A Quantum Fluid Description of the Free Electron Laser

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Abstract

Using the Madelung transformation we show that in a quantum Free Electron Laser (QFEL) the beam obeys the equations of a quantum fluid in which the potential is the classical potential plus a quantum potential. The classical limit is shown explicitly.
INTRODUCTION

In the quantum FEL model, the electron beam is described as a macroscopic matter-wave [1, 2, 3, 4]. When slippage due to the difference between the light and electron velocities is neglected, the electron beam-wave interaction is described by the following equations for the dimensionless radiation amplitude \( A(\bar{z}) \) and the matter wave field \( \Psi(\theta, \bar{z}) \) [5]:

\[
\begin{align*}
    i \frac{\partial \Psi(\theta, \bar{z})}{\partial \bar{z}} &= -\frac{1}{2\rho} \frac{\partial^2}{\partial \theta^2} \Psi(\theta, \bar{z}) - i\bar{\rho} \left[ A(\bar{z}) e^{i\theta} - \text{c.c.} \right] \Psi(\theta, \bar{z}) \\
    \frac{dA(\bar{z})}{d\bar{z}} &= \int_{0}^{2\pi} d\theta |\Psi(\theta, \bar{z})|^2 e^{-i\theta} + i\delta A(\bar{z}).
\end{align*}
\] (1)

The electron beam is therefore described by a Schrödinger equation for a matter-wave field \( \Psi \) in a self-consistent pendulum potential proportional to \( A \), where \( |A|^2 = |a|^2/(N\bar{\rho}) \), \( |a|^2 \) is the average number of photons in the interaction volume \( V \), and \( |\Psi|^2 \) is the space-time dependent electron density, normalized to unity. In Eqs. (1) and (2) we have adopted the universal scaling used in the classical FEL theory [6, 7, 8], i.e. \( \theta = (k + k_w)z - ckt \) is the electron phase, where \( k_w = 2\pi/\lambda_w \) and \( k = \omega/c = 2\pi/\lambda \) are the wiggler and radiation wavenumbers, \( \bar{z} = z/L_g \) is the dimensionless wiggler length, \( L_g = \lambda_w/4\pi\rho \) is the gain length, \( \rho = \gamma^{-1} a_w/(4ck_w)^{2/3}(e^2n/m_e\epsilon_0)^{1/3} \) is the classical FEL parameter, \( \gamma_r = \sqrt{(\lambda/2\lambda_w)(1 + a_w^2)} \) is the resonant energy in \( mc^2 \) units, \( a_w \) is the wiggler parameter and \( n \) is the electron density. Finally, \( \bar{\rho} = (\gamma - \gamma_0)/\rho\gamma_0 \) is the dimensionless electron momentum and \( \delta = (\gamma_0 - \gamma_r)/\rho\gamma_0 \) is the detuning parameter, where \( \gamma_0 \approx \gamma_r \) is the initial electron energy in \( mc^2 \) units.

Whereas the classical FEL equations in the above universal scaling do not contain any explicit parameter (see ref. [8]), the quantum FEL equations [1] and [2] depend on the quantum FEL parameter

\[
\bar{\rho} = \left( \frac{mc\gamma_r}{\hbar k} \right) \rho.
\] (3)

From the definition of \( A \), it follows that \( \bar{\rho}|A|^2 = |a|^2/N \) is the average number of photons emitted per electron. Hence, since in the classical steady-state high-gain FEL \( A \) reaches a maximum value of the order of unity, \( \bar{\rho} \) represents the maximum number of photons emitted per electron, and the classical regime occurs for \( \bar{\rho} \gg 1 \). Note also that in Eq. (1) \( \bar{\rho} \) appears as a “mass” term, so one expects a classical limit when the mass is large. As we shall see, when \( \bar{\rho} < 1 \) the dynamical behavior of the system changes substantially from a classical to a quantum regime.
QUANTUM FLUID DESCRIPTION

We now perform a Madelung-like transformation [9], writing the wavefunction as

$$\Psi = R \exp (i\bar{\rho}S)$$

which allows us to rewrite the Maxwell-Schrodinger equations, eq. (1) and (2), as a system of quantum fluid equations

$$\frac{\partial R}{\partial \bar{z}} = -\frac{\partial R}{\partial \theta} \frac{\partial S}{\partial \theta} - \frac{R}{2} \frac{\partial^2 S}{\partial \theta^2}$$  \hspace{1cm} (4)

$$\frac{\partial S}{\partial \bar{z}} = -\frac{1}{2} \left( \frac{\partial S}{\partial \theta} \right)^2 - V(\theta, \bar{z})$$  \hspace{1cm} (5)

$$\frac{dA}{d\bar{z}} = \int_{0}^{2\pi} R^2 e^{-i\theta} d\theta + i\delta A$$  \hspace{1cm} (6)

where the potential, $V$ in eq. (5) is defined as the sum of a classical term and a quantum term i.e.

$$V(\theta, \bar{z}) = V_C + V_Q$$

where

$$V_C = -i \left( Ae^{i\theta} - c.c. \right)$$  \hspace{1cm} (7)

is the classical component of the potential and

$$V_Q = -\frac{1}{2\bar{\rho}^2 R} \frac{\partial^2 R}{\partial \theta^2}$$  \hspace{1cm} (8)

is the quantum component of the potential, which becomes negligible as $\bar{\rho} \to \infty$.

Defining fluid density and velocity variables

$$n = R^2 = |\Psi|^2, \quad u = \frac{\partial S}{\partial \theta}$$

we can also rewrite Eq. (4)-(6) in an alternative fluid form as

$$\frac{\partial n}{\partial \bar{z}} + u \frac{\partial n}{\partial \theta} (nu) = 0$$  \hspace{1cm} (9)

$$\frac{\partial u}{\partial \bar{z}} + u \frac{\partial u}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$  \hspace{1cm} (10)

$$\frac{dA}{d\bar{z}} = \int_{0}^{2\pi} n e^{-i\theta} d\theta + i\delta A.$$  \hspace{1cm} (11)

It can be seen that Eq. (9) is a continuity equation and Eq. (10) is a Newton-like equation for a fluid. Note that integrating Eq. (9) with respect to $\theta$, then the normalization condition becomes

$$\int_{0}^{2\pi} n(\theta, \bar{z})d\theta = 1,$$
which is satisfied if \( n \) and \( u \) are periodic functions of \( \theta \) between 0 and \( 2\pi \).

A straightforward calculation shows that Eq.(9)-(11) admit two constants of motion,

\[
\langle \bar{p} \rangle + |A|^2 = C_1
\]

and

\[
\frac{\langle \bar{p}^2 \rangle}{2} - i(Ab^* - \text{c.c.}) - \delta |A|^2 = C_2
\]

where \( \langle \bar{p} \rangle = \langle u \rangle = \int_0^{2\pi} d\theta nu \) is the average momentum, \( \langle \bar{p}^2 \rangle = \langle u^2 + 2V_Q \rangle = \int_0^{2\pi} d\theta n(u^2 + 2V_Q) \) is the momentum variance and

\[
b = \int_0^{2\pi} ne^{-i\theta} d\theta
\]

is the bunching. These constants of motion are well-known in the classical FEL model [8] and describe energy conservation and a gain-spread relation. Notice the quantum contribution to the momentum variance proportional to the average quantum potential.

**FOURIER EXPANSION AND LINEAR ANALYSIS**

If \( R \) and \( S \) are periodic functions of \( \theta \), they can be expanded in a Fourier series:

\[
R(\theta, \bar{z}) = \sum_m r_m(\bar{z}) e^{im\theta},
\]

\[
S(\theta, \bar{z}) = \sum_n s_n(\bar{z}) e^{in\theta}
\]

with \( r_m = s^*_m \) and \( s_m = r^*_m \), since \( R \) and \( S \) are real variables. Multiplying Eq.(5) by \( R \) and using (14) and (15) in Eqs.(4)-(6), we obtain:

\[
\sum_m r_{k-m} \frac{ds_m}{d\bar{z}} = -\frac{1}{2} \sum_{m,n} n(n - m) r_{k-m} s_n s_{n-m}^* + i(\bar{A}r_{k-1} - A^*r_{k+1}) - \frac{k^2}{2\rho^2}r_k
\]

\[
\frac{dr_k}{d\bar{z}} = \frac{1}{2} \sum_m (k^2 - m^2) r_m s^*_{m-k}
\]

\[
\frac{dA}{d\bar{z}} = \sum_m r_m r^*_{m-1} + i\delta A
\]

Eqs.(16)-(18) are our working equations which can be numerically solved as it will be shown elsewhere.

Eqs.(16)-(18) admit an equilibrium solution with no field \( (A = 0) \) and unbunched electron beam \( (n = 1/2\pi, \text{i.e. } r_n = \delta_{n0} \text{ and } s_n = 0) \). Linearizing Eqs.(16)-(18) around this
equilibrium in the first order of the variables $A$, $r_1$ and $s_1$, we obtain:

$$\frac{dA}{dz} = 2r_1 + i\delta A \quad (19)$$

$$\frac{dr_1}{dz} = \frac{s_1}{2} \quad (20)$$

$$\frac{ds_1}{dz} = iA - \frac{1}{2\bar{\rho}^2}r_1 \quad (21)$$

Looking for solutions proportional to $\exp(i\lambda z)$, we obtain the well-known cubic equation of the quantum FEL [1]

$$(\lambda - \delta) \left( \lambda^2 - \frac{1}{4\bar{\rho}^2} \right) + 1 = 0. \quad (22)$$

which reduces to the classical dispersion relation in the limit $\bar{\rho} >> 1$.

**CONCLUSIONS**

It has been shown that the quantum FEL model can be rewritten in a form where the electron beam is described a quantum fluid coupled to the electromagnetic field. The evolution of the quantum fluid is determined by a self-consistent potential which consists of a classical and quantum contribution. In the limit where $\bar{\rho} \gg 1$ the quantum contribution to the potential becomes negligible and the force equation reduces to that of a Newtonian fluid. Using a Fourier expansion, linear stability analysis of these quantum fluid equations produced a dispersion relation identical to that derived from the Schrodinger equation. These results show that there are interesting connections between the quantum FEL and quantum plasma instabilities.

[1] R. Bonifacio, M.M. Cola, N. Piovella, and G.R.M. Robb, Europhys. Lett. **69**, 55 (2005).

[2] R. Bonifacio, N. Piovella, G.R.M. Robb, Nucl. Instrum. and Meth. in Phys. Res. A **543**, 645 (2005).

[3] R. Bonifacio, N. Piovella, G.R.M. Robb, and A. Schiavi, Phys. Rev. ST Accel. Beams **9**, 090701 (2006).

[4] A. Serbeto, J. T. Mendonça, K. H Tsui, R. Bonifacio, Physics of Plasmas **15**, 013110 (2008).

[5] G. Preparata, Phys. Rev.A **38**, 233 (1988).

[6] R. Bonifacio, C. Pellegrini and L. Narducci, Opt. Commun. **50**, 373 (1984).
[7] R. Bonifacio, L. De Salvo, P. Pierini, N. Piovella, and C. Pellegrini, Phys. Rev. Lett. 73, 70 (1994).

[8] R. Bonifacio, F. Casagrande, G. Cerchioni, L. De Salvo Souza, P. Pierini and N. Piovella, Rivista del Nuovo Cimento 13, No. 9 (1990).

[9] A. Messiah, Quantum Mechanics, Wiley (1966).