REGULARITY RESULTS FOR THE SOLUTIONS OF A NON-LOCAL MODEL OF TRAFFIC FLOW

Florent Berthelin*
Laboratoire J. A. Dieudonné, UMR 7351 CNRS
Université Côte d’Azur, LJAD, CNRS, Inria, France
and
Université de Nice Sophia-Antipolis, Parc Valrose
06108 Nice cedex 2, France

Paola Goatin
Inria Sophia Antipolis - Méditerranée
Université Côte d’Azur, Inria, CNRS, LJAD, 2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex, France

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Abstract. We consider a non-local traffic model involving a convolution product. Unlike other studies, the considered kernel is discontinuous on \( \mathbb{R} \). We prove Sobolev estimates and prove the convergence of approximate solutions solving a viscous and regularized non-local equation. It leads to weak, \( C([0,T], L^2(\mathbb{R})) \), and smooth, \( W^{2,2N}([0,T] \times \mathbb{R}) \), solutions for the non-local traffic model.

1. Introduction. We consider the non-local traffic model introduced in \([4, 8]\) to account for the reaction of drivers to downstream traffic conditions. It consists in the following scalar conservation law, where the traffic velocity depends on a weighted mean of the density:

\[
\partial_t \rho + \partial_x (\rho v(\rho \ast \omega)) = 0,
\]

where

\[
(\rho \ast \omega)(t,x) = \int_0^\eta \rho(t,x+\gamma) \omega(\gamma) \, d\gamma = \int_x^{x+\eta} \rho(t,y) \omega(y-x) \, dy.
\]

We make the following assumptions for \( k = 1, 2, 3 \):

(\( A^k_\omega \)) \( \omega \in C^k([0,\eta]) \) is non-negative with support in \([0,\eta]\) and is non-increasing on \([0,\eta]\).

(\( A^k_v \)) \( v \in C^k(\mathbb{R}^+) \) with \( v, v', \ldots, v^{(k)} \) bounded.

For traffic flow applications, it is reasonable to assume that \( v \) is non-increasing, even if monotonicity is not required in this paper. We also recall that a similar model, considering a weighted mean of downstream speeds, has been recently introduced in \([7]\). More generally, model (1) belongs to the class of conservation laws

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* Corresponding author: Florent Berthelin.

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with non-local flux functions, which appear in several applications, see for example [3, 6, 9, 10, 16]. We remark that most of the available well-posedness results concern equations involving smooth convolution kernels [1, 2], and are based on the construction of finite-volume approximations and the use of Kružkov’s doubling of variable technique [13]. In particular, these results rely on the concept of entropy solutions. Only recently, alternative proofs based on fixed point theorems have been proposed for specific cases [11, 15], allowing to get rid of the entropy requirement.

In general, solutions to non-local equations may be discontinuous [14], despite the expected regularizing effect of the convolution product. Therefore, given any initial datum \( \rho^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \), the solutions to the Cauchy problem for (1) are usually intended in the following weak form

**Definition 1.1.** A function \( \rho \in (L^\infty \cap L^1)(\mathbb{R}^+ \times \mathbb{R}) \) is a solution of (1) with initial datum \( \rho^0 \) if

\[
\int_0^{+\infty} \int_{-\infty}^{+\infty} (\rho \partial_t \varphi + \rho v * \partial_x \varphi)(t, x) \, dx \, dt + \int_{-\infty}^{+\infty} \rho^0(x) \varphi(0, x) \, dx = 0,
\]

for all \( \varphi \in C_c^\infty(\mathbb{R}^2) \).

In this paper, we are interested in deriving regularity properties of solutions to (1). To this end, we will consider approximate solutions satisfying the viscous and regularized non-local equation

\[
\partial_t \rho_\varepsilon + \partial_x (\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon)) = \varepsilon \partial_{xx}^2 \rho_\varepsilon,
\]

where, for any \( \varepsilon \in [0, 1] \), the smooth function \( \omega_\varepsilon \) is an extension of \( \omega \) with the following regularities:

\( (A^k_\omega) \) \( \omega_\varepsilon \in C^k(\mathbb{R}) \) is non-negative with a support in \([-\varepsilon, \eta + \varepsilon]\), is non-decreasing on \([-\varepsilon, x_\varepsilon]\), for some \( x_\varepsilon \in [-\varepsilon, 0] \), is non-increasing on \([x_\varepsilon, \eta + \varepsilon]\) and \( \omega_\varepsilon = \omega \) on \([0, \eta]\).

We set \( W_\varepsilon := \omega_\varepsilon(x_\varepsilon) \) and we assume that \( \lim_{\varepsilon \to 0} W_\varepsilon = \omega(0) \). Without loss of generality we can assume

\[
W_\varepsilon \leq 2\omega(0).
\]

\( (B^k_\omega) \) \( \omega^{(j)}_\varepsilon(\varepsilon) = \omega^{(j)}(\varepsilon + \varepsilon) = 0 \) for \( j = 1, \ldots, k \) and \( |\omega'_\varepsilon(u)| \leq 2W_\varepsilon/\varepsilon \) on \([-\varepsilon, 0]\) and \( |\omega'_\varepsilon(u)| \leq 2\omega(\eta)/\varepsilon \) on \([\eta, \eta + \varepsilon]\).

**Remark 1.** Given \( \omega \) satisfying \( (A^k_\omega) \), we can construct a function \( \omega_\varepsilon \) satisfying \( (A^k_\omega) \) and \( (B^k_\omega) \). To construct such extensions, for example in the simplest case where the derivatives of \( \omega \) vanish at 0 and \( \eta \), we use the function \( \varphi \) which is zero for \( x \leq -1, 1 \) for \( x \geq 0 \), non-decreasing and of class \( C^\infty \) and we use the function \( \omega_\varepsilon(x) = \omega(0)(x/\varepsilon) \) for \( x < 0 \) and \( \omega(0)(\eta - x)/\varepsilon \) for \( x > \eta \).

The considered smooth solutions \( \rho_\varepsilon \) are infinitely differentiable and the functions and its derivatives tend towards 0 when \( x \) goes to \( \pm \infty \). Indeed, in the proofs, we only need that it is true for \( \rho_\varepsilon, \partial_x \rho_\varepsilon, \partial_{xx} \rho_\varepsilon \) and \( \partial_{xxx} \rho_\varepsilon \).

Notice that a similar approximation was used in [5] to establish a convergence property for the singular limit where the (smooth) convolution kernel is replaced by a Dirac delta, in the viscous case. Here, we will study the properties of smooth solutions \( \rho_\varepsilon \) of this equation corresponding to a fixed initial datum \( \rho^0 \), and then we will recover properties for \( \rho \) passing to the limit as \( \varepsilon \to 0 \).

We have the following result.
Theorem 1.2. We assume \((A^2_\omega)-(A^3_\omega)\). Let \(\rho_\varepsilon\) be smooth solution of (4) with initial datum \(\rho^0\). We assume \(\rho^0 \in \mathbf{W}^{1,4}(\mathbb{R}) \cap \mathbf{H}^\varepsilon(\mathbb{R})\). Then, for \(T > 0\) sufficiently small, \(\rho_\varepsilon\) converges in \(\mathbf{L}^2_{\text{loc}}(0,T] \times \mathbb{R}\) to a solution \(\rho \in C([0,T], \mathbf{L}^2(\mathbb{R}))\) to equation (1) with initial datum \(\rho^0\). Furthermore, if \(\rho^0 \in \mathbf{W}^{1,2N}(\mathbb{R}), N \in \mathbb{N}^*, \) then \(\rho \in \mathbf{W}^{1,2N}(0,T] \times \mathbb{R}, \) and if \(\rho^0 \in \mathbf{W}^{1,4N}(\mathbb{R}) \cap \mathbf{H}^1(\mathbb{R}) \cap \mathbf{W}^{2,2N}(\mathbb{R}), \) then \(\rho \in \mathbf{W}^{2,2N}(0,T] \times \mathbb{R}\).

In particular, this provides an alternative proof of existence of weak solutions, locally in time.

To prove this result, in Section 2 we first establish estimates on the non-local term and we derive \(\mathbf{L}^p(\mathbb{R}), \ p > 1\), estimates for \(\rho_\varepsilon\), then we get estimates in \(\mathbf{W}^{1,2N}(\mathbb{R})\) for \(\rho_\varepsilon\) with respect to \(x\). This allows to prove that there exists \(T > 0\) such that the sequence \(\rho_\varepsilon\) is uniformly bounded with respect to \(\varepsilon\) in \(\mathbf{L}^\infty(\mathbb{R})\) on \([0,T]\). Then we prove uniform space estimates in \(\mathbf{W}^{2,2N}(\mathbb{R})\) for \(\rho_\varepsilon\), which allows to derive estimates on \(\partial_\varepsilon \rho_\varepsilon\). The proof of Theorem 1.2 is deferred to Section 3.

Notice that similar regularity of solutions is obtained in [11] in the one-dimensional setting and in [12] for non-local equations in multi space-dimension. However, in [11] regularity results are obtained assuming that the convolution kernel has no jumps, unlike the case we are considering, where the kernel \(\omega\) is discontinuous at \(x = 0\) and possibly at \(x = \eta\). Besides, in [12] the non-local area of integration does not depend on \(x\). Indeed, the above mentioned results are obtained relying on the characteristics method, and they thus need some regularity assumptions to hold along characteristics. In this paper, we are able to overcome these limitations by the kernel regularization \(\omega_\varepsilon\) and the viscosity approximation procedure given by (4).

2. Estimates. Here and in the following sections, we will denote by

\[\|\rho\|_\infty := \|\rho\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R})}\]

and by

\[\|\rho(t,\cdot)\|_\infty := \|\rho(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})}.\]

Moreover, notice that we have

\[(\rho * \omega_\varepsilon)(t,x) = \int_{\mathbb{R}} \rho(t,x + y)\omega_\varepsilon(y)\, dy = \int_{\mathbb{R}} \rho(t,y)\omega_\varepsilon(y - x)\, dy\]

\[= \int_{-\varepsilon}^{\eta + \varepsilon} \rho(t,x + y)\omega_\varepsilon(y)\, dy = \int_{x - \varepsilon}^{x + \eta + \varepsilon} \rho(t,y)\omega_\varepsilon(y - x)\, dy.\]

2.1. Estimates of the non-local term. We start by proving the following estimates on the non-local term.

Proposition 1. 1. We assume \((A^1_\omega)\) and that \(\rho\) is a function such that \(x \mapsto \rho(t,x) \in \mathbf{L}^\infty(\mathbb{R})\) for any \(t \geq 0\). Then, for any \((t,x) \in \mathbb{R}^+ \times \mathbb{R}\), we have

\[|\partial_\varepsilon (\rho * \omega_\varepsilon)(t,x)| \leq 2\|\rho(t,\cdot)\|_\infty W_\varepsilon.\] (6)

2. Let \(p > 1\). We assume \((A^2_\omega) - (B^1_\omega)\) and that \(\rho\) is a \(C^1\) function with \(x \mapsto \partial_\varepsilon \rho(t,x) \in \mathbf{L}^p(\mathbb{R})\) for any \(t \geq 0\). Then, for any \(t \geq 0\), we have

\[\left(\int_{\mathbb{R}} |\partial_\varepsilon^2 (\rho * \omega_\varepsilon)|^p(t,x)\, dx\right)^\frac{1}{p} \leq \eta^{\frac{1}{p}} \left(\int_0^\eta |\omega''(u)|^{p/(p-1)}\, du\right)^{1-\frac{1}{p}} \left(\int_{\mathbb{R}} \rho^p(t,y)\, dy\right)^\frac{1}{p} \]

\[+ (|\omega'(\eta^-)| + |\omega'(0^+)|) \left(\int_{\mathbb{R}} \rho^p(t,x)\, dx\right)^\frac{1}{p}.\] (7)
$+ 2(\omega(t) + W_\varepsilon) \left( \int_\mathbb{R} |\partial_x \rho(t,x)|^p \, dx \right)^{\frac{1}{p}}.$

3. Let $p > 1$. We assume $\textbf{(A}^2\varepsilon\textbf{)} - \textbf{(B}^2\varepsilon\textbf{)}$ that $\rho$ is a $C^2$ function with $x \mapsto \partial_x \rho(t,x), x \mapsto \partial^2_{xx} \rho(t,x) \in L^p(\mathbb{R})$ for any $t \geq 0$. Then, for any $t \geq 0$, we have

\[
\left( \int_\mathbb{R} |\partial^3_{xxx}(\rho \ast \omega_\varepsilon)|^p(t,x) \, dx \right)^{\frac{1}{p}}
\leq \eta^{\frac{2}{p}} \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} \, du \right)^{1 - \frac{2}{p}} \left( \int_\mathbb{R} |\partial_x \rho(t,y)|^p \, dy \right)^{\frac{1}{p}}
\leq \eta^{\frac{2}{p}} \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} \, du \right)^{1 - \frac{2}{p}} \left( \int_\mathbb{R} |\partial_x \rho(t,y)|^p \, dy \right)^{\frac{1}{p}}
\leq |\omega'(\eta) - |\omega'(0^+)\rangle \left( \int_\mathbb{R} |\partial_x \rho(t,x)|^p \, dx \right)^{\frac{1}{p}}
\leq 2(\omega(t) + W_\varepsilon) \left( \int_\mathbb{R} |\partial^2_{xx} \rho(t,x)|^p \, dx \right)^{\frac{1}{p}}.
\]  

Proof. 1. From

\[
\partial_x (\rho \ast \omega_\varepsilon)(t,x)
= - \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t,y)\omega_\varepsilon'(y-x) \, dy + \rho(t,x+\eta+\varepsilon)\omega_\varepsilon(\eta+\varepsilon) - \rho(t,x-\varepsilon)\omega_\varepsilon(-\varepsilon)
= - \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t,y)\omega_\varepsilon'(y-x) \, dy = - \int_{-\varepsilon}^{\eta+\varepsilon} \rho(t,u+x)\omega_\varepsilon'(u) \, du,
\]

we obtain

\[
|\partial_x (\rho \ast \omega_\varepsilon)(t,x)| \leq \|\rho(t,\cdot)\|_\infty \int_{-\varepsilon}^\eta |\omega_\varepsilon'(u)| \, du
\leq \|\rho(t,\cdot)\|_\infty \left( \int_{-\varepsilon}^{\eta+\varepsilon} \omega_\varepsilon'(u) \, du - \int_{-\varepsilon}^{\eta+\varepsilon} \omega_\varepsilon'(u) \, du \right)
\leq 2\|\rho(t,\cdot)\|_\infty W_\varepsilon.
\]

2. From

\[
\partial^2_{xx} (\rho \ast \omega_\varepsilon)(t,x)
= \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t,y)\omega_\varepsilon''(y-x) \, dy - \rho(t,x+\eta+\varepsilon)\omega_\varepsilon'(\eta+\varepsilon) + \rho(t,x-\varepsilon)\omega_\varepsilon'(-\varepsilon)
= \int_{x-\varepsilon}^{x+\eta+\varepsilon} \rho(t,y)\omega_\varepsilon''(y-x) \, dy = \int_{-\varepsilon}^{\eta+\varepsilon} \rho(t,x+u)\omega_\varepsilon''(u) \, du
= \int_{-\varepsilon}^{0} \rho(t,x+u)\omega_\varepsilon'(u) \, du + \int_{0}^{\eta} \rho(t,x+u)\omega_\varepsilon'(u) \, du + \int_{\eta}^{\eta+\varepsilon} \rho(t,x+u)\omega_\varepsilon'(u) \, du
= \rho(t,x)\omega_\varepsilon'(0) - \rho(t,x-\varepsilon)\omega_\varepsilon'(-\varepsilon) - \int_{-\varepsilon}^{0} \partial_x \rho(t,x+u)\omega_\varepsilon'(u) \, du
+ \int_{0}^{\eta} \rho(t,x+u)\omega_\varepsilon''(u) \, du
+ \rho(t,x+\eta+\varepsilon)\omega_\varepsilon'(\eta+\varepsilon) - \rho(t,x+\eta)\omega_\varepsilon'(\eta) - \int_{\eta}^{\eta+\varepsilon} \partial_x \rho(t,x+u)\omega_\varepsilon'(u) \, du.
\]
\[\begin{align*}
= \rho(t, x) \omega_{\varepsilon}'(0+) - \rho(t, x + \eta) \omega_{\varepsilon}'(\eta-) + \int_0^\eta \rho(t, x + u) \omega_{\varepsilon}'(u) \, du \\
- \int_{-\varepsilon}^0 \partial_x \rho(t, x + u) \omega_{\varepsilon}'(u) \, du - \int_0^{\eta + \varepsilon} \partial_x \rho(t, x + u) \omega_{\varepsilon}'(u) \, du,
\end{align*}\]
we have
\[\left( \int_R \left| \partial_{xx}^2 (\rho \ast \omega_{\varepsilon}) \right|^p (t, x) \, dx \right)^{\frac{1}{p}} \]
\[\leq \left( \int_R \left| \partial_x \rho(t, x + u) \omega_{\varepsilon}'(u) \right|^p \, dx \right)^{\frac{1}{p}} \]
\[+ \left( \int_R \left| \int_\eta^{\eta + \varepsilon} \partial_x \rho(t, x + u) \omega_{\varepsilon}'(u) \, du \right|^p \, dx \right)^{\frac{1}{p}} \]
\[+ \left( \int_R \left| \int_0^{\eta} \rho(t, x + u) \omega_{\varepsilon}''(u) \, du \right|^p \, dx \right)^{\frac{1}{p}} \]
\[+ \left( \int_R \rho(t, x)^p \omega'(0+) \, dx \right)^{\frac{1}{p}} + \left( \int_R \rho(t, x + \eta)^p \omega'(\eta-) \, dx \right)^{\frac{1}{p}}.\]
Notice that
\[\left( \int_R \left| \partial_x \rho(t, x + u) \omega_{\varepsilon}'(u) \right|^p \, dx \right)^{\frac{1}{p}} \]
\[\leq \left( \int_R \left( \int_{-\varepsilon}^0 |\partial_x \rho(t, x + u)|^p \, du \right) \left( \int_{-\varepsilon}^0 \omega_{\varepsilon}'(u) \, dy \right)^{p/q} \, dx \right)^{\frac{1}{p}} \]
\[\leq \left( \int_{-\varepsilon}^0 \omega_{\varepsilon}'(u)^{p/(p-1)} \, du \right)^{1 - \frac{1}{p}} \left( \int_R \left( \int_{-\varepsilon}^0 |\partial_x \rho(t, x + u)|^p \, du \right) \, dx \right)^{\frac{1}{p}} \]
\[\leq \left( \int_{-\varepsilon}^0 \left( \frac{W_{\varepsilon}}{\varepsilon} \right)^{p/(p-1)} \, du \right)^{1 - \frac{1}{p}} \left( \int_R \left( \int_{-\varepsilon}^\eta |\partial_x \rho(t, y)|^p \, dy \right) \, dx \right)^{\frac{1}{p}} \]
\[\leq 2 \frac{W_{\varepsilon}}{\varepsilon} \varepsilon^{1 - \frac{1}{p}} \left( \int_R \left( \int_y^{y+\varepsilon} \, dx \right) |\partial_x \rho(t, y)|^p \, dy \right)^{\frac{1}{p}} \]
\[\leq 2 \frac{W_{\varepsilon}}{\varepsilon} \varepsilon^{1 - \frac{1}{p}} \left( \int_R |\partial_x \rho(t, y)|^p \, dy \right)^{\frac{1}{p}} = 2W_{\varepsilon} \left( \int_R |\partial_x \rho(t, y)|^p \, dy \right)^{\frac{1}{p}}.\]
using Hölder’s inequality with \( q = p/(p - 1) \) the conjugated exponent of \( p \). Similarly
\[\left( \int_R \left| \int_{\eta}^{\eta + \varepsilon} \partial_x \rho(t, x + u) \omega_{\varepsilon}'(u) \, du \right|^p \, dx \right)^{\frac{1}{p}} \leq 2\omega(\eta) \left( \int_R |\partial_x \rho(t, y)|^p \, dy \right)^{\frac{1}{p}}.\]
Then we get
\[\left( \int_R \left| \partial_{xx}^2 (\rho \ast \omega_{\varepsilon}) \right|^p (t, x) \, dx \right)^{\frac{1}{p}}.\]
Proposition 2. We assume estimates solving the viscous and regularized non-local equation. First, we deal with $L^2$.

Proof. The equation (4) can be rewritten as

$$
\partial_t \rho_\varepsilon + v(\rho_\varepsilon \ast \omega_\varepsilon) \partial_x \rho_\varepsilon + \rho_\varepsilon' \rho_\varepsilon' \partial_x (\rho_\varepsilon \ast \omega_\varepsilon) = \varepsilon \partial_{xx}^2 \rho_\varepsilon.
$$

Furthermore, using Fubini’s Theorem we have

$$
\int_{\mathbb{R}} \left( \int_{x}^{x+\eta} \rho^p(t, y) \, dy \right) \, dx = \int_{\mathbb{R}} \int_{y-\eta}^{y} \rho^p(t, y) \, dx \, dy = \eta \int_{\mathbb{R}} \rho^p(t, y) \, dy,
$$

thus we obtain

$$
\eta \left( \int_{0}^{\varepsilon} |\omega''(u)|^{p/(p-1)} \, du \right)^{p-1} \int_{\mathbb{R}} \rho^p(t, y) \, dy,
$$

then we get the announced formula.

3. Remark that, since $\omega_\varepsilon''(\varepsilon) = \omega_\varepsilon''(\eta + \varepsilon) = 0$, we have

$$
\partial_{xxx}^3 (\rho \ast \omega_\varepsilon)(t, x) = - \int_{-\varepsilon}^{\eta+\varepsilon} \rho(t, x + u) \omega_\varepsilon^{(3)}(u) \, du = \int_{-\varepsilon}^{\eta+\varepsilon} \partial_x \rho(t, x + u) \omega_\varepsilon''(u) \, du
$$

then applying 2., we get

$$
\left( \int_{\mathbb{R}} |\partial_{x}^3 (\rho \ast \omega)|^p(t, x) \, dx \right)^{1/p} \leq \eta^{1/p} \left( \int_{0}^{\varepsilon} |\omega''(u)|^{p/(p-1)} \, du \right)^{1-1/p} \left( \int_{\mathbb{R}} |\partial_x \rho(t, y)|^p \, dy \right)^{1/p}
$$

$$
+ (|\omega'(-\varepsilon)| + |\omega' (0+)|) \left( \int_{\mathbb{R}} |\partial_x \rho(t, x)|^p \, dx \right)^{1/p}
$$

$$
+ 2 (\omega(\eta) + W_\varepsilon) \left( \int_{\mathbb{R}} |\partial_{xx}^2 \rho(t, x)|^p \, dx \right)^{1/p}.
$$

\[\square\]

2.2. $L^p$ estimates for the viscous case. We turn now to estimates on solutions solving the viscous and regularized non-local equation. First, we deal with $L^p$ estimates.

Proposition 2. We assume (A$^1_{\rho}$)–(A$^1_{\omega}$). Let $\rho_\varepsilon$ be smooth solution of (4) with initial datum $\rho^0 \in L^p(\mathbb{R})$. If $\rho_\varepsilon \in L^\infty([0, T] \times \mathbb{R})$ for some $T > 0$, then

$$
\rho_\varepsilon \in L^\infty([0, T], L^p(\mathbb{R})) \cap L^p([0, T] \times \mathbb{R}).
$$

Proof. The equation (4) can be rewritten as

$$
\partial_t \rho_\varepsilon + v(\rho_\varepsilon \ast \omega_\varepsilon) \partial_x \rho_\varepsilon + \rho_\varepsilon' \rho_\varepsilon' \partial_x (\rho_\varepsilon \ast \omega_\varepsilon) = \varepsilon \partial_{xx}^2 \rho_\varepsilon.
$$

(9)
Multiplying (9) by $\rho_\varepsilon^{p-1}$, then integrating with respect to $x$, we obtain
\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx = - \int_{\mathbb{R}} \rho_\varepsilon^{p-1}(t,x)v((\rho_\varepsilon * \omega_\varepsilon)(t,x))\partial_x \rho_\varepsilon(t,x) \, dx \\
= \int_{\mathbb{R}} \rho_\varepsilon^p(t,x)v'((\rho_\varepsilon * \omega_\varepsilon)(t,x))\partial_x (\rho_\varepsilon * \omega_\varepsilon)(t,x) \, dx \\
+ \varepsilon \int_{\mathbb{R}} \rho_\varepsilon^{p-1}(t,x)\partial^2_{xx} \rho_\varepsilon(t,x) \, dx.
\]

We observe that
\[
\int_{\mathbb{R}} \rho_\varepsilon^{p-1} v(\rho_\varepsilon * \omega_\varepsilon) \partial_x \rho_\varepsilon \, dx = \int_{\mathbb{R}} \partial_x \left( \frac{\rho_\varepsilon^p}{p} \right) v(\rho_\varepsilon * \omega_\varepsilon) \, dx \\
= - \int_{\mathbb{R}} \frac{\rho_\varepsilon^p}{p} \partial_x (v(\rho_\varepsilon * \omega_\varepsilon)) \, dx \\
= - \int_{\mathbb{R}} \frac{\rho_\varepsilon^p}{p} v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x (\rho_\varepsilon * \omega_\varepsilon) \, dx,
\]

using the fact that the function $\rho_\varepsilon$ tends to 0 at $\pm \infty$ and that $v$ is bounded, and
\[
\int_{\mathbb{R}} \rho_\varepsilon^{p-1}\partial^2_{xx} \rho_\varepsilon \, dx = -(p-1) \int_{\mathbb{R}} \rho_\varepsilon^{p-2} \partial_x^2 \rho_\varepsilon \, dx \leq 0,
\]
since $\rho_\varepsilon$ and $\partial_x \rho_\varepsilon$ tend to 0 at $\pm \infty$. Therefore
\[
\frac{d}{dt} \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx \leq (1-p) \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) v'((\rho_\varepsilon * \omega_\varepsilon)(t,x))\partial_x (\rho_\varepsilon * \omega_\varepsilon)(t,x) \, dx. \quad (10)
\]

We use (6) to control the right hand side of (10) and we get
\[
\frac{d}{dt} \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx \leq C_{1}^{\varepsilon,p} \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx, \quad (11)
\]
which implies
\[
\int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx \leq e^{C_{1}^{\varepsilon,p}t} \int_{\mathbb{R}} \rho_\varepsilon^p(0,x) \, dx, \quad (12)
\]
with
\[
C_{1}^{\varepsilon,p} = 2(p-1)\|\rho_\varepsilon\|_\infty W_\varepsilon \|v'\|_\infty.
\]

It gives
\[
\sup_{t \in [0,T]} \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx \leq e^{C_{1}^{\varepsilon,p}T} \int_{\mathbb{R}} \rho_\varepsilon^p(0,x) \, dx. \quad (13)
\]

By integration of (12) with respect to $t \in [0,T]$, we get
\[
\int_{0}^{T} \int_{\mathbb{R}} \rho_\varepsilon^p(t,x) \, dx \, dt \leq \frac{1}{C_{1}^{\varepsilon,p}} \left( e^{C_{1}^{\varepsilon,p}T} - 1 \right) \int_{\mathbb{R}} \rho_\varepsilon^p(0,x) \, dx. \quad (14)
\]

\[\square\]

2.3. $W^{1,p}$ estimates for $p = 2N$ in the viscous case. We turn now to Sobolev estimates. Let $N \in \mathbb{N}^*$ and set $p = 2N$.

**Proposition 3.** We assume $(A_2^\varepsilon)$-$(A_2^\varepsilon)$. Let $\rho_\varepsilon$ be smooth solution of (4) with initial datum $\rho^0_\varepsilon \in W^{1,2N}(\mathbb{R})$. If $\rho_\varepsilon \in L^\infty([0,T] \times \mathbb{R})$ for some $T > 0$, then

$\rho_\varepsilon \in L^\infty([0,T], W^{1,2N}(\mathbb{R}))$ and $\rho_\varepsilon, \partial_x \rho_\varepsilon \in L^\infty([0,T], L^{2N}(\mathbb{R})) \cap L^{2N}([0,T] \times \mathbb{R})$. 

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Proof. We differentiate (9) with respect to $x$, it gives
\[
\begin{align*}
\partial_t \partial_x \rho_x + 2v'(\rho_x * \omega_x) \partial_x (\rho_x * \omega_x) \partial_x \rho_x + v(\rho_x * \omega_x) \partial^2_{xx} \rho_x \\
+ \rho_x v''(\rho_x * \omega_x) (\partial_x (\rho_x * \omega_x))^2 + \rho_x v'(\rho_x * \omega_x) \partial^2_{xx} (\rho_x * \omega_x) = \varepsilon \partial^3_{xxx} \rho_x. \quad (15)
\end{align*}
\]
Multiplying this relation by $(\partial_x \rho_x)^{p-1}$, then integrating with respect to $x$, we have
\[
\begin{align*}
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x \rho_x)^p \, dx &+ 2 \int_{\mathbb{R}} v'(\rho_x * \omega_x) \partial_x (\rho_x * \omega_x) (\partial_x \rho_x)^p \, dx \\
&+ \int_{\mathbb{R}} \rho_x v''(\rho_x * \omega_x) (\partial_x (\rho_x * \omega_x))^2 (\partial_x \rho_x)^{p-1} \, dx \\
&+ \int_{\mathbb{R}} \rho_x v'(\rho_x * \omega_x) \partial^2_{xx} (\rho_x * \omega_x) (\partial_x \rho_x)^{p-1} \, dx \\
&= \varepsilon \int_{\mathbb{R}} (\partial_x \rho_x)^{p-1} \partial^3_{xxx} \rho_x \, dx.
\end{align*}
\]
Notice that
\[
\int_{\mathbb{R}} v(\rho_x * \omega_x) \partial^2_{xx} \rho_x (\partial_x \rho_x)^{p-1} \, dx = \frac{1}{p} \int_{\mathbb{R}} v(\rho_x * \omega_x) \partial_x (\partial_x \rho_x)^p \, dx \\
= -\frac{1}{p} \int_{\mathbb{R}} v'(\rho_x * \omega_x) \partial_x (\rho_x * \omega_x) (\partial_x \rho_x)^p \, dx,
\]
using the fact that the function $\partial_x \rho_x$ tends to $0$ at $\pm \infty$ and that $v$ is bounded, and, since $p$ is even,
\[
\int_{\mathbb{R}} (\partial_x \rho_x)^{p-1} \partial^3_{xxx} \rho_x \, dx = -(p-1) \int_{\mathbb{R}} (\partial^2_{xx} \rho_x)^2 (\partial_x \rho_x)^{p-2} \, dx \leq 0,
\]
since $\partial_x \rho_x$ and $\partial^2_{xx} \rho_x$ tend to $0$ at $\pm \infty$. Thus
\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}} (\partial_x \rho_x)^p \, dx &\leq (1 - 2p) \int_{\mathbb{R}} v'(\rho_x * \omega_x) \partial_x (\rho_x * \omega_x) (\partial_x \rho_x)^p \, dx \\
&- p \int_{\mathbb{R}} \rho_x v''(\rho_x * \omega_x) (\partial_x (\rho_x * \omega_x))^2 (\partial_x \rho_x)^{p-1} \, dx \\
&- p \int_{\mathbb{R}} \rho_x v'(\rho_x * \omega_x) \partial^2_{xx} (\rho_x * \omega_x) (\partial_x \rho_x)^{p-1} \, dx \\
=: & I^1_1 + I^2_1 + I^3_1.
\end{align*}
\]
We estimate now each of these terms.
- By (6) we get
\[
|I^1_1| = \left| (1 - 2p) \int_{\mathbb{R}} v'(\rho_x * \omega_x) \partial_x (\rho_x * \omega_x) (\partial_x \rho_x)^p \, dx \right| \\
\leq 2(2p-1) \|v'\|_{\infty} \|\rho_x\|_{\infty} W_\varepsilon \int_{\mathbb{R}} |\partial_x \rho_x|^p \, dx.
\]
- Again by (6) we get
\[
|I^2_1| = \left| p \int_{\mathbb{R}} \rho_x v''(\rho_x * \omega_x)(\partial_x (\rho_x * \omega_x))^2 (\partial_x \rho_x)^{p-1} \, dx \right| \\
\leq p \|v''\|_{\infty} (2 \|\rho_x\|_{\infty} W_\varepsilon)^2 \int_{\mathbb{R}} \rho_x |\partial_x \rho_x|^{p-1} \, dx.
\]
\[ \leq 4\|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon^2 \left( \int_R \rho_\varepsilon^p \, dx + (p - 1) \int_R |\partial_x \rho_\varepsilon|^p \, dx \right), \]

where we have used Young’s inequality \( uv \leq \frac{1}{p} u^p + \frac{1}{q} v^q \) with \( q = p/(p - 1) \).

- Similarly,
  \[ |I_3^p| = \left| p \int_R \rho_\varepsilon v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx}^2 (\rho_\varepsilon \ast \omega_\varepsilon) (\partial_x \rho_\varepsilon)^{p-1} \, dx \right| \leq p\|v''\|_\infty \|\rho_\varepsilon\|_\infty \int_R |\partial_{xx}^2 (\rho_\varepsilon \ast \omega_\varepsilon) |\partial_x \rho_\varepsilon|^{p-1}| \, dx \]
  \[ \leq \|v''\|_\infty \|\rho_\varepsilon\|_\infty \left( \int_R |\partial_{xx}^2 (\rho_\varepsilon \ast \omega_\varepsilon)|^p \, dx + (p - 1) \int_R |\partial_x \rho_\varepsilon|^p \, dx \right). \]

We now observe that, by convexity of the function \( x \mapsto x^p \)
\((u + v + w)^p \leq 3^{p-1} (u^p + v^p + w^p) \) for any \( u, v, w > 0 \) and \( p > 0 \).

Estimate (7) of Proposition 1 and inequality (16) give
\[
\int_R |\partial_{xx}^2 (\rho_\varepsilon \ast \omega)|^p \, dx \leq 3^{p-1} \eta \left( \int_0^\eta |\omega''(u)|^{\eta/(p-1)} \, du \right)^{p-1} \int_R \rho_\varepsilon^p(t, x) \, dx \\
+ 3^{p-1} (|\omega'(\eta^-)| + |\omega'(0+)|)^p \int_R \rho_\varepsilon^p(t, x) \, dx \\
+ 2p^{3p-1} (\omega(\eta) + \rho_\varepsilon^p) \int_R |\partial_x \rho_\varepsilon(t, x)|^p \, dx, \tag{17}
\]

thus
\[
\left| p \int_R \rho_\varepsilon v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx}^2 (\rho_\varepsilon \ast \omega_\varepsilon) (\partial_x \rho_\varepsilon)^{p-1} \, dx \right| \leq C_2^{p,p} \int_R \rho_\varepsilon^p \, dx + C_3^{p,p} \int_R |\partial_x \rho_\varepsilon|^p \, dx,
\]

with
\[
C_2^{p,p} = \|v''\|_\infty \|\rho_\varepsilon\|_\infty 3^{p-1} \left[ \eta \left( \int_0^\eta |\omega''(u)|^{\eta/(p-1)} \, du \right)^{p-1} + (|\omega'(\eta^-)| + |\omega'(0+)|)^p \right]
\]
and
\[
C_3^{p} = \|v''\|_\infty \|\rho_\varepsilon\|_\infty \left[ p - 1 + 2p^{3p-1} (\omega(\eta) + \rho_\varepsilon^p) \right].
\]

These bounds give finally the estimate
\[
\frac{d}{dt} \int_R |\partial_x \rho_\varepsilon(t, x)|^p \, dx \leq C_4^{p,p} \int_R |\partial_x \rho_\varepsilon(t, x)|^p \, dx + C_5^{p,p} \int_R \rho_\varepsilon^p(t, x) \, dx.
\]

with
\[
C_4^{p} = 2(2p - 1)\|v''\|_\infty \|\rho_\varepsilon\|_\infty W_\varepsilon + 4(p - 1)|v''|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon^2 + C_3^{p,p}
\]
and
\[
C_5^{p} = 4\|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon^2 + C_2^{p,p}.
\]

With (11), we get
\[
\frac{d}{dt} \left( \int_R |\partial_x \rho_\varepsilon(t, x)|^{2N} \, dx + \int_R \rho_\varepsilon^{2N}(t, x) \, dx \right) \leq C_6^{p,p} \left( \int_R |\partial_x \rho_\varepsilon(t, x)|^{2N} \, dx + \int_R \rho_\varepsilon^{2N}(t, x) \, dx \right), \tag{18}
\]
with $C_6^{r,p} = \max(C_4^{r,p}, C_5^{r,p} + C_1^{r,p})$, which implies
\[
\left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^2 dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t,x) dx \right) \leq e^{C_6^{r,p}T} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(0,x)|^2 dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(0,x) dx \right).
\]
Then
\[
\sup_{t \in [0,T]} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^2 dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t,x) dx \right) \leq e^{C_6^{r,p}T} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(0,x)|^2 dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(0,x) dx \right).
\]
Integrating (19) with respect to $t$ on $[0,T]$, we get
\[
\int_{0}^{T} \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^2 dx dt + \int_{0}^{T} \int_{\mathbb{R}} \rho_\varepsilon^{2N}(t,x) dx dt \leq \frac{1}{C_6^{r,p}} (e^{C_6^{r,p}T} - 1) \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(0,x)|^2 dx + \int_{\mathbb{R}} \rho_\varepsilon^{2N}(0,x) dx \right).
\]

2.4. $L^\infty$ bound on an interval $[0,T]$. With the previous estimates, we are now able to prove an $L^\infty$ bound for the sequence $\{\rho_\varepsilon\}_\varepsilon$ on an interval $[0,T]$.

**Proposition 4.** We assume $(A_7^2)$-$\mathbf{(A_2^2)}$. Let $\rho_\varepsilon$ be smooth solutions of (4) with initial datum $\rho^0 \in H^1$. Then there exists a constant $T > 0$ such that $\rho_\varepsilon \in L^\infty([0,T] \times \mathbb{R})$ for any $\varepsilon > 0$, $T < T$. Furthermore

$\rho_\varepsilon \in L^\infty([0,T], W^{1,2N}(\mathbb{R}))$ and $\rho_\varepsilon, \partial_x \rho_\varepsilon \in L^\infty([0,T], L^\infty(\mathbb{R})) \cap L^{2N}([0,T] \times \mathbb{R})$

and this sequence is uniformly bounded in these spaces with respect to $\varepsilon$.

**Proof.** Let $\rho_\varepsilon$ be a smooth solution of (4) with the same initial datum $\rho^0 \in H^1$. The relation (18) for $N = 1$ gives
\[
\frac{d}{dt} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^2 dx + \int_{\mathbb{R}} \rho_\varepsilon^2(t,x) dx \right) \leq C \max\{1, \|\rho_\varepsilon(t,\cdot)\|^2_{L^\infty}\} \left( \int_{\mathbb{R}} |\partial_x \rho_\varepsilon(t,x)|^2 dx + \int_{\mathbb{R}} \rho_\varepsilon^2(t,x) dx \right),
\]
for some constant $C$ that does not depend on $\varepsilon$ (since $W_\varepsilon$ is uniformly bounded). If no uniform $L^\infty$-bound on $\rho_\varepsilon$ is available, we can use the Sobolev injection of $H^1(\mathbb{R})$ in $L^\infty(\mathbb{R})$ and get
\[
\frac{d}{dt} \|\rho_\varepsilon(t,\cdot)\|^2_{H^1} \leq C\|\rho_\varepsilon(t,\cdot)\|^2_{H^1} + C\|\rho_\varepsilon(t,\cdot)\|^4_{H^1},
\]
possibly updating the constant $C$. We set $u_\varepsilon(t) = \|\rho_\varepsilon(t,\cdot)\|_{H^1}$, then $u_\varepsilon' \leq C(u_\varepsilon + u_\varepsilon^2)$, which leads to
\[
\frac{u_\varepsilon'}{u_\varepsilon} - \frac{u_\varepsilon'}{1 + u_\varepsilon} \leq C.
\]
We obtain
\[
u(t) \leq \frac{C_0 e^{Ct}}{1 - C_0 e^{Ct}}, \text{ for any } 0 \leq t < -\frac{\ln C_0}{C},
\]
with \( C_0 = \frac{u_0}{1 + u_0} < 1, u_0 = \| \rho^0 \|_{H^1} \). Notice that the initial datum is the same for all the sequence and then \( u_0 \) and \( C_0 \) do not depend on \( \varepsilon \). Setting \( T < \bar{T} := -\frac{\ln C_0}{C} \), we have
\[
\| \rho_\varepsilon(t, \cdot) \|_{H^1}^2 \leq C \| \rho^0 \|_{H^1}^2, \quad \text{for any } \quad 0 \leq t \leq T, \quad \varepsilon > 0.
\]
Therefore, by Sobolev injection, \( \rho_\varepsilon \in L^\infty([0, T] \times \mathbb{R}) \). Using the estimates of Propositions 2 and 3, we get (22) with bounds independents of \( \varepsilon \).

2.5. \( W^{2,p} \) estimate for \( p = 2N \). To pass to the limit, we need also estimates in \( W^{2,p} \), which will provide, in the next section, with the help of the equation, the necessary regularity in time. As in Section 2.3, let \( N \in \mathbb{N}^* \) and set \( p = 2N \).

**Proposition 5.** We assume (A\(_2\))- (A\(_3\)). Let \( \rho_\varepsilon \) be smooth solutions of (4) with initial datum \( \rho^0 \in W^{1,4N}(\mathbb{R}) \cap H^3(\mathbb{R}) \cap W^{2,2N}(\mathbb{R}) \). Let \( T > 0 \) as in Proposition 4. Then
\[
\rho_\varepsilon \in L^\infty ([0, T], W^{2,2N}(\mathbb{R})),
\]
\[
\rho_\varepsilon, \partial_x \rho_\varepsilon, \partial_{xx} \rho_\varepsilon \in L^\infty ([0, T], L^{2N}(\mathbb{R})) \cap L^{2N}([0, T] \times \mathbb{R}),
\]
and this sequence is bounded in these spaces with respect to \( \varepsilon \).

**Proof.** We differentiate (15) with respect to \( x \), which gives
\[
\partial_t \partial_{xx}^2 \rho_\varepsilon + 3v''(\rho_\varepsilon \ast \omega_\varepsilon)(\partial_x(\rho_\varepsilon \ast \omega_\varepsilon))^2 \partial_x \rho_\varepsilon \\
+ 3v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx}^2 \rho_\varepsilon \partial_x \rho_\varepsilon \\
+ 3v(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx} \rho_\varepsilon + v(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xxx} \rho_\varepsilon \partial_x \rho_\varepsilon \partial_x(\partial_{xx} \rho_\varepsilon)^3 \\
+ 3 \rho_\varepsilon v''(\rho_\varepsilon \ast \omega_\varepsilon) \partial_x(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx} \rho_\varepsilon + \rho_\varepsilon v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xxx} \rho_\varepsilon + \epsilon \partial_{xxx}^3 \rho_\varepsilon.
\]

Multiplying this relation by \( (\partial_{xx} \rho_\varepsilon)^{p-1} \), then integrating with respect to \( x \), we obtain
\[
\frac{1}{p} \frac{d}{dt} \int_\mathbb{R} (\partial_{xx} \rho_\varepsilon)^p dx + 3 \int_\mathbb{R} v''(\rho_\varepsilon \ast \omega_\varepsilon)(\partial_x(\rho_\varepsilon \ast \omega_\varepsilon))^2 \partial_x \rho_\varepsilon (\partial_{xx} \rho_\varepsilon)^p dx \\
+ 3 \int_\mathbb{R} v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx}^2 \rho_\varepsilon \partial_x \rho_\varepsilon (\partial_{xx} \rho_\varepsilon)^p dx \\
+ 3 \int_\mathbb{R} v(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx} \rho_\varepsilon \partial_x(\partial_{xx} \rho_\varepsilon)^p dx + \int_\mathbb{R} v(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xxx} \rho_\varepsilon \partial_x(\partial_{xx} \rho_\varepsilon)^p dx \\
+ \int_\mathbb{R} \rho_\varepsilon v''(\rho_\varepsilon \ast \omega_\varepsilon) \partial_x(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx} \rho_\varepsilon (\partial_{xx} \rho_\varepsilon)^p dx \\
+ 3 \int_\mathbb{R} \rho_\varepsilon v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_x(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx} \rho_\varepsilon (\partial_{xx} \rho_\varepsilon)^p dx \\
+ \epsilon \int_\mathbb{R} \partial_{xxx} \rho_\varepsilon (\partial_{xx} \rho_\varepsilon)^p dx.
\]

Now
\[
\int_\mathbb{R} v(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xxx} \rho_\varepsilon (\partial_{xx} \rho_\varepsilon)^{p-1} dx = \frac{1}{p} \int_\mathbb{R} v(\rho_\varepsilon \ast \omega_\varepsilon) \partial_x ((\partial_{xx} \rho_\varepsilon)^p) dx
\]
\[ = -\frac{1}{p} \int_R v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_x (\rho_\varepsilon \ast \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^p \, dx, \]

using the fact that the function \( \partial_{xx}^2 \rho_\varepsilon \) tends to 0 at \( \pm \infty \) and that \( v \) is bounded. Therefore

\[
\frac{d}{dt} \int_R (\partial_{xx}^2 \rho_\varepsilon)^p \, dx = -3p \int_R v''(\rho_\varepsilon \ast \omega_\varepsilon) (\partial_x (\rho_\varepsilon \ast \omega_\varepsilon))^2 \partial_x \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} \, dx
\]

\[
-3p \int_R v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx}^2 (\rho_\varepsilon \ast \omega_\varepsilon) \partial_x \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} \, dx
\]

\[
+ (1 - 3p) \int_R v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_x (\rho_\varepsilon \ast \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^p \, dx
\]

\[
- p \int_R \rho_\varepsilon v(3)(\rho_\varepsilon \ast \omega_\varepsilon) (\partial_x (\rho_\varepsilon \ast \omega_\varepsilon))^3 (\partial_{xx}^2 \rho_\varepsilon)^{p-1} \, dx
\]

\[
-3p \int_R \rho_\varepsilon v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_x (\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx}^2 (\rho_\varepsilon \ast \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} \, dx
\]

\[
- p \int_R \rho_\varepsilon v'(\rho_\varepsilon \ast \omega_\varepsilon) \partial_{xx}^3 (\rho_\varepsilon \ast \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} \, dx
\]

\[
+ \varepsilon p \int_R \partial_{xxx}^4 \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} \, dx
\]

\[=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \]

We estimate now each of these terms.

- Using (6) and Young’s inequality, we get

\[
|J_1| = 3p \left| \int_R v''(\rho_\varepsilon \ast \omega_\varepsilon) (\partial_x (\rho_\varepsilon \ast \omega_\varepsilon))^2 \partial_x \rho_\varepsilon (\partial_{xx}^2 \rho_\varepsilon)^{p-1} \, dx \right|
\]

\[
\leq 3p \|v''\|_\infty (2\|\rho_\varepsilon\|_\infty W^2_\varepsilon)^2 \int_R |\partial_x \rho_\varepsilon| |\partial_{xx}^2 \rho_\varepsilon|^{p-1} \, dx
\]

\[
\leq 12 \|v''\|_\infty \|\rho_\varepsilon\|_\infty^2 W_\varepsilon^2 \left( \int_R |\partial_x \rho_\varepsilon|^p \, dx + (p - 1) \int_R |\partial_{xx}^2 \rho_\varepsilon|^p \, dx \right)
\]

- Setting \( p_1 = 2p = p_2 \) and \( p_3 = p/(p - 1) \). Estimate (24) can be derived applying twice the classical Young’s inequality to \( uvw = u(vw) \). Using now the relation (17) with \( 2p \) at the place of \( p \), we get

\[
|J_2| \leq \frac{3^{2p+1}}{2} \|v''\|_\infty \eta \left( \int_0^\eta |\omega''(u)|^{2p/(2p-1)} \, du \right)^{2p-1} \int_R \rho_\varepsilon^{2p} \, dx
\]
\[ + \frac{3^{2p+1}}{2} \| v' \|_\infty (|\omega'(\eta^-)| + |\omega'(0+)|)^{2p} \int |\partial_x^2 \rho_\varepsilon|^p dx \]
\[ + 3^{2p+1} 2^{p-1} \| v' \|_\infty (\omega(\eta) + 2\omega(0))^{2p} \int |\partial_x \rho_\varepsilon|^2 dx \]
\[ + \frac{3}{2} \| v' \|_\infty \int |\partial_x \rho_\varepsilon|^{2p} dx + 3(p-1) \| v' \|_\infty \int |\partial_{xx}^2 \rho_\varepsilon|^p dx. \]

\[ |J_3| = (3p-1) \left| \int_R v'(\rho_\varepsilon * \omega_\varepsilon) \partial_x (\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^p dx \right| \]
\[ \leq 2(3p-1) \| v' \|_\infty \| \rho_\varepsilon \|_\infty W_\varepsilon \int_R |\partial_{xx}^2 \rho_\varepsilon|^p dx. \]

\[ |J_4| = p \left| \int_R \rho_\varepsilon v^{(3)}(\rho_\varepsilon * \omega_\varepsilon) (\partial_x (\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon))^{p-1} dx \right| \]
\[ \leq p \| v^{(3)} \|_\infty (2 \| \rho_\varepsilon \|_\infty W_\varepsilon)^3 \int_R \rho_\varepsilon |\partial_{xx}^2 \rho_\varepsilon|^{p-1} dx \]
\[ \leq 8 \| v^{(3)} \|_\infty \| \rho_\varepsilon \|_\infty W_\varepsilon^3 \left( \int_R \rho_\varepsilon^p dx + (p-1) \int_R |\partial_{xx}^2 \rho_\varepsilon|^p dx \right) \]
using Young's inequality.

\[ |J_5| = 3p \left| \int_R \rho_\varepsilon v''(\rho_\varepsilon * \omega_\varepsilon) \partial_x (\rho_\varepsilon * \omega_\varepsilon) \partial_{xx}^2 (\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \right| \]
\[ \leq 6p \| v'' \|_\infty \| \rho_\varepsilon \|_{-2}^2 W_\varepsilon \int_R |\partial_{xx}^2 (\rho_\varepsilon * \omega_\varepsilon)| |\partial_{xx}^2 \rho_\varepsilon|^{p-1} dx \]
\[ \leq 6 \| v'' \|_\infty \| \rho_\varepsilon \|_{-2}^2 W_\varepsilon \left( \int_R |\partial_{xx}^2 (\rho_\varepsilon * \omega_\varepsilon)|^p dx + (p-1) \int_R |\partial_{xx}^2 \rho_\varepsilon|^p dx \right) \]
\[ \leq 6 \| v'' \|_\infty \| \rho_\varepsilon \|_{-2}^2 W_\varepsilon \left( 3^p \eta \left( \int_0^\eta |\omega''(u)|^{p/(p-1)} du \right)^{p-1} \int_R \rho_\varepsilon^p dx \right. \]
\[ + 3^p (|\omega'(\eta^-)| + |\omega'(0+)|)^p \int_R \rho_\varepsilon^p dx \]
\[ + 6^p (\omega(\eta) + 2\omega(0)) \int_R |\partial_x \rho_\varepsilon|^p dx \]
\[ + (p-1) \int_R |\partial_{xx}^2 \rho_\varepsilon|^p dx \]
using relation (17).

\[ |J_6| = p \left| \int_R \rho_\varepsilon v'(\rho_\varepsilon * \omega_\varepsilon) \partial_{xxx}^3 (\rho_\varepsilon * \omega_\varepsilon) (\partial_{xx}^2 \rho_\varepsilon)^{p-1} dx \right| \]
\[ \leq p \| \rho_\varepsilon \|_\infty \| v' \|_\infty \int_R |\partial_{xxx}^3 (\rho_\varepsilon * \omega_\varepsilon)| |\partial_{xx}^2 \rho_\varepsilon|^{p-1} dx \]
\[ \leq \| \rho_\varepsilon \|_\infty \| v' \|_\infty \left( \int_R |\partial_{xxx}^3 (\rho_\varepsilon * \omega_\varepsilon)|^p dx + (p-1) \int_R |\partial_{xx}^2 \rho_\varepsilon|^p dx \right). \]
Note that \( \int_0^\infty |\nabla^2 \rho_c(t,x)|^p \, dx \leq 3p \eta \left( \int_0^\infty |\nabla \rho_c(t,x)|^p \, dx \right)^{(p-1)/p} \int_\mathbb{R} |\partial_x \rho_c(t,x)|^p \, dx \\
+ 3^p \left( |\nabla'(\eta^-)| + |\nabla'(0+)| \right) \int_\mathbb{R} |\partial_x \rho_c(t,x)|^p \, dx \\
+ 6^p (\omega(\eta) + 2\omega(0)) \int_\mathbb{R} |\partial_x \rho_c(t,x)|^p \, dx.

Then

\[
|J_6| \leq \|\rho_c\|_{\infty} \|v'\|_{\infty} \left( 3^p \eta \left( \int_0^\infty |\nabla \rho_c(t,x)|^p \, dx \right)^{(p-1)/p} \int_\mathbb{R} |\partial_x \rho_c(t,x)|^p \, dx \\
+ 3^p \left( |\nabla'(\eta^-)| + |\nabla'(0+)| \right) \int_\mathbb{R} |\partial_x \rho_c(t,x)|^p \, dx \\
+ 6^p (\omega(\eta) + 2\omega(0)) \int_\mathbb{R} |\partial_x \rho_c(t,x)|^p \, dx \\
+ (p-1) \int_\mathbb{R} |\partial^3 x \rho_c(t,x)|^p \, dx \right).
\]

\( J_7 = \varepsilon p \int_\mathbb{R} \partial^3 x \rho_c \left( \partial^2 x \rho_c \right)^{p-1} \, dx = -\varepsilon p(p-1) \int_\mathbb{R} \left( \partial^3 x \rho_c \right)^2 \left( \partial^2 x \rho_c \right)^{2(N-1)} \, dx \leq 0,

since \( \partial^2 x \rho_c \) and \( \partial^3 x \rho_c \) tend to 0 at \( \pm \infty \).

The above estimates give an estimate of the form

\[
\frac{d}{dt} \int_\mathbb{R} \left( \partial^2 x \rho_c(t,x) \right)^p \, dx \leq C_7^p \left( \int_\mathbb{R} \left( \partial^2 x \rho_c(t,x) \right)^p \, dx + \int_\mathbb{R} |\partial_x \rho_c(t,x)|^p \, dx \right) \\
+ \int_\mathbb{R} \rho_c^p(t,x) \, dx + \int_\mathbb{R} |\partial_x \rho_c(t,x)|^{2p} \, dx + \int_\mathbb{R} \rho_c^{2p}(t,x) \, dx,
\]

where \( C_7^p = C_7^p \left( p, \|v'\|_{\infty}, \|v''\|_{\infty}, \|v^{(3)}\|_{\infty}, \sup_{\epsilon} \{ \|\rho_c\|_{\infty} W_\epsilon \} \right) \) and

\[
C_7^p = \max \left\{ 3^p \eta \left( \int_0^\infty |\nabla \rho_c(t,x)|^p \, dx \right)^{(p-1)/p}, \right. \\
3^p \left( |\nabla'(\eta^-)| + |\nabla'(0+)| \right)^p, 6^p (\omega(\eta) + 2\omega(0))^p \right\}.
\]

Note that \( C_7^p \) is a constant since \( \|\rho_c\|_{\infty} W_\epsilon \) is bounded with respect to \( \epsilon \) thanks to Proposition 4 and 5. This estimate, combined with (18) and (19), give then

\[
\frac{d}{dt} \left( \int_\mathbb{R} \left| \partial^2 x \rho_c(t,x) \right|^{2N} \, dx + \int_\mathbb{R} |\partial_x \rho_c(t,x)|^{2N} \, dx + \int_\mathbb{R} \rho_c^{2N}(t,x) \, dx \right) \\
\leq C_8^{2N} \left( \int_\mathbb{R} \left| \partial^2 x \rho_c(t,x) \right|^{2N} \, dx + \int_\mathbb{R} |\partial_x \rho_c(t,x)|^{2N} \, dx + \int_\mathbb{R} \rho_c^{2N}(t,x) \, dx \right) \\
+ C_7^{2N} e^{C_9^{2N} t} \left( \int_\mathbb{R} |\partial_x \rho(0,x)|^{4N} \, dx + \int_\mathbb{R} |\rho(0,x)|^{4N} \, dx \right),
\]

where

\[
C_8^{2N} = C_7^{2N} + \sup_{\epsilon} C_6^{2N}, \quad C_9^{2N} = \sup_{\epsilon} C_6^{2N}.
\]
Note that $C^{c,2N}_{c_6}$ is bounded with respect to $\varepsilon$ thanks to Proposition 4 and 5. Since an inequality of the form

\[ u'(t) \leq K_1 u(t) + K_2 e^{K_1 t} \]

implies the estimate

\[ u(t) \leq u(0) e^{K_1 t} + K_2 e^{K_1 t} \int_0^t e^{(K_1 - K_1)s} ds \leq u(0) e^{K_1 t} + \frac{K_2}{K_1} \left( e^{(K_1 + K_1)s} - e^{K_1 t} \right), \]

we get the estimate

\[
\left( \int_R |\partial_{xx}^2 \rho_{\varepsilon}(t,x)|^{2N} dx + \int_R |\partial_x \rho_{\varepsilon}(t,x)|^{2N} dx + \int_R \rho_{\varepsilon}^{2N}(t,x) dx \right) \\
\leq \left( \int_R |\partial_{xx}^2 \rho(0,x)|^{2N} dx + \int_R |\partial_x \rho(0,x)|^{2N} dx + \int_R \rho^{2N}(0,x) dx \right) e^{C_{2N}^2 t} \\
+ \frac{C_{2N}^2}{C_9^2} \left( \int_R |\partial_x \rho(0,x)|^{4N} dx + \int_R |\rho(0,x)|^{4N} dx \right) e^{(C_{2N}^2 + C_{2N}^2)T} \]  

which implies

\[
\sup_{t \in [0,T]} \left( \int_R |\partial_{xx}^2 \rho_{\varepsilon}(t,x)|^{2N} dx + \int_R |\partial_x \rho_{\varepsilon}(t,x)|^{2N} dx + \int_R \rho_{\varepsilon}^{2N}(t,x) dx \right) \\
\leq \left( \int_R |\partial_{xx}^2 \rho(0,x)|^{2N} dx + \int_R |\partial_x \rho(0,x)|^{2N} dx + \int_R \rho^{2N}(0,x) dx \right) e^{C_{2N}^2 T} \\
+ \frac{C_{2N}^2}{C_9^2} \left( \int_R |\partial_x \rho(0,x)|^{4N} dx + \int_R |\rho(0,x)|^{4N} dx \right) e^{(C_{2N}^2 + C_{2N}^2)T}. 
\]

\[ \Box \]

3. Proof of Theorem 1.2. In this section, we pass to the limit as $\varepsilon \to 0$ and we show that the limit function $\rho$ satisfies equation (1).

Using Proposition 2, the sequence $\{\rho_{\varepsilon}\}_\varepsilon$ is bounded in $L^\infty([0,T], L^2(\mathbb{R}))$. Using Proposition 3, the sequence $\{\partial_x \rho_{\varepsilon}\}_\varepsilon$ is bounded in $L^\infty([0,T], L^2(\mathbb{R}))$. Using Propositions 1 and 4, the sequences $\{v(\rho_{\varepsilon} \ast \omega_{\varepsilon})\}_\varepsilon$, $\{v'(\rho_{\varepsilon} \ast \omega_{\varepsilon})\}_\varepsilon$ and $\{\partial_x(\rho_{\varepsilon} \ast \omega_{\varepsilon})\}_\varepsilon$ are bounded in $L^\infty([0,T] \times \mathbb{R})$. Then

\[ \partial_{xx}(\rho_{\varepsilon} \ast \omega_{\varepsilon}) = \partial_{xx} \rho_{\varepsilon} \ast v(\rho_{\varepsilon} \ast \omega_{\varepsilon}) + \rho_{\varepsilon} \ast v'(\rho_{\varepsilon} \ast \omega_{\varepsilon}) \partial_x(\rho_{\varepsilon} \ast \omega_{\varepsilon}) \]

is bounded in $L^\infty([0,T], L^2(\mathbb{R}))$. Using Proposition 5, we also have a bound with respect to $\varepsilon$ for $\partial_{xx}^2 \rho_{\varepsilon}$ in the space $L^\infty([0,T], L^2(\mathbb{R}))$, then

\[ \partial_x \rho_{\varepsilon} = \varepsilon \partial_{xx} \rho_{\varepsilon} - \partial_{xx}(\rho_{\varepsilon} \ast \omega_{\varepsilon}) \in L^\infty([0,T], L^2(\mathbb{R})) \]

uniformly with respect to $\varepsilon$. In particular, $\rho_{\varepsilon} \in C([0,T], L^2(\mathbb{R}))$ and the sequence is bounded in this space. Since $\partial_t \rho_{\varepsilon}, \partial_x \rho_{\varepsilon} \in L^\infty([0,T], L^2(\mathbb{R}))$ with uniform bounds with respect to $\varepsilon$, then $\{\rho_{\varepsilon}\}_\varepsilon$ is bounded in $H^1_{\text{loc}}([0,T] \times \mathbb{R})$. Up to the extraction.
of a subsequence, the sequence \( \{ \rho_\varepsilon \} \) converges to some \( \rho \) in \( L^2_{\text{loc}}([0, T] \times \mathbb{R}) \) and a.e. We have now to prove that the limit \( \rho \in C([0, T], L^2(\mathbb{R})) \) is a solution of (1). Since

\[
(\rho_\varepsilon * \omega_\varepsilon)(t, x) - (\rho * \omega)(t, x) = \int_{-\varepsilon}^{0} \rho_\varepsilon(t, x + y) \omega_\varepsilon(y) \, dy + \int_{0}^{\eta} (\rho_\varepsilon - \rho)(t, x + y) \omega(y) \, dy + \int_{\eta}^{\eta + \varepsilon} \rho_\varepsilon(t, x + y) \omega_\varepsilon(y) \, dy
\]

tends to 0 when \( \varepsilon \) goes to zero, we have

\[
\rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon) \to \rho v(\rho * \omega) \quad \text{a.e.}
\]

Therefore using dominated convergence Theorem, we get \( \rho_\varepsilon v(\rho_\varepsilon * \omega_\varepsilon) \to \rho v(\rho * \omega) \) in \( L^1_{\text{loc}}([0, T] \times \mathbb{R}) \), implying that \( \rho \) is a solution of (1).

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E-mail address: Florent.Berthelin@unice.fr
E-mail address: paola.goatin@inria.fr