We study the diffusion and deformation of classical solitons coupled to thermal noise. The diffusion coefficient for kinks in the $\phi^4$ theory is predicted up to the second order in $kT$. The prediction is verified by numerical simulations. Multiskyrmions in the vector O(3) sigma model are studied within the same formalism. Thermal noise results in a diffusion on the multisoliton collective coordinate space (moduli space). There are entropic forces which tend, for example, to bind pairs of solitons into bi-solitonic molecules.

Solutions or extended objects are an important ingredient in many physical phenomena ranging from bioenergetics to superconductivity or nonlinear optics, see e.g. [1] for a review. Most of these phenomena take place at a finite temperature and involve dissipation and noise. Although it is sometimes possible to study transport properties of solitons starting from first principles [2], it seems that in most cases one has to rely on some kind of an effective diffusive nonlinear equation derived by expansion in derivatives of the order parameter, see e.g. [3] for an example.

In the following we develop a perturbative expansion in powers of temperature, which we use to predict the diffusion coefficient and the deformation of kinks at finite temperatures. Our predictions are confirmed by numerical simulations. Then we generalize the formalism to multisolitons in the planar vector O(3) sigma model. In the absence of thermal noise solitons of the pure sigma model do not interact with each other. Thus they provide a convenient setting for an "in vitro" study of noise induced entropic forces.

I. DIFFUSION OF KINKS

Let us consider a dissipative version of the $\phi^4$ theory in one spatial dimension, which is defined, in appropriate dimensionless units, by the stochastic nonlinear field equation

$$ \Gamma \partial_t \phi = \partial_x^2 \phi + 2[1 - \phi^2] \phi + \eta(t,x) , $$

(1)

where $\Gamma$ is a dissipation coefficient and $\eta(t,x)$ is a gaussian white noise with correlations $< \eta(t,x) >= 0$ ,

$$< \eta(t_1,x_1) \eta(t_2,x_2) >= 2kT \delta(t_1 - t_2) \delta(x_1 - x_2) .$$

(2)

The system is coupled to an ideal heat bath at temperature $T$. At nonzero temperature the field $\phi(t,x)$ performs a random walk in its configuration space. In the absence of noise, at $T = 0$, Eq.(1) admits static kink solutions $\phi(t,x) = F(x) \equiv \tanh(x)$. Antikinks are given by $F(-x)$.

**Spectrum of kink excitations.** Small perturbations around the kink take the form $e^{-\gamma t}/\Gamma u(x)$. Linearization of Eq.(1) with respect to $u(x)$, for $\eta(t,x) = 0$, gives

$$ \gamma u(x) = -\frac{d^2}{dx^2} u(x) + \left[4 - \frac{6}{\cosh^2(x)}\right] u(x) .$$

(3)

The eigenvalues and eigenstates can be tabulated as [3]

$$ \gamma, \quad u(x),$$

$$0, \quad F'(x) = \frac{-1}{\cosh^2(x)} ,$$

$$3, \quad B(x) = \frac{\sinh(x)}{\cosh^2(x)} ,$$

$$4 + k^2, \quad u_k(x) \equiv e^{ikx} \left[1 + \frac{3ik \tanh(x) - 3 \tanh^2(x)}{1 + k^2}\right] ,$$

(4)

where $k$ is a real momentum. The zero mode ($\gamma = 0$) is separated by a gap from the first excited state (breather mode) of $\gamma = 3$. The continuum states are normalized so that

$$ \int_{-\infty}^{+\infty} dx \ u_k^* (x) u_{k'} (x) = 2\pi \frac{4 + k^2}{1 + k^2} \delta(k - k') \equiv N(k) \delta(k - k') .$$

(5)

In the following we will often denote the states $\{4\}$ by $u_a(x)$ with the index $a = 0$ reserved for the zero mode.

**Collective coordinates.** The field in the one kink sector of the theory [4] can be expanded in the complete orthogonal basis $\{4\}$ as

$$ \phi(t,x) = F[x - \xi(t)] + \sum_{a \neq 0} A_a(t) \ u_a[x - \xi(t)] .$$

(6)

Substitution of the above to Eq.(1) and projection on the orthogonal basis $\{4\}$ gives a set of stochastic nonlinear differential equations.
\[ \Gamma N_0 \dot{\xi}(t) - \Gamma \xi(t) \sum_{a \neq 0} A_a(t) M_{a0} + \sum_{b,c \neq 0} A_b(t) A_c(t) P_{bc0} + \sum_{b,c,d \neq 0} A_b(t) A_c(t) A_d(t) R_{bcd0} = \eta_0(t) , \]
\[ \Gamma N_a \dot{\xi}_a(t) + \gamma_a N_a A_a(t) - \Gamma \xi(t) \sum_{b \neq 0} A_b(t) M_{ba} + \sum_{b,c \neq 0} A_b(t) A_c(t) P_{ba} + \sum_{b,c,d \neq 0} A_b(t) A_c(t) A_d(t) R_{bcda} = \eta_a(t) \]

where \( \dot{\cdot} \equiv d/dt \) and the coefficients are

\[ N_a = \int_{-\infty}^{+\infty} dx u_a(x) u^*_a(x) , \]
\[ M_{ab} = \int_{-\infty}^{+\infty} dx u'_a(x) u^*_b(x) , \]
\[ P_{abc} = 6 \int_{-\infty}^{+\infty} dx F(x) u_a(x) u_b(x) u^*_c(x) , \]
\[ R_{abcd} = 2 \int_{-\infty}^{+\infty} dx u_a(x) u_b(x) u_c(x) u_d^*(x) . \]

The \( \eta_a \)'s result from the projections
\[ \eta_a(t) = \int_{-\infty}^{+\infty} dx \eta(t,x) u^*_a(x) . \]

With the correlations (8) and the orthogonality of the basis (9) the correlations of the projected noises are

\[ < \eta_a(t) > = 0 , \]
\[ < \eta_a(t_1) \eta_b(t_2) > = 2kT N_a \delta_{ab} \delta(t_1 - t_2) . \]

**Diffusion coefficient to leading order in kT.**

The natural parameter of expansion at low temperature is \( \sqrt{kT} \). The projected noises, compare Eq. (8), are just of this order. Let us make the customary rescaling \( \sqrt{kT} \rightarrow \varepsilon \sqrt{kT} \). Eqs. (9) can be expanded in powers of \( \varepsilon \) and solved in power series of \( \varepsilon \) with \( \varepsilon \) set to 1 at the end of the calculation. With the expansion of the collective coordinates, \( \xi = \varepsilon \xi^{(1)} + \varepsilon^2 \xi^{(2)} + \ldots \) and \( A_a = \varepsilon A_a^{(1)} + \varepsilon^2 A_a^{(2)} + \ldots \), the equations (9) become, to the leading order in \( \varepsilon \),

\[ \Gamma N_0 \dot{\xi}^{(1)}(t) = \eta_0(t) , \]
\[ \Gamma N_a A_a^{(1)}(t) + \gamma_a N_a A_a^{(1)}(t) = \eta_a(t) . \]

In this approximation the zero mode \( \xi^{(1)}(t) \) and the excited modes \( A_a^{(1)}(t) \) are uncorrelated stochastic processes driven, respectively, by their mutually uncorrelated projected noises.

\( \xi^{(1)}(t) \) is a Markovian Wiener process whose only nonvanishing single connected correlation function is

\[ < \dot{\xi}^{(1)}(t) \dot{\xi}^{(1)}(t') > = \frac{2kT}{\Gamma N_0} \delta(t - t') , \]

which is singular for \( t \rightarrow t' \). According to the first of Eqs. (11), the probability \( P(t, \xi) \), that the kink random walks to \( \xi \) at the time \( t \), satisfies the diffusion equation

\[ \frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial \xi^2} , \]

with the diffusion coefficient \( D = kT/\Gamma N_0 = 3kT/4\Gamma \).

\( \eta_0 \)'s are Ornstein-Uhlenbeck random noises with relaxation times \( \Gamma/\gamma_a \). They represent memory effects. If, for \( t < 0, T = 0 \) and then the system is put in contact with a heat bath of temperature \( T > 0 \), for \( t > 0 \) the correlations of \( A^{(1)} \)'s would grow according to

\[ < [A_a^{(1)}(t)]^* A_b^{(1)}(t') > = \delta_{ab} \frac{kT e^{-\gamma_a |t-t'|}}{N_a \gamma_a} [1 - e^{-\gamma_a |t-t'|}] , \]

as can be deduced from a formal solution of the second of Eqs. (11). All other single connected correlations of \( A^{(1)} \)'s vanish. A given mode achieves the state of equilibrium with the heat bath after its characteristic relaxation time. The longest relaxation time is that of the breather mode.

**The next to leading order correction.**

Further terms in the expansion of collective coordinates can be recursively worked out as

\[ \dot{\xi}^{(2)}(t) = \dot{\xi}^{(1)}(t) \sum_{a \neq 0} A_a^{(1)}(t) \frac{M_{a0}}{\gamma_0} - \sum_{b,c \neq 0} A_b^{(1)}(t) A_c^{(1)}(t) \frac{P_{bc0}}{\gamma_0} , \]
\[ A_a^{(2)}(t) = \sum_{b \neq 0} \frac{M_{ba}}{N_a} \int_0^t dt \ e^{-\gamma_a(t-\tau)} \dot{\xi}^{(1)}(\tau) A_b^{(1)}(\tau) - \sum_{b,c \neq 0} \frac{P_{bc}}{\gamma_0 N_a} \int_0^t dt \ e^{-\gamma_a(t-\tau)} A_b^{(1)}(\tau) A_c^{(1)}(\tau) , \]
\[ \dot{\xi}^{(3)}(t) = \sum_{a \neq 0} [\dot{\xi}^{(1)}(t) A_a^{(2)}(t) + (1 \leftrightarrow 2)] \frac{M_{a0}}{\gamma_0} - \sum_{b,c \neq 0} [A_b^{(1)}(t) A_c^{(2)}(t) + (1 \leftrightarrow 2)] \frac{P_{bc0}}{\gamma_0} - \sum_{b,c,d \neq 0} A_b^{(1)}(t) A_c^{(2)}(t) A_d^{(1)}(t) \frac{R_{bcd0}}{\gamma_0} . \]

In the thermodynamic equilibrium the excited modes develop nonzero expectation values

\[ < A_a^{(2)}(\infty) > = - \frac{kT}{\gamma_a N_a} \left[ \frac{1}{2} P_{BBa} + \int_{-\infty}^{+\infty} dk \ \frac{P_{k-k,a}}{\gamma(k)N(k)} \right] , \]

where the index “\( B \)” stands for the breather mode. In particular the expectation value of the breather mode is negative,

\[ < A_B^{(2)}(\infty) > = - \frac{kT e^{\omega_0}}{\gamma_0} . \]

The kink gets thicker due to the noise. At the same time the expectation value of \( \phi(x, t) \), far from the kink, gets smaller,

\[ < \phi(\infty, \infty) > = 1 - \frac{\delta T}{\delta} + O((kT)^2) . \]
The equilibrium correlations of $\dot{\xi}$, up to the next to the leading order, are
\[
< \dot{\xi}(t)\dot{\xi}(t') > \approx \varepsilon^2 < \dot{\xi}^{(1)}(t)\dot{\xi}^{(1)}(t') > +
\varepsilon^4 < \dot{\xi}^{(2)}(t)\dot{\xi}^{(2)}(t') > + \varepsilon^4 [< \dot{\xi}^{(1)}(t)\dot{\xi}^{(3)}(t') > + (1 \leftrightarrow 3)] = < \dot{\xi}^{(1)}(t)\dot{\xi}^{(1)}(t') > \times
[1 + \frac{3kT}{N_0} \sum_{a \neq 0} |M_{a0}|^2 + \frac{2}{N_0} \sum_{a \neq 0} M_{a0} < A^{(2)}_a(\infty) > + \ldots],
\]

where $\ldots$ denotes terms which are regular as $t \to t'$. In the last equality we have already set $\varepsilon = 1$. The integrals over the continuous part of the spectrum and the summation over the breather mode yielded $< \dot{\xi}(t)\dot{\xi}(t') > = \delta(t - t') \frac{3kT}{N_0} [1 + kT \ 1.8164] + O([kT]^3)$. The diffusion coefficient in the equilibrium state, understood as an average over times much longer than the relaxation time of the breather mode, has been found to be
\[
D(\infty) = \frac{3kT}{4T}[1 + kT \ 1.8164] + O(kT^3).
\]

**Numerical experiment.** The projection of the expansion into $F'(x - y)$,
\[
I(t, y) = \int_{-\infty}^{\infty} dx \ \phi(t, x)F'(x - y),
\]

vanishes for $y = \xi(t)$. At moderate temperatures (small $A$'s) $I(t, y) > 0$ for $y > \xi(t)$ and $I(t, y) < 0$ for $y < \xi(t)$. Thus $\xi(t)$ can be defined as the solution of $I[t, \xi(t)] = 0$. This definition was introduced in Ref. [8]; the coordinate defined in this way is a position the kink would relax to if the noise were suddenly switched off.

We performed numerical simulations of Eq. [9] with $\Gamma = 1$ for a range of temperatures. Fig.1 compares numerical results with the perturbative prediction [18].

In principle, for some periods of time, the solution of $I[t, \xi(t)] = 0$ may happen to be not unique even for arbitrarily small $T > 0$. There is always small but finite density of thermally activated kink-antikink pairs. Even if this happens for a given realization of the stochastic process, one can still find two times $t_1 < t_2$ such that the solution is unique and choose at least one solution $\xi(t)$ valid for any $t_1 < t < t_2$; due to topology the equation has at least one solution for any $t$. The ambiguity manifests itself in the first of Eqs. [10] in the coefficient of $\dot{\xi}$, namely $\Gamma [N_0 - \sum_{a \neq 0} A_a(t) M_{a0}]$. The coefficient vanishes and leads to ambiguity, iff $I'[t, \xi(t)] = 0$ or a few zeros meet at $\xi(t)$. At such a point, the trajectory splits into several branches or a few branches merge into one. The perturbative expansion overcomes this problem, it gives an asymptotic approximation to the diffusion coefficient of the chosen line $\xi(t)$.

### II. Diffusion of multisolitons.

A generalization of the ideas developed for a kink to a soliton in higher dimensions is straightforward. Less trivial is the situation with many solitons, which can overlap or pass through each other. Let us consider a dissipative version of the $O(3)$ sigma model
\[
\Gamma \partial_t \hat{M} = \hat{\Pi}_{\hat{M}} \nabla^2 \hat{M} + \eta(t, \vec{x}) \quad ,
\]

where $\hat{\Pi}_{\hat{M}} \vec{A} = \vec{A} - \hat{M} \vec{\hat{M}} \vec{A}$ for any vector $\vec{A}$ and the magnetization $\hat{M}$ is subject to the constraint $\hat{M} \hat{M} = 1$. The vector noise components are white noises
\[
< \eta^k(t, \vec{x}) > = 0,
\]
\[
< \eta^k(t_1, \vec{x}_1)\eta^l(t_2, \vec{x}_2) >= 2kT \ \Gamma \ \delta^{kl} \delta(t_1 - t_2) \ \delta(\vec{x}_1 - \vec{x}_2) \quad ,
\]

for any $k, l = 1, 2, 3$.

For a positive topological index $n$, in the absence of noise, the solution of Eq. [24] relaxes to one out of the continuous family of degenerate ground states [11].

\[
\hat{M}_0(\vec{x}, c, a_1, b_1) = \left( W + W^* \right) \left( W - W^* \right) \left( 1 - |W|^2 \right) \frac{1 + |W|^2}{1 + |W|^2} ,
\]

\[
W = \frac{(z - a_1)\ldots(z - a_n)}{(z - b_1)\ldots(z - b_n)}
\]

where $\vec{A}$, $c$, $a_i$, $b_i$ and $z = x_1 - i x_2$. There are $(4n + 2)$ real parameters. Any of them can be removed by fixing the boundary conditions at planar infinity and the remaining global $U(1)$ symmetry [8]. After this reduction there are $(4n - 1)$ real parameters left, they parametrize a $(4n - 1)$-dimensional real manifold $M_n$, which is called the moduli space [8].

We will denote the real parameters by $\xi^\alpha$ with $\alpha = 1, \ldots, (4n - 1)$. The static solution $\hat{M}_0(\vec{x}, \xi)$ has $(4n - 1)$ independent zero modes $\partial \hat{M}_0(\vec{x}, \xi) / \partial \xi^\alpha$.

As for kinks, compare Eq. [11], the magnetization can be expanded in excited states
\[
\hat{M}(t, \vec{x}) = \hat{M}_0(\vec{x}, \xi(t)) \sqrt{1 - K^2} + K ,
\]

where
\[
K = \sum_{a \neq 0} A_a(t) \vec{u}_a(\vec{x}, \xi(t)) .
\]

Note that $K \cdot \hat{M}_0 = 0$. The sigma model analogue of the first of Eqs. [13] is
\[
\Gamma \ g_{\alpha\beta}(\xi) \ \dot{\xi}^\beta = \eta_\alpha(t, \xi) .
\]

where $g_{\alpha\beta}$ is a metric tensor on the moduli space
\[
g_{\alpha\beta}(\xi) = \int d^2x \ \frac{\partial \hat{M}_0(\vec{x}, \xi)}{\partial \xi^\alpha} \frac{\partial \hat{M}_0(\vec{x}, \xi)}{\partial \xi^\beta} .
\]

Time correlations of the projected noises
\[ \eta_\alpha(t, \xi) = \int d^2x \, \bar{\eta}(t, \bar{x}) \frac{\partial M_\alpha(\bar{x}, \xi)}{\partial \xi^\alpha} \]  
\hspace{1cm} (27)

can be obtained from the correlations \[^{23}\] as

\[ < \eta_\alpha(t, \xi) >= 0 \]
\[ < \eta_\alpha(t_1, \xi) \eta_\beta(t_2, \xi) >= 2kT \Gamma g_{\alpha\beta}(\xi) \delta(t_1 - t_2) \]  
\hspace{1cm} (28)

The equation (23) and the correlations (28) lead to the following diffusion equation on the moduli space

\[ \frac{\partial P(t, \xi)}{\partial t} = D_\sigma \frac{\partial}{\partial \xi^\alpha} \sqrt{g(\xi)} \frac{\partial}{\partial \xi^\beta} P(t, \xi) \]  
\hspace{1cm} (29)

\[ g(\xi) = \det[g_{\alpha\beta}(\xi)], \quad g^{\alpha\beta} = (g^{-1})_{\alpha\beta} \quad \text{and} \quad D_\sigma = kT/\Gamma \]

is a diffusion coefficient. \( P(t, \xi) \) is the probability of finding the multisoliton at the point \( \xi \) on the moduli space, such that the average of any function \( F(\xi) \) is given by \( < F(\xi) >= \int_{M_2} d\xi \, P(t, \xi) F(\xi) \). If \( \xi \) are restricted to a compact area of the moduli space, the probability would finally relax to the equilibrium distribution \( P_{eq}(\xi) \propto \sqrt{g(\xi)} \). Interpreting this distribution as a Boltzmann distribution, we can identify the entropic potential on the moduli space as

\[ -TS(\xi) = -kT \ln \sqrt{g(\xi)} \]  
\hspace{1cm} (30)

which is not quite unexpected as \( \sqrt{g(\xi)} \) is a surface element on \( M_2 \).

**Example.** To give a clear but nontrivial example of entropic forces, let us restrict our attention to the subspace of \( M_2 \) defined by \( W = \frac{\beta^2}{2} \) with real \( A, B \). This space represents states of two solitons of the overall size \( B \). For \( A > 0 \) \( (A < 0) \) they are located on the real (imaginary) axis at the points \( \pm A \) \(( \pm iA) \). The sigma model is scale invariant, so it is not surprising that \( g(A, B) = g(A/B) \), compare Eqs. (10-12) in Ref. 8.

\[ \sqrt{g(A/B)} = \sqrt[4]{8 \left( \frac{E}{2} (K - \frac{E}{2}) - \left( \frac{K - E}{2} \tan \kappa \right)^2 \right) \sin 2\kappa} \]  
\hspace{1cm} (31)

where \( \kappa = \arctan(B/A) \) and \( K = K[\cos(\kappa)], \) \( E = E[\cos(\kappa)] \) are complete elliptic integrals of the first and second kind respectively. The function \( \sqrt{g} \) has a unique global maximum at \( A/B \approx 1.26 \), compare Fig.2. For a definite scale \( B \), which can be fixed by residual interactions inducing an effective potential on the moduli space \( 8 \), such as \( L - S \) coupling, Zeeman energy and a higher derivative (Skyrme) term, the entropic forces tend to bind the two solitons \( Sk \) into a molecule \( Sk_2 \). At short distance there is a logarithmic repulsion.

**CONCLUSIONS.**

The numerical simulations confirm our method of expansion in powers of \( \sqrt{kT} \), when applied to kinks. We believe the same expansion is valid for multisolitons in the sigma model. Even in the pure sigma model, in which the solitons do not interact by potential forces, the thermal noise induces entropic forces. In particular, two solitons attract each other at large distances and repel at short distances, and so the entropic forces tend to bind them into bi-solitonic molecules.

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\[^{1}\] A. S. Davydov, "Solitons in Molecular Systems", Kluwer Academic Publishers, Dordrecht-Boston-London 1990.
\[^{2}\] A. H. Castro-Neto and A. O. Caldeira, Phys.Rev. E48, 4037 (1993).
\[^{3}\] E. Abrahams and T. Tsuneto, Phys.Rev. 152, 416 (1966).
\[^{4}\] R. Jackiw, Rev.Mod.Phys. 49, 681 (1977).
\[^{5}\] J. Dziarmaga and W. Zakrzewski, cond-mat/9706268
\[^{6}\] A. A. Belavin and A. M. Polyakov, JETP Lett. 22, 245 (1975).
\[^{7}\] N. S. Manton, Phys.Lett. B 154, 397 (1985).
\[^{8}\] R. S. Ward, Phys.Lett. B 158, 424 (1985).
\[^{9}\] J. Dziarmaga, Phys.Rev.Lett. 79 (1997) 2129.
FIGURE 2. The probability distribution (31) to find two skyrmions separated by the distance A as a function of $A/B$. 