) = c_k(1 - x^2)L'_k(x), \quad c_k = \frac{\sqrt{k + 1/2}}{k(k + 1)} , \quad k \geq 1. \]

The Legendre-Galerkin semi-discretization of (1) is then

\[ \int_{-1}^{1} u^N_0(x,t)\psi^N(x) \, dx + \int_{-1}^{1} u^N_x(x,t)\psi^N_x(x) \, dx = 0, \quad \forall \psi^N \in P_0^N, \quad t \in (0, T), \tag{15} \]

with suitable initial conditions. Here, the approximation \(u^N(., t)\) of \(u(., t)\) is sought in \(P_0^N(-1, 1)\), that is in the form

\[ u^N(x,t) = \sum_{k=1}^{N-1} a_{N,k}(t)\tilde{L}_k(x). \]

Introducing \(a_N(t) := (a_{N,1}(t), \ldots, a_{N,N-1}(t))^t\), equations (15) can be written in the matrix form

\[ M_N a^N_N(t) + K_N a_N(t) = 0, \tag{16} \]
where $M_N$ and $K_N$ are positive definite symmetric matrices given by (see [30])

\[
M_N(i,j) = \begin{cases} 
2(2i-1)(2i+3) & \text{if } i=j \\
(2r+3)\sqrt{(2r+1)(2r+5)} & \text{if } |i-j| = 2 \\
0 & \text{otherwise}
\end{cases}
\]

\[
K_N(i,j) = \begin{cases} 
1 & \text{if } i=j \\
0 & \text{otherwise},
\end{cases}
\]

with $r = \min\{i,j\}$.

Taking $\psi^N = u^N_t(x,t)$ in (15), one obtains that the energy of the solution of (15) is preserved along time: $E(u^N_t(t)) = E(u^N_t(0))$, $\forall t \geq 0$.

Let $(\lambda_{N,k}, \phi_{N,k})$, $1 \leq k \leq N-1$, denote the solutions of the eigenproblem: find $\lambda \in \mathbb{R}$, $\phi \in P^0_N$ such that

\[
(\phi, \psi^N)_{H^1_0} = \lambda (\phi, \psi^N)_{L^2}, \quad \forall \psi \in P^0_N.
\]  

Figure 1 illustrates the behaviour of $\sqrt{\lambda_{N,k}}$, for $N = 40$, in comparison with $\sqrt{\lambda_k} = k\pi$. The first $2/\pi$ fraction of them are very well approximated, as proved in [33].

The Fourier expansion of the solution of (15) on the orthonormal basis formed by the discrete normalized (for the $L^2$-norm) eigenfunctions $\phi_{N,k}$ writes

\[
u^N(x,t) = \sum_{k=1}^{N-1} \left( \alpha_{N,k} \cos(\sqrt{\lambda_{N,k}} t) + \frac{\beta_{N,k}}{\sqrt{\lambda_{N,k}}} \sin(\sqrt{\lambda_{N,k}} t) \right) \phi_{N,k}(x),
\]

where, for $1 \leq k \leq N-1$,

\[
\alpha_{N,k} = (u^N_0, \phi_{N,k})_{L^2}, \quad \beta_{N,k} = (u^N_1, \phi_{N,k})_{L^2}.
\]
We also consider its Fourier truncation of order \( M, 1 \leq M \leq N - 1 \),

\[
u^{N,M}(x,t) = \sum_{k=1}^{M} \left( \alpha_{N,k} \cos(\sqrt{\lambda_{N,k}}t) + \frac{\beta_{N,k}}{\sqrt{\lambda_{N,k}}} \sin(\sqrt{\lambda_{N,k}}t) \right) \phi_{N,k}(x).
\]

3.2. Discrete observability. As mentioned in section 2, in order to obtain an approximation of the associated controllability problem, we consider the minimization of (13) for initial data \((u_0^N, u_1^N) \in (P_N^0)^2\) and then by setting \(v_N(t) = u_x^N(1,t)\), where \(u^N\) is the solution of (15) with the initial data \((u_0^N, u_1^N)\) minimizing (13).

Considering the discrete observability and direct inequalities (2) and (3), we redefine the constants \(c_{N,T}\) and \(C_{N,T}\) as the best possible ones:

\[
c_{N,T} = \inf_{(u_0^N, u_1^N) \in (P_N^0)^2} \left( \int_0^T |u_x^N(1,t)|^2 \, dt \right) / E(u^N(0)),
\]

(18)

\[
C_{N,T} = \sup_{(u_0^N, u_1^N) \in (P_N^0)^2} \left( \int_0^T |u_x^N(1,t)|^2 \, dt \right) / E(u^N(0)).
\]

(19)

As mentioned in the introduction, it has been proven ([4]) that the observability and direct inequalities do not hold uniformly: for every \(T > 0\),

\[
c_{N,T} \rightarrow 0 \quad \text{and} \quad C_{N,T} \rightarrow \infty \quad \text{as} \quad N \rightarrow \infty.
\]

Figure 2 shows the behaviour of \(c_{N,T}\) and \(C_{N,T}\) with respect to \(N\) for \(T = 8\). We refer to the Appendix for the numerical method used to compute \(c_{N,T}\) and \(C_{N,T}\).

\[\text{Figure 2. Behaviour of } c_{N,T} \text{ (left) and } C_{N,T} \text{ (right) with } T = 8\]

Let us recall that when using finite difference or finite element methods, the observability constant also vanishes as \(N \rightarrow 0\), but the continuity constant is uniformly bounded from above (for any fixed \(T\)).

If we now consider the Fourier truncation \(u^{N,M}\) of order \(M\), which filters out the high frequency components of \(u^N\), then we can recover uniform observability and direct inequalities. More precisely, if \(T > 4\) and if \(M < \frac{2}{\pi} N\), that is if we keep only the \(2/\pi\) fraction of the lowest frequency components, then

\[
c_T E(u^{N,M}(0)) \leq \int_0^T |u_x^{N,M}(1,t)|^2 \, dt \leq C_T E(u^{N,M}(0)),
\]
where \(c_T > 0\) and \(C_T > 0\) are independent of \(N\) ([4]). In Figure 3 we see the behaviour of the constants \(c_{N,T}\) and \(C_{N,T}\) now defined by

\[
c_{N,T} = \inf_{(u_0^N, u_1^N) \in V_{N,M}^2} \left( \frac{\int_0^T |u_x^N(1,t)|^2 \, dt}{E(u^N(0))} \right),
\]

\[
C_{N,T} = \sup_{(u_0^N, u_1^N) \in V_{N,M}^2} \left( \frac{\int_0^T |u_x^N(1,t)|^2 \, dt}{E(u^N(0))} \right),
\]

where \(V_{N,M} = \text{Span}\{\phi_{N,k}; \, k = 1, \ldots, M\}\) and \(M\) is set to be the integer part of \(\frac{2}{\pi}N\).

Note that for the approximation of the control, we would now consider the minimization of (13) for initial data \((u_0^N, u_1^N) \in V_{N,M}^2\).

4. Spectral filtering. We have just recalled that the lack of uniform observability is due to the high frequency components of the approximate solution of the wave equation. On another hand, spectral approximation methods, like the Legendre Galerkin method, are known to be very precise when exact solutions are very smooth but also to lose part of their efficiency when solutions are not that smooth. In the latter case, the high frequency components of the approximation are again seen as the main source of errors or instabilities, for instance giving rise to the famous Gibbs phenomenon for discontinuous exact solutions.

A way to recover precision is to dissipate the high modes using spectral filters (see for instance [6, 18]). These filters are called spectral filters because the filtered approximate solutions are obtained by modifying their spectral (here, polynomial) expansion. Given a spectral expansion

\[
u^N(x) = \sum_{k=1}^{N-1} \alpha_{N,k} \tilde{L}_k(x),
\]

the filtered expansion \(F_N u^N\) is

\[
F_N u^N(x) = \sum_{k=1}^{N-1} \sigma \left( \frac{k - 1}{N - 1} \right) \alpha_{N,k} \tilde{L}_k(x)
\]

where \(\sigma\) is called a filter function or simply a filter. More precisely, we say that \(\sigma : [0, 1] \rightarrow [0, 1]\) is a \(p\)-th order filter \((p \geq 1)\) if
\(\sigma \in C^{p-1}([0, 1]),\)
\(\sigma(0) = 1, \sigma^{(j)}(0) = 0 \text{ for } 1 \leq j \leq p - 1,\)
\(\sigma(1) = 0, \sigma^{(j)}(1) = 0 \text{ for } 1 \leq j \leq p - 1.\)

Legendre polynomials are the eigenfunctions of a Sturm-Liouville operator arranged in the increasing order of their associated eigenvalues or frequencies. The idea behind this filtering procedure is thus to keep almost unchanged the low frequency components of the expansion while having a progressive damping on the higher modes.

Note that the filtering is generally performed on the expansion in terms of the Legendre polynomials \(L_k(x), k = 0, \ldots, N\), which are of increasing frequency with increasing \(k\), instead of \(\tilde{L}_k(x)\). But as we have \(\tilde{L}_k(x) = 1 + \frac{\cos(\pi \eta)}{2} + \frac{35\sigma_0(\eta) + 70\sigma_0(\eta)^2 - 20\sigma_0(\eta)^3}{(2p+1)!} \int_0^\eta (t(1-t))^{p-1} \, dt\), functions \(\tilde{L}_k(x)\) are still of increasing frequency components with increasing \(k\).

In this work we consider the following filter functions, all discussed in [6]:

- **Cesáro filter** (first order)
  \[\sigma(\eta) = 1 - \eta,\]

- **Lanczos filter** (first order)
  \[\sigma(\eta) = \frac{\sin(\pi \eta)}{\pi \eta},\]

- **Raised cosine filter** (second order)
  \[\sigma(\eta) = \frac{1 + \cos(\pi \eta)}{2},\]

- **Sharpened raised cosine filter** (8-th order)
  \[\sigma(\eta) = \sigma_0(\eta)^4 \left(35 - 84\sigma_0(\eta) + 70\sigma_0(\eta)^2 - 20\sigma_0(\eta)^3\right),\]
  where \(\sigma_0\) denotes the raised cosine filter.

- **Vandeven filter** (p-th order)
  \[\sigma(\eta) = 1 - \frac{(2p-1)!}{(p-1)!} \int_0^\eta (t(1-t))^{p-1} \, dt\]

- **Exponential filter** (p-th order)
  \[\sigma(\eta) = e^{-\alpha \eta^p}, \quad \alpha > 0.\]

The exponential filter does not satisfy condition (iii) of the definition. However, taking \(\alpha = -\log(\epsilon_m)\), where \(\epsilon_m\) is the machine accuracy, the value of \(\sigma(1)\) will be computationally interpreted as a 0. The exponential filter has the advantage to be of arbitrary order, like the Vandeven filter, but compared to the latter it has a much lower computational cost.

Vandeven and exponential filters in the context of Legendre spectral methods were studied numerically in [18]. There, it was observed that they behave similarly, they improve accuracy in polynomial expansions of smooth functions and they may be essential to stability for first order wave equations. It was also observed that increasing the order of the filter improves its accuracy.

Figure 4 shows the graphs of the first four filters, whereas Figure 5 shows the graphs of the last two, for different values of \(p\).
Regarding our observability problem, we now consider the boundary observation of the filtered expansion $\mathcal{F}_N u^N$ of $u^N$, that is

\[(\mathcal{F}_N u^N)_x(1,t) = \sum_{k=1}^{N-1} \sigma \left( \frac{k-1}{N-1} \right) a_{N,k}(t) \tilde{L}_k(1).\]

$(\mathcal{F}_N u^N)_x(1,t)$ is expected to converge toward $u_x(1,t)$ (for an appropriate norm). The discrete controllability problem that we consider reduces to minimizing the functional

\[J_{N,\sigma}(u_0^N, u_1^N) = \frac{1}{2} \int_0^T \left| (\mathcal{F}_N u^N)_x(1,t) \right|^2 \, dt\]

over $(\mathbb{P}_N^0)^2$. An approximation of the exact control then is

\[v_N(t) = (\mathcal{F}_N u^N)_x(1,t),\]

where $u^N$ is the solution of (15) with the initial data $(u_0^N, u_1^N) \in (\mathbb{P}_N^0)^2$ minimizing (20).
We now look for uniform bounds (with respect to \( N \)) on the observability and continuity constants \( c_{N,T} \) and \( C_{N,T} \) defined respectively by

\[
c_{N,T} = \inf_{(u^N_0, u^N_1) \in (P^0_N)^2} \left( \int_0^T \left| (\mathcal{F}_N u^N)_x (1, t) \right|^2 dt \right) / E(u^N(0)) \tag{21}
\]

\[
C_{N,T} = \sup_{(u^N_0, u^N_1) \in (P^0_N)^2} \left( \int_0^T \left| (\mathcal{F}_N u^N)_x (1, t) \right|^2 dt \right) / E(u^N(0)). \tag{22}
\]

Figure 6 shows the behaviour of \( c_{N,T} \) and \( C_{N,T} \) of (21)-(22) for different filters. According to these numerical results, there exists a uniform lower bound on \( c_{N,T} \)

Figure 6. Values of \( c_{N,T} \) (left column) and \( C_{N,T} \) (right column) for different filters and \( T = 8 \). The exponential and Vandeven filters (bottom) are both of order \( p = 4 \) for the spectral filters considered here. However, we did not find any that could give an upper bound on \( C_{N,T} \). Figure 7 shows our best result in this regard. A seemingly uniform bound on \( C_{N,T} \) is obtained with the exponential filter but only up to a threshold value for \( N \) (around 100). Moreover, in the meantime, the lower bound for \( c_{N,T} \) is dangerously low.

5. Mixed Legendre-Galerkin method. As an alternative to the classical semi-discrete Legendre-Galerkin approximation of the homogeneous wave equation (1), we consider a mixed formulation, where \( u \) and \( u_t \) are the unknowns and are approximated directly (instead of \( u \) alone in the classical formulation). This is similar to what was considered in [7] using finite elements instead for the space discretization.
Figure 7. Values of $c_{N,T}$ (left) and $C_{N,T}$ (right) for the exponential filter with $p = 2$ and $T = 8$

Let $z = u_t$. Then a mixed formulation is: find $(u(t), z(t)) \in H^1_0 \times L^2$ satisfying

\[
\frac{d}{dt} \int_{-1}^{1} u(x,t) \tilde{v}(x) \, dx = \int_{-1}^{1} z(x,t) \tilde{v}(x) \, dx \quad \forall \tilde{v} \in L^2
\]

\[
\frac{d}{dt} < z(t), \hat{v} >_{H^{-1}, H^1_0} = -\int_{-1}^{1} u_x(x,t) \hat{v}_x(x) \, dx \quad \forall \hat{v} \in H^1_0,
\]

for $0 \leq t \leq T$ and initial conditions

\[
u(0) = u_0 \in H^1_0, \quad u_t(0) = u_1 \in L^2.
\]

The mixed Legendre-Galerkin method considered here consists in seeking an approximation $(u^N(t), z^N(t)) \in \mathbb{P}_0^N \times \mathbb{P}^{N-2}$, $0 \leq t \leq T$, satisfying

\[
\frac{d}{dt} \int_{-1}^{1} u^N(x,t) \tilde{v}(x) \, dx = \int_{-1}^{1} z^N(x,t) \tilde{v}(x) \, dx, \quad \forall \tilde{v} \in \mathbb{P}^{N-2}
\]

(23)

\[
\frac{d}{dt} < z^N(t), \hat{v} >_{H^{-1}, H^1_0} = -\int_{-1}^{1} u^N_x(x,t) \hat{v}_x(x) \, dx, \quad \forall \hat{v} \in \mathbb{P}_0^N,
\]

(24)

where $u^N_0 \in \mathbb{P}_0^N$ and $u^N_1 \in \mathbb{P}^{N-2}$ are suitable approximations of $u_0$ and $u_1$ respectively. Writing down

\[
u^N(x,t) = \sum_{k=1}^{N-1} a_{N,k}(t) \widetilde{L}_k(x),
\]

\[
z^N(x,t) = \sum_{k=0}^{N-2} b_{N,k}(t) L_k(x),
\]

and $a_N(t) = (a_{N,1}(t), \ldots, a_{N,N-1}(t))^t$ and $b_N(t) = (b_{N,0}(t), \ldots, b_{N,N-2}(t))^t$, (23)-(24) can be rewritten in the matrix form

\[
\begin{pmatrix}
  a_N(t) \\
  b_N(t)
\end{pmatrix} =
\begin{pmatrix}
  D_N & 0 \\
  0 & D_N^{-1}
\end{pmatrix}
\begin{pmatrix}
  0 & M_N \\
  -K_N & 0
\end{pmatrix}
\begin{pmatrix}
  a_N(t) \\
  b_N(t)
\end{pmatrix},
\]

and
where $K_N, D_N \in M_{-1,N-1}(\mathbb{R})$ are given by

$K_N(i,j) = \begin{cases} 
1 & \text{if } i=j \\
0 & \text{otherwise}
\end{cases}$

$M_N(i,j) = \begin{cases} 
\frac{2}{2^j - 1} & \text{if } i=j \\
0 & \text{otherwise}
\end{cases}$

$D_N(i,j) = \begin{cases} 
\frac{\sqrt{2}}{2^j + 1} & \text{if } i=j \\
\frac{-\sqrt{2}}{2^j + 1} & \text{if } i=j+2 \\
0 & \text{otherwise}
\end{cases}$

Here again, the energy of the solutions of (23)-(24) is conserved along time $E_m(u^N(t), z^N(t)) = E_m(u^N(0), z^N(0))$, where

$$E_m(u, z) = \frac{1}{2} \int_{-1}^{1} \left( |z(x)|^2 + |u_x(x)|^2 \right) dx.$$

For the approximation of the control, we would now consider the minimization of (13) for initial data $(u^N_0, u^N_1) \in P^N_0 \times P^{N-2}$, where $u$ (and $z$) is solution of (23)-(25). We thus define the observability and continuity constants $c_{N,T}$ and $C_{N,T}$, respectively, by

$$c_{N,T} = \inf_{(u^N_0, u^N_1)\in P^N_0 \times P^{N-2}} \left( \int_0^T |u^N_x(1,t)|^2 dt \right) / E_m(u^N(0), z^N(0)),$$

$$C_{N,T} = \sup_{(u^N_0, u^N_1)\in P^N_0 \times P^{N-2}} \left( \int_0^T |u^N_x(1,t)|^2 dt \right) / E_m(u^N(0), z^N(0)).$$

Their behaviour are shown in Figure 8. We see that we recover a positive uniform lower bound for $c_{N,T}$, but no uniform upper bound for $C_{N,T}$.

![Figure 8](image-url)
6. Nitsche’s method. Nitsche’s method [29] was first introduced to weakly impose boundary conditions of Dirichlet type. Subsequently, the method was used to (weakly) impose the inter-elements continuity in the discontinuous Galerkin finite element method (DGFEM) [1]. Nitsche’s method has similarities with the penalty method but unlike the latter it is consistent and does not have its ill-conditioning problems at the discrete level. In fact, Nitsche’s method is closely related with a stabilized form of the Lagrange multiplier method even if it does not involve any multiplier [17].

The DGFEM combined with Nitsche’s method for inter-elements continuity and applied to the second order wave equation has been analyzed in [16]. The numerical boundary observability of waves for this particular type of finite element discretization has been studied in [27] where it is shown that there is no uniform numerical boundary observability, unless Fourier filtering or a multi-grid strategy is applied. We also note that a formulation close to Nitsche’s method combined with a space-time finite element approximation of the wave equation with non-homogeneous Dirichlet boundary conditions has been analyzed in [2].

Here we consider Nitsche’s method for the weak imposition of the Dirichlet boundary condition at \( x = 1 \), which is to say at the boundary point where the solutions are observed or controlled. We also performed numerical experiments with Nitsche’s method used at both endpoints, but the results are qualitatively similar.

There are various versions of Nitsche’s method. Here we use a symmetric form which has the property of still leading to a conservative finite dimensional dynamical system. Even in this particular form, Nitsche’s method depends on a parameter which, for the stability of the discrete formulation, has to be chosen carefully.

Let \( H^1_L = \{ \psi \in H^1 \mid \psi(-1) = 0 \} \) and let \( P^N_L \) be the set of polynomials of degree \( N \) vanishing at \( x = -1 \). We have, for instance,

\[
H^1_L = \{ \psi \mid \psi(-1) = 0 \} 
\]

where \( \hat{L}_k(x) = L_k(x) - (-1)^k, \ k \geq 1 \). Let us define the following bilinear form over \( P^N_L \times P^N_L \):

\[
A_N(u^N, \psi^N) = \int_{-1}^{1} u^N_x(x, t)\psi^N_x(x) \, dx - u^N_x(1, t)\psi^N(1) - u^N_x(1, t)\psi^N_x(1) + \gamma N^2 u^N(1, t)\psi^N(1).
\]

Note that this bilinear form is symmetric. The approximation method using the symmetric variant of Nitsche’s method then reduces to seeking an approximation \( u^N(., t) \in P^N_L, t > 0 \), in the form

\[
u^N(x, t) = \sum_{k=1}^{N} a_{N,k}(t)\hat{L}_k(x),
\]

and satisfying

\[
\int_{-1}^{1} u^N_x(x, t)\psi^N(x) \, dx + A_N(u^N, \psi^N) = 0, \ \forall \psi^N \in P^N_L. \tag{26}
\]

The linear system (26) can be written in the matrix form

\[
M_N a^N(t) + K_N a_N(t) = 0,
\]
where \( a_N(t) = (a_{N,1}(t), \ldots, a_{N,N}(t))^t \) and \( (M_N, K_N) \in M_{N \times N}(\mathbb{R})^2 \) are given by

\[
M_N(i, j) = \begin{cases} 
\frac{4j + 4}{2i + 1} & \text{if } i=j \\
-2 & \text{if } i + j \text{ odd} \\
2 & \text{if } i + j \text{ even and } i \neq j 
\end{cases}
\]

\[
K_N = P_N - Q_N - Q_N + \gamma N^2 R_N,
\]

\[
P_N(i, j) = \begin{cases} 
r^2 + r & \text{if } i + j \text{ even} \\
0 & \text{otherwise}
\end{cases}
\]

\[
Q_N(i, j) = \begin{cases} 
j^2 + j & \text{if } i \text{ odd} \\
0 & \text{otherwise}
\end{cases}
\]

\[
R_N(i, j) = \begin{cases} 
4 & \text{if } i, j \text{ are odd} \\
0 & \text{otherwise}
\end{cases}
\]

Analogously to what happens with convergence proofs of approximations based on the DGFEM using Nitsche’s method (see [16]), we need the bilinear form \( A_N \) to be coercive (with respect to a \( H^1 \)-norm) in \( \mathbb{P}_N^2 \times \mathbb{P}_N^2 \). This depends on the value of \( \gamma \). We show next that coercivity is ensured with \( \gamma > 1/2 \). In order to prove this, let us recall the following inverse inequality (see [34]):

\[
|p^N(1)|^2 \leq \frac{(N + 1)^2}{2} |p^N(x)|^2_{H^1_0([-1,1])}, \forall p^N(x) \in \mathbb{P}^N.
\]

The coercivity easily follows, provided \( \gamma > 1/2 \):

\[
A_N(\psi^N, \psi^N) = |\psi^N|_{H^1_0}^2 - 2\psi^N(x)\psi^N(1) + \gamma N^2 |\psi^N(1)|^2 \\
\geq |\psi^N|_{H^1_0}^2 - 2 |N^{-1}\psi^N(1)| |N\psi^N(1)| + \gamma N^2 |\psi^N(1)|^2 \\
\geq |\psi^N|_{H^1_0}^2 - \frac{1}{\epsilon} |N^{-1}\psi^N(1)|^2 - \epsilon |N\psi^N(1)|^2 + \gamma N^2 |\psi^N(1)|^2, \\
\geq (1 - \frac{1}{2\epsilon}) |\psi^N|_{H^1_0}^2 + (\gamma - \epsilon) |N\psi^N(1)|^2, \\
\geq C||\psi^N||^2_{H^1_0},
\]

for \( C > 0 \) if \( \gamma > \epsilon > 1/2 \) and where \( ||\cdot||_{1,N} \) denotes the following norm on \( H^1_0 \):

\[
||\psi||^2_{1,N} := |\psi|_{H^1_0}^2 + N^2 |\psi(1)|^2.
\]

Note that the coercivity constant \( C \) does not depend on \( N \) and that \( A_N \) is coercive, also uniformly with respect to \( N \), if instead of \( ||\cdot||^2_{1,N} \) we take the equivalent norm \( ||\cdot||_{H^1_0} \).

Taking \( \psi^N = u^N(x, t) \) in (26), we obtain that the energy of the numerical solutions of (26), defined by

\[
E_{N, \gamma}(u^N(t)) = \frac{1}{2} \int_{-1}^1 |u^N_i(x, t)|^2 + |u^N(x, t)|^2 dx \\
+ \left( \frac{\gamma}{2} N^2 u^N(1, t) - u^N_1(1, t) \right) u^N(1, t),
\]

is preserved along time : \( E_{N, \gamma}(u^N(t)) = E_{N, \gamma}(u^N(0)) \). Note that we have proved that if \( \gamma > 1/2 \) then \( E^{1/2}_{N, \gamma} \) is a norm in the energy space \( H^1_0 \times L^2 \) for solutions of the wave equation and solutions of (26) as well.
Regarding our observability problem, we define the following discrete observability and continuity constants $c_{N,T}$ and $C_{N,T}$ respectively:

$$c_{N,T} = \inf_{(u_0^N, u_1^N) \in (\mathbb{P}_N^L)^2} \left( \int_0^T |u_x^N(1, t) - \gamma N^2 u^N(1, t)|^2 dt \right) / E_{N,\gamma}(u^N(0)), \quad (27)$$

$$C_{N,T} = \sup_{(u_0^N, u_1^N) \in (\mathbb{P}_N^L)^2} \left( \int_0^T |u_x^N(1, t) - \gamma N^2 u^N(1, t)|^2 dt \right) / E_{N,\gamma}(u^N(0)). \quad (28)$$

Some remarks concerning these choices are in order since the observation is not $u_x^N(1, t)$ any more, but $u_x^N(1, t) - \gamma N^2 u^N(1, t)$ instead.

First, if we use Nitsche’s method and the resulting discrete formulation (26) only to have the approximation $u^N$ involved in the functional $J_N$ (defined in (13)) to be minimized then it is natural to observe $u_x^N(1, t)$. Unfortunately, our numerical results show that if we drop the term $\gamma N^2 u^N(1, t)$ then $c_{N,T}$ is not uniformly bounded from below by a positive constant.

Alternatively, if we also use Nitsche’s method to approximate the controlled wave equation, that is to append the Dirichlet boundary condition $y(1, t) = v(t)$, then we obtain that the associated observability and continuity constants are defined by (27) and (28), and that the controllability problem reduces to minimize the functional

$$J_N^*(u_0^N, u_1^N) = \frac{1}{2} \int_0^T |u_x^N(1, t) + \gamma N^2 u^N(1, t)|^2 dt$$

$$- \int_{-1}^1 y_1 u_0^N dx + \int_{-1}^1 y_0 u_1^N dx. \quad (30)$$

over $(\mathbb{P}_N^L)^2$.

Indeed, let $y^N(., t) \in \mathbb{P}_N^L$ be the numerical approximation of $y(., t)$ satisfying

$$\int_{-1}^1 y_{tt}^N(x, t) \psi^N(x) dx + \int_{-1}^1 y_x^N(x, t) \psi_x^N(x) dx - y_x^N(1, t) \psi_x^N(1) + y_x^N(1, t) \psi_x^N(1)$$

$$+ y_x^N(0, t) \psi_x^N(1) = v^N(t) (-\psi_x^N(1) + \gamma N^2 \psi^N(1)), \forall \psi^N \in \mathbb{P}_N^L, \quad (31)$$

and the initial conditions $y^N(x, 0) = y_0^N$ and $y_x^N(x, 0) = y_1^N$, where $y_0^N$ and $y_1^N$ are projections of $y_0$ and $y_1$, respectively, over $\mathbb{P}_N^L$. Let’s also consider, for the moment, that $v^N(t)$ is a control, that is $y^N(x, T) = y_0^N(x, T) = 0$. Now let, in (31), $\psi^N = u^N$, where $u^N(x, t)$ is the solution of (26), and let’s integrate this relation in time. By integrating by parts twice in time the first integral, we obtain

$$\int_{-1}^1 y_x^N(x, t) u_t^N(x, t) dx dt + \int_{-1}^1 y_x^N(x, t) u_x^N(x, t) dx dt$$

$$- \int_{-1}^1 y_x^N(1, t) u^N(1, t) dt - \int_{-1}^T y_x^N(1, t) u_x^N(1, t) dt + \gamma N^2 \int_{-1}^T y_x^N(1, t) u_x^N(1, t) dt$$

$$= \int_0^T v^N(t) (-u_x^N(1, t) + \gamma N^2 u^N(1, t)) dt + \int_{-1}^1 u_0^N(x) y_1^N(x) - u_1^N(x) y_0^N(x) dx.$$

The left-hand side corresponds to the left-hand side of (26) where $\psi^N = y^N$. Thus, we obtain

$$\int_{-1}^T v^N(t) (-u_x^N(1, t) + \gamma N^2 u^N(1, t)) dt = \int_{-1}^1 u_1^N(x) y_0^N(x) - u_0^N(x) y_1^N(x) dx. \quad (32)$$
Using the same arguments as in the introduction for the continuous wave equation, we obtain that the approximate control of minimal $L^2(0,T)$-norm is then $v^N(t) = u^N_x(1,t) - \gamma N^2 u^N(1,t)$ where $u^N$ is the solution of (26) with the initial data $(u^N_0, u^N_1)$ minimizing $J^N_{\gamma}$ over $(P^N_L)^2$. Let us recall that a uniform positive lower bound $c_{N,T} \geq c_T > 0$ ensures that $J^N_{\gamma}$ is uniformly coercive.

In Figure 9 we show the behaviour of $c_{N,T}$ and $C_{N,T}$ in (27)-(28) for the particular case $\gamma = 0.8$. We obtain numerically an uniform lower bound for $c_{N,T}$ but no upper bound for $C_{N,T}$.

In order to show the influence of the term $\gamma N^2 u^N(1,t)$ in the observation and the resulting definition of $c_{N,T}$ and $C_{N,T}$, we show in Figure 10 that if we drop this term then, in the same situation than in the previous example ($T = 8$, $\gamma = 0.8$), we do not have uniform observability (positive lower bound on $c_{N,T}$) any more, nor we have uniform continuity of the trace (upper bound on $C_{N,T}$).

In this section we show the influence that all the reme-dies (spectral filter, mixed formulation and Nitsche’s method) to the non-uniform observability property have on the computation of the associated control problem.

In this example, we take
\[ y_0(x) = x + 1, \quad y_1(x) = 0. \]

The corresponding values of the exact control \( v \) and the initial data \( (u_0, u_1) \) for the adjoint homogeneous wave equation giving the control \( v = u_x(1, t) \) are

\[
\begin{align*}
    u_0(x) &= 0, \quad u_1(x) = -x/4 - 1/4, \\
    v(t) &= -t/4 + 1/2 \quad \text{if} \quad 0 < t < 4, \\
    v(t) &= -t/4 + 3/2 \quad \text{if} \quad 4 < t < 8.
\end{align*}
\]

In Figure 11 we show the approximation \( v^N(t) \) obtained with the Legendre Galerkin method, without any of the remedies studied here, that is minimizing (13). This approximation is highly oscillatory, even for low values of \( N \), and this pathology amplifies with growing values of \( N \).

In Figure 12 we show the approximations \( v^N(t) \), for \( N = 128 \), obtained with the different remedies studied here:

- the spectral method filtering with Cesaro and exponential filters (minimizing (20));
- the mixed formulation of the equations (minimizing (13));
- and Nitsche’s method (minimizing (29)).

All these remedies to the non-uniform observability property of the classical Legendre Galerkin method give good approximations of the exact control. There is some oscillatory pollution for each of them, but always with small amplitudes.
In order to confirm these good results, in Figure 13 we show the behaviour of the errors $|u_0^N - u_0|_{H^1_0}$, $\|u_1^N - u_1\|_{L^2}$ and $\|v^N - v\|_{L^2(0,T)}$ with respect to $N$. For the classical Legendre Galerkin method, without any remedy, there is no convergence of the approximations. However, for each of the remedies studied here, convergence appears to hold, with approximately the same rate.

8. Appendix : Computation of $c_{N,T}$ and $C_{N,T}$. In order to explain how the values of $c_{N,T}$ and $C_{N,T}$ are computed, let us consider the first order matrix form (16) of the Legendre-Galerkin semi-discretization. Defining $b_N(t) := \begin{pmatrix} a_N(t) \\ a'_N(t) \end{pmatrix}$, it also writes

$$\mathbf{b}_N'(t) = A_N \mathbf{b}_N(t),$$

where $A_N$ is the $(2N-2) \times (2N-2)$ matrix

$$A_N = \begin{pmatrix} 0 & I_{N-1} \\ -M_N^{-1}K_N & 0 \end{pmatrix}.$$

The solution is thus $\mathbf{b}_N(t) = e^{A_N t} \mathbf{b}_N(0)$ and the observation is expressed as

$$u^N_x(1,t) = D_N \mathbf{b}_N(t) = D_N e^{A_N t} \mathbf{b}_N(0),$$

where $D_N$ is a suitable row vector of size $2N-2$. We then have

$$\int_0^T |u^N_x(1,t)|^2 \, dt = \mathbf{b}'_N(0) W_{N,T} \mathbf{b}_N(0)$$
where

\[ W_{N,T} = \int_0^T e^{A_N t} D_N^t D_N e^{A_N t} \, dt \]

On the other hand, the (conserved) energy of the discrete solution can be expressed as

\[ E(u^N(0)) = \frac{1}{2} b_N^T (K_N 0 0 M_N) b_N(0). \]

The values of \( e_{N,T} \) and \( C_{N,T} \) are then, respectively, the minimal and the maximal eigenvalues of the generalized eigenproblem: find \((\lambda, U) \in \mathbb{R} \times \mathbb{R}^{2N-2}\) such that

\[ W_{N,T} U = \lambda \frac{1}{2} \begin{pmatrix} K_N & 0 \\ 0 & M_N \end{pmatrix} U \] as the product of two matrix exponentials.

The following lemma ([9]) gives a way to compute \( W_N(0, T) \) as the product of two matrix exponentials.

**Lemma 8.1.** Let

\[ F(t) = \exp \left[ \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} t \right]. \]

Then,

\[ F(t) = \begin{pmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{pmatrix} \]

with \( F_{11}(t) = e^{A_{11} t} \), \( F_{22}(t) = e^{A_{22} t} \) and

\[ F_{12}(t) = \int_0^t e^{A_{11} (t-s)} A_{12} e^{A_{22} s} \, ds. \]

Taking \( A_{11} = -A_{22}^T, A_{12} = D_N^t D_N \) and \( A_{22} = A_N \), we obtain

\[ W_{N,T} = F_{22}(T)^T F_{12}(T). \]

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Figure 13. Errors $|u_0^N - u_0|_{H_0^1}$, $\|u_1^N - u_1\|_{L^2}$ and $\|v^N - v\|_{L^2(0,T)}$ for the classical Legendre Galerkin method, and the remedies studied here: exponential spectral filtering (with $p=6$), the mixed formulation and Nitsche’s method ($\gamma = 1$), for $N = 32, 64, 128, 256$. 