On the deformation quantization of super-Poisson brackets

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Abstract

We show that for every vector bundle $E$ over any given symplectic manifold there exists an explicitly given super Poisson bracket on the space of sections of the dual Grassmann bundle associated to $E$ built out of the symplectic structure of $M$, a fibre metric on $E$ and a connection in $E$ compatible with the given fibre metric. Moreover, we construct a deformation quantization for this space of sections by means of a Fedosov type procedure.

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1 Introduction

In the usual programme of deformation quantization (cf. [3]) the quantum mechanical multiplication is considered as a formal associative deformation (a so-called star product) of the pointwise multiplication of the classical observables, viz. the algebra of smooth complex-valued functions on a given symplectic manifold. The deformation is such that to first order in $\hbar$ the commutator of the deformed product is proportional to the Poisson bracket. The difficult question of existence of these star products for every symplectic manifold was settled independently by DeWilde and Lecomte [6] and Fedosov [7], [8].

In the theory of supermanifolds the algebra $C^0$ of classical superobservables can be considered as the space of sections of the complexified dual Grassmann bundle of an $n$-dimensional vector bundle $E$ (see e. g. [2]) over a symplectic manifold $(M, \omega)$, i.e.

$$C^0 := \mathcal{C}\Gamma(\Lambda^*E), \quad (1)$$

where the multiplication is the pointwise wedge product. Clearly, $C^0$ is a $\mathbb{Z}_2$-graded supercommutative algebra, i.e.

$$\phi \wedge \psi = (-1)^{d_1d_2} \phi \wedge \psi \quad \text{for} \quad \phi, \psi \in \Gamma(\Lambda^*E),$$

where $\phi$ of degree $d_1$ and $\psi$ of degree $d_2$. A super-Poisson bracket for $C^0$ is by definition a $\mathbb{Z}_2$-graded bilinear map $M_1 : C^0 \times C^0 \to C^0$ which is superanticommutative, i.e.

$$M_1(\psi, \phi) = -(-1)^{d_1d_2} M_1(\phi, \psi),$$

satisfies the superderivation rule

$$M_1(\phi, \psi \wedge \chi) = M_1(\phi, \psi) \wedge \chi + (-1)^{d_1d_2} \psi \wedge M_1(\phi, \chi),$$

and the super Jacobi identity, i.e.

$$(-1)^{d_1d_2} M_1(M_1(\phi, \psi), \chi) + \text{cycl.} = 0 \quad \text{where} \quad \chi \in C^0 \quad \text{is of degree} \quad d_3.$$

It is general not difficult to find super-Poisson brackets of purely algebraic type, i.e. which vanish when one of their arguments is restricted to a smooth complex-valued function, by means of a fibre metric $q$ in $E$ (see e. g. [3], p. 123, eqn 5-1):

$$M'_1(\phi, \psi) = q^{AB}(j(e_A)\phi) \wedge (i(e_B)\psi) \quad (2)$$

where $q^{AB}$ are the components of the induced fibre metric in the dual bundle $E^*$ in the dual base to a local base $(e_A)$, $1 \leq A \leq \dim E$, of sections of $E$, and $i(e_B)$ and $j(e_A)$ denote the usual interior product left antiderivation and right antiderivation, respectively. The definition does not depend on the choice of that local base.

In case $M$ is $\mathbb{R}^{2m}$ with the standard Poisson bracket one can combine the standard bracket with the above super-Poisson bracket to get

$$M_1(\phi, \psi) = \frac{\partial \phi}{\partial q^i} \wedge \frac{\partial \psi}{\partial p_i} - \frac{\partial \phi}{\partial p_i} \wedge \frac{\partial \psi}{\partial q^i} + q^{AB}(j(e_A)\phi) \wedge (i(e_B)\psi) \quad (3)$$

However, for nontrivial bundles it does not seem to be so obvious to generalize this bracket in the sense that it is equal to -at least in degree zero- the Poisson bracket of the base space $M$ when restricted to the sections of degree zero.

The main results of this paper are the following: firstly we construct an explicitly given super-Poisson bracket $M_1$ for any vector bundle over any symplectic manifold which is equipped with an arbitrary nondegenerate fibre metric.
and a compatible connection such that for two smooth complex-valued functions $f, g$ we have $M_1(f, g) = \{f, g\} + \text{terms of higher Grassmann degree}$ where $\{\ , \}$ is the Poisson bracket on $M$. This superbracket generalizes the above flat space superbracket (3), contains in a simple polynomial manner the curvature of the connection in $E$, and does not yet seem to have occurred in the literature as far as I know.

Secondly, we show by Fedosov’s quantization procedure that the Grassmann multiplication in $C_0$ can formally be deformed into an associative $\mathbb{Z}_2$-graded multiplication $\ast$ on the space of formal power series in $\hbar$ with coefficients in $C_0$,

$$C := C_0[[\hbar]],$$

such that the term proportional to $\hbar$ in that multiplication is equal to $(i/2)M_1$.

The paper is organized as follows: We first transfer Fedosov’s Weyl algebra bundle to our situation by simply tensoring with the dual Grassmann bundle $\Lambda E^\ast$. The fibrewise multiplication has also a component in $\Lambda E^\ast$ built by means of the fibre metric in $E$. Then Fedosov’s procedure can completely be imitated without further difficulties: we show the existence of a Fedosov connection $D$ of square zero whose kernel in the space of antisymmetric degree zero is in linear $1-1$ correspondence to $C$ which immediately gives rise to the desired quantum deformation (Theorem 2.3).

In the third part we explicitly compute the super-Poisson bracket $M_1$ as the term proportional to $(i\hbar)/2$ by means of Fedosov’s recursion formulae (Theorem 3.1).

Notation: In all of this paper the Einstein sum convention is used that two equal indices are automatically summed up over their range unless stated otherwise. Moreover, we widely make use of Fedosov’s notation in [8] with the following exceptions: we use the symbol $\nabla$ to denote the covariant derivative and not Fedosov’s $\partial$ and describe the occurring symmetric tensor fields with $\vee$ products (see e.g. [8], p. 209-226) and use the symmetric substitution operator $i_s$ instead of Fedosov’s functions of $y$ and derivatives with respect to $y$.

2 The Fedosov quantization procedure for dual Grassmann bundles

Let $(M, \omega)$ be a $2m$-dimensional symplectic manifold and $E$ an $n$-dimensional real vector bundle over $M$ with a fixed positive definite fibre metric $q$. For the computations that will follow we shall use co-ordinates $(x^1, \ldots, x^{2m})$ in a chart $U$ of $M$. The base fields $\frac{\partial}{\partial x^i}$ will be denoted by $\partial_i$ ($1 \leq i \leq 2m$) for short. For computations in $E$ we shall use a local base $(e_A)$, ($1 \leq A \leq n$) of sections of $E$ over $U$. Denote the dual base in the dual bundle $E^\ast$ of $E$ by $(e^A)$, ($1 \leq A \leq n$). Let $\Lambda \in \Gamma(\Lambda^2TM)$ denote the Poisson structure of $(M, \omega)$, i.e. the Poisson bracket of two smooth real valued functions $f, g$ is given by $\{f, g\} := \Lambda(df, dg)$. Denoting the components of $\omega$ and $\Lambda$ in that
chart by $\omega_{ij} := \omega(\partial_i, \partial_j)$ and $\Lambda^{ij} := \Lambda(dx^i, dx^j)$ we use the sign conventions of \[ where $\Lambda^{ik}\omega_{jk} = \delta^i_1$. Fix a torsion-free symplectic connection $\nabla^M$ in the tangent bundle of $M$. This is well-known to always exist which can be seen by Heff’s formula $\omega(\nabla^M_X Y, Z) := \omega(X, Y, Z) + \frac{1}{2}((\nabla^M X)Y) - \frac{1}{2}((\nabla^M Y)X)(X, Z)$ where $X, Y, Z$ are arbitrary vector fields on $M$ and $\nabla$ is an arbitrary torsion-free connection on $M$ (see \[ ]). Fix a connection $\nabla^E$ in $E$ which is compatible with $q$, i.e. $X(q(e_1, e_2)) = q(\nabla^E_X e_1, e_2) + q(e_1, \nabla^E_X e_2)$ for an arbitrary vector field $X$ on $M$ and sections $e_1, e_2$ of $E$.

We are now forming the Fedosov algebra $W \otimes \Lambda$:

\[
W \otimes \Lambda := \left( \times_{s,t=0}^{\infty} \Gamma(\mathcal{C}(\wedge^s T^* M \otimes \Lambda E^s \otimes \Lambda T^* M)) \right)[[\hbar]]
\]

This is to say that the elements of $W \otimes \Lambda$ are formal sums $\sum_{s,t=0}^{\infty} w_{st} h^t$ where the $w_{st}$ are smooth sections in the complexification of the vector bundle $\wedge^s T^* M \otimes \Lambda E^s \otimes \Lambda T^* M$. In what follows we shall sometimes use the following factorized sections $F := f \otimes \phi \otimes \alpha h^{t_1}$ and $G := g \otimes \psi \otimes \beta h^{t_2}$ where $f \in \Gamma(\wedge^{s_1} T^* M)$, $g \in \Gamma(\wedge^{s_2} T^* M)$, $\phi \in \Gamma(\Lambda^{d_1} E^s)$, $\psi \in \Gamma(\Lambda^{d_2} E^s)$, $\alpha \in \Gamma(\Lambda^{a_1} T^* M)$, and $\beta \in \Gamma(\Lambda^{a_2} T^* M)$. Let $\text{deg}_s, \text{deg}_g, \text{deg}_a, \text{deg}_h$ be the obvious degree maps from $W \otimes \Lambda$ to itself, i.e. those $\mathbb{C}$-linear maps for which the above factorized sections $f \otimes \phi \otimes \alpha h^{t_1}$ are eigenvectors to the eigenvalues $s_1, d_1, a_1, t_1$ respectively and which we refer to as the symmetric degree, the $E$-degree, the antisymmetric degree, and the $h$-degree, respectively. Moreover, let $P_E$ and $P_h$ be the corresponding maps $(-1)^{\text{deg}_E}$ and $(-1)^{\text{deg}_h}$ which we refer to as the $E$-parity and the $h$-parity, respectively. We say that a $\mathbb{C}$-linear endomorphism $\Phi$ of $W \otimes \Lambda$ is of $\zeta$-degree $k$ ($\zeta = s, a, E, h$) iff $[\text{deg}_\zeta, \Phi] = k \Phi$. Analogously, $\Phi$ is said to be of $\zeta$-parity $(-1)^k$ ($\zeta = E, h$) iff $P_\zeta \Phi P_\zeta = (-1)^k \Phi$. Let $C$ denote the complex conjugation of sections in $W \otimes \Lambda$.

We shall sometimes write $W$ for the space of elements of $W \otimes \Lambda$ having zero antisymmetric degree and $W \otimes \Lambda^a$ for the space of those elements having antisymmetric degree $a$. The space $W \otimes \Lambda$ is an associative algebra with respect to the usual pointwise product where we do not use the graded tensor product of the two Grassmann algebras involved. More precisely, for the above factorized sections the pointwise or undeformed multiplication is simply given by

\[
(f \otimes \phi \otimes \alpha h^{t_1})(g \otimes \psi \otimes \beta h^{t_2}) := (f \wedge g) \otimes (\phi \wedge \psi) \otimes (\alpha \wedge \beta) h^{t_1+t_2}.
\]

Note that the above four degree maps are derivations and the above two parity maps are automorphisms of this multiplication. Moreover, $W \otimes \Lambda$ is supercommutative in the sense that

\[
GF = (-1)^{d_1 d_2 + a_1 a_2} FG
\]

A linear endomorphism $\Phi$ of $W \otimes \Lambda$ of $E$-parity $(-1)^{d'}$ and antisymmetric degree $a'$ is said to be a superderivation of type $((-1)^{d}, a')$ of the undeformed algebra $W \otimes \Lambda$ iff $\Phi(FG) = (\Phi F)G + (-1)^{d'd + a' a} F(\Phi G)$. Let $\sigma$ denote the linear map

\[
\sigma : W \otimes \Lambda \to \Gamma(\Lambda E^* \otimes \Lambda T^* M)[[\hbar]]
\]
which projects onto the component of symmetric degree zero and clearly is a homomorphism for the undeformed multiplication.

We now combine the two covariant derivatives $\nabla^M_X$ in $TM$ and $\nabla^E_X$ in $E$ into a covariant derivative $\nabla_X$ in $TM \otimes E$ in the usual fashion and extend it canonically to a connection $\nabla$ in $\mathcal{W} \otimes \Lambda$. On the above factorized sections we get in a chart:

$$\nabla(f \otimes \phi \otimes \alpha) = \left( (\nabla^M_X f) \otimes \phi + f \otimes (\nabla^E_X \phi) \right) \otimes (dx^i \wedge \alpha) + f \otimes \phi \otimes \alpha. \quad (9)$$

In order to define a deformed fibrewise associative multiplication consider the following insertion maps for a vector field $X$ on $M$ and a section $e$ of $E$: let $i_s(X)$ and $i(e)$ denote the usual inner product antiderivations in $\Gamma(\Lambda T^*M)$ and $\Gamma(\Lambda E^*)$, respectively, and extend them in a canonical manner to superderivations of type $(1,-1)$ and $(-1,0)$ of the undeformed algebra $\mathcal{W} \otimes \Lambda$, respectively. Let $j(e)$ be defined by $P_E i(e)$. Moreover, let $i_s(X)$ denote the corresponding inner product derivation (or symmetric substitution, [9], p. 209-226) in $\times_{\mathbb{R}}^\infty \Gamma(TM)$, again extended to a derivation of the undeformed algebra $\mathcal{W} \otimes \Lambda$ in the canonical way. Let $q^{AB}$ denote the components of the induced fibre metric $q^{-1}$ in $E^*$, i.e. $q^{AB} := q^{-1}(e^A, e^B)$. Note that $q^{AB}$ is the inverse matrix to $q(e_A, e_B)$. Then for two elements $F, G$ of $\mathcal{W} \otimes \Lambda$ we can now define the fibrewise deformed multiplication $\circ$:

$$F \circ G := \sum_{k,l=0}^{\infty} \frac{(ih/2)^{k+l}}{k!l!} \left( (\Lambda^{i_1j_1} \cdots \Lambda^{i_kj_k})(i_s(\partial_{i_1}) \cdots i_s(\partial_{i_k})F)(i_s(\partial_{j_1}) \cdots i_s(\partial_{j_k})G) 
+ q^{A_1B_1} \cdots q^{A_kB_k}(j(e_{A_1}) \cdots j(e_{A_k})F)(i(e_{B_1}) \cdots i(e_{B_k})G) \right). \quad (10)$$

Moreover, let $\delta$ and $\delta^*$ be the canonical superderivations of the undeformed algebra $\mathcal{W} \otimes \Lambda$ of type $(1,1)$ and $(1,-1)$, respectively, which are induced by the identity map of $T^*M$ to $T^*M$ where in the case of $\delta$ the preimage of the identity is regarded as being part of $\sqrt{\gamma}T^*M$ and the image as being part of $\Lambda T^*M$, and vice versa in the case of $\delta^*$. On the above factorized sections these maps read in co-ordinates

$$\delta(f \otimes \phi \otimes \alpha) = (i_s(\partial_i)f) \otimes \phi \otimes (dx^i \wedge \alpha) \quad (11)$$
$$\delta^*(f \otimes \phi \otimes \alpha) = (dx^i \wedge f) \otimes \phi \otimes (i_s(\partial_i)\alpha). \quad (12)$$

Define the total degree $\text{Deg}$:

$$\text{Deg} := 2\deg_h + \deg_s + \deg_E \quad (13)$$

A $\circ$-superderivation of type $((-1)^{d'}, d')$ is defined in an analogous manner as for the undeformed multiplication.

We collect some properties of the above structures in the following...
Proposition 2.1  With the above definitions and notations we have the following:
1. $\delta^2 = 0 = (\delta^*)^2$ and $\delta\delta^* + \delta^*\delta = \deg_a + \deg_a$.
2. $\delta\nabla + \nabla\delta = 0$.
3. $\text{Ker}(\delta) \cap \text{Ker}(\deg_a) = C$.
4. $P_E$ is a $\circ$-automorphism and $\deg_a$ is a $\circ$-derivation which equips the Fedosov algebra $(\mathcal{W} \otimes \Lambda, \circ)$ with the structure of a $\mathbb{Z}_2 \times \mathbb{Z}$-graded associative algebra.
5. $\delta$, $\nabla$, and $\text{Deg}$ are $\circ$-superderivations of type $(1,1)$, $(1,1)$, and $(1,0)$, respectively.
6. The parity map $P_\hbar$ and the complex conjugation $C$ are graded $\circ$-antiautomorphisms, i.e. $\Phi(F \circ G) = (-1)^{\mu_1 + \mu_2 + \mu_3} G \circ F$ for $\Phi = P_\hbar, C$.

Proof: 1. Straight forward.
2. This follows from the vanishing torsion of $\nabla^M$.
3. Without the factor $\Lambda E^*$ the kernel of $\delta$ in the space of antisymmetric degree zero consists of the constants, which proves this statement.
4. The associativity of $\circ$ is known, see e.g. [3], p. 123, eqn 5-2, and can be done by a long straight forward computation. We shall sketch a shorter proof: $\circ$ is defined on each fibre (for $m \in M$) $\mathcal{W}_m := (\times_{n=0}^\infty (\mathcal{V} \otimes M^* \otimes \Lambda E^m \otimes \Lambda T_m M^*))[[\hbar]]$ on which we can rewrite the multiplication in the more compact form $(F, G \in \mathcal{W}_m)$

$$F \circ G = \mu(e^{\frac{\theta_3}{\hbar}(\Lambda^A_{ij}(\partial_i) \otimes \partial_j + \Lambda_{ij}^j(e_A) \otimes 1)}(F \otimes G))$$

where the tensor product is over $\mathcal{C}[[\hbar]]$ and $\mu$ denotes the undeformed fibrewise multiplication. Due to the derivation properties of $i(e_A)$, $i(e_B)$, and $j(e_B)$ we get formulas like

$$R \mu \otimes 1 = \mu \otimes 1 \ (R_{13} + R_{23})$$
$$R 1 \otimes \mu = 1 \otimes \mu \ (R_{12} + R_{13})$$
$$S \mu \otimes 1 = \mu \otimes 1 \ (S_{13}(P_E) + S_{23})$$
$$S 1 \otimes \mu = 1 \otimes \mu \ (S_{12} + S_{13}(P_E))$$

where the index notation is borrowed from Hopf algebras and indicates on which of the three tensor factors of $\mathcal{W}_m$ the maps $R$, $S$, and $P_E$ should act, e.g. $R_{23} := 1 \otimes R$, $(P_E)_{2} := 1 \otimes P_E \otimes 1$. These “pull through formulas” can be used to pull through the corresponding formal exponentials. Since all the maps $i_A(\partial_i)$ commute with $j(e_A)$ and $i(e_B)$ and since the $j(e_A)$ commute with all $i(e_B)$ whereas $j(e_A)$ and $j(e_B)$ anticommute as well as $i(e_A)$ and $i(e_B)$ we can conclude that all the six maps $R_{12}$, $R_{13}$, $R_{23}$, $S_{12}$, $S_{13}(P_E)_{2}$, and $S_{23}$ pairwise
commute. This is the essential step for associativity. The gradation properties are immediate.

5. The derivation properties of $\delta$ and $\deg$ are clear, for the corresponding statement for $\nabla$ the fact that $\nabla^M$ preserves the Poisson structure $\Lambda$ and that $\nabla^E$ preserves the dual fibre metric $q^{-1}$ is crucial.

6. Straight forward.

Q.E.D.

Due to the first part of this proposition we can construct a $\mathbb{C}[[\hbar]]$-linear endomorphism $\delta^{-1}$ of the Fedosov algebra in the following way: on the above factorized sections $F$ we put

$$\delta^{-1} F := \begin{cases} \frac{1}{s_1 + a_1} \delta^* F & \text{if } s_1 + a_1 \geq 1 \\ 0 & \text{if } s_1 + a_1 = 0 \end{cases}$$

(15)

Since $W \otimes \Lambda$ is an $\mathbb{Z}_2 \times \mathbb{Z}$-graded associative algebra we can form the $\mathbb{Z}_2 \times \mathbb{Z}$-graded super Lie bracket which reads on the above factorized sections:

$$[F, G] := ad(F)G := F \circ G - (-1)^{d_1 d_2 + a_1 a_2} G \circ F$$

(16)

It follows from the associativity of $\circ$ that $ad(F)$ is $\circ$-superderivation of the Fedosov algebra $(W \otimes \Lambda, \circ)$ of type $((-1)^{d_1}, a_1)$. Note that the map $\frac{1}{\hbar} ad(F)$ which we shall often use in what follows is always well-defined because of the super-commutativity of the undeformed multiplication $[\hbar]$. Consider now the curvature tensors $R^M$ of $\nabla^M$ and $R^E$ of $\nabla^E$, i.e. for three vector fields $X, Y, Z$ on $M$ and a section $e$ of $E$ we have $R^M(X, Y)Z = \nabla^M_X \nabla^M_Y Z - \nabla^M_{[X,Y]} Z$ and $R^E(X, Y)e = \nabla^E_X \nabla^E_Y e - \nabla^E_{[X,Y]} e$. Define elements $R^{(M)}$ and $R^{(E)}$ of the Fedosov algebra which are contained in $\Gamma(\nabla^T M \otimes \Lambda^2 T^* M)$ and $\Gamma(\Lambda^2 E^* \otimes \Lambda^2 T^* M)$, respectively, as follows where $V, W$ are vector fields on $M$ and $e_1, e_2$ are sections of $E$:

$$R^{(M)}(V, W, X, Y) := \omega(V, R^M(X, Y)W)$$

(17)

$$R^{(E)}(e_1, e_2, X, Y) := -q(e_1, R^E(X, Y)e_2).$$

(18)

Note that this is well-defined: since $\nabla^M$ preserves $\omega$ and $\nabla^E$ preserves $q$ it follows that $R^{(M)}$ is symmetric in $V, W$ and $R^{(E)}$ is antisymmetric in $e_1, e_2$. In co-ordinates these two elements of the Fedosov algebra can be written in the form $R^{(M)} = (1/4)R^{(M)}_{kij} dx^k \otimes dx^j \otimes 1 \otimes dx^i \wedge dx^j$ and $R^{(E)} = (1/4)R^{(E)}_{ABij} 1 \otimes e^A \wedge e^B \otimes dx^i \wedge dx^j$. Set

$$R := R^{(M)} + R^{(E)}.$$

(19)

Then the following Proposition is immediate:

**Proposition 2.2** With the above definitions and notations we have:

1. $\nabla^2 = \frac{1}{\hbar} ad(R)$. 

2. $P_E(R) = R$, $P_h(R) = R$ and $C(R) = R$.
3. $\delta R = 0$.
4. $\nabla R = 0$.

**Proof:**
1. Straightforward computation.
2. Obvious.
3. This is a consequence of the vanishing torsion of $\nabla^M$ (first Bianchi identity).
4. This is a reformulation of the second Bianchi identity for linear connections in arbitrary vector bundles. Q.E.D.

We shall now make the ansatz for a Fedosov connection $D$, i.e. we are looking for an element $r \in W \otimes \Lambda^1$ of even $E$-parity, i.e. $P_E(r) = r$, such that the map

$$D := -\delta + \nabla + \frac{i}{\hbar}ad(r)$$

has square zero, i.e. $D^2 = 0$. The following properties of $D$ for any $r$ are crucial:

**Lemma 2.1** Let $r$ be an arbitrary element of $W \otimes \Lambda^1$ of even $E$-parity. Then
1. $D^2 = \frac{i}{\hbar}ad(-\delta r + \nabla r + R + \frac{i}{\hbar}r \circ r)$.
2. $D(-\delta r + \nabla r + R + \frac{i}{\hbar}r \circ r) = 0$.

**Proof:** This is straightforward using Proposition 2.2 and the fact that $r \circ r = \frac{1}{2}[r, r]$ for the above elements $r$ of even $E$-parity and odd antisymmetric degree. Q.E.D.

For an arbitrary element $w \in W \otimes \Lambda$ we shall make the following decomposition according to the total degree $\text{Deg}$:

$$w = \sum_{k=0}^{\infty} w^{(k)} \text{ where } \text{Deg}(w^{(k)}) = kw^{(k)}$$

Note that each $w^{(k)}$ is always a finite sum of sections in some $\Gamma(\wedge^k T^*M \otimes \Lambda E^* \otimes \wedge T^*M)$. The subspaces of all elements of $W \otimes \Lambda$, $W$, $W \otimes \Lambda^a$, and $C$ of total degree $k$ will be denoted by $W^{(k)} \otimes \Lambda$, $W^{(k)}$, $W^{(k)} \otimes \Lambda^a$, and $C^{(k)}$, respectively.

As in Fedosov’s paper there is the following

**Theorem 2.1** With the above definitions and notations: Let $r \in W \otimes \Lambda^1$ be defined by the following recursion:

$$r^{(3)} := \delta^{-1} R$$
$$r^{(k+3)} := \delta^{-1} \left( \nabla r^{(k+2)} + \frac{i}{\hbar} \sum_{l=1}^{k-1} r^{(l+2)} \circ r^{(k-l+2)} \right)$$
Then $r$ has the following properties: it is real $(C(r) = r)$, depends only on $h^2$ $(P_h(r) = r)$, has even $E$-parity, and is in the kernel of $\delta^{-1}$. Moreover, the corresponding Fedosov derivation $D = -\delta + \nabla + (i/h)\text{ad}(r)$ has square zero.

**Proof:** The behaviour of $r$ under the parity transformations and complex conjugation immediately follows from the fact that they commute with $\delta^{-1}$ and from their (anti)homomorphism properties (Prop. 2.1, 3., 5.; Prop. 2.2, 2.). Let $A := -\delta r + \nabla r + R + \frac{1}{2} r \circ r := -\delta r + R + B$. Recall the equation $\delta \delta^{-1} + \delta^{-1} \delta = 1$ on the subspace of the Fedosov algebra where $\text{deg}_a + \text{deg}_b$ have nonzero eigenvalues. Clearly, $A^{(2)} = -\delta r^{(3)} + R = 0$ because $\delta R = 0$ (Prop. 2.3.) hence $R = \delta \delta^{-1} R$. Suppose $A^{(l)} = 0$ for all $2 \leq l \leq k + 1$. By Lemma 2.3.2, we have $0 = (DA)^{(k+1)} = -\delta A^{(k+2)} = -\delta B^{(k+2)}$. Hence $B^{(k+2)} = \delta \delta^{-1} B^{(k+2)} = \delta r^{(k+3)}$ proving $A^{k+2} = 0$ which inductively implies $D^2 = 0$ since we had already shown that $r$ is of even $E$-parity. Q.E.D.

We shall now compute the kernel of the Fedosov derivation. More precisely, define

$$W_D := \text{Ker}(D) \cap \text{Ker} (\text{deg}_a).$$

As in Fedosov’s paper \cite{Fedosov} we have the important characterization:

**Theorem 2.2** With the above definitions and notations: $W_D$ is a subalgebra of the Fedosov algebra $(\mathcal{W} \otimes A, \circ)$. Moreover, the map $\sigma$ \cite{Fedosov} restricted to $W_D$ is a $\mathcal{C}[[\hbar]]$-linear bijection onto $\mathcal{C}$.

**Proof:** The kernel of a superderivation is always a subalgebra. Since $D$ and $\sigma$ are $\mathcal{C}[[\hbar]]$-linear the subalgebra $W_D$ is a $\mathcal{C}[[\hbar]]$-submodule of $W$.

Let $w \in W$. Decompose $w = w_0 + w_+$ where $w_0 := \sigma(w)$ and $w_+ := (1 - \sigma)(w)$. We shall prove by induction over the total degree $k$ that $w \in W$ is in $W_D$ iff for all nonnegative integers $k$ $w_0^{(k)}$ is arbitrary in $\mathcal{C}^{(k)}$ and $w_+^{(k)}$ is uniquely given by the equation

$$w_+^{(k)} = \delta^{-1} \left( \nabla w^{(k-1)} + \frac{k-2}{2} \sum_{l=1}^{k-2} [r^{(l+2)}, w^{(k-1-l)}] \right) =: (Aw)^{(k)}$$

where of course an empty sum is defined to be zero and $w_+^{(0)} = 0$. Note that $Dw = -\delta w + Aw$ and that the $\mathcal{C}$-linear map $A$ does not lower the total degree of $w$.

Now the equation $(Dw)^{(k)} = 0$ is equivalent to the inhomogeneous equation $\delta w^{(k+1)} = (Aw)^{(k)}$. A necessary condition for this equation to be solvable for $w_+^{(k+1)}$ clearly is $\delta ((Aw)^{(k)}) = 0$. But this is also sufficient since then $(Aw)^{(k)} = \delta \delta^{-1} (Aw)^{(k)}$ and we have the particular solution $w_+^{(k+1)} = \delta^{-1} (Aw)^{(k)}$ (since $\sigma \delta^{-1} = 0$) which satisfies \cite{Fedosov}. To this particular solution any solution to the homogeneous equation $\delta w^{(k')} = 0$ can be added which precisely is the space $\mathcal{C}^{(k)}$. It remains to show that conversely every initial piece $w' := w_0^{(0)} + w_0^{(1)} + w_+^{(1)} + \cdots + w_0^{(k)} + w_+^{(k)}$ where $w_0^{(0)}$ was arbitrarily chosen in $\mathcal{C}^{(0)}$, $w_+^{(0)}$ is determined
by (24) for all \(0 \leq l \leq k\), and \((Dw')(l) = 0\) for all \(-1 \leq l \leq k - 1\) can be continued to \(w'' := w' + w_0^{(k+1)} + w_+^{(k+1)}\) with \(w_0^{(k+1)}\) and \(w_+^{(k+1)}\) determined by (23), and \((Dw'')(k) = 0\). By induction, this will eventually lead to \(w \in W_D\) characterized by the above properties. Indeed, since \(D^2 = 0\) we have \(0 = (D^2w')(k-1) = -\delta((Aw')(k)) = -\delta((Aw')(k))\). Define \(w_0^{(k+1)}\) by \(\delta^{-1}((Aw')(k))\) and choose any \(w_0^{(k+1)} \in C^{(k+1)}\). It follows at once that \(w_+^{(k+1)}\) satisfies (24) and that we get \((Dw'')(k) = 0\) which proves the induction and the Theorem. \(\text{Q.E.D.}\)

Let

\[
\tau : C \to W_D \subset W
\]

be the inverse of the restriction of \(\sigma\) to \(W_D\). For \(\phi \in \Gamma(\Lambda E^*)\) we shall speak of \(\tau(\phi)\) as the Fedosov-Taylor series of \(\phi\) and refer to the components \(\tau(\phi)^{(k)}\) as the Fedosov-Taylor coefficients. We collect some of the properties of \(\tau\) in the following

**Proposition 2.3** With the above definitions and notations:

1. \(\tau\) commutes with \(P_E, P_h,\) and \(C\).

2. Let \(\phi = \sum_{d=0}^{n} \phi^{(d)} \in \Gamma(\Lambda E^*)\) where \(n := \dim E\). Then \(\text{Deg}(\phi^{(d)}) = d\phi^{(d)} = \text{deg}_E(\phi^{(d)})\).

Moreover

\[
\begin{align*}
\tau(\phi)^{(0)} & = \phi^{(0)} \\
\tau(\phi)^{(1)} & = \delta^{-1}(\nabla \phi^{(0)}) + \phi^{(1)} \\
\vdots & \quad \vdots \\
\tau(\phi)^{(n)} & = \delta^{-1} \left( \nabla (\tau(\phi)^{(n-1)}) + \frac{i}{\hbar} \sum_{l=1}^{n-2} \left[ r^{(l+2)}, \tau(\phi)^{(n-1-l)} \right] \right) + \phi^{(n)} \\
\tau(\phi)^{(n+1)} & = \delta^{-1} \left( \nabla (\tau(\phi)^{(n)}) + \frac{i}{\hbar} \sum_{l=1}^{n-1} \left[ r^{(l+2)}, \tau(\phi)^{(n-l)} \right] \right) \\
\vdots & \quad \vdots \\
\tau(\phi)^{(k+1)} & = \delta^{-1} \left( \nabla (\tau(\phi)^{(k)}) + \frac{i}{\hbar} \sum_{l=1}^{k-1} \left[ r^{(l+2)}, \tau(\phi)^{(k-l)} \right] \right)
\end{align*}
\]

where \(k \geq n\). The Fedosov-Taylor series \(\tau(\phi)\) depends only on \(\hbar^2\).

3. For any nonnegative integer \(k\) the map \(\phi \mapsto \tau(\phi)^{(k)}\) is a polynomial in \(\hbar\) whose coefficients are differential operators from \(\Gamma(\Lambda E^*)\) into some \(\Gamma(\nabla^* T^* M \otimes \Lambda E^*)\) of order \(k\).
Proof: Since $r$ is invariant under the parity maps and complex conjugation, it follows that $D$ commutes with these three maps, hence $W_D$ is stable under these maps. Since $\sigma$ obviously commute with them, so does the inverse of its restriction to $W_D$, $\tau$. The rest is a consequence of the preceding Theorem and a straightforward induction. Q.E.D.

Define the following $\mathbb{C}[[\hbar]]$-bilinear multiplication on $\mathbb{C}$: for $\phi, \psi \in \mathbb{C}$

$$\phi \ast \psi := \sigma(\tau(\phi) \circ \tau(\psi)).$$

(30)

We shall call $\ast$ the Fedosov star product associated to $(M, \omega, \nabla^M, E, q, \nabla^E)$. For $\phi, \psi \in \Gamma(\Lambda^*E)$ the star product $\phi \ast \psi$ will be a formal power series in $\hbar$ which we shall write in the following form:

$$\phi \ast \psi = \sum_{t=0}^{\infty} \left( \frac{i\hbar}{2} \right)^t M_t(\phi, \psi).$$

(31)

We list some important properties of the Fedosov star product in the following

Theorem 2.3 With the above definitions and notations:

1. The Fedosov star product is associative and $\mathbb{Z}_2$-graded, i.e. $P_E$ is an automorphism of $(\mathbb{C}, \ast)$. The map $P_\hbar$ and the complex conjugation $C$ are graded antiautomorphisms of $(\mathbb{C}, \ast)$.

2. The $\mathbb{C}$-bilinear maps $M_t$ are all bidifferential, real, vanish on the constant functions in each argument for $t \geq 1$, and have the following symmetry property:

$$M_t(\psi, \phi) = (-1)^t (-1)^{d_1d_2} M_t(\phi, \psi).$$

(32)

3. The term of order 0 is equal to the pointwise Grassmann multiplication. Hence $(\mathbb{C}, \ast)$ is a formal associative deformation of the supercommutative algebra $(\mathbb{C}_0, \wedge)$.

Proof: Basically, every stated property is easily derived from the definitions (30) and (31) and the corresponding behaviour of the fibrewise multiplication $\circ$ under $P_E$, $P_\hbar$, and $C$. The reality of the $M_t$ follows easily from the graded anti-homomorphism property of $C$ once eqn (32) is proved by means of the graded antihomomorphism property of the $\hbar$-parity. Since $\tau(1)$ is easily seen to be equal to 1 we have $1 \ast \psi = \psi = \psi \ast 1$, and the $M_t$ must vanish on 1 for $t \geq 1$. Finally, each $M_t$ obviously depends on only a finite number of Fedosov-Taylor coefficients whence it must be bidifferential. Q.E.D.
3 Computation of the super-Poisson bracket

In this section we are going to compute an explicit expression for the term $M_1$ of
the Fedosov star product defined in the last section (compare (31) and Theorem
2.3). Only by means of the graded associativity of the deformed algebra $(C, \ast)$
we can derive the following

**Lemma 3.1** Let $\phi, \psi, \chi$ be sections in $C_0$ of $E$-degree $d_1, d_2, d_3$, respectively.
Then

\[ M_1(\psi, \phi) = -(-1)^{d_1d_2} M_1(\phi, \psi) \]  

(33)

\[ M_1(\phi, \psi \wedge \chi) = M_1(\phi, \psi) \wedge \chi + (-1)^{d_1d_2} \psi \wedge M_1(\phi, \chi) \]  

(34)

\[ 0 = (-1)^{d_1d_3} M_1(M_1(\phi, \psi), \chi) + \text{cycl.} \]  

(35)

Hence $M_1$ is a super-Poisson bracket on $C_0$.

**Proof:** The first property is a particular case of (32). Consider now the graded commutator $[\phi, \psi] := \phi \ast \psi - (-1)^{d_1d_2} \psi \ast \phi$ on $C$. Because of the graded associativity
of $\ast$ we have the superderivation property $[\phi, \psi \ast \chi] = [\phi, \psi] \ast \chi + (-1)^{d_1d_2} \psi \ast [\phi, \chi]$.
Writing this out with the $M_1$ and taking the term of order $\hbar$ we get the second
property. For the third, take the term of order $\hbar^2$ in the super Jacobi identity
for the graded commutator.

Q.E.D.

Before we are going to compute $M_1$ directly it is useful to introduce the following
notions:

For $\phi$ in $C_0$ let $\phi_1$ and $\rho$ denote the component of symmetric degree one and
$\hbar$-degree zero of the Fedosov-Taylor coefficient $\tau(\phi)$ and the section $r$ (Theorem
2.1), respectively. Note that $\phi_1$ is a smooth section in the bundle $T^*M \otimes \Lambda E^*$.
Denote by $\Lambda_0E^*$ the subbundle of the dual Grassmann bundle consisting of
elements of even degree. Then $\rho$ is a smooth section in $T^*M \otimes \Lambda_0E^* \otimes T^*M$.
Consider now the bundle $TM \otimes \Lambda_0E^* \otimes T^*M$. There is an obvious fibrewise
associative multiplication $\bullet$ in that bundle which comes from the identification
of $TM \otimes T^*M$ with the bundle of linear endomorphism of $TM$: let $X, Y$ be
vector fields on $M$, $\phi, \psi \in \Lambda_0E^*$, and $\alpha, \beta$ one-forms on $M$. Then

\[ (X \otimes \phi \otimes \alpha) \bullet (Y \otimes \psi \otimes \beta) := (\alpha(Y))X \otimes (\phi \wedge \psi) \otimes \beta. \]  

(36)

Let $\hat{R}^E$ be the section in $\Gamma(TM \otimes \Lambda^2E^* \otimes T^*M)$ whose components in a bundle
chart read

\[ \hat{R}^E := \frac{1}{4} \Lambda_{ABkj} \partial_i \otimes e^A \wedge e^B \otimes dx^j =: \partial_i \otimes (\hat{R}^E)^i_j \otimes dx^j, \]  

(37)

and let $\hat{\rho} \in \Gamma(TM \otimes \Lambda_0E^* \otimes T^*M)$ be defined by

\[ \hat{\rho} := \partial_i \otimes \Lambda_{ik}^i \partial_k \rho =: \partial_i \otimes \hat{\rho}_i \otimes dx^i. \]  

(38)
Note that we can form arbitrary power series in $\hat{R}^E$ by using the multiplication since $\hat{R}^E$ is nilpotent.

We have the following

**Lemma 3.2** With the above notations and definitions:

\[
M_1(\phi, \psi) = \Lambda^{ij}(i_\epsilon(i_\nabla \phi_1))(i_\epsilon(i_\nabla \psi_1)) + q^{AB}(i(e_A)(\phi))(i(e_B)(\psi))
\]

\[
\phi_1 = dx^j((1 - \hat{\rho})^{-1})_j^i \nabla^E_{\delta_i} \phi
\]

\[
\hat{\rho} = 1 - (1 - 2\hat{R}^E)^{1/2}
\]

where $(1 - \hat{\rho})^{-1}$ and $(1 - 2\hat{R}^E)^{1/2}$ denote the corresponding power series with respect to the $\bullet$ multiplication.

**Proof:** The first equation is a straightforward computation.

For the second, use the Fedosov recursion for $\tau(\phi)$, (Proposition 2.3), note that $\phi_1^{(k)}$ is zero for $k \geq n + 2$ and that only the component $\rho$ of $r$ matters since both $\tau(\phi)$ and $r$ depend only on $\hbar^2$, sum over the total degree which yields the equation

\[
\phi_1 = \delta^{-1}\nabla^E \phi + dx^j(\hat{\rho})^j_\epsilon (i_\epsilon(i_\nabla \phi_1))
\]

which proves the second equation.

For the third, use the Fedosov recursion for $r$, (Theorem 2.1), take the component of symmetric degree 1 and $\hbar$-degree zero, sum over the total degree, and arrive at the quadratic equation

\[
\hat{\rho} - \hat{R}^E = \frac{1}{2} \hat{\rho} \bullet \hat{\rho}.
\]

Since $r$ and hence $\rho$ does not contain components of symmetric degree zero, there is only one solution to this equation, namely the above third equation.

Q.E.D.

This Lemma immediately implies the desired formula for the super-Poisson bracket:

**Theorem 3.1** The super-Poisson bracket $M_1$ obtained by the Fedosov star product takes the following form:

\[
M_1(\phi, \psi) = \Lambda^{ij}((1 - 2\hat{R}^E)^{-1/2})_j^i \nabla^E_{\delta_i} \phi \nabla^E_{\delta_j} \psi
\]

\[
+ q^{AB}(i(e_A)(\phi))(i(e_B)(\psi))
\]

**Proof:** Clear from the Lemma ! Q.E.D.

**Remarks 3.1** 1. In case $(M, \omega)$ is Kähler there exist star products of Wick type on $M$ (see [3]): they are characterized by the property that for any two complex-valued smooth functions $f, g$ on $M$ the star product $f \star_g g$ is made
out of bidifferential operators which differentiate \( f \) in holomorphic directions only and \( g \) in antiholomorphic directions only. It seems to me very likely that super analogues of these star products can readily be formulated for any complex holomorphic hermitean vector bundle over \( M \).

2. If the dual Grassmann bundle \( \Lambda E^\ast \) is replaced by the symmetric power \( \vee E^\ast \) and the fibre metric \( q \) by some antisymmetric bilinear form on the fibres covariantly constant by some connection in \( E \) the whole construction can presumably carried through as well.

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