Configuration entropy and confinement-deconfinement transition in higher-dimensional hard wall model

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We consider a higher-dimensional hard wall model with an infrared (IR) cut-off in asymptotically AdS space and investigate its thermodynamics via the holographic renormalization method. We find a relation between the confinement temperature and the IR cut-off for any dimension. It is also shown that the entropy of \( p \)-branes with the number of coincident branes (the number of the gauge group) \( N \) jumps from leading order in \( O(N^0) \) at the confining low temperature phase to \( O(N^{1/2}) \) at the deconfining high temperature phase like \( D3 \)-branes (\( p = 3 \)) case. On the other hand, we calculate the configuration entropy (CE) of various magnitudes of an inverse temperature at an given IR cut-off scale. It is shown that as the inverse temperature grows up, the CE above the critical temperature decreases and AdS black hole (BH) is stable while it below the critical temperature is constant and thermal AdS (ThAdS) is stable. In particular, we also find that the CE below the critical temperature becomes constant and its magnitude increases as a dimension of AdS space increases.

I. INTRODUCTION

The AdS/CFT duality has been originally based on the holographic principle which states that the description of a volume of space is able to be thought of as encoded on a boundary of such a region and conjectured to be a relationship between a gravitational theory in the bulk and a conformal field theory in the boundary \( \mathcal{B} \). In particular, it has been suggested that BH thermodynamics in AdS is the Hawking-Page transition between large BH and ThAdS, which in the CFT is dual to the confinement-deconfinement transition on a sphere \( \mathcal{S} \).

The Dirichlet branes (D-branes) have been defined by Dirichlet boundary conditions in string theory, which has opened up a new window to explore the black hole entropy \( S \). In extremal five-dimensional black hole, they have shown that its horizon area is non-vanishing and the Bekenstein-Hawking entropy is derived by counting the degeneracy of Bogomolny-Prasad-Sommerfield soliton bound states \( \mathcal{B} \). It has been extensively studied for extremal five-dimensional rotating charged BH \( \mathcal{E} \), non-extremal five-dimensional BH \( \mathcal{S} \) and non-extremal six-dimensional black string \( \mathcal{O} \). The entropies of \( Dp \)-branes have been investigated \( \mathcal{I} \).

The AdS/QCD hard wall model has been suggested in search in order to obtain the holographic dual theory from QCD \( \mathcal{Q} \) via the holographic renormalization method \( \mathcal{R} \). They have shown that it is able to describe the confinement-deconfinement transition for the gauge theory from BH thermodynamics in AdS \( \mathcal{A} \). Recently, it has been found that the above result is consistent with that of the analysis of stability through employing the CE in hard wall model \( \mathcal{E} \) and in soft wall model \( \mathcal{S} \). Furthermore the AdS/CFT correspondence holds in different dimensions \( \mathcal{S} \) but their works \( \mathcal{O} \) focus on the case of the five-dimensional AdS. Thus, it is intriguing that the issue is generalized to a variety of AdS space.

The paper is organized as follows: In the next section we briefly review the holographic renormalization method for the \( d \)-dimensional gravity action with the negative cosmological constant and investigate the relationship between the confinement temperature and the IR cut-off for the hard wall model at finite temperature. In particular we calculate the entropy of \( p \)-branes at the confining temperature phase and the deconfining temperature phase. In the next section, we also introduce the CE and explore thermodynamic instability for the hard wall model. In the last section we give our discussion.

II. THERMODYNAMICS FOR THE HARD WALL MODEL

The \( d \)-dimensional gravity action \( I \) with the negative cosmological constant \( \mathcal{A} \) is written as

\[
I = I_B + I_{0B} + I_A \tag{2.1}
\]

where the bulk action \( I_B \) is given as

\[
I_B = -\frac{1}{16\pi G_d} \int_M d^d x \sqrt{-g}(R - 2\Lambda). \tag{2.2}
\]

Here \( G_d, R \), and \( \Lambda \) denote the \( d \)-dimensional Newton constant, Ricci scalar, and negative cosmological constant \( \mathcal{A} \). The boundary action \( I_{0B} \) is

\[
I_{0B} = -\frac{1}{8\pi G_d} \int_{\partial M} d^{d-1} x \sqrt{-\gamma} \Theta, \tag{2.3}
\]

which is introduced to obtain equations of motion well behaved at the boundary of the action where \( \gamma \) is the determinant of the metric of the boundary \( \gamma_{ab} \) and \( \Theta \) is the trace of extrinsic curvature. Then the boundary...
energy-momentum tensor becomes

\[
\frac{2}{\sqrt{-\gamma}} \frac{\delta I_{\text{BH}}}{\delta \gamma^{ab}} = \Theta_{ab} - \gamma_{ab} \Theta. \tag{2.4}
\]

The counterterm action \( I_{\text{ct}} \) is added to the action to remove the divergence appearing as the boundary goes to infinity

\[
I_{\text{ct}} = \frac{1}{8\pi G_d} \int_{\partial M} d^{d-1}x \sqrt{-\gamma} \left\{ \frac{d^2 - 2}{L} R - \frac{L}{2(d-3)} \mathcal{F}(d-4) + \frac{L}{2(d-3)^2} (d-5) \right\}, \tag{2.5}
\]

where \( R \) is the boundary Ricci scalar which only depends on the induced metric \( \gamma_{ab} \) and \( \mathcal{F}(d) \) is step function, 1 when \( d \geq 1 \), 0 otherwise.

ThAdS with the line element becomes

\[
ds^2 = \frac{r^2}{L^2} (-dt^2 + dx_i^2) + \frac{L^2}{r^2} dr^2, \tag{2.6}
\]

and substituting with \( r = L^2 / z \)

\[
ds^2 = \frac{L^2}{z^2} (-dt^2 + dx_i^2 + dz^2). \tag{2.7}
\]

The Schwarzschild BH in AdS space is given as

\[
ds^2 = \frac{r^2}{L^2} \left[ - \left( 1 - \frac{r^d-1}{r^d-1} \right) dt^2 + dx_i^2 \right] + \frac{L^2}{r^2} \left( 1 - \frac{h_{d-1}}{h_{d-1}} \right) dr^2, \tag{2.8}
\]

and substituting with \( r = L^2 / z \)

\[
ds^2 = \frac{L^2}{z^2} \left[ - \left( 1 - \frac{z^d-1}{z^d-1} \right) dt^2 + dx_i^2 + \left( 1 - \frac{z^d-1}{z^d-1} \right) dz^2 \right], \tag{2.9}
\]

which leads to the Hawking temperature of AdS BH \( T_H = (d-1)/(4\pi z_h) \) (\( T_H \equiv 1/\beta \)) where \( z_h \) is AdS BH horizon radius.

The bulk action \( I_{\text{BH}} \) [2.2] for both geometries [2.7] and [2.9] becomes

\[
I_{\text{BH}} = \frac{d-2}{8\pi G_d L^2} \int d^d x \sqrt{-g}, \tag{2.10}
\]

which has the interval \( 0 < z < z_0 \) via the IR cut-off in the hard wall model [21] [22]. Here the inverse of \( z_0 \) may be interpreted as a IR energy cut-off in the dual gauge theory side. Furthermore the bulk action [2.10] becomes singular as the coordinate \( z \) goes to zero. We may restrict ourselves to the regularized action densities, which for AdS and AdS BH are written as

\[
\mathcal{I}_{\text{AdS}}^{\text{BH}} = \frac{(d-1)L^{d-2}}{k_d^2} \int_0^{z_m} dz \frac{\pi z}{z^d 1 - \frac{1}{z_0}} \tag{2.11}
\]

and

\[
\mathcal{I}_{\text{BH}}^{\text{BH}} = \frac{(d-1)L^{d-2}}{k_d^2} \int_0^{z_m} dz \frac{\pi z}{z^d 1 - \frac{1}{z_0}}, \tag{2.12}
\]

with \( k_d^2 = 8\pi G_d \) and \( \beta = \pi z_h \sqrt{1 - (\epsilon/z_h)^{d-1}} \) [13]. Here \( \epsilon \) is the ultraviolet regulator and \( z_m \equiv \min(z_0, z_h) \) denotes the minimum value of \( z_0 \) and \( z_h \).

From Eqs. [2.3], [2.7] and [2.9], the boundary action densities are obtained as

\[
\mathcal{I}_{\text{AdS}}^{\text{BH}} = \frac{(d-1)L^{d-2} \delta^d}{k_d^2} \frac{1}{e^{d-1}}, \tag{2.13}
\]

\[
\mathcal{I}_{\text{BH}}^{\text{BH}} = \frac{(d-1)L^{d-2} \delta^d}{k_d^2} \frac{1}{e^{d-1} - \frac{1}{2z_0}}, \tag{2.14}
\]

The counterterm action \( I_{\text{ct}} \) [2.5] for both geometries [2.7] and [2.9] results in a simple form

\[
I_{\text{ct}} = \frac{d-2}{k_d^2} \frac{L^{d-2}}{L} \int_{\partial M} d^{d-1}x \sqrt{-\gamma}, \tag{2.15}
\]

which leads to the counterterm action densities

\[
\mathcal{I}_{\text{ct}}^{\text{BH}} = \frac{(d-2)L^{d-2} \delta^d}{k_d^2} \frac{1}{e^{d-1}}, \tag{2.16}
\]

\[
\mathcal{I}_{\text{ct}}^{\text{BH}} = \frac{(d-2)L^{d-2} \delta^d}{k_d^2} \frac{1}{e^{d-1} - \frac{1}{2z_0}}, \tag{2.17}
\]

Thus, from the d-dimensional gravity action \( I \) [2.1], the action densities become

\[
\mathcal{I}_{\text{AdS}} = -\frac{L^{d-2} \delta^d}{k_d^2} \frac{1}{z_0}, \tag{2.18}
\]

\[
\mathcal{I}_{\text{BH}} = -\frac{L^{d-2} \delta^d}{k_d^2} \frac{1}{z_m - \frac{1}{2z_0}}. \tag{2.19}
\]

The difference between the action densities is defined as

\[
\Delta \mathcal{I} = \lim_{\epsilon \to 0} \left[ \mathcal{I}_{\text{BH}} - \mathcal{I}_{\text{AdS}} \right], \tag{2.20}
\]

and

\[
\Delta \mathcal{I} = \frac{4\pi L^{d-2} z_h}{(d-1)k_d^2} \frac{1}{2z_h} \text{ for } z_0 < z_h, \tag{2.21}
\]

\[
\Delta \mathcal{I} = \frac{4\pi L^{d-2} z_h}{(d-1)k_d^2} \left( \frac{1}{z_0} - \frac{1}{2z_h} \right) \text{ for } z_0 > z_h. \tag{2.22}
\]
which leads to the critical temperature
\[ T_c = \frac{2}{\pi z_0}. \] (2.23)

After employing \( \log Z = -I \) as the partition function, one can get the following thermal relation \( F = -T \log Z = TI \). The free energies for AdS case and AdS BH are given as
\[ F_{\text{AdS}} = -\frac{L^{d-2}}{k_d^2 z_0^{d-3}} = -\frac{L^{d-2}}{8\pi G_5 z_0^{d-1}}, \] (2.24)
\[ F_{\text{BH}} = -\frac{\pi^{d-1}L^{d-2}}{2k_d^2}T^{d-1} = -\frac{(\pi L)^{d-2}}{16G_d}T^{d-1}, \] (2.25)
which become respectively
\[ F_{\text{AdS}} = -\frac{L^p}{k_{p+2}^2 z_0^{p+1}} = -\frac{L^p}{8\pi G_{p+1} z_0^{p+1}} \quad \text{for} \quad (T_c > T), \] (2.26)
\[ F_{\text{BH}} = -\frac{\pi^{p+1}L^p}{2k_{p+2}^2}T^{p+1} = -\frac{(\pi L)^p}{16G_{p+2}}T^{p+1} \quad \text{for} \quad (T_c < T), \] (2.27)
where \( d = p + 2 \) and \( p \) is dimension of brane.

The supergravity solution reduces to \( p + 2 \)-dimensional AdS space (AdS\(_{p+2}\)) times spheres (i.e. AdS\(_5\) × S\(_5\) for D3-branes). The entropy of the non-dilatonic near-extremal \( p \)-branes is calculated from the leading order supergravity solution. The free energy of CFT side [24] is expected to have the form
\[ F = -\alpha_p N^{\frac{d-2}{2}}T^{p+1}, \] (2.28)
where \( \alpha_p \) is a constant of order unity.

In the D3-branes case (total dimension \( D = 10 \) and \( p = 3 \), AdS\(_5\) × S\(_5\)) the free energy (2.27) is consistent with that in [4]
\[ F_{\text{BH}} = -\frac{\pi^2}{8}N^2T^4, \] (2.29)
where the five-dimensional Newton constant \( G_5 \) and AdS radius \( L \) are
\[ G_5 = \frac{8\pi^3g^2\alpha'/L^5}{} \quad \text{and} \quad L^4 = 4\pi gN\alpha'^2. \] (2.30)
In the M5-branes case (total dimension \( D = 11 \) and \( p = 5 \), AdS\(_7\) × S\(_7\)), the free energy (2.27) becomes the free energy of AdS BH [24]
\[ F_{\text{BH}} = \frac{2^6\pi^3}{3^3}N^3T^6, \] (2.31)
where the seven-dimensional Newton constant \( G_7 \) and AdS radius \( L \) are
\[ G_7 = \frac{3^7\pi^2L^5}{210N^3} \quad \text{and} \quad L^9 = \frac{G_7N^3}{2^{15/2}\pi^4}. \] (2.32)
In the M2-branes case (total dimension \( D = 11 \) and \( p = 2 \), AdS\(_3\) × S\(_7\)), the free energy (2.27) becomes the free energy of AdS BH [24]
\[ F_{\text{BH}} = -\frac{2^7/2\pi^2}{3^4}N^{3/2}T^3, \] (2.33)
where the four-dimensional Newton constant \( G_4 \) and AdS radius \( L \) are
\[ G_4 = \frac{3^4L^2}{2^{15/2}N^3} \quad \text{and} \quad L^9 = \frac{G_4N^3}{2^{7/2}\pi^4}. \] (2.34)
On the other hand, the expectation value of the energy is given as
\[ < E > = -\partial_3 \log Z = \partial_3 I \sim -F. \] (2.35)
After employing the Gibbs-Duhem relation \( S = \beta < E > + \log Z = \beta < E > - I \), the entropies for AdS case and AdS BH are given as
\[ S_{\text{AdS}} = 0 \sim N^0 \quad \text{for} \quad (T_c > T), \] (2.36)
\[ S_{\text{BH}} = \frac{(d-1)(\pi L)^{d-2}}{16G_d}T^{d-2}, \] (2.37)
\[ = \frac{(p+1)(\pi L)^p}{16G_{p+2}}T^p \quad \text{for} \quad (T_c < T), \] (2.38)
where after employing the relation (2.29) between the expectation value of the energy and the free energy, and substituting the Gibbs-Duhem relation with the free energy (2.27), we can read
\[ S_{\text{BH}} \sim N^{\frac{d-2}{2}}T^p \sim N^{\frac{d-4}{2}}. \] (2.39)
It is shown that the entropy of \( p \)-branes jumps from \( O(N^0) \) at the confining low temperature phase to \( O(N^{\frac{d-2}{2}}) \) at the deconfining high temperature phase like D3-branes (\( p = 3 \)) case, which is consistent with the jump in the entropy describing the change of degrees of freedom in the confinement-deconfinement phase transition of QCD [13, 16].

Now, we will check the above relation through explicitly calculating for specific cases. Adopting the five-dimensional Newton constant \( G_5 \) and AdS radius \( L \) (2.30), the entropy (2.28) in the D3-branes case is given as
\[ S_{\text{BH}} = \frac{1}{2}\pi^2N^2T^3 \sim N^2T^3 \sim N^2 \quad \text{for} \quad p = 3. \] (2.40)
In the same way, employing the seven-dimensional Newton constant \( G_7 \) (2.32) and AdS radius \( L \) (2.32) in the M5-branes case, and the four-dimensional Newton constant \( G_4 \) (2.34) and AdS radius \( L \) (2.34) in the M2-branes
After adopting the relation between the mass and the
and the mass for AdS BH becomes

\[ M_{\text{BH}} = \frac{(d - 2)L^{d-2}}{2k_d^2z^{d-1}}. \tag{3.2} \]

After adopting the relation between the mass and the
density \( \int_0^{z_h} dz \rho(z) = M \), one can read the regularized
density from the ThAdS mass [17]

\[ \rho_{\text{AdS}} = \frac{(d - 1)L^{d-2}}{k_d} \frac{1}{z^{d}} \cos \left( \frac{2\pi \epsilon^{d-1}}{z^{d-1}} \right), \tag{3.3} \]

which leads to

\[ \int_{\epsilon}^{z_h} \rho_{\text{AdS}} dz = -\frac{L^{d-2}}{2\pi k_d^2 \epsilon^{d-1}} \sin \left( \frac{2\pi \epsilon^{d-1}}{z^{d-1}} \right), \tag{3.4} \]

which reduces to the mass of ThAdS \( M_{\text{ThAdS}} \) in the

\[ \rho_{\text{BH}} = -\frac{(d - 1)(d - 2)L^{d-2}}{4\pi k_d^2 \epsilon^{d-1}} \sin \left( \frac{2\pi \epsilon^{d-1}}{z^{d-1}} \right), \tag{3.5} \]

which becomes the mass of AdS BH \( M_{\text{BH}} \) in the

When we consider the energy density \( \rho(z) \) as the function of the position \( z \) in \( d \)-dimensional space and its

\[ \rho(k) = \left( \frac{1}{\sqrt{2\pi}} \right)^d \int \rho(z)e^{-ik\cdot z}d^dx, \tag{3.7} \]

and the modal fraction is defined as

\[ \mathcal{F}(k) = \frac{\rho(k)}{\int |\rho(k)|^2 d^dk}. \tag{3.8} \]

One may define the CE \[ CE \] as

\[ S[\mathcal{F}] = -\int_{-\infty}^{\infty} \mathcal{F}(k) \log[\mathcal{F}(k)]d^dk. \tag{3.9} \]

As shown in Table I, we calculate the entropies for the variety of the dimension of AdS space from \( d = 4 \) to \( d = 9 \)
and plot them as you see in Fig.1. Since the temperature

\[ T = (d - 1)/(4\pi \epsilon \chi), \]

which correctly match with the above result \( S_{\text{BH}} \sim N^{d+1}/(2.41) \).

Finally after employing thermal relation \( C = -\beta \partial_\beta S \),

the specific heat for AdS case and AdS BH are given as

\[ C_{\text{AdS}} = 0 \text{ for } (T_c > T), \tag{2.43} \]

\[ C_{\text{BH}} = \frac{p(p + 1)(\pi L)^d}{16G_d} T^{d-2}, \tag{2.44} \]

\[ = \frac{p(p + 1)(\pi L)^d}{16G_{p+2}} T^{d} \text{ for } (T_c < T). \tag{2.45} \]

In particular, since AdS BH for \( T_c < T \) always has

positive specific heat it is thermodynamically stable.

III. CE OF THADS AND ADS BHs

After employing thermal relation \( \partial_\beta I = M \), the mass
for ThAdS is obtain as

\[ M_{\text{ThAdS}} = -\frac{L^{d-2}}{k_d^2 z^{d-1}}, \tag{2.33} \]

and the mass for AdS BH becomes

| \( z_h \) | \( d = 4 \) | \( d = 5 \) | \( d = 6 \) | \( d = 7 \) | \( d = 8 \) | \( d = 9 \) | Type |
|---|---|---|---|---|---|---|
| 0.01 | 17.95581731 | 17.95595568 | 17.95597083 | 17.95597251 | 17.95597269 | 17.95597271 | AdS BH |
| 0.05 | 16.34637369 | 16.34631777 | 16.34653292 | 16.34653460 | 16.34653469 | 16.34653473 | AdS BH |
| 0.1 | 15.65323222 | 15.65337059 | 15.65338574 | 15.65338741 | 15.65338760 | 15.65338762 | AdS BH |
| 0.5 | 14.55461993 | 14.55475830 | 14.55477345 | 14.55477514 | 14.55477521 | 14.55477533 | AdS BH |
| 0.7 | 14.04379431 | 14.04393286 | 14.04397483 | 14.04394950 | 14.04394969 | 14.04394971 | AdS BH |
| \( z_c \) | 13.70732207 | 13.70746044 | 13.70747539 | 13.70747726 | 13.70747745 | 13.70747747 | AdS BH |
| \( z_h > z_c \) | 13.56214063 | 13.50450336 | 13.46952571 | 13.44674434 | 13.43023742 | 13.41785718 | ThAdS |

Here, \( z_c \) denotes the horizon radius at the critical temperature.
as the temperature increases, which correctly coincides with the expected result that the evaporation with arising from emitting AdS BH radiation increases faster at higher temperatures. These results are similar to that in [17].

DISCUSSION

For any dimension we considered the hard wall model in the context of the AdS/CFT duality and explicitly presented the critical temperature $T_c = \frac{2\pi}{\sqrt{z_0\pi}} \frac{1}{2.23}$ via the holographic renormalization method. Then the entropy of $p$-branes jumps from leading order in $O(N^0)$ at the confining low temperature phase below the critical temperature to $O(N^{\frac{2}{p+1}})$ at the deconfining high temperature phase above the critical temperature like $D3$-branes ($p=3$) case [16]. It was found for the hard wall model that AdS BH above the critical temperature $2.23$ becomes stable while ThAdS below the critical temperature is stable. We also obtained the same results through thermodynamic analysis of calculating the CE. It was shown that it is well held to describe the confinement-deconfinement transition for the gauge theory from BH thermodynamics in AdS for the higher-dimensional hard wall model beyond the five-dimensional case [16]. In particular, we found that as the temperature grows up, the CE has constant until the critical temperature while it decreases beyond the critical temperature. This result is consistent with that in [17].

Our research here focused on the higher-dimensional hard wall model. Thus, it will be intriguing that the issue is generalized to the higher-dimensional soft wall model. Adopting the CE, one can explore not only its instability but also conventional hadrons and exotica ones [26].

On the other hand, it is possible to investigate the stability of the AdS black string solution via the CE. It provides a new window in order to check the Gubser-Mitra conjecture [27, 28]. We hope that such kind issues will be carried out in near the future.

DECLARATION OF COMPETING INTEREST

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