The upper triangular solutions to the three-state constant quantum Yang-Baxter equation

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June 16, 1993

Abstract

In this article we present all nonsingular upper triangular solutions to the constant quantum Yang-Baxter equation

$$R_{k_1k_2}^{l_1l_3}R_{k_1k_3}^{l_2l_3} = R_{k_2k_3}^{l_2l_3}R_{k_1k_2}^{l_1l_3}$$

in three state case. The upper triangular ansatz implies 729 equations for 45 variables. Fortunately many of the equations turned out to be simple allowing us to start breaking the problem into smaller ones. In the end we had a total of 552 solutions, but many of them were either inherited from two-state solutions or subcases of others. The final list contains 35 nontrivial solutions, most of them new.

1 Introduction

In this paper we continue our work on systematically solving the constant quantum Yang-Baxter equation

$$R_{j_1j_2}^{k_1k_3}R_{k_1k_3}^{l_1l_3} = R_{j_2j_3}^{k_2k_3}R_{j_1k_3}^{l_1l_3}R_{k_1k_2}^{l_1l_2},$$

(1)

In our previous paper [1] (for more details, see [2]) we solved completely the two-state problem, which involved 64 equations in 16 variables.

In general (1) contains $N^6$ equations for $N^4$ unknowns, so in the present case when $N = 3$ we have 729 equations for 81 variables. At the moment this is too complicated for an exhaustive study and one must proceed step by step with suitable ansatze. The present ansatz is based on the two-state results: it turned out [1, 3, 4] that all two-state nonsingular solutions were either

- Upper triangular: $R_{ij}^{kl} = 0$, if $j > l$ or $j = l$, $i > k$, or
- Even weight: $R_{ij}^{kl} = 0$, if $k + l \neq i + j$ (mod 2).

(In thinking $R_{ij}^{kl}$ as an $N^2 \times N^2$ matrix we use the convention that the right-hand indices define the block.) Thus upper triangularity turns out to be a good ansatz for finding two-state solutions, and we hope it is equally productive when the number of states is three.
2 Simple solutions

Let us first of all note that the YBE does have some ‘easy’ classes of solutions. That is, the solutions are easy to construct from previous solutions or for some other reason. [This does not mean that the other related structures (e.g. algebras) are necessarily less interesting.] In this section we will describe two such classes and solutions belonging to them are not mentioned again when we discuss the results of our search.

2.1 Solutions inherited from lower number of states

A solution to (1) with a lower number of states can always be dressed to become a higher-state solution. There are at least two ways to do this.

2.1.1 Diagonal dressing

Let $\tilde{R}$ be an $M$-state solution of (1) and $N > M$. Let $\mu$ be a selection of $M$ numbers from $\{1, \ldots, N\}$, and define the $N$-state $R$-matrix as follows:

$$R_{ij}^{kl} = \begin{cases} 
\tilde{R}_{ij}^{kl}, & \text{when } i, j, k, l \in \mu, \\
s_{ij}\delta_{i}^{k}\delta_{j}^{l}, & \text{otherwise.}
\end{cases} \quad (2)$$

When this is substituted into (1) we find further conditions on $s$ as follows:

$$\tilde{R}_{ij}^{kl} (s_{mi}s_{mj} - s_{mk}s_{ml}) = 0,$$
$$\tilde{R}_{ij}^{kl} (s_{im}s_{ml} - s_{km}s_{mj}) = 0,$$
$$\tilde{R}_{ij}^{kl} (s_{im}s_{jm} - s_{km}s_{lm}) = 0,$$ \quad (3)

$\forall i, j, k, l \in \mu, m \notin \mu$, no sum. [It is interesting to note that the conditions are exactly like those obtained in a different context in [5].] A trivial solution to (3) is given by $s_{ij} = 1$, but there can also be others (depending on the form of $\tilde{R}$).

In the present case we have $M = 2$ and $N = 3$ and there is only one index outside $\mu$. As an example let us take $\mu = \{1, 3\}$ and

$$\tilde{R} = \begin{pmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & p & 1 - pq & \cdot \\
\cdot & \cdot & q & \cdot \\
\cdot & \cdot & \cdot & 1
\end{pmatrix},$$

(4)

(for matrices we always write ‘.’ in place of a ‘0’, for better readability) and then the only remaining condition from (3) is $s_{12}s_{21} = s_{23}s_{32}$. Thus we can extend (4) to a three-state solution.
with the four additional parameters $a, b, x, y$:

$$R = \begin{pmatrix}
1 & \ldots & \ldots & \ldots \\
. & a & . & . \\
. & . & p & . \\
. & . & . & 1 - pq \\
. & . & by & . \\
. & . & . & x \\
. & . & . & b \\
. & . & . & q \\
. & . & . & ay \\
. & . & . & 1
\end{pmatrix}.$$ \hfill (5)

### 2.1.2 Block dressing

The starting point is as above but the new higher-state solution is constructed as follows:

$$R^{kl}_{ij} = \begin{cases}
\tilde{R}^{kl}_{ij}, & \text{when } i, j, k, l \in \mu, \\
\delta^k F^l_j, & \text{when } j, l \in \mu, i, k \notin \mu \\
G^k_i \delta^l_j, & \text{when } i, k \in \mu, j, l \notin \mu \\
\delta^k_i \delta^l_j, & \text{otherwise.}
\end{cases} \hfill (6)$$

This ansatz leads to the conditions

$$\begin{align*}
(F \otimes F) \tilde{R} &= \tilde{R}(F \otimes F), \\
(1 \otimes F) \tilde{R}(G \otimes 1) &= (G \otimes 1) \tilde{R}(1 \otimes F), \\
\tilde{R}(G \otimes G) &= (G \otimes G) \tilde{R}, \\
[F, G] &= 0.
\end{align*} \hfill (7)$$

The most important case of this is when $G = F^{-1}$, in which case the first equation of (7) is sufficient. An example of such an solution is

$$R := \begin{pmatrix}
1 & 1 & . & . & . & -1 & c_9 \\
. & -1 & . & . & . & d_6 & . \\
. & . & 1 & . & . & . & c_9 \\
. & . & . & g_6 & . & d_6 g_6 & . \\
. & . & . & . & f_5 & . & . \\
. & . & . & . & . & g_6 & . \\
. & . & . & . & . & 1 & -c_9 \\
. & . & . & . & . & . & -1 \\
. & . & . & . & . & . & 1
\end{pmatrix}, \hfill (8)$$
which decomposes as
\[ \tilde{R} := \begin{pmatrix}
1 & 1 & -1 & c_9 \\
. & 1 & . & c_9 \\
. & . & 1 & -c_9 \\
. & . & . & 1
\end{pmatrix}, \quad G := \begin{pmatrix}
-1 & d_6 \\
. & -1 \\
. & . \\
. & . & . & . & . & g_9
\end{pmatrix}. \tag{9} \]

### 2.2 Solutions generated by commuting matrices

The following solutions can be said to be inherited from lower dimension, when dimension means
the number of pairs of indices. (This hierarchal structure is more obvious when one considers
extensions, see e.g. \[\text{[6]}\].)

The result is simply the following: Let \( \{N(\alpha), M(\alpha) | \alpha \in I\} \) be a set of commuting \( N \times N \)
matrices, then it is easy to show that
\[ R_{ij}^{kl} = \sum_{\alpha \in I} N(\alpha)_{ik}^l M(\alpha)_{lj} \tag{10} \]
is an \( N \)-state solution of (9).

A commuting set of matrices can be simultaneously brought to the Jordan canonical form,
but the Jordan form does not have to be diagonal. A two-state example is provided by
\[ R = \begin{pmatrix}
1 & a & b & c \\
. & 1 & b & . \\
. & . & 1 & a \\
. & . & . & 1
\end{pmatrix}, \tag{11} \]
it can be decomposed (this is not unique) as
\[ N(1) = \begin{pmatrix}
1 & a \\
. & 1
\end{pmatrix}, \quad M(1) = \begin{pmatrix}
1 & . \\
. & 1
\end{pmatrix}, \quad N(2) = \begin{pmatrix}
b & c \\
. & b
\end{pmatrix}, \quad N(1) = \begin{pmatrix}
. & 1 \\
. & .
\end{pmatrix}. \tag{12} \]

When the number of states is three there is still more freedom and there are more varied solutions
of this type, but we will not include such solutions in the list of results. An example is
\[ R := \begin{pmatrix}
1 & 1 & . & a_5 & a_6 & c_9 & a_8 & a_9 \\
. & 1 & . & d_8 a_5 & . & c_9 & d_8 a_8 & . \\
. & . & 1 & . & . & . & c_9 & . \\
. & . & . & 1 & . & d_8 a_5 & d_8 a_6 & . \\
. & . & . & . & 1 & . & d_8^2 a_5 & . \\
. & . & . & . & . & 1 & . & 1 \\
. & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & 1
\end{pmatrix} \]
3 Symmetries

It is well known that the set of equations (1) is invariant under the continuous transformation

$$R \to \kappa(Q \otimes Q)R(Q \otimes Q)^{-1},$$

(13)

where $Q$ is any nonsingular $N \times N$ matrix and $\kappa$ a nonzero number. They are also invariant under the index reflections

$$R_{ij}^{kl} \to R_{kl}^{ij},$$

(14)

$$R_{ij}^{kl} \to R_{lk}^{ji},$$

(15)

the first of these is the usual matrix transposition and the second follows from $R \to PRP$, where $P_{ij}^{kl} = \delta_i^k \delta_j^l$ is the permutation operator.

For special choices of $Q$ one can obtain other discrete transformations. In general, if $\sigma$ is a permutation of the set $\{1, \ldots, N\}$, then $Q_i^j = \delta_i^{\sigma(j)}$ yields a relabeling of indices $R_{ij}^{kl} \to R_{\sigma(i)\sigma(j)}^{\sigma(k)\sigma(l)}$. An important special case of this is given by $Q_i^j = \delta_{N-j+1}$ which, when followed by transposition, yields reflection across the secondary diagonal.

In the present paper we restrict the $R$-matrix to be an upper triangular matrix, and then it is natural to use only upper triangular transformation matrices $Q$, which we will call $U$ from now on. Under such transformations the upper triangular nature of $R$ is preserved. Furthermore, one easily finds that the diagonal blocks transform as

$$R_{im}^{km} \to U_i^{ij} R_{ij}^{km} U_k^{ij}^{-1}.$$  

(16)

Thus, in order to fix the remaining rotational freedom we will just need to impose conditions on the diagonal blocks.

Normally one would require the canonical form of an upper triangular matrix be the Jordan canonical form, but this may require transformation matrices that are not upper triangular. For this reason we must relax the definition of what is a canonical form. It turns out that using just upper triangular transformation matrices one can bring all other upper triangular matrices into one of the following five ‘semi-canonical’ forms

$$C_1 = \begin{pmatrix} a & . & . \\ . & b & . \\ . & . & c \end{pmatrix}, \quad C_2 = \begin{pmatrix} a & b & . \\ . & a & . \\ . & . & c \end{pmatrix}, \quad C_3 = \begin{pmatrix} a & b & . \\ . & a & . \\ . & . & a \end{pmatrix},$$

$$C_4 = \begin{pmatrix} a & . \\ . & b & c \\ . & . & b \end{pmatrix}, \quad C_5 = \begin{pmatrix} a & b & . \\ . & a & b \\ . & . & a \end{pmatrix}.$$  

(17)

This is the basis of our classification scheme.
4 How the equations were solved

4.1 Breakdown into smaller sets

As usual we must use the available symmetries to divide the problem into several smaller ones. In particular we want to fix the continuous symmetries related to upper triangular transformation matrices. The detailed breakdown is given in the Appendix, here we give just the general idea.

We start with the top diagonal block and transform it into the semi-canonical Jordan form \( \mathbf{C} \) (17). We may therefore assume that the upper block is of the type \( \mathbf{C}_5 \) in (17). We work through the cases in opposite order, \( \mathbf{C}_5 \) first. Using a reflection across the antidiagonal when necessary we may assume that the lowest diagonal block cannot be reflected up and transformed into anything that has already been analyzed. Using the notation that \( \sim \mathbf{C}_n \) is anything that can be transformed to \( \mathbf{C}_n \) using upper triangular matrices, we can write the first division by giving the upper and lower blocks:

1. up \( \mathbf{C}_5 \), down anything

2. up \( \mathbf{C}_4 \), down anything but \( \sim \mathbf{C}_5 \),

3. up \( \mathbf{C}_3 \), down \( \sim \mathbf{C}_3 \), or \( \sim \mathbf{C}_4 \), or \( \sim \mathbf{C}_1 \),

4. up \( \mathbf{C}_2 \), down \( \sim \mathbf{C}_4 \), or \( \sim \mathbf{C}_1 \),

5. up \( \mathbf{C}_1 \), down \( \sim \mathbf{C}_1 \).

In the third and fourth cases note that an antidiagonal reflection takes \( \mathbf{C}_4 \) into \( \mathbf{C}_2 \).

This the starting point. Further complications arise when \( c = a \) in \( \mathbf{C}_3 \) or \( a = b = c \) in \( \mathbf{C}_1 \), because then they commute with an upper triangular transformation matrix having units on the diagonal. In those cases one can use the remaining freedom to operate on the other diagonal blocks. In the appendix we have given the more detailed breakdown.

4.2 Solving the equations

For each subcase we still have to solve the 729 equations. Some of these equations vanish automatically because of the upper triangular ansatz. It turns out that there are also many simple equations that factor to linear factors. Since the present equation solver routines cannot handle well very large set of equations, we decided to start the solution process by interactively splitting the process into branches and subbranches based on the simple factorizable equations. [All algebraic work was done using REDUCE \footnote{REDUCE}]. The algorithm was the following:

1. Initialize, in particular record the expression that cannot vanish (Ex: \( A \neq 0 \)).

2. Are there simple factorizable equations?
Yes: Choose a simple equation (Ex. $B(A - 1)(C - D) = 0$) and find its solutions. Go to 3.

No: Go to 4

3. Are there any solutions that do not break the nonvanishing condition?

Yes: Add the first solution as an additional rule to the current branch and create a new branch (which inherits the current assignments and nonvanishing condition) for each additional allowed branch, if any. With each later branch add the condition that the assignment of the previous branch cannot hold. Go to 2. (Ex: \{(B \Rightarrow 0), A\}, \{(A \Rightarrow 1), AB\}, \{(C \Rightarrow D), AB(A - 1)\})

No: Go to 5

4. Are there equations you want to solve by hand?

Yes: Do it, but if you have to assume something, create also a branch where the assumption does not hold. Go to 3.

No: Solve the remaining equations using ‘groesolve’ [8] and output the allowed solutions. Go to 5.

5. Is this the last branch?

Yes: End.

No: Take the next branch. Go to 2,

A log of the interactive solution process was saved using the unix ‘tee’ utility, and checked later. Altogether the printout extended well over 1000 pages. In the end we had 552 solutions, but naturally most of them turned out to be of the simple type of Sec. 2 or subcases of other solutions.

5 The nontrivial solutions

As expected, the problem of classifying the solutions is almost as time consuming as finding them. We must characterize the solutions by something that is invariant under the possible transformations. The method we finally chose was to classify first by the number of different eigenvalues of the $R$-matrix. In counting the different eigenvalues one must be careful, because some solutions only seem to have the required number of eigenvalues. [An example is provided by $diag(1, \xi_1, \xi_1^2, \xi_2, \xi_1 \xi_2, \xi_1 \xi_2^2, \xi_1 \xi_2^3, \xi_2)\), $\xi_3^3 = 1$, which seems to have six different eigenvalues. Since there are only three different cubic roots of unity there are actually at most three different eigenvalues, arranged in several ways.]
First a word about notation. The zeroes are represented by dots, for better readability, \( \xi \) is a cubic root of unity in general, \( \eta \) is a cubic root \( \neq 1 \), (i.e. \( \xi^3 = 1, \eta^2 = -1 - \eta \)), \( \epsilon = \pm 1 \), other greek symbols are also roots of some polynomial equation given in the text. Symbols with different subscripts have independent values. Small Latin letters \( x, y, z, p, q, k \ldots \) are free parameters. Capital letters have special properties, given in text.

We would like to repeat again that the simple solutions of Sec. 2 are not included. Also, solutions related by the discussed transformations are not mentioned separately, neither are those obtained from previous solutions by restricting parameters.

5.1 9 different eigenvalues

First we have a solution with the full number of 9 different diagonal elements.

\[
R_{9,1} = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . \\
p & . & 1 - q\eta^2 & . & . & . & . & . & . \\
. & p^2 & . & (1 - q\eta)x & . & (1 - q\eta)(1 - q\eta^2) & . & . & . \\
. & . & q^2p^{-1}\eta^2 & . & . & -q(1 - q\eta^2)/x & . & . & . \\
. & . & . & q\eta & . & -\eta^2q(1 - q\eta) & . & . & . \\
. & . & . & . & q^2p^{-2}\eta & . & . & . & . \\
. & . & . & . & . & . & k & . & . \\
. & . & . & . & . & . & . & B & . \\
. & . & . & . & . & . & . & . & q^2 \\
\end{pmatrix}
\]

Here \( x \) is a free parameter that can be scaled to 1, since it must be nonzero. The \( q = \eta p^2 \) subcase of this solution was presented in [9].

5.2 8 different eigenvalues

The first solution is the multiparameter version of the standard GL(3) solution, as generalized in [10] and [11] to the following that has maximum of 4 free parameters, \( p, c, q, k \).

\[
R_{8,1} = \begin{pmatrix}
1 & . & . . & . & . & . & . & . & . \\
p & . & 1 - q & . & . & . & . & . & . \\
c & . & . & 1 - q & . & . & . & . & . \\
. & . & q\eta & . & A & . & . & . & . \\
. & . & . & . & qk^{-1} & . & 1 - q & . & . \\
. & . & . & . & Aq^{-1} & . & . & 1 - q & . \\
. & . & . & . & . & . & . & k & . \\
. & . & . & . & . & . & . & . & B \\
\end{pmatrix}
\]

where \( A, B \in \{1, -q\} \).
If one imposes further relations among the diagonal entries there can be other nonzero matrix elements elsewhere. The first such solution is

$$R_{8,2} = \begin{pmatrix}
1 & . & . & x & . & . & -xp^2\mu^{-1} & . \\
. & p & . & 1-\mu & . & . & . & . \\
. & . & p^{-1} & . & . & 1-\mu & . & . \\
. & . & . & p^{-1}\mu & . & . & . & . \\
. & . & . & . & 1 & . & . & . \\
. & . & . & . & . & 1-\mu & . & . \\
. & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & \mu^3
\end{pmatrix}$$

where $\mu$ is a root of $(\mu^3 - 1)(\mu^2 + 1) = 0$. In fact $R_{8,2}$ has 8 different eigenvalues when $\mu^2 = -1$ and 7 when $\mu^3 = 1$, but we keep these two branches together. If $x = 0$ we get a subcase of $R_{8,1}$. Other solutions of this type will appear later.

The following is a rather asymmetric solution:

$$R_{8,3} = \begin{pmatrix}
1 & . & . & x & . & . & . & . \\
. & p & . & 1-\xi & . & . & . & . \\
. & . & p^{-1} & . & . & . & . & . \\
. & . & . & p^{-1}\xi & . & . & . & . \\
. & . & . & . & 1 & . & . & . \\
. & . & . & . & . & 1-\xi & . & . \\
. & . & . & . & . & . & p^2 & . \\
. & . & . & . & . & . & . & \xi
\end{pmatrix}$$

### 5.3 7 different eigenvalues

The following solution is related to $R_{9,1}$: if in the diagonal of $R_{9,1}$ the parameter $p$ is restricted to be a cubic root of unity we can have more nonzero matrix elements:

$$R_{7,1} = \begin{pmatrix}
1 & . & . & xy & . & . & xy\xi^2(1-q\eta) & . \\
. & \xi & . & 1-q\eta^2 & . & . & . & y\xi(1-q\eta^2) \\
. & . & \xi^2 & . & (1-q\eta)x & . & (1-q\eta)(1-q\eta^2) & . \\
. & . & . & q\xi^2\eta^2 & . & . & . & -y\eta \\
. & . & . & . & q\eta & . & -q(1-q\eta^2)/x & . \\
. & . & . & . & . & q\xi & -\eta^2q(1-q\eta) & . \\
. & . & . & . & . & . & q^2\xi\eta & . \\
. & . & . & . & . & . & . & q^2
\end{pmatrix}$$
Here we have two scalable parameters $x$ and $y$, and they can both be scaled to 1, because $x \neq 0$ and $y = 0$ leads just to a special case of $R_{9,1}$.

A restriction on the diagonal elements of $R_{8,1}$ allows

$$R_{7,2} = \begin{pmatrix}
1 & \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 - q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p & \cdot & 1 - q & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}$$

where $A \in \{1, -q\}$.

### 5.4 6 different eigenvalues

The following solution is related to $R_{9,1}$ and $R_{7,1}$

$$R_{6,1} = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p & \cdot & 1 - q^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p^2 & \cdot & (1 - q^2)x & \cdot & (1 + q)(1 - q^2) & \cdot & \cdot & \cdot \\
\cdot & \cdot & q & \cdot & \cdot & \cdot & q(1 - q^2)/x & \cdot & \cdot \\
\cdot & \cdot & \cdot & p & \cdot & \cdot & (1 - q^2) & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & q^4 & \cdot & \cdot & q^2p^{-1} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\end{pmatrix}$$

The $p = -q$ subcase was found in [12].

The next solution is related to $R_{8,2}$: if we have $\mu = 1$ in $R_{8,2}$ there is other freedom available:

$$R_{6,2} = \begin{pmatrix}
1 & \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & p \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & p^2 \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}$$

Here $x$ and $y$ can be scaled to 1, if they are nonzero.
5.5 5 different eigenvalues

There were no new nontrivial solutions with 5 different eigenvalues.

5.6 4 different eigenvalues

First we have another relative of $R_{8,1}$

$$R_{4,1} = \begin{pmatrix} 1 & . & x & . & . & . & . & . \\ . & \mu^2 & 1 - \epsilon i & . & . & . & -i\mu^2 ye \\ . & . & \mu & . & . & 1 - \epsilon i & . & . \\ . & . & i\mu^2 \epsilon & . & . & . & . & z \\ . & . & . & -i \epsilon & . & . & . & . \\ . & . & . & . & \mu^3 & . & 1 - \epsilon i & . \\ . & . & . & . & i\mu^3 \epsilon & . & . & . \\ . & . & . & . & . & i \mu \epsilon & . & -i \epsilon \end{pmatrix}$$

where $\mu^4 = 1$.

Up to now the diagonal blocks have been of type $C_1$. The next two represent a pattern that we will meet several times later on, the diagonal and (1,3) blocks are of type $C_3$ (more or less) and in addition there are some nonzero entries on the fourth off-diagonal.

$$R_{4,2} = \begin{pmatrix} 1 & x & . & . & . & . & . & . \\ . & p & . & q & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & p^{-1} & y & . & . & . \\ . & . & . & . & \epsilon & . & . & . \\ . & . & . & . & . & p^{-1} & . & . \\ . & . & . & . & . & . & 1 & x \\ . & . & . & . & . & . & p & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix}$$

$$R_{4,3} = \begin{pmatrix} 1 & x & . & . & . & . & . & . \\ . & p & . & . & x - y & . & -zp^2 & . \\ . & . & 1 & . & . & . & . & -y \\ . & . & p^{-1} & z & . & y - x & . & . \\ . & . & . & \epsilon & . & . & . & . \\ . & . & . & . & p^{-1} & . & . & . \\ . & . & . & . & . & 1 & y & . \\ . & . & . & . & . & p & . & . \\ . & . & . & . & . & . & 1 & . \end{pmatrix}$$
5.7 3 different eigenvalues

The next solution reduces to $R_{8,2}$ if $y=0$, c.f. also $R_{7,1}$
\[ R_{3,3} = \begin{pmatrix} 1 & . & . & . & x & . & x\xi_1^2(1 - \xi_2) & . \\ . & \xi_1 & . & . & . & . & . & . \\ . & . & \xi_2 & . & . & . & . & . \\ . & . & . & \xi_1^2 & y & . & . & . \\ . & . & . & . & -x_2(1 - \xi_2)y^{-1} & . & . & . \\ \end{pmatrix} \]

The following reduces to \( R_{8,3} \) if \( y = 0 \)

\[ R_{3,4} = \begin{pmatrix} 1 & . & . & . & x & . & . & . \\ . & \xi_1 & 1 - \xi_2\xi_1 & . & . & . & . & . \\ . & . & \xi_2 & . & . & . & . & . \\ . & . & . & \xi_1^2 & . & . & . & . \\ . & . & . & . & 1 & . & . & . \\ \end{pmatrix} \]

5.8 2 different eigenvalues

Now most of the solution have the previously mentioned structure: the diagonal and (1,3) blocks are like \( C_3 \) and in addition there are some nonzero entries on the fourth off-diagonal. We list these solutions in decreasing number of nonzero elements on the fourth diagonal.

\[ R_{2,1} = \begin{pmatrix} 1 & x & . & p & . & a & . & q \\ . & \epsilon_1 & . & . & k & . & a\epsilon_1 + k\epsilon_1^2(\epsilon_2 - 1) & . \\ . & 1 & . & . & . & a & . & . \\ . & . & \epsilon_1 & x\epsilon_1 + k\epsilon_1^2(\epsilon_2 - 1) & . & k & . & . \\ . & . & . & \epsilon_2 & . & . & k^2p^{-1} & . \\ \end{pmatrix} \]
\[
R_{2.2} = \begin{pmatrix}
1 & y - x & p & -y + x & \frac{1}{2}(-2y^2 + x^2) \\
& e & -x & -ye & \\
& 1 & & & \\
& & e & ye & x \\
& & -1 & & x^2p^{-1} \\
& & & e & \\
& & & 1 & y + x \\
& & & & 1
\end{pmatrix}
\]

\[
R_{2.3} = \begin{pmatrix}
1 & x & a & -x & -xy \\
& e & -y + x & -ye & \\
& 1 & & & -y \\
& & e & ye & y - x \\
& & -1 & & \\
& & & e & \\
& & & 1 & y \\
& & & & 1
\end{pmatrix}
\]

\[
R_{2.4} = \begin{pmatrix}
1 & x & p & ae & q \\
& e & x - k & a & \\
& 1 & & ae & \\
& & e & ke & \\
& & -1 & & \\
& & & e & \\
& & & 1 & x \\
& & & & 1
\end{pmatrix}
\]

\[
R_{2.5} = \begin{pmatrix}
1 & x & p & -x & -xy \\
& e_1 & & -xe_1 & \\
& 1 & & & -y \\
& & e_1 & xe_1 & \\
& & e_2 & & \\
& & e_1 & & \\
& & 1 & y & \\
& & & e_1 & \\
& & & & 1
\end{pmatrix}
\]
For the following two the diagonal blocks are of type $C_2$. If the parameter $b$ were zero one could make an index shuffle so that they would fit into the previously mentioned band form.

\[
R_{2.6} = \begin{pmatrix}
1 & x & | & -x & -x^2 & | & . & . & a \\
. & 1 & . & -x & . & . & -b \\
. & . & \epsilon_1 & . & -x\epsilon_1 & . & . \\
. & . & 1 & x & . & . & b \\
. & . & . & 1 & . & . & . \\
. & . & . & . & \epsilon_1 & . & . \\
. & . & . & . & . & \epsilon_1 & . \\
. & . & . & . & . & . & \epsilon_2 \\
\end{pmatrix}
\]

\[
R_{2.7} = \begin{pmatrix}
1 & x & | & -x & x(x-2c) & | & . & . & -bx \\
. & 1 & . & x-2c & . & . & -b \\
. & . & \epsilon_1 & . & . & -c\epsilon_1 & . \\
. & . & 1 & -x+2c & . & . & b \\
. & . & . & 1 & . & . & . \\
. & . & . & . & \epsilon_1 & . & . \\
. & . & . & . & . & \epsilon_1 & . \\
. & . & . & . & . & . & \epsilon_2 \\
\end{pmatrix}
\]

The following are rather sparse, but have more freedom on the diagonal

\[
R_{2.8} = \begin{pmatrix}
1 & . & . & | & x & a & | & b & c \\
. & \epsilon_1 & . & . & . & . & . & . & . \\
. & . & \epsilon_1 & . & . & . & . & . & . \\
. & . & . & \epsilon_1 & . & . & . & . & . \\
. & . & . & . & \epsilon_1 & . & . & . & . \\
. & . & . & . & . & \epsilon_1 & . & . & . \\
. & . & . & . & . & . & \epsilon_1 & . & . \\
. & . & . & . & . & . & . & \epsilon_2 \\
\end{pmatrix}
\]
\[ R_{2.9} = \begin{pmatrix}
1 & \ldots & x & \ldots & a \\
. & \epsilon_3 & . & . & . \\
. & . & \epsilon_1 & . & . \\
. & . & . & \epsilon_3 & . \\
. & . & . & \epsilon_4 & . \\
. & . & . & \epsilon_2 & . \\
. & . & . & . & \epsilon_1 \\
. & . & . & . & \epsilon_2 \\
. & . & . & . & \epsilon_5
\end{pmatrix} \]

### 5.9 One eigenvalue

The next four solutions have basically the $C_3$ structure, with some extra elements.

\[ R_{1.1} = \begin{pmatrix}
1 & x & \ldots & -x & -xy \\
. & 1 & . & . & . \\
. & . & -y + x & -x & q \\
. & . & 1 & . & . \\
. & . & . & -y & . \\
. & . & 1 & . & . \\
. & . & . & . & 1 \\
. & . & . & . & 1 \\
. & . & . & 1 & . \\
. & . & . & . & 1
\end{pmatrix} \]

\[ R_{1.2} = \begin{pmatrix}
1 & x & \ldots & p & a \\
. & 1 & . & . & . \\
. & . & p - q & q & k \\
. & . & 1 & . & . \\
. & . & . & p & . \\
. & . & 1 & . & . \\
. & . & . & . & 1 \\
. & . & . & 1 & . \\
. & . & . & . & 1 \\
. & . & . & . & 1
\end{pmatrix} \]
\[ R_{1,3} = \begin{pmatrix}
1 & x & \ldots & -x & -xy \\
.1 & . & \ldots & . & -y -p \\
. . & 1 & \ldots & . & -y \\
. . . & 1 & y & . & p \\
. . . & 1 & . & . & q \\
. . . . & 1 & . & . & . \\
. . . . . & 1 & . & . & \end{pmatrix} \]

\[ R_{1,4} = \begin{pmatrix}
1 & x & \ldots & -x & -xy \\
.1 & . & \ldots & . & -2y + x & p \\
. . & 1 & \ldots & . & -y \\
. . . & 1 & . & . & . \\
. . . & . & \ldots & 1 & 2y - x & . \\
. . . & . & . & 1 & . & \end{pmatrix} \]

The next one is like \( C_2 \). The labeling change \( 2 \leftrightarrow 3 \) would make it almost \( C_3 \) but would not keep the solution upper triangular.

\[ R_{1,5} = \begin{pmatrix}
1 & x & -x & -x^2 & -k + xq & p & k & a \\
.1 & . & -x & . & . & q & b \\
. . & 1 & . & \ldots & . & -p \\
. . . & 1 & x & . & . & p - q & -b \\
. . . & . & \ldots & 1 & . & . & . \\
. . . . & . & . & 1 & . & \end{pmatrix} \]

The next three solutions have mixed diagonal blocks. It could be possible to transform the upper block to \( C_2 \) or \( C_3 \) but we chose a different and apparently simpler form defined by the
requirement that the above mentioned relabeling \(2 \leftrightarrow 3\) keeps the matrix upper triangular.

\[
R_{1.6} = \begin{pmatrix}
1 & x & p & \ldots & -p & k & -pq \\
. & 1 & . & \ldots & . & -y \\
. & . & 1 & \ldots & \ldots & -x \\
. & . & . & \ldots & . & 1 \\
. & . & . & \ldots & . & 1 \\
. & . & . & \ldots & . & 1
\end{pmatrix}
\]

\[
R_{1.7} = \begin{pmatrix}
1 & x & p & \ldots & -x & -xy & -2px - k \\
. & 1 & . & \ldots & . & -y & \ldots \\
. & . & 1 & \ldots & \ldots & -x & \ldots \\
. & . & . & \ldots & . & 1 & \ldots \\
. & . & . & \ldots & . & 1 & \ldots \\
. & . & . & \ldots & . & 1 & \ldots 
\end{pmatrix}
\]

\[
R_{1.8} = \begin{pmatrix}
1 & x & q & \ldots & -x & -x^2 & -k \\
. & 1 & . & \ldots & . & -x & \ldots \\
. & . & 1 & \ldots & \ldots & -x & \ldots \\
. & . & . & \ldots & . & 1 & \ldots \\
. & . & . & \ldots & . & 1 & \ldots \\
. & . & . & \ldots & . & 1 & \ldots 
\end{pmatrix}
\]

Finally we have two solutions whose diagonal blocks are of type \(C_5\).

\[
R_{1.9} = \begin{pmatrix}
1 & 1 & . & -1 & -1 & x & y & y - x & p \\
. & 1 & 1 & . & -1 & -1 & y & -z + y \\
. & . & 1 & . & -1 & -1 & . & . & y \\
. & . & . & 1 & . & -1 & -1 & . & . \\
. & . & . & . & 1 & 1 & . & . & . \\
. & . & . & . & . & 1 & 1 & . & . \\
. & . & . & . & . & . & 1 & . & . \\
\end{pmatrix}
\]
In the next one the parametrization is rather complicated and it would be interesting to understand its origin. \([x = 1 \text{ yields a symmetric solution.}]\)

\[
R_{1.10} = \begin{pmatrix}
1 & 2 & -2 & -4x & 8x & 4 & -4x(7 + x) \\
. & 1 & 2 & -2x & 4 - 6x - 2x^2 & 2x(1 + x) & 6x(2 + x)(-1 + x) \\
. & . & 1 & 2 - 4x & . & . & -4x(1 - 2x) \\
. & . & . & 1 & -2 & -2x(1 + x) & -2x(1 + 3x)(-2 + x) \\
. & . & . & . & 1 & -2x & 4x(1 - 2x) \\
. & . & . & . & . & 2 - 4x & . \\
. & . & . & . & . & . & 1
\end{pmatrix}
\]

6 Discussion

In this paper we have given all upper triangular nonsingular solutions to the constant quantum Yang-Baxter equation (1), modulo upper triangular transformations (13) and reflections (14,15) and minus simple solutions of Sec. 2.

There is not much that we can say now about the solutions, here are just some random observations. (i) Some of solutions \(R_{9.1}, R_{8.1} - R_{8.3}, R_{7.1}, R_{6.1}, R_{6.2}, R_{4.1} \) and \(R_{3.1} - R_{3.4}\) satisfy also the weight condition \(R_{ij}^{kl} = 0, \text{ if } k+l \neq i+j \text{ (mod 3)}\). Several others satisfy the same condition (mod 2), e.g. \(R_{2.1} - R_{2.5}\). (ii) The upper triangular property is of course sensitive to index relabeling, for example 2 ↔ 3 often breaks it. Sometimes one can find an upper triangular transformation after which the solutions stays upper triangular even after 2 ↔ 3 exchange. This happens for the solutions \(R_{8.3}, R_{4.2}, R_{4.4}, R_{3.4}, R_{2.4}, R_{2.5}, R_{2.9}, R_{1.4}, R_{1.6}, R_{1.7} \) and \(R_{1.8}\), and they are presented in this special form. (iii) Some solutions show interesting parametric relations, in particular \(R_{9.1}, R_{7.1}, R_{6.1} \) and \(R_{1.10}\).

Now that we have these constant solutions several natural questions come up for further study, for example, can one add a spectral parameter, can one construct a corresponding universal R-matrix? The corresponding algebraic structures also need investigation.

Note for the readers of the preprint version: The bibliography may contain omissions, if you know additional references where explicit three-state upper triangular solutions have been presented, please let me know by e-mail to hietarin@utu.fi
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Appendix

In this appendix we give the breakdown of the original problem to smaller subsets. The primary classification proceeds by the upper triangular blocks on the diagonal. The “number of solutions” given below is the number before any trivial solutions or subcases were eliminated.

A Upper block of type $C_5$

We use scaling freedom to put all nonzero entries in $C_5$ to 1. The solution procedure produced 8 solutions, which were immediately reduced to 3 basic solutions.

B Upper block of type $C_4$

Let us scale so that $b = c = 1$ in $C_4$. In principle we could exclude lower blocks of type $C_5$ from the very beginning, but no such solutions were found anyway.

1. $a$ arbitrary, $c_5 \neq 0$ or $f_7 \neq 0$: 3 solutions, all have $a = 1$.
2. $a \neq 1$, $c_5 = f_7 = 0$: 22 solutions
3. $a = 1$, $c_5 = f_7 = 0$: 1) $b_7 \neq 0$, 4 solutions. 2) $b_7 = 0$, $a_8 = 1$, 9 solutions. 3) $b_7 = 0$, $a_8 = 0$: 35 solutions.

C Upper block of type $C_3$

We can scale $a = b = 1$

C.1 $c_5 \neq 0$ or $f_7 \neq 0$

no solutions

C.2 $c_5 = f_7 = 0, c \neq 1$

33 solutions

C.3 $c_5 = f_7 = 0, c = 1$

Now $C_3$ commutes with any upper triangular matrix with units on the diagonal. Using this we can put the lower block also into a semicanonical form. Solving some of the equations reveals that the diagonal entries of the lower block are also all = 1. If the lower block is of type $C_5$ a reflection takes the system to one studied before, same holds if the lower block is of type $C_2$. If the lower block is of type $C_4$, we get one solution.
What remains is a lower block of type $C_3$ with $a = c$, but possibly with $b = 0$. This again commutes with UT transformation matrix with units on the diagonal, and thus we will use this rotational freedom to put the center block in a semicanonical form.

1. Center block of type 5: no solutions.
2. Center block of type 2: 3 solutions.
3. Center block of type 3: 42 solutions.

D Upper block of type $C_2$

Now lower block must be of type $C_4$ or $C_1$, all others can be reflected to cases studied before.

1. $f_7 \neq 0$: No solutions.
2. $c_5 \neq 0$ or $f_7 = 0$: 2 solutions.
3. $c_3 \neq 1$, $c_5 = f_7 = 0$: 11 solutions
4. $c_3 = 1$, $c_5 = f_7 = 0$: In this case one quickly finds that the lower block must be

$$
\begin{pmatrix}
k & h & g \\
. & k & . \\
. & . & l
\end{pmatrix}.
$$

Here we need to analyse only the case $h = g = 0$ due to the following: 1) If $k = l$, $h \neq 0$ we can use using upper triangular transformations to put $g = 0$. Then lower block is of type $C_2$ which reflects to an upper block of type $C_4$ done earlier. 2) If $k = l$, $h = 0$, $g \neq 0$ lower block is of type $C_3$, done earlier. 3) If $k \neq l$, $h \neq 0$ we can transform lower block to type $C_2$ without changing the upper block. 4) In the remaining cases the lower block is diagonal, there were 15 solutions of this type.

E Upper block is nonunit diagonal, $c_5 \neq 0$, $f_7 \neq 0$

Let us normalize the diagonal elements as $(1, b, c)$. Here we may also assume that the lower block is diagonalizable by an upper triangular matrix, but maybe not yet in diagonal form. In most cases solving the first few equations yields directly a diagonal lower block, exceptions to this are discussed separately.

In general one finds that if $c_5 \neq 0$ or $f_7 \neq 0$ then $c_3 = b_2^2$, this implies in particular that in this section we must have $b_2 \neq 1$. After that one also finds quickly that the lower block must be

$$
\begin{pmatrix}
h & . & g \\
. & k & . \\
. & . & l
\end{pmatrix}.
$$
1. $b^3 \neq 1, b^2 \neq 1$: 3 solutions.

2. $b = -1$: As mentioned above we must now have $c_3 = 1$. If now $l = h$ in (8) the system can be reflected to one of the cases studied before. If $l \neq h$ one can use the transformation

$$
\begin{pmatrix}
1 & y \\
1 & 1 \\
& 1
\end{pmatrix},
$$

which commutes with $diag(1,-1,1)$, to put $g = 0$. One finds 4 solutions.

3. $b^3 = 1, b \neq 1$: 5 solutions.

**F  Upper block is nonunit diagonal, $c_5 \neq 0, f_7 = 0$**

The same comments as above holds in this case and we just give the results.

1. $b^3 \neq 1, b^2 \neq 1$: 4 solutions.

2. $b = -1$: 4 solutions.

3. $b^3 = 1, b \neq 1$: 6 solutions.

**G  Upper block is nonunit diagonal, $c_5 = 0, f_7 \neq 0$**

As before, the lower block must be as in (8). If $l = h$ we can reflect the system to case F or C, else we eliminate $g$ and reflect to case F.

**H  Upper block is nonunit diagonal, $c_5 = 0, f_7 = 0$**

**H.1  The diagonal elements $(1, b, c)$ are all different**

76 solutions

**H.2  $c = 1, b \neq 1$**

Again one finds that the lower block must be of type (8) and as in H.2 one can show that the lower block must in fact be diagonal. If its diagonal elements are all different we reflect it to H.1. Thus we get subcases according to the diagonal entries of the lower block:

1. Lower block = $diag(l,k,l)$: 21 solutions

2. Lower block = $diag(k,k,l)$ or $diag(k,l,l)$: 37 solutions.

3. Lower block = $diag(l,l,l)$: 19 solutions.
H.3  \( b = 1, c \neq 1 \)

Now we may assume that the lower block is transformable to a diagonal form, which has at most two different elements and is not \((h,l,h)\), all other cases can be reflected to ones studied before. Then one finds that the lower block must be

\[
\begin{pmatrix} h & g & . \\ . & k & . \\ . & . & l \end{pmatrix}.
\]

Since the upper block stays invariant under transformations

\[
\begin{pmatrix} 1 & y & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix},
\]

we can transform the case \( h \neq k \) to a diagonal form while the case \( h = l \) reflects to \( C_4 \).

1. Lower block is \( \text{diag}(h,h,l) \): 36 solutions.
2. Lower block is \( \text{diag}(h,l,l) \): 18 solutions.
3. Lower block is \( \text{diag}(l,l,l) \): 36 solutions.

H.4  \( b = c \neq 1 \)

Again one can transform the lower block to something that reflects to one of the cases before, except for two cases:

1. Lower block is \( \text{diag}(h,h,l) \): 17 solutions.
2. Lower block is \( \text{diag}(l,l,l) \): 40 solutions

I  Upper and lower blocks are unit matrices

In this case we have the full transformation freedom left and we use it to bring the center block into the semicanonical form (recall \( C_2 \) reflects to \( C_4 \))

1. Center block type \( C_5 \): no solutions.
2. Center block type \( C_2 \): 2 solutions.
3. Center block type \( C_3 \): 13 solutions.
4. Center block type \( C_1 \): Let us call the diagonal elements of the center block as \((d,f,g)\). 1) \( d, f, g \) all different, 4 solutions. 2) \( g = d, f \neq d \), 6 solutions. 3) \( d = f, f \neq g \), 9 solutions. 4) Center block unit matrix, 9 solutions.