On the \((1, 1)\)-tensor bundle with Cheeger–Gromoll type metric

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Abstract. The main purpose of the present paper is to construct Riemannian almost product structures on the \((1, 1)\)-tensor bundle equipped with Cheeger–Gromoll type metric over a Riemannian manifold and present some results concerning these structures.

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1. Introduction

A Riemannian metric on the tangent bundle of a Riemannian manifold had been defined by Musso and Tricerri [7] who, inspired by the paper of Cheeger and Gromoll [3], called it the Cheeger–Gromoll metric. The metric was defined by Cheeger and Gromoll; yet, Musso and Tricerri wrote down its expression, constructed it in a more ‘comprehensible’ way, and gave it the name. The Levi–Civita connection of the Cheeger–Gromoll metric and its Riemannian curvature tensor were calculated by Sekizawa in [14] (for more details, see [6]). In [8], Peyghan and his collaborators considered the Cheeger–Gromoll type metric on \((1, 1)\)-tensor bundle and examined some of its geometric properties. In [10], the same authors investigated the \((1, 1)\)-tensor sphere bundle of Cheeger–Gromoll type. Also in [9], the homogeneous lift metric was introduced and studied on the \((1, 1)\)-tensor bundle of the Riemannian manifold. In this paper, our aim is to study some almost product structures on the \((1, 1)\)-tensor bundle endowed with the Cheeger–Gromoll type metric. We get, sequentially, conditions for the \((1, 1)\)-tensor bundle endowed with the Cheeger–Gromoll type metric and some almost product structures to be locally decomposable Riemannian manifold and Riemannian almost product \(\mathcal{W}_3\)-manifold. Finally, we consider a product conjugate connection on the \((1, 1)\)-tensor bundle with the Cheeger–Gromoll type metric and give some results related to the connection.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class \(C^\infty\). Also, we denote by \(\mathfrak{T}_q^p(M)\) the set of all tensor fields of type \((p, q)\) on \(M\), and by \(\mathfrak{T}_q^p(T_1^1(M))\) the corresponding set on the \((1, 1)\)-tensor bundle \(T_1^1(M)\).
2. Preliminaries

Let $M$ be a differentiable manifold of class $C^\infty$ and finite dimension $n$. Then the set $T^1_1(M) = \cup_{P \in M} T^1_1(P)$ is, by definition, the tensor bundle of type $(1, 1)$ over $M$, where $\cup$ denotes the disjoint union of the tensor spaces $T^1_1(P)$ for all $P \in M$. For any point $\tilde{P}$ of $T^1_1(M)$ such that $\tilde{P} \in T^1_1(M)$, the surjective correspondence $\tilde{P} \to P$ determines the natural projection $\pi : T^1_1(M) \to M$. The projection $\pi$ defines the natural differentiable manifold structure of $T^1_1(M)$, that is, $T^1_1(M)$ is a $C^\infty$-manifold of dimension $n + n^2$. If $x^j$ are local coordinates in a neighborhood $U$ of $P \in M$, then a tensor $t$ at $P$ which is an element of $T^1_1(M)$ is expressible in the form $(x^j, t^j)$, where $t^j$ are components of $t$ with respect to the natural base. We may consider $(x^j, t^j) = (x^j, x^j) = (x^j)$, $j = 1, \ldots, n$, $\tilde{j} = n + 1, \ldots, n + n^2$, $J = 1, \ldots, n + n^2$ as local coordinates in a neighborhood $\pi^{-1}(U)$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $A = A^j \frac{\partial}{\partial x^j} \otimes dx^i$ be the local expressions in $U$ of a vector field $X$ and a $(1, 1)$ tensor field $A$ on $M$, respectively. Then the vertical lift $V^A$ of $A$ and the horizontal lift $H_X$ of $X$ are given, with respect to the induced coordinates, by

$$V^A = \begin{pmatrix} V^A_{i,j} \\ V^A_{j,i} \end{pmatrix} = \begin{pmatrix} 0 \\ A^j_i \end{pmatrix},$$

and

$$H_X = \begin{pmatrix} H^X_{i,j} \\ H^X_{j,i} \end{pmatrix} = \begin{pmatrix} X^j \\ X^i (\Gamma^m_{sj} t^i_m - \Gamma^m_{sm} t^i_j) \end{pmatrix},$$

where $\Gamma^m_{ij}$ are the coefficients of the connection $\nabla$ on $M$.

Let $\varphi \in \mathcal{T}_0^1(M)$. The global vector fields $\gamma \varphi$ and $\tilde{\varphi} \varphi \in \mathcal{T}_0^1(T^1_1(M))$ are respectively defined by

$$\gamma \varphi = \begin{pmatrix} 0 \\ t^i_j \varphi^i_m \end{pmatrix}, \quad \tilde{\varphi} \varphi = \begin{pmatrix} 0 \\ t^i_m \varphi^i_j \end{pmatrix}$$

with respect to the coordinates $(x^i, x^j)$ in $T^1_1(M)$, where $\varphi^i_j$ are the components of $\varphi$.

The Lie bracket operation of vertical and horizontal vector fields on $T^1_1(M)$ is given by

$$[H^X, H^Y] = \tilde{H} [X, Y] + (\tilde{\gamma} - \gamma) R(X, Y),$$

$$[H^X, V^A] = V(\nabla_X A),$$

$$[V^A, V^B] = 0,$$

for any $X, Y \in \mathcal{T}_0^1(M)$ and $A, B \in \mathcal{T}_1^1(M)$, where $R$ is the curvature tensor field of the connection $\nabla$ on $M$ defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ and $(\tilde{\gamma} - \gamma) R(X, Y) = \left( t^i_j R^k_{lij} X^i X^j - t^i_j R^k_{klm} i X^j X^j \right)$ (for details, see [2, 11]).

3. Riemannian almost product structures on the $(1, 1)$-tensor bundle with Cheeger–Gromoll type metric

An $n$-dimensional manifold $M$ in which a $(1, 1)$ tensor field $\varphi$ satisfying $\varphi^2 = id$, $\varphi \neq \pm id$ is given is called an almost product manifold. A Riemannian almost product
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manifold \((M, \varphi, g)\) is a manifold \(M\) with an almost product structure \(\varphi\) and a Riemannian metric \(g\) such that

\[
g(\varphi X, Y) = g(X, \varphi Y) \tag{3.1}\]

for all \(X, Y \in \mathfrak{X}(M)\). Also, the condition (3.1) is referred to as a purity condition for \(g\) with respect to \(\varphi\). The almost product structure \(\varphi\) is integrable, i.e. the Nijenhuis tensor \(N_{\varphi}\) determined by

\[
N_{\varphi}(X, Y) = [\varphi X, \varphi Y] - \varphi [\varphi X, Y] - \varphi [X, \varphi Y] + [X, Y]
\]

for all \(X, Y \in \mathfrak{X}(M)\) is zero then the Riemannian almost product manifold \((M, \varphi, g)\) is called a Riemannian product manifold. A locally decomposable Riemannian manifold can be defined as a triple \((M, \varphi, g)\) which consists of a smooth manifold \(M\) endowed with an almost product structure \(\varphi\) and a pure metric \(g\) such that \(\nabla \varphi = 0\), where \(\nabla\) is the Levi–Civita connection of \(g\). It is well-known that the condition \(\nabla \varphi = 0\) is equivalent to decomposability of the pure metric \(g\) \([13]\), i.e. \(\Phi_{\varphi}g = 0\), where \(\Phi_{\varphi}\) is the Tachibana operator \([16, 17]\): \((\Phi_{\varphi}g)(X, Y, Z) = (\varphi X)(g(Y, Z)) - X(g(\varphi Y, Z)) + g((L_Y \varphi)X, Z) + g(Y, (L_Z \varphi)X)\).

Let \(T^1_1(M)\) be the (1,1)-tensor bundle over a Riemannian manifold \((M, g)\). For each \(P \in M\), the extension of scalar product \(g\) (marked by \(G\)) is defined on the tensor space \(\mathfrak{p}^{-1}(P) = T^1_1(P)\) by \(G(A, B) = g_{ij}g^{jl}A^i_jB^l_j\) for all \(A, B \in \mathfrak{p}^1_1(P)\). The Cheeger–Gromoll type metric \(CGg\) is defined on \(T^1_1(M)\) by the following three equations:

\[
CGg(HX, HY) = V(g(X, Y)), \tag{3.2}
\]

\[
CGg(V_A, HY) = 0, \tag{3.3}
\]

\[
CGg(V_A, VB) = \frac{1}{\alpha}(G(A, B) + G(A, t)G(B, t)) \tag{3.4}
\]

for any \(X, Y \in \mathfrak{X}(M)\) and \(A, B \in \mathfrak{p}^1_1(M)\), where \(r^2 = G(t, t) = g_{ij}g^{jl}t^j_l\) and \(\alpha = 1 + r^2\). For the Levi–Civita connection of the Cheeger–Gromoll type metric \(CGg\), we give the next theorem.

**Theorem 1.** Let \((M, g)\) be a Riemannian manifold and \(\tilde{\nabla}\) be the Levi–Civita connection of the tensor bundle \(T^1_1(M)\) equipped with the Cheeger–Gromoll type metric \(CGg\). Then the corresponding Levi–Civita connection satisfies the following relations:

1. \(\tilde{\nabla}_{HX} HY = H(\nabla_X Y) + \frac{1}{2}(\tilde{\gamma} - \gamma)R(X, Y),\)

2. \(\tilde{\nabla}_{HX} VB = \frac{1}{2\alpha}H(g^{lj}R(t_b, B_j)X + g_{ai}(t^a(g^{-1} \circ R(, X)\tilde{B}^i) + V(\nabla_X B),\)

3. \(\tilde{\nabla}_{VA} HY = \frac{1}{2\alpha}(g^{lj}R(t_b, A_l)Y + g_{ai}(t^a(g^{-1} \circ R(, Y)\tilde{A}^i)),\)

4. \(\tilde{\nabla}_{VA} VB = \frac{1}{\alpha}(CGg(V_A, V_t)^t_B + CGg(V_B, V_t)^t_A + \frac{\alpha + 1}{\alpha}CGg(V_A, VB)^t - \frac{1}{\alpha}CGg(V_A, V_t)^t_CGg(V_B, V_t)^t_B,\)

\}
for all $X, Y \in \mathfrak{S}^1_0(M)$ and $A, B \in \mathfrak{S}^1_1(M)$, where $A_I = (A^i_I), \tilde{A}^i = (g^{bl}A^l_i) = (A^l_i), t_I = (t^a_I)$, $t^a = (t^a_b), R(, X)Y \in \mathfrak{S}^1_1(M), g^{-1} \circ R(, X)Y \in \mathfrak{S}^1_0(M)$ (see [8]).

**Theorem 2.** Let $\tilde{\gamma}$ be a Riemannian manifold and $T^1_1(M)$ be its $(1, 1)$-tensor bundle equipped with the Cheeger–Gromoll type metric $CG$ and the almost product structure $D$. The triple $(T^1_1(M), D, CG)$ is a Riemannian almost product manifold.

We now give conditions for the Cheeger–Gromoll type metric $CG$ to be decomposable with respect to the almost product structure $D$. We calculate

$$
(\Phi_D CG)(\tilde{X}, \tilde{Y}, \tilde{Z}) = (D \tilde{X})\tilde{X}(CG)(\tilde{Y}, \tilde{Z}) - \tilde{X}(CG)(D \tilde{Y}, \tilde{Z}) + CG((L_I V_0^B)\tilde{X}, \tilde{Z}) + CG(L_I^B(\tilde{Y}, (L_I^Z D)\tilde{X})
$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{S}^1_0(T^1_1(M))$. For pairs $\tilde{X} = HX, V A$ and $\tilde{Y} = HY, V B$ and $\tilde{Z} = HZ, V C$, we get

$$
(\Phi_D CG)(HX, VB, HZ) = 2CG(V B, (\tilde{\gamma} - \gamma)R(Z, X)),
(\Phi_D CG)(HX, HY, VC) = 2CG((\tilde{\gamma} - \gamma)R(Y, X), VC),
$$

Otherwise $= 0$. (3.5)

Therefore, we have the following result.
Theorem 3. Let \((M, g)\) be a Riemannian manifold and let \(T^1_1(M)\) be its \((1,1)\)-tensor bundle equipped with the Cheeger–Gromoll type metric \(CG_g\) and the almost product structure \(D I\). The triple \((T^1_1(M), D I, CG_g)\) is a locally decomposable Riemannian manifold if and only if \(M\) is flat.

Remark 1. Let \((M, g)\) be a flat Riemannian manifold. In this case the \((1,1)\)-tensor bundle \(T^1_1(M)\) equipped with the Cheeger–Gromoll type metric \(CG_g\) over the flat Riemannian manifold \((M, g)\) is unflat (see [8]).

Let \((M, \varphi, g)\) be a non-integrable almost product manifold with a pure metric. A Riemannian almost product manifold \((M, \varphi, g)\) is a Riemannian almost product \(W_3\)-manifold if \(\sigma_{X,Y,Z}g((\nabla_X \varphi)Y, Z) = 0\), where \(\sigma\) is the cyclic sum by \(X, Y, Z\) [15]. In [12], the authors proved that \(\sigma_{X,Y,Z}g((\nabla_X \varphi)Y, Z) = 0\) is equivalent to \((\Phi_{\varphi} g)(X, Y, Z) + (\Phi_{\varphi} g)(Y, Z, X) + (\Phi_{\varphi} g)(Z, X, Y) = 0\). We compute

\[
A(\tilde{X}, \tilde{Y}, \tilde{Z}) = (\Phi_{DI} CG_g)(\tilde{X}, \tilde{Y}, \tilde{Z}) + (\Phi_{DI} CG_g)(\tilde{Y}, \tilde{Z}, \tilde{X}) + (\Phi_{DI} CG_g)(\tilde{Z}, \tilde{X}, \tilde{Y})
\]

for all \(\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{N}_0^1(T^1_1(M))\). By means of (3.5), we have \(A(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0\) for all \(\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{N}_0^1(T^1_1(M))\). Hence we state the following theorem.

Theorem 4. Let \((M, g)\) be a Riemannian manifold and \(T^1_1(M)\) be its \((1,1)\)-tensor bundle equipped with the Cheeger–Gromoll type metric \(CG_g\) and the almost product structure \(D I\). The triple \((T^1_1(M), D I, CG_g)\) is a Riemannian almost product \(W_3\)-manifold.

Remark 2. Let \((M, g)\) be a Riemannian manifold and let \(T^1_1(M)\) be its \((1,1)\)-tensor bundle equipped with the Cheeger–Gromoll type metric \(CG_g\). Another almost product structure on \(T^1_1(M)\) is defined by the formulas

\[
\begin{align*}
J^H X &= -H X, \\
J^V A &= V A,
\end{align*}
\]

for any \(X \in \mathfrak{N}_0^1(M)\) and \(A \in \mathfrak{N}_1^1(M)\). The Cheeger–Gromoll type metric \(CG_g\) is pure with respect to \(J\), i.e. the triple \((T^1_1(M), J, CG_g)\) is a Riemannian almost product manifold. Also, by using \(\Phi\)-operator, we can say that the Cheeger–Gromoll type metric \(CG_g\) is decomposable with respect to \(J\) if and only if \(M\) is flat. Finally the triple \((T^1_1(M), J, CG_g)\) is another Riemannian almost product \(W_3\)-manifold.

Gil-Medrano and Naveira [5] proved that both distributions of the almost product structure on the Riemannian almost product manifold \((M, F, g)\) are totally geodesic if and only if \(\sigma_{X,Y,Z}g((\nabla_X F)Y, Z) = 0\) for any \(X, Y, Z \in \mathfrak{N}_0^1(M)\). As a consequence of Theorem 4, we have the following.

COROLLARY 1

Both distributions of the Riemannian almost product manifold \((T^1_1(M), D I, CG_g)\) are totally geodesic.
4. Product conjugate connections on the (1, 1)-tensor bundle with Cheeger–Gromoll type metric

Let $F$ be an almost product structure and $\nabla$ be a linear connection on an $n$-dimensional Riemannian manifold $M$. The product conjugate connection $\nabla^{(F)}$ of $\nabla$ is defined by

$$\nabla^{(F)}_X Y = F(\nabla_X F Y)$$

for all $X, Y \in \mathbb{V}_0^1(M)$. If $(M, F, g)$ is a Riemannian almost product manifold, then $\nabla^{(F)}(g)(FY, FZ) = (\nabla_X g)(Y, Z)$, i.e. $\nabla$ is a metric connection with respect to $g$ if and only if $\nabla^{(F)}$ is so. From this, we can say that if $\nabla$ is the Levi–Civita connection of $g$, then $\nabla^{(F)}$ is a metric connection with respect to $g$ [1].

By the almost product structure $D I$ and the Levi–Civita connection $\tilde{\nabla}$ given by Theorem 1, we write the product conjugate connection $\tilde{\nabla}^{(D I)}$ of $\tilde{\nabla}$ as follows:

$$\tilde{\nabla}^{(D I)}_X Y = D I(\tilde{\nabla} X, D I Y)$$

for all $\tilde{X}, \tilde{Y} \in \mathbb{V}_0^1(T^1_1(M))$. Also note that $\tilde{\nabla}^{(D I)}$ is a metric connection of the Cheeger–Gromoll type metric $\tilde{C}g$. The standard calculations give the following theorem.

**Theorem 5.** Let $(M, g)$ be a Riemannian manifold and let $T^1_1(M)$ be its (1,1)-tensor bundle equipped with the Cheeger–Gromoll type metric $\tilde{C}g$ and the almost product structure $D I$. Then the product conjugate connection (or metric connection) $\tilde{\nabla}^{(D I)}$ satisfies

(i) $\tilde{\nabla}^{(D I)}_X H Y = H(\nabla_X Y) - \frac{1}{2}(\tilde{\nabla}^{(L)} - \gamma)R(X, Y),$

(ii) $\tilde{\nabla}^{(D I)}_X V B = - \frac{1}{2a} H(g^{lj} R(t_l, B_j)X + g_{ai} (t^a (g^{-1} \circ R( , X) B_i))) + V(\nabla_X B),$

(iii) $\tilde{\nabla}^{(D I)}_X V Y = \frac{1}{2a} H(g^{lj} R(t_l, B_j)X + g_{ai} (t^a (g^{-1} \circ R( , Y) A_i)))$, 

(iv) $\tilde{\nabla}^{(D I)}_X V B = - \frac{1}{a} Cg(V A, t V B) + Cg(V B, t V A) + \frac{a+1}{a} Cg(V A, V B) + \frac{a-1}{a} Cg(V A, t V B) + \frac{a-1}{a} Cg(V B, V A) + \frac{a+1}{a} Cg(V B, t V A).$

**Remark 3.** Let $(M, g)$ be a flat Riemannian manifold and let $T^1_1(M)$ be its (1,1)-tensor bundle equipped with the Cheeger–Gromoll type metric $\tilde{C}g$ and the almost product structure $D I$. Then the Levi–Civita connection $\tilde{\nabla}$ of $\tilde{C}g$ coincides with the product conjugate connection (or metric connection) $\tilde{\nabla}^{(D I)}$ constructed by the Levi–Civita connection $\tilde{\nabla}$ of $\tilde{C}g$ and the almost product structure $D I$.

The relationship between curvature tensors $R_{\nabla}$ and $R_{\nabla^{(F)}}$ of the connections $\nabla$ and $\nabla^{(F)}$ is as follows: $R_{\nabla^{(F)}}(X, Y, Z) = F(R_{\nabla}(X, Y, F Z))$ for all $X, Y, Z \in \mathbb{V}_0^1(M)$ [1]. Using the almost product structure $D I$ and curvature tensor of the Cheeger–Gromoll type metric $\tilde{C}g$ (for curvature tensor of $\tilde{C}g$, see [8]), by means of $\tilde{R}_{\nabla^{(D I)}}(\tilde{X}, \tilde{Y}, \tilde{Z}) = D I(\tilde{R}_{\nabla}(\tilde{X}, \tilde{Y}, D I \tilde{Z}))$, the curvature tensor $\tilde{R}_{\nabla^{(D I)}}$ of the product conjugate connection (or metric connection) $\tilde{\nabla}^{(D I)}$ can be easily written. Also note that another metric connection of the Cheeger–Gromoll type metric $\tilde{C}g$ can be constructed by using the almost product structure $J$. 

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The torsion tensor $\tilde{T}^{(DI)}_I$ of the product conjugate connection $\tilde{\nabla}^{(DI)}$ (or metric connection of $CG\, g$) has the following properties:

\[ \tilde{T}^{(DI)}_I(HX, HY) = -2(\tilde{\gamma} - \gamma) R(X, Y), \]

\[ \tilde{T}^{(DI)}_I(HX, V_B) = -\frac{1}{\alpha} H(g^{bj} R(t_b, B_j) X + g_{ai} (t^a(g^{-1} \circ R(\cdot, X) \tilde{B}^i))), \]

\[ \tilde{T}^{(DI)}_I(V_A, HY) = \frac{1}{\alpha} H(g^{bj} R(t_b, A_j) Y + g_{ai} (t^a(g^{-1} \circ R(\cdot, Y) \tilde{A}^i))), \]

\[ \tilde{T}^{(DI)}_I(V_A, V_B) = 0. \]

The last equation led to the following result.

**Theorem 6.** Let $(M, g)$ be a Riemannian manifold and let $T^1_1(M)$ be its $(1,1)$-tensor bundle equipped with the Cheeger–Gromoll type metric $CG\, g$ and the almost product structure $DI$. The product conjugate connection (or metric connection) $\tilde{\nabla}^{(DI)}$ constructed by the Levi–Civita connection $\tilde{\nabla}$ of $CG\, g$ and the almost product structure $DI$ is symmetric if and only if $M$ is flat.

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