On a class of random walks with reinforced memory

Erich Baur∗

September 11, 2019

Abstract

This paper deals with different models of random walks with a reinforced memory of preferential attachment type. We consider extensions of the Elephant Random Walk introduced by Schütz and Trimper [31] with stronger reinforcement mechanisms, where, roughly speaking, a step from the past is remembered proportional to some weight and then repeated with probability $p$. With probability $1-p$, the random walk performs a step independent of the past. The weight of the remembered step is increased by an additive factor $b \geq 0$, making it likelier to repeat the step again in the future. A combination of techniques from the theory of urns, branching processes and $\alpha$-stable processes enables us to discuss the limit behavior of reinforced versions of both the Elephant Random Walk and its $\alpha$-stable counterpart, the so-called Shark Random Swim introduced by Businger [12]. We establish phase transitions, separating subcritical from supercritical regimes.

Keywords: Reinforced random walks, preferential attachment, memory, stable processes, branching processes, Pólya urns.

AMS Subject Classification: 60G50; 60G52; 60K35; 05C85.

1 Introduction

In the last decades, there have been a constant interest in (usually non-Markovian) random walks with reinforcement. Arguably the most important class is formed by edge (or vertex) reinforced random walks. We point to the survey of Pemantle [27] or the more recent works [16, 22, 29] with references therein, just to mention a few.

Loosely speaking, an edge reinforced random walk crosses an edge with a probability proportional to a weight associated to that edge, which increases after each visit. Edge reinforced

∗Bern University of Applied Sciences, Switzerland. Email: erich.baur@bfh.ch.
random walks have found several applications in statistical physics and Bayesian statistics, see, e.g., \cite{17, 28, 29}.

In this paper, we shall be interested in another class of random walks with reinforcement, where at each time \( n \) and with a certain probability \( p \), a step from the past is selected according to some weight (which may change over time) and then repeated, whereas with the complementary probability \( 1 - p \), a new step independent of the past is performed. One of the practical interests in such walks comes from the fact that they serve as toy models for anomalous diffusion, describing many phenomena in physics, chemistry and biology \cite{24, 25}.

A prominent example in this class is the Elephant Random Walk (ERW for short) introduced by Schütz and Trimper \cite{31} (equal weights, symmetric ±1 steps), which has drawn a lot of attention in recent years, see \cite{1, 3, 4, 5, 6, 7, 8, 14, 15, 18, 26}, though this is a non-exhaustive list. It is the purpose of this paper to extend both the ERW and its \( \alpha \)-stable version, the Shark Random Swim (SRS for short) introduced by Businger \cite{12}, to models with a stronger (linear) reinforcement mechanism.

Our motivation stems from the desire to describe models of preferential attachment or “rich get richer”-type (see \cite{4} for the origin of such models), which should play an important role in the understanding of evolving networks like social networks or, maybe most prominently, neural networks. The latter are nowadays ubiquitously used to solve problems from computer vision or machine translation and lead in combination with reinforcement learning algorithms to stunning results, see, for example, the recent success of AlphaGo \cite{32}.

The ERW, SRS and our extensions fit into a general framework, which we describe first in a somewhat informal manner, referring to Section 2 for precise definitions. We shall propose two models of a random walk with reinforcement, which differ in their reinforcement mechanism.

We fix a memory parameter \( p \in (0, 1) \), a reinforcement parameter \( b \geq 0 \) measuring the strength of the reinforcement, and a sequence \( (\xi_i, i \in \mathbb{N}) \) of i.i.d. random variables used to model the increments of the random walks. Moreover, we initialize time-evolving weights \( k_n(\cdot) \) by setting \( k_n(i) = 1 \) for all \( i = 1, \ldots, n \) and all \( n \in \mathbb{N} \).

The first step in both models is given by \( \xi_1 \). At time \( n \geq 2 \), we select one of the preceding times \( I_n \in \{1, \ldots, n-1\} \) chosen at random proportional to the weights \( k_{n-1}(\cdot) \). With probability \( p \), the random walk repeats the step \( \xi_{I_n} \) performed at time \( I_n \), whereas with the complementary probability \( 1 - p \), the walk performs a new step \( \xi_n \) independent of the past.

Now in the first model which we call the memory-reinforced random walk, the weight of the selected time \( I_n \) is updated to \( k_n(I_n) = k_{n-1}(I_n) + b \) if and only if the walk decided to repeat the step \( \xi_{I_n} \) (i.e., with probability \( p \)), whereas in the second model called the strongly memory-reinforced random walk, the weight of the selected time \( I_n \) is always updated to \( k_n(I_n) = k_{n-1}(I_n) + b \).
One may shed light on the two models from a different perspective, which is interesting from the point of view of modeling. Namely, we may interpret the memory-reinforced random walk as a model with certain memory lapses, in the sense that the walk remembers and repeats a previous step only with probability $p$. With probability $1 - p$, it performs a new step (say, due to a memory loss).

In contrast, we may view the strongly memory-reinforced walk as a model with a perfect memory, in the sense that the random walk always remembers a previous step, leading to a reinforcement effect in its memory. However, then – say, due to bad experiences with the past or due to an unwillingness to repeat previous faults – the walk decides to repeat the step with probability $p$ only, whereas with probability $1 - p$, it forgets the past and decides to perform a new step.

We particularize these models to two choices of the sequence $(\xi_i, i \in \mathbb{N})$: For independent symmetric $\pm 1$-variables (or their $d$-dimensional generalizations), the first model constitutes what we call the reinforced Elephant Random Walk, whereas the second model leads to the strongly reinforced Elephant Random Walk.

For isometric $d$-dimensional $\alpha$-stable random variables, where $\alpha \in (0, 2]$, the first and second model give rise to the reinforced Shark Random Swim and to the strongly reinforced Shark Random Swim, respectively.

The naming of these models is explained by the fact that in the case $b = 0$ corresponding to a uniform choice of a previous step, we obtain the original ERW and SRS, respectively. Although we will always allow $b = 0$ for completeness (and therefore rediscover along the way results from [3, 6, 12, 14]), we are here interested in the “truly” reinforced case $b > 0$, for which both models are non-Markovian.

The core part of this work deals with the long-time behavior of the strongly reinforced SRS. The mere reinforced SRS would again require a different approach (see Remark 6), which we leave for further investigation. We shall however first discuss both the reinforced and strongly reinforced ERW, for which we establish representations in terms of finite-color urns. The non-Markovian nature of the random walks is handled by keeping track of the number of times a step is repeated. Results of Janson on Pólya urns then allow us to derive the asymptotic behavior of the random walks.

For the mere reinforced ERW model covered by Theorem 3.1, we shall observe a phase transition at $p_* := 1/(2 + b)$. In the subcritical regime $p < p_*$, we prove convergence in law of the (properly normalized) reinforced ERW towards a Gaussian limit process. In the critical regime $p = p_*$, a scaled Brownian motion appears in the limit, and a non nontrivial presumably non-Gaussian process in the supercritical regime $p > p_*$. Provided $0 \leq b < 1$, the strongly reinforced ERW discussed in Theorem 3.2 exhibits a phase
transition at $p_{**} := (1 - b)/2$. Depending on whether $p < p_{**}$, $p = p_{**}$ or $p > p_{**}$, we obtain results similar to the case of the mere reinforced ERW. One should note that if $b \geq 1$, then $p_{**} \leq 0$, implying somewhat surprisingly that in this case, for every choice of $p \in (0, 1)$, the strongly reinforced ERW is supercritical and behaves superdiffusively.

Changing over to the strongly reinforced SRS, we note that a finite-color urn is inappropriate for modeling purposes, since the step variables take infinitely many values. Although there is a growing literature on infinite-color urns, see, e.g., the recent work [23], we follow a different route inspired by an idea of Kürsten [20]. He observed a connection between the original ERW and clusters sizes of a Bernoulli bond percolation on random recursive trees. His ideas were further elaborated by Businger in [12] to understand the SRS, and here, we consider percolation on a family of preferential attachment trees to model the strongly reinforced SRS. One of the main difficulties compared to [12] stems from the fact that the tree processes representing the percolation clusters are no longer branching processes. For controlling their sizes, we are guided by ideas from Bertoin and Bravo [10], who were interested in supercritical percolation on a family of large preferential attachment trees. Their tree model differs from ours, but their techniques prove useful also in our setting.

Taking inspiration from the recent works of Businger [12, 13], we make use of the connection to cluster sizes and prove a phase transition for the strongly reinforced SRS at $\alpha \kappa = 1$, where $\kappa := (b + p)/(b + 1)$. More specifically, in the subcritical case $\alpha \kappa < 1$, we prove in Theorem 4.1 weak convergence of finite-dimensional laws towards a non-Lévy $\alpha$-stable process. In the critical case $\alpha \kappa = 1$, we establish in Theorem 4.2 convergence towards an $\alpha$-stable Lévy process. The case $\alpha \kappa > 1$ treated in Theorem 4.3 covers the supercritical regime. We stress that for $\alpha = 2$, our results show that the strongly reinforced SRS behaves like the strongly reinforced ERW, which should not come as a surprise.

The rest of this work is organized as follows. In Section 2, we introduce the general setting of (strongly) memory-reinforced random walks, which we specify in Section 3 to the ERW. Section 3.1 then establishes the connection to urns and discusses the long-time behavior of the reinforced ERW models. In Section 4, we change over to the strongly reinforced SRS, for which we need more preparation: First, in Section 4.1 we explain the connection to (percolation on) preferential attachment trees, which are then constructed in continuous-time in Section 4.2. Using methods from branching processes, we gain in Section 4.3 control over large cluster sizes, which enables us to finally discuss the asymptotic behavior the strongly reinforced SRS in Section 4.4. Appendix A contains the proofs of some auxiliary results on branching processes.

Some final words concerning notation: For us, $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For two sequences $(r_n), (s_n)$ of positive reals, we write $r_n \lesssim s_n$ if there exists a constant $C$ (which might
depend on $b$ and $p$) such that

$$r_n \leq C s_n \quad \text{for all } n \in \mathbb{N}.$$ 

## 2 Random walks with reinforced memory

We fix a memory parameter $p \in (0, 1)$ and a (real-valued) reinforcement parameter $b \geq 0$. Let $(\xi_i, i \in \mathbb{N})$ be a sequence of i.i.d. random variables in $\mathbb{R}^d$. This sequence will be used to model the steps of the random walk. (Later on, we will consider the variables $\xi_i$ under two particular laws.)

We further let $(\epsilon_i, i \geq 2)$ be a sequence of i.i.d. Bernoulli random variables with success probability $p$. Call a time $i \geq 2$ a memory time if $\epsilon_i = 1$, and a fresh time if $\epsilon_i = 0$. A memory time will correspond to a time where the random walk repeats one of its preceding steps, whereas a fresh time will represent a time where the random walk performs independently of the past a new step.

We shall consider two models of a random walk with reinforced memory. Both models depend on time-evolving weights $k_n(\cdot)$, which we initialize by setting

$$k_n(i) = 1 \quad \text{for } i = 1, \ldots, n \quad \text{and all } n \in \mathbb{N}.$$

We specify the random walks by defining their increments $\zeta_i$, $i \geq 1$. First, we set $\zeta_1 = \xi_1$, and then for $n \geq 2$, we select a previous time $I_n \in \{1, \ldots, n-1\}$ according to

$$\mathbb{P}(I_n = i) = \frac{k_{n-1}(i)}{\sum_{j=1}^{n-1} k_{n-1}(j)}, \quad i = 1, \ldots, n-1.$$

Now if $n$ is a memory time (i.e., if $\epsilon_n = 1$), we let $\zeta_n = \xi_{I_n}$, whereas if $n$ is a fresh time (i.e., if $\epsilon_n = 0$), we let $\zeta_n = \xi_n$.

It remains to update the weights $k_n(\cdot)$, and here, the difference between the two models comes into play: In the first model, we update the weight of the selected time $I_n = i$ if and only if $n$ is a memory time (i.e., if $\epsilon_n = 1$), by setting

$$k_n(i) = k_{n-1}(i) + b. \quad (1)$$

In the second model, we always update the weight of the selected time $I_n = i$ according to (1), no matter whether $n$ is a memory time or not. The other weights remain unchanged, in both models.

Letting $S_0 := 0$ and

$$S_n := \zeta_1 + \ldots + \zeta_n, \quad n \geq 1,$$
we call the process \((S_n, n \in \mathbb{N}_0)\) either the memory-reinforced random walk or the strongly memory-reinforced random walk, depending on which update rule is applied (first or second model).

To summarize in words, if \(n\) is a fresh time, the random walk performs a step independently of the past, whereas if \(n\) is a memory time, the walk repeats the step performed at time \(I_n\). In the memory-reinforced case, the weight \(k_n(I_n)\) of the selected time is increased by the amount \(b\) if and only if \(n\) is a memory time, whereas in the strongly memory-reinforced case, \(k_n(I_n)\) is always increased by the amount \(b\).

**Remark 1.** If \(b = 0\), then the weights remain equal to one, and both models agree. More precisely, in case of a memory time, the increment \(\zeta_n\) is chosen uniformly at random among the previous increments – our memory-reinforced random walk thus corresponds for \(b = 0\) to what is called step reinforced random walk in [8]. More generally, the parameter \(b\) captures the strength of the reinforcement: The larger \(b\) is, the heavier the weight of the chosen time becomes, and the likelier it is to remember this time again in the future.

### 3 The (strongly) reinforced Elephant Random Walk

By choosing \((\xi_i, i \in \mathbb{N})\) to be an i.i.d. sequence with law

\[
P(\xi_1 = 1) = P(\xi_1 = -1) = \frac{1}{2},
\]

the two models of reinforcement described in the last section give rise to what we call the reinforced Elephant Random Walk and the strongly reinforced Elephant Random Walk, respectively (or, for short, the (strongly) reinforced ERW).

To make a clear distinction to the strongly reinforced ERW, we shall sometimes refer to the first model as the “mere reinforced ERW”. The fact that we consider here only one-dimensional symmetric \(\pm 1\)-steps \(\xi_i\) is just to keep the presentation simple; see Remark 5.

We stress that in the case of our interest \(b > 0\), both ERW with reinforcement are non-Markovian even in dimension \(d = 1\). This is in contrast to the original one-dimensional ERW corresponding to \(b = 0\): Indeed, there, if the elephant is at position \(k \in \mathbb{Z}\) at time \(n\), then it performed \((n+k)/2\) steps to the right and \((n-k)/2\) steps to the left up, and more information from the past is irrelevant for predicting the \((n+1)\)th step. (In dimensions greater or equal to two, the ERW model is non-Markovian for any \(b \geq 0\).)

In the following section, we depict the connection to urn models. We shall observe phase transitions for both the mere reinforced ERW (Theorem 3.1) and the strongly reinforced ERW (Theorem 3.2). For what follows, we always fix parameters \(p \in (0,1)\) and \(b \geq 0\) without mentioning this every time.
3.1 Three-color urns with random replacement

In the description of the following urn models we use the terminology (and often notation) of Janson [19], to which we refer for more details on urns.

Remark 2. For ease of understanding, we will phrase our descriptions of the following urns as if the reinforcement parameter $b$ were a positive integer. This allows us to interpret $b$ as a number of balls, whereas otherwise, we would have to consider urns containing a certain mass of each color, rather than balls. However, all the results on urns we are going to use in the proofs of our Theorems 3.1 and 3.2 are also correct for positive real $b$, see [19, Remark 4.2].

3.1.1 A model for the reinforced ERW

We consider an urn $X_n = (B_n, G_n, R_n)$, $n \in \mathbb{N}$, with balls of three different types, with mean replacement matrix given by

$$A = \begin{pmatrix}
(b+1)p & (b+1)p & 0 \\
(1-p)/2 & (1-p)/2 & (1-p)/2 \\
(1-p)/2 & (1-p)/2 & (b+1)p + (1-p)/2
\end{pmatrix}. \tag{2}
$$

The coefficient $a_{ij}$ of the matrix $A$ represents the mean number of balls of type $i$, which are added to the urn if in the $n$th step a ball of type $j$ is drawn. Note that every drawn ball is returned to the urn – the name “mean replacement matrix” might therefore look a bit irritating, but it is standard in this context.

For concreteness, say that balls of type 1 are black, balls of type 2 are green and balls of type 3 are red. Then, $B_n$, $G_n$, and $R_n$ represent the number of black, green and red balls after $n$ draws, $n \geq 1$.

Roughly speaking, an increase by $b+1$ of the number of black balls represents a step to the right of the reinforced ERW due to a memory time, green balls represent steps to the right due to a fresh time, and an increase by $b+1$ or by 1 of the number of red balls models a step to the left due to a memory time or due to a fresh time, respectively.

More precisely, let us first look at the first column of the matrix $A$, which describes in mean what happens if a black ball is drawn in the $n$th step. With probability $p$ (which corresponds to $n$ being a memory time of the reinforced ERW), $b+1$ black balls are added to the urn (entry (1,1)), whereas with the complementary probability $1-p$ (corresponding to $n$ being a fresh time), a green or a red ball is added with probability $1/2$ each (entry (2,1) or entry (3,1)).

The second column is exactly the same as the first column and captures what happens if a green ball is drawn. Last, if in the $n$th step a red ball is picked, then, with probability $p$, $b+1$ red balls are added to the urn (first summand of entry (3,3)), whereas with the complementary
probability $1 - p$, a green or a red ball is added with probability $1/2$ each (entry (2,3) or second summand of entry (3,3)).

We start the urn at time 1 with the random initial configuration consisting of one green or red ball with probability $1/2$ each. Then it follows from the dynamics of the urn that the number of steps to the right of the reinforced ERW until time $n$ is distributed as

$$\frac{B_n}{b+1} + G_n.$$ 

In other words, we have for the position $S_n$ of the reinforced ERW at time $n$ that

$$S_n = 2 \left( \frac{B_n}{b+1} + G_n \right) - n. \quad (3)$$

Of course, the last display may be strengthened to an equality in law of processes (in $n$).

With this correspondence at hand, we are in position to describe the limiting behavior of the reinforced elephant in the Skorokhod space $D([0, \infty))$ of right-continuous functions with left-hand limits.

**Theorem 3.1.** Let $p \in (0,1)$, $b \geq 0$, and let $(S_n, n \in \mathbb{N}_0)$ be the reinforced ERW with parameters $b$ and $p$. Moreover, set $p_* := 1/(2 + b)$, and let $\kappa := (b + 1)p/(bp + 1)$. Then the following convergences in law hold for $n \to \infty$:

a) **Subcritical case:** If $p < p_*$,

$$\left( \frac{S_{\lfloor t n \rfloor}}{\sqrt{n}}, t \geq 0 \right) \Rightarrow (W_t, t \geq 0),$$

where $(W_t, t \geq 0)$ is a continuous $\mathbb{R}$-valued mean-zero Gaussian process started from $W_0 = 0$, with covariances

$$\mathbb{E}[W_s W_t] = \frac{bp + 1}{(1 - (2 + b)p)(b + 1)} s \left( \frac{t}{s} \right)^\kappa + \frac{(pb^3 + (3p - p^2)b^2 + b)}{(bp + 1)^2(b + 1)} s, \quad 0 < s \leq t.$$

b) **Critical case:** If $p = p_*$,

$$\left( \frac{S_{\lfloor t n \rfloor}}{\sqrt{n \ln n}}, t \geq 0 \right) \Rightarrow \sqrt{\frac{p}{1 - p}} (B_t, t \geq 0),$$

where $(B_t, t \geq 0)$ is a one-dimensional Brownian motion.

c) **Supercritical case:** If $p > p_*$,

$$\left( \frac{S_{\lfloor t n \rfloor}}{n^\kappa}, t \geq 0 \right) \Rightarrow t^\kappa Y$$

for some nontrivial random variable $Y = Y(b, p)$.  

8
Remark 3. As already mentioned, the case \( b = 0 \) corresponds to the original Elephant Random Walk, and we recover results from [5, 14]. (However, we stress that the memory is differently parameterized in the cited papers, namely by \( q = (p + 1)/2 \).) In particular, the expression \( g(p, 0) \) in the subcritical case simplifies to
\[
g(p, 0) = \frac{1}{1 - 2p} = \frac{1}{3 - 4q}.
\]

In the supercritical case \( c \), we have \( \kappa > 1/2 \). If \( b = 0 \), the limiting random variable \( Y \) under \( c \) is known to be non-Gaussian (see [6]), and we strongly suspect that for \( b > 0 \), this is the case, too. In this regard, we point to Remark 3.20 of [19] and to Theorem 3.26 therein, where some information on the moments of \( Y \) is given. It seems, however, unclear how to calculate them explicitly.

Proof. The proofs are essentially consequences of results in [19], but the calculations are a bit involved. First, we find that the eigenvalues of the mean replacement matrix (2) are given by \( \lambda_1 = bp + 1 \), \( \lambda_2 = (b + 1)p \), and \( \lambda_3 = 0 \). Corresponding right eigenvectors with \( L^1 \)-norm equal to one are
\[
v_1 = \frac{1}{2(bp + 1)} \begin{pmatrix} (b + 1)p, 1 - p, bp + 1 \end{pmatrix}, \quad v_2 = \frac{1}{2} \begin{pmatrix} -1, 0, 1 \end{pmatrix}, \quad v_3 = \frac{1}{2} \begin{pmatrix} -1, 1, 0 \end{pmatrix},
\]
where we write \( ' \) for the transpose. (Our vectors are always column vectors.) A corresponding dual basis of left eigenvectors \( u_1, u_2, u_3 \) (i.e., \( u_i' \cdot v_j = \delta_{ij} \)) is given by
\[
u_1 = \begin{pmatrix} 1, 1, 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1, -1, 1 \end{pmatrix}, \quad u_3 = \frac{1}{bp + 1} \begin{pmatrix} p - 1, (2b + 1)p + 1, p - 1 \end{pmatrix}.
\]
Solving the equation \( \frac{\lambda_2}{\lambda_1} = \frac{1}{2} \), we find according to a well-known criterion (see [2] with references therein), that a phase transition occurs at \( p_* = 1/(2 + b) \).

a) As for the subcritical case \( p < p_* \), it is readily checked that we are in the setting of [19, Theorem 3.31(i)]. We deduce that
\[
\left( n^{-1/2}(X_{[tn]} - tn\lambda_1 v_1), t \geq 0 \right)
\]
converges in distribution towards a continuous \( \mathbb{R}^3 \)-valued mean-zero Gaussian process \( V = (V_t, t \geq 0) \) with \( V_0 = 0 \). In order to analyze the covariance structure of \( V \), we first note that the mean number of balls which is added to the urn from one step to the next equals \( m = bp + 1 \). Remark 5.7 in [19] then implies that
\[
\mathbb{E} \left[ V_s V_t' \right] = (bp + 1) \Sigma e^{\frac{\ln(t/s)}{b + 1} A'}, \quad 0 < s \leq t,
\]
where \( \Sigma \) is the \( 3 \times 3 \)-matrix given by
\[
\Sigma := \int_0^\infty (P_{\lambda_2} + P_{\lambda_3}) e^{sA} B e^{sA'}(P_{\lambda_2} + P_{\lambda_3}) e^{-(bp + 1)s} ds.
\]
Here, \( P_{\lambda_2} = v_2 u_2' \) and \( P_{\lambda_3} = v_3 u_3' \) are the projections onto the sum of the generalized eigenspaces corresponding to \( \lambda_2 \) and \( \lambda_3 \), and

\[
B := v_1 E[\theta_1 \theta_1'] + v_1 E[\theta_2 \theta_2'] + v_3 E[\theta_3 \theta_3'] = \begin{pmatrix}
\frac{(b+1)^2 p}{2} & 0 & 0 \\
0 & \frac{1-p}{2} & 0 \\
0 & 0 & \frac{(b+1)^2 p}{2} + \frac{1-p}{2}
\end{pmatrix},
\]

(5)

where \( v_1 = (v_{11}, v_{12}, v_{13})' \) and \( \theta_j = (\theta_{1j}, \theta_{2j}, \theta_{3j})' \) is the (random) vector describing how many balls of type 1 ("black"), 2 ("green") and 3 ("red") are added to the urn if a ball of type \( j \in \{1, 2, 3\} \) is picked.

Using that \( P_{\lambda_2} e^{sA} = e^{\lambda s} P_{\lambda_2} \) and similarly for \( P_{\lambda_3} \), we integrate and obtain

\[
\Sigma = \frac{1}{1 - (b + 2)p} P_{\lambda_2} B P_{\lambda_2}' + \frac{1}{1 - p} (P_{\lambda_2} B P_{\lambda_3}' + P_{\lambda_3} B P_{\lambda_2}') + \frac{1}{bp + 1} P_{\lambda_3} B P_{\lambda_3}'.
\]

Going back to (4), it follows that for \( 0 < s \leq t \)

\[
E[V_s V_t'] = (bp + 1) s \left( \left( \frac{t}{s} \right)^{\frac{(b+1)p}{bp+1}} \frac{P_{\lambda_2} B P_{\lambda_2}'}{1 - (b + 2)p} \frac{P_{\lambda_3} B P_{\lambda_2}'}{1 - p} + \frac{P_{\lambda_3} B P_{\lambda_3}'}{bp + 1} \frac{P_{\lambda_3} B P_{\lambda_3}'}{1 - p} \right).
\]

(6)

By (3), we have

\[
S_{[tn]} = 2 \left( \frac{B_{[tn]}}{b + 1} + G_{[tn]} \right) - [tn] = 2 \left( \frac{B_{[tn]} + p(b + 1)[tn]/2}{b + 1} + G_{[tn]} - \frac{(1 - p)[tn]}{2} \right)
\]

\[
= 2 \left( \frac{B_{[tn]} - [tn] \lambda_1 v_{11}}{b + 1} + G_{[tn]} - [tn] \lambda_1 v_{12} \right).
\]

By the continuous mapping theorem, we deduce that \((n^{-1/2} S_{[tn]}), t \geq 0\) converges in law in \( D([0, \infty)) \) to a process \( W = (W_t, t \geq 0) \) given by

\[
W_t = \frac{2}{b + 1} V_t^{(1)} + 2 V_t^{(2)}.
\]

where \( V_t^{(i)} \) denotes the \( i \)th component of \( V_t \). In particular,

\[
E[W_s W_t] = \frac{4}{(b + 1)^2} E[V_s^{(1)} V_t^{(1)}] + \frac{4}{b + 1} E[V_s^{(1)} V_t^{(2)}] + \frac{4}{b + 1} E[V_s^{(2)} V_t^{(1)}] + 4 E[V_s^{(2)} V_t^{(2)}].
\]

Upon evaluating the matrix products \( P_{\lambda_j} B P_{\lambda_j} \) for \( i, j \in \{2, 3\} \), Claim a) now follows from a small calculation using (6).

b) Applying [19, Theorem 3.31(ii)], we deduce that

\[
((n^t \ln n)^{-1/2} (X_{[nt]} - n^t \lambda_1 v_1), t \geq 0)
\]
converges in law as $n \to \infty$ towards a continuous $\mathbb{R}^3$-valued mean-zero Gaussian process $\tilde{V} = (\tilde{V}_t, t \geq 0)$ with $\tilde{V}_0 = 0$ and covariance matrix

$$
\mathbb{E} \left[ \tilde{V}_s \tilde{V}'_t \right] = (P_{\lambda_2}BP'_{\lambda_2}) \cdot s = \frac{1}{4p} \begin{pmatrix} 1-p & 0 & -(1-p) \\ 0 & 0 & 0 \\ -(1-p) & 0 & 1-p \end{pmatrix} \cdot s,
$$

where we have used the critical relation $b = (1/p) - 2$. The limiting process $\tilde{W} = (\tilde{W}_t, t \geq 0)$ of $((n^t \ln n)^{-1/2}S_{[n]'}, t \geq 0)$ is related to $\tilde{V}$ by $\tilde{W}_t = \frac{2}{b+1}\tilde{V}'_t + 2\tilde{V}^{(2)}_t$. From this and the last display, the claim readily follows.

c) Using [19, Theorem 3.24], we see that

$$(n^{-\kappa}(X_{[tn]} - tn\lambda_1 v_1), t \geq 0)$$

converges almost surely to $(t^\kappa \tilde{W}, t \geq 0)$, where $\tilde{W} = (\tilde{W}_1, \tilde{W}_2, \tilde{W}_3)'$ is a (nonzero) random vector in the eigenspace of $A$ associated to $\lambda_2$. Claim c) now follows, with

$$Y = \frac{2}{b+1}\tilde{W}_1 + 2\tilde{W}_2.$$ 

3.1.2 A model for the strongly reinforced ERW

Similarly to the last section, the strongly reinforced ERW may be modeled in terms of a three-color urn $X_n = (B_n, G_n, R_n), n \in \mathbb{N}$, with random replacement, here with mean replacement matrix

$$A = \begin{pmatrix} b & b & 0 \\ (1+p)/2 & (1+p)/2 & (1-p)/2 \\ (1-p)/2 & (1-p)/2 & b + (1+p)/2 \end{pmatrix}. \quad (7)$$

However, in this model, the interpretation of balls of different colors will not be the same as in the urn model from the last section. Here, the number of black balls increases by $b$ if a step to the right was remembered, green balls represent steps to the right, and an increase of the number of red balls means that a step to the left was remembered and/or performed.

More precisely, if in the $n$th step, a black is drawn, always $b$ black balls are added to the urn (entry (1,1)). In addition, with probability $p$ (corresponding to $n$ being a memory time for the strongly reinforced ERW), a green ball is added to the urn (part of entry (2,1)), representing the event that the elephant repeats the remembered step to the right. With the complementary probability $1-p$ (corresponding to $n$ being a fresh time), a green or a red ball is added with probability $1/2$ each (part of entry (2,1) or entry (3,1)).
The second column is again a copy of the first one and describes in mean what happens if a green ball is drawn. It remains to interpret the third column: If in the $n$th step a red ball is picked, then always $b$ red balls are added; moreover, with probability $p$, another red ball is added, whereas with the complementary probability $1 - p$, another red ball or a green ball is added with probability $1/2$ each.

Assuming again that we start the process at time 1 with one green or one red ball with equal probability, the number of steps to the right until time $n$ is now modeled by the number of green balls $G_n$, so that we have for the position $S_n$ of the strongly reinforced ERW at time $n$

$$S_n = d 2G_n - n. \quad (8)$$

For $(S_n, n \in \mathbb{N}_0)$, we find the following limit behavior (again in the space $D([0, \infty)))$:

**Theorem 3.2.** Let $p \in (0, 1)$, $b \geq 0$, and let $(S_n, n \in \mathbb{N}_0)$ be the strongly reinforced ERW with parameters $b$ and $p$. Moreover, set $p_* := (1 - b)/2$, and let $\kappa := (b + p)/(b + 1)$. Then the following convergences in law hold for $n \to \infty$:

a) **Subcritical case:** If $p < p_*$,

$$\left( \frac{S_{\lfloor tn \rfloor}}{\sqrt{n}}, t \geq 0 \right) \Rightarrow (W_t, t \geq 0),$$

where $(W_t, t \geq 0)$ is a continuous $\mathbb{R}$-valued mean-zero Gaussian process started from $W_0 = 0$, with covariances

$$\mathbb{E} [W_s W_t] = \frac{(1 - b^2)p}{(1 - b - 2p)(b + p)} s \left( \frac{t}{s} \right)^\kappa + \frac{(1 + p)b}{b + p} s, \quad 0 < s \leq t.$$

b) **Critical case:** If $p = p_*$,

$$\left( \frac{S_{\lfloor tn \rfloor}}{\sqrt{n^2 \ln n}}, t \geq 0 \right) \Rightarrow \sqrt{\frac{2p^2}{1 - p}} (B_t, t \geq 0),$$

where $(B_t, t \geq 0)$ is a one-dimensional Brownian motion.

c) **Supercritical case:** If $p > p_*$,

$$\left( \frac{S_{\lfloor tn \rfloor}}{n^\kappa}, t \geq 0 \right) \Rightarrow t^\kappa Y$$

for some nontrivial random variable $Y = Y(b, p)$.
Remark 4. In case c), $\kappa > 1/2$, and we suspect the limiting random variable $Y$ again to be non-Gaussian, see Remark 3. Note that when $b \geq 1$, we have $p_{**} \leq 0$, so that for each choice of $p \in (0,1)$, Case c) applies. In other words, if $b \geq 1$, the strongly reinforced ERW behaves always superdiffusively. Informally, if we set $b = \infty$, then $\kappa = 1$, and the elephant goes deterministically in the direction of its first step. Note also that in the case $b = 0$, the expressions for the covariances under a) and b) agree indeed with those given in Theorem 3.1.

Proof. We always refer to the urn with mean replacement matrix $A$ specified in (7). We may assume that $b > 0$, since the case $b = 0$ is already covered by Theorem 3.1. The eigenvalues of $A$ are given by $\lambda_1 = b + 1$, $\lambda_2 = b + p$, and $\lambda_3 = 0$. Right eigenvectors of $L^1$-norm equals one corresponding to $\lambda_1, \lambda_2$ and $\lambda_3$, respectively, are

$$v_1 = \frac{1}{2(b+1)} (b, 1, b+1)' , \quad v_2 = \frac{1}{2(b+p)} (b, p, -(b+p))' , \quad v_3 = \frac{1}{2} (1,-1,0)' ,$$

A dual basis of corresponding left eigenvectors is given by

$$u_1 = (1, 1, 1)' , \quad u_2 = (1, 1, -1)'$$

and

$$u_3 = \frac{1}{(b+1)(b+p)} ((1+p)b+2p, -(2b+1+p)b, (1-p)b)' .$$

Solving $\frac{\lambda_2}{\lambda_1} = \frac{1}{2}$, we find $p_{**} = (1-b)/2$.

a) If $p < p_{**}$ (which is only possible if $b < 1$), we may again apply [19, Theorem 3.31(i)] to obtain convergence of

$$(n^{-1/2}(X_{\lfloor tn \rfloor} - tn\lambda_1 v_1, t \geq 0)$$

towards a continuous $\mathbb{R}^3$-valued mean-zero Gaussian process $V = (V_t, t \geq 0)$ with $V_0 = 0$. In each step, $b + 1$ balls are added to the urn. Similarly to the proof of the subcritical case in Theorem 3.1, we compute for $0 < s \leq t$

$$\mathbb{E} [V_s V_t'] = (b+1) s \left( \left( \left( \frac{t}{s} \right)^{b+p} \frac{P_{\lambda_2}BP_{\lambda_2}'}{1-b-2p} + \frac{P_{\lambda_3}BP_{\lambda_3}'}{1-p} \right) + \frac{P_{\lambda_3}BP_{\lambda_3}'}{b+1} + \frac{P_{\lambda_2}BP_{\lambda_3}'}{1-p} \right) . \quad (9)$$

where $P_{\lambda_2} = v_2 u_2'$, $P_{\lambda_3} = v_3 u_3'$, and, with a definition analogous to [5]

$$B = \frac{1}{4} \begin{pmatrix} 2b^2 & b(1+p) & b(1-p) \\ b(1+p) & 2 & b(1-p) \\ b(1-p) & b(1-p) & 2(b^2 + (1+p)b + 1) \end{pmatrix} .$$

By (8), we have

$$S_{\lfloor tn \rfloor} = 2G_{\lfloor tn \rfloor} - \lfloor tn \rfloor = 2 \left( G_{\lfloor tn \rfloor} - \lfloor tn \rfloor \lambda_1 v_1 \right) . \quad (10)$$
Thus, \((n^{-1/2}S_{[tn]}, t \geq 0)\) converges in law in \(D([0, \infty))\) to a process \(W = (W_t, t \geq 0)\) given by \(W_t = 2V_t^{(2)}\). The claim then follows from (9).

b) It follows from [19, Theorem 3.31(ii)] that
\[
\left( (n^t \ln n)^{-1/2} (X_{[tn]} - n^t \lambda_1 v_1), t \geq 0 \right)
\]
converges in law as \(n \to \infty\) towards a continuous \(\mathbb{R}^3\)-valued mean-zero Gaussian process \(\tilde{V} = (\tilde{V}_t, t \geq 0)\) with \(\tilde{V}_0 = 0\) and covariance matrix

\[
\mathbb{E} \left[ \tilde{V}_s \tilde{V}_t^\prime \right] = (P_{\lambda_2}BP_{\lambda_2}^\prime) s = \begin{pmatrix}
\frac{(1-2p)^2}{2(1-p)} & p(1-2p) & 2p-1 \\
p(2p-1) & \frac{p^2}{2(1-p)} & -p \\
\frac{2p-1}{2} & \frac{p}{2} & \frac{1-p}{2}
\end{pmatrix} s, \quad 0 < s \leq t
\]

where we have used that in the critical case \(b = 1-2p\). For the limiting process \(\tilde{W} = (\tilde{W}_t, t \geq 0)\) of \(((n^t \ln n)^{-1/2}S_{[tn]}, t \geq 0)\), it remains to observe that \(\tilde{W}_t = 2\tilde{V}_t^{(2)}\), as under a).

c) If \(p > p^{**}\), it holds that \(\lambda_2 > \frac{1}{2} \lambda_1\). Similarly to part c) of Theorem 3.1 we may apply [19, Theorem 3.24] to deduce that
\[
(n^{-\kappa}(X_{[tn]} - tn \lambda_1 v_1), t \geq 0)
\]
converges almost surely to \((t^*W, t \geq 0)\) as \(n\) tends to infinity, where \(\tilde{W} = (\tilde{W}_1, \tilde{W}_2, \tilde{W}_3)^\prime\) is a (nonzero) random vector lying in the eigenspace of \(A\) associated to \(\lambda_2\). Using (10) and (11), the claim follows with \(Y = 2\tilde{W}_2\).

**Remark 5.** Let us mention that both models immediately generalize to higher dimensions \(d \geq 2\), and we could as well consider a sequence \((\xi_i, i \in \mathbb{N})\) of i.i.d. steps taking values \(v_1, \ldots, v_{2d} \in \mathbb{R}^d\) according to some probability vector \(p = (p_1, \ldots, p_{2d})^T\), i.e.,

\[
\mathbb{P}(\xi_1 = v_i) = p_i, \quad i = 1, \ldots, 2d.
\]

Indeed, an urn with balls of \(4d - 1\) different colors would be sufficient to model the position of the (strongly) reinforced ERW. For results on the original multi-dimensional ERW based on a martingale approach, see Bercu and Laulin [7].

4 The strongly reinforced Shark Random Swim

Instead of independent \(\pm 1\)-steps, we shall consider in this section a sequence \((\xi_i, i \in \mathbb{N})\) of independent \(\mathbb{R}^d\)-valued isotropic stable random variables specified by

\[
\mathbb{E} \left[ e^{i\langle \theta, \xi_1 \rangle} \right] = e^{-\|\theta\|^\alpha}, \quad \theta \in \mathbb{R}^d,
\]

(12)
where the stability parameter $\alpha$ takes values in $(0, 2]$.

If $d = 1$, this simply means that the $\xi_i$’s are symmetric $\alpha$-stable random variables with scale parameter one. Our arguments are however not limited to the one-dimensional case.

Under the above sequence of $\alpha$-stable random variables, the corresponding (strongly) memory-reinforced random walk $(S_n, n \in \mathbb{N}_0)$ gives rise to what we call the (strongly) reinforced Shark Random Swim, the (strongly) reinforced SRS for short.

In order to model how many times a certain step is repeated, we will establish a connection to percolation on a family of preferential attachment trees. As we explain in Remark 6, this technique is well-suited for modeling the strongly reinforced SRS, to which we restrict ourselves from now on.

The asymptotic behavior of the strongly reinforced SRS will depend on how the stability parameter $\alpha$ relates to the parameter

$$\kappa = \kappa(b, p) = \frac{b + p}{b + 1}.$$  \hfill (13)

which we fix from now on once for all in this way. We point out that the same parameter appears also in Theorem 3.2 which should be compared with our results for the strongly reinforced SRS in the case $\alpha = 2$.

Unless stated otherwise, we shall again assume that $p \in (0, 1)$ and $b \geq 0$. In the case $b = 0$, we will come across results of Businger [12, 13].

4.1 Connection to preferential attachment trees

We will construct on the positive integers $\mathbb{N}$ an increasing tree, which follows a preferential attachment mechanism. In order to clearly illustrate the connection to the strongly reinforced SRS, we first give a discrete-in-time construction of our tree, although we shall work later on primarily with a continuous-in-time construction (see Section 4.2).

In order to describe the building dynamics, we use the same weights $k_n(\cdot)$ as for the memory-reinforced random walk models, starting from

$$k_n(i) = 1 \quad \text{for } i = 1, \ldots, n \quad \text{and all } n \in \mathbb{N}.$$  

We denote by $T_1$ the tree with a single node labeled 1, to which we attach a half-edge, see Figure 1, so that the degree of the root node 1 is equal to one from the very beginning.

Then, for $n \geq 2$, given $T_{n-1}$ has been built, we attach node $n$ to a randomly chosen node $I_n \in \{1, \ldots, n - 1\}$ from the tree $T_{n-1}$ according to

$$\mathbb{P}(I_n = i) = \frac{k_{n-1}(i)}{\sum_{j=1}^{n-1} k_{n-1}(j)}, \quad i = 1, \ldots, n - 1.$$ \hfill (14)
Finally, we update the weight of the parent node $I_n = i$ of $n$ by setting

$$k_n(i) = k_{n-1}(i) + b.$$  

We note that in the case $b = 0$, the above construction produces a tree $\mathcal{T}_n$ uniformly distributed among all increasing trees on the integers 1, \ldots, $n$, a so-called *random recursive tree*.

The weights are intimately related to the degree of a vertex, an observation which will be crucial in what follows. Namely, denote by $d_n(i)$ the degree of vertex $i \in \{1, \ldots, n\}$ in $\mathcal{T}_n$, i.e., the number of edges with endpoint $i$. Then it holds that

$$k_n(i) = b(d_n(i) - 1) + 1. \tag{15}$$

Note that this relation is also true for the root node 1, thanks to the half-edge attached to it.

We now superpose a Bernoulli bond percolation with parameter $0 < p < 1$ on $\mathcal{T}_n$. However, following an idea from [10], rather than deleting edges, we shall cut each edge of $\mathcal{T}_n$ at its midpoint with probability $1 - p$, independently of the others edges. Writing $\mathcal{T}_n^{(p)}$ for the resulting combinatorial structure at time $n$, $\mathcal{T}_n^{(p)}$ is a forest consisting of trees with edges and half-edges. We call these trees *percolation clusters* of $\mathcal{T}_n^{(p)}$.

We write $c_{1,n}, c_{2,n}, \ldots$ for the sequence of percolation clusters increasingly ordered according to the label of their root node, with $c_{i,n} := \emptyset$ if $\mathcal{T}_n^{(p)}$ contains less than $i$ clusters. See the right hand side of Figure 1. In particular, $c_{1,n}$ is the (root) cluster rooted at node 1. Of course, we should rather write $c_{i,n}^{(p)}$, but we drop $p$ from the notation. Moreover, we write $|c_{i,n}|$ for the size of the $i$th cluster, i.e., the number of its nodes.

To make the connection to the strongly reinforced SRS, we assign additionally “spins” to the nodes of $\mathcal{T}_n^{(p)}$, following an idea of Kürsten [20]. More precisely, as it is indicated on the right hand side of Figure 1, we equip all the nodes of the $i$th cluster $c_{i,n}$ with spin $\xi_i$, with $(\xi_i; i \in \mathbb{N})$ a sequence of i.i.d. stable random variables with characteristic function (12). We now claim that the position $S_n$ of the strongly reinforced Shark Random Swim at time $n$ satisfies

$$S_n = d \sum_{i=1}^{n} |c_{i,n}| \xi_i. \tag{16}$$

Indeed, it readily follows from the described tree dynamics that the spin attached to the node labeled $i$ corresponds to the $i$th step of the shark: If $j > i$ and node $j$ is connected to node $i$ by an intact edge, this means in terms of the shark that time $j$ is a memory time, where the $i$th step is repeated. Node $j$ is then equipped with the spin of its parent $i$. If, instead, the edge connecting $j$ to $i$ is cut, this means that $j$ is a fresh time, and consequently, $j$ is equipped with a new (independent) spin.

For what follows it is crucial to notice that it makes no difference if we first build the tree $\mathcal{T}_n$ and then superpose percolation to obtain $\mathcal{T}_n^{(p)}$, or if we dynamically decide for each new vertex
Figure 1: On the left: An instance of $T_5$. On the right: The same tree after superposing percolation on $T_5$, i.e., an instance of $T_5^{(p)}$. Here, the edges between the vertices 1, 3 and 2, 4 are cut. Consequently, three percolation clusters $c_{1,5}, c_{2,5}, c_{3,5}$ of sizes $|c_{1,5}| = |c_{2,5}| = 2$, $|c_{3,5}| = 1$ arise, which are colored black, red and blue. They are equipped with the spins $\xi_1, \xi_2$ and $\xi_3$. Since there are only three clusters, we let $c_{i,5} = \emptyset$ for $i \geq 4$.

If the edge connecting $i$ to its parent will be kept intact (with probability $p$), or cut at its midpoint (with probability $1 - p$).

Remark 6. Clearly, in a similar way one could model the mere reinforced SRS. However, recall that the latter has only a partial memory, leading at each time $n$ to a weight increase with probability $p$ only. In particular, when building the tree with superposed percolation corresponding to the mere reinforced SRS, the weight of the parent node of a newly inserted edge may only be increased by the amount $b$ if the edge is kept intact. Therefore, in this case, one cannot first build the tree and then superpose percolation. Our techniques seem therefore less adequate to discuss the mere reinforced SRS.

In order to make use of (16), we first have to gain information on the cluster sizes $|c_{i,n}|$ of the above preferential attachment tree when $n \to \infty$. To this aim, we will first give an alternative continuous-in-time description of the above preferential attachment tree, which will then allow us to use techniques from branching processes. This is the content of the following section.

4.2 Preferential attachment trees in continuous time

This section is based on ideas from Bertoin and Bravo [10].

We let grow the preferential attachment tree $T_n$ introduced in the last section in continuous time as follows. We start from the root node 1 (with a half-edge attached to it) at time 0. Then, assuming that a tree with $n \geq 1$ vertices has been constructed, we equip each vertex $i \in \{1, \ldots, n\}$ with an independent exponential clock $\rho_i$ with parameter $b(d_n(i) - 1) + 1$, where
we write now $d_n(i)$ for the degree of vertex $i$ when there are $n$ vertices present. The first clock rings at time $\min_{i \in \{1, \ldots, n\}} \rho_i$, and then the vertex labeled $n + 1$ is attached to the vertex $v_n = \arg \min_{i = 1, \ldots, n} \rho_i$. Since the sum of the degrees in the preferential attachment tree with $n$ vertices is $2(n - 1) + 1$ (the $+1$ coming from the half-edge attached to the root), a simple calculation shows
\[
\min_{i = 1, \ldots, n} \rho_i \overset{d}{=} \text{Exp} \left( b(n - 1) + n \right),
\] (17)
where $\text{Exp}(s)$ denotes the exponential distribution with parameter $s > 0$.

We shall write $T(t)$ for the tree constructed in this way at time $t \geq 0$. Define $\tau_n$ to be the first instance when there are $n$ vertices in the tree, i.e.,
\[
\tau_n := \inf \{ t \geq 0 : |T(t)| = n \}.
\]
Recalling (15), it follows from the above dynamics that $T(\tau_n)$ is a version of the preferential attachment tree $T_n$ constructed in Section 4.1.

The fact that the $(n + 1)$st vertex arrives after an exponential waiting time of parameter $b(n - 1) + n$ suggests to consider the process
\[
Y(t) := b(|T(t)| - 1) + |T(t)|, \quad t \geq 0.
\]

**Lemma 4.1.** The process $(Y(t), t \geq 0)$ is a pure birth branching process starting from $Y(0) = 1$, which has only jumps of size $b + 1$, and with unit birth rate per unit population size. Moreover, the process $(e^{-(b+1)t}Y(t), t \geq 0)$ is a square-integrable martingale, whose terminal value $W$ is $\text{Gamma}(\frac{1}{b+1}, \frac{1}{b+1})$-distributed.

The proof can be found in Appendix A. We note for later use that
\[
Y(\tau_n) = b(n - 1) + n.
\] (18)

We now superpose percolation on $T(t)$ as follows: We assign to each edge $e_i$ connecting a vertex $i \geq 2$ to its parent an independent uniform variable $U_i$. If $U_i > p$, we cut $e_i$ at its midpoint, if $U_i \leq p$ we let $e_i$ intact. We obtain a combinatorial structure $T^{(p)}(t)$ with the same set of vertices as $T(t)$, and the subset of edges $e_i$ of $T(t)$ for which $U_i \leq p$, together with half-edges; more precisely, two half-edges for each edge $e_i$ of $T(t)$ for which $U_i > p$. We agree that cutting edges preserves the degrees of the vertices.

The subtrees of $T^{(p)}(t)$ which are spanned by vertices connected to each other by a path of intact edges form what we call the percolation clusters of $T^{(p)}(t)$. We write
\[
T^{(p)}_1(t), T^{(p)}_2(t), T^{(p)}_3(t), \ldots
\]
for these subtrees enumerated in the increasing order of their birth times, with the convention that $T^{(p)}_{i+1}(t) = \emptyset$ if the number of edges that has been cut up to time $t$ is less than $i$, for $i \geq 1$. (We stress that the $T^{(p)}_i(t)$ are combinatorial structures formed by vertices, edges and half-edges and are therefore not subtrees in the strict graph theoretic sense, but we stick to that wording.)

In particular, $T^{(p)}_1(t)$ is the subtree rooted at vertex 1, and, more generally, if $U_j$ is the $i$th variable among $U_2, U_3, \ldots$ to be greater than $p$, then $T^{(p)}_{i+1}(t)$ is the subtree of $T^{(p)}(t)$ rooted at node $j$. Note moreover that

$$\sum_{i=1}^{\infty} |T^{(p)}_i(t)| = |T(t)|,$$

where $|T^{(p)}_i(t)|$ denotes the number of vertices of $T^{(p)}_i(t)$.

From the construction, we readily obtain the following connection to the clusters $c_{i,n}$ of percolation with parameter $p$ on the preferential attachment tree $T_n$:

**Corollary 4.1.** There is the equality in distribution

$$(|c_{1,n}|, |c_{2,n}|, \ldots) =_d \left( |T^{(p)}_1(\tau_n)|, |T^{(p)}_2(\tau_n)|, \ldots \right).$$

It will be useful to introduce a notation for the birth time of the $i$th subtree $T^{(p)}_i$. We set

$$b_1 := 0 \quad \text{and} \quad b_i := \inf\{t \geq 0 : T^{(p)}_i(t) \neq \emptyset\}, \quad i \geq 2.$$  \hspace{1cm}(19)

Of course we should rather write $b^{(p)}_i$, but we skip the parameter $p$ for ease of reading. We warn, however, that $b_i$ should never be confused with the reinforcement parameter $b$.

We further denote by $H^{(p)}_i(t)$ the number of half-edges attached to the vertices of $T^{(p)}_i(t)$. In particular, we have $H^{(p)}_i(b_i) = 1$ and $H^{(p)}_i(t) = 0$ for $0 \leq t < b_i$. (Recall that $H^{(p)}_1(b_1) = 1$ follows from the construction of the tree $T(t)$.)

It should be clear from the description that the processes $(|T^{(p)}_i(b_i + t)|, t \geq 0)$ for $i \geq 1$ are independent and identically distributed; in particular,

$$(|T^{(p)}_i(b_i + t)|, t \geq 0) =_d (|T^{(p)}_i(t)|, t \geq 0).$$  \hspace{1cm}(20)

Although we are primarily interested in the size processes $|T^{(p)}_i(\cdot)|$, it is much more natural to look at

$$Y^{(p)}_i(t) := \left( b(|T^{(p)}_i(t)| - 2 + H^{(p)}_i(t)) + |T^{(p)}_i(t)| \right) \mathbb{1}_{\{b_i \leq t\}}.$$  \hspace{1cm}(21)

Indeed, the processes $Y^{(p)}_i$ are easy to control, thanks to the following lemma.

**Lemma 4.2.** The processes $(Y^{(p)}_i(b_i + t), t \geq 0)$, $i \geq 1$, are i.i.d. pure birth branching processes starting from $Y^{(p)}_1(b_1) = 1$, with unit birth rate per unit population size and reproduction law given by the law of $b + \epsilon_p$, where $\epsilon_p$ is Bernoulli-distributed with success probability $p$. Moreover, the following properties hold:
• $\mathbb{E} \left[ Y_i^{(p)}(b_i + t) \right] = e^{(b+p)t}$ and $\mathbb{E} \left[ Y_i^{(p)}(b_i + t)^2 \right] = \frac{(b+1)(b+2p)}{b+p} (e^{2(b+p)t} - e^{(b+p)t})$.

• The process $(e^{-(b+1)t}Y_i^{(p)}(b_i + t), t \geq 0)$ is a martingale bounded in $L^k$ for any $k \in \mathbb{N}$, whose terminal value $W_i$ is almost surely strictly positive, with

$$\mathbb{E}[W_i] = 1 \quad \text{and} \quad \mathbb{E} [W_i^2] = \frac{(b+1)(b+2p)}{b+p}.$$ 

We stress that the variables $Y_i^{(p)}(t)$ are linked to $Y(t)$ via $\sum_{i=1}^{\infty} Y_i^{(p)}(t) = Y(t)$.

\textbf{Remark 7.} Analogously, one sees that the martingale $(e^{-(b+1)t}Y(t), t \geq 0)$ from Lemma 4.1 is bounded in $L^k$ as well, for any $k \in \mathbb{N}$ (and not merely in $L^2$); however, for our purpose, square-integrability will be sufficient.

We will close this section with an upper and lower bound on the birth times $b_i$ defined in (19). The following lemma extends [12, Lemma 8].

\textbf{Lemma 4.3.} Let $(x_n, n \in \mathbb{N})$ be a sequence of positive integers with $\lim_{n \to \infty} x_n = \infty$ and $x_n \leq n$. Then there exists a sequence $(\varepsilon_n, n \in \mathbb{N})$ of positive reals with $\varepsilon_n \downarrow 0$ as $n \to \infty$ and a sequence of events $(E_n, n \in \mathbb{N})$ with $\lim_{n \to \infty} \mathbb{P}(E_n) = 1$, such that on $E_n$, the following bounds hold for the birth times $b_i$ with $x_n \leq i \leq n$, provided $n$ is large enough:

$$\tau_n - b_i \leq t_{n,i}^+ := \frac{1}{b+1} \left( \ln n - \ln(i-1) + \ln(1-p) + \varepsilon_n \right),$$

$$\tau_n - b_i \geq t_{n,i}^- := \frac{1}{b+1} \left( \ln n - \ln(i+1) + \ln(1-p) - \varepsilon_n \right).$$

The proofs of Lemmas 4.2 and 4.3 are postponed to Appendix A.

Let us point at a useful consequence of the above lemma. For $i, n \in \mathbb{N}$, define the random variables

$$X_i(n) := |T_i^{(p)}(b_i + t_{n,i}^-)|,$$

$$\overline{X}_i(n) := |T_i^{(p)}(b_i + t_{n,i}^+)|,$$

with the convention that $|T_i^{(p)}(s)| := 0$ if $s < b_i$. It follows that on the event $E_n$, for $n$ sufficiently large and $i \geq x_n$, we have

$$X_i(n) \leq |T_i^{(p)}(\tau_n)| \leq \overline{X}_i(n).$$

The obvious advantage of working with the variables $X_i(n)$ is that they are independent, and so are the variables $\overline{X}_i(n)$ (in contrast to the variables $|T_i^{(p)}(\tau_n)|$, $i \geq 1$). As regards their laws, we have by (20), for $i \in \mathbb{N}$ fixed,

$$(X_i(n), n \in \mathbb{N}) =_d \left( |T_i^{(p)}(t_{n,i}^-)|, n \in \mathbb{N} \right), \quad (\overline{X}_i(n), n \in \mathbb{N}) =_d \left( |T_i^{(p)}(t_{n,i}^+)|, n \in \mathbb{N} \right).$$
We will need to bound the moments of $X_i(n)$ and $\overline{X}_i(n)$ (in fact, a bound on the fourth moment will be sufficient). Using that $\overline{X}_i(n)$ stochastically dominated by $Y_1^{(p)}(\tau_{n,i})$, see (21), the following corollary is an immediate consequence of Lemma 4.2.

**Corollary 4.2.** Let $p \in (0, 1)$, $b \geq 0$, and $\kappa = (b + p)/(b + 1)$. For each $\ell \in \mathbb{N}$, there exists a constant $C_\ell = C_\ell(b, p)$ such that for all $i, n \in \mathbb{N},$

$$\mathbb{E}[X_i(n)^\ell] \leq \mathbb{E}[\overline{X}_i(n)^\ell] \leq C_\ell \left(\frac{n}{\ell}\right)^{\ell \kappa}.$$  

4.3 Cluster sizes of percolation on preferential attachment trees

4.3.1 Size of the root cluster

In this section we study the size of the root cluster $c_{1,n}$ of $T_n^{(p)}$. We work in the setting of Section 4.2, with the variables and notation defined there. As always, we assume $p \in (0, 1)$, $b \geq 0$, and $\kappa = \kappa(b, p) = (b + p)/(b + 1)$.

We recall that $|c_{1,n}| = d |T_1^{(p)}(\tau_n)|$, and the latter is linked to $Y_1^{(p)}(\tau_n)$ via (21). We first establish a limit result for $Y_1^{(p)}(\tau_n)$.

**Lemma 4.4.** We have the convergence

$$\lim_{n \to \infty} \frac{Y_1^{(p)}(\tau_n)}{n^\kappa} = \hat{Z}_1 \quad \text{a.s. and in } L^2,$$

where $\hat{Z}_1$ is a strictly positive random variable with

$$\mathbb{E}[\hat{Z}_1] = \frac{\Gamma \left(\frac{1}{b+1}\right)}{\Gamma \left(1 + \frac{p}{b+1}\right)} \quad \text{and} \quad \mathbb{E}[\hat{Z}_1^2] = \frac{(b + 1)^2 \Gamma \left(\frac{1}{b+1}\right)}{(b + p) \Gamma \left(\frac{b+2p}{b+1}\right)}.$$

**Remark 8.** Unfortunately, we were not able to identify the law of $\hat{Z}_1$. One can, however, be a bit more precise about $\hat{Z}_1$, see Display (22) in the proof below. If $b = 0$, $\hat{Z}_1$ is known to follow the Mittag-Leffler-distribution with parameter $p$, see [12, Lemma 3].

**Proof.** All the following convergences hold almost surely and in $L^2$. By (18) and Lemma 4.1

$$\lim_{n \to \infty} e^{-(b+1)\tau_n} (b(n - 1) + n) = W.$$  

Since $(e^{-(b+1)\tau_n})^\kappa = e^{-(b+p)\tau_n}$, Lemma 4.2 shows that (with $W_1$ from there)

$$\lim_{n \to \infty} \frac{Y_1^{(p)}(\tau_n)}{n^\kappa} = W_1 \left(\frac{W}{b + 1}\right)^{-\kappa} =: \hat{Z}_1.$$  

(22)
To see why the first and second moment of $\hat{Z}_1$ take the stated form, we note that $W$ is independent of $\hat{Z}_1$ (of course, $W$ and $W_1$ are not independent). Thus, for any $\alpha \geq 0$,

$$\mathbb{E}\left[\hat{Z}_1^\alpha\right] = \frac{\mathbb{E}\left[W_1^\alpha\right]}{\mathbb{E}\left[(\frac{1}{b+1}W)^\alpha\right]}.$$ 

Specifying to $\alpha = 1$ and $\alpha = 2$, we obtain the claim from what we know about $W_1$ and $W$.

In order to gain information about the root cluster, we need to control the half-edges.

**Lemma 4.5.** For $b > 0$ we have the convergence in $L^2$

$$\lim_{n \to \infty} \frac{H_1^{(p)}(\tau_n) - \frac{1-p}{1+b}Y_1^{(p)}(\tau_n)}{n^\kappa} = 0.$$

**Proof.** First note that the two processes

$$H_1^{(p)}(t) - (1-p) \int_0^t Y_1^{(p)}(s)ds, \quad t \geq 0,$$

and

$$Y_1^{(p)}(t) - (b+p) \int_0^t Y_1^{(p)}(s)ds, \quad t \geq 0,$$

are both martingales with respect to the natural filtration, and thus also the process

$$L(t) := H_1^{(p)}(t) - \frac{1-p}{b+p}Y_1^{(p)}(t), \quad t \geq 0,$$

is a (càdlàg) martingale. Writing $\langle L \rangle_t$ for the bracket process of $L$, the process $L(t)^2 - \langle L \rangle_t$ is a local martingale. It follows from the definition of $Y_1^{(p)}$ given in (21) that the jump sizes $|L(t) - L(t^-)|$ of $L$ satisfy

$$\min(p, 1-p) \leq |L(t) - L(t^-)| \leq 1 + 1/b.$$

Using $H_1^{(p)}(t) \leq (1/b)Y_1^{(p)}(t) + 2$, we then deduce from Lemma 4.4 that the mean number of jumps of $L$ up to time $\tau_n$ is bounded from above by

$$\frac{1}{\min(p, 1-p)} \left( \mathbb{E}\left[H_1^{(p)}(\tau_n)\right] + \frac{1}{b} \mathbb{E}\left[Y_1^{(p)}(\tau_n)\right] \right) \lesssim n^\kappa.$$

By the Burkholder-Davis-Gundy inequality, it follows that

$$\mathbb{E}\left[ (L(\tau_n) - L(0))^2 \right] \lesssim \mathbb{E}[\langle L \rangle_{\tau_n}] \lesssim n^\kappa.$$

In particular,

$$\lim_{n \to \infty} \mathbb{E}\left[ \frac{|L(\tau_n)|^2}{n^\kappa} \right] = 0,$$

proving the lemma. 

\[\Box\]
We arrive at the following result for the size of the root cluster $c_{1,n}$.

**Proposition 4.1.** We have the convergence in $L^2$

$$\lim_{n \to \infty} \frac{|c_{1,n}|}{n^\kappa} = Z_1,$$

where $Z_1 = \frac{p}{b+p} \hat{Z}_1$, with $\hat{Z}_1$ as in Lemma 4.4.

**Proof.** By (21),

$$|c_{1,n}| = \frac{Y_1^{(p)}(\tau_n) - bH_1^{(p)}(\tau_n) + 2b}{b+1},$$

and the claim follows from a combination of Lemmas 4.4 and 4.5. \qed

### 4.3.2 Sizes of the remaining clusters

We now describe how we control the sizes of the percolation clusters rooted at nodes different from the root. We will apply our results from this section in the supercritical case $\alpha \kappa > 1$.

We first look at a different but related quantity. Namely, let us write $T_{i,n}^{(p)}$ for the subtree of $T_n^{(p)}$ rooted at node $i$, i.e., $T_{i,n}^{(p)}$ is the combinatorial structure spanned by the vertices $j \geq i$ which are connected to node $i$ after superposing percolation on $T_n$. Note that $T_{i,n}^{(p)}$ equals the root cluster $c_{1,n}$, whereas for $i \geq 2$, $T_{i,n}^{(p)}$ is a percolation cluster of $T_n^{(p)}$ if and only if the edge connecting $i$ to its parent has been cut. Let us also write $\eta(n,i)$ for the number of nodes $j \geq i$ which are connected to node $i$ in $T_n$, that is, before superposing percolation. Then it holds that

$$|T_{i,n}^{(p)}| = d \ |c_{1,\eta(n,i)}|.$$  \hspace{1cm} (23)

Indeed, as already observed, it does not matter if we first build the tree and then perform percolation or if we decide successively if a new arriving edge is cut or not.

The distribution of $\eta(n,i)$ can be modeled by means of a Pólya urn with diagonal (deterministic) replacement matrix $\begin{pmatrix} b+1 & 0 \\ 0 & b+1 \end{pmatrix}$. We stress that $b \geq 0$ needs not to be an integer: Indeed, one might define the urn process as a Markov process taking values in $\{(x,y) \in \mathbb{R}^2 : x, y > 0\}$ with transitions from $(x,y)$ to $(x+b+1,y)$ with probability $x/(x+y)$, and from $(x,y)$ to $(x+y+b+1)$ with probability $y/(x+y)$. However, for simplicity, let us depict the connection to $\eta(n,i)$ as if we would add $b+1$ balls at each step.

We start from one green ball and $(i-1)(b+1)$ red balls. The single green ball corresponds to the weight $k_i(i)$ of node $i$, and the number of red balls corresponds to the sum $\sum_{j=1}^{i-1} k_i(j)$ of weights of the vertices labeled $1, \ldots, i-1$ just after the $i$th node has been inserted. We then draw repeatedly a ball uniformly at random and return it together with $b+1$ balls of the same color (corresponding to an additional weight increase of $b+1$, namely $b$ for the parent node and
1 for the newly inserted node). The number $G_{n-i}$ of green balls after $n-i$ draws corresponds to the sum of weights of the vertices connected to node $i$ in $T_n$ (before percolation), and we have

$$\eta(n, i) \equiv d \frac{G_{n-i} - 1}{b+1} + 1 = \frac{G_{n-i} + b}{b+1}. \quad (24)$$

We need to control $\eta(n, i)$ when $i$ is fixed and $n$ is large. First, as an immediate consequence of [21, Corollary 3.1], we obtain for the first and second moment of $\eta(n, i)$ the bounds

$$\mathbb{E}[\eta(n, i)] \lesssim \frac{n}{i}, \quad \mathbb{E}[\eta(n, i)^2] \lesssim \frac{n^2}{i^2}. \quad (25)$$

Moreover, Theorem 3.2 of [21] shows that for fixed $i \in \mathbb{N}$, there is the almost sure convergence

$$\lim_{n \to \infty} \frac{\eta(n, i)}{n} = \text{Beta} \left( \frac{1}{b+1}, i-1 \right), \quad (26)$$

where Beta($r, s$) denotes a Beta-distributed random variable with parameters $r, s$ (with the convention Beta($r, 0$) = 1). Note, however, that [21, Theorem 3.2] is formulated for the number $\tilde{G}_n$ of green ball drawings after $n$ draws. But $G_n$ is clearly related to $\tilde{G}_n$ via $G_n = \tilde{G}_n (b+1)+1$, and an application of (24) yields (26).

Let us now come back to the percolation clusters $c_{1,n}, c_{2,n}, \ldots$ of $T_n^{(p)}$. As already remarked, for $i \geq 2$, $T_{i,n}^{(p)}$ is a percolation cluster if and only if the edge between $i$ and its parent has been cut. Setting $C_{i,n} := T_{i,n}^{(p)}$ if the latter is a percolation cluster, and $C_{i,n} := \emptyset$ otherwise, we obtain:

**Corollary 4.3.** Let $p \in (0, 1)$, $b \geq 0$, $\kappa = (b+p)/(b+1)$, and let $i \in \mathbb{N}$. Then we have the $L^2$-convergence

$$\lim_{n \to \infty} \frac{|C_{i,n}|}{n^\kappa} = Z_i,$$

where $Z_i$ is equal in distribution to

$$\epsilon_i \cdot \beta_i^\kappa \cdot Z_1,$$

with $\epsilon_1 = 1$ and $\epsilon_i$ for $i \geq 2$ a Bernoulli-distributed random variable with success probability $1-p$, $\beta_i$ a Beta($\frac{1}{b+1}, i-1$)-distributed random variable independent of $\epsilon_i$, and $Z_1$ as in Proposition 4.1, independent of $\epsilon_i$ and $\beta_i$.

**Proof.** For $i = 1$, this is simply the statement of Proposition 4.1. For general $i \in \mathbb{N}$, we have by (23) the equality in distribution

$$|C_{i,n}| = d \epsilon_i |T_{i,n}^{(p)}| = d \epsilon_i |c_{1,\eta(n,i)}| \cdot \quad (27)$$

The claim then follows from (26), together with Proposition 4.1.
Clearly, unless \( i = 1 \), we do not have \(|c_{i,n}| = |C_{i,n}|\) in general. However, for our purposes, the much weaker relation (44) will be sufficient.

Finally, for future use we record that

\[
\sum_{i=1}^{\infty} \mathbb{E}[Z_i^n] < \infty \quad \text{if and only if} \quad \alpha \kappa > 1.
\]

Indeed, recall that for \( q \geq 0 \), the \( q \)th moment of a Beta\((s,t)\)-distributed random variable is given by \( B(s+q,t)/B(s+t) \), where \( B(s,t) \) denotes the Beta-function with parameters \( s, t > 0 \). A small calculation then leads to (28).

### 4.4 Asymptotic behavior of the strongly reinforced SRS

We are now in position to discuss the long-time behavior of the strongly reinforced Shark Random Swim \( (S_n, n \in \mathbb{N}_0) \). We remind that this means that \( (S_n, n \in \mathbb{N}_0) \) is the strongly memory-reinforced random walk defined in terms of an i.i.d. sequence \( (\xi_i, i \in \mathbb{N}) \) of isotropic \( \alpha \)-stable random variables with characteristic function (12).

As we will show, the strongly reinforced SRS exhibits a phase transition at \( \alpha \kappa = 1 \). The stability parameter \( \alpha \) may take values in \((0, 2]\), and \( \kappa = \kappa(b, p) = (b + p)/(b + 1) \).

If \( \alpha \kappa \leq 1 \), we prove weak convergence of finite-dimensional laws towards an \( \alpha \)-stable stochastic process. In the subcritical case \( \alpha \kappa < 1 \), the limit process is non-Lévy, whereas in the critical case \( \alpha \kappa = 1 \), it is an \( \alpha \)-stable Lévy process. In the supercritical case \( \alpha \kappa > 1 \) we prove convergence in probability to a (nontrivial) stochastic process.

In order to prove our results, we make use of the representation of \( (S_n, n \in \mathbb{N}_0) \) in terms of cluster sizes established in (16). More precisely, recalling Corollary 4.1, we shall work in the continuous setting of Section 4.2 and use the abbreviation

\[
|c_{i,n}| := |T_i^{(p)}(\tau_n)|
\]

for the size of the \( i \)th percolation cluster of \( T^{(p)} \) stopped at time \( \tau_n \).

For fixed \( i \in \mathbb{N} \), we may view \( n \mapsto |c_{i,n}| \) as an increasing \( \mathbb{N}_0 \)-valued process in \( n \). It readily follows that (16) can be strengthened to an equality in law for processes, namely

\[
(S_n, n \in \mathbb{N}_0) =_d \left( \sum_{i=1}^{n} |c_{i,n}| \xi_i, n \in \mathbb{N}_0 \right).
\]

Since \( c_{i,n} = \emptyset \) for \( i > n \), we may as well sum up to infinity in the above sum.
4.4.1 The subcritical case $\alpha \kappa < 1$

Given $p \in (0, 1)$ and $b \geq 0$, we define for $x > 0$ the random (almost surely càdlàg) function

$$f(x) := \left| T_1^{(p)} \left( \frac{1}{b+1} (-\ln x + \ln(1-p)) \right) \right|,$$

recalling that we set $|T_1^{(p)}(s)| = 0$ if $s < 0$.

Bounding $|T_1^{(p)}(s)|$ from above by $Y_1^{(p)}(s)$, Lemma 4.2 shows that

$$\int_0^\infty \mathbb{E} [f(x)\alpha] \, dx = \int_0^{1-p} \mathbb{E} [f(x)\alpha] \, dx \lesssim \int_0^{1-p} x^{-\alpha \kappa} dx < \infty \quad \text{if } \alpha \kappa < 1. \quad (30)$$

We let $\mathcal{X} = (\mathcal{X}_t, t \geq 0)$ denote a $d$-dimensional symmetric $\alpha$-stable stochastic process with $\mathcal{X}_0 := 0$, whose marginals $(\mathcal{X}(t_1), \ldots, \mathcal{X}(t_k))$ for $0 < t_1 < t_2 < \cdots < t_k$ have characteristic function

$$\mathbb{E} \left[ \exp \left( i \sum_{j=1}^k \mathcal{X}(t_j) \cdot \theta_j \right) \right] = \exp \left( - \int_0^\infty \mathbb{E} \left[ \left\| \sum_{j=1}^k f(x/t_j)\theta_j \right\|^{\alpha} \right] \, dx \right), \quad \theta_1, \ldots, \theta_k \in \mathbb{R}^d. \quad (31)$$

The existence of such a process in the subcritical case $\alpha \kappa < 1$ follows from Kolmogorov’s existence theorem. We refer to Chapter 3 of Samorodnitsky and Taqqu [30] for more information on stable processes and for a proof of the existence (given, however, in a much more general setting).

**Theorem 4.1.** Let $p \in (0, 1)$, $b \geq 0$ and $\kappa = (b + p)/(b + 1)$. Assume $0 < \alpha < 1/\kappa$. Then, as $n \to \infty$, the finite-dimensional marginals of the process

$$\left( \frac{S_{\lfloor tn \rfloor}}{n^{1/\alpha}}, t \geq 0 \right)$$

converge in law to those of an $\alpha$-stable stochastic process $\mathcal{X} = (\mathcal{X}(t), t \geq 0)$ specified by (31).

**Proof.** For the ease of reading, we restrict ourselves to the two-dimensional marginals. The general case of $k$-dimensional marginals works in the same way, but is heavier in notation. We fix $t_2 > t_1 > 0$ and $a_1, a_2 \in \mathbb{R}$. We use the abbreviations $n_1 := \lfloor t_1 n \rfloor$, $n_2 := \lfloor t_2 n \rfloor$, interpreting $n_1$, $n_2$ as functions in $n$. Let $F : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. By first conditioning on $|c_{1,n_1}|, \ldots, |c_{1,n_1}|, |c_{1,n_2}|, \ldots, |c_{n_2,n_2}|$ and then integrating out, we obtain, with $\xi = d \xi_1$,

$$\mathbb{E} \left[ F \left( \frac{1}{n^{1/\alpha}} \left( a_1 S_{n_1} + a_2 S_{n_2} \right) \right) \right] = \mathbb{E} \left[ F \left( \frac{1}{n^{1/\alpha}} \left( \sum_{i=1}^{n_2} |a_1| c_{i,n_1} + a_2 |c_{i,n_2}| \right)^{1/\alpha} \right) \right].$$

From the last two displays and standard properties of symmetric stable random variables (see, e.g., Theorem 3.1.2 of [30]), the claim follows if we show the convergence in probability

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n_2} |a_1| c_{i,n_1} + a_2 |c_{i,n_2}|^{\alpha} = \int_0^\infty \mathbb{E} \left[ |a_1 f(x/t_1) + a_2 f(x/t_2)|^{\alpha} \right] \, dx. \quad (32)$$
We split the sum in the last display into
\[
\sum_{i=1}^{n_2} |a_1 c_{i,n_1}| + a_2 |c_{i,n_2}|^\alpha = \sum_{i=1}^{\lfloor \ln n \rfloor - 1} |a_1 c_{i,n_1}| + a_2 |c_{i,n_2}|^\alpha + \sum_{i=\lfloor \ln n \rfloor}^{n_2} |a_1 c_{i,n_1}| + a_2 |c_{i,n_2}|^\alpha. \tag{33}
\]

The first sum we bound by
\[
\sum_{i=1}^{\lfloor \ln n \rfloor - 1} |a_1 c_{i,n_1}| + a_2 |c_{i,n_2}|^\alpha \leq (|a_1| + |a_2|)^\alpha \sum_{i=1}^{\lfloor \ln n \rfloor} |c_{i,n_2}|^\alpha.
\]

Clearly, $|c_{i,n_2}|$ is stochastically dominated by $|c_{1,n_2}|$, and therefore, by Proposition 4.1
\[
\frac{1}{n} \sum_{i=1}^{\lfloor \ln n \rfloor} \mathbb{E}[|c_{i,n_2}|^\alpha] \lesssim \frac{\ln n}{n} \mathbb{E}[|c_{1,n_2}|^\alpha] \lesssim \ln n n^{\alpha \kappa - 1}.
\]

Since $\alpha \kappa < 1$, we deduce that the first sum in (33) is negligible. It remains to show that upon dividing by $n$, the second sum in (33) converge in probability to the integral in (32).

Set $x_n := \lfloor \ln(n/t_2) \rfloor$. We let $(E_n, n \in \mathbb{N})$ be a sequence of events as specified in Lemma 4.3. Then $\mathbb{P}(E_{n_1} \cap E_{n_2}) \to 1$ as $n \to \infty$. On the event $E_{n_1} \cap E_{n_2}$, we have for $i \geq \lfloor \ln n \rfloor$ the lower and upper bounds
\[
X_i(n_1) \leq |c_{i,n_1}| \leq X_i(n_1), \quad X_i(n_2) \leq |c_{i,n_2}| \leq X_i(n_2),
\]
where we recall that $X_i(n) = |T_i^{(p)}(b_i + t_{i,n}^-)|$ and $X_i(n) = |T_i^{(p)}(b_i + t_{i,n}^+)|$, with
\[
t_{n,i}^- = \frac{1}{b+1} (\ln n - \ln(i+1) + \ln(1-p) - \varepsilon_n),
\]
\[
t_{n,i}^+ = \frac{1}{b+1} (\ln n - \ln(i-1) + \ln(1-p) + \varepsilon_n).
\]

Note that for two sequences $r = (r_n)$, $s = (s_n)$ of reals, we have, with $\|r\|_\alpha := (\sum_n |r_n|^\alpha)^{1/\alpha}$ denoting the $L^\alpha$-norm,
\[
\|r\|_\alpha - \|s\|_\alpha \leq \|r + s\|_\alpha \leq \|r\|_\alpha + \|s\|_\alpha.
\]

From this, it is readily seen that our claim (32) follows if we prove the convergences in probability
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lfloor \ln n \rfloor}^{n_2} |a_1 X_i(n_1) + a_2 X_i(n_2)|^\alpha = \int_0^\infty \mathbb{E}[|a_1 f(x/t_1) + a_2 f(x/t_2)|^\alpha] \, dx \tag{34}
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lfloor \ln n \rfloor}^{n_2} |X_i(n_1) - X_i(n_1)|^\alpha = \lim_{n \to \infty} \frac{1}{n} \sum_{i=\lfloor \ln n \rfloor}^{n_2} |X_i(n_2) - X_i(n_2)|^\alpha = 0. \tag{35}
\]
We first show convergence of the expectations. To that end, recall the random function $f$ defined at the beginning of this section. As for the expectation of the sum in (34), we have
\[
\frac{1}{n} \sum_{i=\lfloor \ln n \rfloor}^{n} \mathbb{E} \left[ |a_1 \bar{X}_i(n_1) + a_2 \bar{X}_i(n_2)|^\alpha \right] = \frac{1}{n} \sum_{i=\lfloor \ln n \rfloor}^{n} \mathbb{E} \left[ \left| a_1 T_1^{(p)}(t_{n_1,i}) + a_2 T_1^{(p)}(t_{n_2,i}) \right|^\alpha \right]
\]
\[
= \frac{1}{n} \sum_{i=\lfloor \ln n \rfloor}^{n} \mathbb{E} \left[ a_1 f \left( e^{-\varepsilon_1} \frac{i-1}{n_1} \right) + a_2 f \left( e^{-\varepsilon_2} \frac{i-1}{n_2} \right) \right]^\alpha.
\]
The sum on the right is a classical Riemann sum and converges upon letting $n \to \infty$ towards
\[
\int_0^\infty \mathbb{E} \left[ |a_1 f(x/t_1) + a_2 f(x/t_2)|^\alpha \right] dx.
\]

Analogously,
\[
\frac{1}{n} \sum_{i=\lfloor \ln n \rfloor}^{n} \mathbb{E} \left[ |\bar{X}_i(n_1) - \bar{X}_i(n_1)|^\alpha \right] = \frac{1}{n} \sum_{i=\lfloor \ln n \rfloor}^{n} \mathbb{E} \left[ \left| f \left( e^{-\varepsilon_1} \frac{i-1}{n_1} \right) - f \left( e^{+\varepsilon_1} \frac{i+1}{n_1} \right) \right|^\alpha \right],
\]
and the right hand side converges to zero as $n \to \infty$. The second sum in (35) is handled in the same way.

It remains to check that the variance of $\sum_{i=\lfloor \ln n \rfloor}^{n} |a_1 \bar{X}_i(n_1) + a_2 \bar{X}_i(n_2)|^\alpha$ is of order $o(n^2)$. Using independence, the fact that $2\alpha \leq 4$ and Corollary 4.2 for the last step, the variance is upper bounded by
\[
\sum_{i=\lfloor \ln n \rfloor}^{n} \text{Var} \left( |a_1 \bar{X}_i(n_1) + a_2 \bar{X}_i(n_2)|^\alpha \right) \lesssim \sum_{i=\lfloor \ln n \rfloor}^{n} \mathbb{E} \left[ \bar{X}_i(n_2)^4 \right]^{\alpha/4} \lesssim \sum_{i=\lfloor \ln n \rfloor}^{n} \left( \frac{n}{\ln n} \right)^{2\alpha \kappa}.
\]
Since $\alpha \kappa < 1$, the right hand side of (36) is of order $o(n^2)$, as wanted. Altogether, this proves (32) and hence the theorem. \hfill \Box

### 4.4.2 The critical case $\alpha = 1/\kappa$

Recall the definition of the random variable $Z_1$ from Proposition 4.1.

We let $\mathcal{X} = (\mathcal{X}_t, t \geq 0)$ denote a $d$-dimensional symmetric $\alpha$-stable stochastic process with $\mathcal{X}_0 := 0$, whose marginals $(\mathcal{X}(t_1), \ldots, \mathcal{X}(t_k))$ for $0=: t_0 < t_1 < t_2 < \cdots < t_k$ have characteristic function
\[
\mathbb{E} \left[ \exp \left( i \sum_{j=1}^{k} \mathcal{X}(t_j) \cdot \theta_j \right) \right] = \exp \left( -\frac{1-p}{b+1} \mathbb{E}[Z_1^\alpha] \sum_{j=1}^{k} (t_j - t_{j-1}) \left\| \sum_{i=j}^{k} \theta_i \right\|^\alpha \right), \quad \theta_1, \ldots, \theta_k \in \mathbb{R}^d.
\]

We see the characteristic function of the marginals of a $d$-dimensional $\alpha$-stable Lévy process, scaled by the factor $(\frac{1-p}{b+1} \mathbb{E}[Z_1^\alpha])^{1/\alpha}$.
Theorem 4.2. Let $p \in (0,1)$, $b \geq 0$ and $\kappa = (b + p)/(b + 1)$. Assume $\alpha = 1/\kappa$. Then, as $n \to \infty$, the finite-dimensional marginals of the process
\[ \left( \frac{S_{[n^t]}}{(n^t \ln n)^\kappa}, t \geq 0 \right) \]
converge in law to those of an $\alpha$-stable Lévy process $\mathcal{X} = (\mathcal{X}(t), t \geq 0)$ specified by (37).

Proof. The proof is in spirit similar to the proof of the subcritical case. For ease of reading, we look again at the two-dimensional marginals only. We fix $t_2 > t_1 > 0$ and set $n_1 = n_1(n) := \lfloor n^{t_1} \rfloor$, $n_2 = n_2(n) := \lfloor n^{t_2} \rfloor$. Let $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Then, by conditioning on the cluster sizes and integrating,
\[
\mathbb{E} \left[ F \left( \frac{S_{n_1}}{(n_1 \ln n)^\kappa}, \frac{S_{n_2}}{(n_2 \ln n)^\kappa} \right) \right] = \mathbb{E} \left[ F \left( \frac{1}{(n_1 \ln n)^\kappa} \left( \sum_{i=1}^{n_1} |c_{i,n_1}|^\alpha \right)^{\frac{1}{\alpha}}, \frac{1}{(n_2 \ln n)^\kappa} \left( \left( \sum_{i=1}^{n_1} |c_{i,n_1}|^\alpha \right)^{\frac{1}{\alpha}} \xi + \left( \sum_{i=n_1+1}^{n_2} |c_{i,n_2}|^\alpha \right)^{\frac{1}{\alpha}} \xi' \right) \right],
\]
where $\xi$ and $\xi'$ are i.i.d. copies of $\xi_1$. The stated joint weak convergence of the marginals now follows if we show the following convergences in probability (recall that $\alpha \kappa = 1$):
\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n_1} |c_{i,n_1}|^\alpha}{n_1 \ln n} = t_1 \frac{1 - p}{b + 1} \mathbb{E}[Z_1^\alpha], \quad \lim_{n \to \infty} \frac{\sum_{i=n_1+1}^{n_2} |c_{i,n_2}|^\alpha}{n_2 \ln n} = t_1 \frac{1 - p}{b + 1} \mathbb{E}[Z_1^\alpha].
\]
and
\[
\lim_{n \to \infty} \frac{\sum_{i=n_1+1}^{n_2} |c_{i,n_2}|^\alpha}{n_2 \ln n} = (t_2 - t_1) \frac{1 - p}{b + 1} \mathbb{E}[Z_1^\alpha]. \tag{38}
\]
We show only (38), the first two convergences follow from identical arguments.

We work again on the events $E_n$ from Lemma 4.3 defined with respect to $x_n := \lfloor \ln n \rfloor$. For large $n$, we estimate
\[
\sum_{i=n_1+1}^{n_2} |c_{i,n_2}|^\alpha \mathbb{1}_{E_{n_2}} \geq \sum_{i=n_1+1}^{n_2/x_n} \mathcal{X}_i(n_2)^\alpha \mathbb{1}_{E_{n_2}}, \tag{39}
\]
and
\[
\sum_{i=n_1+1}^{n_2} |c_{i,n_2}|^\alpha \mathbb{1}_{E_{n_2}} \leq \sum_{i=n_1+1}^{n_2/x_n} \mathcal{X}_i(n_2)^\alpha + \sum_{i=n_2/x_n}^{n_2} \mathcal{X}_i(n_2)^\alpha. \tag{40}
\]
Display (38) follows if we show that both right hand sides of (39) and (40) converge in probability upon dividing by $n_2 \ln n$ to
\[
(t_2 - t_1) \frac{1 - p}{b + 1} \mathbb{E}[Z_1^\alpha].
\]
We restrict ourselves to the upper bound (40), the convergence of the lower bound (39) can be shown in the same way. First, by Corollary 4.2 and the fact that \( \alpha \kappa = 1 \),

\[
\sum_{i=n_2/x_n}^{n_2} \mathbb{E} \left[ X_i(n_2)^{\alpha} \right] \lesssim \sum_{i=n_2/x_n}^{n_2} \frac{n_2}{i},
\]

and the right hand side is of order \( o(n_2 \ln n) \) as \( n \to \infty \), by the choice of \( x_n \). It remains to prove the convergence in probability

\[
\lim_{n \to \infty} \frac{1}{n_2 \ln n} \sum_{i=n_1}^{n_2/x_n} X_i(n_2)^{\alpha} = (t_2 - t_1) \frac{1 - p}{b + 1} \mathbb{E}[Z_1^\alpha].
\]

We use again the second moment method. As far as the convergence of the expectations is concerned, we notice that for \( i = i(n) \) between \( n_1 \) and \( n_2/x_n \), we have \( t_{n_2,i}^+ \geq c \ln \ln n \) for some constant \( c > 0 \). Arguments entirely similar to those leading to Proposition 4.1 then show that in \( L^2 \) (and hence in \( L^\alpha \)), uniformly in \( i \) with \( n_1 \leq i \leq n_2/x_n \),

\[
\lim_{n \to \infty} \frac{1}{n_2 \ln n} \sum_{i=n_1}^{n_2/x_n} \frac{1}{i} = t_2 - t_1,
\]

we deduce from the uniform \( L^\alpha \)-convergence in (42) that

\[
\frac{1}{n_2 \ln n} \sum_{i=n_1}^{n_2/x_n} \mathbb{E} \left[ X_i(n_2)^{\alpha} \right] = (t_2 - t_1) \frac{1 - p}{b + 1} \mathbb{E}[Z_1^\alpha].
\]

It remains to show that the variance of the sum \( \sum_{i=n_1}^{n_2/x_n} X_i(n_2)^{\alpha} \) is of order \( o(n_2^2 \ln^2 n) \). Using independence, we obtain with Corollary 4.2

\[
\text{Var} \left( \sum_{i=n_1}^{n_2/x_n} X_i(n_2)^{\alpha} \right) = \sum_{i=n_1}^{n_2/x_n} \text{Var} \left( X_i(n_2)^{\alpha} \right) \lesssim \sum_{i=n_1}^{n_2} \mathbb{E} \left[ X_i(n_2)^4 \right]^{2\alpha/4} \lesssim \sum_{i=n_1}^{n_2} \left( \frac{n_2}{i} \right)^2,
\]

and the right hand side of (43) is in fact of order \( o(n_2^2) \) as \( n \to \infty \).

**Remark 9.** If in the critical case \( \alpha = 2 \), we deduce from (37) that the limiting process \( X \) is a Brownian motion scaled by the factor

\[
\sqrt{2 \frac{1 - p}{b + 1} \mathbb{E}[Z_1^2]}.
\]

The choice \( \alpha = 2 \) implies \( b = 1 - 2p \) (which is, of course, only possible if \( p \leq 1/2 \)). From what we know about \( Z_1 \), we see that the above scaling factor simplifies to \( \sqrt{\frac{4p}{1-p}} \). Up to a factor \( \sqrt{2} \), this is exactly what we find in the critical case for the strongly reinforced ERW, see Theorem 3.2 (b). The factor \( \sqrt{2} \) comes from the fact that if \( \alpha = 2 \), the steps of the shark are normally distributed with variance 2 (and not 1).
4.4.3 The supercritical case $\alpha \kappa > 1$

We recall from Corollary 4.3 that the random variables $Z_i, i \in \mathbb{N}$, are defined as the $L^2$-limits of $|C_{i,n}|/n^\kappa$ upon letting $n \to \infty$, where $|C_{i,n}|$ denotes the size of the percolation cluster of $T_{n}^{(p)}$ rooted at node $i$, with $|C_{i,n}| = 0$ if there is no such cluster. Clearly, it holds that

$$\sum_{i=1}^{n} |C_{i,n}| \xi_i = d \sum_{i=1}^{n} |C_{i,n}| \xi_i,$$

and for proving the following theorem, we will define $S_n$ via the right hand side of (44), i.e., we set $S_n := \sum_{i=1}^{n} |C_{i,n}| \xi_i$. As we will see, the random variable

$$Z := \sum_{i=1}^{\infty} Z_i \xi_i$$

appears in the limit of the strongly reinforced SRS defined in the above way.

**Theorem 4.3.** Let $p \in (0, 1), b \geq 0, \kappa = (b + p)/(b + 1)$, and $t \geq 0$. Let $\alpha$ satisfy $1/\kappa < \alpha \leq 2$, and let $Z$ be given by (45). Then $|Z| < \infty$ almost surely, and we have the convergence in probability

$$\lim_{n \to \infty} \frac{S_{\lfloor tn \rfloor}}{n^\kappa} = t^\kappa Z.$$

**Proof.** Obviously, we may suppose that $t = 1$. Since conditionally on $Z_1, \ldots, Z_n$, $\sum_{i=1}^{n} Z_i \xi_i = d (Z_1^\alpha + \ldots + Z_n^\alpha)^{1/\alpha} \xi_1$, we deduce from (28) that $|Z| < \infty$ almost surely. Now let $F : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. We have

$$\mathbb{E} \left[ F \left( \frac{S_n}{n^\kappa} - Z \right) \right] = \mathbb{E} \left[ F \left( \sum_{i=1}^{\infty} \left( \frac{|C_{i,n}|}{n^\kappa} - Z_i \right) \xi_i \right) \right]$$

$$= \mathbb{E} \left[ F \left( \left( \sum_{i=1}^{\infty} \left| \frac{|C_{i,n}|}{n^\kappa} - Z_i \right|^{\alpha} \right)^{1/\alpha} \xi \right) \right],$$

where $\xi = d \xi_1$, and the last equality follows from conditioning on $|C_{1,n}|, \ldots, |C_{n,n}|$ and on the sequence $(Z_i, i \in \mathbb{N})$. In particular, the theorem follows once we have established that

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} \mathbb{E} \left[ \left| \frac{|C_{i,n}|}{n^\kappa} - Z_i \right|^{\alpha} \right] = 0. \quad (46)$$

To that aim, we first observe from Corollary 4.3 that for each fixed $i \geq 1$ and $0 < \alpha \leq 2$,

$$\lim_{n \to \infty} \mathbb{E} \left[ \left| \frac{|C_{i,n}|}{n^\kappa} - Z_i \right|^{\alpha} \right] = 0.$$
It only remains to argue why the order of sum and limit in (46) can be interchanged. In this regard, it suffices to show that the sum in (46) is bounded uniformly in \( n \). First, for \( 1 \leq i \leq n \), we estimate
\[
E \left[ \left| C_{i,n} \right|^\alpha \right] \leq 2^\alpha \left( E \left[ \left| C_{i,n} \right|^\alpha \right] + E[Z_i^\alpha] \right).
\]
The sum over the terms \( E[Z_i^\alpha] \) is bounded, see (28).

For the sum over the terms \( E \left[ \left| C_{i,n} \right|^\alpha \right] / n^{\alpha \kappa} \), the equality in law (27) gives for \( i \geq 2 \)
\[
E \left[ \left| C_{i,n} \right|^\alpha \right] = (1 - p) \sum_{\ell=1}^{n-i+1} E \left[ \left| C_{1,\ell} \right|^\alpha \right] P(\eta(n,i) = \ell).
\]
For \( \ell \) sufficiently large, say \( \ell \geq \ell_0 \), we have by Proposition 4.1
\[
E \left[ \left| C_{1,\ell} \right|^\alpha \right] \leq \ell_0^{1+\alpha} + E[\eta(n,i)^{\alpha \kappa}] .
\]
Using \( \alpha \kappa \leq 2 \) and (25) for the second moment of \( \eta(n,i) \), we deduce that
\[
E \left[ \left| C_{i,n} \right|^\alpha \right] \lesssim \ell_0^{1+\alpha} + \alpha \kappa \frac{\log(n)}{n^{\alpha \kappa}}.
\]
Since \( \alpha \kappa > 1 \), this implies that the sum \( \frac{1}{n^{\alpha \kappa}} \sum_{i=1}^n E \left[ \left| C_{i,n} \right|^\alpha \right] \) is uniformly bounded in \( n \), which proves what was left to show.

\[ \square \]

A Appendix

Proof of Lemma 4.1 The fact that \( (Y(t), t \geq 0) \) is a pure birth process with the stated properties is a consequence of (17) and of the dynamics of \( (T(t), t \geq 0) \). Standard properties of branching processes (see, e.g., [3]) show that \( (e^{-(b+1)t}Y(t), t \geq 0) \) is a square-integrable martingale, and it follows from Lemma 3 in [11] that its (a.s. and \( L^2 \))-limit is Gamma\((1/(b+1), 1/(b+1))\)-distributed.

Proof of Lemma 4.2 The i.i.d. property of the processes \( (Y_1^{(p)}(b_i + \cdot), t \geq 0), i \geq 1 \), is obvious from the construction. We shall therefore prove everything for \( i = 1 \), in which case \( b_i = b_1 = 0 \).

Clearly, the sum of degrees of vertices of \( T_1^{(p)}(t) \) is equal to
\[
2(|T_1^{(p)}(t)| - 1) + H_1^{(p)}(t).
\]
It now follows from the construction of the preferential attachment tree \( T(t) \) at the beginning of Section 4.2 (recall in particular the parameters of the exponential clocks) that \( (Y_1^{(p)}(t), t \geq 0) \)
is a pure birth process with the stated birth rate and reproduction law. It is then well-known (see again [3]) that $(e^{-t}Y(t), t \geq 0)$ is a martingale, whose terminal value $W_1$ is almost surely strictly positive. By Kolmogorov’s forward equation (see once more [3]) we compute for $t > 0$

$$
\mathbb{E} \left[ Y_1^{(p)}(t) \right] = e^{(b+p)t}, \quad \mathbb{E} \left[ Y_1^{(p)}(t)^2 \right] = \frac{(b+1)(b+2p)}{b+p} \left( e^{2(b+p)t} - e^{(b+p)(b_1+t)} \right).
$$

This proves square-integrability of $(e^{-(b+p)t}Y_1^{(p)}(t), t \geq 0)$, and the claim about the first and second moment of $W_1$ follows from the last display.

It remains to show boundedness in $L^k$ for $k \geq 3$, that is, we have to show that there exists a constant $c_k < \infty$ such that

$$
\mathbb{E} \left[ Y_1^{(p)}(t)^k \right] \leq c_k e^{k(b+p)t} \quad \text{for all } t \geq 0. \quad (47)
$$

In order to prove this, we adapt [3, Proof of Lemma 3] to our situation. First, we note that the generator $\mathfrak{G}$ of $(Y_1^{(p)}(t), t \geq 0)$ is given for any smooth function $f : (0, \infty) \to \mathbb{R}$ by

$$
\mathfrak{G}f(x) = x(1-p)(f(x+b) - f(x)) + xp(f(x+b+1) - f(x)).
$$

Specifying to $f(x) = x^\ell$ for some integer $\ell \geq 3$,

$$
\mathfrak{G}f(x) = x(1-p) \sum_{j=0}^{\ell-1} \left( \begin{array}{c} \ell \\ j \end{array} \right) x^j b^{\ell-j} + xp \sum_{j=0}^{\ell-1} \left( \begin{array}{c} \ell \\ j \end{array} \right) x^j (b+1)^{\ell-j}
\quad = \ell(b+p)x^\ell + (1-p) \sum_{j=0}^{\ell-2} \left( \begin{array}{c} \ell \\ j \end{array} \right) x^{j+1} b^{\ell-j} + p \sum_{j=0}^{\ell-2} \left( \begin{array}{c} \ell \\ j \end{array} \right) x^{j+1} (b+1)^{\ell-j}. \quad (48)
$$

We prove now by induction that (47) holds for all $k \in \mathbb{N}$. We already know it for $k = 1$ and $k = 2$, so let us assume that for some $\ell \geq 3$, (47) holds for all $k = 1, \ldots, \ell-1$. Kolmogorov’s forward equation reads

$$
\frac{d}{dt} \mathbb{E} \left[ f(Y_1^{(p)}(t)) \right] = \mathbb{E} \left[ \mathfrak{G} f(Y_1^{(p)}(t)) \right].
$$

In combination with (48), and using (47) for $k = 1, \ldots, \ell-1$, we deduce that for some $\gamma > 0$ depending on $b$, we have

$$
\frac{d}{dt} \ln \mathbb{E} \left[ Y_1^{(p)}(t)^\ell \right] \leq (b+p)\ell + \gamma \frac{e^{(\ell-1)(b+p)t}}{\mathbb{E} \left[ Y_1^{(p)}(t) \right]}.
$$

By Jensen’s inequality,

$$
\mathbb{E} \left[ Y_1^{(p)}(t)^\ell \right] \geq e^{\ell(b+p)t} \quad \text{for all } t \geq 0,
$$

so that
\[ \int_0^\infty \frac{e^{(\ell-1)(b+p)t}}{\mathbb{E}[Y_i(t)^\ell]} \, dt \leq \int_0^\infty e^{-(b+p)t} \, dt = \frac{1}{b+p}. \]

Going back to (49) and integrating, we obtain
\[ \mathbb{E}[Y_i(t)^\ell] \leq e^{\gamma_1 \frac{1}{b+p} e^{(b+p)t}} \text{ for all } t \geq 0. \]

Thus, (47) does hold for \(k = \ell\) as well, as wanted. \(\square\)

**Proof of Lemma 4.3.** We fix a small \(\varepsilon > 0\) and a sequence \((x_n, n \in \mathbb{N})\) of positive integers with \(\lim_{n \to \infty} x_n = \infty\) and \(x_n \leq n\). Recalling Lemma 4.1 and the notation from there, we define for each \(k \in \mathbb{N}\) the event
\[ E^1_k := \left\{ W(1-\varepsilon) \leq e^{-(b+1)\tau_k} \left( (b+1)k - b \right) \leq W(1+\varepsilon) \right\}. \]

Lemma 4.1 ensures that \(\lim_{n \to \infty} \mathbb{P}(\bigcap_{k=x_n}^\infty E^1_k) = 1\). On \(E^1_k\), it holds for \(k\) sufficiently large that
\[ \tau_k \leq \frac{1}{b+1} (\ln k - \ln W + \ln(b+1) - \ln(1-\varepsilon)), \]
\[ \tau_k \geq \frac{1}{b+1} (\ln k - \ln W + \ln(b+1) - 2\ln(1+\varepsilon)). \tag{50} \]

Writing \(D(k)\) for the number of subtrees present at time \(\tau_k\), i.e.,
\[ D(k) = \max \left\{ i \geq 1 : T_i^{(p)}(\tau_k) \neq \emptyset \right\}, \]
we deduce from the construction of \(T^{(p)}(t)\) that \(D(k)\) has the same law as \(1 + \sum_{i=1}^{k-1} \epsilon_{i,1-p}\), where \(\epsilon_{i,1-p}, i \geq 1\), are i.i.d. Bernoulli random variables with success probability \(1-p\). Consequently, an application of the law of large numbers shows that if we define
\[ E^2_k := \left\{ k(1-p)(1-\varepsilon) \leq D(k) \leq k(1-p)(1+\varepsilon) \right\}, \]
then \(\lim_{n \to \infty} \mathbb{P}(\bigcap_{k=x_n}^\infty E^2_k) = 1\). On \(E^2_k\) it holds by construction that
\[ b_{\lceil k(1-p)(1+\varepsilon) \rceil} \geq \tau_k. \]

Using (50), we find that on the event \(E^1_k \cap E^2_k\), for \(k\) large enough and provided \(\varepsilon\) is sufficiently small,
\[ b_k \geq \tau_{\lfloor k(1-p)(1+\varepsilon) \rfloor} \geq \frac{1}{b+1} (\ln(k-1) - \ln W + \ln(b+1) - \ln(1-p) - 3\ln(1+\varepsilon)). \]

Letting
\[ E_n := \bigcap_{k=x_n}^\infty (E^1_k \cap E^2_k), \]
34
we have by the properties of $E_k^1$ and $E_k^2$ that $\lim_{n \to \infty} \mathbb{P}(E_n) = 1$.

On the event $E_n$, it holds by construction that for all $n$ large and $i$ with $x_n \leq i \leq n$,

$$\tau_n - b_i \leq \frac{1}{b+1} (\ln n - \ln(i-1) + \ln(1-p) + 3\ln(1+\varepsilon) - \ln(1-\varepsilon)).$$

Entirely similar, one sees that on $E_n$

$$\tau_n - b_i \geq \frac{1}{b+1} (\ln n - \ln(i+1) + \ln(1-p) + 2\ln(1-\varepsilon) - 2\ln(1+\varepsilon)).$$

Now notice that

$$\max\{3\ln(1+\varepsilon) - \ln(1-\varepsilon), 2\ln(1+\varepsilon) - 2\ln(1-\varepsilon)\} \downarrow 0$$

if $\varepsilon \downarrow 0$. Since $\varepsilon > 0$ can be chosen arbitrarily small, we can clearly construct a sequence $(\varepsilon_n)$ with $\varepsilon_n \downarrow 0$ such that on $E_n$, the stated bounds hold.

\[\square\]

**Acknowledgments.** I warmly thank Silvia Businger for explaining her work and for her help, and Jean Bertoin for valuable comments.

**References**

[1] Alves, G. A., de Araújo, Cressoni, J. C., da Silva, L. R., da Silva, and M. A. A., Viswanathan, G.M. Superdiffusion driven by exponentially decaying memory. *Journal of Statistical Mechanics: Theory and Experiment*. Volume 2014 (2014).

[2] Athreya, K. A., Karlin, S. Embedding of urn schemes into continuous time Markov branching processes and related limit theorems. *Ann. Math. Statist.* 39 (1968), 1801–1817.

[3] Athreya, K.B., Ney, P.E. *Branching Processes*. Dover Books on Mathematics (2004).

[4] Barabási, A.-L., Albert, R. Emergence of scaling in random networks. *Science* 286 (1999), 509–512.

[5] Baur, E., Bertoin, J. Elephant random walks and their connection to Pólya-type urns. *Phys. Rev. E* 49 052134 (2016).

[6] Bercu, B. A martingale approach for the elephant random walk. *J. Phys. A: Math. Theor.* 51 015201 (2017).

[7] Bercu, B., Laulin, L. On the Multi-dimensional Elephant Random Walk. *J. Stat. Phys.* 175(6) (2019), 1146–1163.
[8] Bertoin, J. Noise reinforcement for Lévy processes. Preprint, arXiv:1810.08364 (2018).

[9] Bertoin, J. A Version of Herbert A. Simons Model With Slowly Fading Memory and Its Connections to Branching Processes. J. Stat. Phys. 176(679) (2019).

[10] Bertoin, J., Uribe Bravo, G. Supercritical percolation on large scale-free random trees. Ann. Appl. Probab. 25-1 (2015), 81–103.

[11] Bertoin, J., Goldschmidt, C. Dual Random Fragmentation and Coagulation and an Application to the Genealogy of Yule Processes. Mathematics and Computer Science III (2012).

[12] Businger, S. The Shark Random Swim (Lévy Flight with Memory). J. Stat. Phys. 172(3) (2018), 701–717.

[13] Businger, S. Nested Occupancy Scheme in a Random Environment and a Lévy-flight with Memory. PhD Thesis, University of Zurich (2018).

[14] Coletti, C. F., Gava, R., and Schütz, G. M. Central Limit Theorem for the Elephant Random Walk. J. Math. Phys. 58(5) (2017).

[15] Coletti, C. F., Gava, R., and Schütz, G. M. A strong invariance principle for the elephant random walk. J. Stat. Mech. Theory Exp. 12 (2017), 123207.

[16] Cotar, C., Thacker, D. Edge- and vertex-reinforced random walks with super-linear reinforcement on infinite graphs. Ann. Probab. 45(4) (2017), 2655–2706.

[17] Diaconis, P. and Rolles, S.W.W Bayesian analysis for reversible Markov chains. Ann. Statist. 34(3) (2006), 1270–1292.

[18] Gut, A., Stadtmüller, U. Variations of the elephant random walk. Preprint, arXiv:1812.01915 (2018).

[19] Janson, S. Functional limit theorems for multitype branching processes and generalized Pólya urns. Stoch. Proc. Appl. 110(2) (2004), 177–245.

[20] Kürsten, R. Random recursive trees and the elephant random walk. Phys. Rev. E 93 032111 (2016).

[21] Mahmoud, H. Pólya Urn Models. CRC Press, Boca Raton FL (2009).

[22] Mailler, C., Uribe Bravo, G. Random walks with preferential relocations and fading memory: a study through random recursive trees. Preprint, arXiv:1810.02735 (2018).

[23] Mailler, C., Marckert, J.-F. Measure-valued Pólya processes. Electron. J. Probab. 22(26) (2017), 33 pp.
[24] Metzler, R., Klafter, J. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. 339 (2000), 1–77.

[25] Oliveria, F. A., Ferreira, R. M. S., Lapas, L. C., and Vainstein, M. H. Anomalous Diffusion: A Basic Mechanism for the Evolution of Inhomogeneous Systems. Front. Phys. 19 (2019), 18 pp.

[26] Paraan, F. N. C., Esguerra, J. P. Exact moments in a continuous time random walk with complete memory of its history. Phys. Rev. E 74, 032101 (2006).

[27] Pemantle, R. A survey of random processes with reinforcement. Probab. Surv. 4 (2007), 1–79.

[28] Sabot, C., Tarrès, P. Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model. JEMS 17(9) (2015), 2353–2378.

[29] Sabot, C., Zeng, X. A random Schrödinger operator associated with the Vertex Reinforced Jump Process on infinite graphs. J. Amer. Math. Soc. 32 (2019), 311–349.

[30] Samorodnitsky, G., Taqqu, M. S. Stable Non-Gaussian Random Processes: Stochastic models with infinite variance. Chapman and Hall/CRC (2000).

[31] Schütz, G. M., Trimper, S. Elephants can always remember: Exact long-range memory effects in a non-Markovian random walk. Phys. Rev. E 70 045101(R) (2004).

[32] Silver, D., Huang, A. et al. Mastering the game of Go with deep neural networks and tree search. Nature 529 (2016), 484–489.