EXISTENCE FOR A QUASISTATIC VARIATIONAL-HEMIVARIATIONAL INEQUALITY

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Abstract. This paper deals with an evolution inclusion which is an equivalent form of a variational-hemivariational inequality arising in quasistatic contact problems for viscoelastic materials. Existence of a weak solution is proved in a framework of evolution triple of spaces via the Rothe method and the theory of monotone operators. Comments on applications of the abstract result to frictional contact problems are made. The work extends the known existence result of a quasistatic hemivariational inequality by S. Migórski and A. Ochal [SIAM J. Math. Anal., 41 (2009) 1415-1435]. One of the linear and bounded operators in the inclusion is generalized to be a nonlinear and unbounded subdifferential operator of a convex functional, and a smallness condition of the coefficients is removed. Moreover, the existence of a hemivariational inequality is extended to a variational-hemivariational inequality which has wider applications.

1. Introduction. In this paper, we are concerned with the solvability of an evolution variational-hemivariational inequality modeling quasistatic contact problems for viscoelastic materials. The inequality is studied in the following equivalent form of a nonlinear evolution inclusion:

\[ Au'(t) + Bu(t) + L^* \partial_{+} J(Lu(t)) \ni f(t) \quad \text{a.e. } t \in [0, T], \]

associated with an initial condition \( u(0) = u_0 \). Here \( A \) and \( B \) are operators from a reflexive Banach space \( V \) to its dual \( V^* \), and \( A \) is a linear and bounded operator, \( B \) is the subdifferential of a convex functional; \( L \) is a linear and continuous operator, \( L^* \) is its dual, \( \partial_{+} J \) denotes the Clarke subdifferential of a locally Lipschitz continuous functional \( J \), and \( f : [0, T] \to V^* \) is prescribed.

The evolution inclusion (1) has been considered by Migórski and Ochal [12] in 2009 when the leading operators \( A \) and \( B \) are linear and bounded. In that paper (1) is equivalent to a quasistatic hemivariational inequality. Compared with dynamic problems, the acceleration term \( u''(t) \) is missing. Under a smallness condition (see
[12, assumption \((H_1)\) and the condition of the coefficients in Theorem 17], the existence of a weak solution was obtained in a framework of evolution triple of spaces by a vanishing acceleration approach ([12, Theorem 13, Corollary 14 and Theorem 17]). An application of the abstract theorem to quasistatic viscoelastic contact problems with nonmonotone boundary conditions is given.

In this paper, the inclusion (1) is an equivalent form of a variational-hemivariational inequality, because \(B\) is the subdifferential of a convex functional \(\Phi\). In particular, if \(\Phi = \Phi_1 + \Phi_2\) such that \(\Phi_1\) is convex and G-differentiable, and \(\Phi_2\) is convex and lower semi-continuous. Then, (1) corresponds to the following quasistatic variational-hemivariational inequality:

\[
\langle Au'(t) + B_1 u(t) - f(t), v - u(t) \rangle + \Phi_2(v) - \Phi_2(u(t)) + J^\circ (Lu(t); L(v - u(t))) \geq 0,
\]

for all \(v \in V\) and a.e. \(t \in [0, T]\), where \(B_1(u) = \partial \Phi_1(u) = \Phi'_1(u)\).

Inequalities, including variational and hemivariational inequalities, were initially developed in monographs [5, 9, 18] to study contact problems. In the past several decades, there is considerable literature devoted to the mathematical theory of these inequalities and their applications to dynamic, static and quasistatic contact problems for elastic and viscoelastic materials; see [12, 17, 14, 26, 25, 24, 23, 6, 7, 19, 11, 13, 15, 16, 10] and the references therein. The study of inclusion (1), as is stated in [12], is motivated by a quasistatic model of contact problems for viscoelastic materials. In the model, \(u\) is the displacement, \(u'\) is the velocity, \(A\) and \(B\) correspond to the viscosity operator and elasticity operator, respectively, \(f\) represents the given body forces and surface traction, and \(\partial_{\lambda}J\) is related to a nonmonotone frictional boundary condition.

We remark that an interesting question concerning the solvability of (1) was proposed in [12] in the case that \(A\) and \(B\) are nonlinear. In general, the study of quasistatic inequalities is more complicated than dynamic ones, because losing the estimation of \(u''(t)\) results in the limit procedure related to nonlinear operators becoming quite difficult. To the best of our knowledge, the aforementioned open problem to (1) hasn’t been solved in the past years. Recently, we have considered an inclusion similar to (1) in [19], where \(A\) is maximal monotone, \(B\) is linear and bounded, and \(J\) is a convex functional of the velocity \(u'(t)\). The existence of weak solutions is proved by the Rothe method and a vanishing acceleration technique. Motivated by [12, 19] and the literature in this area, we study the inclusion (1) in case that \(A\) is linear and \(B\) is a nonlinear subdifferential operator via the theory of monotone operators and Rothe method. The work extends the known existence result of a quasistatic hemivariational inequality in [12]. We establish an existence theorem for a quasistatic variational-hemivariational inequality, where \(B\) is a nonlinear and unbounded subdifferential operator. Moreover, a smallness condition of the coefficients is removed.

We also recall that, in general, the elasticity operator \(B\) is supposed to be linear, continuous, symmetric and monotone in non-static contact problems; see, e.g. [14, 11, 13, 15]. If the (Clarke) subdifferential operator \(\partial_{\lambda}J\) acts on \(u'(t)\), the related dynamic and quasistatic contact problems has been studied in the case that \(B\) is nonlinear and Lipschitz continuous; see, e.g. [7, 10]. In these work, the contraction fixed point theorem is used to prove the existence and uniqueness of a weak solution under a smallness condition of the coefficients. Nevertheless, this method is invalid to inclusion (1).
Our work is organized as follows. In Section 2, we review some standard results of convex analysis, nonsmooth analysis and monotone operators. Section 3 presents the assumptions and the main theorem. Section 4 and 5 are devoted to the proof of the main result by the Rothe method and the theory of monotone operators. In section 6, we make some short comments on the applicability of the abstract result to viscoelastic contact problems.

2. Notations and preliminaries. Let $X$ stand for a real and reflexive Banach space with its dual space $X^\ast$. By $\langle \cdot, \cdot \rangle_{X^\ast \times X}$ and $\| \cdot \|_X$ we denote the duality pairing between $X$ and $X^\ast$, and the norm on $X$, respectively. We begin by summarizing some definitions and standard results on convex analysis, nonsmooth analysis and monotone operators (cf. e.g. [17, 3, 2, 1]).

Let $\phi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a convex functional and $D(\phi) := \{ u \in X : \phi(u) < +\infty \}$ denote its effective domain. We say $\phi$ is proper if $\phi$ does not identically equal positive infinity. Let $\Gamma_0(X)$ denote the set of proper, convex and lower semicontinuous functionals on $X$. The subdifferential of $\phi \in \Gamma_0(X)$ at $u \in X$ is defined by

$$\partial \phi(u) := \{ w \in X^\ast : \phi(v) - \phi(u) \geq \langle w, v - u \rangle_{X^\ast \times X}, \text{ for all } v \in X \}. \quad (3)$$

**Definition 2.1.** Let $X$ be a Banach space and let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional. The generalized (Clarke) directional derivative of $\varphi$ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) := \limsup_{y \to x, \lambda \downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}$$

and the generalized gradient (Clarke subdifferential) of $\varphi$ at $x$, denoted by $\partial \varphi(x)$, is a subset of a dual space $X^\ast$ given by

$$\partial \varphi(x) := \{ x^\ast \in X^\ast : \langle x^\ast, v \rangle_{X^\ast \times X} \leq \varphi^0(x; v) \text{ for all } v \in X \}.$$ 

In particular, if $\varphi : X \to \mathbb{R}$ is convex and continuous, then $\varphi$ is locally Lipschitz and the Clarke subdifferential of $\varphi$ coincides with the subdifferential in the sense of convex analysis (cf. [2, Propositions 2.2.6–2.2.7]).

We recall some definitions concerning set-valued mappings (see e.g., [14, Chapter 3]).

**Definition 2.2.** A multivalued operator $G : D(G) \subset X \to 2^{X^\ast}$ is said to be monotone, if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in D(G), \, y_1 \in G(x_1), \, y_2 \in G(x_2).$$

Moreover, $G$ is said to be maximal monotone if and only if $G$ is monotone and there is no monotone extension in $X \times X^\ast$, i.e., for $x \in X, \, y \in X^\ast$ if

$$\langle y_1 - y, x_1 - x \rangle \geq 0, \quad \forall x_1 \in D(G), \, y_1 \in G(x_1),$$

then we have $x \in D(G)$ and $y \in G(x)$.

It is well known that the subdifferential operator $\partial \phi$ defined by (3) is maximal monotone (cf. [17, Proposition 1.20], [27, Proposition 32.17]).

**Definition 2.3.** A multivalued operator $G : X \to 2^{X^\ast}$ is said to be pseudomonotone if the following conditions are satisfied:

(i) for each $u \in X$, $G(u) \subset X^\ast$ is nonempty, bounded, closed and convex;
(ii) the restriction of $G$ to each finite-dimensional subspace $E$ of $X$ is upper semi-continuous as an operator from $E$ to $X^*$ endowed with weak topology;

(iii) let $u_n \in X$ and $u_n^* \in X^*$ with $u_n^* \in G(u_n)$, from $u_n \to u$ weakly in $X$ and $\limsup (u_n^*, u_n - u)_X \leq 0$, it follows: to each element $v \in X$, there exists a $u^*(v) \in G(u)$ such that

$$\liminf_{n \to \infty} (u_n^*, u_n - v)_X \geq (u^*(v), u - v)_X.$$  

**Lemma 2.4.** ([1, Theorem 2.127]) Let $T : X \to 2^{X^*}$ be a maximal monotone operator with $w_0 \in D(T)$. Let $G : X \to 2^{X^*}$ be a bounded, pseudomonotone operator which is $w_0$-coercive, i.e.,

$$\liminf_{\eta \in G(v), \|v\| \to \infty} \frac{\langle \eta, v - w_0 \rangle}{\|v\|} \to +\infty. \quad (4)$$

Then $T + G$ is surjective, i.e., the range of $T + G$ is $X^*$.

Besides, we recall a discrete Gronwall’s inequality (cf. [6, Lemma 7.26, p. 160]) which will be used in the sequel.

**Lemma 2.5.** Let $T > 0$ be given. For a positive integer $m$, we define $h = T/m$. Assume that $\{e_s\}_{s=0}^{m-1}, \{g_s\}_{s=0}^{m-1}$ are two sequences of nonnegative numbers satisfying

$$e_s \leq cg_s + ch \sum_{n=0}^s e_n \quad \text{for} \quad s = 0, 1, \cdots, m - 1, \quad (5)$$

where $c$ is a positive constant. Then if the step-size $h$ is sufficiently small,

$$e_s \leq c(g_s + h \sum_{n=0}^s g_n) \quad \text{for} \quad s = 0, 1, \cdots, m - 1.$$  

In particular, if $g_s \equiv g_0$, we have $e_s \leq cg_0(1 + T)$, for $s = 0, 1, \cdots, m - 1$.

3. **Hypothesis and the main result.** Let $V$ and $Z$ be real, separable and reflexive Banach spaces, and $V^*$, $Z^*$ denote their dual spaces, respectively. Let $H$ be a real and separable Hilbert space identified with its dual $H^*$. Suppose that $V \subset Z \subset H \subset Z^* \subset V^*$ where the embeddings are dense and continuous. Note that in this case $(V, H, V^*)$ and $(Z, H, Z^*)$ are evolution triple of spaces (c.f. [27, P. 416]). We further assume that the embedding $V \subset Z$ is compact. For brevity, the aforementioned embedding operators are omitted. For example, if we write $u \in V$, it means $u \in Z$ as well. Given a fixed number $0 < T < +\infty$ and a Banach space $X$. We denote by $L^2(0, T; X)$ the $X$-valued Lebesgue-Bochner space endowed with the norm $\|u\|_{L^2(0, T; X)} = \left( \int_0^T \|u(t)\|_X^2 \, dt \right)^{1/2}$. We have that $L^2(0, T; V) \subset L^2(0, T; Z) \subset L^2(0, T; H) \subset L^2(0, T; Z^*) \subset L^2(0, T; V^*)$. Moreover, define $W := \{v \in L^2(0, T; V) \mid v' \in L^2(0, T; V^*) \}$, where the time derivative is understood in the sense of vector-valued distributions. Then $W$ is a separable and reflexive Banach space endowed with the norm $\|v\|_W = \|v\|_{L^2(0, T; V)} + \|v'\|_{L^2(0, T; V^*)}$. By the known Aubin-Lions compactness lemma, the embedding $W \subset L^2(0, T; Z)$ is compact as $V \subset Z$ is compact. Throughout this paper the notation $\langle \cdot, \cdot \rangle_{Y^* \times Y}$ stands for the duality pairing between a Banach spaces $Y$ and its dual $Y^*$, and the subscript is always omitted when no confusion arises.

The hypotheses on the data of the evolution inclusion (1) are as follows:
**H(A)** Let \( A \) be a linear and bounded operator from \( V \) to \( V^* \) satisfying
\[
\|A(v)\|_{V^*} \leq a_1\|v\|_V, \quad \forall \ v \in V, \quad \text{for} \ a_1 > 0.
\tag{6}
\]

(i) \( A \) is symmetric, i.e., \( \langle Au, v \rangle_{V^* \times V} = \langle Av, u \rangle_{V^* \times V}, \quad \forall \ v, u \in V. \)

(ii) \( A \) is positive, i.e., there exists a constant \( a_2 > 0 \) such that
\[
\langle Av, v \rangle_{V^* \times V} \geq a_2\|v\|_V^2, \quad \forall \ v \in V.
\]

Note that \( A \) is weakly sequentially continuous as \( A \in \mathcal{L}(V, V^*) \). By H(A)(i), \( A \) is a potential operator and the potential \( \Psi, \) given by \( \Psi(u) = \frac{1}{2} \langle Au, u \rangle, \) is G-differentiable and \( Au = \Psi'_G(u) \forall u \in V. \) In addition, by H(A)(ii) \( \Psi \) is convex and \( \partial \Psi(u) = \Psi'_G(u) \) (cf. [28, Chaper 41, Potential operators; Chapter 42, Example 42.12]). A concrete example of differential operator corresponding to \( A \) is the negative Laplacian.

**H(B)** Let \( B : V \to 2^{V^*} \) be given by \( B(v) = \partial \Phi(v) \) where the functional \( \Phi : V \to \mathbb{R} \cup \{+\infty\} \) is proper, convex and lower semicontinuous such that
\[
\Phi(v) \geq b_1\|v\|_V^2 - b_2 \quad \forall \ v \in V, \quad \text{with constants} \ b_1, \ b_2 > 0.
\tag{7}
\]

We remark that \( B \) is not necessary bounded since \( \Phi \) takes value \(+\infty\). For example, let \( \Phi(u) = \|u\|_V^p + I_K(u) \) where \( p \geq 2 \) and \( I_K \) is the indicator function on a bounded, convex and closed subset \( K \) of \( V. \) Note that the functional \( \|u\|_V^p \) is G-differentiable, and (7) holds with \( b_1 = 1, \ b_2 = 0 \) in case \( p = 2, \) and \( b_1 = b_2 = 1 \) in case \( p > 2. \) However, \( B \) is unbounded since \( \partial I_K \) is unbounded.

Let \( E \) be a given Banach space such that \( Z \) is continuously embedded into \( E. \) Denote the embedding operator by \( L, \) and let \( L^* \) stand for its dual which is continuous from \( E^* \) to \( Z^* \).

**H(J)** The functional \( J : E \to \mathbb{R} \) is locally Lipschitz continuous and there exist constants \( c_1, c_2 > 0 \) such that
\[
\|\eta\|_{E^*} \leq c_1\|v\|_E + c_2, \quad \forall \ \eta \in \partial_{cl}J(v), \quad v \in E.
\tag{8}
\]

**H0** \( f \in L^2(0, T; V^*), \ u_0 \in D(B). \)

Moreover, we denote by \( c_3 \) the embedding constant of \( V \) to \( E \) for brevity.

**Definition 3.1.** A function \( u \in L^\infty(0, T; V) \) is said to be a weak solution to (1) if \( u' \in L^2(0, T; V) \) and there exist \( \xi, w \in L^2(0, T; V^*) \) such that
\[
\begin{cases}
Au'(t) + \xi(t) + w(t) = f(t) & \text{a.e.} \ t \in [0, T] \\
u(0) = u_0,
\end{cases}
\tag{9}
\]

with \( \xi(t) \in B(u(t)) \) and \( w(t) \in L^\infty\partial_{cl}J(Lu(t)) \) a.e. \( t \in [0, T]. \)

Note that the initial condition \( u(0) = u_0 \) in \( V \) make sense since the embedding \( W^{1,2}(0, T; V) \subset C([0, T]; V) \) is continuous.

**Theorem 3.2.** Suppose that H(A), H(B), H(J) and H0 hold. Then the evolution inclusion (1) admits at least one weak solution.

4. **Discrete Problems and the a Priori estimate.** We use the implicit time-discretization technique (also known as Rothe method) to prove theorem 3.2, which has been proved to be effective in dealing with evolution inclusions and inequality problems, see, e.g., [21, 22, 20, 8].
Theorem 4.1. Under the hypotheses of Theorem 3.2, the discrete problem \((D)\) and Lemma 2.4 to inclusion (11), we have the following conclusion.

Let \(m \in \mathbb{Z}^+\) and \((t_i)_{0 \leq i \leq m}\) be a subdivision of \([0, T]\) whose step-size \(h = T/m\). For \(n \in \{0, 1, \ldots, m - 1\}\), let \(t^n_0 = 0, t^n_m = nh, t^n_i = T\) and the time interval \(I^n = (t^n_i, t^n_{i+1})\).

Set \(u^n_0 = u_0\): for \(n = 0, 1, \ldots, m - 1\), we aim to find \((z^{n+1}_h, \xi^{n+1}_h, \eta^{n+1}_h) \in V \times V \times E^*\) such that

\[
(D^n) \quad \begin{cases}
A z^{n+1}_h + \xi^{n+1}_h + L^* \eta^{n+1}_h = f^n_h, \\
\xi^{n+1}_h \in B(u^{n+1}_h), \quad \eta^{n+1}_h \in \partial \varphi(J(L u^{n+1}_h)).
\end{cases}
\]

where \(f^n_h = \frac{1}{h} \int_{t^n_i}^{t^n_{i+1}} f(t) \, dt\), \(z^{n+1}_h = \frac{1}{h}(u^{n+1}_h - u^n_h)\) and thus \(u^{n+1}_h = u^n_h + h z^{n+1}_h = u^n_h + h \sum_{j=0}^{n} z^{j+1}_h\). For \(n = 0, 1, \ldots, m - 1\), we define \(F^n_h : V \to 2^{V^*}\) as follows.

\[
F^n_h(v) := Av + L^* \partial \varphi(J(L(u^n_h + hv))), \quad \forall v \in V.
\]

**Theorem 4.1.** Take any \(v_0 \in D(B)\) and let the step-size \(h\) be sufficiently small. Then \(F^n_h\) is bounded, pseudomonotone and \(v_0\)-coercive from \(V\) to \(V^*\) for each \(n, n = 0, 1, \ldots, m - 1\).

**Proof.** The boundedness of \(F^n_h\) follows from (6) and (8). In view of the properties of the Clarke’s subdifferential, the pseudomonotonicity of \(F^n_h\) is quite standard since the embedding \(V \subset Z\) is compact and \(A\) is weakly sequentially continuous and strongly monotone. It remains to show the \(v_0\)-coercivity condition. To this end, taking \(\eta \in \partial \varphi(J(L(u^n_h + hv)))\) and using \(H(A)(ii)\) and (8), we have

\[
\langle Av + L^* \eta, v - v_0 \rangle \geq (a_2 - hc_1 c_3^2) \|v\|^2 - \lambda_0 \|v\| \|v\| - c_1 c_3^2 \|u^n_h\| \|v\| - c_2 c_3 \|v_0\| \|v\|, \quad (10)
\]

where \(\lambda_0 = a_1 \|v_0\| \|v\| + c_1 c_3^2 \|u^n_h\| \|v\| + c_1 c_3^2 \|v_0\| \|v\| + c_2 c_3\). It follows that \(F^n_h\) is \(v_0\)-coercive as \(h < a_2/c_1 c_3^2\). The proof is complete. \(\square\)

We are now in a position to show the existence of weak solutions to the discrete problem \((D^n)\) for \(n = 0, 1, 2, \ldots, m - 1\). To this end, we rewrite \((D^n)\) to the following equivalent form: find \(z^{n+1}_h \in V\) such that

\[
B(u^n_h + h z^{n+1}_h) + F^n_h(z^{n+1}_h) \ni f^n_h.
\]

(11)

Since \(B\) is a maximal monotone operator, so is \(B(u^n_h + hv)\). Applying Theorem 4.1 and Lemma 2.4 to inclusion (11), we have the following conclusion.

**Theorem 4.2.** Under the hypotheses of Theorem 3.2, the discrete problem \((D^n)\) admits at least one solution for each \(n, n = 0, 1, \ldots, m - 1\).

Let \((z^{n+1}_h, \xi^{n+1}_h, \eta^{n+1}_h)\) be a solution to \((D^n)\) for \(n = 0, 1, \ldots, m - 1\). The next step is to find the a priori estimates. We define the vector-valued functions \(z_h, u_h, \chi_h, \xi_h, \eta_h, f_h, \hat{u}_h\) as follows.

\[
z_h(t) = z^{n+1}_h, \quad u_h(t) = u^{n+1}_h, \quad \chi_h(t) = A z^{n+1}_h, \quad \xi_h(t) = \xi^{n+1}_h,
\]

\[
\eta_h(t) = \eta^{n+1}_h, \quad f_h(t) = f^n_h, \quad \hat{u}_h(t) = u^n_h + \frac{t - nh}{h}(u^{n+1}_h - u^n_h),
\]

for \(t \in I^n_h, n = 0, 1, \ldots, m - 1\). Moreover, let \(\hat{u}_h(0) = u_0\).

Recall that \(\|f_h\|_{L^2(0, T; V^*)} \leq \|f\|_{L^2(0, T; V^*)}\) uniformly for \(h\), and \(f_h \to f\) in \(L^2(0, T; V^*)\) as \(h \to 0\) (see e.g., [20, Lemma 3.1]).
Since \((z_h^{n+1}, \xi_h^{n+1}, \eta_h^{n+1})\) is a solution to \((D_h)\) for each \(n, n = 0, 1, \cdots m - 1\), by definitions (12) and (13) we have
\[
\chi_h(t) + \xi_h(t) + w_h(t) = f_h(t) \text{ in } V^*, \quad \text{a.e. } t \in [0, T],
\]
where \(w_h(t) = L^* \eta_h(t) \text{ a.e. } t \in [0, T]\). Besides, for each fixed \(h\), we see that the vector-valued functions \(\chi_h, \xi_h, w_h\) and \(f_h\) belong to \(L^2(0, T; V^*)\) by their definitions. Thus, the equation (14) also implies
\[
\chi_h + \xi_h + w_h = f_h \text{ in } L^2(0, T; V^*).
\]

**Theorem 4.3.** The quantities below are uniformly bounded with respect to \(h\) by a constant \(C\) which depends on \(c_1, c_2, c_3, a_2, b_1, b_2, u_0\) and \(f\).
\[
\|u_h\|_{L^\infty(0, T; V)}, \quad \|\tilde{u}_h\|_{C([0, T]; V)}, \quad \|z_h\|_{L^2(0, T; V)}
\]
\[
\|\xi_h\|_{L^2(0, T; V^*)}, \quad \|\chi_h\|_{L^2(0, T; V^*)}, \quad \|\eta_h\|_{L^2(0, T; E^*)}.
\]

**Proof.** Multiplying the equation in \((D_h)\) by \(h z_h^{n+1}\), we have
\[
h \langle A z_h^{n+1} + \xi_h^{n+1} + L^* \eta_h^{n+1} - f_h^{n}, z_h^{n+1} \rangle = 0.
\]
The condition \(H(A)(ii)\) implies
\[
\langle A z_h^{n+1}, z_h^{n+1} \rangle \geq a_2 \|z_h^{n+1}\|^2_V.
\]
On the other hand, we have
\[
\|u_h^{n+1}\|^2_V = \|u_0 + h \sum_{j=0}^{n} z_h^{j+1}\|^2_V
\]
\[
\leq 2(\|u_0\|^2_V + (h \sum_{j=0}^{n} \|z_h^{j+1}\|^2_V)^2)
\]
\[
\leq 2(\|u_0\|^2_V + h^2(n + 1) \sum_{j=0}^{n} \|z_h^{j+1}\|^2_V)
\]
\[
\leq 2\|u_0\|^2_V + 2hT \sum_{j=0}^{n} \|z_h^{j+1}\|^2_V.
\]
Thus, by means of (8), we further get
\[
\langle L^* \eta_h^{n+1}, z_h^{n+1} \rangle = \langle \eta_h^{n+1}, L z_h^{n+1} \rangle \geq -(c_1\|Lu_h^{n+1}\|_E + c_2\|Lz_h^{n+1}\|_E
\]
\[
\geq -\frac{1}{2\rho} (c_1^2 c_3^2 \|u_h^{n+1}\|^2_V + c_2^2) - \rho c_3^2 \|z_h^{n+1}\|^2_V
\]
\[
\geq -\rho c_3^2 \|z_h^{n+1}\|^2_V - \frac{1}{\rho} c_1^2 c_3^2 T h \sum_{j=0}^{n} \|z_h^{j+1}\|^2_V - \frac{1}{2\rho} (2c_1^2 c_3^2 \|u_0\|^2_V + c_2^2).
\]
Since \(\xi_h^{n+1} \in Bu_h^{n+1}\) and \(B = \partial \Phi\), we have
\[
h \langle \xi_h^{n+1}, z_h^{n+1} \rangle = \langle \xi_h^{n+1}, u_h^{n+1} - u_h^n \rangle \geq \Phi(u_h^{n+1}) - \Phi(u_h^n).
\]
Moreover, using the Young inequality, we find
\[
\langle f_h^{n}, z_h^{n+1} \rangle \leq \frac{1}{4\rho} \|f_h^{n}\|^2_V + \rho \|z_h^{n+1}\|^2_V.
\]
Finally, summing (17) over \( n \) from 0 to \( s \) (\( 0 \leq s \leq m - 1 \)), and using inequalities (18) to (21), we have

\[
\Phi(u_h^{s+1}) + (a_2 - \rho c_3^2 - \rho) h \sum_{n=0}^{s} \|z_h^{n+1}\|_V^2 \leq \frac{1}{\rho} c_1^2 c_3^2 T h \sum_{n=0}^{s} h \sum_{j=0}^{n} \|z_h^{j+1}\|_V^2 + \Phi(u_0) + h(s + 1) \frac{1}{2\rho} (2c_1^2 c_3^2 \|u_0\|_V^2 + c_2^2) + \frac{h}{4\rho} \sum_{n=0}^{s} \|f_h^n\|_V^2.
\]  

(22)

Using (7) and taking \( \rho \) being sufficiently small, we see that there exist positive constants \( \alpha, \beta, \gamma, \delta \) such that

\[
b_1 \|u_h^{s+1}\|_V^2 + \alpha h \sum_{n=0}^{s} \|z_h^{n+1}\|_V^2 \leq \beta h \sum_{n=0}^{s} h \sum_{j=0}^{n} \|z_h^{j+1}\|_V^2 + \delta,
\]  

(23)

where \( \delta \) depends on \( c_1, c_2, c_3, b_2, u_0 \) and \( f \). Notice that the inequality (23) still holds if \( b_1 \|u_h^{s+1}\|_V^2 \) is removed. Thus, using Lemma 2.5 by taking \( e_s = h \sum_{n=0}^{s} \|z_h^{n+1}\|_V^2 \), \( c = \beta/\alpha \), and \( g_s = \delta/\beta \), we deduce that

\[
h \sum_{n=0}^{s} \|z_h^{n+1}\|_V^2 \leq C, \quad \forall \ 0 \leq s \leq m - 1.
\]  

(24)

where the constant \( C \) is dependent of \( c_1, c_2, c_3, a_2, b_2, u_0 \) and \( f \), but independent of \( h \). Then, it follows from (23) we see that \( \{u_h^{s+1}\}_{s=0}^{m-1} \) is bounded in \( V \). Thus we now have the following three quantities are uniformly bounded with respect to \( h \).

\[
\|\tilde{u}_h\|_{C([0,T];V)}, \quad \|u_h\|_{L^\infty([0,T];V)}, \quad \|z_h\|_{L^2([0,T];V)}
\]  

(25)

Using (6), (8) we find \( \chi_h \) and \( \eta_h \) are uniformly bounded in \( L^2(0,T;V^*) \) and \( L^2(0,T;E^*) \), respectively. Consequently, \( w_h \) is uniformly bounded in \( L^2(0,T;V^*) \). Therefore, using the uniform boundedness of \( \chi_h, w_h \) and \( f_h \), from (15) we finally get \( \xi_h \) is uniformly bounded in \( L^2(0,T;V^*) \). This completes the proof of Theorem 4.3.

\[\square\]

Lemma 4.4. \( \|u_h - \tilde{u}_h\|_{L^2(0,T;V)} \to 0 \) as \( h \) goes to zero.

Proof. A straightforward calculation gives

\[
\|u_h - \tilde{u}_h\|_{L^2(0,T;V)}^2 = h \sum_{n=0}^{m-1} \|u_h^{n+1} - u_h^n\|_V^2 \int_0^1 \rho^2 d\rho
\]  

(26)

\[
= \frac{h}{2} \sum_{n=0}^{m-1} \|u_h^{n+1} - u_h^n\|_V^2 + \frac{h}{3} \int_0^1 \rho^2 d\rho
\]  

(27)

\[
= \frac{h^2}{3} \|z_h\|_{L^2(0,T;V)}^2.
\]  

Thus, Lemma 4.4 follows from (25). \[\square\]
5. Limit procedure and the proof of the main theorem. Based on Theorems 4.2 and 4.3, in this section we aim to prove Theorem 3.2 by passing to the limit in equation (15).

Proof of Theorem 3.2. First of all, based on Theorem 4.3 and Lemma 4.4, letting $h \to 0$ and after taking a subsequence, we have

\begin{align*}
  u_h, \hat{u}_h &\to u \quad \text{weakly in } L^2(0, T; V) \\
  u_h &\to u \quad \text{weakly in } L^\infty(0, T; V) \\
  z_h &\to z = u' \quad \text{weakly in } L^2(0, T; V) \\
  \chi_h &\to \chi \quad \text{weakly in } L^2(0, T; V^*) \\
  \eta_h &\to \eta \quad \text{weakly in } L^2(0, T; E^*).
\end{align*}

In fact, by Theorem 4.3, it is readily to get (30)–(32), $u_h \to u$ and $\hat{u}_h \to \hat{u}$ weakly in $L^2(0, T; V)$, $u_h \to u_1$ and $\hat{u}_h \to \hat{u}_1$ weakly star in $L^\infty(0, T; V)$, and $z_h \to z$ weakly in $L^2(0, T; V)$. Then, Lemma 4.4 implies $u = \hat{u}$ immediately, i.e., (27) holds. Moreover, since $\hat{u}'_h = z_h$, we have $z = u'$, i.e., (29) holds. Finally, for all $v \in L^2(0, T; V^*) \subset L^1(0, T; V^*)$, we find

$$
\lim_{h \to 0} \langle v, u_h - u_1 \rangle_{L^2(0,T;V^*) \times L^2(0,T;V)} = \lim_{h \to 0} \langle v, u_h - u_1 \rangle_{L^1(0,T;V^*) \times L^\infty(0,T;V)} = 0.
$$

It follows that $u = u_1$ in $L^2(0, T; V)$ as $u_h \to u$ weakly in $L^2(0, T; V)$, and thus $u(t) = u_1(t)$ in $V$ a.e. $t \in [0, T]$. Further, we have $u = u_1$ in $L^\infty(0, T; V)$ because of $u_1 \in L^\infty(0, T; V)$. We also have $\hat{u} = \hat{u}_1$ in $L^\infty(0, T; V)$ by the same reasoning. Therefore, $\eta_1 = u = \hat{u} = \hat{u}_1$, and (28) holds.

Next, we take the limit in (15) and use $f_h \to f$ in $L^2(0, T; V^*)$ and (30)–(32) to find

$$
\chi + \xi + w = f \quad \text{in } L^2(0, T; V^*),
$$

where $w(t) = L^*\eta(t)$.

To finish the proof of Theorem 3.2, it remains to show that $u(0) = u_0$, $\chi(t) = Au(t)$, $\xi(t) = Bu(t)$, and $w(t) = L^*\partial_{v,J}J(Lu(t))$ a.e. $t \in [0, T]$.

Let $A$ be the Nemitsky operator corresponding to $A$ defined by $(A\varphi)(t) = Av(t)$ for $v \in L^2(0, T; V)$. Then $A$ is a linear and bounded operator from $L^2(0, T; V)$ to $L^2(0, T; V^*)$ as $A \in L(V, V^*)$, and thus $A$ is also weakly sequentially continuous [27, Proposition 21.27]. So, we have $Az_h \to Au'$ weakly in $L^2(0, T; V^*)$ due to $z_h \to u'$ weakly in $L^2(0, T; V)$ by (29). Recalling that $Az_h = \chi_h$, using (30) we have $\chi = Au'$, i.e., $\chi(t) = Au'(t)$ a.e. $t \in [0, T]$, as the weak limit of $\chi_h$ is unique.

On the other hand, from (27), (29) and $\hat{u}_h' = z_h$, it follows that $\hat{u}_h \to u$ weakly in $W^{1,2}(0, T; V)$. Note that $W^{1,2}(0, T; V) \subset W$, and the embedding $W \subset L^2(0, T; Z)$ is compact by the well-known Aubin-Lions theorem [1, Theorem 2.141]. Then, we have $\tilde{u}_h \to u$ in $L^2(0, T; Z)$, and thus $u_h \to u$ in $L^2(0, T; Z)$ by Lemma 4.4 and $L^2(0, T; V) \subset L^2(0, T; Z)$. Further, $Lu_{th} \to Lu$ in $L^2(0, T; E)$ as $L$ is continuous. Passing to a subsequence we have $Lu_{th}(t) \to Lu(t)$ in $E$ a.e. $t \in [0, T]$. Using (29) and the upper semicontinuity of the Clarke subdifferential operator, we have $\eta(t) \in \partial_{v,J}J(Lu(t))$ a.e. $t \in [0, T]$ by using Proposition 2 in [12].

Recalling that $\hat{u}_h \to u$ weakly in $W^{1,2}(0, T; V)$ and $W^{1,2}(0, T; V) \subset C([0, T]; V)$, we obtain $\hat{u}_h(t) \to u(t)$ weakly in $V$ for all $t \in [0, T]$. In particular, as $\hat{u}_h(T) = u_n^m$.
and \( \hat{u}_h(0) = u_0 \), we have
\[
0 = u(0), \quad u^m_h \to u(T) \text{ weakly in } V. \quad (34)
\]

Finally, we aim to prove that \( \xi(t) \in B(u(t)) \) a.e. \( t \in [0,T] \). Multiplying the equation (15) by \( u_h - u \) to get
\[
\langle \chi_h + \xi_h + u_h - u \rangle = \langle f_h, u_h - u \rangle. \quad (35)
\]

Since \( Lu_h \to Lu \) in \( L^2(0,T;E) \), from (32) we obtain
\[
\langle w_h, u_h - u \rangle = \langle L^* \eta_h, u_h - u \rangle = \langle \eta_h, Lu_h - Lu \rangle \to 0 \text{ as } h \to 0. \quad (36)
\]

From (27) and \( f_h \to f \) in \( L^2(0,T;V^*) \), we get
\[
\langle f_h, u_h - u \rangle \to 0 \text{ as } h \to 0. \quad (37)
\]

Moreover, using \( H(A) \) we have
\[
\langle \chi_h, u_h \rangle = h \sum_{n=0}^{m-1} \langle Az_h^{n+1}, u_h^{n+1} \rangle
\]
\[
= \sum_{n=0}^{m-1} \langle A(u_h^{n+1} - u_h^n), u_h^{n+1} \rangle
\]
\[
= \sum_{n=0}^{m-1} \langle Au_h^{n+1}, u_h^{n+1} - u_h^n \rangle
\]
\[
\geq \sum_{n=0}^{m-1} \langle \Psi(u_h^{n+1}) - \Psi(u_h^n) \rangle
\]
\[
= \Psi(u_h^m) - \Psi(u_h^0)
\]
\[
= \Psi(u^m_h) - \Psi(u_0). \quad (38)
\]

Since \( u \in W^{1,2}(0,T;V) \), the potential \( \Psi(u(t)) \) is absolutely continuous with respect to \( t \) and \( \frac{d}{dt} \Psi(u(t)) = \langle Au(t), u'(t) \rangle \) a.e. \( t \in [0,T] \) (see, e.g. [21, Lemma 1]), we have
\[
\langle Au, u' \rangle = \int_0^T \langle Au(t), u'(t) \rangle dt = \Psi(u(T)) - \Psi(u_0), \quad (39)
\]
where \( u(0) = u_0 \) is used. By the same way, we find
\[
\langle A\tilde{u}_h', \tilde{u}_h \rangle = \langle A\tilde{u}_h, \tilde{u}_h \rangle = \Psi(\tilde{u}(T)) - \Psi(\tilde{u}(0)) = \Psi(u^m_h) - \Psi(u_0). \quad (40)
\]

Then, using (30), (39) and \( \chi = Au' \) we have
\[
\lim_{h \to 0} \langle \chi_h, u \rangle = \langle \chi, u \rangle = \langle Au', u \rangle = \langle Au, u' \rangle = \Psi(u(T)) - \Psi(u_0). \quad (41)
\]

On the other hand, from \( \tilde{u}_h' = z_h, \chi_h = Az_h \) and (40), it follows
\[
\langle \chi_h, \tilde{u}_h \rangle = \langle Az_h, \tilde{u}_h \rangle = \langle A\tilde{u}_h, \tilde{u}_h \rangle = \Psi(u^m_h) - \Psi(u_0). \quad (42)
\]

Using (30), (34), (40), (42), and Lemma 4.4, we see that
\[
\liminf_{h \to 0} \langle \chi_h, u_h \rangle = \liminf_{h \to 0} \langle \chi_h, u_h - \hat{u}_h + \tilde{u}_h \rangle
\]
\[
= \liminf_{h \to 0} \langle \chi_h, u_h - \hat{u}_h \rangle + \liminf_{h \to 0} \langle \chi_h, \tilde{u}_h \rangle
\]
\[
= \liminf_{h \to 0} (\Psi(u^m_h) - \Psi(u_0)) \geq \Psi(u(T)) - \Psi(u_0). \quad (43)
\]
Now, from (41) and (43) we obtain
\[
\lim_{h \to 0} \inf (\chi_h, u_h - u) = \lim_{h \to 0} \inf (\chi_h, u_h) - \lim_{h \to 0} (\chi_h, u) \geq 0.
\]  
(44)

Combining (35), (36), (37) and (44), we have
\[
\limsup_{h \to 0} (\xi_h, u_h - u) \leq 0.
\]  
(45)

Thus, from (31) it follows
\[
\limsup_{h \to 0} (\xi_h, u_h) \leq (\xi, u).
\]  
(46)

Since \(B\) is a maximal monotone operator from \(V\) to \(V^*\), it generates a Nemitsky operator \(\mathcal{B}\) from \(L^2(0, T; V)\) to \(L^2(0, T; V^*)\) which is maximal monotone too; see, e.g., [4, Lemma 21]. As \(\xi_h(t) \in Bu_h(t)\) a.e. \(t \in [0, T]\), we have \(\xi_h \in Bu_h\). This, together with \(\xi_h \rightharpoonup \xi\) weakly in \(L^2(0, T; V^*)\), \(u_h \to u\) weakly in \(L^2(0, T; V)\), (46) and the maximal monotonicity of \(\mathcal{B}\) implies \(\xi \in Bu\), i.e., \(\xi(t) \in Bu(t)\) a.e. \(t \in [0, T]\) (cf. [4, Lemma 21], [3, Proposition 1.3.65]). The proof of Theorem 3.2 is complete. □

**Remark 1.** Theorem 3.2 is established without a smallness condition related to the coefficients \(a_2\) and \(c_1\) (cf. [12, assumption \((H_1)\) and the condition of the coefficients in Theorem 17]).

**Remark 2.** Theorem 3.2 could be developed in case that \(A\) and \(\partial_d J\) in (1) are replaced by \(A(t)\) and \(\partial_d J(t)\), respectively. We refer readers to [19, 20] for the technique of dealing with time-dependent operators in Rothe method.

6. Remarks on applications to frictional contact problems. Consider an open bounded subset \(\Omega \subset \mathbb{R}^d\) \((d = 2, 3)\) which represents a deformable viscoelastic body. The boundary \(\Gamma\) of \(\Omega\) is supposed to be Lipschitz continuous, and consists of three mutually disjoint measurable parts \(\Gamma_D, \Gamma_C\) and \(\Gamma_N\) such that \(m(\Gamma_D) > 0\) and \(\Gamma_C\) is the contact boundary. Denote by \(\mathcal{S}_d\) the space \(\mathbb{R}^{d \times d}\) of symmetric matrices of order \(d\). Let \(V = \{v \in H^1(\Omega; \mathbb{R}^d) : v = 0 \text{ on } \Gamma_D\}\), \(Z = H^{1/2}(\Omega; \mathbb{R}^d), H = L^2(\Omega; \mathbb{R}^d), \text{ and } E = L^2(\Gamma_C; \mathbb{R}^d)\). We put \(L = \gamma \in L(Z, E)\), where \(\gamma\) is the trace operator. Denote by \(\varepsilon(u) = \frac{1}{2} \nabla u + \frac{1}{2} (\nabla u)^T\) the linearized (small) strain tensor. We consider the quasistatic motion equation: \(-\text{Div } \sigma = f\), where \(\sigma\) is the stress tensor and \(\text{Div}\) is the divergence operator. We refer readers to [12] for details of the aforementioned spaces, physical setting, weak formulations of the model and mathematical derivation of the inclusion (1). More information can be also found in the monographs [14, 6] and the references therein. For short, we merely make some comments on the operator \(B\) in inclusion (1).

First of all, suppose the stress tensor \(\sigma\) is described by the Kelvin-Voigt constitutive law for viscoelastic material, i.e., \(\sigma = \mathcal{C}(\varepsilon(u')) + \mathcal{G}(\varepsilon(u))\) where \(\mathcal{C}\) and \(\mathcal{G}\) denote the viscosity and elasticity operators, respectively. We claim that \(\mathcal{G}\) is nonlinear in this work. A concrete example is as follows. Let \(\mathcal{G}\), independent of time \(t\), be given by
\[
\mathcal{G}(\varepsilon(u)) : \varepsilon(v) = (\varepsilon(u), \varepsilon(v))_{L^2(\Omega; \mathbb{R}^d)} + \tau (\|\varepsilon(u)\|_{L^2(\Omega; \mathbb{R}^d)}^{p-2} \varepsilon(u), \varepsilon(v))_{L^2(\Omega; \mathbb{R}^d)}
\]  
(47)

for all \(u, v \in V\) where \(1 < p < 2\) and \(\tau > 0\) is a parameter. We define \(B : V \to V^*\) as
\[
\langle B(u), v \rangle_{V^* \times V} = \mathcal{G}(\varepsilon(u)) : \varepsilon(v) \quad \forall u, v \in V.
\]  
(48)

In this case the convex functional \(\Phi\) is given by
\[
\Phi(u) = \|u\|^2_V + \tau \int_{\Omega} |\varepsilon(u)|_{L^2(\mathbb{R}^d)}^p dx,
\]  
(49)
which is G-differentiable and $\partial \Phi (u) = \Phi'_G(u) = B(u)$ for all $u \in V$. Obviously, the condition (7) holds with $b_1 = 1$ and $b_2 = 0$. Note that the operator $\mathcal{G}$ defined by (47) is neither linear nor Lipschitz continuous, and the second part on the right-hand side of (47) can be seen a nonlinear perturbation.

Secondly, we consider the case $\Phi = \Phi_1 + \Phi_2$ introduced in (2) where $\Phi_1 : V \to \mathbb{R}$ is convex and G-differentiable; for example, $\Phi_1$ takes the functional in (49). Let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. We define $\Phi_2(u) = \varphi (Lu)$ for all $u \in V$. Then $\partial \Phi_2(u) = L^* \varphi_2(Lu)$ by the chain rule. It follows that $Bu = \Phi'_G(u) + L^* \partial \varphi_2(Lu)$. Therefore, (2) gives

$$\langle Au'(t) + B_1 u(t) - f(t), v - u(t) \rangle + \varphi (Lv) - \varphi (Lu(t)) + J'(Lu(t); L(v - u(t))) \geq 0,$$

for all $v \in V$ and a.e. $t \in [0, T]$. Instead of a quasistatic hemivariational inequality in [12], this is a variational-hemivariational inequality which can be used to deal with various monotone and nonmonotone conditions on the contact boundary $\Gamma_C$ since both a convex functional $\varphi$ and a locally Lipschitz functional $J$ on $L^2(\Gamma_C; \mathbb{R}^d)$ are involved.

Moreover, the fact that the convex functional $\Phi$ can take the value $+\infty$ is significant in applications. For example, let $J = 0$ and $\varphi$ be the indicator function on an appropriate convex and closed subset $K$ of $L^2(\Gamma_C; \mathbb{R}^d)$). Then the weak formulation of a viscoelastic unilateral contact problem with the Signorini boundary condition is included in the last inequality.

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