UNCONSTRAINED HIGHER SPINS OF MIXED SYMMETRY

I. BOSE FIELDS

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Abstract

This is the first of two papers devoted to the local “metric-like” unconstrained Lagrangians and field equations for higher-spin gauge fields of mixed symmetry in flat space. Here we complete the previous constrained formulation of Labastida for Bose fields. We thus recover his Lagrangians via the Bianchi identities, before extending them to their “minimal” unconstrained form with higher derivatives of the compensator fields and to yet another, non-minimal, form with only two-derivative terms. We also identify classes of these systems that are invariant under Weyl-like symmetries.

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1 INTRODUCTION

The theory of Higher-Spin Gauge Fields is an enticing and still largely unexplored corner of Field Theory, that has marked consistently the frontier of our understanding since the 1930’s [1]. This difficult subject has attracted over the years several leading specialists, but despite many efforts a number of key questions remain unanswered. Surprisingly, perhaps, a few of these still concern the free theory, and the present work is devoted to clarifying some of the corresponding issues. Furthermore, our current understanding of higher-spin interactions is quite incomplete: classic no-go theorems showed in fact, long ago, that they are naively inconsistent, at least for massless fields in flat space [2], but several important examples of higher-spin vertices were found nonetheless over the years [3, 4]. Crucial long-term efforts by Vasiliev, initially with Fradkin [5, 6], have actually provided paradigmatic examples of classically consistent interactions between infinitely many totally symmetric massless higher-spin fields in curved (A)dS backgrounds. The systematics of such constructions, however, is not fully understood at this time, while Vasiliev’s “unfolded” formulation is not Lagrangian and appears rather remote from ordinary lower-spin field theories. It is thus natural to try and bridge the apparent gap between these methods and the ordinary constructions for low spins, and this paper is meant as a step in that direction. In addition, and perhaps more importantly, higher spins are a key element of String Theory [7], whose massive spectra contain infinitely many of them. Hence one might say that higher-spin interactions have presently two vastly different realizations. In the first, provided by the Vasiliev setting, their gauge symmetry is unbroken, while in the second, provided by String Theory, it is somehow spontaneously broken. Unfortunately, we do not understand precisely how to bridge the gap between the two, or even how to attach a precise meaning to the breaking mechanism at work in String Theory. Still, it is fair to stress that, while the massive higher-spin modes of String Theory are usually ignored, they are clearly instrumental in granting some of its most spectacular properties, such as modular invariance or open-closed duality.

A deeper understanding of these massive modes can not forego a closer look at higher-spin fields of mixed symmetry, that constitute the vast majority of them, and this provides at least part of the motivations behind the present work. This is the first of two papers devoted to some open issues concerning free higher-spin gauge fields of mixed symmetry. It deals mostly with multi-symmetric bosonic fields of the type \( \varphi_{\mu_1 \ldots \mu_{s_1}; \nu_1 \ldots \nu_{s_2}; \ldots} \), while the companion paper [8] deals mostly with multi-symmetric fermionic fields of the type \( \psi_{\mu_1 \ldots \mu_{s_1}; \nu_1 \ldots \nu_{s_2}; \ldots} \). Both these types of fields possess several “families” of symmetric index sets, are defined in generic \( D > 5 \) dimensional space times and are inevitable ingredients for \( D > 5 \). Dealing with these reducible \( gl(D) \) tensors is a necessary complication if one wishes to establish quantitative links between Higher-Spin Gauge Theory and String Theory, which, as anticipated, is one of the key problems today. Indeed these types of reducible \( gl(D) \) tensors, rather than the more familiar Young-projected tensors, accompany in String Theory products of bosonic oscillators, and therefore it is both natural and convenient to formulate the theory directly in terms of them. Let us also remark that the simplest instance of a reducible field of this type is actually rather familiar. It is simply a two-tensor, which one would usually split aforesaid into a symmetric tensor and a Kalb-Ramond two-form. We also describe how to adapt the formalism to reducible multi-antisymmetric fields or multi-forms, or to more general types of bosonic fields where some collections of space-time indices are fully symmetric and some are fully antisymmetric. The relative simplicity of this type of extension was pointed out in [9, 10, 11, 12], and the result is clearly of interest in comparisons with superstring models, where these more general types of fields accompany products of fermionic and bosonic oscillators.
Allowing mixed symmetry complicates matters to a considerable extent: for instance, the gauge symmetry is typically reducible, so that gauge transformations of the gauge transformations emerge, a feature already displayed by the relatively simple example, which we just mentioned, of a two-form Kalb-Ramond field. Hence, one can not forego the need for a proper notation to deal concisely and effectively with the general case. Our present choice, explained in some detail in the Appendices, is a natural extension of the index-free description used in previous work on fully symmetric (spinor) tensors by some of us. Still, it is far more complicated, and involves explicit “family” indices, needed to identify particular subsets of the tensor indices to be treated differently by the various operators entering the field equations or the Lagrangians. For instance, while the bosonic gauge fields can still be simply denoted by \( \varphi \), as in the fully symmetric case, their gauge transformations,

\[
\delta \varphi = \partial^i \Lambda_i ,
\]

now involve explicit family indices. These are needed to identify the various sets of space-time indices taking part in the gradient, which are to be fully symmetrized only with others of the same type. At the same time, the gauge parameters \( \Lambda_i \) carry one lower family index and bear one less space-time index of the \( i \)-th family than the corresponding gauge fields.

Free higher-spin bosonic gauge fields first took a Lagrangian form in the late seventies, thanks to the work of Frönsdal [13], who displayed the gauge symmetry emerging for massless totally symmetric tensors from the massive Singh-Hagen [14] construction. He thus identified some surprising algebraic constraints on gauge parameters and gauge fields: in his formulation the former are in fact to be traceless, while the latter are to be doubly traceless. In the eighties, the development of free String Field Theory [15] made it imperative to look more closely at tensors of mixed symmetry, and important progress was shortly made by a number of authors [16] (although some relevant work on the subject dates back to previous years [17]). Most notably, Labastida [19, 20] arrived at bosonic Lagrangians that generalize what Frönsdal had previously attained for fully symmetric tensors, requiring that the complete construction result in self-adjoint kinetic operators, as was the case for the simpler, symmetric theory. This condition revealed the need for peculiar constraints on gauge fields and parameters, that in our present notation read

\[
T_{(ij} \Lambda_{k)} = 0 , \quad T_{(ij} T_{kl)} \varphi = 0 ,
\]

where \( T_{ij} \) denotes a trace involving a pair of space-time indices belonging to the \( i \)-th and \( j \)-th families. Hence, quite differently from what Frönsdal’s case could naively suggest, not all traces of the gauge parameters and not all double traces of the gauge fields are forced to vanish in Labastida’s formulation. Further developments along these lines may be found in [21].

As shown in a number of previous works, the need for algebraic constraints on gauge fields and parameters can actually be foregone, arriving at formulations that are more akin, in spirit, to the more familiar cases of low spins. This can be achieved in two quite distinct, albeit related, ways. The first is to allow for non-local gauge-dependent additions to the field equations [22, 23], that have the virtue of linking the result to the linearized higher-spin curvatures introduced by de Wit and Freedman [24]. These results were extended to fields of mixed symmetry in [10, 11, 12], and more recently the setting was shown to also encompass the important case of massive

\[3\] W. Siegel called to our attention [18], where free actions for Bose and Fermi fields were built within a BRST-type setting. These works develop some of the ideas that originally led from light-cone to covariant string fields. The approach, however, is rather remote from the line of developments that lies at the heart of this work and that can be largely traced back to the previous works of Frönsdal and Labastida.
symmetric fields [25]. Let us stress that, in sharp contrast with the Singh-Hagen construction, the non-local formulation needs no additional auxiliary fields to this end. The second way is to allow for additional fields in more conventional local Lagrangian formulations [26, 27]. In their “minimal” form, Fronsdal’s constraints can then be foregone at the modest price of introducing only two additional fields and some higher-derivative terms that are anyway harmless: for a fully symmetric spin-$s$ field $\varphi_{\mu_1...\mu_s}$, these are a spin-$(s-3)$ compensator $\alpha_{\mu_1...\mu_{s-3}}$, that first emerges for $s = 3$, and a spin-$(s-4)$ Lagrange multiplier $\beta_{\mu_1...\mu_{s-4}}$, that first emerges for $s = 4$.

Or, as recently shown in [25], also a further spin-$(s-2)$ compensator and a further Lagrange multiplier, if one insists on having not more than two derivatives in the resulting Lagrangians. It should be stressed, however, that these results, and a similar one obtained in [28] restricting the “triplets” [29, 23, 30] of String Field Theory, were all preceded by what should be regarded as the first instance of a local unconstrained formulation. This non-minimal form, obtained by Pashnev, Tsulaia, Buchbinder and others [31] with BRST techniques, makes use of $O(s)$ different fields to describe unconstrained spin-$s$ modes, and was connected to the minimal on-shell formulation in [32]. Let us also emphasize that most previous works on higher-spin fields, and our constructions in particular, proceed along the lines of the metric formalism for gravity, and this is also true for the world-line constructions of [33]. An important alternative is the analogue of the vielbein formalism, first introduced in [34] and developed further in a number of later works, including those in [35] that are directly connected to mixed-symmetry fields. This framework has already led to key developments, since it is the basis of the Vasiliev construction [5, 6], where the interactions are driven by an infinite-dimensional extension of the tangent-space Lorentz algebra. Still, attaining a deeper understanding of the metric-like formalism appears of interest today, since it has the potential of clarifying the geometrical origin of higher-spin interactions.

The main purpose of this paper is thus to extend the minimal “metric-like” formulation of unconstrained higher spins of [23, 32, 26, 27] to the case of mixed symmetry. As we shall see, in general the structure of the Labastida constraints for Bose fields would suggest to introduce compensators $\alpha_{ijk}$, bearing a triple of family indices and fully symmetric under their interchange, and Lagrange multipliers $\beta_{ijkl}$, bearing a quadruple of family indices and fully symmetric under their interchange. Actually, matters are slightly more complicated, since this natural choice, motivated also by the analogy with the symmetric construction of [23, 32, 26, 27], would lead to the emergence of gauge invariant combinations of the higher traces of the $\alpha_{ijk}$, that therefore can not be regarded as independent compensators. This difficulty reflects an intrinsic problem of the Labastida constraints (1.2), that in fact are not independent. A simple way out is then to relate the $\alpha_{ijk}$ to new fields, here termed $\Phi_i$, whose gauge transformations are proportional to the parameters $\Lambda_i$, that enter Lagrangians and field equations only via their symmetrized traces and are such that

$$\alpha_{ijk} (\Phi) = \frac{1}{3} T_{(ij} \Phi_k) .$$

(1.3)

While the introduction of these additional fields, compensators and Lagrange multipliers alike, reflects nicely the nature of the constraints of eq. (1.2), our approach differs somewhat, in spirit, both from the original work of [20] and from the more recent work of [12], since we are guided throughout by the Bianchi identities and their traces. This choice proves very convenient when building the unconstrained theory, and particularly so for unconstrained Fermi fields, for which in the companion paper [8] we can present for the first time complete “metric-like” local Lagrangians together with their constrained counterparts. Our “minimal” setting to arrive at an unconstrained gauge symmetry stands out for its relative simplicity, but brings about higher-
derivative terms involving the compensator fields. While this is not a problem, as can be clearly seen for example in [27], we shall also discuss how to complicate matters a bit in order to obtain more conventional Lagrangians where all terms contain at most two derivatives, generalizing the construction of [25].

As first shown by Labastida [20], the constrained formulation of mixed-symmetry fields brings about a surprise with respect to Fronsdal’s case. Once, as in eq. (1.2), not all double traces of the gauge field $\varphi$ are forced to vanish, the Lagrangian can (and indeed does) include additional terms involving higher field traces, whose number grows with the number of families. In a similar fashion, the unconstrained formulation for Bose fields also involves a number of higher traces of a proper gauge-invariant tensor built from $\varphi$ and $\alpha_{ijk}(\Phi)$ which grows with the number of families. For this reason, in Section 2 we begin by constructing Lagrangians and field equations for the simplest generalization of the fully symmetric case, two-family fields of the type $\varphi_{\mu_1...\mu_s;\nu_1...\nu_s}$, and study their reduction to the constrained Labastida formulation. This analysis will also display the emergence of Weyl-like symmetries for these higher-spin fields in sporadic low-dimensional cases. We also add some comments on the Lagrangian field equations of [20], in a way that sheds some light on a subtlety originating from the constrained nature of the gauge field $\varphi$. We conclude Section 2 with the explicit examples of reducible rank-$(s,1)$ fields of the type $\varphi_{\mu_1...\mu_s;\nu}$ and reducible rank-$(4,2)$ fields of the type $\varphi_{\mu_1...\mu_4;\nu_1\nu_2}$, that are relatively simple but suffice to illustrate a number of key points of our construction. In Section 3 we extend the discussion to multi-family bosonic fields, identifying precisely via the Bianchi identities their unconstrained Lagrangians. When restricted to the case of constrained fields, our result confirms the Labastida construction, up to some changes of notation and to some typos that we correct.

In Section 4 we describe how the unconstrained formulation of bosonic fields can be adapted to tensors transforming in irreducible representations of the Lorentz group. Whereas the application to String Theory is more closely related to reducible tensors, as we have stressed, it is in fact also interesting to trace how the theory develops along lines closer to what one usually does for low spins. In Section 5 we then describe how the minimal higher-derivative unconstrained Lagrangians can be reduced systematically to others with only two derivatives. In Section 6 we show how the present formalism can be simply adapted to describe multi-forms or even more general types of fields needed to encompass massive superstring excitations. Finally, Section 7 contains our conclusions, and the paper closes with a number of Appendices where our notation is carefully spelled out.

For the reader’s convenience, let us summarize again the main results contained in this paper:

- the formalism previously developed for symmetric fields is extended via the introduction of “family indices” that identify the subsets of space-time indices acted upon by traces, gradients and other operations;
- the Labastida construction is linked to the Bianchi identities, in strict analogy with what happens for symmetric fields, and for linearized gravity in particular. In the multi-symmetric case, the construction rests on the subclass of their traces that are most anti-symmetric in their family indices or, more precisely, that are subject to two-column Young projections;
- formulations capable of bypassing the Labastida constraints on gauge fields and parameters are obtained via a single type of compensators $\alpha_{ijk}$, here to be expressed in terms of other more basic fields $\Phi_i$ in order to account for the linear dependence of the Labastida
constraints, and of a single type of Lagrange multipliers $\beta_{ijkl}$. The resulting Lagrangians display a novel type of gauge symmetry allowing certain redefinitions of the $\beta_{ijkl}$ and, as in the one-family or symmetric case, contain higher derivatives of the compensators. However, we also describe how to systematically modify the construction in order to obtain Lagrangians with only two derivatives that contain a minimal number of additional fields:

- a rich pattern of sporadic cases is exhibited where Weyl-like symmetries emerge, generalizing the well-known property of two-dimensional gravity;
- the formalism applies, with minor modifications, to both multi-symmetric tensors and multi-forms, and thus encompasses all types of bosonic fields that play a role in massive string spectra.

2 TWO-FAMILY BOSONIC FIELDS

In this section we derive Lagrangians and field equations for two-family bosonic fields of the type $\varphi_{\mu_1...\mu_s;\nu_1...\nu_s}$. We aim at a “minimal” unconstrained formulation, and thus initially we allow higher-derivative terms involving the compensators. We defer to Section 5 a discussion of how to recast these results in a form involving only two-derivative compensator terms. While the results for two-family fields are still relatively handy, this analysis has the virtue of displaying quite clearly the key differences with respect to the simpler case of fully symmetric tensors discussed in [26, 27].

2.1 THE LAGRANGIANS

In the following, all space-time indices will be left implicit as in the symmetric case of [26, 27], so that a generic two-family gauge field $\varphi_{\mu_1...\mu_s;\nu_1...\nu_s}$ will be simply denoted by $\varphi$. While our notation is explained in detail in Appendix A, for the sake of clarity let us begin by summarizing a few properties that we shall need repeatedly in the following. The key novelty is the need for family indices specifying the sets of space-time indices acted upon by gradients, divergences and traces. Thus, for instance, the divergence and the gradient involving space-time indices of the first family are here denoted by

$$\partial_1 \varphi \equiv \partial_\lambda \varphi^\lambda_{\mu_1...\mu_{s-1};\nu_1...\nu_s},$$
$$\partial^1 \varphi \equiv \partial_{(\mu_1 \varphi_{\mu_{2+1}}...\mu_{s+1});\nu_1...\nu_s},$$

where, here as in the rest of the paper, a pair of round parentheses enclosing a set of indices indicates their total symmetrization with the minimal possible number of terms and normalized with a unit overall coefficient, rather than with unit strength. Furthermore, we associate upper family indices to operators like the gradient, which add space-time indices, and lower family indices to operators like the divergence, which remove them. In a similar fashion, we also
introduce “diagonal” and “mixed” traces and metric tensors, so that for example

\[ T_1 \equiv \phi_{\lambda_1 \ldots \mu_{s_1}}^{\nu_1 \ldots \nu_{s_2}}, \]

\[ T_2 \equiv \phi_{\lambda_1 \ldots \mu_{s_1}}^{\lambda \nu_1 \ldots \nu_{s_2}}, \]

\[ \eta_1 \equiv \eta_{\mu_1 \mu_2} \phi_{\nu_1 \ldots \nu_{s_2}}, \]

\[ \eta_2 \equiv \frac{1}{2} \left( \eta_{\mu_1 \nu_1} \phi_{\mu_2 \ldots \nu_{s_2+1}} + \eta_{\mu_2 \nu_1} \phi_{\mu_1 \ldots \nu_{s_2+1}} + \cdots \right), \]

where the peculiar factor $\frac{1}{2}$ in the last expression allows a more convenient presentation of a number of results, including the Lagrangians. Further, in order to simplify the combinatorics of partial integrations we shall also introduce a suitably normalized scalar product, which is described in detail in Appendix A. As we have stressed, in these expressions and elsewhere in this paper, round parentheses (brackets) enclose fully (anti)symmetric space-time or family indices. In addition, vertical bars separate, whenever this is needed for clarity, indices belonging to different sets.

The starting point to derive a gauge invariant Lagrangian is the gauge transformation

\[ \delta \phi = \partial^i \Lambda_i, \]

where, as anticipated in the Introduction, the gauge parameters bear a lower family index, consistently with the fact that $\Lambda_i$ carries only $(s_i - 1)$ space-time indices belonging to the $i$-th family. This concise notation displays directly the reducible nature of the gauge symmetry of mixed-symmetry fields: eq. (2.4) clearly allows the “gauge-for-gauge” transformations

\[ \delta \Lambda_i = \partial^j \Lambda_{ij}, \]

where the new parameters are antisymmetric under the interchange of their indices. At two families, the process stops here, while in the general multi-family case to be discussed in Section 3 the chain of gauge-for-gauge transformations continues further.

One can now introduce the Fronsdal-Labastida operator [19, 20]

\[ F = \partial^i \partial^j \phi + \frac{1}{2} \partial^i \partial^j T_{ij}, \]

that, as in the symmetric or one-family case, is not fully invariant under the transformation of eq. (2.4). One indeed finds

\[ \delta F = \frac{1}{6} \partial^i \partial^j \partial^k T_{ijk} \Lambda_k, \]

where the remainder on the right-hand side is proportional to the Labastida constraints on the gauge parameters

\[ T_{ijk} \Lambda_k = 0, \]

and can be eliminated introducing compensators $\alpha_{ijk}$, fully symmetric under the interchange of their three family indices and such that

\[ \delta \alpha_{ijk} = \frac{1}{3} T_{ijk} \Lambda_k, \]

so that for an arbitrary number of families they carry $s_1, \ldots, (s_i - 1), \ldots, (s_j - 1), \ldots, (s_k - 1), \ldots, s_n$ space-time indices. In analogy with the one-family case of [26, 27], one can then introduce the gauge invariant tensor

\[ A = F - \frac{1}{2} \partial^i \partial^j \partial^k \alpha_{ijk}, \]
that plays the role of a basic kinetic tensor for the unconstrained theory. However, as anticipated in the Introduction, allowing independent compensators $\alpha_{ijk}$ would lead to some difficulties, since the very nature of the transformations (2.9) would imply the existence of gauge-invariant constructs built from the $\alpha_{ijk}$ alone. For instance,

$$ \delta \left[ T_i(j \alpha_{klm}) - T_j(i \alpha_{klm}) \right] = 0. \quad (2.11) $$

On the other hand, this gauge invariant combination vanishes identically if the $\alpha_{ijk}$ are expressed in terms of other independent compensators $\Phi_i$ according to

$$ \alpha_{ijk} \equiv \alpha_{ijk}(\Phi) = \frac{1}{3} T_{(ij} \Phi_{k)} , \quad (2.12) $$

where the $\Phi_i$ transform proportionally to the gauge parameters:

$$ \delta \Phi_k = \Lambda_k . \quad (2.13) $$

Indeed, given a pair of traces, the combination

$$ T_{i(j} T_{kl)} = T_{ij} T_{kl} + T_{ik} T_{jl} + T_{il} T_{jk} \quad (2.14) $$

is also totally symmetric in $(ijkl)$. This reflects a special property of products of identical tensors, that can only build Young diagrams in family-index space with even numbers of boxes in each row.

Notice also that the composite compensator $\alpha_{ijk}(\Phi)$ of eq. (2.12) would emerge directly if the Stueckelberg-like shift

$$ \phi \rightarrow \phi - \partial^i \Phi_i \quad (2.15) $$

were performed in the Labastida tensor $F$, while the fact that only the combination (2.12) is present reflects the original, constrained Labastida gauge symmetry. Under the gauge-for-gauge transformations (2.5), these $\Phi_i$ fields shift like the gauge parameters would, and thus like ordinary gauge fields. This is as it should be: if they were inert as the higher-spin field $\phi$, part of the gauge transformations would be ineffective, in contradiction with the manifest possibility of removing the $\Phi_i$ undoing the Stueckelberg shift. Hence, the $\alpha_{ijk}(\Phi)$, that for brevity from now on will be often simply called $\alpha_{ijk}$, can be fully eliminated without affecting the constrained Labastida gauge symmetry, in complete analogy with the minimal unconstrained case.

The Bianchi identities satisfied by $F$,

$$ \partial_i F - \frac{1}{2} \partial^j T_{ij} F = - \frac{1}{12} \partial^j \partial^k \partial^l T_{(ij} T_{kl)} \phi , \quad (2.16) $$

are another crucial ingredient of the construction, and as in the symmetric case the “classical anomaly” present in this expression determines directly the Labastida constraints on the gauge field $\phi$,

$$ T_{(ij} T_{kl)} \phi = 0 . \quad (2.17) $$

The Bianchi identities for the $A$ tensor have the form

$$ \partial_i A - \frac{1}{2} \partial^j T_{ij} A = - \frac{1}{4} \partial^j \partial^k \partial^l C_{ijkl} , \quad (2.18) $$

---

4Our conventions are spelled out in Appendix A. Further details on these matters can be found, for instance, in [36].

5M.A. Vasiliev stressed to us the role of this shift in the symmetric case of [26]. In the following we shall see how to formulate the procedure in various ways for mixed-symmetry tensors.
where
\[ C_{ijkl} = \frac{1}{3} \left\{ T(ij T_{kl}) \varphi - 3 \partial(i \alpha_{jkl}) - \frac{3}{2} \partial^m \left( T(ij \alpha_{kl})m - T_m(i \alpha_{jkl}) \right) \right\} , \tag{2.19} \]
or more simply
\[ C_{ijkl} = \frac{1}{3} T(ij T_{kl}) \left( \varphi - \partial^i \Phi_i \right) , \tag{2.20} \]
are the proper gauge-invariant extensions of the symmetrized double traces of \( \varphi \). Notice that arriving at this rather concise form, that in the one-family case would reduce to the simpler expression
\[ C_{1111} \rightarrow \phi'' - 4 \partial \cdot \alpha - \partial \alpha' \equiv C \tag{2.21} \]
of \([26, 27]\), is straightforward if \( \Phi_i \) is introduced via (2.15). It is nonetheless instructive to trace the key steps of the derivation working in terms of \( \alpha_{ijk} \), since this requires both the symmetries of \( C_{ijkl} \) and those of the pre-factor and is a prototype of a number of similar calculations needed to reproduce our results. Naively the Bianchi identity for \( A \) would read
\[ \partial_i A - \frac{1}{2} \partial^j T_{ij} A = -\frac{1}{12} \partial^j \partial^k \partial^l \left( T(ij T_{kl}) \varphi - 3 \partial(i \alpha_{jkl}) - 3 \partial^m T_m(i \alpha_{jkl}) \right) , \tag{2.22} \]
or
\[ \partial_i A - \frac{1}{2} \partial^j T_{ij} A = -\frac{1}{4} \partial^j \partial^k \partial^l \left( C_{ijkl} + \partial^m D_{i,jkl,m} \right) , \tag{2.23} \]
a result that would seem to differ from eq. (2.19) due to the presence of the remainder
\[ D_{i,jkl,m} = \frac{1}{2} \left( T(ij \alpha_{kl})m - T_m(i \alpha_{jkl}) \right) - T_m(i \alpha_{jkl}) . \tag{2.24} \]
Its family indices \( jklm \), however, are projected according to a hooked Young diagram, so that this expression actually contracts to zero against the four derivatives present in the resulting expression, and consequently the Bianchi identities take the form (2.18).

Notice that, once the \( \alpha_{ijk} \) are expressed in terms of the \( \Phi_i \), the traces of the \( C_{ijkl} \) satisfy the algebraic constraints
\[ Y_{\{5,1\}} T_{mn} C_{ijkl} = 0 , \tag{2.25} \]
where \( Y_{\{5,1\}} \) is a Young projector onto the irreducible \( \{5,1\} \) representation of the permutation group acting on the family indices. The reason, as we already stressed, is that identical \( T \) tensors can only build Young diagrams in family-index space with even numbers of boxes in each row. This is a manifest property of the first contribution to \( C_{ijkl} \), \( T(ij T_{kl}) \varphi \), and becomes a property of all \( C_{ijkl} \) once the \( \alpha_{ijk} \) are expressed in terms of independent compensators \( \Phi_i \) as in (2.20). Some non-trivial identities follow, as for instance
\[ T_i(j \ C_{klmn}) = T_{ij} C_{klmn} \ . \tag{2.26} \]
Let us stress again that these subtleties are by no means a peculiarity of the unconstrained formalism, but reflect the fact that the Labastida constraints (2.17) are nicely covariant but not independent, so that different traces of these equations can indeed turn out to be proportional. The reason behind the need for relating the \( \alpha_{ijk} \) to the \( \Phi_i \) fields can be similarly traced to the lack of independence of the Labastida constraints (2.8) on the gauge parameters.

In complete analogy with the one-family case, the Bianchi identities suggest to begin from the trial Lagrangian
\[ \mathcal{L}_0 = \frac{1}{2} \langle \varphi, A - \frac{1}{2} 
eta^{ij} T_{ij} A \rangle , \tag{2.27} \]
whose gauge variation, up to a total divergence, is
\[
\delta L_0 = -\frac{1}{2} \langle \Lambda_i, \partial_i A - \frac{1}{2} \partial^j T_{ij} A - \frac{1}{2} \eta^{jk} \partial_i T_{jk} A \rangle - \frac{1}{24} \langle \partial_i \partial_j \partial_k \Lambda_l, C_{ijkl} \rangle + \frac{1}{8} \langle T_{jk} \Lambda_i, \partial_i T_{jk} A \rangle.
\] (2.28)

It is at this point that the new subtleties of the two-family case first emerge, since the last term in (2.28) is not directly related to the gauge transformation of the \(\alpha_{ijk}\) compensators. Rather, it contains a reducible tensor in index space, \(T_{ij} \Lambda_k\), that however can be Young projected into the \(\{3\}\) and \(\{2,1\}\) representations, so that
\[
\delta L_0 = -\frac{1}{24} \langle \partial_i \partial_j \partial_k \Lambda_l, C_{ijkl} \rangle + \frac{1}{72} \langle T_{(ij} \Lambda_k), \partial_i T_{jk} A \rangle + \frac{1}{24} \langle T_{jk} \Lambda_i, (2 \partial_i T_{jk} - \partial_{(j} T_{k)i} A) \rangle,
\] (2.29)

where in the last term we have actually projected only the right entry of the scalar product. Out of the last two contributions, only the first can be eliminated by a term involving the compensators, but interestingly this complication is accompanied by another novelty. Indeed the trace of the Bianchi identity now reads
\[
\partial_i T_{jk} A - \frac{1}{2} \partial_{(j} T_{k)i} A = \frac{1}{2} \partial^l T_{il} T_{jk} A - \frac{1}{4} T_{jk} \partial^l \partial^m \partial^n C_{ilmn},
\] (2.30)

and thus, in sharp contrast with the one-family case, contains divergences of single traces of \(A\), with family indices nicely projected in the “hooked” \(\{2,1\}\) representation. Notice that this expression actually admits two independent Young projections. The first is the symmetric \(\{3\}\),
\[
\partial^l T_{(ij} T_{kl)} A = 9 \Box \partial^l C_{ijkl} + 3 \partial^l \partial^m \partial_{(i} C_{jk)lm} + \frac{1}{2} \partial^l \partial^m \partial^n T_{(ij} C_{kl)m},
\] (2.31)

that, as in the one-family case, is not particularly interesting. It relates in fact the symmetrized double trace of the kinetic tensor \(A\) to the gauge invariant constraints (2.19), consistently with an algebraic identity satisfied by \(A\) that can be simply extracted from the trace rules collected in Appendix B, according to which
\[
T_{(ij} T_{kl)} A = 3 \left\{ 3 \Box C_{ijkl} + \partial^m (\partial_{(i} C_{jkl)m} - \partial_m C_{ijkl}) + \frac{1}{2} \partial^m \partial^n T_{mn} C_{ijkl} \right\}.
\] (2.32)

On the other hand, the \(\{2,1\}\) projection is a novel feature of the mixed-symmetry case, that first emerges at two families and links in a non-trivial fashion divergences and gradients of the traces of \(A\):
\[
\partial_i T_{jk} A - \frac{1}{2} \partial_{(j} T_{k)i} A = \frac{1}{6} \partial^l \left( 2 T_{il} T_{jk} - T_{i(j} T_{k)i} \right) A
\]
\[+ \frac{1}{4} \partial^l \partial^m \left( 2 \partial_i C_{jklm} - \partial_{(j} C_{kl)m} \right) - \frac{1}{12} \partial^l \partial^m \partial^n \left( 2 T_{jk} C_{ilmn} - T_{i(j} C_{k)lmn} \right).
\] (2.33)

As a result, the gauge variation (2.28) can be turned into
\[
\delta L_0 = -\frac{1}{24} \langle \partial_i \partial_j \partial_k \Lambda_l, C_{ijkl} \rangle + \frac{1}{72} \langle T_{(ij} \Lambda_k), \partial_{(i} T_{jk) A} \rangle + \frac{1}{24} \langle T_{jk} \Lambda_i, \partial^l (2 T_{il} T_{jk} - T_{i(j} T_{k)i}) A \rangle,
\] (2.34)
where the first term only contains the constraints and therefore can be compensated via Lagrange multipliers. In a similar fashion, the rest can be also partly compensated by a pair of new terms. Only the first of these, however,

\[ \mathcal{L}_1 = -\frac{1}{24}\langle \alpha_{ijk}, \partial_{(i} T_{jk)} A \rangle, \]  

has a direct analogue in the one-family case, while the second,

\[ \mathcal{L}_2 = \frac{1}{72}\langle \varphi, \eta^{ij} \eta^{kl} (2 T_{ij} T_{kl} - T_{i(k} T_{l)j} A) \rangle, \]  

is again a genuine novelty of the two-family case, since it involves double traces and a \{2,1\} Young projection for the family indices, automatically promoted to a \{2,2\} Young projection due to the presence of a pair of identical \( T \) tensors. It should be appreciated that this extension of the symmetry is a further manifestation of the phenomenon mentioned after eq. (2.25).

The resulting Lagrangian is still not gauge invariant, but its gauge variation,

\[
\delta (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) = -\frac{1}{24}\langle \partial_{i} \partial_{j} \partial_{k} \bigg[ \Lambda_{(l} - \frac{1}{9} \eta^{mn} (2 T_{mn} \Lambda_{|l|} - T_{|l|m} \Lambda_{n}) \bigg], \mathcal{C}_{ijkl} \rangle - \frac{1}{864}\langle (2 T_{jk} T_{lm} - T_{j(i} T_{m)k}) \Lambda_{i}, \partial_{i} (2 T_{jk} T_{lm} - T_{j(i} T_{m)k} A) \rangle,
\]

(2.37)
can be compensated following steps similar to the previous ones. The key observation is, once more, that the combination of two identical \( T \) tensors above is not only \{2,1\}, but actually \{2,2\} Young projected. As a result, the left-hand side of the second scalar product contains only two irreducible Young components, as summarized by the diagrams

\[
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\]

In the general multi-family case, one would connect the \{3,2\} Young projection to compensators and the \{2,2,1\} Young projection to a double trace of the Bianchi identity (2.18). However, with only two families the \{2,2,1\} projection simply does not exist, since it would involve anti-symmetrizations over three distinct family indices. As a result, in order to complete the two-family construction, one need only reconstruct the gauge transformations of some compensator traces in the \{3,2\} projection. After some algebra, one finds that

\[
(2 T_{ij} T_{kl} - T_{i(k} T_{l)j}) \Lambda_{m} \overset{(3,2)}{\rightarrow} \frac{3}{4} \left(3 T_{ij} \delta \alpha_{klm} + 3 T_{kl} \delta \alpha_{ijm} - T_{(ij} \delta \alpha_{kl)} m \right),
\]

(2.38)
which indeed fixes the remaining compensator terms.

Summarizing, in the two-family case the terms to be compensated with Lagrange multipliers do not receive further corrections, and as a result the unconstrained gauge invariant Lagrangian is

\[
\mathcal{L} (\varphi, \Phi_{i}, \beta_{ijkl}) = \frac{1}{2}\langle \varphi, A \rangle - \frac{1}{2} \eta^{ij} T_{ij} A + \frac{1}{36} \eta^{ij} \eta^{kl} (2 T_{ij} T_{kl} - T_{i(k} T_{l)j} A) - \frac{1}{24}\langle \alpha_{ijk}(\Phi), \partial_{(i} T_{jk)} A \rangle - \frac{1}{12} \eta^{lm} (2 \partial_{(i} T_{jk)} T_{lm} - \partial_{(i} T_{l|j} T_{k) m}) A + \frac{1}{8}\langle \beta_{ijkl}, \mathcal{C}_{ijkl} \rangle,
\]

(2.39)
where for the sake of clarity we have stressed once more that here the $\alpha_{ijkl}$ are to be regarded as functions of the $\Phi_i$. Furthermore, the $\beta_{ijkl}$ are Lagrange multipliers, whose gauge transformations,

$$
\delta \beta_{ijkl} = \frac{1}{4} \left\{ \partial_{(i} \partial_{j} \partial_k \Lambda_l) + \frac{1}{9} \partial^m \partial_{(i} \partial_{j]} \left( 2 T_{|kl|} \Lambda_m - T_{m|k} \Lambda_l \right) - \frac{1}{9} \eta^{mn} \partial_{(i} \partial_{j]} \partial_k \left( 2 T_{mn} \Lambda_l + T_{l|m} \Lambda_n \right) \right\}, \tag{2.40}
$$

can be obtained from the term of (2.37) that contains $C_{ijkl}$. Notice, however, that when this transformation is inserted in the Lagrangian (2.39), the $\eta^{mn}$ in the last term gives rise to a contribution involving the trace $T_{mn} C_{ijkl}$. Only the resulting $\{4,2\}$ projection of the last term is then effective on the $C_{ijkl}$, on account of the constraint (2.25). Hence, eq. (2.40) could well be presented in a different form, restricting the last term directly to its $\{4,2\}$ Young projection. Interestingly, this also implies that, if the $\alpha_{ijkl}$ are expressed in terms of the independent compensators $\Phi_i$ via eq. (2.12), the Lagrangian (2.39) possesses a further gauge symmetry, related to shifts of the Lagrange multipliers of the type

$$
\delta \beta_{ijkl} = \eta^{mn} L_{ijkl, mn}, \tag{2.41}
$$

where $L_{ijkl, mn}$ is $\{5,1\}$ projected in its family indices. Indeed, this shift would produce the contribution

$$
\delta \mathcal{L} = \frac{1}{16} \left( L_{ijkl, mn} T_{mn} C_{ijkl} \right), \tag{2.42}
$$

an expression that vanishes on account of the constraints (2.25) on the $C_{ijkl}$, that hold once the $\alpha_{ijkl}$ are expressed in terms of the $\Phi_i$.

It is actually possible to recast the Lagrangian (2.39) in an alternative form that will soon prove convenient to derive its field equations. To this end, let us consider a field $\phi$ that is not subject to the double trace constraints (2.17), but whose gauge parameters $\Lambda_i$ are still constrained according to (2.8). A convenient presentation of the corresponding Lagrangian is then

$$
\mathcal{L}_C (\phi, \gamma_{ijkl}) = \frac{1}{2} \left( \phi, \mathcal{F}(\phi) \right) - \frac{1}{2} \eta^{ij} T_{ij} \mathcal{F}(\phi) + \frac{1}{36} \eta^{ij} \eta^{kl} \left( 2 T_{ij} T_{kl} - T_{i(k} T_{l)j} \right) \mathcal{F}(\phi) \right) + \frac{1}{24} \left( \gamma_{ijkl}, T_{ijkl} \phi \right), \tag{2.43}
$$

that differs from the Labastida form of [20] simply because here gauge invariance requires projected traces and Lagrange multipliers $\gamma_{ijkl}$. These fields enforce the double trace constraints, and their gauge transformations are actually those given for the $\beta_{ijkl}$ in eq. (2.40),

$$
\delta \gamma_{ijkl} = \frac{1}{4} \left\{ \partial_{(i} \partial_{j} \partial_k \Lambda_l) + \frac{1}{9} \partial^m \partial_{(i} \partial_{j]} \left( 2 T_{|kl|} \Lambda_m - T_{m|k} \Lambda_l \right) - \frac{1}{9} \eta^{mn} \partial_{(i} \partial_{j]} \partial_k \left( 2 T_{mn} \Lambda_l + T_{l|m} \Lambda_n \right) \right\}. \tag{2.44}
$$

The relation between the two Lagrangians of eqs. (2.39) and (2.43) is then

$$
\mathcal{L} (\varphi, \Phi_i, \beta_{ijkl}) = \mathcal{L}_C (\varphi - \partial^i \Phi_i, \beta_{ijkl} - \Delta_{ijkl}(\Phi)), \tag{2.45}
$$

where

$$
\Delta_{ijkl}(\Phi) = \frac{1}{4} \partial_{(i} \partial_{j} \partial_k \left[ \Phi_{|l]} - \frac{1}{9} \eta^{mn} \left( 2 T_{mn} \Phi_{|l]} - T_{l|}(m \Phi_n) \right) \right]. \tag{2.46}
$$
Finally, in the one-family case the Lagrangian (2.39) reduces to the result of [26, 27], that in this notation would read

\[ \mathcal{L} = \frac{1}{2} \langle \varphi, A - \frac{1}{2} \eta A' \rangle - \frac{1}{8} \langle \alpha, \partial \cdot A' \rangle + \frac{1}{8} \langle \beta, \mathcal{C} \rangle, \]  

(2.47)

where “primes” denote traces and the symbol \( \partial \cdot \), to be identified with \( \partial_1 \), is used to denote a divergence.

One could also work with formally independent \( \alpha_{ijk} \), adding to the Lagrangian (2.39) new \textit{gauge invariant} multipliers \( \ell_{ijk} \), according to

\[ \mathcal{L}_4 = \langle \ell_{ijk} , \frac{1}{3} T_{(ij} \Phi_{k)} - \alpha_{ijk} \rangle. \]  

(2.48)

Let us stress, however, that in this case the \{5,1\} part of the last term in the gauge transformation of the \( \beta_{ijkl} \) would be also effective, since it would couple to the gauge invariant combination

\[
Y_{\{5,1\}} T_{mn} \mathcal{C}_{ijkl} \sim \partial^p \left\{ (T_{n(i} T_{jk} \alpha_{lm)p}) + T_{n(i} T_{jk} \alpha_{ln})p) \right\} \\
- 2 (T_{(ij} T_{kl} \alpha_{m)p} + T_{(ij} T_{kl} \alpha_{n)p}) + (T_{p(i} T_{jk} \alpha_{lm)n} + T_{p(i} T_{jk} \alpha_{ln)m}) \\
- 2 (T_{p(i} T_{n} j \alpha_{klm}) + T_{p(i} T_{m} j \alpha_{klm}) + (T_{pm} T_{(ij} \alpha_{klm}) + T_{pn} T_{(ij} \alpha_{klm}) \}
\]

(2.49)

that would not vanish when working with independent \( \alpha_{ijk} \). On shell, however, one would return anyway to \( \mathcal{C}_{ijkl} \) tensors subject to the proper constraints (2.25), that as we stressed are driven by the double traces of \( \varphi \) that they contain. Notice that in this way of formulating the theory the symmetry of eq. (2.41) would extend to a simultaneous redefinition of the \( \beta_{ijkl} \) and \( \ell_{ijk} \),

\[
\delta \beta_{ijkl} = \eta^{mn} L_{ijkl, mn}, \\
\delta \ell_{ijk} = \frac{1}{4} \eta^{mn} \eta^{pq} \left( \partial_{(i} L_{jk)mn,pq} - \partial_{(m} L_{n)ijk,pq} \right). \]  

(2.50)

There is an issue of completeness for the Lagrangian (2.39), since further terms could be added to it, in principle at least. Considerations of this type were already made in [27] for the one-family case, with the conclusion that no significant additions were actually possible. The reason was that this type of terms would contribute to the field equation proportionally to the constraint \( \mathcal{C} \), and would thus be ineffective by virtue of the field equation for the Lagrange multiplier \( \beta \). In the mixed-symmetry case, however, there are more possibilities, simply because not all double traces of \( \varphi \) are subject to constraints for general mixed-symmetry fields. A potentially interesting term of a new type is for instance

\[
\langle S^n_{(i} S^j_{n|} Y_{\{2,2\}} T_{kl)} T_{mn} A, \mathcal{C}_{ijkl} \rangle,
\]

(2.51)

where the \( S^i_{ij} \) operators, a key novelty of the \( N \)-family case, are defined in Appendix A. Their commutators close on a \( \text{gl}(N) \) algebra, and their net effect for \( i \neq j \) is to displace indices from one family to another. Thanks to the presence of the \( S^i_{ij} \) operators, the double trace above can carry a \{2,2\} “window” projection, that is not proportional to the constraint tensors and yet can appear in a combination that is totally symmetric in \( (ijkl) \). As a result, terms of this type do give a contribution to the \( \varphi \) equation that does not vanish when the equation for the Lagrange multiplier is enforced. They are still irrelevant, however, since their net effect is a mere redefinition of the multipliers by gauge invariant quantities. More generally, such a wide
freedom to redefine the multipliers can be generalized to include terms which are not necessarily themselves gauge invariant. This has interesting consequences both for the Lagrangian and for the field equations, including the possibility of working from the beginning with gauge invariant Lagrange multipliers, an option which calls for a reshuffling of the terms in $L$, and will be discussed in Section 3.1.

2.2 The field equations

Varying the Lagrangian of eq. (2.39) and using repeatedly the trace rules collected in Appendix B yields the equation of motion for the gauge field $\varphi$,

$$ E_\varphi : A - \frac{1}{2} \eta^{ij} T_{ij} A + \frac{1}{36} \eta^{ij} \eta^{kl} \left( 2 T_{ij} T_{kl} - T_i (k T_l) j \right) A + \frac{1}{8} \eta^{ij} \partial^k \partial^l C_{ijkl} + \frac{1}{96} \eta^{ij} \partial^m \partial^n \left( 2 T_{ij} C_{klmn} - T_i (k C_l) jmn \right) + \frac{1}{2} \eta^{ij} \eta^{kl} B_{ijkl} = 0, $$

(2.52)

where

$$ B_{ijkl} \equiv \beta_{ijkl} - \frac{1}{12} \left\{ T_{ij} \partial_k \partial_l \varphi - 3 \Box \partial_i \alpha_{jkl} - \partial^m \partial_i \partial_j \alpha_{kl} m \right\} + \frac{1}{144} \eta^{mn} \left\{ 2 T_{mn} T_{ij} \partial_k \partial_l \varphi - T_m (i | j \partial_k \partial_l \varphi) \right\} - \frac{1}{96} \eta^{mn} \Box \left\{ 3 T_{mn} \partial_i \alpha_{jkl} - 2 T_m (i \partial_j \alpha_{kl}) n + T_{ij} \partial_k \alpha_{l mn} \right\} + \frac{1}{96} \eta^{mn} \partial^p \left\{ T_{mn} \partial_i \partial_j \alpha_{kl} p - T_m (i \partial_j \partial_k \alpha_l) np + T_{ij} \partial_k \partial_l \alpha mnp \right\}. $$

(2.53)

Notice that for fully symmetric tensors eq. (2.52) reduces to

$$ E_\varphi : A - \frac{1}{2} \eta A' + \frac{1}{4} \eta \partial^2 C + \eta^2 B = 0, $$

(2.54)

where $\partial^2 = \frac{1}{2} \partial \partial$ and $\eta^2 = \frac{1}{2} \eta \eta$ are defined according to the rules of [26, 27], and the reader may easily sort out these terms in the original expression.

The considerations of the previous section have an interesting link with the nature of the $B_{ijkl}$. Indeed, the space-time tensors collected in (2.53) are invariant under the gauge transformations of eqs. (2.4), (2.9) and (2.40), as in the one-family case of [26, 27], only if the gauge transformation (2.40) is restricted, in its $\eta$ - dependent terms, to the $\{4, 2\}$ projection, thus eliminating all $\{5, 1\}$ contributions, which is the case precisely if the $\alpha_{ijk}$ are expressed in terms of the independent compensators $\Phi_i$ as in eq. (2.12). Let us also stress that $B_{ijkl}$ varies under the new gauge transformation (2.41) as

$$ \delta B_{ijkl} = \eta^{mn} L_{ijkl, mn}, $$

(2.55)

but the $\varphi$ field equation is properly gauge invariant nonetheless, since as we have pointed out the three $\eta$'s that would accompany $L_{ijkl, mn}$ in the variation of eq. (2.52) simply can not build a $\{5, 1\}$ projection. Indeed, this argument also shows that the combination $\eta^{ij} \eta^{kl} B_{ijkl}$, that appears in the $\varphi$ equation, is in any case invariant under the complete transformation of
eq. (2.40). This clearly reflects the gauge symmetry of eq. (2.52), that the Lagrangian (2.39) guarantees regardless of the possibility, discussed in the previous subsection, of introducing the Lagrange multipliers $\ell_{ijk}$ in order to allow for independent $\alpha_{ijk}$.

In addition, varying in (2.39) the Lagrange multipliers yields the double-trace constraints

$$E_\beta : \frac{1}{8} C_{ijkl} = 0. \quad (2.56)$$

On the other hand, the field equations for the compensators $\Phi_i$ are

$$E_\Phi : \eta^{jk} \left\{ - \frac{1}{12} T_{(ij} \partial_{k)} A + \frac{1}{144} \eta^{lm} \left( 2 T_{lm} T_{(ij} \partial_k A - T_{(i | T_{m | j} \partial_k A) } \right) \right. \nonumber$$

$$+ \frac{1}{8} \Box \partial^l C_{ijkl} + \frac{1}{48} \partial^l \partial^m \partial^i C_{jk} \right\} = 0. \quad (2.57)$$

The rewriting of the Lagrangian (2.39) based on the redefinitions in eqs. (2.45) and (2.46) is particularly convenient to derive these results. Indeed, varying eq. (2.43) one obtains

$$\delta \mathcal{L}_C = \langle \delta \phi, E_\phi \rangle + \langle \delta \gamma_{ijkl}, (E_\gamma)_{ijkl} \rangle = \langle \delta \varphi, E_\varphi \rangle + \langle \delta \beta_{ijkl}, (E_\beta)_{ijkl} \rangle$$

$$+ \langle \delta \Phi_i, \partial_i E_\phi + \partial^j \partial^k \partial^l (E_\gamma)_{ijkl} + \frac{1}{3} \eta^{jk} \partial^l \partial^m \left( 2 \partial_{i} (E_\gamma)_{jklm} - \partial_{(j} (E_\gamma)_{k)lm} \right) \rangle$$

$$- \frac{1}{9} \eta^{jk} \partial^l \partial^m \partial^n \left( 2 T_{jk} (E_\gamma)_{ilmn} - T_{i(j} (E_\gamma)_{k)lmn} \right), \quad (2.58)$$

so that

$$E_\varphi = E_\phi,$$

$$E_\Phi = \partial_i E_\phi + \partial^j \partial^k \partial^l (E_\gamma)_{ijkl} + \frac{1}{3} \eta^{jk} \partial^l \partial^m \left( 2 \partial_{i} (E_\gamma)_{jklm} - \partial_{(j} (E_\gamma)_{k)lm} \right),$$

$$- \frac{1}{9} \eta^{jk} \partial^l \partial^m \partial^n \left( 2 T_{jk} (E_\gamma)_{ilmn} - T_{i(j} (E_\gamma)_{k)lmn} \right)$$

$$E_\beta = E_\gamma, \quad (2.59)$$

where in these expressions $\phi$ and the $\gamma_{ijkl}$ are to be expressed in terms of $\varphi$ and the $\beta_{ijkl}$ via eqs. (2.45) and (2.46). In this fashion one can obtain all field equations starting from the $\varphi$-dependent terms of eqs. (2.52) and (2.56). In particular, the $B_{ijkl}$ tensors can be obtained from

$$B_{ijkl} = \gamma_{ijkl} - \frac{1}{12} \partial_{(i} \partial_{j} T_{kl)} \phi + \frac{1}{144} \eta^{mn} \partial_{(i} \partial_{j)} \left( 2 T_{[k l]} T_{mn} - T_{m | k T_{i n} } \right) \phi, \quad (2.60)$$

that after the shift (2.45) becomes manifestly gauge invariant since this property holds for both $\gamma_{ijkl}$ and $\phi$. The apparent contradiction with the considerations made after eq. (2.53) is resolved observing that eq. (2.60) describes a different form of the $B_{ijkl}$, that in general is not fully expressible in terms of $\varphi$, the $\alpha_{ijk}$ and the $\beta_{ijkl}$. The choice (2.53) for the $B_{ijkl}$ can be recovered, however, if the $\eta$ part of the gauge transformation of the $\beta_{ijkl}$, and consequently of the shift.
(2.46) of the $\gamma_{ijkl}$, is restricted to its $(4,2)$ component, the only relevant one on account of the shift symmetry (2.41).

The field equations for the compensators can be derived from eqs. (2.59), which provide a condition that is tantamount to the conservation of external currents coupling to $\varphi$. Furthermore, as in the symmetric case of [27], it is also possible to rewrite the Lagrangian in the form

$$\mathcal{L} = \frac{1}{2} \langle \varphi, E_\varphi \rangle + \frac{1}{2} \langle \Phi_i, (E_\Phi)_i \rangle + \frac{1}{2} \langle \beta_{ijkl}, (E_\beta)_{ijkl} \rangle,$$

(2.61)

taking into account the fact that the $\phi$ terms present in $B_{ijkl}$ are the adjoints of the $C_{ijkl}$ terms in eq. (2.52), as one can recognize varying eq. (2.43).

Let us conclude this section by stressing that these field equations were simply obtained varying the Lagrangians of eq. (2.39) with respect to gauge fields, compensators and Lagrange multipliers. This was possible since, in our formulation, all these fields are unconstrained. On the other hand, when deriving the field equations of the Labastida theory one is confronted with gauge fields $\varphi$ that are subject to the double-trace constraints (2.17). This was indeed noticed in [20], where the author however seems to conclude that the problem does not present itself because he apparently expected his constrained Einstein-like tensors

$$E_\varphi = \mathcal{F} - \frac{1}{2} \eta^{ij} T_{ij} \mathcal{F} + \frac{1}{36} \eta^{ij} \eta^{kl} (2T_{ij} T_{kl} - T_{i(k} T_{l)j}) \mathcal{F}$$

(2.62)

to have vanishing symmetrized double traces, as in Fronsdal’s case. However, the $S^i_j$ operators bring about a surprise, since already for two families one finds

$$T_{(ij} T_{kl)} \mathcal{E}_\varphi = -\frac{1}{36} S^m_{(i} S^n_{j|} (2T_{kl}) T_{mn} - T_{m(k} T_{l)n}) \mathcal{F},$$

(2.63)

and as a result the actual Lagrangian equations do require a projection. We shall see an explicit example of this fact in Section 2.4 in the case of a reducible rank-(4,2) tensor.

Alternatively, the structure of the unconstrained formulation suggests to recast even the variation

$$\delta \mathcal{L} = \langle \delta \varphi, \mathcal{E}_\varphi \rangle$$

(2.64)

of the constrained Labastida Lagrangian defining its field equations in the form

$$\mathcal{E}_\varphi + \frac{1}{2} \eta^{ij} \eta^{kl} B_{ijkl} = 0,$$

(2.65)

with an additional contribution $B_{ijkl}$ that is totally symmetric in its family indices, simply because such a term would be orthogonal to $\delta \varphi$ in view of eq. (2.17). Notice that this is precisely the type of expression that one would obtain gauge fixing eq. (2.52). Now, once combined with its symmetrized double traces, the modified field equation (2.65) would rebuild the non-trivial projection precisely because, according to eq. (2.63), the symmetrized double traces of $\mathcal{E}_\varphi$ do not vanish identically beyond one family.

### 2.3 On-shell Reduction to $\mathcal{F} = 0$

We can now turn to examine whether eq. (2.52) can be reduced to the non-Lagrangian Labastida form

$$\mathcal{F} = 0,$$

(2.66)
with $\mathcal{F}$ defined in (2.6). We will show that (2.66) can always be recovered either directly from the equations of motion or after using additional local symmetries that emerge in special cases. It is perhaps the case to elaborate on the meaning of the procedure, so as to clarify the spirit of the ensuing analysis.

Eq. (2.66) has the net effect of reducing the $gl(D)$ tensors appearing in the Lagrangians to their $so(D - 2)$ counterparts. However, for particular classes of $gl(D)$ tensors, the relevant $so(D - 2)$ representations may simply not exist. And indeed, a general result of representation theory (see, for instance, the first reference in [36], § 10-6) implies that, for $so(n)$ groups, if the total number of boxes in the first two columns of a tableau exceeds $n$, the corresponding transverse tensor vanishes. According to this theorem, one can thus conclude that in $D = 2$ eq. (2.66) can never describe propagating degrees of freedom (beyond the scalar case), while in $D = 3$ only the vector representation $(1, 0)$ would be non trivial on-shell. In a similar fashion, in $D = 4$ non-vanishing $so(2)$ polarizations are only carried by symmetric tensors of type $(s, 0)$ and by the 2-form (dual to a scalar), while in $D = 5$ one can also add to the latter cases the $(s, 1)$ tensors. Finally, for $D \geq 6$ all $gl(D)$ representations would reduce via (2.66) to non-trivial representations of $so(D - 2)$.

It should be stressed, however, that even the cases with no propagating degrees of freedom are of some interest, since they include a number of topological models that generalize the rich and instructive example of two-dimensional gravity. And indeed a neat pattern of theories of this type will emerge from our analysis, together with other cases that are specific of mixed symmetry fields, as we shall illustrate in detail in what follows.

In order to introduce the general features of our procedure, let us begin by recalling how the same problem was dealt with in [26, 27] for the simpler case of symmetric tensors. The Lagrangian of eq. (2.47) yields the single-family counterparts of the field equations (2.52), (2.56) and (2.57). These include the double-trace constraint

$$C \equiv \varphi'' - 4 \partial \cdot \alpha - \partial \alpha' = 0, \quad (2.67)$$

the field equation for the single Lagrange multiplier $\beta_{1111}$ present in this class of models, in the following simply called $\beta$ as in [26, 27]. In a similar fashion, after making use of eq. (2.67) the field equation for the physical gauge field $\varphi$ takes the form

$$\mathcal{A} - \frac{1}{2} \eta \mathcal{A}' + \frac{1}{2} \eta \eta \mathcal{B} = 0, \quad (2.68)$$

where

$$\mathcal{B} \equiv \beta - \frac{1}{2} \partial \cdot \partial \cdot \varphi' + \Box \partial \cdot \alpha + \frac{1}{2} \partial \partial \cdot \partial \cdot \alpha \quad (2.69)$$

is the gauge-invariant completion of $\beta$.

In order to reduce (2.68) to the non-Lagrangian equation $\mathcal{A} = 0$, one can begin by noticing that $\mathcal{A}$ becomes doubly traceless once the constraint (2.67) is enforced. Taking multiple traces of (2.68) one can then generate equations involving only $\mathcal{B}$ and its traces that imply that the whole tensor $\mathcal{B}$ vanishes on shell [26]. As a result, it is generally a simple matter to turn eq. (2.68) into $\mathcal{A} = 0$, and finally into $\mathcal{F} = 0$, with a proper gauge choice eliminating the compensator $\alpha$. Alternatively, one could decompose (2.68) into a set of relations involving the irreducible
o(1, d − 1) components of the tensors appearing in (2.68), letting for instance 
\[ B = B_T + \eta B_T' + \ldots + \eta^k B_T^{[k]} + \ldots \] (2.70)
In view of the double tracelessness of \( \mathcal{A} \), eq. (2.68) would then imply the system
\[
\begin{cases}
\mathcal{A}_T = 0 , \\
(D - 6 + 2s) \mathcal{A}_T' = 0 , \\
B_T^{[k]} = 0 , & k = 0, \ldots, \lfloor \frac{s-4}{2} \rfloor ,
\end{cases}
\] (2.71)
that would lead in all cases to the two results one is after, namely
\[ \mathcal{A} = 0 , \quad B = 0 , \] (2.72)
but for a single exception.
This corresponds to two-dimensional linearized gravity, and is characterized by the values 
\( s = 2 \) and \( D = 2 \). It actually requires some discussion, since in this case the pre-factor in the 
second equation vanishes. What happens, however, is a familiar story: \( \mathcal{A}_T' \) is left undetermined, 
but the field equation (2.68) acquires at the same time the additional gauge symmetry
\[ \delta \varphi_{\mu\nu} = \eta_{\mu\nu} \Omega . \] (2.73)
This is the linearized Weyl symmetry, that as is well known plays a key role in String Theory. 
More precisely, the Einstein-Hilbert action is a total divergence in two dimensions, and as a 
result the Einstein tensor vanishes, so that any shift of the form
\[ \delta \varphi_{\mu\nu} = \xi_{\mu\nu} \] (2.74)
is actually a symmetry, that in this case can nonetheless be parametrized via Weyl shifts and 
linearized diffeomorphisms. We are stressing the role of the Weyl symmetry since, as we shall 
see shortly, it affords important generalizations for higher-spin fields of mixed symmetry.

All these features of the reduction have direct counterparts in our present, more general 
setting. For tensors of mixed symmetry one can proceed along similar lines, but several new 
features emerge, together with technical complications that require more sophisticated tools. To 
begin with, not all double traces of \( \mathcal{A} \) are removed in general, even when the constraint equations 
\( \mathcal{C}_{ijkl} = 0 \) are enforced. As a result, the symmetrized double traces of the Einstein-like tensors do 
not vanish at two families, and one can not separate \( \mathcal{A} \) and the \( B_{ijkl} \) by simply taking multiple 
traces of eq. (2.52). In addition, the \( B_{ijkl} \) do not 
fully vanish on-shell. Rather, as we shall see, 
some of their components are not even determined, but still do not contribute to the \( \varphi \) field 
equation, due to the shift symmetry discussed in section 2.2. More generally, the key novelties of 
two-family gauge fields reflect the non-Abelian nature of the \( S_{ij} \) operators of Appendix A, that 
do not appear in the Lagrangians and in the corresponding field equations but are ubiquitous 
in their traces.
As for symmetric tensors, let us begin by considering the equation of motion (2.52) for the 
gauge field \( \varphi \),
\[ E_{\varphi} \equiv \mathcal{A} - \frac{1}{2} \eta^{ij} \mathcal{A}_{ij} + \frac{1}{12} \eta^{ij} \eta^{kl} \mathcal{A}_{ij}^{(n)} + \frac{1}{2} \eta^{ij} \eta^{kl} B_{ijkl} = 0 , \] (2.75)
\[ ^{6}\text{Here } B_{ijkl}^{[k]} \text{ denotes the traceless part of the } k \text{-th trace of } B, \text{ up to an overall coefficient. In addition, as in } \text{[26, 27], } \eta^{k} \text{ denotes the product of } k \text{ Minkowski metric tensors, symmetrized via the minimum possible number of terms, so that for example } \eta \eta = 2 \eta^{2}.\]
where we are enforcing the constraints (2.56), and where for convenience we are introducing the shorthand notation

\[ A'_{ij} = T_{ij} A, \]
\[ A''_{ij,kl} = \frac{1}{3} \left( 2T_{ij} T_{kl} - T_i (k T_l) j \right) A. \]  

(2.76)

Notice that in the two-family case \( A''_{ij,kl} \) is effectively traceless, because all triple traces of \( A \) can be related to symmetrized double traces and thus vanish on account of the constraint equations (2.32) and (2.56), as can be seen making use of the results in Appendix C.

In order to proceed, it is useful to disentangle the traceless part of the \( B_{ijkl} \) tensors from their traces, via a decomposition similar to the one in (2.70), and thus letting

\[ B_{ijkl} = B^{(T)}_{ijkl} + \eta^{mn} \tilde{B}_{ijkl, mn}. \]  

(2.77)

The scalar product provides an interesting tool to this effect, since

\[ \langle \tilde{B}_{ijkl, mn}, T_i T_k T_m E_\varphi \rangle \sim \left\| \eta^{ij} \eta^{kl} \eta^{mn} \tilde{B}_{ijkl, mn} \right\|^2 \]  

(2.78)

shows that the expression on the right-hand side vanishes on-shell. While this does not imply directly that the \( \tilde{B}_{ijkl, mn} \) tensors vanish, it does prove that they certainly decouple from \( A \) and its non-vanishing traces, so that the equations of motion reduce to

\[ A - \frac{1}{2} \eta^{ij} A'_{ij} + \frac{1}{12} \eta^{ij} \eta^{kl} A''_{ij,kl} + \frac{1}{2} \eta^{ij} \eta^{kl} B^{(T)}_{ijkl} = 0, \]
\[ \eta^{ij} \eta^{kl} \eta^{mn} \tilde{B}_{ijkl, mn} = 0. \]  

(2.79)

The second, in particular, sets to zero the \( \{6\} \) and \( \{4,2\} \) projections in the family indices of \( \tilde{B}_{ijkl, mn} \), leaving undetermined its \( \{5,1\} \) component, since the product of three \( \eta \)'s does not admit this last projection. This is just the expected behavior, since we have seen in Section 2.2 that the trace parts of the \( B_{ijkl} \) transform under the additional gauge symmetry (2.55), and consequently can be set to zero via the corresponding shift.

On the other hand, taking one trace of the first of eqs. (2.79) gives

\[ (D - 2) E_{ij} + S^k (i E_j)_k \equiv \mathcal{O} [D - 2]_{ij}^{kl} E_{kl} = 0, \]  

(2.80)

which defines the \( \mathcal{O}[\lambda] \) operators, and where

\[ E_{ij} = A'_{ij} - \frac{1}{3} \eta^{kl} A''_{ij,kl} - 2 \eta^{kl} B^{(T)}_{ijkl}. \]  

(2.81)

Notice that in deriving this expression we are taking into account that eq. (2.32), together with eq. (2.56), forces the double trace of \( A \) to coincide with its \( \{2,2\} \) projection defined in (2.76).

From eq. (2.80) one thus obtains that, outside the kernel of \( \mathcal{O}[D - 2] \) (that is empty for \( D \geq 6 \), as we shall see in the next section),

\[ A'_{ij} - \frac{1}{3} \eta^{kl} A''_{ij,kl} - 2 \eta^{kl} B^{(T)}_{ijkl} = 0. \]  

(2.82)

A further trace of this relation then yields

\[ (D - 3) A''_{ij,kl} + S^m (i A''_{ij}) m,kl + 6 \left[ DB^{(T)}_{ijkl} + S^m (i B^{(T)}_j) mkl \right] = 0, \]  

(2.83)
from which one can conclude that both $A''_{ij,kl}$ and $B^{(T)}_{ijkl}$ vanish in general, up to a few exceptional cases that we shall soon identify. Substituting these in (2.82) gives

$$A'_{ij} = 0,$$

so that eq. (2.75) finally becomes

$$A = 0,$$

that after shifting away the compensators is tantamount to the reduction to the Labastida form (2.66).

As we already stressed, the current analysis leaves some loose ends whenever eqs. (2.80) and (2.83) are not invertible, so that either $A'$ or $A''$ are not determined. The next section is thus devoted to a systematic study of the kernel of the relevant operators. This will also clarify the nature of some poles that we shall find in Section 2.4.3 in the current exchanges of a rank-(4,2) field. The final upshot of the discussion will be that interesting patterns of Weyl-like symmetries emerge in sporadic low-dimensional cases, precisely as needed to gauge away the undetermined quantities.

### 2.3.1 Weyl-like symmetries

Let us therefore analyze the kernel of the operator $O$ that first appeared in eq. (2.80),

$$O \left[ D - 2 \right]_{ij}^{kl} = \frac{D - 2}{2} \delta_i^{(k} \delta_j^{l)} + \frac{1}{2} S^{(k} (i \delta_j^{l)}),$$

whose structure is determined by the properties of the $S_{ij}$ operators. In the two family case, there are actually four such operators, $S^1_{11}$, $S^1_{12}$, $S^2_{11}$ and $S^2_{22}$, that generate a $gl(2)$ algebra. Their properties can be exhibited most conveniently introducing two commuting vectors, $u^\mu$ and $v^\nu$, to cast generic two-family fields in the form

$$\varphi \left( u, v \right) = \frac{1}{s_1! s_2!} u^{\mu_1} \cdots u^{\mu_{s_1}} v^{\nu_1} \cdots v^{\nu_{s_2}} \varphi_{\mu_1 \cdots \mu_{s_1}; \nu_1 \cdots \nu_{s_2}}.$$

In this notation

$$S^1_1 = u \cdot \frac{\partial}{\partial u}, \quad S^1_2 = u \cdot \frac{\partial}{\partial v},$$

$$S^2_1 = v \cdot \frac{\partial}{\partial u}, \quad S^2_2 = v \cdot \frac{\partial}{\partial v},$$

so that the pairs $(u, \frac{\partial}{\partial u})$ and $(v, \frac{\partial}{\partial v})$ play the role of the conventional creation and annihilation operators for a couple of quantum-mechanical oscillators. In particular, $S^1_{11}$ and $S^2_{22}$ are diagonal on all fields of the form (2.87), and simply count their space-time indices, according to

$$S^1_{11} \varphi(u,v) = s_1 \varphi(u,v), \quad S^2_{22} \varphi(u,v) = s_2 \varphi(u,v).$$

On the other hand, $S^2_{11}$ and $S^1_{22}$ are not diagonal, but

$$S^1_{12} \varphi(u,v) = \frac{u^{\mu_1} \cdots u^{\mu_{s_1+1}} v^{\nu_1} \cdots v^{\nu_{s_2-1}}}{(s_1+1)! (s_2-1)!} \varphi_{(\mu_1 \cdots \mu_{s_1+1}; \nu_1 \cdots \nu_{s_2-1})},$$

$$S^2_{11} \varphi(u,v) = \frac{u^{\mu_1} \cdots u^{\mu_{s_1-1}} v^{\nu_1} \cdots v^{\nu_{s_2+1}}}{(s_1-1)! (s_2+1)!} \varphi_{(\mu_1 \cdots \mu_{s_1-1}; \nu_1 \cdots \nu_{s_2+1})},$$

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so that they effectively displace one index from one set to the other, raising or lowering the corresponding eigenvalues of \( S^1_1 \) and \( S^2_2 \). Hence, we can not refrain from turning to the suggestive notation

\[
L_+ = S^1_2, \quad L_- = S^2_1, \quad L_3 = \frac{1}{2} \left( S^1_1 - S^2_2 \right),
\]

(2.92)
since these three operators clearly satisfy the very familiar angular-momentum commutation relations

\[
[L_3, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = 2 L_3.
\]

(2.93)

Moreover, as usual

\[
L_- L_+ = L^2 - L^2_3 - L_3.
\]

(2.94)

This is almost Schwinger’s oscillator trick as reviewed for example in [37], although the present construction relies on a pair of vectors \( u^\mu \) and \( v^\nu \), and as a result builds a number of distinct irreducible multiplets characterized by different symmetry properties. The origin of this fact can be appreciated considering the product of \( p u^\mu \)'s and \( q v^\nu \)'s, that clearly gives rise to a number of components on which the permutation group of space-time indices acts irreducibly. These components reflect the decomposition of \( \{p\} \otimes \{q\} \), and a standard property of Young diagrams implies that \( L_+ \) annihilates terms of the type

\[
Y_{\{p,q\}} u^\mu_1 \ldots u^\mu_p v^\nu_1 \ldots v^\nu_q,
\]

(2.95)
simply because it forces a symmetrization beyond a given line \(^7\). Here \( Y_{\{p,q\}} \) denotes the Young projector corresponding to the \( \{p,q\} \) diagram, whose first row has the minimal possible length in the present example. For the combination (2.95), that as we have said is annihilated by \( L_+ \), the eigenvalue \( l \) of the "total angular momentum" \( L^2 \) coincides with the eigenvalue \( m = \frac{p - q}{2} \) of \( L_3 \), while a whole multiplet of irreducible polynomials in \( u \) and \( v \) with lower values of \( m \) can be built from (2.95) by successive applications of \( L_- \).

Using the previous observations, for a rank-(\( s_1, s_2 \)) field \( \varphi \) it is possible to recast the system defined by eq. (2.80) in the form

\[
\begin{align*}
(D + 2 s_1 - 6) E_{11} + 2 L_- E_{12} &= 0, \\
(D + s_1 + s_2 - 4) E_{12} + L_+ E_{11} + L_- E_{22} &= 0, \\
(D + 2 s_2 - 6) E_{22} + 2 L_+ E_{12} &= 0.
\end{align*}
\]

(2.96)

In order to study the non-trivial solutions of (2.96) it is necessary to decompose the unknowns into their irreducible \( gl(D) \) components, independent expressions on which permutations of space-time indices act irreducibly. This step is essential to arrive at an ordinary “square” algebraic system. Actually, this procedure splits the system of equations into a chain of independent sub-systems for the irreducible components of \( E_{11}, E_{12} \) and \( E_{22} \). These correspond to eigenstates of the operator \( L^2 \) introduced in eq. (2.94) so that, for instance, the decomposition of \( E_{12} \) can be presented in the form

\[
E_{12} = \sum_{n=0}^{s_2 - 1} (E_{12})^{\{s_1 + s_2 - n - 2, n\}},
\]

(2.97)

\(^7\)In Section 4 we shall see how to adapt the construction to gauge fields satisfying conditions of this type, that we shall term irreducible. Strictly speaking, however, one could also consider more general types of \( gl(D) \)-irreducible fields that are not associated to the endpoints of chains. From this algebraic vantage point, at two families these fields are simultaneous eigenstates of \( L^2 \) and \( L_+ \), and thus also of \( L_- L_+ \).
where each term in the sum can be related to components annihilated by $L_+$:

\[
(E_{12})^{\{s_1+s_2-n-2,n\}} = (L_-)^{s_2-n} (\hat{E}_{12})^{\{s_1+s_2-n-2,n\}},
\]
\[
L_+ (\hat{E}_{12})^{\{s_1+s_2-n-2,n\}} \equiv 0.
\]

Notice that, for all irreducible components of $E_{12}$ originating from a given field $\varphi$, $m$ takes a fixed value, that is related to $s_1$ and $s_2$ according to

\[
m = \frac{s_1 - s_2}{2},
\]

while the value of $\ell$ for the irreducible component $(E_{12})^{\{s_1+s_2-n-2,n\}}$ is

\[
\ell = \frac{s_1 + s_2}{2} - (n + 1),
\]

so that

\[
\frac{s_1 - s_2}{2} \leq \ell \leq \frac{s_1 + s_2}{2} - 1,
\]

where the allowed range is spanned by values of $\ell$ that differ by integers. Being eigenstates of $L_2$, the various irreducible components of $E_{12}$ are also eigenstates of $L_- L_+$, so that

\[
L_- L_+ (E_{12})^{\{s_1+s_2-n-2,n\}} = (s_1 - n) (s_2 - n - 1) (E_{12})^{\{s_1+s_2-n-2,n\}},
\]

where we have used the standard $sl(2)$ formula

\[
L_- L_+ V_{\ell,m} = (\ell - m) (\ell + m + 1) V_{\ell,m}.
\]

In order to proceed with the analysis of (2.96), it will prove convenient to distinguish various cases corresponding to various possible values of $s_2$, while taking into account that, as we have stressed, the $E_{ij}$ bear fewer indices than the original field $\varphi$. The ensuing discussion is inevitably somewhat technical, and the reader may wish to skip it, confining the attention to the two tables at the end of it, where the results are collected. The naive reduction to the Labastida form indeed fails only for a number of low-dimensional theories that in any case would reduce to trivial $SO(D-2)$ representations. As we stressed at the beginning of this section, these systems are nonetheless rather interesting, since they generalize the familiar example of two-dimensional gravity, that brought about many surprises over the years. Here are anyway the relevant cases ($s_1 \geq s_2$):

**s$_2 = 0$:** This is the symmetric case of [26, 27], and the system either does not exist (for $s_1 = 0,1$) or degenerates to the first equation (for $s_1 \geq 2$) without $E_{12}$. In all cases the equations of motion are invertible in any dimension $D$, with the already recalled exception of two-dimensional gravity.

**s$_2 = 1$:** In this case $E_{22}$ does not exist, and one is left at most with the first two equations, so that the system reduces to

\[
(D + 2 s_1 - 6) E_{11} + 2 L_- E_{12} = 0,
\]
\[
(D + s_1 - 3) E_{12} + L_+ E_{11} = 0,
\]

In addition, with $s_2 = 1$ $E_{12}$ only bears indices of the first family, and as a result contains only an irreducible component, the fully symmetric one. Hence, the first equation sets to
zero the “hooked” \( \{ s_1 - 2, 1 \} \) component of \( E_{11} \) whenever its coefficient does not vanish, while in general the symmetric components give rise to a \( 2 \times 2 \) system of equations. Furthermore, for the only available component of \( E_{12} \) the values of \( \ell \) and \( m \) coincide, so that it is annihilated by \( L_- L_+ \), and this observation will prove convenient in the detailed analysis of eq. (2.104) presented in the following.

For \( s_1 = 1 \) only the second equation is actually present, and in \( D = 2 \) it degenerates to an identity, so that \( E_{12} \) is left undetermined.

For \( s_1 = 2 \) and \( D = 1 \) the second equation reduces to \( L_+ E_{11} = 0 \). Thus, applying \( L_+ \) to the first equation and using eqs. (2.93) and (2.94) gives \( E_{12} = 0 \), and finally \( E_{11} = 0 \).

For \( s_1 = 2 \) and \( D = 2 \) the first equation gives \( L_- E_{12} = 0 \), while acting with \( L_- \) on the second gives

\[
L_- L_+ E_{11} = 0. \tag{2.105}
\]

On the other hand, since \( l \) can only assume the same value as \( m \) (\( E_{11} \) is a vector in this case), this relation reduces to an identity, and the equations of motion are not invertible.

For \( s_1 = 2 \) in \( D \geq 3 \), or for \( s_1 \geq 2 \) and any value of \( D \), one must consider the full system (2.104). Eliminating \( E_{11} \) via the first equation and substituting in the second gives

\[
(D + 2s_1 - 4)(D + s_1 - 5) E_{12} = 0, \tag{2.106}
\]

that always implies \( E_{12} = 0 \), but for three special cases. These are \( s_1 = 4 \) in \( D = 1 \), \( s_1 = 3 \) in \( D = 2 \) and \( s_1 = 2 \) in \( D = 3 \), where the value of \( E_{12} \) is left undetermined by the equations of motion.

For \( s_1 > 2 \) and \( s_2 = 2 \), still in \( D = 2 \), the system becomes

\[
(2s_1 - 4)E_{11} + 2L_- E_{12} = 0, \\
s_1 E_{12} + L_+ E_{11} + L_- E_{22} = 0, \tag{2.108}
\]

so that one can still conclude that \( E_{12} \) is irreducible, and thus only contains the component with \( l = m \). On the other hand, in this case one can determine \( E_{11} \) and \( E_{22} \). In fact, applying \( L_+ \) to the first equation gives

\[
L_+ E_{11} = -E_{12}, \tag{2.109}
\]

\( s_2 \geq 2 \): In this case one must consider the full set of three equations, but for \( s_1 = 2, s_2 = 2 \) in \( D = 2 \) eq. (2.96) simplifies to

\[
\begin{align*}
2L_- E_{12} &= 0, \\
2E_{12} + L_+ E_{11} + L_- E_{22} &= 0, \\
2L_+ E_{12} &= 0,
\end{align*} \tag{2.107}
\]

so that \( E_{12} \) only contains its \( \{1,1\} \) irreducible component. Since in this case \( E_{11} \) and \( E_{22} \) are symmetric, one can also conclude that \( E_{12} = 0 \). On the other hand, the same equation does not determine the other variables, \( E_{11} \) and \( E_{22} \), but only provides a linear relation between them, and thus the solution is not determined.

For \( s_1 > 2 \) and \( s_2 = 2 \), still in \( D = 2 \), the system becomes

\[
\begin{align*}
(2s_1 - 4)E_{11} + 2L_- E_{12} &= 0, \\
s_1 E_{12} + L_+ E_{11} + L_- E_{22} &= 0, \\
2L_+ E_{12} &= 0,
\end{align*} \tag{2.108}
\]

so that one can still conclude that \( E_{12} \) is irreducible, and thus only contains the component with \( l = m \). On the other hand, in this case one can determine \( E_{11} \) and \( E_{22} \). In fact, applying \( L_+ \) to the first equation gives

\[
L_+ E_{11} = -E_{12}, \tag{2.109}
\]
that upon substitution in the second equation provides a relation between $L_\perp E_{22}$ and $E_{12}$. Applying $L_\perp$ to this relation one finally obtains

$$2 L_3 E_{22} + L_\perp L_+ E_{22} = 0.$$  \hfill (2.110)

Since $E_{22}$ is a symmetric tensor of rank $s_1 - 2$ the last equation reads

$$2 s_1 E_{22} = 0,$$  \hfill (2.111)

so that all $E_{ij}$ tensors actually vanish.

Finally, for $s_1 \geq 2$ and $s_2 \geq 2$ in $D \neq 2$, and for $s_1 > 2$ and $s_2 > 2$ in $D = 2$, one must consider the full system (2.96). It is then convenient to solve it directly by substitution, letting

$$E_{11} = -\frac{2}{D + 2 s_1 - 6} L_\perp E_{12},$$

$$E_{22} = -\frac{2}{D + 2 s_2 - 6} L_+ E_{12},$$  \hfill (2.112)

and thus reducing the second of (2.96) to a set of relations for the irreducible components of $E_{12}$ via the decomposition (2.97),

$$4 \left( s_1 + s_2 - n + \frac{D}{2} - 3 \right) \left( 2 - n - \frac{D}{2} \right) (E_{12})^{(s_1 + s_2 - n - 2, n)} = 0,$$  \hfill (2.113)

where

$$n = 0, \ldots, s_2 - 1.$$  \hfill (2.114)

As a first observation, eq. (2.113) implies that non-trivial solutions of (2.96) do not exist in odd dimensions, $D = 2k + 1$. Furthermore, neither of the two equations

$$s_1 + s_2 - n + \frac{D}{2} - 3 = 0,$$

$$2 - n - \frac{D}{2} = 0,$$  \hfill (2.115)

can be satisfied for $D \geq 6$. In $D = 4$ the second equation gives $n = 0$, that corresponds to the symmetric component of $E_{12}$, while in $D = 2$ there are non-trivial solutions for $n = 1$, that selects the $\{s_1 + s_2 - 3, 1\}$ component of $E_{12}$. In both cases, because of (2.112), the available components of $E_{11}$ and $E_{22}$ are the same as those of $E_{12}$.

It should be appreciated that, in all cases where the system (2.96) sets to zero the $E_{ij}$ tensors, it is still possible that the equations of motion be not invertible at the level of the double trace. At two families the relevant equations to this effect are given in (2.83), and in order to perform this analysis it is convenient to split them into their three projections in the family indices: the $\{2, 2\}$, the $\{4\}$ and the $\{3, 1\}$. In particular, the $\{2, 2\}$ projection only contains $A''_{ij, kl}$ and reads

$$(D - 3) A''_{ij, kl} + \frac{1}{2} \left[ S^m (i A''_{j})_{m, kl} + S^m (k A''_{l})_{m, ij} \right] = 0,$$  \hfill (2.116)

while only the tensors $B^{(T)}_{ijkl}$ appear in the $\{4\}$:

$$D B^{(T)}_{ijkl} + \frac{1}{2} S^m (i B^{(T)}_{jkl})_{m} = 0.$$  \hfill (2.117)
Finally, the \( \{3, 1\} \) projection,
\[
S^m (i A^j)_m, kl - S^m (k A^l)_m, ij + 6 \left[ S^m (i B^{(T)} j)_m kl - S^m (k B^{(T)} l)_m ij \right] = 0 , \tag{2.118}
\]
relates the two types of tensors.

Let us now recall that the \( \{2, 2\} \) Young projection defining \( A'' \) implies the relation
\[
A''_{11, 22} + 2 A''_{12, 12} = 0 , \tag{2.119}
\]
otherwise, for a two-family tensor \( \varphi \) of rank \((s_1, s_2)\), (2.116) reduces to the single equation
\[
(D + s_1 + s_2 - 7) A''_{11, 22} = 0 , \tag{2.120}
\]
so that \( A'' \) vanishes in general, the only exceptions being provided by the \((4, 2)\) case in \( D = 1 \), the \((3, 3)\) case in \( D = 1 \), the \((3, 2)\) case in \( D = 2 \) and the \((2, 2)\) case in \( D = 3 \). On the other hand, even in the latter cases, before making definite statements one must take into account the conditions originating from the other two projections in eqs. (2.117) and (2.118).

Since the number of possibilities is relatively small, one can analyze the behavior of the solutions explicitly for the cases of interest. It is thus possible to verify that in the \((3, 2)\) case in \( D = 2 \) the double trace of \( A \) actually vanishes. On the other hand, in \( D = 1 \) the only effective irreducible component of any tensor is the symmetric one, whose equations, as we know, are invertible, while the other possibilities simply do not exist. In conclusion, the only case in which the equations of motion leave the double trace of \( A \) undetermined is the \((2, 2)\) in \( D = 3 \) so that, in particular, the \( D = 1 \) pole in the propagator of a rank-\((4, 2)\) field that we shall find in Section 2.4.3 lacks a true physical meaning.

The relevant results are summarized in Table 1: the first pair of columns lists the reducible \( gl(D)\)-tensors whose traces are left undetermined by the equations of motion. In a similar fashion, the second pair of columns lists the corresponding cases of reducible \( gl(D)\)-tensors for which \( A'' \) has a similar behavior.

| \( \mathcal{A}' \) | \( \mathcal{A}'' \) |
|---|---|
| \( D \) | \( s_1 \) | \( s_2 \) | \( s_1 \) | \( s_2 \) |
| 2 | 2 | 0 | \( \geq 3 \) | \( \geq 3 \) |
| 2 | 1 | 1 | \( \geq 2 \) | \( \geq 2 \) |
| 2 | 2 | 1 | | |
| 2 | 3 | 1 | | |
| 2 | 2 | 2 | | |

Table 1: \( gl(D)\)-reducible fields with Weyl-like symmetries

Indeed, it is possible to be slightly more precise, identifying the irreducible components of \( \varphi \) whose field equations can not be brought directly to the Labastida form. For \( A'' \) this is straightforward, since the only relevant case can be associated to a rank-\((2, 2)\) field, whose
double trace is fully contained in its irreducible $\{2,2\}$ component. In order to obtain the same type of information for $A'$, as a first step one should sort out the irreducible components of $A'$ that are not fixed by the dynamics. To this end, one should notice that its traceless part corresponds to the traceless part of the $E_{ij}$ tensors, whose undetermined irreducible components were already identified in the previous discussion.

For instance, for the $(s_1,s_2)$ case in $D = 4$, with both families containing at least two indices, we saw after eq. (2.115) that the symmetric component of $E_{12}$ was in the kernel of $O$. Therefore, the corresponding irreducible component of $\eta A'$, and then of $A$, might be the symmetric one, or the $\{s_1 + s_2 - 1,1\}$ or the $\{s_1 + s_2 - 2,2\}$. On the other hand, we know that the symmetric case is always invertible in $D = 4$, while an equivalent conclusion can be drawn for the $\{s,1\}$-component, on account of the results presented in Table 1. Hence, the lack of invertibility of the equations of motion must be ascribed to the $\{s_1 + s_2 - 2,2\}$-component. In a similar fashion, when both families contain more than two indices in $D = 2$, the $\{s_1 + s_2 - 3,1\}$ component of $A'$ is not determined. By a similar reasoning, it is then possible to conclude that the undetermined traces are contained in the $\{s_1 + s_2 - 3,3\}$-component, since the other two available ones, $\{s_1 + s_2 - 1,1\}$ and $\{s_1 + s_2 - 1,1\}$, were already excluded by our previous discussion. In the other cases the same type of analysis, together with a more explicit computation concerning $(2,2)$ fields in $D = 2$ that we shall return to later, lead finally to the results collected in Table 2.

| $A'$ | $A''$ |
|------|------|
| $D$  | $s_1$ | $s_2$ | $s_1$ | $s_2$ |
| 2    | 2    | 0    |
| 2    | 2    | 1    |
| 2    | 3    | 1    |
| 2    | $s$  | 3    |
| 3    | 2    | 1    | 2    | 2    |
| 4    | $s$  | 2    |

Table 2: $gl(D)$-irreducible fields with Weyl-like symmetries

We have thus completed the classification of the Lagrangians whose field equations are not directly reducible to the Labastida form, and now we would like to elaborate on their meaning.

According to a theorem that we already recalled at the beginning of this section, all fields appearing in Table 2 would correspond to vanishing irreducible representations of $so(D-2)$, if the Labastida equation (2.66) were to hold. On the other hand, the example of two-dimensional gravity suggests that a novel class of symmetries might emerge in the previous cases, and indeed, the very fact that these do not fix some of the traces of $A$ implies that it is possible to redefine arbitrarily the corresponding quantities.

Moreover, in even closer analogy with two-dimensional gravity, at least for a subclass of the theories of Table 2 the Lagrangian could well be a total derivative. More precisely, referring for simplicity to the constrained case, where the Lagrangians take the simple form

$$\mathcal{L} = \frac{1}{2} \langle \varphi, E_\varphi \rangle,$$

with $E_\varphi$ Labastida’s Einstein-like tensor (2.62), it is simple to see that some theories can be
topological *only if* the tensor defining the equations of motion vanishes identically:

\[ E_\varphi : \mathcal{E}_\varphi + \frac{1}{2} \eta^{ij} \eta^{kl} B_{ijkl} \equiv 0. \]  

(2.122)

Whenever \( A \) does not vanish identically \(^8\), eq. (2.122) can only hold in the cases collected in Table 2, the only ones in which the condition \( E_\varphi = 0 \) does not imply \( A = 0 \). It is then possible to see directly that for irreducible fields of type \( \{1\} \) in \( D = 1 \), \( \{2\} \) and \( \{1,1\} \) in \( D = 2 \) the Einstein-like tensor does vanish identically.

In addressing the remaining possibilities, it will be convenient to treat separately two-column tensors. These types of fields with only two columns possess in fact the peculiar feature, stressed in [19], that *no constraints* arise in their case, for both the gauge field and the gauge parameters, so that in a sense they are closer to spin-2 fields than to ordinary higher-spin ones. In particular, their equations of motion do not contain the \( B_{ijkl} \) tensors, so that the condition that the theory be topological rests solely on the vanishing of the Einstein-like tensor \( \mathcal{E}_\varphi \).

For this class of fields, we can actually go beyond two families and provide a characterization of a wider class of topological theories, observing that the Lagrangians for \( N \)-families, that we shall discuss in Section 3, when restricted to \( \{p,q\} \) two-column fields can be recast in the compact form (see for instance [11] and the first paper in [4])

\[ \mathcal{L} \sim \delta_{[\mu_1 \ldots \mu_{p+q+1}]}^{[\nu_1 \ldots \nu_{p+q+1}]} \partial_{\mu_1} \varphi \mu_2 \nu_1 ; \ldots ; \mu_{q+1} \nu_q ; \mu_{q+2} ; \ldots ; \mu_{p+1} \partial^{\nu_{q+1}} \varphi \mu_{p+2} \nu_{q+2} ; \ldots ; \mu_{p+q+1} \nu_{p+1} ; \nu_{p+2} ; \ldots ; \nu_{p+q+1} , \]  

(2.123)

where we are now using explicit space-time indices, and where the symbol

\[ \delta_{[\mu_1 \ldots \mu_{p+q+1}]}^{[\nu_1 \ldots \nu_{p+q+1}]} \]  

(2.124)

denotes the usual product of antisymmetrized Kronecker delta’s. One can prove that the expression in eq. (2.123) is proportional to the Labastida Lagrangian observing that, because of the antisymmetrizations, it defines a gauge-invariant quadratic polynomial containing \( \Box \varphi \), and recalling that the Labastida Lagrangian is the unique polynomial with these features, up to total derivatives. It is then possible to recognize rather directly that all expressions of the form (2.123) vanish identically whenever the total number of indices belonging to the two families is such that

\[ p + q \leq D . \]  

(2.125)

Summarizing, in a given space-time dimension \( D \) all theories describing irreducible two-column fields whose total number of indices is not larger than \( D \) are topological, provided the representation exist.

In order to complete the discussion, one should also analyze the cases with more than two columns present in Table 2, namely the \( \{s \geq 3, 2\} \) in \( D = 4 \), the \( \{s, 3\} \) and the \( \{3, 1\} \) in \( D = 2 \). We verified that for a \( \{3,1\} \) field the equations of motion do not vanish identically, so that the model is not topological. It would be interesting to come to a definite conclusion also in the other cases.

At any rate, whenever the field equations can not be directly reduced to the Labastida form \( \mathcal{A} = 0 \), we can now show that a wider gauge symmetry emerges. Consequently, the portion of \( \mathcal{A} \)

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\(^8\)This actually happens in particularly degenerate cases, for instance for a vector in \( D = 1 \) or for a two form in \( D = 2 \). However, this behavior is quite rare, as can be seen for instance working in de Donder gauge, so that \( \mathcal{A} \) reduces to \( \Box \varphi \). The point is that only in low-enough dimensions de Donder conditions and the irreducibility properties suffice to force \( \Box \varphi \) to vanish.
that is not set to zero directly by the field equations can be eliminated by a choice of gauge. In order to discuss the origin of these shift symmetries, it is convenient to first perform the partial gauge fixing that removes the $\Phi_i$ compensators. Moreover, one can notice that the equation of motion (2.75) always sets to zero the traceless part of $\mathcal{F}$.

A shift symmetry that does not affect the traceless part of the Fronsdal-Labastida tensor $\mathcal{F}$ can be identified starting from a variation of the gauge field $\varphi$ of the form

$$\delta \varphi = \eta^{ij} \Omega_{ij},$$ (2.126)

that generalizes to the mixed-symmetry case the linearized version of the Weyl symmetry of two-dimensional gravity. The corresponding variation of the Fronsdal-Labastida tensor (2.6) reads

$$\delta \mathcal{F} = \eta^{ij} \mathcal{F}_{ij}(\Omega) + \frac{1}{2} \partial^i \partial^j \left[ (D - 2) \Omega_{ij} + S^k_{(i} \Omega_{j)k} \right],$$ (2.127)

where $\mathcal{F}_{ij}(\Omega)$ is the Fronsdal-Labastida tensor for $\Omega_{ij}$. Hence, if the parameters satisfy the relations

$$(D - 2) \Omega_{ij} + S^k_{(i} \Omega_{j)k} = 0$$ (2.128)

the corresponding shift of the $\mathcal{F}$ tensor is simply

$$\delta \mathcal{F} = \eta^{ij} \mathcal{F}_{ij}(\Omega).$$ (2.129)

The reader will not fail to notice that the condition (2.128) involves precisely the operator $O$ introduced in eq. (2.86) at the beginning of this section. Eq. (2.128) also simplifies the constraint on the $\Omega_{ij}$ parameters induced by the request of gauge invariance for the Labastida constraints (2.17), since on account of eq. (2.128) the condition

$$T_{(ij} T_{kl)} \delta \varphi = T_{(ij} \mathcal{O} [(D - 2)]_{kl}^{mn} \Omega_{mn} + \eta^{mn} T_{(ij} T_{kl)} \Omega_{mn} = 0$$ (2.130)

takes the simple form

$$T_{(ij} T_{kl)} \Omega_{mn} = 0.$$ (2.131)

If eq. (2.128) is satisfied, the Einstein-like tensor (2.62) varies as

$$\delta \mathcal{E}_{\varphi} = -\frac{1}{2} \left\{ \eta^{ij} - \frac{1}{3} Y_{(2,2)} \left( \eta^{ij} \eta^{kl} \right) T_{kl} \right\} \left\{ (D - 2) \mathcal{F}_{ij}(\Omega) + S^k_{(i} \mathcal{F}_{j)}(\Omega) \right\}$$

$$- \frac{1}{2} \eta^{ij} \eta^{kl} T_{ij} \Omega_{kl} + \frac{1}{12} \eta^{ij} \eta^{kl} \eta^{mn} \left( Y_{(2,2)} T_{ij} T_{kl} \right) \Omega_{mn},$$ (2.132)

but the $S^i_j$ operators commute with the Fronsdal-Labastida operator, and therefore eq. (2.128) implies that

$$(D - 2) \mathcal{F}_{ij}(\Omega) + S^k_{(i} \mathcal{F}_{j)}(\Omega) = 0,$$ (2.133)

so that at two-families one is left with

$$\delta \mathcal{E}_{\varphi} = -\frac{1}{2} \eta^{ij} \eta^{kl} Y_{(4)} T_{ij} \Omega_{kl} + \frac{1}{12} \eta^{ij} \eta^{kl} \eta^{mn} Y_{(2,2)} \left( Y_{(2,2)} T_{ij} T_{kl} \right) \Omega_{mn}. $$ (2.134)

Note that this identifies a symmetry, since the remainder can be adsorbed in a redefinition of the $B_{ijkl}$ in the gauge fixed-version of eq. (2.75), that translates into a shift of the $\beta_{ijkl}$ Lagrange multipliers. Equivalently, this variation would not alter the Lagrangian of the constrained theory, simply because these transformations contract to zero in the scalar products against a field $\varphi$ subject to the Labastida constraints (2.17).
We have thus displayed Weyl-like symmetries of the field equations for the models listed in the first two columns of Table 1. By a direct calculation, one can also identify precisely which irreducible components are shifted, thus recovering the results of Table 2. For instance, whenever eq. (2.112) applies, in order to satisfy eq. (2.128) the shift (2.126) must take the form
\[ \delta \varphi = \frac{2}{D + 2s_1 - 6} \eta^{11} L_- \Omega_{12} + 2 \eta^{12} \Omega_{12} - \frac{2}{D + 2s_2 - 6} \eta^{22} L_+ \Omega_{12}, \] (2.135)
with a non trivial doubly traceless \( \Omega_{12} \), that can only exist for the models of Table 1. As we already explained in the preceding pages, eq. (2.135) affects a single \( gl(D) \)-irreducible component of \( \varphi \). We can now see, in fact, that the corresponding \( \delta \varphi \) is an eigenvector of \( L_- L_+ \), as can be recognized taking into account the relations
\[ [L_- L_+, \eta^{11}] = 2 \eta^{12} L_+ , \]
\[ [L_- L_+, \eta^{12}] = 2 \eta^{12} + \eta^{11} L_- + \eta^{22} L_+ , \]
\[ [L_- L_+, \eta^{22}] = 2 \eta^{22} + 2 \eta^{12} L_- . \] (2.136)
Therefore in \( D = 2 \) and, as we have seen in eq. (2.115), with an \( \{ s_1 + s_2 - 3, 1 \} \)-projected parameter \( \Omega_{12} \) one finds
\[ L_- L_+ \delta \varphi = (s_1 - 2)(s_2 - 3) \delta \varphi , \] (2.137)
so that \( \delta \varphi \) is indeed irreducible. Comparing this result with (2.103) then shows that the shift affects only the \( \{ s_1 + s_2 - 3, 3 \} \) component of a rank\((s_1, s_2)\) gauge field \( \varphi \). In a similar fashion, in \( D = 4 \) and with a symmetric parameter \( \Omega_{12} \) one finds
\[ L_- L_+ \delta \varphi = (s_1 - 1)(s_2 - 2) \delta \varphi , \] (2.138)
so that \( \delta \varphi \) can only affect the \( \{ s_1 + s_2 - 2, 2 \} \) component. With a similar procedure it is also possible to show that only the \( \{ 3, 1 \} \) irreducible component of the reducible rank\((2, 2)\) field in \( D = 2 \) presents a shift symmetry, thus concluding the analysis that leads to Table 2.

Finally, the presence in Table 2 of a \( \{ 2, 2 \} \) field in \( D = 3 \) rests on a new gauge symmetry for \( \varphi \). Indeed, one can consider a shift of the form
\[ \delta \varphi = \eta^{ij} \eta^{kl} \Omega_{ij, \ kl} , \] (2.139)
under which the Fronsdal-Labastida tensor transforms as
\[ \delta F = \eta^{ij} \eta^{kl} F_{ij, \ kl} (\Omega) + \eta^{ij} \partial^{k} \partial^{l} [(D - 3) \Omega_{ij, \ kl} + S^{m}_{(k \Omega_{l})m, \ ij}] . \] (2.140)
If the parameters \( \Omega_{ij, \ kl} \) satisfy
\[ (D - 3) \Omega_{ij, \ kl} + S^{m}_{(i \Omega_{j})m, \ kl} = 0 , \] (2.141)
only the double trace of \( F \) is affected, since
\[ \delta F = \eta^{ij} \eta^{kl} F_{ij, \ kl} (\Omega) \] (2.142)
while, as we have already seen in eq. (2.133), eq. (2.141) implies
\[ (D - 3) F_{ij, \ kl} (\Omega) + S^{m}_{(i \mathcal{F}_{j})m, \ kl} (\Omega) = 0 . \] (2.143)
In general, taking into account eq. (2.143), the variation of the Einstein tensor under (2.139) would become

\[ \delta E = -\frac{1}{2} \eta^{ij} \eta^{kl} \eta^{mn} T_{ij} \mathcal{F}_{kl, mn}(\Omega) + \frac{2}{3} \left( Y_{(2,2)} \eta^{ij} \eta^{kl} \right) \eta^{mn} T_{ij} \mathcal{F}_{kl, mn}(\Omega) + \frac{1}{12} \left( Y_{(2,2)} \eta^{ij} \eta^{kl} \right) \eta^{mn} \eta^{pq} T_{ij} T_{kl} \mathcal{F}_{mn, pq}(\Omega), \]

(2.144)

that could be canceled by a suitable shift of the \( B_{ijkl} \) tensors. To this end, one should recall that with only two families the product of three or more \( \eta \)'s can be always rewritten with two of them explicitly symmetrized, as explained in Appendix C. Eq. (2.141) takes a form that is very close to that of eq. (2.83), and actually the two differ only due to the presence of the \( B_{ijkl} \) tensors. In the \( \{2,2\} \) case the latter are actually not present, so that the two equations coincide, and one can conclude that the presence of double traces that are not fully determined reflects the new Weyl-like gauge symmetry of eq. (2.139). Actually, in the \( \{2,2\} \) case the analysis is far simpler, since the \( \Omega_{ij, kl} \) are scalars and thus no traces are available. For this reason, the double-trace constraints are also identically satisfied.

2.4 Examples: reducible tensors of ranks \((s,1)\) and \((4,2)\)

In the previous sections, Lagrangians and field equations were presented in a concise notation capable of encompassing results valid for all types of two-family bosonic fields, independently of the Lorentz group labels that they carry. We thus feel it appropriate to complement the analysis with explicit details on a few types of models that stand out for their relative simplicity and nonetheless can illustrate some key subtleties of the construction. To this end, we shall also return momentarily to the standard notation with space-time indices. Hopefully, this discussion will also make the contents of the other sections more concrete and more accessible for the interested readers. In this spirit, at time we shall take the freedom to repeat and stress, in a more concrete context, some points already made in the previous sections.

2.4.1 Lagrangians and field equations

In order to proceed, it is convenient to resort to a shorthand notation for the possible traces. Here a trace with respect to a pair of indices belonging to the same family will be denoted by a “prime” with a suffix that identifies the family, so that for instance

\[ \eta^{\mu_1 \mu_2} \varphi_{\mu_1 \mu_2 \mu_3 \mu_4; \nu_1 \nu_2} \equiv \varphi^{\mu_3 \mu_4; \nu_1 \nu_2}. \]

(2.145)

Two-family fields, however, admit a different type of trace that brings together two indices belonging to the two different families. This will be denoted by a “hatted” prime, so that for instance

\[ \eta^{\mu_1 \nu_1} \varphi_{\mu_1 \mu_2 \mu_3 \mu_4; \nu_1 \nu_2} \equiv \varphi^{i \mu_2 \mu_3 \mu_4; \nu_1 \nu_2}. \]

(2.146)

In Section 4 we shall explain how to adapt the theory to the case of Young-projected fields, but here we keep restricting our attention to unprojected fields, as in the previous section. Our first class of examples, in particular, concerns unprojected tensors of rank \((s,1)\), so that the gauge fields are of the form

\[ \varphi \equiv \varphi_{\mu_1 \ldots \mu_s; \nu}, \]

(2.147)
while the corresponding gauge transformations read
\[
\delta \varphi = \partial^1 \Lambda_1 + \partial^2 \Lambda_2 \equiv \partial_{\mu_1} \Lambda_1^{(1) \mu_2 \cdots \mu_s; \nu} + \partial_{\nu} \Lambda_2^{(2) \mu_1 \cdots \mu_s}.
\] (2.148)

The \( \Phi_i \) are now the two compensators
\[
\Phi_1 \equiv \Phi^{(1)}_{\mu_1 \cdots \mu_{s-1}; \nu},
\]
\[
\Phi_2 \equiv \Phi^{(2)}_{\mu_1 \cdots \mu_s},
\] (2.149)

that enter the Lagrangians only via their symmetrized traces
\[
\alpha_{111} = T_{11} \Phi_1 \equiv \alpha^{(1)}_{\mu_1 \cdots \mu_{s-3}; \nu},
\]
\[
\alpha_{112} = \frac{1}{3} \left( T_{11} \Phi_2 + 2 T_{12} \Phi_1 \right) \equiv \alpha^{(2)}_{\mu_1 \cdots \mu_{s-2}},
\] (2.150)

and there are finally two Lagrange multipliers,
\[
\beta_{1111} \equiv \beta^{(1)}_{\mu_1 \cdots \mu_{s-4}; \nu},
\]
\[
\beta_{1112} \equiv \beta^{(2)}_{\mu_1 \cdots \mu_{s-3}}.
\] (2.151)

In the conventional space-time notation, the Lagrangians (2.39) for this class of fields read
\[
\mathcal{L} = \frac{1}{2} \varphi^{\mu_1 \cdots \mu_s; \nu} \left( A_{\mu_1 \cdots \mu_s; \nu} - \frac{1}{2} \eta_{\mu_1 \mu_2} A_{\nu \mu_3 \cdots \mu_s; \nu} - \frac{1}{2} \eta_{\nu \mu} A_{\mu_1 \mu_3 \cdots \mu_s} \right)
\]
\[
- \frac{3}{4} \left( \begin{array}{c} s \\ 3 \end{array} \right) \alpha_{\mu_1 \cdots \mu_{s-3}; \nu} \partial^\lambda A_{\nu \mu_1 \cdots \mu_{s-2}; \lambda}
\]
\[
- \frac{1}{4} \left( \begin{array}{c} s \\ 2 \end{array} \right) \alpha^{(2) \mu_1 \cdots \mu_{s-2}} \partial^\lambda \left( A_{\nu \mu_1 \cdots \mu_{s-2}; \lambda} + 2 A_{\nu \lambda \mu_1 \cdots \mu_{s-2}} \right)
\]
\[
+ 3 \left( \begin{array}{c} s \\ 4 \end{array} \right) \beta^{(1) \mu_1 \cdots \mu_{s-4}; \nu} \mathcal{C}^{(1)}_{\mu_1 \cdots \mu_{s-4}; \nu} + 3 \left( \begin{array}{c} s \\ 3 \end{array} \right) \beta^{(2) \mu_1 \cdots \mu_{s-3}} \mathcal{C}^{(2)}_{\mu_1 \cdots \mu_{s-3}},
\] (2.152)

where the two \( \mathcal{C}^{(i)} \) tensors are the space-time manifestations of \( \mathcal{C}_{1111} \) and \( \mathcal{C}_{1112} \):
\[
\mathcal{C}^{(1) \mu_1 \cdots \mu_{s-4}; \nu} = \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} \left( \varphi_{\mu_1 \cdots \mu_{s-4}; \nu} - \partial_{\mu_1} \Phi^{(1) \mu_2 \cdots \mu_s; \nu} - \partial_{\nu} \Phi^{(2) \mu_1 \cdots \mu_s} \right),
\]
\[
\mathcal{C}^{(2) \mu_1 \cdots \mu_{s-3}} = \eta^{\mu_1 \mu_2} \eta^{\mu_3 \nu} \left( \varphi_{\mu_1 \cdots \mu_{s-3}; \nu} - \partial_{\mu_1} \Phi^{(1) \mu_2 \cdots \mu_s; \nu} - \partial_{\nu} \Phi^{(2) \mu_1 \cdots \mu_s} \right).
\] (2.153)

Let us stress that the Lagrangians of eq. (2.152) contain more terms involving compensators or multipliers with respect to their counterparts for the symmetric case proposed in [26, 27]. Still, the Einstein-like tensor only contains simple traces of \( \mathcal{A} \), because in this class of models the two available double traces,
\[
T_{11} T_{11} \mathcal{A} \equiv A_{\nu \mu_1 \cdots \mu_{s-4}; \nu},
\]
\[
T_{11} T_{12} \mathcal{A} \equiv A_{\nu \mu_1 \cdots \mu_{s-3}},
\] (2.154)

are actually symmetrized double traces, that as such do not enter eq. (2.39). In this respect the \((s,1)\) fields are a bit too simple to display all novelties of the mixed symmetry case, but nonetheless, as we shall see shortly, they can convey interesting lessons.

The equations of motion for the two Lagrange multipliers \( \beta^{(1)} \) and \( \beta^{(2)} \) read
\[
\mathcal{C}^{(1) \mu_1 \cdots \mu_{s-4}; \nu} = 0,
\]
\[
\mathcal{C}^{(2) \mu_1 \cdots \mu_{s-3}} = 0,
\] (2.155)
and reduce the $\varphi$ field equation to

$$E_\varphi = A_{\mu_1...\mu_s;\nu} - \frac{1}{2} \eta(\mu_1 \mu_2 A^{\nu}_{...\mu_s};\nu) - \frac{1}{2} \eta\nu(\mu_1 \Lambda^{\phi}_{...\mu_s})$$

$$+ \eta(\mu_1 \nu \eta_{\mu_2 \mu_3} B^{(1)}_{...\mu_s};\nu) + \eta\nu(\mu_1 \eta_{\mu_2 \mu_3} B^{(2)}_{...\mu_s}) = 0,$$

(2.156)

where the two $B^{(i)}$ tensors are gauge invariant completions of the two Lagrange multipliers. We shall refrain from discussing further the field equations for the compensators that, as we have seen in eq. (2.59), simply guarantee the conservation of external currents.

Aside from the previous class of models, it will prove instructive to also discuss in some detail the case of unprojected tensors of rank $(4,2)$,

$$\varphi \equiv \varphi_{\mu_1 \mu_2 \mu_3 \mu_4;\nu_1 \nu_2},$$

(2.157)

since in this relatively simple setting not all double traces are automatically symmetrized. In this case the gauge transformations are

$$\delta \varphi = \delta^1 \Lambda_1 + \delta^2 \Lambda_2 \equiv \delta(\mu_1 \Lambda^{(1)}_{\mu_2 \mu_3 \mu_4;\nu_1 \nu_2} + \delta(\nu_1 \Lambda^{(2)}_{\mu_1 \mu_2 \mu_3 \mu_4;\nu_2}),$$

(2.158)

and there are two compensators, as is the cases for all two-family fields, that in the present example have the following Lorentz structure:

$$\Phi_1 \equiv \Phi^{(1)}_{\mu_1 \mu_2 \mu_3;\nu_1 \nu_2},$$

$$\Phi_2 \equiv \Phi^{(2)}_{\mu_1 \mu_2 \mu_3 \mu_4;\nu}.$$

(2.159)

A general two-family field would allow four distinct symmetrized traces of the $\Phi_i$, and thus four distinct $\alpha_{ijk}$. In this respect, the present example is still a bit degenerate, since one can at most remove two indices from the second family, so that one of the symmetrized traces is impossible and only three $\alpha_{ijk}$ fields exist:

$$\alpha_{111} = T_{11} \Phi_1 \equiv \alpha^{(1)}_{\mu_1;\nu_1 \nu_2},$$

$$\alpha_{112} = \frac{1}{3} T_{11} \Phi_2 + 2 T_{12} \Phi_1 \equiv \alpha^{(2)}_{\mu_1 \mu_2;\nu},$$

$$\alpha_{122} = \frac{1}{3} (T_{22} \Phi_1 + 2 T_{12} \Phi_2) \equiv \alpha^{(3)}_{\mu_1 \mu_2 \mu_3}.$$

(2.160)

In a similar fashion, the present $(4,2)$ example only allows three symmetrized double traces, and thus the three Labastida constraints

$$T_{11} T_{11} \varphi \equiv \varphi^{\mu \nu \rho \nu}_{\mu \nu \rho \nu} = 0,$$

$$T_{11} T_{12} \varphi \equiv \varphi^{\mu \nu i}_{\mu \nu i} = 0,$$

$$T_{12} T_{12} \varphi \equiv \varphi^{\mu \nu i}_{\mu \nu i} \mu_{1 \mu_2} + 2 \varphi^{\mu \nu i}_{\mu_1 \mu_2} = 0,$$

(2.161)

rather than the five available for generic two-family fields, but the interesting novelty with respect to the previous class of models is the existence of a single non-symmetrized double trace, that therefore plays an interesting role both in the Lagrangian and in the corresponding field equations. This non-symmetrized double trace bears family indices subject to a $\{2,2\}$ projection, is present for all generic two-family models and in the present example reads

$$T_{11} T_{22} - T_{12} T_{12} \varphi \equiv \varphi^{\mu \nu i}_{\mu_1 \mu_2} - \varphi^{\mu \nu i}_{\mu_1 \mu_2}.$$

(2.162)
The structure of the Labastida constraints for this model clearly reflects itself in the constraint tensors \( \mathcal{C} \) and in the Lagrange multipliers \( \beta \) that couple to them:

\[
\begin{align*}
\beta_{1111} & \equiv \beta^{(1)}_{\nu_1^2} , \\
\beta_{1112} & \equiv \beta^{(2)}_{\mu;\nu} , \\
\beta_{1122} & \equiv \beta^{(3)}_{\mu_1\mu_2} .
\end{align*}
\] (2.163)

The Lagrangian for unprojected fields of rank \((4,2)\) thus reads

\[
\mathcal{L} = \frac{1}{2} \varphi^{\mu_1\ldots\mu_4;\nu_1\nu_2} (\mathcal{E}_\varphi)_{\mu_1\ldots\mu_4;\nu_1\nu_2}
- 3 \alpha^{(1)}_{\mu;\nu_1\nu_2} \partial^\lambda \left\{ A^{\nu_1}_{\mu_1\mu_2;\nu_1\nu_2} - \frac{1}{6} \eta_{\nu_1\nu_2} \left( A^{\nu_1}_{\mu_1\mu_2;\nu_1\nu_2} - A^{\nu_1}_{\mu_1\mu_2;\nu_1\nu_2} \right) \right\}
- 3 \alpha^{(2)}_{\mu_1\mu_2;\nu} \partial^\lambda \left\{ (A^{\nu_1}_{\mu_1\mu_2;\nu} + 2 A^{\lambda}_{\mu_1\mu_2;\nu}) + \frac{1}{6} \eta_{\nu_1\nu_2} \left( A^{\nu_1}_{\mu_1\mu_2;\nu} - A^{\nu_1}_{\mu_1\mu_2;\nu} \right) \right\}
- \alpha^{(3)}_{\mu_1\mu_2\nu_1\nu_2} \partial^\lambda \left\{ (A^{\nu_1}_{\mu_1\mu_2;\nu_1\nu_2} + 2 A^{\lambda}_{\mu_1\mu_2;\nu_1\nu_2}) - \frac{1}{6} \eta_{\nu_1\nu_2} \left( A^{\nu_1}_{\mu_1\mu_2;\nu_1\nu_2} - A^{\nu_1}_{\mu_1\mu_2;\nu_1\nu_2} \right) \right\}
+ 3 \beta^{(1)}_{\nu_1\nu_2} C^{(1)}_{\nu_1\nu_2} + 24 \beta^{(2)}_{\mu;\nu} C^{(2)}_{\mu;\nu} + 18 \beta^{(3)}_{\mu_1\mu_2} C^{(3)}_{\mu_1\mu_2},
\] (2.164)

with

\[
\mathcal{E}_\varphi = A^{\mu_1\ldots\mu_4;\nu_1\nu_2} - \frac{1}{2} \eta_{\mu_1\mu_2} A^{\mu_1\mu_2;\nu_1\nu_2} - \frac{1}{2} \eta_{\nu_1\nu_2} A^{\mu_1\mu_2;\nu_1\nu_2} - \frac{1}{2} \eta_{\nu_1\nu_2} A^{\mu_1\mu_2;\nu_1\nu_2}
+ \frac{1}{18} \left( 2 \eta_{\nu_1\nu_2} \eta_{\mu_1\mu_2} - \eta_{\nu_1\nu_2} \eta_{\mu_1\mu_2} \right) \left( A^{\mu_1\mu_2;\nu_1\nu_2} - A^{\nu_1}_{\mu_1\mu_2;\nu_1\nu_2} \right) .
\] (2.165)

In the following we shall often use the shorthand notation

\[
a_{\mu_1\mu_2} \equiv A^{\mu_1\mu_2;\nu_1\nu_2} - A^{\nu_1}_{\mu_1\mu_2}
\] (2.166)

to denote this \((2,2)\)-projected trace of \( A \) that appears in \( \mathcal{E}_\varphi \). The equations for the Lagrange multipliers are simply

\[
\begin{align*}
C^{(1)}_{\nu_1\nu_2} &= 0 , \\
C^{(2)}_{\mu;\nu} &= 0 , \\
C^{(3)}_{\mu_1\mu_2} &= 0 ,
\end{align*}
\] (2.167)

and after enforcing them the equation for the gauge field \( \varphi^{\mu_1\ldots\mu_4;\nu_1\nu_2} \) becomes

\[
E_\varphi = \mathcal{E}_\varphi + \eta_{\mu_1\mu_2} \eta_{\mu_3\mu_4} B^{(1)}_{\nu_1\nu_2} + \eta_{\nu_1\nu_2} \eta_{\mu_1\mu_2} B^{(2)}_{\mu_1\mu_2} + \left( \eta_{\nu_1\nu_2} \eta_{\mu_1\mu_2} + \eta_{\nu_1\nu_2} \eta_{\mu_1\mu_2} \right) B^{(3)}_{\mu_1\mu_2} = 0 .
\] (2.168)

In order to appreciate the relative simplicity of the previous results, it is worth comparing eq. (2.168) with the corresponding Labastida equation. This, as we have stressed, differs from the result quoted in [20], since it must include some higher traces that are needed in order that it satisfy the same double trace constraints as the Labastida gauge field. The subtlety draws
its origin from the presence of the non-vanishing double trace \( (2.166) \), that also brings about non-vanishing symmetrized double traces of the Einstein tensor,

\[
\begin{align*}
\mathcal{E}'_{\mu'}^{\mu} v_{\nu_1} v_{\nu_2} &= -\frac{4}{9} a_{\nu_1 \nu_2}, \\
\mathcal{E}'_{\mu;\nu}^{\mu} &= \frac{1}{9} a_{\mu\nu}, \\
\mathcal{E}'_{\mu'}^{\mu'} v_{\mu_1} v_{\mu_2} + 2 \mathcal{E}'_{\mu}^{\mu} v_{\mu_1} v_{\mu_2} &= -\frac{2}{9} a_{\mu_1 \mu_2},
\end{align*}
\]  

where the \( \{2, 2\} \) projected trace \( a_{\mu_1 \mu_2} \) can appear on account of eq. (2.63).

For completeness, we have presented these results for the unconstrained theory, but at any rate their Labastida counterparts can be simply recovered from the previous expressions removing the \( \Phi_i \). As we anticipated, the proper Lagrangian Labastida equation is indeed more complicated than one would naively expect, and reads

\[
E_\phi = \mathcal{E}_\phi + \frac{4}{3(D^2 - 8)} \eta(\mu_1 \mu_2 \eta_{\mu_3 \mu_4}) a_{\nu_1 \nu_2} - \frac{1}{3(D^2 - 8)} \eta(\nu_1 | (\mu_1 \eta_{\mu_2 \mu_3} a_{\mu_4}) | \nu_2) \\
+ \frac{2}{9(D^2 - 8)} \left( \eta_{\nu_1 \nu_2} \eta(\mu_1 \mu_2) + \eta_{\nu_1} (\mu_1 \eta_{\mu_2} | \nu_2) \right) a_{| \mu_3 \mu_4} = 0,
\]

where the coefficients depend explicitly on the space-time dimension. This result should be compared with the relatively simple and universal form of eq. (2.168). Let us stress that for higher-rank fields the relative complication of the constrained theory would be even more sizable, because all possible combinations of the non-vanishing double trace with reshuffled indices would contribute.

We can now illustrate explicitly the on-shell reduction of these types of fields. In the ensuing discussion, we shall often make use of the shorthand notation

\[ \rho_k \equiv D + k, \]

where \( D \) denotes the space-time dimension.

### 2.4.2 On-shell reduction

As we have stressed, the constraints (2.155) force the two types of double traces of the kinetic tensor \( A_{\mu_1 \ldots \mu_s;\nu} \) of a generic \((s, 1)\) field \( \varphi_{\mu_1 \ldots \mu_s;\nu} \) to vanish, so that these fields entail no new subtleties, in this respect, when compared to their fully symmetric counterparts. They are thus particularly convenient to illustrate how the \( B_{ijkl} \) fields are not fully determined, and how nonetheless this peculiarity does not affect the field equation for \( \varphi \). We can conveniently begin by describing the reduction procedure for a relatively simple rank-six tensor of \((5,1)\) type, before moving to analyze generic \((s,1)\) fields.

After enforcing the constraints (2.155), the equation of motion (2.156) for a \((5,1)\)-field \( \varphi_{\mu_1 \ldots \mu_5;\nu} \) becomes

\[
E_\varphi = A_{\mu_1 \ldots \mu_5;\nu} - \frac{1}{2} \eta(\mu_1 \mu_2 A'_{\mu_3 \mu_4 \mu_5};\nu) - \frac{1}{2} \eta_{\mu_1} A'_{\mu_2 \mu_3 \mu_4 \mu_5} \\
+ \eta(\mu_1 \mu_2) B^{(1)}_{\mu_3 \mu_4 \mu_5;\nu} + \eta(\mu_1 \mu_2 \mu_3) B^{(2)}_{\mu_4 \mu_5} = 0.
\]
In this relatively simple model, and actually in the whole family of \((s,1)\) examples that we shall shortly discuss, one can compute explicitly all possible traces of \((2.172)\) in order to extract the conditions on the \(B_{ijkl}\) tensors, that in this case are the pair \(B^{(1)}_{\mu;\nu}\) and \(B^{(2)}_{\mu_1\mu_2}\), as we saw in Section 2.4.1. Taking the two independent double traces of \((2.172)\) yields

\[
E_{\varphi'}^{\prime}(\mu;\nu) : \rho_2 B^{(1)}_{\mu;\nu} + 4 B^{(2)}_{\mu\nu} + 2 \eta_{\mu\nu} B^{(2)'}_{\mu} = 0 ,
\]

\[
E_{\varphi'}^{\prime}(\mu_1;\mu_2) : B^{(1)}_{(\mu_1;\mu_2)} + \rho_4 B^{(2)}_{\mu_1\mu_2} + \eta_{\mu_1\mu_2} \left( B^{(1)} + \frac{B^{(2)}}{2} \right) = 0 ,
\]

while there is only one possible triple trace,

\[
E_{\varphi'}^{\prime}(\mu_1;\mu_2) : B^{(1)} + 2 B^{(2)'}_{\mu} = 0 .
\]

Clearly, one can at most relate the traces of \(B^{(1)}\) and \(B^{(2)}\) to one another, so that in terms of \(B^{(2)'}_{\mu}\) the system \((2.173)\) finally becomes

\[
E_{\varphi'}^{\prime}(\mu;\nu) : \frac{\rho_2}{2} B^{(1)}_{\mu;\nu} + 4 B^{(2)}_{\mu\nu} + 2 \eta_{\mu\nu} B^{(2)'}_{\mu} = 0 ,
\]

\[
E_{\varphi'}^{\prime}(\mu_1;\mu_2) : B^{(1)}_{(\mu_1;\mu_2)} = 0 ,
\]

\[
E_{\varphi'}^{\prime}(\mu_1;\mu_2) : B^{(1)}_{(\mu_1;\mu_2)} + \rho_4 B^{(2)}_{\mu_1\mu_2} - \eta_{\mu_1\mu_2} B^{(2)'}_{\mu} = 0 .
\]

The solution of eqs. \((2.175)\) is therefore

\[
B^{(1)}_{\mu;\nu} = -\frac{2}{\rho_0} \eta_{\mu\nu} B^{(2)'}_{\mu} ,
\]

\[
B^{(2)}_{\mu_1\mu_2} = \frac{1}{\rho_0} \eta_{\mu_1\mu_2} B^{(2)'}_{\mu} .
\]

These results display quite clearly two features of the reduction process that are foreign to the symmetric case:

- in general, the field equations do not force all components of \(B_{ijkl}\) to vanish. In this case, for instance, only the traceless parts of \(B^{(1)}\) and \(B^{(2)}\) are forced to vanish, while their traces are simply subject to a linear relation. As a result, one can take, say, \(B^{(2)'}_{\mu}\) as an independent variable, but as we shall see shortly this undetermined quantity does not enter the resulting equation of motion for \(\varphi\);

- eq. \((2.173)\) allows two independent projections that are to be treated separately. In general, there are as many independent conditions as the irreducible projections of all traces of the original \(\varphi\)-equation. This is an explicit example of the decomposition \((2.97)\).

It is instructive to see explicitly how the undetermined trace in \((2.175)\) disappears from the final result. Substituting eqs. \((2.176)\) in eq. \((2.172)\) one indeed obtains for the reduced \(\varphi\)-equation

\[
E_{\varphi} \equiv A_{\mu_1\mu_23\mu_4\mu_5;\nu} - \frac{1}{2} \eta_{\mu_1\mu_2} A^{(\mu}_{3\mu_4\mu_5;\nu}) - \frac{1}{2} \eta_{\nu(\mu_1} A^{\prime}_{\mu_23\mu_4\mu_5}) - \frac{2}{\rho_0} \eta_{\mu_1\mu_2 \mu_3 \mu_4} B^{(2)}_{\mu} + \frac{2}{\rho_0} \eta_{\nu(\mu_1 \eta_{\mu_2\mu_3 \mu_4 \mu_5})} B^{(2)'}_{\mu} = 0 ,
\]

where the reader should notice that an overall factor 2 was generated in the second \(B\) term, precisely as needed in order that \(B^{(2)}\) disappear from the resulting expression. This overall
factor draws its origin from the standard symmetrization, that overcounts by a factor two the minimal number of permutations needed to define a symmetric pair of $\eta$ tensors. In other words, one is facing a manifestation of the rule according to which $\eta \eta = 2 \eta^2$, that was spelled out for the symmetric case in [27]. Having eliminated $B_{ijkl}$ from the $\varphi$ equation, the reduction of (2.177) to $A_{\mu_1, ..., \mu_s; \nu} = 0$ can finally proceed as in the constrained case.

This result can be understood comparing the overall number of possible projections for the successive traces of the $\varphi$ field equation with the number of unknown tensorial quantities arising from $B_{ijkl}$. One could also count, more conveniently, the available traces of the various quantities, that here are simply bound to allow two independent projections if the $\nu$ index is still present and one otherwise. In the previous example, for instance, we had to face a single triple trace involving two distinct unknowns, the traces of $B^{(1)}$ and $B^{(2)}$. In addition, we faced two distinct double traces involving the two distinct tensors $B^{(1)}$ and $B^{(2)}$. We thus ended up with one undetermined quantity, say $B^{(2)/\nu}$, that however did not affect the reduced equation for the gauge field $\varphi_{\mu_1, ..., \mu_s; \nu}$.

Moving on to the general case, one can note that an $(s, 1)$ field $\varphi_{\mu_1, ..., \mu_s; \nu}$ admits two independent types of $k$-th traces, for $k \leq \lceil \frac{s+1}{2} \rceil$. In the first, here denoted by $\varphi^{[k]}$, all traced indices belong to the first set, so that the $\nu$ index is left untouched, while in the second, here denoted by $\varphi^{[k-1]}$, one of the traces involves the $\nu$ index of the second family. It is also convenient to distinguish two cases when analyzing the equations starting from the highest trace, the $n$-th one, with $n = \lceil \frac{s+1}{2} \rceil$. If $s$ is odd the $n$-th trace is unique and yields a scalar equation, but as in the $\{5, 1\}$ examples this involves two unknowns, the two highest traces $B^{(1)}[n-3]/i$ and $B^{(2)}[n-2]$ of $B^{(1)}$ and $B^{(2)}$. On the other hand, if $s$ is even there are two distinct $n$-th traces leading to two distinct vector equations, but these involve three independent unknowns, $B^{(1)}[n-3]/i$, $B^{(2)}[n-2]$ and $B^{(2)}[n-2]$. Moving backwards to the lower traces of the $\varphi$ field equation, at every step the new unknowns, three lower traces, $B^{(1)}[k-3]/i$, $B^{(1)}[k-2]$ and $B^{(2)}[k-2]$, exceed by one unit the available traces of the $\varphi$ field equation, $E_{\varphi}^{[k-1]/i}$ and $E_{\varphi}^{[k]/i}$. The last step, however, is slightly different, since the two double traces of the $\varphi$ field equation match at this order the two new entries, $B^{(1)}$ and $B^{(2)}$. All in all, for an $(s, 1)$ field one thus finds an excess of $\lceil \frac{s+1}{2} \rceil$ unknowns. Still, in the general case the $\varphi$ field equation can be cast in a form that does not involve any of these undetermined quantities. This can be seen from the explicit solution for $B^{(1)}$ and $B^{(2)}$, that can be obtained in closed form for generic $\{s, 1\}$ fields. For brevity, here we limit ourselves to quoting the final results,

$$
B^{(1)}_{\mu_1, ..., \mu_{s-4}; \nu} = 2 \sum_{l=0}^{n-3} \frac{(-1)^l}{l+1} \prod_{i=0}^{l} \frac{1}{\rho(2n-5-i)} \eta^l \eta_{\mu_1, \mu_2, ..., \mu_{l+1}} B^{(2)}_{[l+1] \mu_5, ..., \mu_{s-4}}, \tag{2.178}
$$

$$
B^{(2)}_{\mu_1, ..., \mu_{s-3}} = \sum_{l=0}^{n-3} \frac{(-1)^l}{l+1} \prod_{i=0}^{l} \frac{\eta^{l+1} \eta_{\mu_1, ..., \mu_{l+1}} B^{(2)}_{[l+1] \mu_{l+2}, ..., \mu_{s-3}}}{\rho(2n-5-i)},
$$

where $\eta^l$ denotes a symmetrized product of $l$ Minkowski metrics with the minimal number of terms, consistently with the notation of Appendix A and of [27]. Substituting eqs. (2.178) in the $\varphi$ equation of motion finally gives

$$
0 = A_{\mu_1, ..., \mu_s; \nu} - \frac{1}{2} \eta_{\mu_1, \mu_2} A^i_{\nu, \mu_3, ..., \mu_s; \nu} - \frac{1}{2} \eta_{\nu} (\mu_1 A^i_{\mu_2, ..., \mu_s}) \tag{2.179}
$$

$$
+ \sum_{l=0}^{n-3} \frac{(-1)^l}{l+1} \left\{ \frac{2}{l+1} \left( \frac{l+2}{2} \right) - (l+2) \right\} \eta^l \eta_{\mu_1, \mu_2, ..., \mu_{l+2}, \mu_5} B^{(2)}_{[l+1] \mu_5, ..., \mu_{s-3}}.
$$
As we have stressed, the theory does not fully determine the traces of the $B^{(i)}$ tensors, but the reader should appreciate that they all cancel out manifestly in this expression. Hence, in this class of models, this novel phenomenon does not hamper the reduction of the $\varphi$ field equation, that after all ought to be possible for physical reasons.

Now we would like to elaborate again on the origin of this peculiar feature, that actually presents itself for all mixed-symmetry fields and whose origin can be traced to the symmetry principle discussed in Section 2.1. Indeed, as we have seen, aside from some degenerate cases of low ranks, the Lagrangians of mixed-symmetry fields possess a symmetry allowing shifts of the $\varphi$ field equation, that after all ought to be possible for physical reasons.

The symmetry (2.180) has a simple origin: as we have stressed, while nicely covariant with respect to the family structure, the Labastida constraints are not independent, since higher traces of two different constraints can give rise to the same condition. For instance, referring to the previous example one can see that the two tensors $C^{(1)}$ and $C^{(2)}$ are not fully independent, since indeed

$$C^{(1)}_{\mu_1...\mu_5} = C^{(2)}_{\nu}\mu_1...\mu_5.$$  

(2.181)

In this respect, the symmetry (2.180) is precisely as needed to remove all left-over components of $B^{(1)}$ and $B^{(2)}$. Indeed, specializing eq. (2.180) to the case of rank– (5, 1) $\varphi$ fields gives

$$\delta B_{1111}^{(1)} = 2\eta^{12} L_{1111,12},$$  

(2.182)

$$\delta B_{1112}^{(2)} = \eta^{11} L_{1112,11},$$

where in this case the $\{5, 1\}$ projection requires that the two gauge parameters above satisfy the relation

$$L_{1111,12} = -2 L_{1112,11}.$$  

(2.183)

Translating these results in the conventional space-time notation then yields

$$\delta B^{(1)}_{\mu;\nu} = -2\eta_{\mu\nu} L,$$

$$\delta B^{(2)}_{\mu\nu\mu_2} = \eta_{\mu_1\mu_2} L,$$

(2.184)

where $L$ denotes the single scalar parameter left over by eq. (2.183). Clearly, these transformations allow to eliminate the undetermined quantities prior to the substitution in eq. (2.172).

For more general tensors, the reduction of the field equations to the Labastida form proceeds along similar lines, albeit with higher technical complications that are inevitable when working in an explicit space-time notation. It is instructive to dwell upon some of these, and therefore we now turn to illustrate the case of (4, 2) tensors, where some non-vanishing double traces of $A$
are present even after enforcing the $C_{ijkl}$ constraints. Our aim will be, again, to reach the non-Lagrangian Labastida equation $\mathcal{F} = 0$ starting from (2.168). However, an intermediate stage of our analysis will be also of some interest, since it will recover the proper Lagrangian form of the Labastida equation (2.170), that differs from the result presented in [20] by the addition of some terms needed to project the Einstein-like tensor in such a way that its symmetrized double traces vanish identically. The analysis of this example terminates with the derivation of the current-current exchange for this type of field, an interesting application of the formalism that we refrained from presenting for the previous class of fields for brevity.

In order to accomplish our task, we can proceed by direct substitutions, as in the $(s, 1)$ cases. Starting from (2.168), let us therefore begin by considering the three symmetrized double traces

$$
E_\phi^{\mu\nu} = -\frac{4}{9} a_{\nu_1\nu_2} + \rho_0 \rho_2 B^{(1)}_{\nu_1\nu_2} + 4 \rho_2 B^{(2)}_{(\nu_1\nu_2)} + 8 B^{(3)}_{\nu_1\nu_2} + 2 \rho_4 \eta_{\nu_1\nu_2} B^{(3)\rho\nu} = 0,
$$

$$
E_\phi^{\rho\nu} = \frac{1}{9} a_{\mu\nu} + \rho_2 B^{(1)}_{\mu\nu} + \rho_2 \rho_4 B^{(2)}_{\mu;\nu} + 2 B^{(2)}_{(\mu;\nu)} + (3 \rho_3 + 1) B^{(3)\mu\nu}
+ \rho_4 \eta_{\mu\nu} B^{(2)\rho} + \rho_4 \eta_{\mu\nu} B^{(3)\rho\nu} = 0,
$$

$$
E_\phi^{\rho\nu} + 2 E_\phi^{\rho\nu} = -\frac{2}{9} a_{\mu_1\mu_2} + 4 B^{(1)}_{\mu_1\mu_2} + 2 (3 \rho_3 + 1) B^{(2)}_{(\mu_1;\mu_2)}
+ (3 \rho_3^2 + 1) B^{(3)}_{\mu_1\mu_2} + \rho_4 \eta_{\mu_1\mu_2} \left( B^{(1)\rho\nu} + 4 B^{(2)\rho} + B^{(3)\rho\nu} \right) = 0,
$$

and the two triple traces of (2.168),

$$
E_\phi^{\mu_1\mu_2\nu_3} = \rho_0 \rho_2 B^{(1)}_{\nu_3} + 8 \rho_2 B^{(2)}_{\rho} + 2 \rho_2^2 B^{(3)}_{\rho\nu} = 0,
$$

$$
E_\phi^{\rho_1\rho_2\nu_3} = \rho_2 B^{(1)}_{\nu_3} + 2 \rho_2 \rho_3 B^{(2)}_{\rho} + \rho_2 \rho_5 B^{(3)}_{\rho\nu} = 0.
$$

As in eq. (2.166), $a_{\mu_1\mu_2}$ denotes the $\{2,2\}$-projected double trace of $A$, that as we stressed is not subject to any constraints. Notice that the last two equations suffice to express, for instance, the traces of $B^{(1)}$ and $B^{(2)}$ in terms of the third one, as

$$
B^{(1)\rho} = -2 B^{(3)\rho},
$$

$$
B^{(2)\rho} = -\frac{1}{2} B^{(3)\rho}.
$$

Substituting this result in the system (2.185) it is simple to see that, again, only the traceless parts of the $B^{(i)}$ tensors survive. Aside from $B^{(2)}$, whose antisymmetric part is set to zero by the equations, all tensors involved are actually symmetric. As a result, one is finally led to a system for the symmetric parts, consisting of four equations for five unknowns:

$$
\rho_0 \rho_2 B^{(1)}_{\nu_1\nu_2} + 4 \rho_2 B^{(2)}_{(\nu_1\nu_2)} + 8 B^{(3)}_{\nu_1\nu_2} = \frac{4}{9} a_{\nu_1\nu_2},
$$

$$
\rho_2 B^{(1)\mu\nu} + \frac{1}{2} (\rho_2 \rho_4 + 4) B^{(2)}_{(\mu;\nu)} + (3 \rho_3 + 1) B^{(3)\mu\nu} = -\frac{1}{9} a_{\mu\nu},
$$

$$
4 B^{(1)}_{\mu_1\mu_2} + 2 (3 \rho_3 + 1) B^{(2)}_{(\mu_1;\mu_2)} + (3 \rho_3^2 + 1) B^{(3)}_{\mu_1\mu_2} = \frac{2}{9} a_{\mu_1\mu_2}.
$$

These equations can be used to relate the traceless part of each of the $B^{(i)}$ to the non-vanishing
double trace tensor $a$, according to

\[ B^{(1)}_{\nu_1 \nu_2} = \frac{4}{3 (3D^2 - 8)} a_{\nu_1 \nu_2}, \]

\[ B^{(2)}_{\mu; \nu} = -\frac{2}{3 (3D^2 - 8)} a_{\mu \nu}, \quad (2.189) \]

\[ B^{(3)}_{\mu_1 \mu_2} = \frac{2}{9 (3D^2 - 8)} a_{\mu_1 \mu_2}. \]

Substituting in the unconstrained $\varphi$ field equation (2.168) one reaches the correct Lagrangian form of the Labastida equation (2.170). At this point one can also set to zero the left-over tensor $\varphi$ account of eqs. (2.189), and the original field equation:

\[ \{ \text{so far}, \text{which is also the real novelty of this example}, \text{the double trace tensor} \}_{\text{in this case the parameters}} \text{that in the conventional space-time notation read} \]

\[ L \text{ in this case the parameters so as to display an explicit realization of the phenomenon discussed in Section 2.2. Indeed, in this case the parameters $L_{ijkl, mn}$ of eqs. (2.180) are subject to the two constraints} \]

\[ Y_{(6)} L_{ijkl, mn} \sim L_{(ijkl, mn)} = 0, \quad \]

\[ Y_{(4,2)} L_{ijkl, mn} \sim 12 L_{ijkl, mn} - 3 \left( L_{m(ijk, l)n} + L_{n(ijk, l)m} \right) + 2 L_{mn(ij, kl)} = 0, \quad (2.192) \]

That in our (4,2) case reduce to

\[ L_{1111, 22} + 8 L_{1112, 12} + 6 L_{1122, 11} = 0, \]

\[ L_{1111, 22} - 2 L_{1112, 12} + L_{1122, 11} = 0. \quad (2.193) \]

Making use of the previous relations to express everything in terms of $L_{1111, 22}$, one is finally led to the transformations

\[ \delta B_{1111} = \eta^{22} L_{1111, 22}, \]

\[ \delta B_{1112} = \frac{1}{2} \eta^{12} L_{1111, 22}, \]

\[ \delta B_{1122} = -\frac{1}{2} \eta^{11} L_{1111, 22}, \quad (2.194) \]

that in the conventional space-time notation read

\[ \delta B^{(1)}_{\nu_1 \nu_2} = \eta_{\nu_1 \nu_2} L, \]

\[ \delta B^{(2)}_{\mu \nu} = \frac{1}{4} \eta_{\mu \nu} L, \]

\[ \delta B^{(3)}_{\mu_1 \mu_2} = -\frac{1}{2} \eta_{\mu_1 \mu_2} L, \quad (2.195) \]
since $L_{1111,22}$ is actually a scalar parameter in the (4, 2) model, here denoted by $L$ for brevity. Comparing with eq. (2.187), one can see directly that the three traces of the $B^{(i)}$ can be gauged away by these transformations, which provides again a different perspective on their decoupling from the field equation for $\varphi$.

### 2.4.3 Current exchanges for rank-(4, 2) gauge fields

The results of the previous section afford an interesting generalization, along the lines of [13, 27], when external currents are present. Let us stress that our unconstrained formulation requires that the external currents be conserved, a feature to be contrasted with their partial conservation in the constrained Fronsdal or Labastida constructions. Referring to the (4, 2) case, our aim is now to invert the field equations in the presence of external sources in order to derive an explicit expression for the massless current exchange. Many of the results will follow the pattern already visible in [27], with an important novelty related to the non-trivial role of double traces, that after all already emerged in the previous discussion of the source-free case.

Let us begin by considering the field equation in the presence of an external source $J$

$$E_\varphi = E_\varphi + \eta(\mu_1\mu_2 \eta_{\mu_3\mu_4}) B^{(1)}_{\nu_1\nu_2} + \eta(\nu_1 | (\mu_1 \eta_{\mu_2\mu_3} B^{(2)}_{\mu_4}) | \nu_2) + (\eta_{\nu_1\nu_2} \eta(\mu_1 \mu_2 | + \eta_{\nu_1} (\mu_1 \eta_{\mu_2 \nu_2}) B^{(3)} | \mu_3 \mu_4) = J_{\mu_1...\mu_4;\nu_1\nu_2}. \tag{2.196}$$

As in the previous section, its two independent triple traces, that in this more general setting give

$$E_\varphi^{\mu\nu\rho\sigma} : \rho_0 B^{(1)}_{\nu} + 8 B^{(2)}_{\nu} + 2 \rho_2 B^{(3)}_{\mu} = \frac{1}{\rho_2} J^{\mu\nu\rho\sigma},$$

$$E_\varphi^{\mu\nu\rho\sigma} : \rho_3 B^{(2)}_{\nu} + \rho_5 B^{(3)}_{\mu} = \frac{1}{\rho_2} J^{\mu\nu\rho\sigma}, \tag{2.197}$$

provide a convenient starting point for the analysis. One is thus facing, again, a system of equations that is not fully determined, but allows nonetheless to relate two of the traces of $B_{ijkl}$ to the third and to the external currents, according to

$$B^{(2)}_{\nu} = -\frac{1}{2} B^{(3)}_{\mu} - \frac{1}{2 \rho_4 \rho_1 \rho_2} \left( J^{\mu\nu\rho\sigma} - \rho_0 J^{\mu\nu\rho\sigma} \right),$$

$$B^{(1)}_{\nu} = -2 B^{(3)}_{\mu} + \frac{1}{\rho_4 \rho_1 \rho_2} \left( \rho_3 J^{\mu\nu\rho\sigma} - 4 J^{\mu\nu\rho\sigma} \right). \tag{2.198}$$

As we saw in the previous section, the shift symmetry of eq. (2.180) actually allows to gauge away the first terms above. The other terms involving the external current, however, cannot be eliminated and contribute to the current exchanges.

Making use of eqs. (2.198) in the non-homogeneous version of the relations (2.185) for the
double traces of $E_\nu$ finally leads to the system

\[
E_\nu^{(i)} : - \frac{4}{9} a_{\nu\nu} + \rho_0 \rho_2 B_T^{(1)}_{\nu} + 4 \rho_2 B_T^{(2)}_{\nu} + 8 B_T^{(3)}_{\nu} \\
= \mathcal{J}_\nu^{(i)} + \frac{1}{\rho_0} \mathcal{J}_\nu^{(i)} \nabla_{\nu},
\]

\[
E_\nu^{(j)} : \frac{1}{9} a_{\mu\nu} + \rho_0 B_T^{(1)}_{\mu\nu} + \frac{\rho_2 \rho_4 + 4}{2} B_T^{(2)}_{\mu\nu} + \rho_2 \rho_4 B_T^{(3)}_{\mu\nu} + (3 \rho_3 + 1) B_T^{(3)}_{\mu\nu} \\
= \mathcal{J}_\mu^{(j)} - \frac{1}{\rho_0} \mathcal{J}_\mu^{(j)},
\]

\[
E_\nu^{(i)} + 2 E_\nu^{(j)} : - \frac{2}{9} a_{\mu_1\mu_2} + 4 B_T^{(1)}_{\mu_1\mu_2} + 2 (3 \rho_3 + 1) B_T^{(2)}_{\mu_1\mu_2} + (3 \rho_3 + 1) B_T^{(3)}_{\mu_1\mu_2} \\
= \left( \mathcal{J}_\mu^{(i)} + 2 \mathcal{J}_\mu^{(j)} \right)_{\mu_1\mu_2} - \frac{1}{\rho_0} \eta_{\mu_1\mu_2} \left( \mathcal{J}_\mu^{(i)} + 2 \mathcal{J}_\mu^{(j)} \right)_{\mu_1\mu_2},
\]

while the non-homogeneous counterpart of eq. (2.190) is

\[
E_\nu^{(i)} - E_\nu^{(j)} : \frac{3 \rho_1 \rho - 4 + 2}{18} a_{\mu_1\mu_2} - 2 \left( B_T^{(1)}_{\mu_1\mu_2} - B_T^{(2)}_{\mu_1\mu_2} + B_T^{(3)}_{\mu_1\mu_2} \right) \\
= \left( \mathcal{J}_\mu^{(i)} - \mathcal{J}_\mu^{(j)} \right)_{\mu_1\mu_2} - \frac{1}{\rho_0} \eta_{\mu_1\mu_2} \left( \mathcal{J}_\mu^{(i)} - \mathcal{J}_\mu^{(j)} \right)_{\mu_1\mu_2}.
\]

As in the previous section, we have separated the traceless parts of the $B^{(i)}$ and correspondingly the right-hand sides involve the traceless combinations

\[
\begin{align*}
\tilde{J}^{(1)}_{\nu\nu} &= \mathcal{J}_{\nu\nu}^{(i)} - \frac{1}{\rho_0} \mathcal{J}_{\nu\nu}^{(i)} \mathcal{J}_{\nu\nu}^{(i)}, \\
\tilde{J}^{(2)}_{\mu\nu} &= \mathcal{J}_{\mu\nu}^{(i)} - \frac{1}{\rho_0} \mathcal{J}_{\mu\nu}^{(i)} \mathcal{J}_{\mu\nu}^{(i)}, \\
\tilde{J}^{(3)}_{\mu_1\mu_2} &= \left( \mathcal{J}_{\mu_1\mu_2}^{(i)} + 2 \mathcal{J}_{\mu_1\mu_2}^{(j)} \right)_{\mu_1\mu_2} - \frac{1}{\rho_0} \eta_{\mu_1\mu_2} \left( \mathcal{J}_{\mu_1\mu_2}^{(i)} + 2 \mathcal{J}_{\mu_1\mu_2}^{(j)} \right)_{\mu_1\mu_2}, \\
\tilde{J}^{(4)}_{\mu_1\mu_2} &= \left( \mathcal{J}_{\mu_1\mu_2}^{(i)} - \mathcal{J}_{\mu_1\mu_2}^{(j)} \right)_{\mu_1\mu_2} - \frac{1}{\rho_0} \eta_{\mu_1\mu_2} \left( \mathcal{J}_{\mu_1\mu_2}^{(i)} - \mathcal{J}_{\mu_1\mu_2}^{(j)} \right)_{\mu_1\mu_2},
\end{align*}
\]

so that one is actually dealing with a triply traceless effective current. As pointed out in the discussion of the homogeneous equations, the system (2.199) effectively splits into a $3 \times 3$ system for the symmetric components of the $B^{(i)}$ and a single equation for the antisymmetric component of $B^{(2)}$. Eqs. (2.199) and (2.200) can then be solved, so that, for instance

\[
\tilde{a}_{\mu_1\mu_2} = \frac{12}{\rho - 4 \rho_1 \rho_2} \left\{ \mathcal{J}_{\mu_1\mu_2}^{(i)} - \mathcal{J}_{\mu_1\mu_2}^{(j)} + \frac{1}{3} \left( \mathcal{J}_{\mu_1\mu_2}^{(i)} + 2 \mathcal{J}_{\mu_1\mu_2}^{(j)} \right)_{\mu_1\mu_2} \right\} + \frac{2 (3 D^2 - 8)}{\rho - 4 \rho_1 \rho_2} \left( \mathcal{J}_{\mu_1\mu_2}^{(i)} - \mathcal{J}_{\mu_1\mu_2}^{(j)} \right)_{\mu_1\mu_2} - \frac{6}{\rho - 4 \rho_1 \rho_2} \eta_{\mu_1\mu_2} \left( \mathcal{J}_{\mu_1\mu_2}^{(i)} - \mathcal{J}_{\mu_1\mu_2}^{(j)} \right)_{\mu_1\mu_2}.
\]

Rather than displaying the corresponding solutions for the $B^{(i)}$ directly in terms of the external current $\tilde{J}_{\mu_1\mu_2}^{(i)}$, it is instructive to elaborate on their form prior to eliminating $a$, since these connect directly with the current exchange amplitudes for the Labastida formulation, that are
determined by the properly projected field equation (2.170). Eqs. (2.199) lead to
\[ B^{(1)}_{\nu_1 \nu_2} = \frac{4}{3 (3 D^2 - 8)} a_{\nu_1 \nu_2} + C \left\{ (3 D^3 + 24 D^2 + 40 D - 16) j^{(1)}_{\nu_1 \nu_2} - 8 (3 D^2 + 12 D + 4) j^{(2)}_{\nu_1 \nu_2} + 16 \rho_1 j^{(3)}_{\nu_1 \nu_2} \right\}, \]
\[ B^{(2)}_{(\mu; \nu)} = -\frac{2}{3 (3 D^2 - 8)} a_{\mu \nu} + 2 C \left\{ -(3 D^2 + 12 D + 4) j^{(1)}_{\mu \nu} + (3 D^3 + 12 D^2 + 16 D - 8) j^{(2)}_{\mu \nu} - \rho_2 (3 D - 2) j^{(3)}_{\mu \nu} \right\}, \]
\[ B^{(3)}_{\mu_1 \mu_2} = \frac{2}{9 (3 D^2 - 8)} a_{\mu_1 \mu_2} + C \left\{ 8 \rho_1 j^{(1)}_{\mu_1 \mu_2} - 4 \rho_2 (3 D - 2) j^{(2)}_{\mu_1 \mu_2} + \rho_2 (D^2 + 2 D - 4) j^{(3)}_{\mu_1 \mu_2} \right\}, \]
with
\[ C = \frac{1}{\rho_0 \rho_4 \rho_6 (3 D^2 - 8)} \quad (2.204) \]
and the \( j^{(i)} \) defined as in eq. (2.201).

Substituting these expressions in (2.196) one obtains
\[ E_\varphi = \mathcal{E}_\varphi + \frac{4}{3 (3 D^2 - 8)} \eta_{(\mu_1 \mu_2 \eta_{\mu_3 \mu_4} a_{\nu_1 \nu_2} - \frac{1}{3 (3 D^2 - 8)} \eta_{\nu_1 |(\mu_1 \eta_{\mu_2 \mu_3} a_{\mu_4})|\nu_2} + \frac{2}{9 (3 D^2 - 8)} \left( \eta_{\nu_1 \nu_2} \eta_{(\mu_1 \mu_2 | + \eta_{\nu_1 |(\mu_1 \eta_{\mu_2 \nu_2}) a_{|\mu_3 \mu_4} = \mathcal{K}_{\mu_1 \ldots \mu_4; \nu_1 \nu_2}. \right) \]
where \( \mathcal{E}_\varphi \) is the tensor defined by eq. (2.165) and \( \mathcal{K}_{\mu_1 \ldots \mu_4; \nu_1 \nu_2} \) is a constrained effective current whose symmetrized double traces vanish. Notice that this expression reduces to the inhomogeneous Lagrangian equation of the Labastida formulation if the compensator is eliminated by a gauge choice.

In terms of \( \mathcal{K} \), eq. (2.202) reduces to the simpler form
\[ a_{\mu_1 \mu_2} = \frac{2 (3 D^2 - 8)}{\rho_4 \rho_1 \rho_0 \rho_2} \left( \mathcal{K}_{\mu \nu} - \mathcal{K}_{\nu \mu} \right)_{\mu_1 \mu_2}, \quad (2.206) \]
and eliminating the various traces of \( \mathcal{A} \) one can finally build the complete current-exchange
amplitude

\[ A_{\mu_1...\mu_4;\nu_1\nu_2} = \mathcal{K}_{\mu_1...\mu_4;\nu_1\nu_2} - \frac{1}{\rho_4} \eta_{(\mu_1\mu_2]} Y_{(2,2)} \mathcal{K}'_{\mu_4\mu_3;\nu_1\nu_2} \\
- \frac{1}{\rho_4 \rho_6} \eta_{(\mu_1\mu_2]} Y_{(2,2)} \left\{ (D^2 - 12) \mathcal{K}'_{\mu_4\mu_3;\nu_1\nu_2} - 2 \rho_4^2 \mathcal{K}'^j_{[\mu_3\mu_4]} (\nu_1;\nu_2) \right\} \\
+ 4 \mathcal{K}'_{[\mu_3\mu_4]} \nu_1\nu_2 \right\} - \frac{1}{\rho_4 \rho_6} \eta_{(\mu_1\mu_2]} Y_{(2,2)} \left\{ - \rho_2^2 \mathcal{K}'_{\mu_2\mu_5;\mu_4]} (\nu_1;\nu_2) + (D^2 - 4) \mathcal{K}'_{[\mu_2\mu_3\mu_4]} (\nu_1;\nu_2) \right\} \\
- \frac{1}{\rho_4 \rho_6} \eta_{(\nu_1]} (\mu_1 \mu_2) Y_{(2,2)} \left\{ - \rho_2^2 \mathcal{K}'_{\mu_2\mu_5;\mu_4]} (\nu_1;\nu_2) + (D^2 + 4D - 8) \mathcal{K}'_{\mu_1\mu_2\mu_3\mu_4} (\nu_1;\nu_2) \right\} \\
- \frac{1}{3 \rho_4 \rho_6} \left\{ - 4 \eta_{(\mu_1\mu_2] \eta_{\mu_3\mu_4]} (\mathcal{K}'_{\mu_4\mu_5} - \mathcal{K}'_{\mu_1\mu_2}) \nu_1\nu_2 + \eta_{(\nu_1]} (\mu_1 \mu_2 \mu_3 \mu_4] (\mathcal{K}'_{\mu_4\mu_5} - \mathcal{K}'_{\mu_1\mu_2}) \nu_1\nu_2 \right\} \\
- \frac{1}{3} \left( \eta_{\nu_1\nu_2} \eta_{(\mu_1\mu_2]} + \eta_{(\nu_1\mu_1 \eta_{\mu_2]} \nu_2) \right) (\mathcal{K}'_{\mu_1\mu_2} - \mathcal{K}'_{\mu_3\mu_4}] \nu_1\nu_2 \\
- \frac{3 D^2 - 8}{6} \left( 2 \eta_{\nu_1\nu_2} \eta_{(\mu_1\mu_2]} - \eta_{(\nu_1\mu_1 \eta_{\mu_2]} \nu_2) \right) (\mathcal{K}'_{\mu_1\mu_2} - \mathcal{K}'_{\mu_3\mu_4}] \nu_1\nu_2 \right\} \right] \right\} \] (2.207)

where the \( Y_i \) are Young projectors and the coefficients that appear on the right-hand side formally build, as was shown to be the case for all symmetric tensors in [27], a traceless current in \( D - 2 \) dimensions. Notice that these expressions present poles at the special dimensions where an additional Weyl-like develops as discussed in Section 2.3.1.

Let us conclude this section by stressing again that the analysis was performed using the minimal Lagrangian of eq. (2.39), while in principle this setting could be modified by the addition of terms involving the constraints. These further couplings, however, would merely induce redefinitions of the Lagrange multipliers, that as such would have no effect on the current exchanges, as the reader can verify.

## 3 General Bosonic Fields

Having discussed in some detail two-family fields, the simplest class of mixed-symmetry gauge bosons, we can now turn to the Lagrangians for the general case of arbitrary numbers of index families. In this section we begin by reconsidering the original result of Labastida, that we approach via a different route, dictated by the Bianchi identities and their traces. We then turn to a detailed derivation of the unconstrained Lagrangians, and conclude with a description of the corresponding field equations and of their reduction to \( \mathcal{F} = 0 \).

### 3.1 The Lagrangians

The traces of the Bianchi identities are a useful tool to derive the Lagrangians of mixed-symmetry fields with arbitrary numbers of index families. In order to illustrate their role, let us begin by showing how one can recover in this fashion the constrained Labastida theory.
After enforcing the double trace constraints
\[ T_{(ij}T_{kl)} \varphi = 0 \] (3.1)
the Bianchi identities take in general the form already foreseen in the two-family case,
\[ \partial_i \mathcal{F} - \frac{1}{2} \partial^j T_{ij} \mathcal{F} = 0 \] (3.2)
where, of course, here the range for the family indices is meant to be arbitrary. Taking a trace of eq. (3.2) gives
\[ \partial_i T_{jk} \mathcal{F} - \frac{1}{2} \partial_{(j} T_{k)i} \mathcal{F} - \frac{1}{2} \partial^l T_{il} T_{jk} \mathcal{F} = 0, \] (3.3)
a relation that admits both a symmetric and a “hooked” \( \{2,1\} \) projection for its family indices. Let us stress, however, that as we saw already for two-family fields the symmetric projection does not contain a divergence, but only the gradient of an expression that vanishes identically, since for the Labastida theory
\[ T_{(ij}T_{kl)} \mathcal{F} = 0. \] (3.4)
Indeed, as we already observed in Section 2.1, eq. (3.1) implies that \( \mathcal{F} \) satisfies the same double trace constraints as the gauge field \( \varphi \), so that the symmetric projection of eq. (3.3) does not convey any new information, but merely recovers an algebraic property of the double trace of \( \mathcal{F} \) that could be derived directly, and rather simply, from eq. (B.14). To this end, one would only need to recall that, for a pair of traces \( T_{ij}T_{kl} \), the symmetrization over three of the four indices induces the full symmetrization. In contrast, the hooked \( \{2,1\} \) projection of the first trace (3.3) of the Bianchi identities,
\[ (2 \partial_i T_{jk} - \partial_{(j} T_{k)i}) \mathcal{F} = \frac{1}{3} \partial^l \left( 2 T_{il} T_{jk} - T_{i(j} T_{k)l} \right) \mathcal{F}, \] (3.5)
plays an important role in the construction, since it relates in a non-trivial fashion terms with divergences to others with gradients.

A further trace of eq. (3.3) gives
\[ 2 \partial_i T_{jk} T_{lm} \mathcal{F} - \left( \partial_{(j} T_{k)i} T_{lm} + \partial_{(l} T_{m)i} T_{jk} \right) \mathcal{F} = \partial^n T_{in} T_{jk} T_{lm} \mathcal{F}, \] (3.6)
an expression that in principle would admit four distinct projections, the symmetric \( \{5\} \), the \( \{4,1\} \), the \( \{3,2\} \) and the \( \{2,2,1\} \). However, a \( \{2,2,1\} \) projected expression obtains simply replacing the double traces of \( \mathcal{F} \) on the left-hand side with their \( \{2,2\} \) projections, which is clearly possible since the remainder vanishes on account of eq. (3.4). In other words, the left-hand side of eq. (3.6) is actually \( \{2,2,1\} \) projected, modulo the Labastida constraints (3.1). This fact has an important consequence: out of the available projections for the double traces of the Bianchi identities, only one, the \( \{2,2,1\} \) projection, relates in a non-trivial fashion divergences and gradients. This result can be related to a general fact, whose proof is deferred to Appendix C, according to which all projections of multiple traces of \( \mathcal{F} \) corresponding to Young diagrams with more than two columns vanish on account of the double trace constraint (3.4).

The pattern just identified extends to the higher traces of the Bianchi identities. At any given order, only a single Young projection plays a dynamical role, and relates divergences of \( p \) traces and gradients of \( p + 1 \) traces of \( \mathcal{F} \), with all traces subject to projections of the “window”
{2, \ldots, 2} type, while the others vanish on account of eq. (3.4). The uniqueness of this relevant type of projections at any given order suggests to resort to the shorthand notation
\begin{align*}
\mathcal{F}'_{ij} &= T_{ij} \mathcal{F}, \\
\mathcal{F}''_{ij;kl} &= \frac{1}{3} \left( 2 T_{ij} T_{kl} - T_i (k T_l)_{j} \right) \mathcal{F}, \\
&\vdots \\
\mathcal{F}^{[p]}_{i_1 j_1, \ldots, i_p j_p} &= Y^{\{2^p\}}_{i_1 j_1} \cdots T_{i_p j_p} \mathcal{F},
\end{align*}
(3.7)
where for brevity we have also let
\begin{equation}
Y^{\{2^p\}} = Y^{\{2, \ldots, 2\}}_{i_1 j_1} \cdots \eta^{i_1 j_1} \cdots \eta^{i_p j_p} \mathcal{F}^{[p]}_{i_1 j_1, \ldots, i_p j_p}.
(3.8)
\end{equation}

For a general \(N\)-family constrained field \(\varphi\), the Lagrangian is thus necessarily of the form
\begin{equation}
\mathcal{L} = \frac{1}{2} \langle \varphi, \mathcal{F} + \sum_{p=1}^{N} k_p \eta^p \mathcal{F}^{[p]} \rangle,
(3.9)
\end{equation}
where
\begin{equation}
\eta^p \mathcal{F}^{[p]} = \eta^{i_1 j_1} \cdots \eta^{i_p j_p} \mathcal{F}^{[p]}_{i_1 j_1, \ldots, i_p j_p}.
(3.10)
\end{equation}
We shall see shortly that the relations between the divergence of \(\mathcal{F}^{[p]}\) and the gradient of \(\mathcal{F}^{[p+1]}\) provided by the left-over two-column traces of the Bianchi identities fix uniquely the \(k_p\) that guarantee the gauge invariance of \(\mathcal{L}\), with the end result that
\begin{equation}
k_p = \frac{(-1)^p}{p!(p+1)!}.
(3.11)
\end{equation}

This recovers the complete Labastida Lagrangian, with some minor differences with respect to the original presentation. First, in [20] the higher traces of \(\mathcal{F}\) do not carry specific projections like here, but we have seen that, at any given order, only one projection actually survives the Labastida constraints. This specification will play an important role in the unconstrained theory. In addition, our \(\eta\) tensors are conveniently rescaled, as explained in Appendix A, which brings about a factor \(2^p\) when compared to [20]. The last difference has to do with what is probably just a misprint in [20], where the factor \((-1)^p\) is not indicated. For brevity, we shall defer the missing details of this derivation to the ensuing discussion of the unconstrained theory, to which we now turn. From our vantage point, both settings rest on similar steps determined by the Bianchi identities. The key differences concern the \(\mathcal{C}_{ijkl}\) constraints and the composite compensators \(\alpha_{ijk}(\Phi)\), that are only present in the unconstrained theory.

Unconstrained Lagrangians for generic mixed-symmetry fields can be constructed starting from the Bianchi identities for the \(\mathcal{A}\) tensor,
\begin{equation}
\partial_i \mathcal{A} - \frac{1}{2} \partial^j T_{ij} \mathcal{A} = - \frac{1}{4} \partial^j \partial^k \partial^l \mathcal{C}_{ijkl}.
(3.12)
\end{equation}
Taking \(p\) traces one then obtains
\begin{align*}
2 \partial_k T_{i_1 j_1} \cdots T_{i_p j_p} \mathcal{A} &- \left( \partial_{(i_1} T_{j_1)} k \cdots T_{i_p j_p} + \cdots + \partial_{(i_p} T_{j_p)} k \cdots T_{i_{p-1} j_{p-1}} \right) \mathcal{A} \\
- \partial^l T_{i_1 j_1} \cdots T_{i_p j_p} T_{kl} \mathcal{A} &= - \frac{1}{2} T_{i_1 j_1} \cdots T_{i_p j_p} \partial^l \partial^m \partial^n \mathcal{C}_{klmn},
(3.13)
\end{align*}
where the left-hand side is not projected to begin with. The constrained case, however, provided two clear indications:

- the relevant projections of the Bianchi identities, to all orders, arise from the \( \{2^p, 1\} \) Young diagram;
- all projections of the multiple traces of \( \mathcal{A} \) corresponding to Young diagrams with more than two columns can be expressed in terms of the \( \mathcal{C}_{ijkl} \) constraints.

If we now construct the \( \{2^p, 1\} \) projection of the Bianchi identities, the gradient term produces directly the \( \{2^{p+1}\} \)-projected trace \( \mathcal{A}_{i_1 j_1, \ldots, i_p j_p, kl}^{[p+1]} \), while the terms bearing a divergence need further care. The key observation is that

\[
Y_{\{2^p, 1\}} \left[ 2 \partial_k T_{i_1 j_1} \ldots T_{i_p j_p} \mathcal{A} - \left( \partial_{(i_1} T_{j_1)} k \ldots T_{i_p j_p} + \ldots \right) \mathcal{A} \right] = (p + 2) Y_{\{2^p, 1\}} \partial_k \mathcal{A}^{[p]}_{i_1 j_1, \ldots, i_p j_p},
\]

and is proved in Appendix C. The factor \((p+2)\) plays a crucial role in the following derivation. It draws its origin from the column antisymmetrization in \( p+1 \) indices, that has precisely the effect of bringing together the \( p+1 \) terms on the left-hand side of eq. (3.14), the first of which bears an overall coefficient 2. In conclusion, the remaining operations recover the \( \{2^p, 1\} \) projection of the divergence \( \partial_k \mathcal{A}^{[p]}_{i_1 j_1, \ldots, i_p j_p} \), but with an overall factor \( p+2 \), as indicated in eq. (3.14), so that the end result is

\[
(p + 2) Y_{\{2^p, 1\}} \partial_k \mathcal{A}^{[p]}_{i_1 j_1, \ldots, i_p j_p} - \partial^l A^{[p+1]}_{i_1 j_1, \ldots, i_p j_p, kl} = - \frac{1}{2} Y_{\{2^p, 1\}} T_{i_1 j_1} \ldots T_{i_p j_p} \partial^l \partial^m \partial^n C_{klmn} .
\]

It should be appreciated that these arguments bring the general case of mixed-symmetry fields very close to the far simpler symmetric construction of [26, 27], since at any order in the traces one is left with a single type of relevant consequences of the Bianchi identities.

Eqs. (3.15) determine completely the structure of the unconstrained Lagrangian, as we now show starting from a trial Lagrangian built from all traces of \( \mathcal{A} \) not related to the constraints. We shall see shortly how to fix the coefficients in order to obtain a gauge invariant result. Our starting point is thus

\[
\mathcal{L}_0 = \frac{1}{2} \left( \varphi , \sum_{p=0}^{N} k_p \eta^p A^{[p]} \right),
\]

where \( k_0 = 1 \), and for brevity we are using the shorthand notation of eq. (3.10). Up to partial integrations, the resulting gauge variation reads

\[
\delta \mathcal{L}_0 = - \sum_{p=0}^{N} \frac{1}{2^{p+1}} \left( \mathcal{T}^p \mathcal{A} , k_p \partial \cdot \mathcal{A}^{[p]} + (p+1) k_{p+1} \partial \mathcal{A}^{[p+1]} \right)
\]

\[
= - \sum_{p=0}^{N} \frac{1}{2^{p+1}} \left( T_{i_1 j_1} \ldots T_{i_p j_p} \mathcal{A}^k , k_p \partial_k \mathcal{A}^{[p]}_{i_1 j_1, \ldots, i_p j_p} + (p+1) k_{p+1} \partial^l \mathcal{A}^{[p+1]}_{kl, i_1 j_1, \ldots, i_p j_p} \right).
\]

The right entries of the scalar product admit only the \( \{3, 2^{p-1}\} \) and \( \{2^p, 1\} \) projections, which induce corresponding projections on the left entries, so that the gauge variation of the
Lagrangian can be turned into the form

$$\delta L_0 = - \sum_{p=0}^{N} \frac{1}{2p+1} \left< Y_{(2p,1)} T^p A, k_p Y_{(2p,1)} \partial \cdot A^{[p]} + (p+1) k_{p+1} \partial A^{[p+1]} \right>$$

$$- \sum_{p=1}^{N} \frac{1}{2p+1} \left< Y_{(3,2p-1)} T^p A, k_p Y_{(3,2p-1)} \partial \cdot A^{[p]} \right>, \quad (3.18)$$

here displayed in a concise but hopefully still clear notation. Notice that no \(\{3,2p-1\}\) component originates from the gradient term, whose free indices belong to \(A^{[p+1]}\) that is two-column projected. Moreover, the first row of (3.18) is closely related to the multiple traces of the Bianchi identities of eq. (3.15), that in this compact notation read

$$\sum_{p=0}^{N} \frac{1}{2p+1} \left< Y_{(2p,1)} T^p A, k_p Y_{(2p,1)} \partial \cdot A^{[p]} \right> = - \frac{1}{2} Y_{(2p,1)} T^p \partial \partial \partial C. \quad (3.19)$$

Hence, if the coefficients satisfy the recursion relation

$$\frac{k_{p+1}}{k_p} = - \frac{1}{(p+1)(p+2)}, \quad (3.20)$$

whose solution is given in eq. (3.11), all \(A\) tensors disappear from the first line of the gauge variation (3.18), that reduces to

$$\delta L_0 = - \frac{1}{8} \left< \delta \beta, C \right> - \sum_{p=1}^{N} \frac{1}{2p+1} \left< Y_{(3,2p-1)} T^p A, k_p Y_{(3,2p-1)} \partial \cdot A^{[p]} \right>, \quad (3.21)$$

where

$$\delta \beta_{ijkl} = \frac{1}{2} \sum_{p=0}^{N} \frac{k_p}{p+2} \partial (i \partial_j \partial_k) \rho_{n_1 n_2} \ldots \rho_{n_p n_p} Y_{(2p,1)} T_{n_1 n_2} \ldots T_{n_p n_p} A_{t \ell} \quad (3.22)$$

identifies the gauge transformations of the Lagrange multipliers, that we shall add shortly. This is a particularly compact expression for their gauge variation, but it is also possible to move the divergences to the right of the \(\eta\)’s to recover the result displayed in Section 2.1, together with additional contributions that appear starting from three families.

The rest of the gauge variation can be canceled adding to \(L_0\) a sum of terms involving the composite compensators \(\alpha_{ijk}(\Phi)\) of eq. (2.9), whose structure follows the pattern that clearly emerged in the two-family case. Their identification, however, requires a non-trivial identity, that will be derived in Appendix C and reads

$$\left< T_{ij_{1j_{1}} \ldots T_{ij_{p} p} A_{k}} Y_{(3,2p-1)} T_{ij_{1} j_{1}} \ldots T_{ij_{p} p} A_{k} \right> = \frac{p}{p+2} \left< T_{ij_{1} j_{1}} \ldots T_{ij_{p} p} A_{k}, \partial (k A_{[i_{1 j_{1}} \ldots i_{j_{p} p}]}) \right>. \quad (3.23)$$

The symmetrization in \((i_{1 j_{1}} k)\) induces a corresponding symmetrization in the left entry of the scalar product, that is then manifestly related to traces of the \(\delta \alpha_{i_{1 j_{1}} k}\). As a result, the gauge variation can be presented in the rather compact form

$$\delta L_0 = - \frac{1}{8} \left< \delta \beta, C \right> - \frac{3}{4} \sum_{p=1}^{N} \left< \delta \alpha, \frac{p k_p}{p+2} \eta^{p-1} Y_{(3,2p-1)} \partial \cdot A^{[p]} \right>, \quad (3.24)$$

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so that the un constrained Lagrangian for unprojected $N$-family bosonic gauge fields is finally

$$\mathcal{L} = \frac{1}{2} \langle \varphi, \mathcal{A} + \sum_{p=1}^{N} k_p \eta^p \mathcal{A}^{[p]} \rangle + \frac{3}{4} \langle \alpha, \sum_{p=1}^{N} \frac{p k_p}{p+2} \eta^{p-1} Y_{(3,2^p-1)} \partial \cdot \mathcal{A}^{[p]} \rangle + \frac{1}{8} \langle \beta, \mathcal{C} \rangle \quad (3.25)$$

where the $k_p$ coefficients are given in eq. (3.11), or more explicitly

$$\mathcal{L} = \frac{1}{2} \langle \varphi, \mathcal{A} - \frac{1}{2} \eta^{ij} \mathcal{A}^{ij} + \frac{1}{12} \eta^{ij} \eta^{kl} \mathcal{A}^{ijk;l} - \frac{1}{144} \eta^{ij} \eta^{kl} \eta^{mn} \mathcal{A}^{ij;kl;mn} + \ldots \rangle$$

$$- \frac{1}{24} \langle \alpha_{ijk}, \partial(i, \mathcal{A}^{jk}) - \frac{1}{4} \partial(i, \mathcal{A}^{jk});im + \frac{1}{40} \partial(i, \mathcal{A}^{jk});lm;np + \ldots \rangle + \frac{1}{8} \langle \beta_{ijkl}, \mathcal{C}_{ijkl} \rangle \quad (3.26)$$

The first few terms of this general result are

$$\mathcal{L} = \frac{1}{2} \langle \varphi, \mathcal{A} - \frac{1}{2} \eta^{ij} A^{'ij} + \frac{1}{12} \eta^{ij} \eta^{kl} A^{''ij;kl} - \frac{1}{144} \eta^{ij} \eta^{kl} \eta^{mn} A^{''ij;kl;mn} + \ldots \rangle$$

$$- \frac{1}{24} \langle \alpha_{ijk}, \partial(i, A^{jk}) - \frac{1}{4} \partial(i, A^{jk});im + \frac{1}{40} \partial(i, A^{jk});lm;np + \ldots \rangle$$

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and agree nicely with the result of Section 2.1. Once the $\alpha_{ijk}$ and $\beta_{ijkl}$ are removed from this expression, the remainder reproduces the result of Labastida, up to some notational changes and up to the oscillating signs, that are probably missing in [20] due to a misprint.

We can not refrain from associating to these expressions simple generating functions for the coefficients. This can be attained starting from the family of contour integrals

$$\mathcal{B}_{k}[z] = (k+1)! \int_{\gamma} \frac{d\zeta}{2\pi i} \zeta^k e^{z \zeta} = \sum_{p=0}^{\infty} \frac{(-1)^p (k+1)!}{p! (p+k)!} z^p = (k+1)! \frac{J_k(\sqrt{z})}{(\sqrt{z})^k} \quad (3.28)$$

where the contour $\gamma$ encircles the origin and $J_k$ denotes a Bessel function of order $k$. One can thus write the general Lagrangians rather concisely in the form

$$\mathcal{L} = \frac{1}{2} \langle \varphi, \mathcal{B}_0[\eta T] \mathcal{A} - \frac{1}{8} \langle \alpha, \mathcal{B}_2[\eta T] \partial \cdot T \mathcal{A} \rangle + \frac{1}{8} \langle \beta, \mathcal{C} \rangle \rangle \quad (3.29)$$

where all $\eta$'s are meant to be placed to the left of all traces, while the latter are meant to be projected as in eq. (3.26), so that for $N$ families the sums actually terminate after the $N$-th trace. In a similar spirit, the gauge transformation (3.22) of Lagrange multipliers $\beta_{ijkl}$ can be written concisely as

$$\delta \beta_{ijkl} = \frac{1}{4} \partial(i) \partial(j) \partial(k) \mathcal{B}_1[\eta T] \Lambda_{|l|} \quad (3.30)$$

where the Young projector in eq. (3.22) is left implicit.

As was the case for two-family fields, it is possible to present the general Lagrangians (3.26) in an alternative way that will soon prove convenient to derive the field equations. To this end,
Let us consider again a field $\phi$ not subject to the double trace constraints \((2.17)\), but whose gauge parameters $\Lambda_i$ are still constrained according to \((2.8)\). The corresponding Lagrangians can be cast in the form

\[
\mathcal{L}_C (\phi, \gamma_{ijkl}) = \frac{1}{2} \left( \phi, \sum_{p=0}^N \frac{(-1)^p}{p!(p+1)!} \eta^{i_1j_1} \cdots \eta^{i_pj_p} \mathcal{F}^{[p]}(\phi)_{i_1j_1, \ldots, i_pj_p} \right) + \frac{1}{24} \left( \gamma_{ijkl}, T_{(ij} T_{kl)} \phi \right),
\]

\[(3.31)\]

that differ from the Labastida Lagrangians of \([20]\) simply because gauge invariance demands the simultaneous presence of projected traces and Lagrange multipliers $\gamma_{ijkl}$. The latter fields enforce the double trace constraints, and their gauge transformations are actually those given for the $\beta_{ijkl}$ in eq. \((3.22)\). The relation between the two Lagrangians of eqs. \((3.26)\) and \((3.31)\) is then

\[
\mathcal{L} (\varphi, \Phi_i, \beta_{ijkl}) = \mathcal{L}_C (\varphi - \partial^i \Phi_i, \beta_{ijkl} - \Delta_{ijkl}(\Phi)),
\]

\[(3.32)\]

where

\[
\Delta_{ijkl}(\Phi) = \frac{1}{2} \sum_{p=0}^N \frac{(-1)^p}{p!(p+2)!} \partial_{(j} \partial_{k} \frac{\eta^{i_1j_1} \cdots \eta^{i_pj_p} Y_{(2p,1)} T_{m_1n_1} \cdots T_{m_pn_p} \Phi_{1)}}{\partial T_{l)},
\]

\[(3.33)\]

or more compactly

\[
\Delta_{ijkl}(\Phi) = \frac{1}{4} \partial_{(j} \partial_{k} \frac{\eta^{i_1j_1} \cdots \eta^{i_pj_p} F^{[p]}(\phi)_{i_1j_1, \ldots, i_pj_p}}{\partial T_{l)},
\]

\[(3.34)\]

Actually, even in the general multi-family setting there is a wide freedom to conveniently redefine the Lagrange multipliers $\beta_{ijkl}$ or $\gamma_{ijkl}$. For instance, one could also start from the Lagrangians

\[
\tilde{\mathcal{L}}_C (\phi, B_{ijkl}) = \frac{1}{2} \left( \phi, \sum_{p=0}^N \frac{(-1)^p}{p!(p+1)!} \eta^{i_1j_1} \cdots \eta^{i_pj_p} \tilde{\mathcal{F}}^{[p]}(\phi)_{i_1j_1, \ldots, i_pj_p} \right) + \frac{1}{24} \left( B_{ijkl}, T_{(ij} T_{kl)} \phi \right),
\]

\[(3.35)\]

where the $\tilde{\mathcal{F}}^{[p]}$ are projected traces of $\mathcal{F}$ deprived of the $\{4, 2^{p-1}\}$ components of the last terms that appear in eq. \((B.11)\), so that

\[
\tilde{\mathcal{F}}^{[p]}(\phi)_{i_1j_1, \ldots, i_pj_p} = Y_{(2p)} \left\{ (p+1) \prod_{r=1}^p T_{i_rj_r} \phi - (p+1) \sum_{n=1}^p \partial_{i_n} \partial_{j_n} \prod_{r \neq n}^p T_{i_rj_r} \phi \right\}
\]

\[
- \partial^k \left[ \partial_k \prod_{r=1}^p T_{i_rj_r} \phi - \sum_{n=1}^p \partial_{(i_n T_{j_n} k) \prod_{r \neq n}^p T_{i_rj_r} \phi} \right] + \frac{1}{2} \partial^k \partial^l Y_{(2^{p+1})} T_{kl} \prod_{r=1}^p T_{i_rj_r} \phi.
\]

\[(3.36)\]

Interestingly, the $\tilde{\mathcal{F}}^{[p]}$ satisfy the two-column projected Bianchi identities

\[
(p+2) Y_{(2p,1)} \partial_k \tilde{\mathcal{F}}^{[p]}(\phi)_{i_1j_1, \ldots, i_pj_p} - \partial^l \tilde{\mathcal{F}}^{[p+1]}(\phi)_{i_1j_1, \ldots, i_pj_p, k} = 0,
\]

\[(3.37)\]

that are free from the classical anomalies related to the constraints. As a result, the gauge invariance of the first series of terms in the Lagrangians \((3.35)\) is essentially manifest, and
consequently the Lagrange multipliers, that we now denoted directly $B_{ijkl}$ as was the case for their gauge-invariant contributions to the $\varphi$ field equations (2.52), are also gauge invariant. In addition, this way of presenting the Lagrangians makes the terms quadratic in $\varphi$ manifestly self adjoint. An unconstrained Lagrangian, that however differs from (3.26) by a field redefinition of the multipliers $\beta_{ijkl}$, obtains rather simply in this case as

$$\tilde{L}(\varphi, \Phi_i, B_{ijkl}) = \tilde{L}_C(\varphi - \partial^i \Phi_i, B_{ijkl}).$$

(3.38)

Notice that the constrained gauge invariance of the original Lagrangian guarantees that $\tilde{L}_C$ depends on the symmetrized traces of the $\Phi_i$, and thus on the composite compensators $\alpha_{ijk}$ but not on the naked $\Phi_i$, as in the other constructions.

Let us stress, to conclude, that the terms involving the various types of Lagrange multipliers present in eqs. (3.26), (3.31) and (3.35) have the structure that surfaced in the two-family case. The arguments presented there still apply, so that transformations of the type

$$\delta \beta_{ijkl} = \eta^{mn} L_{ijkl, mn}$$

(3.39)

generalize to all these presentations and to arbitrary numbers of index families the local symmetry described in Section 2.1.

3.2 The field equations

As we have seen in Section 2.2 for the case of two-family gauge fields, recasting the Lagrangian (3.26) in the form (3.31) is particularly convenient when deriving the field equations for $\varphi$, the $\Phi_i$ and the $\beta_{ijkl}$. Starting from eq. (B.11), the $\phi$ variation of (3.31) thus yields

$$E_{\varphi} : \sum_{p=0}^{N} \frac{(-1)^p}{p!(p+1)!} \eta^{i_1 j_1} \cdots \eta^{i_p j_p} \mathcal{F}^{[p]}(\phi)_{i_1 j_1, \ldots, i_p j_p} + \frac{1}{2} \eta^{ij} \eta^{kl} B_{ijkl}$$

$$- \frac{1}{4} \sum_{p=0}^{N} \frac{(-1)^p}{p!(p+1)!} \eta^{i_1 j_1} \cdots \eta^{i_p j_p} \partial^k \partial^l Y_{\{4, 2^{p-1}\}} T_{kl} \phi^{[p]}_{i_1 j_1, \ldots, i_p j_p} = 0,$$

(3.40)

where

$$B_{ijkl} = \gamma_{ijkl} - \frac{1}{2} \sum_{p=0}^{N-1} \frac{(-1)^p}{p!(p+3)!} \eta^{m_1 n_1} \cdots \eta^{m_p n_p} \partial_{(i} \partial_{j} \phi^{[p+1]}_{kl), m_1 n_1, \ldots, m_p n_p},$$

(3.41)

or more concisely

$$B_{ijkl} = \gamma_{ijkl} - \frac{1}{12} \delta_{2}[\eta T] \partial_{(i} \partial_{j} T_{kl}) \phi.$$

(3.42)

The arguments summarized in Appendix C show that, due to the $\{4, 2^{p-1}\}$ projections of the multiple traces of $\phi$, the second line of eq. (3.40) is proportional to the double-trace constraints, and therefore can be eliminated on-shell. In conclusion, using the approach already described for two-family fields and thus performing the shifts of eq. (3.32), the equations of motion of the unconstrained theory can be finally cast in the form

$$E_{\varphi} : \sum_{p=0}^{N} \frac{(-1)^p}{p!(p+1)!} \eta^{i_1 j_1} \cdots \eta^{i_p j_p} A^{[p]}_{i_1 j_1, \ldots; i_p j_p} + \frac{1}{2} \eta^{ij} \eta^{kl} B_{ijkl} = 0,$$

$$E_{\beta} : \frac{1}{8} C_{ijkl} = 0,$$

(3.43)

(3.44)
where the first can also be written more concisely

$$E_\varphi : \delta_0[\eta T]A + \frac{1}{2} \eta^{ij} \eta^{kl} B_{ijkl} = 0,$$

(3.45)

Finally, the equations of motion for the compensators $\Phi_i$ are as usual the conservation condition for external currents,

$$\partial_i E_\varphi + \sum_{p=0}^N \frac{2}{p!(p+2)!} \eta^{m_1 n_1} \ldots \eta^{m_p n_p} Y_{\{2p+1\}} T_{m_1 n_1} \ldots T_{m_p n_p} \partial^j \partial^k \partial^l (E_\beta)_{ijkl} = 0,$$

(3.46)

or more concisely

$$E_\Phi : \partial_i E_\varphi + \delta_1[\eta T] \partial^i \partial^k \partial^l C_{ijkl} = 0.$$

(3.47)

As we stressed in the previous section, these results could have been derived even more directly starting from the Lagrangian (3.35) and taking into account the self-adjointness of the portion of its terms that are quadratic in $\phi$.

### 3.3 Comments on the on-shell reduction to $\mathcal{F} = 0$

The next step would be the reduction of (3.43) to $\mathcal{A} = 0$ and thus, after a partial gauge fixing, to the non-Lagrangian Labastida form $\mathcal{F} = 0$. The operators on which these more general systems rest are the $gl(N)$ counterparts of those exhibited in Section 2.3, and the experience developed with two-family fields suggests the emergence, in low enough dimensions, of a similar type of phenomena related to Weyl-like symmetries. In the following we shall discuss their general structure and we shall also illustrate them in a class of significant, if relatively simple, examples with three or more index families. On the other hand, one can argue rather simply that in the $D \to \infty$ limit $N$-family gauge fields exhibit a universal behavior, free of special poles, that can be captured ignoring altogether the intricacies introduced by the $S^{i j}$ operators. In this limit, in fact, the reduction proceeds directly along the lines of the symmetric case, and the field equations reduce manifestly and directly to $\mathcal{F} = 0$. Moreover, there are all reasons to expect that, even for higher values of $N$, special behaviors are confined to dimensions $D \leq 2 N + 1$, where the corresponding models are at most dual to other representations characterized by lower values of $N$, although at the moment we do not have a complete argument to this effect.

#### 3.3.1 Weyl-like symmetries

In discussing the reduction of the Lagrangian equations for two-family fields in Section 2.3, we have combined them with their traces, keeping track explicitly both of $\mathcal{A}$, which is the kinetic operator for the physical field $\varphi$, and of the $B_{ijkl}$, which simply relate the $\beta_{ijkl}$ Lagrange multipliers to the other fields. However, in the sporadic cases where the procedure actually proved problematic, new Weyl-like symmetries surfaced, while keeping track explicitly of the $B_{ijkl}$ played essentially no role in their identification. This fact can be turned to our own advantage when trying to extend the analysis to $N$-family fields, since in this more general setting one is inevitably confronted with higher technical complications, so that a smooth track is highly preferable. To wit, the successive traces of the original $\varphi$ field equation soon become unwieldy, due to the proliferation of the $S^{i j}$ operators, and thus it is convenient to leave out from the start some inessential details. For instance, the crucial option of factoring out the
\[ O[D-2] \] operator in the first step of the reduction procedure is not available with three or more families. Nonetheless, we can illustrate an iterative procedure to characterize the Weyl-like symmetries that emerge whenever the Lagrangian field equations leave some of the traces of \( \mathcal{A} \) undetermined.

As we already stressed in our discussion of two-family fields, the equation of motion (3.43) sets to zero the traceless part of \( \mathcal{A} \), while in special circumstances some of its traces may be left undetermined. In order to get a handle on this phenomenon for \( N \)-family fields, it is useful to study the symmetries of eq. (3.43) under transformations of the type

\[
\delta \mathcal{A} = \eta^{ij} \Omega^{(1)}_{ij},
\]

that in general shift the Einstein-like tensor of eq. (3.16) as

\[
\delta \mathcal{E} = \sum_{p=0}^{N} \frac{(-1)^p}{p!(p+1)!} \eta^{i_1j_1} \ldots \eta^{i_pj_p} \left[ Y_{(2^p)} T_{i_1j_1} \ldots T_{i_pj_p}, \eta^{kl} \right] \Omega_{kl}^{(1)} + \sum_{p=0}^{N} \frac{(-1)^p}{p!(p+1)!} \eta^{i_1j_1} \ldots \eta^{i_pj_p} \eta^{\{kl\}} \left( Y_{(2^p)} T_{i_1j_1} \ldots T_{i_pj_p} \right) \Omega_{kl}^{(1)}. \tag{3.49}
\]

Taking into account the \( \{2^p\} \) Young projection in the family indices, the commutator of eq. (B.9) reduces to

\[
[Y_{(2^p)} T_{i_1j_1} \ldots T_{i_pj_p}, \eta^{kl}] \Omega_{kl}^{(1)} = (D-p+1)Y_{(2^p)} \sum_{n=1}^{p} \prod_{r \neq n} T_{i_rj_r} \Omega_{injn}^{(1)} \tag{3.50}
\]

as can be seen resorting to the techniques of Appendix C. Once the traces in the last term of eq. (3.50) are then moved to the left of the \( S^{ij} \) and the two-column projection of the second line of eq. (3.49) is combined with the first, one is left with

\[
\delta \mathcal{E} = \sum_{p=1}^{N} \frac{(-1)^p}{p!(p+1)!} \eta^{i_1j_1} \ldots \eta^{i_pj_p} Y_{(2^p)} \sum_{n=1}^{p} \prod_{r \neq n} T_{i_rj_r} \left\{ (D-2) \Omega_{injn}^{(1)} + S_k^{(i_n} \Omega_{jn)k}^{(1)} \right\} \\
+ \sum_{p=2}^{N} \frac{(-1)^p}{(p!)^2} \eta^{i_1j_1} \ldots \eta^{i_pj_p} Y_{(4,2^{p-2})} \sum_{n=1}^{p} \left( Y_{(2^{p-1})} \prod_{r \neq n} T_{i_rj_r} \right) \Omega_{injn}^{(1)}. \tag{3.51}
\]

The arguments of Appendix C now guarantee that the \( \{4,2^{p-2}\} \) projection present in eq. (3.51) can be recast in a form where the family indices carried by a pair of \( \eta \) tensors are fully symmetrized. Hence, the terms in the second line can be compensated by a shift of the \( \mathcal{B}_{ijkl} \), in sharp contrast with the whole first line, that vanishes if one tries to symmetrize three or more indices. Therefore, these last terms cannot be canceled redefining the \( \mathcal{B}_{ijkl} \), unless their traceless parts vanish so that they actually embody at least an additional \( \eta \) tensor. As a result, in order to identify a symmetry of the equation of motion (3.43), one ought to distinguish two possibilities. The first one is that

\[
O[D-2]_{ij} \Omega_{kl}^{(1)} \equiv (D-2) \Omega_{ij}^{(1)} + S^{(i} \Omega_{j)k}^{(1)} = 0, \tag{3.52}
\]
which recovers the condition already met for two-family gauge fields in eq. (2.128), while a consistent alternative is

\begin{equation}
(D - 2) \Omega^{(1)}_{ij} + S^k \left( \partial_i \Omega^{(1)}_j \right)_k = \eta^{kl} \Delta^{(1)}_{ij, kl}, \tag{3.53}
\end{equation}

that implies

\begin{equation}
\delta \mathcal{A} = \eta^{ij} \eta^{kl} \Omega^{(2)}_{ij, kl}, \tag{3.54}
\end{equation}

provided the parameters \( \Omega^{(2)}_{ij, kl} \) satisfy some conditions that we are about to specify. This can be justified recalling that the commutator with an \( \eta \) tensor of the \( S \) operators, and thus of the whole \( \mathcal{O}[\lambda] \) operators, is still proportional to an \( \eta \) tensor. As a result, inverting eq. (3.53) forces indeed \( \Omega^{(1)} \) to contain an additional \( \eta \).

Before moving to analyze eq. (3.54), it is worth exploring further the first alternative, recalling a result that we already used in Section 2.3. Namely, the double-trace constraints (3.4), or their on-shell analogs for the \( \mathcal{A} \) tensor, bring about the further condition

\begin{equation}
T_{(ij} T_{kl)} \delta \mathcal{A} = T_{(ij} [\mathcal{O}[D - 2]_{kl}^{mn} \Omega^{(1)}_{mn} + \eta^{mn} T_{(ij} T_{kl)} \Omega^{(1)}_{mn} = 0. \tag{3.55}
\end{equation}

The reader should now appreciate that, if eq. (3.52) holds, this becomes essentially a double-trace constraint for the \( \Omega^{(1)}_{mn} \). These remarks are meant to stress that one really ought to classify shift symmetries of the whole set of conditions that a kinetic operator \( \mathcal{A} \) must satisfy, not just of the field equations. In this respect, it is instructive to describe the effect of the transformation (3.48) on the Bianchi identities, that must be preserved. Starting from the shift of eq. (3.48), one thus finds

\begin{equation}
\partial_i \delta \mathcal{A} - \frac{1}{2} \partial^j T_{ij} \delta \mathcal{A} = - \frac{1}{2} \mathcal{O}[D - 2]_{ij}^{kl} \Omega^{(1)}_{kl} + \eta^{kl} \left( \partial_i \Omega^{(1)}_{kl} - \frac{1}{2} \partial^j T_{ij} \Omega^{(1)}_{kl} \right). \tag{3.56}
\end{equation}

Therefore, quite interestingly, if eq. (3.52) holds, the \( \Omega^{(1)}_{mn} \) must also satisfy the Bianchi identities, so that they can be regarded as genuine Fronsdal-Labastida kinetic operators. Actually, as we already saw for two families, a shift of the type

\begin{equation}
\delta \phi = \eta^{ij} \omega_{ij}, \tag{3.57}
\end{equation}

for the basic field \( \phi \) leads generically to the variation

\begin{equation}
\mathcal{F}(\phi) \rightarrow \mathcal{F}(\phi) + \frac{1}{2} \partial^i \partial^j \mathcal{O}[D - 2]_{ij}^{kl} \omega_{kl} + \eta^{ij} \mathcal{F}(\omega_{ij}) \tag{3.58}
\end{equation}

for the Fronsdal-Labastida tensor. As a result, whenever eq. (3.52) holds, a condition that, we should stress, depends only on the index structure of the tensors involved, one recovers precisely a variation like (3.48), with

\begin{equation}
\Omega^{(1)}_{ij} = \mathcal{F}(\omega_{ij}). \tag{3.59}
\end{equation}

In other words, this line of reasoning finally connects the transformation (3.48) of \( \mathcal{A} \) to an operation effected acting on the fundamental field \( \phi \), as pertains to a symmetry transformation proper. The end result is a link between the special types of fields for which \( \mathcal{A}' \) is not determined by the Lagrangian field equation and a genuine Weyl-like symmetry of the action, that generalizes the results of Section 2.3 to fields with an arbitrary number of index families.

To reiterate, a key lesson of the previous analysis is that the relevant symmetry operations are to preserve, at the same time, the Bianchi identities, the double-trace constraints that the
\( \mathcal{A} \) tensor satisfies on shell and its equation of motion. Keeping this in mind, we can now turn our attention to higher-order shifts of the type

\[
\delta \mathcal{A} = \eta^{k_1 l_1} \ldots \eta^{k_p l_p} \Omega^{(p)}_{k_1 l_1, \ldots, k_p l_p},
\]

where the parameters contain a priori all available independent projections in family indices compatible with those admitted by a product of \( \eta \) tensors, are thus fully symmetric under interchanges of pairs of symmetric indices. These transformations generalize eq. (3.54) and naturally emerge in an iterative fashion, as we shall see shortly. Indeed, for a shift of this form the simplest of the three basic conditions that we just stated, the Bianchi identities, imply the relations

\[
\partial_i \delta \mathcal{A} - \frac{1}{2} \partial^j T_{ij} \delta \mathcal{A} = -\frac{p}{2} \eta^{k_1 l_1} \ldots \eta^{k_{p-1} l_{p-1}} \partial^j \left[ (D - 2) \Omega^{(p)}_{i j, k_1 l_1, \ldots, k_{p-1} l_{p-1}} + \sum_{n=1}^{p-1} \Omega^{(p)}_{k_n (i, j) l_n, \ldots, k_{r \neq n} l_r \neq n, \ldots} + S^m_{i} \Omega^{(p)}_{j m, k_1 l_1, \ldots, k_{p-1} l_{p-1}} \right] + \eta^{k_1 l_1} \ldots \eta^{k_p l_p} \left[ \partial_i \Omega^{(p)}_{k_1 l_1, \ldots, k_p l_p} - \frac{1}{2} \partial^j T_{ij} \Omega^{(p)}_{k_1 l_1, \ldots, k_p l_p} \right] = 0,
\]

so that they are preserved if

\[
(D - 2) \Omega^{(p)}_{i j, k_1 l_1, \ldots, k_{p-1} l_{p-1}} + \sum_{n=1}^{p-1} \Omega^{(p)}_{k_n (i, j) l_n, \ldots, k_{r \neq n} l_r \neq n, \ldots} + S^m_{i} \Omega^{(p)}_{j m, k_1 l_1, \ldots, k_{p-1} l_{p-1}} = 0,
\]

and if \( \Omega^{(p)} \) satisfies them as well. Alternatively, the type for reasoning presented after eq. (3.53) leads one to consider the next class of transformations in the sequence, that obtains replacing \( p \) with \( p + 1 \) in (3.60).

As was the case for the order-\( \eta \) shift determined by \( \Omega^{(1)} \), the non-trivial solutions of eq. (3.62) connect naturally to Weyl-like symmetries, since if eq. (3.62) holds

\[
\delta \varphi = \eta^{k_1 l_1} \ldots \eta^{k_p l_p} \omega^{(p)}_{k_1 l_1, \ldots, k_p l_p}
\]

translates into

\[
\mathcal{F}(\varphi) \rightarrow \mathcal{F}(\varphi) + \eta^{k_1 l_1} \ldots \eta^{k_p l_p} \mathcal{F}(\omega^{(p)}_{k_1 l_1, \ldots, k_p l_p}),
\]

and as a result one is led to identify \( \Omega^{(p)} \) with \( \mathcal{F}(\omega^{(p)}) \). In a similar fashion, if eq. (3.62) admits non-trivial solutions, one can show that the double-trace constraints on \( \varphi \) translate essentially into double-trace constraints on the \( \Omega^{(p)} \). Finally, after some algebra one can also show that an invariance of the equation of motion obtains corresponding to this Weyl-like symmetry. This can be actually done most conveniently restricting the attention to parameters \( \Omega^{(p)} \) that are two-column projected. In fact, the mere assumption that eq. (3.62) admit a non-trivial solution suffices to eliminate all \( S^i_{j} \) from the variation of the Einstein tensor, and then all parameters \( \Omega^{(p)} \) that carry projections with more than two columns can be seen to simply call for a redefinition the \( \mathcal{B}_{ijkl} \). On the other hand, for two-column projected \( \Omega^{(p)} \) the invariance of the equation of motion is less evident, and rests on the precise form that eq. (3.62) takes in this case,

\[
\mathcal{O}[D - p + 1]_{i j}^{\ mn} \Omega^{(p)}_{m n, k_1 l_1, \ldots, k_{p-1} l_{p-1}} = 0.
\]
Let us also point out that the order-$p$ equation (3.62) clearly admits solutions of the type \( \eta \Omega^{(p+1)} \), simply because its structure only feels the space-time dimension \( D \) and the \( gl(D) \) tensorial properties of the quantities involved. However, we would like to emphasize that the eigenvalue equation at level \( p + 1 \) is really different, and thus yields genuinely new solutions. We saw this fact explicitly for two families when we solved in Section 2.3 the reducibility conditions for the double trace of \( S \) for a \( \{2, 2\} \) field in three dimensions.

We defer to a future work a more extensive analysis of these systems, since here we have only confined ourselves to characterizing the relations that signal the onset of the type of phenomena that we investigated in detail in Section 2.3. It would be interesting to solve explicitly eqs. (3.62), or their simpler counterparts (3.65) for two-column parameters, extending to the more intricate \( sl(N) \) weight lattices the results obtained with the \( sl(2) \) analysis. However, we can not refrain from displaying a simple class of explicit solutions. To this end, let us consider two-column gauge fields, that as we saw in Section 2.3 provide an interesting playground to exhibit some of the properties of multi-family tensors. For instance, for \( \{2^N\} \)-projected irreducible fields of the type \( \varphi_{\mu_1 \mu_2^1 \ldots \mu_N^1 \mu_N^2} \), the previous equations greatly simplify, to the extent that only a single independent trace emerges at any order. In Section 4 we shall describe in detail how to deal with this type of irreducible fields, but for the time being suffice it to say that the relevant conditions are

\[
S^k_1 \Omega_{1k} = - \Omega_{11} \quad \text{for } k \text{ fixed and } k \neq 1, \tag{3.66}
\]

that generalize to

\[
S^k_1 \Omega_{1k,22,\ldots,pp} = - \Omega_{11,22,\ldots,pp} \quad \text{for } k \text{ fixed and } k > p \tag{3.67}
\]

for the higher shifts of eq. (3.60). As a result, the \( \frac{N(N+1)}{2} \) conditions of eq. (3.52) effectively reduce to the single equation

\[
(D - 2) \Omega_{11} + 2 \sum_{k=2}^{N} S^k_1 \Omega_{1k} = 0, \tag{3.68}
\]

and finally to

\[
(D - 2N) \Omega_{11} = 0, \tag{3.69}
\]

that clearly admits a non-trivial solution for \( D = 2N \). In a similar fashion, one can recognize that all equations of the form (3.65) are equivalent to the conditions

\[
(D - 2N + p - 1) \Omega_{11,22,\ldots,pp} = 0, \tag{3.70}
\]

so that the \( p \)-th trace of a \( \{2^N\} \)-projected field \( \varphi_{\mu_1 \mu_2^1 \ldots \mu_N^1 \mu_N^2} \) is not determined by the Lagrangian field equation in \( D = 2N - p + 1 \).

4 Irreducible Bose fields of mixed symmetry

The theory of higher-spin fields of mixed symmetry discussed in the previous sections is essentially all one needs to compare with String Theory. This is directly true for the bosonic string, whose fields, as we have stressed, accompany products of mutually commuting string oscillators, so that they are naturally reducible symmetric tensors, while a convenient discussion of the bosonic excitations of superstrings would also require the extension to the case of multi-forms.
This will actually be described in Section 6, and as we shall see it is quite simple. For definiteness, we thus continue momentarily to refer to multi-symmetric tensors, and describe how the previous results can be adapted to irreducible tensor fields, that are more along the lines of what one is used to for low spins.

The condition of irreducibility basically states that the symmetrization of a given line of the corresponding Young tableau with any index belonging to one of the lower lines gives a vanishing result. This type of operation can be simply described via the $S^{ij}$ operators defined in eq. (A.8) of Appendix A: an irreducible higher-spin tensor field thus satisfies the conditions

$$S^{ij} \varphi = 0 \quad (i < j).$$

The Fronsdal-Labastida operator $F$ commutes with the $S^{ij}$ operators, a property that can be proved directly making use of the commutation relations listed in Appendix A. As a result, the constrained dynamical equations, and actually the whole constrained Einstein-like tensor, take exactly the same form for irreducible fields. On the other hand, both the compensator fields and the Lagrange multipliers adapt to the given irreducible components in a way that we now describe. To begin with, let us consider the gauge transformations, that for a reducible field are given in eq. (2.4) so that, as we have seen in detail so far, the theory relies on one independent gauge parameter for any index family. In the irreducible case they should rather read

$$\delta \varphi = Y \partial^i \Lambda_i,$$

where $Y$ is the Young projector corresponding to the irreducible symmetry of $\varphi$, that commutes with $F$ on account of the previous discussion. Indeed, $Y$ projects onto the kernel of the relevant $S^{ij}$, to which $\varphi$ belongs. In general, irreducible fields involve fewer independent gauge parameters than their reducible counterparts, and how this number is reduced was already stated in [19]: the relevant irreducible gauge parameters can be associated to all admissible Young diagrams obtained stripping one box from the diagram corresponding to the gauge field. This result reflects the structure of eq. (4.2), whose right-hand side determines the variation of the field via a tensor product with gradients carrying additional types of Lorentz indices, belonging to the different families, up to a proper projection. The differences with respect to the reducible case are sizable. For instance, a reducible rank-(4,2) bosonic field of the type $\varphi_{\mu_1\mu_2\mu_3\mu_4,\nu_1\nu_2}$ admits two independent gauge parameters $^9\Lambda_{\mu_1\mu_2\mu_3\nu_1\nu_2}^{(1)}$ and $\Lambda_{\mu_1\mu_2\mu_3\nu_1\nu_2}^{(2)}$, as many as those admitted by an irreducible $\{4,2\}$ field, but these are themselves irreducible. Another significant example is provided by reducible fields of rank $(s,s)$, that admit two independent reducible gauge parameters while their irreducible counterparts admit a single irreducible gauge parameter of rank $(s,s-1)$. More generally, if an irreducible gauge field is characterized by a Young diagram containing a number of identical rows, only a single gauge parameter is associated to all of them.

It is perhaps instructive to recover these results in a slightly different way. They follow in fact rather directly from the structure of the $S^{ij}$ operators that implement the irreducibility condition, and from the requirement that eq. (4.1) be invariant under gauge transformations, that can be cast in the form

$$\partial^k \left( S^{ij} \Lambda_k + \delta^i_k \Lambda_j \right) = 0 \quad (i < j),$$

which forces the entry to define a gauge-for-gauge transformation:

$$S^{ij} \Lambda_k + \delta^i_k \Lambda_j = \partial^l \Lambda^i_{[kl]j} \quad (i < j).$$

\(^9\)In the following we shall continue to abide to this notation, introduced in Section 2.4, adapting it to the irreducible case.
Up to the last irrelevant term, one thus obtains a set of constraints that are conveniently analyzed starting from the highest available value of $k$. In this case, in fact, $i$ is necessarily less than $k$, being constrained to be less than $j$, so that eq. (4.4) reduces to the more familiar condition that the last gauge parameter in the chain, $\Lambda_k$, be irreducible,

$$S_{ij}^k \Lambda_k = 0.$$  \hfill (4.5)

The remaining contributions can then be used to determine the other parameters corresponding to lower values of $k$ in terms of this solution and of additional independent ones that emerge, one for each step, from the homogeneous parts of eqs. (4.5). Let us stress that the independent parameters solve further irreducibility conditions, and therefore can only exist if the corresponding Young diagrams are admissible. In conclusion, one thus obtains an irreducible gauge parameter for each admissible Young diagram built stripping one box from the original diagram for the gauge field, precisely as stated in [19]. The independent parameters lead by construction to irreducible gauge transformations where no explicit Young projectors as that present in eq. (4.2) are needed. Let us see how this works, referring to an irreducible gauge parameter adapted to an irreducible projection of the original reducible compensate $\Phi_k$.

$$\text{The independent parameters lead by construction to irreducible gauge transformations where no explicit Young projectors as that present in eq. (4.2) are needed. Let us see how this works, referring to an irreducible gauge parameter adapted to an irreducible projection of the original reducible compensate $\Phi_k$.}$$

To begin with, $\Lambda_1$ could have three irreducible components, a $\{3,2\}$, a $\{4,1\}$ and a $\{5\}$, while $\Lambda_2$ could have a $\{4,1\}$ and a $\{5\}$. Now the second of eqs. (4.6) eliminates directly the $\{5\}$ component of $\Lambda_2$, and then the first eliminates the $\{5\}$ component of $\Lambda_1$. The two $\{4,1\}$ components are then connected by the first equation, while the $\{3,2\}$ component is a zero mode of the first equation, and as such is not constrained. In space-time notation the conclusion is that, when adapted to an irreducible $\{4,2\}$ field, the original reducible gauge transformation (2.158) can be cast in the form

$$\delta \varphi_{\mu_1\mu_2\mu_3\mu_4;\nu_1\nu_2} = \partial_{(\mu_1} \Lambda^{(1)\{3,2\}}_{\mu_2\mu_3\mu_4;\nu_1\nu_2} + \left( \partial_{(\mu_1} \Lambda^{(1)\{4,1\}}_{\mu_2\mu_3\mu_4;\nu_1\nu_2} - \partial_{\nu_1} \Lambda^{(1)\{4,1\}}_{(\mu_1\mu_2\mu_3;\mu_4)\nu_2} \right) . \hfill (4.7)$$

Let us stress again that no explicit Young projector is needed here, since both combinations are already $\{4,2\}$ projected.

In a similar fashion, the unconstrained counterpart $A$ of $F$ is annihilated by the $S_{ij}^k$ operators for $i < j$ only if the compensators satisfy a nested set of conditions that are the direct counterpart of those in eq. (4.4),

$$\partial^i \partial^j \partial^k \left( S_{m,n}^m \alpha_{ijk} + \delta_{m}^{(i} \alpha_{jk)n} \right) = 0 \quad (m < n). \hfill (4.8)$$

These, in their turn, reflect the fact that the independent compensators $\Phi_i$ satisfy the same constraints of eq. (4.4) satisfied by the $\Lambda_i$, consistently with their gauge transformation (2.13). Solving eqs. (4.8) shows that there is a single independent compensator $\alpha_{ijk}$ for any allowed irreducible projection of the original reducible compensators, so that the irreducible $\{4,2\}$ $A$ tensor can be cast in the form

$$A^{(4,2)}_{\mu_1...\mu_4;\nu_1\nu_2} = F^{(4,2)}_{\mu_1...\mu_4;\nu_1\nu_2} - 3 \partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} \alpha^{(1)\{2,1\}}_{\mu_4;\nu_1\nu_2} + \partial_{(\nu_1} \partial_{(\mu_1} \partial_{\mu_2} \alpha^{(1)\{2,1\}}_{\mu_3;\mu_4)\nu_2)} - 3 \partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} \alpha^{(1)\{3\}}_{\mu_4)\nu_1\nu_2} + 2 \partial_{(\nu_1} \partial_{(\mu_1} \partial_{\mu_2} \alpha^{(1)\{3\}}_{\mu_3\mu_4)\nu_2)} - 3 \partial_{\nu_1} \partial_{\nu_2} \partial_{(\mu_1} \alpha^{(1)\{3\}}_{(\mu_2\mu_3\mu_4).} \hfill (4.9)$$

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The other key ingredient of the construction are the Labastida double-trace constraints or, in our unconstrained setting, the corresponding $C_{ijkl}$ tensors. The key irreducibility constraints on these quantities are

$$S^m_n C_{ijkl} + \delta^m_{(i} C_{jkl)n} = 0 \quad (m < n),$$

as can be seen either directly from eq. (2.20) or from the Bianchi identities. Solving them shows that in the irreducible $\{T,2\}$ case there is a single independent quantity of this type, say $C^{(1)}\nu_1\nu_2$ of eq. (2.164), and thus also a single Lagrange multiplier. In fact, in space-time notation the multiplier terms of the Lagrangian for an irreducible $\{4,2\}$ field are simply

$$3 \left( \beta^{(1)(2)} \nu_1\nu_2 - 2 \beta^{(2)(2)} \nu_1\nu_2 + \beta^{(3)(2)} \nu_1\nu_2 \right) C^{(1)}\nu_1\nu_2,$$

so that the single surviving Lagrange multiplier $\tilde{\beta}$ is the particular combination of the three $\beta^{(i)(2)}$ above.

Similar steps allow to express the Lagrangian in terms of the independent compensators $\alpha^{(1)(2,1)}_{\mu; \nu_1\nu_2}$ and $\alpha^{(1)(3)}_{\mu_1\mu_2\mu_3}$ of eq. (4.9), and determine the independent irreducible components of the multiple traces of $\mathcal{A}$ and of their divergences. In particular, for two families the relevant conditions are

$$S^m_n T_{ij} \mathcal{A} + \delta^m_{(i} T_{j)n} \mathcal{A} = 0,$$

$$S^m_n T_{ij} T_{kl} \mathcal{A} + \delta^m_{(i} T_{j)n} T_{kl} \mathcal{A} + \delta^m_{(k} T_{l)n} T_{ij} \mathcal{A} = 0,$$

$$S^m_n \partial_i T_{jk} \mathcal{A} + \delta^m_i \partial_n T_{jk} \mathcal{A} + \delta^m_{(j} T_{k)n} \partial_i \mathcal{A} = 0.$$  \hspace{1cm} (4.12)

In conclusion, the Lagrangian for an irreducible $\{4,2\}$ gauge field, when expressed in terms of independent quantities, can be presented in the form

$$\mathcal{L} = \frac{1}{2} \varphi^{\{4,2\} \mu_1...\mu_4; \nu_1\nu_2} \left\{ \mathcal{A}^{\{4,2\} \mu_1...\mu_4; \nu_1\nu_2} - \frac{1}{2} \eta_{(\mu_1\mu_2} \mathcal{A}^{\nu,\{2,2\} \mu_3\mu_4); \nu_2\nu_2} \right\}$$

$$- \frac{1}{2} \eta_{(\mu_1\mu_2} A^{\nu,\{3,1\} \mu_3\mu_4); \nu_1\nu_2} + \frac{1}{4} \eta_{(\nu_1(\mu_1} A^{\nu,\{3,1\} \mu_2\mu_3; \mu_4)} \nu_2)$$

$$- \frac{1}{2} \eta_{(\mu_1\mu_2} A^{\nu,\{4\} \mu_3\mu_4); \nu_1\nu_2} + \frac{3}{4} \eta_{(\nu_1(\mu_1} A^{\nu,\{4\} \mu_2\mu_3\mu_4)} \nu_2) - 3 \eta_{\nu_1\nu_2} A^{\nu,\{4\} \mu_1...\mu_4}$$

$$+ \frac{1}{18} \left( 2 \eta_{\nu_1\nu_2} \eta_{(\mu_1\mu_2} - \eta_{\nu_1(\mu_1} \eta_{\mu_2) \nu_2} \right) \left( A^{\nu\nu}-A^{\nu\nu}|_{\mu_3\mu_4} \right)$$

$$- 4 \alpha^{(1)(2,1)}_{\mu; \nu_1\nu_2} \partial^\lambda \left\{ A^{\nu,\{2,1\} \lambda_{\mu; \nu_1\nu_2} - \frac{1}{18} Y_{\{2,1\} \nu_1\nu_2} \left( A^{\nu\nu}_\mu - A^{\nu\nu}_\mu \right) \mu \lambda \right\}$$

$$- 10 \alpha^{(1)(3)}_{\mu_1\mu_2\mu_3} \partial^\lambda \left\{ A^{\nu,\{3\} \lambda_{\mu_1; \mu_2\mu_3} - \frac{1}{18} \eta_{(\mu_1\mu_2} \left( A^{\nu\nu}_\mu - A^{\nu\nu}_\mu \right) |_{\mu_3}) \lambda \right\}$$

$$+ 3 \tilde{\beta}^{\nu_1\nu_2} C^{(1)}\nu_1\nu_2.$$  \hspace{1cm} (4.13)

Alternatively, one could work with the original Einstein-like tensor of eq. (2.165), that would maintain the same form for projected gauge fields, as we stressed above, but whose traces would not be independent. The procedure that we have illustrated with this example is of general import, since its basic ingredients, the irreducibility conditions and their natural generalizations to multiple traces, clearly hold for generic multi-family gauge fields.
5 Elimination of higher derivatives

In this section we would like to describe a systematic way of eliminating the higher-derivative terms appearing in our unconstrained Lagrangians while preserving, insofar as possible, the simplicity of the construction.

5.1 The problem

Let us begin by stressing once more that all terms involving more than two derivatives enter our Lagrangians in connection with compensator fields, that can be eliminated by a suitable partial gauge fixing. And indeed all tests available in the free theory, and in particular the current-exchange amplitudes for symmetric tensors of [27, 40], confirm the obvious feeling that their presence does not bring about any complications. Still, it is both interesting and natural to try and bring these systems closer to their lower-spin counterparts.

An unconstrained formulation of free symmetric higher spins without higher-derivative terms and with a fixed number of extra fields was first attained in [28]. It is an interesting off-shell variant of the on-shell truncation of the “triplets” [29] of String Field Theory [15] obtained in [23, 32] (see also [41] for some recent developments in the “frame-like” formalism). Our aim in this section is to try and extend these types of results to mixed-symmetry fields, starting however from the alternative construction of [25], that was tailored to the compensator constructions of [26, 27].

The idea underlying our procedure is the replacement of the $\alpha_{ijk}$ of (2.9) with other fields whose dimensions are at least as high as that of the gauge potentials $\varphi$. Let us stress that such fields are not pure gauge, but for instance in the solution for the one-family case given in [25] they vanish on shell.

It is simple to construct kinetic tensors similar to $A$ but free of higher derivatives, and indeed several options are available, but it is less straightforward to make sure that the resulting Lagrangians do not propagate additional degrees of freedom. Let us begin with a brief analysis of symmetric tensors, aimed at clarifying the origin of some potential difficulties. A close scrutiny of this simpler case will provide clues for the more general approach that will be presented in the next section. For fully symmetric bosonic fields, there are in principle several options to compensate the gauge transformation of the Fronsdal tensor \[ \delta \mathcal{F} = \frac{1}{2} \partial \partial \partial \Lambda'. \] (5.1)

In the first part of this section, for brevity, we are hiding the single-valued family indices, so that here gradients are denoted by $\partial$ rather than $\partial^1$, divergences are denoted by $\partial \cdot$ rather than $\partial_1$ and the only available trace, $T_{11}$, is denoted by a “prime”. The choice made in [23, 32, 26]\footnote{A reader familiar with some previous papers, such as [26, 27], will recognize that there we found it more convenient to define products of mutually commuting objects, and of derivatives in particular, with different normalizations. For example, in this case we would have introduced the convenient symbol $\partial^3$ to denote the product of three $\partial$'s, up to an overall factor 6. For the sake of clarity and for an easier comparison with the multi-family case, however, here we prefer to conform, insofar as possible, to the notation of the preceding sections.} was to resort to a compensator $\alpha$, the one-family analogue of the $\alpha_{ijk}$ but treated as an independent...
field, with gauge transformation

\[ \delta \alpha = \Lambda'. \]  \hspace{1cm} (5.2)

On the other hand, three other choices could provide in principle viable alternatives avoiding the introduction of higher derivatives:

\[ \theta(1) : \quad \delta \theta(1) = \partial \Lambda' \rightarrow A_{\theta(1)} = \mathcal{F} - \frac{1}{2} \partial \partial \theta(1), \]  \hspace{1cm} (5.3)

\[ \theta(2) : \quad \delta \theta(2) = \partial \partial \Lambda' \rightarrow A_{\theta(2)} = \mathcal{F} - \frac{1}{2} \partial \theta(2), \]  \hspace{1cm} (5.3)

\[ \theta(3) : \quad \delta \theta(3) = \partial \partial \partial \Lambda' \rightarrow A_{\theta(3)} = \mathcal{F} - \frac{1}{2} \partial \theta(3). \]  \hspace{1cm} (5.3)

Notice that in all these cases the additional field is not manifestly pure gauge, but one can aim nonetheless at constructions where this condition is enforced on-shell, for instance, via a term of the form

\[ \langle \chi, \theta(1) - \partial \alpha \rangle, \]  \hspace{1cm} (5.4)

where for definiteness we are focusing on the case of \( \theta(1) \), and where \( \chi \) is a gauge-invariant Lagrange multiplier.

The three kinetic tensors \( A_{\theta(i)} \) indeed do not contain higher derivatives and satisfy the Bianchi identities

\[ \theta(1) : \quad \partial \cdot A_{\theta(1)} - \frac{1}{2} \partial A'_{\theta(1)} = -\frac{1}{4} \partial \{ \partial \partial \varphi'' + 2 \Box \theta(1) - 2 \partial \partial \theta(1) - \partial \partial \theta'(1) \} \equiv -\frac{1}{2} \partial C(1), \]  \hspace{1cm} (5.5)

\[ \theta(2) : \quad \partial \cdot A_{\theta(2)} - \frac{1}{2} \partial A'_{\theta(2)} = -\frac{1}{4} \{ \partial \partial \varphi'' - 2 \Box \theta(2) + \partial \partial \theta'(2) \} \equiv -C(2), \]  \hspace{1cm} (5.5)

\[ \theta(3) : \quad \partial \cdot A_{\theta(3)} - \frac{1}{2} \partial A'_{\theta(3)} = -\frac{1}{4} \{ \partial \partial \varphi'' + 2 \partial \theta(3) - \partial \theta'(3) \} \equiv -\frac{3}{2} C(3). \]  \hspace{1cm} (5.5)

These relations already display a delicate feature of this approach, related to the request that the resulting Lagrangians only describe the propagation of massless, irreducible spin-s degrees of freedom. For the sake of comparison, let us recall that in the theory described by the minimal Lagrangian (2.47)

- the equation for \( \varphi \) can be reduced to the Fronsdal form;
- the compensator \( \alpha \) can be set to zero with a gauge choice;
- on-shell the Lagrange multiplier \( \beta \) can be expressed in terms of the field \( \varphi \), which guarantees that it does not carry additional degrees of freedom.

If one were to retrace our usual procedure, introducing in the Lagrangian quantities of the form \( \langle \beta(i), C(i) \rangle, \ i = 1, 2, 3, \) with \( \beta(i) \) independent Lagrange multipliers, these fields would enter the equation for \( \varphi \) in combinations involving their divergences, in a way that depends on the corresponding choice for \( \theta(i) \). Thus, differently from the usual case, the equation for \( \varphi \) would not fix directly the multipliers in terms of the gauge potential, simply because not all gradients can be factored in eq. (5.5). In addition, the presence of D’Alembertian operators in \( C(1) \) and \( C(2) \) would have the unpleasant consequence of introducing in the equations for the compensators \( \theta(1) \) and \( \theta(2) \) wave operators acting on the multipliers \( \beta(1) \) and \( \beta(2) \).

An alternative approach that is devoid of these difficulties was proposed in [25], and is based on the following two observations:
• rather than introducing potentially propagating multipliers, one can look for combinations of fields already present in the Lagrangian and possessing the required gauge transformations;

• in order to guarantee that no spurious degrees of freedom propagate, one can add an independent Lagrange multiplier $\beta$ to enforce the usual double trace constraint $\langle \beta, C \rangle$ of [26, 27], together with an additional one for the constraint relating the $\theta(i)$ to $\alpha$, as in eq. (5.4). In this construction, these additional multipliers would be gauge invariant, while their equations of motion would provide the conditions needed to recover the Fronsdal form.

Actually, while the second step can always be performed, only $\theta(1)$ can build a combination transforming as an effective multiplier. In this case the complete Lagrangian without higher derivatives is then

$$\mathcal{L} = \frac{1}{2} \langle \varphi, A_{\theta(1)} \rangle - \frac{1}{2} \eta A'_{\theta(1)} + \frac{1}{4} \langle \theta(1), A'_{\theta(1)} \rangle + \frac{1}{4} \langle \varphi' - \theta(1), C(1) \rangle + \langle \chi, \theta(1) - \partial \alpha \rangle + 3 \langle \beta, \varphi'' - 4 \partial \cdot \alpha - \partial \alpha' \rangle,$$

where $(\varphi' - \theta(1))$ is precisely the combination of the original fields playing the role of an effective Lagrange multiplier for $C(1)$. Notice that the terms of this Lagrangian involving $A_{\theta(1)}$ have the same form as in eq. (2.47), since they are still driven by the Bianchi identity. Furthermore, out of the three fields of eq. (5.3) only $\theta(1)$ can build a direct local coupling to $A'_{\theta(1)}$, that already contains two derivatives, precisely because it has dimension one. Indeed, in [25] it was shown that the resulting equations of motion set to zero on-shell some of the additional fields and relate the others to $\varphi$, so that the system reduces correctly to the Fronsdal theory after a partial gauge fixing. The field content of (5.6) is apparently the minimal one allowing a local description of unconstrained symmetric bosons of all spins without higher derivatives.

For fields of mixed symmetry no simple solution of this type is apparently available, so that one is led to consider more general possibilities for the structure of the compensators. In order to appreciate the difficulties that one is confronted with in the general case, let us try a straightforward extension of the approach of [25], defining the gauge invariant combination

$$A_\theta = \mathcal{F} - \frac{1}{2} \partial^i \partial^j \theta_{ij},$$

where

$$\delta \theta_{ij} = \frac{1}{3} \partial^k T_{(ij} A_k).$$

As for symmetric fields, the subtleties in the construction of the full Lagrangian can be traced to the structure of the Bianchi identity satisfied by $A_\theta$,

$$\partial_i A_\theta - \frac{1}{2} \partial^j T_{ij} A_\theta = \frac{1}{2} \partial^j C^\theta_{ij}.$$

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12For instance, the Lagrangian of [28] provides an elegant off-shell truncation of the “triplets” of [29], and contains the higher-spin field $\varphi$, the compensator $\alpha$, two auxiliary fields $C$ and $D$ and two Lagrange multipliers $\lambda_1$ and $\lambda_2$. It can actually be reduced to (5.6) in two steps. First, solving the equations for $C$ and $\lambda_1$ and substituting back, so as to eliminate $C$, $D$ and $\lambda_1$, and finally making the identifications $\partial \alpha = \theta(1)$, $\lambda_2 = \beta$ and adding the constraint $\langle \chi, \theta(1) - \partial \alpha \rangle$.  

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The gauge invariant constraint tensors $C_{ij}^{\theta}$ generalize the $C_{(1)}$ of the previous example, and are given by

$$C_{ij}^{\theta} = \frac{1}{6} \partial^k \partial^l T_{ij} T_{kl} \varphi + \Box \theta_{ij} - \partial^k \partial_k \theta_{ij}$$

$$- \frac{1}{8} \partial^k \partial^l \left( 2 T_{ij} \theta_{kl} - 2 T_{kl} \theta_{ij} + T_{i(k} \theta_{l)j} + T_{j(k} \theta_{l)i} \right),$$

so that, as expected, a D’Alembertian operator is again present in the constraint tensor associated to the “lower derivative” compensators $\theta_{ij}$.

On account of the results displayed here and in the previous sections, in the general case of $N$-family fields, one can consider the trial Lagrangian

$$L_0 = \frac{1}{2} \langle \varphi, \sum_{p=0}^N \frac{(-1)^p}{p!(p+1)!} \eta^{i_1 j_1} \cdots \eta^{i_p j_p} A_{i_1 j_1, \ldots, i_p j_p} \rangle$$

$$+ \frac{3}{4} \langle \theta_{ij}, \sum_{p=0}^{N-1} \frac{(-1)^p}{p!(p+3)!} \eta^{i_1 j_1} \cdots \eta^{i_p j_p} A_{i_1 j_1, i_1 j_1, \ldots, i_p j_p} \rangle$$

$$+ \langle \chi_{ij}, \theta_{ij} - \partial^k \alpha_{ijk}(\Phi) \rangle,$$

where we already included the constraints on the $\theta_{ij}$ compensators. The gauge variation of eq. (5.11) generates the remainder

$$\delta L_0 = \frac{1}{4} \left( \sum_{p=0}^N \frac{(-1)^p}{p!(p+2)!} \partial_{(i} \eta^{m_1 n_1} \cdots \eta^{m_p n_p} Y_{[2p,1]} T_{m_1 n_1} \cdots T_{m_p n_p} \Lambda_{j)} \right) C_{ij}^{\theta},$$

and again, in strict analogy with what was already seen in the one-family case, one can try to look for combinations of the fields already present in the Lagrangian possessing the gauge transformation in (5.12). However, in sharp contrast with the previous example, with two or more families the generalization of the composite multipliers of the symmetric case transform as

$$\delta (T_{ij} \varphi - \theta_{ij}) = \partial_{(i} \Lambda_{j)} + \frac{1}{3} \left( 2 T_{ij} \Lambda_k - T_{k(i} \Lambda_{j)} \right),$$

with a relative factor between the two terms that does not allow to cancel the first term in (5.12).

In order to recover a formulation of mixed-symmetry gauge fields that is free of higher-derivative terms, one is therefore led to consider more general possibilities for the compensators $\theta_{ij}$. A systematic procedure leading to a successful choice for general multi-family fields is outlined in the next section.

### 5.2 A GENERAL SOLUTION

As we have seen, the remainder or “classical anomaly” in the Bianchi identities is the crucial ingredient when one tries to build unconstrained Lagrangians, while the main virtue of our basic choice (2.12) is that in the Bianchi identity (2.18) the remainder does not contain the D’Alembertian operator. This grants from the very beginning that no pathologies related to the behavior of the multipliers $\beta_{ijkl}$ defined in (2.40) can present themselves. In the search for unconstrained Lagrangians without higher-derivative terms, one should thus be ready to consider more complicated gauge invariant completions of the Fronsdal-Labastida tensor, if their Bianchi
identities are free of D’Alembertian operators acting on the compensators. On the other hand, since the standard form (2.7) of the variation of $F$ does not seem to provide a good guidance, it might be helpful to try and re-express it in alternative ways, so as to suggest which additional fields are actually needed.

A conceptually simple possibility is to separate in the gauge parameters $\Lambda_i$ two contributions, letting

$$\Lambda_i = \Lambda_i^{(t)} + \eta^{jk} \Lambda_i^{(p)jk}, \quad (5.14)$$

where the $\Lambda_i^{(t)}$ satisfy the conditions

$$T_{(ij} \Lambda_i^{(t)k)} \equiv 0, \quad (5.15)$$

so that they are effectively the gauge parameters of the Labastida theory, while the $\Lambda_i^{(p)jk}$ carry the full amount of gauge symmetry that one would like to add, and are such that

$$T_{(ij} \eta^{lm} \Lambda_k^{(p)lm} \equiv T_{(ij} \Lambda_k^{(p)}) \cdot \quad (5.16)$$

Working in terms of $\Lambda_i^{(t)}$ and $\Lambda_i^{(p)jk}$, the gauge variation of the Labastida tensor reads

$$\delta F = \frac{1}{6} \partial^i \partial^j \partial^k \left\{ 3 D \Lambda_i^{(p)jk} + 2 S^l_{(i} \Lambda_j^{(p)k)} + \eta^{lm} T_{(ij} \Lambda_k^{(p)lm} \right\}, \quad (5.17)$$

a result that calls for the introduction of compensator fields $\theta_{ij}$ such that

$$\delta \theta_{ij} = \partial^k \Lambda_i^{(p)jk} \cdot \quad (5.18)$$

One is thus led to define the rather unconventional gauge invariant kinetic tensor

$$A_\theta = F - \frac{1}{2} \left[ (D - 2) \partial^i \partial^j + \partial^k \partial^l S_{ij} \right] \theta_{ij} - \eta^{ij} \mathcal{F}_{ij}(\theta), \quad (5.19)$$

where in particular the $\mathcal{F}_{ij}(\theta)$ are Labastida tensors for the collection of $\theta_{ij}$ fields. According to our previous discussion, despite the rather involved form of $A_\theta$, it is the structure of its Bianchi identity that should tell us whether such a choice for the compensators might prove useful in the construction of a suitable gauge invariant Lagrangian. The explicit computation gives the gratifying result

$$\partial_i A_\theta - \frac{1}{2} \partial^j T_{ij} A_\theta = -\frac{1}{12} \partial^j \partial^k \partial^l T_{(ij} T_{kl)} \left( \varphi - \eta^{mn} \theta_{mn} \right), \quad (5.20)$$

where all contributions involving a D’Alembertian operator acting on $\theta_{ij}$ have disappeared thanks to the presence of the Labastida tensors $\mathcal{F}_{ij}(\theta)$.

The previous result is clearly a strong hint that the definition (5.19) can lead to the desired solution. That this is actually the case is confirmed by the following observations, that also suggest a simpler route leading to complete Lagrangians. Let us in fact recall that the completion of the gauge symmetry of the theory, leading from the Fronsdal-Labastida tensor $F$ to the basic unconstrained tensor $A$ defined in (2.10), could be attained working directly at the level of the gauge field $\varphi$. As observed in Section 2.1, the tensor $A$ can indeed be regarded as the result of a Stueckelberg-like substitution performed in $F$

$$\varphi \rightarrow \varphi - \partial^i \Phi_i, \quad (5.21)$$
where the fields $\varphi$ and $\Phi_i$ possess the gauge transformations

$$
\delta \varphi = \partial^i \Lambda_i,
\delta \Phi_i = \Lambda_i.
$$

(5.22)

Now, in order to better understand the meaning of the $\theta_{ij}$ and of their gauge transformations (5.18), it is worth stressing the otherwise obvious fact that under the substitution (5.21) any function of $\varphi$ would be gauge invariant. One might thus wonder whether the choice (5.21) is really the conceptually simplest option for our purposes, since when performing this shift in the Fronsdal-Labastida tensor one overlooks the fact that the theory already possesses a constrained gauge invariance, that is somehow built anew in terms of the additional fields.

In other words, the Stueckelberg-like shift (2.15) can make any theory gauge invariant, even one that does not possess, to begin with, a gauge symmetry. Our aim here is rather to enlarge to the unconstrained level a constrained gauge symmetry that is already present. In this sense, the substitution (5.21) is really somewhat unnatural, while it looks more logical, if technically more involved, to decompose the gauge variation of $\varphi$, following eq. (5.14), as

$$
\delta \varphi = \partial^i \Lambda^{(t)}_i + \eta^{ij} \partial^k \Lambda^{(p)}_{ijk},
$$

(5.23)

and then to exploit the transformation properties (5.18) of the $\theta_{ij}$ in order to define an improved form of the substitution (2.15),

$$
\varphi \rightarrow \varphi - \eta^{ij} \theta_{ij},
$$

(5.24)

that indeed turns the Fronsdal-Labastida tensor (2.6) into the kinetic tensor $A_\theta$ defined in (5.19). The reader will not fail to notice that this type of shift was already encountered when we discussed Weyl-like symmetries. Here, however, we do not require that the operator $O[D-2]$ of eq. (2.86) annihilate $\theta_{ij}$, but we simply assign to these fields a specific gauge transformation.

Indeed, the key difference between (5.24) and the naive Stueckelberg shift (2.15) is that the combination $\varphi - \eta^{ij} \theta_{ij}$ transforms precisely as the Fronsdal-Labastida field, even in the presence of unconstrained gauge parameters $\Lambda_i$. The substitution can now be effected in any of the two Labastida-like Lagrangians that we have presented in Section 3 in eqs. (3.31) and (3.35). For instance, referring to eq. (3.31) and adding a further constraint forcing the $\theta_{ij}$ to be pure gauge, in the same spirit as in (5.6), leads to

$$
\mathcal{L} = \frac{1}{2} \langle \varphi - \eta^{mn} \theta_{mn} , A_\theta + \sum_{p=1}^N \frac{(-1)^p}{p!(p+1)!} \eta^p A_\theta^{[p]} \rangle \\
+ \frac{1}{8} \langle \beta_{ijkl} , T_{(ij} T_{kl)} (\varphi - \eta^{mn} \theta_{mn}) \rangle + \langle \chi_{ij} , \theta_{ij} - \partial^k \Pi_{ijk}^{lmn} \alpha_{lmn}(\Phi) \rangle.
$$

(5.25)

This Lagrangian is invariant under the gauge transformations that we already defined, and that we collect here:

$$
\delta \varphi = \partial^i \Lambda_i, \\
\delta \theta_{ij} = \delta^k \Lambda^{(p)}_{ijk}, \\
\delta \alpha_{ijk} = \frac{1}{3} T_{(ij} \Lambda_{k)},
$$

(5.26)

$$
\delta \beta_{ijkl} = \frac{1}{2} \sum_{p=0}^N \frac{(-1)^p}{p!(p+2)!} \partial_{(i} \partial_j \partial_{k]} \eta^{m_1 n_1} \cdots \eta^{m_p n_p} Y_{(2p,1)}^{\{1\}} T_{m_1 n_1} \cdots T_{m_p n_p} \Lambda^{(t)}_{l)},
$$

$$
\delta \chi_{ij} = 0.
$$
In (5.25), $\Pi_{lmn}^{ijk}$ is the projector defining the solution of eq. (5.16) for $\Lambda^{(p)}_{ijk}$ in the form

$$\Lambda^{(p)}_{ijk} = \Pi_{lmn}^{ijk} T_{lmn} \Lambda_n.$$  (5.27)

Computing the projector $\Pi_{lmn}^{ijk}$ represents the main technical difficulty of this construction, and indeed we were not able to obtain for it an explicit closed form in the general case, although it is rather straightforward, if lengthy, to compute it explicitly in specific cases of interest. Thus, for instance, in the one-family case of symmetric tensors the explicit relation between $\Lambda^{(p)}_{ijk}$ and $\Lambda^{[k+1]}$ is

$$\Lambda^{(p)} = \sum_{k=0}^{[s/2]} \frac{(-1)^k k!}{\prod_{i=0}^k \sum_{j=0}^i \left[ D + 2(s - 2j - 3) \right]} \eta^k \Lambda^{[k+1]},$$  (5.28)

where $\Lambda^{[k+1]}$ denotes the $(k+1)$-th trace of $\Lambda$ and $\eta^k$ is a product of $k$ Minkowski metric tensors written with unit overall normalization and with the minimal number of terms needed to be totally symmetric.

Finally, the reduction of the equations of motion to the Fronsdal-Labastida form $\mathcal{F} = 0$ would follow steps similar to those illustrated for the minimal higher-derivative Lagrangians.

6 Multi-form gauge fields

So far we have discussed in some detail the properties of multi-symmetric gauge fields of the type $\varphi_{\mu_1 \ldots \mu_s_1 ; \nu_1 \ldots \nu_s_2 ; \ldots}$, with arbitrary numbers of “families” of fully symmetric index sets. As we have stressed, the interest of a general theory for these higher-spin fields lies to a large extent in their direct link with the massive excitations of the bosonic string. Fields of this type are in fact natural partners of generic products of bosonic string oscillators of the type $\alpha_{\mu_1} \ldots \alpha_{\mu_s_1} \alpha_{\nu_1} \ldots \alpha_{\nu_s_2} \ldots$. The resulting theory, however, is not fully conventional for two reasons. First, as we have stressed, these types of fields carry reducible representations of the Lorentz group. Moreover, they are vastly redundant, since fields of this type carrying a single index for each family would suffice to build arbitrary irreducible representations of the Lorentz group. Nonetheless, in comparing with the superstring it is worthwhile to study yet another class of fields, multi-forms of the type $\varphi_{[\mu_1 \ldots \mu_s_1 ; [\nu_1 \ldots \nu_s_2 ; \ldots}$, where now the sets of Lorentz indices belonging to a given family are totally antisymmetric, rather than totally symmetric as before. While this is clearly another step in the direction of redundancy, it is a fact that fields of this type accompany arbitrary products of fermionic oscillators in massive superstring excitations. It is therefore both interesting and useful to also have at one’s disposal a general theory of reducible multi forms, and even for more general types of fields combining some fully symmetric and some fully antisymmetric index families, that exhaust all possible types of massive superstring excitations.

In this section we thus describe briefly how the present theory can be adapted to the case of multi-form gauge fields. The mixed case can then be treated combining the prescriptions we are about to spell out with the results of the preceding sections. It was originally pointed out in [12] that this variant of the Labastida theory can be built in a remarkably simple fashion. With some changes of notation and conventions, we shall soon confirm this conclusion within our formalism.

The key step in the transition to multi-forms is to modify the definition of the few ingredients of the construction, the gradient $\partial^i$, the divergence $\partial_i$, the trace $T_{ij}$ and the metric tensor in
family space $\eta^{ij}$, making the first two anticommuting and the last two antisymmetric. Once this is done, the previous results change as follows.

To begin with, the gauge transformations maintain the same form of eq. (2.4), but now the gauge-for-gauge transformations are associated to symmetric parameters,

$$\delta \Lambda_i = \partial^j \Lambda_{(ij)} ,$$

consistent with the fact that they exist already for a single family, i.e. for the usual form fields.

In addition, the Fronsdal-Labastida operator maintains the form of eq. (2.6), but its gauge variation becomes

$$\delta F = \frac{1}{6} \partial^i \partial^j \partial^k T_{[ij} \Lambda_{k]} ,$$

since the derivatives are now anticommuting. As a result, the trace constraints on the gauge parameters now become

$$T_{[ij} \Lambda_{k]} = 0 ,$$

so that the $\alpha_{ijk}$ are now fully antisymmetric in their family indices. Furthermore, the Bianchi identity now becomes

$$\partial_i F - \frac{1}{2} \partial^j T_{ij} F = - \frac{1}{12} \partial^j \partial^k \partial^l T_{[ij} \Lambda_{kl]} \varphi ,$$

so that one is now led to constrain the fully antisymmetrized double trace of $\varphi$. In deriving these results, we have made use of the multi-form variants of eqs. (A.9) and (A.10), that read

$$\{ \partial_i , \partial^j \} = \square \delta_i^j ,$$

$$[T_{ij} , \partial^k ] = - \partial_{[i} \delta_{j]}^k .$$

Proceeding as in the previous sections, one can now introduce the compensators, or better the corresponding $\Phi_i$, and define the $A$ tensor, which is the starting point to build unconstrained Lagrangians. The last ingredients are the Lagrange multipliers, $\beta_{ijkl}$, that are now fully antisymmetric in their family indices. As a result, the $\alpha_{ijk}$ and the $\beta_{ijkl}$ start to play a role at three and four families, respectively. A further key ingredient are the traces of the Bianchi identity, the first of which now reads

$$\partial_i T_{jk} A - \frac{1}{2} \partial_{[j} T_{k]} A = \frac{1}{2} \partial^l T_{il} T_{jk} A - \frac{1}{4} T_{jk} \partial^l \partial^m \partial^n C_{ilmn} ,$$

so that its content is precisely as before. There is again a dynamical portion relating divergences and gradients, that is still associated to the $\{2,1\}$ projection in family space. However, the projection is now effected in the antisymmetric basis, so that in the present example there is manifest antisymmetry in $j$ and $k$, rather than manifest symmetry as in eq. (2.30). As a technical note, we can add that the Young tableau is now filled in a different fashion, with $j$ and $k$ along the vertical, so that the diagram is effectively “flipped” about its diagonal with respect to eq. (2.30).

For higher traces, one needs the analogue of eq. (3.15), that reads

$$(p + 2) Y_{(p+1,p)} \partial_k \mathcal{A}^{[p]}_{i_1 j_1 ; \ldots ; i_p j_p} - \partial^l \mathcal{A}^{[p+1]}_{i_1 j_1 ; \ldots ; i_p j_p ; k l} =$$

$$= - \frac{1}{2} Y_{(p+1,p)} T_{i_1 j_1} \ldots T_{i_p j_p} \partial^l \partial^m \partial^n C_{k l m n} ,$$

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Now semi-colons separate antisymmetric index pairs, since the previous \( \{2^p\} \) projection is replaced by a flipped diagram, that in our notation is of \( \{p,p\} \) type and is defined in the antisymmetric basis. These results translate into corresponding ones for the Lagrangians, that now read

\[
\mathcal{L} = \frac{1}{2} \left\langle \varphi, \sum_{p=0}^{N} \frac{(-1)^p}{p!(p+1)!} \eta^{i_1j_1} \cdots \eta^{i_pj_p} A^{[p]}_{i_1j_1; \ldots; i_pj_p} \right\rangle \\
- \frac{1}{4} \left\langle \alpha_{ijk}, \sum_{p=0}^{N-1} \frac{(-1)^p}{p!(p+3)!} \eta^{i_1j_1} \cdots \eta^{i_pj_p} \partial_{[i} A^{[p+1]}_{jk];i_1j_1; \ldots; i_pj_p} \right\rangle \\
+ \frac{1}{8} \left\langle \beta_{ijkl}, C_{ijkl} \right\rangle.
\]

(6.9)

Finally, the \( S^i_j \) operators maintain the same algebra, while the irreducibility conditions discussed in Section 4 are now to be defined via the columns of the Young diagrams, and amount to the conditions the any antisymmetrization beyond a given column vanishes.

This presentation of the formalism is particularly effective for two-column fields, that as we have stressed repeatedly are somehow close analogs of the spin-two metric fluctuation, since they need neither compensators nor Lagrange multipliers. Interesting, in the multi-antisymmetric description their Frénel-Labastida operators take the particularly simple form

\[
\mathcal{F} = \frac{1}{2} T_{ij} \partial^i \partial^j \varphi,
\]

(6.10)

so that they are manifestly related to the corresponding curvatures via a single trace.

\section{Conclusions}

In this paper we have described in some detail the general properties of free mixed-symmetry bosonic gauge fields described by multi-symmetric tensors of the type \( \varphi_{\mu_1 \ldots \mu_s; \nu_1 \ldots \nu_s; \ldots} \) or by corresponding multi-forms, or in fact by fields combining arbitrary numbers of families of symmetric or antisymmetric indices. Here we have discussed massless fields in a Minkowski background, but as for lower spins the treatment can be directly extended to the massive case via the harmonics of Kaluza-Klein circle reductions. The extension to \((A)dS\) has not been carried out here, but is expected to be possible, proceeding along the lines of [43, 27, 40]. These general mixed-symmetry fields are needed to describe all representations of the Poincaré group for \( D > 5 \), and in particular are key ingredients of all massive string spectra. Whereas their dynamics is still poorly understood, it is difficult to escape the feeling that they are directly responsible for the most spectacular properties of String Theory. And, we should add, that they might even pave the way to possible generalizations of the string framework.

The theory of mixed-symmetry higher-spin fields was touched upon by a number of authors in the Seventies and Eighties [17, 16], and most notably by Labastida [19, 20], who managed to generalize the Frénel construction of [13] to tensors of this type, after identifying the proper constraints on gauge parameters and gauge fields. His work was then pursued further in [21]. In this paper we have extended the Labastida construction to an unconstrained formulation, along the lines of what was previously done for Frénel’s case in [23, 32, 26, 27] via a single compensator \( \alpha \) and a single Lagrange multiplier \( \beta \). We have followed as closely as possible
the index-free notation that proved so powerful in the symmetric case, but for the introduction of “family indices”. These are the counterpart, in our language, of the non-Abelian oscillator algebra of [19, 20], and in fact the two notations can be turned into one another almost verbatim. Still, in our opinion the present notation has the advantage of bringing these systems, despite the complications introduced by their general nature, closer to more conventional field theories.

For pedagogical reasons, we have started from two-family gauge fields, that follow most closely the pattern that emerged in the symmetric case. The Bianchi identities were again the key ingredients of our construction that, in its minimal form, together with the gauge fields \( \varphi \), also involves compensator fields \( \alpha_{ijk} \) and Lagrange multipliers \( \beta_{ijkl} \). The result was a rather streamlined and compact derivation of the general Lagrangians, from which the constrained Labastida construction can be recovered almost by inspection. Still, the unconstrained theory brings about a number of surprises when compared to the symmetric or single-family case. These have to do, one way or another, with the lack of mutual independence of the Labastida constraints, which makes the \( \alpha_{ijk} \) not independent as well, and forces one to relate them to other more fundamental compensator fields here called \( \Phi_i \), that only enter via the combinations \( T_{\langle ij\Phi_k \rangle} \) due precisely to the constrained Labastida symmetry. In addition, the Lagrangians enjoy a local symmetry under shifts of the Lagrange multipliers that, as a consequence, are not fully determined by the field equations. Another key consequence of the non-Abelian structure underlying these free theories is a rich pattern of sporadic cases where Weyl-like symmetries emerge, generalizing the well-known property of two-dimensional gravity even to cases where the Lagrangians are not topological. Despite these subtleties, however, we have described in full generality how the field equations of two-family fields can be reduced on shell and we have exemplified these results in a number of cases, with or without external currents. Moreover, we have illustrated the key steps of the general reduction procedure for \( N \)-family fields. While the bulk of this paper was devoted to multi-symmetric reducible gauge fields, we have also described how to adapt the formalism to the cases of irreducible fields or multi-forms. Finally, we have shown in full generality how the higher-derivative terms involving the compensators that are present in our unconstrained formulation can be eliminated at the expense of a mild enlargement of the field content, generalizing the construction presented for symmetric bosons in [25].

The companion paper [8] will contain a similar discussion of Fermi fields. There the corresponding Lagrangians will appear for the first time in their general “metric-like” form, since Labastida only obtained in [38] field equations generalizing those of Fang and Fröndsal [39], while the subsequent literature [21] only contains partial results in this respect. A key issue for future research is clearly to attain a better understanding of higher-spin interactions, and above all of their systematics. Much was recently done in this respect by a number of authors [4], but a decisive progress in this respect is clearly expected to be far more difficult. We hope that this “metric-like form” of the theory of free higher-spin fields will provide useful insights in this respect.

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A Notation and conventions

In this paper we use the “mostly plus” convention for the space-time signature and resort to a compact notation eliminating all space-time indices from tensor relations. The fields of interest are here multi-symmetric tensors \( \varphi_{\mu_1...\mu_{n_1};\nu_1...\nu_{n_2};...} \), whose sets of symmetric indices are here referred to as “families”, or corresponding multi-forms. There are a number of sign differences between the cases of multi-symmetric and multi-antisymmetric fields, but for definiteness in this Appendix, as in most of the present paper, we refer explicitly to multi-symmetric fields. The key modifications needed in the case of multi-forms are spelled out in Section 6.

While fully symmetric under the interchange of pairs of indices belonging to the same set, a field like \( \varphi_{\mu_1...\mu_{n_1};\nu_1...\nu_{n_2};...} \) has no prescribed symmetry relating different sets, and is thus a reducible \( gl(D) \) tensor. As a result, it is perhaps less familiar than Young projected tensors, but is a most convenient object to study and plays also a natural role in String Theory. For instance, in the bosonic string multi-symmetric tensors of this type accompany generic products of string oscillators \( \alpha^{i_1}_1 \ldots \alpha^{i_{n_1}}_1 \alpha^{i_2}_2 \ldots \alpha^{i_{n_2}}_2 \ldots \), that are only symmetric under interchanges of pairs of identical oscillators, just as multi-forms accompany in superstring similar products of fermionic oscillators. Recovering more conventional field theories from the present formulation requires suitable projections, discussed in Section 4: the simplest example to this effect, as we anticipated in the Introduction, is a field \( \varphi_{\mu_1;\nu} \), which combines a spin-2 field \( \varphi(\mu;\nu) \) and a Kalb-Ramond field \( \varphi[\mu;\nu] \). In this paper a generic multi-(anti)symmetric gauge field of this type is simply denoted by \( \varphi \).

The Lagrangians and field equations of multi-symmetric fields involve traces, gradients and divergences, as well as Minkowski metric tensors related to one or two of the previous index sets. As a result, “family indices” are needed in order to specify the sets to which some tensor indices belong. These family indices are here denoted by small-case Latin letters, and the Einstein convention for summing over pairs of them is used throughout. It actually proves helpful to be slightly more precise: upper family indices are thus reserved for operators, like a gradient, that add space-time indices, while lower family indices are used for operators, like a divergence, that remove them. As a result gradients, divergences and traces of a field \( \varphi \) are denoted concisely by \( \partial^i \varphi \), \( \partial_i \varphi \) and \( T_{ij} \varphi \). This shorthand notation suffices to identify the detailed meaning of these symbols, so that for instance

\[
\partial^i \varphi \equiv \partial_{(\mu_1^0 | \varphi;...;\mu_{s+1}^0)} \varphi \, ; \, ... \, , \\
\partial_i \varphi \equiv \partial^{\lambda} \varphi \, ; \, \lambda \mu_2^0...\mu_{s}^i \, ; \, ... \\
T_{ij} \varphi \equiv \varphi \, ; \, \lambda \mu_2^0...\mu_{s}^i \, ; \, ... 
\] (A.1)

In addition, as in [22, 23, 32, 26, 27, 25], we work with symmetrizations that are not of unit strength, but involve the minimum possible number of terms, and we use round brackets to denote them. Thus, for instance, the product \( T_{\alpha ij} T_{\beta kl} \) here stands for \( T_{ij} T_{kl} + T_{ik} T_{jl} + T_{il} T_{jk} \). In addition, we use square brackets to denote antisymmetrizations. In the mixed-symmetry case, it is also necessary to introduce a mixed metric tensor

\[
\eta_{ij} \varphi \equiv \frac{1}{2} \sum_{n=1}^{s+1} \eta_{n} w_{0} (\mu_1^0 | \varphi;...;\mu_{s+1}^0 \, ; \, ... \, ; \, | \mu_{i}^j \, ; \, ... \, ) \, ; 
\] (A.2)

that here is suitably rescaled in order that its diagonal terms retain the conventional normalization.
In order to further simplify the combinatorics, it proves very convenient to introduce the scalar product
\[
\langle \varphi, \chi \rangle \equiv \frac{1}{s_1! \cdots s_n!} \varphi_{\mu_1 \cdots \mu_{s_1} ; \cdots ; \mu_{s_1} \cdots \mu_{s_n}} \chi^{\mu_1 \cdots \mu_{s_1} ; \cdots ; \mu_{s_1} \cdots \mu_{s_n}} \equiv \frac{1}{s_1! \cdots s_n!} \varphi \chi. \tag{A.3}
\]
Inside the brackets it is then possible to integrate by parts and to turn \(\eta\)'s into traces without introducing any \(s_i\)-dependent combinatoric factors, since
\[
\langle \varphi, \partial^i \chi \rangle \equiv \frac{1}{s_1! \cdots s_n!} \varphi \partial^i \chi = -\frac{s_i}{s_1! \cdots s_n!} \partial_i \varphi \chi \equiv -\langle \partial_i \varphi, \chi \rangle, \tag{A.4}
\]
\[
\langle \varphi, \eta^{ij} \chi \rangle \equiv \frac{1}{s_1! \cdots s_n!} \varphi \eta^{ij} \chi = \frac{1}{2} \frac{s_i s_j}{s_1! \cdots s_n!} (T_{ij} \varphi) \chi \equiv \frac{1}{2} \langle T_{ij} \varphi, \chi \rangle, \tag{A.5}
\]
where the reader should keep track of the somewhat unusual factor \(\frac{1}{2}\), originating again from our choice of normalization for \(\eta_{ij}\) in eq. (A.2). However, in the main body of this paper we are ignoring, for simplicity, the overall factor \(\prod_{i=1}^N s_i!\), that should accompany the Lagrangian of a multi-symmetric tensor \(\varphi_{\mu_1 \cdots \mu_{s_1} ; \cdots ; \mu_{s_1} \cdots \mu_{s_N}}\) to grant it the conventional normalization.

These matrix elements are quite convenient to derive Lagrangians and field equations for mixed-symmetry fields, but they would be as convenient for symmetric tensors. In this notation the fully symmetric, or one-family, Lagrangian would simply read
\[
\mathcal{L} = \frac{1}{2} \langle \varphi, A - \frac{1}{2} \eta A' \rangle - \frac{1}{8} \langle \alpha, \partial \cdot A' \rangle + \frac{1}{8} \langle \beta, C \rangle, \tag{A.6}
\]
up to an overall \(s!\), to be compared with the corresponding expression of [27]
\[
\mathcal{L} = \frac{1}{2} \varphi \left( A - \frac{1}{2} \eta A' \right) - \frac{3}{4} \binom{s}{3} \alpha \partial \cdot A' + 3 \binom{s}{4} \beta C, \tag{A.7}
\]
that contains explicit combinatoric factors.

As in the symmetric case, in order to take full advantage of the compact notation, it is convenient to collect a number of identities that are used recurrently in this type of analysis. These, however, have a more complicated structure than their symmetric counterparts of [27], since they involve a genuinely new type of operation. This turns a tensor with \(s_i\) indices in the \(i\)-th group into others with \(s_i + 1\) indices in the \(i\)-th group and \(s_j - 1\) indices in the \(j\)-th group, according to
\[
S_{ij}^i \varphi \equiv \varphi_{(\mu_1 \cdots \mu_{s_i+1} ; \cdots ; \mu_{s_i+1}) \mu_{s_i} \cdots \mu_{s_j} ; \cdots}. \tag{A.8}
\]
The new rules one needs follow from the algebra of the various operators, and can be also derived from a realization in terms of bosonic oscillators, along the lines of bosonic String Field Theory.
and of [19, 20]:

\[
\begin{align*}
[\partial_i, \partial^j] &= \Box \delta^j_i, \\
[T_{ij}, \partial^k] &= \partial_i (\delta_j^k), \\
[\partial_k, \eta^{ij}] &= \frac{1}{2} \partial^{(i} \delta^{j)}_k, \\
[T_{ij}, \eta^{kl}] &= \frac{D}{2} \delta_i^{(k} \delta^j_l) + \frac{1}{2} \left( \delta_i^{(k} S^{l)}_j + \delta_j^{(k} S^{l)}_i \right), \\
[S^{i}^j, \eta^{kl}] &= \eta^{i(k} \delta^j_l), \\
[T_{ij}, S^{k^l}] &= T_{l(i} \delta^j_j), \\
[S^{i}^j, \partial^k] &= \partial^i \delta^j_k, \\
[\partial_k, S^{i}^j] &= \partial_j \delta^k_i, \\
[S^{i}^j, S^{k^l}] &= \delta_j^k S^{i}_j - \delta_l^i S^{k}^j.
\end{align*}
\]

Notice that the $S^{i}^j$ operators, the key novelty of the mixed-symmetry case, close into a $gl(N)$ algebra if $N$ index families are present. As a result, one can well say that this class of free theories rests somehow on a non-Abelian structure. These commutation relations give rise to the rules collected in Appendix B.

Finally, in this paper we use extensively a number of standard tools related to the symmetric group. These include, in particular, the Young projectors $Y$, that allow to separate irreducible components in family-index space and can be built combining contributions from different Young tableaux $Y_{\tau}$. In general these Young tableaux can be identified associating integer labels to the tensor indices to be projected and allowing all their arrangements within the given graph such that these integers grow from left to right and from top to bottom. In some cases, however, this simple procedure can actually fail to produce an orthogonal decomposition, which can still be attained by a further Graham-Schmidt orthogonalization. This difficulty is not present if, for any pair of tableaux, there is at least a couple of indices belonging to a row of the first that lie in the same column within the second, and vice versa. Let us stress that this difficulty is never to be faced in our constructions, as a result of the particular symmetry properties of our basic objects.

Outside Section 6, in this paper Young tableaux are defined in the symmetric basis, so that the projector corresponding to a tableau $\tau$ containing $n$ boxes takes the form

\[
Y_{\tau} = \frac{\lambda(\tau)}{n!} S A,
\]

where $S$ and $A$ are the corresponding products of “row symmetrizers” and “column antisymmetrizers”. On the other hand, the results in Section 6 are dealt with more conveniently in the antisymmetric basis, where the roles of $S$ and $T$ are interchanged. Here $\lambda(\tau)$ denotes the dimension of the associated representation of the symmetric group, that can be computed for instance counting the standard ways of filling the boxes of the corresponding diagram with the numbers 1, 2, \ldots, $n$, in increasing order from left to right and from top to bottom. In general, diagrams and tableaux are specified by ordered lists of the lengths of their rows, so that, for instance, the \{3, 2\} graph is

\[
\begin{array}{c|c|c}
\hline
| & | \\
\hline
| | \\
\hline
\end{array}
\]

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Similar techniques are also used extensively in Section 4 to identify irreducible $gl(D)$ and Lorentz tensors. We refrain from adding further details, since these and other related standard facts are discussed extensively in the literature, and in particular in [36].

**B SOME USEFUL IDENTITIES**

Using repeatedly the commutators presented in Appendix A, one can recover the $N$-family counterpart of eq. (A.1) of [27], that collects the key identities for the one-family case:

\[
[\partial_k, \partial^{i_1} \ldots \partial^{i_p}] = \sum_{n=1}^{p} \delta^{i_n} \bigotimes_{r \neq n} \partial^{i_r},
\]

\[
[T_{ij}, \partial^{k_1} \ldots \partial^{k_p}] = \sum_{m<n} \frac{\delta^{k_m}}{(i_j)} \bigotimes_{r \neq m,n} \partial^{k_r} + \sum_{n=1}^{p} \prod_{r \neq n} \partial^{k_r} \delta^{k_n}_{(i_j)},
\]

\[
[\partial_k, \eta^{i_1j_1} \ldots \eta^{p,j_p}] = \frac{1}{2} \sum_{n=1}^{p} \prod_{r \neq n} \eta^{i_r,j_r} \delta^{(i_n,j_n)}_{k},
\]

\[
[T_{ij}, \eta^{k_1l_1} \ldots \eta^{kpl_p}] = \frac{D}{2} \sum_{n=1}^{p} \delta^{(k_n)} \bigotimes_{r \neq n} \delta^{(l_n)}_{j} \prod_{r \neq m,n} \eta^{k_n,l_n} + \frac{1}{2} \sum_{m<n} \left( \delta^{l_m}_{k_i} \delta^{(k_n,l_n)}_{i_j} \right) \prod_{r \neq m,n} \eta^{k_i,l_r} + \frac{1}{2} \sum_{n=1}^{p} \prod_{r \neq n} \left( \delta^{(k_n)} \delta^{l_n}_{j} \right).
\]

One often applies (B.1) to expressions that only contain contracted indices. It is thus convenient to rewrite them explicitly for cases where the various expressions are contracted against tensors $Q$ possessing identical manifest symmetries for their family indices

\[
[\partial_k, \partial^{i_1} \ldots \partial^{i_p}] Q_{i_1 \ldots i_p} = p \bigotimes \partial^{i_1} \ldots \partial^{i_p-1} Q_{k i_1 \ldots i_{p-1}},
\]

\[
[\partial_k, \eta^{i_1j_1} \ldots \eta^{p,j_p}] Q_{i_1 j_1 \ldots i_p} = p \eta^{i_1j_1} \ldots \eta^{p-1,j_p-1} \partial^{i} Q_{k l i_1 \ldots i_{p-1}},
\]

\[
[T_{kl}, \partial^{i_1} \ldots \partial^{i_p}] Q_{i_1 \ldots i_p} = p (p-1) \bigotimes \partial^{i_1} \ldots \partial^{i_{p-2}} Q_{k l i_1 \ldots i_{p-2}} + p \partial^{i_1} \ldots \partial^{i_{p-1}} \partial (k Q l)_{i_1 \ldots i_{p-1}},
\]

\[
[T_{kl}, \eta^{i_1j_1} \ldots \eta^{p,j_p}] Q_{i_1 j_1 \ldots i_p} = p D \eta^{i_1j_1} \ldots \eta^{p-1,j_p-1} Q_{k l i_1 \ldots i_{p-1}} + \eta^{i_1j_1} \ldots \eta^{p-1,j_p-1} \left\{ p (p-1) Q_{i_1 (k,l) j_1 \ldots i_{p-1} j_p} + p S^m (k Q l)_{m i_1 j_1 \ldots i_{p-1} j_p} \right\},
\]

while another useful relation is given by

\[
[T_{i_1j_1} \ldots T_{i_pj_p}, \eta^{k_1l_1}] Q_{kl} = D \sum_{n=1}^{p} \prod_{r \neq n} T_{i_rj_r} \Omega_{i_nj_n} + \sum_{m<n} \prod_{r \neq m,n} T_{i_rj_r} (T_{im} (i_n \Omega_{j_n})_{jm} + T_{jm} (i_n \Omega_{j_n})_{im}) + \sum_{n=1}^{p} S^k (i_n) \prod_{r \neq n} T_{i_rj_r} \Omega_{j_n k}.
\]
Of course, identifying all family indices one can recover, as a special case, the symmetric rules collected in eq. (A.1) of [27].

The previous results are particularly useful when one tries to compute the traces and divergences of the tensors $F$ and $A$ that are needed, for instance, to compute the equations of motion for the Lagrangians (2.39) and (3.26). Starting from the Fronsdal-Labastida tensor

$$F = \Box \varphi - \partial^i \partial_i \varphi + \frac{1}{2} \partial^i \partial_j T_{ij} \varphi,$$

one can thus obtain

$$\prod_{r=1}^{p} T_{ijr} F = (p + 1) \Box \prod_{r=1}^{p} T_{ijr} \varphi - 2 \sum_{n=1}^{p} \partial_{i_n} \partial_{j_n} \prod_{r \neq n}^{p} T_{ijr} \varphi$$

$$+ \sum_{n=1}^{p} \sum_{m < n} (\partial_{i_n} \partial_{(m} T_{jm)} \varphi_{j_n} + \partial_{j_n} \partial_{(m} T_{jm)} \varphi_{i_n}) \prod_{r \neq m, n}^{p} T_{ijr} \varphi$$

$$- \partial^k \left[ \partial_k \prod_{r=1}^{p} T_{ijr} \varphi - \sum_{n=1}^{p} \partial_{i_n} T_{jm} \varphi_{j_n} \prod_{r \neq n}^{p} T_{ijr} \varphi \right] + \frac{1}{2} \partial^k \partial^l T_{kl} \prod_{r=1}^{p} T_{ijr} \varphi$$

and

$$\partial_k \prod_{r=1}^{p} T_{ijr} F = \sum_{n=1}^{p} \partial_k (\partial_{i_n} T_{jm} \varphi_{j_n}) \prod_{r \neq n}^{p} T_{ijr} \varphi - 2 \sum_{n=1}^{p} \partial_k \partial_{i_n} \partial_{j_n} \prod_{r \neq n}^{p} T_{ijr} \varphi$$

$$+ \sum_{n=1}^{p} \sum_{m < n} \partial_k \left( \partial_{i_n} \partial_{(m} T_{jm)} \varphi_{j_n} + \partial_{j_n} \partial_{(m} T_{jm)} \varphi_{i_n} \right) \prod_{r \neq m, n}^{p} T_{ijr} \varphi$$

$$- \partial^l \left[ \partial_k \partial_l \prod_{r=1}^{p} T_{ijr} \varphi - \sum_{n=1}^{p} \partial_k \partial_{i_n} T_{jm} \varphi_{j_n} \prod_{r \neq n}^{p} T_{ijr} \varphi \right] + \Box \partial^l T_{kl} \prod_{r=1}^{p} T_{ijr} \varphi$$

$$+ \frac{1}{2} \partial^l \partial^m \partial_k T_{lm} \prod_{r=1}^{p} T_{ijr} \varphi.$$

Notice that these two expressions do not contain the $S_{ij}^k$ operators, that as a result do not appear in the field equations, but only emerge in their reduction procedure, and in particular in the propagators. Let us also stress that eq. (B.11) is the relation needed to fix the coefficients (3.11) using the condition of self-adjointness, following the original derivation of [20]. In particular, in the two-family case the relevant identities are

$$T_{ij} F = 2 \Box T_{ij} \varphi - 2 \partial_i \partial_j \varphi + \partial^k (T_{k(i} \partial_{j)} \varphi - T_{ij} \partial_k \varphi) + \frac{1}{2} \partial^k \partial^l T_{ij} T_{kl} \varphi,$$

$$T_{ij} T_{kl} F = 3 \Box T_{ij} T_{kl} \varphi - 3 \left( T_{ij} \partial_k \partial_l \varphi + T_{kl} \partial_i \partial_j \varphi - \frac{1}{3} T_{ij} \partial_k \partial_l \varphi \right)$$

$$- \partial^m T_{ij} T_{kl} \partial_m \varphi + \partial^m \left( T_{ij} T_{m(k} \partial_{l)} \varphi + T_{kl} T_{m(i} \partial_{j)} \varphi \right) + \frac{1}{2} \partial^m \partial^n T_{ij} T_{kl} T_{mn} \varphi.$$
\[
T_{ij} \partial_k \mathcal{F} = \Box T_{ij} \partial_k \varphi - 2 \partial_i \partial_j \partial_k \varphi + \Box \partial^l T_{ij} T_{kl} \varphi \\
+ \partial^l \left( T_{(i} \partial_j) \partial_k \varphi - T_{ij} \partial_k \partial_l \varphi \right) + \frac{1}{2} \partial^l \partial^m T_{ij} T_{lm} \partial_k \varphi,
\]
(B.15)

\[
T_{ij} T_{kl} \partial_m \mathcal{F} = 2 \Box T_{ij} T_{kl} \partial_m \varphi + \Box \left( T_{ij} T_{m(k} \partial_{l)} \varphi + T_{kl} T_{m(i} \partial_{j)} \varphi \right) \\
- 3 \left( T_{ij} \partial_k \partial_l \partial_m \varphi + T_{kl} \partial_i \partial_j \partial_m \varphi - \frac{1}{3} T_{(ij} \partial_k \partial_l) \partial_m \varphi \right) \\
- \partial^n T_{ij} T_{kl} \partial_m \partial_n \varphi + \partial^n \left( T_{ij} T_{n(k} \partial_{l)} \partial_m \varphi + T_{kl} T_{n(i} \partial_{j)} \partial_m \varphi \right) \\
+ \Box \partial^n T_{ij} T_{kl} T_{mn} \varphi + \frac{1}{2} \partial^n \partial^p T_{ij} T_{kl} T_{np} \partial_m \varphi.
\]
(B.16)

In a similar fashion, starting from the unconstrained gauge invariant tensor \(A\),
\[
A = \mathcal{F} - \frac{1}{2} \partial^i \partial^j \partial^k \alpha_{ijk},
\]
(B.17)

one can obtain the corresponding expression
\[
\prod_{r=1}^p T_{irjr} A = \prod_{r=1}^p T_{irjr} \mathcal{F} - 3 \Box \sum_{n=1}^p \sum_{m \neq n} \prod_{r=1}^p T_{irjr} \partial_{(im} \alpha_{jmjn)} \\
- 3 \sum_{n=1}^p \sum_{m < n} \prod_{r=1}^p T_{irjr} \left( \partial_{im} \partial_{j(n} \alpha_{j|m}) |jn) + \partial_{jn} \partial_{im} | \partial_{im} \partial_{j(n} \alpha_{j|m}) |jn) \right) \\
- 3 \Box \partial^k \sum_{n=1}^p \prod_{r=1}^p T_{irjr} \alpha_{imjn} - 3 \partial^k \sum_{n=1}^p \sum_{m < n} \prod_{r=1}^p T_{irjr} \partial_{im} | \partial_{im} \alpha_{jn} | kn \\
- \frac{3}{2} \partial^k \partial^l \sum_{n=1}^p \prod_{r=1}^p T_{irjr} \partial_{(im} \alpha_{jn)} | kl - \frac{1}{2} \partial^k \partial^l \partial^m \prod_{r=1}^p T_{irjr} \alpha_{klm},
\]
(B.18)

that restricting again the attention to two index families reduce to
\[
T_{ij} A = T_{ij} \mathcal{F} - 3 \Box \partial^k \alpha_{ijk} - \frac{3}{2} \partial^k \partial^l \partial_{(i} \alpha_{j)kl} - \frac{1}{2} \partial^k \partial^l \partial^m T_{ij} \alpha_{klm},
\]
(B.19)

\[
T_{ij} T_{kl} A = T_{ij} T_{kl} \mathcal{F} - 3 \Box \partial_{(i} \alpha_{jkl)} - 3 \partial^m \left( \partial_{i} \partial_{(k} \alpha_{l)} jm + \partial_{j} \partial_{(k} \alpha_{l)} jm \right) \\
- \frac{3}{2} \partial^m \partial^n \left( T_{ij} \partial_{(k} \alpha_{l)} mn + T_{kl} \partial_{(i} \alpha_{j)} mn \right) \\
- 3 \Box \partial^m \left( T_{ij} \alpha_{klm} + T_{kl} \alpha_{ijm} \right) - \frac{1}{2} \partial^m \partial^n \partial^p T_{ij} T_{kl} \alpha_{mnp}.
\]
(B.20)
Other useful identities are

\[ T_{ij} \partial_k A = T_{ij} \partial_k F - 3 \Box^2 \alpha_{ijk} - 3 \Box \partial^l \partial_{(i} \alpha_{jk)} l - \frac{3}{2} \Box \partial^l \partial^m T_{ij} \alpha_{klm} \]
\[ - \frac{3}{2} \partial^l \partial^m \partial_k \partial_{(i} \alpha_{jk)} l m - \frac{1}{2} \partial^l \partial^m \partial^n T_{ij} \partial_k \alpha_{lmn} , \quad (B.21) \]
\[ T_{ij} T_{kl} \partial_m A = T_{ij} T_{kl} \partial_m F - 3 \Box \left( \partial_m \partial_{(i} \alpha_{jkl)} + \partial_i \partial_{(k} \alpha_l )_{jm} + \partial_j \partial_{(k} \alpha_l )_{im} \right) \]
\[ - 3 \Box \partial^m \left( \partial_m \partial_{(i} \alpha_{jkl)} + \partial_i \partial_{(k} \alpha_l )_{jm} + \partial_j \partial_{(k} \alpha_l )_{im} \right) - 3 \Box^2 \left( T_{ij} \alpha_{klm} + T_{kl} \alpha_{ijm} \right) \]
\[ - 3 \Box \partial^n \left( T_{ij} \partial_m \alpha_{klm} + T_{kl} \partial_m \alpha_{ijm} \right) - 3 \Box \partial^n \left( T_{ij} \partial_{(k} \alpha_l )_{mn} + T_{kl} \partial_{(i} \alpha_j )_{mn} \right) \]
\[ - \frac{3}{2} \Box \partial^n \partial^p T_{ij} T_{kl} \alpha_{mnp} - \frac{3}{2} \Box \partial^n \partial^p \left( T_{ij} \partial_m \partial_{(k} \alpha_l )_{np} + T_{kl} \partial_m \partial_{(i} \alpha_j )_{np} \right) \]
\[ - \frac{1}{2} \Box \partial^n \partial^p \partial^q T_{ij} T_{kl} \partial_m \alpha_{npq} . \quad (B.22) \]

They could be used to compute directly the field equations (2.52) and (2.57), when combined with eqs. (B.13), (B.14), (B.15) and (B.16).

### C Proof of some results used in Section 3

The construction of Section 3 is based on the key result that all Young projections of multiple traces of \( A \) with more than two columns can be related to the constraint tensors \( C_{ijkl} \). In fact, these projections can be realized via a sum of Young tableaux involving at least one symmetrization over four family indices, since from a product of identical traces one can only build Young diagrams with even numbers of boxes in each row. Aside from the two simple cases,

\[ T_{(ab} T_{cd)} \quad \text{and} \quad T_{(a|b} T_{j|c} T_{d)} , \quad (C.1) \]

that are manifestly related to the constraints, either directly via eq. (2.32) or via the enlargement of cycles from three to four family indices, that is automatic for a pair of trace tensors, one ought to consider the non-trivial case where the four symmetrized indices are spread over four distinct traces:

\[ T_{i(a|b} T_{j|c} T_{k|d)} l . \quad (C.2) \]

Even these types of terms, however, can be related to the constraints via eq. (2.32), since

\[ T_{i(a} T_{j|b} T_{k|c} T_{d)} l = T_{(ab} T_{cd} T_{ij} T_{kl}) - T_{(ab} T_{c(i} T_{d)|j} T_{kl)} - T_{(ab} T_{cd} T_{(ij} T_{kl)} , \quad (C.3) \]

where the first and last terms are manifestly related to the constraints, while the second term is like the last one in eq. (C.1). These results clearly imply that, in the constrained Labastida setting, any similar combination of traces of \( F \) tensors vanishes on account of eq. (3.4).

Alternatively, one could start from eq. (2.32) to generate expressions of the type

\[ T_{i_1 j_1} \ldots T_{i_p j_p} T_{(ab} T_{cd)} A \quad (C.4) \]

computing further traces. These admit all the Young projections allowed for the more general expression

\[ T_{i_1 j_1} \ldots T_{i_p j_p} T_{ab} T_{cd} A , \quad (C.5) \]
aside from the two-column one. Acting on each individual irreducible component of (C.4), however, the permutation group can generate the entire corresponding irreducible subspaces, which suffices to extend the statement to (C.5). The conclusion, as above, is that any product of traces of $A$ corresponding to a Young diagram with more than two columns can be linked to the constraints.

In addition, we need the two key identities (3.15) and (3.23), that we list again for convenience:

\[
Y_{\{2p,1^q\}} \left[ 2 \partial_k \prod_{r=1}^{p} T_{i_rj_r} - \sum_{n=1}^{p} \partial_{(i_n T_{jn})k} \prod_{r \neq n}^{p} T_{i_rj_r} \right] A = (p + 2) Y_{\{2p,1^q\}} \partial_k A[^p]_{i_1j_1, \ldots, i_pj_p}, \quad (C.6)
\]

\[
\left( \prod_{r=1}^{p} T_{i_rj_r} \Lambda_k, Y_{\{3,2p-1\}} \left[ \partial_k A[^p]_{i_1j_1, \ldots, i_pj_p} - \frac{p}{p+2} \partial_{(k A[^p]_{i_1j_1}, i_2j_2, \ldots, i_pj_p)} \right] \right) = 0. \quad (C.7)
\]

In the remainder of this Appendix we would like to prove these last two results.

**Proof of Eq. (C.6)**

A crucial observation for this proof is that one can compute the $\{2p,1^q\}$ projection of an expression containing $p$ traces, say $T_{i_1j_1} \ldots T_{i_pj_p} \partial_k$, via the single standard Young tableau

\[
\begin{array}{cccc}
1 & 1 \\
\vdots & \vdots \\
\vdots & \vdots \\
\vdots & \vdots \\
1 & p \\
\end{array}
\]

which corresponds to the choice of standard labeling $i_n = 2n - 1$, $j_n = 2n$, $(n = 1, \ldots, p)$, $k = 2p + 1$. This choice indeed guarantees that all other standard Young tableaux vanish, due to the symmetry properties of the $T_{ij}$. The first step in computing the projection associated to (C.8) is then to enforce the antisymmetrization in $k$ and the $i_m$. In order to prove eq. (C.6), it is sufficient to compare the results obtained antisymmetrizing the two expressions in eq. (C.6),

\[
\begin{aligned}
&\left[ 2 \partial_k \prod_{r=1}^{p} T_{i_rj_r} - \sum_{n=1}^{p} \partial_{(i_n T_{jn})k} \prod_{r \neq n}^{p} T_{i_rj_r} \right] A \longrightarrow (p + 2) \partial_k T_{i_1j_1} \ldots T_{i_pj_p} A, \\
&\partial_k A[^p]_{i_1j_1, \ldots, i_pj_p} \longrightarrow \partial_k A[^p]_{i_1j_1, \ldots, i_pj_p}.
\end{aligned}
\]

Moreover, one should note that in the expansion

\[
\partial_k T_{i_1j_1} \ldots T_{i_pj_p} A = \partial_k \left( Y_{\{2p\}} T_{i_1j_1} \ldots T_{i_pj_p} + Y_{\{4,2p-1\}} T_{i_1j_1} \ldots T_{i_pj_p} + \ldots \right) A \quad (C.10)
\]

in all available irreducible representations, only the first term, that defines $A[^p]$, can survive the antisymmetrization of all the $i_m$ indices, so that

\[
\partial_k A[^p]_{i_1j_1, \ldots, i_pj_p} = \partial_k T_{i_1j_1} \ldots T_{i_pj_p} A. \quad (C.11)
\]

As a result, after the first antisymmetrization the two terms in (C.9) become proportional, and this property remains true for the full projection, which finally proves the identity (C.6).
**Proof of eq. (C.7)**

The presence of the scalar product makes it possible to prove eq. (C.7) computing the projection associated to a single Young tableaux, as in the previous case. In fact, while the full $\{3, 2^{p-1}\}$ projection results from a sum of different tableaux, one can choose them in such a way that only one of them contributes to the scalar product. In particular, one can reduce the full $Y_{\{3, 2^{p-1}\}}$ to the projection associated to

\[
\begin{array}{ccc}
  i_1 & j_1 & k \\
  \vdots & \vdots & \vdots \\
  i_p & j_p \\
\end{array}
\] (C.12)

because any symmetrization involving three indices of the set $\{i_m, j_n\}$ in the left entry of the scalar product extends to a symmetrization over four indices, on account of the properties of products of identical tensors. Then, recalling that by definition $A^{[p]}_{i_1 j_1, \ldots, i_p j_p}$ is already projected according to

\[
\begin{array}{ccc}
  i_1 & j_1 & \\
  \vdots & \vdots & \\
  i_p & j_p \\
\end{array}
\] (C.13)

one can recognize that the operations needed to build the projection associated to the tableau (C.12) differ from those implied by the tableau (C.13) only in the symmetrization of the three indices $(i_1, j_1, k)$. More precisely, denoting the tableau (C.12) by $\tau_1$ and the tableau (C.13) by $\tau_2$ and using the notation of eq. (A.18) one can recognize that acting on products of traces

\[
Y_{\tau_1} = 2 \frac{\lambda(\tau_1)}{(2p + 1)} S_{\tau_1} A_{\tau_1},
\] (C.14)

where the product $S_{\tau_2}$ of “row symmetrizers” does not include the operator $S_{(i_1, j_1)}$ because this symmetrization is already induced by the others. When applied to $T_{i_1 j_1} \ldots T_{i_p j_p} \partial_k$, the product of the two Young projectors $Y_{\tau_1}$ and $Y_{\tau_2}$ then gives

\[
Y_{\tau_1} Y_{\tau_2} = \frac{\lambda(\tau_1)}{(2p + 1)} S_{(i_1, j_1, k)} S_{\tau_2} A_{\tau_2} Y_{\tau_2} = \frac{\lambda(\tau_1)}{2(2p + 1) \lambda(\tau_2)} S_{(i_1, j_1, k)} (Y_{\tau_2})^2. 
\] (C.15)

The ratio that appears in eq. (C.15) is

\[
\frac{\lambda(\tau_1)}{(2p + 1) \lambda(\tau_2)} = \frac{p}{p + 2}
\] (C.16)

so that

\[
Y_{\tau_1} \partial_k A^{[p]}_{i_1 j_1, \ldots, i_p j_p} = \frac{p}{p + 2} \partial(k A^{[p]}_{i_1 j_1, i_2 j_2, \ldots, i_p j_p}),
\] (C.17)

which finally proves eq. (C.7), since $Y_{\{3, 2^{p-1}\}}$ can be replaced with $Y_{\tau_1}$ within the scalar product.
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