A new approach for calculating the Nambu-Gorkov propagator in color superconductivity theory

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Abstract

In this article, we propose a new approach to calculate the Nambu-Gorkov propagator intuitively with some linear algebra techniques in presence of the scalar diquark condensates. With the help of energy projective operators, we can obtain relatively simple expressions for the quark propagators, which greatly facilitate the calculations in solving the Schwinger-Dyson equation to obtain the gap parameters.

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1 Introduction

At sufficiently low temperature and high density, the hadrons may be crushed into quark matter, which may exist in the central region of compact stars, where the densities are above the nuclear density and the temperatures are of the order of tens of keV \[1, 2\]. The color superconducting quark matter may also be created in future low-energy heavy ion colliders. The essence of color superconductivity is the pairing between the quarks when there exists an attractive interaction at the Fermi surface, just like the BCS mechanism in QED. Perturbatively, the quark-quark interaction is dominated by the one-gluon exchange, which is simply the one-photon exchange in QED multiplying the group factor $T^A_{ij}T^A_{kl}$, the group factor can be decomposed into an anti-triplet 3 with attraction and a sextet 6 with repulsion,

\[
T^A_{ij}T^A_{kl} = -\frac{N_c + 1}{4N_c}(\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}) + \frac{N_c - 1}{4N_c}(\delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl}).
\]

The quarks, unlike the electrons, have color and flavor as well as spin degrees of freedom, many different patterns of pairing are possible, which lead to the copious phase structures in color superconductivity theory, such as the color-flavor locking (CFL) state \[3\], gapless color-flavor locking (gCFL) state \[4\], 2SC state, gapless 2SC (gS2C) state \[5\], 2SCus state and Larkin-Ovchinnikov-Fudde-Ferrell (LOFF) state \[6\], etc. Intense theoretical works have been done to investigate the different regions of the QCD phase diagram in past years \[2\]. The scaling formula relating the

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transition temperature and the energy gap to the chemical potential and the running coupling constant have been derived systematically at sufficiently high density, which may serve a rigorous proof of the existence of the color superconducting phase. At the region of moderate baryon density or below, it is too difficult to warrant a first principle investigation with the present analytical and numerical techniques. Furthermore, the charge neutrality condition as well as the $\beta$ equilibrium have to be taken into account to form bulk matter inside the compact stars, which induce mismatches between the Fermi surfaces of the pairing quarks and puts severe constraint in determining the true ground states. We have to resort to the effective actions, for example, the QCD-inspired Nambu-Jona-Lasinio (NJL) like models \[8\], or solving the Schwinger-Dyson equation (SDE) for the gaps consistently \[7\]. In solving the SDE, the manipulations in color-flavor-spin spaces is tedious and cumbersome. In this article, we propose a new approach to calculate the quark propagator in presence of the scalar diquark condensates.

The article is arranged as follows: we introduce the Nambu-Jona-Lasinio model for illustrating the formation of the scalar diquark condensate in section II; in section III, we propose a new approach to calculate the Nambu-Gorkov propagator explicitly in presence of the scalar diquark condensates, which is of special importance in solving the Schwinger-Dyson equation; section IV is reserved for conclusion.

2 Nambu-Jona-Lasinio model for diquark condensate

At low temperature and high (or moderate) baryon density, the quarks can pair with each other at their Fermi surfaces with sufficiently strong attractions and form Cooper pairs. The colored Cooper pairs (represented by the field $\Delta^a$) have interactions with the quarks, its vacuum expectation values (the diquark condensates) induce the energy gaps for the quarks in the superconducting phases. The formation of the diquark condensates can not be derived from the first principle of QCD, we resort to the QCD-inspired NJL model or extended-NJL model to illustrate this subject \[8\]. The NJL model or extended-NJL model have the same symmetries as QCD and can describe the spontaneous breakdown of chiral symmetry in the vacuum and its restoration at high temperature and high density.

For simplicity, we can consider the two-flavor Lagrangian density with $SU(2)_L \times SU(2)_R$ chiral symmetry, the extension to the three-flavor NJL model with $SU(3)_L \times SU(3)_R$ chiral symmetry is straightforward and easy,

$$\mathcal{L} = \bar{q}(i\gamma^\mu \partial_\mu - m_0 + \gamma^0 \mu)q + G_S[(\bar{q}q)^2 + (\bar{q}i\gamma_5 \tau q)^2] + G_D[(i\bar{q}^C \varepsilon^b \gamma_5 q)(i\bar{q}^b \varepsilon \gamma_5 q^C)],$$

(1)

here $q^C = Cq^T$, $q^T = q^TC$ with $C = i\gamma^2 \gamma^0$. The $m_0$ is the current quark mass and the $\mu$ is the chemical potential. The quark field $q = q_{\alpha,i}$ with $\alpha = 1, 2, 3$ are color index and $i = 1, 2$ are flavor index, $\tau = (\tau^1, \tau^2, \tau^3)$ are Pauli matrices, $(\varepsilon)^{ik} = \varepsilon^{ik}$
and \((\epsilon^b)^{\alpha\beta} = \epsilon^{\alpha\beta b}\) are totally antisymmetric tensors in the flavor and color spaces respectively.

We can introduce the auxiliary fields and perform the standard bosonization to obtain the following linearized model in the mean-field approximation,

\[
L = \bar{q}(i\gamma^\mu \partial_\mu - m_0 + \gamma^0 \mu)q - \bar{q}(\sigma + i\gamma^5 \tau \pi)q - \frac{1}{2} \Delta^b i\bar{q} \epsilon^b \gamma_5 q - \frac{1}{2} \Delta^b i\bar{q} \epsilon^b \gamma_5 q^C + \cdots ,
\]

with the bosonic fields

\[
\Delta^b \sim i\bar{q} \epsilon^b \gamma_5 q, \quad \Delta^*^b \sim i\bar{q} \epsilon^b \gamma_5 q^C, \quad \sigma \sim \bar{q} q, \quad \pi \sim i\bar{q} \gamma^5 \tau q.
\]

The conditions \(\langle \sigma \rangle \neq 0\) and \(\Delta^b \neq 0\) indicate that the chiral symmetry and the color symmetry are spontaneously broken respectively. Here only the red and green quarks participate in the condensate, while the blue quarks (with color index \(b\)) do not. We can introduce the constituent quark mass

\[
m = m_0 + \langle \sigma \rangle.
\]

The spontaneously chiral symmetry breaking induced by the formation of chiral condensate in the vacuum and the dynamical chiral symmetry breaking induced by the strong color attraction at low energy region can all result in constituent masses for the quarks, for the \(u\) and \(d\) quarks, \(m_{u,d} \approx 330\text{MeV}\), for the \(s\) quark, \(m_s \approx 500\text{MeV}\). The constituent quark mass \(m\) changes according to the variations of the baryon density, in this article, we take the mass matrix for the \(u, d\) and \(s\) quarks to be

\[
\begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & m + \delta m
\end{bmatrix},
\]

here the \(m\) denotes the degenerate mass and the the mass breaking term \(\delta m\) can be taken into account by modifying the corresponding chemical potential for the \(s\) quark.

3 Nambu-Gorkov propagator with diquark condensates

We introduce the Nambu-Gorkov formulation for the basis \(q_{ru}, q_{gu}, q_{rd}, q_{gd}\),

\[
\Psi^T = (q_{ru}, q_{gu}, q_{rd}, q_{gd}; q^C_{ru}, q^C_{gu}, q^C_{rd}, q^C_{gd}),
\]

\[
\bar{\Psi} = (\bar{q}_{ru}, \bar{q}_{gu}, \bar{q}_{rd}, \bar{q}_{gd}; \bar{q}^C_{ru}, \bar{q}^C_{gu}, \bar{q}^C_{rd}, \bar{q}^C_{gd}),
\]

3
and write down the inverse Nambu-Gorkov propagator in momentum space,

\[ G^{-1} = \begin{pmatrix} [G_0^+]^{-1} & \Delta^- \\ \Delta^+ & [G_0^-]^{-1} \end{pmatrix}, \]

with

\[ [G_0^\pm]^{-1} = \begin{pmatrix} [G_0^\pm]_{ru}^{-1} & 0 & 0 & 0 \\ 0 & [G_0^\pm]_{gu}^{-1} & 0 & 0 \\ 0 & 0 & [G_0^\pm]_{rd}^{-1} & 0 \\ 0 & 0 & 0 & [G_0^\pm]_{gd}^{-1} \end{pmatrix}, \]

\[ \Delta^- = -i\Delta\gamma_5 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \Delta^+ = \gamma_0(\Delta^-)^\dagger\gamma_0, \]

\[ [G_0^\pm]^{-1}_{\alpha i} = (p_0 \pm \mu_{\alpha i})\gamma_0 - \gamma \cdot p - m. \]

At moderate baryon density, the quark masses have contribution from the chiral condensate and can not be neglected. We have to use the following energy projective operators for massive free particles [9], which are used extensively in solving the Bethe-Salpeter equation [10],

\[ \Lambda^{\pm}(\vec{p}) = \frac{1}{2}(1 \pm \gamma_0(\vec{\gamma} \cdot \vec{p} + m)) E, \]

\[ \tilde{\Lambda}^{\pm}(\vec{p}) = \frac{1}{2}(1 \pm \gamma_0(\vec{\gamma} \cdot \vec{p} - m)) E, \]

with the energy \( E = \sqrt{\vec{p}^2 + m^2} \) and the basic properties,

\[ \Lambda^{\pm}(\vec{p})\Lambda^{\pm}(\vec{p}) = \Lambda^{\pm}(\vec{p}), \]

\[ \Lambda^{\pm}(\vec{p})\Lambda^{\mp}(\vec{p}) = 0, \]

\[ \Lambda^{\pm}(\vec{p}) + \Lambda^{\mp}(\vec{p}) = 1, \]

\[ \gamma_0\Lambda^{\pm}(\vec{p})\gamma_0 = \tilde{\Lambda}^{\mp}(\vec{p}), \]

\[ \gamma_5\Lambda^{\pm}(\vec{p})\gamma_5 = \tilde{\Lambda}^{\pm}(\vec{p}). \]

With those projective operators, we re-write the matrix elements for the normal quark propagator in the following form which can greatly facilitate the calculations as the terms concerning the operators \( \Lambda^+ \) and \( \Lambda^- \) decouple from each other when inverting the matrix,

\[ [G_0^\pm]^{-1} = \gamma_0(p_0 - E^\mp)\Lambda_+ + \gamma_0(p_0 + E^\pm)\Lambda_-, \]

\[ G_0^\pm = \frac{\gamma_0\Lambda_+}{p_0 + E^\pm} + \frac{\gamma_0\tilde{\Lambda}_-}{p_0 - E^\mp}. \]
here \( E^\pm = E \pm \mu \).

In the following, we invert the matrix in Eq.(7) with some linear algebra techniques to derive the Nambu-Gorkov propagator \( G \) intuitively. For the CFL and gCFL states, the \( G^{-1} \) is a \( 72 \times 72 \) matrix in the color-flavor-spin spaces, it is very difficult to obtain the Nambu-Gorkov propagator \( G \) with simple expressions, the present approach can greatly facilitate the calculations,

\[
G = \frac{1}{\left( \begin{array}{cc} [G_0^+]^{-1} & \Delta^- \\ \Delta^+ & [G_0^-]^{-1} \end{array} \right)},
\]

\[
= \frac{1}{\left( \begin{array}{cc} [G_0^+]^{-1} & 0 \\ 0 & [G_0^-]^{-1} \end{array} \right) + \left( \begin{array}{cc} 0 & \Delta^- \\ \Delta^+ & 0 \end{array} \right)},
\]

\[
= \frac{1}{\left( \begin{array}{cc} [G_0^+]^{-1} & 0 \\ 0 & [G_0^-]^{-1} \end{array} \right) \left\{ 1 + \left( \begin{array}{cc} G_0^+ & 0 \\ 0 & G_0^- \end{array} \right) \left( \begin{array}{cc} 0 & \Delta^- \\ \Delta^+ & 0 \end{array} \right) \right\}},
\]

\[
= \frac{1}{\left( \begin{array}{cc} [G_0^+]^{-1} & 0 \\ 0 & [G_0^-]^{-1} \end{array} \right) \left\{ 1 - \left( \begin{array}{cc} 0 & G_0^+ \Delta^- \\ G_0^- \Delta^+ & 0 \end{array} \right) \right\} + \left( \begin{array}{cc} G_0^+ & 0 \\ 0 & G_0^- \end{array} \right) \left( \begin{array}{cc} 0 & G_0^+ \Delta^- \\ G_0^- \Delta^+ & 0 \end{array} \right) + \cdots },
\]

\[
= \frac{1}{\left( \begin{array}{cc} [G_0^+]^{-1} & 0 \\ 0 & [G_0^-]^{-1} \end{array} \right) \left\{ 1 - \left( \begin{array}{cc} G_0^+ \Delta^- G_0^- \Delta^+ & 0 \\ 0 & G_0^- \Delta^+ G_0^+ \Delta^- \end{array} \right) \right\}},
\]

\[
= \frac{1}{\left( \begin{array}{cc} [G_0^+]^{-1} & 0 \\ 0 & [G_0^-]^{-1} \end{array} \right) \left\{ 1 - \left( \begin{array}{cc} 0 & G_0^+ \Delta^- \\ G_0^- \Delta^+ & 0 \end{array} \right) \right\} + \left( \begin{array}{cc} G_0^+ \Delta^- G_0^- \Delta^+ & 0 \\ 0 & G_0^- \Delta^+ G_0^+ \Delta^- \end{array} \right) + \cdots },
\]

\[
= \left( \begin{array}{cc} [G_0^+]^{-1} - \Delta^- G_0^- \Delta^+ & 0 \\ 0 & [G_0^-]^{-1} - \Delta^+ G_0^+ \Delta^- \end{array} \right),
\]

\[
= \left( \begin{array}{cc} G^+ & -G_0^+ \Delta^+ G^- \\ -G_0^- \Delta^+ G^+ & G^- \end{array} \right) = \left( \begin{array}{cc} G^+ & \Xi^- \\ \Xi^+ & G^- \end{array} \right),
\]

\[
(16)
\]

\[
(17)
\]

\[
(18)
\]
with
\[
G^+ = \frac{1}{[G_0^+]^{-1} - \Delta^- G_0^- \Delta^+}, \tag{19}
\]
\[
G^- = \frac{1}{[G_0^-]^{-1} - \Delta^+ G_0^+ \Delta^-}. \tag{20}
\]

In the 2SC and g2SC phases with the basis $\Psi$ in the color-flavor spaces (see Eq.(6)), the matrix $\Delta^+$ and $\Delta^-$ (see Eqs.(8-10)) are skew diagonal, the following matrix is strictly diagonal,
\[
\begin{pmatrix}
G_0^+ \Delta^- G_0^- \Delta^+ & 0 \\
0 & G_0^- \Delta^+ G_0^+ \Delta^-
\end{pmatrix}, \tag{21}
\]
the diagonal and skew-diagonal matrix elements decouple from each other in Eq.(16).

We can read out the matrix elements $G_{ru}^\pm$, $G_{rd}^\pm$, $G_{gu}^\pm$, $G_{gd}^\pm$, $\Xi_{ru,gd}^\pm$, $\Xi_{rd,gu}^\pm$, $\Xi_{gd,ru}^\pm$, $\Xi_{gu,rd}^\pm$ from Eqs.(17-20) straightforwardly with the help of the energy projective operators.

For example,
\[
G_{ru}^\pm = \frac{1}{[G_0^+]^{-1}_{ru} - \Delta(-i\gamma_5) [G_0^+]^{-1}_{gd} \Delta(-i\gamma_5)},
\]
\[
= \frac{1}{\gamma_0(p_0 - E_{ru}^\pm)\Lambda_+ + \gamma_0(p_0 + E_{ru}^\pm)\Lambda_- + \Delta\gamma_5(\frac{\gamma_0\Lambda_+}{p_0 + E_{gd}^\pm} + \frac{\gamma_0\Lambda_-}{p_0 - E_{gd}^\pm})\Delta\gamma_5},
\]
\[
= \frac{p_0 - E_{ru}^\pm}{(p_0 + E_{ru}^\pm)(p_0 - E_{gd}^\pm) - \Delta^2 \gamma_0\Lambda_+} + \frac{p_0 + E_{ru}^\pm}{(p_0 - E_{ru}^\pm)(p_0 + E_{gd}^\pm) - \Delta^2 \gamma_0\Lambda_-},
\]
\[
G_{gd}^\pm = G_{ru}^\pm (ru \leftrightarrow gd),
\]
\[
\Xi_{ru,gd}^\pm = \frac{-i\Delta\gamma_5}{(p_0 + E_{ru}^\pm)(p_0 - E_{gd}^\pm) - \Delta^2 \Lambda_+} + \frac{-i\Delta\gamma_5}{(p_0 - E_{ru}^\pm)(p_0 + E_{gd}^\pm) - \Delta^2 \Lambda_-},
\]
\[
\Xi_{ru,gd}^\pm = \Xi_{ru,gd}^\pm (ru \leftrightarrow gd). \tag{22}
\]

We can write down other matrix elements analogously with simple substitution.

In the CFL and gCFL phases, the scalar diquark condensates can be parameterized as
\[
\langle q^a_\alpha C_\gamma q^\beta_j \rangle \sim \Delta_1 \varepsilon^{\alpha\beta\gamma} \varepsilon_{ij1} + \Delta_2 \varepsilon^{\alpha\beta\gamma} \varepsilon_{ij2} + \Delta_3 \varepsilon^{\alpha\beta\gamma} \varepsilon_{ij3}, \tag{23}
\]

\(^2\)The results in Eqs.(18-20) certainly can be derived by the standard linear algebra manipulations, in this article, we propose to separate the diagonal and skew-diagonal matrix elements in Eq.(16) intuitively, the matrix in the denominator in Eq.(17) is block diagonal, for the 2SC and g2SC states, the blocks $G_0^\pm \Delta^- G_0^- \Delta^+$ and $G_0^- \Delta^+ G_0^+ \Delta^-$ are strictly diagonal, the matrix can be inverted easily. The general strategy be, choose the suitable basis for the matrixes $\Delta^+$ and $\Delta^-$ to diagonalize or skew-diagonalize them in the color-flavor spaces, the resulting blocks $G_0^\pm \Delta^- G_0^- \Delta^+$ and $G_0^- \Delta^+ G_0^+ \Delta^-$ are strictly diagonal or mostly diagonal with some non-diagonal blocks. The matrix $G^{-1}$ can also be diagonalized by analytical plus numerical calculations, which is performed in Ref.[11]. Furthermore, there are also other works using the energy projective operators to invert the propagator $G^{-1}$, for example, Ref.[12].
the condensates are Lorentz scalars, antisymmetric in Dirac indices, antisymmetric in color (with the strongest attraction between quarks) and antisymmetric in flavor. The gap parameters describe \( d - s, s - u \) and \( u - d \) Cooper pairs, respectively. We can introduce the Nambu-Gorkov formulation in the three-flavor case,

\[
\Psi^T = (q_{ru}, q_{gd}, q_{bs}, q_{rd}, q_{gr}, q_{rs}, q_{bu}, q_{gs}, q_{bd}; q_{ru}^C, q_{gd}^C, q_{bs}^C, q_{rd}^C, q_{gr}^C, q_{rs}^C, q_{bu}^C, q_{gs}^C, q_{bd}^C),
\]

\[
\bar{\Psi} = (q_{ru}, q_{gd}, q_{bs}, q_{rd}, q_{gr}, q_{rs}, q_{bu}, q_{gs}, q_{bd}; q_{ru}^C, q_{gd}^C, q_{bs}^C, q_{rd}^C, q_{gr}^C, q_{rs}^C, q_{bu}^C, q_{gs}^C, q_{bd}^C).
\]  

(24)

The inverse Nambu-Gorkov propagator is a \( 18 \times 18 \) matrix in the color-flavor spaces,

\[
\Delta^- = -i\gamma_5 \begin{pmatrix} 0 & \Delta_3 & \Delta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_3 & 0 & \Delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_2 & \Delta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Delta_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\Delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\Delta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Delta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Delta_1 \end{pmatrix},
\]

(25)

\[
= -i\gamma_5 \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix},
\]

(26)

\[
G_0^\pm = \begin{pmatrix} G_{0ru}^\pm & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{0gd}^\pm & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{0bs}^\pm & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{0rd}^\pm & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{0gr}^\pm & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{0rs}^\pm & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_{0bu}^\pm & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{0gs}^\pm & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{0bd}^\pm \end{pmatrix},
\]

(27)

where \( A \) is \( 3 \times 3 \) block matrix, \( B, C \) and \( D \) are \( 2 \times 2 \) block matrixes. The matrix

\[
\begin{pmatrix} G_0^+ & \Delta^- G_0^- \Delta^+ & 0 \\ 0 & G_0^- \Delta^+ G_0^+ & \Delta^- \end{pmatrix}
\]

(28)

is block diagonal, the matrix \( G_0^+ \Delta^- G_0^- \Delta^+ \) and \( G_0^- \Delta^+ G_0^+ \Delta^- \) are diagonal for the six basis \((rd, gu, rs, bu, gs, bd)\) and have non-diagonal blocks concerning the matrix \( A \) for the three basis \((ru, gd, bs)\). The diagonal matrix elements of the inverse matrix of

\[
\begin{pmatrix} [G_0^+]^{-1} & -\Delta^- G_0^- \Delta^+ \\ 0 & [G_0^-]^{-1} & -\Delta^+ G_0^+ \Delta^- \end{pmatrix}
\]

(29)
are the inverse of the corresponding matrix elements. We can read out the matrix elements $G_{rd}^{\pm}, G_{gu}^{\pm}, G_{rs}^{\pm}, G_{bu}^{\pm}, G_{gs}^{\pm}, G_{bd}^{\pm}$ directly from the Eqs.(17-20) with the help of the energy projective operators. For example,

$$
G_{rd}^{\pm} = \frac{1}{[G_{0}^{-1}]_{rd}^{\pm} - \Delta_{3}^{\pm}(i\gamma_{5})} \left[ G_{0}^{-1} \right]_{gu}^{\pm} \Delta_{3}(i\gamma_{5})
$$

$$
= \frac{p_{0} - E_{gu}^{\pm}}{(p_{0} + E_{rd}^{\pm}) (p_{0} - E_{gu}^{\pm}) - \Delta_{3}^{2} \gamma_{0} \Lambda_{+}^{\pm} + \frac{p_{0} + E_{gu}^{\pm}}{(p_{0} - E_{rd}^{\pm}) (p_{0} + E_{gu}^{\pm}) - \Delta_{3}^{2} \gamma_{0} \Lambda_{-}^{\pm}},
$$

$$
G_{gu} = C_{gu}(gu \leftrightarrow rd),
$$

$$
\Xi_{rd,gu}^{\pm} = \frac{-i\Delta_{3} \gamma_{5}}{(p_{0} + E_{rd}^{\pm}) (p_{0} - E_{gu}^{\pm}) - \Delta_{3}^{2} \Lambda_{+}^{\pm} + \frac{-i\Delta_{3} \gamma_{5}}{(p_{0} - E_{rd}^{\pm}) (p_{0} + E_{gu}^{\pm}) - \Delta_{3}^{2} \Lambda_{-}^{\pm}},
$$

$$
\Xi_{gu,rd}^{\pm} = \Xi_{rd,gu}^{\pm}(rd \leftrightarrow gu). \tag{30}
$$

We can write down the corresponding ones for the $G_{rs}^{\pm}, G_{bu}^{\pm}, G_{gs}^{\pm}, G_{bd}^{\pm}, \Xi_{rs,gu}^{\pm}, \Xi_{ba,rs}^{\pm}, \Xi_{gs,bu}^{\pm}, \Xi_{bd,gs}^{\pm}$ analogously.

For the non-diagonal blocks concerning the $A$ in Eqs.(28-29), in general one may expect a linear transformation for the basis $(ru, gd, bs)$ can result in diagonal matrices for both the block $A$ and the corresponding $[G_{0}^{-1}]$, however, it is not the case. The color superconducting phases may exist in the core of compact stars, where the bulk matter should satisfy the charge neutrality condition and the $\beta$-equilibrium. The gravitation force is much weaker than the electromagnetic and the color forces, any electric charges or color charges can forbid the formation of bulk matter. For a neutral two-flavor color superconductor, the quark chemical potentials in color and flavor spaces can be expressed in terms of baryon chemical potential $\mu$, electrical chemical potential $\mu_{e}$, and color chemical potentials $\mu_{3}$ and $\mu_{8}$,

$$
\mu_{\alpha i, \beta j} = \mu_{e} Q_{ij} \delta_{\alpha \beta} + \mu_{3} T_{\alpha \beta}^{3} \delta_{ij} + \frac{2}{\sqrt{3}} \mu_{8} T_{\alpha \beta}^{8} \delta_{ij}, \quad (31)
$$

while for the three-flavor quark system, the chemical potentials are taken as

$$
\mu_{\alpha i, \beta j} = \mu_{e} Q_{ij} \delta_{\alpha \beta} + \mu_{3} T_{\alpha \beta}^{3} \delta_{ij} + \frac{2}{\sqrt{3}} \mu_{8} T_{\alpha \beta}^{8} \delta_{ij} - \frac{\delta m^{2}}{2\mu_{d}} \delta_{ij} \delta_{\alpha \beta}. \quad (32)
$$

From above equations for the chemical potentials, we can see that they are far from being equal. The charge neutrality condition puts strong constraints on both the two-flavor and three-flavor systems and results in the g2SC and gCFL phases. In the gCFL phase, the up block $A$ for the matrix $\Delta^{-}$ in Eq.(25) and the corresponding one for $[G_{0}^{-1}]$ in Eq.(27) can not be diagonalized with the same basis. Fortunately, we can solve the SDE for the gaps $\Delta_{1}, \Delta_{2}, \Delta_{3}$ with the six basis $(rd, gu, rs, bu, gs, bd)$.

The explicit and simple expressions for the quark propagators are of great importance in solving the gap equations and determining the gap parameters $\Delta, \Delta_{1}, \Delta_{2}, \Delta_{3}$.

$$
G^{-1}(p) - G_{0}^{-1}(p) = ig^{2} \int \frac{d^{4}k}{(2\pi)^{4}} \Gamma_{\mu}^{A} G(k) \Gamma_{\nu}^{B} D_{AB}^{\mu \nu}(k-p), \quad (33)
$$
where

$$\Gamma^A_\mu = \begin{pmatrix} \gamma_\mu T^A & 0 \\ 0 & C(\gamma_\mu T^A)^T C^{-1} \end{pmatrix}.$$  \hspace{1cm} (34)$$

The gluon propagator \(D_{\mu \nu}^{AB}\) has complex energy and chemical potentials dependence, include the effects of Landau damping and Debye screening induced by the medium \[13\]. Furthermore, the contributions come from the scalar diquark condensates should also be taken into account in the color superconducting phases. The gluon self-energy has been investigated in the g2SC and gCFL phases. The results show that the gauge bosons connected with the broken generators have imaginary Meissner screening masses \[15\], the chromomagnetic instability should be taken into account in a consistent analysis. The explicit expressions for the quark propagators can greatly facilitate the calculations, especially in performing the Matsubara frequency summation. At low baryon density, the coupling constant \(g^2\) can not be factorized out, \(g^2D_{\mu \nu}^{AB}\) should be approximated by some phenomenologically potentials just as the conventional SDE at zero baryon density \[14\].

In the CFL state with the same chemical potential (or in the limit \(\mu \to \infty\)) and degenerate mass \(m (\delta m \to 0)\), we can perform a linear transformation to diagonalize the matrix \(A\) and \(\text{diag}(G^\pm_{0ru}, G^\pm_{0gd}, G^\pm_{0bs})\) simultaneously,

$$\begin{pmatrix} ru \\ gd \\ bs \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{\sqrt{6}}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{6}}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$  \hspace{1cm} (35)$$

With the new basis \((x_1, x_2, x_3)\), the matrix \(A\) takes the following form

$$\begin{pmatrix} 2\Delta & 0 & 0 \\ 0 & -\Delta & 0 \\ 0 & 0 & -\Delta \end{pmatrix}.$$  \hspace{1cm} (37)$$

The matrix \(G^\pm_0 \Delta^- G^\pm_0 \Delta^+\) and \(G^\pm_0 \Delta^+ G^\pm_0 \Delta^-\) are strictly diagonal for the nine basis \((x_1, x_2, x_3, rd, gu, rs, bu, gs, bd)\), we can read out the \(G^\pm\) for the basis \((x_1, x_2, x_3)\) from Eqs.(17-20) directly.

$$G^\pm_1 = \frac{1}{\left[G^\pm_0\right]^{-1}_{1} - 2\Delta(-i\gamma_5)\left[G^\pm_0\right]_{1} 2\Delta(-i\gamma_5)},$$

$$G^\pm_2 = G^\pm_3 = G^\pm_1(2\Delta \to \Delta),$$

$$\Xi_{1,1}^\pm = -\frac{i2\Delta\gamma_5}{p_0^2 - (E^+_1)^2 - 4\Delta^2} \tilde{\Lambda}^+ + \frac{-i2\Delta\gamma_5}{p_0^2 - (E^-_1)^2 - 4\Delta^2} \tilde{\Lambda}^-,$$

$$\Xi_{2,2}^\pm = \Xi_{\tilde{3},3}^\pm = \Xi_{1,1}^\pm(2\Delta \to \Delta).$$  \hspace{1cm} (38)$$
The block matrix for the $G$ and $\Xi$ with the basis $(ru, gd, bs)$ can be written as

$$
\frac{1}{3} \begin{pmatrix}
G_1^+ + 2G_2^+ & G_1^+ - G_2^+ & G_1^+ - G_2^+ \\
G_1^+ - G_2^+ & G_1^+ + 2G_2^+ & G_1^+ - G_2^+ \\
G_1^+ - G_2^+ & G_1^+ - G_2^+ & G_1^+ + 2G_2^+
\end{pmatrix},
$$

(39)

and

$$
\frac{1}{3} \begin{pmatrix}
\Xi_1^+ + 2\Xi_2^+ & \Xi_1^+ - \Xi_2^+ & \Xi_1^+ - \Xi_2^+ \\
\Xi_1^+ - \Xi_2^+ & \Xi_1^+ + 2\Xi_2^+ & \Xi_1^+ - \Xi_2^+ \\
\Xi_1^+ - \Xi_2^+ & \Xi_1^+ - \Xi_2^+ & \Xi_1^+ + 2\Xi_2^+
\end{pmatrix}.
$$

(40)

From above matrixes, we can see that for the normal propagators, due to the presence of the scalar diquark condensates, all the matrix elements $\langle \eta_{\alpha i}\bar{\eta}_{\beta j}\rangle$ are non-zero, while only the diagonal elements in the color-flavor spaces are non-zero in the ordinary vacuum.

4 Conclusion

In this article, we propose a new approach to obtain the Nambu-Gorkov propagator intuitively with some linear algebra techniques in presence of the scalar diquark condensates. With the help of energy projective operators, we obtain relatively simple expressions for the quark propagators, which can greatly facilitate the calculations in solving the SDE to obtain the gap parameters.

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References

[1] M. G. Alford, K. Rajagopal, F. Wilczek, Phys. Lett. B422 (1998) 247; R. Rapp, T. Schafer, E. V. Shuryak, M. Velkovsky, Phys. Rev. Lett. 81 (1998) 53.

[2] K. Rajagopal and F. Wilczek, hep-ph/0011333; D. K. Hong, Acta. Phys. Polon. B32 (2001) 1253; M. Alford, Ann. Rev. Nucl. Part. Sci. 51 (2001) 131; T. Schäfer, hep-ph/0304281; D. H. Rischke, Prog. Part. Nucl. Phys. 52 (2004) 197; M. Buballa, hep-ph/0402234; H. C. Ren, hep-ph/0404074; M. Huang, hep-ph/0409167; I. A. Shovkovy, nucl-th/0410091 and references therein.
[3] M. Alford, K. Rajagopal, F. Wilczek, Nucl. Phys. B537 (1999) 443; T. Schafer, Nucl. Phys. B575 (2000) 269; N. J. Evans, J. Hormuzdiar, S. D. H. Hsu, M. Schwetz, Nucl. Phys. B581 (2000) 391; I. A. Shovkovy, L. C. R. Wijewardhana, Phys. Lett. B470 (1999) 189.

[4] M. Alford, C. Kouvaris, K. Rajagopal, Phys. Rev. Lett. 92 (2004) 222001; M. Alford, C. Kouvaris, K. Rajagopal, Phys. Rev. D71 (2005) 054009; K. Iida, T. Matsuura, M. Tachibana, T. Hatsuda, Phys. Rev. Lett. 93 (2004) 132001; S. B. Ruster, I. A. Shovkovy, D. H. Rischke, Nucl. Phys. A743 (2004) 127; K. Fukushima, C. Kouvaris, K. Rajagopal, Phys. Rev. D71 (2005) 034002.

[5] I. Shovkovy, M. Huang, Phys. Lett. B564 (2003) 205; M. Huang, I. Shovkovy, Nucl. Phys. A729 (2003) 835.

[6] M. Alford, J. A. Bowers, K. Rajagopal, Phys. Rev. D63 (2001) 074016; J. A. Bowers, K. Rajagopal, Phys. Rev. D66 (2002) 065002; J. Kundu, K. Rajagopal, Phys. Rev. D65 (2002) 094022; A. K. Leibovich, K. Rajagopal, E. Shuster, Phys. Rev. D64 (2001) 094005; J. A. Bowers, J. Kundu, K. Rajagopal, Phys. Rev. D64 (2001) 014024.

[7] N. J. Evans, J. Hormuzdiar, S. D. H. Hsu, M. Schwetz, Nucl. Phys. B581 (2000) 391; I. A. Shovkovy, L. C. R. Wijewardhana, Phys. Lett. B470 (1999) 189; D. K. Hong, V. A. Miransky, I. A. Shovkovy, L. C. R. Wijewardhana, Phys. Rev. D61 (2000) 056001, Erratum-ibid. D62 (2000) 059903; H. Abuki, Prog. Theor. Phys. 110 (2003) 937.

[8] U. Vogl and W. Weise, Prog. Part. Nucl. Phys. 27, 195 (1991); S. P. Klevansky, Rev. Mod. Phys. 64, 649 (1992); T. Hatsuda and T. Kunihiro, Phys. Rept. 247, 221 (1994); R. Alkofer, H. Reinhardt and H. Weigel, Phys. Rept. 265, 139 (1996).

[9] M. Huang, P. F. Zhaung, W. Q. Chao, hep-ph/0110046

[10] W. Greiner, J. Reinhardt, ”Quantum Electrodynamics”, Springer-Verlag Berlin Heidelberg, 1992.

[11] S. B. Ruster, V. Werth, M. Buballa, I. A. Shovkovy, D. H. Rischke, Phys. Rev. D72 (2005) 034004; S. B. Ruster, I. A. Shovkovy, D. H. Rischke, Nucl. Phys. A743 (2004) 127; K. Fukushima, C. Kouvaris, K. Rajagopal, Phys. Rev. D71 (2005) 034002.

[12] T. D. Fugleberg, Phys. Rev. D67 (2003) 034013; D. Ebert, K. G. Klimenko, V. L. Yudichev, Phys. Rev. C72 (2005) 015201; D. Blaschke, D. Ebert, K. G. Klimenko, M. K. Volkov, V. L. Yudichev, Phys. Rev. D70 (2004) 014006.

[13] M. L. Bellac, ”Thermal Field Theory”, Cambridge University Press, 1996.
[14] C. D. Roberts and A. G. Williams, Prog. Part. Nucl. Phys. 33 (1994) 477; P. C. Tandy, Prog. Part. Nucl. Phys. 39 (1997) 117; C. D. Roberts and S. M. Schmidt, Prog. Part. Nucl. Phys. 45 (2000) S1; C. D. Roberts, [nucl-th/0304050]; P. Maris and C. D. Roberts, Int. J. Mod. Phys. E12 (2003) 297.

[15] M. Huang and I. A. Shovkovy, Phys. Rev. D70 (2004) 051501; M. Huang and I. A. Shovkovy, Phys. Rev. D70 (2004) 094030; R. Casalbuoni, R. Gatto, M. Mannarelli, G. Nardulli and M. Ruggieri, Phys. Lett. B B605 (2005) 362; M. Alford and Q. H. Wang, arXiv:hep-ph/0501078.