Counting horoballs and rational geodesics

Sa’ar Hersonsky Frédéric Paulin

August 3, 2018

Abstract

Let $M$ be a geometrically finite pinched negatively curved Riemannian manifold with at least one cusp. We study the asymptotics of the number of geodesics in $M$ starting from and returning to a given cusp, and of the number of horoballs at parabolic fixed points in the universal cover of $M$. In the appendix, due to K. Belabas, the case of $\text{SL}(2,\mathbb{Z})$ and of Bianchi groups is developed.

1 Introduction

Let $M$ be a non elementary geometrically finite pinched negatively curved Riemannian manifold with at least one cusp. A geodesic line starting from a given cusp $e$ is rational if it converges to $e$, and irrational if it accumulates inside $M$. Motivated by problems arising from diophantine approximation, we developed in [HP] a theory of approximation of irrational geodesics by rational ones.

As introduced in [HP], the depth $D(r)$ of a rational line $r$ is the length of the subsegment of $r$ between the first and last meeting point with the boundary of the maximal Margulis neighborhood of the cusp $e$. We proved in [HP] that the set of depths of rational lines is a discrete subset of $\mathbb{R}$ with finite multiplicities. So we may define the depth counting function $N_e : \mathbb{R} \rightarrow \mathbb{N}$, with $N_e(x)$ the number of rational geodesics whose depth is less than $x$.

Let $\tilde{M}$ be a fixed universal cover of $M$, with covering group $\Gamma$, and let $x_0$ be a base point in $\tilde{M}$. Recall that (see for example [Bou]) the Poincaré series of a discrete group $G$ of isometries of $\tilde{M}$ is

$$ P(s) = \sum_{g \in G} e^{-s d(x_0, gx_0)} $$

for any $s$ in $\mathbb{R}$. This series converges if $s > \delta_G$ and diverges if $s < \delta_G$ for some $\delta_G$ which is independant of $x_0$. Moreover, $0 < \delta_G < +\infty$ and $\delta_G$ is called the critical exponent of $G$. We say that $G$ is divergent if $P(\delta_G)$ diverges.

Choose a parabolic fixed point $\xi_0$ on the boundary $\partial \tilde{M}$ of $\tilde{M}$, corresponding to $e$. Let $\Gamma_0$ be its stabilizer. Recall that if $\tilde{M}$ is a rank 1 symmetric space (of non compact type), or if $\Gamma_0$ is divergent, then $\delta_G > \delta_{\Gamma_0}$ (see [DOP]. They also give an interesting example where equality holds).

Theorem 1.1 If $\delta_\Gamma > \delta_{\Gamma_0}$, then $\limsup_{x \to +\infty} \frac{\log N_e(x)}{x} = \delta_\Gamma$.

AMS codes: 53 C 22, 11 J 06, 30 F 40, 11 J 70. Keywords: rational geodesic, negative curvature, cusp, horoball.
This result on the asymptotic growth of the depths of rational lines is easily deduced from the following statement. We define the relative Poincaré series of \((\Gamma, \Gamma_0)\) as
\[
P_0(s) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0} e^{-s d(H_0, \gamma H_0)}
\]
for any \(s\) in \(\mathbb{R}\), where \(H_0\) is any fixed horosphere centered at \(\xi_0\). It is easy to see that \(d(H_0, \gamma H_0)\) depends only on the double coset of \(\gamma\), and that the convergence or divergence of the relative Poincaré series does not depend on the choice of \(H_0\).

**Theorem 1.2** If \(\delta_\Gamma > \delta_{\Gamma_0}\), then \(P_0(s)\) converges if and only if \(P(s)\) converges.

These results apply in particular to the arithmetic hyperbolic manifolds or orbifolds. For example, let \(\mathcal{O}\) be the ring of integers of a number field \(K\), having finite group of units \(\mathcal{O}^*\) (i.e. \(K\) is \(\mathbb{Q}\) or an imaginary quadratic number field \(\mathbb{Q}(\sqrt{-d})\)). Let \(N\) be the norm on \(K\), i.e. \(N(x) = x\) if \(K = \mathbb{Q}\) and \(N(x) = |x|^2\) if \(K = \mathbb{Q}(\sqrt{-d})\). Applying Theorem 1.1 to the modular orbifold \(\mathbb{H}^2 / \text{PSL}_2(\mathbb{Z})\) if \(K = \mathbb{Q}\) or to the Bianchi orbifold \(\mathbb{H}^3 / \text{PSL}_2(\mathcal{O})\) if \(K = \mathbb{Q}(\sqrt{-d})\), one gets

**Corollary 1.3** Let \(\varphi_{\mathcal{O}}(x)\) be the cardinal of the set of \(\frac{p}{q}\) mod \(\mathcal{O}\) with \(p, q\) in \(\mathcal{O}\), \((p, q) = 1\) and \(0 < N(q) \leq x\). Then
\[
\limsup_{x \to +\infty} \frac{\log \varphi_{\mathcal{O}}(x)}{\log x} = 2.
\]

This result is well-known for \(K = \mathbb{Q}\), where \(\varphi_{\mathbb{Z}}(x) = \sum_{k=0}^{x} \phi(k)\) and \(\phi\) is the Euler function. A much precise result than Corollary 1.3 is given in the Appendix, due to K. Belabas. Using the techniques of the Appendix when \(r_1 + r_2 = 1\), one can hope for an analogous result for other number fields, in connection with the counting of horoballs in \((\mathbb{H}^2)^{r_1} \times (\mathbb{H}^3)^{r_2}\) under the irreducible lattice \(\text{SL}_2(\mathcal{O})\) having \(\mathbb{Q}\)-rank 1 and \(\mathbb{R}\)-rank \(r_1 + r_2\), where \(r_1\) (resp. \(r_2\)) is the number of real (resp. complex) embeddings of \(K\). This will be developed later.

**Acknowledgement.** We are grateful to K. Belabas who provided the proof of Proposition 4.1, as well as to E. Fouvry and R. de la Bretèche. S. Hersonsky is grateful to the IHES and to the University of Orsay for their hospitality and financial support.

## 2 Definitions

We will use the notations and definitions of [11], that we recall here briefly for the sake of completeness. We refer to [11] for proofs and comments on these notions.

Let \(M\) be a (smooth) complete Riemannian \(n\)-manifold with pinched negative sectional curvature \(-b^2 \leq K \leq -a^2 < 0\). Fix a universal cover \(\tilde{M}\) of \(M\), with covering group \(\Gamma\).

The boundary \(\partial M\) of \(M\) is the set of asymptotic classes of geodesic rays in \(\tilde{M}\). The space \(\tilde{M} \cup \partial \tilde{M}\) is endowed with the cone topology. The limit set \(\Lambda(\Gamma)\) is the set \(\overline{\Gamma x} \cap \partial \tilde{M}\), for any \(x\) in \(\tilde{M}\). Let \(C\Lambda(\Gamma)\) be the convex hull of the limit set of \(\Gamma\).

A point \(\xi\) in \(\Lambda(\Gamma)\) is a conical limit point of \(\Gamma\) if it is the endpoint of a geodesic ray in \(\tilde{M}\) which projects to a geodesic in \(M\) that is recurrent in some compact subset. A point
\( \xi \) in \( \Lambda(\Gamma) \) is a **bounded parabolic point** if it is fixed by some parabolic element in \( \Gamma \), and if the quotient \( (\Lambda(\Gamma) - \{\xi\})/\Gamma_\xi \) is compact, where \( \Gamma_\xi \) is the stabilizer of \( \xi \).

We assume that the group \( \Gamma \) is **geometrically finite**, i.e. that every limit point of \( \Gamma \) is conical or bounded parabolic (see \cite{Bow} for more details). We also assume that \( \Gamma \) is **non elementary**, i.e. that its limit set contains at least 3 points.

Assume that \( M \) has at least one **cusp** \( e \), i.e. an asymptotic class of minimizing geodesic rays in \( M \) along which the injectivity radius goes to 0. We say that a geodesic ray **converges** to \( e \) if some subray belongs to the class \( e \).

Choose a parabolic fixed point \( \xi_0 \) on the boundary \( \partial \widetilde{M} \) of \( \widetilde{M} \), which is the endpoint of a lift of a geodesic ray converging to \( e \). Let \( \Gamma_0 \) be its stabilizer in \( \Gamma \). Let \( H_0 \) be the horosphere centered at \( \xi_0 \) such that the horoball \( H B_0 \) bounded by \( H_0 \) is the maximal horoball centered at \( \xi_0 \) such that the quotient of its interior by \( \Gamma_0 \) embeds in \( M \) under the canonical map \( \widetilde{M} \rightarrow M \). The subset \( \text{int}(H B_0)/\Gamma_0 \) of \( M \) is called the maximal Margulis neighborhood of the cusp \( e \).

Since the convergence or divergence of the Poincaré series does not depend on the base point \( x_0 \), we may assume that \( x_0 \) belongs to \( H_0 \cap \text{int}(\Lambda(\Gamma)) \). Since the convergence or divergence of the relative Poincaré series does not depend on the choice of the horosphere, we will use this \( H_0 \) in the expression of \( P_0(s) \) in all that follows.

Any rational geodesic \( r \) has a lift starting from \( \xi_0 \), which is unique modulo the action of \( \Gamma_0 \). The endpoint of any such lift is the center of an horosphere \( \gamma H_0 \) for some \( \gamma \) in \( \Gamma \). It follows from its definition that the depth of \( r \) is \( d(H_0, \gamma H_0) \). We proved in \cite[Lemma 2.7]{HP} that the map \( r \mapsto \Gamma_0 \gamma \Gamma_0 \) from the set of rational geodesics to the set of double cosets \( \Gamma_0 \gamma \Gamma_0 \) is a bijection. In particular the number \( N_e(x) \) of rational geodesics with depth at most \( x \) is

\[
N_e(x) = \text{Card}\{\Gamma_0 \gamma \Gamma_0 \in \Gamma_0 \gamma \Gamma_0 \mid d(H_0, \gamma H_0) \leq x}\).
\]

The fact that the relative Poincaré series \( P_0(s) \) converges for \( s > \limsup_{x \to +\infty} \frac{\log N_e(x)}{x} \) and diverges if \( s < \limsup_{x \to +\infty} \frac{\log N_e(x)}{x} \) is then easily seen. In particular, Theorem \ref{thm:1} follows from Theorem \ref{thm:1.2}.

Let \( \mathcal{O} \) be the ring of integers of a number field \( K \), where \( K \) is either \( \mathbb{Q} \) or an imaginary quadratic number field \( \mathbb{Q}(\sqrt{-d}) \), with \( d \) a positive square free integer. We use the upper half-space models for the real hyperbolic spaces. Consider the cusp \( e \) in the orbifolds \( \mathbb{H}^2/\text{PSL}_2(\mathbb{Z}) \) if \( K = \mathbb{Q} \), or \( \mathbb{H}^3/\text{PSL}_2(\mathcal{O}) \) otherwise, corresponding to the parabolic fixed point \( +\infty \). We proved in \cite[Section 2.3]{HP} (with the obvious adaptation to the case of orbifolds) that the rational lines \( r \) are in one-to-one correspondence with the fractions \( \frac{p}{q} \) modulo the additive group \( \mathcal{O} \), with the depth of \( r \) being \( \log |q|^2 \), if this fraction is written with relative prime numerator and denominator. Hence Corollary \ref{cor:1.3} follows from Theorem \ref{thm:1.4}.

### 3 Proofs

With the notations and assumptions of the previous section, we start with two lemmata.

**Lemma 3.1** There exists a constant \( C_1 \geq 0 \), such that every double coset \( \Gamma_0 \gamma \Gamma_0 \) in \( \Gamma_0 \gamma \Gamma_0 \) has a representative \( \gamma \) which satisfies

\[
|d(H_0, \gamma H_0) - d(x_0, \gamma x_0)| \leq C_1.
\]
Proof. Choose the identity as the representative of the trivial double coset. Let $\gamma$ be in $\Gamma - \Gamma_0$. Let $p_0$ in $H_0$ and $p_1$ in $\gamma H_0$ be such that the segment $[p_0, p_1]$ is the (unique) common perpendicular to $H_0$ and $\gamma H_0$. In particular, $d(H_0, \gamma H_0) = d(p_0, p_1)$, and $p_0, p_1$ lie on the geodesic line between the centers of $H_0$ and $\gamma H_0$, so that $p_0, p_1$ both belong to the $\Gamma_0$-invariant subset $H_0 \cap C\Lambda(\Gamma)$. Since $\xi_0$ is a bounded parabolic fixed point, the quotient $(H_0 \cap C\Lambda(\Gamma)) / \Gamma_0$ is compact, hence has diameter bounded by $C'_1 \geq 0$.

Since $x_0$ belongs to $H_0 \cap C\Lambda(\Gamma)$, there exists $\alpha$ in $\Gamma_0$ so that $d(p_0, \alpha x_0) \leq C'_1$ and $\beta$ in $\Gamma_0$ so that $d(\gamma^{-1} p_1, \beta x_0) \leq C'_1$.

Since $\alpha x_0$ lies on $H_0$ and $\gamma \beta x_0$ on $\gamma H_0$, we have
\[ d(H_0, \gamma H_0) = d(p_0, p_1) \leq d(\alpha x_0, \gamma \beta x_0). \]

Conversely, by the triangular inequality,
\[ d(\alpha x_0, \gamma \beta x_0) \leq d(\alpha x_0, p_0) + d(p_0, p_1) + d(p_1, \gamma \beta x_0) \leq 2C'_1 + d(H_0, \gamma H_0). \]

Hence the representative $\alpha^{-1} \gamma \beta$ of the double coset $\Gamma_0 \gamma \Gamma_0$ satisfies the condition of the Lemma with $C_1 = 2C'_1$.

Note that by discreteness, the set of representatives as in the Lemma of a given double coset is finite. From now on, we will denote by the same letter a double coset and such a representative.

The following proposition is well-known (see for example the proof of [DOP, Lemme 4]).

Lemma 3.2 There exists a constant $C_2 \geq 0$ (depending only on the upperbound on the curvature of $M$) such that the following holds. Let $H, H'$ be horospheres in $\bar{M}$ bounding disjoint horoballs. Let $[p, p']$ be the common perpendicular segment, with $p$ in $H$ and $p'$ in $H'$. For every $x$ in $H$ and $x'$ in $H'$,
\[ |d(x, x') - (d(x, p) + d(p, p') + d(x, p))| \leq C_2. \]

Figure 1: The quasi-geodesic.

Proof. Up to replacing $H, H'$ by inside concentric horospheres at distance 1, we may assume that $d(H, H') \geq 1$. By the convexity of horoballs, the piecewise geodesic $[x, p] \cup$
\[ [p, p'] \cup [p', x'] \text{ has angles at least } \frac{\pi}{2} \text{ at } p \text{ and at } p'. \] Thus, since \( d(p, p') \geq 1 \), it is a quasi-geodesic, and the result follows for example from \([C1]\), Chapter 3.

**Proof of Theorem 1.2** If \( f, g \) are maps from an interval \( I \) in \( \mathbb{R} \) to \( \mathbb{R} \cup \{+\infty\} \), write \( f \asymp g \) if there exists a finite constant \( c > 0 \) such that \( \frac{1}{c}g(s) \leq f(s) \leq cg(s) \) for all \( s \) in \( I \).

We write the Poincaré series as follows.

\[
P(s) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \sum_{\alpha, \beta \in \Gamma_0} e^{-s d(x_0, \alpha \gamma \beta x_0)}.
\]

Note that \( d(x_0, \alpha \gamma \beta x_0) = d(\alpha^{-1} x_0, \gamma x_0) \). The representatives \( \gamma \) of the (non trivial) double cosets have been chosen so that \( \gamma x_0 \) lies at distance less than a constant \( C_1' \) from the endpoint on \( \gamma H_0 \) of the common perpendicular segment between \( H_0 \) and \( \gamma H_0 \). Applying Lemma 3.1 and Lemma 3.2 we obtain

\[
P(s) \asymp \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \sum_{\alpha, \beta \in \Gamma_0} e^{-s d(H_0, \gamma H_0)} \left( \sum_{\alpha \in \Gamma_0} e^{-s d(x_0, \alpha x_0)} \right)^2.
\]

If \( s > \delta_{\Gamma_0} \), then the Poincaré series of \( \Gamma_0 \) converges at \( s \). Hence \( P \asymp P_0 \) on the interval \( ]\delta_{\Gamma_0}, +\infty[ \). Theorem 1.2 follows.

### 4 Appendix, by K. Belabas

For all undefined objects and unproved results in what follows, see \([Nar]\) for instance.

Let \( K \) be a number field, \( N = N_K \) the norm on \( K \), \( \zeta_K \) the Dedekind zeta function and \( \text{Res}_K = \text{Res}(\zeta_K, s = 1) \) the residue of \( \zeta_K \) at the unity, \( \mathcal{O} = \mathcal{O}_K \) the ring of integers of \( K \), \( \mathcal{O}^* \) the units of \( \mathcal{O} \), \( h_K \) the class number of \( K \), \( W_K \) the number of roots of unity in \( K \), \( R_K \) the regulator of \( K \), and \( D_K \) the discriminant of \( K \). Let \( \mu \) be the Möbius function, defined by \( \mu(\mathcal{O}) = 1 \), \( \mu(I) = (1)^k \) if the ideal \( I \) in \( \mathcal{O} \) is the product of \( k \) distinct prime ideals, and \( \mu(I) = 0 \) if \( I \) is divisible by the square of a prime ideal. It satisfies \( \sum_{I \mid J} \mu(I) = 1 \) if \( J = \mathcal{O} \), and 0 otherwise. Assume that \( \mathcal{O}^* \) is finite (\( K \) is \( \mathbb{Q} \) or an imaginary quadratic field), so that \( \text{Card}(\mathcal{O}^*) = W_K \) and \( R_K = 1 \).

For \( n \) a positive integer, define

\[
\phi(x) = \phi_{\mathcal{O}}(x) = \text{Card}\left\{ \frac{p}{q} \cap \mathcal{O}, (p, q) \in \mathcal{O} \times (\mathcal{O} \setminus \{0\}), (p, q) = 1 \text{ and } N(q) \leq x \right\}.
\]

The following proposition was explained to us by K. Belabas, who said it might be already known. For the sake of completeness, we include its proof here.

**Proposition 4.1** There exists \( \epsilon > 0 \) such that

\[
\phi(x) = \frac{\text{Res}_K}{2h_K \zeta_K(2)} x^2 + O(x^{2-\epsilon}).
\]
Proof (K. Belabas). Since two irreducible fractions are equal if and only if the numerators and denominators are multiplied by the same unit, one has

\[ \phi(x) = \frac{1}{W_K} \text{Card}\{(q,p \mod (q)), q \neq 0, (p,q) = 1 \text{ and } N(q) \leq x\}. \]

Hence \( W_K \phi(x) \) is equal to

\[ \sum_{(q,p \mod (q)), q\neq 0, N(q) \leq x, (p,q)=1} 1 = \sum_{(q,p \mod (q)), q\neq 0, N(q) \leq x} \sum_{I | (p,q)} \mu(I) = \sum_I \mu(I)f(I) \]

where \( I \) ranges over the ideals of \( \mathcal{O} \) and

\[ f(I) = \sum_{(q,p \mod (q)), q\neq 0, I | (p,q), N(q) \leq x} 1 = \sum_{q\in I, q\neq 0, N(q) \leq x} \sum_{p\in I/(q)} 1 = W_K \sum_{(q)\subset I, N(q) \leq x} \frac{N(q)}{N(I)} \]

since \( N(J) = \text{Card} \mathcal{O}/J \), where \( (q) \) is a non zero principal ideal, and since a generator of a principal ideal is uniquely defined up to units.

Lemma 4.2 If \( S(x) = \text{Card}\{(q) \subset I, N(q) \leq x\}, \) then \( S(x) = \frac{\text{Res}_K}{h_K N(I)} x^2 + O((\frac{x}{N(I)})^{1-\epsilon}) \).

Proof. Note that \( (q) \subset I \) if and only if \( (q) = IJ \) for some ideal \( J \) in \( \mathcal{O} \), and \( N(IJ) = N(I)N(J) \). Hence

\[ S(x) = \sum_{J \in [I]^{-1}, N(J) \leq \frac{x}{N(I)}} 1 \]

where \([I]^{-1}\) is the inverse of the class of \( I \) in the class group. With \( \epsilon = 1/[K : \mathbb{Q}] \), the result then follows from [Nar, theo. 7.6 page 361] (for example) for the main term and from [Iat] for the error term.

Lemma 4.3 If \( T(x) = \sum_{(q)\subset I, N(q) \leq x} N(q) \), then \( T(x) = \frac{\text{Res}_K}{2h_K N(I)} x^2 + O(\frac{x^2}{N(I)}) \).

Proof. This is immediate by applying Fubini and using the previous lemma.

Now, since \( \zeta_K(2-\epsilon) \) converges for \( \epsilon < 1 \), and since by [Nar, page 326],

\[ \sum_I \frac{\mu(I)}{N^s(I)} = \frac{1}{\zeta_K(s)}, \]

one gets from Lemma 4.3

\[ \phi(x) = \sum_I \mu(I) \frac{\text{Res}_K}{2h_K N^2(I)} x^2 + O(x^{2-\epsilon}) = \frac{\text{Res}_K}{2h_K \zeta_K(2)} x^2 + O(x^{2-\epsilon}). \]

Since \( \text{Res}_K = \frac{2^{r_1(2x)^2 h_K R_K}}{W_K \sqrt{|D_K|}} \), one has for \( K = \mathbb{Q}(\sqrt{-d}) \), with \( D = d \) if \( d \equiv 3 \mod 4 \) and \( D = 4d \) otherwise, with \( w = 4, 6 \) if \( d = 1, 3 \) and \( w = 2 \) otherwise,

\[ \phi_{\mathbb{Q}(\sqrt{-d})}(x) = \frac{\pi}{w \zeta_{\mathbb{Q}(\sqrt{-d})}(2) \sqrt{D}} x^2 + o(x^2). \]

For \( K = \mathbb{Q} \), the formula gives the well-known result (see [Nar] for instance)

\[ \phi_{\mathbb{Z}}(x) = \frac{3}{\pi^2} x^2 + o(x^2). \]
References

[Bou] M. Bourdon, \textit{Structure conforme au bord et flot géodésique d’un CAT(-1) espace}, L’Ens. Math. \textbf{41} (1995) 63-102.

[Bow] B. Bowditch, \textit{Geometrical finiteness with variable negative curvature}, Duke Math. J. \textbf{77} (1995), 229-274.

[DOP] F. Dal’bo, J.-P. Otal, M. Peigné, \textit{Séries de Poincaré des groupes géométriquement fini}, preprint Univ. Rennes, July 1998.

[GH] E. Ghys, P. de la Harpe, eds. \textit{Sur les groupes hyperboliques d’après Mikhael Gromov}, Prog. in Math. \textbf{83}, Birkhäuser 1990.

[HP] S. Hersonsky, F. Paulin, \textit{Diophantine approximation for negatively curved manifolds, I}, preprint Univ. Orsay, Sept. 1999.

[Nar] W. Narkiewicz, \textit{Elementary and analytic theory of algebraic numbers}, 2nd ed., Polish Scien. Pub. (Springer Verlag) 1990.

[Tat] T. Tatzuawa, \textit{On the number of integral ideals in algebraic number fields whose norms do not exceed } $x$, Sci. Papers Coll. Gen. Edu. Univ. Tokyo \textbf{23} (1973) 73-86.

Department of Mathematics
Ben Gurion University
BEER-SHEVA, Israel
\textit{e-mail:} saarh@math.bgu.ac.il

Laboratoire de Mathématiques UMR 8628 CNRS
Equipe de Topologie et Dynamique (Bât. 425)
Université Paris-Sud
91405 ORSAY Cedex, FRANCE.
\textit{e-mail:} Frederic.Paulin@math.u-psud.fr