Solvability of the Hamiltonians related to exceptional root spaces: rational case

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Abstract

Solvability of the rational quantum integrable systems related to exceptional root spaces $G_2, F_4$ is re-examined and for $E_{6,7,8}$ is established in the framework of a unified approach. It is shown the Hamiltonians take algebraic form being written in a certain Weyl-invariant variables. It is demonstrated that for each Hamiltonian the finite-dimensional invariant subspaces are made from polynomials and they form an infinite flag. A notion of minimal flag is introduced and minimal flag for each Hamiltonian is found. Corresponding eigenvalues are calculated explicitly while the eigenfunctions can be computed by pure linear algebra means for arbitrary values of the coupling constants. The Hamiltonian of each model can be expressed in the algebraic form as a second degree polynomial in the generators of some infinite-dimensional but finitely-generated Lie algebra of differential operators, taken in a finite-dimensional representation.

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1 Introduction

Up to now the Hamiltonian Reduction Method which also is called the Projection Method [1, 2] provides a unique opportunity to construct non-trivial multidimensional completely integrable quantum (and classical) Hamiltonians. These Hamiltonians are associated with root systems, they are related with the Laplace-Beltrami operators on symmetric spaces. Their rational (trigonometric) versions have remarkable properties: (i) their eigenvalues are known explicitly being at most the second degree polynomial in quantum numbers, (ii) any eigenfunction has a form of the ground state eigenfunction multiplied by a polynomial in the Cartesian (or exponential in Cartesian) coordinates. These two specific types of the Hamiltonians appear naturally in this approach – with rational and trigonometric potentials, correspondingly, characterized by the second-order poles in potentials on the boundaries of the configuration space. However, soon after a discovery of these Hamiltonians it became clear [2] there exists a straightforward generalization of these Hamiltonians to the case of arbitrary coupling constants without breaking any nice property. It led to a loss of immediate group theoretical interpretation. However, a property of solvability remained to hold. It gave a hint on existence of a more general formalism where above-mentioned Hamiltonians with arbitrary coupling constants appear naturally. An idea was to connect solvability with possible existence of an intrinsic hidden algebraic structure [3, 4, 5]. It turns out to be true. For arbitrary coupling constants these Hamiltonians admit an algebraic form (where polynomial coefficient functions occur in front of derivatives) and they are related with elements of the universal enveloping algebra of some algebras of differential operators acting in the space of invariants of the corresponding root space. Such an algebra was called the hidden algebra of the Hamiltonian. It was found that for all $A_n, B_n, C_n, D_n, BC_n$ rational and trigonometric models this algebra is the same (!) – it is the maximal affine subalgebra of the $gl_n$-algebra realized by the first order differential operators in $\mathbb{R}^n$ and taken in symmetric representation [4, 5]. Thus, one can state that all these models are nothing but different appearances of a single model characterized by the hidden algebra $gl_n$. Similar situation holds for the SUSY generalizations of above models - all of them turned out to be associated to the hidden superalgebra $gl(n|n-1)$, see [5]).

However, one can naturally expect that the situation is drastically different for the Hamiltonians related to the exceptional algebras - each Hamiltonian is characterized by its own hidden algebra which is different for different Hamiltonians. A first indication stemmed from a study of the $G_2$ rational and trigonometric models,
where the both models were characterized by the same hidden algebra, but this algebra turned out to be a certain infinite-dimensional, finitely-generated algebra of the differential operators \( g^{(2)} \), which is a subalgebra of the algebra of the differential operators on the real plane, \( g^{(2)} \subset \text{diff}(2, \mathbb{R}) \) \([6, 7]\). Later, it was shown that the similar situation holds for the rational and trigonometric \( F_4 \) models \([8]\): both models possess the same hidden algebra of the differential operators \( f^{(4)} \), which is a subalgebra of the algebra of the differential operators on \( \mathbb{R}^4 \), \( f^{(4)} \subset \text{diff}(4, \mathbb{R}) \).

For all previous studies a crucial point was to find a set of variables in which the Hamiltonian under investigation takes an algebraic form and reveal their meaning. A simple observation made in \([4]\) was to consider the variables which incorporate all symmetries of the problem in hand. In particular, these symmetries contain (or sometimes even coincide with) the Weyl group of the associated root system. A natural idea is to take invariants of the fixed degrees of the Weyl group as variables. Many years ago V.I. Arnold \([9]\) pointed out that the contravariant flat metric on the space of orbits of any Coxeter group written in terms of the polynomial invariants has polynomial matrix elements. It implies that the coefficient functions in front of the second derivatives in the Laplace-Beltrami operator are polynomials in invariants of the Weyl group. This result was rediscovered (and then generalized) later in \([4]\), \([5]\), \([6]\) and \([8]\) for \( A_n \), \( BC_n \), \( G_2 \) and \( F_4 \) algebras, respectively. It was shown that the algebraic structure persists for the whole Laplace-Beltrami operator: the coefficient functions in front of the first derivatives are polynomials as well. Further generalization was that similar statement is valid for the entire set of the rational Hamiltonians which are a combination of Laplace-Beltrami operator and a potential – after the gauge (similarity) transformation with the ground state eigenfunction the coefficient functions in front of the first derivatives remain polynomials\(^5\). In the present paper a certain universal prescription about a choice of coordinates is given.

Recently, it was found that the existence of algebraic forms and knowledge of hidden algebraic structure of the above-mentioned Hamiltonians allows to consider their perturbations in a constructive way. One can develop an algebraic perturbation theory, where all corrections can be obtained by pure linear algebra means \([12, 13, 14]\).

The goal of the present article is to carry out a study of the solvability of the

\(^5\) Similar results we also obtained in above-mentioned works for the trigonometric \( A_n \), \( BC_n \), \( G_2 \) and \( F_4 \) Hamiltonians when the trigonometric Weyl invariants (which mean the Weyl invariants periodic in each variable) are used as coordinates. Even for elliptic \( A_1 \) and \( BC_n \) models the elliptic Weyl invariants allow to get the polynomials coefficients in front of derivatives (see \([10]\) and \([11]\) for details)
rational models related to the exceptional root spaces. We introduce a general formalism which allows to study all these models on equal footing (Section 2). In Sections 3, 4 the solvability of the $G_2$ and $F_4$ models is re-examined in a new formalism, while in the Section 5–7 the solvability is established for $E_6, 7, 8$ rational models. In Conclusion the results are summarized. Almost all results were obtained with the help of MAPLE 7 and MAPLE 8 programs together with the package COXETER created by J. Stembridge.

2 Generalities

Let us consider a quantum system described by rational Hamiltonian associated with a root system $\mathcal{R}$ of algebra $g$ of rank $N$:

$$\mathcal{H} = \frac{1}{2} \sum_{k=1}^{N} \left[ -\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 \right] + \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} g_{|\alpha|} |\alpha|^2 \frac{1}{(\alpha \cdot x)^2}, \quad (2.1)$$

where $\alpha \in \mathcal{R}_+$ are positive roots of the system $\mathcal{R}$ which are vectors in $\mathbb{R}^N$, $x = (x_1, \ldots, x_N)$ is a set Cartesian coordinates, $|\alpha|^2 = \sum_1^N \alpha_k^2$, and the scalar product $(\alpha \cdot x) = \sum_1^N \alpha_k x_k$, $\omega$ is a parameter. Coupling constants $g_{|\alpha|}$ are assumed to be equal for roots of the same length. Hence for the $A_n$ case there is a single coupling constant, for the $BC_n$ case there are three coupling constants, etc. For some algebras ($G_2$, $E_6$, $E_7$) it is convenient to embed the roots into the vector space of higher dimension $\mathbb{R}^{N+n}$ ($n = 1$ or 2). In general, the Hamiltonians of this type describe a quantum particle in multidimensional space, although for $A_n$ and $G_2$ cases these Hamiltonians allow another interpretation as the Hamiltonians describing many-body systems, while for $B_n, C_n, D_n$ they correspond to many-body systems on the space with mirror.

Consider the spectral problem for the Hamiltonian $\mathcal{H}$,

$$\mathcal{H} \Psi(x) = E \Psi(x), \quad (2.2)$$

where the configuration space is the Weyl chamber and $\Psi$ should be from the corresponding Hilbert space. Let us make a gauge rotation of the Hamiltonian taking the ground state eigenfunction $\Psi_0$ as a gauge factor

$$h = -2(\Psi_0(x))^{-1}(\mathcal{H} - E_0)\Psi_0(x), \quad (2.3)$$
where $E_0$ is the ground state energy of the Hamiltonian (2.1) (see e.g. [2]),

$$E_0 = \omega \left( \frac{N}{2} + \sum_{\alpha \in \mathbb{R}_+} g_{|\alpha|} \right). \tag{2.4}$$

It should be mentioned that for the $A_n$ and $G_2$ cases the energy $E_0$ does include the ground state energy of center-of-mass motion.

The ground state eigenfunction of (2.1) has a form

$$\Psi_0 = \Delta_g \exp \left( -\frac{\omega t_2^{(\Omega)}}{2} \right), \tag{2.5}$$

where

$$\Delta_g = \prod_{\mathcal{R}_+} (\alpha_k, y)^{\nu_{|\alpha|}}$$

and $\nu_{|\alpha|}$ are defined through $g_{|\alpha|} = \nu_{|\alpha|}(\nu_{|\alpha|} - 1)$ and assumed to be equal for roots of the same length. For example, if all roots are of the same length as for $A_n$ case, we put $\nu_{|\alpha|} = \nu$. In the case of roots of two different lengths as for $G_2$ case we denote $\nu_{|\alpha|} = \nu$ for the short roots and $\nu_{|\alpha|} = \mu$ for the long roots. $t_2^{(\Omega)}$ is the invariant of the degree two (for definition see below).

A new spectral problem arises

$$h\varphi(x) = -\epsilon \varphi(x), \tag{2.6}$$

with a new spectral parameter $\epsilon = 2(E - E_0)$. If in (2.2) the boundary condition means normalizability of the eigenfunction $\Psi(x)$, then for (2.6) it requires the normalizability of $\varphi(x)$ with the weight factor $\Psi_0^2(x)$. By construction, the lowest eigenvalue $\epsilon_0 = 0$ and the lowest eigenfunction is $\varphi_0 = \text{const}$. Our goal is to find, by a change of variables, an algebraic form of the operator $h$ (if it exists).

**Definition:** a linear differential operator with polynomial coefficients is called algebraic.

In order to find these new variables we assume that they respect the symmetries of the Hamiltonian [4]. In our case it means the invariance under the group of automorphisms $A_g$ of the $g$ root space. This group includes or coincides with the Weyl group $W_g$. The algebraically independent invariant polynomials of the lowest possible degrees $a$ generate the algebra $S^{A_g}$ of $A_g$-invariant polynomials. The powers $a$ are the degrees of the group $W_g$. A particular form of these polynomials (denoted
below as \( t_a^{(Ω)}(x) \) can be found by averaging elementary monomials \( (ω · x)^a \) over some group orbit \( Ω \),

\[
t_a^{(Ω)}(x) = \sum_{ω ∈ Ω} (ω · x)^a , \tag{2.7}
\]

(see e.g. [15]), where \( x \)'s are some formal variables. There exists a certain ambiguity in connecting these variables with the variables appearing in the Hamiltonian under study. It is worth mentioning that for any Lie-algebra \( g \) there exists a second degree invariant \( t_2^{(Ω)}(x) \) which does not depend on chosen orbit. It is namely this invariant which defines the exponent in the ground state eigenfunction (see above). One of natural connections is to identify \( x \)'s with Cartesian coordinates. Later on we will use the expressions (2.7) as new variables in Hamiltonians under study and will call them the orbit variables.

In general, making averaging over different orbits in (2.7) we obtain algebraically related invariants, except for a case when averaging over a specific orbit leads to vanishing results for certain invariant(s). Without loss of generality, these orbits can be discarded. In fact, the variables which were successfully used to solve the cases \( A_n \) and \( BC_n \) in [4] and [5], correspondingly, as well as for \( G_2 \) and \( F_4 \) (see [6] and [8], correspondingly) can be easily obtained through the formulas (2.7).

The invariants of the fixed degrees are defined up to a polynomial in invariants of the lower degrees. This ambiguity plays an important role in obtaining the algebraic forms of Hamiltonians – these forms depend on the choice of variables, special combinations of variables only result in the simplest form which correspond to preservation of the minimal flag (see below).

Now we introduce a notion of exact-solvability. Let us assume that the operator \( h \) possesses infinitely-many finite-dimensional invariant subspaces \( V_n, \quad n = 0, 1, \ldots \), which can be ordered forming an infinite flag

\[
V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots ,
\]

(or, filtration) \( V \). Therefore one can say that the operator \( h \) preserves the flag \( V \).

**Definition**

- An operator \( h \) which preserves an infinite flag of explicitly defined finite-dimensional spaces \( V \) is called exactly-solvable operator with flag \( V \). We assume that the flag \( V \) is dense: between any two subsequent spaces there is no space of intermediate dimension which might belong to the flag.
• If given $h$ preserves several flags and among them there is a flag for which \( \dim \mathcal{V}_n \) is maximal for any given $n$, such a flag is called minimal.

Below we will deal with certain linear spaces of polynomials in several variables.

**Definition**

Consider the triangular linear space of polynomials in $k$ variables

\[
\mathcal{P}_n^{(f_1, \ldots, f_k)} = \langle s_1^{p_1} s_2^{p_2} \ldots s_k^{p_k} | 0 \leq f_1 p_1 + f_2 p_2 + \ldots + f_k p_k \leq n \rangle, \tag{2.8}
\]

where $f$’s are positive integer numbers and $n$ is integer. **Characteristic vector** is a vector with components which are equal to the coefficients (weights) $f_i$ in front of $p_i$:

\[
\vec{f} = (f_1, f_2, \ldots, f_k). \tag{2.9}
\]

In other words, $\vec{f}$ defines an action of $C^*$ on the space $C[s_1, \ldots, s_k]$ of polynomials in $k$ variables. The flag is defined using the induced grading.

The characteristic vector is defined up to a multiplicative integer factor which we choose to be minimal. In most of the examples $f_1 = 1$. Taking a sequence of the spaces characterized by growing integer numbers $n$ we arrive at a flag which has $\mathcal{P}_n^{(f_1, \ldots, f_k)}$ as generating linear space. In the examples $n$ takes consecutive integer values, $n = 0, 1, 2, \ldots$. We call such a flag $\mathcal{P}^{(f_1, \ldots, f_k)}$.

All Hamiltonians we are going to study are of quite special type. All their flags of invariant subspaces, that we were able to find so far, are the flags of polynomials of the form (2.8). Among these flags there always exists a minimal flag of a special form – comparing to other flags every $f_i$ takes its minimal value (!). For example, it was found that the minimal flag for $G_2$ models (both rational and trigonometric) is $\mathcal{P}^{(1, 2)}$ and the characteristic vector (2.9) is $(1, 2)$ [6].

Our final goal will be a search for minimal flags. It is worth to mention that one of the situations, when several flags of invariant subspaces can exist, occurs for the operator written in different variables while invariant subspaces remain to be polynomial one. Minimal flags have a remarkable property – they are preserved by corresponding trigonometric Hamiltonians if the latter are written in appropriate variables.

A general strategy of our study is the following: (i) as a first step we gauge rotate the ground state eigenfunction, (ii) choosing a certain orbit we construct a particular set of variables which lead to an algebraic form of the gauge-rotated Hamiltonian,
exploiting ambiguity in definition of the invariants of the fixed degrees we search for variables preserving a minimal flag.

3 The rational $G_2$ model

The rational $G_2$ model was introduced for the first time by Wolfes [17] and later on was obtained in the Hamiltonian Reduction method [1, 2]. This model allows an interpretation as a model of a three-identical particles with two- and three-body interactions. It was extensively studied in [6], [12].

The root system of the $G_2$ algebra is defined in 3-dimensional space with a constraint to the hyperplane $y_1 + y_2 + y_3 = 0$. Six positive roots are

$\{e_2 - e_1, e_3 - e_2, -e_1 + e_3, e_1 - 2e_2 + e_3, -2e_1 + e_2 + e_3, -e_1 - e_2 + 2e_3\}$.

The Hamiltonian of the rational $G_2$ model can be written in the form

$$H^{(r)}_{G_2} = \frac{1}{2} \sum_{k=1}^{3} \left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2\right] + V_{G_2}(x),$$

$$V_{G_2}(x) = g_s \sum_{k<l} \frac{1}{(x_k - x_l)^2} + 3g_l \sum_{k<l \neq m} \frac{1}{(x_k + x_l - 2x_m)^2},$$

(3.1)

where $\omega$ is a frequency parameter and $g_s = \nu(\nu - 1) > -\frac{1}{4}$, $g_l = \mu(\mu - 1) > -\frac{1}{4}$ are coupling constants associated with the two-body and three-body interactions. The existence of two coupling constants reflects the fact that the root system contains two sets of roots, long and short: $R_{\text{short}}$ with roots of length 2 and $R_{\text{long}}$ with roots of length 6. Parameter $\omega$ in (3.1) is the only dimensional parameter in the Hamiltonian and the eigenvalues should be proportional to it. A connection to the root system with the model (3.1) appears when the center-of-mass coordinate is separated out and the relative motion is studied. Let us introduce the Perelomov coordinates of the relative motion [18]

$$y_{1,2,3}^{\text{Perelomov}} = x_{1,2,3} - \frac{1}{3}X, \quad X = x_1 + x_2 + x_3,$$

(3.2)

where $x_i$ are the Cartesian coordinates of particles and $X$ is the center-of-mass coordinate. Another convenient set of relative variables is given by the standard Jacobi coordinates

$$y_1^{\text{Jacobi}} = x_2 - x_1, \quad y_2^{\text{Jacobi}} = x_3 - x_2, \quad y_3^{\text{Jacobi}} = x_1 - x_3.$$
Both sets of the relative coordinates obey a condition \( y_1 + y_2 + y_3 = 0 \). Thus the relative motion can be studied in three-dimensional \( y \)-space with the constraint \( y_1 + y_2 + y_3 = 0 \). Transition from (3.2) to (3.3) corresponds to the interchange of two sets of roots – \( R_{\text{short}} \) and \( R_{\text{long}} \). One can identify the variables \( y \) either with \( y_{\text{Perelomov}} \) or \( y_{\text{Jacobi}} \). Due to the symmetry of the Hamiltonian with respect to the interchange of long and short roots a duality between the two sets of variables, \( y_{\text{Perelomov}} \) and \( y_{\text{Jacobi}} \), appears. It will be demonstrated below.

The ground state of relative motion is given by

\[
\Psi_0^{(r)}(y) = (\Delta_1^{(r)}(y))^\nu(\Delta_2^{(r)}(y))^\mu \exp\left\{ -\frac{\omega}{2} \sum_{k=1}^{3} y_k^2 \right\}, \quad y_1 + y_2 + y_3 = 0,
\]

where \( \Delta_1^{(r)}(y) \) and \( \Delta_2^{(r)}(y) \) are Vandermonde determinants

\[
\Delta_1^{(r)}(x) = \prod_{R_{\text{short}}} (\alpha_k \cdot y) = \prod_{j<i} (y_i - y_j),
\]

\[
\Delta_2^{(r)}(x) = \prod_{R_{\text{long}}} (\alpha_k \cdot y) = \prod_{i<j; i,j \neq k} (y_i + y_j - 2y_k).
\]

Let us make a gauge rotation of the Hamiltonian (3.1) with the ground state eigenfunction (3.4):

\[
h_{G_2}^{(r)} = -2(\Psi_0^{(r)}(y))^{-1}(\mathcal{H}_{G_2}^{(r)} - E_0)\Psi_0^{(r)}(y),
\]

and separate out the center-of-mass motion.

Following the symmetries of the Hamiltonian (3.1) let us define the \( G_2 \) Weyl-invariant polynomials by averaging over the simplest orbit \( \Omega(e_2 - e_1) \), generated by \( (e_2 - e_1) \). It has six elements:

\[
t_a^{(\Omega)}(y) = \frac{1}{6} \sum_{k=1}^{6} (\omega_k \cdot y)^a, \quad \omega_k \in \Omega(e_2 - e_1),
\]

(cf. (2.7)), where \( a = 2, 6 \) are the degrees of the \( G_2 \)-algebra and \( \omega_k, k = 1, 2, \ldots, 6 \) are the orbit elements. Explicitly,

\[
t_2^{(\Omega)}(y) = 2(y_1^2 + y_2^2 + y_1y_2),
\]
\[ t_6^{(\Omega)}(y) = 105y_1^4y_2^2 + 66y_1^5y_2 + 66y_1y_2^5 + 22y_1^6 + 22y_2^6 + 100y_1^3y_2^3 + 105y_1^2y_2^4 , \]

Taking different orbits in the formula of averaging (3.6) we obtain different invariants. They are related to each other as

\[ t_2^{(\Omega')} = t_2^{(\Omega)} , \]
\[ t_6^{(\Omega')} = t_6^{(\Omega)} + A^{(6,\Omega)}(t_2^{(\Omega)})^3 . \] (3.7)

up to multiplicative factors. The general invariants of the lowest degree of the \( G_2 \)-algebra can be found through invariants obtained by averaging, they are algebraically related to \( t_a^{(\Omega)} \). In fact, an arbitrary set of invariants generating the \( S^{W_2} \) algebra can be found through a particular orbit invariants and has a form:

\[ t_2 = t_2^{(\Omega)}(y) , \]
\[ t_6 = t_6^{(\Omega)}(y) + A^{(6)}(t_2^{(\Omega)}(y))^3 . \] (3.8)

It can be shown that by taking \( t_{2,6} \) as new variables in (3.5) we always arrive at the algebraic form of the gauge-rotated rational \( G_2 \) Hamiltonian for any value of the parameter \( A^{(6)} \). One may assume that such a property should correspond to the flag with characteristic vector \((1,3)\), which is invariant with respect to transformation (3.8). The existence of this flag can be easily confirmed.

A set of flags preserved by the Hamiltonian in coordinates (3.8) can be obtained by analyzing its action on monomials \( \phi_{\vec{p}} = t_2^{p_1}t_6^{p_2} \) labelled by vectors \( \vec{p} = (p_1,p_2) \). The monomial \( \phi_{\vec{p}} \) is mapped into a sum of monomials \( \phi_{\vec{p}-\vec{d}_i} \). There are among \( d_i \) two zero vectors \((0,0)\) corresponding to terms \(-4\omega t_1\partial t_1 \) and \(-12\omega t_2\partial t_2 \), which determine eigenvalues of the Hamiltonian. Other shift vectors are \((-1,0)\), \((2,-1)\) and \((5,-2)\). It means that the minimal flag for the general coordinates (3.8) is \((2,5)\).

However, one can tune the value of the parameter \( A^{(6)} \) to eliminate the term \((5,-2)\) in the Hamiltonian. Remarkably there exist two such values of this parameter,

\[ A^{(6)} = -9/4 \quad \text{and} \quad A^{(6)} = -11/4 , \] (3.9)

resulting in smaller flag \((1,2)\) (see below the equation (3.13)). Following the definition of Section 2 this flag \( \mathcal{P}^{(1,2)} \) is minimal. 6

6It is remarkable that this flag is invariant under the action of the trigonometric \( G_2 \) Hamiltonian, when this Hamiltonian is written in appropriate variables.
Making the substitution
\[ \tau_2 = t_2^{(\Omega)}(y), \]
\[ \tau_6 = t_6^{(\Omega)}(y) - \frac{9}{4}(t_2^{(\Omega)}(y))^3 . \tag{3.10} \]
we arrive at the Hamiltonian
\[ h_{G_2}^{(r,1)} = 4\tau_2 \partial_{\tau_2 \tau_2}^2 + 24\tau_6 \partial_{\tau_2 \tau_6}^2 + 18\tau_2^2 \partial_{\tau_6 \tau_6}^2 \]
\[ - \left\{ 4\omega \tau_2 - 4[1 + 3(\mu + \nu)] \right\} \partial_{\tau_2} - \left[ 12\omega \tau_6 - 9(1 + 2\nu)\tau_2^2 \right] \partial_{\tau_6} . \tag{3.11} \]
This is an \textit{algebraic} form of the rational \( G_2 \) model. It can be easily checked that the Hamiltonian (3.11) has infinitely-many finite-dimensional invariant subspaces
\[ P_{n}^{(1,2)} = \langle \tau_2^{p_1} \tau_6^{p_2} | 0 \leq p_1 + 2p_2 \leq n \rangle , \quad n = 0, 1, \ldots , \tag{3.12} \]
with the characteristic vector
\[ \vec{f} = (1, 2) , \tag{3.13} \]
forming the minimal flag \( P^{(1,2)} \) of the rational \( G_2 \) model which we denote \( P^{(G_2)} = P^{(1,2)} \) (see a discussion above).

The Hamiltonian (3.11) admits a representation in terms of a non-linear combination of the generators of some infinite-dimensional finitely-generated algebra \( g^{(2)} \) (for definition and discussion see [6]). It leads to the \( g^{(2)} \)-\textit{Lie-algebraic} form of the rational \( G_2 \) model. This form depends on a subset of the generators of the \( gl_2 \times R^3 \)-subalgebra of algebra \( g^{(2)} \) only and does not contain a raising generator of \( g^{(2)} \). The generators of \( g^{(2)} \) with excluded raising generator have infinitely-many common finite-dimensional spaces which coincide to the finite-dimensional invariant subspaces of the Hamiltonian (3.11).

We already studied the Hamiltonian (3.5) corresponding to the first choice of variables, when \( A^{(6)} = -9/4 \) in (3.8). Now we explore the second choice, when \( A^{(6)} = -11/4 \). One can easily represent these variables in terms of the variables of the first choice
\[ \tau_2(y) = t_2^{(\Omega)}(y) = \tau_2 , \]
\[ \tau_6(y) = -t_6^{(\Omega)}(y) + \frac{11}{4}(t_2^{(\Omega)}(y))^3 = -\tau_6 + \frac{1}{2}\tau_2^3 , \tag{3.14} \]
where \( y \)'s are the Perelomov coordinates (3.2). Making the change of variables (3.14) in (3.11) the Hamiltonian (3.5) takes the form

\[
h_{G_2}^{(r,2)} = 4\tilde{\tau}_2\partial_{\tilde{\tau}_2}\tilde{\tau}_2 + 24\tilde{\tau}_0\partial_{\tilde{\tau}_2}\tilde{\tau}_6 + 18\tilde{\tau}_2^2\tilde{\tau}_0\partial_{\tilde{\tau}_6}\tilde{\tau}_6 \\
- \{4\omega\tilde{\tau}_2 - 4[1 + 3(\mu + \nu)]\} \partial_{\tilde{\tau}_2} - \left[12\omega\tilde{\tau}_6 - 9(1 + 2\mu)\tilde{\tau}_2^2\right] \partial_{\tilde{\tau}_6}. \quad (3.15)
\]

(cf. (3.11)). This is another algebraic form of the rational \( G_2 \) model, which is dual to (3.11): the Hamiltonian (3.15) differs from (3.11) by permutation \( \nu \leftrightarrow \mu \) only. It reflects a symmetry between sets of short and long roots. It is evident that the Hamiltonian (3.15) preserves the flag \( P(1, 2) \) and admits a representation in terms of the generators of the algebra \( g^{(2)} \) but now being written in the coordinates \( \tilde{\tau} \)'s where \( \nu \leftrightarrow \mu \).

The duality between short and long roots also occurs when we make the identification of \( y \) variables either with the Perelomov coordinates (3.2) or with the Jacobi coordinates (3.3). It appears in a relation between the \( \tau \)-variables (of the first choice of \( A^{(6)} \), see (3.9)) in the Perelomov coordinates and the \( \tilde{\tau} \)-variables (of the second choice of \( A^{(6)} \), see (3.9)) in the Jacobi coordinates and visa versa, e.g.

\[
\tau_2(y^{Jacobi}) = 3\tilde{\tau}_2(y^{Perelomov}), \quad \tau_6(y^{Jacobi}) = 27\tilde{\tau}_6(y^{Perelomov}). \quad (3.16)
\]

It is worth emphasizing that the coordinates \( \tilde{\tau} \)'s (3.14) have a remarkable property: they are unique coordinates which can be ‘trigonometrized’ – they coincide with the rational limit of certain trigonometric coordinates, \( \lim_{\alpha \to 0} \sin \alpha x = x \), in which the trigonometric \( G_2 \) model gets an algebraic form (see [6]). The \( \tau_{2,6} \)-coordinates do not have such a property \(^7\).

It is interesting to analyze a transformation preserving the flag \( \mathcal{P}_n^{(G_2)} \). The most general polynomial transformation which preserve the linear space \( \mathcal{P}_n^{(1,2)} \) (3.12) is

\[
\tau_2 \to \tau_2, \\
\tau_6 \to \tau_6 + a^{(2)}\tau_2^2, \quad (3.17)
\]

\(^7\)It is worth noting that in the coordinates \((\tau_2, \sqrt{\tau_6})\) the gauge-rotated rational \( G_2 \) Hamiltonian (3.9) can be rewritten in terms of the generators of the maximal affine subalgebra of the \( gl(3) \)-algebra [6]. Thus, this Hamiltonian possesses two different hidden algebras: the \( g^{(2)} \) algebra acting on the configuration space parameterized by the \( \tau(\tilde{\tau}) \)-coordinates and the \( gl(3) \)-algebra acting on the configuration space parameterized by the \((\tau_2, \sqrt{\tau_6}) \) coordinates (see [6] and Eqs.(2.6)-(2.7) therein).
where $a^{(2)}$ is an arbitrary number of the dimension $[\omega^{-1}]$ for any $n$. It is evident that this transformation preserves the whole flag $P^{(G_2)}$. We find that there exist two algebraic operators which preserve the same flag $P^{(G_2)}$. However, the variables $\tau$'s and $\tilde{\tau}$'s in which these operators have been written are related one to each other through a transformation of the type (3.17) (see the relation (3.14)). Hence these two algebraic operators are non-equivalent. One suspects that the fact of the existence of two non-equivalent algebraic forms of the rational $G_2$ model (which look very much alike) reflects a certain intrinsic degeneracy of the model and should not hold for the trigonometric case. A study of the trigonometric $G_2$ model confirms that there exists a single $g^{(2)}$-Lie-algebraic form.

Thus, the operators (3.11), (3.15) possess infinitely many finite-dimensional invariant subspaces. These invariant subspaces coincide with the finite-dimensional representation spaces of the algebra $g^{(2)}$. It is worth noting that for $\omega = \nu = \mu = 0$ both operators $h^{(r,\lambda(2))}_{G_2}$ coincide and represent the flat space Laplacian written in the $g^{(2)}$-Lie-algebraic form. It is evident that in this case there are no polynomial eigenfunctions in $\tau(\tilde{\tau})$ coordinates.

Both operators (3.11), (3.15) are triangular in the basis of monomials $\tau^{n_1}_2 \tau^{n_2}_6$ ($\tilde{\tau}^{n_1}_2 \tilde{\tau}^{n_2}_6$). Therefore, the spectrum of (3.11), (3.15), $h^{(r)}_{G_2} \varphi = -2\epsilon \varphi$, can be found explicitly and is equal to

$$\epsilon_{n_1,n_2} = \omega(2n_1 + 6n_2) \ , \quad (3.18)$$

where $n_i$ are non-negative integers $n_1, n_2 = 0, 1, \ldots$ (coefficients 2 and 6 are degrees of $G_2$). The spectrum does not depend on the coupling constants $g_l$, $g_s$ (but the reference point for energy (2.6) does), it is equidistant and corresponds to the spectrum of two harmonic oscillators with frequencies $2\omega$ and $6\omega$. Degeneracy of the spectrum is related to the number of partitions of integer number $n$ to two weighted integers $n_1 + 3n_2$. The spectrum of the original rational $G_2$ Hamiltonian (3.1) is $E_n = E_0 + \epsilon_n$. It is worth noting that the Hamiltonian (3.15) (as well as (3.11)) possesses a remarkable property: there exists a family of eigenfunctions which depend on the single variable $\tilde{\tau}(\tau)$. These eigenfunctions are the associated Laguerre polynomials. This property allows to construct a quasi-exactly-solvable generalization of the rational $G_2$ model. It will be done elsewhere.

It is worth mentioning that the boundaries of configuration space are determined by zeros of the ground state wave function (3.2). In $\tau$-variables it is the algebraic curve

$$(\Delta_1 \Delta_2 (\tau))^2 = -27\tau_6(\tau_6 - 1/2\tau_2^3) = 0 \ , \quad (3.19)$$
and in $\tilde{\tau}$-variables it has the same form

$$\left(\Delta_1 \Delta_2 \left(\tilde{\tau}\right) \right)^2 = -27 \tilde{\tau}_6 (\tilde{\tau}_6 - 1/2 \tilde{\tau}_2^3) = 0 .$$  \hspace{1cm} (3.20)

In $\tau_6$, $\tilde{\tau}_6$ variables associated with the Perelomov $y$-coordinates the product of discriminants looks especially simple

$$\left(\Delta_1 \Delta_2 \left(y\right) \right)^2 = 27 \tau_6(y) \tilde{\tau}_6(y) ,$$  \hspace{1cm} (3.21)

where symmetry between short and long roots is seen explicitly. Also note [15]

$$\left[ \det \left( \frac{\partial \tau_i}{\partial y_k} \right) \right]^2 = 16 \left(\Delta_1 \Delta_2 \right)^2 .$$  \hspace{1cm} (3.22)

### 4 The rational $F_4$ model

The Hamiltonian of the rational $F_4$ model written in the basis of the standard $F_4$ roots has the form (see [8]),

$$\mathcal{H}^{(r)}_{F_4} = \frac{1}{2} \sum_{i=1}^{4} \left( -\partial^2 x_i + \omega^2 x_i^2 \right) + g_l \sum_{j>i}^{4} \left( \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right)$$

$$+ \frac{g_s}{2} \sum_{i=1}^{4} \frac{1}{x_i^2} + 2g_s \sum_{\nu' s=0,1} \frac{1}{\nu' \sum_{s=0,1} \left[ x_1 + (-1)^{\nu_2} x_2 + (-1)^{\nu_3} x_3 + (-1)^{\nu_4} x_4 \right] } ,$$

where $g_l = \nu(\nu - 1)$, $g_s = \mu(\mu - 1)$ are coupling constants related to sets of long and short roots, $\mathcal{R}_{long}$ and $\mathcal{R}_{short}$, correspondingly, and $\omega$ is a frequency. Its ground state can be written as

$$\Psi_0^{(r)}(x) = \left( \Delta_+ \Delta_- \right)^\nu (\Delta_0 \Delta)^\mu \exp \left( -\frac{\omega}{2} \sum_{i=1}^{4} x_i^2 \right) ,$$  \hspace{1cm} (4.2)

where

$$\Delta_+ \Delta_- = \prod_{\mathcal{R}_{long}} (\alpha_k \cdot x) = \prod_{j<i} (x_i + x_j) \prod_{j<i} (x_i - x_j) ,$$

\[\text{8There is a misprint in [8] in the definition of coupling constants for the Hamiltonian (3.1): it should read } g_l = \mu(\mu - 1) > -1/4 , \ g = (1/2) \nu(\nu - 1) > -1/8 . \]
\[ \Delta_0 \Delta = \prod_{R_{\text{short}}} (\alpha_k \cdot x) \]

\[ = \prod_{i=1}^{4} x_i \prod_{\nu_s=0,1} \left( \frac{x_1 + (-1)^{-\nu_2} x_2 + (-1)^{-\nu_3} x_3 + (-1)^{-\nu_4} x_4}{2} \right). \quad (4.3) \]

The ground state energy is

\[ E_0 = 2\omega(1 + 6\mu + 6\nu). \quad (4.4) \]

Let us make gauge rotation of the Hamiltonian (4.1) with the ground state eigenfunction (4.2) as a gauge factor

\[ h_F^{(r)} = -2(\Psi^{(r)}_0(x))^{-1}(H^{(r)}_F - E_0)\Psi^{(r)}_0(x). \quad (4.5) \]

As new variables we take Weyl invariant polynomials of the lowest degrees of the group \( W_{F_4} \) found by averaging elementary polynomials \((\omega \cdot x)^a\) over the 24-element orbit \( \Omega \) generated by the root \((e_1 + e_2)\)

\[ t^{(\Omega)}_a(x) = \frac{1}{12} \sum_{\omega \in \Omega} (\omega \cdot x)^a, \quad a = 2, 6, 8, 12. \quad (4.6) \]

(cf. (2.7)). In this case the powers \( a = 2, 6, 8, 12 \) are the degrees of the group \( W_{F_4} \). As we mentioned in Section 2 the invariants of the fixed degrees \( a \) are defined ambiguously, up to some non-linear combinations of invariants of the lower degrees

\[ t^{(\Omega)}_2 \rightarrow t^{(\Omega)}_6, \]
\[ t^{(\Omega)}_6 \rightarrow t^{(\Omega)}_6 + A^{(6)}(t^{(\Omega)}_2)^3, \]
\[ t^{(\Omega)}_8 \rightarrow t^{(\Omega)}_8 + A^{(8)}(t^{(\Omega)}_2)^4 + A^{(8)}(t^{(\Omega)}_2)^4t^{(\Omega)}_6, \]
\[ t^{(\Omega)}_{12} \rightarrow t^{(\Omega)}_{12} + A^{(12)}(t^{(\Omega)}_2)^6 + A^{(12)}(t^{(\Omega)}_2)^6t^{(\Omega)}_6 + A^{(12)}(t^{(\Omega)}_2)^4t^{(\Omega)}_8 + A^{(12)}(t^{(\Omega)}_2)^2t^{(\Omega)}_6 + A^{(12)}(t^{(\Omega)}_2)^2, \quad (4.7) \]

where \( A^{(6,8,12)} \) are arbitrary parameters. It can be shown that operator \( h_F^{(r)} \) is algebraic and it preserves a flag of polynomials for arbitrary values of these parameters. There are two flags \((2, 6, 8, 12)\) and \((2, 6, 8, 11)\) which are invariant under transformations (4.7). They are similar to the flags \((1, 3)\) and \((2, 5)\) for the \( G_2 \) case.

Then we must search for a minimal flag. From a technical point of view it can be found by eliminating certain terms in the Hamiltonian, by choosing appropriate
parameters $A$'s. In \cite{8} the minimal flag, denoted as $\mathcal{P}^{(F_4)}$, is found. This flag is generated by the spaces of quasi-homogeneous polynomials

$$\mathcal{P}_{n}^{(1,2,2,3)} = \{ \tau_1^{p_1} \tau_3^{p_3} \tau_4^{p_4} \tau_6^{p_6} | 0 \leq p_1 + 2p_3 + 2p_4 + 3p_6 \leq n \},$$

(4.8)

with the characteristic vector $\vec{f}$

$$\vec{f} = (1, 2, 2, 3),$$

(4.9)

which give rise to the flag $\mathcal{P}^{(1,2,2,3)}$. Hence, $\mathcal{P}^{(F_4)} = \mathcal{P}^{(1,2,2,3)}$ is the minimal flag for the rational $F_4$ model.

The characteristic vector (4.9) coincides with the highest root among short roots in the $F_4$ root system written in the basis of simple roots. Note that the most general polynomial transformation preserving the linear space $\mathcal{P}_{n}^{(1,2,2,3)}$ is

$$s_1 \rightarrow s_1,$$

$$s_2 \rightarrow s_2 + a_2 s_1^2 + b_2 s_3,$$

$$s_3 \rightarrow s_3 + a_3 s_1^2 + b_3 s_2,$$

$$s_4 \rightarrow s_4 + a_4 s_1^3 + b_4 s_1 s_2 + c_4 s_1 s_3,$$

(4.10)

where $a, b, c$ are arbitrary numbers.

Explicitly the set of variables preserving the minimal flag $\mathcal{P}^{(1,2,2,3)}$ found in Ref. \cite{8} is:

$$\tau_2 = t_2^{(\Omega)},$$

$$\tau_6 = \frac{1}{12} t_6^{(\Omega)} - \frac{1}{12} (t_2^{(\Omega)})^3,$$

$$\tau_8 = \frac{1}{80} t_8^{(\Omega)} - \frac{1}{30} t_2^{(\Omega)} t_6^{(\Omega)} + \frac{1}{48} (t_2^{(\Omega)})^4,$$

$$\tau_{12} = \frac{1}{720} t_{12}^{(\Omega)} - \frac{5}{288} (t_2^{(\Omega)})^2 t_8^{(\Omega)} + \frac{1}{27} (t_2^{(\Omega)})^3 t_6^{(\Omega)} - \frac{29}{1440} (t_2^{(\Omega)})^6 - \frac{1}{1080} (t_6^{(\Omega)})^2,$$

(4.11)

In these variables the algebraic form of the gauge-rotated Hamiltonian (4.5) is:

$$h_{F_4}^{(\mathfrak{r},1)} = 4 \tau_2 \frac{\partial^2}{\partial \tau_2^2} + \frac{2}{3} (10 \tau_2 \tau_8 + \tau_2^2 \tau_6) \frac{\partial^2}{\partial \tau_6^2} + 2 (\tau_2 \tau_{12} + 2 \tau_8 \tau_6) \frac{\partial^2}{\partial \tau_8^2}.$$
\[
\begin{align*}
&+ 24\tau_6 \frac{\partial}{\partial \tau_2} \frac{\partial}{\partial \tau_6} + 32\tau_8 \frac{\partial}{\partial \tau_2} \frac{\partial}{\partial \tau_8} + 48\tau_{12} \frac{\partial}{\partial \tau_2} \frac{\partial}{\partial \tau_{12}} \\
&+ \frac{8}{3}(\tau_2^2 \tau_8 + 6\tau_{12}) \frac{\partial}{\partial \tau_6} \frac{\partial}{\partial \tau_8} + 4(\tau_2^2 \tau_{12} + 8\tau_8^2) \frac{\partial}{\partial \tau_6} \frac{\partial}{\partial \tau_{12}} \\
&+ 4(3\tau_6 \tau_{12} + 2\tau_2 \tau_8^2) \frac{\partial}{\partial \tau_8} \frac{\partial}{\partial \tau_{12}} + 6(2\tau_s^2 \tau_6 + \tau_2 \tau_8 \tau_{12}) \frac{\partial^2}{\partial \tau_7^2} \\
&- 4[\omega \tau_2 - 2(6\nu + 6\mu + 1)] \frac{\partial}{\partial \tau_2} - [12\omega \tau_6 - \tau_2^2(4\nu + 2\mu + 1)] \frac{\partial}{\partial \tau_6} \\
&- 4[4\omega \tau_8 - \tau_6(1 + 3\nu)] \frac{\partial}{\partial \tau_8} - 4[6\omega \tau_{12} - \tau_2 \tau_8(2 + 3\nu)] \frac{\partial}{\partial \tau_{12}}
\end{align*}
\] (4.12)

The variables (4.11) are remarkable as they are the rational limits of certain trigonometric variables in which the trigonometric $F_4$ model takes an algebraic form [8].

However, the set of variables (4.11) does not exhaust all the possible sets of variables leading to the minimal flag (4.9). Similarly to what happens with the rational $G_2$ model, there exists one more set of variables

\[
\begin{align*}
\tilde{\tau}_2 &= t_2^{(\Omega)} = \tau_2, \\
\tilde{\tau}_6 &= -\frac{1}{12} t_6^{(\Omega)} + \frac{1}{8} (t_2^{(\Omega)})^3 = -\tau_6 + \frac{1}{24} t_2^3, \\
\tilde{\tau}_8 &= \frac{1}{80} t_8^{(\Omega)} - \frac{13}{240} t_2^{(\Omega)} t_6^{(\Omega)} + \frac{3}{64} (t_2^{(\Omega)})^4 = \tau_8 - \frac{1}{4} t_2 \tau_6 + \frac{1}{192} t_2^4, \\
\tilde{\tau}_{12} &= -\frac{1}{720} t_{12}^{(\Omega)} + \frac{61}{17280} (t_6^{(\Omega)})^2 + \frac{109}{5760} (t_2^{(\Omega)})^2 t_8^{(\Omega)} - \frac{847}{17280} (t_2^{(\Omega)})^3 t_6^{(\Omega)} + \frac{109}{3840} (t_2^{(\Omega)})^6 \\
&= -\tau_{12} + \frac{1}{8} t_2^2 \tau_8 + \frac{3}{8} \tau_6^2 - \frac{1}{32} \tau_2 \tau_6^3 + \frac{1}{2304} t_2^6,
\end{align*}
\] (4.13)

(cf. (4.11)) leading to an algebraic form of $h_{F_4}^{(r)}$ which preserves the minimal flag $\mathcal{P}^{(F_4)}$. The explicit expression for the Hamiltonian in the variables (4.13) gets the same form as (4.12) with $\nu$ and $\mu$ exchanged:

\[
h_{F_4}^{(r,2)}(\tilde{\tau}) = h_{F_4}^{(r,1)}(\tau; \nu \leftrightarrow \mu)
\] (4.14)

Clearly the Hamiltonian continues to be algebraic under the transformations (4.10).

We were able to find the one-parametric algebra of differential operators for which $\mathcal{P}_n^{(F_4)}$ is the space of finite-dimensional irreducible representation (see Appendix B in [8]). Furthermore, the finite-dimensional representation spaces arise for different
integer values of the parameter. They form an infinite non-classical flag which coincides with $P^{(F_4)}$. We call this algebra $f^{(4)}$. Like the algebra $g^{(2)}$ introduced in \[6\] in relation to the $G_2$ models, the algebra $f^{(4)}$ is infinite-dimensional yet finitely-generated. The rational $F_4$ Hamiltonian in either algebraic form \[11.12\] or \[11.14\] can be rewritten in terms of the generators of this algebra.

The variables \[11.11\] and \[11.13\] can not be related by the transformation \[4.10\]. Therefore these two $f^{(4)}$-Lie-algebraic forms are non-equivalent from the point of view of the transformation \[4.10\]. We interpret this fact by a certain intrinsic degeneracy of the rational $F_4$ model. When the trigonometric $F_4$ model is considered one can show that there exists the only one $f^{(4)}$-Lie-algebraic form (up to a transformation \[4.10\]) \[3\].

Thus, the operators \[11.12\], \[11.14\] have infinitely many finite-dimensional invariant subspaces. The finite-dimensional representations of the algebra $f^{(4)}$. If $\omega = \nu = \mu = 0$, the operators $h^{(r,1)(2)}_{F_4}$ coincide and become the flat space Laplacian written in the $g^{(2)}$-Lie-algebraic form, with no polynomial eigenfunctions in $\tau(\tilde{\tau})$-coordinates.

The operator \[11.12\] (as well as \[11.14\]) is triangular in the basis of monomials $\tau_2^{p_1} \tau_6^{p_2} \tau_8^{p_3} \tau_{12}^{p_4}$. One can find the spectrum of \[11.12\], $h^{(r,1)}_{F_4} \varphi = -2 \epsilon \varphi$, explicitly

$$\epsilon_{n_1,n_2,n_3,n_4} = \omega(2n_1 + 6n_2 + 8n_3 + 12n_4) \text{,}$$

where $n_i = 0, 1, \ldots$ are non-negative integers. Degeneracy of the spectrum is equal to the number of partitions of an integer number $n$ to four weighted integers $n_1 + 3n_2 + 4n_3 + 6n_4$. The spectrum does not depend on the coupling constants $g_l$, $g_s$, it is equidistant and coincides (with different degeneracy) with the spectrum of the harmonic oscillator as well as with that of the rational $D_4$ model. Finally, the energies of the original rational $F_4$ Hamiltonian \[11.11\] are $E_n = E_0 + \epsilon_n$. It is worth noting that the Hamiltonian \[11.12\] possesses a remarkable property: there exists a family of eigenfunctions which depend on the single variable $\tau_2$. These eigenfunctions are the associated Laguerre polynomials. As an illustration the first eigenfunctions are presented in the Appendix A. Again, this property allows to construct a quasi-exactly-solvable generalization of the rational $F_4$ model. It will be done elsewhere.

Configuration space of the rational $F_4$ model \[11.11\] is defined by zeros of the ground state eigenfunction, i.e. by zeros of the pre-exponential factor in \[11.12\]. These zeros also define boundaries of the Weyl chamber (see e.g. \[2\]). The pre-exponential
factor (4.2) at $\nu = \mu = 2$ can be written as a product of two factors. The first is
\[
(\Delta_+ \Delta_-(\tau))^2 = -192 \tau_{12}^2 + 256 \tau_8^3,
\]
(4.16)
it corresponds to the rational $D_4$ model which occurs for $g_s = 0$ in (4.1). The second one
\[
(\Delta_0 \Delta(\tau))^2 = \frac{1}{4096}(-192 \tau_{12}^2 + 256 \tau_8^3 + 144 \tau_6^2 \tau_{12} - 27 \tau_6^4 - 192 \tau_6 \tau_8^2 + 48 \tau_2^2 \tau_8 \tau_{12}
+ 30 \tau_2^2 \tau_6 \tau_8 - 12 \tau_2^3 \tau_6 \tau_{12} + \frac{1}{2} \tau_2^3 \tau_6^3 + \tau_2^4 \tau_8^2 - \frac{1}{2} \tau_2^5 \tau_6 \tau_8 + \frac{1}{6} \tau_2^6 \tau_{12})^2,
\]
(4.17)
corresponds to the case of the degenerate $F_4$ model, $g_t = 0$ (which is equivalent to the $D_4$ model in dual variables, see [8]). Thus, a boundary of the configuration space of the rational $F_4$ model is the union of the algebraic hyper-surfaces (4.16)–(4.17) of degree 3 and 7 respectively, being the reduced algebraic hyper-surface of degree 10.

In agreement with the general theory (see e.g. [15])
\[
\left[\det \left( \frac{\partial \tau_a}{\partial x_k} \right) \right]^2 = 4096 (\Delta_+ \Delta_-)^2 (\Delta_0 \Delta)^2.
\]
(4.18)

To conclude a discussion of the rational $F_4$ model, one can state that the rational $F_4$ model admits two non-equivalent algebraic and $f^{(4)}$-Lie-algebraic forms, correspondingly. However, it does not manifest anything non-trivial. The existence of these two set of variables is related to a phenomenon of duality discussed in [8]: it can be easily checked that taking the variables (4.11) and substituting in (4.6) the variables $x$’s by $z$’s,
\[
z_{1,2} = x_1 \pm x_2 \quad , \quad z_{3,4} = x_3 \pm x_4
\]
(4.19)
we get the variables (4.13).
5 The rational $E_6$ model

The Hamiltonian of the rational $E_6$ model is built using the root system of the $E_6$ algebra. A convenient way to represent the Hamiltonian is to write it in an 8-dimensional space \( \{x_1, x_2, \ldots, x_8\} \) while imposing two constraints $x_7 = x_6$, $x_8 = -x_6$,

$$
\mathcal{H}_{E_6} = -\frac{1}{2} \Delta^{(8)} + \frac{\omega^2}{2} \sum_{i=1}^{8} x_i^2 + V_{E_6},
$$

where

$$
V_{E_6}(x) = g \sum_{j<i=1}^{5} \left[ \frac{1}{(x_i + x_j)^2} + \frac{1}{(x_i - x_j)^2} \right]
$$

$$
+ g \sum_{\{\nu_j\}} \left[ \frac{1}{2} \left( -x_8 + x_7 + x_6 - \sum_{j=1}^{5} (-1)^{\nu_j} x_j \right) \right]^2.
$$

is root-generated part of the potential with a coupling constant $g = \nu (\nu - 1)$. The configuration space is given by the principal $E_6$ Weyl chamber.

In order to resolve the constraints, we introduce new variables

\begin{align*}
    y_i &= x_i, \quad i = 1 \ldots 5 \\
    y_6 &= x_6 + x_7 - x_8, \quad \text{(using the constraint } y_6 = 3x_6) \\
    y_7 &= x_6 - x_7, \quad \text{(using the constraint } y_7 = 0) \\
    y_8 &= x_6 + x_8, \quad \text{(using the constraint } y_8 = 0)
\end{align*}

in which the Laplacian becomes

$$
\Delta^{(8)} = \Delta^{(5)} + 3 \frac{\partial^2}{\partial y_6^2} + 2 \left[ \frac{\partial^2}{\partial y_7^2} + \frac{\partial^2}{\partial y_8^2} + \frac{\partial^2}{\partial y_7 \partial y_8} \right],
$$

while the potential part of (5.1) depends on \( \{y_1 \ldots y_6\} \) only:

\begin{align*}
    V &= \frac{\omega^2}{2} \left\{ \sum_{i=1}^{5} y_i^2 + \frac{y_6^2}{3} \right\} + g \sum_{j<i=1}^{5} \left[ \frac{1}{(y_i + y_j)^2} + \frac{1}{(y_i - y_j)^2} \right] \\
    &+ g \sum_{\nu_j, j=1}^{5} \left[ \frac{1}{2} (y_6 - \sum_{j=1}^{5} (-1)^{\nu_j} y_j) \right]^2.
\end{align*}
In this formalism imposing constraints implies a study of eigenfunctions having no dependence on \( y_7, y_8 \). Hence, \( y_7, y_8 \)-dependent part of the Laplacian standing in square brackets in (5.4) can be dropped off.

The ground state eigenfunction has a form

\[
\Psi_0 = (\Delta_+^{(5)} \Delta_-^{(5)})^{\nu} \Delta_{E_6}^{\nu} e^{-\frac{1}{2} \omega \left( \sum_{i=1}^{5} y_i^2 + \frac{y_6^2}{4} \right)}, \quad E_0 = 3\omega(1 + 12\nu) \tag{5.6}
\]

where

\[
\Delta^{(5)} = \prod_{j<i=1}^{5} (y_i \pm y_j)
\]

\[
\Delta_{E_6} = \prod_{\{\nu_j\}} \left( y_6 + \sum_{j=1}^{5} (-1)^{\nu_j} y_j \right)
\]

with \( g = \nu(\nu - 1) \).

In order to find variables leading to the algebraic form of gauge-rotated Hamiltonian,

\[
h_{E_6}^{(r)}(y_1 \ldots y_6) = -2\Psi_0^{-1}(\mathcal{H}_{E_6} - E_0)(y_1 \ldots y_6)\Psi_0, \tag{5.7}
\]

let us define a basis in the form of the Weyl-invariant polynomials averaged over the 27-element orbit generated by the vector \( e_6 \),

\[
t^{(r)}_a = \sum_{k=1}^{27} (\omega_k \cdot y)^a, \quad \omega_k \in \Omega(e_6), \tag{5.8}
\]

(cf. (2.7)), where \( a = 2, 5, 6, 8, 9, 12 \) are the degrees of the \( E_6 \) Weyl group and \( \omega_k, k = 1, 2, \ldots 27 \) are the orbit elements. The orbit variables \( y \) in (5.8) are identified with variables \( y_1 \ldots y_6 \) in (5.3) and in (5.8). The Weyl-invariant polynomials of the fixed degree are defined ambiguously

\[
\begin{align*}
t^{(r)}_2 & \rightarrow t_2^{(r)}, \\
t^{(r)}_5 & \rightarrow t_5^{(r)}, \\
t^{(r)}_6 & \rightarrow t_6^{(r)} + A^{(6)}(t_2^{(r)})^3, \\
t^{(r)}_8 & \rightarrow t_8^{(r)} + A^{(8)}(t_2^{(r)})^2 t_6^{(r)} + A^{(8)}(t_2^{(r)})^4, \\
t^{(r)}_9 & \rightarrow t_9^{(r)} + A^{(9)}(t_2^{(r)}) t_5^{(r)}, \\
t^{(r)}_{12} & \rightarrow t_{12}^{(r)} + A^{(12)}(t_2^{(r)}) (t_5^{(r)})^2 + A^{(12)}(t_2^{(r)})^2 t_8^{(r)} + A^{(12)}(t_2^{(r)})^3 t_6^{(r)}
\end{align*} \tag{5.9}
\]
\[ + A_4^{(12)} (t_2^{(\Omega)})^6 + A_5^{(12)} (t_6^{(\Omega)})^2. \]

where \( A_i^{(6,8,9,12)} \) are parameters. For general values of these parameters one gets the algebraic Hamiltonian preserving the flag 2, 5, 6, 8, 9, 12 and even smaller one 1, 2, 3, 4, 4, 6. Our goal, as before, is to tune these parameters in such a way that the Hamiltonian in its algebraic form preserves the minimal flag. Quite cumbersome analysis results to a one-parametric set of variables

\[
\begin{align*}
\tau_2 &= \frac{4}{3} t_2^{(\Omega)}, \\
\tau_5 &= \frac{576}{5} t_5^{(\Omega)}, \\
\tau_6 &= 3456 t_6^{(\Omega)} - 24 (t_2^{(\Omega)})^3, \\
\tau_8 &= \frac{248832}{5} t_8^{(\Omega)} + 48 (t_2^{(\Omega)})^4 - \frac{55296}{5} t_2^{(\Omega)} t_6^{(\Omega)}, \\
\tau_9 &= \frac{663552}{7} t_9^{(\Omega)} - \frac{27648}{5} t_5^{(\Omega)} t_2^{(\Omega)}, \\
\tau_{12} &= \frac{9551488}{5} t_{12}^{(\Omega)} - \frac{5568}{5} (t_2^{(\Omega)})^6 + 294912 t_6^{(\Omega)} (t_2^{(\Omega)})^3 + A_1^{(12)} t_2^{(\Omega)} (t_5^{(\Omega)})^2 \\
&\quad - 1658880 (t_2^{(\Omega)})^2 t_8^{(\Omega)} - \frac{5308416}{5} (t_6^{(\Omega)})^2,
\end{align*}
\]

(5.10)

where \( A_1^{(12)} \) is a parameter, leading to the minimal flag, which we denote \( \mathcal{P}^{(E_6)} \). This flag is spanned by the spaces

\[
\mathcal{P}_n^{(1,1,2,2,2,3)} = \langle \tau_2^{p_2} \tau_5^{p_5} \tau_6^{p_6} \tau_8^{p_8} \tau_9^{p_9} \tau_{12}^{p_{12}} | 0 \leq p_2 + p_5 + 2p_6 + 2p_8 + 2p_9 + 3p_{12} \leq n \rangle,
\]

with the characteristic vector,

\[
\vec{f} = (1, 1, 2, 2, 2, 3).
\]

(5.11)

with the characteristic vector, (cf. (3.13) and (4.9)), which coincides with the \( E_6 \) highest root, confirming Kac’s conjecture.

The most general polynomial transformation which preserves a linear space \( \mathcal{P}_n^{(E_6)} = \mathcal{P}_n^{(1,1,2,2,2,3)} \) for any \( n \) is of the form

\[
\begin{align*}
s_2 &\to s_2 + a_2 s_5, \\
s_5 &\to s_5 + a_5 s_2,
\end{align*}
\]

(5.12)
\[ s_6 \rightarrow s_6 + a_{6,1}s_2^2 + a_{6,2}s_5^2 + b_{6,1}s_8 + b_{6,2}s_9 + b_{6,3}s_2s_5 , \]
\[ s_8 \rightarrow s_8 + a_{8,1}s_2^2 + a_{8,2}s_5^2 + b_{8,1}s_6 + b_{8,2}s_9 + b_{8,3}s_2s_5 , \]
\[ s_9 \rightarrow s_9 + a_{8,1}s_2^2 + a_{8,2}s_5^2 + b_{8,1}s_6 + b_{8,2}s_8 + b_{8,3}s_2s_5 , \]
\[ s_{12} \rightarrow s_{12} + a_{12,1}s_2^3 + a_{12,2}s_5^3 + b_{12,1}s_2^2s_5 + b_{12,2}s_2s_5^2 + c_{12,1}s_2s_6 + c_{12,2}s_2s_8 + c_{12,3}s_2s_9 + d_{12,1}s_5s_6 + d_{12,2}s_5s_8 + d_{12,3}s_5s_9 , \] (5.13)

where \{a, b, c, d\} are arbitrary numbers. Surprisingly, there is an overlap of non-linear transformations (5.10) and (5.13). Namely, a variation of the parameter \(A_1^{(12)}\) in (5.10) corresponds to varying the parameter \(b_{12,2}\) in the transformation (5.13).

Therefore there exists a non-trivial one-parametric set of invariants of fixed degrees leading to an algebraic form of the Hamiltonian \(h_{E_6}^{(r)}\) and simultaneously preserving a minimal flag! Thus, the parameter \(A_1^{(12)}\) can be chosen following our convenience.

We set \(A_1^{(12)} = 0\), which makes the coefficient functions in the algebraic form of the Hamiltonian (see below (5.14)) the polynomials of lowest degree. However, it is still an open question for which value(s) of this parameter the invariants (5.10) can be ‘trigonometrized’ leading to an algebraic form of the trigonometric \(E_6\) model.

Finally, the rational Hamiltonian \(h_{E_6}^{(r)}(y_1 \ldots y_6)\) can be written as

\[ h_{E_6}^{(r)}(y_1 \ldots y_6) = A_{a,b} \frac{\partial^2}{\partial \tau_a \partial \tau_b} + B_{a} \frac{\partial}{\partial \tau_a} , \] (5.14)

where summation over \(a, b = 2, 5, 6, 8, 9, 12\) with \(a \leq b\) is carried out with the coefficient functions:

\[ A_{2,2} = 8\tau_2 , \quad A_{2,5} = 20\tau_5 , \quad A_{2,6} = 24\tau_6 , \]
\[ A_{2,8} = 32\tau_8 , \quad A_{2,9} = 36\tau_9 , \quad A_{2,12} = 48\tau_{12} , \]
\[ A_{5,5} = 4\tau_8 , \quad A_{5,5} = 54\tau_6 + 243\tau_2^2\tau_5 , \quad A_{5,8} = 48\tau_5\tau_6 + 90\tau_2\tau_9 , \]
\[ A_{5,9} = 12\tau_{12} + 162\tau_2\tau_5^2 , \quad A_{5,12} = 36\tau_6\tau_9 + 81\tau_2\tau_5\tau_8 , \]
\[ A_{6,6} = 1080\tau_5^2 + 270\tau_2\tau_8 + 162\tau_2^2\tau_6 , \quad A_{6,8} = 144\tau_{12} + 2754\tau_2\tau_5^2 + 324\tau_2^2\tau_8 , \]
\[ A_{6,9} = 234\tau_5\tau_8 + 405\tau_2^2\tau_9 , \quad A_{6,12} = 567\tau_2\tau_5\tau_9 + 72\tau_8^2 + 540\tau_5^2\tau_6 + 486\tau_2^2\tau_{12} , \]
\[ A_{8,8} = 4374\tau_2^2\tau_5^2 + 48\tau_6\tau_8 + 504\tau_5\tau_9 + 216\tau_2\tau_{12} , \]
\[ A_{8,9} = 540\tau_5^3 + 72\tau_6\tau_9 + 378\tau_2\tau_5\tau_8 , \]
\[ A_{8,12} = \frac{3}{2} . \]
worth noting that the Hamiltonian (5.14) possesses a remarkable property: there
As an illustration the first eigenfunctions are presented in the Appendix B. It is
Finally, the energies of the original rational $E$ is equidistant and corresponds to the spectrum of a set of the harmonic oscillators.

One can find the spectrum of (5.14), $E$ studied elsewhere. The rational algebra $G$ introduced in [6] and [8], respectively, in relation to the $G_2$ and $F_4$ models, the algebra $e^{(6)}$ is infinite-dimensional yet finitely-generated. It will be described and studied elsewhere. The rational $E_6$ Hamiltonian in the algebraic form (6.13) with coefficients (5.15), (5.16) can be rewritten in terms of the generators of this algebra.

The operator (5.14) is triangular in the basis of monomials $\tau_2^{p_1} \tau_5^{p_2} \tau_6^{p_3} \tau_9^{p_4} \tau_9^{p_5} \tau_12^{p_6}$. One can find the spectrum of (5.14), $h^{(r)}_{E_6} \varphi = -2\epsilon \varphi$, explicitly

$$\epsilon_{n_1,n_2,n_3,n_4,n_5,n_6} = \omega(2n_1 + 5n_2 + 6n_3 + 8n_4 + 9n_5 + 12n_6), \quad (5.17)$$

where $n_i$ are non-negative integers. Degeneracy of the spectrum is related to the number of partitions of an integer number $n$, $n = 0, 1, 2, \ldots$ to $2n_1 + 5n_2 + 6n_3 + 8n_4 + 9n_5 + 12n_6$. The spectrum does not depend on the coupling constant $g$, it is equidistant and corresponds to the spectrum of a set of the harmonic oscillators. Finally, the energies of the original rational $E_6$ Hamiltonian (5.1) are $E = E_0 + \epsilon$. As an illustration the first eigenfunctions are presented in the Appendix B. It is worth noting that the Hamiltonian (5.14) possesses a remarkable property: there

\begin{align*}
A_{8,12} &= 729 \tau_2^2 \tau_5 \tau_9 + 162 \tau_9^2 + 72 \tau_6 \tau_{12} + 270 \tau_5^2 \tau_8 + 108 \tau_2 \tau_8^2 + 972 \tau_2 \tau_5^2 \tau_6, \\
A_{9,9} &= 216 \tau_2 \tau_5 \tau_9 + 12 \tau_8^2 + 144 \tau_5^2 \tau_6, \\
A_{9,12} &= -162 \tau_2 \tau_5 \tau_9 + 270 \tau_5^2 \tau_9 + 189 \tau_2 \tau_8 \tau_9 + 144 \tau_5 \tau_6 \tau_9, \\
A_{12,12} &= \frac{1215}{2} \tau_2^2 \tau_6^2 - 324 \tau_2 \tau_5 \tau_6 \tau_9 + 36 \tau_6 \tau_8^2 - 648 \tau_5^2 \tau_{12} + 432 \tau_5^2 \tau_6 \\
+ 594 \tau_5 \tau_8 \tau_9 + 162 \tau_2 \tau_8 \tau_{12}, \quad (5.15)
\end{align*}

and

\begin{align*}
B_2 &= -4 \omega \tau_2 + 24(1 + 12\nu), \quad B_3 = -10 \omega \tau_3, \quad B_6 = -12 \omega \tau_6 + 405(1 + 6\nu) \tau_2^2, \\
B_8 &= -16 \omega \tau_8 + 96(1 + 3\nu) \tau_6, \quad B_9 = -18 \omega \tau_9 + 216(2 + 3\nu) \tau_2 \tau_5, \\
B_{12} &= -24 \omega \tau_{12} + 108(7 - 6\nu) \tau_6^2 + 324(1 + 2\nu) \tau_2 \tau_8. \quad (5.16)
\end{align*}

There is a one-parametric algebra of differential operators (in six variables) for which $P^{(1,1,2,2,2,3)}_n$ (see (5.11)) is a finite-dimensional irreducible representation space. Furthermore, the finite-dimensional representation spaces appear for different integer values of the algebra parameter. They form an infinite non-classical flag which coincides with $P^{(E_6)} (5.13)$. We call this algebra $e^{(6)}$. Like the algebras $g^{(2)}, f^{(4)}$ introduced in [3] and [8], respectively, in relation to the $G_2$ and $F_4$ models, the algebra $e^{(6)}$ is infinite-dimensional yet finitely-generated. It will be described and studied elsewhere. The rational $E_6$ Hamiltonian in the algebraic form (6.13) with coefficients (5.15), (5.16) can be rewritten in terms of the generators of this algebra.

The operator (5.14) is triangular in the basis of monomials $\tau_2^{p_1} \tau_5^{p_2} \tau_6^{p_3} \tau_9^{p_4} \tau_9^{p_5} \tau_12^{p_6}$. One can find the spectrum of (5.14), $h^{(r)}_{E_6} \varphi = -2\epsilon \varphi$, explicitly

$$\epsilon_{n_1,n_2,n_3,n_4,n_5,n_6} = \omega(2n_1 + 5n_2 + 6n_3 + 8n_4 + 9n_5 + 12n_6), \quad (5.17)$$

where $n_i$ are non-negative integers. Degeneracy of the spectrum is related to the number of partitions of an integer number $n$, $n = 0, 1, 2, \ldots$ to $2n_1 + 5n_2 + 6n_3 + 8n_4 + 9n_5 + 12n_6$. The spectrum does not depend on the coupling constant $g$, it is equidistant and corresponds to the spectrum of a set of the harmonic oscillators. Finally, the energies of the original rational $E_6$ Hamiltonian (5.1) are $E = E_0 + \epsilon$. As an illustration the first eigenfunctions are presented in the Appendix B. It is worth noting that the Hamiltonian (5.14) possesses a remarkable property: there
exists a family of eigenfunctions which depend on the single variable \( \tau_2 \). These eigenfunctions are the associated Laguerre polynomials. This property admits to construct a quasi-exactly-solvable generalization of the rational \( E_6 \) model, as will be done elsewhere. Due to enormous technical difficulties we were unable to describe explicitly the boundary of the configuration space in the Weyl-invariant variables \( \tau \)'s similarly to what was done in \( G_2 \) and \( F_4 \) cases.

6 The rational \( E_7 \) model

The Hamiltonian of the rational \( E_7 \) model is built using the root system of the exceptional \( E_7 \) algebra. A convenient way to represent the Hamiltonian is to write it in the 8-dimensional space \( \{x_1, x_2, \ldots, x_8\} \) and impose the constraint \( x_8 = -x_7 \),

\[
H_{E_7} = -\frac{1}{2} \Delta^{(8)} + \frac{\omega^2}{2} \sum_{i=1}^{8} x_i^2 + V_{E_7},
\]

(6.1)

where \( \omega \) is a frequency and the root generated part of the potential depends on a single constant \( g = \nu(\nu - 1) \):

\[
V_{E_7} = g \sum_{j < i = 1}^{6} \left[ \frac{1}{(x_i + x_j)^2} + \frac{1}{(x_i - x_j)^2} \right] + g \frac{1}{(x_7 - x_8)^2} + g \sum_{\nu_j}^{6} \frac{1}{\left[ \frac{1}{2} (x_8 - x_7) - \sum_{j=1}^{6} (-1)^{\nu_j} x_j \right]^2},
\]

(6.2)

with \( \nu_j = 0, 1 \), and \( \sum_{j=1}^{6} \nu_j = \text{odd} \). The configuration space is given by the principal \( E_7 \) Weyl chamber.

Let us introduce the new variables

\[
y_i = x_i , \quad i = 1 \ldots 6 \\
y_7 = x_7 - x_8 , \quad \text{using the constraint } y_7 = 2x_7 \\
Y = \frac{1}{2} (x_7 + x_8) , \quad \text{using the constraint } Y = 0
\]

In these variables the Laplacian becomes

\[
\Delta^{(8)} = \Delta^{(6)}_{y} + 2 \frac{\partial^2}{\partial y_7^2} + \frac{1}{2} \frac{\partial^2}{\partial Y^2},
\]

(6.3)

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and the potential part of (6.1) depends only on \(\{y_1 \ldots y_7\}\):

\[
V = \frac{\omega^2}{2} \left\{ \sum_{i=1}^{6} y_i^2 + \frac{y_7^2}{2} \right\} + g \sum_{j<i=1}^{6} \left[ \frac{1}{(y_i + y_j)^2} + \frac{1}{(y_i - y_j)^2} \right] + \frac{g}{y_7^2}
+ g \sum_{\nu_j}^{6} \left[ \frac{1}{2} \left( y_7 - \sum_{j=1}^{6} (-1)^{\nu_j} y_j \right) \right]^2. 
\]

(6.4)

In this formalism imposing constraints means the restriction to the eigenfunctions having no dependence on \(Y\). Hence, \(Y\)-dependent part of the Laplacian, the last term in (6.3), can be dropped off.

The ground state eigenfunction has a form

\[
\Psi_0 = (\Delta_+^{(6)} \Delta_-^{(6)} y_7)^{\nu} \Delta_{E_7}^{\nu} e^{g \left\{ \sum_{i=1}^{6} y_i^2 + \frac{y_7^2}{2} \right\}} , \quad E_0 = \frac{7}{2} \omega (1 + 18 \nu) , \quad \nu \geq 0, \quad 1 \leq \nu_j \leq 56, \quad \sum_{j=1}^{6} \nu_j = \text{odd} \quad \text{and} \quad g = \nu (\nu - 1). 
\]

(6.5)

where

\[
\Delta_+^{(6)} = \prod_{j<i=1}^{6} (y_i \pm y_j) , \\
\Delta_{E_7} = \prod_{\{\nu_j\}} \left( y_7 + \sum_{j=1}^{6} (-1)^{\nu_j} y_j \right) ,
\]

with \(\nu_j = 0, 1\) and \(\sum_{j=1}^{6} \nu_j = \text{odd}\) and \(g = \nu (\nu - 1)\).

In order to find variables leading to algebraic form of gauge-rotated Hamiltonian,

\[
h_{E_7}^{(r)}(y_1 \ldots y_7) = -2 \Psi_0^{-1} (H_{E_7} - E_0) (y_1 \ldots y_7) \Psi_0 , \quad \nu \geq 0, \quad g = \nu (\nu - 1) .
\]

(6.6)

let us take the Weyl-invariant polynomials, obtained by averaging over the 56-dimensional orbit \(\Omega\) generated by the vector \((e_7 - e_6)\),

\[
t_a^{(\Omega)}(x) = \sum_{k=1}^{56} (\omega_k \cdot x)^a , \quad \omega_k \in \Omega(e_7 - e_6) ,
\]

(6.7)

(cf. (2.7)), where \(a = 2, 6, 8, 10, 12, 14, 18\) are the degrees of the \(E_7\) invariants and \(\omega_k, k = 1, 2, \ldots 56\) are the orbit elements. The orbit variables \(t_a^{(\Omega)}\) are functions of
$y_1 \ldots y_7$. The invariants of a fixed degree are defined ambiguously, up to non-linear transformations, similar to (3.8), (4.7), (5.9)

$t_2^{(Ω)} \mapsto t_2^{(Ω)}$,

$t_6^{(Ω)} \mapsto t_6^{(Ω)} + A_6(t_6^{(Ω)})^3$,

$t_8^{(Ω)} \mapsto t_8^{(Ω)} + A_1^{(8)}t_2^{(Ω)}t_6^{(Ω)} + A_8^{(t_2^{(Ω)})^4}$,

$t_{10}^{(Ω)} \mapsto t_{10}^{(Ω)} + A_1^{(10)}(t_2^{(Ω)})^2t_6^{(Ω)} + A_2^{(t_2^{(Ω)})t_8^{(Ω)} + A_3^{(t_2^{(Ω)})^5}}$,

$t_{12}^{(Ω)} \mapsto t_{12}^{(Ω)} + A_1^{(12)}(t_6^{(Ω)})^2 + A_2^{(t_2^{(Ω)})t_10^{(Ω)}} + A_3^{(t_2^{(Ω)})^2t_8^{(Ω)}} + A_4^{(t_2^{(Ω)})^3t_6^{(Ω)}} + A_5^{(t_2^{(Ω)})^6}$,

$t_{14}^{(Ω)} \mapsto t_{14}^{(Ω)} + A_1^{(14)}(t_2^{(Ω)})^2t_6^{(Ω)}t_8^{(Ω)} + A_2^{(t_2^{(Ω)})t_12^{(Ω)}} + A_3^{(t_2^{(Ω)})^2t_{10}^{(Ω)}} + A_4^{(t_2^{(Ω)})^3t_8^{(Ω)}} + A_5^{(t_2^{(Ω)})^6t_6^{(Ω)}} + A_7^{(t_2^{(Ω)})^6}$,

$t_{18}^{(Ω)} \mapsto t_{18}^{(Ω)} + A_1^{(18)}(t_6^{(Ω)})^3 + A_2^{(t_2^{(Ω)})t_12^{(Ω)}} + A_3^{(t_2^{(Ω)})t_{10}^{(Ω)}} + A_4^{(t_2^{(Ω)})t_8^{(Ω)}} + A_5^{(t_2^{(Ω)})^2t_6^{(Ω)}} + A_6^{(t_2^{(Ω)})t_8^{(Ω)}} + A_7^{(t_2^{(Ω)})^3t_{12}^{(Ω)}} + A_8^{(t_2^{(Ω)})^4t_{10}^{(Ω)}} + A_9^{(t_6^{(Ω)})^2} + A_{10}^{(t_6^{(Ω)})^4t_8^{(Ω)}} + A_{11}^{(t_6^{(Ω)})^5t_6^{(Ω)}} + A_{12}^{(t_6^{(Ω)})^5t_6^{(Ω)}} + A_{13}^{(t_2^{(Ω)})^9}$.

Our goal is to tune the parameters $A_1$’s so as to get the algebraic form of the Hamiltonian (if it exists) and a minimal flag. After very cumbersome analysis we discovered a two-parametric set of variables (for simplicity we omit subscript $(Ω)$ in variables $t_i^{(Ω)}$),

$$
\tau_2 = \frac{1}{3}t_2^{(Ω)} , \quad \tau_6 = -\frac{16}{3}t_6^{(Ω)} + \frac{1}{108}t_2^{3(Ω)} , \quad \tau_8 = -\frac{16}{5}t_8^{(Ω)} + \frac{16}{45}t_2^{(Ω)}t_6^{(Ω)} - \frac{1}{2592}t_4^{(Ω)} ,
$$

$$
\tau_{10} = \frac{64}{315}t_{10}^{(Ω)} - \frac{4}{105}t_2^{(Ω)}t_8^{(Ω)} - \frac{1}{466560}t_2^{5(Ω)} + \frac{1}{405}t_2^{2(Ω)}t_6^{(Ω)} ,
$$

$$
\tau_{12} = \frac{1024}{2233}t_{12}^{(Ω)} + \frac{5}{419904}t_2^{6(Ω)} - \frac{64}{3645}t_2^{3(Ω)}t_6^{(Ω)} + \frac{184}{405}t_2^{2(Ω)}t_8^{(Ω)} - \frac{256}{405}t_6^{(Ω)} ,
$$

$$
\tau_{14} = \frac{4096}{2233}t_{14}^{(Ω)} - \frac{43}{634230}t_2^{4(Ω)}t_6^{(Ω)} + \frac{832}{35235}t_2^{2(Ω)}t_6^{(Ω)} - \frac{256}{1305}t_2^{3(Ω)}t_6^{(Ω)} - \frac{202}{246645}t_2^{3(Ω)}t_8^{(Ω)}
$$

$$
+ \frac{341}{5114430720}t_2^{7(Ω)} - \frac{18176}{43065}t_2^{2(Ω)}t_6^{(Ω)} + \frac{8768}{246645}t_2^{2(Ω)}t_{10}^{(Ω)} ,
$$

$$
\tau_{18} = \frac{262144}{3687}t_{18}^{(Ω)} + \frac{635120768}{505211175}t_2^{2(Ω)}t_6^{(Ω)} - \frac{534492928}{5846015025}t_2^{3(Ω)}t_6^{(Ω)} + \frac{49527971}{210456540900}t_6^{2(Ω)}
$$

$$
- \frac{4002704}{5846015025}t_2^{5(Ω)}t_8^{(Ω)} + \frac{192754688}{285805179}t_2^{3(Ω)}t_{12}^{(Ω)} - \frac{9332528}{233840601}t_2^{4(Ω)}t_{10}^{(Ω)} - \frac{2735503}{1697121548176}t_2^{9} .
$$
where $A_2^{(12)}, A_3^{(18)}$ are parameters, leading to an algebraic form of the Hamiltonian and in which the flag is minimal. We denote the minimal flag $\mathcal{P}(E_7)$. This flag is generated by the polynomials
\[ \mathcal{P}_n^{(1,2,2,2,3,3,4)} = \langle \tau_2^{p_2} \tau_6^{p_6} \tau_8^{p_8} \tau_{10}^{p_{10}} \tau_{12}^{p_{12}} \tau_{14}^{p_{14}} \tau_{18}^{p_{18}} | 0 \leq p_2 + 2p_6 + 2p_8 \\
+ 2p_{10} + 3p_{12} + 3p_{14} + 4p_{18} \leq n \rangle, \quad (6.10) \]

with the characteristic vector
\[ \vec{f} = (1, 2, 2, 2, 3, 3, 4), \quad (6.11) \]

(cf. \[6.13\], \[6.19\], \[5.21\]). It again coincides with the highest root of the \(E_7\) algebra confirming the Kac’s conjecture. Therefore \(\mathcal{P}_n^{(E_7)} = \mathcal{P}_n^{(1,2,2,2,3,3,4)}\).

It is worth to emphasize that the most general polynomial transformation which preserve each linear space \(\mathcal{P}_n^{(1,2,2,2,3,3,4)}\) is
\[ s_2 \rightarrow s_2 , \]
\[ s_6 \rightarrow s_6 + a_6 s_2^2 + b_6,1 s_8 + b_6,2 s_{10} , \]
\[ s_8 \rightarrow s_8 + a_8 s_2^2 + b_8,1 s_6 + b_8,2 s_{10} , \]
\[ s_{10} \rightarrow s_{10} + a_{10} s_2^2 + b_{10,1} s_6 + b_{10,2} s_8 , \]
\[ s_{12} \rightarrow s_{12} + a_{12} s_2^3 + b_{12,1} s_2 s_6 + b_{12,2} s_2 s_{10} + b_{12,3} s_2 s_{10} + c_{12} s_{14} , \]
\[ s_{14} \rightarrow s_{14} + a_{14} s_2^3 + b_{14,1} s_2 s_6 + b_{14,2} s_2 s_{10} + b_{14,3} s_2 s_{10} + c_{14} s_{12} , \]
\[ s_{18} \rightarrow s_{18} + a_{18} s_2^4 + b_{18,1} s_2^2 s_6 + b_{18,2} s_2^2 s_{10} + b_{18,3} s_2^2 s_{10} + c_{18} s_2 s_{12} + c_{18,2} s_2 s_{14} \]
\[ + d_{18,1} s_6^2 + d_{18,2} s_6^2 + d_{18,3} s_{10} + d_{18,4} s_6 s_8 + d_{18,4} s_6 s_{10} + d_{18,4} s_6 s_{10} , \quad (6.12) \]

where \(\{a, b, c, d\}\) are arbitrary numbers. It is worth to mention that there exist two exceptional sub-transformations which are common for \(6.8\) and \(6.12\) – when we vary the parameter \(b_{12,3}\) in \(s_{12} (A_2^{(12)} \text{ in } t_{12}^{(\Omega)})\) and \(d_{18,4}\) in \(s_{18} (A_3^{(18)} \text{ in } t_{18}^{(\Omega)})\) keeping all other parameters fixed. Thus, we conclude that there exist a two-parametric set of invariants of the fixed degrees leading to algebraic form of the Hamiltonian \(h_{E_7}^{(r)}\) and moreover preserving the minimal flag. Hence the parameters \(A_2^{(12)}\) and \(A_3^{(18)}\) can be chosen at our will. We set them equal to zero, \(A_2^{(12)} = A_3^{(18)} = 0\), which fixes the coefficient functions in the algebraic form of the Hamiltonian (see below \[6.13\]) in the form of polynomials of lowest degrees. Since an algebraic form of the trigonometric \(E_7\) model is not known so far, it is an open question for which value(s) of these parameters the invariants \(6.8\) would be ‘trigonometrized’ leading to an algebraic form of the trigonometric \(E_7\) model (if such a form exists).
Finally, the Hamiltonian $h_{E_7}^{(r)}$ can be written as

$$h_{E_7}^{(r)} = A_a \frac{\partial^2}{\partial \tau_a \partial \tau_b} + B_a \frac{\partial}{\partial \tau_a} ,$$

(6.13)

where $a, b = 2, 6, 8, 10, 12, 14, 18$ and $a \leq b$ with the coefficient functions:

\begin{align*}
A_{2,2} & = 1672 , \quad A_{2,6} = 487 \tau_6 , \quad A_{2,8} = 647 \tau_8 , \quad A_{2,10} = 80 \tau_{10} , \\
A_{2,12} & = 96 \tau_{12} , \quad A_{2,14} = 112 \tau_{14} , \quad A_{2,18} = 144 \tau_{18} , \\
A_{6,6} & = -872^2 \tau_6 - 160 \tau_2 \tau_8 + 17280 \tau_{10} , \quad A_{6,8} = 1920 \tau_2 \tau_{10} + 96 \tau_{12} - 16 \tau_2^2 \tau_8 , \\
A_{6,10} & = -1127 \tau_1 - 24 \tau_2^2 \tau_{10} , \quad A_{6,12} = -1287^2 + 3456 \tau_1 \tau_{10} - 1440 \tau_2 \tau_{14} - 24 \tau_2^2 \tau_{12} , \\
A_{6,14} & = -128 \tau_8 \tau_{10} - 36 \tau_{18} - 32 \tau_2^2 \tau_{14} , \quad A_{6,18} = -23040 \tau_2 \tau_{10} - 40 \tau_2^2 \tau_{18} \\
& \quad + 3200 \tau_8 \tau_{14} - 4224 \tau_{10} \tau_{12} , \\
A_{8,8} & = 336 \tau_1 + 288 \tau_2^2 \tau_{10} + 8 \tau_6 \tau_8 + 12 \tau_2 \tau_{12} , \quad A_{8,10} = -1672 \tau_{14} + 16 \tau_6 \tau_{10} , \\
A_{8,12} & = -1344 \tau_1 \tau_{10} + 480 \tau_2 \tau_6 \tau_{10} - 108 \tau_{18} - 16 \tau_2 \tau_8^2 + 12 \tau_6 \tau_{12} - 192 \tau_2^2 \tau_{14} , \\
A_{8,14} & = -960 \tau_7^2 - 16 \tau_8 \tau_{14} + 20 \tau_6 \tau_{14} - 5 \tau_2 \tau_{18} , \\
A_{8,18} & = -3456 \tau_2^2 \tau_{10} - 576 \tau_2 \tau_{10} \tau_{12} + 448 \tau_2 \tau_8 \tau_{14} + 24 \tau_6 \tau_{18} - 9984 \tau_1 \tau_{14} , \\
\dot{A}_{10,10} & = \frac{4 \tau_8 \tau_{10}}{3} + \frac{\tau_{18}}{6} , \quad A_{10,12} = 960 \tau_7^2 - 32 \tau_2 \tau_8 \tau_{10} - 20 \tau_6 \tau_{14} + \tau_2 \tau_{18} , \\
\dot{A}_{10,14} & = -\frac{8 \tau_8 \tau_{14}}{3} + 4 \tau_{10} \tau_{12} , \quad A_{10,18} = 288 \tau_2 \tau_{10} \tau_{14} - \frac{8 \tau_8 \tau_{18}}{3} + 16 \tau_6 \tau_{14} + 384 \tau_6 \tau_{10} , \\
A_{12,12} & = 11520 \tau_2 \tau_{10} + 192 \tau_2^2 \tau_{10} \tau_8 - 24 \tau_2 \tau_8 \tau_{12} + 36 \tau_2^2 \tau_{18} - 336 \tau_2 \tau_6 \tau_{14} \\
& \quad - 16 \tau_6 \tau_8^2 - 1056 \tau_8 \tau_{14} + 1152 \tau_10 \tau_{12} + 576 \tau_6 \tau_{10} , \\
A_{12,14} & = 288 \tau_2^2 \tau_{10} + 24 \tau_2 \tau_{10} \tau_{12} - 64 \tau_2 \tau_8 \tau_{14} - 16 \tau_6 \tau_8 \tau_{10} - 6 \tau_6 \tau_{18} + 18 \tau_{10} \tau_{14} , \\
A_{12,18} & = 2688 \tau_2^2 + 73728 \tau_2 \tau_{10} \tau_{14} - 672 \tau_6 \tau_{10} \tau_{12} + 544 \tau_6 \tau_8 \tau_{14} + 384 \tau_2 \tau_{12} \tau_{14} \\
& \quad - 288 \tau_2 \tau_8 \tau_{18} + 9216 \tau_7^2 \tau_{10} + 2592 \tau_2 \tau_{10} \tau_{18} - 27648 \tau_2 \tau_6 \tau_{10}^2 , \\
A_{14,14} & = 32 \tau_2 \tau_{10} \tau_{14} - \frac{2 \tau_8 \tau_{18}}{3} + 4 \tau_{12} \tau_{14} - 64 \tau_6 \tau_{10}^2 , \\
A_{14,18} & = 23040 \tau_2 \tau_{10} + 64 \tau_2^2 \tau_{14} + 128 \tau_2 \tau_{14} - 384 \tau_2 \tau_{10} \tau_{18} - 608 \tau_7 \tau_8 \tau_{14} \\
& \quad + 56 \tau_3 \tau_{10} \tau_{18} + 64 \tau_8 \tau_{10} \tau_{12} + 4 \tau_{12} \tau_2 \tau_8 , \\
A_{18,18} & = 55296 \tau_2 \tau_{10}^2 - 768 \tau_6 \tau_{10} \tau_{18} + 64 \tau_8^2 \tau_{18} + 285696 \tau_2 \tau_{14} + 4608 \tau_6 \tau_{10} \tau_{18} \\
& \quad + 11520 \tau_2 \tau_2 \tau_{10} \tau_{12} + 384 \tau_{10} \tau_{12} - 64 \tau_6 \tau_8 \tau_{14} + 128 \tau_2 \tau_{14} \tau_{18} - 640 \tau_8 \tau_{12} \tau_{14} \\
& \quad - 15360 \tau_2 \tau_8 \tau_{10} \tau_{14} , \quad (6.14)
\end{align*}

and

$$B_2 = -4 \omega \tau_2 + 56(1 + 18 \nu) , \quad B_6 = -12 \omega \tau_6 - 24(1 + 10 \nu) \tau_2^2 , \quad 29$$
There exists one-parametric algebra of differential operators (in seven variables) for which $\mathcal{P}_n^{(1,2,2,3,3,4)}$ (see (6.10)) is a finite-dimensional irreducible representation. Furthermore, the finite-dimensional representation spaces appear for different integer values of the algebra parameter. They form an infinite non-classical flag which coincides with $\mathcal{P}^{(E_7)}$ (6.10). We call this algebra $e^{(7)}$. Like the algebras $g^{(2)}, f^{(4)}$ introduced in [6] and [8], respectively, in relation to the $G_2$ and $F_4$ models, the algebra $e^{(7)}$ is infinite-dimensional yet finitely-generated. It will be described elsewhere. The rational $E_7$ Hamiltonian in the algebraic form (6.13) with coefficients (6.14), (6.15) can be rewritten in terms of the generators of this algebra.

The operator (6.13) is triangular in the basis of monomials $\tau_2^p \tau_6^q \tau_8^r \tau_{10}^s \tau_{12}^t \tau_{14}^u \tau_{18}^v$. One can find the spectrum of (6.13), $h^{(r)}_{E_7} \varphi = -2\epsilon \varphi$, explicitly,

$$\epsilon_{n_1, n_2, n_3, n_4, n_5, n_6, n_7} = 2 \omega(n_1 + 3n_2 + 4n_3 + 5n_4 + 6n_5 + 7n_6 + 9n_7),$$

(6.16)

where $n_i$ are non-negative integers. Degeneracy of the spectrum is related to the number of partitions of integer number $n$, $n = 0, 1, 2, \ldots$ to $n_1 + 3n_2 + 4n_3 + 5n_4 + 6n_5 + 7n_6 + 9n_7$. The spectrum does not depend on the coupling constant $\gamma$, it is equidistant and corresponds to the spectrum of a set of harmonic oscillators. Finally, the energies of the original rational $E_7$ Hamiltonian (6.1) are $E = E_0 + \epsilon$. As an illustration the first eigenfunctions are presented in the Appendix C. The Hamiltonian (6.13) possesses a remarkable property: there exists a family of eigenfunctions which depend on the single variable $\tau_2$. These eigenfunctions are the associated Laguerre polynomials. This property allows to construct a quasi-exactly-solvable generalization of the rational $E_7$ model. It will be done elsewhere. Due to technical difficulties we were unable to find explicitly the boundary of the configuration space (in other words, the boundaries of the $E_7$ Weyl chamber) in the Weyl-invariant variables $\tau$’s similar to what was previously done for $G_2$ and $F_4$ cases (see Section 3 and 4, correspondingly).
7 The rational $E_8$ model

The Hamiltonian of the rational $E_8$ model is built using the root system of the exceptional algebra $E_8$. The Hamiltonian is defined in the 8-dimensional space \( \{ x_1, x_2, \ldots x_8 \} \)

\[
H_{E_8} = -\frac{1}{2} \Delta^{(8)} + \frac{\omega^2}{2} \sum_{i=1}^{8} x_i^2 + V_{E_8},
\]

where \( \omega \) is the oscillator parameter and \( V_{E_8} \) is the root-generated potential with the coupling constant \( g = \nu(\nu - 1) \):

\[
V_{E_8} = g \sum_{j<i=1}^{8} \frac{1}{(x_i + x_j)^2} + \frac{1}{(x_i - x_j)^2} + g \sum_{\nu_j} \left[ \frac{1}{2} \left( x_8 + \sum_{j=1}^{7} (-1)^{\nu_j} x_j \right)^2 \right] ,
\]

where \( \nu_j = 0, 1 \) and \( \sum_{j=1}^{7} \nu_j = \text{even} \). The configuration space is given by the principal $E_8$ Weyl chamber.

The ground state eigenfunction of the Hamiltonian (7.1) is of the form

\[
\Psi_0 = (\Delta_+^{(8)} \Delta_-^{(8)})^\nu \Delta_{E_8}^\nu \epsilon^{-\frac{1}{2}\omega \sum_{i=1}^{8} x_i^2} , \quad E_0 = 4\omega(1 + 30\nu) ,
\]

where

\[
\Delta_+^{(8)} = \prod_{j<i=1}^{8} (x_i \pm x_j) ,
\]

\[
\Delta_{E_8} = \prod_{\{\nu_j\}} \left( x_8 + \sum_{j=1}^{7} (-1)^{\nu_j} x_j \right) ,
\]

with \( \nu_j = 0, 1 \), \( \sum_{j=1}^{7} \nu_j = \text{even} \) and \( g = \nu(\nu - 1) \).

In order to find variables leading to algebraic form of gauge-rotated Hamiltonian,

\[
h^{(r)}_{E_8}(x_1 \ldots x_8) = -2\Psi_0^{-1}(H_{E_8} - E_0)(x_1 \ldots x_8)\Psi_0 ,
\]

let us define a basis in the form of Weyl-invariant polynomials, averaged over one of the smallest orbit, of length 240, generated by some positive root,

\[
\epsilon_a^{(n)} = \sum_{k=1}^{240} (\omega_k \cdot x)^a , \quad \omega_k \in \Omega(\text{a positive root}) ,
\]

with \( a = 0, 1, \ldots, 8 \).
(cf. (2.7)), where \( a = 2, 8, 12, 14, 18, 20, 24, 30 \) are degrees of the lowest \( E_8 \) invariants and \( \omega_k, k = 1, 2, \ldots, 240 \) are the orbit elements. The orbit variables \( t_a^{(\Omega)} \) are functions of \( x_1 \ldots x_8 \). In general, the invariants of fixed degree are defined ambiguously, up to a certain non-linear transformation, cf. \( (3.8), (4.7), (5.9), (6.8) \)

\[
\begin{align*}
t_2^{(\Omega)} &\rightarrow t_2^{(\Omega)}, \\
t_8^{(\Omega)} &\rightarrow t_8^{(\Omega)} + A^8(t_2^{(\Omega)})^4, \\
t_{12}^{(\Omega)} &\rightarrow t_{12}^{(\Omega)} + A_1^{(12)}(t_2^{(\Omega)})^2t_8^{(\Omega)} + A_2^{(12)}(t_2^{(\Omega)})^6, \\
t_{14}^{(\Omega)} &\rightarrow t_{14}^{(\Omega)} + A_1^{(14)}t_2^{(\Omega)}t_{12}^{(\Omega)} + A_2^{(14)}(t_2^{(\Omega)})^3t_8^{(\Omega)} + A_3^{(14)}(t_2^{(\Omega)})^7, \\
t_{18}^{(\Omega)} &\rightarrow t_{18}^{(\Omega)} + A_1^{(18)}t_2^{(\Omega)}(t_8^{(\Omega)})^2 + A_2^{(18)}(t_2^{(\Omega)})^2t_{14}^{(\Omega)} + A_3^{(18)}t_2^{(\Omega)}(t_2^{(\Omega)})^3t_{12}^{(\Omega)} + A_4^{(18)}t_2^{(\Omega)}(t_2^{(\Omega)})^5t_8^{(\Omega)} + A_5^{(18)}(t_2^{(\Omega)})^9, \\
t_{20}^{(\Omega)} &\rightarrow t_{20}^{(\Omega)} + A_1^{(20)}t_8^{(\Omega)}t_{12}^{(\Omega)} + A_2^{(20)}t_2^{(\Omega)}t_{18}^{(\Omega)} + A_3^{(20)}(t_2^{(\Omega)})^2(\Omega) + A_4^{(20)}(t_2^{(\Omega)})^3t_{14}^{(\Omega)} + A_5^{(20)}(t_2^{(\Omega)})^4t_{12}^{(\Omega)} + A_6^{(20)}(t_2^{(\Omega)})^6t_8^{(\Omega)} + A_7^{(20)}(t_2^{(\Omega)})^{10}, \\
t_{24}^{(\Omega)} &\rightarrow t_{24}^{(\Omega)} + A_1^{(24)}(t_2^{(\Omega)})^2 + A_2^{(24)}(t_8^{(\Omega)})^3 + A_3^{(24)}t_2^{(\Omega)}t_8^{(\Omega)}t_{14}^{(\Omega)} + A_4^{(24)}(t_2^{(\Omega)})^2t_{20}^{(\Omega)} + A_5^{(24)}t_2^{(\Omega)}t_8^{(\Omega)}t_{12}^{(\Omega)} + A_6^{(24)}(t_2^{(\Omega)})^3t_{18}^{(\Omega)} + A_7^{(24)}(t_2^{(\Omega)})^4t_8^{(\Omega)} + A_8^{(24)}(t_2^{(\Omega)})^5t_{14}^{(\Omega)} + A_9^{(24)}t_2^{(\Omega)}t_{12}^{(\Omega)} + A_{10}^{(24)}t_2^{(\Omega)}t_8^{(\Omega)} + A_{11}^{(24)}t_2^{(\Omega)}t_{12}^{(\Omega)}, \\
t_{30}^{(\Omega)} &\rightarrow t_{30}^{(\Omega)} + A_1^{(30)}t_{12}^{(\Omega)}t_{18}^{(\Omega)} + A_2^{(30)}(t_8^{(\Omega)})^2t_{14}^{(\Omega)} + A_3^{(30)}t_2^{(\Omega)}(t_{14}^{(\Omega)})^2 + A_4^{(30)}t_2^{(\Omega)}t_8^{(\Omega)}t_{20}^{(\Omega)} + A_5^{(30)}t_2^{(\Omega)}t_8^{(\Omega)}t_{12}^{(\Omega)} + A_6^{(30)}(t_2^{(\Omega)})^2t_{12}^{(\Omega)}t_{14}^{(\Omega)} + A_7^{(30)}(t_2^{(\Omega)})^2t_8^{(\Omega)}t_{18}^{(\Omega)} + A_8^{(30)}(t_2^{(\Omega)})^3t_{24}^{(\Omega)} + A_9^{(30)}(t_2^{(\Omega)})^3(t_8^{(\Omega)})^2 + A_{10}^{(30)}(t_2^{(\Omega)})^3(t_8^{(\Omega)})^3 + A_{11}^{(30)}t_2^{(\Omega)}t_8^{(\Omega)}t_{14}^{(\Omega)} + A_{12}^{(30)}(t_2^{(\Omega)})^5t_{20}^{(\Omega)} + A_{13}^{(30)}(t_2^{(\Omega)})^5t_8^{(\Omega)}t_{12}^{(\Omega)} + A_{14}^{(30)}(t_2^{(\Omega)})^6t_{18}^{(\Omega)} + A_{15}^{(30)}(t_2^{(\Omega)})^7(t_8^{(\Omega)})^2 + A_{16}^{(30)}(t_2^{(\Omega)})^8t_{14}^{(\Omega)} + A_{17}^{(30)}(t_2^{(\Omega)})^9t_{12}^{(\Omega)} + A_{18}^{(30)}(t_2^{(\Omega)})^{11}t_8^{(\Omega)} + A_{19}^{(30)}(t_2^{(\Omega)})^{15}. \\
&\hspace{1cm}(7.6)
\end{align*}
\]

The transformation (7.6) depends on 48 parameters. Our goal is to find parameters \( A \)'s such that (i) the Hamiltonian has the algebraic form, (ii) a set of polynomial invariant subspaces forming a flag occurs and (iii) the flag is minimal. After extremely tedious and cumbersome analysis we discovered a nine-parametric set of variables (see discussion below), for simplicity we omit the subscript \( (\Omega) \) in variables \( t_a^{(\Omega)} \),

\[
\tau_2 = \frac{1}{15}t_2,
\]

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\[
\begin{align*}
\tau_8 &= \frac{1}{30} t_8 - \frac{13}{16200000} t_2^4, \\
\tau_{12} &= \frac{16}{21} t_{12} + A_{1(12)} t_2^2 t_8 + \frac{373}{2551500000} t_2^6, \\
\tau_{14} &= \frac{64}{1155} t_{14} - \frac{2531}{229635000000} t_2^7 - \frac{568}{51975} t_{12} t_8 + \frac{103}{1417500} t_2^3 t_8, \\
\tau_{18} &= \frac{256}{4095} t_{18} + A_{1(18)} t_2^2 t_{14} + \frac{3706}{8353125} t_2^3 t_{12} + A_{1(18)} t_2^2 t_8 - \frac{4051}{1530900000} t_2^5 t_8 \\
&\quad + \frac{330961}{826686000000000} t_2^9, \\
\tau_{20} &= -\frac{575025}{4096} t_{20} - \frac{35320910625}{60576512} t_2 t_{18} + \frac{18641008}{779137734375} t_2^3 t_{14} \\
&\quad + \frac{1}{38335} \left(- \frac{138176}{1575} t_8 - \frac{103942624}{1196015625} t_2^2 t_{12} + \frac{323371}{6538218750} t_2^3 t_8^2 \right) t_2^2 t_8 \\
&\quad - \frac{7626178350000000000000}{10249681} t_2^{10} + \frac{1}{353063812500000} t_2^{12} t_8, \\
\tau_{24} &= A_{2(24)} t_8^3 + A_{3(24)} t_2 t_8 t_{14} + A_{4(24)} t_2^2 t_{20} + \frac{32768}{13101165} t_{24} + \frac{52162303808}{878607651796875} t_2^3 t_{18} \\
&\quad - \frac{2857817967448}{3663018665947265625} t_2^5 t_{14} - \frac{197632}{362005875} t_2^{12} \\
&\quad + \frac{1}{4367055} \left(- \frac{31134375}{21206803851} t_8^2 + \frac{11785468451047}{36777480468750} t_2^2 t_{12} + \frac{101271432653}{2994007} t_2^3 t_8 \right) t_2^2 t_8 \\
&\quad - \frac{1}{13661608781250000}{38183226373283} t_2^4 t_8 - \frac{3688634181093750000000000}{38183226373283} t_2^8 t_8 \\
&\quad + \frac{8764194814278750000000000000000000}{8764194814278750000000000000000000} t_2^{12}, \\
\tau_{30} &= A_{2(30)} t_8^3 t_{14} + A_{3(30)} t_2 t_8 t_{14} + A_{4(30)} t_2^2 t_{20} + \frac{4194304}{114489375} t_{30} \\
&\quad - \frac{87361458176}{412072580390625} t_2^3 t_{24} + \frac{98943157092328832}{1454023854444947265625} t_2^5 t_{20} \\
&\quad + \frac{1}{7632625} \left(- \frac{82608128}{945} t_{12} + \frac{7604210276224}{14671762875} t_2^2 t_8 - \frac{44731593575760671656}{6158701713076171875} t_2^6 \right) t_{18} \\
&\quad + \frac{1}{7632625} \left(- \frac{387292030976}{128638125} t_2^2 t_{12} + \frac{835852048357354832}{8558783998319091796875} t_2^4 t_8 + \frac{1}{162900154624} \left(- \frac{2931829717454144}{271859135625} t_2^6 \right) t_{14} \\
&\quad + \frac{81160947912515625}{7632625} t_2^3 t_{12} + \frac{1}{7632625} \left(- \frac{2931829717454144}{271859135625} t_2^6 \right) t_2^2 t_8 \right)
\end{align*}
\]
which lead to the flag which we think is minimal. We denote this flag \( \mathcal{P}(E_8) \). The flag \( \mathcal{P}(E_8) \) is generated by the spaces of polynomials

\[
\mathcal{P}_n^{(1,3,5,5,7,7,9,11)} = \langle t_2^{p_2} s_8^{p_8}, t_8^{p_8}, s_8^{p_8}, t_8^{p_8}, t_2^{p_2}, t_8^{p_8}, t_2^{p_2}, t_8^{p_8}, t_2^{p_2} \rangle \quad 0 \leq p_2 + 3p_8 + 5p_{12} + 5p_{14} + 7p_{18} + 7p_{20} + 9p_{24} + 11p_{30} \leq n ,
\]

(7.7)

with the characteristic vector

\[
\vec{f} = (1, 3, 5, 5, 7, 7, 9, 11) ,
\]

(7.9)

(cf. (3.13), (4.9), (5.12), (6.11)). Hence \( \mathcal{P}(E_8) = \mathcal{P}^{(1,3,5,5,7,7,9,11)} \). The characteristic vector does not coincide with the highest root \( \vec{f}_{\text{highest root}} = (2, 2, 3, 3, 4, 4, 5, 6) \) suggested by Kac. We were not able to find variables in which the flag \( \vec{f}_{\text{highest root}} \) would be preserved by the rational \( E_8 \) Hamiltonian.

The most general polynomial transformation which preserves each linear space \( \mathcal{P}_n^{(1,3,5,5,7,7,9,11)}, n = 0, 1, 2, \ldots \) is

\[
s_2 \rightarrow s_2 ,
\]

\[
s_8 \rightarrow a_{8,1}s_8 + a_{8,2}s_2^3 ,
\]

\[
s_{12} \rightarrow a_{12,1}s_{12} + a_{12,2}s_{14} + a_{12,3}s_8^2s_8 + a_{12,4}s_8^5 ,
\]

\[
s_{14} \rightarrow a_{14,1}s_{14} + a_{14,2}s_{12} + a_{14,3}s_8^2s_8 + a_{14,4}s_8^5 ,
\]

\[
s_{18} \rightarrow a_{18,1}s_{18} + a_{18,2}s_{20} + a_{18,3}s_8^2s_8 + a_{18,4}s_8^2s_8 + a_{18,5}s_8^2s_8 + a_{18,6}s_8^4s_8 + a_{18,7}s_8^7 ,
\]

\[
s_{20} \rightarrow a_{20,1}s_{20} + a_{20,2}s_{18} + a_{20,3}s_8^2s_8 + a_{20,4}s_8^2s_8 + a_{20,5}s_8^2s_8 + a_{20,6}s_8^4s_8 + a_{20,7}s_8^7 ,
\]

\[
s_{24} \rightarrow a_{24,1}s_{24} + a_{24,2}s_8^3 + a_{24,3}s_8s_8s_{14} + a_{24,4}s_8s_{18} + a_{24,5}s_8^2s_8 + a_{24,6}s_8^2s_8 + a_{24,7}s_8^4s_8 + a_{24,8}s_8^6s_8 + a_{24,9}s_8^9 ,
\]

\[
s_{30} \rightarrow a_{30,1}s_{30} + a_{30,2}s_8^2s_{14} + a_{30,3}s_8s_8s_{14} + a_{30,4}s_8^2s_{14} + a_{30,5}s_8s_{12}s_{14} + a_{30,6}s_8^2s_{12} ,
\]

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\[ +a_{30,7}s_2 s_8 s_{20} + a_{30,8}s_2 s_8 s_{18} + a_{30,9} s_5 s_2 s_{24} + a_{30,10} s_5^2 s_8 + a_{30,11} s_2 s_8 s_{14} \\
+ a_{30,12} s_2 s_8 s_{12} + a_{30,13} s_5 s_2 s_{20} + a_{30,14} s_5^2 s_{18} + a_{30,15} s_2 s_8^2 + a_{30,16} s_2 s_{14} \\
+ a_{30,17} s_2 s_{12} + a_{30,18} s_8 s_8 + a_{30,19} s_2^{11} \]  

(7.10)

where \( a \)'s are parameters. There exist nine exceptional sub-transformations for which (7.6) and (7.10) coincide – when we vary the parameter \( a_{12,3} \) in \( s_2 (A_{12}^{(12)}) \) and \( a_{18,3} \) and \( a_{18,4} \) in \( s_8 (A_{18}^{(18)}) \) and \( A_{18}^{(18)} \) in \( i_{18}^{(1)} \), correspondingly, and \( a_{24,2}, a_{24,3} \) and \( a_{24,5} \) in \( s_2 A_{24}^{(24)}, A_{3}^{(24)} \) and \( A_{4}^{(24)} \) in \( i_{24}^{(24)} \), correspondingly, and \( a_{30,2}, a_{30,4} \) and \( a_{30,7} \) in \( s_2 A_{30}^{(30)}, A_{3}^{(30)} \) and \( A_{4}^{(30)} \) in \( i_{30}^{(2)} \), correspondingly. Of course, this coincidence appears when all other parameters of the transformation are kept fixed. Thus, one can draw a conclusion about existence of nine-parametric set of invariants of the fixed degrees leading to algebraic form of the Hamiltonian \( h_{E_8}^{(r)} \) and moreover preserving the flag \( P^{(1,3,5,7,9,11)} \). Hence, the parameters can be chosen by following our convenience. In a simply-minded way we set all of them equal to zero,

\[ A_{1}^{(12)} = A_{2}^{(18)} = A_{2}^{(24)} = A_{3}^{(24)} = A_{3}^{(30)} = A_{4}^{(30)} = 0 \]

which fixes the coefficient functions in the algebraic form of the Hamiltonian (see below (7.11)) in form of polynomials of lowest degrees. It is an open question for what value(s) of these parameters the invariants (7.6) are ‘trigonometrized’ leading to an algebraic form of the trigonometric \( E_8 \) model, if such a form exists.

Finally, the Hamiltonian \( h_{E_8}^{(r)}(x_1 \ldots x_8) \) can be written as

\[ h_{E_8}^{(r)}(x_1 \ldots x_8) = A_{ab} \frac{\partial^2}{\partial \tau_a \partial \tau_b} + B_{ab} \frac{\partial}{\partial \tau_b} \]  

(7.11)

where \( a, b = 2, 8, 12, 14, 18, 20, 24, 30 \) and \( a \leq b \) with the coefficient functions:

\[ A_{2,2} = 8 \tau_2 \, , \, A_{2,8} = 32 \tau_8 \, , \, A_{2,12} = 48 \tau_{12} \, , \, A_{2,14} = 56 \tau_{14} \, , \]

\[ A_{2,18} = 72 \tau_{18} \, , \, A_{2,20} = 80 \tau_{20} \, , \, A_{2,24} = 96 \tau_{24} \, , \, A_{2,30} = 120 \tau_{30} \, , \]

\[ A_{8,8} = \frac{21}{5} \tau_{14} + \frac{7}{25} \tau_2 \tau_{12} \, , \, A_{8,12} = -\frac{27}{7} \tau_2^5 \tau_8 + \frac{2421}{5} \tau_2^2 \tau_{14} + 81 \tau_{18} - \frac{14664}{7} \tau_2 \tau_{8}^2 - \frac{12}{5} \tau_2^3 \tau_{12} \, , \]

\[ A_{8,14} = 75 \tau_{20} + \frac{32}{15} \tau_8 \tau_{12} + \frac{2}{5} \tau_2 \tau_{18} \, , \, A_{8,18} = \frac{2349}{2} \tau_2^2 \tau_{20} + \frac{16640}{21} \tau_2^3 \tau_8 - 18 \tau_2^5 \tau_{14} + 270 \tau_{24} + \frac{3744}{49} \tau_2 \tau_{8}^2 + \frac{88}{35} \tau_2 \tau_8 \tau_{12} + \frac{16}{75} \tau_2 \tau_{12}^2 - \frac{12232}{35} \tau_2 \tau_8 \tau_{14} - \frac{27}{5} \tau_2^3 \tau_{18} \, , \]

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\[ A_{8,20} = \frac{3}{5} r_2 r_{24} + \frac{32}{375} r_{12} r_{14}, \quad A_{8,24} = r_{30} - \frac{43264}{6615} r_{2} r_{8} + \frac{2288}{525} r_{2} r_{8} r_{14} + \frac{16}{1125} r_{2} r_{12} r_{14}, \]
\[ A_{8,30} = \frac{48}{5} r_{12} r_{24} - \frac{1555840}{1323} r_{8} r_{20} - \frac{27}{5} r_{2} r_{30} + \frac{43264}{945} r_{7} r_{14} + \frac{43264}{19845} r_{2} r_{8} r_{18}, \]
\[ A_{12,12} = \frac{1185120}{49} r_{2} r_{20} + \frac{11880 r_{8} r_{14} + 810}{49} r_{7} r_{8} - \frac{5400}{7} r_{2} r_{8} r_{12} - \frac{43011}{7} r_{2} r_{14} + \frac{9 r_{7} r_{12} - 7155}{7} r_{2} r_{18}, \]
\[ A_{12,14} = \frac{164250}{7} r_{2} r_{20} + 2560 \tau_3 + 4050 \tau_{24} - \frac{240}{7} r_{4} r_{2} - \frac{144}{7} r_{7} r_{8} r_{12} - \frac{22752}{7} r_{2} r_{8} r_{14} - \frac{6}{7} r_{2} r_{18}, \]
\[ A_{12,18} = -\frac{2688}{5} r_{2} r_{12} r_{14} + 12600 r_{8} r_{20} - \frac{4415680}{49} r_{2} r_{8} + \frac{2218185}{14} r_{2} r_{20} + 504 r_{14} + \frac{270}{7} r_{7} r_{14} - \frac{72}{35} r_{2} r_{12} - \frac{53640}{343} r_{6} r_{8} - \frac{264}{49} r_{2} r_{8} r_{12} + \frac{1891584}{49} r_{2} r_{3} r_{14} + \frac{10112}{7} r_{2} r_{8} r_{12} + \frac{20040}{7} r_{2} r_{8} r_{18} + \frac{81}{7} r_{5} r_{18} - \frac{381375}{14} r_{2} r_{24}, \]
\[ A_{12,20} = 45 r_{30} - \frac{48}{35} r_{2} r_{8} r_{14} - \frac{144}{175} r_{2} r_{12} r_{14} - \frac{9}{7} r_{7} r_{24} + \frac{4800}{7} r_{2} r_{8} r_{20} - \frac{315008}{2205} r_{8} r_{14} + \frac{384}{25} r_{2} r_{7} r_{14}, \]
\[ A_{12,24} = 54 r_{2} r_{20} + 408 r_{14} r_{20} - \frac{10112}{35} r_{2} r_{12} + \frac{43264}{3087} r_{5} r_{3} + \frac{1024}{15} r_{8} r_{18} + \frac{117}{7} r_{2} r_{24} - \frac{1504}{735} r_{7} r_{18} + \frac{21408}{49} r_{2} r_{8} r_{18} - \frac{5760}{175} r_{2} r_{14} + \frac{11152}{525} r_{2} r_{7} r_{14} - \frac{64}{7} r_{6} r_{8} r_{14} - \frac{60764416}{46305} r_{2} r_{14} + \frac{3328}{175} r_{2} r_{12} r_{14} - \frac{96}{7} r_{2} r_{12} r_{18} + \frac{6600}{7} r_{2} r_{12} r_{14} - \frac{1968}{7} r_{2} r_{30}, \]
\[ A_{12,30} = -\frac{3056}{7} r_{2} r_{20} + 360000 r_{9} r_{20} - 27000 r_{20}^2 + \frac{17305600}{9261} r_{4} r_{14} + \frac{131456}{1029} r_{5} r_{8} r_{14} - \frac{346112 r_{2} r_{3} r_{12}}{3087} - \frac{43264 r_{3} r_{2} r_{18}}{9261} - \frac{648 r_{7} r_{2} r_{24}}{7} - \frac{71552}{945} r_{2} r_{7} r_{12} r_{14} - \frac{31800 r_{2} r_{12} r_{14} + \frac{48}{7} r_{2} r_{30} + \frac{55377920}{3969} r_{5} - \frac{16640}{7} r_{8} r_{2} r_{14} }{7} r_{7} r_{8} r_{30} + 1728 r_{2} r_{14} r_{24} + \frac{81}{7} r_{5} r_{30} + \frac{55377920}{3969} r_{5} - \frac{16640}{7} r_{8} r_{12} r_{20}. \]
\[
A_{14,14} = -1575 r_2^2 r_3 + 346112 r_2 r_3 - 28080 r_2^2 r_3 + 7444736 r_2^2 r_3^2,
\]
\[
A_{14,18} = 1408 r_2 r_3^2 r_12 + 23680 r_2^2 r_3 r_12 + 80 r_2^3 r_12 - 2928 + 4 r_2^3 r_12 + 75 r_2^2 r_3,
\]
\[
A_{14,20} = -450 r_2^3 r_20 + 1575 r_2^2 r_3 + 35080 r_2 r_3 - 102808 + 1902848 r_2^2 r_3,
\]
\[
A_{14,24} = -16 r_2^2 r_3 + 512 r_2^2 r_3 - 18 r_2^2 r_3 + 2944 r_2^2 r_3 - 1856 r_2^2 r_3 - 64 r_2^4 r_2 + 32 r_2^2 r_3 + 16 r_2 r_3 + 9 r_2 r_3 - 61696 r_2 r_3 + 48 r_2 r_3 - 3 r_2 r_3,
\]
\[
A_{14,30} = -1664 r_2 r_3 - 4352 r_2 r_3 + 2154496 r_2^2 r_3 - 9328 r_2^2 r_3 + 8696 r_2^2 r_3 + 32 r_2 r_3 + 1664 r_2^2 r_3 - 3969 r_2^2 r_3 - 7 r_2^2 r_3,
\]
\[
A_{18,18} = -56700 r_2 r_20 + 1020 r_14^2 r_3 - 960 r_2^3 r_14 - 19125440 r_2 r_3 + 726968320 r_2^2 r_3 - 720 r_2^5 r_3 - 250047 r_2^5 r_3,
\]
\[
A_{18,20} = 15504 r_2^2 r_3 + 24064 r_2^2 r_3 + 559980 r_2^2 r_3 + 68 r_2 r_14 r_20 - 83200 r_2 r_3 + 720 r_2^2 r_14 r_20,
\]
\[
A_{18,24} = 5 r_2^2 r_3 + 2368 r_2^2 r_3 + 164100 r_2^2 r_3 + 166400 r_2^2 r_3 + 945 r_2^2 r_3 - 2 r_2 r_3 - 1760 r_2 r_3 + 156 r_2 r_3 + 375058 r_2^2 r_14 r_20 + 4 r_2^3 r_14 r_20 - 2100000 r_2^2 r_3 - 21677520 r_2^2 r_3 - 1053508 r_2^2 r_3 - 147 r_2^2 r_3 + 156 r_2 r_3 + 156 r_2 r_3 - 2100000 r_2^2 r_3 - 21677520 r_2^2 r_3 - 1053508 r_2^2 r_3 - 147 r_2^2 r_3 + 156 r_2 r_3 + 156 r_2 r_3 - 2100000 r_2^2 r_3 - 21677520 r_2^2 r_3 - 1053508 r_2^2 r_3 - 147 r_2^2 r_3 + 156 r_2 r_3 + 156 r_2 r_3 - 2100000 r_2^2 r_3 - 21677520 r_2^2 r_3 - 1053508 r_2^2 r_3 - 147 r_2^2 r_3 + 156 r_2 r_3 + 156 r_2 r_3
\[-\frac{352}{5} \tau_2^4 \tau_{14} + \frac{240}{7} \tau_2^4 \tau_8 \tau_{20} - \frac{1088}{7} \tau_2^4 \tau_8 \tau_{24} + \frac{1408}{525} \tau_2^4 \tau_8 \tau_{12} \tau_{14} + 48 \tau_2^2 \tau_1 \tau_{12} \tau_{20} - \frac{5032}{5} \tau_2^2 \tau_{14} \tau_{20}, \]

\[A_{18,24} = \frac{3328}{23625} \tau_2^2 \tau_{12}^2 \tau_{14} - \frac{3724}{7} \tau_2^2 \tau_{14} \tau_{20} + \frac{136}{15} \tau_2^2 \tau_{12} \tau_{24} - \frac{12784}{15} \tau_2 \tau_{18} \tau_{20} - \frac{26172032}{15435} \tau_2^5 \tau_8 \tau_{14} \]

\[26850304 \tau_2^3 \tau_8 \tau_{14}^2 + \frac{3048064}{441} \tau_2^3 \tau_8 \tau_{20} - \frac{4432}{175} \tau_2^3 \tau_8 \tau_{14} \tau_{18} + \frac{45456}{7} \tau_2^6 \tau_8 \tau_{20} + \frac{647296}{11025} \tau_2^2 \tau_8 \tau_{14}^2 \]

\[-\frac{692224}{19845} \tau_2^2 \tau_8 \tau_{14}^2 - \frac{592}{5} \tau_2^4 \tau_{12} \tau_{20} + \frac{94592}{1575} \tau_8 \tau_{14} \tau_{18} - \frac{8808}{35} \tau_2^3 \tau_{14} \tau_{20} + \frac{287744}{2205} \tau_2^3 \tau_8 \tau_{18} \]

\[+ \frac{256}{45} \tau_8 \tau_{12} \tau_{20} + \frac{14344}{7} \tau_2^4 \tau_8 \tau_{24} + \frac{1280}{21} \tau_2 \tau_8 \tau_{30} + \frac{13568}{3675} \tau_2^3 \tau_8 \tau_{12} \tau_{14} \]

\[+ \frac{4352}{945} \tau_2 \tau_8 \tau_{12} \tau_{18} + \frac{40960}{21} \tau_2^2 \tau_{24} - \frac{1888}{225} \tau_2^2 \tau_{18} - \frac{108 \tau_2^5 \tau_{30} \tau_8}{\tau_{14} \tau_{20}} \]

\[\tau_2^3 \tau_8 + \frac{47071232}{27783} \tau_2^3 \tau_8 + \frac{256}{1125} \tau_2 \tau_8 \tau_{14} + \frac{4512}{35} \tau_2^3 \tau_{14} + 200 \tau_2^2, \]

\[A_{18,30} = -\frac{74054656}{46305} \tau_2^2 \tau_8 \tau_{14}^2 + \frac{743104}{147} \tau_8 \tau_{14} \tau_{24} - \frac{5448}{7} \tau_2^4 \tau_8 \tau_{30} + \frac{692224}{297675} \tau_8 \tau_{12} \tau_{14} \]

\[-\frac{3328}{315} \tau_2^2 \tau_{12} \tau_{20} + \frac{30457856}{83349} \tau_2^2 \tau_8 \tau_{12} + \frac{737884160}{27783} \tau_2 \tau_{13} \tau_{20} - \frac{24960}{7} \tau_8 \tau_{18} \tau_{20} \]

\[+ \frac{448640}{9} \tau_2^2 \tau_8 \tau_{24} + \frac{260368000}{1029} \tau_2^5 \tau_8 \tau_{20} + \frac{692224}{9261} \tau_8 \tau_{12} \tau_{18} - \frac{21112832}{\tau_2^3 \tau_8 \tau_{18}} \]

\[-\frac{10771764224}{194481} \tau_2^4 \tau_8 \tau_{14}^2 + \frac{216}{5} \tau_2^2 \tau_{12} \tau_{30} - \frac{326400 \tau_2^6 \tau_{14} \tau_{20}}{7} + \frac{3757312}{441} \tau_2 \tau_8 \tau_{14} - \frac{9720 \tau_2^4 \tau_{14} \tau_{24}}{\tau_8 \tau_{14} \tau_{20}} \]

\[-\frac{79740 \tau_2 \tau_{20} \tau_{24}}{7} - \frac{1536 \tau_2 \tau_{14} \tau_{30} - \frac{57600}{7} \tau_2^4 \tau_{18} \tau_{20} + \frac{128}{5} \tau_8 \tau_{14} \tau_{20} - 1416 \tau_2^2 \tau_{18} \tau_{24} \tau_{20} \]

\[+ \frac{294080 \tau_2^2 \tau_8 \tau_{14} \tau_{20}}{21} + \frac{229924864 \tau_2^2 \tau_8 \tau_{12} \tau_{14}}{416745} - \frac{8000 \tau_2 \tau_7 \tau_8 \tau_{12} \tau_{24}}{21} + \frac{330636 \tau_2^3 \tau_8 \tau_{14} \tau_{18}}{2205} \]

\[+ \frac{422528 \tau_2 \tau_8 \tau_{12} \tau_{20} + \frac{95637667840 \tau_2^3 \tau_8 \tau_5}{7} + \frac{248400 \tau_2^3 \tau_8 \tau_{20}}{5250987} + \frac{19793453056 \tau_2^4 \tau_8 \tau_{14}}{1323}, \]

\[A_{20,20} = -\frac{4096 \tau_2 \tau_8 \tau_{14} \tau_{20}^2}{525} - \frac{32 \tau_8 \tau_{30}}{15} + \frac{692224 \tau_8 \tau_{14} \tau_{18}}{59535} + \frac{304 \tau_1 \tau_{14} \tau_{24} + \frac{224 \tau_2 \tau_1 \tau_{14} \tau_{20}}{64 \tau_8 \tau_{20}}}{5} \tau_8 \tau_{14} \tau_{20} \]

\[+ \frac{1792}{5625} \tau_3 \tau_4 - \frac{976 \tau_2 \tau_{20}}{3375} \tau_2 \tau_8 \tau_{14} \tau_{18} + \frac{379136}{7875} \tau_2^3 \tau_8 \tau_{14} + \frac{6656}{118125} \tau_2 \tau_7 \tau_{14}^2 \]

\[A_{20,24} = \frac{1664}{63} \tau_2 \tau_8 \tau_{24} + \frac{48}{25} \tau_8 \tau_{18} \tau_{24} - \frac{32}{3} \tau_2 \tau_8 \tau_{30} - \frac{16}{225} \tau_8 \tau_{12} \tau_{20} + \frac{68864}{1575} \tau_8 \tau_{14} \tau_{20} + \frac{1792}{5625} \tau_3 \tau_4 - \frac{976 \tau_2 \tau_{20}}{3375} \tau_2 \tau_8 \tau_{14} \tau_{18} + \frac{379136}{7875} \tau_2^3 \tau_8 \tau_{14} + \frac{6656}{118125} \tau_2 \tau_7 \tau_{14}^2 \]

\[= \frac{38}{3} \tau_2 \tau_8 \tau_{14} + \frac{52}{25} \tau_8 \tau_{12} \tau_{20} - \frac{264}{15} \tau_8 \tau_{14} \tau_{20} + \frac{128}{5} \tau_8 \tau_{18} \tau_{24} + \frac{352}{7} \tau_2 \tau_{12} \tau_{24}. \]
\[ A_{20,30} = -\frac{11008}{21} \tau_{8 \tau_{20}} + \frac{30240 \tau_{2 \tau_{20}}^2 - 227072}{21} \tau_{2 \tau_{8 \tau_{14}}} \tau_{2 \tau_{14 \tau_{20}}} - \frac{11318912}{6615} \tau_{2 \tau_{8 \tau_{14}}} \tau_{2 \tau_{14 \tau_{24}}} + \frac{692224}{3969} \tau_{8 \tau_{24}}, \]

\[ A_{24,24} = -\frac{11208704}{2083725} \tau_{2 \tau_{8 \tau_{24}}} + \frac{35456}{315} \tau_{8 \tau_{4 \tau_{14}}} - \frac{256}{5} \tau_{2 \tau_{8 \tau_{30}}} - \frac{36608}{354375} \tau_{2 \tau_{8 \tau_{12}}} \tau_{2 \tau_{8 \tau_{14}}} \tau_{2 \tau_{8 \tau_{18}}}, \]

\[ A_{24,30} = \frac{337378902016}{26254935} \tau_{2 \tau_{8 \tau_{14}}} \tau_{2 \tau_{8 \tau_{24}}} + \frac{928}{25} \tau_{2 \tau_{8 \tau_{14}}} \tau_{2 \tau_{18 \tau_{30}}} - \frac{23168}{525} \tau_{2 \tau_{8 \tau_{14}}} \tau_{2 \tau_{14 \tau_{24}}} - \frac{151715392}{231525} \tau_{2 \tau_{8 \tau_{24}}}, \]
\[ A_{30,30} = - \frac{578880}{7} \tau_2 \tau_8^2 \tau_{24} + \frac{153485443072}{236294415} \tau_2^4 \tau_{12} \tau_{14} - \frac{21139200}{7} \tau_2 \tau_8 \tau_{20}^2 \]
\[ - \frac{556201984}{27783} \tau_2^2 \tau_8^3 \tau_{30} - \frac{346112}{3969} \tau_2^2 \tau_8 \tau_{12} \tau_{30} + \frac{63870677811200}{33081281} \tau_2^2 \tau_8^5 \tau_{14} + \frac{1827200}{21} \tau_2 \tau_{14}^2 \tau_{20} \]
\[ + \frac{346112}{147} \tau_2 \tau_8 \tau_{18} \tau_{24} - \frac{57309224960}{750141} \tau_2^2 \tau_8^4 \tau_{24} - \frac{14400}{7} \tau_2 \tau_8 \tau_{20} \tau_{30} - \frac{2043099136000}{1750329} \tau_2 \tau_8 \tau_{12} \tau_{14} \tau_{20} \]
\[ + \frac{432 \tau_2 \tau_8 \tau_{24} \tau_{30}}{96640} - \frac{96640}{7} \tau_8 \tau_{20} \tau_{30} - \frac{31610}{63} \tau_2^3 \tau_{12} \tau_{20}^2 + \frac{16704 \tau_2 \tau_{14} \tau_{20} \tau_{24}}{35721} - \frac{3258990592}{3750705} \tau_2^2 \tau_8^3 \tau_{14} \]
\[ + \frac{116390681152}{5250987} \tau_2 \tau_8 \tau_{14} \tau_{20} + \frac{1307238400}{27783} \tau_2 \tau_8^2 \tau_{20} + \frac{119062528}{59535} \tau_2^2 \tau_8 \tau_{12} \tau_{14} \tau_{20} + \frac{3441737728}{3750705} \tau_2 \tau_8^3 \tau_{14} \]
\[ - \frac{472478875648}{78764805} \tau_2 \tau_8^3 \tau_{14} \tau_{18} - \frac{1583808512}{8037225} \tau_2 \tau_8^2 \tau_{12} \tau_{14}^2 - \frac{71991296}{27783} \tau_2 \tau_8^3 \tau_{12} \tau_{20} \]
\[ + \frac{4230234112}{27783} \tau_2 \tau_8^2 \tau_{14} \tau_{24} + \frac{2903879680}{83349} \tau_2 \tau_8^2 \tau_{18} \tau_{20} + \frac{3058432}{735} \tau_2 \tau_8^3 \tau_{14} \tau_{30} \]
\[ - \frac{147142400}{147} \tau_2 \tau_8 \tau_{20} \tau_{24} + \frac{1384448}{6615} \tau_8 \tau_{12} \tau_{14} \tau_{24} + \frac{1536204800}{147} \tau_2 \tau_8^4 \tau_{14} \tau_{20} \]
\[ + \frac{13312}{21} \tau_2 \tau_8 \tau_{18} \tau_{30} - \frac{452727808}{19845} \tau_2 \tau_8 \tau_{14} \tau_{20} - \frac{6656}{7} \tau_2 \tau_{12} \tau_{18} \tau_{24} \]
\[ - \frac{46419987660800}{992436543} \tau_2 \tau_8^7 + \frac{97332232192}{47258883} \tau_5 \tau_{18} - \frac{3840 \tau_{18} \tau_{20}^2}{5} \tau_{14} \tau_{30} \, , \] (7.12)

and

\[ B_2 = -4 \omega \tau_2 + 32(1 + 30 \nu) \, , \quad B_8 = -16 \omega \tau_8 - \frac{21}{10} (1 + 18 \nu) \tau_2^3 \, , \]
\[ B_{12} = -24 \omega \tau_{12} - \frac{480}{7} (17 + 207 \nu) \tau_2 \tau_8 + \frac{9}{2} (1 + 18 \nu) \tau_2^5 \, , \]
\[ B_{14} = -28 \omega \tau_{14} - \frac{16}{15} (1 + 30 \nu) \tau_{12} - \frac{8}{7} (23 + 354 \nu) \tau_2^2 \tau_8 \, , \]

40
\(B_{18} = -36 \omega \tau_{18} + \frac{12}{7} (25 + 282 \nu) \tau_{2}^4 \tau_{8} + \frac{4}{5} (19 + 198 \nu) \tau_{2}^2 \tau_{12} - \frac{8}{5} (1037 + 7470 \nu) \tau_{2} \tau_{14}
\)
\[ + \frac{1024}{21} (83 + 435 \nu) \tau_{8}^2 , \]
\(B_{20} = -40 \omega \tau_{20} + \frac{512}{63} (2 + 39 \nu) \tau_{2}^2 \tau_{8}^2 - \frac{376}{75} (1 + 18 \nu) \tau_{2}^2 \tau_{14} - \frac{32}{75} (1 + 30 \nu) \tau_{18} , \)
\(B_{24} = -48 \omega \tau_{24} + \frac{512}{4725} (23 - 195 \nu) \tau_{2} \tau_{8} \tau_{12} + \frac{256}{1575} (859 + 3480 \nu) \tau_{8} \tau_{14} - \frac{16}{105} (15331 + 76230 \nu) \tau_{2} \tau_{20} \)
\[ + \frac{16}{1575} (115 + 6678 \nu) \tau_{2}^2 \tau_{18} - \frac{256}{735} (191 + 5590 \nu) \tau_{2}^2 \tau_{8}^2 + \frac{8}{75} (161 + 4338 \nu) \tau_{2} \tau_{14} , \]
\(B_{30} = -60 \omega \tau_{30} - \frac{76544 \tau_{2} \tau_{12} \tau_{14}}{14175} + \frac{64}{945} (8935 + 74646 \nu) \tau_{2} \tau_{8} \tau_{14} + \frac{256}{19845} (3637 + 49140 \nu) \tau_{2} \tau_{8} \tau_{18}
\[ - \frac{128}{1323} (128693 + 531090 \nu) \tau_{8} \tau_{20} + \frac{346112}{39535} (7 - 30 \nu) \tau_{8}^2 \tau_{12} - \frac{4}{7} (839 + 2646 \nu) \tau_{2}^2 \tau_{24}
\]
\[- \frac{13312}{27783} (2861 + 37362 \nu) \tau_{8}^2 \tau_{8}^3 - 16 (151 + 450 \nu) \tau_{2}^4 \tau_{20} + \frac{64}{4725} (5837 - 5670 \nu) \tau_{14}^2 . \] (7.13)

It can be found one-parametric algebra of differential operators (in eight variables) for which \(\mathcal{P}_{n}^{(1,3,5,7,9,11)}\) (see (7.8)) is a finite-dimensional irreducible representation space. Furthermore, the finite-dimensional representation spaces appear for different integer values of the algebra parameter. They form an infinite non-classical flag which coincides to \(\mathcal{P}_{E_{8}}^{(8)}\) (7.3). We call this algebra \(E_{8}\). Like the algebras \(g^{(2)}\), \(f^{(4)}\) introduced in [6] and [8], respectively, in relation to the \(G_{2}\) and \(F_{4}\) models, the algebra \(E_{8}\) is infinite-dimensional but finitely-generated. It will be described elsewhere. The rational \(E_{8}\) Hamiltonian in the algebraic form (7.11) with coefficients (7.12) , (7.13) can be rewritten in terms of the generators of this algebra.

The operator (7.11) is triangular in the basis of monomials \(r_{2} p_{r}^{2} r_{8} p_{12}^{4} r_{14}^{8} p_{18}^{p_{20}} r_{24}^{p_{21}} r_{30}^{p_{30}}\). One can easily find the spectrum of (7.11),
\(\hbar_{E_{8}}^{(n)} \phi = -2 \epsilon \varphi\), explicitly
\(\epsilon_{n_{1},n_{2},n_{3},n_{4},n_{5},n_{6},n_{7},n_{8}} = 2 \omega (n_{1} + 4 n_{2} + 6 n_{3} + 7 n_{4} + 9 n_{5} + 10 n_{6} + 12 n_{7} + 15 n_{8}) , \) (7.14)
where \(p_{i}\) are non-negative integers. Degeneracy of the spectrum is related to the number of partitions of integer number \(n = 0, 1, 2, \ldots, \) to \(n_{1} + 4 n_{2} + 6 n_{3} + 7 n_{4} + 9 n_{5} + 10 n_{6} + 12 n_{7} + 15 n_{8} = 0, 1, 2, \ldots\) to \(n_{1} + 4 n_{2} + 6 n_{3} + 7 n_{4} + 9 n_{5} + 10 n_{6} + 12 n_{7} + 15 n_{8} = 0, 1, 2, \ldots\) to \(n_{1} + 4 n_{2} + 6 n_{3} + 7 n_{4} + 9 n_{5} + 10 n_{6} + 12 n_{7} + 15 n_{8} = 0, 1, 2, \ldots\). The spectrum does not depend on the coupling constants \(g\), it is equidistant and corresponds to the spectrum of a harmonic oscillator. Finally,
the energies of the original rational $E_8$ Hamiltonian (7.1) are $E = E_0 + \epsilon$. As an illustration the first eigenfunctions are presented in the Appendix D.

It is worth noting that the Hamiltonian (7.11) has a remarkable property: there exists a family of eigenfunctions which depend on the single variable $\tau_2$. These eigenfunctions are the associated Laguerre polynomials. This property admits to construct a quasi-exactly-solvable generalization of the rational $E_8$ model. It will be done elsewhere. Due to enormous technical difficulties we were unable to find explicitly the boundary of the configuration space (in other words, the boundaries of the $E_8$ Weyl chamber) in the Weyl-invariant variables $\tau$'s similar to what was previously done for $G_2$ and $F_4$ cases (see Section 3 and 4, correspondingly).

8 Conclusion

We have found in uniform way that all the rational integrable models associated with the root systems of exceptional Lie algebras are exactly-solvable, thus continuing the analysis of the rational (and trigonometric) systems related to the $A_n$ and $BC_n$ root systems as well as their supersymmetric generalizations carried out in Ref. [4, 5, 16, 20]. Our method contains two important ingredients: (i) gauging away the ground state eigenfunction and (ii) taking specific Weyl-invariant polynomials (functions) as variables. After this procedure each Hamiltonian takes an algebraic form thus becoming a linear differential operator with polynomial coefficients. Furthermore, it has infinitely-many invariant subspaces of polynomials, which form a minimal flag. The known characteristic vectors of the minimal flags are collected in the Table. The meaning of the Weyl-invariant variables (3.10), (3.14), (4.11), (4.13), (5.10), (6.9), (7.7) in which the flag of invariant subspaces is minimal is unclear so far. Namely, why in these Weyl-invariant variables the minimal flag is preserved. We show that unlike the rational $A_n$ and $BC_n$ models which all are of the hypergeometric type every rational $G_2, F_4, E_{6,7,8}$ model is special. Each of them is characterized by its own hidden infinite-dimensional algebra which deserve a separate investigation. It will be done elsewhere.

Algebraic forms of the rational $G_2, F_4, E_{6,7,8}$ models which we found allow to construct quasi-exactly-solvable generalizations [21] of these models similar to those of the paper [22]. It is already done in [23]. It concludes an analysis of rational models associated with classical (crystallographic) root systems, while the analysis of the rational systems related with dihedral (non-crystallographic) root systems $H_3, H_4, I_2(m)$ is still waiting to be done.
The presented work complements previous studies where the algebraic and the Lie-algebraic forms as well as the corresponding flags were found for the rational and trigonometric Olshanetsky-Perelomov Hamiltonians of $A-D$ series (and their supersymmetric generalizations) $[4, 5]$, $G_2$ $[6]$ and $F_4$ $[8]$ models. In order to conclude a study of the whole set of Olshanetsky-Perelomov integrable systems appearing in the Hamiltonian reduction method it is necessary to perform the same analysis for remaining $E_{6,7,8}$ integrable trigonometric models. We consider it as a challenging task for future.

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| Model | Rational | Trigonometric |
|-------|----------|---------------|
| $A_n$ | $(1, 1, \ldots, 1)$ \(\frac{n}{n}\) | $(1, 1, \ldots, 1)$ \(\frac{n}{n}\) |
| $BC_n$ | $(1, 1, \ldots, 1)$ \(\frac{n}{n}\) | $(1, 1, \ldots, 1)$ \(\frac{n}{n}\) |
| $G_2$ | $(1, 2)$ | $(1, 2)$ |
| $F_4$ | $(1, 2, 2, 3)$ | $(1, 2, 2, 3)$ |
| $E_6$ | $(1, 1, 2, 2, 2, 3)$ | $(1, 1, 2, 2, 2, 3)$ |
| $E_7$ | $(1, 2, 2, 2, 3, 3, 4)$ | ? |
| $E_8$ | $(1, 3, 5, 5, 7, 7, 9, 11)$ | ? |

Table 1: Characteristic vectors of rational and trigonometric models associated with classical (crystallographic) root spaces
A First eigenfunctions of the rational $F_4$ models

In this Appendix we present explicit expressions for the first eigenfunctions of the rational $F_4$ model at $n = 0, 1, 2$.

- **$n = 0$ (one eigenstate)**
  \[
  \phi_0 = 1, \\
  \epsilon_0 = 0.
  \]

- **$n = 1$ (one eigenstate)**
  \[
  \phi_1 = \tau_2 - \frac{2}{\omega}(6\mu + 6\nu + 1), \\
  \epsilon_1 = 2\omega.
  \]

- **$n = 2$ (three eigenstates)**
  \[
  \phi_{2}^{(1)} = \tau_2^2 - \frac{6}{\omega}(4\mu + 4\nu + 1)\tau_2 + \frac{6}{\omega^2}(4\mu + 4\nu + 1)(6\mu + 6\nu + 1), \\
  \epsilon_{2}^{(1)} = 4\omega,
  \]
  \[
  \phi_{2}^{(2)} = \tau_6 - \frac{1}{4\omega}(2\mu + 4\nu + 1)\tau_2^2 + \frac{3}{4\omega^2}(2\mu + 4\nu + 1)(4\mu + 4\nu + 1)\tau_2 \\
  + \frac{1}{2\omega^3}(2\mu + 4\nu + 1)(6\mu + 6\nu + 1)(4\mu + 4\nu + 1), \\
  \epsilon_{2}^{(2)} = 6\omega,
  \]
  \[
  \phi_{2}^{(3)} = \tau_8 - \frac{1}{\omega}(3\nu + 1)\tau_6 + \frac{1}{8\omega^2}(3\nu + 1)(2\mu + 4\nu + 1)\tau_2^2 \\
  - \frac{1}{4\omega^3}(3\nu + 1)(2\mu + 4\nu + 1)(4\mu + 4\nu + 1)\tau_2 \\
  + \frac{1}{8\omega^4}(3\nu + 1)(2\mu + 4\nu + 1)(6\mu + 6\nu + 1)(4\mu + 4\nu + 1), \\
  \epsilon_{2}^{(3)} = 8\omega.
  \]

B First eigenfunctions of the rational $E_6$ model

- **$n = 0$ (one eigenstate)**
  \[
  \phi_0 = 1, \\
  \epsilon_0 = 0.
  \]
• $n = 1$ (one eigenstate)

$$
\begin{align*}
\phi_{1,1} &= \tau_2 - \frac{6}{\omega} (1 + 12\nu), \quad \epsilon_{1,1} = 2\omega, \\
\phi_{1,2} &= \tau_5, \quad \epsilon_{1,2} = 5\omega.
\end{align*}
$$

• $n = 2$ (six eigenstates)

$$
\begin{align*}
\phi_{2,1} &= \tau_2^2 - \frac{16}{\omega} \tau_2^2 (1 + 9\nu) + \frac{48}{\omega^2} \tau_2 (1 + 9\nu)(1 + 12\nu), \quad \epsilon_{2,1} = 4\omega, \\
\phi_{2,2} &= -4\tau_6 \frac{1}{\omega} \tau_2^2 (1 + 6\nu) - \frac{3240}{\omega^4} \tau_2 (1 + 6\nu)(1 + 9\nu)(1 + 12\nu), \quad \epsilon_{2,2} = 6\omega, \\
\phi_{2,3} &= \left(\tau_2 - \frac{8}{\omega} (2 + 9\nu)\right) \tau_5, \quad \epsilon_{2,3} = 7\omega, \\
\phi_{2,4} &= \tau_8 - \frac{24}{\omega} (1 + 3\nu) \tau_6 + \frac{1215}{\omega^2} (1 + 3\nu)(1 + 6\nu) \tau_2^2 - \frac{6480}{\omega^3} (1 + 3\nu)(1 + 6\nu)(1 + 9\nu) \tau_2 \\
&\quad + \frac{9720}{\omega^4} (1 + 3\nu)(1 + 6\nu)(1 + 9\nu)(1 + 12\nu), \quad \epsilon_{2,4} = 8\omega, \\
\phi_{2,5} &= \tau_9 - \frac{54}{\omega} (2 + 3\nu) \tau_2 \tau_5 + \frac{216}{\omega^2} (2 + 3\nu)(2 + 9\nu) \tau_5, \quad \epsilon_{2,5} = 9\omega, \\
\phi_{2,6} &= \tau_5^2 - \frac{2}{\omega} \tau_8 + \frac{24}{\omega^2} (1 + 3\nu) \tau_6 - \frac{810}{\omega^3} (1 + 3\nu)(1 + 6\nu) \tau_2^2 \\
&\quad + \frac{3240}{\omega^4} (1 + 3\nu)(1 + 6\nu)(1 + 9\nu) \tau_2 - \frac{3888}{\omega^5} (1 + 3\nu)(1 + 6\nu)(1 + 9\nu)(1 + 12\nu), \quad \epsilon_{2,6} = 10\omega.
\end{align*}
$$

C  First eigenfunctions of the rational $E_7$ model

• $n = 0$ (one eigenstate)

$$
\phi_0 = 1, \quad \epsilon_0 = 0.
$$

• $n = 1$ (one eigenstate)

$$
\phi_1 = \tau_2 - \frac{14}{\omega} (1 + 18\nu), \quad \epsilon_1 = 2\omega.
$$
• $n = 2$ (four eigenstate)

\[
\phi_{2,1} = \tau_2^2 - \frac{36}{\omega} (1 + 14\nu) \tau_2 + \frac{252}{\omega^2} (1 + 14\nu)(1 + 18\nu) \quad , \quad \epsilon_{2,1} = 4\omega ,
\]

\[
\phi_{2,2} = \tau_6 + \frac{6}{\omega} (1 + 10\nu) \tau_2^2 - \frac{108}{\omega^2} (1 + 10\nu)(1 + 14\nu) \tau_2 + \frac{504}{\omega^3} (1 + 10\nu)(1 + 14\nu)(1 + 18\nu) \quad ,
\]

\[
\epsilon_{2,2} = 6\omega ,
\]

\[
\phi_{2,3} = \tau_8 - \frac{5}{\omega^2} (1 + 6\nu) \tau_6 - \frac{15}{\omega^2} (1 + 6\nu)(1 + 10\nu) \tau_2^2 + \frac{180}{\omega^3} (1 + 6\nu)(1 + 10\nu)(1 + 14\nu) \tau_2
\]

\[
- \frac{630}{\omega^4} (1 + 6\nu)(1 + 10\nu)(1 + 14\nu)(1 + 18\nu) \quad ,
\]

\[
\epsilon_{2,3} = 8\omega ,
\]

\[
\phi_{2,4} = \tau_{10} + \frac{1}{2\omega} (1 + 2\nu) \tau_8 - \frac{5}{4\omega^2} (1 + 2\nu)(1 + 6\nu) \tau_6 - \frac{5}{2\omega^3} (1 + 2\nu)(1 + 6\nu)(1 + 10\nu) \tau_2^2
\]

\[
+ \frac{45}{2\omega^4} (1 + 2\nu)(1 + 6\nu)(1 + 10\nu)(1 + 14\nu) \tau_2
\]

\[
- \frac{63}{\omega^5} (1 + 2\nu)(1 + 6\nu)(1 + 10\nu)(1 + 14\nu)(1 + 18\nu) \quad ,
\]

\[
\epsilon_{2,4} = 10\omega ,
\]

\[\]

D First eigenfunctions of the rational $E_8$ model

• $n = 0$ (one eigenstate)

\[
\phi_0 = 1 , \quad \epsilon_0 = 0 .
\]

• $n = 1$ (one eigenstate)

\[
\phi_1 = \tau_2 - \frac{8}{\omega} (1 + 30\nu) , \quad \epsilon_1 = 2\omega .
\]

• $n = 2$ (one eigenstate)

\[
\phi_2 = \tau_2^2 - \frac{20}{\omega} (1 + 24\nu) \tau_2 + \frac{80}{\omega^2} (1 + 24\nu)(1 + 30\nu) , \quad \epsilon_2 = 4\omega .
\]

47
• $n = 3$ (two eigenstates)

\[
\begin{align*}
\phi_{3,1} &= \tau_2^3 - \frac{36}{\omega} (1 + 20 \nu) \tau_2^2 + \frac{360}{\omega^2} (1 + 20 \nu)(1 + 24 \nu) \tau_2 - \frac{960}{\omega^3} (1 + 20 \nu)(1 + 24 \nu)(1 + 30 \nu), \\
\epsilon_{3,1} &= 6 \omega,
\end{align*}
\]

\[
\begin{align*}
\phi_{3,2} &= \tau_3 + \frac{21}{40 \omega} (1 + 18 \nu) \tau_2^3 - \frac{189}{20 \omega^2} (1 + 18 \nu)(1 + 20 \nu) \tau_2^2 + \\
&\frac{63}{\omega^3} (1 + 18 \nu)(1 + 20 \nu)(1 + 24 \nu) \tau_2 - \frac{126}{\omega^4} (1 + 18 \nu)(1 + 20 \nu)(1 + 24 \nu)(1 + 30 \nu), \\
\epsilon_{3,2} &= 8 \omega.
\end{align*}
\]

• $n = 4$ (two eigenstates)

\[
\begin{align*}
\phi_{4,1} &= \tau_2^4 - \frac{8}{\omega} (7 + 120 \nu) \tau_2^3 + \frac{144}{\omega^2} (1 + 20 \nu) (7 + 120 \nu) \tau_2^2 - \\
&\frac{960}{\omega^3} (1 + 20 \nu)(1 + 24 \nu)(7 + 120 \nu) \tau_2 + \frac{1920}{\omega^4} (1 + 20 \nu)(1 + 24 \nu)(1 + 30 \nu)(7 + 120 \nu), \\
\epsilon_{4,1} &= 8 \omega,
\end{align*}
\]

\[
\begin{align*}
\phi_{4,2} &= \tau_2 \tau_8 + \frac{21}{40 \omega} (1 + 18 \nu) \tau_2^3 - \frac{24}{\omega} (1 + 10 \nu) \tau_8 - \frac{21}{\omega^2} (1 + 15 \nu)(1 + 18 \nu) \tau_2^2 + \\
&\frac{252}{\omega^3} (1 + 15 \nu)(1 + 18 \nu)(1 + 20 \nu) \tau_2^2 - \frac{1260}{\omega^4} (1 + 15 \nu)(1 + 18 \nu)(1 + 20 \nu)(1 + 24 \nu) \tau_2 + \\
&\frac{2016}{\omega^5} (1 + 15 \nu)(1 + 18 \nu)(1 + 20 \nu)(1 + 24 \nu)(1 + 30 \nu), \\
\epsilon_{4,2} &= 10 \omega.
\end{align*}
\]
References

[1] M. A. Olshanetsky and A. M. Perelomov, “Quantum completely integrable systems connected with semi-simple Lie algebras”,
Lett. Math. Phys. 2 (1977) 7–13

[2] M. A. Olshanetsky and A. M. Perelomov, “Quantum integrable systems related to Lie algebras”,
Phys. Rep. 94 (1983) 313

[3] A. V. Turbiner, “Lie algebras and linear operators with invariant subspace”, in Lie algebras, cohomologies and new findings in quantum mechanics (N. Kamran and P. J. Olver, eds.), AMS Contemporary Mathematics, vol. 160, pp. 263–310, 1994;
funct-an/9301001
“Lie-algebras and Quasi-exactly-solvable Differential Equations”, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol.3: New Trends in Theoretical Developments and Computational Methods, Chapter 12, CRC Press (N. Ibragimov, ed.), pp. 331-366, 1995
hep-th/9409068

[4] W. Rühl and A. V. Turbiner, “Exact solvability of the Calogero and Sutherland models”,
Mod. Phys. Lett. A10 (1995) 2213–2222
hep-th/9506105

[5] L. Brink, A. Turbiner and N. Wyllard, “Hidden Algebras of the (super) Calogero and Sutherland models”,
Journ. Math. Phys. 39 (1998) 1285-1315
hep-th/9705219

[6] M. Rosenbaum, A. Turbiner and A. Capella, “Solvability of the $G_2$ integrable system”,
Intern. Journ.Mod.Phys. A13, (1998) 3885-3904
solv-int/9707005

[7] A. Turbiner, “Hidden Algebra of Three-Body Integrable Systems”,
Mod.Phys.Lett. A13 (1998) 1473-1483
solv-int/9805003

[8] K.G. Boreskov, J.C. Lopez V. and A.V. Turbiner,
"Solvability of $F_4$ integrable system",
Int.Journ.Mod.Phys. A16 (2001) 4769-4801
hep-th/0108021
[9] V.I. Arnold, “Wave front evolution and equivariant Morse lemma”,
Comm.Pure Appl. Math. 29 (1976) 557-582

[10] A.V. Turbiner, ”Lame Equation, sl(2) and Isospectral Deformation”,
Journ.Phys. A22 (1989) L1-L3

[11] D. Gomez-Ullate, A. González-López and M.A. Rodríguez, ”Exact solutions of a
new elliptic Calogero-Sutherland model”,
Phys.Lett.B 511 (2001) 112-118;
Y. Brihaye, B. Hartmann, ”Multiple algebraisations of an elliptic
Calogero-Sutherland model”,
J.Math.Phys. 44 (2003) 1576

[12] A.V. Turbiner,”Quantum many-body problems in Fock space: algebraic forms,
perturbation theory, finite-difference analogs”,
Plenary talk given at International Workshop on Mathematical Physics,
Mexico-City, Febr.17-20, 1999

[13] A.V. Turbiner, ”Quantum Many–Body Problems and Perturbation Theory”,
Russian Journ. of Nuclear Phys. 65(6), 1168-1176 (2002);
Physics of Atomic Nuclei 65(6), 1135-1143 (2002)
(English translation)
hep-th/0108160

[14] A.V. Turbiner, ”Perturbations of integrable systems and Dyson-Mehta integrals”,
Invited talk given at the Workshop ”Superintegrable systems in classical and
quantum mechanics”, Montreal, Canada, September, 2002;
Preprint ICN-UNAM 03-08 (july 2003), pp.17
to be published in Proceedings,
hep-th/0309109

[15] N. Bourbaki, in ”Groups et Algebras de Lie” (Hermann, Paris,1968) Chaps. IV–VI,
(V-5-4, prop.5)

[16] O. Haschke and W. Ruehl, “Is it possible to construct exactly solvable models ?”,
Lect.Notes Phys. 539 (2000) 118-140
hep-th/9809152

[17] J. Wolfes, “On the three-body linear problem with three-body interaction,”
J.Math.Phys. 15 (1974) 1420-1424

[18] A. M. Perelomov, “Algebraic approach to the solution of one-dimensional model of
N interacting particles,” Teor. Mat. Fiz. 6 (1971) 364 (in Russian);
English translation: Sov. Phys. – Theor. and Math. Phys. 6 (1971) 263.
[19] K.G. Boreskov, A.V. Turbiner and J.C. Lopez Vieyra
“Solvability of $E_6$ trigonometric integrable system” (in progress)

[20] O. Haschke and W. Ruehl, “The construction of trigonometric invariants for Weyl
groups and the derivation of corresponding exactly solvable Sutherland models ?”,
*Mod. Phys. Lett.* A14 (1999) 937-949
[math-ph/9904002]

[21] A.V. Turbiner, “Quasi-Exactly-Solvable Problems and the $SL(2, R)$ Group”,
*Comm.Math.Phys.* 118, 467-474 (1988)

[22] A. Minzoni, M. Rosenbaum and A. Turbiner, “Quasi-Exactly-Solvable Many-Body
Problems”,
*Mod. Phys. Lett.* A11 (1996) 1977-1984
[hep-th/9606092]

[23] “Quasi-Exactly Solvable Hamiltonians related to root spaces”,
*J.Nonlin.Math.Phys.* 12, 660-675 (2005) Supplement 1
*Special Issue in Honour of Francesco Calogero on the Occasion of His 70th Birthday*
[hep-th/0407204]