SHARP THRESHOLDS FOR THE RANDOM-CLUSTER AND ISING MODELS

BY BENJAMIN GRAHAM AND GEOFFREY GRIMMETT
University of British Columbia and Cambridge University

A sharp-threshold theorem is proved for box-crossing probabilities on the square lattice. The models in question are the random-cluster model near the self-dual point \( p_{sd}(q) = \sqrt{q}/(1 + \sqrt{q}) \), the Ising model with external field, and the colored random-cluster model. The principal technique is an extension of the influence theorem for monotonic probability measures applied to increasing events with no assumption of symmetry.

1. Introduction. The method of “sharp threshold” has been fruitful in probabilistic combinatorics (see [20, 27] for recent reviews). It provides a fairly robust tool for showing the existence of a sharp threshold for certain processes governed by independent random variables. Its most compelling demonstration so far in the field of physical systems has been the proof in [9] that the critical probability of site percolation on the Voronoi tessellation generated by a Poisson process on \( \mathbb{R}^2 \) equals \( \frac{1}{2} \).

Each of the applications alluded to above involves a product measure. It was shown in [16] that the method may be extended to nonproduct probability measures satisfying the FKG lattice condition. The target of this note is to present two applications of such a sharp-threshold theorem to measures arising in statistical physics, namely those of the random-cluster model and the Ising model. In each case, the event in question is the existence of a crossing of a large box, by an open path in the case of the random-cluster model, and by a single-spin path in the case of the Ising model. A related but more tentative and less complete result has been obtained in [16] in the first case, and the second case has been studied already in [7] and [23, 24].

Our methods for the Ising model can be applied to a more general model termed here the colored random-cluster model (CRCM), see Section 8. This model is related to the so-called fractional Potts model of [26], and the fuzzy Potts model and the divide-and-color model of [5, 13, 21, 22].

The sharp-threshold theorem used here is an extension of that given for product measure in [15, 37], and it makes use of the results of [16]. It is stated, with an outline of the proof, in Section 5. The distinction of the current sharp-threshold
Theorem is that it makes no assumption of symmetry on either the event or measure in question. Instead, one needs to estimate the maximum influence of the various components, and it turns out that this may be done in a manner which is very idiomatic for the models in question. The sharp-threshold theorem presented here may find further applications in the study of dependent random variables.

2. The models.

2.1. The random-cluster model. The random-cluster model on a connected graph $G$ has two parameters: an edge-weight $p$ and a cluster-weight $q$. See Section 3 for a formal definition. When $q \geq 1$ and $G$ is infinite, there is a critical value $p_c(q)$ that separates the subcritical phase of the model [when $p < p_c(q)$ and there exist no infinite clusters] and the supercritical phase. It has long been conjectured that, when $G$ is the square lattice $\mathbb{Z}^2$,

$$p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}, \quad q \geq 1. \quad (2.1)$$

This has been proved rigorously in three famous cases. When $q = 1$, the random-cluster model is bond percolation, and the exact calculation $p_c(1) = \frac{1}{2}$ was shown by Kesten [28]. When $q = 2$, the model is intimately related to the Ising model, and the calculation of $p_c(2)$ is equivalent to that of Onsager and others concerning the Ising critical temperature (see [1, 3] for a modern treatment of the Ising model). Formula (2.1) has been proved for sufficiently large values of $q$ (currently $q \geq 21.61$) in the context of the proof of first-order phase transition, see [19, 29–31].

We recall that, when $q \in \{2, 3, \ldots\}$, the critical temperature $T_c$ of the $q$-state Potts model on a graph $G$ satisfies

$$p_c(q) = 1 - e^{-1/T_c}. \quad (2.2)$$

A fairly full account of the random-cluster model, and its relation to the Potts model, may be found in [19].

Conjecture (2.1) is widely accepted. Physicists have proceeded beyond a “mere” calculation of the critical point, and have explored the behavior of the process at and near this value. For example, it is believed that there is a continuous (second-order) phase transition if $1 \leq q < 4$, and a discontinuous (first-order) transition when $q > 4$, see [6]. Amongst recent progress, we highlight the stochastic Löwner evolution process $SLE_{16/3}$ associated with the cluster boundaries in the critical case when $q = 2$ and $p = \sqrt{2}/(1 + \sqrt{2})$, see [35, 36].

The expression in (2.1) arises as follows through the use of planar duality. When the underlying graph $G$ is planar, it possesses a (Whitney) dual graph $G_d$. The random-cluster model on $G$ with parameters $p, q$ may be related to a dual random-cluster model on $G_d$ with parameters $p_d, q$, where

$$\frac{p_d}{1 - p_d} = \frac{q(1 - p)}{p} \quad (2.3)$$
The mapping \( p \mapsto p_d \) has a fixed point \( p = p_{sd}(q) \), where
\[
p_{sd}(q) := \frac{\sqrt{q}}{1 + \sqrt{q}}
\]
is termed the self-dual point. The value \( p = p_{sd}(q) \) is especially interesting when \( G \) and \( G_d \) are isomorphic, as in the case of the square lattice \( \mathbb{Z}^2 \). See [19], Chapter 6. We note for future use that
\[
p < p_{sd}(q) \quad \text{if and only if} \quad p_d > p_{sd}(q).
\]

Henceforth, we take \( G = \mathbb{Z}^2 \). The inequality
\[
p_c(q) \geq p_{sd}(q), \quad q \geq 1,
\]
was proved in [17, 38] using Zhang’s argument (see [18], page 289). Two further steps would be enough to imply the complementary inequality \( p_c(q) \leq p_{sd}(q) \): firstly, that the probability of crossing a box \([-m, m]^2\) approaches 1 as \( m \to \infty \), when \( p > p_{sd}(q) \); and secondly, that this implies the existence of an infinite cluster. The first of these two claims is proved in Theorem 3.1.

Kesten’s proof for percolation, [28], may be viewed as a proof of the first claim in the special case \( q = 1 \). The second claim follows for percolation by RSW-type arguments, see [32–34] and [18], Section 11.7. Heavy use is made in these works of the fact that the percolation measure is a product measure, and this is where the difficulty lies for the random-cluster measure.

We prove our main theorem (Theorem 3.1 below) by the method of influence and sharp threshold developed for product measures in [15, 25]. This was adapted in [16] to monotonic measures applied to increasing events, subject to a certain hypothesis of symmetry. We show in Section 5 how this hypothesis may be removed, and we apply the subsequent inequality in Section 6 to the probability of a box-crossing, thereby extending to general \( q \) the corresponding argument of [10].

### 2.2. Ising model

We shall consider the Ising model on the square lattice \( \mathbb{Z}^2 \) with edge-interaction parameter \( \beta \) and external field \( h \). See Section 4 for the relevant definitions. Write \( \beta_c \) for the critical value of \( \beta \) when \( h = 0 \), so that
\[
1 - e^{-2\beta_c} = p_{sd}(2),
\]
where \( p_{sd}(2) \) is given as in (2.1). Two notions of connectivity are required: the usual connectivity relation \( \leftrightarrow \) on \( \mathbb{Z}^2 \) viewed as a graph, and the relation \( \leftrightarrow_* \), termed \( * \)-connectivity, and obtained by adding diagonals to each unit face of \( \mathbb{Z}^2 \).

Let \( \pi_{\beta, h} \) denote the Ising measure on \( \mathbb{Z}^2 \) with parameters \( \beta, h \).

Higuchi proved in [23, 24] that, when \( \beta \in (0, \beta_c) \), there exists a critical value \( h_c = h_c(\beta) \) of the external field such that:

(a) \( h_c(\beta) > 0 \),
(b) when $h > h_c$, there exists $\pi_{\beta,h}$-almost-surely an infinite $\ast$-cluster of $\mathbb{Z}^2$, and the radius of the $\ast$-connected $+$-cluster at the origin has exponential tail.

(c) when $0 < h < h_c$, there exists $\pi_{\beta,h}$-almost-surely an infinite $\ast$-connected $-$-cluster of $\mathbb{Z}^2$, and the radius of the $+$-cluster at the origin has exponential tail.

A further approach to Higuchi’s theorem has been given recently by van den Berg [7]. A key technique of the last paper is a sharp-threshold theorem of Talagrand [37] for product measures. The Ising measure $\pi_{\Lambda,\beta,h}$ on a box $\Lambda$ is of course not a product measure, and so it was necessary to encode it in terms of a family of independent random variables. We show here that the influence theorem of [16] may be extended and applied directly to the Ising model to obtain the necessary sharp threshold result. (The paper [7] contains results for certain other models encodable in terms of product measures, and these appear to be beyond the scope of the current method.)

2.3. Colored random-cluster model. The Ising model with external field is a special case of a class of systems that have been studied by a number of authors, and which we term colored random-cluster models (CRCM). Sharp-threshold results may be obtained for such systems also. Readers are referred to Section 8 for an account of the CRCM and the associated results.

3. Box-crossings in the random-cluster model. The random-cluster measure is given as follows on a finite graph $G = (V, E)$. The configuration space is $\Omega = \{0, 1\}^E$. For $\omega \in \Omega$, we write $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ for the set of “open” edges, and $k(\omega)$ for the number of connected components in the open graph $(V, \eta(\omega))$. Let $p \in [0, 1]$, $q \in (0, \infty)$, and let $\phi_{p,q}$ be the probability measure on $\Omega$ given by

$$\phi_{p,q}(\omega) = \frac{1}{Z} \prod_{e \in E} p^{\omega(e)} (1 - p)^{1 - \omega(e)} q^{k(\omega)}, \quad \omega \in \Omega,$$

where $Z = Z_{G,p,q}$ is the normalizing constant. We shall assume throughout this paper that $q \geq 1$, so that $\phi_{p,q}$ satisfies the so-called FKG lattice condition

$$\mu(\omega_1 \lor \omega_2) \mu(\omega_1 \land \omega_2) \geq \mu(\omega_1) \mu(\omega_2), \quad \omega_1, \omega_2 \in \Omega.$$  

Here, as usual,

$$\omega_1 \lor \omega_2(e) = \max\{\omega_1(e), \omega_2(e)\},$$

$$\omega_1 \land \omega_2(e) = \min\{\omega_1(e), \omega_2(e)\}$$

for $e \in E$. As a consequence of (3.2), $\phi_{p,q}$ satisfies the FKG inequality. See [19] for the basic properties of the random-cluster model.

Consider the square lattice $\mathbb{Z}^2$ with edge-set $E$, and let $\Omega = \{0, 1\}^E$. Let $\Lambda = \Lambda_n = [-n, n]^2$ be a finite box of $\mathbb{Z}^2$, with edge-set $E_\Lambda$. For $b \in \{0, 1\}$ define

$$\Omega^b_\Lambda = \{\omega \in \Omega : \omega(e) = b \text{ for } e \notin E_\Lambda\}.$$
On $\Omega^b_\Lambda$ we define a random-cluster measure $\phi^{b}_{\Lambda,p,q}$ as follows. For $p \in [0, 1]$ and $q \in [1, \infty)$, let

$$
(3.3) \quad \phi^{b}_{\Lambda,p,q}(\omega) = \frac{1}{Z^{b}_{\Lambda,p,q}} \left\{ \prod_{e \in E_\Lambda} p^{\omega(e)} (1 - p)^{1 - \omega(e)} \right\} q^{k(\omega, \Lambda)}, \quad \omega \in \Omega^b_\Lambda,
$$

where $k(\omega, \Lambda)$ is the number of clusters of $(\mathbb{Z}^2, \eta(\omega))$ that intersect $\Lambda$. The boundary condition $b = 0$ (resp., $b = 1$) is usually termed “free” (resp., “wired”). It is standard that the weak limits

$$
\phi^{b}_{p,q} = \lim_{n \to \infty} \phi^{b}_{\Lambda_n,p,q}
$$

exist, and that they are translation-invariant, ergodic, and satisfy the FKG inequality. See [19], Chapter 4.

For $A, B \subseteq \mathbb{Z}^2$, we write $A \leftrightarrow B$ if there exists an open path joining some $a \in A$ to some $b \in B$. We write $x \leftrightarrow \infty$ if the vertex $x$ is the endpoint of some infinite open path. The percolation probabilities are given as

$$
\theta^{b}(p, q) = \phi^{b}_{p,q}(0 \leftrightarrow \infty), \quad b = 0, 1.
$$

Since each $\theta^{b}$ is nondecreasing in $p$, one may define the critical point by

$$
p^{c}(q) = \sup\{p : \theta^{1}(p, q) = 0\}.
$$

It is known that $\phi^{0}_{p,q} = \phi^{1}_{p,q}$ if $p \neq p^{sd}(q)$, and we write $\phi_{p,q}$ for the common value. In particular, $\theta^{0}(p, q) = \theta^{1}(p, q)$ for $p \neq p^{c}(q)$. It is conjectured that $\phi^{0}_{p,q} = \phi^{1}_{p,q}$ when $p = p^{c}(q)$ and $q \leq 4$.

Let $B_k = [0, k] \times [0, k - 1]$, and let $H_k$ be the event that $B_k$ possesses an open left–right crossing. That is, $H_k$ is the event that $B_k$ contains an open path having one endvertex on its left side and one on its right-hand side.

**Theorem 3.1.** Let $q \geq 1$. We have that

$$
(3.4) \quad \phi_{p,q}(H_k) \leq 2\rho_k^{p^{sd}p}, \quad 0 < p < p^{sd}(q),
$$

$$
(3.5) \quad \phi_{p,q}(H_k) \geq 1 - 2\nu_k^{p^{sd}p}, \quad p^{sd}(q) < p < 1,
$$

for $k \geq 1$, where

$$
(3.6) \quad \rho_k = [2q \eta_k / p]^{c/q}, \quad \nu_k = [2q \eta_k / p^{d}]^{c/q}
$$

and

$$
(3.7) \quad \eta_k = \phi^{0}_{p^{sd}(q), q}(0 \leftrightarrow \partial \Lambda_k / 2) \to 0 \quad \text{as } k \to \infty.
$$

Here, $c$ is an absolute positive constant, and $p^{d}$ satisfies (2.3).
When \( k \) is odd, we interpret \( \partial \Lambda_{k/2} \) in (3.7) as \( \partial \Lambda_{\lfloor k/2 \rfloor} \).

In essence, the probability of a square-crossing has a sharp threshold around the self-dual “pivot” \( p_{\text{sd}}(q) \). Related results were proved in [16], but with three relative weaknesses, namely: only nonsquare rectangles could be handled, the “pivot” of the threshold theorems was unidentified, and there was no result for infinite-volume measures. The above strengthening is obtained by using the threshold Theorem 5.1 which makes no assumption of symmetry on the event or measure in question. The corresponding threshold theorem for product measure leads to a simplification of the arguments of [10] for percolation, see [20], Section 5.8.

Since \( \phi_{\Lambda_n,p,q} \leq \varphi_{p,q} \leq \phi_{\Lambda_n,p,q}^1 \) and \( H_k \) is an increasing event, Theorem 3.1 implies certain inequalities for finite-volume probabilities also.

No estimate for the rate at which \( \eta_k \to 0 \) is implicit in the arguments of this paper, and indeed one of the targets of the current work is to show that no estimate is necessary for sharp threshold. It is expected that \( \eta_k \to 0 \) at a rate that depends on whether or not the phase transition is continuous: one expects that \( \eta_k \) decays as a power when \( 1 \leq q < 4 \), and as an exponential when \( q > 4 \) (see [19], Section 6.4).

This would imply a threshold of order either \( 1/\log k \) or \( 1/k \) in (3.4)–(3.5). That the radius \( R \) of the open cluster at the origin is \( \phi_{p_{\text{sd}}(q),q}^0 \)-a.s. finite is a consequence of the (a.s.) uniqueness of the infinite open cluster whenever it exists. See [19], Theorem 6.17(a), for a proof of the relevant fact that

\[
\theta(p_{\text{sd}}(q),q) = 0, \quad q \geq 1. \tag{3.8}
\]

We shall prove a slightly more general result than Theorem 3.1. Let \( B_{k,m} = [0,k] \times [0,m] \) and let \( H_{k,m} \) be the event that there exists an open left–right crossing of \( B_{k,m} \).

**Theorem 3.2.** Let \( q \geq 1 \). We have that

\[
\phi_{p_1,q}(H_{k,m}) [1 - \phi_{p_2,q}(H_{k,m})] \leq \rho_k^{p_2 - p_1}, \quad 0 < p_1 < p_2 \leq p_{\text{sd}}(q), \tag{3.9}
\]

\[
\phi_{p_1,q}(H_{k,m}) [1 - \phi_{p_2,q}(H_{k,m})] \leq \nu_{m+1}^{p_2 - p_1}, \quad p_{\text{sd}}(q) \leq p_1 < p_2 < 1, \tag{3.10}
\]

for \( k, m \geq 1 \), where \( \rho_k \) (resp., \( \nu_k \)) is given in (3.6) with \( p = p_1 \) (resp., \( p = p_2 \)), and \( \phi_{p_{\text{sd}}(q),q} \) is to be interpreted as \( \phi_{p_{\text{sd}}(q),q}^0 \).

**4. Box-crossings in the Ising model.** Let \( \Lambda \) be a box of \( \mathbb{Z}^2 \). The spin-space of the Ising model on \( \Lambda \) is \( \Sigma_\Lambda = \{-1, +1\}^\Lambda \), and the Hamiltonian is

\[
H_\Lambda(\sigma) = -\beta \sum_{e = (x,y) \in \mathcal{E}_\Lambda} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x,
\]

where \( \beta > 0, h \geq 0 \). The relevant Ising measure is given by

\[
\pi_{\Lambda,\beta,h}(\sigma) \propto e^{-H_\Lambda(\sigma)}, \quad \sigma \in \Sigma_\Lambda.
\]
and it is standard that the (weak) limit measure \( \pi_{\beta,h} = \lim_{\Lambda \rightarrow \mathbb{Z}^2} \pi_{\Lambda,\beta,h} \) exists. We shall also need the \(+\) boundary-condition measure \( \pi^+_{\beta,0} \) given as the weak limit of \( \pi_{\Lambda,\beta,0} \) conditional on \( \sigma_x = +1 \) for \( x \in \partial \Lambda \). (Here, \( \partial \Lambda \) denotes as usual the boundary of \( \Lambda \), that is, the set of \( x \in \Lambda \) possessing a neighbor not belonging to \( \Lambda \)). By the FKG inequality or otherwise, \( \pi^+_{\beta,0}(\sigma_0) \geq 0 \), and the critical value of \( \beta \) when \( h = 0 \) is given by
\[
\beta_c = \sup\{\beta : \pi^+_{\beta,0}(\sigma_0) = 0\}.
\]
As remarked in Section 2, \( 1 - e^{-2\beta_c} = p_{sq}(2) \). It is well known that there exists a unique infinite-volume measure for the Ising model on \( \mathbb{Z}^2 \) if either \( h \neq 0 \) or \( \beta < \beta_c \), and thus \( \pi_{\beta,h} \) is this measure. By Holley’s theorem, (see [19], Section 2.1, e.g.), \( \pi_{\beta,h} \) is stochastically increasing in \( h \).

Let \( \theta^+_{(\beta,h)}(0 \leftrightarrow \infty) \), \( \theta^-_{(\beta,h)}(0 \leftrightarrow \infty) \), where the relation \( \leftrightarrow \) (resp., \( \leftrightarrow_r \)) means that there exists a path of \( \mathbb{Z}^2 \) each of whose vertices has state \( +1 \) (resp., a \( \leftrightarrow_r \)-connected path of vertices with state \( -1 \)).

The next theorem states the absence of coexistence of such infinite components, and its proof (given in Section 7) is a simple application of the Zhang argument for percolation (see [18], Section 11.3).

**Theorem 4.1.** We have that
\[
\theta^+_{(\beta,h)}(0 \leftrightarrow \infty) = 0, \quad \beta \geq 0, \quad h \geq 0.
\]

There exists \( h_c = h_c(\beta) \in [0, \infty) \) such that
\[
\theta^+_{(\beta,h)}(0 \leftrightarrow \infty) = \begin{cases} 
0, & \text{if } 0 \leq h < h_c, \\
> 0, & \text{if } h > h_c.
\end{cases}
\]

Recall from [23, 24] that \( h_c(\beta) > 0 \) if and only if \( \beta < \beta_c \). It is proved in [24] that
\[
\theta^\pm_{(\beta,h_c(\beta))} = 0,
\]
but we shall not make use of this fact in the proofs of this paper. Indeed, one of the main purposes of this article is to show how certain sharp-thresholds for box-crossings may be obtained using a minimum of background information on the model in question.

Let \( H_{k,m} \) be the event that there exists a left–right + crossing of the box \( B_{k,m} = [0,k] \times [0,m] \). Let \( x^+ = \max\{x, 0\} \).

**Theorem 4.2.** Let \( 0 \leq \beta < \beta_c \) and \( R > 0 \). There exist \( \rho_{i,+} = \rho_{i,+}(\beta) \) and \( \rho_{i,-} = \rho_{i,-}(\beta, R) \) satisfying
\[
\rho_{i,+} + \rho_{i,-} \rightarrow 0 \quad \text{as } i \rightarrow \infty,
\]

such that: for \( 0 \leq h_1 \leq h_c \leq h_2 < R \),
\[
\pi_{\beta,h_1}(H_{k,m})[1 - \pi_{\beta,h_2}(H_{k,m})] \leq \rho_{k,+}^{h_c-h_1} \rho_{m,-}^{h_2-h_c}, \quad k, m \geq 1.
\]
The proof of this theorem shows also that
\[
\pi_{\beta,h_1}(H_{k,m})[1 - \pi_{\beta,h_2}(H_{k,m})] \leq \rho_{k,+}^{h_2-h_1}, \quad h_1 \leq h_2 \leq h_c.
\]
\[
\pi_{\beta,h_1}(H_{k,m})[1 - \pi_{\beta,h_2}(H_{k,m})] \leq \rho_{m,-}^{h_2-h_1}, \quad h_c \leq h_1 \leq h_2.
\]

As in Theorem 3.1, the proof neither uses nor implies any estimate on the rate at which \( \rho_{i,\pm} \to 0 \). The \( \rho_{i,\pm} \) are related to the tails of the radii of the + cluster and the \(-\ast\)-cluster at the origin. More explicitly,

\[
\rho_{i,+} = [2(1 + e^{8\beta})\pi_{\beta,h_c}(0 \leftrightarrow \partial \Lambda_{i/2})]^B^+, \tag{4.4}
\]
\[
\rho_{i,-} = [2(1 + e^{8\beta+2R})\pi_{\beta,h_c}(0 \leftrightarrow \ast \partial \Lambda_{i/2})]^B^-, \tag{4.5}
\]

where 
\[
B^+ = 2c \xi_{\beta,h_c}, \quad B^- = 2c \xi_{\beta,R},
\]
and \( \xi_{\beta,h} \) is given in the forthcoming (7.4). Equation (4.2) holds by Theorem 4.1 with \( h = h_c(\beta) \). It is in fact a consequence of (4.1) that \( \rho_{i,\pm} \to 0 \) as \( i \to \infty \).

### 5. Influence and sharp threshold.

Let \( S \) be a finite set. Let \( \mu \) be a measure on \( \Omega = \{0, 1\}^S \) satisfying the FKG lattice condition (3.2), and assume that \( \mu \) is positive in that \( \mu(\omega) > 0 \) for all \( \omega \in \Omega \). It is standard that, for a positive measure \( \mu \), (3.2) is equivalent to the condition that \( \mu \) be monotone, which is to say that the one-point conditional measure \( \mu(\sigma_x = 1|\sigma_y = \eta_y \text{ for } y \neq x) \) is nondecreasing in \( \eta \). Furthermore, (3.2) implies that \( \mu \) is positively associated, in that increasing events are positively correlated. See, for example, [19], Chapter 2.

For \( p \in (0, 1) \), let \( \mu_p \) be given by

\[
\mu_p(\omega) = \frac{1}{Z_p} \left\{ \prod_{s \in S} p^{\omega(s)} (1 - p)^{1-\omega(s)} \right\} \mu(\omega), \quad \omega \in \Omega, \tag{5.1}
\]
where \( Z_p \) is chosen in such a way that \( \mu_p \) is a probability measure. It is easy to check that each \( \mu_p \) satisfies the FKG lattice condition.

Let \( A \) be an increasing event, and write \( 1_A \) for its indicator function. We define the (conditional) influence of the element \( s \in S \) on the event \( A \) by

\[
J_{A,p}(s) = \mu_p(A|1_s = 1) - \mu_p(A|1_s = 0), \quad s \in S, \tag{5.2}
\]

where \( 1_s \) is the indicator function that \( \omega(s) = 1 \). Note that \( J_{A,p}(s) \) depends on the choice of \( \mu \). The conditional influence is not generally equal to the (absolute) influence of [25],

\[
I_{A,p}(s) = \mu_p(1_A(\omega^s) \neq 1_A(\omega_s)),
\]

where the configuration \( \omega^s \) (resp., \( \omega_s \)) is that obtained from \( \omega \) by setting \( \omega(s) = 1 \) [resp., \( \omega(s) = 0 \)].
Theorem 5.1. There exists a constant $c > 0$ such that the following holds. For any such $S$, $\mu$, and any increasing event $A \neq \emptyset, \Omega$,

\[
\frac{d}{dp} \mu_p(A) \geq \frac{c \xi_p}{p(1 - p)} \mu_p(A)(1 - \mu_p(A)) \log[1/(2m_{A,p})],
\]

where $m_{A,p} = \max_{s \in S} J_{A,p}(s)$ and $\xi_p = \min_{s \in S} \{\mu_p(1_s)(1 - \mu_p(1_s))\}$.

Corollary 5.1. In the notation of Theorem 5.1,

\[
\mu_{p_1}(A)(1 - \mu_{p_2}(A)) \leq \kappa B_{p_2 - p_1}, \quad 0 < p_1 \leq p_2 < 1,
\]

where

\[
\kappa = 2 \sup_{p \in (p_1, p_2)} \inf_{s \in S} J_{A,p}(s).
\]

The corresponding inequality for product measures may be found in [37], Corollary 1.2. Throughout this note, the letter $c$ shall refer only to the constant of Theorem 5.1.

Proof of Theorem 5.1. It is proved in \[8, 16\] that

\[
\frac{d}{dp} \mu_p(A) = \frac{1}{p(1 - p)} \sum_{s \in S} \mu_p(1_s)(1 - \mu_p(1_s)) J_{A,p}(s).
\]

Let $K = [0, 1]^S$ be the “continuous” cube, endowed with Lebesgue measure $\lambda$, and let $B$ be an increasing subset of $K$. The influence $I_B(s)$ of an element $s$ is given in [11] as

\[
I_B(s) = \lambda(1_B(s) \neq 1_B(\psi(s)),
\]

where $\psi(s)$ (resp., $\psi_s$) is the member of $K$ obtained from $\psi \in K$ by setting $\psi(s) = 1$ [resp., $\psi(s) = 0$]. The conclusion of [11] may be expressed as follows. There exists a constant $c > 0$, independent of all other quantities, such that: for any increasing event $B \subseteq K$,

\[
\sum_{s \in S} I_B(s) \geq c \lambda(B)(1 - \lambda(B)) \log[1/(2m_B)],
\]

where $m_B = \max_{s \in S} I_B(s)$. The main result of [11] is a lower bound on $m_B$ that is easily seen to follow from (5.5).

Equation (5.5) does not in fact appear explicitly in [11], but it may be derived from the arguments presented there, very much as observed in the case of the discrete cube from the arguments of [25]. See [15], Theorem 3.4. The factor of 2 on the right-hand side of (5.5) is of little material consequence, since the inequality is important only when $m_B$ is small, and, when $m_B < \frac{1}{3}$ say, the 2 may be removed.
with an amended value of the constant $c$. The literature on influence and sharp-threshold can seem a little disordered, and a coherent account may be found in [20]. The method used there introduces the factor 2 in a natural way, and for this reason we have included it in the above.

It is shown in [16] (see the proof of Theorem 2.10) that there exists an increasing subset $B$ of $K$ such that $\mu_p(A) = \lambda(B)$, and $J_{A,p}(s) \geq I_B(s)$ for all $s \in S$. Inequality (5.3) follows by (5.4)–(5.5).

**Proof of Corollary 5.1.** By (5.3),

$$
\left(\frac{1}{\mu_p(A)} + \frac{1}{1 - \mu_p(A)}\right)\mu'_p(A) \geq B \log(\kappa^{-1}), \quad p_1 < p < p_2,
$$

whence, on integrating over $(p_1, p_2)$,

$$
\frac{\mu_{p_2}(A)}{1 - \mu_{p_2}(A)} / \frac{\mu_{p_1}(A)}{1 - \mu_{p_1}(A)} \geq \kappa^{B(p_2 - p_1)}.
$$

The claim follows.

**6. Proofs of Theorems 3.1 and 3.2.** Note first that a random-cluster measure has the form of (5.1) with $S = E$ and $\mu(\omega) = q^{k(\omega)}$, and it is known and easily checked that $\mu$ satisfies the FKG lattice condition when $q \geq 1$ (see [19], Section 3.2, e.g.). We shall apply Theorem 5.1 to a random-cluster $\phi_p,q$ measure with $q \geq 1$. It is standard (see [19], Theorem 4.17(b)) that

$$
p \leq \frac{p}{p + q(1 - p)} \leq \phi_{p,q}(1_e) \leq p,
$$

whence

$$
\phi_{p,q}(1_e)[1 - \phi_{p,q}(1_e)] \geq \frac{p(1 - p)}{q}.
$$

We may thus take

$$
B = \frac{c}{q}
$$

in Corollary 5.1.

Let $q \geq 1$, $1 \leq k, m < n$, and consider the random-cluster measures $\phi_{n,p} = \phi_{\Lambda_n,p,q}$ on the box $\Lambda_n$. For $e \in \mathbb{E}^2$, write $J_{k,m,n}^b(e)$ for the (conditional) influence of $e$ on the event $H_{k,m}$ under the measure $\phi_{n,p}^b$. We set $J_{k,m,n}^b(e) = 0$ for $e \notin \mathbb{E}_{\Lambda_n}$.

**Lemma 6.1.** Let $q \geq 1$. We have that

$$
\sup_{e \in \mathbb{E}^2} J_{k,m,n}^0(e) \leq \frac{q}{p} \eta_k, \quad 0 < p \leq p_{sd}(q), \ 1 \leq k, m < n,
$$

$$
\sup_{e \in \mathbb{E}^2} J_{k,m,n}^1(e) \leq \frac{q}{p_{sd}} \eta_{m+1}, \quad p_{sd}(q) \leq p < 1, \ 1 \leq k, m < n,
$$
where \( p_d \) satisfies (2.3) and
\[
\eta_k = \phi_{p_d(q), q}^0 (0 \leftrightarrow \partial \Lambda_{k/2}) \to 0 \quad \text{as } k \to \infty.
\]

**Proof.** For any configuration \( \omega \in \Omega \) and vertex \( z \), let \( C_z(\omega) \) be the open cluster at \( z \), that is, the set of all vertices joined to \( z \) by open paths.

Suppose first that \( 0 < p \leq p_d(q) \), and let \( e = \langle x, y \rangle \) be an edge of \( \Lambda_n \). We couple the two conditional measures \( \phi_{n, p}^0(\cdot|\omega(e) = b), b = 0, 1 \), in the following manner. Let \( \Omega_n \) be the configuration space of the edges in \( \Lambda_n \), and let \( T = \{ (\pi, \omega) \in \Omega_n^2 : \pi \leq \omega \} \) be the set of all ordered pairs of configurations. There exists a measure \( \mu^e \) on \( T \) such that:

(a) the first marginal of \( \mu^e \) is \( \phi_{n, p}^0(\cdot|1_e = 0) \),
(b) the second marginal of \( \mu^e \) is \( \phi_{n, p}^0(\cdot|1_e = 1) \),
(c) for any subset \( \gamma \) of \( \Lambda_n \), conditional on the event \( \{ (\pi, \omega) : C_x(\omega) = \gamma \} \), the configurations \( \pi \) and \( \omega \) are \( \mu^e \)-almost-surely equal on all edges having no endvertex in \( \gamma \).

The details of this coupling are omitted. The idea is to build the paired configuration \( (\pi, \omega) \) edge by edge, beginning at the edge \( e \), in such a way that \( \pi(f) \leq \omega(f) \) for each edge \( f \) examined. The (closed) edge-boundary of the cluster \( C_x(\omega) \) is closed in \( \pi \) also. Once this boundary has been uncovered, the configurations \( \pi \), \( \omega \) on the rest of space are governed by the same (conditional) measure, and may be taken equal. Such an argument has been used in [2] and [19], Theorem 5.33(a), and has been carried further in [4].

We claim that
\[
J_{0, n, m}^0 (e) \leq \phi_{n, p}^0 (D_x|1_e = 1),
\]
where \( D_x \) is the event that \( C_x \) intersects both the left and right sides of \( B_{k,m} \). This is proved as follows. By (5.2),
\[
J_{k, m, n}^0 (e) = \mu^e (\omega \in H_{k,m}, \pi \notin H_{k,m}) \\
\leq \mu^e (\omega \in H_{k,m} \cap D_x) \\
\leq \mu^e (\omega \in D_x) = \phi_{n, p}^0 (D_x|1_e = 1),
\]
since, when \( \omega \notin D_x \), either both or neither of \( \omega, \pi \) belong to \( H_{k,m} \). By (6.5),
\[
J_{k, m, n}^0 (e) \leq \phi_{n, p}^0 (D_x) \frac{\phi_{n, p}^0 (D_x)}{\phi_{n, p}^0 (1_e)}.
\]

On \( D_x \), the radius of the open cluster at \( x \) is at least \( \frac{1}{2} k \). Since \( \phi_{n, p}^0 \leq \phi_{p, q} \) and \( \phi_{p, q} \) is translation-invariant,
\[
\phi_{n, p}^0 (D_x) \leq \phi_{p, q} (x \leftrightarrow x + \partial \Lambda_{k/2}) = \phi_{p, q} (0 \leftrightarrow \partial \Lambda_{k/2}).
\]
By (3.8),
\[
\phi_{p,q}(0 \leftrightarrow \partial \Lambda_{k/2}) \leq \phi_{p_{sd}(q),q}^0(0 \leftrightarrow \partial \Lambda_{k/2}) \to 0 \quad \text{as } k \to \infty,
\]
and, by (6.1) and (6.6), the conclusion of the lemma is proved when \( p \leq p_{sd}(q) \).

Suppose next that \( p_{sd}(q) \leq p < 1 \). Instead of working with the open paths, we work with the dual open paths. Each edge \( e_d = \langle u, v \rangle \) of the dual lattice traverses some edge \( e = \langle x, y \rangle \) of the primal, and, for each configuration \( \omega \), we define the dual configuration \( \omega_d \) by \( \omega_d(e_d) = 1 - \omega(e) \). Thus, the dual edge \( e_d \) is open if and only if \( e \) is closed. It is well known (see [19], Equation (6.12), e.g.) that, with \( \omega \) distributed according to \( \phi_{p_{sd}(q)}^{1,n} \), \( \omega_d \) has as law the random-cluster measure, denoted \( \phi_{n,p,d}^{1} \), on the dual of \( \Lambda_n \) with free boundary condition. The event \( H_{k,m} \) occurs if and only if there is no dual open path traversing the dual of \( B_{k,m} \) from top to bottom. We may therefore apply the above argument to the dual process, obtaining thus that
\[
J_{k,m,n}^{1}(e) \leq \frac{\phi_{n,p,d}(V_u)}{\phi_{n,p,d}(1_e)},
\]
where \( V_u \) is the event that \( C_u \) intersects both the top and bottom sides of the dual of \( B_{k,m} \).

On the event \( V_u \), the radius of the open cluster at \( u \) is at least \( \frac{1}{2}(m + 1) \). Since \( \phi_{n,p,d} \leq \phi_{p_{sd},q} \),
\[
\phi_{n,p,d}(V_u) \leq \phi_{p_{sd},q}(u \leftrightarrow u + \partial \Lambda_{(m+1)/2}) = \phi_{p_{sd},q}(0 \leftrightarrow \partial \Lambda_{(m+1)/2}).
\]
As above, by (2.4),
\[
\phi_{p_{sd},q}(0 \leftrightarrow \partial \Lambda_{(m+1)/2}) \leq \phi_{p_{sd},q}^0(0 \leftrightarrow \partial \Lambda_{(m+1)/2}) = \eta_{m+1},
\]
and this completes the proof when \( p \geq p_{sd}(q) \).

**Proof of Theorem 3.2.** This follows immediately from Corollary 5.1 by (6.2) and Lemma 6.1.

**Proof of Theorem 3.1.** By planar duality,
\[
\phi_{p,q}^0(H_k) = 1 - \phi_{p_{sd}(q)}^1(H_k),
\]
where \( p, p_d \) are related by (2.3), see [19], Theorems 6.13, 6.14. Since \( \phi_{p_{sd},q}^0 \leq \phi_{p_{sd},q}^1 \),
\[
\phi_{p_{sd},q}^0(H_k) \leq \frac{1}{2} \leq \phi_{p_{sd},q}^1(H_k),
\]
and Theorem 3.1 follows from Theorem 3.2.
7. Proof of Theorems 4.1 and 4.2. Only an outline of the proof of Theorem 4.1 is included here, since it follows the “usual” route (see [18], Section 11.3, or [19], Section 6.2, for examples of the argument). The measure $\pi_{\beta, h}$ is automorphism-invariant, ergodic, and has the finite-energy property. By the main result of [12], the number $N^+$ (resp., $N^-$) of infinite + clusters (resp., infinite − *-connected clusters) satisfies

\[ \pi_{\beta, h}(N^\pm = 0) = 1 \quad \text{or} \quad \pi_{\beta, h}(N^\pm = 1) = 1. \]

Assume that $\theta^+(\beta, h)\theta^-(\beta, h) > 0$, which is to say that $\pi_{\beta, h}(N^+ = N^- = 1) = 1$. One may find a box $\Lambda$ sufficiently large that, with $\pi_{\beta, h}$-probability at least $\frac{1}{2}$: the top and bottom of its boundary $\partial \Lambda$ are $+$ connected to infinity off $\Lambda$, and the left and right sides are $-\ast$-connected to infinity off $\Lambda$. Since $N^+ = 1$ almost surely, there is a $+$ path connecting the two infinite $+$ paths above, and this contradicts the fact that $N^- = 1$ almost surely.

We turn to the proof of Theorem 4.2. For the moment, let $\pi_{\beta, h}$ be the Ising measure on a finite graph $G = (V, E)$ with parameters $\beta \geq 0$ and $h \geq 0$. It is well known that $\pi_{\beta, 0}$ satisfies the FKG lattice condition (3.2) on the partially ordered set $\Sigma^V = \{-1, +1\}^V$. We identify $\Sigma^V$ with $\{0, 1\}^V$ via the mapping $\sigma_x \mapsto \omega_x = \frac{1}{2} (\sigma_x + 1)$, and we choose $p$ by

\[ \frac{p}{1 - p} = e^{2h}. \]

Then $\pi_{\beta, h}$ may be expressed in the form (5.1), and we may thus apply the results of Section 5. By conditioning on the states of the neighbors of $x$,

\[ \frac{e^{2h - \Delta \beta}}{e^{-\Delta \beta} + e^{2h - \Delta \beta}} \leq \pi_{\beta, h}(1_x) \leq \frac{e^{2h + \Delta \beta}}{e^{-\Delta \beta} + e^{2h + \Delta \beta}}, \]

where $\Delta$ is the degree of the vertex $x$, and $1_x$ is the indicator function that $\sigma_x = +1$. Therefore,

\[ \pi_{\beta, h}(1_x)[1 - \pi_{\beta, h}(1_x)] \geq \min \left\{ \frac{e^{2h}}{(e^{-\Delta \beta} + e^{2h - \Delta \beta})^2}, \frac{e^{2h}}{(e^{-\Delta \beta} + e^{2h + \Delta \beta})^2} \right\} \]

\[ = \frac{e^{2h + 2\Delta \beta}}{(1 + e^{2h + 2\Delta \beta})^2}. \]

This bound will be useful with $\Delta = 4$, and we write

\[ \xi_{\beta, h} = \frac{e^{2h + 8\beta}}{(1 + e^{2h + 8\beta})^2}. \]

Note that $\xi_{\beta, h}$ is decreasing in $h$.

We follow the argument of the proof of Theorem 5.1. Let $\beta \in [0, \beta_c)$, $h > 0$, and $1 \leq k, m \leq r < n$, and consider the Ising measure $\pi_{n, h} = \pi_{\Lambda_n, \beta, h}$ on the box $\Lambda_n = [-n, n]^2$. For $x \in \mathbb{Z}^2$, write $J_{k, m, n}(x)$ for the (conditional) influence of $x$ on the event $H_{k, m}$ under the measure $\pi_{n, h}$. We set $J_{k, m, n}(x) = 0$ for $x \notin \Lambda_n$. 


LEMMA 7.1. Uniformly in $x \in \mathbb{Z}^2$,
\[
J_{k,m,n}(x) \leq (1 + e^{8\beta - 2h})
\times \left[ \pi_{n,h}(B_{k,m} \leftrightarrow \partial \Lambda_r) + \sup_{x \in \Lambda_r} \pi_{n,h}(x \leftrightarrow x + \partial \Lambda_{k/2}) \right].
\] (7.5)

\[
J_{k,m,n}(x) \leq (1 + e^{8\beta + 2h})
\times \left[ \pi_{n,h}(B_{k,m} \leftrightarrow \partial \Lambda_r) + \sup_{x \in \Lambda_r} \pi_{n,h}(x \leftrightarrow x + \partial \Lambda_{m/2}) \right].
\] (7.6)

**Proof.** Let $h > 0$. Let $C^+_x$ be the set of all vertices joined to $x$ by a path of vertices all of whose states are $+1$ (thus, $C^+_x = \varnothing$ if $\sigma_x = -1$). We may couple the conditioned measures $\pi_{n,h}(\cdot | \sigma_x = b)$, $b = \pm 1$, such that the Ising equivalents of (a)–(c) hold as in Section 6. As in (6.6),
\[
J_{k,m,n}(x) \leq \frac{\pi_{n,h}(D^+_x)}{\pi_{n,h}(1_x)},
\] (7.7)

where $D^+_x$ is the event that $C^+_x$ intersects both the left and right sides of $B_{k,m}$. On $D^+_x$, the radius of $C^+_x$ is at least $\frac{1}{2}k$.

For $x \notin \Lambda_r$, $\pi_{n,h}(D^+_x) \leq \pi_{n,h}(B_{k,m} \leftrightarrow \partial \Lambda_r)$.

For $x \in \Lambda_r$, we shall use the bound $\pi_{n,h}(D^+_x) \leq \pi_{n,h}(x \leftrightarrow x + \partial \Lambda_{k/2})$.

Combining the above inequalities with (7.2), we obtain (7.5).

Let $C^-_x$ be the set of all vertices joined to $x$ by a *-connected path of vertices all of whose states are $-1$. The event $H_{k,m}$ occurs if and only if there is no *-*connected path from the top to the bottom of $B_{k,m}$. Therefore, the conditional influence of $x$ on $H_{k,m}$ equals that of $x$ on this new event. As in (7.7),
\[
J_{k,m,n}(x) \leq \frac{\pi_{n,h}(V^-_x)}{\pi_{n,h}(1 - 1_x)},
\] (7.8)

where $V^-_x$ is the event that $C^-_x$ intersects both the top and bottom of $B_{k,m}$. The above argument leads now to (7.6). $\Box$

**Proof of Theorem 4.2.** Let $R > h_c$ and $\delta > 0$, and let $k, m \leq r < n$. We set
\[
\kappa_{n,r,+}^\delta = 2(1 + e^{8\beta}) \left[ \pi_{n,h_c - \delta}(B_{k,m} \leftrightarrow \partial \Lambda_r) + \sup_{x \in \Lambda_r} \pi_{n,h_c - \delta}(x \leftrightarrow x + \partial \Lambda_{k/2}) \right],
\]
\[
\kappa_{n,r,-}^\delta = 2(1 + e^{8\beta + 2R})
\times \left[ \pi_{n,h_c + \delta}(B_{k,m} \leftrightarrow \partial \Lambda_r) + \sup_{x \in \Lambda_r} \pi_{n,h_c + \delta}(x \leftrightarrow x + \partial \Lambda_{m/2}) \right].
\]
Let $0 < h_1 < h_c < h_2 \leq R$, and choose $\delta < \min\{h_c - h_1, h_2 - h_c\}$. By (7.1), (7.3), Lemma 7.1 and Theorem 5.1, $f_n(h) = \pi_{n,h}(H_{k,m})$ satisfies
\begin{equation}
\frac{1}{f_n(h)(1 - f_n(h))} \cdot \frac{df_n}{dh} \geq B_+ \log(1 / \kappa_{n,r,+}^\delta), \quad h_1 \leq h \leq h_c - \delta,
\end{equation}
where $B_+ = 2c\xi_{\beta,h}$, see (7.4). The corresponding inequality for $h_c + \delta \leq h \leq R$ holds with $\kappa_{n,r,+}^\delta$ replaced by $\kappa_{n,r,-}^\delta$, and $B_+$ replaced by $B_- = 2c\xi_{\beta,R}$.

We integrate (7.9) over the intervals $(h_1, h_c - \delta)$ and $(h_c + \delta, h_2)$, add the results, and use the fact that $f_n(h)$ is nondecreasing in $h$, to obtain that
\begin{align*}
\log\frac{f_n(h)}{1 - f_n(h)} &\geq (h_c - \delta - h_1)B_+ \log(1 / \kappa_{n,r,+}^\delta) \\
+ (h_2 - h_c - \delta)B_- \log(1 / \kappa_{n,r,-}^\delta).
\end{align*}
Take the limits as $n \to \infty$, $r \to \infty$, and $\delta \to 0$ in that order, and use the monotonicity in $h$ of $\pi_{\beta,h}$, to obtain the theorem. $\square$

8. The colored random-cluster model. There is a well known coupling of the random-cluster and Potts models that provides a transparent explanation of how the analysis of the former aids that of the latter. Formulated as in [14] (see also the historical account of [19]), this is as follows. Let $p \in (0, 1)$ and $q \in \{2, 3, \ldots\}$. Let $\omega$ be sampled from the random-cluster measure $\phi_{p,q}$ on the finite graph $G = (V, E)$. To each open cluster of $\omega$, we assign a uniformly chosen element of $\{1, 2, \ldots, q\}$, these random spins being independent between clusters. The ensuing spin-configuration $\sigma$ on $G$ is governed by a Potts measure, and pair-spin correlations in $\sigma$ are coupled to open connections in $\omega$. This coupling has inspired a construction that we describe next.

Let $p \in (0, 1)$, $q \in (0, \infty)$, and $\alpha \in (0, 1)$. Let $\omega$ have law $\phi_{p,q}$. To the vertices of each open cluster of $\omega$, we assign a random spin chosen according to the Bernoulli measure on $\{0, 1\}$ with parameter $\alpha$. These spins are constant within clusters, and independent between clusters. We call this the colored random-cluster model (CRCM). With $\sigma$ the ensuing spin-configuration, we write $\kappa_{p,q,\alpha}$ for the measure governing the pair $(\omega, \sigma)$, and $\pi_{p,q,\alpha}$ for the marginal law of $\sigma$. When $q \in \{2, 3, \ldots\}$ and $q\alpha$ and $q(1 - \alpha)$ are integers, the CRCM is a vertex-wise contraction of the Potts model from the spin-space $\{1, 2, \ldots, q\}^V$ to $\Sigma = \{0, 1\}^V$.

The CRCM has been studied in [26] under the name “fractional fuzzy Potts model,” and it is inspired in part by the earlier work of [13, 21, 22], as well as the study of the so-called “divide-and-colour model” of [5].

The following seems to be known, see [13, 21, 22, 26], but the short proof given below may be of value.

**Theorem 8.1.** The measure $\pi_{p,q,\alpha}$ is monotone for all finite graphs $G$ and all $p \in (0, 1)$ if and only if $q\alpha, q(1 - \alpha) \geq 1$. 
We identify the spin-vector $\sigma \in \Sigma$ with the set $A = \{v \in V : \sigma_v = 1\}$. Let $\pi_h = \pi_{p,q,\alpha,h}$ be the probability measure obtained from $\pi_{p,q,\alpha}$ by including an external field with strength $h \in \mathbb{R}$,

(8.1) \[ \pi_h(A) \propto e^{h|A|} \pi_{p,q,\alpha}(A), \quad A \subseteq V. \]

It is an elementary consequence of Theorem 8.1 and (8.1) that, when $q\alpha, q(1 - \alpha) \geq 1$, $\pi_h$ is a monotone measure, and $\pi_h$ is increasing in $h$. When $q = 2$ and $\alpha = \frac{1}{2}$, $\pi_h$ is the Ising measure with external field. The purpose of this section is to extend the arguments of Section 4 to the CRCM with external field.

There is a special case of the CRCM with an interesting interpretation. Let $\omega$ be sampled from $\phi_{p,q}$ as above, and let $\sigma = (\sigma_v : v \in V)$ be a vector of independent Bernoulli ($\gamma$) variables. Let $B$ be the event that $\sigma$ is constant on each open cluster of $\omega$. The pair $(\omega, \sigma)$, conditional on $B$, is termed the massively colored random-cluster measure (MCRCM). The law of $\sigma$ is simply $\pi_{p,2q,\alpha/2,h}$ where $h = \log[\gamma/(1 - \gamma)]$.

Just as $\pi_{p,q,\alpha}$ and $\phi_{p,q}$ may be coupled via $\kappa_{p,q,\alpha}$, so we can couple $\pi_h$ with an “edge-measure” $\phi_h = \phi_{p,q,\alpha,h}$ via the following process. With $B$ given as above, and $(\omega, \sigma) \in B$, denote by $\sigma(C)$ the common spin-value of $\sigma$ on an open cluster $C$ of $\omega$. Let $\kappa_h = \kappa_{p,q,\alpha,h}$ be the probability measure on $\Omega \times \Sigma$ given by

(8.2) \[ \kappa_h(\omega, \sigma) \propto \phi_{p,q}(\omega) 1_B(\omega, \sigma) \prod_C [\alpha e^{h|C|} \sigma(C)(1 - \alpha)^{1 - \sigma(C)}], \]

where the product is over the open clusters $C$ of $\omega$, and $|C|$ is the number of vertices of $C$. The marginal and conditional measures of $\kappa_h$ are easily calculated. The marginal on $\Sigma$ is $\pi_h$, and the marginal on $\Omega$ is $\phi_h = \phi_{p,q,\alpha,h}$ given by

(8.3) \[ \phi_h(\omega) \propto \phi_{p,q}(\omega) \prod_C [\alpha e^{h|C|} + 1 - \alpha], \quad \omega \in \Omega. \]

Note that $\phi_0 = \phi_{p,q}$. Given $\omega$, we obtain $\sigma$ by labeling the open clusters with independent Bernoulli spins in such a way that the odds of cluster $C$ receiving spin 1 are $\alpha e^{h|C|}$ to $1 - \alpha$.

By (8.1), or alternatively by summing $\kappa_h(\omega, \sigma)$ over $\omega$, we find that

(8.4) \[ \pi_h(A) \propto e^{h|A|}(1 - p)^{|\Delta A|} Z_{A,q}(1 - \alpha), \quad A \subseteq V, \]

where $\Delta A$ is the set of edges of $G$ with exactly one endvertex in $A$, and $Z_{B,q}$ is the partition function of the random-cluster measure on the subgraph induced by $B \subseteq V$ with edge-parameter $p$ and cluster-weight $q$. It may be checked as in the proof of Theorem 8.1 that, for given $p, q, \alpha$, the measure $\pi_h$ is bounded above (resp., below) by a product measure with parameter $a(h)$ [resp., $b(h)$] where

(8.5) \[ a(-h) \to 0, \quad b(h) \to 1, \quad \text{as } h \to \infty. \]

The measure $\phi_h$ has a number of useful properties, following.
PROPOSITION 8.1. Let \( q \alpha, q(1 - \alpha) \geq 1 \).

(i) The probability measure \( \phi_h \) is monotone.

(ii) The marginal measure of \( \kappa_h \) on \( \Omega \), conditional on \( \sigma_x = b \), satisfies

\[
\kappa_h(\cdot|\sigma_x = 1) \geq_{st} \kappa_h(\cdot|\sigma_x = 0), \quad h \geq 0,
\]

\[
\kappa_h(\cdot|\sigma_x = 1) \leq_{st} \kappa_h(\cdot|\sigma_x = 0), \quad h \leq 0.
\]

(iii) If \( p_1 \leq p_2 \) and the ordered three-item sequence \( (0, h_1, h_2) \) is monotonic, then \( \phi_{p_1,q,\alpha,h_1} \leq_{st} \phi_{p_2,q,\alpha,h_2} \).

(iv) We have that \( \phi_{p,q,\alpha,h} \leq_{st} \phi_{p,Q} \), where \( Q = Q(h) \) is defined by

\[
Q(h) = \begin{cases} 
q\alpha, & h > 0, \\
q, & h = 0, \\
q(1 - \alpha), & h < 0.
\end{cases}
\]

We assume henceforth that \( q \alpha, q(1 - \alpha) \geq 1 \), and we consider next the infinite-volume limits of the above measures. Let \( G \) be a subgraph of the square lattice \( \mathbb{Z}^2 \) induced by the vertex-set \( V \), and label the above measures with the subscript \( V \).

By standard arguments (see [19], Chapter 4), the limit measure \( \phi_h = \lim_{V \uparrow \mathbb{Z}^2} \phi_{V,h} \) exists, is independent of the choice of the \( V \), and is translation-invariant and ergodic. By an argument similar to that of [19], Theorem 4.91, the measures \( \pi_{V,h} \) have a well-defined infinite-volume limit \( \pi_h \) as \( V \uparrow \mathbb{Z}^2 \). Furthermore, the pair \( (\phi_h, \pi_h) \) may be coupled in the same manner as on a finite graph. That is, a finite cluster \( C \) of \( \omega \) receives spin 1 with probability \( \alpha e^{-h|C|}/(|\alpha e^{-h|C|} + 1 - \alpha| \) An infinite cluster receives spin 1 (resp., 0) if \( h > 0 \) (resp., \( h < 0 \)). When \( h = 0 \), the spin of an infinite cluster has the Bernoulli distribution with parameter \( \alpha \).

Since \( \phi_h \) is translation-invariant, so is \( \pi_h \). As in [19], Theorem 4.10, \( \pi_h \) is positively associated, and the proof of [19], Theorem 4.91, may be adapted to obtain that \( \pi_h \) is ergodic. By a simple calculation, the \( \pi_{V,h} \) have the finite-energy property, with bounds that are uniform in \( V \) (see [19], Equation (3.4)), and therefore so does \( \pi_h \). Adapting the notation used in Section 4 for the Ising model, let

\[
\theta^1(p, q, \alpha, h) = \pi_h(0 \leftrightarrow \infty),
\]

\[
\theta^0(p, q, \alpha, h) = \pi_h(0 \leftrightarrow_* \infty).
\]

As in Theorem 4.1, and with an essentially identical proof,

\[
\theta^1(p, q, \alpha, h) \theta^0(p, q, \alpha, h) = 0.
\]

By the remark after (8.1) and [19], Theorem 4.10, \( \pi_h \) is stochastically increasing in \( h \), whence there exists \( h_c = h_c(p, q, \alpha) \in \mathbb{R} \cup \{ \pm \infty \} \) such that

\[
\theta^1(p, q, \alpha, h) = \begin{cases} 
0, & \text{if } h < h_c, \\
> 0, & \text{if } h > h_c.
\end{cases}
\]
By comparisons with product measures [see the remark prior to (8.5)], we have that $|h_c| < \infty$.

We call a probability measure $\mu$ on $\Sigma$ subcritical (resp., supercritical) if the $\mu$-probability of an infinite 1-cluster is 0 (resp., strictly greater than 0); we shall use the corresponding terminology for measures on $\Omega$. There is a second type of phase transition, namely the onset of percolation in the measure $\phi_h$. An infinite edge-cluster under $\phi_h$ forms part of an infinite vertex-cluster under $\pi_h$. Let $p_c(q)$ be the critical point of the random-cluster measure $\phi_{p,q}$ on $\mathbb{Z}^2$, as usual. By Proposition 8.1(iv), $\phi_h$ is subcritical for all $h$ when $p < p_c(q \min\{\alpha, 1 - \alpha\})$; in particular, for such $p$, $\phi_h$ is subcritical for $h$ lying in some open neighborhood of $h_c$. On the other hand, suppose that $\phi_0 = \phi_{p,q}$ is supercritical. By the remarks above, $\theta_1 > 0$ for $h > 0$, and $\theta_0 > 0$ for $h < 0$. By (8.6), $\theta^1$ is discontinuous at $h = h_c = 0$. By Proposition 8.1(iii), $\phi_h \geq \pi_h$ st $\phi_0$, whence $\theta^1$ is discontinuous at $h = h_c = 0$ whenever $p > p_c(q)$.

With $k, m \in \mathbb{N}$, let $H_{k,m}$ be the event that there exists a left–right 1-crossing of the box $B_{k,m}$. A result corresponding to Theorem 4.2 holds, subject to a condition on $\phi_h$ with $h$ near $h_c$. This condition has not, to our knowledge, been verified for the Ising model, although it is expected to hold. In this sense, the next theorem does not quite generalize Theorem 4.2.

**Theorem 8.2.** Let $R \geq 0$. When $h_c \neq 0$, we require in addition that $R \leq |h_c|$. Suppose that $\phi_h$ is subcritical for $h \in [h_c - R, h_c + R]$. There exist $\rho_{i,1} = \rho_{i,1}(p, q, \alpha, R)$ and $\rho_{i,0} = \rho_{i,0}(p, q, \alpha, R)$ satisfying

$$\rho_{i,1}\rho_{i,0} \to 0 \quad \text{as } i \to \infty,$$

such that: for $h_1 \in [h_c - R, h_c], h_2 \in [h_c, h_c + R],

$$\pi_{h_1}(H_{k,m})[1 - \pi_{h_2}(H_{k,m})] \leq \rho_{k,1}^{h_c-h_1} \rho_{m,0}^{h_2-h_c}, \quad k, m \geq 1.$$

As in the proof of Theorem 4.2, the first step is to establish bounds on the one-point marginals of $\pi_h$. This may be strengthened to a finite-energy property, but this will not be required here. The proof is deferred to the end of the section.

**Lemma 8.1.** Let $G = (V, E)$ be a finite graph with maximum vertex-degree $\Delta$. Then

$$\frac{ae^h}{ae^h + 1 - \alpha} (1 - p)^\Delta \leq \pi_h(\sigma_x = 1) \leq 1 - \frac{1 - \alpha}{ae^h + 1 - \alpha} (1 - p)^\Delta.$$

Consider the subgraph of $\mathbb{Z}^2$ induced by $\Lambda_n = [-n, n]^d$, and let $x \in \Lambda_n$. Objects associated with the finite domain $\Lambda_n$ are labeled with the subscript $n$. For $b = 0, 1$, let $\pi^{b}_{n,h}$ (resp., $\phi^{b}_{n,h}$) be the marginal measure on $\Sigma_n$ (resp., $\Omega_n$) of the coupling $\kappa_{n,h}$ conditioned on $\sigma_x = b$. 


By Proposition 8.1, $\phi_{n,h}^1 \geq_{st} \phi_{n,h}^0$ when $h \geq 0$, and $\phi_{n,h}^1 \leq_{st} \phi_{n,h}^0$ when $h \leq 0$. It is convenient to work with a certain coupling of the pairs $(\phi_{n,h}^0, \pi_{n,h}^0)$ and $(\phi_{n,h}^1, \pi_{n,h}^1)$. Recall that $C_x(\omega)$ denotes the open cluster at $x$ in the edge-configuration $\omega \in \Omega$.

**Lemma 8.2.** Let $h \in \mathbb{R}$. There exists a probability measure $\kappa_{n,h}^{01}$ on $(\Omega_n \times \Sigma_n)^2$ with the following properties. Let $(\omega^0, \sigma^0, \omega^1, \sigma^1)$ be sampled from $(\Omega_n \times \Sigma_n)^2$ according to $\kappa_{n,h}^{01}$.

(i) For $b = 0, 1$, $\omega^b$ has law $\phi_{n,h}^b$.

(ii) For $b = 0, 1$, $\sigma^b$ has law $\pi_{n,h}^b$.

(iii) If $h \leq 0$, $\omega^0 \geq \omega^1$. If $h \geq 0$, $\omega^1 \geq \omega^0$.

(iv) The spin configurations $\sigma^0$ and $\sigma^1$ agree at all vertices $y \notin C_x(\omega^0) \cup C_x(\omega^1)$.

**Proof.** Assume first that $h \geq 0$. There exists a probability measure $\Phi_n$ on $\Omega_n^2$, with support $D_1 = \{ (\omega^0, \omega^1) \in \Omega_n^2 : \omega^0 \leq \omega^1 \}$, whose first (resp., second) marginal is $\phi_{n,h}^0$ (resp., $\phi_{n,h}^1$). By sampling from $\Phi_n$ in a sequential manner beginning at $x$, and proceeding via the open connections of the upper configuration, we may assume in addition that $(\omega^0, \omega^1) \in D_2$, where $D_2$ is the set of pairs such that $\omega^0(e) = \omega^1(e)$ for any edge $e$ having at most one endpoint in $C_x(\omega^1)$. Let $(\omega^0, \omega^1) \in D = D_1 \cap D_2$.

The spin vectors $\sigma^b$ may be constructed as follows:

(a) attach spin $b$ to the cluster $C_x(\omega^b)$,

(b) attach independent Bernoulli spins to the other $\omega^b$-open clusters in such a way that the odds of cluster $C$ receiving spin 1 are $ae^{h|C|}$ to $1 - a$.

We may assign spins $\sigma^b$ to the open clusters of the $\omega^b$ in such a way that: $\sigma^b$ has law $\pi_{n,h}^b$, and $\sigma_y^0 = \sigma_y^1$ for $y \notin C_x(\omega^1)$. Write $\kappa_{n,h}^{01}$ for the joint law of the ensuing pairs $(\omega^0, \sigma^0)$, $(\omega^1, \sigma^1)$.

When $h \leq 0$, let $\kappa_{n,h}^{01}$ be the coupling as above, with the differences that: $\omega^0 \geq \omega^1$, and $\sigma_y^0 = \sigma_y^1$ for $y \notin C_x(\omega^0)$. □

We seek next a substitute for Lemma 7.1 in the current setting. Let $J_{k,m,n}(x)$ be the conditional influence of vertex $x$ on the event $H_{k,m}$, with reference measure $\pi_{n,h}$ on $\Lambda_n$.

Let $(\omega^0, \sigma^0, \omega^1, \sigma^1)$ be sampled according to the measure $\kappa_{n,h}^{01}$ of Lemma 8.2. Define random clusters $C_x^H, C_x^V \subseteq \mathbb{Z}^2$ as follows,

$$C_x^H(\omega^0, \sigma^0, \omega^1, \sigma^1) := \{ z \in \mathbb{Z}^2 : \exists y \in C_x(\omega^0), \ y \leftrightarrow z \text{ in } \sigma^1 \},$$

$$C_x^V(\omega^0, \sigma^0, \omega^1, \sigma^1) := \{ z \in \mathbb{Z}^2 : \exists y \in C_x(\omega^1), \ y \leftrightarrow z \text{ in } \sigma^0 \}.$$
Notice that, if \( h \geq 0 \) (resp., \( h \leq 0 \)), \( C^H_x \) (resp., \( C^V_x \)) is the spin-1 cluster (resp., spin-0 *-cluster) at \( x \) under \( \sigma^1 \) (resp., \( \sigma^0 \)). It may be checked as before that:

\[
J_{k,m,n}(x) \leq \kappa_{n,h}^0 \text{ if } (C^H_x \text{ contains a horizontal crossing of } B_{k,m}),
\]

\[
J_{k,m,n}(x) \leq \kappa_{n,h}^0 \text{ if } (C^V_x \text{ contains a vertical *-crossing of } B_{k,m}).
\]

The notation \( C^H_x, C^V_x \) is introduced in order to treat the cases \( h > 0 \) and \( h < 0 \) simultaneously.

**Lemma 8.3.** Let \( R \) be as in Theorem 8.2.

(i) If \( \theta^1(p, q, \alpha, h_c) = 0 \), and \( \phi_h \) is subcritical for \( h \in [h_c - R, h_c] \), there exists \( \nu_{k,1} \) satisfying \( \nu_{k,1} \to 0 \) as \( k \to \infty \) such that

\[
\limsup_{n \to \infty} \sup_{h \in [h_c - R, h_c]} \sup_{x \in \Lambda_n} J_{k,m,n}(x) \leq \nu_{k,1}.
\]

(ii) If \( \theta^0(p, q, \alpha, h_c) = 0 \), and \( \phi_h \) is subcritical for \( h \in [h_c, h_c + R] \), there exists \( \nu_{m,0} \) satisfying \( \nu_{m,0} \to 0 \) as \( m \to \infty \) such that

\[
\limsup_{n \to \infty} \sup_{h \in [h_c, h_c + R]} \sup_{x \in \Lambda_n} J_{k,m,n}(x) \leq \nu_{m,0}.
\]

**Proof.** We prove part (i) only, the proof of (ii) being similar. If \( [h_c - R, h_c] \subseteq [0, \infty) \), let \( \phi = \phi_{h_c} \); if \( [h_c - R, h_c] \subseteq (-\infty, 0] \), let \( \phi = \phi_{h_c - R} \). By Proposition 8.1, and the assumptions of (i),

(a) \( \phi_{n,h} \leq_{st} \phi \) for \( n \geq 1 \) and \( h \in [h_c - R, h_c] \),

(b) \( \phi \) is subcritical,

(c) \( \pi_{n,h} \) is subcritical, and \( \pi_{n,h} \leq_{st} \pi_{n,h_c} \) for \( h \in [h_c - R, h_c] \).

By Lemma 8.1, there exists \( L > 0 \) such that

\[
\pi_{n,h}(\sigma_x = 1) \pi_{n,h}(\sigma_x = 0) \geq L
\]

for all \( n \geq 1, x \in \Lambda_n \), and \( h \in [h_c - R, h_c + R] \). Let

\[
A_x(\omega) = \sup\{r \geq 0 : x \leftrightarrow x + \partial \Lambda_r\}
\]

denote the radius \( \text{rad}(C_x) \) of the edge cluster \( C_x = C_x(\omega) \) at \( x \), and note that \( \phi(A_x < \infty) = 1 \).

Let \( r \geq \max\{k, m\} \) and \( x \in \Lambda_r \). By (8.7) and the positive association of \( \pi_{n,h}^1 \), and as in (6.6),

\[
J_{k,m,n}(x) \leq \kappa_{n,h}^0(\text{rad}(C^H_x) \geq k/2)
\]

\[
\leq \sum_{a=0}^\infty \phi_{n,h}(A_x = a) \alpha_{n,h}^1(x, a, k/2)
\]

\[
\leq \frac{1}{L} \sum_{a=0}^\infty \phi_{n,h}(A_x = a) \alpha_{n,h}(x, a, k/2),
\]
where
\[ \alpha_{n,h}^\kappa(x, a, b) = \pi_{n,h}^\kappa(x + \Lambda a \leftrightarrow x + \partial \Lambda b | \sigma_y = 1 \text{ for } y \in x + \Lambda a). \]

Since \( \alpha_{n,h}(x, a, b) \) is nondecreasing in \( a \), and furthermore \( \phi_{n,h} \leq_{st} \phi \) and \( \phi \) is translation-invariant,

\[
(8.10) \quad \sup_{x \in \Lambda_r} J_{k,m,n}(x) \leq \frac{1}{L} \sum_{a=0}^{\infty} \phi(A_0 = a) \sup_{x \in \Lambda_r} \{ \alpha_{n,h}(x, a, k/2) \}.
\]

By (8.9) and the fact that \( \pi_{n,h} \leq_{st} \pi_{n,h_c} \),

\[
(8.11) \quad \alpha_{n,h}(x, a, k/2) \leq \min \left\{ 1, \frac{1}{L} \pi_{n,h_c}(x + \Lambda a \leftrightarrow x + \partial \Lambda k/2) \right\}.
\]

Suppose now that \( x \in \Lambda_n \setminus \Lambda_r \). Then

\[
J_{k,m,n}(x) \leq \kappa_{n,h}^{01}(C_x^H \cap B_{k,m} \neq \emptyset)
\leq \sum_{a=0}^{\infty} \phi_{n,h}(A_x = a) \beta_{n,h}(x, a)
\leq \frac{1}{L} \sum_{a=0}^{\infty} \phi_{n,h}(A_x = a) \beta_{n,h}(x, a),
\]

where
\[
\beta_{n,h}^\kappa(x, a) = \pi_{n,h}^\kappa(x + \Lambda a \leftrightarrow B_{k,m} | \sigma_y = 1 \text{ for } y \in x + \Lambda a)
\]
is a nondecreasing function of \( a \). Since \( \phi_{n,h} \leq_{st} \phi \), and \( \phi \) is translation-invariant,

\[
J_{k,m,n}(x) \leq \frac{1}{L} \sum_{a=0}^{\infty} \phi(A_0 = a) \beta_{n,h}(x, a).
\]

As above,
\[
\beta_{n,h}(x, a) \leq \frac{1}{L|\Lambda a|} \pi_{n,h}(x + \Lambda a \leftrightarrow B_{k,m})
\leq \frac{1}{L|\Lambda a|} \pi_{n,h}(B_{k,m} \leftrightarrow \partial \Lambda r - a) \quad \text{if } a \leq r,
\]

whence

\[
(8.12) \quad J_{k,m,n}(x) \leq \frac{1}{L} \sum_{a=0}^{\infty} \phi(A_0 = a) \min \left\{ 1, \frac{1}{L|\Lambda a|} \pi_{n,h_c}(B_{k,m} \leftrightarrow \partial \Lambda r - a) \right\},
\]

where the minimum is interpreted as 1 when \( a > r \).
We add (8.10)–(8.11) and (8.12), and take the limit \( n \to \infty \), to obtain by the bounded convergence theorem that
\[
\limsup_{n \to \infty} \sup_{x \in \Lambda_n} J_{k,m,n}(x) \leq 1 \cdot \sum_{a=0}^{\infty} \phi(A_0 = a) \min \left\{ 1, \frac{1}{L|\Lambda_a|} \pi_{h_c}(x + \Lambda_a \leftrightarrow \partial \Lambda_k/2) \right\}
\]
\[
+ \sum_{a=0}^{\infty} \phi(A_0 = a) \min \left\{ 1, \frac{1}{L|\Lambda_a|} \pi_{h_c}(B_{k,m} \leftrightarrow \partial \Lambda_{r-a}) \right\}.
\]

We now send \( r \to \infty \). Since \( \theta^1(p, q, \alpha, h_c) = 0 \) by assumption, the last summand tends to 0. By the bounded convergence theorem,
\[
\limsup_{n \to \infty} \sup_{x \in \Lambda_n} J_{k,m,n}(x) \leq \nu_{k,1},
\]
where
\[
\nu_{k,1} = \frac{1}{L} \sum_{a=0}^{\infty} \phi(A_0 = a) \min \left\{ 1, \frac{1}{L|\Lambda_a|} \pi_{h_c}(x + \Lambda_a \leftrightarrow \partial \Lambda_k/2) \right\}.
\]
By the bounded convergence theorem again, \( \nu_{k,1} \to 0 \) as \( k \to \infty \). Since (8.10)–(8.11) and (8.12) are uniform in \( h \in [h_c - R, h_c] \), one may include the supremum over \( h \) in (8.13), as required for the lemma. \( \square \)

**Proof of Theorem 8.2.** Let \( f_n(h) = \pi_{n,h}(H_{k,m}) \). By (5.3) and Lemma 8.1,
\[
(8.14) \quad \frac{1}{f_n(h)[1 - f_n(h)]} \frac{d}{dh} f_n(h) \geq cL \log \left[ \frac{1}{2 \max_x J_{k,m,n}(x)} \right],
\]
with \( L \) as in the proof of Lemma 8.3. Let
\[
\xi_{n,k,1} = \sup_{h \in [h_c - R, h_c]} \sup_{x \in \Lambda_n} 2J_{k,m,n}(x), \quad \xi_{n,m,0} = \sup_{h \in [h_c, h_c + R]} \sup_{x \in \Lambda_n} 2J_{k,m,n}(x).
\]
By (8.14),
\[
\log \frac{f_n(h)}{1 - f_n(h)} \bigg|^{h_2}_{h_1} \geq (h_c - h_1)cL \log(\xi_{n,k,1}^{-1}) + (h_2 - h_c)cL \log(\xi_{n,m,0}^{-1}),
\]
whence
\[
f_n(h_1)[1 - f_n(h_2)] \leq \xi_{n,k,1}^{cL(h_c - h_1)} \xi_{n,m,0}^{cL(h_2 - h_c)}.
\]
Take the limit as \( n \to \infty \) and use Lemma 8.3. \( \square \)

**Proof of Proposition 8.1.** A strictly positive measure \( \mu \) on \( \Omega = \{0, 1\}^E \) is monotone if and only if: for all \( \omega \in \Omega \) with \( \omega(e) = \omega(f) = 0, e \neq f \),
\[
(8.15) \quad \mu(\omega^e \triangleright \omega^f) \mu(\omega) \geq \mu(\omega^e) \mu(\omega^f),
\]
see, for example, [19], Theorem 2.19. Given two strictly positive measures \( \mu_1 \) and \( \mu_2 \), at least one of which is monotone, it is sufficient for \( \mu_1 \leq_{st} \mu_2 \) that:

\[
\frac{\mu_1(\omega^e)}{\mu_1(\omega)} \leq \frac{\mu_2(\omega^e)}{\mu_2(\omega)}, \quad \omega \in \Omega, \ e \in E.
\]

This is proved in [19], Theorem 2.6. Condition (8.16) is nontrivial only when \( \omega(e) = 0 \).

We shall prove (i) by checking that \( \phi_h \) satisfies (8.15). Write \( C(\omega) \) for the set of open clusters under \( \omega \), and let \( f_h(k) = \alpha e^{hk} + 1 - \alpha \). Substituting (8.3) into (8.15), we must check

\[
\phi_{p,q}(\omega^e,f) \phi_{p,q}(\omega) \prod_{C \in C(\omega^e)} f_h(|C|) \prod_{C \in C(\omega)} f_h(|C|).
\]

On using the monotonicity of \( \phi_{p,q} \), and on canceling the factors \( f_h(|C|) \) for \( C \in C(\omega) \cap C(\omega^e) \), we arrive at the following three cases.

(i) There are clusters \( C_1, C_2 \in C(\omega) \), such that \( C_1 \cup C_2 \in C(\omega^e) = C(\omega^f) \). It suffices that

\[
q f_h(a) f_h(b) \geq f_h(a + b), \quad a = |C_1|, \ b = |C_2|,
\]

and this is easily checked for \( a, b \geq 0 \) since \( q \alpha, q(1 - \alpha) \geq 1 \).

(ii) There are clusters \( C_1, C_2, C_3 \in C(\omega) \), such that \( C_1 \cup C_2 \in C(\omega^e) \) and \( C_2 \cup C_3 \in C(\omega^f) \). It suffices that

\[
f_h(a + b + c) f_h(b) \geq f_h(a + b) f_h(b + c), \quad a = |C_1|, \ b = |C_2|, \ c = |C_3|,
\]

and this is immediate.

(iii) There are clusters \( C_1, C_2, C_3, C_4 \in C(\omega) \) such that \( C_1 \cup C_2 \in C(\omega^e) \) and \( C_3 \cup C_4 \in C(\omega^f) \). In this case, inequality (8.17) simplifies to a triviality.

It may be checked similarly that the marginal measure of \( \kappa_h(\cdot | \sigma_x = b) \) on \( \Omega \) is monotone if either \( h \geq 0 \), \( b = 1 \) or \( h \leq 0 \), \( b = 0 \). One uses the expression

\[
\kappa_h(\omega | \sigma_x = b) \propto \phi_{p,q}(\omega) e^{h|C_x(\omega)|} \prod_{C \in C(\omega) \setminus \{C_x(\omega)\}} f_h(|C|), \quad \omega \in \Omega.
\]

Parts (ii) and (iii) then follow by checking (8.16) with appropriate \( \mu_i \). Part (iv) follows from part (iii) by taking the limit as \( |h| \to \infty \). Many of the required calculations are rather similar to part (i), and we omit further details. □

PROOF OF THEOREM 8.1. We identify the spin-vector \( \sigma \in \Sigma \) with the set \( A = \{v \in V : \sigma_v = 1\} \). In order that \( \pi = \pi_{p,q,\alpha} \) be monotone, it is necessary and sufficient [see inequality (8.15)] that

\[
\pi(A^x) \pi(A) \geq \pi(A^x) \pi(A^y), \quad A \subseteq V, \ x, y \in V \setminus A, \ x \neq y.
\]
Let $A \subseteq V$, $x, y \in V \setminus A$, $x \neq y$. Let $a$ be the number of edges of the form $\langle x, z \rangle$ with $z \in A$, let $b$ be the number of edges of the form $\langle x, z \rangle$ with $z \notin A$ and $z \neq x, y$, and let $e$ be the number of edges joining $x$ and $y$.

We write $A^x = A \cup \{x\}$, etc. By (8.4) with $h = 0$,

$$\frac{\pi(A^x)}{\pi(A)} = (1 - p)^{b+e-a} \frac{Z_{A^x,q\alpha}Z_A,q(1-\alpha)}{Z_{A,q}Z_A,q(1-\alpha)} = \frac{\alpha}{1 - \alpha} \cdot \frac{\phi_{A,q}(1-\alpha)(I_x)}{\phi_{A^x,q\alpha}(I_x)},$$

where $I_x$ is the event that $x$ is isolated, and $\phi_{A,q}$ is the random-cluster measure on the subgraph induced by vertices of $A$ with edge-parameter $p$ and cluster-weight $q$. Similarly,

$$\frac{\pi(A^{xy})}{\pi(A^y)} = \frac{\alpha}{1 - \alpha} \cdot \frac{\phi_{A^{xy},q\alpha}(I_x)}{\phi_{A^{xy},q\alpha}(I_y)} \cdot \frac{\phi_{A,q}(1-\alpha)(I_x)}{\phi_{A,q}(1-\alpha)(I_y)}.$$

The ratio of the left to the right-hand sides of (8.18) is

$$(8.19) \quad \frac{\phi_{A^x}(I_x)}{\phi_{A^x}(I_x)} \cdot \frac{\phi_{A^{xy}}(I_x)}{\phi_{A^{xy}}(I_x)} = \frac{\phi_{A^{xy},q\alpha}(I_x | I_y)}{\phi_{A^{xy},q\alpha}(I_x)} \cdot \frac{\phi_{A,q}(1-\alpha)(I_x | I_y)}{\phi_{A,q}(1-\alpha)(I_x)}.$$ 

Inequality (8.18) holds by the positive association of random-cluster measures with cluster-weights at least 1.

That the conditions are necessary for monotonicity follows by an example. Suppose $0 < q\alpha < 1$ and $q(1 - \alpha) \geq 1$. Let $G$ be a cycle of length four, with vertices (in order, going around the cycle) $u, x, v, y$. Take $A = \{u, v\}$ above, so that $e = 0$. The final ratio in (8.19) equals 1, and the penultimate is strictly less than 1. □

**Proof of Lemma 8.1.** By Proposition 8.1(iv) and inequality (6.1),

$$\phi_h(I_x) \geq \phi_{p,q}(I_x) \geq (1 - p)^\Delta,$$

where $I_x$ is the event that $x$ is isolated. Conditional on $I_x$, the spin of $x$ under the coupling $\kappa_h$ has the Bernoulli distribution with parameter $\alpha e^h / [\alpha e^h + 1 - \alpha]$. □

**Acknowledgments.** The second author acknowledges the hospitality of the Mathematics Department at the University of British Columbia, the Institut Henri Poincaré–Centre Emile Borel, Paris, and the Section de Mathématiques at the University of Geneva, where this work was largely done. His stay in Geneva was supported by the Swiss National Science Foundation. Rob van den Berg proposed the elimination of equation (4.1) from the proofs.

**REFERENCES**

[1] Aizenman, M., Barsky, D. J. and Fernández, R. (1987). The phase transition in a general class of Ising-type models is sharp. *J. Statist. Phys.* 47 343–374. MR894398

[2] Aizenman, M., Chayes, J. T., Chayes, L. and Newman, C. M. (1988). Discontinuity of the magnetization in one-dimensional $1/|x - y|^2$ Ising and Potts models. *J. Statist. Phys.* 50 1–40. MR939480
[3] Aizenman, M. and Fernández, R. (1986). On the critical behavior of the magnetization in high-dimensional Ising models. *J. Statist. Phys.* **44** 393–454. MR857063

[4] Alexander, K. S. (1998). On weak mixing in lattice models. *Probab. Theory Related Fields* **110** 441–471. MR1626951

[5] Bálint, A., Camia, F. and Meester, R. (2009). Sharp phase transition and critical behaviour in 2D divide and colour models. *Stochastic Process. Appl.* **119** 937–965. MR2499865

[6] Baxter, R. J. (1982). *Exactly Solved Models in Statistical Mechanics*. Academic Press, London. MR690578

[7] van den Berg, J. (2008). Approximate zero-one laws and sharpness of the percolation transition in a class of models including two-dimensional Ising percolation. *Ann. Probab.* **36** 1880–1903. MR2440926

[8] Bezuidenhout, C. E., Grimmett, G. R. and Kesten, H. (1993). Strict inequality for critical values of Potts models and random-cluster processes. *Comm. Math. Phys.* **158** 1–16. MR1243713

[9] Bollobás, B. and Riordan, O. (2006). The critical probability for random Voronoi percolation in the plane is 1/2. *Probab. Theory Related Fields* **136** 417–468. MR2257131

[10] Bourgain, J., Kahn, J., Kalai, G., Katznelson, Y. and Linial, N. (1992). The influence of variables in product spaces. *Israel J. Math.* **77** 55–64. MR1194785

[11] Burton, R. M. and Keane, M. (1989). Density and uniqueness in percolation. *Comm. Math. Phys.* **121** 501–505. MR990777

[12] Chayes, L. (1996). Percolation and ferromagnetism on $\mathbb{Z}^2$: The $q$-state Potts cases. *Stochastic Process. Appl.* **65** 209–216. MR1425356

[13] Edwards, R. G. and Sokal, A. D. (1988). Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm. *Phys. Rev. D (3)* **38** 2009–2012. MR965465

[14] Friedgut, E. and Kalai, G. (1996). Every monotone graph property has a sharp threshold. *Proc. Amer. Math. Soc.* **124** 2993–3002. MR1371123

[15] Graham, B. T. and Grimmett, G. R. (2006). Influence and sharp-threshold theorems for monotonic measures. *Ann. Probab.* **34** 1726–1745. MR2271479

[16] Kähler, J., Winnacker, D. and Ziegler, G. (1989). The stochastic random-cluster process and the uniqueness of random-cluster measures. *Ann. Probab.** 23** 1461–1510. MR1379156

[17] Grimmett, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **321**. Springer, Berlin. MR1707339

[18] Grimmett, G. (2009). *The Random-Cluster Model*. Springer, Berlin. Corrected reprint available at http://www.statslab.cam.ac.uk/~grg/books/rcm.html.

[19] Grimmett, G. (2010). *Probability on Graphs*. Cambridge Univ. Press, Cambridge. Available at http://www.statslab.cam.ac.uk/~grg/books/pgs.html.

[20] Häggström, O. (1999). Positive correlations in the fuzzy Potts model. *Ann. Appl. Probab.* **9** 1149–1159. MR1728557

[21] Häggström, O. (2001). Coloring percolation clusters at random. *Stochastic Process. Appl.* **96** 213–242. MR1865356

[22] Higuchi, Y. (1993). Coexistence of infinite $(\ast)$-clusters. II. Ising percolation in two dimensions. *Probab. Theory Related Fields* **97** 1–33. MR1240714

[23] Higuchi, Y. (1993). A sharp transition for the two-dimensional Ising percolation. *Probab. Theory Related Fields* **97** 489–514. MR1246977

[24] Kahn, J., Kalai, G. and Linial, N. (1988). The influence of variables on Boolean functions. In *Proceedings of 29th Symposium on the Foundations of Computer Science* 68–80. Computer Science Press, Washington, DC.
[26] Kahn, J. and Weininger, N. (2007). Positive association in the fractional fuzzy Potts model. Ann. Probab. 35 2038–2043. MR2353381
[27] Kalai, G. and Safra, S. (2006). Threshold phenomena and influence: Perspectives from mathematics, computer science, and economics. In Computational Complexity and Statistical Physics. St. Fe Inst. Stud. Sci. Complex. 25–60. Oxford Univ. Press, New York. MR2208732
[28] Kesten, H. (1980). The critical probability of bond percolation on the square lattice equals $1/2$. Comm. Math. Phys. 74 41–59. MR575895
[29] Kotecký, R. and Shlosman, S. B. (1982). First-order phase transitions in large entropy lattice models. Comm. Math. Phys. 83 493–515. MR649814
[30] Laanait, L., Messager, A., Miracle-Solé, S., Ruiz, J. and Shlosman, S. (1991). Interfaces in the Potts model. I. Pirogov–Sinai theory of the Fortuin–Kasteleyn representation. Comm. Math. Phys. 140 81–91. MR1124260
[31] Laanait, L., Messager, A. and Ruiz, J. (1986). Phases coexistence and surface tensions for the Potts model. Comm. Math. Phys. 105 527–545. MR852089
[32] Russo, L. (1978). A note on percolation. Z. Wahrsch. Verw. Gebiete 43 39–48. MR0488383
[33] Russo, L. (1981). On the critical percolation probabilities. Z. Wahrsch. Verw. Gebiete 56 229–237. MR618273
[34] Seymour, P. D. and Welsh, D. J. A. (1978). Percolation probabilities on the square lattice. Ann. Discrete Math. 3 227–245. MR0494572
[35] Smirnov, S. (2006). Towards conformal invariance of 2D lattice models. In International Congress of Mathematicians II 1421–1451. Eur. Math. Soc., Zürich. MR2275653
[36] Smirnov, S. (2010). Conformal invariance in random-cluster models. I. Holomorphic fermions in the Ising model. Ann. Math. 172 1441–1473.
[37] Talagrand, M. (1994). On Russo's approximate zero-one law. Ann. Probab. 22 1576–1587. MR1303654
[38] Welsh, D. J. A. (1993). Percolation in the random cluster process and $Q$-state Potts model. J. Phys. A 26 2471–2483. MR1234408