Abstract

The Mallows model, introduced in the seminal paper of Mallows (1957), is one of the most fundamental ranking distribution over the symmetric group $S_m$. To analyze more complex ranking data, several studies considered the Generalized Mallows model (Fligner and Verducci, 1986; Doignon et al., 2004; Marden, 1995). Despite the significant research interest of ranking distributions, the exact sample complexity of estimating the parameters of a Mallows and a Generalized Mallows Model is not well-understood.

The main result of the paper is a tight sample complexity bound for learning Mallows and Generalized Mallows Model. We approach the learning problem by analyzing a more general model which interpolates between the single parameter Mallows Model and the $m$ parameter Mallows model. We call our model Mallows Block Model – referring to the Block Models that are a popular model in theoretical statistics. Our sample complexity analysis gives tight bound for learning the Mallows Block Model for any number of blocks. We provide essentially matching lower bounds for our sample complexity results.

As a corollary of our analysis, it turns out that, if the central ranking is known, one single sample from the Mallows Block Model is sufficient to estimate the spread parameters with error that goes to zero as the size of the permutations goes to infinity. In addition, we calculate the exact rate of the parameter estimation error.

Keywords: Ranking distributions, Mallows model, Generalized Mallows, Exponential family

1. Introduction

The Mallows model is one of the most fundamental ranking distribution since it was introduced in the seminal paper of Mallows (1957). The model has two parameters, the central ranking $\pi_0 \in S_m$ and the spread parameter $\phi \in [0, 1]$. Based on these, the probability of observing a ranking $\pi \in S_m$ is proportional to $\phi^d(\pi, \pi_0)$, where $d$ is a ranking distance, such as the number of discordant pairs, a.k.a Kendall’s tau distance.

To capture more complicated distributions over rankings, several studies considered the generalized Mallows model (Fligner and Verducci, 1986; Doignon et al., 2004; Marden, 1995), which assigns a different spread parameter $\phi_i \in [0, 1]$ to each alternative $i$. Now the probability of observing $\pi \in S_m$ decreases exponentially in a weighted sum over the discordant pairs, where the weights are determined by the spread parameters of discordant items. Statistical estimation of the distri-
bution and the parameters of the Mallows model has been of interest in a wide range of scientific areas including theoretical statistics (Mukherjee, 2016), machine learning (Lu and Boutilier, 2011; Awasthi et al., 2014; Chen et al., 2009; Meila and Bao, 2010), social choice (Caragiannis et al., 2016), theoretical computer science (Liu and Moitra, 2018) and many more, as we discuss in Section 1.2.

Despite this extensive literature, to the best of our knowledge, no optimal results are known on the sample complexity of learning the parameters of a Mallows or a generalized Mallows model. In this work, we fill this gap by proving: (1) an upper bound on the number of samples needed by some simple estimators to accurately estimate the parameters of the Mallows model, (2) an essentially matching lower bound on the sample complexity of any accurate estimator. Using our tight sample analysis, we are able to quantify in the finite sample regime some results that were only known in the asymptotic regime (e.g., Mukherjee (2016)).

Additionally, we introduce the Mallows Block model, which interpolates between the simple Mallows and the generalized Mallows models. The definition of the Mallows Block model is similar in spirit to the (fundamental in theoretical statistics) Stochastic Block model (Klopp et al., 2017), which admits similar statistical properties. Also, Berthet et al. (2016) recently introduced the Ising Block model, which is conceptually similar to the Stochastic Block Model. As we prove, the Mallows Block model combines two nice properties: (a) like the generalized Mallows model, it describes a wider range of distributions over rankings than the Mallows model; and (b) it allows accurate estimation of the spread parameters even from one sample, as it has been proved in (Mukherjee, 2016) for the Mallows model. We analyze the sample complexity of the Mallows Block model by proving essentially tight upper and lower bounds when the block structure is known.

1.1. Results and Techniques

In this work, we fully determine the sample complexity of learning Mallows and Generalized Mallows distributions, in a unified way, via the definition of the Mallows Block model. In a nutshell, we show how to estimate the parameters of these distributions in a (sample and time) efficient way, and how this implies efficient density estimation in KL-divergence and in total variation distance. Our approach is general and exploits properties of the exponential family. As we illustrate in Section 3, the use of these properties might useful in proving the exact learning rates for other complicated exponential families, such as the Ising model.

Learning in KL-divergence. Our learning algorithm for the spread parameters essentially finds the maximum likelihood solution, but in a provably computationally efficient way. The sample complexity analysis of the consistency of our estimator is based on some known and some novel results about exponential families. As we see in Theorem 1.4, the KL-divergence of two distributions in an exponential family is equal to the square difference of their parameters multiplied by the variance of a corresponding distribution inside the exponential family. If we put this together with Theorem 4, where we obtain a new strong concentration inequality for distributions in an exponential family, we get a systematic way of proving upper bounds on the number of samples required to learn an exponential family in KL-divergence. Thus, we depart from the (only known) upper bounds on density estimation in total variation distance. We apply our technique to the Mallows Block model and get tight upper bounds of \( O \left( \frac{d}{\epsilon^2} + \log (m) \right) \) samples, where \( d \) is the (known) number of blocks in the Mallows Block model. We sketch the statement of this result below, for a formal statement see Theorem 15.
**Informal Theorem 1** Given \( n = \tilde{\Omega} \left( \frac{d}{\varepsilon^2} + \log(m) \right) \) samples from a Mallows \( d \)-Block distribution \( P \), we can learn a distribution \( \hat{P} \) such that \( D_{KL}(P \parallel \hat{P}) \leq \varepsilon^2 \) and hence \( d_{TV}(P, \hat{P}) \leq \varepsilon \).

**Parameter Estimation.** Extending a result of Caragiannis et al. (2016), we show that a logarithmic number of samples is both sufficient and necessary to estimate the central ranking of a generalized Mallows distribution (Theorem 12). Then, using our results on exponential families, we show that estimating the spread parameter \( \phi \) of a Mallows distribution boils down to obtaining a lower bound on the KL-divergence between two Mallows distributions with the same central ranking and parameters \(|\phi - \phi'| = \Theta(\varepsilon)|. With such a lower bound on the KL-divergence, we can apply the concentration inequality of Theorem 4, and show that once we learn the central ranking, with additional \( O \left( \frac{d}{m^* \varepsilon^2} \right) \) i.i.d. samples, we can estimate the parameter vector \( \phi \) of the underlying Mallows Block model within \( \ell_2 \) error at most \( \varepsilon \). Here, \( d \) denotes the number of blocks of the Mallows Block model and \( m^* \) is the minimum size of any block. We put everything together in the following informal theorem and refer to Theorem 13 for a formal statement.

**Informal Theorem 2** Given \( n = \tilde{\Omega} \left( \frac{d}{m^* \varepsilon^2} + \log(m) \right) \) samples from a Mallows \( d \)-Block distribution \( P \) with parameters \( \pi^* \) and \( \phi^* \), we can estimate \( \hat{\pi} \) and \( \hat{\phi} \) so that \( \hat{\pi} = \pi^* \) and \( \| \hat{\phi} - \phi^* \|_2 \leq \varepsilon \).

A key observation in the proof of Theorem 13 is that the sufficient statistics for a generalized Mallows model with known central ranking are provided by an \( m \)-variate distribution where the \( i \)-th coordinate is an independent truncated geometric distribution. Truncated geometric distributions interpolate between Bernoulli and geometric distributions. The sufficient statistics of the Mallows Block model correspond to sums of truncated geometric distributions, which interpolate between Binomial and Negative Binomial distributions. We hence believe that the study of sums of truncated geometric distribution may be of independent interest. We should also highlight that in our approach, only the lower bound on the variance depends on Kendall’s tau distance. Once we have such a bound for other exponential families, we can immediately apply our technique, e.g., to Mallows models with Spearman’s Footrule and Spearman’s Rank Correlation, as in (Mukherjee, 2016).

**Learning from one sample.** Arguably, the most interesting corollary of our tight analysis is that a single sample from a Mallows \( d \)-Block model with known central ranking is enough to estimate \( \phi \) within error \( O \left( \sqrt{d/m^*} \right) \), where again \( m^* \) is the minimum size of any block in the Mallows Block model. This result provides the exact rate of an asymptotic result by Mukherjee (2016). The formal version of the following informal theorem can be found in Corollary 14.

**Informal Theorem 3** Given a single sample from a Mallows \( d \)-Block distribution \( P \) with known central ranking \( \pi^* \) and spread parameters \( \phi^* \), we can estimate \( \hat{\phi} \) so that \( \| \hat{\phi} - \phi^* \|_2 \leq \tilde{O} \left( \sqrt{\frac{d}{m^*}} \right) \).

**Lower Bounds.** On the lower bound side, we use Fano’s inequality and show that \( \Omega(\log(m)) \) samples are necessary even for learning a simple Mallows distribution in total variation distance (Lemma 10). Then, we show that \( \Omega \left( \frac{d}{\varepsilon^2} \right) \) samples are necessary for learning a Mallows \( d \)-Block distribution in total variation distance. For a formal statement of the following informal theorem we refer to Lemma 17.

**Informal Theorem 4** Any distribution estimation \( \hat{P} \) that is based only on \( o \left( \frac{d}{\varepsilon^2} + \log(m) \right) \) samples from a Mallows \( d \)-Block distribution \( P \) satisfies \( d_{TV}(P, \hat{P}) \geq \varepsilon \).
Interestingly, our lower bound uses a general way to compute the total variation distance of two distributions that belong to the same exponential family (Theorem 6). This theorem states that the total variation of two distributions in the same exponential family is equal to the distance between their parameters times the absolute deviation of a corresponding distribution in the family. This should be compared with Theorem 1.4 on the KL-divergence between two distributions in the same exponential family. Using Theorem 6, our lower bound boils down to showing that for some range of parameters, the absolute deviation is within a constant from the standard deviation. With this proven, we get that the total variation distance is within a constant factor from the square root of the KL-divergence, and Fano’s inequality can be applied.

**Open Problems.** An open problem that naturally arises from the definition of the Mallows Block model is the possibility of estimating the spread parameters, even from a single sample, of the Mallows Block model when the block structure is unknown. Such results are known for the fundamental Stochastic Block model in theoretical statistics (Klopp et al., 2017). Recently, Berthet et al. (2016) introduced the Ising Block model and proved some similar results. Another interesting question is about the minimum number of samples required to recover the block structure of the Mallows Block Model. Again, similar results are known for the Stochastic Block Model (Mossel et al., 2018).

Another research direction is to obtain lower bounds on the variance of the distance to the central ranking for other notions of distance, such as Spearman’s Footrule and Spearman’s Rank Correlation. Then, we can apply our general approach and obtain tight bounds on the sample complexity of learning such models and on the quality of parameter estimation from a single sample, as in (Mukherjee, 2016).

### 1.2. Related work

There has been a significant volume of research work on algorithmic and learning problems related to our work. In the consensus ranking problem, a finite set \( \{\pi_1, \ldots, \pi_n\} \) of rankings is given, and we want to compute the ranking \( \arg\min_{\pi \in S_m} \sum_{i=1}^n d(\pi, \pi_i) \). This problem is known to be NP-hard (Bartholdi et al., 1989), but it admits a polynomial-time 11/7-approximation algorithm problem (Ailon et al., 2005) and a PTAS (Kenyon-Mathieu and Schudy, 2007). When the rankings are i.i.d. samples from a Mallows distribution, consensus ranking is equivalent to computing the maximum likelihood ranking, which does not depend on the spread parameter. Intuitively, the problem of finding the central ranking should not be hard, if the probability mass is concentrated around the central ranking. Meila et al. (2012) came up with a branch and bound technique which relies on this observation. Braverman and Mossel (2009) proposed a dynamic programming approach that computes the consensus ranking efficiently, under the Mallows model. Caragiannis et al. (2016) showed that the central ranking can be recovered from a logarithmic number of i.i.d. samples from a Mallows distribution (see also Theorem 12).

Mukherjee (2016) considered learning the spread parameter of a Mallows model based on a single sample, assuming that the central ranking is known. He studied the asymptotic behavior of his estimator and proved consistency. We strengthen this result by showing that our parameter estimator, based on single sample, can achieve optimal error for Mallows Block model (Corollary 14).

There has been significant work either on learning a Mallows model based on partial information, e.g. partial rankings or pairwise comparisons (Adkins and Fligner, 1998; Lu and Boutilier, 2011; Busa-Fekete et al., 2014), or on learning generalizations of the Mallows model, such as learning mixture of Mallows models (Liu and Moitra, 2018). Among these works, (Awasthi et al., 2014;
Liu and Moitra, 2018) seem the most relevant to our paper, since they considered learning mixtures of single parameter Mallows models in a learning setup that is similar in spirit to ours: find a model that is close to the underlying one either in the parameter space or in total variation distance based on as few sample as possible. However, the sample complexity of learning mixtures is necessarily much higher and a high degree polynomial of $1/\varepsilon$ and $m$. Hence their results do not compare with our optimal sample complexity analysis even for the simple Mallows model case.

The parameter estimation of the Generalized Mallows Model has been examined from a practical point of view by Meilă et al. (2007) but no theoretical guarantees for the sample complexity have been provided. Several ranking models are routinely used in analyzing ranking data (Marden, 1995; Agarwal, 2016), such as Plackett-Luce model (Plackett, 1975; Luce, 1959), Babington-Smith model (Joe and Verducci, 1993) and spectral analysis based methods (Kondor and Dempsey, 2012; Sibony et al., 2015) and non-parametric methods (Lebanon and Mao, 2007). However, to our best knowledge, none of these ranking methods have been analyzed from point of distribution learning. Hajek et al. (2014) considered the problem of learning parameters of Plackett-Luce model and they came up with high probability bounds for their estimator that is tight in a sense that there is no algorithm which can achieve lower estimation error with fewer examples.

2. Preliminaries and Notation

Small bold letters $x$ refer to real vectors in finite dimension $\mathbb{R}^d$ and capital bold letters $A$ refer to matrices in $\mathbb{R}^{d \times \ell}$. We denote by $x_i$ the $i$th coordinate of $x$, and by $A_{ij}$ the $(i, j)$th coordinate of $A$. For any $x, y \in \mathbb{R}^d$ we define $L(x, y) = \{z \in \mathbb{R}^d \mid z = tx + (1-t)y, \; t \in [0, 1]\}$.

Metrics between distributions. Let $p$, $q$ be two probability measures in the discrete probability space $(\Omega, \mathcal{A})$ then the total variation distance between $p$ and $q$ is defined as $d_{TV}(p, q) = \frac{1}{2} \sum_{x \in \Omega} |p(x) - q(x)| = \max_{A \in \mathcal{A}} |p(A) - q(A)|$, and the KL-divergence between $p$ and $q$ is defined as $D_{KL}(p || q) = \sum_{x \in \Omega} p(x) \ln \left( \frac{p(x)}{q(x)} \right)$.

Exponential Families. In this section we summarize the basic definitions and properties of the exponential families of distributions. We follow the formulation and the expressions of (Keener, 2011; Nielsen and Garcia, 2009) where we also refer for complete proofs of the statements presented in this section. Let $\mu$ be a measure on $\mathbb{R}^d$ and also $h : \mathbb{R}^d \rightarrow \mathbb{R}_+$, $T : \mathbb{R}^d \rightarrow \mathbb{R}^k$ be measurable functions. We define the logarithmic partition function $\alpha_{T,h} : \mathbb{R}^k \rightarrow \mathbb{R}_+$ as $\alpha(\eta) = \alpha_{T,h}(\eta) = \ln \left( \int \exp (\eta^T T(x)) h(x) \, d\mu(x) \right)$. We also define the range of natural parameters $\mathcal{H}_{T,h}$ as $\mathcal{H}_{T,h} = \{ \eta \in \mathbb{R}^k \mid \alpha_{T,h}(\eta) < \infty \}$. The exponential family $\mathcal{E}(T, h)$ with sufficient statistics $T$, carrier measure $h$ and natural parameters $\eta$ is the family of distributions $\mathcal{E}(T, h) = \{ \mathcal{P}_\eta \mid \eta \in \mathcal{H}_{T,h} \}$ where the probability distribution $\mathcal{P}_\eta$ has density

$$p_\eta(x) = \exp \left( \eta^T T(x) - \alpha(\eta) \right) h(x).$$

(2.1)

Truncated Geometric Distribution. We say that a random variable $Z$ follows the truncated geometric distribution $\mathcal{T}\mathcal{G}(\phi, k)$ with parameters $k \in \mathbb{N} \cup \{\infty\}$ and $\phi \in [0, 1]$ if it has the following probability mass function $p(i) = \phi^i / \sum_{j=0}^k \phi^j$ for $i \in [0, k]$ and 0 otherwise.

For $k = 2$ the distribution $\mathcal{T}\mathcal{G}(\phi, 2)$ is a Bernoulli distribution with success probability $\phi/(1 + \phi)$. For $k = \infty$ and $\phi \in [0, 1)$ the distribution $\mathcal{T}\mathcal{G}(\phi, k)$ is a geometric distribution $\mathcal{G}(\phi)$. Observe
that if we fix $k$ then $E_k = \{TG(\phi, k) \mid \phi \in [0, 1]\}$ is an exponential family with natural parameter $\theta = \ln(\phi)$. Again the domain of $\phi$ changes to $\phi \in [0, 1)$ for $k = \infty$.

**Basic Properties of Exponential Families.** We summarize in the next theorem the fundamental properties of exponential families. For a proof of this theorem we refer to the Appendix A.

**Theorem 1** Let $E(T, h)$ be an exponential family parametrized by $\eta \in \mathbb{R}^k$ and for simplicity let $\alpha(\cdot) = \alpha_{T, h}(\cdot)$ and $H = H_{T, h}$ then the following hold.

1. For all $\eta \in H$, it holds that
   \[ \mathbb{E}_{x \sim P_\eta} [T(x)] = \nabla \alpha(\eta). \] (2.2)

2. For all $\eta \in H$, it holds that
   \[ \text{Var}_{x \sim P_\eta} [T(x)] = \nabla^2 \alpha(\eta). \] (2.3)

3. For all $\eta \in H$, $s \in \mathbb{R}^d$, it holds that
   \[ \mathbb{E}_{x \sim P_\eta} [\exp (s^T T(x))] = \exp (\alpha(\eta + s) - \alpha(\eta)). \] (2.4)

4. For all $\eta, \eta' \in H$, and for some $\xi \in L(\eta, \eta')$ it holds that
   \[ D_{KL}(P_{\eta'} || P_{\eta}) = -(\eta' - \eta)^T \nabla \alpha(\eta) + \alpha(\eta') - \alpha(\eta) = (\eta' - \eta)^T \nabla^2 \alpha(\xi) (\eta' - \eta). \] (2.5)

### 2.1. Ranking Distributions

In this section we review the basic definitions of exponential families over permutations. We define the single parameter Mallows model and its generalization.

**Single Parameter Mallows Model.** The Mallows model or, more specifically, Mallows $\phi$-distribution is a parametrized, distance-based probability distribution that belongs to the family of exponential distributions $M_1 = \{P_{\phi, \pi_0} \mid \phi \in [0, 1], \pi_0 \in S_m\}$ with probability mass function $p_{\phi, \pi_0}(\pi) = \phi^{d(\pi, \pi_0)} / Z(\phi, \pi_0)$ where $\phi$ and $\pi_0$ are the parameters of the model: $\pi_0 \in S_m$ is the location parameter also called center ranking and $\phi \in [0, 1]$ the spread parameter. Moreover, $d(\cdot, \cdot)$ is a distance metric on permutations, which for our paper will be the Kendall tau distance, that is, the number of discordant item pairs $d_K(\pi, \pi') = \sum_{1 \leq i < j \leq m} 1 \{(\pi(i) - \pi(j)) (\pi'(i) - \pi'(j)) < 0\}$.

The normalization factor in the definition of the model is equal to $Z(\phi, \pi_0) = \sum_{\pi \in S_m} \phi^{d(\pi, \pi_0)}$. When the distance metric $d$ is the Kendall tau distance we have $Z(\phi, \pi_0) = Z(\phi) = \prod_{i=1}^{m-1} \sum_{j=0}^i \phi^j$. Observe that the family of distributions as stated is not an exponential family because of the location parameter $\pi_0$. If we fix the permutation parameter then the family $M_1(\pi_0) = \{P_{\phi, \pi_0} \mid \phi \in [0, 1]\}$ is an exponential family with natural parameter $\theta = \ln \phi$.

**Generalized Mallows Model.** One of the most famous generalizations of Mallows model is the one introduced by Fligner and Verducci (1986) with the name Generalized Mallows Model. We define $V_j(\sigma, \pi) = (\sigma_i - \pi_i) (\sigma_j - \pi_j)$ to be the number of discordant item pairs involving item $j$, i.e. $V_j(\sigma, \pi) = \sum_{1 \leq i < j \leq m} 1 \{(\sigma_i - \sigma_j) (\pi_i - \pi_j) < 0\}$. The generalized Mallows family of distribution $M_m = \{P_{\phi, \pi_0} \mid \phi \in [0, 1]^m, \pi_0 \in S_m\}$ with parameters $\pi_0 \in S_m$ and $\phi = (\phi_1, \ldots, \phi_m) \in [0, 1]^m$ is defined as the probability measure over $S_m$ with probability mass function $p_{\phi, \pi_0}(\pi) = \prod_{i=1}^m \phi_i^{V_i(\pi, \pi_0)} / Z(\phi, \pi_0)$. One important property of the generalized mallows model when the distance metric $d$ is the Kendall
tau distance is that the random variables \( Y_i = V_i(X, \pi) \) where \( X \sim P_{\phi, \pi_0} \) are independent. This follows from the following decomposition lemma of the partition function \( Z(\phi) \). For the proof of Lemma 2 we refer to the Appendix A.

**Lemma 2** When \( d = \) Kendall tau distance, we have that \( Z(\phi, \pi_0) = Z(\phi) = \prod_{i=1}^{m} Z_i(\phi_i) \), where \( Z_i(x) = \sum_{j=0}^{i-1} x^j \).

In Section 5 we introduce the Mallows Block Model that interpolates between the single parameter and the generalized Mallows model.

### 2.2. Fano’s Inequality

In this section we present Fano’s inequality which is our main technical tool for proving lower bounds on the sample complexity of learning Mallows Block Models. For this, let \( \mathcal{X} \) denote some finite set.

**Maximum Risk of an Estimator.** Let \( \mathcal{F} \) be a family of distributions and assume that we have access to \( n \) i.i.d. samples \( x = (x_1, \ldots, x_n) \sim f^n \in \mathcal{F} \). Let \( \hat{f} : \mathcal{X}^n \rightarrow \Delta_\mathcal{X} \). Then the maximum risk of \( \hat{f} \) with respect to the family \( \mathcal{F} \) is equal to

\[
R_n(\hat{f}, \mathcal{F}) = \sup_{f \in \mathcal{F}} \mathbb{E}_{x \sim f^n} \left[ d_{TV}(\hat{f}(x), f) \right].
\]

**Minimax Risk.** Let \( \mathcal{F} \) be a family of distributions and assume that we have access to \( n \) i.i.d. samples \( x = (x_1, \ldots, x_n) \sim f^n \in \mathcal{F} \). Let also \( \Omega = \{ \hat{f} : \mathcal{X}^n \rightarrow \Delta_\mathcal{X} \} \). Then we define the minimax risk of the family \( \mathcal{F} \) as

\[
R_n(\mathcal{F}) = \inf_{\hat{f} \in \Omega} R_n(\hat{f}, \mathcal{F}).
\]

We can now state Fano’s Inequality as presented by Yu (1997).

**Theorem 3 (Lemma 3 in (Yu, 1997))** Let \( \mathcal{F} \) be a finite family of densities such that

\[
\inf_{f, g \in \mathcal{F} : f \neq g} d_{TV}(f, g) \geq \alpha, \quad \sup_{f, g \in \mathcal{F} : f \neq g} D_{KL}(f || g) \leq \beta,
\]

then it holds that

\[
R_n(\mathcal{F}) \geq \frac{\alpha}{2} \left( 1 - \frac{n\beta + \ln 2}{\ln |\mathcal{F}|} \right).
\]
3. Concentration Inequality and Total Variation of Exponential Families

We shall prove a concentration inequality for the sufficient statistics of an exponential family. This concentration inequality will be the basic building block for the general learning algorithm for exponential inequalities that we will present in the next section. Then we prove an exact formula for the total variation distance between two distributions that belong to the same exponential family.

**Theorem 4** Let \(\mathcal{E}(T, h)\) be an exponential family with natural parameter \(\eta \in \mathbb{R}\), logarithmic partition function \(\alpha\) and range of parameters \(\mathcal{H}\). Then the following concentration inequality holds for all \(\eta, \eta' \in \mathcal{H}\)

\[
\mathbb{P}_{x \sim \mathcal{P}_\eta} \left( \frac{1}{n} \sum_{i=1}^{n} T(x_i) \leq \mathbb{E}_{y \sim \mathcal{P}_{\eta'}} [T(y)] \right) \leq \exp \left( - \frac{1}{4} D_{KL}(\mathcal{P}_{\eta'}||\mathcal{P}_\eta) n \right). \tag{3.1}
\]

**Proof** We give the proof for \(\eta' > \eta\) and the case \(\eta' < \eta\) can be handled respectively. Let \(s > 0, \eta' > \eta\) and for simplicity \(p = \mathbb{P}_{x \sim \mathcal{P}_\eta} \left( \frac{1}{n} \sum_{i=1}^{n} T(x_i) \geq \mathbb{E}_{y \sim \mathcal{P}_{\eta'}} [T(y)] \right)\) then it holds that

\[
p = \frac{\mathbb{P}_{x \sim \mathcal{P}_\eta} \left[ \exp \left( s \sum_{i=1}^{n} T(x_i) \right) \right]}{\exp \left( s \mathbb{E}_{y \sim \mathcal{P}_{\eta'}} [T(y)] \right)} \leq \exp \left( - \frac{1}{s} \mathbb{E}_{x_1 \sim \mathcal{P}_\eta} \left[ \exp \left( s \sum_{i=1}^{n} T(x_i) \right) \right] \right) \tag{Markov’s Inequality}
\]

\[
= \frac{\mathbb{E}_{x \sim \mathcal{P}_\eta} \left[ \exp \left( s \sum_{i=1}^{n} T(x_i) \right) \right]}{\mathbb{E}_{y \sim \mathcal{P}_{\eta'}} \left[ \exp \left( s \mathbb{E}_{y \sim \mathcal{P}_{\eta'}} [T(y)] \right) \right]} = \left( \frac{\exp (\alpha(\eta + s) - \alpha(\eta))}{\exp (s \hat{\alpha}(\eta'))} \right)^n = \exp \left( - \left( s \hat{\alpha}(\eta') - \alpha(\eta + s) + \alpha(\eta) \right) n \right) \quad \text{(By (2.2), (2.4))}
\]

Now we define the function \(f(s) = s \hat{\alpha}(\eta') - \alpha(\eta + s) + \alpha(\eta)\). The second derivative of \(f\) is \(f''(s) = -\hat{\alpha}(\eta + s)\). From (2.3) we conclude that \(\hat{\alpha}(\eta + s) \geq 0\) and hence \(f''(s) \leq 0\) which implies that \(f\) is a concave function. Hence \(f\) achieves its maximum for at \(s^*\) such that \(f'(s^*) = 0\). But \(f'(s) = \hat{\alpha}(\eta') - \hat{\alpha}(\eta + s)\) which implies that for \(s^* = \eta' - \eta\) it holds that \(f'(s^*) = 0\). Therefore the optimal bound of the above form is achieved for \(s = \eta' - \eta\). Hence we have the following

\[
p \leq \exp \left( - \left( s^* \hat{\alpha}(\eta') - \alpha(\eta + s^*) + \alpha(\eta) \right) n \right) \tag{3.2}
\]

which concludes the proof. \(\blacksquare\)

The following useful corollary of Theorem 3.1 can be obtained if we apply Pinsker’s inequality to the right hand side of (3.1).

**Corollary 5** Let \(\mathcal{E}(T, h)\) be an exponential family with natural parameter \(\eta \in \mathbb{R}\), logarithmic partition function \(\alpha\) and range of parameters \(\mathcal{H}\). Then the following concentration inequality holds for all \(\eta, \eta' \in \mathcal{H}\)

\[
\mathbb{P}_{x \sim \mathcal{P}_\eta} \left( \frac{1}{n} \sum_{i=1}^{n} T(x_i) \geq \mathbb{E}_{y \sim \mathcal{P}_{\eta'}} [T(y)] \right) \leq \exp \left( - 2d_{TV}^2(\mathcal{P}_{\eta'}, \mathcal{P}_\eta) n \right). \tag{3.2}
\]
We now move to proving an exact formula for \( d_{TV}(P_\eta, P_\eta') \). For the proof of Theorem 6 we refer to the Appendix B.

**Theorem 6** Let \( \mathcal{E}(T, h) \) be an exponential family with natural parameters \( \eta \). If \( P_\eta, P_\eta' \in \mathcal{E}(T, h) \), with then for some \( \xi \in \mathcal{L}(\eta, \eta') \) it holds that

\[
d_{TV}(P_\eta, P_\eta') = \mathbb{E}_{x \sim \mathcal{P}_\xi} \left[ \text{sign}(P_\eta(x) - P_\eta'(x)) \left( \eta - \eta' \right)^T \left( T(x) - \mathbb{E}_{y \sim \mathcal{P}_\xi} [T(y)] \right) \right].
\]

To give some intuition about Theorem 6, consider the single dimensional case with \( \eta' \rightarrow \eta \) and \( \eta \geq \eta' \). In this case, it is easy to see that the sign of \( \left( T(x) - \mathbb{E}_{y \sim \mathcal{P}_\xi} [T(y)] \right) \) and \( \left( P_\eta(x) - P_\eta'(x) \right) \) are the same and hence the expression becomes \( \left( \eta - \eta' \right) \mathbb{E}_{x \sim \mathcal{P}_\xi} \left[ \left| T(x) - \mathbb{E}_{y \sim \mathcal{P}_\xi} [T(y)] \right| \right] \). This gives the intuition that the total variation of two distribution in the same exponential family, with parameters sufficiently close, is equal to the distance between their parameters times the absolute deviation of a corresponding distribution in the family. This should be compared with Theorem 1.4, on the KL-divergence between two distributions in the same exponential family. The single dimensional version Theorem 1.4 states that the KL-divergence is equal to the square difference of their parameters multiplied by the variance of a corresponding distribution inside the exponential family. Since the standard deviation is greater than the absolute deviation this conclusion resembles the well known Pinsker’s inequality. Furthermore, in a lot of exponential families, e.g. Gaussian distributions, the absolute deviation is only a constant fraction away from the standard deviation which indicates the existence of a converse Pinsker’s inequality in these settings.

### 4. Warm-up: Learning Single Parameter Mallows Model

In this section we give a simple algorithm and prove its sample complexity for learning the parameters \((\phi, \pi_0)\) of a single parameter distribution \( P_{\phi, \pi_0} \in \mathcal{M}_1 \) given i.i.d. samples \( \pi_1, \ldots, \pi_n \) from \( \mathcal{P} \).

We also provide bounds for learning the distribution \( P_{\phi, \pi_0} \) in total variation distance. As we will see if the central ranking \( \pi_0 \) is known then an accurate estimation of \( \phi \) is possible hence giving an alternative proof of a phenomenon proved by Mukherjee (2016).

#### 4.1. Parameter Estimation

For the single parameter Mallows model the sample complexity of estimating the central ranking has been identified in Caragiannis et al. (2016) as we see in the next theorem. We focus on the case where the ranking distance is the Kendall tau distance \( d_K \).

**Theorem 7** (Caragiannis et al., 2016) For any \( \pi_0 \in S_n \) and any \( \phi \in [0, 1 - \gamma] \), there exists a polynomial time estimator \( \hat{\pi} \) such that given \( n = \Theta \left( \frac{1}{\gamma} \log(m/\delta) \right) \) i.i.d. samples \( \pi_1, \ldots, \pi_n \sim P_{\phi, \pi_0} \) satisfies \( \mathbb{P}_{\pi \sim P_{\phi, \pi_0}} (\hat{\pi} \neq \pi_0) \leq \delta \). Moreover, if \( n = o(\log(m/\delta)) \) then for any estimator \( \hat{\pi} \) there exists a distribution \( P_{\phi, \pi_0} \) such that \( \mathbb{P}_{\pi \sim P_{\phi, \pi_0}} (\hat{\pi} \neq \pi_0) > \delta \).

Hence it remains to estimate the parameter \( \phi \) if we have the knowledge of the central ranking \( \pi_0 \). As we explained in the definition of Mallows model when the central ranking is known the family of distributions \( \mathcal{M}_1(\pi_0) \) is a single parameter exponential family. The sufficient statistic of this family is \( T(\pi) = d_K(\pi, \pi_0) \). The natural parameter of \( \mathcal{M}_1(\pi_0) \) is the parameter \( \theta = \ln \phi \) and logarithmic partition function \( \alpha(\theta) = \ln \left( Z(e^\theta) \right) \).
Theorem 8  For any \( \pi_0 \in S_m, \phi^* \in [0, 1-\gamma], \varepsilon, \delta > 0 \) there exist estimators \( \hat{\pi}, \hat{\phi} \) that can be computed in polynomial time from i.i.d. samples \( \pi \sim \mathcal{P}_{\phi^*, \pi_0}^n \) such that if \( n \geq \Omega \left( \frac{\log(1/\delta)}{m \varepsilon^2} + \frac{\log(m/\delta)}{\gamma} \right) \), then

\[
P_{\pi \sim \mathcal{P}_{\phi^*, \pi_0}^n} \left( (\hat{\pi} = \pi_0) \land \left( \hat{\phi} \in [\phi^* - \varepsilon, \phi^* + \varepsilon] \right) \right) \geq 1 - \delta.
\]

In the case where \( \pi_0 \) is known for \( \phi^* \in [0, 1] \) then there exists an estimator \( \hat{\phi} \) that can be computed in polynomial time such that if \( n \geq \Omega \left( \frac{\log(1/\delta)}{m \varepsilon^2} \right) \), then

\[
P_{\pi \sim \mathcal{P}_{\phi^*, \pi_0}^n} \left( \hat{\phi} \in [\phi^* - \varepsilon, \phi^* + \varepsilon] \right) \geq 1 - \delta.
\]

Theorem 8 follows from the more general Theorem 13 and hence we postpone its proof for the Section 5. One interesting thing to point out though from Theorem 8 is that in the case where \( \pi_0 \) is known, Theorem 8 provides accuracy for the parameter \( \phi \) that goes to 0, even with \( n = 1 \) sample, as the size of the permutation goes to infinity, i.e. \( m \to \infty \). This was observed before by Mukherjee (2016) but no explicit rates as the ones we provide, were provided. We summarize our result for \( n = 1 \) sample in the following corollary, which immediately follows from Theorem 8.

Corollary 9  For any known \( \pi_0 \in S_m, \) any \( \phi^* \in [0, 1] \) and \( \delta > 0 \), there exists an estimator \( \hat{\phi} \) that can be computed in polynomial time from one sample \( \pi \sim \mathcal{P}_{\phi^*, \pi_0}^n \) such that

\[
P_{\pi \sim \mathcal{P}_{\phi^*, \pi_0}^n} \left( \hat{\phi} \in [\phi^* - \varepsilon, \phi^* + \varepsilon] \right) \geq 1 - \delta, \text{ where } \varepsilon = O \left( \sqrt{\frac{\log(1/\delta)}{m}} \right).
\]

4.2. Learning in KL and TV Distance

The upper bound on the number of samples that we need to learn the distribution \( \mathcal{P}_{\phi^*, \pi_0}^n \) in KL and TV distance follows from Theorem 4 and Theorem 7 as we show in the more general Theorem 15. To finish this section we focus on proving the lower bound for learning in TV distance. The lower bound for learning the single parameter \( \phi \) follows again from the corresponding lower bound of Section 5 and hence the term \( \frac{1}{m} \) in the sampling complexity necessary. In the next lemma we prove that the term \( \log(m) \) is also necessary.

Lemma 10  For any \( n = o(\log(m)) \) it holds that

\[
\mathcal{R}_n(M_1) \geq 1/16.
\]

For the proof of Lemma 10 we refer to the Appendix C.
5. Learning Mallows Block Model

We start this section with properties of the Generalized Mallows Model as it is defined in Section 2. Then we move to the definition of the Mallows Block Model and the presentation of our main results. We remind the reader that the generalized Mallows family of distribution is \( \mathcal{M}_m = \{ \mathcal{P}_{\phi, \pi_0} \mid \phi \in [0, 1]^m, \pi_0 \in S_m \} \) with parameters \( \pi_0 \in S_m \) and \( \phi = (\phi_1, \ldots, \phi_m) \in [0, 1]^m \) is defined as the probability measure over \( S_m \) with probability mass function that using Lemma 2 is equal to

\[
\mathcal{P}_{\phi, \pi_0}(\pi) = \prod_{i=1}^{m} \frac{\phi_i^{V_i(\pi, \pi_0)}}{Z_i(\phi_i)}.
\]  

(5.1)

We define now the random variables \( Y_i = V_i(\pi, \pi_0) \) where \( \pi \sim \mathcal{P}_{\phi, \pi_0} \) which are the sufficient statistics for \( \mathcal{P}_{\phi, \pi_0} \) when \( \pi_0 \) is known. It is easy to observe from (5.1) that the probability mass function of the vector \((Y_1, \ldots, Y_m)\) is

\[
P(Y_1 = y_1, \ldots, Y_m = y_m) = \left( \frac{\phi_1^{y_1}}{Z_1(\phi_1)} \right) \cdots \left( \frac{\phi_m^{y_m}}{Z_m(\phi_m)} \right) = \mathbb{P}(Y_1 = y_1) \cdots \mathbb{P}(Y_m = y_m)
\]

(5.2)

and hence the random variables \( Y_i \) are independent. Observe also from the probability mass function and the definition of the truncated geometric distribution in Section 2 that \( Y_i \sim T\mathcal{G}(\phi_i, i - 1) \). To formally summarize this observation we define \( \mathcal{P}_\phi \) to be the multivariate distribution \((Z_1, \ldots, Z_m)\), where \( Z_i \sim T\mathcal{G}(\phi_i, i - 1) \). The following lemma relates the distribution \( \mathcal{P}_\phi \) with the distribution \( \mathcal{P}_{\phi, \pi_0} \) when the central ranking \( \pi_0 \) is known. For the proof we refer to the Appendix D.

**Lemma 11** Let \( \pi_0 \in S_m \) and \( \phi, \pi \in [0, 1]^m \). Let also \( R_{\phi} \) be the support of the distribution \( \mathcal{P}_\phi \) and \( R_{\phi, \pi_0} \) the support of the distribution \( \mathcal{P}_{\phi, \pi_0} \). Then there exists a bijective map \( h : R_{\phi, \pi_0} \rightarrow R_{\phi} \) such that for any \( \sigma \in R_{\phi, \pi_0} \) it holds that \( \mathbb{P}_{\mathcal{P}_{\phi, \pi_0}}(\pi = \sigma) = \mathbb{P}_{\mathcal{P}_\phi}(y = h(\sigma)) \). In particular,

\[
d_{TV}(\mathcal{P}_{\phi, \pi_0}, \mathcal{P}_{\phi'}) = d_{TV}(\mathcal{P}_\phi, \mathcal{P}_{\phi'}) \quad \text{and} \quad D_{KL}(\mathcal{P}_{\phi, \pi_0} || \mathcal{P}_{\phi', \pi_0}) = D_{KL}(\mathcal{P}_\phi || \mathcal{P}_{\phi'})
\]

The above lemma reduces the problem of learning the Generalized Mallows distribution \( \mathcal{P}_{\phi, \pi_0} \) to the learning of the central ranking \( \pi_0 \) and the distribution \( \mathcal{P}_\phi \).

**Mallows Block Model.** The motivation of Mallows Block Model is to incorporate setting where some group of alternatives have the same probability of being misplaced hence they have the same parameter \( \phi_i \), but not all alternatives have the same probability of being misplaced as in the single parameter Mallows model. As we will explore in this section, the knowledge of the groups of alternatives with the same parameter can significantly decrease the number of samples needed to learn the parameters of the model. In the extreme case, when the size of the groups of alternatives is large enough, we can get very good rates even from just one samples from the distribution as we already discussed in Corollary 9. The Mallows Block Model with \( d \) parameters is the family of distributions

\[\mathcal{M}_d(B) = \{ \mathcal{P}_{\phi, \pi_0, B} \mid \phi \in [0, 1]^d, \pi_0 \in S_m \}\]

where \( B = \{B_1, \ldots, B_d\} \) is a partitioning of the set \([m]\). Each distribution \( \mathcal{P}_{\phi, \pi_0, B} \) is defined as a probability measure over \( S_m \) with the following probability mass function

\[
p_{\phi, \pi_0, B}(\pi) = \frac{1}{Z(\phi, \pi_0, B)} \prod_{i=1}^{d} \phi_i^{\sum_{j \in B_i} V_j(\pi, \pi_0)}.
\]

(5.3)
Again using Lemma 2 we have that \( Z(\phi, \pi_0, B) = Z(\phi, B) = \prod_{i=1}^{d} \left( \prod_{j \in B_i} Z_j(\phi_i) \right) \). The sufficient statistics of \( \mathcal{P}_{\phi, \pi_0, B} \), when \( \pi_0, B \) are known, is the \( d \) dimensional vector \( T(\pi, \pi_0, B) \) where \( T_i(\pi, \pi_0, B) = \sum_{j \in B_i} V_j(\pi, \pi_0) \). We define the distribution \( \mathcal{P}_{\phi, B} \) to be the distribution of the random vector \((Z_1, \ldots, Z_m)\) where \( Z_j \sim \mathcal{G}(\phi_i, j - 1) \) are independent and \( i \) satisfies \( j \in B_i \).

One important parameter of the Mallows Block Model are the sizes of the sets \( B_i \) in the partition \( B \) of \([m]\). For this reason we define \( m_i = |B_i| \) and \( m^* = \min_{i \in [d]} |B_i| \).

### 5.1. Parameter Estimation in Mallows Block Model

We start with the estimation of the central ranking. Since the single parameter Mallows model is a special case of the Mallows Block Model the lower bound of Caragiannis et al. (2016) presented in Theorem 7 still holds, and thus \( \Omega(\log(m)) \) samples are necessary. The upper bound we present in Theorem 12. Its proof is deferred to Appendix D.

**Theorem 12** For any \( \pi_0 \in S_m \), any \( \phi \in [0, 1 - \gamma)^d \), any known partition \( B \) of \([m]\), there exists a polynomial time computable estimator \( \hat{\pi} \) such that given \( n = \Theta\left( \frac{1}{\gamma} \log(m/\delta) \right) \) i.i.d. samples \( \pi = (\pi_1, \ldots, \pi_n) \sim \mathcal{P}_{\phi, \pi_0, B} \) satisfies \( \mathbb{P}_{\pi \sim \mathcal{P}_{\phi, \pi_0, B}} (\hat{\pi} \neq \pi_0) \leq \delta \). Moreover, if \( n = \omega\left( \log(m/\delta) \right) \) then for any estimator \( \hat{\pi} \) there exists a distribution \( \mathcal{P}_{\phi, \pi_0, B} \) such that \( \mathbb{P}_{\pi \sim \mathcal{P}_{\phi, \pi_0, B}} (\hat{\pi} \neq \pi_0) > \delta \).

What remains is to estimate the vector of parameters \( \phi \) assuming the knowledge of the central ranking \( \pi_0 \). As we explained in the definition of Mallows Block Model when the central ranking is known the family of distributions \( \mathcal{M}_d(B, \pi_0) \) is an exponential family. The sufficient statistics of this family are \( T_i(\pi, \pi_0, B) = \sum_{j \in B_i} V_j(\pi, \pi_0) \). The natural parameters of \( \mathcal{M}_d(B, \pi_0) \) is the vector of parameters \( \theta \in \mathbb{R}^d \) where \( \theta_i = \ln(\phi_i) \) and logarithmic partition function \( \alpha(\theta, B) = \ln(Z(\phi, B)) \). We may simplify the notation \( \alpha(\theta, B) \) to \( \alpha(\theta) \) when \( B \) is clear from the context.

**Theorem 13** For any \( \pi_0 \in S_m \), \( \phi^* \in [0, 1 - \gamma)^d \), any fixed partition \( B \) of \([m]\) with \(|B| = d \) and any \( \varepsilon, \delta > 0 \) there exist estimators \( \hat{\phi}, \hat{\pi} \) that can be computed in polynomial time from i.i.d. samples \( \pi \sim \mathcal{P}_{\phi^*, \pi_0, B} \) such that if \( n \geq \Omega \left( \frac{d \log(d/\delta)}{m^* \varepsilon^2} + \frac{\log(m/\delta)}{\gamma} \right) \), where \( m^* = \min_{i \in [d]} |B_i| \), then

\[
\mathbb{P}_{\pi \sim \mathcal{P}_{\phi^*, \pi_0, B}} \left( \left( \hat{\pi} = \pi_0 \right) \land \left( \left\| \hat{\phi} - \phi^* \right\|_2 \leq \varepsilon \right) \right) \geq 1 - \delta.
\]

In the case where \( \pi_0 \) is known and \( \phi^* \in [0, 1]^d \) then there exists an estimator \( \hat{\phi} \) that can be computed in polynomial time such that if \( n \geq \Omega \left( \frac{d \log(d/\delta)}{m^* \varepsilon^2} \right) \) then

\[
\mathbb{P}_{\pi \sim \mathcal{P}_{\phi^*, \pi_0, B}} \left( \left\| \hat{\phi} - \phi^* \right\|_2 \leq \varepsilon \right) \geq 1 - \delta.
\]

As a corollary of Theorem 13 we also have that when \( \pi_0 \) is known even one sample is sufficient to consistently learn all the parameters \( \phi^* \) as the size of the smaller block of \( B \) goes to infinity.

**Corollary 14** Let \( \pi_0 \in S_m \), \( \phi^* \in [0, 1]^d \), \( \delta > 0 \) and a partition \( B \) of \([m]\) with \(|B| = d \), there exist an estimator \( \hat{\phi} \) that can be computed in polynomial time from a sample \( \pi \sim \mathcal{P}_{\phi^*, \pi_0, B} \) such that

\[
\mathbb{P}_{\pi \sim \mathcal{P}_{\phi^*, \pi_0, B}} \left( \left\| \hat{\phi} - \phi^* \right\|_2 \geq \varepsilon \right) \geq 1 - \delta
\]

where \( \varepsilon = O \left( \sqrt{d/m^*} \cdot \sqrt{\log(d/\delta)} \right) \) and \( m^* = \min_{i \in [d]} |B_i| \).
Proof of Theorem 13: (Sketch) From Theorem 12 we focus on the estimation of the parameters $\phi^*$. We describe the intuition for the single parameter Mallows Model and we defer the full proof to Appendix E. Let $\phi^* = \phi^* \in [0, 1]$. Once the central ranking is known the distribution is an exponential family and let $T(\pi)$ be its sufficient statistics. It is not hard to prove that $\mathbb{E}_{\pi \sim \mathcal{P}_{\phi, \pi_0}} [T(\pi)]$ is an increasing function of $\phi$. Therefore, it follows with a simple argument, that the better we estimate $\mathbb{E}_{\pi \sim \mathcal{P}_{\phi^*, \pi_0}} [T(\pi)]$ the better we can estimate $\phi^*$. Now the main idea of our proof is to use the general concentration inequality of Theorem 4 to bound the accuracy that we can estimate $\mathbb{E}_{\pi \sim \mathcal{P}_{\phi^*, \pi_0}} [T(\pi)]$. As it is clear from the form of the concentration inequality (3.1), to get good enough concentration we have to prove a strong lower bound on the KL-divergence of two distributions in the family. From (2.3) this reduces to proving a lower bound on the variance of a distribution in the family with parameter $\psi$ that is very close to $\phi^*$. Such a good lower bound is not always possible to prove and we have to consider some cases. But in the main case a very careful lower bound of the variance in combination with (3.1) gives the sample complexity upper bound. □

5.2. Learning in KL-divergence and Total Variation Distance

In this section we will describe how we can use the concentration inequality that we proved in Section 3 to learn a distribution $\mathcal{P}_{\phi^*, \pi_0, B}$ in KL-divergence from i.i.d. samples. We also prove a lower bound that matches the upper bound up to a $\log(d)$ factor.

Theorem 15 For any $\pi_0 \in S_m$, $\phi^* \in [0, 1]^d$, any fixed partition $B$ of $[m]$ with $|B| = d$ and any $\varepsilon, \delta > 0$ there exist estimators $\hat{\phi}$ that can be computed in polynomial time from i.i.d. samples $\pi \sim \mathcal{P}_{\phi^*, \pi_0, B}^n$ such that if $n \geq \Omega \left( \frac{d^2}{\varepsilon^2} \log(d/\delta) + \log(m) \right)$, then

$$\mathbb{P}_{\pi \sim \mathcal{P}_{\phi^*, \pi_0, B}^n} \left( \text{D}_{\text{KL}} \left( \mathcal{P}_{\phi^*, \pi_0, B} \| \mathcal{P}_{\phi^*, \pi_0, B} \right) \leq \varepsilon^2 \right) \geq 1 - \delta$$

and hence $\mathbb{P}_{\pi \sim \mathcal{P}_{\phi^*, \pi_0, B}^n} \left( \text{d}_{\text{TV}} \left( \mathcal{P}_{\phi^*, \pi_0, B}, \mathcal{P}_{\phi^*, \pi_0, B} \right) \leq \varepsilon \right) \geq 1 - \delta$.

Furthermore, for any $m \in \mathbb{N}$ there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and all functions $\mathcal{P} : S_m^m \rightarrow \Delta S_m$ with $n = o \left( \frac{d}{\varepsilon^2} \right)$ there exists $\pi_0 \in S_m$, partition $B$ of $[m]$ and $\phi^* \in [0, 1]^d$ such that

$$\mathbb{P}_{\pi \sim \mathcal{P}_{\phi^*, \pi_0, B}^n} \left( \text{d}_{\text{TV}} \left( \mathcal{P}_{\phi^*, \pi_0, B}, \mathcal{P}(\pi) \right) \geq 2\varepsilon \right) \geq 1/3.$$

The proof of Theorem 15 is based on two lemmas, one for the upper bound and one for the lower bound, that we present here and the Lemma 10 that we presented in Section 4. For the proofs of Lemma 16 and Lemma 17 we refer to the Appendix D.

Lemma 16 For any $\pi_0 \in S_m$, $\phi^* \in [0, 1]^d$, any fixed partition $B$ of $[m]$ with $|B| = d$ and any $\varepsilon, \delta > 0$ there exist estimators $\hat{\phi}$ that can be computed in polynomial time from i.i.d. samples $\pi \sim \mathcal{P}_{\phi^*, \pi_0, B}^n$ such that if $n \geq \Omega \left( \frac{d^2}{\varepsilon^2} \log(d/\delta) + \log(m) \right)$, then

$$\mathbb{P}_{\pi \sim \mathcal{P}_{\phi^*, \pi_0, B}^n} \left( \text{D}_{\text{KL}} \left( \mathcal{P}_{\phi^*, \pi_0, B} \| \mathcal{P}_{\phi^*, \pi_0, B} \right) \leq \varepsilon^2 \right) \geq 1 - \delta.$$

Lemma 17 For any $m \in \mathbb{N}$, $d \leq m$, there exists a partition $B$ of $[m]$ and an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and $n = o \left( \frac{d}{\varepsilon^2} \right)$, it holds that

$$\mathcal{R}_n(\mathcal{M}_d(B)) \geq 2\varepsilon.$$
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Appendix A. Proofs of Theorem 1, Lemma 2 and Fano’s Inequality

Proof of Theorem 1: For the parts 1., 2. and 3. we refer the reader to (Keener, 2011; Nielsen and Garcia, 2009). We present here the proof of 4. because it is makes the use of the Taylor’s Theorem in the last step comparing to the usual expression that appears in the literature.

\[
D_{KL}(P_\eta||P_{\eta'}) = \int p_\eta(x) \ln \frac{p_\eta(x)}{p_{\eta'}(x)} d\mu(x)
\]

\[
= \int p_\eta(x) \left( (\eta - \eta')^T T(x) + \alpha(\eta') - \alpha(\eta) \right) d\mu(x)
\]

\[
= (\eta - \eta')^T \sum_{x \sim P_\eta} [T(x)] + \alpha(\eta') - \alpha(\eta)
\]

\[
\overset{(2.2)}{=} -(\eta' - \eta)^T \nabla \alpha(\eta) + \alpha(\eta') - \alpha(\eta)
\]

\[
= (\eta' - \eta)^T \nabla^2 \alpha(\xi) (\eta' - \eta)
\]

where the last step follows from the multidimensional Taylor’s Theorem for some \( \xi \in L(\eta, \eta') \). ■

Proof of Lemma 2: We use the simple but profound one-to-one correspondence between every permutation \( \sigma \in S_m \) and the vector of numbers \( (V_1(\sigma, \pi), V_2(\sigma, \pi), \ldots, V_m(\sigma, \pi)) \), where \( V_j(\sigma, \pi) \in [0, j - 1] \). According to Knuth (1997) this correspondence was first proved by Marshall Hall. Let \( \Omega_m^l = [l] \times [l + 1] \times \cdots \times [m - 1] \). This one-to-one correspondence allows as to write the partition function \( Z(\phi) \) in the following way

\[
Z(\phi) = \sum_{y \in \Omega_m^l} \prod_{i=1}^{m} \phi_{y_j}^{y_i}
\]

\[
= \sum_{y_1 \in [0]} \phi_{y_1}^{y_1} \left( \sum_{y \in \Omega_m^l \setminus y_1} \prod_{i=2}^{m} \phi_{y_j}^{y_i} \right)
\]

\[
= \left( \sum_{y_1 \in [0]} \phi_{y_1}^{y_1} \right) \left( \sum_{y \in \Omega_m^l \setminus \{y_1\}} \prod_{i=2}^{m} \phi_{y_j}^{y_i} \right)
\]

continuing this process recursively the lemma follows. ■
Appendix B. Omitted Proofs from Section 3

Proof of Theorem 6:
\[ d_{TV}(\mathcal{P}_\eta, \mathcal{P}_{\eta'}) = \sum_{x \in X} |h(x) \exp (\eta^T T(x) - \alpha(\eta)) - h(x) \exp (\eta'^T T(x) - \alpha(\eta'))| \]
\[ = \sum_{x \in X} \text{sign} (\mathcal{P}_\eta(x) - \mathcal{P}_{\eta'}(x)) h(x) \left( \exp (\eta^T T(x) - \alpha(\eta)) - \exp (\eta'^T T(x) - \alpha(\eta')) \right) \]

now let \( b(x) = \text{sign} (\mathcal{P}_\eta(x) - \mathcal{P}_{\eta'}(x)) h(x) \), for every \( x \in X \) we can define the function \( g_x(\eta) = b(x) \exp (\eta^T T(x) - \alpha(\eta)) \) and hence
\[ d_{TV}(\mathcal{P}_\eta, \mathcal{P}_{\eta'}) = \sum_{x \in X} g_x(\eta) - g_x(\eta') \]
additionally we define the function \( f(\eta) = \sum_{x \in X} g_x(\eta) \) and hence
\[ d_{TV}(\mathcal{P}_\eta, \mathcal{P}_{\eta'}) = f(\eta) - f(\eta') \]
now from the multidimensional Mean Value Theorem on \( f \) there exists \( \xi \in L(\eta, \eta') \) such that
\[ d_{TV}(\mathcal{P}_\eta, \mathcal{P}_{\eta'}) = (\eta - \eta')^T \nabla \eta f(\eta)|_{\eta=\xi} \]
\[ = \sum_{x \in X} b(x) (\eta - \eta')^T \left( \nabla \eta \left( \exp (\eta^T T(x) - \alpha(\eta)) \right) \right)|_{\eta=\xi} \]
\[ = \sum_{x \in X} b(x) (\eta - \eta')^T \left( T(x) \right) \exp (\xi^T T(x) - \alpha(\xi)) \]
\[ \stackrel{(2.2)}{=} \sum_{x \in X} \text{sign} (\mathcal{P}_\eta(x) - \mathcal{P}_{\eta'}(x)) (\eta - \eta')^T \left( T(x) - \mathbb{E}_{y \sim \mathcal{P}_\xi} [T(y)] \right) \exp (\xi^T T(x) - \alpha(\xi)) \]
\[ = \mathbb{E}_{x \sim \mathcal{P}_\xi} \left[ \text{sign} (\mathcal{P}_\eta(x) - \mathcal{P}_{\eta'}(x)) (\eta - \eta')^T \left( T(x) - \mathbb{E}_{y \sim \mathcal{P}_\xi} [T(y)] \right) \right] \]
and the lemma follows. \( \blacksquare \)

Appendix C. Omitted Proofs of Section 4

Proof of Lemma 10: Our goal is to apply Fano’s Inequality (Theorem 3), hence we have to define a family of distributions with an upper bound on their KL-divergence and a lower bound on their total variation distance.

We define the permutations \( \pi_1, \ldots, \pi_\ell \), with \( \ell = \left\lfloor \frac{m}{2} \right\rfloor \), using the cycle notation of permutations
\[ \pi_1 = (1 \ 2) \ , \ \pi_2 = (3 \ 4) \ , \ \cdots \ , \ \pi_i = ((2i - 1) \ (2i)) \ , \ \cdots \ , \ \pi_\ell = ((m - 1) \ m) . \]

For all the distributions that we define we use \( \phi = 1/2 \). Hence our family of distribution is the following
\[ \mathcal{F} = \{ \mathcal{P}_{\phi, \pi_1}, \ldots, \mathcal{P}_{\phi, \pi_\ell} \} . \]
First we compute the upper bound on the KL-divergence of any pair of the above distributions

\[
D_{KL}(P_{\phi,\pi_i} || P_{\phi,\pi_j}) = \sum_{\pi \in S_m} \frac{\phi d_K(\pi,\pi_i)}{Z(\phi)} \ln \left( \frac{\phi d_K(\pi,\pi_i)}{\phi d_K(\pi,\pi_j)} \right)
\]

\[
= \sum_{\pi \in S_m} \frac{\phi d_K(\pi,\pi_i)}{Z(\phi)} \ln \left( d_K(\pi,\pi_i) - d_K(\pi,\pi_j) \right)
\]

\[
= \ln \left( \frac{1}{\phi} \right) \cdot \mathbb{E}_{\pi \sim P_{\phi,\pi_i}} \left[ d_K(\pi,\pi_j) - d_K(\pi,\pi_i) \right]
\]

Now because of triangle inequality of the Kendall tau distance we have that \(d_K(\pi,\pi_j) \leq d_K(\pi,\pi_i) + d_K(\pi_i,\pi_j)\) and from the definition of \(\pi_i, \pi_j\) we also get that \(d_K(\pi_i,\pi_j) = 2\), hence \(d_K(\pi,\pi_j) - d_K(\pi,\pi_i) \leq 2\) and using also that \(\phi = 1/2\) we have the following bound

\[
D_{KL}(P_{\phi,\pi_i} || P_{\phi,\pi_j}) \leq 2 \ln(2).
\] (C.1)

To lower bound the total variation distance between any two distributions in \(\mathcal{F}\) we use the following claim proved in (Liu and Moitra, 2018).

**Claim 18 (Claim 1 of (Liu and Moitra, 2018))** For any \(\pi, \pi' \in S_m\) with \(\pi \neq \pi'\) and any \(\phi_1, \phi_2 \in [0, 1 - \gamma]\) we have

\[
d_{TV}(P_{\phi_1,\pi}, P_{\phi_2,\pi'}) \geq \frac{\gamma}{2}.
\]

Therefore from the above claim we immediately get that for any \(i, j \in [m]\) it holds that

\[
d_{TV}(P_{\phi,\pi_i}, P_{\phi,\pi_j}) \geq \frac{1}{4}.
\] (C.2)

We can now apply Theorem 3 with \(\alpha = 1/4\) and \(\beta = 2 \ln(2)\) and we get

\[
\mathcal{R}_n(\mathcal{F}) \geq \frac{1}{8} \left( 1 - \frac{n \cdot 2 \ln 2 + \ln 2}{\ln(m) - \ln 2} \right)
\]

from which we get that if \(n = o(\log(m))\) then \(\mathcal{R}_n(\mathcal{F}) \geq \frac{1}{16}\) hence we cannot learn \(P_{\phi,\pi_0}\) \(\epsilon\)-close in total variation distance unless \(n = O(\log(m))\). ■

**Appendix D. Omitted Proof of Section 5**

**Proof of Lemma 11:** The bijective map can be given as \(h(\sigma) = (V_1(\sigma, \pi), \ldots, V_m(\sigma, \pi))\). Based on Lemma 2, we know that \((V_1(X, \pi), \ldots, V_m(X, \pi))\) are independent random variables if \(X \sim \mathcal{M}(\pi, \phi)\), thus their joint distribution can be written as in (5.2) which is equivalent to the definition of Generalized Mallows model. The second part of the claim readily follows from the existence of the bijective map \(h\) that preserves the probability mass. ■

**Proof of Theorem 12:** The lower bound comes from the lower bound that is given for single parameter Mallows model in Theorem 3.7 of (Caragiannis et al., 2016). The proof of upper bound
for Mallows Block model follows closely the proof of Theorem 3.6 of (Caragiannis et al., 2016). Let us assume that we are given i.i.d. samples $\pi_1, \ldots, \pi_n$ where $n \geq \frac{1}{2 e} \log\frac{2^2}{\delta}$ from $\mathcal{P}_{\phi, \pi_0, B}$ with $c = \min_{i,j \in [m]: \pi_0(i) < \pi_0(j)} p_{i,j} - p_{j,i}$ where $p_{i,j}$ is the pairwise marginal for item $i$ and $j$ under $\mathcal{P}_{\phi, \pi_0, B}$, i.e. $p_{i,j} = \sum_{\pi \in S_m: \pi(i) < \pi(j)} \mathcal{P}_{\phi, \pi_0, B}(\pi)$. Then let us define a ranking $\hat{\pi}$ such that $\hat{\pi}(i) < \hat{\pi}(j) \iff n_{i,j} > n_{j,i}$ where $n_{i,j}$ is the number of ranking in the sample for which $\pi_1(i) < \pi_1(j)$. Then, using the union bound, we have

$$
\mathbb{P}(d_K(\pi_0, \hat{\pi}) > 0) \leq \binom{m}{2} 2e^{-2c^2 n} \leq m^2 e^{-2c^2 n} = \delta
$$

What remains is to show that $c$ is constant. For any $i \in B^c$ and $j \in B^f$, it easy to see that $p_{i,j} - p_{j,i} = \Omega((1 - \frac{\phi_j + \phi_i}{2})(1 + \frac{\phi_j + \phi_i}{2}))$ which concludes the proof. ■

**Lemma 19** Let $\mathcal{E}(T, h)$ be an exponential family with sufficient statistics $T$ and carrier measure $h$. For any $\mathcal{P}_\eta \in \mathcal{E}(T, h)$ let $\mathcal{D}_\eta$ be the distribution of the corresponding sufficient statistics, i.e. $\mathcal{D}_\eta$ is the distribution of $T(x)$ when $x \sim \mathcal{P}_\eta$. Then for all $\eta, \eta' \in \mathcal{H}_{T, h}$

$$
d_{\text{TV}}(\mathcal{P}_\eta, \mathcal{P}_{\eta'}) = d_{\text{TV}}(\mathcal{D}_\eta, \mathcal{D}_{\eta'}) \quad \text{and} \quad D_{\text{KL}}(\mathcal{P}_\eta || \mathcal{P}_{\eta'}) = D_{\text{KL}}(\mathcal{D}_\eta || \mathcal{D}_{\eta'}) .
$$

**Proof of Lemma 19:** We prove the statement for discrete distributions since this is the version of the lemma that we are going to use later in this section but with the same arguments we can prove the lemma for continuous distributions too. Let $R$ be the support of the exponential family $\mathcal{E}(T, h)$, $R_T = \{t \mid \exists x \in R : T(x) = t\}$ and let also

$$
Q[t] = \sum_{x \in R} 1\{T(x) = t\} .
$$

We have that

$$
D_{\text{KL}}(\mathcal{P}_\eta || \mathcal{P}_{\eta'}) = \sum_{x \in R} p_\eta(x) \ln \left( \frac{p_\eta(x)}{p_{\eta'}(x)} \right)
= \sum_{x \in R} h(x) \exp(\eta^T T(x) - \alpha(\eta)) \ln \left( \frac{p_\eta(x)}{p_{\eta'}(x)} \right)
= \sum_{t \in R_T} \left( \sum_{x: T(x) = t} h(x) \exp(\eta^T T(x) - \alpha(\eta)) \right) \ln \left( \frac{p_\eta(x)}{p_{\eta'}(x)} \right)
= \sum_{t \in R_T} (Q[t] h(x) \exp(\eta^T T(x) - \alpha(\eta)) \ln \left( \frac{Q[t] p_\eta(x)}{Q[t] p_{\eta'}(x)} \right)
= \sum_{t \in R_T} d_\eta(t) \ln \left( \frac{d_\eta(t)}{d_{\eta'}(t)} \right) = D_{\text{KL}}(\mathcal{D}_\eta || \mathcal{D}_{\eta'}) .
$$
d_{TV}(\mathcal{P}_\eta, \mathcal{P}_{\eta'}) = \frac{1}{2} \sum_{x \in \mathbf{R}} |p_\eta(x) - p_{\eta'}(x)|

= \frac{1}{2} \sum_{x \in \mathbf{R}} |h(x) \exp(\eta^T T(x) - \alpha(\eta)) - h(x) \exp(\eta'^T T(x) - \alpha(\eta))|

= \frac{1}{2} \sum_{t \in \mathbf{R}_T} \sum_{x: T(x) = t} |h(x) \exp(\eta^T T(x) - \alpha(\eta)) - h(x) \exp(\eta'^T T(x) - \alpha(\eta))|

= \frac{1}{2} \sum_{t \in \mathbf{R}_T} Q[t] |h(x) \exp(\eta^T T(x) - \alpha(\eta)) - h(x) \exp(\eta'^T T(x) - \alpha(\eta))|

= \frac{1}{2} \sum_{t \in \mathbf{R}_T} |Q[t] h(x) \exp(\eta^T T(x) - \alpha(\eta)) - Q[t] h(x) \exp(\eta'^T T(x) - \alpha(\eta))|

= \frac{1}{2} \sum_{t \in \mathbf{R}_T} |d_\eta(t) - d_{\eta'}(t)| = d_{TV}(\mathcal{D}_\eta, \mathcal{D}_{\eta'}). 

\textbf{Proof of Lemma 16:} First observe that from Theorem 12 we can use O(\log(m/\delta)) samples to learn the central ranking \pi_0. Once we know \pi_0 we use Lemma 11 and hence we can assume that our samples are coming from the distribution \mathcal{P}_{\phi^*, B} and we want to learn \mathcal{P}_{\phi^*, B} in KL-divergence. But applying Lemma 19 implies that we can assume sample access to the distribution \mathcal{D}_{\phi^*, B} of the sufficient statistics of \mathcal{P}_{\phi^*, B} and we want to learn \mathcal{D}_{\phi^*, B} in KL-divergence. From the definition of \mathcal{P}_{\phi^*, B} we have that the sufficient statistics of \mathcal{P}_{\phi^*, B} is the vector T(z) with \sum_{j \in B_i} z_j.

Let also \mathcal{D}_{\phi^*, i, B} be the distribution of Ti(z), since the coordinates of T(x) are all independent we get

\begin{equation}
\text{D}_{KL}(\mathcal{D}_{\phi, i, B} \parallel \mathcal{D}_{\phi^*, i, B}) = \sum_{i \in [d]} \text{D}_{KL}(\mathcal{D}_{\phi, i, B} \parallel \mathcal{D}_{\phi^*, i, B}) \tag{D.1}
\end{equation}

hence it suffices to learn every \mathcal{D}_{\phi^*, i, B} in KL-divergence with accuracy \varepsilon/d and then we would have learned \mathcal{P}_{\phi^*, B} in KL-divergence with accuracy \varepsilon.

From the above discussion we have that \mathcal{D}_{\phi^*, i, B} is a distribution in an single parameter exponential with natural parameter \theta_i = \ln(\phi_i), let \alpha_i be the logarithmic partition function of the family of \mathcal{D}_{\phi, i, B}. From (2.5) we have that

\begin{equation}
\text{D}_{KL}(\mathcal{D}_{\phi^*, i, B} \parallel \mathcal{D}_{\phi^*, i, B}) = - (\theta'_i - \theta^*_i) \hat{\alpha}_i(\theta^*_i) + \alpha_i(\theta'_i) - \alpha_i(\theta^*_i).
\end{equation}

We define

\begin{equation}
f(x) = -(x - \theta^*_i) \hat{\alpha}_i(\theta^*_i) + \alpha_i(x) - \alpha_i(\theta^*_i)
\end{equation}

and we have that

\begin{equation}
f'(x) = -\hat{\alpha}_i(\theta^*_i) + \hat{\alpha}_i(x)
\end{equation}

\begin{equation}
f''(x) = \hat{\alpha}_i(x) \geq 0.
\end{equation}

Hence f is a convex function with minimum value at \alpha = \theta^*_i. Hence f is a decreasing function for \alpha \leq \theta^*_i and an increasing function for \alpha \geq \theta^*_i.
Observe also that by the definition of \( D_{\phi^*, B}^i \) and the description of the truncated geometric distribution as discussed in Section 2 it holds that \( \alpha_i(\theta_i) = \sum_{j \in B_i} \ln (Z_j(\phi_i)) \). But it is easy to see from the definition of \( Z_j \) that \( Z_j(\phi_i) \geq 1 \) and hence \( \alpha_i(\theta_i) \geq 0 \) for all \( \theta_i \in (-\infty, 0] \). This observation implies \( \lim_{x \to -\infty} f(x) = +\infty \) which can also be written as

\[
\lim_{\phi_i \to 0} \text{KL} \left( D_{\phi^*, B}^i \mid \mid D_{\phi^*, B}^i \right) = +\infty \tag{D.2}
\]

for \( \phi_i^* > 0 \). The truncated geometric distribution \( T G(\phi, k) \) satisfies the symmetry property \( T G(1/\phi, k) = k - T G(\phi, k) \). From this symmetry together with (D.2) we get that

\[
\lim_{\phi_i \to \infty} \text{KL} \left( D_{\phi^*, B}^i \mid \mid D_{\phi^*, B}^i \right) = +\infty \tag{D.3}
\]

for \( \phi_i^* < +\infty \). We can now define the following set

\[
Q_i = \left\{ \theta \in (-\infty, \infty) \mid \text{KL} \left( D_{\phi^*, B}^i \mid \mid D_{\phi^*, B}^i \right) \leq \varepsilon/d \right\}.
\]

Because of the convexity of \( f \) we know that \( Q_i \) is an interval such that \( \theta_i^* \in Q_i \). From \( Q_i \) we can define the following parameters

\[
\theta_i^-=\inf Q_i \quad \text{and} \quad \theta_i^+ = \sup Q_i.
\]

Observe that because of (D.2) and (D.3) \( Q_i \) is a closed interval and hence \( Q_i = [\theta_i^-, \theta_i^+] \) where \( \theta_i^- \) and \( \theta_i^+ \) are finite numbers not equal to \( \pm \infty \). Let \( \phi_i^- = \theta_i^- \) and \( \phi_i^+ = \theta_i^+ \). Because of the convexity of \( f \) and (D.2), (D.3) we can easily get that

\[
\text{KL} \left( D_{\phi_i^-, B}^i \mid \mid D_{\phi_i^*, B}^i \right) = \varepsilon/d \tag{D.4}
\]

\[
\text{KL} \left( D_{\phi_i^+, B}^i \mid \mid D_{\phi_i^*, B}^i \right) = \varepsilon/d \tag{D.5}
\]

Now we apply the same procedure as in the beginning of the proof of Theorem 13 and we define the estimator \( \hat{\theta} (r (\pi)) \) that satisfies

\[
\text{P}_{\pi \sim P^{\theta^*, \pi_0, B}} \left( \theta (r (\pi)) \notin [\theta_i^-, \theta_i^+] \right) \leq 2 \exp \left( -\min_{\theta \in (\theta_i^-, \theta_i^+)} \text{KL} \left( P_\theta^i \mid \mid P_\theta^i \right) n \right)
\]

using (D.4) and (D.5) and the fact that \( Q_i = [\theta_i^-, \theta_i^+] \) this implies

\[
\text{P}_{\pi \sim P^{\theta^*, \pi_0, B}} \left( \theta (r (\pi)) \notin Q_i \right) \leq 2 \exp \left( -\frac{\varepsilon}{d} n \right).
\]

Let now \( \phi (r (\pi)) = \exp \left( \theta (r (\pi)) \right) \), because of the definition of \( Q_i \) we get that

\[
\text{P}_{\pi \sim P^{\theta^*, \pi_0, B}} \left( \text{KL} \left( D_{\phi (r (\pi)), B}^i \mid \mid D_{\phi^*, B}^i \right) \geq \varepsilon \right) \leq 2 \exp \left( -\frac{\varepsilon}{d} n \right).
\]

If we now apply a union bound over all \( i \in [d] \) and (D.1) we get that

\[
\text{P}_{\pi \sim P^{\theta^*, \pi_0, B}} \left( \text{KL} \left( P_{\phi (r (\pi)), \pi_0, B} \mid \mid P_{\phi^*, \pi_0, B} \right) \geq \varepsilon \right) \leq 2d \exp \left( -\frac{\varepsilon}{d} n \right).
\]
Hence for \( n \geq \frac{d}{\varepsilon} \ln (2d/\delta) \) then

\[
\Pr_{\pi \sim \mathcal{P}_{\Phi^*}} \left( D_{KL} \left( \mathcal{P}_{\Phi^*(\pi)}, \pi_0, B \right) \geq \varepsilon \right) \leq \delta
\]

and the lemma follows. ■

**Proof of Lemma 17:** Our goal is to apply Fano’s Inequality (Theorem 3), hence we have to define a family of distributions with an upper bound on their KL-divergence and a lower bound on their total variation distance.

We fix a partition \( B \) of \([m]\) in equal parts, i.e. \( |B_i| = m/d \) for all \( i \in [d] \). We define the following set of parameters \( \Phi \)

\[
\mathcal{G} = \left\{ \phi_i \mid \phi_i \in \left\{ \frac{1}{2}, \frac{1}{2} - \frac{c \varepsilon}{\sqrt{m}} \right\} \right\}
\]

where \( c \) is going to be determined later. Based on the Gilbert-Varshamov bound we have that there exists a binary code with at least \( 2^{d/8} \) codewords with minimum Hamming distance at least \( d/8 \).

Let \( Q \) be such a code, for each codeword \( q \in Q \) we define vector \( \phi(q) \) such that

\[
\phi_i(q) = \begin{cases} 
\frac{1}{2} - \frac{c \varepsilon}{\sqrt{m}} & \text{if } q_i = 0 \\
\frac{1}{2} & \text{if } q_i = 1
\end{cases}
\]

Let \( \mathcal{G}' = \{ \phi(q) \mid q \in Q \} \) and \( \pi_0 \) be the identity permutation, we define the following set of distributions

\[
\mathcal{F} = \{ \mathcal{P}_{\Phi, \pi_0, B} \mid \phi \in \mathcal{G}' \}.
\]

Because of Lemma 11 we can focus for the rest of the proof in the distribution \( \mathcal{P}_{\Phi, B} \). But as we have explained the distribution \( \mathcal{P}_{\Phi, B} \) is an \( m \) dimensional distribution where the \( i \)th coordinate follows the distribution \( T \mathcal{G}(\phi_i, i - 1) \). If we take any \( \mathcal{P}_{\Phi, B}, \mathcal{P}_{\Phi', B} \in \mathcal{F} \) then by the definition of \( \mathcal{F} \) we have that \( |\phi_i - \phi_i'| \leq c \frac{\varepsilon}{\sqrt{m}} \) and \( \phi_i, \phi_i' \geq 1/4 \). We can therefore apply (2.5) and (2.3) to get that for some parameters \( \psi_i \in [\phi_i, \phi_i'] \cup [\phi_i', \phi_i] \)

\[
D_{KL} \left( \mathcal{P}_{\Phi, \pi_0, B} \parallel \mathcal{P}_{\Phi', \pi_0, B} \right) = \sum_{j \in [m]} \left( \ln (\phi_j) - \ln (\phi_j') \right)^2 \sum_{z \sim T \mathcal{G}(\psi_j, j - 1)} \text{Var}_{z}[z]
\]

but applying the Lemma 21 and the Mean Value Theorem we get that

\[
D_{KL} \left( \mathcal{P}_{\Phi, \pi_0, B} \parallel \mathcal{P}_{\Phi', \pi_0, B} \right) \leq \sum_{j \in [m]} (\phi_j - \phi_j')^2 \frac{1}{\psi_j^2} \left( \frac{1}{2} \psi_j^2 - \psi_j \right)^2
\]

but we know that \( \psi_j \in [1/4, 1/2] \) and \( |\phi_i - \phi_i'| \leq c \frac{\varepsilon}{\sqrt{m}} \) and therefore

\[
D_{KL} \left( \mathcal{P}_{\Phi, \pi_0, B} \parallel \mathcal{P}_{\Phi', \pi_0, B} \right) \leq 32c^2 \sum_{j \in [m]} \frac{\varepsilon^2}{m} \leq 32c^2 \cdot \varepsilon^2. \tag{D.6}
\]
We now lower bound the total variation distance between any two distributions in \( \mathcal{F} \). Because of the definition of \( \mathcal{F} \) we have that for any \( \phi, \phi' \in \mathcal{G} \) they differ in at least \( d/8 \) coordinates. Hence there are at least \( d/8 \) different \( i \in [d] \) such that \( \phi_i = \frac{1}{2} \) and \( \phi'_i = \frac{1}{2} - \frac{c}{\sqrt{m}} \) or \( \phi_i = \frac{1}{2} - \frac{c}{\sqrt{m}} \) and \( \phi'_i = \frac{1}{2} \). Therefore for at least \( d/16 \) of those coordinates we will have that also that all \( \phi_i \)'s are the same and all \( \phi'_i \)'s are the same. Let \( A \) be this set of coordinates of \( \phi \) excluding the coordinates \( i \leq 4 \), we define \( K = \bigcup_{a \in A} B_a \) and \( k = |K| \). From the definition of \( B \) we have that \( k = \frac{d}{16} \). Without loss of generality we assume that \( \phi \triangleq \phi_i = \frac{1}{2} \) and \( \phi' \triangleq \phi'_i = \frac{1}{2} - \frac{c}{\sqrt{m}} \). Now we fix \( \phi, \phi' \in \mathcal{G} \) and we define \( \mathcal{T}_\phi \) to be a copy of the distribution \( \mathcal{P}_{\phi,B} \) where we keep only the coordinates in \( K \) and \( \mathcal{T}_{\phi'} \) to be a copy of the distribution \( \mathcal{P}_{\phi',B} \) where we keep only the coordinates in \( K \). Because of the definition of \( \mathcal{P}_{\phi,B} \) we have that \( \mathcal{T}_\phi \) is a distribution over vectors \( (y_1, \ldots, y_k) \) where the all the \( y_i \)'s are independent and \( y_i \sim \mathcal{T}_G (\phi, k_i) \) for some \( k_i \in K \). The same way we have that \( \mathcal{T}_{\phi'} \) is a distribution over vectors \( (y'_1, \ldots, y'_k) \) where the all the \( y'_i \)'s are independent and \( y'_i \sim \mathcal{T}_G (\phi', k_i) \) for some \( k_i \in K \).

From the definition of total variation distance we have that

\[
\text{d}_{TV} (\mathcal{P}_{\phi,B}, \mathcal{P}_{\phi',B}) \geq \text{d}_{TV} (\mathcal{T}_\phi, \mathcal{T}_{\phi'}) .
\]

Also we define \( \mathcal{T}_\phi \) to be the distribution of \( \sum_{i \in [k]} y_i \), where \( (y_1, \ldots, y_k) \sim \mathcal{T}_\phi \) and \( \mathcal{T}'_\phi \) to be the distribution of \( \sum_{i \in [k]} y'_i \), where \( (y'_1, \ldots, y'_k) \sim \mathcal{T}_{\phi'} \). We have that

\[
\text{d}_{TV} (\mathcal{T}_\phi, \mathcal{T}_{\phi'}) \geq \text{d}_{TV} (\mathcal{T}_\phi, \mathcal{T}'_{\phi'})
\]

and hence

\[
\text{d}_{TV} (\mathcal{P}_{\phi,B}, \mathcal{P}_{\phi',B}) \geq \text{d}_{TV} (\mathcal{T}_\phi, \mathcal{T}_{\phi'}) .
\]

It is easy to see now that \( \mathcal{T}_\phi \) is a member of a single parameter exponential family with natural parameter \( \theta = \ln (\phi) \). We prove the following claim.

We now want to apply Theorem 6 to lower bound the quantity \( \text{d}_{TV} (\mathcal{T}_\phi, \mathcal{T}_{\phi'}) \). By the definition of \( \mathcal{T}_\phi \), the sufficient statistics of \( \mathcal{T}_\phi \) is \( \sum_{i \in [k]} y_i \). Hence let \( \theta = \ln (\phi) \), \( \theta' = \ln (\phi') \) and since \( \theta > \theta' \) by the definition of \( \phi, \phi' \) we have that

\[
\text{d}_{TV} (\mathcal{T}_\phi, \mathcal{T}_{\phi'}) = \mathbb{E} \left[ \frac{\text{sign} (\mathcal{T}_\phi (y) - \mathcal{T}_{\phi'} (y)) \left( \sum_{i \in [k]} y_i - \mathbb{E}_{z} \left[ \sum_{i \in [k]} z_i \right] \right)}{\left( \theta - \theta' \right)} \right] . \tag{D.7}
\]

where \( y_i \sim \mathcal{T}_G (\psi, k_i - 1) \) and independently \( z_i \sim \mathcal{T}_G (\psi, k_i - 1) \). Now from the proof of Theorem 6 in Section B, we have that for every \( y \), the sign of \( \left( \sum_{i \in [k]} y_i - \mathbb{E}_{z} \left[ \sum_{i \in [k]} z_i \right] \right) \) is equal to the sign of \( \frac{d\mathcal{T}_\phi(y)}{dx} \bigg|_{x=\psi} \). Hence if \( \frac{d\mathcal{T}_\phi(y)}{dx} \bigg|_{x=\psi} \neq 0 \), then from the definition of \( \phi' \) there exists an \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \leq \varepsilon_0 \) it holds that \( \frac{d\mathcal{T}_\phi(y)}{dx} \bigg|_{x=\psi} \) does not change sign for all \( \psi \in [\phi, \phi'] \).

In this case we have that

\[
\text{sign} \left( \mathcal{T}_\phi(y) - \mathcal{T}_{\phi'}(y) \right) \left( \sum_{i \in [k]} y_i - \mathbb{E}_{z} \left[ \sum_{i \in [k]} z_i \right] \right) = \left( \sum_{i \in [k]} y_i - \mathbb{E}_{z} \left[ \sum_{i \in [k]} z_i \right] \right) . \tag{D.8}
\]

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To be able to use D.8 we need to prove that \( \frac{dT_x(y)}{dx} \bigg|_{x=\phi} \neq 0 \) for every \( y \), which is equivalent with
\[
\sum_{i \in [k]} y_i \neq \mathbb{E}_z \left[ \sum_{i \in [k]} z_i \right]
\]
where \( z_i \sim T G (\phi, k_i - 1) \). We prove this by showing that \( \mathbb{E}_z \left[ \sum_{i \in [k]} z_i \right] \) is not an integer. From Lemma 21 and the fact that \( \phi = 1/2 \) we have that
\[
\mathbb{E}_z \left[ \sum_{i \in [k]} z_i \right] = k - \sum_{i \in [k]} k_i \frac{1}{2k_i - 1}
\]
and but the choice of \( T_\phi \), we have that \( k_i \geq 5 \) and hence \( 0 < \sum_{i \in [k]} k_i \frac{1}{2k_i - 1} \leq 2 \sum_{i=5}^{\infty} i \frac{1}{2i - 1} = 3/4 \), which implies
\[
\mathbb{E}_z \left[ \sum_{i \in [k]} z_i \right] \in (k, k + 3/4).
\]
Therefore as we described above it follows that for all \( x, y \) it holds that \( \frac{dT_x(y)}{dx} \bigg|_{x=\phi} \neq 0 \) and hence by (D.8) we have that
\[
d_{TV}(T_\phi, T_{\phi'}) = \mathbb{E}_{y \sim T G (\psi, k_i - 1)} \left[ \sum_{i \in [k]} y_i - \mathbb{E}_{z \sim T G (\psi, k_i - 1)} \left[ \sum_{i \in [k]} z_i \right] \right] (\theta - \theta')
\]
where \( \psi \in [\phi', \phi] \). We now use the following technical claim which was first presented in Tukey (1946).

**Claim 20 ((Tukey, 1946))** *For any set \( x_1, \ldots, x_n \) of independent random variables it holds that*
\[
\mathbb{E} \left[ \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \mathbb{E} [x_i] \right] \geq \frac{1}{2\sqrt{2n}} \sum_{i=1}^{n} \mathbb{E} [|x_i - \mathbb{E} [x_i]|].
\]

**Proof of Claim 20**: The inequality as presented in (Tukey, 1946) holds for random variables with zero median, whereas the random variables that we want to use \( z_i = x_i - \mathbb{E} [x_i] \) have zero mean. To handle this situation we can use the symmetrization argument from the last page of (Birnbaum et al., 1944). Tukey’s inequality together with the symmetrization lemma of (Birnbaum et al., 1944) give the following
\[
\mathbb{E} \left[ \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \mathbb{E} [x_i] \right] \geq \frac{1}{2n} \frac{n!!}{(n-1)!!} \sum_{i=1}^{n} \mathbb{E} [|x_i - \mathbb{E} [x_i]|].
\]
Now using standard asymptotic formulas of the gamma function we can see that
\[
\frac{n!!}{(n-1)!!} \geq \sqrt{\frac{n}{2}}
\]
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and the lemma follows. ■

Applying Claim 20 to (D.11) we get that

$$d_{TV}(T_\phi, T_{\phi'}) \geq (\theta - \theta') \frac{1}{2\sqrt{2k}} \sum_{i \in [k]} E_{y_i \sim T_G(\psi, k_i-1)} \left| y_i - E_{z_i \sim T_G(\psi, k_i-1)} [z_i] \right|.$$

Hence it remains to lower bound the absolute deviation of a truncated geometric distribution with parameter $\psi \in [\phi', \phi]$. From Lemma 21 we have that $E_{z_i \sim T_G(\psi, k_i-1)} [z_i] \leq \frac{\psi}{1-\psi}$ and from the choice of the values of $\phi, \phi'$ we have that $\frac{\psi}{1-\psi} \leq 1$ hence $0 \leq E_{z_i \sim T_G(\psi, k_i-1)} [z_i] \leq 1$. Therefore

$$E_{y_i \sim T_G(\psi, k_i-1)} \left| y_i - E_{z_i \sim T_G(\psi, k_i-1)} [z_i] \right| = \frac{1}{Z_{k_i-1}(\psi)} \sum_{j=0}^{k_i-1} \frac{\psi^j}{Z_{k_i-1}(\psi)} \left| j - E_{z_i \sim T_G(\psi, k_i-1)} [z_i] \right|$$

$$= \frac{2}{Z_{k_i-1}(\psi)} \sum_{j=0}^{k_i-1} \frac{\psi^j}{Z_{k_i-1}(\psi)} \left| j - E_{z_i \sim T_G(\psi, k_i-1)} [z_i] \right|$$

$$= \frac{2}{Z_{k_i-1}(\psi)} \sum_{j=0}^{k_i-1} \frac{\psi^j}{Z_{k_i-1}(\psi)} \left[ z_i - E_{z_i \sim T_G(\psi, k_i-1)} [z_i] \right] - E_{z_i \sim T_G(\psi, k_i-1)} [z_i]$$

$$= \frac{2}{Z_{k_i-1}(\psi)} \left[ 1 - \psi^{k_i} \right] \left( \frac{\psi}{1-\psi} - \frac{k_i \cdot \psi^{k_i}}{1-\psi} \right)$$

$$= \frac{2}{(1-\psi^{k_i})^2} \left( \psi + (k_i - 1) \psi^{k_i+1} - k_i \psi^{k_i} \right)$$

$$\geq \frac{1}{\sqrt{2}}$$

where for the last inequality we have used the fact that $\psi \in [\phi', \phi]$ and the actual values of $\phi', \phi$ together with the fact that $k_i \geq 2$. Applying this lower bound to (D.12) we get that

$$d_{TV}(T_\phi, T_{\phi'}) \geq (\theta - \theta') \frac{\sqrt{k}}{4} \left( \frac{\ln(\phi) - \ln(\phi')}{\phi - \phi'} \right) (\phi - \phi') \frac{\sqrt{k}}{4}.$$

using Mean Value Theorem and the fact that $\phi = 1/2$, we get that

$$d_{TV}(T_\phi, T_{\phi'}) \geq (\phi - \phi') \frac{\sqrt{k}}{2}.$$

but from the definition of $\phi'$ we also have

$$d_{TV}(T_\phi, T_{\phi'}) \geq c \frac{\varepsilon \sqrt{k}}{\sqrt{2}} = c \frac{\varepsilon}{2}.$$
Therefore we get that for any $\mathcal{P}_{\phi, \pi_0, B}, \mathcal{P}_{\phi', \pi_0, B} \in \mathcal{F}$ it holds
\[ d_{TV}(\mathcal{P}_{\phi, \pi_0, B}, \mathcal{P}_{\phi', \pi_0, B}) \geq c\frac{\varepsilon}{2} \quad (D.13) \]

Using (D.6) and (D.13), we can now apply Theorem 3 with $\alpha = c\frac{\varepsilon}{2}$ and $\beta = 32c^2\varepsilon^2$ and we get
\[ \mathcal{R}_n(\mathcal{F}) \geq c\frac{\varepsilon}{4} \left( 1 - \frac{n \cdot 32c^2\varepsilon^2 + \ln 2}{\ln (|\mathcal{F}|)} \right). \]

But from the definition of $\mathcal{F}$ and the Gilbert-Varshamov bound we get that $|\mathcal{F}| \geq 2^{d/8}$ and hence
\[ \mathcal{R}_n(\mathcal{F}) \geq c\frac{\varepsilon}{4} \left( 1 - \frac{n \cdot 32c^2\varepsilon^2 + \ln 2}{d/8} \right). \]

Hence we set $c = 8$ and we conclude that for any $n \leq \frac{d}{32\varepsilon}$ we have $\mathcal{R}_n(\mathcal{F}) \geq 2\varepsilon$ hence we cannot learn $\mathcal{P}_{\phi, \pi_0, B}$ $\varepsilon$-close in total variation distance unless $n = \Omega\left(\frac{d}{\varepsilon^2}\right)$. ■

Appendix E. Proof of Theorem 13

The estimation $\hat{\pi}$ of $\pi_0$ follows from Theorem 12, hence we focus on the estimation $\hat{\phi}$ of $\phi^*$. Throughout the proof we assume that $B$ is fixed and hence when drop it from the notation when it is not necessary. From Lemma 11 and the expression of the sufficient statistics for the Mallows Block Model we can conclude that
\[ T_i(\pi, \pi_0) = \sum_{j \in B_i} Y_j \quad (E.1) \]

where $Y_j$ are independent random variables with $Y_j \sim \mathcal{T}_G(\phi_i, j - 1)$. Hence we conclude that the random variables $T_i(\pi, \pi_0)$ are independent and we can estimate them independently. Therefore we focus in the estimation of each $\phi_i$ separately. Before continuing we define the distribution $\mathcal{P}_{\phi_i}^j$ to be the probability distribution of $T_i(\pi, \pi_0)$ where $\pi \sim \mathcal{P}_{\theta_{\pi_0}, B} \quad (1)$ with $\theta_i = t$. Also we define
\[ Z^i(\phi_i, B) = \prod_{j \in B_i} Z_j(\phi_i, B) \quad (E.2) \]

and also $\alpha_i(\theta_i, B) = \ln\left( Z^i\left( \exp(\theta_i), B \right) \right)$. Again we may drop the $B$ from the notation since it is fixed throughout the proof.

We fix some $i \in [d]$, and we drop the subscript $i$ from $\theta_i$, $\phi_i$ since it is clear from the context. We define the function $h(\theta) = \mathbb{E}_{\pi \sim \mathcal{P}_{\theta_{\pi_0}, B}}[T_i(\pi, \pi_0)]$, from Theorem 1 we have that $h(\theta) = \tilde{c}_{\phi_i}(\theta)$ and also that $h'(\theta) = \tilde{c}_{\phi_i}(\theta) > 0$ and hence the function $h(\theta)$ is strictly increasing with respect to $\theta$. Therefore $h$ is an injective function and hence given any real number $r$ in the image of $h$ we can find $\theta$ such that $\left| \theta(r) - \theta \right| \leq \gamma$ in $O(\log(1/\gamma))$ time, where $\theta(r)$ is well defined from the equation $h(\theta(r)) = r$ since $h$ is injective.

---

1. Observe here that we index the distribution with the natural parameter $\theta$ instead of the parameter $\phi$ as we defined it in Section 5. We may do this indexing in the rest of the proof when it will be clear from the context whether we refer to the natural parameter or the parameter $\phi$.  

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Let us assume now that we observe \( n \) i.i.d. samples from the distribution \( \mathcal{P}_{\theta^*, \pi_0, \mathbf{B}} \in \mathcal{M}_r(\mathbf{B}, \pi_0) \). Then according to the discussion in the previous paragraph we have that in order to get an estimation for \( \theta^* \) is suffices to find a real value \( r(\pi) \) such that \( h(\theta(r(\pi))) = r(\pi) \) and \( |\theta^* - \theta(r(\pi))| \leq \varepsilon \). For this purpose we are going to use \( r = \frac{1}{n} \sum_{i=1}^n T_i(\pi_i, \pi_0) \). Now from Theorem 4, the independence of \( T_i \)'s and E.1 we have that for any \( \theta_-, \theta_+ \leq 0 \)

\[
\Pr_{\pi \sim \mathcal{P}_{\theta^*, \pi_0, \mathbf{B}}} \left[ r(\pi) \notin [h(\theta_-), h(\theta_+)] \right] \leq 2 \exp \left( - \min_{\theta \in \{\theta_-, \theta_+\}} D_{\text{KL}} \left( \mathcal{P}_\theta || \mathcal{P}_{\theta^*} \right) n \right)
\]

Then since \( h \) is strictly increasing we have that

\[
\theta(r(\pi)) \in [\theta_-, \theta_+] \iff r \in [h(\theta_-), h(\theta_+)]
\]

which together with Theorem 4 implies

\[
\Pr_{\pi \sim \mathcal{P}_{\theta^*, \pi_0, \mathbf{B}}} \left[ \theta(r(\pi)) \notin [\theta_-, \theta_+] \right] \leq 2 \exp \left( - \min_{\theta \in \{\theta_-, \theta_+\}} D_{\text{KL}} \left( \mathcal{P}_\theta || \mathcal{P}_{\theta^*} \right) n \right) \quad (E.3)
\]

For the rest of the proof we are going to take two cases that should be treated a bit differently. The first case is \( \phi_i^* = \exp(\theta_i^*) > 2\varepsilon \) and the second case is \( \phi_i^* \leq 2\varepsilon \), where \( \varepsilon \) is the accuracy that we want to estimate the parameter \( \phi_i^* \).

**Case** \( \phi_i^* > 2\varepsilon \). Since our goal is to estimate \( \phi_i^* = \exp(\theta_i^*) \) we choose \( \theta_- = \log(\phi_i^* - \varepsilon) \) and \( \theta_+ = \log(\phi_i^* + \varepsilon) \). We focus on showing a lower bound in the KL divergence \( D_{\text{KL}} \left( \mathcal{P}_{\theta_-} || \mathcal{P}_{\theta_i^*} \right) \) and a lower bound on \( D_{\text{KL}} \left( \mathcal{P}_{\theta_+} || \mathcal{P}_{\theta_i^*} \right) \) follows the same way and hence we can apply (E.3).

From (2.5) we have that that for some \( \xi \in [\theta_-, \theta_+] \) it holds that

\[
D_{\text{KL}} \left( \mathcal{P}_{\theta_-} || \mathcal{P}_{\theta_i^*} \right) = (\ln(\phi_i^*) - \ln(\phi_i^* - \varepsilon))^2 \hat{\alpha}_i(\xi)
\]

\[
= \left( \frac{\log(\phi_i^*) - \log(\phi_i^* - \varepsilon)}{\varepsilon} \right)^2 \varepsilon^2 \hat{\alpha}_i(\xi)
\]

\[
= \frac{1}{q^2} \varepsilon^2 \hat{\alpha}_i(\xi)
\]

for some \( q \in [\phi_i^* - \varepsilon, \phi_i^*] \) by the Mean Value Theorem. Hence we have

\[
D_{\text{KL}} \left( \mathcal{P}_{\theta_-} || \mathcal{P}_{\theta_i^*} \right) \geq \frac{1}{(\phi_i^*)^2} \varepsilon^2 \hat{\alpha}_i(\xi), \quad (E.4)
\]

for some \( \xi \in [\theta_-, \theta_i^*] \) and we define \( \psi = \exp(\xi) \). Also from (2.3) we have that

\[
\hat{\alpha}_i(\xi) = \text{Var}_{\pi \sim \mathcal{P}_{\phi, \pi_0, \mathbf{B}}} \left[ T_i(\pi, \pi_0) \right] = \text{Var}_{z \sim \mathcal{P}_{\phi_i^*}^1} \left[ z \right] = \sum_{j \in B_i} \text{Var}_{Y_j \sim \mathcal{T}_{\mathbb{G}(\xi, j-1)}} \left[ Y_j \right] \quad (E.5)
\]

where \( \phi_i^* \) is the vector that is equal with \( \phi^* \) except that at the \( i \)th coordinate it has \( \xi \). We therefore need some expressions for the mean and the variance of truncated geometric distributions. We summarize these expressions in the following Lemma.
Lemma 21  Let $k \in \mathbb{N}$, $\phi \in (0,1)$ then

\[
E_{Z \sim T_{\mathcal{G}(\phi,k)}}[Z] = \frac{\phi}{1-\phi} - (k+1) \frac{\phi^{k+1}}{1-\phi^{k+1}} \quad \text{and} \quad \text{Var}_{Z \sim T_{\mathcal{G}(\phi,k)}}[Z] = \frac{\phi}{(1-\phi)^2} - \frac{(k+1)^2 \phi^{k+1}}{(1-\phi^{k+1})^2}.
\]

Proof [Proof of Lemma 21] During the proof of this lemma we shall use the fact

\[
\sum_{\ell=1}^{k} \phi^\ell = \phi^{1-\phi^{k+1-i}} \frac{1-\phi^{k+1-i}}{1-\phi} \quad \text{(E.6)}
\]

at multiple points. In particular,

\[
E_{Z \sim T_{\mathcal{G}(\phi,k)}}[Z] = \frac{1}{\sum_{j=1}^{k} \phi^j} \sum_{i=1}^{k} i \phi^i = \frac{1-\phi}{1-\phi^{k+1}} \sum_{i=1}^{k} i \phi^i = \frac{1-\phi}{1-\phi^{k+1}} \sum_{i=1}^{k} \sum_{j=1}^{k} \phi^j
\]

\[
= \frac{1-\phi}{1-\phi^{k+1}} \sum_{i=1}^{k} \phi^i \frac{1-\phi^{k+1-i}}{1-\phi} = \frac{1}{1-\phi^{k+1}} \sum_{i=1}^{k} \left[ \phi^i - \phi^{k+1} \right]
\]

\[
= \frac{1}{1-\phi^{k+1}} \left[ \frac{1-\phi^{k+1}}{1-\phi} - 1 - k \phi^{k+1} \right] = \frac{1}{1-\phi^{k+1}} \left[ \frac{\phi - \phi \phi^{k+1}}{1-\phi} - (k+1) \phi^{k+1} \right]
\]

\[
= \frac{\phi}{1-\phi} - (k+1) \frac{\phi^{k+1}}{1-\phi^{k+1}}
\]

where we have used (E.6) in the second, third and fifth step.

Now we prove compute the variance. Note that

\[
\sum_{i=1}^{j} (2i-1) = \left( 2 \sum_{i=1}^{j} i \right) - j = j(j+1) - j = j^2, \quad \text{(E.7)}
\]
and thus

\[
\mathbb{E}_{Z \sim T \mathcal{G}(\phi,k)} \left[Z^2\right] = \sum_{i=1}^{k} i^2 \frac{\phi^i}{\sum_{j=1}^{k} \phi^j} \overset{\text{(E.6)}}{=} \frac{1 - \phi}{1 - \phi^{k+1}} \sum_{i=1}^{k} i^2 \phi^i
\]

\[
= \frac{1 - \phi}{1 - \phi^{k+1}} \sum_{i=1}^{k} \left[ (2i - 1) \sum_{j=i}^{k} \phi^j \right]
\]

\[
= 2 \frac{1 - \phi}{1 - \phi^{k+1}} \sum_{i=1}^{k} \left[ i \sum_{j=i}^{k} \phi^j \right] - \frac{1 - \phi}{1 - \phi^{k+1}} \sum_{i=1}^{k} \sum_{j=i}^{k} \phi^j
\]

\[
\overset{\text{(E.6)}}{=} \frac{2}{1 - \phi^{k+1}} \sum_{i=1}^{k} \left[ i \phi^i - i \phi^{k+1} \right] - \frac{\mathbb{E}_{Z \sim T \mathcal{G}(\phi,k)} \left[Z\right]}{1 - \phi^{k+1}}
\]

Consequently,

\[
\Var_{Z \sim T \mathcal{G}(\phi,k)} \left[Z\right] = \mathbb{E}_{Z \sim T \mathcal{G}(\phi,k)} \left[Z^2\right] - \left( \mathbb{E}_{Z \sim T \mathcal{G}(\phi,k)} \left[Z\right] \right)^2
\]

\[
= \frac{\mathbb{E}_{Z \sim T \mathcal{G}(\phi,k)} \left[Z\right]}{1 - \phi^{k+1}} - \frac{\mathbb{E}_{Z \sim T \mathcal{G}(\phi,k)} \left[Z\right]}{1 - \phi^{k+1}} \frac{1}{1 - \phi^{k+1}}
\]

\[
= \frac{\phi}{1 - \phi^{k+1}} \left[ \frac{(k + 1) \phi^{k+1}}{1 - \phi^{k+1}} \right] - \frac{k(k + 1) \phi^{k+1}}{1 - \phi^{k+1}}
\]

\[
= \frac{\phi}{(1 - \phi)^2} \left[ \frac{(k + 1) \phi^{k+1}}{1 - \phi^{k+1}} \right] - \frac{k(k + 1) \phi^{k+1}}{1 - \phi^{k+1}}
\]

\[
= \frac{\phi}{(1 - \phi)^2} - \frac{k(k + 1) \phi^{k+1}}{(1 - \phi^{k+1})^2}
\]

and the lemma follows.

Using Lemma 21 and (2.3) we get that

\[
\tilde{\alpha}_i(\xi) = \sum_{j \in B_i} Y_j \mathbb{Var}_{Y_j \sim T \mathcal{G}(\xi,j-1)} \left[Y_j\right] = m_i \frac{\psi}{(1 - \psi)^2} - \sum_{j \in B_i} \frac{j^2 \psi^j}{(1 - \psi^j)^2}
\]

where we remind that \(m_i = |B_i|\). As we explained already in order to apply the concentration inequality that we proved in Section 3 we have to lower bound the expression of the variance and for this we have to prove the following technical claim.
Claim 22  Let \( x \in [0, 1] \) and \( y \in \mathbb{R}_+ \) and we define the function \( g(y) = y^2 \frac{x^y}{(1-x^y)^2} \). The function \( g \) is a decreasing function of \( y \).

Proof of Claim 22: We first compute the derivative of \( g \) with respect to \( y \) and we get

\[
g'(y) = \frac{y x^y}{(1-x^y)^2} \left( 2(1-x^y) + y \ln(x) + y \ln(x^y) \right).
\]

The sign of \( g'(y) \) is therefore determined by the sign of the following quantity

\[
h(z) = 2(1-z) + \ln(z) + z \ln(z)
\]

where we have replaced \( z = x^y \) and the only restriction that we have is \( z \in [0, 1] \). If we compute the derivative of \( h \) we have

\[
h'(z) = -1 + \frac{1}{z} + \ln(z).
\]

But we know that \( \ln(x) \leq x - 1 \) and hence \( \ln(1/z) \leq 1/z - 1 \) which implies \( h'(z) \geq 0 \). Since \( z \in [0, 1] \) we get that \( h(z) \leq h(1) \) but \( h(1) = 0 \) and hence \( h(z) \leq 0 \). From this we get \( g'(y) \leq 0 \) and therefore \( g \) is a decreasing function of \( y \).

From Claim 22 we get that

\[
\frac{i^2 \psi^i}{(1-\psi)^2} \geq \frac{(i+1)^2 \psi^{i+1}}{(1-\psi^{i+1})^2}
\]

and therefore the following lower bound in the variance of the sufficient statistics

\[
\alpha_i(\xi) \geq m_i \frac{\psi}{(1-\psi)^2} - m_i \frac{4\psi^2}{(1-\psi^2)^2}
\]

where we have replaced all the terms in the sum in the expression (E.8) with \( i \geq 2 \) with \( i = 2 \). The case \( i = 1 \) corresponds to a trivial delta distribution that does not contribute in any part of the proof of this section. Since \( \psi \in [\phi_i^* - \varepsilon, \phi_i^*] \) we get that

\[
\alpha_i(\xi) \geq m_i \frac{\psi}{(1-\psi)^2} - 4m_i \psi \Rightarrow \alpha_i(\xi) \geq \frac{1}{4} m_i (\phi_i^* - \varepsilon).
\]

Now we use (E.4) and (E.5) together with the above lower bound and we get

\[
D_{KL} \left( P_{\theta_-}^i || P_{\theta_+}^i \right) \geq \frac{1}{4} (m - 1) \varepsilon^2 \psi
\]

where \( \psi \in [\phi_i^* - \varepsilon, \phi_i^*] \). Using exactly the same argument we can also prove the same for \( P_{\theta_+}^i \) and \( \psi \in [\phi_i^*, \phi_i^* + \varepsilon] \) and therefore we get

\[
\min_{\theta \in \{\theta_-, \theta_+\}} D_{KL} \left( P_{\theta}^i || P_{\theta}^{i*} \right) \geq \frac{1}{4} m_i \varepsilon^2 \min \left\{ \frac{\phi_i^* - \varepsilon}{(\phi_i^*)^2}, \frac{\phi_i^*}{(\phi_i^* + \varepsilon)^2} \right\}.
\]
Since the function \( x \mapsto \frac{x - \varepsilon}{x + \varepsilon} \) and the function \( x \mapsto \frac{x}{(x + \varepsilon)^2} \) are decreasing functions of \( x \) for \( x \in [2\varepsilon, 1] \) and assuming that \( \varepsilon \leq 3/4 \) we have that
\[
\min_{\theta \in \{\theta_-, \theta_+\}} D_{KL}(P_{\theta}^i || P_{\theta}^i) \geq \frac{1}{16} m_i \varepsilon^2.
\]
(E.9)

We can now apply (E.9) to (E.3) and we get
\[
\pi \sim P_{\theta^* \cdot, \pi_0, \beta}^n \quad (\theta(r(\pi)) \notin [\theta_-, \theta_+]) \leq 2 \exp \left( -\frac{1}{16} m_i \varepsilon^2 n \right).
\]
Hence for \( n \geq 16 \frac{\ln(2/\delta)}{m_i \varepsilon^2} \) we have that
\[
\pi \sim P_{\theta^* \cdot, \pi_0, \beta}^n \quad (\theta(r(\pi)) \notin [\theta_-, \theta_+]) \leq \delta.
\]

**Case \( \phi_i^* \leq 2\varepsilon \).** For this case we will set \( \theta_+ = \theta^*_i \) and assuming that \( \theta_+ = \phi_i^* + k \varepsilon \), where \( k \in \mathbb{N} \) to be determined later. Hence, from (2.5) we have that
\[
D_{KL}(P_{\theta_+}^i || P_{\theta^*_i}^i) = (\ln(\phi_i^* + k \varepsilon) -\ln(\phi_i^*)) \cdot \alpha_i (\ln (\phi_i^* + k \varepsilon)) + \alpha_i (\ln (\phi_i^*)) - \alpha_i (\ln (\phi_i^* + k \varepsilon)).
\]

Our first goal is to show that for \( \phi_i^* \leq 2\varepsilon \) the right hand side of the KL-divergence is a decreasing function of \( \phi_i^* \). We set
\[
f(x) \triangleq (\ln (x + k \varepsilon) - \ln (x)) \cdot \alpha_i (\ln (x + k \varepsilon)) + \alpha_i (\ln (x)) - \alpha_i (\ln (x + k \varepsilon))
\]
we get that
\[
f'(x) = \left( \frac{1}{x + k \varepsilon} - \frac{1}{x} \right) \cdot \alpha_i (\ln (x + k \varepsilon)) + (\ln (x + k \varepsilon) - \ln (x)) \cdot \frac{\alpha_i (\ln (x + k \varepsilon))}{x + k \varepsilon} + \frac{\alpha_i (\ln (x))}{x} - \frac{\alpha_i (\ln (x + k \varepsilon))}{x + k \varepsilon}
\]
\[
= \frac{\alpha_i (\ln (x))}{x} \left( \frac{\alpha_i (\ln (x + k \varepsilon))}{\alpha_i (\ln (x + k \varepsilon))} - \frac{\alpha_i (\ln (x))}{\alpha_i (\ln (x + k \varepsilon))} \right) + \frac{x}{x + k \varepsilon} \left( \ln \left( \frac{x}{x + k \varepsilon} \right) \right)
\]
We use now the easy to check facts that (1) the function \( z \mapsto z \ln (z) \) is a decreasing function of \( z \) for \( z \leq 1/e \), (2) the function \( x \mapsto \frac{x}{x + k \varepsilon} \) is an increasing function of \( x \), (3) we pick \( k \) such that for \( x \in [0, 2 \varepsilon] \) we have that \( \frac{x}{x + k \varepsilon} \leq \frac{1}{e} \) and hence we get that
\[
f'(x) \leq -\frac{\alpha_i (\ln (x + k \varepsilon))}{x} \left( \frac{\alpha_i (\ln (x + k \varepsilon))}{\alpha_i (\ln (x + k \varepsilon))} - \frac{\alpha_i (\ln (x))}{\alpha_i (\ln (x + k \varepsilon))} \right) - \frac{2}{(k + 2)} \left( \ln \left( \frac{k + 2}{2} \right) \right).
\]

Now we want to lower bound the term \( \frac{\alpha_i (\ln (x + k \varepsilon))}{\alpha_i (\ln (x + k \varepsilon))} \) in the parentheses in the last upper bound of \( f'(x) \). From (2.2) and (2.3) we have that
\[
\frac{\alpha_i (\ln (x + k \varepsilon))}{\alpha_i (\ln (x + k \varepsilon))} = \frac{\mathbb{E}_{z \sim P_{\ln (x + k \varepsilon)}^i} [z] - \mathbb{E}_{z \sim P_{\ln (x)}^i} [z]}{\text{Var}_{z \sim P_{\ln (x + k \varepsilon)}^i} [z]} \triangleq \frac{E}{D}
\]
To lower bound this expression we use the following simple claim.
Claim 23  Let $x \in [0, 1]$ and $y \in \mathbb{R}_+$ and we define the function $g(y) = y \frac{x^y}{1-x^y}$. The function $g$ is an decreasing function of $y$.

Proof of Claim 22: We first compute the derivative of $g$ with respect to $y$ and we get

$$g'(y) = \frac{x^y}{(1-x^y)^2} \left((1-x^y) + y \ln(x)\right).$$

The sign of $g'(y)$ is therefore determined by the sign of the following quantity

$$h(z) = (1-z) + \ln(z)$$

where we have replaced $z = x^y$ and the only restriction that we have is $z \in [0, 1]$. But we know that $\ln(x) \leq x - 1$ and hence $h(z) \leq 0$ which implies $g'(y) \leq 0$ and the claim follows. ■

From Lemma 21 and Claim 23 we have that

$$E_{z \sim P_{i \ln(x+k\varepsilon)}} [z] = \frac{m_i(x+k\varepsilon)}{(1-(x+k\varepsilon))} - \sum_{j \in B_i} \frac{j(x+k\varepsilon)^j}{(1-(x+k\varepsilon)^j)}$$

$$\geq \frac{m_i(x+k\varepsilon)}{(1-(x+k\varepsilon))} - \frac{2m_i(x+k\varepsilon)^2}{(1-(x+k\varepsilon)^2)}$$

$$= \frac{m_i(x+k\varepsilon)}{(1+x+k\varepsilon)}$$

where again we have excluded the trivial case $j = 1$ that does not contribute to the above expression.

It is also direct from Lemma 21 that

$$E_{z \sim P_{i \ln(x)}} [z] \leq \frac{m_i x}{(1-x)}$$

From these two bounds, the fact that $x \in [0, 2\varepsilon]$ and the assuming that $\varepsilon \leq \frac{1}{10k}$ we conclude that

$$E \geq \frac{m_i k\varepsilon}{1+k \varepsilon} - \frac{m_i 2\varepsilon}{1-2\varepsilon} = m_i \varepsilon \left( \frac{k}{1+k \varepsilon} - \frac{2}{1-2\varepsilon} \right) \geq m_i \varepsilon \frac{k-2}{1+k \varepsilon}$$

$$\geq m_i \varepsilon (k-2) \frac{10}{11}$$

Also directly from Lemma 21, the fact that $x \in [0, 2\varepsilon]$ and assuming $\varepsilon \leq \frac{1}{10(k+2)}$ we get that

$$D = \Var_{z \sim P_{i \ln(x+k\varepsilon)}} [z] \leq \frac{m_i (k+2)\varepsilon}{(1-(k+2)\varepsilon)^2} \leq m_i (k+2) \varepsilon \frac{100}{81}$$

Putting all these together we get

$$\frac{\hat{c}_i \left( \ln \left( x+k \varepsilon \right) \right) - \hat{c}_i \left( \ln \left( x \right) \right)}{\hat{c}_i \left( \ln \left( x+k \varepsilon \right) \right)} \geq \frac{81}{110} \frac{k-2}{k+2}$$

Hence we have the following upper bound on $f'(x)$

$$f'(x) \leq -\frac{\hat{c}_1 \left( \ln \left( x+k \varepsilon \right) \right)}{x} \left( \frac{81}{110} \frac{k-2}{k+2} - \frac{2}{(k+2)} \left( \ln \left( \frac{k+2}{2} \right) \right) \right)$$
for $k = 14$ we have that

$$f'(x) \leq -\frac{\alpha_1 (\ln (x + k\varepsilon))}{x} \left( \frac{243}{440} - \frac{1}{8} \ln (8) \right) \leq 0$$

Therefore we have that for $k = 14$ and $\varepsilon \leq \frac{1}{10k}$ the function $f$ is a decreasing function of $x$ and hence $f(x) \geq f(2\varepsilon)$ for $x \in [0, 2\varepsilon]$. Let $\theta' = \exp (\ln (2\varepsilon))$ and $\theta'_i = \exp (\ln ((k + 2)\varepsilon))$ then we have

$$D_{\text{KL}} \left( P_{\theta^i} \| P_{\theta'_i} \right) \geq \frac{(k + 2)^2}{16} m_i \varepsilon^2.$$  \hspace{1cm} (E.10)

Hence we can now use (E.9) to bound the right hand side and we get

$$D_{\text{KL}} \left( P_{\theta^i} \| P_{\theta'_i} \right) \geq \frac{(k + 2)^2}{16} m_i \varepsilon^2.$$ \hspace{1cm} (E.11)

and now we can use (E.3) to get that for $n \geq \frac{16}{(k + 2)^2} \ln (2/\delta)$ and $\phi^* \in [0, 2\varepsilon]$ it holds

$$\mathbb{P}_{\pi \sim P_{\theta^i, \pi_0, B}} (\theta(r) \notin [0, \theta_2]) \leq \delta.$$  \hspace{1cm} (E.12)

Now if we combine the results that we have for the two regimes $\phi^*_i \geq 2\varepsilon$, $\phi^*_i < 2\varepsilon$ and given that we computing $\hat{\phi}_i$ such that $|\hat{\phi}_i - \exp (\theta_i(r(\pi)))| \leq \varepsilon$ we get that for any $n \geq \frac{16}{(k + 2)^2} \ln (2/\delta)$ it holds that

$$\mathbb{P}_{\pi \sim P_{\theta^i, \pi_0, B}} (|\hat{\phi}_i - \phi^*_i| \leq \varepsilon) \leq \delta.$$  \hspace{1cm} (E.13)

for any $i \in [d]$. Our goal of course is to compute an estimate $\hat{\phi}$ such that the total $\ell_2$ error from all coordinates is less than $\varepsilon$. To do so we estimate each $\phi^*_i$ with accuracy $\varepsilon' = \varepsilon/\sqrt{d}$ and with error probability $\delta' = \delta/d$. Therefore we have that for any $n \geq \frac{d}{16} \ln (d/\delta)$ is holds that

$$\mathbb{P}_{\pi \sim P_{\theta^i, \pi_0, B}} \left( |\hat{\phi}_i - \phi^*_i| \geq \frac{\varepsilon}{\sqrt{d}} \right) \leq \frac{\delta}{d}$$

and therefore using union bound over all coordinates we get that

$$\mathbb{P}_{\pi \sim P_{\theta^i, \pi_0, B}} \left( \left\| \hat{\phi} - \phi^* \right\| \geq \varepsilon \right) \leq \delta$$

and Theorem 13 follows.