Inverse kinetic theory approach to turbulence theory

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Abstract

A fundamental aspect of turbulence theory is related to the identification of realizable phase-space statistical descriptions able to reproduce in some suitable sense the stochastic fluid equations of a turbulent fluid. In particular, a major open issue is whether a purely Markovian statistical description of hydrodynamic turbulence actually can be achieved. Based on the formulation of a deterministic inverse kinetic theory (IKT) for the 3D incompressible Navier-Stokes equations, here we claim that such a Markovian statistical description actually exists. The approach, which involves the introduction of the local velocity probability density for the fluid (local pdf) - rather than the velocity-difference pdf adopted in customary approaches to homogeneous turbulence - relies exclusively on first principles. These include - in particular - the exact validity of the stochastic Navier-Stokes equations, the principle of entropy maximization and a constant H-theorem for the Shannon statistical entropy. As a result, the new approach affords an exact equivalence between Lagrangian and Eulerian formulations which permit local pdf’s which are generally non-Maxwellian (i.e., non-Gaussian). The theory developed is quite general and applies in principle even to turbulence regimes which are non-stationary and non-uniform in a statistical sense.

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I. INTRODUCTION

In this and in an accompanying paper [1] the possibility of formulating inverse kinetic theory (IKT) approaches to hydrodynamic turbulence theory is investigated. By definition, these are meant to be as phase-space models able to deliver a prescribed set (or subset) of fluid equations to be expressed in terms of appropriate moment equations of a suitable statistical equation, denoted as inverse kinetic equation (IKE). Depending of the subset of fluid equations to be considered different (and possibly non-unique) IKT approaches can in principle be developed. In the present paper, in particular, a Markovian statistical model of turbulence is obtained, based on the formulation of a deterministic inverse kinetic equation for the stochastic-averaged Navier-Stokes equations. The theory permits the explicit construction of the local position-velocity joint probability distribution function (local pdf) which advances in time the corresponding stochastic-averaged fluid fields and which can be shown to characterize uniquely an incompressible isothermal fluid.

A. The unsolved problem

The theoretical approach to the turbulence problem in incompressible fluids is one of the outstanding intellectual challenges of contemporary physics. Turbulence theory has been pioneered by Kolmogorov (K41, [2]) who - using simple heuristic arguments based on dimensional analysis - first shed light on the understanding homogeneous turbulence (HT, i.e., stationary, spatially homogeneous and isotropic turbulence). This led to the interpretation of HT as a self-similar energy cascade in which turbulent eddies transport the kinetic energy of the fluid from a prescribed injection scale to a suitable dissipation scale (the so-called inertial range). This lead to the well-known Kolmogorov ”5/3-power law” conjecture [2] - later confirmed by experiments and numerical results [3] - that in the inertial range the cascade is characterized by energy spectrum $E(k)$ of the form $E(k) = K_{Ko} \Pi^{2/3} k^{-5/3}$, where $K_{Ko}$ is an universal constant called Kolmogorov’s constant, $k$ is the wavenumber, and $\Pi$ is the nonlinear cascade of energy, to be identified with the dissipation rate of the fluid. Subsequently the problem of homogeneous turbulence was approaches with the goal of quantifying the processes underlying the spectrum. Various theories were developed, for this purpose, to understand fluid turbulence, based on attempts to introduce suitable models for
the statistics of turbulent flows [4, 5]. These phenomenological theories can be developed, in principle, choosing for the description of fluids either Eulerian or Lagrangian viewpoints. The two approaches, if fluid dynamics were fully understood, should be completely equivalent. Unfortunately, at least for the treatment of turbulent fluids, we are still quite far from reaching this goal. The main historical reason of this situation can be understood by looking at the customary statistical approach based on so-called velocity probability density function (pdf)- method for an incompressible fluid (for a review see for example [5]). In the Lagrangian treatment of turbulence [denoting the so-called Lagrangian turbulence (LT)] the Lagrangian path \( R(r,t) \) and the velocity \( U(r,t) \) of a fluid element, initially starting at the position \( r \), are determined by the equations:

\[
\frac{dR(r,t)}{dt} = U(r,t),
\]

\[
\frac{dU(r,t)}{dt} = A(r,t; g),
\]

where the vector field \( A(r,t; g) \), to be suitably related to the Navier-Stokes equations, is assumed to depend on an appropriate statistical distribution \( g \). Usually this is identified with the joint position-velocity probability distribution (pdf) of the particle for the increments

\[
x(t) = R(t, y) - U(0, t) t - r,
\]

\[
u(t) = U(t, y) - U(0, y)
\]

defined as

\[
\begin{align*}
g(x, u, t) &= \langle \delta(x - x(t)) \delta(u - u(t)) \rangle, \\
f(x, u, 0) &= \delta(x) \delta(u).
\end{align*}
\]

where the brackets denote a suitable stochastic average over a stationary statistical ensemble. For HT, the velocity increment pdf [5] can be shown to obey a Fokker-Planck statistical equation [5]. The fundamental reason why this happens is that a statistical theory of turbulence, relying solely of first principles, is yet not available. For this reason in the past various statistical models, based on heuristic assumptions about the statistics of \( A(t, y; g) \) and of the related pdf \( g \), have been introduced. In turbulence theory \( g \) is usually identified with the velocity-difference probability density function (pdf), traditionally adopted for the description of homogeneous turbulence. In these cases, however, the corresponding Fokker-Planck equation for \( g \) generally does not define a closed system of moment equations (which should actually coincide with the Navier-Stokes equations themselves). This raises the critical issue of the closure of the statistical models of fluid turbulence and the difficulty of achieving equivalent Eulerian and Lagrangian formulations. In the past several statistical
models have been proposed for the determination of $g$, which include
the mapping-closure method, the test-function method and the field-theoretical approach [6]. However, a critical aspect to turbulence theory is the possible appearance of coherent structures (like vortices and convective cells). In fluid turbulence the signature of the presence of coherent structures is provided by the existence of non-Gaussian features in the probability density. During the last few years lots of efforts have been put into the formulation of more sophisticated phenomenological theories which can take into account these facts (c.f. the review article [7]). These approaches are based on a theoretical analysis of the Navier-Stokes equation or on the advanced data analysis of the experimentally obtained Lagrangian path’s of particles (see for example [8]). Nevertheless, despite the progress achieved in modelling key features of the basic phenomenology, still missing is a consistent, theory-based, statistical description of fluid turbulence. Clearly, such formulation - if achievable at all - should rely exclusively on a rigorous, deductive formulation of the turbulence-modified fluid equations following from the fluid equations [9]. Based on a recently proposed inverse kinetic theory (IKT) for incompressible fluids (Ellero et al., 2004-2008 [10, 11, 12, 13]), here we intend to formulate a deterministic IKT for the stochastic-averaged Navier-Stokes equations. Key features of the new theory are: 1) the formulation is based on the local pdf, rather than the velocity-increment pdf usually adopted in turbulence theory (see for example [3]); 2) the pdf is advanced in time by means of a Markovian Vlasov-type kinetic equation; 3) the kinetic equation implies the exact validity of the stochastic-averaged Navier-Stokes equations (see below) and as a consequence the kinetic equation satisfies exact closure conditions; 4) the theory displays a complete equivalence of the Lagrangian and Eulerian viewpoints [13]; 5) the theory is based on first principles, i.e., besides the Navier-Stokes equations, the validity of the principle of entropy maximization and a constant $H$-theorem to be imposed on the Shannon statistical entropy. The theoretical setting here adopted is based on the definition of a ”restricted” phase-space representation of the fluid [13], whereby the phase space ($\Gamma$) is identified with the direct-product space $\Gamma = \Omega \times V$, where $\Omega, V \subseteq \mathbb{R}^3$, $\Omega$ is an open set which coincides with the fluid domain (i.e., the bounded sub-domain of $\mathbb{R}^3$ occupied by the fluid) and $V$ is the velocity space.
II. STOCHASTIC-AVERAGED FLUID EQUATIONS

For definiteness here we consider an incompressible isothermal fluid immersed in a fluid domain $\Omega$ (to be identified with a bounded subdomain of $\mathbb{R}^3$ with closure $\overline{\Omega} = \Omega \cup \delta\Omega$). In the set $\Omega \times I$, $I$ denoting an appropriate finite time interval $I \subset \mathbb{R}$, the relevant fluids $\{\rho = \rho_0 > 0, V, p\}$, i.e., respectively the (constant) mass density, fluid velocity and pressure describing the fluid, are assumed to be strong solutions of the so-called incompressible Navier-Stokes equations (INSE) \cite{11}. The starting point of the new statistical description here introduced is the assumption that the fluid equations are stochastic in some sense to be suitably specified. This permits us to define a stochastic decomposition for all the relevant quantities, i.e., the fluid fields $Z(r,t) \equiv \{V, p\}$ as well the volume force density $f(r,t)$. Namely we let, $Z(r,t) = \langle Z(r,t) \rangle + \delta Z(r,t)$ and $f(r,t) = \langle f(r,t) \rangle + \delta f(r,t)$, where $\langle \rangle$ is a suitable stochastic-averaging operator. Here, by proper definition of the operator $\langle \rangle$, $\langle Z(r,t) \rangle$, $\langle f(r,t) \rangle$ and $\delta Z(r,t)$ and $\delta f(r,t)$ represent respectively the averaged parts and the stochastic fluctuations. In the sequel the precise definition of the operator $\langle \rangle$ is not required, however, we shall assume that it commutes with all the differential operators appearing in the previous fluid equations (i.e., $\frac{\partial}{\partial t}$, $\nabla$ and $\nabla^2$). As a consequence a suitable set of stochastic equations can be obtained. In particular, the equations for the stochastic-averaged fields $\langle Z(r,t) \rangle$ read

$$\begin{align*}
\nabla \cdot \langle V \rangle &= 0, \\
\langle N \rangle \langle V \rangle + \langle \delta N \delta V \rangle &= 0
\end{align*}$$

(stochastic-averaged INSE). Here the notation is standard \cite{11}. Thus, $N$ is the nonlinear Navier-Stokes differential operator $N\mathbf{V} = \frac{\partial}{\partial t} \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{1}{\rho_0} [\nabla p - f] - \nu \nabla^2 \mathbf{V}$, where $f$ is the force density, assumed to be smooth real vector field, and $\rho_0$ and the kinematic viscosity $\nu$ are real positive constants, both to be considered non-stochastic. Moreover, $\langle N \rangle$ and $\delta N$ are respectively the operators $\langle N \rangle \langle V \rangle = \frac{\partial}{\partial t} \langle V \rangle + \langle V \rangle \cdot \nabla \langle V \rangle + \frac{1}{\rho_0} [\nabla \langle p \rangle - \langle f \rangle] - \nu \nabla^2 \langle V \rangle$ and $\delta N \delta V = \delta \mathbf{V} \cdot \nabla \delta \mathbf{V} + \frac{1}{\rho_0} [\nabla \delta p - \delta f] - \nu \nabla^2 \delta \mathbf{V}$, while $\Pi_R(r,t) \equiv \langle \delta \mathbf{V} \delta \mathbf{V} \rangle$ is the Reynolds stress tensor.
III. DETERMINISTIC IKT FORMULATION FOR THE STOCHASTIC-AVERAGED INSE

The discovery of the IKT for INSE \[^{10}\] suggests us to seek an inverse kinetic equation for the set of stochastic-averaged equations defined by Eqs.\[^{11}\]. In particular we look for a Markovian inverse kinetic equation (IKE) of the Vlasov-type \[^{13}\] which in Eulerian form reads

\[
L(\langle Z \rangle)f(x, t; \langle Z \rangle) = 0 ,
\]

(5)

\[
L(\langle Z \rangle) f \equiv \frac{\partial}{\partial t} f + \frac{\partial}{\partial x} \cdot \{X(\langle Z \rangle)f\} ,
\]

(6)

(Eulerian IKE). Here \(f(x, t; \langle Z \rangle)\) denotes the Eulerian local pdf for Eqs.\[^{11}\], which advances in time the stochastic-averaged fluid fields \(\langle Z(r,t) \rangle\) and \(L(\langle Z \rangle)\) is the corresponding streaming operator. We intend to construct \(f(x, t; \langle Z \rangle)\) and \(L(\langle Z \rangle)\) by imposing a suitable set of prescriptions. Besides the requirement of validity of the fluid equations \[^{11}\], these include in particular:

- \(x = (r,v)\) is the state vector spanning the restricted phase space \(\Gamma = \Omega \times V\[^{13}\]\) and the vector field \(X\) has the form \(X(\langle Z \rangle) = \{v, F(\langle Z \rangle)\}\), where \(F(\langle Z \rangle) \equiv F(x, t; f, \langle Z \rangle)\) is an appropriate mean-field force, to be assumed generally functionally dependent on the local pdf.

- By appropriate choice of the mean field force \(F\) and of the local pdf, the moment equations can be prescribed in such a way to satisfy identically Eqs.\[^{11}\]. For this purpose we assume that \(f\) is a strictly positive, suitably smooth in \(\Gamma \times I\) and summable both in the phase-space \(\Gamma\) and in the velocity space \(V\). In particular, we require that the (Shannon) entropy integral

\[
S(f) = -\int_{\Gamma} dxf(x, t; \langle Z \rangle) \ln f(x, t; \langle Z \rangle)
\]

(7)

exists and that there results identically in \(\Gamma \times I\) :

\[
\langle Z(r,t) \rangle = \int d^3v G(x,t)f(x, t; \langle Z \rangle),
\]

(8)

where \(\langle Z(r,t) \rangle = 1, \langle V(r,t) \rangle\) and \(p_1(r,t)\) are the velocity moments \(G(x,t) = 1, v, \frac{1}{3} (u)^2\). Moreover, \(u = v - V(r,t)\) is the relative velocity and \(p_1(r,t) = P_0(t) + \langle p(r,t) \rangle\) is the
kinetic pressure and $P_0(t)$ a smooth real function (pseudo-pressure) to be suitably defined (see below).

- The form of the pdf $f$ is chosen in such a way to satisfy the principle of entropy maximization (PEM) [14], i.e., imposing the variational equation $\delta S(f) = 0$, with $\delta^2 S(f) < 0$, while requiring that $f$ belongs to a suitable functional class $\{f(x, t; \langle Z \rangle)\}$, to be determined based solely on the information available on $f(x, t; \langle Z \rangle)$. In a turbulent fluid this is manifestly provided by the knowledge of the stochastic-averages of the fluid fields, $\langle Z(r, t) \rangle$. In this case, imposing the constraints placed by the moments [8] it is immediate to prove that PEM yields necessarily as a particular equilibrium solution of the inverse kinetic equation the local Maxwellian distribution function ($\textit{kinetic equilibrium}$)

$$f_M(x, t; \langle Z \rangle) = \frac{1}{(\pi)^{3/2} v_{th}^3} \exp \left\{ -X^2 \right\}, \quad (9)$$

where $X^2 = \frac{(w)^2}{v_{th}^2}, v_{th}^2 = 2p_1/\rho_0$ and $\langle u \rangle = v - \langle V(r, t) \rangle$. However, in principle arbitrary non-Maxwellian initial kinetic distributions are possible. These are potentially relevant, in particular, for direct numerical simulations, in which the kinetic distribution function is simulated numerically by means of test particles. In such a case, in fact, small numerical errors may imply that locally the kinetic distribution function may actually be non-Maxwellian.

- Eq.(5) implies the construction of a suitable classical dynamical system, defined by a phase-space map

$$x_o \to x(t) = T_{t, t_o} x_o, \quad (10)$$

where $T_{t, t_o}$ is the evolution operator [12] generated by the initial-value problem

$$\begin{cases} \frac{d}{dt} x = X(x, t), \\ x(t_o) = x_o, \end{cases} \quad (11)$$

to be viewed as the $\textit{Lagrangian}$ (or Langevin) equations for Eq. (5).

- The equivalence between Eulerian and Lagrangian representations. In fact the Eulerian IKE can also be represented in the equivalent $\textit{Lagrangian form}$ [13]

$$J(x(t), t) f(x(t), t; \langle Z \rangle) = f(x_o, t_o; \langle Z_o \rangle) \equiv f_o(x_o; \langle Z_o \rangle), \quad (12)$$
(Lagrangian IKE) where \( f(x(t), t; \langle Z \rangle) \) is the Lagrangian representation of the pdf, \( x(t) \) is the solution of the initial-value problem \( \Pi, f_o(x_o; \langle Z_o \rangle) \) is a suitably smooth initial pdf and \( J(x(t), t) = \left| \frac{\partial x(t)}{\partial x_o} \right| \) is the Jacobian of the map \( x_o \to x(t) \).

The following theorem can be proven:

**Theorem - Markovian IKT for the stochastic-averaged INSE**

Let us assume that: 

1. \( A_1 \) Eqs. (4) admit a smooth strong solution in \( \Gamma \times I \);  
2. \( A_2 \) the mean-field force \( F(x,t; f, \langle Z \rangle) \) associated to the fluid fields \( \langle Z \rangle \) reads: 
   \[
   F(x,t; f, \langle Z \rangle) = F_0 + F_1, 
   \]
   (13)

where \( F_0, F_1 \) read respectively

\[
F_0(x,t; f, \langle Z \rangle) = \frac{1}{\rho_o} \left[ \nabla \cdot \Pi - \nabla p_1 + \langle f_R \rangle \right] + \frac{1}{2} \langle \mathbf{u} \cdot \nabla \mathbf{v} \rangle + \frac{1}{2} \langle \nabla \cdot \mathbf{V} \rangle + \nu \nabla^2 \mathbf{V},
\]

(14)

\[
F_1(x,t; f; \langle Z \rangle) = \frac{1}{2} \langle \mathbf{u} \rangle \left( \frac{1}{p_1} A + \frac{1}{p_1} \nabla \cdot \mathbf{Q} - \frac{1}{p_1^2} [\nabla \cdot \Pi] \cdot \mathbf{Q} \right) + \frac{v_{th}^2}{2p_1} \nabla \cdot \Pi \left( \frac{u^2}{v_{th}^2} - \frac{3}{2} \right),
\]

(15)

where \( A \equiv \frac{\partial}{\partial t} (P_0(t) + \langle p \rangle) + \langle \mathbf{V} \rangle \cdot \nabla (P_0(t) + \langle p \rangle) ; \)  
3. \( A_3 \) \( f(x,t; \langle Z \rangle) \) satisfies suitable initial and boundary condition consistent with the initial-boundary value problem (4) (see Ref. [11]);  
4. \( A_4 \) the initial pdf, \( f(x,t_o; \langle Z_o \rangle) \), is suitably smooth an strictly positive;  
5. \( A_5 \) \( f(x,t; \langle Z \rangle) \) admits the moments (7) and (8) as well as \( Q = \int d^3 \mathbf{u} u^2 f \) and \( \Pi = \int d^3 \mathbf{v} \mathbf{v} f \), to be assumed suitably smooth;  
6. \( A_6 \) the pseudo-pressure \( P_0(t) \) is determined in such a way that there results identically in \( I \)

\[
\frac{\partial}{\partial t} S(f(t)) = 0
\]

(constant H-theorem);  
7. \( A_7 \) in the fluid equations (4) and in Eqs. (14) and (15) the Reynolds stress tensor \( \Pi_R \equiv \langle \nabla \mathbf{V} \nabla \mathbf{V} \rangle \) is considered a prescribed function of \( (\mathbf{r}, t) \).

It follows that:  

8. \( B_1 \) the velocity-moment equations of IKE (3) evaluated for the weight functions \( G(x,t) = v \frac{1}{3} u^2 \) coincide with the stochastic INSE equations;  
9. \( B_2 \) the Maxwellian pdf (9) is a particular solution of IKE. In particular, the local Maxwellian distribution function (9) is a particular solution of the IKE (3) if an only if the fluid fields \( \{ \langle \mathbf{V} \rangle, \langle p \rangle \} \) satisfy the stochastic-averaged Eqs. (4). In such a case there results identically \( Q = 0, \Pi = 0 \);  
10. \( B_3 \) \( f(x,t; \langle Z \rangle) \) is strictly positive in \( \Gamma \times I \). Hence it is a probability density.
PROOF - The proof is immediate. In fact: B₁) First, invoking Eqs. (13), (14) and (15), it follows that the velocity-moment equations of IKE Eq. (5) for the weight functions \( G(r, v, t) = 1, v, \frac{1}{3} u^2 \) read respectively:

\[
\nabla \cdot \langle V \rangle = 0, \quad (17)
\]

\[
\frac{\partial}{\partial t} \langle V \rangle + \langle V \nabla \cdot V \rangle + \frac{1}{\rho_o} [\nabla \langle p_1 \rangle - \langle f \rangle] - \nu \nabla^2 \langle V \rangle = 0, \quad (18)
\]

\[
\nabla \cdot [\langle V \rangle \langle p_1 \rangle] + \langle V \rangle \cdot \nabla \langle p_1 \rangle = 0 \quad (19)
\]

[which manifestly coincide with the fluid equations Eqs. (4)]. Hence, also thanks to A6 the pdf advances in time uniquely the stochastic-averaged fluid fields \( \langle Z(r, t) \rangle \). B₂) Second, invoking A1 it is immediate to prove that \( f_M(x, t; \langle Z \rangle) \) is a particular solution of the inverse kinetic equation (5). In fact, substituting (9) in the inverse kinetic equation (5) it follows:

\[
L(\langle Z \rangle) f_M = \left\{ \frac{\partial}{\partial t} \langle V \rangle + v \cdot \nabla \langle V \rangle \right\} \cdot \left\{ \frac{\langle u \rangle}{\langle p_1 \rangle} \rho_o f_M + \right.
\]

\[- \ln \left\{ \frac{\langle u \rangle}{\langle v_{th}^2 \rangle} - \frac{3}{2} \right\} f_M - \left. \nabla \cdot \langle V \rangle \right\} \]  

\[
= \int \Gamma d x f(x, t; \langle Z \rangle) \cdot \left\{ \frac{\langle u \rangle}{\langle p_1 \rangle} \rho_o f_M + f_M \frac{\partial}{\partial v} \cdot F(x, t; f_M, \langle Z \rangle) \right\} = 0. \quad (20)
\]

Thanks to Eqs. (13), (14) and (15), there results:

\[
L(\langle Z \rangle) f_M = \left\{ \frac{\partial}{\partial t} \langle V \rangle + v \cdot \nabla \langle V \rangle + \frac{1}{\rho_o} [\nabla \langle p_1 \rangle + \langle f \rangle] - \right.
\]

\[- \langle u \cdot \nabla V \rangle - \nu \nabla^2 \langle V \rangle \right\} \cdot \left\{ \frac{\langle u \rangle}{\langle p_1 \rangle} \rho_o f_M + f_M \nabla \cdot \langle V \rangle = 0 \right.\]

which implies Eqs. (4). Instead, if we assume that in \( \Gamma \times I, f \equiv f_M(x, t; \langle Z \rangle) \) is a particular solution of the inverse kinetic equation, which fulfills identically the constraint equation (16), it follows that the fluid fields \( \langle V \rangle, \langle p \rangle \) are necessarily solutions of the INSE equations. If, instead, \( f \equiv f_M(x, t; \langle Z \rangle) \) is a particular solution of Eq. (5), thanks to B₁ it follows that the stochastic INSE equations are necessarily fulfilled. B₃) Finally, due to Eq. (5) the entropy production rate reads

\[
\frac{\partial}{\partial t} S(f(t)) = - \int_\Gamma d x \frac{\partial}{\partial t} f(x, t; \langle Z \rangle) \left[ 1 + \ln f(x, t; \langle Z \rangle) \right] = \quad (21)
\]

\[
= - \int_\Gamma d x f(x, t; \langle Z \rangle) \frac{\partial}{\partial v} \cdot F(x, t; f, \langle Z \rangle). \]
Hence, thanks to Eqs. (13), (14), (15), the constraint Eq. (16) requires

\[
\frac{\partial}{\partial t} S (f(t)) = - \frac{3}{2} \int_\Omega d^3 r \frac{1}{P_0(t)} \langle p \rangle \left[ A + \nabla \cdot Q - \frac{1}{p_1} [\nabla \cdot \Pi] \cdot Q \right] = 0, \tag{22}
\]

which delivers an ordinary differential equation for the pseudo-pressure. Assuming that the fluid fields \( \langle V \rangle, \langle p \rangle \) and the moments \( Q \) and \( \Pi \) are suitably smooth, this equation can always be fulfilled in a finite time interval \( I \). As a direct consequence, it follows that \( f(x,t;\langle Z \rangle) \) is manifestly strictly positive in \( \Gamma \times I \).

IV. CONCLUSIONS

In this paper a statistical model of hydrodynamic turbulence has been formulated by means of a deterministic inverse kinetic theory (IKT) for the stochastic-averaged INSE. Basic aspects of the new theory are: A) that the IKT satisfies exactly the relevant fluid equations, represented by Eqs. (4), while B) the pdf advances in time uniquely the stochastic-averaged fluid fields \( \langle Z \rangle \). As a main consequence, the theory fulfills identically a closure condition. In fact, there exists, by construction, a subset of the moment equations which is closed and manifestly coincides with the same set of stochastic fluid equations. The result B) holds provided the Reynolds stress tensor can be considered prescribed (to be identified, for example, with a suitably weighted velocity-space integral of the local pdf). While the form of the tensor remains in principle unspecified, this limitation shall be lifted in an accompanying paper [1]. The present theory displays several remarkable new features. In particular, unlike customary statistical approaches, it is based on the introduction of the local position-velocity joint probability density function (local pdf). In addition, the kinetic equation advancing in time the pdf is a Markovian Vlasov-type kinetic equation which admits a straightforward equivalent Lagrangian formulation. Under suitable prescriptions the form of this equation can be proven to be unique [12]. Further key property is that the kinetic equation admits in general, besides a local Maxwellian kinetic equilibrium, also non-Gaussian solutions. In our view the theory provides a convenient setting both for the investigation of theoretical aspects of turbulence theory. These include - besides the mathematical formulation of turbulence problems - several important physical applications, such as, for example, the connection with Fokker-Planck statistical descriptions [1] and the Lagrangian dynamics of particles in turbulent flows and its generalization to incompressible thermofluids [13].
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