Fisher-KPP equation with small data and the extremal process of branching Brownian motion

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November 3, 2021

Abstract

We consider the limiting extremal process $X$ of the particles of the binary branching Brownian motion. We show that after a shift by the logarithm of the derivative martingale $Z$, the rescaled "density" of particles, which are at distance $n + x$ from a position close to the tip of $X$, converges in probability to a multiple of the exponential $e^x$ as $n \to +\infty$. We also show that the fluctuations of the density, after another scaling and an additional random but explicit shift, converge to a 1-stable random variable. Our approach uses analytic techniques and is motivated by the connection between the properties of the branching Brownian motion and the Bramson shift of the solutions to the Fisher-KPP equation with some specific initial conditions initiated in [9, 10] and further developed in the present paper. The proofs of the limit theorems for $X$ rely crucially on the fine asymptotics of the behavior of the Bramson shift for the Fisher-KPP equation starting with initial conditions of "size" $0 < \varepsilon \ll 1$, up to terms of the order $[(\log \varepsilon^{-1})]^{-1-\gamma}$, with some $\gamma > 0$.

1 Introduction

The BBM connection to the Fisher-KPP equation

The standard binary branching Brownian motion (BBM) on $\mathbb{R}$ is a process that starts at the initial time $t = 0$ with one particle at the position $x = 0$. The particle performs a Brownian motion until a random, exponentially distributed with parameter 1, time $\tau > 0$ when it splits into two off-spring particles. Each of the new particles starts an independent Brownian motion at the branching point. They have independent, again exponentially distributed with parameter 1, clocks attached to them that ring at their respective branching times. At the branching time, the particle that is branching produces two independent off-spring, and the process continues, so that at a time $t > 0$ we have a random number $N_t$ of Brownian particles. It will be convenient for us to assume that the variance of each individual Brownian motion is not $t$ but $2t$.

A remarkable observation by McKean [19] is a connection between the location of the maximal particle for the BBM and the Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2,$$

(1.1)
with the initial condition \( u(0, x) = \mathbb{1}(x < 0) \). Let \( x_1(t) \geq x_2(t) \geq \ldots \geq x_{N_t}(t) \) be the positions of the BBM particles at time \( t \). McKean has discovered an exact formula

\[
    u(t, x) = \mathbb{P}[x_{1}(t) > x].
\]

(1.2)

More generally, given a sufficiently regular function \( g(x) \), the solution to the Fisher-KPP equation (1.1) with the initial condition \( u(0, x) = g(x) \) can be written as

\[
    u(t, x) = 1 - \mathbb{E}_x \left[ \prod_{j=1}^{N_t} (1 - g(x_j(t))) \right] = 1 - \mathbb{E}_0 \left[ \prod_{j=1}^{N_t} (1 - g(x + x_j(t))) \right]
\]

\[
    = 1 - \mathbb{E}_0 \left[ \prod_{j=1}^{N_t} (1 - g(x - x_j(t))) \right] = 1 - \mathbb{E}_{-x} \left[ \prod_{j=1}^{N_t} (1 - g(-x_j(t))) \right],
\]

(1.3)

so that (1.1) is a special case of (1.3) with \( g(x) = \mathbb{1}(x < 0) \). Here, \( \mathbb{E}_x \) refers to a BBM that starts at \( t = 0 \) with a single particle at the position \( x \in \mathbb{R} \) and not at \( x = 0 \). The slightly unusual form of (1.3) in the right side, with the process starting at \(-x\), will be convenient when we look at the limiting measure of the BBM, also with a flipped sign, as in (1.10) below.

McKean’s interpretation has been used by Bramson [7, 8] to establish the following result on the long term behavior of the solutions to the Fisher-KPP equation. It has been known since the pioneering work of Fisher [12] and Kolmogorov, Petrovskii and Piskunov [16] that the Fisher-KPP equation admits traveling wave solutions of the form \( u(t, x) = U_c(x - ct) \) that satisfy

\[
    -cU'' = U''' + U - U^2, \quad U_c(-\infty) = 1, \quad U_c(+\infty) = 0,
\]

(1.4)

for all \( c \geq c_* = 2 \). We will denote by \( U(x) \) the traveling wave \( U_2(x) \) that corresponds to the minimal speed \( c_* = 2 \):

\[
    -2U' = U'' + U - U^2, \quad U(-\infty) = 1, \quad U(+\infty) = 0.
\]

(1.5)

Solutions to (1.5) are defined up to a translation in space. We fix the particular translate by requiring that the traveling wave has the asymptotics

\[
    U(x) \sim (x + k_0)e^{-x}, \quad \text{as } x \to +\infty,
\]

(1.6)

with the pre-factor in front of the right side equal to one, and some \( k_0 \in \mathbb{R} \) that is not, to the best of our knowledge, explicit. Let now \( u(t, x) \) be the solution to (1.1) with the initial condition \( g(x) \) such that \( 0 \leq g(x) \leq 1 \) for all \( x \in \mathbb{R} \), and \( g(x) \) is compactly supported on the right – there exists \( L_0 \) such that \( g(x) = 0 \) for all \( x \geq L_0 \). It was already shown in [16], for the particular example of \( g(x) = \mathbb{1}(x \leq 0) \), that there exists a reference frame \( m_{kpp}(t) \) such that \( m_{kpp}(t)/t \to 2 \) as \( t \to +\infty \) and

\[
    u(t, x + m_{kpp}(t)) \to U(x) \text{ as } t \to +\infty,
\]

(1.7)

uniformly on semi-infinite intervals of the form \( x \geq K \), for each \( K \in \mathbb{R} \) fixed. Bramson has refined this result, showing that there exists a constant \( \hat{s}[g] \) that is known as the Bramson shift corresponding to the initial condition \( g \), such that

\[
    u(t, x + m(t)) \to U(x + \hat{s}[g]) \text{ as } t \to +\infty,
\]

(1.8)

uniformly on semi-infinite intervals of the form \( x \geq K \), for each \( K \in \mathbb{R} \) fixed, with

\[
    m(t) = 2t - \frac{3}{2} \log t.
\]

(1.9)
Note that we have chosen the sign of \( \hat{s}[\varphi] \) in (1.8) in the way that makes the shift positive for "small" initial conditions that we will consider later. In that sense, at a time \( t \gg 1 \), the solution is located at the position \( m(t) \). A shorter probabilistic proof of this convergence was given recently in [25], and PDE proofs of various versions on Bramson’s results have been obtained in [14, 18, 23, 26], with further refinements in [13, 15, 24] and especially in the recent fascinating paper [5].

The limiting extremal process of BBM and its connection to the Bramson shift

Motivated by the above discussion, one may consider not just the maximal particle but the statistics of the full BBM process re-centered at the location \( m(t) \), and ask if it can also be connected to the solutions to the Fisher-KPP equation. Let \( x_1(t) \geq x_2(t) \geq \ldots \) be the positions of the BBM particles at time \( t \), and consider the BBM measure seen from \( m(t) \):

\[
X_t = \sum_{k \leq N_t} \delta_{m(t)-x_k(t)}. \tag{1.10}
\]

Recall that \( N_t \) is the number of particles alive at time \( t \). It was shown in [1, 2, 3, 10] that there exists a point process \( X \) so that

\[
X_t \Rightarrow X = \sum_k \delta_{\chi_k} \quad \text{as } t \to +\infty, \tag{1.11}
\]

with \( \chi_1 \leq \chi_2 \leq \ldots \). That is, \( \chi_1 \) corresponds to the position of the maximal particle in the BBM, \( \chi_2 \) to the second largest, and so on. In what follows, we will call \( X \) the limiting extremal process or simply extremal process. Sometimes, it is also referred to as the decorated Poisson point process, see e.g. [1]. The properties of the limit measure \( X \) are closely related to the long time limit of the derivative martingale introduced in [17]:

\[
Z_t = \sum_{k \leq N_t} (2t-x_k(t))e^{-(2t-x_k(t))} \to Z \quad \text{as } t \to +\infty, \quad \mathbb{P}\text{-a.s.} \tag{1.12}
\]

It turns out that there exists a direct connection between \( X, Z \) and the Bramson shift via the Laplace transform of \( X \). As explained in Appendix C of [10], see also [3], the results of [17] imply that for any test function \( \psi \geq 0 \) that is compactly supported on the right, we have the identity

\[
\mathbb{E} \left[ e^{-X(\psi)} \right] = \mathbb{E} \left[ e^{-Ze^{-\hat{s}[\hat{\psi}]} \hat{\psi}(x)} \right], \quad \hat{\psi}(x) = 1 - e^{-\psi(x)}. \tag{1.13}
\]

Here, for a measure \( \mu \) and a function \( f \) we use the notation

\[
\mu(f) = \int_{\mathbb{R}} f(x)\mu(dx).
\]

Note that \( 0 \leq \hat{\psi}(x) \leq 1 \), and \( \hat{\psi}(x) \) is also compactly supported on the right, so that its Bramson shift \( \hat{s}[\hat{\psi}] \) is well-defined and finite. In addition to a special case of (1.13) with \( \hat{\psi}(x) = 1(x \leq 0) \), it was shown in [17] that for each \( y \in \mathbb{R} \) we have

\[
\mathbb{E} \left[ e^{-Ze^{-\psi}} \right] = 1 - U(y). \tag{1.14}
\]

This, together with (1.13), characterizes the Laplace transform of \( X \): for any test function \( \psi \geq 0 \) compactly supported on the right, we have the duality identity

\[
\mathbb{E} \left[ e^{-X(\psi)} \right] = 1 - U(\hat{s}[\hat{\psi}]), \quad \hat{\psi} = 1 - e^{-\psi}. \tag{1.15}
\]
Let us note that the normalization of the traveling wave in (1.6) implies, in particular, that extra shifts appear neither in (1.14), nor in (1.15). A helpful discussion of the normalization constants in these identities can be found in Chapter 2 of [6]. We also explain this in Section 2.

It is important to note that, in fact, the results in [1], [3] and in Appendix C of [10] imply the conditional version of (1.13), see Lemma 2.1 below:

$$\mathbb{E} \left[ e^{-\mathcal{X}(\psi)} | Z \right] = e^{-Ze^{-\bar{s}[\psi]}}. \tag{1.16}$$

In principle, (1.16) completely characterizes the conditional distribution of the measure $\mathcal{X}$ in terms of its conditional Laplace transform. However, the Bramson shift is a very implicit function of the initial condition, and making the direct use of (1.16) is by no means straightforward. One of the goals of the present paper is precisely to make use of this connection to obtain new results about the extremal process $\mathcal{X}$.

Let us illustrate what kind of results on the Bramson shift we may need on the example of the asymptotic growth of $\mathcal{X}$, Theorem 1.1 in [11], originally conjectured in [10]. This result says that there exists $A_0 > 0$ so that

$$\frac{1}{A_0 Z} \mathcal{X}((\infty, x]) \to 1, \quad \text{as} \quad x \to +\infty, \quad \text{in probability.} \tag{1.17}$$

As a consequence of the results of the present paper it turns out that the constant $A_0 = 1/\sqrt{4\pi}$, so that we actually have

$$\left( \frac{Zxe^{x}}{\sqrt{4\pi}} \right)^{-1} \mathcal{X}((\infty, x]) \to 1, \quad \text{as} \quad x \to +\infty, \quad \text{in probability.} \tag{1.18}$$

The proof of (1.17) in [11] uses purely probabilistic tools. In order to relate this result to the Bramson shift and the realm of PDE, we can do the following. Consider the shifted and rescaled version of the measure $\mathcal{X}$, as in (1.17):

$$Y_n(dx) = n^{-1}e^{-n} \mathcal{X}_n(dx), \tag{1.19}$$

where

$$\mathcal{X}_n = \sum_k \delta_{\chi_k-n}, \tag{1.20}$$

so that

$$\frac{1}{Zne^n} \mathcal{X}((\infty, n]) = \frac{1}{Zne^n} \mathcal{X}_n((\infty; 0]) = \frac{1}{Z} Y_n((\infty; 0]). \tag{1.21}$$

We may analyze the conditional on $Z$ Laplace transform of $Y_n$ using (1.16): given a non-negative function $\phi_0(x)$ compactly supported on the right, we have

$$\mathbb{E} \left[ e^{-Y_n(\phi_0)} | Z \right] = \mathbb{E} \left[ e^{-n^{-1}e^{-n} \mathcal{X}_n(\phi_0)} | Z \right] = \mathbb{E} \left[ e^{-\mathcal{X}(\psi_n)} | Z \right] = \exp \left\{ -Ze^{-\bar{s}[\psi_n]} \right\}, \tag{1.22}$$

with

$$\phi_n(x) = n^{-1}e^{-n} \phi_0(x-n), \quad \psi_n(x) = 1 - \exp \{-\phi_n(x)\}. \tag{1.23}$$

Note that $\psi_n$ is also compactly supported on the right, so that its Bramson shift is well-defined. Furthermore, for $n \gg 1$ the function $\psi_n(x)$ is small: it is of the size $O(n^{-1}e^{-n})$, as is $\phi_n(x)$. Thus, (1.22) relates the understanding of the conditional on $Z$ weak limit of $Y_n$ to the asymptotics of the Bramson shift for small initial conditions for the Fisher-KPP equation (1.1), and this is the
strategy we will exploit in this paper to obtain limit theorems for the process $X$. Let us stress that a
connection between the limiting statistics of BBM and the Bramson shift for small initial conditions
was already made in [10], though with a slightly different objective in mind, and in a different way.

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in mind, and in a different way.

The Bramson shift for small initial conditions

We now state the result for the Bramson shift of the solutions to the Fisher-KPP equation

$$
\frac{\partial u_\varepsilon}{\partial t} = \frac{\partial^2 u_\varepsilon}{\partial x^2} + u_\varepsilon - u_\varepsilon^2, \quad (1.24)
$$

with a small initial condition

$$
u_\varepsilon(0, x) = \varepsilon \phi_0(x), \quad (1.25)
$$

that we will need for studying the limiting behavior of $X$. Here, $\varepsilon \ll 1$ is a small parameter, and the
function $\phi_0(x)$ is non-negative, bounded and compactly supported on the right: there exists $L_0 \in \mathbb{R}$
such that $\phi_0(x) = 0$ for $x \geq L_0$. We will use the notation $x_\varepsilon = s[\varepsilon \phi_0]$ for the Bramson shift of $\varepsilon \phi_0$:

$$
|u_\varepsilon(t, x + m(t))| \to U(x + x_\varepsilon) \to 0 \text{ as } t \to +\infty. \text{ uniformly on compact intervals in } x, \quad (1.26)
$$

We chose the sign of $x_\varepsilon$ in (1.26) so that $x_\varepsilon > 0$ for $\varepsilon > 0$ sufficiently small.

In order to obtain results on the limiting behavior of $Y_n$ and on fluctuations of $Y_n$ around its
limit, we will need a rather fine asymptotics for the shift $x_\varepsilon$. To formulate this result, let us define
the constants

$$
\bar{c} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{z/\varepsilon} \phi_0(z) dz, \quad (1.27)
$$

and

$$
\bar{c}_1 = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} x e^{z/\varepsilon} \phi_0(z) dx, \quad (1.28)
$$

that depend on the initial condition $\phi_0$, as well as the universal constants

$$
g_\infty = \int_0^1 e^{x^2/4} \int_{-\infty}^{\infty} e^{-y^2/4} dy dz - 2 \int_0^1 e^{x^2/4} \int_{-\infty}^{\infty} \frac{1}{y^2} e^{-y^2/4} dy, \quad (1.29)
$$

and

$$
m_1 = \frac{3}{2} g_\infty + k_0 + \frac{1}{2}, \quad (1.30)
$$

that do not depend on $\phi_0$. Here, $k_0$ is the constant that appears in the asymptotics (1.6) for $U(x)$. The
following theorem, which is the main PDE result of this paper, allows us to obtain convergence
in law of the fluctuations of $Y_n$.

**Theorem 1.1** Under the above assumptions on $\phi_0$, we have the asymptotics

$$
x_\varepsilon = \log \varepsilon^{-1} - \log \log \varepsilon^{-1} - \log \bar{c} - 2 \frac{\log \log \varepsilon^{-1}}{\log \varepsilon^{-1}} - \left( m_1 - \log \bar{c} + \frac{\bar{c}}{\bar{c}} \right) \frac{1}{\log \varepsilon^{-1}} + O\left( \frac{1}{(\log \varepsilon^{-1})^{1+\gamma}} \right), \quad (1.31)
$$

as $\varepsilon \downarrow 0$, with some $\gamma > 0$. 


The first two terms in the right side of (1.31) have been predicted in [10] in addressing a different BBM question, using an informal Tauberian type argument that we were not able to make rigorous. The rest of the terms have not been predicted, to the best of our knowledge. The proof in the current paper does not seem to be directly related to the arguments of [10] but the general approach to the statistics of BBM via the Bramson shift asymptotics for small initial conditions comes from [10]. We postpone a heuristic discussion of the specific terms in (1.31) until Section 3, where we introduce a related problem in the self-similar variables for which the asymptotics is somewhat more transparent. The corresponding results are summarized in Propositions 3.1 and 3.2 that imply (1.31) immediately. We also discuss there some scaling arguments that allows to predict the form of the terms in (1.31).

Law of large numbers for the extremal process

Theorem 1.1 can be used to obtain results on the limiting behavior of \( Y_n(dx) \). We start with the law of large numbers for the extremal process.

To formulate the convergence results we need to introduce an additional notation. Let \( C_c \) (resp. \( C_c^+ \)) be the space of continuous (resp. non-negative continuous) compactly supported functions on \( \mathbb{R} \), and \( C_{bc} \) (resp. \( C_{bc}^+ \)) be the space of bounded continuous (resp. non-negative bounded continuous) functions on \( \mathbb{R} \) compactly supported on the right. Let \( \mathcal{M}_v \) (resp. \( \mathcal{M}_v^+ \)) be the space of signed (resp. non-negative) Radon measures \( \zeta \) on \( \mathbb{R} \) such that \( |\zeta|((-\infty,0]) < \infty \), equipped with the topology \( \tau_v \) generated by

\[
\zeta_n \xrightarrow{n \to \infty} \zeta \iff \zeta_n(f) \xrightarrow{n \to \infty} \zeta(f) \quad \forall f \in C_{bc}.
\]

Now, as a corollary of Theorem 1.1, we have the following version of (1.17). We set

\[
\mu(dx) = \frac{1}{\sqrt{4\pi}} e^x \, dx, \tag{1.32}
\]

so that

\[
\bar{c} = \mu(\phi_0). \tag{1.33}
\]

**Theorem 1.2** Conditionally on \( Z \), we have

\[
Y_n(dx) \xrightarrow{n \to \infty} Z \mu(dx) \quad \text{in } \mathcal{M}_v^+ \quad \text{in probability.} \tag{1.34}
\]

In other words, \( Y_n(dx) \) looks like an exponential shifted by \( \log Z \) to the left. This theorem also explicitly identifies the constant \( A_0 \) in (1.17). Accordingly, we can reformulate Theorem 1.2 as follows: consider the measures \( X_n^* \) shifted by \( \log Z \):

\[
X_n^* \equiv \sum_k \delta_{\chi_k-n+\log Z}, \tag{1.35}
\]

and

\[
Y_n^*(dx) = n^{-1} e^{-n} X_n^*(dx). \tag{1.36}
\]

This gives the following version of Theorem 1.2:

**Corollary 1.3** We have

\[
Y_n^*(dx) \xrightarrow{n \to \infty} \mu(dx), \quad \text{in } \mathcal{M}_v^+ \quad \text{in probability.} \tag{1.37}
\]
As we have mentioned, Theorem 1.2 and Corollary 1.3 are not really new, except for identifying the constant $A_0$, even though the approach via the asymptotics of the Bramson shift produces an analytic rather than a probabilistic proof. Note in order to prove above conditional ”laws of large numbers”, the full expansion from Theorem 1.1 is not needed, since as we will see from the proof, the first three terms in the expansion do the job.

**Weak convergence of the fluctuations of the extremal process**

Let us now explain how Theorem 1.1 can be used to obtain the limiting behavior of the fluctuations of $Y_n(dx)$. Theorem 1.2 indicates that one should consider a rescaling of the signed measure

$$Y_n(dx) - Z\mu(dx) = Y_n(dx) - \frac{1}{\sqrt{4\pi}} e^{(x + \log Z)} dx. \quad (1.38)$$

It turns out that there is an extra small deterministic shift of the exponential profile that needs to be performed before the rescaling: a better object to rescale is not as in (1.38) but

$$Y_n(dx) - \frac{1}{\sqrt{4\pi}} e^{(x + \log Z + e_n)} dx, \quad (1.39)$$

with a small extra deterministic correction

$$e_n = \log \left( 1 + \frac{2 \log n}{n} \right) \approx \frac{2 \log n}{n}. \quad (1.40)$$

The properly rescaled measures are

$$V_n(\phi_0) = n \left( Y_n(\phi_0) - (1 + \frac{2 \log n}{n}) Z\mu(\phi_0) \right),$$

$$V_n^*(\phi_0) = n \left( Y_n^*(\phi_0) - (1 + \frac{2 \log n}{n}) \mu(\phi_0) \right). \quad (1.41)$$

We denote by

$$\nu(dx) = \frac{1}{4\pi} xe^x dx, \quad (1.42)$$

and let $\{R_t, t \geq 0\}$ be a spectrally positive 1-stable stochastic process with the Laplace transform

$$E \left[ e^{-\lambda R_t} \right] = e^{t\lambda \log \lambda}, \quad \forall \lambda > 0, \ t \geq 0,$$

such that $\{R_t, t \geq 0\}$ is independent of $Z$. We use the notation $\Rightarrow$ for the convergence in distribution. Here is the main probabilistic result of this paper.

**Theorem 1.4** (i) Conditionally on $Z$, we have

$$V_n \Rightarrow L_Z(dx) \quad \text{in} \ M_v, \ \text{as} \ n \to \infty. \quad (1.43)$$

Here, $L_Z$ is a random measure such that

$$L_Z(dx) = R_Z\mu(dx) + Z(m_1\mu(dx) + \nu(dx)), \quad (1.44)$$

and $m_1$ is the constant defined in (1.30).

(ii) We also have

$$V_n^*(dx) \Rightarrow L_1(dx) \quad \text{in} \ M_v, \ \text{as} \ n \to \infty. \quad (1.45)$$

Here, $L_1(dx)$ is a random measure such that

$$L_1(dx) = R_1\mu(dx) + (m_1\mu(dx) + \nu(dx)). \quad (1.46)$$
Note that, conditionally on \( Z \), for any test function \( \phi_0 \in C^+_bc \), \( L_Z(\phi_0) \) is a spectrally positive 1-stable random variable with the Laplace transform:

\[
E \left[ e^{-\lambda L_Z(\phi_0)} \right| Z] = e^{-Z\mu(\phi_0)\lambda \log \lambda + \lambda Z\mu(\phi_0) \log \mu(\phi_0) - \lambda Z(m_1\mu(\phi_0) + \nu(\phi_0))}, \quad \forall \lambda > 0.
\] (1.47)

We recall that, in the study of BBM, both the 1-stable process behavior and a small deterministic correction have appeared in a related but different context for the convergence (1.12) of \( Z_t \) to \( Z \) as \( t \to +\infty \) in [21, 22]. There, the correction is of the form \( \log t/\sqrt{t} \), and, due to the diffusive nature of the space-time scaling, is close in spirit to the term \( e_n \sim \log n/n \) that appears in (1.39). However, the exact translation of the corrections appearing here and in [21, 22] is not quite clear. The \( \log t/t \) correction to the front location has also been observed in [5] and proved in [13] in the PDE context. The 1-stable like tails for \( Z \) itself have been observed already in [4] for the BBM, and in [20] for the branching random walk.

The proof of Theorem 1.4 is based on the aforementioned connection between the Laplace transform of the extremal process and the Bramson shift. In particular, it utilizes the detailed asymptotics of the Bramson shift for small initial conditions in (1.31) in Theorem 1.1. This is explained in Section 2. On the other hand, as we discuss in the same section, the law of large numbers in Theorem 1.2 and Corollary 1.3 follows just from the first three terms in the right side of (1.31).

**Organization of the paper**

The paper is organized as follows. Section 2 explains how the the probabilistic Theorem 1.4 follows from the corresponding asymptotics for the Bramson shift \( x_\epsilon \) in Theorem 1.1, as well as the duality identities (1.14) and (1.15). The rest of the paper is devoted to the proof of Theorem 1.1.

Section 3 introduces the self-similar variables that are used throughout the proofs, and reformulates the required results as Propositions 3.1 and 3.2. Theorem 1.1 is their immediate consequence, with the asymptotics in Proposition 3.2 being the crux of the matter. We also explain in that section why the various logarithmic terms in Theorem 1.1 are natural, and how the values of most of the specific constants that appear in the asymptotics for \( x_\epsilon \) arise from simple scaling arguments.

Section 4 contains the analysis of the linear Dirichlet problem that appears throughout the PDE approach to the Bramson shift [14, 23, 24]. More specifically, we describe an approximate solution to the adjoint linear problem that is one reason for the logarithmic correction in \( e_n \) in (1.40). We should stress that this is not the only contribution to the \( \log n/n \) term – the second one comes from the nonlinear term in the Fisher-KPP equation.

In Section 5 we derive some preliminary bounds used later in the proof of Proposition 3.2. Section 6 contains the proof of Proposition 3.2, with some intermediate steps proved in Section 7. The proof is rather technical and we postpone the discussion of the details until then. Finally, Appendix A contains the proof of Lemma 2.1 used in the proof of Theorem 1.4 in Section 2.

We use the notation \( C, C', C_0, C'_0, \) etc. for various constants that do not depend on \( \epsilon \), and can change from line to line.

**Acknowledgment.** We are deeply indebted to Lisa Hartung for illuminating discussions during the early stage of this work, and are extremely grateful to Éric Brunet and Julien Berestycki for generously sharing their deep understanding of various aspects of the BBM and the Bramson shift, without which this work would not be possible. We also thank the anonymous referee for suggestions on a revision of the original version of the manuscript. The work of JMR was supported by the European Research Council under the European Union’s Seventh Framework Programme (FP/2007-2013) ERC Grant Agreement n. 321186 - ReaDi, and from the ANR NONLOCAL project (ANR-14-CE25-0013). The work or LM and LR was supported by a US-Israel BSF grant, LR was partially
supported by NSF grants DMS-1613603 and DMS-1910023, and ONR grant N00014-17-1-2145. LM was partially supported by ISF grant No. 1704/18.

2 The proof of the probabilistic statements

In this section, we explain how Theorem 1.4 follows from Theorem 1.1.

The duality identity

We first briefly discuss the duality identities (1.14) and (1.15), and, in particular, the fact that no extra shift of the wave is needed in these relations, as soon as the wave normalization (1.6) is fixed. To see this, note that in the formal limit \( \psi(x) \to \infty \cdot \mathbb{1}(x \leq y) \), we have \( \hat{\psi}(x) = \mathbb{1}(x \leq 0) \), and (1.15) reduces it to

\[
\mathbb{P}[\chi_1 \leq y] = 1 - U(\hat{s}[\chi_0] + y), \quad \chi_0(x) = \mathbb{1}(x \leq 0). \tag{2.1}
\]

This is simply the definition of the Bramson shift, combined with the probabilistic interpretation (1.2) of the solution to (1.1) with the initial condition \( u(0, x) = \mathbb{1}(x \leq 0) \). Thus, no extra shift is needed neither in (2.1), nor in (1.15). The generalization of (2.1) to (1.15) is explained in Appendix C of [10]. Some normalization constants appear in the discussion there, but as we see, they are not needed once the wave is normalized by (1.6) rather than by

\[
\int_{-\infty}^{\infty} x \tilde{U}'(x) \, dx = 0, \tag{2.2}
\]

as in [10].

As for the derivative martingale identity (1.14), it is simply an immediate consequence of expressions (10) and (11) in [17], combined with the normalization (1.6) that implies that \( C = 1 \) in both of these equations in [17].

We will need the following conditional result.

**Lemma 2.1** For any \( \phi_0 \in C_{bc}^+ \), we have

\[
\mathbb{E}[e^{-X(\phi_0)} | Z] = \exp \left\{ - Ze^{-\hat{s}[\psi_0]} \right\}, \tag{2.3}
\]

and hence

\[
\mathbb{E}[e^{-Y_n(\phi_0)} | Z] = \exp \left\{ - Ze^{-\hat{s}[\psi_n]} \right\}. \tag{2.4}
\]

As we have mentioned, the proof of this lemma is essentially contained in [1], [3] and in Appendix C of [10]. We present it for the convenience of the reader in Appendix A.

**Proof of Theorem 1.2**

Let us first explain how the law of large numbers stated in Theorem 1.2 and Corollary 1.3 follows just from the first three terms in the right side of the asymptotics (1.31) in Theorem 1.1. To get the weak convergence of the measures \( Y_n(dx) \), it is sufficient to get the convergence of \( Y_n(\phi_0) \) for any \( \phi_0 \in C_{bc}^+ \). Thus, we will check convergence of the Laplace transforms of \( Y_n(\phi_0) \) (conditionally on \( Z \)).

For an arbitrary \( \phi_0 \in C_{bc}^+ \), we set, as in (1.23),

\[
\phi_n(x) = n^{-1}e^{-n}\phi_0(x - n), \quad \tilde{\phi}_n(x) = n^{-1}e^{-n}\phi_0(x), \tag{2.5}
\]
and
\[ \psi_n(x) = 1 - \exp\{-\phi_n(x)\}, \quad \tilde{\psi}_n(x) = 1 - \exp\{-\tilde{\phi}_n(x)\} \]
with
\[ \psi_0(x) = 1 - \exp\{-\phi_0(x)\}. \tag{2.7} \]
Note that both \( \psi_n \) and \( \tilde{\psi}_n \) look like "small step" initial conditions:
\[ \psi_n(x) \approx \phi_n(x), \quad \tilde{\psi}_n(x) \approx \tilde{\phi}_n(x) \text{ as } n \to +\infty. \]
The first three terms in (1.31) say that, with \( \epsilon_n = n^{-1} \exp(-n) \), the Bramson shift appearing in the right side of (2.4) has the asymptotics
\[ \hat{s}[\psi_n] = -n + \hat{s}[\tilde{\psi}_n] = -n + \log(n e^n) - \log(n + \log n) - \log \bar{c} + o(1) \to -\log \mu(\phi_0), \text{ as } n \to +\infty, \tag{2.8} \]
with the measure \( \mu \) defined in (1.32), and \( \tilde{\psi}_n \) as in (2.6).

As a consequence of Lemma 2.1 and (2.8), we obtain
\[ \lim_{n \to \infty} \mathbb{E} \left[ e^{-Y_n(\phi_0)} | Z \right] = \lim_{n \to \infty} e^{-Z e^{-\hat{s}[\psi_n]}} = e^{-Z \mu(\phi_0)}. \tag{2.9} \]
Since \( \phi_0 \in \mathcal{C}_{bc}^+ \) was arbitrary, this implies that, conditionally on \( Z \),
\[ Y_n \Rightarrow Z \mu, \quad \text{in } \mathcal{M}_n^+, \text{ as } n \to \infty. \tag{2.10} \]
Therefore, conditionally on \( Z \), we have
\[ Y_n(dx) \to Z \mu(dx) \quad \text{in } \mathcal{M}_n^+, \text{ as } n \to \infty, \text{ in probability,} \tag{2.11} \]
and this finishes the proof of Theorem 1.2. \( \square \)

Corollary 1.3 is an immediate consequence of Theorem 1.2 and its proof.

**Proof of Theorem 1.4**

Unlike the law of large numbers, the fluctuations result in Theorem 1.4 requires the full asymptotics in Theorem 1.1. We will prove only part (i) of Theorem 1.4 since the proof of part (ii) goes along the same lines. Once again, to get the weak convergence of measures \( V_n(dx) \), it is sufficient to get the convergence of \( V_n(\phi_0) \) for any \( \phi_0 \in \mathcal{C}_{bc}^+ \), and we will check the convergence of the Laplace transforms of \( V_n(\phi_0) \) (conditionally on \( Z \)). Fix an arbitrary \( \phi_0 \in \mathcal{C}_{bc}^+ \). We have
\[ \mathbb{E} \left[ e^{-V_n(\phi_0)} | Z \right] = \mathbb{E} \left[ \exp \left( - n \left( Y_n(\phi_0) - \left( 1 + \frac{2 \log n}{n} \right) Z \mu(\phi_0) \right) \right) | Z \right] \]
\[ = \mathbb{E} \left[ \exp \left( - n Y_n(\phi_0) \right) | Z \right] \exp \left( (n + 2 \log n) Z \mu(\phi_0) \right) \tag{2.12} \]
\[ = \exp \left[ - Z e^{-\hat{s}[\eta_n]} \right] \exp \left[ (n + 2 \log n) Z \bar{c} \right]. \]
We used expression (1.27) for \( \bar{c} \) and identity (2.4) above, with \( \phi_0 \) replaced by \( n \phi_0 \), and
\[ \eta_n(x) = 1 - e^{-n \phi_0(x-n)}. \]
Let us also introduce
\[ \zeta_n(x) = n \phi_n(x) = e^{-\eta_n(\phi_0(x-n), \quad x_n = \hat{s}[\zeta_n], \quad \tilde{x}_n = \hat{s}[\eta_n], \]

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so that (2.12) can be written as

\[
E[e^{-V_n(\phi_0)} | Z] = \exp \left[ -Z \left( e^{-\bar{x}_n} - (n + 2 \log n)\bar{c} \right) \right].
\] (2.13)

Note that by Theorem 1.1 with \( \varepsilon_n = e^{-n} \) we get

\[
\bar{x}_n = x_n + O(e^{-n} \|\phi_0\|_{\infty}) = -n + \log \varepsilon_n^{-1} - \log \varepsilon_n^{-1} - \log \bar{c} - \frac{2 \log \log \varepsilon_n^{-1}}{\log \varepsilon_n^{-1}}
\]

\[- \left( m_1 - \log \bar{c} + \frac{\bar{c}_1}{\bar{c}} \right) \frac{1}{\log \varepsilon_n^{-1}} + O\left( \frac{1}{(\log \varepsilon_n^{-1})^{1+\delta}} \right)
\]

\[- \left( m_1 - \log \bar{c} + \frac{\bar{c}_1}{\bar{c}} \right) \frac{1}{\log \varepsilon_n^{-1}} + O\left( \frac{1}{(\log \varepsilon_n^{-1})^{1+\delta}} \right)
\]

\[= - \log n - \log \bar{c} - \frac{2 \log n}{n} - \frac{1}{n} \left( m_1 - \log \bar{c} + \frac{\bar{c}_1}{\bar{c}} \right) + O(n^{1-\delta}).
\] (2.14)

Using this in (2.13) gives

\[
E[e^{-V_n(\phi_0)} | Z] = \exp \left[ -Z \left( e^{-\bar{x}_n} - \bar{c}n - 2\bar{c} \log n \right) \right]
\]

\[= \exp \left[ -\bar{c}Zn \left\{ \exp \left[ \frac{2 \log n}{n} + \frac{1}{n} \left( m_1 - \log \bar{c} + \frac{\bar{c}_1}{\bar{c}} \right) + O(n^{-1-\delta}) \right] - 1 - \frac{2 \log n}{n} \right\} \right]
\]

\[= \exp \left[ -\bar{c}Zn \left\{ \frac{1}{n} \left( m_1 - \log \bar{c} + \frac{\bar{c}_1}{\bar{c}} \right) + O(n^{-1-\delta}) \right\} \right]
\]

\[\rightarrow \exp \left[ -\bar{c}Z \left( m_1 - \log \bar{c} + \frac{\bar{c}_1}{\bar{c}} \right) \right] = \exp \left[ Z \left( \bar{c} \log \bar{c} - \bar{c} m_1 - \bar{c}_1 \right) \right], \text{ as } n \to +\infty.
\] (2.15)

By the definition of \( L_1, \bar{c} \) and \( \bar{c}_1 \) we get

\[
E[e^{-L_1(\phi_0)} | Z] = \exp \left[ Z\bar{c} \log \bar{c} - Z(\bar{c}m_1 + \bar{c}_1) \right],
\]

and since \( \phi_0 \in C_{bc}^+ \) was arbitrary we are done. \( \square \)

### 3 The solution asymptotics in self-similar variables

The rest of the paper contains the proof of Theorem 1.1. Instead of looking directly at the Bramson shift of a solution to the Fisher-KPP equation with a small initial condition, we will relate it to the asymptotics of the solution to a problem in the self-similar variables with an initial condition shifted far to the right.

**Proposition 3.1** Let \( r_\ell \) be the solution to

\[
\frac{\partial r_\ell}{\partial \tau} - \frac{\eta}{2} \frac{\partial^2 r_\ell}{\partial \eta^2} = \frac{1}{2} e^{-\tau/2} \frac{\partial r_\ell}{\partial \eta} + e^{3\tau/2 - \eta \exp(\tau/2)} r_\ell^2 = 0, \quad \tau > 0, \quad \eta \in \mathbb{R},
\] (3.1)

with the initial condition \( r_\ell(0, \eta) = \psi_0(\eta - \ell) \), where \( \psi_0(\eta) = e^\eta \phi_0(\eta) \). Then, for each \( \ell > 0 \), the function \( r_\ell(\tau, \eta) \) has the asymptotics

\[
r_\ell(\tau, \eta) \sim r_\infty(\ell) \eta e^{-\eta^2/4}, \quad \text{as } \tau \to +\infty, \text{ for } \eta > 0.
\] (3.2)

Furthermore, the Bramson shift that appears in Theorem 1.1 is given by

\[
x_\varepsilon = \log \varepsilon^{-1} - \log r_\infty(\ell_\varepsilon), \quad \text{with } \ell_\varepsilon = \log \varepsilon^{-1}.
\] (3.3)
As we will see in the proof of this proposition, the rather complicated equation (3.1) is simply the FKPP equation (1.1) written in the self-similar variables, with an additional rescaling of the function $u(t,x)$. Working with this form of the FKPP equation proved to be very useful in the PDE studies of the Bramson shift [14, 23, 24] and we adopt this approach here as well. The reason, discussed in Section 4, is that the nonlinear term serves as a very strong absorption for $\eta < 0$. Hence, (3.1) behaves in many ways as a linear equation on the half-line $\eta > 0$ with the Dirichlet boundary condition at $\eta = 0$. This simple-minded idea would not give the precision we need in Theorem 1.1 but gives a very useful intuition behind the lengthy computations in the present paper. The asymptotics (3.2) is essentially an immediate consequence of the results of [14, 23, 24] and the connection (3.3) between $r_\infty(\ell)$ and the Bramson shift $x_\varepsilon$ follows directly from the changes of variables required to pass from (1.1) to (3.1). This is explained in detail below, in the short proof of Proposition 3.1.

The second result, at the core of the proof of Theorem 1.1, gives the asymptotics for $r_\infty(\ell)$ as $\ell \to +\infty$.

**Proposition 3.2** The function $r_\infty(\ell)$ satisfies the following asymptotics:

$$r_\infty(\ell) = \bar{c}\ell + 2\bar{c}\log \ell + m_1\bar{c} + \bar{c}_1 - \bar{c}\log \bar{c} + O(\ell^{-\delta}),$$

with the constants $\bar{c}$, $\bar{c}_1$ and $m_1$ as in (1.27), (1.28) and (1.30).

Using (3.3), we obtain from Proposition 3.2 that

$$x_\varepsilon = \ell_\varepsilon - \log r_\infty(\ell_\varepsilon) = \ell_\varepsilon - \log \left(\bar{c}\ell_\varepsilon + 2\bar{c}\log \ell_\varepsilon + m_1\bar{c} - \bar{c}\log \bar{c} + \bar{c}_1 + O(\ell_\varepsilon^{-\delta})\right)$$

$$= \ell_\varepsilon - \log \ell_\varepsilon - \log \bar{c} - 2\frac{\log \ell_\varepsilon}{\ell_\varepsilon} - \frac{m_1}{\ell_\varepsilon} + \frac{\log \bar{c}}{\ell_\varepsilon} - \frac{\bar{c}_1}{\ell_\varepsilon} + O(\ell_\varepsilon^{-1-\delta}),$$

which proves Theorem 1.1. The rest of the paper is essentially devoted to the proof of Proposition 3.2.

**Common sense scaling arguments**

In order to verify that the constants in Proposition 3.2 are plausible, let us see assume that we have the asymptotics

$$r_\infty(\ell) = \bar{c}\ell + m_0\bar{c}\log \ell + m_1\bar{c} + m_2\bar{c}\log \bar{c} + m_3\bar{c}_1 + O(\ell^{-\delta}), \quad \text{as } \ell \to +\infty,$$

and see what simple arguments say about the possible values of the coefficients $m_0$, $m_1$, $m_2$ and $m_3$. First, consider a shifted initial condition $\phi_0^L(x) = \phi_0(x - L)$. Then, the shift $x_\varepsilon$ given by (3.5) should also change by $L$, so that

$$x_\varepsilon^L = x_\varepsilon - L.$$

Note that

$$\bar{c}_L = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^x \phi_0(x - L)dx = e^L\bar{c},$$

$$\bar{c}_1^L = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} xe^x \phi_0(x - L)dx = e^L\bar{c}_1 + L e^L\bar{c}.$$
Using (3.6) gives

\[ x^L_\varepsilon = \log \varepsilon^{-1} - \log r^L_\infty (\ell_\varepsilon) \]

\[ = \log \varepsilon^{-1} - \log \varepsilon^{-1} - \log c - m_0 \frac{\log \log \varepsilon^{-1}}{\log \varepsilon^{-1}} - \frac{m_1}{\log \varepsilon^{-1}} - \frac{m_2 \log \bar{c}}{\log \varepsilon^{-1}} - \frac{m_3 c^L_1}{\log \varepsilon^{-1}} + O(\varepsilon^{-1-\delta}) \]

\[ = \log \varepsilon^{-1} - \log \varepsilon^{-1} - \log \bar{c} - L - m_0 \frac{\log \log \varepsilon^{-1}}{\log \varepsilon^{-1}} - \frac{m_1}{\log \varepsilon^{-1}} - \frac{m_2 \log \bar{c}}{\log \varepsilon^{-1}} - \frac{m_2 L}{\log \varepsilon^{-1}} \]

\[ - m_3 \frac{\ell \bar{c} + L \ell \bar{c}}{c \bar{c} \log \varepsilon^{-1}} + O(\varepsilon^{-1-\delta}) = x_\varepsilon - L - \frac{m_2 L}{\log \varepsilon^{-1}} - \frac{m_3 L}{\log \varepsilon^{-1}} + O(\varepsilon^{-1-\delta}). \]

This means that for (3.7) to hold we must have

\[ m_3 = -m_2. \] (3.8)

Note that in (3.4) we have \( m_2 = -1 \) and \( m_3 = 1 \), so that (3.8) holds.

The second invariance is to consider an initial condition \( \phi_0^\lambda(x) = \lambda \phi_0 \). This is equivalent to keeping \( \phi_0 \) intact and replacing \( \varepsilon \) by \( \varepsilon \lambda = \varepsilon \lambda \). If we replace \( \phi_0 \) by \( \lambda \phi_0 \) in (3.5) and keep \( \varepsilon \) unchanged, this corresponds to replacing \( \bar{c} \) by \( \lambda \bar{c} \) and \( \bar{c}_1 \) by \( \lambda \bar{c}_1 \), which gives

\[ x^\lambda_\varepsilon = \log \varepsilon^{-1} - \log \varepsilon^{-1} - \log \bar{c} - \log \lambda - m_0 \frac{\log \log \varepsilon^{-1}}{\log \varepsilon^{-1}} - \frac{m_1}{\log \varepsilon^{-1}} - \frac{m_2 \log \bar{c}}{\log \varepsilon^{-1}} - \frac{m_2 \log \lambda}{\log \varepsilon^{-1}} \]

\[ - m_3 \frac{\bar{c}_1}{\bar{c} \log \varepsilon^{-1}} + O(\varepsilon^{-1-\delta}) = x_\varepsilon - \log \lambda - \frac{m_2 \log \lambda}{\log \varepsilon^{-1}} + O(\varepsilon^{-1-\delta}). \] (3.9)

If, instead, we replace \( \varepsilon \) by \( \varepsilon \lambda = \varepsilon \lambda \) and keep \( \phi_0 \) intact, we get from (3.5)

\[ x^\lambda_\varepsilon = \log \varepsilon^{-1} + \log \lambda^{-1} - \log(\log \varepsilon^{-1} + \log \lambda^{-1}) - \log \bar{c} - m_0 \frac{\log(\log \varepsilon^{-1} + \log \lambda^{-1})}{\log \varepsilon^{-1} + \log \lambda^{-1}} \]

\[ - \frac{m_1}{\log \varepsilon^{-1} + \log \lambda^{-1}} - \frac{m_2 \log \bar{c}}{\log \varepsilon^{-1} + \log \lambda^{-1}} - \frac{m_3 \bar{c}_1}{\bar{c}(\log \varepsilon^{-1} + \log \lambda^{-1})} + O(\varepsilon^{-1-\delta}) \] (3.10)

\[ = x_\varepsilon - \log \lambda - \frac{\log \lambda^{-1}}{\log \varepsilon^{-1}} + O(\varepsilon^{-1-\delta}). \]

Comparing (3.9) and (3.10) we see that we should have

\[ m_2 = -1, \] (3.11)

that, in view of (3.8), implies that \( m_3 = 1 \).

**The self-similar variables and the proof of Proposition 3.1**

The conclusion of Proposition 3.1 follows from a series of changes of variables that we now describe. We first go into the moving frame, writing solution to (1.24)-(1.25) as

\[ u_\varepsilon(t, x) = \tilde{u}_\varepsilon(t, x - 2t + \frac{3}{2} \log(t + 1)). \] (3.12)

The function \( \tilde{u}_\varepsilon(t, x) \) satisfies

\[ \frac{\partial \tilde{u}_\varepsilon}{\partial t} - \left( 2 - \frac{3}{2(t + 1)} \right) \frac{\partial \tilde{u}_\varepsilon}{\partial x} = \frac{\partial^2 \tilde{u}_\varepsilon}{\partial x^2} + \tilde{u}_\varepsilon - \tilde{u}_\varepsilon^2. \] (3.13)
Next, we take out the exponential decay factor, writing
\[ \tilde{u}_\varepsilon(t, x) = e^{-x} z_\varepsilon(t, x), \] (3.14)
which gives
\[ \frac{\partial z_\varepsilon}{\partial t} - \frac{3}{2(t+1)} \left( z_\varepsilon - \frac{\partial z_\varepsilon}{\partial x} \right) = \frac{\partial^2 z_\varepsilon}{\partial x^2} - e^{-x} z_\varepsilon^2. \] (3.15)
As (3.15) is a perturbation of the standard heat equation, it is helpful to pass to the self-similar variables:
\[ z_\varepsilon(t, x) = v_\varepsilon(\log(t+1), \frac{x}{\sqrt{t+1}}). \] (3.16)
The function \( v_\varepsilon(\tau, \eta) \) is the solution of
\[ \frac{\partial v_\varepsilon}{\partial \tau} - \frac{\eta}{2} \frac{\partial v_\varepsilon}{\partial \eta} - \frac{\partial^2 v_\varepsilon}{\partial \eta^2} = v_\varepsilon + 3 \frac{3}{2} e^{-\tau/2} \frac{\partial v_\varepsilon}{\partial \eta} + e^{\tau/2} \varepsilon^3 \varepsilon^3 \eta \exp(\tau/2) v_\varepsilon^2 = 0, \] (3.17)
with the initial condition
\[ v_\varepsilon(0, \eta) = \varepsilon \eta \phi_0(\eta). \] (3.18)
In order to get rid of the pre-factor \( \varepsilon \) in the initial condition (3.18), and also to adjust the zero-order term in (3.17), it is convenient to represent \( v_\varepsilon(\tau, \eta) \) as
\[ v_\varepsilon(\tau, \eta) = \varepsilon v_1(\tau, \eta) e^{\tau/2}. \] (3.19)
Here, \( v_1(\tau, \eta) \) is the solution of
\[ \frac{\partial v_1}{\partial \tau} - \frac{\eta}{2} \frac{\partial v_1}{\partial \eta} - \frac{\partial^2 v_1}{\partial \eta^2} = v_1 + 3 \frac{3}{2} e^{-\tau/2} \frac{\partial v_1}{\partial \eta} + \varepsilon^{3\tau/2} \varepsilon^{3\tau/2} \eta \exp(\tau/2) v_1^2 = 0, \] (3.20)
with the initial condition
\[ v_1(0, \eta) = e^{\eta} \phi_0(\eta). \] (3.21)
The next, and last, in this chain of preliminary transformations is to eliminate the pre-factor \( \varepsilon \) in the last term in (3.20). We choose
\[ \beta(\tau) = e^{-\tau/2} \log \varepsilon, \] (3.22)
so that
\[ \varepsilon e^{3\tau/2-\eta} \exp(\tau/2) = e^{3\tau/2-(\eta-\beta(\tau))} \exp(\tau/2), \] (3.23)
and make a change of the spatial variable:
\[ v_1(\tau, \eta) = r_\varepsilon(\tau, \eta - \beta(\tau)). \] (3.24)
The function \( r_\varepsilon \) satisfies:
\[ \frac{\partial r_\varepsilon}{\partial \tau} - \frac{\eta}{2} \frac{\partial r_\varepsilon}{\partial \eta} - \frac{\partial^2 r_\varepsilon}{\partial \eta^2} = r_\varepsilon + 3 \frac{3}{2} e^{-\tau/2} \frac{\partial r_\varepsilon}{\partial \eta} + e^{3\tau/2-\eta} \exp(\tau/2) r_\varepsilon^2 = 0, \] (3.25)
with the initial condition
\[ r_\varepsilon(0, \eta) = \psi_0(\eta - \ell_\varepsilon), \] (3.26)
with \( \ell_\varepsilon \) as in (3.3), and
\[
\psi_0(\eta) = e^{\eta} \phi_0(\eta).
\] (3.27)
This, with a slight abuse of notation, is exactly (3.1). Note that \( r_\varepsilon \) depends on \( \varepsilon \) only through \( \ell_\varepsilon \) as it appears in the initial condition. We will interchangeably, with some abuse of notation use \( r_\varepsilon(t, x) \) and \( r_t(t, x) \) for the same object.

As far as the asymptotics of \( r_\varepsilon(\tau, \eta) \) and its connection to the Bramson shift are concerned, it was shown in [23] that there exists a constant \( v_\infty(\varepsilon) > 0 \) so that the solution \( v_\varepsilon(\tau, \eta) \) of (3.17) has the asymptotics
\[
v_\varepsilon(\tau, \eta) \sim v_\infty(\varepsilon) e^{-\eta^2/4} e^{\tau/2}, \text{ as } \tau \to +\infty, \text{ for } \eta > 0.
\] (3.28)
and the Bramson shift is given by
\[
x_\varepsilon = -\log v_\infty(\varepsilon).
\] (3.29)
The corresponding long-time asymptotics for the function \( v_1(\tau, \eta) \), the solution to (3.20) is
\[
v_1(\tau, \eta) \sim \bar{v}_\infty(\varepsilon) e^{-\eta^2/4}, \text{ as } \tau \to +\infty, \text{ for } \eta > 0,
\] (3.30)
with
\[
\bar{v}_\infty(\varepsilon) = \varepsilon v_\infty(\varepsilon),
\] (3.31)
and the asymptotics for \( r_\varepsilon \) is
\[
r_\varepsilon(\tau, \eta) = v_1(\tau, \eta + \beta(\tau)) \sim \bar{v}_\infty(\varepsilon)(\eta + \beta(\tau)) e^{-\eta^2/4} e^{\tau/2} \sim \bar{v}_\infty(\varepsilon) e^{-\eta^2/4}, \text{ as } \tau \to +\infty, \text{ for } \eta > 0,
\] (3.32)
so that
\[
r_\infty(\ell_\varepsilon) = \bar{v}_\infty(\varepsilon) = \varepsilon v_\infty(\varepsilon),
\] (3.33)
and the Bramson shift is
\[
x_\varepsilon = -\log v_\infty(\varepsilon) = \log \varepsilon^{-1} - \log r_\infty(\ell_\varepsilon).
\] (3.34)
This finishes the proof of Proposition 3.1. \( \square \)

4 Connection to the linear Dirichlet problem

Before embarking on the proof of Proposition 3.2, let us recall the intuition that leads to the long-time asymptotics (3.2) for the solution of (3.1), and also explain how the leading term asymptotics in (3.4) comes about. The key point is that we may think of (3.1) as a linear equation with the factor
\[
e^{3\tau/2 - \eta \exp(\tau/2)} r_\ell(\tau, \eta)
\] (4.1)
in the last term in its right side playing the role of an absorption coefficient multiplying \( r_\ell(\tau, \eta) \). Disregarding our lack of information about \( r_\ell(\tau, \eta) \) that enters (4.1), we expect that when \( \tau \gg 1 \) this term is ”extremely large” for \( \eta < 0 \) and ”extremely small” for \( \eta > 0 \). Thinking again of (3.1) as a linear equation for \( r_\ell(\tau, \eta) \), the former means that \( r_\ell(\tau, \eta) \) is very small for \( \eta < 0 \), while the latter indicates that \( r_\ell(\tau, \eta) \) essentially solves a linear problem for \( \eta > 0 \). The drift term in (3.1) with the pre-factor \( e^{-\eta^2/2} \) is also very small at large times. Thus, if we take some \( T \gg 1 \), then for \( \tau \geq T \), a good approximation to (3.1) is the linear Dirichlet problem
\[
\frac{\partial \zeta_\ell}{\partial \tau} - \frac{\eta}{2} \frac{\partial \zeta_\ell}{\partial \eta} - \frac{\partial^2 \zeta_\ell}{\partial \eta^2} = 0, \quad \tau > T, \quad \eta > 0
\]
\[
\zeta_\ell(\tau, 0) = 0,
\]
\[
\zeta_\ell(T, \eta) = r_\ell(T, \eta).
\] (4.2)
In other words, one would solve the full nonlinear problem on the whole line only until a large
time \( T \gg 1 \), and for \( \tau > T \) simply solve the linear Dirichlet problem (4.2). It is easy to see that
\[
\tilde{\zeta}(\eta) = \eta e^{-\eta^2/4},
\] (4.3)
is a steady solution to (4.2). In addition, the operator
\[
\mathcal{L}u = \frac{\partial^2 u}{\partial \eta^2} + \frac{\eta}{2} \frac{\partial u}{\partial \eta} + u, \quad \eta > 0,
\] (4.4)
with the Dirichlet boundary condition at \( \eta = 0 \) has a discrete spectrum. It follows that \( \zeta_\ell(\tau, \eta) \) has
the long time asymptotics
\[
\zeta_\ell(\tau, \eta) \sim \zeta_\infty(\ell) \eta e^{-\eta^2/4}, \quad \tau \to +\infty.
\] (4.5)
As the integral
\[
\int_0^\infty \eta \zeta_\ell(\tau, \eta) d\eta = \int_0^\infty \eta \zeta_\ell(T, \eta) d\eta
\] (4.6)
is conserved, the coefficient \( \zeta_\infty(\ell) \) is determined by the relation
\[
\zeta_\infty(\ell) \int_0^\infty \eta^2 e^{-\eta^2/4} d\eta = \int_0^\infty \eta \zeta_\ell(T, \eta) d\eta,
\] (4.7)
so that
\[
\zeta_\infty(\ell) = \frac{1}{\sqrt{4\pi}} \int_0^\infty \eta \zeta_\ell(T, \eta) d\eta = \frac{1}{\sqrt{4\pi}} \int_0^\infty \eta r_\ell(T, \eta) d\eta.
\] (4.8)
As we expect \( \zeta_\ell(\tau, \eta) \) and \( r_\ell(\tau, \eta) \) to be close if \( T \) is sufficiently large, we should have an approximation
\[
\zeta_\infty(\ell) \approx r_\infty(\ell),
\] (4.9)
if \( T \gg 1 \). This, in turn, implies that
\[
r_\infty(\ell) = \lim_{\tau \to +\infty} \frac{1}{\sqrt{4\pi}} \int_0^\infty \eta r_\ell(\tau, \eta) d\eta.
\] (4.10)
This informal argument is made rigorous in [23].

The limit in the right side of (4.10) is an implicit functional of the initial conditions for the nonlinear problem (3.1), and the evolution of the solution in the initial time layer, before the linear approximation kicks in, is difficult to control, so that there is no explicit expression for \( r_\infty(\ell) \). In the present setting, however, the initial condition \( r_\ell(0, \eta) = \psi_0(\eta - \ell) \) in (3.1) is shifted to the right by \( \ell \gg 1 \). Therefore, at small times the solution is concentrated at \( \eta \gg 1 \), a region where the factor in front of the nonlinear term in (3.1)
\[
\exp \left( \frac{3\tau}{2} - \eta e^{\tau^2/2} \right) \ll 1
\] (4.11)
is very small even for \( \tau = O(1) \). Hence, solutions to the nonlinear equation (3.1) with the initial conditions (3.26) should be well approximated, to the leading order, by the linear problem
\[
\frac{\partial \tilde{r}_\ell}{\partial \tau} - \frac{\eta}{2} \frac{\partial \tilde{r}_\ell}{\partial \eta} - \frac{\partial^2 \tilde{r}_\ell}{\partial \eta^2} - \tilde{r}_\ell + \frac{3}{2} e^{-\tau^2/2} \frac{\partial \tilde{r}_\ell}{\partial \eta} = 0, \quad \tilde{r}_\ell(0, \eta) = r_\ell(0, \eta),
\] (4.12)
even for small times. However, the solution "does not yet know" for "small" \( \tau \) that there is a large
dissipative term in the nonlinear equation, or the Dirichlet boundary condition in the linear version,
and evolves "as if (4.12) is posed for $\eta \in \mathbb{R}$". This leads to exponential growth in $\tau$ until the solution spreads sufficiently far to the left, close to $\eta = 0$ and "discovers" the Dirichlet boundary condition (or the nonlinearity in the full nonlinear version). During this "short time" evolution we have

$$\frac{d}{d\tau} \int_{0}^{\infty} \eta \tilde{r}_{\ell}(\tau, \eta) d\eta = \frac{3}{2} e^{-\tau/2} \int_{0}^{\infty} \tilde{r}_{\ell}(\tau, \eta) d\eta. \quad (4.13)$$

Unlike the first moment, the total mass in the right side does not grow as $\ell$ gets larger – the shift of the initial condition to the right increases the first moment but not the mass. Thus, the first moment of $r_{\ell}(\tau, \eta)$ will only change by a factor that is $o(1)$ during the "short time" evolution, so that it is conserved to the leading order in $\ell$. The "long time" evolution following this initial time layer is well approximated by the linear Dirichlet problem (4.2) that preserves the first moment. Thus, altogether, the first moment will not change to the leading order if $\ell \gg 1$ is large, so that

$$\lim_{\tau \to +\infty} \int_{0}^{\infty} \eta r_{\ell}(\tau, \eta) d\eta = (1 + o(1)) \int_{0}^{\infty} \eta r_{\ell}(0, \eta) d\eta, \quad \text{as } \varepsilon \to 0,$$

which leads to the explicit expression for $r_{\infty}(\ell)$ in terms of the initial first moment:

$$r_{\infty}(\ell) = (1 + o(1)) \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} \eta e^{-\eta/4} \phi_{0}(\eta - \ell) d\eta = (1 + o(1)) \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{\eta/4} \phi_{0}(\eta) d\eta \quad (4.14)$$

which is the leading order term in (3.4). This very informal argument is behind the reason why we can describe the Bramson shift so explicitly for $\varepsilon \ll 1$, which corresponds to $\ell \gg 1$.

**An approximate solution to the adjoint linear problem**

In order to improve on the approximate conservation law (4.13), and to understand where the log $\ell$ term in (3.4) comes from, let us make the following observation. Let us set

$$Q_{k}(\tau, \eta) = \eta + k\bar{\psi}(\eta) e^{-\tau/2}, \quad (4.16)$$

with

$$\bar{\psi}(\eta) = \int_{0}^{\eta} e^{z^{2}/4} \int_{-\infty}^{\infty} e^{-y^{2}/4} dydz, \quad (4.17)$$

and consider a solution to the forced linear Dirichlet problem that now takes into account the additional drift of the size $e^{-\tau/2}$ that appears in (3.1) but was neglected in (4.2)

$$\frac{\partial p}{\partial \tau} - \frac{\eta}{2} \frac{\partial p}{\partial \eta} - \frac{\partial^{2} p}{\partial \eta^{2}} - p + ke^{-\tau/2} \frac{\partial p}{\partial \eta} = f(\tau, \eta), \quad p(\tau, 0) = 0. \quad (4.18)$$

Here is an analog of the conservation law (4.6) that now holds for the more precise than (4.2), but still linear, equation (4.18). We will be mostly interested below in the case $k = 3/2$ that corresponds to the linear part of (3.1).

**Lemma 4.1** We have

$$\frac{d}{d\tau} \int_{0}^{\infty} Q_{k}(\tau, \eta)p(\tau, \eta) d\eta = k^{2} e^{-\tau} \int_{0}^{\infty} \frac{\partial \bar{\psi}(\eta)}{\partial \eta} p(\tau, \eta) d\eta + \int_{0}^{\infty} f(\tau, \eta) Q_{k}(\tau, \eta) d\eta. \quad (4.19)$$
Proof. Note that
\[
\frac{d}{d\tau} \int_0^\infty Q_k(\tau,\eta)p(\tau,\eta)d\eta = \int_0^\infty \left( \frac{\partial Q_k}{\partial \tau} \right) p + Q_k \left[ \frac{\eta}{2} \frac{\partial p}{\partial \eta} + \frac{\partial^2 p}{\partial \eta^2} + p - k e^{-\tau/2} \frac{\partial p}{\partial \eta} + f \right] d\eta
\]
\[
= \int_0^\infty \left( p \left[ \frac{\partial Q_k}{\partial \tau} + \frac{\partial^2 Q_k}{\partial \eta^2} - \frac{\partial}{\partial \eta} \left( \frac{\eta}{2} Q_k \right) + Q_k + k e^{-\tau/2} \frac{\partial Q_k}{\partial \eta} \right] + f Q_k \right) d\eta.
\]
It is easy to check that the function \( \bar{\psi}(\eta) \) is a solution to
\[
\frac{\eta}{2} \frac{\partial \bar{\psi}}{\partial \eta} - \frac{\partial^2 \bar{\psi}}{\partial \eta^2} - 1 = 0, \quad \bar{\psi}(0) = 0.
\]
With this, we can compute that the function \( Q_k(\tau,\eta) \) satisfies
\[
\frac{\partial Q_k}{\partial \tau} - \frac{\partial}{\partial \eta} \left( \frac{\eta}{2} Q_k \right) + \frac{\partial^2 Q_k}{\partial \eta^2} + Q_k + k e^{-\tau/2} \frac{\partial Q_k}{\partial \eta} = k e^{-\tau/2} - \frac{k}{2} e^{-\tau/2} \bar{\psi}(\eta) - \frac{k}{2} e^{-\tau/2} \bar{\psi}(\eta) e^{-\tau/2}
\]
\[
- \frac{k}{2} \bar{\psi}(\eta) e^{-\tau/2} + k \frac{\partial^2 \bar{\psi}(\eta)}{\partial \eta^2} - k \bar{\psi}(\eta) e^{-\tau/2} + k^2 \frac{\partial \bar{\psi}(\eta)}{\partial \eta} e^{-\tau/2}
\]
\[
= k \left( 1 - \frac{\eta}{2} \frac{\partial \bar{\psi}(\eta)}{\partial \eta} + \frac{\partial^2 \bar{\psi}(\eta)}{\partial \eta^2} \right) e^{-\tau/2} + k \bar{\psi}(\eta) e^{-\tau} = k^2 \bar{\psi}(\eta) e^{-\tau}.
\]
Using this in (4.20) gives (4.19). \( \square \)

Note that for \( 0 \leq \eta \leq 1 \) we have
\[
\bar{\psi}(\eta) \leq C \eta.
\]
For the asymptotics of \( \bar{\psi}(\eta) \) for large \( \eta \gg 1 \), note that
\[
\int_z^\infty e^{-y^2/4} dy = -\int_z^\infty \frac{2}{y} \left( e^{-y^2/4} \right)' dy = \frac{2}{z} e^{-z^2/4} - \int_z^\infty \frac{2}{y^2} e^{-y^2/4} dy.
\]
It follows that for \( \eta \geq 1 \) we have
\[
\bar{\psi}(\eta) = \bar{\psi}(1) + \int_1^\eta e^{z^2/4} \int_z^\infty e^{-y^2/4} dy dz = \bar{\psi}(1) + 2 \log \eta - 2 \int_1^\eta e^{z^2/4} \int_z^\infty \frac{1}{y^2} e^{-y^2/4} dy
d\]
\[
= 2 \log \eta + g(\eta), \quad \eta \geq 1,
\]
where \( g(\eta) \) is a bounded function such that
\[
g_\infty = \lim_{\eta \to \infty} g(\eta) = \int_0^1 e^{z^2/4} \int_z^\infty e^{-y^2/4} dy dz - 2 \int_1^\infty e^{z^2/4} \int_z^\infty \frac{1}{y^2} e^{-y^2/4} dy.
\]
This is how the constant \( g_\infty \) appears in (1.29) and in Theorem 1.1. In addition, the log \( \ell \) term in (3.4) will partly arise from the log \( \eta \) asymptotics of \( \bar{\psi}(\eta) \). However, there is an extra nonlinear contribution at that order, that can not be seen just from the above linear argument.

5 An upper bound for the first moment

In this section, we prove an upper bound for \( r_\infty(\ell) \), as the first step toward the proof of Proposition 3.2.

Lemma 5.1 There exists \( K > 0 \) so that we have
\[
r_\infty(\ell) = \limsup_{\tau \to +\infty} \frac{1}{\sqrt{4\pi}} \int_0^\infty \eta r_\ell(\tau, \eta) d\eta \leq \bar{c} \ell + K \log \ell \quad \text{for} \quad \ell \geq 2,
\]
with \( \bar{c} \) as in (1.27).
Reduction to a Dirichlet problem

We first bound the solution to (3.1) by a solution to the linear Dirichlet problem, up to a relatively small error. We start with two observations. First, the solution of the original KPP problem (1.24) satisfies \( u(t, x) \leq 1 \), hence the function \( v_\varepsilon(\tau, \eta) \) defined in (3.16) satisfies
\[
v_\varepsilon(\tau, \eta) \leq e^{\eta \varepsilon^{1/2}}.
\]
Retracing our changes of variables, we deduce that \( v_1(\tau, \eta) \) defined in (3.19) satisfies
\[
v_1(\tau, \eta) = \varepsilon^{-1} v_\varepsilon(\tau, \eta) e^{-\tau/2} \leq \varepsilon^{-1} e^{-\tau/2 + \eta \varepsilon^{1/2}},
\]
and \( r_\varepsilon(t, x) \) introduced in (3.24) obeys
\[
r_\varepsilon(\tau, \eta) = v_1(\tau, \eta + \beta(\tau)) \leq \varepsilon^{-1} e^{(\eta - \eta^{1/2} \log \varepsilon^{-1})} e^{-\tau/2} e^{-\tau/2} = e^{\eta \varepsilon^{1/2} - \eta^{1/2}}.
\]
It follows that at the boundary \( \eta = 0 \) we have
\[
0 < r_\varepsilon(\tau, 0) \leq e^{-\tau/2}, \quad \text{for all } \tau > 0,
\]
so that for \( \tau \gg 1 \) the function \( r_\varepsilon \) does satisfy an approximate Dirichlet boundary condition at \( \eta = 0 \). However, the bound (5.4) is very poor for \( \tau = O(1) \) – recall that the initial condition is located at distance \( \ell \gg 1 \) away from the origin, so the solution remains small near \( \eta = 0 \) for some time \( \tau \gg 1 \). In particular, as a first step, we can bound \( r_\varepsilon(\tau, \eta) \) from above by the solution to the linear problem on the whole line:
\[
\begin{align*}
\frac{\partial \bar{R}_\ell}{\partial \tau} &- \frac{\eta}{2} \frac{\partial \bar{R}_\ell}{\partial \eta} - \frac{\partial^2 \bar{R}_\ell}{\partial \eta^2} - \bar{R}_\ell + \frac{3}{2} e^{-\tau/2} \frac{\partial \bar{R}_\ell}{\partial \eta} = 0, \quad \eta \in \mathbb{R}, \\
\bar{R}_\ell(0, \eta) &= r_\varepsilon(0, \eta).
\end{align*}
\]
A change of variables
\[
\bar{R}_\ell(\tau, \eta) = Q_\ell(\tau, \eta - \frac{3}{2} \tau e^{-\tau/2})
\]
leads to the standard heat equation in the self-similar variables
\[
\begin{align*}
\frac{\partial Q_\ell}{\partial \tau} - \frac{\eta}{2} \frac{\partial Q_\ell}{\partial \eta} - \frac{\partial^2 Q_\ell}{\partial \eta^2} - Q_\ell &= 0, \quad \eta \in \mathbb{R}, \\
Q_\ell(0, \eta) &= r_\varepsilon(0, \eta).
\end{align*}
\]
Thus, the function \( \bar{R}_\ell(\tau, \eta) \) can be written explicitly as
\[
\bar{R}_\ell(\tau, \eta) = e^\tau \int G(e^\tau - 1, (\eta - \frac{3}{2} \tau e^{-\tau/2}) e^{\eta^{1/2} - y}) r_\varepsilon(0, y) dy
\]
\[
= e^\tau \int G(e^\tau - 1, \eta e^{\eta^{1/2} - \frac{3}{2} \tau} - y) r_\varepsilon(0, y) dy.
\]
Here, \( G(t, x) \) is the standard heat kernel:
\[
G(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/(4t)}.
\]
As \( r_\ell(0, \eta) = \psi_0(\eta - \ell) \), and \( \psi_0(\eta) \) satisfies \( \psi_0(\eta) = 0 \) for \( \eta > L_0 \) and \( \psi_0(\eta) \leq Ce^\eta \) for \( \eta < 0 \), we have

\[
\begin{align*}
\bar{R}_\ell(\tau, 0) &= e^\tau \int G(e^\tau - 1, y + \frac{3}{2}\tau) r_\ell(0, y) dy = e^\tau \int G(e^\tau - 1, y + \ell + \frac{3}{2}\tau) \psi_0(y) dy \\
&\leq Ce^\tau \int_{-\infty}^{L_0} G(e^\tau - 1, y + \ell + \frac{3}{2}\tau) dy = Ce^\tau e^{-\ell - (3/2)\tau} \int_{-\infty}^{L_0 + \ell + (3\tau/2)} G(e^\tau - 1, y) dy.
\end{align*}
\]

Note that for any \( L \in \mathbb{R} \) we have

\[
\begin{align*}
\int_{-\infty}^{L} G(t, y) e^y dy &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{L} \exp \left( -\frac{y^2}{4t} + y \right) dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{L/\sqrt{t}} \exp \left( -\frac{y^2}{4} + y\sqrt{t} - t + t \right) dy \\
&= \frac{e^t}{\sqrt{4\pi}} \int_{-\infty}^{L/\sqrt{t}} \exp \left( -\left(\frac{y}{2} - \sqrt{t}\right)^2 \right) dy = \frac{e^t}{\sqrt{4\pi}} \int_{-\infty}^{L/\sqrt{t} - 2\sqrt{t}} \exp \left( -\frac{y^2}{4} \right) dy.
\end{align*}
\]

We take \( \delta > 0 \) sufficiently small, and consider two cases. First, if

\[
\tau < \tau_1 = \log \ell - \log(2 - \delta),
\]
then we have, from (5.9) and (5.10), for \( 0 < \tau < \tau_1 \):

\[
\bar{R}_\ell(\tau, 0) \leq Ce^{-\ell} e^{-\tau/2} e^{\tau^2} \leq Ce^{-\ell} e^{-\tau_1/2} e^{\tau^2} = Ce^{-\ell} e^{-\tau_1/2} e^{\ell^2} = Ce^{-\ell} e^{-\tau_1} e^{\ell^2} \leq Ce^{-\ell} e^{-\tau_1/4} e^{\ell^2/4}. \tag{5.11}
\]

On the other hand, if \( \tau > \tau_1 \), then, taking

\[
t = e^\tau - 1, \quad L = L_0 + \ell + \frac{3\tau}{2}, \tag{5.12}
\]

in (5.10), we see that for \( \ell \) sufficiently large, the upper limit of integration

\[
\frac{L}{\sqrt{t}} - 2\sqrt{t} = \frac{1}{\sqrt{e^\tau - 1}} \left( L_0 + \ell + \frac{3\tau}{2} - 2e^\tau + 2 \right) = \frac{1}{\sqrt{e^\tau - 1}} \left( L_0 + (2 - \delta)e^{\tau_1} + \frac{3\tau}{2} - 2e^\tau + 2 \right)
\]

\[
\leq -\frac{\delta}{2} e^{\tau/2}
\]

is very negative. In particular, the integral in the right side of (5.10) can be estimated as

\[
\begin{align*}
\int_{-\infty}^{L/\sqrt{t} - 2\sqrt{t}} \exp \left( -\frac{y^2}{4} \right) dy &\leq C \left| \frac{L}{\sqrt{t}} - 2\sqrt{t} \right|^{-1} \exp \left( -\frac{1}{4} \left( \frac{L}{\sqrt{t}} - 2\sqrt{t} \right)^2 \right) \\
&= C \left| \frac{L}{\sqrt{t}} - 2\sqrt{t} \right|^{-1} e^{-L^2/(4t^3)} e^{L^2 e^{\tau}}.
\end{align*}
\]

Then, we have from (5.9), (5.10), (5.13) and (5.14)

\[
\begin{align*}
\bar{R}_\ell(\tau, 0) &\leq Ce^{-\ell - \tau/2} e^{\tau} \left| \frac{L}{\sqrt{t}} - 2\sqrt{t} \right|^{-1} e^{-L^2/(4t^3)} e^{\tau} e^{-t} \\
&\leq C e^{-\ell/2} e^{-\tau/2} e^{L_0 + \ell + 3\tau/2} e^{-\left( L_0 + \ell + 3\tau/2 \right)^2 / (4(e^{\tau} - 1))} \leq C_\delta e^{\tau/2} e^{-(L_0 + \ell + 3\tau/2)^2 / (4(e^{\tau} - 1))} \leq C e^{-\tau/2} \leq C e^{-\tau/4} e^{-\tau/4} \leq Ce^{-\tau/4} e^{-\tau/4},
\end{align*}
\]

provided that \( \tau \geq \tau_1 \) and

\[
\tau \leq (L_0 + \ell + 3\tau/2)^2 / (4(e^{\tau} - 1)). \tag{5.16}
\]
In particular, we can take
\[ \tau \leq \tau_2 = 2 \log \ell - \log \log \ell - 3, \] (5.17)
as long as \( \ell \) is sufficiently large, because then we have
\[ 4\tau(e^\tau - 1) \leq 4\tau e^\tau \leq 2 \log \ell \frac{4\ell^2}{e^3 \log \ell} \leq \ell^2, \] (5.18)
so that (5.16) holds. It follows that
\[ r_\ell(\tau,0) \leq C\ell^{-1/4}e^{-\tau/4}, \quad 0 \leq \tau \leq \tau_2. \] (5.19)

Taking into account (5.4), we deduce that we also have
\[ r_\ell(\tau,0) \leq e^{-\tau/2} \leq e^{-\tau_2/4}e^{-\tau/4} = \frac{C(\log \ell)^{1/4}}{\ell^{1/2}}e^{-\tau/4} \leq \frac{C}{\ell^{1/4}}e^{-\tau/4}, \quad \text{for } \tau > \tau_2. \] (5.20)

It follows that the function \( r_\ell(\tau,\eta) \), the solution to (3.1), is bounded from above by the solution to the linear half-line problem
\[
\begin{align*}
\frac{\partial \tilde{r}_\ell}{\partial \tau} - \frac{\eta}{2} \frac{\partial \tilde{r}_\ell}{\partial \eta} - \frac{\partial^2 \tilde{r}_\ell}{\partial \eta^2} - \tilde{r}_\ell + \frac{3}{2} e^{-\tau/2} \frac{\partial \tilde{r}_\ell}{\partial \eta} &= 0, \quad \eta > 0, \\
\tilde{r}_\ell(\tau,0) &= C\ell^{-1/4}e^{-\tau/4},
\end{align*}
\] (5.21)
with the initial condition
\[ \tilde{r}_\ell(0,\eta) = \psi_0(\eta - \ell). \] (5.22)

In order to deal with the small but non-zero boundary condition in (5.21), we make yet another change of variables:
\[ \tilde{r}_\ell(\tau,\eta) = \bar{q}_\ell(\tau,\eta) + C\ell^{-1/4}e^{-\tau/4}g(\eta), \] (5.23)
with a smooth function \( g(\eta) \) such that \( g(0) = 1, g(\eta) \geq 0 \) and \( g(\eta) = 0 \) for \( \eta \geq 1 \). This leads to
\[
\begin{align*}
\frac{\partial \bar{q}_\ell}{\partial \tau} + \frac{3}{2} e^{-\tau/2} \frac{\partial \bar{q}_\ell}{\partial \eta} &= \mathcal{L}\bar{q}_\ell + C\ell^{-1/4}G(\tau,\eta)e^{-\tau/4}, \quad \eta > 0, \\
\bar{q}_\ell(\tau,0) &= 0,
\end{align*}
\] (5.24)
with a uniformly bounded function \( G(\tau,\eta) \) that is supported in \( \eta \in [0,1] \) and is independent of \( \ell \). We recall that the operator \( \mathcal{L} \) is defined in (4.4). This change of variable does not affect the asymptotics of the first moment:
\[ \limsup_{\tau \to +\infty} \int_0^\infty \eta r_\ell(\tau,\eta) d\eta \leq \limsup_{\tau \to +\infty} \int_0^\infty \eta \bar{r}_\ell(\tau,\eta) d\eta = \limsup_{\tau \to +\infty} \int_0^\infty \eta \bar{q}_\ell(\tau,\eta) d\eta. \] (5.25)

The initial condition for \( \bar{q}_\ell \) is
\[ \bar{q}_\ell(0,\eta) = \tilde{r}_\ell(0,\eta) - C\ell^{-1/4}g(\eta). \] (5.26)

It is easy to see that
\[ \int_0^\infty \eta \bar{q}_\ell(\tau,\eta) d\eta \leq \int_0^\infty \eta \bar{q}_\ell(\tau,\eta) d\eta + C_0 \ell^{-1/4}. \] (5.27)

Here, \( C_0 \) is a constant that depends neither on \( \ell \) nor on \( \phi_0 \), and \( \bar{p}_\ell(\tau,\eta) \) is the solution to the homogeneous problem
\[
\begin{align*}
\frac{\partial \bar{p}_\ell}{\partial \tau} + \frac{3}{2} e^{-\tau/2} \frac{\partial \bar{p}_\ell}{\partial \eta} &= \mathcal{L}\bar{p}_\ell, \quad \eta > 0, \\
\bar{p}_\ell(\tau,0) &= 0, \\
\bar{p}_\ell(0,\eta) &= \psi_0(\eta - \ell).
\end{align*}
\] (5.28)
Let us now use Lemma 4.1, with $k = 3/2$: multiply (5.28) by $Q_{3/2}(\tau, \eta)$ and integrate:

$$\frac{d}{d\tau} \int_{0}^{\infty} Q_{3/2}(\tau, \eta) \bar{p}_\ell(\tau, \eta) d\eta = \frac{9}{4} e^{-\tau} \int_{0}^{\infty} \bar{p}_\ell(\tau, \eta) \frac{\partial \tilde{\psi}(\eta)}{\partial \eta} d\eta \leq C e^{-\tau} \int_{0}^{\infty} \frac{\bar{p}_\ell(\tau, \eta)}{1 + \eta} d\eta \tag{5.29}$$

Note, however, that integrating (5.28) gives a simple upper bound

$$\frac{d}{d\tau} \int_{0}^{\infty} \bar{p}_\ell(\tau, \eta) d\eta = - \frac{\partial \bar{p}_\ell(\tau, 0)}{\partial \eta} + \frac{1}{2} \int_{0}^{\infty} \bar{p}_\ell(\tau, \eta) d\eta \leq \frac{1}{2} \int_{0}^{\infty} \bar{p}_\ell(\tau, \eta) d\eta, \tag{5.30}$$

implying a trivial and very poor upper bound

$$\int_{0}^{\infty} \bar{p}_\ell(\tau, \eta) d\eta \leq C_0 e^{\tau/2}. \tag{5.31}$$

We remind that we use the notation $C_0$ for various constants that do not depend on $\varepsilon$. Using this estimate in the right side of (5.29) gives

$$\int_{0}^{\infty} Q_{3/2}(\tau, \eta) \bar{p}_\ell(\tau, \eta) d\eta \leq \int_{0}^{\infty} Q_{3/2}(0, \eta) \bar{p}_\ell(0, \eta) d\eta + C_0. \tag{5.32}$$

Recalling the definition (4.16) of $Q_k(\tau, \eta)$ and passing to the limit $\tau \to +\infty$ gives

$$\bar{p}_\infty(\ell) = \lim_{\tau \to +\infty} \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} \eta \bar{p}_\ell(\tau, \eta) d\eta \leq \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} \left[ \eta + \frac{3}{2} \bar{\psi}(\eta) \right] \psi_0(\eta - \ell) d\eta + C_0$$

$$= \frac{1}{\sqrt{4\pi}} \int_{-\ell}^{\infty} \eta \psi_0(\eta) d\eta + \frac{\ell}{\sqrt{4\pi}} \int_{-\ell}^{\infty} e^{\eta} \psi_0(\eta) d\eta + \frac{3}{2} \int_{-\ell}^{\infty} \bar{\psi}(\eta) \psi_0(\eta) \frac{d\eta}{\sqrt{4\pi}} + C_0 \tag{5.33}$$

$$\leq c \ell + 3c \log \ell + C_1, \text{ for } \ell \geq 2,$$

with $c$ as in (1.27), and a constant $C_1$ that may depend on $\phi_0$. We used (4.25) in the last inequality above. The conclusion of Lemma 5.1 now follows.

## 6 The proof of Proposition 3.2

The proof of Proposition 3.2 is much more involved than the analysis used to obtain the crude upper bound for the first moment in Lemma 5.1. In this section, we outline the main steps and state the auxiliary results needed in the proof.

We start with (3.25):

$$\frac{\partial r_\ell}{\partial \tau} - \frac{\eta}{2} \frac{\partial r_\ell}{\partial \eta} - \frac{\partial^2 r_\ell}{\partial \eta^2} - r_\ell + \frac{3}{2} e^{-\tau/2} \frac{\partial r_\ell}{\partial \eta} + e^{3\tau/2 - \eta \exp(\tau/2)} r_\ell^2 = 0, \tag{6.1}$$

with the initial condition $r_\ell(0, \eta) = \psi_0(\eta - \ell)$. Recall that we are interested in

$$r_\infty(\ell) = \lim_{\tau \to +\infty} \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} \eta r_\ell(\tau, \eta) d\eta = \lim_{\tau \to +\infty} \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} Q(\tau, \eta) r_\ell(\tau, \eta) d\eta. \tag{6.2}$$

Here, $Q(\tau, \eta) = Q_{3/2}(\tau, \eta)$ is the approximate solution to the adjoint linear problem, defined in (4.16) with $k = 3/2$. Multiplying (6.1) by $Q(\tau, \eta)$ and integrating in $\eta$ gives, according to Lemma 4.1:

$$\frac{d}{d\tau} \int_{0}^{\infty} Q(\tau, \eta) r_\ell(\tau, \eta) d\eta = - \int_{0}^{\infty} e^{3\tau/2 - \eta \exp(\tau/2)} r_\ell^2(\tau, \eta) Q(\tau, \eta) d\eta + E_\ell(\tau), \tag{6.3}$$

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so that

$$r_\infty(\ell) = Q_\ell + Y_\ell + \mathcal{E}_\ell,$$  \hspace{1cm} (6.4)

with

$$Q_\ell = \frac{1}{\sqrt{4\pi}} \int_0^\infty Q(0, \eta) r_\ell(0, \eta) d\eta,$$

$$Y_\ell = -\frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty e^{3\tau/2 - \eta \exp(\tau/2)} r_\ell^2(\tau, \eta) Q(\tau, \eta) d\tau d\eta,$$

$$\mathcal{E}_\ell = \frac{1}{\sqrt{4\pi}} \int_0^\infty E_\ell(\tau) d\tau.$$  \hspace{1cm} (6.5)

The error term

$$E_\ell(\tau) = E_{(1)}^\ell(\tau) + E_{(2)}^\ell(\tau)$$  \hspace{1cm} (6.6)

has two contributions. The first

$$E_{(1)}^\ell(\tau) = r_\ell(\tau, 0) \frac{\partial Q(\tau, 0)}{\partial \eta}$$  \hspace{1cm} (6.7)

comes from the boundary at $\eta = 0$ since we do not have $r_\ell(\tau, 0) = 0$ but only that $r_\ell(\tau, 0)$ is small. The second comes from the error term in Lemma 4.1, because $Q(\tau, \eta)$ is only an approximate solution to the adjoint problem and not an exact one:

$$E_{(2)}^\ell(\tau) = \frac{9}{4} e^{-\tau} \int_0^\infty \frac{\partial \bar{\psi}(\eta)}{\partial \eta} r_\ell(\tau, \eta) d\eta \leq C e^{-\tau} \int_0^\infty \frac{r_\ell(\tau, \eta) d\eta}{1 + \eta}. \hspace{1cm} (6.8)$$

The linear term and the error term in (6.4) are quite straightforward to evaluate and estimate, respectively. The main difficulty will be in finding the precise asymptotics of the nonlinear term in the right side of (6.4).

The error term bound

The error term in (6.4) is bounded by the following lemma.

**Lemma 6.1** There exists $C > 0$ so that

$$\mathcal{E}_\infty(\ell) \leq \frac{C}{\ell^{1/8}}. \hspace{1cm} (6.9)$$

**Proof.** As we have seen in the proof of Lemma 5.1 – see (5.19) and (5.20), we have an upper bound

$$r_\ell(\tau, 0) \leq \bar{r}_\ell(\tau, 0) \leq \frac{C}{\ell^{1/4}} e^{-\tau/4}$$  \hspace{1cm} (6.10)

hence $E_{(1)}^\ell$ can be bounded as

$$\int_0^\infty E_{(1)}^\ell(\tau) d\tau \leq \frac{C}{\ell^{1/4}}. \hspace{1cm} (6.11)$$

We now estimate $E_{(2)}^\ell$. As in the proof of the upper bound in Lemma 5.1, see (5.23)-(5.28), we deduce from (6.10) that

$$E_{(2)}^\ell(\tau) \leq \frac{C}{\ell^{1/4}} e^{-\tau} + C e^{-\tau} \int_0^\infty \frac{\bar{\psi}(\tau, \eta)}{1 + \eta} d\eta, \hspace{1cm} (6.12)$$

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where \( \bar{p}_\ell(\tau, \eta) \) is the solution to (5.28):

\[
\frac{\partial \bar{p}_\ell}{\partial \tau} + \frac{3}{2} e^{-\tau/2} \frac{\partial \bar{p}_\ell}{\partial \eta} + \frac{\partial^2 \bar{p}_\ell}{\partial \eta^2} + \frac{\eta}{2} \frac{\partial \bar{p}_\ell}{\partial \eta} + \bar{p}_\ell, \quad \eta > 0,
\]

\( \bar{p}_\ell(\tau, 0) = 0, \)

\( \bar{p}_\ell(0, \eta) = \psi_0(\eta - \ell) = e^{\eta - \ell} \phi_0(\eta - \ell). \)

The simple-minded bound (5.31) implies that if we fix any \( T > 0 \), then

\[
e^{-\tau} \int_0^\infty \frac{\bar{p}_\ell(\tau, \eta)}{1 + \eta} d\eta \leq Ce^{-\tau/2} \leq Ce^{-\tau/4} e^{-T/4} \quad \text{for all } \tau > T.
\]

For short times \( \tau < T \) we can take \( L > 0 \) and use (5.31) to write

\[
e^{-\tau} \int_0^\infty \frac{\bar{p}_\ell(\tau, \eta)}{1 + \eta} d\eta \leq Ce^{-\tau} \left( \sup_{\eta \in [0, L]} \bar{p}_\ell(\tau, \eta) \right) \log L + \frac{C}{L} e^{-\tau/2}
\]

\[
\leq Ce^{-\tau} \left( \sup_{\eta \in [0, L]} \bar{R}_\ell(\tau, \eta) \right) \log L + \frac{C}{L} e^{-\tau/2}.
\]

Here, \( \bar{R}_\ell(\tau, \eta) \) is the solution to the whole line problem (5.5):

\[
\frac{\partial \bar{R}_\ell}{\partial \tau} - \frac{\eta}{2} \frac{\partial \bar{R}_\ell}{\partial \eta} - \frac{\partial^2 \bar{R}_\ell}{\partial \eta^2} - \bar{R}_\ell + \frac{3}{2} e^{-\tau/2} \frac{\partial \bar{R}_\ell}{\partial \eta} = 0, \quad \eta \in \mathbb{R},
\]

\( \bar{R}_\ell(0, \eta) = r_\ell(0, \eta), \)

(6.16)

given explicitly by (5.7):

\[
\bar{R}_\ell(\tau, \eta) = e^{\tau} \int G(e^\tau - 1, \eta e^{\tau/2} - \frac{3}{2} \tau - y) r_\ell(0, y) dy.
\]

(6.17)

In the proof of Lemma 5.1 we only looked for the bound on \( \bar{R}_\ell(\tau, \eta) \) at \( \eta = 0 \) but now we need to consider \( \eta \in [0, L] \). Note that for \( T \) and \( L \) not too large, the function \( \bar{R}(\tau, \eta) \) is increasing in \( \eta \) for \( \eta \in [0, L] \) and \( 0 < \tau < T \). Hence, we have

\[
\bar{R}_\ell(\tau, \eta) \leq \bar{R}_\ell(\tau, L) = e^{\tau} \int G(e^\tau - 1, Le^{\tau/2} - \frac{3}{2} \tau - y) r_\ell(0, y) dy, \quad \text{for } \eta \in [0, L] \text{ and } 0 < \tau < T.
\]

(6.18)

As in (5.9) and (5.11), we get, as \( \psi_0(y) = 0 \) for \( y \geq L_0 \):

\[
\bar{R}_\ell(\tau, L) = e^{\tau} \int G(e^\tau - 1, y + \frac{3}{2} \tau - Le^{\tau/2}) r_\ell(0, y) dy = e^{\tau} \int G(e^\tau - 1, y + \ell + \frac{3}{2} \tau - Le^{\tau/2}) \psi_0(y) dy
\]

\[
\leq Ce^{\tau} \int_{-\infty}^{L_0} G(e^\tau - 1, y + \ell + \frac{3}{2} \tau - Le^{\tau/2}) e^y dy
\]

\[
= Ce^{\tau} e^{-\ell - (3/2) \tau + Le^{\tau/2}} \int_{-\infty}^{L_0 + \ell + (3\tau/2) - Le^{\tau/2}} G(e^\tau - 1, y) e^y dy
\]

\[
\leq Ce^{-\ell - 2 + Le^{\tau/2}} e^\tau e^{-1} \leq Ce^{-\ell - T/2 + Le^{\tau/2}} e^\tau - 1.
\]

(6.19)

Let us take \( T = (1/2) \log \ell \) and \( L = \ell^{1/2} \), which gives

\[
\bar{R}_\ell(\tau, \eta) \leq Ce^{-\ell - \log \ell/4 + \ell^{3/4} + \ell^{1/2} - 1} \leq Ce^{-\ell/2}.
\]

(6.20)
Using this in (6.15), we obtain for \( \tau < T \):

\[
e^{-\tau} \int_0^\infty \frac{p_t(\tau, \eta)}{1 + \eta} d\eta \leq Ce^{-\tau/2} \log \ell + C \frac{e^{-\tau/2}}{\ell^{1/2}} \text{ for all } \tau < T,
\]

while (6.14) becomes

\[
e^{-\tau} \int_0^\infty \frac{p_t(\tau, \eta)}{1 + \eta} d\eta \leq Ce^{-\tau/4} \ell^{-1/8} \text{ for all } \tau > T.
\]

It follows that

\[
\int_0^\infty \bar{E}^{(2)}(\tau) d\tau \leq \frac{C}{\ell^{1/8}},
\]

and the conclusion of Lemma 6.1 follows. □

The linear contribution

The term \( Q_\ell \) in (6.5) can be computed explicitly from the asymptotics for \( Q(0, \eta) \) as \( \eta \to \infty \) coming from (4.25):

\[
Q(0, \eta) = \eta + 3 \log \eta + \frac{3}{2} g_\infty + O\left(\frac{1}{\eta}\right) \text{ as } \eta \to +\infty,
\]

with \( g_\infty \) defined in (4.26). This gives

\[
Q_\ell = \frac{1}{\sqrt{4\pi}} \int_0^\infty (\eta + 3 \log \eta + \frac{3}{2} g_\infty) \varphi_0(\eta - \ell) d\eta + O\left(\frac{1}{\ell}\right) = \bar{c}_\ell + 3 \bar{c} \log \ell + \frac{3}{2} g_\infty \bar{c} + \bar{c}_1 + O(\ell^{-1}),
\]

with \( \bar{c} \) and \( \bar{c}_1 \) defined in (1.27) and (1.28), respectively. In particular, this is how the term \( \bar{c}_1 \) appears in Theorem 1.1.

The nonlinear contribution

It remains to find the asymptotics of the term \( Y_\ell \) in (6.5), and that computation is quite long. The first step is a series of simplifications. We start by approximating \( Q(\tau, \eta) \) in (6.5) by \( \eta \): set

\[
\bar{Y}_\ell = -\frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty \exp(\tau/2) \tau^2 \varphi_0^2(\tau, \eta) \eta d\tau d\eta.
\]

Lemma 6.2 There exists \( \gamma > 0 \) so that

\[
Y_\ell - \bar{Y}_\ell = O(\ell^{-\gamma}), \text{ as } \ell \to +\infty.
\]

This lemma is proved in Section 7.5. The next approximation involves going back to the original space-time variables and restricting the spatial integration to ”relatively short” distances \( x \leq (t + 1)^\delta \) with \( \delta > 0 \) small.

Lemma 6.3 There exists \( \delta > 0 \) and \( \gamma > 0 \) so that

\[
|\bar{Y}_\ell - \bar{Y}_\ell^{(1)}| = O(\ell^{-\gamma}), \text{ as } \ell \to +\infty,
\]

where

\[
\bar{Y}_\ell^{(1)} = -\frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^{(t+1)^\delta} xe^{-x^2 \varphi_0^2(t, x)} \frac{dxdt}{(1 + t)^{3/2}}.
\]
and \( z_\ell(t, x) \) is the solution to
\[ \frac{\partial z_\ell}{\partial t} - \frac{\partial^2 z_\ell}{\partial x^2} - \frac{3}{2(t + 1)} (z_\ell - \frac{\partial z_\ell}{\partial x}) + e^{-x} z_\ell^2 = 0, \]
with \( z_\ell(0, x) = \psi_0(x - \ell) \).

Note that (6.30) is simply (3.15) with a slightly different notation. The functions \( z_\ell(t, x) \) and \( r_\ell(\tau, \eta) \) are related by
\[ z_\ell(t, x) = \sqrt{t + 1} r_\ell(\log(t + 1), x/\sqrt{t + 1}). \]

This lemma is proved in Section 7.1 below. It is easy to see why (6.28) should be true: we expect that the function \( z_\ell(t, x) \) behaves roughly as
\[ z_\ell(t, x) \sim r_\infty(\ell)x \sim c\ell x, \text{ for } x \gg 1 \text{ and } t \gg 1. \]

With this input, a back of the envelope computation shows that the integral in (6.29) over \( x \geq (t + 1)^\delta \) is, indeed, small, due to the exponential decay factor.

The next approximation is to discard the “short times” from the time integration, looking only at times \( t \geq \ell^{2-\alpha} \) with some \( \alpha > 0 \) sufficiently small. This is, again, quite expected: as the integration is now over the region \( x \leq (t + 1)^\delta \), and \( z_\ell(t, x) \) is initially supported at \( x = \ell \), it takes the time of roughly \( t \sim \ell^2 \) to populate the domain of integration, so shorter times can be discarded.

**Lemma 6.4** Let \( \alpha > 0 \) sufficiently small. Then, there exists \( \gamma > 0 \) so that
\[ |\bar{Y}_\ell^{(2)} - \bar{Y}_\ell^{(1)}| = O(\ell^{-\gamma}), \text{ as } \ell \to +\infty, \]
where
\[ \bar{Y}_\ell^{(2)} = -\frac{1}{\sqrt{4\pi}} \int_{\ell^{2-\alpha}}^\infty \int_0^{(t + 1)^\delta} xe^{-x} z_\ell^2(t, x) \frac{dxdt}{(1 + t)^{3/2}}, \]
This lemma is proved in Section 7.2 below.

Lemmas 6.3 and 6.4 allow us to focus on the integration in the region \( 0 < x < (t + 1)^\delta \) and times \( t > \ell^{2-\alpha} \), and consider \( z_\ell(t, x) \) as the solution to (6.30) for \( 0 < x < (t + 1)^\delta \), with the initial condition \( z_\ell(0, x) = 0 \) and the boundary condition at \( x = (t + 1)^\delta \) coming from the “outer” solution in the self-similar variables:
\[ z_\ell(t, (t + 1)^\delta) = (t + 1)^{1/2} r_\ell(\log(t + 1), (t + 1)^{-1/2+\delta}). \]

The analysis of the behavior of \( r_\ell(\tau, \eta) \) for \( \eta \sim O(1) \) and \( \tau \to +\infty \) that we have done so far, would only give information for \( x \sim \sqrt{t} \), not \( x \sim t^\delta \) with \( \delta < 1/2 \), as this point corresponds to
\[ \eta \sim t^{-(1/2-\delta)} = \exp(-(1/2 - \delta)\tau) \ll 1. \]

To bridge this gap, we need the following crucial lemma that is proved in Section 7.6. It refines the informal asymptotics in (6.32) by an extremely important Gaussian factor that interpolates the short time and the long time behavior.

**Lemma 6.5** There exists \( \alpha > 0 \) sufficiently small and a constant \( K > 0 \) so that for all \( t > \ell^{2-\alpha} \) we have
\[ \frac{z_\ell(t, (t + 1)^\delta)}{(t + 1)^\delta} \leq (c\ell + K \ell^{1-\delta}) e^{-\ell^2/4(t + 1)}, \]
and for all \( \ell^{2-\alpha} < t < \ell^{2+\alpha} \) we have
\[ \frac{z_\ell(t, (t + 1)^\delta)}{(t + 1)^\delta} \geq (c\ell - K \ell^{1-\delta}) e^{-\ell^2/4(t + 1)}. \]
Lemma 6.5, proved in Section 7.6 allows us to look at upper and lower solutions \( z_\pm \) for \( z_\ell \) in the region \( x \leq (1 + t)\delta \) as the solutions to
\[
\frac{\partial z_\pm}{\partial t} - \frac{\partial^2 z_\pm}{\partial x^2} - \frac{3}{2(t + 1)} \left( z_\pm - \frac{\partial z_\pm}{\partial x} \right) + e^{-x} z_\pm^2 = 0, \quad x < (t + 1)\delta, \tag{6.38}
\]
with the boundary condition
\[
z_\pm(t, (1 + t)\delta) = (\bar{c}\ell \pm K\ell^{1-\delta})e^{-t^2/(4(t+1))}(1 + t)\delta. \tag{6.39}
\]
The initial condition for \( z_-(t, x) \) at \( t = \ell^{2-\alpha} \) is \( z_-(\ell^{2-\alpha}, x) = 0 \) for all \( x < (1 + \ell)\delta \), and for \( z_+(t, x) \) it is \( z_+(\ell^{2-\alpha}, x) = Ce^{-c\alpha'} \) for all \( x < (1 + \ell)\delta \), with some \( \alpha' > 0 \), which comes from (7.14) below. We have then the inequality
\[
\bar{Y}_\ell^+ \leq \bar{Y}_\ell^{-(2)} \leq \bar{Y}_\ell^-, \tag{6.40}
\]
with
\[
\bar{Y}_\ell^\pm = -\frac{1}{\sqrt{4\pi}} \int_{\ell^{2-\alpha}}^\infty \int_0^{(t+1)\delta} x e^{-x} z_\pm^2(t, x) \frac{dx dt}{(1 + t)^{3/2}}. \tag{6.41}
\]
The last step in the proof of Proposition 3.2. is to prove the following.

**Lemma 6.6** There exists \( \delta > 0 \) so that
\[
\bar{Y}_\ell^\pm = -\bar{c} \log \ell - \bar{c} \log \bar{c} + k_0 \bar{c} + \frac{\bar{c}}{2} + O(\ell^{-\delta}) \quad \text{as} \quad \ell \to +\infty, \tag{6.42}
\]
with the constant \( k_0 \) as in (1.6).

The proof of Lemma 6.6 is presented in Sections 7.3 and 7.4.

Now, putting together (6.25) and (6.42) gives
\[
r_\infty(\ell) = \bar{c}\ell + 2\bar{c} \log \ell + \frac{3}{2} g_\infty \bar{c} + \bar{c}_1 - \bar{c} \log \bar{c} + k_0 \bar{c} + \frac{\bar{c}}{2} + O(\ell^{-\delta}), \tag{6.43}
\]
with some \( \delta > 0 \), finishing the proof of Proposition 3.2.

### 7 Proofs of auxiliary lemmas

This section contains the proofs of the auxiliary lemmas used in Section 6 to prove Proposition 3.2, except for Lemma 6.5 that is proved in Section 7.6.

#### 7.1 The proof of Lemma 6.3

First, we go back to the original, non-self-similar variables: write
\[
r_\ell(\tau, \eta) = w_\ell(e^\tau - 1, \eta e^{\tau/2}),
\]
so that the function \( w_\ell(t, x) \) satisfies
\[
\frac{\partial w_\ell}{\partial t} - \frac{\partial^2 w_\ell}{\partial x^2} - \frac{w_\ell}{t + 1} + \frac{3}{2(t + 1)} \frac{\partial w_\ell}{\partial x} + \sqrt{1 + 1} e^{-x} w_\ell^2 = 0, \tag{7.1}
\]
with the boundary condition
\[
w_\ell(t, (1 + t)\delta) = (\bar{c}\ell \pm K\ell^{1-\delta})e^{-t^2/(4(t+1))}(1 + t)\delta.
\]
with the initial condition \( w_\ell(0, x) = \psi_0(x - \ell) \). To revert back even more, we write \( w_\ell = z_\ell / \sqrt{t + 1} \), with the function \( z_\ell \) that satisfies (6.30) with \( z_\ell(0, x) = \psi_0(x - \ell) \). Then, we can re-write \( \tilde{Y}_\ell \) as

\[
\tilde{Y}_\ell = -\int_0^\infty \int_0^{(t+1)^3/2} e^{-x} w_\ell^2(t, x) \frac{dx dt}{\sqrt{t + 1} \sqrt{t + 1}}.
\]

Next, we split \( \tilde{Y}_\ell \) into two terms, corresponding to small and large \( x \):

\[
\tilde{Y}_\ell = -\int_0^\infty \int_{(t+1)^3/2}^{(t+1)^3} e^{-x} z_\ell^2(t, x) \frac{dx dt}{(t + 1)^3/2} - \int_0^\infty \int_{(t+1)^3/2}^{(t+1)^3} e^{-x} z_\ell^2(t, x) \frac{dx dt}{(t + 1)^3/2} = \tilde{Y}_\ell^{(1)} + \tilde{Y}_\ell^{(c)},
\]

with \( \delta \in (0, 1/2) \) to be chosen, and split the second term again:

\[
\tilde{Y}_\ell^{(c)} = -\int_0^T \int_{(t+1)^3/2}^{(t+1)^3} e^{-x} z_\ell^2(t, x) \frac{dx dt}{(t + 1)^3/2} - \int_T^\infty \int_{(t+1)^3/2}^{(t+1)^3} e^{-x} z_\ell^2(t, x) \frac{dx dt}{(t + 1)^3/2} = \tilde{Y}_\ell^{(c1)} + \tilde{Y}_\ell^{(c2)},
\]

with a large \( T \) to chosen. We estimate the term \( \tilde{Y}_\ell^{(c2)} \) using the simple upper bound

\[
z_\ell(t, x) \leq C(t + 1)^{3/2},
\]

so that

\[
|\tilde{Y}_\ell^{(c2)}| = \int_T^\infty \int_{(t+1)^3/2}^{(t+1)^3} e^{-x} z_\ell^2(t, x) \frac{dx dt}{(t + 1)^3/2} \leq C \int_T^\infty (t + 1)^{3/2} e^{-(t+1)^3/2} dt \leq C(1 + T)^3 \exp(-T^\delta).
\]

To estimate \( \tilde{Y}_\ell^{(c1)} \) we slightly refine (7.5) to

\[
z_\ell(t, x) \leq C(t + 1)^{3/2} \tilde{z}(t, x - \frac{3}{2} \log(t + 1)).
\]

Here, \( \tilde{z}(t, x) \) is the solution to the heat equation on the whole line

\[
\frac{\partial \tilde{z}}{\partial t} = \Delta \tilde{z}, \quad \tilde{z}(0, x) = \psi_0(x - \ell).
\]

It follows that

\[
z_\ell(t, x) \leq C(t + 1)e^{-c\ell}, \quad \text{for } 0 \leq t \leq \ell \text{ and } 0 < x < \ell/2,
\]

so that for \( 0 \leq t \leq \ell \) we have

\[
\int_{(t+1)^3/2}^{(t+1)^3} e^{-x} z_\ell^2(t, x) dx \leq \int_0^{\ell/2} e^{-x} z_\ell^2(t, x) dx + \int_{\ell/2}^{(t+1)^3/2} e^{-x} z_\ell^2(t, x) dx \leq C(t + 1)^3 e^{-c\ell}.
\]

Thus, if we take \( T = \ell \), then

\[
|\tilde{Y}_\ell^{(c1)}| = \int_0^\ell \int_{(t+1)^3/2}^{(t+1)^3} e^{-x} z_\ell^2(t, x) \frac{dx dt}{(t + 1)^3/2} \leq C \int_0^\ell (t + 1)^3 e^{-c\ell} dt \leq C(\ell + 1)^4 \exp(-c\ell).
\]

Using \( T = \ell \) also in (7.6), we obtain

\[
|\tilde{Y}_\ell^{(c)}| \leq C(\ell + 1)^4 \exp(-c\ell),
\]

finishing the proof of Lemma 6.3.
7.2 The proof of Lemma 6.4

The difference
\[ \bar{Y}_\ell^{(1)}(t) - \bar{Y}_\ell^{(2)}(t) = -\frac{1}{\sqrt{4\pi}} \int_0^{\ell^{2-\alpha}} \int_0^{(t+1)^{\delta}} x e^{-x} z_\ell^2(t, x) \frac{dx dt}{(1 + t)^{3/2}} \]  
(7.13)
can be estimated using (7.7) and a variant of (7.9):
\[ z_\ell(t, x) \leq C(t + 1)^{3/2} e^{-c\ell^\alpha}, \text{ for } t \leq \ell^{2-\alpha} \text{ and } x \leq (t + 1)^{\delta} \leq \ell^{(2-\alpha)\delta}. \]  
(7.14)

This gives
\[ |\bar{Y}_\ell^{(1)}(t) - \bar{Y}_\ell^{(2)}(t)| \leq C \int_0^{\ell^{2-\alpha}} \int_0^{(t+1)^{\delta}} x e^{-x} (1 + t)^{3/2} e^{-c\ell^\alpha} dx dt \leq C \ell^5 e^{-c\ell^\alpha}, \]  
(7.15)
and (6.33) follows.

7.3 The proof of Lemma 6.6: an informal argument

Let us explain informally why Lemma 6.6 is true, before giving the proof in Section 7.4 below. We expect that the functions \( z_\pm(t, x) \), solutions to (6.38)-(6.39), converge as \( t \to +\infty \) to \( \bar{z}_\pm(x) \), the solutions to
\[ -\bar{z}_\pm'' + e^{-x} \bar{z}_\pm^2 = 0, \]  
(7.16)
with the boundary condition
\[ \bar{z}_\pm(x)(+\infty) = r_\pm := c\ell \pm K\ell^{1-\delta}, \]  
(7.17)
with the constant \( K \) as in Lemma 6.5. Note that \( \bar{z}_\pm(x) \) has an explicit form
\[ \bar{z}_\pm(x) = r_\pm \bar{z}_0(x - \log r_\pm), \]  
(7.18)
with \( \bar{z}_0 \) that solves
\[ -\bar{z}_0'' + e^{-x} \bar{z}_0^2 = 0, \]  
(7.19)
with the slope at infinity
\[ \bar{z}_0(x)(+\infty) = 1. \]  
(7.20)

More precisely, we expect that \( z(t, x) \) is well approximated by taking
\[ r_\pm(t) = r_\pm e^{-\ell^2/(4t)}, \]  
(7.21)
leading to the asymptotics
\[ z_\pm(t, x) \approx r_\pm(t)\bar{z}_0(x - \log r_\pm(t)) = \bar{z}_\pm(x + \ell^2/4t)e^{-\ell^2/(4t)}, \]  
(7.22)
that holds for all \( t > 0 \), in the sense that

\[
\tilde{Y}_\ell^\pm = -\frac{1}{\sqrt{4\pi\ell}} \int_0^\infty \int_0^{(t+1)\delta} x e^{-x} z_\pm^2 \left( x + \frac{\ell^2}{4t} \right) e^{-\ell^2/(2t)} \frac{dxdt}{(t+1)^{3/2}} + O(\ell^{-\gamma})
\]

\[
= -\frac{1}{\sqrt{4\pi\ell}} \int_0^\infty \int_0^\infty x e^{-x} z_\pm^2 \left[ x + \frac{1}{4t} \right] e^{-1/(2t)} \frac{dxdt}{t^{3/2}} + O(\ell^{-\gamma})
\]

\[
= -\frac{1}{\sqrt{4\pi\ell}} \int_0^\infty \int_0^\infty x e^{-x} z_\pm^2 \left[ x + \frac{s}{4} \right] e^{-s/2} \frac{dxds}{\sqrt{s}} + O(\ell^{-\gamma})
\]

\[
= -\frac{1}{\sqrt{4\pi\ell}} \int_0^\infty \int_0^\infty \left( x - \frac{s}{4} \right) e^{-x} z_\pm^2(x) e^{-s/4} \frac{dxds}{\sqrt{s}} + O(\ell^{-\gamma})
\]

\[
(7.23)
\]

\[
= -\frac{r_\pm^2}{\sqrt{\pi} \ell} \int_0^\infty \int_0^\infty (x - s) e^{-x} z_0^2(x - \log r_\pm) e^{-s} \frac{dxds}{\sqrt{s}} + O(\ell^{-\gamma})
\]

\[
= -\frac{r_\pm}{\sqrt{\pi} \ell} \int_0^\infty \int_{s - \log r_\pm}^\infty (x + \log r_\pm - s) e^{-x} z_0^2(x) e^{-s} \frac{dxds}{\sqrt{s}} + O(\ell^{-\gamma})
\]

\[
= -\frac{r_\pm}{\sqrt{\pi} \ell} \int_0^\infty G(\log r_\pm - s) e^{-s} \frac{ds}{\sqrt{s}} + O(\ell^{-\gamma}) \quad \text{as } \ell \to +\infty,
\]

with some \( \gamma > 0 \), and

\[
G(q) = \int_{-q}^\infty (x + q) e^{-x} z_0^2(x) dx.
\]

Note that

\[
|G(q)| \leq C(1 + |q|).
\]

(7.25)

It follows that

\[
\left| \int_{\log r_\pm/2}^\infty G(\log r_\pm - s) e^{-s} \frac{ds}{\sqrt{s}} \right| \leq C \int_{\log r_\pm/2}^\infty |\log r_\pm - s| e^{-s} \frac{ds}{\sqrt{s}} \leq \frac{C}{r_\pm} \leq \frac{C}{\ell^{\gamma}}
\]

(7.26)

with \( \gamma > 0 \) sufficiently small. Therefore, we have

\[
\tilde{Y}_\ell^\pm = -\frac{r_\pm}{\sqrt{\pi} \ell} \int_0^{\log r_\pm/2} G(\log r_\pm - s) e^{-s} \frac{ds}{\sqrt{s}} + O(\ell^{-\gamma}) \quad \text{as } \ell \to +\infty.
\]

(7.27)

To analyze the asymptotic behavior of \( G(r) \) for \( r \gg 1 \), note that the solution to (7.19) with the normalization (7.20) is given by

\[
\tilde{z}(x) = e^x U(x),
\]

(7.28)

where \( U(x) \) is the Fisher-KPP minimal speed wave, solution to (1.5), with the normalization (1.6). Hence, the function \( \tilde{z}_0(x) \) has the asymptotics

\[
\tilde{z}_0(x) = x + k_0 + O(e^{-\omega x}), \quad \text{as } x \to +\infty,
\]

(7.29)

with \( k_0 \) as in (1.6), and some \( \omega > 0 \), and

\[
\tilde{z}_0(x) \sim e^x \quad \text{as } x \to -\infty.
\]

(7.30)

In addition, we have from (7.19)-(7.20) that

\[
\int_{-\infty}^\infty e^{-x} \tilde{z}_0^2(x) dx = 1,
\]

(7.31)
and
\[ \int_{-\infty}^{\infty} xe^{-x} z_{0}^{2}(x)\,dx = \int_{-\infty}^{\infty} x z_{0}''(x)\,dx = \lim_{m \to +\infty} \int_{-m}^{m} x z_{0}''(x)\,dx = \lim_{m \to +\infty} \left( m z_{0}'(m) + m z_{0}'(-m) - z_{0}(m) + z_{0}(-m) \right) = -k_{0}. \]  

Therefore, we can write \( G(q) \) as
\[ G(q) = \int_{-\infty}^{\infty} (x + q)e^{-x} z_{0}^{2}(x)\,dx - \int_{-\infty}^{-q} (x + q)e^{-x} z_{0}^{2}(x)\,dx = q - k_{0} - \int_{-\infty}^{-q} (x + q)e^{-x} z_{0}^{2}(x)\,dx. \]  

For the last integral in the right side, we deduce from (7.30) that
\[ \int_{-\infty}^{-q} (x + q)e^{-x} z_{0}^{2}(x)\,dx = \int_{-\infty}^{-q} (x + q)e^{x}\,dx + O(e^{-(1+\delta)q}) = -e^{-q} + O(e^{-(1+\delta)q}), \quad \text{as } q \to +\infty. \]  

It follows that for \( s < \log r_{\pm}/2 \) we have
\[ G(\log r_{\pm} - s) = \log r_{\pm} - s - k_{0} - \frac{1}{r_{\pm}} e^{s} + O(e^{-(1+\delta)(\log r_{\pm} - s)}) = \log r_{\pm} - s - k_{0} + O(\ell^{-\gamma}). \]  

Hence, we have
\[ Y_{\ell}^\pm = -\frac{r_{\pm}}{\sqrt{\pi} \ell} \int_{0}^{\log r_{\pm}/2} [\log r_{\pm} - s - k_{0}] e^{-s} \frac{ds}{\sqrt{s}} + O(\ell^{-\gamma}) \]
\[ = -\frac{r_{\pm}}{\sqrt{\pi} \ell} \int_{0}^{\infty} \left[ \log r_{\pm} - s - k_{0} \right] e^{-s} \frac{ds}{\sqrt{s}} + O(\ell^{-\gamma}) + O(\ell^{-\gamma}) \]
\[ = -\frac{r_{\pm}}{\sqrt{\pi} \ell} \left( [\log r_{\pm} - k_{0}] \Gamma(1/2) - \Gamma(3/2) \right) + O(\ell^{-\gamma}) = -\frac{r_{\pm}}{\ell} \left( \log r_{\pm} - k_{0} - \frac{1}{2} \right) + O(\ell^{-\gamma}) \]
\[ = -c \log \ell - c \log c + c k_{0} + \frac{c}{2} + O(\ell^{-\gamma}). \]

We used above the fact that \( \Gamma(1/2) = \sqrt{\pi} \) and \( \Gamma(3/2) = \sqrt{\pi}/2 \).

### 7.4 The proof of Lemma 6.6

Let us now proceed with the actual proof of Lemma 6.6. The computation in the previous section relied crucially on approximation (7.22), and the main step is to justify it. The function \( z_{\pm}(t,x) \) satisfies (6.38)-(6.39) (we drop the sub-script + in this section):
\[ \frac{\partial z}{\partial t} - \frac{\partial^{2} z}{\partial x^{2}} - \frac{3}{2(t+1)} \left( z - \frac{\partial z}{\partial x} \right) + e^{-x}z^{2} = 0, \quad x < (t+1)^{\delta}, \]  

for \( t \geq \ell^{2-\alpha} \), with the boundary condition
\[ z(t, (t+1)^{\delta}) = r_{\pm} e^{-\ell^{2}/(4(t+1))} (t+1)^{\delta}, \]  

with \( r_{\pm} = \bar{c} \ell + K \ell^{1-\delta} \), and the initial condition
\[ z(t = \ell^{2-\alpha}, x) = e^{-\ell^{2}c' \alpha}, \quad x < (t+1)^{\delta}, \]  

as in the upper bound (7.14), with some \( \alpha' > 0 \).
Our goal is to find a more explicit super-solution than the solution to (7.37)-(7.39). Motivated by (7.18), let us define

$$\psi(t, x) = e^{\zeta(t)} \bar{z}_0(x - \zeta(t)),$$

(7.40)

with

$$\zeta(t) = \log r_+ + \log \left(1 + \frac{\log \ell}{\ell^3}\right) - \frac{\ell^2}{4(t + 1)},$$

(7.41)

so that

$$\psi(t, x) = r_+ \left(1 + \frac{\log \ell}{\ell^3}\right) e^{-\ell^2/(4(t+1))} \bar{z}_0 \left(x - \log r_+ - \log \left(1 + \frac{\log \ell}{\ell^3}\right) + \frac{\ell^2}{4(t + 1)}\right),$$

(7.42)

with $\bar{z}_+(x)$, as in (7.18).

**Lemma 7.1** There exists $C > 0$ sufficiently large, and $\lambda > 0$ sufficiently small, so that

$$z(t, x) \leq \psi(t, x) + C \frac{\ell^2}{(t + 1)^\gamma}, \quad \text{for } t > t_m = \ell^{2-\alpha} \text{ and } |x| < (t + 1)^\delta.$$  

(7.43)

**The proof of Lemma 7.1**

Our goal will be to show that we can choose $\lambda > 0$, $\gamma > 0$ and $C > 0$ so that

$$s(t, x) = z(t, x) - \psi(t, x) \leq \bar{s}(t, x) = \frac{C}{\ell^{1/4}(t + 1)^\lambda} \cos \left(\frac{x}{(t + 1)^{\delta + \gamma}}\right), \quad |x| < (t + 1)^\delta.$$  

(7.44)

Let us first show that with the choice of the shift $\zeta(t)$ as in (7.41), at the boundary $x = (t + 1)^\delta$ we have

$$\psi(t, (t + 1)^\delta) \geq z(t, (t + 1)^\delta),$$

(7.45)

so that

$$s(t, (t + 1)^\delta) \leq 0 \leq \bar{s}(t, (t + 1)^\delta) \text{ for } t \geq T_0.$$  

(7.46)

At this point we have

$$\psi(t, (1 + t)^\delta) = r_+ \left(1 + \frac{\log \ell}{\ell^3}\right) e^{-\ell^2/(4(t+1))} \bar{z}_0 \left((t + 1)^\delta - \log r_+ - \log \left(1 + \frac{\log \ell}{\ell^3}\right) + \frac{\ell^2}{4(t + 1)}\right).$$  

(7.47)

It follows from (7.29) that there exists $k_1$ so that

$$\bar{z}_0(x) \geq x + k_1 \text{ for all } x \geq 0.$$  

(7.48)

Therefore, if

$$\frac{\ell^2}{4(t + 1)} \geq \log r_+ + \log \left(1 + \frac{\log \ell}{\ell^3}\right) - k_1,$$

then, as $\bar{z}_0(x)$ is increasing, we automatically have from (7.47) that

$$\psi(t, (1 + t)^\delta) \geq r_+ e^{-\ell^2/(4(t+1))} \left((t + 1)^\delta - \log r_+ - \log \left(1 + \frac{\log \ell}{\ell^3}\right) + \frac{\ell^2}{4(t + 1)} + k_1\right) \geq r_+ e^{-\ell^2/(4(t+1))}(t + 1)^\delta.$$  

(7.49)
On the other hand, if
\[
\frac{\ell^2}{4(t + 1)} < \log r + \log \left(1 + \frac{\log \ell}{\ell^\delta}\right) - k_1,
\] (7.50)
and \(\ell\) is sufficiently large, then we have
\[
t > d \frac{\ell^2}{\log \ell},
\] (7.51)
with a sufficiently small \(d > 0\), and \((t + 1)\) dominates the other terms in the argument inside \(\bar{z}_0\) in the right side of (7.47). In particular, if (7.50) holds, since \(\bar{z}_0\) is increasing, we can use (7.48) to obtain
\[
\left(1 + \frac{\log \ell}{\ell^\delta}\right)\bar{z}_0 ((t + 1)\delta - 1 - \log r_+ - k_1) \geq (t + 1)\delta;
\] (7.52)
because
\[
\frac{\log \ell}{\ell^\delta} ((t + 1)\delta - 1 - \log r_+ - k_1) - 1 - \log r_+ - k_1 \geq c\ell^{\delta/2} \log \ell - C \log \ell > 0.
\] (7.53)
We have used (7.51) above. Therefore, if we choose \(\zeta(t)\) as in (7.41), then the comparison at the boundary (7.45), indeed, holds.

To look at the other boundary point, \(x = -(t + 1)\delta\), note that the function \(\bar{z}(x) = Ae^x\) is a super-solution to (7.37) for all \(A > 1\) sufficiently large, and at \(x = (t + 1)\delta\) we have, due to Lemma 6.5:
\[
z(t, (t + 1)\delta) \leq C\ell((t + 1)\delta - \ell^2/4(t + 1)) \leq C(t + 1)^{1/2+\delta} \leq Ae^{(t+1)\delta} = \bar{z}((t + 1)\delta),
\] (7.54)
if \(A\) is sufficiently large. It follows that
\[
z(t, x) \leq Ae^x, \quad x < (t + 1)\delta,
\]
and, in particular, we deduce that
\[
z(t, -(t + 1)\delta) \leq Ae^{-(t+1)\delta} \leq \frac{C}{\ell^{1/4}(t + 1)\lambda} \leq \bar{s}(t, -(t + 1)\delta), \quad \text{for } t \geq \ell^{2-\alpha},
\] (7.55)
and thus
\[
s(t, -(t + 1)\delta) \leq \bar{s}(t, -(t + 1)\delta), \quad t \geq \ell^{2-\alpha}.
\] (7.56)
At the initial time \(t_{in} = \ell^{2-\alpha}\) we have the comparison
\[
s(t_{in}, x) \leq z(t_{in}, x) \leq Ce^{-\ell t_{in}} \leq \frac{C}{\ell^{1/4}(1 + t_{in})\lambda} \leq \bar{s}(t_{in}, x), \quad \text{for } x < (t_{in} + 1)\delta.
\] (7.57)
Having established the comparison at the boundary and the initial time, we now show that \(\bar{s}(t, x)\)
We used the fact that the function $\zeta(t)$ is increasing in $t$ and $\bar{z}_0(x)$ is increasing in $x$ in the last step above. Note that

$$\dot{\zeta}(t)e^{\zeta(t)} = r_+\left(1 + \frac{\log \ell}{\ell^8}\right)\frac{\ell^2}{4(t+1)^2}e^{-\ell^2/4(t+1)} \leq \frac{C\ell^3}{(t+1)^2}e^{-\ell^2/(4(t+1))},$$

hence

$$\dot{\zeta}(t)e^{\zeta(t)}\frac{\partial \bar{z}_0(x - \zeta(t))}{\partial x} \leq \frac{C\ell^3}{(t+1)^2}e^{-\ell^2/(4(t+1))} \leq \frac{C}{\ell^{1/2}(t+1)^{1/4}}.$$ (7.60)

The second term in the right side of (7.58) for $x < (t + 1)^{\delta}$ is bounded by

$$\frac{3}{2(t+1)}e^{\zeta(t)}\bar{z}_0(x - \zeta(t)) \leq \frac{Cl}{t+1}e^{-\ell^2/(4(t+1))}\left((t + 1)^{\delta} + \frac{\ell^2}{t + 1}\right)$$

$$= C\left(\frac{\ell}{(t+1)^{1-\delta}} + \frac{\ell^3}{(t+1)^2}\right)e^{-\ell^2/(4(t+1))}$$

$$\leq C\left(\frac{1}{\ell^{1/4}(t+1)^{3/8-\delta}} \cdot \frac{\ell^{5/4}}{(t+1)^{5/8}} + \frac{1}{\ell^{1/2}(t+1)^{1/4}} \cdot \frac{\ell^{7/2}}{(t+1)^{7/4}}\right)e^{-\ell^2/(4(t+1))} \leq \frac{C}{\ell^{1/4}(t+1)^{1/4}}.$$ (7.61)

Therefore, the function $s(t, x) = z(t, x) - \psi(t, x)$ satisfies

$$\frac{\partial s}{\partial t} - \frac{\partial^2 s}{\partial x^2} - \frac{3}{2(t+1)}\left(s - \frac{\partial s}{\partial x}\right) + e^{-x}(\psi(t, x) + z(t, x))s \leq \frac{C}{\ell^{1/4}(t+1)^{1/4}}, \quad x < (t + 1)^{\delta}. \quad (7.62)$$

On the other hand, recall from [23] that for $\gamma > 0$, the function $\tilde{s}(t, x)$ satisfies

$$\frac{\partial \tilde{s}(t, x)}{\partial t} = -\frac{\lambda}{t+1}\tilde{s}(t, x) + g(t, x), \quad (7.63)$$

with $g(t, x)$ such that

$$|g(t, x)| \leq \frac{C(\delta + \gamma)|x|}{\ell^{1/4}(t+1)^{\lambda+\delta+\gamma+1}} \leq \frac{C(\delta + \gamma)\tilde{s}(t, x)}{(t+1)^{\gamma+1}}, \quad \text{for } |x| \leq (t + 1)^{\delta}. \quad (7.64)$$
In addition, we have
\[
\frac{\partial^2 \bar{s}(t,x)}{\partial x^2} = -\frac{1}{(t+1)^{2\delta + 2\gamma}} \bar{s}(t,x),
\] (7.65)
and
\[
\frac{3}{2(t+1)} \left( \frac{\partial \bar{s}(t,x)}{\partial x} - \bar{s}(t,x) \right) \geq -\frac{C}{t+1} \bar{s}(t,x), \quad \text{for } |x| \leq (t+1)^\delta.
\] (7.66)

We conclude that
\[
\frac{\partial \bar{s}}{\partial t} - \frac{\partial^2 \bar{s}}{\partial x^2} - \frac{3}{2(t+1)} \left( \frac{\partial \bar{s}}{\partial x} - \bar{s} \right) + e^{-x}(\psi(t,x) + z(t,x)) \bar{s} \\
\geq -\frac{\lambda}{1 + t} \bar{s}(t,x) + g(t,x) + \frac{1}{(1+t)^{2\delta + 2\gamma}} \bar{s}(t,x) - \frac{C}{(t+1)^2} \bar{s}(t,x) \geq \frac{C}{(1+t)^{2\delta + 2\gamma}} \bar{s}(t,x)
\] (7.67)
provided that \( \delta > 0 \) and \( \gamma > 0 \) are sufficiently small. Now, putting together the boundary comparisons (7.46) and (7.56), the initial time comparison (7.57), as well as (7.62) and (7.67), we deduce from the comparison principle that if \( \lambda > 0 \), \( \delta > 0 \) and \( \gamma > 0 \) are all sufficiently small, then
\[ s(t,x) \leq \bar{s}(t,x) \text{ for } t > t_{in} = \ell^{2-\alpha} \text{ and } |x| < (t+1)^\delta. \] (7.68)

This implies (7.44), and (7.43) follows, finishing the proof of Lemma 7.1. \( \square \)

The end of the proof of Lemma 6.6

Using (7.43) in (6.41) gives
\[
-\overline{Y}^{(+)}_{\ell} = \frac{1}{\sqrt{4\pi}} \int_{t^{2-\alpha}}^{\infty} \int_{t}^{t+1} xe^{-x} z_{\ell}^2(t,x) \frac{dxdt}{(t+1)^{3/2}} \\
\leq \frac{1}{\sqrt{4\pi}} \int_{t^{2-\alpha}}^{\infty} \int_{t}^{t+1} xe^{-x} \left( \psi(t,x) + \frac{C}{\ell^{1/4}(t+1)^\lambda} \right)^2 \frac{dxdt}{(t+1)^{3/2}} = I + II + III,
\] (7.69)
with the three terms coming from the expansion
\[
\left( \psi(t,x) + \frac{C}{\ell^{1/4}(t+1)^\lambda} \right)^2 = \psi^2(t,x) + \frac{2C}{\ell^{1/4}(t+1)^\lambda} \psi(t,x) + \frac{C^2}{\ell^{1/2}(t+1)^{2\lambda}}.
\]

We have, clearly,
\[ III \leq \frac{C}{\ell^{1/2}}. \] (7.70)

For the second term, we recall that for \( x > 0 \) we have
\[
\psi(t,x) = e^{\zeta(t)} \bar{z}_0(x - \zeta(t)) \leq C \ell e^{-\ell^2/(4(t+1))} \bar{z}_0 \left( x + \frac{\ell^2}{4(t+1)} \right) \\
\leq C \ell e^{-\ell^2/(4(t+1))} \left( x + 1 + \frac{\ell^2}{4(t+1)} \right),
\] (7.71)
and use this to write

\[
II = \frac{C}{\ell^{1/4}} \int_{\ell^{2-\alpha}}^{\infty} \int_0^{(t+1)^{\delta}} xe^{-x} \varphi(t, x) \frac{dx dt}{(t + 1)^{3/2 + \lambda}} \leq \frac{C}{\ell^{1/4}} \int_0^{(t+1)^{\delta}} xe^{-x} \ell e^{-\ell^2/(4(t+1))} \left(x + 1 + \frac{\ell^2}{4(t+1)}\right) \frac{dx dt}{(t + 1)^{3/2 + \lambda}} \leq \frac{C}{\ell^{1/4}}. \quad (7.72)
\]

For the main term, we need to compute more precisely, and we use expression (7.42) for \(\varphi(t, x)\):

\[
I = \frac{1}{4\pi} \int_{\ell^{2-\alpha}}^{\infty} \int_0^{(t+1)^{\delta}} xe^{-x} \varphi^2(t, x) \frac{dx dt}{(1 + \ell)^{3/2}} \leq \left(1 + \frac{\log \ell}{\ell^6}\right)^2 \int_{\ell^{2-\alpha}}^{\infty} \int_0^{(t+1)^{\delta}} xe^{-x} \ell e^{-\ell^2/(4(t+1))} \left(x + 1 + \frac{\ell^2}{4(t+1)}\right) \frac{dx dt}{(t + 1)^{3/2 + \lambda}} \leq \frac{C}{\ell^{1/4}}. \quad (7.73)
\]

This is simply expression (7.23) that we have computed in Section 7.3, leading to (7.36):

\[
-\tilde{Y}_t^+ \leq \tilde{c} \log \ell + \tilde{c} \log \tilde{c} \tilde{k}_0 - \frac{\tilde{c}}{2} + O(\ell^{-\gamma}) \quad (7.74)
\]

This finishes the proof of the upper bound in Lemma 6.6.

Proceeding as in the proof of the upper bound, with some minor modifications, we can obtain a matching lower bound

\[
-\tilde{Y}_t^- \geq \tilde{c} \log \ell + \tilde{c} \log \tilde{c} \tilde{k}_0 - \frac{\tilde{c}}{2} + O(\ell^{-\gamma}) \quad (7.75)
\]

which finishes the proof of Lemma 6.6.

### 7.5 The proof of Lemma 6.2

We need to show that

\[
\tilde{Y}_t = -\int_0^{\infty} \int_0^{\infty} e^{3\tau/2 - \eta \exp(\tau/2) r_\tau^2(\tau, \eta)} \tilde{\psi}(\eta) e^{-\tau/2} d\tau d\eta = O(\ell^{-\gamma}), \text{ as } \ell \to +\infty. \quad (7.76)
\]

As in (7.2), we re-write this in terms of \(z_t(t, x)\) as

\[
\tilde{Y}_t = -\int_0^{\infty} \int_0^{(t+1)} e^{-x} w_t^2(t, x) \tilde{\psi}\left(\frac{x}{\sqrt{t+1}}\right) \frac{dt}{t+1} \frac{dx}{\sqrt{t+1}} \leq -\int_0^{\infty} \int_0^{(t+1)} e^{-x} z_t^2(t, x) \tilde{\psi}\left(\frac{x}{\sqrt{t+1}}\right) \frac{dx dt}{(t+1)^{3/2}}. \quad (7.77)
\]

We sketch the argument that can be made precise using Lemma 7.1 in a straightforward way. Let us use approximation (7.22) again:

\[
z_t(t, x) \approx \tilde{z}_\pm(x)e^{-\ell^2/(4(t+1))}, \quad (7.78)
\]
and insert this into (7.77). This would give

\[
\tilde{Y}_t \approx - \int_0^\infty \int_0^\infty e^{-x^2 / (2(t+1))} \psi \left( \frac{x}{\sqrt{t+1}} \right) dx dt / (t+1)^{3/2}.
\]

(7.79)

Recall that \(\tilde{\psi}(\eta) \leq C\eta\), so we can roughly estimate

\[
|\tilde{Y}_t| \leq Cr_\pm \int_0^\infty \int_{-\infty}^\infty e^{-x^2 / (2(t+1))} \left( |x| + \log r_\pm \right) dx dt / (t+1)^{3/2}.
\]

(7.80)

This finishes the proof of Lemma 6.2. □

### 7.6 The proof of Lemma 6.5

We now turn to the proof of Lemma 6.5. Let us go back to (6.30):

\[
\frac{\partial z_l}{\partial t} - \frac{\partial^2 z_l}{\partial x^2} - \frac{3}{2(t+1)} \left( z_l - \frac{\partial z_l}{\partial x} \right) + e^{-x} z_l^2 = 0,
\]

(7.81)

and undo yet another change of variables:

\[
v(t, x) = (t + 1)^{-3/2} z_l(t, x + \frac{3}{2} \log(t + 1)).
\]

(7.82)

The function \(v(t, x)\) satisfies simply

\[
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + e^{-x} v^2 = 0
\]

(7.83)

\[
v(0, x) = \psi_0(x - \ell).
\]

Note that

\[
z_l(t, (t + 1)^{\delta}) = \frac{(t + 1)^{3/2} v(t, (t + 1)^{\delta} - \frac{3}{2} \log(t + 1))}{(t + 1)^{\delta}},
\]

(7.84)

so that our point of interest for \(v(t, x)\) is

\[
x_l(t) = (t + 1)^{\delta} - \frac{3}{2} \log(t + 1).
\]

(7.85)

We perform the standard parabolic scaling

\[
p(s, \xi) = v(\ell^2 s, \ell \xi),
\]

(7.86)

so that the function \(p(s, \xi)\) solves

\[
\frac{\partial p}{\partial s} - \frac{\partial^2 p}{\partial \xi^2} + \ell^2 e^{-\ell \xi} p^2 = 0
\]

(7.87)

\[
p(0, \xi) = \psi_0(\ell(\xi - 1)).
\]
In the new variables, the point of interest is
\[
x_\ell(l^2s) = \frac{x_\ell(l^2s)}{\ell} = \frac{(l^2s + 1)^\delta - (3/2)\log(l^2s + 1)}{\ell} \approx \frac{s^\delta}{\ell^{1-2\alpha}},
\]
(7.88)
and the important time scales are \(\ell^{-\alpha} \ll s \ll \ell^\alpha\), corresponding to \(\ell^{2-\alpha} \ll t \ll \ell^{2+\alpha}\). In particular, we have
\[
|\xi_\ell(s)| = (1 + O(\ell^{-\gamma})) \frac{s^\delta}{\ell^{1-2\alpha}}, \quad \text{for } \ell^{-\alpha} \ll s \ll \ell^\alpha.
\]
(7.89)

Let us start with the proof of the lower bound (6.37). The maximum principle implies that
\[
p(s, \xi) \leq C_0 = \|\psi_0\|_{L^\infty}.
\]
Therefore, if we take \(0 < \gamma \ll \delta\) and impose the Dirichlet boundary condition at \(\xi = \ell^{-(1-\gamma)}\), then we have
\[
p(s, \xi) \geq q(s, \xi), \quad \text{for } s \geq \ell^{-\alpha},
\]
(7.90)
where \(q(s, \xi)\) is the solution to
\[
\begin{align*}
&\frac{\partial q}{\partial s} - \frac{\partial^2 q}{\partial \xi^2} + C_0 l^2 e^{-\ell^\gamma} q = 0, \quad s > 0, \quad \xi > \ell^{-(1-\gamma)}, \\
&q(0, \xi) = \psi_0(l(\xi - 1)), \\
&q(0, \ell^{-(1-\gamma)}) = 0.
\end{align*}
\]
(7.91)
We see from (7.88) that, as \(\gamma < \delta\), we have
\[
\xi_\ell(s) \geq \frac{s^\delta}{2\ell^{1-2\alpha}} \geq \frac{\ell^{-\alpha\delta}}{2\ell^{1-2\alpha}} > \ell^{-(1-\gamma)}, \quad \text{for } s \geq \ell^{-\alpha},
\]
(7.92)
provided that \(\gamma\) is sufficiently small, so that we have not lost our point of interest in this approximation. An explicit formula for the solution to (7.91) gives
\[
q(s, \xi_\ell(s)) = \frac{e^{-C_0 l^2 e^{-\ell^\gamma}s}}{\sqrt{4\pi s}} \int_0^{+\infty} \left( e^{-(\tilde{\xi}_\ell(s) - \zeta)^2/(4s)} - e^{-(\tilde{\xi}_\ell(s) + \zeta)^2/(4s)} \right) \psi_0(l(\zeta - 1)) d\zeta, \quad \text{for } s \geq \ell^{-\alpha},
\]
(7.93)
with
\[
\tilde{\xi}_\ell(s) = \xi_\ell(s) - \ell^{-(1-\gamma)} \approx \frac{s^\delta}{\ell^{1-2\alpha}} - \frac{1}{\ell^{1-\gamma}} \approx \frac{s^\delta}{\ell^{1-2\alpha}}, \quad \text{for } s \geq \ell^{-\alpha}.
\]
(7.94)
The restriction on \(s\) in (7.93) comes from the requirement that (7.92) holds, so that \(\xi_\ell(s)\) is in the region where \(q(s, \xi)\) is defined.

The integrand in (7.93) is very small for \(||\zeta - 1|| \gg \ell^{-1}\) because of the exponential decay of the initial condition \(\psi_0(x)\). With this in mind, we write the difference of the exponentials in (7.93) as
\[
e^{-(\tilde{\xi}_\ell(s) - \zeta)^2/(4s)} - e^{-(\tilde{\xi}_\ell(s) + \zeta)^2/(4s)} = e^{-(\tilde{\xi}_\ell(s) - \zeta)^2/(4s)}(1 - e^{-\tilde{\xi}_\ell(s)\zeta/s}).
\]
(7.95)
As \(\zeta \approx 1\), we have
\[
\frac{\tilde{\xi}_\ell(s)}{s} \approx \frac{1}{\ell^{1-2\alpha} s^{1-\delta}} \ll 1 \quad \text{for } s \gg \ell^{(1-2\delta)/(1-\delta)}.
\]
(7.96)
Using the inequality
\[
1 - e^{-u} \geq u - u^2
\]
(7.97)
for sufficiently small \( u \) we have then
\[
q(s, \xi_\ell(s)) \geq \frac{e^{-s\xi_\ell^2}e^{-\gamma\xi_\ell}}{\sqrt{4\pi s^{3/2}}} (1 + o(\ell^{-1})) \int_0^{+\infty} e^{-(\xi_\ell(s) - \zeta)^2/(4s)} \zeta \left( 1 - \frac{\zeta \xi_\ell}{s} \right) \psi_0(\ell(\zeta - 1)) d\zeta. \tag{7.98}
\]

The \( o(\ell^{-1}) \) correction in the pre-factor comes from \( \zeta \) that violate (7.96), so that we can not use (7.97). Their contribution is extremely small since \( \psi_0(x) \) is decaying exponentially as \( x \to -\infty \) and has compact support for \( x > 0 \). Using the straightforward approximations for the Gaussian and the factor inside the parenthesis inside the integral in (7.98), as well as for the exponential factor in front of the integral, we obtain
\[
q(s, \xi_\ell(s)) \geq \frac{e^{-s\xi_\ell^2}e^{-\gamma\xi_\ell}}{\sqrt{4\pi s^{3/2}}} (1 + O(\ell^{-1})) \int_0^{+\infty} e^{-(\xi_\ell(s) - \zeta)^2/(4s)} \zeta \psi_0(\ell(\zeta - 1)) d\zeta
\]
\[
\geq \frac{\xi_\ell(s) e^{-1/(4s)}}{\sqrt{4\pi s^{3/2}} (1 + O(\ell^{-1}))} \int_0^{+\infty} \zeta \psi_0(\ell(\zeta - 1)) d\zeta
\]
\[
= \frac{\xi_\ell(s) e^{-1/(4s)}}{\sqrt{4\pi s^{3/2}} (1 + O(\ell^{-1}))} \int_{-\ell}^{+\infty} \psi_0(\zeta) d\zeta = \frac{c\xi_\ell(s) e^{-1/(4s)}}{s^{3/2} \ell}(1 + O(\ell^{-1})). \tag{7.99}
\]

Going back to (7.89) and (7.94), we note that
\[
\tilde{\xi}_\ell(s) = \xi_\ell(s) - \ell^{-(1-\gamma)} = (1 + O(\ell^{-\gamma})) \frac{s^\delta}{\ell^{1-\delta}} - \frac{1}{\ell^{1-\gamma}} \approx \frac{s^\delta}{\ell^{1-\delta}} = (1 + O(\ell^{-\gamma})) \xi_\ell(s), \quad \text{for } s \geq \ell^{-\alpha}. \tag{7.100}
\]

Using this in (7.99) gives
\[
q(s, \xi_\ell(s)) \geq \frac{c\xi_\ell(s) e^{-1/(4s)}}{s^{3/2} \ell}(1 + O(\ell^{-\gamma})), \quad \text{for } s \geq \ell^{-\alpha}. \tag{7.101}
\]

Unrolling the changes of variables (7.84), (7.86), and (7.88), and using the bound (7.90), as well as the approximation (7.89), we can re-write (7.101) as
\[
\frac{z_\ell(t, (t+1)^\delta)}{(t+1)^\delta} = \frac{(t+1)^{3/2}v(t, (t+1)^\delta} - (3/2) \log(t+1)}{(t+1)^\delta}
\]
\[
= \frac{(t+1)^{3/2}p(\ell^{-2}t, \ell^{-1}(t+1)^\delta - (3/2)\ell^{-1} \log(t+1))}{(t+1)^\delta} = \frac{(t+1)^{3/2}p(\ell^{-2}t, \xi_\ell(\ell^{-2}t))}{(t+1)^\delta}
\]
\[
\geq \frac{(t+1)^{3/2}q(\ell^{-2}t, \xi_\ell(\ell^{-2}t))}{(t+1)^\delta} \geq \frac{(t+1)^{3/2}c\xi_\ell(\ell^{-2}t) e^{-\ell^2/(4t)} \ell^3}{(t+1)^\delta (t+1)^{3/2} \ell}(1 + O(\ell^{-\gamma}))
\]
\[
= \frac{c\ell^{-\delta} \ell^2 e^{-\ell^2/(4t) \ell^3}}{(t+1)^\delta \ell^{1-2\delta}}(1 + O(\ell^{-\gamma})) = c\ell(1 + O(\ell^{-\gamma})) e^{-\ell^2/(4t) \ell^3}, \quad \text{for } \ell^{2-\alpha} \leq t \leq \ell^{2+\alpha}, \tag{7.102}
\]

which is the lower bound (6.37).

For the upper bound (6.36), we again look at the solution \( p(s, \xi) \) of (7.87). A simple upper bound for \( p(s, \xi) \) is by the solution to the heat equation on the whole real line:
\[
\frac{\partial \tilde{p}}{\partial s} - \frac{\partial^2 \tilde{p}}{\partial \xi^2} = 0, \quad x \in \mathbb{R}, \tag{7.103}
\]
\[
\tilde{p}(0, \xi) = \psi_0(\ell(\xi - 1)).
\]

Accordingly, we set
\[
\overline{\psi}_0(\xi) = \tilde{p}(s_\alpha, \xi) = \frac{1}{\sqrt{4\pi s_\alpha}} \int_{\mathbb{R}} e^{-(\xi - \zeta)^2/4s_\alpha} \psi_0(\ell(\zeta - 1)) d\zeta, \quad s_\alpha = \ell^{-\alpha}, \tag{7.104}
\]
with a sufficiently small $\alpha$ to be chosen, depending on $\delta$. We also have the upper bound

$$p(s, \xi) \leq e^{\ell \xi}, \quad (7.105)$$

that follows immediately from the maximum principle. In particular, we have

$$p(s, -\ell^{-(1-\gamma)}) \leq e^{-\ell^{\gamma}}, \quad (7.106)$$

whence

$$p(s, \xi) \leq e^{-\ell^{\gamma}} + \bar{p}_1(s, \xi). \quad (7.107)$$

Here, $\bar{p}_1(s, \xi)$ is the solution to and $v(\tau, \xi)$ solves

$$\frac{\partial \bar{p}_1}{\partial s} - \frac{\partial^2 \bar{p}_1}{\partial \xi^2} = 0, \quad s > s_\alpha, \ \xi > -\ell^{-(1-\gamma)},$$

$$\bar{p}_1(s_\alpha, \xi) = \psi_0(\xi), \quad \bar{p}_1(s, -\ell^{-(1-\gamma)}) = 0. \quad (7.108)$$

Let us note the following properties of the initial condition $\psi_0$. First, $(7.104)$ implies that it is localized near $\xi = 1$, in the sense that

$$0 < \psi_0(\xi) \leq C \ell^{-\alpha/2} \ell^{-1} \exp \{ -\ell^{\alpha-2\beta} \}, \quad \text{for } |\xi - 1| \geq \ell^{-\beta}. \quad (7.109)$$

Hence, as soon as $\xi$ departs from a very small neighborhood of 1, $\psi_0(\xi)$ is exponentially small in $\ell$. Furthermore, the mass of $\psi_0(\xi)$ is

$$\int_{\mathbb{R}} \psi_0(\xi) d\xi = \frac{\bar{c}}{\ell}, \quad (7.110)$$

and its first moment is

$$\int_{\mathbb{R}} \xi \psi_0(\xi) d\xi = \frac{\bar{c}}{\ell} (1 + O(\frac{1}{\ell})), \quad (7.111)$$

because of $(7.109)$.

It follows that the function $\bar{p}_1(s, \xi)$ can then be estimated along the same lines as $q(s, \xi)$ in the proof of the lower bound. This eventually leads to $(6.36)$. □

## A Proof of Lemma 2.1

Fix an arbitrary $\phi_0 \in \mathcal{C}_c^+$ and let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration generated by $\{\mathcal{A}_t\}_{t \geq 0}$. Then, by the definition $(1.10)$ of $\{\mathcal{A}_t\}_{t \geq 0}$, the Markov property and $(1.3)$, we get (with some ambiguity of notation we set $\mathbb{E} = \mathbb{E}_0$: the expectation for BBMs starting with one particle at 0)

$$\mathbb{E}\left[e^{-\mathcal{A}_t(\phi_0)} \mid \mathcal{F}_s\right] = \mathbb{E}\left[\prod_{j=1}^{N_t} e^{-\phi_0(2t-(3/2) \log t - x_j(t))} \mid \mathcal{F}_s\right]$$

$$= \prod_{j=1}^{N_s} \mathbb{E}_{x_j(s)} \left[ \prod_{i=1}^{N_t-s} \left( 1 - (1 - e^{-\phi_0(2(t-s)-(3/2) \log(t-s) - x_i(t-s) + 2s + (3/2) \log((t-s)/t)))} \right) \right] \quad (A.1)$$

$$= \prod_{j=1}^{N_s} \left( 1 - u(t-s, m(t-s) - x_j(s) + 2s + \frac{3}{2} \log(1 - \frac{s}{t})) \right), \quad \mathbb{P} \text{- a.s.}$$

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Here, \( u(t, x) \) is the solution to (1.1) with the initial condition \( u(0, x) = \psi_0(x) = 1 - \exp(-\phi_0(x)) \). Next, we use (1.8) to get

\[
\lim_{t \to \infty} \prod_{j=1}^{N_s} \left( 1 - u(t - s, m(t - s) - x_j(s) + 2s + \frac{3}{2} \log(1 - \frac{s}{t}) \right) = \prod_{j=1}^{N_s} \left( 1 - U(s\psi_0 - x_j(s) + 2s) \right)
\]

\[
= \exp \left\{ - \sum_{j=1}^{N_s} - \log(1 - U(s\psi_0 - x_j(s) + 2s) \right\}.
\]

Following the derivations in (2.4.9)-(2.4.12) in [6] (see also (23)-(25) in [17]) and using (1.6) which gives \( C = 1 \) in the above references we obtain

\[
\lim_{s \to \infty} \exp \left\{ - \sum_{j=1}^{N_s} - \log(1 - U(s\psi_0 - x_j(s) + 2s) \right\} = \exp \left\{ - Z e^{-\hat{j}[\psi_0]} \right\}, \quad \mathbb{P} - \text{a.s.} \quad (A.3)
\]

Fix an arbitrary bounded continuous function \( h \). Putting together (A.1), (A.2), and (A.3), we get,

\[
\lim_{s \to \infty} \lim_{t \to \infty} \mathbb{E} \left[ h(Z_s) e^{-\mathcal{X}_t(\phi_0)} \right] = \lim_{s \to \infty} \lim_{t \to \infty} \mathbb{E} \left[ h(Z_s) \mathbb{E} \left[ e^{-\mathcal{X}_t(\phi_0)} | \mathcal{F}_s \right] \right] = \lim_{s \to \infty} \mathbb{E} \left[ h(Z_s) e^{-\sum_{j=1}^{N_s} \log(1 - U(s[\psi_0] - x_j(s)) + 2s)} \right] = \mathbb{E} \left[ h(Z) e^{-Ze^{-\hat{j}[\psi_0]}} \right]. \quad (A.4)
\]

We used the bounded convergence theorem in the last equality.

Recall, that by Theorem 2.1 in [1], the pair \( (\mathcal{X}_t, Z_t) \) converges in distribution to \( (\mathcal{X}, Z) \) as \( t \to \infty \), where \( \mathcal{X}_t \) (resp. \( \mathcal{X} \)) is just the measure \( \mathcal{X}_t \) (resp. \( \mathcal{X} \)) shifted by \( \log Z \). This together with a.s. convergence of \( Z_t \to Z \) implies also convergence in distribution of \( (\mathcal{X}_t, Z_t) \) to \( (\mathcal{X}, Z) \) as \( t \to \infty \), and thus

\[
\mathbb{E} \left[ h(Z) e^{-\mathcal{X}(\phi_0)} \right] = \lim_{t \to \infty} \mathbb{E} \left[ h(Z_t) e^{-\mathcal{X}_t(\phi_0)} \right].
\]

From this we get

\[
\left| \mathbb{E} \left[ h(Z) e^{-\mathcal{X}(\phi_0)} \right] - \mathbb{E} \left[ h(Z) e^{-Ze^{-\hat{j}[\psi_0]}} \right] \right| = \lim_{t \to \infty} \mathbb{E} \left[ h(Z_t) e^{-\mathcal{X}_t(\phi_0)} \right] - \mathbb{E} \left[ h(Z) e^{-Ze^{-\hat{j}[\psi_0]}} \right] \leq \lim_{t \to \infty} \mathbb{E} \left[ (h(Z_t) - h(Z_s)) e^{-\mathcal{X}_t(\phi_0)} \right] + \lim_{t \to \infty} \mathbb{E} \left[ h(Z_s) e^{-\mathcal{X}_t(\phi_0)} \right] - \mathbb{E} \left[ h(Z) e^{-Ze^{-\hat{j}[\psi_0]}} \right], \quad \forall s > 0.
\]

Therefore, we have

\[
\mathbb{E} \left[ h(Z) e^{-\mathcal{X}(\phi_0)} \right] - \mathbb{E} \left[ h(Z) e^{-Ze^{-\hat{j}[\psi_0]}} \right] \leq \limsup_{s \to \infty} \lim_{t \to \infty} \mathbb{E} \left[ h(Z_t) - h(Z_s) \right]
\]

\[
+ \left| \limsup_{s \to \infty} \lim_{t \to \infty} \mathbb{E} \left[ h(Z_s) e^{-\mathcal{X}_t(\phi_0)} \right] - \mathbb{E} \left[ h(Z) e^{-Ze^{-\hat{j}[\psi_0]}} \right] \right| = 0,
\]

where convergence to zero of the first term follows from a.s. convergence of \( Z_t \to Z \) and the bounded convergence theorem. The second term converges to zero by (A.4). Hence, we have

\[
\mathbb{E} \left[ h(Z) e^{-\mathcal{X}(\phi_0)} \right] = \mathbb{E} \left[ h(Z) e^{-Ze^{-\hat{j}[\psi_0]}} \right],
\]

for any \( \phi_0 \in C^+_c \) and any bounded continuous function \( h \), which implies (2.3) for any \( \phi_0 \in C^+_c \). The extension to \( \phi_0 \in C^+_bc \) follows via approximation and a kind of continuity of \( s[\psi] \) in functions in \( C^+_bc \) and bounded by 1, made precise in Lemma A.1 below □.
Lemma A.1 Let $\psi \in \mathcal{C}_c^+$, $\psi(\cdot) \leq 1$, and $g$ be a continuous function such that $g(x) = 0$ for $x \leq 0$, $g(x) = 1$ for $x \geq 1$ and $g(x) = x$ for $x \in [0, 1]$. Define $\psi_n(x) = g(x + n)\psi(x) \in \mathcal{C}_c^+$, then $\hat{s}[\psi_n] \to \hat{s}[\psi]$ as $n \to \infty$.

Proof. As $\psi_n(x) \leq \psi(x)$, the comparison principle implies immediately that

$$\hat{s}[\psi] \leq \hat{s}[\psi_n], \quad (A.5)$$

and we only need to verify an opposite bound. Let $u(t, x)$, $u_n(t, x)$ and $\tilde{u}_n(t, x)$ be the solutions to (1.1) with the initial conditions

$$u(0, x) = \psi(x), \quad u_n(t, x) = \psi_n(x), \quad \tilde{u}_n(0, x) = \tilde{\psi}_n(x) := \psi(x) - \psi_n(x). \quad (A.6)$$

The function $v_n = u_n + \tilde{u}_n$ satisfies

$$\frac{\partial v_n}{\partial t} = \frac{\partial^2 u_n}{\partial x^2} + u_n - u_n^2 + \frac{\partial^2 \tilde{u}_n}{\partial x^2} + \tilde{u}_n - \tilde{u}_n^2 \geq \frac{\partial^2 v_n}{\partial x^2} + v_n - v_n^2, \quad (A.7)$$

with the initial condition $v_n(0, x) = u(0, x)$. It follows from the maximum principle that

$$v_n(t, x) \geq u(t, x) \text{ for all } t \geq 0 \text{ and } x \in \mathbb{R},$$

and, in particular,

$$u(t, x + m(t)) \leq u_n(t, x + m(t)) + \tilde{u}_n(t, x + m(t)). \quad (A.8)$$

Passing to the limit $t \to +\infty$, we obtain

$$U(x + \hat{s}[\psi]) \leq U(x + \hat{s}[\psi_n]) + U(x + \hat{s}[\tilde{\psi}_n]), \quad (A.9)$$

for each $n \in \mathbb{N}$ fixed and all $x \in \mathbb{R}$. Dividing by $x \exp(-x)$ and passing to the limit $x \to +\infty$, keeping $n \in \mathbb{N}$ fixed, gives

$$e^{-\hat{s}[\psi]} \leq e^{-\hat{s}[\psi_n]} + e^{-\hat{s}[\tilde{\psi}_n]}, \quad (A.10)$$

As $\tilde{\psi}_n(x) = 0$ for all $x \geq -n + 1$, we know that $\hat{s}[\tilde{\psi}_n] \to +\infty$ as $n \to +\infty$. Using this and passing to the limit $n \to +\infty$ in (A.10) leads to

$$\hat{s}[\psi] \geq \limsup_{n \to +\infty} \hat{s}[\psi_n]. \quad (A.11)$$

This, together with (A.5) finishes the proof. □

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