Regulator-based risk statistics for portfolios

Xiaochuan Deng · Fei Sun

Abstract The portfolio are a critical factor not only in risk analysis, but also in insurance and financial applications. In this paper, we consider a special class of risk statistics from the perspective of regulator. This new risk statistic can be used for the quantification of portfolio risk. By further developing the properties related to regulator-based risk statistics, we are able to derive dual representation for them. Finally, examples are also given to demonstrate the application of this risk statistic.

Keywords risk statistics · regulator · representation

Mathematics Subject Classification (2010) 91B30 · 91B32 · 91B70

1 Introduction

Research on risk is a hot topic in both quantitative and theoretical research, and risk models have attracted considerable attention. The quantitative calculation of risk involves two problems: choosing an appropriate risk model and allocating the risk to individual institutions. This has led to further research on risk statistics.

In their seminal paper, Burgert and Rüschendorf (2006) firstly introduced the concepts of the scalar multivariate coherent and convex risk measures, see also Rüschendorf (2013). However, the traditional risk statistics failed to capture sufficiently of the regulator-based risk. Namely, the regulators almost only focus on the loss of investment rather than revenue. Especially, the axiom of translative invariance in coherent and convex risk statistics will definitely fail when we only deal with the regulator-based risk. Thus, the study of regulator-based risk statistics is particularly interesting.

Evaluating the risk of a portfolio consisting of several financial positions, Jouini et al. (2004) pointed out that a set-valued risk measure is more appropriate than a scalar risk measure, especially in the case where several different kinds of currencies are involved when one is determining capital requirements for the portfolio. They first studied the class of set-valued coherent risk measures by proposing some axioms. Hamel (2009) introduced set-valued convex risk measures by an axiomatic approach. For more studies on set-valued risk measures, see Hamel and Heyde (2010), Hamel et al. (2011), Hamel et al. (2013), Labuschagne and Offwood-Le Roux (2014), Ararat et al. (2014), Farkas et al. (2015), Molchanov and Cascos (2016), and the references therein. A natural set-valued risk statistic can be considered as an empirical (or a data-based) version of a set-valued risk measure.

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From the statistical point of view, the behaviour of a random variable can be characterized by its observations, the samples of the random variable. Heyde, Kou and Peng (2007) and Kou, Peng and Heyde (2013) first introduced the class of natural risk statistics, the corresponding representation results are also derived. An alternative proof of the representation result of the natural risk statistics was also derived by Ahmed, Filipovic and Svindland (2008). Later, Tian and Suo (2012) obtained representation results for convex risk statistics, and the corresponding results for quasiconvex risk statistics were obtained by Tian and Jiang (2015). However, all of these risk statistics are designed to quantify the risk of a single financial position (i.e. a random variable) by its samples. A natural question is how to quantify the risk of a portfolio by its samples, especially in the situation where different kinds of currencies are possibly involved in the portfolio.

The main focus of this paper is a new class of risk statistics for portfolios, named regulator-based risk statistics by an axiomatic approach. By further developing the properties related to regulator-based risk statistics, we are able to derive dual representation for them. This new class of risk statistics can be considered as an extension of those introduced by Cont et al. (2013), Chen et al. (2018) and Sun et al. (2018) from scalar and multivariate risk setting to set-valued multivariate risks setting. Finally, examples are also given to illustrate this new class of risk statistics.

The remainder of this paper is organized as follows. In Section 2, we briefly introduce the preliminaries. In Section 3, we state the main result of regulator-based risk statistic, including the representation result. In Section 4, we investigate the alternative regulator-based risk statistics. Section 5 discusses the main proof in this paper. Finally, in Section 6, examples of regulator-based risk statistic are also given.

2 Preliminaries

In this section, we briefly introduce the preliminaries that are used throughout this paper. Let $d \geq 1$ be a fixed positive integer. The space $\mathbb{R}^{d \times n}$ represents the set of financial risk positions. With positive values of $X \in \mathbb{R}^{d \times n}$ we denote the gains while the negative denote the losses. The behavior of the $d$-dimensional random vector $D = (X_1, \ldots, X_d)$ under different scenarios is represented by different sets of data observed or generated under those scenarios because specifying accurate models for $D$ is usually very difficult. Here, we suppose that there always exist $l$ scenarios. Let $n_j$ be the sample size of $D$ in the $j$th scenario, $j = 1, \ldots, l$. Let $n := n_1 + \cdots + n_l$. More precisely, suppose that the behavior of $D$ is represented by a collection of data $X = (X_1, \ldots, X_d) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_l}$, where $X_i = (X_{i,1}, \ldots, X_{i,l}) \in \mathbb{R}^{n_i}$, $X_{i,j} = (x_{i,j,1}, \ldots, x_{i,j,n_i}) \in \mathbb{R}^{n_i}$ is the data subset that corresponds to the $j$th scenario with respect to $X_i$. For each $j = 1, \ldots, l$, $h = 1, \ldots, n_{i,j}$, $X_{i,j}^h := \{x_{i,j,1}^h, \ldots, x_{i,j,n_i}^h\}$ is the data subset that corresponds to the $h$th observation of $D$ in the $j$th scenario, and can be based on historical observations, hypothetical samples simulated according to a model, or a mixture of observations and simulated samples.

In this paper, an element $z$ of $\mathbb{R}^d$ is denoted by $z := (z_1, \ldots, z_d)$. An element $X$ of $\mathbb{R}^{d \times n}$ is denoted by $X := (X_1, \ldots, X_d) := (x_{1,1}^1, \ldots, x_{n_1}^1, \ldots, x_{1,1}^d, \ldots, x_{n_l}^d)$. The $d \times n$ dimensional financial positions in $\mathbb{R}^{d \times n}$ have a strong realistic interpretation. This is indeed the case if we consider realistic situations where investors have access to different markets and form multi-asset portfolios in the presence of frictions such as transaction costs, liquidity problems, irreversible transfers, etc.

Let $K$ be a closed convex polyhedral cone of $\mathbb{R}^d$ with $K \supseteq \mathbb{R}^d_+$ := \{ (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d \} \text{ and } K \cap \mathbb{R}^d_+ = \emptyset$ where $\mathbb{R}^d_+$ := \{ (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0, 1 \leq i \leq d \}$. Let $K^+$ be the positive dual cone of $K$, that is $K^+ := \{ u \in \mathbb{R}^d : u^T v \geq 0 \text{ for any } v \in K \}$, where $u^T$ means the transpose of $u$. For any $X = (X_1, \ldots, X_d), Y = (Y_1, \ldots, Y_d) \in \mathbb{R}^{d \times n}, X + Y$ stands for $(X_1 + Y_1, \ldots, X_d + Y_d)$ and $aX$ stands for $(aX_1, \ldots, aX_d)$ for any $a \in \mathbb{R}$. For any $z := (z_1, \ldots, z_d) \in \mathbb{R}^d$, denote $K_{1,n} := \{ (z_{1,1}^1, z_{2,1}^2, \ldots, z_{d,1}^d) : z \in K \}$ where $1_{n} := (1, \ldots, 1) \in \mathbb{R}^n$. We denote the positive dual cone of $(K_{1,n})^+ \in \mathbb{R}^{d \times n}$ by $(K_{1,n})^+$, i.e. $(K_{1,n})^+ := \{ w \in \mathbb{R}^{d \times n} : \text{w}Z^T \geq 0 \text{ for any } z \in K \}$. The partial order respect to $K$ is defined as $X \leq_{K_{1,n}} Y$, which means $Y - X \in K_{1,n}$. 

Let $M := \mathbb{R}^m \times \{0\}^{d-m}$ the linear subspace of $\mathbb{R}^d$ for $1 \leq m \leq d$. The introduction of $M$ was considered by Jouini et al. (2004) and Hamel (2009). Denote $M_+ := M \cap \mathbb{R}_+^d$ where $\mathbb{R}_+^d := \{ (x^1, \ldots, x^d) \in \mathbb{R}^d : x^i \geq 0, 1 \leq i \leq d \}$ and $M^+ := \{0\}^m \times \mathbb{R}^{d-m}$. Thus, a regulator can only accept security deposits in the first $m$ reference instruments. Denote $K_M := K \cap M$ by the closed convex polyhedral cone in $M$, $K_M^+ := \{ u \in M : u^t z \geq 0 \text{ for any } z \in K_M \}$ the positive dual cone of $K_M$ in $M$, $int K_M$ the interior of $K_M$ in $M$. We denote $Q_M^+ := \{ A \subset M : A = \text{clco}(A + K_M) \}$ and $Q_{M^+}^+ := \{ A \subset K_M : A = \text{clco}(A + K_M) \}$, where the clco($A$) represents the closed convex hull of $A$.

The cone $K$ models proportional frictions between the markets and contains those reference vectors which can be transferred (with paying transaction costs) into positions in $\mathbb{R}_+^d$, see Hamel (2009). The cone $K$ is also introduced to play the role of the solvency set of all positions that can be liquidated without any debt, or equivalently, it allows to define a liquidation value function as we need it to take into account the interdependencies between currencies, e.g. with respect to transaction costs. In this paper, any financial position belongs to $K$ should be regarded as those who need not to pay the capital requirements.

By Chen and Hu (2017), a set-valued risk statistic is any map $\rho$
$$\rho : \mathbb{R}_+^{d \times n} \rightarrow 2^M$$
which can be considered as an empirical (or a data-based) version of a set-valued risk measure. The axioms related to this set-valued risk statistic are organized as follows,

- **A0 Normalized:** $K_M \subseteq \rho(0)$ and $\rho(0) \cap -\text{int} K_M = \emptyset$;
- **A1 Monotonicity:** for any $X, Y \in \mathbb{R}_+^{d \times n}$, $X - Y \in \mathbb{R}_+^{d \times n} \cap K_{1_n}$ implies that $\rho(X) \supseteq \rho(Y)$;
- **A3 M-translative invariance:** for any $X \in \mathbb{R}_+^{d \times n}$ and $z \in \mathbb{R}^d$, $\rho(X - z 1_n) = \rho(X) + z$;
- **A4 Convex:** for any $X, Y \in \mathbb{R}_+^{d \times n}$ and $\lambda \in [0, 1]$, $\rho(\lambda X) + (1 - \lambda) \rho(Y)$;
- **A5 Subadditivity:** $\rho(X + Y) \supseteq \rho(X) + \rho(Y)$ for any $X, Y \in \mathbb{R}_+^{d \times n}$.

We end this section with more notations. A function $\rho : \mathbb{R}_+^{d \times n} \rightarrow 2^M$ is called proper if $\text{dom} \rho := \{ X \in \mathbb{R}_+^{d \times n} : \rho(X) \neq \emptyset \} \neq \emptyset$ and $\rho(X) \neq M$ for all $X \in \text{dom} \rho$. $\rho$ is said to be closed if $\text{graph} \rho$ is a closed set with respect to the product topology on $\mathbb{R}_+^{d \times n} \times M$.

### 3 Regulator-based risk statistics

In this section, we state the main results of this paper, the representation results of regulator-based risk statistics. However, our viewpoint is not the same as Chen and Hu (2017). Instead, we start from the viewpoint of regulators who only care the positions which need to pay capital requirements. Thus, for any $X \in \mathbb{R}_+^{d \times n}$, we define $X \wedge K_{1_n} 0$ as
$$X \wedge K_{1_n} 0 := \begin{cases} X, & X \notin K_{1_n}, \\ 0, & X \in K_{1_n}. \end{cases} \quad (3.1)$$

Thus, the positions which belongs to $K$ regarded as $0$ position. Firstly, we show the axioms related to regulator-based risk statistics.

**Definition 31** A regulator-based risk statistic is a function $\varphi : \mathbb{R}_+^{d \times n} \rightarrow Q_{M^+}^+$ which satisfies the following properties,

- **R0 Normalized:** $K_M \subseteq \varphi(0)$ and $\varphi(0) \cap -\text{int} K_M = \emptyset$;
- **R1 Cash losses:** for any $z \in K_M$, $z \in \varphi(-z 1_n)$;
- **R2 Monotonicity:** for any $X, Y \in \mathbb{R}_+^{d \times n}$, $X - Y \in \mathbb{R}_+^{d \times n} \cap K_{1_n}$ implies that $\varphi(X) \supseteq \varphi(Y)$;
- **R3 Loss-dependence:** for any $X \in \mathbb{R}_+^{d \times n}$, $\varphi(X) = \varphi(X \wedge K_{1_n} 0)$;
- **R4 Convex:** for any $X, Y \in \mathbb{R}_+^{d \times n}$ and $\lambda \in [0, 1]$, $\rho(\lambda X) + (1 - \lambda) \rho(Y)$.

**Remark 31** The property of **R1** means any fixed negative risk position $-z$ can be canceled by its positive quality $z$; **R2** says that if $X_1$ is bigger than $X_2$ under the partial order under $K$, then the $X_1$ need less capital requirement than $X_2$, so $\varphi(X_1)$ contain $\varphi(X_2)$; **R3** means the regulator-based risk statistics start only from the viewpoint of regulators who only care the positions which need to pay capital requirements, while the positions that belong to $K$ regarded as $0$ position.
Definition 32  Let $Y \in \mathbb{R}^{d \times n}$, $u \in M$. Define a function $S_{(Y,u)}(X) : \mathbb{R}^{d \times n} \to 2^M$ as

$$S_{(Y,u)}(X) := \{ z \in M : X^T Y \leq u^T z \}.$$ 

In order to derive the representation result for regulator-based risk statistics, we recall the Legendre-Fenchel conjugate theory for set-valued introduced by Hamel (2009).

Lemma 31 (Hamel (2009) Theorem 2) Let $R : \mathbb{R}^{d \times n} \to Q^t_M$ be a set-valued closed convex function. Then the Legendre-Fenchel conjugate and the biconjugate of $R$ can be defined, respectively, as

$$-R^*(Y, u) := \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} \left( R(X) + S_{(Y,u)}(-X) \right), \quad Y \in \mathbb{R}^{d \times n}, u \in \mathbb{R}^d;$$

and

$$R(X) = R^{**}(X) := \bigcap_{(Y,u) \in \mathbb{R}^{d \times n} \times K_M^+ \setminus \{0\}} \left\{ -R^*(Y, u) + S_{(Y,u)}(X) \right\}, \quad X \in \mathbb{R}^{d \times n}.$$

Definition 33 (Indicator function) For any $Z \subseteq \mathbb{R}^{d \times n}$, the $Q^t_M$-valued indicator function $I_Z : \mathbb{R}^{d \times n} \to Q^t_M$ is defined as

$$I_Z(X) := \begin{cases} \text{cl}K_M, & X \in Z, \\ \phi, & X \notin Z. \end{cases}$$

Remark 32 The conjugate of $Q^t_M$-valued indicator function $I_Z$ is

$$-(I_Z)^*(Y, u) := \text{cl} \bigcup_{X \in Z} S_{(Y,u)}(-X).$$

Remark 33 The regulator-based risk statistics $\rho$ do not have the property of cash additive, which is also said to be translation invariance, see Hamel (2009). However, they have the property of cash sub-additive studied by El Karoui and Ravanelli (2009) and Sun and Hu (2019). Indeed, from the Theorem 6.2 of Hamel and Heyde (2010), $\rho$ satisfies the Fatou property. Then, consider any $X \in \mathbb{R}^{d \times n}$ and $z \in K_M$, for any $\varepsilon \in (0, 1)$, we have

$$\rho\left((1 - \varepsilon)X - z1_n\right) = \rho\left((1 - \varepsilon)X + \varepsilon\left(-\frac{z}{\varepsilon}\right)1_n\right) \supseteq (1 - \varepsilon)\rho(X) + \varepsilon\rho\left(-\frac{z}{\varepsilon}1_n\right) \supseteq (1 - \varepsilon)\rho(X) + z$$

where the last inclusion is due to the property $\textbf{R1}$. Using the arbitrariness of $\varepsilon$, we have the following lemma.

Lemma 32 Assume that $\rho$ is a regulator-based risk statistic. For any $z \in \mathbb{R}^d$, $X \in \mathbb{R}^{d \times n},$

$$\rho(X - z1_n) \supseteq \rho(X) + z$$

which also implies

$$\rho(X + z1_n) \subseteq \rho(X) - z.$$ 

Proof. From the Remark 33 one can steadily show Lemma 32. □

Remark 34 From Lemma 32, the regulator-based risk statistics have the property of cash sub-additive. Even though regulator-based risk statistics are special case of cash sub-additive risk statistics, it still needed to be highlighted and studied separately because that the property of loss-dependence have its unique meaning in the finance practice. In addition, the property $\textbf{R1}$ was not implied from cash sub-additive and have its rationality to define risk statistics. So it is a great meaning to study this special class of cash sub-additive risk statistics: regulator-based risk statistics.
Proposition 31 Let $\varrho : \mathbb{R}^{d \times n} \to Q_{M^+}^I$ be a proper closed convex regulator-based risk statistic with $u \in \{ - \sum_{j=1}^{l} \sum_{h=1}^{n_j} Y_h^{1,j}, \ldots , - \sum_{j=1}^{l} \sum_{h=1}^{n_j} Y_h^{d,j} \} + M^+ \} \cap K_M^+ \{0\}$. Then
\[
-\varrho^* (Y, u) = \begin{cases} 
cl_X \bigcup_{X \in \mathbb{R}^{d \times n}} S(Y, u)(-X), Y \in -\mathbb{R}^{d \times n} \cap (K^+1_n), \\
M, \\
\text{elsewhere.}
\end{cases} \tag{3.2}
\]

Next, we state the main result of this paper, the representation result of regulator-based risk statistics.

Theorem 31 If $\varrho : \mathbb{R}^{d \times n} \to Q_{M^+}^I$ is a proper closed convex regulator-based risk statistic, then there is a $-\alpha : (-\mathbb{R}^{d \times n} \cap K^+1_n) \times K_M^+ \{0\} \to Q_{M^+}^I$, that is not identically $M$ on the set \[ W = \left\{ (Y, u) \in (-\mathbb{R}^{d \times n} \cap K^+1_n) \times K_M^+ \{0\} : u \in \left\{ - \sum_{j=1}^{l} \sum_{h=1}^{n_j} Y_h^{1,j}, \ldots , - \sum_{j=1}^{l} \sum_{h=1}^{n_j} Y_h^{d,j} \} + M^+ \right\} \right\}, \]
such that for any $X \in \mathbb{R}^{d \times n}$,
\[ \varrho(X) = \bigcap_{(Y, u) \in W} \left\{ - \alpha(Y, u) + S(Y, u) \{X \land K_{1_n} 0\} \right\}. \tag{3.3} \]

4 Alternative versions of regulator-based risk statistics

In this section, we develop another framework of regulator-based risk statistics. This framework is a little different from the previous one. However, almost all the arguments are the same as those in the previous section. Thus, we only state the corresponding notations and results, and omit all the proofs and relevant explanations.

We replace $M$ by $\tilde{M} \in \mathbb{R}^{d \times n}$ which is a linear subspace of $\mathbb{R}^{d \times n}$. We also replace $K$ by $\tilde{K} \in \mathbb{R}^{d \times n}$ which is a is a closed convex polyhedral cone with $\tilde{K} \supseteq \mathbb{R}^{d \times n}$. The partial order respect to $\tilde{K}$ is defined as $X \leq \tilde{K} Y$, which means $Y - X \in \tilde{K}$. Let $M_{\tilde{K}} := M \cap \mathbb{R}^{d \times n}$. Denote $\tilde{K}_{\tilde{M}} := \tilde{K} \cap \tilde{M}$ by the closed convex polyhedral cone in $\tilde{M}$, $\tilde{K}_{\tilde{M}}^+ := \{ \tilde{u} \in M : \tilde{u}^T \tilde{z} \geq 0 \text{ for any } \tilde{z} \in \tilde{K}_{\tilde{M}} \}$ the positive dual cone of $\tilde{K}_{\tilde{M}}$ in $\tilde{M}$, $\text{int} \tilde{K}_{\tilde{M}}$ the interior of $\tilde{K}_{\tilde{M}}$ in $\tilde{M}$. We denote $Q_{M_{\tilde{K}}} := \{ \tilde{A} \in \tilde{M} : \tilde{A} = \text{clco}(\tilde{A} + \tilde{K}_{\tilde{M}}) \}$ and $Q_{M_{\tilde{K}}^+} := \{ \tilde{A} \subset \tilde{K}_{\tilde{M}} : \tilde{A} = \text{clco}(\tilde{A} + \tilde{K}_{\tilde{M}}) \}$. We still start from the viewpoint of regulators who only care the positions which need to pay capital requirements. Thus, for any $X \in \mathbb{R}^{d \times n}$, we define $X \land \tilde{K} 0$ as
\[ X \land \tilde{K} 0 := \begin{cases} X, X \notin \tilde{K}, \\
0, X \in \tilde{K}. \tag{4.1} \end{cases} \]

Then, the axioms related to regulator-based risk statistics become as follows.

Definition 41 A regulator-based risk statistic is a function $\tilde{\varrho} : \mathbb{R}^{d \times n} \to Q_{M_{\tilde{K}}}^I$ which satisfies the following properties,

Q0 Normalized: $\tilde{K}_{\tilde{M}} \subseteq \tilde{\varrho}(0)$ and $\tilde{\varrho}(0) \cap -\text{int} \tilde{K}_{\tilde{M}} = \phi$;
Q1 Cash losses: for any $\tilde{z} \in \tilde{K}_{\tilde{M}}, \tilde{z} \in \tilde{\varrho}(-\tilde{z})$;
Q2 Monotonicity: for any $X_1, X_2 \in \mathbb{R}^{d \times n}, X_1 \neq X_2 \in \mathbb{R}^{d \times n} \cap \tilde{K}$ implies that $\tilde{\varrho}(X_1) \supseteq \tilde{\varrho}(X_2)$;
Q3 Loss-dependence: for any $X \in \mathbb{R}^{d \times n}$, $\tilde{\varrho}(X) = \tilde{\varrho}(X \land \tilde{K} 0)$;
Q4 Convex: for any $X, Y \in \mathbb{R}^{d \times n}$, $\lambda \in [0, 1], \tilde{\varrho}(\lambda X) + (1 - \lambda) Y \supseteq \lambda \tilde{\varrho}(X) + (1 - \lambda) \tilde{\varrho}(Y)$.

We need more notations. Let $Y \in \mathbb{R}^{d \times n}$, $\tilde{\varrho} \in \tilde{M}$. Define a function $S_{(Y, \tilde{\varrho})}(\cdot) : \mathbb{R}^{d \times n} \to 2^{\tilde{M}}$ as
\[ S_{(Y, \tilde{\varrho})}(X) := \{ \tilde{z} \in \tilde{M} : X^T \tilde{Y} \leq \tilde{u}^T \tilde{z} \}. \]
Let $\tilde{R} : \mathbb{R}^{d \times n} \to Q_{\tilde{M}}^t$ be a set-valued closed convex function. Then the Legendre-Fenchel conjugate and the biconjugate of $\tilde{R}$ can be defined, respectively, as

$$-\tilde{R}^*(Y, u) := \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} \left( \tilde{R}(X) + S_{(Y, \tilde{u})}(-X) \right), \quad Y \in \mathbb{R}^{d \times n}, \tilde{u} \in \mathbb{R}^{d \times n};$$

and

$$\tilde{R}(X) = \tilde{R}^{**}(X) := \bigcap_{(Y, \tilde{u}) \in \mathbb{R}^{d \times n} \times \tilde{R}_{\tilde{M}}^+} \left[ -\tilde{R}^*(Y, \tilde{u}) + S_{(Y, \tilde{u})}(X) \right], \quad X \in \mathbb{R}^{d \times n}.$$

For any $\tilde{Z} \subseteq \mathbb{R}^{d \times n}$, the $Q_{\tilde{M}}^t$-valued indicator function $I_{\tilde{Z}} : \mathbb{R}^{d \times n} \to Q_{\tilde{M}}^t$ is defined as

$$I_{\tilde{Z}}(X) := \begin{cases} \text{cl} \tilde{K}_{\tilde{M}}, & X \in \tilde{Z}, \\ \phi, & X \notin \tilde{Z}. \end{cases}$$

The conjugate of $Q_{\tilde{M}}^t$-valued indicator function $I_{\tilde{Z}}$ is

$$-(I_{\tilde{Z}})^*(Y, \tilde{u}) := \text{cl} \bigcup_{X \in \tilde{Z}} S_{(Y, \tilde{u})}(-X).$$

Assume that $\tilde{\phi}$ is a regulator-based risk statistic. For any $\tilde{z} \in \mathbb{R}_{\tilde{M}^+}^{d \times n}$, $X \in \mathbb{R}^{d \times n}$,

$$\tilde{\phi}(X - \tilde{z}) \supseteq \tilde{\phi}(X) + \tilde{z}$$

which also implies

$$\tilde{\phi}(X + \tilde{z}) \subseteq \tilde{\phi}(X) - \tilde{z}.$$

Next, we state the main results of this section.

**Proposition 41** Let $\tilde{\phi} : \mathbb{R}^{d \times n} \to Q_{\tilde{M}}^t$ be a proper closed convex regulator-based risk statistic with

$$\tilde{u} \in \left\{ \left( -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_{h}^{l,j}, \cdots, -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_{h}^{d,j} \right) + \tilde{M}^p \right\} \cap \tilde{K}_{\tilde{M}^p}^+ \setminus \{0\}.$$ Then

$$-\tilde{\phi}^*(Y, \tilde{u}) = \begin{cases} \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y, \tilde{u})}(-X), & Y \in -\mathbb{R}_{\tilde{M}^+}^{d \times n} \cap \tilde{K}_{\tilde{M}^p}^+, \\ \phi, & \text{elsewhere}. \end{cases} \quad (4.2)$$

**Theorem 41** If $\tilde{\phi} : \mathbb{R}^{d \times n} \to Q_{\tilde{M}}^t$ is a proper closed convex regulator-based risk statistic, then there is a $-\alpha : (-\mathbb{R}_{\tilde{M}^+}^{d \times n} \cap \tilde{K}_{\tilde{M}^p}^+) \times \tilde{K}_{\tilde{M}^p}^+ \setminus \{0\} \to Q_{\tilde{M}^p}^t$, that is not identically $\tilde{M}$ on the set

$$\tilde{W} = \left\{ (Y, \tilde{u}) \in (-\mathbb{R}_{\tilde{M}^+}^{d \times n} \cap \tilde{K}_{\tilde{M}^p}^+) \times \tilde{K}_{\tilde{M}^p}^+ \setminus \{0\} : \tilde{u} \in \left( -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_{h}^{l,j}, \cdots, -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_{h}^{d,j} \right) + \tilde{M}^p \right\},$$

such that for any $X \in \mathbb{R}^{d \times n}$,

$$\tilde{\phi}(X) = \bigcap_{(Y, \tilde{u}) \in \tilde{W}} \left\{ -\alpha(Y, \tilde{u}) + S_{(Y, \tilde{u})}(X \wedge 0) \right\}. \quad (4.3)$$
5 Proofs of main results

Proof of Proposition 3.1. If \( Y \notin -\mathbb{R}^{d \times n} \cap (K^+\mathbb{1}_n) \), there exit an \( \bar{X} \in \mathbb{R}^{d \times n} \cap (K\mathbb{1}_n) \) such that \( \bar{X}^t Y > 0 \). Using the definition of \( S_{(Y,u)} \), we have \( S_{(Y,u)}(-t\bar{X}) = \{ z \in M : -t\bar{X}^t Y \leq u^t z \} \) for \( t > 0 \). Thus,

\[
cl \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X) \supseteq \bigcup_{t > 0} S_{(Y,u)}(-t\bar{X}) = M.
\]

The last equality is due to \( -\bar{X}^t Y \to -\infty \) when \( t \to +\infty \). Using the definition of \( S_{(Y,u)} \), we conclude that \( cl \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X) \subseteq M \). Hence

\[
cl \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X) = M \quad \text{whenever} \quad Y \notin -\mathbb{R}^{d \times n} \cap (K^+\mathbb{1}_n).
\]

It is easy to check that for any \( X \in \mathbb{R}^{d \times n} \) and \( v \in M \),

\[
S_{(Y,u)}(-X - v\mathbb{1}_n) = \{ z \in M : -X^t Y \leq u^t z + Y^t (v\mathbb{1}_n) \} = \{ z - v \in M : -X^t Y \leq u^t (z - v) + (Y + u\mathbb{1}_n)^t (v\mathbb{1}_n) \} + v
\]

when \( u \notin (\sum_{j=1}^l \sum_{h=1}^{n_j} Y_{h,j}^1, \cdots, \sum_{j=1}^l \sum_{h=1}^{n_j} Y_{h,j}^d) + u \in M^\perp \), we have \( S_{(Y,u)}(-X - v\mathbb{1}_n) = S_{(Y,u)}(-X) + v \). However, when \( u \notin (\sum_{j=1}^l \sum_{h=1}^{n_j} Y_{h,j}^1, \cdots, \sum_{j=1}^l \sum_{h=1}^{n_j} Y_{h,j}^d) + M^\perp \), we find \( v \in M \), such that for any \( z \in M \),

\[
-X^t Y \leq u^t z + (Y + u\mathbb{1}_n)^t (v\mathbb{1}_n).
\]

Thus, we have

\[
z + v \in S_{(Y,u)}(-X - v\mathbb{1}_n).
\]

Therefore

\[
\bigcup_{v \in M} (z + v) \subseteq \bigcup_{v \in M} S_{(Y,u)}(-X - v\mathbb{1}_n).
\]

Thus,

\[
M \subseteq \bigcup_{v \in M} S_{(Y,u)}(-X - v\mathbb{1}_n).
\]

From the definition of \( S_{(Y,u)} \), the inverse inclusion is always true. So we conclude that

\[
M = \bigcup_{v \in M} S_{(Y,u)}(-X - v\mathbb{1}_n).
\]

It is also easy to check that

\[
-g^*(Y,u) = cl \bigcup_{X \in \mathbb{R}^{d \times n}, v \in M} \left( g(X + v\mathbb{1}_n) + S_{(Y,u)}(-X - v\mathbb{1}_n) \right) = cl \bigcup_{X \in \mathbb{R}^{d \times n}, v \in M} \left( g(X + v\mathbb{1}_n) + M \right) = M
\]

where the last equality comes from that the \( M \) is a linear space and \( g(X) \subseteq M \). We now show that \( -g^*(Y,u) = cl \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X) \). To this end, from \( -g^*(Y,u) = cl \bigcup_{X \in \mathbb{R}^{d \times n}} \left( g(X) + S_{(Y,u)}(-X) \right) \), we derive
Proof of Theorem 31. The proof is straightforward from Lemma 31 and Proposition 31.

Example 61

The coherent risk measure AV@R was studied by Föllmer and Schied (2011) in detail. They have given several representations and many properties like law invariance and the Fatou property. Hamel et al. (2013) first introduced set-valued AV@R, where the representation result is derived. Moreover, they have given several representations and many properties like law invariance and the Fatou property. Hamel et al. (2011) in detail. They have given several representations and many properties like law invariance and the Fatou property.

Using the arbitrariness of \( z \), we have

\[
\varrho(X) + S_{(Y,u)}(-X) \subseteq \inf_{z \in R^d} \left\{ \frac{1}{\alpha}(-\alpha X \wedge \alpha 1_n)^+ - z \right\} + R^n_+.
\]

It is clear that \( \varrho \) satisfies the cash-loss, monotonicity, loss-dependence properties and convexity, so \( \varrho \) is a regulator-based risk statistic. We call such a risk statistic a set-valued loss average value at risk.

6 Examples

In this section, we construct several examples for regulator-based risk statistics.

Example 61

The coherent risk measure AV@IR was studied by Föllmer and Schied (2011) in detail. They have given several representations and many properties like law invariance and the Fatou property. Hamel et al. (2013) first introduced set-valued AV@R, where the representation result is derived. Moreover, they also proved that it is a set-valued coherent risk measure. We now define the set-valued loss average value at risk. For any \( X \in R^{d \times n} \) and \( 0 < \alpha < 1 \), we define \( \varrho(X) \) as

\[
\varrho(X) := AV@IR^\alpha_{\alpha}(X)
\]

:= \inf_{z \in R^d} \left\{ \frac{1}{\alpha}(-\alpha X \wedge \alpha 1_n)^+ - z \right\} + R^n_+.
\]

It is clear that \( \varrho \) satisfies the cash-loss, monotonicity, loss-dependence properties and convexity, so \( \varrho \) is a regulator-based risk statistic. We call such a risk statistic a set-valued loss average value at risk.
The shortfall risk measures were first introduced by Föllmer and Schied (2002) with the help of loss function, and they proved that it is a special case of convex risk measures. Ararat and Hamel (2014) introduced the set-valued shortfall risk measures with vector loss function $\ell$. We now define the loss loss shortfall risk statistics. For any $X \in \mathbb{R}^{d \times n}$, we define

$$ g_\ell(X) := \{ z \in \mathbb{R}^d \mid \ell \left( X \wedge C_1n, 0 \right) - z_1n \in x^0 - C \}, $$

where $x^0$ is the threshold level and $C$ is the threshold set such that $0 \in \mathbb{R}^{d \times n}$ is the boundary point of $C$. When we take $x^0 = \ell(0)$, $g_\ell$ satisfies the cash-loss, monotonicity, loss-dependence properties and convexity, so $g_\ell$ is a regulator-based risk statistic. We call such a risk statistic a loss shortfall risk statistic.

The notion of $\omega$-divergence was first introduced by Ben-Tal and Teboulle (1986), and further it was extended by Ben-Tal and Teboulle (2007) with the name optimized certainty equivalent as a convex risk measure. Ararat and Hamel (2014) introduced set-valued divergence risk measures with vector loss function $\ell$, and proved it is a set-valued convex risk measure. We now define the loss divergence risk statistics. For any $X \in \mathbb{R}^{d \times n}$, we define

$$ g_{g,r}(X) := \bigcap_{r,\omega \in \mathbb{R}^n \setminus \{0\}} \{ z \in \mathbb{R}^d \mid \omega^\top z \geq \omega^\top \delta_{g,r}(X \wedge C_1n, 0) + \inf_{x \in C} \omega^\top \text{diag}(r)x \} $$

where

$$ \delta_{g,r}(X) = (\delta_{g_1,r_1}(X_1), \ldots, \delta_{g_d,r_d}(X_d)) $$

with

$$ \delta_{g_i,r_i}(X_i) := \inf_{z_i \in \mathbb{R}} \left( z_i + r_i E[\ell_i(-X_i - z_i1n)] \right) - r_i x_i^0, $$

where $x^0$ is the threshold level and $C$ is the threshold set such that $0 \in \mathbb{R}^{d \times n}$ is the boundary point of $C$. When we take $x^0 = \ell(0)$, it is easy to check that $g_{g,r}$ satisfies the cash-loss, monotonicity, loss-dependence properties and convexity, so $g_{g,r}$ is a regulator-based risk statistic. We call such a risk statistic a loss divergence risk statistic.

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