1. Introduction

Let \( K \) be a number field and \( \mathcal{O}_K \) its ring of integers. Let \( \mathcal{E} \) be an \( \mathcal{O}_K \)-module of rank \( N + 1 \) in \( \mathbb{P}(\mathcal{E}^\vee) \), the projective space representing lines in \( \mathcal{E}^\vee \). For all closed subvarieties \( X \subseteq \mathbb{P}(\mathcal{E}_K^\vee) \) of dimension \( d \) let \( \deg(X) \) be its degree with respect to the canonical line bundle \( \mathcal{O}(1) \) of \( \mathbb{P}(\mathcal{E}_K^\vee) \). If \( \mathcal{E} \) is endowed with the structure of hermitian vector bundle over \( \text{Spec}(\mathcal{O}_K) \) we can define the Arakelov degree \( \widehat{\deg}(\mathcal{E}) \) and the Faltings height \( h_{\mathcal{E}}(X) \) (see 3.1).

Let \( \mathcal{V} \) be the direct limit of the set \( \mathcal{V}_K \) of \( \mathcal{O}_K \)-modules with an identity \( \mathcal{E}_K \cong K^{N+1} \) as \( K \) varies in the set of number fields. Let \( X \subseteq \mathbb{P}^N \) be a closed irreducible projective variety of dimension \( d \) defined over \( \mathbb{Q} \). We define

\[
\hat{h}(X) = \inf_{\mathcal{E} \in \mathcal{V}} \left( \frac{h_{\mathcal{E}}(X)}{(d + 1)\deg(X)} - \frac{\widehat{\deg}(\mathcal{E})}{N + 1} \right).
\]

According to [1, Théorème 1] if the Chow point of \( X \) is semistable (see 2.4) then there exists a non-negative constant \( C \) such that

\[
\hat{h}(X) \geq -C. \tag{1.1}
\]

This can be shown using the fact that for \( \mathcal{E} \in \mathcal{V} \) the expression \( h_{\mathcal{E}}(X)/(d + 1)\deg(X) \) is bounded below in terms of the height of a system of generators of the rings of invariants \( (\text{Sym} \mathcal{E})^{SL(\mathcal{E})} \) (see also [12]). Moreover, one can even show that \( h_{\mathcal{E}}(X)/(d + 1)\deg(X) \) is bounded below by the average of the successive minima of \( \mathcal{E} \) (see [22]). These results give a lower bounds of \( \hat{h}(X) \) that depends on the field \( K \) of definition of \( \mathcal{E} \) and \( X \). The independence of the constant \( C \) from the field of definition has been shown by Zhang in [26], using his absolute successive minima [27, 5.2], and independently by Bost in [2], using the ring of invariants.

We will prove in this section a conjecture of Zhang [26, 4.3] that the height \( \hat{h}(X) \) of a semistable variety \( X \) is non-negative, as it is true in the function field case (see [5], [1]). The idea of the proof is the same of that for the main theorem of [22], the main difference

relies in the fact that here we make direct use of Zhang’s absolute successive minima of $X$ (defined in [25]) at the place of Zhang’s minima of the projective space.

**Theorem 1.1.** Let us suppose that $X$ is semistable. Then

\[ \hat{h}(X) \geq 0. \]  

(1.2)

In order to prove this inequality we have to introduce the **degree of contact**. This is a birational invariant first considered in the context of Geometric Invariant Theory (see [19]), which has recently found important applications in the domain of diophantine approximations. We refer to [19], and [8, §3] for exhaustive and detailed discussions of the main properties the degree of contact (see also [9], [10] and the references therein).

As a byproduct, we are able to show, for all hermitian $\mathcal{O}_K$-module $\mathcal{E}$, new lower bounds for the normalized Faltings height $h_\mathcal{E}(X)$, (see (3.1)). For instance, when $X$ is a generic $K^3$ surface, i.e. a $K^3$ surface whose Picard group has rank one, then $X$ is semistable (see Corollary 2.10 and [18]), so Theorem 1.1 holds true. This implies that the height $h_\mathcal{E}(X)$ is non-negative for all hermitian $\mathcal{O}_K$-module $\mathcal{E}$. However, Corollary 3.5 provides a lower bound for the Faltings height in terms of a linear combination of Zhang’s absolute minima which is stronger than that given by the semistability of $X$.

The seminal work of Zhang [26] has even contributed to lay down the basis for the proof by Phong and Sturm [21] of a conjecture formulated by Donaldson in his work [7] on Yau’s conjecture. However, Donaldson, Phong and Sturm were concerned about one direction of Yau’s conjecture, namely on the proof of the semistability of $X$ under the assumption of the existence of a Kähler-Einstein metric. It would be interesting to see if the methodology developed in this note could be useful to better understand the other direction of Yau’s conjecture.

## 2. Degree of Contact

2.1. Let $E$ be a finite dimensional vector space defined over a number field $K$. A **weight function** is a map $w : E \to \mathbb{R} \cup \{-\infty\}$ satisfying:

1. $w(x) = -\infty$, if and only if $x = 0$,
2. for all $t \in K^*$ and all $x \in E$, $w(t \cdot x) = w(x)$,
3. for all $x, y \in E$, $w(x + y) \leq \max\{w(x), w(y)\}$.

For all non-negative real numbers $\alpha$ the set $F^\alpha = \{x \in E : w(x) \leq \alpha\}$ is a subspace of $E$, and $F^\alpha \subseteq F^\beta$ whenever $\alpha \leq \beta$. Varying $\alpha$, we get then an exhaustive ($\cup_{\alpha \in \mathbb{R}} F^\alpha = E$) and separated ($\cap_{\alpha \in \mathbb{R}} F^\alpha = \{0\}$) filtration $\mathcal{F}$ of subspaces of $E$. We say that a basis $l_0, \cdots, l_N$ of $E$ is **adapted to the filtration** $\mathcal{F}$, if for all $\alpha \in \mathbb{R}$

\[ F^\alpha = \bigoplus_{w(l_j) \leq \alpha} K \cdot l_j. \]
We number the elements of this basis so that \( w(l_0) \geq w(l_1) \geq \cdots \geq w(l_N) \). For \( h = 0, \ldots, N \) define \( r_h := w(l_h) \) and \( r = (r_0, \ldots, r_N) \). The vector \( r \) is obviously independent of the chosen basis. This construction gives a bijective map between the set of weight functions on \( E \) and the set of couples \((\mathcal{F}, r)\), with \( \mathcal{F} \) a filtration of subspaces of \( E \), and \( r = (r_0, \ldots, r_N) \in \mathbb{R}_{\geq 0}^{N+1} \) with \( r_0 \geq \cdots \geq r_N \). Indeed, to such a pair \((\mathcal{F}, r)\) we associate a weight function \( w \) on \( E \) as follows: Put \( w(0) = -\infty \) and for \( x \in E \setminus \{0\} \) define \( w(x) \) as the smallest \( r_i \) such that \( x \in F^{r_i} \). Equivalently, given a basis \( l_0, \ldots, l_N \) of \( E \) adapted to the filtration \( \mathcal{F} \), write \( x = x_0 l_0 + \cdots x_N l_N \). Then \( w(x) \) is the greatest \( r_i \) for which \( x_i \neq 0 \). Notice that the value \( w(x) \) does not change much if we dilate \( r \) or perturb it a little bit.

We can always find an integer valued weight function \( \tilde{w} \) supported on the same filtration \( \mathcal{F} \) of \( w \), such that for some sufficiently small \( \varepsilon > 0 \) and some positive integer \( m \) we have \( mw \leq \tilde{w} \leq m(1 + \varepsilon)w \). This enables us to reduce most of the computations to weight functions with integer values.

### 2.2. Weight functions satisfy several functorial relations

Let \( w \) be a weight function on \( E \) with non-negative integer weights \( r \in \mathbb{Z}^{N+1} \) and associated filtration \( \mathcal{F} \). Consider a subspace \( F \subseteq E \). The restriction \( w|_F \) of \( w \) on \( F \) defines a weight function on \( F \). Further, \( w \) induces a weight function on the quotient \( E/F \), mapping \( l \) to the minimum of the weights of the elements \( x \in E \) with \( \pi(x) = l \), where \( \pi : E \to E/F \) is the canonical projection. An element \( h \in E^\vee \) is a linear functional \( h : E \to K \). If \( h \neq 0 \) we define the weight of \( h \) as minus the weight of the line \( E/\ker(h) \).

Given two vector spaces \( E_1 \) and \( E_2 \) over \( K \), endowed with weight functions \( w_1, w_2 \) respectively, we define a weight function \( w \) on \( E_1 \oplus E_2 \) by
\[
(2.1) \quad e_1 \oplus e_2 \mapsto \max\{w_1(e_1), w_2(e_2)\}.
\]

Moreover, on the tensor product \( E_1 \otimes E_2 \) we define a weight function \( w \) by \( w(e_1 \otimes e_2) = w_1(e_1) + w_2(e_2) \). Let \( m \) be a positive integer. The symmetric group of order \( m \) operates on the \( m \)-th tensor power \( E^\otimes m \) by permuting the factors. The \( m \)-th symmetric power of \( E \), denoted by \( \text{Sym}^m E \), is the maximal subspace of \( E^\otimes m \) invariant under this operation. Whence by restriction we can canonically define a weight function on \( \text{Sym}^m E \). The exterior power \( \wedge^m E \) is the quotient of \( E^\otimes m \) by \( \text{Sym}^m E \), therefore on this space there exists a canonical induced weight function.

### 2.3. Let \( w \) be a weight function on \( E \) with integer weights \( r \in \mathbb{Z}^{N+1} \), and let \( \mathcal{F} \) be the associated filtration. Let \( X \) be a projective absolutely irreducible scheme of dimension \( d \) and defined over \( K \) embedded into the projective space \( \mathbb{P}(E^\vee) \) representing lines of the dual vector space \( E^\vee \). If \( m \) is large enough, say \( m \geq m_0 \), the cup product map
\[
(2.2) \quad \varphi_m : \text{Sym}^m(E) \to H^0(X, \mathcal{O}(m))
\]
is surjective. Therefore $H^0(X, \mathcal{O}(m))$ can be identified with a quotient of $\text{Sym}^m(E)$. As in §2.2 $w$ induces then a weight function on $H^0(X, \mathcal{O}(m))$, and on the one-dimensional space $\wedge^h(X, \mathcal{O}(m)) H^0(X, \mathcal{O}(m))$. We denote by $w(X, m)$ the weight of this line, which is well defined by the homogeneity property (2) of weight functions. There exists an integer $e_w(X)$ such that when $m$ goes to infinity
\begin{equation}
(2.3) \quad w(X, m) = e_w(X) \frac{m^{d+1}}{(d+1)!} + O(m^d),
\end{equation}
(see [19, Proposition 2.11], or [8, §3]). The number $e_w(X)$ is called degree of contact of $X$ with respect to the weighted filtration associated to the weight function $w$. We extend this definition by linearity to cycles, and by approximating to real weights $r \in \mathbb{R}^{N+1}$.

In the last years appeared several articles in diophantine approximations that make a wide use of the degree of contact (see [8], [9], and [10]). In these articles the main properties of the degree of contact are discussed in detail. We refer to them for a further thorough analysis of the degree of contact (see also [19], [17], [16]).

2.4. Each suitably generic element $h = (h_0, \ldots, h_d) \in \mathbb{P}(E)^{d+1}$ defines naturally a $(N - d - 1)$-dimensional linear subspace $L_h \subseteq \mathbb{P}(E^n)$. Consider the set $Z(X)$ of all $(d + 1)$-tuples $h \in \mathbb{P}(E)^{d+1}$, such that $L_h(\overline{Q}) \cap X(\overline{Q}) \neq \emptyset$, where $L_h$ has dimension $N - d - 1$. Then $Z(X)$ is an irreducible hypersurface of multidegree $(\text{deg}(X), \ldots, \text{deg}(X))$ (see [15, Thm. IV, p. 41]). The hypersurface $Z(X)$ turns out to be given by an up to a constant unique polynomial element $F_X \in V$, where
\begin{equation}
(2.4) \quad V = \left(\left(\text{Sym}^{\text{deg}(X)} E\right)^{\otimes (d+1)}\right)^{\vee}.
\end{equation}
This is a so called (Cayley-Bertini-van der Waerden-) Chow form of $X$. By definition it has that property that $F_X(h_0, \ldots, h_d) = 0$ if and only if $X$ and the hyperplanes given by the vanishing of the linear forms $h_i (i = 0, \ldots, d)$ have a point in common over $\overline{\mathbb{Q}}$.

2.5. As in §2.2 $w$ induces a weight function on $V$, again denoted by $w$. From [19, Proposition 2.11] (see also [8, Theorem 4.1]) we know that the degree of contact corresponds to minus the weight of the Chow point:
\begin{equation}
(2.5) \quad e_w(X) = -w(F_X).
\end{equation}
Indeed, using the terminology of [19], the “n.l.c. of $r^V_n$,” (the degree of contact) corresponds to the “$\lambda$-weight $a_V$ of $\phi_V$” (minus the weight of the Chow form in our notation).

We say that the variety $X$ is Chow-semistable (or simply semistable) if for all weight functions $w$ on $E$ we have
\[
\frac{e_w(X)}{(d+1) \text{deg}(X)} \leq \frac{1}{N+1} \sum_{i=0}^{N} r_i.
\]
According to the Hilbert-Mumford criterion (see [19]), this is equivalent to say that the Zariski closure of the orbit of a representative of $F_X$ in $V$ under $SL(E)$ does not contain $0$.

2.6. Let $l_0, \ldots, l_N$ be a basis of $E$ adapted to the filtration associated to the weight function $w$, which identifies then $E$ with $K^{N+1}$, and define $T = \binom{N+1}{d+1}$. Given the blocks of variables $h_p = (h_{p0}, \ldots, h_{pN})$ ($p = 0, \ldots, d$) we define for each subset $I_k = \{i_{k0}, \ldots, i_{kd}\}$ of $\{0, \ldots, N\}$ with $i_{k0} < \cdots < i_{kd}$ the bracket $[I_k] = [i_{k0} \cdots i_{kd}] = \det(h_{p,i_{kq}})_{p,q=0,\ldots,d}$, for $k = 1, \ldots, T$. From [15, Thm. IV, p.41] it follows that the Chow form $F_X$ can be expressed as a polynomial in terms of such brackets. We expand $F_X$ as a sum of monomials of brackets

$$F_X = \sum_{j=(j_1,\cdots,j_T)\in J} a_j [I_1]^{j_1} \cdots [I_T]^{j_T}$$

where $a_j \neq 0$, and $|j| = \deg(X)$ for $j \in J$. Then if $w$ is a weight function given by the weights $r_0 \geq \cdots \geq r_N \geq 0$ we have

$$e_w(X) = \min_{j \in J} \sum_{i=1}^T j_i \left( \sum_{k \in I_i} r_k \right).$$

2.7. Suppose that the hyperplanes defined by the vanishing of the linear forms $l_{N-d}, \ldots, l_N$ do not have a common point on $X$ defined over $\mathbb{Q}$. If $X$ is linear, this means precisely that the restrictions to $X$ of the linear forms $l_{N-d}, \ldots, l_N$ are linearly independent. We notice that for given real numbers $0 \leq c_0 \leq \cdots \leq c_N$ and for $j \in J$ we have

$$\sum_{i=1}^T j_i \left( \sum_{k \in I_i} c_k \right) \leq \deg(X) \left( c_{N-d} + \cdots + c_N \right). \quad (2.6)$$

Further, due to our assumption that $X$ and the zero set of $l_{N-d}, \ldots, l_N$ do not meet, we have $F_X(l_{N-d}, \ldots, l_N) \neq 0$. Therefore, $F_X$ must contain the monomial $[N-d \cdots N]^{\deg(X)}$. This implies that among the terms $\sum_{i=1}^T j_i \left( \sum_{k \in I_i} c_k \right)$, with $j \in J$ we have $\deg(X) (c_{N-d} + \cdots + c_N)$. But this is the largest among all the terms (2.6), whence

$$\max_{j \in J} \sum_{i=1}^T j_i \left( \sum_{k \in I_i} c_k \right) = \deg(X) \left( c_{N-d} + \cdots + c_N \right).$$
Remember that \( w \) is a weight function given by the basis \( l_0, \ldots, l_N \) and weights \( r_0 \geq \cdots \geq r_N \geq 0 \). We define \( e_i = r_0 - r_i \), for \( i = 0, \ldots, N \). We have
\[
e_w(X) = \min_{j \in J} \sum_{i=1}^T j_i \left( \sum_{k \in I_i} r_k \right)
\]
\[
= \deg(X)(d + 1)r_0 - \max_{j \in J} \sum_{i=1}^T j_i \left( \sum_{k \in I_i} c_k \right)
\]
\[
= \deg(X)(d + 1)r_0 - \deg(X) \left( r_0 - r_{N-d} + \cdots + r_0 - r_N \right)
\]
(2.7)
\[
= \deg(X) \left( r_{N-d} + \cdots + r_N \right).
\]

2.8. Let \( X \) be an absolute irreducible projective surface and \( C \) a pseudo ample divisor on \( X \), i.e. a divisor such that the linear series \(|C|\) has no fixed components and the associated map \( \phi_C : X \to \mathbb{P}(H^0(X, mC)^\vee) \) is birational onto its image, for \( m \) sufficiently large. For \( E = H^0(X, mC) \) with \( \dim(E) = N+1 \) let \( w \) be a weight function on \( E \) with associated filtration
\[
(2.8) \quad \mathcal{F} : \quad E = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_N \supseteq \{0\}.
\]
Suppose that there exists a blow up \( \pi : B \to X \) on which \( C \) has a proper transform \( \tilde{C} \), and such that for each \( i = 0, \ldots, N \) the pullbacks of the sections in \( V_i \) generate an invertible sub-sheaf \( O_B(C_i) \) of \( O(\tilde{C}) \). According to [11, 4.4] \( O_B(C_i) = \pi^*O(C) \otimes \pi^{-1}(J_i) \), where \( J_i \) is the ideal sheaf defining the base locus of \(|V_i|\). This means that the number \( C_i^2 \) is the degree of the projection of \( X \) onto \( \mathbb{P}(V_i^\vee) \). Define
\[
e_{ij} = C^2 - C_i \cdot C_j, \quad e_j = C^2 - C_j^2.
\]
These numbers are independent of the choice of \( B \). Note that for all \( i = 0, \ldots, N \) the number \( e_i \) measures the drop in degree on projection to \( \mathbb{P}(V_i^\vee) \). Let us identify \( X \) with the image of the birational map \( \phi_C \). Let \( r_0 \geq \cdots \geq r_N \geq 0 \) be the weights associated to the weight function \( w \). Then for all sequences of integers \( J = (j_0, \ldots, j_l) \) with \( 0 = j_0 < j_1 < \cdots < j_l = N \) [10, Proposition 2.10] yields
\[
e_w(X) \leq \sum_{k=0}^{l-1} (r_{j_k} - r_{j_{k+1}}) \left( e_{j_k} + e_{j_kj_{k+1}} + e_{j_{k+1}} \right) =: S_J.
\]
(2.9)
Assume now that \( X \) is a K3 surface over \( K \), \( C \) a pseudo ample divisor on \( X \), and \( m = 1 \). Then
\[
h^0(X, C') = \frac{C^2}{2} + 2 \quad \text{and} \quad h^1(X, C') = 0.
\]
(2.10)
Further $C$ has no base points and $\phi_C(X)$ is projectively normal ([23, 2.6], [23, 3.2], [23, 6.1]). If we assume that no curve is contained in the base locus of $V_i$, for all $i = 0, \cdots , N$, and that $r_N = 0$, then from [18, Lemma 5] and [10, Proposition 2.11] we have

$$e_w(X) \leq -4r_0 + 6 \sum_{i=0}^{N} r_i,$$

(2.11)

We are moreover able to prove the following tightening of (2.11).

**Proposition 2.9.** Assume now that $X$ is a $K3$ surface over $K$, $C$ a pseudo ample divisor on $X$, and that no curve is contained in the base locus of $V_i$, for all $i = 0, \cdots , N$. Then if $r_N = 0$ we have

$$e_w(X) \leq \min \left\{ -4r_0 + 6 \sum_{i=0}^{N} r_i, 2(N-1)r_0 \right\}.$$  

**Proof.** Remember that for $i = 0, \cdots , N$, the number $e_i$ is the amount by which the degree of $X$ in $\mathbb{P}(E^\vee)$ is greater that that of its image under the projection onto $\mathbb{P}(V_i^\vee)$. In any case

$$e_i \leq C^2.$$  

By Riemann-Roch (2.10) we have $N + 1 = h^0(X, C) = C^2/2 + 2$, which implies

$$e_i \leq 2(N-1).$$  

(2.12)

Let us now consider the inequality (2.9) with $J = (0, N)$. Then from (2.12) we get

$$e_w(X) \leq S_J = (r_0 - r_N)(e_0 + e_{0N} + e_N) \leq 2(N-1)r_0$$

Together with [10, Proposition 2.11] this concludes the proof.  

We recover here the main result of [18].

**Corollary 2.10.** Let $X$ be a $K3$ surface whose Picard group has rank 1 and $C$ be a primitive divisor class on $X$. Then $X$ is semistable.

**Proof.** Let $w$ be any weight function on $E = H^0(X, C)$ and $r_0 \geq \cdots \geq r_N \geq 0$ be the associated weights. We can assume without restriction that $r_N = 0$. Let us first suppose that

$$r_0 \geq \frac{3}{N+1} \sum_{i=0}^{N} r_i.$$  

Together with Riemann-Roch’s formula (2.10) and Proposition 2.9 this implies

$$\frac{e_w(X)}{(\dim X + 1) \deg(X)} \leq \frac{-4r_0 + 6 \sum_{i=0}^{N} r_i}{6(N-1)} \leq \frac{1}{N+1} \sum_{i=0}^{N} r_i.$$  

We recover here the main result of [18].
We assume now
\[ r_0 \leq \frac{3}{N + 1} \sum_{i=0}^{N} r_i. \]
Then again Riemann-Roch’s formula (2.10) and Proposition 2.9 imply
\[ \frac{e_w(X)}{(\dim X + 1) \deg(X)} \leq \frac{2(N - 1)r_0}{6(N - 1)} \leq \frac{1}{N + 1} \sum_{i=0}^{N} r_i, \]
which concludes the proof. \(\square\)

**Remark 2.11.** As remarked in [18], since the generic member of the moduli space of \(K3\) surfaces has Picard group of rank 1, this result covers almost all \(K3\) surfaces.

### 3. ARAKELOV GEOMETRY

**3.1.** Let \(K\) be a number field and let \(O_K\) be its ring of integers, and let \(S_\infty\) be the set of complex embeddings of \(K\). If \(\mathcal{M}\) is a torsion-free \(O_K\)-module of finite rank such that, for all \(\sigma \in S_\infty\), the corresponding complex vector space \(M_\sigma = \mathcal{M} \otimes_{O_K} \mathbb{C}\) is equipped with a norm \(| \cdot |_\sigma\), we may think of \(\mathcal{M}\) as a free \(\mathbb{Z}\)-module equipped with the norm \(| \cdot |\) on \(M_\sigma = \mathcal{M} \otimes_{O_K} \mathbb{C} \simeq \bigoplus_{\sigma \in S_\infty} M_\sigma\) defined by \(|\sum_{\sigma \in S_\infty} x_\sigma| = \sup_{\sigma \in S_\infty} |x_\sigma|_\sigma\) for \(x_\sigma \in M_\sigma, \sigma \in S_\infty\). In particular, consider an hermitian vector bundle \(E = (E, h)\) over \(\text{Spec}(O_K)\) in the sense of [14]. In other words, \(E\) is a torsion-free \(O_K\)-module of rank \(N + 1 < \infty\), and for all \(\sigma \in S_\infty\), \(E_\sigma\) is equipped with a hermitian scalar product \(h\), compatible with the isomorphism \(E_\sigma \simeq E_\overline{\sigma}\) induced by complex conjugation. We will then denote by \(\| \cdot \|_\sigma\) the norm on \(E_\sigma\) and \(\| \cdot \|\) the norm on \(E \otimes_{\mathbb{Z}} \mathbb{C}\) as above. If \(N = 0\) then the *Arakelov degree* of \(E\) is defined by
\[ \hat{\deg}(E) = \log(\#(E / s \cdot O_K)) - \sum_{\sigma \in S_\infty} \log \|s\|_\sigma, \]
where \(s\) is any non zero element of \(E\). In general, we define the *normalized Arakelov degree* of \(E\) as \(\hat{\deg}_n(E) = \frac{1}{[K : \mathbb{Q}]} \hat{\deg}(\det(E))\).

Let \(E^\vee\) be the dual \(O_K\)-module of \(E\), and let \(\mathbb{P}(E^\vee)\) be the associated projective space representing lines in \(E^\vee\). Consider a closed subvariety \(X \subseteq \mathbb{P}(E^\vee)\), where \(E = E \otimes K\), of dimension \(d\), and let \(\deg(X)\) be its (algebraic) degree with respect to the canonical line bundle \(O(1)\) on \(\mathbb{P}(E^\vee)\). Let \(h_E(X) \in \mathbb{R}\) be the normalized Faltings height of the Zariski closure \(\overline{X}\) of \(X\) in \(\mathbb{P}(E)\), denoted by \(h_F(X)[K : \mathbb{Q}]\) in [3] (3.1.1), (3.1.5)). Let \(\overline{O}(1)\) be the canonical line bundle equipped with the metric induced by \(h\), then
\[
(3.1) \quad h_E(X) = \frac{1}{[K : \mathbb{Q}]} \hat{\deg}(\hat{c}_1(\overline{O}(1))^{d+1}|\overline{X}) \in \mathbb{R},
\]
where \((\cdot, \cdot)\) is the bilinear pairing defined in loc. cit.
3.2. Let $L$ be a finite field extension of $K$, and let $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_N$ be hermitian line sub-bundles of $\overline{\mathcal{E}}$ such that $(\oplus_i \mathcal{L}_i)_L$ generates $\mathcal{E}_L$. There exist points $P_0, \ldots, P_N \in \mathbb{P}(\mathcal{E}_L^\vee)$ associated to the line bundles above such that $h(P_i) = -\deg(\overline{\mathcal{L}}_i)$ for $i = 0, \ldots, N$, [25 Theorem 5.2]. This implies that all $\overline{\deg}(\overline{\mathcal{L}}_i)$ are non-positive. Assume that these line bundles are ordered by increasing Arakelov degree $\deg(\overline{\mathcal{L}}_0) \leq \cdots \leq \deg(\overline{\mathcal{L}}_N) \leq 0$. For $i = 0, \ldots, N$ define $s_i = -\deg(\overline{\mathcal{L}}_i) + \deg(\overline{\mathcal{L}}_N)$, and put $s = (s_0, \ldots, s_N) \in \mathbb{R}^{N+1}$.

Let $x_0, \ldots, x_N$ be nonzero sections of the line bundles $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_N$, respectively, that give an embedding $X \to \mathbb{P}^N$. Further, let $w$ be a weight function on $E = \mathcal{E}_K$ with weights $r_0 \geq \cdots \geq r_N = 0$, and $r = (r_0, \ldots, r_N)$. We get the following variation of [22 Theorem 1]:

**Theorem 3.3.** Assume there exists a continuous function $\psi : \mathbb{R}^{N+1} \to \mathbb{R}$ such that $\psi(tx) = t\psi(x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^{N+1}$, and such that

$$e_w(X) \leq \psi(r).$$

Then the following inequality holds:

$$h(E)(X) \geq (d+1) \deg(X) \overline{\deg}(\overline{\mathcal{L}}_N) - \psi(s).$$

**Proof:** This is a reinterpretation of the proof of of [26 Theorem 4.4] (see [26 4.8]) using the language of [22 Theorem 1].

**Corollary 3.4.** Let $X$ be a surface and $\mathcal{E}$ be a $\mathcal{O}_K$-module with $\mathcal{E}_K = H^0(X, \mathcal{O}(C))$, where $C$ is a pseudo ample divisor. Then

$$h(E)(X) \geq (d+1) \deg(X) \overline{\deg}(\overline{\mathcal{L}}_N) - \sum_{k=0}^m \left( \overline{\deg}(\overline{\mathcal{L}}_{j_{k+1}}) - \overline{\deg}(\overline{\mathcal{L}}_j) \right) \left( e_{j_k} + e_{j_kj_{k+1}} + e_{j_{k+1}} \right).$$

**Proof.** Straightforward consequence of Theorem 3.3 and (2.9).

**Corollary 3.5.** Assume that $X$ is a $K3$ surface, and $\mathcal{E}$ be a $\mathcal{O}_K$-module with $\mathcal{E}_K = H^0(X, \mathcal{O}(C))$, where $C$ is a pseudo ample divisor. For $i = 0, \ldots, N$, let $V_i$ be the linear space of the filtration $\mathcal{E}$. Suppose that no curve is contained in the base locus of $V_i$, for all $i = 0, \ldots, N$. Then

$$\frac{1}{2} h(E)(X) \geq \max \left\{ (N-1) \left( \overline{\deg}(\overline{\mathcal{L}}_0) + 2 \overline{\deg}(\overline{\mathcal{L}}_N) \right), \right.$$

$$(3.2) \left. -2 \overline{\deg}(\overline{\mathcal{L}}_0) + 2 \overline{\deg}(\overline{\mathcal{L}}_N) \right) + 3 \sum_{i=0}^N \overline{\deg}(\overline{\mathcal{L}}_i) \right\}.$$

**Proof:** From the assumption that no curve is contained in the base locus of $V_i$, for all $i = 0, \ldots, N$, given in (2.8) we know that we can estimate the degree of contact with the
help of the estimate (2.9). By Theorem 3.3 and Riemann-Roch (2.10) we get
\[ h^E(X) \geq 6(N - 1) \widehat{\deg}(L_N) - \min \left\{ -4s_0 + 6 \sum_{i=0}^{N} s_i, 2(N - 1)s_0 \right\}, \]
where \( s_i = \widehat{\deg}(L_N) - \widehat{\deg}(L_i) \). Expanding of this formula we get (3.2), which concludes the proof.
\[ \square \]

**Corollary 3.6.** Assume that the linear space defined by the vanishing of the last \( d + 1 \) sections \( x_{N-d}, \ldots, x_N \) does not meet \( X \). Then
\[ h^E(X) \geq \deg(X) \sum_{i=N-d}^{N} \widehat{\deg}(L_i) \]

**Proof.** Follows from the identity (2.7) and Theorem 3.3 \( \square \)

3.7. We recall the definition of the normalized height \( \hat{h}(X) \) form the introduction:
\[ \hat{h}(X) = \inf_{E \in V} \left( \frac{h^E(X)}{(d + 1) \deg(X)} - \frac{\widehat{\deg}(E)}{N + 1} \right), \]
where \( V \) is the direct limit of the set \( V_K \) of \( \mathcal{O}_K \)-modules \( E \) with an identity \( E_K \cong K^{N+1} \) as \( K \) varies in the set of number fields. We will prove in this section that under some conditions on \( X \) the height \( \hat{h}(X) \) is non-negative, as it is true in the function field case (see \[5\], \[1\]). The idea of the proof is the same of \[22\], the main difference relies in the direct use of the minima of \( X \) at the place of the minima of the projective space.

3.8. According to \[26\] (5.2) (see also \[6\], Théorème 3.1]) we know that for all \( \varepsilon > 0 \) the set
\[ X \left( \frac{h^E(X)}{\deg(X)} + \varepsilon \right) = \left\{ x \in X(\overline{K}) \mid \frac{h^E(x)}{\deg(X)} \leq \frac{h^E(X)}{\deg(X)} + \varepsilon \right\} \]
is Zariski dense.

**Proposition 3.9.** Assume that \( X \) is not contained in any proper linear subspace of \( \mathbb{P}^n(E^\vee) \). Then for all \( \varepsilon > 0 \) there exist a finite field extension \( L \) of \( K \) and \( N + 1 \) hermitian line subbundles \( L_0, \ldots, L_N \) of \( E \) such that

1. \( \oplus_{i=0}^{N} L_i \) generates generically \( E \) over \( L \),
2. The arithmetic degrees of this subbundles satisfy
\[ -\frac{1}{N + 1} \sum_{i=0}^{N} \widehat{\deg}(L_i) \leq \frac{h^E(X)}{\deg(X)} + \varepsilon. \]
Proof. Consider the \((N + 1)\)-fold Segre immersion \(\varphi\) of \(X^{N+1} \subseteq \mathbb{P}(E^\vee)^{N+1}\) into the projective space \(\mathbb{P}((E^\vee)^{\otimes (N+1)})\). From \cite{3} (2.3.19) we have
\[
h_{(E)^{\otimes (N+1)}} (\varphi_* (X^{N+1})) = (N + 1) \deg(X)^N h_E(X),
\]
and
\[
\deg (\varphi_* (X^{N+1})) = \deg(X)^{N+1}.
\]
Applying (3.3) to \(\varphi_* (X^{N+1})\) we have that the set \(\varphi_* (X^{N+1}) \left( \frac{h_E(\varphi_* (X))}{\deg(\varphi_* (X))} + \varepsilon \right)\) is Zariski dense. Since \(\varphi\) is an isomorphism, this set is in homeomorphic to
\[
X^{N+1} \left( (N + 1) \frac{h_E(X)}{\deg(X)} + \varepsilon \right),
\]
which is therefore Zariski dense, too. Further, notice that the condition (1) is obviously open, since it corresponds to the non-vanishing of the maximal exterior power of the direct sum of the line subbundles. Hence (1) defines a Zariski open subset of
\[
X^{N+1} \left( (N + 1) \frac{h_E(X)}{\deg(X)} + \varepsilon \right),
\]
which is itself Zariski dense. This implies that the set of points with (1) and (3) is Zariski dense, hence non-empty. Let \(P = (P_0, \ldots, P_N)\) be a point there, and \(L\) be the field \(K(P_0, \ldots, P_N)\), then \(P \in \mathbb{P}(E^\vee)^{N+1}(L)\). Further, we let \(L_i\) the line subbundle of \(E\) associated to \(P_i, i = 0, \ldots, N\). By construction they satisfy (1) and (2). This concludes the proof of the Proposition. \(\square\)

Proof of Theorem \cite{7} According to \cite{26} Proposition 4.2 it suffices to prove that, for any number field \(K, h_E(X)/(d + 1) \deg X \geq 0\) for any hermitian \(\mathcal{O}_K\)-module \(\overline{E}\) such that \(\det \mathcal{E} = 0\). By Proposition \cite{3.9} for any \(\varepsilon > 0\) there exist a finite field extension \(L\) of \(K\) and \(N + 1\) hermitian line subbundles \(\overline{L}_0, \ldots, \overline{L}_N\) of \(\overline{E}\) such that \(\oplus_{i=0}^N \mathcal{L}_i\) generates \(\mathcal{E}\) over \(L\) and (3.4) holds. For each \(i = 0, \ldots, N\) let \(x_i\) be a section of \(\mathcal{L}_i\). These sections give an embedding \(X \to \mathbb{P}^N\). Assume that the line bundles are ordered by increasing arithmetic degree
\[
\widehat{\deg} (\overline{L}_0) \leq \cdots \leq \widehat{\deg} (\overline{L}_N) \leq 0.
\]
Since \(X\) is semistable with respect to the sections \(x_0, \ldots, x_N\) and to any \((N + 1)\)-tuple of integers with \(r_0 \geq \cdots \geq r_N \geq 0\), we have
\[
\frac{e_w(X)}{(d + 1) \deg(X)} \leq \frac{1}{N + 1} \sum_{i=0}^N r_i.
\]
Hence, from Theorem 3.3 we get
\[
\frac{h_E(X)}{(d + 1) \deg(X)} \geq \deg(Z_N) + \frac{1}{N + 1} \sum_{i=0}^{N} \left( \deg(Z_i) - \deg(Z_N) \right)
\]
\[
= \frac{1}{N + 1} \sum_{i=0}^{N} \deg(Z_i).
\]
Since $Z_0, \ldots, Z_N$ satisfy (3.4) this yields
\[
\frac{h_E(X)}{(d + 1) \deg(X)} \geq -\frac{h_E(X)}{\deg(X)} - \varepsilon,
\]
whence
\[
\frac{h_E(X)}{(d + 1) \deg(X)} \geq -\frac{\varepsilon}{d + 2}.
\]
But $\varepsilon > 0$ can be chosen arbitrarily small, hence this concludes the proof of (1.2).

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