Ordinary QED formulated in the Feynman’s space-time picture is equivalent to a one dimensional field theory. In the large N limit there is no phase transition in such a theory. In this letter, we show a phase transition does exist in a generalization of QED characterized by the addition of the curvature of the world line (rigidity) to the Feynman’s space-time action. The large distance scale of the disordered phase essentially coincides with ordinary QED, while the ordered phase is strongly coupled. Although rigid QED exhibits the typical pathologies of higher derivative theories at the classical level, we show that both phases of the quantum theory are free of ghosts and tachyons. Quantum fluctuations prevent taking the naive classical limit and inheriting the problems of the classical theory.
I-The Rigid Model Of QED

The second quantized continuum theory of QED is essentially uniquely determined by the demand of renormalizability. A naive generalization would therefore be difficult. The current theory seems to exist in one phase, the weak coupling phase, where the Coulomb coupling constant increases indefinitely at small distance scales indicating a breakdown of the theory at the Landau singularity. The quest of a strong phase of continuum QED remains therefore an open question.

The first quantized description of the Feynman’s space-time formulation, however, admits the renormalizable generalization of adding the scale invariant curvature of the world line (rigidity) to the usual relativistic point particle-Maxwell action where the matter sector is described by $x^\mu, \mu = 1, 2, ...D$ and the gauge sector by the usual Maxwell $U(1)$ gauge field $A_\mu$. In [1], [2] we have shown that the curvature term arises out of a loop splitting regularization of electrodynamics and conjectured the existence of a phase transition due to the long range Coulomb interactions. Here we prove this conjecture and also show this higher derivative theory is free of ghosts and tachyons that typically plague higher derivative theories.

The conjecture that there is a phase transition can be easily understood by analogy with spin systems. The arc-length gauge fixing condition [3] of the relativistic point particle resembles the sigma model constraint $\sigma^2 = 1$ where the velocity of the particle corresponds to the direction of the spin. Our action resembles a modified sigma model in one dimension with long range interactions [2] where the curvature term plays the role of the kinetic term of the spin field. Since spin systems with sufficiently long range interactions may exhibit a phase transition in one dimension, we conjecture likewise for rigid QED. The large $N$ is a successfull approximation for non-linear sigma models and we applied it to rigid QED where $N$ is the space-time dimension $D$. We found a critical line in the plane of the Coulomb coupling verses the curvature coupling below which there is a disordered phase and above which is a new ordered strongly coupled phase.

The higher derivative nature of QED should cause a serious pause as any higher derivative theory is typically pathological. Indeed, the arc-length plus the curvature term theory has classical runaway solutions which are tachyonic. Whether a higher derivative regulated quatum theory has such pathological behaviour is more subtle and depends on details of the continuum limit. A free scalar field theory on the lattice with spacing $\frac{1}{\Lambda}$, has higher derivatives and ghosts. However these ghosts have mass of order $\Lambda$ and decouple in the continuum limit as $\Lambda \to \infty$. In the less trivial case of rigid QED we will show that the ghosts have mass of order $\Lambda$ and similarly decouple from the continuum limit. The necessity of the decoupling mechanism is associated with the absence of fine tuning of the curvature and the Coulomb coupling constants. This is a manifestation of the fact that we have phase transition with respect to a critical line that separates two distinct regions [4].

The effective action obtained after the Guassian integration of the gauge field
sector is:

\[ I_{\text{eff}} = \mu_0 \int_0^1 d\lambda \sqrt{x^2} + \frac{1}{t} \int_0^1 d\lambda \sqrt{\frac{\dot{x}^2 \ddot{x}^2 - (\dot{x} \ddot{x})^2}{\dot{x}^2}} + \frac{1}{2t} \int_0^1 d\lambda d\lambda' \dot{x}(\lambda) \dot{x}(\lambda') V(|x - x'|) \]  

(1a)

where the first term is the arc-length \( ds = d\lambda \sqrt{x^2} \) of a point particle of bare mass \( \mu_0 \), the second term is the curvature \( k(s) = \left| \frac{d^2 x(s)}{ds^2} \right| \) of the world line defined to be the length of the acceleration, \( t \) is a dimensionless coupling constant (scale invariance of the curvature term) and \( V \) is the long range Coulomb potential:

\[ V(|x - x'|, a) = \frac{2g}{\pi} \frac{1}{|x(\lambda) - x(\lambda')|^2 + a^2} . \]  

(1b)

We have introduced the cut-off "\( a \)" to avoid the singularity at \( \lambda = \lambda' \) and define \( g = t.\alpha_{\text{Coulomb}} = t.\frac{e^2}{4\pi} \). In the arc-length gauge \( \dot{x}^2 = 1 \) we can gauge fix the action and obtain:

\[ I_{g,f} = \frac{1}{2} \mu_0 L + \frac{1}{2t} \int_0^L d\lambda (e^{-1} \dot{x}^2 + e + \omega (\dot{x}^2 - 1)) + \frac{1}{2t} \int_0^L d\lambda d\lambda' \dot{x}(\lambda) \dot{x}(\lambda') V(|x - x'|, a) \]  

(2)

where \( e \) is an einbein to remove the square root of the acceleration, \( \omega \) is a lagrange multiplier that enforces the constraint \( \dot{x}^2 = 1 \), and \( L \) is the length of the path. The partition function is:

\[ Z = \int D\omega D\dot{x} D\lambda exp(-I_{g,f}) . \]  

(3)

II-Large D analysis, absence of Coulomb interactions

The effective action is obtained by integrating over \( x^\nu, \nu = 1, \ldots D \) we have:

\[ S_{0_{\text{eff}}} = \frac{1}{2t} \int d\lambda e(\lambda) - \omega(\lambda) + tDtrlnA \]  

(4)

where \( A \) is the operator

\[ A = \partial^2 e^{-1} \partial^2 - \partial \omega \partial . \]  

(5)

In the large D limit the stationary point equations resulting from varying \( \omega \) and \( e \)
respectively are:

\[ 1 = t D t r G \] (6a)

\[ 1 = t D t r (e^{-2(\partial^2 G)}) \] (6b)

where the world line Green's function is defined by:

\[ G(\lambda, \lambda') = \langle \lambda | (-\partial^2) A^{-1} | \lambda' \rangle \] (7)

The stationary points are:

\[ \omega(\lambda) = \omega^*, \quad \langle \lambda | e^{-1} | \lambda' \rangle = \int \frac{dp}{2\pi} \frac{e^{i(\lambda-\lambda')}}{|p|} \] (8)

where \( \omega^* \) is a constant. Thus eqs.(6) becomes the single mass gap equation [5]:

\[ 1 = Dt \int \frac{dp}{2\pi} \frac{1}{|p| + \omega^*} \] (9)

which yields

\[ \omega^* = \Lambda e^{-\frac{\omega^*}{Dt}} \] (10)

where \( \Lambda = \frac{1}{a} \) is an U.V. cut-off and \( \omega^* \) is now the mass associated with the propagator:

\[ \langle \dot{x}^\mu(p) \dot{x}^\nu(-p) \rangle = Dt \frac{\delta^{\mu\nu}}{|p| + \omega^*} \] (11)

To obtain a non-zero phase transition temperature the mass gap equation must be infra-red finite for \( \omega^* = 0 \). Therefore without Coulomb long range interactions the theory exists only in the disordered phase \( t > t_c \) and the U.V stable fixed point is \( t_c = 0 \). From (10) it is evident that the beta function of the pure curvature theory is asymptotically free indicating the absence of the curvature term at large distance scales. In contrast to the naive classical limit the theory is therefore well behaved and free of ghosts. In sec. IV we will calculate the poles of the Green's function in the presence of Coulomb interactions using Large D limit nd show the absence of ghosts and tachyons in both the ordered and disordered phases of the theory.
III-Phase transition in the presence of Coulomb interactions

The integration is no longer Gaussian. Thus we consider

\[ x^\nu(\lambda) = x^\nu_0(\lambda) + x^\nu_1(\lambda) \]

and expand the action (2) to quadratic order in \( x^\nu_1(\lambda) \) about the background straight line \( x_0 \). The \( x \)-integration is now Gaussian and the new effective action \( S_{\text{eff}} \) is given by (4) and (5) with the new operator \( A \) that includes the Coulomb potential contributions. Using the stationary solutions (8) \( A_{\text{new}} \) is given by:

\[
\text{tr} \ln A_{\text{new}} = \int \frac{dp}{2\pi} \text{tr} \ln [p]^3 + p^2 \omega^* + p^2 V_0(p) + V_1(p)] \tag{12}
\]

where

\[
V_0(p) = \frac{2g}{\alpha} e^{-a|p|}, \quad V_1(p) = \frac{2g}{\alpha^2} [e^{-a|p|}(|p| + \frac{1}{\alpha}) - \frac{1}{\alpha}] . \tag{13}
\]

The new mass gap equation (6) is:

\[
1 = D t \int \frac{dp}{2\pi} \frac{p^2}{p^2(|p| + \omega^*) + p^2 V_0(p) + V_1(p)} \tag{14}
\]

The critical line is defined by eq.(14) at \( \omega^* = 0 \):

\[
1 = \frac{Dt}{\pi} \int_0^1 dy \frac{y^2}{y^2 + 2g(y^2e^{-y} + ye^{-y} + e^{-y} - 1)} \tag{15}
\]

where we made the change of variable \( y = \alpha p \) and introduced the U.V cut-off \( \Lambda = \frac{1}{\alpha} := \Lambda_0^* \). Notice that eq.(15) is finite except at \( g = 0 \) (absence of Coulomb interactions). The critical curve distinguishing the two phases in the (t,g) plane is shown in Fig.1. The order parameter of the theory is the mass gap equation (14) where \( \omega^* \) is the parameter that distinguishes that two phases. In the disordered phase \( \omega^* > 0 \), while in the ordered phase it is straightforward to show that \( \omega^* = 0 \). In the disordered phase the coupling constants t and g are completely fixed by dimensional transmutation in terms of the cut-off \( \Lambda \) and \( \omega^* \). Thus they cannot be fine tuned. This is an important property that is vital in proving the absence

* In fact there exist a \( g^* \) for which any choice of \( \Lambda a = c \) leads to phase transition as long as \( g < g^* \). We choose \( \Lambda = \Lambda_{\text{Planck}} \), therefore \( c \geq 1 \).
of ghosts in our model. From (15) we can immediately examine whether there is a phase transition in the pure Coulomb theory i.e ordinary QED. The curvature term would then be absent. This corresponds to the absence of the $y^3$ term in (15). If we choose the cut-off of the theory $\Lambda$ to be at the Compton wave length i.e $\Lambda_{\text{Compton}} = \frac{2\pi}{a}$ one finds in this particular case that the integral (15) diverges implying an absence of a phase transition.

IV- The Absence of Ghosts and Tachyons

The issue of ghosts and tachyons can be examined by considering the space-time propagator:

$$P(x_0, y_0) = \int_0^\infty dL P(x_0, y_0, L)$$

(16)

where

$$P(x_0, y_0, L) = \langle \delta(x(0) - x_0)\delta(x(L) - y_0) \rangle.$$  

(17)

According to the Kallen-Lehmann representation there will be no ghosts if the residue of the momentum space poles of the Fourier transform of $P(x_0, y_0)$ is positive. Since the average in (17) is with respect to a Gaussian integral it can be done exactly. The result of the Fourier transform of the propagator (16) is

$$P(k) = \int_0^\infty dLe^{-\frac{m^2}{2}L+tk^2}G^*(0,L)$$

(18)

where

$$G^*(0,L) = G(0,L) - G(0,0)$$

(19)

and $G(0,L)$ defined in (7) is given by:

$$G(0, L) = \frac{-D}{2\pi i} \oint_C dp \frac{e^{pl}}{p^2(p + \omega^*) + p^2V_0(p) + V_1(p)}.$$  

(20)

the curve C is a contour containing the poles of the integrand. In the limit $a \rightarrow 0$ ($\Lambda \rightarrow \infty$) we expand the potential terms (13) in ”a” neglecting terms that approach
zero as $a \to o$. Then (19) is easily evaluated we obtain:

$$G^*(0, L) = \frac{1}{\beta} \left[ -\frac{L}{2\Omega} + \frac{1}{\Omega^2} (1 - e^{-|\Omega|L}) \right]$$  \hspace{1cm} (21)

where

$$\beta = 1 - \frac{4g}{3}, \quad \Omega = \frac{\omega^* + \Lambda g}{\beta}. \hspace{1cm} (22)$$

Using (21) and (22) in (18) we finally obtain

$$P(k) = \frac{1}{|\Omega|} e^{\eta} e^{-\epsilon \gamma(\epsilon, \eta)}$$  \hspace{1cm} (23)

where $\gamma$ is the incomplete $\gamma$ function and

$$\epsilon = \frac{s}{|\Omega|} > 0, \quad \eta = \frac{Dtk^2}{\beta \Omega^2}, \quad s = \frac{1}{2} (\mu_0 + \frac{Dtk^2}{\beta \Omega}) . \hspace{1cm} (24)$$

The poles of $\gamma$ are:

$$\epsilon = 0, \quad \epsilon = -n, \quad n = 1, 2, ...$$

using (22) these correspond to physical particles of squared masses:

$$\frac{g}{Dt} \mu_0 \Lambda ; \quad \frac{2ng^2}{|\beta|Dt} \Lambda^2 ; \quad n = 1, 2, .. \hspace{1cm} (25)$$

From the pole structure of the $\gamma$ function, the residue of the first particle is positive while for the rest it goes like $(-1)^n$. Thus the odd $n$'s are ghosts with mass of order $\Lambda$. However these ghosts decouple from the theory when we take the continuum limit for the following reason: Renormalization condition implies that the physical mass of our point particle is related to the bare mass by [6]:

$$\mu_0 = \frac{m_{phys}^2}{\Lambda} = \frac{m_{phys}^2}{\Lambda}$$

Therefore our particle with positive residue has a finite mass $m$, while the ghost particles have mass of order $\Lambda$. The value of the ghost mass cannot be reduced to a finite value because of the absence of fine tuning of the coupling constants.
This is related to the fact that we have a phase transition with two different regions specified by $t(g) < t_c(g_c)$ for the ordered phase and $t(g) > t_c(g_c)$ for the disordered phase. In both phases we have dimensional transmutation where the coupling constants are determined by the cut-off $\Lambda$ so they cannot be fine tuned. As $\Lambda \to \infty$ these ghosts become infinitely heavy thus decoupling from the theory. The case when $\beta = 0$ in (22) gives only the physical particle of mass $m$.

In conclusion, we have a consistent generalization of QED that includes the rigidity of the point particle without spoiling renormalizability. This Rigid model of QED has two distinct phases: I) a disordered phase which at large distances is characterized by the absence of rigidity and essentially coincides with ordinary QED while at short distances the electron and the positron transmute into a Majorana-Dirac fermions [1]. II) an ordered phase which is strongly coupled characterized by the presence of the curvature term at all distance scales. Even though the curvature of the world line of the particle is a higher derivative term in the classical theory, and naively indicates ghosts and tachyons in the mass spectrum, we show that the regulated quantum theory, is free of both ghosts and tachyons. This remarkable situation in the ordered phase, is ultimately related to the fact that the phase transition implies that the dependence on the coupling constants in the theory is non-analytic. Therefore we cannot analytically continue to the troubled classical theory. The absence of ghosts and tachyons in the disordered phase is easily understood by the fact that the curvature coupling constant runs and approaches zero at large distances.

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References

[1] M. Awada and D. Zoller, Phys.Lett B299 (1993) 151, also see the detailed version: Cincinnati preprint Jan. 1993-105. To appear in Int. Journal of Physics. A
[2] M.Awada, M.Ma, and D.Zoller Mod.phys. Lett A8,(1993), 2585
[3] A. Polyakov, Les Houches lecture notes 1988, ed. E.Brezin and J.Zinn-justin.
[4] A. Polyakov, private communications.
[5] R. Pisarski, Phys. Rev. D34, 670 (1986).
[6] A. Polyakov, Gauge fields, and Strings, Vol.3, harwood academic publishers
Long Range Ordered Phase, Order Parameter: $\omega = 0$

Short Range Disordered Phase, Order Parameter: $\omega > 0$