Noise-induced synchronization of self-organized systems: Hegselmann-Krause dynamics in infinite space*

Wei Su  Jin Guo  Xianzhong Chen  Ge Chen

Abstract

It has been well established the theoretical analysis for the noise-induced synchronization of the local-rule based Hegselmann-Krause (HK) dynamics in finite space. However, when system states are allowed in the infinite space, severe mathematical difficulties arise, and the problem remains open. In this paper, we completely resolved the case when system states are allowed in the infinite space, and also the critical noise strength is given.

Keywords: Noise, synchronization, Hegselmann-Krause dynamics, self-organized systems

1 Introduction

In the past decades, self-organized systems based on local rule have been used to investigate the collective behavior in natural and social systems, and several models of self-organized systems have been proposed, including the famous Boid model and Vicsek model [1, 2]. One of the central issues in the study of self-organized systems is the synchronization of collective behavior. Due to difficulty for analysis, most of theoretical researches of these models as well as their synchronization in previous studies have omitted the influence of random noises [3, 4]. But, as Sagués said in [5], "Natural systems are undeniably subject to random fluctuations, arising from either environmental variability or thermal effects", and effect of random noise has been considered as one of the key factors in some original collective models, such as Vicsek model [2]. Mathematical analysis about Vicsek model subject to noises was firstly carried out by [6]. Beyond that, to our best knowledge the essential mathematical study on how random noises affect the collective behavior of self-organized systems is very rare, though wide attention has never ceased from various fields [7, 10, 11].

Very recently, a theoretical analysis of noise-induced consensus was rigorously established based on the widely known Hegselmann-Krause (HK) model of opinion dynamics [12]. In HK model, each agent possesses a bounded confidence, and updates its opinion value by averaging those of its neighbors who locate within its confidence region. Though simple-looking, HK model captures a quite fundamental local-based rule of evolution which ubiquitously exists in

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W. Su, J. Guo, X. Chen are with School of Automation and Electrical Engineering, University of Science and Technology Beijing & Key Laboratory of Knowledge Automation for Industrial Processes, Ministry of Education, Beijing 100083, China, suwei@amss.ac.cn, guojin@amss.ac.cn, cxz@ustb.edu.cn. G. Chen is with National Center for Mathematics and Interdisciplinary Sciences & Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, chenge@amss.ac.cn.
self-organized systems, such as Boid model and Vicsek model, and was fully exploited in its deterministic case \cite{17,20}. In \cite{16}, it for the first time established a rigorous analysis that random noise could enable the system to reach consensus (called quasi-consensus due to noise), and obtained that half confidence threshold is the critical noise strength that could induce consensus. Later HK model was proved to be able to find the truth in a group under the drive of noise \cite{21}.

The noisy HK models in previous studies are subject to a practical assumption in the original noise-free HK opinion dynamics that the state space of opinion is bounded, and the disturbance of noise will not exceed the opinion bound. The boundedness assumption about opinion space is a key to the proof of noise-induced consensus of HK dynamics in \cite{16}, since when state space is bounded, the system has a uniformly positive probability for *any* given initial state to reach quasi-consensus in a finite period and subsequently an almost sure quasi-consensus in finite time. However, while generalizing the state space of HK model to be unbounded which possesses more significance in natural and physical self-organized systems, some substantial difficulties arise for analysis of noise-induced synchronization. When state space is unbounded, the usual methods fail to guarantee the existence of a uniformly positive probability for the system of any given initial state to reach synchronization in a finite period, which is the crux of almost sure synchronization in finite time.

In this paper, via exploring the topological property of system, and further with the help of Law of the Iterated Logarithm for independent variables, we finally obtain the uniformly positive probability for the system with *any* given initial state to reach a quasi-synchronization in an almost surely finite time. We can show that given any initial state, the system will forms a state, whose graph consists of all complete subgraphs, with a uniformly positive probability in a finite period. Afterwards, given any initial state whose graph consists of complete subgraphs, we prove that the system will achieve quasi-synchronization with a uniformly positive probability in an almost surely finite time. Combining the two conclusions leads to the final answer.

The rest of the paper is organized as follows: Section 2 presents some preliminaries and Section 3 gives the main results of the paper; Section 4 shows some simulation results to verify the main theoretical conclusions and some concluding remarks are given in Section 5.

## 2 Model and definitions

Denote $\mathcal{V} = \{1, 2, \ldots, n\}$ as the set of $n$ agents, $x_i(t) \in (-\infty, \infty), i \in \mathcal{V}, t \geq 0$ as the state of agent $i$ at time $t$. The update role of HK dynamics takes:

$$x_i(t + 1) = \frac{1}{|\mathcal{N}_i(x(t))|} \sum_{j \in \mathcal{N}_i(x(t))} x_j(t) + \xi_i(t + 1), \ i \in \mathcal{V},$$

(2.1)

where

$$\mathcal{N}_i(x(t)) = \{ j \in \mathcal{V} \mid |x_j(t) - x_i(t)| \leq \epsilon \}$$

(2.2)

is the neighbor set of $i$ at $t$ and $\epsilon > 0$ represents the confidence threshold of the agents. Here, $|\cdot|$ can be the cardinal number of a set or the absolute value of a real number accordingly.

In \cite{16}, the state space is assumed to be bounded, i.e. $x_i(t) \in [0, 1], i \in \mathcal{V}, t \geq 0$. If there is no noise, it is proved that for any given initial opinion value $x(0) \in [0, 1]^n$, the evolutionary opinion values $x(t), t \geq 0$ of the noise-free HK model cannot exceed the initial boundary opinions. However, in the presence of noise, mathematically the evolutionary opinion values can be driven to run outside the initial boundary opinions, and even the opinion space $[0, 1]$. In \cite{16}, to limit the
noisy opinion values in $[0,1]$, it forcibly assume $x_i(t+1) = 0$ or $1$ when $\frac{1}{|N_i(x(t))|} \sum_{j \in N_i(x(t))} x_j(t) + \xi_i(t+1)$ is less than $0$ or larger than $1$.

To proceed, some preliminary definitions are needed.

**Definition 2.1.** Let $G_V(t) = \{V,E(t)\}$ be the graph of $V$ at time $t$, and $(i,j) \in E(t)$ if and only if $|x_i(t) - x_j(t)| \leq \epsilon$. A graph $G_V(t)$ is called a complete graph if and only if $(i,j) \in E(t)$ for any $i \neq j$; and $G_V(t)$ is called a connected graph if and only if $G_V(t)$ for any $i \neq j$, there is a road $(i,i_1), \ldots, (i_k,j)$ in $E(t)$.

The definition of quasi-synchronization of the noisy model (2.1)-(2.2) takes [16]:

**Definition 2.2.** Denote $d_V(t) = \max_{i,j \in V} |x_i(t) - x_j(t)|$ and $d_V = \limsup_{t \to \infty} d_V(t)$.

(i) if $d_V \leq \epsilon$, we say the system (2.1)-(2.2) will reach quasi-synchronization.
(ii) if $P\{d_V \leq \epsilon\} = 1$, we say almost surely (a.s.) the system (2.1)-(2.2) will reach quasi-synchronization.
(iii) if $P\{d_V \leq \epsilon\} = 0$, we say a.s. the system (2.1)-(2.2) cannot reach quasi-synchronization.
(iv) let $T = \min\{t : d_V(t') \leq \epsilon \text{ for all } t' \geq t\}$. If $P\{T < \infty\} = 1$, we say a.s. the system (2.1)-(2.2) reaches quasi-synchronization in finite time.

### 3 Main Results

For simplicity, we first present a result of quasi-synchronization for independent and identically distributed (i.i.d.) noises, then generalize the results with independent noises by a sufficient and a necessary condition.

**Theorem 3.1.** Suppose the noises $\{\xi_i(t)\}_{i \in V, t \geq 1}$ are zero-mean and non-degenerate random variables with independent and identical distribution, and $E\xi_i^2(1) < \infty$. Let $x(0) \in (-\infty, \infty)^n$ and $\epsilon > 0$ be arbitrarily given, then

(i) if $P\{\xi(1) \leq \epsilon/2\} = 1$, then a.s. the system (2.1)-(2.2) will reach quasi-synchronization in finite time;
(ii) if $P\{\xi(1) > \epsilon/2\} > 0$ and $P\{\xi(1) < -\epsilon/2\} > 0$, then a.s. the system (2.1)-(2.2) cannot reach quasi-synchronization.

Conclusion (i) shows that if noise strength is no more than $\epsilon/2$ a.s., the system will a.s. achieve quasi-synchronization in finite time; Conclusion (ii) states that when noise strength has a positive probability to exceed $\epsilon/2$, the system will not reach quasi-synchronization. This implies $\epsilon/2$ is the critical noise strength to induce a quasi-synchronization. (i) and (ii) can be directly derived from the following Theorems 3.2 and 3.6, which present a sufficient condition and a necessary condition for independent noises respectively.

**Theorem 3.2.** Suppose $\{\xi_i(t), i \in V, t \geq 1\}$ are independent and satisfy: i) $P\{|\xi_i(t)| \leq \delta\} = 1$ with $0 < \delta \leq \epsilon/2$; ii) there exist constants $a \in (0,\delta), p \in (0,1)$ such that $P\{\xi_i(t) \geq a\} \geq p$, $P\{0 \leq \xi_i(t) \leq a\} \geq p$ and $P\{|\xi_i(t) - a| \geq p\} \geq p$, $P\{-a \leq \xi_i(t) \leq 0\} \geq p$. Then, for any initial state $x(0) \in (-\infty, \infty)^n$ and $\epsilon > 0$, the system (2.1)-(2.2) will a.s. reach quasi-synchronization in finite time and $d_V \leq 2\delta$ a.s.
Lemma 3.3. Suppose \( \{z_i, i = 1, 2, \ldots\} \) is a nonnegative nondecreasing (nonincreasing) sequence. Then for any \( s \geq 0 \), the sequence \( \{g_s(k) = \frac{1}{k} \sum_{i=s+1}^{s+k} z_i, k \geq 1\} \) is monotonically nondecreasing (nonincreasing) for \( k \).

In what follows, the ever appearing time symbols \( t \) (or \( T \), etc.) all refer to the random variables \( t(\omega) \) (or \( T(\omega) \), etc.) on the probability space \( (\Omega, \mathcal{F}, P) \), and will be still written as \( t \) (or \( T \), etc.) for simplicity.

Lemma 3.4. For the system (2.1)-(2.2) with conditions of Theorem 3.2 i), if there exists a finite time \( 0 \leq T < \infty \) such that \( d_\nu(T) \leq \epsilon \), then we have a.s. \( d_\nu(t) \leq 2\delta \) for \( t > T \).

Proof. Denote \( \tilde{x}_i(t) = |\mathcal{N}(i,x(t))|^{-1} \sum_{j \in \mathcal{N}(i,x(t))} x_j(t), t \geq 0 \), and this denotation remains valid for the rest of the text. If \( d_\nu(T) \leq \epsilon \), by (2.2) we have

\[
\tilde{x}_i(T) = \frac{1}{n} \sum_{j=1}^{n} x_j(T), \quad i \in \mathcal{V}.
\tag{3.1}
\]

Since \( |\xi_i(t)| \leq \delta \) a.s., we obtain a.s.

\[
d_\nu(T + 1) = \max_{1 \leq i, j \leq n} |x_i(T + 1) - x_j(T + 1)|
\leq \max_{1 \leq i, j \leq n} (|\xi_i(T + 1)| + |\xi_j(T + 1)|) \leq 2\delta \leq \epsilon.
\tag{3.2}
\]

Repeating (3.1) and (3.2) yields the conclusion. \( \square \)

Lemma 3.5. For system (2.1)-(2.2) with conditions of Theorem 3.2 i), if there exists a finite stopping time \( 0 \leq T < \infty \) and subsets \( \mathcal{V}_k \subset \mathcal{V}, k = 1, \ldots, m(1 \leq m \leq n) \) such that \( \bigcup_{k=1}^{m} \mathcal{V}_k = \mathcal{V}, d_{\nu_k}(T) \leq \epsilon, 1 \leq k \leq m \), and for \( k_1 \neq k_2, \mathcal{V}_{k_1} \cap \mathcal{V}_{k_2} = \emptyset \), \( |x_i(T) - x_j(T)| > \epsilon \) for \( i \in \mathcal{V}_{k_1}, j \in \mathcal{V}_{k_2} \), then there exist constants \( 0 < p_0 \leq 1, L_0 > 0 \) and a finite stopping time series \( T_i \) which is \( \sigma((i-1)L_0 + \sum_{j=1}^{i-1} T_j) + 1, \ldots \) measurable, \( i = 1, \ldots, m - 1 \) such that \( P\{d_\nu(T + (m - 1)L_0 + T_1 + \ldots + T_{m-1} \leq 2\delta \} \geq p_0 \).

Proof. Without loss of generality, suppose \( T = 0 \) a.s. Then at the initial moment, the system forms \( m \) subgroups with complete graphs, and by (2.1), \( d_{\nu_k}(1) \leq 2\delta \leq \epsilon, k = 1, \ldots, m \). Before one subgroup enters the neighbor region of another one, for each \( i \in \mathcal{V} \), we have

\[
x_i(t + 1) = \frac{1}{|\mathcal{V}_k|} \sum_{j \in \mathcal{V}_k} x_j(t) + \xi_i(t + 1)
= \frac{1}{|\mathcal{V}_k|} \sum_{j \in \mathcal{V}_k} x_j(0) + \sum_{l=1}^{t} \frac{\sum_{j \in \mathcal{V}_k} \xi_j(l)}{|\mathcal{V}_k|} + \xi_i(t + 1), \quad i \in \mathcal{V}_k, 1 \leq k \leq m.
\tag{3.3}
\]

Order the subgroups from the bottom up as \( 1, 2, \ldots, m \), and consider the subgroups \( \mathcal{V}_1(1) \) and \( \mathcal{V}_m(1) \). Let

\[
y_k(t + 1) = \frac{1}{|\mathcal{V}_k|} \sum_{j \in \mathcal{V}_k} x_j(0) + \sum_{l=1}^{t} \frac{\sum_{j \in \mathcal{V}_k} \xi_j(l)}{|\mathcal{V}_k|}, \quad t \geq 0, k = 1, \ldots, m.
\]

Then for \( t \geq 1 \),

\[
y_m(t) - y_1(t) = \sum_{j \in \mathcal{V}_m} x_j(0) - \sum_{j \in \mathcal{V}_1} x_j(0) + \sum_{l=1}^{t-1} \left( \frac{\sum_{j \in \mathcal{V}_m} \xi_j(l)}{|\mathcal{V}_m|} - \frac{\sum_{j \in \mathcal{V}_1} \xi_j(l)}{|\mathcal{V}_1|} \right).
\]
Since $\xi(t) = \{\xi_i(t), i \in V\}, t \geq 1$ are independent, the $\sigma$-algebras $\sigma(\xi(t)), t \geq 1$ are independent. By Law of the Iterated Logarithm (Theorem 10.2.1 of [22]), we have that

$$\limsup_{t \to \infty} (y_m(t) - y_1(t)) = \infty, \text{ a.s.,} \quad \liminf_{t \to \infty} (y_m(t) - y_1(t)) = -\infty, \text{ a.s.} \quad (3.4)$$

Notice that $|\sum_{i \in V_m} \xi_i(t) - \sum_{i \in V_1} \xi_i(t)| \leq 2\Delta \leq \epsilon$ a.s., by (3.4), there exists a $\sigma_t$-time $0 \leq T_0 < \infty$ where $\sigma_t = \sigma(\xi(1), \ldots, \xi(t))$ that

$$0 < y_m(T_0) - y_1(T_0) \leq \epsilon, \text{ a.s.} \quad (3.5)$$

Combining (3.3) and (3.5), we obtain that there a.s. exists a $\sigma_t$-time $T_1 \leq T_0$ such that at least two subgroups with complete graphs for the first time reach the neighbor region of one another and become a complete or connected graph. Denote $\mathcal{V}_c(t)$ as the new emerging subgroups with connected graphs at each moment $t$, and design the following protocol:

$$\begin{cases}
\xi_i(t+1) \in [a, \delta], & \text{if } \min_{j \in \mathcal{V}_c(t)} x_j(t) \leq \bar{x}_i(t) \leq \min_{j \in \mathcal{V}_c(t)} x_j(t) + \frac{d\xi_j(t)}{2}; \\
\xi_i(t+1) \in [-\delta, -a], & \text{if } \min_{j \in \mathcal{V}_c(t)} x_j(t) + \frac{d\xi_j(t)}{2} < \bar{x}_i(t) \leq \max_{i \in \mathcal{V}_c(t)} x_j(t).
\end{cases} \quad (3.6)$$

For all $\mathcal{V}_c(t)$, we know that $d\mathcal{V}_c(t) \leq n\epsilon$, thus by (2.1) and Lemma 3.3, we know that under the protocol (3.6), there must exist a constant $L_0 \leq \left\lceil \frac{(n-1)\epsilon}{2a} \right\rceil$ such that $d\mathcal{V}_c(T_1 + L_0) \leq \epsilon$ a.s. (This also means protocol (3.6) occurs $L_0$ times). Since there exist $m \leq n$ subgroups with complete graphs at the initial moment, by following the above procedure, we obtain that under the protocol (3.6), the whole group $\mathcal{V}$ will a.s. form a complete graph in a finite time $\bar{T} \leq \sum_{j=1}^{m-1} T_j + (m-1)L_0$ where $T_j = \sigma(\xi((j-1)L_0 + \sum_{j=1}^{m-1} T_j + 1), \ldots)$-measurable, and during this process, protocol (3.6) occurs no more than $(m-1)L_0$ times. By independence of $\xi_i(t), i \in \mathcal{V}, t \geq 1$, we know that

$$P\{\text{protocol (3.6) occurs } (m-1)L_0 \text{ times}\} \geq \frac{n(n-1)}{2}L_0 > 0.$$ 

Let $p_0 = p^{n(n-1)L_0}$ and consider Lemma 3.3, we obtain the conclusion. □

Proof of Theorem 3.2: For each $t \geq 0$ and any given $x_i(t) \in (-\infty, \infty), i \in \mathcal{V}$, it is easy to check that there exist disjointed subsets $\mathcal{V}_k(t), k = 1, \ldots, m (1 \leq m \leq n)$ such that $\mathcal{V} = \bigcup_{k=1}^{m} \mathcal{V}_k(t)$ and such that $\mathcal{V}_c(t) = \mathcal{V}_k(t), \mathcal{E}_k(t)$ is either a connected graph or a complete graph. If $G_{\mathcal{V}_k}(0)$ is a complete graph, by Lemma 3.4, the conclusion holds. Otherwise, for each $\mathcal{V}_k(t), t \geq 0, k = 1, \ldots, m$, design the following protocol:

(i) $G_{\mathcal{V}_k}(t)$ is a connected graph, then

$$\begin{cases}
\xi_i(t+1) \in [a, \delta], & \text{if } \min_{j \in \mathcal{V}_k(t)} x_j(t) \leq \bar{x}_i(t) \leq \min_{j \in \mathcal{V}_k(t)} x_j(t) + \frac{d\xi_j(t)}{2}; \\
\xi_i(t+1) \in [-\delta, -a], & \text{if } \min_{j \in \mathcal{V}_k(t)} x_j(t) + \frac{d\xi_j(t)}{2} < \bar{x}_i(t) \leq \max_{i \in \mathcal{V}_k(t)} x_j(t).
\end{cases} \quad (3.7)$$

(ii) $G_{\mathcal{V}_k}(t)$ is a complete graph, then

$$\begin{cases}
\xi_i(t+1) \in [0, a], & \text{if } \min_{j \in \mathcal{V}_k(t)} x_j(t) \leq \bar{x}_i(t) \leq \min_{j \in \mathcal{V}_k(t)} x_j(t) + \frac{d\xi_j(t)}{2}; \\
\xi_i(t+1) \in [-a, 0], & \text{if } \min_{j \in \mathcal{V}_k(t)} x_j(t) + \frac{d\xi_j(t)}{2} < \bar{x}_i(t) \leq \max_{i \in \mathcal{V}_k(t)} x_j(t).
\end{cases} \quad (3.8)$$

For a connected graph $G_k(t)$, by (2.1) and Lemma 3.3, we know that under protocol (3.7) the minimum opinion value of $\mathcal{V}_k(t)$ increases by at least $a$, the maximum opinion value of
\( \mathcal{V}_k(t) \) decreases by at least \( a \), and \( d_{\mathcal{V}_k(t)} \) reduces by at least \( 2a \) after each step. For a complete graph \( \mathcal{G}_k(t) \), by (2.1) and Lemmas 3.3, 3.4 we know that under protocol (3.8) the aggregated opinions fluctuates with amplitude no more than \( a \) after each step. Then under protocols (3.7) and (3.8), a subgroup with complete graph never enter the neighbor region of a subgroup with connected graph though it can meet another subgroup with complete graph. Since \( d_{\mathcal{V}_k(t)} \leq |\mathcal{V}_k(t)|\epsilon \leq n\epsilon \) when \( \mathcal{G}_k(t) \) is a connected graph, we can get that under protocol (3.7), \( \mathcal{G}_k(t) \) will become a complete graph after no more than \( \left\lceil \frac{(n-1)^2\epsilon}{2a} \right\rceil \) steps. Considering that during this period, two subgroups with complete graph may meet and become a connected graph, we know that under protocols (3.7) and (3.8), all subgroups will become complete graphs after no more than \( \left\lceil \frac{(n-1)^2\epsilon}{2a} \right\rceil \). 

By independence of \( \xi_i(t), i \in \mathcal{V}, t > 0 \), we know that \( P\{\text{protocols (3.7) and (3.8) occur} \} \geq p^{\left\lceil \frac{n(n-1)^2\epsilon}{2a} \right\rceil} > 0 \), implying for any given \( x(0) \in (-\infty, \infty)^n \), there exists a constant \( L \leq \left\lfloor \frac{(n-1)^2\epsilon}{2a} \right\rfloor \) such that

\[
P\{G_{\mathcal{V}}(L) \text{ consists of complete graphs} \} \geq p^{\left\lfloor \frac{n(n-1)^2\epsilon}{2a} \right\rfloor} > 0. \tag{3.9}
\]

Denote \( C(L) = \{ \Omega : G_{\mathcal{V}}(L) \text{ consists of complete graphs} \} \), then by Lemma 3.5 there exists a finite time \( T_1 \) which is \( \sigma(\xi(1), \ldots) \)-measurable, and a constant \( 0 < p_0 < 1 \) such that

\[
P\{d_{\mathcal{V}}(L + T_1) \leq \epsilon \} = P\{d_{\mathcal{V}}(T_1) \leq \epsilon | C(L) \} \cdot P\{C(L) \} \geq p_0p^{\left\lceil \frac{n(n-1)^2\epsilon}{2a} \right\rceil} > 0,
\]

and hence

\[
P\{d_{\mathcal{V}}(L + T_1) > \epsilon \} \leq 1 - p_0p^{\left\lceil \frac{n(n-1)^2\epsilon}{2a} \right\rceil} < 1. \tag{3.10}
\]

For a finite time \( T \), define \( \mathcal{U}(T) = \{ \omega : d_{\mathcal{V}}(L + T) > \epsilon \}, U = \{ \omega : (2.7) - (2.2) \text{ does not reach quasi-synchronization in finite time} \} \). By (3.10),

\[
P\{\mathcal{U}(T_1) \} \leq 1 - p_0p^{\left\lfloor \frac{n(n-1)^2\epsilon}{2a} \right\rfloor} < 1.
\]

Since \( x(0) \) is arbitrarily given in \( (-\infty, \infty)^n \), considering the independence of \( \sigma(T_1) \) and \( \sigma(T_1 + 1, \ldots) \) and following the procedure of (3.10), we know there exist a finite time sequence \( T_1 \leq T_2 \leq \ldots < \infty \) such that

\[
P\{\mathcal{U}(T_{m+1}) | \mathcal{U}(T_m) \} \leq 1 - p_0p^{\left\lfloor \frac{n(n-1)^2\epsilon}{2a} \right\rfloor}, \quad m \geq 1.
\]

Notice by Lemma 3.4 that once there is a finite time \( T \) when \( d_{\mathcal{V}}(T) \leq \epsilon \), it will hold \( d_{\mathcal{V}} \leq 2\delta \leq \epsilon \), thus \( \mathcal{U}(T_{j+1}) \subset \mathcal{U}(T_j), j \geq 1 \) and hence

\[
P\{U\} \leq P\{ \bigcap_{m=1}^{\infty} \mathcal{U}(T_m) \} = \lim_{m \to \infty} P\{ \bigcap_{j=1}^{m} \mathcal{U}(T_j) \} = \lim_{m \to \infty} \prod_{j=1}^{m-1} P\{U(T_{j+1}) | \bigcap_{i \leq j} \mathcal{U}(T_i) \} \cdot P\{U(T_1)\} = \lim_{m \to \infty} \prod_{j=1}^{m-1} P\{U(T_{j+1}) | \mathcal{U}(T_j) \} \cdot P\{U(T_1)\} \leq \lim_{m \to \infty} (1 - p_0p^{\left\lfloor \frac{n(n-1)^2\epsilon}{2a} \right\rfloor})^m = 0,
\]
here, the first equation holds since \( \{ \bigcap_{j=1}^{m} U(T_j), m \geq 1 \} \) is a decreasing sequence and \( P \) is a probability measure. As a result

\[
P\{\text{(2.1) } - \text{ (2.2)} \text{ reach quasi-synchronization in finite time}\} = 1 - P\{U\} = 1.
\]

This completes the proof. \( \Box \)

Next we will present the necessary part of the noise induced synchronization, which shows that when the noise strength has a positive probability of exceeding \( \epsilon/2 \), the system a.s. cannot reach quasi-synchronization.

**Theorem 3.6.** Let \( x(0) \in (-\infty, \infty)^n \), \( \epsilon > 0 \) are arbitrarily given. Assume the zero-mean random noises \( \{\xi_i(t), i \in V, t \geq 1\} \) are i.i.d. with \( \mathbb{E}\xi_i^2(1) < \infty \) or independent with \( \sup_{t \in \mathbb{R}} |\xi_i(t)| < \infty \), a.s.. If there exists a lower bound \( q > 0 \) such that \( P\{\xi_i(t) > \epsilon/2\} \geq q \) and \( P\{\xi_i(t) < -\epsilon/2\} \geq q \), then a.s. the system (2.1)-(2.2) cannot reach quasi-synchronization.

**Proof.** We only need to prove the independent case, while the i.i.d. case can be obtained similarly. For the independent case, we only need to prove that, for any constant \( T_0 \geq 0 \), there exists \( t \geq T_0 \) a.s. such that \( d_Y(t) > \epsilon \), i.e.

\[
P\left\{ \bigcup_{T_0=0}^{\infty} \{d_Y(t) \leq \epsilon, t \geq T_0\} \right\} = 0.
\]

Given any \( T_0 \geq 0 \), by independence of \( \xi_i(t), i \in V, t \geq 1 \), it has

\[
P\{d_Y(T_0 + 1) > \epsilon\} \geq P\{\min_{i \in V} \xi_i(T_0 + 1) < -\frac{\epsilon}{2}, \max_{i \in V} \xi_i(T_0 + 1) > \frac{\epsilon}{2}\} \geq q^2.
\]

Hence, \( P\{d_Y(T_0 + 1) \leq \epsilon\} \leq 1 - q^2 < 1 \). Similarly,

\[
P\left\{ d_Y(t) \leq \epsilon \bigg| \bigcap_{T_0 \leq l < t} \{d_Y(l) \leq \epsilon\} \right\} \leq 1 - q^2.
\]

Thus

\[
P\{d_Y(t) \leq \epsilon, t \geq T_0\} = P\left\{ \bigcap_{t=T_0}^{\infty} \{d_Y(t) \leq \epsilon\} \right\} = \lim_{m \to \infty} P\{\bigcap_{t=T_0}^{m} \{d_Y(t) \leq \epsilon\}\}
\]

\[
= \lim_{m \to \infty} \prod_{t=T_0}^{m} P\{d_Y(t) \leq \epsilon \big| \bigcap_{l < t} \{d_Y(l) \leq \epsilon\}\}
\]

\[
\leq \lim_{m \to \infty} (1 - q^2)^m = 0.
\]

This completes the proof. \( \Box \)

4 Simulations

In this part, we will present some simulation results to verify the main theoretical results in this paper. First, we present a fragmentation of noise-free HK model. Take \( n = 20, \epsilon = 5 \) and the initial states randomly generating on \( [0, 50] \). Figure 1 shows four clusters form. Then add independent noises which are uniformly distributed on \([-\delta, \delta]\) to the agents. According to Theorem 3.2 when \( \delta \geq 0.5\epsilon \), the system almost surely achieve quasi-synchronization. Let \( \delta = 0.2\epsilon \), then Figure 2 clearly displays the quasi-synchronization picture. Next we consider the case when noise strength exceeds the critical value. For a better demonstration, we simply show a synchronized system will divide in the presence of larger noise. Let \( \delta = 0.6\epsilon \), and Figure 3 shows the separation of the system.
Figure 1: Opinion evolution of system (2.1)-(2.2) of 20 agents without noise. The initial system states are randomly generated on $[0, 50]$, confidence threshold $\epsilon = 5$, noise strength $\delta = 0$.

Figure 2: Opinion evolution of system (2.1)-(2.2) of 20 agents with noise uniformly distributed on $[-1, 1]$. The initial system states are randomly generated on $[0, 50]$, confidence threshold $\epsilon = 5$, noise strength $\delta = 0.2\epsilon$. 
5 Conclusions

In this paper, we mainly established a rigorous theoretical analysis for noise-induced synchronization of HK model in infinite state space. By investigating the graph property of initial opinion values, we completely solved this open problem. The analysis skill about the graph property of HK model will provides further tools for studying synchronization problem of noisy HK-based dynamics. Also conclusions in this paper offer insights into exploring noise-induced order of more self-organized systems or cellar automation.

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