(In)finiteness of Spherically Symmetric Static Perfect Fluids

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This work is concerned with the finiteness problem for static, spherically symmetric perfect fluids in both Newtonian Gravity and General Relativity. We derive criteria on the barotropic equation of state guaranteeing that the corresponding perfect fluid solutions possess finite/infinite extent. In the Newtonian case, for the large class of monotonic equations of state, and in General Relativity we improve earlier results. Moreover, we are able to treat the two cases in a completely parallel manner, which is accomplished by using a relativistic version of Pohozaev’s identity in the proof of the relativistic criterion. This identity and further generalizations are presented in detail.

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I. INTRODUCTION

In this paper we consider non-rotating stellar models, i.e., we consider static, self-gravitating perfect fluids with a barotropic equation of state \( \rho(p) \) relating density and pressure. The basic equations are the Euler-Poisson equations in Newtonian theory and the Euler-Einstein equations in General Relativity. We are concerned with globally regular solutions on \( \mathbb{R}^3 \) consisting of a perfect fluid region and possibly a vacuum region. We focus on spherically symmetric solutions in particular.

In Newtonian theory spherical symmetry is no restriction; static stellar models are necessarily spherically symmetric (see [12] for an overview). In General Relativity, although conjectural, a general theorem establishing spherical symmetry of solutions does not exist so far, although some progress has been made [5, 6, 2]. Some more remarks follow below.

Existence and uniqueness of spherically symmetric solutions has been proven satisfactorily in Newtonian theory (for rather general equations of state [18]) as well as in General Relativity (for smooth equations of state [17]). For results regarding topology see [1].

Mathematical issues apart, perfect fluid solutions are of interest in astrophysics. Primarily, perfect fluid solutions represent stellar models (we refer to the classic [9]; a recent review addressing some issues is [4]). But also other astrophysical objects as globular clusters possess descriptions in terms of perfect fluid solutions [13].

The main question we pose in this work is the (in)finiteness question. Under which conditions on the equation of state does the corresponding perfect fluid solution possess finite or infinite extent? We briefly recall some criteria existing in the literature: The polytropic equations of state \( p = K \rho^{(n+1)/n} \) \((K > 0, n > 0 \text{ constants})\) have been studied extensively, both analytically and numerically. In Newtonian theory finiteness is guaranteed for \( n < 5, K \) arbitrary (for a general discussion see [14]), in General Relativity the case is considerably more complex [15]. Criteria for more general classes of equations of state are, e.g., the following: In Newtonian theory and in General Relativity, spherical symmetry presupposed, the perfect fluid solution is finite, if \( \rho|_{p=0} > 0 \). This result can be subsumed under the criterion guaranteeing finiteness of the fluid configuration if \( \int_{0}^{p} dp' \rho(p')^{-2} < \infty \) holds. In the case \( \rho|_{p=0} = 0 \) monotonicity of the equation of state must be assumed here. As a counterpart to this criterion, under the same assumptions, there exists the following theorem: If \( \int_{0}^{p} dp' \rho^{-1}(p') \neq \infty \) (Newtonian theory) or \( \int_{0}^{p} dp'(|\rho(p') + c^{-2}p'|^{-1} \neq \infty \) (General Relativity), then the fluid solution must necessarily extend to infinity (see [17] for a good presentation of these criteria). Note incidentally that the last two quantities will play an important role for our considerations as well (see below). For equations of state of the type \( p = K \rho^{(n+1)/n}(1 + O(\rho^{1/n}) \) (as \( \rho \to 0 \)) with \( 1 < n < 3 \) finiteness of the fluid solution has
be proven in \[14\] (some detail follow below). Note that these criteria involve the behavior of the equation of state for small \( p \) only. More general criteria, however, must be based on the behavior of the equation of state \( \rho(p) \) for all \( p \). To show that consider the polytropic equation of state for \( n = 5 \) in Newtonian theory as an example. Perturbing \( \rho(p) \) in a small neighborhood of some finite value of \( p \) suffices to produce either finite or infinite fluid configurations. Criteria which are capable of dealing with such phenomena have been derived in \[20\] and \[21\]. Some of those criteria will be reproduced in this paper (see, e.g., theorems \[IV.3\] and \[VII.2\]); the other criteria we present here can be regarded as generalizations or modifications in a certain sense. In \[21\] the assumptions on the equation of state and the solutions are kept rather general, several theorems are formulated for Sobolev functions. Occasionally we give cross references to \[21\].

The paper is divided into three main parts. Part one (Newton): In sections II–IV we treat the \( \rho(p) \) question for perfect fluid solutions in Newtonian theory. Section II is concerned with the Euler-Poisson equations; in section II some quantities are studied which are necessary to formulate the \( \rho(p) \) criteria in section IV (Theorem IV.3 and theorem \[IV.6\]). The crucial tool for the proofs of the theorems is Pohozaev’s identity [16]. Part two (General Relativity): Sections V–VII deal with general relativistic perfect fluid solutions. The presentation parallels the Newtonian case in order to facilitate comparison; section V: The Euler-Einstein equations, section VI: Definitions. The \( \rho(p) \) criteria we have derived both for the Newtonian and the relativistic case. Part three (Pohozaev-like identities): In section IX we present a powerful method of deriving “Pohozaev-like” identities. In particular we treat the relativistic Pohozaev identity which has been used for the proof of theorem \[VII.6\] in section VII.

II. NEWTON: BASICS

Newtonian static perfect fluids are regular solutions to the Euler-Poisson equations (on \( (\mathbb{R}^3, \delta_{ij}) \)), given a barotropic equation of state \( \rho(p) \) relating the pressure and the density.

\begin{equation}
\begin{aligned}
\Delta u(x) &= 4\pi \rho(x) \\ \partial_i p(x) &= -\rho \partial_i u(x)
\end{aligned}
\end{equation}

(1a)

(1b)

Here, \( u \) denotes the Newtonian potential, \( \partial_i = \frac{\partial}{\partial x^i} \) and \( \Delta = \partial^i \partial_i \).

In section III we discuss which classes of equations of state we consider in this paper (see definition III.1 ff.). In all cases under consideration the potential \( u \) can be viewed as a function of \( p \); integrating (1b) we obtain

\begin{equation}
u(p) - u_S = - \int_0^p dp' \rho^{-1}(p') =: -\Gamma(p) \quad \text{where} \quad u_S := u|_{p=0}.
\end{equation}

(2)

We distinguish solutions of (1) according to whether they extend to infinity or not. A solution with finite extent (surface \( \{p = 0\} \) ) possesses the surface potential \( u_S \). At the surface the interior solution is joined to an exterior vacuum solution (standard junction conditions). Solutions extending to infinity satisfy \( p > 0 \) everywhere.

We define the “normalized” potential \( v \) as \( v := u - u_S \). Obviously, \( v(p) \) is monotonic with \( v|_{p=0} = 0 \). Consequently, both \( p \) and \( \rho \) can be viewed as functions of \( v \), and, moreover, \( p(v) = -\int_0^v \rho(v')dv' \). For monotonic equations of state \( \rho(p) \), \( \rho(v) \) is monotonic as well (the converse also being true).

Remark. The quantities \( p(x), \rho(x) \) vanish at infinity, i.e., \( p(x), \rho(x) \to 0 (||x|| \to \infty) \) for solutions of (1). In spherical symmetry this can be proven rather easily (see, e.g., [12]). Using the standard asymptotic condition for \( u(x) \), it follows that \( u(x) \to 0 (||x|| \to \infty) \). As a consequence, \( v = u \), if the fluid has infinite extent.

Remark. Newtonian static perfect fluid solutions are necessarily spherically symmetric. For a short overview on the topic of symmetries of stellar models see [12] (and references therein).

Remark. The potential \( u(r) \) \( (r = ||x||) \) is monotonically increasing. This follows directly from the maximum principle applied to (1a). Hence, \( u(r) \) assumes its minimal value \( u_c \) at the center. Correspondingly, \( p(r) \) is a strictly decreasing function (with maximum \( p_c \)), which results from monotonicity of \( \Gamma(p) \) in (2). \( p_c (u_c, \ldots) \) is a suitable parameter to characterize the one-parameter family of regular solutions to (1).

III. NEWTON: THE QUANTITIES \( I_i \)

Throughout this paper we consider barotropic equations of state \( \rho(p) \) given at least on an interval \([0, p_{max}]\). We assume that \( \rho \) is positive for all \( p \in [0, p_{max}] \). Recall that perfect fluid configurations are always finitely extended
in the case \( \rho|_{p=0} > 0 \). Therefore we focus on equations of state with \( \rho|_{p=0} = 0 \). “Microscopically stable” matter (see, e.g., [19]) is described by monotonic equations of state, i.e., by increasing functions \( \rho(p) \). Often we will restrict ourselves to this class of equations of state.

To formulate our theorems in section [V] we now introduce some quantities. Note that all these definitions depend on the equation of state only.

**Definition III.1.** Consider a barotropic equation of state \( \rho(p) \) as described above with \( \rho(p) \) piecewise \( C^0 \) on \([0,p_{\text{max}}]\). Let \( \Gamma(p) \) be defined as

\[
\Gamma(p) := \int_0^p dp' \rho^{-1}(p') .
\]  

**Assumption 1.** Throughout this paper we assume that \( \Gamma(p) \) exists for some \( p > 0 \). Obviously, \( \Gamma(p) \) is \( C^0[0,p_{\text{max}}] \).

**Assumption 2.** We require the limit \( \lim_{p \to 0} \rho^{-1}p \) to exist.

**Remark.** Note that assumptions [I] and [II] are independent. However, if the equation of state \( \rho(p) \) is monotonic (at least on \([0,\epsilon]\) for some \( \epsilon > 0 \)), then assumption [II] follows from assumption [I]. Note further that \( \lim_{p \to 0} \rho^{-1}p = 0 \), if [I] and [II] hold. For the details we refer the reader to appendix [A].

**Remark.** Provided that \( \rho(p) \) is at least piecewise \( C^1 \) we may investigate \( \frac{dp}{d\rho} \). From \( \lim_{p \to 0} \rho^{-1}p = 0 \) we conclude \( \lim_{p \to 0} (\frac{dp}{d\rho})^{-1} = 0 \), if this limit exists. For details see appendix [A].

**Definition III.2.** Consider an equation of state \( \rho(p) \) satisfying assumptions [I] and [II]. We define the following quantities:

\[
I_{-1}(p) := 7 \int_0^p dp' \Gamma(p') - 6 \Gamma(p)p \tag{4a}
\]

\[
I_0(p) := \Gamma(p) - 6 \rho^{-1}p \tag{4b}
\]

\[
I_1(p) := 6p \rho^{-2} \frac{dp}{dp} - 5p^{-1} \tag{4c}
\]

\[
I_2(p) := 5p \rho^{-2} \frac{d^2 \rho}{dp^2} + p \rho^{-3} (\frac{dp}{d\rho})^2 \tag{4d}
\]

We require \( \rho(p) \) to be piecewise \( C^0 \) on \([0,p_{\text{max}}]\); in order to define \( I_1 \) let \( \rho(p) \) be \( C^0[0,p_{\text{max}}] \) and piecewise \( C^1 \); for \( I_2 \) we require differentiability of one order higher.

**Remark.** From the comments above we conclude that \( I_0(p) \) is well-defined and piecewise continuous on \([0,p_{\text{max}}]\). Moreover, \( I_{0|p=0} = 0 \).

**Proposition III.3.** Let \( \{I_i \leq 0\}, \{I_i = 0\}, \) and \( \{I_i \geq 0\} \) denote the sets of all equations of state \( \rho(p) \) (as in definition III.2) such that, for all \( p \leq p_{\text{max}}, I_i \leq 0, I_i = 0, \) or \( I_i \geq 0 \) respectively. Then

\[
\{I_1 \leq 0\} \subset \{I_0 \leq 0\} \subset \{I_{-1} \leq 0\} \tag{5a}
\]

\[
\{I_1 = 0\} \equiv \{I_0 = 0\} \equiv \{I_{-1} = 0\} \tag{5b}
\]

\[
\{I_1 \geq 0\} \subset \{I_0 \geq 0\} \subset \{I_{-1} \geq 0\} . \tag{5c}
\]

**Proof.** \( I_i \) is the integral of \( I_{i+1} \) (\( i = -1,0 \)), i.e.

\[
\frac{d}{dp} I_{-1}(p) = I_0(p), \quad \frac{d}{dp} I_0(p) = I_1(p) . \tag{6}
\]

Together with \( I_{0|p=0} = 0 \) and \( I_{-1|p=0} = 0 \), the claim is proven. \( \blacksquare \)

**Remark.** \( \{I_i \equiv 0\} \) (\( i = -1,0,1 \)) corresponds to the set of polytropic equations of state \( p = K \rho^{\frac{1}{n-1}} \) \( (K = \text{const}) \) with polytropic index \( n = 5 \). This can be obtained easily by solving the differential equation given by \( I_1 \equiv 0 \).

**Remark.** We also have the relations \( \{I_2 \equiv 0\} \supset \{I_1 \equiv 0\} \) and \( \{I_2 \leq 0\} \subset \{I_1 \leq 0\} \). For a proof see [I] [II].

**Comments.** In this paper we will only be concerned with the quantities \( I_0 \) and \( I_{-1} \). In [21] it has been shown that \( I_1 \) plays the main role in a theorem related to part \( \Lambda \) of theorem [V.5] and [V.6]; basically it states that by controlling the sign of \( I_1 \) it becomes possible to get rid of the AFMD requirement (see section [V] for the context). \( I_2 \) itself is not as relevant as its relativistic counterpart; some comments follow in section [V].
IV. NEWTON: CRITERIA

In the present section we prove the main theorems in the Newtonian case (theorems IV.5 and IV.6). These theorems formulate criteria on the equation of state ensuring (in)finiteness of the corresponding static perfect fluid solutions. The main ingredient for the proofs is Pohozaev’s identity (proposition IV.2).

Remark. For the following we assume that the equation of state \( \rho(p) \) (and thus \( \rho(v) \)) is at least piecewise \( C^0 \), that the function \( p(r) \) is continuous and piecewise \( C^1 \), \( \rho(r) \) piecewise \( C^0 \), and \( u(r) \) \( C^1 \) and piecewise \( C^2 \). These requirements are sensible: In \( [18] \), beyond existence and uniqueness, it has been shown that for the large class of so-called admissible equations of state (including step functions, polytropic behavior, etc.) the (in general non-classical) solutions are sensible: In \( [18] \), beyond existence and uniqueness, it has been shown that for the large class of so-called admissible solutions (Asymptotic conditions). A solution of (1) is called AFMD (asymptotically flat with mass decay conditions), if for some \( \epsilon > 0 \)

\[ M/R \geq \epsilon \] 

that

\[ \rho \equiv \rho(v) \equiv \rho(u) \] 

The identities (10a) plus (10b) can be combined to give

\[ \int_{\mathbb{R}^3} d^3x (\rho v + 6p) = 0 \] 

This is because, for the finite body case, \( \int d^3x \rho u = \int d^3x \rho (v + u_S) = \int d^3x \rho v + u_S M \), and \( u_S = \rho M / R \).

Theorem IV.5. \( [22], [23] \). Consider an equation of state \( \rho(p) \) being piecewise \( C^0 \) and satisfying assumptions [1] and [16].

Definition IV.1. \( [1] \). From \( v \), its derivatives, and from \( p \), \( \rho \) we define the following symmetric tensor \( \sigma_{ij} \) on \( \mathbb{R}^3 \):

\[ \sigma_{ij} := -2v_{v,ij} + 6v_{i}v_{j} - 2\delta_{ij}v_k v^k + 8\pi \delta_{ij} (\rho v + 4p) \]  

(7)

\[ \sigma_{ij} \] is divergence free, \( \partial^j \sigma_{ij} = 0 \); its trace \( \sigma_i^i = 16\pi (\rho v + 6p) \). Note that by \( [1] \) \( v_{ij} \) can be written as first derivatives. Pohozaev’s identity is a direct consequence. In its present form it is due to \( [20] \).

Proposition IV.2. (Pohozaev identity). With \( \xi^j \) a dilation, i.e., \( \xi^j = x^j \) we have the following identity:

\[ \partial^i (\sigma_{ij} \xi^j) = 16\pi (\rho v + 6p) \]  

(8)

Definition IV.3. (Asymptotic conditions). A solution of (1) is called AFMD (asymptotically flat with mass decay conditions), if for some \( \epsilon > 0 \)

\[ u = O(\|x\|^{-1}) \] 

\[ \rho = O(\|x\|^{-3-\epsilon}) \] 

(9)

Consequently, using Euler’s equation (1b), \( p = O(\|x\|^{-4-\epsilon}) \). For a generalization in terms of Sobolev spaces see [21].

Proposition IV.4. (Pohozaev integrated). Consider a static perfect fluid solution in Newtonian theory, i.e., let \( v(r) = u(r) - u_S \), \( p(r) \), \( \rho(r) \) be a solution of (1). Furthermore, assume AFMD. Then,

\[ \int_{\mathbb{R}^3} d^3x (\rho v + 6p) = \frac{M^2}{R} , \]  

for solutions with finite extent, and,

\[ \int_{\mathbb{R}^3} d^3x (\rho v + 6p) = 0 , \]  

if the fluid extends to infinity.

In (10a), \( M \) denotes the mass, and \( R \) the radius of the finite fluid object.

Proof. Integrating (8) for fluid solutions with finite extent we get for the l.h. side

\[ \int_{\text{Ball}(R)} d^3x \partial^i (\sigma_{ij} \xi^j) = 4\pi R^2 \sigma_{ij} |_{R} x^i R^{-1} x^j = 16\pi R^3 v_{r} \]  

(11)

The exterior solution is vacuum, i.e., \( u = -M/r \) (\( r \geq R \)), so that \( v_{r} |_{R} = u_{r} |_{R} = MR^{-2} \). For solutions extending to infinity, \( \int_{\text{Ball}(R)} d^3x \partial^i (\sigma_{ij} \xi^j) = 4\pi \sigma_{ij} x^i x^j \). Making use of the AFMD conditions, this expression can be shown to converge to zero as \( r \to \infty \). 

Remark. The identities (10a) plus (10b) can be combined to give

\[ \int_{\mathbb{R}^3} d^3x (\rho v + 6p) = 0 \]  

(12)

This is because, for the finite body case, \( \int d^3x \rho u = \int d^3x \rho (v + u_S) = \int d^3x \rho v + u_S M \), and \( u_S = -M/R \).
A Let \((v = u - u_S, \rho, p)\) be an AFMD solution to the Euler-Poisson equations \((\ref{1})\) with \(p \leq \tilde{p}\) for some \(\tilde{p} > 0\). If \(I_0 \leq 0 (I_0 \neq 0)\) for all \(p \in [0, \tilde{p}]\), then the solution has finite extent.

B If \(I_0 \geq 0 (I_0 \neq 0)\) for all \(p \in [0, \tilde{p}]\), then there is no AFMD solution to the Euler-Poisson equations \((\ref{1})\) satisfying \(p \leq \tilde{p}\).

Proof. \(I_0(p) = \Gamma(p) - 6\rho^{-1}p = -(v(p) + 6\rho^{-1}p)\). Hence, the l.h. side of \((\ref{10})\) equals \(-\int d^4x I_0(p(x))\). The rest follows easily from proposition \((\ref{IV.4})\).\]

Remark. Since \(p(r)\) is a monotonically decreasing function, \(p \leq \tilde{p}\) corresponds to \(p \leq p_c\), where \(p_c\) is the central pressure of the perfect fluid solution.

Remark. In the proof only the quantity \(pI_0\) was used. Thus, proposition \((\ref{V.3})\) also holds without assuming \(\ref{3}\).

Remark. Recall that \(I_0 \equiv 0\) corresponds to \(p = \frac{4}{3} \rho_{-1/2} \rho^2 (\rho_{-1} = \text{const})\). The associated static perfect fluid solutions are known explicitly \((\ref{3})\) and read

\[v(r) = u(r) = -\frac{M}{\sqrt{4\pi\rho_{-1/2} M^4 + r^2}} \quad (M \geq 0).\] (13)

Obviously, \((\ref{3})\) are AFMD solutions extending to infinity.

Theorem IV.6. Consider an equation of state \(\rho(p)\) being \(C^0\), piecewise \(C^1\) and monotonic, and assume \(\ref{4}\).

A Let \((v = u - u_S, \rho, p)\) be an AFMD solution to the Euler-Poisson equations \((\ref{1})\) with \(p \leq \tilde{p}\) for some \(\tilde{p} > 0\). If \(I_{-1} \leq 0 (I_{-1} \neq 0)\) for all \(p \in [0, \tilde{p}]\), then the solution has finite extent.

B If \(I_{-1} \geq 0 (I_{-1} \neq 0)\) for all \(p \in [0, \tilde{p}]\), then there is no AFMD solution to the Euler-Poisson equations \((\ref{1})\) satisfying \(p \leq \tilde{p}\).

Proof. The proof is based on Pohozaev’s identity in the form encountered in proposition \((\ref{IV.4})\). According to the remarks at the end of section II spherical symmetry is no restriction.

\[
-\int_{\text{Ball}(\hat{r})} d^3x (\rho v + 6p) = -4\pi \int_0^{\hat{r}} dr r^2 \rho r I_0(p(r)) = 4\pi \int_{p_c}^{\rho(p(r))} dp \rho v^{-1} r I_0(p)
\]

\[
= 4\pi \left[ r^2 v^{-1} \right]_{p_c}^{\rho(p(r))} - 4\pi \int_{p_c}^{\rho(p(r))} dp \frac{d}{dp} (r^2 v^{-1}) I_{-1}
\]

\[
= 4\pi \left[ r^2 v^{-1} I_{-1}(p(r)) \right]_0^{\rho(p(r))} - 4\pi \int_0^{\hat{r}} dr \frac{d}{dr} (r^2 v^{-1}) I_{-1}(p(r))
\]

Here, the subscript \(c\) refers to the central value of the quantity, e.g., \(p_c = p|\_{r=0}\). Regularizing and solving the Euler Poisson equations in a small neighborhood of \(r = 0\), we obtain \(v(r) = v_\xi + 4\pi \rho_c r^2 + o(r^2)\); this implies that \(r^2 v^{-1} I_{-1}(p(r))\) vanishes as \(r \to 0\). At \(r = \hat{r}\) we have the boundary term \(r^2 v^{-1} I_{-1}(p(\hat{r}))\). For solutions with finite extent we may choose \(\hat{r} = R\) in order to obtain \(R^2 v^{-1} |_{p = 0} = R^4 M^{-1} I_{-1} |_{p = 0} = 0\). For AFMD solutions extending to infinity the boundary term vanishes as \(\hat{r} \to \infty\): Note that \(I_{-1}(p)\) is \(C^1\) at \(p = 0\) and \(I_{-1} |_{p = 0} = 0\) as well as \(\frac{d}{dp} I_{-1} |_{p = 0} = I_0 |_{p = 0} = 0\). Therefore, \(I_{-1}(p) = pf(p)\), where \(f(p)\) is some function with \(f |_{p = 0} = 0\). Consequently, \(r^2 v^{-1} I_{-1}(p(\hat{r})) = -r^2 \rho(\hat{r}) \frac{p(\hat{r})}{p(\hat{r})} f(p(\hat{r}))\), which must go to zero as \(\hat{r} \to \infty\), if the AFMD conditions are assumed. Now, we define

\[
a(r) := r^{-2} \frac{d}{dr} (r^2 v^{-1}) = 4r^{-1} v^{-1} - 4\pi \rho v^{-2}.
\]

The function \(a(r)\) is strictly positive, as we will show in the following. We investigate the function \(b(r) := r^2 v^{-1} a(r) = 4r^2 v^{-1} - 4\pi \rho^2\). In a neighborhood of \(r = 0\), i.e., in the limit \(r \to 0\), \(b(r)\) is given by the positive function \(b_{\text{approa}}(r) = 4r^2 \frac{d}{dr} \rho_c - 4\pi r^3 \rho_c = \frac{4}{3} \rho_c \geq 0\). Differentiating \(b(r)\) we obtain \(b_r(r) = 4\pi r^2 \rho - 4\pi r^3 \rho_r > 0\), which results from \(\rho_r = \frac{d}{dr} \rho |_{r < 0} < 0\). We conclude that \(b(r)\) is monotonic and \(b(r) > 0\) for all \(r > 0\). Therefore, \(a(r) > 0\) for all \(r > 0\).
Combining (10) and (14) we obtain
\[ \int_{\mathbb{R}^3} d^3x \ a(r) I_{-1}(p(x)) = -\frac{M^2}{R}, \quad \text{for solutions with finite extent, and,} \] (16a)
\[ \int_{\mathbb{R}^3} d^3x \ a(r) I_{-1}(p(x)) = 0, \quad \text{if the fluid extends to infinity,} \] (16b)
provided that the solution is AFMD. From these equations the claim of the theorem follows easily.

Remark. (Relationship between theorem \[ IV.5 \] and theorem \[ IV.6 \].) For the large class of monotonic, piecewise \( C^1 \) equations of state the \( I_{-1} \)-theorem \[ IV.6 \] comprises the \( I_0 \)-theorem \[ IV.5 \]. This is simply because \( I_{-1}(p) \) is the integral of \( I_0(p) \) (compare proposition \[ III.3 \]).

Remark. One might conjecture that it is possible to achieve further improvements of theorem \[ IV.6 \] by introducing higher integrals of \( I_0(p) \), i.e., \( I_{-2}(p), I_{-3}(p) \), etc. In particular one might think of an extension of theorem \[ IV.6 \] where \( I_{-1}(p) \) is replaced by such an integral. However, already the \( I_{-3} \)-analogue of theorem \[ IV.6 \] seems to be wrong (see the following example). So we may say that theorem \[ IV.6 \] is almost the “best we can get”, at least in the sense that probably no further “integral extension” exists.

Example. Consider the equation of state given by \( \rho(v) = \rho_0 v^6 (1 - |v|^2)^3 \), \( p(v) = -\int \rho(v) dv \). This equation of state is differentiable and monotonic (at least up to \( |v| = 0.87 \)). Applying the \( I_{-1} \)-theorem \[ IV.6 \] we find that \( I_{-1}(v) \) is greater than zero up to \( v_{\text{crit}} = 0.76 \). For all solutions with central potential \( |v_c| \) less than \( v_{\text{crit}} \), theorem \[ IV.6 \] guarantees infinite extent and infinite mass. The hypothetical \( I_{-3} \)-criterion would predict infinite extent for all \( |v_c| \) in the considered range, i.e., for all \( |v_c| \leq v_{\text{max}} = 0.87 \). Now, the numerical observations are the following: Every solution corresponding to central potential \( |v_c| \) less than a critical value \( |v_{\text{crit}}| = 0.86 \) has infinite extent and infinite mass. However, for \( |v_c| > |v_{\text{crit}}| \) the solution possesses finite extent! The numerical results are in accord with theorem \[ IV.6 \] but they contradict the hypothetical \( I_{-3} \)-analogue.

\section{V. EINSTEIN: BASICS}

We will now investigate static perfect fluids in General Relativity. Several quantities and equations which have been relevant in the Newtonian description possess direct analogues in General Relativity. To facilitate comparison we will use the same symbols for corresponding quantities; moreover, wherever permitted, we will investigate the Newtonian limit.

The metric for a static spacetime can be written as
\[ ds^2 = -V^2(x) dt^2 + g_{ij}(x) dx^i dx^j. \] (17)
Here, \( g \) is a Riemannian metric on the 3-space \( M \), \( V \) a (positive) scalar function on \( M \). \( M \) can be viewed as a hypersurface orthogonal to a timelike Killing vector whose norm is \( V \).

A static perfect fluid solution is a regular solution to the Euler-Einstein system, i.e., \((V, g_{ij})\) must satisfy
\[ \Delta V = 4\pi (\rho + 3p) V \] (18a)
\[ R_{ij} = V^{-1} \nabla_i \nabla_j V + 4\pi (\rho - p) g_{ij} \] (18b)
\[ \nabla_i p = -V^{-1} (\rho + p) \nabla_i V. \] (18c)
The covariant derivative \( \nabla_i \) and the Laplacian \( \Delta \) refer to \( g \). Note that (18c) is not an independent equation; it can be derived from the Bianchi identity, using (18b) and (18a). Equations (18) are to be understood in connection with an equation of state \( \rho(p) \) relating the (energy) density and the pressure.

Integrating (18c) we see that the function \( V \) can be regarded as a function of \( p \).
\[ \log Y(p) := \log V(p) - \log V_S = -\int_0^p dp' (\rho(p') + p')^{-1} =: -\Gamma(p) \] (19)
\( Y := V/V_S \) is the “normalized” quantity. For solutions with finite extent, \( V_S \) is the value of \( V \) at the surface \( \{ p = 0 \} \) (such that \( Y = 1 \) there); at the surface the interior metric (17) is joined to an exterior vacuum solution (by standard junction conditions), i.e., a Schwarzschild metric in spherical symmetry. By (19) we see that \( \rho \) and \( p \) can be viewed as functions of \( Y \). If \( \rho(p) \) is monotonic, then \( \rho(Y) \) is monotonic as well. \( p(Y) \) and \( \rho(Y) \) are related by
\[ -\int_1^Y dY' \rho(Y') = pY. \]
Proposition VI.3. Using the same notation as in proposition III.3 we have

\[ \rho \rightarrow 0, \rho'(r) \rightarrow 0 \text{ and } V(r) \rightarrow 1 \text{ as } r \rightarrow \infty \text{ (where } r \text{ is an appropriate radial coordinate).} \]

Hence, \( V = Y \), if the fluid extends to infinity. Like in Newtonian theory \( p(r) \) is a decreasing and \( V(r) \) is an increasing function; their central values are again denoted by \( p_c, V_c \).

Remark. The Newtonian equations follow from (17) and (18). Writing \( V = c^2 \exp(\frac{\rho}{\rho'}) \) we may regard \( u \) as the generalization of the Newtonian potential. E.g., (18a) becomes \( \nabla_i p = -(\rho + \frac{\rho'}{c^2}) \nabla u \). In the limit \( c \rightarrow \infty \) the metric (17) coincides with the Minkowski metric, and, e.g., (18c) becomes \( \partial_i p = -\rho \partial_i u \). Analogously, in the Newtonian limit, the normalized \( Y \) is related to the normalized Newtonian potential \( v \), \( Y = V/V_S = \exp(\frac{\rho - \rho_Y}{c^2}) = \exp(\frac{\rho}{c^2}) \).

VI. EINSTEIN: THE QUANTITIES \( I_i \)

Definition VI.1. Consider a barotropic equation of state \( \rho(p) \) \((p \leq p_{max})\) with \( \rho(p) \) piecewise \( C^0 \) on \([0, p_{max}]\). Let \( \Gamma(p) \) be defined as

\[ \Gamma(p) := \int_0^p dp' (\rho(p') + p')^{-1}. \]  

(20)

Assumption 3. We require that \( \Gamma(p) \) exists. Obviously, \( \Gamma(p) \) is \( C^0[0, p_{max}] \)

Assumption 4. We assume that the limit \( \lim_{p \rightarrow 0} \rho^{-1} p \) exists.

Definition VI.2. Consider an equation of state \( \rho(p) \) as above. We define the following quantities:

\[ J_0(p) := \frac{1}{2p} [1 - e^{-\Gamma}] H_0(p) - 3p^{-1} \int_0^p dp' e^{-\Gamma(p')} \frac{1}{\rho(p') + p'} H_0(p') \]  

(21a)

where \( H_0(p) := \rho(p) + \rho(p)e^{-\Gamma(p)} + 6pe^{-\Gamma(p)} \)

\[ I_0(p) := 1 - e^{-\Gamma(p)} - 6e^{-\Gamma(p)} p \rho^{-1} \]  

(21c)

\[ I_1(p) := 6e^{-\Gamma(p)} \rho^{-2} \frac{dp}{dp} - 5e^{-\Gamma(p + \rho)} \]  

(21d)

\[ I_2(p) := 5(\rho + p) \frac{ds}{dp} + \kappa^2 + 10\kappa \quad (\kappa = \frac{\rho + p}{\rho + 3p \frac{dp}{dp}}) \]  

(21e)

Remark. In complete analogy with section III, \( J_0|_{p=0} = 0 \) and \( I_0|_{p=0} = 0 \). The comments concerning assumption 1 and 2 in section III apply here as well.

Remark. (The Newtonian limit). In the Newtonian limit, i.e., \( c \rightarrow \infty \), \( \Gamma(p) = \int_0^p dp' (\rho(p') + p')^{-1} \) coincides with the Newtonian \( \Gamma(p) = \int_0^p dp' \rho^{-1}(p') \). According to (21b), \( H_0(p) = \rho + p \exp(-\Gamma/c^2) + 6p/c^2 \exp(-\Gamma/c^2) \) becomes \( H_0(p) = 2\rho \) in the limit. Therefore, \( J_0(p) = c^2/(2\rho) [1 - e^{-\Gamma/c^2}] H_0(p) - 3p^{-1} \int_0^p dp' (\cdots) \) approximates \( J_0(p) = \Gamma - 6p^{-1} \rho \), which coincides with (41). Note that also \( I_0(21c) \) coincides with \( J_0 \) in the limit \( c \rightarrow \infty \). Analogously, \( I_1 \rightarrow J_1 \) and the relativistic \( I_2 \approx \Gamma \) approximates the Newtonian \( I_2 \).

Proposition VI.3. Using the same notation as in proposition III.3 we have

\[ \{ I_1 \leq 0 \} \subset \{ I_0 \leq 0 \} \subset \{ J_0 \leq 0 \} \]  

(22a)

\[ \{ I_1 \equiv 0 \} \equiv \{ I_0 \equiv 0 \} \equiv \{ J_0 \equiv 0 \} \]  

(22b)

\[ \{ I_1 \geq 0 \} \subset \{ I_0 \geq 0 \} \subset \{ J_0 \geq 0 \}. \]  

(22c)

Proof. For the first inclusions note that \( I_0 \) is the integral of \( I_1 \), i.e., \( \frac{dp}{dp} I_0(p) = I_1(p), \) and \( I_0|_{p=0} = 0 \). Second, with \( I_0\rho = \rho - \rho Y - 6pY \) we obtain \( H_0 = 2\rho - I_0\rho \), whereby \( J_0 \) becomes \( J_0 = \frac{1}{2p} [(1 + Y)I_0\rho - 6 \int_Y^Y I_0dpdY \). Note that \( Y \in (0,1] \). Therefore, \( I_0 \leq 0 \) implies \( J_0 \leq 0 \) and the other inclusions hold as well.

Remark. \( \{ J_0 \equiv 0 \} \) defines the so-called Buchdahl equation of state \( \frac{\rho}{\rho'} \), i.e., \( p = \frac{1}{2}(\rho^{1/5} - \rho^{1/5})^{-1} \rho^{6/5} \rho < \rho_{-} \).

Remark. The following relations also hold: \( \{ I_2 \leq 0 \} \supset \{ I_1 \equiv 0 \} \) and \( \{ I_2 \leq 0 \} \subset \{ I_1 \leq 0 \} \). For a proof see II.

Comments. In the subsequent sections we will only be concerned with the quantities \( I_0 \) and especially \( J_0 \), \( I_1 \) appears in a theorem which basically states that a solution (not necessarily AFMD) with \( I_1 \leq 0 \) must have finite extent. See II. In General Relativity spherical symmetry of perfect fluid solutions is a rather involved topic II. \( I_2 \) plays the main part in a “symmetry theorem”. From \( I_2 \leq 0 \) it can be concluded that the (asymptotically flat) solution must necessarily be spherically symmetric II.
VII. EINSTEIN: CRITERIA

In analogy to the Newtonian case we formulate criteria for (in)finiteness of static perfect fluid solutions. The proofs involve relativistic generalizations of Pohozaev’s identity.

Remark. The differentiability assumptions for \( p(r), V(r), \) etc. are analogous to the Newtonian case.

Definition VII.1. A solution of (18) is called AFMD, if for some \( \epsilon > 0 \)

\[
1 - V = O^\infty(\|x\|^{-1}) \quad g_{ij} - \delta_{ij} = O^\infty(\|x\|^{-1}) \quad \rho = O^\infty(\|x\|^{-3-\epsilon})
\]

in suitable coordinates \( \{x^i\} \). By (18a), \( p = O^\infty(\|x\|^{-4-\epsilon}) \). For a generalization in terms of Sobolev spaces see [21].

Theorem VII.2. [21]. Consider an equation of state \( p(p) \) being piecewise \( C^0 \) and satisfying assumptions \( 3 \) and \( 4 \).

A Let \( (Y = V/V_S, \rho, p) \) be an AFMD solution to the Euler-Einstein equations (18) with \( p \leq \tilde{p} \) for some \( \tilde{p} > 0 \). If \( I_0 \leq 0 \) (\( I_0 \neq 0 \)) for all \( p \in [0, \tilde{p}] \), then the solution has finite extent.

B If \( I_0 \geq 0 \) (\( I_0 \neq 0 \)) for all \( p \in [0, \tilde{p}] \), then there is no AFMD solution to the Euler-Einstein equations (18) satisfying \( p \leq \tilde{p} \).

Sketch of proof. The proof of A is based on an inequality which can be understood as some kind of "Pohozaev inequality". To establish B a version of the positive mass theorem is used.

Remark. For the Buchdahl case \( I_0 \equiv 0 \), i.e., \( p = \frac{1}{3}(\rho^{1/5} - \rho^{1/5})^{-1} \rho^{6/5} \) \( (\rho < \rho_-) \), we have the following explicit form for the corresponding static perfect fluid solutions [8]:

\[
V(r) = 1 - \frac{M}{\sqrt{\frac{4\pi}{5}\rho_-M^4 + r^2 + M}} \quad g_{ij} = \left( \frac{2}{1 + V(r)} \right)^4 \delta_{ij}.
\]

Here, \( M \) is bounded by \( 3M^{-2} < 16\pi\rho_- \).

Definition VII.3. From \( Y \), its derivatives, and \( p, \rho \) we define the following symmetric tensor \( \sigma_{ij} \):

\[
\sigma_{ij} = \frac{1 - Y^2}{Y} \nabla_a \nabla_c Y + 6Y_a Y_c - 2g_{ac}Y_d Y^d + g_{ac}[-4\pi(1 - Y^2)(\rho + p) - 16\pi p Y(1 - Y)] - 16\pi g_{ij} \int_1^Y dY' H_0(Y'),
\]

where \( H_0(Y) \) is again \( H_0(Y) = \rho + \rho Y + 6pY \).

Proposition VII.4. (Relativistic Pohozaev identity). Assume spherical symmetry. The metric \( g_{ij} \) can be written as \( ds^2 = h(r)dr^2 + r^2d\Omega^2 \), and, moreover, there exists the conformal Killing vector \( \xi_a dx^a = r\sqrt{h}dr \) ("asymptotic dilation"). We have the following identity:

\[
\nabla_i (\sigma_{ij} \xi^j) = -8\pi \frac{1}{\sqrt{h}} [(1 - Y)H_0(Y) + 6 \int_1^Y dY' H_0(Y')] \quad (26)
\]

Proof. \( \nabla_a (\sigma_{ab} \xi_b) = \frac{1}{\sqrt{h}} g_{ab} \). The tensor \( \sigma_{ij} \) is divergence free, \( \nabla_i \sigma^{ij} = 0 \); its trace is \( -8\pi [(1 - Y)H_0(Y) + 6 \int_1^Y dY' H_0(Y')] \).

Proposition VII.5. (Integrated version). Consider a spherically symmetric static perfect fluid solution, i.e., let \( V(r), p(r), \rho(r) \) (with \( g_{ij}(r) \)) be a solution of (18). Assume AFMD. Then,

\[
\int_{\mathbb{R}^3} d^3 x (-\rho J_0) = \frac{M^2}{R} \left( 1 - \frac{2M}{R} \right)^{-1}, \quad \text{for solutions with finite extent, and,}
\]

\[
\int_{\mathbb{R}^3} d^3 x (-\rho J_0) = 0, \quad \text{if the fluid extends to infinity.}
\]

In (27a), \( M \) denotes the mass, and \( R \) the radius of the finite fluid object; \( \{x^i\} \) are the Cartesian coordinates associated with the spherical coordinates \( \{r, \theta, \phi\} \), so that \( d^3 x = r^2 dr d\Omega \).
Proof. From (26) we have $\nabla^i(\sigma_j \xi^j) = -16\pi \frac{1}{\sqrt{h}} \rho J_0$. We integrate the l.h.s. of this equation for fluids with finite extent first.

$$\int_{\text{Ball}(R)} \sqrt{g} d^3x \nabla^i(\sigma_j \xi^j) = 4\pi R^2 \sigma_j |_R \xi^j R^{-1} \frac{1}{\sqrt{h(R)}} x^i = 16\pi R^3 Y^2 |_R h(R)^{-1}$$

The exterior solution is Schwarzschild, i.e., $V^2(r) = (1 - 2M/r)$ and $h(r) = V^{-2}(r)$ ($r \geq R$), so that $Y^2 |_R = V^{-1} | R = V^{-2}M R^{-2}$ (with $V_S = V(R)$). Hence, in (28), $Y^2 |_R h(R)^{-1} = M^2 R^{-4} (1 - 2M/R)^{-1}$ and (27) is established. For solutions extending to infinity, again $\int_{\text{Ball}(r)} \sqrt{g} d^3x \nabla^i(\sigma_j \xi^j) = 4\pi \rho \sigma_j x^i x^j h(r)^{-1}$. Structurally, $\sigma_j$ is mainly built up by terms such as $Y^2 g_{ij}$ or $p_{ij}$ . Taking account of $g_{ij} x^i x^j = r^2 h(r)$ we obtain $\sigma_j x^i x^j r h(r)^{-1} \sim r^3 Y^2_j$, or $r^3 \rho$, or the like. By the AFMD conditions (23) these terms converge to zero as $r \to \infty$. In similar expressions appearing in $\sigma_j$, it is helpful to use $h(r) = (1 - 2m(r)/r)^{-1}$ (where $m(r) = 4\pi \int_0^r drr^2 \rho$), which follows from the field equations (18b).

Eventually, by (23), $\sigma_j x^i x^j r h(r)^{-1} \to 0$ ($r \to \infty$).

**Theorem VII.6.** Assume spherical symmetry. Consider an equation of state $\rho(p)$ being piecewise $C^0$ and satisfying assumptions 1 and 2.

- **A** Let $(Y = V^{-1}_S V, p, \rho)$ be an AFMD solution to the Euler-Einstein equations (18) with $p \leq \tilde{p}$ for some $\tilde{p} > 0$. If $J_0 \leq 0$ ($J_0 \neq 0$) for all $p \in [0, \tilde{p}]$, then the solution has finite extent.

- **B** If $J_0 \geq 0$ ($J_0 \neq 0$) for all $p \in [0, \tilde{p}]$, then there is no AFMD solution to the Euler-Einstein equations (18) satisfying $p \leq \tilde{p}$.

**Proof.** The theorem is a direct consequence of proposition VII.3. \( \blacksquare \)

**Remark.** (Relationship between theorem VII.2 and theorem VII.6.) The $I_0$-theorem VII.2 does not rely on the existence of a conformal Killing vector; spherical symmetry need not be presupposed. In spherical symmetry, however, the $I_0$-theorem is covered by the wider $J_0$-theorem VII.6. See proposition VI.3.

**VIII. EXAMPLES AND DISCUSSION**

**Example.** (The Generalized Buchdahl family of equations of state). Consider the following family of equations of state,

$$p(\rho) = \frac{1}{n+1} \frac{\rho^{n+1}}{\rho^* - \rho^*} \quad (\rho < \rho_-).$$

The case $n = 5$ corresponds to the Buchdahl equation of state (see section VII). Equivalently, (29) can be represented by

$$\rho(Y) = \rho_- (1 - Y)^n \quad p(Y) = \frac{\rho_-}{n+1} Y^{-1} (1 - Y)^{n+1} \quad (Y \in (0, 1]).$$

From (21a) and (21b) we can compute $J_0(Y)$ which results in

$$J_0(Y) = \left[1 - \frac{6}{n+1}\right] (1 - Y) \left[1 - \frac{1}{2} (1 - Y) (1 - \frac{6}{n+2})\right].$$

The last bracket is greater than zero for all $Y \in [0, 1]$, so we obtain

- $n > 5 \implies J_0 > 0$
- $n = 5 \implies J_0 \equiv 0$
- $n < 5 \implies J_0 < 0$.

Consequently, theorem VII.6 applies, i.e., we get finiteness of the fluid configuration for $n < 5$ and infiniteness for $n > 5$. Note incidentally that this result could also have been obtained using the weaker $I_0$-criterion VII.2.

**Remark.** Note that VII.2 also holds for the polytropes in Newtonian theory. We see that in this sense the Generalized Buchdahl family is the relativistic analogue of the (Newtonian) polytropes.
Example. (The polytropes in General Relativity). The polytropic equations of state, \( p(\rho) = (n+1)^{-1}\rho^{-1/n}\rho^{(n+1)/n} \), have the following \([\rho(Y), p(Y)]\) form:

\[
\rho = \rho_-(n+1)^n(Y^{-\frac{1}{n+1}} - 1)^n \quad p = (n+1)^n\rho_-(Y^{-\frac{1}{n+1}} - 1)^{n+1}
\]

With some support of a computer algebra program one can show that

- \( n > 5 \Rightarrow J_0 > 0 \) (\( \forall Y \))
- \( n = 5 \Rightarrow J_0 > 0 \) (\( \forall Y \))
- \( n < 5 \Rightarrow J_0 < 0 \) (\( Y \geq Y_0(n) \))

Here, \( Y_0(n) \) is some value of \( Y \), where \( J_0(Y) \) changes sign; e.g., \( Y_0|_{n=5} \approx 0.64 \). For \( n \to 5^- \), \( Y_0(n) \) approximates 1. Obviously, for solutions with central “potential” \( Y_0 > Y_0 \), theorem VII.6 applies (predicting finiteness of solutions).

Remark. Applying the \( I_0 \)-criterion (theorem VII.3) to the last example we obtain qualitatively similar results. However, since it is weaker, the breakdown of the \( I_0 \)-criterion occurs somewhat earlier. Note further that for certain values \( Y_0 < Y_0 \) one indeed finds solutions which are not finite but still AFMD [13].

Example. (Asymptotically polytropic equation of state). In Newtonian theory asymptotically polytropic equations of state can be written as \( \rho(v) = K|v|^n(1 + O(|v|^m)) \) (for some \( m > 0 \)). For \( m \geq 1 \) they coincide with the quasipolytropic equations of state [14, 21] (see below). In analogy to above, up to a certain central density, theorems IV.5 and IV.6 ensure finiteness/infiniteness (for \( n < 5/n > 5 \) respectively). Again, for certain values exceeding the critical central density, one finds (AFMD) solutions extending to infinity or finite solutions (for \( n < 5 \) or \( n > 5 \) respectively). Compare partly with the remarks at the end of section IV. Note that these examples indicate that the breakdown of the \( I_0 \) or \( I_{-1} \)-criterion is not at all artificial; on the contrary it seems to be an important feature, anticipating the appearance of the counterintuitive behavior we observe for higher densities.

Example. (Degenerate matter). An equation of state satisfying \( p = K\rho^{(n+1)/n}(1 + O(\rho^{1/n})) \) (as \( \rho \to 0 \)) is called quasipolytropic [21]. Note the following important result: Spherically symmetric perfect fluid solutions corresponding to quasipolytropic equations of state with \( n < 3 \) have finite extent [14]. Degenerate matter is usually described by quasipolytropic equations of state with \( n = 3/2 \), so we know that the corresponding star model is finite. In this connection it is instructive to investigate how restrictive the \( J_0 \)-criterion is. Consider the equation of state of a completely degenerate, ideal Fermi gas (see, e.g., [19]). For degenerate electrons (white dwarfs) \( J_0 \) is negative for all admissible densities (and far beyond), i.e., as long as the energy density is dominated by the rest mass of the ions. For neutron stars (degenerate neutrons) \( J_0 \leq 0 \) is valid up to densities of \( 10^{17}\text{g cm}^{-3} \) which is beyond the top end of neutron star densities. As an example for a more realistic neutron star model take the Harrison-Wheeler equation of state (see, e.g., [14] for an introduction). \( J_0 \leq 0 \) holds for \( \rho \leq 7 \times 10^{11}\text{g cm}^{-3} \). This is slightly beyond the density region where “neutron drip” occurs. Sometimes the Harrison-Wheeler equation of state is replaced by different equations of state for such densities, so the outcome is satisfying enough also in this case. Summing up we see that the \( J_0 \)-theorem VII.6 is rather generally applicable also for these (already known) situations.

IX. POHOZAEV IDENTITIES

The present section provides the necessary background material to understand how “Pohozaev-like” identities come into existence: We outline a method of constructing such identities. In particular, we re-derive Pohozaev’s identity [10] in the Newtonian case (see equation (8)) and we construct its direct analogue for the relativistic case (see (26)).

Basics. Let \((M, g)\) be a Riemannian manifold, \( \dim M = 3 \), with Ricci tensor \( R_{ij} \) and curvature scalar \( R \). Assume that \( M \) possesses a conformal Killing vector \( \xi \), \( \nabla_{(a}\xi_{b)} = \frac{1}{3}R^{c}_{ab}g_{ab} \). Consider the conformal rescaling

\[
\tilde{g}_{ij} = \Omega^2 g_{ij}.
\]

We define

\[
\tilde{R}_{ij} := \tilde{\nabla}^lR_{lij} - \frac{1}{2}\tilde{g}_{ij}\tilde{R}, \quad \tau_{ij} := \Omega^{-1}\tilde{R}_{ij}.
\]

The tensor \( \tilde{\tau} \) is divergence free (with respect to \( \tilde{\nabla} \)) and symmetric, thus \( \nabla_i\tau^{ij} = \Omega^{-1}\tau_k^i g^{ij}\nabla_i\Omega \), wherefore

\[
\nabla_i(\tau^{ij}\xi_j) = \tau^k_i \left[ \frac{1}{3}(\nabla^l\xi_l) + \Omega^{-1}g^{ij}(\nabla_i\Omega)\xi_j \right].
\]
Here, $\Omega$ abbreviates $\Omega = 2\Omega_R$. If this expression can be evaluated at $x = 0$, then we assume spherical symmetry. This is because we are only interested in the trace of $\hat g_{ij}$, i.e., $\Omega = \Omega(v)$.

\[ \hat g_{ij} = \Omega^2 g_{ij} \quad g_{ij} = \delta_{ij}. \]  

From (36) we get the following for the tensor $\tau_{ij}$ (we use the notation $\Omega' = \frac{d\Omega}{dv}$):

\[ \tau_{ij} = (2\Omega^{-1}\Omega'' - \Omega')v_iv_j - \Omega'v_{ij} + (\Omega'' - \Omega^{-1}\Omega'^2)v^k v_{j,k} \delta_{ij} + \Omega'\Delta v \delta_{ij}. \]  

On $(\mathbb{R}^3, \delta_{ij})$ we may use the dilation $\xi_i = x_i$ as the conformal Killing vector, i.e., $\partial_{(a} x_{b)} = \delta_{ab}$, and (37) becomes

\[ \partial_i(\tau^{ij} x_j) = \tau^k_k + \Omega^{-1} \tau^k_k x^i \Omega_{i*,k}. \]  

Based on (41) the idea is to modify the tensor $\tau_{ij}$ in order to get a r.h. side of the form (38). We define

\[ \sigma_{ij} := \tau_{ij} + d_{ij} \]  

\[ d_{ij} := d_1 \delta_{ij} + d_2 \frac{v_i v_j}{x^2} + d_3 v_i v_j. \]  

In (42b) the $d_i$ are functions $d_i(v)$, $\xi_i = x_i$ is the conformal Killing vector, $r$ its norm. Hence, $d_{ij}$ is a tensor consisting of functions of $v$, derivatives of $v$ and the conformal Killing vector. The ansatz (42b) for $d_{ij}$ is general provided that we assume spherical symmetry. This is because we are only interested in the trace of $d_{ij}$ and the expression $x^j \partial^i d_{ij}$ (see below). Terms like $v_{ij}$, $v^k v_{ij}$, or $x_i x_j$ can be subsumed under the ansatz (42b).

Replacing $\tau_{ij}$ by $\sigma_{ij}$ in (41) we obtain $\partial_i(\sigma^{ij} x_j) = \tau^k_k + \Omega^{-1} \tau^k_k x^i \Omega_{i*,k} + d^k_k + x^j \partial^i d_{ij}$, i.e., explicitly,

\[ \partial_i(\sigma^{ij} x_j) = 8\pi \rho \Omega'(v) + 3d_1(v) + 4\pi \rho d_2(v) + \]  

\[ + v_r^2 [\Omega + d_2 - d_3] + rv_r [8\pi \rho \Omega^{-1}\Omega'^2 + 2d_1' + 8\pi \rho d_3] + \]  

\[ + rv_r^2 [\Omega^{-1}\Omega' + d_3']. \]

Here, $\Omega$ abbreviates $\hat \Omega = 2\Omega'' - \Omega^{-1}\Omega'^2$. Without loss of generality we have restricted ourselves to spherical symmetry.

For a large class of $\Omega(v)$ and the free functions $d_i(v)$, the r.h. side of equation (38) is exactly of the form (38). Integrating the total divergence on the l.h. side we obtain

\[ \frac{1}{4\pi} \int_{\text{Ball}(r)} d^3x \partial^i(\sigma_{ij} x^j) = r^3 v_r^2 \Omega^{-1}\Omega'^2 + 2r^2 v_r \Omega' + r^3 [d_1 + \frac{v_r}{r} d_2 + v_r^2 d_3]. \]  

If this expression can be evaluated at $r = R$ for finite fluid solutions, then (38) is a candidate for a sensible Pohozaev-like identity. For AFMD solutions the existence of the limit $r \to \infty$ of (44) should be investigated.

**Example.** (Pohozaev identity). We choose $\Omega(v) = v^2$. Note that $\Omega = 0$. Choose $d_2 = 0$ and $d_3 = 0$, so (38) becomes

\[ \partial_i(\sigma^{ij} x_j) = 16\pi \rho v + 3d_1(v) + rv_r [32\pi \rho + d_1']. \]  

The last bracket vanishes if $d_1(v) = 32\pi \rho$, so that we obtain

\[ \partial_i(\sigma^{ij} x_j) = 16\pi (\rho v + 6p), \]  

which coincides with the Pohozaev identity (8). Moreover, as expected, $\sigma_{ij}$ (as given by (42) which is essentially (38)) coincides with (8).
Remark. An even wider class of identities could be constructed admitting more general conformal factors in (33), e.g., let \( \Omega = \Omega(v, v_r) \). In this way one could possibly obtain identities adapted to specific problems (i.e., classes of equations of state).

Relativistic Pohozaev identity. The relativistic case can be treated in analogy to the Newtonian one. However, in order to simplify the presentation we confine ourselves to a certain choice of the conformal factor \( \Omega \) from the beginning. The metric for a spherically symmetric 3-space can be written as

\[
g_{ij} dx^i dx^j = h(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (h > 0) ,
\]

Such a metric is conformally flat and possesses the conformal Killing vector \( \xi^a \),

\[
\xi_a dx^a = r \sqrt{h} dr , \quad \nabla_a \xi_b = \frac{1}{\sqrt{h}} g_{ab} .
\]

For perfect fluid solutions \( h(r) = (1 - \frac{2m(r)}{r})^{-1} \), where \( m(r) = 4\pi \int_0^r dr' r'^2 \rho \), so that for AFMD solutions - \( \xi \) can be viewed as an “asymptotic dilation”.

We make the following choice for the conformal factor \( \Omega \):

\[
\Omega(Y) = (1 - Y)^2
\]

From (36) we calculate the tensor \( \tau_{ac} \),

\[
\tau_{ac} = \frac{1 - Y^2}{Y} \nabla_a \nabla_c Y + 6Y_a Y_c - 2g_{ac} Y_{d} Y_{d} +
+ g_{ac} [ -4\pi (1 - Y^2) (\rho + p) - 16\pi p Y (1 - Y) ]
\]

With \( \sigma_{ij} := \tau_{ij} + d_{ij} \) and the simplified ansatz \( d_{ij} = d_1(Y) g_{ij} \) we modify equation (33) in order to get

\[
\nabla_i (\sigma^{ij} \xi_j) = \frac{1}{\sqrt{h}} \tau_k^i + \Omega^{-1} \tau_k^i \sigma^{ij} \xi_j + \frac{1}{\sqrt{h}} d_k^i + \xi^i \nabla^j d_{ij}
\]

\[
\quad = \frac{1}{\sqrt{h}} (\tau_k^i + d_k^i) + h^{-1} Y_r \xi_r \left[ \frac{d}{dY} d_1(Y) + 16\pi \rho + 16\pi \rho Y + 96\pi p Y \right] .
\]

Equation (51b) suggests the choice \( \frac{d}{dY} d_1(Y) = -16\pi H_0(Y) = -16\pi \rho - 16\pi \rho Y - 96\pi p Y \). Consequently, \( d_1(Y) = -16\pi \int dY H_0(Y) \), and

\[
\nabla_i (\sigma^{ij} \xi_j) = -8\pi \left[ (1 - Y) H_0(Y) + 6 \int_1^Y dY' H_0(Y') \right]
\]

with \( H_0(Y) = \rho + \rho Y + 6p Y \).

Equation (52) is the relativistic counterpart of Pohozaev's identity (see (20)). Recall that the tensor \( \sigma_{ij} \) is given by \( \sigma_{ij} = \tau_{ij} + d_{ij} \), i.e.,

\[
\sigma_{ij} = (50) - 16\pi g_{ij} \int_1^Y dY' H_0(Y') .
\]

Remark. In order to simplify the presentation we have chosen a particular conformal factor in (49). Obviously, without specifying \( \Omega(Y) \), via a more general ansatz for \( d_{ij} \) (as in (122)), we would arrive at relativistic analogues of (33). Since we have not used such identities in the paper, we refrain from presenting the (lengthy) formulae here.

**APPENDIX A: MATHEMATICAL PROPERTIES OF \( \rho(p) \)**

Assumptions 1 and 2 in section IV are independent. From the existence of \( \Gamma(p) \), by (A2), we may conclude that \( \rho p^{-1} \) cannot be bounded as \( p \to 0 \). However, \( \lim_{p \to 0} \rho^{-1} p \) need not exist as is shown by the following example. Define a \( C^\infty \) function \( s(p) \) with the properties \( 0 \leq s \leq 1 \), \( s(p_\alpha) = 1 \), \( \text{supp} s \subseteq \bigcup_n [p_n - p_n^2, p_n + p_n^2] \); where \( p_n = 2^{-n} \) \( (n \in \mathbb{N}) \). Consider the equation of state \( \rho(p) = p (s(p) + p^2) \) \( (0 < \epsilon < 1) \). \( \rho \) is in \( C^\infty(0, 1) \) and in \( C^\theta[0, 1] \), with
\( \rho(p = 0) = 0. \) Furthermore, \( \Gamma(p) \) exists, as can easily be seen. However, the limit \( \rho^{-1}p \) as \( p \to 0 \) does not exist. Conversely, take the equation of state \( \rho(p) = -p \log p \) \((p \leq p_{\text{max}} < 1)\). The limit \( \lim_{p \to 0} \rho^{-1}p \) exists. However, 
\[
\int_0^p dp' \rho^{-1}(p' \log(p'))^{-1} = \log(-\log p) - \log(-\log \epsilon) \quad \text{diverges as } \epsilon \to 0, \quad \text{so that } \Gamma(p) \text{ does not exist.}
\]

If the equation of state \( \rho(p) \) is monotonic on \([0, \epsilon)\), then assumption \( \square \) follows from assumption \( 
\]

To show this note that 
\[
\Gamma(p) = \int_0^p dp' \rho^{-1}(p') \geq \rho^{-1}(p) \int_0^p dp' = \rho^{-1}(p)p. \tag{A1}
\]

This is because \( \rho^{-1}(p') \geq \rho^{-1}(p) \forall p' \leq p. \) Letting \( p \to 0 \) the claim is established.

Evidently, if \( \square \) and \( \square \) hold, \( \lim_{p \to 0} \rho^{-1}p = 0 \). For \( \rho|_{p=0} > 0 \) this is obvious, so let us indirectly assume that \( \rho|_{p=0} = 0 \) with \( \lim_{p \to 0} \rho^{-1}p = c \neq 0 \). Choose \( \bar{p} > 0 \), such that \( \rho^{-1}p > c/2 \forall p < \bar{p} \).

\[
\infty > \int_0^{\bar{p}} \rho^{-1}(p)dp = \int_0^{\bar{p}} \rho^{-1}(p)pp^{-1}dp > \frac{c}{2} \int_0^{\bar{p}} pp^{-1}dp = \infty \tag{A2}
\]

A contradiction.

If it exists, \( \lim_{p \to 0} (\frac{dp}{dp'})^{-1} = 0 \); this is shown by simply applying de l’Hospital’s rule. Note, however, that there are monotonic equations of state satisfying assumption \( \square \) (and, consequently, assumption \( \square \)), whose limit \( \lim_{p \to 0} \frac{dp}{dp'} \) is not defined. As an example, take \( \rho(p) = \int_0^p p^{-1/(n+1)}(1 - s(p))dp \) (where \( s(p) \) is defined as above).

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