Probing the dark side of gravitational clustering: weak lensing statistics at large smoothing angle

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ABSTRACT

The weak lensing surveys have the potential to probe directly the clustering statistics of dark matter in the universe. Recent studies have shown that it is possible to predict analytically the whole probability distribution function (pdf) and the bias associated with the collapsed objects in the highly non-linear regime using the hierarchical ansatz. We extend such studies to the quasi-linear regime where the hierarchical ansatz is replaced by the tree-level perturbative calculations to an arbitrary order. It is shown how the generating function techniques can be coupled with the perturbative calculations to compute the complete pdf and the bias in the quasi-linear regime for the weak-lensing convergence field. We study how these quantities depend on the smoothing angle and the source red-shift in different realistic cosmological scenarios. We show that it is possible to define a reduced convergence whose statistics is similar to underlying 3D mass distribution for small smoothing angle but it resembles projected mass distribution for large smoothing angles. We have also compared our perturbative results with log-normal model for pdf and bias and found a good agreement between the two analytical results.

Key words: methods: analytical – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

The theoretical study of the weak gravitational lensing to probe the statistics of large scale structures in the universe has received much attention (Mellier 1999; Bartelmann & Schneider 1991) in recent past.

Based on the earlier works done by several authors (Gunn 1967; Blandford et al. 1991; Miralda-Escudé 1991; Kaiser 1992), current development has focused mainly on two different directions. Much progress have been achieved in the field of numerical simulations of the weak lensing, in particular the construction of shear maps which provide invaluable tools in testing the analytical results. The numerical simulations typically employ N-body simulations, through which ray tracing experiments are conducted (Schneider & Weiss 1988; Jaroszynski et al. 1990; Lee & Paczynski 1991; Babul & Lee 1991; Bartelmann & Schneider 1991; Blandford et al. 1991). Building on the earlier works of Wambsganss et al. (1995, 1997); Wambsganss, Cen & Ostriker (1998) the most detailed numerical study of lensing was done by Wambsganss et al. (1998). Other recent studies using ray tracing experiments have been conducted by Premadi, Martel & Matzner (1998), van Waerbeke, Bernardeau & Mellier (1999), Bartelmann et al. (1998), Couchman, Barber & Thomas (1998) and Jain, Seljak & White (2000).

On the analytical front, Villumsen (1996), Stebbins (1996), Bernardeau, van Waerbeke & Mellier (1997) and Kaiser (1992) have focused on the use of perturbative techniques to study weak lensing. More recently it has been shown that the hierarchical ansatz for the higher order correlation functions can be used to describe the statistical properties of the weak lensing convergence fields for the case of small smoothing angles (Hui 1999; Munshi & Coles 1999; Munshi & Jani 1999; Valageas 1999a,b; Munshi 2000; Munshi & Coles 2000a,b).

In this paper we will show that while perturbative calculations have already been used to compute the lower order moments of convergence field (see e.g. Bernardeau et al. 1997; van Waerbeke et al. 1999, for detailed review on perturbative calculations in the context of weak lensing surveys see Bernardeau 1999); when coupled with generating function techniques it can also reproduce the complete pdf and bias functions associated with the smoothed projected density field. We also show that these results can simply be obtained by replacing the generating function of the tree hierarchy in non-linear regime with its quasi-linear counterpart. While the non-linear generating function corresponds to the three dimensional generating function for the matter correlation hierarchy, in the quasi-linear regime it has to be replaced by a generating
function which represent the smoothed two-dimensional projected density field. These generating functions have already been studied in great detail by (Bernardeau 1992, 1994, 1995, 1996) for the case of top-hat smoothing function. Using the recent analytical results from the study of statistics of the convergence field for small smoothing angles, we will introduce a reduced convergence field $\eta$ and study how its statistical properties (including the lower order moments) are related with its counterpart used to describe the projected density field.

2 THE STATISTICS OF THE CONVERGENCE FIELD FOR LARGE SMOOTHING ANGLES

The weak lensing convergence $\kappa$ is simply the projected mass distribution and its statistics carries valuable information about the geometry and dynamics of the background universe as well as the physics of collision-less clustering.

$$\kappa(\gamma) = \int_0^\infty \text{d}x \omega(x) \delta(r(x)\gamma).$$

(1)

Throughout our discussion we will be placing the sources at a fixed red-shift (an approximation not too difficult to modify for more realistic description). The weight function can be expressed as $\omega(x) = 3/2a c^{-3} H_0^2 \Omega_{0} r(x)(x_s(x_T - x))/r(z_s)$. Where $x_s$ is the comoving radial distance to the source placed at a red-shift $z_s$. We will be focusing on the quasi-linear regime where it is possible to expand the density contrast using a perturbative series.

$$k_0^{(1)}(\gamma) + k_0^{(2)}(\gamma) + \cdots = \int_0^\infty \text{d}x \omega(x) \left( \delta^{(1)}(r(x)\gamma) + \delta^{(2)}(r(x)\gamma) + \cdots \right).$$

(2)

It will be useful to work in Fourier domain and the Fourier decomposition of $\delta$ can be written as:

$$\kappa(\gamma) = \int_0^\infty \text{d}x \omega(x) \int \frac{\text{d}^3k}{(2\pi)^3} \exp(i \mathbf{k} \cdot \mathbf{r}) \kappa_0, \quad \kappa_0 = \kappa_0(\mathbf{k}),$$

(3)

where we have used $k_0$ and $k_1$ to denote the components of wave vector $\mathbf{k}$, parallel and perpendicular to the line of sight direction $\gamma$. In the small angle approximation however, one assumes that $k_1$ is much larger compared to $k_0$. We will denote the angle between the line of sight direction $\gamma$ and the wave vector $\mathbf{k}$, by $\theta$. The smoothing angle will be denoted by $\theta_0$. Using the definitions that we have introduced above we can now express the smoothed projected two-point correlation function (Linder 1954; Peebles 1980; Kaiser 1992, 1998):

$$\langle \kappa(\gamma_1) \kappa(\gamma_2) \rangle = \int_0^\infty \text{d}x \omega(x) \frac{\omega(x)}{r^2(x)} \int \frac{\text{d}l}{(2\pi)^2} \exp(i \theta) P \left( \frac{l}{r(x)} \right) W_2(l \theta_0),$$

(4)

where we have introduced a new notation $l = r(x) k_\perp$ which denotes the scaled wave vector projected on the surface of the sky. The average of the two-point correlation function $\langle \kappa_\psi \rangle$, smoothed over an angle $\theta_0$ with a top-hat smoothing window $W_2(l \theta_0)$ is useful to quantify the fluctuations in $\kappa_\psi$ which is often used to reconstruct the matter power spectrum $P(k)$ (Jain & Seljak 1997).

$$\langle \kappa_\psi \rangle = \int_0^\infty \text{d}x \omega(x) \frac{\omega(x)}{r^2(x)} \int \frac{\text{d}l}{(2\pi)^2} P \left( \frac{l}{r(x)} \right) W_2(l \theta_0).$$

(5)

Using the standard perturbative techniques, it is possible to compute the normalized cumulants or $s_N$ parameters, for the convergence field in the large smoothing angle regime.

2.1 The probability distribution function of $\kappa(\theta_0)$

To construct the probability distribution function (pdf) it is necessary to have the $s_N$ parameters for the convergence maps computed to an arbitrary order. These computations have already been done by Bernardeau (1996) and van Waerbeke et al. (1999).

$$\langle \kappa^2(\gamma) \rangle = C_2[I_{\theta_0}^{\text{PT}}],$$

(6)

$$\langle \kappa^4(\gamma) \rangle = s_4^{\text{PT}} C_4[I_{\theta_0}^{\text{PT}}],$$

(7)

$$\langle \kappa^N(\gamma) \rangle = s_N^{\text{PT}} C_N[I_{\theta_0}^{\text{PT}}],$$

(8)

where the $s_N$ parameters correspond to the case of 3D gravitational dynamics with 2D perturbations (Bernardeau 1995),

$$s_1^{\text{PT}} = \frac{36}{7} - \frac{3}{2(n + 2)}$$

(9)

$$s_4^{\text{PT}} = \frac{2540}{49} - 33(n + 2) + \frac{21}{4}(n + 2)^2$$

(10)

and $C_n$ denotes the line of sight integration.

$$C_n[I_{\theta_0}^{\text{PT}}] = \int_0^\infty \frac{\omega(x)}{r^{n+1}(x)} I_{\theta_0}^{\text{PT}} \text{d}x.$$  

(11)

1 Throughout this paper we will use lower case letters for quantities related to both the convergence $\kappa$ and its reduced counterpart $\eta$. We will show that the statistics associated with the reduced convergence map $\eta(\theta_0) = s_{\eta}(\theta_0) = s_{\eta}(\theta_0) = 1 + \frac{\omega(x)}{r^{n+1}(x)}$ is very similar to the projected two dimensional density fields. The Moments of these fields will be represented by a superscript PT to denote their perturbative origin, where as moments associated with the convergence map $\kappa$ will be denoted by the superscript $\kappa$.
The spectral index \( n \) is however same as that of the three dimensional (local) power spectral index. These results have already been tested using numerical simulation by Munshi et al. (1999). Typically for a degree scale smoothing, we will use \( n = -1.3 \). The projection effects are incorporated in the line of sight integration. Such a separation of dynamical part and the geometrical part occurs only when we replace the actual initial power spectra with the local power law spectra (Bernardeau 1995).

\[
I_{\phi \eta} = \int \frac{d\Omega}{(2\pi)^2} P\left( \frac{1}{\bar{r}(x)}, x \right) W_2(\theta_\eta).
\]

Finally we can write down the normalized cumulants for the convergence field as:

\[
s_N = s_N^{\phi \eta} C_N \left[ \kappa_{\theta_\eta}^{N-1} \right] \frac{1}{C_2 \left[ I_{\phi \eta} \right]}.
\]

The generating function approach developed by Balian & Schaeffer (1989) and later extended by Bernardeau & Schaeffer (1992) was used to compute the whole hierarchy of \( s_N \) parameters for the three dimensional matter distribution and the projected matter distribution (Bernardeau 1995). The generating function is related to the vertices which appear in the tree representation of the correlation hierarchy which appear in the quasi-linear evolution of gravitational clustering.

\[
G^{\phi \eta}(\tau) = 1 - \tau + \frac{\nu_1}{2!} \tau^2 - \frac{\nu_1}{3!} \tau^3 + \cdots.
\]

Such a hierarchy also appears in the highly non-linear regime but with different tree amplitudes. The generating function for the \( s_N \) parameters can similarly be represented as (Balian & Schaeffer 1989):

\[
\phi^{\phi \eta}(\tau) = \sum_{N=1}^{\infty} (-1)^{N-1} \frac{s_N^{\phi \eta}}{N!} \tau^N.
\]

The function \( \phi^{\phi \eta}(\tau) \) satisfies the constraint \( s_1 = s_2 = 1 \) necessary for the normalization of the PDF. These two generating function are known to be related to each other by the following set of equations (Balian & Schaeffer 1989; Bernardeau & Schaeffer 1992):

\[
\phi^{\phi \eta}(\tau) = y \phi^{\phi \eta}(\tau) - \frac{1}{2} \frac{d}{d\tau} \phi^{\phi \eta}(\tau)
\]

\[
\tau = -y \frac{d}{d\tau} \phi^{\phi \eta}(\tau).
\]

Hence once we know the generating function \( G(\tau) \) it is possible to compute the whole hierarchy of the projected \( s_N \) parameters. It was shown that the generating function takes a particularly simple form (Bernardeau 1995):

\[
G^{\phi \eta}(\tau) = \left( 1 - \frac{\tau}{\kappa_0} \right)^{-\kappa_0},
\]

with \( \kappa_0 = (\sqrt{13} - 1)/2 \) for 2D perturbative dynamics of gravitational clustering in a 3D universe. This particular result hold for unsmoothed density field but it is possible to incorporate the effect of smoothing into account if we assume that the smoothing window is a top-hat function (Bernardeau 1994, 1995).

\[
G^{\eta \eta}_\sigma(\tau) = G^{\phi \eta} \left[ \frac{\sigma \left( R_0 [1 + G^{\phi \eta}(\tau)]^{1/2} \right)}{\sigma(R_0)} \right].
\]

For the variance \( \sigma \) at length scale \( R_0 \) will use the local power spectral index \( n \) so that \( \sigma(R_0) \propto R_0^{-(\alpha+3)/2} \).

The PDF and the VPF can be related to each other by the following equation (Balian & Schaeffer 1989):

\[
P^{\phi \eta}(\delta) = \int_{-\infty}^{\infty} \frac{dy}{2\pi i} \exp \left[ \frac{(1 + \delta)y - \phi^{\phi \eta}(y)}{\tilde{\xi}_2} \right].
\]

The above expressions are clearly suitable for the density field and similar calculations can be done for convergence maps and the generating functions for the cumulants of the convergence maps can then directly be related to its counterpart for the density field. The details of such calculations can be found in Munshi & Jain (2000) and Valageas (1999b). Our notations follow that of Munshi & Jain (1999) more closely. These studies clarified that it is useful to define a reduced convergence map \( \eta \) smoothed with an angle \( \theta_\eta \):

\[
\eta(\theta_\eta) = \frac{\kappa(\theta_\eta) - \kappa_{\text{min}}}{\kappa_{\text{min}}} = 1 + \frac{\kappa(\theta_\eta)}{\kappa_{\text{min}}}
\]

where the minimum value of the convergence along any line of sight direction can be defined by the following equation:

\[
\kappa_{\text{min}} = -\int_{z_s}^{\infty} d\chi \phi(\chi).
\]

Under certain approximation \( \eta \) will have the exactly same one-point statistics as the density field \( 1 + \delta \).

\[
\phi^{\phi \eta}(y) = \phi^{\phi \eta}(y).
\]

Notice that the quantity \( k_{\text{min}} \) does not depend on the smoothing angle \( \theta_\eta \) and hence will only depend on the source red-shift \( z_s \) and the definition of reduced convergence remains same both in the quasi-linear and the highly non-linear regime and is not connected with the convergence or divergence of the perturbative series.
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Figure 1. The Probability distribution function associated with different realistic scenarios. The smoothing angle is fixed at $\theta_0 = 1^\circ$. The source is placed at $z_s = 1$ for each of these models. The slope of correlation function $\gamma = 1.7$ was assumed. Dots represent results from the log-normal fitting form with same variance and $k_{\text{min}}$.

Figure 2. The Probability distribution function associated with different realistic scenarios. The smoothing angle is fixed at $\theta_0 = .5^\circ$ in the left panel and at $\theta_0 = .25^\circ$ in the right panel. The source is placed at $z_s = 1$ for each of these models. The slope of correlation function $\gamma = 1.7$ was assumed. Dots represent results from the log-normal fitting formula with same variance and $k_{\text{min}}$.

In general the expression connecting the cumulant generator for $\eta$, i.e. $\Phi_\eta$ can be related to $\phi$ by the following approximation.

$$
\Phi_\eta(\gamma) = \frac{1}{\int_0^{\chi_s} \omega(\chi) d\chi} \int_0^{\chi_s} d\chi \left[ \frac{\langle \kappa^2(\theta_0) \rangle}{\int_0^{\chi_s} \omega(\chi) d\chi \int \frac{2}{\gamma^2(\chi)} W_2^2(\theta_0) \int d_2 l (2\pi)^2 P \left( \frac{l}{r(\chi)} \right) W_2^2(\theta_0) \int d_2 l (2\pi)^2 P \left( \frac{l}{r(\chi)} \right) W_2^2(\theta_0) \int d_2 l (2\pi)^2 P \left( \frac{l}{r(\chi)} \right) W_2^2(\theta_0) \right] \phi^{\text{PT}} \left[ \omega(\chi) \int_0^{\chi_s} \omega(\chi) d\chi \int \frac{2}{\gamma^2(\chi)} W_2^2(\theta_0) \int d_2 l (2\pi)^2 P \left( \frac{l}{r(\chi)} \right) W_2^2(\theta_0) \right].
$$

Clearly this expression does not depend on whether we are in the quasi-linear regime or in the highly non-linear regime. Only changes which are to be made is that of $\phi$. In quasi-linear regime the quasi-linear phi can be computed from quasi-linear generating function for tree vertices defined earlier and in highly non-linear regime it needs to replaced with its non-linear counterpart. The simplification mentioned above can be achieved by replacing the red-shift dependence of different integrands with their values at a median red-shift. Numerical computations seems to suggest that such an approximation works remarkably well (Valageas 1999b; Munshi & Jain 1999).

It is possible to have different approximate asymptotic expressions for different limiting values of $\eta$. However here we have performed a direct numerical integration and results for different cosmologies and various smoothing angles are presented in the Figs 1 and 2. The different curves represent various versions of (standard, lambda, omega and $\tau$) cold dark matter models. The details of various parameters characterising these dark matter models can be found in Munshi & Jain (1999).
As in the case of quasi-linear regime we now get:

\[
s_i^* = \frac{s_i^{PT}}{\kappa_{\text{min}}}.
\]

\[
s_2^* = \frac{s_2^{PT}}{\kappa_{\text{min}}^2}.
\]

\[
s_3^* = \frac{s_3^{PT}}{\kappa_{\text{min}}^{N-2}}.
\]

Although these expressions are very similar to their small angle counterparts (Munshi & Jain 1999; Valageas 1999a), however it is not very difficult to notice that there is a sharp difference between the two. For larger smoothing angle the statistics of reduced convergence map follows that of projected density field which in turn is related to the results associated with the two dimensional perturbations in a three dimensional universe. However in case of very small smoothing angles when we are probing the non-linear regime, the statistics of the convergence maps is that of full three dimensional density field. The intermediate regime is more difficult to model and it interpolates these two regimes in a smooth manner. Numerical investigations of $s_3^*$ was done by Gaztanaga & Bernardeau (1998) for SCDM model for various smoothing angle. We find a good agreement between our results and their numerical measurements for angular scales as small as $\theta = 15'$. Which means that our analytical results for pdf and bias (based on tree-level calculations), may even be valid for much smaller angular scales. A detailed comparison however will require a large simulation patch and is not available at present.

### 2.2 The bias associated with the convergence field $\kappa(\theta_0)$

While it is clear that one-point studies of convergence field is interesting it was shown by Munshi (2000) that such results can also be obtained for the case of two-point quantities such as bias associated with the convergence map $b(\kappa)$ and its moment, the cumulant correlators $c_{pq}$ (see Munshi, Melott & Coles 1999, for detailed descriptions of generalized cumulant correlators based on various hierarchical models).

\[
p(\kappa_1, \kappa_2) \, d\kappa_1 \, d\kappa_2 = p(\kappa_1) p(\kappa_2) \left( 1 + b(\kappa_1) \xi_{\kappa_1} b(\kappa_2) \right) \, d\kappa_1 \, d\kappa_2.
\]

We have used shorthand notations $\kappa_1 = \kappa(\gamma_1)$, $\kappa_2 = \kappa(\gamma_2)$ and $\xi_{\kappa_1} = \langle \kappa(\gamma_1) \kappa(\gamma_2) \rangle$ to simplify the above expression. A similar expression can be obtained for the reduced convergence map $\eta$.

Our studies in the highly non-linear regime have shown that the hierarchical ansatz can be used to understand how the ‘hot-spots’ in convergence maps are correlated and how such correlations can be used to study the similar correlation for the underlying matter distribution and hence the bias associated with the collapsed objects. We will show that a similar study is also possible in the quasi-linear regime with appropriate change in the generating function as was the case for non-linear regime.

The normalized cumulant correlators or $c_{pq}^\eta$ parameters have been calculated for dark matter density perturbations using direct perturbative calculations for low orders and for arbitrary orders using the generating function techniques. We can define these parameters by the following relation

\[
\langle \kappa^p(\gamma_1) \kappa^q(\gamma_2) \rangle = c_{pq} \langle \kappa^1 \rangle^{(p+q-2)} \langle \kappa(\gamma_1) \kappa(\gamma_2) \rangle.
\]

These parameters were computed by Bernardeau (1996) for the density field and were found to obey a generic factorization rule $c_{pq} = c_{p1} c_{q1}$ for the case of 3D perturbations. Similar computations were later generalized for the case of 2D perturbations.

\[
c_{21}^{PT} = \frac{24}{7} - \frac{1}{2} (n + 2),
\]

\[
c_{31}^{PT} = \frac{1473}{49} - \frac{195}{14} (n + 2) + \frac{3}{2} (n + 2)^2.
\]

For projected fields we will get a prefactor resulting from the line of sight integration as was the case for the one-point cumulants.

\[
\langle \kappa^p(\gamma_1) \kappa^q(\gamma_2) \rangle = c_{pq} \langle \kappa^1 \rangle^{(p+q-2)} I_{s_{12}} = c_{p1} c_{q1} \langle \kappa^1 \rangle^{(p+q-2)} I_{s_{12}},
\]

where we have introduced a new quantity $I_{s_{12}}$ which incorporates the line of sight integration effects for the two-point quantities, such as the cumulant correlators and bias which we are considering here.

\[
I_{s_{12}} = \frac{d^4}{(2\pi)^4} P \left( \frac{l}{r(x)} \right) W_2^2(l\theta_0) \exp(l\theta_{12}).
\]

It can be shown that the role of the generating function for the normalized cumulant correlators for the (unsmoothed) density field $1 + \delta_{\text{PT}}^\rho$ is played by the quantity $\gamma$ itself.

\[
\beta^{\rho}(y) = \sum_{p=1}^{\infty} c_{p1} \langle \kappa^1 \rangle^p y^p.
\]

In a very similar way we can define the generating function $c_{pq}$ for the convergence map $\kappa$ by the following relation:

\[
\beta(\gamma_1, \gamma_2) = \sum_{p=1}^{\infty} c_{pq} \langle \kappa^1 \rangle^p \gamma_1^p \gamma_2^q = \sum_{p=1}^{\infty} c_{p1} \langle \kappa^1 \rangle^p \sum_{q} c_{q1} \gamma_1^q \gamma_2^q = \beta(\gamma_1) \beta(\gamma_2).
\]
however such a factorization is not possible for the projected density fields, without some simplifying assumptions and in general we have to use the generating function for $c_{pq}^{PT}$ which we denote as $\beta$. Using relations we have already presented above we can finally write:

$$\beta_{x}(y_1, y_2) = \sum_{p,q} \frac{\rho_{pq}^{PT}}{\rho_{pq}} \frac{1}{\langle \kappa^{2} \rangle_{\ell}} \int_{0}^{\kappa} d\chi \omega_{\beta}^{p+q} \left[ \int \frac{d\Omega}{(2\pi)^2} P \left( \frac{1}{\ell(\eta)} \right) W^{2}(l(\eta)) \exp[i\eta l(\eta)] \right]$$

$$\times \left[ \int \frac{d\Omega}{(2\pi)^2} P \left( \frac{1}{\ell(\chi)} \right) W^{2}(l(\chi)) \exp[i\eta l(\chi)] \right]^{p+q-2} y_1^{p} y_2^{q}.$$  \hspace{1cm} (36)

As was the case with one point cumulants we can use the reduced convergence $\eta(\eta)$ to simplify the analysis because as pointed out earlier the statistics of reduced convergence map is very similar to that of the underlying projected density field.

$$\beta_{x}(y_1, y_2) = \int_{0}^{\kappa} d\chi \omega^{2}(\chi) d\chi \frac{\langle \kappa^{2} \rangle_{l}^{2}}{\langle \kappa \rangle_{l}^{2}} \left[ \int \frac{d\Omega}{(2\pi)^2} P \left( \frac{1}{\ell(\eta)} \right) W^{2}(l(\eta)) \exp[i\eta l(\eta)] \right]$$

$$\times \beta^{PT} \left( y_1 \omega(\chi) \left[ \int \frac{d\Omega}{(2\pi)^2} P \left( \frac{1}{\ell(\chi)} \right) W^{2}(l(\chi)) \right] \int_{0}^{\kappa} d\chi \omega(\chi) d\chi \right)$$

$$\times \beta^{PT} \left( y_2 \omega(\chi) \left[ \int \frac{d\Omega}{(2\pi)^2} P \left( \frac{1}{\ell(\chi)} \right) W^{2}(l(\chi)) \right] \int_{0}^{\kappa} d\chi \omega(\chi) d\chi \right).$$  \hspace{1cm} (37)

If we use the same techniques which we have used to simplify the integrals associated with one point cumulants (see Munshi & Jain 1999; Munshi 2000) where we replaced the red-shift dependence in integrands with their values at a median red-shift we can write:

$$\beta_{x}(y_1, y_2) = \beta_{x}(y_1) \beta_{x}(y_2) = \beta^{PT}(y_1) \beta^{PT}(y_2).$$  \hspace{1cm} (38)

This is exactly the same result which we have obtained for small smoothing angle, however results we have derived here are valid for large smoothing angles and hence the generating function $\beta^{PT}(y)$ corresponds to the case of perturbative regime. If we neglect the effect of smoothing the generating function $\beta^{PT}(y)$ is simply the function $\tau(y)$ (Bernardeau 1996) however for smoothed convergence maps we can write (Bernardeau 1996):

$$\beta^{PT}(y) = \tau(y) \frac{\sigma(R_0)}{\sigma(R_0[1 + G^{PT}(\gamma)])^{1/2}}.$$  \hspace{1cm} (39)

Finally the bias $b(\eta)$ can be expressed as:

$$b(\eta) = \int_{-\infty}^{\infty} dy \frac{\beta(\eta)}{2\pi} \exp \frac{\eta y - \phi^{PT}(y)}{\bar{\epsilon}_{l}} \int_{-\infty}^{\infty} dy \exp \frac{\eta y - \phi^{PT}(y)}{\bar{\epsilon}_{l}}.$$  \hspace{1cm} (40)

The generating function for $\eta$ and projected density field are same under certain simplifying approximation, it shows that the bias associated with the reduced convergence is very similar to the statistics of the projected density field. This is in direct contrast with what was obtained using the nonlinear theory for small smoothing angle where the statistics of hot-spots in projected map are more closely related to the underlying three dimensional density field and not to the projected density field. Finally the bias associated with the convergence $\kappa$ can be expressed by the following equation.

$$b_{x}(\kappa) = \frac{b_{x}(\eta)}{\bar{\kappa}_{mn}}.$$  \hspace{1cm} (41)

Clearly it is possible to integrate the exact equation which represents joint moments exactly and hence compute the exact bias function for large smoothing angle from $\beta(\gamma)$, but it was found by Munshi (2000) that simplification of replacing all the integrals by their approximate cation of replacing all the integrals by their approximate (Bernardeau 1996), but it was found by Munshi (2000) that simplification of replacing all the integrals by their approximate

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3 DISCUSSION

Perturbative calculations have already been used to compute the lower order hierarchy of $s_N$ parameters associated with the convergence maps in quasi-linear regime. On the other hand hierarchical ansatz was shown to be useful in highly non-linear regime. In this particular paper we have shown how the recent development in highly non-linear regime can also be used to study the quasi-linear regime by introducing a reduced convergence field to construct the full pdf and bias associated with convergence maps in the quasi-linear regime. We have shown that a reduced convergence field in fact is related directly with the two-dimensional projected density field. It is known from earlier studies that the generating function corresponding to the two-dimensional density field is connected with the two dimensional spherical collapse. Which indicates that such a generating function now can be used to predict the pdf and bias of smooth convergence maps in the quasi-linear regime. A quick comparison with corresponding results in highly non-linear regime shows that while statistics of projected density for small smoothing is intimately related with the full three dimensional density field at larger smoothing angle it is related to the projected two dimensional density field.

From observational point of view our results will have direct significance. While most recent observational studies cover rather small patches of the sky necessary for quasi-linear approximations to be valid, it is hoped that larger and larger patches of the sky will be covered in future, where one can apply perturbative techniques. At smaller angular scales it is well known that noise due to intrinsic ellipticity of galaxies will play a major role in determination of the cosmological parameters, however such noise will be less significant for studies at larger angular
scales where our results will have direct relevance. Although signal variance will start declining with larger and larger smoothing angle and direct determination of weak lensing signal at degree scale might seem difficult at present, in the intermediate length scale $\theta_0 = 15'$ where the hierarchical ansatz starts to break down our analysis using perturbative techniques will provide much needed theoretical insight.

Most of the presently available numerical maps for cosmic shear are generated from high resolution numerical simulation and are generally correspond to few degree patches in the sky which are suitable for testing analytical predictions for smaller smoothing angles, and recent comparison of various statistics were found to be very accurately reproduced by non-linear theory. Such studies for larger smoothing angles will have to wait till much larger numerical maps are available. There have been some attempts in this direction using higher order Lagrangian theories which replaces the exact gravitational dynamics. Such models although less accurate can reproduce the basic features of convergence maps in larger smoothing angles and are definitely very cost effective. For reproducing the statistics of convergence maps made using higher order Lagrangian perturbation theory we just need to change the exact generating function which we have used in this study by corresponding smoothed generating function for perturbative Lagrangian dynamics for the case of projected perturbations and it can then be used to compare with numerical studies. It can be shown that the frozen flow approximation as proposed by Matarrese et al. (1992) will produce a log-normal pdf in the quasi-linear regime (Munshi, Sahni & Starobinsky 1994), hence such an approximation can also be used to generate convergence maps.

Results presented here can be generalized very easily for the case of projected galaxy survey. A detailed study will be presented elsewhere.

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APPENDIX: LOG-NORMAL DISTRIBUTION

Lognormal distribution provides a phenomenological description of one-point probability distribution function (see Coles & Jones 1991) and its generalisation to compute the bias function associated with overdense region (Taruya et al. 2002). Although inherently local in nature we found that it can also be used as a good fitting function for both one-point probability distribution function (see Bernardeau & Kofman 1995 for similar comparison in 3D for density Pd) and the bias function associated with the convergence maps. For the projected density contrast \( \delta \) (which is related to the reduced convergence field \( \eta = 1 + \delta \)) we can write the pdf as:

\[
P(\delta) d\delta = \frac{1}{\sqrt{2\pi \Sigma^2}} \exp\left[-\frac{\Lambda^2}{2\Sigma}\right] \frac{1}{(1+\delta)} d\delta.
\]  

(1)

Similarly the joint probability distribution function can be written as (Kayo et al. 2001):

\[
P(\delta_1, \delta_2) d\delta_1 d\delta_2 = \frac{1}{2\pi \sqrt{(\Sigma^2 - X^2)}} \exp\left[-\frac{\Sigma (\Lambda_1^2 + \Lambda_2^2) - 2X \Lambda_1 \Lambda_2}{2(\Sigma^2 - X^2)}\right] \frac{1}{(1+\delta_1)(1+\delta_2)} d\delta_1 d\delta_2
\]  

(2)

where we have introduced the following notations:

\[
X = \ln(1 + \xi_{nl}), \quad \Sigma = (1 + \sigma_{nl}), \quad \Lambda_i = \ln[(1 + \delta_i) \sqrt{(1 + \sigma_{nl})}]
\]  

(3)

\( \xi_{nl} \) and \( \xi_{lin} \) denotes the non-linear and linear correlation functions and can be computed using suitable modelling of non-linear evolution e.g. Peacock & Dodds (1996). To compute the one-point and two-point pdf’s for \( \kappa \) from above expressions we have to apply the transformation \( \eta \rightarrow \kappa \) (which involve computation of \( \kappa_{min} \)) discussed before. A detailed comparison of bias from log-normal model and from hierarchical ansatz for the case of density field will be presented elsewhere.

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