Reserved-Length Prefix Coding

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Abstract—Huffman coding finds an optimal prefix code for a given probability mass function. Consider situations in which one wishes to find an optimal code with the restriction that all codewords have lengths that lie in a user-specified set of lengths (or, equivalently, no codewords have lengths that lie in a complementary set). This paper introduces a polynomial-time dynamic programming algorithm that finds optimal codes for this reserved-length prefix coding problem. This has applications to quickly encoding and decoding lossless codes. In addition, one modification of the approach solves any quasiarithmetic prefix coding problem, while another finds optimal codes restricted to the set of codes with $g$ codeword lengths for user-specified $g$ (e.g., $g = 2$).

I. INTRODUCTION

A source emits symbols drawn from the alphabet $\mathcal{X} = \{1, 2, \ldots, n\}$. Symbol $i$ has probability $p_i$, thus defining probability mass function vector $p$. We assume without loss of generality that $p_i > 0$ for every $i \in \mathcal{X}$, and that $p_i \leq p_j$ for every $i > j$ ($i, j \in \mathcal{X}$). The source symbols are coded into binary codewords. The codeword $c_i$ corresponding to symbol $i$ has length $l_i$, thus defining length vector $l$.

It is well known that Huffman coding [1] yields a prefix code minimizing

$$\sum_{i \in \mathcal{X}} p_i l_i$$

given the natural coding constraints: the integer constraint, $l_i \in \mathbb{Z}_+$, and the Kraft (McMillan) inequality [2]:

$$\kappa(l) \triangleq \sum_{i \in \mathcal{X}} 2^{-l_i} \leq 1. \quad (1)$$

Since an exchange argument (e.g., [3, pp. 124-125]) easily shows that an optimal code exists which has monotonic non-decreasing lengths, we can assume without loss of generality that such minimum-redundancy codes have $l_i \geq l_j$ for every $i > j$ ($i, j \in \mathcal{X}$).

There has been much work on solving this problem with other objectives and/or additional constraints [4]. One especially useful constraint [5], [6] is that of length-limited coding, in which

$$l_i \in \{1, 2, \ldots, l_{\text{max}}\} \forall i$$

for some $l_{\text{max}}$. A constraint that has received less attention is the reserved-length constraint:

$$l_i \in \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{|\Lambda|}\} \forall i$$

for $\lambda_i \in \mathbb{Z}_+ \forall i$. In this case, instead of restricting the range of codeword lengths to an interval as in length-limited coding, it is restricted to an arbitrary set of lengths. (As demonstrated in the next section, there is no loss of generality in assuming this set to be finite). The problem is well-formed if and only if $\lambda_{|\Lambda|} \geq \log_2 n$.

This problem was proposed in the 1980s but, due to the lack of a solution, never published [7]. A practical application is that of fast data decompression. Perhaps the greatest bottleneck in fast Huffman decoding is the determination of codeword length from input bits, which can be done using a lookup table, a linear search, or a decision tree, depending on the complexity of the code involved [5]. The average time taken by a linear search or an optimal decision tree increases with the number of possible codeword lengths, so limiting the number of possible codeword lengths can make decoding faster; if the resulting increase in expected codeword length is small or zero, this can be an effective way of trading off compression and speed, with no compression on one end of the spectrum and optimal compression on the other end.

Consider the optimal prefix code for random variable $Z$ drawn from the Zipf distribution with $n = 2^{12}$, that is,

$$\mathbb{P}[Z = i] = \frac{1}{i \sum_{j=1}^{n} j^{-1}}$$

which is approximately equal to the distribution of the $n$ most common words in the English language [8, p. 89]. This code has codewords of 13 different lengths, with an average length of about 8.78 bits. If one were to restrict this code to only allow codewords of lengths in $\{5, 9, 14\}$, the resulting optimal restricted code would have an average length of about 9.27 bits. Although suboptimal, this restricted code would decode more quickly than the optimal unrestricted code.

An $O(n^4)$-time $O(n^3)$-space dynamic programming approach, introduced shortly, finds optimal reserved-length binary prefix codes. Variants of this algorithm solve a related length constraint and any case of the quasiarithmetic coding problem introduced by Campbell [9], extending the result of [10].

II. PRELIMINARIES AND ALGORITHM

Many prefix coding problems — most notably binary Huffman coding and binary length-limited “Huffman” coding — must return an optimal code in which the Kraft inequality (1) is satisfied with equality, that is, for which $\kappa(l) = 1$. For
nonbinary problems, although the corresponding inequality is not always satisfied with equality, a simple modification to the problem changes this, causing the inequality to always be equal for optimal codes [1], [11]. This is not the case for the reserved-length problem. For example, if \( n = 3 \) and the allowed lengths are 1 and 3, then the optimal code must have lengths 1, 3, and 3, resulting in a code for which \( \kappa(l) = 0.75 \). Moreover, it is not clear how to determine \( \kappa(l) \) for the optimal code other than to calculate the optimal code itself. The Huffman coding and most common length-limited approaches rely on \( \kappa(l) = 1 \), so these methods cannot be used to find an optimal code here.

The Kraft inequality is often explained in terms of a coding tree. A binary coding tree is a rooted binary tree in which the leaves represent items to be coded. Along the path to a leaf, if the highest edge goes to the leftmost child, the highest bit of the codeword is a 0; otherwise, it is a 1. For a finite code tree, the Kraft inequality is an equality if and only if every node has 0 or 2 children, that is, if it is full. This assumption needs to be relaxed for finding an optimal reserved-length prefix code.

One approach that does not require \( \kappa(l) = 1 \) is dynamic programming. Many prefix coding solutions use dynamic programming techniques [4], e.g., finding optimal codes for which all codewords end with a ‘1’ bit [12], a situation in which, necessarily, a finite code cannot have \( \kappa(l) = 1 \). For the current problem, the dynamic programming algorithm should find, for increasing tree heights, a set of candidate trees from which to choose, and it should terminate when the longest feasible length is encountered. First, however, we have to find this longest feasible length, since we didn’t specify that \( \Lambda \), the set of allowed lengths, needed to be upper-bounded by any function of \( n \) or even finite.

**Theorem 1:** Any codeword \( l_i \) of an optimal reserved-length code either satisfies \( l_i \leq n - 2 \) or \( l_i = \lambda_\infty \), where \( \lambda_\infty \) is the smallest element of \( \Lambda \) that satisfies \( \lambda_\infty > n - 2 \).

**Proof:** We first show that no partial Kraft sum of \( x \) items

\[
\kappa(l, x) = \sum_{i=1}^{x} 2^{-l_i}
\]

can be in the open interval \( (1 - 2^{-x}, 1) \), and, furthermore, if the longest codeword is of length \( l_x > x - 1 \), the sum cannot be in \( (1 - 2^{-x+1} + 2^{-l_x}, 1) \). This is shown by induction on codeword lengths of nondecreasing order. Clearly

\[
\kappa(l, 2) = 2^{-l_1} + 2^{-l_2} \notin (3/4, 1)
\]

satisfies this. Suppose the Kraft sum for \( x - 1 \) items cannot fall in \( (1 - 2^{-x+1}, 1) \), that is, for any code for which \( \kappa(l, x - 1) < 1 \), \( \kappa(l, x - 1) \leq 1 - 2^{-x+1} \). Since the \( x \)th term is a power of two, the partial sum of a code is no greater than \( 1 - 2^{-x+1} + 2^{-x} = 1 - 2^{-x} \) for \( \kappa(l, x) < 1 \). Moreover, if \( l_x \geq x \), the partial sum is less than or equal to \( 1 - 2^{-x+1} + 2^{-l_x} \).

Now suppose there is an optimal code for \( n \) items which includes codeword lengths \( l_\mu \) and \( l_\nu \), where \( n - 2 < l_\mu < l_\nu \). Assume without loss of generality that \( l_\mu \) and \( l_\nu \) are the longest codeword lengths and \( l_\nu = l_n \) (i.e., \( l_\nu \) is the longest codeword length). Note that \( l_\nu \geq n \) and the Kraft sum cannot equal 1 for any code in which the longest codeword has length equal to or exceeding \( n \); it is well known that the deepest full tree is a terminated unary tree, one with depth \( n - 1 \). Thus

\[
\kappa(l) < 1 - 2^{-n}.
\]

Consider a code with lengths \( l'_i = l_i \) for \( i < n \) and \( l'_n = l_\mu, l_\nu \). We show that a prefix code exists with these lengths and thus achieves greater compression, rendering \( l \) suboptimal. If \( l_\nu = l_\mu + 1 \), then

\[
kappa'(l) = \kappa(l) - 2^{-l_\mu - 1} + 2^{-l_\nu} \leq 1 - 2^{-n} + 2^{-l_\nu} 
\]

since \( n \leq l_\mu + 1 \). Otherwise, \( l_n = l_\nu \geq n \), and

\[
kappa'(l) = \kappa(l) - 2^{-l_\nu} + 2^{-l_\nu} \leq 1 - 2^{-n+1} + 2^{-l_\nu} \leq 1
\]

since \( n - 1 \leq l_\nu \).

Since an optimal tree exists which has monotonic nondecreasing lengths, optimal codeword lengths can be fully specified by the number of leaves on each of the allowed levels of the code tree. For such an optimal tree, given any “allowed level” \( \lambda_m \), the lengths with \( l_i \leq \lambda_m \) have a partial Kraft sum

\[
\kappa_{\lambda_m}(l) = \kappa(l, v_m)
\]

for \( v_m \) such that \( v_m \leq \lambda_m \) and either \( l_1 + v_m \leq \lambda_m \) or \( v_m = n \). This Kraft sum is a multiple of \( 2^{-\lambda_m} \), so there exists an \( \eta_m \) such that \( \kappa(l, v_m) = 1 - \eta_m 2^{-\lambda_m} \), and this \( \eta_m \) is the number of internal nodes on level \( \lambda_m \) of any coding tree corresponding to the codeword lengths.

In an optimal coding tree, if \( \Delta_m \) is defined to be \( \lambda_{m+1} - \lambda_m \), then, for any \( v_m < n \),

\[
\eta_m 2^{\Delta_m} - (2^{\Delta_m} - 2) \leq n - v_m
\]

(2) internal nodes next minus leaves under \( m \) single-node expansion factor

This can be seen by observing that, if a code violates this, we can produce a code with the same lengths for \( l_1 \) through \( v_m \) and assign \( l_{v_m+1} = \lambda_m \) and \( l_i = \lambda_{m+1} \) for \( i > v_m + 1 \), and the new code would have no length exceeding that of the original code; in fact, \( l_{v_m+1} \) is strictly shorter, so the original code could not be optimal. For \( \lambda_{m+1} = \lambda_m + 1 \), this condition is identical to

\[
2^{\eta_m} \leq n - v_m
\]

(3) which is a looser necessary condition for optimality. For similar reasons, no optimal tree will have a partial tree with \( v_m = n - 1 \) for any \( m \), since using an internal node on level \( \lambda_m \) for the final item results in an improved tree.

Such properties can be used to construct a dynamic programming algorithm. In describing this algorithm, we use the following notation (with mnemonics in boldface):

\[
v_m : \text{Used up leaves at or above level } \lambda_m
\]

\[
\eta_m : \text{Nodes internal at level } \lambda_m
\]

\[
\Upsilon[m, v, \eta] : \text{Leaves above level } \lambda_m
\]

\[
L[m, v, \eta] : \text{Partial sum of a code}
\]

The idea for the algorithm is to calculate the optimal \( L[m, v_m, \eta_m] \) given feasible values of partial trees \( v_m < n \);
n − 1) and to separately keep track of the best finished tree (\(v_m = n\)) as the algorithm progresses. The trees grow by level (\(\Lambda_m\) for increasing \(m\)), while the algorithm calculates all feasible values of \(v_m\) (which are in \([0, n − 2]\) for partial trees) and \(\eta_m\) (which are in \([0, \lfloor n/2\rfloor]\) for partial trees due to (3); if \(\eta_m > n/2\), at least one node on a lower level could be shortened to length \(\lambda_m\), resulting in a strictly improved code). Thus there are \(O(n^2)\) values per level, and we can try all feasible combinations, calculating \(L\) for all combinations of partial trees — saving optimal combinations — and finished trees — saving only the best finished tree encountered up to this point. Clearly, \(v_m\) must be nondecreasing. This, along with the bounds on \(\eta_m\), are used to try the aforementioned combinations. In cases where \(|\Lambda|\) is much smaller than \(n\), additional constraints can be made, based on (2), but such constraints do not improve computational complexity in the general case, so we do not discuss them here.

After finishing level \(\lambda_{|\Lambda|}\), the optimal tree is rebuilt via backtracking. Assuming arithmetic operations are constant-time, complexity of the dynamic programming Algorithm 1 is \(O(|\Lambda|n^3)\)-time and \(O(|\Lambda|n^2)\)-space. Because \(|\Lambda| < n\) without loss of generality, if we assume arithmetic operations are constant time, time complexity should be \(O(n^3)\) and space complexity \(O(n^2)\).

A simple example of this algorithm at work is in finding an optimal code for Benford’s law [13], [14] with the restriction that all codeword lengths must be powers of two. In this case, \(p_i\) is \(\log_{10}(i + 1) − \log_{10}(i)\) for \(i\) from 1 to \(n = |\mathcal{X}| = 9\), and \(\Lambda = \{1, 2, 4, 8\}\) is a sufficient range of lengths to allow, due to Theorem 1. The calculated values for each feasible partial \(L[m, v, \eta]\) are shown in Table I.

On the first level, \(\lambda_1\), average length is identical to the level number, and, if, for example, \(\lambda_1 = 1\), the nodes at the level can include zero ((\(v, \eta) = (0, 2)\)), one ((\(1, 1)\)), or two ((\(2, 0)\)) terminating nodes, which are the only nontrivial entries in a two-dimensional grid for this level, which is indicated by the first grid in Table I. From each nontrivial entry in the level \(\lambda_1\) grid, all allowed combinations of terminating and expanding are considered until the second (level \(\lambda_2\)) grid is arrived at, and the algorithm proceeds similarly until all allowed levels are accounted for. All trees with \(v = n\) (all leaves accounted for) are compared with the best one so far in order to find an optimal tree. In the Benford’s law example, this is a tree with two codewords of length two and seven codewords of length four. Note that the strict inequality of line 29 means that, if there are multiple optimal length vectors, the algorithm selects one of minimal maximum length.

Note that a similar approach could be used for nonbinary trees, although an efficient exponentiation procedure should be used in place of shifting in lines 15 and 47 of Algorithm 1. Codeword construction changes (lines with “\(c_i\)” and the aforementioned expansion bounds (lines with “\(|\mathcal{X}|/2\)”) also need adjustment for nonbinary cases. These alternations do not worsen computational complexity.

### III. Extensions and Conclusion

The aforementioned method yields a prefix code minimizing expected length for a known finite probability mass function under the given constraints. However, there are many varied instances in which expected length is not the proper value to minimize [4]. Many such problems are in a certain family of generalizations of the Huffman problem introduced by Campbell in [15].

While Huffman coding minimizes \(\sum i \in \mathcal{X} p_i l_i\), Campbell’s quasiarithmetic formulation adds a continuous (strictly) monotonic increasing cost function \(\varphi(l) : \mathbb{R}_+ \rightarrow \mathbb{R}_+\). The value to minimize is then

\[
L(p, l, \varphi) \equiv \varphi^{-1}\left(\sum_{i \in \mathcal{X}} p_i \varphi(l_i)\right).
\]

Convex \(\varphi\) have been solved for [10]. For nonconvex functions, it suffices to replace line 16 in the algorithm,

\[
L[i − 1, v, \eta] + (\lambda' − \lambda'')(1 − F_v)
\]

with

\[
L[i − 1, v, \eta] + (\varphi(\lambda') − \varphi(\lambda''))(1 − F_v).
\]

The exchange argument still holds, resulting in a monotonic solution, and \(\Lambda\) still has cardinality less than \(n\), so the algorithm proceeds similarly for identical reasons, and thus with the same complexity. A nonbinary coding extension is similar to that used to minimize expected length.

We earlier stated that one purpose for reserving lengths is to allow faster decoding by having fewer codewords. However, if this is the objective, the problem remains of how to select the codeword lengths to use. We might, for example, restrict our solution to having two codeword lengths, but not put any restrictions on what these codeword lengths should be. Such a problem was examined analytically in [16] for \(n\) approaching infinity. Here, we consider solving the problem for fixed \(n\).

One approach to the two-length problem would be to try all feasible combinations of codeword lengths. We then have to find a feasible set, hopefully one relatively small so as not to drastically increase the complexity of the problem.

First note that, if only one codeword length is used, then \(\lambda_2 = \lambda_1 = \lfloor \log_2 n \rfloor\). Otherwise, we begin by observing that, for the best tree, the number of internal nodes and leaves on the first allowed level \(\lambda_1\) must each be greater than 0 (or else only one codeword length could be used) and combined be no greater than \(n − 1\) (or else a better code exists with all codewords having one length). Thus \(\lambda_1 \leq \log_2 (n − 1)\), or, put another way, \(\lambda_1 \leq \lfloor \log_2 n \rfloor − 1\). At the same time, the second allowed level cannot have 2\(n − 2\) or more combined internal nodes and leaves; otherwise an improved tree can be found by decreasing \(\lambda_2\) by one, since no more than \(n − 1\) leaves can be on this level. Because these nodes are all descendants of all least one internal node on the first allowed level, this results in \(2\lambda_2 − \lambda_1 < 2n − 2\), which leads to \(\lambda_2 − \lambda_1 \leq \lfloor \log_2 (n − 1) \rfloor \leq \lfloor \log_2 n \rfloor\). Combining these results, we find that \(\lambda_2 \leq 2\lfloor \log_2 n \rfloor − 1\).
Algorithm 1 Dynamic programming algorithm for reserved-length prefix coding

Require: \( p \) of size \( n = |X| \), \( \Lambda \) for which (without loss of generality) \( \lambda_{|\Lambda|-1} \leq |X| - 2 \)

1: \( F_0 \leftarrow 0 \)
2: for \( i \leftarrow 1, |X| \) do
3: \( F_i \leftarrow F_{i-1} + p_i \) {Calculate cumulative distribution function}
4: end for
5: for all \( 0 \leq m < |\Lambda|, 0 \leq v \leq |X| - 2, 0 \leq \eta \leq |X|/2 \) do
6: \( L[m, v, \eta] \leftarrow \infty \) {Initialize partial tree costs}
7: end for
8: \( L_{\text{min}} \leftarrow \infty \) {Best total tree cost so far}
9: \( L[0, 0, 1] \leftarrow 0 \) {Trivial tree cost}
10: \( \lambda'' \leftarrow 0 \) {Previous level}
11: for \( m \leftarrow 1, |\Lambda| \) {Level by level} do
12: \( \lambda' \leftarrow \lambda_m \) {Current level}
13: for all \( (v, \eta) \in [0, |X|-2] \times [0, |X|/2] \) {Find optimal partial trees with given \( m \) from \( (m-1, v, \eta) \)} do
14: if \( L[m-1, v, \eta] < \infty \) then
15: \( \eta' \leftarrow \eta \ll (\lambda' - \lambda'') \) {Total nodes on new level \( \lambda_m \)}
16: \( L' \leftarrow L[m-1, v, \eta] + (\lambda' - \lambda'')(1 - F_v) \) {Cost on new level \( \lambda_m \)}
17: if \( m < |\Lambda| \) {Build partial trees (for which \( m < |\Lambda| \)} then
18: \( v_{\text{min}} \leftarrow \max(v, 2(v + \eta') - |X|) \) {Range of potential \( v_m \)}
19: \( v_{\text{max}} \leftarrow \min(v + \eta', |X| - 2) \)
20: for \( v' \leftarrow v_{\text{min}}, v_{\text{max}} \) {Compare cost for all potential \( v_m < |X| \)} do
21: if \( L[m, v', \eta' - v' + v] > L' \) then
22: \( \bar{L}[m, v', \eta' - v' + v] \leftarrow L' \) {New optimal partial cost for \( (m, v_m \eta_m) = (m, v', \eta' - v' + v) \)}
23: \( \bar{\chi}[m, v', \eta' - v' + v] \leftarrow \nu \) {Save with \( v_{m-1} \) for backtracking}
24: end if
25: end for
26: end if
27: end if
28: if \( |X| \leq v + \eta' \) then
29: if \( L' < L_{\text{min}} \) {Find best finished tree} then
30: \( L_{\text{min}} \leftarrow L' \) {Best finished tree cost}
31: \( (m_{\text{min}}, v_{\text{min}}, \eta_{\text{min}}, \chi_{\text{min}}) \leftarrow (m, |X|, \eta' - |X| + v, v) \) {Save optimal values with \( \chi_{\text{min}} = v_{m-1} \) for backtracking}
32: end if
33: end if
34: end for
35: \( \lambda'' \leftarrow \lambda' \) {Current level now previous level}
36: end for
37: \( (m, v, \eta, \chi) \leftarrow (m_{\text{min}}, v_{\text{min}}, \eta_{\text{min}}, \chi_{\text{min}}) \) {Backtrack to find optimal tree}
38: \( c \leftarrow (1 \ll \lambda_m) - \eta \) {1 greater than integer representation of final codeword}
39: while \( m > 1 \) {Rebuild best tree} do
40: if \( v < |X| \) then
41: \( \chi \leftarrow v - \bar{\chi}[m, v, \eta] \) {Number of leaves above level}
42: end if
43: for \( j \leftarrow v \) down to \( v - \chi + 1 \) do
44: \( \lambda_{c+j-v-1} \leftarrow \lambda_{c+j-v} \) {Assign lengths/codewords (where \( \{x\}_y \) denotes the \( y \)-bit representation of \( x \))}
45: end for
46: \( c \leftarrow (m - \lambda_{m-1}) \) {Start codewords of length \( \lambda_{m-1} \)}
47: \( (m, v, \eta) \leftarrow (m - 1, v - \chi, (\eta + \chi) \gg (\lambda_m - \lambda_{m-1})) \) {Calculate new \( (m, v, \eta) \) from old using \( \chi \)}
48: end while
49: for \( j \leftarrow v \) down to \( 1 \) do
50: \( (l_j, c_j) \leftarrow (\lambda_1, (j-1)\lambda_1) \) {Shortest lengths/codewords}
51: end for
This result, while not the strictest bound possible, is sufficient for us to determine that the number of codeword length combinations one would have to try would be $O(|\Lambda|^2 n)$. Thus, since $|\Lambda| = 2$ in all cases and only $O(1)$ data need be kept between combinations, the algorithm has only an $O(n^2)$ space and an $O(n^3 \log^2 n)$ time requirement, smaller than even the general version of the reserved length problem. For example, the optimal two-length code for the Benford distribution has two codewords of length two and seven codewords of length four. This is the code found above to be optimal for lengths restricted to powers of two. This two-length code has average codeword length $3.04\ldots$, very near to that of the optimal unrestricted Huffman code, which has average codeword length $2.92\ldots$

The two-length problem’s solution can be easily generalized to that of a $g$-length problem, which can be optimally solved with $O(n^2 g)$ space and $O(n^3 (\log^2 n) g^3)$ time in similar fashion. In fact, all $g^i$-length problems, for $g^i \leq g$, can be solved with this complexity, allowing for a selection of the desired trade-off between number of codeword lengths (speed) and expected codeword length (compression efficiency). Modifications can enact additional restrictions on codeword lengths (e.g., a limit on maximum length) in a straightforward fashion.

We thus find that this dynamic programming method is quite general, solving three problems that previously had no proposed polynomial-time solutions: the reserved-length problem, Campbell’s quasiarithmetic problem, and the $g$-length problem.

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