Tunable spin/charge Kondo effect at a double superconducting island connected to two spinless quantum wires

Domenico Giuliano(1,2), Luca Lepori(3), and Andrea Nava(4)

(1) Dipartimento di Fisica, Università della Calabria Arca vacata di Rende I-87036, Cosenza, Italy
(2) I.N.F.N., Gruppo collegato di Cosenza, Areavacata di Rende I-87036, Cosenza, Italy
(3) Istituto Italiano di Tecnologia, Graphene Labs, Via Morego 30, I-16163, Genova, Italy
(4) International School for Advanced Studies (SISSA), Via Bonomea 265, I-34136 Trieste, Italy

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We propose that a pertinently engineered double superconducting island connected to two spinless one-dimensional conducting leads can work as a tunable spin/charge-Kondo system. We evidence how, by tuning a single gate voltage applied to the island, it is possible to make the system switch from the spin-, to the charge-Kondo phase, passing across an intermediate phase, in which the Kondo impurity is effectively irrelevant for the low-temperature behavior of the system. Eventually, we evidence how to probe the various phases by measuring the dependence on the temperature of the dc conductance tensor of our system, by emphasizing the features that should allow to identify the onset of the so far quite elusive charge-Kondo effect.

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I. INTRODUCTION

The Kondo effect has been experimentally seen for the first time as an upturn in the resistivity of metals doped with magnetic impurity, as the temperature $T$ goes below the nonuniversal Kondo temperature $T_K$, typically of the order of a few Kelvin, or less. On the theoretical side, the Kondo effect was readily explained in terms of a dynamical screening of the single-impurity magnetic moment by means of the spin density of itinerant electrons in the metal. As $T \to 0$, the impurity moment is fully screened, which allows for trading the impurity for a local (Kondo) spin singlet, acting as a scattering center that forbids electrons from accessing the impurity site. The Kondo spin singlet provides one of the few theoretically well-understood examples of strongly-correlated states of matter. For this reason, soon after its discovery and its theoretical explanation, Kondo effect began to be used as a paradigmatic testbed for a number of remarkable analytical, as well as numerical, methods to study strongly-correlated systems, including the well-celebrated Wilson’s numerical renormalization group (RG) technique. On top of that, it has been found how a peculiar realization of the effect, such as e.g. the “overscreened” one, in which more than one itinerant electron “channel” contributes to screen the magnetic impurity, yields to a novel phase of matter, which, differently from the local spin singlet, cannot be described within the “standard” Landau’s Fermi liquid framework.

Recently, a renewed interest has arisen in the Kondo effect, due to the possibility of realizing it in a controllable way in mesoscopic systems with tunable parameters, such as semiconducting quantum dots with metallic leads, in which Kondo effect is expected to appear in an upturn of the conductance, rather than of the resistivity, across the dot connected to the leads, or with superconducting leads, in which Kondo effect should be evidenced by a change in the behavior of the subgap (Josephson) supercurrent across the dot, when the leads are held at a fixed phase difference $\varphi$. In addition, since the effect is merely due to spin dynamics, it has been proposed that a “spin-Kondo” effect can take place in systems with itinerant, low-energy excitations carrying spin, but not charge, such as XXZ spin-1/2 chains, which can be for instance realized by loading cold atoms on a pertinently designed optical lattice, or frustrated $J_1-J_2$ spin chains with $J_2/J_1$ tuned at the “critical” value at the phase transition between the spin liquid- and the dimerized-phase of the system. In fact, a tunable realization of the spin-Kondo effect has recently been proposed at junctions of quantum spin chains, or of one-dimensional arrays of Josephson junctions. Finally, it is worth mentioning the possibility, that has been recently put forward, that a remarkable “topological” Kondo effect might arise, in which the impurity spin is realized by means of Majorana modes emerging at the interface between topological superconductors and normal conductors (or the empty space) and, more generally, the striking similarity between the Kondo physics and the hybridization between a Majorana mode and the itinerant electrons in a metal connected to the topological superconductor.

A key point about Kondo effect is that, besides all the spin dynamics underneath, in order for the effect to take place one generally needs an impurity with a twofold degenerate groundstate, which is able to switch from one state to the other via quantum number exchange processes with itinerant particles from the medium into which the impurity is embedded. On this respect, a number of proposals have been put forward in which Kondo effect is associated to charge, rather than to spin degeneracy in the impurity groundstate. Such a charge-based version of Kondo effect is typically dubbed “charge-Kondo” (CK) effect, to distinguish it from the “standard” “spin-Kondo” (SK) effect. CK effect was originally proposed as a possible mechanism, related to the “negative-$U$” Anderson model, able to...
induce a charge-dual version of the highly-correlated, heavy fermion groundstate\textsuperscript{32}. CK effect has later on been theoretically studied in dots connected to bulk leads\textsuperscript{33}, in single-electron transistors\textsuperscript{34–36}, as well as in generalizations of the negative-$U$ Anderson model\textsuperscript{37–39}, also involving optical lattice systems\textsuperscript{40}.

Notwithstanding the great interest in CK effect, witnessed by the large number of papers on the topic, a clear-cut experimental verification of the effect is still lacking (differently from what happens for SK effect). For this reason, in the last years there has been an increasing interest in realizing CK effect in a controlled way, in systems with tunable parameters. For instance, it has been proposed to realize CK effect in mesoscopic superconductors coupled to normal metals\textsuperscript{41}, in negative-$U$ quantum dots with superconducting electrodes\textsuperscript{42}, and even in double quantum-dot, in which the effect should be mediated by the Coulomb repulsion between the electrons at the double dot\textsuperscript{43}. In general, defining an appropriate tunable device to probe CK effect, possibly in comparison with the more “standard” SK effect, is still an open challenge, also in view of the potential relevance of CK effect to explain the physics of e.g. superconductivity in PbTe doped with Tl\textsuperscript{32}, or of impurities formed at dots in LaSrO$_3$/SrTiO$_3$-interfaces\textsuperscript{44}.

In this paper, we propose to realize CK effect at a “minimal” tunable device in which one may in principle switch from SK, to CK effect by just acting onto a limited number of system’s parameters (ideally, one parameter only). Our system, which we sketch in Fig.\textsuperscript{1} in its “minimal” version, consists of two spinless conducting fermionic channels (the “leads”), connected to a “tunable” effective Kondo impurity $K$ whose Kondo-like coupling to lead electrons can be either SK-like, or CK-like, depending on the specific values of the tuning parameters. In particular, we propose to realize the tunable Kondo impurity by means of a pertinently designed double superconducting island. Our design allows for changing in a controlled way the magnitude and the sign of the electronic interaction at the superconducting island. In the language of the Anderson impurity model Hamiltonian, which provides a reliable low-energy description of the island, this corresponds to tuning the system across a transition from the positive-$U$, to the negative-$U$ regime. While, for both signs of $U$, the island groundstate keeps twofold degenerate, thus triggering the onset of Kondo physics when connected to the leads, the nature of the degenerate groundstate doublet strongly depends on the sign of $U$. Eventually, this implies a transition from the SK to the CK regime at the change in the sign of the electronic interaction strength\textsuperscript{45}.

Within our device, the SK and the CK phase do not overlap in parameter space with each other. On one hand, this avoids the simultaneous presence of both effects which, though making the physical scenario richer, does not possibly allow for a clear-cut detection of the latter effect against the former one\textsuperscript{42}. In fact, the separate detection of either effect is even more favored by the fact that the SK and the CK phase are separated, in parameter space, by an intermediate, “disconnected lead” phase, in which the impurity plays no relevant role for the low-$T$ physics of the system.

To probe the various phases of our system, we propose to look at the dependence on $T$ of the dc conductance across the impurity when a voltage bias is applied to one lead at one side of the impurity and the current is measured at the other side, either within the same lead (“diagonal” conductance), or in the other one (“off-diagonal” conductance). Eventually, we show how a synoptic comparison of the $T$-dependent diagonal and off-diagonal conductances offers a simple, though effective, way of detecting the SK and the CK effect in out system. Moreover, when tuned within the CK phase, the Kondo impurity triggers off-diagonal conduction via a peculiar realization of Crossed Andreev reflection between the two leads. In analogy to the devices discussed in Refs.\textsuperscript{16,17}, this suggests that, in a “dual” setup, in which a Cooper pair is injected into the leads through the impurity, our system might effectively work as a “long-distance electronic entangler”, with potential applications to realizing large-distance entangled two-particle states.

The paper is organized as follows:

- In section \textsuperscript{II} we introduce the lattice model Hamiltonian for our system. In particular, we discuss in detail how to engineer the double superconducting island at the center of the system, so to make it work as a spin-Kondo, or charge-Kondo impurity, by acting onto a pertinent tuning parameter;
- In section \textsuperscript{III} we derive the effective low-energy, long-wavelength, continuum Hamiltonian of our system in its various phases: the spin-Kondo phase, the charge-Kondo phase and the decoupled lead phase;
- In section \textsuperscript{IV} we resort to a perturbative renormalization group analysis, to recover how the system scales with the temperature $T$ towards the fixed point corresponding to each one of its phases;
- In section \textsuperscript{V} we discuss the dependence of the conductances on $T$ in each phase, paying particular attention to the onset of the nonperturbative Kondo regime in the SK and in the CK phase. Eventually, we highlight how an appropriate measurement of the dc conductances as a function of $T$ provides an effective mean to map out the phase diagram of our system;
- In section \textsuperscript{VI} we summarize our result by also discussing about a possible practical realization of our system and by eventually highlighting possible further developments of our work.
The two quantum wires, represented as one-dimensional lattices, while the tunable Kondo impurity is realized by connecting the ladder to a double superconducting island with pertinently chosen parameters (see the main text).

• In the various appendices, we report the mathematical details of our derivation.

II. MODEL HAMILTONIAN

Our device is sketched in Fig. 1. To model it, we resort to a lattice Hamiltonian for two leads, which we represent as two $2\ell + 1$-site chains, with $\ell$ eventually sent to $\infty$. Therefore, the lattice Hamiltonian for the leads, $H_{0,\text{Lat}}$, is given by

$$H_{0,\text{Lat}} = \sum_{a=1,2} \left\{ -J_a \sum_{j=-\ell}^{\ell-1} \left[ c_{j,a}^\dagger c_{j+1,a} + c_{j+1,a}^\dagger c_{j,a} \right] - \mu_a \sum_{j=-\ell}^{\ell} c_{j,a}^\dagger c_{j,a} \right\},$$

with $\{c_{j,a}, c_{j,a}^\dagger\}$ being single-fermion annihilation/creation operators at site-$j$ of lead $a$, obeying the standard anticommutation relations $\{c_{j,a}, c_{j',a'}^\dagger\} = \delta_{j,j'} \delta_{a,a'}$. $J_a, \mu_a$ respectively correspond to the single-fermion hopping strength and to the on-site chemical potential. For the sake of simplicity, in the following we choose the parameters entering Eq. (1) independent of $a$. In fact, this only quantitatively affects the final result and, in any case, one may readily discuss the case of different parameters in the two leads by using the approach discussed in e.g. Ref. [48] in the general case of a ladder of interacting quantum wires.

We now show how it is possible to realize a tunable Kondo Hamiltonian by connecting the two-leg ladder to a double superconducting island (DSI), with pertinently chosen parameters. Specifically, we consider an adapted version of the topological Kondo Hamiltonian introduced in Ref. [49], and then widely studied in Refs. [28–30, 50–55]. In Fig. 2 we draw a sketch of the proposed device. It consists of two mesoscopic s-wave superconducting island with two spinless quantum wires deposited onto each of them. According to Refs. [56, 57], we expect four localized real Majorana modes to emerge at the endpoints of the wires. Let $\gamma_1, \gamma_2$ and $\gamma_3, \gamma_4$ be the two Majorana modes arising at wire 1 and 2, respectively. To construct the tunable Kondo Hamiltonian, we assume that the $\gamma_1$ and $\gamma_3$ are tunnel-coupled to respectively lead-1 and lead-2. Also, we assume that the length of each wire deposited on the island and the distance between the wires are large enough to suppress the direct tunneling between the two Majorana modes at each wire. Yet, the two Majorana modes at the same island are assumed to be coupled to each other via the capacitive charging energy between each island and the ground. Finally, we assume a nonzero direct cross-capacitance coupling between the two superconducting island and a Josephson coupling allowing for Cooper pair exchange between the islands and an underneath superconducting island $S$. As a result, the total Hamiltonian for the double-island system is given by

$$H_{\text{Island}} = H_{I,1} + H_{I,2} + H_C + H_S.$$  

In Eq. (2), $H_{I,1}$ and $H_{I,2}$ describe the two islands coupled to $S$. They are defined so that

$$H_{I,1} + H_{I,2} = -E_J \cos \chi_1 + E_C \left[ 2N_1 + n_1 - Q_1 \right]^2 - E_J \cos \chi_2 + E_C \left[ 2N_2 + n_2 - Q_2 \right]^2,$$

with $n_i = \frac{1}{2} \left[ 1 + e^{i \gamma_{2i-1} - i \gamma_{2i}} \right]$, so that $2N_i + n_i$ is the total charge (in units of $e$) lying at island-$i$, including a possible quasiparticle occupying the Dirac level made out of the two Majorana modes, and with $Q_i$ being the backgate voltage,
determined by the voltage across the capacitor. \( E_C = e^2 / (2C) \) is the charging energy of each island, and \(-E_J \cos(\chi_i)\) corresponds to the Josephson coupling between island-\(i\) and the superconductor underneath, with the phase difference \(\chi_i\) canonically conjugate to the number of Cooper pairs \((N_i)\). \(H_C\) in Eq. \(2\) describes the cross-capacitive coupling between the two islands. It is given by

\[
H_C = E_{CC} [2N_1 + n_1 - Q_1] [2N_2 + n_2 - Q_2] ,
\]

with \(E_{CC}\) being the cross-charging energy. Finally, \(H_S\) describes the superconducting island \(S\), which we assume to be large enough for it to be able to absorb/emit Cooper pair at no additional cost of energy. In the following, we assume that the parameters of the two mesoscopic islands have been chosen so that they lie within the “charging” regime, in which \(E_C/E_J \gg 1\). In this case, Coulomb blockade prevents Cooper pairs from tunneling across the island, except if the backgate potential is tuned at the degeneracy point between states with different total charge at the island. In our specific case, the single-fermion state associated to the pair of real Majorana modes can be combined into Dirac complex fermion operators, \(a_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)\), and \(a_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)\), thus allowing for low-energy charge tunneling processes across the impurity involving a single quasiparticle, rather than a Cooper pair. Therefore, setting \(Q_i\) at each island so that \(Q_i = 2\hat{N}_i + \frac{1}{2}\), with integer \(\hat{N}_i\), allows for defining the low-energy subspace at the double junction as spanned by the four state with \(\hat{N}_i\) Cooper pairs at island-\(i\), with the mode corresponding to \(a_i\) full, or empty. Listing those states, together with the corresponding energy eigenvalues, we obtain the set

\[
\begin{align*}
|\hat{N}_1, \hat{N}_2, 0, 0\rangle, & \quad \epsilon_{0,0} = \frac{1}{2}E_C + \frac{1}{4}E_{CC} \\
|\hat{N}_1, \hat{N}_2, 1, 1\rangle, & \quad \epsilon_{1,1} = \frac{1}{2}E_C + \frac{1}{4}E_{CC} \\
|\hat{N}_1, \hat{N}_2, 1, 0\rangle, & \quad \epsilon_{1,0} = \frac{1}{2}E_C - \frac{1}{4}E_{CC} \\
|\hat{N}_1, \hat{N}_2, 0, 1\rangle, & \quad \epsilon_{0,1} = \frac{1}{2}E_C - \frac{1}{4}E_{CC} 
\end{align*}
\]

with \(|\hat{N}_1, \hat{N}_2, \nu_1, \nu_2\rangle\) denoting the state with \(\hat{N}_i\) Cooper pairs at island-\(i\) and \(\nu_i\) additional quasiparticle in the level determined by the Majorana modes (clearly, \(\nu_i = 0, 1\)), and with the energies measured with respect to a common reference level. As we discuss in the following, the level structure summarized in Eqs. \(4\) is enough to induce an effective Kondo Hamiltonian, except that, in order to recover both SK and CK effect, one has to have a window of values of parameters corresponding to an attractive inter-island interaction (that is, \(E_{CC}\) has to become < 0). This is the main motivation for introducing \(S\), which is coupled to the two island by a small Josephson term that allows to form a Cooper pair in the superconductor through the annihilation of the two Dirac fermions in the islands, and vice versa. At low energies, only particles populating the \(a_i\)-levels are involved. Therefore, the corresponding processes are described by the hopping Hamiltonian

\[
H_S = -\tau \left( a_1^\dagger a_2 e^{2i\varphi} + a_2^\dagger a_1^\dagger e^{-2i\varphi} \right) .
\]
Differently from island-1 and -2, S hosts no Majorana modes. Therefore, at low energies, charges can enter and exit it only as Cooper pairs. To fix the number of Cooper pairs at S, we assume that it has a finite capacitive energy $E_S$, which enters the corresponding Hamiltonian given by

$$H^{(0)}_S = -E_{J,S} \cos \chi_S + E_{Q,S} [2N_S - Q_S]^2 \ .$$  

(7)



Tuning $Q_S = 2N_S$ and assuming $E_{Q,S}/E_{J,S} > 1$, Coulomb blockade pins at $N_S$ the number of Cooper pairs at S. In this case, charges can tunnel from the islands to S, and vice versa, only in virtual processes. This implies a corresponding lowering of the energy of the states with $\nu_1 = \nu_2 = 0$ and $\nu_1 = \nu_2 = 1$ by an amount that appears at second order in $\tau$ and is given by $\epsilon_r = -\frac{C_{ST}^2}{4e^2}$. Taking this result and the level diagram in Eqs. 6 altogether, the double island dynamics is described by the effective Hamiltonian

$$H_{\text{Island}} = \delta \left( \langle 0,0 | \langle 0,0 | + | 1,1 \rangle \langle 1,1 | - | 1,0 \rangle \langle 0,1 | - | 0,1 \rangle \langle 0,1 | \right) \ .$$  

(8)

To simplify the notation, in Eq. 8, we have omitted the labels associated to the number of Cooper pairs on the two island. The right-hand side of Eq. 8 depends only on the tuning parameter $\delta = \frac{E_{CC}}{\tau} - \epsilon_r$, which we will use as a control parameter to switch from IS- to IC-Kondo effect. For the following discussion, it is useful to rewrite $H_{\text{Island}}^{\text{Eff}}$ in terms of the Dirac complex fermion operators $a_1, a_2$ and of their Hermitian conjugates as

$$H_{\text{Island}}^{\text{Eff}} = \delta \{ 1 - 2[a_1^\dagger a_1 + a_2^\dagger a_2] + 4a_1^\dagger a_1 a_2^\dagger a_2 \} \ .$$  

(9)

Formally, the coupling between the hybrid island and the leads is described by the tunneling Hamiltonian $H_t$, which we model in analogy to what is done in Ref. 49, as

$$H_t = -t \sum_{a=1,2} \left( c_{0,a}^\dagger c_{a,a} + c_{0,a}^\dagger a_{a} e^{-i \omega a} \right) + \text{h.c.} \ .$$  

(10)

In Eq. 10, the term $c_{0,a}^\dagger a_{j}$ describes the transfer of a fermion from the $i$-th island to the central site of the corresponding lead, with the corresponding depletion of the level $a_j$. The term $c_{0,a}^\dagger a_{a} e^{-i \omega a}$ represents an alternative process through which a fermion is created in the level $a_j$ and another one is created in the corresponding lead along with the annihilation of a Cooper pair in the island by the operator $e^{-i \omega a}$. Noticeably, this process induces a transition to a state with an higher number of Cooper pairs in the islands, which we rule out on projecting onto the low-energy groundstate manifold of the islands. Therefore, we drop it henceforth from the tunneling Hamiltonian and describe the DSI coupled to the ladder by means of the boundary Hamiltonian $H_B$ given by

$$H_B = H_{\text{Island}}^{\text{Eff}} - t \sum_{a=1,2} \left( c_{0,a}^\dagger a_{a} + a_{a}^\dagger c_{0,a} \right) \ .$$  

(11)

In addition to the direct tunneling between the leads and the DSI, a local density-density interaction Hamiltonian may arise, as well. The corresponding Hamiltonian can be simply modelled as by

$$H_{\text{DI}} = \sum_{a,b=1,2} \mu_{a,b} c_{0,a}^\dagger c_{0,a} a_{b}^\dagger a_{b} \ .$$  

(12)

In the following, we use the boundary Hamiltonian $H_B = \hat{H}_B + H_{\text{DI}}$ to discuss the crossover between the SK and the CK regime at the impurity by also pointing out the remarkable emergence of an intermediate “disconnected lead” (DL) phase, with peculiar properties.

III. EFFECTIVE IMPURITY HAMILTONIAN IN THE VARIOUS REGIMES

A first step to describe the impurity dynamics in our system is to resort to pertinent approximations for $\hat{H}_B$ in different windows of values of the various parameters. In fact, we see that $\delta$ is the only scale related to the isolated impurity. The other relevant scales are the tunneling strength $t$ and the local density-density interaction strengths, the $\mu_{a,b}$’s in Eq. 12 which, consistently with our symmetry assumption, we choose so that $\mu_{1,1} = \mu_{2,2} = \mu_4$, and $\mu_{1,2} = \mu_{2,1} = \mu_{od}$. A first important limit corresponds to $|\delta| \to \infty$. In this limit, the low-energy manifold of the system is twofold degenerate. In particular, for $\delta \to +\infty$, the two degenerate groundstates correspond to the “mini-domain walls” of Ref. 53, that is, to the $|1,0)$ and to the $|0,1)$-eigenstates of the DSI, while, for $\delta \to -\infty$, the two degenerate states correspond to the $|0,0)$ and to the $|1,1)$-eigenstates of the DSI.
Leaving aside, for the time being, the density-density interaction encoded in $H_{\text{DI}}$ in Eq. (12), we see that, at finite values of the $t_j$’s, tunneling processes between the degenerate groundstates are accounted for by resorting to an effective, Kondo-like description of the interaction of the DSI with the leads. To do so, we employ the Schrieffer-Wolff procedure which we illustrate the details in appendix [3]. In the symmetric case $t_j = t_2 = t$, the leading boundary operator describing the residual dynamics within the low-energy subspace of the states of the DSI is either a SK version of the approach used in Refs. [17, 19, 60], we eventually get the full set of RG equations, given by (apart for

$$H_{\text{K,S}} = J_S \tilde{S}_0 \cdot \tilde{S} ,$$

(13)

with $J_S = 2t^2/\delta$ and the impurity spin operator $\tilde{S}$ and the lead spin density operator $\tilde{S}_j$ respectively defined as

$$S^\alpha = \frac{1}{2} \sum_{u, u' = 1, 2} a_{u'}^\dagger \tau^\alpha u,u ' a_{u'} , \quad S_j^\alpha = \frac{1}{2} \sum_{u, u' = 1, 2} c_{j,u'}^\dagger \tau^\alpha u,u ' c_{j,u'} ,$$

(14)

with $\tau^\alpha, \alpha = x, y, z$ being the Pauli matrices.

• For $\delta < 0$, the charge-Kondo (CK) Hamiltonian $H_{\text{K,C}}$, given by

$$H_{\text{K,C}} = J_C \tilde{T}_0 \cdot \tilde{T} ,$$

(15)

with $J_C = 2t^2/|\delta|$ and the impurity charge-isospin operator $\tilde{T}$ and the lead charge-isospin density operator $\tilde{T}_j$ respectively defined as

$$T^\alpha = \frac{1}{2} \sum_{u, u' = 1, 2} \tilde{a}_{u'}^\dagger \tau^\alpha u,u ' \tilde{a}_{u'} , \quad T_j^\alpha = \frac{1}{2} \sum_{u, u' = 1, 2} \tilde{c}_{j,u'}^\dagger \tau^\alpha u,u ' \tilde{c}_{j,u'} ,$$

(16)

with $\tilde{a}_1 = a_1, \tilde{a}_2 = a_2^\dagger$, and $\tilde{c}_{j,1} = c_{j,1}, \tilde{c}_{j,2} = c_{j,2}^\dagger$.

Turning on $H_{\text{DI}}$ we see that, in the SK regime, it modifies the effective impurity Hamiltonian as

$$H_{\text{K,S}} \rightarrow \tilde{H}_{\text{K,S}} = J_S \tilde{S}_0 \cdot \tilde{S} + \frac{\mu_d - \mu_{\text{od}}}{2} \tilde{S}_0^z \tilde{S}_0^z + \frac{\mu_d + \mu_{\text{od}}}{2} \sum_{\alpha = 1, 2} c_{0,a}^\dagger \tau^\alpha c_{0,a} ,$$

(17)

that is, the effective isotropic Kondo Hamiltonian acquires a nonzero anisotropy along the $z$-direction as soon as $\mu_d \neq \mu_{\text{od}}$ plus a local scattering potential term, which does not substantially affect Kondo physics. At variance, in the complementary CK regime, on turning on $H_{\text{DI}}$, we obtain

$$H_{\text{K,C}} \rightarrow \tilde{H}_{\text{K,C}} = J_C \tilde{T}_0 \cdot \tilde{T} + \frac{\mu_d + \mu_{\text{od}}}{2} \{ \tilde{T}_0^z \tilde{T}_0^z + \tilde{T}_0^+ \tilde{T}_0^+ \} ,$$

(18)

that is, again a nonzero anisotropy along the $z$-direction in the (charge) Kondo interaction terms, plus effective, local field contributions coupled to both the impurity- and the itinerant-fermion effective charge-isospin operator at $j = 0$. All the terms appearing at the right-hand side of both Eqs. (17, 18) are “standard” contributions arising in the Kondo problem and, accordingly, their effect, at least in the simplest case of a perfectly screened spin-1/2 impurity, is basically well understood. In the following, we therefore discuss the corresponding Kondo physics along the standard approach to Kondo problem in metal.

It is also interesting to address in detail what happens at small values of $|\delta|$. At $\delta = 0$, the leads are fully decoupled from the impurity. This is the central point of the DL region which, nevertheless, is expected to extend to finite values of $\delta$, due to the effects of the competition between the interaction energy at the DSI (measured by $\delta$ itself) and the hybridization between the $a_{\alpha}$ modes and the leads (measured by $t_{\alpha,}^{\text{DD}}$). To verify how the DSI extends to $\delta \neq 0$, till the system crosses over towards either the SK, or the CK, phase, we now resort to an “all-inclusive” RG analysis, considering the RG equations for all the running couplings associated to $\tilde{H}_B$. To do so, we define the dimensionless couplings as $\tilde{\tau} = (D_{n})^{-d_f} (\tilde{\mu}_d)$, $\tilde{\mu}_{\text{od}} = A(d_{\text{od}}) (\tilde{\delta})$, with $d_f = \frac{3}{2}$. On employing a pertinent adapted version of the approach used in Refs. [17, 19, 60], we eventually get the full set of RG equations, given by (apart for
irrelevant boundary terms, which can in principle be generated along the RG procedure, and which we neglect in the following\)

\[
\frac{d\bar{T}(D)}{d\ln \left(\frac{D_0}{D}\right)} = (1 - d_f)\bar{T}(D) + \bar{\mu}_d(D)\bar{T}(D) \\
\frac{d\bar{\mu}_d(D)}{d\ln \left(\frac{D_0}{D}\right)} = [\bar{T}(D)]^2 - \tilde{\delta}(D)\bar{\mu}_d(D) \\
\frac{d\bar{\mu}_d(D)}{d\ln \left(\frac{D_0}{D}\right)} = -\tilde{\delta}(D)\bar{\mu}_d(D) \\
\frac{d\tilde{\delta}(D)}{d\ln \left(\frac{D_0}{D}\right)} = \tilde{\delta}(D) - \frac{\bar{\mu}_d(D)\bar{\mu}_d(D)}{4}.
\] (19)

To infer from Eqs.(19) the condition for the crossover between the DL and either the SK, or the CK, phase, we simplify the right-hand side of Eqs.(19) by neglecting nonlinear terms in the various coupling strengths. Accordingly, the only actually running couplings are now \(\bar{T}(D)\) and \(\tilde{\delta}(D)\), respectively given by

\[
\bar{T}(D) = D_0^{1 - d_f} \left(\frac{at}{v}\right) ; \quad \tilde{\delta}(D) = D_0 \left(\frac{D_0}{D}\right) \left(\frac{a\tilde{\delta}}{v}\right).
\] (20)

The derivation of Eqs.(19) relies on the small-coupling assumption for the various boundary interaction strengths. This corresponds to requiring that \(|a\bar{T}/v| < 1\), a condition that, if one consider the running coupling strength in Eq.(20), only holds up to \(D \sim D_\ast\), with \(D_\ast \sim D_0 \left(\frac{27}{a^2}\right)^{\frac{1}{1-d_f}}\). In order for the Schrieffer-Wolff transformation leading to the effective Kondo Hamiltonian to apply, the condition \(t/\tilde{\delta} < 1\) must hold. From Eqs.(20), one sees that this happens at any scale if \(|\tilde{\delta}| > t\) (at the reference scale \(D = D_0\)). However, due to the nontrivial renormalization of the running parameters, the condition can also be satisfied if \(|\tilde{\delta}| < t\). To recover this condition, we note that \(\tilde{\delta}(D)\) takes over \(\bar{T}(D)\) at a scale \(D_{\text{Cross}}\), determined by

\[
\tilde{\delta}(D_{\text{Cross}}) = \bar{T}(D_{\text{Cross}}) \Rightarrow D_{\text{Cross}} = D_0 \left(\frac{\tilde{\delta}}{t}\right)^{\frac{1}{1 - d_f}}.
\] (21)

Therefore, in order for the Kondo regime to set in, one has to have that \(D_{\text{Cross}} > D_\ast\), which implies the condition

\[
\frac{a|\tilde{\delta}|}{v} > \left(\frac{at}{v}\right)^{\frac{1}{1-d_f}}.
\] (22)

Once the condition in Eq.(22) is satisfied, the impurity dynamics is either described by \(H_{K,S}\) in Eq.(13), or by \(H_{K,C}\) in Eq.(15), depending on whether \(\delta > 0\), or \(\delta < 0\).

At variance, in the DL region the boundary dynamics is described, as we discuss in detail in appendix C, by the effective local density-density interaction Hamiltonian given by

\[
H_{DL} = \kappa c_{0,1}^{\dagger} c_{0,1} c_{0,2}^{\dagger} c_{0,2} + \sum_{a=1,2} \lambda_a c_{0,a}^{\dagger} c_{0,a},
\] (23)

with \(\kappa\) being the effective local inter-lead density-density interaction strength and \(\lambda_1, \lambda_2\) being “residual” intra-wire single-body potential scattering strengths.

In the following, we use the results we derived in this section to recover, after resorting to an appropriate low-energy, long-wavelength continuum limit for the fermionic fields in the leads, a detailed RG analysis of the system’s boundary dynamics in the three phases discussed above.

**IV. RENORMALIZATION GROUP ANALYSIS OF THE IMPURITY DYNAMICS**

We now resort to a perturbative RG analysis, to recover the fixed point (that is, the phase) to which the system flows in the various regions we discuss in the previous section. In doing so, a necessary preliminary steps consists in expanding the lattice fermionic fields, \(c_{j,a}\), by retaining only low-energy, long-wavelength excitations around the Fermi points \(\pm k_F = \pm \arcsin \left(\frac{\mu}{2F}\right)\). In doing so, we obtain
FIG. 3: Boundary phase diagram of our system in the \( -a\delta \) plane: for \( |\delta| < v^{\frac{1}{df}} \) the decoupled lead phase sets in. For \( \delta > v^{\frac{1}{df}} \), the spin-Kondo, and the charge-Kondo phase respectively set in.

\[
c_{j,a} \approx \sqrt{a} \{ \psi_{R,a}(x_j) + \psi_{L,a}(x_j) \},
\]
with \( a \) being the lattice step (which we set to 1 henceforth, except when explicitly required for the sake of the presentation clarity), \( x_j = aj \) and \( \psi_{R,a}(x) \), \( \psi_{L,a}(x) \) being chiral fields described by the 1+1-dimensional Hamiltonian \( H_0 \), given by

\[
H_0 = -iv \sum_{a=1,2} \int_{-\ell}^{\ell} dx \{ \psi_{R,a}^\dagger(x) \partial_x \psi_{R,a}(x) - \psi_{L,a}^\dagger(x) \partial_x \psi_{L,a}(x) \}.
\]

To simplify the following derivation we note that, since we are representing the DIS as a pointlike impurity localized at \( x = 0 \), it is useful to resort to the “even” and the “odd” linear combinations of the chiral fermionic fields, \( \psi_{e,a}(x) \), \( \psi_{o,a}(x) \), respectively given by

\[
\psi_{e,a}(x) = \frac{1}{\sqrt{2}} \{ \psi_{R,a}(x) + \psi_{L,a}(-x) \}
\]
\[
\psi_{o,a}(x) = \frac{1}{\sqrt{2}} \{ \psi_{R,a}(x) - \psi_{L,a}(-x) \}.
\]

Apparently, \( \psi_{o,1}(x) \) and \( \psi_{o,2}(x) \) fully decouple from the impurity dynamics, which is accordingly described, in the three different regions identified in section III, by the boundary Hamiltonians

\[
\hat{H}_{K,S} = 2J_S \hat{\sigma}_c(0) \cdot \hat{S} + 2 \left( \frac{\mu_d - \mu_{od}}{2} \right) \hat{S}_z^2 + 2 \left( \frac{\mu_d + \mu_{od}}{2} \right) \rho_c(0)
\]
\[
\hat{K}_{K,C} = 2J_C \hat{\tau}_c(0) \cdot \hat{T} + \left( \frac{\mu_d + \mu_{od}}{2} \right) \{ 2\tau_c^z(0)\hat{T}_z + 2\tau_c^z(0) + \hat{T}_z \}
\]
\[
H_{DL} = \sum_{a=1,2} 2\lambda_a \rho_{e,a}(0) + 4\kappa \rho_{1,e}(0)\rho_{2,e}(0),
\]

with

\[
\rho_{c,a}(0) = \psi_{c,a}^\dagger(0)\psi_{c,a}(0)
\]
\[
\rho_c(0) = \sum_{a=1,2} \rho_{e,a}(0)
\]
\[
\hat{\sigma}_c(0) = \frac{1}{2} \sum_{a,b=1,2} \psi_{c,a}^\dagger(0)\hat{\sigma}_{a,b}\psi_{c,b}(0)
\]
\[
\hat{\tau}_c(0) = \frac{1}{2} \sum_{a,b=1,2} \psi_{c,a}^\dagger(0)\hat{\tau}_{a,b}\psi_{c,b}(0)
\]

and \( \tilde{\psi}_{e,1}(x) = \psi_{e,1}(x), \tilde{\psi}_{e,2}(x) = \psi_{e,2}(x) \).
In performing the RG analysis, we neglect the non-purely Kondo-like terms at the right-hand side of the first two ones of Eqs. (27). This makes us deal with a generally anisotropic Kondo Hamiltonian, though we do not expect anisotropy to take relevant effects, here. Specifically, in the following we perform the RG analysis along the guidelines of Ref. [61]. To do so, we note that, in view of the fact that only the $\psi_{a,e}$-fields do actually couple to the impurity spin, the fields in the $a$-sector obey the continuity condition at $x = 0$ given by

$$\psi_{a,o}(0^+, \tau) = \psi_{a,o}(0^-, \tau) \quad .$$

(29)

for any value of the Kondo coupling. At zero Kondo coupling, the fields in the $e$-sector satisfy the same boundary conditions as in Eq. (29), that is

$$\psi_{a,e}(0^+, \tau) = \psi_{a,e}(0^-, \tau) \quad .$$

(30)

At variance, when turning on the Kondo coupling, the relevance of the corresponding boundary interaction, combined with the fact that, as we extensively discuss in appendix D, a limited isotropy in the Kondo coupling strength does not prevent the system from reaching the Kondo fixed point, determines a change in the boundary conditions in Eq. (30). In particular, at the SK fixed point one obtains

$$\psi_{a,e}(0^+, \tau) = e^{-2i\delta} \psi_{a,e}(0^-, \tau) \quad ,$$

(31)

with $\delta$ being a nonuniversal phase shift, which, at the strongly coupled fixed point is independent of the momentum $k$ measured with respect to the Fermi momentum of the chiral fermion excitations, and is equal to $\frac{\pi}{2}$ if particle-hole symmetry is unbroken. Eq. (31) simply corresponds to Nozierès Fermi liquid boundary conditions, that is, to the fact that the formation of the local Kondo singlet at the impurity location prevents any other electron from accessing that point. From that, taking also into account a possible breaking of the spin rotational symmetry, one obtains that the leading boundary operator allowed at the Kondo fixed point can be expressed as

$$\tilde{\mathcal{H}}_{K,S} = \alpha_S \psi_{1,e}^\dagger(0) \psi_{1,e}(0) \psi_{2,e}^\dagger(0) \psi_{2,e}(0) + \sum_{a=1,2} \beta_{S,a} \psi_{a,e}^\dagger(0) \psi_{a,e}(0) \quad ,$$

(32)

with $\alpha_S, \beta_{S,a}$ appropriate boundary coupling strengths. A similar construction holds for the CK effect, as well. In this case, the main idea is to construct a faithful mapping of the SK onto the CK-Hamiltonian by trading $\tilde{\vartheta}(0)$ for $\tilde{\varrho}(0)$, given by

$$\tilde{\varrho}(0) = \frac{1}{2} [\psi_{1,e}^\dagger(0), \psi_{2,e}^\dagger(0)] \tilde{\sigma} \left[ \begin{array}{c} \psi_{1,e}(0) \\ \psi_{2,e}(0) \end{array} \right] \quad .$$

(33)

Clearly, from Eq. (33) one can readily trace out the faithful correspondence with the isospin Kondo Hamiltonian, which implies that, on one hand, the noninteracting fixed point is described by the boundary conditions at $x = 0$ given by

$$\tilde{\psi}_{a,e}(0^+, \tau) = \tilde{\psi}_{a,e}(0^-, \tau) \quad ,$$

(34)

and that, at variance, CK fixed point is described by the boundary conditions

$$\tilde{\psi}_{a,e}(0^+, \tau) = e^{-2i\delta} \tilde{\psi}_{a,e}(0^-, \tau) \quad ,$$

(35)

$$\tilde{\psi}_{a,o}(0^+, \tau) = \tilde{\psi}_{a,o}(0^-, \tau) \quad ,$$

and

$$\tilde{\psi}_{a,o}(0^+, \tau) = e^{-2i\delta} \tilde{\psi}_{a,o}(0^-, \tau) \quad .$$

(36)

By a straightforward extension of the analysis we performed at the SK fixed point, we readily infer that, according to Nozierès Fermi liquid theory, the leading boundary perturbation at the CK fixed point is given by

$$\tilde{\mathcal{H}}_{K,C} = \alpha_C \psi_{1,e}^\dagger(0) \psi_{1,e}(0) \psi_{2,e}^\dagger(0) \psi_{2,e}(0) + \sum_{a=1,2} \beta_{C,a} \psi_{a,e}^\dagger(0) \psi_{a,e}(0) \quad ,$$

(37)

with $\alpha_C, \beta_{C,a}$ interaction strengths. Remarkably, the leading boundary Hamiltonian at the both the SK and the CK fixed point, respectively reported in Eq. (32) and in Eq. (36), takes exactly the same form as the leading boundary Hamiltonian in the DL-region — third one of Eqs. (27). What changes from one case to the others is the actual meaning of the fermionic fields entering the boundary interaction. As we discuss in the following, this reflects onto the different
conduction properties of the system at the various fixed points. In any case, the scaling properties of the boundary operator are always the same in the three cases. Before moving to discuss them, we briefly mention the effects of the local magnetic field acting on both the impurity spin and the spin of the conduction electrons in the effective Kondo Hamiltonians on the first two lines of Eqs. (27). In fact, while, in general, a strong applied field may eventually lead to the suppression of Kondo effect, it is by now well established that the Kondo effect instead survives the applied field, as long as $T_K$ is higher than the energy scale associated to the applied magnetic field\textsuperscript{12,62}. The local magnetic field encodes the local density-density interaction at the DSI. This is a minor effect, which is expected to be much smaller than the direct electronic tunneling encoded in $H_B$ in Eq. (11). This, on one hand implies that, being close to the isotropic case, the RG trajectories in the Kondo regions always flow towards the Kondo fixed point (see appendix D for details), on the other hand, that $T_K$ is always much larger than the energy scale associated to the Zeeman term, which implies that this does not spoil Kondo effect. Based on this observations, we now discuss the leading boundary perturbation at both the Kondo- and the DL-fixed points. As we discuss above, basically $H_{DL}$ in Eqs. (27) encompasses both the boundary Hamiltonians in Eqs. (32,36). Therefore, we refer to $H_{DL}$ for our further analysis. As noted in appendix D a simple power counting implies that the dimensionless coupling associated to $\kappa$ is $K(D) = \frac{D}{D_0} \kappa$. On lowering the running cutoff $D$, one therefore obtains that $K(D)$ scales to 0 as $D/D_0$. Thus, we conclude that the inter-wire local density interaction is an irrelevant perturbation at either the SK (CK), or at the DL-fixed point, which is consistent with the expected stability of the various fixed points. The left-over terms are, instead, marginal one-body scattering potential terms. These marginally deform the fixed point dynamics, by changing the phase shift $\delta$ in the $e$-channel, possibly differently in different channels, with minor consequences on the conduction properties of the system, as we discuss in the following.

V. DC CONDUCTANCE TENSOR AND PHASE DIAGRAM OF THE SYSTEM

In this section, referring to the results we derive in detail in appendix E, we discuss how the dependence on the temperature $T$ of the conductance tensor varies from phase to phase and when moving from the perturbative regime towards the fixed points characterizing each phase. As a result, we evidence how it is possible to map out the whole phase diagram of our tunable Kondo device by means of the dc conductance tensor and of its dependence on $T$.

To perform our analysis, in the following we mostly focus onto the conduction properties of the system across the impurity, which is the appropriate measurement to discuss the behavior of a central impurity. However, our approach can be readily extended to computing the conductances of the system when both the voltage bias and the measured current are at the same side of the impurity, which is rather appropriate for a side impurity at an endpoint of the device. Also, we note that, when not connected to the central impurity, our system behaves as two parallel and disconnected perfectly conducting channels, with the corresponding dc conductance tensor, $G^{(0)}$, simply given by $G^{(0)} = \frac{e^2}{2\pi} I$, with $I$ being the 2×2 identity matrix. We now discuss in detail the various regions.

A. DC conductance in the spin-Kondo phases

Within the spin-Kondo phase the impurity dynamics is described by the boundary Hamiltonian $\hat{H}_{K,S}$ in Eq. (17). Leaving aside, for the time being, the one-body potential scattering term, $\hat{H}_{K,S}$ takes the form of the anisotropic Kondo Hamiltonian $H_{K,S}^{\text{anis}}$, given by

$$H_{K,S}^{\text{anis}} = J_{S,\perp} \{ S^+ \sigma^- (0) + S^- \sigma^+ (0) \} + J_{S,\parallel} S^\parallel \sigma^\parallel (0).$$

(37)

Consistently with the microscopic derivation of $H_{K,S}^{\text{anis}}$, we expect that both $J_{S,\perp}$ and $J_{S,\parallel}$ are $> 0$. Therefore, one expects that the RG trajectories induced by $H_{K,S}^{\text{anis}}$ all lie within either region I, or region II, of the diagram in Fig. of appendix D which corresponds to a flow towards the Kondo fixed point. This implies that here the anisotropy in the Kondo coupling only affects the formula for the (nonuniversal) Kondo scale, but not the over-all behavior of the system. Therefore, we simplify our further derivation by neglecting the anisotropy henceforth. Nevertheless, our results are expected to be reliable also in the case $J_{S,\perp} \neq J_{S,\parallel}$.

In the isotropic case, the RG flow induced by the spin-Kondo Hamiltonian is encoded in the running dimensionless coupling $J_S(D) = \frac{a_{JS}}{D}$, with the running scale $D$ to be identified with $kT$, consistently with the analysis we perform in this paper. On integrating the second-order RG equation for the running coupling strength, one obtains (trading the dependence on $D$ for an explicit dependence on $kT$)

$$J_S(T) = \frac{J_{S,0}}{1 - J_{S,0} \ln \left( \frac{D_0}{kT} \right)},$$

(38)
with $D_0$ being the reference energy scale (high-energy cutoff) and $J_{S,0} = J_S(D = D_0)$ (note that, in our specific case, since, in order for the Schrieffer-Wolff transformation leading to $H_{K,S}$ and to $H_{K,C}$ to be effective we have to assume that there are no physical processes involving energies of the order of $|2\delta|$, we must properly set $D_0 = |2\delta|$).

In order to incorporate the nontrivial RG flow in Eq. (38) into the dc conductance tensor $G$, we first of all note that, as we discuss in detail in appendix A, a possible nonzero one-body scattering in $\hat{H}_{K,S}$ may slightly renormalize $G$ by an over-all, nonuniversal factor. This effect does not qualitatively affect the following analysis so, for the sake of simplicity, we just neglect it in the following of this section. Turning on a finite $J_S$, in appendix E.1 we show how the marginal relevance of the Kondo coupling induces a nontrivial RG flow in $G$. Using the temperature $T$ as the running energy scale, we therefore obtain

$$G(T) \approx \frac{e^2}{2\pi} \left[ 1 - \left( \frac{J_{S,0}}{1 - J_{S,0} \ln \left( \frac{kT}{D_0} \right)} \right)^2 \right],$$

that is, the result in Eq. (33) of appendix E.3. As stated in appendix E.1 the result in Eq. (39) is expected to apply from $kT \sim D_0$ all the way down to $kT \sim kT_K = D_0 e^{-\frac{\pi J_{S,0}}{4}}$. From the right-hand side of Eq. (39) we see that turning on $J_S$ implies first of all a reduction in the current flowing through the lead to which the voltage bias is applied (that is, $G_{a,a}(T) < \frac{e^2}{2\pi}$ as soon as $J_S \neq 0$). At the same time, a nonzero conductance arises from lead-1 to lead-2 (and vice versa). This is due to the peculiar features of the Kondo processes mediating the transport at nonzero coupling to the impurity. In terms of electron transmission across the impurity, transport from, say, lead-1 to the same lead does not correspond to a change in the “spin-index” of the transmitted electron. At the onset of Kondo dynamics, together with the former scattering process, which we depict in Fig. (1) as a particle-to-particle transmission within lead-1, impurity spin-flip processes can induce single electron tunneling from lead-1 to lead-2, mediated by the coupling to the impurity spin $\hat{S}$. This process, which we depict in Fig. (1) as a particle-to-particle transmission from lead-1 to lead-2, is what is responsible for a nonzero $G_{1,2}(T)$. This is a (marginally) relevant process as $kT$ goes down from $D_0$ to $kT_K$. Accordingly, we find that the corresponding off-diagonal conductance takes over, when going down in temperature, till one enters the Kondo regime at the scale $T_K$. An important observation is that, since the current induced in lead-2 is due to particle-to-particle transmission processes it takes the same sign as the current in lead-1, as evidenced in Eq. (39) by the fact that, as long as the perturbative RG approach holds (that is, for $T > T_K$), one has that both $\delta G_{1,1}(T)$ and $G_{1,2}(T)$ are $> 0$.

On flowing towards the SK fixed point, the spin density of the lead electrons at $x = 0$ “locks together” with the impurity spin, so to effectively cut the system into two separate parts. As we discuss in appendix E.1 this implies no dc transport in the particle-hole symmetric case (corresponding to a total phase shift $\delta = \frac{\pi}{2}$ in the $e$-linear combinations of the chiral fields in each channel). More generally, for $\delta \neq \frac{\pi}{2}$, one expects a reduction of the diagonal elements of $G$ by a factor $\cos^2(\delta)$ and a simultaneous suppression of the off-diagonal elements. Near by the SK fixed point, finite-$T$ corrections to $G$ are determined by the residual interaction $H_{K,S}$ in Eq. (32). In fact, the term at the right-hand side of Eq. (32) that might potentially contribute a nonzero $G_{1,2}(T)$ is the one $\propto \alpha_S$. Yet, in analogy with the analogous calculations in the DL phase we perform in appendix E.3 we find that $G_{1,2}(T) = 0$, at least to second-order in $\alpha_S$, (in fact, as long as the boundary interaction describes electron scattering off a local singlet, one is expected to obtain $G_{1,2}(T) = 0$ at any order in $\alpha_S$). To find out the second-order correction to the diagonal components of $G$, we may just borrow the results of appendix E.3 with the main difference arising from the nonzero phase shift $\delta$ in the $e$-sector at the SK fixed point. As a result, we obtain

$$\delta G_{1,1}(T) = \delta G_{2,2}(T) = -\frac{\cos(2\delta)\alpha_S^2 e^2}{3\pi v^2}(kT)^2.$$

Thus, we conclude that, to order $\alpha_S^2$, the dc conductance tensor at the spin-Kondo fixed point, $G_{SK}(T)$, is given by

$$G_{SK}(T) = \frac{e^2}{2\pi} \left\{ \cos^2(\delta) - \frac{2\cos(2\delta)\alpha_S^2 (kT)^2}{3\pi v^2} \right\} I,$$

with zero off-diagonal elements. In particular, in the particle-hole symmetric case, one has $2\delta = \pi$, which implies

$$G_{SK}(T) = \frac{e^2}{2\pi} \frac{2\alpha_S^2 (kT)^2}{3\pi v^2} I.$$

To summarize, we have shown that, on lowering $kT$ from $kT \sim D_0$ to $kT \sim kT_K$, the diagonal elements of the dc conductance tensor are suppressed by the marginally relevant Kondo interaction. At the same time, the relevance
of the “effective” spin-flip processes induces nonzero off-diagonal elements in the conductance tensor, which increase as $T$ is lowered. Since a spin-flip process here corresponds to a particle/hole tunneling from one lead as an injected particle/hole to the other one, the currents induced in the two leads by means of a voltage bias applied to either one of them flow towards the same directions. Once the system has flown to the spin-Kondo fixed point, it may, or may not, exhibit a finite diagonal zero-$T$ conductance tensor, depending on whether particle-hole symmetry is broken, or not. The finite-$T$, correction to the dc conductance tensor is diagonal, as well, and $\propto T^2$, consistently with Nozierès Fermi liquid theory.

B. DC conductance in the charge-Kondo phases

Within the CK phase, the impurity dynamics is described by the Hamiltonian $\hat{H}_{K,C}$ in Eq. (36). Besides the CK coupling, $\hat{H}_{K,C}$ contains a Zeeman-like coupling to a local magnetic field of both $\tau^z(0)$ and $T^z$. Out of this two terms, the former one provides an additional phase shift to single-electron scattering amplitudes which is different in different leads. Again, this just quantitatively affects the calculation of the dc conductance, without invalidating the whole RG analysis of the Kondo interaction. The effects of the term $\propto T^z$ are discussed in appendix B. Here, we just mention that this term is not expected to substantially affect the Kondo physics as long as the applied field $B$ is much lower than the energy scale associated to the Kondo temperature $T_K = D_0 e^{-\frac{1}{\mathcal{J}_{C,0}}}$, with $\mathcal{J}_{C,0} = a J_C / v_D$. Thus, again, for the sake of simplicity, we focus onto a simple, isotropic charge-Kondo Hamiltonian, with over-all Kondo coupling equal to $J_C$. In this case, the running coupling strength associated to $J_C$ is given by $\mathcal{J}_C(D) = \frac{4To}{T_D}$. When discussing the dependence on $T$ of $G(T)$ for $D_0 \geq kT \geq kT_K$, we see that, compared to the SK effect, the main difference here lies in the nature of physical processes yielding a nonzero off-diagonal conductance $G_{a,\alpha}(T)$. While scattering from, say, lead-1 to lead-2 is still supported by an impurity spin-flip, this process now corresponds to a switch between local states with a net charge difference equal to $\pm 2e$. Thus, charge is conserved in a single scattering process only modulo 2 and, in particular, inter-lead scattering processes are Andreev-like, with an incoming particle from lead-1 emerging as an outgoing hole in lead-2. At variance, intra-lead scattering processes again correspond to particle-to-particle (hole-to-hole) scattering events.

\[
G(T) \approx \frac{e^2}{2\pi} \left[ 1 - \frac{\mathcal{J}_{C,0}}{1 - \mathcal{J}_{C,0} \ln(\frac{2To}{T_D})} \right]^2 \left[ 1 - \frac{\mathcal{J}_{C,0}}{1 - \mathcal{J}_{C,0} \ln(\frac{2To}{T_D})} \right]^2. \tag{43}
\]

As at the SK fixed point, again, when flowing towards the CK fixed point the charge-isospin density of the lead electrons at $x = 0$ “locks together” with the impurity spin, so to effectively cut the system into two separate parts. Again, this implies no dc transport in the particle-hole symmetric case and, in general, a reduction of the diagonal elements of $G$ by a factor $\cos^2(\delta)$ and a simultaneous suppression of the off-diagonal elements. To account for finite-$T$ corrections to the conductance tensor, we consider the leading boundary perturbation at the CK fixed point, that is, $\hat{H}_{K,C}$ in Eq. (36). Just as in the spin-Kondo case, we therefore obtain, to order $\alpha_C^3$, that the dc conductance tensor at the charge-Kondo fixed point, $G_{CK}(T)$, is given by

\[
G_{CK}(T) = \frac{e^2}{2\pi} \left\{ \cos^2(\delta) - \frac{2 \cos(2\delta)\alpha_C^2(kT)^2}{3\pi v^2} \right\} I. \tag{44}
\]

To summarize the results of this section, we see that on lowering $T$, just as in the SK case, the diagonal elements of the dc conductance tensor are suppressed by means of the marginally relevant Kondo interaction, while the relevance of the “effective” spin-flip processes induces nonzero off-diagonal elements in the conductance tensor. Since, now, a spin-flip process corresponds to a particle/hole from one lead as injected as a hole/particle into the other one, the currents induced in the two leads by means of a voltage bias applied to either one of them flow towards opposite directions and, therefore, $G_{a,\alpha}(T)$ and $G_{a,\alpha}(T)$ have opposite signs. Finite-$T$ contributions to $G$ at the CK fixed point are $\propto T^2$, again consistently with Nozierès Fermi liquid theory.
FIG. 4: Sketch of the possible single-particle transmission processes that can take place in either the SK, or the CK, phase, if an incoming particle from lead-1 hits the effective magnetic impurity. The ket represents the “impurity” state, so that a filled (empty) dot corresponds to the full (empty) state corresponding to the fermionic mode $a_1$ (left-hand dot) or $a_2$ (right-hand dot) in Eq.(9). In particular:

a) In the SK phase, the particle from lead-1 is transmitted as a particle towards the same lead. The impurity state is $|\uparrow\rangle$ before, and after, the scattering process;

b) Still in the SK phase, the particle from lead-1 is transmitted as a particle towards lead-2. The impurity state is $|\downarrow\rangle$ before the scattering process and switches to $|\uparrow\rangle$ after, consistently with total spin conservation;

c) In the CK phase, the particle from lead-1 is transmitted as a particle towards the same lead. The impurity state is $|\uparrow\rangle$ before, and after, the scattering process;

d) Still in the CK phase, the particle from lead-1 is transmitted as a hole towards lead-2 (crossed Andreev reflection). The impurity state is $|\downarrow\rangle$ before the scattering process and switches to $|\uparrow\rangle$ after, consistently with total charge conservation. Remarkably, total charge conservation forbids crossed Andreev reflection with the hole transmitted towards lead-1.

C. DC conductance in the decoupled lead phase

The DL phase is characterized by the irrelevant boundary interaction $H_{DL}$ in Eq.(27). In discussing its effects, as we have done above, we do not consider the effects of the marginal operators that it might contain. In appendix E.3 we show how, in the decoupled lead phase, our system is expected to behave as a pair of perfectly conducting channels, with the off-diagonal conductances equal to 0 and the finite-$T$ corrections to the diagonal conductances that are $\propto T^2$. Collecting the results of appendix E.3 altogether we, therefore, find that, in the DL phase, one gets

$$G(T) = \frac{\kappa^2}{2\pi} \left\{ 1 - \frac{2\kappa^2(kT)^2}{3\pi v^2} \right\} I.$$  \hspace{1cm} (45)

On top of the result in Eq.(45) it is also worth stressing that, since the boundary interaction describing the impurity throughout the decoupled lead phase is irrelevant, there is no “Kondo-like” expected crossover in this region, on lowering $T$. Thus, we may eventually conclude that, lowering $T$ at fixed system’s parameters, the Kondo-like phases are dramatically different from the non-Kondo like one in that first of all the former ones are characterized by a strong dependence on $T$ of the conductance tensor, as $T$ is lowered towards $T_K$, while the latter one just exhibits a mild dependence on $T$, on top of a conductance tensor almost not affected at all by the coupling of the leads to the superconducting island. Secondly, the fixed point (that is, $T = 0$) properties are dramatically different, as well. Indeed, at $T = 0$, when lying within either one of the Kondo phases, the system is expected to behave as a perfect insulator (zero dc conductance tensor). At variance, in the decoupled lead phase, all the effects of the coupling to the impurity are washed out as $T = 0$ and the system behaves as a perfect conductor, with the dc conductance tensor only limited by possible one-body scattering potential terms due to the coupling to the DSI.

The results of this section allow for fully mapping out the phase diagram of the system by looking at the dc conductance tensor of the device as a function of both the temperature $T$ and of the control parameter $\delta$, as we summarize in the following.
D. DC transport properties of the system in the $\delta - T$ parameter space

Referring to the phase diagram of Fig.6, in the following we use as tuning parameter $r = \delta v/(at^2)$. In particular, at $r = \pm 1$, the system undergoes two transitions between phases characterized by completely different zero-T transport properties: perfectly conducting, with no inter-wire conduction, for $-1 < r < 1$, insulating for $|r| > 1$. The three different phases can be well-characterized by looking at the dc conductance tensor as a function of $T$, from $kT \sim D_0$ all the way down to $T = 0$. In Fig.6 we show the expected behavior of the diagonal, as well as of off-diagonal, entries of the conductance tensor in the three phases. We note that, while there is a very mild dependence on $T$ of the diagonal conductance $G_{1,1}(T)$, with zero off-diagonal conductance, in the DL phase ($|r| < 1$), for $r > 1$ (spin-Kondo phase) $G_{1,1}(T)$ drops to 0 on lowering $T$, with a crossover scale determined by $T_K$ and $G_{1,2}(T)$ first rises and then drops to 0, as well, for $T < T_K$. In this phase, in the window of values of $T$ in which both are nonzero, $G_{1,1}(T)$ and $G_{1,2}(T)$ have the same sign. Finally, for $r < -1$ (charge-Kondo phase) $G_{1,1}(T)$ and $G_{1,2}(T)$ behave as in the spin-Kondo phase, but now, in the window of values of $T$ in which both are nonzero, they have opposite sign.

A complementary, alternative analysis can instead be performed by sweeping $r$ (that is, $\delta$) at fixed temperature $T$. Moving along a horizontal line in the $r - T$-plane, crossing the boundary at $r = \pm 1$ from within the DL phase, one enters either one of the Kondo phases. If the sweep is done at $T$ constant and larger than $T_K$ at $r = 1$, since, according to the scaling assumption for the Kondo effect, one expects the conductance to be a scaling function of $T/T_K(r)$, increasing $|r|$ (that is, increasing $T_K(r)$) is in principle equivalent to lowering $T$ towards $T_K$. So, one expects plots similar to the ones in Fig.5, but, now, using $r$ (or $\delta$) as a control parameter, which might in principle be easier to do, since we engineer our device so that $\delta$ can actually be used as a control parameter of the device.

Therefore, we may readily conclude how pertinently changing either $T$, or $\delta$ (or both), one can in principle probe the whole phase diagram of the system and, in particular, the remarkable possibility of switching from SK, to CK effect by acting upon one control parameter only.

VI. CONCLUDING REMARKS

In this paper, we propose how to engineer a tunable Kondo system which, depending on the value of in principle one parameter only, can either work as a spin-Kondo, or as a charge-Kondo impurity. Gauging the control parameter $\delta$, one moves the system from the SK to the CK phase, passing across an intermediate, DL phase, in which the Kondo impurity is effectively irrelevant for the low-temperature conduction properties of the system.

Within linear response theory, we derive the dc diagonal- and off-diagonal-conductance of the system as a function of the temperature throughout the whole phase diagram. As a result, we show how a comparison of the $T$-dependent diagonal and off-diagonal conductances provides an effective mean to identify, and distinguish from each other, the SK, the CK and the DL phase of the system. Therefore, we conclude that our proposed device can work as a controlled framework to realize and study in detail the so far quite elusive charge-Kondo effect.

To engineer our system, we employ a minimal setup, with only two spinless fermionic leads. In principle, nowadays technology allows for realizing spinless, one-dimensional electronic conduction channels at, for instance, semiconductor nanowires with a strong Rashba spin-orbit interaction and Zeeman energy, as well as edge states of a spin-Hall insulator. So, we expect it to be possible to realize our model, in a realistic experiment, with spinless leads, which would rule out unwanted complications on top of the minimal physics we describe here, such as onset of multichannel either SK, or CK, phases which, nevertheless, we plan to study in a future work.

Finally, it is also worth recalling how, within the CK phase, our Kondo impurity triggers off-diagonal conduction via a peculiar Crossed Andreev reflection between the two leads, which suggests that, in a “dual” setup, in which a Cooper pair is injected into the leads through the DSI, our system might realize an efficient long-distance electronic entangler.

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Appendix A: The conductance tensor in the case of a pointlike tunneling term

In this appendix, we briefly review the formula for the conductance in the case in which there is a simple pointlike tunneling interaction at $x = 0$ in the ladder. Specifically, resorting to the continuum formulation, we consider an Hamiltonian of the form
The scattering dynamics due to the scatterer at $x$ is written in the form

$$H_{\text{PL}} = -iv \sum_{a=1,2} \int_{-\ell}^{\ell} dx \left\{ \psi_{R,a}^\dagger(x) \partial_x \psi_{R,a}(x) - \psi_{L,a}^\dagger(x) \partial_x \psi_{L,a}(x) \right\} +$$

$$V_{d,\alpha} \sum_{a=1,2} \left\{ \psi_{R,a}^\dagger(0) \psi_{R,a}(0) + \psi_{L,a}^\dagger(0) \psi_{L,a}(0) \right\} + V_{d,\beta} \sum_{a=1,2} \left\{ \psi_{R,a}^\dagger(0) \psi_{L,a}(0) + \psi_{L,a}^\dagger(0) \psi_{R,a}(0) \right\} +$$

$$V_{o d,\alpha} \sum_{a=1,2} \left\{ \psi_{R,a}^\dagger(0) \psi_{R,a}(0) + \psi_{L,a}^\dagger(0) \psi_{L,a}(0) \right\} + V_{o d,\beta} \sum_{a=1,2} \left\{ \psi_{R,a}^\dagger(0) \psi_{L,a}(0) + \psi_{L,a}^\dagger(0) \psi_{R,a}(0) \right\} . \quad (A1)$$

A generic eigenmode of $H_{\text{PL}}$ with energy eigenvalue $\epsilon$ is written in the form

$$\Gamma_{\epsilon,q} = \sum_{a=1,2} \int_{-\ell}^{\ell} dx \left\{ u_{R,a,\epsilon,q}^\ast(x) \psi_{R,a}(x) + u_{L,a,\epsilon,q}^\ast(x) \psi_{L,a}(x) \right\} , \quad (A2)$$

with $q$ labelling the various independent solutions. On imposing the commutation relation $[\Gamma_{\epsilon,q}, H_{\text{PL}}] = \epsilon \Gamma_{\epsilon,q}$, one obtains the Schrödinger equations in the form

$$\epsilon u_{R,1(2),\epsilon,q}(x) = -iv \partial_x u_{R,1(2),\epsilon,q}(x) + \delta(x) \left\{ V_{d,\alpha} u_{R,1(2),\epsilon,q}(x) + V_{o d,\alpha} u_{L,1(2),\epsilon,q}(x) \right\}$$

\[ + \delta(x) \left\{ V_{d,\beta} u_{R,2(1),\epsilon,q}(x) + V_{o d,\beta} u_{L,2(1),\epsilon,q}(x) \right\} \]

$$\epsilon u_{L,1(2),\epsilon,q}(x) = iv \partial_x u_{L,1(2),\epsilon,q}(x) + \delta(x) \left\{ V_{d,\alpha} u_{L,1(2),\epsilon,q}(x) + V_{o d,\alpha} u_{R,1(2),\epsilon,q}(x) \right\}$$

\[ + \delta(x) \left\{ V_{d,\beta} u_{L,2(1),\epsilon,q}(x) + V_{o d,\beta} u_{R,2(1),\epsilon,q}(x) \right\} . \quad (A3)\]

The scattering dynamics due to the scatterer at $x = 0$ is fully characterized by the S-matrix $S$. When linearizing the dispersion relation of the fermionic fields around the Fermi points, the S-matrix becomes independent of the energy $\epsilon$. Therefore, dropping the dependence on $\epsilon$ of the S-matrix elements and denoting with $r_{(a,b),L/R}$ the amplitude for a particle coming from the left/right hand side within lead $a$ to backscatter within lead $b$ and with $t_{(a,b),L/R}$ the amplitude for a particle coming from the left/right hand side within lead $a$ to propagate towards lead $b$, one obtains

$$S = \begin{bmatrix} r_{(1,1),L} & t_{(1,1),R} & r_{(1,2),L} & t_{(1,2),R} \\ t_{(1,1),L} & r_{(1,1),R} & t_{(1,2),L} & r_{(1,2),R} \\ r_{(2,1),L} & t_{(2,1),R} & r_{(2,2),L} & t_{(2,2),R} \\ t_{(2,1),L} & r_{(2,1),R} & t_{(2,2),L} & r_{(2,2),R} \end{bmatrix} . \quad (A4)$$

The chiral field operators can therefore be rewritten in terms of the energy eigenmodes and eigenfunctions as

$$\begin{bmatrix} \psi_{R,1}(x) \\ \psi_{L,1}(x) \\ \psi_{R,2}(x) \\ \psi_{L,2}(x) \end{bmatrix} = \sum_{\epsilon} \begin{bmatrix} u_{R,1,\epsilon,(1),L}(x) & u_{R,1,\epsilon,(1),R}(x) & u_{R,1,\epsilon,(2),L}(x) & u_{R,1,\epsilon,(2),R}(x) \\ u_{L,1,\epsilon,(1),L}(x) & u_{L,1,\epsilon,(1),R}(x) & u_{L,1,\epsilon,(2),L}(x) & u_{L,1,\epsilon,(2),R}(x) \\ u_{R,2,\epsilon,(1),L}(x) & u_{R,2,\epsilon,(1),R}(x) & u_{R,2,\epsilon,(2),L}(x) & u_{R,2,\epsilon,(2),R}(x) \\ u_{L,2,\epsilon,(1),L}(x) & u_{L,2,\epsilon,(1),R}(x) & u_{L,2,\epsilon,(2),L}(x) & u_{L,2,\epsilon,(2),R}(x) \end{bmatrix} \begin{bmatrix} \Gamma_{\epsilon,(1,L)} \\ \Gamma_{\epsilon,(1,R)} \\ \Gamma_{\epsilon,(2,L)} \\ \Gamma_{\epsilon,(2,R)} \end{bmatrix} . \quad (A5)$$

Using the field operators in Eq. (A5) to write down the dc conductance tensor, we find that, given the symmetry assumptions behind the Hamiltonian in Eq. (A1), the conductances of interest to us are given by

$$G_{1,1} = \frac{e^2}{2\pi} |t_{(1,1),L}|^2 = \frac{e^2}{2\pi} |t_{(1,1),R}|^2 = G_{2,2} = \frac{e^2}{2\pi} |t_{(2,2),L}|^2 = \frac{e^2}{2\pi} |t_{(2,2),R}|^2$$

$$G_{1,2} = \frac{e^2}{2\pi} |t_{(1,2),L}|^2 = \frac{e^2}{2\pi} |t_{(1,2),R}|^2 = G_{2,1} = \frac{e^2}{2\pi} |t_{(2,1),L}|^2 = \frac{e^2}{2\pi} |t_{(2,1),R}|^2 . \quad (A6)$$

In the specific case of a pointlike scatterer described by $V_{d,\alpha} = V_{o d,\alpha} = V$ and by $V_{d,\beta} = V_{o d,\beta} = 0$, we obtain

$$S = \frac{1}{1 + i\rho} \begin{bmatrix} -i\rho & 1 & 0 & 0 \\ 1 & -i\rho & 0 & 0 \\ 0 & 0 & -i\rho & 1 \\ 0 & 0 & 1 & -i\rho \end{bmatrix} , \quad (A7)$$

with $\rho = 2V/v$, which implies

$$G_{1,1} = G_{2,2} = \frac{e^2}{2\pi} \frac{1}{1 + \rho^2} , \quad G_{1,2} = G_{2,1} = 0 . \quad (A8)$$
Appendix B: Schrieffer-Wolff transformation and derivation of the effective Kondo Hamiltonian

It general, the Schrieffer-Wolff procedure, when applied to a generic Hamiltonian $\hat{H}$, allows for recovering a reduced, effective Hamiltonian, acting on a limited subspace of the Hilbert space, typically determined as the subspace spanned by a certain set of low-lying eigenstates of $\hat{H}$. To be specific, let us consider a generic time-independent Schrödinger equation

$$\hat{H} |\Psi\rangle = E |\Psi\rangle,$$

and suppose we want to “project” it onto a pertinent subspace $G$. Let $P_G$ be the projector on $G$. To lowest order in the “off-diagonal” matrix elements connecting $G$ to its orthogonal subspace, we obtain

$$P_G \hat{H} P_G \{ P_G |\Psi\rangle \} + P_G \hat{H} \{ I - P_G |\Psi\rangle \} ( I - P_G |\Psi\rangle \} = E \{ P_G |\Psi\rangle \}$$

$$\{ I - P_G |\Psi\rangle \} \hat{H} \{ I - P_G |\Psi\rangle \} P_G \{ P_G |\Psi\rangle \} + \{ I - P_G |\Psi\rangle \} \hat{H} P_G \{ P_G |\Psi\rangle \} = E \{ I - P_G |\Psi\rangle \}.$$  \hspace{1cm} \text{(B2)}

Putting together Eqs. (B2), one eventually obtains the “projected” Schrödinger equation

$$\begin{align*}
\left\{ P_G \hat{H} P_G + P_G \hat{H} \{ I - P_G |\Psi\rangle \} \{ I - P_G |\Psi\rangle \} \right\} P_G |\Psi\rangle &= E P_G |\Psi\rangle.
\end{align*}$$

Dividing the Hamiltonian as the sum of a non perturbed contribution plus a perturbation term $H = H_0 + H_1$, we can project it onto the ($n$ times degenerate) groundstate subspace of $H_0$

$$H_0 |\psi_i\rangle = E_0 |\psi_i\rangle, \quad i = 1, ... n \, .$$

(B4)

to obtain the Brillouin-Wigner perturbation expansion

$$H_{Eff} = \sum_{i,j} h_{i,j} |\psi_i\rangle \langle \psi_j| \, .$$

(B5)

with

$$h_{i,j} = \langle \psi_i | H_0 |\psi_j\rangle + \langle \psi_i | H_1 |\psi_j\rangle + \sum_k \frac{\langle \psi_i | H_1 |\varphi_k\rangle \langle \varphi_k | H_1 |\psi_j\rangle}{E_0 - E_k} \, ,$$

(B6)

where the sum over $k$ runs on the low energy excited states of the unperturbed Hamiltonian, such that $H_0 |\varphi_i\rangle = E_i |\varphi_i\rangle$. Eqs. (B4) define a systematic procedure. A straightforward implementation of the procedure we illustrate here, allows for recovering the effective Kondo Hamiltonians in Eqs. (B4).

Another effective use of the Schrieffer-Wolff transformation leads, to the residual interaction at both the SK, and the CK fixed point. To illustrate our derivation, we consider the SK fixed point. For the sake of generality, we consider an anisotropic lattice version of the lattice SFK Hamiltonian in the form

$$H_{lat;S} = \sum_{a=1,2} \sum_{j=0}^{\ell-1} \{ -J \sum c_{j,a}^\dagger c_{j+1,a} + c_{j+1,a}^\dagger c_{j,a} \} - J S_z S_z \{ S_0^+ S^- + S_0^- S^+ \} + J S_{\parallel} S_{\parallel}.$$

(B7)

The groundstate at the SK fixed point minimizes the boundary interaction energy encoded in the Hamiltonian in Eq. (B7) as $J S_{\parallel}, J S_{\parallel} \to \infty$. To explicitly provide such a state, in the following, we denote by $| 0 \rangle, | \uparrow \rangle, | \downarrow \rangle, | \uparrow \downarrow \rangle$ the two eigenstates of $S_z$, with $| 0 \rangle, | \uparrow \rangle, | \downarrow \rangle, | \uparrow \downarrow \rangle$ the states in which, respectively, the site $j = 0$ is empty in both chain, is filled with one electron on chain-1 and empty on chain-2, is filled with one electron on chain-2 and empty on chain-1, is filled with one electron in both chains. As a result of the hybridization with the impurity spin, the locally hybridized states, which we list below together with the corresponding energies, are generated

Local singlet : $| S \rangle = \frac{1}{\sqrt{2}} \{ | \downarrow, \uparrow \rangle - | \uparrow, \downarrow \rangle \} \, ; \quad (\epsilon_S = -\frac{1}{4}(S_{\parallel}^2 + 2J S_{\parallel}))$

Local doublet $D : | D_\sigma \rangle = | 0, \sigma \rangle \, ; \quad (\epsilon_D = 0)$

Local doublet $D : | \tilde{D}_\sigma \rangle = | \uparrow, \sigma \rangle \, ; \quad (\epsilon_D = 0)$

Local triplet $T_1 : | T_1 \rangle = | \uparrow, \uparrow \rangle \, ; \quad (\epsilon_{T_1} = \frac{J S_{\parallel}}{4})$

Local triplet $T_{-1} : | T_{-1} \rangle = | \downarrow, \downarrow \rangle (\epsilon_{T_{-1}} = (\epsilon_{T_1} = \frac{J S_{\parallel}}{4})$

Local triplet $T_0 : | T_0 \rangle = \frac{1}{\sqrt{2}} \{ | \downarrow, \uparrow \rangle + | \uparrow, \downarrow \rangle \} \, ; \quad (\epsilon_{T_0} = \frac{1}{4}(-J S_{\parallel} + 2J S_{\parallel}))$.  \hspace{1cm} \text{(B8)}
At large values of $J_{S,\perp}, J_{S,\parallel}$, the system lies within the $|S\rangle$ local state, with the higher-energy states in Eqs. (B8) playing a role in the allowed physical processes only as virtual states. Taking this into account, one can therefore go through a systematic Schrieffer-Wolff transformation, by using as “unperturbed” Hamiltonian $\mathcal{H}_0 = \sum_X \epsilon_X |X\rangle\langle X|$, with $\{|X\rangle\}$ being the set of states listed in Eqs. (B8), and as “perturbing” Hamiltonian $\mathcal{H}_t$, describing the coupling between site-0 and site-1 of each lead, and given by

$$\mathcal{H}_t = t\{c_{1,1}^\dagger c_{0,1}^\dagger + c_{1,2}^\dagger c_{0,2}^\dagger - c_{1,1}^\dagger c_{0,1} - c_{1,2}^\dagger c_{0,2}\} \ . \quad \text{(B9)}$$

As a result one finds that the first nontrivial boundary interaction operator, $\hat{H}_{K,S}$, arises to fourth-order in $\mathcal{H}_t$, and is given by

$$\hat{H}_{K,S} = \sum_{\{X,X\'} = (D_x, \bar{D}_x)} \sum_{\{Y\} = \{T_N\}} \left[ \mathcal{P}_S \frac{\mathcal{H}_t}{\epsilon_S - \epsilon_X} \mathcal{P}_X \frac{\mathcal{H}_t}{\epsilon_S - \epsilon_Y} \mathcal{P}_Y \frac{\mathcal{H}_t}{\epsilon_S - \epsilon_{X'}} \mathcal{P}_{X'} \mathcal{P}_S \right] , \quad \text{(B10)}$$

with $\mathcal{P}_X$ being the projector onto the state spanned by the local state $|X\rangle$. Plugging into Eq. (B10) the explicit expressions for $\mathcal{H}_t$ and for the various energies, one finds

$$\hat{H}_{K,S} = -\zeta_{t,\perp}\{[c_{1,2}, c_{1,1}^\dagger][c_{1,1}, c_{1,2}^\dagger] + [c_{1,1}, c_{1,2}^\dagger][c_{1,2}, c_{1,1}^\dagger]\} - \zeta_{t,\parallel}\{[c_{1,2}, c_{1,2}^\dagger] - [c_{1,1}, c_{1,1}^\dagger]\}^2 , \quad \text{(B11)}$$

with

$$\zeta_{t,\perp} = \frac{t^4}{4\epsilon_S^2(\epsilon_T - \epsilon_S)} , \quad \zeta_{t,\parallel} = \frac{t^4}{4\epsilon_S^2(\epsilon_T - \epsilon_S)} . \quad \text{(B12)}$$

$\hat{H}_{K,S}$ in Eq. (B11) is the lattice version of the operator describing the leading perturbation at Nozierès Fermi-liquid fixed point as derived in e.g. appendix D of Ref. [6]. Once resorting to the continuum field framework by means of e.g. the low-energy, long-wavelength expansions in Eqs. (25,26) and inserting the continuum fields, supplemented with the appropriate fixed point boundary conditions, in Eq. (B12), one recovers the continuum formula for the leading boundary perturbation at the SK fixed point, Eq. (32). Similarly, at the CK fixed point, one obtains Eq. (36) of the main text.

Appendix C: Derivation of $\hat{H}_{DL}$ in the disconnected lead phase

In this appendix, we briefly discuss the derivation of $\hat{H}_{DL}$ in Eq. (23) as the leading boundary interaction describing the impurity within the DL phase. To do so, we start by “artificially” introducing two independent parameters in the DSI Hamiltonian, which we accordingly rewrite as

$$\hat{H}_{\text{Island}} = -2\delta_1 \sum_{a=1,2} a_a^\dagger a_a + 4\delta_2 a_1^\dagger a_1 a_2^\dagger a_2 \ . \quad \text{(C1)}$$

Clearly, in order to recover physically meaningful results, one has to eventually set $\delta_1 = \delta_2 = \delta$. However, Eq. (C1) comes out to be useful in that it allows for exactly accounting for at least part of the $\delta$-depending terms in $\hat{H}_{\text{Island}}$.

Identifying $\delta_2$ as our perturbative parameter, we note that, leaving aside the density-density interaction (that is, setting $\mu_d = \mu_{\text{cd}} = 0$), the system Hamiltonian can be written as $\hat{H}_0 = \sum_a \hat{H}_a$, with

$$\hat{H}_a = -J \sum_{j=-\ell}^{\ell-1} \{c_{j,a}^\dagger c_{j+1,a} + c_{j+1,a}^\dagger c_{j,a}\} - \mu \sum_{j=-\ell}^\ell c_{j,a}^\dagger c_{j,a} - t\{c_{0,a}^\dagger a_a + a_a^\dagger c_{0,a}\} - 2\delta_1 a_a^\dagger a_a \ . \quad \text{(C2)}$$

The right-hand side of Eq. (C2) contains a quadratic Hamiltonian defined over an $2\ell + 2$-site lattice. This can be exactly diagonalized by means of the eigenmodes $\Gamma_{k,a}$, defined as

$$\Gamma_{k,a} = \sum_{j=-\ell}^\ell u_{j,a,k} c_{j,a} + \xi_{k,a} a_a \ . \quad \text{(C3)}$$
and (on imposing periodic boundary conditions at \( j = \pm \ell \)) the wavefunction satisfying the lattice Schrödinger equation

\[
\begin{align*}
\epsilon_k u_{j,k,a} &= -J\{u_{j+1,k,a} + u_{j-1,k,a}\} - \mu u_{j,k,a}, \quad (j \neq 0) \\
\epsilon_k u_{j,k,0} &= -J\{u_{1,k,a} + u_{-1,k,a}\} - \mu u_{0,k,a} - t\xi_{k,a} \\
\epsilon_k \xi_{k,a} &= -2\delta_1 \xi_{k,a} - tu_{0,k,a}.
\end{align*}
\] (C4)

To solve Eqs. (C4), we make the ansatz

\[
\begin{align*}
u_{j,k,a} &= \alpha_{k,a}^e e^{ikj} + \beta_{k,a}^e e^{-ikj}, \quad (-\ell \leq j < 0) \\
u_{j,k,a} &= \alpha_{k,a}^e e^{ikj} + \beta_{k,a}^e e^{-ikj}, \quad (0 < j \leq \ell) ,
\end{align*}
\] (C5)

which yields \( \epsilon_k = -2J \cos(k) - \mu \) and, in addition, the conditions at \( j = 0 \) encoded in

\[
\begin{align*}
u_{0,k,a} &= \alpha_{k,a}^e + \beta_{k,a}^e \\
u_{0,k,a} &= \alpha_{k,a}^e + \beta_{k,a}^e \\
\{2J \cos(k) + \mu\} \nu_{0,k,a} &= J\{\alpha_{k,a}^e e^{-ik} + \beta_{k,a}^e e^{ik} + \alpha_{k,a}^e e^{ik} + \beta_{k,a}^e e^{-ik}\} + t\xi_{k,a} \\
\{2J \cos(k) + \mu + 2\delta_1\} \xi_{k,a} &= tu_{0,k,a}.
\end{align*}
\] (C6)

Once the system in Eqs. (C6) has been explicitly solved, one finds that, at a given \( k \), one obtains

\[
u_{0,k,a} = \frac{2iv\delta_1}{t^2 - 2\mu\delta_1 + 2iv\delta_1}, \quad \xi_{k,a} = \frac{ivt}{t^2 - 2\mu\delta_1 + 2iv\delta_1},
\] (C7)

which, given the relations

\[
c_{0,a} = \sum_k u_{0,k,a}^* \Gamma_{k,a}, \quad a_a = \sum_k \xi_{k,a}^* \Gamma_{k,a}
\] (C8)

implies

\[
a_a \approx t^2 \delta_1 c_{0,a} \Rightarrow a_a^\dagger a_a \approx \frac{t^2}{4\delta_1} \delta_1 c_{0,a}^\dagger c_{0,a}.
\] (C9)

Clearly, Eq. (C9) applies to energy scales lower than \( t \). It implies \( a_a^\dagger a_a \propto c_{0,a}^\dagger c_{0,a} \). Therefore, the whole impurity interaction Hamiltonian can be traded for a density-density interaction one, \( H_{DL} \), of the form

\[
H_{DL} = \kappa c_{0,1}^\dagger c_{0,1}c_{0,2}^\dagger c_{0,2} + \sum_{a=1,2} \lambda_a c_{0,a}^\dagger c_{0,a}c_{0,a},
\] (C10)

with \( \kappa, \lambda_1, \lambda_2 \) parameters respectively corresponding to the inter-wire local density-density interaction and to the residual intra-wire local one-body potentials and, clearly, \( \kappa \propto \delta_2 \). \( H_{DL} \) in Eq. (C10) is the Hamiltonian we used in the main text to discuss the effective impurity dynamics in the DL region of the phase diagram.

**Appendix D: Renormalization group equations for the running coupling in the effective impurity Hamiltonians**

In this appendix, we concisely review the derivation and the solution of the RG equations for the various boundary Hamiltonians describing the impurity dynamics in the various regions of the system.

To begin with, we consider \( H_{KS} \) in Eq. (D1) and \( H_{KC} \) in Eq. (D2). For the sake of the discussion, here we leave aside purely marginal terms (that is, one-body potential scattering terms), whose nonuniversal effects we account for when actually computing the dc conductance tensor of the system. At the same time, we neglect local fields acting on the effective spin impurity, whose effects we briefly discuss by the end of this appendix. Accordingly, in order to encompass in our analysis both cases, we henceforth consider the RG equations for the running couplings associated a generic anisotropic Kondo Hamiltonian \( H_{AN} \), given by

\[
H_{AN} = J_+ \{\psi_1^\dagger(0) \psi_2(0) S^- + \psi_2^\dagger(0) \psi_1(0) S^+\} + J_z \left\{ \frac{\psi_1^\dagger(0) \psi_1(0) - \psi_2^\dagger(0) \psi_2(0)}{2} \right\} S^z,
\] (D1)
with \( \psi_1(x), \psi_2(x) \) being one-dimensional, chiral fermionic fields. \( J_{\perp}, J_{\parallel} \) are respectively the transverse and the longitudinal Kondo coupling strengths. On defining the associated dimensionless running coupling strengths \( J_{\perp}(D) = \frac{J_{\perp}}{aJ} \) and \( J_{\parallel}(D) = \frac{aJ}{\sqrt{D}} \), the derivation of the RG equations for those running coupling has a long story and goes back to the original works on the subject\(^{2,68-70}\). Specifically, on varying the energy cutoff \( D \), one obtains that the corresponding variation of the running couplings is determined by the equations

\[
\frac{dJ_{\parallel}(D)}{d\ln \left( \frac{D}{D_0} \right)} = J_{\parallel}(D) J_{\parallel}(D) \quad \text{and} \quad \frac{dJ_{\perp}(D)}{d\ln \left( \frac{D}{D_0} \right)} = J_{\perp}^2(D) \ .
\]

(D2)

The system in Eqs.\((\mathbf{D2})\) corresponds to the set of the standard Kosterlitz-Thouless RG equations. To solve it, one defines the RG invariant \( H = -J_{\parallel}^2(D) + J_{\perp}^2(D) \). In particular, for \( H = 0 \), one recovers the standard poor man’s result for the RG equations in the case of isotropic Kondo effect\(^{2,70}\). In this case, one readily finds the solution in the form

\[
J_{\perp}(D) = J_{\parallel}(D) = \frac{J_{\parallel}(D_0)}{1 - J_{\perp}(D_0) \ln \left( \frac{D_0}{D} \right)} \ ,
\]

(D3)

with the corresponding Kondo scale \( D_K \) given by

\[
D_K \sim D_0 e^{-\sqrt{\frac{\ln \pi}{2}}} \ .
\]

(D4)

At a generic value of \( H \), as we show in Fig.\(\text{B}\) there are three relevant regions in the half-plane \( J_{\parallel}, J_{\perp} > 0 \). In detail, we have:

- **Region I**: this is defined for \( J_{\parallel}(D_0) > 0 \) and \( H < 0 \). In this case, the integrated RG equations yield

\[
J_{\parallel}(D) = \sqrt{|H|} \left\{ \frac{\sqrt{\mathcal{J}_{\parallel}(D_0)} + \sqrt{|H|} + (\mathcal{J}_{\parallel}(D_0) - \sqrt{|H|}) (\frac{D}{D_0})^{\sqrt{|H|}}}{\mathcal{J}_{\parallel}(D_0) + \sqrt{|H|} - (\mathcal{J}_{\parallel}(D_0) - \sqrt{|H|}) (\frac{D}{D_0})^{\sqrt{|H|}}} \right\} \quad \text{and} \quad J_{\perp}(D) = 2\sqrt{|H|} \left\{ \sqrt{\mathcal{J}_{\parallel}^2(D_0) - |H| (\frac{D}{D_0})^{\sqrt{|H|}}} \right\} .
\]

(D5)

Both running couplings increase as \( D_0/D \) gets large. Eventually, they hit a diverging point at the scale \( D = D_{K_T}^{(1)} \), with

\[
D_{K_T}^{(1)} = D_0 \left\{ \frac{J_{\parallel}(D_0) - \sqrt{|H|}}{J_{\parallel}(D_0) + \sqrt{|H|}} \right\}^{\frac{1}{\sqrt{|H|}}} .
\]

(D6)

- **Region II**: this is defined for \( H > 0 \). In this case, one obtains

\[
J_{\parallel}(D) = \sqrt{H} \tan \left\{ \frac{\sqrt{\mathcal{J}_{\parallel}(D_0)}}{\sqrt{H}} + \sqrt{H} \ln \left( \frac{D_0}{D} \right) \right\} \quad \text{and} \quad J_{\perp}(D) = \frac{\sqrt{H}}{\cos \left\{ \frac{\sqrt{\mathcal{J}_{\parallel}(D_0)}}{\sqrt{H}} + \sqrt{H} \ln \left( \frac{D_0}{D} \right) \right\}} .
\]

(D7)

In the case \( G_{\parallel}(D_0) < 0 \), Eqs.\((\mathbf{D7})\) imply a crossing of the vertical axis at the scale \( D_{\text{cross}} \) defined as

\[
D_{\text{cross}} = D_0 \exp \left[ -\frac{1}{\sqrt{H}} \left| \text{atan} \left( \frac{J_{\parallel}(D_0)}{\sqrt{H}} \right) \right| \right] .
\]

(D8)

Both couplings diverge at the scale \( D = D_{K_T}^{(2)} \), with

\[
D_{K_T}^{(2)} = D_0 \exp \left[ -\frac{1}{\sqrt{H}} \left( \frac{\pi}{2} - \text{atan} \left( \frac{J_{\parallel}(D_0)}{\sqrt{H}} \right) \right) \right] .
\]

(D9)
The only nontrivial interaction term is the direct local density-density coupling, therefore introduce the corresponding running coupling

\[ \mathcal{J}_z(D) = \sqrt{|H|} \left( \frac{|\mathcal{J}_z(D_0)| - \sqrt{|H|} + \sqrt{|H|} \left( \frac{D_0}{D} \right)^2 \sqrt{|H|} \right) \]

\[ \mathcal{J}_\perp(D) = 2\sqrt{|H|} \left( \frac{\sqrt{\mathcal{J}_z^2(D_0) - |H|} \left( \frac{D_0}{D} \right)^{\sqrt{|H|}}}{ -|\mathcal{J}_z(D_0)| + \sqrt{|H|} + \sqrt{|H|} \left( \frac{D_0}{D} \right)^2 \sqrt{|H|} \right) \].

(D10)

In this case, we see that the flow is no more towards a point at \( \infty \), but we rather get

\[ \lim_{D \to \infty} \left[ \mathcal{J}_z(D) \right] = \left[ -\sqrt{|H|} \right] \ . \]

(D11)

From the analysis we perform above, it is natural to associate the onset of the Kondo regime (and the corresponding emergence of a dynamically generated energy scale) to region I and II, while region III is characterized by a flow towards a manifold of “trivial” fixed points, continuously parametrized by \( H \). In Fig.6 we provide a sketch of the RG trajectories for \( \mathcal{J}_1(D) \) and \( \mathcal{J}_z(D) \) by particularly evidencing how, in the “Kondo regions” I and II, both running couplings flow to \( \infty \). While this result is important for building a description of the corresponding Kondo fixed point, we now briefly review what are the possible effects of the “non-Kondo” terms in Eqs.\((17,18)\). A first additional term potentially appearing in the Kondo boundary Hamiltonian is the one-body, local scattering potential, which may just marginally change the single-particle phase shifts at the impurity location and, correspondingly, slightly renormalize the dc conductance tensor with the Kondo interaction turned off, without essentially affecting the Kondo physics. Also, in the CK regime, one may get a term corresponding to an effective, local magnetic field along the \( z \)-direction, either coupled to the impurity spin, or to the electronic spin density at the impurity location (or both). In general, this terms are known not to substantially affect the Kondo physics as long as the applied field \( B \) is much lower than the energy scale associated to the Kondo temperature\((17,62,63)\). In fact, this is the assumption we make here, as the effective \( B \) field is determined by the direct density-density interaction within the wires, which is expected to be much lower than the other energy scales in the system. Therefore, throughout all this paper, we consistently neglect the corresponding contributions to the boundary Hamiltonian effectively describing the impurity dynamics.

To conclude the discussion of this appendix, we now consider \( H_{DL} \) in Eq.\((23)\). Besides the one-body local scattering potentials, the only nontrivial interaction term is the direct local density-density coupling, \( \kappa \). To deal with it, we therefore introduce the corresponding running coupling \( K(D) = \left( \frac{D_0}{D} \right)^{-1} \kappa \). The RG equation for \( K(D) \) to leading order in the running coupling is therefore given by

\[ \frac{dK(D)}{d \ln \left( \frac{D_0}{D} \right)} = -K(D) \ . \]

(D12)

which shows that this term is irrelevant and that, on lowering the cutoff, the corresponding running coupling scales as \( D/D_0 \). Accordingly, its effects can be consistently accounted for within a standard perturbative analysis in the coupling itself, which is what we made, when e.g. discussing the effects of \( H_{DL} \) on the dc-conductance tensor of the system.

Appendix E: DC conductance tensor: details about the derivation in the various regions of the phase diagram of the system

In this appendix, we explicitly compute the DC conductance tensor of our system within linear response theory. The starting point is the formula for the current-density operators in the low-energy, long-wavelength approximation, at imaginary time \( \tau \), given by

\[ j_1(x, \tau) = ev \left\{ \psi^\dagger_{R,1}(x, \tau) \psi_{R,1}(x, \tau) - \psi^\dagger_{L,1}(x, \tau) \psi_{L,1}(x, \tau) \right\} \]

\[ j_2(x, \tau) = ev \left\{ \psi^\dagger_{R,2}(x, \tau) \psi_{R,2}(x, \tau) - \psi^\dagger_{L,2}(x, \tau) \psi_{L,2}(x, \tau) \right\} \ . \]

(E1)

The dc-conductance tensor between points \( x' \) and \( x \), \( G(x, x') \), as the ratio between the current flowing at \( x \) and the voltage bias between the two points, in the zero-frequency, zero-bias limit. As a result of linear response theory, one obtains for the corresponding \( a,b \)-component the result
\[ G_{a,b}(x, x') = \lim_{\Omega \to 0} \left\{ \frac{\Gamma_{a,b}(x, x'; i\Omega)}{|\Omega|} \right\}, \] 

(E2)

with

\[ \Gamma_{a,b}(x, x'; i\Omega) = -\int_0^\beta d\tau e^{i\Omega\tau} \langle T_\tau j_a(x, \tau) j_b(x', 0) \rangle, \] 

(E3)

with \( T_\tau \) being the imaginary time-ordering operator. In the following, we use Eqs.(E3) to derive the conductance tensor throughout the various regions of the phase diagram of the system. In the absence of the impurity, one gets

\[
\begin{align*}
\Gamma_{1,1}(x, x'; i\Omega) &= \Gamma_{2,2}(x, x'; i\Omega) = \frac{e^2|\Omega|}{2\pi} e^{-\frac{\alpha(x-x')}{\pi}} \\
\Gamma_{1,2}(x, x'; i\Omega) &= \Gamma_{2,1}(x, x'; i\Omega) = 0,
\end{align*}
\] 

(E4)

which implies for the dc-conductance tensor \( G^{(0)} \) the expression

\[
G^{(0)} = \frac{e^2}{2\pi} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\] 

(E5)

Connecting the leads to the superconducting island results into either a SK, or a CK Hamiltonian, or to a localized density-density interaction. We now investigate the effects of these three possible Hamiltonians on the dc-conductance tensor, starting with the Kondo-like interactions. To do so, it is useful to rewrite the operators \( j_a(x, \tau) \) in terms of the even- and odd-parity fermion operators \( \psi_{e,a}(x, \tau) \) and \( \psi_{o,a}(x, \tau) \) defined in Eqs.(26). As a result, one obtains

\[
\begin{align*}
\langle T_\tau j_a(x, \tau) \rangle &= \frac{e^2}{2} \sum_{u=\pm 1} \left\{ \psi_{e,a}^\dagger(ux + iv\tau) \psi_{e,a}(ux + iv\tau) + \psi_{o,a}^\dagger(ux + iv\tau) \psi_{o,a}(ux + iv\tau) \right\} \\
n &- \frac{e^2}{2} \sum_{u=\pm 1} u \left\{ \psi_{e,a}^\dagger(ux + iv\tau) \psi_{o,a}(ux + iv\tau) + \psi_{o,a}^\dagger(ux + iv\tau) \psi_{e,a}(ux + iv\tau) \right\} \\
n &= \sum_{r,\tau'=e,o} j(r,\tau'; a)(x, \tau).
\end{align*}
\] 

(E6)

Next, we note that, throughout all the three regions of the boundary phase diagram, the boundary interaction Hamiltonian depends on the \( e \)-fields only and, therefore, so does the corresponding imaginary time action. Therefore, retaining only connected diagrams, at a generic order \( N \) of the perturbative expansion we find that the current-current correlation function is corrected to

\[
\delta^{(N)}(x, \tau; x', \tau') = -\frac{1}{N!} \langle T_\tau j_{e,a}(x, \tau) j_{e,a'}(x', \tau') S_B^N \rangle
\]

\[
-\frac{1}{N!} \langle T_\tau j_{o,a}(x, \tau) j_{o,a'}(x', \tau') S_B^N \rangle,
\]

(E7)

with \( B = (K,S), (K,C), DL \). We now the contributions to the right-hand side of Eq.(E7) in all three the cases of interest, up to second-order in the boundary interaction.

1. Second-order corrections to the conductance tensor in the spin-Kondo phase

The SK interaction is described by the Euclidean action \( S_{K,S} \) which is given by (apart for non-isotropic terms that, as discussed before, just have a minor effect on the physical properties of the system)

\[
S_{K,S} = J_S \int_0^\beta d\tau \sum_{a,b=1,2} \psi_{e,a}^\dagger(i\tau) \hat{S}_{a,b} \psi_{e,b}(i\tau).
\]

(E8)

Typically, whether the Kondo Hamiltonian is isotropic, or not, the first, nontrivial corrections to the conductance arise to second-order in the Kondo coupling. To that order, one obtains that the current-current correlation functions are corrected by \( \delta^{(2)}(x, \tau; x', \tau') \), given by
The conductance tensor corresponding to the SK fixed point we identified throughout the analysis of appendix D. As to compute the corresponding corrections to the conductance tensor, one has to move to Matsubara-Fourier space.

To do so, we note that one obtains

\[
\int_0^\beta d\tau_1 \, g^2(u,\tau;0,\tau_1) = \int_0^\beta d\tau_2 \, g^2(u'x',\tau;0,\tau_2) = 0 ,
\]

and

\[
\int_0^\beta d(\tau - \tau') \, e^{i\Omega(\tau - \tau')} \int_0^\beta d\tau_1 \, d\tau_2 \, g(u,\tau;u'x',\tau')g(u,\tau;0,\tau_1)g(0,\tau_1;0,\tau_2)g(0,\tau_2;u'x',\tau') = \\
\frac{e^{-\frac{i}{\beta \psi^4}(u-u')^2}}{2 \pi v^4} \sum_{i\omega} \{\theta(\omega)\theta(-\Omega)\theta(u)\theta(-u') + \theta(-\omega)\theta(-\Omega + \omega)\theta(-u)\theta(u')\} = \\
\frac{\Omega}{2 \pi v^4} \{\theta(u)\theta(-u') + \theta(-u)\theta(u')\} e^{\frac{i}{\beta v^4}(u-u')^2} .
\]

Considering that, by definition, we construct the dc-conductance tensor \(G_{a,b}\) from the ratio between the current flowing after the impurity (say at the right-hand side of the impurity) in response to a voltage drop \(V\) applied across the impurity (so that we may assume a voltage drop \(V\) fully located at the left-hand side of the impurity), we have to evaluate \(G_{a,b}(x,x')\) for \(x > 0\) and \(x' < 0\). Accordingly, this implies, for the second-order correction to the conductance tensor, \(\delta G^{(2)}\), the result

\[
\delta G^{(2)} = -\frac{e^2}{2\pi} \frac{J_S^2}{v^2} \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \Rightarrow G \approx \frac{e^2}{2\pi} \left[ \begin{array}{cc} 1 - \frac{J_S^2}{\frac{J_S^2}{\psi^2}} & \frac{J_S^2}{\psi^2} \\ \frac{J_S^2}{\psi^2} & 1 - \frac{J_S^2}{\frac{J_S^2}{\psi^2}} \end{array} \right] .
\]

Apparently, the right-hand side of Eq. (E12) only depends on the “bare” squared interaction strength \(J_S^2\). Indeed, as no anomalous scaling dimensions arise for the current operator along the RG trajectories associated to the Kondo problem, the standard mean to incorporate the nontrivial RG trajectory within the formulas for the conductance consists in substituting the bare with the renormalized coupling strength, \(J_S(D)\). In the case of isotropic Kondo interaction, which we are considering here, this is defined according to Eqs. (D3, D4), from which we see that the nontrivial RG flow for \(J_S\) only arises to order \(J_S^2\). Thus, it is expected that it does not appear in Eqs. (E12), as it would correspond to a further correction to \(G\) that would be \(\propto J_S^4\). Nevertheless we may simply account for the nontrivial RG flow by trading \(J_S/\psi\) for the running coupling \(J_S(D)\). On identifying the scale \(D\) with the Boltzmann constant \(k\) times the temperature, we end up with the improved perturbative result

\[
G(T) \approx \frac{e^2}{2\pi} \left[ \begin{array}{cc} 1 - \left[ \frac{J_{S,0}}{1 - J_{S,0} \ln \left( \frac{v}{\psi^2} \right)} \right]^2 & \frac{J_{S,0}}{1 - J_{S,0} \ln \left( \frac{v}{\psi^2} \right)} \\ \frac{J_{S,0}}{1 - J_{S,0} \ln \left( \frac{v}{\psi^2} \right)} & 1 - \left[ \frac{J_{S,0}}{1 - J_{S,0} \ln \left( \frac{v}{\psi^2} \right)} \right]^2 \end{array} \right] ,
\]

which is expected to hold for \(D_0 \geq kT \geq kT_K\). Before moving to the CK effect, it is interesting to verify what is the conductance tensor corresponding to the SK fixed point we identified throughout the analysis of appendix D. As stated in the main text, the only remarkable difference with respect to the \(J = 0\) fixed point consists in the change of the phase shift in the \(e\)-sector, so that \(\psi_{e,a}(0^+;\tau) = e^{-2i\delta} \psi_{e,a}(0^+;\tau)\). Nevertheless, this strongly affect the form of the current operator which, using as a reference phase in the \(e\)-sector the one corresponding to \(x < 0\), becomes

\[
j_a(x,\tau) = \frac{e v}{2} \sum_{u = \pm 1} \{ \psi_{e,a}^{\dagger}(u x + i v \tau) \psi_{e,a}(u x + i v \tau) + \psi_{o,a}^{\dagger}(u x + i v \tau) \psi_{o,a}(u x + i v \tau) \} \\
- \frac{e v}{2} \sum_{u = \pm 1} u \{ \psi_{e,a}^{\dagger}(u x + i v \tau) \psi_{o,a}(u x + i v \tau) e^{2i\delta \theta(u x)} + \psi_{o,a}^{\dagger}(u x + i v \tau) \psi_{e,a}(u x + i v \tau) e^{-2i\delta \theta(u x)} \} .
\]
As a result, we now obtain

\[-\left\langle T_{\tau} f_a(x,\tau) j_{a'}(x',\tau') \right\rangle = -\frac{e^2 v^2}{2} \delta_{a,a'} \sum_{u=\pm 1} \left\{ \left[ 1 + \cos(2\delta) \right] g^2(u x, \tau; u x', \tau') \right\}. \] (E15)

At $2\delta = \pi$ the right-hand side of Eq. (E15) is identically equal to 0, that is, each lead behaves, when considering the conductance across the impurity, as a perfect insulator (see the main text for a discussion about this point).

We now move to discuss the transport properties of the system within the CK phase.

2. Second-order corrections to the conductance tensor in the charge-Kondo phase

As we discuss in the main text, the CK Hamiltonian exactly maps onto the SK one once, in the $c$-sector, one introduces the fields $\tilde{\psi}_{c,1}(x,\tau) = \psi_{c,1}(x,\tau)$, and $\tilde{\psi}_{c,2}(x,\tau) = \psi_{c,2}(x,\tau)$. Taking into account that, in terms of the $\tilde{\psi}_{c/o,a}$ fields, the current-density operators are written as

\[
j_1(x,\tau) = \frac{e v}{2} \sum_{u=\pm 1} \left\{ \tilde{\psi}_{c,1}^\dagger (u x + iv \tau) \tilde{\psi}_{c,1} (u x + iv \tau) + \tilde{\psi}_{o,1}^\dagger (u x + iv \tau) \tilde{\psi}_{o,1} (u x + iv \tau) \right\}
\]

\[
j_2(x,\tau) = \frac{e v}{2} \sum_{u=\pm 1} \left\{ -\tilde{\psi}_{c,2}^\dagger (u x + iv \tau) \tilde{\psi}_{c,2} (u x + iv \tau) + \tilde{\psi}_{o,2}^\dagger (u x + iv \tau) \tilde{\psi}_{o,2} (u x + iv \tau) \right\}
\]

one obtains that the perturbative corrections to the dc conductance tensor due to the CK interaction are given by

\[
\delta G^{(2)} = -\frac{e^2}{2\pi} \frac{J_C^2}{v^2} \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] \Rightarrow G \approx \frac{e^2}{2\pi} \left[ \begin{array}{cc} 1 - \frac{J_C^2}{v^2} & -\frac{J_C^2}{v^2} \\ -\frac{J_C^2}{v^2} & 1 - \frac{J_C^2}{v^2} \end{array} \right]. \] (E17)

From Eq. (E17), by resorting to the running coupling in analogy to what we have done in getting to Eq. (E13), one obtains the finite-$T$ correction to the dc conductance tensor due to the CK interaction for $D_0 \geq kT \geq kT_K$ in the form

\[
G(T) \approx \frac{e^2}{2\pi} \left[ 1 \right. - \left. \frac{J_{C,0}}{1-J_{C,0} \ln \left( \frac{2T}{v} \right)} \right] \left[ 1 \right. - \left. \frac{J_{C,0}}{1-J_{C,0} \ln \left( \frac{2T}{v} \right)} \right]^{1/2} \left[ 1 \right. - \left. \frac{J_{C,0}}{1-J_{C,0} \ln \left( \frac{2T}{v} \right)} \right]^{1/2}, \] (E18)

with $J_{C,0} = aJ_C/v$.

To conclude the discussion of this appendix, we now consider the leading corrections to the dc conductance tensor in the DL region.

3. Second-order corrections to the conductance tensor in the disconnected lead phase

In the DL phase, the corrections to the dc conductance tensor are determined by $H_{DL}$ in Eq. (27). An important observation is that $H_{DL}$ in Eq. (27) contains a “truly marginal” term $= \sum_{a=1,2} \lambda_a \rho_a(0)$, with

\[
\rho_a(0) = \tilde{\psi}_{R,a}(0) \tilde{\psi}_{R,a}(0) + \tilde{\psi}_{L,a}(0) \tilde{\psi}_{L,a}(0) + \tilde{\psi}_{R,a}(0) \tilde{\psi}_{R,a}(0) + \tilde{\psi}_{L,a}(0) \tilde{\psi}_{L,a}(0) = \sqrt{2} \tilde{\psi}_{c,a}(0) \tilde{\psi}_{c,a}(0), \] (E19)

corresponding to a pointlike one-body potential scattering at $x = 0$. As we show in appendix A, a terms as such just renormalizes the value of the dc conductance in each channel by means of a nonuniversal factor. This does not qualitatively affects the conduction properties of the system, compared to the case in which $\lambda_1 = \lambda_2 = 0$ and, accordingly, we will neglect the effects of this term henceforth. In the following, we therefore set $\lambda_1 = \lambda_2 = 0$ and rewrite $H_{DL}$ in terms of the $c$-fields only as
We now compute the corrections to the conductance tensor up to second-order in $H_{DL}$. The first-order correction to the current-current correlation functions is given by

$$
\delta^{(1)} \Gamma_{a,a'}(x, \tau; x', \tau') = \frac{\kappa e^2 v^2}{2} \delta a, a' \sum_{u=\pm 1} \int_0^\beta d\tau_1 g^2(u, \tau; 0, \tau_1)g^2(0, \tau_1; u'x', \tau') .
$$

(E21)

Resorting to Fourier space, one obtains

$$
\int_0^\beta d(\tau-\tau') \int_0^\beta d\tau_1 g^2(u, \tau; 0, \tau_1)g^2(0, \tau_1; u'x', \tau') = \frac{\Omega^2}{(2\pi v^2)^2} \{ \theta(ux)\theta(-u'x')\theta(\Omega) + \theta(-ux)\theta(u'x')\theta(-\Omega) \} e^{-\frac{\Omega(uu'-x'x)}{v^2}},
$$

(E22)

which clearly shows that no corrections to the dc conductance tensor arise to first-order in $\kappa$. Moving to second-order in $\kappa$, one obtains

$$
\delta^{(2)} \Gamma_{a,a'}(x, \tau; x', \tau') = -2\kappa e^2 v^2 \int_0^\beta d\tau_1 d\tau_2 \times
\langle T \{ j_{(e,e);a}(x, \tau)j_{(e,e);a'}(x', \tau') \} \times \psi^\dagger_{e,1}(iv\tau_1)\psi_{e,1}(iv\tau_2)\psi_{e,2}(iv\tau_2)\psi_{e,2}(iv\tau_2) \rangle .
$$

(E23)

It is useful to separately compute the right-hand side of Eq. (E23) in the case $a = a' = 1$ and $a = 1, a' = 2$. For $a = a' = 1$ one obtains

$$
\delta^{(2)} \Gamma_{1,1}(x, \tau; x', \tau') = -\kappa e^2 v^2 \int_0^\beta d\tau_1 d\tau_2 \times
\{ \sum_{u, u'=\pm 1} \{ g(ux, \tau; 0, \tau_1)\}^2 \{ g(0, \tau_1; 0, \tau_2)\}^2 \{ g(0, \tau_2; u'x', \tau')\}^2
- 2 \sum_{u, u'=\pm 1} uu' g(ux, \tau; ux', \tau') g(ux, \tau; 0, \tau_1) g(0, \tau_1; 0, \tau_2) g(0, \tau_2; ux', \tau') \} .
$$

(E24)

Now, moving to Fourier-Matsubara space, one obtains

$$
\int_0^\beta d(\tau-\tau') e^{i\Omega(\tau-\tau')} [ g(ux, \tau; 0, \tau_1)\}^2 \{ g(0, \tau_1; 0, \tau_2)\}^2 \{ g(0, \tau_2; u'x', \tau')\}^2 =
\frac{|\Omega|^3}{(2\pi v^2)^3} \{ \theta(\Omega)\theta(ux)\theta(-u'x') + \theta(-\Omega)\theta(-ux)\theta(u'x') \} e^{-\frac{\Omega(uu'-x'x)}{v^2}},
$$

(E25)

which readily implies that the dc conductance tensor takes no contributions from this term. As for what concerns the other term at the right-hand side of Eq. (E24), in order to compute the corresponding contribution to the dc conductance tensor, we set

$$
G_2(ux, \tau; u'x', \tau') = g(ux, \tau; 0, \tau_1) g(0, \tau_1; 0, \tau_2) g(0, \tau_2; u'x', \tau') .
$$

(E26)

Thus, moving to Fourier-Matsubara space (with the fermionic frequency $\omega$), one gets

$$
\int_0^\beta d\tau e^{i\omega\tau} G_2(ux, \tau; u'x', 0) = \frac{i}{2\pi^2 v^3} \left[ \omega^2 - \frac{\pi^2}{\beta^2} \right] \{ \theta(\omega)\theta(ux)\theta(-u'x') + \theta(-\omega)\theta(-ux)\theta(u'x') \} e^{-\frac{\Omega(uu'-x'x)}{v^2}} .
$$

(E27)

Using Eq. (E25), we get

$$
\delta^{(2)} \Gamma_{1,1}(x, x'; i\Omega) =
-\frac{\kappa e^2 v}{2\pi^3 \beta^2} \sum_{u, u'=\pm 1} uu' \sum_{\omega} \left[ \theta(\omega)\theta(-\omega)\theta(-u'x') + \theta(-\omega)\theta(-\Omega + \omega)\theta(-ux)\theta(u'x') \right] \left[ \omega^2 - \frac{\pi^2}{\beta^2} \right] e^{-\frac{\Omega(uu'-x'x)}{v^2}} =
-\frac{\kappa^2 e^2}{12\pi^4 v^2} |\Omega| \left[ \Omega^2 - \frac{4\pi^2}{\beta^2} \right] \sum_{u, u'=\pm 1} uu' \left[ \theta(\Omega)\theta(-u'x') + \theta(-\Omega)\theta(-ux)\theta(u'x') \right] e^{-\frac{\Omega(uu'-x'x)}{v^2}} .
$$

(E28)
Apparently, Eq. (E28) yields a finite-temperature correction to the diagonal dc conductance that is given by

\[ \delta G_{1,1}(T) = \delta G_{2,2}(T) = -\frac{k^2 e^2}{3\pi^2 v^2} (kT)^2 . \]  

(E29)

At variance, for \( a = 1, a' = 2 \), one obtains

\[ \delta^{(2)} \Gamma_{1,2}(x, \tau; x', \tau') = -\frac{k^2 e^2 a^2}{2} \int_0^\beta d\tau_1 d\tau_2 \sum_{u, u' = \pm 1} \times \]

\[ \langle T_{\tau} \psi_{e,1}(ux + i\tau) \psi_{e,1}(ux + i\tau) \psi_{e,1}(i\tau_1) \psi_{e,1}(i\tau_2) \psi_{e,1}(i\tau_2) \rangle \times \]

\[ \langle T_{\tau} \psi_{e,2}(u' x' + i\tau') \psi_{e,2}(u' x' + i\tau') \psi_{e,2}(i\tau_1) \psi_{e,2}(i\tau_2) \psi_{e,2}(i\tau_2) \rangle . \]  

(E30)

It is easy to show that the right-hand side of Eq. (E30) is \( = 0 \), as a consequence of the identity

\[ \langle T_{\tau} \psi_{e,a}(ux + i\tau) \psi_{e,a}(ux + i\tau) \psi_{e,a}(i\tau_1) \psi_{e,a}(i\tau_2) \psi_{e,a}(i\tau_2) \rangle = 0 . \]  

(E31)

1. J. Kondo, Progress of Theoretical Physics 32, 37 (1964).
2. A. C. Hewson, The Kondo Problem to Heavy Fermions, Cambridge Studies in Magnetism (Cambridge University Press, 1993).
3. P. Nozieres, Journal of Low Temperature Physics 17, 31 (1974).
4. K. G. Wilson, Rev. Mod. Phys. 47, 773 (1975).
5. R. Bulla, T. A. Costi, and T. Pruschke, Rev. Mod. Phys. 80, 395 (2008).
6. I. Affleck and A. W. W. Ludwig, Phys. Rev. B 48, 7297 (1993).
7. L. P. Kouwenhoven and L. Glazman, Physics World 14, 33 (2001).
8. A. Alivisatos, Science 271, 933 (1996).
9. L. P. Kouwenhoven and C. Marcus, Physics World 11, 35 (1998).
10. D. Goldhaber-Gordon, H. Shtrikman, D. Mahalu, D. Abusch-Magder, U. Meirav, and M. A. Kastner, Nature 391, 156 (1998).
11. S. M. Cronenwett, T. H. Oosterkamp, and L. P. Kouwenhoven, Science 281, 540 (1998).
12. Y. Avishai, A. Golub, and A. D. Zaikin, Phys. Rev. B 63, 134515 (2001).
13. M.-S. Choi, M. Lee, K. Kang, and W. Belzig, Phys. Rev. B 70, 020502 (2004).
14. G. Campagnano, D. Giuliano, A. Naddeo, and A. Tagliacozzo, Physica C: Superconductivity 406, 1 (2004).
15. S. Eggert and I. Affleck, Phys. Rev. B 46, 10866 (1992).
16. A. Furusaki and T. Hikihara, Phys. Rev. B 58, 5529 (1998).
17. D. Giuliano, D. Rossini, and A. Trombettoni, Phys. Rev. B 98, 235164 (2018).
18. D. Giuliano, D. Rossini, P. Sodano, and A. Trombettoni, Phys. Rev. B 87, 035104 (2013).
19. D. Giuliano, P. Sodano, and A. Trombettoni, Phys. Rev. A 96, 033603 (2017).
20. N. Laflorencie, E. S. Sorensen, and I. Affleck, Journal of Statistical Mechanics: Theory and Experiment 2008, P02007 (2008).
21. N. Crampé and A. Trombettoni, Nuclear Physics B 871, 526 (2013).
22. A. M. Tsvelik, Phys. Rev. Lett. 110, 147202 (2013).
23. A. M. Tsvelik and W.-G. Yin, Phys. Rev. B 88, 144401 (2013).
24. D. Giuliano, P. Sodano, A. Tagliacozzo, and A. Trombettoni, Nuclear Physics B 909, 135 (2016).
25. D. Giuliano and P. Sodano, Nuclear Physics B 811, 395 (2009).
26. A. Cirillo, M. Mancini, D. Giuliano, and P. Sodano, Nuclear Physics B 852, 235 (2011).
27. D. Giuliano and P. Sodano, EPL (Europhysics Letters) 103, 57006 (2013).
28. B. Béri, Phys. Rev. Lett. 110, 216803 (2013).
29. B. Béri and N. R. Cooper, Phys. Rev. Lett. 109, 156803 (2012).
30. A. Altland, B. Béri, R. Egger, and A. M. Tsvelik, Phys. Rev. Lett. 113, 076401 (2014).
31. I. Affleck and D. Giuliano, Journal of Statistical Physics 157, 666 (2014).
32. A. Taraphder and P. Coleman, Phys. Rev. Lett. 66, 2814 (1991).
33. A. K. Matveev, Phys. Rev. B 51, 1743 (1995).
34. G. Zarãš, G. T. Zimånyi, and F. Wilhelmi, Phys. Rev. B 62, 8137 (2000).
35. A. Buxboim and A. Schiller, Phys. Rev. B 67, 165320 (2003).
36. K. Le Hur, P. Simon, and L. Borda, Phys. Rev. B 69, 045326 (2004).
37. P. S. Cornaglia, H. Ness, and D. R. Grempel, Phys. Rev. Lett. 93, 147201 (2004).
38. M. Dzero and J. Schmalian, Phys. Rev. Lett. 94, 157003 (2005).
39. S. Andergassen, T. A. Costi, and V. Zlatić, Phys. Rev. B 84, 241107 (2011).
FIG. 5: Sketch of $G_{1,1}(T)$ and of $G_{1,2}(T)$ as a function of $T$ in the various phases of the system, determined by different values of $r = \delta v/(at^2)$. From top to bottom:
a) Expected behavior of $G_{1,1}(T)$ (blue curve) and $G_{1,2}(T)$ (red curve) in units of $e^2/(2\pi)$ as a function of $T/T_K$ in the SK region ($r > 1$). The saturation of $G_{1,1}(T)/(e^2/(2\pi))$ at high values of $T/T_K$ may take place to values lower than 1, depending on the presence of potential scattering terms. Breaking of particle-hole symmetry may give rise to a nonzero saturation value of $G_{1,1}(T)/(e^2/(2\pi))$ as $T \to 0$ (see main text for details);
b) Expected behavior of $G_{1,1}(T)$ (blue curve) and $G_{1,2}(T)$ (red curve) in units of $e^2/(2\pi)$ as a function of $T/T_K$ in the CK region ($r < -1$). Again, potential scattering terms can make $G_{1,1}(T)/(e^2/(2\pi))$ saturate to values lower than 1 at high values of $T/T_K$ and breaking of particle-hole symmetry may give rise to a nonzero saturation value of $G_{1,1}(T)/(e^2/(2\pi))$ as $T \to 0$; 
c) Expected behavior of $G_{1,1}(T)$ (blue curve) and $G_{1,2}(T)$ (red curve) in units of $e^2/(2\pi)$ in the DL region ($-1 < r < 1$). Due to the absence of a physically meaningful temperature reference scale in this region, $T$ has rescaled in units of what would be $T_K$ for the specified values of the system’s parameters in that region if Kondo effect were taking place.
FIG. 6: Sketch of the RG trajectories for the running couplings $J_\perp(D)$ and $J_z(D)$. 