NEW SOME HADAMARD’S TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper, we establish new some Hermite-Hadamard’s type inequalities of convex functions of 2−variables on the co-ordinates.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex mapping defined on the interval \( I \) of real numbers and \( a, b \in I \), with \( a < b \), the following double inequality is well known in the literature as the Hermite-Hadamard inequality:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Let us now consider a bidimensional interval \( \Delta =: [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). A mapping \( f : \Delta \to \mathbb{R} \) is said to be convex on \( \Delta \) if the following inequality:

\[
f(tx + (1 - t)z, ty + (1 - t)w) \leq tf(x, y) + (1 - t)f(z, w)
\]

holds, for all \((x, y), (z, w) \in \Delta \) and \( t \in [0, 1] \). A function \( f : \Delta \to \mathbb{R} \) is said to be on the co-ordinates on \( \Delta \) if the partial mappings \( f_y : [a, b] \to \mathbb{R}, \ f_y(u) = f(u, y) \) and \( f_x : [c, d] \to \mathbb{R}, \ f_x(v) = f(x, v) \) are convex where defined for all \( x \in [a, b] \) and \( y \in [c, d] \) (see [3]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. A function \( f : \Delta \to \mathbb{R} \) will be called co-ordinated convex on \( \Delta \), for all \( t, s \in [0, 1] \) and \( (x, y), (u, v) \in \Delta \), if the following inequality holds:

\[
f(tx + (1 - t)y, su + (1 - s)v) \\
\leq tsf(x, u) + s(1 - t)f(y, u) + t(1 - s)f(x, v) + (1 - t)(1 - s)f(y, v).
\]

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [3]). For several recent results concerning Hermite-Hadamard’s inequality for some convex function on the co-ordinates on a rectangle from the plane \( \mathbb{R}^2 \), we refer the reader to ([1]-[6]).

Also, in [3], Dragomir establish the following similar inequality of Hadamard’s type for co-ordinated convex mapping on a rectangle from the plane \( \mathbb{R}^2 \).

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Theorem 1. Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on $\Delta$. Then one has the inequalities:

\begin{align*}
(1.1) \quad f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
\leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) \, dx + \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) \, dy \right] \\
\leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \\
\leq \frac{1}{4} \left[ \frac{1}{b - a} \int_a^b f(x, c) \, dx + \frac{1}{b - a} \int_a^b f(x, d) \, dx \\
+ \frac{1}{d - c} \int_c^d f(a, y) \, dy + \frac{1}{d - c} \int_c^d f(b, y) \, dy \right] \\
\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{align*}

The above inequalities are sharp.

The main purpose of this paper is to establish new Hadamard-type inequalities of convex functions of 2-variables on the co-ordinates.

2. Inequalities for co-ordinated convex functions

Lemma 1. Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then the following equality holds:

\begin{align*}
(2.1) \quad & \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \\
& - \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b [f(x, c) + f(x, d)] \, dx + \frac{1}{d - c} \int_c^d [f(a, y) + f(b, y)] \, dy \right] \\
& = \frac{(b - a)(d - c)}{4} \int_0^1 \int_0^1 (1 - 2t)(1 - 2s) \frac{\partial^2 f}{\partial t \partial s}(ta + (1 - t)b, sc + (1 - s)d) \, dt \, ds.
\end{align*}
Proof. By integration by parts, we get

\[
\int_0^1 \int_0^1 (1-2s)(1-2t) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \, dt \, ds \tag{2.2}
\]

\[
= \int_0^1 (1-2s) \left\{ (1-2t) \frac{1}{a-b} \frac{\partial f}{\partial s} (ta + (1-t)b, sc + (1-s)d) \bigg|_0^1 \\
+ \frac{2}{a-b} \int_0^1 \frac{\partial f}{\partial s} (ta + (1-t)b, sc + (1-s)d) \, dt \right\} \, ds \\
= \int_0^1 (1-2s) \left\{ -\frac{1}{a-b} \frac{\partial f}{\partial s} (a, sc + (1-s)d) - \frac{1}{a-b} \frac{\partial f}{\partial s} (b, sc + (1-s)d) \\
+ \frac{2}{a-b} \int_0^1 \frac{\partial f}{\partial s} (ta + (1-t)b, sc + (1-s)d) \, dt \right\} \, ds
\]

\[
= \frac{1}{b-a} \int_0^1 (1-2s) \left\{ \frac{\partial f}{\partial s} (a, sc + (1-s)d) + \frac{\partial f}{\partial s} (b, sc + (1-s)d) \right\} \, ds \\
- 2 \int_0^1 (1-2s) \frac{\partial f}{\partial s} (ta + (1-t)b, sc + (1-s)d) \, dt \, ds
\]

Thus, again by integration by parts in the right hand side of (2.2), it follows that

\[
\int_0^1 (1-2s) \left\{ \frac{\partial f}{\partial s} (a, sc + (1-s)d) + \frac{\partial f}{\partial s} (b, sc + (1-s)d) \frac{c}{c-d} \right\} \, ds
\]

\[
- 2 \int_0^1 (1-2s) \frac{\partial f}{\partial s} (ta + (1-t)b, sc + (1-s)d) c \, ds \\
= (1-2s) \left\{ f (a, sc + (1-s)d) + f (b, sc + (1-s)d) \right\} \frac{1}{c-d} \bigg|_0^1 \\
+ \frac{2}{c-d} \int_0^1 (f (a, sc + (1-s)d) + f (b, sc + (1-s)d)) \, ds \\
- 2 \int_0^1 \left\{ (1-2s) \frac{f (ta + (1-t)b, sc + (1-s)d)}{c-d} \bigg|_0^1 \\
+ \frac{2}{c-d} \int_0^1 f (ta + (1-t)b, sc + (1-s)d) \, ds \right\} \, dt
\]
\[ \frac{f(a, c) + f(b, c) - f(a, d) + f(b, d)}{c - d} \]

\[ + \frac{2}{c - d} \int_0^1 \left( f(a, sc + (1 - s)d) + f(b, sc + (1 - s)d) \right) ds \]

\[ - \frac{2}{c - d} \int_0^1 \left\{ \frac{f(ta + (1 - t)b, c) - f(ta + (1 - t)b, d)}{c - d} \right\} dt \]

Writing (2.3) in (2.2), using the change of the variable \( x = ta + (1 - t)b \) and \( y = sc + (1 - s)d \) for \( t, s \in [0, 1]^2 \), and multiplying the both sides by \( \frac{(b-a)(c-d)}{4} \), we obtain (2.1), which completes the proof. \( \square \)

**Theorem 2.** Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^2 \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial t \partial s} \) is a convex function on the co-ordinates on \( \Delta \), then one has the inequalities:

\[ \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{d - c} \right| \]

\[ + \frac{4}{(b-a)(c-d)} \int_0^1 \int_0^1 f(x, y) dy dx - A \]

\[ \leq \frac{(b-a)(c-d)}{16} \left( \frac{\partial^2 f}{\partial x \partial t}(a, c) + \frac{\partial^2 f}{\partial x \partial t}(a, d) + \frac{\partial^2 f}{\partial x \partial t}(b, c) + \frac{\partial^2 f}{\partial x \partial t}(b, d) \right) \]

where

\[ A = \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f(x, c) + f(x, d) \ dx + \frac{1}{d-c} \int_c^d f(a, y) + f(b, y) \ dy \right] . \]
Proof. From Lemma 1, we have

\[
\begin{align*}
&\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx - A \right| \\
\leq & \frac{(b-a)(d-c)}{4} \\
& \times \int_0^1 \int_0^1 |(1-2t)(1-2s)| \left| \left( \frac{\partial^2 f}{\partial t \partial s}(a,sc + (1-s)d) \right) + (1-t) \left( \frac{\partial^2 f}{\partial t \partial s}(b,sc + (1-s)d) \right) \right| dt ds.
\end{align*}
\]

Since \( f : \Delta \to \mathbb{R} \) is co-ordinated convex on \( \Delta \), then one has:

\[
\begin{align*}
&\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx - A \right| \\
\leq & \frac{(b-a)(d-c)}{4} \\
& \times \int_0^1 \int_0^1 |(1-2t)(1-2s)| \left\{ \left| t \left( \frac{\partial^2 f}{\partial t \partial s}(a,sc + (1-s)d) \right) + (1-t) \left( \frac{\partial^2 f}{\partial t \partial s}(b,sc + (1-s)d) \right) \right| dt \right\} ds.
\end{align*}
\]

Firstly, by calculating the integral in above inequality, we have

\[
\begin{align*}
&\int_0^1 |1-2t| \left\{ \left| t \left( \frac{\partial^2 f}{\partial t \partial s}(a,sc + (1-s)d) \right) + (1-t) \left( \frac{\partial^2 f}{\partial t \partial s}(b,sc + (1-s)d) \right) \right| dt \right. \\
= & \int_0^1 (1-2t) \left\{ \left| t \left( \frac{\partial^2 f}{\partial t \partial s}(a,sc + (1-s)d) \right) + (1-t) \left( \frac{\partial^2 f}{\partial t \partial s}(b,sc + (1-s)d) \right) \right| dt \right. \\
+ & \int_0^1 (2t-1) \left\{ \left| t \left( \frac{\partial^2 f}{\partial t \partial s}(a,sc + (1-s)d) \right) + (1-t) \left( \frac{\partial^2 f}{\partial t \partial s}(b,sc + (1-s)d) \right) \right| dt \right. \\
= & \frac{1}{4} \left( \left| \frac{\partial^2 f}{\partial t \partial s}(a,sc + (1-s)d) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(b,sc + (1-s)d) \right| \right).
\end{align*}
\]
Thus, we obtain

\[
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right| + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx - A \leq \frac{(b - a)(d - c)}{16} \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s}(a, s c + (1 - s)d) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(b, s c + (1 - s)d) \right| ds.
\]

A similar way for other integral, since \( f : \Delta \to \mathbb{R} \) is co-ordinated convex on \( \Delta \), we get

\[
\int_0^1 |1 - 2s| \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a, sc + (1 - s)d) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(b, sc + (1 - s)d) \right| \right\} ds = \frac{1}{4} \int_0^1 (2s - 1) \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \right\} ds
\]

\[
+ \frac{1}{4} \int_0^1 (2s - 1) \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \right\} ds.
\]

By the (2.5) and (2.6), we get the inequality (2.4).

**Theorem 3.** Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^2 \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right| \), \( q > 1 \), is a convex function on
the co-ordinates on $\Delta$, then one has the inequalities:

\begin{equation}
\frac{\left|f(a, c) + f(a, d) + f(b, c) + f(b, d)\right|}{4} + \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx - A \leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{p}{q}}} \left(\frac{\partial^{2}f}{\partial x \partial t}(a, c) + \frac{\partial^{2}f}{\partial s \partial t}(a, d) + \frac{\partial^{2}f}{\partial x \partial t}(b, c) + \frac{\partial^{2}f}{\partial s \partial t}(b, d)\right)^{\frac{1}{q}}.
\end{equation}

where

\[A = \frac{1}{2} \left[ \frac{1}{b-a} \int_{a}^{b} [f(x, c) + f(x, d)] \, dx + \frac{1}{d-c} \int_{c}^{d} [f(a, y) + f(b, y)] \, dy \right].\]

and $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** From Lemma [1], we have

\begin{equation}
\frac{\left|f(a, c) + f(a, d) + f(b, c) + f(b, d)\right|}{4} + \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx - A \leq \frac{(b-a)(d-c)}{4} \times \int_{0}^{1} \int_{0}^{1} \left| (1-2t)(1-2s) \right| \left| \frac{\partial^{2}f}{\partial t \partial s}(ta + (1-t)b, sc + (1-s)d) \right| \, dt \, ds.
\end{equation}

By using the well known Hölder inequality for double integrals, $f : \Delta \to \mathbb{R}$ is co-ordinated convex on $\Delta$, then one has:

\begin{equation}
\frac{\left|f(a, c) + f(a, d) + f(b, c) + f(b, d)\right|}{4} + \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx - A \leq \frac{(b-a)(d-c)}{4} \left( \int_{0}^{1} \int_{0}^{1} \left| (1-2t)(1-2s) \right|^{p} \, dt \, ds \right)^{\frac{1}{p}} \times \left( \int_{0}^{1} \int_{0}^{1} \left| \frac{\partial^{2}f}{\partial t \partial s}(ta + (1-t)b, sc + (1-s)d) \right|^{q} \, dt \, ds \right)^{\frac{1}{q}}.
\end{equation}
Since \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q \) is convex function on the co-ordinates on \( \Delta \), we know that for \( t \in [0, 1] \)

\[
\left| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right|^q \leq t \left| \frac{\partial^2 f}{\partial t \partial s} (a, sc + (1-s)d) \right|^q + (1-t) \left| \frac{\partial^2 f}{\partial t \partial s} (b, sc + (1-s)d) \right|^q
\]

and

\[
\left| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right|^q \leq ts \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q + (1-t)s \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q
\]

hence, it follows that

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right| + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx - A \leq \frac{(b-a)(d-c)}{4(p+1)^\frac{n}{2}} \\
\times \left( \int_0^1 \int_0^1 \left\{ ts \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q + (1-t)s \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q \right\} dt ds \right) \frac{1}{2} \\
= \frac{(b-a)(d-c)}{4(p+1)^\frac{n}{2}} \\
\times \left( \frac{\left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q}{4} \right) \frac{1}{4}.
\]

\[\Box\]

**Theorem 4.** Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q \geq 1 \), is a convex function on
the co-ordinates on $\Delta$, then one has the inequalities:

\[
(2.8) \quad \left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right| + \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx - A \leq \frac{(b-a)(d-c)}{16} \times \left( \left| \frac{\partial^2 f}{\partial t \partial s} (a,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (a,d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s} (b,d) \right|^q \right)^{\frac{1}{q}}
\]

where

\[
A = \frac{1}{2} \left[ \frac{1}{b-a} \int_{a}^{b} \left[ f(x,c) + f(x,d) \right] \, dx + \frac{1}{d-c} \int_{c}^{d} \left[ f(a,y) + f(b,y) \right] \, dy \right].
\]

**Proof.** From Lemma 1 we have

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right| + \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx - A \leq \frac{(b-a)(d-c)}{4} \times \int_{0}^{1} \int_{0}^{1} \left| (1-2t)(1-2s) \right| \left| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right| \, dt \, ds.
\]

By using the well known power mean inequality for double integrals, $f : \Delta \to \mathbb{R}$ is co-ordinated convex on $\Delta$, then one has:

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right| + \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx - A \leq \frac{(b-a)(d-c)}{4} \times \left( \int_{0}^{1} \int_{0}^{1} \left| (1-2t)(1-2s) \right| \, dt \, ds \right)^{1-\frac{1}{q}} \times \left( \int_{0}^{1} \int_{0}^{1} \left| (1-2t)(1-2s) \right| \left| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right|^q \, dt \, ds \right)^{\frac{1}{q}}.
\]
Since \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q \) is convex function on the co-ordinates on \( \Delta \), we know that for \( t \in [0, 1] \)

\[
\left| \frac{\partial^2 f}{\partial t \partial s}(ta + (1 - t)b, sc + (1 - s)d) \right|^q \\
\leq t \left| \frac{\partial^2 f}{\partial t \partial s}(a, sc + (1 - s)d) \right|^q + (1 - t) \left| \frac{\partial^2 f}{\partial t \partial s}(b, sc + (1 - s)d) \right|^q
\]

and

\[
\left| \frac{\partial^2 f}{\partial t \partial s}(ta + (1 - t)b, sc + (1 - s)d) \right|^q \\
\leq ts \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + t(1 - s) \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q \\
+ (1 - t)s \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + (1 - t)(1 - s) \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q
\]

hence, it follows that

\[
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right|^q \\
\leq \frac{(b - a)(d - c)}{4} \left( \frac{1}{4} \right)^{1 - \frac{q}{4}} \\
\times \left( \int_0^1 \int_0^1 |(1 - 2t)(1 - 2s)| \left\{ ts \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + t(1 - s) \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q \\
+ (1 - t)s \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + (1 - t)(1 - s) \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q \right\} dt ds \right)^{\frac{q}{4}}.
\]
Firstly, by calculating the integral in above inequality, we have

\[
\begin{align*}
&\int_0^1 |1-2t| \left( ts \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + t(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q 
+ (1-t)s \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q \right) dt \\
&= \int_0^1 (2t-1) \left( ts \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + t(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q 
+ (1-t)s \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q \right) dt \\
&\quad + \int_0^1 (2t-1) \left( ts \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + t(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q 
+ (1-t)s \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q \right) dt \\
&= \frac{s}{24} \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + \frac{(1-s)}{24} \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q 
+ \frac{5s}{24} \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + \frac{5(1-s)}{24} \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q \\
&\quad + \frac{5s}{24} \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + \frac{5(1-s)}{24} \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q 
+ \frac{s}{24} \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + \frac{(1-s)}{24} \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q \\
&= \frac{s}{4} \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + \frac{(1-s)}{4} \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q 
+ \frac{s}{4} \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + \frac{(1-s)}{4} \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q.
\end{align*}
\]
Thus, we obtain
\[
\begin{align*}
&\left(2.9\right) \left| f(a,c) + f(a,d) + f(b,c) + f(b,d) \right| \\
&\quad + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dydx - A \leq \frac{(b-a)(d-c)}{16} \left[ \int_0^1 \left| 1 - 2s \right| \left( s \left| \frac{\partial^2 f}{\partial t^2}(a,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(a,d) \right|^q \right. \\&\left. + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,d) \right|^q \right) ds \right]^\frac{1}{2}.
\end{align*}
\]

A similar way for other integral, since \( f : \Delta \rightarrow \mathbb{R} \) is co-ordinated convex on \( \Delta \), we get
\[
\begin{align*}
&\left(2.10\right) \int_0^1 \left| 1 - 2s \right| \left( s \left| \frac{\partial^2 f}{\partial t^2}(a,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(a,d) \right|^q \right. \\&\left. + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,d) \right|^q \right) ds^{\frac{1}{2}} \\&\quad + \frac{1}{2} \left( 2s - 1 \right) \left( s \left| \frac{\partial^2 f}{\partial t^2}(a,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(a,d) \right|^q \right) ds \\&\quad + \frac{1}{2} \left( 2s - 1 \right) \left( s \left| \frac{\partial^2 f}{\partial t^2}(b,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,d) \right|^q \right) ds \\&\quad = \frac{24}{5} \left( 2s - 1 \right) \left( s \left| \frac{\partial^2 f}{\partial t^2}(a,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(a,d) \right|^q \right. \\&\left. + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,d) \right|^q \right) ds \\&\quad = \frac{24}{5} \left( 2s - 1 \right) \left( s \left| \frac{\partial^2 f}{\partial t^2}(a,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(a,d) \right|^q \right. \\&\left. + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,d) \right|^q \right) ds \\&\quad = \frac{24}{5} \left( 2s - 1 \right) \left( s \left| \frac{\partial^2 f}{\partial t^2}(a,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(a,d) \right|^q \right. \\&\left. + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,c) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t^2}(b,d) \right|^q \right) ds.
\end{align*}
\]

By the \(2.9\) and \(2.10\), we get the inequality \(2.8\).

\begin{remark}
Since \( \frac{1}{p} < \frac{1}{(p+1)q} < 1 \), if \( p > 1 \), the estimation given in Theorem 4 is better than the one given in Theorem 3.
\end{remark}
CO-ORDINATED CONVEX FUNCTIONS

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