OBJECT-UNITAL GROUPOID GRADED MODULES

JUAN CALA, PATRIK NYSTEDT, AND H. PINEDO

Abstract. Given a groupoid \( G \) and an associative but not necessarily unital ring \( R \), we introduce the notion of object unital graded ring and construct the category of object unital graded modules which we denote by \( G-R\text{-umod} \). Following the ideas developed in [8], we focus our attention on the forgetful functor \( U: G-R\text{-umod} \to R\text{-mod} \) and on the attempt to determine properties \( X \) for which the following implications are valid: For \( M \in G-R\text{-umod} \), (i) if \( M \) has property \( X \), then \( U(M) \) has property \( X \); (ii) if \( U(M) \) has property \( X \), then \( M \) has property \( X \). Here we treat the cases when \( X \) is: direct summand, projective, injective, free, simple and semisimple. Moreover, graded versions of results concerning classical module theory are established, as well as some structural properties related to the category \( G-R\text{-umod} \).

1. Introduction

Let \( R \) denote an associative, but not necessarily unital, ring and \( M \) a left (resp. right) \( R \)-module. If \( X \subseteq R \) and \( Y \subseteq M \), then we let \( XY \) (or \( YX \)) denote the set of finite sums of elements of the form \( xy \) (or \( yx \)) for \( x \in X \) and \( y \in Y \). Then we say that \( M \) is unitary if the equality \( RM = M \) (resp. \( MR = R \)) holds. We denote by \( R\text{-mod} \) the category having unitary left \( R \)-modules as objects and \( R \)-module homomorphisms as morphisms. Let \( G \) denote a groupoid, that is a small category with the property that all morphisms are isomorphisms. This also can be defined by saying that \( G \) is a set equipped with a unary operation \( G \ni \sigma \mapsto \sigma^{-1} \in G \) (inversion) and a partially defined multiplication \( G \times G \ni (\sigma, \tau) \mapsto \sigma \tau \in G \) (composition) such that for all \( \sigma, \tau, \rho \in G \) the following four axioms hold:

(i) \( (\sigma^{-1})^{-1} = \sigma \); (ii) if \( \sigma \tau \) and \( \tau \rho \) are defined, then \( (\sigma \tau) \rho \) and \( \sigma (\tau \rho) \) are defined and \( (\sigma \tau) \rho = \sigma (\tau \rho) \); (iii) the domain \( d(\sigma) := \sigma^{-1} \sigma \) is always defined and if \( \sigma \tau \) is defined, then \( d(\sigma) \tau = \tau \); (iv) the range \( r(\tau) := \tau^{-1} \tau \) is always defined and if \( \sigma \tau \) is defined, then \( \sigma r(\tau) = \sigma \).

The maps \( d \) and \( r \) have common image denoted by \( G_0 \), which is called the unit space of \( G \). The set \( G_2 = \{ (\sigma, \tau) \in G \times G : \sigma \tau \text{ is defined} \} \) is called the set of composable pairs of \( G \).

Recall from [8] that by a graded ring by \( G \) we mean a ring \( R \) with the property that for every \( \sigma \in G \) there is an additive subgroup \( R_\sigma \) of \( R \) such

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that \( R = \bigoplus_{\sigma \in \mathcal{G}} R_{\sigma} \) and for all \( \sigma, \tau \in \mathcal{G} \) we have either \( R_{\sigma} R_{\tau} \subseteq R_{\sigma \tau} \) if \( (\sigma, \tau) \in \mathcal{G}_2 \), or \( R_{\sigma} R_{\tau} = \{0\} \), otherwise. The set \( H(R) = \bigcup_{\sigma \in \mathcal{G}} R_{\sigma} \) is called the set of homogeneous elements of \( R \). If \( r \in R_{\sigma} \setminus \{0\} \), then we say that \( r \) is of degree \( \sigma \) which is denoted by \( \deg(r) = \sigma \). Any \( r \in R \) has a unique decomposition \( r = \sum_{\sigma \in \mathcal{G}} r_{\sigma} \), where \( r_{\sigma} \in R_{\sigma} \), for \( \sigma \in \mathcal{G} \), and all but a finite number of the \( r_{\sigma} \) are zero.

Some important classes of rings that can be graded by groupoids are matrix rings, crossed product algebras defined by separable extensions and partial skew groupoid rings (see for instance [8], [9] and [12]).

Let \( M \) be a left \( R \)-module which is graded by \( \mathcal{G} \). This means that for all \( \sigma \in \mathcal{G} \) there is an additive subgroup \( M_{\sigma} \) of \( M \) such that \( M = \bigoplus_{\sigma \in \mathcal{G}} M_{\sigma} \) and for all \( \sigma, \tau \in \mathcal{G} \) we have either \( R_{\sigma} M_{\tau} \subseteq M_{\sigma \tau} \) if \( (\sigma, \tau) \in \mathcal{G}_2 \), or \( R_{\sigma} M_{\tau} = \{0\} \) otherwise. If \( N \) is another left \( R \)-module graded by \( \mathcal{G} \), then a left \( R \)-module homomorphism \( f: M \to N \) is said to be graded if for all \( \sigma \in \mathcal{G} \) the inclusion \( f(M_{\sigma}) \subseteq N_{\sigma} \) holds. The collection of left \( R \)-modules graded by \( \mathcal{G} \) and the collection of graded homomorphisms together form the category of graded \( R \)-modules which we denote by \( \mathcal{G} \text{-} R \text{-mod} \), which turns to be an abelian category with enough projectives.

A natural class of functors to study from a categorical perspective are the forgetful functors, which forgets parts of the structure. For instance \( U: \mathcal{G} \text{-} R \text{-mod} \to R \text{-mod} \) which is defined by forgetting the grading. This suggests a natural question:

**Question 1.** For which graded modules \( M \) and for what properties \( X \) do any of the following implications hold

1. \( M \) has property \( X \) \( \Rightarrow \) \( U(M) \) has property \( X \)?;
2. \( U(M) \) has property \( X \) \( \Rightarrow \) \( M \) has property \( X \)?

When \( R \) is unital and \( \mathcal{G} \) is a group, then Question 1 has been investigated for many different properties \( X \) including: direct summand, free, finitely generated, finitely presented, projective, injective, essential, small and flat (see [10] Sections I.2 – I.3). Many of these results have been extended to the category \( \mathcal{G} \text{-} R \text{-mod} \) when \( R \) is a unital ring (see [8], [9]), and \( \mathcal{G} \) is a groupoid. Also cohomological results related to the category \( \mathcal{G} \text{-} R \text{-mod} \) are given in [7] [9] and more recently in [12]. The aim of this article is to establish these and other results in the context when \( R \) is an object unital groupoid graded ring, which generalizes that concept of just being unital (see below).

Recall that a subgroupoid of \( \mathcal{G} \) is a subcategory \( \mathcal{H} \) of \( \mathcal{G} \) which is closed under inverses. The impetus for the investigations in the present article is the following observation from [8] Proposition 2.1.1]: if \( R \) is a unital ring which is graded by \( \mathcal{G} \), then there is a subgroupoid \( \mathcal{H} \) of \( \mathcal{G} \) such that (i) \( \mathcal{H}_0 \) is finite; (ii) \( R = \bigoplus_{\sigma \in \mathcal{H}} R_{\sigma} \); (iii) for all \( e \in \mathcal{H}_0 \) the ring \( R_e \) is non-zero and unital; (iv) \( 1_R = \sum_{\sigma \in \mathcal{H}} 1_{R_{\sigma}} \). In other words, if \( R \) is unital, then we can always assume that \( \mathcal{H}_0 \) is finite. This puts a severe restriction on the groupoids and certainly does not cover many interesting ring constructions.
e.g. infinite direct sum of non-zero unital rings, rings of matrices over an infinite index set, where each matrix only has finitely many non-zero entries, or, even more generally, groupoid algebras over arbitrary groupoids. To rectify this we suggest to replace unitality by the weaker condition of being **object unital**. By this we mean that for all $e \in G_0$ the ring $R_e$ is unital and for all $\sigma \in G$ and all $r \in R_\sigma$ there are $1_{R_{r(\sigma)}} \in R_{r(\sigma)}$ and $1_{R_{d(\sigma)}} \in R_{d(\sigma)}$ such that the equalities $1_{R_{r(\sigma)}} r = r 1_{R_{d(\sigma)}} = r$ hold.

Here is a detailed outline of the article, after the introduction in Section 2, we present some notions and results that will be used throughout the work, moreover we introduce the category $O\mathcal{G} - \text{RING}$ of object unital graded rings, which will be important for our purposes. The principal object for our study is the category $G - \text{R-umod}$ consisting of graded unital modules and graded homomorphism over object unital graded rings is introduced in in Section 3, some properties of the functors $\text{HOM}$ and tensor over this category are considered, also we give an answer of question (1) in the cases when $X$ is one of the properties: direct summand and free, we also investigate when split exact sequences are preserved by the forgetful functor. The answer of question (1) when $X$ is projective or injective is treated in Section 4. Finally in Section 5 we deal with the semisimplicity property.

Since some of the proofs of our results resemble their ungraded counterparts, we have sometimes taken the liberty of omitting the details.

2. Preliminaries

Throughout this work $\mathcal{G}$ denotes a groupoid and $R$ denotes an associative, but not necessarily unital, ring graded by $\mathcal{G}$.

2.1. A monoid related to $\mathcal{G}$. Recall that a monoid is a non-empty set $\mathcal{M}$ equipped with an associative binary operation $*$ and a neutral element $e$. An element $x \in \mathcal{M}$ is called invertible if there is $y \in \mathcal{M}$ such that $x * y = y * x = e$.

**Proposition 2.** Let $\mathcal{P}(\mathcal{G})$ denote the set consisting of non-empty subsets of $\mathcal{G}$. If we for $\Sigma, \Sigma' \in \mathcal{P}(\mathcal{G})$, define

$$\Sigma * \Sigma' = \{ \sigma \tau \mid \sigma \in \Sigma, \tau \in \Sigma', d(\sigma) = r(\tau) \},$$

then:

(a) $(\mathcal{P}(\mathcal{G}), *)$ is a monoid with neutral element $\mathcal{G}_0$.

(b) The element $\Sigma \in \mathcal{P}(\mathcal{G})$ is invertible, if and only if, the maps $d: \Sigma \rightarrow \mathcal{G}_0$ and $r: \Sigma^{-1} \rightarrow \mathcal{G}_0$ are bijections, where $\Sigma^{-1} = \{ \sigma^{-1} \mid \sigma \in \Sigma \}$.

(c) Take $\sigma \in \mathcal{G}$. If we let $\Sigma_\sigma \in \mathcal{P}(\mathcal{G})$ be defined by

$$\Sigma_\sigma = \begin{cases} \{ \sigma, \sigma^{-1} \} \cup (\mathcal{G}_0 \setminus \{d(\sigma), r(\sigma)\}) & \text{if } d(\sigma) \neq r(\sigma), \\
\{ \sigma \} \cup (\mathcal{G}_0 \setminus \{d(\sigma)\}) & \text{otherwise,} \end{cases}$$

then $\Sigma_\sigma$ is invertible.
Proof. (a) Is given in [8, Proposition 2.1.1]. Now we check (b). For \( \Rightarrow \) as observed in [8, Proposition 2.1.1] one has that \( \Sigma \) is invertible, if and only if \( \Sigma \star \Sigma^{-1} = \Sigma^{-1} \star \Sigma = \mathcal{G}_0 \) which implies that the maps \( d, r: \Sigma \to \mathcal{G}_0 \) are surjective. If \( d \) is not injective (resp. \( r \) is not injective), there are \( \sigma, \tau \in \Sigma \) such that \( \sigma \neq \tau \) and \( d(\sigma) = d(\tau) \) (resp. \( r(\sigma) = r(\tau) \)). From this, as is the proof of [8, Proposition 2.1.1], we obtain the contradiction 

\[
\sigma \tau^{-1} \in (\Sigma \star \Sigma^{-1}) \setminus \mathcal{G}_0 \text{ (resp. } \sigma^{-1} \tau \in (\Sigma^{-1} \star \Sigma) \setminus \mathcal{G}_0) \text{). Conversely, if } d: \Sigma \to \mathcal{G}_0 \text{ and } r: \Sigma^{-1} \to \mathcal{G}_0 \text{ are bijections, then } \Sigma \star \Sigma^{-1} = \Sigma^{-1} \star \Sigma = \mathcal{G}_0 \text{ and } \Sigma \text{ is invertible. For the part (c) it is not difficult to show that the maps } d: \Sigma_\sigma \to \mathcal{G}_0 \text{ and } r: \Sigma^{-1}_\sigma \to \mathcal{G}_0 \text{ are bijections.} \qed
\]

2.2. Conventions on rings and modules. Following Fuller in [5] we say that \( R \) has enough idempotents if there exists a set \( \{e_i\}_{i \in I} \) of orthogonal idempotents in \( R \) (called a complete set of idempotents for \( R \)) such that 

\[
R = \bigoplus_{i \in I} Re_i = \bigoplus_{i \in I} e_i R. 
\]

Following Ánh and Márki in [1] we say that \( R \) is locally unital if for all \( n \in \mathbb{N} \) and all \( r_1, \ldots, r_n \in R \) there is an idempotent \( e \in R \) such that for all \( i \in \{1, \ldots, n\} \) the equalities \( er_i = r_i e = r_i \) hold. The ring \( R \) is called \( s \)-unital if for all \( r \in R \) the relation \( r \in Rr \cap rrR \) holds. The following chain of implications hold (see e.g. [11]) for all rings:

\[
(3) \text{ unital } \Rightarrow \text{ enough idempotents } \Rightarrow \text{ locally unital } \Rightarrow s\text{-unital } \Rightarrow \text{idempotent.}
\]

Remark 3. Let \( R \) be a locally unital ring and let \( M \) be a unitary left \( R \)-module. For \( n \in \mathbb{N} \) and all \( m_1, \ldots, m_n \in M \) there is an idempotent \( e \in M \) such that for all \( i \in \{1, \ldots, n\} \) the equalities \( em_i = m_i \) hold. Moreover for any \( R \)-module \( N \) one has that \( RN \) is unital.

Definition 4. We say that \( R \) is object unital if the ring \( R_{r(\sigma)} \) is unital with identity \( 1_{R_{r(\sigma)}} \) and for all \( \sigma \in \mathcal{G} \) and all \( r \in R_\sigma \) the equalities \( 1_{R_{r(\sigma)}} r = r 1_{R_{d(\sigma)}} = r \) hold. We denote by \( \mathcal{U}_{gr}(R) \) the set of identities of each \( R_{r(\sigma)} \), \( \sigma \in \mathcal{G} \), that is, \( \mathcal{U}_{gr}(R) = \{1_{R_{r(\sigma)}} : \sigma \in \mathcal{G}\} \).

Example 5. Let \( T \) be a ring. The groupoid ring of \( T \) over \( \mathcal{G} \), denoted by \( T[\mathcal{G}] \), is defined to be the set of all formal sums \( \sum_{\sigma \in \mathcal{G}} t_\sigma \sigma \), with \( t_\sigma \in T \), \( \sigma \in \mathcal{G} \), and \( t_\sigma = 0 \) for almost all \( \sigma \in \mathcal{G} \). Addition is defined point-wise and multiplication is defined by \( T \)-linear extension of the rule

\[
\sigma \cdot \tau = \begin{cases} 
\sigma \tau, & \text{if } d(\sigma) = r(\tau), \\
0, & \text{otherwise.}
\end{cases}
\]

The grading is, of course, given by \( T[\mathcal{G}]_{t(\sigma)} = T \sigma, \sigma \in \Gamma \), and if \( T \) is unital, then \( T[\Gamma] \) is a graded unital ring with \( \mathcal{U}_{gr}(T[\mathcal{G}]) = \{1_{T e} : e \in \mathcal{G}_0\} \). Two important cases of groupoid rings are the following.

If \( \mathcal{G} \) is a group, then \( T[\mathcal{G}] \) is the usual group ring of \( T \) over \( \mathcal{G} \). On the other hand, if \( \mathcal{G} = I \times I \), where \( I \) is a finite set, and \( \mathcal{G} \) is equipped with the operation \((i,j) \cdot (k,l) = (i,l)\) if \( j = k \), then \( T[\mathcal{G}] \) is the ring of \(|I| \times |I|\) matrices over \( T \).
Example 6. Partial skew groupoid rings Let $G$ be a groupoid such that $G_0$ is infinite, $A$ an associative ring, and let $\alpha = (A_g, \alpha_g)_{g \in G}$ be an unital partial action of $G$ in $A$. That is, for all $g \in G$, $A_g$ is an ideal of $A_{r(g)}$ and $A_{r(g)}$ is an ideal of $A$, moreover there exists a central idempotent $1_g$ of $A$ such that $A_g = A1_g$, $\alpha_g : A_{g^{-1}} \to A_g$ is a ring isomorphism such that $\alpha_g h$ is an extension of $\alpha_g \circ \alpha_h$, for all $(g, h) \in G_2$. The partial skew groupoid ring $A \star_\alpha G$ associated to $\alpha$ is the set of all formal sums $\sum_{g \in G} a_g \delta_g$, where $a_g \in A_g$, with the usual addition and multiplication induced by the following rule

$$(a_g \delta_g)(a_h \delta_h) = \begin{cases} a_g a_h 1_{g^{-1}} \delta_{gh}, & \text{if } (g, h) \in G_2 \\ 0, & \text{otherwise,} \end{cases}$$

for all $g, h \in G$, $a_g \in A_g$ and $a_h \in A_h$. Then $A \star_\alpha G = \bigoplus_{g \in G} A_g \delta_g$ is an object unital ring with $U_{gr}(A \star_\alpha G) = \{1_g \delta_{r(g)} | g \in G\}$.

3. Groupoid graded modules

For the rest of the article, we fix a graded ring $R$ such that $R$ is object unital. Given a graded $R$-module $M = \bigoplus_{\sigma \in G} M_\sigma$, the set $H(M) = \bigcup_{\sigma \in G} M_\sigma$ is called the set of homogeneous elements of $M$. If $m \in M_\sigma$ is a non-zero element, then we say that $m$ is of degree $\sigma$ and write $\text{deg}(m) = \sigma$. Any $m \in M$ has a unique decomposition $m = \sum_{\sigma \in G} m_\sigma$, where $m_\sigma \in M_\sigma$, for $\sigma \in G$, and all but a finite number of the $m_\sigma$ are zero.

If $N$ is an $R$-submodule of $M$, then it is called a graded submodule if $N = \bigoplus_{\sigma \in G} (N \cap M_\sigma)$. An (left, right) ideal of $R$ is called graded if it is graded as a (left, right) submodule of $R$.

The following lemma will be used in the sequel.

Lemma 7. [§ 3.1.1 Lemma] Let $M$, $N$ and $P$ be graded left $R$-modules and suppose that $f : M \to P$, $g : N \to P$ and $h : M \to N$ are $R$-linear maps such that $f = g \circ h$. If $f$ and $g$ (resp. $f$ and $h$) are graded maps, then there is a graded map $h' : M \to N$ (resp. $f = g \circ h'$ (resp. $f = g' \circ h$).

Definition 8. Let $M$ be a graded left (resp. right) $R$-module. We say that $M$ is left (resp. right) graded unital if for all $\sigma \in G$ and all $m_\sigma \in M_\sigma$ the equality $1_{R_{r(\sigma)}} m_\sigma = m_\sigma$ (resp. the equality $m_\sigma 1_{R_{l(\sigma)}} = m_\sigma$) holds.

We denote by $G$-$R$-$\text{umod}$ the subcategory of $G$-$R$-$\text{mod}$ consisting of graded unital modules. Then $G$-$R$-$\text{umod}$ is an abelian subcategory of $G$-$R$-$\text{mod}$.

We give the following.

Proposition 9. Let $M$ be an object in $G$-$R$-$\text{umod}$. Then $M$ is a unitary $R$-module.

Proof. If $m \in M$, then $m = \sum_{\sigma \in G} m_\sigma$, where $m_\sigma \in M_\sigma$, $\sigma \in G$, and all but a finite number of the $m_\sigma$ are nonzero. Since $M$ is graded unital, given $\sigma \in G$ every $m_\sigma \in M_\sigma$ satisfies $1_{R_{r(\sigma)}} m_\sigma = m_\sigma$ and hence $m = \sum_{\sigma \in G} 1_{R_{r(\sigma)}} m_\sigma \in RM$, that is, $M$ is unital. \hfill \square
3.1. **The suspension functor.** For an object \( M = \bigoplus_{\tau \in \mathcal{G}} M_{\tau} \) in \( \mathcal{G}-\text{R-mod} \) and \( \sigma \in \mathcal{G} \) the \( \sigma \)-suspension of \( M \) is a graded submodule of \( M(\sigma) \) with graduation

\[
M(\sigma)_{\tau} = \begin{cases} M_{\tau\sigma}, & \text{if } (\tau, \sigma) \in \mathcal{G}_{2}, \\ \{0\}, & \text{otherwise.} \end{cases}
\]

It is clear that \( M(\sigma) = M \) for all \( \sigma \in \mathcal{G} \), if and only if, \( \mathcal{G} \) is a group.

Notice that \( M(\sigma) \) is an object in \( \mathcal{G}-\text{R-umod} \) for all \( \sigma \in \mathcal{G} \), provided that \( M \) is also in \( \mathcal{G}-\text{R-umod} \). Then suspension functor introduced in \([8, \text{Proposition 2.2.3}]\), restricts to category \( \mathcal{G}-\text{R-umod} \) as follows: For \( \Sigma \in \mathcal{P}(\mathcal{G}) \), let

\[
T_{\Sigma} : \mathcal{G}-\text{R-umod} \to \mathcal{G}-\text{R-umod}
\]

be defined at objects by \( T_{\Sigma}(M) = \bigoplus_{\tau \in \Sigma} M(\sigma) \), for all graded left unital \( R \)-modules \( M \), and for \( f : M \to N \) in \( \mathcal{G}-\text{R-umod} \) is given by

\[
T_{\Sigma}(f) \left( \sum_{\sigma \in \Sigma} m_{\sigma} \right) = \sum_{\sigma \in \Sigma} f(m_{\sigma}).
\]

We have the following.

**Proposition 10.** \([8 \text{ Proposition 2.2.3}]\) With the above notations, we get:

(a) If \( \Sigma, \Sigma' \in \mathcal{P}(\mathcal{G}) \), then \( T_{\Sigma}T_{\Sigma'} = T_{\Sigma \times \Sigma'} \).

(b) If \( \Sigma \in \mathcal{P}(\mathcal{G}) \) is invertible, then \( T_{\Sigma} \) is an auto-equivalence of \( \mathcal{G}-\text{R-umod} \).

**Lemma 11.** Let \( M \in \mathcal{G}-\text{R-umod} \), \( \sigma \in \mathcal{G} \). Then the assertions below hold:

i) \( R(\sigma) = R1_{R(d(\sigma))} \), the left principal ideal generated by \( 1_{R(d(\sigma))} \)

ii) Let \( m \in M_{\sigma} \) and \( Rm \) the cyclic submodule generated by \( m \). Then \( Rm = R(\sigma^{-1})m \) and \( Rm \) is a unital \( \mathcal{G} \)-graded submodule of \( M \).

iii) If \( M \in \mathcal{G}-\text{R-umod} \) is finitely generated, then \( M \) contains a maximal graded submodule.

**Proof.** Fix \( \sigma \in \mathcal{G} \).

i) Since \( 1_{R(d(\sigma))} \in R_{\sigma^{-1}\sigma} = R(\sigma)_{\sigma^{-1}} \), then \( (1_{R(d(\sigma))}) \subseteq R(\sigma) \). Conversely, if \( r = \sum r_{\tau\sigma} \in R(\sigma) \), then

\[
r = \sum r_{\tau\sigma} = \sum r_{\tau\sigma}1_{R(d(\sigma))} = \left( \sum r_{\tau\sigma} \right)1_{R(d(\sigma))}.
\]

Therefore, \( R(\sigma) \subseteq R1_{R(d(\sigma))} \).

ii) Let \( m \in M_{\sigma} \), \( r = \sum_{\tau \in \mathcal{G}} r_{\tau} \in R \). Then \( rm = \left( r1_{R(\sigma)} \right) m \in R(\sigma^{-1})m \). Thus, \( Rm \subseteq R(\sigma^{-1})m \). The other inclusion follows immediately. To see that \( Rm \) is a \( \mathcal{G} \)-graded submodule of \( M \), we must show that if \( \tau \in \mathcal{G} \) then \( (Rm)_{\tau} = M_{\tau} \cap Rm \), where \( (Rm)_{\tau} = R(\sigma^{-1})_{\tau}m \). We have to consider two cases.

**Case 1:** \( (\tau, \sigma^{-1}) \notin \mathcal{G}(2) \). Then \( (Rm)_{\tau} = \{0\} = M_{\tau} \cap Rm \).

**Case 2:** \( (\tau, \sigma^{-1}) \in \mathcal{G}(2) \). In this case \( R(\sigma^{-1})_{\tau}m \subseteq R_{\tau\sigma^{-1}\sigma}m \subseteq M_{\tau d(\sigma)} = M_{\tau} \) and this implies \( (Rm)_{\tau} \subseteq M_{\tau} \cap Rm \). Now, if \( rm \in M_{\tau} \) with \( r \in R \) then

\[
r = \sum_{\lambda \in \mathcal{G}} r_{\lambda} = r_{\tau\sigma^{-1}} \in R(\sigma^{-1})_{\tau}.
\]

Thus, \( rm \in (Rm)_{\tau} \), as desired.
iii) Consider the collection $L$ of all proper graded submodules of $M$. This is a non-empty collection partially ordered by inclusion. Let $K$ be a chain in $L$ and put $N = \sum_{L \in K} L$. Then $N$ is an upper bound for $K$ and $N \in L$. Otherwise $N = M$ and there would be a finite set $I \subseteq H(M)$ such that $N = \sum_{m \in I} Rm$. But then every $m \in I$ would belong to the graded submodule $Rm$ in $K$ and since $K$ is a chain, the finite sum $N = \sum_{m \in I} Rm \in K$, leading to the contradiction $M \in K$. From this, Zorn’s Lemma provides a maximal graded submodule of $M$. \[ \square \]

3.2. Graded homomorphisms and the functors $HOM$ and tensor. Let $M$ and $N$ be graded left $R$-modules. If $f: M \to N$ is $R$-linear and $\Sigma \in P(G)$, then we say that $f$ is a map of degree $\Sigma$ if for all $\lambda \in G$ we have

$$f(M_\lambda) \subseteq \bigoplus_{\sigma \in \Sigma, \ r(\sigma) = d(\lambda)} N_{\sigma \lambda}.$$ 

The collection of maps of degree $\Sigma$ is denoted $HOM_R(M, N)_\Sigma$. In the case that $\Sigma$ is the emptyset we set $HOM_R(M, N)_\emptyset = \{0\}$, moreover if $\Sigma = \{\sigma\}$ for some $\sigma \in G$, then we write $HOM_R(M, N)_\sigma$ instead of $HOM_R(M, N)_{\{\sigma\}}$. In the sequel, we will refer to $f \in HOM_R(M, N)_\sigma$ as a graded map.

**Remark 12.** In the case that $M$ and $N$ are graded right $R$-modules, a map $f: M \to N$ is of degree $\Sigma$ if $f(M_\lambda) \subseteq \bigoplus_{\sigma \in \Sigma, \ d(\sigma) = r(\lambda)} N_{\sigma \lambda}$, and analogously one defines the sets $HOM_R(M, N)_\Sigma$, for all $\Sigma \in P(G)$.

Let $Ab_G$ denote the category of $G$-graded abelian groups. Groups of this type can always, in a natural way, be viewed as graded left $\mathbb{Z}[G_0]$-modules (note that $\mathbb{Z}[G_0]$ - being a graded subring of $\mathbb{Z}[G]$ - is an object unital ring). We call this the trivial grading of the objects in $Ab_G$.

3.2.1. Homomorphisms. Define the functor

$$HOM_R: (G-R\text{-umod})^{op} \times G-R\text{-umod} \to Ab_G$$

by

$$HOM_R(M, N) = \bigoplus_{\sigma \in \Sigma} HOM_R(M, N)_\sigma,$$

for all $M, N$ graded unital left $R$-modules.

As usual, $END_R(M)$ denotes $HOM_R(M, M)$.

**Remark 13.** If $M$ and $N$ are graded left $R$-modules, then

$$HOM_R(M, N) \subseteq \text{hom}_R(M, N).$$

The equality holds in (4) e.g. when $G$ is a finite group or $M$ is finitely generated and $G$ is an arbitrary group. However, equality does not hold in general (for a counterexample in the case when $G$ is a group, see [10, p. 11]).
By Remark 13 one may ask if the equality holds in (4) when \( G \) is a finite groupoid or \( M \) is finitely generated. For this, take \( \varphi \in \text{hom}_{R}(M, N) \) such that \( \varphi = \sum_{\tau \in \mathcal{G}} \varphi_{\tau} \) for some family \( (\varphi_{\tau})_{\tau \in \mathcal{G}} \) of graded maps. Then for any \( \sigma \in \mathcal{G} \) we have that
\[
(5) \quad \varphi(M_{\sigma}) = \sum_{\tau \in \mathcal{G}} \varphi_{\tau}(M_{\sigma}) \subseteq \sum_{\tau \in \mathcal{G}, r(\tau) = d(\sigma)} N_{\sigma \tau},
\]
which does not hold in general. For a concrete example we give the following.

**Example 14.** Let \( I \) denote a set. Consider the groupoid \( \mathcal{G} := I \times I \) defined in Example 5. Now take \( I = \{1, 2\} \), a ring \( A \) and \( S = A \times A \). Then \( S \) is a \( G \)-graded ring by taking
\[
\bullet S_{(1, 2)} = A \times \{0\},
\]
\[
\bullet S_{(2, 1)} = \{0\} \times A,
\]
\[
\bullet S_{\sigma} = \{0\}, \sigma \notin \{(1, 2), (2, 1)\}.
\]
Now let \( f : S^{2} \to S^{2} \) be defined by \( f(a, b, c, d) = (b, a, d, c) \), for all \( a, b, c, d \in A \). If \( \sigma = (1, 2) \), we get that \( f(R_{\sigma}^{2}) = R_{\sigma^{-1}}^{2} \), but \( \sum_{\tau \in \mathcal{G}, r(\tau) = d(\sigma)} R_{\sigma \tau}^{2} = R_{\sigma} \). So (5) does not hold.

For the rest of the article, we fix another object unital ring \( S \).

**Definition 15.** A \( R-S \)-bimodule \( M \) is called graded if there is a family of additive subgroups, \( M_{\sigma}, \sigma \in \mathcal{G}, \) of \( M \) such that \( M = \bigoplus_{\sigma \in \mathcal{G}} M_{\sigma} \), and for all \( \sigma, \tau \in \mathcal{G} \), we have \( R_{\sigma}M_{\tau}S_{\rho} \subseteq M_{\sigma \tau \rho} \), \( d(\sigma) = r(\tau) \) and \( d(\tau) = r(\rho) \), and \( R_{\sigma}M_{\tau}S_{\rho} = \{0\} \) otherwise. Moreover, \( M \) is said to be graded \((R, S)\)-unital, if for all \( \sigma \in \mathcal{G} \) and \( m_{\sigma} \in M_{\sigma} \) the equalities \( 1_{R_{\sigma}(\sigma)}m_{\sigma} = m_{\sigma} = m_{\sigma}1_{S_{d(\sigma)}} \) hold. Let \( \mathcal{G}-R-\text{unmod-}S \) denote the category of graded unital \((R, S)\)-bimodules. The morphisms \( f : M \to N \) are taken to be \( R-S \)-bimodule maps such that \( f(M_{\sigma}) \subseteq N_{\sigma} \) for all \( \sigma \in \mathcal{G} \).

We gather some elementary properties of \( \text{HOM} \). We start with the following result whose proof is straightforward.

**Proposition 16.** Let \( R \) and \( S \) be object unital graded rings. Then
\begin{enumerate}
\item[(a)] Given \( R_{\text{MS}} \) and \( R_{\text{N}} \) the action
\[
(6) \quad r \cdot f : M \ni m \mapsto s \cdot f(m) = f(sm) \in N,
\]
for all \( s \in S \) and \( f \in \text{HOM}_{R}(M, N) \) defines a left \( S \)-module structure on \( \text{HOM}_{R}(M, N) \).
\item[(b)] If \( R_{\text{MS}} \) and \( N_{\text{S}} \), the action
\[
f \cdot r : M \ni m \mapsto f \cdot r(m) = f(rm) \in N,
\]
for all \( r \in R \) and \( f \in \text{HOM}_{S}(M, N) \) defines a right \( R \)-module structure on \( \text{HOM}_{S}(M, N) \).
\end{enumerate}
Proposition 17. Let \( e \in [13] \). For (a) consider the map
\[
\eta: \mathcal{M} \ni m \mapsto \eta_m \in RHOM_R(M, M),
\]
where \( \eta_m(x) = xm \), for all \( x \in R \). First we need to check that \( \eta \) is well defined, that is \( \eta_m \in RHOM_R(R, M) \) for all \( m \in M \). Indeed, it is not difficult to show that if \( m = \sum_{\sigma \in \mathcal{G}} m_{\sigma} \) is the homogeneous decomposition of \( m \in M \), then \( \eta_m = \sum_{\sigma \in \mathcal{G}} \eta_{m_{\sigma}} \) and \( \eta_{m_{\sigma}} \in HOM_R(R, M)_\sigma \), then \( \eta_m \in HOM_R(R, M) \), for all \( m \in M \). Moreover, by Proposition 9 one has that \( M \) is unital, then \( M = RM \) and \( \eta_m \in RHOM_R(R, M) \). Now we check that \( \eta \) is an isomorphism. Take \( m, m' \) in \( M \), then \( rm = rm' \) for all \( m \in M \), since \( M \) is left unital, by Remark 8 there is an idempotent \( e \in R \) such that \( em = m \) and \( em' = m' \), so \( m = m' \). To prove that \( \eta \) is surjective, take \( f \in RHOM_R(R, M) \). Since \( RHOM_R(R, M) \) is an unital left \( R \)-module, there is an idempotent \( e \) of \( R \) such that \( f = e \cdot f \), that is \( f(x) = f(xe) \) for all \( x \in R \). This means that \( \eta_{f(e)} = f \) and therefore \( \eta \) is surjective. Finally it is clear that \( \eta \) is \( R \)-linear.

\[ \]
Proposition 19. There is an \((R,R)\)-bimodule isomorphism
\[
\eta: R \to REND_R(R),
\]
where \(\eta\) is defined by \((7)\).

Proof. By (a) of Proposition 17 one has that \(\eta\) is an isomorphism in the category \(\mathcal{G}\text{-}R\text{-}umod\). Now \(REND_R(R)\) has right \(R\)-module structure given by \((f \cdot r)(x) = f(x) r\), and with this action \(\eta\) is right \(R\)-linear. \(\square\)

3.2.2. Tensor products. If \(M\) is a graded right \(R\)-module and \(N\) is a graded left \(R\)-module, then we may consider \(M \otimes_R N\) as an object in \(\text{Ab}_\mathcal{G}\), where the grading is defined by letting \((M \otimes_R N)_\sigma, \sigma \in \Gamma\), be the \(\mathbb{Z}\)-module generated by all \(m_\tau \otimes n_\rho, d(\tau) = r(\rho), \tau \rho = \sigma, m_\tau \in M_\tau, n_\rho \in N_\rho\). To see that this is well defined, note that \(M \otimes_R N = M \otimes_\mathbb{Z} N/L\) where \(L\) is the graded subgroup of \(M \otimes_\mathbb{Z} N\) generated by the elements of the form \(mr \otimes n - m \otimes rn\). The grading on \(M \otimes_R N\) is therefore induced by the grading on \(M \otimes_\mathbb{Z} N\).

We state some elementary properties concerning \(\text{HOM}\) and \(\otimes\).

Proposition 20. Let \(M\) be a graded unital right \(R\)-module, \(N\) a graded unital \(R\)-module, and \(P\) a graded unital right \(S\)-module. Suppose that \(U_{gr}(R) = U_{gr}(S)\). Then:

(a) \(M \otimes_R N\) is a graded unital right \((R,S)\)-bimodule.

(b) There is an isomorphism in \(\text{Ab}_\mathcal{G}\):
\[
\text{HOM}_S(M \otimes_R N, P) \cong \text{HOM}_R(M, \text{HOM}_S(N, P) R),
\]
which is natural in each variable \(M, N, P\).

Proof. The proof of (a) is analogous to the ungraded case (found e.g. in [13]). (see loc. cit.). For (b) let
\[
\varphi: \text{HOM}_S(M \otimes_R N, P) \ni f \mapsto \varphi_f \in \text{HOM}_R(M, \text{HOM}_S(N, P) R),
\]
where \(\varphi_f: M \ni m \to \varphi_f(m) \in \text{HOM}_S(N, P) R\), and \(\varphi_f(m)(n) = f(m \otimes_R n)\) for all \(n \in N\). It is not difficult to show that \(\varphi_f(m) \in \text{HOM}_S(N, P)\) and \(\varphi_f \in \text{HOM}_R(M, \text{HOM}_S(N, P) R)\), for all \(m \in M\) and \(f \in \text{HOM}_S(M \otimes_R N, P)\). To see that \(\varphi_f(m) \in \text{HOM}_S(N, P) R\), we may assume without loss of generality that \(m\) is of degree \(\tau\), so \(m = m1_{S_{d(\tau)}} = m1_{R_{d(\tau)}}\), which implies \(\varphi_f(m) \cdot 1_{R_{d(\tau)}} = \varphi_f(m)\), and \(\varphi_f(m) \in \text{HOM}_S(N, P) R\). The rest of the proof follows as in the ungraded case. \(\square\)

Remark 21. Item b) of Proposition 20 says that the functors
\[
M \otimes_R - : \mathcal{G} \text{-} R\text{-}umod \to \mathcal{G} \text{-} umod \cdot S
\]
and
\[
\text{HOM}_S(-, P) R: \mathcal{G} \text{-} umod \cdot S \to \mathcal{G} \text{-} R\text{-}umod
\]
form and adjoint pair.
3.3. **Direct summands and exact sequences.** Let \( A \) and \( B \) be objects in an abelian category. Recall that \( B \) is called a direct summand of \( A \) if there is an object \( C \) in the category such that \( A \cong B \oplus C \).

As a consequence from Lemma \ref{lem:summand} we have the next result.

**Corollary 22.** Let \( M \) and \( N \) be graded left \( R \)-modules. If \( N \) is a graded submodule of \( M \), then \( N \) is a direct summand of \( M \) if and only if \( U(N) \) is a direct summand of \( U(M) \).

**Definition 23.** In the category \( \mathcal{G}-R\text{-umod} \) we say that a short exact sequence

\[
0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0
\]

splits if there exists an isomorphism \( h: M \to L \oplus N \) in \( \mathcal{G}-R\text{-umod} \) making commutative the diagram below:

\[
\begin{array}{c}
0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \\
\big\downarrow \quad \big\downarrow h \\
0 \to L \oplus N \xrightarrow{\pi_N} N \to 0
\end{array}
\]

**Proposition 24.** For a short exact sequence in \( \mathcal{G}-R\text{-umod} \)

\[
0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0
\]

the following conditions are equivalent:

i) The sequence splits.

ii) There exists \( \varphi: M \to L \) in \( \mathcal{G}-R\text{-umod} \) such that \( \varphi \circ f = 1_L \).

iii) There exists \( \psi: N \to M \) in \( \mathcal{G}-R\text{-umod} \) such that \( g \circ \psi = 1_N \).

**Proof.** The proofs of i) \( \Rightarrow \) ii), i) \( \Rightarrow \) iii) and iii) \( \Rightarrow \) i) are as in the ungraded case, also the part ii) \( \Rightarrow \) i) is done in a similar way, just taking into account the Five Lemma for abelian categories (\cite{bass} Theorem 5.9)). \( \square \)

3.4. **Free modules.** For the case of unital rings, the notion of free module in \( \mathcal{G}-R\text{-mod} \) was first introduced by the second named author in \cite{Steve} 3.2 as follows: An object \( M \in \mathcal{G}-R\text{-mod} \) is free if there exists a collection \( \{ \sigma_i: i \in I \} \) of elements of \( \mathcal{G} \) such that \( M \cong \bigoplus_{i \in I} R(\sigma_i) \) as graded \( R \)-modules.

In the particular case when \( \mathcal{G} \) is a group, it can be proven (see \cite{Steve} p. 33) that the following conditions are equivalent for \( M \in \mathcal{G}-R\text{-mod} \):

i) \( M \) is free;

ii) \( M \) has a \( R \)-basis of homogeneous elements (of not necessarily distinct degrees).

In the general case when \( \mathcal{G} \) is a groupoid this is no longer valid. Indeed, if we consider the groupoid ring over a field \( \mathbb{K} \) of \( \mathcal{G} = I \times I \), where \( I = \{1, 2\} \),
then \( \mathbb{K}G \cong M_2(\mathbb{K}) \) and if \( \sigma = (1, 2) \in G \), \( \mathbb{K}G(\sigma) = \mathbb{K}(1, 2) \oplus \mathbb{K}(2, 2) \). Thus,

\[
\mathbb{K}G(\sigma) \cong \begin{bmatrix} 0 & \mathbb{K} \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{K} \end{bmatrix}.
\]

If \( a, b \in \mathbb{K} \) are not simultaneously nonzero, then

\[
\begin{bmatrix} -b & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Thus \( \mathbb{K}G(\sigma) \) is, by definition [8, 3.2], free but each nonzero element in \( \mathbb{K}G(\sigma) \) has nontrivial annihilator and so \( R(\sigma) \) cannot have a \( R \)-basis of homogeneous elements.

But the converse holds. Indeed, if \( B \) is a \( R \)-basis consisting of homogeneous elements of a module \( M \in \mathcal{G} \)-\text{mod}, we can define a \( R \)-linear map from \( \bigoplus_{m \in B} R(\sigma_m) \) to \( M = \bigoplus_{m \in B} Rm \), where \( \sigma_m = \deg(m)^{-1} \in G \), sending every \( e_m \) into \( m \in B \), where \( e_m \) is the element whose entries are all zero except at the \( m \)-th coordinate where it takes the value \( 1_{R(\sigma_m)} \). By Lemma [11] this is a well-defined graded surjective map and since the elements of the basis have trivial annihilators, the map is an isomorphism.

The previous discussion shows us that the definition [8, 3.2] makes sense. However, the term free, adopted from the classic module theory, comes precisely from the fact that each element of the basis can not be annihilated under the action of the ring (free of annihilators). With this in mind we give the following.

**Definition 25.** An object \( M \in \mathcal{G} \)-\text{mod} is called **free (of finite type)** by suspension whenever \( M \cong \bigoplus_{i \in I} R(\sigma_i) \), for some (finite) collection \( \{\sigma_i : i \in I\} \), and the isomorphism is given in \( \mathcal{G} \)-\text{mod}.

**Example 26.** If \( R \) is graded unital, then \( R = \bigoplus_{\delta \in \mathcal{G}_0} R(\delta) \) and so \( R \) is free by suspension. Moreover, if \( R \) is unital, it follows from [8, Proposition 2.1.1] that \( R \) is free of finite type.

**Proposition 27.** Every \( M \in \mathcal{G} \)-\text{mod} is the quotient of a free module by suspension.

**Proof.** Let \( \{m_i : i \in I\} \) be a homogeneous generator set of \( M \), with \( \deg(m_i) = \sigma_i, i \in I \). Put \( N = \bigoplus_{i \in I} R(\sigma_i^{-1}) \). The function \( \varphi : N \rightarrow M \) defined by \( R \)-linear extension of \( e_i \mapsto m_i, i \in I \), is an epimorphism in \( \mathcal{G} \)-\text{mod} and, therefore, \( M \cong N / \operatorname{Ker} \varphi \) and the isomorphism is in \( \mathcal{G} \)-\text{mod}.

As in the case of unital left graded modules, we can always prove the following (see [8, Proposition 3.2.2.]).

**Proposition 28.** Let \( M \) be a free graded unital left \( R \)-module (of finite type). Then there is a free graded unital left \( R \)-module \( M' \) (of finite type) such that \( U(M \oplus M') \) is free (of finite type).
4. Projective and injective objects in $\mathcal{G}$-$R$-$\mathbb{u}mod$

4.1. Projective modules. Recall that an object $A$ in an abelian category $\mathcal{A}$ is called projective if the functor $\text{hom}_\mathcal{A}(A, -): \mathcal{A} \to \text{Ab}$ is exact.

As in the ungraded case, we have the following.

**Proposition 29.** An object $P \in \mathcal{G}$-$R$-$\mathbb{u}mod$ is projective if, and only if, for every epimorphism $g \in \text{hom}_{\mathcal{G}$-$R$-$\mathbb{u}mod}(M, N)$ and every $h \in \text{hom}_{\mathcal{G}$-$R$-$\mathbb{u}mod}(P, N)$, there exists $\overline{h} \in \text{hom}_{\mathcal{G}$-$R$-$\mathbb{u}mod}(P, M)$ such that $h = g \circ \overline{h}$, that is, making that the diagram commutes

\[
\begin{array}{ccc}
P & \xrightarrow{\overline{h}} & M \\
\downarrow{h} & & \downarrow{g} \\
0 & \rightarrow & N
\end{array}
\]

Items (i) and (ii) of the next result are standard facts which can be found in \cite{14}.

**Proposition 30.** Let $\mathcal{A}$ be an abelian category. Then:

(i) If $(P_t)_{t \in I}$ is a family of objects in $\mathcal{A}$, then $\bigoplus_{t \in I} P_t$ is projective if and only if each $P_t$ is projective.

(ii) If $0 \to A \to B \xrightarrow{\alpha} C \to 0$ is an exact sequence in $\mathcal{A}$, then the sequence splits if and only if there is $\beta: C \to B$ such that $\alpha \circ \beta = \text{id}_C$.

Using Proposition 30(i), Lemma 11 and Lemma 7 also hold in the category $\mathcal{G}$-$R$-$\mathbb{u}mod$, one can prove the following result in the same way as in \cite[Lemma 3.2.1]{8}

**Lemma 31.** If a graded left $R$-module is free by suspension, then it is projective.

Now we give the graded version of \cite[Proposition 2.2]{2}.

**Lemma 32.** If $e \in R$ is a nonzero homogeneous idempotent, then $Re$ is a projective module. In particular $R$ is projective.

**Proof.** The fact that $Re$ is an object of $\mathcal{G}$-$R$-$\mathbb{u}mod$ is consequence of Lemma 11. Now, consider the diagram

\[
\begin{array}{ccc}
Re & \xrightarrow{\overline{h}} & M \\
\downarrow{h} & & \downarrow{g} \\
0 & \rightarrow & N
\end{array}
\]

where $M, N \in \mathcal{G}$-$R$-$\mathbb{u}mod$, $g$ and $h$ are morphism in $\mathcal{G}$-$R$-$\mathbb{u}mod$ with $g$ surjective and $\overline{h}: r \mapsto rm$ is the morphism in $R$-$\mathbb{m}od$ with $m \in M$ fixed such that $h(e) = g(m)$. Since $e$ is idempotent, $r = re$ for all $r \in Re$ and so, $h(r) = rh(e) = rg(m) = g(rm) = g(\overline{h}(r))$. 
Then \( h = g \circ \overline{h} \) and consequently the diagram commutes. Since \( h \) and \( g \) are morphisms in \( \mathcal{G} \)-\texttt{umod}, \( \overline{h} \) can be assumed in \( \mathcal{G} \)-\texttt{umod} by Lemma 7, and the conclusion that \( Re \) is projective follows from Proposition 29. Finally by Example 26 and item i) of Lemma 11 we get that \( R = \bigoplus_{\sigma \in G_0} R_1 R_{d(\sigma)} \) and \( R \) is projective due to i) of Proposition 30. \( \square \)

**Remark 33.** Notice that if \( A \) is a non unital ring then \( _AA \) is not necessarily projective. Indeed, \( _AA \) is in general locally projective (see [1, Proposition 2]).

**Proposition 34.** Consider the following statements for \( P \in \mathcal{G} \)-\texttt{mod}:

i) \( U(P) \) is projective.

ii) \( U(P) \) is projective in \( R^1 \)-\texttt{mod}, where \( R^1 = R \times \mathbb{Z} \) is the unitalization of \( R \).

iii) \( P \) is projective.

iv) Every short exact sequence in \( \mathcal{G} \)-\texttt{umod}

\[ 0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} P \rightarrow 0 \]  \hspace{1cm} (9)

splits.

v) \( P \) is a direct summand of a free module by suspension.

**Proof.** i) \( \iff \) ii) This follows from (ii) of [2 Proposition 2.4] and the fact that \( R \) is locally unital.

i) \( \implies \) iii) Consider the diagram

\[
\begin{array}{c}
P \\
\downarrow h \\
M \xrightarrow{g} N \xrightarrow{0}
\end{array}
\]

where \( M, N \in \mathcal{G} \)-\texttt{umod}, \( g, h \) are morphism in \( \mathcal{G} \)-\texttt{umod} and \( g \) is surjective. Since \( P \) is projective in \( R \)-\texttt{mod} there is \( \overline{h}: P \rightarrow M \) in \( R \)-\texttt{mod} such that \( h = g \circ \overline{h} \). But then by Lemma 7 the map \( h \) can be considered in \( \mathcal{G} \)-\texttt{umod} and so \( P \) is projective.

iii) \( \implies \) i) This can be done as the second part of the proof of [3 Proposition 3.4.3].

iii) \( \implies \) iv) Consider the following diagram:

\[
\begin{array}{c}
P \\
\downarrow 1_P \\
0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} P \rightarrow 0
\end{array}
\]

If \( P \) is projective, there is \( \varphi: P \rightarrow M \) in \( \mathcal{G} \)-\texttt{mod} such that \( g \circ \varphi = 1_P \).

By Proposition 24 the sequence (9) splits.
iv) \Rightarrow v) By Proposition 27 there is a short exact sequence
\[ 0 \to \ker \varphi \to F \overset{\varphi}{\to} P \to 0 \]
with \( F \) free by suspension. By hypothesis, this sequence split in \( \mathcal{G}\text{-}\text{R-mod} \), so \( P \) is a direct summand of \( F \).

v) \Rightarrow i) By Lemma 31 and Proposition 30 it follows that \( P \) is projective. On the other hand, the proof of Lemma 32 can be used to show that every free module by suspension is projective in \( \text{R-mod} \) (specifically, every \( R(\sigma), \sigma \in \mathcal{G} \)). Therefore, \( U(P) \) being a direct summand of a direct sum of projective modules, is projective in \( \text{R-mod} \). \( \square \)

By Proposition 34 and [8, Proposition 3.5.1 b)], we get:

**Corollary 35.** Let \( M \) be a graded left unital \( \text{R-module} \). Then \( M \) is projective finitely generated if and only if \( U(M) \) is projective finitely generated.

4.2. **Injective modules.** Recall that an object \( A \) in an abelian category \( \mathcal{A} \) is called injective if the functor \( \text{hom}_A(\text{−}, A): \mathcal{A} \to \text{Ab} \) is exact.

We have the next.

**Proposition 36.** An object \( Q \in \mathcal{G}\text{-}\text{R-umod} \) is injective if, and only if, for every monomorphism \( f \in \text{hom}_{\mathcal{G}\text{-}\text{R-umod}}(M, N) \) and every morphism \( h \in \text{hom}_{\mathcal{G}\text{-}\text{R-umod}}(M, Q) \), there exists \( \overline{h} \in \text{hom}_{\mathcal{G}\text{-}\text{R-umod}}(N, Q) \) such that \( h = \overline{h} \circ f \), that is, making the diagram below commutative:

\[
\begin{array}{ccc}
Q & \xrightarrow{f} & N \\
\downarrow{h} & & \downarrow{\overline{h}} \\
0 & \to & M \\
\end{array}
\]

Now we give a description of the injective objects in \( \mathcal{G}\text{-}\text{R-umod} \) analogous to Baer’s criterion (see e.g. [13]). But first we recall the following well-known result about injective objects in abelian categories whose prove can be found in [14].

**Proposition 37.** Let \( (A_i)_{i \in I} \) be a family of objects in an abelian category. Then \( \prod_{i \in I} A_i \) is injective if and only if each \( A_i \) is injective.

**Proposition 38.** The following statements for \( Q \in \mathcal{G}\text{-}\text{R-umod} \) are equivalent:

i) \( Q \) is injective.

ii) Every short exact sequence in \( \mathcal{G}\text{-}\text{R-umod} \)

\[ 0 \to Q \overset{f}{\to} M \overset{g}{\to} N \to 0 \]

splits.

iii) The functor \( \text{HOM}_{\mathcal{G}\text{-}\text{R-umod}}(\text{−}, M): \text{R-mod} \to \text{Ab}_\mathcal{G} \) is exact.
(iv) For every graded left ideal $I$ of $R$, the canonical map

$$HOM_R(R, M) \to HOM_R(I, M)$$

is surjective.

Proof. i) $\Rightarrow$ ii) Consider the following diagram in $G$-$R$-$\text{umod}$:

$$
\begin{array}{cccccc}
Q & \xrightarrow{r} & 0 \\
\downarrow{1_Q} & & & & \\
0 & \rightarrow & Q & \rightarrow & M & \rightarrow & N & \rightarrow & 0
\end{array}
$$

If $Q$ is injective, there is $h: M \rightarrow Q$ in $G$-$R$-$\text{umod}$ such that $h \circ f = 1_Q$ and the sequence (10) splits by Proposition 24.

ii) $\Rightarrow$ i) Consider the diagram in $G$-$R$-$\text{umod}$

$$
\begin{array}{ccc}
Q & \xrightarrow{r} & 0 \\
\downarrow{h} & & \\
0 & \rightarrow & M & \rightarrow & N
\end{array}
$$

where $f$ is a monomorphism. Put $J = (Q \oplus N)/K$, where $K = \{h(m) - f(m) : m \in M\}$ is graded submodule of $Q \oplus N$ and the graduation is induced by the homogeneous components of $M$. Define $\varphi: Q \rightarrow J$ by $q \mapsto \overline{q}$, $q \in Q$. Then $\varphi$ is a monomorphism in $G$-$R$-$\text{umod}$. By hypothesis, there is a morphism $\psi: J \rightarrow Q$ in $G$-$R$-$\text{umod}$ such that $\psi \circ \varphi = 1_Q$. With this define $\overline{h} = \psi \circ \theta: N \rightarrow Q$, where $\theta: N \rightarrow J$ is the morphism of $G$-$R$-$\text{umod}$ given by $n \mapsto \overline{n}$, $n \in N$. Observe that for $m \in M$, $\varphi(h(m)) = \overline{h(m)} = f(m) = \theta(f(m))$. Hence, $\overline{h}$ is a morphism in $G$-$R$-$\text{umod}$ satisfying $\overline{h} \circ f = \psi \circ (\theta \circ f) = (\psi \circ \varphi) \circ h = 1_N \circ h = h$. Therefore, $Q$ is injective.

(i) $\Rightarrow$ (iii). Let $\Sigma \in \mathcal{P}(G)$ and $\sigma \in \Sigma$. By c) of Proposition 2 the element $\Sigma_{\sigma}$ is invertible in $\mathcal{P}(G)$, and thus $T_{\Sigma_{\sigma}}$ is invertible due to Proposition 10(b). Since injectivity is preserved by equivalences the proof follows as in [8, Proposition 3.5.2].

The implication (iii)$\Rightarrow$(iv) is clear.

Finally the proof that (iv) holds implies (i) follows the lines of (iii) $\Rightarrow$(i) in [8, Proposition 3.5.2].

The following is a direct consequence of Proposition 37.

Corollary 39. Let $M$ be a graded left $R$-module. If $U(M)$ is injective, then $M$ is injective.

As observed in [8] the converse of Corollary 39 does not hold in general. For a counterexample when $G$ is a group, see p. 8 of [10].
5. Semisimplicity in $G$-$R$-umod

We recall the following.

**Definition 40.** An object $M \in G$-$R$-umod is **simple** if their only graded submodules are $\{0\}$ and $M$. When $M$ happens to be a sum of simple modules it will be called **semisimple**.

It’s easily seen that every simple object $M \in G$-$R$-umod is semisimple. If $M$ is simple as $R$-module, then $M$ is simple, being $\{0\}$ and $M$ their only $R$-submodules.

In classic module theory, given $M \in R$-mod the following properties are equivalents:

i) $M$ is semisimple;

ii) $M$ is a direct sum of simple modules;

iii) Every submodule of $M$ is a direct summand.

Our purpose is to prove a graded version of this result. Also, we will conclude that semisimplicity is a well behaved property under preimages of the forgetful functor.

The proof of the following result follows the lines of the ungraded case.

**Proposition 41.** The following properties of an object $M \in G$-$R$-umod are equivalent:

i) $M$ is semisimple.

ii) $M$ is a direct sum of simple modules.

**Proof.** From the definition is straightforward that ii) $\Rightarrow$ i). Let $M = \sum_{i \in I} M_i$ be a sum of simples modules. The proof will be completed if we can show that this sum is direct. In order to do this, we will prove the following:

**Assertion 42.** For every graded submodule $N$ of $M$, there exists $J \subseteq I$ such that

$$M = N \oplus \bigoplus_{j \in J} M_j.$$

In fact, consider the non-empty collection

$$\mathcal{M} = \left\{ J \subseteq I : N + \sum_{j \in J} M_j \text{ is direct} \right\}.$$

$\mathcal{M}$ is partially order by inclusion. We will show that $\mathcal{M}$ is inductive and by an application of Zorn’s Lemma the maximal element of $\mathcal{M}$ is exactly $M$.

Let $\mathcal{B}$ be a chain in $\mathcal{M}$. Put $K = \bigcup_{J \in \mathcal{B}} J$. Then $K$ is an upper bound for $\mathcal{B}$ and $K \in \mathcal{M}$. For, if $K \not\in \mathcal{M}$, there will be $j_1 \in J_1, \ldots, j_r \in J_r$ and

$$n \in N, m_1 \in M_{j_1}, \ldots, m_{j_r} \in M_{j_r} \text{ not all zero such that}$$

$$0 = n + \sum_{t=1}^{r} m_{j_t}.$$
Being $\mathcal{B}$ chain, there is $J \in \mathcal{B}$ such that $j_1, \ldots, j_r \in J$ and consequently the sum $N + \sum_{j \in J} M_j$ will not be direct, which is impossible. Therefore, Zorn’s Lemma provides a maximal $J \subseteq I$ with the property that $L = N + \sum_{j \in J} M_j$ is direct. If we can show that $L = M$ we will be end. For this, is enough to see that $M_i \subseteq L$, for every $i \in I$. But this follows immediately since if $M_i \cap L = \{0\}$ then $J \cup \{i\} \in \mathcal{M}$.

Now i) is obtained by taking $N = \{0\}$. □

The next result shows two really important facts about graded submodules of a semisimple graded module.

**Proposition 43.** Let $M = \bigoplus_{i \in I} M_i$ be a sum of simple modules and let $N$ be a graded submodule of $M$. Then:

i) $N$ is a direct summand.

ii) $N \cong \bigoplus_{j \in J} M_j$, for some $J \subseteq I$ and the isomorphism is given in $\mathcal{G}$-$\mathcal{R}$-umod.

*Proof.* i) It follows directly from the proof of Proposition 41.

ii) By i), there is a graded submodule $K$ of $M$ such that $M = N \oplus L$. By an application of Zorn’s Lemma as in the proof of Proposition 41 $M = L \oplus \bigoplus_{j \in J} M_j$ for some $J \subseteq I$. Therefore, $N \cong M/L \cong \bigoplus_{j \in J} M_j$ and the isomorphism is given in $\mathcal{G}$-$\mathcal{R}$-umod. □

As a consequence of Proposition 43 we have as in the ungraded case the following fact.

**Corollary 44.** Every graded submodule and every quotient of a semisimple module is semisimple.

Now we are in conditions to establish the desired equivalence.

**Proposition 45.** For an object $M \in \mathcal{G}$-$\mathcal{R}$-umod the following properties are equivalents:

i) $M$ is semisimple

ii) $M$ is a direct sum of simple modules.

iii) Every graded submodule of $M$ is a direct summand.

*Proof.* By Proposition 43 we have that i) and ii) are equivalent and i) ⇒ iii) follows by Assertion 12. It only remains to prove iii) ⇒ i). By hypothesis,

$$L = \sum\{N: N \text{ is a graded simple submodule of } M\}$$

is a direct summand of $M$. The proof will be over if we show that the complement of $L$ is $\{0\}$. For this, note that every graded submodule of $M$ contains a simple submodule. In fact, since every graded submodule is a sum of homogeneous cyclic modules, is enough to see this assertion is valid for every $Rm$, $m \in H(M)$. Given $m \in H(M)$ the fact that $R$ is graded unital implies that $Rm$ is finitely generated and then by iii) of Lemma 11 there exists $K \subseteq Rm$ a maximal graded submodule of $Rm$. By hypothesis,
$M = K \oplus K'$ with $K'$ graded submodule of $M$. But $Rm = M \cap Rm = K \oplus (K' \cap Rm)$, then $Rm \cap K' \cong Rm/K$ is a simple submodule of $Rm$ due to the maximality of $K$ over $Rm$. Summarizing, $L = M$ is a sum of simple modules. □

**Proposition 46.** Let $M \in \mathcal{G}$-$R$-$\text{mod}$. If $U(M)$ is semisimple then $M$ is semisimple.

**Proof.** Let $N$ be a graded submodule of $M$. If $U(M)$ is semisimple then $U(N)$ is a direct summand, so there is $f : M \to N$ in $R$-$\text{mod}$ such that $f \circ \iota_N = 1_N$, where $\iota_N : N \to M$ is the canonical inclusion. Thus $f$ can be considered in $\mathcal{G}$-$R$-$\text{mod}$ by Lemma 7 and thus $N$ is a direct summand of $M$. □

**Definition 47.** A graded ring $R$ is **semisimple** if it is semisimple in $\mathcal{G}$-$R$-$\text{umod}$.

We finish this work with the following.

**Proposition 48.** For an object unital graded ring $R$ the following properties are equivalent:

i) $R$ is semisimple.

ii) Every graded left ideal $I$ of $R$ is a direct summand.

iii) Every object in $\mathcal{G}$-$R$-$\text{umod}$ is injective.

iv) Every object in $\mathcal{G}$-$R$-$\text{umod}$ is projective.

v) Every object in $\mathcal{G}$-$R$-$\text{umod}$ is semisimple.

**Proof.** i) $\Rightarrow$ ii) Follows from Proposition 45.

ii) $\Rightarrow$ iii) Let $M \in \mathcal{G}$-$R$-$\text{umod}$, $I, J$ be graded left ideals of $R$ such that $R = I \oplus J$ and $g \in \text{HOM}_R(I, M)$. The function $f : R \to M$ defined by $f(i + j) = g(i)$, for every $i \in I$ and every $j \in J$, satisfies $g = f \circ \iota$, where $\iota : I \to R$ is the inclusion, and $f \in \text{HOM}_R(R, M)$. By Baer’s Criteria (Proposition 35), $M$ is injective.

iii) $\Rightarrow$ iv) If $0 \to L \to M \to N \to 0$ is a short exact sequence in $\mathcal{G}$-$R$-$\text{umod}$, the fact that $L$ is injective implies by Proposition 38 that the sequence splits. But this is equivalent to (iv) by Proposition 34.

iv) $\Rightarrow$ v) For every object in $\mathcal{G}$-$R$-$\text{umod}$, any graded submodule induces a short exact sequence that split by hypothesis, turning it into a direct summand.

v) $\Rightarrow$ i) It is clear. □

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ESCUELA DE MATEMÁTICAS, UNIVERSIDAD INDUSTRIAL DE SANTANDER, CARRERA 27 CALLE 9, EDIFICIO CAMILO TORRES APARTADO DE CORREOS 678, BUCARAMANGA, COLOMBIA
E-mail address: jcgalab@gmail.com

UNIVERSITY WEST, DEPARTMENT OF ENGINEERING SCIENCE, SE-46186 TROLLHÄTTAN, SWEDEN
E-mail address: patrik.nystedt@hv.se

ESCUELA DE MATEMÁTICAS, UNIVERSIDAD INDUSTRIAL DE SANTANDER, CARRERA 27 CALLE 9, EDIFICIO CAMILO TORRES APARTADO DE CORREOS 678, BUCARAMANGA, COLOMBIA
E-mail address: hpinedot@uis.edu.co