A Reweighted Total Variation Algorithm with the Alternating Direction Method for Computed Tomography

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Abstract
A variety of alternating direction methods have been proposed for solving a class of optimization problems. The applications in computed tomography (CT) perform well in image reconstruction. The reweighted schemes were applied in $l_1$-norm and total variation minimization for signal and image recovery to improve the convergence of algorithms. In this paper, we present a reweighted total variation algorithm using the alternating direction method (ADM) for image reconstruction in CT. The numerical experiments for ADM demonstrate that adding reweighted strategy reduces the computation time effectively and improves the quality of reconstructed images as well.

Keywords
Computed Tomography, Nonmonotone Alternating Direction Algorithm, Reweighted Algorithm

1. Introduction
In many applications, the problem of recovering a sparse unknown vector $x$ from a set of measurements $b$ is presented. The $l_1$ minimization,

$$\min_x \|x\|_1 \text{ subject to } Ax = b,$$

(1)

where $A$ represents a measurement matrix, has been widely used in solving the problem [1–5]. This method has a unique sparsest solution $x$ under certain conditions [6]. However, the $l_1$-norm regularization provides a less sparsity representation than the $l_0$-norm regularization and may lose some detailed features and contrast. One major improved $l_1$ minimization algorithm for finding the sparsest solution efficiently is the reweighted $l_1$ minimization [7]. Some modifications of the reweighted $l_1$-minimization algorithms have been studied in recent years. A more general form of the reweighted $l_1$-minimization,
in which the weights are raised to a certain power between 0 and 1, was considered in [8,9]. Numerical experiments in the areas of sparse signal recovery, the error correction in sample transition and the total minimization for sparse image gradients have shown that the performance of recovery using reweighted $l_1$ minimization is better than the regular $l_1$ minimization [7].

In computed tomography (CT), algebraic approaches for image reconstruction involves solving a consistent system of linear equations

$$\Phi f + e = u,$$

where $\Phi$ is an $m \times n^2$ projection matrix, $f \in \mathbb{R}^{n^2}$ represents a 2D $n \times n$ image to be reconstructed, $e$ an additive noise with $||e||_2 \leq \varepsilon$ for some known $\varepsilon > 0$, and $u \in \mathbb{R}^m$ the noisy projection data. For limited-view reconstruction, the system is underdetermined with $m \ll n^2$ and has infinitely many solutions. Optimization-based Iterative methods are usually used to find an optimal solution representing the original image.

The image $f$ in System (2) can be reconstructed by solving the following total variation (TV) minimization with an $l_2$-norm constraints,

$$\min_f ||f||_{TV} \quad \text{subject to} \quad ||\Phi f - u||^2 \leq \varepsilon,$$

where $||*||$ stands for $||*||_2$ for simplicity. Tomography images can be approximately modeled to be essentially piecewise constant so the gradients are sparse. As $||f||_{TV}$ is the $l_1$-norm of the gradient of $f$, the TV minimization is also known as the $l_1$-minimization method. A generalized $l_1$ greedy algorithm in the compressed sensing framework [10] was introduced to incorporate the threshold feature of the $l_1$ greedy algorithm [11] and the inversely proportional weights used in the reweighted $l_1$-minimization algorithm for CT. Error analysis of reweighted $l_1$ greedy algorithm for noisy reconstruction was studied in [12]. The reweighted $l_1$-norm was incorporated into non-local TV minimization to strengthen the structural details and the tissue contrast and thus to enhance the CT reconstructing performance [13].

The alternating direction method (ADM) is a variant of the classic augmented Lagrangian method for structured optimization. For a convex optimization problem where variables appearing separately are coupled in the constraint, the minimization with respect to all variables are time-consuming. The ADM minimizes the augmented Lagrangian function with respect to variables separately via iteration to reduce computation time, widely used in $l_1$-norm minimization [14,15]. An ADM with fast convergence under certain strong conditions was proposed in [16]. A unified alternating direction method by majorization minimization was proposed and its convergence was analyzed [17]. The nonmonotone alternating direction method (NADA) was proposed for solving a class of equality-constrained nonsmooth optimization problems and applied to the total variation minimization effectively [18,19]. The advantage of the NADA lies in that the objective function is not required to be differentiable while reducing the computation complexity and improving the efficiency.

In this paper, we present a reweighted TV algorithm using ADM in CT. The rest of this paper is organized as follows. Notations and preliminary are introduced in Section 2. A reweight scheme corporation into the total variation algorithm using NADA is developed in Section 3. Numerical experiments in Section 4 show the adopt of
weights improves the performance of the TV algorithm using NADA in computational time and error measurements. Finally a discussion in Section 5 concludes the paper.

2. Preliminary

In this section we introduce notations and review approaches for solving a TV minimization problem (3) using the reweight scheme and ADM in the literature, respectively.

In the case of sparse signal recovery, it is extremely important to identify the locations of nonzero entries of a sparse vector \( x \). After a few iterations, the larger entries in magnitude are naturally considered to be nonzero entries. The process should concentrate on other entries smaller in magnitude. The idea of the reweighted \( l_1 \) minimization is to make the weights small for the entries of \( x \) larger in magnitude and the weights large for those smaller in magnitude. In other words, the weights are roughly inversely proportional to the magnitude of the previous iterate solution. So reweighted \( l_1 \) minimization speeds up the convergence of recovery. The minimization problem (1) is modified as

\[
\min_x \| Wx \|_1 \text{ subject to } Ax = b,
\]

where at the \( k \)-th iteration \( W = diag\{w^{(k)}_1, \ldots, w^{(k)}_n\} \) is the diagonal weight matrix with each \( w^{(k)}_i = \frac{1}{\varepsilon + ||x^{(k-1)}||} \) and \( \varepsilon > 0 \).

In the case of image reconstruction in CT, the total variation of an image \( f \) is defined as

\[
\| f \|_{TV} = \| \nabla f \|_1 = \sum_{i=1}^{n^2} || D_i f || ,
\]

where \( \nabla f \) is the gradient of \( f \), and \( D_i \) is a forward difference operator at a pixel \( i \) in both horizontal and vertical directions. The gradient \( \nabla f \) of an essentially piecewise constant image \( f \) is sparse so a reweighted scheme could be adopted to speed up the convergence and improve the efficiency of image reconstruction. Thus, the term to be minimized in (3) is revised as \( \| W \nabla f \|_1 \), where at the \( k \)-th iteration \( W = diag\{w^{(k)}_1, \ldots, w^{(k)}_n\} \) with all \( w^{(k)}_i = \frac{1}{\varepsilon + ||D_i f^{k-1}||} \).

Minimization (3) with constraints can be expressed as the minimization of an augmented Lagrangian function as, with Lagrangian multipliers \( \lambda_i \), auxiliary vectors \( v_i \), and regularization parameters \( \beta, \mu > 0 \),

\[
L(f, v, \lambda) = \frac{\beta}{2} || \Phi f - u ||^2 + \sum_{i=1}^{n^2} \{ || v_i || + \lambda_i^T (D_i f - v_i) + \mu || D_i f - v_i ||^2 \} ,
\]

where \( \lambda_i, v_i \in \mathbb{R}^2 \). Given \( f^k, v^k, \) and \( \lambda^k \) in the \( k \)-th iteration, ADM minimizes \( L(f, v, \lambda) \) with respect to \( f \) and \( v \) separately, then updates \( \lambda \). In other words, ADM iterates as

\[
\begin{align*}
    f^{k+1} &= \arg \min_f L(f, v^k, \lambda^k); \\
    v^{k+1} &= \arg \min_v L(f^{k+1}, v, \lambda^k); \\
    \lambda^{k+1} &= \lambda^k + \beta(D f^{k+1} - v^{k+1}).
\end{align*}
\]
3. Reweighted TV Algorithm Using NADA

In this section we develop a new reweighted TV algorithm using NADA for solving the following minimization

$$\min_f ||W\nabla f||_1 \quad \text{subject to} \quad ||\Phi f - u||^2 \leq \varepsilon,$$

(5)

where the $l_1$-norm $||W\nabla f||_1$ is a weighted total variation of $f$, and $W$ is a weight matrix with all positive diagonals.

Dealing with the constraint as usual ADM, we first rewrite the minimization above with a penalty parameter $\mu > 0$ as

$$\min_f f||W\nabla f||_1 + \frac{\mu}{2}||\Phi f - u||^2,$$

or

$$\min_f \sum_{i=1}^{n^2} ||v_i|| + \frac{\mu}{2}||\Phi f - u||^2 \quad \text{subject to} \quad w_iD_if = v_i, \quad 1 \leq i \leq n^2.$$

(6)

Applying Lagrangian vectors $\lambda_i \in \mathbb{R}^2$, for $1 \leq i \leq n^2$, we define

$$L_w(f, v, \lambda) = \frac{\mu}{2}||\Phi f - u||^2 + \sum_{i=1}^{n^2} (||v_i||_2 - \lambda_i^T (w_iD_if - v_i) + \frac{\beta}{2}||w_iD_if - v_i||^2).$$

(7)

Thus, Minimization (6) is converted to minimizing the objective function in (7),

$$\min_f L_w(f, v, \lambda).$$

(8)

For given $f^k, v^k$, and $\lambda^k$, the proposed algorithm iterates as

$$f^{k+1} = \arg \min_f L_w(f, v^k, \lambda^k),$$

(9)

$$v^{k+1} = \arg \min_v L_w(f^{k+1}, v, \lambda^k),$$

(10)

$$\lambda^{k+1} = \arg \min_\lambda L_w(f^{k+1}, v^{k+1}, \lambda).$$

(11)

In this new algorithm, a modified reweighted scheme is incorporated after a few non-weighted iterations in order to speed up the convergence of the algorithm and improve the efficiency. The standard reweighted scheme adopts a matrix $W = \text{diag}\{w_1, \ldots, w_{n^2}\}$, where $w_i$ is essentially inversely proportional to the norm $||D_if^k||$. In this project, the values of $w_i$s need to be rescaled so that the weights of four terms in the objective function $L_w(f, v, \lambda)$ remain close to the case where $W$ is the identity matrix. The standard reweighted scheme is modified to calculating a vector $w$ with $w_i = \frac{1}{\varepsilon + ||D_if^k||}$ for $1 \leq i \leq n^2$, and rescaling $w$ by a factor $r = \frac{2}{\text{mean}(w)}$. The reweighted matrix $W$ in this paper is obtained this way.

Minimizations (9) and (10) are solved by adopting the nonmonotone line search scheme in the framework of NADA though there is an extra reweighted matrix $W$ in the objective function $L_w(f, v, \lambda)$. The details are as follows. After choosing a direction $d^k = -\frac{\partial}{\partial f}L_w(f^k, v^k, \lambda^k)$, search a step size $s_k$ uniformly bounded above such that $L_w(f^k + s_kd^k, v^k, \lambda^k) \leq C_k - s_k\delta ||d^k||^2$, for a scalar $C_k$ and $0 < \delta < 1$. Then update $f^{k+1} = f^k + s_kd^k$ and compute a scalar $C_{k+1}$. DOI: 10.4236/act.2019.81001 4 Advances in Computed Tomography
$L_w(f^{k+1}, v^k, \lambda^k) \leq C_{k+1} \leq C_k$. Combined with solution of (10), the sequence $\{L_w(f^k, v^k, \lambda^k)\}$ is bounded above by a monotonically non-increasing sequence $\{C_k\}$ though $\{L_w(f^k, v^k, \lambda^k)\}$ itself is not decreasing. The advantages of the NADA remain so that $L_w(f, v, \lambda)$ is not required to be differentiable and that the usage of the nonmonotone line search scheme reduces the computation complexity and improves the efficiency.

To find $v^{k+1} = \arg \min_v L_w(f^{k+1}, v, \lambda^k)$ in (10), we solve

$$\frac{\partial L_w}{\partial v_i} = \frac{v_i}{\|v_i\|} + \lambda_i - \beta w_i D_i f + \beta v_i = 0.$$ 

So $v_i (1 + \frac{1}{\beta \|v_i\|}) = w_i D_i f - \frac{\lambda_i}{\beta}$. It follows that $\|v_i\| = \|w_i D_i f - \frac{\lambda_i}{\beta}\| - \frac{1}{\beta}$. Thus, we have

$$v_i = \max\{\|w_i D_i f - \frac{\lambda_i}{\beta}\| - \frac{1}{\beta}, 0\} \frac{w_i D_i f - \frac{\lambda_i}{\beta}}{\|w_i D_i f - \frac{\lambda_i}{\beta}\|}, \quad 1 \leq i \leq n^2. \quad (12)$$

Similarly, the solution of (11) is given by

$$\lambda_i^{k+1} = \lambda_i^k + \beta (w_i D_i f^{k+1} - v^{k+1}), \quad 1 \leq i \leq n^2. \quad (13)$$

Now we present a new algorithm to solve $\min_f L_w(f, v, \lambda)$ using NADA.

**Algorithm (Reweighted Total Variation Algorithm Using NADA)**

1. input $\Phi, u, \varepsilon$
2. initialize $\beta, \mu, f^0, v^0, \lambda^0, \varepsilon, tol, k, maxit$
3. perform a few iterations of regular TV algorithm without weight using NADA
4. while $k < maxit$
   4.1 set a reweighted matrix $W$
   4.2 update $f^{k+1} = \arg \min_f L_w(f, v^k, \lambda^k)$ by NADA
   4.3 update $v^{k+1}$ by (12)
   4.4 update $\lambda^{k+1}$ by (13)
   4.5 if error $< tol$ then output $f^{k+1}$, stop
   4.6 increase $k = k + 1$
   end

4. Numerical Experiments

In this section, the reweighted TV algorithm using NADA for CT reconstruction is implemented in MATLAB. The reweighted TV minimization (5) (or (8)) is compared with the regular TV minimization (3) (or minimization of (4)) for their performances. The numerical experiments are conducted with the 2D Shepp-Logan phantom [20] of size $128 \times 128$ on an Intel Core i7 3.40 GHz PC. The MATLAB code is developed based on the software package TVAL3 [19].

In each test, a random matrix $\Phi \in \mathbb{R}^{m \times n^2}$ ($m \approx 0.3 n^2$) is generated and $u = \Phi f + e$ is set, where the noise $e = 0.05 \cdot \text{mean}(\Phi f) \cdot \text{randn}(m)$. The parameters are taken as $\beta = 2^4$ and $\mu = 2^6$. The values of $\varepsilon$ in the weight matrix is $0.01$. If the relative error $\|f - f_{\text{recon}}\|_F / ||f||_F$ of the
reconstructed image $f_{\text{recon}}$ in Frobenius norm is less than $tol = 0.05$ then the iteration is terminated. For the reweighted TV algorithm, 15 iterations of the standard total variation minimization are set as the initial solution for the reweighted TV-minimization. The average results from 100 tests of the proposed algorithm and non-weighted TV regularization are used in the following table and figure for evaluations.

Let $f_{\text{ave}}$ denote the average of the pixel values of a 2D image $f$. Experimental results are also evaluated using the root-mean-square error (RMSE), the normalized root mean square deviation (NRMSD), and the normalized mean absolute deviation (NMAD) which are defined as follows:

$$RMSE = \frac{||f - f_{\text{recon}}||_F}{\sqrt{m \times n}},$$

$$NRMSD = \frac{||f - f_{\text{recon}}||_F}{||f_{\text{ave}} - f||_F},$$

$$NMAD = \frac{\sum_{m,n} |f(m,n) - f_{\text{recon}}(m,n)|}{\sum_{m,n} |f(m,n)|}.$$

These measurements reflect different aspects of the quality of the recovered images. RMSE evaluates the reconstruction quality on a pixel-by-pixel basis. NRMSD emphasizes large errors in a few pixels of the recovered image. NMAD focuses on small errors in the recovered image.

The original and reconstructed images are shown in Figure 1. The experimental results are summarized in Table 1. The CPU time is measured by MATLAB built-in functions. The values of RMSE, NRMSD, and NMAD for the reweighted TV minimization are improved from the corresponding values for the TV minimization. The CPU time is saved 34.9%. The numerical experiments demonstrate that the reweighted TV minimization is superior to the regular TV minimization.

![Figure 1. Shepp-Logan phantom and reconstructed images.](image)

**Table 1.** Experimental data with shepp-logan phantom.

| Algorithm                  | CPU Time | RMSE   | NRMSD | NMAD  |
|----------------------------|----------|--------|-------|-------|
| TV Using NADA              | 14.4 sec | 0.0085 | 0.0088| 0.0353|
| Reweighted TV using NADA   | 9.4 sec  | 0.0066 | 0.0069| 0.0294|

There are three important parameters in this proposed algorithm: $\beta$, $\mu$, and rescaling factor $r$ for $w$. Similar to a regular NADA without reweight, the values of $\beta$ and $\mu$ should be between $2^{14}$ and $2^{13}$. The rescaling factor $r$ is adopted between $\frac{1}{\text{mean}(w)}$ and $\frac{4}{\text{mean}(w)}$ in the experiments. It is a challenging question how to optimize the reconstructed images by reweighed TV using NADA. More extensive numerical experiments will be performed in future investigation.
5. Conclusion
The reweighted TV minimization has shown its capability of speeding up the convergence of sparse image reconstruction in CT since it was proposed more than ten years ago. In recent years the ADM was widely applied to a class of equality-constrained nonsmooth optimization problems and particularly image reconstruction in CT. As far as the authors are aware, the ADM hasn’t been used to solve the reweighted TV minimization in the literature. This paper develops a reweighted total variation algorithm using the ADM method for image reconstruction in CT, as the major novelty of our work. The usage of the nonmonotone line search scheme reduces complexity and improves efficiency. Numerical simulation indicate that the proposed algorithm can effectively incorporate the reweighted scheme and improve the efficiency for image reconstruction in CT.

Conflicts of Interest
The authors declare no conflicts of interest regarding the publication of this paper.

References
[1] Chen, S., Donoho, D. and Saunders M. (1998) Atomic Decomposition by Basis Pursuit. *SIAM Journal on Scientific Computing, 20*, 33-61. https://doi.org/10.1137/S1064827596304010

[2] Donoho, D. and Logan, B.F. (1992) Signal Recovery and the Large Sieve. *SIAM Journal on Applied Mathematics, 52*, 577-591. https://doi.org/10.1137/0152031

[3] Donoho, D. and Stark, P.B. (1989) Uncertainty Principles and Signal Recovery. *SIAM Journal on Applied Mathematics, 49*, 906-931. https://doi.org/10.1137/0149053

[4] Santosa, F. and Symes, W. (1986) Linear Inversion of Band-Limited Reflection Seismograms. *SIAM Journal on Scientific and Statistical Computing, 7*, 1307-1330. https://doi.org/10.1137/0907087

[5] Tibshirani, R. (1996) Regression Shrinkage and Selection via the Lasso. *Journal of the Royal Statistical Society: Series B, 58*, 267-288. https://doi.org/10.1111/j.2517-6161.1996.tb02080.x

[6] Zhang, H., Yin, W. and Cheng, L. (2015) Necessary and Sufficient Conditions of Solution Uniqueness in $l_1$-Norm Minimization. *Journal of Optimization Theory and Applications, 154*, 109-122. https://doi.org/10.1007/s10957-014-0581-z

[7] Candes, E., Wakin, W. and Boyd, S. (2008) Enhancing Sparsity by Reweighted $l_1$-Minimization. *Journal of Fourier Analysis Application, 14*, 877-905. https://doi.org/10.1007/s00041-008-9045-x

DOI: 10.4236/act.2019.81001
[8] Foucart, S. and Lai, M. (2009) Sparsest Solutions of Underdetermined Linear Systems via $l_q$-Minimization for $0 < q \leq 1$. *Applied and Computational Harmonical Analysis*, 26, 395-407. https://doi.org/10.1016/j.acha.2008.09.001

[9] Lai, M.J. (2010) On Sparse Solutions of Underdetermined Linear Systems. *Journal of Concrete & Applicable Mathematics*, 8, 296-327.

[10] Zhu, J. and Li, X. (2013) A Generalized $l_1$ Greedy Algorithm for Image Reconstruction in CT. *Applied Mathematics and Computation*, 219, 5487-5494. https://doi.org/10.1016/j.amc.2012.11.052

[11] Kozlov, I. and Petukhov, A. (2010) Sparse Solutions for Underdetermined Linear System. In: Freeden, W., Nashed, M.Z. and Sonar, T., Eds., *Handbook of Geomathematics*, Vol. 1, Springer, New York, 1243-1260. https://doi.org/10.1007/978-3-642-01546-5_42

[12] Zhu, J., Li, X., Arroyo, F. and Arroyo, E. (2015) Error Analysis of Reweighted $l_1$ Greedy Algorithm for Noisy Reconstruction. *Journal of Computational and Applied Mathematics*, 286, 93-101. https://doi.org/10.1016/j.cam.2015.02.038

[13] Kim, H., Chen, J., Wang, A., Chiang, C., Held, M. and Poullot, J. (2016) Non-Local Total-Variation (NLTV) Minimization Combined with Reweighted $l_1$-Norm for Compressed Sensing CT Reconstruction. *Physics in Medicine & Biology*, 61, 6878-6891. https://doi.org/10.1088/0031-9155/61/18/6878

[14] Tao, M. and Yang, F. (2009) Alternating Direction Algorithms for Total Variation Deconvolution in Image Reconstruction. TR0918, Department of Mathematics, Nanjing University, Optimization.

[15] Yang, J. and Zhang, Y. (2011) Alternating Direction Algorithms for $l_1$-Problems in Compressive Sensing. *SIAM Journal on Scientific Computing*, 33, 250-278. https://doi.org/10.1137/090777761

[16] Goldstein, T., O’Donoghue, B., Setzer, S. and Baraniuk, R. (2014) Fast Alternating Direction Optimization Methods. *SIAM Journal on Imaging Sciences*, 7, 1588-1623. https://doi.org/10.1137/120896219

[17] Lu, C., Feng, J., Yan, S. and Liu, Z. (2018) A Unified Alternating Direction Method of Multipliers by Majorization Minimization. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 40, 527-541. https://doi.org/10.1109/TPAMI.2017.2689021

[18] Li, C., Yin, W., Jiang, H. and Zhang, T. (2013) An Efficient Augmented Lagrangian Method with Applications to Total Variation Minimization. *Computational Optimization and Applications*, 56, 507-530. https://doi.org/10.1007/s10589-013-9576-1

[19] Li, C., Yin, W. and Zhang, Y. (2010) Users Guide for TVAL3: TV Minimization by Augmented Lagrangian and Alternating Direction Algorithms. *CAAM Report*. DOI: 10.4236/act.2019.81001 8 Advances in Computed Tomography
[20] Kak, A.C. and Slaney, M. (2001) Principles of Computerized Tomographic Imaging. Society of Industrial and Applied Mathematics, Philadelphia.
https://doi.org/10.1137/1.9780898719277