Extended Partial-Wave Analysis of $\pi N$ Scattering Data

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We present results from a comprehensive partial-wave analysis of $\pi^\pm p$ elastic scattering and charge-exchange data, covering the region from threshold to 2.6 GeV in the lab pion kinetic energy, employing a coupled-channel formalism to simultaneously fit $\pi^- p \to \eta n$ data to 0.8 GeV. Our main result, solution SP06, utilizes a complete set of forward and fixed-t dispersion relation constraints applied to the $\pi N$ elastic amplitude. The results of these analyses are compared with previous solutions in terms of their resonance spectra and preferred values for couplings and low-energy parameters.

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I. INTRODUCTION

Most $N$ and $\Delta$ resonances, listed as 3- and 4-star states in the Review of Particle Properties (RPP) \cite{RPP}, have had their existence, masses, and widths determined through single-channel fits to scattering data, with $\pi N$ elastic scattering being the predominant source. The most comprehensive $\pi N$ analyses have been performed by the Karlsruhe-Helsinki (KH) \cite{KH}, Carnegie-Mellon–Berkeley (CMB) \cite{CMB}, and George Washington (GW) \cite{GW} groups.

All of these studies essentially agree on the existence and (most) properties of the 4-star states. For the 3-star and lower states, however, even a statement of existence is problematic. Many states claimed in the KH and CMB fits have not been found in recent GW analyses. This discrepancy clearly impacts the “missing resonance” problem, which has more quark model states predicted than observed. If many 3-star and lower-rated states are not observed in $\pi N$ scattering data (where they were first identified) then many more states are either “missing” or weakly coupled to the $\pi N$ channel.

These problems have motivated a reexamination of the KH analysis \cite{KH} and further improvements to the ongoing GW studies. We have recently added data from the reaction $\pi N \to \eta N$, in order to better describe the $\pi N$ S-wave and the $N(1535)S_{11}$ resonance which have a significant coupling to the $\eta N$ final state \cite{KH}. In the present study, we have extended the energy upper limit from 2.1 GeV to 2.6 GeV, in the lab pion kinetic energy, in order to cover the resonance region more completely. This extended energy range will be carried over to our fits of pion photoproduction and electroproduction data, which are parameterized in terms of the $\pi N$ scattering amplitudes. The extended energy range for photoproduction should allow us to fit all single-pion photoproduction data expected from the present generation of Jefferson Lab experiments.

A description of the coupled-channel analysis of $\pi N$ elastic and $\eta N$ production data, constrained by dispersion relations, is given in Ref. \cite{GW} and will not be repeated here. In this report, we will concentrate on new features seen over our extended energy region and changes to the fit below 2.1 GeV. These are discussed in Section III. Changes to the database are described in Section IV. We have mainly added data from 2.1 to 2.6 GeV, but have also included new measurements at very low energies. Some of these require special attention. Finally, in Section IV we summarize our results.

II. DATABASE

The first two decades (1957 through 1979) of experiments focused on the $\pi N$ system and non-strange baryon resonances produced a large amount of data below 2.6 GeV (9932 $\pi^+ p$, 9637 $\pi^- p$, and 1569 charge-exchange data). These data were used in the canonical KH \cite{KH} and CMB \cite{CMB} analyses. In the present study, we have fitted 13344 $\pi^+ p$, 11967 $\pi^- p$, 2933 charge-exchange, and 257 $\eta$-production data. This increase is primarily due to a second generation of $\pi N$ measurements (both unpolarized and polarized) carried out at high-intensity facilities such as LAMPF, TRIUMF, and PSI (former SIN). These more recent measurements generally have small statistical and systematic uncertainties and, therefore, have a significant influence on fits to the full database.

The evolution of our database is summarized in Table I. Over the course of five previous pion-nucleon analyses \cite{GW}. 

our energy range was extended from 1.1 to 2.1 GeV in the lab pion kinetic energy. Here we have incorporated missed measurements below 2.1 GeV and the existing database to 2.6 GeV (the \( \eta \) production database was not extended) using the Durham RAL Database [11].

Below, we list recent (post-2003) additions below 2 GeV for elastic scattering, charge-exchange scattering, and \( \eta \)-production. As in previous fits, not all of the available data have been used. Some data with very large \( \chi^2 \) contributions have been excluded from our fits. Redundant data are also excluded. These include total elastic cross sections based on differential cross sections already contained in the database. Measurements of \( P \) with uncertainties more than 0.2 are not included as they have little influence in our fits. However, all available data have been retained in the database (the excluded data labeled as “flagged” [12]) so that comparisons can be made through our on-line facility [14]. Some of the data, listed as new, were available in unpublished form at the time of our previous analysis [4]. A complete description of the database and those data not included in our fits is available from the authors [14].

Most recent \( \pi^\pm p \) measurements have been performed at low energies, TRIUMF being the main source. From this laboratory, we have added 274 \( \pi^\pm p \) and 271 \( \pi^- p \) differential cross sections from 20 to 40 MeV. These data cover a broad angular range from 10° (including the Coulomb-nuclear interference region) to 170° [15] and have allowed us to extend our single-energy fits to very low energies (20 MeV) for the first time (see Table II).

The TRIUMF cross sections for \( \pi^\pm \) elastic scattering were measured simultaneously over the full angular range using the CHAOS facility. At low energies, however, the forward (backward) cross sections are determined from measurements of the charged pion (proton). We mention this because the full angular range is difficult to fit with a single systematic uncertainty. The backward angle data disagree with both the KH predictions and predictions based on our FA02 solution. Including these data in our fit did not solve the problem, as can be seen in Fig. II(a). To resolve the conflict between forward, medium and backward scattering measurements, we divided the data into two or three pieces and treated them independently [Fig. II(b)]. Clearly, the angular dependence at backward angles is not reproduced by SP06, nor was it reproduced by our single-energy fit. The reason for this conflict is unclear.

Further low-energy additions include 25 \( \pi^+ p \) and 3 \( \pi^- p \) \( A_p \) data between 50 and 130 MeV, at medium scattering angles, measured at PSI [16]. New total cross sections for charge-exchange measurements between 40 and 250 MeV came from PSI recently [17]. They have very little effect and seem quite well fitted by SP06 without any adjustment. Two BNL–AGS experiments from the Crystal Ball Collaboration have also been analyzed and added to our database. These include 648 charge-exchange data between 520 and 620 MeV [18] and 84 \( \eta \)-production data between 560 and 620 MeV [19]. The angular coverage was 30 to 160° in both cases. Results based on the inclusion of these \( \eta \)-production data are given in Ref. [6].

Finally, ITEP–PNPI \( \pi^\pm p \) experiments have provided 3 \( P \) and 3 \( A \) measurements at 1300 MeV in the backward direction [20]. Previous measurements of the \( \pi^\pm p \) spin-rotation parameter \( A \), by the same collaboration [21], allowed us to resolve a discrepancy between the GW and the CMB/KH predictions (Fig. 2) using the method of Barrelet. These new measurements agree with predictions from our older FA02 and SM95 solutions.

### III. RESULTS AND DISCUSSION

#### A. SP06 versus the FA02 and KH fits

The main result of this work is an energy-dependent solution (SP06), fitting data from threshold to 2.6 GeV, and a set of single-energy solutions (SES) ranging from 20 MeV to 2.575 GeV. Our present and previous energy-dependent solutions are compared in Table II. Results from the KH solutions are listed here as well. A comparison of SP06 and our previous solution FA02, up to the energy limit of FA02, shows that a fit to higher energies is possible without degrading the description of data below 2.1 GeV.

In previous analyses, we have used the systematic uncertainty as an overall normalization factor for angular distributions. With each angular distribution, we associate the pair \((X, \epsilon_X)\): a normalization constant \( X \) and its uncertainty \( \epsilon_X \). The quantity \( \epsilon_X \) is generally associated with the systematic uncertainty (if known.) The modified \( \chi^2 \) function, to be minimized, is then given by

\[
\chi^2 = \sum_i \left( \frac{X \theta_i - \theta_{i \exp}}{\epsilon_i} \right)^2 + \left( \frac{X - 1}{\epsilon_X} \right)^2,
\]

where the subscript \( i \) labels data points within the distribution, \( \theta_{i \exp} \) is an individual measurement, \( \theta_i \) is the calculated
value, and $\epsilon_i$ is the statistical uncertainty. For total cross sections and excitation data, we have combined statistical and systematic uncertainties in quadrature.

Renormalization freedom significantly improves our best-fit results, as shown in Table III. This renormalization procedure was also applied to the KH solutions. Here, however, only the normalization constants were searched to minimize $\chi^2$ (no adjustment of the partial waves was possible). In cases where the systematic uncertainty varies with angle, this procedure may be considered a first approximation. Clearly, this procedure can significantly improve the overall $\chi^2$ attributed to a fit, and has been applied in calculating the $\chi^2$ values of Table I.

In Table III we compare the energy-dependent and SES results over the energy bins used in each single-energy analysis. The quantity $\delta\chi^2$ computes $|\chi^2(\text{SP06}) - \chi^2(\text{SES})|$ divided by the number of data in each single-energy bin, providing a measure of the agreement between an individual SES and the global SP06 results (see Fig. 3). Also listed is the number of parameters varied in each SES. As was emphasized in Ref. 16, the SES are generated mainly to search for missing structures in the global fit.

Figs. 4 through 7 compare the energy-dependent fits SP06 and KA84 2 over the SP06 energy range (KA84 is valid to 10 GeV/c). The SP06 analysis has fitted waves up to $l = 8$, compared to $l = 7$ for FA02. Deviations from the KA84 results are largest in the isospin 3/2 amplitudes. One possible explanation is illustrated in Fig. 2 which compares the KA84 solution to a Barrelet-transformed version versus the double-polarization quantity $A$ for $\pi^+p$ ($I = 3/2$) scattering. This exercise and resulting changes to the KA84 isospin 3/2 amplitudes were discussed in Ref. 22 (see also the comments in Ref. 22). The agreement between SP06 and KA84 for $\pi^-p$ data is much closer, suggesting the absence of a Barrelet ambiguity in the isospin 1/2 amplitudes. Deviations from FA02 are visible mainly near the end point of the FA02 analysis (some examples are given in Fig. 3).

B. Resonance Parameter Extraction

The resonance spectrum of our fit has been extracted in terms of poles and residues found by continuing into the complex energy plane. These are compiled in Tables IV and V. Zeros can be found in a similar manner and have been listed in a previous paper 4. The location of a zero is not directly related to resonance properties, but the close proximity of zeros and poles may indicate cases where a simple Breit-Wigner parameterization is questionable.

The more commonly used, and more model-dependent, Breit-Wigner parameters for resonances are listed in Tables VI and VII. Here, in the FA02 and SM95 fits, a unitary Breit-Wigner plus background form was assumed for the resonant partial wave. Data within an energy bin were then fitted using this representation. The remaining waves were fixed to values found in the full global analysis. Energy ranges over which fits were performed, and $\chi^2$ comparisons are given in Tables VIII and IX. This method is more directly linked to data than a fit to the SES. However, the resulting parameter uncertainties tend to be small, reflecting the statistical error but not the (possibly large) systematic error associated with a separation of resonance and background contributions. The pole and Breit-Wigner representations are compared in Fig. 3.

The onset of resonant behavior, seen in the FA02 $G_{17}$, $G_{19}$, and $H_{19}$ partial waves, is fully developed in SP06, the extension by 500 MeV in $T_\pi$ corresponding to a 200 MeV increase in center of mass energy. We can now also see resonant behavior in the $G_{39}$, $H_{3,11}$, and $I_{1,11}$ waves. A possible resonance is seen in $H_{1,11}$, though the SES scatter is large and the amplitude is small in magnitude.

We have tried to associate each state with its corresponding PDG designation. In some cases, this resulted in a resonance mass far from that of a “named” resonance. One such case is the $N(2000)F_{15}$. We find evidence for a second $F_{15}$ state closer to 1800 MeV. The KH analysis also finds a mass near 1880 MeV, which suggests a name change for this 2-star state may be in order. In SP06, this second $F_{15}$ resonance was found by scanning each partial wave for small structures. Its resonance parameters have been determined through a fit to the full database and are quoted without errors.

The $\Delta(1910)P_{31}$ is also problematic. We find only a single $P_{31}$ state, with a pole position more in line with the (1-star) $\Delta(1750)P_{31}$ than the (4-star) $\Delta(1910)P_{31}$. As can be seen in Fig. 3, the $P_{31}$ resonance signature is particularly subtle for a 4-star state. Small changes in the KH and CMB amplitudes, due to the Barrelet ambiguity, could explain this mass shift. Our Breit-Wigner fits to this structure yielded spurious results, with a mass several hundred MeV above the pole position and a width exceeding 1 GeV, if data were fitted around the assumed 4-star state mass. More reasonable values were obtained when the fitted energy range was expanded. This fit to a Breit-Wigner form is questionable for states with poles so far from the physical axis (see Fig. 3).

The $P_{11}$ partial wave of KA84 and the SES associated with SP06 agree reasonably well over the full range of SP06.
However, this does not lead to agreement on the resonance content. The prominent \( N(1440)P_{11} \) resonance is clearly evident in both analyses, but occurs very near the \( \pi\Delta \) threshold, making a Breit-Wigner fit questionable. Above this energy, the \( P_{11} \) partial wave wraps around the center of the Argand diagram (Fig. 10). As a result, small changes in the amplitude can produce large changes in the phase, though these changes have little influence on the fit to data. States above the \( N(1440)P_{11} \) should be established in reactions where they are more clearly required.

IV. SUMMARY AND CONCLUSIONS

We have fitted the existing \( \pi N \) elastic scattering and charge-exchange database to 2.6 GeV (\( \eta N \) data included to 800 MeV), employing a complete set of dispersion relation constraints, up to \( T_\pi = 1 \) GeV and \( t = -0.4 \) (GeV/c)^2. This extension in \( T_\pi \) has allowed us to search an addition 200 MeV of the resonance region (in center of mass energy).

Some resonance structures, at the limit of our FA02 analysis, are now better defined, while new structures have appeared in the G-, H- and I-waves. Both the SP06 and KH solutions are reasonably well within the spread of the isospin 1/2 SES (as shown in Figs. 4 and 5). However, the KH solutions are less smooth suggesting the existence of additional resonances weakly coupled to the \( \pi N \) channel. In our opinion, such states should be established in reactions where they couple more strongly; the \( \pi N \) database can be fitted without these additional resonances. A comparison of SP06 and KH partial waves with isospin 3/2 is more interesting. The \( P_{31}, D_{33}, \) and \( D_{35} \) waves show large deviations, some of which have been qualitatively explained in Ref. 22.

Other quantities of interest, such as the scattering lengths, the \( \pi N \) coupling constant and sigma term are consistent with values obtained in the FA02 fit. In Fig. 11 we show a quadratic fit to \( \chi^2 \) values from solutions with \( g^2/4\pi \) ranging from 13.70 to 13.85, yielding the value \( g^2/4\pi = 13.76 \pm 0.01 \), in agreement with the FA02 result. Finally, we note that the sigma term was extracted from our FA02 solution, using interior dispersion relations, in Ref. 24. We find this quantity has changed by less than 2 MeV between FA02 and SP06.

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TABLE I: Comparison of present (SP06) and previous (FA02 \[4\], SM95 \[7\], FA93 \[8\], SM90 \[9\], and FA84 \[10\]) energy-dependent partial-wave analyses of elastic $\pi^\pm p$, charge-exchange ($\pi^0 n$), and $\pi^- p \rightarrow \eta n$ ($\eta n$) scattering data. For both SP06 and FA02 solutions, $\eta N$ data have been included to 800 MeV. The older Karlsruhe KA84 and KH80 results \[2\] are included for comparison. $N_{prm}$ is the number of parameters ($I = 1/2$ and $3/2$) varied in the fit. SP06$^*$ gives the SP06 result evaluated over the energy range of our previous fits.

| Solution Range (MeV) | $\chi^2/\pi^+ p$ | $\chi^2/\pi^- p$ | $\chi^2/\pi^0 n$ | $\chi^2/\eta n$ | $N_{prm}$ |
|----------------------|------------------|------------------|------------------|----------------|-----------|
| SP06 2600            | 27155/13344      | 22702/11967      | 6084/2933        | 626/257        | 93/81     |
| KA84 2600            | 48394/13344      | 61845/11967      | 9410/2933        |                |           |
| KH80 2600            | 32468/13344      | 40634/11967      | 8005/2933        |                |           |
| SP06$^*$ 2100        | 22879/11842      | 18701/10561      | 4945/2640        | 626/257        | 93/81     |
| FA02 2100            | 21735/10468      | 18932/9650       | 4136/1690        | 439/173        | 86/70     |
| SM95 2100            | 23593/10197      | 18855/9421       | 4442/1625        | 94/80          |           |
| FA93 2100            | 23552/10106      | 20747/9304       | 4834/1668        | 83/77          |           |
| SM90 2100            | 24897/10031      | 24293/9344       | 10814/2132       | 76/68          |           |
| FA84 1100            | 7416/3771        | 10658/4942       | 2062/717         | 64/57          |           |

FIG. 1: Differential cross sections for $\pi^+ p$ elastic scattering at 26 MeV: (a) unnormalized and (b) normalized data. The Karlsruhe KA84 prediction \[2\] is plotted as a dot-dashed line. Data are taken from Ref. \[15\].
Table II: Single-energy (binned) fits of combined elastic $\pi^+ p$, charge-exchange, and $\pi^- p \rightarrow \eta n$ scattering data. $N_{prm}$ gives the number of parameters varied in each single-energy fit and $\chi^2_\theta$ is given by the energy-dependent fit, SP06, over the same energy interval. $\delta \chi^2 = [\chi^2(SP06) - \chi^2(SES)]$/data quantifies the agreement between individual SES and SP06.

| $T_\pi$ (MeV) | Range (MeV) | $N_{prm}$ | $\chi^2$/data | $\chi^2_\theta$ | $\delta \chi^2$ | $T_\pi$ (MeV) | Range (MeV) | $N_{prm}$ | $\chi^2$/data | $\chi^2_\theta$ | $\delta \chi^2$ |
|--------------|-------------|-----------|----------------|----------------|----------------|--------------|-------------|-----------|----------------|----------------|----------------|
| 20           | 19–21       | 4         | 163/85 194 0.36 |                | 930           | 920–940     | 28          | 338/287 534 0.68 |                | 960           | 950–970     | 32          | 350/332 570 0.66 |
| 30           | 26–34       | 4         | 291/231 329 0.16 |                | 47            | 45–50       | 4           | 181/124 238 0.46 | 1000         | 985–1015    | 36          | 688/442 839 0.34 |
| 66           | 61–70       | 4         | 204/161 213 0.16 |                | 66            | 87–92       | 4           | 126/121 149 0.19 | 1045         | 1040–1050   | 40          | 301/210 406 0.50 |
| 90           | 107–117     | 8         | 131/114 148 0.15 |                | 112           | 121–127     | 8           | 82/63 101 0.30 | 1100         | 1095–1105   | 40          | 266/229 362 0.42 |
| 124          | 139–146     | 9         | 211/160 225 0.09 |                | 142           | 159–167     | 9           | 259/219 318 0.50 | 1150         | 1140–1160   | 42          | 665/446 863 0.44 |
| 170          | 165–175     | 9         | 174/163 200 0.16 |                | 170           | 191–195     | 9           | 97/107 117 0.19 | 1210         | 1200–1220   | 44          | 299/274 378 0.29 |
| 193          | 214–221     | 9         | 106/109 145 0.36 |                | 217           | 235–241     | 9           | 111/115 143 0.28 | 1320         | 1300–1340   | 46          | 824/567 1036 0.37 |
| 238          | 263–271     | 9         | 152/123 181 0.24 |                | 266           | 291–294     | 10          | 155/129 208 0.41 | 1400         | 1385–1415   | 46          | 587/423 871 0.67 |
| 292          | 306–311     | 10        | 158/140 180 0.16 |                | 309           | 332–336     | 11          | 93/58 139 0.79 | 1480         | 1465–1495   | 50          | 626/409 861 0.57 |
| 334          | 351–352     | 11        | 64/109 84 0.18 |                | 352           | 370–411     | 11          | 259/219 318 0.50 | 1570         | 1555–1585   | 54          | 568/478 826 0.54 |
| 390          | 420–430     | 12        | 170/162 215 0.28 |                | 425           | 450–480     | 14          | 266/178 358 0.52 | 1660         | 1645–1675   | 56          | 695/496 976 0.57 |
| 465          | 490–510     | 15        | 382/245 444 0.25 |                | 500           | 511–529     | 17          | 132/125 176 0.35 | 1840         | 1825–1855   | 58          | 423/323 741 0.98 |
| 520          | 530–540     | 19        | 270/247 321 0.21 |                | 535           | 555–565     | 20          | 387/270 601 0.79 | 1870         | 1860–1880   | 58          | 642/441 1005 0.82 |
| 560          | 570–590     | 20        | 439/401 542 0.26 |                | 580           | 595–605     | 20          | 275/274 414 0.51 | 2025         | 2010–2040   | 58          | 397/339 714 0.94 |
| 600          | 620–630     | 21        | 182/162 234 0.32 |                | 625           | 645–675     | 23          | 573/462 727 0.36 | 2075         | 2050–2100   | 58          | 928/425 1270 0.80 |
| 660          | 700–740     | 26        | 383/307 597 0.70 |                | 720           | 735–755     | 26          | 362/257 609 0.96 | 2175         | 2150–2200   | 58          | 1025/486 1373 0.72 |
| 745          | 755–775     | 26        | 375/381 549 0.26 |                | 765           | 776–788     | 27          | 170/116 353 0.72 | 2325         | 2300–2350   | 58          | 662/419 870 0.50 |
| 782          | 790–810     | 27        | 634/441 747 0.26 |                | 800           | 813–827     | 28          | 431/393 518 0.22 | 2425         | 2400–2450   | 58          | 205/186 679 2.55 |
| 820          | 856–885     | 28        | 661/444 880 0.49 |                | 875           | 886–904     | 28          | 238/203 456 1.07 | 2525         | 2500–2550   | 58          | 497/171 889 2.29 |
| 890          | 895–905     | 28        | 515/409 776 0.64 |                | 900           | 913/915     | 29          | 395/342 701 0.72 | 2575         | 2550–2600   | 58          | 385/139 911 3.78 |
TABLE III: Comparison of $\chi^2$/data for normalized (Norm) and unnormalized (Unnorm) data used in the SP06 and FA02 solutions. Karlsruhe KA84 and KH80 results are included for comparison. Values for FA02 correspond to a 2.1 GeV energy limit. SP06, KH80, and KA84 are evaluated up to 2.6 GeV.

| Reaction      | SP06 |        | FA02 |        | KA84 |        | KH80 |        |
|---------------|------|--------|------|--------|------|--------|------|--------|
|               | Norm | Unnorm | Norm | Unnorm | Norm | Unnorm | Norm | Unnorm |
| $\pi^+ p \rightarrow \pi^+ p$ | 2.0  | 6.7    | 2.1  | 9.3    | 3.6  | 10.0   | 2.4  | 8.5    |
| $\pi^- p \rightarrow \pi^- p$ | 1.9  | 6.2    | 2.0  | 7.1    | 5.2  | 13.0   | 3.4  | 10.2   |
| $\pi^- p \rightarrow \pi^0 n$ | 2.1  | 4.5    | 2.4  | 9.5    | 3.2  | 7.8    | 2.7  | 5.9    |
| $\pi^- p \rightarrow \eta n$ | 2.4  | 10.1   | 2.5  | 4.6    |      |        |      |        |

FIG. 2: Spin-rotation parameter $A$ for $\pi^+ p$. The original KA84 solution (dot-dashed line) compared to a Barrelet-transformed solution (dotted line) and our SP06 solution (solid line). Data are taken from Ref. 21.

FIG. 3: Comparison of the SES and global SP06 fits via $\delta \chi^2 = [\chi^2(SP06) - \chi^2(SES)]/\text{data}$ presented in Table III.
TABLE IV: Pole positions from the solution SP06, our previous solution FA02, and a range from the Particle Data Group (RPP) (in square brackets). Real ($W_R$) and imaginary ($-2W_I$) parts are listed for isospin 1/2 baryon resonances. The second sheet pole is labeled by a †. Modulus and phase values are listed for the $\pi N$ elastic pole residue.

| Wave | $W_R$ | $-2W_I$ | Modulus (MeV) | Phase (deg) | Ref |
|------|-------|---------|---------------|-------------|-----|
| $S_{11}$ | 1502 | 95 | 16 | $-16$ | SP06 |
| | 1526 | 130 | 33 | $+14$ | FA02 |
| [1490–1530] | [90–250] | | | | RPP |
| $S_{11}$ | 1648 | 80 | 14 | $-69$ | SP06 |
| | 1653 | 182 | 69 | $-55$ | FA02 |
| [1640–1670] | [150–180] | | | | RPP |
| $P_{11}$ | 1359 | 162 | 38 | $-98$ | SP06 |
| | 1357 | 160 | 36 | $-102$ | FA02 |
| [1350–1380] | [160–220] | | | | RPP |
| $P_{11}^\dagger$ | 1388 | 165 | 86 | $-46$ | SP06 |
| | 1385 | 166 | 82 | $-51$ | FA02 |
| | | | | | RPP |
| $P_{13}$ | 1666 | 355 | 25 | $-94$ | SP06 |
| | 1655 | 278 | 20 | $-88$ | FA02 |
| [1660–1690] | [115–275] | | | | RPP |
| $D_{13}$ | 1515 | 113 | 38 | $-5$ | SP06 |
| | 1514 | 102 | 35 | $-6$ | FA02 |
| [1505–1515] | [105–120] | | | | RPP |
| $D_{15}$ | 1657 | 139 | 27 | $-21$ | SP06 |
| | 1659 | 146 | 29 | $-22$ | FA02 |
| [1655–1665] | [125–150] | | | | RPP |
| $F_{15}$ | 1674 | 115 | 42 | $-4$ | SP06 |
| | 1678 | 120 | 43 | $+1$ | FA02 |
| [1665–1680] | [110–135] | | | | RPP |
| $F_{15}$ | 1807 | 109 | 60 | $-67$ | SP06 |
| | | | | | FA02 |
| | | | | | RPP |
| $G_{17}$ | 2070 | 520 | 72 | $-32$ | SP06 |
| | 2076 | 502 | 68 | $-32$ | FA02 |
| [2050–2100] | [400–520] | | | | RPP |
| $H_{19}$ | 2199 | 372 | 33 | $-33$ | SP06 |
| | 2209 | 564 | 96 | $-71$ | FA02 |
| [2130–2200] | [400–560] | | | | RPP |
| $G_{19}$ | 2217 | 431 | 21 | $-20$ | SP06 |
| | 2238 | 536 | 33 | $-25$ | FA02 |
| [2150–2250] | [350–550] | | | | RPP |
| $H_{1,11}$ | 2203 | 133 | 1 | $-12$ | SP06 |
| | | | | | FA02 |
| | | | | | RPP |
TABLE V: Pole positions for isospin 3/2 baryon resonances. Notation as in Table IV. † A second $P_{33}$ state was not reported in FA02. †† No RPP average given.

| Wave | $W_R$ (MeV) | $-2W_I$ (MeV) | Modulus (MeV) | Phase (deg) | Ref |
|------|-------------|---------------|---------------|-------------|-----|
| $S_{31}$ | 1595 | 135 | 15 | $-92$ | SP06 |
| | 1594 | 118 | 17 | $-104$ | FA02 |
| | [1590–1610] | [115–120] | RPP |
| $P_{31}$ | 1771 | 479 | 45 | $+172$ | SP06 |
| | 1748 | 524 | 48 | $+158$ | FA02 |
| | [1830–1880] | [200–500] | RPP |
| $P_{33}$ | 1211 | 99 | 52 | $-47$ | SP06 |
| | 1210 | 100 | 53 | $-47$ | FA02 |
| | [1209–1211] | [98–102] | RPP |
| $P_{33}$ | 1457 | 400 | 44 | $+147$ | SP06 |
| | | | | | FA02† |
| | [1500–1700] | [200–400] | RPP |
| $D_{33}$ | 1632 | 253 | 18 | $-40$ | SP06 |
| | 1617 | 226 | 16 | $-47$ | FA02 |
| | [1620–1680] | [160–240] | RPP |
| $D_{35}$ | 2001 | 387 | 7 | $-12$ | SP06 |
| | 1966 | 364 | 16 | $-21$ | FA02 |
| | [1840–1960] | [175–360] | RPP |
| $F_{35}$ | 1819 | 247 | 15 | $-30$ | SP06 |
| | 1825 | 270 | 16 | $-25$ | FA02 |
| | [1825–1835] | [265–300] | RPP |
| $F_{37}$ | 1876 | 227 | 53 | $-31$ | SP06 |
| | 1874 | 236 | 57 | $-34$ | FA02 |
| | [1870–1890] | [220–260] | RPP |
| $G_{39}$ | 1983 | 878 | 24 | $-139$ | SP06 |
| | | | | | FA02 |
| | | | | | RPP†† |
| $H_{3,11}$ | 2529 | 621 | 33 | $-45$ | SP06 |
| | | | | | FA02 |
| | [2260–2400] | [350–750] | RPP |
TABLE VI: Resonance couplings from a Breit-Wigner fit to the SP06 solution, our previous solution FA02, and a range from the [RPP] (in square brackets). Masses $W_R$, widths $\Gamma$, and partial width $\Gamma_{\pi N}/\Gamma$ are listed for isospin 1/2 baryon resonances. $\Gamma_{\pi N}/\Gamma$ for $N(1650)S_{11}$ is not varied in the BW fit.

| Resonance      | $W_R$ (MeV) | $\Gamma$ (MeV) | $\Gamma_{\pi N}/\Gamma$ | Ref     |
|----------------|-------------|-----------------|--------------------------|---------|
| $N(1440)P_{11}$| 1485.0±1.2  | 284±18          | 0.787±0.016              | SP06    |
|                | 1468.0±4.5  | 360±26          | 0.750±0.024              | FA02    |
|                | [1420–1470] | [200–450]       | [0.55–0.75]              | RPP     |
| $N(1520)D_{13}$| 1514.5±0.2  | 103.6±0.4       | 0.632±0.001              | SP06    |
|                | 1516.3±0.8  | 98.6±2.6        | 0.640±0.005              | FA02    |
|                | [1515–1525] | [100–125]       | [0.55–0.65]              | RPP     |
| $N(1535)S_{11}$| 1547.0±0.7  | 188.4±3.8       | 0.355±0.002              | SP06    |
|                | 1546.7±2.2  | 178.0±11.6      | 0.360±0.009              | FA02    |
|                | [1525–1545] | [125–175]       | [0.35–0.55]              | RPP     |
| $N(1650)S_{11}$| 1634.7±1.1  | 115.4±2.8       | 1.000                    | SP06    |
|                | 1651.2±4.7  | 130.6±7.0       | 1.000                    | FA02    |
|                | [1645–1670] | [145–185]       | [0.60–0.95]              | RPP     |
| $N(1675)D_{13}$| 1674.1±0.2  | 146.5±1.0       | 0.393±0.001              | SP06    |
|                | 1676.2±0.6  | 151.8±3.0       | 0.400±0.002              | FA02    |
|                | [1670–1680] | [130–165]       | [0.35–0.45]              | RPP     |
| $N(1680)F_{15}$| 1680.1±0.2  | 128.0±1.1       | 0.701±0.001              | SP06    |
|                | 1683.2±0.7  | 134.4±3.8       | 0.670±0.004              | FA02    |
|                | [1680–1690] | [120–140]       | [0.65–0.70]              | RPP     |
| $N(1720)P_{13}$| 1763.8±4.6  | 210±22          | 0.094±0.005              | SP06    |
|                | 1749.6±4.5  | 256±22          | 0.190±0.004              | FA02    |
|                | [1700–1750] | [150–300]       | [0.10–0.20]              | RPP     |
| $N(2000)F_{15}$| 1817.7      | 117.6           | 0.127                    | SP06    |
|                | [2000]      |                 |                         | RPP     |
| $N(2190)G_{17}$| 2152.4±1.4  | 484±13          | 0.238±0.001              | SP06    |
|                | 2192.1±8.7  | 726±62          | 0.230±0.002              | FA02    |
|                | [2100–2200] | [300–700]       | [0.1–0.2]                | RPP     |
| $N(2220)H_{19}$| 2316.3±2.9  | 633±17          | 0.246±0.001              | SP06    |
|                | 2270±11     | 366±42          | 0.200±0.006              | FA02    |
|                | [2200–2300] | [350–500]       | [0.1–0.2]                | RPP     |
| $N(2245)H_{1,11}$| 2247.2±6.2  | 225±23          | 0.014±0.001              | SP06    |
|                | [2550–2750] | [500–800]       | [0.05–0.10]              | RPP     |
| $N(2250)G_{19}$| 2302±6      | 628±28          | 0.089±0.001              | SP06    |
|                | 2376±43     | 924±178         | 0.110±0.004              | FA02    |
|                | [2200–2350] | [230–800]       | [0.05–0.15]              | RPP     |
| $N(2600)I_{1,11}$| 2623±197   | 1311±996        | 0.050±0.018              | SP06    |
|                | [2550–2750] | [500–800]       | [0.05–0.10]              | RPP     |
TABLE VII: Parameters for isospin 3/2 baryon resonances. Notation as in Table VI. \( \Gamma_{\pi N}/\Gamma \) for \( P_{33}(1232) \) is not varied in the BW fit.

| Resonance | \( W_R \) | \( \Gamma \) | \( \Gamma_{\pi N}/\Gamma \) | Ref |
|-----------|----------|----------|-----------------|-----|
| \( \Delta(1232)P_{33} \) | 1233.4±0.4 | 118.7±0.6 | 1.000 | SP06 |
| \( [1231−1233] \) | 118.0±2.2 | 1.000 | FA02 |
| \( \Delta(1620)S_{31} \) | 1615.2±0.4 | 146.9±1.9 | 0.315±0.001 | SP06 |
| \( [1600−1660] \) | 141.0±6.0 | 0.310±0.004 | FA02 |
| \( \Delta(1700)D_{33} \) | 1695.0±1.3 | 375.5±7.0 | 0.156±0.001 | SP06 |
| \( [1670−1750] \) | 364.8±16.6 | 0.150±0.001 | FA02 |
| \( \Delta(1905)F_{35} \) | 1857.8±1.6 | 320.6±8.6 | 0.122±0.001 | SP06 |
| \( [1865−1915] \) | 334±22 | 0.120±0.008 | FA02 |
| \( \Delta(1910)P_{31} \) | 2067.9±2.5 | 543.0±10.1 | 0.239±0.001 | SP06 |
| \( [1900−1950] \) | 534.0±238 | 0.390±0.019 | FA02 |
| \( \Delta(1930)D_{35} \) | 2233±53 | 773±187 | 0.081±0.012 | SP06 |
| \( [1900−2020] \) | 402±198 | 0.040±0.014 | FA02 |
| \( \Delta(1950)F_{37} \) | 1921.3±0.2 | 271.0±1.1 | 0.471±0.001 | SP06 |
| \( [1915−1950] \) | 278.2±3.0 | 0.480±0.002 | FA02 |
| \( \Delta(2400)G_{39} \) | 2643±141 | 895±432 | 0.064±0.022 | SP06 |
| \( [2400] \) | 543.0±10.1 | 0.239±0.001 | SP06 |
| \( \Delta(2420)H_{3,11} \) | 2633±29 | 692±47 | 0.085±0.008 | SP06 |
| \( [2300−2500] \) | 673±47 | 0.075±0.008 | FA02 |
| \( \Delta(2245)H_{1,11} \) | 2050±47 | 773±187 | 0.081±0.012 | SP06 |
| \( [2200−2500] \) | 402±198 | 0.040±0.014 | FA02 |
| \( \Delta(2600)I_{3,11} \) | 2070±47 | 773±187 | 0.081±0.012 | SP06 |
| \( [2300−2500] \) | 402±198 | 0.040±0.014 | FA02 |

TABLE VIII: Comparison of SP06 and BW plus background representations for isospin 1/2 baryon resonance fits (see text and associated Table VI). “Data” refers to the number of scattering data between \( W_{\text{min}} \) and \( W_{\text{max}} \).

| Resonance | \( W_{\text{min}} \) | \( W_{\text{max}} \) | BW fit | SP06 | Data |
|-----------|-------------|-------------|--------|------|-----|
| \( N(1440)P_{11} \) | 1350 | 1550 | 5437 | 5377 | 3104 |
| \( N(1520)D_{13} \) | 1480 | 1560 | 3350 | 3399 | 2068 |
| \( N(1535)S_{11} \) | 1490 | 1590 | 3451 | 3481 | 2195 |
| \( N(1650)S_{11} \) | 1620 | 1770 | 8658 | 8558 | 4678 |
| \( N(1675)D_{15} \) | 1610 | 1730 | 7072 | 7093 | 3932 |
| \( N(1680)F_{15} \) | 1620 | 1730 | 6326 | 6317 | 3443 |
| \( N(1720)P_{13} \) | 1620 | 1820 | 10701 | 10743 | 5837 |
| \( N(2190)G_{17} \) | 2050 | 2250 | 10414 | 10549 | 4908 |
| \( N(2220)H_{19} \) | 2150 | 2350 | 11649 | 11690 | 4660 |
| \( N(2245)H_{1,11} \) | 2050 | 2380 | 17451 | 17508 | 7573 |
| \( N(2250)G_{19} \) | 2050 | 2350 | 16073 | 16095 | 6895 |
| \( N(2600)I_{3,11} \) | 2070 | 2460 | 19554 | 19414 | 7590 |
TABLE IX: Comparison of SP06 and BW plus background representations for isospin 3/2 baryon resonance fits (see text and associated Table VII).

| Resonance     | Wmin (MeV) | Wmax (MeV) | BW fit | SP06 Data | $\chi^2$ | $\chi^2$ |
|---------------|------------|------------|--------|-----------|---------|---------|
| $\Delta(1232)P_{\frac{3}{2}}$ | 1180 | 1270 | 1283 | 1278 | 1016 |
| $\Delta(1620)S_{\frac{1}{2}}$ | 1570 | 1680 | 4696 | 4715 | 2705 |
| $\Delta(1700)D_{\frac{3}{2}}$ | 1550 | 1750 | 9959 | 9992 | 5490 |
| $\Delta(1905)F_{\frac{3}{2}}$ | 1770 | 1920 | 7545 | 7567 | 4039 |
| $\Delta(1910)P_{\frac{1}{2}}$ | 1650 | 2150 | 26540 | 25363 | 13258 |
| $\Delta(1930)D_{\frac{3}{2}}$ | 1770 | 2100 | 16241 | 16176 | 8442 |
| $\Delta(1950)F_{\frac{3}{2}}$ | 1800 | 2000 | 10842 | 10890 | 5437 |
| $\Delta(2400)G_{\frac{3}{2}}$ | 2140 | 2460 | 16855 | 16626 | 6134 |
| $\Delta(2420)H_{\frac{3}{2}}$ | 2150 | 2460 | 16149 | 16138 | 5970 |
FIG. 4: Isospin 1/2 partial-wave amplitudes $J < 3$ ($L_{1/2}, J_{1/2}$) from $T_\pi = 0$ to 2.6 GeV. Solid (dashed) curves give the real (imaginary) parts of amplitudes corresponding to the SP06 solution. The real (imaginary) parts of single-energy solutions are plotted as filled (open) circles. The dotted curve gives the unitarity limit ($\text{Im} T - T^*T$) from SP06. The Karlsruhe KA84 solution [2] is plotted with long dash-dotted (real part) and short dash-dotted (imaginary part) lines. All amplitudes are dimensionless. Vertical arrows indicate resonance $W_R$ values and horizontal bars show full $\Gamma$ and partial widths for $\Gamma_{\pi N}$. The lower BW resonance symbols are associated with the SP06 values of Table VI; upper symbols give RPP [1] values.
FIG. 5: Isospin 1/2 partial-wave amplitudes $J > 3$ ($L_{21,23}$) from $T_N = 0$ to 2.6 GeV. Notation as in Fig. 4.
FIG. 6: Isospin 3/2 partial-wave amplitudes $J < 3$ ($\ell_{1,2,3}$) from $T_\pi = 0$ to 2.6 GeV. The lower BW resonances are associated with the SP06 values of Table VI; upper symbols give RPP [1] values. Notation as in Fig. 4.
FIG. 7: Isospin 3/2 partial-wave amplitudes \( J > 3 \) (\( L_{2,2J} \)) from \( T_{\pi} = 0 \) to 2.6 GeV. Notation as in Fig.
FIG. 8: Comparison of isospin 1/2 and 3/2 partial-wave amplitudes (L_{2J,2J}) from $T_{\pi} = 0$ to 2.6 GeV. Solid (dashed) curves give the real (imaginary) parts of the SP06 amplitudes. The FA02 solution (valid to 2.1 GeV) is plotted with long dash-dotted (real part) and short dash-dotted (imaginary part) lines. All amplitudes are dimensionless.
FIG. 9: Comparison of complex plane (bottom panel) and Breit-Wigner (top panel) parameters for resonances found in the SP06 solution. Plotted are the result for (a) S- and P-wave resonances, (b) D- and F-wave resonances, and (c) G-, H, and I-wave resonances. Complex plane poles are shown as stars (the boxed star denotes a second-sheet pole). \( W_R \) and \( W_I \) give real and imaginary parts of the center-of-mass energy. The full \( \pi N \) partial widths are denoted by thin (thick) bars for each resonance. The branch point for \( \pi \Delta(1232), 1350 - i50 \text{ MeV} \), is represented as a solid triangle. The branch points for \( \eta N, 1487 - i0 \text{ MeV}, \) and \( \rho N, 1715 - i73 \text{ MeV} \), thresholds are shown as a solid diamond and solid square, respectively.
FIG. 10: Argand plots for partial-wave amplitudes from threshold (1080 MeV) to \( W = 2.5 \) GeV. Crosses indicate 50 MeV steps in \( W \). Solid circles correspond to BW \( W_R \) determination presented in Tables VI and VII.
FIG. 11: Best-fit $\chi^2$ as a function of the coupling constant $g^2/4\pi$, where all other parameters were fixed to their optimal (best-fit) values. The solid curve gives the best-fit of $\chi^2$ vs $g^2/4\pi$ assuming $\chi^2 = a + \left(\frac{g^2/4\pi - b}{c}\right)^2$, where $a$, $b$, and $c$ are free parameters.