Path Integral for Relativistic Equations of Motion

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Abstract

A non-Grassmanian path integral representation is given for the solution of
the Klein-Gordon and the Dirac equations. The trajectories of the path inte-
gral are rendered differentiable by the relativistic corrections. The nonrela-
tivistic limit is briefly discussed from the point of view of the renormalization
group.

1 Introduction

The path integral representation for the solution of the Schrödinger equation
is not only a powerful computational method but provides a framework to
the understanding of the propagation in nonrelativistic Quantum Mechanics.

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In a surprising manner the known extensions of this formalism for spin half or relativistic particles are not too satisfactory.

Due to the spin-statistics theorem [1] the second quantized path integral for fermions is usually given in terms of anticommuting, formal Grassmann variables. The first quantized path integral for the Dirac equation can be obtained by means of the Grassmann variables in a manner analogous to the case of the Brownian motion [2]. In order to apply local nonperturbative methods to the path integral one needs a less formal basis in terms of ordinary complex numbers. A Brownian motion on a rectangular space-time lattice with a Poisson distributed helicity flip was used to construct a c-number path summation in ref. [3] which satisfies the Dirac equation. The underlying random walk was elucidated in [4] but without path integration. The application of this formalism is not clear in the more realistic cases.

The spin zero relativistic equation of motion is of second order in the time derivatives which creates problems in the generalization of the nonrelativistic path integral. Only Schwinger’s proper time formalism [5] is known to tackle this problem in the second quantized formalism.

We outline in this paper a modified version of the path integration which reproduces the solutions of the Dirac and the Klein-Gordon equations. It is similar to the standard path integral known from the Schrödinger equation except the lagrangian is matrix valued. Though this feature makes it less useful and attractive than Feynman’s original path integral however the ordinary integration over the coordinates may in the future lead to an eventual discovery of a more attractive path integral formalism for fermions. The characteristic feature of the construction is that the cutoff of the path integral, $\Delta t$, remains explicitly present in the continuum limit, $\Delta t \to 0$. In this respect our construction is formally similar to the second quantized Quantum Field Theory models which require the introduction of a cutoff due to their ultraviolet divergences even if this cutoff, as in our case, can be pushed beyond any finite limit without modifying the physical content of the theories.

The main achievement of the path integral formalism is that the particle is perfectly localised along the trajectories during the propagation. The relevance of this formalism is naturally limited up to the energies of the transition between the relativistic and the nonrelativistic regime. At distances below the Compton wave length the localization becomes vague what can be seen physically in the creation of the particle-anti particle pairs or mathematically in the smearing of the relativistically covariant coordinate operator [6].
and second quantization is needed. Nevertheless our representation of the solutions of the relativistic equations of motion may provide a conceptional insight into the transition between the relativistic and the nonrelativistic region, the origin of the spin in the first quantized description and might be useful as a framework for nonperturbative approximations as a starting point to the relativistic valence approximation. We shall consider free particle for simplicity.

The organization of the paper is the following. We recapitulate some properties of the nonrelativistic path integral in Section 2. Section 3 contains the derivation of the path integral representation for the solution of the massless Dirac equation. The massive case is dealt with in Section 4. The finite time propagator is investigated in Section 5 by means of the Foldy-Wouthuysen transformation. The path integral for the solution of the Klein-Gordon equation is presented in Section 6. Section 7 is a brief qualitative digression to comment our results from the point of view of the renormalization group and to verify the nonrelativistic limit. Finally Section 8 is reserved for the summary.

2 Nonrelativistic path integral

In order to motivate our construction of the relativistic path integration we start with a hand-waving introduction of the nonrelativistic path integral for a free particle in flat space what is based on the assumption that the action is a quadratic functional. The propagator,

\[ \langle x | e^{\frac{\hat{h} \hat{S}}{2m}} | y \rangle = G_0(x, y; t) = e^{\frac{i}{\hbar} S(x, y; t)}, \]  

is always self reproducible, i.e. a fixed point of the convolution,

\[ e^{\frac{i}{\hbar} S(x, y; t_1 + t_2)} = \int dz e^{\frac{i}{\hbar} S(x, z; t_1)} e^{\frac{i}{\hbar} S(z, y; t_2)}. \]  

The assumption is that this is achieved in the simplest manner, i.e. by choosing quadratic expression for \( S(x, y; t) \) in the coordinates. The form of \( S(x, y; t) \) is then further constrained by the kinematical symmetries, i.e. invariance under the translations in space and time, the three-rotations and the space and time inversion.
The translation symmetry restricts the functional form to
\[ S(x, y; t) = \frac{1}{2} (x - y)^j A_{jk} (x - y)^k + (x - y)^j B_j. \]  
(3)

Rotational symmetry imposes \( A_{jk} = A \delta_{j,k} \) and \( B = 0 \) for spin zero particles described by single component wave functions. In the case of spin the rotational symmetry allows \( B = B_S \), where \( S \) is the spin operator but the space inversion symmetry requires \( B = 0 \). The dimension of the action is \( [S(x, y; t)] = ML^2T^{-1} \) so \( [A] = MT^{-1} \). Due to the translation invariance in time there is no other time dimensional parameter in the action than \( t \) so \( A = m/t \). The quantity \( m \) has mass dimension and is real due to the time inversion symmetry.

In the presence of an external potential the action is nonquadratic and the symmetry properties are not enough to determine the propagator. Direct computation yields in the short time limit the well known expression, \[ G(x, y; \Delta t) = \left( \frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{3}{2}} \times \exp \left( \frac{i m (x - y)^2}{2 \hbar \Delta t} - \frac{i}{\hbar} \Delta t V \left( \frac{x + y}{2} \right) \right) \]  
(4)

for the hamiltonian \( H = \frac{p^2}{2m} + V(x) \). The simplest proof of this result is to verify the time evolution of the wave function,
\[ \psi(x, t + \Delta t) = \int d y G(x, y; \Delta t) \psi(y, t). \]  
(5)

In fact, this reproduces the Schrödinger equation as \( \Delta t \to 0 \),
\[ \psi(x, t + \Delta t) = \left[ 1 - \frac{i}{\hbar} \Delta t \left( -\frac{\hbar^2}{2m} \partial^2 + V(x) \right) + \frac{i \hbar \Delta t}{2m} \partial V(x) \cdot \partial + \cdots - \frac{i \hbar^3 \Delta t}{8m^2} (\partial^2)^2 \right] \psi(x, t). \]  
(6)

The first set of dots stands for the contributions which are higher order in the Taylor expansion of the potential \( V(x) \). The second set of dots represents the higher orders in the expansion of the wave function. Let us introduce
\[ \lambda_B = \frac{\hbar}{|p|}, \]  
(7)
the de Broglie wave length divided by $2\pi$, the characteristic scale of the the wave function, The small parameter in the first set of corrections is

\[
\frac{\Delta t \hbar |\partial_V|}{\lambda} \frac{\Delta v \hbar}{\lambda} = \frac{\Delta (p^2)}{2m},
\]

where we used the classical relations

\[
m \Delta v = \Delta p = \partial V \Delta t.
\]

These corrections are small if the kinetic energy does not change too much during the time $\Delta t$ of the propagation. The second kind of corrections are organized according to the small parameter

\[
\frac{\hbar^2 \Delta t}{m^2 \lambda_B^3} \frac{p^2}{m^3} = \frac{|v| \Delta t}{4\pi^3 \lambda_B^4},
\]

indicating that the particle should drift by a small amount compared to its characteristic length during the time $\Delta t$. These kind of corrections can be reexpontentiated and pose no problem for the free nonrelativistic propagation. In our study of the free particle we shall keep track of such kind of corrections only.

The expression (5) is not unique, the method of the renormalization group can be used to identify the family of lagrangians which belong to the universality class of the Schrödinger equation [8].

### 3 Massless fermions

The covariant equation of motion for the wave function $\phi$ of the first quantized massless spin half particle (right handed anti-neutrino) is

\[
(p^0 - c \mathbf{p} \cdot \mathbf{\sigma}) \phi = (i\hbar \partial_0 + i\hbar \mathbf{\sigma} \cdot \mathbf{\sigma}) \phi = 0.
\]

Since the space inversion

\[
\mathcal{P} \mathbf{x} \longrightarrow -\mathbf{x},
\]
does not commute with the Lorentz boosts $P$ can not be represented trivially in the irreducible representation $\phi$ of the special Poincare group. After space inversion one finds

$$ (p^0 + cp \cdot \sigma)\chi = (i\hbar \partial_0 - i\hbar \partial \cdot \sigma)\chi = 0, \quad (13) $$

with $\chi$

$$ P\phi = i\chi, \quad P\chi = i\phi. \quad (14) $$

The equation (13) is covariant only if $\chi$ (left handed neutrino) transforms as a complex conjugate spinor under the Lorentz transformations.

Our requirements about the propagator for finite time are: (i) Self reproducibility under the convolution, achieved by a quadratic action, (ii) Covariance with respect to the special Poincare group and (iii) the restriction of the propagation along the light cone even for the quantum fluctuations, i.e. independently of $\hbar$ up to the distance scale of the cutoff, $c\Delta t$. As in the nonrelativistic case, (i) and the invariance under translations and rotations in space leads to the form (3). Since the space inversion is not expected to be a symmetry, $B_j = B\sigma_j$. The strong kinematical constraint on the propagation which is imposed even on the level of the quantum fluctuations requires that the infinitesimal time propagator should be independent of $\hbar$. Since there is no parameter with mass dimension the only choice is

$$ G(x, y; \Delta t) = N^{-1}(\Delta t) \exp \left( i \frac{A}{2} \left( \frac{x - y}{c\Delta t} \right)^2 - iB \frac{x - y}{c\Delta t} \cdot \sigma \right), \quad (15) $$

where $N$ is fixed by the normalization, $A$ and $B$ are dimensionless numbers. By analogy with the nonrelativistic case the $-i\hbar$ times the exponent in this expression might be called the action. We set $B = 1$ in order reproduce the propagation where $c$ is the speed of light. One finds for the time evolution of the amplitude,

$$ \phi(x, t + \Delta t) = \int dy G(x, y; \Delta t)\phi(y, t) \quad (16) $$

$$ = N^{-1} e^{i \frac{A}{2}\Delta t^2} (2\pi c^2 \Delta t^2)^{\frac{D}{2}} \times \left( 1 - \frac{c\Delta t}{A} \sigma \cdot \partial + \frac{3c^2\Delta t^2}{2A} \left(A^{-1} + i\right) \partial^2 + \cdots \right) \phi(x, t), $$

4The factor $i$ included here is a convention as long as there is no neutral spin half particle.
what agrees with the equation of motion as \( \Delta t \to 0 \) for \( A = 1 \) as long as the characteristic length, \( \lambda_B \), of the amplitude \( \phi \) is long enough c.f. (10),
\[
c \Delta t << \lambda_B .
\] (17)
The normalization yields
\[
\mathcal{N} = e^{i \frac{3}{2}(2\pi ic^2 \Delta t^2)^{\frac{3}{2}}}. 
\] (18)
Thus the path integral
\[
G(x, y; t) = \prod_j \mathcal{N}^{-1} \int dx_j \exp \left\{ i \frac{1}{2} \left( \frac{x - y}{c \Delta t} \right)^2 - i \frac{x - y}{c \Delta t} \cdot \sigma \right\}
\] (19)
over the trajectories for time \( t \) with the end points \( x \) and \( y \) reproduces the propagator for finite time. The time ordering \( T \) in the second line takes care of the noncommutation of the spin mixing at different time slices.

Notice the explicit linear divergence, \( \frac{1}{dt} \), in the action. The role of the divergence is to suppress the term \( \partial^2 \) in the equation of motion. It is impossible to maintain the linear equation of motion without keeping the cutoff, \( \Delta t \), large but finite. This is reminiscent of Quantum Field Theory where the bare lagrangian contains the divergent coupling constants. Eq. (17) assures that the hamiltonian corresponding this diverging bare lagrangian is actually renormalizable and finite,
\[
H = -ich \partial \cdot \sigma .
\] (20)
The path integral (19) can easily be computed by the help of the Fourier transformation
\[
\tilde{G}(p, \Delta t) = \int dz e^{-\frac{1}{2}(x-y)p} G(x, 0; \Delta t) = \exp \left\{ -i \left( \frac{1}{2} \left( \frac{pc\Delta t}{\hbar} \right)^2 - \frac{pc\Delta t}{\hbar} \cdot \sigma \right) \right\},
\] (21)

since the convolution (2) becomes multiplication in the momentum space,
\[
G(x, y; t_1 + t_2) = \int dz \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi \hbar)^6} e^{i(x-z)p_1 + i(x-y)p_2} \tilde{G}(\mathbf{p}_1, t_1) \tilde{G}(\mathbf{p}_2, t_2)
\] (22)
The Fourier transformed propagator for time $t$ is
\begin{equation}
\tilde{G}(p, t) = \exp \left[ -it \left( \frac{1}{2} \left( \frac{pc}{\hbar} \right)^2 \Delta t - \frac{pc}{\hbar} \cdot \sigma \right) \right], \tag{23}
\end{equation}
what yields
\begin{equation}
G(x, y; t) = \mathcal{N}^{-1} \exp \left[ \frac{t}{\Delta t} \left( \frac{i}{2} \left( \frac{x - y}{ct} \right)^2 - i \frac{x - y}{ct} \cdot \sigma \right) \right]. \tag{24}
\end{equation}
The energy of the states described by the plane wave with momentum $p$ is
\begin{equation}
E(p) = \pm |p|c \left( 1 + O \left( \frac{\Delta tc}{\lambda_B} \right) \right). \tag{25}
\end{equation}
Note that the commutativity
\begin{equation}
[\tilde{G}(p, t), p \cdot \sigma] = 0 \tag{26}
\end{equation}
indicates that the helicity is conserved during the propagation.

It is illuminating to rewrite the logarithm of the propagator for finite time as
\begin{equation}
S(x, y; t) = \frac{\tilde{m}(x - y)^2}{2t} - \frac{1}{c} (x - y) \cdot A, \tag{27}
\end{equation}
where $\tilde{m}c^2 = \hbar \omega / 2\pi$, $\omega = 2\pi / \Delta t$, $A = A^a \sigma^a / 2$, and $A^a = \delta^a_j \tilde{m}c^2$. This shows a formal similarity with the propagation of a particle with mass $m$ in the presence of an SU(2) background field, $A$, whose magnetic field strength is
\begin{equation}
B^a_j = \delta^a_j \tilde{m}^2 c^4 \tag{28}
\end{equation}
in nonrelativistic Quantum Mechanics.

In deriving the neutrino equation we had to neglect the term $O(\partial^2)$ what required that the cutoff, $c\Delta t$, be smaller than the characteristic length scale of the amplitude, $\lambda_B$. So long as the cutoff is kept small but finite the path integral contains some undesired, noncovariant contributions at the distance scales below $c\Delta t$. This is obvious from (24) where the exponent is slowly varying at such distance scales thus the propagation in not constrained on the light cone. For longer distances the phase of the exponent is large and its stationary point with respect $x$ yields the "equation of motion"
\begin{equation}
\frac{x - y}{ct} \phi(y, 0) = \sigma \phi(y, 0). \tag{29}
\end{equation}
This guarantees that the particle stays on the light cone because the eigenvalues of the Pauli matrices is ±1.

In order to understand the role of the cutoff better we follow the spread of a wave packet with the Fourier transform

\[ \tilde{\phi}(p) = (2\pi \Delta p^2)^{-\frac{3}{2}} \exp \left( -\frac{p^2}{2\Delta p^2} \right) \phi_0, \]  

at \( t = 0 \) where \( \phi_0^\dagger \phi_0 = 1 \). The wave function at time \( t \) is then

\[ \phi(x, t) = (2\pi \sigma^2)^{-\frac{3}{2}} \int \frac{dp}{(2\pi \hbar)^3} \exp \left[ \frac{i}{\hbar} p \cdot x - it \left( \frac{1}{2} \left( \frac{pc}{\hbar} \right)^2 \Delta t + \frac{pc}{\hbar} \cdot \sigma \right) - \frac{p^2}{2\Delta p^2} \right] \phi_0 
\]

\[ = N^{-\frac{1}{2}} \exp \left( -\frac{1}{2} A(x^2 - 2tcx \cdot \sigma) \right) \phi_0, \]

where \( N \) is a time dependent constant and

\[ A = \frac{1 + it \Delta tc^2 \Delta p^2}{\hbar^2 + (t \Delta tc)^2 \Delta p^2 \hbar}. \]  

The absolute magnitude of the wave function is

\[ \phi^\dagger(x, t)\phi(x, t) = N^{-1}(t) \exp \left( -\frac{x^2}{2\Delta x^2(t)} \right), \]

with

\[ \Delta x^2(t) = \Delta x^2(0) + (tc)^2 \frac{(\Delta tc)^2}{\Delta x^2(0)}, \]

\[ \Delta x^2(0) = \lambda_{\hbar}^2 = \frac{\hbar^2}{\Delta p^2}. \]  

By comparing this result with the spread of the nonrelativistic wave packet we find that that \( \Delta x^2(t) \) corresponds to the usual nonrelativistic expression obtained from (27) and is independent of \( \hbar \) when expressed in terms of \( \Delta x^2(0) \). Furthermore the explicit apparence of the cutoff in the finite
time propagator, (24), induces acausal propagation by the gradual loss of
coherence between the high momentum modes when the initial wave func-
tion varies considerably within the cutoff. This spread is suppressed and the
propagation becomes causal for $\Delta tc \ll \Delta x$.

The lesson to be learned from (34) is that the wave function is constrained
by causality only as $\Delta t \to 0$. The acausal propagation is unaccessible for
observables far from the cutoff and drops out from the ”renormalized” theory,
where $\Delta t = 0$. How does this propagation look like? It turns out to be
similar to the nonrelativistic case. In fact, at short distances, i.e. high
momentum the $O(\partial^2)$ part of the time evolution is the dominant one and we
find a nonrelativistic propagation. According to (24) the mass parameter,
$\tilde{m}$, of this propagation is such that its Compton wave length is just the
cutoff, $c\Delta t$. Another role the short distance modes play is that the phase
of the wave function which diverges for a perfectly localized state, $\lambda B \to 0$,
becomes finite when (17) holds due to the interference between points with
distance $c\Delta t$ in the original wave packet. The phase of the amplitude for
wave packet with finite extent is cutoff independent in this manner.

4 Massive fermions

The mass of a spin half particle is the parameter which controls the mixing
between a particle and its space inverted state,

$$(p^0 - c\mathbf{p} \cdot \sigma)\phi = mc^2 \chi,$$

$$(p^0 + c\mathbf{p} \cdot \sigma)\chi = mc^2 \phi. \quad (35)$$

This mixing can be reproduced by multiplying the massless kernel for the
bispinor $\psi = (\phi, \chi)$ by

$$\exp \left( -\frac{mc^2 \Delta t}{\hbar} \mathbf{p} \right). \quad (36)$$

The kernel for a massive fermion is then

$$G(x, y; \Delta t) = e^{-\frac{3}{2}i(2\pi ic^2\Delta t)^{\frac{3}{2}}}$$

$$\times \exp \left[ i \left( \frac{1}{2} \left( \frac{x - y}{c\Delta t} \right)^2 - \alpha \cdot \left( \frac{x - y}{c\Delta t} \right) \right) \right]$$

$$\exp \left( -\frac{i}{\hbar} \beta mc^2 \Delta t \right), \quad (37)$$
where

\[ \beta = \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
\[ \alpha^j = \gamma^0 \gamma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}. \] (38)

By substituting this expression into (5) we obtain

\[ \psi(x, t + \Delta t) = \left( 1 - \frac{i}{\hbar} \beta mc^2 \Delta t - \Delta t \alpha \cdot \partial - \frac{1}{8} c^2 \Delta t^2 \partial^2 + \cdots \right) \psi(x, t), \] (39)

what goes over the Dirac equation,

\[ i \frac{\hbar}{c} \partial_0 \Psi = (-i \hbar \alpha_j \partial_j + \beta mc) \psi, \] (40)

as long as (17) holds to suppresses the acausal \( O(\partial^2) \) term in (39) and

\[ c \Delta t << \lambda_c = \frac{\hbar}{mc} \] (41)

in order to perform the linearization in the space inversion. The latter condition is needed to arrive at a finite difference equation in time what amounts to the construction of the Poissonian stochastic process in \([4]\).

5 Finite Time Propagator

The propagator for finite time is the initial and the final point dependent form of the path integral

\[ G(x, y; t) = \prod_j \left\{ \int dx_j \mathcal{N}^{-1}(\Delta t) \exp \left[ \frac{i}{2} \left( \frac{x - y}{c \Delta t} \right)^2 - \frac{x - y}{c \Delta t} \cdot \alpha \right] \right\} \]
\[ \times \exp \left( -\frac{i}{\hbar} \beta mc^2 \Delta t \right), \]
\[ = \int D[x(t)] T \exp \left\{ i \int dt \left[ \frac{1}{2} \left( \frac{dx}{dt} \right)^2 \frac{1}{c^2 d\tau} - \frac{1}{c} \frac{dx}{dt} \cdot \alpha \right] \right\} \]
\[ \times \exp \left( -\frac{i}{\hbar} \int d\tau \beta mc^2 \right), \] (42)
and it will be obtained by the help of (2),
\[ G(x, y; t_1 + t_2) = \int dz G(x, z; t_1)G(z, y; t_2). \] (43)

We shall work in Fourier space where the convolution is the multiplication,
\[ \tilde{G}(\mathbf{p}, t_1 + t_2) = \tilde{G}(\mathbf{p}, t_1)\tilde{G}(\mathbf{p}, t_2). \] (44)

The finite time propagator can not be given in terms of the exponential of a simple quadratic expression due the nonlinear time dependent mixing of the chiral spinors. But a partial simplification is gained by the Foldy-Wouthuysen transformation given by the similarity transformation
\[ U_{FW} = \exp \left( -\frac{i}{2mc} \beta \alpha \cdot \mathbf{p} \right) \left( 1 + O \left( \frac{c\Delta t}{\lambda_B} \right) \right) \] (45)
which decouples the large and the small components of the standard representation.

One can perform the Fourier transformation with the result
\[ \tilde{G}(\mathbf{p}, \Delta t) = \exp \left[ -i \left( \frac{1}{2} \left( \frac{|\mathbf{p}|c\Delta t}{\hbar} \right)^2 - \frac{c\Delta t}{\hbar} \alpha \cdot \mathbf{p} \right) \right] \exp \left( -\frac{i}{\hbar} \beta mc^2 \Delta t \right). \] (46)

The convolution (44) suggests that the relativistic-nonrelativistic crossover will be at \( c\Delta t \approx \lambda_C \) where the noncommutativity due to the mass term becomes important. By the help of the generalized Euler relation,
\[ e^{i\mathbf{v} \cdot \alpha} = \cos |\mathbf{v}| + i\alpha \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \sin |\mathbf{v}|, \] (47)
we can write
\[
\tilde{G}(\mathbf{p}, \Delta t) = \exp \left( -i \frac{1}{2} \left( \frac{|\mathbf{p}|c\Delta t}{\hbar} \right)^2 \right) \\
\times \left[ \cos \left( \frac{|\mathbf{p}|c\Delta t}{\hbar} \right) + i\alpha \cdot \frac{\mathbf{p}}{|\mathbf{p}|} \sin \left( \frac{|\mathbf{p}|c\Delta t}{\hbar} \right) \right] \\
\times \exp \left( -\frac{i}{\hbar} \beta mc^2 \Delta t \right). \] (48)
We now introduce the three rotation $R(p)$ which rotates the momentum into the $z$-direction, $R(p)p = \hat{z}$ and thus diagonalize the Fourier transformed infinitesimal time propagator. In the spin half representation

$$R(p) \rightarrow R_{1/2}(p) = e^{-\frac{i}{2} \hat{p} \cdot \sigma},$$

and one finds

$$R_{1/2}(p) \cdot \sigma R_{1/2}^{-1}(p) = |p| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (50)

This allows us to write

$$\tilde{G}(p, \Delta t) = e^{-\frac{i}{2} \left( \frac{c \Delta t}{\lambda_B} \right)^2} \begin{pmatrix} R_{1/2}^{-1} & 0 \\ 0 & R_{1/2}^{-1} \end{pmatrix} \mathcal{G} \begin{pmatrix} R_{1/2} & 0 \\ 0 & R_{1/2} \end{pmatrix}$$

with

$$\mathcal{G} = \begin{pmatrix}
  z \cos \left( \frac{c \Delta t}{\lambda_B} \right) & 0 & +iz^* \sin \left( \frac{c \Delta t}{\lambda_B} \right) & 0 \\
  0 & z \cos \left( \frac{c \Delta t}{\lambda_B} \right) & 0 & -iz^* \sin \left( \frac{c \Delta t}{\lambda_B} \right) \\
  +iz \sin \left( \frac{c \Delta t}{\lambda_B} \right) & 0 & z^* \cos \left( \frac{c \Delta t}{\lambda_B} \right) & 0 \\
  0 & -iz \sin \left( \frac{c \Delta t}{\lambda_B} \right) & 0 & z^* \cos \left( \frac{c \Delta t}{\lambda_B} \right)
\end{pmatrix},$$ \hfill (51)

expressed in the standard representation and

$$z = e^{-i \frac{c \Delta t}{\lambda_B}}.$$ \hfill (53)

The eigenvalues $\lambda$ of $\mathcal{G}$ satisfy the characteristic equation

$$\text{Det} [G(p, \Delta t) - \lambda \text{Id}] = \left[ 1 + \lambda^2 - 2 \cos \left( \frac{mc^2 \Delta t}{\hbar} \right) \cos \left( \frac{|p|c \Delta t}{\hbar} \right) \lambda \right]^2 = 0.$$ \hfill (54)

There are two double eigenvalues,

$$\lambda_{\pm} = \cos \left( \frac{mc^2 \Delta t}{\hbar} \right) \cos \left( \frac{|p|c \Delta t}{\hbar} \right) \pm i \sqrt{1 - \left[ \cos \left( \frac{mc^2 \Delta t}{\hbar} \right) \cos \left( \frac{|p|c \Delta t}{\hbar} \right) \right]^2}$$

$$= \cos \left( \frac{mc^2 \Delta t}{\hbar} \right) \cos \left( \frac{|p|c \Delta t}{\hbar} \right)$$

$$\pm i \sqrt{\sin^2 \left( \frac{|p|c \Delta t}{\hbar} \right) + \cos^2 \left( \frac{|p|c \Delta t}{\hbar} \right) \sin^2 \left( \frac{mc^2 \Delta t}{\hbar} \right)}.$$ \hfill (55)
with unit absolute magnitude. The matrix \( \tilde{G}(p, \Delta t) \) can be transformed into the diagonal matrix

\[
\tilde{G}_d(p, \Delta t) = \exp \left( -i \Delta t \beta \theta - \frac{i}{2} \left( \frac{|p|c \Delta t}{\hbar} \right)^2 \right),
\]

(56)

with \( e^{i \Delta t \theta} = \lambda_+ \) by means of the Foldy-Wouthuysen transformation and \( R(p) \).

The Fourier transform of the \( N \)-th power of the infinitesimal time propagator is

\[
\tilde{G}^N(p, \Delta t) = \tilde{G}(p, t) = \exp \left( -\frac{i p^2 c^2 t \Delta t}{2 \hbar^2} \right)
\times U_{FW} \left( \begin{pmatrix} R_{1/2}^{-1} & 0 \\ 0 & R_{1/2}^{-1} \end{pmatrix} e^{-it \beta \theta} \left( \begin{pmatrix} R_{1/2} & 0 \\ 0 & R_{1/2} \end{pmatrix} U_{FW}^\dagger \right. \right)
\]

(57)

To compute \( \theta \) we write \( \lambda_+ \) as

\[
\lambda_+ = \cos \left( \frac{mc^2 \Delta t}{\hbar} \right) \cos \left( \frac{|p|c \Delta t}{\hbar} \right)
\times \left\{ 1 + \tan \left( \frac{mc^2 \Delta t}{\hbar} \right) i \sqrt{1 + \left[ \frac{\tan \left( \frac{|p|c \Delta t}{\hbar} \right)}{\sin \left( \frac{mc^2 \Delta t}{\hbar} \right)} \right]^2} \right\},
\]

(58)

what yields

\[
\tan \Delta t \theta = \tan \left( \frac{mc^2 \Delta t}{\hbar} \right) \sqrt{1 + \left[ \frac{\tan \left( \frac{|p|c \Delta t}{\hbar} \right)}{\sin \left( \frac{mc^2 \Delta t}{\hbar} \right)} \right]^2}
\]

\[
= \tan \left( \frac{c \Delta t}{\lambda_C} \right) \sqrt{1 + \left[ \frac{\tan \left( \frac{c \Delta t}{\lambda_B} \right)}{\sin \left( \frac{c \Delta t}{\lambda_C} \right)} \right]^2}.
\]

(59)

Assuming that the cutoff is chosen to be sufficiently deeply in the relativistic regime, \( c \Delta t \ll \lambda_C \), and far from the momentum, \( c \Delta t \ll \lambda_B \), one obtains

\[
\theta = \frac{mc^2 \Delta t}{\hbar} \sqrt{1 + \left( \frac{p}{mc} \right)^2}.
\]

(60)
The $N$-th iteration leads then to
\[
\tilde{K}_d(p, t) = \exp \left( -\frac{i}{2N} \left( \frac{pct}{\hbar} \right)^2 \right) \exp(-iN\beta \theta)
\]
\[
= \exp \left( -\frac{i}{2N} \left( \frac{pct}{\hbar} \right)^2 \right)
\]
\[
\times \exp \left[ -i\beta \frac{mc^2t}{\hbar} \sqrt{1 + \left( \frac{p}{mc} \right)^2} + O \left( \sqrt{\frac{\Delta t}{t}} \right) \right],
\]
(61)
where $t = N\Delta t$.

The final expression for the propagator is then
\[
G(x, y; t) = \int \frac{dp}{(2\pi\hbar)^3} U_{FW} \left( \begin{array}{cc}
R_{1/2}^{-1} & 0 \\
0 & R_{1/2}^{-1}
\end{array} \right)
\]
\[
\times \exp \left[ \frac{i}{\hbar} p(x - y) - \frac{i}{\hbar} \beta tmc^2 \sqrt{1 + \left( \frac{p}{mc} \right)^2} \right]
\]
\[
\times \left( \begin{array}{cc}
R_{1/2} & 0 \\
0 & R_{1/2}
\end{array} \right) U_{FW}^\dagger,
\]
(62)
what reproduces the usual relativistic dispersion relation.

## 6 Scalar Particle

The solutions of the equations with second order derivatives in the time have no direct path integral representation. But it is straightforward to bring the
\[
\left( \Box + \frac{1}{\lambda^2} \right) \phi(x) = 0,
\]
(63)
Klein-Gordon equation into a first order equation of motion,
\[
\frac{1}{c} \partial_0 \Phi = \partial_j \Phi_j - m\Phi_d,
\]
(64)
$j = 1, \cdots, d - 1$, for the $d + 1$ component field
\[
\Phi = \left( \begin{array}{c}
\Phi_0 \\
\Phi_j \\
\Phi_d
\end{array} \right) = \left( \begin{array}{c}
\frac{1}{c} \partial_0 \phi \\
\partial_j \phi \\
\frac{1}{\lambda c^2} \phi
\end{array} \right).
\]
(65)
One can verify that the relations
\[
\frac{1}{c} \partial_0 \Phi_d = \frac{1}{\lambda c} \Phi_0
\]
\[
\partial_j \Phi_d = \frac{1}{\lambda c} \Phi_j,
\]
are preserved by (64). In order to bring the linearized equation in the form similar to the Dirac equation we write
\[
i\hbar \frac{\partial}{c} \Phi(x) = (-i\hbar \alpha_j \partial_j + \beta mc) \Phi(x),
\]
with the help of the hermitean matrices
\[
\alpha_j = \begin{pmatrix} 0 & -e_j & 0 \\ -e_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},
\]
where \(e_j\) is the unit vector in the direction \(j\). An important property of the matrices \(\alpha\) what holds for any vector \(u\) is
\[
(\alpha \cdot u)^2 = M_u u^2,
\]
with
\[
M_u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P_u & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
where \(P_u\) is the projection on \(u\). \(M_u\) satisfies \(M_u^2 = M_u\).

Analogously with the Dirac equation we introduce the following kernel
\[
G(x, y; \Delta t) = e^{-\frac{i}{\hbar} \alpha \cdot \frac{(x-y)c}{\Delta t}} - \frac{i}{\hbar} \beta mc^2 \Delta t
\]
\[
\times \exp \left[ i \left( \frac{1}{2} \left( \frac{x-y}{c \Delta t} \right)^2 - \alpha \cdot \left( \frac{x-y}{c \Delta t} \right) \right) \right]
\]
\[
\exp \left( -\frac{i}{\hbar} \beta mc^2 \Delta t \right),
\]
In order to obtain (67) we write in first order in \(\Delta t\)
\[
\Phi(x, t + \Delta t) = \left\{ 1 - \frac{i}{\hbar} \beta mc^2 \Delta t - i e^{\frac{-i}{\hbar} M_u (2 \pi i c^2 \Delta t)^{-\frac{3}{2}}} \right\}
\]
\[ x \int dy e^{i \frac{y \cdot (x - y)}{2t}} \left[ M_u \alpha \cdot \left( \frac{x - y}{|x - y|} \right) \sin \left( \frac{|x - y|}{c \Delta t} \right) \right. \\
\left. + (1 - M_u) \alpha \cdot \left( \frac{x - y}{c \Delta t} \right) \right] \partial + \cdots \right\} \Phi(y, t). \] (72)

The last equation turns out to be

\[ \Phi(x, t + \Delta t) = \left( 1 - \frac{i}{\hbar} \beta mc^2 \Delta t - M_u c \Delta t \alpha \cdot \partial \\
- (1 - M_u) e^{-i \frac{\Delta y}{2} M_u c \Delta t \alpha \cdot \partial + \cdots} \right) \Phi(x, t), \] (73)

leading to

\[ \frac{\partial}{\partial t} \Phi(x, t) = \left( -\frac{i}{\hbar} \beta mc^2 - c \alpha \cdot \partial + \cdots \right) \Phi(x, t). \] (74)

## 7 Scaling laws

Certain aspects of our construction can be better understood by means of ideas borrowed from the renormalization group. We start with the remark that the dimensional analysis alone is enough to understand the singular properties of the propagation in Quantum Mechanics. In fact, the action in (5) contains the term \((x - y)^2/t\) so the phase velocity is diverging at short time,

\[ \left( \frac{\Delta x}{t} \right)^2 \approx \frac{\hbar}{mt}, \] (75)

reflecting the fast spread of the wave function for well localized states. This result is not so surprising by recalling that there is no internal velocity parameter for a free nonrelativistic particle. In fact, the quantities with the dimension of the velocity have to be made up by the help of the observational time scale, \(t\), which leads to the divergence.

We may look at Quantum Mechanics as a Quantum Field Theory in 0+1 dimensions. The ultraviolet divergences make the introduction of a cutoff necessary in Quantum Field Theory where a model is renormalizable if this artificial parameter, the cutoff, can be pushed beyond any limit and can thus be removed. Nevertheless, the precise mathematical construction of the model requires the introduction of the cutoff with arbitrary high but finite value. The cutoff can completely be eliminated in the nonrelativistic
systems without velocity dependent interactions, and the path integral can formally be written in the continuum as

$$\int D[x] e^{\frac{i}{\hbar} \int dt L[x]},$$

(76)

where $L[x]$ is the well defined finite, i.e. cutoff independent lagrangian. The cutoff appears in an explicit manner in the relativistic path integral, as in bare Quantum Field Theory.

There are two scale parameters in our path integrals, the cutoff, $c \Delta t$ and $\lambda_C >> c \Delta t$. We shall first consider the asymptotically ultraviolet regime, for the length scales above the cutoff but safely below the Compton wave length. In this regime the mass can be neglected or treated as a weak perturbation and the propagation is practically governed by the neutrino equation. In particular, it has been pointed out in ref. and that the helicity flip occurs rarely and the particle stays on the light cone in this regime. The insertion of the covariant coordinate operator introduced in corresponds to the multiplication with the coordinate in the path integral. In this manner the acausal spread of the wave packet, when the initial localization is comparable or better than $c \Delta t$ is the path integral analogue of the nonlocal feature of the covariant coordinate operator. The $\hbar$ independence of the spread actually shows the kinematical origin. The spread beyond the Compton wavelength looses its dependence on the cutoff and one recovers the usual nonrelativistic, $\hbar$ dependent behavior. $\hbar$ enters via the mass term and several helicity flips may occur simultaneously in this regime. The Poisson distribution of the helicity flips characterizing the relativistic domain gives rise the Wiener process of the nonrelativistic propagation at the infrared side of the Compton wave length, and.

The dependence of the average velocity on the observational time scale can be traced down by the method of the renormalization group. This amounts to perform the convolution repeatedly and to follow the average velocity as the time of the observation, $t$, increases. The dependence obtained in this manner,

$$\left( \frac{\Delta x}{t} \right)^2 = O \left( \frac{1}{t} \right),$$

(77)

can be interpreted as a scaling relation.
An analogous scaling law results for the free relativistic particles since they lagrangian is quadratic. The massless propagator, (24), is nonvanishing for \( t \gg \Delta t \) when
\[
\left( \frac{\Delta x}{t} \right)^2 \approx c^2, \tag{78}
\]
since the eigenvalue of the Pauli matrices is \( \pm 1 \). For \( t \approx \Delta t \) one finds
\[
\left( \frac{\Delta x}{t} \right)^2 \approx c^2 \frac{\Delta t}{t}. \tag{79}
\]
Thus we recover the usual kinematics for the propagation of the massless particles at time scales longer than the cutoff.

The asymptotic scaling agree for massive and massless particles in the ultraviolet regime. Starting from the infrared end the usual nonrelativistic scaling law must somehow change into the relativistic one for a massive particle where the average velocity reaches the speed of the light, \( ct \approx \lambda C \). To see this on a qualitative level we shall evaluate the average velocity by neglecting the influence of the end points of the trajectories in the path integral,
\[
\langle \left( \frac{\Delta x}{ct} \right)^2 \rangle = \int \Delta x G(\Delta x, 0; t) \left( \frac{\Delta x}{ct} \right)^2. \tag{80}
\]
We employ the semiclassical approximation to the integral in the propagator (62) and in (80). The extremum of the exponent in (62) given by
\[
\frac{|x|}{ct} = \frac{|p|}{mc} \frac{1}{\sqrt{1 + \left( \frac{p}{mc} \right)^2}}, \tag{81}
\]
which shows that the contribution is concentrated in the causal region,
\[
\frac{|x|}{ct} < 1. \tag{82}
\]
The value of the phase factor at the saddle point,
\[
\exp \left( i \frac{ct}{\lambda C} \sqrt{1 - \left( \frac{x}{ct} \right)^2} \right), \tag{83}
\]
is the integrand for the $x$ integration in the range $(S^2)$ what covers the possible saddle points. The integral $(80)$ is rewritten by means of the dimensionless variable $z = x/ct$,

$$< \left( \frac{\Delta x}{ct} \right)^2 > = \frac{\int dz z^2 e^{\frac{i\pi}{\hbar c} \sqrt{1-z^2}}}{\int dz e^{\frac{i\pi}{\hbar c} \sqrt{1-z^2}}} \approx \begin{cases} O(t^0) & \text{for } tc << \lambda C, \\ \frac{\hbar}{mc^2 t} & \text{for } tc >> \lambda C, \end{cases}$$

(84)

in leading order of $\hbar$, indicating a crossover at the relativistic-nonrelativistic transition at $\Delta tc \approx \lambda C$. In fact, the path integral displays different universal behavior in these regimes: The trajectories of the path integral follow the massless relativistic scaling laws for $ct < \lambda C$ and appear to be differentiable, $|\Delta x| \approx c$. The usual nonrelativistic scaling law characteristic of the Brownian motion sets in for $ct > \lambda C$.

8 Summary

A path integral was presented in this paper which reproduces the solutions of the Klein-Gordon and the Dirac equations. The nonrelativistic limit was verified for massive fermions. It was found that the trajectories of the path integral follow the nonrelativistic scaling law characterizing the Brownian motion when $c\Delta t > \lambda C$. When the observational time, $\Delta t$, is decreased down to the value $\Delta t \approx \lambda C/c$ then the average velocity reaches the order of magnitude of $c$ and the relativistic effects become visible as a crossover. In the relativistic regime, $c\Delta t << \lambda C$, the average velocity remains $c$.

The free particle corresponds to a quadratic lagrangian $L = L_1 + L_2$, $L_1 = O(\Delta x)$, $L_2 = O(\Delta^2 x)$ where the linear terms were responsible for the time evolution. The coefficient of the quadratic term is linearly divergent, $L_2 = O((\Delta t)^{-1})$, and appeared as a formal device only. It is worthwhile noting that the explicit appearance of the linear divergent coefficient, $(\Delta t)^{-1}$, does not mean that this path integral is more singular in the continuum limit that its nonrelativistic counterpart. Actually the situation is just the contrary. Note that in both cases the phase angle $\Delta t L = O((\Delta t)^0)$ for the typical trajectories. The diverging coefficient in the relativistic lagrangian suppresses the velocity fluctuations and renders the trajectories differentiable. In such a
manner the relativistic effects "regulate" the Itô calculus [9] and the classical calculus is recovered within the relativistic regime.

The spin-statistic theorem asserts the equivalence of the phase factor appearing in front of the wave function under rotation by $2\pi$ and exchange of two particles. In this manner one may speculate that the proper treatment of the spin half aspect of the propagation which includes the $-1$ factor for each rotation by $2\pi$ will ultimately yield the correct antisymmetrized Green functions in the second quantization without the use of formal Grassmann variables in the path integral.

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