Random Ensembles of Lattices from Generalized Reductions

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Abstract

We propose a general framework to study constructions of Euclidean lattices from linear codes over finite fields. In particular, we prove general conditions for an ensemble constructed using linear codes to contain dense lattices (i.e., with packing density comparable to the Minkowski-Hlawka lower bound). Specializing to number field lattices, we obtain a number of interesting corollaries - for instance, the best known packing density of ideal lattices, and an elementary coding-theoretic construction of asymptotically dense Hurwitz lattices. All results are algorithmically effective, in the sense that, for any dimension, a finite family containing dense lattices is exhibited. For suitable constructions based on Craig’s lattices, this family is significantly smaller, in terms of alphabet-size, than previous ones in the literature.

Keywords: Lattices, sphere packings, random codes, ideal lattices, codes over matrix rings

I. INTRODUCTION

There has been a renewed interest in the search for new constructions of lattices from error-correcting codes due to their various recent applications, such as coding for fading wiretap channels [KOOT15], Gaussian relay networks [HNW15], compound fading channels [CLB16] and index codes [Hua17], to name only a few. For the applications considered in these works, it is desirable to lift a code over a finite field into a lattice that possesses a rich algebraic structure, often inherited from the properties of number fields. In the present work we provide an unified analysis of these constructions and investigate “random-coding”-like results for such lattices. Our focus is on the problem of finding dense structured lattice packings, although our techniques have a much broader scope of applications (as discussed in Section VII).

Indeed, finding the densest packing is a central subject in the Geometry of Numbers, with a variety of well-established connections to Coding Theory. Let \( \Delta_n \) denote the best possible sphere packing density achievable by a Euclidean lattice of dimension \( n \). The celebrated Minkowski-Hlawka theorem (e.g. [Cas97], [GL87]) gives the lower bound \( \Delta_n \geq \zeta(n)/2^{n-1} \) for all \( n \geq 2 \), where \( \zeta(n) = 1 + 1/2^n + 1/3^n + \ldots \) is the Riemann zeta function. Up to very modest asymptotic improvements, this is still, to date, the best lower bound for the best packing density in high dimensions.

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Typical methods for establishing the theorem depend on the construction of random ensembles of lattices and on mean-value results [Sie45], [Rog64]. Rush [Rus89] and Loeliger [Loe97] obtained the lower bound from integer lattices constructed from linear codes in $\mathbb{F}_p^n$, in the limit when $p \to \infty$, with random coding-like arguments. Improvements to the lower bound, in turn, strongly rely on additional (algebraic) structure. For instance, Vance [Van11] showed that the best quaternionic lattice in dimension $m$ (with real equivalent in dimension $4m$) has density at least $3m\zeta(4m)/e 2^{4m-3} \leq \Delta_{4m}$. Using lattices built from cyclotomic number fields, Venkatesh [Ven12] established the bound $\Delta_{2\varphi(m)} \geq m/2^{\varphi(m)-1}$, where $\varphi(m)$ is the Euler’s totient function (since $m$ can grow as fast as $\varphi(m) \log \log \varphi(m)$ this provides the first super-linear improvement). From another perspective, Gaborit and Zémor [GZ07], and Moustrou [Mou16] exploit additional coding-theoretic and algebraic structures to significantly reduce the family size of ensembles containing dense lattices.

**Main Contributions.** In this work we investigate general random lattices obtained from error correcting codes. The objective of this study is two-fold: we provide unified analyses and coding-theoretic proofs of the aforementioned results, as well as a simple condition to verify if any new construction can be used to build ensembles containing dense lattices. We start from the fairly general definition of a *reduction*, i.e. a mapping that takes a lattices into the space $\mathbb{F}_p^n$. For a general reduction we prove the following:

**Theorem 1.** Let $\phi_p : \Lambda \to \mathbb{F}_p^n$ be a family of reductions (surjective homomorphisms), where $\Lambda$ is a lattice of rank $m$. Consider the ensemble

\[ L_p = \left\{ \beta \phi_p^{-1}(C) : C \text{ is a } k \text{-dimensional code in } \mathbb{F}_p^n \right\}, \]

for an appropriate normalization factor $\beta$ so that all lattices have volume $V$. Denote by $N_{\Lambda}(r) = \#(B_r^n \cap \Lambda')$ the number of primitive points of $\Lambda$ inside a ball of radius $r$. If the first minimum of $\Lambda_p = \ker \phi_p$ satisfies

\[ \liminf_{p \to \infty} \left( \frac{\lambda_1(\Lambda_p)}{p^{n/m}} \right) > 0, \]

\[ \lim_{p \to \infty} E_{L_p}[N_{\Lambda}(r)] = (\zeta(m)V)^{-1} \text{vol } B_r, \]

where the average is with respect to the uniform distribution on $L_{k,p}$.

A slightly stronger version of the above result is precisely stated in Theorem 2. We shall refer to a family of reductions that satisfies condition (1) as *non-degenerate*. Non-degeneracy is indeed a very mild condition, and is satisfied, for instance, if $\Lambda_p$ has non-vanishing Hermite parameter (e.g., $\Lambda_p = p\mathbb{Z}^n$). Non-degenerate constructions immediately yield lattices satisfying the Minkowski-Hlawka lower bound (see the discussion in the end of Section III). By choosing specific suitable families of non-degenerate reductions, we can further improve this density and obtain a number of interesting corollaries. We highlight one of them:
Corollary 1. Let \( \mathcal{O}_K \) be the ring of integers of a degree \( n \) number field \( K \) containing \( r(K) \) roots of unity. For any integer \( t \geq 2 \), there exists a \( \mathcal{O}_K \)-lattice with dimension \( t \) and packing density

\[
\Delta \geq \frac{r(K)t\zeta(tn)}{e(1 - e^{-t})2^{tn}}.
\]

This proves for instance, the existence of ideal lattices in any dimension with density better, by a linear factor, than the Minkowski-Hlawka lower bound. This also recovers, for \( t = 2 \) and a judicious choice of number field/degree, the density in [Ven12, Thm. 1] and [Mou16, Thm. 2]. By allowing reductions to codes over matrix rings (rather than the field \( \mathbb{F}_p \)), we provide, in Section \( \text{V} \), a coding-theoretic proof of the existence of dense Hurwitz lattices, as in [Van11].

Here is how Theorem 1 may be interpreted: the density of the kernel-lattice \( \Lambda_p \) is improved by “adjoining” a code \( C \) to it, through the reduction \( \phi_p \). Now if \( \Lambda_p \) itself has a reasonable density, we can improve it up to the Minkowski-Hlawka bound. Building on this idea, we show in Section \( \text{VI} \) that if we start from a suitable reduction so that base lattice \( \Lambda \) is not so thick (in terms of its covering density) and such that the kernels are not so sparse (in terms of packing density), we can bound the required size of \( p \) to be within a finite range. For instance, we show that by starting from the family of Craig’s lattices [CS98], we can build dense lattices from codes with any fixed ratio \( k/n = 1/(1 + 2\delta) \), where \( \delta > 0 \) is any small number, and \( p = O((\log n)^{1/2+\delta}) \). This improves significantly the size of codes required by [Rus89] (where the ratio is set to \( k/n = 2/3 + o(1) \) and \( p = \omega(n^{3/2}) \)) and [Loe97] (\( p = \omega(n^{1/2+\delta}) \)).

As observed in [CS98, pp. 18-19], the works of Rush (and Loeliger) already significantly reduce the family sizes of typical proofs of the Minkowski-Hlawka lower bound. It is worth noting that, in terms of absolute family size, the best result is achieved by [GZ07] by restricting the average to double-circulant codes. We conjecture that such methods, along with our analyses, may be used to further reduce the complexity of finding a dense lattice.

Organization. This work is organized as follows. In Section \( \text{II} \) we describe some basic definitions and notation. In Section \( \text{III} \) we establish our main result on general reductions and several corollaries. In Section \( \text{IV} \) we consider reductions induced by quotients of ideals in the ring of integers of a number field, proving the main corollaries. In Section \( \text{V} \) we construct random Hurwitz and Lipschitz lattices from codes over matrix rings. In Section \( \text{VI} \) we discuss an “algorithmic” version of the main theorem and draw the final conclusions.

II. Preliminaries and Notation

The Euclidean norm of \( \mathbf{x} \in \mathbb{R}^n \) is denoted by \( \|\mathbf{x}\| = (x_1^2 + \ldots + x_n^2)^{1/2} \). The ball of radius \( r \) in \( \mathbb{R}^n \) is denoted by \( \mathcal{B}_r^n = \{ \mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq r \} \). A lattice \( \Lambda \) is a discrete additive subgroup of \( \mathbb{R}^n \). Denote by \( \text{span} \Lambda \) the minimal subspace of \( \mathbb{R}^n \) that contains \( \Lambda \). The rank of \( \Lambda \) is defined to be the dimension of \( \text{span} \Lambda \). The quotient \( (\text{span} \Lambda)/\Lambda \) is compact, and its volume, denoted by \( V(\Lambda) \), is said to be the volume of \( \Lambda \). The first minimum \( \lambda_1(\Lambda) \) of \( \Lambda \) is the shortest Euclidean norm of non-zero vectors in \( \Lambda \). In general, the \( i \)-th minima of \( \Lambda \) are defined as

\[
\lambda_i(\Lambda) = \min \{ r : \dim (\text{span} \{ \mathcal{B}_r^n \cap \Lambda \}) = i \}.
\]
The packing density of a rank $m$ lattice $\Lambda$ is defined as

$$\Delta(\Lambda) = \frac{\text{vol } B_{\Lambda/m}^m}{V(\Lambda)}.$$  

We say that a point in $x \in \Lambda$ is primitive if the intersection between $\Lambda$ and the open line segment $\{\alpha x : \alpha \in (0, 1)\}$ is the empty set. The set of all primitive points in $\Lambda$ is denoted by $\Lambda'$. 

Theorem [11] implies the Minkowski-Hlawka lower bound in the following fashion. From the average result, it follows that it must exist at least one $\Lambda \in \mathcal{L}$ such that $N_\Lambda(r) \leq (\zeta(m)V)^{-1}\text{vol } B_r$. Now if we force the right-hand side to be equal to $2(1 - \varepsilon)$, for some small $\varepsilon > 0$, then, since a lattice has at least two minimum vectors, we must have $N_\Lambda(r) = 0$. Therefore $\Lambda$ can pack balls of radius $r/2$; rearranging the terms gives us, up to $\varepsilon$, the Minkowski-Hlawka bound. If $\Lambda$ is a lattice with guaranteed number of minimum vectors (say, $L$) we can, by similar arguments, achieve density $L(1 - \varepsilon)/2^m$.

A $k$-dimensional vector subspace $C \subset \mathbb{F}_p^n$ is called a (linear) code with parameters $(n, k, p)$ (or simply an $(n, k, p)$-code).

### III. Generalized Reductions

From now on, let $\Lambda$ be a rank $m$ lattice and let $n \leq m$ be an integer.

**Definition 1.** Let $\phi_p : \Lambda \to \mathbb{F}_p^n$ be a surjective homomorphism. Given a linear code $C$, its associated lattice via $\phi_p$ is defined as $\Lambda_p(C) \triangleq \phi_p^{-1}(C)$.

A surjective homomorphism as in the above definition will, from now on, be called a reduction. We shall see that $\Lambda_p(C)$ is indeed a lattice of rank $m$. First observe that $\Lambda_p(C)$ is a subgroup of $\Lambda_p(\mathbb{F}_p^n) = \Lambda$. Since the quotient $\Lambda/\ker(\phi_p) \simeq \mathbb{F}_p^n$ is finite, $\ker(\phi_p) = \Lambda_p(\{0\}) \triangleq \Lambda_p$ is a sub-lattice of $\Lambda$, of rank $m$. From the inclusion $\Lambda_p \subset \Lambda_p(C) \subset \Lambda$, we conclude that the three lattices have the same rank. Moreover, $\Lambda_p(C)/\Lambda_p \simeq C$, and therefore $V(\Lambda_p(C)) = |C|^{-1}p^nV(\Lambda)$.

**Remark 1.** There is an off-topic connection between Definition [11] and combinatorial tilings. If in addition to being surjective, the reduction $\phi_p$ is a bijection when restricted to a set $\mathcal{P} \subset \Lambda$ of cardinality $p^n$, then $\mathcal{P}$ tiles $\Lambda$ by translations of vectors of $\Lambda_p$.

This framework contains classical Construction A [CS98], [Rus89], the constructions in [KOO15], and [VKO14]. We derive sufficient conditions for this general construction to admit a Minkowski-Hlawka theorem. Set $\beta = V^{1/m}/(p^{n-k}V(\Lambda)^{1/m}$ and let

$$\mathbb{L}_p = \{\beta \Lambda_p(C) : C \text{ is an } (n, k, p) - \text{code}\}$$

be the ensemble of all lattices associated to codes of dimension $k$, normalized to volume $V$. Suppose a lattice in $\mathbb{L}_p$ is picked at random by choosing $C$ uniformly. We shall prove a generalized version of the Minkowski-Hlawka theorem for $\mathbb{L}_p$. Instead of functions with bounded support, we will consider a wider classer of functions. Let $W = \text{span}(\Lambda)$ be the minimal subspace of $\mathbb{R}^n$ containing $\Lambda$ (therefore, $\dim(\text{span}(\Lambda_p(C))) = m$).
Definition 2. Let \( f : W \to \mathbb{R} \) be a Riemman-integrable function. We say that \( f \) is semi-admissible if
\[
|f(\mathbf{x})| \leq \frac{b}{(1 + \|\mathbf{x}\|)^{m+\delta}}, \forall \mathbf{x} \in W
\]
where \( b > 0 \) and \( \delta > 0 \) are positive constants.

Notice that any bounded integrable function with compact support is semi-admissible. Of particular interest are indicator functions of bounded convex sets.

Remark 2. If \( f \) and its Fourier transform \( \hat{f} \) are semi-admissible, then \( f \) is said to be admissible. In this paper we will not be concerned about admissible functions, which, nonetheless, play an important role in the sphere-packing literature.

Theorem 2. Let \( (p_j)_{j=1}^{\infty} \) be an increasing sequence of prime numbers such that there exist reductions \( \phi_{p_j} : \Lambda \to \mathbb{F}_p \) and let \( f : W \to \mathbb{R} \) be a semi-admissible function. If the first minimum of \( \Lambda_{p_j} = \Lambda_{p_j}(\{0\}) \) satisfies
\[
\lambda_1(\Lambda_{p_j}) \geq cp_j^{n-m+\alpha},
\]
for some constants \( c, \alpha > 0 \), then
\[
\lim_{p_j \to \infty} \mathbb{E}_{\Lambda_p} \left[ \sum_{x \in \beta \Lambda_{p_j}(C)} f(x) \right] = (\zeta(m)V)^{-1} \int_{\mathbb{R}^m} f(x)dx,
\]
where the average is taken over all \( \beta \Lambda_p(C) \) in the ensemble \( \Lambda_p \) (Equation 2).

Proof: Recall that the set of \( \mathcal{C}_{n,k} \) of all \( (n,k) \)-codes satisfies Loeliger’s balancedness equation \([Loe97]\)
\[
\frac{1}{|\mathcal{C}_{n,k}|} \sum_{C \in \mathcal{C}_{n,k}} \sum_{c \in C \setminus \{0\}} g(c) = \frac{p^k - 1}{p^n - 1} \sum_{v \in \mathbb{F}_p \setminus \{0\}} g(v),
\]
for a function \( g : \mathbb{F}_p^n \to \mathbb{R} \). Now for \( f : W \to \mathbb{R} \),
\[
\mathbb{E} \left[ \sum_{x \in \beta \Lambda_{p_j}(C)} f(x) \right] = \mathbb{E} \left[ \sum_{\substack{x \in \beta \Lambda_{p_j}(C) \\phi_{p_j}(x/\beta) = 0}} f(x) \right] + \mathbb{E} \left[ \sum_{\substack{x \in \beta \Lambda_{p_j}(C) \\phi_{p_j}(x/\beta) \neq 0}} f(x) \right].
\]
From the assumption on \( f \),
\[
\left| \sum_{\substack{x \in \beta \Lambda_{p_j}(C) \\phi_{p_j}(x) = 0}} f(x) \right| \leq \sum_{x \in \Lambda_{p_j}} \left| \sum_{\substack{x \in \beta \Lambda_{p_j} \\phi_{p_j}(x) = 0}} f(\beta x) \right| \leq \sum_{x \in \Lambda_{p_j}} \frac{1}{(1 + \|\beta x\|)^{m+\delta}}.
\]
Since the lattice \( \Lambda_p(C) \) has rank \( m \geq 1 \), the series on the right-hand-side of the above inequality is absolutely convergent for any \( p_j \), and dominated by some constant. Moreover, since, by assumption
\[
\|\beta x\| \geq \beta \lambda_1(\Lambda_{p_j}) \geq cV^{1/m}(V(\Lambda))^{-1/m}p_j^{\alpha} \to \infty,
\]
the last sum tends to zero, as \( p \to \infty \). For the second term of Equation (5), we have:

\[
\mathbb{E} \left[ \sum_{x \in \Lambda' \setminus \Lambda} f(\beta x) \right] \overset{(a)}{=} \frac{p^k - 1}{p^n - 1} \sum_{x \in \Lambda'} f(\beta x) \overset{(b)}{=} (V^{-1}(m)) \int_W f(x) dx
\]

where (a) follows from (5) and (b) from the definition of Riemann integral and from the Möbius inversion formula (see, e.g., [Cas97, Sec. VI. 3.2]).

**Example 1.** If \( \Lambda = \mathbb{Z}^n \) and \( \phi_p \) is the reduction modulo \( p \), we obtain mod-\( p \) lattices as in [Loe97]. It is clear that \( \Lambda_p = p\mathbb{Z}^n \) satisfy the hypothesis of Theorem 2 with \( m = n \) and \( \alpha = k/n \). This implies Theorem 1 of [Loe97].

**Definition 3.** A sequence of surjective homomorphisms \( (\phi_j)_{j=1}^\infty, \phi_j : \Lambda \to \mathbb{F}_p^n \) is said to be non-degenerate if

\[
\lambda(\Lambda_{p_j}) \geq cp_j^\alpha
\]

for some constants \( c, \alpha > 0 \). Similarly, the sequence of associated ensembles (Equation 2) are said to be non-degenerate.

It follows that if the reductions are non-degenerate, the associated ensemble admits the Minkowski-Hlawka theorem.

**Example 2** ("Natural reduction"). If \( m = n \), there is a natural reduction to \( \mathbb{F}_p^n \) as follows. Given a basis \( x_1, \ldots, x_n \) for \( \Lambda \), take \( \phi_p \) to be the linear map defined by \( \phi_p(x_i) = e_i \in \mathbb{F}_p^n \), where \( e_i \) the \( i \)-th canonical vector \((0, \ldots, 0, 1, 0, \ldots, 0)\). It is clear that \( \phi \) is surjective and \( \ker \phi_p = p\Lambda \), therefore the associated sequence of reductions is non-degenerate. This provides a systematic way of constructing good sublattices of a given lattice.

Taking \( f(x) \) to be the indicator function of a ball in Theorem 2 we recover Theorem 1. Another function of interest is \( f(x) = e^{-\tau \|x\|^2} \) for \( \tau > 0 \), yielding the theta series

\[
\Theta_\Lambda(\tau) = \sum_{x \in \Lambda} e^{-\tau \|x\|^2}.
\]

**Corollary 2.** The average theta series of a sequence of non-degenerate ensemble satisfies

\[
\lim_{p_j \to \infty} E_{n, p_j} [\Theta_\Lambda(\tau)] = V^{-1} \left( \frac{\pi}{\tau} \right)^{m/2} + 1.
\]

Corollary 2 can, for instance, be applied to the construction sufficiently flat Gaussian measures for secure communications (cf [LLBS14]).

**Remark 3.** The condition for non-degeneracy can be re-written as

\[
\lim_{p_j \to \infty} \gamma(\Lambda_{p_j}) > 0,
\]
where $\gamma(\Lambda) = \lambda(\Lambda)/V(\Lambda)^{1/m}$ is the Hermite parameter of $\Lambda$. In other words, non-degeneracy is equivalent to non-vanishing Hermite parameter of the kernel.

We close this section with another consequence of Theorem 2. We shall refer to each ratio

$$\Delta_i(\Lambda) = \frac{\text{vol} B_{\lambda_i/2}^m}{V(\Lambda)}, \ i = 1, \ldots, m,$$

(8)

as the $i$-th successive density of a lattice $\Lambda$. For a sequence of non-degenerate ensemble, put $L = \bigcup_{j=1}^{\infty} L_{p_j}$.

**Corollary 3.** For any $\varepsilon > 0$, there exists $\Lambda_{p_j} \in L$ such that

$$\prod_{i=1}^{m} \Delta_i(\Lambda)^{1/m} \geq \frac{2m\zeta(m)(1-\varepsilon)}{e(1-e^{-m})}.$$  

(9)

**Proof:** The proof follows from a method of Rogers [Rog64], choosing $f(x)$ appropriately in the Minkowski-Hlawka theorem. We shall give a complete proof in the next section, in the context of $\mathcal{O}_K$-lattices.

### IV. CONSTRUCTIONS FROM NUMBER FIELDS

Let $K/\mathbb{Q}$ be a number field with degree $n$ and signature $(r_1, r_2)$. Denote its real embeddings by $\sigma_1, \ldots, \sigma_{r_1}$ and its pairs of complex embeddings by $\sigma_{r_1+1}, \ldots, \sigma_{r_1+2r_2+1}, \overline{\sigma_{r_1+2r_2+1}}$.

Let $\mathcal{O}_K$ be the ring of integers of $K$ and $\mathcal{I} \subset \mathcal{O}_K$ be an ideal. An ideal can be identified with a real lattice of dimension $(r_1 + 2r_2)$ via the canonical embedding

$$\sigma : \mathcal{O}_K \rightarrow \mathbb{R}^{r_1+2r_2}$$

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_{r_1}(x), \Re \sigma_{r_1+1}(x), \ldots, \Re \sigma_{r_1+r_2+1}(x), \Im \sigma_{r_1+1}(x), \ldots, \Im \sigma_{r_1+r_2+1}(x)).$$

Lattices constructed from the embedding of ideals $\mathcal{I} \subset \mathcal{O}_K$ are called ideal lattices, and appear in the study of modular forms, coding theory, and cryptography. In this section we study the Minkowski Hlawka theorem for $\mathcal{O}_K$-lattices and related structures.

Let $E = K \otimes_{\mathbb{Q}} \mathbb{R}$ be the Euclidean space generated by $K$. An $\mathcal{O}_K$-lattice is a free $\mathcal{O}_K$ sub-module of $E^t$, for some $t > 0$. In particular, an $\mathcal{O}_K$ lattice is closed under multiplication by elements of $\mathcal{O}_K$. The Euclidean norm in $E$ is induced by the trace form $\langle x, y \rangle = \text{tr}(x \overline{y}) = \sum_{i=1}^{r_1+r_2} \sigma_i(x) \overline{\sigma_i(y)}$. Notice that $K$ is naturally embedded in $E$ (or in looser terms, $K \subset E$).

We discuss the average behavior of a general reduction from algebraic number theory [KOO15], [CLB16], [HNW15], defined as follows.

**Definition 4.** Let $p$ be a splitting prime, and $\mathfrak{p}$ an ideal above $p$. Consider $\pi : \mathcal{O}_K \rightarrow \mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_p$ a projection onto $\mathfrak{p}$ and $\sigma$ the canonical embedding. Let $\Lambda = \sigma(\mathcal{O}_K)^t$ (the canonical embedding is applied componentwise). Take

$$\phi_p : \Lambda \rightarrow \mathbb{F}_p^t$$

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and define \( \Lambda_p(C) = \phi_p^{-1}(C) \subset \mathbb{R}^n_t \).}

A very important caveat is the fact that there must exist an infinite number of primes \( p \) such that the construction above is possible. This follows from Chebotarev’s density theorem (e.g. [Neu99] Cor. 13.6, p. 547), which implies that the natural density of splitting primes in \( K \) is \( 0 < \delta(M) = 1/|K : \mathbb{Q}| \leq 1/n! \).

**Lemma 1.** The ensemble induced by Definition 4 is non-degenerate.

**Proof:** The minimum algebraic norm of an element of \( p \) is greater or equal than \( p \). Hence \( \Lambda_p = \psi(p^t) \) has minimum norm at least \( \sqrt{np^{1/n}} \), finishing the proof.

**Remark 4.** Very similarly, it is possible to prove that the constructions in [VKO14] are non-degenerate.

Suppose that the \( K \) contains \( r(K) \) roots of unity. Let \( \mu \) be a root of unity contained in \( \mathcal{O}_K \). It follows that \( \|\sigma(\mu x)\| = \|\sigma(x)\| \) (e.g. [Aut16] Lem. 3.1]). Therefore, each \( \Lambda \) constructed as in Definition 4 contains at least \( r(K) \) minimal vectors and we automatically obtain the density \( m(1 - \varepsilon)/2^{\phi(m)} \). For \( t \)-dimensional \( \mathcal{O}_K \) lattices, however, there a loss of a linear factor of \( t \) in the enumerator. Nevertheless, we can improve the following density up to that of Corollary 1 using a method by Rogers, recently employed in [Van11] to quaternionic lattices. We describe the details below for the sake of completeness.

**Proof of Corollary 1.** For \( \Lambda_0 \subset \mathcal{O}_K^t \) let the \( i \)-th successive minima of \( \Lambda_0 \) (over \( K \)) be the smallest \( i \)-such that the ball \( B_{n_t}^{\nu_t} \) contains the canonical embedding of \( i \) linearly independent vectors (over \( K \)). More formally

\[
\lambda^K_i(\Lambda_0) = \min\{ r > 0 : \dim_{\mathbb{K}} (\sigma^{-1}\left( (\sigma(\Lambda_0) \cap B_{n_t}^{\nu_t}) \right)) = i \}.
\]

Notice that \( \lambda^K_1(\Lambda_0) = \lambda_1(\sigma(\Lambda_0)) \) and, in general \( \lambda^K_i(\Lambda_0) \geq \lambda_i(\sigma(\Lambda_0)) \). Also, if \( x_1, \ldots, x_l, l = 1, \ldots, s \), are linearly independent over \( K \) and achieve the successive minima of \( \Lambda_0 \), then the embeddings \( \sigma(x_1), \ldots, \sigma(x_l) \) are linearly independent and primitive in \( \sigma(\Lambda_0) \in \mathbb{R}^n \). Now let \( f : \mathbb{R}^n \to \mathbb{R} \) be the following function with limited support:

\[
f(x) = \begin{cases} 
 1/n & \text{if } \|x\| \leq re^{(1-t)/tn} \\
 -\log \left( \frac{\|x\|}{r} \right) & \text{if } re^{(1-t)/tn} \leq \|x\| \leq re^{1/tn} \\
 0 & \text{otherwise.}
\end{cases}
\]

We have

\[
\int_{\mathbb{R}^n} f(x)dx = \frac{e(1 - e^{-t})r^{nt}\text{vol } B_{n_t}^{nt}}{nt}.
\]
Choose \( r \) such that the right-hand side of this last equation does not exceed \( r(K)V\zeta(nt)/n \). According to Theorem 2 it is possible to find \( \Lambda_1 \in \mathbb{L} \) such that

\[
\sum_{x \in \Lambda'} f(x) < \frac{r(K)}{n}.
\]

Let \( \Lambda_0 \) be the associated lattice in \( \mathcal{O}_K^t \) (i.e., such that \( \beta\sigma(\Lambda_0) = \Lambda_1 \)) and let \( v_1, \ldots, v_s \) be linearly independent vectors in \( \Lambda_0 \) achieving the successive minima, \( \|\sigma(v_i)\| = \lambda_t^K(\Lambda_0) \). We have

\[
\sum_{x \in \Lambda'} f(x) \geq \sum_i \sum_{\mu} f(\mu v_i) = r(K)\sum_i f(\sigma(v_i)).
\]

From this we conclude that, for all \( i \), \( \lambda^K_t(\Lambda_0) \geq re^{1/n-1} \) and

\[
\frac{1}{n} - \log\left(\frac{\prod\lambda_i(\Lambda_0)}{r}\right) < \frac{1}{n}.
\]

Therefore, for the \( t \) successive densities (cf Eq. (8))

\[
\left(\prod \Delta^K_i\right)^{1/t} := \prod \left(\frac{\text{vol}(\mathcal{B}_{\lambda^K_i}/2)}{\mathcal{V}(\Lambda)}\right)^{1/t} \geq \frac{r^{nt}\mathcal{B}_{\lambda^K_i}/2}{2^{nt}\mathcal{V}(\Lambda)} \geq \frac{r(K)t\zeta(nt)}{e(1-e^{-1})2^{nt}}.
\]

But in this case, we can find \( \tilde{\Lambda} \) whose packing density (or \( \Delta^K_1 \)) is greater or equal than \( \frac{r(K)t\zeta(nt)}{e(1-e^{-1})2^{nt}} \). More precisely, the lattice \( \tilde{\Lambda} = (\text{diag}(1/\lambda^K_1, \ldots, 1/\lambda^K_t) \otimes I_{n \times n}) \Lambda_1 \).

has minimum \( (\lambda^K_1 \lambda^K_2 \ldots \lambda^K_t)^{1/t} \), which finishes the proof.

V. BALANCED SETS OF CODES OVER MATRIX RINGS

In some contexts, the “natural” underlying alphabet in the reduction \( \phi_p \) is, rather than the field \( \mathbb{F}_p \), the ring \( \mathcal{M}_n(\mathbb{F}_p) \) of \( n \times n \) matrices with entries in \( \mathbb{F}_p \). Although we can identify \( \mathcal{M}_n(\mathbb{F}_p) \) with \( \mathbb{F}_{p^2}^n \), the identification does not carry enough algebraic structure for our purposes. For instance, we cannot guarantee that the constructed lattices are closed under multiplication by units, which is crucial in order to obtain the full density improvements of these lattices, as in [Van11]. For this reason, we study in this section a version of Theorem 2 for codes over matrix rings.

Let \( \mathcal{R} \) be a finite ring and \( \mathcal{R}^* \) its units. Denote by \( (\mathcal{R}^n)^* \) the set of vectors in \( \mathcal{R}^n \) such that at least one coordinate is a unit. A linear code in \( \mathcal{C} \subset \mathbb{R}^n \) is a free \( \mathcal{R} \)-module of \( \mathcal{R}^n \) (with the natural scalar multiplication). Following [Loe97], we define balanced sets of codes as follows.

**Definition 5.** Consider a non-empty set of codes \( \mathcal{C}_b \) of same cardinality. We say that \( \mathcal{C}_b \) is balanced if any \( x \in (\mathcal{R}^n)^* \) is contained in the same number of codes (say, \( L \)) of \( \mathcal{C}_b \).

Let \( M \) be the cardinality of a code in \( \mathcal{C}_b \). From a counting argument, one can see that \( M|\mathcal{C}_b| \geq L|(\mathcal{R}^n)^*| \). The following lemma shows how to bound averages of functions in \( (\mathcal{R}^n)^* \).

\[1\text{This may differ from the literature, where a linear code over a ring is simply an additive subgroup of} \ \mathcal{R}^n. \text{The requirement that a linear code is a free module is necessary for Lemma 2 to hold.} \]
Lemma 2. Let \( g : \mathbb{R}^n \to \mathbb{R}^+ \) be a function. For a code \( C \), we define \( g^*(C) = \sum_{c \in C \cap (\mathbb{R}^n)^*} g(c) \). If \( C_b \) is the set of all codes of rank \( k \) then

\[
E[g^*(C)] \leq \frac{|\mathbb{R}|^k}{|\binom{\mathbb{R}^n}{*}|} g^*(\mathbb{R}^n),
\]

where the expectation is with respect to the uniform distribution on \( C_b \).

**Proof:** For any balanced set of codes with cardinality \( M \), we have

\[
E[g^*(C)] = E \left[ \sum_{c \in C \cap (\mathbb{R}^n)^*} g(c) \right] = E \left[ \sum_{x \in (\mathbb{R}^n)^*} g(x) \mathbb{1}_C(x) \right] = \sum_{x \in (\mathbb{R}^n)^*} E[g(x) \mathbb{1}_C(x)] = \sum_{x \in (\mathbb{R}^n)^*} g(x) \frac{L}{|C_b|} \leq \frac{M}{|\binom{\mathbb{R}^n}{*}|} g^*(\mathbb{R}^n).
\]

We now need to prove that the set of all codes of rank \( k \) is balanced. Let \( y \) be any element in \( \mathbb{R}^n^* \). There exists an invertible linear map \( T(y) = (1, 0, \ldots, 0) = e_1 \). Since \( T \) is rank-preserving \( y \in C \) if and only if \( e_1 \in T(C) \), where \( C \) and \( T(C) \) have same rank. This induces a bijection between the codes that contain \( y \) and the codes that contain \( e_1 \), proving the statement. \( \square \)

**A. Lipschitz and Hurwitz Lattices**

The quaternion skew-field \( \mathbb{H} \) is given by \( \mathbb{H} = \{ a + bi + (c + di)j : a, b, c, d \in \mathbb{R} \} \), with the usual relations \( i^2 = j^2 = -1 \) and \( ij = -ji \). Vance recently [Van11] proved a Minkowski-Hlawka theorem for lattices in \( \mathbb{H} \) over the Hurwitz order. Here we show how to recover a “coding-theoretic” version of this result from generalized reductions.

We first explain how to deduce a slightly simpler case, for the Lipschitz order. The **Lipschitz integers** \( \mathcal{L} \subset \mathbb{H} \) is the (non-maximal) order \( \mathcal{L} = \{ x + yj : x, y \in \mathbb{Z}[i] \} \). Recall that a quaternion has matrix representation

\[
\begin{pmatrix}
x & -\overline{y} \\
y & \overline{x}
\end{pmatrix}.
\]

Let \( p \) be an ideal in \( \mathbb{Z}[i] \) above \( p \) that splits. Let \( \pi : \mathbb{Z}[i] \to \mathbb{Z}[i]/p \) be a projection. We consider the following “single-letter” reduction:

\[
\phi^p_\mathbb{H} : \mathcal{L} \to \mathcal{M}_2(\mathbb{F}_p),
\]

\[
\phi_p(x + yj)^\mathbb{H} = \begin{pmatrix}
\pi(x) & -\pi(\overline{y}) \\
\pi(y) & \pi(\overline{x})
\end{pmatrix}.
\]

We have \( \ker \phi^p_\mathbb{H} = (p\mathbb{Z}[i]) + (p\mathbb{Z}[i])j \). Identifying \( \mathbb{H} \) with \( \mathbb{R}^4 \) in the natural way

\[
\psi(a, b, c, d) \to a + bi + (c + di)j
\]

we obtain a reduction \( \phi_p : \mathbb{Z}^4 \to \mathcal{M}_2(\mathbb{F}_p), \phi_p(x) = \phi^p_p(\psi(x)) \). By abuse of notation, we will also denote by \( \phi^p_\mathbb{H} \) the reduction applied componentwise in vector of \( \mathcal{L}^m \), i.e.,

\[
\phi^p_\mathbb{H}(x_1 + y_1j, \ldots, x_m + y_mj) = (\phi^p_p(x_1 + y_1j), \ldots, \phi^p_p(x_m + y_mj)) \in \mathcal{M}_2(\mathbb{F}_p)^m.
\]
Let $C \in \mathcal{M}_2(\mathbb{F}_p)^m$ be a linear code with cardinality $|C|$. Then $\Lambda_p^H(C) = (\phi_p^H)^{-1}(C)$ is a quaternionic lattice with volume $|C|p^{-m}$. Let $C_X$ be the ensemble of codes associated to some balanced set $C_b$, and $\mathbb{L}_p$ the associated lattice ensembles

$$\mathbb{L}_p = \{ \beta \Lambda^H_p(C) : C \in C_b \},$$

where $\beta = (V/|C|^{-1}p^{4m})^{1/(4m)}$. The following Theorem 3 is the analogous of Theorem 1 for Lipschitz lattices. We need the following lemma

**Lemma 3.** If $\phi_p(x + yj)$ is non-invertible for $x + yi \in \mathcal{L}$, then the norm of $x + yj$ is a multiple of $p$.

**Proof:** If $\det \phi_p(x + yj) = 0$, then $\pi(x\bar{x} + y\bar{y}) = 0$, i.e., $\| (x, y) \|^2 \in \mathfrak{p}$. Since the norm of a Lipschitz quaternion is an integer, and $p$ is above $\mathfrak{p}$, the result follows.

**Theorem 3.** Let $C_b$ be a balanced set of codes with rank $k > m/2$. If $f$ is a semi-admissible function then

$$\lim_{p \to \infty} E_{\mathbb{L}_p}\left[ \sum_{x \in \beta \Lambda^H_p(C_X)} f(\psi(x)) \right] \leq (\zeta(4m)V^{-1}) \int_{\mathbb{R}^{4m}} f(x).$$

**Proof:** The proof is very similar to that of Theorem 1. Here we divide the expectation into invertible and non-invertible elements (we make the change of variable $y = \beta x$, to facilitate), i.e.

$$\sum_{y \in \Lambda^H_p(C_X)} f(\psi(\beta y)) = \sum_{y \in \Lambda^H_p(C_X)} f(\psi(\beta y)) + \sum_{x \in \Lambda^H_p(C_X) \setminus \phi(y) / (\mathcal{M}_2(\mathbb{F}_p)^m)^*} f(\psi(\beta y)).$$

The first term tends to zero as $p \to \infty$ from Lemma 3 since $f$ is semi-admissible and

$$\beta \| \psi(y) \| \geq \beta \sqrt{p} = |C|^{1/4m}p^{-1/2} = p^{k/m - 1/2} \to \infty \text{ as } p \to \infty.$$  

From Lemma 2 we conclude that the second term is upper bounded by the right-hand side of (14) as $p \to \infty$.

For the maximal Hurwitz order

$$\mathcal{H} = \{ a + bi + cj + d(-1 + i + j + ij)/2 : a, b, c, d \in \mathbb{Z} \},$$

the theorem follows by considering reductions from left-prime ideals $\mathfrak{P} \triangleleft \mathcal{H}$. For any rational prime $p$, there exist isomorphisms $\mathcal{H}/p\mathcal{H} \sim \mathbb{F}_p(i, j, k) \sim \mathcal{M}_2(\mathbb{F}_p)$ (e.g. Weddeburn’s Theorem [Voi Thm. 6.16 Lem 9.2.1]), where non-invertible elements in $\mathcal{H}/p\mathcal{H}$ have reduced norm (determinant) proportional to $p$. Notice that in this case we obtain a reduction

$$\phi_p : D_4 \to \mathcal{M}_2(\mathbb{F}_p),$$

where $D_4$ is the checkerboard lattice in dimension four [CS98 Sec. 7.2].
VI. ALGORITHMIC EFFECTIVENESS

Theorem 1 holds in the limit $p_j \to \infty$. However, for each $n$, under some conditions it is possible to find finite ensembles that contain dense lattices. In the literature, this is referred to as effectiveness (e.g. [CS98, p. 18] and [GZ07]). We show conditions for a family of reductions to be effective. We need the following lemma, which is a special case of a classical result in the Geometry of Numbers (see [GL87, p. 141]) and is also valid if $B_r^n$ is replaced by more general sets. We include a proof here for the sake of completeness.

**Lemma 4.** Let $P$ be a fundamental region for $\Lambda$, an let $l_0 = \sup_{x \in P} \|x\|$. For $r > l_0$, we have

$$(r - l_0)^n V_n \leq (\det \Lambda) N_\Lambda(r) \leq (r + l_0)^n V_n. \tag{15}$$

*In particular, we can take $l_0 = \tau(\Lambda)$ to be the covering radius of $\Lambda$.*

**Proof:** We show the set inclusion

$$B_{r-l_0}^n \subset \bigcup_{x \in \Lambda \cap B_r} (x + P(B)) \subset B_r^n.$$

The lemma then follows from a simple volume calculation of the three sets.

If $y = x + p$, $x \in \Lambda \cap B_r$, $p \in P(B)$, then $\|y\| \leq \|x\| + \|p\| \leq r + l_0$, proving the second inclusion. For the first inclusion, let $y \in B_{r-l_0}^n$ and write it as $y = x + p$, with $x \in \Lambda$ and $p \in P$ (this is always true since $P$ is a fundamental region). Then $\|x\| \leq \|y\| + \|p\| \leq r$. \qed

The next proposition essentially says that if the base lattice $\Lambda$ is sufficiently “thin” and the kernel-lattices $\Lambda_{p_j}$ are not so sparse, it is possible to bound $p_j$ in terms of the rate of the underlying code. The conditions are very mild (they are achievable, for instance, by $\mathbb{Z}^n$).

**Proposition 1.** Let the notation be as in Theorem 1 and let $\tau(\Lambda)$ be the covering radius of $\Lambda$. If, for $m$ sufficiently large,

$$\frac{\tau(\Lambda)}{V(\Lambda)^{1/m}} = O(\sqrt{m}) \text{ and } \frac{\lambda_1(\Lambda_{p_j})}{V(\Lambda)^{1/m}} \geq 1.$$  

For any rate $R = k/n$, any small $\varepsilon > 0$ and $p_j \geq (m/2\pi\varepsilon)^{\frac{n}{n-k}}$, there exists a code with parameters $(n, k, p_j)$ such that the lattice $\Lambda_{p_j}(C)$ has density greater or equal than $(1 - \varepsilon)/2^{m-1}$.

**Proof:** In the notation of the proof of Theorem 1, Equation (6) becomes:

$$E\left[ \# (\beta \Lambda_{p_j}^r(C) \cap B_r^n) \right] = E\left[ \# (\beta \Lambda_{p_j}^r \cap B_r^n) \right] + E\left[ \# (\beta (\Lambda_{p_j}^r(C) \setminus \Lambda_p) \cap B_r^n) \right].$$

Suppose wlog that $V = 1$. The first term is zero whenever

$$\frac{\lambda_1(\Lambda_{p_j})}{p(n-k)/m V(\Lambda)^{1/m}} \geq r. \tag{16}$$

The second term satisfies

$$E\left[ \# (\beta (\Lambda_{p_j}^r(C) \setminus \Lambda_p) \cap B_r^n) \right] \leq \frac{p^k - 1}{p^n - 1} \# (\beta (\Lambda_{p_j}^r \setminus \Lambda_p) \cap B_r^n) \leq p^{k-n} (r + \beta \tau(\Lambda))^n \frac{V_m}{\beta^m V(\Lambda)}. \tag{17}$$

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Imposing the right-hand-side of (17) to be slightly less than 2, we obtain a lattice with the required density (see the discussion on Section II). Since $V_m^{1/m} \sim \sqrt{2\pi e/m}$, as $p \to \infty$ this is satisfied by imposing $r \sim \sqrt{m/2\pi e}$. To finish the proof, notice that condition (16) can be satisfied by choosing $p \geq \left( \sqrt{\frac{m}{2\pi e}} + o(1) \right)^{m/k} \left( \frac{V(\Lambda_p)}{\lambda_1(\Lambda_p)^{1/m}} \right)^{1/k}$.

When $m = n$, $\Lambda = \mathbb{Z}^n$ and $\phi_p$ is the ‘mod-$p$’ reduction, we recover the well-known fact that it is possible to construct dense lattices for codes with any positive rate with alphabet size $p = O(m^{1/2})$, where $R = k/n$ is the $(p$-ary) rate of the underlying code. By making the rate tend to one (say $R = 1/(1 + 2\delta)$ for some small $\delta > 0$), we can construct dense lattices with essentially $p = O(m^{1/2+\delta})$, where $\delta$ is any small (but positive) number. This behavior can be further improved by starting the reductions with a lattice which already has a good density, as shown next.

Let $\Lambda = A_k^n$ be a Craig’s lattice [CS98, pp.222-224], where $n+1 = q$ is a prime. From [BB92, Prop. 4.1], a Craig’s lattice is similar to the embedding of the ideal $(1 - \mu_p)\mathbb{Z}[\mu_p]$ in the cyclotomic field $\mathbb{Q}(\mu_p)$. A concrete realization is

$$\Lambda = \sqrt{n} \sigma((1 - \mu_p)\mathbb{Z}[\mu_p]).$$

From this, we have $\Lambda^* \sim A_n^{n/2-l}$,

$$\frac{\lambda_1(\Lambda)}{V(\Lambda)^{1/n}} \geq \frac{\sqrt{2l}}{(n-1)^{(2l-1)/2n}} \text{ and } \frac{\lambda_1(\Lambda^*)}{V(\Lambda^*)^{1/n}} \geq \frac{\sqrt{n-2l}}{(n-1)^{(n-2l-1)/2n}}.$$

Following [CS98 p. 224]'s suggestion, we consider Craig’s lattices with parameter $l = \lceil n/2 \log(n+1) \rceil$ so that, for sufficiently large $n$,

$$\frac{\lambda_1(\Lambda)}{V(\Lambda)^{1/m}} \geq \sqrt{\frac{2\pi}{\log n}} \left( \sqrt{\frac{n}{2\pi e}} + o(1) \right) \text{ and } \frac{\lambda_1(\Lambda^*)}{V(\Lambda^*)^{1/n}} \geq \sqrt{e} + o(1).$$

From Banaszczyk’s transference bound [Ban93]:

$$\frac{\tau(\Lambda)}{V(\Lambda)^{1/m}} \leq \frac{\sqrt{n}}{2\sqrt{e} + o(1)}.$$

Therefore, using a natural reduction, we can satisfy equations (16) and (17) by setting $p = O((\log n)^{1/2+\delta})$.

**Corollary 4.** Let $\Lambda = A_n^{n/2(n+1)}$ and let $\phi_p$ be a natural reduction, as described in Example 2. For any $\varepsilon > 0$, $n$ sufficiently large and fixed rate $k/n = 1/(1 + 2\delta)$, there exists a code $C$ with parameters

$$\left( n, \frac{n}{1 + 2\delta}, O((\log n)^{1/2+\delta}) \right)$$

such that $\Lambda_p(C)$ has density $(1 - \varepsilon)/2^{n-1}$.
VII. Final Discussion

Applications. As observed by Loeliger [Loe97], random ensembles of lattices are not only good in terms of packing density, but are also sphere-bound achieving when used as infinite constellations for the AWGN channel. Indeed, Rush [Rus89] and Loeliger’s [Loe97] Construction A \( \mathbb{Z} \)-lattices are ubiquitous in applications to information transmission over Gaussian channels and networks. However, for other communication problems, such as information transmission in the presence of fading and multiple antennas, it is desirable to enrich the lattices with some algebraic (multiplicative) structure. To this purpose, several recent works such as [HNW15], [KOO15], [VKO14], [CLB16] present different constructions that attach a linear code to an algebraic lattice, but, to date, there is no unified analysis of such ensembles. The generalized reductions described here provide a method for establishing the “goodness” of all such constructions at once. It also provides a simple condition to verify if any new construction is “good” (e.g., sphere-bound achieving). This was indeed the initial motivation of the author.

Further Perspectives. The framework considered in this paper is used to provide simple alternative (coding-theoretic) proofs and improvements on previous refinements on the best packing density. It not only implies the existence of dense lattices, but also of structured lattices, with the structured inherited from the underlying reduction.

The question whether it is possible to improve on the \( cn \log \log n/2^{n-1} \) asymptotic behavior of cyclotomic fields by specializing the reductions (or the family of codes) appropriately is still open. Furthermore, all known lower bounds on \( \Delta_n \) are of the form \( \Delta_n \geq 2^{-n(1+\varepsilon(n))} \), with \( \varepsilon(n) = O(\log n/n) \), which improves only marginally on the Minkowski-Hlawka lower bound. According to Gruber [Gru07, p. 388], Hlawka believed that no essential improvement can be made, probably meaning that the exponent 2 is optimal. Nevertheless, the best known upper bound on \( \Delta_n \), due to Kabatianskii and Leveshenstein, is of the form \( C^{-n} \), where \( C \approx 1.51 \). Closing this gap is a long-standing open problem.

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