Abstract. We will introduce the basics of dyadic harmonic analysis and how it can be used to obtain weighted estimates for classical Calderón-Zygmund singular integral operators and their commutators. Harmonic analysts have used dyadic models for many years as a first step towards the understanding of more complex continuous operators. In 2000 Stefanie Petermichl discovered a representation formula for the venerable Hilbert transform as an average (over grids) of dyadic shift operators, allowing her to reduce arguments to finding estimates for these simpler dyadic models. For the next decade the technique used to get sharp weighted inequalities was the Bellman function method introduced by Nazarov, Treil, and Volberg, paired with sharp extrapolation by Dragičević et al. Other methods where introduced by Hytönen, Lerner, Cruz-Uribe, Martell, Pérez, Lacey, Reguera, Sawyer, Uriarte-Tuero, involving stopping time and median oscillation arguments, precursors of the very successful domination by positive sparse operators methodology. The culmination of this work was Tuomas Hytönen’s 2012 proof of the $A_2$ conjecture based on a representation formula for any Calderón-Zygmund operator as an average of appropriate dyadic operators. Since then domination by sparse dyadic operators has taken central stage and has found applications well beyond Hytönen’s $A_p$ theorem. We will survey this remarkable progression and more in these lecture notes.

Contents

1. Introduction 2
2. Weighted Norm Inequalities 6
2.1. Some basic notation and prototypical operators 7
2.2. Hilbert transform 8
2.3. Maximal function 11
2.4. Why are we interested in these estimates? 13
2.5. First Linear Estimates 13
2.6. Extrapolation and Hytönen’s $A_p$ theorem 14
2.7. Two-weight problem for the Hilbert transform and the maximal function 16
3. Dyadic harmonic analysis 18
3.1. Dyadic intervals, dyadic maximal functions 18
3.2. One-third trick and Lerner’s proof of Buckley’s result 19
3.3. Random dyadic grids on $\mathbb{R}$ 21
3.4. Haar bases 22
3.5. Dyadic analysis on spaces of homogeneous type 24
4. Dyadic operators, weighted inequalities, and Hytönen’s representation theorem 30

2010 Mathematics Subject Classification. Primary 42B20, 42B25 ; Secondary 47B38.
Key words and phrases. Weighted norm estimate, Hilbert transform, commutators, Dyadic operators, $A_p$-weights, Carleson sequences, Bellman functions, sparse operators.
1. Introduction

These notes are based on lectures delivered by the author on August 7-9, 2017 at the CIMPA 2017 Research School – IX Escuela Santaló: Harmonic Analysis, Geometric Measure Theory and Applications, held in Buenos Aires, Argentina. The course was titled "Dyadic Harmonic Analysis and Weighted Inequalities".

The main question of interest in these notes is to decide for a given operator or class of operators and a pair of weights \((u, v)\), if there is a positive constant, \(C_p(u, v, T)\), such that

\[
\|Tf\|_{L^p(v)} \leq C_p(u, v, T) \|f\|_{L^p(u)} \quad \text{for all functions } f \in L^p(u).
\]

The main goals in these lectures are two-fold. First, given an operator \(T\) (or family of operators), identify and classify pairs of weights \((u, v)\) for which the operator(s) \(T\) is(are) bounded on weighted Lebesgue spaces, more specifically from \(L^p(u)\) to \(L^p(v)\) – qualitative bounds –. Second, understand the nature of the constant \(C_p(u, v, T)\) – quantitative bounds –.

We concentrate on one-weight \(L^p\) inequalities for \(1 < p < \infty\), that is the case when \(u = v = w\), for the prototypical operators, dyadic models, and their commutators, although we will state some of the known two-weight results. The operators we will focus on are the Hardy-Littlewood maximal function; Calderón-Zygmund operators \(T\), such as the Hilbert transform.
and their dyadic analogues, specifically the dyadic maximal function, the martingale transform, the dyadic square function, the Haar shift multipliers, the dyadic paraproducts, and the sparse dyadic operators.

The question now reduces to: Given weight $w$ and $1 < p < \infty$, is there a constant $C_p(w, T) > 0$ such that for all functions $f \in L^p(w)$

$$
\|T f\|_{L^p(w)} \leq C_p(w, T) \|f\|_{L^p(w)}.
$$

We have known since the 70’s that the maximal function is bounded on $L^p(w)$ if and only if the weight $w$ is in the Muckenhoupt $A_p$ class [Mu], similar result holds for the Hilbert transform [HMW]. General Calderón-Zygmund operators and dyadic analogues are bounded on $L^p(w)$ [CoFe] when the weight $w \in A_p$ and the same holds for their commutators with functions in the space of bounded mean oscillation (BMO) [Bl, ABKPz]. The quantitative versions of these results were obtained several decades later, in 1993 for the maximal function [Bu1], in 2007 for the Hilbert transform [Pet2], in 2012 for Calderón-Zygmund singular integral operators [Hyt2] and for their commutators [ChPPz]. We will say more about $A_p$ weights and the quantitative versions of these classical results in the following pages.

We will show or at least describe, for the model operators $T$, the validity of a weighted $L^2$ inequality that is linear on $[w]_{A_2}$, the $A_2$ characteristic of the weight, namely there is a constant $C > 0$ such that for all weights $w \in A_2$ and for all functions $f \in L^2(w)$

$$
\|T f\|_{L^2(w)} \leq C [w]_{A_2} \|f\|_{L^2(w)}.
$$

That this holds for all Calderón-Zygmund singular integrals operators was the $A_2$ conjecture. We will also describe several approaches for the corresponding quadratic estimate for the commutator $[b, T] = bT - Tb$ where $b$ is a function in BMO, namely

$$
\|[b, T] f\|_{L^2(w)} \leq C [w]_{A_2}^2 \|b\|_{\text{BMO}} \|f\|_{L^2(w)}.
$$

Dyadic models have been used in harmonic analysis and other areas of mathematics for a long time, Terry Tao has an interesting post in his blog$^1$ regarding the ubiquitous "dyadic model". For a presentation suitable for beginners, see the lecture notes by the author [P1], which describe the status quo of dyadic harmonic analysis and weighted inequalities as of 2000. This millennium has seen new dyadic techniques evolve, become mainstream, and help settle old problems, these lecture notes try to illustrate some of this progress. In particular averaging and sparse domination techniques with and by dyadic operators have allowed researchers to transfer results from the dyadic world to the continuous world. No longer the dyadic models are just toy models in harmonic analysis, they can truly inform the continuous models. Here are some examples where this dyadic paradigm has been useful.

The dyadic maximal function controls the maximal function (the converse is immediate) by means of the one-third trick. Estimates for the dyadic maximal function are easier to obtain and transfer to the maximal function painlessly.

The Walsh model is the dyadic counterpart to Fourier analysis. The first real progress towards proving boundedness of the bilinear Hilbert transform [LTh], result that earned

$^1$https://terrytao.wordpress.com/2007/07/27/dyadic-models/
Christoph Thiele and Michael Lacey the 1996 Salem Prize\textsuperscript{2}, was made by Thiele in his 1995 PhD thesis proving the Walsh model version of such result [Th].

Stefanie Petermichl showed in 2000 that one can write the Hilbert transform as an "average of dyadic shift operators" over random dyadic grids [Pet1]. She achieved this using the well-known symmetry properties that characterize the Hilbert transform. Namely, the Hilbert transform commutes with translations and dilations, and anticommutes with reflections. A linear and bounded operator on $L^2(\mathbb{R})$ with those properties must be a constant multiple of the Hilbert transform. Similarly, the Riesz transforms [Pet3] can be written as averages of suitable dyadic operators. Petermichl proved the $A_2$ conjecture for these dyadic operators using Bellman function techniques [Pet2, Pet3]. These results added a very precise new dyadic perspective to such classic and well-studied operators in harmonic analysis and earned Petermichl the 2006 Salem Prize, first time this prize was awarded to a female mathematician.

The Martingale transform was considered the dyadic toy model "par excellence" for Calderón-Zygmund singular integral operators. For many years one would test the martingale transform first and, if successful, then worry about the continuous versions. In 2000, Janine Wittwer proved the $A_2$ conjecture for the martingale transform using Bellman functions [W1]. The Beurling transform can be written as an average of martingale transforms in the complex plane, and this allowed Stefanie Petermichl and Sasha Volberg [PetV] to prove in 2002 linear weighted inequalities on $L^p(w)$ for $p \geq 2$, and as a consequence deduce an important end-point result in the theory of quasiconformal mappings that had been conjectured by Kari Astala, Tadeusz Iwaniec, and Eero Saksman [AIS].

Surprisingly, all Calderón-Zygmund singular integral operators, can be written as averages of Haar shift dyadic operators of arbitrary complexity and dyadic paraproducts as proven by Tuomas Hytönen [Hyt2]. In 2008, Oleksandra Beznosova proved the $A_2$ conjecture for the dyadic paraproduct [Be2] and, together with Hytönen's dyadic representation theorem, this lead to Hytönen's proof of the full $A_2$ conjecture [Hyt2].

Leading towards Hytönen's result there were a number of breakthroughs that have recently coalesced under the umbrella of "domination by finitely many sparse positive dyadic operators". Andrei Lerner's early results [Le5] played a central role in this development. It is usually straightforward to verify that these sparse operators have desired (quantitative) estimates, it is harder to prove appropriate domination results for each particular operator and function it acts on. This methodology has seen an explosion of applications well-beyond the original $A_2$ conjecture where it originated. Identifying the sparse collections associated to a given operator and function is the most difficult part of the argument and it involves using weak-type inequalities, stopping time techniques, and adjacent dyadic grids.

We will explore some of these examples in the lecture notes with emphasis on quantitative weighted estimates. We will illustrate in a few case studies different techniques that have evolved as a result of these investigations such as Bellman functions, quantitative extrapolation and transference theorems, and reduction to studying dyadic operators either by averaging or by sparse domination.

The structure of the lecture notes remains faithful to the lectures delivered by the author in Buenos Aires except for some minor reorganization. Some themes are touched at the beginning, to wet the appetite of the audience, and are expanded on later sections. Most

\textsuperscript{2}The Salem Prize, founded by the widow of Raphael Salem, is awarded every year to a young mathematician judged to have done outstanding work in Salem’s field of interest, primarily the theory of Fourier series. The prize is considered highly prestigious and many Fields Medalists previously received Salem prize (Wikipedia)
objects are defined as they make their first appearance in the story. Naturally more details are provided than in the actual lectures, some details were in the original slides, but had to be skipped or fast forwarded, those topics are included in these lecture notes. The sections are peppered with historical remarks and references, but inevitably some will be missing or could be inaccurate despite the time and effort spent by the author on them. Thus, the author apologizes in advance for any inaccuracy or omission, and gratefully would like to hear about any corrections for future reference.

In Section 2, we introduce the basic model operators: the Hilbert transform and the maximal function and we discuss their classical $L^p$ and weighted $L^p$ boundedness properties. We show that $A_p$ is a necessary condition for the boundedness of the maximal function on weighted Lebesgue spaces $L^p$. We describe why are we interested on weighted estimates, and more recently on quantitative weighted estimates. In particular we describe the linear weighted $L^2$ estimates saga leading towards the resolution of the $A_2$ conjecture and how to derive quantitative weighted $L^p$ estimates using sharp extrapolation. We finalize the section with a brief summary of the two-weight results known for the Hilbert transform and the maximal function.

In Section 3, we introduce the elements of dyadic harmonic analysis and the basic dyadic maximal function. More precisely we discuss dyadic grids (regular, random, adjacent) and Haar functions (on the line, on $\mathbb{R}^d$, on spaces of homogeneous type). As a first example, illustrating the power of the dyadic techniques, we present Lerner’s proof of Buckley’s quantitative $L^p$ estimates for the maximal function, which reduces, using the one-third trick, to estimates for the dyadic maximal function. We also describe, given dyadic cubes on spaces of homogeneous type, how to construct corresponding Haar bases, and briefly describe the Auscher-Hytönen "wavelets" in this setting.

In Section 4, we discuss the basic dyadic operators: the martingale transform, the dyadic square function, the Haar shifts multipliers (Petermichl’s and those of arbitrary complexity), and the dyadic paraproducts. These are the ingredients needed to state Petermichl’s and Hytönen’s representation theorems for the Hilbert transform and Calderón-Zygmund operators respectively. For each of these dyadic model operators we describe the known $L^p$ and weighted $L^p$ theory and we state both Petermichl’s and Hytönen’s representation theorems.

In Section 5, we sketch Beznosova’s proof of the $A_2$ conjecture for the dyadic paraproduct, this is a Bellman function argument. As a first approach we get a $3/2$ estimate, and with a refinement the linear estimate for the dyadic paraproduct is obtained. Along the way we introduce weighted Carleson sequences, a weighted Carleson embedding lemma, some Bellman function lemmas: the Little Lemma and the $\alpha$-Lemma, and weighted Haar functions needed in the argument, we also sketch the proofs of these auxiliary results.

In Section 6, we discuss weighted inequalities in a case study: the commutator of the Hilbert transform $H$ with a function $b$ in BMO. We summarize chronologically the weighted norm inequalities known for the commutator. We sketch the dyadic proof of the quantitative weighted $L^2$ estimate for the commutator $[b, H]$ due to Daewon Chung, yielding the optimal quadratic dependence on the $A_2$ characteristic of the weight. We discuss a very useful transference theorem of Daewon Chung, Carlos Pérez and the author, and present its proof based on the celebrated Coifman–Rochberg–Weiss argument. The transference theorem allows to deduce quantitative weighted $L^p$ estimates for the commutator of a linear operator with a BMO function, from given weighted $L^p$ estimates for the operator.
In Section 7, we introduce the sparse domination by positive dyadic operators paradigm that has emerged and proven to be very powerful with applications in many areas not only weighted inequalities. We discuss a characterization of sparse families of cubes via Carleson families of dyadic cubes due to Andrei Lerner and Fedja Nazarov. We present the beautiful proof of the $A_2$ conjecture for sparse operators due to David Cruz-Uribe, Chema Martell, and Carlos Pérez. We illustrate with one toy model example, the martingale transform, how to achieve the pointwise domination by sparse operators following an argument by Michael Lacey. Finally we briefly discuss a sparse domination theorem for commutators valid for (rough) Calderón-Zygmund singular integral operators due to Andrei Lerner, Sheldon Ombrosi, and Israel Rivera-Ríos that yields a new quantitative two weight estimates of Bloom type, and recovers all known weighted results for the commutators.

Finally, in Section 8 we present a summary and briefly discuss some very recent progress.

Acknowledgements: I would like to thank Ursula Molter, Carlos Cabrelli, and all the organizers of the CIMPA 2017 Research School – IX Escuela Santaló: Harmonic Analysis, Geometric Measure Theory and Applications, held in Buenos Aires, Argentina from July 31 to August 11, 2017, for the invitation to give the course on which these lecture notes are based. It meant a lot to me to teach in the "Pabellón 1 de la Facultad de Ciencias Exactas", having grown up hearing stories about the mythical Universidad de Buenos Aires (UBA) from my parents, Concepción Ballester and Victor Pereyra, and dear friends 3 who, like us, were welcomed in Venezuela in the late 60’s and 70’s, and to whom I would like to dedicate these lecture notes. Unfortunately the flow is now being reversed as many Venezuelans of all walks of life are fleeing their country and many, among them mathematicians and scientists, are finding a home in other South American countries, in particular in Argentina. I would also like to thank the enthusiastic students and other attendants, as always, there is no course without an audience, you are always an inspiration for us. I thank the kind referee, who made many comments that greatly improved the presentation, and my former PhD student David Weirich, who kindly provided the figures. Last but not least, I would like to thank my husband, who looked after our boys while I was traveling, and my family in Buenos Aires who lodged and fed me.

Acknowledgements: I would like to thank Ursula Molter, Carlos Cabrelli, and all the organizers of the CIMPA 2017 Research School – IX Escuela Santaló: Harmonic Analysis, Geometric Measure Theory and Applications, held in Buenos Aires, Argentina from July 31 to August 11, 2017, for the invitation to give the course on which these lecture notes are based. It meant a lot to me to teach in the "Pabellón 1 de la Facultad de Ciencias Exactas", having grown up hearing stories about the mythical Universidad de Buenos Aires (UBA) from my parents, Concepción Ballester and Victor Pereyra, and dear friends 3 who, like us, were welcomed in Venezuela in the late 60’s and 70’s, and to whom I would like to dedicate these lecture notes. Unfortunately the flow is now being reversed as many Venezuelans of all walks of life are fleeing their country and many, among them mathematicians and scientists, are finding a home in other South American countries, in particular in Argentina. I would also like to thank the enthusiastic students and other attendants, as always, there is no course without an audience, you are always an inspiration for us. I thank the kind referee, who made many comments that greatly improved the presentation, and my former PhD student David Weirich, who kindly provided the figures. Last but not least, I would like to thank my husband, who looked after our boys while I was traveling, and my family in Buenos Aires who lodged and fed me.

2. Weighted Norm Inequalities

In this section, we introduce some basic notation and the model operators: the Hilbert transform and the maximal function and we discuss their classical $L^p$ and weighted $L^p$ boundedness properties. We show that $A_p$ is a necessary condition for the boundedness of the maximal function on weighted $L^p$. We describe why are we interested in weighted estimates, and more recently on quantitative weighted estimates. In particular we describe the linear weighted $L^2$ estimates saga leading towards the resolution of the $A_2$ conjecture and how to

3Dear friends such as Julián and Amanda Araoz, Manolo Bemporad, Mischa and Yanny Cotlar, Rebeca Guber, Mauricio and Gloria Milchberg, Cora Ratto and Manuel Sadosky, Cora Sadosky and Daniel Goldstein, and Cristina Zoltan, some sadly no longer with us.
derive quantitative weighted $L^p$ estimates using sharp extrapolation. We finalize the section with a brief summary of the two-weight results known for the Hilbert transform and the maximal function.

2.1. Some basic notation and prototypical operators. We introduce some basic notation used throughout the lecture notes. We remind the reader the basic spaces (weighted $L^p$ and bounded mean oscillation, BMO), and the prototypical continuous operators to be studied, namely the maximal function, the Hilbert transform and its commutator with functions in BMO. We briefly recall some of the settings where these operators appear.

The weights $u$ and $v$ are locally integrable functions on $\mathbb{R}^d$, namely $u, v \in L^1_{\text{loc}}(\mathbb{R}^d)$, that are almost everywhere positive functions.

Given a weight $u$, a measurable function $f$ is in $L^p(u)$ if and only if

$$\|f\|_{L^p(u)} := \left( \int_{\mathbb{R}^d} |f(x)|^p u(x) \, dx \right)^{1/p} < \infty.$$  

When $u \equiv 1$ we denote $L^p(\mathbb{R}^d) = L^p(u)$ and $\|f\|_{L^p} := \|f\|_{L^p(\mathbb{R}^d)}$.

Given $f, g \in L^1(\mathbb{R}^d)$ their convolution is given by

$$f \ast g(x) = \int_{\mathbb{R}^d} f(x-y) g(y) \, dy.$$  

A locally integrable function $b$ is in the space of bounded mean oscillation, namely $b \in \text{BMO}$, if and only if

$$\|b\|_{\text{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - \langle b \rangle_Q| \, dx < \infty,$$

where $\langle b \rangle_Q := \frac{1}{|Q|} \int_Q b(t) \, dt$, here $Q \subset \mathbb{R}^d$ are cubes with sides parallel to the axes, $|Q|$ denotes the volume of the cube $Q$, and more generally, $|E|$ denotes the Lebesgue measure of a measurable set $E$ in $\mathbb{R}^d$. Note that $L^\infty(\mathbb{R}^d)$, the space of essentially bounded functions on $\mathbb{R}^d$, is a proper subset of BMO (e.g. $\log|x|$ is a function in BMO but not in $L^\infty(\mathbb{R})$).

We will consider linear or sublinear operators $T : L^p(u) \to L^p(v)$. Among the linear operators the Calderón-Zygmund singular integral operators and their dyadic analogues will be most important for us.

The prototypical Calderón-Zygmund singular integral operator is the Hilbert transform on $\mathbb{R}$, given by convolution with the distributional Hilbert kernel $k_H(x) := \text{p.v.}(1/(\pi x))$

$$Hf(x) := k_H \ast f(x) = \text{p.v.} \frac{1}{\pi} \int \frac{f(y)}{x-y} \, dy := \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} \, dy.$$  

The Hilbert transform and its periodic analogue naturally appear in complex analysis and in the study of convergence on $L^p$ of partial Fourier sums/integrals. The Hilbert transform siblings, the Riesz transforms on $\mathbb{R}^d$ and the Beurling transform on $\mathbb{C}$, are intimately connected to partial differential equations and to quasiconformal theory, respectively. Its cousin, the Cauchy integral on curves and higher dimensional analogues, is connected to rectifiability and geometric measure theory.

A prototypical sublinear operator is the Hardy-Littlewood maximal function

$$Mf(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$
here the supremum is taken over all cubes \( Q \subset \mathbb{R}^d \) containing \( x \) and with sides parallel to the axes. The maximal function naturally controls many singular integral operators and approximations of the identity; its weak-boundedness properties on \( L^1(\mathbb{R}^d) \) imply the Lebesgue differentiation theorem. Another sublinear operator that we will encounter in these lectures is the dyadic square function, see Section 4.2.

Given \( T \) a linear or sublinear operator, its \textit{commutator} with a function \( b \) is given by
\[
[b, T](f) := b T(f) - T(bf).
\]
The commutators are important in the study of factorization for Hardy spaces and to characterize the space of bounded mean oscillation (BMO). They also play a central role in the theory of partial differential equations (PDEs).

We refer the reader to [Gr1, Gr2, St] for encyclopedic presentations of classical harmonic analysis, [Duo1] for a more succinct yet deep presentation, and [PW] for an elementary presentation emphasizing the dyadic point of view.

### 2.2. Hilbert transform.

We now recall familiar facts about the Hilbert transform, including its \( L^p \) and one-weight (quantitative) \( L^p \) boundedness properties.

The Hilbert transform is defined by (2.3) on the underlying space and on frequency space the following representation as a Fourier multiplier with Fourier symbol \( m_H \), holds,
\[
\hat{H}f(\xi) = m_H(\xi) \hat{f}(\xi), \quad \text{where} \quad m_H(\xi) := -i \text{sgn}(\xi).
\]
To connect the two representations for the Hilbert transform, on the underlying space and on the frequency space, remember that multiplication on the Fourier side corresponds to convolution on the underlying space. Therefore, \( k_H \), the Hilbert kernel, is given by the inverse Fourier transform of the Fourier symbol \( m_H \),
\[
Hf(x) = k_H * f(x), \quad \text{where} \quad k_H(x) := (m_H)^{\vee}(x) = \text{p.v.} \frac{1}{\pi x},
\]
which is precisely the content of (2.3). Here the \textit{Fourier transform} and \textit{inverse Fourier transform} of a Schwartz function \( f \) on \( \mathbb{R} \) are defined by
\[
\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, dx, \quad (f)^{\vee}(x) := \int_{\mathbb{R}} f(\xi) e^{2\pi i \xi x} \, d\xi.
\]
The Fourier transform is a bijection and an \( L^2 \) isometry on the Schwartz class that can be extended to be an isometry on \( L^2(\mathbb{R}) \), that is \( \| \hat{f} \|_{L^2(\mathbb{R})} = \| f \|_{L^2(\mathbb{R})} \) (Plancherel’s identity), and it can also be extended to be a bijection on the space of tempered distributions. The convolution \( f \ast g \) is a well-defined function on \( L^r(\mathbb{R}) \) when \( f \in L^p(\mathbb{R}) \) and \( g \in L^q(\mathbb{R}) \), provided \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \) and \( p, q, r \in [1, \infty] \). Moreover, on the same range, \textit{Young’s inequality} holds,
\[
\| g \ast f \|_{L^r} \leq \| g \|_{L^q} \| f \|_{L^p}.
\]
In these lecture notes we will explore, in Section 4.3, a third representation for the Hilbert transform in terms of dyadic shift operators discovered by Stefanie Petermichl [Pet1] in 2000.

#### 2.2.1. \( L^p \) boundedness properties of \( H \).

Fourier theory ensures boundedness on \( L^2(\mathbb{R}) \) for the Hilbert transform \( H \). In fact, applying Plancherel’s identity twice and using the fact that \( |m_H(\xi)| = 1 \) a.e., one immediately verifies that \( H \) is an isometry on \( L^2(\mathbb{R}) \), namely
\[
\| Hf \|_{L^2} = \| \hat{H}f \|_{L^2} = \| \hat{f} \|_{L^2} = \| f \|_{L^2}.
\]
Young’s inequality (2.6) for \( p \geq 1, q = 1 \) (hence \( r = p \)), imply that if \( g \in L^1(\mathbb{R}) \) and \( f \in L^p(\mathbb{R}) \) then \( g * f \in L^p(\mathbb{R}) \), moreover
\[
\|g * f\|_{L^p} \leq \|g\|_{L^1} \|f\|_{L^p}.
\]
This would imply boundedness on \( L^p(\mathbb{R}) \) for the Hilbert transform if the Hilbert kernel, \( k_H \), were integrable, but is not. Despite this fact, the following boundedness properties for the Hilbert transform hold (shared by all Calderón-Zygmund singular integral operators).

The Hilbert transform is not bounded on \( L^1(\mathbb{R}) \), it is of weak-type \((1,1)\) (Kolmogorov 1927), that is there is a constant \( C > 0 \) such that for all \( \lambda > 0 \) and for all \( f \in L^1(\mathbb{R}) \)
\[
\left| \{ x \in \mathbb{R} : |Hf(x)| > \lambda \} \right| \leq \frac{C}{\lambda} \|f\|_{L^1}.
\]

The Hilbert transform is bounded on \( L^p(\mathbb{R}) \) for all \( 1 < p < \infty \) (M. Riesz 1927), namely there is a constant \( C_p > 0 \) such that for all \( f \in L^p(\mathbb{R}) \)
\[
\|Hf\|_{L^p} \leq C_p \|f\|_{L^p} \quad \text{(best constant was found by Pichorides in 1972)}.
\]

Note that for \( 1 < p < 2 \) the \( L^p \) boundedness can be obtained by Marcinkiewicz interpolation theorem, from the weak-type \((1,1)\) and the \( L^2 \) boundedness. Then, for \( 2 < p < \infty \), the boundedness on \( L^p(\mathbb{R}) \) can be obtained by a duality argument, suffices to observe that the adjoint of \( H \) is \(-H\), that is the Hilbert transform is almost self-adjoint. However the Marcinkiewicz interpolation did not exist in 1927, Riesz proved instead that boundedness on \( L^p(\mathbb{R}) \) implied boundedness on \( L^{2p}(\mathbb{R}) \), hence boundedness on \( L^2(\mathbb{R}) \) implied boundedness on \( L^4(\mathbb{R}) \), then on \( L^8(\mathbb{R}) \) and by induction on \( L^{2^n}(\mathbb{R}) \). Strong interpolation, which already existed, then gave boundedness on \( L^p(\mathbb{R}) \) for \( 2^n \leq p \leq 2^{n+1} \) and for all \( n \geq 1 \), that is for all \( 2 \leq p < \infty \). Finally a duality argument took care of \( 1 < p < 2 \). In Section 4.3 we will deduce the \( L^p \) boundedness of the Hilbert transform from the \( L^p \) boundedness of dyadic shift operators, see Section 2.6.

Interpolation is an extremely powerful tool in analysis that allows to deduce intermediate norm inequalities given two end-point (weak)norm inequalities. We will not discuss interpolation further in these notes, instead we will focus on extrapolation, that allows us to deduce weighted \( L^p \) norm inequalities for all \( 1 < p < \infty \) given weighted \( L^r \) norm inequalities for one index \( r > 1 \).

Finally it is important to note that the Hilbert transform is not bounded on \( L^\infty(\mathbb{R}) \), however it is bounded on the larger space BMO of functions of bounded mean oscillation (C. Fefferman 1971).

To illustrate the lack of boundedness on \( L^\infty(\mathbb{R}) \) and on \( L^1(\mathbb{R}) \) it is helpful to calculate the Hilbert transform for some simple functions, showing in fact that the Hilbert transform does not map neither \( L^1(\mathbb{R}) \) nor \( L^\infty(\mathbb{R}) \) into themselves. This immediately eliminates the possibility for the Hilbert transform being bounded on either space.

**Example 2.1** (Hilbert transform of an indicator function).
\[
H 1_{[a,b]}(x) = (1/\pi) \log \left( |x - a|/|x - b| \right),
\]
where the indicator function \( 1_{[a,b]}(x) := 1 \) when \( x \in [a,b] \) and zero otherwise, a bounded and integrable function, that is \( 1_{[a,b]} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). However \( \log |x| \) is neither in \( L^\infty(\mathbb{R}) \) nor in \( L^1(\mathbb{R}) \), but it is a function of bounded mean oscillation. The functions \( f \) in \( L^1(\mathbb{R}) \) whose Hilbert transforms \( Hf \) are also in \( L^1(\mathbb{R}) \) constitute the Hardy space \( H^1(\mathbb{R}) \), such functions need to have some cancellation (\( \int f(x) \, dx = 0 \), clearly not shared by the indicator function \( 1_{[a,b]} \).
2.2.2. One-weight inequalities for $H$. The one-weight theory à la Muckenhoupt for the Hilbert transform is well understood, the qualitative theory has been known since 1973 [HMW], the quantitative estimates were settled by Stefanie Petermichl in 2007 [Pet2]. The two-weight problem on the other hand, was studied for a long time but the necessary and sufficient conditions à la Muckenhoupt for pairs of weights $(u,v)$ that ensure boundedness of the Hilbert transform from $L^p(u)$ into $L^p(v)$ were only settled in 2014 by Michael Lacey, Chun-Yen Shen, Eric Sawyer, and Ignacio Uriarte-Tuero [L1, LSSU].

Theorem 2.2 (Hunt, Muckenhoupt, Wheeden 1973). The Hilbert transform is bounded on $L^p(w)$ for $1 < p < \infty$ if and only if the weight $w \in A_p$. In either case there is a constant $C_p(w) > 0$ depending on $p$ and on the weight $w$ such that

$$\|Hf\|_{L^p(w)} \leq C_p(w)\|f\|_{L^p(w)} \quad \text{for all } f \in L^p(w).$$

At this point we remind the reader that a weight $w$ is in the Muckenhoupt $A_p$ class if and only if $[w]_{A_p} < \infty$, where the $A_p$ characteristic of the weight $w$ is defined to be

$$[w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}(x) \, dx \right)^{p-1} \quad \text{for } 1 < p < \infty,$$

the supremum is taken over all cubes $Q$ in $\mathbb{R}^d$ with sides parallel to the axes. We will denote integral averages with respect to Lebesgue measure on cubes or on measurable sets $E$ by $\langle f \rangle_E := \frac{1}{|E|} \int_E f(x) \, dx$. Also given $w$, a weight, $w(E)$ will denote the $w$-mass of the measurable set $E$, that is, $w(E) = \int_E w(x) \, dx$. With this notation

$$[w]_{A_2} := \sup_Q \langle w \rangle_Q (w^{-1})_Q.$$

Note that $w \in A_2$ if and only if $w^{-1} \in A_2$.

Example 2.3. Power weights offer examples of $A_p$ weights on $\mathbb{R}^d$, $w(x) = |x|^\alpha$ is in $A_p$ if and only if $-d \leq \alpha \leq d(p-1)$ for $1 < p < \infty$.

In Theorem 2.2, the optimal dependence of the constant $C_p(w)$ on the $A_p$ characteristic $[w]_{A_p}$ of the weight $w$, was found more than 30 years later.

Theorem 2.4 (Petermichl 2007). Given $1 < p < \infty$, for all $w \in A_p$ and for all $f \in L^p(w)$ we have that

$$\|Hf\|_{L^p(w)} \lesssim [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(w)}.$$

Note that the estimate is linear on $[w]_{A_p}$ for $p \geq 2$, and of power $\frac{1}{p-1}$ for $1 < p < 2$.

Cartoon of the proof. The following is a very brief sketch of Petermichl’s argument. First, write $H$ as an average over dyadic grids of dyadic shift operators [Pet1]. Second, find linear estimates, uniform (on the dyadic grids), for the dyadic shift operators on $L^2(w)$ [Pet2]. Deduce from the first two steps linear estimates on $L^2(w)$ for the Hilbert transform, namely estimates valid for all $w \in A_2$ and for all $f \in L^2(w)$ of the form

$$\|Hf\|_{L^2(w)} \lesssim [w]_{A_2} \|f\|_{L^2(w)}.$$

Third, use a sharp extrapolation theorem [DGPPet] to get estimates for $p \neq 2$ from the linear $L^2(w)$ estimate. $\square$
Same estimates hold for all Calderón-Zygmund singular integral operators, solving the famous $A_2$ conjecture, which was proven by Tuomas Hytönen in 2012, see [Hyt2]. We will say more about Petermichl’s and Hytönen’s landmark results as well as about sharp extrapolation later in Section 2.6 and in Section 4.

2.3. Maximal function. We summarize the $L^p$ and one-weight (quantitative) $L^p$ boundedness properties for the maximal function. We also show that the $A_p$ condition on the weight $w$ is a necessary condition for boundedness of the maximal function on $L^p(w)$.

2.3.1. $L^p$ boundedness properties of $M$. From its definition (2.4), it is clear that the maximal function is bounded on $L^\infty(\mathbb{R}^d)$ with norm one. The maximal function is not bounded on $L^1(\mathbb{R}^d)$, however it is of weak-type $(1,1)$ (Hardy, Littlewood 1930). The next example shows that the maximal function does not map $L^1(\mathbb{R})$ onto itself.

**Example 2.5.** The characteristic function $1_{[0,1]}$ is integrable however its image, under the maximal function, $M1_{[0,1]}$, is not. The diligent reader can verify that $M1_{[0,1]}(x) = 1/(1 - x)$ if $x < 0$, $M1_{[0,1]}(x) = 1$ if $0 \leq x \leq 1$, and $M1_{[0,1]}(x) = 1/x$ if $x > 1$.

Marcinkiewicz interpolation gives boundedness of the maximal function on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ from the strong $L^\infty$ and the weak-type $(1,1)$ boundedness results. We will present an alternate argument in Section 3.2.2 that will cover the weighted $L^p$ estimates as well without reference to neither interpolation nor extrapolation.

2.3.2. One-weight $L^p$ inequalities for $M$. The maximal function is of weak $L^p(w)$ type if and only if $w \in A_p$, moreover the following quantitative result was proven in 1972 by Benjamin Muckenhoupt [Mu], for $p \geq 1$ and for all $w \in A_p$,

$$\|M\|_{L^p(w) \to L^p,\infty(w)} \lesssim_p \|w\|_{A_p}^{1/p},$$

where the quantity on the left-hand-side, $\|M\|_{L^p(w) \to L^p,\infty(w)}$, denotes the smallest constant $C > 0$ such that for all $\lambda > 0$ and for all $f \in L^p(w)$

$$w\{x \in \mathbb{R}^d : Mf(x) > \lambda\} \leq \left(\frac{C}{\lambda}\|f\|_{L^p(w)}\right)^p.$$ 

We say a weight $w$ is in the Muckenhoupt $A_1$ class if and only if there is a constant $C > 0$ such that

$$Mw(x) \leq Cw(x) \quad \text{for a.e. } x \in \mathbb{R}^d.$$ 

The infimum over all possible such constants $C$ is denoted $[w]_{A_1}$. The $A_1$ class of weights is contained in all $A_p$ classes of weights for $p > 1$.

The maximal function is bounded on $L^p(w)$, moreover the following quantitative result was proven in 1993 by Stephen Buckley [Bu1] valid for $p > 1$ and for all $w \in A_p$ and $f \in L^p(w)$,

$$\|Mf\|_{L^p(w)} \lesssim_p [w]_{A_p}^{1/(p-1)}\|f\|_{L^p(w)}.$$ 

Buckley deduced these estimates from quantitative self-improvement integrability results known for $A_p$ weights, the weak $L^{p+\varepsilon}(w)$ boundedness of the maximal function, and Marcinkiewicz interpolation. More precisely, $w \in A_p$ implies $w \in A_{p-\varepsilon}$ with $\varepsilon \sim [w]_{A_p}^{-p'}$ and $[w]_{A_{p-\varepsilon}} \leq 2[w]_{A_p}$, on the other hand Hölder’s inequality implies $A_p \subset A_{p+\varepsilon}$ and $[w]_{A_{p+\varepsilon}} \leq [w]_{A_p}$. Interpolating between weak $L^{p-\varepsilon}(w)$ and weak $L^{p+\varepsilon}(w)$ estimates and keeping track of the constants one gets Buckley’s quantitative estimate (2.8).
In particular when \( p = 2 \) the maximal function obeys a linear estimate on \( L^2(w) \) with respect to the \( A_2 \) characteristic of the weight, namely for all \( w \in A_2 \) and \( f \in L^2(w) \)

\[
\|Mf\|_{L^2(w)} \lesssim [w]_{A_2}\|f\|_{L^2(w)}.
\]

A beautiful proof of Buckley’s quantitative estimate for the maximal function was presented in 2008 by Andrei Lerner [Le1], mixed \( A_p \)-\( A_\infty \) estimates in 2011 by Tuomas Hytönen and Carlos Pérez [HytPz], and extensions to spaces of homogeneous type in 2012 by Tuomas Hytönen and Anna Kairema [HytK]. We will present Lerner’s proof of Buckley’s inequality (2.8) in Section 3.2.2.

2.3.3. \( A_p \) is a necessary condition for \( L^p(w) \) boundedness of \( M \). We would like to demystify the appearance of the \( A_p \) weights in the theory by showing that \( w \in A_p \) is a necessary condition for the maximal function to be bounded on \( L^p(w) \) when \( p > 1 \).

We will show that if the maximal function is bounded on \( L^p(w) \) THEN the weight \( w \) must be in the Muckenhoupt \( A_p \) class.

**Proof.** By hypothesis, there is a constant \( C > 0 \) such that for all \( f \in L^p(w) \),

\[
\|Mf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}.
\]

For all \( \lambda > 0 \), let \( E_\lambda^{Mf} \) be the \( \lambda \)-level set for the maximal function \( Mf \), that is

\[
E_\lambda^{Mf} = \{ x \in \mathbb{R}^d : Mf(x) \geq \lambda \},
\]

then, by Chebychev’s inequality\(^4\), and using the hypothesis we conclude that

\[
w(E_\lambda^{Mf}) = \int_{E_\lambda^{Mf}} w(x) \, dx \leq \frac{1}{\lambda^p} \int_{\mathbb{R}^d} |Mf(x)|^p w(x) \, dx \leq \frac{C_p}{\lambda^p} \|f\|_{L^p(w)}^p.
\]

Fix a cube \( Q \subset \mathbb{R}^d \), for any integrable function \( f \geq 0 \), supported on the cube \( Q \), let \( \lambda := \frac{1}{|Q|} \int_Q f(y) \, dy \). Then \( Mf(x) \geq \lambda \) for all \( x \in Q \) hence \( Q \subset E_\lambda^{Mf} \), moreover

\[
(2.9) \quad \left( \frac{1}{|Q|} \int_Q f(x) \, dx \right)^p w(Q) \leq \lambda^p w(E_\lambda^{Mf}) \leq C_p \int_Q f^p(x) w(x) \, dx.
\]

Consider the specific function \( f = w^{-1/p} \mathbbm{1}_Q \) supported on \( Q \) and chosen so that both integrands coincide, namely \( f = f^p w \). Substitute this specific function \( f \) into (2.9) to obtain the following inequality only pertaining the weight \( w \) and the cube \( Q \),

\[
\frac{1}{|Q|^p} \left( \int_Q w^{1-p}(x) \, dx \right)^{p-1} w(Q) \leq C_p.
\]

Distribute \( |Q| \) and take the supremum over all cubes \( Q \) to conclude that \( [w]_{A_p} \leq C_p \), and hence \( w \in A_p \). There is one technicality, the chosen function may not be integrable, choose instead \( f_\epsilon = \mathbbm{1}_Q(w + \epsilon)^{-1/p} \), run the argument for each \( \epsilon > 0 \) then let \( \epsilon \) go to zero. \( \square \)

We just showed that if the maximal function \( M \) is bounded on \( L^p(w) \) then it is of weak \( L^p(w) \) type. Moreover \( [w]_{A_p}^{1/p} \leq \|M\|_{L^p(w) \to L^{p,\infty}(w)} \) therefore Muckenhoupt’s weak \( L^p(w) \) bound (2.7) is optimal.

\(^4\)Namely, for \( g \in L^1(\mu) \) it holds that \( \mu(\{x \in \mathbb{R}^d : |g(x)| > \lambda\}) \leq \frac{1}{\lambda} \|g\|_{L^1(\mu)} \) for all \( \lambda > 0 \), in other words if \( g \in L^1(\mu) \) then \( g \in L^{1,\infty}(\mu) \), where \( g \in L^{p,\infty}(\mu) \) means \( \|g\|_{L^{p,\infty}(\mu)} := \sup_{\lambda > 0} \lambda^{1/p}\{x \in \mathbb{R}^d : |g(x)| > \lambda\} < \infty \).
2.4. Why are we interested in these estimates? We record a few instances where $L^p$ and weighted $L^p$ estimates are of importance in analysis.

- **Fourier Analysis**: Boundedness of the periodic Hilbert transform on $L^p(\mathbb{T})$ implies convergence on $L^p(\mathbb{T})$ of the partial Fourier sums.
- **Complex Analysis**: $Hf$ is the boundary value of the harmonic conjugate of the Poisson extension to the upper-half-plane of a function $f \in L^p(\mathbb{R})$.
- **Factorization**: Theory of (holomorphic) Hardy spaces $H^p$. Elements of $H^p$ can be defined as those distributions whose image under properly defined maximal functions (or other suitable singular operators or square functions) are in $L^p$.
- **Approximation Theory**: Boundedness properties of the martingale transform (a dyadic analogue of the Hilbert transform) show that Haar functions and other wavelet families are unconditional bases of several functional spaces.
- **PDEs**: Boundedness of the Riesz transforms (analogues of the Hilbert transform on $\mathbb{R}^d$) and their commutators have deep connections to partial differential equations.
- **Quasiconformal Theory**: Boundedness of the Beurling transform (singular integral operator on $\mathbb{C}$) on $L^p(w)$ for $p > 2$ and with linear estimates on $[w]_{A_p}$ implies borderline regularity result.
- **Operator Theory**: Weighted inequalities appear naturally in the theory of Hankel and Toeplitz operators, perturbation theory, etc.

We expand on the weighted estimate needed in quasiconformal theory which propelled the interest in quantitative weighted estimates. This was work by Kari Astala, Tadeusz Iwaniec, and Eero Saksman in 2001, we refer to their paper [AIS] for appropriate definitions. They showed that for $1 < K < \infty$ every weakly $K$-quasi-regular mapping, contained in a Sobolev space $W^{1,q}_{loc}(\Omega)$ with $2K/(K+1) < q \leq 2$, is quasi-regular on $\Omega$, that is to say, it belongs to $W^{1,2}_{loc}(\Omega)$. For each $q < 2K/(K+1)$ there are weakly $K$-quasi-regular mappings $f \in W^{1,q}_{loc}(\mathbb{C})$ which are not quasi-regular. The only value of $q$ that remained unresolved was the endpoint, they conjectured that all weakly $K$-quasi-regular mappings $f \in W^{1,q}_{loc}$ with $q = 2K/(K+1)$ are in fact quasi-regular. They reduced the conjecture to showing [AIS, Proposition 22] that the Beurling transform $T$ satisfies linear bounds in $L^p(w)$ for $n^p > 1^n$, namely

$$\|Tg\|_{L^p(w)} \lesssim_p [w]_{A_p} \|g\|_{L^p(w)},$$

for all $w \in A_p$ and $g \in L^p(w)$.

Fortunately the values of interest for $q$ are $1 < q < 2$ and $p = q^r > 2$. Linear bounds for the Beurling transform and $p \geq 2$ were proven in 2002 by Stefanie Petermichl and Sasha Volberg [PetV]. As a consequence the regularity at the borderline case $q = 2K/(K+1)$ was established. For $1 < p < 2$ the correct estimate for the Beurling transform is of the form

$$\|Tg\|_{L^p(w)} \lesssim_p [w]_{A_p}^{1/(p-1)} \|g\|_{L^p(w)},$$

for all $w \in A_p$ and $g \in L^p(w)$, as shown in [DGPPet].

2.5. First Linear Estimates. Interest in quantitative weighted estimates exploded in this millennium. A chronology of the early linear estimates on $L^2(w)$ for the weight $w$ in the Muckenhoupt $A_2$ class, namely $\|Tf\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)}$, is as follows.

- Maximal function (Buckley ‘93 [Bu1]).
- Martingale transform (Wittwer ‘00 [W1]).
- (Dyadic) square function (Hukovic, Treil, Volberg ‘00 [HTV]; Wittwer ‘02 [W2]).
- Beurling transform (Petermichl, Volberg ‘02 [PetV]).
- Hilbert transform (Petermichl ‘07 [Pet2]).
- Riesz transforms (Petermichl ‘08 [Pet3]).
- Dyadic paraproduct (Beznosova ‘08 [Be2]).

Except for the maximal function, all these linear estimates were obtained using Bellman functions and (bilinear) Carleson estimates for certain dyadic operators (Petermichl dyadic shift operators, martingale transform, dyadic paraproducts, dyadic square function), and then either the operator under study was one of them or had enough symmetries that it could be represented as a suitable average of dyadic operators (Beurling, Hilbert, and Riesz transforms). The Bellman function method was introduced in the 90’s to harmonic analysis by Fedja Nazarov, Sergei Treil, and Sasha Volberg [NT, NTV2], although they credit Donald Burkholder in his celebrated work finding the exact $L^p$ norm for the martingale transform [Bur2]. With their students and collaborators they have been able to use the Bellman function method to obtain a number of astonishing results not only in this area, see Volberg’s INRIA lecture notes [V] and references. In Volberg’s own words\(^5\) "the Bellman function method makes apparent the hidden multiscale properties of Harmonic Analysis problems".

A flurry of work ensued and other techniques were brought into play including stopping time techniques (corona decompositions) and median oscillation techniques. These techniques became the precursors of what is now known as the method of domination by dyadic sparse operators, with important contributions from David Cruz-Uribe, Chema Martell, Carlos Pérez, Andrei Lerner, Tuomas Hytönen, Michael Lacey, Mari Carmen Reguera, Stefanie Petermichl, Fedja Nazarov, Sergei Treil, Sasha Volberg and others. We will say more about sparse domination in Section 7.

The culmination of this work was the celebrated resolution of the $A_2$ conjecture by Tuomas Hytönen [Hyt2] in 2012 where he showed first every Calderón-Zygmund operator could be written as an average of dyadic shift operators of arbitrary complexity, dyadic paraproducts and their adjoints, second the weighted $L^2$ norm of the dyadic shifts depended linearly on the $A_2$ characteristic of the weight and polynomially on the complexity, and third these ingredients implied that the Calderón-Zygmund operator obeyed linear bounds on $L^2(w)$. How about weighted $L^p$ estimates for $1 < p < \infty$?

2.6. Extrapolation and Hytönen’s $A_p$ theorem. There is a, by now, classical technique to obtain weighted $L^p$ estimates from weighted $L^2$ estimates or more generally from weighted $L^r$ estimates, called extrapolation. In this section, we recall the classical Rubio de Francia extrapolation theorem, a quantitative version, due to Oliver Dragičević et al, called "sharp extrapolation", and deduce from the later Hytönen’s $A_p$ theorem.

2.6.1. Rubio de Francia Extrapolation Theorem. José Luis Rubio de Francia introduced in the 80’s his celebrated extrapolation result, a theorem that allowed to transfer estimates from weighted $L^r$ (provided it held for all $A_r$ weights) to weighted $L^p$ for all $1 < p < \infty$ and all $A_p$ weights.

Theorem 2.6 (Rubio de Francia 1981). Given $T$ a sublinear operator and $r \in \mathbb{R}$ with $1 < r < \infty$. If for all $w \in A_r$ there is a constant $C_{T,r,d,w} > 0$ such that

$$
\|Tf\|_{L^r(w)} \leq C_{T,r,d,w}\|f\|_{L^r(w)} \text{ for all } f \in L^r(w).
$$

\(^5\)http://www-sop.inria.fr/apics/ahpi/summerschool11/bellman_lectures_volberg-1.pdf
Then for each $1 < p < \infty$ and for all $w \in A_p$, there is a constant $C_{T,p,r,d,w} > 0$ such that

$$\|Tf\|_{L^p(w)} \leq C_{T,p,r,d,w}\|f\|_{L^p(w)}$$

for all $f \in L^p(w)$.

If we choose $r = 2$, paraphrasing Antonio Córdoba\textsuperscript{6} we will conclude that

There is no $L^p$ just weighted $L^2$.

(Since $w \equiv 1 \in A_p$ for all $p$.)

There are books dedicated to the subject that cover this and many useful variants of this theorem, the classical reference is the out-of-print 1985 book by García-Cuerva and Rubio de Francia [GaRu]. A modern presentation, including quantitative versions of this theorem, is the 2011 book by David Cruz-Uribe, Chema Martell, and Carlos Pérez [CrMPz1].

2.6.2. Sharp extrapolation. In the 80’s and 90’s the interest was on qualitative weighted estimates. Once the interest on quantitative weighted estimates was sparked it was natural to consider quantitative extrapolation theorems, what we call ”sharp extrapolation theorems". This is precisely what Stefanie Petermichl and Sasha Volberg did [PetV] to obtain linear estimates for the Beurling transform and $p \geq 2$, they missed the range $1 < p < 2$ because it was of no interest, and their calculation was very specific to the martingale transforms that properly averaged yielded the Beurling transform. It was soon realized that a general principle was at work [DGPPet]. We state a simplified version of what a quantitative extrapolation theorem says, useful for the purposes of this survey.

**Theorem 2.7** (Dragičević et al 2005). Let $T$ be a sublinear operator, $r \in \mathbb{R}$ with $1 < r < \infty$. If for all $w \in A_r$ there are constants $\alpha, C_{T,r,d} > 0$ such that

$$\|Tf\|_{L^r(w)} \leq C_{T,r,d}[w]_A^\alpha\|f\|_{L^r(w)}$$

for all $f \in L^r(w)$.

Then for each $1 < p < \infty$ and for all $w \in A_p$, there is a constant $C_{T,p,r,d} > 0$ such that

$$\|Tf\|_{L^p(w)} \leq C_{T,p,r,d}[w]_A^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}$$

for all $f \in L^p(w)$. 

The proof follows by now standard arguments involving the celebrated Rubio de Francia algorithm, and inserting whenever possible Buckley’s quantitative bounds (2.8) for the maximal function [Bu1].

An alternative, streamlined proof of the sharp extrapolation theorem, was presented by Javier Duoandikoetxea in [Duo2], extending the result to more general settings including off-diagonal and partial range extrapolation. It was observed [CrMPz1] that one can replace the pair $(Tf,f)$ by a pair of functions $(g,f)$ in the extrapolation theorem, in particular one could consider the pair $(f,Tf)$ instead, as long as one has the corresponding initial weighted inequalities required to jump-start the theorem.

Sharp extrapolation is sharp in the sense that no better power for $[w]_Ap$ can appear in the conclusion that will work for all operators. For some operators it is known that the extrapolated $L^p(w)$ bounds from the known optimal $L^r(w)$ estimates are themselves optimal for all $1 < p < \infty$. However it is not necessarily optimal for a particular given operator. Here are some examples illustrating this phenomenon.

**Example 2.8.** Start with Buckley’s sharp estimate on $L^r(w)$, $\alpha = \frac{1}{r-1}$, for the maximal function, extrapolation will give sharp bounds only for $1 < p \leq r$.

\textsuperscript{6}See page 8 in José García-Cuerva’s eulogy for José Luis Rubio de Francia (1949-1988) [Ga].
Example 2.9. Sharp extrapolation from $r = 2$, $\alpha = 1$, is sharp for the Hilbert, Beurling, Riesz transforms for all $1 < p < \infty$ (for $p > 2$ [PetV], [Pet2], [Pet3]; $1 < p < 2$ [DGPPet]).

Example 2.10. Extrapolation from linear bound on $L^2(w)$ is sharp for the dyadic square function only when $1 < p \leq 2$ ("sharp" [DGPPet], "only" [Le2]). However, extrapolation from square root bound on $L^3(w)$ is sharp for all $p > 1$ [CrMPz2].

2.6.3. Hytönen’s $A_p$ Theorem. Sharp extrapolation was used by Tuomas Hytönen to prove the celebrated $A_p$ theorem, the quantitative weighted $L^p$ estimates for Calderón-Zygmund operators [Hyt2].

Theorem 2.11 (Hytönen 2012). Let $1 < p < \infty$ and let $T$ be any Calderón-Zygmund singular integral operator on $\mathbb{R}^d$, then for all $w \in A_p$ and $f \in L^p(w)$

$$\|Tf\|_{L^p(w)} \lesssim_{T, d, p} [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}} \|f\|_{L^p(w)}.$$  

Cartoon of the proof. Enough to prove the $p = 2$ case thanks to sharp extrapolation. To prove the linear weighted $L^2$ estimate two important steps were required.

First, prove a representation theorem in terms of Haar shift operators of arbitrary complexity, dyadic paraproducts, and their adjoints on random dyadic grids introduced in [NTV3]. This representation hinges on certain reductions obtained in [PzTV].

Second, prove linear estimates on $L^2(w)$ with respect to the $A_2$ characteristic for paraproducts [Be2] and Haar shift operators [LPetR] but with polynomial dependence on the complexity (independent of the dyadic grid) [Hyt2].

We will say more about random dyadic grids, Haar shift operators, and paraproducts, the ingredients in Hytönen’s theorem, in Sections 3 and 4. It is now well understood that the $L^2(w)$ bounds for the Haar shift operators not only depend linearly on the $A_2$ characteristic of $w$ but also depend linearly on the complexity [T].

Sharp extrapolation has also been used to obtain quantitative estimates in other settings. For example, Sandra Pott and Mari Carmen Reguera used sharp extrapolation when studying the Bergman projection on weighted Bergman spaces in terms of the Békollé constant [PoR]. They proved the base estimate on $L^2(w)$ for certain sparse dyadic operators and then showed the Bergman projection could be dominated with these sparse dyadic operators.

2.7. Two-weight problem for the Hilbert transform and the maximal function. We briefly state a necessarily incomplete chronological list of two-weight results for the Hilbert transform, the maximal function, and allied dyadic operators.

2.7.1. Two-weight problem for $H$ and its dyadic model the martingale transform. In the ‘80s, Mischa Cotlar, and Cora Sadosky found necessary and sufficient conditions à la Helson-Szegö solving the two-weight problem for the Hilbert transform. The methods used involved complex analysis and had applications to operator theory [CS1, CS2]. Afterwards various sets of sufficient conditions à la Muckenhoupt were found to be valid also in the matrix-valued context, one of the earliest such sets appeared in 1997 in joint work with Nets Katz [KP], see also the 2005 unpublished manuscript [NTV5]. Necessary and sufficient conditions for (uniform and individual) martingale transform and well-localized dyadic operators were found in 1999 and 2008 respectively by Fedja Nazarov, Sergei Treil, Sasha Volberg [NTV1, NTV4], using Bellman function techniques. We will say more about this in Section 4.1. Long-time sought necessary and sufficient conditions for two-weight boundedness of the Hilbert transform were
found in 2014 by Michael Lacey, Eric Sawyer, Chun-Yen Shen, and Ignacio Uriarte-Tuero [L1, LSSU] for pairs of weights that do not share a common point-mass. Corresponding quantitative estimates were obtained using very delicate stopping time arguments. See also [L2]. Improvements have since been obtained, relaxing the conditions on the weights, by the same authors and Tuomas Hytönen [Hyt4].

2.7.2. Two-weight estimates for the maximal function. In 1982 Eric Sawyer showed in [S1] that the maximal function $M$ is bounded from $L^2(u)$ into $L^2(v)$ if and only if the following testing conditions\footnote{Nowadays called "Sawyer’s testing conditions".} hold for the weights $u$ and $v$: there is a constant $C_{u,v} > 0$ such that for all cubes $Q$

$$\int_Q (M(1_Q u^{-1})(x))^2 v(x) \, dx \leq C_{u,v} u^{-1}(Q)$$

and

$$\int_Q (M(1_Q v^{-1})(x))^2 u^{-1}(x) \, dx \leq C_{u,v} v(Q).$$

Sawyer also identified necessary and sufficient conditions for two-weight inequalities for certain positive operators, the fractional and Poisson integrals [S2], these results were of qualitative type. In 2009, Kabe Moen presented the first quantitative result [Moe], he proved that the two weight operator norm of $M$ is comparable to the constants $C_{u,v}$ in Sawyer’s result. Note that Sawyer’s testing conditions imply the following joint $A_2$ condition:

$$[u,v]_{A_2} := \sup_Q \langle u^{-1} \rangle_Q \langle v \rangle_Q < \infty,$$

where $\langle v \rangle_Q := v(Q)/|Q|$. In 2015, Carlos Pérez and Ezequiel Rela [PzR] considered a particular case when $(u,v) \in A_2$ and $u^{-1} \in A_\infty$ and showed the following so-called mixed-type estimate

$$\|M\|_{L^2(u) \to L^2(v)} \lesssim [u,v]_{A_2}^{1/2} [u^{-1}]_{A_\infty}^{1/2}.$$

In the one-weight setting, when $u = v = w$, one gets the following improved mixed-type estimate

$$\|M\|_{L^2(w) \to L^2(w)} \lesssim [w]_{A_2}^{1/2} [w^{-1}]_{A_\infty}^{1/2} \leq [w]_{A_2}.$$

The $A_\infty$ class of weights is defined to be the union of all the $A_p$ classes of weights for $p > 1$, the classical $A_\infty$ characteristic is given by

$$[w]_{A_\infty} := \sup_Q \langle w \rangle_Q \exp(-\langle \log w \rangle_Q).$$

A weight $w$ is in $A_\infty$ if and only if $[w]_{A_\infty} < \infty$. An equivalent characterization is obtained using instead the Fujii-Wilson characteristic, defined by

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)(x) \, dx.$$

The Fujii-Wilson $A_\infty$ characteristic is smaller than the classical one [BeRe]. For mixed-type estimates of similar nature for Calderón-Zygmund singular integral operators see [HytPz].

For sharp weighted inequalities for fractional integral operators see [LMPzTo].
3. DYADIC HARMONIC ANALYSIS

In this section, we introduce the elements of dyadic harmonic analysis and the basic dyadic maximal function. More precisely we discuss dyadic grids (regular, random, adjacent) and Haar functions on the line, on $\mathbb{R}^d$, and on spaces of homogeneous type. As a first example, illustrating the power of the dyadic techniques, we present Lerner’s proof of Buckley’s quantitative $L^p$ estimates for the maximal function, which reduces, using the one-third trick, to estimates for the dyadic maximal function. We also describe, given dyadic cubes on spaces of homogeneous type, how to construct corresponding Haar bases, and briefly describe the Auscher-Hytönen "wavelets" in this setting.

3.1. Dyadic intervals, dyadic maximal functions. In this section we recall the dyadic intervals and the weighted dyadic maximal function on the line, as well as basic $L^p$ estimates for the dyadic maximal function.

3.1.1. Dyadic intervals. The standard dyadic grid $\mathcal{D}$ on $\mathbb{R}$ is the collection of intervals of the form $[k2^{-j}, (k+1)2^{-j})$, for all integers $k,j \in \mathbb{Z}$. The dyadic intervals are organized by generations: $\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j$, where $I \in \mathcal{D}_j$ if and only if $|I| = 2^{-j}$. Note that the larger $j$ is the smaller the intervals are. For each interval $J \in \mathcal{D}$ denote by $\mathcal{D}(J)$ the collection of dyadic intervals $I$ contained in $J$.

The standard dyadic intervals satisfy the following properties,

- (Partition Property) Each generation $\mathcal{D}_j$ is a partition of $\mathbb{R}$.
- (Nested property) If $I,J \in \mathcal{D}$ then $I \cap J = \emptyset$, $I \subseteq J$, or $J \subseteq I$.
- (One parent property) If $I \in \mathcal{D}_j$ then there is a unique interval $\tilde{I} \in \mathcal{D}_{j-1}$, called the parent of $I$, such that $I \subseteq \tilde{I}$. The parent is twice as long as the child, that is $|\tilde{I}| = 2|I|$.
- (Two children property) Given $I \in \mathcal{D}_j$, there are two disjoint intervals $I_l, I_r \in \mathcal{D}_{j+1}$ (the right and left children), such that $I = I_l \cup I_r$.
- (Tower of dyadic intervals) Each point $x \in \mathbb{R}$ belongs to exactly one dyadic interval $I_j(x) \in \mathcal{D}_j$. The family $\{I_j(x)\}_{j \in \mathbb{Z}}$ forms a "tower" or "cone" over $x$. The union of the intervals in a "tower", $\bigcup_{j \in \mathbb{Z}} I_j(x)$, is a "quadrant".
- (Two-quadrant property) The origin, 0, separates the positive and the negative dyadic interval, creating two "quadrants".

More generally, a dyadic grid on $\mathbb{R}$ is a collection of intervals organized in generations with the partition, nested, and two children properties. In this subsection we reserve the name $\mathcal{D}$ for the standard dyadic grid, however later on we will use $\mathcal{D}$ to denote a general dyadic grid.

The partition and nested properties are common to all dyadic grids, the one parent property is a consequence of these properties. The two children property is responsible for the name "dyadic", the equal length property is a consequence of choosing to subdivide in halves, and is in general not so important, one could subdivide into two children of different lengths, if the ratio is uniformly bounded we have a homogeneous or doubling dyadic grid. One can manufacture dyadic grids on the line where each interval has two equal-length children but there is no distinguished point and only one quadrant. This is because given an interval in the grid, its descendants are completely determined, however we have two choices for the parent, hence four choices for the grandparent, etc. In [LeN] dyadic grids are defined to have one quadrant, such grids have the additional useful property that given any compact set there will be a dyadic interval containing it.
There are many variants, for example, we could subdivide each interval into a uniformly bounded number of children or into arbitrarily finitely many children. In fact, there are regular dyadic structures on $\mathbb{R}^d$ where the role of the intervals is played by cubes with sides parallel to the axes. In this case, each cube in the dyadic grid is subdivided into $2^d$ congruent children, see Section 3.4.2. We will also see that there are dyadic structures in spaces of homogeneous type, where each "cube" may have no more than a fixed number of children, but sometimes it will only have one child (itself) for several generations, see Section 3.5.2. In all cases the dyadic grids provide a hierarchical structure that allows for simplified arguments in this setting, the so-called "induction on scale arguments".

3.1.2. Dyadic Maximal Function. Given a dyadic grid $D$ on $\mathbb{R}^d$ and a weight $u$, the (weighted) dyadic maximal function $M^D_u$ is defined like the maximal function $M$ except that instead of taking the supremum over all cubes in $\mathbb{R}^d$ with sides parallel to the axes we restrict to the dyadic cubes. This is often how one transitions from continuous to dyadic models.

More precisely, the weighted dyadic maximal function with respect to a weight $u$ and a dyadic grid $D$ on $\mathbb{R}^d$ is defined by

$$M^D_u f(x) := \sup_{Q \in D, Q \ni x} \frac{1}{u(Q)} \int_Q |f(y)| u(y) dy.$$ 

Here $u(Q) := \int_Q u(x) dx$. When $u = 1$ a.e. then $M^D_1 =: M^D$.

The dyadic maximal function inherits boundedness properties from the regular maximal function. This is clear once one notices that the dyadic maximal function is trivially pointwise dominated by the maximal function. However these properties are much easier to verify for the dyadic maximal function. We now list three basic boundedness properties of the dyadic maximal function, with a word or two as how one can verify each one of them.

First, the dyadic maximal function, $M^D_u$, is of weak $L^1(u)$ type, with constant one (independent of dimension). This is an immediate corollary of the Calderón-Zygmund lemma (a stopping time), no covering lemmas are required unlike the usual arguments for $M$.

Second, clearly $M^D_u$ is bounded on $L^\infty(u)$ with constant one. Interpolation between the weak $L^1(u)$ and the $L^\infty(u)$ estimates shows that $M^D_u$ is bounded on $L^p(u)$ for all $p > 1$. Moreover the following estimate holds with a constant independent of the weight $v$ and the dimension $d$,

$$\|M^D_u f\|_{L^p(u)} \lesssim p' \|f\|_{L^{p'}(u)} \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1 \text{ and } p > 1.$$  

Third, the dyadic maximal function is pointwise comparable to the maximal function. We explain in Section 3.2.1 why this domination holds in the one-dimensional case ($d = 1$).

3.2. One-third trick and Lerner’s proof of Buckley’s result. We present the one-third trick on $\mathbb{R}$ and how it can be used to dominate the maximal function by a sum of dyadic maximal functions. The one-third trick appeared in print in 1991 in Kate Okikiolu characterization of subsets of rectifiable curves in $\mathbb{R}^d$ [Ok, Lemma 1(b)], see [Cr, Footnote p.32] for fascinating historical remarks on the one-third trick. This was probably well-known among the John Garnett’s school of thought see for example [GJ], and also by the Polish school specifically by Tadeusz Figiel [Fi]. We illustrate how this principle can be used to recover Buckley’s quantitative weighted $L^p$ estimate for the maximal function.
3.2.1. One-Third Trick. The families of intervals $\mathcal{D}^i := \cup_{j \in \mathbb{Z}} \mathcal{D}^i_j$, for $i = 0, 1, 2$, where

$$\mathcal{D}^i_j := \{2^{-i}([0, 1] + m + (-1)^i \frac{j}{2^i}) : m \in \mathbb{Z}\},$$

are dyadic grids satisfying partition, nested, and two equal children properties. We make four observations. First, when $i = 0$ we recover the standard dyadic grid, $\mathcal{D}^0 = \mathcal{D}$. Second, the grids $\mathcal{D}^1$ and $\mathcal{D}^2$ are nested but there is only one quadrant (the line $\mathbb{R}$). Third, the grids, $\mathcal{D}^i$, for $i = 0, 1, 2$ are as "far away" as possible from each other, to be made more precise in Example 3.3. Fourth, given any finite interval $I \subset \mathbb{R}$, for at least two values of $i = 0, 1, 2$, there are $J^i \in \mathcal{D}^i$ such that $I \subset J^i$, $3|I| \leq |J^i| \leq 6|I|$. In particular this implies that given $i \neq k$, $i, k = 0, 1, 2$, there is at least one interval $J \in \mathcal{D}^i \cup \mathcal{D}^k$ such that $I \subset J$ and $3|I| \leq |J| \leq 6|I|$, and furthermore

$$\frac{1}{|I|} \int_I |f(y)| \, dy \leq \frac{6}{|J|} \int_J |f(y)| \, dy.$$ 

This last observation allows us to dominate the maximal function $M$ by its dyadic counterpart. In fact, the following estimate holds,

$$(3.2) \quad Mf(x) \leq 6(M^{\mathcal{D}}f(x) + M^{\mathcal{D}^i}f(x)).$$

More precisely, for $i \neq k$

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| \, dy \leq 6 \sup_{J \in \mathcal{D}^i \cup \mathcal{D}^k \ni J \ni x} \frac{1}{|J|} \int_J |f(y)| \, dy \leq 6 \max\{M^{\mathcal{D}^i}f(x), M^{\mathcal{D}^k}f(x)\} \leq 6\left[M^{\mathcal{D}^i}f(x) + M^{\mathcal{D}^k}f(x)\right].$$

In particular setting $i = 0$ and $k = 1$, we obtain (3.2).

There is an analogue of the one-third trick in higher dimensions. In $\mathbb{R}^d$ one can get by with $3^d$ grids as is very well explained in [LeN, Section 3], with $2^d$ grids [HytPz], or, with $d + 1$ grids and this is optimal, by cleverly choosing the grids, for $\mathbb{R}$ and for the $d$-torus see [Me], for $\mathbb{R}^d$ and $d > 1$ see [C].

3.2.2. Buckley’s $A_p$ estimate for the maximal function. We illustrate the use of dyadic techniques paired with domination to recover Stephen Buckley’s quantitative weighted $L^p$ estimate for the maximal function [Bu1]. Namely for all $w \in A_p$ and $f \in L^p(w)$

$$\|Mf\|_{L^p(w)} \lesssim \|w\|_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}.$$

The beautiful argument we present is due to Andrei Lerner [Le1].

Lerner’s Proof. By the one-third trick suffices to check that for $1 < p < \infty$ there is a constant $C_p > 0$ such that for all $w \in A_p$ and for all $f \in L^p(w)$ then

$$\|M^{\mathcal{D}}f\|_{L^p(w)} \leq C_p \|w\|_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)},$$

independently of the dyadic grid $\mathcal{D}$ chosen on $\mathbb{R}^d$.

For any dyadic cube $Q \subset \mathcal{D}$, let $A_p(Q) = w(Q)(\sigma(Q))^{p-1}/|Q|^p$, where we denote by $\sigma := w^{\frac{1}{p-1}}$ the dual weight of $w$, then

$$\frac{1}{|Q|} \int_Q |f(x)| \, dx = A_p(Q)^{\frac{1}{p-1}} \left[\frac{|Q|}{w(Q)} \left(\frac{1}{\sigma(Q)} \int_Q |f(x)| \sigma^{-1}(x) \sigma(x) \, dx\right)^{p-1}\right]^{\frac{1}{p-1}}.$$
where for all carefully peeling off the maximal functions, we get

**Example 3.3.** The measure zero in parameter space described below. Comparatively speaking this set of dyadic grids is negligible, since it corresponds to a set of

$$\frac{1}{w} \left[ \frac{1}{w(Q)} \int_Q (M^D(f \sigma^{-1})(x) w^{-1}(x) w(x) dx)^{p-1} \right]^{\frac{1}{p-1}}.$$

Taking the supremum over $Q \in \mathcal{D}$ we obtain

$$M^D f(x) \leq \left[ \frac{1}{w} A_p \left( M^D_w (M^D(f \sigma^{-1})^{p-1}w^{-1})(x) \right) \right]^{\frac{1}{p-1}}.$$

Computing the $L^p(w)$ norm on both sides, recalling that $(p-1)p' = p$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and carefully peeling off the maximal functions, we get

$$\|M^D f\|_{L^p(w)} \leq \left[ \frac{1}{w} A_p \left( M^D_w (M^D(f \sigma^{-1})^{p-1}w^{-1}) \right) \right]^{\frac{1}{p-1}} L^p(w) \leq \left[ \frac{1}{w} A_p \left( \|M^D_w\|^p \right) \right]^{\frac{1}{p-1}} \|M^D(f \sigma^{-1})\|_{L^p(\sigma)} \leq \left[ \frac{1}{w} A_p \left( \|M^D_w\|^p \right) \right]^{\frac{1}{p-1}} \|M^D(f \sigma^{-1})\|_{L^p(\sigma)} \leq \frac{1}{w} A_p \left( \|f\| L^p(w) \right),$$

where we used in the last line the uniform bounds (3.1) of $M^D_w$ on $L^p(w)$ and $M^D_\sigma$ on $L^p(\sigma)$.

Notice that in this argument neither extrapolation nor interpolation are used. For extensions to two-weight inequalities and to the fractional maximal function see [Moe].

### 3.3. Random dyadic grids on $\mathbb{R}$.

For the purpose of this section, a dyadic grid on $\mathbb{R}$ is a collection of intervals that are organized in generations, each generation provides a partition of $\mathbb{R}$ and the family has the nested, one parent, and two equal-length children per interval properties. Shifted and scaled regular dyadic grid are dyadic grids. These are not the only ones, there are other dyadic grids, such as the ones defined for the one-third trick: $\mathcal{D}^1$ and $\mathcal{D}^2$. The following parametrization will capture all dyadic grids in $\mathbb{R}$ [Hyt1].

**Lemma 3.1** (Hytönen 2008). For each scaling parameter $r$ with $1 \leq r < 2$, and shift parameter $\beta \in \{0, 1\}^\mathbb{Z}$, meaning $\beta = \{\beta_i\}_{i \in \mathbb{Z}}$ with $\beta_i = 0$ or 1, then $\mathcal{D}^{r, \beta} := \cup_{j \in \mathbb{Z}} \mathcal{D}_j^{r, \beta}$ is a dyadic grid. Where

$$\mathcal{D}_j^{r, \beta} := r \mathcal{D}_j^{\beta}, \text{ and } \mathcal{D}_j^{\beta} := x_j + \mathcal{D}_j, \text{ with } x_j = \sum_{i \geq j} \beta_i 2^{-i}.$$

We shift by a different parameter $x_j$ at each level $j$, in a way that is consistent and preserves the nested property of the grid. Moreover the shift parameter $\beta_j = 0, 1$ for $j \in \mathbb{Z}$ encodes the information whether a base interval at level $j$ will be the right or the left half of its parent.

**Example 3.2.** Shifted and scaled regular grids correspond to the shift parameter $\beta_i = 0$ for all $i < N$ (or $\beta_i = 1$ for all $i < N$) for some integer $N$. These are the grids with two quadrants. Comparatively speaking this set of dyadic grids is negligible, since it corresponds to a set of measure zero in parameter space described below.

**Example 3.3.** The $1/3$-shifted dyadic grids introduced in the previous section correspond to Hytönen’s dyadic grids for $r = 1$. More precisely,

$$\mathcal{D}^i = \mathcal{D}^{1, \beta^i} \text{ for } i \in \{0, 1, 2\},$$

where for all $j \in \mathbb{Z}$, $\beta_0^i \equiv 0$ (or $\equiv 1$), $\beta_1^1 = 1_{2\mathbb{Z}}(j)$, and $\beta_2^1 = 1_{2\mathbb{Z} + 1}(j)$. 


We call these grids \textit{random dyadic grids} because we view the parameters \( \beta_j \) and \( r \) as independent identically distributed random variables. There is a very natural probability space, say \(( \Omega, \mathcal{F} )\) associated to the parameters, \( \Omega = \{ 1, 2 \} \times \{ 0, 1 \}^\mathbb{Z} \). Averaging in this context means calculating the expectation in this probability space, that is
\[
E_{\Omega} f = \int_{\Omega} f(\omega)\,d\mathcal{F}(\omega) = \int_1^2 \int_{\{0,1\}^\mathbb{Z}} f(r, \beta)\,d\mu(\beta)\,\frac{dr}{r},
\]
where \( \mu \) stands for the canonical probability measure on \( \{ 0, 1 \}^\mathbb{Z} \) which makes the coordinate functions \( \beta_j \) independent with \( \mu(\beta_j = 0) = \mu(\beta_j = 1) = 1/2 \).

Random dyadic grids have been used for example in the study of \( T(\mathcal{B}) \) theorems on metric spaces with non-doubling measures \cite{NTV3, HytMa} and of \( \text{BMO} \) from dyadic \( \text{BMO} \) on the bidisc and product spaces of spaces of homogeneous type \cite{PiW, CLW}, inspired by celebrated work of John Garnett and Peter Jones from the 80’s \cite{GJ}. They have also been used in Hytönen’s representation theorem \cite{Hyt2} and in the resolution of the two-weight problem for the Hilbert transform \cite{LSSU, L1}.

3.4. \textbf{Haar bases}. Associated to dyadic intervals (or dyadic cubes) there is a very important collection of step functions, the Haar functions. In this section we recall the Haar bases on \( \mathbb{R} \) and on \( \mathbb{R}^d \), and some of their well-known properties.

3.4.1. \textbf{Haar basis on} \( \mathbb{R} \). The \textit{Haar function} associated to an interval \( I \subset \mathbb{R} \) is defined to be
\[
h_I(x) := |I|^{-1/2} \left( \mathbb{1}_{I_r}(x) - \mathbb{1}_{I_l}(x) \right),
\]
where \( I_r \) and \( I_l \) are the right and left halves respectively of \( I \), and the characteristic function \( \mathbb{1}_I(x) = 1 \) if \( x \in I \), zero otherwise. Haar functions have mean zero, that is, \( \int_{\mathbb{R}} h_I = 0 \), and they are normalized on \( L^2(\mathbb{R}) \).

The Haar functions indexed on any dyadic grid \( \mathcal{D} \), \( \{ h_I \}_{I \in \mathcal{D}} \), form a \textit{complete orthonormal system} of \( L^2(\mathbb{R}) \) (Haar 1910). In particular for all \( f \in L^2(\mathbb{R}) \), with \( \langle f, g \rangle := \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,dx \),
\[
f = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle \, h_I.
\]
You can find a complete proof of this statement in \cite[Chapter 9]{PW}.

The Haar basis is an unconditional basis of \( L^p(\mathbb{R}) \) and of \( L^p(w) \) if \( w \in A_p \) for \( 1 < p < \infty \) \cite{TV}. This is deduced from the boundedness properties of the martingale transform, we will say more about this dyadic operator in Section 4.1.

The Haar basis constitutes the first example of a wavelet basis\footnote{An orthonormal wavelet basis of \( L^2(\mathbb{R}) \) is an orthonormal basis where all its elements are translations and dilations of a fixed function \( \psi \), called the wavelet. More precisely, a function \( \psi \in L^2(\mathbb{R}) \) is a \textit{wavelet} if and only if the functions \( \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \) for \( j, k \in \mathbb{Z} \) form an orthonormal basis of \( L^2(\mathbb{R}) \).} and its corresponding Haar multiresolution analysis provides the canonical example of a multiresolution analysis \cite[Chapters 9-11]{PW}.

3.4.2. \textbf{Dyadic cubes and Haar basis on} \( \mathbb{R}^d \). In \( d \)-dimensional Euclidean space the regular dyadic cubes are cartesian products of regular dyadic intervals of the same generation. More precisely, a cube \( Q \in \mathcal{D}_j(\mathbb{R}^d) \) if and only if \( Q = I_{1} \times \cdots \times I_d \), where \( I_n \in \mathcal{D}_j(\mathbb{R}) \) for \( n = 1, 2, \ldots, d \). Each generation \( \mathcal{D}_j(\mathbb{R}^d) \) is a partition of \( \mathbb{R}^d \) and they form a nested grid, each cube has one parent and \( 2^d \) congruent children, and there are \( 2^d \) quadrants. Had we
used dyadic intervals with just one quadrant then the corresponding dyadic cubes in \( \mathbb{R}^d \) will also have only one quadrant. We denote \( \mathcal{D}(\mathbb{R}^d) \) the collection of all dyadic cubes in all generations, that is, \( \mathcal{D}(\mathbb{R}^d) = \cup_{j \in \mathbb{Z}} \mathcal{D}_j(\mathbb{R}^d) \). For \( Q \in \mathcal{D}(\mathbb{R}^d) \) we denote \( \mathcal{D}(Q) \) the set of dyadic cubes contained in \( Q \).

For each dyadic cube \( Q \) in \( \mathbb{R}^d \) we can associate \( 2^d \) step functions, constant on each children of \( Q \) by taking appropriate tensor products. More precisely, for \( Q \in \mathcal{D}(\mathbb{R}^d) \) and \( \vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_d) \), with \( \epsilon_n = 0 \) or \( 1 \), let

\[
\hat{h}^\vec{\epsilon}(x_1, \ldots, x_d) := h_{\vec{I}}^{\epsilon_1}(x_1) \times \cdots \times h_{\vec{I}}^{\epsilon_d}(x_d),
\]

where for each dyadic interval \( I \) we denote \( h_I^0 := h_I \) and \( h_I^1 = |I|^{-1/2} 1_I \). Note that \( h_{Q}^{\vec{I}} = |Q|^{-1/2} 1_Q \), where \( \vec{I} = (1, 1, \ldots, 1) \). The remaining \((2^d - 1)\) functions are the Haar functions associated to the cube \( Q \). The tensor product Haar functions \( h_Q^\vec{\epsilon} \), for \( \vec{\epsilon} \neq \vec{I} \), are supported on the corresponding dyadic cube \( Q \), they have mean zero, \( L^2 \) norm one, and they are constant on \( Q \)'s children. The collection \( \{ h_Q^\vec{\epsilon} : \vec{\epsilon} \neq \vec{I}, Q \in \mathcal{D}(\mathbb{R}^d) \} \) is an orthonormal basis of \( L^2(\mathbb{R}^d) \), and an unconditional basis of \( L^p(\mathbb{R}^d), 1 < p < \infty \) (the Haar basis). Figure 3.4.2 and Figure 3.4.2 illustrate the Haar functions associated to a square in \( \mathbb{R}^2 \) and to a cube in \( \mathbb{R}^3 \) respectively.

---

**Figure 1.** The three Haar function associated to the unit square in \( \mathbb{R}^2 \). Figure kindly provided by David Weirich [We].

**Figure 2.** The seven Haar functions associated to a cube in \( \mathbb{R}^3 \). Figure kindly provided by David Weirich [We].

The tensor product construction just described seems very rigid, it is very dependent on the geometry of the cubes and on the group structure of the Euclidean space \( \mathbb{R}^d \). Can we do
dyadic analysis on other settings? The answer is a resounding YES!!!! One such setting is on spaces of homogeneous type introduced by Coifman and Weiss in the early 70s. In Section 3.5 we will describe how to construct Haar basis on spaces of homogeneous type given suitable collections of "dyadic cubes" and argue why they constitute an orthonormal basis. This argument can be used to show that the Haar functions introduced in this section constitute an orthonormal basis of $L^2(\mathbb{R}^d)$.

3.5. Dyadic analysis on spaces of homogeneous type. In this section we will define spaces of homogeneous type. We will present a generalization of the dyadic cubes adapted to this setting. Given dyadic cubes we will show how to construct corresponding Haar functions, and briefly discuss the Auscher-Hytönen wavelets on spaces of homogeneous type.

Before we start we would like to quote Yves Meyer.

One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón-Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis.

Yves Meyer in his preface to [DH].

3.5.1. Spaces of homogeneous type (SHT). Let us first define what is a space of homogeneous type in the sense of Coifman and Weiss [CoW].

**Definition 3.4** (Coifman, Weiss 1971). For a set $X$, a triple $(X, \rho, \mu)$ is a space of homogeneous type (SHT) in Coifman-Weiss’s sense if

1. $\rho : X \times X \to [0, \infty)$ is a quasi-metric on $X$, more precisely the following hold:
   - (positive definite) $\rho(x, y) = 0$ if and only if $x = y$;
   - (symmetry) $\rho(x, y) = \rho(y, x) \geq 0$ for all $x, y \in X$;
   - (quasi-triangle inequality) there exists constant $A_0 \geq 1$ such that $\rho(x, y) \leq A_0(\rho(x, z) + \rho(z, y))$ for all $x, y, z \in X$.
2. $\mu$ is a nonzero Borel regular measure with respect to the topology induced by the quasi-metric.
3. Quasi-metric balls are $\mu$-measurable. A quasi-metric ball is the set $B(x, r) := \{ y \in X : \rho(x, y) < r \}$, where $x \in X$ and $r > 0$.
4. $\mu$ is a doubling measure, namely, there exists a constant $D_\mu \geq 1$ (the doubling constant of the measure $\mu$) such that for each quasi-metric ball $B(x, r)$
   \[ 0 < \mu(B(x, 2r)) \leq D_\mu \mu(B(x, r)) < \infty \text{ for all } x \in X, r > 0. \]

---

9Recipient of the 2017 Abel Prize.
10A measurable set $E$ of finite measure is Borel regular if there is a Borel set $B$ such that $E \subset B$ and $\mu(E) = \mu(B)$.
11The topology induced by a quasi-metric is the largest topology $\mathcal{T}$ such that for each $x \in X$ the quasi-metric balls centered at $x$ form a fundamental system of neighborhoods of $x$. Equivalently a set $\Omega$ is open, $\Omega \in \mathcal{T}$, if for each $x \in \Omega$ there exists $r > 0$ such that the quasi-metric ball $B(x, r) \subset \Omega$. A set in $X$ is closed if it is the complement of an open set.
Notice that Condition (4) implies that there are constants \( \omega > 0 \) (known as an \textit{upper dimension} of \( \mu \)) and \( C \geq 1 \) such that for all \( x \in X \), \( \lambda \geq 1 \) and \( r > 0 \)
\[
\mu(B(x, \lambda r)) \leq C \lambda^{\omega} \mu(B(x, r)).
\]
In fact, we can choose \( C = D_\mu \geq 1 \) and \( \omega = \log_2 D_\mu \).

The quasi-metric balls may \textbf{not} be open in the topology induced by the quasi-metric, as Example 3.5 shows. Therefore the assumption that the quasimetric balls are \( \mu \)-measurable is not redundant. The following example illustrate this phenomenon \cite{HytK}

**Example 3.5.** Consider the set \( X = \{-1\} \cup [0, \infty) \), the map \( \rho : X \times X \to [0, \infty) \) given by \( \rho(-1, 0) = \rho(0, -1) = 1/2 \) and \( \rho(x, y) = |x - y| \) otherwise, and the measure \( \mu(E) = \delta_{-1}(E) + m(E \cap [0, \infty)) \), where \( m \) is the Lebesgue measure and \( \delta_{-1} \) the point-mass at \( x = -1 \), that is \( \delta_{-1}(E) = 0 \) if \(-1 \notin E \) and \( \delta_{-1}(E) = 1 \) if \(-1 \in E \). Then \( \rho \) is not a metric since \( \rho(1, -1) = 2 > 3/2 = 1 + 1/2 = \rho(1, 0) + \rho(0, -1) \), however \( \rho \) is a quasi-metric and the measure \( \mu \) is doubling. It is a good exercise to compute both the quasi-triangle constant of \( \rho \) and the doubling constant of \( \mu \). Finally the ball \( B(-1, 1) = \{-1, 0\} \) is not open because it does not contain any ball centered at 0 with positive radius \( r \), since \([0, r] \subset B(0, r) \) and the interval \([0, r] \) is not contained in \( B(-1, 1) \).

A couple of further remarks are in order.

First, a given quasi-metric \( \rho \) may \textbf{not} be Hölder regular. Recall that \( \rho \) is a Hölder regular quasi-metric if there are constants \( 0 < \theta < 1 \) and \( C_0 > 0 \) such that
\[
|\rho(x, y) - \rho(x', y')| \leq C_0 \rho(x, x')^\theta [\rho(x, y) + \rho(x', y')]^{1-\theta} \quad \forall x, x', y \in X.
\]
Metrics are Hölder regular for any \( 0 < \theta \leq 1 \), \( C_0 = 1 \). The quasi-metric in Example 3.5 is not continuous let alone Hölder regular. Quasi-metric balls for Hölder regular quasi-metrics are always open.

Second, Roberto Macías and Carlos Segovia showed in 1979 \cite{MS} that given a space of homogeneous type \((X, \rho, \mu)\) there is an equivalent Hölder regular quasi-metric \( \rho' \) on \( X \) and some \( \theta \in (0, 1) \), and for which the measure \( \mu \) is 1-Ahlfors regular, more precisely,
\[
\mu(B_{\rho'}(x, r)) \sim r^1.
\]
Here are some examples of spaces of homogeneous type.

- \( \mathbb{R}^n \), with the Euclidean metric and the Lebesgue measure.
- \( \mathbb{R}^n \) with the Euclidean metric and an absolutely continuous measure with respect to the Lebesgue measure \( d\mu = w \, dx \) where \( w \) is a doubling weight (for example \( w \) could be an \( A_\infty \) weight).
- Quasi-metric spaces with \( d \)-Ahlfors regular measure: \( \mu(B(x, r)) \sim r^d \) (e.g. Lipschitz surfaces, fractal sets, \( n \)-thick subsets of \( \mathbb{R}^n \)). More concretely, consider for example \( X \) the four-corners Cantor set with the Euclidean metric and the one-dimensional Hausdorff measure, or consider \( X \) the graph of a Lipschitz function \( F : \mathbb{R}^n \to \mathbb{R} \) with the induced Euclidean metric and measure the volume of the set's "shadow", \( \mu(E) = m \left( \{ x \in \mathbb{R}^n : (x, F(x)) \subset E \} \right) \) where \( m \) is the Lebesgue measure on \( \mathbb{R}^n \).
- \( C^\infty \) manifolds with doubling volume measure for geodesic balls.
- Nilpotent Lie groups \( G \) with the left-invariant Riemannian metric and the induced measure (e.g. Heisenberg group where \( X \) is the boundary of the unit ball in \( \mathbb{C}^n \), \( \rho(z, w) = 1 - \bar{z} \cdot w \) and with surface measure).
The 2015 book by Ryan Alvarado and Marius Mitrea [AMi] discusses in more detail many of these examples and relies heavily on the Macías-Segovia philosophy, meaning they consider equivalent classes of quasi-metrics knowing that among them they can choose a representative that is Hölder regular and for which the measure is Ahlfors regular.

3.5.2. Dyadic cubes in SHT. Systems of "dyadic cubes" were built by Hugo Aimar and Roberto Macías, Eric Sawyer and Richard Wheeden, and Guy David in the 80’s [AiM, SW, Da], and by Michael Christ in the 90’s [Chr] on spaces of homogeneous type, and by Tuomas Hytönen and Anna Kairema in 2012 on geometrically doubling quasi-metric spaces [HytK] without reference to a measure.

A geometrically doubling quasi-metric space \((X,d)\) is one such that every quasi-metric ball of radius \(r\) can be covered with at most \(N\) quasi-metric balls of radius \(r/2\) for some natural number \(N\).

**Example 3.6.** Spaces of homogeneous type in the Coifman-Weiss sense are geometrically doubling [CoW].

Systems \(\mathcal{D}\) of dyadic cubes in spaces of homogeneous type or, more generally, on geometrically doubling spaces, are organized in disjoint generations \(\mathcal{D}_k, k \in \mathbb{Z}\), such that \(\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k\) and the following qualitative properties hold.

(a) Each generation \(\mathcal{D}_k\) is a partition of \(X\), so the cubes in a generation are pairwise disjoint and form a covering of \(X\).

(b) The generations are nested, that is there is no partial overlap across generations.

(c) As a consequence, each cube has unique ancestors in earlier generations.

(d) Dyadic cubes have at most \(M\) children for some positive natural number \(M\) (this is a consequence of the geometric doubling property).

(e) There exists a constant \(\delta \in (0,1)\) such that for every dyadic cube in \(\mathcal{D}_k\) there are inner and outer balls of radius roughly \(\delta^k\) (the "sidelength" of the cube).

(f) The outer ball corresponding to a dyadic cube’s child is inside its parent’s outer ball.

Note that since \(\delta \in (0,1)\) the larger \(k\) is the smaller in diameter the cubes are. If \(Q \in \mathcal{D}_k\) then its parent will be the unique cube \(\bar{Q} \in \mathcal{D}_{k-1}\) such that \(Q \subset \bar{Q}\).

Furthermore, cubes can be constructed to have a "small boundary property" [Chr, HytK] which is very useful in applications.

A quantitative and more precise statement of the defining properties for a dyadic system of cubes on geometric doubling metric spaces is encapsulated in the following construction that appeared in [HytK, Theorem 2.2].

**Theorem 3.7** (Hytönen, Kairema 2012). Given \((X,d)\) a geometrically doubling quasi-metric space. Suppose the constants \(C_0 \geq c_0 > 1\) and \(\delta \in (0,1)\) satisfy \(12A_0^3C_0\delta \leq c_0\). Given a set of points \(\{z^k_\alpha : \alpha \in \mathcal{A}_k\}\), where \(\mathcal{A}_k\) is a countable set of indexes, with the properties that \(d(z^k_\alpha, z^k_\beta) \geq c_0\delta^k (\alpha \neq \beta), \min_{\alpha \in \mathcal{A}_k} d(x, z^k_\alpha) < C_0\delta^k\) for all \(x \in X\).

For each \(k \in \mathbb{Z}\) and \(\alpha \in \mathcal{A}_k\) there exists sets \(Q^k_\alpha, \overline{Q^k_\alpha}, Q^k_{\alpha} \subseteq Q^k_\alpha \subseteq Q^k_{\alpha} \subseteq -\) called open, half-open, and closed dyadic cubes— such that:

1. \(Q^k_\alpha\) and \(Q^k_{\alpha}\) are the interior and closure of \(Q^k_\alpha\), respectively;
2. (nested) if \(\ell \geq k\), then either \(Q^k_\beta \subseteq Q^k_\alpha\) or \(Q^k_\alpha \cap Q^k_\beta = \emptyset\).
(3) (partition) \[ X = \bigcup_{\alpha \in \mathcal{A}_k} Q^k_{\alpha} \text{ for all } k \in \mathbb{Z}; \]
(4) (inner/outer balls) \[ B(z^k_{\alpha}, c_1 \delta^k) \subseteq Q^k_{\alpha} \subseteq B(z^k_{\alpha}, C_1 \delta^k) \text{ where } c_1 := (3A_0^2)^{-1}c_0 \text{ and } C_1 := 2A_0C_0; \]
(5) if \( \ell \geq k \) and \( Q^k_{\beta} \subseteq Q^k_{\alpha} \), then \( B(z^k_{\beta}, C_1 \delta^\ell) \subseteq B(z^k_{\alpha}, C_1 \delta^k) \).

The open and closed cubes \( Q^k_{\alpha} \) and \( Q^k_{\alpha} \) depend only on the points \( z^k_{\beta} \) for \( \ell \geq k \). The half-open cubes \( Q^k_{\alpha} \) depend on \( z^k_{\beta} \) for \( \ell \geq \min(k, k_0) \), where \( k_0 \in \mathbb{Z} \) is a preassigned number entering the construction.

The geometrically doubling condition implies that sets of points \( \{x^k_{\alpha} : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\} \) with the required separation properties exist and that the set \( \mathcal{A}_k \) is a countable set of indices for each \( k \in \mathbb{Z} \). The cubes in this construction are built as countable unions of quasi-metric balls, hence once a space of homogeneous type is given, the cubes will be measurable sets.

Figure 3. The 4 Haar functions for a cube with 5 children in SHT. Figures kindly provided by David Weirich [We].

3.5.3. Haar basis on SHT. Given a space of homogeneous type \((X, \rho, \mu)\) with a dyadic structure \(\mathcal{D}\) given by Theorem 3.7, we can construct a system of Haar functions that will be an orthonormal basis of \(L^2(X, \mu)\).

Given a cube \(Q \in \mathcal{D}\), denote by \(\text{ch}(Q)\) the collection of dyadic children of \(Q\), and by \(N(Q)\) its cardinality, that is \(Q\) has \(N(Q)\) children. Let \(S_Q\) be the subspace of \(L^2(X, \mu)\) spanned by those square integrable functions that are supported on \(Q\) and are constant on the children of \(Q\). The subspace \(S_Q\) has dimension \(N(Q)\) as the characteristic functions of the children cubes normalized with respect to the \(L^2\) norm, namely \(\{1_Q'/\sqrt{\mu(Q')} : Q' \in \text{ch}(Q)\}\), form an orthonormal basis for \(S_Q\). The subspace \(S_Q^0\) of \(S_Q\) consisting of those functions that have mean zero, that is \(\int_Q f(x)\,dx = 0\), will have one fewer dimension, namely \(\dim(S_Q^0) = N(Q) - 1\).

Given an enumeration of the children of \(Q\), that is a bijection \(u_Q : \{1, 2, \ldots, N(Q)\} \rightarrow \text{ch}(Q)\), we will define recursively subsets of \(Q\) that are unions of children of \(Q\). More precisely at each stage we will remove one child according to the given enumeration, let \(E_Q^{k+1} \subseteq Q\), let \(E_Q^{k+1} = E_Q^k \setminus u_Q(k)\) for \(k = 1, 2, \ldots, N(Q) - 1\). We can split each of these sets into two disjoint pieces, \(E_Q^{-} := E_Q^{ki+} \cup E_Q^{-}\) where \(E_Q^{ki+} = u_Q(i)\), the child removed (green in
Figure 3.5.2) and $E_Q^{i,-} = E_Q^{i,+}$ (blue in Figure 3.5.2). With this notation, the Haar functions associated to the cube $Q$ and the enumeration $u_Q$ as illustrated in Figure 3.5.2, are supported on $Q$ and are constant on the colored regions: positive on the green regions, negative on the blue regions, and zero on the red regions, thus they are given by

$$h_Q^i(x) = a \mathbb{1}_{E_Q^{i,+}}(x) - b \mathbb{1}_{E_Q^{i,-}}(x), \quad 1 \leq i \leq N(Q) - 1,$$

where the positive constants $a$ and $b$, dependent on the base cube $Q$ and the label $i$, are chosen to enforce $L^2$ normalization and mean zero. More precisely, the unknowns $a, b$ must satisfy the system of two equations:

$$\int_Q |h_Q^i(x)|^2 \, d\mu = a^2 \mu(E_Q^{i,+}) + b^2 \mu(E_Q^{i,-}) = 1$$

$$\int_Q h_Q^i(x) \, d\mu = a \mu(E_Q^{i,+}) - b \mu(E_Q^{i,-}) = 0.$$

Solving the system of equations we get the positive solutions

$$a = \sqrt{\mu(E_Q^{i,-})/(\mu(E_Q^i) \mu(E_Q^{i,+}))}, \quad b = \sqrt{\mu(E_Q^{i,+})/(\mu(E_Q^i) \mu(E_Q^{i,-}))}.$$

Note that the doubling condition on the measure $\mu$ ensures $\mu(Q) > 0$ for all $Q \in \mathcal{D}$, and hence also $\mu(E_Q^i) > 0$ for all labels $i$.

The Haar basis consists of all functions $h_Q^i$ where $Q \in \mathcal{D}$ and $i = 1, 2, \ldots, N(Q) - 1$. Note that a cube may not subdivide for a while, meaning that it could have just one child, itself, for several generations or forever. In the former case we wait until we subdivide to define the subspace $S_Q^0$, in the later case we let $S_Q^0$ be the trivial subspace.

By construction for each $Q \in \mathcal{D}$ the collection $\{h_Q^i : i = 1, \ldots, N(Q) - 1\}$ is normalized on $L^2(X, \mu)$, each Haar function has mean zero, and by the nested property of the dyadic cubes it is easy to verify this is an orthonormal family. No matter what enumeration for $ch(Q)$ we use we will get each time an orthonormal basis of $S_Q^i$. The orthogonal projection onto $S_Q^0$ of a square integrable function $f$ is independent of the orthonormal basis chosen on $S_Q^0$. Given $x \in Q$ choose an enumeration so that $x \in u_Q(1) =: R \in ch(Q)$ then

$$\text{Proj}_{S_Q} f(x) = \langle f, h_Q^i \rangle_{\mu}, \quad h_Q^i(x) = \langle f, h_Q^i \rangle_{R} - \langle f \rangle_Q^{\mu},$$

where $\langle f, g \rangle_{\mu}$ denotes the inner product in $L^2(X, \mu)$ and $\langle f \rangle_Q^{\mu}$ denotes the $\mu$-average of $f$. The first equality holds by support considerations, since $h_i^j(x) = 0$ for all $i > 1$ by the choice of the enumeration, the second equality is now a simple calculation by substitution.

Using a telescoping sum argument one can verify that completeness of the Haar basis on $L^2(\mu)$ hinges on the following limits holding in the $L^2(\mu)$ sense:

$$\lim_{j \to \infty} E_j^{\mu} f = f,$$

$$\lim_{j \to \infty} E_j^{\mu} f = 0,$$

where $E_j f := \langle f \rangle_Q^{\mu}$, with $x \in Q \in \mathcal{D}_j$, or $E_j f = \sum_{Q \in \mathcal{D}_j} \langle f \rangle_Q^{\mu} \mathbb{1}_Q$. That the limits do hold can be justified by martingale theory [Hyt3, Mul], in fact they do hold in $L^p(X, \mu)$ for $1 < p < \infty$. The pointwise convergence a.e. of the averages to $f$ as $j$ goes to infinity is a consequence of the Lebesgue differentiation theorem which holds because the measure is assumed to be Borel regular, see [AMi, Section 3.3].
3.5.4. Random dyadic grids, adjacent dyadic grids, and wavelets on SHT. The counterparts of the random dyadic grids and the one-third trick have been identified in the general setting of geometrically doubling quasi-metric spaces by Tuomas Hytönen and his students and collaborators. Using them, Pascal Auscher and Tuomas Hytönen constructed in 2013 a remarkable orthonormal basis of $L^2(X, \mu)$ [AH1, AH2].

A notion of random dyadic grids can be introduced on geometrically doubling quasi-metric spaces $(X, d)$ by randomizing the order relations in the construction of the Hytönen-Kairema cubes [HytMa, HytK]. In 2014, Tuomas Hytönen and Olli Tapiola modified the randomization to improve upon Auscher-Hytönen wavelets in metric spaces [HytTa]. A different randomization can be found in [NRV].

One can find finitely many adjacent families of Hytönen-Kairema dyadic cubes, $\mathcal{D}^t$ for $t = 1, \ldots, T$, with the same parameters, that play the role of the $1/3$-shifted dyadic grids in $\mathbb{R}$. The main property the adjacent families of dyadic cubes have is that given any ball $B(x, r) \subset X$, with $r \sim \delta^k$, then there is $t \in \{1, 2, \ldots, T\}$ and a cube in the $t$-grid and in the $k$th-generation, $Q \in \mathcal{D}^t_k$, such that $B(x, r) \subset Q \subset B(x, Cr)$, where $C > 0$ is a geometric constant only dependent on the quasi-metric and geometric doubling parameters of $X$ [HytK]. Furthermore, given a $\sigma$-finite measure $\mu$ on $X$, the adjacent dyadic systems can be chosen so that all cubes have small boundaries: $\mu(\partial Q) = 0$ for all $Q \in \cup_{t=1}^T \mathcal{D}^t [HytK]$. The number of indexes $\alpha$ so that $y_k^\alpha \in Y_k$ for each $Q^k_\alpha$ is exactly $N(Q^k_\alpha) - 1$, where recall that $N(Q^k_\alpha)$ denotes the number of children of $Q^k_\alpha$. This is the right number of wavelets per cube $Q^k_\alpha$ if our intuition is to be guided by the constructions of the Haar functions. The precise nature of these wavelets is detailed in [AH1, Theorem 7.1].

Furthermore, the functions $\{\psi^k_{\alpha}\}_{k \in \mathbb{Z}, \alpha \in Y_k}$ form an unconditional basis on $L^p(X)$ for all $1 < p < \infty$ and the following wavelet expansion is valid in $L^p(X)$,

$$f(x) = \sum_{k \in \mathbb{Z}} \sum_{y^k_r \in Y^k} (f, \psi^k_{\alpha}) \psi^k_{\alpha}(x).$$

Hytönen and Tapiola were able to build such wavelets for all $0 < \eta < 1$ in the context of metric spaces [HytTa]. It is still an open problem to construct smooth wavelets that are
compactly supported. These wavelets have been used to study Hardy and BMO spaces on product spaces of homogeneous type, as well as their dyadic counterparts [KLPW].

4. Dyadic operators, weighted inequalities, and Hytönen’s representation theorem

In this section we introduce the model dyadic operators: the martingale transform, the dyadic square function, the dyadic paraproduct, Petermichl’s Haar shift operator, and Haar shift operators of arbitrary complexity. All ingredients in Hytönen’s proof of the $A_2$ conjecture [Hyt2]. We will state the known quantitative one- and two-weight inequalities for these dyadic operators. We end the section with Hytönen’s representation theorem in terms of Haar shift operators of arbitrary complexity, dyadic paraproducts and adjoints of dyadic paraproducts over random dyadic grids, valid for all Calderón-Zygmund operators and key to the resolution of the $A_2$ conjecture.

4.1. Martingale transform. Let $\mathcal{D}$ denote a dyadic grid on $\mathbb{R}$, the Martingale transform is the linear operator formally defined as

$$T_\sigma f(x) := \sum_{I \in \mathcal{D}} \sigma_I \langle f, h_I \rangle h_I(x), \quad \text{where} \quad \sigma_I = \pm 1.$$

This is a constant Haar multiplier in analogy to Fourier multipliers, where here the Haar coefficients are modified multiplying them by uniformly bounded constants, the Haar symbol $\{\sigma_I : I \in \mathcal{D}\}$ (in this case arbitrary changes of sign). The martingale transform is bounded on $L^2(\mathbb{R})$, in fact it is an isometry on $L^2(\mathbb{R})$ by Plancherel’s identity, that is $\|T_\sigma f\|_{L^2} = \|f\|_{L^2}$.

The martingale transform is a good toy model for Calderón-Zygmund singular integral operators such as the Hilbert transform. Suffices to recall that on Fourier side the Hilbert transform is a Fourier multiplier with Fourier symbol $m_H(\xi) = -i \text{sgn}(\xi)$. Compare the Fourier transform of the Hilbert transform and the "Haar transform" of the martingale transform, namely,

$$\hat{H}f(\xi) = -i \text{sgn}(\xi) \hat{f}(\xi) \quad \text{and} \quad \langle T_\sigma f, h_I \rangle = \sigma_I \langle f, h_I \rangle.$$

Unconditionality of the Haar basis on $L^2(\mathbb{R})$ follows from uniform (on the choice of signs $\sigma$) boundedness of the martingale transform $T_\sigma$ on $L^p(\mathbb{R})$. More precisely for all $f \in L^p(\mathbb{R})$

$$\sup_{\sigma} \|T_\sigma f\|_{L^p} \lesssim_p \|f\|_{L^p}.$$

This was proven by Donald Burkholder in 1984, he also found the optimal constant $C_p$ in work that can be described as the precursor of the (exact) Bellman function method [Bur2].

Unconditionality of the Haar basis on $L^p(w)$ when $w \in A_p$ follows from the uniform boundedness of $T_\sigma$ on $L^p(w)$, this was proven in 1996 by Sergei Treil and Sasha Volberg [TV].

4.1.1. Quantitative weighted inequalities for the martingale transform. Quantitative one- and two-weight inequalities are known for the martingale transform. In fact, the $A_2$ conjecture (linear bound) was proven by Janine Wittwer in 2000 and necessary and sufficient conditions for two-weight uniform (on the symbol $\sigma$) $L^2$ boundedness were identified by Fedja Nazarov, Sergei Treil, and Sasha Volberg in 1999. We present now the precise statements.

Sharp linear bounds on $L^2(w)$ when $w$ is an $A_2$ weight are known [W1]. More precisely, for all $\sigma$ there is $C > 0$ such that for all $w \in A_2$ and all $f \in L^2(w)$

$$\|T_\sigma f\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)}.$$
Sharp extrapolation gives optimal bounds on $L^p(w)$ when $w$ is an $A_p$ weight [DGPPet]. More precisely, for all $\sigma$ there is a constant $C_\sigma > 0$ such that for all $w \in A_p$ and $f \in L^p(w)$

$$\|T_\sigma f\|_{L^p(w)} \leq C_\sigma [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(w)}.$$ 

Necessary and sufficient conditions on pairs of weights $(u, v)$ are known ensuring two weight boundedness [NTV1]. More precisely,

**Theorem 4.1** (Nazarov, Treil, Volberg 1999). The martingale transforms $T_\sigma$ are uniformly (on $\sigma$) bounded from $L^2(u)$ to $L^2(v)$ if and only if the following conditions hold simultaneously:

(i) $(u, v)$ is in joint dyadic $\mathcal{A}_2$. Namely $[u, v]_{\mathcal{A}_2} := \sup_{I \in \mathcal{D}} \langle u^{-1}\rangle_I \langle v \rangle_I < \infty$.

(ii) $\{|I| \Delta_I(u^{-1})^2 \langle v \rangle_I |I| \} \subseteq \mathcal{D}$ is a $u^{-1}$-Carleson sequence.

(iii) $\{|I| \Delta_I(v^{-1})^2 \langle u \rangle_I |I| \} \subseteq \mathcal{D}$ is a $v$-Carleson sequence (dual condition).

(iv) The positive dyadic operator $T_0$ is bounded from $L^2(u)$ into $L^2(v)$. Where

$$T_0 f(x) := \sum_{I \in \mathcal{D}} \frac{\alpha_I}{|I|} \langle f \rangle_I \mathbb{1}_I(x),$$

with $\alpha_I := (|\Delta_I v|/\langle v \rangle_I)(|\Delta_I (u^{-1})|/\langle u^{-1} \rangle_I) |I|$, and $\Delta_I v := \langle v \rangle_{I_+} - \langle v \rangle_{I_-}$.

A sequence $\{\lambda_I\}_{I \in \mathcal{D}}$ is $v$-Carleson if and only if there is constant $B > 0$ such that $\sum_{I \in \mathcal{D}(J)} \lambda_I \leq B v(J)$ for all $J \in \mathcal{D}$. The smallest constant $B$ is called the intensity of the sequence. When $u = v = w \in A_2$ then (i)-(iii) hold, and by Example 5.8 the sequence $\{\alpha_I\}_{I \in \mathcal{D}}$ is a $1$-Carleson sequence implying (iv).

In 2008, Nazarov, Treil, and Volberg found necessary and sufficient conditions for two-weight boundedness of martingale transforms and other well-localized operators [NTV4], see also [Vu1].

**4.2. Dyadic square function.** The dyadic square function is the sublinear operator formally defined as

$$(S^\mathcal{D} f)(x) := \left( \sum_{I \in \mathcal{D}} |\langle f, h_I \rangle|^2/|I| \mathbb{1}_I(x) \right)^{1/2}.$$ 

The dyadic square function is an isometry on $L^2(\mathbb{R})$, as a calculation quickly reveals, namely $\|S^\mathcal{D} f\|_{L^2} = \|f\|_{L^2}$. It is also bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, furthermore

$$\|S^\mathcal{D} f\|_{L^p} \sim \|f\|_{L^p}.$$ 

This result plays the role of Plancherel on $L^p$ (Littlewood-Paley theory). It readily implies boundedness of $T_\sigma$ on $L^p(\mathbb{R})$ since $S^\mathcal{D}(T_\sigma f) = S^\mathcal{D} f$, as follows,

$$\|T_\sigma f\|_{L^p} \sim \|S^\mathcal{D}(T_\sigma f)\|_{L^p} = \|S^\mathcal{D} f\|_{L^p} \sim \|f\|_{L^p}.$$ 

A somewhat convoluted argument can be done to prove the $L^p$ boundedness of the dyadic square function. First prove $L^2(w)$ estimates for $A_2$ weights $w$, second extrapolate to get $L^p(w)$ estimates for $A_p$ weights $w$, and third set $w \equiv 1 \in A_p$. Stephen Buckley has a very nice and elementary argument showing boundedness of the dyadic square function on $L^2(w)$ when $w$ is an $A_2$ weight [Bu2] or see [P1, Section 2.5.1]. One can track the dependence on the weight again get a $3/2$ power on the $A_2$ characteristic of the weight [BeCMoP, Section 5], far from the optimal linear dependence discussed in Section 4.2.1.
4.2.1. One-weight estimates for $S^\mathbb{Q}$. Quantitative one-weight inequalities are known for the dyadic square function. The $A_2$ conjecture (linear bound) was proven by Sanja Hukovic, Sergei Treil, and Sasha Volberg in 2000 [HTV] and the reverse estimate was proven by Stefanie Petermichl and Sandra Pott in 2002 [PetPo].

We present now the precise statements. For all weights $w \in A_2$ and functions $f \in L^2(w)$

$$\|A_2^\frac{1}{2}\|f\|_{L^2(w)} \lesssim \|S^\mathbb{Q}f\|_{L^2(w)} \lesssim [w]_{A_2} \|f\|_{L^2(w)}$$

The direct and reverse estimates on $L^2(w)$ for the dyadic square function play the role of Plancherel on $L^2(w)$. We can use these inequalities to obtain $L^2(w)$ bounds for the martingale transform $T_\sigma$ of the form $[w]^{3/2}_{A_2}$. However the optimal bound is linear [W1], as we already mentioned in Section 4.1.1.

Boundedness on $L^2(w)$ for all weights $w \in A_2$ implies by extrapolation boundedness on $L^p(\mathbb{R})$ (and on $L^p(w)$ for all $w \in A_p$). However sharp extrapolation will only yield the optimal power for $1 < p \leq 2$, if one starts with the optimal linear bound on $L^2(w)$. Not only $S^\mathbb{Q}$ is bounded on $L^p(w)$ if $w \in A_p$, moreover for $1 < p < \infty$ and for all $w \in A_p$ and $f \in L^p(w)$

$$\|S^\mathbb{Q}f\|_{L^p(w)} \lesssim_p [w]_{A_p}^{\max\{\frac{1}{2}, \frac{1}{p-1}\}} \|f\|_{L^p(w)}.$$  

The power $\max\{1/2, 1/(p-1)\}$ is optimal. It corresponds to sharp extrapolation starting at $r = 3$ with square root power [CrMPz2]. More precisely, for all $w \in A_3$ and $f \in L^3(w)$,

$$\|S^\mathbb{Q}f\|_{L^3(w)} \lesssim [w]_{A_3}^{\frac{1}{2}} \|f\|_{L^3(w)}.$$  

This estimate is valid more generally for Wilson’s intrinsic square function [Le2, Wil2].

Sharp extrapolation from the reverse estimate on $L^2(w)$ also yields the following reverse estimate on $L^p(w)$ for all $w \in A_p$ and $f \in L^p(w)$,

$$\|f\|_{L^p(w)} \lesssim_p [w]_{A_p}^{\frac{1}{2} \max\{1, \frac{1}{p-1}\}} \|S^\mathbb{Q}f\|_{L^p(w)}.$$  

This estimate can be improved, using deep estimates of Chang, Wilson, and Wolff [CWilW] for all $p > 1$ to the following $1/2$ power of the smaller Fujii-Wilson $A_\infty$ characteristic,

$$\|f\|_{L^p(w)} \lesssim_p [w]_{A_\infty}^{\frac{1}{2}} \|S^\mathbb{Q}f\|_{L^p(w)}.$$  

This estimate is better in the range $1 < p < 2$ where the power is $1/2$ instead of $1/2(p-1)$.

For future reference, we can compute precisely the weighted $L^2$ norm of $S^\mathbb{Q}f$ as follows

$$\|S^\mathbb{Q}f\|_{L^2(w)}^2 = \sum_{I \in \mathbb{D}} \langle f, h_I \rangle |\langle w, h_I \rangle|^2.$$  

4.2.2. Two-weight estimates for $S^\mathbb{Q}$. Two-weight inequalities are understood for the dyadic square function. The necessary and sufficient conditions for two-weight $L^2$ boundedness are known [NTV1]. Qualitative (mixed) estimates have been found by different authors, these estimates reduce to the linear estimate in the one-weight case. We present now the precise statements.

**Theorem 4.2** (Nazarov, Treil, Volberg 1999). The dyadic square function $S^\mathbb{Q}$ is bounded from $L^2(u)$ into $L^2(v)$ if and only if the following conditions hold simultaneously:

(i) $(u, v) \in \mathcal{A}_2$ (joint dyadic $\mathcal{A}_2$).

(ii) $\{\|I\| |\Delta f u^{-1} |^2(v)_I \} \subseteq \mathcal{D}$ is a $u^{-1}$-Carleson sequence with intensity $C_{u,v}$.
Notice (ii) is a localized "testing condition" on test functions $u^{-1}1_{J}$. Also note that the necessary and sufficient conditions (i)-(iii) in Theorem 4.1 for the martingale transform can be now replaced by

(i) $S^{\Omega}$ is bounded from $L^{2}(u)$ into $L^{2}(v)$.
(ii) $S^{\Omega}$ is bounded from $L^{2}(v^{-1})$ into $L^{2}(u^{-1})$.

This is because $(u, v) \in \mathcal{A}_{2}$ if and only if $(v^{-1}, u^{-1}) \in \mathcal{A}_{2}$.

A quantitative version of the boundedness estimate in terms of the constants appearing in the necessary and sufficient conditions is the following

$$\|S^{\Omega}\|_{L^{2}(u)\rightarrow L^{2}(v)} \lesssim ([u, v]_{\mathcal{A}_{2}} + C_{u,v})^{1/2}.$$  

There are similar two-weight $L^{p}$ estimates for continuous square function [LLi1, LLi2], see also [BeCMoP, Theorem 6.2].

If the weights $(u, v) \in \mathcal{A}_{2}$ and $u^{-1} \in A_{\infty}$ then they satisfy the necessary and sufficient conditions in Theorem 4.2 and the following estimate holds [BeCMoP],

$$\|S^{\Omega}\|_{L^{2}(u)\rightarrow L^{2}(v)} \lesssim ([u, v]_{\mathcal{A}_{2}} + [u, v]_{\mathcal{A}_{2}}[u^{-1}]_{A_{\infty}})^{1/2}.$$  

Setting $u = v = w \in A_{2}$ this improves the known linear bound to a mixed-type bound

$$\|S^{\Omega}\|_{L^{2}(w)} \lesssim ([w]_{A_{2}}[w^{-1}]_{A_{\infty}})^{1/2} \lesssim [w]_{A_{2}}.$$  

Same one-weight estimate have been shown to hold for the dyadic square function and for matrix valued weights [HytPetV]. Quantitative weighted estimates from $L^{p}(u)$ into $L^{q}(v)$ in terms of quadratic testing condition are known [Vu2].

4.3. Petermichl’s dyadic shift operator. Given parameters $(r, \beta) \in \Omega = [1, 2] \times \{0, 1\}^{2}$, the Petermichl’s dyadic shift operator $\Pi^{r,\beta}$ (pronounced “Sha”) associated to the random dyadic grid $\mathcal{D}^{r,\beta}$ is defined for functions $f \in L^{2}(\mathbb{R})$ by

$$\Pi^{r,\beta}f(x) := \sum_{I \in \mathcal{D}^{r,\beta}} \langle f, h_{I} \rangle H_{I}(x) = \sum_{I \in \mathcal{D}^{r,\beta}} 2^{-1/2} \sigma(I) \langle f, h_{\tilde{I}} \rangle h_{I}(x),$$

where $H_{I} = 2^{-1/2}(h_{I_{r}} - h_{I_{l}})$ and $\sigma(I)$ is $\pm 1$ depending whether $I$ is the right or left child of $I$’s parent $\tilde{I}$. More precisely, $\sigma(I) = 1$ if $I = (\tilde{I})_{r}$ and $\sigma(I) = -1$ if $I = (\tilde{I})_{l}$.

When $r = 1$ and $\beta_{j} = 0$ for all $j \in \mathbb{Z}$ the corresponding grid is the regular dyadic grid and we denote the associated dyadic shift operator simply $\Pi$.

Petermichl’s dyadic shift operators are isometries on $L^{2}(\mathbb{R})$, that is for all $r, \beta \in \Omega$, $\|\Pi^{r,\beta}f\|_{L^{2}} = \|f\|_{L^{2}}$, and they are bounded operators on $L^{p}(\mathbb{R})$, as can be readily seen using Plancherel’s identity and dyadic square function estimates.

Each operator $\Pi^{r,\beta}$ is a good dyadic model for the Hilbert transform $H$. The images under $\Pi^{r,\beta}$ of the Haar functions are the normalized differences of the Haar functions on its children, namely $\Pi^{r,\beta}h_{J}(x) = H_{J}(x)$. The functions $h_{J}$ and $H_{J}$ can be viewed as localized sines and cosines, in the sense that if we were to extend them periodically, with period the length of the support, we will see two square waves shifted by half the length of the period. More evidence comes from the way the family $\{\Pi^{r,\beta}\}_{(r,\beta)\in \Omega}$ interacts with translations, dilations and reflections. Each dyadic shift operator does not have symmetries that characterize the Hilbert transform, but an average over all random dyadic grids $\mathcal{D}^{r,\beta}$ does. It is a good

\[12\] Recall that any bounded linear operator on $L^{2}(\mathbb{R})$ that commutes with dilations and translations and anticommutes with reflections must be a constant multiple of the Hilbert transform.
exercise to figure out how each individual shift interacts with these rigid motions, they almost commute except that the dyadic grid changes. For example regarding reflections, it can be seen that if we denote by \( R(x) = -x \) then \( R \Pi_{r,\beta} = -\Pi_{r,\beta} R \), where \( \beta = \{1 - \beta_i\}_{i \in \mathbb{Z}} \). The corresponding rules for translations and dilations are slightly more complicated, but what matters is that there is a one-to-one correspondence between the dyadic grids so that when averaging over all dyadic grids the average will have the desired properties, and hence it will be a constant multiple of the Hilbert transform. This is precisely what Stefanie Petermichl proved in 2000, a ground breaking and unexpected new result for the Hilbert transform \[\text{[Pet1]}\]. More precisely she showed that

\[
H = -\frac{8}{\pi} \mathbb{E}_\Omega \Pi r,\beta = -\frac{8}{\pi} \int_{\Omega} \Pi r,\beta d\mathcal{P}(r, \beta).
\]

The result follows after verifying that the averages have the invariance properties that characterize the Hilbert transform \[\text{[Pet1, Hyt1]}\]. Because the shift operators \( \Pi r,\beta \) are uniformly bounded on \( L^p(\mathbb{R}) \) for \( 1 < p < \infty \), this representation will immediately imply that the Hilbert transform \( H \) is bounded on \( L^p(\mathbb{R}) \) in the same range, a result first proved by Marcel Riesz in 1928. Similarly once uniform (on the dyadic grids \( D r,\beta \) ) weighted inequalities are verified for \( \Pi r,\beta \) the inequalities will be inherited by the Hilbert transform. Petermichl proved the linear bounds on \( L^2(w) \) for the shift operators using Bellman function methods and hence she proved the A2 conjecture for the Hilbert transform \[\text{[Pet2]}\].

These results added a very precise new dyadic perspective to such a classic and well-studied operator as the Hilbert transform. Similar representations hold for the Beurling and the Riesz transforms \[\text{[PetV, Pet3]}\], these operators have many invariance properties as the Hilbert transform does. It came as a surprise when Tuomas Hytönen proved in 2012 that there is a representation formula valid for all Calderón-Zygmund singular integral operators \[\text{[Hyt2]}\]. To state Hytönen’s result we need to introduce Haar shift operators of arbitrary complexity and paraproducts.

4.4. Haar shift operators of arbitrary complexity. The Haar shift operators of complexity \((m, n)\) associated to a dyadic grid \( \mathcal{D} \) were introduced by Michael Lacey, Stefanie Petermichl, and Mari Carmen Reguera \[\text{[LPetR]}\], they are defined on \( L^2(\mathbb{R}) \) as follows

\[
\Pi_{m,n} f(x) := \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_m(L), J \in \mathcal{D}_n(L)} c_{I,J}^L \langle f, h_I \rangle h_J(x),
\]

where the coefficients \( |c_{I,J}^L| \leq \frac{\sqrt{|I||J|}}{|L|} \), and \( \mathcal{D}_m(L) \) denotes the dyadic subintervals of \( L \) with length \( 2^{-m}|L| \).

The cancellation property of the Haar functions and the normalization of the coefficients ensures that \( \|\Pi_{m,n} f\|_{L^2} \leq \|f\|_{L^2} \), square function estimates ensure boundedness on \( L^p(\mathbb{R}) \) for all \( 1 < p < \infty \). The martingale transform, \( T_\sigma \), is a Haar shift operator of complexity \((0, 0)\). Petermichl’s \( \Pi r,\beta \) operators are Haar shift operators of complexity \((0, 1)\). The dyadic paraproduct, \( \pi_b \), to be introduced in Section 4.5, is not one of these and nor is its adjoint \( \pi_b^* \).

The following estimates are known for dyadic shift operators of arbitrary complexity. First, Michael Lacey, Stefanie Petermichl, and Mari Carmen Reguera proved the A2 conjecture for the Haar shift operators of arbitrary complexity with constant depending exponentially on the complexity \[\text{[LPetR]}\]. Unlike their predecessors, they did not use Bellman functions, instead
they used stopping time techniques and a two-weight theorem for "well localized operators" of [NTV4]. Second, David Cruz-Uribe, Chema Martell, and Carlos Pérez [CrMPz2] used a local median oscillation technique introduced by Andrei Lerner [Le1, Le2]. The local median oscillation method was quite flexible, they obtained new results such as the sharp bounds for the square function for $p > 2$, for the dyadic paraproduct, also for vector-valued maximal operators, as well as two-weight results, however for the dyadic shift operators the weighted estimates still depended exponentially on the complexity. Third, Tuomas Hytönen [Hyt2] obtained the linear estimates with polynomial dependence on the complexity, needed to prove the $A_2$ conjecture for Calderón-Zygmund singular integral operators.

4.5. Dyadic paraproduct. Quoting from an article on "What is....a Paraproduct?" for the broader public by Arpad Bényi, Diego Maldonado, and Virginia Naibo [BMN]:

"The term paraproduct is nowadays used rather loosely in the literature to indicate a bilinear operator that, although noncommutative, is somehow better behaved than the usual product of functions. Paraproducts emerged in J.-M. Bony’s theory of paradifferential operators [Bo], which stands as a milestone on the road beyond pseudodifferential operators pioneered by R. R. Coifman and Y. Meyer in [CM]. Incidentally, the Greek word παρα (para) translates as beyond in English and au delà de in French, just as in the title of [CM]. The defining properties of a paraproduct should therefore go beyond the desirable properties of the product."

The dyadic paraproduct associated to a dyadic grid $\mathcal{D}$ and to $b \in \text{BMO}^D$ is an operator acting on square integrable functions $f$ as follows

$$\pi_b f(x) := \sum_{I \in \mathcal{D}} \langle f \rangle_I \langle b, h_I \rangle h_I(x),$$

where $\langle f \rangle_I = \frac{1}{|I|} \int_I f(x) \, dx = \langle f, 1_I / |I| \rangle$. A function $b$ is in the space of dyadic bounded mean oscillation, BMO$^D$ if and only if

$$\|b\|_{\text{BMO}^D} := \sup_{J \in \mathcal{D}} \left( \frac{1}{|J|} \int_J |b(x) - \langle b \rangle_J|^2 \, dx \right)^{1/2} < \infty.$$  

Notice that we are using an $L^2$ mean oscillation instead of the $L^1$ mean oscillation used in (2.2) in the definition of BMO, and of course, we are restricting to dyadic intervals. As it turns out, one could use an $L^p$ mean oscillation for any $1 \leq p$ and obtain equivalent norms in BMO thanks to the celebrated John-Nirenberg Lemma [JN].

Formally, expanding $f$ and $b$ in the Haar basis, multiplying and separating the terms into upper triangular, diagonal, and lower triangular parts, one gets that

$$bf = \pi_b f + \pi_f^b f + \pi_f b,$$

in doing so is important to note that $\sum_{I \in \mathcal{D}} I \ni J \langle f, h_I \rangle h_I = \langle f \rangle_J$. It is well known that multiplication by a function $b$ is a bounded operator on $L^p(\mathbb{R})$ if and only if the function is essentially bounded, that is, $b \in L^\infty(\mathbb{R})$. However the paraproduct is a bounded operator on $L^p(\mathbb{R})$ if and only if $b \in \text{BMO}^D$, which is a space strictly larger than $L^\infty(\mathbb{R})$. The $L^2$ estimate can be obtained using, for example, the Carleson embedding lemma, see Section 5.1.

Using a weighted Carleson embedding lemma, one can check that the paraproduct is bounded on $L^2(w)$ for all $w \in A_2$ [P1]. Furthermore, Beznosova proved the $A_2$ conjecture for
paraproducts [Be2], namely
\[ \|\pi_b f\|_{L^2(w)} \leq C[w] A_2 \|b\|_{BMO^a} \|f\|_{L^2(w)}. \]

By extrapolation one concludes that the paraproduct is bounded on \(L^p(w)\) for all \(w \in A_p\) and \(1 < p < \infty\), in particular it is bounded on \(L^p(\mathbb{R})\). In Section 5, we will present Beznosova’s Bellman function argument proving the \(A_2\) conjecture for the dyadic paraproduct. This argument was generalized to \(\mathbb{R}^d\) in [Ch3] and to spaces of homogeneous type in [We]. It was pointed out to us recently [Wic] that the paraproduct is a well-localized operator (for trivial reasons) in the sense of [NTV4] and therefore it falls under their theory.

To finish this brief introduction to the paraproduct, we would like to mention its intimate connection to the \(T(1)\) and \(T(b)\) Theorems of Guy David, Jean-Lin Journé, and Stephen Semmes [DaS, DaJS]. These theorems give (necessary and sufficient) conditions to verify boundedness on \(L^2(\mathbb{R})\) for singular integral operators \(T\) with a Calderón-Zygmund kernel when Fourier analysis, almost-orthogonality (Cotlar’s lemma), or other more standard techniques fail. In the \(T(1)\) theorem the conditions amount to checking some weak boundedness property which is a necessary condition, and checking that the function 1 is "mapped" under the operator and its adjoint, \(T(1)\) and \(T^*(1)\), into \(BMO\). Once this is verified the operator can be decomposed into a "simpler" operator \(S\) with the property that \(S(1) = S^*(1) = 0\), a paraproduct, \(\pi_{T(1)}\), and the adjoint of a paraproduct, \(\pi_{T^*(1)}^*\). The paraproduct terms are bounded on \(L^2(\mathbb{R})\), the operator \(S\) can be verified to be bounded on \(L^2(\mathbb{R})\), and as a consequence so will be the operator \(T\).

We have defined all these model operators in the one-dimensional case, there are corresponding Haar shift operators and dyadic paraproducts defined on \(\mathbb{R}^d\) as well as \(T(1)\) and \(T(b)\) theorems.

4.6. Hytönen’s representation theorem. Let us remind the reader that a bounded operator on \(L^2(\mathbb{R}^d)\) is a Calderón-Zygmund singular integral operator with smoothness parameter \(\alpha > 0\) if it has an integral representation
\[ Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, dy, \quad x \notin \text{supp} \, f, \]
for a kernel \(K(x, y)\) defined for all \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d\) such that \(x \neq y\), and verifying the standard size and smoothness estimates, respectively \(|K(x, y)| < C|x - y|^d\) and
\[ |K(x + h, y) - K(x, y)| + |K(x, y + h) - K(x, y)| \leq C|h|^{\alpha}|x - y|^{d+\alpha}, \]
for all \(|x - y| > 2|h| > 0\) and some fixed \(\alpha \in [0, 1]\).

It is worth remembering that such Calderón-Zygmund singular integral operators are bounded on \(L^p(\mathbb{R}^d)\) for all \(1 < p < \infty\), they are of weak-type \((1, 1)\), and they map \(BMO\) into itself.

We now have all the ingredients to state the celebrated Hytönen’s representation theorem [Hyt2] at least in the one-dimensional case.

**Theorem 4.3** (Hytönen’s 2012). Let \(T\) be a Calderón-Zygmund singular integral operator with smoothness parameter \(\alpha > 0\), then
\[ T(f) = \mathbb{E}_\Omega \left( \sum_{(m, n) \in \mathbb{N}^2} e^{-(m+n)\alpha/2} \mathcal{P}_{m,n}^T(f) + \pi_{T(1)}(f) + \pi_{T^*(1)}^*(f) \right). \]
Where for each pair of random parameters \((r, \beta) \in \Omega\), the operator \(\mathcal{H}^r_{m,n}\) is a Haar shift operators of complexity \((m,n)\), the operator \(\pi^r_{T(1)}\) is a dyadic paraproduct, and the operator \((\pi^r_{T(1)})^*\) is the adjoint of a dyadic paraproduct, all defined on the random dyadic grid \(\mathcal{D}^r_{\beta}\).

The paraproducts and their adjoints in the decomposition depend on the operator \(T\) via \(T(1)\) and \(T^*(1)\). The Haar shift operators in the decomposition also depend on \(T\) although it is not obvious in the notation we used. Indeed, the coefficients \(c^r_{I,J}\), in the definition of the Haar shift multiplier of complexity \((m,n)\) (see Section 4.4), will depend on the given operator \(T\) for each \(L \in \mathcal{D}^r_{\beta}, I \in \mathcal{D}^r_{m}(L)\), and \(J \in \mathcal{D}^r_{n}(L)\). Notice that the exponential nature of the coefficients in the expansion explains why the Haar shift multipliers of arbitrary complexity will need to be bounded with a bound depending at most polynomially on the complexity.

To the author, this is a remarkable result providing a dyadic decomposition theorem for a large class of operators. Once you have such decomposition and \(L^2(w)\) estimates for each of the components (Haar shift operators, paraproducts and their adjoints) that are linear on \([w]_{A_2}\) and that are uniform on the dyadic grids then the \(A_2\) conjecture is resolved in the positive, as Tuomas Hytönen did in his celebrated paper [Hyt2].

5. \(A_2\) theorem for the dyadic paraproduct: A Bellman function proof

As a model example we will present in this section Beznosova’s argument proving the \(A_2\) conjecture for the dyadic paraproduct [Be2]. The goal is to show that for all weights \(w \in A_2\), functions \(b \in \text{BMO}^{\mathcal{D}}\), and functions \(f \in L^2(w)\) the following estimate holds,

\[
\|\pi_b f\|_{L^2(w)} \lesssim [w]_{A_2} \|b\|_{\text{BMO}^{\mathcal{D}}} \|f\|_{L^2(w)}.
\]

We remind the reader that the dyadic paraproduct associated to \(b \in \text{BMO}^{\mathcal{D}}\) is defined by \(\pi_b f(x) := \sum_{I \in \mathcal{D}} \langle f \rangle_I b_I h_I(x)\), where \(b_I = \langle b, h_I \rangle\) and \(\langle f \rangle_I = (1/|I|) \int_I f(y) \, dy\).

To achieve a preliminary estimate, where instead of the linear bound on \([w]_{A_2}\) we get a 3/2 bound, namely \([w]_{A_2}^{3/2}\), we need to introduce a few ingredients: (weighted) Carleson sequences, Beznosova’s Little Lemma, and the Weighted Carleson Lemma. Afterwards, in Section 5.4, we will refine the argument to get the desired linear bound. To achieve the linear bound, we need to add additional ingredients, including the \(\alpha\)-Lemma, introduced by Oleksandra Beznosova in her original proof [Be1]. Both the Little Lemma and the \(\alpha\)-Lemma are proved using Bellman functions and we sketch their proofs, as well as the proof of the Weighted Carleson Lemma.

5.1. Weighted Carleson sequences, Weighted Carleson Lemma, and Little Lemma

In this section we introduce weighted an unweighted Carleson sequences and the weighted Carleson embedding lemma. We also present Beznosova’s Little Lemma that enables us to ensure that given a weight \(w\) and a Carleson sequence \(\{\lambda_I\}_{I \in \mathcal{D}}\) we can create a \(w\)-weighted Carleson sequence by multiplying each term of the given sequence by the reciprocal of \(w^{-1}(I)\).

5.1.1. Weighted Carleson sequences and lemma. Given a weight \(w\), a positive sequence \(\{\lambda_I\}_{I \in \mathcal{D}}\) is \(w\)-Carleson if there is a constant \(A > 0\) such that

\[
\sum_{I \in \mathcal{D}(J)} \lambda_I \leq Aw(J) \quad \text{for all } J \in \mathcal{D},
\]

where \(w(J) = \int_J w(x) \, dx\). The smallest constant \(A > 0\) is called the intensity of the sequence. When \(w = 1\) a.e. we say that the sequence is Carleson (not 1-Carleson).
Example 5.1. If \( b \in \text{BMO}^\mathbb{D} \), then the sequence \( \{ b_I^2 \}_{I \in \mathbb{D}} \) is Carleson with intensity \( \| b \|_{\text{BMO}}^2 \).

Indeed, for any \( J \in \mathbb{D} \), the collection of Haar functions corresponding to dyadic intervals \( I \subset J \), \( \{ h_I \}_{I \in \mathbb{D}(J)} \), forms an orthonormal basis on \( L_0^2(J) = \{ f \in L^2(J) : \int_J f(x) \, dx = 0 \} \). The function \( (b - \langle b \rangle_J)_{J} \) belongs to \( L_0^2(J) \), therefore by Plancherel's inequality,

\[
\sum_{I \in \mathbb{D}(J)} b_I^2 = \sum_{I \in \mathbb{D}(J)} |\langle b, h_I \rangle|^2 = \int_J |b(x) - \langle b \rangle_J|^2 \, dx \leq \| b \|_{\text{BMO}}^2 |J|.
\]

The following weighted Carleson lemma that appeared in [NTV1] will be extremely useful in our estimates, you can find a proof in [MoP] that we reproduce in Section 5.5.3.

**Lemma 5.2 (Weighted Carleson Lemma).** Given a weight \( v \), then \( \{ \lambda_I \}_{I \in \mathbb{D}} \) is a \( v \)-Carleson sequence with intensity \( A \) if and only if for all non-negative \( F \in L^1(v) \) we have

\[
\sum_{I \in \mathbb{D}} \lambda_I \inf_{x \in I} F(x) \leq A \int_R F(x) v(x) \, dx.
\]

The following particular instance of the Weighted Carleson Lemma will be useful.

**Example 5.3.** Let \( \{ \lambda_I \}_{I \in \mathbb{D}} \) be a \( v \)-Carleson sequence with intensity \( A \), let \( f \in L^2(v) \) and set \( F(x) = (M_v^f(x))^2 \) where \( M_v^f \) is the weighted dyadic maximal function, namely

\[
M_v^f(x) := \sup_{I \in \mathbb{D} : x \in I} \langle |f| \rangle_I^v \quad \text{where} \quad \langle |f| \rangle_I^v := \langle |f|_I \rangle_1 / \langle v \rangle_I.
\]

By definition of the dyadic maximal function, \( \langle |f| \rangle_I^v \leq \inf_{x \in I} M_v^f(x) \). Then by the Weighted Carleson Lemma (Lemma 5.2) and the boundedness of \( M_v^f \) on \( L^2(v) \) with operator bound independent of the weight we conclude that

\[
\sum_{I \in \mathbb{D}} \lambda_I \langle |f| \rangle_I^v \leq A \| M_v^f \|_{L^2(v)}^2 \leq A \| f \|_{L^2(v)}^2.
\]

Specializing even further we get another useful result, that establishes the boundedness of the dyadic paraproduct on \( L^2(\mathbb{R}) \) when \( b \in \text{BMO}^\mathbb{D} \).

**Example 5.4.** In particular, if \( v \equiv 1 \) and \( b \in \text{BMO}^\mathbb{D} \), then \( \lambda_I := b_I^2 \) for \( I \in \mathbb{D} \) defines a Carleson sequence with intensity \( \| b \|_{\text{BMO}}^2 \), hence

\[
\| \pi_b f \|_{L^2}^2 = \sum_{I \in \mathbb{D}} \langle \pi_b f, h_I \rangle \langle |f| \rangle_I^2 \leq \sum_{I \in \mathbb{D}} b_I^2 \langle |f| \rangle_I^2 \lesssim \| b \|_{\text{BMO}}^2 \| f \|_{L^2}^2.
\]

5.1.2. Beznosova’s Little Lemma. We will need to create \( w \)-Carleson sequences from given Carleson sequences. The following lemma will come in handy [Be2].

**Lemma 5.5 (Little Lemma).** Let \( w \) be a weight, such that \( w^{-1} \) is also a weight. Let \( \{ \lambda_I \}_{I \in \mathbb{D}} \) be a Carleson sequence with intensity \( A \), the sequence \( \{ \lambda_I / (w^{-1})_I \}_{I \in \mathbb{D}} \) is \( w \)-Carleson with intensity \( 4A \). In other words, for all \( J \in \mathbb{D} \)

\[
\sum_{I \in \mathbb{D}(J)} \frac{\lambda_I}{(w^{-1})_I} \leq 4A w(J).
\]

The proof uses a Bellman function argument that we will present in Section 5.5. Note that the weight \( w \) in the Little Lemma is not required to be in the Muckenhoupt \( A_2 \) class. It does require that the reciprocal \( w^{-1} \) is a weight, of course if \( w \in A_2 \) then \( w^{-1} \) is a weight in \( A_2 \).
Example 5.6. Let \( b \in \text{BMO}^\alpha \) and \( w \in A_2 \). The sequence \( \{ b_I^2 / \langle w \rangle_I \} \) is a \( w^{-1} \)-Carleson, with intensity \( 4 \| b \|^2_{\text{BMO}^\alpha} \). By Example 5.1 the sequence \( \{ b_I^2 \} \) is a Carleson sequence with intensity \( \| b \|^2_{\text{BMO}^\alpha} \), and then applying Lemma 5.5 with the roles of \( w \) and \( w^{-1} \) interchanged we get the stated result.

5.2. The 3/2 bound for the paraproduct on weighted \( L^2 \). We now show that the paraproduct \( \pi_b \) is bounded on \( L^2(w) \) when \( w \in A_2 \) and \( b \in \text{BMO}^\alpha \), with bound \( \| w \|_{A_2}^{3/2} \| b \|_{\text{BMO}^\alpha} \), not yet the optimal linear bound.

Proof. By duality suffices to show that for all \( f \in L^2(w) \) and \( g \in L^2(w^{-1}) \)
\[
|\langle \pi_b f, g \rangle| \lesssim \| w \|_{A_2}^{3/2} \| b \|_{\text{BMO}^\alpha} \| f \|_{L^2(w)} \| g \|_{L^2(w^{-1})}.
\]
By definition of the dyadic paraproduct and the triangle inequality,
\[
|\langle \pi_b f, g \rangle| \leq \sum_{I \in \mathcal{D}} |\langle f \rangle_I \| b_I \| \| g, h_I \|.
\]
First, using the Cauchy-Schwarz inequality, we can estimate as follows,
\[
|\langle \pi_b f, g \rangle| \leq \left( \sum_{I \in \mathcal{D}} \frac{|\langle f \rangle_I|^2 b_I^2}{\langle w^{-1} \rangle_I} \right)^{1/2} \left( \sum_{I \in \mathcal{D}} |\langle g, h_I \rangle|^2 \langle w \rangle_I \langle w^{-1} \rangle_I \right)^{1/2}.
\]
Second, using the fact that \( \| S^g g \|^2_{L^2(w^{-1})} = \sum_{I \in \mathcal{D}} |\langle g, h_I \rangle|^2 \langle w^{-1} \rangle_I \) and the linear bound on \( L^2(v) \) for the square function for \( v = w^{-1} \in A_2 \) with \( [w^{-1}]_{A_2} = \| w \|_{A_2} \), we further estimate by,
\[
|\langle \pi_b f, g \rangle| \leq \left( \sum_{I \in \mathcal{D}} \frac{|\langle f \rangle_{w^{-1}}|^2 I}{\langle w \rangle_I \langle w^{-1} \rangle_I} \right)^{1/2} \| [w^{-1}]_{A_2} \| S^g g \|_{L^2(w^{-1})}
\]
\[
\lesssim \| w \|_{A_2}^{1/2} \left( \sum_{I \in \mathcal{D}} \left( \| f \|_{w^{-1}}^2 \right) \frac{b_I^2}{\langle w \rangle_I} \right)^{1/2} \| w \|_{A_2} \| g \|_{L^2(w^{-1})}.
\]
Third, using the Weighted Carleson Lemma (Lemma 5.2) for the \( w^{-1} \)-Carleson sequence \( \{ b_I^2 / \langle w \rangle_I \} \) (see Example 5.6), together with the fact that \( \| f \|_{L^2(w)} = \| f w \|_{L^2(w^{-1})} \), we get that,
\[
|\langle \pi_b f, g \rangle| \lesssim \| w \|_{A_2}^{3/2} \| b \|_{\text{BMO}^\alpha} \| f \|_{L^2(w)} \| g \|_{L^2(w^{-1})}.
\]
where in the last line we used the boundedness of the dyadic weighted maximal function \( M^\alpha_v \) on \( L^2(v) \) with an operator norm independent of the weight \( v \). This implies that
\[
\| \pi_b f \|_{L^2(w)} \lesssim \| w \|_{A_2}^{3/2} \| b \|_{\text{BMO}^\alpha} \| f \|_{L^2(w)}.
\]
This is precisely what we set out to prove. \( \square \)

5.3. Algebra of Carleson sequences, \( \alpha \)-Lemma, and weighted Haar bases. To get a linear bound instead of the 3/2 power bound we just obtained, we will need a couple more ingredients, some algebra with Carleson sequences, the \( \alpha \)-Lemma, and weighted Haar bases.
5.3.2. Algebra of Carleson sequences. Given weighted Carleson sequences we can create new weighted Carleson sequences by linear operations or by taking geometric means.

Lemma 5.7 (Algebra of Carleson sequences). Given a weight $v$, let $\{\lambda_I\}_{I \in \mathcal{D}}$ and $\{\gamma_I\}_{I \in \mathcal{D}}$ be two $v$-Carleson sequences with intensities $A$ and $B$ respectively then for any $c, d > 0$

(i) The sequence $\{c\lambda_I + d\gamma_I\}_{I \in \mathcal{D}}$ is a $v$-Carleson sequence with intensity at most $cA + dB$.

(ii) The sequence $\{\sqrt{\lambda_I \gamma_I}\}_{I \in \mathcal{D}}$ is a $v$-Carleson sequence with intensity at most $\sqrt{AB}$.

The proof is a simple exercise which we leave to the interested reader. We do need some specific Carleson sequences, and we record them in the next example.

Example 5.8. Let $u, v \in A_\infty$ and $\Delta_I v := \langle v \rangle_{I^+} - \langle v \rangle_{I^-}$. Then

(i) The sequence $\{||\Delta_I v||/\langle v \rangle_I^2 |I|\}_{I \in \mathcal{D}}$ is a Carleson sequence, with intensity $C \log[w]_{A_\infty}$.

(ii) Let $\alpha_I = (||\Delta_I v||/\langle v \rangle_I)(||\Delta_I u||/\langle u \rangle_I)|I|$. The sequence $\{\alpha_I\}_{I \in \mathcal{D}}$ is a Carleson sequence.

(iii) When $v \in A_2$, $u = v^{-1}$ (also in $A_2$) the sequence $\{\alpha_I\}_{I \in \mathcal{D}}$ defined in item (ii) has intensity at most $\log[v]_A$.

Example 5.8(i) was discovered by Robert Fefferman, Carlos Kenig\textsuperscript{13}, and Jill Pipher\textsuperscript{14} in 1991, see [FKP]. The sharp constant $C = 8$ was obtained by Vasily Vasyunin using the Bellman function method [Va]. In fact this example provides a characterization of $A_\infty$ by summation conditions, for many more such characterizations for other weight classes see [Bu2, BeRe]. Example 5.8(ii)-(iii) follow from Example 5.8(i) and from Lemma 5.7(ii).

5.3.2. The $\alpha$-Lemma. The key to dropping from power $3/2$ to linear power in the weighted $L^2$ estimate for the paraproduct is the following lemma, discovered by Beznosova, like the Little Lemma, in the course of writing her PhD Dissertation [Be1], see also [Be2, MoP]. Both lemmas were proved using Bellman functions and we will sketch the arguments in Section 5.5.

Lemma 5.9 ($\alpha$-Lemma). If $w \in A_2$ and $\alpha > 0$, then the sequence

$$\mu_I := \langle w \rangle_I^\alpha \langle w^{-1} \rangle_I^\alpha |I| \left( |\Delta_I w|^2 / \langle w \rangle_I + |\Delta_I w^{-1}|^2 / \langle w^{-1} \rangle_I \right)$$

is a Carleson sequence with intensity at most $C_\alpha \|w\|_{A_2}^\alpha$, and $C_\alpha = \max\{72/(\alpha - 2\alpha^2), 576\}$.

Notice that the algebra of Carleson sequences encoded in Lemma 5.7 together with the R. Fefferman-Kenig-Pipher Example 5.8(iii) give, for $\mu_I$, an intensity of $\|w\|_{A_2}^\alpha \log[w]_{A_2}$, which is larger by a logarithmic factor than the one claimed in the $\alpha$-Lemma. This lesser estimate will improve the $3/2$ estimate to a linear times logarithmic estimate [Be1], the stronger $\alpha$-Lemma will yield the desired linear estimate.

Example 5.10. Let $w \in A_2$ and $b \in \text{BMO}^\infty$. By the $\alpha$-Lemma and the algebra of Carleson sequences we conclude that

(i) $\{v_I := |\Delta_I w|^2 \langle w^{-1} \rangle_I^2 |I|\}_{I \in \mathcal{D}}$ is Carleson with intensity $C_{1/4} \|w\|_{A_2}^2$, $C_{1/4} = 576$.

(ii) $\{b_I \sqrt{\gamma_I} \}_{I \in \mathcal{D}}$ is Carleson with intensity $24 \|w\|_{A_2} \|b\|_{\text{BMO}^\infty}$.

\textsuperscript{13}Carlos Kenig, an Argentinian mathematician, was elected President of the International Mathematical Union in July 2018 in the International Congress of Mathematicians (ICM) held in Brazil and for the first time in the Southern hemisphere.

\textsuperscript{14}Jill Pipher is the president-elect of the American Mathematical Society (AMS), and will begin a two-year term in 2019.
5.3.3. Weighted Haar basis. The last ingredient before we present the proof of the $A_2$ conjecture for the dyadic paraproduct is the weighted Haar basis.

Given a doubling weight $w$ and an interval $I$, the weighted Haar function $h_I^w$ is given by

$$h_I^w(x) := \sqrt{w(I_-)/w(I)w(I_+)} \mathbb{1}_{I_+}(x) - \sqrt{w(I_+)/w(I)w(I_-)} \mathbb{1}_{I_-}(x).$$

The collection $\{h_I^w\}_{I \in \mathcal{D}}$, of weighted Haar functions indexed on $\mathcal{D}$ – a system of dyadic intervals –, is an orthonormal system of $L^2(w)$. In fact, the weighted Haar functions are the Haar functions corresponding to the space of homogeneous type $X = \mathbb{R}$ with the Euclidean metric, the doubling measure $d\mu = w\,dx$, and the dyadic structure $\mathcal{D}$, defined in Section 3.5.3.

There is a very simple formula relating the weighted Haar function and the regular Haar function. More precisely, given $I \in \mathcal{D}$ there exist numbers $\alpha_I^w, \beta_I^w$ such that

$$h_I(x) = \alpha_I^w \langle h_I^w(x) \rangle + \beta_I^w \mathbb{1}_I(x)/\sqrt{|I|}.$$  

The coefficients can be calculated precisely, and they have the following upper bounds:

(i) $|\alpha_I^w| \leq \sqrt{\langle w \rangle_I}$,  

(ii) $|\beta_I^w| \leq |\Delta_I w|/\langle w \rangle_I$ where $\Delta_I w := \langle w \rangle_{I_+} - \langle w \rangle_{I_-}$.

5.4. $A_2$ conjecture for the dyadic paraproduct. We present a proof of Beznosova’s theorem, namely, for all $b \in \text{BMO}^\mathcal{D}$, $w \in A_2$, and $f \in L^2(w)$

$$\|\pi_b f\|_{L^2(w)} \lesssim \|b\|_{\text{BMO}^\mathcal{D}}[w]_{A_2} \|f\|_{L^2(w)}.$$

The proof uses the same ingredients introduced by Oleksandra Beznosova [Be2], and a beautiful argument by Fedja Nazarov, Sasha Reznikov, and Sasha Volberg that yields polynomial in the complexity bounds for Haar shift operators on geometric doubling metric spaces [NRV]. An extension of their result to paraproducts with arbitrary complexity can be found in joint work with Jean Moraes [MoP].

Proof. Suffices by duality to prove that

$$|\langle \pi_b f, g \rangle| \leq C \|b\|_{\text{BMO}^\mathcal{D}}[w]_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}.$$

We introduce weighted Haar functions to obtain two terms to be estimated separately,

$$|\langle \pi_b f, g \rangle| \leq \sum_{I \in \mathcal{D}} |b_I| \langle |f| w^{-1} \rangle_I \langle |g| w^{-1} \rangle_I |\langle h_I^w, f \rangle| \leq \Sigma_1 + \Sigma_2.$$

Explicitly, the sums $\Sigma_1$ and $\Sigma_2$ are obtained replacing $h_I = \alpha_I^w h_I^w + \beta_I^w \mathbb{1}_I/\sqrt{|I|}$, and using the estimates on the coefficients $\alpha_I, \beta_I$, to get

$$\Sigma_1 := \sum_{I \in \mathcal{D}} |b_I| \langle |f| w^{-1} \rangle_I \langle |g| w^{-1} \rangle_I |\langle h_I^w, f \rangle| \sqrt{\langle w \rangle_I},$$

$$\Sigma_2 := \sum_{I \in \mathcal{D}} |b_I| \langle |f| w^{-1} \rangle_I \langle |g| w^{-1} \rangle_I \frac{|\Delta_I w|}{\langle w \rangle_I} \sqrt{|I|}.$$

First Sum $\Sigma_1$. Denote the $L^2(w)$ pairing $\langle h, k \rangle_{L^2(w)} := \langle hw, k \rangle$. To estimate the first sum we observe that the weighted average (with respect to the weight $w^{-1}$) of the function $|f|w$ over a dyadic interval is bounded by the corresponding dyadic weighted maximal function evaluated at any point on the interval, hence by the infimum over the interval, more precisely,
\[
\langle f | w^{-1} \rangle_I \langle w^{-1} \rangle_I \leq \inf_{x \in I} M_{w^{-1}}^2(fw)(x). \]
Then using the definition of \( A_2 \) and the Cauchy-Schwarz inequality, we get
\[
\Sigma_1 = \sum_{I \in \mathcal{D}} \frac{|b_I|}{\sqrt{\langle w \rangle_I}} \frac{\langle |f| w^{-1} \rangle_I}{\langle w^{-1} \rangle_I} \frac{\langle |gw^{-1}, h_I^w \rangle_{L^2(w)} \rangle}{\langle w \rangle_I \langle w^{-1} \rangle_I} \\
\leq |w|A_2 \sum_{I \in \mathcal{D}} \frac{|b_I|}{\sqrt{\langle w \rangle_I}} \inf_{x \in I} M_{w^{-1}}^2(fw)(x) \frac{\langle |gw^{-1}, h_I^w \rangle_{L^2(w)} \rangle}{\langle w \rangle_I \langle w^{-1} \rangle_I} \\
\leq |w|A_2 \left( \sum_{I \in \mathcal{D}} \frac{|b_I|^2}{\langle w \rangle_I} \inf_{x \in I} M_{w^{-1}}^2(fw)(x) \left| \frac{1}{2} \right| \left( \sum_{I \in \mathcal{D}} \frac{\langle |gw^{-1}, h_I^w \rangle_{L^2(w)} \rangle^2}{\langle w \rangle_I \langle w^{-1} \rangle_I} \right) \right) ^{1/2}.
\]
Using the Weighted Carleson Lemma (Lemma 5.2) with \( F(x) = |M_{w^{-1}}^2(fw)(x)|^2 \), with weight \( v = w^{-1} \in A_2 \) recalling \([w]_{A_2} = [w^{-1}]_{A_2}\), and with \( w^{-1}\)-Carleson sequence \( \{b_I^2/\langle w \rangle_I\}_{I \in \mathcal{D}} \) with intensity \( 4\|b\|^2_{BMO^w} \) by the Little Lemma (Lemma 5.1), we get that,
\[
\Sigma_1 \leq 2[w]_{A_2} \|b\|_{BMO^w} \left( \int_{\mathbb{R}} |M_{w^{-1}}^2(fw)(x)|^2 w^{-1}(x) \, dx \right) ^{1/2} \|gw^{-1}\|_{L^2(w)} \\
\leq 4[w]_{A_2} \|b\|_{BMO^w} \|f\|_{L^2(w)} \|g\|_{L^2(w)}.
\]
Where we used in the last inequality the estimate (3.1) for the weighted dyadic maximal function, and noting that \( h \in L^2(w) \) if and only if \( hw \in L^2(w^{-1}) \), moreover \( \|hw\|_{L^2(w^{-1})} = \|h\|_{L^2(w)} \) (we used this twice, for \( h = f \) and for \( h = gw^{-1} \)).

**Second Sum \( \Sigma_2 \).** Using similar arguments to those used for \( \Sigma_1 \) we get
\[
\Sigma_2 \leq \sum_{I \in \mathcal{D}} |b_I| \frac{\langle f | w w^{-1} \rangle_I}{\langle w^{-1} \rangle_I} \frac{\langle g | w^{-1} w \rangle_I}{\langle w \rangle_I} \sqrt{ \left| \Delta_I w \right|^2 \langle w^{-1} \rangle_I^2 / |I| } \\
\leq \sum_{I \in \mathcal{D}} |b_I| \sqrt{\nu_I} \inf_{x \in I} M_{w^{-1}}^2(fw)(x) M_{w}^2(gw^{-1})(x),
\]
where \( |b_I|^2 \) and \( \nu_I := |\Delta_I w|^2 \langle w^{-1} \rangle_I^2 / |I| \) are Carleson sequences with intensities \( \|b\|^2_{BMO^w} \) and \([w]_{A_2}^2\) respectively, by Example 5.1 and Example 5.10(i). Then by the algebra of Carleson sequences the sequence \( |b_I| \sqrt{\nu_I} \) is a Carleson sequence with intensity \( \|b\|^2_{BMO^w} [w]_{A_2} \). Using the Weighted Carleson Lemma (Lemma 5.2) with \( F(x) = M_{w^{-1}}^2(fw)(x) M_{w}^2(gw^{-1})(x) \) and with \( v = 1 \), we conclude that
\[
\Sigma_2 \leq |w|A_2 \|b\|_{BMO^w} \int_{\mathbb{R}} M_{w^{-1}}^2(fw)(x) M_{w}^2(gw^{-1})(x) \, dx.
\]
To finish we use the Cauchy-Schwarz inequality, the fact that \( w^{1/2}(x) w^{-1/2}(x) = 1 \), and estimate (3.1) for the weighted dyadic maximal functions, to get that,
\[
\Sigma_2 \leq |w|A_2 \|b\|_{BMO^w} \left[ \int_{\mathbb{R}} (M_{w^{-1}}^2(fw)(x))^2 w^{-1}(x) \, dx \right] ^{1/2} \left[ \int_{\mathbb{R}} (M_{w}^2(gw^{-1})(x))^2 w(x) \, dx \right] ^{1/2} \\
= |w|A_2 \|b\|_{BMO^w} \|M_{w^{-1}}^2(fw)\|_{L^2(w^{-1})} \|M_{w}^2(gw^{-1})\|_{L^2(w)} \\
\leq 4[w]_{A_2} \|b\|_{BMO^w} \|f\|_{L^2(w)} \|g\|_{L^2(w)}.
All together this implies that $\|\pi_b f\|_{L^2(w)} \leq 8[w]_{A_2}\|b\|_{\text{BMO}}\|f\|_{L^2(w)}$, proving the $A_2$ conjecture for the dyadic paraproduct.

5.5. **Auxiliary lemmas.** We now present the Bellman function proofs (or at least the main ideas) for the Little Lemma (Lemma 5.5) and the $\alpha$-Lemma (Lemma 5.9), to illustrate the method in a very simple setting. For completeness we also present the proof of the Weighted Carleson Lemma (Lemma 5.2). The lemmas in this section hold on $\mathbb{R}^d$ and also on geometrically doubling metric spaces [Ch3, NRV].

5.5.1. **Proof of Beznosova’s Little Lemma.** We wish to prove Lemma 5.5. The proof uses a Bellman function argument, which we now describe. As usual, the argument proceeds in two steps. First, Lemma 5.11, encodes what now is called an induction on scales argument. If we can find a Bellman function with certain properties, then we will solve our problem by induction on scales. This type of arguments shows that if we can find a function with certain size, domain, and dyadic convexity properties tailored to the inequality of interest, we will be able to induct on scales and obtain the desired inequality. Second, Lemma 5.12 will show that such Bellman function exists.

**Lemma 5.11** (Beznosova 2008). Suppose there exists a real valued function of 3 variables $B(x) = B(u,v,l)$, whose domain $\mathcal{D}$ contains points $x = (u,v,l)$

$$\mathcal{D} := \{(u,v,l) \in \mathbb{R}^3 : u, v > 0, \ uv \geq 1 \ and \ 0 \leq l \leq 1\},$$

whose range is given by $0 \leq B(x) \leq u$, and such that the following convexity property holds,

$$B(x) - (B(x_+) + B(x_-))/2 \geq \alpha/4v, \ for \ all \ x, x_\pm \in \mathcal{D} \ with \ x - x_\pm \geq (0,0,\alpha).$$

Then Lemma 5.5. will be proven, more precisely (5.1) holds.

**Proof.** Without loss of generality we may assume that the intensity $A$ of the Carleson sequence \{\lambda_I\}_{I \in \mathcal{D}} in Lemma 5.5 is one, $A = 1$.

Fix a dyadic interval $J$. Let $u_J := \langle w \rangle_J$, $v_J := \langle w^{-1} \rangle_J$ and $\ell_J := \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \lambda_I$, then $x_J := (u_J, v_J, \ell_J) \in \mathcal{D}$. Recall that $\mathcal{D}(J)$ denotes the intervals $I \in \mathcal{D}$ such that $I \subset J$.

Let $x_\pm := x_J \in \mathcal{D}$, then

$$x_J - \frac{x_J^+ + x_J^-}{2} = (0, 0, \alpha_J), \ where \ \alpha_J := \frac{\lambda_J}{|J|}.$$

Hence, by the size and convexity property (5.2), and $|J^+| = |J^-| = |J|/2$,

$$|J| \langle w \rangle_J \geq |J| \ B(x_J) \geq |J^+|B(x_J^+) + |J^-|B(x_J^-) + \lambda_J/4(w^{-1})_J.$$

Repeat the argument this time for $|J^+|B(x_J^+)$ and $|J^-|B(x_J^-)$, use that $B \geq 0$ on $\mathcal{D}$ and keep repeating to get, after dividing by $|J|$, that

$$\langle w \rangle_J \geq \frac{1}{4|J|} \sum_{I \in \mathcal{D}(J)} \frac{\lambda_I}{(w^{-1})_I},$$

which implies (5.1) after multiplying through by $4|J|$. The lemma is proved.

The previous induction on scales argument is conditioned on the existence of a function with certain properties, a Bellman function. We now establish the existence of such function, both lemmas appeared in [Be2].
Lemma 5.12 (Beznosova 2008). The function \( B(u,v,l) := u - \frac{1}{v(1+l)} \) is (i) defined on the domain \( D \) introduced in Lemma 5.11, (ii) \( 0 \leq B(x) \leq u \) for all \( x = (u,v,l) \in D \), and (iii) obeys the following differential estimates on \( D \):

\[
(\partial B/\partial l)(u,v,l) \geq 1/(4v) \quad \text{and} \quad - (du, dv, dl) \overline{d^2 B(u,v,l)} (du, dv, dl) \leq 0,
\]

where \( \overline{d^2 B(u,v,l)} \) denotes the Hessian matrix of the function \( B \) evaluated at \( (u,v,l) \). Moreover, these imply the dyadic convexity condition \( B(x) - (B(x_+) + B(x_-))/2 \geq \alpha/(4v) \).

Proof. Differential conditions can be checked by a direct calculation that we leave as an exercise for the reader. By the Mean Value Theorem and some calculus,

\[
B(x) - \frac{B(x_+) + B(x_-)}{2} = \frac{\partial B}{\partial l}(u,v,l') \alpha - \frac{1}{2} \int_{-1}^{1} (1 - |t|) b''(t) dt \geq \frac{\alpha}{4v},
\]

where \( b(t) := B(x(t)) \) and \( x(t) := \frac{t+2}{2} x_+ + \frac{t-2}{2} x_- \) for \(-1 \leq t \leq 1\).

Note that \( x(t) \in D \) whenever \( x_+ \) and \( x_- \) are in the domain, since \( D \) is a convex domain and \( x(t) \) is a point on the line segment between \( x_+ \) and \( x_- \), and \( l' \) is a point between \( l \) and \( l_+ + l_-/2 \). This proves the lemma.

These two lemmas prove Beznosova’s Little Lemma (Lemma 5.5).

5.5.2. \( \alpha \)-Lemma. We present a very brief sketch of the argument leading to the proof of the \( \alpha \)-Lemma (Lemma 5.9), see [Be2] for \( 0 < \alpha < 1/2 \), and [MoP] for \( \alpha \geq 1/2 \). Recall that we wish to show that if \( w \in A_2 \) and \( 0 < \alpha \), then the sequence

\[
\mu_I := \langle w \rangle_I^\alpha \langle w^{-1} \rangle_I^- \left| I \right| \left( \frac{\left| \Delta_I w \right|^2}{\langle w \rangle_I^2} + \frac{\left| \Delta_I w^{-1} \right|^2}{\langle w^{-1} \rangle_I^2} \right)
\]

for \( I \in D \) is a Carleson sequence with intensity at most \( C_\alpha \langle w \rangle_{A_2}^\alpha \), and \( C_\alpha = \max\{72/(\alpha - 2\alpha^2), 576\} \).

Sketch of the Proof. Use the Bellman function method. Figure out the domain, range and dyadic convexity conditions needed to run an induction on scale argument that will yield the inequality. Verify that the Bellman function \( B(u,v) = (uv)^\alpha \) satisfies those conditions (or at least a differential version, that can then be seen implies the dyadic convexity) for \( 0 < \alpha < 1/2 \).

For \( \alpha \geq 1/2 \) just observe that one can factor out \( \langle w \rangle_I^{-1/4} \langle w^{-1} \rangle_I^{-1/4} \leq \langle w \rangle_{A_2}^{\alpha-1/4} \) and then use the already proven lemma when \( \alpha = 1/4 < 1/2 \). \( \square \)

5.5.3. Weighted Carleson Lemma. Finally we present a proof of the Weighted Carleson Lemma (Lemma 5.2), that states that if \( v \) is a weight, \( \{\alpha_L\}_{L \in \mathbb{D}} \) a \( v \)-Carleson sequence with intensity \( A \), and \( F \) a positive measurable function on \( \mathbb{R} \), then

\[
\sum_{L \in \mathbb{D}} \alpha_L \inf_{x \in L} F(x) \leq A \int_{\mathbb{R}} F(x) v(x) \, dx.
\]

The Weighted Carleson Lemma we present here is a variation in the spirit of other weighted Carleson embedding theorems that appeared before in the literature [NTV1]. The converse is immediately true by choosing \( F(x) = 1_{J}(x) \).
Proof. Assume that \( F \in L^1(v) \) otherwise the first statement is automatically true. Setting \( \gamma_L = \inf_{x \in L} F(x) \), we can write

\[
\sum_{L \in \mathcal{D}} \alpha_L \gamma_L = \sum_{L \in \mathcal{D}} \alpha_L \int_0^\infty \chi(L,t) \, dt = \int_0^\infty \left( \sum_{L \in \mathcal{D}} \chi(L,t) \alpha_L \right) dt,
\]

where \( \chi(L,t) = 1 \) for \( t < \gamma_L \) and zero otherwise, and where we used the monotone convergence theorem in the last equality. Define the level set \( \chi \) where the last equality follows from the layer cake representation. □

Moreover, there is a collection of maximal disjoint dyadic intervals \( \mathcal{P}_t \) that will cover \( E_t \) except for at most a set of \( v \)-measure zero. Finally observe that \( L \subset E_t \) if and only if \( \chi(L,t) = 1 \). All together we can rewrite the integrand in the right-hand-side of (5.3) as

\[
\sum_{L \in \mathcal{D}} \chi(L,t) \alpha_L = \sum_{L \in \mathcal{E}_t} \alpha_L \leq \sum_{L \in \mathcal{P}_t} \sum_{I \in \mathcal{D}(L)} \alpha_I \leq A \sum_{L \in \mathcal{P}_t} v(L) = A v(E_t),
\]

where we used in the second inequality the fact that \( \{\alpha_I\}_{I \in \mathcal{D}} \) is a \( v \)-Carleson sequence with intensity \( A \). Thus we can estimate

\[
\sum_{L \in \mathcal{D}} \alpha_L \inf_{x \in L} F(x) = \sum_{L \in \mathcal{D}} \alpha_L \gamma_L \leq A \int_0^\infty v(E_t) \, dt = A \int_\mathbb{R} F(x) v(x) \, dx,
\]

where the last equality follows from the layer cake representation.

6. Case study: Commutator of Hilbert transform and function in BMO

In this section, we summarize chronologically the weighted norm inequalities known for the commutator \([b,T]\) where \( T \) is a linear operator and \( b \) a function in BMO. In particular we will consider \( T = H \) the Hilbert transform. We sketch a dyadic proof of the first quantitative weighted estimate for the commutator \([b,H]\) due to Daewon Chung [Ch2], yielding the optimal quadratic dependence on the \( A_2 \) characteristic of the weight. We discuss a very useful transference theorem of Chung, Pérez and the author [ChPPz], and present its proof based on the celebrated Coifman–Rochberg–Weiss argument. The transference theorem allows to deduce quantitative weighted \( L^p \) estimates for the commutator of a linear operator with a BMO function, from given quantitative weighted \( L^p \) estimates for the operator.

6.1. \( L^p \) theory for \([b,H]\). Recall that the commutator of a function \( b \in \text{BMO} \) and \( H \) the Hilbert Transform is defined to be

\([b,H](f) := b(Hf) - H(bf)\).

The commutator \([b,H]\) is bounded on \( L^p(\mathbb{R}) \) for \( 1 < p < \infty \) if and only if \( b \in \text{BMO} \) [CoRW]. Moreover, the following estimate is known to hold for all \( b \in \text{BMO} \) and \( f \in L^p(\mathbb{R}) \)

\[
\|[b,H](f)\|_{L^p} \lesssim_p \|b\|_{\text{BMO}} \|f\|_{L^p}.
\]

In fact the operator norm \( \|[b,H]\|_{L^2 \to L^2} \sim \|b\|_{\text{BMO}} \). Observe that \( bH \) and \( Hb \) are NOT necessarily bounded on \( L^p(\mathbb{R}) \) when \( b \in \text{BMO} \). The commutator introduces some key cancellation. This is very much connected to the celebrated \( H^1 \)-BMO duality theorem by Fefferman and Stein [FS], where the Hardy space \( H^1 \) can be defined as those functions \( f \in L^1(\mathbb{R}) \) such that their maximal function \( Mf \) is also in \( L^1(\mathbb{R}) \).
The commutator \([b, H]\) is more singular than \(H\), as evidenced by the fact that, unlike the Hilbert transform, the commutator is not of weak-type \((1, 1)\) \([Pz1]\). In particular the commutator is not a Calderón-Zygmund operator, if it were it would be of weak-type \((1, 1)\), and is not.

6.2. Weighted Inequalities. The first two-weight results for the commutator that we present are of a qualitative nature. The first one is a two-weight result due to Steven Bloom for the commutator of the Hilbert transform with a function in weighted \(BMO\) when both weights are in \(A_p\).

**Theorem 6.1** (Bloom 1985). If \(u, v \in A_p\) then \([b, H] : L^p(u) \rightarrow L^p(v)\) is bounded if and only if \(b \in BMO_\mu\) where \(\mu = u^{-1/p}v^{1/p}\). Where \(b \in BMO_\mu\) if and only if

\[
\|b\|_{BMO_\mu} := \sup_{I \subseteq \mathbb{R}} \frac{1}{\mu(I)} \int_I |b(x) - \langle b \rangle_I| \, dx < \infty.
\]

What is important in this setting is that the hypothesis \(u, v \in A_p\) imply that \(\mu \in A_2\) as can be seen by a direct calculation using Hölder’s inequality. The weighted \(BMO\) space defined by (6.1) was first introduced by Eric Sawyer and Richard Wheeden \([SW]\), and it has been called in the literature, somewhat misleadingly, Bloom \(BMO\). For a "modern" dyadic proof of Bloom’s result see \([HoLW1, HoLW2]\).

The second result is a one-weight result for very general linear operators \(T\) obtained by Josefina Álvarez, Richard Bagby, Doug Kurtz, and Carlos Pérez \([ABKPz]\), they also prove two-weight estimates.

**Theorem 6.2** (Álvarez, Bagby, Kurtz, Pérez 1993). Let \(T\) be a linear operator on the set of real-valued Lebesgue measurable functions defined on \(\mathbb{R}^d\), with a domain of definition which contains every compactly supported function in a fixed \(L^p\) space. If \(w \in A_p\) and \(b \in BMO\) then there is a constant \(C_p(w) > 0\) such that for all \(f \in L^p(w)\) the following inequality holds,

\[
\|[b, T](f)\|_{L^p(w)} \leq C_p(w)\|b\|_{BMO}\|f\|_{L^p(w)}.
\]

The proof uses a classical argument by Raphy Coifman\(^{15}\), Richard Rochberg, and Guido Weiss \([CoRW]\). In Section 6.4.2 we will present a quantitative version of this argument \([ChPPz]\). For a proof of Bloom’s result using this type of argument, see \([Hyt5]\).

The next result is a quantitative weighted inequality obtained by Daewon Chung in his PhD Dissertation \([Ch1, Ch2]\).

**Theorem 6.3** (Chung 2010). For all \(b \in BMO\), \(w \in A_2\) and \(f \in L^2(w)\) the following holds

\[
\|[b, H](f)\|_{L^2(w)} \lesssim \|b\|_{BMO}\|w\|_{A_2}^2\|f\|_{L^2(w)}.
\]

The quadratic power on the \(A_2\) characteristic and the linear bound on the \(BMO\) norm are both optimal powers. The quadratic dependence on the \(A_2\) characteristic is another indication that this operator is more singular than the Calderón-Zygmund singular integral operators for whom the dependence is linear \([Hyt2]\), as we have emphasized throughout these lectures.

\(^{15}\)As I am writing these notes, it has been announced that Coifman won the 2018 Schock Prize in Mathematics for his "fundamental contributions to pure and applied harmonic analysis".
6.3. **Dyadic proof of Chung's Theorem.** We now sketch Chung’s dyadic proof of the quadratic estimate for the commutator [Ch2].

*Sketch of proof.* Chung’s "dyadic" proof is based on using Petermichl’s dyadic shift operators $\Pi^{r,\beta}$ instead of $H$ [Pet1] and proving uniform (on the dyadic grids $D^{r,\beta}$) quadratic estimates for the corresponding commutators $[\Pi^{r,\beta}, b]$. To ease notation we drop the superscripts $r, \beta$ and simply write $\Pi$ for $\Pi^{r,\beta}$, the estimates will be independent of the parameters $r$ and $\beta$.

To achieve this we first recall the decomposition of a product $bf$ in terms of paraproducts and their adjoints,

$$bf = \pi_b f + \pi_b^* f + \pi_f b,$$

notice that the first two terms are bounded on $L^p(w)$ when $b \in \text{BMO}$ and $w \in A_p$, the enemy is the third term. Decomposing the commutator accordingly we get,

$$[b, \Pi](f) = [\pi_b, \Pi](f) + [\pi_b^*, \Pi](f) + \left(\pi_{\Pi f}(b) - \Pi(\pi_f b)\right).$$

(6.2)

Known linear bounds on $L^2(w)$ for the dyadic paraproduct $\pi_b$, its adjoint $\pi_b^*$, and for Petermichl’s dyadic shift operator $\Pi$, see [Be2, Pet2], immediately give by iteration, quadratic bounds for the first two terms on the right-hand-side of (6.2). Surprisingly, the third term is better, it obeys a linear bound, and so do halves of the first two commutators, as shown in [Ch2] using Bellman function techniques, namely

$$\|\pi_{\Pi f}(b) - \Pi(\pi_f b)\|_{L^2(w)} + \|\Pi\pi_b(f)\|_{L^2(w)} + \|\pi_b^* \Pi(f)\|_{L^2(w)} \leq C\|b\|_{\text{BMO}[w]}\|f\|_{L^2(w)}.$$

All together providing uniform (on the random dyadic grids $D^{r,\beta}$) quadratic bounds for the commutators $[b, \Pi^{r,\beta}]$, and hence, averaging over the random grids we get the desired quadratic estimate for $[b, H]$. □

The quadratic estimate and the corresponding extrapolated estimates, namely for all $b \in \text{BMO}$, $w \in A_p$, and $f \in L^p(w)$

$$\|[b, H](f)\|_{L^p(w)} \lesssim_p \left[w\right]_{A_p}^{2\max\left\{\frac{1}{p} - 1\right\}} \|b\|_{\text{BMO}}\|f\|_{L^p(w)},$$

(6.3)

are optimal for all $1 < p < \infty$, as can be seen considering appropriate power functions and power weights [ChPPz].

The "bad guys" are the non-local terms $\pi_b \Pi$, $\Pi \pi_b^*$. A posteriori one realizes the pieces that obey linear bounds are generalized Haar Shift operators and hence their linear bounds can be deduced from general results for those operators.

As a byproduct of Chung’s dyadic proof we get that the extrapolated bounds for the dyadic paraproduct are optimal [P2], namely for all $b \in \text{BMO}$, $w \in A_p$, and $f \in L^p(w)$

$$\|\pi_b f\|_{L^p(w)} \lesssim_p \left[w\right]_{A_p}^{\max\left\{\frac{1}{p} - 1\right\}} \|b\|_{\text{BMO}}\|f\|_{L^p(w)}.$$

*Proof.* By contradiction, if not for some $p$ then $[b, H]$ will have a better bound in $L^p(w)$ than the known optimal bound given by (6.3) for that $p$. □

6.4. **A quantitative transference theorem.** The following theorem provides a mechanism for transferring known quantitative weighted estimates for linear operators to their commutators with BMO functions [ChPPz, P2].
Theorem 6.4 (Chung, Pereyra, Pérez 2012). Given linear operator $T$ and $1 < r < \infty$, such that for all $w \in A_r$ and $f \in L^r(w)$ the following estimate holds

$$
\|Tf\|_{L^r(w)} \lesssim_{T,d} [w]_{A_r} \|f\|_{L^r(w)},
$$

then the commutator of $T$ with $b \in BMO$ is such that for all $w \in A_r$ and $f \in L^r(w)$

$$
\|[b,T](f)\|_{L^r(w)} \lesssim_{r,T,d} [w]^{\alpha+\max\{1,\frac{1}{r-1}\}}_{A_r} \|b\|_{BMO} \|f\|_{L^r(w)}.
$$

The proof follows the classical Coifman–Rochberg–Weiss argument using (i) the Cauchy integral formula; (ii) the following quantitative Coifman–Fefferman result: $w \in RH_q$ with $q = 1 + c_d/[w]_{A_r}$ and $[w]_{RH_q} \leq 2$; (iii) a quantitative version of the estimate: $b \in BMO$ implies $e^{ab} \in A_r$ for $a$ small enough with control on $[e^{ab}]_{A_r}$. We will present the whole argument in the case $r = 2$ in Section 6.4.2. Here the Reverse Hölder-$q$ weight class ($RH_q$) for $1 < q < \infty$ is defined to be all those weights $w$ such that

$$
[w]_{RH_q} := \sup_Q \langle w^q \rangle_Q^{1/q} \langle w \rangle_Q^{-1} < \infty,
$$

where the supremum is taken over all cubes in $\mathbb{R}^d$ with sides parallel to the axes.

A variation on the argument yields corresponding estimates for the higher order commutators $T^k_b := [b,T^{k-1}_b]$ for $k \geq 1$ and $T^0_b := T$. More precisely, given the initial estimate $\|T^0_b f\|_{L^r(w)} \lesssim [w]_{A_r}$, valid for all $w \in A_r$, then the following estimate holds for all $k \geq 1$, $b \in BMO$, $w \in A_r$, and $f \in L^r(w)$

$$
\|T^k_b f\|_{L^r(w)} \lesssim_{r,T,d} [w]^{\alpha+k \max\{1,\frac{1}{r-1}\}}_{A_r} \|b\|_{BMO} \|f\|_{L^r(w)}.
$$

Transference theorems for commutators are useless unless there are operators known to obey an initial $L^r(w)$ bound valid for all $w \in A_r$. We have already mentioned that the class of Calderón-Zygmund singular integral operators obey linear bounds on $L^2(w)$ thanks to Hytönen’s $A_2$ theorem [Hyt2]. We conclude that for all Calderón-Zygmund singular integral operators $T$ their commutators obey a quadratic bound on $L^2(w)$, more precisely,

$$
\|[b,T]f\|_{L^2(w)} \lesssim_{T,d} [w]_{A_2}^2 \|b\|_{BMO} \|f\|_{L^2(w)}.
$$

With a slight modification of the argument one can see [ChPPz] that the correct estimate for the iterated commutators of Calderón-Zygmund singular integral operators and function $b \in BMO$ is

$$
\|[T^k_b f]\|_{L^2(w)} \lesssim_{T,d} [w]_{A_2} \|b\|_{BMO} \|f\|_{L^2(w)}.
$$

There are operators (for example the Hilbert, Riesz, and Beurling transforms) for whom these estimates are optimal in terms of the powers both for the $A_2$ characteristic and the $BMO$ norm. This can be seen testing power functions and weights [ChPPz].

6.4.1. Some generalizations. There are extensions to commutators with fractional integral operators, two-weight problem and more [CrMoe, Cr]. There are mixed $A_2$-$A_\infty$ estimates, where recall that $A_\infty = \cup_{p > 1} A_p$ and $[w]_{A_\infty} \leq [w]_{A_2}$ [HytPz, OrPzRe], more precisely estimates of the form,

$$
\|[b,T]\|_{L^2(w)} \lesssim [w]_{A_2}^{1/2} ([w]_{A_\infty}^{1/2} + [w^{-1}]_{A_\infty}) \|b\|_{BMO}.
$$

There are generalizations to commutators of matrix valued operators and BMO [IKP] as well as to the two-weight setting (both weights in $A_p$, à la Bloom) [HoLW1, HoLW2], and also for biparameter Journé operators [HoPetW]. See also the comprehensive paper [BMMST].
where a systematic use of the Coifman–Rochberg–Weiss trick recovers all known results and some new ones such as boundedness of the commutator of the bilinear Hilbert transform and a function in $BMO$. Pointwise control by sparse operators adapted to the commutator, improving weak-type, Orlicz bounds, and quantitative two weight Bloom bounds was recently obtained [LeOR1, LeOR2]. We will say more about this generalization in Section 7.

6.4.2. Proof of the transference theorem. We now present the proof of the quantitative transference theorem when $r = 2$, following the lines of the Coifman–Rochberg–Weiss argument [CoRW] with a few quantitative ingredients. For $r \neq 2$ see [P2].

Proof in [ChPPz]. "Conjugate" the operator as follows: for any $z \in \mathbb{C}$ define

$$T_z(f) = e^{zb} T(e^{-zb} f).$$

A computation together with the Cauchy integral theorem give (for "nice" functions),

$$[b, T](f) = \frac{d}{dz} T_z(f)|_{z=0} = \frac{1}{2\pi i} \int_{|z| = \epsilon} \frac{T_z(f)}{z^2} dz, \quad \epsilon > 0.$$

Now, by Minkowski’s integral inequality

$$||[b, T](f)||_{L^2(w)} \leq \frac{1}{2\pi \epsilon^2} \int_{|z| = \epsilon} ||T_z(f)||_{L^2(w)}|dz|, \quad \epsilon > 0.$$  

The key point is to find an appropriate radius $\epsilon > 0$. To that effect, we look at the inner norm and try to find bounds depending on $z$. More precisely,

$$||T_z(f)||_{L^2(w)} = ||T(e^{-zb} f)||_{L^2(w e^{2\text{Re} z b})}.$$  

We use the main hypothesis, namely that $T$ is bounded on $L^2(v)$ if $v \in A_2$ with $||T||_{L^2(v)} \leq C[v]_{A_2}$, for $v = w e^{2\text{Re} z b}$. We must check that if $w \in A_2$ then $v \in A_2$ for $|z|$ small enough. Indeed,

$$[v]_{A_2} = \sup_Q \left( \frac{1}{|Q|} \int_Q w^q(x) e^{2\text{Re} z b(x)} dx \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_Q w^{-1}(x) e^{-2\text{Re} z b(x)} dx \right)^{\frac{1}{q'}}.$$  

It is well known that if $w \in A_2$ then $w \in RH_q$ for some $q > 1$ [CoFe]. There is a quantitative version of this result [Pz2], namely if $q = 1 + 1/2(\text{Re} z + |w|_{2,A_2})$ then

$$\left( \frac{1}{|Q|} \int_Q w^q(x) dx \right)^{\frac{1}{q}} \leq \frac{2}{|Q|} \int_Q w(x) dx,$$

similarly for $w^{-1} \in A_2$ and for the same $q$, since $[w]_{A_2} = [w^{-1}]_{A_2}$, we have that

$$\left( \frac{1}{|Q|} \int_Q w^{-q}(x) dx \right)^{\frac{1}{q'}} \leq \frac{2}{|Q|} \int_Q w^{-1}(x) dx.$$  

In what follows $q = 1 + 1/(2d+5|w|_{A_2})$. Using these estimates and Holder’s inequality we have for an arbitrary cube $Q$

$$\left( \frac{1}{|Q|} \int_Q w(x) e^{2\text{Re} z b(x)} dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1} e^{-2\text{Re} z b(x)} dx \right) \leq \left( \frac{1}{|Q|} \int_Q w^{q}(x) dx \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_Q e^{2\text{Re} z q b(x)} dx \right)^{\frac{1}{q'}} \left( \frac{1}{|Q|} \int_Q w^{-q}(x) dx \right)^{\frac{1}{q'}} \left( \frac{1}{|Q|} \int_Q e^{-2\text{Re} z q b(x)} dx \right)^{\frac{1}{q''}}.$$
\[\leq 4 \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w^{-1}(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q e^{2\text{Re} q' b(x)} \, dx \right)^{\frac{1}{q'}} \left( \frac{1}{|Q|} \int_Q e^{-2\text{Re} q' b(x)} \, dx \right)^{\frac{1}{q'}} \]

\[\leq 4 \left[ w \right]_{A_2} \left( e^{2\text{Re} q' b} \right)^{\frac{1}{q'}}_{A_2}. \]

Taking the supremum over all cubes we conclude that

\[\left[ v \right]_{A_2} = \left[ w e^{2\text{Re} b} \right]_{A_2} \leq 4 \left[ w \right]_{A_2} \left[ e^{2\text{Re} q' b} \right]_{A_2}. \]

Now, since \( b \in \text{BMO} \) there are \( 0 < \alpha_d < 1 \) and \( \beta_d > 1 \) such that if \( |2\text{Re} q'| \leq \alpha_d/\|b\|_{\text{BMO}} \) then \( \left[ e^{2\text{Re} q' b} \right]_{A_2} \leq \beta_d, \) see [ChPPz, Lemma 2.2]. Hence for these \( z, \)

\[\left[ v \right]_{A_2} \leq 4 \left[ w \right]_{A_2} \beta_d^2 \leq 4 \left[ w \right]_{A_2} \beta_d. \]

We have shown that if \( |z| \leq \alpha_d/(2q'\|b\|_{\text{BMO}}) \) then \( \left[ v \right]_{A_2} \leq 4 \left[ w \right]_{A_2} \beta_d \) and

\[\|T_z(f)\|_{L^2(w)} = \|T(e^{-zb}f)\|_{L^2(w)} \leq \left[ v \right]_{A_2} \|f\|_{L^2(w)} \leq 4 \left[ w \right]_{A_2} \beta_d \|f\|_{L^2(w)}. \]

Where the first inequality holds since \( \|e^{-zb}f\|_{L^2(w)} = \|e^{-zb}f\|_{L^2(we^{2\text{Re} b})} = \|f\|_{L^2(w)}. \)

Thus choose the radius \( \epsilon := \alpha_d/(2q'\|b\|_{\text{BMO}}), \) and get

\[\|\left[ b, T \right](f)\|_{L^2(w)} \leq \frac{1}{2\pi \epsilon^2} \int_{|z| = \epsilon} \|T_z(f)\|_{L^2(w)} \, dz \leq \frac{1}{2\pi \epsilon^2} \int_{|z| = \epsilon} 4 \left[ w \right]_{A_2} \beta_d \|f\|_{L^2(w)} \, dz = \frac{1}{\epsilon} 4 \left[ w \right]_{A_2} \beta_d \|f\|_{L^2(w)}. \]

Note that \( \epsilon^{-1} \approx \left[ w \right]_{A_2} \|b\|_{\text{BMO}}, \) because \( q' = 1 + 2^{d+5} \left[ w \right]_{A_2} \approx 2^d \left[ w \right]_{A_2}, \)

\[\|\left[ b, T \right](f)\|_{L^2(w)} \leq C_d \left[ w \right]_{A_2}^2 \|b\|_{\text{BMO}}. \]

Which is exactly what we wanted to prove. \( \Box \)

### 7. Sparse operators and sparse families of dyadic cubes

In this section, we discuss the sparse domination by finitely many positive dyadic operators paradigm that has recently emerged as a byproduct of the study of weighted inequalities. This sparse domination paradigm has proven to be very powerful with applications in areas other than weighted norm inequalities. In this section, we introduce the sparse operators and the sparse families of cubes. We discuss a characterization of sparse families of cubes via Carleson families of dyadic cubes due to Andrei Lerner and Fedja Nazarov, however this was well known 20 years earlier by Igor Verbitsky [Ve, Corollary 2, p.23], see also [Ha]. We present the beautiful proof of the \( A_2 \) conjecture for sparse operators due to David Cruz-Uribe, Chema Martell, and Carlos Pérez. We record the sparse domination results for the operators discussed in these notes. We present how to dominate pointwise the martingale transform by a sparse operator following Michael Lacey’s argument, illustrating the technique in a toy model. Finally we briefly discuss a sparse domination theorem for commutators valid for (rough) Calderón-Zygmund singular integral operators due to Andrei Lerner, Sheldy Ombrosi, and Israel Rivera-Ríos that yields new quantitative two weight estimates of Bloom type, and recovers all known weighted results for the commutators.
7.1. Sparse operators. David Cruz-Uribe, Chema Martell, and Carlos Pérez showed in [CrMPz2] the $A_2$ conjecture in a few lines for sparse operators $\mathcal{A}_S$ defined as follows,

$$\mathcal{A}_S f(x) = \sum_{Q \in S} \langle f \rangle_Q 1_Q(x).$$

Here $\mathcal{S}$ is a sparse collection of dyadic cubes. A collection of dyadic cubes $S$ in $\mathbb{R}^d$ is $\eta$-sparse, $0 < \eta < 1$ if there are pairwise disjoint measurable sets $E_Q$ for each $Q \in S$ such that $E_Q \subset Q$ with $\left|E_Q\right| \geq \eta \left|Q\right|$ for all $Q \in S$.

A primary example for us are the Calderón-Zygmund singular integral operators, they and the "rough" Calderón-Zygmund operators have been shown to be pointwise dominated by a finite number of sparse operators [Le4, CR, LeN, L3]. A quantitative form of these estimates can be found in [Le6, HytRoTa]. More recently see sparse domination principles for rough Calderón-Zygmund singular integral operators [CuDiPOu1, CCuDiPOu, HytRoTa, DiPHytLi].

7.2. Sparse vs Carleson families of dyadic cubes. We have seen in Section 4 how Carleson sequences and Carleson embedding lemmas come handy when proving weighted inequalities. There is an intimate connection between Carleson families of cubes and sparse families of cubes. A family of dyadic cubes $S$ in $\mathbb{R}^d$ is called $\Lambda$-Carleson for $\Lambda > 1$ if

$$\sum_{P \in S, P \subset Q} |P| \leq \Lambda |Q| \quad \forall Q \in \mathcal{D}.$$ 

Notice that a family of cubes being $\Lambda$-Carleson is equivalent to the sequence $\{\left|P\right|1_{\mathcal{S}}(P)\}_{P \in \mathcal{S}}$ being Carleson with intensity $\Lambda$. Furthermore the notion is equivalent to the family of cubes being $1/\Lambda$-sparse. These type of conditions are also called Carleson packing conditions.

**Lemma 7.1** (Verbitsky 1996, Lerner, Nazarov 2014). Let $\Lambda > 1$. The family of dyadic cubes $\mathcal{S}$ in $\mathbb{R}^d$ is $\Lambda$-Carleson if and only if $\mathcal{S}$ is $1/\Lambda$-sparse.

Proof. We sketch the beautiful argument in [LeN].

($\Leftarrow$) The family of cubes $\mathcal{S}$ being $1/\Lambda$-sparse means that for all cubes $P \in \mathcal{S}$ there are pairwise disjoint subsets $E_P \subset P$ that have a considerable portion of the total mass of the cube, more precisely $\Lambda |E_P| \geq |P|$. Hence,

$$\sum_{P \in \mathcal{S}, P \subset Q} |P| \leq \Lambda \sum_{P \in \mathcal{S}, P \subset Q} |E_P| \leq \Lambda |Q|.$$ 

Where the last inequality holds because the sets $E_P \subset Q$ and are pairwise disjoint. Therefore the family of cubes $\mathcal{S}$ is $\Lambda$-Carleson.

($\Rightarrow$) Assume now that $\mathcal{S}$ is a $\Lambda$-Carleson family. We say that a family $\mathcal{S}$ has a bottom layer $\mathcal{D}_K$ if for all $Q \in \mathcal{S}$ we have $Q \subset \mathcal{D}_k$ for some $k \leq K$. Assume $\mathcal{S}$ HAS A BOTTOM LAYER $\mathcal{D}_K$. Then consider all cubes in the bottom layer, $Q \in \mathcal{S} \cap \mathcal{D}_K$, and choose any sets $E_Q \subset Q$ with $|E_Q| = \frac{1}{\Lambda} |Q|$. This choice is always possible, because of the nature of the Lebesgue measure, and the sets will automatically be pairwise disjoint because the cubes in a fixed generation $\mathcal{D}_K$ are pairwise disjoint. Then go up layer by layer, meaning we have already selected sets $E_R \subset R$ for all $R \in \mathcal{S} \cap \mathcal{D}_j$ and $k < j \leq K$ with the property that $|E_R| = \frac{1}{\Lambda^j} |R|$, then for each
\( Q \in \mathcal{D}_k, k < K, \) choose any \( E_Q \subset Q \setminus \bigcup_{R \in \mathcal{S}, R \subseteq Q} E_R \) with \( |E_Q| = \frac{1}{\Lambda} |Q| \). Such choice is always possible because for every \( Q \in \mathcal{S} \) we have

\[
\left| \bigcup_{R \in \mathcal{S}, R \subseteq Q} E_R \right| \leq \frac{1}{\Lambda} \sum_{R \in \mathcal{S}, R \subseteq Q} |R| \leq \frac{\Lambda - 1}{\Lambda} |Q| = \left( 1 - \frac{1}{\Lambda} \right) |Q|,
\]

where we used in the inequality the hypothesis that \( \mathcal{S} \) is a \( \Lambda \)-Carleson family. Therefore

\[
|Q \setminus \bigcup_{R \in \mathcal{S}, R \subseteq Q} E_R| \geq \frac{1}{\Lambda} |Q|,
\]

hence there is enough mass left in \( Q \), after removing the sets \( E_R \) corresponding to \( R \) in \( \mathcal{S} \) and proper subcubes of \( Q \), to select a subset \( E_Q \) of \( Q \) with the aforementioned property. Moreover by construction the sets \( E_Q \) are pairwise disjoint, and we are done.

**But, what if there is no bottom layer?** The idea is to run the construction for each \( K \geq 0 \) and pass to the limit! One has to be a bit careful! As Lerner and Nazarov put it: “All we have to do is replace "free choice" with "canonical choice".” The diligent reader can find the details of the argument, including a very illuminating picture, in [LeN, Lemma 6.3 and Figure 8].

### 7.3. \( A_2 \) theorem for sparse operators

We now present David Cruz-Uribe, Chema Martell, and Carlos Pérez’ beautiful proof of the \( A_2 \) conjecture for sparse operators [CrMPz2].

**Theorem 7.2** (Cruz-Uribe, Martell, Pérez 2012). Let \( \mathcal{S} \) be an \( \eta \)-sparse family of cubes then for all \( w \in A_2 \) and \( f \in L^2(w) \) the following inequality holds

\[
\| \mathcal{A}_S f \|_{L^2(w)} \lesssim [w]_{A_2} \| f \|_{L^2(w)}.
\]

**Proof.** For \( w \in A_2 \), \( \mathcal{S} \) and \( \eta \)-sparse family with \( \eta \in (0,1) \), showing (7.1) is equivalent by duality to showing that for all \( f \in L^2(w), g \in L^2(w^{-1}) \)

\[
|\langle \mathcal{A}_S f, g \rangle| \lesssim [w]_{A_2} \| f \|_{L^2(w)} \| g \|_{L^2(w^{-1})}.
\]

By the Cauchy-Schwarz inequality

\[
|E_Q| = \int_{E_Q} w^{\frac{1}{2}} w^{-\frac{1}{2}} \leq (w(E_Q))^{\frac{1}{2}} (w^{-1}(E_Q))^{\frac{1}{2}}.
\]

Using the definition of the sparse operator, some algebra and the definition of an \( \eta \)-sparse family of cubes, namely \( |Q| \leq (1/\eta) |E_Q| \) we get that

\[
|\langle \mathcal{A}_S f, g \rangle| \leq \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \langle g \rangle_Q |Q| \leq \frac{1}{\eta} \sum_{Q \in \mathcal{S}} \frac{\| f \|_{W^{-1}} Q}{\langle w^{-1} \rangle_Q} \frac{\| g \|_{W^{-1}} Q}{\langle w \rangle_Q} \langle w \rangle Q \langle w^{-1} \rangle_Q |E_Q| \leq \frac{[w]_{A_2}}{\eta} \sum_{Q \in \mathcal{S}} \frac{\| f \|_{W^{-1}} Q}{\langle w^{-1} \rangle_Q} (w^{-1}(E_Q))^{\frac{1}{2}} \frac{\| g \|_{W^{-1}} Q}{\langle w \rangle_Q} (w(E_Q))^{\frac{1}{2}}.
\]

Using once more the Cauchy-Schwarz inequality and the fact that for all \( x \in E_Q \subset Q \) it holds that

\[
|\langle h \rangle_Q | \langle v \rangle_Q | \leq M_{v} h(x) \quad \text{therefore} \quad \left| \langle h \rangle_Q \right| \left| \langle v \rangle_Q \right| |v(E_Q)| \leq \int_{E_Q} |M_{v} h(x)|^2 v(x) \, dx,
\]

we conclude that

\[
|\langle \mathcal{A}_S f, g \rangle| \leq \frac{[w]_{A_2}}{\eta} \sum_{Q \in \mathcal{S}} \frac{\| f \|_{W^{-1}} Q}{\langle w^{-1} \rangle_Q} (w^{-1}(E_Q))^{\frac{1}{2}} \left( \sum_{Q \in \mathcal{S}} \frac{\| g \|_{W^{-1}} Q}{\langle w \rangle_Q} (w(E_Q))^{\frac{1}{2}} \right)^{\frac{1}{2}}.
\]
where, in the line before the last, we used the fact that the sets \( S \) are pairwise disjoint and, in the last line, we used estimate (3.1) for the weighted dyadic maximal functions.

\[
\leq \frac{[w]_{A_2}}{\eta} \left[ \sum_{Q \in S} \int_{E_Q} |M_{w^{-1}}^{\emptyset}(fw)(x)|^2 w^{-1}(x) \, dx \right] \frac{1}{2} \left[ \sum_{Q \in S} \int_{E_Q} |M_w^{\emptyset}(gw^{-1})(x)|^2 w(x) \, dx \right] \frac{1}{2}
\]

\[
\leq \frac{[w]_{A_2}}{\eta} \left\| M_{w^{-1}}^{\emptyset}(fw) \right\|_{L^2(w^{-1})} \left\| M_w^{\emptyset}(gw^{-1}) \right\|_{L^2(w)}
\]

\[
\lesssim [w]_{A_2} \left\| fw \right\|_{L^2(w^{-1})} \left\| gw^{-1} \right\|_{L^2(w)} = [w]_{A_2} \left\| f \right\|_{L^2(w)} \left\| g \right\|_{L^2(w^{-1})}.
\]

Where, in the line before the last, we used the fact that the sets \( E_Q \) for \( Q \in S \) are pairwise disjoint and, in the last line, we used estimate (3.1) for the weighted dyadic maximal functions.

Similar argument yields linear bounds on \( L^p(w) \) for \( p > 2 \) and by duality (sparse operators are self-adjoint) we get bounds like \( [w]_{A_p} \frac{1}{p-1} = [w^{-1/(p-1)}]_{A_2} \) when \( 1 < p < 2 \), see [Moe]. In other words, we can get directly the same \( L^p(w) \) bounds that sharp extrapolation will give if we were to extrapolate from the linear \( L^2(w) \) bounds, namely, for all \( w \in A_p \) and \( f \in L^p(w) \)

\[
\| \mathcal{D}_S f \|_{L^p(w)} \lesssim [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \| f \|_{L^p(w)}.
\]

7.4. Domination by Sparse Operators. Many operators can be dominated by finitely many sparse operators, pointwise, in norm, or by forms. The collections \( \mathcal{S} \), \( \mathcal{S}' \) are sparse families tailored to the operator and the particular function \( f \) the operator is acting on. Identifying these sparse families is where most of the work lies, usually done using some sort of weak-(1, 1) inequality that is available a priori, or a specific stopping time designed for the problem at hand. We will illustrate this process for the martingale transform in Section 7.5. Here is the status, in terms of sparse domination, of the operators we have been discussing in these lecture notes. In particular quantitative weighted estimates for corresponding sparse operators, such as (7.2), immediately transfer to the dominated operators, providing new and streamlined proofs of the quantitative weighted inequalities we have been focusing on previous sections.

The martingale transforms and the dyadic paraproduct are locally pointwise dominated by sparse operators [L3]. More precisely, given a cube \( Q_0 \) and \( f \in L^1(\mathbb{R}) \) there are sparse families \( \mathcal{S}, \mathcal{S}' \) such that

\[
|1_{Q_0} T_f (f \mathbb{1}_{Q_0})| \lesssim \mathcal{D}_S |f|, \quad |1_{Q_0} \pi_b(f \mathbb{1}_{Q_0})| \lesssim \mathcal{D}_S' |f|.
\]

We will say more about the martingale transform in Section 7.5.

Calderón-Zygmund operators are pointwise dominated by finitely many sparse operators [CR, Le6, LeN]. More precisely, given \( T \) and \( f \) there are finitely many sparse families \( \mathcal{S}_i \), for \( i = 1, \ldots, N_d \), such that

\[
|Tf| \leq \sum_{i=1}^{N_d} \mathcal{D}_{\mathcal{S}_i} f.
\]

The dyadic square function is pointwise dominated by finitely many sparse-like operators [LLi2]. More precisely, given \( f \) there are finitely many sparse families \( \mathcal{S}_i \), for \( i = 1, \ldots, N_d \), such that

\[
|S^\emptyset f|^2 \leq \sum_{i=1}^{N_d} \sum_{Q \in \mathcal{S}_i} |f|^2 \mathbb{1}_Q.
\]

Notice that the sparse-like operators have been adapted to the square function.
Commilator \([b,T]\) for \(T\) an \(\omega\)-Calderón-Zygmund operator with \(\omega\) satisfying a Dini condition, \(b \in L^1_{\text{loc}}(\mathbb{R})\) can be pointwise dominated by finitely many sparse-like operators and their adjoints [LeOR1, LeOR2]. We will say more about this in Section 7.6.

The finitely many sparse families come from the analogue of the one-third trick for the dyadic grids, usually \(N_d = 3^d\) will suffice.

7.5. **Domination of martingale transform d'après Lacey.** We would like to illustrate how to achieve domination by sparse operators for a toy model operator, the martingale transform \(T_\sigma\) on \(L^2(\mathbb{R})\). Following an argument of Michael Lacey [L3, Section 3].

Given interval \(I_0 \in \mathfrak{D}\) and function \(f \in L^1(\mathbb{R})\) supported on \(I_0\), we need to find a 1/2-sparse family \(\mathcal{S} \subset \mathfrak{D}\), such that for all choices of signs \(\sigma\), there is a constant \(C > 0\) such that

\[
|1_{I_0} T_\sigma f| \leq C\mathcal{A}_{\mathcal{S}}|f|.
\]

**Proof.** Without loss of generality we can assume that \(f \in L^1(\mathbb{R})\) is not only supported on \(I_0\) but also \(\int_{I_0} |f(x)|dx > 0\). We will need the following well-known weak-type estimates.

First, the sharp truncation \(T^2_\sigma\) is of weak-type \((1,1)\) [Bur1], with a constant independent of the choice of signs \(\sigma\), thus

\[
\sup_{\lambda > 0} \lambda \left| \left\{ x \in \mathbb{R} : T^2_\sigma f(x) > \lambda \right\} \right| \leq C\|f\|_{L^1(\mathbb{R})},
\]

where \(T^2_\sigma f = \sup_{I \in \mathfrak{D}} \sum_{J \subset I, \sigma J \cap I' \neq \emptyset} \sigma J \|f_h\|_J\|f_h\|_{I'}\).

Second, the maximal function \(M\) is also of weak-type \((1,1)\), therefore

\[
\sup_{\lambda > 0} \lambda \left| \left\{ x \in \mathbb{R} : Mf(x) > \lambda \right\} \right| \leq C\|f\|_{L^1(\mathbb{R})},
\]

As a consequence there exists a constant \(C_0 > 0\) such that the subset of \(I_0\) defined by

\[
F_{I_0} := \left\{ x \in I_0 : \max \{Mf(x), T^2_\sigma f(x)\} > \frac{1}{2} C_0 \|f\|_{I_0}\right\}
\]

has no more than half the mass of \(I_0\), that is, \(|F_{I_0}| \leq \frac{1}{2}|I_0|\). In fact, suppose no such constant would exist, then for all \(C_0 > 0\) it would hold that

\[
|F_{I_0}| = \left| \left\{ x \in I_0 : \max \{Mf(x), T^2_\sigma f(x)\} > \frac{1}{2} C_0 \|f\|_{I_0}\right\} \right| > \frac{1}{2}|I_0|,
\]

therefore for each \(C_0 > 0\) it must be that either \(|\{x \in I_0 : Mf(x) > \frac{1}{2} C_0 \|f\|_{I_0}\}| > \frac{1}{2}|I_0|\) or \(|\{x \in I_0 : T^2_\sigma f(x) > \frac{1}{2} C_0 \|f\|_{I_0}\}| > \frac{1}{2}|I_0|\). But either of these sets has measure bounded above by \(2C\|f\|_{L^1(\mathbb{R})}/(C_0 \|f\|_{I_0})\), choosing \(C_0\) large enough, so that \(2C\|f\|_{L^1(\mathbb{R})}/(C_0 \|f\|_{I_0}) < 1/4\), a contradiction will be reached. It seems as if the constant \(C_0\) depends on the interval \(I_0\), however once we recall that the function \(f\) is supported on \(I_0\) then all is required is that \(2C/C_0 < 1/4\).

Let \(\mathcal{E}_{I_0}\) be the collection of maximal dyadic intervals \(I \in \mathfrak{D}\) contained in the set \(F_{I_0}\), then we claim that

\[
|T_\sigma f(x)| 1_{I_0} \leq C_0 \|f\|_{I_0} + \sum_{I \in \mathcal{E}_{I_0}} |T^I_\sigma f(x)|
\]

(7.3)

where \(T^I_\sigma f := \sigma_{I'}(f) 1_I + \sum_{J \subset I, J \subset I'} \sigma_{J} f_h h_J\), and \(I'\) is the parent of \(I\).

Repeat for each \(I \in \mathcal{E}_{I_0}\) and the function \(T^I_\sigma f\) which is supported on \(I\), then repeat for each \(I' \in \mathcal{E}_I\), etc. Let \(S_0 = \{I_0\}\), and \(S_j := \cup_{I \in S_{j-1}} \mathcal{E}_I\). Finally let \(\mathcal{S} := \cup_{j=0}^{\infty} S_j\). For each
Let $E_I = I \setminus F_I$, by construction the sets $E_I \subset I$ are pairwise disjoint and $|E_I| \geq \frac{1}{2}|I|$, therefore $\mathcal{S}$ is a $\frac{1}{2}$-sparse family. Moreover

$$|1_{I_0}T_\sigma f| \leq C_0 A_\mathcal{S}|f|,$$

which is what we set out to prove. We are done modulo verifying the claimed inequality (7.3), which we now prove. Note that $|T_\sigma f(x)| \leq 2T_\sigma^2 f(x)$. Thus, if $x \in I_0 \setminus F_{I_0}$ then $|T_\sigma f(x)| \leq C_0 \langle |f| \rangle_{I_0}$, and (7.3) is satisfied.

If $x \in F_{I_0}$ then there is unique $I \in \mathcal{S}_1 = \mathcal{E}_{I_0}$ with $x \in I$, and recalling that $\langle f, h_\tilde{I} \rangle h_\tilde{I}(x) = \langle f \rangle_I - \langle f \rangle_{\tilde{I}}$, we conclude that

$$T_\sigma f(x) = \sum_{J \supset \tilde{I}} \sigma_J(f, h_J)h_J(x) + \sum_{J \subset \tilde{I}} \sigma_J(f, h_J)h_J(x)$$

$$= \sum_{J \supset \tilde{I}} \sigma_J(f, h_J)h_J(x) - \sigma_\tilde{I}(f) + T_\sigma^I f(x).$$

Therefore we find that when $x \in F_{I_0}$ and for all $y \in \tilde{I}$ the following inequality holds

(7.4) $$|T_\sigma f(x)| \leq T_\sigma^y f(y) + Mf(y) + \sum_{I \in \mathcal{E}_{I_0}} T_\sigma^I f(x).$$

In particular, because $I$ is a maximal dyadic interval in $F_{I_0}$, there must be $y_0 \in \tilde{I} \setminus I$ such that $y_0 \notin F_{I_0}$ and therefore $T_\sigma^y f(y_0) + Mf(y_0) \leq \frac{1}{2} C_0 \langle |f| \rangle_{I_0}$. Substituting $y = y_0$ in (7.4), and using this estimate proves the claimed inequality (7.3), and therefore the pointwise localized domination by sparse operators for the martingale transform is proven. \(\square\)

### 7.6 Case study: Sparse operators vs commutators.

Carlos Pérez and Israel Rivera-Ríos proposed the following $L \log L$-sparse operator as a candidate for sparse domination of the commutator.

$$B_\mathcal{S} f(x) = \sum_{Q \in \mathcal{S}} \|f\|_{L \log L, Q} 1_Q(x).$$

The reason for this choice is that $M^2 \sim M_{L \log L}$ is the correct maximal function for the commutator. However they showed that these operators cannot bound pointwise the commutator $[b, T]$ in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Andrei Lerner, Sheldy Ombrosi, and Israel Rivera-Ríos proposed the following sparse-like operator and its adjoint adapted to the commutator with locally integrable function $b$,

$$T_{S, b} f(x) := \sum_{Q \in \mathcal{S}} |b(x) - \langle b \rangle_Q | \langle f \rangle_Q 1_Q(x),$$

$$T_{S^*, b} f(x) := \sum_{Q \in \mathcal{S}} \langle |b - \langle b \rangle_Q | f \rangle_Q 1_Q(x).$$

They showed, in [LeOR1], that finitely many of these operators will provide pointwise domination for the commutator, $[b, T]$, where $T$ is a rough Calderón-Zygmund operator and $b$ a locally integrable function.

**Theorem 7.3** (Lerner, Ombrosi, Rivera-Ríos 2017). Let $T$ be an $\omega$-Calderón-Zygmund singular integral operator with $\omega$ satisfying a Dini condition, $b \in L^1_{\text{loc}}(\mathbb{R}^d)$. For every compactly
supported \( f \in L^\infty(\mathbb{R}^d) \), there are \( 3^n \) dyadic lattices \( \mathcal{D}_k \) and \( \frac{1}{2^n} \)-sparse families \( \mathcal{S}_k \subset \mathcal{D}_k \) such that for a.e. \( x \in \mathbb{R}^d \)

\[
||b, T||_x(f)(x) \lesssim_{d,T} \sum_{k=1}^{3^n} (||\mathcal{S}_{k,b}||_x f(x) + ||\mathcal{S}_{k,b}^*||_x f(x)).
\]

Quadratic bounds on \( L^2(w) \) for the commutator \([b, T]\) will follow from quadratic bounds for these adapted sparse operators [LeOR1]. The following quadratic bounds on \( L^2(w) \) for \( \mathcal{S}_{S,b} , \mathcal{S}_{S,b}^* \) hold,

\[
||\mathcal{S}_{S,b}||_{L^2(w)} + ||\mathcal{S}_{S,b}^*||_{L^2(w)} \lesssim [w]_{A_2}^2 ||b||_{BMO} \parallel f \parallel_{L^2(w)}.
\]

These quadratic bounds, the corresponding extrapolated bounds on \( L^p(w) \)

\[
||\mathcal{S}_{S,b}||_{L^p(w)} + ||\mathcal{S}_{S,b}^*||_{L^p(w)} \lesssim_p [w]_{A_p}^{2 \max \{1, \frac{1}{p-1}\}} ||b||_{BMO} \parallel f \parallel_{L^p(w)},
\]

and much more follow from a key lemma that we now state.

**Lemma 7.4** (Lerner, Ombrosi, Rivera-Ríos 2017). Given \( \mathcal{S} \) an \( \eta \)-sparse family in \( \mathcal{D} \), \( b \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) then there is a larger collection \( \tilde{\mathcal{S}} \subset \mathcal{D} \) which is an \( \frac{\eta}{2(1+\eta)} \)-sparse family, \( \mathcal{S} \subset \tilde{\mathcal{S}} \), such that for all \( Q \in \tilde{\mathcal{S}} \), the following estimate holds

\[
||b(x) - (b)_Q|| \leq 2^{d+2} \sum_{R \in \tilde{\mathcal{S}}, R \subset Q} \Omega(b; R) \chi_R(x), \quad \text{a.e. } x \in Q,
\]

where \( \Omega(b; R) := \frac{1}{|R|} \int_R |b(x) - (b)_R| \, dx \), the mean oscillation of \( b \) on the dyadic cube \( R \).

From this lemma we immediately deduce quantitative Bloom bounds for the sparse-like adjoint operator associated to the commutator [LeOR1]. A similar result holds for \( \mathcal{S}_{S,b} \).

**Corollary 7.5** (Quantitative Bloom). Let \( u, v \in A_p, \mu = u^{1/p}v^{-1/p} \) and \( b \in \text{BMO}_\mu \) then there is a constant \( c_{d,p} > 0 \) such that for all \( f \in L^p(u) \) the following inequality holds,

\[
||\mathcal{S}_{S,b}||_{L^p(v)} \leq c_{d,p} ||b||_{\text{BMO}_\mu} (||u||_{A_p} ||u||_{A_p})^{\max \{1, \frac{1}{p-1}\}} ||f||_{L^p(u)}.
\]

Similarly for \( \mathcal{S}_{S,b}^* \).

**Proof.** First notice that since \( ||b||_{\text{BMO}_\mu} = \sup_Q |Q| \Omega(b; Q)/\mu(Q) \),

\[
\mathcal{S}_{S,b}^* ||f|| = c_{d,p} ||b||_{\text{BMO}_\mu} A_{\mathcal{S}_b} (||f||_\mu)(x),
\]

where \( \mathcal{S}_b \) is the larger sparse family given by Lemma 7.4.

Taking \( L^p(v) \) norm on both sides, and unfolding we conclude that

\[
||\mathcal{S}_{S,b}||_{L^p(v)} \leq c_{d,p} ||b||_{\text{BMO}_\mu} ||A_{\mathcal{S}_b}||_{L^p(v)} ||A_{\mathcal{S}_b}||_{L^p(u)} ||f||_{L^p(u)} \leq c_{d,p} ||b||_{\text{BMO}_\mu} (||u||_{A_p} ||u||_{A_p})^{\max \{1, \frac{1}{p-1}\}} ||f||_{L^p(u)},
\]

where in the last line we used the one-weight estimates on both \( L^p(u) \) and \( L^p(v) \) for the sparse operator \( A_{\mathcal{S}_b} \) given that \( u \) and \( v \) are \( A_p \) weights by assumption. Observing that \( \mathcal{S}_{S,b}^* ||f||(x) \leq \mathcal{S}_{S,b}^* ||f||_x(x) \) we get the desired estimate. \( \square \)
Setting \( u = v = w \in A_p \), then \( \mu \equiv 1 \), \( b \in \text{BMO} \), and we recover the expected one-weight quantitative \( L^p \) estimates for the sparse-like operators dominating the commutator, and hence for the commutator itself, without using extrapolation,

\[
\| T_{\Delta, b} f \|_{L^p(w)} + \| T_{\Delta, b}^* f \|_{L^p(w)} \leq c_{n, p}\| b \|_{\text{BMO}[w]} A_p \| f \|_{L^p(w)}. \]

8. Summary and recent progress

In these lecture notes we have studied weighted norm inequalities through the dyadic harmonic analysis lens. We focused on classical operators such as the Hilbert transform and the maximal function, and dyadic operators such as the dyadic maximal function, the martingale transform, the dyadic square function, Haar shift multipliers, the dyadic paraproduct, and the latest "kid in the block" the dyadic sparse operator. To carry on our program, we discussed dyadic tools such as dyadic cubes (regular, random, adjacent) and Haar functions on \( \mathbb{R}, \mathbb{R}^d \), and more generally on spaces of homogeneous type.

In this millennium the interest shifted from qualitative weighted norm inequalities to quantitative weighted norm inequalities. New techniques were developed to obtain quantitative estimates, including Bellman function and median oscillation techniques, quantitative extrapolation and transference theorems, corona decompositions and stopping times, representation of operators as averages of dyadic operators, and, most recently, domination by dyadic sparse operators. One important landmark in this quest was the proof of the \( A_2 \) conjecture. Some of these techniques are amenable to generalizations to other settings that support dyadic structures such as spaces of homogeneous type.

We tried to illustrate the power of the dyadic methods studying in detail the maximal function and the commutator of the Hilbert transform with a function in \( \text{BMO} \) via their dyadic counterparts, in both cases obtaining the optimal estimates on weighted Lebesgue spaces. We presented a self-contained Bellman function proof of the \( A_2 \) conjecture for the dyadic paraproduct, in order to illustrate these technique. We showed how to pointwise dominate the martingale transform by sparse operators, and we presented the beautiful and simple proof of the \( A_2 \) conjecture for sparse operators. We illustrated the power of pointwise domination techniques by sparse-like operators through a case study: the commutator of Calderón-Zygmund singular integral operators and locally integrable functions, recovering all the quantitative weighted norm inequalities discussed in the notes, and some new ones.

The methods developed in this millennium, initially to study quantitative weighted inequalities for operators defined on \( \mathbb{R}^d \), have proven to be quite flexible and far reaching. There are extensions to metric spaces with geometrically doubling condition, spaces of homogeneous type, and beyond doubling even in a non-commutative setting of operator-valued dyadic harmonic analysis [Hyt4, NRV, KLPW, LoMaPa, DGKLWY, ThTV, CLo]. There are off-diagonal sharp two-weight estimates for sparse operators [FaHyt]. There are generalizations to matrix valued operators [IKP], so far the best weighted \( L^2 \) estimates in this setting are \( 3/2 \) powers for the matrix-valued paraproducts, shift operators, and Calderón-Zygmund operators satisfying a Dini condition [NPetTV], and linear for the square function [HytPetV]. The validity of the \( A_2 \) conjecture in the matrix setting is unknown. Two-weight estimates have been obtained for well localized operators with matrix weights [BiCuTW] and a weighted Carleson embedding theorem with matrix weights is known and proved using a "Bellman function with a parameter" [CuT]. Researchers are busy working towards increasing our knowledge on this setting,
see for example [CuPetPo] where a bilinear Carleson embedding theorem with matrix weight and scalar measure is proved using Bellman function techniques.

More importantly, out of these investigations a domination paradigm by sparse positive dyadic operators has emerged and proven to be very powerful with applications in many areas not only weighted inequalities. The following is a partial and ever-growing list of such applications to: (maximal) rough singular integrals [CuDiPOu1, CCuDiPOu, HytRoTa, DiPHytLi]; singular non-integral operators [BFPet]; multilinear maximal and singular integral operators [CuDiPOu2, LeN, BMu1, Z]; non-homogeneous spaces and operator-valued singular integral operators [CPa, VZ]; uncentered variational operators [deFZ]; variational Carleson operators [DiPDoU]; Walsh-Fourier multipliers [CuDiPLOu]; Bochner-Riesz multipliers [BBLu, LMeR, KL]; maximally truncated oscillatory singular integral operators [KrL1, KrL3]; spherical maximal function [L4]; Radon transform [Ob]; Hilbert transform along curves [CIOu]; pseudodifferential operators [BC]; the lattice Hardy-Littlewood maximal operator [HaLo]; fractional operator with $L^{α,r'}$-Hörmander conditions [IbRiVi]; Rubio de Francia’s Littlewood–Paley square function [GRS]. Sparse $T(1)$ theorems [LMe] and applications in the discrete setting [KrL2, KMe, CuKL] have been found as well as logarithmic bounds for maximal sparse operators [KaL].

We are starting to understand why in certain settings this philosophy does not work. For example very recently it was shown that dominating the dyadic strong maximal function by $(1,1)$-type sparse forms based on rectangles with sides parallel to the axes is impossible [BaCOuR], this is in the realm of multiparameter analysis were many questions still need to be answered. Perhaps a new type of sparse domination in this setting will have to be dreamed.

Not only the methodology is tried on each author’s favorite operator, far reaching extensions and broader understanding is being gained. For example, the convex body domination paradigm [NPetTV] shows that if a scalar operator can be dominated by a sparse operator, then its vector version can be dominated by a convex body valued sparse operator, a transference theorem. Similarly, multiple vector-valued extensions of operators and more can be explained through the very general helicoidal method [BMu2], yet another far-reaching transference methodology.

This is a very active area of research and we hope this lecture notes have helped to impress on the reader its vitality.

References

[Ai] H. Aimar, *Construction of Haar type bases on quasi-metric spaces with finite Assouad dimension*. Anal. Acad. Nac. Cs. Ex., F. y Nat., Buenos Aires 54, 2002.

[AiBI] H. Aimar, A. Bernardis, B. Iaffei, *Multiresolution approximations and unconditional bases on weighted Lebesgue spaces on spaces of homogeneous type*. J. Approx. Theory 148, no. 1 (2007), 12–34.

[AiBN1] H. Aimar, A. Bernardis, L. Nowak, *Dyadic Fefferman-Stein inequalities and the equivalence of Haar bases on weighted Lebesgue spaces*. Proc. Roy. Soc. Edinburgh Sect. A 141, no. 1 (2011), 1–21.

[AiBN2] H. Aimar, A. Bernardis, L. Nowak, *Equivalence of Haar bases associated with different dyadic systems*. J. Geom. Anal. 21, no. 2 (2011), 288–304.

[AiG] H. Aimar O. Gorosito, *Unconditional Haar bases for Lebesgue spaces on spaces of homogeneous type*. Proc. SPIEE 4119, Wavelet Applications in Signal and Image Processing VIII (2000), 556–563.

[AiM] H. Aimar, R. A. Macià, *Weighted norm inequalities for the Hardy-Littlewood maximal operator on spaces of homogeneous type*. Proc. Amer. Math. Soc. 91 (2) (1984), 213–216.

[AMi] R. Alvarado, M. Mitrea, *Hardy spaces on Ahlfors-regular quasi metric spaces. A sharp theory*. Springer Lecture Notes in Mathematics 2142, 2015.
[ABKPz] J. Álvarez, R. J. Bagby, D. S. Kurtz, C. Pérez, Weighted estimates for commutators of linear operators. Studia Math. 104(2) (1993), 195–209.

[AIS] K. Astala, T. Iwaniec, E. Saksman, Beltrami operators in the plane. Duke Math. J. 107. 1 (2001), 27–56.

[AH1] P. Auscher, T. Hytönen, Orthonormal bases of regular wavelets in spaces of homogeneous type. Appl. Comput. Harmon. Anal. 34, no. 2 (2013), 266–296.

[AH2] P. Auscher, T. Hytönen, Addendum to Orthonormal bases of regular wavelets in spaces of homogeneous type. Appl. Comput. Harmon. Anal. 39, no. 3 (2015), 568–569.

[BaCOuR] A. Barron, J. Conde-Alonso, Y. Ou, G. Rey, Sparse domination and the strong maximal function. Available at arXiv:1811.01243.

[BCl] D. Beltran, L. Cladek, Sparse bounds for pseudodifferential operators. To appear in J. Anal. Math. Available at arXiv:1711.02339.

[BBLu] C. Benea, F. Bernicot, T. Luque, Sparse bilinear forms for Bochner Riesz multipliers and applications. Available on arXiv:1605.06401.

[BMu1] C. Benea, C. Muscalu. Multiple vector-valued inequalities via the helicoidal method. Appl. Comput. Harmon. Anal. 34, no. 2 (2013), 266–296.

[BMu2] C. Benea, C. Muscalu, Sparse domination via the helicoidal method. Available at arXiv:1707.05484

[BMN] A. Bényi, D. Maldonado, V. Naibo, What is... a Paraproduct?. Notices Amer. Math. Soc. 57, no. 7 (2010), 858–860.

[BFPet] F. Bernicot, D. Frey, S. Petermichl, Sharp weighted norm estimates beyond Calderón-Zygmund theory. Anal. PDE 9 (2016), 109–1113.

[Be1] O. Beznosova, Bellman functions, paraproducts, Haar multipliers, and weighted inequalities. PhD Dissertation, University of New Mexico, 2008.

[Be2] O. Beznosova, Linear bound for the dyadic paraproduct on weighted Lebesgue space $L^2(w)$. J. Func. Anal. 255 (2008), 994–1007.

[BeCMoP] O. Beznosova, D. Chung, J. Moraes, M. C. Pereyra, On two weight estimates for dyadic operators. Harmonic analysis, partial differential equations, complex analysis, Banach spaces, and operator theory. Vol. 2, 135–169, Assoc. Women Math. Ser. 5. Springer, Cham, 2017.

[BeRe] O. Beznosova, A. Reznikov, Equivalent definitions of dyadic Muckenhoupt and Reverse Holder classes in terms of Carleson sequences, weak classes, and comparability of dyadic $L \log L$ and $A_\infty$ constants. Rev. Mat. Iberoam. 30, Issue 4 (2014), 1191–1190.

[BiCuTW] K. Bickel, A. Culiuc, S. Treil, B. Wick, Two weight estimates for well localized operators with matrix weights. To appear Trans. Amer. Math. Soc. DOI: 10.1090/tran/7400.

[Bl] S. Bloom, A commutator theorem and weighted $BMO$. Trans. Amer. Math. Soc. 340, no. 1 (1993), 253–272.

[Bu1] S. M. Buckley, Estimates for operator norms on weighted spaces and reverse Jensen inequalities. Trans. Amer. Math. Soc. 340, no. 1 (1993), 253–272.

[Bu2] S. M. Buckley, Summation condition on weights. Michigan Math. J. 40 (1993), 153–170.

[Bur1] D.L. Burkholder, Martingale transforms. Ann. Math. Statist. 37, no. 6 (1966), 1494–1504.

[Bur2] D.L. Burkholder, Boundary value problems and sharp inequalities for martingale transforms. Ann. Probab. 12 (1984), 647–702.

[CWWiW] S.-Y. A. Chang, J. M. Wilson, T. H. Wolff, Some weighted norm inequalities concerning the Schrödinger operators. Comment. Math. Helv. 60, no. 2 (1985), 217–246.

[CLW] P. Chen, J. Li, L. A. Ward, BMO from dyadic BMO via expectations on product spaces of homogeneous type. J. Func. Anal. 265, no. 10 (2013), 2420–2451.

[Chr] M. Christ, A $T(\beta)$ theorem with remarks on analytic capacity and the Cauchy integral,. Colloq. Math. 60/61, no. 2 (1990), 601–628.

[Ch1] D. Chung, Commutators and dyadic paraproducts on weighted Lebesgue spaces. PhD Dissertation, University of New Mexico 2008.

[Ch2] D. Chung, Sharp estimates for the commutators of the Hilbert, Riesz and Beurling transforms on weighted Lebesgue spaces. Indiana U. Math. J. 60, Issue 5 (2011), 1543–1588.
[Ch3] D. Chung, *Weighted inequalities for multivariable dyadic paraproducts*. Publ. Mat. 55, no. 2 (2011), 475–499.

[ChPPz] D. Chung, M. C. Pereyra, C. Pérez, *Sharp bounds for general commutators on weighted Lebesgue spaces*. Trans. Amer. Math. Soc. 364 (2012), 1163–1177.

[ClOu] L. Cladek, Y. Ou, *Sparse domination of Hilbert transforms along curves*. Math. Res. Lett. 25, no. 2 (2018), 415–436.

[CoFe] R. Coifman, C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*. Studia Math. 51 (1974), 241–250.

[CoM] R. R. Coifman, Y. Meyer, *Au délà des opérateurs pseudo-différentiels*. Astérisque 57, 1979.

[CoRW] R. R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for Hardy spaces in several variables*. Ann. of Math. 103 (1976), 611–635.

[CoW] R. R. Coifman, G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*. Étude de certaines intégrales singulières. Lecture Notes in Math. 242, Springer-Verlag, Berlin, 1971.

[C] J. M. Conde. *A note on dyadic coverings and nondoubling Calderón-Zygmund theory*. J. Math. Anal. Appl. 397(2) (2013), 785–790.

[CCuDiPOu] J. M. Conde-Alonso, A. Culiuc, F. Di Plinio, Y. Ou, *A sparse domination principle for rough singular integrals*. Anal. PDE 10, no. 5 (2017), 1255–1284.

[CLo] J. M. Conde-Alonso, L. D. López-Sánchez, *Operator-valued dyadic harmonic analysis beyond doubling measures*. Proc. Amer. Math. Soc. 144, no. 9 (2016), 3869–3885.

[CPa] J. M. Conde-Alonso, J. Parcet, *Nondoubling Calderón-Zygmund theory - a dyadic approach-. To appear in J. Fourier Anal. Appl. Available at arXiv:1604.03711.*

[CR] J. M. Conde-Alonso, G. Rey, *A pointwise estimate for positive dyadic shifts and some applications*. Math. Annalen 365 (3-4) (2016), 1111–1135.

[CS1] M. Cotlar, C. Sadosky, *On the Helson-Szegö theorem and a related class of modified Toeplitz kernels*. in Harmonic Analysis in Euclidean spaces, ed. by G. Weiss and S. Wainger, Proc. Symp. Pure Math. 35, Amer. Math. Soc., Providence, R.I., (1979), 383–407.

[CS2] M. Cotlar, C. Sadosky, *On some Lp versions of the Helson-Szegö theorem*. Conference on Harmonic Analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), 306–317, Wadsworth Math. Ser., Wadsworth, Belmont, CA (1983).

[Ct] D. Cruz-Uribe, *Two weight norm inequalities for fractional integral operators and commutators*. Advanced Courses of Mathematical Analysis VI (2017), 25–85.

[CrMoe] D. Cruz-Uribe, K. Moen, *Sharp norm inequalities for commutators of classical operators*. Publ. Mat. 56 (2012), 147–190.

[CrMP21] D. Cruz-Uribe, J. M. Martell, C. Pérez, *Weights, extrapolation and the theory of Rubio the Francia*. Birkhäuser, 2011.

[CrMP22] D. Cruz-Uribe, J. M. Martell, C. Pérez, *Sharp weighted estimates for classical operators*. Adv. Math. 229 (2012), 408–441.

[CuDiPLOu] A. Culiuc, F. Di Plinio, M. T. Lacey, Y. Ou, *Endpoint sparse bound for Walsh-Fourier multipliers of Marcinkiewicz type*. Submitted to Rev. Mat. Iberoam. Available at arXiv:1805.06060.

[CuDiPLOu1] A. Culiuc, F. Di Plinio, Y. Ou, *Uniform sparse domination of singular integrals via dyadic shifts*. Math. Res. Lett. 25, no. 1 (2018), 21–42.

[CuDiPLOu2] A. Culiuc, F. Di Plinio, Y. Ou, *Domination of multilinear singular integrals by positive sparse forms*. J. London Math. Soc. 98, no. 2 (2018), 369–392.

[CuKL] A. Culiuc, R. Kesler, M. T. Lacey, *Sparse Bounds for the discrete cubic Hilbert transform*. To appear in Anal. PDE. Available at arXiv:1612:08881.

[CuPetPo] A. Culiuc, S. Petermichl, S. Pott, *A matrix weighted bilinear Carleson lemma and maximal function*. Available at arXiv:1811.05388.

[CuT] A. Culiuc, S. Treil, *The Carleson Embedding Theorem with matrix weights*. To appear Int. Math. Res. Not. https://doi.org/10.1093/imrn/rnx222. Available at arXiv:1508.01716.

[Da] G. David, *Moreux de graphes lipschitziens et intégrales singulières sur une surface*. Rev. Mat. Iberoam. 4(1) (1988), 73–114.

[DaS] G. David, S. Semmes, *A boundedness criterion for generalized Calderón-Zygmund operators*. Ann. of Math. 20 (1984), 371–397.
[DaJS] G. David, J.-L. Journé, S. Semmes, Opérateurs de Calderón-Zygmund, fonctions paraaccrétives et interpolation. Rev. Mat. Iberoam. 1, no. 4 (1985), 1–56.
[deFZ] F. C. de França Silva and P. Zorin-Kranich, Sparse domination of sharp variational truncations. Available at arXiv:1604.05506.
[DiDuoU] F. Di Plinio, Y. Do, G. Uraltsev. Positive sparse domination of variational Carleson operators. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 18, no. 4 (2018), 1443–1458.
[DiPHytLi] F. Di Plinio, T. Hytönen, K. Li, Sparse bounds for maximal rough singular integrals via the Fourier transform. Submitted to Ann. Inst. Fourier. (Grenoble). Available at arXiv 1706.07111.
[DH] D.G. Deng, Y. Han, Harmonic analysis on spaces of homogeneous type. Springer-Verlag 2009.
[DGPPet] O. Dragičević, L. Grafakos, M. C. Pereyra, S. Petermichl, Extrapolation and sharp norm estimates for classical operators in weighted Lebesgue spaces. Publ. Mat. 49 (2005), 73–91.
[Duo] J. Duoandicoetxe, Fourier Analysis. Graduate Studies in Math. 29, Amer. Math. Soc., Providence, RI, 2001.
[Duo2] J. Duoandicoetxea, Extrapolation of weights revisited: New proofs and sharp bounds. J. Func. Anal. 260, Issue 6, 15 (2011), 186–1901.
[FaHyt] S. Fackler, T. Hytönen, Off-diagonal sharp two-weight estimates for sparse operators. New York J. Math 24 (2018), 21–42.
[DGKLWY] X. T. Duong, R. Gong, M.-J. S. Kuffner, J. Li, B. Wick, D. Yang, Two commutators on spaces of homogeneous type and applications. Available at arXiv:1809.07942.
[FKP] R. Fefferman, C. Kenig, J. Pipher, The theory of weights and the Dirichlet problem for elliptic equations. Ann. of Math. (2) 134, no. 1 (1991), 65–124.
[FaHyt] S. Fackler, T. Hytönen, Off-diagonal sharp two-weight estimates for sparse operators. New York J. Math 24 (2018), 21–42.
[Fi] T. Fiegel, Singular integral operators: A martingale approach, in: Geometry of Banach Spaces. London Math. Soc. Lecture Notes Ser. 158, Cambridge Univ. Press, 1990, 95–110.
[Ha] T. S. Hänninen, Equivalence of sparse and Carleson coefficients for general sets. Available at arXiv:1709.10457.
[HaLo] T. S. Hänninen, E. Lorist, Sparse domination for the lattice Hardy-Littlewood maximal operator. Available at arXiv:1712.02952.
[HoLW1] I. Holmes, M. T. Lacey, B. Wick, Commutators in the two-weight setting. Math. Ann. 367 (2017), 5–80.
[HoLW2] I. Holmes, M. T. Lacey, B. Wick, Bloom’s inequality: Commutators in a two-weight setting. Arch. Math. (Basel) 106(1) (2016) 53–63.
[HoPetW] I. Holmes, S. Petermichl, B. Wick, Weighted little bmo and two-weight estimates for Journé commutators. Anal. PDE 11, no. 7 (2018), 1693–1740.
[HTV] S. Hukovic, S. Treil, A. Volberg, The Bellman function and sharp weighted inequalities for square functions. In “Complex analysis, operators and related topics”, Oper. Theory Adv. Appl. 113 (2000), 97–113, Birkhäuser Basel.
[HMW] R. Hunt, B. Muckenhoupt, R. Wheeden Weighted norm inequalities for the conjugate function and the Hilbert transform. Trans. Amer. Math. Soc. 176 (1973), 227–252.
[Hyt2] T. Hytönen, The sharp weighted bound for general Calderón-Zygmund Operators. Ann. of Math. (2) 175, no. 3 (2012), 1473–1506.

[Hyt3] T. Hytönen, Martingales and harmonic analysis. Preprint, (2013) can be found online at http://www ctr.maths.lu.se/media/MATP29/2016vt2016/maha-eng_1.pdf

[Hyt4] T. Hytönen, The two-weight inequality for the Hilbert transform with general measures. Proc. London Math. Soc. 117, Issue 3 (2018), 483–526.

[Hyt5] T. Hytönen, The Holmes-Wick theorem on two-weight bounds for higher order commutators revisited. Archiv der Mathematik. 107, 4 (2016), 389–395.

[HytK] T. Hytönen, A. Kairema, Systems of dyadic cubes in a doubling metric space. Colloq. Math. 126(1), (2012), 1–33.

[HytL] T. Hytönen, M. T. Lacey, The $A_p$-$A_\infty$ inequality for general Calderón-Zygmund operators. Indiana Univ. Math. J. 61 (2012), 2041–2052.

[HytLi] T. Hytönen, K. Li, Weak and strong $A_p$-$A_\infty$ estimates for square functions and related operators. Proc. Amer. Math. Soc. 146, no. 6 (2018), 2497–2507.

[HytMa] T. Hytönen, H. Martikainen, Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces. J. Geom. Anal. 22 (4) (2012), 1071–1107.

[HytPvz] T. Hytönen, C. Pérez, Sharp weighted bounds involving $A_\infty$. Anal. PDE 6, no. 4 (2013), 777–818.

[HytPetV] T. Hytönen, S. Petermichl, A. Volberg, The sharp square function estimate with matrix weight. Available on arXiv:1702.04569

[HytRoTa] T. Hytönen, L. Roncal, O. Tapiola, Quantitative weighted estimates for rough homogeneous singular integrals. Israel J. Math. 218, no. 1 (2017), 133–164.

[HytTa] T. Hytönen, O. Tapiola, Almost Lipschitz-continuous wavelets in metric spaces via a new randomization of dyadic cubes. J. Approx. Theory 185 (2014), 12–30.

[IbRiVi] G. H. Ibañez-Firnkorn, M. S. Riveros, R. E. Vidal, Sharp bounds for fractional operator with $L^{\alpha,r}$-Hörmander conditions. Available at arXiv:1804.09631.

[IKP] J. Isralowitz, H. K. Kwon, S. Pott, Matrix weighted norm inequalities for commutators and paraproducts with matrix symbols. J. London Math. Soc. (2) 96, no. 1 (2017), 243–270.

[IN] F. John, L. Nirenberg, On functions of bounded mean oscillation. Comm. Pure Appl. Math. 14 (1961), 415–426.

[KLPW] A. Kairema, J. Li, M. C. Pereyra, L. A. Ward, Haar bases on quasi-metric measure spaces, and dyadic structure theorems for function spaces on product spaces of homogeneous type. J. Func. Anal. 271, no. 7 (2016), 1793–1843.

[KaL] G. A. Karagulyan, M. T. Lacey, On logarithmic bounds of maximal sparse operators. Available on arXiv:1802.00954.

[KP] N. H. Katz, M. C. Pereyra, On the two weight problem for the Hilbert transform. Rev. Mat. Iberoamericana 13, 01 (1997), 211–242.

[KL] R. Kesler, M. T. Lacey, Sparse endpoint estimates for Bohner-Riesz multipliers on the plane. Collect. Math. 69, no. 3 (2018), 427–435.

[KMe] R. Kesler, D. Mena, Uniform sparse bounds for discrete quadratic phase Hilbert transform. Anal. Math. Phys. (2017). https://doi.org/10.1007/s13324-017-0195-3.

[KrL1] B. Krause, M. T. Lacey, Sparse bounds for maximal monomial oscillatory Hilbert transforms. Studia Math. 242, no. 3 (2018), 217–229.

[KrL2] B. Krause, M. T. Lacey, Sparse bounds for random discrete Carleson theorems. 50 years with Hardy spaces. 317–332, Oper. Theory Adv. Appl. 261, Birkhäuser/Springer, Cham, 2018.

[KrL3] B. Krause, M. T. Lacey, Sparse bounds for maximally truncated oscillatory singular integrals. To appear Annal. Sci. Scuola Norm. Sup. DOI Number:10.2422/2036-2145.201706_023. Available at arXiv:1701.05249.

[L1] M. T. Lacey, Two weight Inequality for the Hilbert transform: A real variable characterization, II. Duke Math. J. 163, no. 15 (2014), 2821–2840.

[L2] M. T. Lacey, The two weight Inequality for the Hilbert transform: A Primer. Harmonic analysis, partial differential equations, Banach spaces, and operator theory. Vol. 2, 11–84, Assoc. Women Math. Ser. 5, Springer, Cham. 2017.

[L3] M. T. Lacey, An elementary proof of the $A_2$ bound. Israel J. Math. 217 (2017), 181–195.
[L4] M. T. Lacey, Sparse Bounds for Spherical Maximal Functions. To appear in J. d’Analyse Math. Available at arXiv:1702.08594.

[LLi1] M. T. Lacey, K. Li, Two weight norm inequalities for the g function. Math. Res. Lett. 21, no. 03 (2014), 521–536.

[LLi2] M. T. Lacey, K. Li, Two weight norm inequalities for the g function. Math. Res. Lett. 21, no. 03 (2014), 521–536.

[LLi3] M. T. Lacey, K. Li, On A_p–A∞ estimates for square functions. Math. Z. 284 (2016), 1211–1222.

[LMe] M. T. Lacey, D. Mena, The sparse T1 Theorem. Houston J. Math. 43, no. 1 (2017), 111–127.

[LMeR] M. T. Lacey, D. Mena, M. C. Reguera, Sparse Bounds for Bochner-Riesz Multipliers. J. Fourier Anal. Appl. (2017). https://doi.org/10.1007/s00041-017-9590-2.

[LMPzTo] M. T. Lacey, K. Moen, C. Pérez, R. H. Torres, Sharp weighted bounds for fractional integral operators. J. Funct. Anal. 259 (2010), 107–1097.

[LMPzTo] M. T. Lacey, K. Moen, C. Pérez, R. H. Torres, Sharp weighted bounds for fractional integral operators. J. Funct. Anal. 259 (2010), 107–1097.

[LMPzTo] M. T. Lacey, K. Moen, C. Pérez, R. H. Torres, Sharp weighted bounds for fractional integral operators. J. Funct. Anal. 259 (2010), 107–1097.

[LMPzTo] M. T. Lacey, K. Moen, C. Pérez, R. H. Torres, Sharp weighted bounds for fractional integral operators. J. Funct. Anal. 259 (2010), 107–1097.

[LMPzTo] M. T. Lacey, K. Moen, C. Pérez, R. H. Torres, Sharp weighted bounds for fractional integral operators. J. Funct. Anal. 259 (2010), 107–1097.

[LPetR] M. T. Lacey, S. Petermichl, M. C. Reguera, Sharp A2 inequality for Haar shift operators. Math. Ann. 348 (2010), 127–141.

[LSSU] M. T. Lacey, E. Sawyer, C.-Y. Shen, I. Uriarte-Tuero, The two weight inequality for the Hilbert transform, coronas and energy conditions. Duke Math. J. 163, no.15 (2014), 2795–2820.

[LS] M. T. Lacey, S. Spencer, Sparse Bounds for Oscillatory and Random Singular Integrals. New York J. Math. 23 (2017), 119–131.

[LTh] M. T. Lacey, C. Thiele, Lp bounds for the bilinear Hilbert transform. Ann. of Math. 146 (1997), 693–724.

[Le1] A. K. Lerner, An elementary approach to several results on the Hardy-Littlewood maximal operator. Proc. Amer. Math. Soc. 136, no. 8 (2008), 2829–2833.

[Le2] A. K. Lerner, Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals. Adv. Math. 226 (2011), 3912–3926.

[Le3] A. K. Lerner, Mixed A_p–A_r inequalities for classical singular integrals and Littlewood-Paley operators. J. Geom. Anal. 23 (2013), 1343–1354.

[Le4] A. K. Lerner, On an estimate of Calderón-Zygmund operators by dyadic positive operators. J. Anal. Math. 121 (2013), 141–161.

[Le5] A. K. Lerner, A simple proof of the A2 conjecture. Int. Math. Res. Not. 14 (2013), 3159–3170.

[Le6] A. K. Lerner, On pointwise estimates involving sparse operators. New York J. Math. 22 (2016), 341–349.

[Le7] A. K. Lerner, K. Moen, Mixed A_p–A∞ estimates with one supremum. Studia Math. 219, no. 3 (2013), 247–267.

[Le8] A. K. Lerner, F. Nazarov, Intuitive dyadic calculus: the basics. To appear in Expo. Math. Available at arXiv:1508.05639.

[LeOR1] A. K. Lerner, S. Ombrosi, I. Rivera-Ríos, On pointwise and weighted estimates for commutators of Calderón-Zygmund operators. Adv. Math. 319 (2017), 153–181.

[LeOR2] A. K. Lerner, S. Ombrosi, I. Rivera-Ríos, Commutators of singular integrals revisited. To appear in Bull. London Math. Soc. Online Nov 2018. https://doi.org/10.1112/blms.12216

[LoMaPa] L. D. López-Sánchez, J. M. Martell, J. Parcet, Dyadic harmonic analysis beyond doubling measures. Adv. Math. 267 (2014), 44–93.

[MS] R. A. Macías, C. Segovia, Lipschitz functions on spaces of homogeneous type. Adv. Math. 33 (1979), 257–270.

[Me] T. Mei, BMO is the intersection of two translates of dyadic BMO. C. R. Math. Acad. Sci. Paris 336, no. 12 (2003), 1003–1006.

[Moe] K. Moen, Sharp one-weight and two-weight bounds for maximal operators. Studia Math. 194(2)(2009), 163–180.

[MoP] J. C. Moraes, M. C. Pereyra, Weighted estimates for dyadic Paraproducts and t-Haar multiplies with complexity (m, n). Publ. Mat. 57 (2013), 265–294.

[Mu] B. Muckenhoupt, Weighted norm inequalities for the Hardy–Littlewood maximal function. Trans. Amer. Math. Soc. 165 (1972), 207–226.

[MuW] B. Muckenhoupt, R. Wheeden, Weighted bounded mean oscillation and the Hilbert transform. Studia Math. 54 (1975/76), 221–237.

[Mu] B. Muckenhoupt, Weighted norm inequalities for the Hardy–Littlewood maximal function. Trans. Amer. Math. Soc. 165 (1972), 207–226.

[Mu] B. Muckenhoupt, Weighted norm inequalities for the Hardy–Littlewood maximal function. Trans. Amer. Math. Soc. 165 (1972), 207–226.

[Mu] B. Muckenhoupt, Weighted norm inequalities for the Hardy–Littlewood maximal function. Trans. Amer. Math. Soc. 165 (1972), 207–226.

[Naz] F. Nazarov, S. Petermichl, S. Treil, A. Volberg, Convex body domination and weighted estimates with matrix weights. Adv. Math. 318 (2017), 279–306.
[NRTV] F. Nazarov, A. Reznikov, S. Treil, A. Volberg, A Bellman function proof of the $L^2$ bump conjecture. J. Anal. Math. 121 (2013), 255–277.

[NRV] F. Nazarov, A. Reznikov, A. Volberg, The proof of $A_2$ conjecture in a geometrically doubling metric space. Indiana Univ. Math. J. 62, no. 5 (2013), 1503–1533.

[NT] F. Nazarov, S. Treil, The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis. St. Petersburg Math. J. 8 (1997), 721–824.

[NTV1] F. Nazarov, S. Treil, A. Volberg, The Bellman functions and the two-weight inequalities for Haar multipliers. J. Amer. Math. Soc. 12 (1999), 909–928.

[NTV2] F. Nazarov, S. Treil, A. Volberg, Bellman function in stochastic optimal control and harmonic analysis (how our Bellman function got its name). Oper. Theory: Adv. Appl. 129 (2001), 393–424.

[NTV3] F. Nazarov, S. Treil, A. Volberg, The Tb-theorem on non-homogeneous spaces. Acta Math. 190 (2003), 151–239.

[NTV4] F. Nazarov, S. Treil, A. Volberg, Two weight inequalities for individual Haar multipliers and other well localized operators. Math. Res. Lett. 15, no. 3 (2008), 583–597.

[NTV5] F. Nazarov, S. Treil, A. Volberg, Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures. Preprint 2005 posted in 2010, available arXiv:1003.1596.

[Ob] R. Oberlin, Sparse bounds for a prototypical singular Radon transform. Canadian Math. Bull. 12 pages. https://doi.org/10.4153/CMB-2018-007-5.

[Ok] K. Okikiolu, Characterization of subsets of rectifiable curves in $\mathbb{R}^n$. J. London Math. Soc. (2) 46, no. 2 (1992), 336–348.

[OrPzRe] C. Ortiz-Caraballo, C. Pérez, E. Rela, Improving bounds for singular operators via Sharp Reverse Hölder Inequality for $A_{\infty}$. Operator Theory: Advances and Applications 229 (2013), 303–321.

[P1] M. C. Pereyra, Lecture notes on dyadic harmonic analysis. Contemp. Math. 289 (2001), 1–60.

[P2] M. C. Pereyra, Weighted inequalities and dyadic harmonic analysis. *Excursions in harmonic analysis. Volume 2*, 281–306, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York, 2013.

[PW] M. C. Pereyra, L. A. Ward, Harmonic Analysis: from Fourier to wavelets. Student Mathematical Library Series 63, Amer. Math. Soc. 2012.

[Pz1] C. Pérez, Endpoint Estimates for Commutators of Singular Integral Operators. J. Func. Anal. (1) 128 (1995), 163–185.

[Pz2] C. Pérez, A course on singular integrals and weights. Harmonic and Geometric Analysis, Advanced courses in Mathematics C.R.M. Barcelona, Birkhauser, Basel, 2015.

[PzR] C. Pérez, E. Rela, A new quantitative two weight theorem for the Hardy-Littlewood maximal operator. Proc. Amer. Math. Soc. 143 (2015), 641–655.

[PzTV] C. Pérez, S. Treil, A. Volberg, Sharp weighted estimates for dyadic shifts and the $A_2$ conjecture. J. Reine Angew. Math. (Crelle’s Journal) 687 (2014), 43–86.

[Pet1] S. Petermichl, Dyadic shift and a logarithmic estimate for Hankel operators with matrix symbol. C. R. Acad. Sci. Paris Sér. I Math. 330, no. 6 (2000), 455–460.

[Pet2] S. Petermichl, The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical $A_p$ characteristic. Amer. J. of Math. 129 (2007), 1355–1375.

[Pet3] S. Petermichl, The sharp weighted bound for the Riesz transforms. Proc. Amer. Math. Soc. 136(04) (2007), 1237–1249.

[PetPo] S. Petermichl, S. Pott, An estimate for weighted Hilbert transform via square functions. Trans. Amer. Math. Soc. 354 (2002), 281–305.

[PetV] S. Petermichl, A. Volberg, Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular. Duke Math. J. 112, 2, (2002), 281–305.

[PiW] J. Pipher, L. A. Ward, BMO from dyadic BMO on the bidisc. J. London Math. Soc. 77, no. 2 (2008), 524–544.

[PoR] S. Pott, M. C. Reguera, Sharp Bekolle estimates for the Bergman projection. J. Func. Anal. 265 (12) (2013), 3233–3244.

[S1] E. Sawyer, A characterization of a two weight norm inequality for maximal functions. Studia Math. 75, no. 1 (1982), 1–11.

[S2] E. Sawyer, A characterization of two weight norm inequalities for fractional and Poisson integrals. Trans. Amer. Math. Soc. 308, no. 2 (1988), 533–545.
[SW] E. Sawyer, R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*. Amer. J. Math. 114(4) (1992), 813–874.

[St] E. Stein, *Harmonic analysis : real-variable methods, orthogonality, and oscillatory integrals*. First Edition, Princeton University Press, 1993.

[Th] C. Thiele, *Time-Frequency Analysis in the Discrete Phase Plane*, PhD Thesis Yale 1995.

[ThTV] C. Thiele, S. Treil, A. Volberg, *Weighted martingale multipliers in the non-homogeneous setting and outer measure spaces*. Adv. Math. 285 (2015), 1155–1188.

[T] S. Treil, *Sharp $A_2$ estimates of Haar shifts via Bellman function*. "Recent trends in analysis" 187–208, Theta Ser. Adv. Math., Theta, Bucharest, 2013.

[TV] S. Treil, A. Volberg, *Wavelets and the angle between past and future*. J. Func. Anal. 143(2) (1997), 269–308.

[Va] V. Vasyunin, *Cincinnati Lectures on Bellman Functions*. Edited by L. Slavin. Available at arXiv:1508.07668.

[Ve] I. Verbitsky, *Imbedding and multiplier theorems for discrete Littlewood-Paley spaces*. Pacific J. Math. 176, no. 2 (1996), 529–556.

[V] A. Volberg, *Bellman function technique in Harmonic Analysis*. Lectures of INRIA Summer School in Antibes, Preprint (2011) 1-58, available at arXiv:1106.3899

[VZ] A. Volberg, P. Zorin-Kranish, *Sparse domination on non-homogeneous spaces with an application to $A_p$ weights*. Rev. Mat. Iberoam. 34, no. 3 (2018), 1401–1414.

[Vu1] E. Vuorinen, $L^p(\mu) \to L^q(\nu)$ characterization for well localized operators. J. Fourier Anal. Appl. 22(5) (2016), 1059–1075.

[Vu2] E. Vuorinen, Two weight $L^p$-inequalities for dyadic shifts and the dyadic square function. Studia Math. 237(1) (2017), 25–56.

[We] D. Weirich, *Weighted inequalities for dyadic operators over spaces of homogeneous type*. PhD Dissertation, University of New Mexico, 2018.

[Wic] B. Wick, *Personal communication*. April 2016.

[Wil1] M. Wilson, *Weighted inequalities for the dyadic square function without dyadic $A_\infty$*. Duke Math. J. 55 (1987), 19–49.

[Wil2] M. Wilson, *Weighted Littlewood-Paley Theory and Exponential-Square Integrability*. Lecture Notes in Mathematics, 1924. Springer, Berlin, 2008.

[W1] J. Wittwer, *A sharp estimate on the norm of the martingale transform*. Math. Res. Letters 7 (2000), 1–12.

[W2] J. Wittwer, *A sharp estimate on the norm of the continuous square function*. Proc. Amer. Math. Soc. 130(8) (2002), 2335–2342.

[Z] P. Zorin-Kranish, $A_p - A_\infty$ estimates for multilinear maximal and sparse operators. To appear in J. Anal. Math.

Maria Cristina Pereyra, Department of Mathematics and Statistics, 1 University of New Mexico, 311 Terrace St. NE, MSC01 1115, Albuquerque, NM 87131-0001

E-mail address: crisp@math.unm.edu