From thermal to excited-state quantum phase transitions —the Dicke model

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We study the thermodynamics of the full version of the Dicke model, including all the possible values of the total angular momentum $j$, with both microcanonical and canonical ensembles. We focus on how the excited-state quantum phase transition, which only appears in the microcanonical description of the maximum angular momentum sector, $j = N/2$, change to a standard thermal phase transition when all the sectors are taken into account. We show that both the thermal and the excited-state quantum phase transitions have the same origin; in other words, that both are two faces of the same phenomenon. Despite all the logarithmic singularities which characterize the excited-state quantum phase transition are ruled out when all the $j$-sectors are considered, the critical energy (or temperature) still divides the spectrum in two regions: one in which the parity symmetry can be broken, and another in which this symmetry is always well defined.

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I. INTRODUCTION

Quantum phase transitions (QPTs) and critical phenomena play an important role in the study of many-body quantum systems \([1]\). During the last decade, a new kind of phase transition has been studied in depth — the excited-state quantum phase transition (ESQPT) \([2–6]\). In contrast to QPTs, which describe the non-analytical evolution of the ground state energy as a function of a control parameter, ESQPTs refer to a similar non-analytic behavior that takes place at a certain critical energy \(E_c\), when the control parameter responsible for the QPT is kept fixed \([7]\).

ESQPTs have been theoretically studied in many kinds of quantum systems. Paradigmatic examples are the Lipkin-Meshkov-Glick (LMG) \([8, 9]\), theDicke and Tavis Cummings models \([10, 11]\), the interacting boson model \([12]\), the molecular vibron model \([13]\), atom-molecule condensates \([14]\), the kicked-top \([15]\) or the Rabi model \([16]\). Also, a number of experimental results have been recently reported, in molecular systems \([17]\), superconducting microwave billiards \([18]\), and spinor condensates \([19]\). However, and despite the intense research performed during the last couple of years, some important questions still remain open. The most important one is whether the critical energy does separate two different phases in the spectrum. Contrary to what happens in quantum and thermal phase transitions, there are no clear traces of order parameters in ESQPTs. Though many physical observables become singular at the critical point, it seems impossible to find a magnitude which is zero at one side of the transition, and remains different from zero at the other (see, for example \([5]\)). A recent proposal relies on symmetry-breaking. A number of quantum systems showing ESQPTs are characterized by a discrete \(Z_2\) symmetry which can be broken at one side of the transition, but not at the other. This fact entails measurable dynamical consequences if a thermodynamic process is performed from a symmetry-breaking initial condition — the symmetry of the final equilibrium state remains broken only if the final energy is at the corresponding side of the transition, whereas the symmetry is restored on the contrary \([20, 21]\). However, if the initial condition has a well-defined value of this symmetry, nothing similar happens when crossing the critical energy. Another recent proposal consists on how the values of the physical observables change with energy \([22]\). Anyhow, this complex behavior is in contrast with what happens in quantum and thermal phase transitions, where the two different phases are easily distinguishable just by measuring an appropriate
observable.

Notwithstanding, from fundamental physical reasons a link between ESQPTs and thermal phase transitions is expected. ESQPTs occur when the system is kept isolated from any environment, and thus can be described by means of the microcanonical ensemble. On the contrary, thermal phase transitions take place at a certain critical temperature $\beta_c$, and are usually described considering that the critical system is in contact with a thermal bath, that is, by means of the canonical ensemble \[23\]. But, as microcanonical and canonical descriptions become equivalent in the thermodynamical limit $N \to \infty$, where $N$ is the number of particles of the critical system, it is logical to expect that critical energy $E_c$ and critical temperature $\beta_c$ are two manifestations of the same phenomenon. If we describe the critical system by means of the canonical ensemble, we should expect that the critical energy $E_c$ of the ESQPT corresponds to the internal energy $U = -\partial \log Z / \partial \beta$, evaluated at the critical temperature $\beta_c$, being $Z$ the canonical partition function. And if the system is described by means of the microcanonical ensemble, the critical temperature $\beta$ should correspond to the microcanonical temperature $\beta = \partial \rho(E) / \partial E$, evaluated at the critical energy $E_c$, being $\rho(E)$ the density of states. However, all the facts discussed below suggest just the opposite —that thermal and excited-state quantum phase transitions are totally different. Probably, this is due to the fact that ESQPTs take place in systems with a small number of semiclassical degrees of freedom, implying that the size of the corresponding Hilbert space grows as $N^f$, being $f$ the number of degrees of freedom —the larger the number of degrees of freedom, the less important are the consequences of the ESQPT \[5\]). On the contrary, thermal phase transitions require an exponential growth of the size of the Hilbert space with the number of particles, in order to assure that intensive thermodynamical quantities, like the entropy per particle $S/N$ or the Helmholtz potential per particle $F/N$ are well defined in the thermodynamical limit. Hence, it is not clear even if the correspondence between thermal and excited-state quantum phase transitions exists, or if they are different phenomena occurring under different physical circumstances. (We notice that, during the progress of this work, a similar analysis, but with a different aim, was performed in the generalized Dicke model, showing that it shows two different kinds of superradiance \[24\]).

In this work we tackle this task by studying, both analytically and numerically, the Dicke model. It describes a system of $N$ two-level atoms interacting with a single monochromatic electromagnetic radiation mode within a cavity \[25\]. It is well known from the seventies
that this model exhibit a thermal phase transition \[26, 27\]. However, recently it was also found that undergoes an ESQPT \[10\] aside the QPT \[28\]. This kind of QPT has been experimentally observed in several systems \[29\], and the Dicke model itself can be simulated by means of a Bose-Einstein condensate in an optical cavity \[30\]. All these facts make this model the best one to study the link between thermal and excited-state quantum phase transitions.

Up to now, the majority of the works on the Dicke model, including the ones dealing with QPTs and ESQPTs (except \[24\], as we have pointed above), were done in the subspace with maximum pseudo-spin sector \(j = N/2\), in which the ground state is included. This restriction is enough to properly describe the recent experimental results \[30\], and also to study all the consequences of the QPT. Furthermore, ESQPTs have been observed in the subspace with \(j = N/2\), which can be described by means of a semiclassical approximation with just two degrees of freedom in the thermodynamic limit \(N \to \infty\). However, it is well known that this restriction destroys the thermal phase transition \[31\]; the fact that the atomic subspace grows linearly with \(N\) in the \(j = N/2\) sector makes impossible to properly define the entropy \(S\) or the Helmholtz potential \(F\), and therefore precludes the thermal phase transition. In this work we deal with the complete Dicke model, including all the \(j\) sectors. Contrary to the seminal papers on the thermal phase transition \[26, 27\] we study the thermodynamics of this model in the microcanonical ensemble, considering the system isolated instead of being in contact with a thermal bath. This point of view allows us to link the ESQPT with the thermal phase transition. In particular, we show that each \(j\) sector displays the same kind of ESQPT, provided that the coupling constant is large enough (see below for a detailed discussion regarding this condition), but having each one a different critical energy \(E_c\). Paradoxically, this fact, together with the different weight of each \(j\) sector in the spectrum of the complete Dicke model, destroys most of the signatures of the ESQPT, and somehow surprisingly entails the appearance of the typical signatures of thermal phase transitions, like the existence of an order parameter. In particular, we show that the collective contribution of all the \(j\) sectors rules out the logarithmic singularities in the derivatives of the density of states, \(\rho(E)\), and the third component of the angular momentum, \(J_z\), characteristic of the ESQPT. However, one of the most important signatures of the ESQPT survives. The parity symmetry of the Dicke model (see below for details) can be still broken below the critical energy \(E_c\) (or the critical temperature \(\beta_c\)), but it is always well defined above. We show
that the expectation value of a symmetry-breaking observable, like $J_z$, is always zero above $E_c$, but it can be different from zero below. And we prove that the critical temperature $\beta_c$ exactly coincides with the microcanonical temperature $\beta$ evaluated at the critical energy $E_c$. Thus, and even all the $j$ sectors take part in the dynamics of the system, the consequences of this symmetry breaking can be dynamically explored if the system remains isolated from any heat bath —if it is heated by means of the Joule effect. Therefore, as both the ESQPT and the thermal phase transition are characterized by the same fundamental property, the fact that the parity symmetry can be only broken below the critical energy $E_c$, making possible to find symmetry-breaking equilibrium states, we conclude that both phenomena are manifestation of the same underlying physical mechanism.

This paper is organized as follows. In section 2 we present the Dicke model. In section 3 we review the thermodynamics of the Dicke model restricted to the highly-symmetric Dicke states, $| j = N/2, M \rangle$. We compare the results provided by the micro and canonical ensembles, and we analyze the symmetry-breaking character of the ESQPT. In section 4 we perform a similar analysis including all the $j$ sectors. We show that an ESQPT occurs in each $j$ sector, and that the collective contribution of all of them produces the thermal phase transition. We also show that the symmetry-breaking nature of the transition is still present if the system remains isolated from any heat reservoir. In addition, we study the main physical differences between the system in isolation and in contact with a thermal bath. Finally, we extract the more relevant conclusions in the last section.

II. THE DICKE MODEL

The Dicke model describes the interaction of $N$ two-level atoms of splitting $\omega_0$ with a single bosonic mode of frequency $\omega$, by means of a coupling parameter $\lambda$,

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{2\lambda}{\sqrt{N}} J_x (a^\dagger + a),$$

where $a^\dagger$ and $a$ are the usual creation and annihilation operators of photons, and $\vec{J} = (J_x, J_y, J_z)$ is the Schwinger pseudospin representation of the $N$ two-level atom system, that is, the total angular momentum of a system of $N$ 1/2-spin particles. This Hamiltonian has two conserved quantities. The first one is $\Pi = \exp(i\pi [J + J_z + a^\dagger a])$, due to the invariance of $H$ under $J_x \rightarrow -J_x$ and $a \rightarrow -a$; as this is a discrete symmetry, $\Pi$ has only
two different eigenvalues, \( \Pi |E_i, \pm\rangle = \pm |E_i, \pm\rangle \), and it is usually called \textit{parity}. The second one is the total angular momentum \( J^2 \) of the \( N \) \( 1/2 \)-spin particles. This entails that the Hamiltonian \( \Pi \) is block-diagonal in \( J^2 \), and hence each sector is totally independent from the others. The main dynamical consequence is that each \( j \) sector evolves independently in any protocol keeping the Dicke model isolated from any heat bath. Furthermore, as the recent experimental realizations of this model involve only the sector of maximum angular momentum, \( j_{\text{max}} = N/2 \), the great majority of the papers published during the last couple of years are devoted to this case.

This model shows QPTs, ESQPTs and thermal phase transitions. In the following paragraphs we summarize the known results.

\section*{III. THE CASE WITH \( j = N/2 \)}

In this section, for the sake of completeness, we review the thermodynamics of the Dicke model restricted to the highly-symmetric Dicke states, \( |j = N/2, M\rangle \). This configuration corresponds to a two-level system in which \( N \) bosons can occupy either the upper or the lower level \([31]\). It has been recently explored by means of a Bose Einstein condensate in an optical cavity \([30]\). First of all, we present the density of states \( \rho(E) \), which is computed by means of the microcanonical ensemble, and later we show the same \( \rho(E) \) but considering the calculation in the canonical ensemble. These are well established results. Finally, we compare both approaches and get some conclusions.

\subsection*{A. Microcanonical ensemble}

Let’s consider that the system is thermally isolated and that we perform the following procedure: first, we freeze the system keeping fixed all the external parameters of the Hamiltonian, up to it is equilibrated at \( T \sim 0 \). This entails that the ground state, which always correspond with the sector of maximum angular momentum \( j = N/2 \), is the only populated energy level. Second, we perform a quench, abruptly changing one of the external parameters. Then, if the system remains thermally isolated from the environment, the unitary evolution is totally captured by the sector with \( j = N/2 \). Hence, all the thermodynamic results after the system is equilibrated at the final values of the external parameters
should be obtained from a microcanonical calculation with fixed $j = N/2$. This calculation can be completed by means of a semiclassical approximation, following different methods \[11, 32, 33\]. Here, we follow the method in ref. \[33\].

Considering $\omega = \omega_0 = 1$, the density of states reads

$$\rho(E, J) = \begin{cases} 
2J & \text{if } E/N > 1/2, \\
\frac{E}{J} + 1 + \frac{2}{\pi} \int_{E/J}^{y_+} dy \ \text{acos} \left( \frac{\sqrt{y - E/J}}{2\lambda^2(1 - y^2)} \right) & \text{if } -1/2 \leq E/N \leq 1/2, \\
\frac{2}{\pi} \int_{y_-}^{y_+} dy \ \text{acos} \left( \frac{\sqrt{y - E/J}}{2\lambda^2(1 - y^2)} \right) & \text{if } E/N < -1/2,
\end{cases}$$

(2)

where

$$y_- = -\frac{J + \sqrt{J} \sqrt{J + 8E\lambda^2 + 16J\lambda^2}}{4J\lambda^2},$$

(3)

and

$$y_+ = -\frac{J + \sqrt{J} \sqrt{J + 8E\lambda^2 + 16J\lambda^2}}{4J\lambda^2}.$$  

(4)

For the third component of the angular momentum, we obtain

$$\frac{J_z(E, J)}{J} = \begin{cases} 
0 & \text{if } E/N > 1/2, \\
\frac{E^2}{2J^2 - \frac{1}{2}} \frac{J}{\rho(E, J)} + \frac{2J}{\pi \rho(E, J)} \int_{E/J}^{y_+} dy y \ \text{acos} \left( \frac{\sqrt{y - E/J}}{2\lambda^2(1 - y^2)} \right) & \text{if } -1/2 \leq E/N \leq 1/2, \\
\frac{2J}{\pi \rho(E, J)} \int_{y_-}^{y_+} dy y \ \text{acos} \left( \frac{\sqrt{y - E/J}}{2\lambda^2(1 - y^2)} \right) & \text{if } E/N < -1/2,
\end{cases}$$

(5)

Finally, for the first component of the angular momentum and considering that the parity is totally broken in the initial state,

$$\frac{J_x(E, J)}{J} = \begin{cases} 
0 & \text{if } E/N > -1/2, \\
\pm\frac{2J}{\pi \rho(E, J)} \int_{y_-}^{y_+} dy (1 - y^2) \ \text{acos} \left( \frac{\sqrt{y - E/J}}{2\lambda^2(1 - y^2)} \right) & \text{if } E/N < -1/2,
\end{cases}$$

(6)

where the sign depends on the initial state. This expression has been obtained taking into account only one of the two disjoint parts in which the semiclassical phase space is divided for $\lambda > \lambda_c$ and $E < -N/2$ \[20\]. If the initial state has a well-defined parity, both parts of the semiclassical phase space are populated, giving rise to $\langle J_x \rangle = 0$.

These results show that an ESQPT happens at $E_c/N = -1/2$ \[10, 11, 20, 33\]. There are singular points for both $\rho(E, J)$ and $J_z(E, J)$ —the derivatives of both magnitudes show a
logarithmic divergence at $E_c$. The reason for this behavior is the following: the density of states, Eq. (2), is proportional to the size of the phase space available to the system,

$$\rho(E, J) = C \int dq_1 dq_2 dp_1 dp_2 \delta [E - H(q_1, q_2; p_1, p_2)],$$  \hspace{1cm} (7)

where $q_1$ and $q_2$ denote the semiclassical coordinates; $p_1$ and $p_2$, the semiclassical momenta, and $C$ is a normalization constant (see, for example, [33]). The key point is that despite this semiclassical system is finite, it describes the quantum Dicke model in the thermodynamical limit, $N \to \infty$, and it has just $f = 2$ degrees of freedom. Furthermore, every quantum system showing an ESQPT is equivalent to a semiclassical system with a finite number of degrees of freedom (see, for example, ref. [5]). As a consequence, non-analyticities in the quantum density of states are linked to stationary points in the corresponding semiclassical model; and the geometric properties of such stationary points determine the nature of the corresponding singularities. In particular, systems with $f = 1$ semiclassical degrees of freedom show logarithmic singularities in the density of states as well as in certain physical observables at the critical energy of the ESQPT, $E_c$, whereas systems with $f = 2$ degrees of freedom show the same kind of singularities in the derivatives of the same magnitudes [5]. Results for an higher number of degrees of freedom have been recently published, showing that the larger $f$, the higher derivative in which the logarithmic singularity takes place [34].

Also, if the parity symmetry is broken in the initial state, $J_x(E, J)$ acts like an order parameter for the ESQPT; that is, it shows a finite jump at $E_c$, from $\langle J_x \rangle \neq 0$ to $\langle J_x \rangle = 0$ [20]. On the contrary, initial conditions with well-defined positive (or negative) parity do not suffer any change when crossing the ESQPT.

Another singular point is located at $E_c/N = 1/2$, whilst its critical character is controversial [11, 33]. Above this energy, $\rho(E) = 1$ and $\langle J_z \rangle = 0$, due to the ergodic character of the atomic motion (now the whole phase space is accessible to the system). Despite this point is not usually identified as an ESQPT, it has some of the features of a second order phase transition. First, there exists an order parameter identifying two different phases: for $E < N/2$, $\langle J_z \rangle \neq 0$, whereas $\langle J_z \rangle = 0$ for $E > N/2$. Second, there is a discontinuity in the derivative of $\rho(E)$, that is, in the second derivative of the cumulated level density $N(E)$. We will come back to this discussion in Sec. [IV] Numerical results illustrating these facts are shown later.
B. Canonical ensemble

The same kind of calculation can be performed in the canonical ensemble, considering that the system weakly interacts with a thermal bath which commutes with $J^2$. Following ref. [31] we can obtain the partition function

$$Z(N, \beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \exp\left(-\beta \omega y^2\right) \int_{-\infty}^{\infty} dx \exp\left(-\beta \omega x^2\right) Z_g(N, \beta), \quad (8)$$

where

$$Z_g(N, \beta) = \sum_{m=-N/2}^{N/2} \exp\left(-\beta m \sqrt{\omega^2 + \frac{4\lambda^2 x^2}{N}}\right). \quad (9)$$

The final result is

$$Z(N, \beta) = \sqrt{\frac{1}{\pi \beta \omega}} \int_{-\infty}^{\infty} dx \exp\left[-\beta \left(\omega x^2 + \frac{n}{2} \sqrt{\omega^2 + \frac{4\lambda^2 x^2}{n}}\right)\right] \frac{\exp\left(\beta(n + 1) \sqrt{\omega^2 + \frac{4\lambda^2 x^2}{n}}\right) - 1}{\exp\left(\beta \sqrt{\omega^2 + \frac{4\lambda^2 x^2}{n}}\right) - 1}. \quad (10)$$

There is no way to write this integral in terms of simple analytical functions, but it can be evaluated numerically to obtain results for precise values of all the external parameters $\omega$, $\omega_0$ and $\lambda$. Furthermore, other thermodynamic results can be obtained from the partition function

$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta}, \quad (11)$$

$$\langle J_z \rangle = -\frac{1}{\beta} \frac{\partial \log Z}{\partial \omega_0}. \quad (12)$$

In all the cases $\langle J_x \rangle = 0$.

It has been shown that there is no thermal phase transition under these circumstances [31]. In other words, microcanonical and canonical ensembles give rise to totally different results. If the system remains thermally isolated there exists a critical energy $E_c = -N/2$ at which a non-analiticity occurs, giving rise to a number of dynamical (and observable) consequences [10, 20]. On the other hand, if the system is put in contact with a thermal bath, everything changes smoothly with the temperature $\beta$; in particular, nothing happens at the critical temperature $\beta_c$, $\langle E(\beta_c) \rangle = -N/2$. 
C. Results

In this subsection, we compare the results of both the microcanonical and the canonical calculations, for a system with $\omega = \omega_0 = 1$, $\lambda = 3\lambda_c = 1.5$, and $N = 10^5$. All the results are plotted versus the scaled energy $\langle E \rangle / N$. For the canonical calculation, this energy is obtained directly from Eq. (11).

![Graph of $J_z$](image1.png)

**FIG. 1.** (Color online) $J_z$ for the microcanonical ensemble (green solid points), and the canonical ensemble (dashed red line). The vertical dashed line shows the energy of the ESQPT.

![Graph of $dJ_z/dE$](image2.png)

**FIG. 2.** (Color online) $dJ_z/dE$ for the microcanonical ensemble (green solid points), and the canonical ensemble (dashed red line). The vertical dashed line shows the energy of the ESQPT.
In Figs. 1 and 3 we depict the results for $\langle J_z \rangle$, $\langle dJ_z/dE \rangle$, and $\langle J_x \rangle/N$ respectively. In the first two cases, we show both the microcanonical (solid green points) and the canonical (dashed red line) calculations; in Fig. 3 we show just the microcanonical calculation, because $\langle J_x \rangle = 0$ in the canonical ensemble. In all the cases we show the critical energy of the ESQPT, $E_c/N = -1/2$, by means of a vertical dashed line.

As a general result, we can observe that the behavior of the Dicke model in the $j = N/2$ sector is totally different depending whether it is thermally isolated or in contact with a thermal bath. In the first case, we can see neat signatures of the ESQPT (a singular point in $\langle dJ_z/dE(E_c) \rangle$, or the crossing from $\langle J_x(E) \rangle \neq 0$ to $\langle J_x(E) \rangle = 0$, if the parity symmetry is broken in the initial state). In the second one, no traces of such phenomena are present. The reason behind this result is that the microcanonical density of states grows linearly with $N$. This entails that the thermodynamic magnitudes that should be extensive, like the entropy, $S$, or the Helmholtz free energy, $F$, grow with log $N$, and therefore $S/N \to 0$ and $F/N \to 0$ in the thermodynamic limit. The main consequence is that the different ensembles are not equivalent in the thermodynamic limit, and that thermodynamics in this system is far from usual, and hence the results for the different statistical ensembles do not coincide.
IV. THE FULL DICKE MODEL

In this section, we perform a similar analysis as the former one, but now including all the \( j \) sectors in the calculation.

A. Microcanonical ensemble

If we consider that the system is thermally isolated, we can follow the same procedure than for the case with \( j = N/2 \), taking into account that each \( j \) sector is totally independent from the others. In other words, we can rely on the semiclassical approximation for each \( j \) sector, and then collect all these results. Note, however, that the semiclassical approximation only gives good results for large values of the total number of two-level atoms, \( N \). Hence, our procedure is questionable for sectors with low values of \( j \), and, in particular, for the \( j = 0 \) sector. This issue is discussed in detail later on.

Formally, the Dicke Hamiltonian does not depend on the value of \( j \), and hence the full case consists just of \( N/2 \) copies of the \( j = N/2 \) case, weighted by the corresponding degeneracy. However, there is a subtle detail that we have to take into account. The full Hamiltonian reads

\[
H = \omega a^\dagger a + \omega_0 J_z + \frac{2\lambda}{\sqrt{N}} J_x (a + a^\dagger).
\]  

(13)

Nevertheless, the Hamiltonian for the case with \( j = N/2 \) is slightly different,

\[
H = \omega a^\dagger a + \omega_0 J_z + \frac{2\lambda}{\sqrt{2J}} J_x (a + a^\dagger).
\]  

(14)

As we can see, the only difference is in the interacting term: the coupling constant \( \lambda \) is normalized in a different way. For the case \( j = N/2 \), this difference is irrelevant, since \( N = 2J \). However, if we want to take advantage from this case to describe the full Hamiltonian, we have to take into account this small difference. In particular, we can consider that each \( j \) sector is a copy of the \( j = N/2 \) case, but with the coupling constant normalized to \( \lambda_J = \lambda \sqrt{2J/N} \). In other words, the full Hamiltonian consists of the sum of \( N/2 \) copies of the \( j = N/2 \) Hamiltonian, but with different effective couplings.

With this in mind, we proceed to discuss the presence of ESQPTs in each \( j \) sector. From the results derived in the section IIIA, we conclude:
1. ESQPT appears if $\lambda > \lambda_c = \sqrt{\omega_0/2}$. This entails that each $j$ sector shows an ESQPT if $\lambda_j > \lambda_c^{(J)}$, being

$$\lambda_c^{(J)} = \sqrt{\frac{N\omega_0}{4J}}. \quad (15)$$

Therefore, the $j = 0$ sector does not exhibit an ESQPT in any case ($\lambda_c^{(J)} \to \infty$). The lower values of $j$ require so large coupling constants for having an ESQPTs, that these transitions are going to be restricted to the larger values of $j$ in all the practical cases.

2. The critical energy for each sector is located at $E_c^J/N = -J/N$, and the energy of the other singular point at $E^J_*/N = J/N$. Thus, the lower $j$, the smaller is the energy band between these two singular points. If $j \to 0$ with a coupling constant large enough for the ESQPT to occur, the band shrinks to a single point located at $E/N = 0$.

3. For any finite value of the coupling strength in the superradiant phase, $\lambda > \lambda_c$, the dynamics of the full Hamiltonian is the result of collecting all the $j$ sectors, with both critical and non-critical behavior.

Considering that each $j$ sector is totally independent from the others, the density of states for the full Hamiltonian can be obtained as

$$\rho(E) = \sum_{J=0}^{N/2} g(N, J)\rho(E, J), \quad (16)$$

being $g(N, J)$ the degeneracy of each $j$ sector, and $\rho(E, J)$ is given by Eq. (2).

The degeneracy is obtained as the number of ways in which a set of $N$ 1/2-spin particles can give rise to a total angular momentum $j$. The result is

$$g(N, J) = \frac{1 + 2J}{1 + J + N/2} \left( \begin{array}{c} N \\ N/2 - J \end{array} \right). \quad (17)$$

In the thermodynamic limit $N \to \infty$, the discrete variable $j = J/N$ can be approximated by the continuum $x \in [0, 1/2]$, and $g(N, J)$ can be expanded in series giving rise to

$$g(N, x) = \frac{(2Nx + 1) \exp \left[ \frac{2N+1}{2} \log \left( \frac{2N+1}{N+1+2Nx} \right) + (N - 2Nx) \log \left( \frac{N+1+2Nx}{N-2Nx} \right) \right]}{\sqrt{N - 2Nx(N + 2 + 2Nx)} \sqrt{2\pi N}}. \quad (18)$$

And hence, the total density of states is given by

$$\rho(E, N) = \int_0^{1/2} dx \, g(N, x)\rho(E, Nx). \quad (19)$$
We can apply the same procedure to the expected values of $J_z$ and $J_x$, obtaining

$$J_z(E, N) = \frac{1}{\rho(E, N)} \int_0^{1/2} dx \, g(N, x) \rho(E, Nx) J_z(E, Nx),$$  \hspace{1cm} (20)$$

with $J_z(E, Nx)$ given by Eq. (5). And

$$J_x(E, N) = \frac{1}{\rho(E, N)} \int_0^{1/2} dx \, g(N, x) \rho(E, Nx) J_x(E, Nx),$$  \hspace{1cm} (21)$$

with $J_x(E, Nx)$ given by Eq. (6). All these integrals have to be performed numerically since it is not possible to get analytical expressions.

As it has been pointed before, this procedure assumes that all the $j$ sectors can be properly described by means of the semiclassical approximation, and this is not completely true. Therefore, the goodness of the final result critically depends on the shape of Eq. (18). If the subsequent integrals are dominated by sectors with $j$ large enough, we can rely on our procedure; if they are dominated by the lowest $j$ sectors, the procedure is not going to work. So, prior to present the numerical results, we study here the shape of the function $g(N, x)$. In particular, we analyze: i) the sector with the largest degeneracy, $j_{\text{max}}$; and ii) the ratio between the degeneracy of this sector and the one with $j = 0$, $r = g(N, 0)/g(N, J_{\text{max}}/N)$. To be sure that our approximation is correct, we require that, on the one hand $j_{\text{max}} \rightarrow \infty$ and, on the other hand $r \rightarrow 0$, both in the thermodynamical limit $N \rightarrow \infty$.

![Graph](image_url)

**FIG. 4.** (Color online) Scaling behavior for the degeneracy of the different $j$ sectors. Left panel, $j_{\text{max}}$ versus the system size: red circles, numerical results; green solid line, numerical fit to a power law $j_{\text{max}} \propto N^\alpha$. Right panel, $r$ versus the system size, with the same symbols. Both cases are depicted in a double logarithmic scale.
Results are shown in Fig. 4 (see caption for details). From them, we can infer the following conclusions. First, \( j_{\text{max}} \sim C \sqrt{N} \). In other words, the maximally degenerated sectors grows with the square root of the system size, and therefore can be properly described by the semiclassical approximation. Second, the ratio \( r \sim C / \sqrt{N} \). That is, the relevance of the \( j = 0 \) decreases with the square root of the system size. Therefore, we can expect that the semiclassical approximation properly works in the thermodynamical limit. We can conclude that the contribution of lower \( j \) sectors is negligible in the thermodynamical limit.

B. Canonical ensemble

Let’s consider that the system is in contact with a thermal bath, so the total Hamiltonian (system + environment) reads

\[
H = H_{\text{Dicke}} + H_{\text{bath}} + H_I,
\]

where \( H_I \) is the interacting term between the system (the Dicke model) and its environment. If we assume that \([H_I, J^2] \neq 0\) and \([H_I, \Pi] \neq 0\), we have to take into account both parities and all the possible values of the angular momentum to derive the thermodynamics of the Dicke model. As it is indicated in [31], this is equivalent to a set of \( N \) fermions occupying either the lower or the upper level of a two-level system. Under such circumstances, the partition function can be explicitly obtained; this calculation was completed around 40 years ago [27]. Here, we summarize the main results.

The partition function can be exactly derived, giving rise to

\[
Z(N, \beta) = \frac{2^N}{\sqrt{\pi} \beta \omega} \int_{-\infty}^{\infty} dx \exp\left(-\beta \omega x^2\right) \left[ \cosh\left(\frac{\beta \sqrt{N \omega_0^2 + 16 \lambda^2 x^2}}{2 \sqrt{N}}\right)\right]^N.
\]

This integral cannot be solved in terms of simple analytical functions. Exact results have to be derived by means of numerical integration. The same procedure can be used to obtain the expected values of the relevant observables of the system. For example, we can obtain \( J_x \) and \( J_z \) considering

\[
J_\alpha(N, \beta) = \frac{1}{Z(N, \beta)} \text{Tr} [J_\alpha \exp(-\beta H)],
\]
where $\alpha = x, y, z$ is a label. From this equation it is straightforward to obtain

$$J_z(N, \beta) = -\frac{\omega_0 2^{N-1}}{Z(N, \beta)} \sqrt{\frac{N^3}{\pi \beta \omega}} \int_{-\infty}^{\infty} dx \exp \left( -\beta \omega x^2 \right) \left[ \cosh \left( \frac{\beta \sqrt{N \omega_0^2 + 16 \lambda^2 x^2}}{2 \sqrt{N}} \right) \right]^{N-1} \sinh \left( \frac{\beta \sqrt{N \omega_0^2 + 16 \lambda^2 x^2}}{2 \sqrt{N}} \right) \sqrt{N \omega_0^2 + 16 \lambda^2 x^2}^{N-1} \left( \frac{\beta \sqrt{N \omega_0^2 + 16 \lambda^2 x^2}}{2 \sqrt{N}} \right)$$

(25)

$$J_x(N, \beta) = -\frac{N \lambda 2^{N-1}}{Z(N, \beta)} \sqrt{\frac{1}{\pi \beta \omega}} \int_{-\infty}^{\infty} dx \exp \left( -\beta \omega x^2 \right) \left[ \cosh \left( \frac{\beta \sqrt{N \omega_0^2 + 16 \lambda^2 x^2}}{2 \sqrt{N}} \right) \right]^{N-1} \sinh \left( \frac{\beta \sqrt{N \omega_0^2 + 16 \lambda^2 x^2}}{2 \sqrt{N}} \right) \sqrt{N \omega_0^2 + 16 \lambda^2 x^2}^{N-1} \left( \frac{\beta \sqrt{N \omega_0^2 + 16 \lambda^2 x^2}}{2 \sqrt{N}} \right)$$

(26)

Note that the last integral is an odd function in the $x$ variable, so $J_x(N, \beta) = 0$. The same happens for any other symmetry-breaking observable, like, for example $q = (a + a^\dagger)/2$. Also, both $\langle E \rangle$ and $\langle J_z \rangle$ can be obtained directly from the partition function making use of Eqs. (11) and (12).

Since phase transitions are defined in the thermodynamic limit, $N \to \infty$, we can apply Laplace’s method to evaluate the partition function. Defining $y^2 = x^2/N$ we can write

$$Z(N, \beta) = \frac{\sqrt{N}}{\sqrt{\pi \beta \omega}} \int_{-\infty}^{\infty} dy \exp \left\{ N \left[ -\beta \omega y^2 + \log \left( 2 \cosh \left( \frac{\beta \omega_0}{2} \sqrt{1 + \frac{16 \lambda^2 y^2}{\omega_0^2}} \right) \right) \right] \right\}.$$  

(27)

As a consequence,

$$\lim_{N \to \infty} Z(N, \beta) = \sqrt{\frac{2}{\beta |\Psi'(y_0)|}} \exp \left[ N \Psi(y_0) \right],$$

(28)

where

$$\Psi(y) = -\beta \omega y^2 + \log \left( 2 \cosh \left( \frac{\beta \omega_0}{2} \sqrt{1 + \frac{16 \lambda^2 y^2}{\omega_0^2}} \right) \right).$$

(29)

and $y_0$ is the value of $y$ which maximizes $\Psi(y)$.

A phase transition normally happens when the position of the maximum $y_0$ changes at a certain critical temperature $\beta_c$. The easiest way to obtain $y_0$ is solving $\Psi'(y_0) = 0$, and evaluating $\Psi(y_0)$ for all the solutions. For the Dicke model, the trivial solution $y_0 = 0$ exists for all the temperatures and the values of the system parameters. Under certain circumstances, there also exists another solution,

$$4 \lambda^2 \omega \tanh \left( \frac{\beta \omega_0}{2} \sqrt{1 + \frac{16 \lambda^2 y_0^2}{\omega_0^2}} \right) = \omega_0 \sqrt{1 + \frac{16 \lambda^2 y_0^2}{\omega_0^2}}.$$

(30)

Defining $z = \omega_0 \sqrt{1 + 16 \lambda^2 y_0^2/\omega_0^2}$, the former equation reads,

$$\tanh \left( \frac{\beta z}{2} \right) = \frac{\omega_0}{4 \lambda^2} z.$$

(31)
It is important to note that, by definition, $z > 1$.

As $-1 < \tanh(z) < 1 \ \forall z$, the former equation only has solutions if

$$\lambda > \lambda_c = \frac{\sqrt{\omega \omega_0}}{2}. \quad (32)$$

Furthermore, the only way for Eq. (31) having a solution for $z > 1$ is that $\tanh\left(\frac{\beta z}{2}\right) > \frac{\omega \omega_0}{4\lambda^2} z$ at $z = 1$; if this condition does not hold, the right side of the equation is larger than the left for any $z > 1$. Therefore, if

$$\beta < 2 \tanh^{-1}\left(\frac{\omega \omega_0}{4\lambda^2}\right), \quad (33)$$

the only solution of the problem is the trivial one $y_0 = 0$. On the contrary, if $\beta$ exceeds this value, there exists a non-trivial solution $\tilde{y}_0 \neq 0$. Evaluating $\Psi(0)$ and $\Psi(\tilde{y}_0)$ we can see that $\Psi(\tilde{y}_0) > \Psi(0)$ in all the cases. Therefore, the position of the maximum $y_0$ changes at the critical temperature

$$\beta_c = 2 \tanh^{-1}\left(\frac{\omega \omega_0}{4\lambda^2}\right), \quad (34)$$

entailing that the partition function becomes non-analytic at the critical temperature $\beta_c$.

Summarizing, if $\lambda < \lambda_c$, there is no thermal phase transition. At $\lambda = \lambda_c$, the phase transition takes place at $\beta \to \infty$, that is, at $T \to 0$; it constitutes a QPT. If $\lambda > \lambda_c$, there exists a thermal phase transition at a critical temperature $T_c = 1/\beta_c$. The values for $\langle E \rangle$ and $\langle J_z \rangle$ in the thermodynamical limit can be easily obtained making use of Eqs. (11) and (12).

**C. Numerical results: different $j$ sectors**

Prior to study the ESQPT and the thermal phase transition, we give a glimpse about the behavior of the different $j$ sectors. In Fig. 5 we plot the results for the sectors $j = 2N/16, 3N/16, \ldots, 8N/16$, with $\omega = \omega_0 = 1, \lambda = 1.5$ and $N = 10^5$. In particular, we deal with six different magnitudes: the density of states, $\rho(E)$; the derivative of the density of states $\rho'(E)$; the third component of the angular momentum, $\langle J_z \rangle$; the derivative of the third component of the angular momentum $\langle dJ_z/dE \rangle$; the temperature, $\beta$; and the first component of the angular momentum, $\langle J_x \rangle$. All this magnitudes are calculated by means of the microcanonical formalism; $\beta$ is the microcanonical temperature

$$\beta = \frac{\partial \log \rho(E)}{\partial E}. \quad (35)$$
FIG. 5. (Color online) Microcanonical calculation for fixed and different values of $j$, for $N = 10^5$. Panel (a), density of states, $\rho(E)$; panel (b), derivative of the density of states $\rho'(E)$; panel (c), third component of the angular momentum, $\langle J_z \rangle$; panel (d), derivative of the third component of the angular momentum $\langle dJ_z/dE \rangle$; panel (e), temperature, $\beta$; panel (f), first component of the angular momentum, $\langle J_x \rangle$. Different colors show different values of $j$, $j = 2N/16, 3N/16, \ldots, 8N/16$.

We can see that the ESQPT occurs at a different energy for each different $j$ sectors. This is clearly seen in panels (b), (d), (e) and (f). The first three cases show logarithmic sin-
gularities associates with the derivatives of the density of states or the third component of the angular momentum \[11\]. It is worth to mention that this singularity is also present in the microcanonical temperature \(\beta\). Also, note that \(\beta\) is not a monotonous function of the energy; this is a clear signature of the anomalous thermodynamic behavior of each \(j\) sector. The fourth panel shows the finite jump of the first component of the angular momentum, provided that the initial state has the parity symmetry broken \[20\].

All these facts give important hints to understand the behavior of the full Hamiltonian, including all the \(j\) sectors. If the system remains thermally isolated and follows a non-trivial time evolution, for example resulting from a time-dependent protocol \(\lambda(t)\), both the total angular momentum, \(J^2\), and the parity, \(\Pi\), are conserved. This entails that the evolution of every \(j - \Pi\) sector is totally independent from the others. The main consequences of this fact are the following: i) every \(j\) sector is affected by its ESQPT, showing the dynamical consequences reported in \[10, 20\]; ii) the behavior of the total system is the sum of all the sector, weighted by the corresponding degeneracies \(g(N, x)\). In the next section we study the link between all these features and the thermal phase transition, well known since more than 40 years ago \[27\].

D. Numerical results: ESQPT versus thermal phase transition

In order to compare the physics of the isolated Dicke model (for which \(J^2\) and \(\Pi\) are conserved quantities) and the Dicke model in contact with a thermal bath (for which \(J^2\) and \(\Pi\) are not conserved), we proceed as follows. On the one hand, we obtain the microcanonical results, depending on the energy \(E\), following the same procedure than in previous section. On the other, the canonical calculation depends on \(\beta\), and the energy is derived from Eq. \[11\]. It predicts a critical temperature, given by Eq. \[31\], and hence we can obtain the corresponding values for the critical energy,

\[
\langle E_c \rangle = - \frac{\partial \log Z}{\partial \beta} \bigg|_{\beta_c},
\]

the critical value of \(J_z\)

\[
\langle J_{z,c} \rangle = \frac{1}{\beta} \frac{\partial \log Z}{\partial \omega_0} \bigg|_{\beta_c},
\]
and the derivative of $J_z$
\[
\langle \frac{dJ_{z,c}}{dE} \rangle = \frac{d}{dE} \beta \left. \frac{1}{\beta} \partial \log Z \right|_{\beta_c} = \beta \left. \frac{1}{\beta} \partial \log Z \frac{\partial \beta}{\partial \omega_0} \right|_{\beta_c}.
\] (38)

With the values of the external parameters used in this work, $\omega = \omega_0 = 1$ and $\lambda = 1.5$, we obtain

\[
\beta_c = 0.223144,
\]
\[
\langle E_c \rangle / N = -0.0551039,
\]
\[
\langle J_{z,c} \rangle = -0.0551039 = \langle E_c \rangle / N.
\] (41)

The derivative of $J_z$ is not defined at the critical temperature $\beta_c$; it jumps from 0 to 1.

![FIG. 6. (Color online) Temperature $\beta$ vs. energy $\langle E \rangle / N$ obtained by means of the microcanonical (solid green line) and the canonical (dashed red line) ensemble. The vertical dashed line shows the critical energy $\langle E_c \rangle / N$. The inset shows the same results around this critical value.](image)

In Fig. 6 we plot the temperature $\beta$ in terms of the energy $\langle E \rangle / N$. We display the microcanonical result by means of a solid (green online) line, and the canonical result by means of a dashed (red online) line. The critical value for the energy is shown by a vertical dashed (blue online) line, and the inset shows a zoom around the critical energy. Microcanonical calculation is done with $N = 10^5$ particles. The canonical calculation is performed in the thermodynamical limit, by means of Laplace’s method. The results are pretty different from the ones obtained with the different $j$ sectors. First, we can see that $\beta$ is a monotonous
function of the energy, as one expects from standard thermodynamics. Second, microcanonical and canonical ensembles give rise to the same results; particularly, both display the same critical behavior. However, we can also see an important difference. When the system is put in contact with a thermal bath, the region with \( \langle E \rangle / N > 0 \) is unreachable. In the canonical formalism, the limit \( T \to \infty (\beta \to 0) \) corresponds with \( \langle E \rangle / N \to 0 \). Hence, if we heat the system by means an external source of heat, we are restricted to the region with \( \langle E \rangle / N < 0 \). On the contrary, if the system remains isolated from any environment, and we heat the system by means of a mechanical procedure, for example performing fast cycles between \( \lambda_i \) and \( \lambda_f \), we can reach any final energy value. Note that \( \langle E \rangle / N = 0 \) acts like a second critical energy, since the curve \( \beta(E) \) shows a singularity at this point.

Another remarkable fact is that the logarithmic singularities shown in panel (e) of Fig. 5 are washed out —despite results shown in Fig. 6 consist of collecting all the \( j \) sectors shown in panel (e) of Fig. 5, weighted by the corresponding degeneracy according to Eq. (18). On the other hand, the second singular point, taking place at \( E_J^*/N = J/N \) in each \( j \) sector, still occurs, at \( E_z/N = 0 \).

![FIG. 7. (Color online) \( j_z \) vs. energy \( \langle E \rangle / N \) obtained by means of the microcanonical (solid green line) and the canonical (dashed red line) ensemble. The vertical dashed line shows the critical energy \( \langle E_c \rangle / N \). The inset shows the same results around this critical value.](image)

Results for the third component of the angular momentum, \( J_z/N \), are shown in Fig. 7. We can see the same kind of non-analiticity at the critical energy \( E_c/N \sim -0.055 \) than for the temperature \( \beta \), despite the behavior for each \( j \) sector, shown in panel (c) of Fig. 5.
is totally different. Furthermore, both microcanonical and canonical calculations give the same results below $E_*/N = 0$. At this value, the microcanonical ensemble shows a second singular point, and $J_z/N = 0$ for $E/N > E_*/N$. It is worth to remark that, despite the consequences of the ESQPT are not so clear for this magnitude, the minimum appearing in each $j$ sector just above the critical energy $E^J_c/N$ is not visible in the figure, giving rise to an approximately flat region $J_z/N \sim -0.055$ for $E < E_c$. However, a zoom around $E_c$ shows that this minimum still exists for finite systems (see below for more details).

FIG. 8. (Color online) Derivative of $J_z$ vs. energy $\langle E \rangle /N$ obtained by means of the microcanonical (solid green line) and the canonical (dashed red line) ensemble. The vertical dashed line shows the critical energy $\langle E_c \rangle /N$. The inset shows the same results around this critical value.

Results for the energy derivative of $J_z$ are shown in Fig. 8. Again, microcanonical and canonical ensembles give the same results, below $E_*/N = 0$. In this case, we can see a finite jump at the critical energy $E_c$; the logarithmic singularities, shown in panel (d) of Fig. 5 are also ruled out.

Finally, results for the first component of the angular momentum, $J_x/N$ are shown in Fig. 9. In this case, we only depict the microcanonical result, since the calculation in the canonical ensemble results in $J_x = 0$. Calculations have been done considering that the parity symmetry is totally broken in the initial state, and therefore the integrals over the phase space are restricted to one of the two disjoint regions for $E^J/N < E^J_c/N$ in each $j$ sector. (If we perform the calculations on the other disjoint region, we obtain the same curve, but with negative values for $J_x/N$). This observable shows a behavior that is qualitatively
FIG. 9. (Color online) $J_x$ vs. energy $\langle E \rangle / N$ obtained by means of the microcanonical ensemble. The vertical dashed line shows the critical energy $\langle E_c \rangle / N$. The inset shows the same results around this critical value.

different than the previous ones. The main signature of the ESQPT is still present, but with a different quantitative behavior. $J_x$ is still an order parameter: it changes from $J_x \neq 0$ for $E < E_c$ to $J_x = 0$ for $E > E_c$. The main difference to what occurs in all the $j$ sectors is that $J_x$ is continuous at the critical energy.

From all these results, we infer the following conclusions:

1. Microcanonical and canonical ensembles are equivalent, below the singular point located at $E_s / N = 0$. This energy constitutes an unreachable limit if the system is put in contact with a thermal bath. It corresponds to $\beta \to 0$ (or $T \to \infty$). On the contrary, there is no such a limit if the system remains isolated.

2. The main signatures of the ESQPT are ruled out when we collect all the $j$ sectors: the logarithmic singularities in the derivatives of $\rho$ and $J_z$ are present when the system is neither isolated (microcanonical calculation) nor in contact with a thermal bath (canonical calculation). As these singularities are linked to stationary points in the corresponding semiclassical phase space, we can conclude that the relevance of such classical structures vanish when all the $j$ sectors are taken into account. A possible explanation is that, in this case, the number of effective degrees of freedom become infinite, since we have an infinite number of $j$ sectors (each one with $f = 2$ degrees of
freedom) in the thermodynamical limit.

3. Notwithstanding, the other main signature of the ESQPT, the fact $J_z$ acts like an order parameter if the initial condition has the parity symmetry broken, survives. In other words, despite the effective number of semiclassical degrees of freedom becomes infinite, the fact that the corresponding phase space is divided in two disjoint regions below the critical energy in each $j$ sector is still relevant for the complete Hamiltonian. Hence, we conclude that both the thermal and the excited-state quantum phase transitions have the same origin.

E. Results: finite size scaling

Results shown in previous sections have been obtained by means of two different procedures: the thermodynamical limit has been explicitly considered for the canonical calculations, whilst microcanonical results are been obtained with a large but finite system. Here we perform a finite-size scaling for the microcanonical calculation, in order to check if it gives rise to the same critical behavior in the thermodynamical limit.

![FIG. 10. (Color online) Finite size scaling for the critical energy, obtained with $J_z$ (left panel) and $J_x$. Both cases are depicted in a double logarithmic scale. The solid lines represent the least-square fits to straight lines, showing a power-law scaling with the system size.](image)

Results for the finite-size precursor of the critical energy $E_c^{(N)}$ are shown in Fig. 10. We plot the difference between this precursor and the critical energy obtained by means the canonical calculation, $E_c^{(N)} - E_c$ versus the size of the system, in a double logarithmic scale.
We also show a straight line representing the power-law behavior \( E_c^{(N)} - E_c \propto N^{-\alpha} \), with \( \alpha(J_z) \approx 0.47 \), and \( \alpha(J_x) \approx 0.41 \). Calculations have been performed as follows. In the left panel, \( E_c^{(N)} \) is estimated as the energy corresponding to the minimum of \( J_z/N \). Though not explicitly shown, this minimum becomes less pronounced as the system-size grows, vanishing in the thermodynamical limit. In the right panel, \( E_c^{(N)} \) is identified as the energy at which \( J_x/N \) becomes less than 0.01. This bound is arbitrary, but we are not interested in quantitative results for each system size \( N \), but in their scaling with the system size. From the results shown in Fig. 10 we can conclude that the finite size precursor \( E_c^{(N)} \) tends to the critical energy \( E_c \), with a power-law finite-size scaling.

![FIG. 11. (color online) Finite size scaling for the critical energy value of \( J_z \), in a double logarithmic scale. The solid lines represent the least-square fits to straight lines, showing a power-law scaling with the system size.](image)

In Fig. 11 we show the same results for the critical value of the third component of the angular momentum, \( J^{(N)}_{z,c} - J_{z,c} \). Though in this case the scaling is not so clean, we still can conclude that \( J^{(N)}_{z,c} - J_{z,c} \propto N^{-\alpha} \), with \( \alpha \approx 0.40 \).

V. CONCLUSIONS

In this work we have analyzed the relationship between the thermal phase transition and the ESQPT in the Dicke model. First of all, we have studied the thermodynamics of the model by means of microcanonical and canonical ensembles, and we have found that both approaches are incompatible if we consider just the maximum spin representation, i.e. \( j = N/2 \). The reason is that the size of the Hilbert space grows linearly with the number of
atoms, $N$, instead of exponentially. The main consequence is that extensive thermodynamic magnitudes, like the entropy $S$ or the Helmholtz potential $F$, do not scale with the number of particles $N$; hence, thermodynamics is anomalous in this model, and the different ensembles are not equivalent in the thermodynamic limit, $N \to \infty$. In order to get a correct description of the thermodynamics properties it is necessary to include all the $j$ sectors.

To perform the microcanonical calculation including all the $j$ sectors, we have considered that all them can be adequately described by means of the semiclassical approximation. In order to check this approximation we have analyzed the scaling behaviour of the largest degeneracy, $j_{\text{max}}$, as well as the ratio between the degeneracy of this sector and the one with $j = 0$. Our findings show that the contribution of lower $j$ sectors is negligible in the thermodynamical limit, so the major contribution come from the sectors with $j$ large enough. This makes possible the employ of the semicclassical approximation in the thermodynamical limit.

We have shown that each $j$ sector is equivalent to the one with $j = N/2$, but with a smaller effective coupling strength. The main consequence is that, despite all of them have an ESQPT if the global coupling strength $\lambda$ is large enough, for any finite value $\lambda > \lambda_c$ there are a large number of $j$ sectors which are in the normal phase. To illustrate this fact, we have computed different magnitudes for different $j$ values: the density of states $\rho(E)$, the derivative of the density of states $\rho'(E)$, the third component of the angular momentum $\langle J_z \rangle$, the derivative of the third component of the angular momentum $\langle dJ_z/dE \rangle$, the temperature, $\beta$ and the first component of the angular momentum, $\langle J_x \rangle$.

We have found that the ESQPT converges into the well known thermal phase transition, when all the $j$ sectors are taken into account. This fact entails that the main signatures of the ESQPT, in particular the logarithmic singularities characteristic of the critical energy $E_c$, are ruled out. However, $\langle J_x \rangle$ still changes from a value different from zero below the critical energy or the critical temperature, to zero above them. This fact shows that the superradiant phase is characterized by the possibility of breaking the parity symmetry; in other words, $J_x$ plays the role of an order parameter, provided that the initial condition has the parity symmetry broken. All this findings are well supported by a finite size scaling. From all these facts, we conclude that the ESQPT and the thermal phase transition are different manifestations of the same phenomenon.

Finally, we have also discussed the main physical differences between the Dicke model in
isolation and in contact with a thermal bath. Despite both the microcanonical and canonical
descriptions mainly coincide in the thermodynamic limit, one important difference remains.
If the system is in contact with a thermal bath, that is, if it is described by means of
the canonical ensemble, the energy $E_*/N = 0$ constitutes an upper bound; this energy
implies $T \to \infty$, and thus cannot be exceeded in any experiment. On the contrary, if the
system remains thermally isolated and is heated by means of the Joule effect, for example by
quenching $\lambda_i \to \lambda_f \to \lambda_i$ repeatedly, the limit $E_*/N = 0$ can be exceeded; in other words,
$E/N > 0$ are accessible in the microcanonical description.

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