Perturbations of a Schwarzschild black hole. 
Notes on Zerilli’s approach

Gianluca Cruciani*

ICRA-International Centre for Relativistic Astrophysics
I-00185 Rome (Italy)

December 8, 2005

PACS: 0420C
Ref.: Il Nuovo Cimento B, 120, n. 10-11, 1045-1053 (2005).

Abstract

Modelling the free fall (and radiative phenomenology) of a massive particle, charged or not, in a static and spherically symmetric black hole is a classic, good relativistic dare that produced a remarkable series of papers, mainly in the seventies of the past century. Some formal topics about the mathematical machinery required to perform the task are unfortunately still not very clear; however, with the help of modern computer algebra techniques, some results can at least be tested and corrected.

1 Introduction

When a massive particle falls in a black hole, it induces changes in the spacetime metric that, although considered as an “ordinary” perturbation, can produce a phenomenology hardly comparable to what it would be in a flat background. This fact, even if dealing with elementary concepts of General Relativity, found its first serious applications in a celebrated paper of 1957 by T. Regge and J. A. Wheeler [1] that somehow founded the theory of the stability of the Schwarzschild black hole. However, due to hard complications in the calculations, this investigation came to a first end, after a number of intermediate stages (in particular [2, 3, 4, 5]), only in 1970-1974 with the work of F. J. Zerilli, who, in collaboration with Wheeler and R. Ruffini [6, 7], attacked the problem of considering the perturbation of the e.m. and gravitational fields produced in the falling in greater detail. In [8], in particular, a realistic treatment is made, considering the first-order stress-energy tensor contributions of the perturbing particle, but the ideas lying at the base of the angular decoupling of the equations still remain cryptical in some of their aspects; it is now possible, also by the help of quite popular computer algebra means (Mathematica by S. Wolfram and the tensor calculus-dedicated package MathTensor by L. Parker and S. Christensen), to account for some hidden aspects of the logic path followed by the authors and fix some errors.

*cruciani@icra.it
2 Formulating the problem

The perturbation analysis is performed by writing the Einstein’s system (in geometric units: \( G = c = 1 \)) as

\[
G_{\mu\nu}(h) = 8\pi T_{\mu\nu}(h)
\]

where \( h \) is the perturbation tensor accounting for the presence of a massive particle, eventually endowed with an electric charge, that acts directly on the spacetime geometry in addition to the Schwarzschild metric tensor: \( g_{\mu\nu} = g_{\mu\nu}^S + h_{\mu\nu} \) where \( g_{\mu\nu}^S \) corresponds to the line element

\[
ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

with \( \nu(r) = -\lambda(r) = \ln(1 - 2m/r) \), \( m \) being the mass of the BH.

Historically speaking, almost all the ideas developed to perform a pure metric perturbation analysis (i.e., with \( T_{\mu\nu} = 0 \)) in the original paper of 1957 were adopted by the successive works, in particular the “harmonic syntax” of the angular terms and the distinction between two different kinds of perturbations, belonging to different choices among the angular operators that generate the multipole expansion itself.

3 Nature and aims of the angular bases

To write the equations in a smart basis, accounting for the angular properties of a static, spherically symmetric problem (by this fact allowing the separation of the radial parts), a set of harmonic “objects” is originally introduced by Regge-Wheeler, by a mechanism of parity splitting and repeated derivations of the scalar harmonics; those objects are seen to form a useful basis for rank-2 tensors in the Euclidean 3-dim. space and can successively be modified to form a basis of the Minkowski spacetime. They are called “tensor multipole” (TM) basis. Later, Frank J. Zerilli modified the original choice in two of the ten elements and noticed that the way those tensors are constructed must be consistent with the procedure adopted by Jon Mathews [2], consisting in a chain of external products of elements, starting from the spherical vector basis, aimed to the forming of a set which, in its spatial Euclidean version, transforms under the irreducible representations of \( SO(3) \). This set is called “tensor harmonic” (TH) basis.

The procedure to obtain the TM basis resides on the combined action of the time and radial projection, \( P_t, P_r \), the covariant derivative \( \nabla \) and orbital angular momentum \( L = -i r \times \nabla \) operators on the scalar harmonics, that turn out to be linear combinations of the THs. Exploring the relationships between bases that originate from external products of unit vectors and those derived from operator compositions, one is soon aware of the different characteristics of those sets concerning parity and with respect to scalar products in \( \mathcal{E}_c \otimes \mathcal{E}_c \) and in \( \mathcal{L}_2(S^2) \) (respectively, the Euclidean complexified tensor product space and the square-integrable functions’ Hilbert space on the unit 2-sphere). This topic was formerly treated in a couple of papers ([10, 11]) that unfortunately contain a certain amount of errors and misprints. Some of those results (those, in particular, about the construction of a new TH basis of \( \mathcal{E}^3_c \) with improved features) are revisited and presented in a correct, implementable form in Appendix A.
4 Scalar, vector and tensor harmonics in $E^3$

To reach a satisfying definition of rank-2 THs, it is natural to hierarchically construct them from simpler objects belonging to the same family: scalar and vector harmonics.

The basis of the Hilbert space of the square-integrable complex functions on the unit sphere embedded in $E^3$, whose elements possess the property of being eigenfunctions of $L^2$ and $L_z$, is indicated by the double-indexed function $Y_{JM}(\theta, \phi)$ (the scalar spherical harmonics), and its full expression normally adopted (after having made explicit the Legendre polynomials that appear in it) is:

$$Y_{JM}(\theta, \phi) = (-1)^{M/2} \sqrt{\frac{2J+1}{4\pi}} \sqrt{\frac{(J-|M|)!}{(J+|M|)!}} (\cos^2 \theta - 1)^{|M|/2} \cdot \left\{ \frac{\partial^{|M|}}{\partial (\cos \theta)^{|M|}} \left[ \frac{1}{2^J J!} \frac{\partial^J}{\partial (\cos \theta)^J} (\cos^2 \theta - 1)^J \right] \right\} e^{iM\phi}$$

(2)

in which the spherical coordinates are, conventionally, $\theta \in [0, \pi]$ (the polar angle referred to the z-axis) and $\phi \in [0, 2\pi]$ (the azimuthal angle referred to the x-axis of a rectangular cartesian frame).

In fact, it is of common experience that the practical problem of separating the angular parts in a set of equations that must be reduced to a pure radial form can be efficaciously treated once a gauge choice is made that transforms those angular parts in algebraic expressions depending only on spherical harmonics of same $(J, M)$ and their derivatives (usually up to the second order in $\theta$). All combinations of those objects can be rewritten in terms of, at most, two different harmonics, in $(J, M)$ and $(J-1, M)$, making use of well-known recurrent relations between Legendre polynomials of different $J$, leading to the following formulas:

$$\frac{\partial Y_{JM}}{\partial \theta} = J \cot \theta Y_{JM} - l(J, M) \csc \theta Y_{J-1, M}$$

(3)

$$\frac{\partial^2 Y_{JM}}{\partial \theta^2} = \{J^2 \cot^2 \theta - [J (J+1) - M^2] \csc^2 \theta\} Y_{JM} + l(J, M) \cot \theta \csc \theta Y_{J-1, M}$$

(4)

with $l(J, M) = \sqrt{(2J+1)(J^2-M^2)}$. Those two expressions can be joined together, once noticed that $\frac{\partial^2 Y_{JM}}{\partial \phi^2} = -M^2 Y_{JM}$, in a second order partial differential equation satisfied by the $Y_{JM}$:

$$\frac{\partial^2 Y_{JM}}{\partial \theta^2} + \csc^2 \theta \frac{\partial^2 Y_{JM}}{\partial \phi^2} + \cot \theta Y_{JM} \frac{\partial Y_{JM}}{\partial \theta} + J (J+1) Y_{JM} = 0.$$  

(5)

This relation will be used later to write in a more compact form the Zerilli’s THs.

The second step on the way to the THs was made by Blatt and Weisskopf [9], who introduced a basis of the space of the complex vector fields on $S^2$, which is built as:

$$Y^I_{JM}(\theta, \phi) = \langle 1, m, 1, n \mid J, M \rangle Y_{Im}(\theta, \phi) e_n$$

(6)
where $e_i$ symbolizes the generic element of the basis of the complexified Euclidean space $E_3^c$ composed by the simultaneous eigenvectors of the spin operators $S$, $S_z$ with $S = 1$:

$$
e_1 = -\frac{(\hat{x} + i \hat{y})}{\sqrt{2}},$$
$$e_0 = \hat{z},$$
$$e_{-1} = \frac{(\hat{x} - i \hat{y})}{\sqrt{2}}$$

and the symbol $\langle | \rangle$ is the bracketed Dirac notation of a Clebsch-Gordan coefficient.

After this definition, the rank-2 tensor space $E_3^c \otimes E_3^c$ can be provided a proper basis by considering orthonormalized external products of the $e_i$; keeping the formalism of [4]:

$$t^j_m = \sum_{\mu=-1}^{1} \langle 1, \mu, 1, 1-m | j, m \rangle e_{m-\mu} \otimes e_{\mu}$$

Now, the harmonic tensor spherical basis, spanning the space of the finite-dimensional irreducible representations of $SO(3)$, can be defined as:

$$Y^{j l}_{JM}(\theta, \phi) = \sum_{m=-j}^{j} \langle L, M - m, j, m | J, M \rangle t^j_m Y^{l M}_{-m}(\theta, \phi)$$

The orthonormality in $L^2(S^2)$ of these objects is guaranteed once a scalar product in this tensor space is defined as (the overbar meaning complex conjugation):

$$(T, S) = \int T : S d\Omega \equiv \int_0^{2\pi} \int_0^{\pi} T^{\rho\sigma} S_{\rho\sigma} \sin \theta d\theta d\phi.$$ (10)

It is worth noting that such a frame is not orthogonal in $E_3^c \otimes E_3^c$ endowed with the scalar product $(T, S) = T : S$, for fixed values of $J, M$.

5 Tensor multipoles

It is not difficult to directly write Zerilli’s $TM$ covariant basis in $M^4$ once a proper definition of $L$ acting on scalar functions is adopted:

$$L f = -i r \wedge \nabla f^\text{comp} = -i r E_{\mu}^\rho f_{-\rho}$$

where $E_{\mu}^\nu$ belongs to a rank-2 component of the well-known Levi-Civita’s tensor $\epsilon$:

$$E_{\mu}^\nu = \eta_{\mu\rho} \epsilon^{\rho\nu\theta\phi} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sin \theta \\
0 & 0 & -\frac{1}{\sin \theta} & 0
\end{pmatrix},$$

$\eta_{\mu\nu}$ being the covariant Minkowski metric in spherical coordinates. With this specification, the $TM$s are defined as follows (the symbol “$|$” means that only the symmetric part of the
The explicit matrix form, with rows and columns labelled as \((t, r, \theta, \phi)\), of these objects, with the help of \([5]\), can be cast as:

\[
\begin{align*}
TM_1 &= \begin{pmatrix} Y_{JM} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad TM_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_{JM} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
TM_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ i r U & i r V & * & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix} \quad TM_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
TM_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
TM_6 &= \begin{pmatrix} 0 & 0 & \frac{r}{\sin \theta} V & -r \sin \theta U \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix} \\
TM_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i r}{\sin \theta} V & -i r \sin \theta U \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
TM_8 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r^2 W}{\sin \theta} & r^2 \sin \theta W \\ 0 & 0 & * & -r^2 \sin^2 \theta W \end{pmatrix} \\
TM_9 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
TM_{10} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

where \(m(J) = \sqrt{\frac{2}{J(J+1)}}\), \(n(J) = \sqrt{\frac{2}{(J-1)(J+1)(J+2)}}\).
where “*” denotes a symmetric component of a tensor.

Under a geometric point of view, the $T_M$ where "\( \ast \) denotes a symmetric component of a tensor.

Under a geometric point of view, the $T_M$ can be classified as:

- Space longitudinal (i.e. orthogonal to the unit 2-sphere) elements: $T_{M_2}, T_{M_5}, T_{M_7}$ (in Zerilli’s notation “a”, “b”, “c”);
- Space transverse (i.e. tangent to the unit 2-sphere) elements: $T_{M_8}, T_{M_9}, T_{M_{10}}$ (“d”, “f”, “g”);
- Time addition elements: $T_{M_1}, T_{M_3}, T_{M_4}, T_{M_6}$ (“a_0”, “a_1”, “b_0”, “c_0”).

Being the ideas underlying the formation of this set mainly based on the euclidean properties of transformation under $L$ and $\nabla$, orthonormalization problems were expected to arise for the components belonging to the $E_3^c \rightarrow M_4^c$ extension: in fact, it is straightforward to see that the last three cited multipoles ($T_{M_{3,4,6}}$) have their norm (defined by (10)) equal to -1.

5.1 Regge-Wheeler perturbation tensors

In the $T_M$ basis, $h^m, h^e$, the well-known Regge-Wheeler perturbation tensors [11] that split [1] in magnetic and electric parts, totally decoupled from each other, can be written as:

\[
\begin{align*}
\text{h}^m &= \frac{2}{m(J) r} \left[ -h_0(t,r) T_{M_6} + i h_1(t,r) T_{M_7} \right] - \frac{i}{n(J) r} h_2(t,r) T_{M_8} \\
\text{h}^e &= e^{\nu(r)} H_0(t,r) T_{M_1} + e^{\lambda(r)} H_2(t,r) T_{M_2} - i \sqrt{2} H_1(t,r) T_{M_3} \\
&\quad + \frac{2}{m(J) r} \left[ -i h_0(t,r) T_{M_4} + h_1(t,r) T_{M_5} \right] \\
&\quad + \frac{1}{n(J)} G(t,r) T_{M_9} + \sqrt{2} \left[ K(t,r) - \frac{1}{m^2(J)} G(t,r) \right] T_{M_{10}}
\end{align*}
\]

and, once the celebrated gauge choices ($h_2 = 0$ for $h^m$ and $h_0 = h_1 = G = 0$ for $h^e$) that share the same names are applied, they reduce to:

\[
\begin{align*}
\text{h}^m &= \frac{2}{m(J) r} \left[ -h_0(t,r) T_{M_6} + i h_1(t,r) T_{M_7} \right] \\
\text{h}^e &= e^{\nu(r)} H_0(t,r) T_{M_1} + e^{\lambda(r)} H_2(t,r) T_{M_2} + \sqrt{2} \left[ -i H_1(t,r) T_{M_3} + K(t,r) T_{M_{10}} \right]
\end{align*}
\]
making evident their belonging to different classes of transformations of \(SO(3)\).
It is easier to explain their decoupling in those terms than invoking an “opposite parity”
feature which is hardly recognizable, once we refer to the eigenvalues (if existing) of the
operator \(P\) that acts on a function defined on \(S^2\) as \(P(f(\theta, \phi)) = f(\pi - \theta, \phi + \pi)\): in the case
of \(h^e\), it is manifestly \(P(h^e_{\mu\nu}) = P(Y_{JM}) = (-1)^J Y_{JM}\) for any choice of \(\mu, \nu\), while \(h^m\) has
no defined parity, since it is immediately seen from (3) that \(P(\partial_\theta Y_{JM}) = (-1)^{J+1} Y_{JM}\) while
\(\partial_\phi Y_{JM} = i M Y_{JM}\) so that \(P(\partial_\phi Y_{JM}) = P(Y_{JM}) = (-1)^J Y_{JM}\).

A Harmonics and multipoles in the euclidean case

In 1976 M. Daumens and P. Minnaert (Université de Bordeaux-France) produced two papers
[10, 11] about the relationship between tensor harmonics and tensor multipoles in both the
Euclidean and the Minkowski space. Concerning the euclidean 3-dim. space, to stress the
fact that no conceptual contradictions arise using one basis instead of the other in separating
the angular dependance of systems like (1), once a unique definition of the quantum index \(J\)
is adopted, they derived an alternative \(TH\) basis which had the property of being traceless
in all its components but one and parity-defined in the traditional sense, contrary to Zerilli’s
\(TM\)s, and, contrary to the formerly defined \(TH\)s, orthogonal also with respect to scalar
product in \(E_3^c \otimes E_3^c\). Unfortunately, some of the main formulas and relations concerning
this topic contain errors and/or misprints. The correct form of this harmonic basis for rank-2
tensors in a Euclidean three-dimensional space is presented here in a fully implementable form,
illustrating the linear combinations of the \(TH\)s and \(TM\)s previously defined that generate it.

Called \(v\) the element of \(S^2_c\) whose coordinates are \((\frac{-1}{\sqrt{3}})^{J+j}((-1)^{L+1}, i, 1)\), the basis can be
defined as:

\[
X_{JM}^{j\ell}(\theta, \phi) = \sum_{L=|J-j|}^{J+j} \sqrt{3} v \cdot e_{sign(l)} (|j|, |l|, J, -|l|, L, 0) Y_{JM}^{j\ell}(\theta, \phi)
\]

where \(0 \leq j \leq 2, -j \leq \ell \leq j\).

It is readily seen that those harmonics have the following parity:

\[
X_{JM}^{j\ell}(-\vec{u}) = \{sign(l) + [1 - sign^2(l)] (-1)^{j+\ell}\} X_{JM}^{j\ell}(\vec{u})
\]

\(\vec{u}\) being a generic element of \(S^2\). Every multipole of order \(j\) is found to be a linear combination
of tensor harmonics of the same order, following this scheme (that corrects [10] - Table 1):

\[
\begin{align*}
X_{JM}^{00} & = Y_{JM}^0 \\
(X_{JM}^{11} X_{JM}^{00} X_{JM}^{1-1}) & = \begin{pmatrix}
Y_{JM}^{1J+1} & Y_{JM}^{1J} \\
Y_{JM}^{1J} & Y_{JM}^{1J-1}
\end{pmatrix} A \\
(X_{JM}^{22} X_{JM}^{21} X_{JM}^{20} X_{JM}^{2-2}) & = \begin{pmatrix}
Y_{JM}^{2J+2} & Y_{JM}^{2J+1} \\
Y_{JM}^{2J} & Y_{JM}^{2J-1} Y_{JM}^{2J-2}
\end{pmatrix} B
\end{align*}
\]
where $A$ and $B$ are two orthogonal matrices, confirming that the structure of a linear transformation with Clebsch-Gordan coefficients effectively corresponds to a rotation:

$$A = \begin{pmatrix}
0 & -\sqrt{\frac{J+1}{2J+1}} & \sqrt{\frac{J}{2J+1}} \\
1 & 0 & 0 \\
0 & \sqrt{\frac{J+1}{2J+1}} & \sqrt{\frac{J}{2J+1}}
\end{pmatrix} \quad (19)$$

$$B = \begin{pmatrix}
\sqrt{\frac{J(J-1)}{2(2J+1)(2J+3)}} & -\sqrt{\frac{2J(J+2)}{(2J+1)(2J+3)}} & \sqrt{\frac{3(J+1)(J+2)}{2(2J+1)(2J+3)}} & 0 & 0 \\
0 & 0 & 0 & -\sqrt{\frac{J+2}{2J+1}} & \sqrt{\frac{J-1}{2J+1}} \\
\sqrt{\frac{3(J-1)(J+2)}{(2J-1)(2J+3)}} & -\sqrt{\frac{3(J+1)}{(2J-1)(2J+3)}} & -\sqrt{\frac{J(J+1)}{(2J-1)(2J+3)}} & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{J-1}{2J+1}} & \sqrt{\frac{J+2}{2J+1}} \\
\sqrt{\frac{(J+1)(J+2)}{2(2J-1)(2J+1)}} & \sqrt{\frac{2J(J+1)(J-1)}{(2J-1)(2J+1)}} & \sqrt{\frac{3(J-1)}{2(2J-1)(2J+1)}} & 0 & 0
\end{pmatrix} \quad (20)$$

Finally, the linear combinations of those $THs$ that generate Zerilli’s $TMs$ can be easily found by a simple matricial calculus procedure:

a) called $J$ the jacobian matrix of the transformation from the cartesian rectangular coordinates $(x, y, z)$ to the polar spherical $(\theta, \phi)$ on $S^2$:

$$J = \begin{pmatrix}
\sin \theta \cos \phi & \cos \theta \cos \phi & \sin \phi \\
\sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\
\cos \theta & -\sin \theta & 0
\end{pmatrix} \quad (21)$$

and $M$ the Euclidean 3-dim. covariant metric tensor:

$$M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \sin^2 \theta
\end{pmatrix} \quad (22)$$

the operator linking the spherical representation $T_s$ of a tensor to its cartesian one, $T_c$, can be expressed, in the matrix algebra representation, as:

$$T_s = \mathcal{O}_{cs}(T_c) = J^{-1} T_c J M \quad (23)$$

b) called $TM^{(3)}_i$ the Euclidean 3-dim. tensors introduced in [4], counterparts of Zerilli’s multipoles of the previous section (as matrices, they are obtained by deleting the first row and column from the $(2, 5, 7, 8, 9, 10)$-labelled ones);
it is readily found that:

\[ T M_1^{(3)} = "a^{(3)}" = -\frac{1}{\sqrt{3}} \mathcal{O}_{cs}(X_{J_M}^{00}) + \sqrt{\frac{2}{3}} \mathcal{O}_{cs}(X_{J_M}^{20}) \]
\[ T M_2^{(3)} = "b^{(3)}" = \mathcal{O}_{cs}(X_{J_M}^{2-1}) \]
\[ T M_3^{(3)} = "c^{(3)}" = \mathcal{O}_{cs}(X_{J_M}^{21}) \]
\[ T M_4^{(3)} = "d^{(3)}" = \mathcal{O}_{cs}(X_{J_M}^{22}) \]
\[ T M_5^{(3)} = "e^{(3)}" = \mathcal{O}_{cs}(X_{J_M}^{2-2}) \]
\[ T M_6^{(3)} = "f^{(3)}" = -\sqrt{\frac{2}{3}} \mathcal{O}_{cs}(X_{J_M}^{00}) - \frac{1}{\sqrt{3}} \mathcal{O}_{cs}(X_{J_M}^{20}) \]

References

[1] T. Regge, J. A. Wheeler, *Phys. Rev.*, 108, 1063 (1957).
[2] J. Mathews, *J. Soc. Ind. Appl. Math.*, 10, 768 (1962).
[3] L. A. Edelstein, C. V. Vishveshwara, *Phys. Rev. D*, 1, 3514 (1969).
[4] F. J. Zerilli, *J. Math. Phys.*, 11, 2203 (1970).
[5] F. J. Zerilli, *Phys. Rev. Lett.*, 24, 737 (1970).
[6] M. Johnston, R. Ruffini, F. J. Zerilli, *Phys. Rev. Lett.*, 31, 1317 (1973).
[7] M. Johnston, R. Ruffini, F. J. Zerilli, *Phys. Lett.*, 49B, 185 (1974).
[8] F. J. Zerilli, *Phys. Rev. D*, 2, 2141 (1970).
[9] J. M. Blatt, V. F. Weisskopf, *Theoretical Nuclear Physics*, (Appendix), Wiley, New York, 1952.
[10] M. Daumens, P. Minnaert, *J. Math. Phys.*, 17, 1903 (1976).
[11] M. Daumens, P. Minnaert, *J. Math. Phys.*, 17, 2085 (1976).