An Entropy Functional for Riemann-Cartan Space-Times

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Abstract By viewing space-time as a continuum elastic medium and introducing an entropy functional for its elastic deformations, T. Padmanabhan has shown that general relativity emerges from varying the functional and that the latter suggests holography for gravity and yields the Bekenstein-Hawking entropy formula. In this paper we extend this idea to Riemann-Cartan space-times by constructing an entropy functional for the elastic deformations of space-times with torsion. We show that varying this generalized entropy functional permits to recover the full set of field equations of the Cartan-Sciama-Kibble theory. Our generalized functional shows that the contributions to the on-shell entropy of a bulk region in Riemann-Cartan space-times come from the boundary as well as the bulk and hence does not suggest that holography would also apply for gravity with spin in space-times with torsion. It is nevertheless shown that for the specific cases of Dirac fields and spin fluids the system does become holographic. The entropy of a black hole with spin is evaluated and found to be in agreement with Bekenstein-Hawking formula.

Keywords Riemann-Cartan space-time · Entropy functional · Cartan-Sciama-Kibble field equations · Black holes · Holography

1 Introduction

The study of black hole thermodynamics is certainly one of the most promising ways towards a deep understanding of the quantum nature of space and time. The pioneering works of Bekenstein [1, 2] and Hawking [3, 4], based respectively on the study of the properties of black holes geometry rising from general relativity and a semiclassical combination of the latter with quantum field theory, would later give birth to the novel concept of space-time holography [5, 6]. It is then interesting to investigate the role of any possible extension of general relativity in the study of black hole thermodynamics in the hope of learning more about the nature of space-time at the quantum level.
One of the old extensions of general relativity is the Einstein-Cartan theory of gravity. In this theory the intrinsic spin of a particle is naturally included in the geometric description of space-time using Riemann-Cartan geometry instead of the Riemannian geometry, i.e. by introducing torsion besides the metric as another degree of freedom for space-time. In this theory the Einstein field equations are replaced by what are commonly known as the Cartan-Sciama-Kibble (CSK) field equations, first discovered by Cartan [7], then independently rediscovered later by Kibble [8] and Sciama [9].

Ironically, the initial idea that led É. Cartan to generalize the Riemannian geometry and introduce the concept of torsion for space-time came from an analogy with the macroscopic concept of torsion in the physics of continuum media. This relation between space-time and continuum media stayed a mere analogy, though, up until 1967 when A.D. Sakharov proposed that general relativity may after all be just a low-energy approximation to the dynamics of space-time in the same sense that elasticity is an approximation to the microstructure of solids [10–12]. Recently, various authors have investigated the idea of an eventual elasticity of space-time either by applying concepts from elasticity theory to explore its dynamics [13–15] or by bringing novel interpretations to some fundamental concepts of modern cosmology such as inflation [16–18] and cosmic strings [19–21] by generalizing the three dimensional theory of defects to space-time.

Recently still, T. Padmanabhan has taken up this idea of elastic space-time from a thermodynamic viewpoint and introduced an entropy functional to be associated with the elastic deformations [22–24]. When extremized in accordance with the second law of thermodynamics, the entropy functional yields the equations of general relativity (see also [25]). Furthermore, the functional implies that the contributions to the on-shell entropy of a bulk region of space-time reside only on the boundary of the region in accordance with the holographic principle. In addition, the entropy of spinless black holes is found to be proportional to the area of their event horizon in exact agreement with Bekenstein-Hawking formula. The ability of this approach to reproduce both the fundamental equations of general relativity and main results from the thermodynamics of black holes suggests that extending the approach into Riemann-Cartan space-times may provide a simple way to include spin in the study of black hole thermodynamics. The question then is whether it would be possible to construct a generalized functional that reproduces the full set of the CSK field equations as well as familiar results from black hole thermodynamics. It is our aim in this paper to show that this is indeed possible.

The paper is organized as follows. In Sect. 2 we introduce an entropy functional for space-times with torsion and show how the CSK field equations emerge from varying the functional with respect to the deformation vector field. In Sect. 3 we use the CSK field equations to obtain a general form for the on-shell functional in which the boundary contributions are separated from the bulk ones. We then examine two specific cases of matter fields with spin embedded in space-time with torsion for which the functional takes exactly the form obtained by Padmanabhan for Riemannian space-times. Moving on to space-times with an event horizon, we show in Sect. 4 how the Bekenstein-Hawking entropy formula for a black hole with spin is recovered in our approach. We conclude this work with a discussion section to highlight and comment our main results.

2 The CSK Field Equations from an Entropy Functional

The motivation behind the introduction of an entropy functional by Padmanabhan [22–24] was to consider, in the spirit of Sakharov, space-time as a continuum medium subject at