The algebra of polynomial integro-differential operators is a holonomic bimodule over the subalgebra of polynomial differential operators

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Abstract

In contrast to its subalgebra $A_n := K(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n)$ of polynomial differential operators (i.e. the $n$'th Weyl algebra), the algebra $I_n := K(x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, f_1, \ldots, f_n)$ of polynomial integro-differential operators is neither left nor right Noetherian algebra; moreover it contains infinite direct sums of nonzero left and right ideals. It is proved that $I_n$ is a left (right) coherent algebra if $n = 1$; the algebra $I_n$ is a holonomic $A_n$-bimodule of length $3^n$ and has multiplicity $3^n$, and all $3^n$ simple factors of $I_n$ are pairwise non-isomorphic $A_n$-bimodules. The socle length of the $A_n$-bimodule $I_n$ is $n + 1$, the socle filtration is found, and the $m$'th term of the socle filtration has length $\binom{n}{m}2^{n-m}$. This fact gives a new canonical form for each polynomial integro-differential operator. It is proved that the algebra $I_n$ is the maximal left (resp. right) order in the largest left (resp. right) quotient ring of the algebra $I_n$.

Key Words: the algebra of polynomial integro-differential operators, the Weyl algebra, the socle, the socle length.

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1 Introduction

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \ldots\}$ is the set of natural numbers; $K$ is a field of characteristic zero and $K^*$ is its group of units; $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra over $K$; $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives ($K$-linear derivations) of $P_n$; $\text{End}_K(P_n)$ is the algebra of all $K$-linear maps from $P_n$ to $P_n$; the subalgebras $A_n := K(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n)$ and $I_n := K(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, f_1, \ldots, f_n)$ of the algebra $\text{End}_K(P_n)$ are called the $n$'th Weyl algebra and the algebra of polynomial integro-differential operators respectively.

The Weyl algebras $A_n$ are Noetherian algebras and domains. The algebras $I_n$ are neither left nor right Noetherian and not domains. Moreover, they contain infinite direct sums of nonzero left and right ideals \footnote{V. V. Bavula}. The algebra $A_n$ is isomorphic to its opposite algebra $A_n^{op}$ via the $K$-algebra involution:

$$A_n \rightarrow A_n, \quad x_i \mapsto \partial_i, \quad \partial_i \mapsto x_i, \quad i = 1, \ldots, n.$$ 

Therefore, every $A_n$-bimodule is a left $A_{2n}$-module and vice versa. Inequality of Bernstein \footnote{I. G. Macdonald} states that each nonzero finitely generated $A_{2n}$-module has Gelfand-Kirillov dimension which is greater or equal to $n$. A finitely generated $A_{2n}$-module is holonomic if it has Gelfand-Kirillov dimension $n$. The holonomic $A_{n}$-modules share many pleasant properties. In particular, all holonomic modules have finite length, each nonzero submodule and factor module of a holonomic module is holonomic.

The aim of the paper is to prove Theorem 2.5. In particular, to show that the algebra $I_n$ is a holonomic $A_{2n}$-bimodule of length $3^n$ and has multiplicity $3^n$, i.e. a holonomic left $A_{2n}$-module of length $3^n$ and has multiplicity $3^n$. All $3^n$ simple factors of $I_n$ are pairwise non-isomorphic $A_n$-bimodules. We also found the socle filtration of the $A_{2n}$-module $I_n$. It turns out that the socle length of the $A_{2n}$-module is $n + 1$, and the length, as an $A_{2n}$-module, of the $m$'th socle factor is $\binom{n}{m}2^{n-m}$ (Theorem 2.5(4)) where $m = 0, 1, \ldots, n$. A new $K$-basis for the algebra $I_n$ is found which gives a new canonical form for each polynomial integro-differential operator, see \footnote{V. V. Bavula}. By the
very definition, \( \mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i) \simeq \mathbb{I}_1^\otimes n \) where \( \mathbb{I}_1(i) := K(x_i, \partial_i, \int_i) \) and \( A_n = \bigotimes_{i=1}^n A_1(i) = A_1^\otimes n \) where \( A_1(i) := K(x_i, \partial_i) \). So, the properties of the algebras \( \mathbb{I}_n \) and \( A_n \) are ‘determined’ by the properties of the algebras \( \mathbb{I}_1 \) and \( A_1 \).

At the beginning of Section 2 we collect necessary facts on the algebras \( \mathbb{I}_n \). Then we prove Theorem 2.5 in the case when \( n = 1 \) and prove some necessary results that are used in the proof of Theorem 2.5 (in the general case) which is given at the end of the section.

In Section 3 it is proved that the algebra \( \mathbb{I}_n \) is left (right) coherent if \( n = 1 \) (Theorem 3.1).

In Section 4 it is proved that the algebra \( \mathbb{I}_n \) is the maximal left (resp. right) order in its largest left (resp. right) quotient ring (Theorem 4.3).

## 2 Proof of Theorem 2.5

At the beginning of this section, we collect necessary (mostly elementary) facts on the algebra \( \mathbb{I}_1 \) from 11 that are used later in the paper.

The algebra \( \mathbb{I}_1 \) is generated by the elements \( \partial, H := \partial x \) and \( \int \) (since \( x = \int H \)) that satisfy the defining relations (Proposition 2.2, 11):

\[
\partial \int = 1, \quad [H, \int] = \int, \quad [H, \partial] = -\partial, \quad H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial, \quad (1)
\]

where \([a, b] := ab - ba\) is the commutator of elements \( a \) and \( b \). The elements of the algebra \( \mathbb{I}_1 \),

\[
e_{ij} := \int e^j - \int e^{j+1}, \quad i, j \in \mathbb{N},
\]

(2)

satisfy the relations \( e_{ij}e_{kl} = \delta_{jk}e_{il} \) where \( \delta_{jk} \) is the Kronecker delta function and \( \mathbb{N} := \{0, 1, \ldots\} \) is the set of natural numbers. Notice that \( e_{ij} = \int e_{00}^j e^i_j \). The matrices of the linear maps \( e_{ij} \in \text{End}_K(K[x]) \) with respect to the basis \( \{x^s := x_1^s\}_{s \in \mathbb{N}} \) of the polynomial algebra \( K[x] \) are the elementary matrices, i.e.

\[
e_{ij} * x^a = \begin{cases} x^i & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}
\]

Let \( E_{ij} \in \text{End}_K(K[x]) \) be the usual matrix units, i.e. \( E_{ij} * x^a = \delta_{ja}x^i \) for all \( i, j, s \in \mathbb{N} \). Then

\[
e_{ij} = j! E_{ij},
\]

(3)

\( K e_{ij} = K E_{ij}, \) and \( F := \bigoplus_{i,j \geq 0} K e_{ij} = \bigoplus_{i,j \geq 0} K E_{ij} \simeq M_\infty(K), \) the algebra (without 1) of infinite dimensional matrices. \( F \) is the only proper ideal (i.e. \( \neq 0, \mathbb{I}_1 \)) of the algebra \( \mathbb{I}_1 \).

**Z-grading on the algebra \( \mathbb{I}_1 \) and the canonical form of an integro-differential operator 11, 2.** The algebra \( \mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} \mathbb{I}_{1,i} \) is a \( Z \)-graded algebra (\( \mathbb{I}_{1,i} \mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j} \) for all \( i, j \in \mathbb{Z} \)) where

\[
\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^i D_1 = D_1 \partial^i & \text{if } i < 0, \end{cases}
\]

the algebra \( D_1 := K[H] \bigoplus_{i \in \mathbb{N}} K e_{ii} \) is a commutative non-Noetherian subalgebra of \( \mathbb{I}_1 \), \( H e_{ii} = e_{ii}H = (i + 1)e_{ii} \) for \( i \in \mathbb{N} \) (and so \( \bigoplus_{i \in \mathbb{N}} K e_{ii} \) is the direct sum of non-zero ideals \( K e_{ii} \) of the algebra \( D_1 \)) \( (\int^i D_1) D_1 \simeq D_1, \int^i d \mapsto d; D_1 (\partial^i) \simeq D_1, d \partial^i \mapsto d \) for all \( i \geq 0 \) since \( \partial^i \int^i = 1 \). Notice that the maps \( \cdot \int^i : D_1 \to D_1 \int^i, d \mapsto \int^i d \) and \( \partial^i : D_1 \to \partial^i D_1, d \mapsto \partial^i d \), have the same kernel \( \bigoplus_{i \geq 1} K e_{ij} \).

Each element \( a \) of the algebra \( \mathbb{I}_1 \) is the unique finite sum

\[
a = \sum_{i > 0} a_{-i} \partial^i + a_0 + \sum_{i \geq 0} \int^i a_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij}
\]

(4)
where \( a_k \in K[H] \) and \( \lambda_{ij} \in K \). This is the canonical form of the polynomial integro-differential operator \([1]\).

**Definition.** Let \( a \in \mathbb{I}_1 \) be as in \([4]\) and let \( a_F := \sum \lambda_{ij} e_{ij} \). Suppose that \( a_F \neq 0 \) then

\[
\text{deg}_F(a) := \min\{n \in \mathbb{N} \mid a_F \in \bigoplus_{i,j=0}^n Ke_{ij}\}
\]

(5)

is called the \( F \)-degree of the element \( a \); \( \text{deg}_F(0) := -1 \).

Let \( v_i := \begin{cases} f^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ \partial^{|i|} & \text{if } i < 0. \end{cases} \) Then \( \mathbb{I}_{1,i} = D_1 v_i = v_i D_1 \) and an element \( a \in \mathbb{I}_1 \) is the unique finite sum

\[
a = \sum_{i \in \mathbb{Z}} b_i v_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij}
\]

(6)

where \( b_i \in K[H] \) and \( \lambda_{ij} \in K \). So, the set \( \{H^i \partial^j, H^j \partial^i, e_{it} \mid i \geq 1; j \geq 1; t \geq 0 \} \) is a \( K \)-basis for the algebra \( \mathbb{I}_1 \). The multiplication in the algebra \( \mathbb{I}_1 \) is given by the rule:

\[
\int H = (H-1) \int f, \quad H \partial = \partial (H-1), \quad e_{ij} = e_{i+1,j}, \quad \partial e_{ij} = e_{i-1,j}, \quad e_{ij} \partial = \partial e_{i,j+1}.
\]

\[
He_{ii} = e_{ii}H = (i+1)e_{ii}, \quad i \in \mathbb{N},
\]

where \( e_{-1,j} := 0 \) and \( e_{i,-1} := 0 \).

The factor algebra \( B_1 := \mathbb{I}_1/F \) is the simple Laurent skew polynomial algebra \( K[H][\partial, \partial^{-1}; \tau] \) where the automorphism \( \tau \in \text{Aut}_{K-\text{alg}}(K[H]) \) is defined by the rule \( \tau(H) = H + 1, \quad [1] \). Let

\[
\pi : \mathbb{I}_1 \rightarrow B_1, \quad a \mapsto \pi : a + F,
\]

(7)

be the canonical epimorphism.

The Weyl algebra \( A_2 \) is equipped with the, so-called, filtration of Bernstein, \( A_2 = \bigcup_{i \geq 1} A_{2,i} \) where \( A_{2,i} := \bigoplus \{K x_1^{\alpha_1} x_2^{\alpha_2} \partial_1^{\beta_1} \partial_2^{\beta_2} \mid \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \leq i \} \). The polynomial algebra \( P_2 := K[x_1, x_2] \simeq A_2/(A_2 \partial_1 + A_2 \partial_2) \) is a simple left \( A_2 \)-module with \( \text{End}_{A_2}(P_2) = \ker \partial_2(\partial_1)^r \ker \partial_1(\partial_2) = K. \)

The standard filtration \( \{A_{2,i} : i \in \mathbb{N}\} \) of the \( A_2 \)-module \( P_2 \) coincides with the filtration \( \{P_{2,i} := \sum_{\alpha_1, \alpha_2 \geq 0} \{K x_1^{\alpha_1} x_2^{\alpha_2} \mid \alpha_1 + \alpha_2 \leq i \} \}_{i \in \mathbb{N}} \) on the polynomial algebra \( P_2 \) by the total degree, i.e. \( P_{2,i} = A_{2,i} \cdot 1 \) for all \( i \geq 0 \), and so \( \dim_K(A_{2,i}) = (i+1)(i+2)/2 \). Therefore, \( P_2 \) is a holonomic \( A_2 \)-module with multiplicity \( c(P_2) = 1 \) and \( \text{End}_{A_2}(P_2) \simeq K. \) The Weyl algebra \( A_1 \) admits the \( K \)-isomorphism:

\[
\xi : A_1 \rightarrow A_1, \quad x \mapsto \partial, \quad \partial \mapsto -x.
\]

(8)

Then \( 1 \otimes \xi \) is an automorphism of the Weyl algebra \( A_2 \). The twisted by the automorphism \( 1 \otimes \xi \) \( A_2 \)-module \( P_2 \)

\[
P_2^{1 \otimes \xi} \simeq K[x_1, \partial_2] \simeq A_2/(A_2 \partial_1 + A_2 x_2)
\]

(9)

is a simple holonomic \( A_2 \)-module with multiplicity 1 and \( \text{End}_{A_2}(P_2^{1 \otimes \xi}) \simeq K. \)

The Weyl algebra \( A_1 \) is isomorphic to its opposite algebra \( A_1^{op} \) via

\[
A_1 \rightarrow A_1^{op}, \quad x \mapsto \partial, \quad \partial \mapsto x.
\]

(10)

In particular, each \( A_1 \)-bimodule \( A_1 M_{A_1} \) is a left \( A_2 \)-module: \( A_1 M_{A_1} = A_1 \otimes A_1^{op} M \simeq A_1 \otimes A_1^1 M = A_2 M. \)

**Lemma 2.1**

1. \( A_1 F_{A_1} = A_1 e_{00} A_1 \simeq A_1(A_1/\partial \otimes A_1/x A_1) A_1. \)
2. \( A_2 F \cong A_2/(A_2 \partial_1 + A_2 \partial_2) \cong K[x_1, x_2] \) is a simple holonomic \( A_2 \)-module with multiplicity 1 with respect to the filtration of Bernstein of the algebra \( A_2 \) and \( \text{End}_{A_2}(F) \cong K \).

**Proof.** \( A_1/(A_1 \partial \otimes A_1/xA_1)_{A_1} \cong A_1 \otimes A_1/(A_1 \partial \otimes A_1/xA_1) \partial \cong A_2/(A_2 \partial_1 + A_2 \partial_2) \cong K[x_1, x_2] \) is a simple holonomic \( A_2 \)-module with multiplicity 1 with respect to the filtration of Bernstein of the algebra \( A_2 \) and \( \text{End}_{A_2}(F) \cong K \). Clearly, \( A_1 F_{A_1} = A_1 e_{00} A_1 \) and the \( A_1 \)-bimodule homomorphism
\[
A_1/A_1 \partial \otimes A_1/xA_1 \to A_1 e_{00} A_1, \quad (1 + A_1 \partial_1) \otimes (1 + xA_1) \mapsto e_{00},
\]
is an epimorphism. Therefore, it is an isomorphism by the simplicity of the first \( A_1 \)-bimodule. \( \square \)

**Proposition 2.2**

1. \( A_1 (\mathbb{I}_1/(A_1 + F))_{A_1} \cong A_1/A_1 \partial \otimes A_1/\partial A_1 \).

2. \( A_2 (\mathbb{I}_1/(A_1 + F)) \cong A_2 (A_1/A_1 \partial \otimes A_1/A_1 x) \cong A_2/(A_2 \partial_1 + A_2 x_2) \cong K[x_1, \partial_2] \) is a simple holonomic \( A_2 \)-module with multiplicity 1 with respect to the filtration of Bernstein and \( \text{End}_{A_2}(K[x_1, \partial_2]) \cong K \).

3. \( A_1 (\mathbb{I}_1/(A_1 + F)) \cong (A_1/A_1 \partial)^{(N)} \cong K[x]^{(N)} \) is a semi-simple left \( A_1 \)-module and \( (\mathbb{I}_1/(A_1 + F))_{A_1} \cong (A_1/\partial A_1)^{(N)} \cong K[x]^{(N)} \) is a semi-simple right \( A_1 \)-module.

**Proof.** 1 and 2. Notice that \( A_2 (A_1/A_1 \partial \otimes A_1/A_1 x) \cong A_2 (A_2/(A_2 \partial_1 + A_2 x_2)) \cong K[x_1, \partial_2] \) is a simple holonomic \( A_2 \)-module with multiplicity 1 with respect to the filtration of Bernstein and \( \text{End}_{A_2}(K[x_1, \partial_2]) \cong K \). The natural filtration of the polynomial algebra \( Q' := K[x_1, \partial_2] = \bigcup_{i \geq 0} Q_{\leq i} \) by the total degree of the variables, i.e., \( Q_{\leq i} := \bigoplus_{t+i \leq j} \partial x^j \), is a standard filtration for the \( A_2 \)-module \( Q' = A_2 \cdot 1 \) since \( Q_{\leq i} = A_2 \cdot 1 \) for all \( i \geq 0 \). In particular, \( \dim_K(Q_{\leq i}) = \frac{(i+1)(i+2)}{2} \) for all \( i \geq 0 \). By (11), the \( A_1 \)-bimodule \( Q := \mathbb{I}_1/(A_1 + F) \) is the direct sum
\[
Q = \bigoplus_{i \geq 1} Q_i
\]
of its vector subspaces
\[
(Q_i)_K[H] \cong \int K[H]/x^i K[H] \cong \int K[H]/(H(H+1) \cdots (H+i-1)) \cong K[H]/(H(H+1) \cdots (H+i-1))
\]
(12)

Since \( x^i = \int H(\partial)^i = \int H(i+1) \cdots (i+1-i) \) and \( \partial^j \int = 1 \) such that \( x Q_i \subseteq Q_{i+1}, Q_i x \subseteq Q_{i-1}, \partial Q_i \subseteq Q_{i-1} \) and \( Q_i \partial \subseteq Q_{i-1} \) for all \( i \geq 1 \) where \( Q_0 := 0 \). Then \( A_2 \)-module \( Q \) has the finite dimensional ascending filtration \( Q = \bigcup_{i \geq 0} Q_{\leq i} \) where \( Q_{\leq i} := \bigoplus_{t+i \leq j} \partial x^j \) and
\[
\dim_K(Q_{\leq i}) = \sum_{j=0}^i (j+1) = \frac{(i+1)(i+2)}{2} \quad \text{for all } i \geq 0.
\]

Since \( \partial Q_{\leq i} = Q_{\leq i} \partial = 0 \), the simple filtered \( A_2 \)-module (treated as \( A_1 \)-bimodule) \( A_1 Q_{\leq i} A_1 = A_1/A_1 \partial \otimes A_1/\partial A_1 \) can be seen as a filtered \( A_2 \)-submodule of \( Q \) via \( (1 + A_1 \partial) \otimes (1 + \partial A_1) \mapsto \int + A_1 + F \). In particular, for all \( i \geq 0 \), we have the inclusions \( Q_i \subseteq Q_{\leq i} \) which are, in fact, equalities since \( \dim_K(Q_{\leq i}) = \dim_K(Q_i) \). Then,
\[
A_1 (\mathbb{I}_1/(A_1 + F))_{A_1} \cong A_1 (A_1/A_1 \partial \otimes A_1/\partial A_1)_{A_1} \cong A_2 (A_1/A_1 \partial \otimes A_1/A_1 x) \cong K[x_1, \partial_2].
\]

It is obvious that the \( A_2 \)-module \( K[x_1, \partial_2] \) is a simple \( A_2 \)-module with multiplicity 1 and \( \text{End}_{A_2}(K[x_1, \partial_2]) \cong K \).

3. Statement 3 follows from statement 1. \( \square \)

A linear map \( \varphi \) acting in a vector space \( V \) is called a **locally nilpotent map** if \( V = \bigcup_{i \geq 1} \ker(\varphi^i) \), i.e. for each element \( v \in V \) there exists a natural number \( i \) such that \( \varphi^i(v) = 0 \).
It follows from Proposition 2.2 and (12) that
\[ \ker_{i_{1}}/(A_{1} + F)(\partial) \cap \ker_{i_{1}}/(A_{1} + F)(\partial) = K(\int + A_{1} + F), \]
and that the maps \( \partial : \mathbb{I}_{1}/(A_{1} + F) \rightarrow \mathbb{I}_{1}/(A_{1} + F) \), \( u \mapsto \partial u \), and \( \partial : \mathbb{I}_{1}/(A_{1} + F) \rightarrow \mathbb{I}_{1}/(A_{1} + F) \), \( u \mapsto u \partial \), are locally nilpotent since
\[ \partial \cdot x_{1}^{i}x_{2}^{j} = ix_{1}^{i-1}x_{2}^{j}, \quad x_{1}^{i}x_{2}^{j} \cdot \partial = -jx_{1}^{i}x_{2}^{j-1}. \]

Recall that the socle \( \text{soc}_{A}(M) \) of a module \( M \) over a ring \( A \) is the sum of all the simple submodules of \( M \), if they exist, and zero, otherwise.

**Theorem 2.3**
1. The \( A_{1} \)-bimodule \( \mathbb{I}_{1} \) is a holonomic \( A_{2} \)-module of length 3 with simple non-isomorphic factors \( F \simeq A_{2}K[x_{1}, x_{2}], A_{1}A_{1} \) and \( A_{2}K[x_{1}, \partial_{2}] \). Each factor is a simple holonomic \( A_{2} \)-module with multiplicity 1 and its \( A_{2} \)-module endomorphism algebra is \( K \).
2. \( \text{soc}_{A_{2}}(\mathbb{I}_{1}) = A_{1} \oplus F \).
3. The short exact sequence of \( A_{2} \)-modules
\[ 0 \rightarrow A_{1} \oplus F \rightarrow \mathbb{I}_{1} \rightarrow \mathbb{I}_{1}/(A_{1} + F) \rightarrow 0 \]
is non-split.

**Proof.**
1. Statement 1 follows from Lemma 2.1, Proposition 2.2 and (12).
2. Suppose that the short exact sequence of \( A_{1} \)-bimodules splits, we seek a contradiction. Then, by Proposition 2.2 (1) and (13), there is a nonzero element, say \( u = \int + a + f \in \mathbb{I}_{1} \) with \( a \in A_{1} \) and \( f \in F \) such that \( \partial u = 0 \) and \( u \partial = 0 \). The first equation implies \( 1 + \partial a = -\partial f \in A_{1} \cap F = 0 \), and so \( \partial u = -1 \) in \( A_{1} \), a contradiction.
3. Statement 2 follows from statement 3. \( \square \)

**New basis for the algebra** \( I_{n} \). It follows from (11), (12) and (15) that
\[ \mathbb{I}_{1} = \bigoplus_{i,j \geq 0} Kx^{i}\partial^{j} \oplus \bigoplus_{k,l \geq 0} Ke_{kl} \oplus \{ K \int^{s} H^{t} \mid s \geq 1, t = 0, 1, \ldots, s-1 \}. \]

This gives a new \( K \)-basis for the algebra \( \mathbb{I}_{1} \): \{ \( x^{i}\partial^{j}, e_{kl}, \int^{s} H^{t} \mid i, j, k, l \geq 0; s \geq 1; t \geq 0, 1, \ldots, s-1 \). By taking \( n \)'th tensor product of this basis we obtain a new \( K \)-basis for the algebra \( \mathbb{I}_{n} = \mathbb{I}_{1}^{\otimes n} \).

**Lemma 2.4**
1. The \( A_{1} \)-bimodule \( \mathbb{I}_{1}/A_{1} \) is a holonomic \( A_{2} \)-module of length 2 with simple non-isomorphic factors \( F \simeq A_{2}K[x_{1}, x_{2}] \) and \( A_{2}K[x_{1}, \partial_{2}] \). Each factor is a simple holonomic \( A_{2} \)-module with multiplicity 1 and its \( A_{2} \)-module endomorphism algebra is \( K \).
2. \( \text{soc}_{A_{2}}(\mathbb{I}_{1}/A_{1}) = F \).
3. The short exact sequence of \( A_{2} \)-modules
\[ 0 \rightarrow F \rightarrow \mathbb{I}_{1}/A_{1} \rightarrow \mathbb{I}_{1}/(A_{1} + F) \rightarrow 0 \]
is non-split.
4. The short exact sequence of left \( A_{1} \)-modules (17) splits and so \( A_{1}(\mathbb{I}_{1}/A_{1}) \simeq K[x]^{(N)} \) is a semi-simple left \( A_{1} \)-module.
5. The short exact sequence of right \( A_{1} \)-modules (17) does not split, and so \( (\mathbb{I}_{1}/A_{1})A_{1} \) is not a semi-simple right \( A_{1} \)-module.
Proof. Statement 1 follows from Theorem 2.3(1).

3. Suppose that the short exact sequence of $A_1$-bimodules (17) splits, we seek a contradiction.

Then, by Proposition 2.2(1) and (13), there is a nonzero element, say $u = f + f + A_1 \in \mathcal{I}/A_1$ with $f \in F$ such that $0 = u = 1 + f \partial f$ and $0 = u \partial f = 1 - e_{00} + f \partial$ in $\mathcal{I}/A_1$. The first equality gives $\partial f = 0$ in $\mathcal{I}/A_1$, and so $f = \sum_{i=0}^{A_1} A_i e_{0i}$ for some $A_i \in K$. Then the second equality gives $e_{00} = f \partial = \sum_{i=0}^{A_1} \lambda_i e_{0i} \partial = \sum_{i=0}^{A_1} \lambda_i e_{0i+1}$, a contradiction.

2. Statement 2 follows from statement 3.

4. Let $L$ be the last sum in the decomposition (16), i.e.

$$\mathcal{I} = (A_1 \oplus L).$$

Then $A_1 \oplus L$ is a left $A_1$-submodule of $A_1 \mathcal{I}$. Since $\partial f = 1$, $x = f H$ and $f H = (H - 1) f$. Notice that $A_1 \oplus L$ is a right $A_1$-submodule of $\mathcal{I}$ since $f + e_{00} \not\in A_1 \oplus L$. By (15), $A_1(\mathcal{I}/A_1) \simeq K$-module splits, we seek a contradiction. In the factor module $\mathcal{I}/(A_1 + F)$, $(f + f + A_1) \partial = 0$ since $f + e_{00} \not\in A_1 + F$. Then the spliteness implies that $(f + f + A_1) \partial = 0$ in $\mathcal{I}/A_1$ for some element $f \in F$, equivalently, $-e_{00} + f \partial \in A_1 \cap F = 0$ in $\mathcal{I}$, i.e. $f \partial = e_{00}$, this is obviously impossible (since $e_{i,j} \partial = e_{i,j+1}$), a contradiction. □

Let $M$ be a module over a ring $R$. The socle $soc_R(M)$, if nonzero, is the largest semi-simple submodule of $M$. The socle of $M$, if nonzero, is the only essential semi-simple submodule. The socle chain of the module $M$ is the ascending chain of its submodules:

$$soc^0_R(M) := soc_R(M) \subseteq soc^1_R(M) \subseteq \cdots \subseteq soc^i_R(M) \subseteq \cdots$$

where $soc^i_R(M) := \varphi_{i-1}^{-1}(soc_R(M)/soc^{i-1}_R(M))$ where $\varphi_i : M \rightarrow M/soc^{i-1}_R(M)$, $m \mapsto m + soc^{i-1}_R(M)$. Let $soc^i_R(M) := \bigcup_{i=0}^{n} soc^i_R(M)$. If $M = soc^i_R(M)$ then

$$1. soc_R(M) = 1 + \min\{i \geq 0 | M = soc^i_R(M)\}$$

is called the socle length of the $R$-module $M$. So, a nonzero module is semi-simple iff its socle length is 1.

Theorem 2.5

1. The $A_n$-bimodule $\mathcal{I}_n$ is a holonomic $A_2n$-module of length $3^n$ with pairwise non-isomorphic simple factors and each of them is the tensor product $\bigotimes_{i=1}^{n} M_i$ of simple $A_2(i)$-modules $M_i$ as in Theorem 2.3 for $i = 1, \ldots, n$. Each simple factor $\bigotimes_{i=1}^{n} M_i$ is a simple holonomic $A_2n$-module and has multiplicity 1 (with respect to the filtration of Bernstein on the algebra $A_2n$) and its $A_2n$-module endomorphism algebra is $K$.

2. $soc_{A_{2n}}(\mathcal{I}_n) = \bigotimes_{i=1}^{n} soc_{A_2(i)}(\mathcal{I}_1(i)) = \bigotimes_{i=1}^{n} (A_1(i) \oplus F(i))$.

3. The socle length of the $A_{2n}$-module $\mathcal{I}_n$ is $n + 1$. For each number $m = 0, 1, \ldots, n$,

$$soc^m_{A_{2n}}(\mathcal{I}_n) = \bigotimes_{1 \leq i_1 + \cdots + i_n = m} soc^i_{A_2(i)}(\mathcal{I}_1(i))$$

where all $i_s \in \{0, 1\}$ and $soc^i_{A_2(i)} = \begin{cases} A_1(i) \oplus F(i) & \text{if } j = 0, \\ \mathcal{I}_1(i) & \text{if } j = 1. \end{cases}$

4. For each number $m = 0, 1, \ldots, n$,

$$soc^m_{A_{2n}}(\mathcal{I}_n)/soc^{m-1}_{A_{2n}}(\mathcal{I}_n) = \bigotimes_{1 \leq i_1 + \cdots + i_n = m} soc^i_{A_2(i)}(\mathcal{I}_1(i))/soc^{i-1}_{A_2(i)}(\mathcal{I}_1(i))$$
and its length (as an $A_{2n}$-module) is $(\binom{n}{m})2^{n-m}$ where all $i_s \in \{0,1\}$ and $soc^{-1} := 0$.

5. The left $A_{2n}$-module $\mathbb{I}_n$ has multiplicity $3^n$ with respect to the filtration of Bernstein of the Weyl algebra $A_{2n}$.

**Remark.** The sum of lengths of all the factors in statement 4 is $3^n$ as

$$3^n = (1 + 2)^n = \sum_{m=0}^{n} \binom{n}{m}2^{n-m}.$$ 

**Proof.** 1. By Theorem 2.3.(1), each of the tensor multiples $\mathbb{I}_1(i)$ in $\mathbb{I}_n = \bigotimes_{i=1}^{n} \mathbb{I}_1(i)$ has the $A_{2}(i)$-module (i.e. the $A_{1}(i)$)-bimodule) filtration of length 3 with factors $M_i$ as in Theorem 2.3.(1). By considering the tensor product of these filtrations, the $A_{2n}$-module $\mathbb{I}_n = \bigotimes_{i=1}^{n} M_i$ (i.e. the $A_{n}$-bimodule) has a filtration (of length $3^n$) with factors $\bigotimes_{i=1}^{n} M_i$. It is obvious that each $A_{2n}$-module $\bigotimes_{i=1}^{n} M_i$ is isomorphic to a twisted $A_{2n}$-module $\sigma P_{2n}$ by an automorphism $\sigma$ of the Weyl algebra $A_{2n}$ that preserves the filtration of Bernstein on the algebra $A_{2n}$ where

$$P_{2n} = K[x_1, \ldots, x_{2n}] \simeq A_{2n}/\sum_{i=1}^{2n} A_{2n}\partial_i.$$ 

This statement is obvious for $n = 1$, then the general case follows at once. Since the $A_{2n}$-module $P_{2n}$ is simple, holonomic with multiplicity 1 (since $\epsilon(\sigma P_{2n}) = \epsilon(P_{2n}) = 1$) and $End_{A_{2n}}(P_{2n}) \simeq K$, then so are all the $A_{2n}$-modules $\bigotimes_{i=1}^{n} M_i$. This finishes the proof of statement 1.

2. Statement 2 follows from statement 3.

3. To prove statement 3 we use induction on $n$. The initial step when $n = 1$ is true due to Theorem 2.3.(1). Suppose that $n > 1$ and the statement holds for all $n' < n$. Let $\{s^0 = A_1 \bigoplus F, s^1 = \mathbb{I}_1\}$ be the socle filtration for $A_n \mathbb{I}_n$, and let $\{s^0, s^1, \ldots, s^{n-1}\}$ be the socle filtration for $A_{n-1} \mathbb{I}_{n-1}$. We are going to prove that

$$\{s^0 := s^0 \otimes s^0, s^1 := s^0 \otimes s^1 + s^1 \otimes s^0, \ldots, s^{n-1} := s^0 \otimes s^{n-1} + s^1 \otimes s^{n-2}, s^m := s^1 \otimes s^{n-1}\}$$

is the socle filtration for $A_n \mathbb{I}_n$. Notice that $A_n = A_1 \otimes A_{n-1}$, $\mathbb{I}_n = \mathbb{I}_1 \otimes \mathbb{I}_{n-1}$ and $\{s^0 \otimes s^i\}_{i=0}^{n-1}$ is the socle filtration for $A_{n-1}(s^0 \otimes \mathbb{I}_{n-1})A_{n-1} = s^0 \otimes (A_{n-1} \mathbb{I}_{n-1} A_{n-1})$ since the $\mathbb{I}_{n-1}$-bimodules $s^0 \otimes s^i/s^i$ are semi-simple. Since, for each number $m = 0, 1, \ldots, n$, the $A_n$-subbimodule

$$\mathcal{S}^m := s^m/s^{m-1} = s^0 \otimes (s^m/s^{m-1}) \bigoplus (s^1/s^0) \otimes (s^{m-1}/s^{m-2})$$

(whence $\overline{s}^0 = s^0 \otimes s^0$) of $\mathbb{I}_n/s^{m-1}$ is semi-simple, in order to finish the proof of statement 3 it suffices to show that $\mathcal{S}^m$ is an essential $A_n$-subbimodule of $\mathbb{I}_n/s^{m-1}$. Let $a$ be a nonzero element of the $A_n$-bimodule $\mathbb{I}_n/s^{m-1}$. We have to show that $A_n a A_n \cap \mathcal{S}^m \neq 0$. If $a \in s^0 \otimes \mathbb{I}_{n-1} + s^{m-1}$ then

$$0 \neq \mathbb{I}_{n-1} a \mathbb{I}_{n-1} \cap s^0 \otimes (s^m/s^{m-1}) \subseteq \mathcal{S}^m$$

(since $\{s^0 \otimes s^i\}_{i=0}^{n-1}$ is the socle filtration for $A_{n-1}(s^0 \otimes \mathbb{I}_{n-1})A_{n-1}$). If $a \notin s^0 \otimes \mathbb{I}_{n-1} + s^{m-1}$ then using the explicit basis $\{x_1^j x_2^k\}_{j \geq 0}$ for the $A_1$-bimodule $s^1/s^0$ (Proposition 2.2.(1)) and the action of the element $\partial$ on it (see (14)), we can find natural numbers, say $k$ and $l$, such that, by (13), the element

$$a' := \partial^k a \partial^l = \int \otimes u_1 + v_2 \otimes u_2 + \cdots + v_s \otimes u_s,$$

such that $0 \neq u_1 \in \mathbb{I}_{n-1}/s^{m-1}$ (in particular, $a'$ is a nonzero element of $\mathbb{I}_n/s^{m-1}$); $u_2, \ldots, u_s$ are linearly independent elements of $\mathbb{I}_{n-1}$; $v_2, \ldots, v_s$ are linearly independent elements of $s^0$. If the elements $u_1, u_2, \ldots, u_s$ are linearly independent then

$$a'' := \partial a' = 1 \otimes u_1 + (\partial v_2) \otimes u_2 + \cdots + (\partial v_s) \otimes u_s$$
is a nonzero element of $s^0 \otimes \mathbb{I}_{n-1}$, and so, by the previous case $\mathbb{I}_n a \mathbb{I}_n \cap \mathfrak{m}^m \neq 0$.

If the elements $u_1, u_2, \ldots, u_s$ are linearly dependent then $u_1 = \sum_{i=2}^s \lambda_i u_i$ for some elements $\lambda_i \in K$ not all of which are zero ones, say $\lambda_2 \neq 0$. The element $a'$ can be written as $a' = (\lambda_2 f + v_2) \otimes u_2 + \cdots + (\lambda_s f + v_s) \otimes u_s$.

$$a'' := \partial a' = (\lambda_2 + \partial v_2) \otimes u_2 + \cdots + (\lambda_s + \partial v_s) \otimes u_s.$$ 

We claim that $a'' \neq 0$. Suppose that $a'' = 0$, we seek a contradiction. Then $\lambda_2 + \partial v_2 = 0, \ldots, \lambda_s + \partial v_s = 0$ in $A_1 \bigoplus F$ (since the elements $u_2, \ldots, u_s$ are linearly independent). The first equality yields $0 \neq \lambda_2 = \partial b$ in the Weyl algebra $A_1$ for some element $b \in A_1$. This is clearly impossible. Therefore, $a''$ is a nonzero element of $s^0 \otimes \mathbb{I}_{n-1}$, and so, by the previous case, $\mathbb{I}_n a \mathbb{I}_n \cap \mathfrak{m}^m \neq 0$.

4. The equality follows from statement 3. To prove the claim about the length note that $(\binom{n}{m})$ is the number of vectors $(i_1, \ldots, i_n) \in \{0, 1\}^n$ with $i_1 + \cdots + i_n = m$; and for each choice of $(i_1, \ldots, i_n)$ the length of the $A_{2n}$-module $\bigotimes_{s=1}^n \text{soc}^s_{A_{2n}(i)}(\mathbb{I}_1(i))$/soc$_{A_{2n}(i)}^1(\mathbb{I}_1(i))$ is $2^{n-m}$. Therefore, the length of the $A_{2n}$-module soc$_{A_{2n}}^m (\mathbb{I}_m)/$soc$_{A_{2n}}^{m-1}(\mathbb{I}_m)$ is $(\binom{n}{m})2^{n-m}$.

5. Statement 5 follows from statement 1 and the additivity of the multiplicity on the holonomic modules.

\[\square\]

3 The algebra $\mathbb{I}_n$ is coherent iff $n = 1$

The aim of this section is to prove Theorem 3.1

A module $M$ over a ring $R$ is finitely presented if there is an exact sequence of modules $R^m \rightarrow R^n \rightarrow M \rightarrow 0$. A finitely generated module is a coherent module if every finitely generated submodule is finitely presented. A ring $R$ is a left (resp. right) coherent ring if the module $\mu R$ (resp. $R \mu$) is coherent. A ring $R$ is a left coherent ring iff, for each element $r \in R$, ker$R(r)$ is a finitely generated left $R$-module and the intersection of two finitely generated left ideals is finitely generated, Proposition 13.3. \[6\]. Each left Noetherian ring is left coherent but not vice versa.

**Theorem 3.1** The algebra $\mathbb{I}_n$ is a left coherent algebra iff the algebra $\mathbb{I}_n$ is a right coherent algebra iff $n = 1$.

**Proof.** The first ‘iff’ is obvious since the algebra $\mathbb{I}_n$ is self-dual $\mathbb{I}_n$, i.e. is isomorphic to its opposite algebra $\mathbb{I}_n^op$. If $n = 1$ the algebra is a left coherent algebra $\mathbb{I}_n$ [2]. If $n \geq 2$ then the algebra $\mathbb{I}_2$ is not a left coherent algebra since, by Lemma 3.2, ker$_1(\langle H_1 - H_2 \rangle) = \ker_{e_1}(\langle H_1 - H_2 \rangle) \otimes \mathbb{I}_{n-2} \simeq \mathbb{I}_n(P_2 \otimes \mathbb{I}_{n-2})^{(0)}$ is an infinite direct sum of nonzero $\mathbb{I}_n$-modules, hence it is not finitely generated. Therefore, the algebra $\mathbb{I}_n$ is not a left coherent algebra, by Proposition 13.3. \[6\]. \[\square\]

**Lemma 3.2** ker$_1(\langle H_1 - H_2 \rangle) = \ker_{e_2}(\langle H_1 - H_2 \rangle) = \bigoplus_{i,j,k \in \mathbb{N}} Ke_{ij}(1)e_{kj}(2) \simeq (i_2P_2)^{(0)}$.

**Proof.** The algebra $B_2 = \mathbb{I}_2/\mathfrak{a}_2$ is a domain $\mathbb{I}_2$ where $\mathfrak{a}_2 := F(1) \otimes I_1(2) + I_1(1) \otimes F(2)$ and $H_1 - H_2 \not\in \mathfrak{a}_2$. Therefore, $\mathcal{K} := \ker_{e_1}(\langle H_1 - H_2 \rangle) = \ker_{e_2}(\langle H_1 - H_2 \rangle)$. Let $F_2 := F(1) \otimes F(2)$. Notice that $i_2(\mathfrak{a}_2/F_2)i_2 \simeq F(1) \otimes B_1(2) \bigoplus B_1(1) \otimes F(2)$ is a direct sum of two $\mathbb{I}_2$-bimodules. It follows from the presentation $F(1) \otimes B_1(2) = \bigoplus_{i,j \in \mathbb{N}, k \in \mathbb{Z}} e_{ij}(1) \otimes \partial^k_2 K[H_2]$ that ker$F(1)\otimes B_1(2)(\langle H_1 - H_2 \rangle) = 0$. Similarly, ker$B_1(1)\otimes F(2)(\langle H_1 - H_2 \rangle) = 0$ (or use the $(1, 2)$-symmetry). Therefore, $\mathcal{K} = \ker_{e_2}(\langle H_1 - H_2 \rangle) = \bigoplus_{i,j,k \in \mathbb{N}} Ke_{ij}(1)e_{kj}(2)$ $\simeq \bigoplus_{j \in \mathbb{N}} (i_2P_2) \simeq i_2(P_2)^{(0)}$. \[\square\]

8
4 The algebra \( \mathbb{I}_n \) is maximal order

The aim of this section is to prove Theorem 4.3.

Let \( R \) be a ring. An element \( x \in R \) is right regular if \( xr = 0 \) implies \( r = 0 \) for \( r \in R \). Similarly, a left regular element is defined. A left and right regular element is called a regular element.

The sets of regular/left regular/right regular elements of a ring \( R \) are denoted respectively by \( C_R(0), C_R(0) \) and \( C_R(0) \). For an arbitrary ring \( R \) there exists the largest (w.r.t. inclusion) left regular denominator set \( S_l,0 = S_l,0(R) \) in the ring \( R \) (regular means that \( S_l,0(R) \subseteq C_R(0) \)), and so \( Q_l(R) := S_l,0^{-1}R \) is the largest left quotient ring of \( R \) (Theorem 2.1, [4]). Similarly, for an arbitrary ring \( R \) there exists the largest right regular denominator set \( S_r,0 = S_r,0(R) \) in \( R \), and so \( Q_r(R) := RS_r^{-1} \) is the largest right quotient ring of \( R \).

The aim of this section is to prove Theorem 4.3.

\[ \text{Lemma 4.1} \]

Let \( R \) be an arbitrary ring. A subring \( S \) (not necessarily with 1) of the largest right quotient ring \( Q_r,0(R) \) of the ring \( R \) is called a right order in \( Q_r,0(R) \) if each element \( q \in Q_r,0(R) \) has the form \( rs^{-1} \) for some elements \( r, s \in S \). A subring \( S \) (not necessarily with 1) of the largest left quotient ring \( Q_l,0(R) \) of the ring \( R \) is called a left order in \( Q_l,0(R) \) if each element \( q \in Q_l,0(R) \) has the form \( s^{-1}r \) for some elements \( r, s \in S \).

Let \( R \) be a ring. A subring \( S \) (not necessarily with 1) of the largest right quotient ring \( Q_r,0(R) \) of the ring \( R \) is called a right order in \( Q_r,0(R) \) if each element \( q \in Q_r,0(R) \) has the form \( rs^{-1} \) for some elements \( r, s \in S \). A subring \( S \) (not necessarily with 1) of the largest left quotient ring \( Q_l,0(R) \) of the ring \( R \) is called a left order in \( Q_l,0(R) \) if each element \( q \in Q_l,0(R) \) has the form \( s^{-1}r \) for some elements \( r, s \in S \).

\[ \text{Lemma 4.2} \]

Let \( R \) be an arbitrary ring. A subring \( S \) (not necessarily with 1) of the largest right quotient ring \( Q_r,0(R) \) of the ring \( R \) is called a right order in \( Q_r,0(R) \) if each element \( q \in Q_r,0(R) \) has the form \( rs^{-1} \) for some elements \( r, s \in S \). A subring \( S \) (not necessarily with 1) of the largest left quotient ring \( Q_l,0(R) \) of the ring \( R \) is called a left order in \( Q_l,0(R) \) if each element \( q \in Q_l,0(R) \) has the form \( s^{-1}r \) for some elements \( r, s \in S \).

\[ \text{Theorem 4.3} \]

The algebra \( \mathbb{I}_n \) is a maximal left order in \( Q_l(\mathbb{I}_n) \) and a maximal right order in \( Q_r(\mathbb{I}_n) \).

\[ \text{Proof.} \] Suppose that \( \mathbb{I}_n \subseteq S \) and \( S \sim \mathbb{I}_n \) for some right order \( S \) in \( Q_r(\mathbb{I}_n) \). Then \( aSb \subseteq \mathbb{I}_n \) for some elements \( a, b \in \mathbb{I}_n \cap Q_r(\mathbb{I}_n)^* \), by Lemma 4.1 where \( Q_r(\mathbb{I}_n)^* \) is the group of units of the algebra \( Q_r(\mathbb{I}_n) \). By Theorem 2.8, [4], \( \mathbb{I}_n \cap Q_r(\mathbb{I}_n)^* = S_{r,0}(\mathbb{I}_n) \). Then, by Corollary 2.2 (3), \( \mathbb{I}_n = \mathbb{I}_n aSb \mathbb{I}_n = (\mathbb{I}_n aSb) S_{r,0}(\mathbb{I}_n) = \mathbb{I}_n S_{r,0}(\mathbb{I}_n) = S \), i.e. \( \mathbb{I}_n = S \). Then the algebra \( \mathbb{I}_n \) is a maximal right order in \( Q_r(\mathbb{I}_n) \). Since the algebra \( \mathbb{I}_n \) admits an involution [1], the algebra \( \mathbb{I}_n \) is also a maximal left order in \( Q_l(\mathbb{I}_n) \). □
References

[1] V. V. Bavula, The algebra of integro-differential operators on a polynomial algebra, *Journal of the London Math. Soc.* DOI:10.1112/jlms/jdq081, (Arxiv:math.RA: 0912.0723).

[2] V. V. Bavula, The algebra of integro-differential operators on an affine line and its modules, (Arxiv:math.RA: 1011.2997).

[3] V. V. Bavula, An analogue of the Conjecture of Dixmier is true for the algebra of polynomial integro-differential operators, (Arxiv:math.RA: 1011.3009).

[4] V. V. Bavula, The largest left quotient ring of a ring, (Arxiv:math.RA:1101.5107).

[5] I. N. Bernstein The analytic continuation of generalized functions with respect to a parameter, *Funct. Anal. and Appl.* 6 (1972), no. 4, 26–40.

[6] B. Stenström, Rings of quotients. An introduction to methods of ring theory. Springer-Verlag, New York-Heidelberg, 1975.

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