On a convexity problem

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Abstract

This work is a continuation of what was done in [3] and it is strongly connected to the work done in [1].

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1. Introduction

Let \( n \in \mathbb{N} \) and let \( \Pi_n \) denote the set of all polynomials of degree \( \leq n \). The fundamental Bernstein polynomials of degree \( n \) are given by:

\[
b_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \quad k = 0, 1, \ldots, n.
\]

In ([8], Problem 2, pp. 164), I. Raşa ([8], Problem 2, pp. 164), came up with the following problem: Prove or disprove the following inequality:

\[
\sum_{i=0}^{n} \sum_{j=0}^{n} \left[ b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y) \right] f \left( \frac{i+j}{2n} \right) \geq 0, \quad (1)
\]

for any convex function \( f \in C[0,1] \) and any \( x, y \in [0,1] \). In [7], by using a probabilistic approach, J. Mrowiec, T. Rajba and S. Wąsowicz, gave a positive answer to the above problem and proved the following generalization of inequality (1).
Theorem 1.1 ([7], Theorem 12). Let $m, n \in \mathbb{N}$ with $m \geq 2$. Then,
\[ \sum_{i_1, \ldots, i_m=0}^{n} [b_{n,i_1}(x_1)\ldots b_{n,i_m}(x_1) + \ldots + b_{n,i_1}(x_m)\ldots b_{n,i_m}(x_m)] - mb_{n,i_1}(x_1)\ldots b_{n,i_m}(x_m)] f \left( \frac{i_1 + \ldots + i_m}{mn} \right) \geq 0, \quad (2) \]
for any convex function $f \in C[0,1]$ and any $x_1, \ldots, x_m \in [0,1]$.

An elementary proof of (1), was given recently by Abel in [2], where it is shown that a type (1) inequality holds also for the Mirakyan-Favard-Szász ([2], Theorem 5) and for the Baskakov operators ([2], Theorem 6).

In [3], we proved a type (1) inequality for a large class of operators defined in the following way. Let $I$ be one of the intervals $[0, \infty)$ or $[0, 1]$. Let $g_n : I \times D \rightarrow \mathbb{C}$, $D = \{z \in \mathbb{C} | |z| \leq R\}$, $R > 1$ be a function with the property that for any fixed $x \in I$, the function $g_n(x, \cdot)$ is an analytic function on $D$,
\[ g_n(x, z) = \sum_{k=0}^{\infty} a_{n,k}(x)z^k \]
\[ a_{n,k}(x) \geq 0, \forall k \geq 0 \]
\[ g_n(x, 1) = 1, \forall x \in I. \]

In what follows, let $I = [0, \infty)$. The case $I = [0, 1]$ follows in the same way. Let $\mathcal{F}$ be a linear set of functions defined on the interval $I$ and let $\{A_t\}_{t \in I}$ be a set of real linear positive functionals defined on $\mathcal{F}$ with the property that for any $f \in \mathcal{F}$, the series
\[ L_{n,A}(f)(x) := \sum_{k=0}^{\infty} a_{n,k}(x)A_\frac{k}{n}(f). \]
is convergent for any $x \in I$. The identity (1) defines a positive linear operator. The function $g_n$ will be referred to as the generating function for the operator $L_{n,A}$ relative to the set of functionals $\{A_t\}_{t \in I}$.

In what follows, we assume that the linear positive functionals $\{A_t\}_{t \in I}$ are such that $L_{n,A}$ is well defined for any $f \in \mathcal{F}$ and any $x \in I$, the set of all real polynomials $\Pi \subseteq \mathcal{F}$ and every functional $A_t$ has the following properties:

i) $A_t(e_0) = 1, t \in I$
ii) \( A_t(e_1) = at + b, t \in I \), where \( a \) and \( b \) are two real numbers independent of \( t \) and \( e_i(x) = x^i, x \in I, i \in \mathbb{N} \).

In [3], we obtained the following result: if

\[
\left[ \frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n} ; A_t(f) \right] \geq 0
\]

and

\[
\frac{d^k}{dz^k} \left[ \frac{g_n(x,z) - g_n(y,z)}{z-1} \right]_{z=0}^2 \geq 0,
\]

for any \( k \in \mathbb{N} \) and all \( x, y \in I \), then \( A(f) \geq 0 \). Here, for \( x, y \in I \) fixed, the functional \( A \) is defined by

\[
A(f) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [a_n,i(x)a_n,j(y) + a_n,i(y)a_n,j(x) - 2a_n,i(x)a_n,j(y)] A_{i+j}^{n}(f),
\]

The following result ([3], Corollary 3.2) is useful to verify inequality (6). Let \( x, y \in I \) be two distinct numbers. Assume that conditions i) and ii) above hold,

\[
\frac{g_n(x,z) - g_n(y,z)}{z-1} = \sum_{k=0}^{\infty} \beta_{n,k}(x,y) z^k
\]

and \( \text{sgn } \beta_{n,k}(x,y) \) is the same for all \( k \in \mathbb{N} \), then (6) is satisfied.

For \( m \in \mathbb{N}, m \geq 2 \) and \( x \in I^m, x = (x_1, \ldots, x_m) \), we define the functionals:

\[
C_m(f) = \sum_{i_1, \ldots, i_m=0}^{\infty} [a_{n,i_1}(x_1) \ldots a_{n,i_m}(x_1) + \ldots + a_{n,i_1}(x_m) \ldots a_{n,i_m}(x_m)
- ma_{n,i_1}(x_1) \ldots a_{n,i_m}(x_m)] A_{i_1+\ldots+i_m}^{n}(f)
\]

In [3], Theorem 4.1, we have proved the following result: If (5) and (6) hold, then

\[
C_m(f) \geq 0
\]

for any \( m \in \mathbb{N}, m \geq 2 \).

Applications, such as Bernstein type operators, Mirakyan-Favard-Szász type operators, Meyer-König and Zeller type operators, were considered in [3].
Let us assume that the generating functions \( g_n, n \in \mathbb{N}^* \) are of the form
\[
g_n(x, z) = \phi^n(x, z),
\]
where \( \phi : I \times D \to \mathbb{C} \) is such that \( \phi(x, \cdot) \) is an analytic function and the function \( g_n \) given by (9) satisfies conditions (3). Under these assumptions, we have
\[
\sum_{i_1 + \ldots + i_m = k} a_{n,i_1}(x) \ldots a_{n,i_m}(x) = a_{nm,k}(x).
\]
(10)

The above identity implies that
\[
C_m(f) = \sum_{k=1}^{m} L_{mn,A}(f)(x_k) - m \sum_{i_1, \ldots, i_m = 0}^{\infty} a_{i_1}(x_1) \ldots a_{i_m}(x_m) f\left(\frac{i_1 + \ldots + i_m}{mn}\right)
\]
(11)

Let us assume that the sequence \((L_{n,A})_{n \in \mathbb{N}^*}\) preserves convexity. More precisely, we assume that for every convex function \( f \in \mathcal{F} \), \( L_{n,A}(f), n \in \mathbb{N}^* \) is convex too. Under this assumption, we have
\[
L_{nm,A}(f)\left(\frac{x_1 + \ldots + x_m}{m}\right) \leq \sum_{k=1}^{m} L_{nm,A}(f)(x_k)
\]
(12)

For the Bernstein operators, in [1], the following problem was studied:

Prove that
\[
B_{2n}(f)\left(\frac{x + y}{2}\right) \geq \sum_{i=0}^{n} \sum_{j=0}^{n} b_{n,i}(x)b_{n,j}(x)f\left(\frac{i+j}{2n}\right),
\]
(13)

for all convex \( f \in C[0,1] \) and \( x, y \in [0,1] \).

A probabilistic solution was found by A. Komisarski and T. Rajba. [5]. In [1], U. Abel and I. Raşa gave an analytic proof to the following theorem.

**Theorem 1.2 ([1], Theorem 1).** Let \( n, m \in \mathbb{N} \). If \( f \in C[0,1] \) is a convex function, then the inequality
\[
B_{mn}(f)\left(\frac{1}{m} \sum_{\nu=1}^{m} x_{\nu}\right) \geq \sum_{i_1=0}^{n} \ldots \sum_{i_m=0}^{n} \left(\prod_{\nu=1}^{m} b_{n,i_{\nu}}(x_{\nu})\right)f\left(\frac{1}{mn} \sum_{\nu=1}^{m} i_{\nu}\right)
\]
is valid for all \( x_1, \ldots, x_m \in [0,1] \).
The purpose of this paper is to give sufficient conditions for the generating functions \( g_n, n \in \mathbb{N} \), such that the functional \( \mathbb{B}_m : \mathcal{F} \to \mathbb{R} \),

\[
\mathbb{B}_m(f) = L_{mn,A}(f) \left( \frac{x_1 + \ldots + x_m}{m} \right) - \sum_{i_1,\ldots,i_m=0}^{\infty} a_{i_1}(x_1)\ldots a_{i_m}(x_m) A_{i_1+\ldots+i_m}(f)
\]

is nonnegative for any function \( f \in \mathcal{F} \) for which

\[
\left[ \begin{array}{c} k+n, k+1+n, k+2+n \vdots A_r(f) \end{array} \right] \geq 0
\]

and for any \( x = (x_1,\ldots,x_m) \in I^m \) and any \( k \in \mathbb{N} \). It is immediate to see, from (11), that if \( \mathbb{B}_m(f) \geq 0 \), then \( C_m(f) \geq 0 \) as well.

2. Main results

**Theorem 2.1.** Let \( f \in \mathcal{F} \) be such that inequality (15) holds. If

\[
\frac{d^k}{dz^k} \left[ \frac{g_{nm}(x_1+\ldots+x_m,z) - g_n(x_1,z)\ldots g_n(x_m,z)}{z-1} \right] \bigg|_{z=0} \geq 0
\]

for any \( k \in \mathbb{N} \) and any \( x = (x_1,\ldots,x_m) \in I^m \), then

\[ \mathbb{B}_m(f) \geq 0. \]

If the reverse of (16) holds for any \( k \in \mathbb{N} \) and any \( x = (x_1,\ldots,x_m) \in I^m \), then

\[ \mathbb{B}_m(f) \leq 0. \]

**Proof.** We note that

\[ \mathbb{B}_m(e_0) = \mathbb{B}_m(e_1) = 0. \]

On the other hand, we have

\[
\mathbb{B}_m(f) = L_{mn,A}(f) \left( \frac{x_1 + \ldots + x_m}{m} \right) - \sum_{k=0}^{\infty} \alpha_{n,k}(x) A_{\frac{k}{mn}}(f),
\]

where

\[
\alpha_{n,k}(x) = \sum_{i_1+\ldots+i_m=k} a_{n,i_1}(x_1)\ldots a_{n,i_m}(x_m).
\]
So
\[ B_m(f) = \sum_{k=0}^{\infty} \left[ a_{mn,k} \left( \frac{x_1 + \ldots + x_m}{m} \right) - \sum_{i_1 + \ldots + i_m = k} a_{n,i_1}(x_1) \ldots a_{n,i_m}(x_m) \right] f \left( \frac{k}{m+n} \right). \]

We note that
\[ g_{mn} \left( \frac{x_1 + \ldots + x_m}{m}, z \right) - g_n(x_1, z) \ldots g_n(x_m, z) = \sum_{k=0}^{\infty} \left[ a_{mn,k} \left( \frac{x_1 + \ldots + x_m}{m} \right) - \sum_{i_1 + \ldots + i_m = k} a_{n,i_1}(x_1) \ldots a_{n,i_m}(x_m) \right] z^k. \tag{17} \]

From (17), we get
\[ a_{mn,k} \left( \frac{x_1 + \ldots + x_m}{m} \right) - \sum_{i_1 + \ldots + i_m = k} a_{n,i_1}(x_1) \ldots a_{n,i_m}(x_m) = \frac{1}{2\pi} \int_0^{2\pi} \left[ g_{mn} \left( \frac{x_1 + \ldots + x_m}{m}, e^{i\theta} \right) - g_n(x_1, e^{i\theta}) \ldots g_n(x_m, e^{i\theta}) \right] e^{-ik\theta} d\theta, \tag{18} \]
for any \( k \in \mathbb{N}. \) From (18), by using the same technique as in the proof of Theorem 4.1, [3], we get
\[
B_m(f) = \frac{2}{nm} \sum_{k=2}^{\infty} B_m \left( \left| - k - \frac{1}{mn} \right| \right) \left[ \frac{k-2}{mn}, \frac{k-1}{mn}, \frac{k}{mn}; A_t(f) \right], \tag{19}
\]
where
\[
B_m \left( \left| - k - \frac{1}{mn} \right| \right) = \frac{1}{nm} \frac{1}{(k-2)!} d^{k-2} E_m^2(x, z) \bigg|_{z=0} \tag{20}
\]
and
\[ E_m(x, z) = g_{mn} \left( \frac{x_1 + \ldots + x_m}{m}, z \right) - g_n(x_1, z) \ldots g_n(x_m, z). \tag{21} \]

Equations (19), (20) and (7) conclude our proof.

In what follows we are interested in whether there exists a large class of linear positive operators for which \( A(f) \geq 0, \) whenever (5) and (6) are satisfied and \( B_m(f) \geq 0 \) or \( B_m(f) \leq 0. \)
Mastroianni type operators

We denote by \( C_2([0, \infty)) \) the function space

\[
C_2([0, \infty)) := \left\{ f \in C([0, \infty)) : \exists \lim_{x \to \infty} \frac{f(x)}{1 + x^2} < \infty \right\}.
\]

Let \((\varphi_n)_{n \in \mathbb{N}}\) be a sequence of real functions defined on \([0, \infty)\), \(\varphi_n \in C^\infty[0, \infty)\), \(n \in \mathbb{N}\) that are strictly monotone and satisfy the following conditions:

\[
\varphi_n(0) = 1, \quad n \in \mathbb{N} \quad (22)
\]
\[
(-1)^n \varphi_n^{(k)}(x) \geq 0, \quad n \in \mathbb{N}^*, \quad k \in \mathbb{N}, \quad x \geq 0 \quad (23)
\]
\[
\forall (n, k) \in \mathbb{N} \times \mathbb{N}, \exists p(n, k) \in \mathbb{N} \quad (24)
\]
\[
\exists \alpha_{n, k} : [0, \infty) \to \mathbb{R} \quad \text{such that} \quad \forall x \geq 0, \forall i \in \mathbb{N}^*, \quad \varphi_n^{(i+k)}(x) = (-1)^k \varphi_{p(n, k)}^{(i)}(x) \alpha_{n, k}(x) \quad (25)
\]

G. Mastroianni, in \([6]\), introduced for any \(n \in \mathbb{N}^*\), the operators \(M_n : C_2([0, \infty)) \to C([0, \infty))\), defined by

\[
M_n(f)(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \varphi_n^{(k)}(x) f \left( \frac{k}{n} \right).
\]

Let \((A_t)_{t \in I}\) be a set of linear positive functionals defined on the linear set of functions \(\mathcal{F}\), satisfying conditions i) and ii) above and such that for every \(f \in \mathcal{F}\), the series

\[
M_{n,A}(f)(x) := \sum_{k=0}^{\infty} (-1)^k \frac{x^k \varphi_n^{(k)}(x)}{k!} A_k \left( \frac{k}{n} \right) \quad (25)
\]

converges. We will assume that \(\Pi_2 \subseteq \mathcal{F}\).

**Remark.** If \(\mathcal{F} = C_2([0, \infty))\), then \(M_{n,A}(f)\) is well defined.

**Lemma 2.1.** If for any \(x \in [0, \infty)\), the function \(g_n(x, \cdot) = \varphi(x(1 - \cdot))(\text{ is analytic in } D = \{z \in \mathbb{C} : |z| < R\}, \ R > 1\), then \(g_n\) is a generating function for \(M_{n,A}\).

**Proof.** We have

\[
\frac{d^k}{dz^k} g_n(x, z) = (-1)^k x^k \varphi_n^{(k)}(x(1 - z))
\]

and therefore

\[ g_n(x, z) = \sum_{k=0}^{\infty} (-1)^k x^k \frac{\varphi_n^{(k)}(x)}{k!} z^k. \]

\[ \blacksquare \]

**Theorem 2.2.** Let \( x, y \in [0, \infty), \ x \neq y \). If

\[ \frac{g_n(x, z) - g_n(y, z)}{z - 1} = \sum_{k=0}^{\infty} \beta_{n,k}(x, y) z^k, \]

then \( \text{sgn} \beta_{n,k}(x, y) \) is the same for all \( k \in \mathbb{N} \).

**Proof.** We have

\[ \frac{g_n(x, z) - g_n(y, z)}{z - 1} = -\sum_{p=0}^{\infty} \frac{(-1)^p x^p \varphi_n^{(p)}(x) - (-1)^p y^p \varphi_n^{(p)}(y)}{p!} \sum_{m=0}^{\infty} z^m. \]

It follows that

\[ \beta_{n,k}(x, y) = -\sum_{p=0}^{k} \frac{(-1)^p x^p \varphi_n^{(p)}(x) - (-1)^p y^p \varphi_n^{(p)}(y)}{p!}. \quad (26) \]

Let us consider the function \( h_{n,k} : [0, \infty) \to \mathbb{R} \) defined by

\[ h_{n,k}(t) = -\sum_{p=0}^{k} \frac{(-1)^p t^p \varphi_n^{(p)}(t)}{p!}. \]

We have

\[ h'_{n,k}(t) = -\sum_{p=0}^{k} \frac{(-1)^p (p t^{p-1} \varphi_n^{(p)}(t) - (-1)^p t^p \varphi_n^{(p+1)}(t))}{p!} \]

\[ = \sum_{p=0}^{k-1} \frac{(-1)^p t^p \varphi_n^{(p+1)}(t)}{p!} - \sum_{p=0}^{k} \frac{(-1)^p t^p \varphi_n^{(p+1)}(t)}{p!} \]

\[ = \frac{(-1)^{p+1} t^p \varphi_n^{(p+1)}(t)}{p!} \geq 0, \ \forall t \in [0, \infty), \ \forall p \in \mathbb{N}. \]
But

\[ \beta_{n,k}(x, y) = h_{n,k}(x) - h_{n,k}(y) \]

and therefore

\[ \text{sgn}\beta_{n,k}(x, y) = \text{sgn}(x - y), \quad \forall x, y \in [0, \infty), \]

which concludes our proof.

**Corollary 2.1.** Let \( M_{n,A} \) be a family of Mastroianni type operators and let \( f \in \mathcal{F} \). If

\[ \left[ k, \frac{k+1}{n}, \frac{k+2}{n}; A_t(f) \right] \geq 0, \quad \forall k \in \mathbb{N}, \]

then for all the functionals \( C_m \), given by (8), with

\[ a_{n,i_k}(x_i) = \frac{(-1)^i_k x_i \varphi_n(i_k)(x_i)}{i_k!}, \]

we have \( C_m(f) \geq 0 \).

**Examples**

1. **Bernstein type operators** are Mastroianni type operators with the functions \((\varphi_n)_{n \in \mathbb{N}}\) defined by \( \varphi_n(x) = (1 - x)^n \) and the generating functions \( g_n(x, t) \) given by

\[ g_n(x, t) = (1 - x + tx)^n. \]

2. **Mirakyan-Favard-Szász type operators**, \( S_{n,A} \),

\[ S_{n,A}(f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} A_{n,k}^\frac{k}{n}(f) \]

are obtained for \( \varphi_n(x) = e^{-nx}, x \geq 0 \) and \( g_n(x, z) = e^{-nx(1-z)} \).

3. **Baskakov type operators**, \( V_{n,A} \),

\[ V_{n,A}(f)(x) = (1 + x)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left( \frac{x}{1+x} \right)^k A_{n,k}^\frac{k}{n}(f) \]

are obtained for \( \varphi_n(x) = (1+x)^{-n}, n \in \mathbb{N}^* \) and \( g_n(x, z) = (1+x-xz)^{-n} \).
4. Szász-Schurer type operators, $s_{n,p,A}$,

$$S_{n,p,A}(f)(x) = e^{-(n+p)x} \sum_{k=0}^{\infty} \frac{(n+p)^k}{k!} A_k(f)$$

are obtained for $\varphi_n(x) = e^{-(n+p)x}$ and $g_n(x, z) = e^{-(n+p)x(1-z)}$.

We note that in the above examples the generating functions are of the following form:

$$g_n(x, z) = g^n_1(x, z),$$

where $g_1(x, z)$ is the generating function for the operator $L_{1,A}$. In these case $E_m(x, z)$ given by (21) can be written in the following form

$$E_m(x, z) = g^{nm}_1 \left( x_1 + \ldots + x_m \right) - (g_1(x_1, z)\ldots g_1(x_m, z))^n$$

and by using Theorem 2.1, we get the following result

**Theorem 2.3.** Let $f \in F$ be a function with the property that

$$\left[ \frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; A_t(f) \right] \geq 0, \forall k \in \mathbb{N}.$$ 

If

$$\frac{d^k}{dz^k} \left[ g^{nm}_1 \left( \frac{x_1 + \ldots + x_m}{m}, z \right) - (g_1(x_1, z)\ldots g_1(x_m, z))^n \right]_{z=0} \geq 0$$

for all $k \in \mathbb{N}$ and all $x = (x_1, \ldots, x_m) \in I^m$, then $B_m(f) \geq 0$. If the reverse of inequality (27) is satisfied for all $k \in \mathbb{N}$ and all $x = (x_1, \ldots, x_m) \in I^m$, then $B_m(f) \leq 0$.

**Concluding remarks**

We mention below a few consequences of Theorem 2.3

1. The Bernstein type operators verify (27). In this case $g_n(x, z) = 1 - x + zx$ and inequality (27) follows from Gusić, [4], Theorem 1 (see also [3], Equation (2)), where the following representation is given

$$\left( \sum_{\nu=1}^{m} a_\nu \right)^m - m^m \sum_{\nu=1}^{m} a_\nu = \sum_{1 \leq i < j \leq m} (a_i - a_j)^2 P_{i,j}(a_1, \ldots, a_m).$$
In (28), $P_{i,j}$ are some homogeneous polynomials of degree $n - 2$ with non-negative coefficients. Identity (28) was used by Abel and Raşa in [1] for the classical Bernstein operators.

2. For $g_1(x, z) = e^{-x(1-z)}$, we get

$$B_m(f) = C_m(f), \ m \in \mathbb{N}^*.$$ 

3. In the case of Baskakov type operators, we have

$$g_1(x, z) = \frac{1}{1 + x - xz}.$$ 

Using now (28), it follows that the reverse of inequality (27) is satisfied. Therefore, if $f \in \mathcal{F}$ and

$$\left[ \frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; A_t(f) \right] \geq 0, \forall k \in \mathbb{N},$$

then the Baskakov type operators satisfy the following inequalities

$$V_{n,A}(f) \left( \frac{x_1 + \ldots + x_m}{m} \right) \leq \sum_{i_1=0}^{\infty} \ldots \sum_{i_m=0}^{\infty} \prod_{\nu=1}^{m} a_{n,i_{\nu}}(x_{\nu}) A_{\sum_{\nu=1}^{m} i_{\nu}/m}$$

and

$$\sum_{i_1,\ldots,i_m=0}^{\infty} [a_{n,i_1}(x_1) \ldots a_{n,i_m}(x_1) + \ldots + a_{n,i_1}(x_1 \ldots a_{n,i_m}(x_m)] A_{i_1+\ldots+i_m}(f)$$

$$\geq m \sum_{i_1,\ldots,i_m=0}^{\infty} a_{n,i_1}(x_1) \ldots a_{n,i_m}(x_m) A_{i_1+\ldots+i_m}(f)$$

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