Electroweak Contributions to Thermal Gravitino Production

Insititut für Theoretische Physik, Universität Wien

Max-Planck-Institut für Physik, (Werner-Heisenberg-Institut)

Diplomarbeit zur Erlangung des Grades
Magister der Naturwissenschaften
(diploma thesis)

Verfasser: Josef Pradler
Matrikelnummer: 0009878
Studienrichtung: Physik
Eingereicht am: 23.10.2006
Betreuer: O. Univ.-Prof. Dr. Alfred Bartl
Dr. Frank Daniel Steffen
Dedicated to my parents.
List of Figures

2.1 Gravitino Feynman rules ........................................... 24
2.2 Feynman rules for the gauge interactions ....................... 25
2.3 Feynman rules for light Gravitinos ............................... 25

3.1 Leading scattering processes ......................................... 29
3.2 Hard Thermal Loop self-energies ................................. 39
3.3 Goldstino self-energy ............................................... 41

4.1 Renormalization Group running in the MSSM ................... 49
4.2 Relic gravitino abundance ......................................... 52
4.3 Upper bounds on the reheating temperature ..................... 53

5.1 Upper limits on the gaugino masses ......................... 55
5.2 Probing the viability of leptogenesis .......................... 58
List of Tables

2.1 MSSM gauge fields .............................................. 20
2.2 MSSM matter fields ............................................. 21
3.1 Squared Matrix elements ................................. 30
3.2 Multiplicity coefficients ................................. 35
Chapter 1

Introduction

The Standard Model of elementary particle physics provides a most successful description of the electroweak and strong interactions among all presently observed particles. However, the observational fact that most of the matter of the Universe resides in the form of cold non-baryonic dark matter provides an impressive evidence for physics beyond the Standard Model [1]. In fact, the nature and identity of dark matter is one of the most pressing questions in the natural sciences.

Remarkably, supersymmetry (SUSY) offers an attractive solution of the dark matter problem. Supersymmetry assigns to each particle a superpartner whose spin differs by 1/2. One can distinguish between “normal” matter, i.e., Standard Model fields, and their supersymmetric partners by assigning to each particle the $R$-parity quantum number $R = (-1)^{3B+L+2S}$, where $B$, $L$, and $S$ denote the baryon number, the lepton number, and the spin of the corresponding particle, respectively. For conserved $R$-parity, the lightest supersymmetric particle (LSP) is stable. Thus, the LSP is a compelling candidate for dark matter, provided that it does not have electromagnetic or strong interactions.

The gravitino $\tilde{G}$ is a particularly attractive candidate for such an LSP. Any supersymmetric theory containing gravity predicts the existence of the gravitino, a spin-3/2 particle which acquires a mass from the spontaneous breakdown of supersymmetry. As the superpartner of the graviton, it is extremely weakly interacting. Hence, if the gravitino is the LSP, it can be dark matter.

At high temperatures, gravitinos are generated in inelastic scattering processes with particles that are in thermal equilibrium with the hot primordial SUSY plasma. Assuming that inflation governed the earliest moments of the Universe, any initial population of gravitinos must be diluted away by the exponential expansion during the slow-roll phase.
We consider the regeneration of gravitinos that starts after completion of reheating. The calculation of the production rate of these thermally produced gravitinos requires a consistent finite-temperature approach. A result that is independent of arbitrary cutoffs was derived for supersymmetric quantum chromodynamics (QCD) in Ref. [2]. Following this approach, we provide for the first time the complete Standard Model gauge group result to leading order in the gauge couplings. For gravitino dark matter scenarios, this allows us to calculate the relic density of thermally produced gravitinos, $\Omega^{TP}_G$. The comparison of this density with the observed dark matter density is crucial to decide whether gravitinos can explain dark matter.

If the gravitino is not the LSP, it can decay at late times. The late decays of thermally produced gravitinos can then affect the abundances of light elements during big bang nucleosynthesis (BBN). Thus, the new results will also serve as central input parameters for deriving cosmological constraints for scenarios with unstable gravitinos.

Thermal gravitino production becomes very efficient if the reheating temperature $T_R$ of the Universe after inflation is high. Thus, gravitinos play an important role in models where thermal leptogenesis explains the matter-antimatter asymmetry of the Universe. At the dawn of the Large Hadron Collider era, we face the exciting possibility to confirm SUSY directly in experiments. In fact, we show in this thesis that a conceivable determination of the gravitino mass at future colliders will allow for a unique test of the viability of thermal leptogenesis in the laboratory.

This thesis is organized as follows. Chapter 2 discusses the general supergravity Lagrangian. We show how to obtain an effective low-energy theory. This allows us to derive the complete set of Feynman rules that is necessary for the subsequent calculations. In Chapter 3 we identify all processes for thermal gravitino production to leading order in the Standard Model gauge couplings. This yields the complete $\text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y$ result for the thermal gravitino production rate. Chapter 4 covers phenomenological implications of the new result for cosmology. In particular, for gravitino dark matter scenarios, we provide $\Omega^{TP}_G$ and derive an upper limit on the reheating temperature of the Universe. In Chapter 5 we propose the collider test of the viability of thermal leptogenesis. Here, we take into account also the non-thermal gravitino production from decays of the next-to-lightest supersymmetric particle (NLSP) into the gravitino LSP.
Chapter 2

From Supergravity to Gravitino Phenomenology

The natural scale of supergravity is the Planck scale. For our purposes, however, we need to find a low-energy version which is reconcilable with phenomenology. On this quest we shall keep track of the gravitino interactions. This chapter therefore provides a trail from supergravity to gravitino phenomenology.

After briefly sketching how the gravitino emerges in supersymmetric theories, we quote the general form of the supergravity Lagrangian\(^1\) in four spacetime dimensions and discuss its properties. We explain the conditions for spontaneous supersymmetry breaking. For a concrete model of gravity-mediated supersymmetry breaking, we carry out the transition to the low-energy effective theory relevant for gravitino phenomenology. We relate the low-energy supergravity Lagrangian with the Minimal Supersymmetric Standard Model (MSSM) in the high-energy limit of unbroken electroweak symmetry. Having identified the couplings of the gravitino to the fields of the MSSM, we provide the complete set of Feynman rules necessary for the calculations in this thesis.

\(^1\)We only consider unextended $N = 1$ supersymmetry.
2.1 Local Supersymmetry

Supersymmetry is a spacetime symmetry. It relates the bosonic and fermionic degrees of freedom in a supermultiplet of particles:

\[ Q | \text{boson} \rangle \simeq | \text{fermion} \rangle, \quad Q | \text{fermion} \rangle \simeq | \text{boson} \rangle, \quad (2.1) \]

The anti-commuting Weyl spinor \( Q \), which is the generator of the SUSY transformation, extends the Poincaré algebra to a graded Lie algebra. The generators \( Q \) fulfill the fundamental commutator \([\cdot, \cdot]\) and anti-commutator \(\{\cdot, \cdot\}\) relations\(^2\)

\[ \{Q_\alpha, \overline{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu, \quad (2.2a) \]
\[ \{Q_\alpha, Q_\beta\} = \{\overline{Q}_\dot{\alpha}, \overline{Q}_\dot{\beta}\} = 0, \quad (2.2b) \]
\[ [P_\mu, Q_\alpha] = [P_\mu, \overline{Q}_\dot{\beta}] = 0. \quad (2.2c) \]

Let \( \epsilon \) and \( \eta \) be two infinitesimal Weyl spinors which parameterize the supersymmetry transformations. We then can write (2.2a) in terms of a commutator:

\[ [\epsilon Q, \eta \overline{Q}] = 2 \epsilon \sigma^\mu \eta P_\mu. \quad (2.3) \]

In a globally supersymmetric theory, the parameters \( \epsilon \) and \( \eta \) are spacetime independent, corresponding to a rigid translation. Gauging supersymmetry corresponds to making the supersymmetry parameters local, i.e., dependent on \( x \). Then, the commutator of two local supersymmetry transformations yields translations \( \sim \epsilon(x) \sigma^\mu \eta(x) \) which differ from point to point. Invariance under these general coordinate transformations is exactly what we expect from a theory of gravity. Local supersymmetry is therefore referred to as supergravity (SUGRA)\(^3\).

The gravitino is a central element of SUGRA because it is the gauge field of local supersymmetry transformations. Since supergravity is a non-renormalizable theory, operators of mass dimension five and higher occur. The gravitino thus is an extremely weakly interacting particle with couplings suppressed by inverse powers of the reduced Planck mass

\[ M_P = \frac{1}{\sqrt{8\pi G_N}} = 2.4 \times 10^{18} \text{ GeV}, \quad (2.4) \]

where \( G_N \) is Newton’s gravitational constant [5].

\(^2\)The conventions on spinor notation are found in Appendix A

\(^3\)Good reviews about supergravity are [3] and [4]. For further reading see references therein.
2.2 The Supergravity Lagrangian

Our starting point of our analysis is the Lagrangian given in Appendix G in the book of Wess and Bagger [6]. We rewrite it in terms of four-component spinors and adopt it to our conventions given in Appendix A. This involves a change of signature in the Lorentz metric from \((-,-,+,+)\), used by Wess and Bagger, to the signature more commonly used in high-energy physics \((+,-,-,-)\). With our conventions, we can carry out this task in a safe way. We explicitly restore units of \(M_P\) so that the supergravity Lagrangian in the four-component formalism then reads:

\[
\frac{1}{e^2} L = -\frac{M_P^2}{2} R + g_{i\bar{j}} \bar{\psi}_L \gamma^\mu \psi_L D^\mu \phi^i - \frac{1}{2} g^2 \left( (\text{Ref})^{-1} \right)^{ab} D_a D_b \\
+ g_{i\bar{j}} \bar{\chi}_L \gamma^\mu \gamma^\nu \psi_L D^\mu \psi_L + e_{\mu \nu \rho \sigma} \bar{\psi}_L \gamma^\mu \psi_L D^\nu \psi_L \\
- \frac{1}{4} \text{Ref}_{ab} F^a_{\mu \nu} F^b_{\mu \nu} + \frac{1}{8} e_{\mu \nu \rho \sigma} \text{Im} f_{ab} F^a_{\mu \rho} F^b_{\sigma} \\
+ \frac{i}{2} \text{Ref}_{ab} \bar{\chi}^a \gamma^\mu \chi^b - e^{-1} \frac{1}{2} \text{Im} f_{ab} D_{\mu} \left[ e\bar{\chi}^a \gamma^\mu \chi^b \right] \\
+ \left[ - \sqrt{2} g \partial_i D_a \bar{\chi}^a \chi^i + \frac{1}{4} \sqrt{2} g \left( (\text{Ref})^{-1} \right)^{ab} \partial_i f_{bc} D_a \bar{\chi}^a \chi^i \\
- \frac{i}{16} \sqrt{2} \partial_i f_{ab} [\gamma^\mu, \gamma^\nu] \chi^i F_{\mu \nu} - \frac{1}{2 M_P^2} g D_a \bar{\chi}^a \gamma^\mu \psi_\mu \\
- \frac{i}{2 M_P} \sqrt{2} g_{i\bar{j}} \bar{\psi}_L \gamma^\mu \gamma^\nu \chi^i \chi^j + \text{h.c.} \right] \\
- \frac{i}{8 M_P} \text{Ref}_{ab} \bar{\psi}_L [\gamma^m, \gamma^n] \chi^a \chi^b F_{mn} \\
- e^{K/2 M_P^2} \left[ \frac{1}{4 M_P^2} W^{*} \bar{\psi}_L \gamma^\mu D_{\mu} \psi_L + \frac{1}{2 M_P} \sqrt{2} D_i W \bar{\psi}_L \gamma^\mu \chi^i \\
+ \frac{1}{2} D_i D_j W \bar{\chi}^i \chi^j + \frac{1}{4} g^{i\bar{j}} D_j W \bar{\psi}_L \gamma^\mu \chi^i + \text{h.c.} \right] \\
- e^{K/ M_P^2} \left[ g^{i\bar{j}} (D_i W) (D_j W^*) - \frac{3}{M_P^2} |W|^2 + \mathcal{O}(M_P^{-2}) \right]. \tag{2.5}
\]
2.2. The Supergravity Lagrangian

Let us briefly introduce the building blocks of this Lagrangian:

**Matter sector** Matter fermions are described in terms of left-handed four-spinors

\[ \chi^i_L = \begin{pmatrix} (\chi_\alpha)_{W/B} \\ 0 \end{pmatrix}, \tag{2.6} \]

where \((\chi_\alpha)_{W/B}\) stands for the two-component Weyl spinor. Here and in the following, the subscript \(W/B\) indicates the quantities used in the book by Wess and Bagger [6]. Since the matter sector of the supergravity Lagrangian is built up in terms of left-chiral superfields, only left-handed four-spinors \(\chi^i_L\) appear in the resulting Lagrangian in the four-component notation. The corresponding scalar superpartners are denoted as \(\phi^i\). The index \(i\) runs over all chiral superfields.

**Gauge sector** The gauge multiplet consists of gauge bosons \(A^a_\mu\) and their superpartners, the gauginos

\[ \lambda^a = \begin{pmatrix} -i(\lambda^a_\alpha)_{W/B} \\ i(\lambda^{a\dot{\alpha}})_{W/B} \end{pmatrix}, \tag{2.7} \]

which are Majorana fields. Both are in the adjoint representation of the gauge group, \(a, b, \cdots = 1, \ldots, \text{dim} G\). The associated field strengths of the gauge fields \(A^a_\mu\) are written as \(F^a_\mu\). The auxiliary fields \(D_a\) are generalizations of the \(D\)-terms in the vector supermultiplets of a globally supersymmetric theory.

**Gravity sector** The graviton \(e_\mu^m\) shows up implicitly as the determinant of the vielbein \(e = \det e^m_\mu\) and the curvature scalar \(R\). Flat spacetime indices of the local Lorentz frame are denoted by \(m, n, \ldots\); Einstein indices by \(\mu, \nu, \ldots\). The gravitino is a spin-3/2 field which is written in terms of the Majorana vector-spinor,

\[ \psi_\mu = \begin{pmatrix} -i(\psi_\mu\alpha)_{W/B} \\ i(\psi_\mu^{\dot{\alpha}})_{W/B} \end{pmatrix}. \tag{2.8} \]

Since it is the superpartner of the graviton, the gravitino is massless for exact local supersymmetry. Note, that we include factors of \(i\) in the definition of the gravitino as well as for the gauginos in (2.7).

**Gauge kinetic function** The gauge kinetic function \(f_{ab}(\phi)\) is a dimensionless analytic function in the scalars \(\phi\). In the superfield approach, it multiplies the kinetic term for the vector supermultiplet. In the component version (2.5) of the supergravity Lagrangian, it therefore shows up as a prefactor in the kinetic terms of the gauginos.
and gauge bosons,

\[
\begin{align*}
\frac{i}{2} \text{Re} f_{ab} \lambda^a \gamma^\mu \partial_\mu \lambda^b, \\
-\frac{1}{4} \text{Re} f_{ab} F^a_{\mu\nu} F^{b,\mu\nu},
\end{align*}
\]

(2.9a)

(2.9b)

respectively. Because of gauge invariance, it transforms under gauge transformations as the symmetric product of adjoint representations of the gauge group \(G\).

Note that derivatives of the dimensionless gauge kinetic function \(f_{ab}(\phi)\) with respect to the scalars \(\phi^i\),

\[
\partial_i f_{ab} \equiv \frac{\partial f_{ab}}{\partial \phi^i},
\]

(2.10)

have negative mass dimension. In the course of spontaneous symmetry breaking, it is reasonable to consider models where \(\partial_i f_{ab} = \mathcal{O}(M_P^{-1})\). Therefore, we have not explicitly written out terms in the Lagrangian (2.5) which are proportional to \(\propto M_P^{-1} \partial_i f_{ab}\) since they are of order \(\mathcal{O}(M_P^{-2})\).

**Superpotential** The superpotential is an analytic function in the chiral scalar fields. It has mass dimension three and its general form is restricted by gauge invariance,

\[
W = W_h(h) + \frac{1}{2} \mu_{ij}(h) \phi^i \phi^j + \frac{1}{6} y_{ijk}(h) \phi^i \phi^j \phi^k + \mathcal{O}(M_P^{-1}).
\]

(2.11)

Here, we distinguish between the *observable* part of the superpotential and the *hidden* superpotential \(W_h(h)\). The latter depends only on hidden scalar fields \(h\), which have no or only very small couplings to ordinary matter and gauge fields of the *observable sector*. The simplest choice is a superpotential with separated hidden and observable sectors

\[
W = W_h(h) + W_o(\phi).
\]

(2.12)

Let us remark here that a hidden field dependence \(\mu_{ij}(h)\) which mixes observable and hidden sector can be crucial in models offering a dynamical solution to the \(\mu\) problem [7, 8].

**Kähler structure** The Kähler potential \(K(\phi, \phi^*)\) is a real-valued function in the chiral scalar fields and has mass dimension two. It has the generic form:

\[
K = K_h(h, h^*) + \alpha_{ij}(h, h^*) \phi^i \phi^{*i} + [Z_{ij}(h, h^*) \phi^i \phi^j + h.c.] + \mathcal{O}(M_P^{-1}),
\]

(2.13)

where \(K_h\) denotes the hidden part of the Kähler potential that is independent of observable fields.
In fact, the theory is endowed with a rich Kähler structure. One can think of the scalars \( \phi^i \) as coordinates of a Kähler manifold whose metric \( g_{ij}^* \) can be expressed by the second derivatives of the Kähler potential \( K(\phi, \phi^*) \):

\[
g_{ij}^* = \frac{\partial^2 K}{\partial \phi^i \partial \phi^j} \equiv \partial_i \partial_j K \tag{2.14}
\]

with its inverse \( g^{ij*} \), i.e., \( g^{ij*} g_{jk} = \delta^i_k \). The Kähler connection is given by

\[
\Gamma^k_{ij} = g^{kl} \partial g_{jl}^*/\partial \phi^i . \tag{2.15}
\]

For example, the Kähler metric enters in the chiral sector in terms of a prefactor for the kinetic terms of the chiral fermions

\[
ig_{ij}^* \gamma^\mu p_\mu \chi^i_L . \tag{2.16}
\]

The Kähler covariant derivatives of the superpotential read

\[
D_i W = W_i + M_p^{-2} K_i W , \tag{2.17a}
\]

\[
D_i D_j W = W_{ij} + M_p^{-2} (K_{ij} W + K_i D_j W + K_j D_i W) - \Gamma^k_{ij} D_k W + O(M_p^{-3}) , \tag{2.17b}
\]

where \( K_i = \partial_i K, W_i = \partial_i W, K_{ij} = \partial_i \partial_j K, \) and \( W_{ij} = \partial_i \partial_j W \). Beside being invariant under local supersymmetry- and gauge transformations, (2.5) is inert under Kähler transformations with arbitrary holomorphic functions \( F(\phi) \)

\[
K(\phi, \phi^*) \rightarrow K(\phi, \phi^*) + F(\phi) + F^*(\phi^*) , \tag{2.18a}
\]

\[
W \rightarrow e^{-F(\phi)/M_p^2} W , \tag{2.18b}
\]

provided that the spinor fields undergo \( F \)-dependend Weyl rotations

\[
\chi^i_L \rightarrow e^{\frac{i}{2} \text{Im} F/M_p^2} \gamma_5 \chi^i_L , \tag{2.19a}
\]

\[
\chi^a \rightarrow e^{-\frac{i}{2} \text{Im} F/M_p^2} \gamma_5 \chi^a , \tag{2.19b}
\]

\[
\psi_{\mu} \rightarrow e^{\frac{i}{2} \text{Im} F/M_p^2} \gamma_5 \psi_{\mu} . \tag{2.19c}
\]

It is easily seen that the metric \( g_{ij}^* \) and the Kähler connection \( \Gamma^k_{ij} \) remain unchanged under a Kähler transformation (2.18). More generally, the isometries of the Kähler manifold can be expressed by the real Killing potentials \( D_a(\phi, \phi^*) \). The corresponding (holomorphic) Killing vector fields are given by

\[
X^i_a = -ig^{ij*} \partial_j D_a , \tag{2.20a}
\]

\[
X^a_{ij} = ig^{ij*} \partial_i D_a . \tag{2.20b}
\]
In the Lagrangian (2.5) we used this to express the Killing vector fields in terms of the Killing potentials. In principle, the Killing potentials $D_a$ can be found by solving the corresponding Killing equation. We quote their explicit form later in (2.35), where we restrict ourselves to flat Kähler manifolds.

The physical fields $A^a_\mu$ and $\lambda^a$ in the vector multiplet are written with upper gauge indices $a, b, \ldots$ while the Killing potentials $D_a$ have lower indices. To bring them in canonical form, they are lowered or raised with $\text{Ref}_{ab}$ or $[(\text{Ref})^{-1}]^{ab}$, respectively.

Furthermore, the supergravity Lagrangian above is given for a simple gauge group only. An extension to a non-simple gauge group $G = \prod_\alpha G_\alpha$ like $SU(3)_c \times SU(2)_L \times U(1)_Y$ requires to introduce an additional index. We will do so in Sec. 2.6.

The covariant derivatives are defined as
\begin{align}
D_\mu \phi^i &= \partial_\mu \phi^i + ig g^{ij} \partial_j D_a A^a_\mu, \\
D_\mu \chi^i_L &= \partial_\mu \chi^i_L + \Gamma^i_{jk} D_\mu \phi^j \chi^k_L \\
&\quad + ig \partial_k \left( g^{ij} \partial_j D_a \right) A^a_\mu \chi^k_L + \mathcal{O}(M_p^{-2}), \\
D_\mu \chi^a &= \partial_\mu \chi^a - g f^{abc} A^b_\mu \chi^c + \mathcal{O}(M_p^{-2}), \\
D_\mu \psi_\nu &= \partial_\mu \psi_\nu + \mathcal{O}(M_p^{-2}).
\end{align}

The Lagrangian (2.5) is invariant under local supersymmetry transformations, parametrized by means of an anticommuting Majorana spinor $\zeta(x) = \begin{pmatrix} (\zeta(x)_a)_{W/B} \\ (\bar{\zeta}(x)_{\dot{a}})_{W/B} \end{pmatrix}$ (2.22)
of mass dimension $-1/2$, namely,
\begin{align}
\delta \zeta e^m_\mu &= \frac{1}{M_p} (\bar{\zeta} \gamma^m \psi_{\mu R} - \bar{\zeta} \gamma^m \psi_{\mu L}), \\
\delta \zeta \phi^i &= \sqrt{2} \bar{\zeta} \chi^i_L, \\
\delta \zeta \chi^i_L &= -i \sqrt{2} \gamma^i \zeta R D_\mu \phi^i - \Gamma^i_{jk} \delta \zeta \phi^j \chi^k_L \\
&\quad - \sqrt{2} e^{K/2M_p^2} g^{ij} D_\mu W^* \chi^j_L + \mathcal{O}(M^{-1}) \\
\delta \zeta A^a_\mu &= \bar{\zeta} \gamma^a \gamma_{\mu R} - \bar{\zeta} \gamma^a \chi^a_\mu \\
\delta \zeta \lambda^a_L &= \frac{1}{4} \bar{\zeta} \gamma^a \left[ \gamma^\mu, \gamma^\nu \right] \zeta_\nu - ig \left[ (\text{Ref})^{-1} \right]^{ab} D_a \zeta_L + \mathcal{O}(M^{-1}) \\
\delta \zeta \psi_\mu &= 2 M_p D_\mu \zeta + \frac{i}{M_p} e^{K/2M_p^2} (W \gamma_\mu \zeta_R + W^* \gamma_\mu \zeta_L) + \mathcal{O}(M^{-1})
\end{align}

\footnote{We drop the contributions from the spin-connection $\omega_\mu^{\,mn}$ in the covariant derivatives of the fermion fields since we will only consider flat spacetime in this thesis for which $\omega_\mu^{\,mn} \rightarrow 0$.}
Let us stress that the Lagrangian (2.5) depends on two arbitrary functions of the chiral scalars, namely, the Kähler function

$$G(\phi, \phi^*) = \frac{K(\phi, \phi^*)}{M_B^2} + \ln \frac{|W(\phi)|^2}{M_B^2},$$

and the gauge kinetic function $f_{ab}(\phi)$. This is a consequence of the Kähler-Weyl invariance and can be seen by virtue of (2.18) with $F(\phi) = \ln W(\phi)$. Within the framework of supergravity, there is no mechanism which tells us one particular form of $G$ and $f_{ab}$ from the other. On the other hand, phenomenological considerations will suggest a certain minimal choice as described below.

### 2.3 Supersymmetry Breaking

Particles within the same supermultiplet are degenerate in mass, because

$$[P^2, Q_\alpha] = [\overline{P}^2, \overline{Q}_\dot{\alpha}] = 0.$$  

Since we do not yet have experimental evidence for supersymmetry, we know that it has to be a broken symmetry if realized in nature.

The condition for the spontaneous breakdown of a symmetry is a ground state that does not respect this symmetry. In the case of local supersymmetry, we find from the supersymmetry transformations (2.23) that the only Lorentz invariant ways to achieve this are\footnote{We do not consider the possibility of fermion-antifermion condensates which require the presence a strong gauge coupling force.}:

$$\langle \delta \zeta \chi^i_L \rangle \propto \langle F^i \rangle \zeta \neq 0 \quad \text{F-term breaking} \quad (2.26a)$$

and/or

$$\langle \delta \zeta \lambda^a \rangle \propto \langle D_\alpha \rangle \xi \neq 0 \quad \text{D-term breaking}. \quad (2.26b)$$

Here, $\langle \cdot \rangle$ denotes the vacuum expectation value (VEV) and $F^i$ is given by

$$F^i = e^{K/2M_B^2} g^{ij^*} D_j^* W^*. \quad (2.27)$$

This is a generalization of the auxiliary fields $F_i^{\text{glob}} = -\partial_s W^*$ which are part of the chiral supermultiplets in a globally supersymmetric theory.

Many models of spontaneous symmetry breaking have been proposed. Common to these models is that supersymmetry breaking occurs in the hidden sector of particles.
Supersymmetry breaking is then communicated to the observable sector either at tree level or radiatively. In the course of spontaneous SUSY breaking, the gravitino acquires a mass $m_G$. Depending on the SUSY breaking scheme, $m_G$ can range from the eV scale up to scales beyond the TeV region [9].

### 2.4 Gravity-Mediated Supersymmetry Breaking

An appealing scenario of supersymmetry breaking is gravity-mediation. We show how phenomenological considerations can lead to certain minimal choices of the gauge kinetic function and the Kähler potential in such a scenario. Furthermore, we illustrate the super-Higgs mechanism and show how soft supersymmetry-breaking terms arise in the low-energy limit of supergravity. Finally, we also comment on the way Yukawa couplings emerge.

Supersymmetry is broken by VEVs $\langle F^m \rangle$ of some scalar fields $h$ which are gauge singlets. The hidden fields $h$ appear in non-renormalizable terms of mass dimension higher than four, which are suppressed by powers of $M_P$. We expect soft terms of order

$$m_{\text{soft}} \sim \frac{\langle F \rangle}{M_P}$$

(2.28)

to arise. This is intuitive since $m_{\text{soft}}$ has to vanish in the limiting case of unbroken supersymmetry $\langle F \rangle \to 0$. In order to obtain $m_{\text{soft}}$ in the electroweak range, supersymmetry breaking has to occur on rather high scales, $\sqrt{\langle F \rangle} \sim O(10^{10} \text{ GeV})$.

In the following we will sketch such a scenario and derive a low-energy limit of the supergravity Lagrangian (2.5). The phenomenological imperative for any concrete supersymmetry-breaking scenario is that it delivers a supersymmetric theory equipped with terms that softly break it. The soft terms obtained from our chosen gravity mediation scenario will have specific sizes which might differ significantly from other breaking scenarios. Nevertheless, their structure will be the same. In the end, we therefore can treat the soft parameters as free parameters (to be determined experimentally) and forget about the details of the underlying breaking scenario. What we will have gained is a consistent set of Feynman rules for the MSSM in the high-energy limit together with a set of gravitino rules which share the same convention.

Let us start with the gauge kinetic function. The trivial choice $f_{ab} = \delta_{ab}$ would render the kinetic term of the gauginos (2.9a) and the kinetic term of the gauge bosons (2.9b) renormalizable. This form, however, is phenomenologically disfavored since we cannot produce gaugino masses at the tree level.\(^6\) The candidate for the gaugino mass term in

\(^6\)A tree-level gaugino mass can also arise from $D$-term breaking which we do not consider in this
2.4. Gravity-Mediated Supersymmetry Breaking

(2.5) is
\[ \frac{1}{4} e^{K/2M_p^2} g^{ij} \partial^2 D_j W^* \partial_{ij} \bar{\lambda}^a R_{\lambda L}^b + h.c. , \] (2.29)
which vanishes for \( f_{ab} = \delta_{ab} \). We can obtain a diagonal gaugino mass matrix with the generic ansatz:

\[ f_{ab} = \delta_{ab} f(h) . \] (2.30)

After some hidden field \( h \) has acquired a VEV \( \langle h \rangle \) at some high scale, we obtain canonically normalized gauge kinetic terms by a rescaling

\[ \hat{\lambda}^a = \sqrt{\langle \text{Re} f \rangle} \lambda^a \] (2.31a)
\[ \hat{A}_\mu^a = \sqrt{\langle \text{Re} f \rangle} A_\mu^a. \] (2.31b)

Rewriting (2.5) in terms of \( \hat{\lambda}^a \) and \( \hat{A}_\mu^a \), we see immediately that \( 1/\sqrt{\langle \text{Re} f \rangle} \) appears in the covariant derivatives as prefactor of the gauge coupling \( g \). This is remarkable because it suggests that \( \text{Re} f \) plays the role of a coupling constant. The rescaling

\[ \hat{g} = \frac{g}{\sqrt{\langle \text{Re} f \rangle}} \] (2.32)

then completely hides any dependence on \( \text{Re} f_{ab} \) in (2.5). If we further assume that \( \langle h \rangle \) is real, all terms proportional to \( \text{Im} f_{ab} \) vanish in (2.5). For our purposes, this assumption is safe since only terms of \( O(M_p^{-1}) \) with a gravitino in the vertex will be considered; and an operator \( \propto M_p^{-1} \text{Im} f_{ab} \psi_\mu \) is absent in the Lagrangian (2.5).

The term (2.29) contains also the superpotential \( W \) and the Kähler potential \( K \). The general form of the Kähler potential is specified in (2.13). There, a diagonal choice \( \alpha_{ij}(h,h^*) = \alpha(h,h^*) \delta_{ij} \) is phenomenologically favored. A non-diagonal form of \( \alpha_{ij} \) leads to off-diagonal terms in the the scalar mass matrix in the low-energy limit. This can induce flavor-changing neutral currents for which strong bounds exist [9].

In the case of the MSSM emerging as a low-energy effective theory, we can have a contribution like \( [Z(h,h^*)H_1H_2 + h.c.] \) in the Kähler potential where \( H_1 \) and \( H_2 \) denote the the two \( SU(2)_L \) Higgs doublets. The hidden field dependence of \( Z \) is closely connected to the \( \mu \) problem [7, 8], but is beyond the scope of this thesis. A particularly appealing choice of the Kähler potential is therefore given by

\[ K = K_0(h,h^*) + \sum_i \phi_i^i \phi_i^i + [Z(h,h^*)H_1H_2 + h.c.] \] (2.33)

with \( \alpha_{ij} = \delta_{ij} \) [see Eqn. (2.13)], corresponding to a flat Kähler manifold in the observable sector. With this choice, the Kähler metric (2.14) becomes trivial,

\[ g_{ij}^* = \delta_{ij}^* \] , (2.34)
and leads to canonical kinetic terms in (2.16). Moreover, the connection coefficients (2.15) vanish: $\Gamma^k_{ij} = 0$. The Killing potentials $D_a$ then coincide with the $D$-terms of a globally supersymmetric Yang-Mills theory:

$$D_a = \phi^i T_{a,ij} \phi^j.$$  

(2.35)

The generators of the Lie algebra of the gauge group are denoted by $T_a$. They are chosen to be hermitian and do obey the commutator relation

$$[T_a, T_b] = i f^{abc} T_c$$  

(2.36)

with the structure constants $f^{abc}$.

**Super-Higgs mechanism**

In supergravity, an analogon to the Higgs mechanism of electroweak-symmetry breaking exists. When supergravity is spontaneously broken, the corresponding massless Goldstone fermion, or goldstino, is absorbed by the gravitino which acquires thereby its $\pm 1/2$ helicity components.

Mass terms for the gravitino, i.e., terms in (2.5) involving the gravitino field, which are quadratic in the fermionic fields and do not have derivative couplings are

$$- \frac{1}{4 M_P^2} e^{K/2M_P^2} W^* \overline{\psi}_R \gamma^\mu \psi_L \nu - \frac{1}{2 M_P} e^{K/2M_P^2} \sqrt{2} D_i W \overline{\psi}_\mu \gamma^\mu \chi^i_L + h.c.$$  

(2.37)

Since hidden sector fields do not share any gauge interactions, there are no $D$-terms contributing to supersymmetry breaking via $\langle D_a \rangle \neq 0$. Therefore, we do not consider

$$- \frac{1}{2 M_P} g D_a \overline{\chi}^a_R \gamma^\mu \psi_\mu + h.c.$$  

(2.38)

as a potential mass term.

Because of their hidden field dependence, the gauge kinetic function, the superpotential, and Kähler potential will get vacuum expectation values $\langle f_{ab} \rangle$, $\langle K_h \rangle$, and $\langle W_h \rangle$, respectively. From the first term in (2.37) we can read off the gravitino mass after spontaneous symmetry breaking:

$$m_G = \frac{1}{M_P^2} e^{\langle K_h \rangle/2M_P^2} \langle W_h^* \rangle.$$  

(2.39)

The second term in (2.37) mixes the gravitino $\psi_\mu$ with chiral fermions $\chi^i_L$. Let us define the spinor

$$\eta_L = D_i W \chi^i_L$$  

(2.40)

\footnote{In case of $D$-term breaking, the definition (2.40) of the goldstino has to be extended in order to remove the mixing in (2.38).}
and apply the supersymmetry transformation (2.23). We see that $\eta$ changes by a shift
\[ \delta\xi_{\eta L} = -\sqrt{2}e^{K/M_P^2}g^{ij^*}(D_iW)(D_j^*W^*) + \cdots = -3\sqrt{2}m_\xi\xi + \cdots, \quad (2.41) \]
where in the last equality we have assumed a vanishing cosmological constant as explained below. This implies that
\[ \eta = D_iW\chi^i_L + D_i^*W^*(\chi^i_L)^c \quad (2.42) \]
is the Goldstone fermion. Indeed, with the choice $\zeta = \eta\sqrt{2}/(6m_\xi)$, we can choose a unitary gauge where $\eta$ transforms to zero. This removes the mixing term in (2.37).

**Soft terms from the scalar potential**

The scalar potential in the supergravity Lagrangian (2.5) reads
\[ V = e^{K/M_P^2}\left[g^{ij^*}(D_iW)(D_j^*W^*) - 3\frac{|W|^2}{M_P^2}\right] + \frac{1}{2}g^2D^2a \]
\[ = g^{ij^*}F^iF^j^* - 3e^{K/M_P^2}\frac{|W|^2}{M_P^2} + \frac{1}{2}g^2D^2a, \quad (2.43) \]
where the rescaled coupling (2.32) is used. We have already seen that successful symmetry breaking is achieved if some of the auxiliary fields $F^i$ (2.27) get a non-zero vacuum expectation value
\[ \langle F^i \rangle = \left\langle e^{K/2M_P^2}g^{ij^*}(D_jW)(D_j^*W^*) \right\rangle \neq 0. \quad (2.44) \]
While the scalar potential
\[ V_{\text{glob}} = FF^* + 1/2D^2 \quad (2.45) \]
is positive semi-definite in global supersymmetry, the minus sign of the second term in (2.43) offers the appealing possibility to break supersymmetry with vanishing cosmological constant:
\[ \langle V \rangle = \left\langle g^{ij^*}(D_iW)(D_j^*W^*) - 3\frac{|W|^2}{M_P^2} \right\rangle = 0. \quad (2.46) \]
We therefore can work in the limit of flat Minkowski spacetime and neglect the interactions of the graviton: $R \to 0$ and $e \to 1$. Moreover, we drop the distinction between flat and curved spacetime indices. Because of notational habit, we denote from now on Minkowski spacetime indices with $\mu, \nu = 0, \ldots, 3$.

For the sake of simplicity, we further assume that we do not have any mixing between the observable and hidden sectors in the superpotential and the Kähler potential. For the superpotential, this corresponds to the choice (2.12). For the Kähler potential, we write
\[ K = K_h(h, h^*) + \sum_i \phi^i\phi^{*i}. \quad (2.47) \]
It is instructive to get a feeling for the scales involved:

\[
\langle h \rangle \sim \mathcal{O}(M_\mathcal{P}), \quad \langle W_h \rangle \sim \mathcal{O}(M_\mathcal{P}^2), \quad \langle K_h \rangle \sim \mathcal{O}(M_\mathcal{P}^2), \quad \langle \partial_m W_h \rangle \sim \mathcal{O}(M_\mathcal{P}). \tag{2.48}
\]

The second and third relations come from the requirement that we want to obtain a gravitino mass \(m_\tilde{G} \lesssim \text{few TeV}\). This requirement is crucial because \(m_\tilde{G}\) will govern the size of the soft terms. The fourth relation is a consequence of the second.

In the following we will use indices \(m, n \ldots\) for fields \(h\) which we allow to acquire a VEV and indices \(i, j \ldots\) for observable fields \(\phi\). Using the definitions (2.17) for the Kähler covariant derivatives, the scalar potential (2.43) without the D-term becomes

\[
V_F = e^{K/M_\mathcal{P}^2} \left[ \sum_i \left( \frac{\partial W_o}{\partial \phi^i} + \frac{\phi^{*i}}{M_\mathcal{P}} (W_o + W_h) \right)^2 + \sum_m \left( \frac{\partial W_h}{\partial h^m} + \frac{1}{M_\mathcal{P}} \frac{\partial K_h}{\partial h^m} (W_o + W_h) \right)^2 - 3 \frac{|W_o + W_h|^2}{M_\mathcal{P}^2} \right]. \tag{2.49}
\]

We expand this to

\[
V_F = e^{K_h/M_\mathcal{P}^2} \left[ \sum_i \left( \frac{\partial W_o}{\partial \phi^i} + \frac{|W_h|^2}{M_\mathcal{P}^4} \phi^{*i} \phi^i + \frac{W_h^* \partial W_o}{M_\mathcal{P}^2} \frac{\partial \phi^i}{\partial \phi^*} + \frac{W_h \partial W_o^*}{M_\mathcal{P}^2} \frac{\partial \phi^{*i}}{\partial \phi^*} \right) + \sum_m \left( \frac{1}{M_\mathcal{P}^4} \frac{\partial K_h}{\partial h^m} \frac{\partial K_h}{\partial h^m} (W_o W_h^* + W_o^* W_h) + \frac{W_o^*}{M_\mathcal{P}^2} \frac{\partial K_h}{\partial h^m} \frac{\partial W_h^*}{\partial h^m} \frac{\partial h^m}{\partial h^m} + \frac{W_h}{M_\mathcal{P}^2} \frac{\partial K_h}{\partial h^m} \frac{\partial W_o^*}{\partial h^m} \right) - \frac{3}{M_\mathcal{P}^2} \frac{W_o W_h^*}{M_\mathcal{P}^2} - 3 \frac{W_h^* W_h}{M_\mathcal{P}^2} \right] + \mathcal{O}(M_\mathcal{P}^2) + \mathcal{O}(M_\mathcal{P}^{-1}). \tag{2.50}
\]

To get from (2.49) to (2.50), we have expanded the exponential of the observable sector in powers of \(M_\mathcal{P}^{-2}\).

Now we can perform flat limit of supergravity where one sends the Planck mass \(M_\mathcal{P}\) to infinity but holds the gravitino mass \(m_\tilde{G}\) (2.39) fixed. Terms of \(\mathcal{O}(M_\mathcal{P}^2)\) in (2.50) contain relative signs and have to be fine-tuned to zero for vanishing cosmological constant \(\langle V \rangle = 0\). With a rescaling of the observable superpotential,

\[
\tilde{W}_o = W_o \langle W_h^* \rangle \langle K_h \rangle^{2M_\mathcal{P}^2}, \tag{2.51}
\]

this yields

\[
V_F = \sum_i \left[ \left( \frac{\partial \tilde{W}_o}{\partial \phi^i} \right)^2 + m_G^2 \phi^{*i} \phi^i + m_\tilde{G} \left( \phi^{*i} \frac{\partial \tilde{W}_o}{\partial \phi^i} + \text{h.c.} \right) \right] - \left( 3m_\tilde{G} \tilde{W}_o + \text{h.c.} \right) + \sum_m \left[ m_\tilde{G} \langle \frac{\partial K_h}{\partial h^m} \rangle \left( \frac{1}{M_\mathcal{P}^4} \langle \frac{\partial K_h}{\partial h^m} \rangle + \frac{1}{\langle W_h \rangle} \langle \frac{\partial W_h^*}{\partial h^m} \rangle \right) \tilde{W}_o + \text{h.c.} \right]. \tag{2.52}
\]
In the first term of (2.52) we recover the scalar potential of global supersymmetry (2.45), where \( F^i_{\text{glob}} = -\partial_i W^* \). (We do not carry along the D-terms which are already shown to coincide with the globally supersymmetric case in (2.35).) From (2.27), we find

\[
\langle F^m \rangle = m\bar{G} M_P^2 \left( \frac{1}{M_P^2} \left\langle \frac{\partial K_h}{\partial h^m} \right\rangle + \frac{1}{\langle W_h \rangle} \left\langle \frac{\partial W_h^*}{\partial h^m} \right\rangle \right).
\] (2.53)

Let us identify the remaining parts in (2.52) with the commonly used notation:\

\[
\begin{align*}
m_0 &= m\bar{G}, \\
A &= m\bar{G} \sum_m \left\langle \frac{\partial K_h}{\partial h^m} \right\rangle \left\langle F^m \right\rangle M_P^2, \\
B &= m\bar{G} \sum_m \left\langle \frac{\partial K_h}{\partial h^m} \right\rangle \left\langle F^m \right\rangle - m\bar{G}.
\end{align*}
\] (2.54)

In Sec. 2.6 we will write down the superpotential of the MSSM. It is then not hard to show that \( A \) corresponds to the trilinear scalar coupling and that \( B \) is a bilinear mass parameter; \( m_0 \) is a universal scalar mass. The values of the above soft-supersymmetry breaking parameters are in the electroweak range. Since the gravitino mass \( m\bar{G} \) governs the size of the soft terms, the choices made in (2.48) turn out to be phenomenologically favorable.

**Gaugino mass term**

To complete the set of soft breaking parameters (2.54), we discuss now the gaugino mass term (2.29). Following the logic above, it is straightforward to perform the flat limit. One then finds in terms of the rescaled gaugino fields (2.31a):

\[
\frac{1}{2} M_{ab} \hat{\lambda}^a_R \hat{\lambda}^b_L + h.c.,
\] (2.55)

with the mass matrix [cf. (2.30)]:

\[
M_{ab} = \frac{1}{2} \left\langle \frac{\langle F^m \rangle}{\langle W(h) \rangle} \frac{\partial f(h)}{\partial h^m} \right\rangle \delta_{ab}.
\] (2.56)

From (2.53) together with (2.48), one finds that \( \langle F^m \rangle \sim \mathcal{O}(M_P) \) so that the gaugino masses (2.56) are in the electroweak range for

\[
\left\langle \text{Ref}(h) \right\rangle = \mathcal{O}(1) \quad \text{and} \quad \left\langle \frac{\partial f(h)}{\partial h^m} \right\rangle \sim \mathcal{O}(M_P^{-1}).
\] (2.57)

For example, a straightforward choice is \( f(h) = h/M_P \).

---

8The relation \( B = A - m\bar{G} \) is just a consequence of the simple breaking scenario which we have chosen. In general, one treats \( A \) and \( B \) as independent parameters.
The second relation in (2.57) justifies our dropping of terms \( \propto M_P^{-1} \partial_i f_{ab} \) in the Lagrangian (2.5), because these terms are of \( O(M_P^{-2}) \).

The flat limit, i.e., \( M_P \to \infty \) and \( m_G \) fixed, amounts to integrating out fields in the Lagrangian which are suppressed by powers of \( M_P \). We therefore expect that the resulting Lagrangian is valid at a high energy scale. Thus, the soft parameters (2.54) and (2.56) should be interpreted as boundary conditions for the renormalization group (RG) equations [10].

**Yukawa interactions and chiral fermion mass terms**

As far as spontaneous supersymmetry breaking is concerned, there is another term in (2.5) to address, namely,

\[
-\frac{1}{2} \epsilon^{K/2M_P^2} \mathcal{D}_i D_j \overline{W}_{ab} \chi^a_L \chi^j_L + \text{h.c.} \, .
\]  

The definition for the Kähler covariant derivatives is given in (2.17). Recalling the dimensional considerations (2.48), one finds that the only contribution for the observable sector fields that survives in the flat limit is

\[
-\frac{1}{2} \frac{\partial^2 \overline{W}_o}{\partial \phi^i \partial \phi^j} \chi^i_L \chi^j_L + \text{h.c.} \, .
\]  

Here we made use of the rescaling of the observable superpotential (2.51). This rescaling can be absorbed into the parameters of the superpotential (2.11),

\begin{align}
\hat{\mu}_{ij} &= \mu_{ij} \frac{\langle W^*_h \rangle}{\langle |W_h| \rangle} e^{\langle K_h \rangle/2M_P^2} , \\
\hat{y}_{ijk} &= y_{ijk} \frac{\langle W^*_h \rangle}{\langle |W_h| \rangle} e^{\langle K_h \rangle/2M_P^2} .
\end{align}

We see that the bilinear part of the superpotential (2.11) produces mass terms for chiral fermion fields with mass parameter \( \hat{\mu}_{ij} \) while Yukawa couplings \( \hat{y}_{ijk} \) arise from the trilinear part of the observable superpotential.

**2.5 The Free Gravitino Field**

In the preceding section we have seen that the gravitino acquires a mass \( m_G \) through the super-Higgs mechanism. From (2.5) together with (2.39), we find the Lagrangian for the free gravitino field

\[
\mathcal{L}_\psi = \epsilon^{\mu\nu\rho\sigma} \overline{\psi}_{L\mu} \gamma^\nu \partial^\rho \psi_{L\sigma} - \frac{1}{4} m_G^2 \left( \overline{\psi}_R [\gamma^\mu, \gamma^\nu] \psi_{L\mu} + \text{h.c.} \right)
\]  

\[ \text{In fact, a Weyl rotation (2.19a) of the fermion fields with } F = -\frac{\langle K_h \rangle}{2M_P^2} - M_P^2 \ln \frac{\langle W^*_h \rangle}{\langle |W_h| \rangle} \text{ is necessary.} \]
which can be rewritten as
\[
\mathcal{L}_{\text{free}}^\psi = -\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \overline{\psi} \gamma_\mu \gamma_\nu \partial_\rho \psi_\sigma - \frac{1}{4} m_G^2 \overline{\psi} \gamma_\mu [\gamma^\nu, \gamma^\rho] \psi_\nu + \text{tot. div.} \ .
\] (2.62)

Variation of (2.62) yields the Rarita-Schwinger equation [11]
\[
-\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \partial_\rho \psi_\sigma - \frac{1}{4} m_G^2 [\gamma^\mu, \gamma^\nu] \psi_\nu = 0 \ .
\] (2.63)

Since the gravitino satisfies the constraints
\[
\gamma^\mu \psi_\mu = 0 \ ,
\] (2.64a)
\[
\partial^\mu \psi_\mu = 0 \ ,
\] (2.64b)
the Rarita-Schwinger equation (2.63) can be shown to reduce to the Dirac equation for each vector component \( \mu \) of the gravitino,
\[
(i\partial - m_G) \psi_\mu = 0 \ .
\] (2.65)

The polarization tensor for a gravitino with four-momentum \( P \) is given by [2]
\[
\Pi_{\mu\nu}(P) = \sum_s \overline{\psi}^{(s)}(P) \psi^{(s)}(P)
\]
\[
= - (\overrightarrow{P} + m_G) \left( g_{\mu\nu} - \frac{P_\mu P_\nu}{m_G^2} \right) - \frac{1}{3} \left( \gamma^\mu + \frac{P_\mu}{m_G} \right) \left( \overrightarrow{P} - m_G \right) \left( \gamma^\nu + \frac{P_\nu}{m_G} \right) \ ,
\] (2.66)

where the sum is performed over the four gravitino helicities \( s = \pm 3/2, \pm 1/2 \). The polarization tensor obeys
\[
\gamma^\mu \Pi_{\mu\nu}(P) = 0 \ ,
\] (2.67a)
\[
P^\mu \Pi_{\mu\nu}(P) = 0 \ ,
\] (2.67b)
\[
(\overrightarrow{P} - m_G) \Pi_{\mu\nu}(P) = 0 \ .
\] (2.67c)

For energies much higher than the gravitino mass, it can be shown that the \( \Pi_{\mu\nu} \) splits into two parts [2],
\[
\Pi_{\mu\nu}(P) \simeq -\overrightarrow{P} g_{\mu\nu} + \frac{2}{3} \overrightarrow{P} \frac{P_\mu P_\nu}{m_G^2} \ .
\] (2.68)

The first term represents the sum over the helicity \( \pm 3/2 \) states of the gravitino whereas the second part corresponds to the sum over the \( \pm 1/2 \) helicities of the goldstino (2.40).
2.6 The MSSM in the High-Energy Limit

In the calculation of the gravitino production rate we will assume a primordial plasma with the particle content of the Minimal Supersymmetric Standard Model (MSSM). Since we will consider thermal production far above the electroweak symmetry-breaking scale $O(100 \text{ GeV})$, we can work with an unbroken electroweak symmetry group $SU(2)_L \times U(1)_Y$.

We now collect our previous results and extend the above considerations to the standard model gauge group, i.e., we introduce an additional index $\alpha$ that keeps track of the different factors

$$\mathcal{G} = \prod_{\alpha=1}^{3} \mathcal{G}_\alpha = U(1)_Y \times SU(2)_L \times SU(3)_c$$

so that henceforth we use the assignment

$$\alpha = 1 \text{ for } U(1)_Y, \quad \alpha = 2 \text{ for } SU(2)_L, \quad \alpha = 3 \text{ for } SU(3)_c.$$

Accordingly, the gauge couplings $g_\alpha$ are given by

$$g_1 \equiv g', \quad g_2 \equiv g, \quad g_3 \equiv g_s,$$

with the $U(1)_Y$ hypercharge coupling $g'$, and the weak and strong coupling constants $g$ and $g_s$, respectively.

In terms of the rescaled quantities (2.31) and (2.32), we find from (2.3) for the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge interactions

$$\mathcal{L}_{\text{gauge}} = \sum_{\alpha=1}^{3} \mathcal{L}_{\text{gauge}}^{(\alpha)}$$

with

$$\mathcal{L}_{\text{gauge}}^{(\alpha)} = + D_\mu^{(\alpha)\phi} D^{(\alpha)\mu} \phi^i - \frac{1}{2} g_\alpha \left( \phi^i T^{(\alpha)\phi^j}_{a,ij} \right)^2$$

$$+ i \bar{\chi}^i \gamma^\mu \mathcal{D}_\mu^{(\alpha)} \chi^L - \frac{1}{4} F^{(\alpha)\mu \nu} F^{(\alpha)\mu \nu} + \frac{i}{2} \bar{\chi}^{(\alpha)\alpha} \gamma^\mu \mathcal{D}_\mu^{(\alpha)} \chi^{(\alpha)\alpha}$$

$$- \sqrt{2} g_\alpha \bar{\chi}^{(\alpha)\alpha} \phi^i T^{(\alpha)\phi^j}_{a,ij} \chi^L - \sqrt{2} g_\alpha \bar{\chi}^{(\alpha)\alpha} \phi^j T^{(\alpha)\phi^i}_{a,ij} \chi^L.$$

Note that we have dropped the hats which were introduced originally to indicate the rescalings, i.e., we set

$$\hat{g} \rightarrow g, \quad \hat{A} \rightarrow A, \quad \hat{\lambda} \rightarrow \lambda.$$
2.6. The MSSM in the High-Energy Limit

Table 2.1: Gauge fields of the MSSM

| Name          | Gauge bosons $A^{(a)}_\mu$ | Gauginos $\lambda^{(a)}$ | $(SU(3)_c, SU(2)_L)_Y$ |
|---------------|-----------------------------|---------------------------|-------------------------|
| B-boson, bino | $A^{(1)}_\mu = B_\mu \delta^{a1}$ | $\lambda^{(1)} = \bar{B} \delta^{a1}$ | $(1, 1)_0$ |
| W-bosons, winos | $A^{(2)}_\mu = W^a_\mu$ | $\lambda^{(2)} = \bar{W}^a$ | $(1, 3)_0$ |
| gluon, gluino | $A^{(3)}_\mu = G^a_\mu$ | $\lambda^{(3)} = \bar{g}^a$ | $(8, 1)_0$ |

With the $D$-term (2.35) and the vanishing Kähler connection for trivial Kähler manifolds, the covariant derivatives (2.21) in the Lagrangian (2.73) become

\[
D^{(a)}_\mu \phi^i = \partial_\mu \phi^i + i g_\alpha A^{(a)}_\mu T^{(a)}_{\alpha ij} \phi^j,
\]

(2.75a)

\[
D^{(a)}_\mu \chi^i_L = \partial_\mu \chi^i_L + i g_\alpha A^{(a)}_\mu T^{(a)}_{\alpha ij} \chi^j_L,
\]

(2.75b)

\[
D^{(a)}_\mu \lambda^{(a)} = \partial_\mu \lambda^{(a)} - g_\alpha f^{(a)abc} A^{(a)}_\mu \lambda^{(a)} c.
\]

(2.75c)

The field strength tensor $F^{(a)}_{\mu\nu}$ reads

\[
F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu - g f^{(a)abc} A^{(a)}_\mu A^{(a)}_\nu A^{(a)}_\nu.
\]

(2.76)

The corresponding gauge fields and their superpartners are listed in Table 2.6. Matter fields with gauge couplings are in the fundamental (anti-fundamental) representation of the corresponding gauge group, namely, 2 ($\overline{2}$) for $SU(2)_L$ and 3 ($\overline{3}$) for $SU(3)_c$. The $SU(2)_L$ doublet structure for the MSSM matter fields is reviewed in Table 2.6. The strongly interacting particles gather in color triplets, i.e., squarks and quarks indicated in Table 2.6 carry an additional color index. Gauge singlets are denoted by 1 and $(\cdot, \cdot)_0$ for the strong/weak and $U(1)_Y$ interactions, respectively. Note that the normalization for the hypercharges is such that the electric charge $Q$ is given by $Q = T_3 + Y/2$ where $T_3$ denotes the weak isospin eigenvalue $= \pm 1/2$ for upper/lower entries in the doublets of Table 2.6, respectively; $T_3 \equiv 0$ for $SU(2)_L$ singlets $(\cdot, 1)_Y$. The family index $I$ in Table 2.6 refers to one out of three generations of,

leptons

\[
\begin{align*}
\nu^-_L & = (\nu_e^-, \nu_\mu^-, \nu_\tau^-) \\
L^- & = (e^-_L, \mu^-_L, \tau^-_L) \\
e^-_R & = (e^-_R, \mu^-_R, \tau^-_R) \\
\end{align*}
\]

sleptons

\[
\begin{align*}
\bar{\nu}_L^- & = (\bar{\nu}_e^-, \bar{\nu}_\mu^-, \bar{\nu}_\tau^-) \\
\bar{L}_- & = (\bar{e}_L^-, \bar{\mu}_L^-, \bar{\tau}_L^-) \\
\bar{e}_R^- & = (\bar{e}_R^-, \bar{\mu}_R^-, \bar{\tau}_R^-) \\
\end{align*}
\]
Table 2.2: Matter fields of the MSSM

| Name             | Bosons $\phi^i$ | Fermions $\chi^i_L$ | $(SU(3)_c, SU(2)_L)_Y$ |
|------------------|-----------------|----------------------|------------------------|
| Sleptons, leptons $I = 1, 2, 3$ | $\bar{L}^I = \begin{pmatrix} \bar{\nu}_L^I \\ \bar{e}_L^I \end{pmatrix}$ | $L^I = \begin{pmatrix} \nu_L^I \\ e_L^I \end{pmatrix}$ | $(1, 2)_{-1}$ |
| Squarks, quarks $I = 1, 2, 3$ | $\bar{Q}^I = \begin{pmatrix} \bar{u}_R^I \\ \bar{d}_R^I \end{pmatrix}$ | $Q^I = \begin{pmatrix} u_R^I \\ d_R^I \end{pmatrix}$ | $(3, 2)_{-1/3}$ |
|                  | $\bar{U}^I = \begin{pmatrix} \bar{u}_R^I \\ \bar{c}_R^I \end{pmatrix}$ | $U^I = \begin{pmatrix} u_R^I \\ c_R^I \end{pmatrix}$ | $(3, 1)_{-2/3}$ |
|                  | $\bar{D}^I = \begin{pmatrix} \bar{d}_R^I \\ \bar{s}_R^I \end{pmatrix}$ | $D^I = \begin{pmatrix} d_R^I \\ s_R^I \end{pmatrix}$ | $(3, 1)_{+2/3}$ |
|                  | $H_d = \begin{pmatrix} H_d^0 \\ H_d^+ \end{pmatrix}$ | $\bar{H}_d = \begin{pmatrix} \bar{H}_d^0 \\ \bar{H}_d^- \end{pmatrix}$ | $(1, 2)_{-1}$ |
|                  | $H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}$ | $\bar{H}_u = \begin{pmatrix} \bar{H}_u^+ \\ \bar{H}_u^- \end{pmatrix}$ | $(1, 2)_{+1}$ |

for $I = 1, 2, 3$, respectively.

All matter fields are written in terms of left-handed four-spinors since they stem from left-chiral supermultiplets. For example, a right handed tau lepton $\tau_R^-$ with hypercharge $-2$ is written in terms of its charge conjugate $\tau_R^{*-c}$ with hypercharge $+2$.

The generators in (2.73) for the standard model gauge group read

\begin{align}
T^{(1)}_{a, ij} &= \frac{1}{2} Y_i \delta_{ij} \delta_{a1} \quad \text{for } U(1)_Y, \\
T^{(2)}_{a, ij} &= \frac{1}{2} \sigma_{a, ij} \quad \text{for } SU(2)_L, \\
T^{(3)}_{a, ij} &= \frac{1}{2} \lambda_{a, ij} \quad \text{for } SU(3)_c,
\end{align}

where $Y_i$ is the hypercharge of the corresponding particle $\phi^i$ or $\chi^i_L$ (see Table 2.6). The Pauli sigma matrices $\sigma_a$ ($a = 1, 2, 3$) are given in (A.7) in the Appendix and $\lambda_a$ ($a = 1, \ldots, 8$) denote the Gell-Mann matrices. The chosen basis (2.77) for the generators implies that the structure constants $f^{(a)abc}$ for the non-abelian groups are totally
antisymmetric. Since U(1)$_Y$ is abelian, the commutator relation (2.36) is trivial, i.e.,

$$f^{(1)}_{abc} \equiv 0 \quad \text{for } U(1)_Y.$$  \hfill (2.78)

The superpotential of the MSSM is given by

$$\tilde{W}_o = W_{\text{MSSM}} = \tilde{U}^* y_u \tilde{Q} \cdot H_u - \tilde{D}^* y_d \tilde{Q} \cdot H_d - \tilde{E}^* y_e \tilde{L} \cdot H_d + \mu H_u \cdot H_d. \hfill (2.79)$$

The doublet structure is tied together as $\tilde{Q} \cdot H_u = \varepsilon_{ij} \tilde{Q}_i H_{uj}^+$, with $\varepsilon_{ij}$ given in (A.4a). Furthermore, $\tilde{U}^* y_u \tilde{Q}$ is meant to be a matrix multiplication in family space, $\tilde{U}^* y_u \tilde{Q} = \tilde{U}^* y_{u}^{Ij} \tilde{Q}^J$. The couplings $y_{u,d,e}$ and the bilinear parameter $\mu$ in (2.79) are understood to be the rescaled quantities on the left-hand sides in (2.60). We see from (2.59) that $y_{u,d,e}$ yield Yukawa couplings $\sim y_{\Phi} \chi_L^c \chi_L$ and that $\mu$ gives mass to the higgsinos. Neglecting all Yukawa couplings except the one for the top quark,

$$y_e \simeq 0, \quad y_d \simeq 0, \quad (y_u^{Ij}) \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \end{pmatrix}, \hfill (2.80)$$

the superpotential (2.79) reduces to

$$W_{\text{MSSM}} \simeq y_t \tilde{t}_R \tilde{t}_L H_u^0 - y_t \tilde{t}_R \tilde{b}_L H_d^+ + \mu \left( H_u^+ H_d^- - H_u^0 H_d^0 \right). \hfill (2.81)$$

In this thesis, however, we will not consider contributions to the thermal gravitino production rate coming from top-Yukawa interactions; we leave such an analysis for future work.

### 2.7 Gravitino Interactions

Let us now collect the terms in the supergravity Lagrangian (2.5) which describe the gravitino interactions with ordinary matter fields. In the calculation of the gravitino production rate, we will only consider external gravitinos which are subject to the constraint (2.64a). Therefore, the relevant interaction Lagrangian extracted from (2.5) is

$$L_{\psi, \text{int}}^{(\alpha)} = -\frac{i}{\sqrt{2}M_P} \left[ D_{\mu}^{(\alpha)} \phi^* \overline{\psi}_i \gamma^\mu \gamma^\nu \chi_L \gamma^\nu \gamma^\mu \psi_i - D_{\mu}^{(\alpha)} \phi \overline{\chi}_L \gamma^\mu \gamma^\nu \psi_{\nu} \right]$$

$$- \frac{i}{8M_P} \overline{\psi}_i \left[ \gamma^{\rho} \gamma^{\sigma} \right] \gamma^\mu \lambda^{(\alpha)a} \gamma^\rho \mu \psi_i. \hfill (2.82)$$

Here we have used the Kähler metric (2.34) and the definition for the rescaled fields (2.31), but dropping the hats as in (2.73). Note that each operator in (2.82) is suppressed by $M_P^{-1}$. 

22
2.8 Effective Theory for light Gravitinos

In the discussion of the super-Higgs mechanism, we have seen that the goldstino degrees of freedom become the helicity $\pm 1/2$ components of the gravitino. Indeed, these longitudinal helicity $\pm 1/2$ components become dominant if the energies involved are much higher than the gravitino mass $m_G$. The dynamics of the Goldstone fermion is given by the derivative coupling of the goldstino to the supercurrent. The correct effective Lagrangian for light gravitinos with goldstino-matter couplings in non-derivative form has been found in [12]. For a single external goldstino the non-derivative form is equivalent to the derivative form to all orders in perturbation theory and reads

\[
\mathcal{L}_{\psi, \text{light}}^{(\alpha)} = i \frac{m_{\phi^i}^2 - m_{\chi^i}^2}{\sqrt{3} M_\phi m_G} \left( \bar{\psi} \chi_L^i \phi^i - \bar{\chi}_L^i \psi \phi^i \right) - \frac{M_\alpha}{4 \sqrt{6} M_P m_G} \bar{\psi} [\gamma^\mu, \gamma^\nu] \lambda^{(\alpha) a} F_{\mu\nu}^{\alpha a} - i \frac{g_\alpha M_\alpha}{\sqrt{6} M_P m_G} \phi^{\alpha T_{(a)} a \Lambda} \phi^i \bar{\psi} \gamma_5 \lambda^{(\alpha) a} .
\] (2.83)

where the Majorana goldstino field is denoted by $\psi$.\(^{10}\) Note that all vertices are proportional to supersymmetry-breaking mass terms. The coupling in the first term is proportional to the squared masses $m_{\phi^i}^2$ and $m_{\chi^i}^2$ of the corresponding matter fields $\phi^i$ and $\chi_L^i$. The couplings in the remaining terms are linear in the gaugino masses $M_\alpha$. Therefore, at high energies and temperatures $T$, contributions involving the goldstino-fermion-scalar vertex are suppressed relative to the gaugino contributions due to the higher mass dimension of the coupling.

2.9 Feynman Rules

We are now in a position to provide all Feynman rules necessary for the calculation of the thermal gravitino production rate. The gauginos $\lambda^a$, the gravitino $\psi_\mu$, and the goldstino $\psi$ are Majorana fermions. Since these fields are self-conjugate, they yield different Wick contractions from those of Dirac fields. We therefore use the method proposed in [13] and introduce a continuous fermion flow, i.e., an arbitrary orientation of each fermion line. Proceeding against the fermion flow then allows one to form chains of Dirac matrices such that the relative sign of interfering diagrams can be obtained in the same manner as one does for Dirac fermions. We therefore have two analytical expressions for each vertex corresponding to the two possible orientations of the fermion flow.

\(^{10}\)The definitions for the gaugino fields in [12] already contain the factors of $i$ of our convention (2.7). For the goldstino $\psi$, we have included them in the transition to the four-component formalism.
For one direction of the fermion flow, the gravitino Feynman rules derived from (2.82) are given in Fig. 2.0. The gravitino is represented as a double solid line, scalars $\phi^i$ are given by dashed lines and chiral fermions $\chi^i_L$ are given by solid lines. Gauge bosons $A^{(\alpha)\,a}$ are shown as wiggled lines and the corresponding gauginos are depicted as wiggled lines with additional straight solid lines. All momenta are understood to flow into the vertex.

Figure 2.1: Feynman rules for the gravitino from (2.82) for one direction of the fermion flow. The wiggled lines represent gauge fields $A^{(\alpha)\,a}$ while the corresponding superpartners $\lambda^{(\alpha)\,a}$ are depicted by wiggled lines with additional straight solid lines. Scalars $\phi^i$, chiral fermions $\chi^i_L$, and the gravitino are represented respectively by dashed, solid, and double-solid lines. All momenta flow into the vertex.

Figure 2.1 shows the relevant Feynman rules for the gauge interactions of the gauge group $G_\alpha$ derived from (2.73) for one direction of the fermion flow. For U(1)$_Y$, or $\alpha = 1$, there is no self-coupling of the gauge bosons $A^{(1)\,\mu} = B_\mu$. Therefore, the triple gauge boson vertex and its supersymmetric counterpart in the second line of Fig. 2.1 are absent.

For a light gravitino, its interactions are dominated by goldstino dynamics. The corresponding Feynman rules derived from the effective Lagrangian (2.83) are shown in Fig. 2.2. The double solid lines now represent the Majorana field $\psi$.

The complete set of the relevant Feynman rules including vertices with both directions of the fermion flow are given in Appendix B.
2.9. Feynman Rules

Figure 2.2: Relevant Feynman rules for the supersymmetric gauge interactions derived from (2.73). As in Fig. 2.0, it is understood that the gauge bosons and gauginos are associated with the corresponding gauge group ($\alpha$). All momenta are ingoing.

\begin{align*}
\text{a, } \mu & \quad \quad -ig_a T_{a,ij}^{(\alpha)} \gamma^\mu P_L \\
\text{a, } \nu & \quad \quad -ig_a T_{a,ij}^{(\alpha)} (P^\mu - K^\mu) \\
\text{c, } \sigma, K & \quad \quad g_{\alpha} f^{(\alpha)abc} \left[ g^\sigma (P^\rho - K^\rho) \\
& \quad \quad + g^\nu (Q^\nu - P^\nu) \\
& \quad \quad + g^{\sigma\nu} (K^{\sigma\nu} - Q^{\sigma\nu}) \right] \\
\text{b, } \rho, Q & \quad \quad g_{\alpha} f^{(\alpha)abc\gamma\mu} \\
\text{i} & \quad -i\sqrt{2} g_a T_{a,ij}^{(\alpha)} P_L \\
\text{j} & \quad -i\sqrt{2} g_a T_{a,ij}^{(\alpha)} P_R
\end{align*}

Figure 2.3: Feynman rules for the effective theory of light gravitinos from (2.83) for one direction of the fermion flow. The double solid line now denotes the Majorana spinor $\psi$. Again, all momenta are understood to be ingoing.

\begin{align*}
\text{i} & \quad \quad \psi \quad -m_{\psi}^2/m^2 \delta_{ij} P_L \\
\text{j} & \quad \quad \psi \quad m_{\psi}^2/m^2 \delta_{ij} P_R \\
\text{b, } \rho, P & \quad \quad \psi \quad -M_a/2\sqrt{6M_p m_G} \delta_{ab} [\bar{\psi}, \gamma^\rho] \\
\text{c, } \sigma & \quad \quad \psi \quad i g_{\alpha} M_a/2\sqrt{6M_p m_G} [\gamma^\rho, \gamma^\sigma] f^{(\alpha)abc} \\
\text{i} & \quad \quad \psi \quad -g_{\alpha} M_a/\sqrt{6M_p m_G} T_{a,ij}^{(\alpha)} \gamma_5
\end{align*}
Chapter 3

Thermal Gravitino Production

Thermal gravitino production in a consistent thermal field theory approach has been worked out for supersymmetric quantum chromodynamics in Ref. [2]. Taking into account also the electroweak processes, we extend the calculation to the full standard model gauge group SU(3)$_c \times$ SU(2)$_L \times$ U(1)$_Y$ to leading order in the gauge couplings. We also correct an error in the SU(3)$_c$ result of Ref. [2].

3.1 The Braaten–Yuan Prescription

In the previous chapter we have seen that gravitino interactions are suppressed by inverse powers of $M_P$. Thus, the dominant contributions to gravitino production and annihilation processes to leading order in the gauge couplings are inelastic $2 \rightarrow 2$ reactions with one external gravitino. For one of these $2 \rightarrow 2$ scattering processes, the net gravitino production rate at finite temperature reads [14]

\[
\frac{d\Gamma_{\tilde{G}}}{d^3p} = \frac{1}{2(2\pi)^3 E} \int \frac{d\Omega_p}{4\pi} \int \left[ \prod_{i=1}^{3} \left( \frac{d^3p_i}{(2\pi)^3 2E_i} \right) \right] (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P) \times \left\{ f_1(E_1)f_2(E_2)[1 \pm f_3(E_3)][1 - f_{\tilde{G}}(E)]|M(1 + 2 \rightarrow 3 + \tilde{G})|^2 - [1 \pm f_1(E_1)][1 \pm f_2(E_2)]f_3(E_3)f_{\tilde{G}}(E)|M(3 + \tilde{G} \rightarrow 1 + 2)|^2 \right\},
\]

(3.1)

where $E$ and $P$ are the energy and four-momentum of the gravitino, respectively. The corresponding squared matrix element $|M|^2$ is weighted with the phase space distribution functions $f_i(E_i)$ of the particles involved in the scattering; ± applies for final-state bosons/fermions and corresponds to Bose enhancement/Pauli blocking, respectively. In expression (3.1), an average $d\Omega_p/4\pi$ over the directions of the gravitino momentum is
3.1. The Braaten–Yuan Prescription

taken. The squared matrix elements $|M|^2$ are assumed to be summed over initial and final polarizations and to be weighted with the appropriate multiplicities.

At high temperatures, all particles except the gravitino are in thermal equilibrium so that $f_1$, $f_2$, and $f_3$ are given by the equilibrium distributions,

\[ f_B(E_i) = \frac{1}{e^{E_i/T} - 1} \quad \text{for bosons}, \]
\[ f_F(E_i) = \frac{1}{e^{E_i/T} + 1} \quad \text{for fermions}, \]

where $T$ denotes the temperature of the thermal bath.

Assuming that inflation governed the earliest moments of the Universe, any initial population of gravitinos must be diluted away by the exponential expansion during the slow-roll phase. We consider the thermal production (or regeneration) of gravitinos that starts after completion of reheating at the temperature $T_R$. Accordingly, the gravitino phase space density $f_G$ is much smaller than the equilibrium distribution $f_F$. We thus can set $(1 - f_G) \simeq 1$ and neglect gravitino disappearance processes $3 + \tilde{G} \to 1 + 2$. The production rate (3.1) then becomes

\[
\frac{d\Gamma_G}{d^3p} = \frac{1}{2(2\pi)^3E} \int \frac{d\Omega_p}{4\pi} \int \left[ \prod_{i=1}^{3} \frac{d^3p_i}{(2\pi)^32E_i} \right] (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P)
\times f_{F/B}(E_1)f_{F/B}(E_2)[1 \pm f_{F/B}(E_3)]|M(1 + 2 \to 3 + \tilde{G})|^2. \tag{3.3}
\]

Note that a naive application of perturbation theory can lead to logarithmically singular contributions to the production rate. If a massless gauge boson with three-momentum $|k| = k$ is exchanged in the $t$- or $u$-channel, the corresponding squared matrix element is divergent for $k \to 0$. A rigorous method to deal with such situations is the Braaten–Yuan prescription [14] which requires the weak coupling limit, $g \ll 1$: One introduces an intermediate momentum scale $k^*$ such that $gT \ll k^* \ll T$. This scale seperates soft gauge bosons with momentum transfers of order $gT$ from hard ones with momentum transfers of order $T$. The gravitino production rate is then given by the sum

\[
\frac{d\Gamma_G}{d^3p} = \left. \frac{d\Gamma_G}{d^3p} \right|_{\text{hard}} + \left. \frac{d\Gamma_G}{d^3p} \right|_{\text{soft}}. \tag{3.4}
\]

The hard part is obtained conveniently by computing the squared matrix elements $|M((1 + 2 \to 3 + \tilde{G}))|^2$ in standard zero-temperature perturbation theory. Whenever a massless gauge boson is exchanged in the $t$- or $u$-channel, $k^*$ will be introduced as an infrared momentum cutoff. The resulting production rate will then be of the form

\[
\left. \frac{d\Gamma_G}{d^3p} \right|_{\text{hard}} = \left. \frac{d\Gamma_G}{d^3p} \right|_{k^*<k} = A_{\text{hard}} + B \ln \left( \frac{T}{k^*} \right). \tag{3.5}
\]
In the region of soft momentum transfer, \( k < k^* \), one employs the finite-temperature version of the optical theorem. At \( T = 0 \), the imaginary part of the self-energy is related to the decay width of an unstable particle. In a thermal plasma, however, even stable particles can disappear and be produced in inelastic scattering off particles in the thermal background. Accordingly, the discontinuity in the self-energy gives the rate at which a non-equilibrium distribution of the corresponding particle approaches thermal equilibrium \( [15] \). The soft part of the thermal production (or regeneration) rate of gravitinos can thus be expressed in terms of the thermal gravitino self-energy \( \Sigma_{\tilde{G}}(P) \),

\[
\frac{d\Gamma_{\tilde{G}}}{d^3 p} \bigg|_{\text{soft}} = -\frac{1}{(2\pi)^3} \frac{f_F(E)}{E} \text{Im} \Sigma_{\tilde{G}}(E + i\varepsilon, p)\bigg|_{k<k^*}.
\]  

(3.6)

In the naive consideration of the production rate, the singular behavior stems from long-range forces. The physical cutoff is provided by the screening of the interactions due to the cooperative motion of particles in the thermal bath \( [14] \). Instead of using bare gauge boson propagators, hard thermal loop (HTL) resummed gauge boson propagators have to be used in the region of soft momentum transfer, \( k < k^* \) \( [16] \). For gravitino energies \( E \gtrsim T \), there is no need to consider HTL-resummed vertices. The soft part of the thermal gravitino production rate (3.6) will then be of the form

\[
\frac{d\Gamma_{\tilde{G}}}{d^3 p} \bigg|_{\text{soft}} = A_{\text{soft}} + B \ln \left( \frac{k^*}{m_{\text{th}}} \right)
\]  

(3.7)

with \( m_{\text{th}} \) denoting the thermal mass of the corresponding gauge boson.

Indeed, in the sum (3.4) the logarithmic dependence on \( k^* \) cancels out and yields a finite result. The Braaten–Yuan prescription \( [14] \) together with the HTL-resummation technique \( [16] \) allows us to calculate the gravitino production rate in a gauge-invariant way.

### 3.2 Hard Contribution

From the Feynman rules for the gravitino and the supersymmetric gauge interactions presented respectively in Figs. 2.0 and 2.1, we find the leading processes for thermal gravitino production. Figure 3.2 shows these processes for one factor \( \mathcal{G}_\alpha \) (2.69) of the standard model gauge group \( \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y \).

Note that the processes A, B, and F are not present for \( \alpha = 1 \), i.e., for \( \text{U}(1)_Y \), since there is no self-coupling of gauge bosons for an abelian gauge group; thereby also the supersymmetrized version of the relevant three-gauge boson vertex is absent. The matter
3.2. Hard Contribution

- **A (BBF):** $A^{(\alpha)}a + A^{(\alpha)}b \rightarrow \lambda^{(\alpha)c} + \tilde{G}$ (for $U(1)_Y$)
  
- **B (BBF):** $A^{(\alpha)}a + \lambda^{(\alpha)b} \rightarrow A^{(\alpha)}c + \tilde{G}$ (crossing of A, for $U(1)_Y$)

- **C (BBF):** $\phi^i + A^{(\alpha)}b \rightarrow \chi^j_L + \tilde{G}$

- **D (BBF):** $A^{(\alpha)}a + \chi^i_L \rightarrow \phi^j + \tilde{G}$ (crossing of C)

- **E (BBF):** $\phi^a + \chi^j_L \rightarrow A^{(\alpha)}a + \tilde{G}$ (crossing of C)

- **F (FFF):** $\lambda^{(\alpha)}a + \lambda^{(\alpha)b} \rightarrow \lambda^{(\alpha)c} + \tilde{G}$ (for $U(1)_Y$)

- **G (FFF):** $\chi^i_L + \lambda^{(\alpha)a} \rightarrow \chi^j_L + \tilde{G}$

- **H (BBF):** $\phi^j + \lambda^{(\alpha)a} \rightarrow \phi^j + \tilde{G}$

- **I (FFF):** $\chi^i_L + \chi^j_L \rightarrow \lambda^{(\alpha)a} + \tilde{G}$ (crossing of G)

- **J (BBF):** $\phi^j + \phi^a \rightarrow \lambda^{(\alpha)a} + \tilde{G}$ (crossing of H)

**Figure 3.1:** The $2 \rightarrow 2$ scattering processes for gravitino production. Processes A, B, and F are not present for $U(1)_Y$ since $f^{(1)abc} \equiv 0$. 

29
Table 3.1: Squared matrix elements for the $2 \rightarrow 2$ scatterings in terms of the Mandelstam variables $s$ and $t$; $M_\alpha$ denote the gaugino masses. Processes A, B and F are not present for $U(1)_Y$. Sums over initial and final spins have been performed.

| Label | Class | Process | $|M_i|^2 / g_\alpha^2 M_P^{-2} \left(1 + \frac{M_\alpha^2}{3m_G^2}\right)$ |
|-------|-------|---------|---------------------------------------------------------------|
| A     | BBF   | $A^{(\alpha)a} + A^{(\alpha)b} \rightarrow \lambda^{(\alpha)c} + \bar{G}$ | $4 \left(s + 2t + 2\frac{t^2}{s}\right) |f^{(\alpha)abc}|^2$ |
| B     | BFB   | $A^{(\alpha)a} + \lambda^{(\alpha)b} \rightarrow A^{(\alpha)c} + \bar{G}$ | $-4 \left(t + 2s + 2\frac{t^2}{s}\right) |f^{(\alpha)abc}|^2$ |
| C     | BBF   | $\phi^i + A^{(\alpha)b} \rightarrow \chi^L_i + \bar{G}$ | $2s |T_{a,ij}^{(\alpha)}|^2$ |
| D     | BFB   | $A^{(\alpha)a} + \chi^L_i \rightarrow \phi^i + \bar{G}$ | $-2t |T_{a,ij}^{(\alpha)}|^2$ |
| E     | BFB   | $\phi^{*i} + \chi^L_i \rightarrow A^{(\alpha)a} + \bar{G}$ | $-2t |T_{a,ij}^{(\alpha)}|^2$ |
| F     | FFF   | $\lambda^{(\alpha)a} + \lambda^{(\alpha)b} \rightarrow \lambda^{(\alpha)c} + \bar{G}$ | $-8 \frac{(s^2 + s + t + 2t^2)}{s(t + s + t)} |f^{(\alpha)abc}|^2$ |
| G     | FFF   | $\chi^L_i + \lambda^{(\alpha)a} \rightarrow \chi^L_i + \bar{G}$ | $-4 \left(s + \frac{s^2}{t}\right) |T_{a,ij}^{(\alpha)}|^2$ |
| H     | BFB   | $\phi^i + \lambda^{(\alpha)a} \rightarrow \phi^i + \bar{G}$ | $-2 \left(t + 2s + 2\frac{t^2}{s}\right) |T_{a,ij}^{(\alpha)}|^2$ |
| I     | FFF   | $\chi^L_i + \chi^L_j \rightarrow \lambda^{(\alpha)a} + \bar{G}$ | $-4 \left(t + \frac{t^2}{s}\right) |T_{a,ij}^{(\alpha)}|^2$ |
| J     | BBF   | $\phi^i + \phi^{*i} \rightarrow \lambda^{(\alpha)a} + \bar{G}$ | $2 \left(s + 2t + 2\frac{t^2}{s}\right) |T_{a,ij}^{(\alpha)}|^2$ |

fields $\phi^i$ and $\chi^L_i$ coupling to the gauge bosons $A^{(\alpha)a}$ of the corresponding group $G_\alpha$ can be read from Table 2.6.

The squared matrix elements for the processes shown in Fig. 3.2 are given in Table 3.2 in terms of the Mandelstam variables

$$s = (P_1 + P_2)^2, \quad (3.8a)$$
$$t = (P_1 - P_3)^2, \quad (3.8b)$$

where the four-momenta $P_1$, $P_2$, and $P_3$ are associated with the particles in the order in which they are written down in the column “Process i” of Table 3.2. The gaugino masses are written as $M_\alpha$.

The Dirac traces occurring in the evaluation of the squared matrix elements have been performed using the computer program FORM [17]. Polarizations sums over final and
initial states have been carried out (no averaging). In process A, there are two gauge bosons with four-momenta $P_1$ and $P_2$ in the initial state. The simple replacement of the corresponding polarization sum $\sum_{\text{pol}} \varepsilon^\mu_a \varepsilon^\nu_b \rightarrow -g^{\mu\nu} \delta_{ab}$ would amount to the inclusion of longitudinal polarizations. Since we work in the high-energy limit of unbroken electroweak symmetry, not only the gluons but also the electroweak gauge bosons are massless. Therefore, we have used the proper polarization sum

$$\sum_{\text{pol.}} \varepsilon^\mu_a (P_{1/2}) \varepsilon^\nu_b (P_{1/2}) = \left[ -g^{\mu\nu} + \frac{2}{s} (P_1^\mu P_2^\nu + P_2^\mu P_1^\nu) \right] \delta_{ab},$$  \hspace{1cm} (3.9)

in which the unphysical modes are already subtracted [18]. The Feynman rules used in the evaluation of the Feynman diagrams are given in Appendix B.

The squared amplitudes shown in Table 3.2 have to be weighted with appropriate multiplicities:

**Processes A, B, F** The sum over the gauge-group indices $a, b, c = 1, \ldots, \dim G_\alpha$ has to be performed:

$$\sum_{a,b,c} |f^{(\alpha)}_{a,b,c}|^2 = N^{2}_\alpha - 1 \quad \text{for } SU(N_\alpha).$$  \hspace{1cm} (3.10)

For processes A and F, an additional factor of $1/2$ occurs because of identical particles in the initial state. Note that the structure constants vanish for $U(1)_Y$.

**Processes C, D, E, G, H** In the corresponding initial state, there can be

$$2n_2 \equiv 28,$$  \hspace{1cm} (3.11a)

distinct isospin-doublets for $SU(2)_L$ or

$$2n_3 \equiv 24,$$  \hspace{1cm} (3.11b)

color-triplets for $SU(3)_c$; cf. Table 2.6. The factor 2 takes into account the corresponding conjugate multiplet as the initial state. The sum over the isospin/color degrees of freedom reads

$$\sum_{i,j,a} |T_{\alpha,i,j}^{(\alpha)}|^2 = \frac{1}{2} (N^{2}_\alpha - 1) \quad \text{for } SU(N_\alpha).$$  \hspace{1cm} (3.12)

\[1\text{This is correct since there is no factor included in the definition for the hard production rate (3.1), which counts the internal degrees of freedom.}\]
For $U(1)_Y$, we have $T^{(1)}_{a,ij} = \delta_{ij} \delta_{a1} Y_i^2 / 4$ with $Y_i$ given in Table 2.6. All matter fields of the MSSM carry non-vanishing hypercharge. Thus, the correct multiplicity factor for the $U(1)_Y$ contributions is obtained by summing over all squared hypercharges of either scalars $\phi^i$ or fermions $\chi^i_L$ listed in Table 2.6. We find

$$2 n_1 \equiv 2 \sum_{\phi^i/\chi^i_L} \frac{Y_i^2}{4} = 22,$$

(3.13)

where the factor of 2 accounts for the corresponding antiparticles.

**Processes I, J** Since I and J describe the same physical processes after replacing the incoming fields by their conjugates, the multiplicities (3.11a), (3.11b), and (3.13) apply without the factor of 2.

The diagrams of Fig. 3.2 fall into three separate classes depending on the number of bosons and fermions involved in the initial and final state. The BBF processes A, C, and J have two bosons in the initial state and one fermion in addition to the gravitino in the final state. Correspondingly, B, D, E, and H are BFB processes and F, G, and I are FFF processes. The hard part of the thermal gravitino production rate for the full standard model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ can then be written as

$$\frac{d\Gamma_G}{d^3p} \bigg|_{\text{hard}} = \frac{1}{2(2\pi)^3 E} \int d\Omega_p \int \prod_{i=1}^3 \frac{d^3p_i}{2(2\pi)^3 E_i} (2\pi)^4 \delta^4(P_1 + P_2 - P - P_3)$$

$$\times \sum_{\alpha=1}^3 \left( f_{BBF} |M^{(\alpha)}_{BBF}|^2 + f_{BBF} |M^{(\alpha)}_{BBF}|^2 + f_{FFF} |M^{(\alpha)}_{FFF}|^2 \right) \Theta(|p_1 - p_3| - k^*)$$

(3.14)

with the shorthand notation

$$f_{BBF} = f_B(E_1) f_f(E_2) \left[ 1 + f_B(E_3) \right],$$

(3.15a)

$$f_{BBF} = f_B(E_1) f_B(E_2) \left[ 1 - f_F(E_3) \right],$$

(3.15b)

$$f_{FFF} = f_F(E_1) f_F(E_2) \left[ 1 - f_F(E_3) \right].$$

(3.15c)
3.2. Hard Contribution

Contributions from SU(2)$_L$ ($\alpha = 2$) and SU(3)$_c$ ($\alpha = 3$)

For the weighted sums, we find

$$|M_{BFB}^{(\alpha)}|^2 = \left(1 + \frac{M_{\alpha}^2}{3m_G^2}\right) \frac{4g_{\alpha}^2(N_{\alpha}^2 - 1)}{M_P^2} \left[\left(-t - 2s - \frac{s^2}{t}\right)(N_{\alpha} + \frac{n_{\alpha}}{2}) - t n_{\alpha}\right], \quad (3.16a)$$

$$|M_{BBF}^{(\alpha)}|^2 = \left(1 + \frac{M_{\alpha}^2}{3m_G^2}\right) \frac{2g_{\alpha}^2(N_{\alpha}^2 - 1)}{M_P^2} \left[\left(s + 2t + \frac{2t^2}{s}\right)(N_{\alpha} + \frac{n_{\alpha}}{2}) + s n_{\alpha}\right], \quad (3.16b)$$

$$|M_{FFF}^{(\alpha)}|^2 = \left(1 + \frac{M_{\alpha}^2}{3m_G^2}\right) \frac{4g_{\alpha}^2(N_{\alpha}^2 - 1)}{M_P^2} \left[-\frac{(s^2 + st + t^2)^2}{st(s + t)}N_{\alpha} - \left(t + 2s + \frac{t^2}{s} + 2\frac{s^2}{t}\right)\frac{n_{\alpha}}{2}\right]. \quad (3.16c)$$

Equation (3.16c) can be rewritten as

$$|M_{FFF}^{(\alpha)}|^2 = \left(1 + \frac{M_{\alpha}^2}{3m_G^2}\right) \frac{4g_{\alpha}^2(N_{\alpha}^2 - 1)}{M_P^2} \left[-\frac{(s^2 + st + t^2)^2}{st(s + t)}N_{\alpha} - \left(t + 2s + \frac{t^2}{s} + 2\frac{s^2}{t}\right)\frac{n_{\alpha}}{2}\right]. \quad (3.17)$$

Since $s + t + u = \sum m_i^2$, where $m_i$ are the masses of the external particles, and $u = (P_1 - P)^2$, we can write $s + 2t = t - u$ in the high-energy limit, $T \gg m_i$. Therefore,

$$\pm \frac{s^2}{t} + \frac{s^2}{s + t} = \pm \frac{s^2}{t} - \frac{s^2}{u}. \quad (3.18)$$

Since the difference $t - u$ and $1/t - 1/u$ is odd under exchange of $P_1$ and $P_2$, the integral over such terms will be zero, as long as the remaining integrand and the measure is even under this transformation. Therefore, there is no contribution to the integral from $s + 2t$ in (3.16b) and we can further substitute $s$ by $-2t$ in the last term of (3.16b). In Eq. (3.17) we use the following replacements

$$\frac{s^2}{t} + \frac{s^2}{s + t} \rightarrow 0, \quad (3.19a)$$

$$\frac{-s^2}{t} + \frac{s^2}{s + t} \rightarrow -\frac{2s^2}{t}. \quad (3.19b)$$

For the gauge groups SU(2)$_L$ ($\alpha = 2$) and SU(3)$_c$ ($\alpha = 3$), we can therefore write the weighted sums of the squared matrix elements of Table 3.2 in terms of three distinct
matrix elements,

\[
|M_B^{(\alpha)}|_2^2 = \left(1 + \frac{M_\alpha^2}{3 m_G^2}ight) \frac{4 g_\alpha^2 (N_\alpha^2 - 1)}{M_P^2} \left[|M_1|^2 \left(N_\alpha + \frac{n_\alpha}{2}\right) - |M_2|^2 n_\alpha\right], \quad (3.20a)
\]

\[
|M_B^{(\alpha)}|_2^2 = \left(1 + \frac{M_\alpha^2}{3 m_G^2}ight) \frac{2 g_\alpha^2 (N_\alpha^2 - 1)}{M_P^2} \left[|M_3|^2 \left(N_\alpha + \frac{n_\alpha}{2}\right) - |M_2|^2 n_\alpha\right], \quad (3.20b)
\]

\[
|M_F^{(\alpha)}|_2^2 = \left(1 + \frac{M_\alpha^2}{3 m_G^2}ight) \frac{4 g_\alpha^2 (N_\alpha^2 - 1)}{M_P^2} \left[|M_1|^2 - |M_3|^2\right] \left(N_\alpha + \frac{n_\alpha}{2}\right), \quad (3.20c)
\]

with \(n_\alpha\) as defined in (3.11) and

\[
|M_1|^2 = -t - 2s - \frac{2s^2}{t}, \quad (3.21a)
\]

\[
|M_2|^2 = t, \quad (3.21b)
\]

\[
|M_3|^2 = \frac{t^2}{s}. \quad (3.21c)
\]

Recall from (3.10) and (3.12) that \(N_\alpha\) denotes the dimension of the fundamental representation of the corresponding gauge group, i.e., \(N_2 = 2\) and \(N_3 = 3\).

**Contribution from U(1)_Y (\alpha = 1)**

In analogy to the considerations which lead to (3.20), we obtain for the contributions from the U(1)_Y interactions

\[
|M_B^{(1)}|_2^2 = \left(1 + \frac{M_1^2}{3 m_G^2}ight) \frac{4 g_1^2 n_1}{M_P^2} \left(|M_1|^2 - 2 |M_2|^2\right), \quad (3.22a)
\]

\[
|M_B^{(1)}|_2^2 = \left(1 + \frac{M_1^2}{3 m_G^2}ight) \frac{2 g_1^2 n_1}{M_P^2} \left(|M_3|^2 - 2 |M_2|^2\right), \quad (3.22b)
\]

\[
|M_F^{(1)}|_2^2 = \left(1 + \frac{M_1^2}{3 m_G^2}ight) \frac{4 g_1^2 n_1}{M_P^2} \left(|M_1|^2 - |M_3|^2\right), \quad (3.22c)
\]

with \(n_1\) as defined in (3.13) and \(g_1 = g'\). Note that \(M_1\) in the prefactor of (3.22) denotes the Bino mass while the squared matrix elements \(|M_i|^2\) are the ones of (3.21).

For the calculation of the gravitino production rate (3.14), various integrations are necessary. Since the structure of the integrands is identical for all factors \(G_\alpha\) of the standard
model gauge group, we introduce the following notation

\[
\frac{d\Gamma_{\tilde{G}}}{d^3p}{\bigg|}_{\text{hard}} = \sum_{\alpha=1}^{3} \frac{g_\alpha^2}{M_P^2} \left(1 + \frac{M_\alpha^2}{3m_{\tilde{G}}^2}\right) \sum_{i=1}^{3} \left[ c_{\text{BFB},i}^{(\alpha)} I_{\text{BFB}}^{[M_i]^2} \right], \quad (3.23a)
\]

\[
\frac{d\Gamma_{\tilde{G}}}{d^3p}{\bigg|}_{\text{hard}} = \sum_{\alpha=1}^{3} \frac{g_\alpha^2}{M_P^2} \left(1 + \frac{M_\alpha^2}{3m_{\tilde{G}}^2}\right) \sum_{i=1}^{3} \left[ c_{\text{FFF},i}^{(\alpha)} I_{\text{FFF}}^{[M_i]^2} \right], \quad (3.23b)
\]

\[
\frac{d\Gamma_{\tilde{G}}}{d^3p}{\bigg|}_{\text{hard}} = \sum_{\alpha=1}^{3} \frac{g_\alpha^2}{M_P^2} \left(1 + \frac{M_\alpha^2}{3m_{\tilde{G}}^2}\right) \sum_{i=1}^{3} \left[ c_{\text{BBF},i}^{(\alpha)} I_{\text{BBF}}^{[M_i]^2} \right]. \quad (3.23c)
\]

The coefficients \(c_{\text{BFB},i}^{(\alpha)}\) are given in Table 3.2. The other factors are given by

\[
c_{\text{BBF},1}^{(\alpha)} = c_{\text{FFF},2}^{(\alpha)} = c_{\text{BFB},3}^{(\alpha)}, \quad (3.24a)
\]

\[
c_{\text{BBF},2}^{(\alpha)} = c_{\text{FFF},1}^{(\alpha)} = c_{\text{BFB},2}^{(\alpha)}, \quad (3.24b)
\]

\[
c_{\text{BBF},3}^{(\alpha)} = -c_{\text{FFF},3}^{(\alpha)} = c_{\text{BFB},1}^{(\alpha)}. \quad (3.24c)
\]

The calculation of the integrals \(I_{\text{BFB}}^{[M_i]^2}\), \(I_{\text{FFF}}^{[M_i]^2}\), and \(I_{\text{BBF}}^{[M_i]^2}\) is presented in detail in Appendix C. Here are the results:

\[
I_{\text{FFF}}^{[M_i]^2} = \left\{ \begin{array}{c} 1/2 \end{array} \right\} \frac{T^3 f_f(E)}{192\pi^4} \left[ \ln \left( \frac{2T}{\kappa^*} \right) + \frac{17}{6} - \gamma + \zeta(2) - \left\{ \ln 2 \right\} \right]
\]

\[
+ \frac{1}{256\pi^6} \int_0^\infty dE_3 \int_0^{E_1 + E_3} dE_1 \ln \left( \frac{|E_1 - E_3|}{E_3} \right)
\]

\[
\times \left\{ -\Theta(E_1 - E_3) \frac{d}{dE_1} \left[ f_{\text{BBF}}^{(\alpha)} \frac{E_2}{E_3} (E_1^2 + E_3^2) \right]
\]

\[
+ \Theta(E_3 - E_1) \frac{d}{dE_1} \left[ f_{\text{FFF}}^{(\alpha)} (E_1^2 + E_3^2) \right]
\]

\[
+ \Theta(E - E_1) \frac{d}{dE_1} \left[ f_{\text{FFF}}^{(\alpha)} \left( \frac{E_1^2 E_2^2}{E_3^2} - E_3^2 \right) \right] \right\}, \quad (3.25)
\]
\[ I_{\{\text{BFB}\}}^{M_{2}^{2}} = \frac{1}{256\pi^{6}} \int_{0}^{\infty} dE_{3} \int_{0}^{E+E_{3}} dE_{2} f_{\{\text{BFB}\}} \times \left\{ \Theta(E - E_{3}) \frac{E_{2}^{2}E_{3}^{2}}{E^{2}} \left( \frac{E_{3}}{3} - E_{1} \right) + \Theta(E_{3} - E) \left( \frac{E}{3} - E_{2} \right) \right. \\
+ \Theta(E_{3} - E_{2})\Theta(E - E_{3}) \frac{E_{2}^{2} - E_{3}^{2}}{3E^{2}} [(E_{2} - E_{3})(E_{2} + 2E_{3}) - 3(E_{2} + E_{3})E] \\
- \Theta(E_{2} - E)\Theta(E_{3} - E) \frac{(E_{2} - E)^{3}}{3E^{2}} \\
+ \Theta(E - E_{2})\Theta(E_{3} - E) \frac{(E_{2} - E)^{3}}{3E^{2}} \\
- \Theta(E_{2} - E_{3})\Theta(E_{3} - E) \frac{E_{2}^{2} - E_{3}^{2}}{3E^{2}} [(E_{2} - E_{3})(E_{2} + 2E_{3}) - 3(E_{2} + E_{3})E] \right\} , \tag{3.26} \]

\[ I_{\{\text{FFF}\}}^{M_{3}^{2}} = \frac{1}{256\pi^{6}} \int_{0}^{\infty} dE_{3} \int_{0}^{E+E_{3}} dE_{2} f_{\{\text{FFF}\}} \times \left\{ \Theta(E - E_{3}) \frac{1}{E^{2}} \frac{E_{2}^{2}E_{3}^{2}}{E + E_{3}} \\
+ \Theta(E_{3} - E) \frac{E_{2}^{2}}{E + E_{3}} \\
- \Theta(E_{3} - E_{2})\Theta(E_{3} - E_{2}) \frac{E_{2}^{2} - E_{3}^{2}}{E^{2}} [E_{2}(E_{3} - E) - E_{3}(E_{2} + E)] \\
+ \Theta(E_{3} - E)\Theta(E_{2} - E_{3}) \frac{E_{2}^{2} - E_{3}^{2}}{E^{2}} [E_{2}(E_{3} - E) - E_{3}(E_{2} + E)] \right\} , \tag{3.27} \]

where \( \gamma = 0.57722 \ldots \) is the Euler-Mascheroni constant, \( \zeta(x) \) is Riemann’s Zeta function with \( \zeta(2) = \pi^2/6 \) and \( \zeta'(2)/\zeta(2) = -0.56996 \ldots \). In the above expressions, \( E_{1} + E_{2} = E_{3} + E \) is understood.

Only the part \( \propto 1/t \) in the squared matrix element \( |M_{i}|^{2} \) given in (3.21a) exhibits a singular behavior for \( t \to 0 \). Therefore, the integrals contributing to the logarithmic dependence on \( k^{*} \) are \( I_{\{\text{BFB}\}}^{M_{2}^{2}} \). The logarithm is extracted analytically using integration by parts. Details are given in Appendix C; see Eq. (C.31) and subsequent steps. In the calculation of \( I_{\{\text{BFB}\}}^{M_{2}^{2}} \) and \( I_{\{\text{FFF}\}}^{M_{3}^{2}} \), the cutoff \( k^{*} \) is set to zero from the very beginning.

The hard part of the gravitino production rate (3.14) is then given by the sum

\[ \frac{d\Gamma_{G}^{\text{(BFB)}}}{d^{3}p} \bigg|_{\text{hard}} = \frac{d\Gamma_{G}^{\text{(BFB)}}}{d^{3}p} \bigg|_{\text{hard}} + \frac{d\Gamma_{G}^{\text{(FFF)}}}{d^{3}p} \bigg|_{\text{hard}} + \frac{d\Gamma_{G}^{\text{(BFB)}}}{d^{3}p} \bigg|_{\text{hard}} . \tag{3.28} \]
We correct an error in the SU(3)$_c$ result of Ref. [2].

\[ T^3(N + n_f)[\text{Li}_2(-e^{-E/T}) - \pi^2/6] \] (3.29)

given as part of $I_{BFB}$ in (C.14) of Ref. [2]. Although such a contribution appears as a surface term in our calculation, it is canceled by another surface term. The crucial spots to look at are Eqs. (C.42) and (C.49) in our Appendix C. The authors of [2] agree with our finding and will publish an erratum. The phenomenological implications will be discussed in chapters 4 and 5.

### 3.3 Soft Contribution

As mentioned in Sec. 3.1, hard thermal loop (HTL) resummed propagators have to be used for soft gauge boson momentum transfers. Since diagrams of nominally higher order in the loop expansions can contribute to same order in the coupling constant $g$ at high temperatures, this corresponds to an improved perturbation theory where a certain subset of diagrams, the HTL self-energies, are resummed [19].

The HTL self-energies which contribute to the gauge boson polarization tensor $\Pi^{(\alpha)}_{\mu\nu}$ are given by one-loop diagrams such as the ones shown in Fig. 3.1. They carry soft external momenta $K = (k_0, k)$ with $k_0$ and $k = |k|$ of order $gT$ and hard internal loop momenta of order $T$. In the HTL approximation of hard loop momenta, only a small part of the integration region in the one-loop diagrams contributes. Hard thermal loops are gauge invariant and the corresponding self-energy satisfies the Ward identity $K^\mu \Pi^{(\alpha)}_{\mu\nu}(K) = 0$. They are exclusively due to thermal fluctuations and are therefore ultraviolet finite [16].

At finite temperatures, the thermal bath constitutes a privileged rest frame. The polarization tensor $\Pi^{(\alpha)}_{\mu\nu}$ can be decomposed into two independent propagating modes which are chosen to be the longitudinal and the transverse parts of the self-energy of the gauge bosons $A^{(\alpha)}_{\mu}$. Both are physical and read respectively

\[
\Pi^{(\alpha)}_L(K) = \Pi^{(\alpha)}_{00}(K),
\]
\[
\Pi^{(\alpha)}_T(K) = \frac{1}{2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \Pi^{(\alpha)}_{ij}(K).
\] (3.30a, 3.30b)

In the high temperature limit, $k_0, k \ll T$, to leading order in the gauge couplings $g_\alpha$,}

\footnote{For a direct comparison with Ref. [2], it should be stressed that our definition for the production rate differs by a factor of $f(E)/(2\pi)^3$.}
these components of the self-energy are given by \[20\]

\[
\Pi_L^{(\alpha)}(k_0, k) = -3 m_\alpha^2 \left(1 - \frac{k_0}{2k} \ln \frac{k_0 + k}{k_0 - k}\right),
\]

(3.31a)

\[
\Pi_T^{(\alpha)}(k_0, k) = \frac{3}{2} m_\alpha^2 \frac{k_0^2}{k^2} \left[1 - \left(1 - \frac{k^2}{k_0^2}\right) \frac{k_0}{2k} \ln \frac{k_0 + k}{k_0 - k}\right],
\]

(3.31b)

where \(m_\alpha\) denotes the thermal masses of the corresponding gauge boson \(A_\mu^{(\alpha)}\). Note that the self-energy components (3.31) depend on the gauge boson momenta in a non-trivial way and that they have an imaginary part for \(k_0^2 < k^2\).

In the static limit, \(k_0 \to 0\), the longitudinal part of the self-energy reduces to \(\Pi_L^{(\alpha)}(0, k) = -3 m_\alpha^2\) which is analogous to the Debye screening of static electric fields in a QED plasma with inverse screening length \(\lambda_D^{-1} = \sqrt{3} m_\alpha\). There is no static magnetic screening since \(\Pi_T^{(\alpha)}(0, k) = 0\). Nevertheless, the transverse self-energy \(\Pi_T\) approaches zero sufficiently slow so that quantities in which the magnetic divergence is only logarithmic are regularized [14].

In SUSY extensions of the Standard Model, the hard thermal loops shown in Fig. 3.1 contribute to the self-energy tensor and thereby to the thermal gauge boson mass \(m_\alpha\),

\[
m_\alpha^2 = m^2_{\text{gauge bosons}} + m^2_{\text{gauginos}} + m^2_{\text{fermions}} + m^2_{\text{scalars}}.
\]

(3.32)

For the non-abelian gauge group \(\text{SU}(N_\alpha)\), the contributions are given by [16, 21]

\[
m^2_{\alpha, \text{gauge bosons}} = \frac{g_\alpha^2 T^2}{9} \sum_{b,c} |f^{(\alpha)abc}|^2 = N_\alpha \frac{g_\alpha^2 T^2}{9},
\]

(3.33a)

\[
m^2_{\alpha, \text{gauginos}} = \frac{g_\alpha^2 T^2}{18} \sum_{b,c} |f^{(\alpha)abc}|^2 = N_\alpha \frac{g_\alpha^2 T^2}{18},
\]

(3.33b)

\[
m^2_{\alpha, \text{fermions}} = n_\alpha \frac{g_\alpha^2 T^2}{18} \sum_{i,j} |T_{\alpha,ij}|^2 = n_\alpha \frac{g_\alpha^2 T^2}{36},
\]

(3.33c)

\[
m^2_{\alpha, \text{scalars}} = n_\alpha \frac{g_\alpha^2 T^2}{9} \sum_{i,j} |T_{\alpha,ij}|^2 = n_\alpha \frac{g_\alpha^2 T^2}{18}.
\]

(3.33d)

For \(\alpha = 2\) and \(\alpha = 3\) with \(n_\alpha\) defined in (3.11), this determines the thermal masses of the wino and the gluino, respectively. They read

\[
m_\alpha^2 = \frac{g_\alpha^2 T^2}{6} \left(N_\alpha + \frac{n_\alpha}{2}\right).
\]

(3.34)

Analogously, one finds for \(\text{U}(1)_Y\):

\[
m_1^2 = \frac{g_1^2 n_1 T^2}{6}
\]

(3.35)
3.3. Soft Contribution

(a) Gauge boson contributions for SU(3)$_c$ and SU(2)$_L$

(b) Gaugino contributions

(c) Fermion contributions

(d) Scalar contributions

**Figure 3.2:** HTL self-energy contributions to the gauge boson polarization tensor $\Pi^{(\alpha)}_{\mu\nu}$. The internal loop momenta are hard, i.e., of order $T$. In the resummation these contributions become relevant for soft gauge boson momenta $K \sim gT$.

which determines the thermal bino mass. In the U(1)$_Y$ case, $m^2_{1, \text{gauge bosons}} = 0$ and $m^2_{1, \text{gauginos}} = 0$ since the diagrams of Fig. 3.1a are absent.

In covariant gauge, the HTL-resummed gauge boson propagator has the form [22, 23]

$$i\Delta^{(\alpha)}_{\mu\nu}(K) = i \left( A_{\mu\nu} \Delta^{(\alpha)}_T + B_{\mu\nu} \Delta^{(\alpha)}_L + C_{\mu\nu} \xi \right),$$

(3.36)

with the tensorial quantities

$$A_{\mu\nu} = -g_{\mu\nu} - \frac{1}{k^2} \left[ K^2 v_\mu v_\nu - K \cdot v (K_\mu v_\nu + K_\nu v_\mu) + K_\mu K_\nu \right],$$

(3.37a)

$$B_{\mu\nu} = v_\mu v_\nu - \frac{K \cdot v}{K^2} (K_\mu v_\nu + K_\nu v_\mu) + \left( \frac{K \cdot v}{K^2} \right)^2 K_\mu K_\nu,$$

(3.37b)

$$C_{\mu\nu} = \frac{K_\mu K_\nu}{(K^2)^2}.$$  

(3.37c)
The velocity of the thermal bath is denoted by \( v \) and the gauge fixing parameter by \( \xi \). The transverse and longitudinal propagators are

\[
\Delta_T^{(\alpha)}(k_0, k) = \frac{1}{k_0^2 - k^2 - \Pi_T^{(\alpha)}(k_0, k)}, \tag{3.38a}
\]

\[
\Delta_L^{(\alpha)}(k_0, k) = \frac{1}{k^2 - \Pi_L^{(\alpha)}(k_0, k)}, \tag{3.38b}
\]

which have spectral representations [22]

\[
\Delta_T^{(\alpha)}(k_0, k) = \int_{-\infty}^{\infty} d\omega \frac{1}{k_0 - \omega} \rho_T^{(\alpha)}(\omega, k). \tag{3.39}
\]

For \(|\omega| < k\), the spectral densities \( \rho_L^{(\alpha)} \) are given by

\[
\rho_T^{(\alpha)}(\omega, k) = \frac{3}{4m_G^2} \frac{x}{(1 - x^2)} \left[ A_T(x)^2 + (z + B_T(x))^2 \right], \tag{3.40a}
\]

\[
\rho_L^{(\alpha)}(\omega, k) = \frac{3}{4m_G^2} \frac{x}{2} A_L(x)^2 + (z + B_L(x))^2, \tag{3.40b}
\]

with \( x = \omega/k \) and \( z = k^2/m_\alpha^2 \) and

\[
A_T(x) = \frac{3}{4} \pi x, \quad B_T(x) = \frac{3}{4} \left( 2 \frac{x^2}{1 - x^2} + x \ln \frac{1 + x}{1 - x} \right), \tag{3.41a}
\]

\[
A_L(x) = \frac{3}{2} \pi x, \quad B_L(x) = \frac{3}{2} \left( 2 - x \ln \frac{1 + x}{1 - x} \right). \tag{3.41b}
\]

Let us now turn to the calculation of the soft part of the gravitino production rate. In the previous section we have seen that the squared matrix elements for the \( 2 \to 2 \) scatterings given in Table 3.2 contain the factor

\[
|\mathcal{M}_i|^2 \propto \left( 1 + \frac{M_5^2}{3m_G^2} \right). \tag{3.42}
\]

The first term results from the helicity \( \pm 3/2 \) states of the gravitino, the second term from the helicity \( \pm 1/2 \) states of the gravitino which represent the goldstino components. For the gravitino self-energy \( \Sigma_G \), it has been shown up to two loop order in the gauge couplings that one obtains the same factor (3.42); cf. [2]. Indeed, we employ the effective theory for light gravitinos given in (2.83) to calculate the production rate for the helicity \( \pm 1/2 \) components of the gravitino in terms of the imaginary part of the goldstino self-energy \( \Sigma_\psi^{(\alpha)}(P) \), namely,

\[
E \left. \frac{d^2 \Gamma_\psi^{(\alpha)}}{d^3 p} \right|_{\text{soft}} = -\frac{1}{(2\pi)^3} f_F(E) \text{Im} \Sigma_\psi^{(\alpha)}(E + i\varepsilon, p)|_{k<k^*}. \tag{3.43}
\]
This leads to the full rate by replacing $M_{G}^{2}/3m_{G}^{2}$ with the prefactor \((3.42)\). Note that we have introduced the cutoff $k^*$ since we consider here only soft three-momentum transfers.

The leading contribution to the self-energy is given by the gauge boson–gaugino loop shown in Fig. 3.2; see the discussion below (2.83). Here the HTL-resummed gauge boson propagator \((3.36)\) is indicated by the blob. Accordingly, one finds for the leading contribution to the self-energy\(^{3}\)

\[
-i \Sigma_{\psi}(P) = \frac{X^{(a)}M_{G}^{2}}{24m_{G}^{2}M_{P}^{2}} \sum_{s=\pm 1/2} \int \frac{d^{4}K}{(2\pi)^{4}} \text{tr} \left\{ \overline{u}^{s}(P) \left[ K, \gamma^{\mu} \right] \frac{iQ}{Q^{2}} i \Delta^{(a)}_{\mu\nu} \left[ \gamma^{\mu}, K \right] u^{s}(P) \right\} .
\]

\[(3.44)\]

Working in the high-energy limit, we have neglected the masses in the gaugino propagators. The polarization sum for the goldstino spinors $u^{s}(P)$ and $\overline{u}^{s}(P)$ is included. The gauge bosons carry soft momenta $K$ and the gaugino momenta are given by $Q = P - K$.

The multiplicity factors $X^{(a)}$ count the number of gauge bosons/gauginos in the loop for the corresponding gauge group $G_{\alpha}$, namely,

\[
X^{(a)} = N_{a}^{2} - 1 \quad \text{for} \quad SU(N_{a}) ,
\]

\[(3.45a)\]

\[
X^{(1)} = 1 \quad \text{for} \quad U(1)_{Y} .
\]

\[(3.45b)\]

One computes \((3.44)\) in the imaginary time formalism where the energies of the particles are given by their Matsubara frequencies. For the gauge bosons, the frequencies are

\(^{3}\text{We commit a certain abuse of terminology since a self-energy contribution is usually referred to as the expression which one obtains from the amputated diagram of Fig. 3.2. The inclusion of the spinors } u(P) \text{ and } \overline{\tau}(P) \text{ is necessary to interpret } \(3.43\) \text{ in terms of a probability (see [15])}.


\[\text{Figure 3.3: The leading contribution to the imaginary part of the goldstino self-energy in the effective theory of light gravitinos. The blob indicates the HTL-resummed gauge boson propagator } (3.36).\]
even,  \( k_0 = 2\pi inT \), and the integral over \( k_0 \) turns into a sum over discrete energies, i.e.,
\[
\int \frac{dk_0}{2\pi} \rightarrow iT \sum_{n=-\infty}^{\infty}.
\] (3.46)

After performing the polarization sum, the self-energy reads
\[
\Sigma^{(\alpha)}(P) = \frac{4}{3} \frac{X^{(\alpha)}m_\alpha^2 T}{m_G^2 M_p^2} \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \frac{1}{Q^2} \left( D_L \Delta_L + D_T \Delta_T \right),
\] (3.47)

with the Dirac traces:
\[
D_T = \frac{1}{32} \text{tr} \{ \slashed{P} \slashed{K}, \gamma_\nu \} \slashed{Q} \slashed{K}, \gamma_\mu \} A_{\mu\nu} \}
\] (3.48a)
\[
D_L = \frac{1}{32} \text{tr} \{ \slashed{P} \slashed{K}, \gamma_\nu \} \slashed{Q} \slashed{K}, \gamma_\mu \} B_{\mu\nu} \}
\] (3.48b)

Note that the dependence on the gauge fixing parameter \( \xi \) drops out since (3.37c) contracted with the gauge boson momentum \( K \) vanishes in the Dirac trace.

Using the spectral representations of the propagators (3.39), the summation over the Matsubara frequencies can be performed conveniently with the Saclay method \[24\]. The result reads after analytic continuation from discrete energy values \( p_0 \) to continuous real goldstino energies \( E \):
\[
\frac{d\Gamma^{(\alpha)}_\psi}{d^3p} \bigg|_{\text{soft}} = \frac{X^{(\alpha)}m_\alpha^2 M_\alpha^2 T}{48\pi^4 M_p^2 m_G^2} f_F(E) \int_0^{k^*} dk k^3 \int^{k^*}_k d\omega \omega \\
\times \left[ \rho_L(\omega, k) \left(1 - \frac{\omega^2}{k^2}\right) + \rho_T(\omega, k) \left(1 - \frac{\omega^2}{k^2}\right)^2 \right].
\] (3.49)

The structure of the integrand is identical to the one obtained for the axion production rate \[14\] where the analytic dependence on the cutoff \( k^* \) is extracted. Thus, after performing the integrations, one finds
\[
\frac{d\Gamma^{(\alpha)}_\psi}{d^3p} \bigg|_{\text{soft}} = f_F(E) \frac{X^{(\alpha)}m_\alpha^2 M_\alpha^2 T}{32\pi^4 M_p^2 m_G^2} \left[ \ln \left(\frac{k^*}{m_\alpha^2}\right) - 1.379 \right].
\] (3.50)

The replacement of \( M_\alpha^2/3m_G^2 \) with the factor (3.42) for the full theory yields the final result for the soft part of the gravitino production rate (3.6)
\[
\frac{d\Gamma^{(\alpha)}_G}{d^3p} \bigg|_{\text{soft}} = f_F(E) \sum_{\alpha=1}^3 \left(1 + \frac{M_\alpha^2}{3m_G^2} \right) \frac{3X^{(\alpha)}m_\alpha^2 T}{32\pi^4 M_p^2} \left[ \ln \left(\frac{k^*}{m_\alpha^2}\right) - 1.379 \right]
\] (3.51)

with the multiplicites \( X^{(\alpha)} \) given in (3.45) and the thermal masses \( m_\alpha \) given in (3.34) and (3.35).

Adding the results for the hard and the soft part of the gravitino production rate, one finds that the logarithmic dependence on \( k^* \) cancels out. We will do so by calculating the Boltzmann collision term which is the crucial quantity for all further calculations.
3.4 The Boltzmann Collision Term

The quantity we are interested in is the gravitino number density

\[ \tilde{n}_G = 4 \int \frac{d^3 p}{(2\pi)^3} f_G(E, t). \tag{3.52} \]

Its evolution with cosmic time is governed by the Boltzmann equation

\[ \frac{d\tilde{n}_G}{dt} + 3H \tilde{n}_G = C_G. \tag{3.53} \]

The second term on the left-hand side accounts for the dilution of gravitinos due to the expansion of the Universe, which is described by the Hubble parameter \( H \). For negligible gravitino disappearance processes, the collision term \( C_G \) on the right-hand side of the Boltzmann equation describes gravitino production processes. It is obtained by integrating the thermal gravitino production rate

\[ C_G = \int \frac{d^3 p}{E} \left[ E \frac{d\Gamma_G}{d^3 p} \right] = \int d^3 p \left[ \frac{d\Gamma_G^1}{d^3 p} \bigg|_{\text{hard}} + \frac{d\Gamma_G^1}{d^3 p} \bigg|_{\text{soft}} \right]. \tag{3.54} \]

Consider first the soft part (3.51). The corresponding collision term reads

\[ C_G,_{\text{soft}}^{(\alpha)} = \left( 1 + \frac{M^2_\alpha}{3m^2_G} \right) \frac{3X^{(\alpha)}m^2_\alpha T}{8\pi^3 M_P^2} \left[ \ln \left( \frac{k^*}{m_\alpha^2} \right) - 1.379 \right] \int_0^\infty dE \frac{E^2}{e^{E/T} + 1}. \tag{3.55} \]

The final integration over the Fermi-Dirac distribution function is easily performed. The result reads

\[ C_G,_{\text{soft}}^{(\alpha)} = \sum_{\alpha=1}^3 \left( 1 + \frac{M^2_\alpha}{3m^2_G} \right) \frac{9X^{(\alpha)}\zeta(3)m^2_\alpha T^4}{16\pi^3 M_P^2} \left[ \ln \left( \frac{k^*}{m_\alpha^2} \right) - 1.379 \right]. \tag{3.56} \]

Now we turn to the hard part (3.28). The production rates for the BFB, FFF, and BBF processes are given in (3.23). The numerical integrations are performed using VEGAS [25]. We get:

\[ C_G,_{\text{hard}}^{\text{BFB}} = \sum_{\alpha=1}^3 \frac{g^2_\alpha T^6}{M_P^2} \left( 1 + \frac{M^2_\alpha}{3m^2_G} \right) \times \left\{ c_\text{BFB,1}^{(\alpha)} \left[ \frac{\zeta(3)}{32\pi^3} \left( \ln \left( \frac{2T}{k^*} \right) + 0.9930 \right) - 11.1362 \times 10^{-4} \right] + c_\text{BFB,2}^{(\alpha)} \left[ -1.3284 \times 10^{-4} \right] \right\}. \tag{3.57} \]

\(^4\)The factor of four accounts for the internal degrees of freedom of the massive gravitino.
3.4. The Boltzmann Collision Term

\[ C_{G, \text{hard}}^{\text{FFF}} = \sum_{\alpha=1}^{3} \frac{g_\alpha^2 T^6}{M_\alpha^2} \left( 1 + \frac{M_\alpha^2}{3m_\alpha^2 G} \right) \]
\[ \times \left\{ c_{\text{FFF},1}^{(\alpha)} \left[ \frac{\zeta(3)}{64\pi^2} \left( \ln \left( \frac{2T}{k^*} \right) + 1.6862 \right) - 6.9992 \times 10^{-4} \right] + c_{\text{FFF},3}^{(\alpha)} \left[ 0.5039 \times 10^{-4} \right] \right\} \] (3.58)

\[ C_{G, \text{hard}}^{\text{BBF}} = \sum_{\alpha=1}^{3} \frac{g_\alpha^2 T^6}{M_\alpha^2} \left( 1 + \frac{M_\alpha^2}{3m_\alpha^2 G} \right) \]
\[ \times \left\{ -c_{\text{BBF},2}^{(\alpha)} \left[ -1.2975 \times 10^{-4} \right] + c_{\text{BBF},3}^{(\alpha)} \left[ 0.8647 \times 10^{-4} \right] \right\} \] (3.59)

In the sum of the soft and hard parts, the logarithmic dependence on \( k^* \) cancels out as anticipated. Plugging in the squared thermal masses (3.34) and (3.35), the final result can be brought into the following form

\[ C_{G} = \sum_{\alpha=1}^{3} \left( 1 + \frac{M_\alpha^2}{3m_\alpha^2 G} \right) \frac{3 \zeta(3) T^6}{16\pi^3 M_\alpha^2} c_\alpha g_\alpha^2 \ln \left( \frac{k_\alpha}{g_\alpha} \right) \] (3.60)

This is one of the main results of this thesis. The coefficients are given by \( c_\alpha = (11, 27, 72) \) and the scales in the logarithms by \( k_\alpha = (1.266, 1.312, 1.271) \). These numbers are associated with the gauge groups U(1)Y, SU(2)L, and SU(3)c, respectively. The temperature \( T \) provides the scale for the evaluation of the gaugino mass parameters \( M_\alpha = (M_1, M_2, M_3) \) and the gauge couplings \( g_\alpha = (g', g, g_s) \).

The error in the hard production rate of Ref. [2] manifests itself in a larger coefficient \( k_3 = 1.271 \) for the SU(3)c contribution than 1.163 obtained from [2].

Recall that the Braaten–Yuan prescription [14] relies on the weak coupling limit \( g \ll 1 \). Because of the large value of the strong coupling constant, e.g., \( g_s(100 \text{ GeV}) \simeq 2.5 \), the results for the production rates and the corresponding collision term require high temperatures \( T \gg 10^6 \text{ GeV} \), where, for example, \( g_s(10^6 \text{ GeV}) \simeq 0.99 \) and \( g_s(10^9 \text{ GeV}) \simeq 0.88 \).

Since we neglect gravitino disappearance processes, the collision term (3.60) acts as pure source term in the Boltzmann equation (3.53). For small temperatures, the logarithm turns negative for the SU(3)c part. Thus, the result becomes unphysical and shall not be trusted when extrapolating to small temperatures.
Chapter 4

Gravitino Cosmology

The new result for the Boltzmann collision term (3.60) has important implications for gravitino cosmology. In this chapter, we compute the gravitino yield which describes the primordial gravitino abundance. This quantity is crucial for phenomenological considerations of stable and unstable gravitinos. For gravitino dark matter scenarios, we calculate the relic gravitino density from thermal production. This consequently yields an upper bound on the reheating temperature of the Universe.

4.1 Gravitino Yield from Thermal Production

The observed isotropy and homogeneity of the Universe on large scales allows us to express the overall geometry of the Universe in terms of the Robertson-Walker metric with the line element

\[ ds^2 = dt^2 - R(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right), \]

(4.1)

where \((t, r, \theta, \phi)\) are comoving coordinates. By a rescaling of \(r\), the curvature parameter \(k\) can be assigned the discrete values \(k = 1, -1,\) or \(0\), corresponding to spatially closed, open, or flat geometries. The evolution of the scale factor \(R\) is described by the Friedmann equation

\[ H^2 = \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G_N}{3} \rho - \frac{k}{R^2}, \]

(4.2)

which defines the Hubble parameter \(H\). The total energy density of the Universe is denoted by \(\rho\). The derivative of the scale factor \(R\) with respect to cosmic time \(t\) is written as \(\dot{R}\).
In the radiation dominated epoch of the Universe, $\rho$ is given in good approximation by

$$\rho = g_\ast \frac{\pi^2}{30} T^4, \quad (4.3)$$

where $T$ is the photon temperature and $g_\ast$ denotes the effectively massless degrees of freedom, i.e., those species with mass $m_i \ll T_i$,

$$g_\ast = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4. \quad (4.4)$$

The Hubble rate in the radiation dominated epoch is given by ($k = 0$)

$$H(T) = \sqrt{\frac{g_\ast \pi^2}{90} T^2 M_P}, \quad (4.5)$$

where $M_P$ is the reduced Planck mass (2.4). During this epoch, time and temperature are related via $H(T) = 1/(2t)$.

The entropy density of the Universe, defined as $s \equiv (\rho + p)/T$, is dominated by the contribution of relativistic particles for which $p = \rho/3$ holds. Hence, one finds

$$s = g_\ast S \frac{4\pi^2}{90} T^3, \quad (4.6)$$

where

$$g_\ast S = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3. \quad (4.7)$$

In the previous section, the Boltzmann equation (3.53) has been written in terms of the gravitino number density $n_{\tilde{G}}$. It is useful to scale out the expansion by dividing the gravitino number density $n_{\tilde{G}}$ by the entropy density $s$. This defines the yield variable:

$$Y_{\tilde{G}} \equiv \frac{n_{\tilde{G}}}{s}. \quad (4.8)$$

With the conservation of entropy per comoving volume, $sR^3 = \text{const.}$, the Boltzmann equation (3.53) can be rewritten as

$$\frac{dY_{\tilde{G}}}{dt} = \frac{C_{\tilde{G}}}{s}. \quad (4.9)$$

Using $dt = -dT/[H(T)T]$, the gravitino yield is obtained by integrating

$$dY_{\tilde{G}} = -\frac{C_{\tilde{G}}(T)dT}{s(T)H(T)T}. \quad (4.10)$$
4.2 Gravitino Dark Matter

We consider thermal gravitino production beginning after completion of the reheating phase where the temperature of the primordial plasma is the reheating temperature $T_R$. We assume that any initial gravitino population has been diluted away by inflation, i.e., $Y_G(T_R) = 0$. Hence, the gravitino yield at the temperature $T'$ is

$$Y_G(T') = -\int_{T_R}^{T'} dT \frac{C_G(T)}{s(T)H(T)}.$$  \hfill (4.11)

Unstable gravitinos have typically long lifetimes because their interactions are suppressed by $M_P$. In particular, when the gravitino is lighter than $\lesssim 20$ TeV, its lifetime becomes longer than $\gtrsim 1$ s [26]. Hence, unstable gravitinos may decay during and/or after big-bang nucleosynthesis where $t_{BBN} \approx 1$ s and $T_{BBN} \approx 1$ MeV. Thus, the yield of gravitinos from thermal production prior to their decay is obtained for $T' = T_{BBN}$ ($m_G \lesssim 20$ TeV). Note that the $T^6$ dependence of the collision term (3.60) cancels out in the integrand of (4.11). Furthermore, recall that we consider scenarios in which $T_R \gtrsim 10^6$ GeV $\gg T_{BBN}$ (see Section 3.4). Thus, with the collision term (3.60), we can solve the Boltzmann equation to good approximation analytically. We find

$$Y_G(T_{BBN}) \simeq \sum_{\alpha=1}^{3} \left(1 + \frac{M_\alpha^2(T_R)}{3m_G^2} \right) y_\alpha g_\alpha(T_R)^2 \ln \left( \frac{k_\alpha}{g_\alpha(T_R)^2} \right) \left( \frac{T_R}{10^{10} \text{ GeV}} \right)$$ \hfill (4.12)

with $y_\alpha = (0.653, 1.604, 4.276) \times 10^{-12}$ for $U(1)_Y$, $SU(2)_L$, and $SU(3)_c$, respectively. The scales $k_\alpha$ in the logarithms are given at the end of Sec. 3.4. Here, we have used that after reheating, at temperature $T_R$, all particles of the MSSM are in thermal equilibrium and relativistic, for which $g_s(T_R) = g_s(T_R) = 915/4$.

The yield (4.12) is the starting point for studies of cosmological constraints in scenarios with unstable gravitinos. In the remainder of this chapter we will consider the case of a stable gravitino.

4.2 Gravitino Dark Matter

In the following, we focus on scenarios in which the gravitino is the LSP and stable due to R-parity conservation. Gravitinos, once produced, will thus contribute to the present value of the energy density $\rho$ since they do not decay. Because of the large redshift, the gravitino energy density at the present time $t_0$ is $\rho_G(t_0) = m_G n_G(t_0)$. We have seen that the thermal production of gravitinos is efficient only during the very early hot radiation dominated epoch so that

$$Y_G(T_0) \simeq Y_G(T_{BBN})$$ \hfill (4.13)
4.2. Gravitino Dark Matter

for stable gravitinos. Here,

$$T_{0} = 2.725 K = 2.348 \times 10^{-13} \text{GeV}$$  \hfill (4.14)

is the present temperature of the Universe \[5\].

Thus, the present day density parameter of thermally produced gravitinos is given by:

$$\Omega_{T}^{\text{TP}} h^2 = \frac{\rho_{T}(t_0)}{\rho_c(t_0)} h^2 = \frac{m_{\tilde{G}} Y_{\tilde{G}}(T_0) s(T_0) h^2}{\rho_c(T_0)} .$$  \hfill (4.15)

Here, the dimensionless quantity $h$ is used to parameterize the Hubble constant $H_0 = H(T_0) = 100 \, h \, \text{km s}^{-1} \, \text{Mpc}^{-1}$ and the present value of the critical density reads \[1\] \[5\]

$$\rho_c(t_0)/h^2 = 8.096 \times 10^{-47} \, \text{GeV}^4 .$$  \hfill (4.16a)

The entropy density $s(T_0)$ is obtained from (4.6) with $g_{*S}(T_0) = 43/11$.

Thus, from (4.12), (4.13), and (4.15) we find the result for the relic density of thermally produced gravitinos to leading order in the Standard Model gauge couplings:

$$\Omega_{T}^{\text{TP}} h^2 = \sum_{\alpha=1}^{3} \left( 1 + \frac{M_{\alpha}(T_R)^2}{3 m_{\tilde{G}}^2} \right) \omega_{\alpha} g_{\alpha}(T_{R})^2 \ln \left( \frac{k_{\alpha}}{g_{\alpha}(T_{R})} \right) \left( \frac{m_{\tilde{G}}}{100 \, \text{GeV}} \right) \left( \frac{T_{R}}{10^{10} \, \text{GeV}} \right)$$  \hfill (4.17)

with $\omega_{\alpha} = (0.018, 0.044, 0.117)$ for U(1)$_{Y}$, SU(2)$_{L}$, and SU(3)$_{c}$, respectively.

The relic gravitino density $\Omega_{T}^{\text{TP}} h^2$ is essentially linear in the reheating temperature $T_{R}$.

Recall that the production of the helicity $\pm 3/2$ states of gravitinos is described by the first term in the generic factor $(1 + M_{\alpha}^2/3 m_{\tilde{G}}^2)$.

Thus, the relative weights for the helicity $\pm 3/2$ production of gravitinos for the different factors of the Standard Model gauge group are basically given by $\omega_{\alpha} g_{\alpha}(T_{R})^2$—which is model-independent. The production of the helicity $\pm 1/2$ states depends on the ratio of squared gaugino masses $M_{\alpha}^2$ to gravitino mass $m_{\tilde{G}}$. Hence, the thermal production becomes more efficient for light gravitinos.

In order to calculate numbers from (4.17), we need to evaluate the gauge couplings $g_{\alpha}$ and the gaugino mass parameters $M_{\alpha}$ at the scale provided by the reheating temperature $T_{R}$.

At the one-loop level, the renormalization group (RG) equations for the gauge couplings read

$$\frac{dg_{\alpha}(Q)}{d \ln Q/\bar{Q}_0} = \frac{\beta_{\alpha}^{(1)}}{16\pi^2} g_{\alpha}(Q)$$  \hfill (4.18)

\[1\] $h = 0.73^{+0.04}_{-0.03}$ \[5\]
4.2. Gravitino Dark Matter

Figure 4.1: One-loop renormalization group running of the gauge couplings (a) and the gaugino mass parameters (b) in the MSSM. The point of gauge coupling unification is denoted as $M_{\text{GUT}}$. The runnings of the gaugino masses are inferred from (4.21) for universal $M_{1,2,3} = m_{1/2}$ at $M_{\text{GUT}}$ with a value of $m_{1/2} = 400$ GeV. They are given in (b) as solid lines. Dashed lines in (b) show a non-universal scenario with $0.5 M_{1,2} = M_3 = m_{1/2}$ at $M_{\text{GUT}}$, i.e., $x_{1,2} = 2$.

where $Q$ is the scale of evaluation and $Q_0$ is some input scale. One can solve (4.18) analytically:

$$g_\alpha(T_R) = \left[ g_\alpha(m_Z)^{-2} - \frac{\beta_\alpha^{(1)}}{8\pi^2} \ln \left( \frac{T_R}{m_Z} \right) \right]^{-1/2}$$

(4.19)

with $Q_0 = m_Z \simeq 91.19$ GeV [5] and $Q = T_R$. The beta-function coefficients $\beta_\alpha^{(1)} = (11, 1, -3)$ correspond to $g_\alpha = (g', g, g_s)$ in the MSSM, respectively. In Fig. 4.0a the point of gauge coupling unification is referred to as the grand unification (GUT) scale $M_{\text{GUT}} \simeq 2 \times 10^{16}$ GeV. It is defined as the point where the GUT-normalized hypercharge coupling,

$$\hat{g}_1 = \sqrt{\frac{5}{3}} g'$$

(4.20)

the weak coupling, $g$, and the strong coupling, $g_s$, meet.

The one-loop RG equations for the gaugino masses $M_\alpha$ are given by the expression analogous to (4.18) and the same coefficients $\beta_\alpha^{(1)}$ as above. Often, universal boundary conditions are considered for the RG equations in which the gaugino masses unify at

49
4.3 Upper Bounds on the Reheating Temperature

Since our result for the relic gravitino abundance depends on $M_\alpha$, we will also consider scenarios in which the gaugino masses do not unify at $M_{\text{GUT}}$. We can parameterize this by writing

$$\frac{1}{x_1} \frac{M_1(Q)}{g_1(Q)^2} = \frac{1}{x_2} \frac{M_2(Q)}{g_2(Q)^2} = \frac{M_3(Q)}{g_3(Q)^2}$$

which holds at any scale $Q$.\(^2\) For the gaugino masses, we choose the input scale $Q_0$ to be $M_{\text{GUT}}$. This defines the gaugino mass parameter $m_{1/2}$, namely,

$$m_{1/2} \equiv M_3(M_{\text{GUT}}) = m_{1/2} \frac{M_1(M_{\text{GUT}})}{x_1} = M_1(M_{\text{GUT}}) / x_1,$$

so that for a unifying scenario, i.e., for $x_{1,2} = 1$, we have $m_{1/2} = M_1(M_{\text{GUT}}) = M_2(M_{\text{GUT}}) = M_3(M_{\text{GUT}})$. As illustrated for $m_{1/2} = 400$ GeV, fixing $m_{1/2}$ determines all gaugino mass parameters $M_\alpha$.

Taking the RG evolution into account, we compute the relic gravitino density. Figure 4.1 shows $\Omega^{\text{TP}} h^2$ as a function of the reheating temperature $T_R$ for gravitino masses $m_G = 1, 10, 50,$ and $300$ GeV. We consider two representative values of the gaugino mass parameter $m_{1/2} = 400$ GeV and $1500$ GeV in Figs. 4.1a and 4.1b, respectively. The solid curves represent the gravitino density for universal gaugino masses $M_{1,2,3} = m_{1/2}$ at $M_{\text{GUT}}$ while the dashed curves show a non-universal scenario where $0.5 M_{1,2} = M_3 = m_{1/2}$ at $M_{\text{GUT}}$, i.e., $x_{1,2} = 2$. The corrected result for the SU(3)$_c$ contribution is given by the dotted lines. The grey band indicates the dark matter density $\Omega_{\text{dm}} h^2 = 0.105^{+0.007}_{-0.010}$.

and thus shows the parameter region in which the thermally produced gravitinos provide the observed dark matter density.

We find that electroweak processes enhance $\Omega^{\text{TP}} G$ by about 20% for universal gaugino masses at $M_{\text{GUT}}$. In non-universal cases, $M_{1,2} > M_3$ at $M_{\text{GUT}}$, the electroweak contributions are more important. For $0.5 M_{1,2} = M_3$ at $M_{\text{GUT}}$, they provide about 40% of $\Omega^{\text{TP}} G h^2$. Moreover, with our new $k_3$ value—see Eqs. (3.60) and (4.17)—we find an enhancement of about 30% of the SU(3)$_c$ contribution to relic density in comparison to the result given in [2].

4.3 Upper Bounds on the Reheating Temperature

In the gravitino dark matter scenario we can derive an upper bound on the reheating temperature $T_R$ from $\Omega^{\text{TP}} G \leq \Omega_{\text{dm}}$, once $m_{1/2}$ is specified. Figure 4.2 shows the upper

\(^2\) Up to small two loop effects and possible (unknown) threshold effects close to $M_{\text{GUT}}$ [9].
4.3. Upper Bounds on the Reheating Temperature

limits on $T_R$ for $m_{1/2} = 400\,\text{GeV}$ and $m_{1/2} = 1500\,\text{GeV}$, respectively. The solid and dashed curves give the upper bounds on $T_R$ inferred from our SU(3)$_c \times$ SU(2)$_L \times$ U(1)$_Y$ result of $\Omega^\text{TP}_G h^2$ for universal ($M_{1,2,3} = m_{1/2}$) and non-universal ($0.5 M_{1,2} = M_3 = m_{1/2}$) gaugino masses at $M_{\text{GUT}}$, respectively. The dotted curves show the SU(3)$_c$ limits for $M_3 = m_{1/2}$ at $M_{\text{GUT}}$. We have adopted

$$\Omega^\text{max}_\text{dm} h^2 = 0.126$$

(4.24)
as a nominal $3\sigma$ upper limit on $\Omega_{\text{dm}}h^2$.

Note that for higher values of $m_{1/2}$ the bounds on $T_R$ are more stringent; cf. (4.17). For small $m_{\tilde{G}}$, the thermal gravitino production is very efficient. Then $\Omega^\text{TP}_G h^2 \sim M_a^2 / 3m_{\tilde{G}}$ since the production of helicity $\pm 1/2$ states dominates. For large values of $m_{\tilde{G}}$, the upper limit on $T_R$ becomes more severe again. Then $\Omega^\text{TP}_G h^2 \sim m_{\tilde{G}}$ since the production of the helicity $\pm 3/2$ states constitutes the dominant part. The bounds on $T_R$ will become more severe when one includes non-thermal production processes such as late-decays of the next-to-lightest supersymmetric particle (NLSP) into the gravitino. This, however, depends on the details of the realized SUSY model while the bounds in Fig. 4.2 are rather model independent.
Figure 4.2: The relic gravitino density from thermal production, $\Omega_{\text{TP}}^2$, as a function of $T_R$. The solid and dashed curves show the SU(3)$_c \times$ SU(2)$_L \times$ U(1)$_Y$ results for universal ($M_{1,2,3} = m_{1/2}$) and non-universal ($0.5 M_{1,2} = M_3 = m_{1/2}$) gaugino masses at $M_{\text{GUT}}$, respectively. The dotted curves show the new result of the SU(3)$_c$ contribution for $M_3 = m_{1/2}$ at $M_{\text{GUT}}$. The grey band indicates the dark matter density $\Omega_{\text{dm}}^2$. 
4.3. Upper Bounds on the Reheating Temperature

\[ T_R \text{[GeV]} \]
\[ m_e \text{[GeV]} \]
\[ m_1/2 = 400 \text{ GeV} \]
\[ \Omega_{\text{max}} \]
\[ dm \text{h}^2 = 0 \]
\[ 10^{-2} \times 10^9 \]
\[ 1 \times 10^9 \]
\[ 5 \times 10^8 \]
\[ m_{\tilde{G}} \text{[GeV]} \]
\[ m_{1/2} = 1500 \text{ GeV} \]

Figure 4.3: Upper bounds on \( T_R \) from \( \Omega_{\text{TP}}^{\text{max}} \leq \Omega_{\text{dm}}^{\text{max}} \). The solid and dashed curves show the limits inferred from the full SU(3)\(_c\) × SU(2)\(_L\) × U(1)\(_Y\) result (4.17) for \( M_{1,2,3} = m_{1/2} \) and \( 0.5 M_{1,2} = M_3 = m_{1/2} \) at \( M_{\text{GUT}} \), respectively. The dotted curves show the limit on \( T_R \) for our new result of the SU(3)\(_c\) contribution.
Chapter 5

Testing Leptogenesis at Colliders

The smallness of the neutrino masses can be understood naturally in terms of the see-saw mechanism \[27, 28\] once the Standard Model is extended with right-handed neutrinos which have heavy Majorana masses and only Yukawa couplings. For a reheating temperature after inflation, \(T_R\), which is larger or not much smaller than the masses of the heavy neutrinos, these particles are produced in thermal reactions in the early Universe. The CP-violating out-of-equilibrium decays of the heavy neutrinos generate a lepton asymmetry that is converted into a baryon asymmetry by sphaleron processes \[29\]. This mechanism, known as thermal leptogenesis, can explain the cosmic baryon asymmetry for \(T_R \gtrsim 3 \times 10^9\) GeV \[30\].

One will face severe cosmological constraints on \(T_R\) if supersymmetry is discovered. We have seen that gravitinos are produced efficiently in the hot primordial plasma. Because of their extremely weak interactions, unstable gravitinos with \(m_\tilde{G} \lesssim 5\) TeV have long lifetimes, \(\tau_\tilde{G} \gtrsim 100\) s, and decay after BBN. The associated decay products affect the abundances of the primordial light elements. Demanding that the successful BBN predictions are preserved, bounds on the abundance of gravitinos before their decay can be derived which imply \(T_R \lesssim 10^8\) GeV for \(m_\tilde{G} \lesssim 5\) TeV \[26\]. Thus, the temperatures needed for thermal leptogenesis are excluded.

Let us therefore consider SUSY scenarios in which a gravitino with \(m_\tilde{G} \gtrsim 10\) GeV is the LSP and stable due to R-parity conservation. These scenarios are particularly attractive for two reasons: (i) the gravitino LSP can be dark matter and (ii) thermal leptogenesis can still be a viable explanation of the baryon asymmetry \[31\].
5.1 Collider Predictions of Leptogenesis

Thermal leptogenesis requires $T_R \geq 3 \times 10^9 \text{ GeV}$ [30]. This condition together with the constraint $\Omega_G^{\text{TP}} \leq \Omega_{\text{dm}}^{\text{max}}$ [see Eqn. (4.24)] leads to upper limits on the gaugino masses. The SU(3)$_c$ result for $\Omega_G^{\text{TP}}$ implies limits on the gluino mass [2, 32]. With our SU(3)$_c \times$ SU(2)$_L \times$ U(1)$_Y$ result, the limits on the gluino mass $M_3$ become more stringent because of the new $k_3$ value (3.60) and the additional electroweak contributions. Moreover, as a prediction of thermal leptogenesis, we obtain upper limits on the electroweak gaugino mass parameters $M_{1,2}$. At the Large Hadron Collider (LHC) and the International Linear Collider (ILC), these limits will be probed in measurements of the masses of the neutralinos and charginos, which are typically lighter than the gluino. If the superparticle spectrum does not respect these bounds, one will be able to exclude standard thermal leptogenesis.

Figure 5.0 shows the gaugino mass bounds for $T_R = 10^9, 3 \times 10^9, \text{ and } 10^{10} \text{ GeV}$ evolved to $M_{\text{GUT}}$, i.e., in terms of limits on the gaugino mass parameter $m_{1/2}$. With the observed superparticle spectrum, one will be able to evaluate the gaugino mass parameters $M_{1,2,3}$ at $M_{\text{GUT}}$ using the SUSY renormalization group equations [33, 34, 35, 36]. While the
5.2 Decays of the Next-to-Lightest Supersymmetric Particle

With a gravitino LSP of $m_{\tilde{G}} \geq 10$ GeV, the next-to-lightest SUSY particle (NLSP) has a long lifetime of $\tau_{\text{NLSP}} \geq 10^6$ s [37, 38]. After decoupling from the primordial plasma, each NLSP decays into one gravitino LSP and Standard Model particles. The resulting relic density of these non-thermally produced gravitinos is given by

$$\Omega_{\tilde{G}}^{\text{NTP}} h^2 = \frac{m_{\tilde{G}}}{m_{\text{NLSP}}} \Omega_{\text{NLSP}} h^2,$$

(5.1)

where $m_{\text{NLSP}}$ is the mass of the NLSP and $\Omega_{\text{NLSP}} h^2$ is the relic density that the NLSP would have today, if it had not decayed. As shown below, more severe limits on $m_{1/2}$ are obtained with $\Omega_{\tilde{G}}^{\text{NTP}} h^2$ taken into account. Moreover, since the NLSP decays take place after BBN, the emitted Standard Model particles can affect the abundance of the primordial light elements. Successful BBN predictions thus imply bounds on $m_{\tilde{G}}$ and $m_{\text{NLSP}}$ [37, 38]. From these cosmological constraints it has been found that thermal leptogenesis remains viable only in the cases of a charged slepton NLSP or a sneutrino NLSP [32, 39].

5.3 Collider Tests of Leptogenesis

Thermal leptogenesis will predict a lower bound on the gravitino mass $m_{\tilde{G}}$ once the masses of the Standard Model superpartners are known. With a charged slepton as the lightest Standard Model superpartner, it could even be possible to identify the gravitino as the LSP and to measure its mass $m_{\tilde{G}}$ at future colliders [40, 41, 42, 43]. Confronting the measured $m_{\tilde{G}}$ with the predicted lower bound will then allow us to decide about the viability of thermal leptogenesis.

To be specific, let us assume that the analysis of the observed spectrum [35, 36] will point to the universality of the soft SUSY breaking parameters at $M_{\text{GUT}}$ and, in particular, to the minimal supergravity (mSUGRA) scenario with the gaugino mass parameter
3. Collider Tests of Leptogenesis

$m_{1/2} = 400$ GeV, the scalar mass parameter $m_0 = 150$ GeV, the trilinear coupling $A_0 = -150$, a positive higgsino mass parameter, $\mu > 0$, and the mixing angle $\tan \beta = 30$ in the Higgs sector. A striking feature of the spectrum will then be the appearance of the lighter stau $\tilde{\tau}_1$ with $m_{\tilde{\tau}_1} = 143.4$ GeV as the lightest Standard Model superpartner [44]. In the considered gravitino LSP case, $10$ GeV $\leq m_{\tilde{G}} < m_{\tilde{\tau}_1}$, this stau is the NLSP and decays with a lifetime of $\tau_{\tilde{\tau}_1} \geq 10^6$ s into the gravitino. For the identified mSUGRA scenario and the considered reheating temperatures, the cosmological abundance of the $\tilde{\tau}_1$ NLSP prior to decay can be computed from $\Omega_{\tilde{\tau}_1} h^2 = \Omega_G h^2 \simeq 3.83 \times 10^{-3}$, which is provided by the computer program micrOMEGAs [45]. For given $m_{\tilde{G}}$, this abundance determines $\Omega_{\tilde{\tau}_1} h^2$ and the release of electromagnetic (EM) and hadronic energy in $\tilde{\tau}_1$ NLSP decays governing the cosmological constraints [37, 38].

Figure 5.1 allows us to probe the viability of thermal leptogenesis in the considered mSUGRA scenario. From the constraint $\Omega_{\tilde{\tau}} h^2 \leq \Omega_{\text{max}}$, we obtain the solid curves which provide the upper limits on $m_{1/2}$ for $T_R = 10^9$, $3 \times 10^9$, and $10^{10}$ GeV. The dashed line indicates the $m_{1/2}$ value of the considered scenario. The vertical solid line is given by $m_{\tilde{\tau}_1} = 143.4$ GeV which limits $m_{\tilde{G}}$ from above. In the considered scenario, the $m_{1/2}$ value exceeds the $m_{1/2}$ limits for $T_R \geq 10^{10}$ GeV. Thus, temperatures above $10^{10}$ GeV can be excluded. Temperatures above $3 \times 10^9$ GeV and $10^9$ GeV remain allowed for $m_{\tilde{G}}$ values indicated by the dark-shaded (dark-green) and medium-shaded (light-green) regions, respectively. The $m_{\tilde{G}}$ values indicated by the light-shaded (grey) region are excluded by BBN constraints for late $\tilde{\tau}_1$ NLSP decays.

Here thermal leptogenesis, $T_R \geq 3 \times 10^9$ GeV, predicts $m_{\tilde{G}} \geq 130$ GeV and thus a $\tilde{\tau}_1$ lifetime of $\tau_{\tilde{\tau}_1} > 10^{11}$ s [37, 38]. If decays of long-lived $\tilde{\tau}_1$‘s can be analyzed at colliders giving evidence for the gravitino LSP [40, 41, 42, 43], there will be the possibility to determine $m_{\tilde{G}}$ in the laboratory: From a measurement of the lifetime $\tau_{\tilde{\tau}_1}$ governed by the decay $\tilde{\tau}_1 \rightarrow \tilde{G}\tau$, $m_{\tilde{G}}$ can be extracted using the supergravity prediction for the associated partial width,

$$
\tau_{\tilde{\tau}_1} \simeq \Gamma^{-1}(\tilde{\tau}_1 \rightarrow \tilde{G}\tau) = \frac{48\pi m_{\tilde{G}}^2 M_P^2}{m_{\tilde{\tau}_1}^5} \left(1 - \frac{m_{\tilde{G}}^2}{m_{\tilde{\tau}_1}^2}\right)^{-4}
$$

1Thermal leptogenesis requires right-handed neutrinos and thus an extended mSUGRA scenario. This could manifest itself in the masses of the third generation sleptons [33]. Since the effects are typically small, we leave a systematic investigation of extended scenarios for future work.

2We use the conservative BBN bounds considered in [38]. The average EM energy release in one $\tilde{\tau}_1$ NLSP decay is assumed to be $E_\tau/2$, where $E_\tau$ is the energy of the tau emitted in the dominant 2-body decay $\tilde{\tau}_1 \rightarrow \tilde{G}\tau$ (cf. Fig. 16 of Ref. [38]). With an EM energy release below $E_\tau/2$, the grey band can become smaller. For less conservative BBN constraints and/or enhanced EM energy release, the excluded $m_{\tilde{G}}$ region becomes larger.

57
Figure 5.2: Probing the viability of thermal leptogenesis. The solid curves show the limits on the gaugino mass parameter $m_{1/2}$ from $\Omega^\text{TP}_G + \Omega^\text{NTP}_G \leq \Omega^\text{max}_\text{dm}$ for $T_R = 10^9$, $3 \times 10^9$, and $10^{10}$ GeV. The dashed line indicates the $m_{1/2}$ value of the considered scenario. The vertical solid line is given by the $\tilde{\tau}_1$ NLSP mass which limits the gravitino LSP mass from above: $m_{\tilde{G}} < m_{\tilde{\tau}_1} = 143.4$ GeV. The $m_{\tilde{G}}$ values at which temperatures above $3 \times 10^9$ GeV and $10^9$ GeV remain allowed are indicated by the dark-shaded (dark-green) and medium-shaded (light-green) regions, respectively. The $m_{\tilde{G}}$ values within the light-shaded (grey) region are excluded by BBN constraints.

Moreover, for $m_{\tilde{G}} \geq 0.1 m_{\tilde{\tau}_1}$, $m_{\tilde{G}}$ can be inferred kinematically from the energy of the tau, $E_\tau$, emitted in the 2-body decay $\tilde{\tau}_1 \rightarrow \tilde{G} \tau$ [40, 42]:

$$m_{\tilde{G}} = \sqrt{m_{\tilde{\tau}_1}^2 - m_\tau^2 - 2m_{\tilde{\tau}_1} E_\tau}.$$  \hspace{1cm} (5.3)

While $m_{\tilde{G}}$ within the dark-shaded (dark-green) region will favor thermal leptogenesis, any $m_{\tilde{G}}$ outside of the medium-shaded (light-green) region will require either non-standard mechanisms lowering the $T_R$ value needed for thermal leptogenesis or an alternative explanation of the cosmic baryon asymmetry.
Chapter 6

Conclusions

The starting point of this thesis has been a discussion of the general supergravity Lagrangian in four spacetime dimensions. We have shown how supersymmetry can be broken spontaneously. For a simple gravity-mediation scenario, we have explicitly carried out the transition to a softly-broken supersymmetric theory as the low-energy limit of supergravity. Relating the obtained effective theory with the MSSM in the high-energy limit of unbroken electroweak symmetry and identifying the relevant gravitino interactions, we have derived all Feynman rules necessary for the calculation of the regeneration of gravitinos after inflation.

We have used the Braaten–Yuan [14] prescription together with the hard thermal loop resummation technique [16] to calculate the thermal gravitino production rate. We have considered the regeneration of gravitinos that starts after completion of reheating. As one of the main results of this thesis, we present the Boltzmann collision term (3.60) to leading order in the SU(3)$_c$ × SU(2)$_L$ × U(1)$_Y$ couplings. It acts as the source term in the Boltzmann equation which governs the time evolution of the gravitino number density in the thermal bath. The collision term has been obtained in a gauge invariant way within the framework of thermal field theory and does not depend on arbitrary cutoffs. Our result includes for the first time the SU(2)$_L$ × U(1)$_Y$ sector. Moreover, we correct an error in the previously known SU(3)$_c$ result which was obtained in Ref. [2].

With direct implications for gravitino cosmology, we obtain the gravitino yield (4.12) from thermal production by solving the Boltzmann equation. The yield parametrizes the primordial abundance of gravitinos in the early Universe. Hence, it is a crucial quantity for both scenarios with stable gravitinos and scenarios with unstable gravitinos. Focusing on gravitino dark matter scenarios, we have obtained the relic abundance of thermally produced gravitinos, $\Omega^\text{TP}_G$. It depends on the reheating temperature $T_R$, the gravitino
mass $m_{\tilde{G}}$, and the gaugino masses $M_\alpha$. We find that electroweak processes enhance $\Omega_{\tilde{G}}^{\text{TP}}$ by about 20\% for universal gaugino masses at the grand unification scale $M_{\text{GUT}}$. For non-universal scenarios, the electroweak contributions become more important. Furthermore, with our new SU(3)$_c$ result, we find an enhancement of the SU(3)$_c$ contribution to $\Omega_{\tilde{G}}^{\text{TP}}$ by about 30\% as compared to the relic density obtained in [2]. We also give an update for the upper bounds on the reheating temperature depending on the specified gaugino mass parameter $m_{1/2}$.

With the result for the relic density of thermally produced gravitino LSPs, new gravitino and gaugino mass bounds emerge as a prediction of thermal leptogenesis which requires $T_R \geq 3 \times 10^9$ GeV. If supersymmetry is realized in nature, these bounds will be accessible at the LHC and the ILC. For certain SUSY parameter regions, a charged slepton will be the NLSP. We have studied a scenario in which the lighter stau is the NLSP. Here we have also taken into account the non-thermal gravitino production from the stau decays into the gravitino LSP. There exists then the exciting possibility to identify the gravitino as the LSP and to measure its mass. Confronting the measured gravitino mass with the predicted bounds will then allow for a unique test of the viability of thermal leptogenesis in the laboratory.
Appendix A

Conventions and Spinor Notation

The flat-space Lorentz metric is given by\(^1\)

\[ \eta_{\mu \nu} = \eta^{\mu \nu} \equiv \text{diag}(+1, -1, -1, -1). \]  \tag{A.1}

We fix the sign of the completely antisymmetric tensor \( \varepsilon_{\mu \nu \rho \sigma} \) by choosing

\[ \varepsilon_{0123} \equiv -1. \]  \tag{A.2}

Greek indices \( \mu, \nu, \cdots = 0, \ldots, 3 \) denote space-time indices.

A.1 Weyl Spinors

A two-component complex undotted Weyl spinor (left-handed Weyl spinor) \( \xi_\alpha \) transforms in the \((\frac{1}{2}, 0)\) matrix representation of the Lorentz group SO(3,1), i.e. under SL(2,\( \mathbb{C} \)), while the dotted Weyl spinor (right-handed Weyl spinor) \( \overline{\xi}_\dot{\alpha} \) is in the conjugate representation \((0, \frac{1}{2})\). Both spinors are related by hermitian conjugation, i.e. \( (\xi_\alpha)^\dagger = \overline{\xi}_{\dot{\alpha}} \) and \( (\overline{\xi}_{\dot{\alpha}})^\dagger = \xi_\alpha \). Explicitly, for a Lorentz transformation \( M \in \text{SL}(2, \mathbb{C}) \):

\[ \xi'^\alpha = M_{\alpha}^\beta \xi_\beta, \quad (\overline{\xi}'_{\dot{\alpha}} = M^{*\dot{\alpha}}_{\dot{\beta}} \overline{\xi}_{\dot{\beta}}), \]  \tag{A.3a}

\[ \xi'^\dot{\alpha} = M^{-1} \dot{\alpha} \beta \xi_\beta, \quad (\overline{\xi}'^\dot{\alpha} = (M^*)^{-1} \dot{\beta} \overline{\xi}_{\dot{\beta}}). \]  \tag{A.3b}

\(^1\)Though we make a distinction between Einstein and Lorentz indices in the beginning of Chapter 2, greek indices stand for flat spacetime indices in the other chapters of this thesis.
Spinor indices are pulled by the Lorentz invariant $\varepsilon$-tensors

\begin{align}
\varepsilon_{\alpha\beta} &\equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha\beta} &\equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\varepsilon_{\dot{\alpha}\dot{\beta}} &\equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\dot{\alpha}\dot{\beta}} &\equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\end{align}

namely,

\begin{align}
\xi_\alpha = \varepsilon_{\alpha\beta} \xi_\beta, \quad \xi^\alpha = \varepsilon^{\alpha\beta} \xi_\beta, \\
\bar{\xi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}}.
\end{align}

Furthermore, we define the Pauli sigma matrices (index 1,2,3) with lower Lorentz indices:

\begin{align}
\sigma_0 &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2 &\equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 &\equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align}

The standard convention for the contraction of anticommuting Weyl spinors is

\begin{align}
\xi_\eta &\equiv \xi^\alpha \eta_\alpha = \varepsilon^{\alpha\beta} \eta_\alpha \xi^\beta = \eta^\xi, \\
\bar{\xi}_{\bar{\eta}} &\equiv \bar{\xi}_{\dot{\alpha}} \bar{\eta}_{\dot{\beta}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\eta}_{\dot{\alpha}} \bar{\xi}_{\dot{\beta}} = \bar{\eta}^{\bar{\xi}}.
\end{align}

Note the spinor index structure of the sigma matrices $\sigma^{\mu}_{\alpha\dot{\alpha}}$. One defines

\begin{align}
\bar{\sigma}^{\mu}_{\dot{\alpha}\alpha} &\equiv \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon_{\alpha\beta} \sigma^{\mu}_{\beta\dot{\beta}},
\end{align}

as well as

\begin{align}
\sigma^{\mu\nu}_{\alpha\dot{\beta}} &\equiv \frac{1}{4} \left( \sigma^{\mu}_{\alpha\dot{\alpha}} \sigma^{\nu}_{\dot{\beta}\beta} - \sigma^{\nu}_{\alpha\dot{\alpha}} \sigma^{\mu}_{\beta\dot{\beta}} \right), \\
\bar{\sigma}^{\mu\nu}_{\dot{\alpha}\alpha} &\equiv \frac{1}{4} \left( \bar{\sigma}^{\mu}_{\dot{\alpha}\dot{\alpha}} \sigma^{\nu}_{\alpha\beta} - \bar{\sigma}^{\nu}_{\dot{\alpha}\dot{\alpha}} \sigma^{\mu}_{\alpha\beta} \right).
\end{align}

### A.2 Four-component Spinors

In the Weyl basis, the Dirac $\gamma$ matrices read

\begin{align}
\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}.
\end{align}
They satisfy the Clifford algebra

\[ \{ \gamma_\mu, \gamma_\nu \} = 2 \eta_{\mu\nu} \quad (A.12) \]

and anticommute with \( \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \), i.e. \( \{ \gamma_\mu, \gamma_5 \} = 0 \). In this representation\(^2\):

\[ \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (A.13) \]

We can write a Dirac spinor in terms of a left-handed and a right-handed Weyl spinor

\[ \psi_{(D)} = \begin{pmatrix} \xi_\alpha \\ \eta^{\dot{\alpha}} \end{pmatrix}, \quad (A.14) \]

and its adjoint spinor as

\[ \bar{\psi}_{(D)} \equiv \psi_{(D)}^\dagger \gamma^0 = \begin{pmatrix} \eta^\alpha \\ \xi_{\dot{\alpha}} \end{pmatrix} \quad (A.15) \]

With the chiral projectors \( P_L = \frac{1}{2}(1 + \gamma_5) \) and \( P_R = \frac{1}{2}(1 - \gamma_5) \) left-handed and right-handed four-spinors are given as

\[ \psi_L \equiv P_L \psi_{(D)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_\alpha \\ \eta^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix} \quad (A.16a) \]

and

\[ \psi_R \equiv P_R \psi_{(D)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_\alpha \\ \eta^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} 0 \\ \eta^{\dot{\alpha}} \end{pmatrix} \quad (A.16b) \]

respectively. For the adjoints of the chiral spinors one finds \( \bar{\psi}_L = \bar{\psi}_{(D)} P_R \) and \( \bar{\psi}_R = \bar{\psi}_{(D)} P_L \).

By virtue of the charge conjugation matrix \( C \) an equivalent realization of the Clifford algebra is given by the transposed \( \gamma \) matrices:

\[ C^{-1} \gamma_\mu C = -\gamma_\mu^T, \quad (A.17) \]

with

\[ C^\dagger = C^T = C^{-1} = -C \quad \text{and} \quad C^2 = -1. \quad (A.18) \]

\(^2\)The sign-convention for \( \gamma_5 \) results from the choice \((A.7)\) which allowed for a clean transition in Lorentz signatures.
The matrix $C$ can be written as

$$C = i \gamma^2 \gamma^0 = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\alpha\beta} \end{pmatrix}, \quad (A.19)$$

so that the charge-conjugated Dirac spinor of $(A.14)$ then reads

$$\psi^c_{(D)} \equiv C\overline{\psi}^T_{(D)} = \begin{pmatrix} \eta_\alpha \\ \xi_\alpha \end{pmatrix} \cdot (A.20)$$

A Majorana spinor is equal to its own charge-conjugate, i.e., $\psi_{(M)} = \psi^c_{(M)}$, so that it can be written as

$$\psi_{(M)} = \begin{pmatrix} \xi_\alpha \\ \overline{\xi}_\alpha \end{pmatrix}. \quad (A.21)$$
Appendix B

Feynman rules

Here we provide the complete set of the Feynman rules which is necessary for the calculations performed in Chapter 3. The method of a continuous fermion flow is addressed in Section 2.9. For details, see [13].

The momentum $P$ always flows from the left to the right for the external lines and propagators shown below. Furthermore, momenta are assumed to flow towards the vertices.

External Lines

- Gauginos $\lambda^{(\alpha)}$ and matter fermions $\chi_L$

- Gauge bosons $A^{(\alpha)}$

- Gravitinos $\psi_\mu$

\[
\begin{align*}
\bullet & \quad \text{Gauginos } \lambda^{(\alpha)} \text{ and matter fermions } \chi_L \\
\bullet & \quad \text{Gauge bosons } A^{(\alpha)} \\
\bullet & \quad \text{Gravitinos } \psi_\mu
\end{align*}
\]
Propagators

- Gauginos $\lambda^{(\alpha)}$
  \[
  \frac{i(\not{P} + M_\alpha)}{P^2 - M^2_\alpha} \delta^{ab}
  \]

- Matter fields $\chi_L$
  \[
  \frac{i(\not{P} + m_\chi)}{P^2 - m^2_\chi} \delta^{ij}
  \]

- Scalars $\phi$
  \[
  \frac{i}{P^2 - m^2_\phi} \delta^{ij}
  \]

- Gauge bosons $A^{(\alpha)}$
  \[
  i \left[ -\frac{g_{\mu\nu}}{P^2} + (1 - \xi) \frac{P_{\mu} P_{\nu}}{(P^2)^2} \right] \delta^{ab}
  \]

Relevant Gauge Vertices from Eq. (2.73)

1Note that for unbroken electroweak symmetry, the only non-vanishing masses for the chiral matter fermions arise from the higgsino mass parameter $\mu$; see (2.59) and (2.79).
Gravitino Vertices from Eq. (2.82)
Gravitino Vertices from the Effective Theory (2.83)

Note that in the effective theory for light gravitinos, the external gravitinos are treated as Majorana fermions.
Appendix C

Hard Production Rate

In this Appendix, we derive the hard part of the gravitino production rate for the full SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. We will follow the approach of [2] but show in detail how to perform the integrations and point out where differences between our $SU(3)_c$ result and the one obtained in [2] emerge.

Recall the definition for the production rate \((3.14)\)

\[
\frac{d\Gamma_{\tilde{G}}}{d^3p} \bigg|_{\text{hard}} = \frac{1}{(2\pi)^3 2E} \int \frac{d\Omega_p}{4\pi} \int \left[ \prod_{i=1}^{3} \frac{d^3p_i}{(2\pi)^3 2E_i} \right] (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P)
\]

\[
\times \sum_{\alpha=1}^{3} \left( f_{\text{BFB}} |M_{\text{BFB}}^{(\alpha)}|^2 + f_{\text{BBF}} |M_{\text{BBF}}^{(\alpha)}|^2 + f_{\text{FFF}} |M_{\text{FFF}}^{(\alpha)}|^2 \right) \Theta(|p_1 - p_3| - k^+),
\]

(C.1)

where the shorthand notation for the products of the quantum statistical distribution functions is given in \((3.15a)\). In the following, we show in detail how to obtain

\[
\frac{d\Gamma_{\tilde{G}}}{d^3p} \bigg|_{\text{hard}} = \frac{d\Gamma_{\tilde{G}}}{d^3p} \bigg|_{\text{hard}}^{(\text{BFB})} + \frac{d\Gamma_{\tilde{G}}}{d^3p} \bigg|_{\text{hard}}^{(\text{FFF})} + \frac{d\Gamma_{\tilde{G}}}{d^3p} \bigg|_{\text{hard}}^{(\text{BBF})}.
\]

(C.2)

C.1 Production Rate for BFB Processes

To keep better track of the various contributions, we use the notation

\[
\frac{d\Gamma_{\tilde{G}}}{d^3p} \bigg|_{\text{hard}}^{(\text{BFB})} = \sum_{\alpha=1}^{3} \frac{g_\alpha^2}{M_P^2} \left( 1 + \frac{M_\alpha^2}{3m_e^2 G} \right) \sum_{i=1}^{3} \left[ C_{\text{BFB},i}^{(\alpha)} I_{\text{BFB}}^{M_i^2} \right],
\]

(C.3)

which was introduced in Eq. \((3.23)\). The integrals $I_{\text{BFB}}^{M_i^2}$ will be calculated below. The thermal bath provides a distinguished frame of reference. It is the frame, in which
C.1. Production Rate for BFB Processes

The quantum statistical distribution functions $f_B$ and $f_F$ have their simple form (3.2a). Hence, we perform all integrations in the rest frame of the plasma.

C.1.1 Contribution from $|M_1|^2$

We have to compute the integral

$$I_{|M_1|^2} = \frac{1}{(2\pi)^3 2E} \int \frac{d\Omega_p}{4\pi} \int \left[ \prod_{i=1}^{3} \frac{d^3p_i}{(2\pi)^3 2E_i} \right] (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P)$$

$$\times f_{BFB} |M_1|^2 \Theta(|p_1 - p_3| - k^*) \, .$$

(C.4)

The squared matrix element

$$|M_1|^2 = -t - 2s - \frac{2s^2}{t}$$

(C.5)

becomes singular for $t = (P_1 - P_3)^2 \to 0$. It proves useful to relate the phase space integrations to the reference three momentum $k \equiv p_1 - p_3$. The Lorentz invariant measures can then be written as

$$\frac{d^3p_1}{2E_1} = \delta(P_1^2) \Theta(E_1) dE_1 d^3p_1$$

$$= \int d^3k \delta^3(k + p_3 - p_1) \delta(P_1^2) \Theta(E_1) dE_1 d^3p_1$$

$$= \delta(E_1^2 - |k + p_3|^2) \Theta(E_1) dE_1 d^3k$$

(C.6)

and

$$\frac{d^3p_2}{2E_2} \delta^4(P_1 + P_2 - P_3) = \frac{\delta(E_2 - |p_2|)}{2|p_2|} \Theta(E_2) dE_2 d^3p_2$$

$$\times \delta(E_1 + E_2 - E - E_3) \delta^3(p_1 + p_2 - p - p_3)$$

$$= \frac{\delta(E + E_3 - E_1 - |p + p_3 - p_1|)}{2|p + p_3 - p_1|} \Theta(E + E_3 - E_1)$$

$$\times \delta((E + E_3 - E_1)^2 - |p - k|^2) \Theta(E + E_3 - E_1) \, .$$

(C.7)

In order to perform the angular integrations, we are free to choose

$$k = k \ (0, 0, 1)^T,$$

$$p = E \ (0, \sin \theta, \cos \theta)^T,$$

$$p_3 = E_3 \ (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)^T \, ,$$

(C.8)
where we have made the approximation that the typical energies in the thermal bath are much higher than the rest masses of the particles involved in the scattering. This yields

\[ s = (P_1 + P_2)^2 = (P + P_3)^2 = 2EE_3(1 - \sin \theta \sin \phi \sin \tilde{\theta} - \cos \theta \cos \tilde{\theta}) , \]  
\[ t = (P_1 - P_3)^2 = (E_1 - E_3)^2 - k^2 , \]

and allows us to write \(|M_1(t, s)|^2\) as \(|M_1(E, E_1, E_2, k, \theta, \tilde{\theta}, \phi)|^2\). For the \(\delta\)-functions in (C.6) and (C.7), we find

\[
\delta((E + E_3 - E_1)^2 - |p - k|^2) = \frac{1}{2kE} \delta \left( \cos \tilde{\theta} - \frac{E^2 + k^2 - (E + E_3 - E_1)^2}{2kE} \right), \\
\delta(E_1^2 - |k + p_3|^2) = \frac{1}{2kE_3} \delta \left( \cos \theta - \frac{E_1^2 - E_3^2 - k^2}{2kE_3} \right). \]  

Equation (C.4) then reads

\[
I_{\text{BFB}}^{[M_1]^2} = \frac{1}{2^{14}\pi^9E^2} \int (-d \cos \tilde{\theta})d\tilde{\phi} \int dE_1 \int dE_3(-d \cos \theta)d\phi \int dk d\Omega_k \times \delta \left( \cos \theta - \frac{E_1^2 - E_3^2 - k^2}{2kE_3} \right) \delta \left( \cos \tilde{\theta} - \frac{E^2 + k^2 - (E + E_3 - E_1)^2}{2kE} \right) \\
\times f_{\text{BFB}} \left| M_1(E, E_1, E_2, k, \theta, \tilde{\theta}, \phi) \right|^2 \Theta(k - k^*) \Theta(E_1) \Theta(E_3) \Theta(E + E_3 - E_1) . \]  

The integrations over \(\cos \theta\) and \(\cos \tilde{\theta}\) yield restrictions on the integration range. Using also the \(\Theta\)-functions in (C.11) and positivity of \(k\) we find

\[
\cos \tilde{\theta} < 1 \Leftrightarrow k > E_1 - E_3 , \\
\cos \tilde{\theta} > -1 \Leftrightarrow E_3 - E_1 < k < E_1 + E_3 , \\
\cos \theta < 1 \Leftrightarrow E_1 - E_3 < k < 2E + E_3 - E_1 , \\
\cos \tilde{\theta} > -1 \Leftrightarrow k > E_3 - E_1 . \]  

The relations (C.12) are all fulfilled with the inclusion of \(\Theta(k - |E_1 - E_3|)\) and \(\Theta(2E + E_3 - E_1 - k)\).

Furthermore, it is easy to integrate out all angles and we find

\[
I_{\text{BFB}}^{[M_1]^2} = \frac{3}{2^{12}\pi^6E^2} \int dE_1 dE_3 dk f_{\text{BFB}}^{[M_1]^2} \Omega 
\]

where

\[
g_{\text{BFB}}^{[M_1]^2} \equiv ((E_1 - E_3)^2 - k^2) \left[ -1 + \frac{2}{3} \frac{E_1^2 + E_3^2 + 2EE_2}{k^2} - \frac{(E_3 + E_1)^2(E + E_2)^2}{k^4} \right], \]  

(C.14)
C.1. Production Rate for BFB Processes

\[ \Omega = \Theta(k - k^*)\Theta(k - |E_1 - E_3|) \]
\[ \times \Theta(E_1 + E_3 - k)\Theta(2E + E_3 - E_1 - k) \]
\[ \times \Theta(E_1)\Theta(E_3)\Theta(E + E_3 - E_1) . \]  \hspace{1cm} (C.15)

Here, \( E_2 = E + E_3 - E_1 \) is understood. Using

\[ \Theta(E_1 + E_3 - k) = 1 - \Theta(k - E_1 - E_3) \]

and

\[ \Theta(k - E_1 - E_3) = \Theta(k - E_1 - E_3) \Theta(k - |E_1 - E_3|) , \]

we can rewrite (C.15) as

\[ \Omega = \left[ \Theta(k - k^*)\Theta(k - |E_1 - E_3|)\Theta(2E + E_3 - E_1 - k) \right. \]
\[ - \Theta(k - k^*)\Theta(k - E_1 - E_3)\Theta(2E + E_3 - E_1 - k) \]
\[ \times \Theta(E_1)\Theta(E_3)\Theta(E + E_3 - E_1) . \]  \hspace{1cm} (C.16)

Now we insert \( 1 = \Theta(k^* - E_1 - E_3) + \Theta(E_1 + E_3 - k^*) \) in the second term in the brackets of (C.16) and use

\[ \Theta(k - k^*)\Theta(k - E_1 - E_3)\Theta(k^* - E_1 - E_3) = \Theta(k - k^*)\Theta(k^* - E_1 - E_3) , \]

and

\[ \Theta(k - k^*)\Theta(k - E_1 - E_3)\Theta(E_1 + E_3 - k^*) = \Theta(k - E_1 - E_3)\Theta(E_1 + E_3 - k^*) . \]

This leaves us with a sum of three integrals which we have to compute:

\[ I_{BFB}^{[M_1]^2} = g_{BFB,1}^{[M_1]^2} + g_{BFB,2}^{[M_1]^2} + g_{BFB,3}^{[M_1]^2} , \]  \hspace{1cm} (C.17)

namely,

\[ g_{BFB,1}^{[M_1]^2} = \frac{3}{2^{12} \pi^6 E^2} \int_0^\infty dE_1 \int_0^\infty dE_3 f_{BFB}(E + E_3 - E_1) \]
\[ \times \int dk \Theta(k - k^*)\Theta(k - |E_1 - E_3|)\Theta(2E + E_3 - E_1 - k) g_{BFB}^{[M_1]^2} , \]  \hspace{1cm} (C.18)

\[ g_{BFB,2}^{[M_1]^2} = -\frac{3}{2^{12} \pi^6 E^2} \int_0^\infty dE_1 \int_0^\infty dE_3 f_{BFB}(E + E_3 - E_1) \Theta(E_1 + E_3 - k^*) \]
\[ \times \int dk \Theta(k - E_1 - E_3)\Theta(2E + E_3 - E_1 - k) g_{BFB}^{[M_1]^2} , \]  \hspace{1cm} (C.19)
\[ g_{\text{BFB,3}}^{M_1^2} = - \frac{3}{2^{12} \pi^6 E^2} \int_0^\infty dE_1 \int_0^\infty dE_3 f_{\text{BFB}}(E + E_3 - E_1) \Theta(k^* - E_1 - E_3) \]
\[ \times \int dk \Theta(k - k^*) \Theta(2E + E_3 - E_1 - k) g_{\text{BFB}}^{M_1^2}, \tag{C.20} \]

**Calculation of \( g_{\text{BFB,1}}^{M_1^2} \)**

The \( \Theta \)-functions for in the integral \( g_{\text{BFB,1}}^{M_1^2} \) can be further manipulated. We multiply (C.18) by

\[ 1 = \Theta(k^* - |E_1 - E_3|) + \Theta(|E_1 - E_3| - k^*), \]

which allows us to split the integral into two parts

\[ g_{\text{BFB,1}}^{M_1^2} = g_{\text{BFB,11}}^{M_1^2} + g_{\text{BFB,12}}^{M_1^2}. \tag{C.21} \]

Note that

\[ \Theta(k - k^*) \Theta(k - |E_1 - E_3|) \Theta(k^* - |E_1 - E_3|) \]
\[ = \Theta(k - k^*) \Theta(k^* - |E_1 - E_3|), \]

and

\[ \Theta(k - k^*) \Theta(k - |E_1 - E_3|) \Theta(|E_1 - E_3| - k^*) \]
\[ = \Theta(k - |E_1 - E_3|) \Theta(|E_1 - E_3| - k^*), \]

so that we find

\[ g_{\text{BFB,11}}^{M_1^2} = \frac{3}{2^{12} \pi^6 E^2} \int_0^\infty dE_3 \int_0^\infty dE_1 f_{\text{BFB}}(E + E_3 - E_1) \Theta(k^* - |E_1 - E_3|) \]
\[ \times \int dk \Theta(k - k^*) \Theta(2E + E_3 - E_1 - k) g_{\text{BFB}}^{M_1^2}, \tag{C.22} \]

and

\[ g_{\text{BFB,12}}^{M_1^2} = \frac{3}{2^{12} \pi^6} \int_0^\infty dE_3 \int_0^\infty dE_1 f_{\text{BFB}}(E + E_3 - E_1) \Theta(|E_1 - E_3| - k^*) \]
\[ \times \int dk \Theta(k - |E_1 - E_3|) \Theta(2E + E_3 - E_1 - k) g_{\text{BFB}}^{M_1^2}. \tag{C.23} \]

The integration over \( dk \) in \( g_{\text{BFB,11}}^{M_1^2} \) yields a lengthy expression. From \( \Theta(k^* - |E_1 - E_3|) \) in (C.22) we see that the \( E_1 \) integration only contributes for

\[ E_3 - k^* < E_1 < E_3 + k^*. \]
Hence, in the limit \( k^* \to 0 \), we can set \( E_1 = E_3 \) in \( f_{FBF} \) and find

\[
g_{BFB,11}^{[M_1]} = \frac{1}{48\pi^6} f_F(E) \int_0^\infty dE_3 E_3^2 f_B(E_3)(1 + f_B(E_3)). \tag{C.24}
\]

This integral can be evaluated analytically by choosing the proper series expansion for the exponentials which reduces the integral to a Laplace transformation on the summands. The transformed series can then be resummed [46]. For (C.24) we find

\[
g_{BFB,11}^{[M_1]} = \frac{T^3 f_F(E)}{48\pi^6} \int_0^\infty dx \frac{x^2 e^x}{(e^x - 1)^2}
\]

\[
= \frac{T^3 f_F(E)}{48\pi^6} \sum_{n=1}^\infty n \int_0^\infty dx x^2 e^{-nx}
\]

\[
= \frac{T^3 f_F(E)}{24\pi^6} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{T^3 f_F(E) \zeta(2)}{24\pi^6}, \tag{C.25}
\]

where \( \zeta(x) \) is the Riemann Zeta function.

Now consider \( g_{BFB,12}^{[M_1]} \). In order to remove the absolute value in the \( \Theta \)-function of (C.23) we insert \( 1 = \Theta(E_1 - E_3) + \Theta(E_3 - E_1) \) and thereby split \( g_{BFB,12}^{[M_1]} \) into two parts,

\[
g_{BFB,12}^{[M_1]} = g_{BFB,121}^{[M_1]} + g_{BFB,122}^{[M_1]}, \tag{C.26}
\]

where

\[
g_{BFB,121}^{[M_1]} = \frac{3}{2^{12}\pi^6 E^2} \int_0^\infty dE_3 \int_0^\infty dE_1 f_{BFB} \Theta(E + E_3 - E_1) \Theta(E_1 - E_3 - k^*)
\]

\[
\times \int dk \Theta(k - E_1 + E_3) \Theta(2E + E_3 - E_1 - k) g_{BFB}^{[M_1]}, \tag{C.27}
\]

\[
g_{BFB,122}^{[M_1]} = \frac{3}{2^{12}\pi^6 E^2} \int_0^\infty dE_3 \int_0^\infty dE_1 f_{BFB} \Theta(E_3 - E_1 - k^*)
\]

\[
\times \int dk \Theta(k - E_3 + E_1) \Theta(2E + E_3 - E_1 - k) g_{BFB}^{[M_1]}. \tag{C.28}
\]

Performing the \( k \) integration yields

\[
g_{BFB,121}^{[M_1]} = \frac{1}{2^{8}\pi^6 E^2} \int_0^\infty dE_3 \int_0^{E_3 + E_1} dE_1 f_{BFB} \frac{(E_1^2 + E_3^2) E_2^2}{E_1 - E_3}, \tag{C.29}
\]

\[
g_{BFB,122}^{[M_1]} = -\frac{1}{2^{8}\pi^6 E^2} \int_0^\infty dE_3 \int_0^{E_3 - k^*} dE_1 f_{BFB} \frac{E^2 (E_1^2 + E_3^2)}{E_1 - E_3}. \tag{C.30}
\]

Since

\[
\frac{d}{dE_1} \ln \left( \frac{|E_1 - E_3|}{E_3} \right) = \frac{1}{E_1 - E_3}, \tag{C.31}
\]

\[74\]
an integration by parts will extract the logarithmic dependence on \( k^* \) in (C.29) and (C.30). Let us consider first equation (C.29) which is then given by the sum of the surface term and the remaining integral, namely,

\[
|M_1|^2_{\text{BFB, 121}} = |M_1|^2_{\text{surface, BFB, 121}} + |M_1|^2_{\text{partial, BFB, 121}}. \tag{C.32}
\]

The surface term is

\[
g_{\text{BFB, 121}}^{|M_1|^2_{\text{surface}}} = \frac{1}{2^{8/3} \pi^6 E^2} \int_0^\infty dE_3 \left[ \ln \left( \frac{E_1 - E_3}{E_3} \right) f_{\text{BFB}}(E_1^2 + E_3^2) \right]_{E_3=E_3\pm E_1 \pm E^*} \int_0^\infty dE_3 \ln \left( \frac{k^*}{E_3} \right) \frac{E_3^2 e^{E_3/T}}{(e^{E_3/T} - 1)^2}, \tag{C.33}
\]

where we used \( E_2 = E + E_3 - E_1 \) and set \( k^* \to 0 \) in the distribution functions after evaluating the borders. For the analytic integration of (C.33) we write the logarithm as \( \ln(k^*/E_3) = \ln k^* - \ln E_3 \). The integration of the first term is then the same as in (C.25).

For the second part we have to evaluate an integral of the form

\[
\int_0^\infty dx \ x^2 \ln x \frac{e^x}{(e^x - 1)^2}
= \sum_{n=1}^\infty \int_0^\infty dx \ x^2 \ln x \ e^{-nx}
= \sum_{n=1}^\infty \left( \frac{3}{n^2} - \frac{2 \gamma}{n^2} - \frac{2 \ln n}{n^2} \right) = \frac{\pi^2}{2} - \frac{\gamma \pi^2}{3} - 2\zeta'(2). \tag{C.34}
\]

Thus, for (C.33) we find

\[
g_{\text{BFB, 121}}^{|M_1|^2_{\text{surface}}} = \frac{f_F(E) T^3}{3 \cdot 2^7 \pi^4} \left[ \ln \left( \frac{T}{k^*} \right) + \frac{3}{2} - \gamma + \frac{\zeta'(2)}{\zeta(2)} \right], \tag{C.35}
\]

where we used \( \zeta(2) = \pi^2/6 \). In the remaining integral, we can set \( k^* \to 0 \) and obtain

\[
g_{\text{BFB, 121}}^{|M_1|^2_{\text{partial}}} = -\frac{1}{2^{8/3} \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_1 \ln \left( \frac{E_1 - E_3}{E_3} \right) \times \Theta(E_1 - E_3) \frac{d}{dE_1} \left[ f_{\text{BFB}}E_2^2(E_1^2 + E_3^2) \right]. \tag{C.36}
\]

Now we turn to the second integral (C.30). Note that for the surface term

\[
g_{\text{BFB, 122}}^{|M_1|^2_{\text{surface}}} = \frac{1}{2^{8/3} \pi^6} \int_0^\infty dE_3 \left[ \ln \left( \frac{E_1 - E_3}{E_3} \right) f_{\text{BFB}}(E_1^2 + E_3^2) \right]_{E_1=E_3\pm k^*} \tag{C.37}
\]

we get an additional contribution from the lower integral border,

\[
\lim_{E_1 \to 0} \left[ \ln \left( \frac{E_1 - E_3}{E_3} \right) f_B(E_1)(E_1^2 + E_3^2) \right] = -E_3 T. \tag{C.38}
\]
In fact, the upper border yields the same contribution as in (C.35), so that we can write

\[
g_{\text{BFB, 122}}^{[M_1]^2 \text{surface}} = g_{\text{BFB, 121}}^{[M_1]^2 \text{surface}} + g_{\text{BFB, 122}}^{[M_1]^2 \text{rest}}
\]  
(C.39)

with (in the limit \(k^* \to 0\))

\[
g_{\text{BFB, 122}}^{[M_1]^2 \text{rest}} = -\frac{T}{2^8 \pi^6} \int_0^\infty dE_3 E_3 f_F(E + E_3)[1 + f_B(E_3)]
\]

\[
= -\frac{T f_F(E)}{2^8 \pi^6} \int_0^\infty dE_3 E_3 [f_B(E_3) + f_F(E + E_3)].
\]  
(C.40)

This can be written as with \(E_3 = x T\) and \(a \equiv E/T\) as

\[
g_{\text{BFB, 122}}^{[M_1]^2 \text{rest}} = -\frac{T^3 f_F(E)}{2^8 \pi^6} \sum_{n=1}^\infty \int_0^\infty dx x \left[ e^{-nx} - (-1)^n e^{-nx} e^{-na} \right]
\]

\[
= -\frac{T^3 f_F(E)}{2^8 \pi^6} \sum_{n=1}^\infty \left[ 1 n^2 - (-1)^n e^{-na} n^2 \right],
\]  
(C.41)

and be resummed to

\[
g_{\text{BFB, 122}}^{[M_1]^2 \text{rest}} = \frac{T^3 f_F(E)}{2^8 \pi^6} \left[ \text{Li}_2 \left( -e^{-E/T} \right) - \zeta(2) \right].
\]  
(C.42)

The dilogarithm \(\text{Li}_2(x)\) is defined as

\[
\text{Li}_2(x) = -\int_0^x dt \frac{\ln(1 - t)}{t}.
\]  
(C.43)

The authors of [2] find a contribution like (C.42) for the BFB-part of the gravitino production rate, but do not specify from where it emerges. In contrast, we find below in (C.49) that \(g_{\text{BFB, 122}}^{[M_1]^2 \text{rest}}\) exactly cancels out and thus we do not get (C.42) as part of the production rate.

For the remaining integral, we find for \(k^* \to 0\) after an integration by parts

\[
g_{\text{BFB, 122}}^{[M_1]^2 \text{partial}} = \frac{1}{2^8 \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_1 \ln \left( \frac{|E_1 - E_3|}{E_3} \right)
\]

\[
\times \Theta(E_3 - E_1) \frac{d}{dE_1} \left[ f_{\text{BFB}} E^2(E_1^2 + E_3^2) \right].
\]  
(C.44)

**Calculation of \(g_{\text{BFB}, 2}^{[M_1]^2}\)**

In equation (C.19), the \(\Theta\)-functions allow for a contribution of the integrand only if

\[
E_1 + E_3 < 2E + E_3 - E_1 \Leftrightarrow E_1 < E.
\]
We can take the limit $k^* \to 0$ and perform the $k$ integration which gives

$$g_{\text{BFB,}2}^{|M_1|^2} = \frac{1}{2^8 \pi^6 E^2} \int_0^\infty dE_3 \int_0^E dE_1 f_{\text{BFB}}(E_1 - E) \left[(E + E_1)E_3 + E_1(E - E_1)\right].$$ \hspace{1cm} (C.45)

This can be rewritten with $E_2 = E + E_3 - E_1$ as

$$g_{\text{BFB,}2}^{|M_1|^2} = -\frac{1}{2^8 \pi^6 E^2} \int_0^\infty dE_3 \int_0^E dE_1 f_{\text{BFB}} \frac{1}{E_1 - E_3} \left(E_2^2 E_2^3 - E^2 E_3^2\right).$$ \hspace{1cm} (C.46)

An integration by parts yields a surface term and we write

$$g_{\text{BFB,}2}^{|M_1|^2} = g_{\text{BFB,}2}^{|M_1|^2 \text{ surface}} + g_{\text{BFB,}2}^{|M_1|^2 \text{ partial}}.$$ \hspace{1cm} (C.47)

For the lower border $E_1 = 0$ of the surface term

$$g_{\text{BFB,}2}^{|M_1|^2 \text{ surface}} = -\frac{1}{2^8 \pi^6 E^2} \int_0^\infty dE_3 \left[f_{\text{BFB}} \ln \left(\frac{|E_1 - E_3|}{E_3}\right) \right]_{E_1 = E} \times (E_2^2 (E + E_3 - E_1)^2 - E^2 E_3^2)\bigg|_{E_1 = 0},$$ \hspace{1cm} (C.48)

the Bose-distribution $f_B(E1)$ becomes singular. In the limit $E_1 \to 0$, we obtain

$$g_{\text{BFB,}2}^{|M_1|^2 \text{ surface}} = \frac{T}{2^8 \pi^6 E^2} \int_0^\infty dE_3 E_3 f_F(E + E_3)[1 + f_B(E_3)] = -g_{\text{BFB,}121}^{|M_1|^2 \text{ rest}}.$$ \hspace{1cm} (C.49)

We see that the arising dilogarithm (and $\zeta$-function) from the integral (C.40) cancels with (C.49).

In the partial term, we include $\Theta(E - E_1)$ in order to change the upper integral border and find

$$g_{\text{BFB,}2}^{|M_1|^2 \text{ partial}} = \frac{1}{2^8 \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E + E_3} dE_1 \ln \left(\frac{|E_1 - E_3|}{E_3}\right)$$

$$\times \Theta(E - E_1) \frac{d}{dE_1} \left[f_{\text{BFB}}(E_2^2 E_2^3 - E^2 E_3^2)\right].$$ \hspace{1cm} (C.50)

**Calculation of $g_{\text{BFB,}3}^{|M_1|^2}$**

The $k$ integration in (C.20) can be carried out directly and leads a lengthy result. From the $\Theta$-function $\Theta(k^* - E_1 - E_3)$ in (C.20) it follows that $E_1 < k^*$ and $E_2 < k^*$. Thus, one finds that the integration over $k$ yields an expression of order $k^*$ and hence

$$g_{\text{BFB,}3}^{|M_1|^2} = 0 \quad \text{for} \quad k^* \to 0.$$ \hspace{1cm} (C.51)
Result for $I_{BFB}^{[M_1]}$

Collecting the results, we find for the contributions from $|M_1|^2$

$$I_{BFB}^{[M_1]} = g_{BFB,11}^{[M_1]^2} + 2 \cdot g_{BFB,121}^{[M_1]^2 \text{surface}} + g_{BFB,121}^{[M_1]^2 \text{partial}} + g_{BFB,122}^{[M_1]^2 \text{partial}} + g_{BFB,2}^{[M_1]^2 \text{partial}}$$

$$= \frac{T^3 f_T(E)}{192 \pi^4} \left[ \ln \left( \frac{T}{k^*} \right) + \frac{17}{6} - \gamma + \frac{\zeta'(2)}{\zeta(2)} \right]$$

$$+ \frac{1}{256 \pi^6} \int_0^\infty dE_3 \int_0^{E_3 + E_3} dE_1 \ln \left( \frac{|E_1 - E_3|}{E_3} \right)$$

$$\times \left\{ - \Theta(E_1 - E_3) \frac{d}{dE_1} \left[ f_{BFB} \frac{E_2^2}{E_2^2} (E_1^2 + E_3^2) \right]$$

$$+ \Theta(E_3 - E_1) \frac{d}{dE_1} \left[ f_{BFB}(E_1^2 + E_3^2) \right]$$

$$+ \Theta(E - E_1) \frac{d}{dE_1} \left[ f_{BFB} \frac{E_1^2 E_2^2}{E_2^2} - E_3^2 \right] \right\}.$$  (C.52)

This concludes our analysis of $|M_1|^2$ and we can turn to the integrals containing $|M_2|^2$.

C.1.2 Contribution from $|M_2|^2$

The expression to evaluate is

$$I_{BFB}^{[M_2]} = \frac{1}{(2\pi)^3 2E} \int \frac{d\Omega_p}{4\pi} \int \prod_{i=1}^3 d^3 p_i \frac{d^3 p_1}{(2\pi)^3 2E_i} (2\pi)^4 \delta^4(P_1 + P_2 - P - P_3)$$

$$\times f_{BFB}|M_2|^2 \Theta(|p_1 - p_3| - k^*) .$$  (C.53)

Since there is no singular behavior of $|M_2|^2 = t$ for $t \to 0$, we set $k^* \to 0$ from the very beginning. We use the center of mass momentum $q = p + p_3$ as a reference momentum which yields in an analogous manner to (C.6) and (C.7) the integration measures

$$\frac{d^3 p_1}{2E_1} \delta^4(P_1 + P_2 - P - P_3)$$

$$= \delta((E + E_3 - E_2)^2 - |q - p_2|^2) \Theta(E + E_3 - E_2) ,$$  (C.54)

and

$$\frac{d^3 p_3}{2E_3} = \delta(E_3^2 - |q - p|^2) \Theta(E_3) dE_3 d^3 q .$$  (C.55)

Rotational invariance allows us to choose a frame where

$$q = q (0, 0, 1)^T ,$$

$$p = p (0, \sin \theta, \cos \theta)^T ,$$

$$p_2 = p_2 (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)^T .$$  (C.56)
C.1. Production Rate for BFB Processes

Again, we have assumed that the typical energies in the plasma are much higher as compared to the rest masses of the external particles which are involved in the scattering. The squared matrix element $|M_2|^2 = t$ in system (C.56) reads

$$|M_2|^2 = -2E_2 \left( 1 - \cos \theta \cos \tilde{\theta} - \sin \theta \sin \tilde{\theta} \sin \phi \right). \quad (C.57)$$

For the $\delta$-functions we find

$$\delta((E + E_3 - E_2)^2 - |q - p_2|^2) = \frac{1}{2qE_2} \delta \left( \cos \theta - \frac{E^2 + q^2 - (E + E_3 - E_2)^2}{2qE_2} \right),$$

$$\delta(E_3^2 - |q - p|^2) = \frac{1}{2qE} \delta \left( \cos \tilde{\theta} - \frac{E^2 - E_3^2 + q^2}{2qE} \right). \quad (C.58)$$

Equation (C.53) thus reads

$$I_{BFB}^{\mid M_2 \mid^2} = \frac{1}{2^{11} \pi^6 E^2} \int \left( -d \cos \tilde{\theta} \right) d\tilde{\theta} \left( -d \cos \theta \right) d\phi dE_2 dE_3 dq d\Omega_q$$

$$\times \delta \left( \cos \theta - \frac{E^2 + q^2 - (E + E_3 - E_2)^2}{2qE_2} \right) \delta \left( \cos \tilde{\theta} - \frac{E^2 - E_3^2 + q^2}{2qE} \right)$$

$$\times f_{BFB} \mid M_2 \mid^2 \Theta(E_2) \Theta(E_3) \Theta(E + E_3 - E_2) \ . \quad (C.59)$$

From the integration over the $\delta$-functions, we find the phase-space restrictions

$$\cos \theta < 1 \Leftrightarrow 2E_2 - E_3 - E < q < E + E_3 ,$$

$$\cos \theta > -1 \Leftrightarrow q > E + E_3 - 2E_2 ,$$

$$\cos \tilde{\theta} < 1 \Leftrightarrow E - E_3 < q < E + E_3 ,$$

$$\cos \tilde{\theta} > -1 \Leftrightarrow q > E_3 - E , \quad (C.60)$$

which yield the $\Theta$-functions $\Theta(q - |E - E_3|), \Theta(E + E_3 - q)$ and $\Theta(q - |2E_2 - E_3 - E|)$. We can perform the remaining angular integrations and find for Eq. (C.59)

$$I_{BFB}^{\mid M_2 \mid^2} = \frac{1}{2^{11} \pi^6 E^2} \int dE_2 dE_3 dq f_{BFB} g_{BFB}^{\mid M_2 \mid^2} \Omega \quad (C.61)$$

with

$$g_{BFB}^{\mid M_2 \mid^2} = \left( (E_3 + E)^2 - q^2 \right) \left[ -1 + \frac{E_3^2 - 2E_2E_3 - E^2 + 2E_2E}{q^2} \right] \quad (C.62)$$

and

$$\Omega = \Theta(q - |2E_2 - E_3 - E|) \Theta(q - |E - E_3|) \times \Theta(E + E_3 - q) \Theta(E + E_3 - E_2) \Theta(E_2) \Theta(E_3) \ . \quad (C.63)$$

79
Now we use
\[
\Theta(q - |2E_2 - E_3 - E|) = 1 - \Theta(|2E_2 - E_3 - E| - q)
\]
and
\[
\Theta(|2E_2 - E_3 - E| - q)\Theta(E + E_3 - q) = \Theta(|2E_2 - E_3 - E| - q)
\]
to split (C.61) into two parts, namely,
\[
I_{\text{BFB}}^{M_2} = g_{\text{BFB}, 1} + g_{\text{BFB}, 2}, \tag{C.64}
\]
with
\[
g_{\text{BFB}, 1}^{M_2} = \frac{1}{211 \pi^6 E^2} \int_0^\infty dE_2 \int_0^\infty dE_3 \Theta(E + E_3 - E_2) \times \int dq \Theta(q - |E - E_3|) \Theta(E + E_3 - q) f_{\text{BFB}} g_{\text{BFB}}^{M_2} \tag{C.65}
\]
\[
g_{\text{BFB}, 2}^{M_2} = -\frac{1}{211 \pi^6 E^2} \int_0^\infty dE_2 \int_0^\infty dE_3 \Theta(E + E_3 - E_2) \times \int dq \Theta(q - |E - E_3|) \Theta(|2E_2 - E_3 - E| - q) f_{\text{BFB}} g_{\text{BFB}}^{M_2}. \tag{C.66}
\]
We can take away the absolute value in \(\Theta(q - |E - E_3|)\) if we insert
\[
1 = \Theta(E - E_3) + \Theta(E_3 - E), \tag{C.67}
\]
which splits again the integrals (C.65) and (C.66) into
\[
g_{\text{BFB}, 1}^{M_2} = g_{\text{BFB}, 11}^{M_2} + g_{\text{BFB}, 12}^{M_2}, \tag{C.68}
\]
\[
g_{\text{BFB}, 2}^{M_2} = g_{\text{BFB}, 21}^{M_2} + g_{\text{BFB}, 22}^{M_2}. \tag{C.69}
\]
Let us consider first (C.68) which then reads
\[
g_{\text{BFB}, 11}^{M_2} = \frac{1}{211 \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E + E_3} dE_2 f_{\text{BFB}} \int_{E - E_3}^{E + E_3} dq \Theta(E - E_3) g_{\text{BFB}}^{M_2}, \tag{C.70}
\]
\[
g_{\text{BFB}, 12}^{M_2} = \frac{1}{211 \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E + E_3} dE_2 f_{\text{BFB}} \int_{E_3 - E}^{E + E_3} dq \Theta(E_3 - E) g_{\text{BFB}}^{M_2}. \tag{C.71}
\]
Integration over \(q\) gives
\[
g_{\text{BFB}, 11}^{M_2} = \frac{1}{28 \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E + E_3} dE_2 \Theta(E - E_3) f_{\text{BFB}} E_3^2 \left( \frac{E_3 - 3}{3} - E_1 \right), \tag{C.72}
\]
\[
g_{\text{BFB}, 12}^{M_2} = \frac{1}{28 \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E + E_3} dE_2 \Theta(E_3 - E) f_{\text{BFB}} E_3^2 \left( \frac{E_3 - 3}{3} - E_2 \right). \tag{C.73}
\]
Now we turn to \( g_{BFB,2}^{[M_2]^2} \). The insertion (C.67) yields

\[
g_{BFB,21}^{[M_2]^2} = -\frac{1}{2^{11}\pi^6E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 f_{BFB} \int_0^{E+E_3} \int_0^{E+E_3-2E_2} dE_3' dE_2' f_{BFB} \Theta(E_3 - E_2') \Theta(E - E_3') dE_3'' dE_2'' f_{BFB} dE_3' dE_2' f_{BFB} \Theta(E_3 - E_2') \Theta(E - E_3') g_{BFB}^{[M_2]^2}, \tag{C.74}
\]

\[
g_{BFB,22}^{[M_2]^2} = -\frac{1}{2^{11}\pi^6E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 f_{BFB} \int_0^{E+E_3} \int_0^{E+E_3-2E_2} dE_3' dE_2' f_{BFB} \Theta(E_3 - E_2') \Theta(E - E_3') dE_3'' dE_2'' f_{BFB} \Theta(E_3 - E_2') \Theta(E - E_3') g_{BFB}^{[M_2]^2}. \tag{C.75}
\]

We proceed in the same manner as in (C.67) and insert

\[
1 = \Theta(E_3 + E - 2E_2 - q) + \Theta(2E_2 - E_3 - E - q), \tag{C.76}
\]

which leads to four integrals

\[
g_{BFB,21}^{[M_2]^2} = g_{BFB,211}^{[M_2]^2} + g_{BFB,212}^{[M_2]^2}, \tag{C.77}
\]

\[
g_{BFB,22}^{[M_2]^2} = g_{BFB,221}^{[M_2]^2} + g_{BFB,222}^{[M_2]^2}. \tag{C.78}
\]

Note that

\[
\Theta(q - E + E_3) \Theta(E + E_3 - 2E_2 - q) \Rightarrow E_2 < E_3,
\]

\[
\Theta(q - E + E_3) \Theta(2E_2 - E - E_3 - q) \Rightarrow E < E_2,
\]

\[
\Theta(q + E - E_3) \Theta(E + E_3 - 2E_2 - q) \Rightarrow E_2 < E,
\]

\[
\Theta(q + E - E_3) \Theta(2E_2 - E - E_3 - q) \Rightarrow E_3 < E_2,
\]

so that we include the corresponding \( \Theta \)-functions, and then integrate over \( q \), namely,

\[
g_{BFB,211}^{[M_2]^2} = -\frac{1}{2^{11}\pi^6E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 f_{BFB} \int_0^{E+E_3} \int_0^{E+E_3-2E_2} dE_3' dE_2' f_{BFB} \Theta(E_3 - E_2') \Theta(E - E_3') g_{BFB}^{[M_2]^2}
\]

\[
= \frac{1}{3} \frac{1}{2^{11}\pi^6E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 \Theta(E_3 - E_2) \Theta(E - E_3)
\]

\[
\times f_{BFB}(E_2 - E_3) \left( (E_2 - E_3)(E_2 - E_3 - 3E_2 + 3E_3) - 3(E_2 + E_3)E \right), \tag{C.79}
\]

\[
g_{BFB,212}^{[M_2]^2} = -\frac{1}{2^{11}\pi^6E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 f_{BFB} \int_0^{2E_2 - E - E_3} \int_0^{E+E_3} dE_3' dE_2' f_{BFB} \Theta(E_2 - E) \Theta(E - E_3) g_{BFB}^{[M_2]^2}
\]

\[
= -\frac{1}{3} \frac{1}{2^{11}\pi^6E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 \Theta(E_2 - E) \Theta(E - E_3) f_{BFB}(E_2 - E)^3, \tag{C.80}
\]

\[\text{81}\]
\[ g_{\text{BFB, 221}}^{|M_2|^2} = -\frac{1}{2^{11} \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 f_{\text{BFB}} \int_{E_3-E}^{E+E_3-2E_2} dq \Theta(E - E_2) \Theta(E_3 - E) g_{\text{BFB}}^{|M_2|^2} \]
\[ = \frac{1}{3} \frac{1}{2^{8} \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 \Theta(E - E_2) \Theta(E_3 - E) f_{\text{BFB}}(E_2 - E)^3, \]  
\[ \text{(C.81)} \]

\[ g_{\text{BFB, 222}}^{|M_2|^2} = -\frac{1}{2^{11} \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 f_{\text{BFB}} \int_{2E_2-E_3}^{E+E_3} dq \Theta(E_2 - E_3) \Theta(E_3 - E) g_{\text{BFB}}^{|M_2|^2} \]
\[ = \frac{1}{3} \frac{1}{2^{8} \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 \Theta(E_2 - E_3) \Theta(E_3 - E) \]
\[ \times f_{\text{BFB}}(E_2 - E_3) [(E_2 - E_3)(E_2 + 2E_3) - 3(E_2 + E_3)E], \]  
\[ \text{(C.82)} \]

Result for \( I_{\text{BFB}}^{|M_2|^2} \)

For the contributions from \( |M_2|^2 \) we thus find

\[ I_{\text{BFB}}^{|M_2|^2} = g_{\text{BFB, 11}}^{|M_2|^2} + g_{\text{BFB, 12}}^{|M_2|^2} + g_{\text{BFB, 211}}^{|M_2|^2} + g_{\text{BFB, 212}}^{|M_2|^2} + g_{\text{BFB, 221}}^{|M_2|^2} + g_{\text{BFB, 222}}^{|M_2|^2} \]
\[ = \frac{1}{256 \pi^6} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 f_{\text{BFB}} \]
\[ \times \left\{ + \Theta(E - E_3) \frac{E_3^2}{E^2} \left( \frac{E_3}{3} - E_1 \right) + \Theta(E_3 - E) \left( \frac{E}{3} - E_2 \right) \right. \]
\[ + \Theta(E_3 - E_2) \Theta(E - E_3) \frac{E_2 - E_3}{3E^2} [(E_2 - E_3)(E_2 + 2E_3) - 3(E_2 + E_3)E] \]
\[ - \Theta(E_2 - E) \Theta(E - E_3) \frac{(E_2 - E)^3}{3E^2} \]
\[ + \Theta(E - E_2) \Theta(E_3 - E) \frac{(E_2 - E)^3}{3E^2} \]
\[ - \Theta(E_2 - E_3) \Theta(E_3 - E) \frac{E_2 - E_3}{3E^2} [(E_2 - E_3)(E_2 + 2E_3) - 3(E_2 + E_3)E] \left\}. \right. \]  
\[ \text{(C.83)} \]

This concludes the calculation for the BFB-processes since there is no contribution from the matrix element \( |M_3|^2 \) because of \( c_{\text{BFB, 3}}^{(\alpha)} = 0 \).
C.2 Production Rate for FFF Processes

We now turn to the processes where all in- and outgoing particles for the $2 \rightarrow 2$ scatterings are fermions. We have to compute

$$
\frac{d\Gamma_e}{d^3p} = \sum_{\alpha=1}^{3} \frac{g_{\alpha}^2}{M_F^2} \left( 1 + \frac{M_{\alpha}^2}{3m_{\alpha}^2} \right) \sum_{i=1}^{3} \left[ \gamma_{\alpha}^{(i)} \Gamma_{FFF,i}^{M_1^2} \right].
$$

(C.84)

C.2.1 Contribution from $|M_1|^2$

Let us first discuss matrix element $|M_1|^2$. The solution of the integral

$$
\int_{|M_1|^2} \frac{d\Omega_p}{4\pi} \int \left[ \prod_{i=1}^{3} \frac{d^2p_i}{(2\pi)^32E_i} \right] (2\pi)^4\delta^4(P_1 + P_2 - P - P_3) \times f_{FFF}|M_1|^2\Theta(|p_1 - p_3| - k^*)
$$

(C.85)

is obtained as in section C.1.1. Instead of $f_{BFB}$, we now have to integrate over the statistical factor $f_{FFF}$ (3.15a).

The analogous expression to (C.24) is

$$
g_{\text{FFF},11}^{M_1^2} = \frac{1}{48\pi^6} f_F(E) \int_0^\infty dE_3 \int_{E_3}^{E_3+E} f_F(E_3)(1 - f_F(E_3))
$$

(C.86)

and the $E_3$ integration yields

$$
g_{\text{FFF},11}^{M_1^2} = \frac{T^3 f_F(E)}{48\pi^6} \int_0^\infty dx \frac{x^2e^x}{(e^x + 1)^2}
$$

$$
= -\frac{T^3 f_F(E)}{48\pi^6} \sum_{n=1}^{\infty} (-1)^n n \int_0^\infty dx x^2 e^{-nx}
$$

$$
= -\frac{T^3 f_F(E)}{48\pi^6} \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2} = \frac{T^3 f_F(E)\zeta(2)}{48\pi^6}.
$$

(C.87)

Now consider $g_{\text{FFF},12}$ which we have written in section C.1.1 as two parts

$$
g_{\text{FFF},12}^{M_1^2} = g_{\text{FFF},121}^{M_1^2} + g_{\text{FFF},122}^{M_1^2},
$$

(C.88)

namely, [cf. (C.29), (C.30)],

$$
g_{\text{FFF},121}^{M_1^2} = \frac{1}{28\pi^6 E_1^2} \int_0^{E_3+E} dE_3 \int_{E_3+k^*}^{E_3+E_1} dE_1 f_{FFF}(E_1^2 + E_3^2) E_2^2 E_3^2 (E_1 - E_3)
$$

(C.89)

$$
g_{\text{FFF},122}^{M_1^2} = -\frac{1}{28\pi^6 E_1^2} \int_0^{E_3-k^*} dE_3 \int_0^{E_3-k^*} dE_1 f_{FFF}(E_1^2 + E_3^2) E_2^2 (E_1^2 + E_3^2)
$$

(C.90)
An integration by parts will exhibit the logarithmic $k^*$ dependence in the surface term so that we find for (C.89)

$$|M_1|^2 = |M_1|^2_{\text{surface}} + |M_1|^2_{\text{partial}},$$  \hspace{1cm} (C.91)

where the surface term reads

$$|M_1|^2_{\text{surface}} = \frac{1}{2^8 \pi^6 E^2} \int_{0}^{\infty} dE_3 \left[ \ln \left( \frac{|E_1 - E_3|}{E_3} \right) f_{\text{FFF}}(E_1^2 + E_3^2) \right] f_{\text{FFF}}(E_1^2 + E_3^2) E_1 = E_3 + E,$$

$$= - \frac{f_F(E)}{2^8 \pi^6} \int_{0}^{\infty} dE_3 \ln \left( \frac{k^*}{E_3} \right) \frac{E_3^2 e^{E_3/T}}{(e^{E_3/T} + 1)^2}. \hspace{1cm} (C.92)$$

Again, $E_2 = E + E_3 - E_1$ is understood and we have set $k^* \to 0$ in the distribution functions after evaluating the borders. The integral is solved in an analogous manner as in (C.33) so that we find

$$|M_1|^2_{\text{surface}} = \frac{f_F(E) T^3}{3 \cdot 2^8 \pi^4} \left[ \ln \left( \frac{2T}{k^*} \right) + 3 - \frac{1}{2} - \gamma + \frac{\zeta'(2)}{\zeta(2)} \right]. \hspace{1cm} (C.93)$$

Note that in contrast to (C.35) there is a factor of two in the numerator of the logarithm.

For the remaining part, we find for $k^* \to 0$

$$|M_1|^2_{\text{partial}} = - \frac{1}{2^8 \pi^6 E^2} \int_{0}^{\infty} dE_3 \int_{0}^{E + E_3} E_1 \ln \left( \frac{|E_1 - E_3|}{E_3} \right) \times \Theta(E_3 - E_1) \frac{d}{dE_1} f_{\text{FFF}} E_2^2 (E_1^2 + E_3^2) \right]. \hspace{1cm} (C.94)$$

The second integral (C.90) does not yield any additional contribution from the lower border as in (C.38). Thus,

$$|M_1|^2_{\text{rest}} = 0, \hspace{1cm} (C.95)$$

and we easily obtain

$$|M_1|^2_{\text{surface}} = |M_1|^2_{\text{surface}}. \hspace{1cm} (C.96)$$

Furthermore, we find for $k^* \to 0$

$$|M_1|^2_{\text{partial}} \int_{0}^{\infty} dE_3 \int_{0}^{E + E_3} E_1 \ln \left( \frac{|E_1 - E_3|}{E_3} \right) \times \Theta(E_3 - E_1) \frac{d}{dE_1} f_{\text{FFF}} E_2^2 (E_1^2 + E_3^2) \right]. \hspace{1cm} (C.97)$$

An integration by parts of

$$g_{\text{FFF}, 122} = - \frac{1}{2^8 \pi^6 E^2} \int_{0}^{\infty} dE_3 \int_{0}^{E} E_1 f_{\text{FFF}} \frac{1}{E_1 - E_3} (E_1^2 E_2^2 - E_2 E_3^2) \hspace{1cm} (C.98)$$
does not yield surface term; see (C.49). Hence, we obtain

\[ g_{\text{FFF}, 2} |M_1|^2 = \frac{1}{28\pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_1 \ln \left( \frac{|E_1 - E_3|}{E_3} \right) \]

\[ \times \Theta(E - E_1) \frac{d}{dE_1} \left[ f_{\text{FFF}}(E_1^2 - E_2^2 E_3^2) \right]. \]  

(C.99)

Again, \( g_{\text{FFF}, 3} |M_1|^2 = 0 \), since the argument preceding (C.51) is independent of the quantum statistical distribution functions.

Result for \( I_{\text{FFF}} |M_1|^2 \)

For the FFF processes, we find in total from the contribution of \( |M_1|^2 \):

\[ I_{\text{FFF}} |M_1|^2 = g_{\text{FFF}, 11} + 2 \cdot g_{\text{FFF}, 121} + g_{\text{FFF}, 122} + g_{\text{FFF}, 11} |M_1|^2 + g_{\text{FFF}, 12} + g_{\text{BFB}, 2} \]

\[ = \frac{T^2 f_F(E)}{384\pi^4} \left[ \ln \left( \frac{2T}{k^*} \right) + \frac{17}{6} - \gamma + \zeta'(2) - \zeta(2) \right] \]

\[ + \frac{1}{256\pi^6} \int_0^\infty dE_3 \int_0^{E+E_3} dE_1 \ln \left( \frac{|E_1 - E_3|}{E_3} \right) \]

\[ \times \left\{ - \Theta(E - E_3) \frac{d}{dE_1} \left[ f_{\text{BFB}}(E_1^2 + E_3^2) \right] \right. \]

\[ + \Theta(E_3 - E_1) \frac{d}{dE_1} \left[ f_{\text{BFB}}(E_1^2 + E_3^2) \right] \]

\[ + \Theta(E - E_1) \frac{d}{dE_1} \left[ f_{\text{BFB}}(E_1^2 - E_3^2) \right] \right\}. \]  

(C.100)

Note that \( |M_2|^2 \) does not contribute to the FFF processes because \( c_{\text{BFB} 2}^{(\alpha)} = 0 \) so that we can immediately turn to the integrals containing \( |M_3|^2 \).

C.2.2 Contribution from \( |M_3|^2 \)

We now encounter for the first time also the matrix element \( |M_3|^2 \), namely,

\[ I_{\text{FFF}} |M_3|^2 = \frac{1}{(2\pi)^3 2E} \int d\Omega_p \int \frac{d^3 p_i}{(2\pi)^3 2E_i} \left( \frac{2\pi}{4\pi} \right)^4 \delta^4(P_1 + P_2 - P_3 - P) \]

\[ \times f_{\text{FFF}} |M_3|^2 \Theta(|P_1 - P| - k^*). \]  

(C.101)

Recall from (3.21) that \( |M_3|^2 = t^2/s \) so that we can set \( k^* \to 0 \) and employ the formalism developed in section C.1.2. We choose (C.56) as the frame of reference for the integrations.
It follows that

\[ s = (E + E_3)^2 - q^2 \]  \hspace{1cm} \text{(C.102)}

and with \( t \) as in \((C.57)\) we find for the squared matrix element

\[ |M_3|^2 = \frac{1}{(E + E_3)^2 - q^2} \left( -2EE_2 \left( 1 - \cos \theta \cos \tilde{\theta} - \sin \theta \sin \tilde{\theta} \sin \phi \right) \right)^2 \]  \hspace{1cm} \text{(C.103)}

The angular integrations are straightforward to compute and we get

\[ I |M_3|^2_{\text{FFF}} = \frac{3}{2^{13/2} \pi^6 E^2} \int dE_2 dE_3 dq f_{\text{FFF}} |M_3|^2 \Omega \]  \hspace{1cm} \text{(C.104)}

with

\[ g_{\text{BFB}} |M_3|^2 = ((E_3 + E)^2 - q^2) \left[ 1 - \frac{2}{3} \frac{(2E_2^2 - 6E_2E_3 + 3E_3 + 2E_2E - E^2)}{q^2} \right. \]

\[ + \left. \frac{(E_3 - E)^2(E_1 - E_2)^2}{q^4} \right] \]  \hspace{1cm} \text{(C.105)}

and \( \Omega \) as in \((C.63)\). The manipulations of the \( \Theta \)-functions are the same as for \( |M_2|^2 \) so we can merely present the results. Analog to \((C.70)\), \((C.71)\), \((C.79)\), \((C.80)\), \((C.81)\) and \((C.82)\), we find

\[ g_{\text{FFF}, 11} |M_3|^2 = \frac{1}{2^{13/2} \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 \Theta(E - E_3) f_{\text{FFF}} \frac{E_1^2E_3^2}{E + E_3}, \]  \hspace{1cm} \text{(C.106)}

\[ g_{\text{FFF}, 12} |M_3|^2 = \frac{1}{2^{13/2} \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 \Theta(E_3 - E) f_{\text{FFF}} \frac{E_2^2E_3^2}{E + E_3}, \]  \hspace{1cm} \text{(C.107)}

\[ g_{\text{FFF}, 211} |M_3|^2 = -\frac{1}{2^{13/2} \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 \Theta(E - E_3) \Theta(E_3 - E_2) \]

\[ \times f_{\text{FFF}}(E_2 - E_3) \left[ E_2(E_3 - E) - E_3(E_3 + E) \right], \]  \hspace{1cm} \text{(C.108)}

\[ g_{\text{BFB}, 222} |M_3|^2 = \frac{1}{2^{13/2} \pi^6 E^2} \int_0^\infty dE_3 \int_0^{E+E_3} dE_2 \Theta(E_3 - E) \Theta(E_2 - E_3) \]

\[ \times f_{\text{FFF}}(E_2 - E_3) \left[ E_2(E_3 - E) - E_3(E_3 + E) \right]. \]  \hspace{1cm} \text{(C.109)}

while

\[ g_{\text{FFF}, 212} |M_3|^2 = 0, \]  \hspace{1cm} \text{(C.110)}

\[ g_{\text{FFF}, 221} |M_3|^2 = 0. \]  \hspace{1cm} \text{(C.111)}
C.3 Production Rate for BBF Processes

The BFB processes contain the contributions with two incoming bosons and the gravitino and another fermion as outgoing particles. In total,

\[
\frac{d\Gamma_{G}^{(BBF)}}{d^{3}p}\bigg|_{\text{hard}} = \sum_{\alpha=1}^{3} \frac{g_{\alpha}^{2}}{M_{p}^{2}} \left( 1 + \frac{M_{3}^{2}}{3m_{G}^{2}} \right) \sum_{i=1}^{3} \left[ c_{BBF,i}^{(\alpha)} I_{BBF}^{[M_{i}]^{2}} \right].
\]

(C.113)

Note that all \( c_{BBF,1}^{(\alpha)} = 0 \) so that we are left with the computation of \( |M_{2}|^{2} \) and \( |M_{3}|^{2} \).

C.3.1 Contribution from \( |M_{2}|^{2} \)

We have discussed in great detail how to perform the integrations for the non-singular matrix elements \( |M_{2}|^{2} \) and \( |M_{3}|^{2} \) in sections C.1.2 and C.2.2, respectively. Again, all that changes is a different combination of quantum mechanical distribution functions,

\[
I_{BBF}^{[M_{2}]^{2}} = \frac{1}{(2\pi)^{3}2E} \int \frac{d^{3}p_{i}}{4\pi} \int \left[ \prod_{i=1}^{3} \frac{d^{3}p_{i}}{(2\pi)^{3}2E_{i}} \right] (2\pi)^{4}\delta^{4}(P_{1} + P_{2} - P_{3} - P) f_{BBF} |M_{2}|^{2},
\]

(C.114)
so that we can immediately present the result:

\[
J_{BBF}^{\vert M_2 \vert^2} = \frac{1}{256\pi^6} \int_0^\infty dE_3 \int_0^{E_3+E_1} dE_2 f_{BBF} \times \left\{
\Theta(E - E_3) \frac{E_2^2}{E^2} \left( \frac{E_3}{3} - E_1 \right) + \Theta(E_3 - E) \left( \frac{E}{3} - E_2 \right) \\
+ \Theta(E_3 - E_2) \Theta(E - E_3) \frac{E_2 - E_3}{3E^2} \left[ (E_2 - E_3)(E_2 + 2E_3) - 3(E_2 + E_3)E \right] \\
- \Theta(E_2 - E) \Theta(E - E_3) \frac{(E_2 - E)^3}{3E^2} \\
+ \Theta(E - E_2) \Theta(E_3 - E) \frac{(E_2 - E)^3}{3E^2} \\
- \Theta(E_2 - E_3) \Theta(E_3 - E) \frac{E_2 - E_3}{3E^2} \left[ (E_2 - E_3)(E_2 + 2E_3) - 3(E_2 + E_3)E \right] \right\}.
\]

\[(C.115)\]

### C.3.2 Contribution from \( \vert M_3 \vert^2 \)

For the squared matrix element \( \vert M_3 \vert^2 \) we can set as usual \( k^* \to 0 \), i.e.,

\[
J_{BBF}^{\vert M_3 \vert^2} = \frac{1}{(2\pi)^3 2E} \int_0^{\infty} \frac{d\Omega_p}{4\pi} \int \left[ \prod_{i=1}^{3} \frac{d^3p_i}{(2\pi)^3 2E_i} \right] (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P) f_{BBF} \vert M_3 \vert^2,
\]

\[(C.116)\]

and find in complete analogy to section C.2.2:

\[
J_{BBF}^{\vert M_3 \vert^2} = \frac{1}{256\pi^6} \int_0^\infty dE_3 \int_0^{E_3+E_1} dE_2 f_{BBF} \times \left\{
\Theta(E - E_3) \frac{1}{E^2} \frac{E_2^2}{E_3^2} \\
+ \Theta(E_3 - E) \frac{1}{E^2} \frac{E_2^2}{E_3^2} \\
- \Theta(E_2 - E_3) \Theta(E_3 - E_2) \frac{E_2 - E_3}{E^2} \left[ E_2(E_3 - E) - E_3(E_3 + E) \right] \\
+ \Theta(E_3 - E) \Theta(E_2 - E_3) \frac{E_2 - E_3}{E^2} \left[ E_2(E_3 - E) - E_3(E_3 + E) \right] \right\}.
\]

\[(C.117)\]
Appendix D

Published Work from this Thesis

The new SU(3)_{c} × SU(2)_{L} × U(1)_{Y} result for the collision term and its implication on gravitino dark matter scenarios allow for a collider test probing the viability of thermal leptogenesis. As a summary of this thesis, the proposed method has been published in Physical Review D [47].

An e-print of the paper is available on the arXiv server:
http://lanl.arxiv.org/abs/hep-ph/0608344
Bibliography

[1] D. N. Spergel et al., Wilkinson microwave anisotropy probe (WMAP) three year results: Implications for cosmology, astro-ph/0603449.

[2] M. Bolz, A. Brandenburg and W. Buchmüller, Thermal production of gravitinos, Nucl. Phys. B606 (2001) 518–544 [hep-ph/0012052].

[3] D. G. Cerdeño and C. Muñoz, An introduction to supergravity, Prepared for 6th Hellenic School and Workshop on Elementary Particle Physics; Corfu, Greece, 6-26 Sep 1998.

[4] H. P. Nilles, Supersymmetry, supergravity and particle physics, Phys. Rept. 110 (1984) 1.

[5] Particle Data Group Collaboration, W. M. Yao et al., Review of particle physics, J. Phys. G33 (2006) 1–1232.

[6] J. Wess and J. Bagger, Supersymmetry and supergravity, Princeton, USA: Univ. Pr. (1992) 259 p.

[7] G. F. Giudice and A. Masiero, A natural solution to the μ problem in supergravity theories, Phys. Lett. B206 (1988) 480–484.

[8] A. Brignole, L. E. Ibañez and C. Muñoz, Soft supersymmetry-breaking terms from supergravity and superstring models, hep-ph/9707209.

[9] S. P. Martin, A supersymmetry primer, hep-ph/9709356.

[10] M. Drees, R. Godbole and P. Roy, Theory and phenomenology of sparticles: An account of four-dimensional N=1 supersymmetry in high energy physics, Hackensack, USA: World Scientific (2004) 555 p.

[11] W. Rarita and J. S. Schwinger, On a theory of particles with half-integral spin, Phys. Rev. 60 (1941) 61.
[12] T. Lee and G.-H. Wu, *Interactions of a single goldstino*, Phys. Lett. B447 (1999) 83–88 [hep-ph/9805512].

[13] A. Denner, H. Eck, O. Hahn and J. Kublbeck, *Feynman rules for fermion-number violating interactions*, Nucl. Phys. B387 (1992) 467–484.

[14] E. Braaten and T. C. Yuan, *Calculation of screening in a hot plasma*, Phys. Rev. Lett. 66 (1991) 2183–2186.

[15] H. A. Weldon, *Simple rules for discontinuities in finite-temperature field theory*, Phys. Rev. D28 (1983) 2007.

[16] E. Braaten and R. D. Pisarski, *Soft amplitudes in hot gauge theories: A general analysis*, Nucl. Phys. B337 (1990) 569.

[17] J. A. M. Vermaseren, *New features of FORM*, math-ph/0010025.

[18] R. Cutler and D. W. Sivers, *Quantum chromodynamic gluon contributions to large p(t)-reactions*, Phys. Rev. D17 (1978) 196.

[19] R. D. Pisarski, *Scattering amplitudes in hot gauge theories*, Phys. Rev. Lett. 63 (1989) 1129.

[20] H. A. Weldon, *Covariant calculations at finite temperature: The relativistic plasma*, Phys. Rev. D26 (1982) 1394.

[21] M. Bolz, *Thermal production of gravitinos*. PhD thesis, DESY, 2000.

[22] R. D. Pisarski, *Renormalized gauge propagator in hot gauge theories*, Physica A158 (1989) 146–157.

[23] O. K. Kalashnikov and V. V. Klimov, *Polarization tensor in QCD for finite temperature and density*, Sov. J. Nucl. Phys. 31 (1980) 699.

[24] R. D. Pisarski, *Computing finite-temperature loops with ease*, Nucl. Phys. B309 (1988) 476.

[25] T. Hahn, *Cuba: A library for multidimensional numerical integration*, Comput. Phys. Commun. 168 (2005) 78–95 [hep-ph/0404043].

[26] K. Kohri, T. Moroi and A. Yotsuyanagi, *Big-bang nucleosynthesis with unstable gravitino and upper bound on the reheating temperature*, Phys. Rev. D73 (2006) 123511 [hep-ph/0507245].
[27] P. Minkowski, $\mu \rightarrow e \gamma$ at a rate of one out of 1-billion muon decays?, *Phys. Lett. B* 67 (1977) 421.

[28] T. Yanagida, *Horizontal gauge-symmetry and masses of neutrinos*, *Prog. Theor. Phys.* 64 (1980) 1103.

[29] M. Fukugita and T. Yanagida, *Baryogenesis without grand unification*, *Phys. Lett. B* 174 (1986) 45.

[30] W. Buchmüller, P. Di Bari and M. Plümacher, *Leptogenesis for pedestrians*, *Ann. Phys.* 315 (2005) 305–351 [hep-ph/0401240].

[31] M. Bolz, W. Buchmüller and M. Plümacher, *Baryon asymmetry and dark matter*, *Phys. Lett. B* 443 (1998) 209–213 [hep-ph/9809381].

[32] M. Fujii, M. Ibe and T. Yanagida, *Upper bound on gluino mass from thermal leptogenesis*, *Phys. Lett. B* 579 (2004) 6–12 [hep-ph/0310142].

[33] H. Baer, M. A. Diaz, P. Quintana and X. Tata, *Impact of physical principles at very high energy scales on the superparticle mass spectrum*, *JHEP* 04 (2000) 016 [hep-ph/0002245].

[34] G. A. Blair, W. Porod and P. M. Zerwas, *The reconstruction of supersymmetric theories at high energy scales*, *Eur. Phys. J.* C 27 (2003) 263–281 [hep-ph/0210058].

[35] R. Lafaye, T. Plehn and D. Zerwas, *SFITTER: SUSY parameter analysis at LHC and LC*, hep-ph/0402482.

[36] P. Bechtle, K. Desch and P. Wienemann, *FITINO, a program for determining MSSM parameters from collider observables using an iterative method*, *Comput. Phys. Commun.* 174 (2006) 47–70 [hep-ph/0412012].

[37] J. L. Feng, S. Su and F. Takayama, *Supergravity with a gravitino LSP*, *Phys. Rev. D* 70 (2004) 075019 [hep-ph/0404231].

[38] F. D. Steffen, *Gravitino dark matter and cosmological constraints*, *JCAP* 0609 (2006) 001 [hep-ph/0605306].

[39] D. G. Cerdeño, K.-Y. Choi, K. Jedamzik, L. Roszkowski and R. Ruiz de Austri, *Gravitino dark matter in the CMSSM with improved constraints from BBN*, *JCAP* 0606 (2006) 005 [hep-ph/0509275].
[40] W. Buchmüller, K. Hamaguchi, M. Ratz and T. Yanagida, *Supergravity at colliders*, Phys. Lett. B588 (2004) 90–98 [hep-ph/0402179].

[41] A. Brandenburg, L. Covi, K. Hamaguchi, L. Roszkowski and F. D. Steffen, *Signatures of axinos and gravitinos at colliders*, Phys. Lett. B617 (2005) 99–111 [hep-ph/0501287].

[42] H. U. Martyn, *Detecting metastable staus and gravitinos at the ILC*, hep-ph/0605257.

[43] F. D. Steffen, *Collider signatures of axino and gravitino dark matter*, hep-ph/0507003.

[44] A. Djouadi, J.-L. Kneur and G. Moultaka, *SuSpect: A fortran code for the supersymmetric and higgs particle spectrum in the MSSM*, hep-ph/0211331.

[45] G. Belanger, F. Boudjema, A. Pukhov and A. Semenov, *micrOMEGAs: A program for calculating the relic density in the MSSM*, Comput. Phys. Commun. 149 (2002) 103–120 [hep-ph/0112278].

[46] A. D. Wheelon, *On the summation of infinite series in closed form*, Journal of Applied Physics 25 (1954) 113.

[47] J. Pradler and F. D. Steffen, *Thermal gravitino production and collider tests of leptogenesis*, Phys. Rev. D75 (2007) 023509 [hep-ph/0608344].
Acknowledgements

I would like to thank Professor Alfred Bartl for strong support and for the possibility to conduct my diploma thesis abroad at the Max-Planck-Institute in Munich. Frank Daniel Steffen deserves my special thanks for proposing the interesting topic which resulted in a close collaboration. It is a great pleasure to work under his responsible and encouraging supervision. Furthermore, I want to thank Georg Raffelt for his friendly advice.

I am thankful to Marco Zagermann for helpful discussions on supergravity and to Professor Wilfried Buchmüller and Arnd Brandenburg for correspondence on the gravitino production rate. For brief but inspiring discussions, I would like to thank Professor Antonio Masiero.

I want to thank my office mates Andreas Biffar, Florian Hahn-Woernle, Max Huber, Alexander Laschka, Felix Rust, and Tobias Schlüter for having spent a most pleasant and diverse time with them.

Thanks to Simon Gröblacher, Erik Hörtnagl, Ulrich Matt, and Robert Prevedel for friendship and not least for providing me with accommodation during my visits in Vienna.

Last but not least I wish to thank my family for their trust in me. I deeply thank my parents for their support.
Revision History:

March 15, 2007

- Cover sheet: date of hand in of the thesis added
- Appendix A: Footnote added
- Appendix D: Reference updated

June 01, 2007

- Typos corrected in Eqs. (2.5), (2.20), and (2.23)