OLD RECURRENCE FORMULAE FOR GROWTH SERIES OF COXETER GROUPS

JAN DYMARA (WROCŁAW)

Abstract. Several classical formulae for the growth series of a Coxeter group are proved in a new way, using the structure of the Coxeter complex, the Davis complex, or the Tits non-complex.

1. Introduction

Throughout this paper \((W, S)\) is a Coxeter system. This means that \(W\) is the group generated by the finite set \(S\), subject to relations of the form \((st)^{m_{st}} = 1\), where \(m_{ss} = 1\) and for \(t \neq s\) we have \(m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}\) (\(\infty\) means no relation). We will usually say “Coxeter group \(W\)” instead of mentioning the whole system. We denote by \(\ell(w)\) the word length of an element \(w \in W\) with respect to the generating set \(S\). For any subset \(T \subseteq S\) the subgroup \(W_T\) of \(W\) generated by \(T\) is also a Coxeter group, with relations being the relevant relations of \(W\). The word length in \(W_T\) agrees with the restriction to \(W_T\) of the word length of \(W\); we denote both by \(\ell\).

The growth series of \(W\) is the formal power series \(W(t) = \sum_{w \in W} t^{\ell(w)}\). We will prove several well-known formulae for this function. For infinite \(W\):

\[
\sum_{T \subseteq S} \frac{(-1)^{|T|}}{W_T(t)} = 0.
\]

If \(W\) is finite, it has a unique element of longest length \(m\). Then

\[
\sum_{T \subseteq S} \frac{(-1)^{|T|}}{W_T(t)} = \frac{t^m}{W(t)}.
\]

A subset \(T \subseteq S\) is called spherical if \(W_T\) is finite. We denote by \(S\) the set of all spherical subsets of \(S\). Let \(\chi_T = \sum_{U \subseteq S} (-1)^{|U|}\). Then

\[
\sum_{T \in S} \frac{(-1)^{|T|}\chi_T}{W_T(t)} = \frac{1}{W(t)};
\]

\[
\sum_{T \in S} \frac{(-1)^{|T|}}{W_T(t)} = \frac{1}{W(t^{-1})}.
\]

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These formulae are classical and quite popular, cf. [Bou, Ex. IV.26], [D, Chapter Seventeen], [ChD]; variants of Coxeter group growth series are investigated even nowadays, cf. [OS], [BGM, Prop. 8.3]. The usual proofs use an inclusion–exclusion principle for appropriate subsets of \( W \), or induction on the cardinality of \( S \). Our goal is to use spaces on which \( W \) acts to establish (1), (2), (3) and (4) in an almost uniform way.

2. SOME PROPERTIES OF COXETER GROUPS

The basic reference is [Bou], a newer one [D].

- For any \( T \subseteq S \), and any \( w \in W \), the coset \( wW_T \) has a unique shortest element, say \( u \). Moreover, if \( w = uw' \), then \( \ell(w) = \ell(u) + \ell(w') \). ([D, Lemma 4.3.1])

- If \( W \) is finite, then it has a unique element \( w_0 \) of longest length \( m = \ell(w_0) \). Then, for any \( w \in W \) we have \( \ell(w) + \ell(w^{-1}w_0) = \ell(w_0) \). ([D, Lemma 4.6.1]). Consequently, in \( W \) there are as many elements of length \( k \) as of length \( m - k \). More succinctly, \( W(t) = t^m W(t^{-1}) \).

- For any \( w \in W \), the set \( \text{In}(w) = \{ s \in S \mid \ell(ws) < \ell(w) \} \) is spherical. ([D, Lemma 4.7.2]).

3. BASIC SPACES

The basic construction ([D], Chapter Five) is performed as follows. Let \( Y \) be a space (topological, or a simplicial complex) with a collection of subspaces \((Y_s)_{s \in S}\) (called panels). For \( y \in Y \) we put \( S(y) = \{ s \in S \mid y \in Y_s \} \). Then \( \mathcal{U}(W, Y) = W \times Y / \sim \), where \((w, y) \sim (w', y') \iff y = y' \) and \( w^{-1}w' \in W_{S(y)} \). The image of \( \{w\} \times Y \) in \( \mathcal{U}(W, Y) \) will be called a chamber and denoted \( wY \). The set of all chambers, \( \text{Ch} (\mathcal{U}(W, Y)) \), is in bijective correspondence with \( W \), provided \( Y \neq \bigcup_{s \in S} Y_s \). Via this bijection the length function is transported to the set of chambers: \( \ell(wY) = \ell(w) \). The \( W \)-action on \( \mathcal{U}(W, Y) \) is defined by \( w(u, y) = (wu, y) \). For \( T \subseteq S \) we put \( Y_T = \bigcap_{t \in T} Y_t \) (we call it the face of type \( T \)) and \( Y^T = \bigcup_{t \in T} Y_t \); this notation and terminology equivariantly extends to all chambers. We will use the notation \( \sigma < X \) to indicate that \( \sigma \) is a simplex of a simplicial complex \( X \).

We will use three spaces arising from this construction.

1. The Coxeter complex \( X_\Delta = \mathcal{U}(W, \Delta) \), where \( \Delta \) is an \((|S| - 1)\)-dimensional simplex, and \((\Delta_s)_{s \in S}\) is the collection of its codimension-1 faces.

2. The Davis complex \( X_D = \mathcal{U}(W, D) \). The chamber model \( D \) (the Davis chamber) is the subcomplex of the first barycentric subdivision \( \Delta' \) of \( \Delta \) spanned by the barycentres of faces of \( \Delta \) that are of spherical type (for
finite $W$ one should include also the barycentre of the empty face—we shall not consider this case). Alternatively, it is the subcomplex $X_D$ of the first barycentric subdivision $X'_{\Delta}$ spanned by barycentres of simplices of $X_{\Delta}$ that are contained in finitely many chambers. It is a locally finite complex. The panels are $D_s = D \cap \Delta_s$, and the faces $D_T = D \cap \Delta_T$. An important property: for spherical $T$ the face $D_T$ is contractible; indeed, it is a cone with apex at the barycentre of $\Delta_T$. For non-spherical $T$ we have $D_T = \emptyset$. For any simplex $\sigma < D$ we define its type $S(\sigma) = \{ s \in S \mid \sigma < D_s \}$; this definition extends by invariance to all simplices in $X_D$.

3. The Tits non-complex $X'_{\Delta} = \mathcal{U}(W, \Delta')$. Here $\Delta'$ is obtained from $\Delta$ by removing the closed simplices whose types are non-spherical. It is not a simplicial complex. For faces $\sigma$ of $\Delta$ of spherical type we still use the notation $\sigma < \Delta'$, even though, in the geometric realisation, some part of $\sigma$ may not be a subset of $\Delta'$.

For infinite $W$, all of the above spaces are contractible; $X_D$ and $X'_{\Delta}$ even carry good CAT(0) metrics. Though they are generally very useful, these properties will not be relied upon in this note.

4. **Euler characteristic generating function**

Let $X$ be either the Coxeter complex or the Davis complex for the Coxeter system $(W, S)$. (The case of the Tits non-complex is slightly different and will be dealt with in the last section.) For a simplex $\sigma < X$ we define its length $\ell(\sigma)$ as the minimum of $\ell(C)$, where $C$ runs through the chambers of $X$ that contain $\sigma$. There are finitely many chambers $C$ of $X$ of any given length, and each of them contains finitely many simplices. Therefore the following formal power series—the Euler characteristic generating function of $X$—is well defined:

$$
\chi_t(X) = \sum_{\sigma < X} (-1)^{\dim \sigma} t^{\ell(\sigma)}.
$$

We will calculate this series in two ways: grouping simplices according to type, and by local summation in $X$. Comparing the results we will obtain the formulae (1),(2) and (3).

5. **Summation according to type**

We consider $X = \mathcal{U}(W, Y)$ for $Y = \Delta$ or $D$. Let $\sigma < X$ be a simplex of type $T$. Then the chambers $C$ that contain $\sigma$ correspond to elements of some coset $uW_T$. Assume that $u$ is the shortest element in that coset. Then
\[ \ell(\sigma) = \ell(u), \] and

\[ \sum_{C > \sigma} t^{\ell(C)} = \sum_{w \in uW_T} t^{\ell(u)} = \sum_{v \in W_T} t^{\ell(u)} t^{\ell(v)} = t^{\ell(u)} W_T(t) = t^{\ell(\sigma)} W_T(t). \]

Multiplying both sides by \((-1)^{\dim \sigma}\) and summing over all simplices of type \(T\) we get

\[ \sum_{C \in \text{Ch}(X)} t^{\ell(C)} \sum_{\sigma < C \atop S(\sigma) = T} (-1)^{\dim \sigma} = W_T(t) \sum_{\sigma < X \atop S(\sigma) = T} (-1)^{\dim \sigma} t^{\ell(\sigma)}, \]

which implies

\[ \frac{W(t)}{W_T(t)} \sum_{\sigma < Y \atop S(\sigma) = T} (-1)^{\dim \sigma} = \sum_{\sigma < X \atop S(\sigma) = T} (-1)^{\dim \sigma} t^{\ell(\sigma)}. \]

Finally, we sum over all \(T \subseteq S\):

\[ W(t) \sum_{T \subseteq S} \frac{1}{W_T(t)} \sum_{\sigma < Y \atop S(\sigma) = T} (-1)^{\dim \sigma} = \chi_t(X). \]

We will calculate the inner sum separately in each case.

The Coxeter complex. For any proper subset \(T \subset S\) there is one \(\sigma < \Delta\) of type \(T\), namely \(\Delta_T\). It has dimension \(|S| - |T| - 1\). Thus

\[ \chi_t(X_\Delta) = W(t) \sum_{T \subset S} \frac{(-1)^{|S| - |T| - 1}}{W_T(t)}. \]

The Davis complex. Recall that the face \(D_T\) is a cone with apex at the barycentre of \(\Delta_T\); let \(L'_T\) be its base \(D_T \cap \partial \Delta_T\). The simplices \(\sigma < D\) of type \(T\) are the interior simplices of this cone (including the apex), so that

\[ \sum_{\sigma < D \atop S(\sigma) = T} (-1)^{\dim \sigma} = 1 - \chi_t(L'_T). \]

Simplices in \(L'_T\) correspond to chains \(U_1 \subset \ldots \subset U_k\), where \(U_i\) are spherical and properly contain \(T\). This means that \(L'_T\) is, as the notation suggests, the barycentric subdivision of the following simplicial complex \(L_T\). The vertices of \(L_T\) correspond to spherical sets of the form \(T \cup \{u\}\) (with \(u \in S - T\)); vertices \(T \cup \{u_1\}, \ldots, T \cup \{u_k\}\) span a simplex if \(T \cup \{u_1, \ldots, u_k\}\) is spherical. Thus, simplices of \(L_T\) correspond to spherical \(U\) containing \(T\), and

\[ 1 - \chi_t(L'_T) = 1 - \chi(L_T) = \sum_{T \subseteq U \subseteq S} (-1)^{|U| - |T|} = (-1)^{|T|} \chi_T \]

for \(\chi_T = \sum_{T \subseteq U \subseteq S} (-1)^{|U|}. \) Finally, returning with this to (9) we get

\[ \chi_t(X_D) = W(t) \sum_{T \subseteq S} \frac{(-1)^{|T|} \chi_T}{W_T(t)}. \]
6. Local summation

We continue with the cases $Y = \Delta$ or $D$. Now we compute the sum $\sum_{\sigma < X} (-1)^{\dim \sigma} \ell(\sigma)$ by arranging its summands according to chambers: we consider each $\sigma$ as part of the chamber realising $\ell(\sigma)$. We get

\begin{equation}
\chi_t(X) = \sum_{C \in \text{Ch}(X)} \ell(C) \sum_{\sigma < C \atop \ell(\sigma) = \ell(C)} (-1)^{\dim \sigma} = \sum_{w \in W} \ell(w) \sum_{\sigma < wY \atop \ell(\sigma) = \ell(w)} (-1)^{\dim \sigma}.
\end{equation}

Lemma 6.1. Let $\sigma < wY$. Then $\ell(\sigma) < \ell(w)$ if and only if $\sigma < (wY)^{\text{In}(w)}$.

Proof. Let $T$ be the type of $\sigma$. The chambers containing $\sigma$ correspond to elements of the coset $wW_T$.

Suppose $\ell(\sigma) < \ell(w)$. It follows that $w$ is not the shortest element of the coset $wW_T$. Therefore $\ell(wt) < \ell(w)$ for some $t \in T$. This $t$ also belongs to $\text{In}(w)$. Finally, $\sigma < wY \cap wtY = (wY)_t$.

Conversely, if $\sigma < (wY)^{\text{In}(w)}$, then $\sigma < (wY)_s$ for some $s \in \text{In}(w)$. But then $\sigma < wsY$ and $\ell(\sigma) \leq \ell(ws) < \ell(w)$. The Lemma is proved. \qed

By the Lemma, the inner sum in (14) is performed over simplices of $wY$ that are not in $(wY)^{\text{In}(w)}$. Thus

\begin{equation}
\sum_{\sigma < wY \atop \ell(\sigma) = \ell(w)} (-1)^{\dim \sigma} = \chi(wY, (wY)^{\text{In}(w)}) = \chi(Y, Y^{\text{In}(w)})
\end{equation}

\[= \chi(Y) - \chi(Y^{\text{In}(w)}) = 1 - \chi(Y^{\text{In}(w)}),\]

the last equality due to contractibility of $Y$.

We split the calculation of $\chi(Y^{\text{In}(w)})$ into three cases.

1) The generic case: $\text{In}(w)$ is a proper non-empty subset of $S$, of some cardinality $k$. Notice that for any $T \subseteq \text{In}(w)$ (a fortiori, spherical) the face $Y_T$ is contractible. Using the inclusion–exclusion principle we get

\begin{equation}
\chi(Y^{\text{In}(w)}) = \sum_{s \in \text{In}(w)} \chi(Y_s) - \sum_{s, t \in \text{In}(w) \atop s \neq t} \chi(Y_{s,t}) + \ldots + (-1)^{k-1} \chi(Y_{\text{In}(w)})
\end{equation}

\[= k - \binom{k}{2} + \binom{k}{3} - \ldots + (-1)^{k-1} \frac{k}{k} = 1 - (1 - 1)^k = 1.\]

2) The set $\text{In}(w)$ is empty. Then $\chi(Y^{\text{In}(w)}) = \chi(\emptyset) = 0$. This happens exactly when $w = 1$.

3) The case $\text{In}(w) = S$. This happens only if $W$ is finite and $w$ is the element of longest length (say length $m$). Then $Y^S$ is a triangulation of the $(|S| - 2)$-dimensional sphere, $\chi(Y^S) = 1 - (-1)^{|S| - 1}$. 

Plugging these results into (15), and then into (14), we get for infinite $W$

(17) \[ \chi_t(X_\Delta) = 1, \quad \chi_t(X_D) = 1, \]

while for finite $W$ with element of longest length $m$

(18) \[ \chi_t(X_\Delta) = 1 + (-1)^{|S|-1} t^m. \]

Comparing these results with (10) and (13) we get (1), (2) and (3).

7. The Tits non-complex

Consider now the case $X = X_f^f$. For $\sigma < X$ we define $L(\sigma)$ as the maximum of $\ell(C)$ over all chambers containing $\sigma$. This is finite, since the type of $\sigma$ is spherical. Then we define $\chi^f(X) = \sum_{\sigma < X} (-1)^{\dim \sigma} t^{L(\sigma)}$. As before, we will calculate this sum in two ways.

First, we group the summands according to the type of $\sigma$. Let $\sigma < X$ be a simplex of (spherical!) type $T$. Then the chambers $C$ that contain $\sigma$ correspond to elements of some finite coset $uW_T$. Assume that $u$ is the longest element in that coset. Then $L(\sigma) = \ell(u)$, and

(19) \[ \sum_{C > \sigma} t^{\ell(C)} = \sum_{w \in uW_T} t^{\ell(w)} = \sum_{v \in W_T} t^{\ell(u)} t^{-\ell(v)} = t^{\ell(u)} W_T(t^{-1}) = t^{L(\sigma)} W_T(t^{-1}). \]

Multiplying both sides by $(-1)^{\dim \sigma} (= (-1)^{|S|-|T|-1})$ and summing over all simplices of type $T$ (there is just one in each chamber) we get

(20) \[ (-1)^{|S|-|T|-1} \sum_{C \in \text{Ch}(X)} t^{\ell(C)} = W_T(t^{-1}) \sum_{\sigma < X \atop S(\sigma) = T} (-1)^{\dim \sigma} t^{L(\sigma)}, \]

which implies

(21) \[ \frac{W(t)}{W_T(t^{-1})} (-1)^{|S|-|T|-1} = \sum_{\sigma < X \atop S(\sigma) = T} (-1)^{\dim \sigma} t^{L(\sigma)}. \]

Summing this equality over all spherical types $T$ we get

(22) \[ W(t) \sum_{T \in S} \frac{(-1)^{|S|-|T|-1}}{W_T(t^{-1})} = \chi^f(X_f^f). \]

Second, we group summands in chambers according to $L$:

(23) \[ \chi^f(X) = \sum_{\sigma < X} (-1)^{\dim \sigma} t^{L(\sigma)} = \sum_{w \in W} t^{\ell(w)} \sum_{\sigma < w \Delta f \atop L(\sigma) = \ell(w)} (-1)^{\dim \sigma}. \]
Every $\sigma < w^{\Delta f}$ is of the form $w^{\Delta T}$ for some spherical $T$; moreover, $L(\sigma) = \ell(w)$ if and only if $w$ is the longest element in the coset $wW_T$, which happens exactly when $T \subseteq \text{In}(w)$. Therefore

$$\sum_{\substack{\sigma < w^{\Delta f} \\ L(\sigma) = \ell(w)}} (-1)^{\dim \sigma} = \sum_{T \subseteq \text{In}(w)} (-1)^{|S| - |T| - 1} = (-1)^{|S| - 1}(1 - 1)^{|\text{In}(w)|}. \tag{24}$$

Thus, the only non-zero summand corresponds to $w$ with empty $\text{In}(w)$, that is to $w = 1$. We get

$$\chi^t(X_{\Delta f}^t) = (-1)^{|S| - 1}. \tag{25}$$

Comparing (25) with (22), and switching $t$ and $t^{-1}$, we get (4).

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Instytut Matematyczny
Uniwersytet Wrocławski
pl. Grunwaldzki 2/4,
50-384 Wrocław
Poland
dymara@math.uni.wroc.pl