Algebraic Extensions of Normed Algebras

Disclaimer: This dissertation does not contain plagiarised material; except where otherwise stated all theorems are the author’s.

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Chapter 1  
Basic Definitions and a Discussion of the Main Problem

1. Introduction

This dissertation is concerned with adjoining roots of polynomials over normed algebras. It was originally intended to investigate the Galois theory of topological algebras but this was quickly found to be too ambitious. Nevertheless we borrow ideas from field theory to form definitions and try to obtain some analogies.

First we show how to adjoin the root of one monic polynomial to a commutative normed algebra with unity, $A$. This construction is then iterated to adjoin roots of arbitrary sets of monic polynomials, and then to obtain an extension closed under taking of roots of monic polynomials (integrally closed).

There is a similarity with the chain of extensions of $\mathbb{Q}$ (say) to include a root of a single irrational number (say a root of $x^2 - 2 \in \mathbb{Q}[x]$), to a field, $\mathbb{R}$, containing the roots of an uncountable set of polynomials irreducible over $\mathbb{Q}$, to an algebraically closed field, $\mathbb{C}$. For normed algebras this can be achieved entirely by a certain type of extension, introduced by Arens and Hoffman in [1]. The simple extension is then iterated using deeper set theory to obtain the ‘standard normed extensions’ of chapter 5 which contain roots of any set of monic polynomials over $A$. Finally transfinite methods are used on these extensions to obtain an integrally closed normed extension. To illustrate the resemblance:

\[
\begin{array}{ccc}
\text{algebraically closed} & C & \text{C} \text{ integrally closed} \\
\uparrow & \circ & \uparrow \\
\mathbb{R} & B & \text{a standard extension} \\
\uparrow & \circ & \uparrow \\
\mathbb{Q}(\sqrt{2}) & A_\lambda & \text{an Arens-Hoffman extension} \\
\uparrow & \circ & \uparrow \\
\mathbb{Q} & A & \\
\end{array}
\]

The context of this work in function analysis is described in chapter 7.

A theme is to determine when various properties (principally completeness and semisimplicity) of the original algebra are preserved by the types of extension discussed.

Along the way we are led to consider, in appendix 2, following suggestions in [1], a Gelfand theory for normed rather than Banach algebras.

An application of Arens-Hoffman extensions not directly connected with algebraic extension is included in chapter 3.

Arens-Hoffman extensions are not the only type of extensions possible. For example in the category of uniform algebras there is a more natural construction available (due to Cole). We attempt to compare the two methods in the case of a simple extension. Finally in chapter 6 we show that Cole’s method can be used to construct integrally closed uniform algebra extensions.

2. Basic Definitions

An algebra is a ring, $A$, which is also a vector space over some field, $F$, such that for all $a, b \in A, \lambda \in F$ we have $\lambda(ab) = (\lambda a)b = a(\lambda b)$. A normed algebra is an algebra over $\mathbb{R}$ or $\mathbb{C}$ with a map $\| \cdot \|: A \to [0, +\infty)$ such that $(A, \| \cdot \|)$ is a normed linear space and for all $a, b \in A \|ab\| \leq \|a\|\|b\|$. If the ring $A$ has a unity, 1, and $\|1\| = 1$ then we call $A$ unital, and we call $A$ commutative if it is commutative as a ring. We assume that the reader is familiar with normed spaces and with the basic facts of Gelfand theory (listed in appendix 1).
Given an algebra, $A$, and an element $a \in A$ there are at least three senses in which $a$ might be called algebraic:

(i) there is a non-zero $f(x) \in \mathbb{F}[x] : f(a) = 0$,

(ii) there is a non-zero $\gamma(x) \in A[x] : \gamma(a) = 0$,

(iii) there is $n \in \mathbb{N}$ and $a_0, \ldots, a_{n-1} \in A : a_0 + \cdots + a_{n-1}a^{-1} + a^n = 0$.

Plainly (i) and (iii) are special cases of (ii) when $A$ has a unity. We shall never mean sense (i). If (ii) holds then $a$ is algebraic over $A$. In (iii), $a$ is said to satisfy an equation of integral dependence (over $A$) or to be integral over $A$. In fact we shall mainly be concerned with integral dependence in this dissertation.

Given $\gamma(x) \in A[x]$ we can ask if there exists $a \in A$ such that $\gamma(a) = 0$. If such an $a$ exists it is called a root or zero of $\gamma(x)$. Sometimes a $\gamma(x)$ will have no zeros in $A$ and we try to extend $A$ so as to include one. To be more precise we make the following definition:

**DEFINITION 1.1:** Let $A$ and $B$ be algebras. $B$ is an extension of $A$ if there exists a monomorphism $\theta: A \to B$. $B$ is a unital extension of $A$ if $A$ has a unity, 1, and $\theta(1)$ is the unity for $B$. If $A$ and $B$ are normed then $B$ is an isometric extension of $A$ if there exists an isometric monomorphism $\theta: A \to B$. We shall say that $B$ is a topological extension if there is a continuous monomorphism $\theta: A \to B$ which is bounded below (meaning that for some $m > 0$ we have $m\|a\| \leq \|\theta(a)\|$ $(a \in A)$). In each case $\theta$ is referred to as an embedding.

We shall sometimes abbreviate ‘$B$ is a unital extension of $A’$ to $B : A$.

The insistence on isometric embedding is explained by Lemma 1.3 and because we only distinguish normed algebras up to isometric isomorphism.

**DEFINITION 1.2:** Let $A$ and $B$ be algebras and $\theta: A \to B$ be an embedding. Let $\gamma(x) = \sum_{k=0}^{p} c_kx^k \in A[x]$ and $b \in B : \theta(\gamma(b)) := \sum_{k=0}^{p} \theta(c_k)b^k = 0$. Then $b$ is a root of $\gamma(x)$.

**LEMMA 1.3** (the embedding lemma [well-known]): Let $A, B$ be normed unital algebras and $\theta: A \to B$ be a unital isometric monomorphism. Then there exists a normed algebra, $C$, and a unital isometric isomorphism $\phi: B \to C$ such that the inclusion map, $\iota: A \to C$ is an isometric embedding and the following diagram is commutative:

$$
\begin{array}{ccc}
B & \xrightarrow{\phi} & C \\
\uparrow{\theta} & & \uparrow{\iota} \\
A & & \\
\end{array}
$$

**Proof.** Omitted. (An exercise in set theory.)

Clearly there are many variants on this.

### 3. The Main Problem

Our first problem is, given an algebra, $A$, and a set of polynomials, $P \subseteq A[x]$, to find an extension algebra, $B$, in the same category as $A$ and such that $\forall \alpha(x) \in P \exists b \in B : \alpha(b) = 0$. Here we are identifying $\alpha(x)$ with its image under the map $A[x] \to B[x]$ induced by the implicit embedding $\theta: A \to B$. Usually we shall make the embedding explicit and write $\theta(\alpha)(x)$ for this.

Our second problem is to discuss which properties of the original algebra are preserved in $B$. This may cause us to attempt to construct new types of extension algebra. As hinted above we shall only consider unital algebras and then only attempt to adjoin roots of monic polynomials. We now provide some justification for this.

**Complexification and Unitisation**

If $A$ is a real normed algebra then it is known (see [2] p. 69) that there exists a complex normed algebra, $A_C$, and an isometric monomorphism $\theta: A \to A_C$ (regarding $A_C$ as an algebra over $\mathbb{R}$). Similarly, if $A$ has no
identity we can embed $A$ into a unital normed algebra, $A + F$ ([2] p. 15). Therefore we could take extensions of algebra made complex and unital in which our set of polynomials have roots.

We are of course neglecting questions such as how the final algebra might behave under changing the order of the composite embedding or how the different methods of complexification or unitisation behave with respect to different types of algebraic extension, but these would distract us too much.

The reader may also wonder why we do not assume that $A$ is complete, in view of the availability of a standard completion to embed into. The main reason is that we would lose the interesting results of chapter 3.

**Commutativity**

This is not so easily swept aside except to say that the whole problem seems much more natural in the commutative context. We can, if the coefficients of all polynomials in $P$ commute, consider extensions of the (maximal or otherwise) commutative subalgebra of $A$ generated by these and the unity, but it seems difficult treat the problem of extending the whole algebra suitably.

**Restriction to Monic Polynomials**

First we make the trivial observation that this is really equivalent to the assumption that all polynomials to be considered have an invertible leading coefficient. The reason why we do not consider adjoining roots of polynomials with arbitrary leading coefficients is that a general method would tackle the problem of adjoining inverses to $A$. (I.e. to adjoin an inverse of $a \in A$ we would look to the unital extension generated by $ax - 1$.) We refer the reader to [3] p. 256 for an article on this subject, which is really another problem. (It was proved by Šilov that if $a \in A$, a commutative unital Banach algebra, then there exists a unital extension, $B$, containing an inverse for $a$ if and only if $a$ is not a topological zero divisor. So algebraic extension by some polynomials is not possible. A much more detailed discussion containing some results on this problem can be found in [4]).

Throughout the rest of this dissertation $A$ will denote a complex, commutative, unital normed algebra. All extensions and homomorphisms will be unital unless otherwise stated.
Chapter 2
Arens-Hoffman Extensions

Let $R$ be a commutative ring with unity and $\alpha(x) \in R[x]$ be monic. Let $(\alpha(x))$ denote the principal ideal in $R[x]$ generated by $\alpha(x)$. It is well-known that $\alpha(x)$ has a root in the factor ring $B := R[x] / (\alpha(x))$ and that $\nu': R \to B: a \mapsto (\alpha(x)) + a$ defines an embedding. The simplicity of this construction is both appealing and practical. In the case when $R = A$ is a normed algebra we hope that there exists a norm on $B$ so that $B : A$ is an isometric extension.

The purpose of this section is relate the work of Arens and Hoffman in [1] where it is shown that not only can this be done but that the resulting extensions have good properties. For example if $A$ is complete then so is $B$.

A second concern is whether $B$ is semisimple† if $A$ is a semisimple Banach algebra. Arens and Hoffman generalise the notion of semisimplicity to normed algebras and prove their results for these. They make the following definition

DEFINITION 2.1: Let $A$ be a unital normed algebra. $A$ is tractable if the intersection of its closed maximal ideals is $\{0\}$. The motivation for this is explained in appendix 2. Tractability is a stronger condition than semisimplicity and when $A$ is complete they are equivalent since then it is well-known that maximal ideals are automatically closed.

Throughout the rest of this chapter $A$ will denote a commutative unital normed algebra and $\alpha(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$ a polynomial over $A$. $B$ will denote $A[x] / (\alpha(x))$. The set of all continuous characters (see appendix 2) on $A$ will be denoted $\Omega(A)$.

1. The Main Theorem

Arens and Hoffman’s solution is to define a norm on the induced polynomial algebra, $A[x]$, so that $(\alpha(x))$ is closed. We then have a norm for $B$ in the standard way (see for example [5] p. 316):

$$\forall \beta(x) \in A[x] \quad \| (\alpha(x)) + \beta(x) \|_B := \inf_{\gamma(x) \in (\alpha(x))} \| \gamma(x) + \beta(x) \|.$$  

The reader will remember from the proof of this that the function above still defines a seminorm if $(\alpha(x))$ is not closed. Most of the proof below establishes that a norm can be found for which this seminorm is faithful.

LEMMA 2.2‡: ([6] p. 128) Let $R$ be a commutative ring with unity and

$$f(x), g(x) \in R[x]: f(x) = \sum_{j=0}^{n} a_j x^j, \text{ with } g(x) = \sum_{j=0}^{m} b_j x^j \text{ monic } (b_m = 1).$$

Then there are unique $q(x), r(x) \in R[x]$ such that

$$\deg(r(x)) < \deg(g(x)) \quad \text{and} \quad f(x) = q(x)g(x) + r(x) \quad (\ast)$$

Proof. ([6] p. 128): If $\deg(f(x)) < \deg(g(x))$ then we can take $q(x) = 0, r(x) = f(x)$ so we may assume $n \geq m$.

† See appendix 1.
‡ With the convention $\deg(0) = -\infty.$
If \( n = 0 \) then \( f(x) = a_0, g(x) = b_0 = 1 \neq 0 \). So we have
\[
f(x) = a_0 = a_0 g(x) + 0
\]
and the result holds. Let \( p > 0 \) and suppose the result is true for all \( n < p \). Put \( f_1(x) = f(x) - a_p x^{p-m} g(x) \). Then \( \deg(f_1(x)) < p \) so by hypothesis there are \( q_1(x), r(x) \in R[x] \) (\( \deg(r(x)) < \deg(g(x)) \)):
\[
f_1(x) = q_1(x) g(x) + r(x)
\]
\[
\Rightarrow f(x) - a_p x^{p-m} g(x) = q_1(x) g(x) + r(x)
\]
\[
\Rightarrow f(x) = q_2(x) g(x) + r(x)
\]
where \( q_2(x) = a_p x^{p-m} + q_1(x) \). \( r(x) \) has degree less than \( m \) so the existence result now follows from the principle of mathematical induction.
To show uniqueness, let \( f(x), g(x), q(x), r(x) \in R[x] \) satisfy (*) and \( q_1(x), r_1(x) \in R[x] \) also having \( \deg(r_1(x)) < \deg(g(x)) \) and \( f(x) = q_1(x) g(x) + r_1(x) \). Then
\[
(q(x) - q_1(x)) g(x) = r_1(x) - r(x).
\]
So if \( q(x) \neq q_1(x) \) then, since \( g(x) \) is monic, the degree of the right hand side is at least \( m \), a contradiction. So \( q(x) = q_1(x) \) and \( r_1(x) = r(x) \).

**THEOREM 2.3:** There exists a norm on \( B \) such that \( B \) is an isometric extension of \( A \) and such that \( B \) is complete if \( A \) is.

**Proof.** ([1]): Let
\[
t > 0 : \|a_0\| + \|a_1\| t + \cdots + \|a_{n-1}\| t^{n-1} \leq t^n
\]
and set for \( \gamma(x) = \sum_{k=0}^{p} c_k x^k \in A[x] , \)
\[
\| \gamma(x) \| := \sum_{k=0}^{p} \| c_k \| t^k
\]
It is easy to check that this defines a norm on \( A[x] : \)
Let \( \gamma(x), \delta(x) \in A[x] : \gamma(x) = \sum_{k=0}^{p} c_k x^k, \delta(x) = \sum_{k=0}^{q} d_k x^k . \) Let \( r = \max(p,q) \) and \( c_k = d_l = 0 \ \forall k > p, l > q . \) Then
\[
\| \gamma(x) + \delta(x) \| = \sum_{k=0}^{r} \| c_k + d_k \| t^k \leq \sum_{k=0}^{r} (\| c_k \| + \| d_k \|) t^k = \sum_{k=0}^{r} \| c_k \| t^k + \sum_{k=0}^{r} \| d_k \| t^k
\]
\[
= \| \gamma(x) \| + \| \delta(x) \| .
\]

\( A[x] \) is an algebra so we need to check that \( \| \cdot \| \) is also submultiplicative:
\[
\| \gamma(x) \delta(x) \| = \left| \sum_{k=0}^{p+q} \left( \sum_{j=0}^{k} c_j d_{k-j} \right) x^k \right|
\]
\[
= \sum_{k=0}^{p+q} \left( \sum_{j=0}^{k} c_j d_{k-j} \right) t^k
\]
\[
\leq \sum_{k=0}^{p+q} \sum_{j=0}^{k} \| c_j \| \| d_{k-j} \| t^j t^{k-j}
\]
\[
= \| \gamma(x) \| \| \delta(x) \| .
\]
Let \( \lambda \in \mathbb{C} \). Then
\[
\| \lambda \gamma(x) \| = \left| \sum_{k=0}^{p} \lambda c_k x^k \right| = \sum_{k=0}^{p} |\lambda||c_k||t^k = \sum_{k=0}^{p} |\lambda||c_k||t^k = |\lambda| \sum_{k=0}^{p} ||c_k||t^k = |\lambda| \| \gamma(x) \| .
\]
Finally,
\[\|\gamma(x)\| = 0 \iff \sum_{k=0}^{p} \|c_k\|t^k = 0 \iff \|c_k\| = 0 \quad (k = 0, \ldots, p)\]
\[\iff c_k = 0 \quad (k = 0, \ldots, p)\]
\[\iff \gamma(x) = 0.\]

We form the quotient algebra, \(B\), as usual and define
\[\|(\alpha(x)) + \delta(x)\| = \inf_{\gamma(x) \in A[x]} \|\gamma(x)\alpha(x) + \delta(x)\|.
\]

With the special choice of \(t\), this seminorm is faithful. Suppose \((\alpha(x)) + \delta(x) \in B\). By the lemma, there is a unique \(q(x) \in A[x]\) and \(\beta(x) = \sum_{k=0}^{n-1} b_k x^k \in A[x]: \delta(x) = q(x)\alpha(x) + \beta(x)\).

We show that \(\forall \gamma(x) \in A[x] \|\alpha(x)\gamma(x) + \beta(x)\| \geq \|\beta(x)\|\). It then follows that
\[\|(\alpha(x)) + \delta(x)\| = \inf_{\gamma(x) \in A[x]} \|\gamma(x)\alpha(x) + \delta(x)\| = \inf_{\gamma(x) \in A[x]} \|(\gamma(x) - q(x))\alpha(x) + \delta(x)\|
\]
\[= \inf_{\gamma(x) \in A[x]} \|\gamma(x)\alpha(x) + \beta(x)\|
\]
\[\geq \|\beta(x)\|.
\]

But \(\|(\alpha(x)) + \delta(x)\| = \|(\alpha(x)) + \beta(x)\| \leq \|\beta(x)\|\) so we shall have
\[\|(\alpha(x)) + \delta(x)\| = \|\beta(x)\|.
\]

(In particular, if \(\delta(x) \notin (\alpha(x))\) then \(\|(\alpha(x)) + \delta(x)\| > 0\).) Let \(\gamma(x) \in A[x]\). To make the following writing easier we shall set \(a_n = 1, b_j = a_j = c_l = 0 \quad \forall l > p, j > n, b_n = 0\). Thus

\[\|\alpha(x)\gamma(x) + \beta(x)\| = \left\| \sum_{k=0}^{n-1} b_k x^k + \left( \sum_{k=0}^{n} a_k x^k \right) \left( \sum_{k=0}^{p} c_k x^k \right) \right\|
\]
\[= \left\| \sum_{k=0}^{n-1} b_k x^k + \sum_{j=0}^{n+p} \left( \sum_{k=0}^{j} a_{j-k} c_k \right) x^j \right\|
\]
\[= \sum_{j=0}^{n+p} \left\| b_j + \sum_{k=0}^{j} a_{j-k} c_k \right\| t^j
\]
\[= \|b_0 + a_0 c_0\| + \|b_1 + a_1 c_0 + a_0 c_1\| + \cdots + \|b_{n-1} + a_{n-1} c_0 + \cdots + a_0 c_{n-1}\| t^{n-1}
\]
\[+ \|c_0 + a_0 c_{n-1} + \cdots + a_0 c_n\| t^n + \|c_1 + a_1 c_{n-1} + \cdots + a_0 c_{n+1}\| t^{n+1}
\]
\[+ \cdots + \|c_p\| t^{n+p}
\]
\[\geq (\|b_0\| - \|a_0\| \cdot \|c_0\|) + (\|b_1\| - \|a_1\| \cdot \|c_0\| - \|a_0\| \cdot \|c_1\|)
\]
\[+ \cdots + (\|b_{n-1}\| - \|a_{n-1}\| \cdot \|c_0\| - \cdots - \|a_0\| \cdot \|c_{n-1}\|) t^{n-1}
\]
\[+ (\|c_0\| - \|a_{n-1}\| \cdot \|c_1\| - \cdots - \|a_0\| \cdot \|c_n\|) t^n
\]
\[+ (\|c_1\| - \|a_{n-1}\| \cdot \|c_2\| - \cdots - \|a_0\| \cdot \|c_{n+1}\|) t^{n+1} + \cdots + \|c_p\| t^{n+p}
\]
\[= \|b_0\| + \|b_1\| t + \cdots + \|b_{n-1}\| t^{n-1}
\]
\[+ \|c_0\| (t^n - \|a_{n-1}\| t^{n-1} - \cdots - \|a_1\| t - \|a_0\|)
\]
\[+ \cdots + t^j \|c_j\| (t^n - \|a_{n-1}\| t^{n-1} - \cdots - \|a_0\|) \geq 0
\]

We see that terms after \(\|\beta(x)\| = \|b_0\| + \|b_1\| t + \cdots + \|b_{n-1}\| t^{n-1}\) are of the form
\[t^j \|c_j\| (t^n - \|a_{n-1}\| t^{n-1} - \cdots - \|a_0\|) \geq 0
\]
(for some \( l \in \{0, \ldots, n\} \)). So this really does define a norm, \( \| \cdot \|_B \), on \( A[x] / (\alpha(x)) \). In particular the natural epimorphism \( \nu : A[x] \to A[x] / (\alpha(x)) \) is continuous so
\[
\nu^{-1}(\{0\}) = \{ \delta(x) \in A[x] : \| (\alpha(x)) + \delta(x) \| = 0 \} = (\alpha(x))
\]
is closed. The unital embedding
\[
\nu : A \to A[x] / (\alpha(x)); a \mapsto (\alpha(x)) + ax^0
\]
is an isometry since the unique polynomial of degree less than \( n \) which is equal to \( ax^0 \) modulo \( (\alpha(x)) \) is \( ax^0 \) so that \( \| (\alpha(x)) + ax^0 \| = \| ax^0 \|_{A[x]} = |a| \).

It remains to check that \( B \) is complete if \( A \) is. Let \( \left( (\alpha(x)) + \gamma^{(m)}(x) \right)_{m \in \mathbb{N}} \) be a Cauchy sequence in \( B \). By the lemma, we may assume that \( \deg(\gamma^{(m)}(x)) < n \) \( \forall m \in \mathbb{N} \). So let
\[
\gamma^{(m)}(x) = \sum_{k=0}^{n-1} c_k^{(m)} x^k \quad \forall m \in \mathbb{N}.
\]
Then
\[
\left\| \left( (\alpha(x)) + \gamma^{(l)}(x) \right) - \left( (\alpha(x)) + \gamma^{(m)}(x) \right) \right\| = \left\| (\alpha(x)) + \left( \gamma^{(l)}(x) - \gamma^{(m)}(x) \right) \right\|
\]
\[
= \left\| \gamma^{(l)}(x) - \gamma^{(m)}(x) \right\|
\]
\[
= \sum_{k=0}^{n-1} \| c_k^{(l)} - c_k^{(m)} \| x^k
\]
\[
\to 0 \quad \text{as } l, m \to +\infty.
\]
\[
\Rightarrow \left( c_k^{(m)} \right)_{m \in \mathbb{N}} \text{ is Cauchy (} k = 0, \ldots, n - 1 \right)\
\Rightarrow \sum_{k=0}^{n-1} c_k \in A : \lim_{m \to +\infty} c_k^{(m)} = c_k \quad (k = 0, \ldots, n - 1)
\]
So we clearly have \( \lim_{m \to \infty} \gamma^{(m)}(x) = \sum_{k=0}^{n-1} c_k x^k =: \gamma(x) \) whence
\[
(\alpha(x)) + \gamma^{(m)}(x) \to (\alpha(x)) + \gamma(x) \quad \text{as } n \to +\infty \text{ in } B.
\]
Having chosen a specific value of \( t \) for the extension we shall refer to it as the parameter of the extension. The theorem shows that a continuum of parameters is possible. Although we shall not investigate whether or not the Arens-Hoffman norm topology is the only norm topology on \( B \) which extends that on \( A \) we stop to prove a related remark from [7] to the effect that there is only one topological solution specified by the family of Arens-Hoffman norms.

**PROPOSITION 2.4 ([7])**: All the norms constructed for \( B \) (indexed by \( t > 0 \) where \( t^n \geq \| a_0 \| + \| a_1 \| t + \cdots + \| a_{n-1} \| t^{n-1} \)) are equivalent.

**Proof.** Let \( t_j > 0 \) (\( j = 1, 2 \)) with \( t_j^n \geq \| a_0 \| + \| a_1 \| t_j + \cdots + \| a_{n-1} \| t_j^{n-1} \) (\( j = 1, 2 \)) and denote the corresponding norms on \( B \) by \( \| \cdot \|_{j,B} \) (\( j = 1, 2 \)). Let \( t_1 \leq t_2 \) without loss of generality. It is enough to show that \( \exists K_1, K_2 > 0 : \forall \beta(x) \in A[x] \quad (\deg(\beta(x)) < n = \deg(\alpha(x))) \)
\[
K_1 \| (\alpha(x)) + \beta(x) \|_{2,B} \leq \| (\alpha(x)) + \beta(x) \|_{1,B} \leq K_2 \| (\alpha(x)) + \beta(x) \|_{2,B}
\]
Let \( \beta(x) = \sum_{k=0}^{n-1} b_k x^k \). Then
\[
\| (\alpha(x)) + \beta(x) \|_{1,B} = \sum_{k=0}^{n-1} \| b_k \| t_k \leq \sum_{k=0}^{n-1} \| b_k \| t_k = \| (\alpha(x)) + \beta(x) \|_{2,B}
\]
so we can take \( K_2 = 1 \). Now let \( r = t_1 / t_2 \) and set \( K_1 = \min(r^0, r, \ldots, r^{n-1}) \). Then \( K_1 > 0 \) and
\[
\| (\alpha(x)) + \beta(x) \|_{1,B} = \sum_{k=0}^{n-1} \| b_k \| t_k r^k \geq K_1 \sum_{k=0}^{n-1} \| b_k \| t_k = K_1 \| (\alpha(x)) + \beta(x) \|_{2,B}.
\]
2. Maximal Ideal Spaces

We now turn to the remaining results in [1]. These concern the spaces of continuous characters on \(A\) and its Arens-Hoffman extension. We recommend that the reader looks at appendix two for the definition and motivation for studying these spaces. In particular it is explained there why we are studying the space of closed maximal ideals rather than the maximal ideals. (The title of the section is therefore a misnomer, but intended to make the reader think along the right lines with respect to the application of these results.)

The content of the following lemma is not actually needed but it introduces some useful notation.

**LEMMA 2.5 ([1]):** Let \(\alpha(x)\) be a non-constant monic polynomial over \(A\) and let \(A[x]\) be given one of the norms above (an ‘Arens-Hoffman’ norm) with parameter \(t\). Then, letting \(\Delta_t\) denote the closed disc centre 0 of radius \(t\) in \(C\), we have (up to a homeomorphism) \(\Omega(A[x]) = \Omega(A) \times \Delta_t\). Also: \(A[x]\) is tractable if and only if \(A\) is tractable.

**Proof.** (We fill in some of the detail behind the identification made in [1].) Set \(\Upsilon: \Omega(A) \times \Delta_t \rightarrow \Omega(A[x])\); \((h, \lambda) \mapsto \Upsilon(h, \lambda)\) where, for \(\gamma(x) = \sum_{k=0}^{p} c_k x^k \in A[x]\),

\[
\Upsilon(h, \lambda)(\gamma(x)) = h(\gamma)(\lambda) := \sum_{k=0}^{p} h(c_k)\lambda^k.
\]

\(\Upsilon\) is well-defined: Let \(\gamma(x) \in A[x]\) \((\gamma(x) = \sum_{k=0}^{p} c_k x^k)\). Then it is routine to verify that \(\Upsilon(h, \lambda)\) is a homomorphism. To show it is continuous:

\[
|\Upsilon(h, \lambda)(\gamma(x))| = \left| \sum_{k=0}^{p} h(c_k)\lambda^k \right|
\leq \sum_{k=0}^{p} |h(c_k)||\lambda|^k
\leq \sum_{k=0}^{p} \|c_k\| t^k = \|\gamma(x)\|.
\]

(We used the fact, which follows from Lemma A2.7, that for characters on Banach algebras, the continuous characters on normed algebras have norms at most 1.)

It is also easily checked that \(\Upsilon\) is a bijection; for \(H \in \Omega(A[x])\) we have \(\Upsilon^{-1}(H) = (H \circ \nu', H(x))\).

Finally we show that \(\Upsilon\) is a homeomorphism when \(\Omega(A) \times \Delta_t\) is given the product topology.

\(A\) is unital and so by A2.6, \(\Omega(A)\) is compact in the weak \(*\)-topology. \(\Delta_t\) is compact by the Heine-Borel theorem and so by Tychonoff’s theorem \(\Omega(A) \times \Delta_t\) is compact. Again from a result in appendix 2, \(\Omega(A[x])\) is Hausdorff. It is therefore sufficient to prove that \(\Upsilon\) is continuous.

Now \(\Omega(A[x])\) has the weak topology induced by \(A[x]^*\) (the Gelfand transform: see appendix 2) so \(\Upsilon\) is continuous if and only if for every \(\gamma(x) \in A[x]\) \ \(\gamma(x)^* \circ \Upsilon\) is continuous.

Let \(\gamma(x) = \sum_{k=0}^{p} c_k x^k \in A[x]\).

\[
\gamma(x)^* \circ \Upsilon: (h, \lambda) \mapsto \gamma(x)^* (\Upsilon(h, \lambda)) = \Upsilon(h, \lambda)(\gamma(x)) = \sum_{k=0}^{p} h(c_k)\lambda^k.
\]

Let \((h_\mu, \lambda_\mu)_{\mu \in M}\) be a net in \(\Omega(A) \times \Delta_t\) with \((h_\mu, \lambda_\mu) \rightarrow (h, \lambda)\). So (by definition of the product topology)

\[
h_\mu \rightarrow h, \quad \lambda_\mu \rightarrow \lambda
\]

\(\Omega(A)\) has the weak topology induced by \(A^*\) so we have that

\[
\forall a \in A \quad \hat{a}(h_\mu) = h_\mu(a) \rightarrow \hat{a}(h) = h(a)
\]
In particular,
\[ h_\mu(c_k) \to h(c_k) \quad (k = 0, \ldots, p), \quad \lambda_\mu \to \lambda \quad \text{in } \Delta_t \subseteq \mathbb{C} \]
\[ \Rightarrow \sum_{k=0}^p h_\mu(c_k)\lambda_k^k = (\gamma(x) \circ \Upsilon(h_\mu, \lambda_\mu)) \to (\gamma(x) \circ \Upsilon(h, \lambda)) \quad \text{in } \mathbb{C}, \text{ by the continuity of addition and multiplication there}. \]
So \( \Upsilon \) is continuous.

It remains to prove that \( A[x] \) is tractable if and only if \( A \) is tractable. Now \( A \) and \( A[x] \) are commutative, unital normed algebras so they are tractable if and only if the Gelfand transforms are injective (Theorem A2-9). First suppose \( A[x] \) is tractable. Let \( a \in A \) with \( \hat{a} = 0 \). Set \( \gamma(x) = a \). Then for each \( H \in \Omega(A[x]) \) \( \exists h_0 \in \Omega(A), \lambda_0 \in \mathbb{C} : H = \Upsilon(h_0, \lambda_0) \) and so \( (\gamma(x)^\gamma)(H) = (\gamma(x)^\gamma)(\Upsilon(h_0, \lambda_0)) = \Upsilon(h_0, \lambda_0)(a) = h_0(a) = \hat{a}(h_0) = 0 \).
whence \( \gamma(x)^\gamma = 0 \).

But \( A[x] \) is tractable so \( \gamma(x) = a\hat{a}^0 = 0 \) so \( a = 0 \). Therefore \( A \) is tractable. Conversely let \( A \) be tractable. Let \( \gamma(x) = \sum_{k=0}^p c_k x^k \in A[x] \) with \( \gamma(x)^\gamma = 0 \). By hypothesis we have that for every \( H \in \Omega(A[x]) \) \( H(\gamma(x)) = 0 \). So \( \forall h \in \Omega(A), \lambda \in \Delta_t \) \( \Upsilon(h, \lambda)(\gamma(x)) = \sum_{k=0}^p h(c_k)\lambda^k = 0 \). By the uniqueness of the Taylor coefficients of the analytic function \( \sum_{k=0}^p h(c_k)\lambda^k \) on \( \Delta_t^p \) we have \( h(c_k) = 0 \) \( k = 0, \ldots, p \). But \( h \) is arbitrary and \( A \) is tractable so \( c_k = 0 \) \( (k = 0, \ldots, p) \). So \( \gamma(x) = 0 \) and \( A[x] \) is tractable. \( \square \)

Our next result of Arens-Hoffman, again quoted directly from [1], is the characterisation of the character space of the extension. We supply a proof for the convenience of the reader. To ease notation we shall write \( \bar{x} \) for the coset \( (\alpha(x)) + x \) from now on.

**THEOREM 2.6:**
\[ \Omega(B) = \{ H \in \Omega(A[x]) : H(\alpha(x)) = 0 \} \]
\[ = \{(h, \lambda) \in \Omega(A) \times \Delta_t : h(\alpha)(\lambda) = \Upsilon(h, \lambda)(\alpha(x)) = 0 \}. \]

*Proof.* We can make a similar identification \( \Pi : \Omega(B) \to \Omega(A) \times \Delta_t \). Let \( H \in \Omega(B) \). Define \( h = H \circ \nu \) (recall that \( \nu \) was the natural homomorphism, \( \nu : A[x] \to B \), restricted to the constants, and is isometric).
So \( h \) is a continuous homomorphism \( A \to \mathbb{C} \). It is non-zero since \( h(1) = H((\alpha(x)) + 1) = H(1_B) \neq 0 \).

Let \( \lambda = H(\bar{x}) \quad (= (H \circ \nu)(x)) \). Then \( \lambda \in \Delta_t \) since
\[ |\lambda| \leq ||\bar{x}||_B \leq ||x||_A[x] = t. \]
So we have \( (h, \lambda) \in \Omega(A) \times \Delta_t \) and moreover
\[ H \circ \nu = \Upsilon(h, \lambda) \quad (\dagger) \]
because if \( \gamma(x) = \sum_{k=0}^p c_k x^k \in A[x] \) then
\[ H \left((\alpha(x)) + \gamma(x)\right) = H(\gamma(\bar{x})) = H \left( \sum_{k=0}^p c_k \bar{x}^k \right) \]
\[ = H \left( \sum_{k=0}^p \nu'(c_k)\nu(x)^k \right) \]
\[ = \sum_{k=0}^p (H \circ \nu')(c_k)(H \circ \nu)(x)^k \]
\[ = \sum_{k=0}^p h(c_k)\lambda^k = \Upsilon(h, \lambda)(\gamma(x)). \]

We now check that \( \Pi : H \mapsto (H \circ \nu', (H \circ \nu)(x)) \) is injective and that
\[ \text{im } \Pi = \{(h, \lambda) \in \Omega(A) \times \Delta_t : \Upsilon(h, \lambda)(\alpha(x)) = 0 \}. \]
\( \Pi \) is obviously injective because it determines the image of \( H \) on the generators (over \( A \)), 1 and \( \bar{x} \), of \( B \).
Let \((h, \lambda) \in \im \Pi; \exists H \in \Omega(B) : (h, \lambda) = (H \circ \nu', (H \circ \nu)(x))\). From (†), we have \(\Upsilon(h, \lambda)(\alpha(x)) = (H \circ \nu)(\alpha(x)) = H(\nu(\alpha(x))) = H(0) = 0\).

For the reverse inclusion, suppose \(\Upsilon(h, \lambda)(\alpha(x)) = 0\) for \((h, \lambda) \in \Omega(A) \times \Delta_t\). Define
\[
H : B \to \mathbb{C}; \ (\alpha(x)) + \gamma(x) \mapsto \Upsilon(h, \lambda)(\gamma(x)).
\]
This is well-defined: let \(\gamma(x), \delta(x) \in A[x]\) with \((\alpha(x)) + \gamma(x) = (\alpha(x)) + \delta(x)\).
\[
\Rightarrow \quad \gamma(x) - \delta(x) \in (\alpha(x))
\Rightarrow \quad \Upsilon(h, \lambda)(\gamma(x) - \delta(x)) = 0
\Rightarrow \quad \Upsilon(h, \lambda)(\delta(x)) = \Upsilon(h, \lambda)(\delta(x))
\]

We check that \(H \in \Omega(B)\) and that \(\Pi(H) = (h, \lambda)\). Let \(\sum_{k=0}^{p} c_k x^k = \gamma(x) \in A[x]\). We shall write \(\gamma(\bar{x})\) for \(\sum_{k=0}^{p} c_k x^k\). Let \(\delta(x) \in A[x], \zeta \in \mathbb{C}\).
\[
H(\delta(\bar{x}) + \zeta \gamma(\bar{x})) = \Upsilon(h, \lambda)(\delta(x) + \zeta \gamma(x)) = \Upsilon(h, \lambda)(\delta(x)) + \zeta \Upsilon(h, \lambda)(\gamma(x)) = H(\delta(x)) + \zeta H(\gamma(x)).
\]
\[
H(\delta(x)\gamma(\bar{x})) = \Upsilon(h, \lambda)(\delta(x)\gamma(x)) = \Upsilon(h, \lambda)(\delta(x)) \Upsilon(h, \lambda)(\gamma(x)) = H(\delta(x))H(\gamma(x)).
\]
Let \(\beta(x)\) be the monic polynomial of degree less than \(n\) for which \(\beta(x) - \gamma(x) \in (\alpha(x))\) \((\Leftrightarrow \beta(\bar{x}) = \gamma(\bar{x}))\).

Then \(\|(\alpha(x)) + \gamma(x)\|_B = \|\beta(x)\|_{A[x]}\) (see Theorem 2.3) so that
\[
|H(\gamma(\bar{x}))| = |H(\beta(\bar{x}))| = |\Upsilon(h, \lambda)(\beta(x))| \leq \|\beta(x)\|_{A[x]} = \|\gamma(\bar{x})\|.
\]
So \(H\) is continuous. Finally \(H(1) = \Upsilon(h, \lambda)(1) = h(1) \neq 0\) so \(H\) is a character of \(B\). Moreover, \(\Pi(H) = (H \circ \nu', (H \circ \nu)(x))\) and:
\[
\forall a \in A \ (H \circ \nu')(a) = H(ax^0) = \Upsilon(h, \lambda)(ax^0) = h(a) \quad \Rightarrow \quad H \circ \nu' = h,
\]
\[
(H \circ \nu)(x) = H(\bar{x}) = \Upsilon(h, \lambda)(x) = \lambda.
\]
Thus \(\Upsilon(h, \lambda) = \Pi(H)\).

Finally we show that \(\Pi\) is a homeomorphism onto a closed subset of \(\Omega(A) \times \Delta_t\). Again it is enough to show \(\Pi\) is continuous since \(\Omega(B)\) is compact and \(\Omega(A) \times \Delta_t\) is Hausdorff by Lemma A2.6. By the definition of the product topology this is equivalent to showing that the maps
\[
\begin{align*}
\Omega(B) \to \Omega(A); \ H &\mapsto H \circ \nu', \\
\Omega(B) \to \Delta_t; \ H &\mapsto H(\bar{x})
\end{align*}
\]
are continuous. The latter is simply \((\bar{x})^\lambda\) with a restricted codomain and continuous by definition of the topology on \(\Omega(B)\). To show that the former is continuous, let \((H_\lambda)_{\lambda \in A}\) be a net in \(\Omega(B)\) with \(\lim_{\lambda} H_\lambda = H \in \Omega(B)\). Let \(a \in A\). Then \(\nu'(\hat{a}) \in \hat{B}\) so
\[
\hat{a}(H_\lambda \circ \nu') = (H_\lambda \circ \nu')(\hat{a}) = \nu'(\hat{a})(H_\lambda) \to \nu'(\hat{a})(H) = (H \circ \nu')(a) = \hat{a}(H \circ \nu).
\]
Since \(a\) was arbitrary \(H_\lambda \circ \nu' \to H \circ \nu'\) in \(\Omega(A)\).

\[
\square
\]

3. Discriminants and Tractability

For the next result we need to define the discriminant of a polynomial over an arbitrary commutative ring. The definition below is not standard but according to Arens and Hoffman ‘agrees with classical usage’.  

10
DEFINITION 2.7: Let \( R \) be ring and \( f(x), g(x) \in R[x] \). The resultant of \( f(x) \) and \( g(x) \) is, for \( f(x) = \sum_{k=0}^{n} a_kx^k, g(x) = \sum_{k=0}^{m} b_kx^k \) \((m, n > 0)\),

\[
\begin{array}{ccccccc}
 a_n & a_{n-1} & \cdots & a_1 & & & \\
 0 & a_n & \cdots & \cdots & & & \\
 \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \\
 b_m & b_{m-1} & \cdots & \cdots & \cdots & \cdots & 0 \\
 0 & b_m & \cdots & \cdots & \cdots & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \cdots \\
 0 & 0 & \cdots & b_m & b_{m-1} & \cdots & b_0 \\
\end{array}
\]

\[
\text{res}(f, g) := a_0^{m-n} a_n^{n} 
\]

It can be shown (see [6] p. 325) that if \( R = F \), a field, \( \text{res}(f, g) = 0 \) if and only if \( a_n = b_m = 0 \) or \( f(x) \) and \( g(x) \) have a non-constant common factor in \( F[x] \). In particular we have the familiar criterion from field theory that \( f \) has a repeated root (in some extension) if and only if \( \text{res}(f, f') = 0 \). We will only apply these results over the field \( \mathbb{C} \). Since the first \( a_n \) can be taken outside the determinant, if \( a_n \neq 0 \) one can apply the criterion for repeated roots to \( a_n^{-1}\text{res}(f, f') \) in fields: \( f \) has a repeated root iff \( a_n^{-1}\text{res}(f, f') = 0 \). This motivates the definition

DEFINITION 2.8: Let \( R \) be ring with unity and \( f(x) \in R[x] \). The discriminant of \( f(x) = \sum_{k=0}^{n} a_kx^k \) is

\[
\begin{array}{ccccccc}
 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & & \\
 0 & a_n & a_{n-1} & \cdots & \cdots & & \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
 n & (n-1)a_{n-1} & \cdots & \cdots & \cdots & \cdots & a_1 \\
 0 & na_n & (n-1)a_{n-1} & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots \\
 0 & 0 & \cdots & \cdots & \cdots & \cdots & a_1 \\
\end{array}
\]

\[
\text{discr}(f) := a_n^{-1}\text{res}(f, f') \text{ when } R \text{ is a field.}
\]

EXAMPLE 2.9: In the commutative ring \( R \) we have, for \( ax^2 + bx + c \in R[x] \),

\[
\text{discr}(ax^2 + bx + c) = \begin{vmatrix} 1 & b & c \\ 2 & b & 0 \\ 0 & 2a & b \end{vmatrix} = b^2 - 2(b^2 - 2ac) = 4ac - b^2,
\]

which is reassuring (though the 1 might not lie in \( R \) here).

The following theorem is taken directly from [1].

THEOREM 2.10: Let \( A \) be tractable and \( d = \text{discr}(\alpha) \) not be a zero divisor or zero. Then \( B \) is tractable.
Proof. See [1].

The following corollary (Arens-Hoffman) is easy to apply and so useful for later examples. We reproduce the proof given in [1] to illustrate the use of discriminants.

**COROLLARY 2.11:** Let \( A \) be tractable and \( \alpha(x) = x^n + a_0 \in A[x] (n \geq 2) \). Then \( B \) is tractable if and only if \( a_0 \) is not a zero divisor or zero.

**Proof.**

\[
\begin{array}{c|c|c}
 & 0 & n-1 \\
\hline
1 & 0 & \\
\vdots & \vdots & \\
0 & \cdots & 1 \\
n & \cdots & 0 \\
\vdots & \vdots & \\
0 & \cdots & n \\
\end{array}
\]

\[d = \text{discr}(\alpha) = \det(( -1)^n a_0)^{n-1} \cdot 
\begin{array}{c|c}
 & 0 \\
\hline
n & \cdots \\
0 & \cdots n \\
\end{array} = n^n a_0^{n-1}.
\]

So if \( a_0 \) is not a zero divisor, \( n^n a_0^{n-1} \) is not a zero divisor and by the theorem above, \( B \) is tractable.

Conversely suppose \( d \) is a zero divisor; there exists \( a \in A - \{0\} : a a_0 = 0 \). We show that \( a \bar{x} \in \cap M(B) \), where \( M(B) \) denotes the set of closed maximal ideal of \( B \), so that \( B \) is not tractable.

Let \( H \in M(B) \). By Theorem 2.6, \( \exists (h, \lambda) \in \Omega(A) \times \Delta_f: \Pi(H) = \Upsilon(h, \lambda) \) and \( \Upsilon(h, \lambda)(\alpha(x)) = 0 \). Therefore \( \Upsilon(h, \lambda)(x^n + a_0) = \lambda^n + h(a_0) = 0 \). Also

\[
H(a \bar{x}) = \Upsilon(h, \lambda)(a \bar{x}) = h(a) \lambda.
\]

Now \( \forall h \in \Omega(A) \) \( h(a a_0) = h(a) h(a_0) = 0 \)

\[\Rightarrow \forall h \in \Omega(A) \quad h(a) \lambda^n = h(a). - h(a_0) = 0. \]

So

\[
\begin{cases}
\lambda = 0 & \Rightarrow H(a \bar{x}) = 0 \\
\lambda \neq 0 & \Rightarrow H(a \bar{x}) = h(a) \lambda = h(a) \lambda^n \lambda^{-(n-1)} = 0
\end{cases}
\]

\( H \) was arbitrary so

\[0 \neq a \bar{x} \in H \in \Omega(B) \cap \ker H = \cap M(B) \]

The last main result of the Arens-Hoffman paper is a result saying that although the Arens-Hoffman extension obtained may not be tractable, it can always be embedded in a tractable normed algebra in which \( \alpha(x) \) has a root.

Before we can state the theorem we again need to quote some algebraic facts. It is known that if \( F \) is a field and \( f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n \in \mathbb{I}[x] \) then the the sums, \( q_k = q_k(a_0, \ldots, a_{n-1}) \), of the \( k \)-th powers of roots of \( f(x) \) \( (k = 0, \ldots, n - 1) \) are polynomials in the coefficients of \( f(x) \) and therefore lie in \( F \) (see [6] Theorem 2.20 p. 139). Given a commutative ring, \( A \), and \( \alpha(x) \in A[x] \) we define \( q_k(a_0, \ldots, a_{n-1}) \)
to be the corresponding polynomial in $a_0, \ldots, a_{n-1}$. Also set $K(B) = \cap M(B)$ (clearly itself a closed ideal of $B$). Then

LEMMA 2.12: Let the parameter $t$ for the Arens-Hoffman norm on $B$ be large enough that $t^k \geq \|q_k\|$ ($k = 0, \ldots, n - 1$). Then the $A$ embeds isometrically in $B/K(B)$, via the natural epimorphism of $B$ onto this factor algebra.

Proof. Let the natural epimorphism be $\nu_B: B \to B/K(B)$. It is shown in [1] that the homomorphism $\nu'' := \nu_B \circ \nu'$ is isometric.

THEOREM 2.13 ([1]): Let $A$ be a tractable normed algebra. Then there is a tractable normed algebra extending $A$ isometrically and in which $\alpha(x)$ has a root. If $A$ is complete (so that tractability is equivalent to semisimplicity) then $B$ is a semisimple Banach algebra.

Proof. By the preceding lemma, $\nu''$ is an isometric monomorphism into the normed algebra $C := B/K(B)$. By Theorem 2.3, $B$ is complete when $A$ is so $C$ will be a Banach algebra when $A$ is. It is obvious that $\alpha(x)$ has a root in $C$; it is $K(B) + \bar{x}$ since

$$\alpha(K(B) + \bar{x}) = K(B) + \alpha(\bar{x}) = K(B)$$

It therefore remains to check that $C$ is always tractable. That a normed algebra factored by its intersection of closed maximal ideals is tractable is true in general and proved in Proposition A2.14.
Chapter 3
Examples and an Application of Arens-Hoffman Extensions

1. Uniform Algebras

Before going on to look at general questions about algebraic extensions we introduce some examples of normed algebras which will make useful illustrations. The material in this section is all standard. An important class is given by the following:

DEFINITION 3.1: Let $X$ be a non-empty compact Hausdorff space and $A$ be a subalgebra of $C(X)$, the set of all continuous maps $X \to \mathbb{C}$. $A$ is a uniform algebra if

(i) $A$ separates the points of $X$ (meaning that $\forall \kappa_1, \kappa_2 \in X \quad \kappa_1 \neq \kappa_2 \Rightarrow \exists f \in A : f(\kappa_1) \neq f(\kappa_2)$),

(ii) the constant map, $1 \in A$,

(iii) $A$ is uniformly closed (i.e. closed with respect to the supremum norm $\|f\|_{\infty} := \sup_{\kappa \in X} |f(\kappa)|$).

A more general definition can be given where $X$ is replaced by a locally compact Hausdorff space (see [8] p. 36) but we shall not need this.

EXAMPLE 3.2: Let $\Delta = \{z \in \mathbb{C} : |z| \leq 1\}$. A standard example of a uniform algebra is the disc algebra, $A_\Delta := \{f \in C(\Delta) : f \text{ is analytic on } \Delta^o\}$. (We use $E^o$ to denote the interior of a subset, $E$, of a topological space.)

An extremely useful object associated to any uniform algebra is defined through

THEOREM 3.3: Let $A$ be a uniform algebra on $X$. Then there exists a unique minimal closed set, $\hat{S}(A) \subseteq X$ with the property that every $f \in A$ attains its supremum on $\hat{S}(A)$.

Proof. We refer the reader to [8] p. 37. \qed

DEFINITION 3.4: The set $\hat{S}(A)$ is called the Šilov boundary of $A$.

EXAMPLE 3.5: For the disc algebra, $\hat{S}(A) \subseteq \partial \Delta$ by the maximum modulus principle. On the other hand, let $\theta \in (-\pi, \pi]$ and set $f(z) = (\frac{1}{2})(1 + e^{-i\theta}z)$. Then $f \in A$ but it is easy to see that $|f(z)| = 1 = \|f\|$ only occurs for $z = e^{i\theta}$. Therefore $\hat{S}(A) = \partial \Delta$.

The next standard result will frequently be referred to.

THEOREM 3.6: Let $A$ be a uniform algebra on $X$ and $\varepsilon: X \to \Omega(A); \kappa \mapsto \varepsilon_\kappa$ where $\forall \kappa \in X \quad \varepsilon_\kappa: A \to \mathbb{C}; f \mapsto f(\kappa)$. Then $\varepsilon$ is a homeomorphism onto a closed subset of $\Omega(A)$.

Proof. The map is well-defined: let $\kappa \in X$.

$$
\begin{align*}
\forall f, g \in A, \lambda \in \mathbb{C} & \quad \varepsilon_\kappa(f + \lambda g) = (f + \lambda g)(\kappa) = f(\kappa) + \lambda g(\kappa) = \varepsilon_\kappa(f) + \lambda \varepsilon_\kappa(g) \\
\varepsilon_\kappa(fg) &= (fg)(\kappa) = f(\kappa)g(\kappa) = \varepsilon_\kappa(f)\varepsilon_\kappa(g) \\
\varepsilon_\kappa(1) &= 1 \neq 0,
\end{align*}
$$

so $\varepsilon_\kappa \in \Omega(A)$.

$\varepsilon$ is continuous: Let $(\kappa_\lambda)_{\lambda \in \Lambda}$ be a net in $X$ with $\kappa_\lambda \to \kappa \in X$. Let $f \in A$. Since $f$ is continuous,

$$
\hat{f}(\varepsilon_{\kappa_\lambda}) = \varepsilon_{\kappa_\lambda}(f) = f(\kappa_\lambda) \to f(\kappa) = \hat{f}(\varepsilon_\kappa).
$$

Since the topology on $\Omega(A)$ is the one induced by $\hat{A}$ this shows that $\varepsilon_{\kappa_\lambda} \to \varepsilon_\kappa$ as required.

Let $\kappa, \kappa' \in X$ and suppose $\varepsilon_\kappa = \varepsilon_{\kappa'}$. This means that

$$
\forall f \in A \quad f(\kappa) = \varepsilon_\kappa(f) = \varepsilon_{\kappa'}(f) = f(\kappa').
$$
Since $A$ separates the points of $X$ this implies that $\kappa = \kappa'$. So $\varepsilon$ is injective. But $\Omega(A)$ is Hausdorff (see A1.4) and $X$ is compact so the result follows.

We therefore have that each $f \in A\hat{f}$ (which is defined on $\Omega(A) \xrightarrow{\varepsilon} X$) extends $f$ in the sense that $\hat{f} \circ \varepsilon = f$.

**DEFINITION 3.7:** A uniform algebra on $X$ is *natural* (on $X$) if the above map, $\varepsilon$, is surjective.

**EXAMPLE 3.8:** The disc algebra, $A_\Delta$, is natural. Let $\omega \in \Omega(A_\Delta)$. Set $v = \omega(z)$ where $z = \text{id}_\Delta$. Then $\omega = \varepsilon_v$. To prove this we quote without proof the standard result that $A_\Delta = P(\Delta)$, the uniform closure of the algebra $\text{Pol}_C(\Delta)$, the polynomial maps $\Delta \to \mathbb{C}$ with complex coefficients. So for all $f \in A \exists (p_n) \subseteq \text{Pol}_C(\Delta)$ : $p_n \to f$ ($n \to +\infty$). Now $\omega$ is continuous so $\omega(f) = \lim_{n \to +\infty} \omega(p_n)$.

But for $p(z) = \sum_{k=0}^{m} a_k z^k \in \text{Pol}_C(\Delta)$ we have

$$
\omega(p(z)) = \sum_{k=0}^{m} a_k \omega(z)^k = p(\omega(z)).
$$

Thus

$$
\omega(f) = \lim_{n \to +\infty} p_n(\omega(z)) = f(v) = \varepsilon_v(f).
$$

Finally one last standard property of uniform algebras we shall need:

**PROPOSITION 3.9:** Let $A$ be a uniform algebra on $X$. Then $A$ is semisimple.*

*Proof. By Theorem 3.6, $X \xrightarrow{\varepsilon} \Omega(A)$ so by A1.3,

$$
\bigcap_{\kappa \in X} \ker \varepsilon_{\kappa} = \{ f \in A : \forall \kappa \in X \varepsilon_{\kappa}(f) = f(\kappa) = 0 \} = \{0\}.
$$



2. Completions of Normed Algebras and Tractability

The novel condition of tractability featured heavily in the Arens-Hoffman paper but does not seem to be commonly used in the literature. One might at first think that this is because completing† a tractable normed algebra gives a semisimple Banach algebra. But we shall show (drawing on results in chapter 1) that this is not always true. This is our first application of Arens-Hoffman extensions to questions other than of algebraic extension. This work is due to Lindberg.

There are also lots of other questions which spring to mind. Dr. Feinstein was immediately led to ask whether the operations of forming an Arens-Hoffman extension of a normed algebra and taking its completion commute. This is stated in [9] for quadratic Arens-Hoffman extensions; we now prove the general case which has apparently not been discussed elsewhere. The first part of Lemma 3.10 is a purely algebraic result.

**LEMMA 3.10:** Arens-Hoffman extensions satisfy the following universal property. Suppose $\theta: A_2 \to B_2$ is a unital algebra extension and that $A_1$ is another unital algebra. Let $\alpha_1(x) \in A_1[x]$ be monic and $B_1 = A_1[x]/(\alpha_1(x))$. Let $\phi_0$ be a homomorphism $A_1 \to A_2$ and $y \in B_2$ be a root of $\alpha_2 := \phi_0(\alpha)(x)$. Then there is a unique homomorphism $\phi: B_1 \to B_2$ such that

$$
\begin{array}{ccc}
B_1 & \xrightarrow{\phi} & B_2 \\
\uparrow{\phi_0} & & \uparrow{\theta} \\
A_1 & \xrightarrow{\phi_0} & A_2
\end{array}
$$

* See appendix 1.
† See appendix 2.

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is commutative and such that \( \phi(\bar{x}) = y \).

If \( B_1: A_1, B_2: A_2 \) are normed and unital with \( B_1 \) having an Arens-Hoffman norm then \( \phi \) is continuous if \( \theta, \phi_0 \) are continuous.

Proof. Let \( n = \deg(\alpha_1(x)) \). Recall that every coset of \( B_1 \) has a unique representative of degree less than \( n \). For \( \beta(x) = \sum_{k=0}^{n-1} b_k x^k \in A_1[x] \) define

\[
\phi(\beta(\bar{x})) := \sum_{k=0}^{n-1} (\theta \circ \phi_0)(b_k)y^k
\]

\[
= \theta'(\beta)(y)
\]

where \( \theta' := \theta \circ \phi_0 \). Thus \( \phi \circ \nu' = \theta \circ \phi_0 \), and in particular \( \phi \) maps the unity of \( B_1 \) to the unity of \( B_2 \).

Let \( \beta(x) = \sum_{k=0}^{n-1} b_k x^k \in A_1[x] \), \( \gamma(x) = \sum_{k=0}^{n-1} c_k x^k \in A_1[x] \), and \( \lambda \in \mathbb{C} \). We have

\[
\phi(\beta(\bar{x}) + \lambda \gamma(\bar{x})) = \phi \left( \sum_{k=0}^{n-1} (c_k + \lambda d_k) \bar{x}^k \right)
\]

\[
= \sum_{k=0}^{n-1} \theta'(c_k + \lambda d_k)y^k
\]

\[
= \sum_{k=0}^{n-1} (\theta'(c_k) + \lambda \theta'(d_k))y^k
\]

\[
= \phi(\beta(\bar{x})) + \lambda(\phi(\gamma(\bar{x}))
\]

To show that \( \theta \) is multiplicative remember from Lemma 2.1 that there is a unique \( r(x) \in A_1[x] : \deg(r(x)) < n \) and \( \exists q(x) \in A_1[x] : \)

\[
r(x) = \beta(x)\gamma(x) + q(x)\alpha_1(x)
\]

(\dagger)

Applying the natural epimorphism, \( A_1[x] \to B_1 \), to (\dagger) we obtain \( r(\bar{x}) = \beta(\bar{x})\gamma(\bar{x}) \). Thus \( \phi(\beta(\bar{x})\gamma(\bar{x})) = \phi(r(\bar{x})) = \theta'(r)(y) \).

On the other hand \( \theta' \) induces a homomorphism of the polynomial algebras:

\[
\theta'_\# : A_1[x] \to B_2[x] ; \delta(x) \mapsto \theta'(\delta(x)).
\]

Applying \( \theta'_\# \) to (\dagger) gives \( \theta'(r)(x) = \theta'(\beta(x))\theta'(\gamma(x)) + \theta'(q(x))\theta'(\alpha_1(x)) \). Following this by the evaluation-at-\( y \) homomorphism gives \( \theta'(r)(y) = \theta'(\beta(y))\theta'(\gamma(y)) = \phi(\beta(\bar{x})), \phi(\gamma(\bar{x})) \) as required, since we assumed \( \theta(\phi_0(\alpha_1))(y) = 0 \).

Uniqueness of \( \phi \) is obvious since \( 1, \bar{x} \) generate \( B_1 \) over \( A \).

If \( A_1, A_2, B_2 \) are normed and \( \theta, \phi_0 \) are continuous (bounded), let \( t > 0 \) be a suitable parameter for the Arens-Hoffman norm on \( B_1 \). By assumption \( \theta \) is also continuous so \( \theta' \) is continuous. For arbitrary \( \beta(x) = \sum_{k=0}^{n-1} b_k x^k \in A_1[x] \) we have

\[
\|\phi(\beta(\bar{x}))\| = \left\| \sum_{k=0}^{n-1} \theta'(b_k)y^k \right\|
\]

\[
\leq \sum_{k=0}^{n-1} \|\theta'(b_k)\|\|y\|^k
\]

\[
\leq M \sum_{k=0}^{n-1} \|b_k\|t^k
\]

\[
= M \|\beta(\bar{x})\|,
\]

where \( M = \|\theta'\|\max_{k=0,\ldots,n-1} \left( \frac{\|\phi(\bar{x})\|}{t} \right)^k > 0 \). So \( \phi \) is continuous. \( \square \)
COROLLARY 3.11: Let $A_1, A_2$ be commutative unital algebras and $\phi_0$ be an algebra homomorphism $A_1 \to A_2$. Let $\alpha_i(x) \in A_1[x]$ be monic, $\alpha_2(x) = \phi_0(\alpha_1(x))$, and $\nu_i'$ be the canonical embeddings $A_i \to B_i := A_i[x]/(\alpha_i(x)) : a \mapsto (\alpha_i(x)) + a$ $(i = 1, 2)^*$. Then there is a unique algebra homomorphism $\phi : B_1 \to B_2$ with $\phi(\bar{x}) = \bar{x}$ and such that the following diagram commutes:

$$
\begin{array}{ccc}
B_1 & \xrightarrow{\phi} & B_2 \\
\uparrow{\nu_1'} & & \uparrow{\nu_2'} \\
A_1 & \xrightarrow{\phi_0} & A_2
\end{array}
$$

If $A_1, A_2$ are normed (unital), $B_1, B_2$ being given Arens-Hoffman type norms, and if $\phi_0$ is continuous, then $\phi$ is continuous.

LEMMA 3.12: Let $A_i$ $(i = 1, 2)$ be commutative unital normed algebras and $\phi_0$ be an isometric monomorphism $A_1 \to A_2$. Let $\alpha_i(x) \in A_1[x]$ be a monic polynomial, $\alpha_2(x) = \phi_0(\alpha_1(x))$. Then we can form the Arens-Hoffman extensions $B_i = A_i[x]/(\alpha_i(x))$ $(i = 1, 2)$ of $A_i$ $(i = 1, 2)$ using the same norm parameter. Moreover if $\phi$ is the unique homomorphism extending $\phi_0$ in the above sense and sending $\bar{x} \mapsto \bar{x}$ then $\phi$ is an isometry and has dense image if $\phi_0$ does.

Proof. Let $t > 0$ satisfy $t^n \geq \sum_{k=0}^{n-1} \|a_k\|^k$ where $\alpha(x) = a_0 + \cdots + a_{n-1}x^{n-1} + x^n$. Now $\alpha_2(x) = \phi_0(\alpha_0) + \cdots + \phi_0(a_{n-1})x^{n-1} + x^n$ and $\sum_{k=0}^{n-1} \|\phi_0(a_k)\|^k = \sum_{k=0}^{n-1} \|a_k\|^k \leq t^n$ so the same parameter for the norm of $B_2$ can be used. In the same way, $\phi$ must be an isometry: $\forall \beta(x) = \sum_{k=0}^{n-1} b_kx^k \in A_1[x]$ we have

$$
\|\phi(\bar{x})\| = \left\|\sum_{k=0}^{n-1} \phi_0(b_k)\bar{x}^k\right\| = \sum_{k=0}^{n-1} \|\phi_0(b_k)\|^k = \sum_{k=0}^{n-1} \|b_k\|^k = \|\beta(\bar{x})\|.
$$

Now suppose $\text{im} \phi_0$ is dense in $A_2$. Let $\varepsilon > 0$, and $\sum_{k=0}^{n-1} c_k' \bar{x}^k \in B_2$. Let $M = \max_{k=0, \ldots, n-1}(t^k)$. By assumption, $\exists c_0, \ldots, c_{n-1} \in A_1 : \|\phi_0(c_k) - c_k'\| < \frac{\varepsilon}{nM}$ $(k = 0, \ldots, n-1)$. Thus

$$
\left\|\phi\left(\sum_{k=0}^{n-1} c_k\bar{x}^k\right) - \sum_{k=0}^{n-1} c_k'\bar{x}^k\right\| = \left\|\sum_{k=0}^{n-1} (\phi_0(c_k) - c_k')\bar{x}^k\right\|
\leq \sum_{k=0}^{n-1} \|\phi_0(c_k) - c_k'\|^k t^k
< \sum_{k=0}^{n-1} \left(\frac{\varepsilon}{nM}\right)^k t^k
\leq \varepsilon,
$$

which shows that $\text{im} \phi$ is dense in $B_2$.

We are now ready to prove:

THEOREM 3.13: Let $A_\alpha$ be an Arens-Hoffman extension of $A$ where $\alpha(x)$ is a monic polynomial over the commutative unital normed algebra $A$ as usual. Then with the same Arens-Hoffman norm parameter, $t$, we can form the Arens-Hoffman extension of the completion, $A^\sim$, of $A$. Let this be denoted $(A^\sim)_\alpha$. Moreover $(A^\sim)_\alpha = (A_\alpha)^\sim$.

Proof. Let $(A^\sim, i)$ be a realisation of the completion of $A$. By the lemma above we can form the Arens-Hoffman extension $(A^\sim)_\alpha := A^\sim[x]/(i(\alpha(x)))$ using the same value of $t$ for the norm function. Moreover, by Lemma 3.12, $i$ extends to an isometric monomorphism $A_\alpha \to (A^\sim)_\alpha$ with dense image. The result follows from the uniqueness of completions.

Having established Theorem 3.12 it makes sense to consider which implications hold between the following statements:

(a) $A$ is tractable

* Strictly we should use different symbols for the inderminates of the two polynomial algebras.
Corollary 2.11, \( \bar{A} \) is not semisimple. From this definition we read off that \( f \) is semisimple. The same observation shows that \( A \) is semisimple. It will be proved in chapter 5 (Corollary 5.8, though it is obvious from Theorem 2.6, or see [7] for a stronger result) that this implies that \( A \) is semisimple. The key result used here is Corollary 2.11. The idea of the proof (due to Lindberg who gives a general class of counterexamples in [9]) is to choose \( A \) and \( \alpha(x) = x^2 - a_0 \in A[x] \) so that on completion \( A \) remains tractable but \( a_0 \) becomes a zero divisor so that \( (A^-)_\alpha \) is not tractable (which is equivalent to semisimplicity in Banach algebras).

Let \( A_\Delta \) denote the disc algebra. Set \( K = \{ \frac{1}{2}, 1 \}, J = K \cup \{ 0 \} \). \( A_\Delta \) is tractable by Proposition 3.9. Define \( f: J \to \mathbb{C} \) by \( f(w) = \begin{cases} \frac{1}{w} & w \neq 0 \\ 0 & w = 0 \end{cases} \) when \( J \) is continuous on the compactum, \( J \), so by the Stone-Weierstrass theorem, there exists a sequence, \( \{ f_n \} \), of (real) polynomials which converges uniformly to \( f \) on \( J \). Consider the subalgebra \( A := A_\Delta | J := \{ g|J : g \in A_\Delta \} \). Its closure, \( \bar{A} \), in \( C(J) \) is a realisation of \( \bar{A} \).

For \( w \in J \) we have \( \varepsilon_w \in \Omega(\bar{A}) \). As in Proposition 3.9 we have \( \cap M(\bar{A}) \subseteq \bigcap_{w \in J} \ker \varepsilon_w = \{ 0 \} \) so that \( \bar{A} \) is semisimple. The same observation shows that \( A \) is tractable. Now let \( f_0 \) be the inclusion map \( J \to \mathbb{C} \). We have \( f_n f_0 \to \bar{1} \) uniformly on \( K \), \( 0 \) uniformly on \( \{ 0 \} \).

So \( f_n f_0 \to \bar{1} \) (n \to +\infty) where \( \bar{1} \) is the characteristic function of \( K \) (in \( J \)). We have \( \varepsilon_1 := 1 - \varepsilon_0 \in \bar{A} \). From this definition we read off that \( \varepsilon_1 f_0 = 0_J \). Since \( \varepsilon_1, f_0 \neq 0 \), \( f_0 \) is a zero divisor in \( \bar{A} \). Hence by Corollary 2.11, \( \bar{A}[x]/(x^2 - f_0) \) is not tractable. This means (since it is complete by Theorem 2.3) that it is not semisimple.

But by Theorem 3.13 we have \( \bar{A}[x]/(x^2 - f_0) \) is not semisimple.

The completion of a tractable normed algebra need not be semisimple. In the above example, \( A[x]/(x^2 - f_0) \) is tractable because \( f_0 \) is not a zero divisor in \( A \). (To see this suppose \( f_0 g_0 = 0 \) for \( g_0 \in A \). Let \( g_0 = g|J \) where \( g \in A_\Delta \). We have \( (zg)|J = 0 \) whence \( zg = 0 \) by the identity principle. So \( g \) is zero on \( \Delta - \{ 0 \} \) and therefore everywhere (by continuity).)

Answers to these can be found from results in [9] and [7]. We first indicate why \( (b) \Rightarrow (a) \).

**Theorem 3.14:** Let \( \alpha(x) \) be a monic polynomial over \( A \) and suppose \( (A_\alpha)^- \) is semisimple. Then \( A \) is tractable.

**Proof.** By Theorem 3.13 we have that \( (A^-)_\alpha \) is semisimple. It will be proved in chapter 5 (Corollary 5.8, though it is obvious from Theorem 2.6, or see [7] for a stronger result) that this implies that \( A^- \) is semisimple. It now follows from Proposition A2.6 that \( A \) is tractable.

Finally we give an example to show \( (a) \not\Rightarrow (b) \).

**Theorem 3.15 ([9]):** There exists a tractable normed algebra, \( A \), and a monic polynomial, \( \alpha(x) \in A[x] \) such that \( (A_\alpha)^- \) is not semisimple.

**Proof.** The key result used here is Corollary 2.11. The idea of the proof (due to Lindberg who gives a general class of counterexamples in [9]) is to choose \( A \) and \( \alpha(x) = x^2 - a_0 \in A[x] \) so that on completion \( A \) remains tractable but \( a_0 \) becomes a zero divisor so that \( (A^-)_\alpha \) is not tractable (which is equivalent to semisimplicity in Banach algebras). We first indicate why \( (b) \Rightarrow (a) \).
Chapter 4
Cole’s Construction

1. Algebraic Extension of Uniform Algebras

We describe a construction for adjoining roots of monic polynomials to a uniform algebra. It was formulated by Cole in [10] for square roots but generalises easily to arbitrary monic polynomials. The generalisation is very slight and appears to be known to Lindberg (it is implicit in [11]) although the details do not appear in the literature. It is interesting for us because it shows we do not have to leave the category of uniform algebras to adjoin the roots whereas Arens-Hoffman extensions only guarantee the existence of commutative Banach algebras with the desired properties.

Perhaps more immediately striking is that this method allows for roots of arbitrary sets of monic polynomials to be added all at once.

Throughout the rest of this chapter $A$ will denote a uniform algebra on the compact Hausdorff space, $X$. $U$ will be a fixed, non-empty set of monic polynomials over $A$. When using $\alpha(x)$ as an index we will drop the indeterminate. Let each $\alpha(x) \in U$ be given by $\alpha(x) = f_0^{(\alpha)} + \cdots + f_{n(\alpha)-1}^{(\alpha)} x^{n(\alpha)-1} + x^{n(\alpha)}$ for $(n(\alpha))_{\alpha \in U} \subseteq \mathbb{N}$ and $f_0^{(\alpha)}, \ldots, f_{n(\alpha)-1}^{(\alpha)} \in A$.

THEOREM 4.1: There exists a uniform algebra, $A^U$, and an isometric monomorphism, $\pi^*: A \rightarrow A^U$ such that $\forall \alpha(x) \in U \exists f \in A^U : \pi^*(\alpha)(f) = 0$.

Proof. (Cf. [10].) Let $Y = X \times \mathbb{C}^U$ and give $Y$ the product topology. Let

$$\pi: Y \rightarrow X$$
$$p_\alpha: Y \rightarrow \mathbb{C}; (\kappa, \lambda) \mapsto \lambda_\alpha$$

be the canonical projections. Let $\pi^*$ be the induced homomorphism $A \rightarrow C(Y); f \mapsto f \circ \pi$. Since $\pi$ is surjective $\pi^*$ is isometric. Set

$$X^U := \left\{ y \in Y : \forall \alpha \in U \left( \pi^*(\alpha)(p_\alpha) \right)(y) = 0 \right\}$$

$$= \left\{ y \in Y : \forall \alpha \in U \left( \pi^*(f_0^{(\alpha)}) + \cdots + \pi^*(f_{n(\alpha)-1}^{(\alpha)}) p_\alpha^{n(\alpha)-1} + p_\alpha^{n(\alpha)} \right)(y) = 0 \right\}$$

$$= \left\{ (\kappa, \lambda) \in Y : \forall \alpha \in U \ f_0^{(\alpha)}(\kappa) + \cdots + f_{n(\alpha)-1}^{(\alpha)}(\kappa) \lambda_\alpha^{n(\alpha)-1} + \lambda_\alpha^{n(\alpha)} = 0 \right\}.$$ 

By the fundamental theorem of algebra (and the axiom of choice) $X^U$ is non-empty; for every $\kappa \in X$ we can choose $\lambda_\alpha \in \mathbb{C}: f_0^{(\alpha)}(\kappa) + \cdots + f_{n(\alpha)-1}^{(\alpha)}(\kappa) \lambda_\alpha^{n(\alpha)-1} + \lambda_\alpha^{n(\alpha)} = 0$ ($\alpha \in U$). $X^U$ is Hausdorff because it is a subspace of a Hausdorff space (products of Hausdorff spaces are Hausdorff).

To show $X^U$ is compact we require a lemma. Suppose $c_0, \ldots, c_{m-1} \in \mathbb{C}, \lambda \in \mathbb{C}, |\lambda| \geq 1, c_0 + \cdots + c_{m-1} \lambda^{m-1} + \lambda^m = 0$. By assumption, $-\lambda^m = c_0 + \cdots + c_{m-1} \lambda^{m-1}$.

Hence $|\lambda|^m = |c_0 + \cdots + c_{m-1} \lambda^{m-1}|$.

From which $|\lambda|^m \leq |c_0| + \cdots + |c_{m-1}| |\lambda|^{m-1} \leq |c_0| |\lambda|^{m-1} + \cdots + |c_{m-1}| |\lambda|^{m-1}$ (since $|\lambda| \geq 1$).

And so $|\lambda| \leq |c_0| + \cdots + |c_{m-1}|$.

Hence $\forall (\kappa, \lambda) \in X^U \forall \alpha \in U \ |\lambda_\alpha| \leq 1$ or $|\lambda_\alpha| \leq \sum_{k=0}^{n(\alpha)-1} |f_k^{(\alpha)}(\kappa)| \leq \sum_{k=0}^{n(\alpha)-1} \|f_k^{(\alpha)}\|$. Hence

$$X^U \subseteq X \times \prod_{\alpha \in U} \Delta_{\max(1, \|f_0^{(\alpha)}\| + \cdots + \|f_{n(\alpha)-1}^{(\alpha)}\|)}.$$
which is compact by Tychonoff’s theorem. Now by definition, \( X^U \) is the intersection of a family of zero sets of continuous functions and so closed. It follows that \( X^U \) is compact.

From now on we shall now write \( \pi^* \), \( p_\alpha \) for their restrictions to \( X^U \).

Now define \( A^U \) to be the closed subalgebra of \( C(X^U) \) generated by \( \pi^*(A) \cup \{ p_\alpha : \alpha \in U \} \). So \( \pi^* \) is an embedding into this uniformly closed subalgebra of \( C(X^U) \). Clearly \( p_\alpha \) is a root of \( \alpha(x) \) for each \( \alpha(x) \in U \). So to complete the proof it remains to check that \( A^U \) separates the points of \( X^U \) but this is obvious because \( A \) separates the points of \( X \).

We include one result on the general properties of such extensions. It is explained in [10], at least for square roots, that the operations of passing to the maximal ideal space of a uniform algebra and forming a Cole type of extension commute. We prove a special case here.

**THEOREM 4.2:** Let \( A \) be natural. Then \( A^U \) is natural on \( X^U \).

Proof. (cf. [8] p. 195) Let the canonical embeddings \( X \to \Omega(A) \), \( X^U \to \Omega(A^U) \) be denoted by \( \varepsilon, \eta \) respectively. We must show that \( \eta \) is surjective, given that \( \varepsilon \) is. Let \( \varphi \in \Omega(A^U) \). Consider \( \omega := \varphi \circ \pi^* : A \to \mathbb{C} \). It is a unital homomorphism since it is a composition of them; in other words \( \omega \in \Omega(A) \). By hypothesis \( \exists \kappa \in X : \omega = \varepsilon_\kappa \). Thus

\[
\forall \kappa \in X \quad \varphi(\pi^*(\kappa)) = h(\kappa)
\]

Now \( \forall \alpha \in U \pi^*(f_0^{(\alpha)}) + \cdots + \pi^*(f_{n(\alpha)-1}^{(\alpha)})p_\alpha^{n(\alpha)-1} + p_\alpha = 0 \), so applying \( \varphi \) to this equation gives

\[
f_0^{(\alpha)}(\kappa) + \cdots + f_{n(\alpha)-1}^{(\alpha)}(\kappa)\varphi(p_\alpha)^{n(\alpha)-1} + \varphi(p_\alpha)^{n(\alpha)} = 0
\]

Hence \( y := (\kappa, \lambda) \in X^U \) where \( \lambda = (\varphi(p_\alpha))_{\alpha \in U} \).

Let \( g \in A^U \). There is a sequence of polynomials with complex coefficients in the generating elements, \( \pi^*(A) \cup \{ p_\alpha : \alpha \in U \} \), with \( g_n \to g \ (n \to +\infty) \). Explicitly let \( g_n = G_n(\pi^*(h_1), \ldots, \pi^*(h_{l(n)}), p_{\alpha_1}, \ldots, p_{\alpha(l(n)}) \) for a non-decreasing sequence \( (l(n))_{n \in \mathbb{N}} \subseteq \mathbb{N} \) and \( (h_n) \subseteq A, (\alpha_n) \subseteq U \). Then

\[
\varphi(g) = \varphi \left( \lim_{n \to +\infty} g_n \right)
\]

\[
= \lim_{n \to +\infty} \varphi(g_n)
\]

\[
= \lim_{n \to +\infty} G_n(h_1(\kappa), \ldots, h_{l(n)}(\kappa), \varphi(p_{\alpha_1}), \ldots, \varphi(p_{\alpha_{l(n)}}))
\]

\[
= \lim_{n \to +\infty} G_n(\pi^*(h_1)(y), \ldots, \pi^*(h_{l(n)})(y), p_{\alpha_1}(y), \ldots, p_{\alpha_{l(n)}}(y))
\]

\[
= \lim_{n \to +\infty} g_n(y) = g(y).
\]

So \( \varphi = \eta g \). \( \square \)

2. Comparison of Extensions

We would now like to compare the two methods of extending a uniform algebra so as to include a root of the polynomial \( \alpha(x) \in A[x] \). To do this systematically we modify the appropriate definitions from field theory.

**DEFINITION 4.3:** Let \( A_1, A_2, B_1, B_2 \) be commutative algebras with unities and \( \theta_k : A_k \to B_k \) be unital monomorphisms \((k = 1, 2)\). Let \( \phi_k : A_1 \to A_2 \) be a unital isomorphism. Then \( B_1 : A_2 \) and \( B_2 : A_2 \) are **isomorphic extensions** if there exists a unital isomorphism \( \phi : B_1 \to B_2 \) making the following diagram commute:

\[
\begin{array}{ccc}
B_1 & \xrightarrow{\phi} & B_2 \\
\uparrow{\theta_1} & & \uparrow{\theta_2} \\
A_1 & \xrightarrow{\phi_0} & A_2
\end{array}
\]
If \( A_1 = A_2 = A \) and \( \phi_0 = \text{id}_A \) then we shall say that \( B_1, B_2 \) are isomorphic over \( A \) and write \( B_1 \cong_A B_2 \).

If \( A_1, A_2, B_1, B_2 \) are normed unital algebras and \( \theta_1, \theta_2 \) are continuous and \( \phi_0, \phi \) are homeomorphisms, \( B_1: A_1, B_2: A_2 \) will be called topologically isomorphic extensions. If all maps are isometries then \( B_1: A_1, B_2: A_2 \) will be said to be isometrically isomorphic extensions.

All the remaining results in this chapter are apparently new.

**EXAMPLE 4.4:** Let \( A_\Delta \) be the disc algebra and \( \alpha(x) = x^2 - z \) where \( z = \text{id}_\Delta \). Let \( A' \) denote the extension generated by Cole’s method; \( A' \) is the closed subalgebra of \( C(X') \) generated by \( \pi^* (A_\Delta) \cup \{ p_0 \} \) in the notation from above where \( X' = \{ (u, v) \in \Delta \times C : v^2 = u \} = \{ (v^2, v) : v \in \Delta \} \).

Let \( A_\alpha \) denote the corresponding Arens-Hoffman extension. Lemma 3.10 shows that

\[
\phi : A_\alpha \to A' ; \quad f + g \bar{x} \mapsto \pi^* (f) + \pi^* (g) p_\alpha
\]

is a homomorphism whose image is clearly a dense subspace of \( A' \). It will follow very easily from a result in the next section that \( \phi \) is a topological isomorphism \( A_\alpha \to A' \) over \( A_\Delta \) (the hard part of this is to show that \( \phi \) is surjective).

We could therefore renorm \( A_\alpha \) so as to make \( \phi \) an isometry. But the resulting norm will not be an Arens-Hoffman norm for any choice of norm parameter \( t > 0 \).

To see this, suppose \( t > 0 \) is such that \( \| f + g \bar{x} \| = \| f \| + \| g \| t \) defines a norm on \( A_\alpha \) so that \( \phi \) is an isometry. We must have

\[
t = \| \bar{x} \| = \| \phi(\bar{x}) \| = \| p_\alpha \| = \sup_{(v^2, v) : v \in \Delta} | p_\alpha (v^2, v) | = \sup_{v \in \Delta} | v | = 1.
\]

But now let \( f = (\frac{1}{2})(z + 1), g = (\frac{1}{2})(z - 1) \). We must have

\[
\| f + g \bar{x} \| = \| \pi^* (f) + \pi^* (g) p_\alpha \|
\]

\[
= \sup_{v \in \Delta} \left| (\pi^* (f) + \pi^* (g) p_\alpha) (v^2, v) \right|
\]

\[
= \sup_{v \in \Delta} \left| f (v^2) + v g (v^2) \right|
\]

\[
= \left| f (e^{2i\theta}) + e^{i\theta} g (e^{2i\theta}) \right|
\]

for some \( \theta \in (-\pi, \pi) \) by the maximum modulus principle. Thus

\[
\| f + g \bar{x} \|\leq \left| f (e^{2i\theta}) \right| + \left| g (e^{2i\theta}) \right| = \left( 1/2 \right) \left[ | e^{2i\theta} + 1 | + | e^{2i\theta} - 1 | \right]
\]

Now \( \begin{cases} | e^{2i\theta} + 1 | < 2 & \text{unless } \theta = 0, \pi, \text{ and} \\ | e^{2i\theta} - 1 | < 2 & \text{unless } \theta = \pm \frac{\pi}{4}, \end{cases} \)

so \( \dagger \) shows that \( 2 = \| f \| + \| g \| = \| f + g \bar{x} \| < (2 + 2)/2 = 2 \), a contradiction.

\[ \square \]

3. The Cole-Feinstein Conjecture

In this section \( A \) will be a uniform algebra on the compactum \( X \) (\( \langle A, X \rangle \) is a uniform algebra’). We shall write \( \langle A^\alpha, X^\alpha \rangle \) for the simple Cole extension \( \langle A^{\{\alpha\}}, X^{\{\alpha\}} \rangle \). The corresponding Arens-Hoffman extension, \( A[x]/(\alpha(x)) \), will be denoted \( A_\alpha \) or \( B \) as usual.

It is easy to give examples where \( A^\alpha \neq A_\alpha \). For instance, \( \tilde{A} \) of Theorem 3.15 is obviously a uniform algebra on \( J \) but there was a monic polynomial over \( \tilde{A} \) for which the Arens-Hoffman extension was not semisimple so it could not be isomorphic to a Cole type of extension because those are semisimple (Proposition 3.9).
More generally, by Corollary 2.11, \( A[x]/(x^2 - f_0) \) cannot be isomorphic to a uniform algebra if \( f_0 \in A \) is a zero divisor or zero.

However Theorem 2.13 states that factoring the Arens-Hoffman extension by its radical, \( K(B) \), leads to a semisimple isometric extension of \( A \). (Remember that uniform algebras are complete so \( B \) is always complete and so semisimplicity is equivalent to tractability then.) The proper question to ask is therefore whether or not we always have \( B/K(B) =: C = A^\alpha \). B. J. Cole and J. F. Feinstein conjectured that this is true but we shall show that there are counterexamples.

Recall that there is a continuous injection \( \varepsilon: X \to \Omega(A) \) and that (modulo II)

\[
\Omega(B) = \{(h, \lambda) \in \Omega(A) \times \Delta_t: \Upsilon(h, \lambda)(\alpha(x)) = 0\}
\]

\[
= \{(h, \lambda) \in \Omega(A) \times \Delta_t: \sum_{k=0}^{n-1} h(a_k)\lambda^k = 0\},
\]

where \( t > 0 \) is the Arens-Hoffman norm parameter. We exploit the strong resemblance of this with

\[
X^\alpha = \{ (\kappa, \lambda) \in X \times \mathbb{C}: a_0(\kappa) + \cdots + a_{n-1}(\kappa)\lambda^{n-1} + \lambda^n = 0 \}
\]

Now \( t \) only had to satisfy

\[
t^n \geq \sum_{k=0}^{n-1} \|a_k\|^k
\]

so we can assume \( t \geq \max(1, \|a_0\| + \cdots + \|a_{n-1}\|) \). Therefore by the lemma in the proof of Theorem 4.1 we have that for every \( (\kappa, \lambda) \in X^\alpha | |\lambda| \leq t \). We summarise:

**Lemma 4.5:** \( \rho: X^\alpha \to \Omega(B); (\kappa, \lambda) \mapsto \Pi^{-1}(\varepsilon_\kappa, \lambda) \) is a homeomorphism onto a closed subspace of \( \Omega(B) \).

**Proof.** Since \( \Pi \) is a homeomorphism it is enough to check that the map \( (\kappa, \lambda) \mapsto (\varepsilon_\kappa, \lambda) \) is a homeomorphism onto a closed subspace of \( \Omega(A) \times \Delta_t \). (The map is well defined by the comments above.) The codomain is Hausdorff and \( X^\alpha \) is compact so we only need to check injectivity and continuity. Since \( \varepsilon \) is continuous and injective the map composed with each of the canonical projections onto \( \Omega(A), \Delta_t \) has these properties and the result follows.

\( \rho \) induces a map \( \rho^*: C(\Omega(B)) \to C(X^\alpha); h \mapsto h \circ \rho \). This map is clearly a (unital) homomorphism. We first show that it restricts to a map \( \bar{B} \to A^\alpha \) where \( \bar{B} \subseteq C(\Omega(B)) \) is the image of the Gelfand transform, \( \Gamma: B \to B \). By the fundamental isomorphism theorem, \( \Gamma \) induces an isomorphism \( \bar{\Gamma}: B/K(B) \to B \) of norm \( \|\bar{\Gamma}\| \leq 1 \).

**Lemma 4.6:** \( \rho^*(\bar{B}) \subseteq A^\alpha \).

**Proof.** A typical element of \( B \) is \( b = \beta(\bar{x}) \) where \( \beta(x) = \sum_{k=0}^{n-1} b_k x^k \in A[x] \). We must show that \( \rho^*(\bar{b}) = \bar{b} \circ \rho \in A^\alpha \).

Let \( (\kappa, \lambda) \in X^\alpha \). Please refer to Theorem 2.6 for the notation used.

\[
\left( \beta(\bar{x}) \circ \rho \right)(\kappa, \lambda) = \beta(\bar{x}) \left( \Pi^{-1}(\varepsilon_\kappa, \lambda) \right)
\]

\[
= \left( \Pi^{-1}(\varepsilon_\kappa, \lambda) \right) \left( \beta(\bar{x}) \right)
\]

\[
= (\Pi^{-1}(\varepsilon_\kappa, \lambda) \circ \nu) \left( \beta(x) \right)
\]

\[
= \Upsilon(\varepsilon_\kappa, \lambda) \left( \beta(x) \right) \quad \text{(by Theorem 2.6)}
\]

\[
= \sum_{k=0}^{n-1} \varepsilon_\kappa(b_k)\lambda^k
\]

\[
= \sum_{k=0}^{n-1} b_k(\kappa)\lambda^k
\]

\[
= \sum_{k=0}^{n-1} \pi^*(b_k)(\kappa, \lambda)p_{\alpha}(\kappa, \lambda)^k
\]

\[
= \left( \sum_{k=0}^{n-1} \pi^*(b_k)p_{\alpha}^k \right)(\kappa, \lambda)
\]

Thus \( \rho^*(\bar{b}) = \sum_{k=0}^{n-1} \pi^*(b_k)p_{\alpha}^k \in A^\alpha \).
We can therefore consider $\rho^*$ to be a contractive homomorphism $\hat{B} \to A^\alpha$. Moreover since $A^\alpha$ is generated by $\im \tau^* \cup \{p_\alpha\}$, $\rho^*$ has dense image. The last line of the above proof allows us to summarise with a commutative diagram:

\[
\begin{array}{ccc}
B/K(B) & \xrightarrow{\hat{\nu}^*} & \hat{B} \\
& \searrow & \nearrow \\
& \downarrow & \uparrow \\
& A
\end{array}
\]

There is one immediate positive result:

**PROPOSITION 4.7:** Let $A$ be natural on $X$. Then $A^\alpha = \hat{B}^-\rangle$ (isometric isomorphism).

**Proof.** In this case $\varepsilon$ is a homeomorphism so $\rho$ is surjective. Hence for $b \in B$, $\|\rho^*(\hat{b})\| = \|\hat{b} \circ \rho\| = \|\hat{b}\|$ and $\rho^*$ is an isometry. Now the unique continuous linear extension of $\rho^*$ to $\rho^* : \hat{B}^-\rangle \to A^\alpha$ is therefore also isometric. But $\hat{B}^-\rangle, A^\alpha$ are complete and $\im \rho^* = A^\alpha$ so $\rho^*$ is surjective (considered as a map $\hat{B}^-\rangle \to A^\alpha$).

It is therefore desirable to know when $\hat{B}$ is closed in $C(\Omega(B))$ (i.e. complete in the sup-norm). For natural uniform algebras this has been worked out by Heuer and Lindberg in [12]. (First recall

**DEFINITION 4.8:** $a \in A$ is a **topological divisor of zero** if $\exists (u_n) \subseteq A$: $u_n a \to 0$ $(n \to +\infty)$ with $\forall n \in \mathbb{N} \|u_n\| = 1$)

**THEOREM 4.9** ([12] p. 338): Let $A$ be a natural uniform algebra and $B = A_\alpha$ an Arens-Hoffman extension. Let $d := \text{discr}(\alpha(x))$ not be a zero divisor. Then $\hat{B}$ is sup-norm complete if and only if $d$ is not a topological divisor of zero.

We therefore have:

**COROLLARY 4.10:** Let $A$ be natural and $d$ not be a zero divisor. Then $A^\alpha = \hat{B} \iff d$ is not a topological divisor of zero.

To give examples of this situation there is a further equivalent condition available:

**THEOREM 4.11** ([13] p. 137): Let $A$ be a uniform algebra and $d \in A$. Then $d$ is a topological zero divisor if and only if $d$ has a zero on $\hat{S}(A)$.

Before going on, note that the condition on $d$ in Corollary 4.10 is quite strong; if $d$ is not a zero divisor then by Theorem 2.10, $B$ is already semisimple. Therefore we can prove the following

**THEOREM 4.12:** Let $A$ be a natural uniform algebra. Then $B$ and $A^\alpha$ are topologically isomorphic extensions of $A$ if $d$ is not a topological zero divisor.

**Proof.** Suppose $d$ is not a topological divisor of zero. By Theorem 2.10, $K(B) = \{0\}$ so that $B/K(B) = B$ (to isometric isomorphism). By Proposition 4.7 and Theorem 4.9, $A^\alpha = \hat{B}^-\rangle$ and $(\hat{B})^-\rangle = B$. By Banach’s isomorphism theorem, $\hat{\Gamma}$ is a topological isomorphism $B/K(B) \to B$.

Thus for example, for the disc algebra, $A_\Delta$, and $\alpha(x) = x^2 - z$ we have $\text{discr}(\alpha(x)) = 4z$ (see example 2.9) which does not vanish on $\hat{S}(A_\Delta) = \partial \Delta$ (example 3.5) so that Cole’s extension is topologically isomorphic to the Arens-Hoffman extension.

We have not yet found a counterexample to the Cole-Feinstein conjecture although we do now know that if $T$ is a topological isomorphism such that the following commutes

\[
\begin{array}{ccc}
B/K(B) & \xrightarrow{T} & A^\alpha \\
& \hat{\nu}^* & \nearrow \pi^* \\
& A
\end{array}
\]
and $T(K(B) + x) = p_a$ then $T = \rho^* \circ \tilde{\Gamma}$. This forces $\rho^*$ to be both injective and surjective. $\rho^*$ is always a contraction so if $\hat{B}$ is not uniformly closed (i.e. closed in the sup norm) then (when $A$ is natural) we would have a continuous algebraic isomorphism $\hat{B} \to (\hat{B})^-$ which seems strange but not impossible. However for our situation this cannot happen and I am very grateful to Dr. Feinstein for pointing out the following result to me which is an elementary exercise in uniform algebras:

**LEMMA 4.13 (Dr. Feinstein):** Suppose that $(B_1, \| \cdot \|_1)$ is a commutative semisimple complex unital Banach algebra and that $(B_2, \| \cdot \|_\infty)$ is a uniform algebra. Suppose that $T: B_1 \to B_2$ is an algebra isomorphism. Then $B_1$ is topologically isomorphic to its the Gelfand transform. In particular $\hat{B}_1$ is complete in the sup norm.

**Proof.** Since $B_2$ is a uniform algebra we have for each $f \in B_2$ $\|f\|_\infty = \lim_{n \to +\infty} \|f^n\|_\infty^{\frac{1}{n}}$. Now $T$ induces a second norm, $|\cdot|$ on $B_1$ defined by

$$\forall b \in B_1 \quad |b| := \|T(b)\|_\infty = \lim_{n \to +\infty} \|T(b^n)\|_\infty^{\frac{1}{n}}$$

$$= \lim_{n \to +\infty} \|T(b^n)\|_\infty^{\frac{1}{n}}$$

$$= \lim_{n \to +\infty} |b^n|^\frac{1}{n}.$$  

(†)

In this way, $T^{-1}$ is an isometric isomorphism $(B_2, \| \cdot \|_\infty) \to (B_1, |\cdot|)$. Since $B_2$ is complete, $(B_1, |\cdot|)$ is complete. Hence by the (commutative) uniqueness of norm theorem, $|\cdot|$ and $\| \cdot \|_1$ are equivalent norms.

Now $\hat{B}_1$ has the sup norm inherited from $C(\Omega(B_1))$ and by Corollary A1.8 for $b \in B_1$ we have

$$\|b\| = \lim_{n \to +\infty} \|b^n\|_1^{\frac{1}{n}}.$$  

Since $|\cdot|$ and $\| \cdot \|_1$ are equivalent norms it follows easily from (†) that the Gelfand transform is continuous and bounded below as a map from $(B_1, |\cdot|)$ and so also from $(B_1, \| \cdot \|_1)$.

We now come to our main result allowing for counterexamples to the Cole-Feinstein conjecture to be found very easily.

**THEOREM 4.14:** Let $A$ be a natural uniform algebra and $d = \text{discr}(\alpha(x))$ be a topological divisor of zero in $A$, but not zero or a zero divisor. Then there does not exist an (algebraic) isomorphism $B/K(B) \to A^\alpha$.

**Proof.** Suppose $A$ and $d$ satisfy the hypotheses above. Since $d$ is not zero or a zero divisor, $B$ is semisimple by Theorem 2.10. So $K(B) = \{0\}$ and $B$ is isometrically isomorphic to $B/K(B)$ where the latter space has the quotient norm. Suppose for a contradiction that there is an isomorphism $T: B \to A^\alpha$. By Lemma 4.13, $\hat{B}$ must be complete and this contradicts Theorem 4.9.

So for example letting $A = A_\Delta$, the disc algebra, and $f_0 = z - 1, \alpha(x) = x^2 - f_0$ we have that $d = 4(z - 1)$ is not a zero divisor or zero but is a topological zero divisor (by Theorem 4.11). So by the last theorem, $B/K(B)$ and $A^\alpha$ are essentially different here. [By Proposition 4.7 there is however an embedding $B/K(B) \to A^\alpha$.]

We close this chapter by remarking that a condition on $\alpha(x)$ commonly occurring in the literature (‘separability’) on the type of polynomial considered excludes examples like this one.

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Chapter 5
Integral Closure of Banach Algebras

1. Introduction

In field theory it is well-known that algebraic closures exist. These are field extensions which contain roots of all polynomials over themselves. In other words there is no need to find further extensions in order to adjoin roots. In the same way one can obtain extensions which are closed under taking roots of monic polynomials. More precisely:

**DEFINITION 5.1:** Let $A$ be a commutative algebra with identity. $A$ is *integrally closed* if for every monic $\alpha(x) \in A[x]$ $\exists \xi \in A : \alpha(\xi) = 0$. If $B : A$ is an algebra extension and $B$ is integrally closed then $B$ is an *integral closure* of $A$.

Note that we do not ask (here) that every monic polynomial ‘splits’ completely into linear factors in an integral closure; for results on this problem see [7]. Nor do we ask that integral closure be in any sense minimal here.

The main theorem in this section is that every Banach algebra has an integral closure (in the same category). At the moment we can only adjoin roots of finitely many polynomials at a time so our first task is to extend this method. Lindberg has made a study of this in [11] so we are reporting his results.

2. Standard Extensions

Let $U \subseteq A[x]$ be any non-empty set of monic polynomials where $A$ is a commutative algebra with unity. We follow Lindberg by obtaining an algebra extension in which every polynomial in $U$ has a root and then consider the case when $A$ is normed.

**DEFINITION 5.2 ([11]):** Let $B : A$ be an algebra extension and $U \subseteq A[x]$ a set of monic polynomials. $B : A$ is called a *standard extension* if there exists a well-ordering, $\leq$, on $U$ (with $\alpha_0 = \inf U$) and intermediate subalgebras, $(B_\alpha)_{\alpha \in U}$ such that

(i) $B = \bigcup_{\alpha \in U} B_\alpha$
(ii) $\forall \alpha, \beta \in U \ \alpha \leq \beta \Rightarrow B_\beta : B_\alpha$ (with respect to inclusion)
(iii) $\forall \alpha \in U \ B_\alpha \cong_{A_\alpha} A_\alpha[x]/(\alpha(x))$ where

\[
A_\alpha := \begin{cases}
\bigcup_{\beta < \alpha} B_\beta & \alpha_0 < \alpha \\
A & \alpha_0 = \alpha.
\end{cases}
\]

(Warning: we are using $A_\alpha$ to mean something different to in previous chapters.) Recall that the notation in (iii) means that there is an isomorphism $\phi_\alpha : B_\alpha \to A_\alpha[x]/(\alpha(x))$ satisfying $\phi_\alpha(a) = \nu'_\alpha(a)$ ($a \in A_\alpha$) where $\nu'_\alpha$ is the usual embedding $A_\alpha \to A_\alpha[x]/(\alpha(x))$. It is clear that Arens-Hoffman extensions are standard extensions and that the embedding lemma can be used to construct standard extensions from any finite sequence of Arens-Hoffman extensions. We should also warn the reader that the definition in [11] is slightly more general in that it allows the polynomial, $\alpha(x)$, generating each intermediate subalgebra, $B_\alpha$, to have coefficients in $A_\alpha$ rather than $A$.

**THEOREM 5.3 ([11]):** Let $A$ be a commutative algebra with unity and $\emptyset \neq U \subseteq A[x]$ a set of monic polynomials. Then there exists a standard extension, $B_U : A$, in which every $\alpha(x) \in U$ has a root.

Proof. (We follow the outline in [11] and supply extra details for the convenience of the reader.) By the well-ordering principle there exists a well-ordering, $\leq$, on $U$. Let $\alpha_0 = \inf U$. We shall need to refer to some
results from set theory in the course of this proof. These are given in appendix 3 to avoid cluttering this proof, but the reader may wish to glance at it now to check notations and conventions.

First we require a set, \( S \), with \( \# S > \# A \) and \( A \subseteq S \). (For example \( \mathcal{P}(A) \cup A \) could be used for \( S \).) Let \( \mathcal{Q} \) be the set of all maps from initial segments of \( U \) to \( \mathcal{P}(S) \):

\[
\mathcal{Q} := \text{Map}(U, \mathcal{P}(S)) \cup \bigcup_{\alpha \in U} \text{Map}([\alpha_0, \alpha), \mathcal{P}(S)).
\]

(\( U \) is an initial segment by convention.) We now let

\[
M := \left\{ f \in \mathcal{Q} : \begin{align*}
(i) & \forall \alpha \in \text{dom } f \ f(\alpha) : A \text{ w.r.t. inclusion} \\
(ii) & \forall \alpha, \beta \in \text{dom } f \ \alpha \leq \beta \Rightarrow f(\alpha) : f(\beta) \text{ w.r.t. inclusion} \\
(iii) & \forall \beta \in \text{dom } f \ f(\beta) \equiv f(\beta)[x]/(\beta(x)) \text{ where} \\
& \tilde{f}(\beta) := \bigcup_{\alpha < \beta} f(\alpha) \text{ if } \alpha_0 < \beta \text{ and } \tilde{f}(\beta) := A \text{ if } \alpha_0 = \beta
\end{align*} \right\}
\]

Note that (i) implies that \( f(\alpha) \) carries an algebraic structure. Really what we have written is not adequate, but it is consistent with the universal abuse of language when speaking of ‘the algebra \( A' \). An algebra is strictly a 4-tuple, \((A, S, S, S_{\times A})\), where \( S, S \subseteq (A \times A) \times A, S_{\times A} \subseteq (\mathbb{F} \times A) \times A \) define addition, multiplication and the action of the field on \( A \). (If \( A \) is normed there is also a fifth component, \( S_{\parallel \cdot} \subseteq A \times \mathbb{R}_+ \))
determining the norm map.) So it would be more correct to consider a suitable subset of maps, \( f \), into

\[ S \times \mathcal{P}(S \times S \times S) \times \mathcal{P}(S \times S) \times \mathcal{P}(\mathbb{F} \times S \times S), \]

but this level of formality is not appropriate here.

\( M \neq \emptyset \) for the trivial map, \( \emptyset, \) the single element of \( \text{Map}([\alpha_0, \alpha), \mathcal{P}(S)) \), belongs to \( M \). (The reader can be assured that the ‘inductive step’ in the proof below will make it clear that there are non-trivial elements in \( M \).) For \( f, g \in M \) define

\[
f \preceq g \iff \text{dom } f \subseteq \text{dom } g \land g|_{\text{dom } f} = f.
\]

This clearly defines a partial ordering on \( M \). Suppose \( \emptyset \neq L \subseteq M \) is totally ordered. Now \( \bigcup_{f \in L} \text{dom } f =: D \) is an initial segment of \( U \) for if \( D \subset U \) we have \( D = [\alpha_0, \alpha) \) where \( \alpha = \inf(U - D) \). In either case we can define \( g : D \to \mathcal{P}(S) \) by \( \alpha \to f(\alpha) \) where \( f \in L, \alpha \in \text{dom } f \). This is well-defined by the assumption on \( L \) and easily checked to be a member of \( M \). Since \( g \) is an upper bound for \( L \), \((M, \preceq)\) is inductively ordered. By Zorn’s lemma there is a maximal element, \( f \in M \).

Suppose \( \text{dom } f \subset U \). We shall show that this leads to a contradiction and then use \( f \) to define \( B_U \).

Let \( \alpha = \inf(U - \text{dom } f) \) so that \( \text{dom } f = [\alpha_0, \alpha) \). Set

\[
A_\alpha := \begin{cases} 
\bigcup_{\beta < \alpha} f(\beta) & \alpha_0 < \alpha \\
A & \alpha_0 = \alpha.
\end{cases}
\]

Thus \( A_\alpha \) is an algebra extension of \( A \) when it is given the obvious structure (which is well-defined by the conditions in \( M \)).

We claim that \# \( A_\alpha \) = \# \( A \). Now since \( U \subseteq A[x] \) and \# \( A[x] \) = \# \( A \) (Lemma A3.4) it is enough (Lemma A3.3) to show that \( \forall \beta < \alpha_0 \ f(\beta) = \# A \). This we do by transfinite induction.

Let \( J = \{ \beta : \# f(\beta) = \# A \} \) and suppose \( [\alpha_0, \beta) \subseteq J \).

Now \( f(\beta) \equiv f(\beta)[x]/(\beta(x)) \) so \( \# f(\beta) = \# f(\beta)[x]/(\beta(x)) \). Recall from chapter 2 that since \( \tilde{f}(\beta) \) is a commutative ring with unity this factor algebra is identifiable setwise with \( \text{deg}(\beta(x))^{-1} \bigoplus_{k=0} \tilde{f}(\beta) \). This last set has cardinality \( \# \tilde{f}(\beta) \) by Lemma A3.2 (all our algebras will be infinite). But by definition of \( \tilde{f}(\beta) \), the inductive hypothesis, and Lemma A3.3 again, \( \# \tilde{f}(\beta) = \# A \). Therefore \( \beta \in J \) and the claim follows from the principle of transfinite induction.

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Now form $B'_\alpha := A_\alpha[x]/(\alpha(x))$. By the claim and by the same reasoning as for the factor algebra considered just above, $\#B'_\alpha = \#A$. Let $\nu'$ be the usual embedding of $A_\alpha$ into this algebra. We shall need a variant of the embedding lemma to complete the diagram below:

$$
\begin{array}{c}
B'_\alpha \quad \cdots \quad B_\alpha \subseteq S \\
\uparrow \nu' \quad \nearrow \subseteq \\
A_\alpha \\
\uparrow \subseteq \\
A
\end{array}
$$

By assumption, $\#S > \#A = \#A_\alpha$ so $\#(S - A_\alpha) > \#A = \#B'_\alpha \geq \#(B'_\alpha - \im \nu')$. Let $i$ be an injection $B'_\alpha - \im \nu' \to S - A_\alpha$. Define the subset $f_1(\alpha) := \im i \cup A_\alpha$ and let it have the algebraic structure determined by the induced bijection $B'_\alpha \to f_1(\alpha)$. Thus $f_1(\alpha)$ is a commutative algebra unitally extending $A_\alpha$.

Define $f_1(\beta) = f(\beta)$ for $\beta < \alpha$. If $U = [\alpha_0, \alpha]$ then dom $f_1$ is an initial segment of $U$. In the case $U \supset [\alpha_0, \alpha]$ we have dom $f_1 = [\alpha_0, \inf(U - [\alpha_0, \alpha])]$, also an initial segment of $U$. It is routine to check that $f_1$ satisfies (i),(ii),(iii) so that $f_1 \in M$. However we have $f_1 \succ f$, a contradiction.

Therefore dom $f_0 = U$ and we define

$$
B_U := \bigcup_{\alpha \in U} f(\alpha)
$$

In this way $B_U$ is a standard extension of $A$ relative to $(B_\alpha)_{\alpha \in U}$ where $\forall \alpha \in U B_\alpha := f(\alpha)$. We have that $\forall \alpha(x) \in U \alpha(x)$ has a root in $B_\alpha \subseteq B_U$ by condition (iii).

So we have solved the algebraic problem. We next consider the normability of such extensions.

**THEOREM 5.4 ([11]):** Let $(A, \| \cdot \|_A)$ be a commutative unital normed algebra and $B$ a standard extension of $A$ with respect to $(B_\alpha)_{\alpha \in U}$ where $U \neq \emptyset$ is a well-ordered set of monic polynomials over $A$. Let the isomorphisms $B_\alpha \cong A_\alpha A_\alpha[x]/(\alpha(x))$ be $\phi_\alpha: B_\alpha \to A_\alpha[x]/(\alpha(x)) \ (\alpha \in U)$. Then there exists a norm, $\| \cdot \|_B$, on $B$ and values of the Arens-Hoffman norm parameters for the $A_\alpha[x]/(\alpha(x)) \ (\alpha \in U)$ where the norm on $(A_\alpha, \| \cdot \|_A)$ is the restriction of $\| \cdot \|_B$ to $A_\alpha$, such that $\phi_\alpha$ is an isometry for all $\alpha \in U$.

**Proof.** This is an application of the transfinite recursion theorem, which we use informally. By assumption the following commutes for all $\alpha \in U$:

$$
\begin{array}{c}
B_\alpha \xrightarrow{\phi_\alpha} A_\alpha[x]/(\alpha(x)) \\
\uparrow \subseteq \\
\not\exists \nu'_\alpha \\
A_\alpha
\end{array}
$$

where $\nu'_\alpha$ is our usual embedding of an algebra into its Arens-Hoffman extension; it will be an isometry whatever value is chosen for the norm parameter.
Let \( \gamma \in U \) and suppose that for all \( \alpha < \gamma \), \( B_\alpha \) has a norm \( \| \cdot \|_\alpha \) and \( t_\alpha > 0 \) (\( \alpha < \gamma \)) are such that

(i) \( \forall \alpha < \beta < \gamma \) \( \| \cdot \|_\beta \) extends \( \| \cdot \|_\alpha \)

(ii) \( \forall \beta < \gamma \) \( \phi_\beta \) is an isometry when \( A_\beta \) has norm the restriction of \( \| \cdot \|_\beta \) and \( A_\beta[x] / (\beta(x)) \) is given an Arens-Hoffman norm with parameter \( t_\beta \).

Now \( A_\gamma = \bigcup_{\beta < \gamma} B_\beta \) so by assumption (i), \( (B_\beta, \| \cdot \|_\beta)_{\beta < \gamma} \) induce a well-defined norm on \( A_\gamma \). Choosing \( t_\gamma > 0 \) according to the condition in Theorem 2.3 makes \( A_\gamma[x] / (\gamma(x)) \) into an Arens-Hoffman extension when given an Arens-Hoffman norm. We can now define a norm, \( \| \cdot \|_\gamma \), on \( B_\gamma \) by declaring \( \phi_\gamma \) to be an isometry. By the comments in the first paragraph the system \( (B_\beta, \| \cdot \|_\beta)_{\beta < \gamma} \) satisfies (i), (ii) above but with ‘\( \gamma \)’ replaced with ‘\( \leq \gamma \)’. The result now follows.*

\[
\square
\]

3. Integral Closure

We are now in a position to prove the main theorem of this chapter. It is a simplification of a result stated in [11] and the proof is too. Similar methods are used in [14] but this has not been our main source.

**THEOREM 5.5**: Let \((A, \| \cdot \|_A)\) be a commutative unital Banach algebra. Then there is a Banach algebra extension, \((C, \| \cdot \|_C)\), which is integrally closed.

**Proof.** Let \( \omega_1 \) denote the first uncountable ordinal, and 0 the first ordinal. Let, for \( \gamma \in [0, \omega_1] \), \( p(\gamma) \) be the proposition 'There exist Banach algebras, \((C_\gamma, \| \cdot \|_\gamma)\), such that:

(i) \( (C_0, \| \cdot \|_0) = (A, \| \cdot \|_A) \)

(ii) \( \forall \alpha \leq \beta < \gamma \) \( (C_\beta, \| \cdot \|_\beta) \) is a unital Banach algebra extension of \((C_\alpha, \| \cdot \|_\alpha)\) with respect to \( \subseteq \)

(iii) \( \forall \alpha < \beta < \gamma \) \( \forall \) monic \( \mu(x) \in C_\alpha[x] \) \( \exists \xi \in C_\beta : \mu(\xi) = 0. \)

Suppose \( \gamma < \omega_1 \) and that \( p(\gamma) \) holds. We must show that we can choose a Banach algebra, \((C_\gamma, \| \cdot \|_\gamma)\), so that the family \((C_\alpha, \| \cdot \|_\alpha)_{\alpha < \gamma}\) satisfies (i)-(iii) above. This is obvious if \( \gamma = 0 \) so let \( 0 < \gamma \).

Set \( A_\gamma = \bigcup_{\alpha < \gamma} C_\alpha \) and give \( A_\gamma \) the [well-defined] algebraic operations and norm, \( \| \cdot \|_A \), induced by the subalgebras \((C_\alpha, \| \cdot \|_\alpha)_{\alpha < \gamma}\).

By Theorem 5.4 we can form a standard normed extension, \((B_\gamma, \| \cdot \|_B)\) of \((A_\gamma, \| \cdot \|_A)\) in which every monic polynomial over \( A_\gamma \) has a root. Recall that in this construction the unital embedding was given by inclusion. Therefore every monic polynomial over \( C_\alpha \) (\( \alpha < \gamma \)) has a root in \( B_\gamma \).

\( B_\gamma \) may not be complete so we form its completion, \( B_\gamma^* \). By the embedding lemma, there exists a Banach algebra, \((C_\gamma, \| \cdot \|_C)\), containing \( A_\gamma \) as a subalgebra, extending its norm and such that there is an isometric isomorphism \( \phi: B_\gamma^* \rightarrow C_\gamma \) with \( \phi \circ j = \iota \) where \( j \) is the embedding of \( B_\gamma \) in \( B_\gamma^* \) and \( \iota \) is the inclusion map \( B_\gamma \rightarrow C_\gamma \).

Let \( \alpha < \gamma \) and \( \mu(x) \in C_\alpha[x] \) be monic. So \( \mu(x) \) is a monic polynomial over \( A_\gamma \) and so there is \( \xi \in B_\gamma : \mu(\xi) = 0 \). But \( B_\gamma \subseteq C_\gamma \) so \( \mu(\gamma + 1) \) is satisfied by this family of Banach algebras.

Therefore \( p(\omega_1) \) holds; let \((C_\alpha, \| \cdot \|_\alpha)_{\alpha < \omega_1}\) be a family of Banach algebras satisfying \( p(\omega_1) \).

Set \( C = \bigcup_{\alpha < \omega_1} C_\alpha \) and give \( C \) the usual induced algebraic structure and norm. It remains to check that \( C \) is complete and integrally closed.

Let \((c_n) \subseteq C \) be a Cauchy sequence. So \( \exists n \in \mathbb{N} \) \( \alpha_n < \omega_1 : \forall n \in \mathbb{N} \) \( c_n \in C_{\alpha_n} \). As in the proof of Theorem A3.20, \( \exists \alpha < \omega_1 : \forall n \in \mathbb{N} \) \( \alpha_n < \alpha \). So \((c_n) \subseteq C_\alpha \). But \( C_\alpha \) is complete so \( \exists c \in C_\alpha : c_n \rightarrow c \) (\( n \rightarrow +\infty \)). Thus \( C \) is complete.

Finally let \( \mu(x) \) be a monic polynomial over \( C \). It has finitely many coefficients so \( \exists \alpha < \omega_1 : \mu(x) \in C_\alpha[x] \). But then \( \mu(x) \) has a root in \( C_{\alpha + 1} \subseteq C \). So \( C \) is integrally closed.

---

* The reader may be more familiar with this type of informal inductive definition when \( U \) is replaced by an initial segment of some ordinal number. However there is no essential difference in view of result A3.16 of appendix three.
4. Maximal Ideal Spaces

This chapter could really have been called ‘standard extensions’ and though their main application here was to form an integral closure of an arbitrary Banach algebra they generalise Arens-Hoffman extensions. One can ask if the results of chapter two also generalise. The first thing to note is that the method of obtaining a standard normed extension from a set of monic polynomials, \( U \subseteq A[x] \), will not generally give a Banach algebra, even if \( A \) is Banach. The details of exactly when what has been called \( B_U \) is Banach (given \( A \) is complete) have been worked out in [11].

We turn our attention instead to determining the space of continuous characters (so again our section title is a deliberate misnomer) of a standard normed extension and are able to generalise Theorem 2.6 directly. This refines a remark in [11] (for our definitions) that the map induced by taking restrictions \( \Omega(B_U) \to \Omega(A) \) is surjective. (The reasoning is different: we obtain \( \Omega(B_U) \) explicitly whereas Lindberg makes use of the fact (which there is not space to discuss here) that standard extensions are examples of ‘integral extensions’.)

For rest of this chapter \( U \) will be a non-empty set of monic polynomials over \( A \):

\[
\forall \alpha \in U \quad \alpha(x) = a_0^{(\alpha)} + \cdots + a_{n(\alpha)-1}^{(\alpha)} x^{n(\alpha)-1} + x^{n(\alpha)}.
\]

\( \leq \) will be a well-ordering on \( U \) with least element \( a_0^{(\alpha)} \).

THEOREM 5.6: Let \( B := B_U \) be the generalised Arens-Hoffman extension of the normed algebra \( A \) as in Theorem 5.4. Then (up to homeomorphism)

\[
\Omega(B) = \{(h, \lambda) \in \Omega(A) \times C^U : \forall \alpha \in U \quad h(\alpha)(\lambda_{\alpha}) = 0\}
\]

Proof. Let \( P \) denote the set on the right-hand side of this equation.

Let \( t_\alpha > 0 \) be the parameter for the Arens-Hoffman norm on \( A_\alpha[x]/(\alpha(x)) \) (\( \alpha \in U \)). We see from the inductive construction that it is possible to choose \( t_\alpha \geq 1 \), \(|a_0^{(\alpha)}| + \cdots + |a_{n(\alpha)-1}^{(\alpha)}||\forall \alpha \in U\). Let all notation be as in Theorem 5.4, so that \( \phi_\alpha : B_\alpha \to A_\alpha[x]/(\alpha(x)) \) is an isometric isomorphism (\( \alpha \in U \)). Define

\[
\Pi(B) \to P; \quad h \mapsto (h, \lambda)
\]

where \( h = H|_A \) and \( \forall \alpha \in U \lambda_\alpha = H(\phi_\alpha^{-1}(\bar{x})) \). For convenience we shall set \( \xi_\alpha := \phi_\alpha^{-1}(\bar{x}) \) (\( \forall \alpha \in U \)). This is well-defined since \( H|_A \) is a continuous unital homomorphism \( A \to C \) and for \( \alpha \in U \) we have

\[
h(\alpha)(\lambda_\alpha) = h(a_0^{(\alpha)} + \cdots + a_{n(\alpha)-1}^{(\alpha)} x^{n(\alpha)-1} + x^{n(\alpha)})
\]

\[
= H\left(a_0^{(\alpha)} + \cdots + a_{n(\alpha)-1}^{(\alpha)} x^{n(\alpha)-1} + x^{n(\alpha)}\right)
\]

\[
= (H \circ \phi_\alpha^{-1})(a_0^{(\alpha)} + \cdots + a_{n(\alpha)-1}^{(\alpha)} x^{n(\alpha)-1} + x^{n(\alpha)})
\]

An easy induction shows that \( B \) is generated (over \( A \)) by \( \{1\} \cup \{\xi_\alpha : \alpha \in U\}. \) Therefore \( \Pi \) is injective.

\( \Pi \) is surjective: Let \( (h, \lambda) \in P \). We use transfinite recursion (informally) to construct a suitable lift of \( h \) to \( B \).

Let \( \beta \in U \) and suppose that \( (H_\alpha)_{\alpha < \beta} \) is a family of continuous characters (\( \forall \alpha < \beta \quad H_\alpha \in \Omega(B_\alpha) \)):

(i) \( \forall \alpha < \beta \quad H_\alpha|_A = h \),

(ii) \( \forall \alpha < \beta \quad H_\alpha(\xi_\alpha) = \lambda_\alpha \),

(iii) \( \forall \alpha < \gamma < \beta \quad H_\alpha = H_\gamma|_{B_\alpha} \).

* It is not possible that the \( \xi_\alpha \) will not all be distinct so we can use set notation here.
Now define \( h_b : A_\beta \to \mathbb{C} ; b \mapsto H_\alpha(b) \) where \( b \in B_\alpha \) (or if \( \beta = \alpha_0 \) define \( h_\beta = h \)). This is well-defined by (iii) and plainly \( h_\beta \in \Omega(A_\beta) \).

By assumption \( h(a_0^{(\beta)}) + \cdots + h(a_{\alpha(\beta)-1}^{(\beta)})\lambda_\beta^{n(\beta)-1} + \lambda^{n(\beta)} = 0 \). This implies (as in the proof of Theorem 4.1) that
\[
|\lambda_\beta| \leq 1 \text{ or } |\lambda_\beta| = \left| h(a_0^{(\beta)}) \right| + \cdots + \left| h(a_{\alpha(\beta)-1}^{(\beta)}) \right| \leq \left\| a_0^{(\beta)} \right\| + \cdots + \left\| a_{\alpha(\beta)-1}^{(\beta)} \right\|
\]
(since \( \|h\| \leq 1 \) by A2.7). So \( |\lambda_\beta| \leq \beta \).

Therefore \( (h_\beta, \lambda_\beta) \in \Omega(A_\beta) \times \Delta_\beta \) and by Theorem 2.6, there exists \( \omega \in \Omega(A_\beta[x]/(\beta(x))) : (h_\beta, \lambda_\beta) = (\omega \circ \nu_\beta(\omega(x))) \).

Define \( H_\beta = \omega \circ \phi_\beta \). Thus \( H_\beta \in \Omega(B_\beta) \). It is simple to check that \( H_\beta \) is consistent with (i), (ii), (iii):
(i) Let \( a \in A \). \( H_\beta(a) = \omega((\beta(x)) + a) = h_\beta(a) = h(a) \) so \( H_\beta|_A = h \)
(ii) \( H_\beta(\xi_\beta) = \omega(\bar{x}) = \lambda_\beta \)
(iii) Suppose \( \alpha < \beta \). Let \( b \in B_\alpha \subseteq A_\beta \). Then \( H_\beta(b) = \omega(\phi_\beta(b)) = \omega(\nu_\beta(b)) = h_\beta(b) = H_\alpha(b) \).

By the transfinite recursion theorem† there exist continuous characters \( H_\alpha \in \Omega(B_\alpha) (\alpha \in U) \) such that
(i) \( \forall \alpha \in U \quad H_\alpha|_A = h \)
(ii) \( \forall \alpha \in U \quad H_\alpha(\xi_\alpha) = \lambda_\alpha \)
(iii) \( \forall \alpha, \beta \in U \quad \alpha < \beta \Rightarrow H_\alpha = H_\beta|_{B_\alpha} \).

Therefore we can define \( H : B = \bigcup_{\alpha \in U} B_\alpha \to \mathbb{C} ; b \mapsto H_\alpha(b) \) where \( \alpha \in U, b \in B_\alpha \). Thus \( H \) is a well-defined unital homomorphism \( B \to \mathbb{C} \) and bounded since \( |H(b)| = |H_\alpha(b)| \leq \|b\| \). So \( H \in \Omega(B) \). Moreover we have \( \Pi(H) = (h, \lambda) \).

\( \Omega(B) \) is compact by A2.6 and \( P \) is clearly Hausdorff so it remains to check that \( \Pi \) is continuous; the proof of this is similar to the corresponding part of Theorem 2.6.

Our first application of this is to prove the result mentioned at the start of this section.

**COROLLARY 5.7.** Let \( B : A \) be a standard normed extension relative to \( (B_\alpha)_{\alpha \in U \subseteq A[x]} \). Then the following are surjections:
\[
\Omega(B) \to \Omega(A) : H \mapsto H|_A \\
\Omega(B) \to \Omega(B_\alpha) : H \mapsto H|_{B_\alpha} \quad (\alpha \in U)
\]

**Proof.** This is clear for the first map. For the second, let \( H_\alpha \in \Omega(B_\alpha) \).

Let
\[
\lambda_\beta = \begin{cases} \lambda_\beta(\xi_\beta) & \text{if } \beta \leq \alpha \\
\lambda & \in \mathbb{C} : H_\alpha(\beta)(\lambda_\beta) = 0 & \text{for some specific choices} \text{ if } \beta > \alpha.
\end{cases}
\]

Thus \( H := \Pi^{-1}(H|_A, \lambda) \) and \( H_\alpha \) agree on the generators, \( \{1\} \cup \{\xi_\beta : \beta \leq \alpha\} \), of \( B_\alpha \) over \( A \) so \( H_\alpha = H|_{B_\alpha} \).

**COROLLARY 5.8.** Let \( B \) be a standard normed extension of \( A \). Then \( K(A) = A \cap K(B) \).

**Proof.**
\[
K(A) = \bigcap_{H \in \Omega(B)} \ker H|_A = \bigcap_{H \in \Omega(B)} A \cap \ker H = A \cap K(B).
\]

The second application concerns the similarity with Cole’s construction in the case when \( (A, X) \) is a uniform algebra for we have (in the above notation and the notation of chapter 4) a map \( \rho : X^U \to \Omega(B) ; (\kappa, \lambda) \mapsto \Pi^{-1}(\kappa_\alpha, \lambda) \). \( \rho \) is a homeomorphism onto its image in \( \Omega(B) \) and induces a homomorphism \( \rho^* : C(\Omega(B)) \to C(X^U) \) as in chapter 4. Moreover we have the corresponding

**LEMMA 5.9.** \( \rho^*(\hat{B}) \subseteq A^U \).

† In this case the objects are sufficiently simple for a formal application of that theorem. In the notation of A3.17 we can take \( W = U \) and
\[
X = \bigcup_{\text{initial segments, } J \text{ of } U} \left\{ (H_\alpha)_{\alpha \in J} : \alpha \in J \mapsto \Omega(B_\alpha) : \text{statements similar to (i), (ii), (iii) hold} \right\}.
\]

30
Proof. The proof is a straightforward induction. Let \( J = \{ \beta \in U : \forall b \in B_\beta \, \rho^*(\hat b) \in A^U \} \) and assume \([\alpha_0, \alpha] \subseteq J\).

Let \( b \in B_\alpha \). Let \( n = n(\alpha) \), the degree of \( \alpha(x) \). Then there are \( b_0, \ldots, b_{n-1} \in A_\alpha : b = b_0 + \cdots + b_{n-1} \xi_{\alpha}^{n-1} \).

So
\[
\rho^*(\hat b) = \rho^*(\hat b_0) + \cdots + \rho^*(\hat b_{n-1}) \rho^*(\xi_{\alpha})^{n-1}
\]
and by the inductive hypothesis it is enough to show that \( \rho^*(\xi_{\alpha}) \in A^U \). Now \( \forall (\kappa, \lambda) \in X^U \)
\[
\rho^*(\xi_{\alpha}) (\kappa, \lambda) = \xi_{\alpha} (\Pi^{-1}(\varepsilon_{\kappa}, \lambda)) = \Pi^{-1}(\varepsilon_{\kappa}, \lambda)(\xi_{\alpha}) = H_{\alpha}(\xi_{\alpha})
\]
where (as in the proof of Theorem 5.6) \( H_{\alpha} := \Pi^{-1}(\varepsilon_{\kappa}, \lambda)|_{B_\alpha} = \omega \circ \phi_{\alpha} \) for \( \omega \in \Omega \left(A_\alpha[x]/(\alpha(x))\right) \) where in particular we have \( \omega(\bar x) = \lambda_\alpha \). Therefore \( H_{\alpha}(\xi_{\alpha}) = \omega(\bar x) = \lambda_\alpha = p_\alpha(\kappa, \lambda) \). So (as expected) \( \rho^*(\xi_{\alpha}) = p_\alpha \in A^U \).

Thus \( \alpha \in J \) so \( J = U \).

Moreover the following is commutative, because \( \forall f_0 \in A \), \( (\kappa, \lambda) \in X^U \)
\[
\left( \rho^* \circ \hat f_0 \right) (\kappa, \lambda) = \Pi^{-1}(\varepsilon_{\kappa}, \lambda)(f_0) = \varepsilon_{\kappa}(f_0) = f_0(\kappa) = \pi^*(f_0)(\kappa, \lambda).
\]

COROLLARY 5.10: Let \( (A, X) \) be a natural uniform algebra. Then \( \rho^* \) is isometric and \( A^U = \overline{B}^\sigma \).

Proof. This follows from exactly the same reasoning as for simple extensions.

As a last application of Theorem 5.6 we generalise in Theorem 5.12 Arens’ and Hoffman’s Theorem 2.10 from chapter 2.

LEMMA 5.11: Let \( B \) be a standard extension of \( A \) relative to \( (B_\alpha)_{\alpha \in U} \) and \( d \in A \) not be zero or a zero divisor in \( A \). Then \( d \) is not a zero divisor in \( B \).

Proof. We use induction again but for some variation in the presentation suppose \( d \) is a zero divisor in \( B \) but not in \( A \). Recall that \( B = \bigcup_{\alpha < \beta} B_\alpha \). Let \( \alpha \) be the least element of \( U \) such that \( dd' = 0 \) for some \( d' \in B_\alpha - \{0\} \).

As in the proof of Lemma 5.9 \( \phi_\alpha : B_\alpha \rightarrow A_\alpha[x]/(\alpha(x)) \) is an isomorphism over \( A_\alpha \) so there are elements \( d_0, \ldots, d_{n(\alpha) - 1} \in A_\alpha : d' = d_0 + \cdots + d_{n(\alpha) - 1} \xi_{\alpha}^{n(\alpha) - 1} \). Thus we have
\[
0 = \phi_\alpha(dd') = dd_0 + \cdots + dd_{n(\alpha) - 1} \bar x^{n(\alpha) - 1}
\]
By the uniqueness (discussed at the beginning of chapter 2) of this representation of the coset \( (n(\alpha) \) the degree of \( \alpha(x) \)) we have \( dd_j = 0 \) (\( j = 0, \ldots, n(\alpha) - 1 \)). By assumption \( d \) is not a zero divisor in \( A_\alpha \) so \( d_j = 0 \) (\( j = 0, \ldots, n(\alpha) - 1 \)). So we have the contradiction \( d' = 0 \).

THEOREM 5.12: Let \( B \) be the standard normed extension of \( A \) generated by \( U \). Suppose that for every \( \alpha \in U \) \( d_{\alpha} := \text{discr} (\alpha(x)) \) is not zero or a zero divisor. Then \( B \) is tractable if \( A \) is.

Proof. Let \( A \) be tractable. Let \( J = \{ \beta \in U : B_\beta \text{ is tractable} \} \) and suppose \( [\alpha_0, \beta] \subseteq J \).

Let \( a \in K(A_\beta) := \cap M(A_\beta) \). Now \( \exists \alpha < \beta : a \in B_\alpha \) (or \( a \in A \); it makes no difference in the following argument but where we write \( B_\alpha \) we do mean \( \{B_\alpha \text{ or } A\} \). Let \( H_\alpha \in \Omega(B_\alpha) \).
Now $A_\beta$ is a standard normed extension of $A$ relative to $(B_\gamma)_{\gamma < \beta}$ so by Corollary 5.7,

$$\exists H \in \Omega(A_\beta) : H_\alpha = H|_{B_\alpha}.$$  

$\Rightarrow H_\alpha(a) = H(a) = 0$. Thus $a \in K(B_\alpha) = (0)$ since $B_\alpha$ is tractable by hypothesis.

So $A_\beta$ is tractable. Now $d_\beta$ is not zero or a zero divisor in $A_\beta$ by Lemma 5.11 so by Theorem 2.10, $A_\beta[x]/(\beta(x))$, which is isometrically isomorphic to $B_\beta$, is tractable.

By the principle of transfinite induction $J = U$ and it now follows from Corollary 5.7 again (with a proof as in the case of $A_\alpha$ that $B$ is tractable.

In fact the converse is also true and the proof is another straightforward induction. However it relies on the result being true in the simple case (Corollary 9.3 of [7]) so we omit this.
Chapter 6
Integral Closure of Uniform Algebras

1. Introduction

In Cole’s thesis the main example features a method for constructing uniform algebra extensions closed under taking square roots and Lindberg remarks in [11] that the same techniques can be used to construct an integrally closed uniform algebra which isometrically extends any given uniform algebra, \((A, X)\). We give the details explicitly here.

2. Inverse and Direct Systems

The basic result is that there is a canonical way of forming transfinite extensions of uniform algebras and it is this pre-existing construction which Cole exploits in [10]. In anticipation of the definition to be given below, the result is that ‘direct limits’ of uniform algebras always exist (we shall prove a weaker result here). For the general categorical definitions of direct and inverse systems and their limits we refer the reader to [15], but it is easy to see what they are from the definitions below.

Throughout the rest of this chapter, \(v\) will denote a fixed ordinal number greater than zero.

**DEFINITION 6.1**: Let \((X_\alpha)_{\alpha<\upsilon}\) be a family of topological spaces and \((\pi_{\alpha,\beta})_{\alpha\leq\beta<\upsilon}\) a family of continuous maps where \(\forall \alpha \leq \beta < \upsilon \pi_{\alpha,\beta} \in C(X_\beta, X_\alpha)\). We say that \(((X_\alpha)_{\alpha<\upsilon}, (\pi_{\alpha,\beta})_{\alpha\leq\beta<\upsilon})\) is an inverse system of topological spaces if

(i) \(\forall \alpha < \upsilon \pi_{\alpha,\alpha} = \text{id}_{X_\alpha}\)

(ii) \(\forall \alpha \leq \beta \leq \gamma < \upsilon \pi_{\alpha,\gamma} = \pi_{\alpha,\beta} \circ \pi_{\beta,\gamma}\).

An inverse limit for the system above is a pair \((X_\upsilon, (\pi_{\alpha,\upsilon})_{\alpha<\upsilon})\) where \(X_\upsilon\) is a topological space and \(\forall \alpha < \upsilon \pi_{\alpha,\upsilon} \in C(X_\upsilon, X_\alpha)\) are such that

\[\forall \alpha \leq \beta < \upsilon \quad \pi_{\alpha,\upsilon} = \pi_{\alpha,\beta} \circ \pi_{\beta,\upsilon}\]

and such that the following universal property is satisfied: Whenever \(Y_\upsilon\) is a topological space and \(\exists_{\alpha<\upsilon} \rho_{\alpha,\upsilon} \in C(Y_\upsilon, X_\alpha)\): \(\forall \alpha \leq \beta < \upsilon \rho_{\alpha,\upsilon} = \pi_{\alpha,\beta} \circ \rho_{\beta,\upsilon}\) then there is a unique \(\rho_\upsilon \in C(Y_\upsilon, X_\upsilon)\):

\[\forall \alpha < \upsilon \quad \pi_{\alpha,\upsilon} \circ \rho_\upsilon = \rho_{\alpha,\upsilon}\]

This is written

\[X_\upsilon = \lim_{\leftarrow} X_\alpha.\]

In general there is no guarantee that an inverse limit will exist or be non-trivial. However we have the important result below:

**THEOREM 6.2**: Let \(((X_\alpha)_{\alpha<\upsilon}, (\pi_{\alpha,\beta})_{\alpha\leq\beta<\upsilon})\) be an inverse system of non-empty compact Hausdorff spaces. Then an inverse limit exists and is a non-empty compact Hausdorff space.

*Proof.* (From [16].) Define \(K = \times_{\alpha<\upsilon} X_\alpha\) and let \(\pi_\alpha : K \to X_\alpha\) be the canonical projection \((\alpha < \upsilon)\). Set

\[X_\upsilon := \{\kappa \in K : \forall \alpha \leq \beta < \upsilon (\pi_{\alpha,\beta} \circ \pi_{\beta})(\kappa) = \pi_{\alpha}(\kappa)\}\]

\[\pi_{\alpha,\upsilon} := \pi_{\alpha} |_{X_\upsilon} \quad (\alpha < \upsilon).\]
We show that \((X, \pi_{\alpha,\upsilon})_{\alpha < \upsilon}\) is an inverse limit for the system.

A product of Hausdorff spaces is Hausdorff so \(K\) is Hausdorff and \(K\) is compact by Tychonoff’s theorem.

We have that \(X = \underset{\alpha < \upsilon}{\bigcap} F_{\alpha}\) where \(\forall \beta < \upsilon F_{\beta} = \{ \kappa \in K : \forall \alpha \leq \beta (\pi_{\alpha,\beta} \circ \pi_{\beta})(\kappa) = \pi_{\alpha}(\kappa) \}\). By exercise 3C of [17] the set on which two continuous maps into a Hausdorff space agree is closed so the \(F_{\beta}\) are closed.

Hence \(X\) is a closed and therefore compact subset of \(K\).

To check that \(X\) is non-empty it is enough, since it is now known to be compact and therefore has the finite intersection property, to show that \(\forall \beta_1, \ldots, \beta_n < \upsilon \bigcap_{k=1}^{n} F_{\beta_k} \neq \emptyset\).

We first need to know that \(\forall \beta < \upsilon F_{\beta} \neq \emptyset\).

Let \(\beta < \upsilon\) and choose \(K_{\beta} \in X_{\beta}\) and for \(\alpha \leq \beta\) let \(K_{\alpha} = \pi_{\alpha,\beta}(K_{\beta})\). Let \(K_{\alpha}\) be arbitrary in \(X_{\alpha}\) for \(\beta < \alpha < \upsilon\). Then \(K\in K\) and \(\forall \alpha < \beta (\pi_{\alpha,\beta} \circ \pi_{\beta})(\kappa) = \pi_{\alpha,\beta}(\kappa) = \pi_{\alpha}(\kappa)\) for \(\kappa \in F_{\beta}\).

Now let \(\beta = \max_{k=1, \ldots, n}(\beta_k)\). Let \(K \in F_{\beta}\). We have

\[
\forall \alpha \leq \beta (\pi_{\alpha,\beta} \circ \pi_{\beta})(\kappa) = \pi_{\alpha}(\kappa)
\]

and so \(\forall j \in \{1, \ldots, n\}, \alpha \leq \beta_j (\pi_{\alpha,\beta_j} \circ \pi_{\beta_j})(\kappa) = (\pi_{\alpha,\beta_j} \circ \pi_{\beta_j} \circ \pi_{\beta})(\kappa)\)

whence \(\forall j \in \{1, \ldots, n\}, \alpha \leq \beta_j (\pi_{\alpha,\beta_j} \circ \pi_{\beta_j})(\kappa) = (\pi_{\alpha,\beta} \circ \pi_{\beta})(\kappa) = \pi_{\alpha}(\kappa)\)

and so \(K \in \bigcap_{j=1}^{n} F_{\beta_j}\).

Thus \(\emptyset \neq \bigcap_{j=1}^{n} F_{\beta_j}\).

It remains to check that \((X, \pi_{\alpha,\upsilon})_{\alpha < \upsilon}\) satisfies the rest of the conditions in 6.1 above. The maps are clearly continuous by the definition of the product topology. Now suppose that \((Y, \rho_{\alpha,\upsilon})_{\alpha < \upsilon}\) is a topological space and family of continuous maps with \(\forall \alpha \leq \beta < \upsilon \rho_{\alpha,\upsilon} = \pi_{\alpha,\beta} \circ \rho_{\beta,\upsilon}\).

Define

\[
\rho_{\upsilon}: Y_{\upsilon} \to X_{\upsilon}; y \mapsto (\rho_{\alpha,\upsilon}(y))_{\alpha < \upsilon}
\]

Then \(\forall \alpha \leq \beta < \upsilon (\pi_{\alpha,\beta} \circ \pi_{\beta})(\rho_{\upsilon}(y)) = \pi_{\alpha,\beta}(\rho_{\beta,\upsilon}(y)) = \rho_{\alpha,\upsilon}(y) = \pi_{\alpha}(\rho_{\upsilon}(y))\) so \(\rho_{\upsilon}\) is well-defined and has \(\forall \alpha < \upsilon \pi_{\alpha,\upsilon} \circ \rho_{\upsilon} = \rho_{\alpha,\upsilon}\). \(\rho_{\upsilon}\) is continuous since \(\forall \alpha < \upsilon \pi_{\alpha,\upsilon} \circ \rho_{\upsilon}\) is continuous and \(K\) has the initial topology induced by \((\pi_{\alpha})_{\alpha < \upsilon}\) so \(X_{\upsilon}\) has initial topology induced by \((\pi_{\alpha}|_{X_{\alpha}})_{\alpha < \upsilon} = (\pi_{\alpha,\upsilon})_{\alpha < \upsilon}\). Uniqueness of \(\rho_{\upsilon}\) is clear. \(\square\)

**Definition 6.3:** Let \((\{A_{\alpha}\}_{\alpha < \upsilon}, (\theta_{\alpha,\beta})_{\alpha \leq \beta < \upsilon})\) be a family of algebras and homomorphisms where \(\forall \alpha \leq \beta < \upsilon \theta_{\alpha,\beta}: A_{\alpha} \to A_{\beta}\). This is a direct system if

(i) \(\forall \alpha < \upsilon \theta_{\alpha,\alpha} = \text{id}_{A_{\alpha}}\)

(ii) \(\forall \alpha \leq \beta \leq \gamma < \upsilon \theta_{\alpha,\gamma} = \theta_{\beta,\gamma} \circ \theta_{\alpha,\beta}\).

A direct limit of this system is \((A_{\upsilon}, (\theta_{\alpha,\upsilon})_{\alpha < \upsilon})\) for an algebra \(A_{\upsilon}\) and homomorphisms \(\theta_{\alpha,\upsilon}: A_{\alpha} \to A_{\upsilon}\) such that

\[
\forall \alpha \leq \beta < \upsilon \theta_{\alpha,\upsilon} = \theta_{\beta,\upsilon} \circ \theta_{\alpha,\beta}
\]

and whenever \(B_{\upsilon}\) is an algebra with homomorphisms \(\phi_{\alpha,\upsilon}: A_{\alpha} \to B_{\upsilon} (\alpha < \upsilon)\) satisfying

\[
\forall \alpha \leq \beta < \gamma \phi_{\alpha,\upsilon} = \phi_{\beta,\upsilon} \circ \theta_{\alpha,\beta}
\]

then

\[
\exists \theta_{\upsilon} \in \text{Hom}(A_{\upsilon}, B_{\upsilon}); \forall \alpha < \upsilon \phi_{\alpha,\upsilon} = \theta_{\upsilon} \circ \theta_{\alpha,\upsilon}.
\]

We write

\[
A_{\upsilon} = \text{lim}_{\alpha \to \upsilon} A_{\alpha}.
\]

It is clear what these definitions in other specific categories of algebras should be. In fact for our purposes it will be sufficient to restrict attention to the category, \(U\), of uniform algebras and continuous unital homomorphisms between them of norms at most 1 (i.e. multiplicative contractions).
Now let \(((A_\alpha, X_\alpha)_{\alpha \leq \beta \leq < \nu}, (\pi_{\alpha, \beta})_{\alpha \leq \beta \leq < \nu})\) be a family of uniform algebras such that \(((X_\alpha)_{\alpha \leq < \nu}, (\pi_{\alpha, \beta})_{\alpha \leq \beta \leq < \nu})\) is an inverse system. Then

\[ \forall \alpha \leq \beta \leq \nu \quad \pi_{\alpha, \beta} \text{ induces } \pi_{\alpha, \beta}^* \in \text{Hom}_U(A_\alpha, A_\beta) \text{ defined by } \]

\[ \pi_{\alpha, \beta}^*: A_\alpha \to A_\beta: f \mapsto f \circ \pi_{\alpha, \beta}. \]

The success of Cole's method rests (among other things) on:

**THEOREM 6.4:** Let \(((A_\alpha, X_\alpha)_{\alpha < \nu}, (\pi_{\alpha, \beta})_{\alpha \leq \beta < \nu})\) be a family of uniform algebras and suppose

\[ (X_\nu, (\pi_{\alpha, \nu})_{\alpha < \nu}) \]

is an inverse limit of the inverse system \(((X_\alpha)_{\alpha < \nu}, (\pi_{\alpha, \beta})_{\alpha \leq \beta < \nu})\) where all the \(\pi_{\alpha, \beta}\) are surjective \((\alpha \leq \beta < \nu)\).

Let \(A_\nu\) be the closed subalgebra of \(C(X_\nu)\) generated by \(I_0 := \bigcup_{\alpha < \nu} \pi_{\alpha, \nu}(A_\alpha). \) Then \((A_\nu, (\pi_{\alpha, \nu})_{\alpha < \nu})\) is a direct limit of the dual system \(((A_\alpha)_{\alpha < \nu}, (\pi_{\alpha, \beta})_{\alpha \leq \beta < \nu})\) (in \(U\)) and \((A_\nu, X_\nu)\) is a uniform algebra.

**Proof.** (Adapted from the sketch in [10].) It is immediate that \((A_\nu, X_\nu)\) is a uniform algebra; \(X_\nu\) is a non-empty compact Hausdorff space by Theorem 6.2 and \(A_\nu\) is already a uniformly closed algebra. It contains the constants because \(\pi_{\alpha, \nu}^*(1) = 1 \in A_\nu. \) \(A_\nu\) separates the points of \(X_\nu:\) let \(\kappa, \kappa' \in X_\nu\) with \(\kappa \neq \kappa'. \) There exists \(\alpha < \nu: \kappa_\alpha \neq \kappa'_\alpha. \) But \(A_\alpha\) is a uniform algebra so for some \(f \in A_\alpha, \) \(f(\kappa_\alpha) \neq f(\kappa'_\alpha). \) We have \(\pi_{\alpha, \nu}(f) \in A_\nu\) and \(\pi_{\alpha, \nu}^*(f)(\kappa) = (f \circ \pi_{\alpha, \nu})(\kappa) = f(\kappa_\alpha) \neq f(\kappa'_\alpha) = \pi_{\alpha, \nu}^*(f)(\kappa'). \)

We now verify that \((A_\nu, (\pi_{\alpha, \nu})_{\alpha < \nu})\) is a direct limit of the dual system in \(U. \) By Lemma 9 of [16] (p. 210), whose proof we do not reproduce here, the surjectivity of the maps \((\pi_{\alpha, \beta})_{\alpha \leq \beta < \nu}\) implies that \(\pi_{\alpha, \nu}\) is surjective \((\alpha < \nu)\). Hence \(\pi_{\alpha, \nu}^*\) is isometric \((\alpha < \nu)\). So for \(\alpha < \nu\) \(\|\pi_{\alpha, \nu}^*\| = 1. \) Also

\[ \forall \alpha \leq \beta < \nu \quad \pi_{\beta, \nu}^* \circ \pi_{\alpha, \beta}^* = (\pi_{\alpha, \beta} \circ \pi_{\beta, \nu})^* = \pi_{\alpha, \nu}^*. \]

It therefore remains to check that \((A_\nu, (\pi_{\alpha, \nu})_{\alpha < \nu})\) solves the universal problem in definition 6.3.

Let \(((B_\nu, Y_\nu), (\phi_{\alpha, \nu})_{\alpha < \nu})\) be another pair of uniform algebras and family of homomorphic contractions, \(\phi_{\alpha, \nu}: A_\alpha \to B_\nu (\alpha < \nu),\) with

\[ \forall \alpha \leq \beta < \nu \quad \phi_{\alpha, \nu} = \phi_{\beta, \nu} \circ \pi_{\alpha, \beta}^*. \]

We must show

\[ \exists \pi_{\nu}^* \in \text{Hom}(A_\nu, B_\nu): \forall \alpha < \nu \quad \phi_{\alpha, \nu} = \pi_{\nu}^* \circ \pi_{\alpha, \nu}^*. \tag{\dagger} \]

The uniqueness part is clear because \(I_0\) generates \(A_\nu. \) Also notice that \(I_0\) is a subspace of \(A_\nu:\) let \(g_1, g_2 \in I_0, \lambda \in C. \)

There exist \(\alpha, \beta < \nu: g_1 \in \pi_{\alpha, \nu}(A_\alpha), g_2 \in \pi_{\beta, \nu}(A_\beta). \)

So there are \(f_1 \in A_\alpha, f_2 \in A_\beta: g_1 = \pi_{\alpha, \nu}(f_1), g_2 = \pi_{\beta, \nu}(f_2). \)

We have \(\lambda g_1 = \pi_{\alpha, \nu}(\lambda f_1) \in I_0. \) Now let \(\alpha \leq \beta\) without loss of generality. So \(\pi_{\alpha, \nu}^* = \pi_{\beta, \nu}^* \circ \pi_{\alpha, \beta}^*\) whence \(g_1 + g_2 = \pi_{\beta, \nu}^* \big( \pi_{\alpha, \beta}^*(f_1) + f_2 \big) \in I_0. \)

So if we can define \(\pi_{\nu}^*\) on \(I_0\) to be a homomorphism with norm at most 1 satisfying \((\dagger)\) then (since \(B_\nu\) is a Banach space) there is a unique continuous linear extension to a map \(A_\nu \to B_\nu\) with norm at most 1. By continuity the extension will be a homomorphism.

Set \(T_0: I_0 \to B_\nu: g \mapsto \phi_{\alpha, \nu}(f)\) where \(g = \pi_{\alpha, \nu}^*(f), f \in A_\alpha, \alpha < \nu. \) \(T_0\) is well-defined: suppose \(\alpha \leq \beta < \nu\) and \(g = \pi_{\alpha, \nu}^*(f) = \pi_{\beta, \nu}^*(f')\) for \(f \in A_\alpha, f' \in A_\beta. \) Now \(\pi_{\alpha, \nu}^* = \pi_{\beta, \nu}^* \circ \pi_{\alpha, \beta}^*\) so \(\pi_{\alpha, \nu}^*(f) = \pi_{\beta, \nu}^*(\pi_{\alpha, \beta}^*(f)) = \pi_{\beta, \nu}^*(f').\) But \(\pi_{\beta, \nu}^*\) is surjective so \(\pi_{\beta, \nu}^*\) is a monomorphism (in fact isometric, as noted above). So \(f' = \pi_{\alpha, \beta}^*(f) = \phi_{\beta, \nu}^*(f').\)

Similar calculations show that \(T_0\) is a homomorphism. We also have that \(T_0\) is a contraction for if \(g = \pi_{\alpha, \nu}^*(f), f \in A_\alpha\) then

\[ \|T_0(g)\| = \|\phi_{\alpha, \nu}(f)\| \leq \|f\| = \|\pi_{\alpha, \nu}^*(f)\| = \|g\|. \]

So the required map \(\pi_{\nu}^*: A_\nu \to B_\nu\) exists. \qed
3. Universal Root Algebras

We can now apply the machinery above to obtain a ‘universal root algebra’ (an integrally closed extension algebra) by mimicking Cole’s work. This is almost certainly repeating the work of Lindberg.

THEOREM 6.5 Let \((A, X)\) be a uniform algebra. Then there exists an integrally closed uniform algebra which is an isometric extension of \(A\).

**Proof.** Let \(\omega_1\) denote the first uncountable ordinal. Now let \(\gamma < \omega_1\) and suppose we have chosen uniform algebras, \((A_\alpha, X_\alpha)_{\alpha < \gamma}\), and continuous surjections, \((\pi_{\alpha, \beta})_{\alpha \leq \beta < \gamma}\), such that

\(\forall (X_\alpha)_{\alpha < \gamma}, (\pi_{\alpha, \beta})_{\alpha \leq \beta < \gamma}\) is an inverse system

(i) \((X_\alpha)_{\alpha < \gamma}, (\pi_{\alpha, \beta})_{\alpha \leq \beta < \gamma}\) is an inverse system

(ii) \((A_0, X_0) = (A, X)\)

(iii) \(\forall \beta < \gamma, \exists \alpha < \beta: \beta = \alpha + 1\) then \((A_{\beta}, X_{\beta})\) is the Cole-extension as in Theorem 4.1 taken with respect to \(U_\alpha\), the set of all monic polynomials over \(A_\alpha\) and \(\forall \alpha \leq \beta < \gamma \pi_{\alpha, \beta}\) is the associated projection \((X_\alpha)^{U_\alpha} = X_\alpha \times C^{U_\alpha} \to X_\alpha\).

(iv) \(\forall \beta < \gamma, \pi_{\beta, \omega}\) is a limit ordinal then \(A_\beta\) is the direct limit of \((A_\alpha, X_\alpha)_{\alpha < \beta}\) and \((\pi_{\alpha, \nu})_{\alpha < \nu}\) are as described in Theorem 6.4.

There are two cases. If \(\gamma = \beta + 1\) for some \(\beta < \gamma\) then we (are forced to) define \((A_\gamma, X_\gamma)\) to be the uniform algebra generated according to Theorem 4.1 by the set of all monic polynomials over \(A_\beta\). For \(\alpha \leq \gamma\) set

\[\pi_{\alpha, \gamma} = \begin{cases} \text{id}_{X_\gamma}, & \alpha = \gamma \\ \text{the coordinate projection } \pi_{\beta, \gamma}: X_\gamma \to X_\beta & \alpha = \beta \\ \pi_{\alpha, \beta} \circ \pi_{\beta, \gamma} & \alpha < \beta. \end{cases}\]

All maps on the right hand side are surjective and continuous so the same is true of \(\pi_{\alpha, \gamma}(\alpha \leq \gamma)\).

The other case is when \(\gamma\) is a non-zero limit ordinal. Define \((A_\gamma, X_\gamma, (\pi_{\alpha, \gamma})_{\alpha < \gamma})\) to be the uniform algebra constructed out of the existing system according to Theorem 6.4 (and let \(\pi_{\gamma, \gamma} = \text{id}_{X_\gamma}\), the identity map on \(X_\gamma\)).

In each case \((X_\alpha)_{\alpha < \gamma}, (\pi_{\alpha, \beta})_{\alpha \leq \beta < \gamma}\) is an inverse system of compact Hausdorff spaces and continuous surjections and (i)-(iv) are plainly satisfied by the extended system of uniform algebras. By the transfinite recursion theorem we obtain such a family of uniform algebras with \(\gamma = \omega_1\).

It is clear that \(\omega_1\) is a limit ordinal. Therefore by (iv),

\[A_{\omega_1} = \bigcup_{\alpha < \omega_1} \pi^*_{\alpha, \omega_1}(A_\alpha)\]

the closure being taken in \(C(X_{\omega_1})\). This follows from the facts that \(\bigcup_{\alpha < \omega_1} \pi^*_{\alpha, \omega_1}(A_\alpha)\) is a subalgebra (similar to the result seen in the proof of Theorem 6.4) and that the closure of a subalgebra is a subalgebra. We also saw in the proof of Theorem 6.4 that \(\pi^*_{\alpha, \omega_1}\) is an isometry \((\alpha < \omega_1)\). Hence \(\pi^*_{\alpha, \omega_1}(A_\alpha)\) is closed \(\forall \alpha < \omega_1\) (since \(A_\alpha, C(X_{\omega_1})\) are complete).

Also the relation \(\forall \alpha \leq \beta < \omega_1 \pi^*_{\alpha, \omega_1} = \pi^*_{\beta, \omega_1} \circ \pi^*_{\alpha, \beta}\) shows that \(\forall \alpha \leq \beta < \omega_1 \pi^*_{\alpha, \omega_1}(A_\alpha) \subseteq \pi^*_{\beta, \omega_1}(A_\beta)\).

Hence by Theorem A3.20, \(\bigcup_{\alpha < \omega_1} \pi^*_{\alpha, \omega_1}(A_\alpha)\) is closed. Therefore

\[A_{\omega_1} = \bigcup_{\alpha < \omega_1} \pi^*_{\alpha, \omega_1}(A_\alpha).\]

Now if \(\mu(x)\) is a monic polynomial over \(A_{\omega_1}\) then it is monic over \(\pi^*_{\alpha, \omega_1}(A_\alpha)\) for some \(\alpha < \omega_1\) and therefore has a root in \(\pi^*_{\alpha, \omega_1}(A_\beta)\) where \(\beta = \alpha + 1\). So \(A_{\omega_1}\) is integrally closed. Finally note that \(\pi^*_{0, \omega_1}\) gives an isometric embedding \(A \hookrightarrow A_{\omega_1}\). \[\square\]

We therefore have refined Theorem 5.5 to the category of uniform algebras. One can also investigate the inheritance properties of these ‘universal root extensions’. For example Theorem 4.2 generalises: if \(A\) is natural on \(X\) then \(A_{\omega_1}\) is natural on \(X_{\omega_1}\). This can be proved in exactly the same way as the corresponding result for the square-root closed extensions mentioned before; see for example [18] p. 103.
Chapter 7
Conclusion

1. Conclusions

We have explored two methods for extending uniform algebras by roots of monic polynomials, one of which applies to general commutative normed unital algebras. We have seen that integrally closed extensions always exist in both categories of algebras.

The two constructions have well-understood properties, and, as we have seen lead to essentially different solutions.

We have taken a simplified view of the literature. There have been many contributions (mainly by Lindberg) on properties of Banach algebras with respect to Arens-Hoffman extensions which we have not considered. For example (for appropriate initial algebras) naturality, normality, regularity, the Šilov boundary, indecomposability and involutions. (See [7].)

Another topic not covered here is on 'integral extensions'. Standard extensions are examples of integral extensions and they can be used to investigate the purely algebraic questions on them; see [11].

2. Context

Apart from the minor application in Corollary 3.16, Arens-Hoffman extensions’ mainly feature in the literature in connection with Galois theory for Banach algebras where they play a very important role. However in some versions of this they are not sufficient (there are ‘Galois extensions which are not Arens-Hoffman extensions’ mentioned in [19]).

Arens-Hoffman extensions also feature in some examples in the literature on function algebras. For example, Karahanjan uses them in [14] to construct an example of a uniform algebra with certain properties. This was the area of Cole’s work in [10] where he used a universal root algebra to disprove a certain conjecture.

3. Further Questions

In addition to the properties mentioned in the first section there are other inheritance properties of Arens-Hoffman extensions it might be interesting to investigate; for example the popular $C^*$ property. Also the work in section 5.4 suggests that many of these results will lift to standard extensions.

The corresponding results for extension of uniform algebras by Cole’s methods have not been studied as much in the literature so it is important to know precisely when such extensions can be realised via Arens-Hoffman extensions.

We would also like to be able (if possible) to generalise the result at the end of chapter two so that every tractable normed algebra can be embedded in a tractable normed algebra extension which contains roots of an arbitrary large set of monic polynomials (which may have to be restricted in some other way).
Appendix 1
Assumed Gelfand Theory

Gelfand theory is a tool used to investigate normed algebras. It identifies arbitrary normed algebras with complex function algebras via a map, $\Gamma$, to be defined below. We list the definitions and results which are used in the dissertation. This material is standard and proofs can be found in many introductory books on functional analysis, for example [5] chapters 12 and 13 or [20] chapters 1 and 2.

Let $A$ be a commutative unital Banach algebra.

**DEFINITION A1.1:** The maximal ideal space and character (carrier) space of $A$ are respectively
\[
M(A) := \{I \subseteq A: I \text{ is a maximal ideal of } A\}, \quad \text{and} \quad \Omega(A) := \text{Hom}_\mathbb{C}(A, \mathbb{C}) - \{0\}.
\]

The topologies on these two sets are defined below. Recall that $A^* = \mathcal{B}(A, \mathbb{C})$ has a topology called the weak *-topology, the initial topology induced by the evaluation-at-$a$ linear functionals ($a \in A$).

**THEOREM A1.2:** $\Omega(A) \subseteq A^*$.

**THEOREM A1.3:** $M(A) \neq \emptyset$ and $\mu: \Omega(A) \to M(A); \omega \mapsto \ker \omega$ is a bijection.

The topology on $\Omega(A)$ is the subspace topology relative to the weak *-topology. The topology on $M(A)$ is the one obtained by using $\mu$ to identify the two sets.

**THEOREM A1.4:** $\Omega(A)$ is a compact Hausdorff subspace of $A^*$ in the weak *-topology.

**DEFINITION A1.5:** The Gelfand transform of $A$ is the map $\Gamma: A \to C(\Omega(A)); a \mapsto \hat{a}$ where
\[
\forall a \in A, \omega \in \Omega(A) \quad \hat{a}(\omega) := \omega(a).
\]

**THEOREM A1.6:**
(i) $\Gamma$ is a contractive (norm-decreasing) homomorphism
(ii) $\forall a \in A \| \hat{a} \| = r(a) = \lim_{n \to +\infty} \|a^n\|^{\frac{1}{n}}$
(iii) $\forall a \in A \text{ im } \hat{a} = \sigma(a)$
(iv) $\ker \Gamma = \cap M(A)$.

**DEFINITION A1.7:** $A$ is semisimple if $\cap M(A) = \{0\}$.

**COROLLARY A1.8:**
(i) $\Gamma$ is injective $\iff$ $A$ is semisimple
(ii) $\Gamma$ is isometric $\iff$ $\forall a \in A \|a^2\| = \|a\|^2$. 

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Appendix 2
Gelfand Theory for Normed Algebras

While Gelfand theory is usually expounded for Banach algebras it is interesting to study its generalisation to normed algebras in its own right as well for its use as a tool to investigate incomplete normed algebras. Most of the results presented below seem to be well-known but not to be found explicitly stated in the literature. They are obtained by modifying the standard proofs for the complete case or by considering the algebra’s completion. Accordingly we begin by recalling the definition of this.

**DEFINITION A2.1:** Let $A$ be a normed algebra. Then a completion of $A$ is a pair $(\hat{A}, i)$ where $\hat{A}$ is a Banach algebra and $i$ is an isometric monomorphism $A \to \hat{A}$ with dense image.

There is a well known construction for normed spaces. One defines $\hat{A} := S/\sim$ where $S = \{ s \in A^\mathbb{N} : s$ is a Cauchy sequence $\}$ and $\sim$ is the equivalence relation on $S$ given by

$$s \sim s' \iff \lim_{n \to +\infty} \|s_n - s'_n\| = 0$$

where $s = (s_n), s' = (s'_n)$

It is standard that completions are unique up to isometric isomorphism. By this we mean that if $(B, j)$ is another completion then there is an isometric isomorphism $\phi: \hat{A} \to B$ such that the following is commutative:

$$\hat{A} \xrightarrow{\phi} B$$

$$\uparrow i \quad \uparrow j$$

$$A$$

For $a, b \in S$ ($a = (a_n), b = (b_n)), \lambda \in \mathbb{C}$ the normed space structure is given by

$$[(a_n)] + [(b_n)] := [(a_n + b_n)]$$

$$\lambda[(a_n)] := [(\lambda a_n)]$$

$$\|[(a_n)]\| := \lim_{n \to +\infty} \|a_n\|$$

It is a standard exercise to check that

$$[(a_n)][(b_n)] := [(a_n b_n)]$$

properly defines a multiplication on $\hat{A}$ in such a way that $i$ is an algebra homomorphism. It is also clear that normed algebra completions are unique to isometric isomorphism. They are clearly commutative if $A$ is commutative. If $A$ is unital then so is $\hat{A}$ and $i$ is too (i.e. $1_{\hat{A}} = i(1_A)$).

In the rest of this appendix $A$ will denote a complex normed algebra, not necessarily commutative or unital.

Although the applications in the dissertation are only to unital algebras it is not much more difficult to develop the Gelfand theory for non-unital normed algebras. There is an existing theory for non-unital Banach algebras (see for example [20] chapter 1) in which the maximal ideals are replaced by the modular maximal ideals.

**DEFINITION A2.2:** Let $R$ be a commutative algebra and $J$ be a maximal ideal of $R$. $J$ is called modular if for some $u \in J$ we have $\forall r \in R \ r - ru \in J$.

Clearly all ideals are modular if $R$ has a unity.
When $A$ is not complete it is no longer automatic that a complex homomorphism is continuous. We therefore only consider the continuous characters on $A$:

$$\Omega(A) := \{\omega \in \text{Hom}_\mathbb{C}(A, \mathbb{C}) : \omega \text{ is continuous}, \omega \neq 0\}$$

In this way $\Omega(A) \subseteq A^*$ and we can again exploit the topology on $A^*$.

**Lemma A2.3 (Gelfand-Mazur):** Let $A$ be a normed, unital division algebra. Then there exists an isometric isomorphism $A \to \mathbb{C}$.

**Proof.** This will be as for Banach algebras if we can show that again in normed algebras every isomorphism to a character on $\tilde{A}$ exists but may not be isometric. (The map is simple so $\lambda_a$ where $\lambda_a$ is the unique $\lambda \in \mathbb{C} : a = \lambda 1_A$.)

One comment is in order: if the normed algebra has a unity but is not unital (i.e. $1 < \|1_A\|$) then the topological isomorphism of the theorem still exists but may not be isometric. (The map is $A \to \mathbb{C} : a \mapsto \lambda_a$ where $\lambda_a$ is the unique $\lambda \in \mathbb{C} : a = \lambda 1_A$.)

**Lemma A2.4:** Let $A$ be a commutative normed algebra and let

$$M(A) := \{I \subseteq A : I \text{ is a closed, modular maximal ideal of } A\}.$$

Then $M(A) \neq \emptyset \iff \Omega(A) \neq \emptyset$ and in this case $\mu : \Omega(A) \to M(A) ; \omega \mapsto \ker \omega$ is a bijection.

**Proof.** First suppose $\Omega(A) \neq \emptyset$. Let $\omega \in \Omega(A)$. Since $\omega \neq 0$ its kernel is a proper ideal of $A$. Suppose $J$ is an ideal of $A$ satisfying $\ker \omega \subseteq J \subseteq A$. Now $\omega$ is a ring epimorphism $A \to \mathbb{C}$ so $\omega(J)$ is an ideal of $\mathbb{C}$. But $\mathbb{C}$ is simple so $\omega(J) = \{0\}$ or $\mathbb{C}$.

If $\omega(J) = \{0\}$ then $J \subseteq \omega^{-1}(\omega(J)) = \omega^{-1}(\{0\}) = \ker \omega$ which implies that $\ker \omega = J$, a contradiction.

The other case can’t occur because if $\omega(J) = \mathbb{C}$ then $0 = \mathbb{C}/\omega(J) \cong A/J$ (since $\omega$ is an epimorphism). But $J \neq A$ so $A/J \neq 0$, a contradiction.

Thus $\ker \omega$ is maximal.

$\ker \omega$ is closed by the continuity of $\omega$.

Let $a \in A - \ker \omega$ and put $u = a/\omega(a)$. Then we have that

$$\forall b \in A \quad b - ub \in \ker \omega$$

and so $\ker \omega$ is modular. Thus $\ker \omega \in M(A)$. Note that this also shows that $\mu$ is well-defined and exists whenever $\Omega(A) \neq \emptyset$. Also, the result that $\mu$ is always injective when it exists does not require the completeness of $A$ so we omit the detail (see for example [5] p. 321).

Suppose now that $M \in M(A)$. Since $A$ is commutative, $A/M$ is a field; the unity is $M + u$ where $u \in A : \forall a \in A \quad a - ua \in M$. By Lemma A2.3 there is a topological isomorphism $\phi : A/M \to \mathbb{C}$. The map $\omega := \phi \circ \nu$ where $\nu$ is the natural epimorphism is therefore a continuous character of $A$. (It is standard that the natural epimorphism of a normed space onto its quotient by a closed subspace is continuous.)

We have now shown $M(A) \neq \emptyset \iff \Omega(A) \neq \emptyset$. It remains to check that when either set is non-empty, $\mu$ is surjective. But for $M \in M(A)$, setting $\omega = \phi \circ \nu$ as described above gives a continuous character on $A$ with $M \subseteq \ker \omega$. Since $\ker \omega \subseteq A$ and $M$ is maximal we have $M = \ker \omega = \mu(\omega)$ as required. 

**Lemma A2.5:** Let $A$ be a normed algebra with $\Omega(A)$ non-empty. Every continuous character on $A$ extends to a character on $\tilde{A}$.

**Proof.** Let $\omega \in \Omega(A)$. We must show that $\exists \tilde{\omega} \in \tilde{A} : \tilde{\omega} \circ i = \omega$ where $(\tilde{A}, i)$ as is above. But if $i$ is dense in $\tilde{A}$, $\omega \circ i^{-1}$ is a bounded linear map $\text{im } i \to \mathbb{C}$ with norm $\|\omega\|$ (where $i^{-1}$ is the (induced) isometric isomorphism $\text{im } i \to A$). $\mathbb{C}$ is complete so by the standard continuous linear extension theorem, $\exists \tilde{\omega} \in \tilde{B}(\tilde{A}, \mathbb{C}) : \tilde{\omega}|_{\text{im } i} = \omega$.

This continuous extension is clearly also a unital homomorphism satisfying $\tilde{\omega} \circ i = \omega$. 

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As for Banach algebras we define, when \( \Omega(A) \neq \emptyset \), for \( a \in A \) the map
\[
\hat{a} : \Omega(A) \to \mathbb{C}; \omega \mapsto \omega(a).
\]
Let the set of these maps be \( \hat{A} \).

**LEMMA A2.6:** Let \( A \) be a normed algebra with \( \Omega(A) \) non-empty. Let \( \tau_i \) be the initial topology on \( \Omega(A) \) induced by \( \hat{A} \). Then \( \Omega(A) \) is a locally compact Hausdorff space and is compact if \( A \) is unital.

*Proof.* First we observe that \( \tau_i = \tau_{\Omega(A)}^* \) where \( \tau^* \) is the weak *-topology on \( A^* \). This is because \( \tau_i \) is the initial topology induced by the restrictions to \( \Omega(A) \) of the set of maps inducing the topology \( \tau^* \) (see [5] exercise 19-C).

So it is immediate that \( \Omega(A) \) is Hausdorff, since \( \tau^* \) is known to be. Now let \( \omega \in \Omega(A) \).

By Lemma A2.5, \( \exists \hat{\omega} \in \Omega(A) : \hat{\omega} \circ i. \) By the result for Banach algebras, \( \hat{\omega} \in \Omega(A) \Rightarrow \| \hat{\omega} \| \leq 1. \) So \( \| \omega \| = \| \hat{\omega} \circ i \| \leq \| \hat{\omega} \| \| i \| \leq 1. \) Therefore \( \Omega(A) \subseteq S^* := B_{A^*}[0,1]. \) By the Banach-Alaoglu theorem, \( S^* \) is compact with respect to \( \tau_{\Omega(A)}^* = \tau_i. \) The rest of the proof (that \( \Omega(A) \cup \{ 0 \} \) is closed in \( S^* \) so that \( \Omega(A) \) is locally compact) is exactly the same as for Banach algebras (see for example [20] *1.3.5*) so we omit this.

**COROLLARY A2.7:** \( \forall \omega \in \Omega(A) \| \omega \| = 1. \)

**LEMMA A2.8:** Let \( A \) be a normed algebra with \( \Omega(A) \) non-empty. Then \( \hat{A} \) is a subalgebra of \( C_0(\Omega(A)) \) which separates the points of \( \Omega(A) \) strongly.

*Proof.* Let \( \Gamma : A \to C_0(\Omega(A)); a \mapsto \hat{a}. \) It is clear that \( \hat{A} = \text{im} \Gamma \) separates the points of \( \Omega(A) \) strongly: if \( \omega_1, \omega_2 \in \Omega(A) \), \( \omega_1 \neq \omega_2 \Rightarrow \exists a \in A : \omega_1(a) = \hat{a}(\omega_1) \neq \hat{a}(\omega_2) = \omega_2(a). \) That \( \hat{A} \) vanishes at no point of \( \Omega(A) \) is built into the definition of characters.

The verifications that \( \Gamma \) is a homomorphism and that \( \forall a \in A \hat{a} \) is continuous and vanishes at infinity are the same as for Banach algebras so we omit the detail. Since \( \hat{A} = \text{im} \Gamma \), it is automatically an algebra.

To summarise our progress: we have established a norm decreasing homomorphism, \( \Gamma \), from the normed algebra \( A \) into a \( C_0(X) \) for some locally compact Hausdorff space, and that in fact it is given in the same way as for Banach algebras but restricting attention to the continuous characters of \( A \). We shall still call \( \Gamma \) the Gelfand transform. However it is now longer true that \( \Gamma \) is injective iff \( A \) is semisimple and the relevant condition on \( A \) is given by the following definition, given in [1]:

**DEFINITION A2.9 ([1]):** Let \( A \) be a normed unital algebra. \( A \) is called *tractable* if the intersection of its closed maximal ideals is \( \{ 0 \}. \)

(Recall that the existence of the unit implies maximal ideals exist.) In fact by lemma A2.4 we can generalise (for commutative \( A \) at least) this to

**DEFINITION A2.10** Let \( A \) be a normed algebra. \( A \) is *tractable* if the intersection of its closed modular maximal ideals, \( M(A) \), is \( \{ 0 \}. \)

There is a definition of modular ideals for non-commutative normed algebras but we shall not refer to it. (The definition implies that \( M(A) \) is non-empty (unless \( A = 0 \)) since the empty intersection of subsets of \( A \) is \( A \) itself.) It is consistent with the unital case because then every maximal ideal is modular with ‘modular unit’ \( 1_A. \) We have

**THEOREM A2.11:** Let \( A \) be a normed algebra for which \( \Omega(A) \neq \emptyset \) and let \( \Gamma \) be the Gelfand transform. Then \( \Gamma \) is injective iff \( A \) is tractable.

*Proof.* By Lemma A2.4 there is a bijection \( \Omega(A) \to M(A); \omega \mapsto \ker \omega. \) We have
\[
\ker \Gamma = \{ a \in A : \hat{a} = 0 \}
= \{ a \in A : \forall \omega \in \Omega(A) \omega(a) = 0 \}
= \bigcap_{\omega \in \Omega(A)} \{ a \in A : \omega(a) = 0 \}
= \bigcap_{\omega \in \Omega(A)} \ker \omega
= \cap M(A).
\]
and the result follows.

A result concerning tractability, which we state because it is needed in chapter 3, is:

**PROPOSITION A2.12**: Let $A$ be a commutative unital normed algebra and $(\hat{A}, i)$ a completion. Then if $\hat{A}$ is semisimple, $A$ is tractable.

*Proof.* Suppose $\hat{A}$ is semisimple and that $a \in \bigcap M(A)$. Let $\omega \in \Omega(\hat{A})$. $i$ is a unital homomorphism so $\omega \circ i \in \Omega(A)$. So by assumption $(\omega \circ i)(a) = \omega(i(a)) = 0$. Therefore $i(a) \in \bigcap M(\hat{A}) = (0)$. But $i$ is injective so $a = 0$. □

The converse to this is false; see chapter 3 for an example.

**DEFINITION A2.13**: Let $A$ be a normed algebra. $K(A) := \cap M(A)$ is the radical of $A$.

The last result of this section confirms the result asserted in [1] that factoring a normed algebra by ker $\Gamma$ (where it exists) gives a tractable normed algebra. We prove it in the commutative, unital case; it is applied in the proof of Theorem 2.13.

**PROPOSITION A2.14**: Let $A$ be a commutative unital normed algebra and $M(A)$ be as above. Then $A/K(A)$ is a tractable normed algebra.

*Proof.* By the fundamental isomorphism theorem, $\Gamma$ induces an isomorphism $\tilde{\Gamma}: A/K(A) \rightarrow \hat{A}$ with norm $\|\Gamma\|$. Now $\hat{A} \subseteq C(\Omega(A))$ and we have the evaluation characters $\varepsilon_\omega: \hat{A} \rightarrow \mathbb{C}; f \mapsto f(\omega)$ ($\omega \in \Omega(A)$)

It follows that $\hat{A}$ is tractable because if $f \in K(\hat{A})$ then $\forall \omega \in \Omega(A) \varepsilon_\omega(f) = f(\omega) = 0$ so $f = 0$.

Now $\tilde{\Gamma}$ is an algebraic isomorphism so it induces a bijection between the maximal ideals of $A/K(A)$ and $\hat{A}$ given by $J \mapsto \tilde{\Gamma}(J)$. $\tilde{\Gamma}$ is continuous so for every closed maximal ideal, $J$, of $\hat{A}$, $\tilde{\Gamma}^{-1}(J)$ is a closed maximal ideal of $A/K(A)$. Therefore

$$K(A/K(A)) = \bigcap M(A/K(A)) \subseteq \bigcap \{\tilde{\Gamma}^{-1}(J) : J \in M(\hat{A})\} = \tilde{\Gamma}^{-1}\left(\bigcap M(\hat{A})\right) = \tilde{\Gamma}^{-1}(\{0\}) = \{0\}$$ □
Appendix 3
Set Theory

The constructions in chapters 5 and 6 require powerful set theory and here we set out the relevant definitions, results, notations and other conventions. All the results in this chapter are well known and we shall omit all proofs of the more standard ones; we refer the reader to [21] for these.

DEFINITION A3.1: Let $(W, \leq)$ be a well-ordered set and $\alpha, \beta \in W$ with $\alpha \leq \beta$. The segment $(\alpha, \beta)$ is $\{w \in W : \alpha < w < \beta\}$ while the interval $[\alpha, \beta] := \{w \in W : \alpha \leq w \leq \beta\}$. Let $0 = \inf W$. An initial segment of $W$ is a subset of the form $[0, \alpha)$ or $W$, the former being proper.

The following three lemmas are needed in chapter 5.

LEMMA A3.2: Let $A$ be an infinite set. Then $\# A = \#(A \times A)$.
Proof. A solution to this can be found in [21].

COROLLARY A3.3: Let $A$ be an infinite set and $U$ be a well-ordered set with $0 < \# U \leq \# A$. Suppose that $\forall \alpha \in U A_\alpha$ is a set with $\#A_\alpha = \#A$. Then

$$\# \left( \bigcup_{\alpha \in U} A_\alpha \right) = \#A.$$  

Proof. Obviously $\#A \leq \# \left( \bigcup_{\alpha \in U} A_\alpha \right)$. Now for $a \in \bigcup_{\alpha \in U} A_\alpha$, $\exists \alpha \in U : a \in A_\alpha$. Also there is a bijection $\phi_\alpha : A_\alpha \to A$. We have $(\alpha, \phi_\alpha(a)) \in U \times A$ and there is an injection $\bigcup_{\alpha \in U} A_\alpha \to U \times A$. The result now follows from the Cantor-Schröder-Bernstein theorem since $\#(U \times A) \leq \#(A \times A) = \#A$.

COROLLARY A3.4: Let $A$ be an infinite ring. Then $\# A = \#A[x]$.
Proof. Clearly $\#A \leq \#A[x]$. Define

$$\theta : A[x] \to \bigcup_{n \in \mathbb{N}} A^n ; \alpha(x) = a_0 + \cdots + a_n x^n \mapsto \begin{cases} (a_0, \ldots, a_n) \in A^{n+1} & a_n \neq 0 \\ (0) \in A & \alpha(x) = 0. \end{cases}$$

$\theta$ is an injection so $\#A[x] \leq \# \left( \bigcup_{n \in \mathbb{N}} A^n \right) \leq \#(\mathbb{N} \times A)$ as in the proof of Corollary A3.3. And $\#(\mathbb{N} \times A) \leq \#(A \times A) = \#A$ by Lemma A3.2.

DEFINITION A3.5: A set, $S$, is transitive if $\forall s \in S s \subseteq S$.

DEFINITION A3.6: The set $\alpha$ is an ordinal if $\alpha$ is transitive and $\alpha = \emptyset$ or $(\alpha, \in)$ is well-ordered.

PROPOSITION A3.7: A set, $S$, is an ordinal $\iff \exists$ a well-ordering on $S$ such that $\forall s \in S \{0, s\} = s$.

PROPOSITION A3.8: Let $\alpha, \beta$ be ordinals. Then $\alpha \in \beta \iff \alpha \subset \beta$.

DEFINITION A3.9: Whenever $\alpha, \beta$ are ordinals $\alpha < \beta :\iff \alpha \subset \beta$.

DEFINITION A3.10: Let $(S, \leq), (T, \subseteq)$ be partially ordered sets and $\phi : S \to T$ be bijective. $\phi$ is an order isomorphism if $\forall s_1, s_2 \in S s_1 \leq s_2 \iff \phi(s_1) \subseteq \phi(s_2)$.

PROPOSITION A3.11: If $\phi : \alpha \to \beta$ is an order isomorphism of ordinals then $\alpha = \beta$.  

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PROPOSITION A3.12: Let $E$ be a non-empty set of ordinals. Then $(E, \subseteq)$ is well-ordered.

PROPOSITION A3.13: Let $E$ be a set of ordinals. Then $\cup E$ is an ordinal.

PROPOSITION A3.14: Let $\alpha$ be an ordinal. Then $\alpha + 1 := \alpha \cup \{\alpha\}$ is an ordinal.

DEFINITION A3.15: Let $E$ be a set of ordinals and $\beta \in E$. $\beta$ is a non-limit ordinal in $E$ if $\exists \alpha \in E : \beta = \alpha + 1$. Otherwise $\beta$ is a limit ordinal in $E$.

THEOREM A3.16 (The Counting Theorem): Let $W$ be a well-ordered set. Then there is a unique ordinal which is order isomorphic to $W$.

THEOREM A3.17 (The Principle of Transfinite Induction): Let $W$ be a well-ordered set and $J \subseteq W$ be such that $\forall w \in W \ [0, w) \subseteq J \Rightarrow w \in J$. Then $J = W$.

THEOREM A3.18 (The Transfinite Recursion Theorem): Let $W$ be a well-ordered set and $X$ a set. Let $Q = \cup I : I$ is an initial segment of $\text{Map}(I, X)$

Let $f \in \text{Map}(Q, X)$. Then $\exists U \in \text{Map}(W, X) : \forall a \in W U(a) = f (U|_{[0,a)})$.

THEOREM A3.19: Let $S \neq \emptyset$. Then there exists a well-ordering on $S$.

The applications require the existence of an uncountable well-ordered set which has the property that all its proper initial segments are countable. By A3.19.16 there certainly exist uncountable ordinals. By A3.12 there is a unique first uncountable ordinal, which we denote $\omega_1$. It is easy to see that $\omega_1$ has the desired property. The sort of situation in which it becomes useful is illustrated by the result below (which is also used in chapter 6).

THEOREM A3.20: Let $(X, d)$ be a metric space and $(F_\alpha)_{0 \leq \alpha < \omega_1}$ be a family of closed sets in $X$ such that $\forall \alpha \leq \beta < \omega_1 F_\alpha \subseteq F_\beta$. Then $F := \bigcup_{\alpha < \omega_1} F_\alpha$ is closed.

Proof. Let $(x_n)_{n \in \mathbb{N}} \subseteq F$ and $x_n \rightarrow x \in X (n \rightarrow +\infty)$. Now $\forall n \in \mathbb{N} \exists \alpha_n < \omega_1 : x_n \in F_{\alpha_n}$.

Now $\exists \alpha < \omega_1 : \forall n \in \mathbb{N} \alpha_n \leq \alpha$ for if this were not true then $\forall \alpha < \omega_1 \exists n \in \mathbb{N} : \alpha < \alpha_n$. Hence $[0, \omega_1) \subseteq \bigcup_{n \in \mathbb{N}} [0, \alpha_n)$. But by the definition of $\omega_1$, $[0, \alpha_n)$ is countable $\forall n \in \mathbb{N}$. Therefore $\omega_1 = [0, \omega_1)$ is countable since it is contained in a countable union of countable sets. But this is a contradiction.

Therefore $(x_n) \subseteq F_\alpha$ and since $F_\alpha$ is closed $x \in F_\alpha \subseteq F$. \qed
References

[1] Arens, R. and Hoffman, K. (1956) Algebraic Extension of Normed Algebras. Proc. Am. Math. Soc., 7, 203-210

[2] Bonsall, F. F. and Duncan, J. (1973) Complete Normed Algebras. Germany: Springer-Verlag.

[3] Bollobás, B. (1975) Adjoining Inverses to Commutative Banach Algebras, In: Williamson, J. H. (ed.) Algebras in Analysis. Norwich: Academic Press Inc. (London) Ltd. 256-257

[4] Lindberg, J. A. (1964) Factorization of Polynomials over Banach Algebras. Trans. Am. Math. Soc., 112, 356-368

[5] Simmons, G. F. (1963) Introduction to Topology and Modern Analysis. Singapore: McGraw-Hill Book Co.

[6] Jacobson, N. (1996) Basic Algebra I. 2nd ed. United States of America: W.H. Freeman and Company.

[7] Lindberg, J. A. (1964) Algebraic Extensions of Commutative Banach Algebras. Pacif. J. Math., 14, 559-583

[8] Stout, E. L. (1973) The Theory of Uniform Algebras. Tarrytown-on-Hudson, New York: Bogden and Quigley Inc.

[9] Lindberg, J. A. (1963) On the Completion of Tractable Normed Algebras. Proc. Am. Math. Soc., 14, 319-321

[10] Cole, B. J. (1968) One-Point Parts and the Peak-Point Conjecture, Ph.D. Thesis, Yale University.

[11] Lindberg, J. A. (1973) Integral Extensions of Commutative Banach Algebras. Can. J. Math., 25, 673-686

[12] Heuer, G. A. and Lindberg, J. A. (1963) Algebraic Extensions of Continuous Function Algebras. Proc. Am. Math. Soc., 14, 337-342

[13] Rickart, C. E. (1960) Banach Algebras. United States of America: D. Van Nostrand Company, Inc.

[14] Karahanjan, M. I. (1979) Some Algebraic Characterisations of the Algebra of All Continuous Functions on a Locally Connected Compactum. Math. USSR Sb., 35, 681-696

[15] Rotman, J. J. (1979) An Introduction to Homological Algebra. London: Academic Press Inc.

[16] Leibowitz, G. M. (1970) Lectures on Complex Function Algebras. United States of America: Scott, Foresman and Company.

[17] Kelley, J. L. (1955) General Topology. New-York: Van Nostrand.

[18] Feinstein, J. F. (1989) Derivations from Banach Function Algebras, Ph.D. Thesis, University of Leeds

[19] Zame, W. R. (1984) Covering Spaces and the Galois Theory of Commutative Banach Algebras. J. Funct. Anal., 27, 151-171

[20] Murphy, G. J. (1990) C*-Algebras and Operator Theory. Boston: Academic Press.

[21] Halmos, P. R. (1970) Naive Set Theory. New-York: Springer-Verlag.