Approximate Analytical Solution for a Coupled System of Fractional Nonlinear Integrodifferential Equations by the RPS Method

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In this work, a modified residual power series method is implemented for providing efficient analytical and approximate solutions for a class of coupled system of nonlinear fractional integrodifferential equations. The proposed algorithm is based on the concept of residual error functions and generalized power series formula. The fractional derivative is described under the Caputo concept. To illustrate the potential, accuracy, and efficiency of the proposed method, two numerical applications of the coupled system of nonlinear fractional integrodifferential equations are tested. The numerical results confirm the theoretical predictions and depict that the suggested scheme is highly convenient, is quite effective, and practically simplifies computational time. Consequently, the proposed method is simple, accurate, and convenient in handling different types of fractional models arising in the engineering and physical systems.

1. Introduction

The theory of fractional calculus is indeed a generalization of the standard calculus that deals with differentiation and integration of a noninteger order, which is utilized to describe various real-world phenomena arising in natural sciences, applied mathematics, and engineering fields with great applications for these tools, including nonlinear oscillation of earthquakes, fractional fluid-dynamic traffic, economics, solid mechanics, viscoelasticity, and control theory [1–7]. The major cause behind this is that modeling of a specific phenomenon does not depend only at the time instant but also the historical state, so the differential and integral operators for integer and fractional cases are found to be a superb tool in describing the hereditary and memory properties for different engineering and physical phenomena [8–12]. However, several mathematical forms of the abovementioned issues contain nonlinear fractional integrodifferential equations (FIDEs), and other nonlinear models can be found in [13–18]. Since most fractional differential and integrodifferential equations cannot be solved analytically, it is necessary to find an accurate numerical and analytical method to deal with the complexity of fractional operators involving such equations [19–24].

This paper aims to introduce a recent analytical as well as numerical method based on the use of fractional residual power series (RPS) technique for obtaining the approximate solution for a class of coupled system of fractional integrodifferential equations in the following form:

\[
\mathcal{D}^\beta v_1(t) = \lambda_1 \int_a^b K_1(t, \xi)J_1(v_1(\xi), v_2(\xi))d\xi
\]

\[
+ \lambda_2 \int_a^t K_2(t, \xi)J_2(v_1(\xi), v_2(\xi))d\xi + f_1(t),
\]

\[
\mathcal{D}^\beta v_2(t) = \lambda_3 \int_a^b H_1(t, \xi)T_1(v_1(\xi), v_2(\xi))d\xi
\]

\[
+ \lambda_4 \int_a^t H_2(t, \xi)T_2(v_1(\xi), v_2(\xi))d\xi + f_2(t).
\]

(1)
This is subject to the following initial conditions:

\begin{equation}
\begin{align*}
\nu_1(0) &= \nu_{1,0}, \\
\nu_2(0) &= \nu_{2,0},
\end{align*}
\end{equation}

where $0 < \beta \leq 1$.

The residual power series method (RPSM) has a wide range of applications, especially in simulating nonlinear issues in a fractional meaning, which has been developed and modified over recent years as a powerful mathematical treatment indispensable in dealing with the emerging realistic system in physics, engineering, and natural sciences [25–28]. More specifically, it is a modern analytical and approximation technique that relies on the expansion of the fractional power series and residual error functions, which was first proposed in 2013 to provide analytical series solutions to perturbation. Therefore, it attracted the attention of many researchers.

Freihat et al. [29] have used the fractional power series for solving the fractional stiff system and introduced some basic theorems related to RPS generalization in the sense of Caputo fractional derivative. The $(2 + 1)$-dimensional time-fractional Burgers–Kadomtsev–Petviashvili equation has been solved by the RPS method [30]. In [31], analytic approximations solutions for nonlinear coupled fractional Jaulent–Miodek equations with energy-dependent Schrödinger potential have been obtained using the RPS and q-homotopy analysis methods. Moreover, this method was successfully applied for solving both linear and nonlinear ordinary, partial, and fuzzy differential equations [32–37]. Therefore, such adaptives can be used as an approximate technique in solving several nonlinear problems arising in engineering and sciences.

The outline of this paper is organized as follows: In the next section, we review some basic definitions and theories related to fractional differentiation and fractional power series representations. In Section 3, the solution by the RPS technique is provided. In Section 4, numerical application is performed to show accuracy and efficiency of the RPS method. Finally, we give concluding remarks in Section 5.

### 2. Basic Mathematical Concepts

In this section, basic definitions and results related to fractional calculus are given and fractional power series concept is also represented.

**Definition 1.** The Riemann–Liouville fractional integral operator of order $\beta$, over the interval $[a, b]$ for a function $\nu \in L_1[a, b]$, is defined by

\begin{equation}
\mathcal{I}_a^\beta \nu(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\nu(\epsilon)}{(x - \epsilon)^{1-\beta}} d\epsilon, \quad 0 < \epsilon < x, \; \beta > 0,
\end{equation}

where $\Gamma(\beta)$ is the gamma function.

**Definition 2.** For $\beta > 0$, $a, t, \beta \in \mathbb{R}$. The Caputo fractional derivative of order $\beta$ is given by

\begin{equation}
\mathcal{D}_a^\beta \nu(x) = \frac{1}{\Gamma(n-\beta)} \int_a^x \frac{\nu^{(n)}(\epsilon)}{(x - \epsilon)^{\beta+n-1}} d\epsilon.
\end{equation}

**Definition 3.** A fractional power series (FPS) representation at $t = a$ has the following form:

\begin{equation}
\sum_{n=0}^{\infty} \nu_m (x - a)^{\mu n} = \nu_0 + \nu_1 (x - a)^{\beta} + \nu_2 (x - a)^{2\beta} + \cdots,
\end{equation}

where $0 \leq n - 1 < \beta \leq n$ and $x \geq a$ and $\nu_m$ is the coefficient of the series.

**Theorem 1.** Suppose that $\nu(x)$ has the following FPS representation at $t = a$:...
where \( n - 1 < \beta \leq n, a < x < a + R, \varphi(x) \in C[a,a + R], \) and \( \mathcal{D}^{m\beta}_{a^+} \varphi(x) \in C[a,a + R] \) for \( m = 0, 1, 2, \ldots \) Then, the coefficients \( v_m \) will be in the form \( v_m = (\mathcal{D}^{m\beta}_{a^+} \varphi(a)/(\Gamma(m\beta + 1))) \) such that \( \mathcal{D}^{m\beta}_{a^+} = \mathcal{D}^{\beta}_{a^+}, \mathcal{D}^{\beta}_{a^+}, \ldots, \mathcal{D}^{\beta}_{a^+} \) (m-times).

3. Fractional RPS Method for the Coupled System of IDEs

The purpose of this section is to construct FPS solutions for the coupled system of nonlinear fractional integro-differential equations (1) and (2) by substituting the FPS expansion among the truncated residual functions.

The RPS algorithm proposing the solution of equations (1) and (2) about \( x_0 = 0 \) gives the following FPS expansion:

\[
v_i(x) = \sum_{m=0}^{\infty} v_{i,m} x^{m\beta}/\Gamma(m\beta + 1), \quad i = 1, 2. \quad (7)
\]

The truncated series form of equation (7) can be given by the following kth-FPS approximate solution:

\[
v_{i,k}(x) = \sum_{m=0}^{k} v_{i,m} x^{m\beta}/\Gamma(m\beta + 1), \quad i = 1, 2. \quad (8)
\]

Clearly, if \( v_i(0) = v_{i,0}, i = 1, 2, \) then expansion (8) can be written as

\[
v_{i,k}(x) = v_{i,0} + \sum_{m=1}^{k} v_{i,m} x^{m\beta}/\Gamma(m\beta + 1), \quad i = 1, 2. \quad (9)
\]

Define the residual function for equations (1) and (2) as follows:

\[
\text{Res}_1(x) = \mathcal{D}^{\alpha}_{a^+} v_1(t) - \lambda_1 \int_a^b K_1(t, \xi) J_1(v_1(\xi), v_2(\xi)) d\xi - \lambda_2 \int_a^b K_2(t, \xi) J_2(v_1(\xi), v_2(\xi)) d\xi - f_1(t),
\]

\[
\text{Res}_2(x) = \mathcal{D}^{\alpha}_{a^+} v_2(t) - \lambda_3 \int_a^b H_1(t, \xi) T_1(v_1(\xi), v_2(\xi)) d\xi - \lambda_4 \int_a^b H_2(t, \xi) T_2(v_1(\xi), v_2(\xi)) d\xi - f_2(t).
\]

Also, define the kth-residual function as

\[
\text{Res}_{1,k}(x) = \mathcal{D}^{\alpha}_{a^+} v_{1,k}(t) - \lambda_1 \int_a^b K_1(t, \xi) J_1(v_{1,k}(\xi), v_{2,k}(\xi)) d\xi - \lambda_2 \int_a^b K_2(t, \xi) J_2(v_{1,k}(\xi), v_{2,k}(\xi)) d\xi - f_{1,k}(t),
\]

\[
\text{Res}_{2,k}(x) = \mathcal{D}^{\alpha}_{a^+} v_{2,k}(t) - \lambda_3 \int_a^b H_1(t, \xi) T_1(v_{1,k}(\xi), v_{2,k}(\xi)) d\xi - \lambda_4 \int_a^b H_2(t, \xi) T_2(v_{1,k}(\xi), v_{2,k}(\xi)) d\xi - f_{2,k}(t).
\]

According to the RPS algorithm [27–32], we have the following relations:

(i) \( \lim_{k \to \infty} \text{Res}_{1,k}(x) = \text{Res}_1(x) = 0, \) for each \( x \in (0, 1), i = 1, 2. \]

(ii) \( \mathcal{D}^{m\beta}_{a^+} \text{Res}_0(x) = \mathcal{D}^{m\beta}_{a^+} \text{Res}_{1,k}(0) = 0, \) for each \( m = 0, 1, 2, \ldots, k, i = 1, 2. \)

For obtaining the coefficients \( v_{i,m}, m = 0, 1, 2, \ldots, k, i = 1, 2, \) one can solve the solution of the following relation:

\[
\mathcal{D}^{(k-1)\beta}_{a^+} \text{Res}_{1,k}(0) = 0, \quad k = 1, 2, 3, \ldots, i = 1, 2. \quad (12)
\]

Algorithm 1. To find the coefficients \( v_{i,m}, m = 1, 2, 3, \ldots, k, i = 1, 2, \) in equation (9), perform the following steps:

(i) Step 1: substitute expansion (9) function \( v_{i,k}(x) \) into \( k \)-th residual function (11) such that

\[
\text{Res}_1(t) = \mathcal{D}^{\alpha}_{a^+} (v_{1,0} + \sum_{m=1}^{k} v_{1,m} (\xi^{m\beta}/\Gamma(m\beta + 1)))) - \lambda_1 \int_a^b K_1(t, \xi) J_1(v_{1,0} + \sum_{m=1}^{k} v_{1,m} (\xi^{m\beta}/\Gamma(m\beta + 1)))),
\]

\[
\text{Res}_2(t) = \mathcal{D}^{\alpha}_{a^+} (v_{2,0} + \sum_{m=1}^{k} v_{2,m} (\xi^{m\beta}/\Gamma(m\beta + 1)))) - \lambda_4 \int_a^b H_2(t, \xi) T_2(v_{2,0} + \sum_{m=1}^{k} v_{2,m} (\xi^{m\beta}/\Gamma(m\beta + 1)))) d\xi - f_{2,k}(t).
\]

(ii) Step 2: find the relation of fractional formula \( D^{(k-1)\beta}_{a^+} \) of \( i\)-th residual function at \( x = x_0, i = 1, 2. \)

(iii) Step 3: for \( k = 1 \), obtain the relation through the fact \( \text{Res}_{1,1}(x)|_{x=0} = 0, i = 1, 2. \) For \( k = 2, \) obtain the relation through the fact \( D^{(k-1)\beta}_{a^+} \text{Res}_{1,2}(x)|_{x=0} = 0, i = 1, 2. \) For \( k = 3, \) obtain the relation through the fact \( D^{(k-1)\beta}_{a^+} \text{Res}_{1,3}(x)|_{x=0} = 0, i = 1, 2. \ldots \) For \( k = m, \) obtain the relation through the fact \( D^{(k-1)\beta}_{a^+} \text{Res}_{1,m}(x)|_{x=0} = 0, i = 1, 2. \)

(iv) Step 4: solve the obtained algebraic fractional system \( D^{(k-1)\beta}_{a^+} \text{Res}_{i,k}(x_0), k = 1, 2, 3, \ldots, i = 1, 2. \)

(v) Step 5: substitute the values of \( v_{i,m} \) back into equation (8) and then stop.

4. Numerical Applications and Simulation

This section aims to test two applications of the system of nonlinear fractional IDEs to show the efficiency, accuracy, and applicability of the proposed method. In this section, all calculations are performed using Wolfram Mathematica 10.

Example 1. Consider the following nonlinear fractional integro-differential equation:
The relative error as a ratio between the absolute error and the exact value leads to the determination of the matter of the study. Therefore, the comparison of absolute and relative errors provides clear and explicit evidence regarding the subject estimates related to significance and units, as well as does not calculate the absolute error, which produces some vague solutions and estimating errors of these approximations. For approximate methods for finding digital or approximate solutions and estimating errors of these approximations. For such purpose, a few decimal numbers are often recorded to calculate the absolute error, which produces some vague estimates related to significance and units, as well as does not provide clear and explicit evidence regarding the subject matter of the study. Therefore, the comparison of absolute error with the exact value leads to the determination of the relative error as a ratio between the value of absolute error and the exact, which gives some importance and reduces ambiguity for a deeper understanding of the behavior of the approximate solutions. By using the RPS method of Example 1, the numerical results of \( v_1 (t) \) and \( v_2 (t) \) are shown in Tables 1 and 2 at \( \beta = 1 \) and \( k = 8 \). The results obtained in Tables 1 and 2 show that the error estimate using the proposed method is very small and that the solutions correspond well to each other. In general, it should be noted that increasing the number of iterations \( k \) will lead to an improvement in numerical solutions and approaching the exact value.

Tables 3 and 4 show the sixth approximate solutions of Example 1 at different values of \( \beta \) such that

\[
\mathcal{D}^a v_1 (x) + \int_0^x t (v_1 (t) - v_2 (t)) dt + \int_0^1 t x^5 (v_1 (t) + v_2 (t)) dt = e^x x + 2 x^5 \frac{2 x^5}{e} + e^{-x} (1 + x),
\]

\[
\mathcal{D}^a v_2 (x) + \int_0^x t (v_1 (t) + v_2 (t)) dt + \int_0^1 t x^5 (v_1 (t) - v_2 (t)) dt = 2 e^x (-1 + x) + \frac{2 x^5}{e} - e^{-x} (2 + x).
\]

This is subject to the following initial conditions:

\[
v_1 (0) = v_2 (0) = 1.
\]

The exact solution of this coupled system is \( v_1 (x) = e^x \) and \( v_2 (x) = e^{-x} \).

Using the RPS algorithm, the \( k \)-th residual functions \( \text{Res}_1 (x) \) and \( \text{Res}_2 (x) \) are given by

\[
\text{Res}_1 (x) = \mathcal{D}^a v_{1,m} (x) + \int_0^x t (v_{1,m} (t) - v_{2,m} (t)) dt + \int_0^1 t x^5 (v_{1,m} (t) + v_{2,m} (t)) dt - \left( e^x x + 2 x^5 \frac{2 x^5}{e} + e^{-x} (1 + x) \right),
\]

\[
\text{Res}_2 (x) = \mathcal{D}^a v_{2,m} (x) + \int_0^x t (v_{1,m} (t) + v_{2,m} (t)) dt + \int_0^1 t x^5 (v_{1,m} (t) - v_{2,m} (t)) dt - \left( 2 e^x (-1 + x) + \frac{2 x^5}{e} - e^{-x} (2 + x) \right),
\]

where \( v_{1,m} (x) \) has the following form:

\[
v_{1,m} (x) = 1 + \sum_{m=1}^{k} v_{1,m} \frac{x^m}{\Gamma(m\beta + 1)}, \quad i = 1, 2.
\]

Consequently,

\[
\text{Res}_1 (x) = \mathcal{D}^a \left( 1 + \sum_{m=1}^{k} v_{1,m} \frac{x^m}{\Gamma(m\beta + 1)} \right) + \int_0^x t \left( \sum_{m=1}^{k} (v_{1,m} - v_{2,m}) \frac{x^m}{\Gamma(m\beta + 1)} \right) dt + \int_0^1 t x^5 \left( 2 + \sum_{m=1}^{k} (v_{1,m} + v_{2,m}) \frac{x^m}{\Gamma(m\beta + 1)} \right) dt - \left( e^x x + 2 x^5 \frac{2 x^5}{e} + e^{-x} (1 + x) \right),
\]

\[
\text{Res}_2 (x) = \mathcal{D}^a \left( 1 + \sum_{m=1}^{k} v_{2,m} \frac{x^m}{\Gamma(m\beta + 1)} \right) + \int_0^x t \left( \sum_{m=1}^{k} (v_{1,m} + v_{2,m}) \frac{x^m}{\Gamma(m\beta + 1)} \right) dt + \int_0^1 t x^5 \left( \sum_{m=1}^{k} (v_{1,m} - v_{2,m}) \frac{x^m}{\Gamma(m\beta + 1)} \right) dt - \left( 2 e^x (-1 + x) + \frac{2 x^5}{e} - e^{-x} (2 + x) \right).
\]
Table 1: Numerical results of exact $v_1(t)$ and approximate $v_{1,8}(t)$ solutions for Example 1 at $\beta = 1$ and $k = 8$ over the interval $[0, 1]$.  

| $t$  | Exact solution $v_1(t)$ | Approximate solution $v_{1,8}(t)$ | Absolute error $|v_1(t) - v_{1,8}(t)|$ |
|------|-------------------------|----------------------------------|----------------------------------|
| 0    | 1.0                     | 1.0                              | 0.0                              |
| 0.1  | 1.1051709               | 1.1051708                        | $8.4700 \times 10^{-8}$          |
| 0.2  | 1.2214028               | 1.2214000                        | $2.7581 \times 10^{-6}$          |
| 0.3  | 1.3498588               | 1.3498375                        | $2.1307 \times 10^{-5}$          |
| 0.4  | 1.4918247               | 1.4917333                        | $9.1364 \times 10^{-5}$          |
| 0.5  | 1.6487213               | 1.6484375                        | $2.8377 \times 10^{-4}$          |
| 0.6  | 1.8221188               | 1.8214000                        | $7.1880 \times 10^{-4}$          |
| 0.7  | 2.0137527               | 2.0121708                        | $1.5819 \times 10^{-3}$          |
| 0.8  | 2.2255409               | 2.2224000                        | $3.1409 \times 10^{-3}$          |
| 0.9  | 2.4596031               | 2.4538375                        | $5.7656 \times 10^{-3}$          |
| 1.0  | 2.7182818               | 2.7083333                        | $9.9485 \times 10^{-3}$          |

Table 2: Numerical results of exact $v_2(t)$ and approximate $v_{2,8}(t)$ solutions for Example 1 at $\beta = 1$ and $k = 8$ over the interval $[0, 1]$.  

| $t$  | Exact solution $v_2(t)$ | Approximate solution $v_{2,8}(t)$ | Absolute error $|v_2(t) - v_{2,8}(t)|$ |
|------|-------------------------|----------------------------------|----------------------------------|
| 0    | 1.0                     | 1.0                              | 0.0                              |
| 0.1  | 0.9048374               | 0.9048375                        | $8.1900 \times 10^{-8}$          |
| 0.2  | 0.8187308               | 0.8187333                        | $2.5802 \times 10^{-6}$          |
| 0.3  | 0.7408182               | 0.7408375                        | $1.9279 \times 10^{-5}$          |
| 0.4  | 0.67033201              | 0.6704000                        | $7.9954 \times 10^{-5}$          |
| 0.5  | 0.5488116               | 0.5494000                        | $5.8836 \times 10^{-4}$          |
| 0.6  | 0.4965853               | 0.4978375                        | $1.2522 \times 10^{-3}$          |
| 0.7  | 0.4493290               | 0.4517333                        | $2.4043 \times 10^{-3}$          |
| 0.8  | 0.4065697               | 0.4108375                        | $4.2678 \times 10^{-3}$          |
| 0.9  | 0.367894                | 0.3750000                        | $7.1206 \times 10^{-3}$          |
| 1.0  | 0.367894                | 0.3750000                        | $7.1206 \times 10^{-3}$          |

Table 3: Numerical results of sixth approximate solution $v_{1,6}(t)$ for Example 1 at different values of fractional order $\beta, k = 6$, and $t \in [0, 1]$.  

| $t$  | $\beta = 1$ | $\beta = 0.9$ | $\beta = 0.8$ | $\beta = 0.7$ |
|------|-------------|---------------|---------------|---------------|
| 0.0  | 1.0         | 1.0           | 1.0           | 1.0           |
| 0.1  | 1.0510833   | 1.073961      | 1.1073884     | 1.1601510     |
| 0.2  | 1.1246667   | 1.1716823     | 1.2392129     | 1.3365098     |
| 0.3  | 1.2212500   | 1.2956381     | 1.3974229     | 1.5539285     |
| 0.4  | 1.3413333   | 1.4446154     | 1.5801578     | 1.7560644     |
| 0.5  | 1.4854167   | 1.6181748     | 1.7860776     | 1.9951767     |
| 0.6  | 1.6540000   | 1.8160345     | 2.0142021     | 2.2519736     |
| 0.7  | 1.8475833   | 2.0380217     | 2.2637874     | 2.5254605     |
| 0.8  | 2.0666667   | 2.2840401     | 2.5342526     | 2.8148458     |
| 0.9  | 2.3117500   | 2.5540495     | 2.8251333     | 3.1194830     |
| 1.0  | 2.5833333   | 2.8480518     | 3.1360514     | 3.4383292     |

Table 4: Numerical results of sixth approximate solution $v_{2,6}(t)$ for Example 1 at different values of fractional order $\beta, k = 6$, and $t \in [0, 1]$.  

| $t$  | $\beta = 1$ | $\beta = 0.9$ | $\beta = 0.8$ | $\beta = 0.7$ |
|------|-------------|---------------|---------------|---------------|
| 0.0  | 1.0         | 1.0           | 1.0           | 1.0           |
| 0.1  | 0.9048333   | 0.8780779     | 0.8460694     | 0.8088467     |
| 0.2  | 0.8186666   | 0.7855478     | 0.7499395     | 0.7123611     |
| 0.3  | 0.7404999   | 0.7071736     | 0.6734520     | 0.6391004     |
| 0.4  | 0.6693333   | 0.6386210     | 0.6084174     | 0.5772768     |
| 0.5  | 0.6041666   | 0.5772055     | 0.5505285     | 0.5214466     |
| 0.6  | 0.5349999   | 0.5209163     | 0.4969643     | 0.4681588     |
| 0.7  | 0.4878333   | 0.4681027     | 0.4456563     | 0.4160595     |
| 0.8  | 0.4346666   | 0.4173375     | 0.3949800     | 0.3628482     |
| 0.9  | 0.3835000   | 0.3673449     | 0.3435991     | 0.3076204     |
| 1.0  | 0.3333333   | 0.3169592     | 0.2937874     | 0.2494581     |
We are given a differential equation:

\[ D_\alpha^\beta v_1(x) + \int_0^x v_2(\xi) d\xi + \int_0^1 \xi v_1(\xi) d\xi = -\cos(1) + \cos(x) + \sin(1) + \sin(x), \]

\[ D_\alpha^\beta v_2(x) + \int_0^x v_1(\xi) d\xi + \int_0^1 \xi v_2(\xi) d\xi = \cos(1) - \cos(x) + \sin(1) - \sin(x). \]  

\( \beta \in \{1, 0.9, 0.8, 0.7\} \) with step size 0.1 and \( k = 6 \). From these tables, it can be concluded that the RPS algorithm and the approximate solutions are consistent with each other and with the exact solutions for all values of \( t \in [0, 1] \). Here, it is worth noting that the closer the value of the fractional derivative approaching the integer case \( \beta = 1 \), the closer the approximate solution is to the exact solution.

**Example 2.** Consider the following fractional integro-differential equation:

\[ D_\alpha^\beta v_1(x) + \int_0^x v_2(\xi) d\xi + \int_0^1 \xi v_1(\xi) d\xi = -\cos(1) + \cos(x) + \sin(1) + \sin(x), \]

\[ D_\alpha^\beta v_2(x) + \int_0^x v_1(\xi) d\xi + \int_0^1 \xi v_2(\xi) d\xi = \cos(1) - \cos(x) + \sin(1) - \sin(x). \]  

(18)
Table 8: Numerical results of approximate solution \( v_{2,\beta}(t) \) of Example 2 for different values of \( \beta \), \( k = 6 \), and \( t \in [0,1] \).

| \( t \) | \( \beta = 0.9 \) | \( \beta = 0.8 \) | \( \beta = 0.7 \) |
|---|---|---|---|
| 0.0 | 1.0 | 1.0 | 1.0 |
| 0.1 | 1.0036839 | 0.9995798 | 0.9902709 |
| 0.2 | 0.9918177 | 0.9770693 | 0.9526430 |
| 0.3 | 0.9678131 | 0.9409453 | 0.9018037 |
| 0.4 | 0.9329689 | 0.8938412 | 0.8413882 |
| 0.5 | 0.8881139 | 0.8372777 | 0.7733064 |
| 0.6 | 0.8338592 | 0.7722922 | 0.6987749 |
| 0.7 | 0.7706901 | 0.6996578 | 0.6186510 |
| 0.8 | 0.8338592 | 0.7722922 | 0.6987749 |
| 0.9 | 0.9918177 | 0.9770693 | 0.9526430 |
| 1.0 | 1.0036839 | 0.9995798 | 0.9902709 |

This is subject to the following initial conditions:

\[
v_1(0) = v_2(0) = 1.
\]

The exact solution of this coupled system is \( v_1(x) = \sin x \) and \( v_2(x) = \cos x \). Using the RPS algorithm, the \( k \)-th residual functions \( R_{2,k}(x) \) and \( R_{3,k}(x) \) are given by

\[
R_{2,k}(x) = D^a v_{1,\beta}(x) + \int_0^x v_{2,\beta}(t) \, dt + \int_0^1 v_{1,\beta}(t) \, dt - (\cos(x) + \sin(x) - \cos(1) + \sin(1)),
\]

\[
R_{3,k}(x) = D^a v_{2,\beta}(x) + \int_0^x v_{1,\beta}(t) \, dt + \int_0^1 v_{2,\beta}(t) \, dt - (\cos(x) - \sin(x) + \sin(1) + \cos(1)),
\]

where \( v_{1,\beta}(x) \) and \( v_{2,\beta}(x) \) have the following form:

\[
v_{1,\beta}(x) = \sum_{m=1}^k v_{1,\beta,m} \frac{x^{m\beta}}{\Gamma(m\beta + 1)}
\]

\[
v_{2,\beta}(x) = 1 + \sum_{m=1}^k v_{2,\beta,m} \frac{x^{m\beta}}{\Gamma(m\beta + 1)}
\]

The absolute errors are listed in Tables 5 and 6. The results obtained by the RPS method show that the exact solutions are in good agreement with approximate solutions at \( \beta = 1 \), \( k = 6 \), and step size 0.1. Tables 7 and 8 show approximate solutions at different values of \( \beta \) such that \( \beta \in [0.9, 0.8, 0.7] \) and \( k = 6 \) with step size 0.1. From these tables, one can find that the RPS method provides us with an accurate approximate solution, which is in good agreement with the exact solutions for all values of \( t \in [0,1] \). Also, it is worth noting that the closer the value of the fractional derivative approaching the integer case \( \beta = 1 \), the closer the approximate solution is to the exact solution.

5. Concluding Remarks

In this work, a class of a coupled system of nonlinear fractional integrodifferential equations of fractional order has been discussed by using the RPS method under the Caputo fractional derivative. The RPS algorithm has been given to optimize the approximate solution by minimizing a residual error function with the help of generalized Taylor formula. To demonstrate the consistency with the theoretical framework, two illustrative examples have been provided. The obtained results indicated that the approximate solutions are coinciding with the exact solution and with each other for different values of the fractional order over the selected nodes and parameters. From our results, we can conclude that the RPS method is a systematic and suitable scheme to address many fractional initial value problems with great potential in scientific applications. The calculations have been performed by using Wolfram-Mathematica 10.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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