GLOBAL GRADIENT ESTIMATES FOR THE $p(\cdot)$-LAPLACIAN

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Abstract. We consider Calderón-Zygmund type estimates for the non-homogeneous $p(\cdot)$-Laplacian system

$$-\text{div}(|Du|^{p(\cdot)}-2Du) = -\text{div}(|G|^{p(\cdot)}-2G),$$

where $p$ is a variable exponent. We show that $|G|^{p(\cdot)} \in L^q(\mathbb{R}^n)$ implies $|Du|^{p(\cdot)} \in L^q(\mathbb{R}^n)$ for any $q \geq 1$. We also prove local estimates independent of the size of the domain and introduce new techniques to variable analysis. The paper is an extension of the local estimates of Acerbi-Mingione [2].

1. Introduction

In recent years there has been an extensive interest in the field of variable exponent spaces $L^{p(\cdot)}$. Different from the classical Lebesgue spaces $L^p$, the exponent is not a constant but a function $p = p(x)$.

The increasing interest was motivated by the model for electrorheological fluids [23, 24]. Those are smart materials whose viscosity depends on the applied electric field. This is modeled via a dependence of the viscosity on a variable exponent. Electrorheological fluids can for example be used in the construction of clutches and shock absorbers.

Further applications of the variable exponent spaces can be found in the area of image reconstruction. Here, the change of the exponent is used to model different smoothing properties according to the edge detector. This can be seen as a hybrid model of standard diffusion and the TV-model introduced by [3].

A model problem for image reconstruction as well as a starting point for the study of electrorheological fluids is the $p(\cdot)$-Laplacian system. We consider local, weak solutions $u \in W^{1, p(\cdot)}(\Omega)$ of the non-homogeneous $p(\cdot)$-Laplacian system

$$-\text{div}(|Du|^{p(\cdot)}-2Du) = -\text{div}(|G|^{p(\cdot)}-2G).$$

where $\Omega \subset \mathbb{R}^n$ is an open set and $u : \Omega \to \mathbb{R}^N$. Note that the specific form of the right hand side is no restriction, but allows an easier formulation of our results.

Our main result is that $L^q$ integrability of $|G|^{p(\cdot)}$ implies $L^q$ integrability of $|Du|^{p(\cdot)}$. We present local and global versions of this result, see Subsection 4.5. For the exponent we assume the vanishing log-Hölder continuity

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introduced in [1] (see [2,2] for the definition). Our result is an extension of the results of Acerbi and Mingione in [2], where the authors prove the local version of the higher integrability. The main difference to [2] is that our estimates have a controllable dependence of the size of the ball, where higher integrability is considered. This allows to extend the result to the whole spaces as well as to countable families of balls. The proof of these estimates require a finer analysis of the underlying $p(\cdot)$-structure. This refinement simplifies the proof significantly.

Higher integrability of the non-linear $p$-Laplace (which corresponds to a constant exponent $p$) was introduced in [10]. The principle is known under the name \emph{Non-linear Calderón-Zygmund Theory}. In the limiting case $q = \infty$, the space $L^\infty$ has to be replaced by BMO as in the linear Calderón-Zygmund theory. Corresponding BMO results for the $p$-Laplace system has been shown in [6,11]. In [10,12] a nonlinear Calderón-Zygmund theory was developed for the (constant) $p$-Stokes equation, which have some implications for the $p$-Navier-Stokes system. Furthermore higher integrability for small exponents and the $p$-Stokes system was shown in [27] and [7, Chapter 7]. One future aim would be to combine these results with the variable exponent technique presented in this work to gain a nonlinear Calderón-Zygmund theory for electrorheological fluids, that includes large exponents and BMO estimates.

This paper is formulated only in terms of the $p(\cdot)$-Laplacian in order to simplify the notations. However, it is possible to work in a more general setting and to consider the equation

$$-\text{div}(A(\cdot,Du)) = -\text{div}(A(\cdot,G)).$$

Our estimates in Section 3 are only based on the following two estimates:

$$|A(x,z)| \leq c_1 |z|^{p(x)-1} + h_1(x),$$
$$|A(x,z) \cdot z| \geq c_2 |z|^{p(x)} - h_2(x)$$

for all $x \in \Omega$ and all $z \in \mathbb{R}^{N\times n}$ and $h_1, h_2 \in L^1(\Omega) \cap L^\infty(\Omega)$. No more additional assumptions on $A$ are needed!

For Section 4 our estimates additionally need

$$|A(x,z) - A(y,z)| \leq c_3 |p(x) - p(y)| \log |z| \left( |z|^{p(x)-1} + |z|^{p(y)-1} \right),$$

(1.2)

$$|z|^{p(x)} \leq c_4 |\xi|^{p(x)} + c_4 \left( A(x,z) - A(x,\xi) \right) \cdot (z - \xi)$$

(1.3)

for all $x \in \Omega$ and $z, \xi \in \mathbb{R}^n$. However, we use some estimates (for our homogeneous comparison solution) which requires more assumptions on $A$, but not on $p(\cdot)$. (See Theorem 4.11 and Theorem 4.13). The necessary assumption for these theorems can be found in the given references. Certainly all estimates mentioned above are valid in case of the $p(\cdot)$-Laplacian; i.e. for $A(x,z) := |z|^{p(\cdot)-2}z$. Note that the same technique allows to treat the cases $A(x,z) = (\gamma + |z|)^{p(\cdot)-2}z$ or $A(x,z) = (\gamma^2 + |z|^2)^{\frac{p(\cdot)-2}{2}}z$ for some $\gamma \geq 0$.

The structure of the paper is as follows. In Section 2 we introduce the necessary notation. In particular, the Lebesgue spaces with variable exponents and the (vanishing) log-Hölder continuity is introduced. In Section 3 we show that the solutions to (1.1) satisfy a Gehring type estimate. This
corresponds to the higher integrability $|Du|^{p(\cdot)} \in L^q$ with $q$ only slightly bigger than one. The proof goes by standard arguments via a Caccioppoli estimate and a reverse Hölder’s inequality. In Section 4 we prove the main results on higher integrability for large exponents. The arguments uses re-
distributional estimates (good-$\lambda$ estimates), which are based on comparison estimates.

2. Notation and Structure

By $c$ we denote a generic constant, whose value may change between appear-
ces even within a single line. By $f \sim g$ we mean that there exists $c$ such that $\frac{1}{c}f \leq g \leq cf$.

For a measurable set $E \subset \mathbb{R}^n$ let $|E|$ be the Lebesgue measure of $E$ and $\chi_E$ its characteristic function. For an open set $\Omega \subset \mathbb{R}^n$ let $L^0(\Omega)$ denote the set of measurable functions $f : \Omega \to \mathbb{R}$ and let $L^1_{\text{loc}}$ denote the set of locally integrable functions (integrable on compact subsets). For $0 < |E| < \infty$ and $f \in L^1(E)$ we define the mean value of $f$ over $E$ by

$$\langle f \rangle_E := \frac{1}{|E|} \int_E f \, dx.$$ 

By $L^{s,\infty}(\mathbb{R}^n) := \{f \in L^0(\mathbb{R}^n) : \|f\|_{s,\infty} < \infty\}$ with $s \in [1, \infty)$ and

$$\|f\|_{s,\infty} := \sup_{\lambda > 0} \|\lambda \chi_{\{|f| > \lambda\}}\|_s = \sup_{\lambda > 0} \lambda \{\|f\| > \lambda\}^{\frac{1}{s}}$$

we denote the Marcinkiewicz spaces.

Let us introduce the spaces of variable exponents $L^{p(\cdot)}$. We use the no-
tation of the recent book [9]. We define $P(\Omega)$ to consist of all $p \in L^0(\Omega)$ with $p : \Omega \to [1, \infty]$ (called variable exponents). For $p \in P(\Omega)$ we define $p^-_\Omega := \inf_{E} p$ and $p^+_\Omega := \sup_{E} p$. For non-localized results we omit the index $\Omega$ of $p^+_\Omega$ and $p^-_\Omega$. Note that the higher integrability results in this article are restricted to the case $1 < p^- \leq p^+ < \infty$.

For $p \in P(\Omega)$ with $p^+ < \infty$ the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as

$$L^{p(\cdot)}(\Omega) := \{f \in L^0(\Omega) : \|f\|_{L^{p(\cdot)}(\Omega)} < \infty\},$$

where

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \frac{|f(x)|^{p(x)}}{\lambda^{p(x)}} \, dx \leq 1 \right\}.$$ 

The generalized Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of those $L^1_{\text{loc}}(\Omega)$-functions whose norm

$$\|f\|_{W^{1,p(\cdot)}(\Omega)} = \|f\|_{L^{p(\cdot)}(\Omega)} + \|Df\|_{L^{p(\cdot)}(\Omega)};$$

is finite, where $Df$ is the distributional derivative of $f$.

If $p$ is constant, then $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ coincide with the classical Lebesgue and Sobolev spaces. The spaces $L^{p(\cdot)}$ where introduced by [22]. Many properties of $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ can be found in [21, 15] and the book [9].
We say that a function \( \alpha : \Omega \to \mathbb{R} \) is \( \log \)-H"older continuous on \( \Omega \) if there exists a constant \( c \geq 0 \) and \( \alpha_{\infty} \in \mathbb{R} \) such that
\[
|\alpha(x) - \alpha(y)| \leq \frac{c}{\log(e + 1/|x - y|)} \quad \text{and} \quad |\alpha(x) - \alpha_{\infty}| \leq \frac{c}{\log(e + |x|)}
\]
for all \( x, y \in \Omega \). The first condition describes the so called local log-H"older continuity and the second the decay condition. The smallest such constant \( c \) is the log-H"older constant of \( \alpha \). The decay condition is always satisfied if \( \Omega \) is bounded. We define \( P_{\text{log}}(\Omega) \) to consist of those exponents \( p \in P(\Omega) \) for which \( \frac{1}{p} : \Omega \to [0,1] \) is log-H"older continuous on \( \Omega \). If \( p \in P(\Omega) \) is bounded, then \( p \in P_{\text{log}}(\Omega) \) is equivalent to the log-H"older continuity of \( p \). However, working with \( 1/p \) gives better control of the constants especially in the context of averages and maximal functions. Therefore, we define \( c_{\text{log}}(p) \) as the log-H"older constant of \( 1/p \). Expressed in \( p \) we have for all \( x, y \in \Omega \)
\[
|p(x) - p(y)| \leq \frac{(p^+)^2 c_{\text{log}}(p)}{\log(e + 1/|x - y|)} \quad \text{and} \quad |p(x) - p_{\infty}| \leq \frac{(p^+)^2 c_{\text{log}}(p)}{\log(e + |x|)}.
\]

In this work cubes are always parallel to the axes and are usually called \( Q \). We write \( \ell(Q) \) for the side length of \( Q \) and \( \text{center}(Q) \) for the center of \( Q \). By \( \gamma Q \) with \( \gamma > 0 \) we mean the cube scaled by the factor \( \gamma \) with the same center as \( Q \).

If \( p \in P_{\text{log}}(\Omega) \) with \( p^- > 1 \), then the Hardy-Littlewood maximal operator \( M \)
\[
(Mf)(x) := \sup_{x \ni Q} \frac{1}{Q} \int |f(y)| \, dy,
\]
is bounded on \( L^{p(\cdot)}(\mathbb{R}^n) \), where the supremum is taken over all cubes (with sides parallel to the axes) containing \( x \). The operator norm of \( M \) depends only on \( c_{\text{log}}(p) \) and \( p^- \). This result goes back to [8, 5]. The most advanced form of this result can be found in [9, Theorem 4.3.8]. The boundedness of \( M \) has many interesting consequences like Sobolev embeddings and the boundedness of singular integrals, see [4, 9].

In the case of higher-integrability of the \( p(\cdot) \)-Laplacian system a slightly stronger condition is needed. It’s local version has been introduced by Acerbi and Mingione in [1]. It’s natural decay counterpart has been introduced in [25]. We say that a function \( \alpha : \Omega \to \mathbb{R} \) is \( \text{vanishing log-H"older continuous} \) on \( \Omega \) if there exists \( \alpha_{\infty} \) such that for every \( \varepsilon > 0 \) there exists \( r, R > 0 \) such that
\[
|\alpha(x) - \alpha(y)| \leq \frac{\varepsilon}{\log(e + 1/|x - y|)}
\]
for all \( x, y \) with \( |x - y| \leq r \) and all \( x, y \) with \( |x|, |y| \geq R \) and
\[
|\alpha(z) - \alpha_{\infty}| \leq \frac{\varepsilon}{\log(e + |z|)}
\]
for all \( z \) with \( |z| \geq R \). We say that \( p \in P_{\text{van}}^{\log}(\Omega) \) if \( \frac{1}{p} \) is vanishing log-H"older continuous. For bounded exponents this is equivalent to the vanishing log-H"older continuity of \( p \) itself.
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The necessity of the extra “vanishing” condition is in analogy to the situation of the higher integrability results for the \( p \)-Laplacian with coefficients, see [19]: here it is necessary that the coefficients are in VMO (vanishing mean oscillation) rather than just in BMO (bounded mean oscillation). As in [19] the vanishing log-Hölder condition can be replaced by smallness of the log-Hölder constant \( c_{\log} \), see Remark [1.8]

3. A Gehring type estimate

In this section we show that local solutions of the \( p(\cdot) \)-Laplacian system satisfy a Caccioppoli estimate. The next step is a reverse Hölder estimate. We present the tools from the Lebesgue and Sobolev spaces of variable exponents, which are necessary for this step. In the end we apply the classical Gehring Lemma to the quantity \( |Du|^{p(\cdot)} \) to derive higher integrability for small exponents.

Let us begin with the Caccioppoli estimate.

Lemma 3.1 (Caccioppoli estimate). Let \( Q \subset \mathbb{R}^n \) be a cube (or ball) with length \( R \) and \( p \in \mathcal{P}(2Q) \) with \( p^+ < \infty \). Then the local weak solution \( u \in W^{1,p}(2Q) \) of (1.1) satisfies

\[
\int_Q |Du|^{p(\cdot)} \, dx \leq c \left( \int_{2Q} \frac{|u - \langle u \rangle_{2Q}|^{p(\cdot)}}{R} \, dx + \int_{2Q} |G|^{p(\cdot)} \, dx \right).
\]

The constant depends only on \( p^+ \).

Proof. The Caccioppoli estimate is proved straightforward by using the test function \( \eta^k(u - \langle u \rangle_{2Q}) \); here \( \chi_Q \leq \eta \leq \chi_{2Q} \) is the usual cut off function with \( |D\eta| \leq \frac{c}{R} \) and \( k > p^+ \) a fixed integer. Therefore we get by (1.1)

\[
\int \eta^k |Du|^{p(\cdot)} \, dx \leq k \int |G|^{p(\cdot)-1} |D\eta||u - \langle u \rangle_{2Q}| \eta^{k-1} \, dx
\]

\[
+ \int \eta^k |G|^{p(\cdot)-1} |Du| \, dx
\]

\[
+ k \int |Du|^{p(\cdot)-1} \eta^{k-1} |D\eta||u - \langle u \rangle_{2Q}| \, dx
\]

=: (I) + (II) + (III).

Now, \( |D\eta| \leq \frac{c}{R} \) and Young’s inequality imply

\[
(I) \leq c \int_{2Q} |G|^{p(\cdot)} \, dx + c \int_{2Q} \frac{|u - \langle u \rangle_{2Q}|^{p(\cdot)}}{R} \, dx,
\]

\[
(II) \leq c \int_{2Q} |G|^{p(\cdot)} \, dx + \int \eta^k |Du|^{p(\cdot)} \, dx,
\]

\[
(III) \leq \delta \int \eta^{(k-1)p'(\cdot)} |D\eta|^{p'(\cdot)} \, dx + c \int_{2Q} \frac{|u - \langle u \rangle_{2Q}|^{p(\cdot)}}{R} \, dx.
\]

Since \( (k-1)p'(\cdot) \geq k \), we can absorb the terms with the factor \( \delta \) on the left hand side of (3.1) to get the claim. □
To deduce the reverse Hölder estimate from our Caccioppoli estimate, we need a Sobolev-Poincaré inequality for variable exponents. The proof is based on a Jensen type inequality, known as the \textit{key estimate for variable exponents}. It first appeared in a simpler form in \cite{8} and was later improved in \cite[Theorem 4.2.4]{9}. We will use a further improvement \cite[Theorem 1]{13}, a further improvement which allows to apply the key estimate to a larger class of functions. The idea of this refinement goes back to Schwarzacher \cite{25}.

**Lemma 3.2** (\textit{p(·)-Jensen’s inequality}). Let $Q \subset \mathbb{R}^n$ be a cube (or ball), $p \in \mathcal{P}^{\log}(Q)$ with $p^+ < \infty$, $m > n$, $\beta \geq 0$ and $K_1 \geq 1$. Then

$$
\left( \frac{\int_Q |f|^p \ dy}{Q} \right)^{p(x)} \leq c \int_Q |f|^{p(y)} \ dy + c \int_Q (e + |y|)^{-m} \ dy
$$

for all $x \in Q$ and all $f \in L^{p(\cdot)}(Q)$ satisfying

$$
\int_Q |f| \ dy \leq K_1 \max \{1, |Q|^{-\beta}\},
$$

where $c$ depends only on $c_{\log}(p), m, n, \beta, K_1, p^+$.

**Remark 3.3.** Let us point out that Lemma 3.2 is valid for all functions $f \in L^1 + L^\infty$, since

$$
\int_Q |f| \ dx \leq 2 \|f\|_{L^1 + L^\infty} \max \{1, |Q|^{-1}\}.
$$

Since $L^{p(\cdot)} + L^\infty \hookrightarrow L^1 + L^\infty$, Lemma 3.2 is a stronger version of Theorem 4.2.4 of \cite{9}, which proves the same result under the condition $f \in L^{p(\cdot)} + L^\infty$.

**Remark 3.4.** Whenever $x \in Q$ satisfies $|x| = \sup_{y \in Q} |y|$, then by the geometry of $(e + |\cdot|)^{-m}$, we have

$$
(e + |x|)^{-m} \leq \int_Q (e + |y|)^{-m} \ dy.
$$

In this case we can remove one term in the estimate of Lemma 3.2.

Based on the new key estimate it was possible to prove an refined version of the Sobolev Poincaré inequality. See \cite[Proposition 8.2.11]{9} and \cite[Corollary 3]{13} for a proof.

**Proposition 3.5** (Sobolev Poincaré). Let $Q \subset \mathbb{R}^n$ be a cube (or ball) with length $R$, $p \in \mathcal{P}^{\log}(Q)$ with $p^+ < \infty$, and $f \in W^{1,p(\cdot)}(Q)$ with $\|Df\|_{L^1 + L^\infty} \leq K_2$. For $s \in [1, \min\{\frac{n}{n-1}, p_Q\})$ and $m > n$ there exists a constant $c$ depending on $s, p_Q, c_{\log}(p), m, n, K_2$ for which

$$
\int_Q \left( \frac{|f - (f)_{Q,R}|}{R} \right)^{p(\cdot) \frac{s}{R}} \ dx \leq c \left( \int_Q |Df|^{p(\cdot)} \ dx \right)^s + c \int_Q h \ dx,
$$

where $h = |f - (f)_{Q,R}|$.
Lemma 3.2 applied to $Df$ (Gehring’s lemma)

Theorem 3.7 (Reverse Hölder inequality)

and

Lemma 3.6

Proposition 3.5.

Let $h(x) := (e + |x|)^{-m}$. It is possible to replace the condition $\|Df\|_{L^1 + L^\infty}$ by the condition of Lemma 3.2 applied to $Df$.

Now, the reverse Hölder estimate follows directly by Lemma 3.1 and Lemma 3.6.

Lemma 3.6 (Reverse Hölder estimate). Let $Q \subset \mathbb{R}^n$ be a cube (or ball) and $p \in P^{\log}(2Q)$. For a local weak solution $u \in W^{1,p}(2Q)$ of (1.1) we have

$$
\frac{1}{2Q} \int_Q |Du|^{p(x)} \, dx \leq c \left( \frac{1}{2Q} \int_{2Q} |Du|^{p(x)/s} \, dx \right)^s + c \frac{1}{2Q} \int_{2Q} |G|^{p(x)} \, dx + c \frac{1}{2Q} \int_{2Q} h \, dx,
$$

for all $s \in [1, \min\{p^-, \frac{n}{n-1}\})$. The constant depends on $p_0^+, c_{\log}(p), m, n, s$ and $\|Du\|_{L^\infty + L^1(2Q)}$.

Let us restate Gehring’s Lemma at this point [17, Section 4].

Theorem 3.7 (Gehring’s lemma). Let $f \in L^s(\Omega)$ and $g \in L^q(\Omega)$. If the reverse Hölder inequality

$$
\left( \frac{1}{2Q} \int_{2Q} |f|^s \, dx \right)^{\frac{1}{s}} \leq C_{Ge} \left( \frac{1}{2Q} \int_{2Q} |f| \, dx + \left( \frac{1}{2Q} \int_{2Q} |g|^s \, dx \right)^{\frac{1}{s}} \right)
$$

is satisfied for an $s > 1$ and all $2Q \subset \Omega$, then there exists an $m_0 > 1$ depending on $c_{Ge}, s, q$ and the dimension such that

$$
\left( \frac{1}{2Q} \int_{2Q} |f|^{s\mu} \, dx \right)^{\frac{1}{s\mu}} \leq c \left( \frac{1}{2Q} \int_{2Q} |f|^s \, dx \right)^{\frac{1}{s}} + c \left( \frac{1}{2Q} \int_{2Q} |g|^{s\mu} \, dx \right)^{\frac{1}{s\mu}},
$$

for all $1 < \mu < m_0$, all $2Q \subset \Omega$. The constant $c$ depends on $n, s, C_{Ge}$.

The following corollary is a consequence of Gehring’s Lemma and Lemma 3.6.

Corollary 3.8. Let $p \in P^{\log}(\Omega)$, $|G|^{p(x)} \in L^q(\Omega)$ with $q > 1$. Let $u \in W^{1,p(\cdot)}(\Omega)$ be a local weak solution of (1.1). Then there exists $m_0 \in (1, q]$ such that for all cubes $Q$ with $2Q \subset \Omega$ and all $\mu \in [1, m_0]$ there holds

$$
\left( \frac{1}{2Q} \int_{2Q} |Du|^{p(\cdot)\mu} \, dx \right)^{\frac{1}{p(\cdot)\mu}} \leq c \left( \frac{1}{2Q} \int_{2Q} |Du|^{p(\cdot)} \, dx + \left( \frac{1}{2Q} \int_{2Q} |G|^{p(\cdot)\mu} \, dx + \right. \right. + \left. \left. \frac{1}{2Q} \int_{2Q} h^\mu \, dx \right)^{\frac{1}{p(\cdot)\mu}},
$$

where $h(x) = (e + |x|)^{-m}$ with $m > n$. The constant $c$ depends on $n, c_{\log}(p), p^-, p^+, m$ and $\|Du\|_{L^1 + L^\infty(\Omega)}$.

Remark 3.9. By a standard covering argument it is possible to replace the pair $(Q, 2Q)$ in Lemma 3.7, Lemma 3.6, Theorem 3.7 and Corollary 3.8 by $(Q, aQ)$ for any $a > 1$. The constant then depends on $a$. 

4. Higher Integrability

In this section we prove the higher integrability of our local solutions. We will derive a local result, with controllable dependence on the size of the ball. Theorem 4.39 such that the global result follows as a corollary. For better readability we split the section into several parts. Firstly, we recall the technique of redistributional estimates (good λ-estimates). Secondly, we split the corresponding level sets into cubes. Thirdly, we define a local comparison problem of p-Laplace type with constant exponent. Fourth, we derive estimates controlling the distance of the local auxiliary problem to our original system. This enables us in our fifth step to prove our main result of higher integrability.

4.1. Redistributional Estimate. The higher integrability of our solutions will be achieved by redistributional estimates also known as good-λ-estimates. Let us briefly describe this well known technique: Assume \( f \) and \( g \) to be integrable, non-negative functions. Moreover, assume that the following redistributional estimate holds: There exists \( \kappa > 1, \varepsilon > 0 \) and \( \delta = \delta(\varepsilon) > 0 \) with \( \delta(\varepsilon) \to 0 \) for \( \varepsilon \to 0 \) such that for all \( \lambda > 0 \)

\[
(4.1) \quad |\{|f| > \kappa \lambda\} \cap \{|g| \leq \varepsilon \lambda\}| \leq \delta |\{|f| > \lambda\}|.
\]

A direct consequence of this estimate is

\[
|\{|f| > \kappa \lambda\}| \leq \delta |\{|f| > \lambda\}| + |\{|g| > \varepsilon \lambda\}|,
\]

which basically shows that the level sets of \( f \) can be controlled in a certain sense by the ones of \( g \). (Later in our setting we will choose \( \kappa = 2^{n+1} c_4 \) with \( c_4 \) from (1.3).) Multiplying this estimate by \( \lambda^{q-1} \) with \( q \in [1, \infty) \) and integrating over \( \lambda \in (0, \infty) \) gives for suitable small \( \delta \) (formally)

\[
\int_0^\infty \lambda^q |\{|f| > \lambda\}| \, d\lambda \leq c \int_0^\infty \lambda^q |\{|g| > \lambda\}| \, d\lambda,
\]

where \( c \) depends on \( \kappa, \varepsilon \) and \( q \). In other words,

\[
\|f\|_q \leq c \|g\|_q.
\]

We will apply this argument to the functions \( f = M_{\Omega}^n(|Du|^{p(\cdot)}) \) and \( g = M_{m_0,\Omega}^n(|G|^{p(\cdot)} + h) \), where \( M_{\Omega}^n \) and \( M_{m_0,\Omega}^n \) are localized, dyadic maximal operators which we will introduce below and \( h(x) := (e + |x|)^{-2n} \).

4.2. Maximal operators and coverings. Let us introduce the localized maximal operators. By \( \Delta \) we denote the standard set of (open) dyadic cubes \( (2^k a) + (0, 2^k)^n \) with \( k \in \mathbb{Z} \) and \( a \in \mathbb{Z}^n \). Now, take an arbitrary, open cube \( Q' \subset \mathbb{R}^n \) and let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear mapping, that maps \((0,1)^n\) onto \( Q' \). Then cubes \( \{T(Q) : Q \in \Delta\} \) are called the \( Q' \)-dyadic cubes. By \( \Delta_{Q'} \) we denote the \( Q' \)-dyadic sub-cubes of \( Q' \), i.e., \( \Delta_{Q'} := \{T(Q) : T(Q) \subset Q', Q \in \Delta\} \). Note that two dyadic cubes from \( \Delta_{Q'} \) are either disjoint or one is a subset of the other. The predecessor of a \( Q' \)-dyadic cube \( Q \) is the unique \( Q' \)-dyadic cube \( Q^{pre} \), which contains \( Q \) and has double the diameter of \( Q \).
For $s \in [1, \infty)$ we define for all $x \in \mathbb{R}^n$

$$(M^s_{Q',s}f)(x) := \sup_{Q \in \Delta_{Q'}: x \in Q} \left( \int_{2Q} |f|^{s} \, dz \right)^{\frac{1}{s}},$$

$$(M^s_{Q'}f)(x) := (M^s_{Q',1}f)(x),$$

where the supremum is taken over all $Q'$-dyadic sub-cubes of $Q'$ which contain $x$ in its closure. In particular, $M^s_{Q',s}f$ is zero outside of $Q'$ and depends only on the values of $f$ on $2Q'$. It is well known that $M^s_{Q',s}$ is bounded from $L^q(2Q')$ to $L^q(\mathbb{R}^n)$ for $q > s$ and from $L^s(\mathbb{R}^n)$ to the Marcinkiewicz space $L^{s,\infty}(\mathbb{R}^n)$.

From now on we fix an open cube $\Omega \subset \mathbb{R}^n$ such that $u$ is a weak local solution of \ref{1.1} on $2\Omega$. Our goal is to prove higher integrability of $|Du|^{p(\cdot)}$ on $\Omega$.

We define our level sets

\begin{equation}
\begin{aligned}
\mathcal{O}_\lambda &:= \{ M^s_\Omega(|Du|^{p(\cdot)}) > \lambda \} \subset \Omega, \\
U^{\kappa,\varepsilon}_\lambda &:= \{ M^s_{\Omega}(|Du|^{p(\cdot)}) > \kappa \lambda, M^s_{\omega,\Omega}(\{|G|^{p(\cdot)} + h\}) \leq \varepsilon \lambda \} \subset \Omega.
\end{aligned}
\end{equation}

Our goal is to show

\begin{equation}
|U^{\kappa,\varepsilon}_\lambda| \leq \delta |\mathcal{O}_\lambda|
\end{equation}

with $\delta = \delta(\varepsilon) \to 0$ for $\varepsilon \to 0$. This together with the arguments similar to the ones in the subsection will give the desired result of higher integrability.

Since we are mainly interested in a result of local higher integrability, it suffices to consider \ref{4.3} for large values of $\lambda$. In particular, we define

\begin{equation}
\lambda_0 := \int_{2\Omega} |Du|^{p(\cdot)} \, dx.
\end{equation}

We can assume without loss of generality that $\lambda_0 > 0$, since otherwise $u$ is locally a constant.

To prove \ref{4.3} we will decompose $\mathcal{O}_\lambda$ into suitable dyadic cubes, which we will construct now. For every $x \in \mathcal{O}_\lambda$, there exists a largest $\Omega$-dyadic cube $Q_x$ with the property $\int_{2Q_x} |Du|^{p(\cdot)} \, dx > \lambda$. In particular, any $Q' \in \Delta_\Omega$ with $Q' \supset Q_x$ satisfies $\int_{2Q'} |Du|^{p(\cdot)} \, dx \leq \lambda$.

The family $\{Q_x : x \in \mathcal{O}_\lambda\}$ covers the set $\mathcal{O}_\lambda$. Since the dyadic cubes have a natural order (if two dyadic cubes intersect, one of them contains the other), the sub-family of maximal cubes still covers $\mathcal{O}_\lambda$. We denote this at most countable sub-family by $\{Q_j\}$. In particular, we have

$$\lambda < \int_{2Q} |Du|^{p(\cdot)} \, dx.$$

Since $\lambda \geq \lambda_0$, we have

$$\int_{2\Omega} |Du|^{p(\cdot)} \, dx = \lambda_0 \leq \lambda < \int_{2Q_j} |Du|^{p(\cdot)} \, dx \leq \frac{[\Omega]}{|Q_j|} \int_{2\Omega} |Du|^{p(\cdot)} \, dx.$$

This implies $|Q_j| < |\Omega|$. In particular, we know that $Q_j$ is a proper $\Omega$-dyadic sub-cube of $\Omega$. Let $Q_{j,\text{pre}}$ denote the $\Omega$-dyadic predecessor of $Q_j$. 
Then \( Q_j^{\text{pre}} \in \Delta \Omega \). Since the \( Q_j \) (former \( Q_x \)) were chosen to be maximal we have \( \int_{2Q_j^{\text{pre}}} |Du|^{p(\cdot)} \, dx \leq \lambda \). This and \( 3Q_j \subset 2Q_j^{\text{pre}} \) implies

\[
(4.5) \quad \lambda < \int_{2Q_j} |Du|^{p(\cdot)} \, dx \leq \frac{3^n}{2^n} \int_{3Q_j} |Du|^{p(\cdot)} \, dx \leq 2^n \int_{2Q_j^{\text{pre}}} |Du|^{p(\cdot)} \, dx \leq 2^n \lambda.
\]

Our goal was to prove the estimate \( |U_{\lambda}^{\kappa,\varepsilon}| \leq \delta |O_\lambda| \). Since the dyadic \( Q_j \) cover the set \( O_\lambda \) it suffices to prove

\[
(4.6) \quad |Q_j \cap U_{\lambda}^{\kappa,\varepsilon}| \leq \delta |Q_j|.
\]

We prove this estimate in the next section. This estimate is obvious if \( Q_j \cap U_{\lambda}^{\kappa,\varepsilon} = \emptyset \). Therefore, we will assume in the following, that

\( Q_j \cap U_{\lambda}^{\kappa,\varepsilon} \neq \emptyset \).

In this case we find \( x_j \in Q_j \cap U_{\lambda}^{\kappa,\varepsilon} \) with \( M_{m_0,\Omega}^*((|G|^{p(\cdot)} + h))(x_j) \leq \varepsilon \lambda \), which implies

\[
(4.7) \quad \left( \int_{3Q_j} (|G|^{p(\cdot)} + h)^{m_0} \, dx \right)^{\frac{1}{m_0}} \leq \left( \frac{3^n}{2^n} \int_{2Q_j^{\text{pre}}} (|G|^{p(\cdot)} + h)^{m_0} \, dx \right)^{\frac{1}{m_0}} \leq \frac{3^n}{2^n} \varepsilon \lambda.
\]

By Corollary 3.3, Remark 3.3, (4.5) and (4.7) we have

\[
(4.8) \quad \left( \int_{2Q_j} |Du|^{p(\cdot)m_0} \, dx \right)^{\frac{1}{m_0}} \leq c \int_{3Q_j} |Du|^{p(\cdot)} \, dx + c \left( \int_{3Q_j} (|G|^{p(\cdot)} + h)^{m_0} \, dx \right)^{\frac{1}{m_0}} \leq c \lambda.
\]

### 4.3. Comparison Problem

If the right hand side \( G \) of our system is locally zero and \( p \) is locally constant, then \( u \) is locally a \( p \)-harmonic function with all its nice regularity properties. If \( G \) is non-zero but "small", then \( u \) is still close to a \( p(\cdot) \)-harmonic function (where \( G = 0 \)). This allows to transfer some of the regularity results from the \( p(\cdot) \)-harmonic function to \( u \). This is the well known comparison principle. Unfortunately, \( p(\cdot) \)-harmonic functions do not have as nice regularity properties as \( p \)-harmonic functions with constant \( p \); so it makes more sense to compare \( u \) to a \( p_j \)-harmonic function, where \( p_j \) is a local constant approximation of \( p(\cdot) \) on \( Q_j \). Certainly, for a good comparison \( p \) should not vary to much.

In the following we will define our comparison system. For every cube \( Q_j \) let \( y_j \) denote a point of \( 2Q_j \) furthest away from the point zero, i.e we choose \( y_j \in 2Q_j \) such that \( |y_j| = \sup_{x \in 2Q_j} |x| \). Now, define \( p_j := p(y_j) \) as an approximation of \( p \). We define our comparison system by

\[
(4.9) \quad \int_{2Q_j} A_j(Dw_j)D\varphi \, dx = 0 \text{ for all } \varphi \in W^{1,p_j}(2Q_j)
\]

\[
w_j = u \text{ on } \partial(2Q_j),
\]

where \( A_j(Dw_j) := |Dw_j|^{p_j-2}Dw_j \). We are looking for solutions \( w_j \) in \( W^{1,p_j}(2Q_j) \).
Note that for small cubes $Q_j$ the choice of $p_j$ is not important and one could take any $p(x)$ with $x \in Q_j$. For example Acerbi and Mingione used in [2] the choice $p^+_{Q_j}$. However, for large cubes this is not a good choice. It is more reasonable to take an exponent, which is close to the average of $p$ over $Q_j$ (actually the best choice is the average defined by the reciprocal). Due to the log-Hölder continuity of $p$ our choice of $p_j$ has this property.

Note that it is a priori not clear that our comparison system (4.9) is well defined. This is due to the fact, that the stated boundary condition on $w_j$ requires $u \in W^{1,p_j}(2Q_j)$. Since $p_j$ might be bigger than $p(\cdot)$ at some parts of $2Q_j$, this does not follow from $u \in W^{1,p(\cdot)}(2Q_j)$. However, it follows from Corollary 3.8 that $u \in W^{1,m_0p(\cdot)}(2Q_j)$ for some $m_0 > 1$. So if $p(\cdot)$ does not vary to much on $2Q_j$ in the sense that $m_0 p_{2Q_j} \geq p_{2Q_j}^+$, then $m_0 p(x) \geq p_j$. This implies $u \in W^{1,p_j}(2Q_j)$ and our comparison system (4.9) is well defined. It is now standard that the system has a unique solution $w_j \in u + W^{1,p_j}(2Q_j)$.

We will later have a similar problem, when passing back from $w_j$ to $u$, since $w_j \in W^{1,p_j}(2Q_j)$ is not enough to deduce $w_j \in W^{1,p(\cdot)}(2Q_j)$. For this, we need to control of $w_j$ in the space $W^{1,p^+_{2Q_j}}(2Q_j)$. This is possible due to the following result of higher integrability (up to the boundary) [18, Theorem 1.1]. More precisely we use a quantitative estimate which is a consequence from (3.4) and Lemma 2.4 in this work.

**Theorem 4.1.** There exists a $m_1 > 1$ such that for all $m_1 \geq \mu \geq 1$ the following holds. If $u \in W^{1,p(\cdot)}(2Q_j)$ and $w_j$ is the solution of (4.9), then there exists a constant $c$ depending on $p_j$ such that

$$
\int_{2Q_j} |Du_j|^{pp_j} \, dx \leq c \int_{2Q_j} |Du|^{pp_j} \, dx.
$$

Let us point out, that the paper [18] is stated for equations only; however, all their arguments used for the estimate below are valid for systems as well. For equations the last Theorem holds for all $1 \leq \mu < \infty$ which was proven in [20, Theorem 5].

The use of this theorem requires higher integrability of $Du$. To close this argument, we will assume in the following that for every $Q_j$ with $Q_j \cap U^{\kappa,\varepsilon}_\lambda \neq \emptyset$ it holds

$$
\sigma p_{2Q_j}^+ \geq p_{2Q_j} \quad \text{with } \sigma := \sqrt{\min \{m_0, m_1\}} > 1.
$$

In this situation Corollary 3.8 implies $u \in W^{1,\sigma^4 p(\cdot)}(2Q_j) \hookrightarrow W^{1,\sigma^4 p_j}(2Q_j)$ and then Theorem 4.1 implies $w_j \in W^{1,\sigma^4 p_j}(2Q_j) \hookrightarrow W^{1,\sigma^4 p(\cdot)}(2Q_j)$.

The above estimates of $u$ in $W^{1,\sigma^4 p_j}(2Q_j)$ and $w$ in $W^{1,\sigma^4 p(\cdot)}(2Q_j)$ depend unfortunately on the size of $Q_j$. We derive in the following three lemmas more precise modular estimates.

**Lemma 4.2.** If $Q_j \cap U^{\kappa,\varepsilon}_\lambda \neq \emptyset$ and $p$ holds (4.10), then

$$
\left( \int_{2Q_j} |Du|^{\sigma^4 p_j} \, dx \right)^{\frac{1}{\sigma^4}} \leq c \lambda.
$$
Proof. We want to apply the key estimate (Lemma 3.2) to the function $|Du|^\sigma p_j$. So let us verify the requirements. By Corollary 3.8, Remark 3.9 and Young’s inequality we deduce

$$
\int_{2Q_j} |Du|^{\sigma p_j} \, dx \leq \int_{2Q_j} |Du|^{m_0 p} \, dx + 1
$$

(4.11)

$$
\leq c \left( \int_{3Q_j} |Du|^{p_j} \, dx \right)^{m_0} + c \int_{3Q_j} |G|^{p_j m_0} + h^{m_0} \, dx + 1
$$

$$
\leq c \max \left\{ |Q_j|^{-m_0}, 1 \right\},
$$

where the last constant depends on $\| |Du|^{p_j} \|_{L^1(3Q_j)}$ and $\| |G|^{p_j} \|_{L^q(3Q_j)}$.

This allows to apply Lemma 3.2 to $|Du|^{\sigma p_j}$ with exponent $\frac{\sigma p_j}{\sigma^j} \geq 1$ at $x = y_j$ (recall $p(y_j) = p_j$ and $\sigma^j = \min \{m_0, m_1\}$) and use Remark 3.4.

Due to (4.8) we can estimate the right-hand side by $\lambda^{m_0}$.

---

Lemma 4.3. If $Q_j \cap U^{\kappa, \varepsilon}_\lambda \neq \emptyset$, then

$$
\left( \int_{2Q_j} |Dw_j|^{\sigma^j p_j} \, dx \right)^{\frac{1}{\sigma^j}} \leq c \lambda.
$$

Proof. The lemma is an immediate consequence of Theorem 4.1 and Lemma 4.2.

---

Lemma 4.4. If $Q_j \cap U^{\kappa, \varepsilon}_\lambda \neq \emptyset$, then

$$
\left( \int_{2Q_j} |Dw_j|^{\sigma^j p} \, dx \right)^{\frac{1}{\sigma}} \leq c \lambda.
$$

Proof. We want to apply the key estimate (Lemma 3.2) to the function $|Dw_j|^{\sigma^j p}$. So let us verify the requirements. By Young’s inequality, Theorem 4.1 and (4.11) we have

$$
\int_{2Q_j} |Dw_j|^{\sigma^j p} \, dx \leq \int_{2Q_j} |Dw_j|^{\sigma p_j} \, dx + 1
$$

$$
\leq c \int_{2Q_j} |Du|^{\sigma p_j} \, dx + 1
$$

$$
\leq c \max \left\{ |Q_j|^{-m_0}, 1 \right\}.
$$
This allows to apply Lemma 3.2 to $|Dw_j|^{|\sigma^2 p|}$ with exponent $\frac{\sigma p_j}{p^2}$ at $x = y_j$ (recall $p(y_j) = p_j$ and $\sigma^4 = \min\{m_0, m_1\}$) and use Remark 3.3

\[
\left( \int_{2Q_j} |Dw_j|^{\sigma^2 p_j} \right)^{\frac{\sigma p_j}{p_j}} \leq c \int_{2Q_j} |Dw_j|^{\sigma^2 p_j} dx + c \int_{2Q_j} (e + |x|)^{-\sigma^2 2n} dx.
\]

The first term on the right hand side is estimated by Lemma 4.3. The second term is controlled by (1.7). \qed

4.4. Comparison Estimate. We show in this subsection that our approximate solution $w_j$ is indeed close to our solution $u$. Obviously, the (small) distance from $A$ to $A_j$ is most important for our estimates. Due to (1.2) we have

\[
|A(x, z) - A_j(z)| \leq c |p(x) - p_j| |\log |z|| |(z)^{p(x)-1} + |z|^{|p(y)|-1}|
\]

for all $x \in 2\Omega$ and $z \in \mathbb{R}^{N \times n}$, in particular, the distance of $A$ and $A_j$ is strongly connected with the distance from $p$ to $p_j$. We begin with some auxiliary estimates.

Due to the special choice of $y_j$, namely $|y_j| = \sup_{x \in 2Q_j} |x|$, and $p(y_j) = p_j$ it follows from (2.1) that

\[
|p(x) - p_j| \leq |p(x) - p_\infty| + |p(y_j) - p_\infty| \leq \frac{2(p^+)^2c_{\log}(p)}{\log(e + |x|)}
\]

for all $x \in 2Q_j$.

Lemma 4.5. Let $Q \subset \mathbb{R}^n$ be a cube (or ball) with side length $R$ and let $p \in \mathcal{P}^{\log}(Q)$. Then for every $s \geq 1$ there exists a constant $c$ depending only on $s$ such that

\[
\left( \int_{Q} |p(\cdot) - p_j|^s \right)^{\frac{1}{s}} \leq \frac{c(p^+)^2c_{\log}(p)}{\log(e + \max\{R, 1/R, |\text{center}(Q)|\})}.
\]

Proof. If $\max\{R, 1/R, |\text{center}(Q)|\} = 1/R$, then $R \leq 1$ and by the local estimate of (2.1)

\[
\left( \int_{Q} |p(\cdot) - p_j|^s \right)^{\frac{1}{s}} \leq p_Q^+ - p_Q^- \leq \frac{2(p^+)^2}{\log(e + 1/(\sqrt{n}R))} \leq \frac{c(p^+)^2c_{\log}(p)}{\log(e + 1/R)},
\]

where the constant depends on $p^+$.

In the following let $R \geq 1$. Then by (4.13)

\[
\left( \int_{Q} |p(\cdot) - p_j|^s \right)^{\frac{1}{s}} \leq (p^+)^2c_{\log}(p) \left( \int_{Q} c \frac{1}{(\log(e + |x|))^s} dx \right)^{\frac{1}{s}}.
\]

If $\sqrt{n}R \leq \frac{1}{2}|\text{center}(Q)|$, then $\max\{R, 1/R, |\text{center}(Q)|\} = |\text{center}(Q)|$ and $|x| \geq \frac{1}{2}|\text{center}(Q)|$ for all $x \in Q$. Hence,

\[
\left( \int_{Q} \frac{c}{(\log(e + |x|))^s} dx \right)^{\frac{1}{s}} \leq \frac{c}{\log(e + \frac{1}{2}|\text{center}(Q)|)} \leq \frac{c}{\log(e + |\text{center}(Q)|)}.
\]
It remains to consider the case $\sqrt{\pi} R \geq \frac{1}{2} |\text{center}(Q)|$ and $R \geq 1$. In this situation we have max $\{R, 1/R, |\text{center}(Q)|\} \leq 2\sqrt{\pi} R$. Let $Q_{R(0)}$ denote the cube of same size as $Q$ but centered at zero. Then with $R \geq 1$

$$
\left( \frac{c}{Q} \left( \log(e + |x|) \right)^{s} \right)^{s} dx \leq \left( \frac{c}{Q_{R(0)}} \left( \log(e + |x|) \right)^{s} \right)^{s} dx
$$

$$
= c \left( R^{-n} \int_{0}^{R} \frac{r^{n-1}}{\left( \log(e + r) \right)^{s}} dr \right)^{\frac{s}{s}}
$$

$$
\leq c \left( R^{-n} \int_{0}^{R} \frac{r^{n-1}}{\left( \log(e + r) \right)^{s}} dr + R^{-n} \int_{R}^{2R} \frac{r^{n-1}}{\left( \log(e + r) \right)^{s}} dr \right)^{\frac{s}{s}}
$$

$$
\leq c \left( R^{-n+\frac{1}{s}} \frac{R^{n-1}}{\left( \log(e + R) \right)^{s}} \right)^{\frac{1}{s}}
$$

$$
\leq c R^{-\frac{1}{s}} + \frac{c}{\left( \log(e + R) \right)^{s}} \leq \frac{c}{\left( \log(e + R) \right)^{s}}.
$$

where we used that $\log(e + r) \geq c \log(e + R)$ for $r \in [R^{\frac{1}{s}}, R]$. This and (4.14) prove the remaining case.

We need another technical lemma that takes care of the logarithmic factor in (4.12).

**Lemma 4.6.** Let $Q \subset \mathbb{R}^{n}$ be a cube and $s \geq 1$. Then there exists a constant $c$ depending only on $s$ such that every $f \in L^1(Q)$ satisfies

$$
\int_{Q} \log \left( e + \frac{|f|}{\int_Q |f| \, dx} \right)^{s} \, dx \leq c.
$$

**Proof.** It suffices to prove the estimate for $f$ with $\int_{Q} |f| \, dx = 1$. We estimate

$$
\int_{Q} \left( \log(e + |f|) \right)^{s} \, dx = \frac{1}{|Q|} \int_{Q} \int_{0}^{\|f\|} s(\log(e + t))^{s-1} \frac{1}{e + t} \, dt \, dx
$$

$$
= \frac{1}{|Q|} \int_{0}^{\infty} s(\log(e + t))^{s-1} \frac{1}{e + t} |Q \cap \{|f| > t\}| \, dt.
$$

We split the domain of integration into $(0, 1)$ and $(1, \infty)$ and use the estimate $t |Q \cap \{|f| > t\}| \leq \int_{Q} |f| \, dx$ to get

$$
\int_{Q} \left( \log(e + |f|) \right)^{s} \, dx \leq \int_{0}^{1} s(\log(e + t))^{s-1} \frac{1}{e + t} \, dt
$$

$$
+ \frac{1}{|Q|} \int_{1}^{\infty} s(\log(e + t))^{s-1} \frac{1}{(e + t)t} \int_{Q} |f| \, dx \, dt
$$

$$
\leq c.
$$

Let us now turn to the closeness of $Dw_{j}$ and $Du$.

**Proposition 4.7.** For every $\kappa > 0$ and $\delta > 0$ there exists a $\delta_{1} > 0$ and $\varepsilon > 0$, such that the following holds for every $\lambda > \lambda_{0}$ (defined in (1.3)).
If \( Q_j \cap U^{r,e}_\lambda \neq \emptyset \) and \( c_{\log}(p|2Q_j) \leq \delta_1 \), then
\[
\int_{2Q_j} (A(\cdot, Du) - A(\cdot, Dw_j)) \cdot (Du - Dw_j) \, dx \leq \delta \lambda.
\]

Here \( \delta_1, \varepsilon \) depends on \( \kappa, \delta, p^+ \) and \( \|Du\|_{L^p(Q)} \).

**Proof.** Since \( p^+ < \infty \), we can choose \( \delta_1 \) (depending on \( p^+ \)) so small such that \( c_{\log}(p|2Q_j) \leq \delta_1 \) implies (4.10): \( p_{2Q_j} \leq p_{Q_j} \).

By \( u - w \in W^{1,p}_0(2Q_j) \cap W^{1,p}_0(Q) \) it follows from the equations for \( u \) and \( w_j \) that
\[
(I) := \int_{2Q_j} (A(\cdot, Du) - A(\cdot, Dw_j)) \cdot (Du - Dw_j) \, dx
\]
\[
= \int_{2Q_j} (A_j(Dw_j) - A(\cdot, Dw_j)) \cdot (Du - Dw_j) \, dx + \int_{2Q_j} A(\cdot, G) \cdot (Du - Dw_j) \, dx
\]
\[
=: (II) + (III).
\]

By Young’s inequality with \( \gamma > 0 \) we have that
\[
(III) \leq c \gamma^{1-(p^-)'} \int_{2Q_j} |G|^p \, dx + \gamma \int_{2Q_j} |Du|^p + |Dw_j|^p \, dx.
\]

With (4.7), (4.5) and Lemma (4.4) it follows that
\[
(III) \leq \varepsilon c \gamma^{1-(p^-)'} \lambda + \gamma c \lambda = (\varepsilon c \gamma^{1-(p^-)'} + c \gamma) \lambda.
\]

The factor in front of \( \lambda \) is small if \( \gamma \) is small and (then) \( \varepsilon \) is small.

It remains to estimate (II). We divide the domain of integration in (II) into the sets
\[
H_1 := \{ x \in 2Q_j : |Dw_j(x)| \geq 1 \},
\]
\[
H_2 := \{ x \in 2Q_j : 1 \geq |Dw_j(x)| \geq h(x) \},
\]
\[
H_3 := \{ x \in 2Q_j : h(x) \geq |Dw_j(x)| \}.
\]

We define
\[
(II_k) := \int_{2Q_j} \chi_{H_k} |A_j(Dw_j) - A(\cdot, Dw_j)| \cdot |Du - Dw_j| \, dx \quad \text{for } k = 1, 2, 3,
\]
then (II) \( \leq (II_1) + (II_2) + (II_3) \).

We begin the easiest term (II_3). By Young’s inequality with \( \gamma > 0 \) we estimate pointwise on \( H_3 \)
\[
|A_j(Dw_j) - A(\cdot, Dw_j)| \cdot |Du - Dw_j|
\]
\[
\leq c (|Dw_j|^{p_j} + |Dw_j|^{p_j}) (|Du| + |Dw_j|)
\]
\[
\leq c \gamma^{1-(p^-)'} (|Dw_j|^{p_j} + |Dw_j|^{p_j}) + \gamma (|Du|^{p_j} + |Du|^{p_j})
\]
\[
\leq c \gamma^{1-(p^-)'} h(\cdot) + \gamma (|Du|^{p_j} + |Du|^{p_j}),
\]
as $h(x) \geq h(x)^\alpha$ for any $\alpha \geq 1$. This, \eqref{1.7}, Lemma \ref{L4.2} and \eqref{4.5} imply
\[
(II_3) \leq c \gamma^{1-(p^-)^*} \int_{2Q_j} h \, dx + \gamma \int_{2Q_j} |Du|^{p_j} + |Du|^{p(\cdot)} \, dx
\]
\[
\leq (c \varepsilon \gamma^{1-(p^-)^*} + \gamma) \lambda.
\]
Again the factor in front of $\lambda$ is small if $\gamma$ is small and (then) $\varepsilon$ is small.

For the remaining terms $(II_1)$ and $(II_2)$ we have to use the closeness of $A_j$ to $A$ (more precisely the smallness of $c\log(p|2Q_j|)$). In particular, by \eqref{1.2} and Young’s inequality we have pointwise on $2Q_j$
\[
|(A_j(Dw_j) - A(\cdot, Dw_j)) \cdot (Du - Dw_j)|
\]
\[
\leq c |p(\cdot) - p_j||\log |Dw_j||((|Dw_j|^{p_j-1} + |Dw_j|^{p(\cdot)-1})(|Du| + |Dw_j|)
\]
\[
\leq c |p(\cdot) - p_j||\log |Dw_j||(|Dw_j|^{p(\cdot)} + |Dw_j|^{p_j} + |Du|^{p(\cdot)} + |Du|^{p_j}).
\]
This implies for $k = 1, 2$
\[
(II_k) \leq \int_{2Q_j} \chi_{H_k}|p(\cdot) - p_j||\log |Dw_j||(|Dw_j|^{p(\cdot)} + |Dw_j|^{p_j} + |Du|^{p(\cdot)} + |Du|^{p_j}) \, dx.
\]
After applying Hölder’s estimate for the exponents $(2\sigma', 2\sigma', \sigma)$ and using Lemma \ref{L4.4}, Lemma \ref{L5.3}, \ref{L5.5} and Lemma \ref{L4.2} we get for $k = 1, 2$
\[
(II_k) \leq \left( \int_{2Q_j} |p(\cdot) - p_j|^{2\sigma'} \right)^{\frac{1}{2\sigma'}} \left( \int_{2Q_j} \chi_{H_k}|\log |Dw_j||^{2\sigma'} \, dx \right)^{\frac{1}{2\sigma'}} \lambda
\]
\[
=: (IV) (V_k) \lambda.
\]
Due to Lemma \ref{L4.3} we have
\[
(IV) \leq \frac{c(p^+)^2c_{\log(p)}}{\log(e + \max\{1/\ell(Q_j), \ell(Q_j), |\text{center}(Q_j)|\})}.
\]
Since $|Dw_j| \geq 1$ on $H_1$, we have
\[
(V_1) \leq \left( \int_{2Q_j} \chi_{H_1}(\log(|Dw_j|^{p_j}))^{2\sigma'} \, dx \right)^{\frac{1}{2\sigma'}}
\]
\[
 \leq \left( \int_{2Q_j} \chi_{H_1}(\log(e + |Dw_j|^{p_j}))^{2\sigma'} \, dx \right)^{\frac{1}{2\sigma'}}.
\]
Using the estimate $\log(e + t) \leq \log(e + t/\lambda) + \log(e + \lambda)$ we get
\[
(V_1) \leq \left( \int_{2Q_j} \chi_{H_1}(\log(e + |Dw_j|^{p_j})/\lambda))^{2\sigma'} \, dx \right)^{\frac{1}{2\sigma'}} + c \log(e + \lambda).
\]
Due to Lemma \ref{L4.3} we have
\[
\int_{2Q_j} |Dw_j|^{p(\cdot)} \, dx \leq c \lambda.
Lemma 4.6 and the previous estimate imply

\[(V_1) \leq c + c \log(e + \lambda) \leq c \log(e + \lambda).\]

From \(\|Du\|_{L^p} \leq c\), we know that \(\int_{2\Omega} |Du| \leq c\). This and (4.5) imply

\[\lambda \leq \frac{c}{|Q_j|}.\]

Therefore we gain

\[(V_1) \leq c \log(e + 1/\ell(Q_j)).\]

Since \((e + |x|)^{-m} = h(x) \leq |Dw_j(x)| \leq 1\) on \(H_2\), we have

\[(V_2) \leq \left( \int_{2Q_j} (\log(e + |x|))^{2/\sigma'} \, dx \right)^{1/2}\]

\[\leq c \log(e + \max\{1/\ell(Q_j), \ell(Q_j), |\text{center}(Q_j)|\}).\]

Overall, we have

\[(V_1) + (V_2) \leq c \log(e + \max\{1/\ell(Q_j), \ell(Q_j), |\text{center}(Q_j)|\}).\]

The estimates for (IV), (V_1) and (V_2) imply

\[(II_1) + (II_2) \leq c (p^+)^2 c \log(p) \lambda.\]

The factor in front of \(\lambda\) is small if \(\delta_1\) is small (for fixed upper bound of \(p^+\)).

Combining the estimates for (III), (II_1), (II_2) and (II_3) proves the claim. \(\square\)

**Remark 4.8.** The condition on \(p\) can be weakened for the last estimate. Indeed, we can replace the smallness of the log Hölder constant by the assumption that the oscillations of \(p\) are small:

\[\left( \int_{2Q_j} |p(\cdot) - p_j|^{2/\sigma'} \, dx \right)^{1/2} \leq \frac{\delta_1}{\log(e + \max\{1/\ell(Q_j), \ell(Q_j), |\text{center}(Q_j)|\}).}\]

Or as a counterpart for the pointwise vanishing condition, the following VMO condition:

\[\left( \int_{2Q_j} |p(\cdot) - p_j|^{2/\sigma'} \, dx \right)^{1/2} \log(e + \max\{1/\ell(Q_j), \ell(Q_j), |\text{center}(Q_j)|\}) \to 0,\]

when \(\ell(Q_j) \to 0\) or \(\text{center}(Q_j) \to \infty\). This is a first step to weaken the pointwise vanishing log Hölder continuity by an integral vanishing oscillation condition on the exponent.

However, up to now we still require the (not small) log Hölder continuity to be able to apply the key estimate (Lemma 3.2).
4.5. Main results. Let us present the main results of this paper on local higher integrability.

**Theorem 4.9.** Let $\Omega \subset \mathbb{R}^n$ be a cube and let $u$ be a solution of (1.1) on $2\Omega$. Further, let $p \in \mathcal{P}^{\log}(2\Omega)$, $1 < p^- \leq p^+ < \infty$, $q \geq 1$, and $|G|^{(\cdot)} \in L^q(2\Omega)$. Then there exists a $\delta_1 > 0$ such that $c_{\log}(p;2\Omega) \leq \delta_1$ implies

$$\left( \int_{\Omega} |Du|^{p(\cdot)^q} \, dx \right)^{\frac{1}{q}} \leq c \int_{2\Omega} |Du|^{p(\cdot)} \, dx + c \left( \int_{2\Omega} (|G|^{(\cdot)} + h)^q \, dx \right)^{\frac{1}{q}}.$$ 

Here $\delta_1$ and $c$ only depend on $\|\|Du|^{p(\cdot)}\|_{1,2\Omega}$, $p^-, p^+, q$. The function $h(x) := (e + |x|)^{-2n}$.

**Theorem 4.10.** Let $\Omega \subset \mathbb{R}^n$ be a cube and let $u$ be a solution of (1.1) on $2\Omega$. Further, let $p \in \mathcal{P}^{\log}(2\Omega)$, $1 < p^- \leq p^+ < \infty$, $q \geq 1$, and $|G|^{(\cdot)} \in L^q(2\Omega)$. Then

$$\left( \int_{\Omega} |Du|^{p(\cdot)^q} \, dx \right)^{\frac{1}{q}} \leq c \int_{2\Omega} |Du|^{p(\cdot)} \, dx + c \left( \int_{2\Omega} (|G|^{(\cdot)} + h)^q \, dx \right)^{\frac{1}{q}},$$

where $c$ depends on $\|\|Du|^{p(\cdot)}\|_{1,2\Omega}$, $p^-, p^+, q$ and $p$ via the vanishing log-Hölder continuity. The function $h(x) := (e + |x|)^{-2n}$.

We postpone the proof of these theorems until the end of this subsection. The above theorems on local higher integrability have a global counterpart.

**Corollary 4.11.** Let $u$ be a solution of (1.1) on $\mathbb{R}^n$ with $Du \in L^{p(\cdot)}(\mathbb{R}^n)$, let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $q \geq 1$, and $G^{(\cdot)} \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. Then there exists $\delta_1 > 0$ such that $c_{\log}(p) \leq \delta_1$ implies

$$\|Du|^{p(\cdot)}\|_q \leq c \|G|^{p(\cdot)} + h\|_q.$$ 

Here $\delta_1$ and $c$ depend on $\|\|Du|^{p(\cdot)}\|_1$ and $q$. The function $h(x) := (e + |x|)^{-2n}$.

**Corollary 4.12.** Let $u$ be a solution of (1.1) on $\mathbb{R}^n$ with $Du \in L^{p(\cdot)}(\mathbb{R}^n)$, let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $q \geq 1$, and $G^{(\cdot)} \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. Then

$$\|Du|^{p(\cdot)}\|_q \leq c \|G|^{p(\cdot)} + h\|_q,$$

where $c$ depends on $\|\|Du|^{p(\cdot)}\|_1$, $p^-, p^+, q$ and $p$ via the vanishing log-Hölder continuity. The function $h(x) := (e + |x|)^{-2n}$.

The two corollaries are immediate consequences of the two theorems above. Just apply Theorem 4.9 and Theorem 4.10 resp., to $\Omega = (-R, R)^n$, multiply by $|\Omega|$ and let $R \to \infty$.

Before we proof Theorem 4.9 and Theorem 4.10 we need a few auxiliary results. First, we need the interior regularity of $p_j$-harmonic functions, which was proven by Uhlenbeck for systems and $p \geq 2$ in [20], (3.2)Theorem]. By duality the estimate holds also in the case $p \leq 2$; see also [14] for a more general version of the estimate below.
Theorem 4.13. The $p_j$-harmonic function $w_j$ satisfies

$$\sup_{\frac{3}{2}Q_j} |Dw_j| \leq c \left( \int_{2Q_j} |Dw_j|^{p_j} \, dx \right)^{\frac{1}{p_j}}.$$ 

This implies

Lemma 4.14. With the same assumptions as in Proposition 4.7 on $Q_j$ we have

$$\left( \int_{2Q'} |Dw_j|^{p(\cdot)} \, dx \right)^{\frac{1}{p_j}} \leq c \lambda,$$

for all $Q' \in \Delta_\Omega$ with $Q' \subsetneq Q_i$ and $Q' \cap U_{\lambda}^{\kappa,\varepsilon} \neq \emptyset$.

Proof. The first estimate follows directly from Theorem 4.13 and Lemma 4.3.

By the key estimate (Lemma 3.2) applied to the exponent $\sigma p_j p(\cdot) \geq 1$ at the point $y_j$ and Remark 3.4 we have

$$\left( \int_{2Q'} |Dw_j|^{p(\cdot)} \, dx \right)^{\sigma p_j p(\cdot)} \leq c \int_{2Q'} |Dw_j|^{\sigma p_j} + c \int_{2Q'} (e + |x|-2n \, dx.$$

This and the first part of the lemma imply

$$\int_{2Q'} |Dw_j|^{p(\cdot)} \, dx \leq c \lambda + c \int_{2Q'} (e + |y|-2n \, dy.$$

Since $Q' \cap U_{\lambda}^{\kappa,\varepsilon} \neq \emptyset$, the last integral is bounded by $c \varepsilon \lambda$ due to (4.7). □

We are now prepared to show our redistributional estimate (4.3).

Proposition 4.15. There exists $\kappa \geq 2^n$ such that for every $\delta > 0$ we find $\varepsilon > 0$ and $\delta_1 > 0$ with the following property for all $\lambda \geq \lambda_0$: If $c_{\log(p|2Q_j)} \leq \delta_1$ for all $Q_j \cap U_{\lambda}^{\kappa,\varepsilon} \neq \emptyset$, then there holds

$$|U_{\lambda}^{\kappa,\varepsilon}| \leq \delta |O_\lambda|.$$

The value of $\varepsilon$ and $\delta_1$ depends on $\delta$, $p^+$, and $\|Du|^{p(\cdot)}\|_{L^1(2\Omega)}$.

Proof. We will choose the exact value of $\varepsilon$, $\kappa$ and $\delta_1$ during the proof. Let $\lambda \geq \lambda_0$. We already know (see (4.6)) that it suffices to prove

$$|Q_j \cap U_{\lambda}^{\kappa,\varepsilon}| \leq \delta |Q_j| \quad \text{for all } j \in \mathbb{N} \text{ with } Q_j \cap U_{\lambda}^{\kappa,\varepsilon} \neq \emptyset.$$

So let us assume in the following that $Q_j \cap U_{\lambda}^{\kappa,\varepsilon} \neq \emptyset$.

Let $y \in Q_j \cap U_{\lambda}^{\kappa,\varepsilon}$. By definition of $U_{\lambda}^{\kappa,\varepsilon}$, see (4.2) we have

$$M_{\kappa}^{\kappa}(\mathcal{M}_{\kappa}(y)) > \kappa \lambda,$$

$$M_{\kappa}^{\lambda}(\mathcal{M}_{\lambda}(y)) \leq \varepsilon \lambda.$$
Therefore, there exists a cube $C_y \in \Delta_{2\Omega}$ with
\[ \int_{2C_y} |Du|^{p(\cdot)} \, dx > \kappa \lambda. \]

First we assume that $\kappa \geq 2^n$. Then obviously $\kappa \lambda > \lambda_0$ and hence by definition of $\lambda_0$ the cube $C_y$ must be a strict dyadic sub-cube of $\Omega$. Hence the predecessor $C_{\text{pre}}^y$ satisfies $C_{\text{pre}}^y \in \Delta_\Omega$ and
\[ \int_{2C_{\text{pre}}^y} |Du|^{p(\cdot)} \, dx \geq 2^{-n} \int_{2C_y} |Du|^{p(\cdot)} \, dx > 2^{-n} \kappa \lambda \geq \lambda. \]

Since $Q_j$ was chosen to be the maximal cube of $\Delta_\Omega$ containing $y$ with this property, it follows that $C_{\text{pre}}^y \subset Q_j$.

By (1.3) and Lemma 4.14 (using $C_y \subset Q_j$, since $C_{\text{pre}}^y \subset Q_j$) we have
\[ \kappa \lambda < \int_{2C_y} |Du|^{p(\cdot)} \, dx \]
\[ \leq c_4 \int_{2C_y} |Dw_j|^{p(\cdot)} \, dx + c_4 \int_{2C_y} (A(\cdot, Du) - A(\cdot, Dw_j)) \cdot (Du - Dw_j) \, dx \]
\[ \leq c + c_4 \int_{2C_y} (A(\cdot, Du) - A(\cdot, Dw_j)) \cdot (Du - Dw_j) \, dx. \]

The constant $c$ depends on $p^+$ and $\|Du|^{p(\cdot)}\|_{L^1(2\Omega)}$. So for $\kappa$ large (which finally fixes $\kappa$) we can absorb $c \lambda$ into $\kappa \lambda$. By multiplication with $|C_y|$ we get
\[ \kappa \lambda |C_y| \leq c \int_{2C_y} (A(\cdot, Du) - A(\cdot, Dw_j)) \cdot (Du - Dw_j) \, dx. \]

The collection of $C_y$ covers the set $Q_j \cap U^{\kappa,\varepsilon}_\lambda$. Since all these cubes are $\Omega$-dyadic, there exists a sub-family $\{C_{j,k}\}_k$ of maximal $\Omega$-dyadic cubes. We sum the previous inequality over these cubes to get
\[ \kappa \lambda |Q_j \cap U^{\kappa,\varepsilon}_\lambda| \leq \kappa \lambda \sum_k |C_{j,k}| \]
\[ \leq c \sum_k \int_{2C_{j,k}} (A(\cdot, Du) - A(\cdot, Dw_j)) \cdot (Du - Dw_j) \, dx \]
\[ \leq c \int_{2Q_j} (A(\cdot, Du) - A(\cdot, Dw_j)) \cdot (Du - Dw_j) \, dx. \]

Due to Proposition 4.7 we can find for every $\delta > 0$ a proper choice of $\delta_1 > 0$ and $\varepsilon > 0$ such that the last integral is bounded by $|2Q_j|\delta \lambda$. Hence,
\[ \kappa \lambda |Q_j \cap U^{\kappa,\varepsilon}_\lambda| \leq |2Q_j|\delta \lambda. \]

In other words (using $\kappa \geq 2^n$)
\[ |Q_j \cap U^{\kappa,\varepsilon}_\lambda| \leq \delta^{-1}2^n |Q_j| \leq \delta |Q_j|. \]

This proves the claim. \(\square\)
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We can now prove our first main result on local higher integrability.

Proof of Theorem 4.9. If $q \in [1, m_0]$, then the claim follows directly from Corollary 3.8. So we can assume in the following $q > m_0 > 1$. Fix $\kappa$ as in Proposition 4.15. Then for $\delta := \frac{1}{2}\kappa^{-q}$ let $\varepsilon$ and $\delta_1$ be chosen (depending on $\delta$) as in Proposition 4.15 such that $|U^{\kappa,\varepsilon}_\lambda| \leq \delta |O|$ for every $\lambda \geq \lambda_0$ with $\lambda_0 = \frac{1}{2\varepsilon} \int_{\Omega} |Du|^{p(\cdot)} \, dx$.

We estimate

\[
\int_{\Omega} |Du|^{p(\cdot)} \, dx = \frac{1}{|\Omega|} \left( \int_{|\lambda_0} \quad + \int_{|\lambda_0}^{\infty} \quad \right) \leq \kappa^q q \lambda^{-1} \{ M^*_\Omega(|Du|^{p(\cdot)} > \lambda) \} \, d\lambda
\]

(4.16)

For $\Lambda \geq \kappa \lambda_0$ define

\[
(I_\Lambda) = \int_{|\lambda_0}^\Lambda \quad q \lambda^{-1} \{ M^*_\Omega(|Du|^{p(\cdot)} > \lambda) \} \, d\lambda,
\]

then $(I) = \lim_{\Lambda \to \infty} (I_\Lambda)$. Substitution gives

\[
(I_\Lambda) := \kappa^q \int_{\lambda_0}^{\Lambda/\kappa} q \lambda^{-1} \{ M^*_\Omega(|Du|^{p(\cdot)} > \kappa \lambda) \} \, d\lambda.
\]

From $|U^{\kappa,\varepsilon}_\lambda| \leq \delta |O|$ for $\lambda \geq \lambda_0$ it follows that

\[
|\{ M^*_\Omega(|Du|^{p(\cdot)} > \kappa \lambda) \}| \leq \left| \{ M^*_\Omega(|G|^{p(\cdot)} + h) > \varepsilon \lambda \} \right| + \delta \left| \{ M^*_\Omega(|Du|^{p(\cdot)} > \lambda) \} \right|.
\]

Hence,

\[
(I_\Lambda) \leq \kappa^q \int_{\lambda_0}^{\Lambda/\kappa} q \lambda^{-1} \{ M^*_\Omega(|G|^{p(\cdot)} + h) > \varepsilon \lambda \} \, d\lambda + \kappa^q \delta \int_{\lambda_0}^{\Lambda/\kappa} q \lambda^{-1} \{ M^*_\Omega(|Du|^{p(\cdot)} > \lambda) \} \, d\lambda + \kappa^q \delta \int_{\lambda_0}^{\Lambda/\kappa} q \lambda^{-1} \{ M^*_\Omega(|Du|^{p(\cdot)} > \lambda) \} \, d\lambda.
\]

Since $\kappa^q \delta = \frac{1}{2}$, the second term is bounded by $\frac{1}{2} (I_\Lambda)$. The last term can be estimated as in (4.16). This implies

\[
(I_\Lambda) \leq 2 \kappa^q \int_{\lambda_0}^{\Lambda/\kappa} q \lambda^{-1} \{ M^*_\Omega(|G|^{p(\cdot)} + h) > \varepsilon \lambda \} \, d\lambda + |\Omega| \lambda_0^q
\]

\[
= 2 \kappa^q \varepsilon^{-q} \int_{\lambda_0}^{\varepsilon \Lambda/\kappa} q \lambda^{-1} \{ M^*_\Omega(|G|^{p(\cdot)} + h) > \lambda \} \, d\lambda + |\Omega| \lambda_0^q
\]

\[
\leq 2 \kappa^q \varepsilon^{-q} \int_{\Omega} \left( M^*_\Omega(|G|^{p(\cdot)} + h) \right)^q \, dx + |\Omega| \lambda_0^q.
\]
The boundedness of the operator $M^*_N$ on $L^q(2\Omega)$ (using $q > m_0$) implies
\[
(I_A) \leq c\kappa^q\varepsilon^{-q} \int_{2\Omega} (|G_p^{(c)} + h|)^q\,dx + |\Omega|\lambda^q_0.
\]
The constant depends on $q$, but the lower bound $q > m_0$ ensures that the operator norm of $M^*_N$ is uniformly bounded. We pass to the limit $\Lambda \to \infty$, combine this with (4.10) and use the definition of $\lambda_0$ to get
\[
\int_{\Omega} |Du|^{p(c)}\,dx \leq c\varepsilon^{-q} \int_{2\Omega} (|G_p^{(c)} + h|)^q\,dx + \left(2\kappa\int_{2\Omega} |Du|^{p(c)}\right)^q\,dx.
\]
This proves the claim. \qed

**Proof of Theorem 4.10.** Let $q \geq m_0$ and choose $\delta_1 > 0$ as in Theorem 4.9. Since $c_{\log}(p|\partial \Omega)$ does not need to be smaller that $\delta_1$, we cannot apply Theorem 4.9 directly. Let $\{Q_j\}_{j \in \mathbb{N}}$ be the Calderón-Zygmund covering introduced at the beginning of this section. We will show the following:

For every $\delta_1 > 0$ there exists an $\varepsilon_0$, such that for all $\varepsilon \leq \varepsilon_0$ and all $Q_j$ with $Q_j \cap U^N_{\lambda,\varepsilon} \neq \emptyset$ we have $c_{\log}(p|2Q_j) \leq \delta_1$. Then Proposition 4.15 can be applied and the result follows as in Theorem 4.9.

By the vanishing log-Hölder continuity, we find $r, R > 0$ such that
\[
|p(x) - p(y)| \leq \frac{\delta_1}{\log(e + 1/|x - y|)}
\]
for all $x, y$ with $|x - y| \leq r$ and all $x, y \in 2\Omega \setminus Q_R(0)$. We can choose $R$ large enough such that additionally
\[
|p(z) - p_\infty| \leq \frac{\delta_1}{\log(e + |z|)}
\]
for all $z \in 2\Omega \setminus Q_R(0)$. Therefore, if $2Q_j \subset 2\Omega \setminus Q_R(0)$, then $c_{\log}(p|2Q_j) \leq \delta_1$. On the other hand if the length of $Q_j$ is smaller than $r$, then $c_{\log}(p|2Q_j) \leq \delta_1$.

It leaves the case when $|Q_j| \geq r^n$ and $2Q_j \cap Q_R(0) \neq \emptyset$. If now $Q_j \cap U^N_{\lambda,\varepsilon} \neq \emptyset$, then there exists a $c$ depending on $r$ and $R$, (but independent of $Q_j$), such that
\[
\lambda |2Q_j| \leq \int_{2\Omega} |Du|^{p(c)}\,dx \leq c\int_{2Q_j} hdx \leq c\varepsilon\lambda |2Q_j|.
\]
This is never the case, whenever $\varepsilon$ is small enough. Therefore the proof is complete. \qed

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