From One-Component KP Hierarchy
to Two-Component KP Hierarchy and Back

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Abstract

We show that the system of the standard one-component KP hierarchy endowed
with a special infinite set of abelian additional symmetries, generated by squared
eigenfunction potentials, is equivalent to the two-component KP hierarchy.

Background Information on the KP Hierarchy and Ghosts
Symmetries.

The starting point of our presentation is the pseudo-differential Lax operator $\mathcal{L}$ obeying
KP evolution equations w.r.t. the multi-time $(t) \equiv (t_1, t_2, \ldots)$ :

$$
\mathcal{L} = D + \sum_{i=1}^{\infty} u_i D^{-i} ; \quad \frac{\partial \mathcal{L}}{\partial t_l} = \left[ \left( \mathcal{L}^l \right)_+, \mathcal{L} \right] , \quad l = 1, 2, \ldots
$$

(1)

The symbol $D$ stands for the differential operator $\partial/\partial x$, whereas $\partial \equiv \partial_x$ will denote
derivative of a function. Equivalently, one can represent Eq.(1) in terms of the dressing
operator $W$ whose pseudo-differential series are expressed in terms of the so called tau-
function $\tau(t)$ :

$$
\mathcal{L} = W D W^{-1} , \quad \frac{\partial W}{\partial t_l} = - \left( \mathcal{L}^l \right)_- W , \quad W = \sum_{n=0}^{\infty} p_n \left( \frac{-[\partial]}{\tau(t)} \right) D^{-n}
$$

(2)

with the notation: $[y] \equiv (y_1, y_2/2, y_3/3, \ldots)$ for any multi-variable $(y) \equiv (y_1, y_2, y_3, \ldots)$
and with $p_k(y)$ being the Schur polynomials. In the present approach a basic notion is
that of (adjoint) eigenfunctions $\Phi(t)$, $\Psi(t)$ of the KP hierarchy satisfying:

$$\frac{\partial \Phi}{\partial t_k} = L^k_+ (\Phi) \quad ; \quad \frac{\partial \Psi}{\partial t_k} = - (L^*)^k_- (\Psi)$$

(3)

The Baker-Akhiezer (BA) “wave” functions $\psi_{BA}(t, \lambda) = W(\exp(\xi(t, \lambda)))$ and its adjoint

$\psi^*_{BA}(t, \lambda) = (W^*)^{-1}(\exp(-\xi(t, \lambda)))$ (with $\xi(t, \lambda) \equiv \sum_{i=1}^{\infty} t_i \lambda^i$) are (adjoint) eigenfunctions satisfying additionally the spectral equations $L^{(s)} (\psi_{BA}^{(s)}(t, \lambda)) = \lambda \psi_{BA}^{(s)}(t, \lambda)$.

Throughout this note we will rely on an important tool provided by the spectral representation of eigenfunctions $\[\Pi\]$. The spectral representation is equivalent to the following statement. $\Phi$ and $\Psi$ are (adjoint) eigenfunctions if and only if they obey the integral representation:

$$\Phi(t) = \int dz \frac{e^{\xi(t', z)}}{z} \frac{\tau(t - [z^{-1}]) \tau(t + [z^{-1}])}{\tau(t) \tau(t')} \Phi(t' + [z^{-1}])$$

(4)

$$\Psi(t) = \int dz \frac{e^{\xi(t', z)}}{z} \frac{\tau(t + [z^{-1}]) \tau(t' - [z^{-1}])}{\tau(t) \tau(t')} \Psi(t' - [z^{-1}])$$

(5)

where $\int dz$ denotes contour integral around origin.

One needs to point out that the proper understanding of Eqs.(3) and (4) (as in the case of original Hirota bilinear identities) requires, following $\[3\]$, expanding of the integrand in $\[2\]$ and $\[6\]$ as formal power series w.r.t. in $t_n'$, $n = 1, 2, \ldots$.

Consider now an infinite system of independent (adjoint) eigenfunctions $\{\Phi_j, \Psi_j\}_{j=1}^{\infty}$ of the standard KP hierarchy Lax operator $L$ and define the following infinite set of the additional “ghost” symmetry flows $\[3\]:

$$\frac{\partial}{\partial t_s} L = \left[ M_s, L \right] \quad , \quad M_s = \sum_{j=1}^{s} \Phi_{s-j+1} D^{-1} \Psi_j$$

(6)

$$\frac{\partial}{\partial t_s} \Phi_k = \sum_{j=1}^{s} \Phi_{s-j+1} S_{k,j} - \Phi_{k+s} \quad , \quad \frac{\partial}{\partial t_s} \Psi_k = \sum_{j=1}^{s} \Psi_j S_{s-j+1,k} + \Psi_{k+s}$$

(7)

$$\frac{\partial}{\partial t_s} F = \sum_{j=1}^{s} \Phi_{s-j+1} S (F \Psi_j) \quad ; \quad \frac{\partial}{\partial t_s} F^* = \sum_{j=1}^{s} \Psi_j S (\Phi_{s-j+1} F^*)$$

(8)

where $s, k = 1, 2, \ldots$, and $F$ ($F^*$) denote generic (adjoint) eigenfunctions which do not belong to the “ghost” symmetry generating set $\{\Phi_j, \Psi_j\}_{j=1}^{\infty}$. Moreover, we used abbreviations $S_{k,j} \equiv S(\Phi_k \Psi_j) = \partial^{-1} (\Phi_k \Psi_j)$ to denote the so called squared eigenfunction potentials (SEP) $\{s, s\}$ for which we find the “ghost” symmetry flows:

$$\frac{\partial}{\partial t_s} S_{k,l} = S_{k,l+s} - S_{k+s,l} + \sum_{j=1}^{s} S_{k,j} S_{s-j+1,l}$$

(9)

Eqs.(8) become for the first “ghost” symmetry flow $\tilde{\partial} \equiv \partial/\partial t_1$:

$$\tilde{\partial} \Phi_k = \Phi_1 S_{k,1} - P_{k+1} \quad , \quad \tilde{\partial} \Psi_k = \Psi_1 S_{1,k} + \Psi_{k+1} \quad ; \quad \tilde{\partial} F = \Phi_1 \partial^{-1} (\Psi_1 F)$$

(10)

It is easy to show that the “ghost” symmetry flows $\tilde{\partial}/\tilde{\partial} t_s$ from Eqs.(9)–(10) commute. This can be done by proving that the $\tilde{\partial}$-pseudo-differential operators $M_s (s)$ satisfy the zero-curvature equations $\partial M_r/\partial t_s - \partial M_s/\partial t_r - [M_s, M_r] = 0$. 

2
Lax Representation for the Ghosts Flows

We now show that the “ghost” symmetry flows from Eqs. (6)-(8) admit their own Lax representation in terms of the pseudo-differential Lax operator \( \mathcal{L} \) w.r.t. multi-time \( (\bar{t}) \equiv (\bar{t}_1 \equiv \bar{x}, \bar{t}_2, \ldots) \). While showing it we will make contact with the structure defining the affine coordinates on the Universal Grassmannian Manifold (UGM) [5, 6]. For this purpose we define objects:

\[
\bar{w}_i = \Phi_i + 1 \Phi_1^i ; \quad i = 0, 1, 2, \ldots
\]

which can be grouped together into the Laurent series expansion:

\[
\bar{w} = \sum_{i=0}^{\infty} \frac{\Phi_{i+1}}{\Phi_1} z^{-i} = 1 + \sum_{i=1}^{\infty} \frac{\Phi_{i+1}}{\Phi_1} z^{-i}
\]

From Eq.(8) we find that the action of the “ghost” symmetry flows on \( \bar{w}_i \) takes a form:

\[
\frac{\partial}{\partial \bar{t}_s} \bar{w}_k = -\bar{w}_k + \sum_{l=0}^{s} \bar{w}_l W_k^{(s-l)}
\]

where the coefficients \( W_k^{(j)} \) are given in terms of SEP-functions as

\[
W_k^{(j)} = S_{k+1,j} - \bar{w}_k S_{1,j} ; \quad j = 1, 2, \ldots, k = 0, 1, 2, \ldots
\]

\[
W_k^{(0)} = \bar{w}_k = \frac{\Phi_{k+1}}{\Phi_1}
\]

They in turn satisfy the following flow equations resulting from those in (9):

\[
\frac{\partial}{\partial \bar{t}_s} W_k^{(j)} = W_k^{(j+s)} - W_k^{(j)} + \sum_{l=1}^{s} W_l^{(s-l)} W_k^{(j)}
\]

which provide an example of the matrix Riccati equations (see e.g. [8]). The coefficients \( W_k^{(j)} \) span the Laurent series:

\[
W^{(j)} = z^j + \sum_{l=1}^{\infty} W_l^{(j)} z^{-l} ; \quad j = 0, 1, 2, \ldots
\]

whose structure is reminiscent of the Laurent series defining the Sato Grassmannian. The connection to the usual KP setup can now be established as follows. We first introduce the well-known notion of a long derivative :

\[
\nabla_s = \frac{\partial}{\partial \bar{t}_s} + z^s = e^{-\xi(\bar{t},z)} \frac{\partial}{\partial \bar{t}_s} e^{\xi(\bar{t},z)}
\]

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which together with Eq.(17) allow us to cast both Eqs.(16) and (13) in a more compact form:

\[ \nabla_s \bar{w} = \sum_{l=0}^{s} \bar{w}_l \bar{W}^{(s-l)} \quad (19) \]

\[ \nabla_s \bar{W}^{(j)} = \bar{W}^{(j+s)} + \sum_{l=1}^{s} \bar{W}_l \bar{W}^{(s-l)} ; \quad j = 1, 2, \ldots \quad (20) \]

From Eq.(20) we obtain recursive expressions for \( \bar{W}^{(j)} \) with \( j > 0 \) in terms of non-negative powers of \( \nabla_1 \) acting on \( \bar{w} \). Indeed, from (20) with \( j = 0 \) one finds \( \bar{W}^{(1)} = \nabla_1 \bar{w} - \bar{w}_1 \bar{w} \) and so on. Finally, by increasing \( j \) one arrives at expansion \( \bar{W}^{(j)} = \sum_{l=0}^{j} \bar{w}_l \nabla_1^{j-l} \bar{w} \), which allows to rewrite Eq.(19) as \( \nabla_s \bar{w} = \sum_{l=0}^{s} \bar{w}_l \nabla_1^{l} \bar{w} \) with some coefficients \( \bar{w}_l \) and \( \bar{W}^{(j)} \). Using Eq.(18) we obtain the standard evolution equation \( \partial \bar{\psi}_{BA}(t, z) / \partial t_s = \mathcal{B}_s \bar{\psi}_{BA}(t, z) \) with \( \mathcal{B}_s = \sum_{l=0}^{s} \bar{w}_l \nabla_1^{l} \bar{w} \) and the wave-function:

\[ \bar{\psi}_{BA}(t, z) = \bar{w} e^{\xi(t, z)} = \bar{W} e^{\xi(t, z)} ; \quad \bar{W} = 1 + \sum_{l=1}^{\infty} \frac{\Phi_{l+1}}{\Phi_1} \bar{D}^l \quad (21) \]

with \( \bar{D} \equiv \partial / \partial \bar{t}_1 \). The evolution operators \( \mathcal{B}_s \) satisfy \( \partial \bar{W} / \partial t_s = \mathcal{B}_s \bar{W} - \bar{W} \bar{D}^s \) and are, therefore, reproduced by the usual relation \( \mathcal{B}_s = \left( \bar{W} \bar{D}^s \bar{W}^{-1} \right)_+ \).

The standard KP Lax operator construction follows now upon defining the \( \bar{\partial} \)-Lax operator \( \bar{\mathcal{L}} = \bar{W} \bar{D} \bar{W}^{-1} = \bar{D} + \sum_{i=1}^{\infty} \bar{w}_i \bar{D}^{-i} \), which enters the hierarchy equations \( \partial \bar{\mathcal{L}} / \partial t_s = \left[ \bar{\mathcal{L}}_+, \bar{\mathcal{L}} \right] \) with \( \bar{\mathcal{B}}_s = \bar{\mathcal{L}}_s^+ \). In this way we arrive at a new integrable system defined in terms of two Lax operators \( \mathcal{L} \) and \( \bar{\mathcal{L}} \) with two different sets of evolution parameters \( t \) and \( \bar{t} \) which we will call double KP system. The double KP system can be viewed as ordinary one-component KP hierarchy Eq.(2) supplemented by infinite-dimensional additional symmetry structure given by Eqs.(1)-3).

Let \( \bar{\tau}(t, \bar{t}) \) be a tau-function associated with the \( \bar{\partial} \)-Lax operator \( \bar{\mathcal{L}} \), then the following results follow from the above discussion:

\[ \bar{\tau}(t, \bar{t}) = \Phi_1(t, \bar{t}) \tau(t, \bar{t}) \quad , \quad \Phi_s \left( \bar{\mathcal{L}} \right) \bar{\tau} = \Phi_{s+1} \Phi_1 ; \quad s = 0, 1, 2, \ldots \quad (22) \]

where the \( \tau(t, \bar{t}) \) is tau-function of the original \( \partial \)-Lax operator \( \mathcal{L} \). Moreover, for any generic eigenfunction \( F \) of \( \mathcal{L} \), which does not belong to the set \{ \Phi \} in (1) and has “ghost” symmetry flows given by Eq.(6), the function \( F = \Phi_1 \) is automatically an eigenfunction of the “ghost” Lax operator \( \bar{\mathcal{L}} \):

\[ \frac{\partial}{\partial t_s} (F / \Phi_1) = \bar{\mathcal{L}}_s^+ (F / \Phi_1) \quad (23) \]

We will also introduce the Darboux-Bäcklund (DB) transformations:

\[ \bar{\mathcal{L}}(n+1) = \left( \frac{1}{\Phi_1^{(n+1)}} \bar{D}^{-1} \Phi_1^{(n+1)} \right) \bar{\mathcal{L}}(n) \left( \frac{1}{\Phi_1^{(n+1)}} \bar{D} \Phi_1^{(n+1)} \right) \quad (24) \]
for the “ghost” KP Lax operator which have an additional property of commuting with the “ghost” symmetries (6). In the Eq.(24) the the DB “site” index \( n \) parametrizes the DB orbit. The convention we adopt is that the index \( n \) labels the particular \( \bar{\partial} \)-Lax operator \( \bar{\mathcal{L}} \) constructed above. In terms of the original isospectral flows the DB transformations take a form:

\[
\mathcal{L}(n+1) = \left( \Phi_1^{(n)} D \Phi_1^{(n)} \right) \mathcal{L}(n) \left( \Phi_1^{(n)} D^{-1} \Phi_1^{(n)} \right)^{-1}
\]

(25)

where \( \mathcal{L}(n) \) is the original Lax operator underlying our construction. In this setting the tau-function \( \bar{\tau} \) appears, according to Eq.(22), to be nothing but the tau-function associated with the Lax operator \( \mathcal{L}(n+1) \) at the site \( n+1 \), namely \( \bar{\tau}_{n+1} = \tau(n+1) \).

We can now present results for the adjoint eigenfunctions \( \Psi_i \) which parallel those in Eqs.(22)-(23) for the eigenfunctions \( \Phi_i \). Defining \( \bar{\omega}^*_k = \Psi_{k+1}/\Psi_1 \) we find that:

\[
\frac{\partial}{\partial \bar{t}} \bar{\omega}^*_k = \bar{\omega}^*_{k+1} + \sum_{l=0}^{\infty} \bar{\omega}^*_l \mathcal{W}^*_{k+1-l} \quad (s=0,1,2,\ldots)
\]

(26)

with the coefficients:

\[
\mathcal{W}^*_{(j)} = S_{j,k+1} - \bar{\omega}^*_k S_{j,1} \quad ; \quad j = 1, 2, \ldots, k = 0, 1, 2, \ldots
\]

(27)

\[
\mathcal{W}^*_{(0)} = \bar{\omega}^*_k = \Psi_{k+1}/\Psi_1
\]

(28)

Let \( \bar{\tau}(t, \bar{t}) \) be a tau-function associated with the \( \bar{\partial} \)-Lax operator \( \bar{\mathcal{L}}(n-2) \) at the DB site \( n-2 \). Then the following results can be shown:

\[
\bar{\tau}(t, \bar{t}) = \Psi_1(t, \bar{t}) \tau(t, \bar{t}) \quad , \quad \frac{p_j}{\bar{\tau}} \left[ \bar{\partial} \right] \bar{\tau} = \frac{\Psi_{s+1}}{\Psi_1} \quad ; \quad s = 0, 1, 2, \ldots
\]

(29)

where \( \tau(t, \bar{t}) \) is the tau-function of the original \( \partial \)-Lax operator \( \mathcal{L} \) (at the DB site \( n \)). Moreover, for any generic adjoint eigenfunction \( F^* \) of \( \mathcal{L} \), which does not belong to the set \( \{ \Psi_j \} \) in (8) and satisfies, therefore, the “ghost” symmetry flows given by Eq.(8), the function \( F^*/\Psi_1 \) is an adjoint eigenfunction of the “ghost” Lax operator \( \mathcal{L}(n-2) \):

\[
\frac{\partial}{\partial t} \left( F^*/\Psi_1 \right) = - \left( \mathcal{L}^*(n-2) \right)^n \left( F^*/\Psi_1 \right)
\]

(30)

Let us list two other important identities which relate the tau-function \( \tau \) to the SEP-functions (using notation of Eqs.(6)-(8)):

\[
\frac{p_j}{\tau} \left[ \bar{\partial} \right] \tau = -S_{1,j} \quad ; \quad \frac{p_j}{\tau} \left[ -\bar{\partial} \right] \tau = S_{j,1} \quad ; \quad j \geq 1
\]

(31)

**Embedding of Double KP System into Two-Component KP Hierarchy**

The two-component KP hierarchy [7] is given by three tau-functions \( \tau_{11}, \tau_{12}, \tau_{21} \) depending on two sets of multi-time variables \( t, \bar{t} \) and obeying the following Hirota bilinear
identities:

\[ \int dz \frac{e^{\xi(t'-z)}}{z^2} \tau_{12}(t, \bar{t} - [z^{-1}], \bar{t} + [z^{-1}]) = \int dz e^{\xi(t'-z)} \tau_{11}(t - [z^{-1}], \bar{t}) \tau_{11}(t' + [z^{-1}], \bar{t}') = \]

\[ \int dz e^{\xi(t'-z)} \tau_{11}(t - [z^{-1}], \bar{t}) \tau_{12}(t' + [z^{-1}], \bar{t}') = \int dz e^{\xi(t'-z)} \tau_{12}(t, \bar{t} - [z^{-1}]) \tau_{11}(t', \bar{t}' + [z^{-1}]) = \]

\[ \int dz e^{\xi(t'-z)} \tau_{12}(t, \bar{t} - [z^{-1}]) \tau_{21}(t', \bar{t}' + [z^{-1}]) = \int dz e^{\xi(t'-z)} \tau_{21}(t - [z^{-1}], \bar{t}) \tau_{11}(t' + [z^{-1}], \bar{t}') = \]

\[ \int dz e^{\xi(t'-z)} \tau_{21}(t - [z^{-1}], \bar{t}) \tau_{12}(t' + [z^{-1}], \bar{t}') = \int dz e^{\xi(t'-z)} \tau_{11}(t, \bar{t} - [z^{-1}]) \tau_{11}(t', \bar{t}' + [z^{-1}]) = \]

\[ \int dz e^{\xi(t'-z)} \tau_{11}(t, \bar{t} - [z^{-1}]) \tau_{11}(t' + [z^{-1}], \bar{t}') = \]

We will now show that the double KP system defined in the previous section in terms of the tau-functions \( \tau, \bar{\tau}, \hat{\tau} \), will satisfy the Hirota identities (32)-(35) upon the identification:

\[ \tau = \tau_{11} ; \; \bar{\tau} = \tau_{12} ; \; \hat{\tau} = \tau_{21} \] (36)

and upon making the obvious identification for the multi-time variables \( t \) and \( \bar{t} \).

As an example of our method we will derive Eq.(33) using the technique which employs the spectral representations (33)-(35). Let \( F \) be a wave-function for the Lax operator:

\[ F = \psi_{BA}(t, \bar{t}, \lambda) = \frac{\tau(t - [\lambda^{-1}], \bar{t})}{\tau(t, \bar{t})} e^{\xi(t, \lambda)} \] (37)

According to Eq.(33):

\[ \frac{F}{\Phi_1} = \frac{\tau(t - [\lambda^{-1}], \bar{t})}{\tau(t, \bar{t})} e^{\xi(t, \lambda)} \] (38)

is an eigenfunction for the Lax operator \( \hat{\mathcal{L}} \) w.r.t. the multi-time \( \bar{t} \) and in view of Eq.(3) admits the spectral representation:

\[ \frac{F}{\Phi_1}(t, \bar{t}) = \int dz \frac{e^{\xi(t'-z)}}{z} \bar{\tau}(t, \bar{t} - [z^{-1}]) \tau(t, \bar{t}' + [z^{-1}]) F(t, \bar{t}' + [z^{-1}]) \] (39)

Substituting Eq.(37) into the r.h.s. of Eq.(33) and Eq.(38) into the l.h.s. of Eq.(39) we obtain:

\[ \bar{\tau}(t, \bar{t}') \tau(t - [\lambda^{-1}], \bar{t}) = \int dz \frac{e^{\xi(t'-z)}}{z} \bar{\tau}(t, \bar{t} - [z^{-1}]) \tau(t - [\lambda^{-1}], \bar{t}' + [z^{-1}]) \] (40)
Choose now $\bar{F}$ to be a wave-function for the Lax operator $\bar{L}$:
\[
\bar{F} = \psi_{BA}(t, \bar{t}, \lambda) = \frac{\bar{\tau}(t, \bar{t} - [\lambda^{-1}])}{\tau(t, t)} e^{\xi(t, \lambda)}
\] (41)
then the function:
\[
\bar{F} \Phi_1 = \frac{\bar{\tau}(t, \bar{t} - [\lambda^{-1}])}{\tau(t, t)} e^{\xi(t, \lambda)}
\] (42)
is an eigenfunction for the Lax operator $\bar{L}$ w.r.t. the multi-time $t$ and in view of Eq.(4) admits the spectral representation :
\[
\bar{F} \Phi_1(t, \bar{t}) = \int dz \frac{e^{\xi(t-t', z)}}{z} \frac{\tau(t - [z^{-1}], \bar{t}) \bar{\tau}(t' + [z^{-1}], \bar{t})}{\tau(t, t) \tau(t', \bar{t})} F(t' + [z^{-1}], \bar{t})
\] (43)
Substituting Eq.(41) into the r.h.s. of Eq.(43) and Eq.(42) into the l.h.s. of Eq.(39) we obtain: 
\[
\bar{\tau}(t, \bar{t} - [\lambda^{-1}]) \tau(t', \bar{t}) = \int dz \frac{e^{\xi(t-t', z)}}{z} \tau(t - [z^{-1}], \bar{t}) \bar{\tau}(t' + [z^{-1}], \bar{t} - [\lambda^{-1}])
\] (44)
Subtracting Eq.(44) from Eq.(40) indeed reproduces Hirota identity (33) with identifications $t' = t - [\lambda^{-1}]$ and $\bar{t}' = \bar{t} - [\lambda^{-1}]$. Proofs of the remaining Hirota identities (32), (34) and (35) follow along similar lines.

**From Two-Component KP Hierarchy to Double KP Hierarchy**

In this section we are going to show that the two-component KP hierarchy, with the two sets of multi-times $t, \bar{t}$, can be regarded as ordinary one-component KP hierarchy w.r.t. to one of the multi-times, e.g. $t$, supplemented by an infinite-dimensional abelian algebra of additional (“ghosts”) symmetries, such that the second multi-time $\bar{t}$ plays the role of “ghost” symmetry flow parameters.

We first observe, by putting $\bar{t} = \bar{t}', \, t = t'$ in Hirota identities (32) and (35), that the tau-function $\tau_{11}(t, \bar{t})$ defines two one-component KP hierarchies $\text{KP}_{11}$ and $\bar{\text{KP}}_{11}$ w.r.t. the multi-time variables $t$ and $\bar{t}$, respectively. This is because in the limits $\bar{t} = \bar{t}', \, t = t'$ Eqs.(32), (35) reduce to the ordinary one-component KP Hirota identities.

Let us define :
\[
\Phi_1(t, \bar{t}) \equiv \frac{\tau_{12}(t, \bar{t})}{\tau_{11}(t, \bar{t})} ; \quad \Psi_1(t, \bar{t}) \equiv \frac{\tau_{21}(t, \bar{t})}{\tau_{11}(t, \bar{t})}
\] (45)
These functions have the following important properties. $\Phi_1$ turns out to be simultaneously an eigenfunction of the KP$_{11}$ hierarchy and an adjoint eigenfunction of the $\bar{\text{KP}}_{11}$ hierarchy. Similarly, $\Psi_1$ is simultaneously an adjoint eigenfunction of the KP$_{11}$ hierarchy and an eigenfunction of the $\bar{\text{KP}}_{11}$ hierarchy. The proof proceeds by showing that $\Phi_1$ and $\Psi_1$ satisfy the corresponding spectral representations (4) and (5) as a result of taking special limits in Hirota identities (33) and (34).
As a consequence of these last properties we conclude that $\tau_{12} = \Phi_1 \tau_{11}$ and $\tau_{21} = \Psi_1 \tau_{11}$ are also tau-functions of KP$_{11}$ and KFP$_{11}$ hierarchies since they can be regarded as DB transformations of $\tau_{11}$.

Next, define:

$$
\Phi_j(t, \bar{t}) \equiv \frac{p_{j-1}(-[\bar{\partial}]) \tau_{12}(t, \bar{t})}{\tau_{11}(t, \bar{t})}; \quad \Psi_j(t, \bar{t}) \equiv \frac{p_{j-1}([\partial]) \tau_{21}(t, \bar{t})}{\tau_{11}(t, \bar{t})}; \quad j \geq 1 \quad (46)
$$

It turns out that $\Phi_j, \Psi_j$ are (adjoint) eigenfunctions of KP$_{11}$. We present the proof for $\Phi_j$ which goes as follows. Substitute $\bar{t}' = \bar{t} - [\lambda^{-1}]$ in (43). Using identity $\int dz F(z)/z(1 - z/\lambda) = F(-1)(\lambda)$ (where the subscript (−) indicates taking the non-positive-power part of the corresponding Laurent series), the r.h.s. of (43) becomes:

$$
\int dz \frac{1}{z - \lambda} \tau_{12}(t, \bar{t} - [z^{-1}]) \tau_{11}(t', \bar{t}' - [\lambda^{-1}] + [z^{-1}]) = \tau_{12}(t, \bar{t} - [\lambda^{-1}]) \tau_{11}(t', \bar{t}) \quad (47)
$$

and the Hirota Eq. (43) simplifies now to:

$$
\tau_{12}(t, \bar{t} - [\lambda^{-1}]) \tau_{11}(t', \bar{t}) = \int dz \frac{\xi(t-t', \bar{z})}{z} \tau_{11}(t - [z^{-1}], \bar{t}) \tau_{12}(t' + [z^{-1}], \bar{t} - [\lambda^{-1}]) \quad (48)
$$

Upon expanding Eq. (48) in $\lambda$ and keeping the term of the order $\lambda^j$ we obtain Eq. (4) for $\Phi \to \Phi_{j+1}$, with $\Phi_{j+1}$ as defined in (46). Consequently, the functions $\Phi_{j+1}$ from Eq. (46) are eigenfunctions of the KP$_{11}$ hierarchy. The proof for $\Psi_j$ goes along the same lines.

Define now functions $S_{1,j}$ as $S_{1,j} \equiv p_j([\partial]) \tau_{11}/\tau_{11}$ for $j \geq 1$ (cf. Eq. (47)). We can also define the functions $S_{k,j}$ for $k > 1$ via:

$$
S_{k+1,j} = \mathcal{W}_{k}^{(j)} + \bar{w}_j \frac{p_j([\partial]) \tau_{11}}{\tau_{11}} \quad (49)
$$

where $\mathcal{W}_{k}^{(j)}$ and $\bar{w}_j$ are the affine coordinates of UGM for KFP$_{11}$ hierarchy satisfying Eqs. (43) and (47). Recall that the latter flow equations can be regarded as an equivalent definition of KP$_{11}$ hierarchy.

Now, one can show that upon substitution of definitions (48) and $\bar{w}_j = \Phi_{j+1}/\Phi_1$ (following from Eq. (46)) into the KP$_{11}$ structure Eqs. (43) and (47), the latter two equations go over into Eqs. (4) and (4) for $\Phi_j$ and $S_{k,j}$, which define an infinite-dimensional additional “ghost” symmetry structure of KP$_{11}$ hierarchy.

This last observation shows that the isospectral evolution parameters $\bar{t}$ of the one-component KFP$_{11}$ hierarchy indeed play the role of parameters of an infinite-dimensional abelian algebra of additional “ghosts” symmetries of the one-component KP$_{11}$ hierarchy.

**Conclusions and Outlook.**

In the present paper we have shown that, given an ordinary one-component KP hierarchy, we can always construct a two-component KP hierarchy, embedding the original one, in the following way. We choose an infinite set of (adjoint-)eigenfunctions of the
one-component KP hierarchy (such a choice is always possible due to our spectral representation theorem [1]), which we use to construct an infinite-dimensional abelian algebra of additional symmetries. The one-component KP hierarchy equipped with such additional symmetry structure turns out to be equivalent to the standard two-component KP hierarchy.

It is an interesting question for further study whether the origin of higher multi-component KP hierarchies can be similarly traced back to one-component KP hierarchy endowed with an appropriate infinite-dimensional abelian additional symmetry structure, generalizing the above construction for two-component KP hierarchy.

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