Cubic Interactions of Bosonic Higher Spin Gauge Fields in $AdS_5$

M.A. VASILIEV

I.E. Tamm Department of Theoretical Physics, Lebedev Physical Institute, Leninsky prospect 53, 119991, Moscow, Russia

Abstract

The dynamics of totally symmetric free massless higher spin fields in $AdS_d$ is reformulated in terms of the compensator formalism for AdS gravity. The $AdS_5$ higher spin algebra is identified with the star product algebra with the $su(2, 2)$ vector (i.e., $o(4, 2)$ spinor) generating elements. Cubic interactions of the totally symmetric bosonic higher spin gauge fields in $AdS_5$, including the interaction with gravity, are formulated at the action level.

1 Introduction

Irreducible relativistic fields in the flat $d$-dimensional space-time classify according to the finite-dimensional representations of the Wigner little algebra $l$. It is well known that $l = o(d-2)$ for the massless case $m = 0$ and $l = o(d-1)$ for $m \neq 0$. From the field-theoretical viewpoint the difference between the massless and massive cases is that, except for the scalar and spinor fields, all massless fields possess specific gauge symmetries reducing a number of independent degrees of freedom.

Since the totally antisymmetric symbol $\epsilon^{a_1 \ldots a_n}$ ($a = 1 \div n$) is $o(n)$ invariant it is enough to consider the representations of $o(n)$ associated with the Young diagrams having at most $\left[ \frac{1}{2} n \right]$ rows. For lower dimensions like $d = 4$ and $d = 5$ only the totally symmetric massless higher spin representations of the little algebra appear, characterized by a single number $s$. An integer spin $s$ massless field is described by a totally symmetric tensor $\varphi_{n_1 \ldots n_s}$ subject to the double tracelessness condition [1] $\varphi^{r, s}_{r, s_{n_1 \ldots n_s}} = 0$ which is nontrivial for $s \geq 4$. 
A quadratic action \([1]\) for a free spin \(s\) field \(\varphi_{n_1...n_s}\) is fixed unambiguously \([2]\) up to an overall factor in the form

\[
S_s = \frac{1}{2}(-1)^s \int d^4x \left\{ \partial_n \varphi_{m_1...m_s} \partial^n \varphi_{m_1...m_s} \right. \\
\left. - \frac{1}{2} s(s-1) \partial_n \varphi^r_{m_1...m_{s-2}} \partial^n \varphi^k_{m_1...m_{s-2}} + s(s-1) \partial_n \varphi^r_{m_1...m_{s-2}} \partial_k \varphi^{nkm_1...m_{s-2}} \\
- s \partial_n \varphi^r_{m_1...m_{s-1}} \partial_r \varphi_{m_1...m_{s-1}} - \frac{1}{4} s(s-1)(s-2) \partial_n \varphi^r_{m_1...m_{s-3}} \partial_k \varphi^t_{km_1...m_{s-3}} \right\}
\]  

by the requirement of gauge invariance under the Abelian gauge transformations

\[
\delta \varphi_{n_1...n_s} = \partial_{t_1} \varepsilon_{n_2...n_s}
\]  

with the parameters \(\varepsilon_{n_1...n_{s-1}}\) being rank \((s-1)\) totally symmetric traceless tensors, \(\varepsilon^r_{m_3...m_{s-1}} = 0\). This formulation is parallel \([3]\) to the metric formulation of gravity and is called formalism of symmetric tensors. Fermionic higher spin gauge fields are described analogously \([4]\) in terms of rank-\((s-1/2)\) totally symmetric spinor-tensors \(\psi_{n_1...n_{s-1/2}}\) subject to the \(\gamma\)-tracelessness condition \(\gamma^{r\alpha}_{\beta} \psi^r_{m_{s-1/2}...n_4} = 0\). A progress on the covariant description of generic (i.e., mixed symmetry) massless fields in any dimension was achieved in \([5, 6]\).

Higher spin gauge symmetry principle is the fundamental concept of the theory of higher spin massless fields. By construction, the class of higher spin gauge theories consists of most symmetric theories having as many as possible symmetries unbroken\(^{1}\). Any more symmetric theory will have more lower and/or higher spin symmetries and therefore will belong to the class of higher spin theories. As such, higher spin gauge theory is of particular importance for the search of a fundamental symmetric phase of the superstring theory. This is most obvious in the context of the so-called Stueckelberg symmetries in the string field theory which have a form of some spontaneously broken higher spin gauge symmetries. Whatever a symmetric phase of the superstring theory is, Stueckelberg symmetries are expected to become unbroken higher spin symmetries in such a phase and, therefore, the superstring field theory has to identify with one or another version of the higher spin gauge theory.

\(^{1}\)We only consider the case of relativistic fields that upon quantization are described by lowest weight unitary representations (lowest weight implies in particular, that the energy operator is bounded from below). Beyond this class some other “partially massless” higher spin gauge fields can be introduced \([7]\) which are either non-unitary or live in the de Sitter space (recall that de Sitter group \(SO(d,1)\) does not allow lowest weight unitary representations).
The problem is to introduce interactions of higher spin fields with some other fields in a way compatible with the higher spin gauge symmetries. Positive results in this direction were first obtained for interactions of higher spin gauge fields in the flat space with the matter fields and with themselves but not with gravity [8]. In the framework of gravity, the nontrivial higher spin gauge theories were so far elaborated [9, 10, 11] (see also [12, 13] for reviews) for \( d = 4 \) which is the simplest nontrivial case since higher spin gauge fields do not propagate if \( d < 4 \). As a result, it was found out that

(i) in the framework of gravity, unbroken higher spin gauge symmetries require a non-zero cosmological constant;

(ii) consistent higher spin theories contain infinite sets of infinitely increasing spins;

(iii) consistent higher spin gauge interactions contain higher derivatives: the higher spin is the more derivatives appear;

(iv) the higher spin symmetry algebras [14] identify with certain star product algebras with spinor generating elements [15].

Some of these properties, like the relevance of the \( \text{AdS} \) background and star product algebras, discovered in the eighties were rather unusual at that time but got their analogues in the latest superstring developments in the context of \( \text{AdS/CFT} \) correspondence [16] and the non-commutative Yang-Mills limit [17]. We believe that this convergency can unlikely be occasional. Let us note that recently an attempt to incorporate the dynamics of higher spin massless into the two-time version of the non-commutative phase space approach was undertaken in [18].

The feature that unbroken higher spin gauge symmetries require a non-zero cosmological constant is of crucial importance in several respects. It explained why negative conclusions on the existence of the consistent higher-spin-gravitational interactions were obtained in [19] where the problem was analyzed within an expansion near the flat background. Also it explains why the higher spin gauge theory phase is not directly seen in the \( \text{M} \) theory (or superstring theory) framework prior its full formulation in the \( \text{AdS} \) background is achieved. The same property makes the \( S \)-matrix Coleman-Mandula-type no-go arguments [20] irrelevant because there is no \( S \)-matrix in the \( \text{AdS} \) space.

A challenging problem of the higher spin gauge theory is to extend the \( 4d \) results on the higher-spin-gravitational interactions to higher dimensions. This is of particular importance in the context of the possible applications of the higher spin gauge theory to the superstring theory \( (d = 10) \) and \( \text{M} \) theory \( (d=11) \). A conjecture on the structure of the higher spin symmetry algebras in any dimension was made in [21] where the idea was put forward.
that analogously to what was proved to be true in $d = 4$ [15] and $d < 4$ (see [13] for references) higher spin algebras in any dimension are certain star product algebras with spinor generating elements.

As a first step towards higher dimensions it is illuminating to analyze the next to $d = 4$ nontrivial case, which is $d = 5$. This is the primary goal of this paper. The case of $AdS_5/CFT_4$ higher spin duality is particularly interesting in the context of duality of the type IIB superstring theory on $AdS_5 \times S^5$ with a constant Ramond-Ramond field strength to the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [16]. It was conjectured recently in [22, 23] that the boundary theories dual to the $AdS_{d+1}$ higher spin gauge theories are free conformal theories in $d$ dimensions. This conjecture is in agreement with the results of [24] where the conserved conformal higher spin currents bilinear in the $d$-dimensional free massless scalar field theory were shown to be in the one-to-one correspondence with the set of the $d + 1$-dimensional bulk higher spin gauge fields associated with the totally symmetric representations of the little algebra. In contrast to the regime $g^2 n \to \infty$ underlying the standard AdS/CFT correspondence [16], as conjectured in [22, 23], the AdS/CFT regime associated with the higher spin gauge theories corresponds to the limit $g^2 n \to 0$. The realization of the higher spin conformal symmetries in the $4d$ free boundary conformal theories was considered recently in [25]. It was shown that they indeed possess the global higher spin conformal symmetries proposed long ago by Fradkin and Linetsky [26] in the context of $4d$ conformal higher spin gauge theories [27], and some their further extensions\(^2\). It was conjectured in [25] that the $4d$ conformal higher spin gauge symmetries can be realized as higher spin gauge symmetries of $AdS_5$ bulk unitary higher spin gauge theories. Analogous conjecture was made in [28] with respect to the minimal infinite-dimensional reduction of the $4d$ conformal higher spin algebra.

The $AdS_5$ case is more complicated compared to $AdS_4$. Naively, one might think that only one-row massless higher spin representations of the little group characterized by a single number $s$ appear. However, there is a catch due to the fact that the classification of massless fields in $AdS_d$ is different [29] from that of the flat space. As a result, more types of massless fields appear in $AdS_5$ which all reduce in the flat limit to the some combinations of the symmetric fields. In the $AdS_5$ case however, they are expected to be described by the dynamical fields having the symmetry properties of the two row Young diagrams. Unfortunately, so far the covariant formalism for the description of such fields in the $AdS$ space, that would extend that developed in [30, 31] for

\(^2\)The conformal higher spin gauge theories of [27] generalize $C^2$ Weyl gravity and are non-unitary because of the higher derivatives in the kinetic terms that give rise to ghosts.
the totally symmetric higher spin fields, was not worked out. This complicates a formulation of the $AdS_5$ higher spin gauge theories for the general case. In particular, based on the results presented in section 4 of this paper, it was argued in [25] that such mixed symmetry higher gauge spin fields have to appear in the $5d$ higher spin algebras with $\mathcal{N} \geq 2$ supersymmetries. For this reason, in this paper we confine ourselves to the simplest purely bosonic $\mathcal{N} = 0$ case of the $AdS_5$ higher spin gauge theory. The $\mathcal{N} = 1$ case will be considered in the forthcoming paper [32]. To proceed beyond $\mathcal{N} = 1$ one has first of all to develop the appropriate formulation of the $AdS_5$ massless higher spin fields that have the symmetry properties of the two-row Young diagrams.

The organization of the paper is as follows. To make it as much self-contained as possible we start in section 2 with a summary of the general features of the approach developed in [33, 9, 30] relevant to the analysis of the $5d$ higher spin gauge theory in this paper. In particular, the main idea of the higher spin extension at the algebraic and Lagrangian level is discussed in section 2.2. In section 2.3 we summarize the main results of [30] on the formulation of the totally symmetric bosonic massless fields in any dimension, introducing covariant notation based on the compensator approach to $AdS_d$ gravity explained in the section 2.1. In section 3 we reformulate $AdS_5$ gravity in the $su(2, 2)$ spinor notations. The correspondence between finite-dimensional representations of $su(2, 2)$ and $o(4, 2)$ relevant to the $AdS_5$ higher spin problem is presented in the section 4. $AdS_5$ higher spin gauge algebras are defined in section 5. $su(2, 2)$ systematics of the $5d$ higher spin massless is given in section 6. The unfolded form of the free equations of motion for all massless totally symmetric tensor fields in $AdS_5$ called Central On-Mass-Shell Theorem is presented in the section 7. The analysis of the $AdS_5$ higher spin action is the content of section 8 where we, first, discuss some general properties of the higher spin action, and then derive the quadratic (section 8.1) and cubic (section 8.2) higher spin actions possessing necessary higher spin symmetries. The reductions to the higher spin gauge theories that describe finite collections of massless fields of any given spin are defined in section 9. Conclusions and some open problems are discussed in section 10. The Appendix contains a detailed derivation of the $5d$ free higher spin equations of motion.
2 Generalities

2.1 AdS\(_d\) Gravity with Compensator

It is well-known that gravity admits a formulation in terms of the gauge fields associated with one or another space-time symmetry algebra [34, 35, 36]. Gravity with the cosmological term in any space-time dimension can be described in terms of the gauge fields \(w^{AB} = -w^{BA} = dx^m w^{AB}_m\) associated with the AdS\(_d\) algebra \(h = o(d-1,2)\) with the basis elements \(t_{AB}\). Here the underlined indices \(m, n \ldots = 0 \div d - 1\) are (co)tangent for the space-time base manifold while \(A, B = 0 \div d\) are (fiber) vector indices of the gauge algebra \(h = o(d-1,2)\).

Let \(r^{AB}\) be the Yang-Mills \(o(d-1,2)\) field strength

\[
r^{AB} = dw^{AB} + w^{AC} \wedge w_C^B, \tag{2.1}
\]

where \(d = dx^m \frac{\partial}{\partial x^m}\) is the exterior differential. One can use the decomposition

\[
w = w^{AB}t_{AB} = \omega^{L,ab}L_{ab} + \lambda e^a P_a \tag{2.2}
\]

\((a, b = 0 \div d - 1)\). Here \(\omega^{L,ab}\) is the Lorentz connection associated with the Lorentz subalgebra \(o(d-1,1)\). The frame field \(e^a\) is associated with the AdS\(_d\) translations \(P_a\) parametrizing \(o(d-1,2)/o(d-1,1)\). Provided that \(e^a\) is non-degenerate, the zero-curvature condition

\[
r^{AB}(w) = 0 \tag{2.3}
\]

implies that \(\omega^{L,ab}\) and \(e^a\) identify with the gravitational fields of AdS\(_d\). \(\lambda^{-1}\) is the radius of the AdS\(_d\) space-time. (Note that \(\lambda\) has to be introduced to make the frame field \(e^a\) dimensionless.)

One can make these definitions covariant with the help of the compensator field [36] \(V^A(x)\) being a time-like \(o(d-1,2)\) vector \(V^A\) normalized to

\[
V^A V_A = 1 \tag{2.4}
\]

(within the mostly minus signature). The Lorentz algebra then identifies with the stability subalgebra of \(V^A\). This allows for the covariant definition of the frame field and Lorentz connection [36, 37]

\[
\lambda E^A = D(V^A) = dV^A + w^{AB}V_B, \tag{2.5}
\]

\[
\omega^{L,AB} = w^{AB} - \lambda(E^A V^B - E^B V^A). \tag{2.6}
\]
According to these definitions
\[ E^AV_A = 0, \]  
\[ D^L V^A = dV^A + \omega^{L, AB} V_B \equiv 0. \]  
\[ (2.7) \]
\[ (2.8) \]
\( V_A \) is the null vector of \( E^A = dx^A E_a^A \). When the matrix \( E^A_n \) has the maximal rank \( d \) it can be identified with the frame field giving rise to the nondegenerate space-time metric tensor
\[ g_{\mu\nu} = E^A_n E^B_n \eta_{AB}. \]  
\[ (2.9) \]

The torsion 2-form is
\[ t^A \equiv D E^A \equiv \lambda^{-1} r^{AB} V_B. \]  
\[ (2.10) \]
(Note that due to (2.7) \( D E^A = D^L E^A \).) The zero-torsion condition
\[ t^A = 0 \]  
\[ (2.11) \]
expresses the Lorentz connection via (derivatives of) the frame field in a usual manner.

With the help of \( V_A \) it is straightforward to build a \( d \)-dimensional generalization of the 4d MacDowell-Mansouri-Stelle-West pure gravity action [35, 36]
\[ S = -\frac{1}{4\lambda^2 \kappa^{d-2}} \int_M \epsilon_{A_1...A_{d+1}} r^{A_1 A_2} \wedge r^{A_3 A_4} \wedge E^{A_5} \wedge \ldots \wedge E^{A_d} V^{A_{d+1}}. \]  
\[ (2.12) \]
Taking into account that
\[ \delta r^{AB} = D \delta w^{AB}, \quad \delta E^A = \lambda^{-1} (\delta w^{AB} V_B + D \delta V^A), \quad V_A \delta V^A = 0 \]  
\[ (2.13) \]
along with the identity
\[ \epsilon^{A_1...A_{d+1}} = V^A_1 V_B \epsilon^{BA_2...A_{d+1}} + \ldots + V^{A_{d+1}} V_B \epsilon^{A_1...A_d B} \]  
\[ (2.14) \]
one finds
\[ \delta S = -\frac{1}{4\lambda^2 \kappa^{d-2}} \int_M \epsilon_{A_1...A_{d+1}} r^{A_1 A_2} \wedge (2(-1)^{d-1} \lambda \delta w^{A_3 A_4} \wedge E^{A_5} \wedge \ldots \wedge E^{A_d} V^{A_{d+1}}) + \delta_1 S, \]  
\[ (2.15) \]
where
\[ \delta_1 S = -\frac{1}{4\lambda^2 \kappa^{d-2}} \int_M \epsilon_{A_1...A_{d+1}} r^{A_1 A_2} \wedge t^{A_3} \wedge ((d-4) \delta w^{A_4 A_5} \wedge E^{A_6} \wedge \ldots \wedge E^{A_d} V^{A_{d+1}}) + \ldots + 4(d-3) \lambda V^{A_4} \wedge E^{A_5} \wedge \ldots \wedge E^{A_d} \delta V^{A_{d+1}}) \]  
\[ (2.16) \]
is the part of the variation that contains torsion.

We shall treat the action $S$ perturbatively with $r^{AB}$ being small. According to (2.3) this implies a perturbation expansion around the $AdS_d$ background. In this framework, the second term in (2.15) and the first term in (2.16) only contribute to the nonlinear corrections of the field equations for the gravitational fields $w^{AB}$.

For the part of $\delta w^{AB}$ orthogonal to $V_C$

$$\delta w^{AB} = \delta \xi^{AB}, \quad \delta \xi^{AB} V_B = 0$$

we obtain

$$\frac{\delta S}{\delta \xi_{A_1 B_2}} = \kappa^{2-d} \epsilon_{A_1 \ldots A_{d+1}} \epsilon^{A_3} \wedge \left( E^{A_4} \wedge E^{A_5} - \frac{d-4}{2\lambda^2} r^{A_4 A_5} \right) \wedge E^{A_6} \wedge \ldots \wedge E^{A_d} V^{A_{d+1}}.$$  

(2.18)

Perturbatively, (i.e. for $r^{AB}$ small) (2.18) is equivalent to the zero-torsion condition (2.11). In what follows we will use the so called 1.5 order formalism. Namely, we will assume that the zero-torsion constraint is imposed to express the Lorentz connection via derivatives of the frame field. The same time we will use an opportunity to choose any convenient form for the variation of the Lorentz connection because any term containing this variation is zero by (2.18) and (2.11).

The generalized Einstein equations originating from the variation

$$\delta w^{AB} = \lambda (\delta \xi^A V^B - \delta \xi^B V^A), \quad \delta \xi^A V_A = 0$$

are

$$\kappa^{2-d} \epsilon_{A_1 \ldots A_{d+1}} r^{A_2 A_3} \wedge \left( E^{A_4} \wedge E^{A_5} - \frac{d-4}{4\lambda^2} r^{A_4 A_5} \right) \wedge E^{A_6} \wedge \ldots \wedge E^{A_d} V^{A_{d+1}} = 0.$$  

(2.20)

The first term is nothing but the left-hand-side of the Einstein equations with the cosmological term. The second term describes some additional interaction terms bilinear in the curvature $r^{AB}$. These terms do not contribute to the linearized field equations. In the $4d$ case the additional terms are absent because the corresponding part of the action is topological having the Gauss-Bonnet form. Note that the additional interaction terms contain higher derivatives together with the factor of $\lambda^{-2}$ that diverges in the flat limit $\lambda \to 0$. Terms of this type play an important role in the higher spin theories to guarantee the higher spin gauge symmetries. Let us note that the form of the equation (2.20) indicates that beyond $d = 4$ the action (2.12) may have other symmetric
vacua\(^3\) (e.g. with \( r^{AB} = \frac{4\lambda^2}{d-4} E^A \wedge E^B \)). We shall not discuss this point here in more detail because its analysis requires the non-perturbative knowledge of the higher spin theory which is still lacking for \( d > 4 \).

From (2.16) it follows that the variation of \( S \) with respect to the compensator \( V^A \) is proportional to the torsion 2-form \( t^A \). This means that, at least perturbatively, there exists such a variation of the fields

\[
\delta V^A = \epsilon^A(x), \quad \delta w^{AB} = \eta^{AB}(r, \epsilon)
\]  

with \( \epsilon^A V_A = 0 \) and some \( \eta^{AB}(r, \epsilon) \) bilinear in \( r^{AB} \) and \( \epsilon^A \) that \( S \) remains invariant. As a result, there is an additional gauge symmetry that allows to gauge fix \( V^A \) to any value satisfying (2.4). It is therefore shown that \( V^A \) does not carry extra degrees of freedom.

The compensator field \( V^A \) makes the \( o(d-1,2) \) gauge symmetry manifest

\[
\delta w^{AB} = D\epsilon^{AB}, \quad \delta V^A = -\epsilon^{AB} V_B.
\]  

Fixing a particular value of \( V^A \) one is left with the mixture of the gauge transformations that leave \( V^A \) invariant, i.e. with the parameters satisfying

\[
0 = \delta V^A = \epsilon^A(x) - \epsilon^{AB} V_B.
\]  

Since the additional transformation (2.21) contains dependence on the curvature \( r^{AB} \), this property is inherited by the leftover symmetry with the parameters satisfying (2.23).

The fact that there is an additional symmetry (2.21) is not a big surprise in the framework of the theory of gravity formulated in terms of differential forms, having explicit invariance under diffeomorphisms. That this should happen is most clear from the observation that the infinitesimal space-time diffeomorphisms induced by an arbitrary vector field \( \epsilon^\nu(x) \) admit a representation

\[
\delta w^{AB}_{\underline{m}} = \epsilon^\nu \partial_{\underline{m}} w^{AB}_{\underline{n}} + \partial_{\underline{m}}(\epsilon^\nu) w^{AB}_{\underline{n}} = \epsilon^\nu r^{AB}\underline{m} - D_m \epsilon^{AB},
\]  

\[
\delta V^A = \epsilon^\nu \partial_{\underline{m}} V^A = \epsilon^\nu E^A_{\underline{m}} + \epsilon^{AB} V_B,
\]  

where

\[
\epsilon^{AB} = -\epsilon^\nu w^{AB}_{\underline{m}}.
\]  

The additional gauge transformation (2.21) with \( \epsilon^A = \epsilon^\mu E^A_{\underline{m}} \) can therefore be understood as a mixture of the diffeomorphisms and \( o(d-1,2) \) gauge transformations.

\(^3\)I am grateful to K.Alkalaev for the useful discussion of this point.
Another useful interpretation of the formula (2.24) is that, for the vacuum solution satisfying (2.3), diffeomorphisms coincide with some gauge transformations. This observation explains why the space-time symmetry algebras associated with the motions of the most symmetric vacuum spaces reappear as gauge symmetry algebras in the “geometric approach” to gravity and its extensions.

2.2 General Idea of the Higher Spin Extension

The approach to the theory of interacting higher spin gauge fields developed originally in [33, 9] for the $d = 4$ case is a generalization of the “geometric” approach to gravity sketched in section 2.1. The idea is to describe the higher spin gauge fields in terms of the Yang-Mills gauge fields and field strengths associated with an appropriate higher spin symmetry algebra $g$ being some infinite-dimensional extension of the finite-dimensional $AdS_d$ space-time symmetry algebra $h = o(d − 1, 2)$.

Let the 1-form $\omega(x) = dx^a \omega_a(x)$ be the gauge field of $g$ with the field strength (curvature 2-form)

$$R = d\omega + \omega \wedge \star \omega,$$

(2.27)

where $\star$ is some associative product law leading to the realization of $g$ via commutators. (This is analogous to the matrix realization of the classical Lie algebras with the star product instead of the matrix multiplication. A particular realization of the star product relevant to the $5d$ higher spin dynamics is given in section 3). An infinitesimal higher spin gauge transformation is

$$\delta^g \omega = D\epsilon,$$

(2.28)

where $\epsilon(x)$ is an arbitrary infinitesimal symmetry parameter taking values in $g$,

$$Df = df + [\omega, f]_\star$$

(2.29)

and

$$[a, b]_\star = a \star b - b \star a.$$  

(2.30)

The higher spin curvature has the standard homogeneous transformation law

$$\delta^g R = [R, \epsilon]_\star.$$  

(2.31)

The higher spin equations of motion will be formulated in terms of the higher spin curvatures and therefore admit a zero-curvature vacuum solution.
with $R = 0$. Since the space-time symmetry algebra $h$ is assumed to belong to $g$, a possible ansatz is with all vacuum gauge fields vanishing except for $\omega_0$ taking values in $h$

$$\omega_0 = w_0^{AB} t_{AB} = \omega_0^{L ab} L_{ab} + \lambda h_a P^a .$$  \hspace{1cm} (2.32)

Provided that $h^a$ is nondegenerate, the zero-curvature condition

$$R(\omega_0) = (d \omega_0^{AB} + \omega_0^A C \wedge \omega_0^{CB}) t_{AB} = 0$$  \hspace{1cm} (2.33)

implies that $\omega_0^{L ab}$ and $h^a$ identify with the gravitational fields of $AdS_d$. Let us note that throughout this paper we use notation $\omega_0^{L ab}$ and $h^a$ for the background $AdS$ fields satisfying (2.32) but $\omega^{L ab}$ and $e^a$ for the dynamical gravitational fields.

Suppose there is a theory invariant under the gauge transformations (2.28). Global symmetry is the part of the gauge transformations that leaves invariant the vacuum solution $\omega_0$. The global symmetry parameters therefore satisfy

$$0 = D_0 \epsilon^{gl} ,$$  \hspace{1cm} (2.34)

where

$$D_0 f = df + [\omega_0 , f]_* .$$  \hspace{1cm} (2.35)

The vacuum zero-curvature equation (2.33) guarantees that (2.34) is formally consistent. Fixing a value of $\epsilon^{gl}(x_0)$ at some point $x_0$, (2.34) allows one to reconstruct $\epsilon^{gl}(x)$ in some neighbourhood of $x_0$. Since $D_0$ is a derivation, the star commutator of any two solutions of (2.34) gives again some its solution. The global symmetry algebra therefore coincides with the algebra of star commutators at any fixed space-time point $x_0$, which is $g$. An important comment is that this conclusion remains true also in case the theory is invariant under a deformed gauge transformation of the form

$$\delta \omega = D \epsilon + \Delta (R, \epsilon) ,$$  \hspace{1cm} (2.36)

where $\Delta (R, \epsilon)$ denotes some $R$-dependent terms, i.e. $\Delta (0, \epsilon) = 0$. Indeed, all additional terms do not contribute to the invariance condition (2.34) once the vacuum solution satisfies (2.33). In fact, as is clear from the discussion in the section 2.1, the deformation of the gauge transformations (2.36) takes place in all theories containing gravity and, in particular, in the higher spin gauge theories.

Let us use the perturbation expansion with

$$\omega = \omega_0 + \omega_1 ,$$  \hspace{1cm} (2.37)
where $\omega_1$ is the dynamical (fluctuational) part of the gauge fields of the higher spin algebra $g$. Since $R(\omega_0) = 0$ we have

$$R = R_1 + R_2,$$  \hspace{1cm} (2.38)

where

$$R_1 = d\omega_1 + \omega_0 \star \wedge \omega_1 + \omega_1 \star \wedge \omega_0, \quad R_2 = \omega_1 \star \wedge \omega_1.$$  \hspace{1cm} (2.39)

The Abelian lowest order part of the transformation (2.28) (equivalently, (2.36)) has the form

$$\delta_0 \omega_1 = D_0 \epsilon.$$  \hspace{1cm} (2.40)

From (2.31) and (2.33) it follows that

$$\delta_0 R_1 = 0.$$  \hspace{1cm} (2.41)

The idea is to construct the higher spin action from the higher spin curvatures $R$ in the form analogous to the gravity action (2.12)

$$S = \int U_{\Omega \Lambda} \wedge R^\Omega \wedge R^\Lambda,$$  \hspace{1cm} (2.42)

with some $(d-4)$ - form coefficients $U_{AB}$ built from the frame field and compensator. ($\Omega, \Lambda$ label the adjoint representation of $g$). To clarify whether this is possible or not, one has to check first of all if it is true for the free field action, i.e. whether some action of the form

$$S^s_2 = \int U^s_{0 \Omega \Lambda} \wedge R_1^\Omega \wedge R_1^\Lambda,$$  \hspace{1cm} (2.43)

describes the free field dynamics of a field of a given spin $s$. As long as $g$ is not known, a form of $R_1$ has itself to be fixed from this requirement. In fact, the form of $R_1$ provides an important information on the structure of $g$ fixing a pattern of the decomposition of $g$ under the adjoint action of $h \subset g$ (up to a multiplicity of the representations associated with a given spin $s$: it is not a priori known how many fields of a given spin are present in a full higher spin multiplet). For the totally symmetric higher spin gauge fields described by the action (1.1) this problem was solved first for case $d = 4$ [33] and then for any dimension both for bosons [30] and for fermions [21]. The results of [30] are summarized in the section 2.3.

As a result of (2.41) any action of the form (2.43) is invariant under the Abelian free field higher spin gauge transformations (2.40). However, for generic coefficients, it not necessarily describes a consistent higher spin dynamics. As this point is of the key importance for the analysis of the higher
spin dynamics let us explain it in somewhat more detail. The set of 1-forms contained in $\omega$ decomposes into subsets $\omega_n^s$ associated with a given spin $s$. The label $n$ enumerates different subsets associated with the same spin. (For the case $d = 4$ $s$ is indeed a single number while for generic fields in higher dimensions $s$ becomes a vector associated with the appropriate weight vector of the $AdS_d$ algebra $O(d - 1, 2)$). Any subset $\omega_n^s$ forms a representation of the space-time subalgebra $h \subset g$. It further decomposes into representations of the Lorentz subalgebra of $h$, denoted $\omega_{s,t}^n$. For the case of totally symmetric representations discussed in the Introduction, there is a single integer parameter $t = 0, 1, \ldots, s - 1$ that distinguishes between different Lorentz components (for definiteness we focus here on the bosonic case of integer spins studied in this paper). True higher spin field identifies with $\omega_{s,0}^n$. It is called dynamical higher spin field. The rest of the fields $\omega_{s,t}^n$ with $t > 0$ express in terms of (derivatives of) the dynamical ones by virtue of certain constraints. At the linearized level, the gauge invariant constraints can be chosen in the form of some linear combinations of the linearized higher spin curvatures

$$\Phi'(R_1) = 0$$

(2.44)

with the coefficients built from the background frame field. By virtue of these constraints all fields $\omega_{n,t}^{s,t}$ turn out to be expressed via $t^{th}$ space-time derivatives of the dynamical field

$$\omega_{n,t}^{s,t} \sim \left( \frac{\partial}{\lambda \partial x} \right)^t (\omega_{n,0}^{s,0}) + \text{pure gauge terms (2.40)} .$$

(2.45)

These expressions contain explicitly the dependence on the $AdS_d$ radius $\lambda^{-1}$ as a result of the definition of the frame field (2.5).

A particular example is provided with the spin 2. Here $\omega_{2,0}^{2,0}$ identifies with the frame field while $\omega_{2,1}^{2,1}$ is the Lorentz connection. (We skip the label $n$ focusing on a particular spin 2 field). The constraint (2.44) is the linearized zero-torsion condition.

For $s > 2$ the fields $\omega_{n,t}^{s,t}$ with $t \geq 2$ appear, containing second and higher derivatives of the dynamical field. These are called extra fields. From this perspective, the requirement that the free action contains at most two space-time derivatives of the dynamical field is equivalent to the condition that the variation of the free action with respect to all extra fields is identically zero

$$\frac{\delta S_2^s}{\delta \omega_{n,t}^{s,t}} \equiv 0 \quad t \geq 2 .$$

(2.46)

It turns out that this extra field decoupling condition fixes a form of the free action (i.e. of $U_{0,\Omega}$) uniquely modulo total derivatives and an overall ($s$- and
$n$- dependent) factor. The Lorentz-type fields $\omega^n_1$ are auxiliary, i.e. they do contribute into the free action but express via the derivatives of the dynamical field by virtue of their field equations equivalent to some of the constraints (2.44).

Once the extra fields are expressed in terms of the derivatives of the dynamical fields, the higher spin transformation law (2.28) (and its possible deformation (2.36)) describes the transformations of the dynamical fields via their higher derivatives. Since $t$ ranges from 0 to $s - 1$ one finds that the higher spin is the higher derivatives appear in the transformation law. Note that this conclusion is in agreement with the general analysis of the structure of the higher spin interactions [8] and conserved higher spin currents [38, 13] containing higher space-time derivatives.

As a first step towards the non-linear higher spin dynamics one can try the action (2.42) with $U_{\Omega \Lambda}$ proportional to the coefficients $U^s_{\Omega \Lambda}$ in the subsector of each field of spin $s$. This action is not invariant under the original higher spin gauge transformations (2.28) since $U_{\Omega \Lambda}$ cannot be an invariant tensor of $g$. Indeed, since the action is built in terms of differential forms without Hodge star operation, its generic variation is

$$\delta S = -2 \int D(U_{\Omega \Lambda}) \wedge \delta \omega^\Omega \wedge R^A.$$  \hspace{1cm} (2.47)

If $U_{\Omega \Lambda}$ would be a $g$-invariant tensor, $S$ would be a topological invariant. This cannot be true since the linearized action (2.43) is supposed to give rise to nontrivial equations of motion. Therefore, $D(U_{\Omega \Lambda}) \neq 0$. The trick is that for some particular choice of $U_{\Omega \Lambda}$ there exists such a deformation of the gauge transformations (2.36) that the action remains invariant at least in the lowest nontrivial order, i.e. the $\omega^\Omega \epsilon$ type terms can be proved to vanish in the variation. (Note that this is just the order at which the difficulties with the higher-spin-gravitational interactions were originally found [19]). In particular, we show in the section 8 that this is true for the $\mathcal{N} = 0$ 5d higher spin theory. This deformation of the gauge transformations is analogous to that resulting via (2.23) from the particular gauge fixing of the compensator field $V^A$ in the case of gravity which, in turn, is described by the spin 2 part of the action (2.42) equivalent to (2.12). A complication of the Lagrangian formulation of the higher spin dynamics is that no full-scale extension of the compensator $V^A$ to some representation of $g$ is yet known. The clarification of this issue is one of the key problems on the way towards the full Lagrangian formulation of the higher spin theory. Note, that the full formulation of the on-mass-shell 4d higher spin dynamics [39, 11] was achieved by virtue of introducing additional compensator-type pure gauge variables [11, 13].
Since the extra fields do contribute into the nonlinear action it is necessary to express them in terms of the dynamical higher spin fields to make the nonlinear action (2.42) meaningful. The expressions (2.45) that follow from the constraints for extra fields effectively induce higher derivatives into the higher spin interactions. The same mechanism induces the negative powers of $\lambda$, the square root of the cosmological constant, into the higher spin interactions with higher derivatives\(^4\). A specific form of the constraints (2.44) plays a crucial role in the proof of the invariance of the action.

The program sketched in this section was accomplished for the 4$d$ case. The free higher spin actions of the form (2.43) were built in [33]. The 4$d$ higher spin algebra $g$ was then found in [14]. In [9] the action (2.42) was found that described properly some cubic higher spin interactions including the gravitational interaction. In this paper we extend these results to the bosonic $\mathcal{N} = 0$ 5$d$ higher spin gauge theory.

### 2.3 Symmetric Bosonic Massless Fields in $AdS_d$

In this section the results of [30] are reformulated in terms of the compensator formalism. According to [30], a totally symmetric massless field of spin $s$ is described by a collection of 1-forms $dx^a_\mu \omega^{a_1...a_{s-1},b_1...b_t}_\mu$ which are symmetric in the Lorentz vector indices $a_i$ and $b_j$ separately ($a,b \ldots 0 \div d - 1$), satisfy the antisymmetry relation

$$\omega^{a_1...a_{s-1},b_1...b_t}_\mu = 0,$$

implying that symmetrization over any $s$ fiber indices gives zero, and are traceless with respect to the fiber indices

$$\omega^{a_1...a_{s-3},c,b_1...b_t}_\mu = 0.$$

(From this condition it follows by virtue of (2.48) that all other traces of the fiber indices are also zero).

The higher spin gauge fields associated with the spin $s$ massless field therefore take values in the direct sum of all irreducible representations of the $d$-dimensional massless Lorentz group $o(d-1,1)$ described by the Young diagrams with at most two rows such that the longest row has length $s - 1$

\(^4\)Note that one can rescale the fields in such a way that the corresponding expression (2.45) will not contain negative powers of $\lambda$ explicitly. However, as a result of such a rescaling, $\lambda$ will appear both in positive and in negative powers in the structure coefficients of the algebra $g$ and, therefore, in the nonlinear action. From this perspective, the appearance of the extra fields for higher spins makes difference compared to the case of pure gravity that allows the Inönü-Wigner flat contraction.
ω∗_{a_1...a_{s-1}} is treated as the dynamical spin $s$ field analogous to the gravitational frame (spin 2). The fields corresponding to the representations with nonzero second row ($t > 0$) are auxiliary ($t = 1$) or “extra” ($t > 1$), i.e. express via derivatives of the dynamical field by virtue of certain constraints analogously to the Lorentz connection in the spin 2 case. Analogously to the relationship between metric and frame formulations of the linearized gravity, the totally symmetric double traceless higher spin fields used to describe the higher spin dynamics in the metric-type formalism [1, 3] identify with the symmetrized part of the field $\omega_{n}^{a_1...a_{s-1}}$

\[ \varphi_{a_1...a_s} = \omega\{a_1...a_s\}. \]  

The antisymmetric part in $\omega_{n}^{a_1...a_{s-1}}$ can be gauge fixed to zero with the aid of the generalized higher spin Lorentz symmetries with the parameter $\epsilon^{a_1...a_{s-1},b}$. That $\varphi_{a_1...a_s}$ is double traceless is a trivial consequence of (2.49).

The collection of the higher spin 1-forms $dx_n \omega_{n}^{a_1...a_{s-1},b_1...b_t}$ with all $0 \leq t \leq s-1$ can be interpreted as a result of the “dimensional reduction” of a 1-form $dx_2 \omega_{n}^{A_1...A_{s-1},B_1...B_{s-1}}$ carrying the irreducible representation of the $AdS_d$ algebra $\sigma(d-1,2)$ described by the traceless two-row rectangular Young diagram of length $s-1$

\[ \omega\{A_1...A_{s-1},A_s\}B_2...B_{s-1} = 0, \quad \omega^{A_1...A_{s-3}C,B_1...B_{s-1}} = 0. \]  

The linearized higher spin curvature $R_1$ has the following simple form

\[ R_1^{A_1...A_{s-1},B_1...B_{s-1}} = D_0(\omega^{A_1...A_{s-1},B_1...B_{s-1}}) = d\omega_1^{A_1...A_{s-1},B_1...B_{s-1}} \\
+ (s-1)(\omega_0^{A_1C} \land \omega_1^{A_2...A_{s-1},B_1...B_{s-1}} + \omega_0^{B_1C} \land \omega_1^{A_1...A_{s-1},C,B_2...B_{s-1}}). \]  

where $\omega_0^{AB}$ is the background $AdS_d$ gauge field satisfying the zero curvature condition (2.33).

In these terms, the Lorentz covariant irreducible fields $dx_2 \omega_{n}^{a_1...a_{s-1},b_1...b_t}$ identify with those components of $dx_2 \omega_{n}^{A_1...A_{s-1},B_1...B_{s-1}}$ that are parallel to $V^A$ in $s-t-1$ indices and transversal in the rest ones. The expressions for the Lorentz components of the linearized curvatures have the following structure

\[ R_1^{a_1...a_{s-1},b_1...b_t} = D^L \omega_{1}^{a_1...a_{s-1},b_1...b_t} + \tau_-(\omega)^{a_1...a_{s-1},b_1...b_t} + \tau_+(\omega)^{a_1...a_{s-1},b_1...b_t} \]  

(2.54)
with
\[ \tau_-(\omega)^{a_1...a_{s-1},b_1...b_t} = \alpha h_c \wedge \omega_1^{a_1...a_{s-1},b_1...b_t}, \]
\[ \tau_+(\omega)^{a_1...a_{s-1},b_1...b_t} = \beta \Pi \left( h_b \wedge \omega_1^{a_1...a_{s-1},b_2...b_t} \right), \]
where \( D^L \) is the background Lorentz covariant differential, \( \Pi \) is the projection operator to the irreducible representation described by the traceless Young diagram of the Lorentz algebra \( o(d-1,1) \) with \( s-1 \) and \( t \) cells in the first and second rows, respectively, and \( \alpha \) and \( \beta \) are some coefficients depending on \( s, t \) and \( d \) and fixed in such a way that
\[ (\tau_-)^2 = 0, \quad (\tau_+)^2 = 0, \quad (D^L)^2 + \{\tau_-, \tau_+\} = 0. \]
For the explicit expressions of \( \alpha, \beta \) and \( \Pi \) we refer the reader to the original paper \[30\]. The explicit spinor version of the formula (2.54) for \( d = 5 \) will be given in section 6.

The quadratic action functional for the massless spin \( s \) field equivalent to the \( AdS_d \) deformation of the action (1.1) has the following simple form \[30\]
\[ S_{2}^s = \frac{1}{2} \chi(s) \int_{M^d} \sum_{p=0}^{s-2} \frac{[(p+1)!]^2}{(d+p-3)!} \varepsilon_{c_1...c_d} h^{c_5} \wedge \ldots \wedge h^{c_d} \]
\[ \wedge R_1^{c_1a_1...a_{s-2},c_3b_1...b_p} \wedge R_1^{c_2a_1...a_{s-2},c_4b_1...b_p}. \]
It is fixed up to an overall normalization factor \( \chi(s) \) by the conditions that it is \( P \)-even and its variation with respect to the “extra fields” is identically zero,
\[ \frac{\delta S_{2}^s}{\delta \omega_2^{a_1...a_{s-1},b_1...b_t}} \equiv 0 \quad \text{for} \quad t \geq 2. \]

Let us explain how one can derive a \( o(d-1,2) \) covariant form of the same action with the aid of the compensator \( V^A \). Taking into account the irreducibility properties (2.52) one finds that the general form of the \( P \)-even action written in terms of differential forms is
\[ S_2^s = \frac{1}{2} \int_{M^d} \sum_{p=0}^{s-2} a(s,p) \varepsilon_{A_1...A_{d+1}} h^{A_5} \wedge \ldots \wedge h^{A_d} V^{A_{d+1}} V_{C_1} \ldots V_{C_2(s-2-p)} \]
\[ R_1^{A_1B_1...B_{s-2}} \wedge R_1^{A_2C_1...C_{s-2-p}D_1...D_p} \wedge R_1^{A_3} \wedge R_1^{A_s}, \]
\[ R_1^{A_1B_1...B_{s-2}} \wedge A_2C_1...C_{s-2-p}D_1...D_p \wedge R_1^{A_3} B_1...B_{s-2}, \]
\[ A_4C_1...C_{s-2-p}D_1...D_p \wedge R_1^{A_3} B_1...B_{s-2},. \]
Consider a general variation of $S^s_2$ with respect to $\omega^{A_1...A_{s-1}}_{B_1...B_{s-1}}$. Using that
\[
\delta R^A_{1...A_{s-1}, B_1...B_{s-1}} = D_0 \delta \omega^A_{1...A_{s-1}, B_1...B_{s-1}},
\]
where $D_0$ is the background derivative, one integrates by parts taking into account that $D_0(V^A) = h^A$, $D_0(h^A) = 0$. With the help of the irreducibility conditions (2.52), the identity (2.14) and the identity
\[
\epsilon_{A_1...A_5 B_6...B_{d+1}} h^C \wedge h^{B_6} \wedge \ldots \wedge h^{B_{d+1}} = (d - 3)^{-1} (\epsilon_{A_1 A_2 A_3 A_5 B_5...B_{d+1}} \delta^C_{A_4} - \epsilon_{A_1 A_2 A_3 A_5 B_5...B_{d+1}} \delta^C_{A_3} - \epsilon_{A_1 A_2 A_3 A_4 B_5...B_{d+1}} \delta^C_{A_3} - \epsilon_{A_1 A_3 A_4 A_5 B_5...B_{d+1}} \delta^C_{A_2} + \epsilon_{A_2 A_3 A_4 A_5 B_5...B_{d+1}} \delta^C_{A_2} + \epsilon_{A_2 A_3 A_4 A_5 B_5...B_{d+1}} \delta^C_{A_1} + \epsilon_{A_2 A_3 A_4 A_5 B_5...B_{d+1}} \delta^C_{A_1}) h^{B_5} \wedge \ldots \wedge h^{B_{d+1}},
\]
which expresses the simple fact that the total antisymmetrization of any set of $d + 2$ vector indices $A_i$ is zero, one finds
\[
\delta S^s_2 = -\frac{\lambda}{d - 3} \int_{M^d} \sum_{p=0}^{k-2} \frac{(s - p)(d - 7 + 2(s - p))}{s - p - 1} a(s, p) - (s - p - 1)a(s, p - 1)
\begin{align*}
& V_{C_1} \ldots V_{C_{(s-p)-3}} \epsilon_{A_1...A_{d+1}} V^{A_4} \wedge h^{A_5} \wedge \ldots \wedge h^{A_{d+1}} \wedge \\
& \left(\delta \omega^A_{1 B_{1}...B_{s-2}} A_{2} C_{1}...C_{s-2-p} D_{1}...D_{p} \wedge R^A_{1 B_{1}...B_{s-2}, C_{s-1-p}...C_{2(s-p)-3}} D_{1}...D_{p} \right) + R^A_{1 B_{1}...B_{s-2}} A_{2} C_{1}...C_{s-2-p} D_{1}...D_{p} \wedge \delta \omega^A_{1 B_{1}...B_{s-2}, C_{s-1-p}...C_{2(s-p)-3}} D_{1}...D_{p}.
\end{align*}
\]
(2.63)
The idea is to require all the terms in (2.63) to vanish except for the term at $p = 0$. This condition fixes the coefficients $a(s, p)$ up to a normalization factor $\tilde{a}(s)$ in the form
\[
a(s, p) = -\tilde{a}(s) (d - 3)^{-1} (d - 5 + 2(s - p - 2))!! (s - p - 1)(s - 2)! (s - p - 2)!.
\]
(2.64)
As a result the variation (2.63) acquires the form
\[
\delta S^s_2 = \tilde{a}(s) \int_{M^d} V_{C_1} \ldots V_{C_{2(s-p)-3}} \epsilon_{A_1...A_{d+1}} V^{A_4} h^{A_5} \wedge \ldots \wedge h^{A_{d+1}} \wedge \\
\left(\delta \omega^A_{1 B_{1}...B_{s-2}} A_{2} C_{1}...C_{s-2-p} D_{1}...D_{p} \wedge R^A_{1 B_{1}...B_{s-2}, C_{s-1-p}...C_{2(s-p)-3}} D_{1}...D_{p} \right) + R^A_{1 B_{1}...B_{s-2}} A_{2} C_{1}...C_{s-2-p} D_{1}...D_{p} \wedge \delta \omega^A_{1 B_{1}...B_{s-2}, C_{s-1-p}...C_{2(s-p)-3}} D_{1}...D_{p}.
\]
(2.65)
This formula implies that the free action (2.60), (2.64) essentially depends only on the $V^A$-transversal parts of
\[
\omega^{A_1...A_{s-1}}_{2 B_1...B_{s-1}} = \omega^{A_1...A_{s-1}, B_1...B_{s-1}} V^{B_1} \ldots V^{B_{s-1}}
\]
(2.66)
and
\[ \omega^n_{A_1...A_{s-1}} B_1 = \omega^n_{A_1...A_{s-1}} B_1 V^{B_2} ... V^{B_{s-1}} . \] (2.67)

These fields identify respectively with the frame–like dynamical higher spin field \( \omega^{a_1...a_{s-1}} \) and the Lorentz connection–like auxiliary field \( \omega^{a_1...a_{s-1}, b} \) expressed in terms of the first derivatives of the frame-like field by virtue of its equation of motion equivalent to the “zero torsion condition”

\[ 0 = T_{A_1...A_{s-1}} \equiv R_{A_1...A_{s-1}} V^{B_1} ... V^{B_{s-1}} . \] (2.68)

Insertion of the expression for \( \omega^{a_1...a_{s-1}, b} \) into (2.60) gives rise to the higher spin action expressed entirely (modulo total derivatives) in terms of \( \omega^{a_1...a_{s-1}} \) and its first derivatives. Since the linearized curvatures (2.54) are by construction invariant under the Abelian higher spin gauge transformations

\[ \delta \omega^{A_1...A_{s-1}} B_1 B_1 ... B_{s-1} = D_0 \epsilon^{A_1...A_{s-1}} B_1 B_1 ... B_{s-1} \] (2.69)

with the higher spin gauge parameters \( \epsilon^{A_1...A_{s-1}} B_1 ... B_{s-1} \), the resulting action possesses required higher spin gauge symmetries and therefore describes correctly the free field higher spin dynamics in \( AdS_d \). In particular, the generalized Lorentz-like transformations with the gauge parameter

\[ \epsilon^{A_1...A_{s-1}} B_1 (x) \] (2.70)
guarantee that only the totally symmetric part of the gauge field (2.66) equivalent to \( \varphi_{m_1...m_s} \) contributes to the action. Analogously, the auxiliary Lorentz-type higher spin field has pure gauge components associated with the generalized Lorentz-type transformations parameter described by the two-row Young diagram with two cells in the second row. These components do not express in terms of the dynamical higher spin field. However, the invariance with respect to the gauge transformations (2.69) guarantees that these pure gauge components do not contribute into the action.

Although the extra fields \( \omega^{a_1...a_{s-1}, b_1...b_t} \) with \( t \geq 2 \) do not contribute to the free action, as we have learned from the four-dimensional case [9] they do contribute at the interaction level. To make such interactions meaningful, one has to express the extra fields in terms of the dynamical ones modulo pure gauge degrees of freedom. This is achieved by imposing constraints [30]

\[ \epsilon^{a_1 b_1} e_1 ... e_{d-4} cf h^{e_1} \land ... \land h^{e_{d-4}} \land \tau_+ (R_1)^{c a_2...a_{s-1}, f b_2...b_t} = 0 \] (2.71)

(total symmetrizations within the groups of indices \( a_i \) and \( b_j \) is assumed). The covariant version of these constraints is

\[ \epsilon^{A_1 B_1} E_1 ... E_{d-4} CFG V^G h E_1 \land ... \land h^{E_{d-4}} \land \tau_+ (R_1)^{C A_2...A_{s-1}, F B_2...B_{s-1}} = 0 . \] (2.72)
The covariant expressions for the operators $\tau_{\pm}$ are complicated and will not be given here for general $d$. For $d = 5$ they are given in section 6 in the spinor formalism.

An important fact is [30] that, by virtue of these constraints, most of the higher spin field strengths vanish on-mass-shell according to the following relationship referred to as the First On-Mass-Shell Theorem

$$R^{a_1...a_{s-1},b_1...b_t}_{t} = X^{a_1...a_{s-1},b_1...b_t} \left( \frac{\delta S_2}{\delta \omega_{\text{dyn}}} \right) \quad \text{for} \quad t < s-1,$$

$$R^{a_1...a_{s-1},b_1...b_s}_{s-1} = h_{a_s} \wedge h_{b_s} C^{a_1...a_{s-1},b_1...b_s} + X^{a_1...a_{s-1},b_1...b_s} \left( \frac{\delta S_2}{\delta \omega_{\text{dyn}}} \right). \quad (2.73)$$

Here $X^{a_1...a_{s-1},b_1...b_t} \left( \frac{\delta S_2}{\delta \omega_{\text{dyn}}} \right)$ are some linear functionals of the left-hand-sides of the free field equations $\frac{\delta S_2}{\delta \omega_{\text{dyn}}} = 0$ for the spin $s$ dynamical one-forms $\omega_{\text{dyn}}^{a_1...a_{s-1}}$. The 0-forms $C^{a_1...a_{s-1},b_1...b_s}$ are described by the traceless two-row rectangular Young diagrams of length $s$ and parametrize those components of the higher spin field strengths that can remain nonvanishing when the field equations and constraints are satisfied. These generalize the Weyl tensor in gravity ($s = 2$) that parametrizes the components of the Riemann tensor allowed to be nonvanishing when the zero-torsion constraint and Einstein equations (requiring the Ricci tensor to vanish) are imposed. The covariant version of (2.73) is

$$R^{A_1...A_{s-1},B_1...B_{s-1}}_{t} = h_{A_s} \wedge h_{B_s} C^{A_1...A_{s-1},B_1...B_s} + X^{A_1...A_{s-1},B_1...B_{s-1}} \left( \frac{\delta S_2}{\delta \omega_{\text{dyn}}} \right). \quad (2.74)$$

with $C^{A_1...A_{s-1},B_1...B_s}$ described by the traceless $V^A$–transversal two-row rectangular Young diagram of length $s$, i.e.

$$C^{\{A_1...A_{s+1}\}B_2...B_s} = 0, \quad (2.75)$$

$$C^{A_1...A_{s-2},C D,B_1...B_s} \eta_{CD} = 0, \quad C^{A_1...A_{s-1},C,B_1...B_s} V_C = 0. \quad (2.76)$$

For completeness, let us present the unfolded equations of motion for all free integer spin massless higher spin fields in $AdS_d$ corresponding to the totally symmetric representations of the Wigner little group (more precisely, totally symmetric lowest weight vacua of the irreducible representations of the $AdS_d$ algebra $o(d-1,2)$). The content of the Central On-Mass-Shell Theorem is that the equations of motion for massless free fields of all spins can be written in the form

$$R^{A_1...A_{s-1},B_1...B_{s-1}}_{t} = h_{A_s} \wedge h_{B_s} C^{A_1...A_{s-1},B_1...B_s}, \quad (2.77)$$
\[ D_0 C^{A_1...A_u,B_1...B_s} = 0 \quad u \geq s, \quad (2.78) \]

where
\[ D_0 = D_0^L + \sigma_- + \sigma_+, \quad (2.79) \]

\( D_0^L \) is the vacuum Lorentz covariant derivation and the operators \( \sigma_\pm \) have the form
\[
\sigma_-(C)^{A_1...A_u,B_1...B_s} = (u - s + 2)E_C C^{A_1...A_u,C,B_1...B_s} + sE_C C^{A_1...A_uB_s,B_1...B_{s-1}C} \]
\[ \quad (2.80) \]

\[
\sigma_+(C)^{A_1...A_u,B_1...B_s} = u\lambda^2 \left( \frac{d + u + s - 4}{d + 2u - 2} \right) E_C C^{A_1A_2...A_u,B_1...B_s} \\
- \frac{s}{d + 2u - 2} \eta^{A_1B_1} E_C C^{A_2...A_u,CB_2...B_s} \\
- \frac{(u - 1)(d + u + s - 4)}{(d + 2u - 2)(d + 2u - 4)} \eta^{A_1A_2} E_C C^{A_3...A_uC,B_1...B_s} \\
+ \frac{s(u - 1)}{(d + 2u - 2)(d + 2u - 4)} \eta^{A_1A_2} E_C C^{A_3...A_uB_1,CB_2...B_s} \\
\quad (2.81) \]

(total symmetrization within the groups of indices \( A_i \) and \( B_j \) is assumed). The set of 0-forms \( C^{A_1...A_u,B_1...B_s} \) consists of all two-row traceless \( V^A \)–transversal Young diagrams with the second row of length \( s \), i.e.
\[ C^{(A_1...A_u,A_{u+1})B_2...B_s} = 0, \quad (2.82) \]

\[ C^{A_1...A_{u-2}CD,B_1...B_s} \eta_{CD} = 0, \quad C^{A_1...A_{u-1}C,B_1...B_s} V_C = 0. \quad (2.83) \]

The equations (2.77) (being a consequence of the First On-Mass-Shell Theorem) and (2.78) are equivalent to the free equations of motion of (totally symmetric) massless fields of all spins in \( AdS_d \) along with some constraints that express an infinite set of auxiliary variables via higher derivatives of the dynamical fields of all spins. The proof of the Central On-Mass-Shell Theorem is analogous to that given in the \( su(2,2) \) notation in section 7 for the 5d case. The Central On-Mass-Shell Theorem plays the key role in many respects and, in particular, for the analysis of interactions as was originally demonstrated in [39] where it was proved for the 4d case.

Note that, as shown in [40], the equations of motion of massless scalar coincide with the sector of equations (2.78) with \( s = 0 \). Analogously, the equations (2.78) with \( s = 1 \) impose the Maxwell equations on the spin 1 potential (1-form) \( \omega \).
3 Compensator Formalism in $su(2, 2)$ Notation

It is well known that the $AdS_5$ (equivalently, $4d$ conformal) algebra $o(4, 2)$ is isomorphic to $su(2, 2)$ and, as such, admits realization in terms of oscillators

$$[a_\alpha, b_\beta]_s = \delta_\alpha^\beta, \quad [a_\alpha, a_\beta]_s = 0, \quad [b_\alpha, b_\beta]_s = 0 \quad (3.1)$$

$\alpha, \beta = 1 \div 4$. Here we use the star product realization of the algebra of oscillators that describes the totally symmetric (i.e., Weyl) ordering

$$(f \ast g)(a, b) = \frac{1}{(\pi)^2} \int d^4u d^4v d^4s d^4t f(a+u, b+t) g(a+s, b+v) \exp 2(s_\alpha t^\alpha - u_\alpha v^\alpha)$$

$$= e^{\frac{1}{2} \left( \frac{\partial^2}{\partial s_\alpha \partial t^\alpha} - \frac{\partial^2}{\partial u_\alpha \partial v^\alpha} \right)} f(a + s, b + u) g(a + v, b + t) \bigg|_{s=u=t=v=0}. \quad (3.2)$$

It is straightforward to see that this star product is associative and gives rise to the commutation relations (3.1) via (2.30). The associative star product algebra with eight generating elements $a_\alpha$ and $b_\beta$ is called Weyl algebra $A_4$. Let us note that the star product algebras relevant to the higher spin gauge theory (in particular, the one used throughout this paper) are treated as the algebras of polynomials or formal power series thus being different from the star product algebras of functions regular at infinity that are relevant to the noncommutative Yang-Mills theory [17]. One important difference concerns the definitions of the invariant trace operations because, as shown in [15], the star product algebras of formal power series possess a uniquely defined supertrace operation but admits no usual trace at all (like the one used in the non-commutative Yang-Mills theory). It is worth to mention that the superstructure underlying the supertrace of the polynomial star product algebras is just appropriate in the context of the spinor interpretation of the generating elements like $a_\alpha$ and $b_\beta$ in the $5d$ higher spin theory studied in this paper.

The Lie algebra $gl_4$ is spanned by the bilinears

$$T^\alpha_\beta = a_\alpha b_\beta \equiv \frac{1}{2} (a_\alpha \ast b_\beta + b_\beta \ast a_\alpha) \quad (3.3)$$

The central component is associated with the generator

$$N = a_\alpha b_\alpha \equiv \frac{1}{2} (a_\alpha \ast b_\alpha + b_\alpha \ast a_\alpha) \quad (3.4)$$

while the traceless part

$$t^\alpha_\beta = a_\alpha b_\beta - \frac{1}{4} \delta^\alpha_\beta N \quad (3.5)$$
spans $sl_4$. The $su(2, 2)$ real form of $sl_4(\mathbb{C})$ results from the reality conditions
\[
\bar{a}_\alpha = b^\beta C_{\beta \alpha}, \quad \bar{b}^\alpha = C^{\alpha \beta} a_\beta,
\] (3.6)
where bar denotes the complex conjugation while $C_{\alpha \beta} = -C_{\beta \alpha}$ and $C^{\alpha \beta} = -C^{\beta \alpha}$ are some real antisymmetric matrices satisfying
\[
C_{\alpha \gamma} C^{\beta \gamma} = \delta^\beta_\alpha.
\] (3.7)

The oscillators $b^\alpha$ and $a_\alpha$ are in the fundamental and the conjugated fundamental representations of $su(2, 2)$ equivalent to the two spinor representations of $o(4, 2)$. A $o(6)$ complex vector $V^A$ ($A = 0 \div 5$) is equivalent to the antisymmetric bispinor $V^{\alpha \beta}$ having six independent components (equivalently, one can use $V_{\alpha \beta} = \frac{1}{2} \varepsilon_{\alpha \beta \gamma \delta} V^{\gamma \delta}$ where $\varepsilon_{\alpha \beta \gamma \delta}$ is the $sl_4$ invariant totally antisymmetric tensor ($\varepsilon_{1234} = 1$)). A $o(4, 2)$ real vector $V^A$ is described by the antisymmetric bispinor $V^{\alpha \beta}$ satisfying the reality condition
\[
\nabla^{\alpha \beta} C_{\gamma \delta} C_{\beta \gamma} = \frac{1}{2} \varepsilon_{\alpha \beta \gamma \delta} V^{\gamma \delta}.
\] (3.8)

One can see that the invariant norm of the vector
\[
V^2 = V_{\alpha \beta} V^{\alpha \beta}
\] (3.9)
has the signature $(++----)$. The vectors with $V^2 > 0$ are time-like while those with $V^2 < 0$ are space-like. To perform a reduction of the representations of the $AdS_5$ algebra $su(2, 2) \sim o(4, 2)$ into representations of its Lorentz subalgebra $o(4, 1)$ we introduce a $su(2, 2)$ antisymmetric compensator $V^{\alpha \beta}$ with positive square (3.9), which is the spinor analog of the compensator $V^A$ of section 2. The Lorentz algebra is identified with its stability subalgebra. (Let us note that $V^{\alpha \beta}$ must be different from the form $C^{\alpha \beta}$ used in the definition of the reality conditions (3.6) since the latter is space-like and therefore has $sp(4; \mathbb{R}) \sim o(3, 2)$ as its stability algebra.)

We shall treat $V^{\alpha \beta}$ as a symplectic form that allows one to raise and lower spinor indices in the Lorentz covariant way
\[
A^\alpha = V^{\alpha \beta} A_\beta, \quad A_\alpha = A^\beta V_{\beta \alpha},
\] (3.10)
Using that the total antisymmetrization over any four indices is proportional to the $\varepsilon$ symbol, we normalize $V^{\alpha \beta}$ so that
\[
V_{\alpha \beta} V^{\alpha \gamma} = \delta_\gamma^\alpha, \quad V_{\alpha \beta} = \frac{1}{2} \varepsilon_{\alpha \beta \gamma \delta} V^{\gamma \delta},
\] (3.11)
\[ \varepsilon_{\alpha\beta\gamma\delta} = V_{\alpha\beta} V_{\gamma\delta} + V_{\beta\gamma} V_{\alpha\delta} + V_{\gamma\alpha} V_{\beta\delta}, \quad (3.12) \]

\[ \varepsilon^{\alpha\beta\gamma\delta} = V^{\alpha\beta} V^{\gamma\delta} + V^{\beta\gamma} V^{\alpha\delta} + V^{\gamma\alpha} V^{\beta\delta}. \quad (3.13) \]

In these terms the Lorentz subalgebra is spanned by the generators symmetric in the spinor indices

\[ L_{\alpha\beta} = \frac{1}{2} (t_{\alpha\beta} + t_{\beta\alpha}) \quad (3.14) \]

while the \( AdS_5 \) translations are associated with the antisymmetric traceless generators

\[ P_{\alpha\beta} = \frac{1}{2} (t_{\alpha\beta} - t_{\beta\alpha}). \quad (3.15) \]

The gravitational fields are identified with the gauge fields taking values in the \( AdS_5 \) algebra \( su(2,2) \)

\[ w = w_{\alpha} a_{\alpha} b_{\beta}. \quad (3.16) \]

The invariant definitions of the frame field and Lorentz connection for a \( x \)-dependent compensator \( V^{\alpha\beta}(x) \) are

\[ E^{\alpha\beta} = D V^{\alpha\beta} \equiv d V^{\alpha\beta} + w_{\alpha} V^{\gamma\beta} + w_{\beta} V^{\alpha\gamma}, \quad (3.17) \]

\[ \omega^{L_{\alpha\beta}} = w_{\alpha} + \frac{1}{2} E^{\alpha\gamma} V_{\gamma\beta}. \quad (3.18) \]

The normalization condition (3.11) implies

\[ E_{\alpha\beta} = - D V_{\alpha\beta}, \quad E_{\alpha} = 0. \quad (3.19) \]

The non-degeneracy condition implies that \( E^{\alpha\beta} \) spans a basis of the 5d 1-forms. The basis \( p \)-forms \( E_p \) can be realized as

\[ E_2^{\alpha\beta} = E_2^{\beta\alpha} = E^{\alpha}_{\gamma} \wedge E^{\beta\gamma}, \quad (3.20) \]

\[ E_3^{\alpha\beta} = E_3^{\beta\alpha} = E^{\alpha}_{\gamma} \wedge E^{\beta\gamma}, \quad (3.21) \]

\[ E_4^{\alpha\beta} = - E_4^{\beta\alpha} = E^{\alpha}_{\gamma} \wedge E^{\beta\gamma}, \quad (3.22) \]

\[ E_5 = E^{\alpha}_{\gamma} \wedge E^{\gamma}_{\alpha}. \quad (3.23) \]

The following useful relationships hold as a consequence of the facts that 5d spinors have four components and the frame field is traceless (3.19)

\[ E^{\alpha\beta} \wedge E^{\gamma\delta} = \frac{1}{2} (V^{\alpha\gamma} E_2^{\beta\delta} - V^{\beta\gamma} E_2^{\alpha\delta} - V^{\alpha\delta} E_2^{\beta\gamma} + V^{\beta\delta} E_2^{\alpha\gamma}), \quad (3.24) \]
The gravitational field \( w \) describes the \( AdS_5 \) geometry provided that \( w = \omega_0 \) satisfies the zero-curvature equation

\[
d\omega_0 + \omega_0 \wedge \ast \omega_0 = 0 \tag{3.29}
\]

and the frame 1-form is non-degenerate. The background frame field and Lorentz connection will be denoted \( h = h_{\alpha \beta}^a \alpha \beta \) and \( \omega_L^a_\alpha \beta \alpha \beta \) respectively. The vacuum values of the \( p \)-forms \( E_{\alpha \beta}^p \) are denoted \( H_{\alpha \beta}^p \).

### 4 su(2, 2) - o(4, 2) Dictionary

To make contact between the tensor and spinor forms of the higher spin dynamics one has to identify in terms of \( o(4, 2) \) the irreducible finite-dimensional representations of \( su(2, 2) \) described by a pair of mutually conjugated traceless \( su(2, 2) \) multispinors

\[
X^{\alpha_1...\alpha_n}_{\beta_1...\beta_m} \oplus X^{\beta_m}_{\alpha_1...\alpha_n} , \quad X^{\alpha_1...\alpha_{n-1}\gamma}_{\beta_1...\beta_{m-1}\gamma} = 0 . \tag{4.1}
\]

The result is that for even \( n + m \) the representation (4.1) is equivalent to the representation of \( o(4, 2) \) described by the traceless tree-row Young diagram having two rows of equal lengths \( \frac{1}{2} |n + m| \) and the third one of length \( \frac{1}{2} |n - m| \).

In other words, the \( o(4, 2) \) form of the representation (4.1) is described by the tensor \( X_{A_1...A_p , B_1...B_q , C_1...C_q} \) with \( p = \frac{1}{2} |n + m| , \quad q = \frac{1}{2} |n - m| \), which is separately symmetric with respect to the indices \( A_i , B_i \) and \( C_i \) and satisfies the conditions

\[
X_{(A_1...A_p , A_{p+1} )B_2...B_p , C_1...C_q} = 0 , \quad X_{A_1...A_p , (B_1...B_q , B_{p+1} )C_2...C_q} = 0 \tag{4.2}
\]

and

\[
\eta^{D_1D_2} X_{D_1D_2A_3...A_p , B_1...B_p , C_1...C_q} = 0 . \tag{4.3}
\]

(From these conditions it follows that all other traces vanish as well.) One example of this identification is provided by the isomorphism between \( X^{\alpha \beta} \)
(with its conjugate $\overline{X}_{\alpha\beta}$) and the 3-form representation of $o(4,2)\, X_{A,B,C}$ being totally antisymmetric in its indices.

For the case of half-integer spins with odd $n + m$, the identification is analogous with the tensor-spinor $X_{A_{1}...A_{p},B_{1}...B_{p},C_{1}...C_{q}}$ carrying the $o(4,2)$ spinor index $\alpha$, $2p = n + m - 1$, $2q = |n - m| - 1$ and the $\gamma$-transversality condition with respect to all indices in addition to the tracelessness condition (4.3).

A particular case of a self-conjugated traceless multispinor

$$X^{\alpha_{1}...\alpha_{n}}_{\beta_{1}...\beta_{n}}, \quad X^{\alpha_{1}...\alpha_{n-1}}_{\beta_{1}...\beta_{n-1}\gamma} = 0 \quad (4.4)$$

is most important for this paper. Such a tensor is equivalent to the representation of $o(4,2)$ described by a length-$n$ rectangular traceless two-row Young diagram, i.e. to

$$\tilde{X}_{A_{1}...A_{n},B_{1}...B_{n}} \quad (4.5)$$

which is separately symmetric in the indices $A_{k}$ and $B_{k}$, has all traces zero and is subject to the condition that symmetrization of any $n + 1$ indices gives zero. One way to see this isomorphism is to compare the dimensions of the representations to make sure that they are both equal to $\frac{(2n+3)(n+1)^{2}(n+2)^{2}}{12}$. It is easy to see that this formula is true from the $sl_{4}$ side. The computation in terms of $o(4,2)$ is more complicated. The dimensionality of the representation of the orthogonal algebra $o(d)$ described by the two-row traceless rectangular diagram of length $s$ is

$$\mathcal{N}(s, d) = \frac{(2s + d - 2)!(s + d - 4)!(s + d - 5)!}{(d - 2)!(d - 4)!s!(s + 1)!(2s + d - 5)!} \quad (4.6)$$

For $n = s$ and $d = 6$ one finds the desired result. For $n = 1$ the isomorphism between the adjoint representations of $su(2,2)$ and $o(4,2)$ is recovered. Note that the analogous analysis of the representation (4.4) of $su(2,2)$ was done in [28] in terms of representations of the 5$d$ Lorentz algebra $o(4,1) \subset o(4,2)$.

In accordance with the analysis of section 2.3 (and of [28]) we conclude that 5$d$ spin $s$ bosonic gauge fields can be described by 1-forms $\omega^{\alpha_{1}...\alpha_{s-1}}_{\beta_{1}...\beta_{s-1}}$ which are traceless multispinors symmetric in the upper and lower indices. Totally symmetric spin $s$ fermionic tensor-spinor representations are described by the gauge fields $\omega^{\alpha_{1}...\alpha_{s-1/2}}_{\beta_{1}...\beta_{s-3/2}}$ and their conjugates.

All other representations in the set (4.1) do not correspond to the sets of gauge fields associated with the totally symmetric tensor(-spinor) fields. These are expected to underly the description of the mixed symmetry $AdS_{5}$ massless fields to be developed. According to [29], such fields are inequivalent to the
totally symmetric higher spin fields in the AdS regime, although reduce in the flat limit to some combinations of the higher spin fields associated with the totally symmetric representations of the flat Wigner little algebra. As argued in [25], the fields $\omega^{\alpha_1...\alpha_p\beta_1...\beta_q}$ with $|p - q| \geq 2$ necessarily appear in the 5d higher spin gauge theories with $\mathcal{N} \geq 2$ extended supersymmetry. This raises the important problem of the development of the formulation of the corresponding massless fields in $AdS_5$ for $d > 4$. This problem is now under investigation. Prior it is solved, we can only study the purely bosonic theory with totally symmetric higher spin fields, which is the subject of this paper, and its $\mathcal{N}=1$ supersymmetric version, which is the subject of the forthcoming paper [32].

5 5d Higher Spin Algebra

The $AdS_5$ higher spin algebras are expected to identify with 4d conformal higher spin algebras studied by Fradkin and Linetsky [26], and their further extensions [25] and reductions [28, 25]. One starts with the Lie superalgebra constructed via supercommutators of the star product algebra (3.2). In [25] it was argued that this algebra as a whole, called $hu(1,1|8)$ [42], may play a key role in a $AdS_5$ higher spin gauge theory. The set of the gauge fields corresponding to the algebra $hu(1,1|8)$ is

$$\omega(a,b|x) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \omega^{\alpha_1...\alpha_m\beta_1...\beta_n}(x) a_{\alpha_1} ... a_{\alpha_m} b^{\beta_1} ... b^{\beta_n}. \quad (5.1)$$

The 5d higher spin field strength has the form

$$R(a,b|x) = d\omega(a,b|x) + \omega(a,b|x) \wedge \omega(a,b|x). \quad (5.2)$$

The higher spin gauge fields in (5.1) contain 1-forms in all representations (4.1). According to the analysis of [28] and section 4 of this paper, only the fields with $n = m$ correspond to usual (i.e., totally symmetric) higher spin fields. Before the free theory of the mixed symmetry $AdS_5$ higher spin gauge fields is elaborated, we confine ourselves to the higher spin algebra associated with the simplest case of the purely bosonic theory of totally symmetric higher spin fields.

We therefore want to have only the gauge fields carrying equal numbers of the upper and lower $su(2,2)$ indices. As a result, the elements of the higher spin algebra should satisfy

$$N_a f = N_b f, \quad (5.3)$$
where
\[ N_a = a_\gamma \frac{\partial}{\partial a_\gamma}, \quad N_b = b_\gamma \frac{\partial}{\partial b_\gamma}. \] (5.4)

This is equivalent to the condition [26]
\[ N \ast f = f \ast N. \] (5.5)

Thus, the bosonic 5d higher spin algebra identifies with the Lie algebra built from the star-commutators of the elements of the centralizer of \( N \) in the star product algebra (3.2). The same algebra (although rewritten in the 4d covariant notations) was interpreted in [26] as the 4d conformal higher spin algebra called \( hsc^\infty(4) \) and was proved to give rise to the gauge invariant cubic interactions of the 4d conformal higher spin theory in [27]. We change the names of some of the higher spin superalgebras in accordance with the notation of [42, 25] to include in our systematics the two-parametric series of matrix extensions of the higher spin superalgebras. In particular we will use the name \( cu(1,0|8) \) for the algebra \( hsc^\infty(4) \) of [26].

The set of the gauge field corresponding to the algebra \( cu(1,0|8) \) is
\[ \omega(a, b| x) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \omega^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}(x) a_{\alpha_1} \ldots a_{\alpha_n} b_{\beta_1} \ldots b_{\beta_n}. \] (5.6)

The \( cu(1,0|8) \) field strength has the form (5.2) and admits analogous expansion
\[ R(a, b| x) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} R^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}(x) a_{\alpha_1} \ldots a_{\alpha_n} b_{\beta_1} \ldots b_{\beta_n}. \] (5.7)

So far we considered complex fields. To impose the reality conditions let us define the involution \( \dagger \) by the relations
\[ (a_\alpha)^\dagger = ib_\beta C_{\beta \alpha}, \quad (b^\alpha)^\dagger = iC^{\alpha \beta} a_{\beta}, \] (5.8)

Since an involution is required to reverse an order of product factors
\[ (f \ast g)^\dagger = g^\dagger \ast f^\dagger \] (5.9)

and to conjugate complex numbers
\[ (\mu f)^\dagger = \bar{\mu} f^\dagger, \quad \mu \in C, \] (5.10)

the definition (5.8) contains an additional factor of \( i \) compared to the complex conjugation (3.6). The involution \( \dagger \) leaves invariant the defining relations (3.1)
of the star product algebra and has the involutive property \((\dagger)^2 = Id\). By (5.9) the action of \(\dagger\) extends to an arbitrary element \(f\) of the star product algebra. Since the star product we use corresponds to the totally symmetric (i.e. Weyl) ordering of the product factors, the result is simply

\[
(f(a, b^\alpha))\dagger = f(ib^\gamma C_\gamma a, iC^\beta \gamma a_\gamma).
\] (5.11)

It is elementary to check directly with (3.2) that (5.11) defines an involution of the star product algebra.

The reality conditions on the elements of the higher spin algebra have to be imposed in a way consistent with the form of the higher spin curvature. This is equivalent to singling out a real form of the higher spin Lie algebra. With the help of any involution \(\dagger\) this is achieved by imposing the reality conditions

\[
f\dagger = -f.
\] (5.12)

This condition defines the real higher spin algebra \(hu(1,0|8)\) for four pairs of oscillators and \(cu(1,0|8)\) as its subalgebra being the centralizer of \(N\). Note that the operator \(N\) is self-conjugated

\[
N\dagger = N.
\] (5.13)

Let us stress that the condition (5.12) extracts a real form of the Lie superalgebra built from the star product algebra but not of the associative star product algebra itself. The situation is very much the same as for the Lie algebra \(u(n)\) singled out from the complex Lie algebra of \(n \times n\) matrices by the condition (5.12) with \(\dagger\) identified with the hermitian conjugation. Antihermi- tian matrices form the Lie algebra but not associative algebra. In fact, the relevance of the reality conditions of the form (5.12) is closely related with this matrix example because it demonstrates that the spin 1 (i.e., purely Yang-Mills) part of the matrix extensions of the higher spin algebras is compact. More generally, these reality conditions guarantee that the higher spin symmetry admits appropriate unitary highest weight representations. Note that in the sector of the \(AdS_5\) algebra \(su(2,2)\) the reality condition (5.12) is equivalent to (3.6).

The higher spin gauge fields \(\omega(a, b|x)\) are required to satisfy the condition analogous to (5.12)

\[
\omega\dagger = -\omega,
\] (5.14)

that gives rise to the component form of the reality condition by virtue of (5.11).
For any fixed $n$ the connection $\omega^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n}(x)$ is reducible because it is not traceless. It decomposes into the set of $n+1$ irreducible components $\omega^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\alpha_k}$ with all $k$ in the interval $n \geq k \geq 0$ ($\omega^{\alpha_1\ldots\alpha_{k-1}\gamma}_{\beta_1\ldots\beta_{k-1}\gamma} = 0$).

As a result, fields of every spin appear in infinitely many copies in the expansion (5.6). The origin of this infinite degeneracy can be traced back to the fact that the algebra $A^0_4$ has infinitely many ideals $I_P(N)$ associated with various central elements $P(N)$ being star-polynomials of $N$, $\{x \in I_P(N) : x = P(N) \ast y, \ y \ast N = N \ast y\}$ [26]. On the one hand this infinite degeneracy makes 5d higher spin gauge theories reminiscent of the superstring theory that contains infinitely many (massive) modes of any given symmetry type. On the other hand a question arises whether it is possible to consider consistent higher spin models with reduced spectra of spins associated with the quotient higher spin algebras. The most interesting reductions are provided with the algebra $hu_0(1,0|8) = cu(1,0|8) / I_N$ called $hs^0(4)$ in [26] and its further reduction $ho_0(1,0|8)$ [25] called $hs(2,2)$ in [28]. ($I_N$ is the ideal spanned by the elements of the form $g = N \ast f = f \ast N$.) The gauge fields of the algebra $hu_0(1,0|8)$ correspond to the set of all integer spins $s \geq 1$ (every spin appears once) while $ho_0(1,0|8)$ describes its reduction to the subalgebra associated with even spins.

As we show both options are allowed in the framework of the cubic analysis of this paper. We start in the section 6 with the analysis of the unreduced case of $cu(1,0|8)$ considering the reduced cases afterwards in section 9. Note that from this perspective our conclusions are somewhat different from those of [27] where it was argued that only the unreduced algebra $cu(1,0|8)$ admits consistent dynamics in the framework of the 4d conformal higher spin theory. From the perspective of AdS/CFT correspondence the most interesting cases are associated either with the maximally reduced models [28, 25] and their supersymmetric extensions or with the unreduced models based on the algebras $hu(m,n|8)$ [25] which, presumably, give rise to all types of $AdS_5$ massless fields.

6 5d Higher Spin Gauge Fields

The $cu(1,0|8)$ linearized higher spin curvature

$$R_1(a,b|x) = d\omega_1(a,b|x) + \omega_0(a,b|x) \ast \omega_1(a,b|x) + \omega_1(a,b|x) \ast \w_0(a,b|x),$$

with

$$\omega_0(a,b|x) = \omega_0^{\alpha\beta}a_\alpha b^\beta, \quad \omega_0^{\alpha\alpha} = 0$$

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satisfying the zero curvature condition (3.29), provides the 5d spinor version of the formula (2.53). Equivalently,

\[ R_1(a, b|x) = d\omega_1(a, b|x) + \omega_0^\alpha_\beta(x)(\frac{\partial}{\partial b^\alpha} b^\beta - a^\alpha \frac{\partial}{\partial a^\beta}) \wedge \omega_1(a, b|x). \]  

(6.3)

The component formula reads

\[ R_1^{\alpha_1...\alpha_{s-1}}_{\beta_1...\beta_{s-1}} = d\omega^{\alpha_1...\alpha_{s-1}}_{\beta_1...\beta_{s-1}} - (s-1)(\omega_0^\alpha_\gamma \wedge \omega_0^{\gamma\alpha_2...\alpha_{s-1}}_{\beta_1...\beta_{s-1}}) \]

\[ - \omega_0^\gamma_{\beta_1} \wedge \omega^{\alpha_1...\alpha_{s-1}}_{\gamma\beta_2...\beta_{s-1}}. \]  

(6.4)

The linearized (Abelian) higher spin gauge transformations are

\[ \delta\omega_0(a, b|x) = D_0 \varepsilon(a, b|x), \]  

(6.5)

where

\[ D_0 = d + \omega_0^\alpha_\beta(\frac{\partial}{\partial b^\alpha} b^\beta - a^\alpha \frac{\partial}{\partial a^\beta}). \]  

(6.6)

is the background covariant derivative. The fact that \( \omega_0 \) satisfies the zero-curvature condition implies

\[ \delta_0 R_1 = 0. \]  

(6.7)

To decompose the representations of the \( AdS_5 \) algebra \( su(2, 2) \sim o(4, 2) \) into representations of its Lorentz subalgebra \( o(4, 1) \) we use the antisymmetric compensator \( V^{\alpha\beta} \). A \( su(2, 2) \) counterpart of the reduction of the tensor higher spin gauge field \( dx^a \omega^{A_1...A_{s-1}, B_1...B_{s-1}}_0 \) carrying the irreducible representation of the \( AdS_d \) algebra \( o(d-1, 2) \) described by the traceless two-row Young diagram of length \( s - 1 \) into a collection of the Lorentz covariant higher spin 1-forms \( dx^a \omega^{a_1...a_{s-1}, b_1...b_t}_0 \) with all \( 0 \leq t \leq s - 1 \) goes as follows. The field \( V^{\alpha\beta} \) is used to raise and lower spinor indices. Then, the Lorentz algebra irreducible components correspond to various types of symmetrization between the two types of indices, i.e. again to all two-row traceless Young diagrams but now in the spinor indices, \( \omega^{b_{1...a_{s-1+q}, b_{1...b_1}}}_{0a_1...a_{s-1}, b_1...b_t} \) with all \( 0 \leq q \leq s - 1 \) (all traces with \( V^{\alpha\beta} \) are zero and symmetrization with respect to any \( s + q \) indices gives zero). The identification with the \( o(4, 1) \) tensor notation is

\[ \omega'_{a_1...a_{s-1+q}, b_1...b_{s-1-t}} \sim \omega^{a_1...a_{s-1}, b_1...b_t}. \]  

(6.8)

For example, the two-row rectangular diagram of length \( s - 1 \) in tensor notation is described by the one-row diagram of length \( 2(s - 1) \) in the spinor
notation, while the two-row rectangular diagram of length \( s - 1 \) in spinor notation corresponds to the one-row diagram of length \( s - 1 \) in the tensor notation. (Particular manifestations of this relationship are those between the vector and traceless antisymmetric second-rank spinor or antisymmetric tensor and symmetric second rank spinor, both underlying the isomorphism between the spinor and vector realization of the 5d space-time symmetry algebras.) Note that the analogous identification of the representations was discussed in the recent paper [28], where the spinor version of the linearized higher spin curvatures has been presented. The difference is that in this paper we use the manifestly \( su(2, 2) \) covariant compensator formalism that simplifies greatly the analysis of the interactions.

In what follows we shall use the two sets of the differential operators in the spinor variables

\[
S^- = V^{\alpha\beta} a_\alpha \frac{\partial}{\partial b_\beta}, \quad S^+ = V_{\alpha\beta} b^\alpha \frac{\partial}{\partial a_\beta}, \quad S^0 = N_b - N_a \quad (6.9)
\]

and

\[
T^+ = a_\alpha b^\alpha, \quad T^- = \frac{1}{4} \frac{\partial^2}{\partial a_\alpha \partial b^\alpha}, \quad T^0 = \frac{1}{4} (N_a + N_b + 4) . \quad (6.10)
\]

They form two mutually commuting \( sl_2 \) algebras

\[
[S^0, S^\pm] = \pm 2S^\pm, \quad [S^-, S^+] = S^0, \quad (6.11)
\]

\[
[T^0, T^\pm] = \mp \frac{1}{2} T^\pm, \quad [T^-, T^+] = T^0, \quad (6.12)
\]

\[
[T^i, S^j] = 0 . \quad (6.13)
\]

(Unusual normalization of the generators \( T^i \) in (6.10), and (6.12) is chosen for the future convenience).

The operators \( T^i \) and \( S^0 \) are independent of the compensator \( V^{\alpha\beta} \) and, therefore, are \( su(2, 2) \) invariant. As a result,

\[
D_0(T^i) = 0, \quad D_0(S^0) = 0 . \quad (6.14)
\]

(These relations have to be understood in the sense that \( D_0(X(f)) = X(D_0(f)) \), where \( X \) is one of the operators \( T^i \) and \( S^0 \), while \( f \) is an arbitrary element of the star product algebra.) A useful consequence of this fact is

\[
R_1(T^i(\omega)) = T^i(R_1(\omega)) . \quad (6.15)
\]
From (6.11) it also follows

\[ [S^+, D_0(S^-)] + [D_0(S^+), S^-] = 0. \]  

(6.16)

According to (5.3) the elements of the higher spin algebra \( cu(1,0|8) \) satisfy

\[ S^0(f) = 0, \quad S^0R(a,b|x) = 0. \]  

(6.17)

As a result, the operators \( S^+ \) and \( S^- \) commute to each other on the higher spin gauge fields and field strengths of \( cu(1,0|8) \).

The \( V^{\alpha\beta} \) - dependent operators \( S^\pm \) are only Lorentz invariant. In accordance with (3.17) and (3.19)

\[ D_0S^- = h^{\alpha\beta}a_{\alpha}^{\beta} \frac{\partial}{\partial b^{\beta}}, \quad D_0S^+ = -h_{\alpha\beta}b_{\alpha}^{\beta} \frac{\partial}{\partial a^{\beta}}. \]  

(6.18)

Let us note that the background covariant derivative \( D_0 \) (6.6) admits the representation

\[ D_0 = D_0^L + \frac{1}{2}[S^-, D_0S^+] = D_0^L - \frac{1}{2}h^{\alpha\beta}(a_{\alpha}^{\beta} \frac{\partial}{\partial a_{\alpha}} - b_{\beta}^{\beta} \frac{\partial}{\partial b^{\beta}}), \]  

(6.19)

where the background Lorentz derivative \( D_0^L \) commutes with all operators \( T^i \) and \( S^i \).

From the star product (3.2) it follows that

\[ N \ast f = \left( T^+ - T^- + \frac{1}{2}(N_b - N_a) \right) f. \]  

(6.20)

According to (5.3), for \( f \in cu(1,0|8) \) this simplifies to

\[ N \ast f = f \ast N = (T^+ - T^-)f. \]  

(6.21)

The decomposition into \( su(2,2) \) irreducible fields is

\[ \omega(a,b) = \sum_{s,n=0}^{\infty} (T^+)^nv_n(T^0)\omega^s_n(a,b), \]  

(6.22)

with

\[ T^0\omega^s_n = \frac{1}{2}(s+1)\omega^s_n, \]  

(6.23)

\[ T^-\omega^s_n(a,b) = 0. \]  

(6.24)
For the future convenience, we fix the normalization coefficients \( v_n(T^0) \) in (6.22) in the form

\[
v_n\left(\frac{1}{2}(s+1)\right) = v\left(\frac{1}{2}(s+1)\right)(2i)^n \sqrt{\frac{(2s+1)!}{n!(n+2s+1)!}}.
\]

(6.25)

Note that the factor of \( i^n \) in (6.25) implies that the different copies of the fields \( \omega_n^a \) with the same spin contained in polynomials of degree \( 4p \) and \( 4p + 2 \) contribute with opposite signs. This is appropriate because the coefficients in front of the corresponding parts of the invariant action will be shown to have opposite signs as well.

Due to (6.15), the linearized curvatures admit the expansion analogous to (6.22)

\[
R_1(a,b) = \sum_n (T^+)^nv_n(T^0)R_{1,n}(a,b)
\]

(6.26)

with

\[
T^- R_{1,n}(a,b) = 0.
\]

(6.27)

Let us now explain how the invariant version of the Lorentz covariant decomposition used in [30, 28] can be defined. Lorentz multispinors associated with the two-row Young diagrams having \( n_1 \) and \( n_2 \) cells in the upper and lower rows respectively \( (n_1 \geq n_2) \) can be described as the polynomials \( \eta(a,b) \) of the spinor variables \( a^\alpha \) and \( b^\beta \) subject to the conditions

\[
N_a \eta(a,b) = n_1 \eta(a,b), \quad N_b \eta(a,b) = n_2 \eta(a,b),
\]

(6.28)

\[
S^- \eta(a,b) = 0,
\]

(6.29)

where the latter condition implies that the symmetrization over any \( n_1 + 1 \) indices gives zero. The tracelessness condition reads in these terms

\[
T^- \eta(a,b) = 0.
\]

(6.30)

The \( su(2,2) \) irreducible higher spin gauge field \( \omega \) admits the following representation in terms of the Lorentz - irreducible higher spin fields

\[
\omega(a,b) = \sum_{t=0}^{s} (S^+)^t \eta^t(a,b),
\]

(6.31)

(Note that the asymmetric form of this formula with respect to \( a^\alpha \) and \( b^\beta \) is a result of a particular basis choice.) Since \( \omega(a,b) \) has equal numbers of \( a \) and \( b \), we set \( 2t = n_1 - n_2 \). For the spin \( s \) we have \( 2(s-1) = n_1 + n_2 \) (cf. (6.23)). For \( s \) fixed, \( t \) ranges from 0 to \( s - 1 \).
One can treat the Lorentz-irreducible 1-forms $\eta_t(a, b)$ as an alternative basis of the higher spin gauge fields. The linearized higher spin curvature 2-forms (6.1) admit the analogous expansion

$$R_1(a, b) = \sum_{t=0} (S^+)^t r_1^t(a, b),$$  

(6.32)

with the Lorentz - irreducible component curvatures $r_1^t(a, b)$ satisfying the Young property

$$S^-r_1^t(a, b) = 0$$  

(6.33)

and the tracelessness condition

$$T^-r_1^t(a, b) = 0.$$  

(6.34)

From the definition of $r_1^t(a, b)$ it follows that

$$r_1^t(a, b) = D_0^L \eta^t(a, b) + \tau_-(\eta^{t+1}(a, b)) + \tau_+(\eta^{t-1}(a, b)),$$  

(6.35)

where $D_0^L$ is the Lorentz covariant derivative and the 5d spinor realization of the operators (2.55) and (2.56) is

$$\tau_+ = \frac{1}{S^0 + 1} D_0(S^-),$$  

(6.36)

$$\tau_- = \frac{1}{2} \left( S^+ [S^-, D_0(S^+)] - D_0(S^+)S^0 - (S^+)^2 \frac{1}{S^0 + 1} D_0(S^-) \right).$$  

(6.37)

One can see that the properties (2.57) are satisfied on the space of functions $\eta$ satisfying (6.29).

Analogous decomposition

$$D_0 = D_0^L + \tau_+ + \tau_-$$  

(6.38)

exists in the original basis of fields $\omega$ satisfying the condition (6.17). The explicit form of $\tau_\pm$ in this basis is

$$\tau_\pm = \frac{1}{4} \left( [S^-, D_0S^+] \pm \frac{1}{\sqrt{1 - 4S^+S^-}} ( [S^-, D_0S^+] + 2D_0(S^+S^-) ) \right).$$  

(6.39)

Derivation of this formula is more complicated. It is based on the fact that the operator $S^-S^+ = S^+S^-$ diagonalizes on the vectors with different $t$ in (6.31) with the eigenvalues $-t(t+1)$ so that the operator $\hat{t}$

$$\hat{t} = \frac{1}{2} \left( \sqrt{1 - 4S^+S^-} - 1 \right)$$  

(6.40)
has eigenvalues $t$. The property that

$$[\hat{t}, \tau_\pm] = \pm \tau_\pm$$  \hspace{1cm} (6.41)

turns out to be equivalent to

$$[S^+ S^-, \tau_\pm] = (1 \mp \sqrt{1 - 4S^+ S^-}) \tau_\pm.$$  \hspace{1cm} (6.42)

Taking into account the fact that the decomposition of $D_0$ into eigenspaces $D_0^L$, $\tau_+$ and $\tau_-$ of $S^+ S^-$ is unique, the problem is to find such operators $\tau_\pm$ on the space of functions satisfying (6.17) that the formulas (6.42) and (6.38) are true. Formula (6.39) solves this problem. For (6.38) this is obvious. The verification of the formula (6.42) is also elementary with the help of identities valid on the subspace of null-vectors of $S^0$

$$[S^+ S^-, [S^-, D_0 S^+]] = -2D_0 (S^+ S^-),$$  \hspace{1cm} (6.43)

$$[S^+ S^-, D_0 (S^+ S^-)] = 2 \left( D_0 (S^+ S^-) + S^+ S^- [S^-, D_0 S^+] \right).$$  \hspace{1cm} (6.44)

Another useful fact is that the operator

$$\tau_0 = S^- D_0 (S^+) - S^+ D_0 (S^-)$$  \hspace{1cm} (6.45)

does not affect the gradation $t$, i.e.

$$[S^+ S^-, \tau_0] = 0.$$  \hspace{1cm} (6.46)

It is less trivial to check that $(\tau^\pm)^2 = 0$. The simplest way is to use the basis of the fields $\eta_i$, i.e. the operators $\tau_\pm$ in the form (6.36) and (6.37).

Let us note that the variables $\omega$ and $\eta$ can be interpreted as different representatives of the same representation of the $sl_2$ algebra spanned by the operators $S^j$. Namely, the variables $\omega$ are associated by (6.23) with the elements having zero eigenvalue of the Cartan element, while the variables $\eta$ are associated by (6.29) with the lowest weight vectors. This suggests the idea that there should be some formulation operating in terms of the representations of this $sl_2$ algebra as a whole.

Equipped with the operators $\tau_\pm$ and $\tau_0$, one can write the spinor form of the constraints (2.71) either as

$$\tau_0 \wedge \tau_+ R_1 = 0$$  \hspace{1cm} (6.47)

in the basis $\omega$ or as

$$\tau_0 \wedge \tau_+ r_1 = 0$$  \hspace{1cm} (6.48)
in the basis $\eta$. To obtain the spinor form of the First On-Mass-Shell theorem one takes into account that, as shown in the beginning of this section (see also [28]), the Weyl tensor, described in terms of tensors by the length $s$ two-row traceless Young diagram $C^{a_1...a_s b_1...b_s}$, is described in terms of spinors by a rank $2s$ totally symmetric multispinor $C^{\alpha_1...\alpha_{2s}}$. Since the First On-Mass-Shell Theorem (2.73) is true for any irreducible higher spin field in the expansion (6.26), it acquires the form

$$R_{1,n}(a,b) \bigg|_{m.s.} = H_{2\alpha \beta} \frac{\partial^2}{\partial a_\alpha \partial b_\beta} \text{Res}_\mu (C_n(\mu a + \mu^{-1} b)),$$

where the label $\bigg|_{m.s.}$ implies the on-mass-shell consideration modulo terms proportional to the left hand sides of the free field equations and constraints (6.47) (equivalently, (6.48)). $\text{Res}_\mu$ singles out the $\mu$–independent part of a Laurent series in $\mu$, i.e.

$$\text{Res}_\mu \left( \sum_{n=-\infty}^{\infty} \alpha_n \mu^n \right) = \frac{1}{2\pi i} \oint d\log \mu \left( \sum_{n=-\infty}^{\infty} \alpha_n \mu^n \right) = \alpha_0 .$$

Note that a function of one spinor variable

$$C_n(\mu a + \mu^{-1} b) = \sum_{k,l} \frac{\mu^{k-l}}{k!l!} C^{\alpha_1...\alpha_k \beta_1...\beta_l a_{\alpha_1}...a_{\alpha_k} b_{\beta_1}...b_{\beta_l}}$$

has totally symmetric coefficients $C^{\alpha_1...\alpha_k \beta_1...\beta_l}$ while $\text{Res}_\mu$ singles out its part containing equal numbers of the oscillators $a$ and $b$ that belongs to $cu(1,0|8)$.

7 Central On-Mass-Shell Theorem

The matter fields and higher spin Weyl tensor can be interpreted as representatives of the $\sigma_-$ cohomology group associated with the so-called twisted adjoint representation of the higher spin algebra. Given automorphism $\tau$ of the higher spin algebra (in fact any associative algebra used to build a Lie superalgebra via supercommutators), one defines the covariant derivative $\tilde{D}$ of a field $C$ taking values in the twisted adjoint representation

$$\tilde{D}C = dC + \omega * C - C * \tau(\omega).$$

The property that $\tau$ is an automorphism guarantees that this definition is consistent with the Bianchi identities. (See [39, 13] for particular examples.
and references.) To have a formulation in terms of Lorentz covariant fields (i.e., finite-dimensional representations of the Lorentz algebra), \( \tau \) is required to leave invariant the Lorentz subalgebra of the full \( \text{AdS} \) algebra. In terms of the compensator formalism this is automatically achieved by using the compensator field for definition of \( \tau \). For the problem under consideration, the appropriate definition is

\[
\tau(a_\alpha) = b^\beta V_{\beta\alpha}, \quad \tau(b^\alpha) = V^{\alpha\beta} a_\beta
\]

implying

\[
\tau(f(a,b|x)) = f(\tau(a),\tau(b)|x).
\]

Let us note that in this section we require the compensator \( V_{\alpha\beta} \) to be a constant so that \( \tau \) commutes with the exterior differential \( d \).

The linearized covariant derivative (6.19) in the adjoint representation can be written as

\[
D_0 = D_L^0 + \frac{1}{2}\{ h^\alpha_\beta a_\alpha b^\beta, \ldots \}_*.
\]

Analogously to the 4d case [39, 13], the twisted linearized covariant derivative results from the replacement of the star commutator to star anticommutator in the part of the covariant derivative associated with the frame 1-form

\[
\tilde{D}_0(C) = D_L^0(C) + \frac{1}{2}\{ h^\alpha_\beta a_\alpha b^\beta, C \}_*.
\]

In fact, this is not surprising because the only nontrivial Lorentz covariant definition of the restriction of \( \tau \) to the \( \text{AdS}_d \) algebra in any dimension is to change a sign of the \( \text{AdS} \) translations. From the perspective of the higher spin symmetry the problem therefore is to find an appropriate extension of this automorphism of the \( \text{AdS} \) algebra to the full higher spin algebra. This is achieved by the definition (7.2) for the case of \( \text{AdS}_5 \). For some specific choice of the compensator, this definition reproduces the twisted adjoint representation used in [28].

The twisted covariant derivative (7.5) has the form

\[
\tilde{D}_0(C) = D_L^0(C) + \sigma_- + \sigma_+,
\]

where

\[
\sigma_- = -\frac{1}{4} h^\alpha_\beta \frac{\partial^2}{\partial a_\beta \partial b^\alpha}, \quad \sigma_+ = h^\alpha_\beta a_\alpha b^\beta.
\]

The operators \( D_L^0 \) and \( \sigma_\pm \) have the properties

\[
(\sigma_\pm)^2 = 0, \quad (D_L^0)^2 + \{\sigma_-,\sigma_+\} = 0, \quad \{D_L^0,\sigma_\pm\} = 0.
\]
Only the operator $D^L_0$ acts nontrivially (differentiates) on the space-time coordinates while $\sigma_{\pm}$ act in the fiber linear space $V$ isomorphic as a linear space to the twisted adjoint representation of the higher spin algebra. Also there is the gradation operator $G = \frac{1}{2}(N_a + N_b)$ such that

$$[G, D^L_0] = 0, \quad [G, \sigma_{\pm}] = \pm \sigma_{\pm}. \quad (7.9)$$

Since $V$ is spanned by polynomials in the spinor variables $a_a$ and $b^b$, the spectrum of $G$ in $V$ is bounded from below.

The important observation is (see, e.g., [40]) that the nontrivial dynamical equations hidden in

$$\tilde{D}_0(C) = 0 \quad (7.10)$$

are in the one-to-one correspondence with the nontrivial cohomology classes of $\sigma_-$. For the case with $C$ being a 0-form, the relevant cohomology group is $H^1(\sigma_-)$. For the more general situation with $C$ being a $p$-form, the relevant cohomology group is $H^{p+1}(\sigma_-)$. From this perspective, the operator $\tau_-$ identifies with $\sigma_-$ in the sector of the higher spin gauge 1-forms.

Indeed, consider the decomposition of the space of fields $C$ into the direct sum of eigenspaces of $G$. Let a field having a definite eigenvalue $k$ of $G$ be denoted $C_k$, $k = 0, 1, 2, \ldots$ Suppose that the dynamical content of the equations (7.10) with the eigenvalues $k \leq k_q$ is found. Applying the operator $D^L_0 + \sigma_-$ to the left hand side of the equations (7.10) at $k \leq k_q$ we obtain taking into account (7.8) that

$$\sigma_-(D^L_0 + \sigma_- + \sigma_+)(C_{k_q+1}) = 0. \quad (7.11)$$

Therefore $(D^L_0 + \sigma_- + \sigma_+)(C_{k_q+1})$ is $\sigma_-$ closed. If the group $H^1(\sigma_-)$ is trivial in the grade $k_q + 1$ sector, any solution of (7.11) can be written in the form $(D^L_0 + \sigma_- + \sigma_+)(C_{k_q+1}) = \sigma_- \tilde{C}_{k_q+2}$ for some field $\tilde{C}_{k_q+2}$. This, in turn, is equivalent to the statement that one can adjust $C_{k_q+2}$ in such a way that $\tilde{C}_{k_q+2} = 0$ or, equivalently, that the part of the equation (7.10) of the grade $k_q + 1$ is some constraint that expresses $C_{k_q+2}$ in terms of the derivatives of $C_{k_q+1}$ (to say that this is a constraint we have used the assumption that the operator $\sigma_-$ is algebraic in the space-time sense, i.e. it does not contain space-time derivatives.) If $H^1(\sigma_-)$ is nontrivial, this means that the equation (7.10) sends the corresponding cohomology class to zero and, therefore, not only expresses the field $C_{k_q+2}$ in terms of derivatives of $C_{k_q+1}$ but also imposes some additional differential conditions on $C_{k_q+1}$. Thus, the nontrivial space-time differential equations described by (7.10) are classified by the cohomology group $H^1(\sigma_-)$. 

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The nontrivial dynamical fields are associated with $H^0(\sigma_-)$ which is always non-zero because it at least contains a nontrivial subspace of $V$ of minimal grade. As follows from the $H^1(\sigma_-)$ analysis of the dynamical equations, all fields in $V/H^0(\sigma_-)$ are auxiliary, i.e. express via the space-time derivatives of the dynamical fields by virtue of the equations (7.10).

In the problem under consideration we are interested in the sector of fields $C(a, b|x)$ that commute to $N$ (i.e. $N_a(C) = N_b(C)$). In this sector the representatives of $H^0(\sigma_-)$ (i.e., fields $C$ satisfying $\sigma_-(C) = 0$) are described by the fields of the form

$$C_0(a, b|x) = \text{Res}_\mu C_0(\mu a + \mu^{-1}b, a_\alpha b^\alpha|x).$$  \hspace{1cm} (7.12)

We see that these are just the fields that appeared in the first on-mass-shell theorem (6.49). The additional dependence on $a_\alpha b^\alpha$ matches the degeneracy of the higher spin fields of $cu(1, 0|8)$ due to traces (i.e., ideals generated by $N$).

Application of the same analysis to the higher spin gauge 1-forms with the operator $\tau_-$ instead of $\sigma_-$ leads to the following interpretation of the results of section 6. The dynamical fields with spins $s \geq 1$ belong to the cohomology group $H^1(\tau_-)$. The exact 1-forms $\omega(a, b|x) = \tau_-(\xi)$ describe pure gauge degrees of freedom in $\omega(a, b|x)$ analogous to the antisymmetric part of the frame field associated with the local Lorentz transformations in gravity. The cohomology group $H^2(\tau_-)$ responsible for nontrivial differential conditions on the higher spin gauge fields is a direct sum of two linear spaces \[ H^2(\tau_-) = V_2^E(\tau_-) \oplus V_2^W(\tau_-). \] (7.13)

The space $V_2^W(\tau_-)$ called Weyl cohomology is spanned by the 2-forms of the form of the right hand side of the equation (6.49), i.e. a generic element of $V_2^W(\tau_-)$ has the form (to simplify formulae, in the rest of this section we confine ourselves to the case of irreducible fields of different spins satisfying $T^-\omega = 0$)

$$H_{2a}^\beta \frac{\partial^2}{\partial a_\alpha \partial b^\beta} \text{Res}_\mu (C(\mu a + \mu^{-1}b)).$$  \hspace{1cm} (7.14)

The space $V_2^E(\tau_-)$ called Einstein cohomology is spanned by the 2-forms of the form

$$H_{2a}^\beta \left( \frac{\partial^2}{\partial a_\alpha \partial b^\beta} R(a, b) + b^\alpha a_\beta r(a, b) \right),$$  \hspace{1cm} (7.15)

where the 0-forms $R(a, b)$ and $r(a, b)$ have themselves the properties of the dynamical fields, i.e.

$$S^\pm R = 0, \hspace{1cm} T^- R = 0, \hspace{1cm} S^\pm r = 0, \hspace{1cm} T^- r = 0.$$  \hspace{1cm} (7.16)
The 0-forms $R$ and $r$ parametrize the right hand sides of the spin $s \geq 2$ equations of motion. They generalize the traceless part of the Ricci tensor and the scalar curvature, respectively. In other words, they correspond to the dynamical equations of motion associated with the irreducible traceless parts of the double traceless Fronsdal fields $\varphi_{a_1 \ldots a_n}$ in the action (1.1). The equation (6.49) sends the right hand sides of the dynamical equations associated with the $R$ and $r$ to zero imposing no other conditions on the dynamical fields because the Weyl cohomology remains arbitrary. This is the content of the first off-mass-shell theorem that states that (6.49) is equivalent to the free equations of motion for all spins $s \geq 2$. Note that the spin 1 Maxwell equations are not contained in the equation (6.49) which merely defines the associated spin 1 field strength as the degree two part of the Weyl cohomology $C(a, b)$. The degree zero part $C(0, 0)$ associated with spin 0 field does not show up in the Weyl cohomology because the scalar field $C(0, 0)$ is not associated with the gauge fields.

The fact that the equation (6.49) sends the 0-forms $C(a, b|x)$ to the higher spin curvatures has two effects. First, what looked like an independent dynamical spin $s \geq 1$ field in the module $C(a, b|x)$ becomes an auxiliary field expressed by (6.49) in terms of the dynamical fields described by the 1-form gauge fields. Second, the fields on the right hand side of (6.49) have to satisfy some differential restrictions as a consequence of the Bianchi identities. For all spins $s \geq 2$ these differential restrictions are equivalent to what looked like independent equations in the condition that the section $C(a, b|x)$ is flat. In other words, the Bianchi identities send to zero the part of the cohomology group $H^1(\tau_{-})$ associated with all spins $s \geq 2$ ($s \geq 3/2$ when fermions are included [32]). For spin 1 only a half of the corresponding part of $H^1(\tau_{-})$ is sent to zero by the Bianchi conditions. This is associated with the Maxwell equation that encodes the Bianchi identities for the field strength expressed in terms of the 1-form potential. The dynamical part of the Maxwell equations is imposed by the covariant constancy condition for the spin 1 part of $C(a, b|x)$, i.e. by setting to zero the rest of the restriction of $H^1(\tau_{-})$ to the spin 1 sector. The equation for spin 0 is the condition that $H^1(\tau_{-}) = 0$ in the spin 0 sector [40] (the situation with spin 1/2 is analogous [32]).

As a result, we arrive at the Central On-Mass-Shell Theorem that states that the equations (6.49), (7.10) describe the equations of motion for free massless fields of all spins along with an infinite set of constraints that express some auxiliary fields via higher derivatives of the dynamical fields associated with the cohomology group $H^1(\tau_{-})$ and the scalar field $c(x) = C(a, b|x)$. Let us note that, by construction, the set of fields $\omega(a, b|x)$ and $C(a, b|x)$ provide
the complete basis for all combinations of derivatives of massless fields of all spins that are allowed to be nonzero by field equations (equivalently, to take arbitrary values at any fixed point \(x_0\) of space-time). The Central On-Mass-Shell Theorem is the starting point for the description of the nonlinear higher spin dynamics in the unfolded form. The equation (6.49) also plays the key role in the analysis of cubic higher spin interactions at the action level.

The proof and the meaning of the tensor form of the Central On-Mass-Shell Theorem (2.77) and (2.78) in any dimension is analogous.

8 5d Higher Spin Action

The aim of this section is to formulate the action for the totally symmetric gauge massless boson fields in \(AdS_5\) that solves the problem of higher-spin-gravitational interactions in the first nontrivial order. The results reported here extend the 4d results of [33, 9] to \(d = 5\).

We shall look for the action of the form

\[
S = S_2 + S_3 + \ldots
\]  

(8.1)

within the perturbation expansion (2.37) with the background gravitational field being of the zero order and the higher spin fields of the first order. \(S_2\) is the quadratic action that describes properly the free higher spin dynamics. \(S_3\) is the cubic part. Higher-order corrections do not contribute to the order under investigation. The gauge transformations are supposed to be of the form (2.36). Equivalently one can expand

\[
\delta \omega = \delta_0 \omega + \delta_1 \omega + \ldots,
\]

(8.2)

where \(\delta_0 \omega\) is the linearized Abelian transformation (2.40) while \(\delta_1 \omega\) contains terms linear in the dynamical fields \(\omega_1\). Recall that the background field \(\omega_0\) is chosen in such a way that \(R_0 = 0\) (thus implying \(AdS_5\) background) so that \(R\) starts with the first order part. As a result, the deformation terms \(\Delta(R, \varepsilon)\) in (2.36) contribute to \(\delta_1 \omega\).

The free higher spin action \(S_2\) is required to be invariant under the linearized higher spin gauge transformations

\[
\delta_0 S_2 = 0.
\]

(8.3)

This means that the part of the variation of the action, which is linear in the dynamical fields, is zero. The first nontrivial part is therefore bilinear in the dynamical fields

\[
\delta_1 S = \delta_0 S_3 + \delta_1 S_2 \sim \omega_1^2 \varepsilon.
\]

(8.4)
Our aim is to find an action $S_3$ that admits a nontrivial deformation of the gauge transformation guaranteeing that the gauge variation (8.4) is zero.

Using the decomposition (2.36) for the gauge variation one can rewrite the condition $\delta_1 S = 0$ in the equivalent form

$$0 = \delta^g S + \Delta S_2 + O(\omega_1^3 \varepsilon), \quad (8.5)$$

where $\delta^g$ is the original higher spin gauge transformation (2.28) that contains the zero-order part of the variation along with some part of the first-order terms. Other possible linear terms in the variation are contained in $\Delta \omega_1$. Since

$$\Delta S_2 = \frac{\delta S_2}{\delta \omega_{dyn}} \Delta \omega_{dyn}, \quad (8.6)$$

a (local) deformation $\Delta \omega_{dyn}$ fulfilling the invariance condition (8.5) exists iff

$$\delta^g S = -Y(\omega_1, \frac{\delta S_2}{\delta \omega_{dyn}}, \varepsilon) + O(\omega_1^3 \varepsilon), \quad (8.7)$$

where $Y(\omega_1, \frac{\delta S_2}{\delta \omega_{dyn}}, \varepsilon)$ is some trilinear local functional, i.e. iff the original gauge variation of $S_2 + S_3$ vanishes on-mass-shell $\frac{\delta S_2}{\delta \omega_{dyn}} = 0$.

Note that a deformation of the gauge variation of the extra and auxiliary fields does not contribute into the variation to the order under consideration because the variation of $S_2$ with respect to these fields is either identically zero by the extra field decoupling condition (2.46) for extra fields or zero by virtue of constraints (i.e., by the 1.5-order formalism) for the Lorentz-type auxiliary fields. This is important because the constraints for the extra and auxiliary fields are not invariant under the original higher spin gauge transformations $\delta^g \omega$. As a result, the higher spin gauge transformation for the extra and auxiliary fields should necessarily be deformed to be compatible with the constraints. This phenomenon does not however affect our consideration because the constraints are formulated in terms of the higher spin curvatures and therefore are invariant under the linearized higher spin gauge transformations in the lowest order. As a result, the deformation of the transformation low for the extra and auxiliary fields due to the constraints is at least of order $\omega_1 \varepsilon$ which was argued to be irrelevant in the approximation under consideration.

Our analysis of the gauge invariance will be based heavily on the First On-Mass-Shell Theorem (2.73) in its spinor form (6.49). Namely, the variation $\delta^g S$ is some bilinear functional of the higher spin curvatures $R$ which can be replaced by the linearized curvatures $R_1$ at the order of interest. Assuming that the constraints for auxiliary and extra fields are satisfied we can use the
representation (2.73) for the linearized curvatures. All terms contained in $X$ are proportional to the left hand sides of the free field equations and, therefore, give rise to some variation of the form (8.7) that can be compensated by an appropriate deformation $\Delta \omega_1$ (that itself is at least linear in the higher spin curvatures). The terms that cannot be compensated this way are those bilinear in the higher spin Weyl tensors $C_{A_1...A_{s-1}B_1...B_{s-1}}$. Therefore, the condition that the higher spin action is invariant under some deformation of the higher spin gauge transformations is equivalent to the condition that the original (i.e. undeformed) higher spin gauge variation of the action is zero once the linearized higher spin curvatures $R_1$ are replaced by the Weyl tensors $C$ according to (2.73) i.e., schematically,

$$
\delta^g S\bigg|_{R=\text{Weyl}} = 0.
$$

(8.8)

Being rather nontrivial, this condition will be shown to admit a solution linking the normalization coefficients in front of the free higher spin action functionals for different fields.

Let us now sketch the general procedure for the search of the $AdS_5$ higher spin action. In accordance with (2.42) we shall look for a Lagrangian 5-form bilinear in the higher spin curvatures with some 1-form $U_{\Omega\Lambda}$ built from the higher spin gauge 1-forms. As no useful extension of the compensator formalism to the full higher spin algebra is known so far, we use a mixed approach with the frame field $E^{\alpha\beta}$ built from the compensator $V^{\alpha\beta}$ and the gravitational fields associated with the $AdS_5$ subalgebra $su(2,2) \subset cu(1,0|8)$. In addition, some explicit dependence on the higher spin gauge fields taking values in $cu(1,0|8)/su(2,2)$ will be allowed. Presumably, such an approach is a result of a partial gauge fixing in a full compensator formalism in the $AdS_5$ higher spin theory to be developed. Note that, perturbatively, $E^{\alpha\beta}$ contains the background gravitational field and, therefore, is of the zero order, while the higher spin fields are of the first order.

In our analysis the higher spin gauge fields will be allowed to take values in some associative (e.g., matrix) algebra $\omega \rightarrow \omega_{\mathfrak{g}}$. The resulting ambiguity is equivalent to the ambiguity of a particular choice of the Yang-Mills gauge algebra in the spin 1 sector. The higher spin action will be formulated in terms of the trace $tr$ in this matrix algebra (to be not confused with the trace in the star product algebra). As a result, only cyclic permutations of the matrix factors will be allowed under the trace operation. Note that the gravitational field is required to take values in the center of the matrix algebra, being proportional to the unit matrix. For this reason, the factors associated with the gravitational field are usually written outside the trace.
Let us consider an action of the form
\[ S = S^E + S^\omega, \]  
where
\[ S^E = \frac{1}{2} \int_{M^5} \left( \alpha E_{\alpha\beta} \frac{\partial^2}{\partial a_{1\alpha} \partial a_{2\beta}} + \beta E_{\alpha\beta} \frac{\partial^2}{\partial b_{1}^{\alpha} \partial b_{2}^{\beta}} + \sum_{ij=1}^{2} \gamma_{ij} E_{\alpha}^{\beta} \frac{\partial^2}{\partial a_{i\alpha} \partial b_{j}^{\beta}} \right) \wedge \]
\[ tr(R(a_1, b_1|x) \wedge R(a_2, b_2|x)) \big|_{a_1=b_1=a_2=b_2=0} \]  
and
\[ S^\omega = \frac{1}{2} \int_{M^5} \tau tr(R(a_1, b_1|x) \wedge R(a_2, b_2|x) \wedge \omega(a_3, b_3|x)) \big|_{a_1=b_1=0}. \]

Here the coefficients \( \alpha, \beta, \gamma_{ij} \) and \( \tau \) are some functions of the Lorentz invariant combinations of derivatives with respect to the spinor variables \( a_{i\alpha} \) and \( b_{j}^{\alpha} \),
\[ \bar{a}_{ij} = V_{\alpha\beta} \frac{\partial^2}{\partial a_{i\alpha} \partial a_{j\beta}}, \quad \bar{b}_{ij} = V_{\alpha\beta} \frac{\partial^2}{\partial b_{i}^{\alpha} \partial b_{j}^{\beta}}, \quad \bar{c}_{ij} = \frac{\partial^2}{\partial a_{i\alpha} \partial b_{j}^{\alpha}} \]  
\( (i, j = 1, 2 \text{ for } (8.10) \text{ and } 1, 2, 3 \text{ for } (8.11)) \). Functions \( \alpha, \beta, \gamma_{ij} \) and \( \tau \) parametrize the ambiguity in all possible contractions of indices of the component higher spin fields and curvatures. Note that the gravitational field is not allowed to appear among the components of the connection \( \omega \) that enters explicitly the action (8.11). Instead, all terms with the gravitational field in front of the curvature terms are collected in the action \( S^E \) (8.10). With this convention, \( S^E \) contributes both to the quadratic and to the cubic parts of the action while \( S^\omega \) only contributes to the interaction part of the action.

Below we show that there exists a consistent cubic higher-spin-gravitational interaction for \( S^\omega = 0 \). Since the aim of this paper is to show that at least some consistent higher-spin-gravitational interaction exists in \( AdS_5 \), we shall mostly focus on this particular case. Note that it is anyway hard to judge on a full structure of the theory from the perspective of the cubic interactions. Indeed, at the cubic level one can switch out interactions among any three elementary (i.e. irreducible at the free field level) fields without spoiling the consistency at this order. This is most obvious from the Noether coupling interpretation of the cubic interactions: setting to zero some of the fields is always consistent with the conservation of currents. It is plausible to speculate that the action \( S^E \) accounts the terms relevant to the higher-spin-gravitational interaction but may miss some other higher spin interactions described by the action \( S^\omega \).
Indeed, as a by product of the consideration below we shall give an example of a consistent higher spin interactions $S^\omega$. Note that even writing down all terms of the form (8.9) there is little chance to have a fully consistent theory beyond the cubic order without introducing more dynamical fields because, as we know from the 4d example [39] (see also [12, 13]), some lower spin fields (e.g. spin 1 and spin 0) have to be added. Note that the actions for spin 1 and spin 0 massless fields do not admit a formulation in the form (2.42). To simplify the presentation we will assume in this paper that these fields are set to zero, that is a consistent procedure at the cubic order. By analogy with the 4d case [10] we expect that an extension of the results of this paper to the full system with the lower spin fields will cause no problem. Let us note that an appropriate reformulation of the Lagrangian spin 0 free field dynamics was developed in [40].

Let us now focus on the structure of the action $S^E$. The ambiguity in the coefficient functions $\alpha, \beta, \gamma_{ij}$ can be restricted by not allowing a contraction of the both of indices of $E^\alpha_\beta$ with the same curvature. Another restriction we impose is that a total number of derivatives in $a_1$ and $b_1$ is equal to the number of derivatives in $a_2$ and $b_2$, i.e. the terms resulting from the products of the polynomials of different powers in $R(a_1, b_1)$ and $R(a_2, b_2)$ are required to vanish. (The most important argument for this ansatz is, of course, that it will be proved to work.) We therefore consider the action of the form (8.10) with the coefficients $\gamma^{11} = \gamma^{22} = 0$. Taking into account that the higher spin gauge fields and curvatures carry equal numbers of lower and upper indices, i.e. $R(\mu a, \mu b) = R(a, b)$, the appropriate ansatz is

$$S^E = \frac{1}{2}A^E_{\alpha,\beta,\gamma}(R, R), \hspace{1cm} (8.13)$$

where the symmetric bilinear $A^E_{\alpha,\beta,\gamma}(f, g) = A^E_{\alpha,\beta,\gamma}(g, f)$ is defined for any 2-forms $f$ and $g$ as

$$A^E_{\alpha,\beta,\gamma}(f, g) = \int_{M^5} \left( \alpha(p, q, t) E^\alpha_\beta \frac{\partial^2}{\partial a_1 \partial a_2} b_{12} + \beta(p, q, t) E^\alpha_\beta \frac{\partial^2}{\partial b_1 \partial b_2} a_{12} \right. \left. + \gamma(p, q, t) \left( E^\alpha_\beta \frac{\partial^2}{\partial a_1 \partial b_2} c_{21} - E^\alpha_\beta \frac{\partial^2}{\partial b_1 \partial a_2} c_{12} \right) \right) \wedge tr(f(a_1, b_1|x) \wedge g(a_2, b_2|x)) \bigg|_{a_i=b_j=0} \hspace{1cm} (8.14)$$

where we use notations

$$p = \bar{a}_{12} \bar{b}_{12} \hspace{0.5cm} q = \bar{c}_{12} \bar{c}_{21} \hspace{0.5cm} t = \bar{c}_{11} \bar{c}_{22} \hspace{1cm} (8.15)$$
The labels $\alpha, \beta, \gamma$ and $E$ in $A_{\alpha,\beta,\gamma}^{E}(f, g)$ refer to the functions $\alpha(p, q, t)$, $\beta(p, q, t)$, $\gamma(p, q, t)$ and the frame field $E^{\alpha\beta}$ that fix a particular form of the bilinear form. Sometimes we will write $A(f, g)$ instead of $A_{\alpha,\beta,\gamma}^{E}(f, g)$.

As explained in section 2.2, nonlinear actions of this form cannot have the invariant trace property, i.e. $A(af, g) \neq A(f, ag)$ for generic $a, f, g \in cu(1, 0|8)$. One can require however a weaker condition

$$A(N * f, g) = A(f, g * N),$$

where $f$ and $g$ are any elements satisfying $f * N = N * f$, $g * N = N * g$. From (8.16) it follows that

$$A(\phi(N) * f, g) = A(f, g * \phi(N)).$$

We will refer to the property (8.16) as the $C$-invariance condition. It will play the key role in the analysis of the invariance of the cubic action in section 8.2. The explicit form of the restrictions on the coefficients $\alpha, \beta, \gamma$ due to (8.16) is given in section 8.1.

The main steps of the rest of the analysis are as follows. First we analyze the quadratic part of the action choosing the functions $\alpha, \beta$ and $\gamma$ to guarantee that the free action $S_2$ describes a sum of compatible with unitarity free field actions for the set of the higher spin fields associated with the higher spin algebra $cu(1, 0|8)$. This is equivalent to the two conditions. First, the extra field decoupling condition requires the variation of the quadratic action with respect to the extra fields to vanish. Second, the quadratic action should decompose into infinite sum of free actions for the different copies of fields of the same spin associated with the spinor traces as discussed below (6.31). This is referred to as the factorization condition.

Note that at the free field level there is an ambiguity in the coefficients $\alpha(p, q, t)$ and $\beta(p, q, t)$ due to the freedom in adding the total derivative terms

$$\delta S^2 = \frac{1}{2} \int_{M^5} d(\Phi(p, q, t) \text{tr}(R_1(a_1, b_1|x) \wedge R_1(a_2, b_2|x))|_{a_1=b_1=a_2=b_2=0})$$

$$= \frac{1}{2} \int_{M^5} \frac{\partial \Phi(p, q, t)}{\partial p} \left(h^{\alpha\beta} \frac{\partial}{\partial b_1^\alpha} \frac{\partial}{\partial b_2^\beta} - h_{\alpha\beta} \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2 \partial \beta_1 \partial \beta_2} \right)$$

$$\wedge \text{tr}(R_1(a_1, b_1|x) \wedge R_1(a_2, b_2|x))|_{a_i=b_j=0},$$

where, using the manifest $su(2, 2)$ covariance of our formalism, the differential $d$ in the first line is replaced by the background $su(2, 2)$ derivative, and the definition of the background frame field (3.17) has been taken into account.
along with the Bianchi identities $D_0(R_1) = 0$. As a result, the variation of the coefficients
\[
\delta \alpha(p, q, t) = \epsilon(p, q, t), \quad \delta \beta(p, q, t) = -\epsilon(p, q, t)
\] (8.19)
does not affect the physical content of the quadratic action, i.e. only the combination $\alpha(p, q, t) + \beta(p, q, t)$ has invariant meaning at the free field level. Modulo the ambiguity (8.19) the factorization condition along with the extra field decoupling condition fix the functions $\alpha, \beta, \gamma$ up to an arbitrary function parametrizing the ambiguity in the normalization coefficients in front of the individual free actions. The proof of this fact is the content of section 8.1.

In the analysis of the cubic interactions, there are two types of terms to be taken into account. Terms of the first type result from the gauge transformations of the gravitational fields and the compensator $V^{\alpha\beta}$ that contribute into the factors in front of the higher spin curvatures in the action (8.10). The proof of the respective invariances goes the same way as in the example of gravity considered in section 2.1 as it is based entirely on the explicit $su(2, 2)$ covariance and invariance of the whole setting under diffeomorphisms (recall that the additional invariance (2.21) was identified in section 2.1 with a mixture of the diffeomorphisms and $su(2, 2)$ gauge transformations.) Also, one has to take into account that the higher spin gauge transformation of the gravitational fields is at least linear in the dynamical fields and therefore has to be discarded in the analysis of $\omega^2\varepsilon$ type terms under consideration.

The nontrivial terms of the second type originate from the variation (2.31) of the higher spin curvatures. According to (8.8) the problem is to find such functions $\alpha, \beta$ and $\gamma$ that
\[
\delta^g S^E(R, R) \big|_{E=h, R=h\wedge hC} = A^h_{\alpha, \beta, \gamma}(R, [R, \varepsilon\star]) \big|_{R=h\wedge hC} = 0
\] (8.20)
for an arbitrary gauge parameter $\varepsilon(a, b|x)$. As shown in section 8.2 this condition fixes the coefficients in the form
\[
\alpha(p, q, t) + \beta(p, q, t) = \varphi_0 \sum_{m, n=0}^{\infty} (-1)^{m+n} \frac{m + 1}{2^{2(m+n+1)}(m + n + 2)!m!(n + 1)!} p^m q^n,
\] (8.21)
\[
\gamma(p, q, t) = \gamma(p + q), \quad \gamma(p) = \varphi_0 \sum_{m=0}^{\infty} (-1)^{m+1} \frac{1}{2^{2m+3}(m + 2)!m!} p^m
\] (8.22)
where $\varphi_0$ is an arbitrary normalization factor to be identified with the (appropriately normalized in terms of the cosmological constant) gravitational coupling constant. Let us note that the sign factors in the coefficients (8.21)
and (8.22) distinguish between the polynomials of the oscillators $a_\alpha$ and $b_\beta$ of degree $4p$ and $4p+2$. Together with the signs due to the factors of $i$ in the normalization coefficients (6.22) this implies that fields of equal spins contribute to the quadratic action with the same sign. The fields of even and odd spins contribute with opposite signs.

As a result, the condition that cubic $AdS_5$ higher spin action possesses higher spin gauge symmetries fixes uniquely the relative coefficients in front of the free actions of fields of all spins in a way compatible with unitarity. Note that the analysis of the interactions does not fix the ambiguity (8.19). Taking into account (8.18) along with the full Bianchi identities, one observes that the ambiguity in $\alpha - \beta$ is equivalent to the ambiguity in the interaction terms $S^\omega$ of the form

$$ S^\omega = \left. \frac{1}{2} \int_{M^5} \Phi(p, q, t) tr \left( [\omega_1, R_\ast(a_1, b_1|x) \wedge R(a_2, b_2|x) ight. 
+ \left. R(a_1, b_1|x) \wedge [\omega_1, R_\ast(a_2, b_2|x) \right] \right|_{\alpha_i = 0 \beta_j = 0} $$

parametrized by an arbitrary function $\Phi(p, q, t)$.

### 8.1 Quadratic Action

The free field part $S_2$ of the action $S$ is obtained from (8.13) by the substitution of the linearized curvatures (6.1) instead of $R$ and $h^\alpha_\beta$ instead of $E^\alpha_\beta$. The resulting action is manifestly invariant under the linearized transformations (6.5) because the linearized curvatures are invariant. We want the free action to be a sum of actions for the irreducible higher spin fields we are working with. This requirement is not completely trivial because of the infinite degeneracy of the algebra due to the traces. The factorization condition requires

$$ S_2 = \sum_{s,n=0}^{\infty} S_2^{s,n}(\omega_n^s), $$

i.e., the terms containing products of the fields $\omega_n^s$ and $\omega_m^s$ should all vanish for $n \neq m$. As follows from (6.22), (6.23) along with (6.12) this is true if

$$ A^E_{\alpha, \beta, \gamma}(f, (T^+)^k g) = A^E_{\alpha_k, \beta_k, \gamma_k}((T^-)^k f, g), \quad \forall k $$

for some $\alpha_k, \beta_k$ and $\gamma_k$.

An elementary computation shows that

$$ A^E_{\alpha, \beta, \gamma}(f, T^+ g) = A^E_{\alpha_1, \beta_1, \gamma_1}(T^- f, g) 
+ \int_{M^5} Q(p, q, t) E^\beta_\alpha \frac{\partial^2}{\partial a_\alpha \partial b_1} \wedge tr \left( f(a_1, b_1|x) \wedge g(a_2, b_2|x) \right) \right|_{\alpha_i = 0 \beta_j = 0}. $$

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where
\[ Q = \left( 1 + p \frac{\partial}{\partial p} \right) (\alpha(p, q, t) + \beta(p, q, t)) + 2 \left( 1 + q \frac{\partial}{\partial q} \right) \gamma(p, q, t) \] (8.27)
and
\[ \alpha_1(p, q, t) = 4 \left( (2 + p \frac{\partial}{\partial p}) \frac{\partial}{\partial p} + (1 + q \frac{\partial}{\partial q}) \frac{\partial}{\partial q} + \left( 2p \frac{\partial}{\partial p} + 2q \frac{\partial}{\partial q} + t \frac{\partial}{\partial t} + 6 \right) \frac{\partial}{\partial t} \right) \times \alpha(p, q, t), \] (8.28)
\[ \beta_1(p, q, t) = 4 \left( (2 + p \frac{\partial}{\partial p}) \frac{\partial}{\partial p} + (1 + q \frac{\partial}{\partial q}) \frac{\partial}{\partial q} + \left( 2p \frac{\partial}{\partial p} + 2q \frac{\partial}{\partial q} + t \frac{\partial}{\partial t} + 6 \right) \frac{\partial}{\partial t} \right) \times \beta(p, q, t), \] (8.29)
\[ \gamma_1(p, q, t) = 4 \left( (1 + p \frac{\partial}{\partial p}) \frac{\partial}{\partial p} + (2 + q \frac{\partial}{\partial q}) \frac{\partial}{\partial q} + \left( 2p \frac{\partial}{\partial p} + 2q \frac{\partial}{\partial q} + t \frac{\partial}{\partial t} + 6 \right) \frac{\partial}{\partial t} \right) \times \gamma(p, q, t). \] (8.30)

The factorization condition therefore requires
\[ Q = \left( 1 + p \frac{\partial}{\partial p} \right) (\alpha(p, q, t) + \beta(p, q, t)) + 2 \left( 1 + q \frac{\partial}{\partial q} \right) \gamma(p, q, t) = 0. \] (8.31)

Then one observes that from (8.31) it follows that the same is true for the coefficients \( \alpha_1, \beta_1 \) and \( \gamma_1 \) (8.28)-(8.30), and, therefore, (8.31) guarantees (8.25) for all \( k \). In the sequel, the factorization condition (8.31) is required to be true. Since the operator \( 1 + q \frac{\partial}{\partial q} \) is invertible, it allows to express \( \gamma \) in terms of \( \alpha \) and \( \beta \).

Let us now analyze the \( C^- \)invariance condition (8.16). Taking into account (6.21) along with the factorization condition (8.25), it amounts to
\[ A_{\alpha,\beta,\gamma}^E(T^-f, g) + A_{\alpha_1,\beta_1,\gamma_1}^E(T^-f, g) = A_{\alpha,\beta,\gamma}^E(f, T^-g) + A_{\alpha_1,\beta_1,\gamma_1}^E(f, T^-g). \] (8.32)

Obviously, this is true iff
\[ A_{\alpha,\beta,\gamma}^E(f, g) = -A_{\alpha_1,\beta_1,\gamma_1}^E(f, g), \] (8.33)
i.e.
\[ \alpha(p, q, t) = -\alpha_1(p, q, t), \quad \beta(p, q, t) = -\beta_1(p, q, t), \quad \gamma(p, q, t) = -\gamma_1(p, q, t). \] (8.34)
This is equivalent to the requirement that the operators $T^+$ and $-T^-$ are conjugated with respect to the bilinear form $A^{E}_{\alpha,\beta,\gamma}(f,g)$

$$A^{E}_{\alpha,\beta,\gamma}(T^\pm f,g) = -A^{E}_{\alpha,\beta,\gamma}(f,T^\mp g).$$

(8.35)

Let us note that the original ansatz for the bilinear form (8.14) satisfies

$$A^{E}_{\alpha,\beta,\gamma}(T^0 f,g) = A^{E}_{\alpha,\beta,\gamma}(f,T^0 g).$$

(8.36)

From (8.28)-(8.30) it is clear that (8.34) reconstructs the dependence of $\alpha(p,q,t)$, $\beta(p,q,t)$ and $\gamma(p,q,t)$ on $t$ in terms of the “initial data” $\alpha(p,q,0)$, $\beta(p,q,0)$ and $\gamma(p,q,0)$.

With the help of (8.35) along with (6.12) it is elementary to compute the relative coefficients of the actions for the different copies of fields in the decomposition (6.22), (6.24). The coefficients (6.25) are fixed so that the linearized actions have the same normalization for different copies of the higher spin fields parametrized by the label $n$

$$S_2 = \sum_{s,n=0}^{\infty} S^s_2(\omega^*_n).$$

(8.37)

In the linearized approximation it is therefore enough to analyze the situation for any fixed $n$. We confine ourselves to the case of $\omega = \omega_0$ assuming in the rest of this section that

$$T^- \omega = 0.$$

(8.38)

Let us now consider the extra field decoupling condition. Since the generic variation of the linearized higher spin curvature is $\delta R_1 = D_0 \delta \omega$, where $D_0$ is the $AdS_5$ background covariant derivative and because the action is formulated in the $AdS_5$ covariant way with the aid of the compensator field $V^\alpha\beta$, integrating by parts one obtains for the generic variation of $S_2$

$$\delta S_2 = \int_{M^5} D_0 \left( \alpha(p,q,0) h^{\alpha\beta}_{\alpha_1\alpha} \frac{\partial^2}{\partial a_{1\alpha} \partial a_{2\beta}} \bar{b}_{12} + \beta(p,q,0) h^{\alpha\beta}_{\beta_1\beta} \frac{\partial^2}{\partial b_{1\beta} \partial b_{2\beta}} \bar{a}_{12} + \gamma(p,q,0) \left( h^{\alpha\beta}_{\alpha_1\alpha} \frac{\partial^2}{\partial a_{1\alpha} \partial b_{2\beta}} \bar{c}_{21} - h^{\alpha\beta}_{\beta_1\beta} \frac{\partial^2}{\partial b_{1\beta} \partial a_{2\beta}} \bar{c}_{12} \right) \right) \wedge \text{tr}(\delta \omega(a_1, b_1 | x) \wedge R_1(a_2, b_2 | x)) \bigg|_{a_i = b_j = 0},$$

(8.39)

where it is taken into account that the $t$-dependent terms trivialize as a consequence of (8.38). The derivative $D_0$ produces the frame field every time it
meets the compensator. (Recall that $D_0(h^{\alpha\beta}) = 0$ because $D_0^2 = R_0 = 0$.)

Taking into account (3.24) and (8.38), one finds

$$\delta S_2 = \frac{1}{2} \int_{M^5} \rho(p, q) \left( \frac{\partial^2}{\partial a_{1\alpha} \partial b_{2\beta}} \bar{e}_{21} + \frac{\partial^2}{\partial a_{2\alpha} \partial b_{1\beta}} \bar{e}_{12} \right) H_{2\alpha}^\beta \wedge \text{tr} \left( \delta \omega(a_1, b_1|x) \wedge R_1(a_2, b_2|x) \right) |_{a_i = b_j = 0}, \quad (8.40)$$

where

$$\rho(p, q) = \left( 1 + p \frac{\partial}{\partial p} \right) \left( \alpha(p, q, 0) + \beta(p, q, 0) - 2\gamma(p, q, 0) \right). \quad (8.41)$$

According to (6.8) the extra fields are associated with the multispinors described by the two-row Young diagrams of the Lorentz algebra having at least four more cells in the upper row than in the lower one. As follows from (6.31), generic variation sharing this property has the form

$$\delta \omega^{ex}(a, b) = (S^+)^2 \xi(a, b). \quad (8.42)$$

To guarantee $(N_a - N_b)\delta \omega^{ex}(a, b) = 0$, the infinitesimal $\xi(a, b)$ is required to satisfy $(N_a - N_b - 4)\xi(a, b) = 0$.

To derive the restriction on the coefficients imposed by the requirement that the extra fields do not contribute to the variation one observes, first, that

$$\left[ S^+_1, q \right] = -\left[ S^+_1, p \right] = u, \quad \left[ S^+_1, u \right] = 0, \quad (8.43)$$

where

$$S^+_1 = b_{1\beta} \frac{\partial}{\partial a_{1\beta}}, \quad u = \bar{a}_{12} \bar{c}_{12} \quad (8.44)$$

and, second, that the double commutator of $S^+_1$ to the differential operator next to $\rho(p, q)$ in (8.40) is zero. As a result, substituting (8.42) into (8.40) one finds that the corresponding variation of the action vanishes provided that

$$\left( \frac{\partial}{\partial p} - \frac{\partial}{\partial q} \right) \rho(p, q) = 0. \quad (8.45)$$

Therefore the extra field decoupling condition requires

$$\rho(p, q) = \rho(p + q). \quad (8.46)$$

From the factorization condition (8.31) and (8.41), (8.46) it follows that

$$\gamma(p, q, 0) = \gamma(p + q, 0), \quad \rho(r) = -2(r \frac{\partial}{\partial r} + 2)\gamma(r, 0). \quad (8.47)$$
The function $\alpha(p, q, 0) + \beta(p, q, 0)$ is fixed in terms of $\gamma(p + q, 0)$ by the factorization condition (8.31)

$$\alpha(p, q, 0) + \beta(p, q, 0) = -2 \int_0^1 du (1 + q \frac{\partial}{\partial q}) \gamma(up + q). \quad (8.48)$$

The function of one variable $\gamma(p + q, 0)$ parametrizes the leftover ambiguity in the coefficients (discarding the trivial ambiguity (8.19)) associated with the ambiguity in the coefficients in front of the free actions of fields with different spins. Indeed, the total homogeneity degree in the variables $p$ and $q$, telling us how many pairs of indices are contracted, equals to $s - 1$. Clearly, this ambiguity cannot be fixed from the analysis of the free action.

### 8.2 Cubic Interactions

Let us now analyze the on-mass-shell invariance condition (8.8) to prove the existence of a nonlinear deformation of the higher spin gauge transformations that leaves the cubic part of the action $S = S^E$ invariant to the order $\omega^2 \varepsilon$. As explained in the beginning of this section this condition amounts to (8.20).

Taking into account (6.26), our aim is to prove that there exist such coefficient functions $\alpha, \beta$ and $\gamma$ satisfying the $C$-invariance condition, factorization condition and extra field decoupling condition that

$$\sum_{m,n} A^h_{\alpha, \beta, \gamma} ((T^+)^m v_m(T^0)^R_{1,m} \langle a, b \rangle \big| m, s \rangle, [\varepsilon, (T^+)^n v_n(T^0)^R_{1,n} \langle a, b \rangle \big| m, s \rangle]_s) = 0 \quad (8.49)$$

for any gauge parameter $\varepsilon \in cu(1,0|8)$ and arbitrary Weyl tensors $C_n(a)$ in the spinor form (6.49) of the First On-Mass-Shell Theorem.

To this end, one first of all observes that the dependence of $v_n(T^0)$ on $T^0$ can be absorbed into (spin-dependent) rescalings of the Weyl tensors $C_n(a)$ which are treated as arbitrary field variables in this consideration. As a result it is enough to prove (8.49) for arbitrary constant coefficients $v_n$.

Now let us show that, once (8.49) is valid for $m = n = 0$, it is automatically true for all other values of $m$ and $n$ as a consequence of the $C$-invariance condition. Indeed, suppose that (8.49) is true for $m_0 \geq m \geq 0$, $n_0 \geq n \geq 0$.

Consider the term with $m = m_0 + 1$. Then, from (6.21) it follows

$$(T^+)^{m_0 + 1} R_{1,m_0+1} \langle a, b \rangle = N \ast ((T^+)^{m_0} R_{1,m_0+1} \langle a, b \rangle) + T^{-(T^+)^{m_0} R_{1,m_0+1} \langle a, b \rangle}. \quad (8.50)$$

The term containing $T^-$ gives zero contribution by the induction assumption since, taking into account (6.27), $T^-$ decreases a number of $T^+$. By virtue of
the \( C^{-}\)invariance condition (8.16) along with the fact that \( N \) belongs to the center of \( cu(1,0|8) \) so that

\[
(N * f) * g = f * (N * g),
\]

the term containing the star product with \( N \) equals to

\[
A_{\alpha,\beta,\gamma}^h \left( (T^+)^{\nu_0} v_{\nu_0} (T^0) R_{1,\nu_0} (a, b) \right) \left|_{m.s.} \right. \left[ N * \varepsilon, (T^+)^{\nu_0} v_{\nu_0} (T^0) R_{1,\nu_0} (a, b) \right]_{m.s.} \right)
\]

which is zero by the induction assumption valid for any \( \varepsilon \). Analogously, one performs induction \( n_0 \to n_0 + 1 \) with the aid of (8.51).

Thus, it suffices to find the coefficients satisfying the \( C^{-}\)invariance condition for \( R = \mathcal{R} \equiv R_{1,0} \). In other words one has to prove that

\[
A_{\alpha,\beta,\gamma}^h (\mathcal{R}, [\varepsilon, \mathcal{R}]_*) = 0
\]

for

\[
\mathcal{R}(a, b) = H_{2\alpha} \frac{\partial^2}{\partial \alpha_\alpha \partial \beta_\beta} \text{Res}_\mu (C(\mu a + \mu^{-1} b)).
\]

Note that because \( T^{-}(\mathcal{R}) = 0 \) the terms containing \( \bar{c}_{11} \) (8.12) and, therefore, \( t \) (8.15) does not contribute into the condition (8.53).

Using the differential (Moyal) form of the star product (3.2) one finds

\[
A_{\alpha,\beta,\gamma}^h (f, \eta * \varepsilon) = \int_{M^5} e^{\frac{i}{2}(\xi_{a_1} - \xi_{a_2})} \left\{ (\bar{b}_{12} + \bar{b}_{13}) h_{\alpha\beta} \left( \frac{\partial^2}{\partial a_{1\alpha} \partial a_{2\beta}} + \frac{\partial^2}{\partial a_{1\alpha} \partial a_{3\beta}} \right) \right.
\]

\[
\times \alpha \left( (\bar{a}_{12} + \bar{a}_{13}) (\bar{b}_{12} + \bar{b}_{13}, (\bar{c}_{12} + \bar{c}_{13}) (\bar{c}_{21} + \bar{c}_{31}), 0 \right)
\]

\[
+ (\bar{a}_{12} + \bar{a}_{13}) h_{\alpha\beta} \left( \frac{\partial^2}{\partial b_{2\alpha} \partial b_{2\beta}} + \frac{\partial^2}{\partial b_{3\alpha} \partial b_{3\beta}} \right)
\]

\[
\times \beta \left( (\bar{a}_{12} + \bar{a}_{13}) (\bar{b}_{12} + \bar{b}_{13}, (\bar{c}_{12} + \bar{c}_{13}) (\bar{c}_{21} + \bar{c}_{31}), 0 \right)
\]

\[
+ (\bar{c}_{21} + \bar{c}_{31}) h_{\alpha\beta} \left( \frac{\partial^2}{\partial a_{1\alpha} \partial b_{2\beta}} + \frac{\partial^2}{\partial a_{1\alpha} \partial b_{3\beta}} \right) - (\bar{c}_{12} + \bar{c}_{13}) h_{\alpha\beta} \left( \frac{\partial^2}{\partial b_{2\alpha} \partial a_{2\beta}} + \frac{\partial^2}{\partial b_{3\alpha} \partial a_{3\beta}} \right)
\]

\[
\times \gamma \left( (\bar{a}_{12} + \bar{a}_{13}) (\bar{b}_{12} + \bar{b}_{13}, (\bar{c}_{12} + \bar{c}_{13}) (\bar{c}_{21} + \bar{c}_{31}), 0 \right)
\]

\[
tr \left( f(a_1, b_1) \eta(a_2, b_2) \varepsilon(a_3, b_3) \right) \bigg|_{a_i = b_i = 0}
\]

provided that \( T^{-} f = 0 \). Let us consider \( A_{\alpha,\beta,\gamma}^h (\mathcal{R}, \mathcal{R} * \varepsilon) \). Rewriting (8.54) as

\[
\mathcal{R}(a_i, b_i) = \text{Res}_\mu e^{\mu a_{ia} \frac{\partial}{\partial c_i} \tau_{\mu_1}^{-1} b_{ia} \frac{\partial}{\partial c_i} H_{\alpha\beta} \frac{\partial^2}{\partial c_i^a \partial c_i^\beta} C(c_i)} \bigg|_{c_i = 0},
\]
of $R$ and renumerating the spinor variables one obtains

$$\bar{\kappa} = \frac{\partial^2}{\partial c_{1a} \partial c_{2}^a}, \quad \bar{u}_i = \frac{\partial^2}{\partial c_i^a \partial a_{3a}}, \quad \bar{v}_i = \frac{\partial^2}{\partial c_{i\alpha} \partial b_{3}^\alpha}$$ (8.57)

along with the identities (3.23) and (3.28) applied to the background fields, one finds

$$A^h_{\alpha,\beta,\gamma}(R, \mathcal{R} \ast \varepsilon) = B \int_{M^5} \bar{k}^2 H_5 \text{Res}_\mu \left( e^{\frac{1}{2}(\mu \bar{\epsilon}_2 - \mu^{-1} \bar{u}_2)} \varphi(Z) \right)$$

$$\times \text{tr} \left( C(c_1) C(c_2) \varepsilon(a_3, b_3) \right) \bigg|_{c_1 = c_2 = a_3 = b_3 = 0},$$ (8.58)

where $B \neq 0$ is some numerical factor,

$$\varphi(Z) = \left( 2\gamma(Z, -Z) - (\alpha(Z, -Z) + \beta(Z, -Z)) \right)$$ (8.59)

and

$$Z = (\mu \bar{k} - \bar{u}_1)(\mu^{-1} \bar{k} + \bar{v}_1).$$ (8.60)

(Note that the dependence on $\mu_1$ in the representation (8.56) for the first factor of $\mathcal{R}$ cancels out while $\mu = \mu_2$ for the analogous representation in the factor of $\mathcal{R}$ in $\mathcal{R} \ast \varepsilon$.)

Analogously, after recycling the product factors under the matrix trace $\text{tr}$ and renumerating the spinor variables one obtains

$$A^h_{\alpha,\beta,\gamma}(R, \varepsilon \ast \mathcal{R}) = B \int_{M^5} \bar{k}^2 H_5 \text{Res}_\mu \left( e^{\frac{1}{2}(\mu \bar{\epsilon}_1 - \mu^{-1} \bar{u}_1)} \varphi(Y) \right)$$

$$\times \text{tr} \left( C(c_1) C(c_2) \varepsilon(a_3, b_3) \right) \bigg|_{c_1 = c_2 = a_3 = b_3 = 0},$$ (8.61)

where

$$Y = (\mu \bar{k} + \bar{u}_2)(\mu^{-1} \bar{k} - \bar{v}_2).$$ (8.62)

The problem therefore is to find such a function $\varphi(Y)$ that

$$\bar{k}^2 \text{Res}_\mu \left( e^{\frac{1}{2}(\mu \bar{\epsilon}_1 - \mu^{-1} \bar{u}_1)} \varphi(Z) - e^{\frac{1}{2}(\mu \bar{\epsilon}_1 - \mu^{-1} \bar{u}_1)} \varphi(Y) \right)$$

$$\times \text{tr} \left( C(c_1) C(c_2) \varepsilon(a_3, b_3) \right) \bigg|_{c_1 = c_2 = a_3 = b_3 = 0} = 0.$$ (8.63)

As a first guess let us try $\varphi(AB) = \text{Res}_\nu \left( e^{\frac{1}{2}(\nu^{-1} A + \nu B)} \right)$. Then the two terms in brackets in (8.63) amount to

$$\text{Res}_{\mu,\nu} \left( e^{\frac{1}{2}(\mu \bar{\epsilon}_1 - \mu^{-1} \bar{u}_1) + \nu^{-1} \bar{k}} - e^{\frac{1}{2}(\nu^{-1} A + \nu B)} \right).$$ (8.64)
These cancel out upon substitution $\nu \leftrightarrow -\mu$. However, this solution is not completely satisfactory because the formula (8.59) requires $\varphi(Z)$ to vanish at $Z = 0$ to have analytic functions $\alpha, \beta, \gamma$.

The following comment is now in order. As discussed in the beginning of this section, throughout this paper we only consider interactions of the higher spin fields with spins $s \geq 2$. From the perspective of the First On-Mass-Shell Theorem in the form (8.56) this implies that $C(c)$ starts from the fourth-order polynomials in the spinor variables $c$, i.e.

$$C(0) = 0, \quad \frac{\partial^2}{\partial c^\alpha \partial c^\beta} C(c)\big|_{c^\alpha = 0} = 0.$$  \hspace{1cm} (8.65)

Since the factor $\bar{k}^2$ in (8.63) contains two differentiations both in $c_1$ and in $c_2$, (8.65) means that adding a constant to $\varphi$ does not affect (8.63). This allows one to cancel out a constant term in $\varphi$ by setting

$$\varphi(A) = \varphi_0 \text{Res}_\nu \left( e^{\frac{i}{2}(\nu^{-1} + \nu A)} - 1 \right).$$  \hspace{1cm} (8.66)

As a result, the on-mass-shell invariance condition solves by

$$2\gamma(A, -A) - \alpha(A, -A) - \beta(A, -A) = \varphi_0 A^{-1} \left( \text{Res}_\nu \left( e^{\frac{i}{2}(\nu^{-1} + \nu A)} - 1 \right) \right)$$

$$= \frac{1}{2} \varphi_0 \int_0^1 du \left( \text{Res}_\nu \left( \nu e^{\frac{i}{2}(\nu^{-1} + \nu A)} \right) \right).$$  \hspace{1cm} (8.67)

Taking into account (8.47) and (8.48) this is solved by

$$\gamma(p) = \frac{1}{4} \varphi_0 \int_0^1 dv v \text{Res}_\nu \left( \nu e^{\frac{i}{2}(\nu^{-1} + \nu vp)} \right)$$  \hspace{1cm} (8.68)

and

$$\alpha(p, q, 0) + \beta(p, q, 0) = 2\gamma(p + q) - \frac{1}{2} \varphi_0 \int_0^1 du \text{Res}_\nu \left( \nu e^{\frac{i}{2}(\nu^{-1} + \nu (up + q))} \right).$$  \hspace{1cm} (8.69)

Expansion of these expressions for $\gamma(p)$ and $\alpha(p, q, 0) + \beta(p, q, 0)$ in the power series gives (8.21) and (8.22). With aid of these power series expansions one can see that the following identities are true

$$\left( p \frac{\partial^2}{\partial p^2} + 3\frac{\partial}{\partial p} + \frac{1}{4} \right) \gamma(p) = 0,$$  \hspace{1cm} (8.70)

$$\left( \left( 2 + p \frac{\partial}{\partial p} \right) \frac{\partial}{\partial p} + \left( 1 + q \frac{\partial}{\partial q} \right) \frac{\partial}{\partial q} + \frac{1}{4} \right) \left( \alpha(p, q, 0) + \beta(p, q, 0) \right) = 0.$$  \hspace{1cm} (8.71)
From (8.28) - (8.30) it follows then that the $C-$invariance condition (8.34) is satisfied with
\[
\alpha(p, q, t) + \beta(p, q, t) = \alpha(p, q, 0) + \beta(p, q, 0), \quad \gamma(p, q, t) = \gamma(p, q, 0). (8.72)
\]
Thus it is shown that the coefficient functions (8.21) and (8.22) satisfy the factorization condition, $C-$invariance condition, extra field decoupling condition and the on-mass-shell invariance condition. The resulting bilinear form (8.14) defines the action (8.13) that properly describes the higher spin dynamics both at the free field level and at the level of cubic interactions. The leftover ambiguity in the coefficients $\alpha(p, q, t) + \beta(p, q, t)$ and $\gamma(p, q, t)$ reduces to the overall factor $\varphi_0$ that encodes the ambiguity in the gravitational constant.

9 Reduced Models

So far we discussed the 5d higher spin algebra $cu(1, 0\vert 8)$ being the centralizer of $N$ in the star product algebra. This algebra is not simple as it contains infinitely many ideals $I_{P(N)}$ spanned by the elements of the form $P(N) \ast f$ for any $f \in cu(1, 0\vert 8)$ and any star-polynomial $P(N)$ [26]. Considering the quotient algebras $cu(1, 0\vert 8) / I_{P(N)}$ is equivalent to “imposing operator constraints” $P(N) = 0$. In this section we focus on the algebra $hu_0(1, 0\vert 8)$ that results from $P(N) = N$ and its further reduction $ho_0(1, 0\vert 8)$. The algebra $hu_0(1, 0\vert 8)$ corresponds to the system of higher spin fields of all integer spins with every spin emerging once. $ho_0(1, 0\vert 8)$ is its reduction to the system of all even spins. Both of these algebras are of interest from the AdS/CFT perspective [28, 25].

The explicit construction of $hu_0(1, 0) = cu(1, 0\vert 8) / N$ via factorization is not particularly useful within the star product setup because $N \ast$ is the second order differential operator (6.20). A useful approach used in [25] consisted of taking projection by considering elements of the form $f \ast F$ where $f$ was an element of $cu(1, 0\vert 8)$ while $F$ was a certain Fock vacuum projector satisfying $N \ast F = 0$. In fact, the left module over $cu(1, 0\vert 8)$ generated from $F$ was shown in [25] to describe 4d conformal fields. In this construction, the factorization of $cu(1, 0\vert 8)$ to $cu(1, 0\vert 8) / N$ was automatic. The Fock vacuum $F$ was 4d Lorentz invariant and had definite scaling dimension. It is not invariant under the $AdS_5$ Lorentz algebra $o(4, 1)$ however, and therefore cannot be used for the $AdS_5$ bulk higher spin gauge theory considered in this paper. On the other hand, from the perspective of this paper the Fock module construction is irrelevant. We therefore relax the property that the projector is a Fock vacuum for certain
oscillators. Instead we shall look for a \( su(2,2) \) invariant operator \( M \) satisfying

\[
N * M = M * N = 0, \quad (9.1)
\]

\[
D_0(M) = 0. \quad (9.2)
\]

To satisfy (9.2) we choose a manifestly \( su(2,2) \) covariant ansatz \( M = M(a_\alpha b_\alpha) \). For any polynomial function \( M \) this would imply that it is a star polynomial of \( N \). From (9.1) it is clear however that \( M \) cannot be a star product function of \( N \). Nevertheless there is a unique (up to a factor) analytic solution for \( M = M(a_\alpha b_\alpha) \) that solves (9.1). Indeed, from (6.20) it follows that the condition (9.1) has the form

\[
-xM(x) + M'(x) + \frac{1}{4}xM''(x) = 0, \quad M' = \frac{\partial M}{\partial x}. \quad (9.3)
\]

This is solved by

\[
M(x) = \int_{-1}^{1} dl(1 - l^2)e^{2lx}, \quad (9.4)
\]

as one can easily see using \( (2x - \frac{\partial}{\partial l})e^{2lx} = 0 \) and integrating by parts. Equivalently

\[
M(x) = \left(1 - \frac{1}{4} \frac{\partial^2}{\partial x^2}\right) \frac{sh(2x)}{x}. \quad (9.5)
\]

Note that \( M(x) \) is even

\[
M(-x) = M(x). \quad (9.6)
\]

Having found the operator \( M \) we can write the action for the reduced system associated with \( \hbar u_0(1,0|8) \) by replacing the bilinear form in the action with

\[
A(f, g) \rightarrow A_0(f, g) = A(f, M * g). \quad (9.7)
\]

Note that \( A_0(f, g) \) is well-defined as a functional of polynomial functions (or, formal power series) \( f \) and \( g \) for any entire function \( M(a_\alpha b_\alpha) \) because, for polynomial \( f \) and \( g \), only a finite number of terms in the expansion of \( M(a_\alpha b_\alpha) \) contributes. The modification of the bilinear form according to (9.7) with any \( M(a_\alpha b_\beta) \) leads to a new invariant action (8.13). The reason why this ambiguity was not observed in our analysis is that we have imposed the factorization condition in a particular basis of the higher spin gauge fields, thus not allowing the transition to the new bilinear form (9.7).

All other conditions, namely, the \( C-\)invariance condition, extra field decoupling condition and the on-mass-shell invariance condition remain valid for
any entire function \( M(a_{\alpha}b_{\beta}) \) inserted into the bilinear form. The factorization condition is relaxed in this section. Note that the \( C - \text{invariance condition} \) guarantees that the bilinear form \( A_0 \) is symmetric

\[
A(f, M \ast g) = A(f \ast M, g). \tag{9.8}
\]

Inserting a particular function \( M(a_{\alpha}b_{\beta}) \) (9.4) we automatically reduce the system to a smaller subset of fields being linear combinations of the different copies of the fields emerged in the original \( cu(1,0|8) \) model. Namely, we can now require all fields in the expansion (5.6) to be traceless. In other words, the representatives of the quotient algebra \( hu_0(1,0|8) \) are identified with the elements \( g \) satisfying the traceless condition

\[
T^{-}g = 0. \tag{9.9}
\]

Indeed, by virtue of (6.21) any polynomial \( \tilde{g}(a,b) \in cu(1,0|8) \) is equivalent to some \( g \) satisfying (9.9) modulo terms containing star products with \( N \) which trivialize when acting on \( M \). The star product \( f \ast g \) of any two elements \( f \) and \( g \) satisfying the tracelessness condition (9.9) does not necessarily satisfy the same condition, i.e. \( T^{-}(f \ast g) \neq 0 \) (otherwise the elements satisfying (9.9) would form a subalgebra rather than a quotient algebra). However the difference is irrelevant inside the action built with the help of the bilinear form \( A_0 \). In particular, the higher spin field strength

\[
(d\omega + \omega \wedge *\omega) \ast M \tag{9.10}
\]

is equivalent to that of the higher spin algebra \( hu_0(1,0|8) \).

Thus, the action

\[
S_{\text{red}}^{E} = \frac{1}{2} A_{\alpha\beta\gamma}^{E} (R \ast M, R) \tag{9.11}
\]

leads to a consistent free field description and cubic interactions for the system of the higher spin fields associated with the higher spin algebra \( hu_0(1,0|8) \). The resulting system describes massless fields of all integer spins \( s \geq 2 \), every spin emerges once. The further reduction to the subalgebra \( ho_0(1,0|8) \subset hu_0(1,0|8) \) associated with the subset of even spins is now trivially obtained by setting to zero all fields of odd integer spins. (For more details on the Lie algebraic definition of the corresponding reduction we refer the reader to [28, 25]). Note that according to the analysis of the section 8.2 one can consider the dynamical system with \( n^2 \) fields of each spin, taking values in the matrix algebra \( \text{Mat}_n \). This system corresponds to the higher spin algebra \( hu_0(n,0|8) \). Its reduction to \( ho_0(n,0|8) \) describes higher spin fields of even spins in the
symmetric representation of $o(n)$ and odd spins in the adjoint representation of $o(n)$. (Therefore, no odd spins for the case of $n = 1$).

Note that the conclusions of this section sound somewhat opposite to those of [27] where it was claimed that the analogous reduction for the $4d$ conformal higher spin theories is inconsistent.

The following comment is now in order. Since $M$ is a particular non-polynomial (although entire) function, one has to be careful in treating it as an element of the star product algebra which in our setup is regarded either as the algebra of polynomials or of formal power series. Since $M(a, b^\alpha)$ is uniquely defined by the property (9.1) and $M \ast M$ formally has the same property, one might expect that $M \ast M = m M$ with some numerical factor $m$. Once this would be true, it would be possible to rescale $M$ to a projection operator. However, this is not possible because the parameter $m$ turns out to be infinite. As this issue may be interesting beyond the particular $5d$ problem studied in this paper, let us consider the general case with the indices $\alpha, \beta \ldots$ ranging from $1$ to $2n$. The equation (9.3) generalizes to

$$-xM(x) + \frac{1}{2} nM'(x) + \frac{1}{4} xM''(x) = 0$$

(9.12)

with the solution

$$M(x) = \int_{-1}^{1} dl (1 - l^2)^{n-1} e^{2lx}.$$  

(9.13)

An elementary computation then shows that

$$(M \ast M)(x) = \int_{-1}^{1} dl \int_{-1}^{1} dl' \int_{-1}^{1} dk \delta\left(k - \frac{l + l'}{1 + ll'}\right) \frac{(1 - k^2)^{n-1}}{(1 + ll')^2} \exp 2kx.$$  

(9.14)

From this formula it follows that the expression $M \ast M$ is ill-defined for any $n$ because the factor $\frac{1}{(1 + ll')^2}$ gives rise to a divergency at the boundary of the integration region.

Therefore, one cannot treat elements like $M$ as elements of the star product algebra. In particular, this concerns the construction suggested in [28] for the description of the $5d$ generating function for the scalar massless field and higher spin Weyl tensors in terms of the fields $\Phi(a, b|x)$ required to satisfy the condition $N \ast \Phi = \Phi \ast N = 0$. From what is explained in this section it is clear that

$$\Phi = M \ast \phi = \phi \ast M , \quad \forall \phi : [\phi, N], = 0.$$  

(9.15)

In particular $M$ itself belongs to this class. There is no problem at the linearized level as far as the variables $\Phi$ are no multiplied, but it is likely to be a problem at the interaction level.
Let us stress again that in the construction presented in this section the appearance of \( M \) in the action functional causes no problem because it is only multiplied with polynomial elements of the higher spin algebra and never with itself.

10 Conclusion

It is shown that 5d higher spin gauge theories admit consistent higher spin interactions at the action level at least in the cubic order and that, in agreement with the conjecture of [21] and the construction of 4d conformal higher spin algebras of [26], 5d higher spin symmetry algebra admits a natural realization in terms of certain star product algebras with spinor generating elements. One difference compared to the 4d case is that the 5d higher spin algebra \( cu(1,0|8) \) contains non-trivial center freely generated by the element \( N \) (3.4). As a result, 5d higher spin algebra \( cu(1,0|8) \) gives rise to the infinite sets of fields of all spins. That every spin appears in infinite number of copies makes the spectrum of the 5d higher spin theories reminiscent of the string theory. On the other hand, we have shown that the factorization of the algebra \( cu(1,0|8) \) with respect to the maximal ideal generated by \( N \), that gives rise to the reduced higher spin algebra \( hu_0(1,0|8) \) in which every integer spin appears in one copy, admits consistent interactions as well. The same is true for the further reduction the algebra \( ho_0(1,0|8) \) discussed in [28], that describes higher spin fields of even spins, as well as for the matrix extensions \( hu_0(n,0|8) \) and \( ho_0(n,0|8) \) that describe either \( n^2 \) fields of every integer spin in the case of \( hu_0(n,0|8) \) or \( \frac{1}{2}n(n+1) \) fields of every even spin and \( \frac{1}{2}n(n-1) \) fields of every odd spin in the case of \( ho_0(n,0|8) \).

The obtained results are expected to admit a generalization to the supersymmetric case. To this end one extends the set of oscillators \( a_\alpha \) and \( b^\beta \) with the set of Clifford elements \( \phi_i \) and \( \bar{\phi}^j \) \((i, j = 1 \ldots N)\) satisfying the commutation relations

\[
\{\phi_i, \phi_j\} = 0, \quad \{\bar{\phi}^i, \bar{\phi}^j\} = 0, \quad \{\phi_i, \bar{\phi}^j\} = \delta_i^j. \tag{10.1}
\]

The supersymmetric extension of the 5d higher spin algebra is then defined as the centralizer of

\[
N_N = a_\alpha b^\alpha - \phi_i \bar{\phi}^i. \tag{10.2}
\]

The \( N = 1 \) supersymmetric 5d higher spin theories will be analyzed in [32]. An extension to \( N \geq 2 \) is more complicated because the condition \([N_N, f]_* = 0\) allows \( f(a, b, \phi, \bar{\phi}|x) \) with \(|N_a - N_\phi| > 1\) that, according to the analysis...
of section 4, corresponds to the higher spin potentials with the symmetry properties of the $o(4,2)$ Young diagrams having tree rows. Such fields are not related to the totally symmetric tensor and tensor-spinor fields described in [30, 21] and are expected to correspond to the mixed symmetry free $AdS_5$ fields which, as shown in [29], are not equivalent to the symmetric fields in the $AdS_5$ background although becoming equivalent to some their combinations in the flat limit. Therefore, to proceed towards $AdS_5$ supersymmetric higher spin gauge theories it is first of all necessary to develop an appropriate free field formulation of the mixed symmetry fields in the $AdS$ background. This problem is now under study.

Once the formulation of the mixed symmetry fields in $AdS_5$ is developed, it will allow one to consider higher spin theories with all $\mathcal{N}$. These theories are expected to be dual to the 4$d$ free supersymmetric conformal higher spin theories analyzed in [25]. In [25] it was suggested that a class of larger CFT$_4$ and $AdS_5$ consistent higher spin theories should exist exhibiting manifest $sp(8)$ symmetry. Such theories result from relaxing the condition that the $AdS_5$ higher spin algebra is spanned by the elements that commute to $N_\mathcal{N}$. Being analogous to the 4$d$ higher spin gauge theories based on the algebras $hu(n, m|4)$, the generalized higher spin gauge theories based on the algebras $hu(n, m|8)$ are expected to be dual to the $hu(n, m|8)$ invariant 4$d$ conformal higher spin gauge theories [25]. The 5$d$ $sp(8)$ invariant higher spin gauge theories are likely to be generating theories for the reduced models based on the centralizers of $N_\mathcal{N}$ in $hu(n, m|8)$ as the ones discussed in this paper. It is tempting to speculate that the reduction of the higher spin algebras $hu(n, m|8)$ is a result of a certain spontaneous symmetry breaking mechanism with some dynamical field $\varphi$ in the adjoint representation of the higher spin algebra that develops a vacuum expectation value $\varphi = N_\mathcal{N} + \ldots$.

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A Free Field Equations

In this appendix we give some details on the derivation of the equations of motion that follow from the quadratic part of the higher spin action. The variation (8.40) can be equivalently rewritten as

\[ \delta S_2 = -\frac{1}{2} \int_{M^5} \rho(p + q) \left( a_1^\gamma \frac{\partial}{\partial b_1^\gamma} \delta_{12} + a_2^\gamma \frac{\partial}{\partial b_2^\gamma} \delta_{21} \right) H_{2 \alpha \beta} \wedge tr(\delta \omega(a_1, b_1|x) \wedge R_1(a_2, b_2|x)) \bigg|_{a_i = b_j = 0}. \]  

(A.1)

Taking into account that the Young symmetrizers \( a_1^\gamma \frac{\partial}{\partial b_1^\gamma} \) and \( a_2^\gamma \frac{\partial}{\partial b_2^\gamma} \) commute to \( p + q \) and using the definition of the component fields (6.29), (6.31) we find

\[ \delta S_2 = -\int_{M^5} \rho(p + q) tr \left( \frac{\partial^2}{\partial a_{1 \alpha} \partial a_{2 \beta}} \delta_{12} \eta^1(a_1, b_1|x) \wedge R_1(a_2, b_2|x) \right. \\
+ \left. \frac{\partial^2}{\partial a_{1 \alpha} \partial a_{2 \beta}} \delta_{21} \eta^1(a_1, b_1|x) \wedge r^1(a_2, b_2|x) \wedge H_{2 \alpha \beta} \right|_{a_i = b_j = 0}. \]  

(A.2)

Now one observes that the terms containing simultaneously \( \eta^1 \) and \( r^1 \) cancel out because \( H_{2 \alpha \beta} \) is symmetric in the spinor indices. Therefore

\[ \delta S_2 = -\int_{M^5} \rho(p + q) \bar{c}_{12} \frac{\partial^2}{\partial a_{1 \alpha} \partial a_{2 \beta}} H_{2 \alpha \beta} \wedge tr(\delta \eta^1(a_1, b_1|x) \wedge R_1(a_2, b_2|x) + \delta \eta^0(a_2, b_2|x)) \bigg|_{a_i = b_j = 0}. \]  

(A.3)

Inserting here the component expansions

\[ \eta^i(a, b|x) = \sum_{u,v=0}^{\infty} \frac{1}{u!v!} \delta(2i - u + v) \eta_{\alpha_1 \ldots \alpha_u \beta_1 \ldots \beta_v}(x) a_{\alpha_1} \ldots a_{\alpha_u} b_{\beta_1} \ldots b_{\beta_v}, \]  

(A.4)

\[ r^i(a, b|x) = \sum_{u,v=0}^{\infty} \frac{1}{u!v!} \delta(2i - u + v) r_{\alpha_1 \ldots \alpha_u \beta_1 \ldots \beta_v}(x) a_{\alpha_1} \ldots a_{\alpha_u} b_{\beta_1} \ldots b_{\beta_v}, \]  

(A.5)

\[ \rho(p) = \sum_{n=0}^{\infty} \frac{\rho_n}{n!} p^n, \]  

(A.6)

where, taking into account (8.22) and (8.47),

\[ \rho_n = (-1)^n \frac{1}{2^{2n+2} (n+1)!}, \]  

(A.7)
and completing the differentiations one gets
\[
\delta S_2 = - \int_{M^5} \sum_{u,v=0}^{\infty} \frac{\rho_{u+v}}{u!v!} (-1)^v H_{2\alpha\beta} \wedge 
\]
\[
tr \left( \delta \eta^{1\alpha_1\ldots\gamma_{u+1}\kappa_1\ldots\kappa_v} \sigma_1\ldots\sigma_u \rho_1\ldots\rho_v \wedge r^{0\beta}_{1\gamma_1\ldots\gamma_{u+1}\rho_1\ldots\rho_v} \right) 
\]
\[
+ r^{0\alpha_1\gamma_1\ldots\gamma_{u+1}\kappa_1\ldots\kappa_v} \sigma_1\ldots\sigma_u \rho_1\ldots\rho_v \wedge \delta \eta^{0\beta}_{\gamma_1\ldots\gamma_{u+1}\rho_1\ldots\rho_v} \right) \bigg|_{\alpha_i = \beta_j = 0} \quad (A.8)
\]

Using (A.7) and the Young properties of the component fields and curvatures one obtains
\[
\delta S_2 = - \frac{1}{2} \phi_0 \int_{M^5} \sum_{n=0}^{\infty} (-1)^n 2^{-2n} \frac{1}{(n-1)!n!} H_{2\alpha\beta} \wedge 
\]
\[
\times \left( \delta \eta^{1\alpha_1\ldots\gamma_n} \sigma_1\ldots\sigma_n\rho_1\ldots\rho_n \wedge r^{0\beta}_{1\gamma_1\ldots\gamma_n} \right) 
\]
\[
+ r^{0\alpha_1\gamma_1\ldots\gamma_n} \sigma_1\ldots\sigma_n\rho_1\ldots\rho_n \wedge \delta \eta^{0\beta}_{\gamma_1\ldots\gamma_n} \right) \bigg|_{\alpha_i = \beta_j = 0} . \quad (A.9)
\]

The free equations of motion corresponding to the variation with respect to Lorentz-type fields \( \eta^1 \) and the frame-type field \( \eta^0 \) have the following component form, respectively,
\[
0 = H_{\alpha_1 \sigma} r^{0}_{1\sigma \alpha_2\ldots\alpha_{m+1}} - \frac{m-1}{2(m+1)} V_{\alpha_1 \beta_1} H_{\gamma \sigma} r^{0}_{1\gamma \sigma \alpha_2\ldots\alpha_{m+1}} \quad (A.10)
\]
and
\[
0 = H_{\alpha_1} \gamma (r^{1}_{1\beta_1\kappa_1\gamma_1\kappa_2\ldots\kappa_m} + \frac{m-1}{3} r^{1}_{1\beta_1\kappa_1\gamma_1\kappa_2\ldots\kappa_m}) 
\]
\[
- m H_{\gamma \beta_1} (r^{1}_{1\beta_1\kappa_1\gamma_1\kappa_2\ldots\kappa_m} + \frac{m-1}{3} r^{1}_{1\beta_1\kappa_1\gamma_1\kappa_2\ldots\kappa_m}) 
\]
\[
+ \frac{m}{m+1} V_{\alpha_1 \beta_2} H^{\gamma \sigma} (r^{1}_{1\beta_1\gamma_1\sigma_1\kappa_2\ldots\kappa_m} + (m-1) r^{1}_{1\beta_2\gamma_1\sigma_1\kappa_2\ldots\kappa_m}) 
\]
\[
+ \frac{1}{6} (m-1)(m-2) r^{1}_{1\beta_1\gamma_1\sigma_1\gamma_2\kappa_3\ldots\kappa_m}) . \quad (A.11)
\]
(As usual, the symmetrization of the indices denoted by the same Greek letters is assumed).

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