Stability and Robustness of a Hybrid Control Law for the Half-bridge Inverter

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Abstract—Hybrid systems combine both discrete and continuous state dynamics. Power electronic inverters are inherently hybrid systems: they are controlled via discrete-valued switching inputs which determine the evolution of the continuous-valued current and voltage state dynamics.

Hybrid systems analysis could prove increasingly useful as large numbers of renewable energy sources are incorporated to the grid with inverters as their interface. In this work, we explore a hybrid systems approach for the stability analysis of power and power electronic systems. We provide an analytical proof showing that the use of a hybrid model for the half-bridge inverter allows the derivation of a control law that drives the system states to desired sinusoidal voltage and current references. We derive an analytical expression for a global Lyapunov function for the dynamical system in terms of the system parameters, which proves uniform, global, and asymptotic stability of the origin in error coordinates. Moreover, we demonstrate robustness to parameter changes through this Lyapunov function. We validate these results via simulation.

Fig. 1: Typically, a grid-forming (GFM) control strategy instructs the inverter to achieve certain set points that have been optimized at a system level. The inner voltage and current control loops, in red, aim to drive the actual inverter variables to these references by translating them into a modulation signal for a pulse-width modulator. In this work, we propose replacing the inner loops with our hybrid control strategy, in green.

Finally, we present simulation results that combine a droop controller with this hybrid systems approach, and benchmark against the well-studied averaged model and inner current and voltage control loops. In the low-inertia grid community, the juxtaposition of droop control with the hybrid switching control can be considered a grid-forming control strategy using a switched inverter model.

I. INTRODUCTION

The power grid is seeing a large portion of its generation replaced by renewable energy. Therefore, the energy conversion process is changing and converter-interfaced generation (CIG) systems, primarily in the form of power electronic inverters, are becoming the primary interface between energy sources and loads.

A. The Need for New Inverter Controls

Inverters are highly controllable devices that convert a DC energy source, like a solar photovoltaic panel, to a grid-compatible AC energy form. Since the number of inverters in the power grid is increasing, and because they react to programmable instructions, the design of control strategies for power grids with large numbers of CIGs is considered a problem of paramount importance in the low-inertia grid research community [1]. Emphasizing this point, in 2016 the European Union funded the MIGRATE (Massive Integration of Power Electronic Devices) project [2] to study fundamental challenges associated with incorporating renewables in the grid. The team was composed of experts from over 20 participating institutions from industry and academia. Furthermore, in 2021, the United States of America’s Department of Energy funded the UNIFI (Universal Interoperability for Grid-Forming Inverters) Consortium [3], consisting of experts from over 40 participating institutions, to answer questions of a similar nature.

Recent work further recognizes the need to consider new control methodologies for inverters [4]. Accentuating this need, the IEEE task force for stability definitions and classification for low-inertia grids has specifically recognized the importance of hybrid systems analysis for the study of power and power electronic systems [5]. Hence, due to the importance that inverter control strategies will play in the future grid, we believe it is imperative to explore the whole landscape of possible controllers, including those that, as in this work, rely on the more physically realistic hybrid systems model of the inverter.

One approach to inverter control is the grid-forming paradigm, in which the inverter is configured as a controllable voltage source with provisions that enable power sharing, among other system-level desired performance requirements. A future grid will likely need a large portion of its generation controlled via grid-forming controls [1].
Several grid-forming control strategies with strong theoretical guarantees have been proposed and validated through simulation and hardware experimentation [6], [7], [8], [9]. Their implementations thus far usually use a sinusoidal pulsed-width modulation (PWM) strategy whose analysis relies on the inverter averaged model and leads to the well-studied nested voltage and current feedback loop structure, shown above in Fig. 1, as the inner loop inverter controls [9]. By providing a fixed template for which new control strategies need to be compatible with, the averaged model and inner control loops constrain how new inverter controls can be implemented. Moreover, in simulation, they ignore the fast switching dynamics, which are on similar time scales as line voltage and current dynamics [5], thus potentially hiding some dynamic interactions. Furthermore, some initial results point to the inner control loops as the primary culprits for instability in system-wide studies [9].

Therefore, in this paper we propose a hybrid systems approach for the modeling, analysis, and design of a controller with a new structure meant to replace the inner voltage and current loops to address these concerns. As we will show in Section III, the controller we present is not dependent on control gains that need to be tuned. This provides a promising path to ensure system stability and performance guarantees for system-wide studies, where studies that use the averaged model and inner PI control loops are dependent on the choice of controller gains. We believe this could provide a way to obtain trustworthy and generalizable system-level results. Specifically, Fig. 1 presents the standard control architecture for an inverter (above), and our proposed hybrid control strategy (below) and how we envision it as an alternative.

B. Hybrid Systems for Power and Power Electronic Systems

Even though power and power electronic systems inherently exhibit hybrid dynamics, the field of hybrid systems has only seen limited use in the context of its stability analysis and control design.

Most of the known work in this space has attempted to model general power system problems within a hybrid dynamical systems framework. For example, some applications include the modeling of continuous-valued state variables (e.g., currents and voltages) in systems with discrete-valued control or protection actions (e.g., circuit breaker operation) [10]. Other work has modeled the variation of system-level inertia in the swing equation in a hybrid dynamical systems framework [11].

In the power electronics literature, hybrid systems theory has been primarily used in the context of DC-DC converters. One of the first reported uses of a hybrid systems framework for power electronic converters was in [12], where the authors derive conditions for a safe set and design a hybrid control law to maintain the system state within this set. Further work derives a set of control laws, also for DC-DC converters, that achieve global, asymptotic stability to the desired operating point [13].

More recently, some of these ideas have started to appear in the control of power electronic inverter, for which the desired reference signal is a time-varying sinusoid instead of a constant setpoint [14]. The authors of [14] derive a model for the half-bridge inverter in error coordinates and show that the solution of an optimization problem leads to the desired behavior. More work in this space has also been pursued through sliding mode control ideas for inverters connected to an infinite bus [15], [16].

C. Summary of Contributions

Specifically, beyond prior results in [14], this work’s contributions include i.) an explicit, closed-form expression for a globally-asymptotically stabilizing hybrid control law for the half-bridge inverter that needs no tuning, ii.) an analytical derivation for a global Lyapunov function that proves uniform, global, asymptotic stability of the origin in error coordinates thus achieving the tracking of the reference signal for a hybrid inverter model, iii.) a method to update the controller to be robust against resistive load changes, iv.) analysis and validation of the controller in simulation, and v.) a demonstration in simulation of the controller’s operation in conjunction with a grid-forming control strategy, suggesting the potential of hybrid systems-based control in this practical context. We further aim to aid understanding of the control scheme by interpreting the controller’s behavior, including finding an underlying switching surface. We note that, while the designed controller itself is a switching policy, we call it a hybrid control law because it is implemented in a hybrid dynamical system environment.

D. Paper Organization

The rest of the paper is organized as follows. Section II presents the notation and problem formulation, Section III presents our theoretical results, and Section IV our experimental results. Finally, Section V concludes the paper and discusses limitations and future work.

II. PROBLEM FORMULATION

In this section, we present our modeling approach for the half-bridge inverter and the control problem we aim to solve. The model we use was first proposed in [14].

A. Notation

Dot notation indicates the time derivative of a variable, i.e., $\dot{x} = \frac{dx}{dt}$. Bold-faced capital letters will indicate matrices. $\sigma(\mathbf{A})$ is the spectrum of $\mathbf{A}$, and $\lambda_\mathbf{A}$ an eigenvalue of $\mathbf{A}$. $\| \cdot \|_2$ is the Euclidean vector norm. The operator $\text{sign}(\cdot)$ returns $+1$ if its scalar argument is greater than or equal to zero, and $-1$ otherwise. The operator $\text{Re}(\cdot)$ takes the real part of its argument.

B. Switched Model

We study the half-bridge inverter shown in Fig. 2, composed of a set of switches with an LC filter at its output. We consider a resistive load, $R$, as the natural first case study. Future work will consider RLC loads, ZIP loads (constant impedance, Z, current, I, and power, P), and grid interconnections.
We define two discrete states: i.) SW1 on and SW2 off, and ii.) SW1 off and SW2 on. We also define the control input to be \( u \in \{ +1, -1 \} \), where \( u = +1 \) denotes the first discrete operating state, and \( u = -1 \) denotes the second. We define the state vector as \( x = [v_C, i_L]^T \in \mathbb{R}^2 \), where \( v_C \) and \( i_L \) are the capacitor voltage and inductor current, respectively. Applying Kirchhoff’s current and voltage laws to model the voltage and current dynamics for each discrete state, we obtain equation (1a). Conveniently, the choice of control input, \( u \in \{ +1, -1 \} \), characterizes the inverter’s terminal voltage polarity in Kirchhoff’s voltage law equations such that the dynamic evolution of both discrete states can be compactly characterized by a single equation, (1a). The model describes a linear, time-invariant (LTI) dynamical system with a binary-valued, discrete control input.

\[
\dot{x} = Ax + Bu
\]

(1a)

\[
A = \begin{bmatrix}
\frac{1}{R_C} & \frac{1}{C} \\
-\frac{1}{L} & -\frac{1}{L} 
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
\frac{V_{DC}}{2L}
\end{bmatrix}
\]

(1b)

C. Designing the Reference

The control objective is for \( v_C \) to track a sinusoidal reference signal. Following [14], we first define an oscillator system that produces the appropriate reference signal. Then, we reformulate the dynamical system in terms of error from the desired reference state.

We denote the reference vector \( x_{ref} = [v_{C,ref} \ i_{L,ref}]^T \). We choose a sinusoidal voltage reference \( v_{C,ref}(t) = V_m \sin(\omega t) \), with amplitude \( V_m \) and frequency \( \omega \), both being design parameters. Kirchhoff’s laws allow us to express the appropriate inductor current reference \( i_{L,ref} \) in terms of these parameters as \( i_{L,ref}(t) = \omega CV_m \cos(\omega t) + \frac{1}{R} V_m \sin(\omega t) \).

To design the reference, we define an oscillator system of the form of (2).

\[
\dot{z} = \Theta z = \begin{bmatrix}
0 & \omega \\
-\omega & 0
\end{bmatrix} z
\]

(2)

Note that \( \Theta \) is a skew-symmetric matrix with eigenvalues \( \lambda_0 \in \{ +j \omega, -j \omega \} \) defining an oscillator. Therefore, we know that, for a given initial condition, which can arbitrarily be chosen as \( z(t = 0) = V_m \begin{bmatrix}
0 \\
1
\end{bmatrix} \), \( z(t) = \begin{bmatrix}
V_m \sin(\omega t) \\
V_m \cos(\omega t)
\end{bmatrix}^T \).

We define our state reference to be a linear transformation of the oscillator state, \( z \), to achieve \( x_{ref} \) by

\[
x_{ref} = \Pi z = \begin{bmatrix}
1 \\
0
\end{bmatrix} \frac{1}{R} \omega C z.
\]

(3)

We further compute the reference dynamics to arrive at (4c).

\[
\dot{x}_{ref} = \Pi \dot{z}
\]

(4a)

\[
= \Pi \Theta z
\]

(4b)

\[
= (\Lambda \Pi + B \Gamma)z,
\]

(4c)

where

\[
\Gamma = \frac{2}{V_{DC}} \begin{bmatrix}
\omega & 0 \\
0 & -\omega \end{bmatrix} \left( 1 - \omega^2 LC \right) = \frac{2}{V_{DC}} \begin{bmatrix}
\Gamma_1 & 0 \\
0 & \Gamma_2
\end{bmatrix}.
\]

(5)

Our subsequent analysis will be simplified by defining the state error from the reference signal as \( e = x - x_{ref} \). Combining equations (1a) and (4c), we have that

\[
\dot{e} = Ax + Bu - (\Lambda \Pi z + B \Gamma z)
\]

(6a)

\[
= A e + B (u - \Gamma z).
\]

(6b)

Together, equations (2) and (6b) define the system dynamics.

Now that we have fully characterized the system dynamics in error coordinates, the problem we are trying to solve is to find a control law \( u(t) \) that drives the error coordinates to the origin, implying that the system states will track the desired reference signals in the original coordinates.

III. GLOBAL ASYMPTOTICALLY-STABLE REFERENCE TRACKING

In this section, we present our theoretical stability results including i.) the derivation of an explicit control law with a proof that this control drives the reference error to the zero state, ii.) an analytical solution to the Lyapunov equation as a function of the system’s parameters, and iii.) a proof showing that we can appropriately adjust our control law to be robust against known changes in load, and still track the desired trajectory.

A. Global Asymptotic Stability Result

**Theorem 1.** Consider the dynamical system (6b). Let \( A \) be Hurwitz, i.e. \( \Re \{ \lambda_A \} < 0 \ \forall \lambda_A \in \sigma(\Lambda) \), and let \( \| \Gamma \|_2 < \frac{1}{\sqrt{\omega m}} \). Let \( P \in \mathbb{R}^{2 \times 2} \) be the symmetric, positive-definite matrix that satisfies the Lyapunov equation, \( A^T P + PA = Q \), for a negative definite \( Q \). The existence of such a \( P \) is guaranteed by the assumption that \( A \) is Hurwitz [17]. Then, the switching policy

\[
u = -\text{sgn}(B^T P e)
\]

(7)

results in the uniform, global, asymptotic stability of the origin for the error dynamics (6b).

**Proof:** We propose the candidate Lyapunov function \( V(e) = e^T P e \), which is globally positive definite [18]. Taking its time derivative, we have that

\[
\dot{V}(e) = e^T (A^T P + PA) e + 2(u - \Gamma z)B^T P e.
\]

(8)
Furthermore, using our knowledge of the oscillator state, \( z(t) \), which we know is stable by construction, we have that
\[
\Gamma z = V_m \frac{2}{V_{DC}} (\Gamma_1 \sin(\omega t) + \Gamma_2 \cos(\omega t)), \tag{9a}
\]
\[
= V_m \left\| \Gamma \right\|_2 z(\omega t + \psi), \quad \psi = \arctan(\frac{\Gamma_1}{\Gamma_2}), \tag{9b}
\]
\[
\Rightarrow \left\| \Gamma z \right\| \leq V_m \left\| \Gamma \right\|_2, \quad \forall t. \tag{9c}
\]
Therefore, the worst case magnitude for \( u - \Gamma z \), i.e., \( \max_z \left\{ u - \Gamma z \right\} \), is obtained when \( \sin(\omega t + \psi) = -1 \). So, we can upper bound \( \dot{V}(e) \) using the Lyapunov equation and (9c) to yield the following.
\[
\dot{V}(e) \leq e^\top Q e + 2(u + V_m \left\| \Gamma \right\|_2)B^\top Pe \tag{10}
\]

We know the term \( e^\top Q e \leq 0 \), \( \forall e \). Therefore, to ensure stability, we force the second term, \( 2(u + V_m \left\| \Gamma \right\|_2)B^\top Pe \), to be also less than or equal to zero using the following. Since by assumption \( \left\| \Gamma \right\|_2 < \frac{1}{V_m} \), and \( u \in \{+1,-1\} \) by design, then the term \( u + V_m \left\| \Gamma \right\|_2 \) will take the sign of \( u \). By choosing a switching policy defined as \( u = -\text{sign}(B^\top Pe) \) under the stated assumptions, the second term will always be less than or equal to zero, and \( \dot{V}(e) \leq 0 \) always holds, implying that \( V(e) \) is a valid, global Lyapunov function for the given dynamics. We can thus conclude that the origin in error coordinates is a uniformly-, globally-, asymptotically-stable equilibrium point. The states will track the desired trajectories in the original coordinates.

It is worth noting that the assumptions required for this theorem are not very restrictive. For a passive RLC circuit as we have here, it can be proven that the eigenvalues of \( A \) will always have strictly negative real parts. Moreover, \( \left\| \Gamma \right\|_2 < \frac{1}{V_m} \) is easily satisfied for a range of realistic parameter values.

It is also worth noting that the control policy (7) is conservative since (10) is a worst-case upper bound for the Lyapunov function: it is only reached when \( \sin(\omega t + \psi) = -1 \). This implies that it is, potentially, inherently robust to some disturbances. In fact, it can be shown that this kind of formulation is actually robust with respect to disturbances [19].

**B. Interpretation of The Control Result**

1) **Main theorem condition:** The main theorem condition is \( \left\| \Gamma \right\|_2 < \frac{1}{V_m} \). It turns out that \( \Gamma_2 = 1 - \omega^2 L C \) is 0 because it is the filter design equation for its resonant frequency. This implies that the theorem condition reduces to
\[
V_m < \frac{1}{\omega L} \frac{R V_{DC}}{2}. \tag{11}
\]
This condition can be interpreted as a requirement on the amplitude of the desired sinusoidal voltage amplitude, \( V_m \), to be strictly less than a scaling of the DC link voltage, \( V_{DC} \).

In particular, a larger \( \omega \), smaller, \( R \), and larger \( L \) make the bound tighter, implying it is harder to control because there is less slack on the bound. This is intuitively reasonable because high frequency implies a faster changing reference to track, a smaller \( R \) implies larger power consumption at the load, and a larger \( L \) implies current is more stiff to change, all of which intuitively make tracking the reference signal more difficult. A larger \( V_{DC} \), meaning a larger buffer in the DC link voltage, will also lead to more slack on the bound.

2) **Control action:** The derived control law is a switching-based control, however, in this context it does not come with the traditional concerns for dynamic systems. For example, chattering, which is usually undesirable, is actually an inherent property of inverters: they are designed to execute extremely fast switching between discrete states to achieve the desired sinusoidal reference tracking, and the filter is designed to remove the higher order harmonics. Moreover \( u \) solely determines the discrete state of the system, and does not inject any signal directly by feedback, therefore there is no potential for high-gain injection either.

3) **An underlying switching surface:** The control law (7) can be interpreted as a switching decision based on whether or not the dynamics are over or under a specific hyperplane in a two dimensional state space. This is because of the use of the sign function: the control law is evaluating if the dynamics in the state space lie over or under the hyperplane defined by \( B^\top Pe = 0 \). If it is over this hyperplane, then the \( \text{sign(\cdot)} \) function will return \( +1 \), and a \( -1 \) otherwise. This means that the hyperplane can further be interpreted as a switching surface, or a sliding mode.

**C. Explicit Lyapunov Function in Terms of \( R, L, \) \( C \)**

In general, finding an explicit Lyapunov function that guarantees stability of a hybrid dynamical system can be difficult, even if each discrete dynamic state is LTI. To do this, researchers often resort to semidefinite optimization programs [20]. However, since the error coordinate dynamics (6b) reduce our system model to a single set of LTI dynamics, we can calculate the appropriate \( P \) matrix that satisfies the Lyapunov equation, \( A^\top P + PA = Q \), by solving a system of linear equations.

In this section, we show that this procedure allows us to express the \( P \) matrix explicitly in terms of arbitrary load and inverter parameters, \( R, L, \) and \( C \).

Noting that \( P \) is symmetric, we have that
\[
P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \tag{12}
\]
where \( p_{12} = p_{21} \).

Selecting \( Q = -I \), the matrix equation \( A^\top P + PA = -I \) evaluates to
\[
\begin{bmatrix} -2p_{11} & -2p_{12} \\ \frac{p_{11}}{C} - \frac{p_{12}}{RC} & \frac{p_{11}}{L} - \frac{p_{12}}{RC} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{13}
\]

We can use this equality to solve for \( p_{11}, p_{12}, \) and \( p_{22} \).

Doing so gives us an explicit solution for \( P \) in (14), as desired.
\[
P = \frac{1}{2} \begin{bmatrix} RC + \frac{RC^2}{L} & -C \frac{L}{R} + RC \\ -C \frac{L}{R} + RC & RL + \frac{L}{R} + RC \end{bmatrix}. \tag{14}
\]
Therefore, a function \( V(e) = e^\top Pe \), with \( P \) given by (14), is guaranteed to be a global Lyapunov function for the dynamical system (6b) for the designed controller.
Remark. While $Q$ is oftentimes parameterized with some coefficient $\alpha > 0$ for tuning purposes in these kinds of proofs, $Q = -\alpha I$ for example, it can be proven that in this scenario, because of the use of the $\text{sign}(\cdot)$ operator, there is no change in the control decision when $\alpha$ is introduced. Therefore, this controller requires no tuning.

What can be done, however, is to make a different choice of $Q$, which will change $P$. This in turn will affect the slope of the switching surface defined by $B^\top Pe = 0$ which can be chosen to prioritize $e_1$ or $e_2$.

**D. Controlling for Known Changes in Load**

One motivation for finding an analytical expression for $P$ is to handle changes in the system parameters. Note that our control policy, (7), relies on knowing $P$. For example, if the load resistance, $R$, or filter parameters, $L$ or $C$, change, the $A$ matrix that defines the inverter dynamics in (1a) will change accordingly, and the switching strategy derived in section III-A may no longer achieve reference tracking if we do not update $P$ to reflect that change.

We have shown that we can use (7) to asymptotically drive the states to our desired sinusoidal reference signals. Moreover, we have shown that for a specific choice of $R$, $L$, $C$ parameters, we can analytically compute a $P$ matrix which makes $V(e) = e^\top Pe$ a global Lyapunov function for the dynamical system, and thus we can implement our control law with such a $P$.

Based on these two arguments, we now make the claim that if the system load changes, that is, $R$ changes, and we know both loading scenarios, we can update our control law, (7), through (14) at the time the load changes. This, however, assumes that the load does not continuously switch between different $R$ values too quickly, i.e. it’s a step disturbance in the load. This will allow the control to still asymptotically drive the error state to the origin [21]. This is a reasonable assumption from the power and power electronics systems perspectives.

We provide validation through simulation for these three theoretical results in the following section.

**IV. Controller Validation Through Simulation**

In this section, we test in simulation the performance of our switching controller. The simulation parameters are shown in Table I. $L$ is designed to have a 30% inductor current ripple, a common practice in power electronics [22], while $C$ completes the filter design to have a resonant frequency of 60 Hz with low gain. $R$ is chosen as the nominal load equivalent to a power draw of 1 kW at the nominal operating voltage.

The inverter operates at a 10 kHz switching frequency, and we implement a simulation update frequency of 1 MHz. We also try a 100 kHz switching frequency for some experiments. These are realistic switching frequencies. We assume they are ideal: operating instantaneously and without losses.

| Parameter                  | Symbol | Value   |
|----------------------------|--------|---------|
| Load resistance            | $R$    | 14.4 Ω  |
| Inverter inductance        | $L$    | 9.6 mH  |
| Inverter capacitance       | $C$    | 0.7 mF  |
| DC supply voltage          | $V_{DC}$ | 1,200 V |
| Target (reference) frequency | $f$   | 60 Hz   |
| Target (reference) angular frequency | $\omega$ | $2\pi f \approx 377 \text{ rad/s}$ |
| Target (reference) magnitude | $V_m$ | $120\sqrt{2}/\sqrt{V \approx 170 V}$ |

**TABLE I: Parameter values used for simulation experiments.**

![Simulated Tracking Results](image)

Fig. 3: We show accurate tracking of the desired reference signal. a) Our controller drives the inverter voltage, $v_C$, to the reference signal voltage, $v_{C,ref}$, when there is a 240 V difference in the initial condition at $t = 0$ s. The second panel shows the reference tracking error over time. b) Our controller drives the reference tracking error towards the origin in the error coordinates state space. The switching surface induced by (7) is shown by the black line.

**A. Asymptotic Stability Under Constant Load**

We first test the performance of our controller at recovering the target reference signal when starting from an off-reference initial condition. As shown in Fig. 3, the capacitor voltage, $v_C$, starts with a large deviation in the initial condition of $v_C(t = 0) = 240$ V, but the controller is able to drive it to track the reference over time. The controller also allows for step changes in voltage amplitude and frequency.

The bottommost panel of Fig. 3 shows the trajectory of the system in the state space defined by the error coordinates. We see how the trajectory evolves over time in the state space and how the hyperplane defined by $B^\top Pe = 0$ indeed is a switching surface and a sliding mode of the error coordinates.
Fig. 4: We show our controller’s performance under known and unknown changing load conditions. a) We change our loading condition to $R_{\text{disturbed}} = 10 \Omega$ and $64.4 \Omega$ at $t = 1.0 \text{s}$. Updating our 10 kHz controller in real time maintains accurate tracking of the reference signal, as evidenced by small error. b) The same load disturbances for a non-updated controller. $R_{\text{disturbed}} = 10 \Omega$ has increased error but good averaged behavior. $R_{\text{disturbed}} = 64.4 \Omega$ causes larger amounts of error. c) We plot the time-averaged absolute tracking error at $t = 4 \text{s}$ for a variety of load conditions when the controller is not updated. These results suggest that the updated controller is robust to parameter changes, and even when updating is not possible the controller may still exhibit reasonable performance.

**B. Stability Under Changes in Load**

If the loading conditions change, we can ensure continued tracking of the reference by updating the control policy (7) through $P$ in (14), as described in section III-C. We validate this result in simulation by changing our initial load of $R = 14.4 \Omega$ to different values during the simulation. As shown in panel a) of Fig. 4, a controller updated through $P$ with the new $R$ continues tracking the reference after a switch to $R_{\text{disturbed}} = 10 \Omega$ or $64.4 \Omega$.

Updating the controller in response to load changes allows us to guarantee continued global, asymptotic stability. Moreover, empirically we find that for small disturbances an unmodified controller may also perform well. However, stability is lost for large disturbances such as a load change to $5 \Omega$, as shown in panels b) and c) of Fig. 4.

**C. Stability Under Extreme Reference Values**

Since the result of Theorem 1 assumes that $\|\Gamma\|_2 < \frac{1}{\sqrt{V_m}}$, and $\|\Gamma\|_2$ depends on $\omega$, our controller is not guaranteed to track a reference signal for all possible combinations of $V_m$ and $\omega$. As shown in Fig. 5, while good tracking performance is possible for a wide range of practical values of $V_m$ and $\omega$, increasing tracking error occurs when either value is too large. This occurs because the assumptions of Theorem 1 no longer hold. As can be seen in the figure, there is very good agreement between the predicted range of values where our controller should be stable and the experimentally determined range of values where we achieve stable reference tracking in practice, supporting our theoretical result.

**D. Layering a Grid-Forming Control Strategy**

Until now, we have only focused on having the inverter track a sinusoid with a constant amplitude and frequency. In practice, inverters will regulate their sinusoidal voltage amplitude and frequency based on a predetermined grid-forming control strategy. In general, these try to optimize for some system-level operation criteria, for example, power sharing, frequency regulation, or a stability metric. Examples in the literature include droop control, virtual oscillator control, synchronous machine emulation, and matching control, among others [6], [7], [8], [9].

These control strategies have been implemented in the past using the inverter averaged model, and the inner voltage and current loop structure [23]. However, in this work, we use
simulations to explore the performance of our hybrid control strategy while layering on top a grid-forming control strategy. We choose droop control as an example.

**Remark.** Droop control is often used in the context of power sharing and frequency regulation in a network of generators or inverters. In this experiment, we use droop control as an established way to change the voltage amplitude and frequency to determine the ability of our proposed controller to handle continuous changes in reference values, rather than to test synchronizing or power sharing capabilities.

Droop control regulates the rms voltage, \( V_{rms} = V_m / \sqrt{2} \), and frequency, \( \omega \), of the reference signal according to the following equations.

\[
\begin{align*}
\omega &= \omega^* + k_p (P^* - P) \quad (15a) \\
V_{rms} &= V_{rms}^* + k_q (Q^* - Q) \quad (15b)
\end{align*}
\]

Here \( P^* \) and \( P \) represent, respectively, the setpoint and measured values for real power supplied by the inverter while \( Q^* \) and \( Q \) represent, respectively, setpoint and measured values for reactive power. \( V_{rms}^* = \sqrt{2}V_m \) and \( \omega^* \) are the setpoints for the reference signal, and \( k_p \) and \( k_q \) are the droop coefficients [9].

For our experiment, we begin with \( V_{rms}^* = 120 \text{ V} \) and \( \omega^* = 120 \pi \text{ rad/s} \), as before. We change the steady-state real power being drawn by the resistive load as given by

\[
P = \frac{V_{rms}^2}{R} = 1.44 \text{ kW}
\]

by changing the load to \( R = 10 \Omega \). Although Theorem 1 does not guarantee stability under continuously changing setpoints provided by droop control, we find that, for values of \( k_p = 0.01 \text{ rad/s kW}^{-1} \) and \( k_q = 0.0025 \text{ V} \text{ var} R^{-1} \), our controller is able to track the changing reference very well, as shown in Fig. 6.

**E. Benchmarking Against the Averaged Inverter Model and PI Control Loops**

To study the performance of our controller, we compare it against the state-of-the-art control architecture: droop control as the grid-forming control strategy with the well-known inner PI control loops for the averaged inverter model. We implement the nested PI control architecture and averaged inverter model as in [9]. Our PI controller gain design is inspired by [24].

We perform the same droop simulation for both cases with the aforementioned droop parameters. Fig. 6 shows our results. It can be seen how before the disturbance at \( t = 0.2 \text{ s} \), both the PI control loops and our hybrid model perfectly track the desired sinusoidal reference signal. After the change in load, we can see how both the voltages lag the 60 Hz reference because droop control instructs the actual voltage signals to slow down. We see how our hybrid approach matches the averaged model and PI controllers very closely.

**Remark.** It is worth mentioning that the averaged model and PI controllers have some benefits in this simulation that would not be seen in practical application. Specifically, this approach does not account for any kind of switching. In practice, the averaging is used for a sinusoidal PWM strategy which determines the ON/OFF states of the transistors. This behavior is not captured in this simulation, thus showing idealized behavior in the simulation that will not be seen in practice, and making the comparison favor the averaged model and PI control approach. Future work will address this comparison in more detail.

V. DISCUSSION AND CONCLUSIONS

As the power grid transitions away from fossil fuel-based generation, it is expected that increasing numbers of DC energy sources will supply power to the grid. The stability results we present are a step to tackle some of the challenges that arise in ensuring reliability of the grid with increased numbers of CIGs.

We demonstrate a controller for a half-bridge inverter that explicitly models the switches instead of making time-averaged assumptions. The controller achieves globally asymptotically stable reference-tracking capabilities, and can handle load changes and changing inputs from a grid-forming control strategy. Furthermore, using a Lyapunov function to guarantee error convergence also suggests further avenues for switching policy design options. For example, a future experiment could allow a small amount of tracking error in return for reducing the switching frequency and reducing wear on the inverter switches [13]. Moreover, because we consider a single-phase system, the results found directly extrapolate to balanced three-phase systems.

In comparison with the state of the art, our proposed controller provides two significant advantages. The first is the safety guarantee. We can mathematically prove that we
can converge to the desired reference signal from anywhere in the state-space. We are unaware of similar claims for the averaged model and PI controllers. A second advantage is there are no control gains that need to be tuned in this design. A significant drawback of the inner PI control loops is the reliance on its gains and the different methodologies to design them. Studies have shown how a poor choice of control gains can lead to instability [9]. Our hybrid controller avoids this tuning process altogether.

Though we believe a hybrid systems approach holds promise for further exploration, we briefly note some limitations of the inverter control design we developed here. First, our stability result relies on the ability to measure \( e \) and respond to changes in the sign of \( \text{B}^\top \text{P}e \), which requires fast sensing and actuation. Furthermore, our stability result is so far only guaranteed for constant resistance loads, whereas most loads include constant inductance or capacitance, or even appear as elements with impedances that vary in response to source voltages. As a step toward resolving this limitation, we have shown the ability of our controller to update in real time to handle changes in resistive load, and empirically show robustness. In both of these cases, it may still be possible to provide rigorous stability guarantees after taking realistic sampling frequencies and control delays into account with further development of the theory. Moreover, ideally we would only have to rely on a sinusoidal voltage reference, as opposed to requiring a current reference as well, which is not always possible in practice. And, while we have shown simulation results for continuously changing set points, these have not yet been studied theoretically. Finally, this analysis is limited to a two-state system which we seek to extend to larger scale network-level studies. Singular perturbation theory and graph theory could prove useful tools for these analyses. We leave considering these research directions to future work.

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