Modeling the solution of some management problems using latent variables

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Abstract. Original approaches to the solution of some common management problems with the use of latent variables are proposed in the work. Calculations of the values of the latent variables in the Rasch model is based on the least squares technique thus allowing to perform computations with the use of standard application packages.

1. Introduction
At present time latent variables and methods of their measurements are more and more widely applied in different areas of economics, social investigations, education and so on. First of all, this is caused by the fact that researchers rather often face the situation when the examined factors either can not be measured directly or their measurement is based on some arbitrary choice of the measurement scale. (For example, a customer, choosing certain article from several uniform ones is permanently facing to solve the problem of correlation of the price for the product with its quantity and quality). In order to assign clearly specified numerical values for these factors just in the middle of 20-th century it was proposed to use latent variables. These variables had such names because, although they cannot be evaluated with the use of direct measurements but, nevertheless, their precise numerical values can be obtained after processing some measurable variables which are named as indicator ones. In particular, due to such a possibility of the objective determinaton of numerical values for the latent variables it becomes possible to assess quantitatively such indicators as a qualitative level, convenience, efficiency degree or attraction, dominancy and some others.

This work is aimed at the description of technique based on Rasch model, for the application of approach for the assessment of latent indices in order to solve some typical management problems.

Let us first consider some basic provisions of the classical dichotomy Rasch model [1, 2], which represent the basis for solution of some certain management problems.

The proposed model is based on the function which connects latent variables θᵢ and βⱼ, implying a sense of the estimates for the objects and assessing criteria, respectively. There is some probability that i-th object will be assessed by the estimate increasing the estimate of j-th criterion:
Latent indices $\beta$ and $\theta$ are dimensionless, and their measurement scale is divided in intervals and is linear.

One of the first areas of the practical application of Rasch model was education since the obtained estimates according to this model made it possible to realize objective measurements of knowledge of the trainees based on the tests. Let us assume that there are $n$ of the learning trainees who should perform $m$ test tasks. Basing on the results of the tests the measurement of some integral index $\theta$ is performed, that characterizes the learning curve of the tested person. At the same time some latent value $\beta$ is estimated characterizing a degree of complexity for the tests tasks.

In the classical dichotomy model the values of $x_{ij}$ of the indicator variables (results of the partial assessments for the persons learning in accordance with the test tasks) required for obtaining of the values of latent variables should be discrete ones implying that:

$$x_{ij} = \begin{cases} 1, & \text{if the tested person is positively assessed for a task } j; \\ 0, & \text{if the tested person is negatively assessed for a task } j. \end{cases}$$

The restraint, which means the requirement of the binarity for matrix (2), is connected with the computational procedures in the classical Rasch model that requires minimization of the likelihood function (MP-method, maximum likelihood estimation). Discreteness of the raw data limits the areas of the practical application of the model since the measurement scales of the indicator variables (2) prove to be continuous ones in many of the application problems. However, this drawback can be eliminated using the least squares technique (LST) for the calculations of the latent variables [3, 4]: $\theta_i$ and $\beta_j$. According to this approach with the use of LST the latent indices were found in such a way that the sum of squared deviations of the empirical measures (indicator variables $x_{ij}$) on the calculated probabilities $p_{ij}$, computed according to the formula (1), would tend to minimum. This condition can be expressed through solution of the optimization problem of the type:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - p_{ij})^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \left( x_{ij} - \frac{e^{\theta_i - \beta_j}}{1 + e^{\theta_i - \beta_j}} \right)^2 \rightarrow \text{min} \quad (3)$$

In such approach, the indicator variables can possess any continuous values from the interval $[0; 1]$. Due to the presence of latent variables in the form of their difference in (3) the reference points for $\theta_i$ and $\beta_j$ will be, in principle, undefined. In Rasch model, the reference point is set to the mean values of the latent variables. In terms of these arguments, expression (3) can be added by a system of limitations:

$$\sum_{i=1}^{m} \theta_i = 0; \quad \sum_{j=1}^{n} \beta_j = 0. \quad (4)$$

However, one can use the other normalization conditions of the desired latent indices, for example, their non-negativity. Besides the use of continuous indicator variables such model has one more important advantage: instead of minimization of rather complicated likelihood function as it was done while using minimization likelihood (ML) method, the solution is reduced to the minimization of relatively simple objective function (3) with the account of the limitations (4). This enables to find the numerical solution applying a large set of the widely-distributed software products, including that one in MS Excel with the use of “Search for solutions” superstructure (Solver) [5, 6].

Now we can proceed to the application of the described model for solving of the number of the typical management problems.
2. Problems of labor management

Now we consider latent variables theory based on the Rasch model as it can be used for the labor management. Let us assume that there is some project involving \( n \) kind of works. There is also a certain group of workers for the accomplishment of the project. Below we consider several models of organization for the system of distribution of the executives by the kinds of work:

1) Formation of subgroups: workers can associate in several groups with certain degree of responsibility which represents the quality of execution as of some separate operations as of the project as a whole. Then within these groups the roles of certain executives are determined.

2) Individual distribution of the works: each of the executors gets one certain work and vice versa, each executor is allocated for a certain work.

3) Group execution: each kind of work can be accomplished by several workers and every one of them has a certain part of work to do.

Let us consider the corresponding models.

Formation of subgroups:

Let us have a complex of \( n \) works that has to be done by \( n \) executors. At first, testing of the executors should be done allowing to estimate how far qualitatively they can accomplish each of the work \( j \). To do this we can use matrix of the continuous indicator variables: \( x_{ij} \) – which represent a quality or the probability of the execution of the task \( j \) by the executor \( i \). This matrix is processed by Rasch method based on LST. As a result, the values of latent parameters can be obtained: \( \theta_i \) – is a capability of the \( i \)-th executive to perform all set of works and \( \beta_j \) – difficulty of execution of \( j \)-th work over all group of the executives. Next, with the obtained values of \( \theta_i \) and \( \beta_j \) in accordance with the expression (1) the probability \( p_{ij} \) of the execution of \( j \)-th work by \( i \)-th executive can be calculated.

These estimates can be used for the determination of preferable executives and complexity of the works. Subgroups in principle can be formed from the executives with the best rating of activities. Nevertheless, for more precise solution of the assigned task mathematical model of a theory of the pair matrix games can be applied [7].

With the use of such approach a certain pair game is formed where the strategies of the first player are the capabilities of the workers concerning accomplishment of the project and the value of game equal to the probability of its accomplishment is maximized. A payoff matrix in this case will be a set of probabilities \( p_{ij} \), obtained from (1). This game can be realized by its reducing to the linear programming task. Let us introduce intermediate variables \( x_i \), \( i = 1,2,...,n \) and compile the task in the form of:

\[
\min \sum_{i=1}^{n} x_i \geq 0, \quad \sum_{j=1}^{m} p_{ij} x_i \geq 1, \quad j = 1,2,...,m; \\
x_i \geq 0, \quad i = 1,2,...,n.
\] (5)

As a result of solution for (5) and obtaining of the optimal values for the variables \( x_i^* \), it is possible to calculate the value of the game \( \nu = \left( \sum_{i=1}^{n} x_i^* \right)^{-1} \) and the probabilities of pure strategies: \( P_i = \nu x_i^* \). The value of game \( \nu \) in this case is interpreted as the probability of execution of the complex of works by all the group of executives, and the calculated probability \( P_i \) represents an optimal share of participation got the executive in the accomplishment of the work package.

The presented technique allows realization of the optimal division of the group of executives into subgroups. To make this, at the first stage the primary, most responsible subgroup of the executives is selected from the whole group and their strategies are the most active ones, meaning that the probabilities \( P_i \) are not equal to zero. At the second stage, a new game is formed again from the rest of
the executives according to the technique described above and then the problem of linear programming is solved. Its solution allows formation of the next subgroup with a less responsibility and so on. Executing the procedure for a desired number of times one can form labor employees with a decreasing labor quality.

Individual distribution of works:

It is a classical task on the assignments [7]: it is required to assign \( n \) executives to \( n \) jobs, so that every executive will be assigned one job.

The role of Rasch model in this case comprises in the fact that we shall use the probability matrix (1) as a gain matrix. Now let us determine the assignment matrix \( y_{ij} \), specified according to the following scheme:

\[
y_{ij} = \begin{cases} 1, & \text{if } i \text{-th executive will accomplish the task } j; \\ 0, & \text{if } i \text{-th executive will not accomplish the task } j. \\
\end{cases}
\]

Let us formulate the optimization task of the following kind:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} y_{ij} \to \max;
\]

\[
\begin{align*}
\sum_{j=1}^{n} y_{ij} &= 1, \ j = 1, 2, \ldots, n; \\
\sum_{i=1}^{n} y_{ij} &= 1, \ i = 1, 2, \ldots, n; \\
y_{ij} &\geq 0, \ y_{ij} - \text{integer}, \\
i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, n.
\end{align*}
\]

Solving (6) it is possible to find a scheme of the optimal arrangement of the executives over the jobs \( y_{ij} \) and basing on these considerations we can obtain the probabilities of the execution for all the works of the project which are selected from (1) over the positions where one can see units for the matrix \( y_{ij} \). Using theorems of summation and multiplication of the probabilities it is possible to determine the probability of accomplishment of the project as a whole basing on the probabilities of execution of separate jobs.

Group execution.

This problem differs from the previous one in that every worker can anyhow execute any work; therefore, the situation is possible when any work can be executed by the whole number of the executives but the role of everyone in this work will be only partial. It is necessary to determine the roles or the shares of participation for every of the workers in each operation in such a way that the aggregate efficiency of accomplishment of the project as a whole would be maximal one. In this case, the number of executives \( n \) should not be obligatory equal to the number of jobs \( m \).

In order to solve the problem assignment matrix \( y_{ij} \) is determined however, its elements will not be the binary ones as in the previous case but continuous variables from the unit scale. As it was previously indicated, probabilities (1) are used as the raw data. Model of the problem is expressed as:

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} y_{ij} \to \max;
\]

\[
\begin{align*}
\sum_{j=1}^{m} y_{ij} &= 1, \ j = 1, 2, \ldots, n; \\
\sum_{i=1}^{n} y_{ij} &= 1, \ i = 1, 2, \ldots, m; \\
y_{ij} &\geq 0, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m.
\end{align*}
\]
Maximum value of the objective function found from solution (7) can be considered as the sum of probabilities of the project accomplishment for all of its participants. If it is divided by the number of participants, then we obtain the estimate of the mean probability of the project accomplishment.

3. Problem of the optimal alternative choice
Assume that a decision making person should make a choice of the most attractive alternatives among \( n \) ones. Let us designate them as \( A_1, A_2, \ldots, A_n \), using \( m \) criteria for their assessment and designate them as \( C_1, C_2, \ldots, C_m \). Let \( U_{ij} \) be a partial estimate of the attraction degree for \( i \)-th alternative by \( j \)-th assessment criterion, normalized over segment \([0;1]\). They are used as the primary data for the calculation of latent variables \( \theta_i \) and \( \beta_j \), equal to the degree of attraction of the alternatives and the levels of criteria feasibility. Note, that for the latent variable \( \beta_j \) the following rule is actual: the less is its value then the more those alternatives in their aggregate satisfy this criterion.

For the described solution of the problem of choice from a set of alternatives we assumed that all of the criteria are of the equal importance. If it is not so, it is possible to introduce their statistical weights \( w_j \) (for example, once again within the scale from 0 to 1). It means that one should account for the weights of criteria. Finally, assessments of the alternatives \( u_{ij} \) are multiplied by their weights \( w_j \), and after that it is possible to use new estimates \( \hat{u}_{ij} = u_{ij}w_j \), accounting for the weights.

4. Assessment of the alternatives by the group of experts
Let us consider the situation, when \( n \) alternatives \( A_1, A_2, \ldots, A_n \) should be assessed by a group of \( m \) experts on the basis of some qualitative criterion. We shall use a degree of attraction for each alternative, obtained by the aggregate of the multiple experts, and \( \beta_j \) – is a degree of exactingness of \( j \)-th expert characterizing his requirements in the estimating of all the group of alternatives. In such approach expression (1) will determine the probability \( p_{ij} \) of assessment for \( i \)-th alternative higher than the level of exactingness for \( j \)-th expert. In this situation the latent variables are determined from the optimization task (3). A quantitative estimate of the alternative by the qualitative criterion in such approach is the latent variable \( \theta_i \).

5. Formation of the investment portfolio on the basis of expert validation
In principle, this problem only slightly differs from the previous one. Let it be \( n \) securities of \( A_1, A_2, \ldots, A_n \), which can be potentially included into investment portfolio. In order to assess each of them into portfolio the estimates of \( m \) experts are attracted which can be designated as \( B_1, B_2, \ldots, B_m \). Every expert must assess the expediency of inclusion for each of the security \( x_{ij} \) into portfolio within the unit scale: \((i – number of security, j – number of the expert)\). Since the indicator variables \( x_{ij} \) can take any of the values within the range from 0 to 1, they can be considered as the probability of inclusion of the corresponding security into portfolio of investments from the viewpoint of the corresponding expert.

Next, we shall determine latent indices: \( \theta_i \) – degree of attraction of security \( B_i \) from the viewpoint of expediency of its inclusion into portfolio; \( \beta_i \) – is a parameter characterizing exactness of expert \( A_i \) to all of the securities. To calculate the latent parameters one should solve (3) and then he obtains a degree of the investment attraction for securities \( \theta_i \), which is a basis for the inclusion of a security into portfolio. Its share \( q_i \) in portfolio is calculated because of normalization \( \theta_i \):

\[
q_i = \frac{\theta_i - \theta_{\min}}{\sum_{i=1}^{n} (\theta_i - \theta_{\min})}.
\]
where $\theta_{\text{min}}$ – is a minimum assessment among all of the aggregate of securities.

Estimates of parameters obtained for the presented problems and tasks with the use of the models will provide a number of advantages before the routine estimates of these parameters. These advantages are presented in [8, 9]. In conclusion, based on Rasch model with the calculations of the variables values on the ground of LST in the work we considered a possibility of utilizing latent variables for solution of some typical management problems. This model, in principle, can be also applied for the assessment of the quality of the accomplished works or production, efficiency of work and other qualitative indicators.

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