Path Integrals and Lorentz Violation in Polymer Quantized Scalar Fields

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We obtain a path integral formulation of polymer quantized scalar field theory, starting from the Hilbert Space framework. This brings the polymer quantized scalar field theory under the ambit of Feynman diagrammatic techniques. The path integral formulation also shows that Lorentz invariance is lost for the Klein-Gordon field.

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I. INTRODUCTION

In describing most situations in nature, quantum field theories defined on a given background spacetime are adequate. When a background spacetime is itself not well-defined, as in the early universe or in the last stages of the evaporation of a black hole, such a description will not do and a more fundamental description seems necessary. According to Loop Quantum Gravity [1], [2], [3], the appropriate description of such situations is in terms of quantum geometry and polymer quantized matter fields. The low energy description using field theory in a given background should then be derivable from this.

In the fundamental description, the quantum geometry lives on graphs and polymer quantized scalar fields live on the vertex sets of the graphs. To make connection with low energy physics, one may neglect the quantum nature of geometry and investigate the dynamics of polymer quantized scalar field in a fixed background. Such an investigation has hitherto not been carried out, possibly because of the complexity of the calculations in the absence of a path integral formulation. In this work, we seek to remedy the situation for polymer quantized real valued scalar fields [4] by developing such a framework. This would make it easier to employ semiclassical approximation schemes for the polymer quantized scalar fields.

A path integral formulation for a polymer quantized theory was first provided by Ashtekar, Campigilia and Henderson in [5] for loop quantum cosmology, a theory with a single degree of freedom. We follow their technique and extend the path integral description to scalar field theories. For simplicity we’ll derive the path integral formulation for a 1+1 dimensional field theory. The extension to higher dimensions is straightforward. Our strategy for this derivation will be to start from a scalar field theory defined on a one dimensional lattice, obtain a path integral formulation for this and finally take the continuum limit. This follows the strategy followed in QFT textbooks such as [6].

The particular theory we consider is the Klein-Gordon field in a Minkowski background. The path integral description sheds light on the issue of Lorentz symmetry of this theory. As the quantization is background-independent, information about the background Lorentz symmetry can only enter through the Hamiltonian, which classically is a function of both field and momentum. In the polymer quantized theory the field is not a well-defined operator, only momentum and holonomy operators are well-defined. Thus to construct a Hamiltonian, a field operator has to be constructed from the holonomy operator by introducing a scale. We’ll see in the path integral formulation that the introduction of this scale results in the disappearance of Lorentz symmetry.

The paper is organized as follows. In section I we’ll recall the technique used in [5] by obtaining a path integral formulation for a polymer quantized simple harmonic oscillator. In section II we briefly recall the scalar field theory and introduce some notation. In section III we’ll derive the path integral formulation. In section IV we conclude with a discussion of the results.

II. PATH INTEGRAL FORMULATION OF THE POLYMER QUANTIZED SHO

A. Simple Harmonic Oscillator in the polymer representation

Like the ordinary Schrodinger representation, the polymer representations of quantum mechanics [7], [8] are based on unitary representations of the Weyl Algebra, which is given by:

\[ U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2) \]
\[ V(\mu_1)V(\mu_2) = V(\mu_1 + \mu_2) \]
\[ U(\lambda)V(\mu) = e^{-i\lambda\mu}V(\mu)U(\lambda) \]

Unlike the Schrodinger representation however, the representation of the Weyl algebra in the polymer representations are not weakly continuous. In the Schrodinger representation \( U(\lambda), V(\mu) \) can be understood as exponentiated position and momentum observables, respectively. Since both their representations cannot be weakly continuous in polymer quantization, both momentum and position cannot be well-defined operators. There are two possible polymer representations depending on which of

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the two subgroups $U(\lambda), V(\mu)$ is taken to be continuously represented. Here we’ll consider the case where the momentum operator is defined. The advantage of this for a path integral formulation, as we will see, is that the momentum integral in the phase space path integral will be gaussian. Thus the integration over moments can be easily performed and a closed-form expression for the configuration space path integral can be obtained.

Let us recall the polymer representation of quantum mechanics in a bit more detail. To construct the Hilbert Space, we first choose a countable set, $\gamma = \{ p_j, \mu_j \in \mathbb{R}\}$ and define a set $\text{Cyl}_\gamma$ of linear combinations of the form: $\text{Cyl}_\gamma := \{ \sum_j f_j e^{ip_j x}, f_j \in \mathbb{C} \}$ The $f_j$ are subject to certain regularity conditions [9]. Then we define the set of functions of $x$, $\text{Cyl} := \cup_\gamma \text{Cyl}_\gamma$. The inner product on this set is chosen to be

$$\langle e^{ip_1 x}, e^{ip_2 x} \rangle = \delta_{p_1, p_2}$$

(1)

$\{ e^{ip x} / p \in \mathbb{R} \}$ form an uncountable basis of this space and we denote them as the kets $| p \rangle$. The completion of Cyl w.r.t this inner product is our requisite Hilbert Space $H_{\text{poly}}$: $\text{Cyl} := H_{\text{poly}}$.

On this Hilbert Space we have the basic operators:

$$\hat{p}| p \rangle = p| p \rangle$$

(2)

and

$$\hat{V}(\lambda)| p \rangle = | p - \lambda \hbar \rangle$$

(3)

As $\hat{V}(\lambda)$ is not weakly continuous in $\lambda$ a position operator cannot be defined. We can however define an approximate position operator by choosing some scale $\mu_0$:

$$\hat{x}| \mu_0 \rangle = \frac{\hat{V}(\mu_0) - \hat{V}(-\mu_0)}{2\mu_0 i}$$

(4)

Once the approximate position operator is defined with a particular choice $\mu_0$, starting from a given $| p_0 \rangle$ and acting on it with $\hat{V}(\mu_0)$ we’ll generate a set of basis vectors $\{ | p_0 + n \mu_0 \hbar \rangle \}$. This gives a proper subspace of the Hilbert Space and the action of observables will leave the subspace invariant. We’ll be working in one such subspace.

The simple harmonic oscillator Hamiltonian in the polymer representation reads:

$$\hat{p}^2 = \frac{2m}{\mu_0 i} \left( \hat{V}(\mu_0) - \hat{V}(-\mu_0) \right)$$

(5)

Note that this choice of polymer Hamiltonian is different from [9]. The path integral formulation would however be the same in both cases.

B. Path Integral formulation

In this section we’ll follow the steps of [10] to derive a path integral expression for a polymer quantized simple harmonic oscillator. As usual, our starting point will be the transition amplitude $\langle p_f | e^{-\hat{H} t/\epsilon} | p_i \rangle$. We divide $t$ into $N$ pieces $\epsilon = t/N$. So

$$e^{-\hat{H} t} = \prod_{n=1}^{N} e^{-\hat{H} \epsilon}$$

Inserting complete basis of the form $\mathbb{1} = \sum | p \rangle \langle p |$ in between each factor we have

$$\langle p_f | e^{-\hat{H} t} | p_i \rangle = \sum_{p_{N-1}, \ldots, p_1} \langle p_f | e^{-\hat{H} \epsilon} | p_{N-1} \rangle \cdots \langle p_1 | e^{-\hat{H} \epsilon} | p_i \rangle$$

(6)

Taking $N$ very large ($\epsilon \ll 1$) and expanding the $n$th term of the series in $\epsilon$ we have

$$\langle p_{n+1} | e^{-\hat{H} \epsilon} | p_{n} \rangle = \delta_{p_{n+1}, p_n} + \frac{i}{\hbar} \langle p_{n+1} | \hat{H} | p_n \rangle + O(\epsilon^2)$$

(7)

The matrix elements of $\hat{H}$ are:

$$\langle p_{n+1} | \hat{H} | p_n \rangle = \frac{p_{n+1}^2}{2m} \langle p_{n+1} | p_n \rangle$$

(8)

$$+ \langle p_{n+1} | \left( \hat{V}(\mu_0) - \hat{V}(-\mu_0) \right)^2 | p_n \rangle$$

(9)

We would have obtained the same expression using the Hamiltonian given in [9]. We’ll make use of the identity:

$$\delta_{\mu', \mu} = \frac{\mu_0}{2\pi} \int_0^{2\pi} dx e^{i \mu \left( y' - p \right)}$$

(10)

From [10], [11] and [12] we obtain:

$$\langle p_{n+1} | e^{-\hat{H} \epsilon} | p_n \rangle = \frac{\mu_0}{2\pi} \int_0^{2\pi} dx_{n+1} e^{-\hat{x}_{n+1}(p_{n+1} - p_n)} \left[ 1 - \frac{i \epsilon}{\hbar} \left( \frac{p_{n+1}^2}{2m} + \sin^2 \left( \frac{\mu_0 x_{n+1}}{\mu_0^2} \right) \right) \right] + O(\epsilon^2)$$

(11)

Substituting this in (10) we have

$$\langle p_f | e^{-\hat{H} t} | p_i \rangle = \sum_{p_{N-1}, \ldots, p_1} \left( \frac{\mu_0}{2\pi} \right)^N \int dx_n \cdots dx_1 e^{-\hat{x} S_N} + O(\epsilon^2)$$

(12)

where

$$S_N = \epsilon \sum_{n=0}^{N-1} x_{n+1} \left( \frac{p_{n+1} - p_n}{\epsilon} \right) + \frac{p_{n+1}^2}{2m} + \frac{\sin^2 \left( \frac{\mu_0 x_{n+1}}{\mu_0^2} \right)}{\mu_0^2}$$

(13)
This is the discrete sum version of the path integral. The next step is to take the $N \to \infty$ limit. However we cannot interpret $\frac{p_{n+1} - p_n}{\epsilon}$ as a derivative as $p$ takes discrete values. So we use:

$$
\epsilon \sum_{n=0}^{N-1} -x_{n+1} - x_n = \epsilon \sum_{n=0}^{N-1} \frac{(x_{n+1} - x_n) + (x_1 p_0 - x_N p_N)}{\epsilon}
$$

Now the limit $N \to \infty$ may be taken. This gives us the path integral expression: $K(p_0, p_N) = \int [Dp] |Dx| e^{i \pi S'}$ where $S'$ given by

$$
S' = \int_0^T d\tau \ p \dot{x} - \left( \frac{p^2}{2m} + \frac{\sin^2 (\mu_0 x_{\tau})}{\mu_0^2} \right)
$$

The subscript $q$ in the path integral expression is to indicate that this expression is different from the usual path integral expression. Here we have a sum over ‘quantum paths’, that is to say $p$ takes only discrete values and $x$ is bounded. However, this can be converted into the familiar form by using the following identity [10]:

$$
\sum_{n \in \mathbb{Z}} \int_0^{2\pi} d\theta \ f(\theta, m) e^{in\theta} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\theta \ f(\theta, x) e^{i\epsilon \theta}
$$

This is true for any continuous function $f(\theta, x)$ with a period of $2\pi$ in $\theta$. Using this we can now replace

$$
\frac{\mu_0}{2\pi} \sum_{n=0}^{\infty} \int_0^{2\pi} d\theta \ x_n \int_{-\infty}^{\infty} dp_n \int_{-\infty}^{\infty} dx_n
$$

Now we have our result for the phase space path integral:

$$
K(p_0, p_N) = \int [Dp] |Dx| e^{i \pi S}
$$

where

$$
S = \int_0^T d\tau \ p \dot{x} - \left( \frac{p^2}{2m} + \frac{\sin^2 (\mu_0 x)}{\mu_0^2} \right)
$$

As the momentum integral is Gaussian one may integrate it out from the phase space partition function to obtain the configuration space partition function

$$
Z = \int |Dx| e^{i \pi S}
$$

with

$$
S = \int_0^T d\tau \ \frac{\dot{x}^2}{2} - \frac{\sin^2 (\mu_0 x)}{\mu_0^2}
$$

We note that in the path integral formulation, going from Schrodinger to polymer representation for any polynomial potential the only change will be a replacement of $x$ by $\sin(\mu_0 x)/\mu_0$ in the action. Further, $x \to x + \frac{2\pi}{\mu_0}$ is a symmetry of the action. We’ll find that these features hold for the scalar field as well.

III. THE POLYMER QUANTIZED SCALAR FIELD

We first describe the polymerised scalar field theory. From now on we’ll set $\hbar = 1$. We’ll follow the notation of [11]. First define a vertex set $V = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ of finitely many, distinct points $\in \mathbb{R}^3$. The corresponding vector space $\text{Cyl}_V$ is generated by basis vectors:

$$
\psi_{V, x, \lambda} := e^{i \sum_j \lambda_j \phi(\bar{x}_j)}
$$

where $\lambda_j$ are non-zero real numbers. Then define $\text{Cyl} := \cup_{V} \text{Cyl}_V$ and on $\text{Cyl}$ define the inner product

$$
\int d\mu(\phi) \psi_{V, x, \lambda}^* \psi_{V, x, \lambda'} = \delta_{\lambda, \lambda'}
$$

The Cauchy completion of $\text{Cyl}$ w.r.t this inner product gives the Hilbert Space $H_{\text{poly}}$: $\tilde{\text{Cyl}} = H_{\text{poly}}$. The basic operators here are $\hat{U}(\lambda, x)$ and $\hat{\pi}(x)$. The former acts as:

If $x$ is not in $\{x_j\}$

$$
\hat{U}(\lambda, x) e^{i \sum_j \lambda_j \phi(x_j)} = e^{i \sum_j \lambda_j \phi(x_j)} e^{i \lambda \phi(x)}
$$

If $x = x_i \in \{x_j\}$ and $\lambda_j + \lambda \neq 0$

$$
\hat{U}(\lambda, x) e^{i \sum_j \lambda_j \phi(x_j)} = e^{i \sum_j (\lambda_j + \lambda \phi(x_j))} e^{i \lambda \phi(x_i)}
$$

where the sum is over the set $\{x_j\} - x_i$, and the action of the latter is given by

$$
\hat{\pi}(x) = \frac{\delta}{\delta \phi(x)}
$$

Just like the position operator for the polymerised quantum mechanics, the field operator here is not well defined. We may define an approximate field operator using a scale $\mu$:

$$
\phi_{\mu}(x) = \hat{U}(\mu, x) - \hat{U}(-\mu, x)
$$

Again, just as in quantum mechanics, the choice of $\mu$ and a basis vector gives a proper subspace of the Hilbert space on which the basic operators act invariantly. We’ll restrict ourselves to one such subspace.

Now we’ll introduce a new notation which we believe helps underline the similarities between polymer and Schrodinger frameworks. Let us consider a state $e^{i \sum_j \lambda_j \phi(x_j)}$. We can specify this state by specifying the vertex set $V$ and the values of $\lambda_i$. But we could alternately specify the state by we defining a field $\pi(x)$ for all $x$ such that

$$
\pi(x_j) = \lambda_j
$$
IV. PATH INTEGRAL FORMULATION OF THE POLYMER QUANTIZED SCALAR FIELD

Now we’ll calculate \( \langle \pi_f | e^{-iHt} | \pi_i \rangle \) by discretizing \( t \) into \( N \) pieces \( e^{-iHt} = \prod_{\mu=0}^{N-1} e^{-i\epsilon_t \hat{H}} \). Our strategy will be to do this first for a scalar field theory which lives on a lattice and finally take the continuum limit. To this end we discretize space with lattice spacing \( \epsilon_x \) and call the entire lattice \( L \) from now on. That is all vertex sets are now subsets of \( L \).

We verify this using (30):

\[
\left( \prod_{x} \sum_{\pi(x)} |\pi'(x)\rangle \langle \pi'(x)| \right) |\pi\rangle = \prod_{x} \sum_{\pi'(x)} \delta_{\pi(x),\pi'(x)} e^{i\pi'(x)\phi(x)} = \prod_{x} e^{i\pi(x)\phi(x)} = |\pi\rangle
\]

Now we’ll calculate \( \langle \pi_f | e^{-i\hat{H}t/N} | \pi_i \rangle \) by discretizing \( t \) into \( N \) pieces \( e^{-i\hat{H}t/N} = \prod_{n=1}^{N} e^{-i\epsilon_t \hat{H}} \). Our strategy will be to do this first for a scalar field theory which lives on a lattice and finally take the continuum limit. To this end we discretize space with lattice spacing \( \epsilon_x \) and call the entire lattice \( L \) from now on. That is all vertex sets are now subsets of \( L \).

So we have

\[
\langle \pi_f | e^{-i\hat{H}t/N} | \pi_i \rangle = \langle \pi_f | \prod_{n=1}^{N} e^{-i\epsilon_t \hat{H}} | \pi_i \rangle
\]

Using (31) this is rewritten as:

\[
\langle \pi_f | e^{-i\hat{H}t/N} | \pi_i \rangle = \prod_{x \in L} \sum_{\pi(x)} \cdots \prod_{x \in L} \sum_{\pi(x)} \langle \pi(x) | e^{-i\epsilon_t \hat{H}} | \pi_{N-1}(x) \rangle \cdots \langle \pi_1(x) | e^{-i\epsilon_t \hat{H}} | \pi_i(x) \rangle
\]

Taking \( N \) very large (\( \epsilon_t << 1 \)) and expanding the \( n \)th factor in \( \epsilon_t \) we have

\[
\langle \pi_{n+1}(x) | e^{-i\epsilon_t \hat{H}} | \pi_n(x) \rangle = \langle \pi_{n+1}(x) | \pi_n(x) \rangle + i \epsilon_t \langle \pi_{n+1}(x) | \hat{H} | \pi_n(x) \rangle + O(\epsilon_t^2)
\]

Our Hamiltonian will be a discretization of

\[ \int dx \frac{\hat{H}}{2} \left( \delta^2(\pi(x)) + (\nabla\phi(x))^2 + m^2 \phi^2(x) \right) \]

We’ll treat these terms one by one. To begin with we’ll consider a Hamiltonian with only the first term and demonstrate how the path integral expression is derived. Then we’ll consider the next two terms and see how they contribute to the action.

So with only the first term the discrete Hamiltonian is

\[ \sum_{x \in L} \epsilon_x \frac{\pi(x)^2}{2x} \]

where \( \pi = \epsilon \hat{\pi} \) is the canonical conjugate to the discrete field \( \phi \). We’ll suppress \( O(\epsilon_t^2) \) terms in the following. We have, therefore

\[
\prod_{L} \langle \pi_{n+1}(x) | e^{-i\epsilon_t \hat{H}} | \pi_n(x) \rangle = \prod_{L} \delta_{\pi_{n+1}(x),\pi_n(x)} + \epsilon_t \sum_{L} \frac{\pi_n(x)^2}{2x} \prod_{L} \delta_{\pi_{n+1}(x),\pi_n(x)}
\]

As we have restricted ourselves to a subspace with a choice of \( \mu \) it follows that the values of the \( \pi \) fields at any point will differ by an integer multiple of \( \mu \), just as the momenta in quantum mechanics differed by factors of \( \mu_0 \). Using this fact, we may write:

\[
\delta_{\pi(x),\pi'(x)} = \frac{\mu_0}{2\pi} \int_0^{2\pi/\mu_0} d\phi(x) e^{i\phi(x)(\pi'(x)-\pi(x))}
\]

This can be used to rewrite (35) as:

\[
\langle \pi_{n+1} | e^{-i\epsilon_t \hat{H}} | \pi_n \rangle = \left( \prod_{L} \frac{\mu_0}{2\pi} \int_0^{2\pi/\mu_0} d\phi_{n+1}(x) e^{i \sum \phi(x)(\pi_{n+1}(x)-\pi_n(x)+(\pi_n(x)-\pi_n(x))^2/2\epsilon_t)} \right)^n
\]

Then we have

\[
\langle \pi_f | e^{-i\hat{H}t} | \pi_i \rangle = \left( \prod_{L} \sum_{\pi_1,\pi_2,\ldots,\pi_N} \int d\phi_1(x) \ldots d\phi_n(x) e^{-iSN.L} \right)
\]
where 

\[ S_{N,L} = \sum_{L} \varepsilon_{x} \left( \epsilon_{t} \sum_{n=0}^{N-1} \frac{\phi(x)(\pi_{n+1}(x) - \pi_{n}(x))}{\epsilon_{t} \varepsilon_{x}} + \frac{(\pi_{n}(x))^{2}}{2\epsilon_{x}^{2}} \right) \]  

(40)

To take the limit \( \epsilon_{t} \) going to zero we use, as before:

\[-\epsilon_{t} \sum_{n=0}^{N-1} \frac{\phi_{n+1}(x)(\pi_{n+1}(x) - \pi_{n}(x))}{\epsilon_{t}} = \epsilon_{t} \sum_{n=0}^{N-1} \frac{\phi_{n}(x)(\pi_{n}(x) - \phi_{n})(x)}{\epsilon_{t}} + (\phi_{1}(x)\pi_{0}(x) - \phi_{N}(x)\pi_{N}(x)) \]  

(41)

And using (40) as before we’ll turn the sums over paths into path integrals. We finally have the following expression for the transition amplitude

\[ \langle \pi_{f}|e^{-iHt}|\pi_{i} \rangle = \prod_{x \in L} \int d\phi(x) \int d\pi_{k}(x) e^{iS_{N,L}} \]  

(42)

where

\[ S_{N,L} = \sum_{L} \varepsilon_{x} \left( -\epsilon_{t} \sum_{n=0}^{N-1} \frac{\pi(x)_{n}}{\epsilon_{t}} (\phi_{n+1}(x) - \phi_{n}(x)) \right) \]

\[ + (\phi_{1}(x)\pi_{0}(x) - \phi_{N}(x)\pi_{N}(x)) - \frac{(\pi_{n}(x))^{2}}{2\epsilon_{x}^{2}} \]  

(43)

Now we can take the \( \epsilon_{t} \to 0 \) limit. This gives us:

\[ \langle \pi_{f}|e^{-iHt}|\pi_{i} \rangle = \prod_{x \in L} \int d\phi(x) \int d\pi_{k}(x) e^{iS_{L}} \]  

where

\[ S_{L} = \sum_{L} \varepsilon_{x} \left( -\int dt \sum_{n=0}^{N-1} \int dt \pi(x) \frac{\pi(x)}{\epsilon_{t}} - \frac{\pi(x)^{2}}{2\epsilon_{x}^{2}} \right) \]  

(44)

In the partition function we can integrate out the \( \pi \) field and obtain the expression

\[ Z = \prod_{x \in L} \int d\phi(x) e^{-i \int dt \sum_{L} \varepsilon_{x} \frac{\pi(x)^{2}}{4} } \]  

(45)

Finally taking the \( \varepsilon_{x} \) going to zero limit we obtain the configuration space path integral expression for the partition function:

\[ Z = \int |D\phi| e^{iS} \]  

(46)

where

\[ S = \int dx dt \phi^{2}/2 \]  

(47)

and

\[ \int |D\phi| = \prod d\phi(x) \]  

(48)

This demonstrates how the path integral for a polymerised scalar field theory is obtained. Now we’ll look at the contributions from the \( \phi^{2} \) and \( m^{2}\phi^{2} \) terms respectively. We start with the former.

The discretized expression for this is:

\[ \sum_{x,y} \epsilon_{x} \delta_{x,y} \nabla_{e_{y}}\phi_{\mu}(x) \nabla_{e_{y}}\phi_{\mu}(y) \]  

(49)

and we’ll be calculating:

\[ \langle \pi_{n+1}| \sum_{x,y} \epsilon_{x} \delta_{x,y} \nabla_{e_{y}}\phi_{\mu}(x) \nabla_{e_{y}}\phi_{\mu}(y)|\pi_{n} \rangle \]  

(50)

where

\[ \phi_{\mu}(x) = \frac{\hat{U}(\mu, x) - \hat{U}(\mu, x)}{2\mu i} \]  

(51)

and

\[ \nabla_{e_{y}}f(x) = f(x + \epsilon_{x}) - f(x) \]  

(52)

Let us evaluate

\[ \sum_{x,y} \epsilon_{x} \delta_{x,y} \nabla_{e_{y}}\phi_{\mu}(x) \nabla_{e_{y}}\phi_{\mu}(y) \]  

(53)

\[ = \sum_{x,y} \epsilon_{x} \delta_{x,y} \nabla_{e_{y}}\phi_{\mu}(x) \nabla_{e_{y}}\phi_{\mu}(y) e^{i \sum_{x \in V} (x_{j}) \phi(x_{j})} \]  

(54)

Let us denote the vertex set associated with |\( \pi_{n} \) ( i.e the set on which \( \pi_{n}(x) \) takes non zero values) as V. For a given \( x,y \) the action of the operator \( \nabla_{e_{y}}\phi_{\mu}(x) \nabla_{e_{y}}\phi_{\mu}(y) \) acting on the state \( e^{i \sum_{x \in V} (x_{j}) \phi(x_{j})} \) will produce different results depending on whether \( x,y \) coincide with \( x_{j} \in V \) or not. Thus (after evaluating the Kronecker delta) the sum can be split into two different sums: (i)A sum over points in L but not in V. (ii) A sum over points in V. (i) is a sum over all points on the lattice except a finite number of points belonging to V while (ii) is a sum over the finite number of points of V. When the continuum limit is taken contribution from (ii) will therefore disappear. So for our purposes we can ignore (ii)(these terms will play a role when quantum corrections to the geometry are taken into account) and work with (i). This term is:

\[ \sum_{L} \frac{\varepsilon_{x}}{4\epsilon_{x}^{2}} e^{i \sum_{x \in V} (x_{j}) \phi(x_{j})} (e^{2i\mu\phi(x+\epsilon_{x})} - 2e^{i\mu\phi(x+\epsilon_{x})} + \phi(x)) \]

\[ + e^{2i\mu\phi(x)} + 2e^{i\mu\phi(x+\epsilon_{x})} - \phi(x)) + 2e^{-i\mu(-\phi(x+\epsilon_{x}) + \phi(x))} \]

\[ + e^{-2i\mu\phi(x+\epsilon_{x})} - 2e^{-i\mu(-\phi(x+\epsilon_{x}) + \phi(x))} + e^{-2i\mu\phi(x) - 4} \]  

(55)
Now let us consider a term like \( i \sum_{x} \pi_{n}(x) \phi(x)\), \( e^{2i\mu \phi(x+\epsilon x)} \) that appears in the above expression. We may rewrite this as

\[
\sum_{x} \frac{\epsilon_x}{-4\epsilon_x^2 \mu^2} \{\{\pi_{n+1}(y)\}\{\pi_{n}(y) + 2\mu \delta_{x+\epsilon_x y}\}\},
\]

Using this and acting on \( \langle \pi_{n+1} \rangle \) from left we have:

\[
\sum_{L} \frac{\epsilon_x}{-4\epsilon_x^2 \mu^2} \{\{\pi_{n+1}(y)\}\{\pi_{n}(y) + 2\mu \delta_{x+\epsilon_x y}\} + \{\{\pi_{n+1}(y)\}\{\pi_{n}(y) - 2\mu \delta_{x+\epsilon_x y}\} - 2\{\{\pi_{n+1}(y)\}\{\pi_{n}(y) + \mu \delta_{x+\epsilon_x y} + \mu \delta_{x,y}\} - 2\{\{\pi_{n+1}(y)\}\{\pi_{n}(y) - \mu \delta_{x+\epsilon_x y} - \mu \delta_{x,y}\} - 4\{\{\pi_{n+1}(y)\}\{\pi_{n}(y)\}\} + 2\{\{\pi_{n+1}(y)\}\{\pi_{n}(y) + \mu \delta_{x+\epsilon_x y} - \mu \delta_{x,y}\} + 2\{\{\pi_{n+1}(y)\}\{\pi_{n}(y) - \mu \delta_{x+\epsilon_x y} + \mu \delta_{x,y}\}\}
\]

Using (56) this becomes

\[
\prod_{y} \frac{2\mu_0}{\pi} \int_{0}^{\pi \mu_0} d\phi \sum_{y} \phi_{n+1}(y)(\pi_{n+1}(y) - \pi(y))
\]

\[
\sum_{x \in L} \frac{\epsilon_x}{-4\epsilon_x^2 \mu^2} \left(e^{2i\mu \phi(x+\epsilon_x)} - 2e^{i\mu \phi(x+\epsilon_x)} + \phi(x)\right)
\]

\[
+ e^{-2i\mu \phi(x+\epsilon_x)} - 2e^{-i\mu \phi(x+\epsilon_x)} + \phi(x) + e^{-2i\mu \phi(x)} - 2e^{-i\mu \phi(x)} - 4 \right)
\]

In the continuum limit (\( \epsilon_x, \epsilon_t \to 0 \)) the sum goes to \( \int dx (\nabla \phi)^2 (\cos \mu \phi)^2 \). So including this term in the Hamiltonian and following the same steps as before will mean that the expression for action in the partition function becomes

\[
S = \int dx dt (\dot{\phi}^2/2 - \pi^2/2 + (\nabla \phi)^2 (\cos \mu \phi)^2/2)
\]

Finally we have the \( n^2 \phi^2 \) term. We discretize this as:

\[
\sum_{x,y} \epsilon_x \delta_{x,y} \dot{\phi}_x(x) \dot{\phi}_x(y)
\]

Then following the same steps as above we have that the contribution from this term in the action will be \( m^2 (\sin (\mu \phi))^2 / 4 \mu^2 \).

Bringing it all together we have the following result:

\[
Z = \int [D\phi] e^{iS}
\]

where

\[
S = \int dx dt \frac{1}{2} (\dot{\phi}^2 - (\nabla \phi)^2 (\cos \mu \phi)^2 - m^2 (\sin (\mu \phi)^2 / \mu^2))
\]

V. DISCUSSION

There are several points to be noted here. First, we note that even though we quantized a non-interacting field, the resulting polymerised theory is not free. In fact it contains interaction terms of all orders. Thus the polymerised Klein-Gordon field cannot be regarded as a collection of simple harmonic oscillators in fourier space.

Secondly, we note that the action for the Klein-Gordon field has a discrete global symmetry: \( \phi \to \phi + \frac{2\pi}{\mu} \). A similar discrete symmetry \( x \to x + \frac{2\pi}{\mu} \) appeared in quantum mechanics as well. The origin of this symmetry can be traced to the fact that in both cases, we chose a single scale while defining the position/field observable, and thereafter restricted ourselves to a subspace of the Hilbert Space generated by the action of the observables. It was shown in [12] that the classical phase space corresponding to a polymer quantum mechanical theory with one degree of freedom is a cylinder when one restricts to observables defined with a single scale, as in here. This explains the periodicity in the action that we have here.

Thirdly, we observe that the polymerised Klein-Gordon theory does not respect Lorentz symmetry. The origin of the Lorentz violation is easy to understand. The Hamiltonian has terms \((\nabla \phi)^2 \) and \( \pi^2 \), the latter equaling \((\phi)^2 \) in the classical theory. In the transition to the polymerised theory, the latter term remains as it is while a scale \( \mu \) enters the polymerisation of the former. This spoils the symmetry between the space and time derivatives that existed in the classical theory. The question can be asked if the Lorentz violation in the polymerised scalar field theory is suppressed at low energies. If one takes the point of view that polymerised field theories in continuum should make contact with known physics at low energies, then one should insist on the suppression of Lorentz violating effects. To answer this question one would need to study the RG flow of this theory. The present work makes such a study feasible.

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