ON A NEW TWO-COMPONENT \emph{b} -FAMILY PEAKON SYSTEM WITH CUBIC NONLINEARITY

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Abstract. In this paper, we propose a two-component \emph{b} -family system with cubic nonlinearity and peaked solitons (peakons) solutions, which includes the celebrated Camassa-Holm equation, Degasperis-Procesi equation, Novikov equation and its two-component extension as special cases. Firstly, we study single peakon and multi-peakon solutions to the system. Then the local well-posedness for the Cauchy problem of the system is discussed. Furthermore, we derive the precise blow-up scenario and global existence for strong solutions to the two-component \emph{b} -family system with cubic nonlinearity. Finally, we investigate the asymptotic behaviors of strong solutions at infinity within its lifespan provided the initial data decay exponentially and algebraically.

1. Introduction. In this paper, we consider the following \emph{b} -family of two-component system with cubic nonlinearity:

\[
\begin{align*}
    m_t + uvm_x + bu_xvm &= 0, \\
    n_t + uvn_x + bnv_xn &= 0,
\end{align*}
\]

(1.1)

where \( m = u - u_{xx}, \ n = v - v_{xx}, \) and \( b \) takes an arbitrary value.

For \( b = 2 \) and \( v \equiv 1, \) the system (1.1) becomes the following celebrated Camassa-Holm (CH) equation \cite{3}

\[
m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx},
\]

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which models the unidirectional propagation of shallow water waves over a flat bottom. Here $u(t, x)$ stands for the fluid velocity at time $t$ in the spatial $x$ direction [3, 20, 31]. The CH equation is also recognized as a model for the propagation of axially symmetric waves in hyperelastic rods [16]. It has a bi-Hamiltonian structure [3, 24] and is completely integrable with algebro-geometric solutions on a symplectic submanifold [38]. Its solitary waves vanishing at both infinities are peaked solitons (peakons) [4], and they are orbitally stable [15]. It is also worth pointing out that the peakons replicate a feature that is characteristic for the waves of great height – waves of the largest amplitude that are exact traveling wave solutions of the governing equations for irrotational water waves, cf. [10, 43]. The Cauchy problem and initial boundary value problem for the CH equation have been studied extensively [7, 8, 17, 22]. It has been shown that this equation is locally well-posed [7, 8, 17, 42] for initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$. Moreover, it has both globally strong solutions [6–8] and blow-up solutions at a finite time [6–9]. On the other hand, it also has globally weak solutions in $H^1(\mathbb{R})$ [2, 14, 46]. In comparison with the KdV equation, the advantage of the CH equation lies in the fact that the CH equation has peakons and models wave breaking [4, 9] (namely, the wave remains bounded while its slope becomes unbounded in finite time [44]).

For $b = 3$ and $v \equiv 1$, the system (1.1) becomes another important integrable equation admitting peakons, that is the following well-known Degasperis-Procesi (DP) equation [19]

$$m_t + um_x + 3u_xm = 0, \quad m = u - u_{xx}.$$ 

The DP equation is regarded as another model for nonlinear shallow water dynamics [11, 13]. It was proved in [18] that the DP equation has a bi-Hamiltonian structure and an infinite number of conservation laws, and admits peakon solutions which are analogous to the CH peakons. The DP equation was already extended to a completely integrable hierarchy in a $3 \times 3$ matrix Lax pair, which possesses involutive representation of solutions under a Neumann constraint on a symplectic submanifold [39], and furthermore it was proven to have algebro-geometric solutions for such a $3 \times 3$ integrable system [30]. The Cauchy problem and initial boundary value problem for the DP equation have been studied extensively in [5, 22, 49, 50]. Although the DP equation is very similar to the CH equation in the aspects of integrability, particularly in the form of equation, there are some significant differences between these two equations. One of the remarkable features of the DP equation is that it has not only (periodic) peakon solutions [18, 50], but also (periodic) shock peakons [34]. Besides, the CH equation is a re-expression of geodesic flow on the diffeomorphism group [12], while the DP equation is regarded as a non-metric Euler equation [21].

For $b = 3$ and $v \equiv u$, the system (1.1) becomes the following Novikov equation which was proposed in [36]:

$$m_t + u^2m_x + 3uu_xm = 0, \quad m = u - u_{xx}.$$ 

We notice that the nonlinear terms in both CH and DP equations are quadratic with slightly different constant coefficients. However, the Novikov equation is an integrable peakon system with cubic nonlinearity. The first cubic nonlinear peakon system is the following FORQ equation (some times called the modified Camassa-Holm equation) [23, 37, 40]

$$m_t + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx},$$
which was shown integrable with Lax pair [40] and already generalized to a completely integrable hierarchy including both negative and positive flows with explicit peaked and cusped solutions [41]. The Novikov equation was found to be the second cubic system possessing peakon solutions. Moreover, the Novikov equation’s well-posedness, blow-up phenomena and global solutions have been studied extensively in [28, 29, 45].

In view of the significance of conservation laws in studying the blow-up phenomena and global existence, we shall summarize two useful conservation laws which are for the CH, DP, and Novikov equations in the Table 1. But unfortunately, the above two conservation laws (the $H^1(R)$ and $L^1(R)$ norms) are unavailable to the system (1.1).

For $b = 3$, the system (1.1) becomes the following Geng-Xue (GX) system (sometimes called the two-component Novikov system) which was proposed in [26]:

$$
\begin{align*}
\begin{cases}
m_t + uvu_x + 3u_x u_{xx} = 0, \\
n_t + uvv_x + 3v_x v_{xx} = 0,
\end{cases}
\end{align*}
$$

where $m = u - u_{xx}$, $n = v - v_{xx}$. The GX system (1.2) is an integrable generalization of the DP equation with cubic nonlinearity, and is associated with a $3 \times 3$ Lax pair to guarantee the integrability. This system also admits bi-Hamiltonian structure and regular single peakon solutions.

Let us now set up Cauchy problem for system (1.1) as follows:

$$
\begin{align*}
\begin{cases}
m_t + uvu_x + buu_x v_{xx} = 0, & t > 0, x \in \mathbb{R}, \\
n_t + uvv_x + bvv_x u_{xx} = 0, & t > 0, x \in \mathbb{R}, \\
m(0,x) = m_0(x), & x \in \mathbb{R}, \\
n(0,x) = n_0(x), & x \in \mathbb{R},
\end{cases}
\end{align*}
$$

(1.3)

Using the Green function $p(x) \triangleq \frac{1}{2} e^{-|x|} (x \in \mathbb{R})$ and the identity $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$, we can rewrite the system (1.3) as follows:

$$
\begin{align*}
\begin{cases}
u_t + (uv)u_x + p * I(u,v) = 0, & t > 0, x \in \mathbb{R}, \\
\nu_t + (uv)v_x + p * J(u,v) = 0, & t > 0, x \in \mathbb{R}, \\
u(0,x) = u_0(x), & x \in \mathbb{R}, \\
\nu(0,x) = v_0(x), & x \in \mathbb{R},
\end{cases}
\end{align*}
$$

(1.4)

where

$$
I(u,v) \triangleq \partial_x (uu_x v_x) + buu_x v_x + (uu_x)_x v_x - \frac{b - 3}{2} (u_x^2)_x v_x,
$$

and

$$
J(u,v) \triangleq \partial_x (u_x v_x v_x) + buv_x v_x + u_x (v_v)_x v_x - \frac{b - 3}{2} u(v_x^2)_x.
$$
By using an approach similar to the one in [47], the analytic solutions to the system (1.3) can readily be proved in both variables, globally in space and locally in time. However, the main goal of this paper is to discuss the peakon solutions of the system (1.1), study the blow-up phenomena and global existence for strong solutions to the system (1.3), and investigate the asymptotic behaviors of strong solutions at infinity within its lifespan.

In order to analyze the blow-up phenomena, here we may make good use of the fine structure of the system (1.3). Applying the transport equation theory and one-dimensional Morse type estimates, an important blow-up criterion is obtained. Then we exploit the characteristic ODE related to the system (1.3) to construct some invariant properties of the solutions, and sufficiently utilize the structure of the system itself, which eventually leads to the precise blow-up scenario and global existence for strong solutions to the system (1.3). Overall, we do not use any conservation laws rather than the symmetrical structure of the system (1.3) in the whole paper.

The rest of our paper is organized as follows. In Section 2, we discuss the peakon solutions of the system (1.1). In Section 3, we state the local well-posedness of the system (1.4). In Section 4, we provide the precise blow-up scenario and global existence for strong solutions to the system (1.3). In Section 5, we investigate the asymptotic behaviors of strong solutions at infinity within its lifespan.

2. Explicit peakon solutions. Regarding integrability of the $b$-family cubic system (1.1), so far we know that it is integrable when $b = 3$ (see the Lax pair in Ref. [26]). For other $b$'s values, within our knowledge, the problem of integrability is still open. But, we may study the single and multi-peakon solutions of the system (1.1). To do so, let us assume that the $b$-family cubic system (1.1) has N-peakon solutions in the following form:

$$
\begin{align*}
&u(t,x) = \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|}, \\
v(t,x) = \sum_{j=1}^{N} r_j(t)e^{-|x-q_j(t)|},
\end{align*}
$$

(2.1)

where

$$
\begin{align*}
\frac{dp_j}{dt} &= p_j \sum_{i,k=1}^{N} p_i r_k \left( (b-1) \text{sgn}(q_j - q_i) - \text{sgn}(q_j - q_k) \right) e^{-|q_j - q_k| - |q_j - q_i|}, \\
\frac{dr_j}{dt} &= r_j \sum_{i,k=1}^{N} p_i r_k \left( (b-1) \text{sgn}(q_j - q_k) - \text{sgn}(q_j - q_i) \right) e^{-|q_j - q_k| - |q_j - q_i|}, \\
\frac{dq_j}{dt} &= \sum_{i,k=1}^{N} p_i r_k e^{-|q_j - q_k| - |q_j - q_i|}.
\end{align*}
$$

(2.2)

For $N = 1$, one can readily get one single peakons:

$$
\begin{align*}
&u(t,x) = c_1 e^{-|x-c_1 ct|}, \\
v(t,x) = c_2 e^{-|x-c_1 ct|},
\end{align*}
$$

(2.3)

where $c_1$ and $c_2$ are two arbitrary constants.
For $N = 2$, we may rewrite the peakon system (2.2) as

\[
\begin{align*}
\frac{dp_1}{dt} &= p_1 \left\{ -p_1 r_2 \text{sgn}(q_1 - q_2) + p_2 r_1 (b - 1) \text{sgn}(q_1 - q_2) \right\} e^{-|q_1 - q_2|} \\
&\quad + p_2 r_2 (b - 2) \text{sgn}(q_1 - q_2) e^{-2|q_1 - q_2|}, \\
\frac{dp_2}{dt} &= p_2 \left\{ -p_1 r_2 (b - 1) \text{sgn}(q_1 - q_2) + p_2 r_1 \text{sgn}(q_1 - q_2) \right\} e^{-|q_1 - q_2|} \\
&\quad - p_1 r_1 (b - 2) \text{sgn}(q_1 - q_2) e^{-2|q_1 - q_2|}, \\
\frac{dr_1}{dt} &= r_1 \left\{ p_1 r_2 (b - 1) \text{sgn}(q_1 - q_2) - p_2 r_1 \text{sgn}(q_1 - q_2) \right\} e^{-|q_1 - q_2|} \\
&\quad + p_2 r_2 (b - 2) \text{sgn}(q_1 - q_2) e^{-2|q_1 - q_2|}, \\
\frac{dr_2}{dt} &= r_2 \left\{ -p_2 r_1 (b - 1) \text{sgn}(q_1 - q_2) + p_1 r_2 \text{sgn}(q_1 - q_2) \right\} e^{-|q_1 - q_2|} \\
&\quad - p_1 r_1 (b - 2) \text{sgn}(q_1 - q_2) e^{-2|q_1 - q_2|}, \\
\frac{dq_1}{dt} &= p_1 r_1 + (p_1 r_2 + p_2 r_1) e^{-|q_1 - q_2|} + p_2 r_2 e^{-2|q_1 - q_2|}, \\
\frac{dq_2}{dt} &= p_2 r_2 + (p_2 r_2 + p_1 r_1) e^{-|q_1 - q_2|} + p_1 r_1 e^{-2|q_1 - q_2|}.
\end{align*}
\]  
(2.4)

If $q_1 > q_2$, the peakon system (2.4) yields

\[
\begin{align*}
\frac{dp_1}{dt} &= p_1 \left\{ -p_1 r_2 + p_2 r_1 (b - 1) \right\} + p_2 r_2 (b - 2) e^{-(q_1 - q_2)} e^{-(q_1 - q_2)}, \\
\frac{dp_2}{dt} &= -p_2 \left\{ -p_2 r_1 + p_1 r_2 (b - 1) \right\} + p_1 r_1 (b - 2) e^{-(q_1 - q_2)}, \\
\frac{dr_1}{dt} &= r_1 \left\{ -p_2 r_1 + p_1 r_2 (b - 1) \right\} + p_2 r_2 (b - 2) e^{-(q_1 - q_2)} e^{-(q_1 - q_2)}, \\
\frac{dr_2}{dt} &= -r_2 \left\{ -p_1 r_2 + p_2 r_1 (b - 1) \right\} + p_1 r_1 (b - 2) e^{-(q_1 - q_2)} e^{-(q_1 - q_2)}, \\
\frac{dq_1}{dt} &= p_1 r_1 + (p_1 r_2 + p_2 r_1) e^{-(q_1 - q_2)} + p_2 r_2 e^{-2(q_1 - q_2)}, \\
\frac{dq_2}{dt} &= p_2 r_2 + (p_2 r_2 + p_1 r_1) e^{-(q_1 - q_2)} + p_1 r_1 e^{-2(q_1 - q_2)},
\end{align*}
\]  
(2.5)

and if $q_1 < q_2$, we have a similar system odd symmetric to equation (2.5) through a very simple transform $(p_1, p_2, r_1, r_2, q_1, q_2) \rightarrow (-p_1, -p_2, -r_1, -r_2, -q_1, -q_2)$. So, let us just consider the system (2.5). Apparently, a keen observation on the system (2.5) leads to

\[
\begin{align*}
\frac{dp_1}{dt} r_2 + p_2 \frac{dr_1}{dt} &= -(b - 2)p_1 r_2 (p_1 r_1 - p_2 r_2) e^{-2(q_1 - q_2)}, \\
\frac{dp_2}{dt} r_1 + p_1 \frac{dr_2}{dt} &= -(b - 2)p_2 r_1 (p_1 r_1 - p_2 r_2) e^{-2(q_1 - q_2)}.
\end{align*}
\]  
(2.6)

Therefore,

\[
\frac{d(\ln |p_1 r_2|)}{dt} = \frac{d(\ln |p_2 r_1|)}{dt},
\]

and there exists a constant $c^*$ such that

\[
\frac{p_1}{p_2} = c^* \frac{r_1}{r_2}.
\]  
(2.7)
Let \( p_1 = c_p p_2, \ r_1 = c_r r_2. \) Then, substituting them into equation (2.4) generates the following crucial relationships:

\[
\begin{cases}
  c_p c_r = -1, \\
  (c_p + c_r)(b - 2) = 0.
\end{cases}
\]

Obviously, when \( b \neq 2, \) we have \( c_p = -1, \ c_r = 1 \) or \( c_p = 1, \ c_r = -1. \), and when \( b = 2, \ c_p = -1/c_r. \) Let us discuss the two peakon solutions for \( b = 2 \) and \( b \neq 2 \) below, separately.

- **For the case** \( b = 2 \), the system (2.4) can be changed into

\[
\begin{align*}
\frac{dp_1}{dt} &= -p_1(p_1 r_2 - p_2 r_1) - p_1 r_2 (q_1 - q_2) e^{-|q_1 - q_2|}, \\
\frac{dp_2}{dt} &= -p_2(p_1 r_2 - p_2 r_1) - p_2 r_2 (q_1 - q_2) e^{-|q_1 - q_2|}, \\
\frac{dr_1}{dt} &= r_1(p_1 r_2 - p_2 r_1) + p_1 r_1 (q_1 - q_2) e^{-|q_1 - q_2|}, \\
\frac{dr_2}{dt} &= r_2(p_1 r_2 - p_2 r_1) + p_2 r_2 (q_1 - q_2) e^{-|q_1 - q_2|}, \\
\frac{dt}{dt} &= p_1 r_1 + p_2 r_2 e^{-|q_1 - q_2|} + p_1 r_1 e^{-|q_1 - q_2|}.
\end{align*}
\]

(2.8)

Through a lengthy procedure to solve the above system, we can arrive at

\[
\begin{align*}
p_1(t) &= e^{\frac{-|A_2 - A_3 A_3|}{A_2 - A_3 A_3}} + \frac{A_1}{A_2 - A_3 A_3} \sqrt{1 + e^{-2|A_2 - A_3 A_3|}} e^{\frac{-A_1 A_1}{A_2 - A_3 A_3}}, \\
p_2(t) &= A_1 p_1(t), \\
r_1(t) &= \frac{A_1}{p_1(t)}, \\
r_2(t) &= \frac{A_1}{p_1(t)}, \\
q_1(t) &= A_2 t - \frac{A_1 + A_2 A_2}{A_2 - A_3 A_3} \ln(e^{-|A_2 - A_3 A_3|} + \sqrt{1 + e^{-2|A_2 - A_3 A_3|}}) \\
&- \frac{A_1 A_1}{A_2 A_2 - A_3 A_3} \ln(1 + e^{-2|A_2 - A_3 A_3|}), \\
q_2(t) &= A_1 A_1 t + \frac{A_3 + A_1 A_1}{A_2 A_2 - A_3 A_3} \ln(e^{-|A_2 - A_3 A_3|} + \sqrt{1 + e^{-2|A_2 - A_3 A_3|}}) \\
&- \frac{A_1 A_1}{A_2 A_2 - A_3 A_3} \ln(1 + e^{-2|A_2 - A_3 A_3|}),
\end{align*}
\]

(2.9)

where \( A_i \ (i = 1, 2, 3) \) and \( c \) are arbitrary constants.

So, we get the following two-peakons for the \( b \)-family two-component system (1.1) with \( b = 2 \):

\[
\begin{align*}
u(t,x) &= p_1(t)(e^{-|x-q_1(t)| + A_1 e^{-|x-q_2(t)|}}, \\
v(t,x) &= \frac{1}{p_1(t)}(A_2 e^{-|x-q_1(t)|} + A_3 e^{-|x-q_2(t)|}).
\end{align*}
\]

(2.10)

where \( p_i(t), r_i(t), q_i(t) \) \( (i = 1, 2) \) are given through equation (2.9). The graphs of \( u(t,x) \) and \( v(t,x) \) are shown in Figure (1).

- **For the case** \( b \neq 2 \), we do not have a general explicit formula for the two-peakon solutions to the system (1.1). But, we do solve for the explicit two-peakon solutions for \( b = 3 \) and \( b = 4 \). Let us display them below.

When \( b = 3 \), solving the system (2.4) generates the following two-peakon solutions to the \( b \)-family two component system (1.1):

\[
\begin{align*}
u(t,x) &= p_1(t)(e^{-|x-q_1(t)|} - e^{-|x-q_2(t)|}), \\
v(t,x) &= r_1(t)(e^{-|x-q_1(t)|} + e^{-|x-q_2(t)|}).
\end{align*}
\]

(2.11)
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Figure 1. two-peakon solutions $u$ and $v$ given by (2.10) with $A_1 = 1$, $A_2 = 2$, $A_3 = 3$ and $c = 1$.

where
\[
\begin{align*}
    p_1(t) &= -p_2(t) = A_2(1 - e^{-2|A_2 A_3 t|+\frac{c}{2}})(1 + e^{-2|A_2 A_3 t|+\frac{c}{2}})\left(1 + e^{-4|A_2 A_3 t|+c}\right)^{\frac{1}{2}}, \\
    r_1(t) &= r_2(t) = A_3(1 - e^{-2|A_2 A_3 t|+\frac{c}{2}})(1 + e^{-2|A_2 A_3 t|+\frac{c}{2}})\left(1 + e^{-4|A_2 A_3 t|+c}\right)^{\frac{1}{2}}, \\
    q_1(t) &= A_2 A_3 t + \frac{1}{2}\text{sgn}(A_2 A_3 t)\left[\ln(1 + e^{-4|A_2 A_3 t|+c}) - \ln 2 - \frac{c}{2}\right] + \frac{A_3}{2}, \\
    q_2(t) &= -A_2 A_3 t - \frac{1}{2}\text{sgn}(A_2 A_3 t)\left[\ln(1 + e^{-4|A_2 A_3 t|+c}) - \ln 2 - \frac{c}{2}\right] + \frac{A_3}{2},
\end{align*}
\]

and $A_i (i = 1, 2, 3)$, $c$ are arbitrary constants. The graphs of $u(t, x)$ and $v(t, x)$ are shown in Figure (2).

When $b = 4$, solving the system (2.4) we obtain the following two-peakon solutions to the $b$-family two component system (1.1):
\[
\begin{align*}
    u(t, x) &= p_1(t)(e^{-|x-q_1(t)|} - e^{-|x-q_2(t)|}), \\
    v(t, x) &= r_1(t)(e^{-|x-q_1(t)|} + e^{-|x-q_2(t)|}),
\end{align*}
\]
where

\[
\begin{align*}
p_1(t) &= -p_2(t) = A_2(1 - e^{-2|A_2 A_3 t + \frac{c}{2}|} \frac{c}{4}) (1 + e^{-2|A_2 A_3 t + \frac{c}{2}|} \frac{c}{4}), \\
r_1(t) &= r_2(t) = A_3(1 - e^{-2|A_2 A_3 t + \frac{c}{2}|} \frac{c}{4}) (1 + e^{-2|A_2 A_3 t + \frac{c}{2}|} \frac{c}{4}), \\
q_1(t) &= A_2 A_3 t - \frac{1}{4} sgn(A_2 A_3 t) c + \frac{A_3}{2}, \\
q_2(t) &= -A_2 A_3 t + \frac{1}{4} sgn(A_2 A_3 t) c + \frac{A_3}{2},
\end{align*}
\]

and \(A_i (i = 1, 2, 3)\), \(c\) are arbitrary constants. The graphs of \(u(t,x)\) and \(v(t,x)\) are shown in Figure (3).

**Remark 2.1.** The two-soliton graphs (2) and (3) for \(b = 3\) and \(b = 4\) do include a singular amplitude \(p_1(t)\) for the function \(u(t,x)\) at some time while the amplitude \(r_1(t)\) for the other function \(v(t,x)\) immediately vanishes to zero, but fortunately both \(u(t,x)\) and \(v(t,x)\) are continuous and bounded all the times. The two-soliton graph (1) for \(b = 2\) has no any singularity since both \(u(t,x)\) and \(v(t,x)\) are continuous at any time.
In order to show the above three two-peakons satisfy the \( b \)-family two-component system (1.1), let us take the case \( b = 4 \) as the representative to prove. Without loss of generality, let \( A_1 = c = 0 \), then the system (2.12) can be rewritten in the following form:

\[
\begin{align*}
    u(t, x) &= p_1(t)(e^{-|x-A_2A_3t|} - e^{-|x+\frac{A_2}{A_3}A_3t|}), \\
    v(t, x) &= r_1(t)(e^{-|x-A_2A_3t|} + e^{-|x+\frac{A_2}{A_3}A_3t|}),
\end{align*}
\]

where

\[
\begin{align*}
    p_1(t) &= -p_2(t) \\
    &= A_2(1 - e^{-2|A_2A_3t|})^{\frac{1}{2}} (1 + e^{-2|A_2A_3t|})^{\frac{1}{2}} \triangleq A_2X^{\frac{1}{2}}Y^{\frac{1}{2}}, \\
    r_1(t) &= r_2(t) \\
    &= A_3(1 - e^{-2|A_2A_3t|})^{\frac{1}{2}} (1 + e^{-2|A_2A_3t|})^{\frac{1}{2}} \triangleq A_3X^{\frac{1}{2}}Y^{\frac{1}{2}}, \\
    q_1(t) &= q_2(t) = A_2A_3t,
\end{align*}
\]
and $A_i (i = 2, 3)$ are arbitrary constants.

Hence, we substitute them into the $b$-family two-component system (1.1) and obtain

$$m_t + uvm_x + 4u_x vm = 2\frac{dp_1}{dt} [\delta(x - A_2 A_3 t) - \delta(x + A_2 A_3 t)] - 2 A_2 A_3 p_1 [\delta'(x - A_2 A_3 t) + \delta'(x + A_2 A_3 t)]$$

$$+ 2p_1 \frac{A_2 A_3}{XY} (e^{-2|X - A_2 A_3 t|} - e^{-2|x| + A_2 A_3 t}) [\delta'(x - A_2 A_3 t) - \delta'(x + A_2 A_3 t)]$$

$$+ 8p_1 \frac{A_2 A_3 \Delta_1}{XY} [\delta(x - A_2 A_3 t) - \delta(x + A_2 A_3 t)]$$

$$= \left( 2 \frac{dp_1}{dt} + \frac{8A_2 A_3 p_1 \Delta_1}{XY} \right) \delta(x - A_2 A_3 t) - \left( 2 \frac{dp_1}{dt} + \frac{8A_2 A_3 p_1 \Delta_1}{XY} \right) \delta(x + A_2 A_3 t)$$

$$+ 2A_2 A_3 p_1 \left( -1 + \frac{e^{-2|X - A_2 A_3 t|} - e^{-2|x| + A_2 A_3 t}}{XY} \right) \delta'(x - A_2 A_3 t)$$

$$- 2A_2 A_3 p_1 \left( 1 + \frac{e^{-2|X - A_2 A_3 t|} - e^{-2|x| + A_2 A_3 t}}{XY} \right) \delta'(x + A_2 A_3 t)$$

$$= 0,$$

and

$$n_t + uvn_x + 4uv_x n = 2\frac{dr_1}{dt} [\delta(x - A_2 A_3 t) + \delta(x + A_2 A_3 t)]$$

$$- 2A_2 A_3 r_1 [-\delta'(x - A_2 A_3 t) + \delta'(x + A_2 A_3 t)]$$

$$+ 2r_1 \frac{A_2 A_3}{XY} (e^{-2|X - A_2 A_3 t|} - e^{-2|x| + A_2 A_3 t}) [\delta'(x - A_2 A_3 t) + \delta'(x + A_2 A_3 t)]$$

$$+ 8r_1 \frac{A_2 A_3 \Delta_2}{XY} [\delta(x - A_2 A_3 t) + \delta(x + A_2 A_3 t)]$$

$$= \left( 2 \frac{dr_1}{dt} + \frac{8A_2 A_3 r_1 \Delta_2}{XY} \right) \delta(x - A_2 A_3 t) + \left( 2 \frac{dr_1}{dt} + \frac{8A_2 A_3 r_1 \Delta_2}{XY} \right) \delta(x + A_2 A_3 t)$$

$$+ 2A_2 A_3 r_1 \left( -1 + \frac{e^{-2|X - A_2 A_3 t|} - e^{-2|x| + A_2 A_3 t}}{XY} \right) \delta'(x - A_2 A_3 t)$$

$$+ 2A_2 A_3 r_1 \left( 1 + \frac{e^{-2|X - A_2 A_3 t|} - e^{-2|x| + A_2 A_3 t}}{XY} \right) \delta'(x + A_2 A_3 t)$$

$$= 0,$$

where

$$\Delta_1 = -\text{sgn}(x - A_2 A_3 t) e^{-2|X - A_2 A_3 t|} + \text{sgn}(x + A_2 A_3 t) e^{-2|x| + A_2 A_3 t}$$

$$+ \left( -\text{sgn}(x - A_2 A_3 t) + \text{sgn}(x + A_2 A_3 t) \right) e^{-2|X - A_2 A_3 t| - 2|x| + A_2 A_3 t},$$

$$\Delta_2 = -\text{sgn}(x - A_2 A_3 t) e^{-2|X - A_2 A_3 t|} + \text{sgn}(x + A_2 A_3 t) e^{-2|x| + A_2 A_3 t}$$

$$+ \left( \text{sgn}(x - A_2 A_3 t) - \text{sgn}(x + A_2 A_3 t) \right) e^{-2|X - A_2 A_3 t| - 2|x| + A_2 A_3 t},$$

$$X = 1 - e^{-2|A_2 A_3 t|},$$

$$Y = 1 + e^{-2|A_2 A_3 t|}.$$
Therefore, both \( u(t, x) \) and \( v(t, x) \) given in equation (2.13) are two-peakon solutions to the \( b \)-family two-component system (1.1). Similarly, we can prove that both \( u(t, x) \) and \( v(t, x) \) given in equations (2.10) and (2.11) are also two-peakon solutions to the \( b \)-family two-component system (1.1) with \( b = 2 \) and \( b = 3 \), respectively.

3. Local well-posedness. Applying the Littlewood-Paley decomposition and transport equation theory in Besov spaces [1], which in combination with the classical Kato semigroup theory [32], one may follow the similar arguments as in [48] to obtain the following local well-posedness result for the system (1.4).

**Theorem 3.1.** Suppose that \((u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})\) with \( s > \frac{5}{2} \). There exists a maximal existence time \( T = T(\|u_0\|_{H^s(\mathbb{R})}, \|v_0\|_{H^s(\mathbb{R})}) > 0 \), and a unique solution \((u, v)\) to the system (1.4) such that

\[
(u, v) \in C([0, T); H^s(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R})).
\]

Moreover, the solution depends continuously on the initial data, that is, the mapping \((u_0, v_0) \mapsto (u, v) : H^s(\mathbb{R}) \times H^s(\mathbb{R}) \to C([0, T); H^s(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R}))\) is continuous.

**Remark 3.1.**
1. Theorem 3.1 also holds true for the system (1.3) if we assume that the initial data \((m_0, n_0)\) belongs to \( H^s(\mathbb{R}) \times H^s(\mathbb{R})\) whenever \( s > \frac{3}{2} \).
2. The maximal existence time \( T \) in Theorem 3.1 can be chosen independent of the regularity index \( s \), which will be shown in Remark 4.1 below.
3. As is well known, the Cauchy problems for the CH, DP and Novikov equations are locally well-posed when the initial data \( u_0 \in H^s(\mathbb{R}) \) as long as \( s > \frac{3}{2} \) [42, 45, 49]. However, the regularity index in Theorem 3.1 cannot be improved to \( s > \frac{3}{2} \), whose reason lies in the following observation:

Let \((u^{(i)}, v^{(i)})\) \( (i = 1, 2) \) be two solutions to the system (1.4) with the initial data \((u_0^{(i)}, v_0^{(i)})\). If we set \( u^{(12)} \triangleq u^{(2)} - u^{(1)} \) and \( v^{(12)} \triangleq v^{(2)} - v^{(1)} \), then

\[
\begin{align*}
\partial_t u^{(12)} + (u^{(1)} v^{(1)}) \partial_x u^{(12)} &= -u_x^{(2)} v^{(1)} u^{(12)} - u_x^{(1)} v^{(2)} u^{(12)} \quad \text{and} \\
\partial_t v^{(12)} + (u^{(1)} v^{(1)}) \partial_x v^{(12)} &= -v_x^{(2)} u^{(1)} v^{(12)} - v_x^{(1)} u^{(2)} v^{(12)}.
\end{align*}
\]
Going the similar line as the proof of Lemmas 3.1-3.2 in [48], via the Morse type inequality, one needs to estimate the \( H^{s-1}(\mathbb{R}) \) norms of the following cross terms of \( u_x^{(i)}, v_x^{(i)}, u_x^{(12)}, v_x^{(12)} \) (\( i = 1, 2 \)) and their first order derivatives:

\[
-p \left( (u_x^{(12)}u_x^{(1)})_x v_x^{(2)} + (u_x^{(1)}u_x^{(12)})_x v_x^{(2)} - \frac{b-3}{2} \left( u_x^{(12)}(u_x^{(1)})_x + u_x^{(2)} \right) \right) v_x^{(2)},
\]

and

\[
-p \left( (v_x^{(12)}v_x^{(1)})_x u_x^{(2)} + (v_x^{(1)}v_x^{(12)})_x u_x^{(2)} - \frac{b-3}{2} \left( v_x^{(12)}(v_x^{(1)})_x + v_x^{(2)} \right) \right) u_x^{(2)},
\]

which ultimately leads to \( s > \frac{5}{7} \). Nevertheless, this case would not happen to the reduced single equations of system (1.4), such as the CH, DP and Novikov equations.

4. Blow-up and global existence. In this section, we will derive the precise blow-up scenario of strong solutions to the system (1.3), and then state its global existence under some assumptions. Let us first prove a crucial blow-up criterion for the system (1.3). For this, we need some a priori estimates of the following transport equation:

\[
(TE) \left\{ \begin{array}{l}
\partial_t f + v \partial_x f = F, \\
| f | t = 0 = f_0.
\end{array} \right.
\]

Lemma 4.1. [1] Let \( s > -\frac{1}{2} \). Assume that \( f_0 \in H^s(\mathbb{R}) \), \( F \in L^1(0, T; H^s(\mathbb{R})) \), and \( \partial_x v \) belongs to \( L^1(0, T; H^{s-1}(\mathbb{R})) \) if \( s > \frac{3}{2} \), or to \( L^1(0, T; H^{\frac{3}{2}}(\mathbb{R}) \cap L^\infty(\mathbb{R})) \) if \( -\frac{1}{2} < s \leq \frac{3}{2} \). If \( f \in L^\infty(0, T; H^s(\mathbb{R})) \cap C([0, T]; S'(\mathbb{R})) \) solves (TE), then \( f \in C([0, T]; H^s(\mathbb{R})) \). Moreover, for all \( s \leq \frac{4}{3} \), there exists a constant \( C = C(s) > 0 \) such that for all \( t \in [0, T] \),

\[
||f(t)||_{H^s} \leq ||f_0||_{H^s} + \int_0^t ||F(\tau)||_{H^s} d\tau + C \int_0^t V(\tau)||f(\tau)||_{H^s} d\tau
\]

with

\[
V(\tau) \triangleq \left\{ \begin{array}{ll}
||\partial_x v(\tau)||_{H^s \cap L^\infty}, & \text{if} \quad \frac{3}{2} < s < \frac{4}{3}, \\
||\partial_x v(\tau)||_{H^{s-1}}, & \text{if} \quad s > \frac{4}{3}.
\end{array} \right.
\]

Lemma 4.2. [27] Let \( 0 < s < 1 \). Assume that \( f_0 \in H^s(\mathbb{R}) \), \( F \in L^1(0, T; H^s(\mathbb{R})) \), and \( v, \partial_x v \in L^1(0, T; L^\infty(\mathbb{R})) \). If \( f \in L^\infty(0, T; H^s(\mathbb{R})) \cap C([0, T]; S'(\mathbb{R})) \) solves (TE), then \( f \in C([0, T]; H^s(\mathbb{R})) \). Moreover, there exists a constant \( C = C(s) > 0 \) such that for all \( t \in [0, T] \),

\[
||f(t)||_{H^s} \leq ||f_0||_{H^s} + C \int_0^t ||F(\tau)||_{H^s} d\tau + C \int_0^t V(\tau)||f(\tau)||_{H^s} d\tau
\]

with

\[
V(t) \triangleq ||v(t)||_{L^\infty} + ||\partial_x v(t)||_{L^\infty}.
\]

In addition, the following one-dimensional Morse-type estimates are also required.
Proposition 4.1. [1] For all \( s > 0 \), there exists a positive constant \( C \) independent of \( f \) and \( g \), such that

\[
\|fg\|_{H^s(\mathbb{R})} \leq C(\|f\|_{H^s(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|g\|_{H^s(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})}),
\]

and

\[
\|f\partial_x g\|_{H^s(\mathbb{R})} \leq C(\|f\|_{H^{s+1}(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}\|\partial_x g\|_{H^s(\mathbb{R})}).
\]

Theorem 4.1. Let \( (m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \) with \( s > \frac{1}{2} \) and \( T \) be the maximal existence time of the solution \( (m, n) \) to the system \((1.3)\). If \( T < \infty \), then

\[
\int_0^T (\|m(\tau, \cdot)\|_{L^\infty} + \|n(\tau, \cdot)\|_{L^\infty})^2 \, d\tau = \infty.
\]

Proof. We will prove the theorem by induction with respect to the regularity index \( s (s > \frac{1}{2}) \) as follows.

**Step 1.** For \( s \in (\frac{1}{2}, 1) \), by Lemma 4.2 and the system \((1.3)\), we have

\[
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} + C \int_0^t \|(u_xv)m)(\tau)\|_{H^s} \, d\tau
\]

\[
+ C \int_0^t (\|uv\|_{L^\infty} + \|(uv)_x\|_{L^\infty}) \|m(\tau)\|_{H^s} \, d\tau,
\]

and

\[
\|n(t)\|_{H^s} \leq \|n_0\|_{H^s} + C \int_0^t \|(u_xn)(\tau)\|_{H^s} \, d\tau
\]

\[
+ C \int_0^t (\|uv\|_{L^\infty} + \|(uv)_x\|_{L^\infty}) \|n(\tau)\|_{H^s} \, d\tau,
\]

where \( C = C(b, s) \) is a positive constant.

Noting that \( u = (1 - \partial_x^2)^{-1} m = p \ast m \) with \( p(x) \equiv \frac{1}{2} e^{-|x|} \) (\( x \in \mathbb{R} \)), \( u_x = (\partial_x p) \ast m \), \( u_{xx} = u - m \) and \( \|p\|_{L^1} = \|\partial_x p\|_{L^1} = 1 \), together with the Young inequality, for all \( s \in \mathbb{R} \), we have

\[
\|u\|_{L^\infty}, \|u_x\|_{L^\infty}, \|u_{xx}\|_{L^\infty} \leq C\|m\|_{L^\infty}
\]

and

\[
\|u\|_{H^s}, \|u_x\|_{H^s}, \|u_{xx}\|_{H^s} \leq C\|m\|_{H^s}.
\]

Similarly, the identity \( v = p \ast n \) ensures

\[
\|v\|_{L^\infty}, \|v_x\|_{L^\infty}, \|v_{xx}\|_{L^\infty} \leq C\|n\|_{L^\infty}
\]

and

\[
\|v\|_{H^s}, \|v_x\|_{H^s}, \|v_{xx}\|_{H^s} \leq C\|n\|_{H^s}.
\]

Then Proposition 4.1 gives

\[
\|u_xv_m\|_{H^s} \leq C\|u_xv\|_{H^s}\|m\|_{L^\infty} + C\|u_xv\|_{L^\infty}\|m\|_{H^s}
\]

\[
\leq C(\|m\|_{H^s}\|n\|_{L^\infty} + \|m\|_{L^\infty}\|n\|_{H^s})\|m\|_{L^\infty}
\]

\[
+ C\|m\|_{L^\infty}\|n\|_{L^\infty}\|m\|_{H^s}
\]

\[
\leq C\|m\|_{L^\infty}\|n\|_{L^\infty}\|m\|_{H^s} + C\|m\|_{L^\infty}^2\|n\|_{H^s},
\]

and

\[
\|uv\|_{L^\infty} + \|(uv)_x\|_{L^\infty} \leq C\|m\|_{L^\infty}\|n\|_{L^\infty}.
\]
Step 2. For the second equation of the system (1.3), we can deal with it in a similar way to (1.3), we get

\[ \|m(t)\|_{H^s} + \|n(t)\|_{H^s} \leq \|m_0\|_{H^s} + C \int_0^t \|m(\tau)\|_{L^\infty} \|n(\tau)\|_{L^\infty} \|m(\tau)\|_{H^s}^2 + \|n(\tau)\|_{L^\infty}^2 \|n(\tau)\|_{H^s} \, d\tau. \]

Likewise,

\[ \|n(t)\|_{H^s} \leq \|n_0\|_{H^s} + C \int_0^t \|n(\tau)\|_{L^\infty}^2 \|m(\tau)\|_{H^s} + \|m(\tau)\|_{L^\infty} \|n(\tau)\|_{L^\infty} \|n(\tau)\|_{H^s} \, d\tau. \]

Thus, we have

\[ \|m(t)\|_{H^s} + \|n(t)\|_{H^s} \leq \|m_0\|_{H^s} + \|n_0\|_{H^s} + C \int_0^t (\|m\|_{L^\infty} + \|n\|_{L^\infty})^2 (\|m\|_{H^s} + \|n\|_{H^s}) \, d\tau. \] (4.7)

Taking advantage of Gronwall’s inequality, one gets

\[ \|m(t)\|_{H^s} + \|n(t)\|_{H^s} \leq (\|m_0\|_{H^s} + \|n_0\|_{H^s}) e^{C \int_0^t (\|m\|_{L^\infty} + \|n\|_{L^\infty})^2 \, d\tau}. \] (4.8)

Therefore, if \( T < \infty \) satisfies \( \int_0^T (\|m(\tau)\|_{L^\infty} + \|n(\tau)\|_{L^\infty})^2 \, d\tau < \infty \), then we deduce from (4.8) that

\[ \limsup_{t\to T} (\|m(t)\|_{H^s} + \|n(t)\|_{H^s}) < \infty, \] (4.9)

which contradicts the assumption that \( T < \infty \) is the maximal existence time. This completes the proof of the theorem for \( s \in (\frac{1}{2}, 1) \).

Step 2. For \( s \in [1, \frac{3}{2}) \), applying Lemma 4.1 to the first equation of the system (1.3), we get

\[ \|m(t)\|_{H^s} \leq \|m_0\|_{H^s} + |b| \int_0^t \|(u_x v m)(\tau)\|_{H^s} \, d\tau \]

\[ + C \int_0^t \|m(\tau)\|_{H^s} \|(uv)_x\|_{H^s \cap L^\infty} \, d\tau. \]

Note that

\[ \|(uv)_x\|_{H^s \cap L^\infty} \leq C \|u_x v + uv_x\|_{H^{s_0}} \leq C \|m\|_{H^{s_0}} \|n\|_{H^{s_0}} \]

with \( s_0 \in (0, \frac{1}{2}) \). Using (4.5) and the fact that \( H^{s_0}(\mathbb{R}) \hookrightarrow H^{s}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) leads to

\[ \|m(t)\|_{H^s} \leq \|m_0\|_{H^s} + C \int_0^t \|m\|_{H^{s_0}} \|n\|_{H^{s_0}} \|m\|_{H^s} + \|m\|_{H^{s_0}}^2 \|n\|_{H^s} \, d\tau. \]

For the second equation of the system (1.3), we can deal with it in a similar way and obtain that

\[ \|n(t)\|_{H^s} \leq \|n_0\|_{H^s} + C \int_0^t \|n\|_{H^{s_0}}^2 \|m\|_{H^s} + \|m\|_{H^{s_0}} \|n\|_{H^s} \, d\tau. \]
Hence,
\[
\|m(t)\|_{H^s} + \|n(t)\|_{H^s} \\
\leq \|m_0\|_{H^s} + \|n_0\|_{H^s} + C \int_0^t (\|m\|_{H^{\frac{s}{2}+\varepsilon_0}} + \|n\|_{H^{\frac{s}{2}+\varepsilon_0}})^2(\|m\|_{H^s} + \|n\|_{H^s}) \, d\tau.
\]
Thanks to Gronwall’s inequality again, we have
\[
\|m(t)\|_{H^s} + \|n(t)\|_{H^s} \\
\leq (\|m_0\|_{H^s} + \|n_0\|_{H^s}) e^{C \int_0^t (\|m\|_{H^{\frac{s}{2}+\varepsilon_0}} + \|n\|_{H^{\frac{s}{2}+\varepsilon_0}})^2 \, d\tau}.
\]
Therefore, if \( T < \infty \) satisfies \( \int_0^T (\|m(\tau)\|_{L^\infty} + \|n(\tau)\|_{L^\infty})^2 \, d\tau < \infty \), then we deduce from the uniqueness of the solution to the system (1.3) and (4.9) with \( \frac{1}{2} + \varepsilon_0 \in (\frac{1}{2}, 1) \) instead of \( s \) that
\[
\|m(t)\|_{H^{\frac{s}{2}+\varepsilon_0}} + \|n(t)\|_{H^{\frac{s}{2}+\varepsilon_0}} \text{ is uniformly bounded in } t \in (0, T).
\]
This along with (4.10) implies that
\[
\limsup_{t \to T} (\|m(t)\|_{H^s} + \|n(t)\|_{H^s}) < \infty,
\]
which contradicts the assumption that \( T < \infty \) is the maximal existence time. This completes the proof of the theorem for \( s \in [1, \frac{3}{2}) \).

**Step 3.** For \( s \in (1, 2) \), differentiating the system (1.3) with respect to \( x \), we have
\[
\partial_t m_x + (uv) \partial_x m_x = -((b+1) u_x v + uv_x) m_x - b(u_x v)_x m \triangleq R_1(t, x)
\]
and
\[
\partial_t n_x + (uv) \partial_x n_x = -((b+1) u v_x + u v_x) n_x - b(u v_x)_x n \triangleq R_2(t, x)
\]
By Lemma 4.2 with \( s - 1 \in (0, 1) \), we get
\[
\|m_x(t)\|_{H^{s-1}} \leq \|\partial_x m_0\|_{H^{s-1}} + C \int_0^t \|R_1(\tau)\|_{H^{s-1}} \, d\tau \\
+ C \int_0^t (\|uv\|_{L^\infty} + ||(uv)_x||_{L^\infty}) \|m_x(\tau)\|_{H^{s-1}} \, d\tau,
\]
and
\[
\|n_x(t)\|_{H^{s-1}} \leq \|\partial_x n_0\|_{H^{s-1}} + C \int_0^t \|R_2(\tau)\|_{H^{s-1}} \, d\tau \\
+ C \int_0^t (\|uv\|_{L^\infty} + ||(uv)_x||_{L^\infty}) \|n_x(\tau)\|_{H^{s-1}} \, d\tau,
\]
Due to Proposition 4.1 and (4.1)-(4.4), we have
\[
\| - ((b+1) u_x v + uv_x) m_x \|_{H^{s-1}} \\
\leq C(\|(b+1) u_x v + uv_x\|_{H^s} \|m\|_{L^\infty} + \|(b+1) u_x v + uv_x\|_{L^\infty} \|m_x\|_{H^{s-1}}) \\
\leq C \|m\|_{L^\infty} \|n\|_{L^\infty} \|m\|_{H^s} + C \|m\|_{L^\infty}^2 \|n\|_{H^s},
\]
and
\[
\| - b(u_x v)_x m \|_{H^{s-1}} \leq C(\|m\|_{H^s} \|u_x v\|_{L^\infty} + \|m\|_{L^\infty} \|(u_x v)_x\|_{H^s}) \\
\leq C \|m\|_{L^\infty} \|n\|_{L^\infty} \|m\|_{H^s} + C \|m\|_{L^\infty}^2 \|n\|_{H^s},
\]
which together with (4.6) yields
\[ \|m_x(t)\|_{H^{s-1}} \leq \|m_0\|_{H^s} + C \int_0^t \|m\|_{L^\infty} \|n\|_{L^\infty} \|m\|_{H^s} + \|m\|^2_{L^\infty} \|n\|_{H^s} \, dt. \]

Likewise,
\[ \|n_x(t)\|_{H^{s-1}} \leq \|n_0\|_{H^s} + C \int_0^t \|n\|^2_{L^\infty} \|m\|_{H^s} + \|m\|_{L^\infty} \|n\|_{L^\infty} \|n\|_{H^s} \, dt. \]

Thus, we have
\[ \|m_x(t)\|_{H^{s-1}} + \|n_x(t)\|_{H^{s-1}} \leq \|m_0\|_{H^s} + \|n_0\|_{H^s} + C \int_0^t (\|m\|_{L^\infty} + \|n\|_{L^\infty})^2 (\|m\|_{H^s} + \|n\|_{H^s}) \, dt. \]

This along with (4.7) with \( s - 1 \in (0, 1) \) instead of \( s \) ensures
\[ \|m(t)\|_{H^s} + \|n(t)\|_{H^s} \leq \|m(0)\|_{H^s} + \|n(0)\|_{H^s} + C \int_0^t (\|m\|_{L^\infty} + \|n\|_{L^\infty})^2 (\|m\|_{H^s} + \|n\|_{H^s}) \, dt. \]

Similar to Step 1, we can easily prove the theorem for \( s \in (1, 2) \).

**Step 4.** For \( s = k \in \mathbb{N} \) and \( k \geq 2 \), differentiating the system (1.3) \( k - 1 \) times with respect to \( t \), we get
\[ \left( \partial_t + (uv) \partial_x \right) \partial_x^{k-1} m = - \sum_{l=0}^{k-2} C_{k-l} \partial_x^{k-l-1}(uv) \partial_x^{l+1} m - b \partial_x^{k-1}(uv \cdot n) \triangleq F_1(t, x), \]
and
\[ \left( \partial_t + (uv) \partial_x \right) \partial_x^{k-1} n = - \sum_{l=0}^{k-2} C_{k-l} \partial_x^{k-l-1}(uv) \partial_x^{l+1} n - b \partial_x^{k-1}(uv \cdot n) \triangleq F_2(t, x), \]

which together with Lemma 4.1 imply
\[ \|\partial_x^{k-1} m(t)\|_{H^1} \leq \|\partial_x^{k-1} m_0\|_{H^1} + \int_0^t \|F_1(\tau)\|_{H^1} \, d\tau + C \int_0^t \|((uv)_x)\|_{H^{1/2}} \|\partial_x^{k-1} m(\tau)\|_{H^1} \, d\tau \]
and
\[ \|\partial_x^{k-1} n(t)\|_{H^1} \leq \|\partial_x^{k-1} n_0\|_{H^1} + \int_0^t \|F_2(\tau)\|_{H^1} \, d\tau + C \int_0^t \|((uv)_x)\|_{H^{1/2}} \|\partial_x^{k-1} n(\tau)\|_{H^1} \, d\tau. \]

Making use of Proposition 4.1 and (4.1)-(4.4) again, one infers
\[ \| - \sum_{l=0}^{k-2} C_{k-l} \partial_x^{k-l-1}(uv) \partial_x^{l+1} m\|_{H^1} \leq C(k) \sum_{l=0}^{k-2} \left( \|\partial_x^{k-l-1}(uv)\|_{L^\infty} \|\partial_x^{l+1} m\|_{H^1} + \|\partial_x^{k-l-1}(uv)\|_{H^1} \|\partial_x^{l+1} m\|_{L^\infty} \right) \]
\[ \leq C(k) \sum_{l=0}^{k-2} \left( \|uv\|_{H^{k-l-\frac{1}{2}+\epsilon_0}} \|m\|_{H^{l+2}} + \|uv\|_{H^{k-1}} \|m\|_{H^{l+\frac{1}{2}+\epsilon_0}} \right). \]
\[
\begin{align*}
&\leq C(k)(\|uv\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|m\|_{H^k} + \|uv\|_{H^{k}}\|m\|_{H^{k-\frac{1}{2}+\varepsilon_0}}) \\
&\leq C(k)(\|u\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|m\|_{H^k} \\
&\quad + C(k)(\|u\|_{H^{k}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^k})\|m\|_{H^{k-\frac{1}{2}+\varepsilon_0}} \\
&\leq C(k)(\|m\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|n\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|m\|_{H^k} + \|m\|^2_{H^{k-\frac{1}{2}+\varepsilon_0}}\|n\|_{H^k}), \quad (4.12)
\end{align*}
\]

\[
|| - b\partial_x^{k-1}(u_x v m)||_{H^s} \\
\leq C(b)(\|u_x v m\|_{H^k} \\
\leq C(b)(\|u_x v\|_{L^\infty} \|m\|_{H^k} + \|u_x v\|_{H^k} \|m\|_{L^\infty} \\
\leq C(b)(\|m\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|n\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|m\|_{H^k} + \|m\|^2_{H^{k-\frac{1}{2}+\varepsilon_0}}\|n\|_{H^k}), \quad (4.13)
\]

and
\[
\|(uv)_{x}\|_{H^{k-\frac{1}{2}} \cap L^\infty} \leq C(\|(uv)_{x}\|_{H^{k-\frac{1}{2}+\varepsilon_0}} \leq C(\|m\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|n\|_{H^{k-\frac{1}{2}+\varepsilon_0}})
\]

where \(\varepsilon_0 \in (0, \frac{1}{2})\) and we used the fact that
\[
H^{k-\frac{1}{2}+\varepsilon_0}(\mathbb{R}) \hookrightarrow H^{\frac{1}{2}+\varepsilon_0}(\mathbb{R}) \hookrightarrow H^{\frac{1}{2}}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \text{with} \quad k \geq 2. \quad (4.14)
\]

Thus, we get
\[
\|\partial_x^{k-1}m(t)\|_{H^s} \\
\leq \|m_0\|_{H^k} + C\int_0^t (\|m\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|n\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|m\|_{H^k} + \|m\|^2_{H^{k-\frac{1}{2}+\varepsilon_0}}\|n\|_{H^k})dt. 
\]

Similarly,
\[
\|\partial_x^{k-1}n(t)\|_{H^s} \\
\leq \|n_0\|_{H^k} + C\int_0^t (\|n\|^2_{H^{k-\frac{1}{2}+\varepsilon_0}}\|m\|_{H^k} + \|m\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|n\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|n\|_{H^k})dt. 
\]

Then, we have
\[
\|\partial_x^{k-1}m(t)\|_{H^s} + \|\partial_x^{k-1}n(t)\|_{H^s} \\
\leq \|m_0\|_{H^k} + \|n_0\|_{H^k} + C\int_0^t (\|m\|_{H^{k-\frac{1}{2}+\varepsilon_0}} + \|n\|_{H^{k-\frac{1}{2}+\varepsilon_0}})^2(\|m\|_{H^k} + \|n\|_{H^k})dt, 
\]

which together with Gronwall’s inequality and (4.10) with \(s = 1\) imply
\[
\|m(t)\|_{H^k} + \|n(t)\|_{H^k} \\
\leq (\|m_0\|_{H^k} + \|n_0\|_{H^k}) e^{C\int_0^t (\|m\|_{H^{k-\frac{1}{2}+\varepsilon_0}} + \|n\|_{H^{k-\frac{1}{2}+\varepsilon_0}})^2dt}. \quad (4.15)
\]

If \(T < \infty\) satisfies \(\int_0^T (\|m(\tau)\|_{L^\infty} + \|n(\tau)\|_{L^\infty})^2d\tau < \infty\), applying Step 3 with \(\frac{1}{2} + \varepsilon_0 \in (1, 2)\) and by induction with respect to \(k \geq 2\), we see that \(\|m(t)\|_{H^{k-\frac{1}{2}+\varepsilon_0}} + \|n(t)\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\) is uniformly bounded in \(t \in (0, T)\). By (4.15), we have
\[
\limsup_{t \to T} (\|m(t)\|_{H^k} + \|n(t)\|_{H^k}) < \infty, \quad (4.16)
\]
which contradicts the assumption that $T < \infty$ is the maximal existence time. This completes the proof of the theorem for $s = k \in \mathbb{N}$ and $k \geq 2$.

**Step 5.** For $s \in (k, k+1)$, $k \in \mathbb{N}$ and $k \geq 2$, differentiating the system (1.3) $k$ times with respect to $x$, we get

\[
(\partial_t + (uv)\partial_x)\partial_x^k m = -\sum_{l=0}^{k-1} C_k^l \partial_x^{l+1}(uv)\partial_x^{l+1} m - b\partial_x^k (u_v m) \triangleq G_1(t, x),
\]

and

\[
(\partial_t + (uv)\partial_x)\partial_x^k n = -\sum_{l=0}^{k-1} C_k^l \partial_x^{l+1}(uv)\partial_x^{l+1} n - b\partial_x^k (u_v n) \triangleq G_2(t, x),
\]

which together with Lemma 4.2 with $s - k \in (0, 1)$ imply

\[
||\partial_x^k m(t)||_{H^{s-k}} \leq ||\partial_x^k m_0||_{H^{s-k}} + C \int_0^t ||G_1(\tau)||_{H^{s-k}} d\tau + C \int_0^t (||uv||_{L^\infty} + ||(uv)_x||_{L^\infty})||\partial_x^k m(\tau)||_{H^{s-k}} d\tau
\]

and

\[
||\partial_x^k n(t)||_{H^{s-k}} \leq ||\partial_x^k n_0||_{H^{s-k}} + C \int_0^t ||G_2(\tau)||_{H^{s-k}} d\tau + C \int_0^t (||uv||_{L^\infty} + ||(uv)_x||_{L^\infty})||\partial_x^k n(\tau)||_{H^{s-k}} d\tau.
\]

By (4.14) and using the procedure similar to (4.12)-(4.13), we have

\[
|| - \sum_{l=0}^{k-1} C_k^l \partial_x^{l+1}(uv)\partial_x^{l+1} m - b\partial_x^k (u_v m)||_{H^{s-k}} \leq C(k)(||m||_{H^{k-\frac{1}{4}+\varepsilon}} ||n||_{H^{k-\frac{1}{4}+\varepsilon}} ||m||_{H^s} + ||m||_{H^{k-\frac{1}{2}+\varepsilon}}^2 ||n||_{H^s}),
\]

and

\[
|| - C_k^0 \partial_x^k (uv) m_x ||_{H^{s-k}} = ||m_x \partial_x (\partial_x^{k-1}(uv))||_{H^{s-k}} \leq C (||m_x||_{H^{s-k+1}} ||\partial_x^{k-1}(uv)||_{L^\infty} + ||m_x||_{L^\infty} ||\partial_x^k (uv)||_{H^{s-k}}) \leq C (||m||_{H^s} ||uv||_{H^{k-\frac{1}{2}+\varepsilon}} + ||m||_{H^{k-\frac{1}{2}+\varepsilon}} ||uv||_{H^s}) \leq C (||m||_{H^{k-\frac{1}{2}+\varepsilon}} ||n||_{H^{k-\frac{1}{2}+\varepsilon}} ||m||_{H^s} + ||m||_{H^{k-\frac{1}{2}+\varepsilon}}^2 ||n||_{H^s}).
\]

Thus, we obtain

\[
||\partial_x^k m(t)||_{H^{s-k}} + ||\partial_x^k n(t)||_{H^{s-k}} \leq ||m_0||_{H^s} + ||n_0||_{H^s} + C \int_0^t (||m||_{H^{k-\frac{1}{2}+\varepsilon}} + ||n||_{H^{k-\frac{1}{2}+\varepsilon}})^2 (||m||_{H^s} + ||n||_{H^s}) d\tau.
\]

This along with (4.7) with $s - k \in (0, 1)$ instead of $s$ lead to

\[
||m(t)||_{H^s} + ||n(t)||_{H^s} \leq ||m_0||_{H^s} + ||n_0||_{H^s} + C \int_0^t (||m||_{H^{k-\frac{1}{2}+\varepsilon}} + ||n||_{H^{k-\frac{1}{2}+\varepsilon}})^2 (||m||_{H^s} + ||n||_{H^s}) d\tau.
\]
By using Gronwall’s inequality, Step 3 with \( \frac{3}{2} + \varepsilon_0 \in (1, 2) \) and the similar argument as shown in Step 4, we can arrive at the desired result.

In summary, the above 5 steps complete the proof of the theorem. \( \square \)

**Remark 4.1.** The maximal existence time \( T \) in Theorem 4.1 can be chosen independent of the regularity index \( s \). Indeed, let \((m_0, n_0) \in H^s \times H^s \) with \( s > \frac{1}{2} \) and some \( s' \in (\frac{1}{2}, s) \). Then Remark 3.1 (1) ensures that there exists a unique \( H^s \times H^s \) (resp., \( H^{s'} \times H^{s'} \)) solution \((m_s, n_s)\) (resp., \((m_{s'}, n_{s'})\)) to the system (1.3) with the maximal existence time \( T_s \) (resp., \( T_{s'} \)). Since \( H^s \hookrightarrow H^{s'} \), it follows from the uniqueness that \( T_s \leq T_{s'} \) and \((m_s, n_s) \equiv (m_{s'}, n_{s'}) \) on \([0, T_s)\). On the other hand, if we suppose that \( T_s < T_{s'} \), then \((m_{s'}, n_{s'}) \in C([0, T_s]; H^{s'} \times H^{s'}) \). Hence \((m_s, n_s) \in L^2(0, T_s; L^\infty \times L^\infty) \), which is a contradiction to Theorem 4.1. Therefore, \( T_s = T_{s'} \).

Now we turn our attention to the precise blow-up scenario for sufficiently regular solutions to the system (1.3). For this, motivated by [6, 35], we first consider the characteristic ordinary differential equation as follows:

\[
\begin{cases}
q_t(t, x) = (uv)(t, q(t, x)), & (t, x) \in (0, T) \times \mathbb{R}, \\
q(0, x) = x, & x \in \mathbb{R},
\end{cases}
\] (4.17)

for the flow generated by \( uv \).

The following lemmas are very crucial to study the blow-up phenomena of strong solutions to the system (1.3).

**Lemma 4.3.** Let \((m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \) with \( s > \frac{1}{2} \) and \( T > 0 \) be the maximal existence time of the corresponding solution \((m, n)\) to the system (1.3). Then Eq.(4.17) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}; \mathbb{R}) \). Moreover, the mapping \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with

\[
q_x(t, x) = \exp \left( \int_0^t (u_x v + uv_x)(s, q(s, x))ds \right) > 0,
\] (4.18)

for all \((t, x) \in [0, T) \times \mathbb{R} \).

**Proof.** Since \((u, v) \in C^1([0, T); H^s(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \) with \( s > \frac{5}{2} \), it follows from the fact \( H^{s-1}(\mathbb{R}) \hookrightarrow Lip(\mathbb{R}) \) with \( s > \frac{5}{2} \) that \( uv \) is bounded and Lipschitz continuous in the space variable \( x \) and of class \( C^1 \) in time variable \( t \). Then the classical ODE theory ensures that Eq.(4.17) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}; \mathbb{R}) \).

Differentiating Eq.(4.17) with respect to \( x \) gives

\[
\begin{cases}
\frac{dq_x(t, x)}{dt} = (u_x v + uv_x)(t, q(t, x))q_x(t, x), & (t, x) \in (0, T) \times \mathbb{R}, \\
q_x(0, x) = 1, & x \in \mathbb{R},
\end{cases}
\]

which leads to (4.18).

On the other hand, \( \forall t < T \), by the Sobolev embedding theorem, we have

\[
\sup_{(s, x) \in [0, T) \times \mathbb{R}} |(u_x v + uv_x)(s, x)| < \infty,
\]

which along with (4.18) implies that there exists a constant \( C > 0 \) such that

\[
q_x(t, x) \geq e^{-Ct}, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]

This implies that the mapping \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) before blow-up. Therefore, we complete the proof of Lemma 4.3. \( \square \)
Lemma 4.4. Let $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > \frac{1}{2}$ and $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to the system (1.3). Then we have

\[ m(t, q(t, x)) = m_0(x) \exp \left( -b \int_0^t (u_x v)(s, q(s, x))ds \right), \tag{4.19} \]

and

\[ n(t, q(t, x)) = n_0(x) \exp \left( -b \int_0^t (u v)(s, q(s, x))ds \right). \tag{4.20} \]

for all $(t, x) \in [0, T) \times \mathbb{R}$.

Moreover, if there exists a $C > 0$ such that for all $(t, x) \in [0, T) \times \mathbb{R}$,

\[ (u v)(t, x) \geq -C \quad \text{and} \quad (u_x v)(t, x) \geq -C, \]

then for all $t \in [0, T)$,

\[ ||m(t, \cdot)||_{L^\infty} \leq C e^{bCt}||m_0||_{H^s} \quad \text{and} \quad ||n(t, \cdot)||_{L^\infty} \leq C e^{bCt}||n_0||_{H^s}. \tag{4.21} \]

Proof. Differentiating the left-hand side of (4.19)-(4.20) with respect to $t$ and making use of (4.17)-(4.18) and the system (1.3), we have

\[
\frac{d}{dt}(m(t, q(t, x))q_x^b(t, x)) = (m_t(t, q) + m_x(t, q)q_x^b(t, x))q_x^b(t, x) + bm(t, q)q_x^{b-1}q_x(t, x)
\]

and

\[
\frac{d}{dt}(n(t, q(t, x))q_x^b(t, x)) = (n_t(t, q) + n_x(t, q)q_x^b(t, x))q_x^b(t, x) + bn(t, q)q_x^{b-1}q_x(t, x)
\]

which yield (4.19) and (4.20). By Lemma 4.3, in view of (4.18)-(4.20), the assumption of the lemma, and the fact $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ as $s > \frac{1}{2}$, we obtain for all $t \in [0, T)$,

\[ ||m(t, \cdot)||_{L^\infty} = ||m(t, q(t, \cdot))||_{L^\infty} \leq C e^{Ct}||m_0||_{H^s} \]

and

\[ ||n(t, \cdot)||_{L^\infty} = ||n(t, q(t, \cdot))||_{L^\infty} \leq C e^{Ct}||n_0||_{H^s}, \]

which complete the proof of the lemma. \qed
Remark 4.2. Under the same assumption of Lemma 4.4, by (4.1) and (4.3), we get

$$||uv(t, \cdot)||_{L^\infty} \leq C ||m(t, \cdot)||_{L^\infty} ||n(t, \cdot)||_{L^\infty} \leq Ce^{Ct},$$

where $C = C(b, ||m_0||_{H^s}, ||n_0||_{H^s})$ is a positive constant.

The following theorem shows the precise blow-up scenario for sufficiently regular solutions to the system (1.3).

**Theorem 4.2.** Let $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > \frac{1}{2}$ and $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to the system (1.3). Then the solution $(m, n)$ blows up in finite time if and only if

$$\liminf_{t \to T} (\inf_{x \in \mathbb{R}} (uv)_x(t, x)) = -\infty \quad \text{or} \quad \liminf_{t \to T} (\inf_{x \in \mathbb{R}} (u_xv)(t, x)) = -\infty.$$

**Proof.** Assume that the solution $(m, n)$ blows up in finite time ($T < \infty$) and there exists a constant $C > 0$ such that

$$(uv)_x(t, x) \geq -C \quad \text{and} \quad (u_xv)(t, x) \geq -C, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$ 

By (4.21), we have

$$\int_0^T (||m(t)||_{L^\infty} + ||n(t)||_{L^\infty})^2 dt \leq C^2 Te^{2bCT} (||m_0||_{H^s} + ||n_0||_{H^s})^2 < \infty,$$

which contradicts to Theorem 4.1.

On the other hand, by (4.1)-(4.4) and the Sobolev embedding theorem, if

$$\liminf_{t \to T} (\inf_{x \in \mathbb{R}} (u_xv)(t, x)) = -\infty \quad \text{or} \quad \liminf_{t \to T} (\inf_{x \in \mathbb{R}} (uv_x)(t, x)) = -\infty,$$

then the solution $(m, n)$ must blow up in finite time. This completes the proof of the theorem. \qed

**Remark 4.3.** Theorem 4.2 covers the corresponding results for the CH, DP and Novikov equations [9, 45, 49].

With Theorem 4.2 in hand, we conclude this section with the global existence for the strong solutions to system (1.3). On the one hand, Table 1 in the Introduction shows that, unlike the CH and Novikov equations, the $H^1(\mathbb{R})$ norm is not conserved for the solution to system (1.3), thus it cannot be applied directly to uniformly control the $L^\infty(\mathbb{R})$ norm of the solution by Sobolev’s embedding theorem. Besides, for the DP equation, one can obtain that the $L^\infty(\mathbb{R})$ norm of its solution will at most increase linearly in finite time, although the $H^1(\mathbb{R})$ norm is not a conservation law in this case [33]. On the other hand, Remark 4.2 and Theorem 4.2 tell us that the strong solution will blow up in finite time if the following assumption is not satisfied:

$$\exists C > 0, \quad \text{such that} \quad ||uv(t, \cdot)||_{L^\infty} \leq Ce^{Ct}, \quad \forall t \in [0, T). \quad (4.22)$$

Moreover, the $L^\infty(\mathbb{R})$ norms of the solutions to the CH, DP and Novikov equations all do satisfy (4.22). Therefore, the above exponential increase condition (4.22) is a reasonable assumption to study the global strong solution to the system (1.3). Indeed, we here could obtain the global existence under a slight weaker condition.
Theorem 4.3. Let \((m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})\) with \(s > \frac{1}{2}\) and \(T > 0\) be the maximal existence time of the corresponding solution \((m, n)\) to the system (1.3). Assume that there exists a constant \(C = C(b, ||m_0||_{H^s}, ||n_0||_{H^s}) > 0\) such that
\[
(uv)(t, x) \leq Ce^{Ct}, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]
Moreover, if both \(m_0(x)\) and \(n_0(x)\) do not change signs for all \(x \in \mathbb{R}\), then \(T = +\infty\), i.e. the solution to the system (1.3) exists globally.

Proof. We may assume that \(m_0(x), n_0(x) \geq 0\) for all \(x \in \mathbb{R}\). Because of (4.19) and (4.20), we have
\[
m(t, x), n(t, x) \geq 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]
Noticing
\[
u(t, x) = (1 - \partial_x^2)^{-1}m(t, x) = (p \ast m)(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|m(t, y)}dy,
\]
then we obtain
\[
u(t, x) = e^{-x} \int_{-\infty}^x e^y m(t, y)dy + \frac{e^x}{2} \int_x^{\infty} e^{-y} m(t, y)dy,
\]
and
\[
u_x(t, x) = -e^{-x} \int_{-\infty}^x e^y m(t, y)dy + \frac{e^x}{2} \int_x^{\infty} e^{-y} m(t, y)dy,
\]
which together with (4.23) imply
\[
u(t, x) + \nu_x(t, x) = e^x \int_x^{\infty} e^{-y} m(t, y)dy \geq 0
\]
and
\[
u(t, x) - \nu_x(t, x) = e^{-x} \int_{-\infty}^x e^y m(t, y)dy \geq 0.
\]
Hence, we have
\[
|\nu_x(t, x)| \leq \nu(t, x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]
Likewise, \(v(t, x) = (1 - \partial_x^2)^{-1}n(t, x) = (p \ast n)(t, x)\) ensures
\[
|v_x(t, x)| \leq v(t, x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]
By (4.24) and (4.25), together with the assumption of the theorem, we infer
\[
(u_xv)(t, x), (uv_x)(t, x) \geq -(uv)(t, x) \geq -Ce^{Ct}, \quad \forall (t, x) \in [0, T) \times \mathbb{R},
\]
which implies
\[
\text{nor} \quad \liminf_{t \to T} \left( \inf_{x \in \mathbb{R}} (u_xv)(t, x) \right) = -\infty \quad \text{nor} \quad \liminf_{t \to T} \left( \inf_{x \in \mathbb{R}} (uv_x)(t, x) \right) = -\infty.
\]
According to Theorem 4.2, \(T = +\infty\). Therefore, we have proven the theorem. \(\square\)
5. **Asymptotic behaviors.** In this section, we investigate the asymptotic behaviors of the strong solution to the system \((1.4)\) at infinity within its lifespan as the initial data decay exponentially and algebraically, respectively.

**Notation.** \(f(x) \sim \mathcal{O}(g(x))\) as \(x \to \infty\) means \(\lim_{x \to \infty} \frac{|f(x)|}{|g(x)|} \leq L\) for some \(L \geq 0\).

**Theorem 5.1.** Let \((u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})\) with \(s > \frac{5}{2}\) and \(T > 0\) be the maximal existence time of the corresponding solution \((u, v)\) to the system \((1.4)\). If there exists some \(\theta \in (0, 1)\) such that
\[
u_0(x), \nu_{0,x}(x), \nu_0(x), \nu_{0,x}(x) \sim \mathcal{O}(e^{-\theta x}), \text{ as } x \to +\infty,
\]
then
\[
u(t, x), u_x(t, x), \nu(t, x), v_x(t, x) \sim \mathcal{O}(e^{-\theta x}), \text{ as } x \to +\infty,
\]
uniformly in the time interval \([0, T]\).

**Proof.** Let us first introduce the following weight function series \(\{\varphi_N(x)\} \ (N = 1, 2, \cdots)\):
\[
\varphi_N(x) = \begin{cases} 
1, & x \leq 0, \\
e^{-\theta x}, & 0 < x < N, \\
e^{\theta N}, & x \geq N,
\end{cases}
\]
with \(\theta \in (0, 1)\). Then a simple calculation yields
\[
0 \leq \varphi'_N(x) \leq \varphi_N(x) \leq \max\{1, e^{\theta x}\}, \text{ a.e. } x \in \mathbb{R}, \tag{5.1}
\]
and
\[
\Phi_N(x) \triangleq \varphi_N(x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_N(y)} \, dy \leq \frac{3 - \theta}{1 - \theta} \text{ for all } x \in \mathbb{R}, \forall N \in \mathbb{N}_+ \tag{5.2}
\]
By the first equation in the system \((1.4)\), we have
\[
(\varphi_N u)_t + \varphi_N uu_x + \varphi_N(p * I(u, v)) = 0. \tag{5.3}
\]
Multiplying \((5.3)\) by \((\varphi_N u)^{2n-1}\) \((n \in \mathbb{N}_+)\) and integrating the resulted equation on \(\mathbb{R}\) with respect to \(x\), one gets
\[
\int_{\mathbb{R}} (\varphi_N u)^{2n-1}(\varphi_N u)_t \, dx = - \int_{\mathbb{R}} (\varphi_N u)^{2n} u_x v \, dx - \int_{\mathbb{R}} (\varphi_N u)^{2n-1} \varphi_N(p * I(u, v)) \, dx.
\]
Note that
\[
\int_{\mathbb{R}} (\varphi_N u)^{2n-1}(\varphi_N u)_t \, dx = \frac{1}{2n} \frac{d}{dt} \|\varphi_N u\|_{L^{2n}}^{2n} = \|\varphi_N u\|_{L^{2n}}^{2n-1} \frac{d}{dt} \|\varphi_N u\|_{L^{2n}},
\]
and
\[
\int_{\mathbb{R}} (\varphi_N u)^{2n} u_x v \, dx \leq \|u_x v\|_{L^\infty} \|\varphi_N u\|_{L^{2n}}^{2n},
\]
and
\[
\int_{\mathbb{R}} (\varphi_N u)^{2n-1} \varphi_N(p * I(u, v)) \, dx \leq \|\varphi_N u\|_{L^{2n}}^{2n-1} \|\varphi_N(p * I(u, v))\|_{L^{2n}},
\]
where we used Hölder’s inequality in the last inequality.

Thus, we deduce
\[
\frac{d}{dt} \|\varphi_N u\|_{L^{2n}} \leq \|u_x v\|_{L^\infty} \|\varphi_N u\|_{L^{2n}} + \|\varphi_N(p * I(u, v))\|_{L^{2n}}. \tag{5.4}
\]
To deal with the second equation in the system (1.4) in a similar way, we obtain
\[
\frac{d}{dt} ||\varphi_N u_x||_{L^{2n}} \leq ||u_{xx}||_{L^\infty} ||\varphi_N u||_{L^{2n}} + ||\varphi_N (p * J(u,v))||_{L^{2n}}. \tag{5.5}
\]

Differentiating the first equation in the system (1.4) with respect to \( x \) and multiplying the obtained equation by \( \varphi_N (x) \), one gets
\[
(\varphi_N u_x)_t + \varphi_N wu_x + \varphi_N u_x (uv) + \varphi_N (\partial_x p * I(u,v)) = 0. \tag{5.6}
\]

Multiplying (5.6) by \( (\varphi_N u_x)^{2n-1} \) \((n \in \mathbb{N}_+)\) and integrating the resulted equation on \( \mathbb{R} \) with respect to \( x \), we have
\[
\int_{\mathbb{R}} (\varphi_N u_x)^{2n-1}(\varphi_N u_x)_t \, dx = - \int_{\mathbb{R}} (uv)(\varphi_N u_x)^{2n-1} \, dx - \int_{\mathbb{R}} (\varphi_N u_x)^{2n} (uv)_x \, dx - \int_{\mathbb{R}} (\varphi_N u_x)^{2n-1} (\varphi_N (\partial_x p * I(u,v))) \, dx. \tag{5.7}
\]

As before, by the Hölder inequality, the right hand side of (5.7) can be bounded by
\[
(||u_{xx}||_{L^{\infty}} ||\varphi_N u||_{L^{2n}} + ||\varphi_N (\partial_x p * I(u,v))||_{L^{2n}} ||\varphi_N u_x||_{L^{2n}}^{2n-1} + ||(uv)_x||_{L^{\infty}} ||\varphi_N u_x||_{L^{2n}}^{2n}.
\]

Hence, we obtain
\[
\frac{d}{dt} ||\varphi_N u_x||_{L^{2n}} \leq ||u_{xx}||_{L^\infty} ||\varphi_N u||_{L^{2n}} + ||(uv)_x||_{L^{\infty}} ||\varphi_N u_x||_{L^{2n}} + ||\varphi_N (\partial_x p * I(u,v))||_{L^{2n}}, \tag{5.8}
\]

as well as
\[
\frac{d}{dt} ||\varphi_N v_x||_{L^{2n}} \leq ||u_{xx}||_{L^\infty} ||\varphi_N v||_{L^{2n}} + ||(uv)_x||_{L^{\infty}} ||\varphi_N v_x||_{L^{2n}} + ||\varphi_N (\partial_x p * J(u,v))||_{L^{2n}}. \tag{5.9}
\]

Set
\[
A(t) \triangleq ||\varphi_N u||_{L^{\infty}} + ||\varphi_N u_x||_{L^{\infty}} + ||\varphi_N v||_{L^{\infty}} + ||\varphi_N v_x||_{L^{\infty}},
\]

and
\[
M = M(s) \triangleq \sup_{t \in [0,T]} (||u(t,\cdot)||_{H^{s}} + ||v(t,\cdot)||_{H^{s}}) \quad \text{with} \quad s > \frac{5}{2}.
\]

In addition, observe that if the function \( f \in L^p \cap L^\infty \) for some \( 1 \leq p < \infty \) such that \( f \in L^q (\forall \ q > p) \), then
\[
\lim_{q \to \infty} ||f||_{L^q} = ||f||_{L^\infty}.
\]

Let \( n \to \infty \) in (5.4)-(5.5) and (5.8)-(5.9). Applying the Sobolev embedding theorem, we obtain
\[
\frac{d}{dt} A(t) \leq 2M^2 A(t) + ||\varphi_N (p * J(u,v))||_{L^\infty} + ||\varphi_N (\partial_x p * I(u,v))||_{L^\infty} + ||\varphi_N (p * J(u,v))||_{L^\infty} + ||\varphi_N (\partial_x p * J(u,v))||_{L^\infty}. \tag{5.10}
\]

On the other hand, in view of the facts
\[
(p * f)(x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} f(y) \, dy \quad \text{and} \quad |\partial_x p * f| \leq p * |f|,
\]
which together with (5.2) imply
\begin{align}
|\varphi_N(p * I(u, v))| & \leq \varphi_N p \left( |u u_x v_x| + |b uu_x v + (uu_x)_x v_x - (b-3)u_x u_{xxx}v| \right) \\
& = \frac{1}{2} \varphi_N(x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_N(y)} \left( |uu_y v_y| + |b uu_y v + (uu_y)_y v_y - (b-3)u_y u_{yy}v| \right) dy \\
& \leq \frac{3-\theta}{2(1-\theta)} \left( |b-3||uu_x v_x||_{L^\infty} + |b||uv||_{L^\infty} + ||uv||_{L^\infty} \right) \|\varphi_N u_x\|_{L^\infty} \\
& + ||uu_x||_{L^\infty} \|\varphi_N v_x||_{L^\infty} \\
& \triangleq RHS.
\end{align}

Thanks to Sobolev’s embedding theorem again, we obtain
\begin{align}
||\varphi_N(p * I(u, v))||_{L^\infty} & \leq \frac{3-\theta}{2(1-\theta)}(|b-3| + |b| + 2)M^2 A(t),
\end{align}
as well as
\begin{align}
||\varphi_N(p * J(u, v))||_{L^\infty} & \leq \frac{3-\theta}{2(1-\theta)}(|b-3| + |b| + 2)M^2 A(t).
\end{align}

Since $\partial_x^2 p * f = p * f - f$ and $|\partial_x p * f| \leq p * |f|$, it follows from (5.11) that
\begin{align}
|\varphi_N(\partial_x p * I(u, v))| & \leq \varphi_N p \left( |uu_x v_x| - uu_x v_x \right) + \varphi_N p \left( buu_x v + (uu_x)_x v_x - (b-3)u_x u_{xxx}v| \right) \\
& \leq RHS + |\varphi_N(x)uu_x v_x| \\
& \leq RHS + ||uu_x||_{L^\infty} \|\varphi_N u_x\|_{L^\infty},
\end{align}

which together with (5.12) and the Sobolev embedding theorem yield
\begin{align}
||\varphi_N(\partial_x p * I(u, v))||_{L^\infty} & \leq \left( \frac{3-\theta}{2(1-\theta)} (|b-3| + |b| + 2) + 1 \right) M^2 A(t).
\end{align}

Similarly,
\begin{align}
||\varphi_N(\partial_x p * J(u, v))||_{L^\infty} & \leq \left( \frac{3-\theta}{2(1-\theta)} (|b-3| + |b| + 2) + 1 \right) M^2 A(t).
\end{align}

Combining (5.10) and (5.12)-(5.15), we get
\begin{align}
\frac{d}{dt} A(t) & \leq CA(t),
\end{align}
where $C = C(b, s, M, \theta)$ is a positive constant.

Thanks to Gronwall’s inequality and (5.1), $\forall N \in \mathbb{N}_+$, $\forall (t, x) \in [0, T) \times \mathbb{R}$, we have
\begin{align}
A(t) & \leq e^{Ct} A(0) \\
& \leq e^{CT} (||u_0 max(1, e^{\theta x})||_{L^\infty} + ||u_{0,x} max(1, e^{\theta x})||_{L^\infty} \\
& + ||v_0 max(1, e^{\theta x})||_{L^\infty} + ||v_{0,x} max(1, e^{\theta x})||_{L^\infty}).
\end{align}
Finally, by the definition of $\varphi_N(x)$ and taking the limit as $N \to \infty$ in the above inequality, for all $(t, x) \in [0, T) \times \mathbb{R}$, we obtain
\[
|e^{\theta t}u(t, x)| + |e^{\theta t}u_x(t, x)| + |e^{\theta t}v(t, x)| + |e^{\theta t}v_x(t, x)| \\
\leq e^{CT}(||u_0\max(1, e^{\theta x})||_{L^\infty} + ||u_0x\max(1, e^{\theta x})||_{L^\infty} \\
+ ||v_0\max(1, e^{\theta x})||_{L^\infty} + ||v_0x\max(1, e^{\theta x})||_{L^\infty}),
\]
which completes the proof of the theorem.

However, the assumption of exponentially decaying initial data in Theorem 5.1 is too restrictive, which enlightens us to consider a slower decay rate. A natural way is to study the asymptotic behavior of the solution to system (1.4) provided the initial data decays algebraically at infinity. Indeed, if we introduce another auxiliary weight function series $\{\psi_N(x)\}$ $(N = 1, 2, \cdots)$ as follows:
\[
\psi_N(x) = \begin{cases} 
1, & x \leq 0, \\
(1 + x)^\alpha, & 0 < x < N, \\
(1 + N)^\alpha, & x \geq N,
\end{cases}
\]
where $\alpha \in (0, 1]$. Then a direct computation leads to
\[
0 \leq \psi'_N(x) \leq \psi_N(x) \leq \max\{1, (1 + x)^\alpha\}, \ a.e. \ x \in \mathbb{R},
\]
and
\[
\Psi_N(x) \triangleq \psi_N(x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\psi_N(y)} \, dy \leq 3 + (1 + \alpha)^2, \ \forall \ x \in \mathbb{R}, \ \forall \ N \in \mathbb{N}_+.
\]
So, making use of $\psi_N(x)$ instead of $\varphi_N(x)$ and going along the similar line as the proof in Theorem 5.1, one can readily get the following result.

**Theorem 5.2.** Let $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > \frac{5}{2}$ and $T > 0$ be the maximal existence time of the corresponding solution $(u, v)$ to the system (1.4). If there exists some $\alpha \in (0, 1]$ such that
\[
u_0(x), u_0x(x), v_0(x), v_0x(x) \sim O((1 + x)^{-\alpha}), \ as \ x \to +\infty,
\]
then
\[
u(t, x), u_x(t, x), v(t, x), v_x(t, x) \sim O((1 + x)^{-\alpha}), \ as \ x \to +\infty,
\]
uniformly in the time interval $[0, T)$.

**Remark 5.1.** By modifying slightly the definitions of the weight function series $\{\varphi_N(x)\}$ and $\{\psi_N(x)\}$, Theorems 5.1-5.2 can also be shown for the corresponding decay assumptions as $x \to -\infty$.

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REFERENCES

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften, Vol. 343, Berlin-Heidelberg-New York: Springer, 2011.

[2] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, *Arch. Rat. Mech. Anal.*, 183 (2007), 215–239.

[3] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Letters*, 71 (1993), 1661–1664.

[4] R. Camassa, D. Holm and J. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.*, 31 (1994), 1–33.

[5] G. M. Coclite and K. H. Karlsen, On the well-posedness of the Degasperis-Procesi equation, *J. Funct. Anal.*, 233 (2006), 60–91.

[6] A. Constantin, Global existence of solutions and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)*, 50 (2000), 321–362.

[7] A. Constantin and C. Foias, Global existence and blow-up for a shallow water equation, *Annali Sc. Norm. Sup. Pisa*, 26 (1998), 303–328.

[8] A. Constantin and J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math.*, 51 (1998), 475–504.

[9] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematica*, 181 (1998), 229–243.

[10] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, *Ann. of Math. (2)*, 173 (2011), 559–568.

[11] A. Constantin, R. Ivanov and J. Lenells, Inverse scattering transform for the Degasperis-Procesi equation, *Nonlinearity*, 23 (2010), 2559–2575.

[12] A. Constantin and B. Kolev, Geodesic flow on the diffeomorphism group of the circle, *Comm. Math. Helv.*, 78 (2003), 787–804.

[13] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Rat. Mech. Anal.*, 192 (2009), 165–186.

[14] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, *Comm. Math. Phys.*, 211 (2000), 45–61.

[15] A. Constantin and W. A. Strauss, Stability of peakons, *Comm. Pure Appl. Math.*, 53 (2000), 603–610.

[16] H. H. Dai, Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod, *Acta Mech.*, 127 (1998), 193–207.

[17] R. Danchin, A few remarks on the Camassa-Holm equation, *Differential Integral Equations*, 14 (2001), 953–988.

[18] A. Degasperis, D. D. Holm and A. N. W. Hone, A new integral equation with peakon solutions, *Theo. Math. Phys.*, 133 (2002), 1463–1474.

[19] A. Degasperis and M. Procesi, Asymptotic integrability, *Symmetry and Perturbation Theory*, Rome, 1998, World Sci. Publishing, River Edge, NJ, 1999, 23–37.

[20] H. R. Dullin, G. A. Gottwald and D. D. Holm, An integrable shallow water equation with linear and nonlinear dispersion, *Phys. Rev. Letters*, 87 (2001), 4501–4504.

[21] J. Escher and B. Kolev, The Degasperis-Procesi equation as a non-metric Euler equation, *Math. Z.*, 269 (2011), 1137–1153.

[22] J. Escher and Z. Yin, Initial boundary value problems for nonlinear dispersive wave equations, *J. Funct. Anal.*, 256 (2009), 479–508.

[23] A. Fokas, On a class of physically important integrable equations, *Physica D*, 87 (1995), 145–150.

[24] A. Fokas and B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Physica D*, 4 (1981), 47–66.

[25] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation, *Physica D*, 95 (1996), 229–243.

[26] X. Geng and B. Xue, An extension of integrable peakon equations with cubic nonlinearity, *Nonlinearity*, 22 (2009), 1847–1856.

[27] G. Gui and Y. Liu, On the global existence and wave-breaking criteria for the two-component Camassa-Holm system, *J. Funct. Anal.*, 258 (2010), 4251–4278.

[28] A. A. Himonas and C. Holliman, The Cauchy problem for the Novikov equation, *Nonlinearity*, 25 (2012), 449–479.
[29] A. A. Himonas and J. Holmes, Hölder continuity of the solution map for the Novikov equation, 
*J. Math. Phys.*, **54** (2013), 061501, 11pp.

[30] Y. Hou, P. Zhao, E. Fan and Z. Qiao, Algebro-geometric solutions for the Degasperis–Procesi 
hierarchy, *SIAM J. Math. Anal.*, **45** (2013), 1216–1266.

[31] R. S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. 
Fluid. Mech.*, **455** (2002), 63–82.

[32] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, in: *Spectral Theory and Differential Equations, Lecture Notes in Math.*, Springer Verlag, Berlin, **448** (1975), 25–70.

[33] Y. Liu and Z. Yin, Global existence and blow-up phenomena for the Degasperis-Procesi 
equation, *Comm. Math. Phys.*, **267** (2006), 801–820.

[34] H. Lundmark, Formation and dynamics of shock waves in the Degasperis-Procesi equation, 
*J. Nonlinear Sci.*, **17** (2007), 169–198.

[35] H. P. McKean, Breakdown of a shallow water equation, *Asian J. Math.*, **2** (1998), 867–874.

[36] V. Novikov, Generalizations of the Camassa-Holm equation, *J. Phys. A*, **42** (2009), 342002, 
14pp.

[37] P. J. Olver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions 
having compact support, *Phys. Rev. E*, **53** (1996), 1900–1906.

[38] Z. Qiao, The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro- 
geometric solution on a symplectic submanifold, *Comm. Math. Phys.*, **239** (2003), 309–341.

[39] Z. Qiao, Integrable hierarchy (the DP hierarchy), 3 by 3 constrained systems, and parametric and stationary solutions, *Acta Applicandae Mathematicae*, **83** (2004), 199–220.

[40] Z. Qiao, A new integrable equation with cuspons and W/M-shape-peaks solitons, *J. Math. 
Phys.*, **47** (2006), 112701, 9pp.

[41] Z. Qiao, New integrable hierarchy, its parametric solutions, cuspons, one-peak solutions, and 
M/W-shape peak solitons, *J. Math. Phys.*, **48** (2007), 082701, 20pp.

[42] G. Rodriguez-Blanco, On the Cauchy problem for the Camassa-Holm equation, *Nonlinear 
Anal.*, **46** (2001), 309–327.

[43] J. F. Toland, Stokes waves, *Topol. Methods Nonlinear Anal.*, **7** (1996), 1–48.

[44] G. B. Whitham, *Linear and Nonlinear Waves*, A Wiley-Interscience Publication. John Wiley 
& Sons, Inc., New York, 1999.

[45] X. Wu and Z. Yin, Well-posedness and global existence for the Novikov equation, *Ann. Sc. 
Norm. Super. Pisa Cl. Sci. (5)*, **11** (2012), 707–727.

[46] Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, *Comm. Pure Appl. 
Math.*, **53** (2000), 1411–1433.

[47] K. Yan and Z. Yin, Analytic solutions of the Cauchy problem for two-component shallow 
water systems, *Math. Z.*, **269** (2011), 1113–1127.

[48] K. Yan, Z. Qiao and Z. Yin, Qualitative analysis for a new integrable two-component Camassa- 
Holm system with peakon and weak kink solutions, *Comm. Math. Phys.*, **336** (2015), 581–617.

[49] Z. Yin, On the Cauchy problem for an integrable equation with peakon solutions, *Illinois J. 
Math.*, **47** (2003), 649–666.

[50] Z. Yin, Global weak solutions to a new periodic integrable equation with peakon solutions, 
*J. Funct. Anal.*, **212** (2004), 182–194.

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