Fractional values of orbital angular momentum in problems of classical physics

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Abstract. Classical field theory problems for which the presence of nontrivial topological Pauli phase is essential (i.e. fractional values of the orbital angular momentum are possible in two-dimensional case) are discussed within the two-dimensional Helmholtz equation. As examples, we consider the “wedge problem” — determination of the field created by a point charge placed between two conducting half-planes — and the Fresnel diffraction from knife edge.

1. Introduction

It is well known that translation generators, i.e. self-adjoint operators of infinitesimal translations on a segment $a \leq q \leq b$, form a one-parameter family, $S_\theta = S_\theta^+$,

$S_\theta \Psi(q) = -i \Psi'(q), \quad \mathcal{D}(S_\theta) = \left\{ \Psi \in \mathcal{H}, \Psi' \in \mathcal{H}; \Psi(b) = e^{i\theta} \Psi(a) \right\}.
\tag{1}$

Here $\mathcal{D}(S_\theta)$ is the domain of the operator $S_\theta$ in the Hilbert space $\mathcal{H} = L^2([a,b])$ of the wave functions $\Psi(q)$ that are square-integrable on the interval $[a,b]$, $\theta$ is the topological phase, $0 \leq \theta \leq 2\pi$, and the derivative $\Psi'(q) = d\Psi/dq$ should be understood in the sense of the Schwartz distribution theory.

In the particular case of two-dimensional rotations, when the generalized coordinate $q$ is the angle $\varphi$, $0 \leq \varphi \leq 2\pi$, and the shift operator $S_\theta$ is the operator of the orbital angular momentum in a plane $L_\theta$, the topological phase arises as a result of rotation about the full angle $\varphi = 2\pi$,

$\Psi(2\pi) = e^{i\theta} \Psi(0), \quad \theta = 2\pi \delta, \quad 0 \leq \delta < 1.
\tag{2}$

This phase separates out the rotation generator $L_\theta$ and, therefore, defines the unitary operator $U_\theta$ of finite rotations, which describes the rotational dynamics of the system,

$U_\theta(\alpha) = e^{i\alpha L_\theta}, \quad U_\theta(\alpha) \Psi(\varphi) = \Psi(\varphi + \alpha).
\tag{3}$

Because of the boundary condition the eigenvalues of the operator $L_\theta$, i.e. orbital angular momenta, are

$L_\theta \Psi_M(\varphi) = M \Psi_M(\varphi), \quad M = \delta + m, \quad 0 \leq \delta < 1, \quad m = 0, \pm 1, \pm 2, \ldots
\tag{4}$

1 On other topological phases in quantum mechanics in a more general context, see the reviews.

2 In units of $\hbar = 1.0546 \cdot 10^{-34}$ J · s.
Its eigenfunctions
\[ \Psi_M(\varphi) = \frac{1}{\sqrt{2\pi}} e^{iM\varphi} \] (5)
form an orthonormal basis in the space of wave functions \( \Psi(\varphi) \in L^2([0, 2\pi]) \) and realize a multivalued (at \( \delta \neq 0 \)) irreducible representations of the two-dimensional rotation group \( SO(2) \) \[8\].

For systems that are invariant with respect to time reversal, more precisely, with respect to reversal of the direction of motion \[9\], we have \[10\]

\[ 1) \theta = 0, \delta = 0; \quad 2) \theta = \pi, \delta = 1/2. \] (6)

Therefore, in such two-dimensional quantum-mechanical systems, only two possibilities can be realized: integer and half-integer quantization of the orbital angular momentum, corresponding to single-valued and two-valued irreducible representations of the group \( SO(2) \) \[8, 10\].

Since the requirement of single-valuedness of wave functions does not belong to the fundamental principles of quantum theory and multivalued wave functions cannot be excluded in advance \[11\], the question arises as to why only integer values of the orbital angular momentum squared, \( l(l+1), l = 0, 1, 2, \ldots \), and its projection onto any axis, \( M = m, 0, \pm 1, \pm 2, \ldots \), are realized in the three-dimensional case \[1\].

Apparently, this problem was first raised and solved by Pauli in \[11\], and then it was studied in detail in \[12\]. The main reason for integer quantization of the orbital angular momentum in three-dimensional space when its projection operators are subject to the commutation relations

\[ L_x L_y - L_y L_x = i L_z \] (7)

with cyclic permutations \((x, y, z)\) is in the requirement of unitary equivalence of the operators \( L_x, L_y, L_z \) \[12\].

Fractional values of the orbital angular momentum arise in classical field theory problems with variables separable in cylindrical coordinates. Such as, for example, the “wedge problem” \[13, 14\] and Fresnel diffraction from knife edge \[15\]. In this paper we consider the specifics of these problems.

2. Two-dimensional Helmholtz equation, boundary conditions on rays

If variables are separable in cylindrical coordinates in a problem of classical field theory \[16\] and the polar angle \( \varphi \) varies in the sector \( a \leq \varphi \leq b \), then it is useful to utilize the full set of eigenfunctions of the rotation generator \( R_\theta \), acting on the Hilbert space of wave functions \( \Psi(\varphi) \in L_2([a, b]) \),

\[ R_\theta \Psi(\varphi) = -i \Psi'(\varphi), \quad \mathcal{D}(R_\theta) = \left\{ \Psi \in L_2([a, b]), \Psi' \in L_2([a, b]); \Psi(b) = e^{i2\pi\delta}\Psi(a) \right\}. \]

The eigenfunctions of this rotation generator in the specified sector,

\[ \Psi_\mu(\varphi) = \frac{1}{\sqrt{b-a}} e^{i\mu\varphi}, \quad \mu = \frac{2\pi}{(b-a)}(\delta + m), \quad 0 \leq \delta < 1, \quad m = 0, \pm 1, \pm 2, \ldots, \] (8)

with the eigenvalues of the angular momentum \( \mu \) form a complete orthonormal basis in \( L_2([a, b]) \), see the monograph \[2\]. If \( (b-a) = 2\pi \), then the rotation generator coincides with the orbital angular momentum operator, \( R_\theta = L_\theta \), and its eigenvalues — with the orbital angular momentum, \( \mu = M \), see \[4\].
3. The Wedge Problem

As a typical example we consider the “wedge problem” [13, 14], i.e. definition of electric field created by a point charge $e$ placed between two intersecting conducting half-planes, see Fig. 1.

Fourier component of the scalar potential $\Phi(\rho, \varphi, z)$,

$$u_{x\varphi}(\rho, \varphi) = \frac{1}{\pi} \int_{-\infty}^{\infty} dz \Phi(\rho, \varphi, z) \cos(xz),$$

satisfies the inhomogeneous two-dimensional Helmholtz equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u_{x\varphi}}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u_{x\varphi}}{\partial \varphi^2} + k^2 u_{x\varphi} = \frac{-4e}{r} \delta(\rho - r) \delta(\varphi - \gamma), \quad k = i \kappa$$

with boundary conditions

$$u_{x\varphi}(\rho, 0) = u_{x\varphi}(\rho, \beta) = 0, \quad u_{x\varphi}(\rho, \varphi) \bigg|_{\rho \to \infty} = 0.$$  

To satisfy the Dirichlet boundary conditions (12), in the set (9) we put

$$b = 2\beta, \quad a = 2\alpha, \quad \alpha = 0, \quad 0 \leq \varphi \leq 2\beta.$$  

If $(b - a) \neq 2\pi$, then the angular momentum $\mu$ is fractional, and in this case we can put $\delta = 0$ in (8) without loss of generality, and it is convenient to use the full set of real functions

$$\sin \left[ \frac{2\pi n}{(b-a)} \varphi \right], \quad \cos \left[ \frac{2\pi n}{(b-a)} \varphi \right], \quad n = 0, 1, 2, \ldots,$$

resulting in standard Fourier decomposition, instead of the set indicated in (8). This set corresponds to a superposition of functions (8) with angular momenta

$$\mu = \frac{2\pi}{(b-a)} m, \quad m = 0, \pm 1, \pm 2, \ldots, \quad (b-a) \neq 2\pi,$$

which are not integer even at $\delta = 0$, unlike the orbital angular momenta (4).
The particular solutions of the homogeneous equation corresponding to (11) that are finite at zero and vanish at infinity are the products

\[ R_n(\rho) \sin \left( \frac{\pi n}{\beta} \varphi \right), \quad n = 1, 2, 3, \ldots; \quad R_n(\rho) = \begin{cases} I_{\frac{n\pi}{\beta}}(\kappa \rho), & \rho < r, \\ K_{\frac{n\pi}{\beta}}(\kappa \rho), & \rho > r, \end{cases} \]

where \( I_\nu(x) \) and \( K_\nu(x) \) are the modified Bessel functions \(^13\). Then we can show \(^14\) that

\[ u_\varphi(\rho, \varphi) = \frac{8e}{\beta} \sum_{n=1}^{\infty} K_{\frac{n\pi}{\beta}}(\kappa r) I_{\frac{n\pi}{\beta}}(\kappa \rho) \sin \left( \frac{\pi n}{\beta} \varphi \right), \quad \rho < r, \]

the field potential is

\[ \Phi(\rho, \varphi, z) = \frac{e}{\beta \sqrt{2r_\rho}} \int_{\eta}^{\infty} \left\{ \frac{1}{\cosh \left( \frac{\pi \xi}{\beta} \right) - \cos \left[ \frac{\pi(\varphi - \gamma)}{\beta} \right]} - \frac{1}{\cosh \left( \frac{\pi \xi}{\beta} \right) - \cos \left[ \frac{\pi(\varphi + \gamma)}{\beta} \right]} \right\} \times \]

\[ \times \frac{\sinh \left( \frac{\pi \xi}{\beta} \right)}{\sqrt{\cosh \xi - \cosh \eta}} d\zeta, \quad \cosh \eta = \frac{\rho^2 + \rho^2 + z^2 - 2r_\rho \cos(\varphi \pm \gamma)}{2r_\rho}, \quad \eta > 0, \]

and the angular momentum in the superposition \(^14\) \( \mu = \frac{\pi}{2} m, \quad m = \pm 1, \pm 2, \ldots \) It is apparent that the potential \( \Phi = 0 \) on the conductor surface, i.e. at \( \varphi = 0, \beta \).

If \( \beta = \pi, \) i.e. the wedge turns into a conducting plane, the integral in \( 15 \) is elementary,

\[ \Phi(\rho, \varphi, z) \bigg|_{\beta=\pi} = e R_+ + \frac{(-e)}{R_-}, \quad R_\pm = \left[ r^2 + \rho^2 + z^2 - 2r_\rho \cos(\varphi \mp \gamma) \right]^{1/2}, \]

which fully corresponds to the result of the method of images. The equalities \(^13\) ensure the presence of “fictitious” half-space of the image, \( \pi \leq \varphi \leq 2\pi, \) and at the same time integer values of the orbital angular momentum \( \mu = M = m, \quad m = \pm 1, \pm 2, \ldots \) in the expansion \(^14\).

If \( \beta = 2\pi, \) then the wedge turns into a conducting half-plane. In this case, the integral in \( 15 \) is also calculated in closed form and gives \(^13\)

\[ \Phi(\rho, \varphi, z) \bigg|_{\beta=2\pi} = \frac{q_+}{R_+} + \frac{(-q_-)}{R_-}, \quad q_\pm = \frac{e}{\pi} \arccos \left[ \frac{-\cos(\varphi \mp \gamma)}{\cosh(\frac{\pi}{2})} \right], \]

and \( R_\pm \) are defined in \(^16\). Such a value of \( \beta \) corresponds to the presence of fictitious space, \( 2\pi < \varphi < 4\pi, \) to the summation of the potentials of a countable number of image charges located there, and to the contribution of half-integer angular momentum values \( \mu = \frac{1}{2} m, \quad m = \pm 1, \pm 2, \ldots \) to the expansion \(^14\).

4. Diffraction from knife edge

As a second example, we discuss the scattering of a plane monochromatic wave with a frequency \( \omega = ck \) by a semi-infinite screen, \( x = 0, \quad y < 0, \) which satisfies Neumann conditions, see Fig. 2.

Consider a particular solution \(^15\) of the homogeneous Helmholtz equation \(^11\), asymptotically, \( k\rho \gg 1, \) turning into a plane wave,

\[ u(\rho, \varphi) = \frac{1}{2} \sum_{n=0}^{\infty} \varepsilon_n (-i)^n J_\frac{n}{2}(k\rho) \cos \left( \frac{n}{2} \varphi \right) = \]

\[ = \frac{1}{\sqrt{4\pi}} e^{-ik\rho \cos \varphi} \Phi \left[ \sqrt{2k\rho \cos \left( \frac{\varphi}{2} \right)} \right] \sim e^{-ik\rho \cos \varphi} \left[ 1 + O \left( \frac{1}{(k\rho)} \right) \right]. \]

(18)
Here \( \varepsilon_n \) is the Neumann factor, \( \varepsilon_0 = 1, \varepsilon_n = 2 \) for \( n > 0 \), \( J_\nu(x) \) is the Bessel function, and the function

\[
\Phi(z) = \int_{-\infty}^{z} e^{it^2} dt = \frac{\sqrt{\pi t}}{3} \left[ 1 + \text{erf} \left( e^{-\frac{i \pi}{4} z} \right) \right]
\]

is connected with Fresnel integrals and is expressed through the probability integral \([17]\) of the complex argument.

Both integer and half-integer values of the orbital angular momentum contribute to the expansion \([18]\), therefore the function \( u(\rho, \varphi) \) is periodic in \( \varphi \) with period \( 4\pi \), and not \( 2\pi \). For a screen placed along the negative \( y \) axis, the area \( -\pi/2 < \varphi < \pi/2 \) is “real space” and the area \( 3\pi/2 < \varphi < 7\pi/2 \) is “fictitious space”. It is necessary in order to place auxiliary sources\(^3\) and satisfy the Neumann boundary conditions,

\[
\frac{\partial}{\partial \varphi} \Psi(\rho, \varphi) \bigg|_{\varphi = -\frac{\pi}{2}} = \frac{\partial}{\partial \varphi} \Psi(\rho, \varphi) \bigg|_{\varphi = \frac{3\pi}{2}} = 0.
\]

Therefore, to obtain the corresponding complete set of functions, we need to set \( \alpha = -\pi/2 \) and \( \beta = 3\pi/2 \) in the first two equalities \([13]\), which leads to the presence of half-integer orbital angular momenta. The solution of the Helmholtz equation satisfying the conditions \([20]\) is \([15]\)

\[
\Psi(\rho, \varphi) = u(\rho, \varphi) + u(\rho, 3\pi - \varphi) = \frac{1}{\sqrt{i\pi}} \left\{ e^{-ik\rho \cos \varphi} \Phi \left[ \sqrt{2k\rho} \cos \left( \frac{\varphi}{2} \right) \right] + e^{ik\rho \cos \varphi} \Phi \left[ -\sqrt{2k\rho} \sin \left( \frac{\varphi}{2} \right) \right] \right\}. \tag{21}
\]

Given this exact expression for the solution \( \Psi(\rho, \varphi) \), one can determine the shadow region and the region of the reflected wave for any value of \( \rho \). So, for \( \rho \to \infty \) the asymptotic behavior of \( \Psi(\rho, \varphi) \) in different areas of “real space” is \([13]\)

\[
\Psi(\rho, \varphi) \simeq_{k\rho \gg 1} \begin{cases} 
 e^{-ik\rho \cos \varphi} + e^{ik\rho \cos \varphi} + f(\rho, \varphi), & -\frac{1}{2}\pi < \varphi < 0, \\
 e^{-ik\rho \cos \varphi} + f(\rho, \varphi), & 0 < \varphi < \pi, \\
 f(\rho, \varphi), & \pi < \varphi < \frac{3}{2}\pi,
\end{cases} \tag{22}
\]

where

\[
f(\rho, \varphi) = \left[ \frac{i}{8\pi k \rho} e^{ik\rho} \left( \frac{1}{\sin \left( \frac{\varphi}{2} \right)} - \frac{1}{\cos \left( \frac{\varphi}{2} \right)} \right) \right]. \tag{22'}
\]

Thus, the function \( \Psi(\rho, \varphi) \) corresponds to a solution of the physical problem posed: a plane wave, \( \exp(-ikx) \), comes from the right and meets the screen. In the region \( -\pi/2 < \varphi < 0 \) the incident wave is reflected from the screen, \( \exp(ikx) \), in the region \( 0 < \varphi < \pi \) it propagates unhindered to the left, and in the “shadow” region \( \pi < \varphi < 3\pi/2 \) there is no plane wave, see Fig. 2.

Both integer and half-integer values of the orbital angular momentum contribute to the solution \( \Psi(\rho, \varphi) \) of the problem of scattering of a plane wave by a semi-infinite screen. However, the question about a superselection rule \([18]\) in the superposition of integer and half-integer orbital angular momenta does not arise here, since the auxiliary sources necessary to satisfy the Neumann boundary conditions are located in “fictitious space”, \( 3\pi/2 < \varphi < 7\pi/2 \).

\(^3\) Compare with equalities \([13]\) and formulae \([16]\) and \([17]\).
5. Conclusions
The question about the connection of the orbital angular momentum quantization in two-dimensional and three-dimensional Euclidean spaces was first discussed by Pauli in the article [11]. In addition, the specific values of the topological phase $\theta$, equal to 0 or $\pi$, in circular quantum dots with different numbers of electrons are completely determined by the Pauli exclusion principle. Therefore, the phase $\theta$ appearing in the boundary condition (2) can rightfully be called the topological Pauli phase.

Fractional values of the orbital angular momentum can appear in problems of classical and quantum physics, in which the variables are separable in cylindrical coordinates. These values are related to the topological phase $\theta$, acquired by the wave function $\Psi(\varphi)$ at full rotation, $0 \leq \varphi \leq 2\pi$, see relation (2). This nontrivial topological phase is associated with multivalued irreducible representations of the two-dimensional rotation group $SO(2)$. Only integer and half-integer values of the orbital angular momentum can be realized in quantum systems that are invariant under time reversal.

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