Kaigorodov spaces
and their Penrose limits

Christophe Patricot ¹

DAMTP,
University of Cambridge,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge CB3 0WA,
U.K.

Abstract

Kaigorodov spaces arise, after spherical compactification, as near horizon limits of M2, M5, and D3-branes with a particular pp-wave propagating in a world volume direction. We show that the uncompactified near horizon configurations $K \times S$ are solutions of $D = 11$ or $D = 10$ IIB supergravity which correspond to perturbed versions of their $AdS \times S$ analogues. We derive the Penrose-Güven limits of the Kaigorodov space and the total spaces and analyse their symmetries. An İnönü-Wigner contraction of the Lie algebra is shown to occur, although there is a symmetry enhancement. We compare the results to the maximally supersymmetric CW spaces found as limits of $AdS \times S$ spacetimes: the initial gravitational perturbation on the brane and its near horizon geometry remains after taking non-trivial Penrose limits, but seems to decouple. One particular limit yields a time-dependent homogeneous plane-wave background whose string theory is solvable, while in the other cases we find inhomogeneous backgrounds.

¹email: C.E.Patricot@damtp.cam.ac.uk
1 Introduction

Penrose showed [1] that any spacetime, in the neighbourhood of a null geodesic containing no conjugate points, has a plane wave limit spacetime. This limit is essentially local, and may be thought of physically as the spacetime seen by an observer approaching the speed of light at a given point, along a particular null geodesic, and recalibrating his clock to its affine parameter. It can thus be understood as a null Lorentz boost together with a (singular) uniform rescaling of the coordinates which leaves the affine parameter along the chosen null geodesic invariant. Güven [2] extended the concept to supergravity theories: given a solution of the supergravity equations of motion, there exists a limit of this solution which also satisfies these equations and has a plane-wave spacetime.

It has long been known that string theory on gravitational wave backgrounds is exact and potentially solvable [3], mainly because all of the curvature invariants of these spaces vanish [4]. The recently discovered [5] (BFHP) maximally supersymmetric plane-wave solution of IIB string theory was shown to be exactly solvable [6]. The fact that this maximally supersymmetric plane-wave, together with its 11-dimensional analogue [7] (KG), arise as Penrose limits of $AdS \times S$ spacetimes [8], suggests one could probe into the string or M-theories of the latter. This idea was vindicated by the BMN proposal [9], or plane-wave/CFT correspondence.

One should not forget however that $AdS \times S$ spacetimes, though maximally supersymmetric solutions of supergravity or IIB string theory by themselves, also essentially arise in these theories as near-horizon limits of extremal branes. These branes also are fundamental dynamical objects which maybe in essence should be thought of as candidates for “world-volumes”. The AdS/CFT correspondance together with the BMN correspondance certainly give an insight into gravity theories on the extremal branes whose near horizon geometry is of $AdS \times S$ type, but what if the geometry of these branes were perturbed?

In [10] and [11], the near-horizon geometries of non-dilatonic extremal branes with a pp-wave propagating in the world-volume were analysed, and it was found that in the most simple case one gets a product of a sphere and a homogeneous space of constant negative cosmological constant which generalizes the 4 dimensional Kaigorodov space [12]. We will show in fact that the near horizon configurations of these particular M2, M5 and D3-pp-branes are also solutions of D=11 supergravity or D=10 IIB supergravity, and are analogous to the $AdS \times S$ spaces. Therefore it seems interesting to find the Penrose limits of these spaces, not
for the mere sake of geometry, but rather to find backgrounds to study string theory which naturally arise from perturbed branes.

Indeed, string theory was explicitly solved in the light-cone gauge not only in the time-independent Cahen-Wallach spaces, but also recently in some time dependent plane-wave backgrounds ([13] [14], and earlier references therein). The quantization of strings and particles gives rise to time-dependent harmonic oscillators, and whenever the plane-wave spaces are homogeneous, it seems the equations of motion can be solved explicitly (for the particle case see [15]). It turns out that one of the plane-wave spaces we derive is very similar to that in [14], and on these grounds should admit an explicitly solvable string theory, though one of the oscillators has negative mass. The article is organized as follows.

In section 2 we first briefly review how Kaigorodov spaces arise as near horizon limits of M2, D3, and M5 pp-branes [10]. We show that the near-horizon geometries of type $K_{n+3} \times S^d$ together with their fluxes, satisfy the equations of supergravity or IIB theory in the respective cases. In fact, these configurations can be thought of as perturbed $AdS \times S$ spaces. We then briefly describe the geometry of the Kaigorodov space $K$. This non-conformally flat homogeneous Einstein spacetime can be interpreted [16] as an $AdS$ space with a propagating gravitational wave. Although it is non-static and has a pp-singularity, we show that it is stably causal in the sense of [17]. In fact, the gravitational wave will be reminiscent in the non-trivial Penrose limit spaces we find. In section 3 we derive the Penrose limit plane-wave spaces of the Kaigorodov space alone, and analyse the symmetries of a particular limit space which is homogeneous. An Inöni-Wigner contraction of the group of bosonic symmetries is shown to occur although there is an enhancement of 1 symmetry in the limit. We then derive in section 4 the Penrose limits of the uncompactified $K_{n+3} \times S^d$ spacetimes and their associated field-strengths, by considering both the null geodesics which wind around the sphere and those which do not. As opposed to the $AdS \times S$ case, we get non-trivial limits even in the non-winding case. These provide homogeneous plane-wave backgrounds with solvable string theories which have vanishing field strength and constant dilaton. The non-winding case yields non-homogeneous plane-waves whose matrix $A_{ij}$ in Brinkman coordinates is not diagonal.
2 Kaigorodov spaces in M/String theory

2.1 Near-horizon limits of perturbed branes

We first review in detail the near horizon geometry of an M2-brane with a gravitational wave propagating in one of its world volume directions, and then give the general expressions of the metrics and derive the field strengths of the near horizon geometries in the 3 cases of the M2, D3, and M5 pp-branes. The pp-wave or brane-wave solutions actually arise as intersections of branes [18] [19].

Following [10], a $D=11$ supergravity solution describing a non-dilatonic extremal M2-brane with a pp-wave is given by

$$
\begin{align*}
    ds^2_{11} &= H^{-2/3}(-K^{-1}dt^2 + K(dx_1 + (K^{-1} - 1)dt)^2 + dx_2^2) + H^{1/3}(dr^2 + r^2d\Omega_7^2), \\
    F_4 &= dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1}, \\
    H &= 1 + Q_1 r^b, \quad K = 1 + Q_2 r^b,
\end{align*}
$$

where $r$ is the distance to the brane in the bulk. The affine change of coordinates $t' + x'/2 = t, t' + 3x'/2 = x_1$ yields a simpler expression, similar to the formalism used in [11]:

$$
\begin{align*}
    ds^2_{11} &= H^{-2/3}(2dt'dx' + (K + 1)dx'^2 + dx_2^2) + H^{1/3}(dr^2 + r^2d\Omega_7^2), \\
    F_4 &= dt' \wedge dx' \wedge dx_2 \wedge dH^{-1},
\end{align*}
$$

and one can change $K$ into $K - 1$ by further coordinate transformation. This shows that from the brane point of view, the gravitational wave propagates along the $t'$ null direction and is uniformly distributed along the world-volume directions, but although this solution is often denoted as a pp-brane, the metric (1) does not possess a covariantly constant null Killing vector: it merely arises as the intersection of a membrane with an 11 dimensional pp-wave.

Note that the wave is not localized on the brane since $K(r)$ depends on $r$, but spacetime is asymptotically flat away from the brane. The near horizon limit $r \to 0$ together with rescalings of the coordinates by powers of $Q_1$ and $Q_2$, and the change of variable $\rho = \ln r$, induce the following metric and field strength:

$$
\begin{align*}
    ds^2_{11} &\sim Q_1^{1/3}(e^{-2\rho}dx'^2 + e^{4\rho}(2dt''dx'' + dx_2''^2) + d\rho^2 + d\Omega_7^2), \\
    F_4 &\sim 6Q_1^{1/2}e^{6\rho}dt'' \wedge dx'' \wedge dx_2'' \wedge d\rho
\end{align*}
$$

This is the metric of the product space of a 4 dimensional Kaigorodov space [12] of negative cosmological constant $\Lambda = -12Q_1^{-1/3}$ and a 7-sphere of radius $R^2 = Q_1^{1/3}$. It will be denoted
The cases of the near horizon geometries of the M-5 supergravity brane and the D-3 type
IIB brane are obtained in a similar way. They yield $K_{n+3} \times S^d$ spaces, with $K_{n+3}$ the
$n + 3$-dimensional generalisation of $K_4$, and $S^d$ the $d$-dimensional sphere. The (negative)
cosmological constant $\Lambda$ of $K_{n+3}$ and the radius of the sphere $R_{S^d}$ depend on the charges of
the initial branes. It turns out these can be expressed quite simply combining results of [10] and [11].

2.2 $K_{n+3} \times S^d$ spaces as solutions of supergravity

In $n + 3 + d = 11, 10$ or $11$ dimensions, for $n = 1, 2, 4$ respectively (M2, D3, M5 pp-
brane), the near horizon geometry is a (topological) product of the Kaigorodov space $K_{n+3}$
of cosmological constant $\Lambda = -4(n + 2)L^2$ and a $d$-sphere $S^d$ of radius $R = 1/(Ln)$, with $L$
depending on the charges of the brane. If we call $Q$ the charge of the extremal brane, in the
sense that its harmonic function is $H(r) = 1 + Q/r^{(d-1)}$, then the Kaigorodov parameter $L$
of the horizon geometry is $L = (1/n)Q^{-1/(n-2)}$, with $d$ the dimension of the $S^d$. After rescaling
the colatitude coordinate $\psi$ of $S^d$ by a factor of $1/(Ln)$, the 3 metrics read:

$$ds_{n+3+d}^2 = e^{-2Ln}\rho dx^2 + e^{4L}\rho (2dxdt + dy^1 dy^1) + \rho^2 + d\psi^2 + (Ln)^{-2}\sin^2(Ln\psi)d\Omega_{d-1}^2$$

(4)

Here $d\Omega_{d-1}^2$ is the surface element of a $d-1$-sphere of unit radius, and $i \in \{1, \ldots, n\}$.
Neither [10] nor [11] give the expressions of the fluxes in the limit. In our coordinate system,
restoring $n = 1, 2, 4$ for the M2, D3, M5-pp-branes respectively, the field strengths are:

$$F_4 = 6Le^{6L}\rho dp \wedge dx \wedge dt \wedge dy^1 \quad (K_4 \times S^7)$$

$$F_5 = 8Le^{8L}\rho dp \wedge dx \wedge dt \wedge d^2y^1 + [8Le^{8L}\rho dp \wedge dx \wedge dt \wedge d^2y^1] \quad (K_5 \times S^5)$$

$$*F_4 = 12Le^{12L}\rho dp \wedge dx \wedge dt \wedge d^4y^1 \quad (K_7 \times S^4)$$

(5)

The field equations of supergravity [20] with the fermionic fields set to zero, in the conventions
used in [21], read:

$$R_{MN} = \frac{1}{12}(F_{MPQR}F_{NPQR} - \frac{1}{12}g_{MN}F_{PQRS}F^{PQRS})$$

(6)

$$dF = 0$$

(7)

$$d(*F) = \frac{1}{2}F \wedge F$$

(8)
Calling $g(K_{n+3})$ the determinant of the metric of the Kaigorodov space $K_{n+3}$, we see that 
\[ \sqrt{-g(K_{n+3})} = e^{2L(n+2)\rho}. \]
Thus the flux $F_4$ is proportional to the volume form on $K_4$ arising from the metric, hence is closed and co-closed. As $F_4 \wedge F_4 = 0$, $F_4$ satisfies (5). The Einstein equation (6) follows since $K_4$ and $S^7$ are Einstein spaces. In detail, with $\mu, \nu$ labelling the coordinates of $K_4$ and $a, b$ those of $S^7$, the right-hand-side of (6) expands to:

\[
R_{\mu\nu} = -18L^2 g_{\mu\nu} + 6L^2 g_{\mu\nu} R_{ab} = 6L^2 g_{ab}
\]

For Kaigorodov spaces $R_{\mu\nu} = -4(n + 2)L^2 g_{\mu\nu}$ (see next section) and for $d$-dimensional spheres of radius $R$, $R_{ab} = (d - 1)R^{-2} g_{ab}$, but here $R = (L\nu)^{-1}$. Hence (6) is satisfied, and $(K_4 \times S^7, F_4)$ is a solution of supergravity.

Similarly, since $\ast[F_4]$ is proportional to the volume form of $K_7$, $F_4 \equiv -\ast(\ast[F_4])$ is closed and co-closed, and proportional to the volume form of $S^4$. In fact $F_4 = +12L\text{Vol}_S^4$, thus $R_{\mu\nu} = -24L^2 g_{\mu\nu}$ (on $K_7$) and $R_{ab} = 48L^2 g_{ab} = (4 - 1)(4L)^2$ (on $S^4$). Hence $(K_7 \times S^4, F_4)$ is a solution of supergravity. $F_5$ is proportional to the self dualized volume form of $K_5$ or $S^5$, and the same computations show that $(K_5 \times S^5, F_5)$ is a solution of the field equations of chiral $N = 2$ $D = 10$ supergravity. These solutions are completetly analogous to the 3 $AdS_{n+3} \times S^d$ configurations, in the same way as the M2, M5 and D3-pp-branes are analogous to their “flat” versions. We now review some important geometric features of Kaigorodov spaces.

### 2.3 Kaigorodov spaces versus $AdS$ spaces

Some general properties of Kaigorodov spaces can be found in [10] and [16]. In $n+3$ spacetime dimensions, letting $L = \frac{1}{2} \sqrt{-\Lambda/(n + 2)}$, their metric reads:

\[
ds^2_{n+3} = e^{-2nL\rho} dx^2 + e^{4L\rho}(2dxdt + \sum_{i=1}^{n} dy^i)^2 + d\rho^2
\]

(9)

These spaces are solutions of Einstein pure gravity with cosmological constant $\Lambda$; they admit $\frac{1}{2}n(n + 3) + 3$ Killing vectors and are homogeneous spaces. They are not static, and are $1/4$ supersymmetric. We show in section 3.1 that they are stably causal in the sense of Hawking [17].

The change of coordinates

\[
z = e^{-\rho/R}, \quad x = Rx^+, \quad t = Rx^-, \quad y^i = Rx^i
\]
where \( R = 1/(2L) \), takes the metric to a horospherical-type form (\( z > 0 \) only)

\[
ds_{n+3}^2 = \frac{R^2}{z^2} \left( 2dx^+dx^- + z^{n+2}(dx^+)^2 + dx^i dx^i + dz^2 \right)
\]  

(10)

In this appendix, starting from this expression of the metric, we find an isometric embedding of the Kaigorodov space \( K_{n+3} \) into a space of signature \((2, n+3)\). In this coordinate system, the metric of the uncompactified \( K_{n+3} \times S^d \) spaces \( (4) \), with the radii of the spheres \( 1/(Ln) \), read:

\[
ds_{n+3+d}^2 = \frac{R^2}{z^2} \left( 2dx^+dx^- + z^{n+2}(dx^+)^2 + dx^i dx^i + dz^2 \right) + \left( \frac{2}{n} \right)^2 R^2 (d\bar{\psi}^2 + (\sin \bar{\psi})^2 d\Omega^2_{d-1})
\]

Podolšký [16] argued that the Siklos spaces (a general family of spacetimes containing the Kaigorodov space) can be viewed as an AdS space-time with a propagating gravitational wave, whose spatial direction rotates at a constant velocity in orthonormal frames parallelly transported along timelike geodesics. He also showed that these spaces have a pp-singularity at \( z = +\infty \): the geodesic deviation equation becomes singular, but the square of the Riemann tensor remains finite. This occurs at \( r = 0 \) in \( (1) \), which corresponds to the brane horizon for us. We shall show in section 3.2 that this pp-singularity, which represents the divergence of tidal forces as one approaches the brane, can remain after taking the Penrose limit.

Clearly \( (10) \) can be interpreted as a wave propagating on the horospheres of \( AdS_{n+3} \), but \( \partial_{x^-} \) is not covariantly constant. However this form of the metric suggests the rescalings:

\[
(x^-, x^+, x^i, z) \mapsto (x^-, \Omega^2 x^+, \Omega x^i, \Omega z)
\]

(11)

whereupon taking the singular limit \( \Omega \to 0 \) the metric becomes that of \( AdS_{n+3} \) in horospherical coordinates

\[
ds_{n+3}^2 = \frac{R^2}{z^2} \left( 2dx^+dx^- + dx^i dx^i + dz^2 \right)
\]

Although \( \partial_{x^-} \) is null, Killing and hence geodetic, this is not a Penrose limit, since the metric is not rescaled by \( \Omega^{-2} \) to yield conservation of the affine parameter along \( \partial_{x^-} \). The corresponding Penrose limit, as explained in the next section, yields flat space. Thus \( (11) \) can be interpreted as an infinite unrescaled boost of the Kaigorodov spacetime, which yields \( AdS \). The dynamical interpretation of this boost is unclear though. (One also gets the \( AdS \) metric by rescaling all the coordinates by \( \Omega \) and letting \( \Omega \to 0 \). In \( (11) \), it is shown that the boundary CFT energy-momentum tensor of the Kaigorodov space is a constant...
null momentum density, therefore it is the Kaigorodov space which should be thought of as an infinitely boosted $AdS$ spacetime, and not the reverse. It will be more relevant in our analysis to consider Kaigorodov spaces just as $AdS$ spacetimes with a gravitational wave perturbation.

In this sense, the near horizon limits: $M/D$-brane $\rightarrow AdS \times S$ and $M/D$-pp-brane $\rightarrow K \times S$ show a nice geometric behaviour of the world-volume gravitational waves under the limiting procedure: indeed the null direction of propagation of the wave on the brane, $t'$ in (1), becomes that of the wave on the Kaigorodov space, ie $t$ in (9) or $x^-$ in (10). In a way these perturbations on the branes can be added either before or after the near horizon limiting procedure, and yield a perturbed $AdS$ spacetime. This may be viewed as a decoupling of the perturbation.

On the Penrose limit point of view, any 10 or 11 dimensional plane-wave spacetime we will obtain can trivially be thought of as flat space or a maximally supersymmetric plane-wave space, with a superposed gravitational wave breaking some of the symmetries. Formally, the analogy between flat membranes and membranes with a gravitational wave remains both after taking the near-horizon limit and then taking a Penrose limit which is not trivial. We now derive the Penrose limits of $K_{n+3}$, and analyse a particular homogeneous plane-wave we get.

3 Penrose Limits of $K_{n+3}$

3.1 Construction

To classify all the possible Penrose limits of $K_{n+3}$, we could follow the method of "celestial spheres" [8], consisting of looking at the orbits of the tangent vectors at a point under the isotropy subgroup, but this requires finding a group representation of the Killing vectors. We shall just mention it before considering a particular limit. Clearly starting from the metric [9] the null geodesics of $K_{n+3}$ which are going to give non-trivial Penrose limit spaces are those for which $\rho$ varies with the affine parameter. Otherwise the $\rho$ dependence of the metric vanishes in the limit and we get flat Minkowski space. In particular, the limit along the null geodesic of tangent vector $\partial_t$, ie along $\partial_{x^-}$ the direction of propagation of the gravitational wave in the horospheres of [10], yields flat space.

Because null hypersurfaces play a crucial role in finding coordinates adapted to taking Penrose
limits \[1\] \[23\], we adopt the Hamilton-Jacobi formulation. The following formalism is not explained in \[23\], and it seems that it provides a general way of finding adapted coordinates of type \[15\]. Let \(S(x, t, \rho, (y^i))\) be such that \(g^{\mu\nu} \partial_{\mu} S \partial_{\nu} S = 0\). Introducing the conserved “momenta” \(p_x, E, p_i\), we find that

\[
S(x, t, \rho, (y^i)) = p_x x + Et + \rho^* + p_i y^i \tag{12}
\]

where

\[
\rho^*(\rho, p_x, E, p_i) = \int \sqrt{e^{-4L\rho}(E^2 e^{-(2Ln+4L)\rho} - 2Ep_x - p_i p_i)} \, d\rho \equiv \int f'(\rho) \, d\rho \tag{13}
\]

Equivalently, the Lagrangian formulation \(L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -m^2\) yields

\[
\dot{x} = E e^{-4L\rho}, \quad \dot{y}^i = p_i e^{-4L\rho},
\]

\[
\dot{t} = e^{-4L\rho}(p_x - E e^{-(4L+2Ln)\rho}),
\]

\[
\dot{\rho}^2 = E^2 e^{-(8L+2Ln)\rho} - (2Ep_x + p_i p_i) e^{-4L\rho} - m^2 \tag{14}
\]

where \(m = 0\) for null geodesics. We see that \(E \neq 0\) is necessary to find null geodesics for which \(\rho\) varies. It is sufficient provided \(\rho\) stays small enough.

We can easily show here that Kaigorodov spaces admit a time function, or indeed, because they are time and space orientable, they are stably causal in the sense of Hawking \[17\]. The following argument is adapted from the 4-dimensional case in \[24\]. We write the metric \(g_{\mu\nu}\) as

\[
d s_{n+3}^2 = -(e^{4L\rho+Ln\rho} dt)^2 + (e^{4L\rho+Ln\rho} dt + e^{-Ln\rho} dx)^2 + (e^{2L\rho} dy^i)^2 + d\rho^2
\]

and require for timelike or null future-directed geodesics:

\[
g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \leq 0 \quad \text{and} \quad e^{4L\rho+Ln\rho} \dot{t} \geq 0.
\]

If \(e^{4L\rho+Ln\rho} \dot{t}\) vanishes at one point, then using equations \[14\] we see that \(\dot{\rho}^2 \geq 0\) implies \(p_x = p_i = E = m = 0\) and the \(x^\mu\)'s must all be constant. Since this is not an acceptable solution, \(\dot{t} > 0\) for every timelike or null future-directed geodesic, and hence \(t(\lambda)\) provides a suitable time function.

We now find the general Penrose limit of \(g_{\mu\nu}\) about the geodesic of tangent vector \(g^{\mu\nu} \partial_{\mu} S = (\text{grad } S)^\mu \equiv \partial_v\), where \(v\) is the affine parameter along the chosen null geodesic and the future null geodesic congruence. Note that \(\partial_v\) is both orthogonal to the null hypersurfaces \(S = u\) and null, hence is tangent to them. It is easier to find an integrable
system of coordinates using covectors or one-forms rather than coordinate vectors, since the integrability conditions of one forms are just that they be total derivatives. We want to put \( (15) \) in the form
\[
ds^2 = 2du(dv + \frac{1}{2} adu + b_\alpha dx^\alpha) + g_{\alpha\beta} dx^\alpha dx^\beta
\]
with \( a, b_\alpha \) and \( g_{\alpha\beta} \) functions of the coordinates. This is equivalent to looking for coordinates \( u, v, x^\alpha \) such that
\[
g^{uu} = 0, \quad g^{uv} = 1, \quad g^{u\alpha} = 0
\]
or coordinate-forms such that
\[
\langle du|du \rangle = 0, \quad \langle du|dv \rangle = 1, \quad \langle du|dx^\alpha \rangle = 0
\]
(16)

A solution to this system is
\[
du = p_x dx + Edt + f'(\rho)d\rho + p_i dy_i,
\]
\[
dv = \frac{d\rho}{f'(\rho)}
\]
\[
dz = \frac{dx}{E} - \frac{e^{-4L\rho}}{f'(\rho)}d\rho
\]
\[
dx^i = \frac{dy^i}{p_i} - \frac{e^{-4L\rho}}{f'(\rho)}d\rho
\]
(17)
The line element then reads:
\[
ds_{^n+3}^2 = 2du(dv + e^{4L\rho}dz) + (E^2 e^{-2Ln\rho} - 2Ep_x e^{4L\rho})dz^2
\]
\[
-2e^{4L\rho} p_i dz dx^i + e^{4L\rho} p_i^2 (dx^i)^2
\]
(18)
where \( \rho \) is a function of \( v \) defined by (17). The Penrose limit is taken \( \Omega \to 0 \) in the following coordinate rescalings:
\[
v \to v, \quad u \to \Omega^2 u, \quad z \to \Omega z, \quad x^i \to \Omega x^i.
\]
The metric is also rescaled by a factor of \( \Omega^{-2} \). Thus the general metric of all non-trivial Penrose limits of \( K_{n+3} \) is
\[
ds_{^n+3}^2 = 2du dv + (E^2 e^{-2Ln\rho} - 2Ep_x e^{4L\rho})dz^2
\]
\[
-2e^{4L\rho} p_i dz dx^i + e^{4L\rho} p_i^2 (dx^i)^2
\]
(19)
Although this formula breaks down when one \( p_i = 0 \), it is easy to see that it is equivalent to cancelling the \( dz dx^i \) cross term and keeping the \( (dx^i)^2 \) term with \( p_i = 1 \). Furthermore, it is
clear that we recover Minkowski space if \( \rho(v) \) is constant, but unlike the AdS case, we also get non-trivial limits.

Explicit solutions can be found by integrating and inverting (17), where \( f'(\rho) \) is defined in (13). As explained in [8], Penrose limits taken along null geodesics related by an isometry are themselves isometric. Thus given a point in the initial space, it is sufficient to look at the limits along one (rescaled) null vector of each orbit of the “celestial sphere” of null vectors under the isotropy subgroup of the point. As \( K_{n+3} \) is homogeneous, one can choose the origin of the coordinate system in (9), whereupon it is easily seen using the expression of the Killing vectors in (28) that the isotropy subgroup is generated by the \( L_i \)'s and \( L_{ij} \)'s, and hence is isomorphic to \( \mathbb{R}^n \rtimes SO(n) \). A simple computation shows that the action of the \( L_i \)'s suffices to independently set all the \( p_i \)'s to zero, but not \( p_x \). One can set \( p_x = 0 \) keeping one \( p_i \neq 0 \), and the equations are then related to those in section 4.2.

For now, we consider the null geodesic with \( p_x = p_i = 0 \). We get

\[
\rho(v) = \frac{1}{L(n+4)} \ln (EL(n+4)v)
\]

and after various rescalings of the coordinates the metric reads:

\[
ds^2_{n+3} = 2dudv + \left( \frac{1}{v} \right)^{\frac{n+4}{n}} dz^2 + v^{\frac{1}{n+4}} dx^i dx^i
\]  

(20)

Note that this metric has a scaling symmetry:

\[
v \mapsto \lambda v, \quad u \mapsto \lambda^{-1} u, \quad z \mapsto \lambda^{\frac{1}{n+4}} z, \quad x^i \mapsto \lambda^{-\frac{1}{n+4}} x^i.
\]  

(21)

The Penrose limit obtained here is valid globally for \( v > 0 \) and \( u, z, x^i \in \mathbb{R} \) : although the limiting process is defined locally, the coordinate system used to obtain (20) describes the whole initial spacetime. However the plane-wave metric is well defined for \( v < 0 \), so the question of coordinate extension through \( v = 0 \) should be raised: the vanishing of the determinant of the metric at \( v = 0 \) merely signals the presence of conjugate points. The coordinate extension is usually done by going to Brinkman coordinates. Indeed, for the maximally supersymmetric solutions, the \( \cos v \) coordinate singularity in Rosen coordinates disappears in Brinkman coordinates [8].
3.2 Geometry and symmetries of the pp-wave

It is simple to express \((20)\) in Brinkman coordinates as the tranverse part of the metric is diagonal. Consider the following change of coordinates

\[
v = 2x^-, \quad u = x^+ - \frac{1}{2(n+4)x^-(n(y^0)^2 - 2 \sum_{i=1}^{n} y_i^2)} , \quad z = (2x^-)^{\frac{n}{n+4}} y^0 , \quad x^i = (2x^-)^{-\frac{2}{n+4}} y^i
\]

where upon \((20)\) becomes

\[
ds_{n+3}^2 = 2dx^+dx^- + \frac{2n+4}{(n+4)^2 x^2} (n(y^0)^2 - \sum_{i=1}^{n} y_i^2)(dx^-)^2 + \sum_{i=0}^{n} dy_i^2 \tag{22}
\]

The dependence on \(n\) cannot be scaled out. It is easily seen, as was expected, that this plane-wave metric satisfies the Einstein vacuum equations with zero cosmological constant: the \((dx^-)^2\) term, written as \(A_{\mu\nu}y^\mu y^\nu\), satisfies \(\text{Tr}(A) = 0\). It describes a gravitational wave propagating in the null direction \(x^+\), distributed along the \(y^\mu\)'s. The null Killing vector \(\partial_{x^+}\) is covariantly constant. We notice that the coordinate singularity at \(v = 2x^- = 0^+\) remains, but although \((22)\) is ill-defined and the components of the Riemann tensor diverge as \(x^- \to 0\), the square of the Riemann tensor vanishes. The singularity is a so-called pp-singularity.

\[
R_{-\alpha-\beta} = \partial_\alpha \partial_\beta \left\{ \frac{2n+4}{(n+4)^2} \frac{1}{x^2} \left( n(y^0)^2 - \sum_{i=1}^{n} y_i^2 \right) \right\} \propto \frac{1}{x^2} \tag{23}
\]

Actually \(x^- \to 0^+\) corresponds to \(\rho \to -\infty\) in Kaigorodov space, so to the pp-singularity initially present in spacetime. In a sense it remains after taking the Penrose limit.

This singularity can be reached in a finite time \([3]\), thus the spacetime is geodesically incomplete. However, the extra scaling symmetry \([21]\), or \(x^+ \mapsto \sigma x^+\) and \(x^- \mapsto \sigma^{-1} x^-\) in \([22]\), makes this spacetime homogeneous. We now analyse its Lie algebra of symmetries, explain why the space is a Lorentzian homogeneous space. We then exhibit an Inönü-Wigner contraction of the Lie algebra of the initial space into a subalgebra of the symmetries of the plane-wave spacetime.

Applying the general procedure of \([8]\) the plane-wave metric \((20)\) admits, in addition to the \(n(n-1)/2\) Killing vectors \(e_{ij}\) generating the \(SO(n)\) symmetry algebra of the \(x^i\)'s, \(2n + 3\) Killing vectors spanning a Heisenberg algebra. Moreover, the scaling symmetry \((21)\) provides
an extra Killing vector, $e^-$. In the Rosen coordinates of (20) these vectors read:

\[
\begin{align*}
e_+ &= \frac{\partial}{\partial u}, \quad e_0 = \frac{\partial}{\partial z}, \quad e_i = \frac{\partial}{\partial x^i}, \\
e_0^* &= \frac{\partial}{\partial u} - \left(\frac{n+4}{3n+4}\right)v^{n+4} \frac{\partial}{\partial z}, \quad e_i^* = x^i \frac{\partial}{\partial u} - \left(\frac{n+4}{n}\right)v^{n+4} \frac{\partial}{\partial x^i} \\
e_{ij} &= x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \\
e_- &= v \frac{\partial}{\partial v} - u \frac{\partial}{\partial u} + \left(\frac{n}{n+4}\right)z \frac{\partial}{\partial z} - \left(\frac{2}{n+4}\right)x^i \frac{\partial}{\partial x^i}.
\end{align*}
\]

(24)

The non trivial commutation relations are ($i, j, k, l = 1, \ldots, n$):

\[
\begin{align*}
[e_\mu, e^*_\nu] &= \delta_{\mu\nu} e_+, \quad \text{for } \mu, \nu = 0, 1, \ldots, n, \\
[e_-, e_+] &= e_+, \quad [e_-, e_0] = -\left(\frac{n}{n+4}\right)e_0, \quad [e_-, e_i] = \left(\frac{2}{n+4}\right)e_i, \\
[e_-, e_0^*] &= \left(\frac{2n+4}{n+4}\right)e_0^*, \quad [e_-, e_i^*] = \left(\frac{n+2}{n+4}\right)e_i^*, \\
[e_{ij}, e_{kl}] &= -\delta_{ik} e_{jl} + \delta_{jk} e_{il} - \delta_{jl} e_{ik} + \delta_{il} e_{jk}, \\
[e_{ij}, e_k] &= \delta_{jk} e_i - \delta_{ik} e_j, \quad [e_{ij}, e^*_k] = \delta_{jk} e^*_i - \delta_{ik} e^*_j.
\end{align*}
\]

(25)

(26)

(27)

Thus $\mathcal{H} = \langle e^+, e_\mu, e^*_\nu \rangle$ is a $2n + 3$ dimensional Heisenberg algebra of central element $e^+$, and $SO(n)$ acts on the $e_i$'s and $e^*_i$'s as on vectors. We notice that $\mathcal{H}$ and $SO(n)$ generate a Lie algebra $\mathcal{G} = \mathcal{H}(2n+3) \rtimes SO(n)$ of dimension $n(n+3)/2 + 3$, which is precisely the same dimension as $\mathcal{K}$, the symmetry algebra of $K_{n+3}$.

However, there is an extra Killing vector $e^-$, which acts on $\mathcal{G}$ as an outer automorphism. The maximal Lie algebra of symmetries of the plane-wave can be written as $\tilde{\mathcal{G}} = \mathcal{H}(2n+3) \rtimes (SO(n) \oplus \mathbb{R})$, as $e^-$ acts non-trivially only on $\mathcal{H}$. There is an enhancement of 1 bosonic symmetry in the Penrose limit, while the fraction of supersymmetry goes from $1/4$ to $1/2$.

In the same way as homogeneity of the CW-spaces relies on the extra Killing vector $\partial_{x^-}$, the existence of the Killing vector $e_+$ or $X = x^+ \partial_+ - x^- \partial_-$ implies that the plane-wave spacetime (22) is (Lorentzian) homogeneous since the other Killing vectors are clearly transitive on the plane of constant $v$ in (20). Homogeneity is not hereditary in Penrose limits, and can be lost as we shall see in the next section. Strictly speaking though, one must remove the hyperplane $x^- = 0$ because it is invariant under the action of $X$, but it precisely corresponds to the pp-singularity of the plane-wave and the initial space. Moreover, $x^- > 0$, $x^+, y^\mu \in \mathbb{R}$ covers the whole initial Kaigorodov space.
### 3.3 Inönü-Wigner contraction

Although symmetry is enhanced, we can try to relate $\mathcal{K}$ to $\mathfrak{g}$, since they have the same dimension. The rescalings of the coordinates by $\Omega$ in the Penrose Limit suggest an Inönü-Wigner contraction \cite{25} of $\mathcal{K}$ into $\mathfrak{g}$. The forthcoming contraction is very similar to \cite{26}

$$\mathfrak{so}(3,2) \oplus \mathfrak{so}(8) \longrightarrow \mathfrak{h}(19) \rtimes (\mathfrak{so}(3) \oplus \mathfrak{so}(6) \oplus \mathbb{R})$$

in the Penrose limit $AdS_4 \times S^7 \longrightarrow CW_{\max SUSY}$, and the other similar types, apart from the fact that we cannot take the outer automorphism of the plane-wave algebra since it stems from a symmetry enhancement.

$\mathcal{K}$ is spanned by the following Killing vectors \cite{10} expressed in the coordinates of (9):

$$K_0 = \frac{\partial}{\partial t}, \quad K_x = \frac{\partial}{\partial x}, \quad K_i = \frac{\partial}{\partial y^i},$$

$$L_i = x \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial t}, \quad L_{ij} = y^i \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial y^i},$$

$$J = \frac{\partial}{\partial \rho} - (n + 4) L t \frac{\partial}{\partial t} + n L x \frac{\partial}{\partial x} - 2 L y^i \frac{\partial}{\partial y^i}$$

(28)

We define $P_\pm = K_0 \pm K_x$ and consider

$$P_+ \rightarrow \Omega^2 P_+, \quad P_- \rightarrow \Omega P_-, \quad J \rightarrow \Omega J,$$

$$K_i \rightarrow \Omega K_i, \quad L_j \rightarrow \Omega L_j, \quad L_{ij} \rightarrow \Omega L_{ij}$$

Taking $\Omega \longrightarrow 0$, the commutation relations of $\mathcal{K}$ (see \cite{10}) yield those of $\mathfrak{g}$, where we further make the following association

$$e_+ = P_+, \quad e_0 = P_-, \quad e_0^* = -\frac{J}{(n + 2)L},$$

$$e_i = -2K_i, \quad e_i^* = L_i, \quad e_{ij} = L_{ij}.$$  \hspace{1cm} (29)

Thus we get the Inönü-Wigner contraction $\mathcal{K} \longrightarrow \mathfrak{h}(2n + 3) \rtimes \mathfrak{so}(n)$. However one could perfectly interchange $P_+$ and $P_-$ and get $P_-$ as the central element of the contracted algebra: it is not clear whether $\mathcal{K}$ really “undergoes” a contraction in the Penrose limit, as the latter is not uniquely defined. The similar feature arises in the maximally supersymmetric cases. Nevertheless, one likes to think of Penrose limits as yielding approximations of spacetimes, and the trivialization of any algebra of symmetries into a Heisenberg type algebra certainly
“happens”. On the dynamical point of view, a precise understanding of what happens group theoretically should help to explain how the degrees of motion of a particle or a string decouple to yield independent harmonic oscillators in the plane-wave limit. We now consider possible limits of the whole supergravity or IIB spacetimes, topological products of Kaigorodov spaces and spheres.

4 Penrose limits of $K_{n+3} \times S^d$ spaces

We want to find a coordinate system which singles out a particular null geodesic congruence of the following metrics, for $n + 3 + d = 10$ or 11:

$$ds^2 = e^{-2Ln\rho}dx^2 + e^{4L\rho}(2dxdt + dy^idy^i) + d\rho^2 + d\psi^2 + (Ln)^{-2}\sin^2(Ln\psi)d\Omega^2_{d-1}$$  \hspace{1cm} (30)

Again we adopt the Hamilton-Jacobi formulation, with $S(x,t,\rho,(y^i),\psi)$ a function of the coordinates satisfying

$$g_{\mu\nu}\partial_{\mu}S\partial_{\nu}S = 0.$$  \hspace{1cm} (31)

where now

$$\rho^*(\rho, p_x, E, p_i, l) = \int \sqrt{e^{-4L\rho}(E^2e^{-2Ln+4L})p - 2Ep_x - (p_i)^2 - l^2} d\rho \equiv \int f'(\rho) d\rho$$  \hspace{1cm} (32)

4.1 Non-winding geodesics

We will first consider the case where the chosen null geodesic of tangent vector $g_{\mu\nu}\partial_{\nu}S = (\text{grad } S)_{\mu} \equiv \partial_{\nu}$ does not wind around the sphere, i.e. $l = 0$, $E \neq 0$. In this case, the spherical part of the metric yields flat space, and the limits of $K_{n+3}$ where found in the previous section. In the particular case $p_x = p_i = 0$, using \cite{22}, the limit metric is simply:

$$ds^2_{n+3+d} = 2dx^+dx^- + \frac{2n+4}{(n+4)^2} \frac{1}{x^2}(n(y^0)^2 - \sum_{i=1}^{n} y^i)^2(dx^-)^2 + \sum_{i=0}^{n} dy^i^2 + ds^2_{\mathbb{R}^d}$$  \hspace{1cm} (33)

The field strength Güven \cite{2} limit $F_{p+1}^* = \lim_{\Omega \to 0} \Omega^{-p} F_{p+1}$ yields $F_4 = 0$ for $K_4 \times S^7$, whatever $p_x, p_1$, provided that $E \neq 0$ and $l = 0$. Indeed

$$F_4 \propto e^{6L\rho} f'(\rho) du \wedge dz \wedge dy^1 \wedge dv$$

so $F_4$ goes as $\Omega$ in the limit and vanishes. The same is true for the field strengths $*[F_4]$ and $F_5$. This merely reflects the fact that Kaigorodov spaces are solutions of Einstein pure
gravity. Indeed, as a consequence, their Penrose limits, and hence the Penrose limits of $K \times S$ spaces along non winding geodesics, satisfy Einstein pure gravity too. Therefore, all the plane-wave spaces of type (33) for $n+3+d = 11$ or 10 are trivial solutions of supergravity (with vanishing field strengths). They describe a gravitational wave propagating in $n + 3$ dimensions. However, they only arise as limits of brane-like solutions for $n = 1, 2$ or 4.

As all generic pp-waves, the spaces (33) preserve 1/2 of the supersymmetries, merely the constant spinors $\epsilon$ satisfying $\Gamma \pm \epsilon = 0$ in the light-cone vielbein formalism (see [27] [28] for example). Since the field strengths vanish, there are no non-constant solutions and these are the only solutions. Comparing with the $AdS \times S$ Penrose limits, we here have non-trivial gravitational waves even in the non-winding case. They are essentially $n + 3$ dimensional, and thus can be viewed as consequences of the waves perturbing the initial branes and their near horizon geometries. Formally one can write (33) as a Minkowski metric (Penrose limit of the $AdS \times S$ along non-winding geodesics) “perturbed” by an $n + 3$ dimensional wave.

In fact these plane-wave spacetimes, as their $n+3$ dimensional analogues (22), are Lorentzian homogeneous spaces. In 10 dimensions, with a constant dilaton field, they can provide a time dependent background on which string theory is exact and solvable [15]. However, the term $n(y^0)^2$ will give rise to a negative mass harmonic oscillator. Fixing $y^0 = 0$ corresponds to the background studied in [14].

4.2 Winding geodesics

Whereas in the previous case the plane-wave obtained propagated in $n + 3$ dimensions only, we might expect as in the $AdS \times S$ cases [8] to find an essentially 11(or 10)-dimensional plane-wave by taking a null geodesic which winds around an equator of the sphere. Analogously to section (3.1) a possible coordinate transformation for $E \neq 0$, $l \neq 0$, $p_x = p_i = 0$ is given by:

\[
\begin{align*}
    du &= Edt + f'(\rho)d\rho + ld\psi \\
    dv &= \frac{d\rho}{f'(\rho)} \\
    dw &= -\frac{d\rho}{f'(\rho)} + \frac{d\psi}{l} \\
    dz &= \frac{dx}{E} - \frac{e^{-4L\rho}}{f'(\rho)}d\rho \\
    dy^i &= dy^i
\end{align*}
\]
This is only valid for \( \rho \leq \frac{1}{2Ln+8L} \ln((\frac{E}{l})^2 \equiv \rho_0 \), because greater values of \( \rho \) are not reached by the chosen geodesic congruence, as can be seen by analysing (32). Again the Hamilton-Jacobi formalism to find null hypersurfaces and therefore possible null geodesic congruences, together with the orthogonality conditions on the coordinate-forms (16), yield this non-trivial (integrable) change of coordinates. The metric then reads

\[
\begin{align*}
\rho(v) &= \frac{1}{2Ln+8L} \ln \cos^2(l(4L+Ln)v) + \rho_0 \\
\end{align*}
\]

To find \( \rho(v) \) one must set \( 0 \leq -l(4L+Ln)v \leq \pi/2 \) during integration, however (35) stays valid for all \( v \in \mathbb{R} \) if we accept \( \rho \) periodic in \( v \) on the range \( (-\infty, \rho_0] \), and the remaining coordinate transforms still make sense. Note that (34) does not cover the whole of \( K_{n+3} \times S^d \), but at any given point (so any \( \rho_0 \)), we can pick \( E \) and \( l \) so as to cover a neighbourhood of that point with null geodesic congruence coordinates which break down when \( \cos(-l(4L+Ln)v) = 0 \). The coordinates of (34) cannot be used to discuss global properties of \( K_{n+3} \times S^d \) spacetimes.

As usual we define the following rescalings of the coordinates, the metric and the field strength:

\[
\begin{align*}
v &\to v, \quad u \to \Omega^2 u, \quad (z,w,y^i) \to (\Omega z, \Omega w, \Omega y^i), \\
g &\to \Omega^{-2} g, \quad F_{p+1} \to \Omega^{-p} F_{p+1},
\end{align*}
\]

The Penrose-Güven limit is obtained by taking \( \Omega \to 0 \). Using (35) and \( f'(\rho) = l \tan(-l(4L+Ln)v) \), (take \( l \geq 0 \)) this yields:

\[
\begin{align*}
ds^2 &= 2dudv + E^2\left(\frac{E^2}{l^2} \cos^2(l(Ln+4L)v)\right)^{-\frac{n}{4+n}} dz^2 + l^2 dw^2 + \sin^2(Lnv)ds^2_{\mathbb{R}^{d-1}} \\
&\quad + \left(\frac{E^2}{l^2} \cos^2(l(Ln+4L)v)\right)^{\frac{2}{4+n}} (-2l^2 dzdw + dy^i dy^i) \\
F_4 &= 6L^2 f'(\rho)e^{6L^2} dv \wedge dw \wedge dz \wedge dy^1 \\
&= 6L^3 \tan(-l(4L+Ln)v)\left(\frac{E^2}{l^2} \cos^2(-l(4L+Ln)v)\right)^{\frac{2}{n+4}} dv \wedge dw \wedge dz \wedge dy^1
\end{align*}
\]

After rescaling the coordinates appropriately by powers of \( E \) and \( l \), we get the following plane-wave spacetime, where the flux term \( F_4 \) is only relevant in the \( n = 1 \) case (limit of
\[ K_3 \times S^7): \]
\[
\begin{align*}
    ds^2 &= 2du dv + \left( \cos v \right)^{-\frac{2n}{n+3}}dz^2 + \left( \cos v \right)^4(2dz dw + dy^i dy^i) \\
    &\quad + dw^2 + \sin^2 \left( (n/(n+4))v \right) ds^2_{R^{d-1}} \\
    F_4 &= \frac{6}{n+4} \tan v \left( \cos v \right)^{\frac{3n}{n+4}} dv \wedge dw \wedge dz \wedge dy^1 
\end{align*}
\] (37)

These Rosen coordinates are valid for \(0 \leq v \leq \pi/2\). Similarly we obtain from (5) the limits of \(*[F_4]\) and \(F_5\) for the \(n = 4\) and \(2\) cases respectively:
\[
\begin{align*}
    *[F_4] &= \frac{12}{n+4} \tan v \left( \cos v \right)^{\frac{12n}{n+4}} dv \wedge dw \wedge dz \wedge d^i y^i \quad (n = 4) \\
    F_5 &= \frac{8}{n+4} \tan v \left( \cos v \right)^{\frac{8n}{n+4}} dv \wedge dw \wedge dz \wedge d^2 y^i \\
    &\quad + *[\frac{8}{n+4} \tan v \left( \cos v \right)^{\frac{8n}{n+4}} dv \wedge dw \wedge dz \wedge d^2 y^i] \quad (n = 2)
\end{align*}
\] (39) (40)

Note that the metrics and the field strengths obtained are independent of \(E\), \(l\), and also the Kaigorodov parameter \(L\). Only the region of the initial space covered by \(0 \leq v \leq \pi/2\) depends on them. The limit spaces do not depend on the charges of the initial pp-branes.

One can easily find the Killing vectors of the metric (37) which generate a Heisenberg algebra of dimension \(2(n+d+1)+1\), together with those which span the \(SO(n)\) and \(SO(d-1)\) algebras respectively (see [8] for example). Counting these symmetries, and since \(K_{n+3} \times S^d\) has \(n(n+3)/2 + 3 + (d+1)d/2\) Killing vectors, there is an enhancement of at least 1 bosonic symmetry in the Penrose limit, and in fact no more. This is true in the three cases. There is no obvious additional scaling symmetry as we had in (20). As suggested in [15], it is often easier to see that in Brinkman coordinates.

### 4.3 Brinkman coordinates

The change of coordinates between Rosen and Brinkman coordinates has been known for a long time. It is reviewed in [8] [15] for example. We relabel the coordinates of (37) by \(z = x_0\), \(w = x_1\), \(y_i = x_{i+1}\) for \(i = 1, \ldots, n\), and also call \(x_i\) for \(i = n + 2, \ldots, n + d\), the coordinates of \(\mathbb{R}^{d-1}\). From now on latin indices range from 0 to \(n + d\). The spatial part of (37), denoted \(C_{ij}(v) dx^i dx^j\), is not diagonal, and cannot be made so by staying in Rosen type coordinates. Thus the change of coordinates involves a particular inverse vielbein field \(Q^i_j\) of \(C_{ij}\) which non-trivially rotates the \(z = x_0\) and \(w = x_1\) coordinates. Formally, the solution
can be written:

\[
v = x^-, \quad y^i = Q^i_j z^j, \\
u = x^+ - \frac{1}{2} C_{ij} \dot{Q}^i_k Q^j_k z^j,
\]

(41)

with \( Q \) satisfying \( Q^T C Q = I \) and \( \dot{Q}^T C Q = \dot{Q}^T C \dot{Q} \), where the dot denotes differentiation with respect to \( x^+ \). This yields the plane-wave metric:

\[
ds^2 = 2 dx^+ dx^- + A_{ij}(x^-) z^i z^j (dx^-)^2 + dz^i dz^j
\]

where \( A_{kl} = -(C_{ij} Q^i_k) Q^j_l \) (42)

A possible solution is given by \( Q(v) \) whose only non-vanishing components are:

\[
Q^0_0(v) = (\tan v)^{-1} (\cos v)^{-\frac{4}{n+4}} \cos \left( \frac{2v}{n+4} \right), \\
Q^0_1(v) = (\tan v)^{-1} (\cos v)^{-\frac{4}{n+4}} \sin \left( \frac{2v}{n+4} \right), \\
Q^1_0(v) = - (\tan v)^{-1} \cos \left( \frac{2v}{n+4} \right) - \sin \left( \frac{2v}{n+4} \right), \\
Q^1_1(v) = - (\tan v)^{-1} \sin \left( \frac{2v}{n+4} \right) + \cos \left( \frac{2v}{n+4} \right), \\
Q^i_i(v) = \sin^{-1} \left( \frac{n v}{n+4} \right) \quad \text{for} \quad i = 2, \ldots, n+1
\]

\[
Q^i_i(v) = \sin^{-1} \left( \frac{n v}{n+4} \right) \quad \text{for} \quad i = n+2, \ldots, n+d.
\]

The expression for \( A_{ij} \) given by (42) turns out complicated (and non-diagonal) for \( i, j \in \{0, 1\} \). For the diagonal terms \( (i, j > 1) \), \( A_{kl}(x^-) = \frac{\sqrt{C_{ii}}}{\sqrt{C_{ii}}} \delta_{ij} \) reads:

\[
A_{ii} = - \left( \frac{2}{n+4} \right)^2 \left( 1 + \frac{n+2}{2} \cos^{-2} x^- \right) \quad \text{for} \quad i = 2, \ldots, n+1
\]

\[
A_{ii} = - \left( \frac{n}{n+4} \right)^2 \quad \text{for} \quad i = n+2, \ldots, n+d.
\]

Then letting \( a, b \in \{0, 1\} \), and defining \( \tilde{A}_{ab}(x^-) = A_{ab}(x^-) + \left( \frac{2}{n+4} \right)^2 \delta_{ab} \), the metric reads:

\[
ds^2 = 2 dx^+ dx^- - \left( \frac{n}{n+4} \right)^2 \left\{ \left( \frac{2}{n+4} \right)^2 \sum_{i=0}^{n+1} (z^i)^2 + \sum_{i=n+2}^{n+1+d} (z^i)^2 \right\} (dx^-)^2 + dz^i dz^j
\]

\[
+ \left\{ \tilde{A}_{ab}(x^-) z^a z^b - \frac{2n+4}{(n+4)^2} \cos^{-2} x^- \sum_{i=2}^{n+1} (z^i)^2 \right\} (dx^-)^2.
\]

(43)

Introducing \( \tilde{A}_{ab} \) seems unnecessary. However, the first line of (43) describes the maximally supersymmetric plane-waves (when supported by appropriate field strengths of course). Indeed, for \( n = 1 \) or \( n = 4 \) in 11 dimensions, \((2/n)^2 = 4 \) or \( 1/4 \), it corresponds to the
Kowalski-Glikman solution [7] (also described in [21]), while for \( n = 2 \) in 10 dimensions, it is the metric of the BFHP maximally supersymmetric type IIB plane-wave [7]. Although we could say of any plane-wave that it consists of the sum of a maximally supersymmetric Cahen-Wallach space and another plane-wave, the argument seems illuminating here in the context of Penrose limits of particular brane solutions. Indeed, the Penrose limits of the \( AdS \times S \) spaces along geodesics which wind round the sphere [8] yield the maximally supersymmetric plane-waves. Here, starting from a \( K_{n+3} \times S^d \) geometry, or an \( AdS_{n+3} \times S^d \) spacetime perturbed with an \((n+3)\)-dimensional gravitational wave, we exhibit a Penrose limit space which can quite naturally be interpreted as a maximally supersymmetric plane-wave together with an additional gravitational wave. When solving the Killing equations [15], we see that \( \partial_{x^-} \) does not admit a Killing vector with non-vanishing \( \partial_{x^-} \) component. Indeed, the diagonal terms \( A_{ii} \) for \( i = 2, \ldots, n+1 \) are neither constants nor proportional to \((1/x^-)^2\). Hence the plane-wave we obtain is not homogeneous. Moreover, \([13]\) tells us that the plane-wave limit has a pp-singularity at \( v = \pi/2 \). As in the non-winding case, this singularity stems from the pp-singularity of the Kaigorodov space, which itself corresponds to the divergence of the tidal forces as one approaches the pp-branes of the initial geometries.

5 Conclusion

The plane-wave spacetimes obtained in this paper should really be thought of as arising from Penrose limits of Kaigorodov spaces, themselves near horizon limits of M2, M5 or D3-pp-branes. In this sense they are dynamically relevant. Moreover \( K_{n+3} \times S^d \) spaces provide themselves interesting solutions of supergravity which are analogous to the \( AdS \times S \) configurations. In [10] and [11] evidence was given that gravity in the Kaigorodov space is dual to a CFT in the infinite momentum frame with constant (null) momentum density, and that therefore one can consider the Kaigorodov space as an infinitely boosted version of \( AdS \). When taking Penrose limits, it seems difficult to keep track of this fact. However, when we think of \( K \) as an \( AdS \) space perturbed by a gravitational wave, the picture seems clearer. The plane-waves [33] and [43] can be naturally interpreted as the corresponding Penrose limits of \( AdS \times S \) spaces along non-winding and winding null geodesics, perturbed by a gravitational wave. This decoupling, similar to the one occurring when taking the near horizon geometries of pp-branes, needs to be investigated further.
Our analysis of various Penrose limits certainly illustrates the variety of possible symmetry enhancements. The example of the Kaigorodov space limit is worth noting: an enhancement of one bosonic symmetry, but nevertheless an İnönü-Wigner contraction of the initial algebra of symmetries to a subalgebra of the plane-wave symmetries. Note that since all plane-wave Lie algebras have the same structure (up to a possible extra outermorphism), the contraction of the algebra of $K_{n+3}$ into $\mathfrak{h}(2n + 3) \times SO(n)$ can be shown to occur systematically in all cases. The loss of homogeneity when taking the limit of $K \times S$ spaces along winding geodesics might seem striking, but is linked to the fact that the limiting process is essentially local. In addition, although the number of Killing vectors is conserved or increases [8], their action often becomes redundant. For example, in the typical $2d + 1$-dimensional plane-wave Heisenberg algebra, possibly enlarged by a semi-direct product of rotations, only $d + 1$ symmetries yield motions of the spacetime in independent dimensions. The question of homogeneity becomes important when one wants to solve string theories in plane-wave backgrounds in view of possibly relating them to a certain CFTs, since the extra conserved quantity simplifies the equations [15]. The precise understanding of what the algebra (and super-algebra) of symmetries incurs in the limit, should help to understand how the degrees of motion of strings and particles moving in a given background decouple to yield, in the bosonic part for example, independent harmonic oscillators, and then better relate their simple dynamics to that of the initial space.

Acknowledgements

I would like to thank Gary Gibbons for motivating discussions and insightful comments, and also the EPSRC, the DAMTP, and the Cambridge European Trust for financial support. I am also grateful to the referees for some useful comments.

A Isometric embedding of the Kaigorodov space

It seems natural to look for an embedding of $K_{d+1}$ which resembles that of $AdS_{d+1}$ in Minkowski space of signature $(2, d)$. Consider the $(2, d)$-signature space of metric ($i \in \{1, \ldots, d - 2\}$):

$$ds^2 = dU dV + dX^+ dX^- + dX^i dX^i + R^d \left( \frac{dX^+}{U^d} + \frac{X^+}{U^{d+2}} du^2 - 2 \frac{X^+}{U^{d+1}} dU dX^+ \right)$$

(44)
One can isometrically embed the hyperboloid-like hypersurface defined by:

\[ UV + X^+X^- + (X^i)^2 = -R^2 \]  \hspace{1cm} (45)

Analogously to the AdS case we define horospherical coordinates on this hypersurface:

\[ z = \frac{R}{U}, \quad x^\pm = X^\pm \frac{z}{R}, \quad x^i = X^i \frac{z}{R} \]

and replace \( V(U, X^\pm, X^i) \) with \( (45) \). The induced metric on the hypersurface is then that of \( K_{d+1} \) in horospherical-like coordinates, for \( z > 0 \):

\[ ds_{d+1}^2 = \frac{R^2}{z^2} \left( 2dx^+dx^- + z^d(dx^+)^2 + \sum_{i=1}^{d-2} dx^i dx^i + dz^2 \right) \]  \hspace{1cm} (46)

One might think that a simpler embedding space could be found by changing the right-hand-side of \((45)\) by a function \( f(U, X^+) \), but in fact it only amounts to a change of coordinates in the embedding space. The embedding space satisfies the Einstein vacuum field equations, is Ricci flat, and its Riemann and Weyl tensors only have the following independent non-vanishing components:

\[ R_{UX+UX+} = C_{UX+UX+} \propto \frac{1}{U^{d+2}} \]

Therefore this space is not a symmetric space (the Riemann tensor is not covariantly constant). As a side remark, solving Killing’s equations of \((44)\) should not be difficult, and finding the Killing vectors which survive on the hypersurface might give a deeper insight into the Lie group of motions of the generalized Kaigorodov spaces, as an intersection of the anti-de-Sitter group and the group of motions of \((44)\).

References

[1] R. Penrose, Any space-time has a plane wave limit, in Differential Geometry and Relativity, ed M. Cahen and M. Flato, D. Reidel Publishing Co., Dordrecht-Boston, Mass., (1976) pp.271-275.

[2] R. Güven, Plane wave limits and T-Duality, Phys. Lett. B482 (2000), pp.255-263, arXiv:hep-th/0005061.

[3] G.T. Horowitz, A.R. Steif, Strings on strong gravitational fields, Phys. Rev. D42 (1990), p. 1950.
[4] G.W. Gibbons, *Quantized fields propagating in plane-wave spacetimes*, Commun. Math. Phys. **45** (1975), pp.191-202.

[5] M. Blau, J. Figueroa-O’Farrill, C. Hull, G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, JHEP 0201 (2002) 047, arXiv:hep-th/0110242

[6] R. Metsaev and A. Tseytlin, *Exactly solvable model of superstring in plane-wave Ramond-Ramond background*, arXiv:hep-th/0201109

[7] J. Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, Phys. Lett. **134B** (1984), pp.194-196.

[8] M. Blau, J. Figueroa-O’Farill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, arXiv:hep-th/0202111

[9] D. Berenstein, J. Maldacena and H. Nastase, *Strings in flat space and pp-waves from N = 4 Super-Yang-Mills*, arXiv:hep-th/0202021

[10] M. Cvetič, H. Lü and C. Pope, *Spacetimes of boosted p-branes, and CFT in infinite-momentum frame*, arXiv:hep-th/9810123

[11] D. Brecher, A. Chamblin and H. Reall, *AdS/CFT in the infinite-momentum frame*, arXiv:hep-th/0012076

[12] V.R. Kaigorodov, *Einstein spaces of maximal mobility*, Soviet Physics Dokl, vol **7** (1963), pp.893-895.

[13] H. Fuji, K. Ito and Y. Sekino, *Penrose limit and string theories on various brane backgrounds*, arXiv:hep-th/0209003

[14] G. Papadopoulos, J.G. Russo and A.A. Tseytlin, *Solvable model of strings in a time-dependent plane-wave background*, arXiv:hep-th/0211289

[15] M. Blau and M. O’Loughlin, *Homogeneous plane-waves*, arXiv:hep-th/0212135

[16] J. Podolský, *Interpretation of the Siklos solutions as exact gravitational waves in the anti-de-Sitter universe*, arXiv:gr-qc/9801052

[17] S.W. Hawking, *The existence of cosmic time functions*, Proc. R. Soc, **A308**, 433 (1969).
[18] A.A. Tseytlin, *Harmonic superpositions of M-branes*, Nucl. Phys. B475 (1996), pp.149-163.

[19] I.R. Klebanov and A.A. Tseytlin, *Intersecting M-branes as four-dimensional black holes*, Nucl. Phys. B475 (1996), pp.179-192.

[20] E. Cremmer, B. Julia and J. Scherk, *Supergravity in eleven dimensions*, Phys. Lett. B76 (1978), pp.409-412.

[21] J. Figueroa-O’Farrill and G.Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, arXiv:hep-th/0105308.

[22] J.H. Schwarz, *Covariant field equations of chiral N=2 D=10 supergravity*, Nucl. Phys. B226 (1986), pp.269-288.

[23] R. Penrose, *Techniques of differential topology in relativity* (Section 7), Philadelphia, Soc. for Industrial and Applied Mathematics (1972).

[24] M.E. Osinovsky, *Stable Causality of highly mobile space-times*, Acta. Phys. Polon. B 5 (1974), pp.74-79.

[25] E. Inönü and E.P. Wigner, *On the Contraction of groups and their representations*, Proc. of the Nat. Ac. of Sc. (USA), vol 39 (1953), pp.510-524.

[26] M. Hatsuda, K. Kamimura and M. Sakaguchi, *Super-pp-wave Algebra from super-AdS × S algebras in eleven-dimensions*, arXiv:hep-th/0204002.

[27] J.P. Gauntlett, C.M. Hull, *pp-waves in 11-dimensions with extra supersymmetry*, JHEP 0206 (2002) 013, arXiv:hep-th/0203255.

[28] M. Cvetič, H. Lü and C. Pope, *M-theory pp-waves, Penrose limits and supernumerary supersymmetries*, arXiv:hep-th/0203229.