On the Complexity of the Empire Colouring Problem

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Let $r$ and $s$ be fixed positive integers. Assume that the $n$ vertices of a planar graph are partitioned into blocks (or empires) each containing exactly $r$ vertices. The $(s,r)$-colouring problem ($s$-COL$_r$) asks for a colouring of the vertices of the graph that uses at most $s$ colours, never assigns the same colour to adjacent vertices in different empires and, conversely, assigns the same colour to all vertices in the same empire, disregarding adjacencies. For $r = 1$ the problem coincides with the classical vertex colouring problem on planar graphs. The generalization for $r \geq 2$ was defined by Percy Heawood in 1890 in the same paper in which he refuted a previous “proof” of the famous Four Colour Theorem.

When $r = 2$ it is well-known that twelve colours are enough to solve any instance of such problem, and in fact four colours suffice if the input graph is a tree. We show that the problem is NP-hard when only three colours are allowed even if the input graph is a forest of paths. We also prove that the problem can be solved in polynomial time on graphs with no induced subgraph of average degree larger than $3/2$. The results generalize to larger values of $r$ and $s$.

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