A Remark of the Sanders-Wang’s Theorem on Symmetry-integrability

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Abstract We extend the integrability analysis for scalar evolution equations of type

$$u_t = u_m + f(u, u_1, \ldots, u_{m-1})$$

from the case that the right hand side is a $\lambda$-homogeneous formal power series to the case that it is a nonhomogeneous formal power series. It is proved that the existence of one nontrivial symmetry implies the existence of infinitely many, more precisely, the orders of the infinite integrable hierarchy must be one of the following cases: $\mathbb{Z}_+ + 1$, $2\mathbb{Z}_+ + 1$, $6\mathbb{Z}_+ \pm 1$, or $6\mathbb{Z}_+ + 1$. Moreover, if the nonlinear part of the equation is a polynomial of order less than $m - 1$, we show that any generalized symmetry is also of polynomial type.

1 Introduction

Stimulated by the great progress of the theory of solitons and integrable systems, the symmetry aspect of PDE systems, which was initiated by Sophus Lie more than one hundred years ago, has been intensively studied by many famous researchers in the last three decades and is still very important. We refer to the book [1] for history remarks and various applications. An interesting problem arising in this period is whether a system of partial differential equations can admit only a finite dimensional space of generalized symmetries. It is a common knowledge that an integrable evolution equation is always a member of an infinite integrable hierarchy whose members are symmetries one for another, as Fokas [2] stated

Another interesting fact regarding the symmetry structure of evolution equations is that in all known cases the existence of one generalized symmetry implies the existence of infinitely many. (However, this has not been proved in general.)

For $\lambda$-homogeneous, with positive $\lambda$, equations of the form

$$u_t = u_m + f(u, u_1, \ldots, u_{m-1}),$$

where $f$ is a formal power series with terms that are at least quadratic, the conjecture has been proved by Sanders and Wang [3]. The exact statement is

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A nontrivial symmetry of a $\lambda$-homogeneous equation is part of a hierarchy starting at order 3, 5, or 7 in the odd case, and at order 2 in the even case.

Note that $\lambda$-homogeneity with positive $\lambda$ implies the equation is of polynomial type and is very restricted. The aim of the present paper is to remove the condition of $\lambda$-homogeneity and show that the orders of the infinite integrable hierarchy must be one of the following cases, Theorem 6: (i) all positive integers, as the Burgers equation; (ii) all odd positive integers, as the KdV equation; (iii) all positive integers congruent to 1 or $-1$ modulo 6, as the potential Sawada-Kotera equation; (iv) all positive integers congruent to 1 modulo 6. In the last case, however, no example has been found as far as we know, cf. [4]. Furthermore, we prove that if the nonlinear part of the equation is a polynomial of order less than $m-1$, then any generalized symmetry is also of polynomial type, Theorem 3.

In contrast to the scalar case, the Fokas’ conjecture for systems of evolution equations has been disproved. An example due to Bakirov of a fourth order system of two coupled evolution equations is proved to possess only one nontrivial symmetry of order six by Beukers, Sanders and Wang [5]. Even for the refined version of the conjecture [6] that a system of $m$ evolution equations requires $m$ higher order symmetries in order to be integrable, a counterexample is given by van der Kamp and Sanders [7].

The outline of the paper is as follows. In Section 2, we introduce some definitions and notations used throughout this paper. In Section 3, we estimate the orders of homogeneous components of generalized symmetries by induction on degrees. In particular for a polynomial evolution equation without submaximal order terms, we obtain an upper bound for the degree of the symmetry with a prescribed linear term. Section 4 contains a proof of the main theorem claimed above. For the reader’s convenience, we provide two appendices which state in our notation some well known results necessary to understand the text.

The present paper is only a very restricted study of the symmetry structure of scalar evolution equations. We apologize to whom have read through it and still not found anything, especially practical examples or physical applications, they are interested in.

2 Basic definitions and notations

Let $\mathbb{R}[u, u_1, u_2, \ldots]$ denote the polynomial ring of infinitely many variables $u = u_0, u_1, u_2, \ldots$ with real coefficients (any fixed element of $\mathbb{R}[u, u_1, u_2, \ldots]$ involves only finitely many variables). Elements of $\mathbb{R}[u, u_1, u_2, \ldots]$ are also called differential polynomials when $u$ is understood as a function of $x$ and $u_i$ is the $i$th order derivative, $i = 0, 1, 2, \ldots$, with respect to $x$. And let $\mathbb{R}[[u, u_1, u_2, \ldots]]$ denote the ring of formal power series in variables $u, u_1, u_2, \ldots$ with real coefficients (any fixed element of $\mathbb{R}[[u, u_1, u_2, \ldots]]$ may depend on infinitely many variables, but its homogeneous components all live in $\mathbb{R}[u, u_1, u_2, \ldots]$). For $k = 1, 2, \ldots$, $M^k$ will stand for the subset of $\mathbb{R}[[u, u_1, u_2, \ldots]]$ consists of elements whose homogeneous components of degree less than $k$ all vanish. For convenience, if $f \in \mathbb{R}[u, u_1, u_2, \ldots]$, write $f^k$ for the $k$th degree homogeneous component of $f$.

If $f \in \mathbb{R}[u, u_1, u_2, \ldots]$ and $f$ is not a constant, define the order of $f$ is the maximal integer $l$ such that $u_l$ appears in the expression of $f$. The nonzero
constants are regarded as being of order zero. And define the order of the zero element is $-\infty$. If $f \in \mathbb{R}[\{u, u_1, u_2, \ldots\}]$, the order of $f$ is defined to be the maximum value of the orders of its homogeneous components, and may be $+\infty$. Denote $O(l)$ the subset of $\mathbb{R}[\{u, u_1, u_2, \ldots\}]$ consists of all elements of order less than or equal to $l$. Note that $O(l)$ is well defined for arbitrary real number $l$. In the light of Lemma 2 in section 3 where $d$ is a real number, which leads to a short proof of Theorem 3, we will work freely in the context of real numbers even though in principle nonnegative integers are sufficient. By definition, $O(l) = \{0\}$ when $l$ is negative, and $O(l)$ forms a linear space for any real number $l$.

Since only the autonomous equations are concerned in this paper, the total derivative operator becomes

$$D = \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i},$$

which is well defined on $\mathbb{R}[\{u, u_1, u_2, \ldots\}]$ because $D$ preserves the set of homogeneous differential polynomials of a fixed degree. Let $F, G \in \mathbb{R}[u, u_1, u_2, \ldots]$, define

$$\{F, G\} = \nu_F G - \nu_G F,$$

where

$$\nu_F = \sum_{i=0}^{\infty} D^i F \frac{\partial}{\partial u_i}, \quad \nu_G = \sum_{i=0}^{\infty} D^i G \frac{\partial}{\partial u_i}.$$  

It is well defined because the orders both of $F$ and of $G$ are finite. In particular, if both $F$ and $G$ are homogeneous differential polynomials, say, of degree $k$ and $l$ respectively, then $\{F, G\}$ is a homogeneous differential polynomial of degree $k+l-1$. Hence we can extend the definition of the bracket onto $\mathbb{R}[\{u, u_1, u_2, \ldots\}]$ by defining

$$\{F, G\} = \sum_{s=0}^{\infty} \sum_{k+l-1=s} \{F^k, G^l\}.$$  

The bracket $\{ , \}$ is a Lie structure. In fact, the Jacobi identity can be easily checked using the equality $[D, \nu_F] = 0$.

Two evolution equations $u_t = F$ and $u_t = G$, where $F$ and $G$ are both of finite order, are called ($t$-independent) symmetries of each other if $\{F, G\} = 0$. We will not work out the definition of generalized symmetries in its most generality. For our purpose, the formalism derived so far is enough. The reason to adopt the notation $\{ , \}$ instead of $[ , ]$ is that, as pointed out by A.M. Vinogradov, see page 10 in [8], the bracket between generalized symmetries of scalar equations coincides with the standard Poisson bracket for first order differential functions which do not depend on $u$.

### 3 Estimates of orders

The following lemma is the key observation of this section.

**Lemma 1.** Let $m, n$ be nonnegative integers, $m \geq 2$ and $G \in \mathcal{M}$. Then

$$G \in O(n) \iff \{u_m, G\} \in O(m + n - 1).$$
Proof. The assertion is trivial for $G = 0$. Now assume $G$ is nonzero and the order of $G$ is $n \geq 0$, it suffices to show that the order of $\{G, u_m\}$ is exact $m + n - 1$. We will prove it using the formula

$$\frac{\partial}{\partial u_j} D^m = \sum_{i=0}^{\min\{j, m\}} \binom{m}{i} D^{m-i} \frac{\partial}{\partial u_{j-i}}.$$ 

By definition,

$$\{G, u_m\} = D^m G - \sum_{i=0}^{n} u_{m+i} \frac{\partial G}{\partial u_i}.$$ 

It is obvious that $\{G, u_m\} \in O(m + n)$. Since $m \geq 2$, we have

$$\frac{\partial}{\partial u_{m+n}} \{G, u_m\} = \frac{\partial G}{\partial u_n} - \frac{\partial G}{\partial u_n} = 0$$

and

$$\frac{\partial}{\partial u_{m+n-1}} \{G, u_m\} = \frac{\partial G}{\partial u_{n-1}} + m D \frac{\partial G}{\partial u_n} - \frac{\partial G}{\partial u_{n-1}} = m D \frac{\partial G}{\partial u_n},$$

where we have adopted the convention $\frac{\partial}{\partial u_{-1}} G = 0$. $G$ is of $n$th order and $G \in M^2$ imply $\frac{\partial G}{\partial u_{n-1}}$ is not a constant, and hence $D \frac{\partial G}{\partial u_{n-1}}$ is nonzero. Therefore, the order of $\{G, u_m\}$ is $m + n - 1$.

**Lemma 2.** Suppose $F, G \in M^1$, $F^1 = u_m$, $G^1 = u_n$, where $m, n \geq 2$. And suppose $\{F, G\} \in M^{s_0+1}$, for some $s_0 \geq 2$. If there is a real number $d$ such that $F^k \in O(m - 1 - (k-1) d)$, $\forall k$, $2 \leq k \leq s_0$, then

$$G^l \in O(n - 1 - (l-1) d), \forall l, 2 \leq l \leq s_0.$$ 

**Proof.** Since $M^{s_0+1} \subset M^{s_0}$, by induction on $s_0$, we may assume

$$G^l \in O(n - 1 - (l-1) d), \forall l, 2 \leq l \leq s_0 - 1.$$ 

Observe that $\{O(m), O(n)\} \subset O(m + n)$ holds for arbitrary real numbers $m, n$ as well as for nonnegative integers and that Lemma holds for arbitrary real number $n$ as well as for nonnegative integer $n$. From $\{F, G\}^{s_0} = \sum_{k+l=1=s_0} \{F^k, G^l\} = 0$, we have

$$\{G^{s_0}, u_m\} = \sum_{k=2}^{s_0-1} \{F^k, G^{s_0-k+1}\} + \{F^{s_0}, u_n\}$$

$$\in \sum_{k=2}^{s_0-1} \{O(m-1-(k-1)d), O(n-1-(s_0-k)d)\}$$

$$+ \{O(m-1-(s_0-1)d), u_n\}$$

$$\subset O(m+n-2-(s_0-1)d).$$

By Lemma $G^{s_0} \in O(n - 1 - (s_0 - 1)d).$ 

Lemma 2 serves two purposes. Consider the equation \( u_t = F = u_m + f \) where \( f \in \mathcal{M}^2 \cap O(m - 1) \). Suppose we have obtained a solution \( G = u_n + g \) of the symmetry equation \( \{ F, G \} = 0 \), where \( g \) lives in \( \mathcal{M}^2 \), then applying Lemma 2 for the case \( d = 0 \), we get \( g \in O(n - 1) \). This is the last step of the main theorem in the next section. And the case \( d > 0 \) of Lemma 2 leads to the following

**Theorem 3.** Suppose \( F = u_m + f, G = u_n + g \), where \( m, n \geq 2, f, g \in \mathcal{M}^2 \). If \( \{ F, G \} = 0 \) and if \( f \) is a differential polynomial of order less than \( m - 1 \), then \( g \) is a differential polynomial of order less than \( n - 1 \).

**Proof.** Since \( f \) is a differential polynomial of order less than \( m - 1 \), it is easy to see that \( f^k \in O(m - 1 - (k - 1) \cdot d) \), \( \forall k \geq 2 \), for sufficient small \( d > 0 \). By Lemma 2, \( g^l \in O(n - 1 - (l - 1) \cdot d) \), \( \forall l \geq 2 \). Hence the order of \( g \) is less than \( n - 1 \). When \( n - 1 - (l - 1) \cdot d < 0 \), \( g^l = 0 \). Therefore \( g \) is also a differential polynomial and its degree is not bigger than \( \frac{m - 1}{d} + 1 \).

The upper bound for the degree in the preceding proof may be not accurate in particular examples. Nevertheless, applying directly the method of estimating orders we have derived may give sharp upper bounds for the degrees of generalized symmetries before their explicit expressions are obtained. For example, suppose \( G = u_{2k+1} + g \), where \( g \in \mathcal{M}^2 \), is a generalized symmetry of the KdV equation \( u_t = u_3 + uu_1 \), then estimating inductively the orders of the homogeneous components of \( g \) yields that \( g^l \in O(2(k - l + 1) + 1) \), hence the degree of \( g \) is not bigger than \( k + 1 \). More generally, for any equation of the form \( u_t = F = u_{2k_0+1} + f \) where \( f \in \mathcal{M}^2 \) admits the estimate \( f^l \in O(2(k_0 - l + 1) + 1) \), e.g. the potential Sawada-Kotera equation \( u_t = u_5 + 5u_1u_3 + \frac{5}{2}u_3^2 \), the same conclusion follows.

### 4 Symmetry-integrability

We now proceed to prove the theorem claimed before:

For any scalar evolution equation of the form

\[
    u_t = u_m + f(u, u_1, \ldots, u_{m-1}),
\]

where \( f \) is a formal power series with terms that are at least quadratic, the existence of one nontrivial symmetry implies the existence of infinitely many. Moreover, the orders of the infinite integrable hierarchy must be one of the following cases: (i) all positive integers; (ii) all odd positive integers; (iii) all positive integers congruent to 1 or \(-1\) modulo 6; (iv) all positive integers congruent to \(1\) modulo 6.

Let us begin with a consequence of the Beukers’ theorem, see Appendix 1.

**Corollary 4.** Let \( m, n \geq 2, k_0 = \left\lfloor \frac{3}{4} \frac{mn}{2} \right\rfloor \), and \( k \geq k_0 \). And let \( F, G \) be two homogeneous differential polynomials of \( k \)th degrees satisfying \( \{ F, u_m \} = \{ G, u_n \} \). Then there exists a unique ”pull back” \( H \), also a homogeneous differential polynomial of \( k \)th degree, s.t. \( F = \{ H, u_m \} \) and \( G = \{ H, u_n \} \).

**Proof.** By the Beukers’ theorem (see (3) and (4)), \( P^{(m)}_k \) and \( P^{(n)}_k \) are relative prime. Thanks to the Gel’fand-Diki˘ı transformation, we have \( \tilde{F}P^{(m)}_k = \tilde{G}P^{(n)}_k \).
Hence $P_k(m) \mid \widetilde{F}$ and $P_k(n) \mid \widetilde{G}$. Set $\widetilde{H} = \frac{\widetilde{F}}{P_k(m)} = \frac{\widetilde{G}}{P_k(n)}$, then the preimage $H$ of $\widetilde{H}$ under the Gel’fand-Dikii transformation is the needed. The uniqueness of $H$ is obvious.

To avoid endlessly repeating the hypothesis, let us denote

$$W_l = \{u_l + f \mid f \in \mathcal{M}^2 \cap O(l - 1)\}$$

for arbitrary $l \geq 2$, and

$$W = \bigcup_{l=2}^{\infty} W_l.$$ From now on in this section, we always assume $F \in W_m$, $G \in W_n$, satisfying $\{F, G\} = 0$, where $m, n \geq 2$ and $m \neq n$. It just means that the equation $u_t = F$ has a nontrivial symmetry $G$.

**Proposition 5.** Suppose $E \in \mathcal{M}$ and $k \geq 2$.

(i) If $\{E, F\}, \{E, G\} \in \mathcal{M}^k$, then $\{\{E, F\}^k, u_n\} = \{\{E, G\}^k, u_m\}$;

(ii) If $\{E, F\} \in \mathcal{M}^{k+1}$ and $\{E, G\} \in \mathcal{M}^k$, then $\{E, G\} \in \mathcal{M}^{k+1}$;

(iii) If $\{E, F\} = 0$, then $\{E, G\} = 0$.

Proof. (i) Since $\{F, G\} = 0$, by the Jacobi identity, we have

$$\{\{E, F\}, G\} = \{\{E, G\}, F\}.$$ Taking the $k$th degree homogeneous components of the two sides of the above equality, we get

$$\{\{E, F\}^k, u_n\} = \{\{E, G\}^k, u_m\}.$$ (ii) By condition, part (i) holds and $\{E, F\}^k = 0$. Thus $\{\{E, G\}^k, u_m\} = 0$. It, see Appendix 1, implies $\{E, G\}^k = 0$.

(iii) It is easy to see that $\{E, G\} \in \mathcal{M}^2$. The conclusion follows from part (ii) by induction since $\bigcap_{l=2}^{\infty} \mathcal{M}^l = 0$.

Consider the linear space of nontrivial symmetries of the equation $u_t = F$

$$\mathcal{F} = \text{span} \{E \in W \mid \{E, F\} = 0\}$$

and the subspaces of $l$th order symmetries with a single linear term, $l = 2, 3, \ldots$.

$$\mathcal{F}_l = \text{span} W_l \cap \mathcal{F}.$$ We know $\dim \mathcal{F}_1 = 0$ or 1, and $\mathcal{F} = \bigoplus_{l=2}^{\infty} \mathcal{F}_l$. By Proposition (iii), $\mathcal{F}$ is a commutative Lie subalgebra of $(\mathbb{R}[u, u_1, u_2, \ldots], \{, \})$. And $\dim \mathcal{F} \geq 2$, since $F$ belongs to $\mathcal{F}$ and we have assumed the existence of $G$. Now the main theorem can be reformulated as follows.

**Theorem 6.** The space $\mathcal{F}$ is infinite dimensional. More precisely,

$$\{l \mid \dim \mathcal{F}_l = 1\} = \mathbb{Z}_{>0} + 1, \text{ or } 2\mathbb{Z}_{>0} + 1, \text{ or } 6\mathbb{Z}_{>0} \pm 1, \text{ or } 6\mathbb{Z}_{>0} + 1.$$
Here is our key observation. Without losing generality, we may assume
\[ \{l \mid \dim \mathcal{F}_l = 1 \} \subset \left\{ \frac{t_{2}^{(m)}}{t_{2}^{(l)}} \right\}, \]
see \( \ref{5} \). In fact, there exists \( F' \in W_{m'} \cap \mathcal{F} \), such that \( t_{2}^{(m')} \mid t_{2}^{(l)} \), for any \( l \) satisfying \( \dim \mathcal{F}_l = 1 \). By Proposition \( \ref{5}(iii) \), we can replace \( F \) by \( F' \) without changing \( \mathcal{F} \). And by \( \ref{5} \),
\[ \left\{ \left| \frac{t_{2}^{(m)}}{t_{2}^{(l)}} \right| \right\} = \begin{cases} \mathbb{Z}_{>0} + 1, & m = 0 \text{ mod } 2; \\ 2\mathbb{Z}_{>0} + 1, & m = 3 \text{ mod } 6; \\ 6\mathbb{Z}_{>0} + 1, & m = 5 \text{ mod } 6; \\ 6\mathbb{Z}_{>0} + 1, & m = 1 \text{ mod } 6. \end{cases} \]

Thus it remains to show
\[ \{l \mid \dim \mathcal{F}_l = 1 \} \supset \left\{ \frac{t_{2}^{(m)}}{t_{2}^{(l)}} \right\}. \]

We have reduced Theorem \( \ref{6} \) to the following

**Theorem 7.** Let \( l \geq 2 \). If \( t_{2}^{(m)} \mid t_{2}^{(l)} \), then \( \dim \mathcal{F}_l = 1 \).

**Proof.** We shall show that there exists \( E \in W_l \) for \( l \neq m \), s.t. \( \{E, F\} = 0 \).
First, let \( E' = u_l \).

Taking the second degree homogeneous component of the equality \( \{F, G\} = 0 \), we get \( \{F^2, u_n\} = \{G^2, u_n\} \), equivalently, \( \widetilde{F^2} t_{2}^{(m)} p_{2}^{(n)} = \widetilde{G^2} t_{2}^{(m)} p_{2}^{(n)} \). Hence \( p_{2}^{(m)}, p_{2}^{(n)} \) divide \( F^2, G^2 \) respectively. Set
\[ \widetilde{E^2} = \frac{\widetilde{F^2}}{p_{2}^{(m)} t_{2}^{(m)} p_{2}^{(l)}}, \quad (1) \]
then \( \{E^3 + E^2, F\}^2 = 0 \), i.e. \( \{E^3 + E^2, F\} \in \mathcal{M}^3 \).

By Proposition \( \ref{6}(ii) \), \( \{E^3 + E^2, G\} \in \mathcal{M}^3 \). Then by Proposition \( \ref{6}(i) \),
\( \{E^3 + E^2, G\}^3 \) respectively.

By Proposition \( \ref{6}(ii) \), \( \{E^3 + E^2, G\} \in \mathcal{M}^3 \). Then by Proposition \( \ref{6}(i) \),
\( \{E^3 + E^2, G\}^3 \) respectively.

Taking the third degree homogeneous component of the equality \( \{F, G\} = 0 \), we get \( \{F^3, G^2\} + \{F^2, u_n\} = \{G^3, u_n\} \), equivalently, \( \{F^3, G^2\} + \{G^3, u_n\} = \widetilde{F^2} P_{3}^{(m)} \). When \( 2 \mid mn \), we obtain \( (x_1 + x_2)(x_2 + x_3)(x_3 + x_1) \mid \{F^2, G^2\} \), see \( \ref{6} \).

Now we need another lemma which is the same as Proposition 5.3 in \( \ref{6} \). For the reader’s convenience, we provide a proof in our notation (without referring to \( \lambda \)-homogeneity) in Appendix 2.

**Lemma 8.** If \( 2 \nmid lm \), then
\[ (x_1 + x_2)(x_2 + x_3)(x_3 + x_1) \mid \{\widetilde{F^2}, \widetilde{F^2}\} \iff x_1 + x_2 \mid \widetilde{F^2} \text{ or } x_1x_2 \mid \widetilde{F^2}. \]

When \( 2 \nmid mn \), using Lemma \( \ref{6} \) we obtain \( x_1 + x_2 \mid \widetilde{F^2} \) or \( x_1x_2 \mid \widetilde{F^2} \) from \( t_2^{(m)} \mid \{\widetilde{F^2}, \widetilde{G^2}\} \). Since \( t_2^{(m)} \mid t_2^{(l)} \), \( l \) is also odd. Using Lemma \( \ref{6} \) again, we obtain
\[ \iota_3 \mid \{\tilde{E}^2, F^2\}. \] Consequently, \( P_3^{(m)} = \iota_3^{(m)} p_3^{(m)} \) divides \( \{\tilde{E}^2, F^3\} + \{\tilde{E}^2, F^2\} \).

Set \( E^3 \) be the quotient of them, then \( \{E^1 + E^2 + E^3, F\} \in \mathcal{M}^4 \).

In sum, let \( k_0 = \left\{ \begin{array}{ll} 3, & 2 \mid mn \\ 4, & 2 \nmid mn \end{array} \right\} \), we have obtained

\[
\mathcal{T} = \begin{cases} E^1 + E^2, & 2 \mid mn \\ E^1 + E^2 + E^3, & 2 \nmid mn \end{cases},
\]

satisfying \( \{\mathcal{T}, F\} \in \mathcal{M}^{k_0} \).

By Proposition \( \mathbf{E}(ii) \), \( \{\mathcal{T}, G\} \in \mathcal{M}^{k_0} \). Then by Proposition \( \mathbf{E}(i) \), we see that \( \{\mathcal{T}, F\}^{k_0}, u_m \} = \{\{\mathcal{T}, G\}^{k_0}, u_m \} \). Now applying Corollary \( \mathbf{D} \) there exists a homogeneous differential polynomial \( E^{k_0} \) of degree \( k_0 \), such that \( \{\mathcal{T}, F\}^{k_0} = \{-E^{k_0}, u_m\} \) and \( \{\mathcal{T}, G\}^{k_0} = \{-E^{k_0}, u_n\} \). Thus \( \{\mathcal{T} + E^{k_0}, F\}, \{\mathcal{T} + E^{k_0}, G\} \) belong to \( \mathcal{M}^{k_0+1} \).

By induction, we can obtain a formal power series solution \( E \) of the symmetry equation \( \{E, F\} = 0 \) satisfying \( E^0 = 0 \) and \( E^1 = u_l \). Finally, by the arguments after Lemma \( \mathbf{E} \), \( E \in W_1 \).

**Remark.** In [3], Sanders and Wang have formulated Proposition \( \mathbf{E}(i), (ii) \) and the induction part of the proof of Theorem \( \mathbf{F} \) in terms of Lie algebraic modules. As we have seen, however, they are all rather simple and the abstract setting is not necessary in our context. It is worse that the abstract setting has concealed Corollary \( \mathbf{D} \) and Proposition \( \mathbf{E}(iii) \), although they seem to be also very simple.

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**Appendix 1: the symbolic method**

The purpose of this appendix is to introduce some basic results of the symbolic method, which is first introduced by Gel’fand and Diki˘ı in [9] and play a key role in Section 4, see [3] [10] [11] for proofs.

Let \( k \) be a natural number. Write \( U^k \) for the set of \( k \)th degree homogeneous differential polynomials in \( \mathbb{R}[[u, u_1, u_2, \ldots]] \). Denote, as usual, \( \mathbb{R}[x_1, \ldots, x_k] \) for the polynomial ring of variables \( x_1, \ldots, x_k \) with real coefficients and \( \Lambda_k \) the set of symmetric polynomials in \( \mathbb{R}[x_1, \ldots, x_k] \). The well known symmetrizing operator, denoted by \( \langle \cdot \rangle \), from \( \mathbb{R}[x_1, \ldots, x_k] \) to \( \Lambda_k \) is defined by

\[
f(x_1, \ldots, x_k) \mapsto \langle f \rangle = \frac{1}{k!} \sum_{\sigma \in S_k} f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}),
\]

where \( S_k \) denotes the \( k \)th symmetry group.
The Gel'fand-Dikiĭ transformation is a linear isomorphism between $U^k$ and $\Lambda_k$. Its action on the monomials in $U^k$ is as follows

$$u^\alpha = u^{\alpha_0}u_1^{\alpha_1} \cdots u_m^{\alpha_m} \mapsto \widetilde{u^\alpha} = \langle x_1^0 \cdots x_{\alpha_0}^0 x_{\alpha_0+1}^1 \cdots x_{\alpha_0+\alpha_1}^1 \cdots x_k^{m-\alpha_m+1} \cdots x_k^m \rangle,$$

where $\sum_{i=0}^m \alpha_i = k$.

For arbitrary $F \in U^k$, $G \in U^l$, we have

\begin{align*}
(i) & \quad \widetilde{DF} = \bar{F} \sum_{i=1}^k x_i; \\
(ii) & \quad \frac{\partial \bar{F}}{\partial u_m} = \frac{k}{m!}\frac{\partial^m \bar{F}}{\partial x_k^m} |_{x_k=0}; \\
(iii) & \quad \bar{v}_G = l \left\langle \bar{F}(x_1, \ldots, x_k) \bar{G} \left( \sum_{i=1}^k x_i, x_{k+1}, \ldots, x_{k+l-1} \right) \right\rangle, \\
\{F, G\} & = l \left\langle \bar{F}(x_1, \ldots, x_k) \bar{G} \left( \sum_{i=1}^k x_i, x_{k+1}, \ldots, x_{k+l-1} \right) \right\rangle - k \left\langle \bar{G}(x_1, \ldots, x_l) \bar{F} \left( \sum_{j=1}^{l-1} x_j, x_{l+1}, \ldots, x_{k+l-1} \right) \right\rangle; \\
(iv) & \quad \{F, u_m\} = \bar{F} P_k^{(m)}; \text{ where } P_k^{(m)} = \left( \sum_{i=1}^k x_i \right)^m - \sum_{i=1}^m x_i^m. 
\end{align*}

When $k, m \geq 2$, $P_k^{(m)}$ is nonconstant. It immediately follows from (iv) that if $F \in M^2$ and $\{F, u_m\} = 0$, $m \geq 2$, then $F = 0$.

**Theorem (Beukers).** The symmetric polynomials $P_k^{(m)}$'s have factorizations $P_k^{(m)} = t_k^{(m)} p_k^{(m)}$, such that (the greatest common divisor)

$$\gcd(t_k^{(m)}, p_k^{(m)}) = \gcd(p_k^{(m)}, p_k^{(n)}) = 1, \quad \forall \, k, n \geq 2,$$

where $t_k^{(m)}, p_k^{(m)} \in \Lambda_k$ and $t_k^{(m)}$'s are as follows.

- **$k = 2$**:
  \begin{align*}
  t_2^{(m)} = \begin{cases} x_1x_2, & m = 0 \mod 2; \\
  x_1x_2(x_1 + x_2), & m = 3 \mod 6; \\
  x_1x_2(x_1 + x_2)(x_1^2 + x_1x_2 + x_2^2), & m = 5 \mod 6; \\
  x_1x_2(x_1 + x_2)(x_1^2 + x_1x_2 + x_2^2)^2, & m = 1 \mod 6. \end{cases}
  \end{align*}

- **$k = 3$**:
  \begin{align*}
  t_3^{(m)} = \begin{cases} 1, & m = 0 \mod 2; \\
  (x_1 + x_2)(x_2 + x_3)(x_3 + x_1), & m = 1 \mod 2. \end{cases}
  \end{align*}

- **$k \geq 4$**:
  \begin{align*}
  t_k^{(m)} = 1. \end{align*}
Appendix 2

Proof of Lemma 8. First of all, note that $x_1 + x_2 \mid \widetilde{E}^2$ is equivalent to $x_1 + x_2 \mid \widetilde{F}^2$ and that $x_1 x_2 \mid E^2$ is equivalent to $x_1 x_2 \mid F^2$, following from (1) in Section 4 and the Beukers’ theorem in Appendix 1.

According to the second formula of (iii) in Appendix 1, we have

$$\{\widetilde{E}^2, \widetilde{F}^2\} = 2\left(\widetilde{E}^2(x_1, x_2)\widetilde{F}^2(x_1 + x_2, x_3) - \widetilde{F}^2(x_1, x_2)\widetilde{E}^2(x_1 + x_2, x_3)\right)$$

Then we have

$$\frac{2}{3}\left(\widetilde{E}^2(x_1, x_2)\widetilde{F}^2(x_1 + x_2, x_3) - \widetilde{F}^2(x_1, x_2)\widetilde{E}^2(x_1 + x_2, x_3)ight) + \widetilde{E}^2(x_2, x_3)\widetilde{F}^2(x_2 + x_3, x_1) - \widetilde{F}^2(x_2, x_3)\widetilde{E}^2(x_2 + x_3, x_1) + \widetilde{E}^2(x_3, x_1)\widetilde{F}^2(x_3 + x_1, x_2) - \widetilde{F}^2(x_3, x_1)\widetilde{E}^2(x_3 + x_1, x_2).$$

Since $\{\widetilde{E}^2, \widetilde{F}^2\}$ is a symmetric polynomial,

$$(x_1 + x_2)(x_2 + x_3)(x_3 + x_1) \mid \{\widetilde{E}^2, \widetilde{F}^2\} \iff x_2 + x_3 \mid \{\widetilde{E}^2, \widetilde{F}^2\}$$

$$(x_2 + x_3) = 0.$$ 

Observe that

$$P_2^{(m)}(x_1, x_2) = (x_1 + x_2)^m - x_1^m - x_2^m = -P_2^{(m)}(x_1 + x_2, x_2).$$

Thus it follows from $\widetilde{E}^2 P_2^{(m)} = \widetilde{F}^2 P_2^{(l)}$ that $\widetilde{E}^2(x_1 + x_2, x_2) P_2^{(m)}(x_1, x_2) = \widetilde{F}^2(x_1 + x_2, x_2) P_2^{(l)}(x_1, x_2).$ Consequently,

$$\widetilde{E}^2(x_1, x_2)\widetilde{F}^2(x_1 + x_2, x_2) = \widetilde{F}^2(x_1, x_2)\widetilde{E}^2(x_1 + x_2, x_2).$$

Changing the variable $x_2$ to $-x_2$, we get

$$\widetilde{E}^2(-x_2, x_1)\widetilde{F}^2(-x_2 + x_1, x_2) = \widetilde{F}^2(-x_2, x_1)\widetilde{E}^2(-x_2 + x_1, x_2).$$

Hence

$$\{\widetilde{E}^2, \widetilde{F}^2\}(x_1, x_2, -x_2) = \frac{2}{3}\left(\widetilde{E}^2(x_2, -x_2)\widetilde{F}^2(0, x_1) - \widetilde{F}^2(x_2, -x_2)\widetilde{E}^2(0, x_1)\right).$$

In addition, since $l, m$ are odd integers, according to (2) and the equality $\widetilde{E}^2 P_2^{(m)} = \widetilde{F}^2 P_2^{(l)}$, we have

$$\widetilde{E}^2(x_2, x_3) P_2^{(m)}(x_2, x_3) P_2^{(l)}(x_2, x_3)$$

and

$$\frac{\widetilde{E}^2(x_2 + x_3, x_1) P_2^{(m)}(x_2 + x_3, x_1)}{x_2 + x_3} = \frac{\widetilde{F}^2(x_2 + x_3, x_1) P_2^{(l)}(x_2 + x_3, x_1)}{x_2 + x_3}.$$ 

Multiplying the above two equations by cross and setting $x_3 = -x_2$, we may obtain

$$x_2 + x_3 \mid \{\widetilde{E}^2, \widetilde{F}^2\} \iff \widetilde{E}^2(x_2, -x_2)\widetilde{F}^2(0, x_1) = \widetilde{F}^2(x_2, -x_2)\widetilde{E}^2(0, x_1)$$

or $$(x_2 + x_3)^3 \mid Q(x_1, x_2, x_3),$$

where $Q(x_1, x_2, x_3)$ is a polynomial.
where
\[ Q(x_1, x_2, x_3) = P_2^{(m)}(x_2, x_3)P_2^{(l)}(x_2 + x_3, x_1) - P_2^{(l)}(x_2, x_3)P_2^{(m)}(x_2 + x_3, x_1). \]

But \((x_2 + x_3)^3 \nmid Q(x_1, x_2, x_3)\), because
\[
\left. \frac{\partial^2 Q}{\partial x_3^2} \right|_{x_3 = -x_2} = 2 \left( \frac{\partial}{\partial x_3} P_2^{(m)}(x_2, x_3) \frac{\partial}{\partial x_3} P_2^{(l)}(x_2 + x_3, x_1) - \frac{\partial}{\partial x_3} P_2^{(l)}(x_2, x_3) \frac{\partial}{\partial x_3} P_2^{(m)}(x_2 + x_3, x_1) \right)_{x_3 = -x_2}
\]
\[ = 2lm \left( -x_2^{m-1} x_1^{l-1} - (-x_2^{l-1}) x_1^{m-1} \right) \neq 0. \]

Finally,
\[
\widetilde{E}^2(x_2, -x_2)F^2(0, x_1) = 0 \iff \widetilde{E}^2(x_2, -x_2) = 0 \text{ or } \widetilde{F}^2(0, x_1) = 0
\]
\[ \iff x_1 + x_2 \mid \widetilde{E}^2 \text{ or } \widetilde{F}^2
\]
\[ \iff x_1 + x_2 \mid \widetilde{F}^2 \text{ or } x_1 x_2 \mid \widetilde{F}^2. \]

In the same manner,
\[
\widetilde{F}^2(x_2, -x_2)\widetilde{E}^2(0, x_1) = 0 \iff x_1 + x_2 \mid \widetilde{F}^2 \text{ or } x_1 x_2 \mid \widetilde{F}^2.
\]

The conclusion follows. \(\square\)

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