Period-Life of a Branching Process with Migration and Continuous Time

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Abstract: The homogeneous branching process with migration and continuous time is considered. We investigated the distribution of the period-life \( \tau \), i.e., the length of the time interval between the moment when the process is initiated by a positive number of particles and the moment when there are no individuals in the population for the first time. The probability generating function of the random process, which describes the behavior of the process within the period-life, was obtained. The boundary theorem for the period-life of the subcritical or critical branching process with migration was found.

Keywords: branching process; migration; continuous time; generating function; period-life

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1. Introduction

Branching processes (BPs) are often used as mathematical models of different real processes, in particular, chemical [1], biological [2], genetic [3], demographic [4], technical [5] and others. In addition, BPs can describe the population dynamics of particles of different natures, in particular, they can be photons, electrons, neutrons, protons, atoms, molecules, cells, microorganisms, plants, animals, individuals, prices, information, etc. This list can be continued. Thus, a BP is quite widely used in various sciences. Since third party factors often exist, there is a need to study different modifications of this process. Among them are BPs with immigration, emigration, or a combination of two processes, namely processes with migration for the case of discrete or continuous time.

The theory of Non-Homogeneous Markov systems first introduced in [6]. The case of the Non-Homogeneous Markov systems in continuous time in its latest results exist in Dimitriou and Georgiou [7].

For the first time, the term period-life (PL) \( \tau \) for the Galton–Watson BP and the Markov BP with immigration was considered by Zubkov A.M. in [8]. He obtained asymptotic formulas for the distribution tails as a function of the PL and found the necessary and sufficient conditions for the process to obtain zero for the corresponding Markov chain. Vatutin V.A. [9] continued the study of PL for the critical case and obtained a limit theorem on the behavior of the process at \( \tau > t \) and provided that the beginning of the PL \( T = 0 \). PL for a critical BP with random migration and discrete time were studied by Yanev N.M. and Mitov K.V. [10]. The distribution PL of the BP with immigration in a limited environment and its behavior in the PL was investigated by Boyko R.V. [11]. Formanov Sh. K., Yasin M.T. [12] obtained boundary theorems for the PL of critical BP Galton–Watson...
with migration. The case of the Bellman–Harris BP with immigration was studied in [13,14]. The distribution of the PL for the subcritical and critical BP with immigration in a random environment was studied in the works [15,16]. However, the asymptotic properties of the PL of the branching process with migration and continuous time (BPMCT) have not been considered to date.

The case of a BPMCT is considered in [17–21]. Chen A. Y. and Renshaw E. [17] have considered a case of the process in which large immigration, i.e., the sum of immigration rates, is infinite; excessively high population levels are avoided by allowing the carrying capacity of the system to be controlled by mass emigration. Rahimov I. and Al-Sabah W.S. [18] have investigated a family of independent, equally distributed with a continuous time Markov BP. The migration was determined as follows: the particles first immigrate and stay in the population for some time, and then emigrate. Srivastava O. P. and Gupta S. [19] have considered a branching process in which the migration and the emigration of the particle occur independently of each other and with the same probability. Pakes A. G [20] has studied the process of when a batch of immigrants arrive in a region at event times of a renewal process and the individuals grow according to a Bellman–Harris branching process. Tribal emigration allows the possibility that all descendants of a group of immigrants collectively leave the region at some instant. Balabaev I. S. [21] considered the Khan–Nagaev process [22] as a nested BP.

In this article, we consider a more general model of the BPMCT [23]. Immigration, emigration, and evolution occur at random moments in time are determined by the intensity of the transition probabilities.

The purpose of this work is to study the PL of the BPMCT and find the distribution of the PL and the boundary distribution in the case of the subcritical or critical process. This model can be used to model real processes, including biological and demographic, which allow migration processes with continuous time.

The structure of this article is as follows. The first part contains a brief overview of PL that studies different types of BP. Then comes a model of the process with migration and continuous time. In the next section, we find the form of the differential equation and the generating function for the random process, which describes the behavior of the process within the PL. The boundary theorem for PL of the subcritical and critical processes is given below. The section of the conclusion emphasizes the obtained results.

2. Description of a Branching Process Model with Migration and Continuous Time

Consider a Markov BP with one type of particles and migration \( \mu(t) \), \( t \in [0, \infty) \). Let \( \mu(t) \) denote the number of particles at time \( t \in [0, \infty) \).

We suppose that at time \( t = 0 \), the process starts with one particle in the system:

\[
\mu(0) = 1.
\]  

(1)

The process \( \mu(t), t \in [0, \infty) \) then \( \Delta t \to 0 \) is given by the transition probabilities:
where \( m \) is a fixed integer, and \( p_k, q_k, r_n \) satisfy the conditions:

\[
p_k \geq 0, \quad k \neq 1, \quad p_1 < 0, \quad \sum_{k=0}^{\infty} p_k = 0,
\]

\[
q_k \geq 0, \quad k \neq 0, \quad q_0 < 0, \quad \sum_{k=0}^{\infty} q_k = 0,
\]

\[
r_n \geq 0, \quad n = \frac{1}{1, m}, \quad r_0 < 0, \quad \sum_{k=0}^{m} r_k = 0.
\]

We note that \( p_k \ (k = 0, 1, \ldots) \) is the intensity of the reproduction particle, \( q_k \ (k = 0, 1, \ldots) \) is the intensity of immigration, and \( r_n \ (n = \frac{1}{0, m}) \) is the intensity of emigration.

We introduce the following notation:

\[
f(s) = \sum_{n=0}^{\infty} p_n s^n, \quad |s| \leq 1, \quad s \in C,
\]

\[
g(s) = \sum_{n=0}^{\infty} q_n s^n, \quad |s| \leq 1, \quad s \in C,
\]

\[
r(s) = \sum_{n=0}^{m} r_n s^{-n}, \quad 0 < |s| \leq 1.
\]

We let \( \hat{F}(t, s) \) be the probability generating functions (PGFs) of a BP with continuous time (without migration) ([24], page 24).

3. Results

In this section, we find a differential equation for the PGF and PGF random process, which describes the behavior of the process within the PL of the BPMCT.

The method of PGF is widely used in the study of processes with continuous time, because in some cases, it can be found in the form of its generation, and then the corresponding probabilities of the process are calculated. The PGF of the process will uniquely determine the distribution of the process and the limiting behavior of the process.

3.1. PGF of the Random Process, Which Describes the Behavior of the Process within the PL

Definition 1. [8] \( \tau \) is the PL of a BP within which immigration begins at the moment \( T \) and has length \( \tau \) if \( P\{\mu(T + \tau) = 0\} = P\{\mu(T - \Delta t) = 0\} = 0 \) and \( P\{\mu(t) = 0\} > 0 \) for all \( t \in [T, T + \tau) \) (the trajectories of the process are assumed to be continuous from the right).
Let:
\[ u(t) = P\{\tau > t\}, \]
and define a random process \( v(t) \), which describes the behavior of the process within the PL:
\[
v(t) = \begin{cases} 
\mu(t), & t \leq \tau, \\
0, & t > \tau,
\end{cases}
\]

obviously that:
\[ v(0) = \mu(0). \]

We define a PGF for \( v(t) \):
\[ N(t, s) = \sum_{k=0}^{\infty} P\{v(t) = k\} s^k. \]

**Theorem 1.** Let \( \tau \) be the PL of the BPMCT, then:
1. The PGF \( N(t, s) \) satisfies the differential equation:
\[
\frac{dN(t, s)}{dt} = \sum_{n=1}^{m} \sum_{k=n}^{m} P\{v(t) = n\} r_k (1 - s^{n-k}),
\]
with the initial condition:
\[ N(0, s) = \frac{q_0 - g(s)}{q_0}. \]

2. The PGF for \( v(t) \) has the form:
\[
N(t, s) = V\left(t + \int_{0}^{s} \frac{du}{f(u)} \right) e^{\int_{0}^{t} \left( g(\hat{F}(u,s)) + r(\hat{F}(u,s)) \right) du - \int_{0}^{t} P\{v(t) = 0\} \left( g(\hat{F}(t-x,s)) + r(\hat{F}(t-x,s)) \right) e^{\int_{0}^{t-x} \left( g(\hat{F}(u,s)) + r(\hat{F}(u,s)) \right) du} dx } \]
\[
+ \int_{0}^{t} \sum_{n=1}^{m} P\{v(t) = n\} \sum_{k=n}^{m} r_k (1 - \hat{F}^{n-k}(t-x,s)) e^{\int_{0}^{t-x} \left( g(\hat{F}(u,s)) + r(\hat{F}(u,s)) \right) du} dx, \]

where \( V(\cdot) \) is some continuous-differentiated function that satisfies:
\[
\begin{aligned}
V\left(\int_{0}^{s} \frac{du}{f(u)} \right) &= \frac{q_0 - g(s)}{q_0}, \\
V(\infty) &= 1.
\end{aligned}
\]

**Proof of Theorem 1.** We prove the first part of the theorem. Consider:
\[
\sum_{k=0}^{\infty} P\{v(t + \Delta t) = k|v(t) = n\} s^k.
\]
For \( n = 0 \):
\[
\sum_{k=0}^{\infty} P\{v(t + \Delta t) = k|v(t) = 0\} s^k = 1s^0 = 1.
\]
If \( n = 1 \) then:

\[
\sum_{k=0}^{\infty} P\{\nu(t + \Delta t) = k | \nu(t) = 1\} s^k = \sum_{k=0}^{\infty} P\{\mu(t + \Delta t) = k | \mu(t) = 1\} s^k
\]

\[
= s + (f(s) + g(s))s + \sum_{k=1}^{m} r_k + r_0 s) \Delta t + o(\Delta t).
\]

In case \( 1 < n \leq m \):

\[
\sum_{k=0}^{\infty} P\{\nu(t + \Delta t) = k | \nu(t) = n\} s^k = \sum_{k=0}^{\infty} P\{\mu(t + \Delta t) = k | \mu(t) = n\} s^k
\]

\[
= s^n + (n s^{n-1} f(s) + s^n g(s) + s^n \sum_{k=0}^{n-1} r_k s^{-k} + \sum_{k=n}^{m} r_k) \Delta t + o(\Delta t).
\]

For \( n > m \):

\[
\sum_{k=0}^{\infty} P\{\nu(t + \Delta t) = k | \nu(t) = n\} s^k = \sum_{k=0}^{\infty} P\{\mu(t + \Delta t) = k | \mu(t) = n\} s^k
\]

\[
= s^n + (n s^{n-1} f(s) + s^n g(s) + s^n r(s)) \Delta t + o(\Delta t).
\]

Hence, we obtain the following:

\[
\sum_{k=0}^{\infty} P\{\nu(t + \Delta t) = k | \nu(t) = n\} s^k = \begin{cases} 
1, & n = 0; \\
\left(1 + (f(s) + g(s))s + \sum_{k=1}^{m} r_k + r_0 s) \Delta t + o(\Delta t), & n = 1; \\
s^n + (n s^{n-1} f(s) + s^n g(s) + \sum_{k=0}^{n-1} r_k s^{-k} + \sum_{k=n}^{m} r_k) \Delta t + o(\Delta t), & 1 < n \leq m; \\
s^n + (n s^{n-1} f(s) + s^n g(s) + \sum_{k=0}^{m} r_k) \Delta t + o(\Delta t), & n > m.
\end{cases}
\]

Consider the PGF \( N(t, s) \).

Let \( T \) be the beginning of the PL of the process \( \mu(t) \), then:

\[
N(0, s) = \sum_{k=0}^{\infty} P\{\nu(0) = k\} s^k = \sum_{k=0}^{\infty} P\{\mu(T) = k | \mu(T - \Delta t) = 0, \mu(0) > 0\} s^k
\]

\[
= \lim_{\Delta t \to 0} \sum_{k=1}^{\infty} \left(\frac{q_k \Delta t + o(\Delta t)}{q_0}\right) s^k = \sum_{k=1}^{\infty} \frac{q_k s^k}{q_0} = \frac{q_0 - g(s)}{q_0}.
\]

Since it is the initial condition, we obtain (4).

Thus, we derive:

\[
N(t + \Delta t, s) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} P\{\nu(t + \Delta t) = k | \nu(t) = n\} P\{\nu(t) = n\} s^k = P\{\nu(t) = 0\}
\]

\[
+ \sum_{n=1}^{m} P\{\nu(t) = n\} \left(s^n + (n s^{n-1} f(s) + s^n g(s) + s^n \sum_{k=0}^{n-1} r_k s^{-k} + \sum_{k=n}^{m} r_k) \Delta t + o(\Delta t)\right)
\]
corresponding inhomogeneous equation: $g$ and rewrite it in the form:

$$
= \sum_{n=0}^{\infty} P\{v(t) = n\} s^n + \sum_{n=1}^{\infty} P\{v(t) = n\} n s^{n-1} f(s) \Delta t + \sum_{n=1}^{\infty} P\{v(t) = n\} g(s) s^n \Delta t
$$

+ \sum_{n=1}^{m} P\{v(t) = n\} \left( r(s) s^n + \sum_{k=n}^{m} r_k (1 - s^{n-k}) \right) \Delta t + o(\Delta t)

= N(t,s) + \frac{\partial N(t,s)}{\partial s} f(s) \Delta t + N(t,s) g(s) \Delta t - g(s) P\{v(t) = 0\} \Delta t

+ N(t,s) r(s) \Delta t - r(s) P\{v(t) = 0\} \Delta t + \sum_{n=1}^{m} P\{v(t) = n\} \sum_{k=n}^{m} r_k (1 - s^{n-k}) \Delta t + o(\Delta t)

= N(t,s) + \left( \frac{\partial N(t,s)}{\partial s} f(s) + N(t,s) (g(s) + r(s)) - (g(s) + r(s)) P\{v(t) = 0\} \right)

+ \sum_{n=1}^{m} P\{v(t) = n\} \sum_{k=n}^{m} r_k (1 - s^{n-k}) \Delta t + o(\Delta t).

Consider $\frac{N(t+\Delta t,s) - N(t,s)}{\Delta t}$ and directing $\Delta t \to 0$, we obtain (5).

We turn to the second part of the theorem. In proving the second part, we will use the notation $g(s) + r(s) = \gamma(s)$.

Consider the equation of the characteristics:

$$
dt = -\frac{ds}{f(s)} = \frac{dN(t,s)}{(N(t,s) - P\{v(t) = 0\}) \gamma(s) + \sum_{n=1}^{m} P\{v(t) = n\} \sum_{k=n}^{m} r_k (1 - s^{n-k})}.
$$

The first integral of this equation:

$$
C_1 = t + \int_{0}^{s} \frac{du}{f(u)}.
$$

We obtain the second integral of this equation:

$$
dt = \frac{dN(t,s)}{(N(t,s) - P\{v(t) = 0\}) \gamma(s) + \sum_{n=1}^{m} P\{v(t) = n\} \sum_{k=n}^{m} r_k (1 - s^{n-k})},
$$

and rewrite it in the form:

$$
\frac{\partial N(t,s)}{\partial t} = (N(t,s) - P\{v(t) = 0\}) \gamma(s) + \sum_{n=1}^{m} P\{v(t) = n\} \sum_{k=n}^{m} r_k (1 - s^{n-k}).
$$

We find the solution of the corresponding homogeneous equation:

$$
N(t,s) = C_2 e^{\int_{u_0}^{t} \gamma(\tilde{t},u_0) du}.
$$

Using the method of the variation of constants, we obtain a partial solution of the corresponding inhomogeneous equation:

$$
N(t,s) = - \int_{0}^{1} P\{v(t) = 0\} \gamma(\tilde{t},t,s) e^{\int_{u_0}^{t-x} \gamma(\tilde{F}(u,s)) du} dx.
$$
Thus, the general solution of the inhomogeneous equation will take the form:

$$N(t, s) = C_2 e^{\int_0^t \gamma(\hat{t}(u, s)) du} - \int_0^t P\{v(t) = 0\} \gamma(\hat{t}(t - x, s)) e^0 \int_0^x \gamma(\hat{t}(u, s)) du dx$$

$$+ \int_0^t \sum_{n=1}^m P\{v(t) = n\} \sum_{k=n}^m r_k (1 - \hat{F}^{n-k}(t - x, s)) e^0 \int_0^x \gamma(\hat{t}(u, s)) du dx.$$

Hence, the second integral of the equation:

$$C_2 = N(t, s) e^{-t \int_0^s \gamma(\hat{t}(u, s)) du} + \int_0^t P\{v(t) = 0\} \gamma(\hat{t}(t - x, s)) e^0 \int_0^x \gamma(\hat{t}(u, s)) du dx$$

$$- t \int_0^t \sum_{n=1}^m P\{v(t) = n\} \sum_{k=n}^m r_k (1 - \hat{F}^{n-k}(t - x, s)) e^0 \int_0^x \gamma(\hat{t}(u, s)) du dx,$$

According to ([25] page 97), we obtain:

$$V\left(t + \int_0^s \frac{du}{f(u)}\right) = N(t, s) e^{-t \int_0^s \gamma(\hat{t}(u, s)) du}$$

$$+ \int_0^t P\{v(t) = 0\} \gamma(\hat{t}(t - x, s)) e^0 \int_0^x \gamma(\hat{t}(u, s)) du dx$$

$$- t \int_0^t \sum_{n=1}^m P\{v(t) = n\} \sum_{k=n}^m r_k (1 - \hat{F}^{n-k}(t - x, s)) e^0 \int_0^x \gamma(\hat{t}(u, s)) du dx,$$

where $V(\cdot)$ is some continuous-differentiated function.

Thus, we obtain a PGF:

$$N(t, s) = V\left(t + \int_0^s \frac{du}{f(u)}\right) e^0 \int_0^t \gamma(\hat{t}(u, s)) du$$

$$- \int_0^t P\{v(t) = 0\} \gamma(\hat{t}(t - x, s)) e^0 \int_0^x \gamma(\hat{t}(u, s)) du dx$$

$$+ \int_0^t \sum_{n=1}^m P\{v(t) = n\} \sum_{k=n}^m r_k (1 - \hat{F}^{n-k}(t - x, s)) e^0 \int_0^x \gamma(\hat{t}(u, s)) du dx.$$

From the initial condition, we obtain:

$$V\left(\int_0^s \frac{du}{f(u)}\right) = \frac{q_0 - s}{q_0}.$$
When $s = 1$, then:

$$N(t, 1) = V \left( t + \int_0^1 \frac{du}{f(u)} \right) e^{\int_0^t \gamma(\hat{F}_t, 1) du}$$

$$- \int_0^t P\{v(t) = 0\} \gamma(\hat{F}(t-x, 1)) e^{\int_0^{t-x} \gamma(\hat{F}(u, 1)) du} dx$$

$$+ \int_0^t \sum_{n=1}^m P\{v(t) = n\} \sum_{k=n}^m r_k \left( 1 - \hat{F}^{n-k}(t-x, 1) \right) e^{\int_0^{t-x} \gamma(\hat{F}(u, 1)) du} dx.$$

From ([24], p. 69), it is known that:

$$\int_0^1 \frac{du}{f(u)} = \infty,$$

then $V(\infty) = 1$. Thus, we obtain (6).

Considering [9], and $g(s) + r(s) = \gamma(s)$ we obtain (5), where $V(\cdot)$ is some continuous-differentiated function that satisfies (6). □

3.2. The Limit Theorem for PL of the Subcritical and Critical BPMCT

Let $\xi(t)$ be BP (without migration) with continuous time ([24], page 24), then we obtain the following result:

**Theorem 2.** Let $M\xi(t) \leq 0$, then:

$$\lim_{t \rightarrow \infty} u(t) = 0. \quad (7)$$

**Proof of Theorem 2.** Consider:

$$u(t) = P\{\tau > t\} = P\{v(t) > 0\} = 1 - N(t, 0)$$

$$= 1 - V \left( t + \int_0^t \frac{du}{f(u)} \right) e^{\int_0^t (g(\hat{F}(u, 0)) + r(\hat{F}(u, 0))) du}$$

$$+ \int_0^t \sum_{n=1}^m P\{v(t) = n\} \sum_{k=n}^m r_k \left( 1 - \hat{F}^{n-k}(t-x, 0) \right) e^{\int_0^{t-x} (g(\hat{F}(u, 0)) + r(\hat{F}(u, 0))) du} dx$$

$$- \int_0^t \sum_{n=1}^m P\{v(t) = n\} \sum_{k=n}^m r_k \left( 1 - \hat{F}^{n-k}(t-x, 0) \right) e^{\int_0^{t-x} (g(\hat{F}(u, 0)) + r(\hat{F}(u, 0))) du} dx$$

$$= 1 - V \left( t \right) e^{\int_0^t (g(\rho(t)) + r(\rho(t))) du}$$

$$+ \int_0^t \sum_{n=1}^m P\{v(t) = n\} \sum_{k=n}^m r_k \left( 1 - \rho(t-x) \right) e^{\int_0^{t-x} (g(\rho(u)) + r(\rho(u))) du} dx$$

$$- \int_0^t \sum_{n=1}^m P\{v(t) = n\} \sum_{k=n}^m r_k \left( 1 - \rho(t-x) \right) e^{\int_0^{t-x} (g(\rho(u)) + r(\rho(u))) du} dx.$$
where \( \rho(t) = P\{\xi(t) = 0\} \).

In the case of the subcritical and critical process \( \xi(t) \), the probability of degeneration is 1 and \( \lim_{t \to \infty} P\{\xi(t) = 0\} = q \). Hence, we obtain (7). □

4. Conclusions

This article investigates a more general model of the BPMCT than in [17–21]. The form of the differential equation and the PGF for the random process \( \nu(t) \), which describes the behavior of the process within the PL, was determined. The boundary theorem for the PL of the subcritical and critical BPMCT has been proven.

This model of the development process can be used to describe the popularization of countries or species in the external territory or to predict epidemics. One of our next works will be to describe the development of the COVID-19 pandemic through an extensive migration process.

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Abbreviations

The following abbreviations are used in this manuscript:

- PL: period-life
- BP: branching process
- BPMCT: branching process with migration and continuous time
- PGF: probability generating functions

References

1. Demetrius, L.; Schuster, P.; Sigmund, K. Polynucleotide evolution and branching processes. Bull. Math. Biol. 1985, 47, 239–262. [CrossRef]
2. Kimmel, M.; Axelrod, D. Branching Processes in Biology; Springer: New York, NY, USA, 2002.
3. Sawyer, S. Branching Diffusion Processes in Population Genetics. Adv. Appl. Probab. 1976, 8, 659–689. [CrossRef]
4. Caron-Lormier, G.; Masson, J.P.; Ménard, N.; Pierre, J.S. A branching process, its application in biology: Influence of demographic parameters on the social structure in mammal groups. J. Theor. Biol. 2005, 238, 564–574. [CrossRef]
5. Gleeson, J.P.; Onaga, T.; Fennell, P.; Cotter, J.; Burke, R.; O’Sullivan, D. Branching process descriptions of information cascades on Twitter. J. Complex Netw. 2021, 8, 1–29.
6. Vassiliou, P.-C.G. Asymptotic behaviour of Markov systems. J. Appl. Prob. 1982, 19, 851–857. [CrossRef]
7. Dimitriou, V.A.; Georgiou, A.C. Introduction and asymptotic behavior of a multi-level manpower planning model in a continuous time setting under potential department contraction. Commun. Stat. Theory Methods 2021, 50, 1173-1199. [CrossRef]
8. Zubkov, A.M. Life-Periods of a Branching Process with Immigration. Theory Probab. Appl. 1972, 17, 174–183. [CrossRef]
9. Vatutin, V.A. A conditional limit theorem for a critical Branching process with immigration. Math. Notes Acad. Sci. USSR 1977, 21, 405–411. [CrossRef]
10. Yanev, N.M.; Mitov, K.V. Lifetimes of Critical Branching Processes with Random Migration. Theory Probab. Appl. 1984, 28, 481–491. [CrossRef]
11. Boiko, R.V. Lifetime of branching process with immigration in limiting environment. Ukr. Math. J. 1983, 35, 242–247. [CrossRef]
12. Formanov, S.K.; Yasin, M.T. Limit theorems for life periods for critical Galton-Watson branching processes with migration. Izv. Akad. Nauk UzSSR Ser. Fiz-Mat. Nauk 1989, 1, 40–44.
13. Badalbaev, I.S.; Ganikhodjaev, A.N. Limit theorem for the branching Bellman-Harris process with immigration under the condition of non-zero. Izv. Akad. Nauk UzSSR Ser. Fiz-Mat. Nauk **1989**, *3*, 8–14.

14. Badalbaev, I.S.; Mashrabbbaev, A. Life spans of a Bellman-Harris branching process with immigration. *J. Sov. Math.* **1987**, *38*, 2198–2210. [CrossRef]

15. Li, D.; Vatutin, V.; Zhang, M. Subcritical branching processes in random environment with immigration stopped at zero. *J. Theor. Probab.* **2020**, *1*, 1–23. [CrossRef]

16. Dyakonova, E.; Li, D.; Vatutin, V.; Zhang, M. Branching processes in random environment with immigration stopped at zero. *J. Appl. Probab.* **2020**, *57*, 237–249. [CrossRef]

17. Chen, A.Y.; Renshaw, E. Markov branching processes regulated by emigration and large immigration. *Stoch. Process. Appl.* **1995**, *57*, 339–359. [CrossRef]

18. Rahimov, I.; Al-Sabah, W.S. Branching processes with decreasing immigration and tribal emigration. *Arab. J. Math. Sci.* **2000**, *6*, 81–97.

19. Srivastava, O.P.; Gupta, S.C. On a continuous-time branching process with migration. *Statistica* **1989**, *49*, 547–552.

20. Pakes, A.G. Some properties of a branching process with group immigration and emigration. *Adv. Appl. Prob.* **1986**, *18*, 628–645. [CrossRef]

21. Balabaev, I.S. Limit theorems for a critical Markov branching process with continuous time and migration. *Uzbek Math. J.* **1994**, *2*, 12–15.

22. Nagaev, S.V.; Khan, L.V. Limit theorems for Galton-Watson branching processes with migration. *Theory Probab. Appl.* **1980**, *25*, 523–534.

23. Yakymyshyn, K. Equation for generation function for branching processes with migration. *Visnyk Lviv Univ. Ser. Mech. Math.* **2017**, *84*, 119–125.

24. Sevastyanov, B.A. *Branching Processes*; Nauka: Moscow, Russia, 1971; p. 436.

25. Zaitsev, V.F.; Polyanin, A.D. *Handbook of First Order Partial Differential Equations*; Fizmatlit: Moscow, Russia, 2003; p. 416.