Spectral triples and wavelets for higher-rank graphs

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Abstract

In this paper, we present a new way to associate a spectral triple to a higher-rank graph \( \Lambda \). Moreover, we prove that this spectral triple has a close connection to the wavelet decomposition of the infinite path space of \( \Lambda \) which was introduced by Farsi, Gillaspy, Kang, and Packer in 2015. We first introduce the concept of stationary \( k \)-Bratteli diagrams, to associate a family of ultrametric Cantor sets to a finite, strongly connected higher-rank graph \( \Lambda \). Then we show that under mild hypotheses, such Cantor sets are \( \zeta \)-regular in the sense of Pearson and Bellissard. Consequently, one can compute the Dixmier trace associated to the Pearson-Bellissard spectral triples of these Cantor sets. We show that in this setting, the measure \( \mu \) induced by the Dixmier trace agrees with the measure \( M \) on the infinite path space \( \Lambda^\infty \) of \( \Lambda \) which was introduced by an Huef, Laca, Raeburn, and Sims. Finally, we investigate the eigenspaces of a family of Laplace-Beltrami operators associated to the Dirichlet forms of the spectral triples. We show that these eigenspaces refine the wavelet decomposition of \( L^2(\Lambda^\infty, M) \) which was constructed by Farsi et al.

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1 Introduction

Both spectral triples and wavelets are algebraic structures which encode geometrical information. In this paper, we expand the correspondence established in [15] between wavelets and spectral triples for the Cuntz algebras $\mathcal{O}_N$ to the setting of higher-rank graphs. To be precise, we associate a Pearson-Bellissard spectral triple [36] to a higher-rank graph (or $k$-graph) $\Lambda$, and relate this spectral triple with the representation of the higher-rank graph $C^*$-algebra $C^*(\Lambda)$ and the associated wavelet decomposition which were introduced in [17]. We also investigate the geometry of ultrametric Cantor sets associated to $\Lambda$ by studying the $\zeta$-function and Dixmier trace of the spectral triple.

Spectral triples were introduced by Connes in [10] as a noncommutative generalization of a compact Riemannian manifold. A spectral triple consists of a representation of a pre-$C^*$-algebra $A$ on a Hilbert space $H$, together with a Dirac-type operator $D$ on $H$, which satisfy certain commutation relations. In the case when $A = C^\infty(X)$ is the algebra of smooth functions on a compact spin manifold $X$, Connes showed [11] that the algebraic structure of the associated spectral triple suffices to reconstruct the Riemannian metric on $X$. Moreover, Connes established in [10] that the Dixmier trace of this spectral triple recovers the Riemannian volume form on $X$. The $\zeta$-function and Dixmier trace associated to a spectral triple also play important roles in the applications of spectral triples to physics, from the standard model [12] to classical field theory [27].

In addition to spin manifolds, Connes studied spectral triples for the triadic Cantor set and Julia set in [10]. Shortly thereafter, Lapidus [33] suggested studying spectral triples $(A, H, D)$ where $A$ is a commutative algebra of functions on a fractal space $X$, and investigating which aspects of the geometry of $X$ are recovered from the spectral triple. Of the many authors (cf. [8, 18, 36]) who have pursued Lapidus’ program, we focus here on the spectral triples introduced by Pearson and Bellissard in [36].

Motivated by a desire to apply the tools of noncommutative geometry to the study of transversals of aperiodic Delone sets [24], Pearson and Bellissard constructed in [36] spectral triples for ultrametric Cantor sets associated to Michon trees. They also showed how to recover geometric information about the Cantor set $\mathcal{C}$ from their spectral triple: using the $\zeta$-function and the Dixmier trace, Pearson and Bellissard reconstructed the ultrametric and the upper box dimension of $\mathcal{C}$. Moreover, they constructed a family of Laplace-Beltrami operators $\Delta_s$, $s \in \mathbb{R}$, on $L^2(\mathcal{C}, \mu)$, where the measure $\mu$ arises from the Dixmier trace. Julien and Savinien subsequently applied the Pearson-Bellissard spectral triples to the study of substitution tilings in [26], by sharpening many of the results from [36] and reinterpreting them using stationary Bratteli diagrams.

In this paper, we extend the Pearson-Bellissard spectral triples to the setting of higher-rank graphs. A $k$-dimensional generalization of directed graphs, higher-rank graphs (also called $k$-graphs) were introduced by Kumjian and Pask in [32] to provide computable, combinatorial examples of $C^*$-algebras. The combinatorial character of $k$-graph $C^*$-algebras has facilitated the analysis of their structural properties, such as simplicity and ideal structure [37, 38, 13, 28, 7], quasidiagonality [9] and KMS states [23, 22, 21]. In particular, results such as [40, 6, 5, 35] show that higher-rank graphs often provide concrete examples of $C^*$-algebras which are relevant to Elliott’s classification program for simple separable nuclear $C^*$-algebras.

In order to associate Pearson-Bellissard spectral triples to $k$-graphs, we introduce a new class of Bratteli diagrams: namely, the stationary $k$-Bratteli diagrams. Where a stationary Bratteli diagram is completely determined by a single square matrix $A$, the stationary $k$-Bratteli diagrams are determined by $k$ matrices $A_1, \ldots, A_k$; see Definition 2.5 below. The space of infinite paths $\partial \mathcal{B}$ of a stationary $k$-Bratteli diagram $\mathcal{B}$ is often an ultrametric Cantor set, enabling us to study its
associated Pearson-Bellissard spectral triple. Indeed, if the matrices $A_1, \ldots, A_k$ are the adjacency matrices for a $k$-graph $\Lambda$, then the space of infinite paths in $\Lambda$ is homeomorphic to the ultrametric Cantor set $\partial B$. In other words, the Pearson-Bellissard spectral triples for stationary $k$-Bratteli diagrams can also be viewed as spectral triples for higher-rank graphs.

The complexity of stationary $k$-Bratteli diagrams, as compared to the stationary Bratteli diagrams studied in [26], requires us to use careful and delicate arguments to analyze the $\zeta$-function and Dixmier trace of our spectral triples. This attention to detail has enabled us to obtain results which, when restricted to the setting of stationary Bratteli diagrams, are stronger than those of [36, 26]. To be precise, we prove in Theorem 3.8 and Theorem 3.9 that the abscissa of convergence, which, when restricted to the setting of stationary Bratteli diagrams, are stronger than those of and Dixmier trace of our spectral triples. This attention to detail has enabled us to obtain results about the matrices $A_1, \ldots, A_k$, forming a family of irreducible matrices in the sense of [23]. Even for the stationary Bratteli diagrams studied in [26], our irreducibility hypothesis is strictly weaker than the hypothesis of primitivity invoked in [26]. Moreover, in Corollary 3.10, we give an explicit formula for the Dixmier trace, identifying it with the canonical measure $M$ which was introduced in [23].

As mentioned earlier, one of our motivations for studying Pearson-Bellissard spectral triples for $k$-graphs was to understand their relationship with the wavelets and representations for $k$-graphs introduced in [17]. Wavelet analysis has many applications in various areas of mathematics, physics and engineering. For example, it has been used to study L-graphs introduced in [17]. Wavelet analysis has many applications in various areas of mathematics, for example, the theory of quantum gravity [14, 2].

Although wavelets were introduced as orthonormal bases or frames for $L^2(\mathbb{R}^n)$ which behaved well under compression algorithms, wavelet decompositions for $L^2(X)$, where $X$ is a fractal space, were defined by Jonsson [25] and Strichartz [41] shortly thereafter. In this fractal setting, the wavelet orthonormal bases reflect the self-similar structure of $X$. A few years later, Jonsson and Strichartz’ fractal wavelets inspired Marcolli and Paolucci [34] to construct a wavelet decomposition of $L^2(\Lambda, \mu)$ for the Cuntz-Krieger algebra $\mathcal{O}_A$, where $A$ is an $N \times N$ matrix, $\Lambda$ denotes the limit set of infinite sequences in an alphabet on $N$ letters, and $\mu$ is a Hausdorff measure on $\Lambda$. Similar wavelets were developed in the higher-rank graph setting by four of the authors of the current paper [17], using a separable representation $\pi$ of the $k$-graph $C^*$-algebra $C^*(\Lambda)$. In particular, this representation gave us a wavelet decomposition of $L^2(\Lambda^\infty, M)$, where $\Lambda^\infty$ denotes the space of infinite paths in the $k$-graph $\Lambda$, and the measure $M$ was introduced by an Huef et al. in [23]. This wavelet decomposition is given by

$$L^2(\Lambda^\infty, M) = \mathcal{V}_0 \oplus \bigoplus_{n \geq 0} \mathcal{W}_n.$$  

Each subspace $\mathcal{W}_n = \{S_\lambda f : f \in \mathcal{W}_0, \lambda \in \Lambda^{(n, \ldots, n)}\}$ is constructed from $\mathcal{W}_0$ by means of generalized “scaling and translation” operators $S_\lambda := \pi(s_\lambda)$ which reflect the (higher-rank) graph structure of $\Lambda$. (See Theorem 4.2 of [17] or Section 4 below.)

One of the main results of this paper, Theorem 4.3, proves that the spectral triples of Pearson and Bellissard [36] are intimately tied to the wavelets of [17]. Recall that a Pearson-Bellissard spectral triple for an ultrametric Cantor set $\mathcal{C}$ gives rise to a family of Laplace-Beltrami operators $\Delta_s$, $s \in \mathbb{R}$, on $L^2(\mathcal{C}, \mu)$ associated to the spectral triple’s Dirichlet form as in Equation (29) below. Julien and Savinien established in [26] that in the Bratteli diagram setting the eigenspaces of $\partial C$ are the matrices $A_1, \ldots, A_k$ for $j \in \mathbb{N}$ in Theorem 4.2 of [17].

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1The subspaces denoted in this paper by $\mathcal{W}_n$ were labeled $\mathcal{W}_{j,\Lambda}$ for $j \in \mathbb{N}$ in Theorem 4.2 of [17].
$\Delta_s$ are parametrized by the finite paths $\gamma$ in the Bratteli diagram. Theorem 4.3 establishes that when $(\mathcal{C}, \mu) = (\Lambda^\infty, M)$, the eigenspaces $E_\gamma$ of the Laplace-Beltrami operators refine the wavelet decomposition of $\mathcal{H}$.

This paper is organized as follows. In Section 2 we recall the basic facts about higher-rank graphs (or $k$-graphs) and we develop the machinery of stationary $k$-Bratteli diagrams (Definition 2.5). This enables us to construct a family of ultrametrics $\{d_\delta : \delta \in (0, 1)\}$ on the infinite path space $\Lambda^\infty$ of a $k$-graph $\Lambda$. In many situations, $\Lambda^\infty$ is a Cantor set (see Proposition 2.4); Section 3 studies the fine structure of the Pearson-Bellissard spectral triples associated to the ultrametric Cantor sets $\{\Lambda^\infty, d_\delta\}$. The major technical achievements of this paper are Theorem 3.8 and Theorem 3.9. Theorem 3.8 establishes that the $\zeta$-function of the spectral triple associated to the ultrametric Cantor set $({\Lambda^\infty, d_\delta})$ has abscissa of convergence $\delta$, and Theorem 3.9 shows that the Dixmier trace of the spectral triple induces a finite probability measure $\mu_\delta$ on $\Lambda^\infty$. Corollary 3.10 then shows that under mild additional hypotheses, all of the probability measures $\mu_\delta$ agree with the Borel probability measure $M$ on $\Lambda^\infty$ which was identified in Proposition 8.1 of [23] and which we used in [17] to construct a wavelet decomposition of $L^2(\Lambda^\infty, M)$.

Finally, Section 4 presents the promised connection between the Pearson-Bellissard spectral triples and the wavelet decomposition of $L^2(\Lambda^\infty, M)$ from [17]. Under appropriate hypotheses we show in Theorem 4.3 that the eigenspaces $E_\gamma$ of the Laplace-Beltrami operator $\Delta_s$ refine the wavelet decomposition of $\mathcal{H}$: namely, for all $n \in \mathbb{N}$, $$W_n = \bigoplus_{nk \leq |\gamma| < (n+1)k} E_\gamma.$$ 

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2 Higher-rank graphs and ultrametric Cantor sets

In this section, we review the basic definitions and results that we will need about directed graphs, higher-rank graphs, (weighted/stationary) Bratteli diagrams, infinite path spaces, and (ultrametric) Cantor sets. Throughout this article, $\mathbb{N}$ will denote the non-negative integers.

2.1 Bratteli diagrams

A directed graph is given by a quadruple $E = (E^0, E^1, r, s)$, where $E^0$ is the set of vertices of the graph, $E^1$ is the set of edges, and $r, s : E^1 \to E^0$ denote the range and source of each edge. A vertex $v$ in a directed graph $E$ is a sink if $s^{-1}(v) = \emptyset$; we say $v$ is a source if $r^{-1}(v) = \emptyset$.

Definition 2.1. [1] A Bratteli diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ is a directed graph with vertex set $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$, and edge set $\mathcal{E} = \bigsqcup_{n \geq 1} \mathcal{E}_n$, where $\mathcal{E}_n$ consists of edges whose source vertex lies in $\mathcal{V}_n$ and whose range vertex lies in $\mathcal{V}_{n-1}$, and $\mathcal{V}_n$ and $\mathcal{E}_n$ are finite sets for all $n$. 


For a Bratteli diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, define a sequence of adjacency matrices $A_n = (f^n(v, w))_{v,w}$ of $\mathcal{B}$ for $n \geq 1$, where

$$f^n(v, w) = \#\left\{ e \in \mathcal{E}_n : r(e) = v \in \mathcal{V}_{n-1}, s(e) = w \in \mathcal{V}_n \right\},$$

where by $\#(Q)$ we denote the cardinality of the set $Q$. A Bratteli diagram is stationary if $A_n = A_1 =: A$ are the same for all $n \geq 1$. We say that $\eta$ is a finite path of $\mathcal{B}$ if there exists $m \in \mathbb{N}$ such that $\eta = \eta_1 \ldots \eta_m$ for $\eta_i \in \mathcal{E}_i$, and in that case the length of $\eta$, denoted by $|\eta|$, is $m$.

**Remark 2.2.** In the literature, Bratteli diagrams traditionally have $s(\mathcal{E}_n) = \mathcal{V}_n$ and $r(\mathcal{E}_n) = \mathcal{V}_{n+1}$; our edges point the other direction for consistency with the standard conventions for higher-rank graphs and their $C^*$-algebras.

It is also common in the literature to require $|\mathcal{V}_0| = 1$ and to call this vertex the root of the Bratteli diagram; we will NOT invoke this hypothesis in this paper.

**Definition 2.3.** Given a Bratteli diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, denote by $X_\mathcal{B}$ the set of all of its infinite paths:

$$X_\mathcal{B} = \{(x_n)_{n \geq 1} : x_n \in \mathcal{E}_n \text{ and } s(x_n) = r(x_{n+1}) \text{ for } n \geq 1 \}.$$

For each finite path $\lambda = \lambda_1 \lambda_2 \ldots \lambda_\ell$ in $\mathcal{B}$ with $r(\lambda) \in \mathcal{V}_0$, $\lambda_i \in \mathcal{E}_i$, define the cylinder set $[\lambda]$ by

$$[\lambda] = \{x = (x_n)_{n \geq 1} \in X_\mathcal{B} : x_i = \lambda_i \text{ for } 1 \leq i \leq \ell \}.$$

The collection $\mathcal{T}$ of all cylinder sets forms a compact open sub-basis for a locally compact Hausdorff topology on $X_\mathcal{B}$ and cylinder sets are clopen; we will always consider $X_\mathcal{B}$ with this topology.

The following proposition will tell us when $X_\mathcal{B}$ is a Cantor set; that is, a totally disconnected, compact, perfect topological space.

**Proposition 2.4.** (Lemma 6.4. of [1]) Let $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ be a Bratteli diagram such that $\mathcal{B}$ has no sinks outside of $\mathcal{V}_0$, and no sources. Then $X_\mathcal{B}$ is a totally disconnected compact Hausdorff space, and the following statements are equivalent:

1. The infinite path space $X_\mathcal{B}$ of $\mathcal{B}$ is a Cantor set;
2. For each infinite path $x = (x_1, x_2, \ldots)$ in $X_\mathcal{B}$ and each $n \geq 1$ there is an infinite path $y = (y_1, y_2, \ldots)$ with $x \neq y$ and $x_k = y_k$ for $1 \leq k \leq n$;
3. For each $n \in \mathbb{N}$ and each $v \in \mathcal{V}_n$ there is $m \geq n$ and $w \in \mathcal{V}_m$ such that there is a path from $w$ to $v$ and

$$\#(r^{-1}(\{w\})) \geq 2.$$

### 2.2 Higher-rank graphs and stationary $k$-Bratteli diagrams

**Definition 2.5.** Let $A_1, A_2, \ldots, A_k$ be $N \times N$ matrices with non-negative integer entries. The stationary $k$-Bratteli diagram associated to the matrices $A_1, \ldots, A_k$, which we will call $\mathcal{B}(A_1, \ldots, A_k)$, is the Bratteli diagram given by a filtered set of vertices $\mathcal{V} = \bigsqcup_{n \in \mathbb{N}} \mathcal{V}_n$ and a filtered set of edges $\mathcal{E} = \bigsqcup_{n \geq 1} \mathcal{E}_n$, where the edges in $\mathcal{E}_n$ go from $\mathcal{V}_n$ to $\mathcal{V}_{n-1}$, such that:

(a) For each $n \in \mathbb{N}$, $\mathcal{V}_n$ consists of $N$ vertices, which we will label $1, 2, \ldots, N$. 


(b) When \( n \equiv i \pmod{k} \), there are \( A_i(p,q) \) edges whose range is the vertex \( p \) of \( \mathcal{V}_{n-1} \) and whose source is the vertex \( q \) of \( \mathcal{V}_n \).

In other words, the matrix \( A_1 \) determines the edges with source in \( \mathcal{V}_1 \) and range in \( \mathcal{V}_0 \); then the matrix \( A_2 \) determines the edges with source in \( \mathcal{V}_2 \) and range in \( \mathcal{V}_1 \); etc. The matrix \( A_k \) determines the edges with source in \( \mathcal{V}_k \) and range in \( \mathcal{V}_{k-1} \), and the matrix \( A_1 \) determines the edges with range in \( \mathcal{V}_k \) and source in \( \mathcal{V}_{k+1} \).

Note that a stationary 1-Bratteli diagram is often denoted a stationary Bratteli diagram in the literature (cf. \([4, 26]\)).

Just as a directed graph has an associated adjacency matrix \( A \) which also describes a stationary Bratteli diagram \( B_A \), the higher-dimensional generalizations of directed graphs known as higher-rank graphs or \( k \)-graphs give us \( k \) commuting matrices \( A_1, \ldots, A_k \) and hence a stationary \( k \)-Bratteli diagram.

We use the standard terminology and notation for higher-rank graphs, which we review below for the reader’s convenience.

**Definition 2.6.** \([32]\) A \( k \)-graph is a countable small category \( \Lambda \) equipped with a degree functor \( d : \Lambda \rightarrow \mathbb{N}^k \) satisfying the factorization property: whenever \( \lambda \) is a morphism in \( \Lambda \) such that \( d(\lambda) = m + n \), there are unique morphisms \( \mu, \nu \in \Lambda \) such that \( d(\mu) = m, d(\nu) = n \), and \( \lambda = \mu \nu \).

We use the arrows-only picture of category theory; thus, \( \lambda \in \Lambda \) means that \( \lambda \) is a morphism in \( \Lambda \). For \( n \in \mathbb{N}^k \), we write

\[
\Lambda^n := \{ \lambda \in \Lambda : d(\lambda) = n \}.
\]

When \( n = 0 \), \( \Lambda^0 \) is the set of objects of \( \Lambda \), which we also refer to as the vertices of \( \Lambda \).

Let \( r, s : \Lambda \rightarrow \Lambda^0 \) identify the range and source of each morphism, respectively. For \( v \in \Lambda^0 \) a vertex, we define

\[
v\Lambda^n := \{ \lambda \in \Lambda^n : r(\lambda) = v \} \quad \text{and} \quad \Lambda^n w := \{ \lambda \in \Lambda^n : s(\lambda) = w \}.
\]

We say that \( \Lambda \) is finite if \( \#(\Lambda^n) < \infty \) for all \( n \in \mathbb{N}^k \), and we say \( \Lambda \) is source-free or has no sources if \( \#(v\Lambda^n) > 0 \) for all \( v \in \Lambda^0 \) and \( n \in \mathbb{N}^k \).

For \( 1 \leq i \leq k \), write \( e_i \) for the \( i \)th standard basis vector of \( \mathbb{N}^k \), and define a matrix \( A_i \in M_{\Lambda^0}(\mathbb{N}) \) by

\[
A_i(v, w) = \#(v\Lambda^e_i w).
\]

We call \( A_i \) the \( i \)th adjacency matrix of \( \Lambda \). Note that the factorization property implies that the matrices \( A_i \) commute.

Despite their formal definition as a category, it is often useful to think of \( k \)-graphs as \( k \)-dimensional generalizations of directed graphs. In this interpretation, \( \Lambda^{e_i} \) is the set of “edges of color \( i \)” in \( \Lambda \). The factorization property implies that each \( \lambda \in \Lambda \) can be written as a concatenation of edges in the following sense: A morphism \( \lambda \in \Lambda \) with \( d(\lambda) = (n_1, n_2, \ldots, n_k) \) can be thought of as a \( k \)-dimensional hyper-rectangle of dimension \( n_1 \times n_2 \times \cdots \times n_k \). Any minimal-length lattice path in \( \mathbb{N}^k \) through the rectangle lying between 0 and \( (n_1, \ldots, n_k) \) corresponds to a choice of how to order the edges making up \( \lambda \), and hence to a unique decomposition or “factorization” of \( \lambda \). For example, the lattice path given by walking in straight lines from 0 to \( (n_1, 0, \ldots, 0) \) to \( (n_1, n_2, 0, \ldots, 0) \) to \( (n_1, n_2, n_3, 0, \ldots, 0) \), and so on, corresponds to the factorization of \( \lambda \) into edges of color 1, then edges of color 2, then edges of color 3, etc.

\[\text{We view } \mathbb{N}^k \text{ as a category with one object, namely 0, and with composition of morphisms given by addition.}\]
For any directed graph $E$, the category of its finite paths $\Lambda_E$ is a 1-graph; the degree functor $d : \Lambda_E \to \mathbb{N}$ takes a finite path $\lambda$ to its length $|\lambda|$. Example 2.7 below gives a less trivial example of a $k$-graph. The $k$-graphs $\Omega_k$ of Example 2.7 are also fundamental to the definition of the space of infinite paths in a $k$-graph.

**Example 2.7.** For $k \geq 1$, let $\Omega_k$ be the small category with

$$\text{Obj} (\Omega_k) = \mathbb{N}^k, \quad \text{Mor} (\Omega_k) = \{(m,n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}, \quad r(m,n) = m, \ s(m,n) = n.$$  

If we define $d : \Omega_k \to \mathbb{N}^k$ by $d(m,n) = n - m$, then $\Omega_k$ is a $k$-graph with degree functor $d$.

**Definition 2.8.** Let $\Lambda$ be a $k$-graph. An infinite path of $\Lambda$ is a $k$-graph morphism $x : \Omega_k \to \Lambda$; we write $\Lambda^\infty$ for the set of infinite paths in $\Lambda$. For each $p \in \mathbb{N}^k$, we have a map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ given by

$$\sigma^p(x)(m,n) = x(m+p,n+p)$$

for $x \in \Lambda^\infty$ and $(m,n) \in \Omega_k$.

**Remark 2.9.**

(a) Given $x \in \Lambda^\infty$, we often write $r(x) := x(0) = x(0,0)$ for the terminal vertex of $x$. This convention means that an infinite path has a range but not a source.

We equip $\Lambda^\infty$ with the topology generated by the sub-basis $\{[\lambda] : \lambda \in \Lambda\}$ of compact open sets, where

$$[\lambda] = \{x \in \Lambda^\infty : x(0,d(\lambda)) = \lambda\}.$$  

Remark 2.5 of [32] establishes that, with this topology, $\Lambda^\infty$ is a locally compact Hausdorff space.

Note that we use the same notation for a cylinder set of $\Lambda^\infty$ and a cylinder set of $X_B$ in Definition 2.3 since we will prove in Proposition 2.10 that $\Lambda^\infty$ is homeomorphic to $X_B^\Lambda$ for a finite, source-free $k$-graph $\Lambda$.

(b) For any $\lambda \in \Lambda$ and any $x \in \Lambda^\infty$ with $r(x) = s(\lambda)$, we write $\lambda x$ for the unique infinite path $y \in \Lambda^\infty$ such that $y(0,d(\lambda)) = \lambda$ and $\sigma^{d(\lambda)}(y) = x$. If $d(\lambda) = p$, the maps $\sigma^p$ and $\sigma^p := x \mapsto \lambda x$ are local homeomorphisms which are mutually inverse:

$$\sigma^p \circ \sigma_\lambda = id_{[s(\lambda)]}, \quad \sigma_\lambda \circ \sigma^p = id_{[\lambda]};$$

although the domain of $\sigma^p$ is $\Lambda^\infty \supseteq [\lambda]$.

Informally, one should think of $\sigma^p$ as “chopping off” the initial segment of length $p$, and the map $x \mapsto \lambda x$ as “gluing $\lambda$ on” to the front of $x$. By “front” and “initial segment” we mean the range of $x$, since an infinite path has no source.

We can now state precisely the connection between $k$-graphs and stationary $k$-Bratteli diagrams.

**Proposition 2.10.** Let $\Lambda$ be a finite, source-free $k$-graph with adjacency matrices $A_1, \ldots, A_k$. Denote by $B^\Lambda$ the stationary $k$-Bratteli diagram associated to the matrices $\{A_i\}_{i=1}^k$. Then $X_B^\Lambda$ is homeomorphic to $\Lambda^\infty$. 

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Proof. Fix $x \in \Lambda^\infty$ and write $1 := (1,1,\ldots,1) \in \mathbb{N}^k$. Then the factorization property for $\Lambda^\infty$ implies that there is a unique sequence

$$(\lambda_i)_i \in \prod_{i=1}^{\infty} \Lambda^1$$

such that $x = \lambda_1 \lambda_2 \lambda_3 \cdots$ with $\lambda_i = x((i-1)1,i1)$. (See the details in Remark 2.2 and Proposition 2.3 of [32]). Since there is a unique way to write $\lambda_i = f_i^1 f_i^2 \cdots f_i^k$ as a composable sequence of edges with $d(f_i^j) = e_j$, we have

$$x = f_1^1 f_2^1 \cdots f_k^1 f_k^2 \cdots f_k^j f_k^3 \cdots,$$

where the $nk + j$th edge has color $j$. Thus, for each $i$, $f_i^j$ corresponds to an entry in $A_j$, and hence

$$f_1^1 f_2^1 \cdots f_k^1 f_k^2 \cdots f_k^j f_k^3 \cdots \in X_{B_A}.$$

Conversely, given $y = (g_t)_t \in X_{B_A}$, we construct an associated $k$-graph infinite path $\tilde{y} \in \Lambda^\infty$ as follows. To $y = (g_t)_t$ we associate a sequence $(\eta_n)_{n \geq 1}$ of finite paths in $\Lambda$, where

$$\eta_n = g_1 \cdots g_{nk}$$

is the unique morphism in $\Lambda$ of degree $(n, \ldots, n)$ represented by the sequence of composable edges $g_1 \cdots g_{nk}$. Recall from [32] Remark 2.2 that a morphism $\tilde{y} : \Omega_k \to \Lambda$ is uniquely determined by $\{\tilde{y}(0,n1)\}_{n \in \mathbb{N}}$. Thus, the sequence $(\eta_n)_n$ determines $\tilde{y}$:

$$\tilde{y}(0,0) = r(y) = \tilde{y}(g_1), \quad \tilde{y}(0,n1) := \eta_n \forall \ n \geq 1.$$

The map $y \mapsto \tilde{y}$ is easily checked to be a bijection which is inverse to the map $x \mapsto f_1^1 f_2^1 \cdots f_k^1 f_k^2 \cdots f_k^j f_k^3 \cdots$.

Moreover, for any $i \in \mathbb{N}$, $0 \leq j \leq k - 1$, and any $\lambda = f_1^i f_2^i \cdots f_k^i f_k^j f_k^3 \cdots f_k^3 \cdots$ with $d(\lambda) = (i-1)1 + (1,1,0,\ldots,0)$, both of these bijections preserve the cylinder set $[\lambda]$. In particular, these bijections preserve the “square” cylinder sets $[\lambda]$ associated to paths $\lambda$ with $d(\lambda) = i1$ for some $i \in \mathbb{N}$. (If $i = 0$ then we interpret $d(\lambda) = 0 \cdot 1$ as meaning that $\lambda$ is a vertex in $V_0 \cong \Lambda^0$.) From the proof of Lemma 4.1 of [17], any cylinder set can be written as a disjoint union of square cylinder sets, and therefore the square cylinder sets generate the topology on $\Lambda^\infty$. We deduce that $\Lambda^\infty$ and $X_{B_A}$ are homeomorphic, as claimed. \qed

Remark 2.11. (a) Thanks to Proposition 2.10 we will usually identify the infinite path spaces $X_{B_A}$ and $\Lambda^\infty$, denoting this space by the symbol which is most appropriate for the context. In particular, the Borel structures on $X_{B_A}$ and $\Lambda^\infty$ are isomorphic, and so any Borel measure on $\Lambda^\infty$ induces a unique Borel measure on $X_{B_A}$ and vice versa.

(b) The bijection of Proposition 2.10 between infinite paths in the $k$-graph $\Lambda$ and in the associated Bratteli diagram $B_A$ does not extend to finite paths. While any finite path in the Bratteli diagram determines a finite path, or morphism, in $\Lambda$, not all morphisms in $\Lambda$ have a representation in the Bratteli diagram. For example, if $e_1$ is a morphism of degree $(1,0,\ldots,0) \in \mathbb{N}^k$ in a $k$-graph $(k > 1)$ with $r(e_1) = s(e_1)$, the composition $e_1 e_1$ is a morphism in the $k$-graph which cannot be represented as a path on the Bratteli diagram. However, the proof of Proposition 2.10 above establishes that “rainbow” paths in $\Lambda$ – morphisms of degree

$$(q+1,\ldots,q+1,q,\ldots,q)$$

for some $q \in \mathbb{N}$ and $1 \leq j \leq k - 1$ – can be represented uniquely as paths of length $kq + j$ in the Bratteli diagram.
2.3 Ultrametrics on $X_B$

Although the Cantor set is unique up to homeomorphism, different metrics on it can induce quite different geometric structures. In this section, we will focus on Bratteli diagrams $B$ for which the infinite path space $X_B$ is a Cantor set. In this setting, we construct ultrametrics on $X_B$ by using weights on $B$. To do so, we first need to introduce some definitions and notation.

**Definition 2.12.** A metric $d$ on a Cantor set $C$ is called an ultrametric if $d$ induces the Cantor set topology and satisfies the so-called strong triangle inequality

\[ d(x, y) \leq \max\{d(x, z), d(y, z)\} \quad \text{for all } x, y, z \in C. \quad (2) \]

**Definition 2.13.** Let $B$ be a Bratteli diagram. Denote by $F_B$ the set of finite paths in $B$ with range in $V_0$. For any $n \in \mathbb{N}$, we write

\[ F^n_B = \{ \lambda \in F_B : |\lambda| = n \}. \]

Given two (finite or infinite) paths $\lambda, \eta$ in $B$, we say $\eta$ is a sub-path of $\lambda$ if there is a sequence $\gamma$ of edges, with $r(\gamma) = s(\eta)$, such that $\lambda = \eta \gamma$.

For any two infinite paths $x, y \in X_B$, we define $x \wedge y$ to be the longest path $\lambda \in F_B$ such that $\lambda$ is a sub-path of $x$ and $y$. We write $x \wedge y = \emptyset$ when no such path $\lambda$ exists.

**Definition 2.14.** (c.f. [36]) A weight on a Bratteli diagram $B$ is a function $w : F_B \to \mathbb{R}^+$ such that

- If $V_0$ denotes the set of vertices at level 0, then $\sum_{v \in V_0} w(v) \leq 1$.
- $\lim_{n \to \infty} \sup\{w(\lambda) : \lambda \in F^n_B\} = 0$.
- If $\eta$ is a sub-path of $\lambda$, then $w(\lambda) < w(\eta)$.

A Bratteli diagram with a weight is often called a weighted Bratteli diagram and denoted by $(B, w)$.

Observe that the third condition implies that for any path $x = (x_n)_n \in B$ (finite or infinite),

\[ w(x_1x_2\ldots x_n) > w(x_1x_2\ldots x_{n+1}) \quad \text{for all } n. \]

The concept above of a weight was inspired by Definition 2.9 of [26] which was in turn inspired by the work of [36]; indeed, if one denotes a weight in the sense of [26] Definition 2.9 by $w'$, and defines $w(\lambda) := w'(s(\lambda))$, then $w$ is a weight on $B$ in the sense of Definition 2.14 above.

**Proposition 2.15.** Let $(B, w)$ be a weighted Bratteli diagram such that $X_B$ is a Cantor set. The function $d_w : X_B \times X_B \to \mathbb{R}^+$ given by

\[ d_w(x, y) = \begin{cases} 1 & \text{if } x \wedge y = \emptyset, \\ 0 & \text{if } x = y, \\ w(x \wedge y) & \text{else.} \end{cases} \]

is an ultrametric on $X_B$. Moreover $d_w$ metrizes the cylinder set topology on $X_B$. 

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2.4 Strongly connected higher-rank graphs

Proof. It is evident from the defining conditions of a weight that $d_w$ is symmetric and satisfies $d_w(x, y) = 0 \Leftrightarrow x = y$. Since the inequality (2) is stronger than the triangle inequality, once we show that $d_w$ satisfies the ultrametric condition (2) it will follow that $d_w$ is indeed a metric.

To that end, first suppose that $d_w(x, y) = 1$; in other words, $x$ and $y$ have no common sub-path. This implies that for any $z \in \mathcal{X}_B$, at least one of $d_w(x, z)$ and $d_w(y, z)$ must be 1, so

$$d_w(x, y) \leq \max\{d_w(x, z), d_w(y, z)\},$$

as desired. Now, suppose that $d_w(x, y) = w(x \land y) < 1$. If $d_w(x, z) \geq d_w(x, y)$ for all $z \in \mathcal{X}_B$ then we are done. On the other hand, if there exists $z \in \mathcal{X}_B$ such that $d_w(x, z) < d_w(x, y)$, then the maximal common sub-path of $x$ and $z$ must be longer than that of $x$ and $y$. This implies that

$$d_w(y, z) := w(y \land z) = w(y \land x) = d_w(x, y);$$

consequently, in this case as well we have $d_w(x, y) \leq \max\{d(x, z), d_w(y, z)\}$.

Finally, we observe that the metric topology induced by $d_w$ agrees with the cylinder set topology. This fact may be known, but because we did not find the proof in the literature, we include it here. Let $B[x, r]$ be the closed ball of center $x$ and radius $r > 0$. We will show first that $B[x, r] \subset [x_1 \cdots x_n]$ for some $n \in \mathbb{N}$. To obtain an easy upper bound on the diameter of $B[x, r]$, choose $y, z \in B[x, r]$ and observe that

$$d_w(y, z) \leq \max\{d_w(x, y), d_w(x, z)\} \leq r.$$  

Taking suprema reveals that $\text{diam } B[x, r] \leq r$.

We now check that $B[x, r] = [x_1 \cdots x_n]$ for some $n \in \mathbb{N}$. By the definition of the weight $w$, there is a smallest $n \in \mathbb{N}$ such that

$$w(x_1 \cdots x_n) \leq \text{diam } B[x, r].$$

If $y \in B[x, r]$, then

$$\text{diam } B[x, r] \geq d_w(x, y) = w(x \land y) = w(x_1 \cdots x_m)$$

for some $m \geq n \in \mathbb{N}$ by Definition 2.14 and the minimality of $n$. It follows that $y \in [x_1 \cdots x_n]$, so that $B[x, r] \subset [x_1 \cdots x_n]$. On the other hand, if $z \in [x_1 \cdots x_n]$ then

$$d_w(z, x) = w(z \land x) \leq w(x_1 \cdots x_n) \leq \text{diam } B[x, r] \leq r.$$  

so $z \in B[x, r]$ by construction, and hence $[x_1 \cdots x_n] \subset B[x, r]$. In other words, $B[x, r] = [x_1 \cdots x_n]$ as claimed, so cylinder sets of $X_B$ and closed balls (which are open in the topology induced by the metric $d_w$) agree. (If $n = 0$ then we interpret $[x_1 \cdots x_n]$ as $[r(x)]$.)

\[\square\]

2.4 Strongly connected higher-rank graphs

When $\Lambda$ is a finite $k$-graph whose adjacency matrices satisfy some additional properties, there is a natural family $\{w_\delta\}_{0 < \delta < 1}$ of weights on the associated Bratteli diagram $\mathcal{B}_\Lambda$ which induce ultrametrics on the infinite path space $X_{\mathcal{B}_\Lambda}$. We describe these additional properties on $\Lambda$ and the formula of the weights $w_\delta$ below.

Definition 2.16. A $k$-graph $\Lambda$ is strongly connected if, for all $v, w \in \Lambda^0$, $v\Lambda w \neq \emptyset$.

In Lemma 4.1 of [23], an Huef et al. show that a finite $k$-graph $\Lambda$ is strongly connected if and only if the adjacency matrices $A_1, \ldots, A_k$ of $\Lambda$ form an irreducible family of matrices. Also,
Proposition 3.1 of \cite{23} implies that if \( \Lambda \) is a finite strongly connected \( k \)-graph, then there is a unique positive vector \( x^\Lambda \in (0, \infty)^{\Lambda^0} \) such that \( \sum_{v \in \Lambda^0} x^\Lambda_v = 1 \) and for all \( 1 \leq i \leq k \),

\[
A_ix^\Lambda = \rho_ix^\Lambda,
\]

where \( \rho_i \) denotes the spectral radius of \( A_i \). We call \( x^\Lambda \) the *Perron-Frobenius eigenvector* of \( \Lambda \). Moreover, an Huef et al. constructed a Borel probability measure \( M \) on \( \Lambda^\infty \) in Proposition 8.1 of \cite{23} when \( \Lambda \) is finite, strongly connected \( k \)-graph. The measure \( M \) on \( \Lambda^\infty \) is given by

\[
M([\lambda]) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)} \quad \text{for } \lambda \in \Lambda, \tag{3}
\]

where \( x^\Lambda \) is the Perron-Frobenius eigenvector of \( \Lambda \) and \( \rho(\Lambda) = (\rho_1, \ldots, \rho_k) \), and for \( n = (n_1, \ldots, n_k) \in \mathbb{N}^k \),

\[
\rho(\Lambda)^n := \rho_1^{n_1} \cdots \rho_k^{n_k}.
\]

We know from Remark 2.11 that every finite path \( \lambda \in \mathcal{B}_\Lambda \) corresponds to a unique morphism in \( \Lambda \). Using this correspondence and the homeomorphism \( X_{\mathcal{B}_\Lambda} \cong \Lambda^\infty \) of Proposition 2.10 Equation (3) translates into the formula

\[
M([\lambda]) = (\rho_1 \cdots \rho_t)^{-(q+1)} (\rho_{t+1} \cdots \rho_k)^{-q} x^\Lambda_{s(\lambda)} \tag{4}
\]

for \( [\lambda] \subseteq X_{\mathcal{B}_\Lambda} \), where \( \lambda \in \mathcal{F}\mathcal{B}_\Lambda \) with \( |\lambda| = qk + t \) and \( x^\Lambda \) is the Perron-Frobenius eigenvector of \( \Lambda \).

In the proof that follows, we rely heavily on the identification between \( \Lambda^\infty \) and \( X_{\mathcal{B}_\Lambda} \) of Proposition 2.10. We also use the observation from Remark 2.11 that every finite path in \( \mathcal{F}\mathcal{B}_\Lambda \) corresponds to a unique finite path \( \lambda \in \Lambda \).

**Proposition 2.17.** Let \( \Lambda \) be a finite, strongly connected \( k \)-graph with adjacency matrices \( A_i \). Then the infinite path space \( \Lambda^\infty \) is a Cantor set whenever \( \prod_i \rho_i > 1 \).

**Proof.** We let \( A = A_1 \ldots A_k \); it is a matrix whose entries are indexed by \( \Lambda^0 \times \Lambda^0 \), and its spectral radius is \( \prod_i \rho_i \). We assume that \( \Lambda^\infty \) is not a Cantor set, and will prove that the spectral radius of \( A \) is at most 1, hence proving the Proposition.

Since \( \Lambda^\infty \) is compact Hausdorff and totally disconnected, but not a Cantor set, it has an isolated point \( x \). We write \( \{\gamma_n\}_{n \in \mathbb{N}} \) for the increasing sequence of finite paths in \( \mathcal{B}_\Lambda \) which are sub-paths of \( x \). If \( n = \ell k + t \), then \( |\gamma_n| = n \) and (thinking of \( \gamma_n \) as an element of \( \Lambda \)) \( d(\gamma_n) = (\ell + 1, \ldots, \ell + 1, \ell, \ldots, \ell) \) with \( t \) occurrences of \( \ell + 1 \). Since \( x \) is an isolated point, there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( \lfloor \gamma_n \rfloor = \{x\} \). Without loss of generality, we can assume that \( N = dk \) is a multiple of \( k \), so that \( d(\gamma_N) = (d, \ldots, d) \). For \( n \geq N \), we write \( \gamma_n = \gamma_N \eta_n \), with \( |\gamma_n| = n \) and \( |\eta_n| = n - N = qk + t \), so that \( d(\eta_n) = (q + 1, \ldots, q + 1, q, \ldots, q) \), with \( t \) occurrences of \( q + 1 \).

By Proposition 2.23, our hypothesis that \( x \) is an isolated point implies that for all \( n \geq N \), \( \eta_n \) is the unique path of degree \( d(\eta_n) \) whose range is \( s(\gamma_N) = r(\eta_n) \). This, in turn, implies that for all \( n \geq N \), we have \( A^qA_1 \ldots A_t(r(\eta_n), z) \) equal to 1 for a single \( z \), and 0 otherwise. In other words, if we consider the column vector \( \delta_v \) which is 1 at the vertex \( v \) and 0 else, we have that

\[
(\delta_{r(\eta_n)})^T A^qA_1 \ldots A_t = (\delta_{s(\eta_n)})^T.
\]

Note that for each \( n \geq N \) with \( n - N = qk + t \), \( s(\eta_{n+1}) \) is the label of the only non-zero entry in row \( s(\eta_n) \) of the matrix \( A_t \). Since each entry in the sequence \( (s(\eta_n))_{n \in \mathbb{N}} \) is completely determined by a finite set of inputs – namely, the previous entry in the sequence, and the entries of the matrices
\(A_i\) — and the set \(\Lambda^0\) of vertices is finite, the sequence \((s(\eta_n))_{n \in \mathbb{N}}\) is eventually periodic. Let \(p\) be a period for this sequence. Then \(kp\) is also a period, so there exists \(J\) such that for all \(n \geq J\) we have

\[(A^p)^T \delta_{s(\eta_n)} = \delta_{s(\eta_n)}.\]

If we average along one period and define

\[\bar{v} = \frac{1}{kp} \sum_{j=J+1}^{J+kp} \delta_{s(\eta_j)},\]

then we can compute that

\[A^T \bar{v} = \frac{1}{kp} \sum_{j=J+1}^{J+kp} \delta_{s(\eta_j)} = \bar{v},\]

so \(\bar{v}\) is an eigenvector of \(A^T\) with eigenvalue 1, with nonnegative entries.

Since \(\Lambda\) is strongly connected by hypothesis, Lemma 4.1 of [23] implies that there exists a matrix \(A_F\) which is a finite sum of finite products of the matrices \(A_i\) and which has positive entries. This matrix \(A_F\) commutes with \(A\), and therefore

\[A^T A_F \bar{v} = A_F A^T \bar{v} = A_F^T \bar{v},\]

and so \(\bar{u} := A_F^T \bar{v}\) is an eigenvector of \(A^T\) with eigenvalue 1. Since \(A_F\) is positive and \(\bar{v}\) is nonnegative, \(\bar{u}\) is positive. Therefore, we can apply Lemma 3.2 of [23] and conclude that \(\prod_i \rho_i = \rho(A) \leq 1.\)

**Remark 2.18.** The proof of Proposition 2.17 simplifies considerably if we add the hypothesis that each row sum of each adjacency matrix \(A_i\) is at least 2. In this case, any finite path \(\gamma\) in the Bratteli diagram has at least two extensions \(\gamma_e\) and \(\gamma_f\). In terms of neighbourhoods, this means that each clopen set \([\gamma]\) contains at least two disjoint non-trivial sets \([\gamma_e],[\gamma_f]\). It is therefore impossible to have a cylinder set \([\gamma]\) consist of a single point. Therefore, there is no isolated point in \(X_{B_A}\), and the path space is a Cantor set.

**Proposition 2.19.** Let \(\Lambda\) be a finite, strongly connected \(k\)-graph with adjacency matrices \(A_i\). For \(\eta \in F_{B_A}\) with \(|\eta| = n \in \mathbb{N}\), write \(n = qk + t\) for some \(q,t \in \mathbb{N}\) with \(0 \leq t \leq k-1\). For each \(\delta \in (0,1)\), define \(w_\delta : F_{B_A} \to \mathbb{R}^+\) by

\[w_\delta(\eta) = \left(\rho_1^{q+1} \cdots \rho_i^{q+1} \cdots \rho_k^{q+1}\right)^{-1/\delta} x^\Lambda_\eta,\]

where \(x^\Lambda\) is the unimodular Perron-Frobenius eigenvector for \(\Lambda\). If the spectral radius \(\rho_i\) of \(A_i\) satisfies \(\rho_i > 1 \forall i\), then \(w_\delta\) is a weight on \(B_A\).

**Proof.** Recall that \(x^\Lambda \in (0,\infty)^{\Lambda^0}\), \(\sum_{v \in \Lambda^0} x^\Lambda_v = 1\) and \(A_i x^\Lambda = \rho_i x^\Lambda\) for all \(1 \leq i \leq k\); thus,

\[\sum_{v \in \mathcal{V}_0} w_\delta(v) = \sum_{v \in \mathcal{V}_0} x^\Lambda_v = 1,\]

and the first condition of Definition 2.14 is satisfied. Since \(\rho_i > 1\) for all \(i\) and \(0 < \delta < 1\),

\[\lim_{q \to \infty} \left(\rho_i^q\right)^{-1/\delta} = \lim_{q \to \infty} \left(\frac{1}{\rho_i^{1/\delta}}\right)^q = 0.\]
Thus the second condition of Definition 2.14 holds. To see the third condition, we observe that it is enough to show that \( w_\delta(\lambda f) > w_\delta(\lambda) \) for any edge \( f \) with \( s(\lambda) = r(f) \). Note that if \( |\lambda| = qk + j \) for \( q \in \mathbb{N} \) and \( 0 \leq j \leq k - 1 \), so that \( s(\lambda) \in \mathcal{V}_{qk+j} \), then

\[
\sum_{f : r(f) = s(\lambda)} w_\delta(\lambda f) = ((\rho_1 \cdots \rho_k)^q \rho_1 \cdots \rho_{j+1})^{-1/\delta} \sum_{v \in \Lambda^0} A_{j+1}(s(\lambda)), v) x^A_v
\]

\[
= ((\rho_1 \cdots \rho_k)^q \rho_1 \cdots \rho_j)^{-1/\delta} \rho_{j+1}^{-1/\delta} x^A_{s(\lambda)}
\]

\[
< w_\delta(\lambda).
\]

Here the second equality follows since \( x^A \) is an eigenvector for \( A_{j+1} \) with eigenvalue \( \rho_{j+1} \), and the final inequality holds because \( \rho_{j+1} > 1 \) and \( 1/\delta > 1 \), and consequently

\[
\rho_{j+1}^{1-1/\delta} = \frac{1}{\rho_{j+1}^{1/\delta-1}} < 1.
\]

Our primary application for the results of this section is the following.

**Corollary 2.20.** Let \( \Lambda \) be a finite, strongly connected \( k \)-graph with adjacency matrices \( A_i \) and let \( \rho_i \) be the spectral radius for \( A_i \), \( 1 \leq i \leq k \). Suppose that \( \rho_i > 1 \) for all \( 1 \leq i \leq k \). Let \((\mathcal{B}_\Lambda, w_\delta)\) be the associated weighted stationary \( k \)-Bratteli diagram given in Proposition 2.19. Then the infinite path space \( X_{\mathcal{B}_\Lambda} \) is an ultrametric Cantor set with the metric \( d_{w_\delta} \) induced by the weight \( w_\delta \).

**Proof.** Combine Proposition 2.19, Proposition 2.17, and Proposition 2.15.

\[\Box\]

## 3 Spectral triples and Hausdorff dimension for ultrametric higher-rank graph Cantor sets

Proposition 8 of [36] (also see Proposition 3.1 of [26]) gives a recipe for constructing an even spectral triple for any ultrametric Cantor set induced by a weighted tree. We begin this section by explaining how this construction works in the case of the ultrametric Cantor sets which we associated to a finite strongly connected \( k \)-graph in the previous section. In Section 3.1 we recall basic facts about spectral triples, and in Section 3.2, we investigate the \( \zeta \)-function and Dixmier trace of the spectral triples coming from the ultrametric Cantor sets that arise from \( k \)-graphs.

To be precise, consider the Cantor set \( \Lambda^\infty \cong X_{\mathcal{B}_\Lambda} \) with the ultrametric induced by the weight \( w_\delta \) of Equation (5). (Because of Proposition 2.10 we will identify the infinite path spaces of \( \Lambda \) and of \( \mathcal{B}_\Lambda \), and use either \( \Lambda^\infty \) or \( X_{\mathcal{B}_\Lambda} \) to denote this space, depending on the context.) Under additional (but mild) hypotheses, Theorem 3.8 establishes that the \( \zeta \)-function of the associated spectral triple has abscissa of convergence \( \delta \). After proving in Theorem 3.9 that the Dixmier trace of the spectral triple induces a well-defined measure \( \mu_\delta \) on \( \Lambda^\infty \), Corollary 3.10 establishes that \( \mu_\delta \) agrees with the measure \( M \) introduced in [23] and used in [17] to construct a wavelet decomposition of \( L^2(\Lambda^\infty, M) \).

Analogues of Theorems 3.8 and 3.9 were proved in Section 3 of [26] for stationary Bratteli diagrams (equivalently, directed graphs) with primitive adjacency matrices. However, even for directed graphs our results in this section are stronger than those of [26], since in this setting, our hypotheses are equivalent to saying that the adjacency matrix is merely irreducible.
3.1 A review of spectral triples and the associated \( \zeta \)-functions

We begin by recalling the definition of a spectral triple.

**Definition 3.1.** Given a pre-\( C^* \)-algebra \( \mathcal{A} \), a faithful \(*\)-representation \( \pi : \mathcal{A} \to B(\mathcal{H}) \), and an unbounded operator \( D \) on \( \mathcal{H} \) such that

\[
(D^2 + 1)^{-1} \in K(\mathcal{H}) \quad \text{and} \quad [D, \pi(a)] \in B(\mathcal{H}) \forall a \in \mathcal{A},
\]

we say that \((\mathcal{A}, \mathcal{H}, D)\) is an \((odd)\) spectral triple. If \( \mathcal{H} \) has a grading operator – a self-adjoint unitary \( \Gamma \) – such that

\[
[\Gamma, \pi(a)] = 0 \forall a \in \mathcal{A} \quad \text{and} \quad \Gamma D = -D \Gamma,
\]

we say that \((\mathcal{A}, \mathcal{H}, D, \Gamma)\) is an \((even)\) spectral triple.

Sometimes the representation \( \pi \) is also included in the notation for a spectral triple.

To any spectral triple, even or odd, we associate a \( \zeta \)-function and Dixmier trace as follows.

**Definition 3.2.** The \( \zeta \)-function associated to a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is given by

\[
\zeta(s) = \frac{1}{2} \text{Tr} (|D|^{-s}) \quad \text{for } s \in \mathbb{C}.
\]

The \( \zeta \)-function of a spectral triple is always a Dirichlet series since \(|D|^{-1}\) is compact and hence has a decreasing sequence of eigenvalues. Thus, Chapter 2 of \[19\] tells us that \( \zeta \) either converges everywhere, nowhere, or in the complex half plane \( \text{Re}(s) \geq s_0 \) for some \( s_0 \), which we call the abscissa of convergence of \( \zeta \).

**Remark 3.3.** To determine the abscissa of convergence of the \( \zeta \)-function, it suffices to evaluate \( \zeta \) at points \( s \in \mathbb{R} \). Since we are primarily interested in the abscissa of convergence of \( \zeta \), throughout this article, we will only consider real arguments for \( \zeta \).

**Definition 3.4.** If the abscissa of convergence \( s_0 \) of the \( \zeta \)-function exists, then the Dixmier trace of an element \( a \in \mathcal{A} \) is given by

\[
\mu(a) = \lim_{s \searrow s_0} \frac{\text{Tr} (|D|^{-s}(a))}{\text{Tr} (|D|^{-s})},
\]

where we take the limit over \( s \in \mathbb{R}, s > s_0 \).

We now review the construction of the spectral triple from \[36\] (see also Section 3 of \[26\]). Let \((\mathcal{B}, w)\) be a weighted Bratteli diagram such that the infinite path space \( X_\mathcal{B} \) is a Cantor set. Let \((X_\mathcal{B}, d_w)\) be the associated ultrametric Cantor space. A choice function for \((X_\mathcal{B}, d_w)\) is a map \( \tau : \mathcal{B} \to X_\mathcal{B} \times X_\mathcal{B} \) such that \( \tau(\gamma) = (\tau_+(\gamma), \tau_-(\gamma)) \in [\gamma] \times [\gamma] \) and \( d_w(\tau_+(\gamma), \tau_-(\gamma)) = \text{diam} [\gamma] \), where

\[
\text{diam} [\gamma] = \sup \{d_w(x, y) \mid x, y \in [\gamma]\}.
\]

We denote by \( \Upsilon \) the set of choice functions for \((X_\mathcal{B}, d_w)\). Note that \( \Upsilon \) is nonempty whenever \( X_\mathcal{B} \) is a Cantor set, because Condition (3) of Proposition \[2.3\] implies that for every finite path \( \gamma \) of \( \mathcal{B} \) we can find two distinct infinite paths \( x, y \in [\gamma] \).

As in \[36, 26\], let \( C_{\text{Lip}}(X_\mathcal{B}) \) be the pre-\( C^* \)-algebra of Lipschitz continuous functions on \((X_\mathcal{B}, d_w)\) and let \( \mathcal{H} = \ell^2(\mathcal{B}) \otimes \mathbb{C}^2 \). For \( \tau \in \Upsilon \), we define a faithful \(*\)-representation \( \pi_\tau \) of \( C_{\text{Lip}}(X_\mathcal{B}) \) on \( \mathcal{H} \) by

\[
\pi_\tau(f) = \bigoplus_{\gamma \in \mathcal{B}} \left( \begin{array}{cc} f(\tau_+(\gamma)) & 0 \\ 0 & f(\tau_-(\gamma)) \end{array} \right).
\]
A Dirac operator $D$ on $\mathcal{H}$ is given by
\[
D = \bigoplus_{\gamma \in FB} \frac{1}{\text{diam}[\gamma]} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
The grading operator $\Gamma$ is given by
\[
\Gamma = 1_{\ell^2(FB)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Then by Proposition 8 of [36], $(C_{\text{Lip}}(X_B), \mathcal{H}, \pi_\tau, D, \Gamma)$ is an even spectral triple for all $\tau \in \Upsilon$.

For the rest of the paper, we assume the following:

**Hypothesis 3.5.** Given a weighted Bratteli diagram $(B, w)$, the weight $w$ satisfies
\[
w(\lambda) = \text{diam}[\lambda] \quad \text{for all } \lambda \in FB. \tag{8}
\]

**Remark 3.6.** Observe that if $B = B_\Lambda$ for a $k$-graph $\Lambda$, then Equation (8) holds for the weights $w_\delta$ of Equation (5) if and only if every vertex $a \in \Lambda^0$ receives at least two edges of each color, i.e. $\sum_{b \in \Lambda^0} A_i(a, b) \geq 2$ for all $a \in \Lambda^0$ and $1 \leq i \leq k$. Since the spectral radius of a nonnegative matrix is at least the minimum of its row sums, Equation (8) implies that $\rho_i \geq 2 > 1$ for all $1 \leq i \leq k$, and hence $\rho = \rho_1 \ldots \rho_k > 1$. Therefore, if the function $w_\delta$ given in Equation (5) satisfies Equation (8), then $w_\delta$ is a weight and it gives rise to an ultrametric Cantor set $X_{B_\Lambda}$ by Corollary 2.20.

When Equation (8) holds, then the $\zeta$-function is given by the formula
\[
\zeta_w(s) = \frac{1}{2} \text{Tr}(|D|^{-s}) = \sum_{\lambda \in FB} w(\lambda)^s. \tag{9}
\]

If, moreover, the abscissa of convergence $s_0$ of the zeta function $\zeta_w$ is finite and the Dixmier trace in (9) exists, then it induces a measure on the infinite path space $X_B$, whose explicit formula on cylinder sets is given by
\[
\mu_w([\gamma]) := \mu_w(\chi[\gamma]) = \lim_{s \searrow s_0} \frac{\sum_{\lambda \in F_B} w(\lambda)^s}{\zeta_w(s)}, \tag{10}
\]
where $F_B = \{ \alpha \in FB : \gamma \text{ is a sub-path of } \alpha \}$ is the set of finite paths which extend a finite path $\gamma$. By abuse of notation, we use the same notation $\mu_w$ for the induced measure as for the Dixmier trace.

Before we begin our analysis of the spectral triples associated to the ultrametric Cantor sets $(X_{B_\Lambda}, d_{w_\delta})$, we first present sufficient conditions for the Dixmier trace to give a well-defined measure on $X_B$.

**Proposition 3.7.** Let $(B, w)$ be a weighted Bratteli diagram and let $(X_B, d_w)$ be the associated ultrametric Cantor set. Suppose that the weight $w$ satisfies Equation (8), and that the $\zeta$-function $\zeta_w(s)$ in (9) has abscissa of convergence $s_0$. If $\mu_w([\gamma]) < \infty$ for all cylinder sets $[\gamma] \in X_B$, then $\mu_w$ determines a unique finite measure on $X_B$. 

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Proof. This proof relies on Carathéodory’s theorem. Notice that

\[ \mathcal{F} := \{[\lambda] : \lambda \in FB\} \]

is closed under finite intersections (if \([\lambda] \cap [\gamma] \neq \emptyset\), then either \(\lambda\) is a sub-path of \(\gamma\) or vice versa, and thus \([\lambda] \cap [\gamma] = [\gamma]\)), and

\[ [\lambda]^c = \bigsqcup_{|\lambda_1|=|\lambda|, \lambda_1 \neq \lambda} [\lambda_1]. \]

In other words, the complement of any element of \(\mathcal{F}\) can be written as a finite disjoint union of elements of \(\mathcal{F}\).

Since \(\mathcal{F}\) generates the topology on \(X_B\), and \(\mu_w([\gamma])\) is finite for all \([\gamma] \in \mathcal{F}\) by hypothesis, Carathéodory’s theorem tells us that in order to show that \(\mu_w\) determines a measure on \(X_B\), we merely need to check that \(\mu_w\) is \(\sigma\)-additive on \(\mathcal{F}\). In fact, since the cylinder sets \([\gamma]\) are clopen, the fact that \(X_B\) is compact means that it is enough to check that \(\mu_w\) is finitely additive on \(\mathcal{F}\).

We remark that since \(s_0\) is the abscissa of convergence of \(\zeta_w(s)\), we must have

\[ \lim_{s \nearrow s_0} \zeta_w(s) = \infty. \]  \hfill (11)

Consequently, since \(\mu_w([\gamma]) = \lim_{s \nearrow s_0} \frac{\sum_{\lambda \in F_s B} w(\lambda)^s}{\zeta_w(s)}\), in calculating \(\mu_w([\gamma])\) we can ignore finitely many initial terms in the sum in the numerator. In other words, for any \(L \in \mathbb{N}\),

\[ \mu_w([\gamma]) = \lim_{s \nearrow s_0} \frac{\sum_{|\lambda| \geq L} \lambda \in F_s B \, w(\lambda)^s}{\zeta_w(s)} \]  \hfill (12)

Thus, suppose that \([\gamma] = \bigsqcup_{i=1}^N [\lambda_i]\). Write \(L = \max_i |\lambda_i|\), and for each \(i\), write \([\lambda_i] = \bigsqcup_{\ell} [\lambda_{i,\ell}]\) where \(|\lambda_{i,\ell}| = L\). If \(\lambda \in F_s B\) with \(|\lambda| \geq L\), then \(\lambda_i\) is a sub-path of \(\lambda\) for precisely one \(i\), and hence

\[ \mu_w([\gamma]) = \lim_{s \nearrow s_0} \frac{\sum_{|\lambda| \geq L} \lambda \in F_s B \, w(\lambda)^s}{\zeta_w(s)} = \lim_{s \nearrow s_0} \sum_i \frac{\sum_{\lambda \in F_s B \, w(\lambda)^s}}{\zeta_w(s)} = \sum_i \mu_w([\lambda_i]). \]

For each fixed \(i\), \(\bigsqcup_{\ell} [\lambda_{i,\ell}] = [\lambda_i]\), so the same argument will show that \(\mu_w([\lambda_i]) = \sum_\ell \mu_w([\lambda_{i,\ell}]\).

Thus,

\[ \mu_w([\gamma]) = \sum_{i, \ell} \mu_w([\lambda_{i,\ell}]) = \sum_i \mu_w([\lambda_i]). \]

Since \(\mu_w\) is finitely additive on \(\mathcal{F}\), Carathéodory’s theorem allows us to conclude that it gives a well-defined finite measure on \(X_B\).

\[ \square \]

3.2 Properties of the \(\zeta\)-function and Dixmier trace

In this subsection, which involves some of the most intricate proofs in this paper, we return to our focus on the even spectral triples \((C_{Lip}(X_{B,\delta}), \mathcal{H}, \pi_\tau, D, \Gamma)\) associated to the weighted stationary \(k\)-Bratteli diagrams \((B_\lambda, w_\delta)\) of Proposition 2.19 above. From now on, the \(\zeta\)-function and Dixmier trace of these spectral triples will be denoted by \(\zeta_\delta\) and \(\mu_\delta\) to emphasize that they depend on the choice of \(\delta \in (0,1)\). Similarly, we write \(d_\delta\) for the ultrametric associated to \(w_\delta\).

We begin by showing that \(\zeta_\delta\) has abscissa of convergence \(\delta\).
**Theorem 3.8.** Let $\Lambda$ be a finite, strongly connected $k$-graph. Fix $\delta \in (0, 1)$ and suppose that Equation $(\mathbf{3})$ holds for the weight $w_\delta$ of Equation $(\mathbf{5})$. Then

$$\zeta_\delta(s) < \infty \text{ if and only if } s > \delta.$$  

**Proof.** In order to explicitly compute $\zeta_\delta(s)$, we first observe that we can rewrite

$$\zeta_\delta(s) = \sum_{\lambda \in F_{B_\Lambda}} w_\delta(\lambda)^s = \sum_{n \in \mathbb{N}} \sum_{\lambda \in F^n B_\Lambda} w_\delta(\lambda)^s = \sum_{q \in \mathbb{N}} \sum_{t=0}^{k-1} \sum_{\lambda \in F^{qk+t} B_\Lambda} w_\delta(\lambda)^s.$$  

Now, write $A := A_1 \cdots A_k$ for the product of the adjacency matrices of $\Lambda$. If $t \in \{0, 1, \ldots, k - 1\}$ is fixed and $n = qk + t$, then the number of paths in $F^n(B_\Lambda)$ with source vertex $b$ and range vertex $a$ is given by

$$A^q A_1 \cdots A_t(a, b),$$  

where $F^n(B_\Lambda)$ is the set of finite paths of $B_\Lambda$ with length $n$. Thus, writing $\rho := \rho_1 \cdots \rho_k$ for the spectral radius of $A$, the formula for $w_\delta$ given in Equation $(\mathbf{5})$ implies that

$$\zeta_\delta(s) = \sum_{t=0}^{k-1} \frac{1}{(\rho_1 \cdots \rho_t)^{s/\delta}} \sum_{a, b \in V_0} A^q A_1 \cdots A_t(a, b) \frac{(x^A_b)^s}{\rho^q s/\delta}. \quad (13)$$  

Since all terms in this sum are non-negative, the series $\zeta_\delta(s)$ converges iff it converges absolutely; hence, rearranging the terms in the sum does not affect the convergence of $\zeta_\delta(s)$. Thus, we can rewrite

$$\zeta_\delta(s) = \sum_{t=0}^{k-1} \sum_{a, b, z \in V_0} A_1 \cdots A_t(z, b) (\rho_1 \cdots \rho_t)^{s/\delta} \sum_{q \in \mathbb{N}} A^q(a, z) \frac{(x^A_z)^s}{\rho^q s/\delta}. \quad (14)$$  

In order to show that $\zeta_\delta(s)$ converges for $s > \delta$, we begin by considering the sum

$$\sum_{q \in \mathbb{N}} \frac{A^q(a, z)}{(\rho^q s/\delta)^q}.$$

Since $A$ has a positive right eigenvector of eigenvalue $\rho$ (namely $x^A$), Corollary 8.1.33 of [20] implies that

$$\frac{A^q(a, z)}{\rho^q} \leq \frac{\max \{ x^A_b \}_{b \in V_0}}{\min \{ x^A_b \}_{b \in V_0}} \forall q \in \mathbb{N} \setminus \{0\}.$$  

Consequently,

$$\sum_{q \in \mathbb{N}} \frac{A^q(a, z)}{\rho^q s/\delta - 1} \leq \delta_{a, z} + \max \{ x^A_b \}_{b \in V_0} \sum_{q \geq 1} \frac{1}{\rho^q (s/\delta - 1)q}.$$

If $s > \delta$, then our hypothesis that $\rho > 1$ implies that $1/\rho^{(s/\delta - 1)} \in (0, 1)$, and thus $\sum_{q \geq 1} \rho^{(1-s/\delta)q}$ converges to $(1 - \rho^{(s/\delta - 1)})^{-1} - 1$. Consequently,

$$\sum_{q \in \mathbb{N}} \frac{A^q(a, z)}{(\rho^q s/\delta)^q} < \infty,$$

and hence $\zeta_\delta(s) < \infty$, for any $s > \delta$ since $V_0$ is a finite set.
To see that \( \zeta_\delta(s) = \infty \) whenever \( s \leq \delta \), we have to work harder. Theorem 8.3.5 part(b) of [20] implies that the Jordan form of \( A \) is

\[
J = \begin{pmatrix}
\rho & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \rho & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_1\rho & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega_1\rho & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \omega_2\rho & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{p-1}\rho & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{p+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{m-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_m
\end{pmatrix},
\]

where \( p \) is the period of \( A \), \( \omega_i \) is a \( p \)th root of unity for each \( i \), each eigenvalue \( \omega_i\rho \) is repeated along the diagonal \( m_i \) times, and \( J_i, i = p + 1, \ldots, m \) are Jordan blocks — that is, upper triangular matrices whose constant diagonal is given by an eigenvalue \( \alpha_i \) of \( A \) (with \( |\alpha_i| < \rho \)) and which have a superdiagonal of 1s as the only other nonzero entries. Thus, for each \( 1 \leq a, b \leq |V_0| \),

\[
J^q(a, b) \in \{0\} \cup \{\rho^q\} \cup \{\rho^q\omega_i^q : 1 \leq i \leq p - 1\} \cup \left\{ \frac{1}{\alpha_i^{\ell}}(\frac{q}{\ell})\alpha_i^q : 0 \leq \ell \leq \text{dim } J_i \right\}. \tag{15}
\]

Consequently,

\[
\left| \frac{1}{\rho^q} J^q(a, b) \right| \in \{0, 1\} \cup \left\{ \beta_i \frac{1}{|\alpha_i|^{\ell}}(\frac{q}{\ell}) : \beta_i = \frac{|\alpha_i|}{\rho} < 1, \ 0 \leq \ell \leq \text{dim } J_i \right\}.
\]

Thanks to [39] and [3] Chapter 2], we know that since \( A \) has a positive eigenvector (namely \( x^A \)) of eigenvalue \( \rho \), \( \lim_{m \to \infty} \frac{1}{\rho^{mp+j}} A^{mp+j} \) exists for all \( 0 \leq j \leq p - 1 \), where \( p \) denotes the period of \( A \). Moreover, if we write

\[
A^{(j)} = \lim_{m \to \infty} \frac{1}{\rho^{mp+j}} A^{mp+j}
\]

for this limit, and \( \tau \) for the maximum modulus of the eigenvalues \( \alpha_i \) of \( A \) with \( |\alpha_i| < \rho \),

\[
\forall \left( \frac{\tau}{\rho} \right)^p < \beta < 1, \ \exists M_{\beta, j} \in \mathbb{R}^+ \ \text{s.t.} \ \forall m \in \mathbb{N}, \ \left| \frac{A^{mp+j}(a, b)}{\rho^{mp+j}} - A^{(j)}(a, b) \right| \leq M_{\beta, j} \beta^m.
\]

Thus, for all \( m \in \mathbb{N} \) and all \( 0 \leq j \leq p - 1 \), and all such \( \beta \),

\[
\frac{A^{mp+j}(a, b)}{\rho^{mp+j}} \geq A^{(j)}(a, b) - M_{\beta, j} \beta^m \quad \text{for all } m \in \mathbb{N}. \tag{17}
\]

Reordering the summands of \( \sum_{q \in \mathbb{N}} A^q(a, b)(\rho^{-s/\delta})^q \), we see that

\[
\sum_{q \in \mathbb{N}} A^q(a, b)(\rho^{-s/\delta})^q = \sum_{j=0}^{p-1} \sum_{m \in \mathbb{N}} A^{mp+j}(a, b)(\rho^{-s/\delta})^{mp+j}.
\]

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Now, fix \( j \in \{0, \ldots, p-1\} \) and consider the sum

\[
\sum_{m \in \mathbb{N}} A^{mp+j}(a, b)(\rho^{-s/\delta})^{mp+j} = \sum_{m \in \mathbb{N}} A^{mp+j}(a, b) \left( \frac{1}{\rho^{s/\delta - 1}} \right)^{mp+j} \\
\geq \frac{1}{\rho^{s/\delta - 1}j} \sum_{m \in \mathbb{N}} (A^{(j)}(a, b) - M_{\beta,j}^m) \left( \frac{1}{\rho^{s/\delta - 1}} \right)^{pm}.
\]

If \( A^{(j)}(a, b) > 0 \), the fact that \( \beta < 1 \) and \( M_{\beta,j} > 0 \) implies that there exists \( M \) such that for \( m > M \), \( A^{(j)}(a, b) > M_{\beta,j}^m \). Consequently, if we define

\[
K = \frac{1}{\rho^{s/\delta - 1}j} \sum_{m=0}^{M} A^{(j)}(a, b) - M_{\beta,j}^m, \\
\] and write \( \nu = A^{(j)}(a, b) - M_{\beta,j}^M > 0 \), the fact that \( \{M_{\beta,j}^m\}_{m \in \mathbb{N}} \) is a decreasing sequence implies that

\[
\sum_{m \in \mathbb{N}} A^{mp+j}(a, b)(\rho^{-s/\delta})^{mp+j} > K + \frac{\nu}{\rho^{s/\delta - 1}j} \sum_{m > M} \left( \frac{1}{\rho^{s/\delta - 1}} \right)^{pm}.
\]

Since \( \rho > 1 \) and \( s \leq \delta \), \( \rho^{(1-s/\delta)p} \geq 1 \); consequently, the series \( \sum_{m > M} (\rho^{(1-s/\delta)p})^m \) diverges to infinity. The fact that \( K, \nu \) are finite now implies that \( \sum_{m \in \mathbb{N}} A^{mp+j}(a, b)(\rho^{-s/\delta})^{mp+j} \) also diverges to infinity if \( A^{(j)}(a, b) > 0 \).

Now, we show that for each \( j \), there must exist some \((a, b) \in \mathbb{V}_0\) such that \( A^{(j)}(a, b) > 0 \). Recall that \( x^\Lambda \) is an eigenvector for \( A \), and consequently for \( A^{mp+j} \). Thus,

\[
\sum_{b \in \mathbb{V}_0} A^{mp+j}(a, b)x^\Lambda_b = \rho^{mp+j}x^\Lambda_a.
\]

Since \( x^\Lambda \) is a positive eigenvector, there exists \( \alpha > 0 \) such that \( x^\Lambda_a > \alpha \) for all \( a \in \mathbb{V}_0 \). Moreover, \( x^\Lambda \) is a unimodular eigenvector, so \( 0 < x^\Lambda_b \leq 1 \) for all \( b \in \mathbb{V}_0 \). Thus the above equation becomes

\[
\rho^{mp+j} \alpha < \rho^{mp+j}x^\Lambda_a = \sum_{b \in \mathbb{V}_0} A^{mp+j}(a, b)x^\Lambda_b \leq \sum_{b \in \mathbb{V}_0} A^{mp+j}(a, b).
\]

Consequently, for each \( a \in \mathbb{V}_0 \) and each \( m \in \mathbb{N} \) there exists at least one vertex \( b \) such that

\[
\frac{A^{mp+j}(a, b)}{\rho^{mp+j}} > \frac{\alpha}{\#(\mathbb{V}_0)}.
\]

Moreover, since \( \#(\mathbb{V}_0) < \infty \), the definition of the limit \( A^{(j)} \) implies that there exists \( N \in \mathbb{N} \) such that whenever \( m \geq N \) we have

\[
A^{(j)}(a, b) > \frac{A^{mp+j}(a, b)}{\rho^{mp+j}} > \frac{\alpha}{2\#(\mathbb{V}_0)} \forall a, b \in \mathbb{V}_0.
\]

Now, fix \( a \) and \( m \geq N \). Choose \( b \in \mathbb{V}_0 \) such that \( \frac{A^{mp+j}(a,b)}{\rho^{mp+j}} \geq \frac{\alpha}{\#(\mathbb{V}_0)} \). It then follows that for this choice of \( b \),

\[
A^{(j)}(a, b) > \frac{A^{mp+j}(a,b)}{\rho^{mp+j}} > \frac{\alpha}{2\#(\mathbb{V}_0)} > \frac{\alpha}{2\#(\mathbb{V}_0)}.
\]
In other words, we have proved that

\[
\forall 1 \leq j \leq p, \forall a \in \mathcal{V}_0, \exists b \in \mathcal{V}_0 \quad \text{s.t.} \quad A^{(j)}(a, b) > \frac{\alpha}{2\#(\mathcal{V}_0)} > 0. \tag{18}
\]

Finally, recalling that the matrices \(A_i\) commute, we observe that

\[
\sum_{z \in \mathcal{V}_0} A^{mp+j}(a, z) A_1 \cdots A_t(z, b) = (A_1 \cdots A_t) A^{mp+j}(a, b) = \sum_{z \in \mathcal{V}_0} A_1 \cdots A_t(a, z) A^{mp+j}(z, b).
\]

Using this, we rewrite

\[
\zeta_\delta(s) = \sum_{a, b, z \in \mathcal{V}_0} \sum_{t = 0}^{k-1} A_1 \cdots A_t(a, z) (x^A_b)^s \rho^{mp+j}(a, z) \sum_{j = 0}^{p-1} \sum_{m \in \mathbb{N}} A^{mp+j}(z, b) \rho^{mp+j}\delta/s.
\]

It now follows from our arguments above that \(\zeta_\delta(s)\) diverges whenever \(s \leq \delta\). To convince yourself of this, it may help to recall that \(x^A_b\) is positive for all vertices \(b\), and that (since \(A_1 \cdots A_t(a, z)\) represents the number of paths of degree \((1, \ldots, 1, 0, \ldots, 0)\) with source \(z\) and range \(a\)) \(\sum_a A_1 \cdots A_t(a, z)\) must be strictly positive for each \(t\) since \(\Lambda\) is source-free. In other words, \(\zeta_\delta(s)\) is computed by taking a bunch of sums that diverge to infinity when \(s \leq \delta\), possibly adding some other positive numbers, multiplying the lot by some positive scalars, and adding the results.

Consequently, \(\delta\) is the abscissa of convergence of the \(\zeta\)-function \(\zeta_\delta(s)\), as claimed. \(\square\)

In the terminology of [36], the following Theorem establishes the \(\zeta\)-regularity of the ultrametric Cantor sets \((X_{B_\Lambda}, d_\delta)\).

**Theorem 3.9.** Let \(\Lambda\) be a finite, strongly connected \(k\)-graph and fix \(\delta \in (0, 1)\). Suppose moreover that Equation (8) holds on the associated ultrametric Cantor set \((X_{B_\Lambda}, d_\delta)\). Then the associated Dixmier trace induces a finite measure \(\mu_\delta\) on \((X_{B_\Lambda}, d_\delta)\), where

\[
\mu_\delta([\gamma]) := \lim_{s \searrow \delta} \frac{\sum_{\lambda \in F_{B_\Lambda}} w_\delta(\lambda)^s}{\zeta_\delta(s)} \quad \text{for} \quad \gamma \in FB_\Lambda.
\]

**Proof.** Thanks to Proposition 3.7, it is enough to show that the limit \(\mu_\delta([\gamma])\) is finite for all \(\gamma \in FB_\Lambda\). To this end, we begin by computing a more explicit expression for \(\zeta_\delta(s)\) when \(s > \delta\). Recall from our computations in (15) of the Jordan form \(J\) of \(A\) that for any \(z, v \in \mathcal{V}_0\) we can find constants \(c_{i,v}^z\) and polynomials \(P_{i,v}^z\) such that for any \(n \in \mathbb{N}\), we have

\[
A^n(z, v) = c_1^z v^\rho + c_2^z v^\rho \omega_1 v + \cdots + c_p^z v^\rho \omega_{p-1} v + \sum_{i=p+1}^m P_{i,v}^z(n(a, v)) \alpha_i^n, \tag{19}
\]

where \(\omega_i\) is a \(p\)th root of unity for all \(i\) (denoting by \(p\) the period of \(A\)) and each \(\alpha_i\) is an eigenvalue of \(A\) with \(|\alpha_i| < \rho\). In a bit more detail, writing \(A = C^{-1}JC\) for some invertible matrix \(C\), we have

\[
c_{i,v}^z = \sum_{j=m_0+\cdots+m_{i-1}+1}^{m_0+\cdots+m_i} C^{-1}(z, j)C(j, v)
\]

and

\[
P_{i,v}^z(n) = \sum_{(a, b), \gamma_p(a, b) \neq 0} C^{-1}(z, a)C(b, v) \frac{1}{\alpha_i^{b-a}} \binom{n}{b-a}.
\]
Recall that since \( J \) is a Jordan block, \( J^n(a, b) = 0 \) unless \( a \leq b \).

Equivalently, setting \( c_{z,v;n} = c_{1,1}^{-n} + c_{2,1}^{-n} \omega_1 + \ldots + c_{p,n}^{-n} \omega_{p-1} \), we have

\[
A^n(z, v) = c_{z,v;n} \rho^n + \sum_{i=p+1}^m P_i^{z,v}(n) \alpha_i^n. \tag{20}
\]

Therefore,

\[
\zeta_\delta(s) = \sum_{l=0}^{k-1} \sum_{a,b,z \in \mathcal{V}_0} A_1 \cdots A_l(z, b) \sum_{q \in \mathbb{N}} A^q(a, z) \frac{(x_b^\Lambda)^s}{\rho^{q s/\delta}} = \sum_{l=0}^{k-1} \sum_{a,b,z \in \mathcal{V}_0} A_1 \cdots A_l(z, b) \sum_{q \in \mathbb{N}} c_{a,z;q} \frac{P^a(z) \alpha_q}{\rho^{q (s/\delta)}} \frac{(x_b^\Lambda)^s}{\rho^{q s/\delta}}.
\]

Now, define

\[
\tilde{c}_q(s) = \sum_{l=0}^{k-1} \sum_{a,b,z \in \mathcal{V}_0} A_1 \cdots A_l(z, b) \frac{P^a(z)}{(x_b^\Lambda)^s} c_{a,z;q} \frac{P^a(z) \alpha_q}{\rho^{q (s/\delta)}} = \sum_{l=0}^{k-1} \sum_{a,b,z \in \mathcal{V}_0} A_1 \cdots A_l(z, b) \frac{P^a(z) \alpha_q}{\rho^{q (s/\delta)}} P^a(z) \alpha_q.
\]

Since \( \mathcal{V}_0 \) is a finite set, we can rewrite \( \zeta_\delta(s) \) as

\[
\zeta_\delta(s) = \sum_{q \in \mathbb{N}} \rho^{q (1-s/\delta)} \tilde{c}_q(s) + \sum_{q \in \mathbb{N}} \tilde{Q}_i(q, s) \frac{\alpha_i}{\rho^{s/\delta}} q. \tag{21}
\]

We now observe that the series \( \sum_{q \in \mathbb{N}} \tilde{c}_q(s, t) \rho^{q (1-s/\delta)} \) converges by the Root Test. To be more precise, the fact that each \( \omega_i \) is a \( \rho \)th root of unity implies that

\[
c_{a,z;q} = c_{a,z;q+p} \quad \forall \ q \in \mathbb{N}.
\]

Moreover, if we consider the limit \( A^{(j)}(z, v) = \lim_{m \to \infty} \frac{A^{m+j}(z, v)}{\rho^{mp+j}} \), Equation \( \tag{20} \) implies that

\[
A^{(j)}(z, v) = c_{z,v;j}. \tag{22}
\]

Consequently, \( c_{z,v;j} \) is a non-negative real number for all \( z, v \in \mathcal{V}_0 \). Furthermore, Equation \( \tag{18} \) implies that for all \( 1 \leq j \leq m \) and all \( z \in \mathcal{V}_0 \), there exists \( v \in \mathcal{V}_0 \) such that \( c_{z,v;j} \neq 0 \). Thus,

\[
\sum_{q \in \mathbb{N}} c_{a,z;q} \rho^{q (1-s/\delta)} = \sum_{j=0}^{p-1} c_{a,z;j} \sum_{l \in \mathbb{N}} \rho^{lp+j} (1-s/\delta) = \sum_{j=0}^{p-1} c_{a,z;j} \rho^{j (1-s/\delta)} \sum_{l \in \mathbb{N}} \rho^{lp (1-s/\delta)} = \sum_{j=0}^{p-1} c_{a,z;j} \rho^{j (1-s/\delta)} \frac{1}{1 - \rho^{(1-s/\delta)}};
\]

the last equality follows because \( \rho^{1-s/\delta} < 1 \) whenever \( s > \delta \), and hence \( \sum_l \left( \rho^{(1-s/\delta)} \right)^l \) is a geometric series with ratio \( \rho^{p (1-s/\delta)} < 1 \). Thus, the first term of \( \zeta_\delta(s) \) in \( \tag{21} \),

\[
\sum_{q \in \mathbb{N}} \tilde{c}_q(s) \rho^{q (1-s/\delta)} = \sum_{l=0}^{k-1} \sum_{a,b,z \in \mathcal{V}_0} A_1 \cdots A_l(z, b) \frac{P^a(z) \alpha_q}{\rho^{q (s/\delta)}} \frac{(x_b^\Lambda)^s}{\rho^{q s/\delta}} c_{a,z;j} \rho^{j (1-s/\delta)} \frac{1}{1 - \rho^{(1-s/\delta)}}.
\]
is finite for any $s > \delta$. Moreover, the fact that $\zeta_\delta(s)$ is finite implies that, for any $s > \delta$,

$$R_\zeta(s) := \zeta_\delta(s) - \sum_{q \in \mathbb{N}} \tilde{c}_q(s) \rho^{p(1-s/\delta)}$$

$$= \sum_{q \in \mathbb{N}} \sum_{t=p+1}^m \tilde{Q}_t(q, s) \left(\frac{\alpha_i}{\rho^{p/\delta}}\right)^q$$

is also finite. Thus, we have

$$(1 - \rho^{p(1-s/\delta)})\zeta_\delta(s) = \sum_{a,b,z} \sum_{t=0}^{k-1} A_1 \cdots A_t(z, b) \frac{(\rho_1 \cdots \rho_t)^{s/\delta}}{\rho^{(1-q_0)s/\delta}} \sum_{j=0}^{p-1} c_{a,z;j} \rho^{j(1-s/\delta)} + (1 - \rho^{p(1-s/\delta)})R_\zeta(s). \quad (23)$$

Recall that $F_\gamma B_\lambda$ is the set of $\lambda \in FB_\lambda$ such that $\lambda$ is an extension of $\gamma$, and the formula for the measure $\mu_\delta$ is given on cylinder sets by

$$\mu_\delta([\gamma]) = \lim_{s \to \delta} \frac{\sum_{\lambda \in F_\gamma B_\lambda} w_\delta(\lambda)^s}{\zeta_\delta(s)} =: \lim_{s \to \delta} N_\gamma(s).$$

To compute $\mu_\delta([\gamma])$, then, we will begin by rewriting $N_\gamma(s)$ along the same lines as the expression of $\zeta_\delta(s)$ in Equation (23). This will simplify the computation of $\mu_\delta([\gamma])$.

Assume that $\gamma$ has $n_0 = q_0 k + t_0$ edges, for some $q_0, t_0 \in \mathbb{N}$, $0 \leq t_0 < k$. For bookkeeping’s sake, it is now useful to distinguish two cases: $t_0 = 0$ and $t_0 \neq 0$. We detail the case $t_0 = 0$ below; the case $t_0 > 0$ is similar but requires more bookkeeping.

For the case $t_0 = 0$, observe that any path $\eta \in F_\gamma B_\lambda$ can be uniquely realized as $\eta = \gamma \eta'$. We will group the paths $\eta$ according to the number of edges in $\eta'$. Following the same arguments that led us to the formula (14) for $\zeta_\delta(s)$, we see that

$$N_\gamma(s) = \sum_{\eta \in F_\gamma B_\lambda} w_\delta(\eta)^s = \sum_{n \in \mathbb{N}} \sum_{\eta \in F_{\gamma n} B_\lambda} w_\delta(\eta)^s$$

$$= \sum_{t=0}^{k-1} \sum_{q \in \mathbb{N}} \sum_{z, b \in V_0} A^q(s(\gamma), z)(x_b^\lambda)^s A_1 \cdots A_t(z, b) \frac{1}{(\rho_1 \cdots \rho_1)^{s/\delta}} \sum_{j=0}^{p-1} c_{s(\gamma), z;j} \rho^{j(1-s/\delta)}$$

Note that the only differences between the expression above and the formulation of $\zeta_\delta(s)$ given in Equation (13) are the initial factor of $\rho^{-q_0 s/\delta}$ and that in the expression for $N_\gamma(s)$, there is no summation on $s(\gamma)$ (this was $a \in V_0$ in Equation (13)).

Therefore, by using the expression for $A^q(s(\gamma), z)$ given by Equation (20) and going through the same formal derivations as we did for $\zeta_\delta(s)$, we get

$$(1 - \rho^{p(1-s/\delta)})N_\gamma(s) = \frac{1}{\rho^{q_0 s/\delta}} \sum_{b, z \in V_0} \sum_{t=0}^{k-1} (x_b^\lambda)^s A_1 \cdots A_t(z, b) \frac{1}{(\rho_1 \cdots \rho_1)^{s/\delta}} \sum_{j=0}^{p-1} c_{s(\gamma), z;j} \rho^{j(1-s/\delta)}$$

$$+ (1 - \rho^{p(1-s/\delta)})R_N(s(\gamma), s), \quad (24)$$

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where for any $a \in V_0$, we set

$$R_{\mathcal{N}}(a, s) = \rho^{-q_0(s/\delta)} \sum_{b,z \in V_0} \sum_{t=0}^{k-1} \frac{A_1 \cdots A_t(z, b)}{(\rho_1 \cdots \rho_t)^{s/\delta}} (x_b^A)^s \sum_{i=p+1}^{m} \sum_{q \in \mathbb{N}} P_{i}^{a,z}(q) \frac{\alpha_{i}}{\rho^{q_{i}/\delta}}. \quad (25)$$

Note that, as before, for any fixed $a \in V_0$ and any $s > \delta$, $R_{\mathcal{N}}(a, s)$ is finite.

Now, for $s > \delta$ and $a \in V_0$, let

$$Y_{a,0}(s) = \sum_{t=0}^{k-1} \sum_{z, b \in V_0} \frac{A_1 \cdots A_t(z, b)}{(\rho_1 \cdots \rho_t)^{s/\delta}} (x_b^A)^s \sum_{j=0}^{p-1} c_{a, z ; j} \rho^{j(1-s/\delta)}.$$

Observe that $Y_{a,0}(s)$ is positive for every $a \in V_0$, since the fact that $A$ is source-free implies that $\sum_{b \in V_0} A_1 \cdots A_t(z, b)$ is nonzero for each $z \in V_0$, $0 \leq t \leq k - 1$, and Equation (22) implies that $c_{a, z ; j} = A^{(j)}(a, z) \in \mathbb{R}_{\geq 0}$ is nonzero for at least one $z$. If we also define

$$Y_s = \sum_{a \in V_0} Y_{a,0}(s) = \sum_{t=0}^{k-1} \sum_{a, z, b \in V_0} \frac{A_1 \cdots A_t(z, b)}{(\rho_1 \cdots \rho_t)^{s/\delta}} (x_b^A)^s \sum_{j=0}^{p-1} c_{a, z ; j} \rho^{j(1-s/\delta)},$$

then we have $\zeta_\delta(s) = Y_s(1 - \rho^{(1-s/\delta)})^{-1} + R_\zeta(s)$. Similarly we have from Equation (24) that

$$\mathcal{N}_\gamma(s) = \frac{Y_{s(\gamma);0}(s)}{\rho^{q_{0,s}/\delta}(1 - \rho^{(1-s/\delta)})} + R_{\mathcal{N}}(s(\gamma); s).$$

It follows that

$$\mu_\delta([\gamma]) = \lim_{s \searrow \delta} \mathcal{N}_\gamma(s) = \lim_{s \searrow \delta} \frac{(1 - \rho^{(1-s/\delta)}) \mathcal{N}_\gamma(s)}{(1 - \rho^{(1-s/\delta)}) \zeta_\delta(s)} = \lim_{s \searrow \delta} \frac{\rho^{-q_0(s/\delta)} d_{s(\gamma);0}(s) + (1 - \rho^{(1-s/\delta)}) R_{\mathcal{N}}(s(\gamma), s)}{Y_s + (1 - \rho^{(1-s/\delta)}) R_\zeta(s)}.$$

The next step in calculating $\mu_\delta([\gamma])$ is to compute

$$\lim_{s \searrow \delta}(1 - \rho^{(1-s/\delta)}) R_\zeta(s) \text{ and } \lim_{s \searrow \delta}(1 - \rho^{(1-s/\delta)}) R_{\mathcal{N}}(s(\gamma), s).$$

For each $i$, the infinite series

$$\sum_{q \in \mathbb{N}} P_{i}^{a,z}(q) \left( \frac{\alpha_{i}}{\rho^{q/\delta}} \right)^q \quad (26)$$

used in the definition of $R_{\mathcal{N}}(a, s)$ and $R_\zeta(s)$ satisfies

$$\left| \sum_{q \in \mathbb{N}} P_{i}^{a,z}(q) \left( \frac{\alpha_{i}}{\rho^{q/\delta}} \right)^q \right| \leq \sum_{q \in \mathbb{N}} \left| P_{i}^{a,z}(q) \right| \left( \frac{\alpha_{i}}{\rho^{q/\delta}} \right)^q \leq \sum_{q \in \mathbb{N}} \left| P_{i}^{a,z}(q) \right| \left( \frac{|\alpha_{i}|}{\rho} \right)^q \quad (27)$$

whenever $s > \delta$, since $\rho > 1$. The eigenvalue $\alpha_{i}$ also satisfies $|\alpha_{i}| < \rho$ by construction. Since $|\alpha_{i}|/\rho < 1$, it now follows that the series (26) converges by the Root Test. Moreover, the fact
that the final series in (27) converges independently of $s$ implies that both $\lim s \searrow \delta R_N(a, s)$ and $\lim s \searrow \delta R_\varsigma(s)$ are finite. Since $\lim s \searrow \delta (1 - \rho^p(1-s/\delta)) = 0$, we have

$$\lim s \searrow \delta (1 - \rho^p(1-s/\delta))R_N(a, s) = 0 = \lim s \searrow \delta (1 - \rho^p(1-s/\delta))R_\varsigma(s).$$

All of the terms in the formulas for $Y_s$ and $Y_{a,0}(s)$ are continuous in the limit $s \searrow \delta$; thus,

$$\mu_\delta([\gamma]) = \frac{\rho^{-q_0} Y_{s(\gamma);0}(\delta)}{Y_\delta}$$

whenever $\gamma \in F \mathcal{B}_\Lambda$ has $|\gamma| = q_0 k$. Since the sums defining $Y_{s(\gamma);0}(\delta)$ and $Y_\delta$ are finite, $\mu_\delta([\gamma])$ is finite in this case.

If $|\gamma| = q_0 k + t_0$ for some $t_0 \neq 0$, we can consider separately the paths in $F \mathcal{B}_\Lambda$ extending $\gamma$ which have length less than $(q_0 + 1)k$, and those which are longer. This perspective gives

$$\mathcal{N}_\gamma(s) = \sum_{a \in \mathcal{V}_0} \sum_{r=0}^{k-t_0-1} \frac{A_{t_0+1} \cdots A_{t_0+r}(s(\gamma), a)}{\rho^{p_0 s/\delta}(p_1 \cdots p_{t_0+r})^{s/\delta}} (x_a^s)^s$$

$$+ \sum_{a, z, b \in \mathcal{V}_0} \sum_{t=0}^{k-1} A_{t_0+1} \cdots A_k(s(\gamma), a)(x_{b}^s)^s A_1 \cdots A_{t}(z, b) (p_1 \cdots p_{t})^{s/\delta} \sum_{q \in \mathbb{N}} A^q(a, z) \rho^{-(q_0+1+q)s/\delta}$$

The first sum, being finite and continuous in the limit as $s \searrow \delta$, will vanish when we multiply $\mathcal{N}_\gamma(s)$ by $(1 - \rho^p(1-s/\delta))$. For the second term, the same manipulations that we performed above on the sum $\sum_{q \in \mathbb{N}} A^q(a, z) \rho^{-q(s/\delta)}$ tell us that if we let (for $v \in \mathcal{V}_0$)

$$\hat{Y}_{v; t_0}(s) := \sum_{a, z, b \in \mathcal{V}_0} A_{t_0+1} \cdots A_k(v, a) \sum_{t=0}^{k-1} \frac{A_1 \cdots A_{t}(z, b)}{\rho_1 \cdots \rho_{t})^{s/\delta}} (x_{b}^s)^s \sum_{j=0}^{p-1} c_{a, z; j} \rho^{j(1-s/\delta)},$$

we have

$$\mu_\delta([\gamma]) = \lim_{s \searrow \delta} \frac{\mathcal{N}_\gamma(s)}{\mathcal{C}_\delta(s)} = \lim_{s \searrow \delta} \frac{(1 - \rho^p(1-s/\delta))\mathcal{N}_\gamma(s)}{(1 - \rho^p(1-s/\delta))\mathcal{C}_\delta(s)}$$

$$= \frac{\hat{Y}_{s(\gamma);t_0}(\delta)}{\rho^{p_0+1} Y_\delta}.$$

where for any $v \in \mathcal{V}_0$, we have

$$\hat{Y}_{v; t_0}(\delta) = \sum_{a, h, z \in \mathcal{V}_0} \sum_{t=0}^{k-1} A_1 \cdots A_t(z, b) (x_{b}^s)^s A_{t_0+1} \cdots A_k(v, a) \sum_{j=0}^{p-1} c_{a, z; j}$$

$$= \sum_{a \in \mathcal{V}_0} A_{t_0+1} \cdots A_k(v, a) Y_{a;0}(\delta).$$

Observe that $\mu_\delta([\gamma])$ only depends on the vertex $s(\gamma)$ and on $|\gamma| = q_0 k + t_0$. Moreover, $\mu_\delta([\gamma])$ is a quotient of finite sums and is hence finite. The fact that the Dixmier trace determines a measure $\mu_\delta$ now follows from Proposition 3.7.

We now proceed to compare the measures $\mu_\delta$ on $X_{\mathcal{B}_\Lambda}$ of Theorem 3.9 with the unique scale-invariant Borel measure $M$ on $\Lambda^\infty \cong X_{\mathcal{B}_\Lambda}$ identified in Proposition 8.1 of [23], which was described in Equation (11) above.
Corollary 3.10. Let $\Lambda$ be a finite, strongly connected $k$-graph with adjacency matrices $A_i$ such that $A = A_1 \cdots A_k$ is irreducible. For any $\delta \in (0, 1)$ such that Equation (5) holds on the associated ultrametric Cantor set $(X_{B_\Lambda}, d_\delta)$, the measure $\mu_\delta$ induced by the Dzimir trace agrees with the measure $M$ introduced in Proposition 8.1 of [23]: for any $\lambda \in FB_\Lambda$,

$$\mu_\delta([\lambda]) = (\rho_1 \cdots \rho_t)^{-q+1}(\rho_{t+1} \cdots \rho_k)^{-q} x_\Lambda^{(\lambda)} = M([\lambda]).$$

In particular, $\mu_\delta$ is a probability measure which is independent of $\delta$.

Proof. To see that $\mu_\delta$ is a probability measure, we compute

$$\mu_\delta(X_{B_\Lambda}) = \sum_{v \in \mathcal{V}_0} \mu_\delta([v]) = \sum_{v \in \mathcal{V}_0} \frac{Y_{v;0}(\delta)}{Y_\delta} = 1,$$

since $Y_s = \sum_{v \in \mathcal{V}_0} Y_{v;0}(s)$ for all $s$. Recall that $Y_{v;0}(\delta) > 0$ for all $v \in \mathcal{V}_0$, and hence $Y_\delta$ is also finite and nonzero.

Moreover, for any path $\gamma \in F_\delta B_\Lambda$ with $|\gamma| \geq k$, write $\gamma = \gamma_0 \gamma'$ with $|\gamma_0| = k$. Since $r(\gamma') \in \mathcal{V}_k = \mathcal{V}_0$, we can identify $\gamma'$ with a path in $FB_\Lambda$. Then Proposition (219) tells us that

$$w_\delta(\gamma) = \rho^{-1/\delta} w_\delta(\gamma').$$

Consequently,

$$\mu_\delta([v]) = \lim_{s \to \delta} \frac{\sum_{\gamma \in F_s B_\Lambda} w_\delta(\gamma)_s}{\zeta_\delta(s)} = \lim_{s \to \delta} \left( \frac{\sum_{r(\gamma) = v, |\gamma| < k} w_\delta(\gamma)_s}{\zeta_\delta(s)} + \frac{\sum_{r(\gamma) = v, |\gamma| \geq k} w_\delta(\gamma)_s}{\zeta_\delta(s)} \right)$$

$$= \lim_{s \to \delta} \left( \frac{\sum_{r(\gamma) = v, |\gamma| < k} w_\delta(\gamma)_s}{\zeta_\delta(s)} + \frac{\rho^{-s/\delta} \sum_{n=1}^{\infty} \sum_{t=0}^{k-1} \sum_{r(\gamma) = v, |\gamma| = nk+t} A(v, z) w_\delta(\gamma')_s}{\zeta_\delta(s)} \right)$$

$$= \lim_{s \to \delta} \left( \frac{\sum_{r(\gamma) = v, |\gamma| < k} w_\delta(\gamma)_s}{\zeta_\delta(s)} + \frac{\rho^{-s/\delta} \sum_{n=0}^{\infty} \sum_{t=0}^{k-1} \sum_{z \in A^0} \sum_{r(\gamma') = z, |\gamma'| = nk+t} A(v, z) w_\delta(\gamma')_s}{\zeta_\delta(s)} \right)$$

$$= \frac{1}{\rho} \sum_{z \in A^0} A(v, z) \mu_\delta([z]).$$

The penultimate equality holds because of the formula (5) for the weight $w_\delta$; the last equality holds because $\lim_{s \to \delta} \zeta_\delta(s) = \infty$, and hence the first sum (having a finite numerator) tends to zero as $s$ tends to $\delta$.

Thus, $(\mu_\delta([v]))_v$ is a positive eigenvector for $A$ with $\ell^1$-norm 1 and eigenvalue $\rho$, and hence must agree with $x^\Lambda$ by the irreducibility of $A$.

Recall from Equation (28) that if $|\gamma| = q_0 k$ (equivalently, if we think of $\gamma \in \Lambda$, we have $d(\gamma) = (q_0, \ldots, q_0)$ we have

$$\mu_\delta([\gamma]) = \frac{Y_{s(\gamma);0}(\delta)}{Y_\delta \rho^{q_0}}.$$ 

Thus, for such $\gamma$, we have

$$\mu_\delta([\gamma]) = \mu_\delta([s(\gamma)]) \rho^{-q_0} = \mu_\delta([s(\gamma)]) (\rho_1 \cdots \rho_k)^{-q_0} = x_\Lambda^\Lambda (\rho_1 \cdots \rho_k)^{-q_0}.$$ 

Comparing this formula with Equation (11) tells us that whenever $|\gamma| = q_0 k$,

$$\mu_\delta([\gamma]) = M([\gamma]).$$
Since \( \mu_\delta \) agrees with \( M \) on the square cylinder sets \([\lambda]\) with \( d(\lambda) = (q_0, \ldots, q_0) \), and we know from the proof of Lemma 4.1 of [17] that these sets generate the Borel \( \sigma \)-algebra of \( \Lambda^\infty \), \( \mu_\delta \) must agree with \( M \) on all of \( \Lambda^\infty \).

\[ \square \]

Remark 3.11. If one could prove that the vector \((\mu_\delta[v])_{v \in \mathcal{V}_0}\) was an eigenvector for each \( A_i \) with eigenvalue \( \rho_i \), then we could use the theory of families of irreducible matrices, developed in [23, Section 3], to remove the hypothesis that \( A \) be irreducible in Corollary 3.10. So far we have been unable to do this.

4 Eigenvectors of Laplace-Beltrami operators and wavelets

In this section, we investigate the relation between the decomposition of \( L^2(\Lambda^\infty, \mu_\delta) \) via the eigenspaces of the Laplace-Beltrami operators \( \Delta_s \) associated to the spectral triples of Section 3 for the ultrametric Cantor set \((X_{\Lambda}, d_{\omega})\) of Corollary 2.20 and the wavelet decomposition of \( L^2(\Lambda^\infty, M) \) given in Theorem 4.2 of [17]. The connection between Laplace-Beltrami operators and wavelets that we identify in this section goes deeper than the frequently-seen connection between wavelet decompositions and Dirac operators. To be precise, the wavelet decomposition of \( L^2(\Lambda^\infty, M) \) arises from a representation of \( C^*(\Lambda) \) (see Definition 4.1). Thus, the results in this section establish a link between representations of higher-rank graphs and the Pearson-Bellissard spectral triples, in addition to identifying the wavelet decomposition of [17] with the eigenspaces of the Laplace-Beltrami operators \( \Delta_s \).

We begin by describing the Laplace-Beltrami operators and their eigenspaces. According to Section 8.3 of [36] and Section 4 of [26], for each \( s \in \mathbb{R} \) the Pearson-Bellissard spectral triple from the previous section gives rise to a Laplace-Beltrami operator \( \Delta_s \) on \( L^2(X_{\Lambda}, \mu) \) via the Dirichlet form \( Q_s \) as follows:

\[
\langle f, \Delta_s(g) \rangle = Q_s(f, g) := \frac{1}{2} \int_{\Upsilon} \text{Tr}(|D|^{-s}[D, \pi_r(f)]*[D, \pi_r(g)]) \, d\nu(\tau), \tag{29}
\]

where \( \Upsilon \) is the set of choice functions and \( \nu \) is the measure on \( \Upsilon \) induced from the Dixmier measure \( \mu_\delta \) on the infinite path space. Thanks to Section 8.1 of [36], we know that \( Q_s \) is a closable Dirichlet form for all \( s \in \mathbb{R} \) and it has a dense domain that is generated by a set of characteristic functions on cylinder sets of \( X_{\Lambda} \). Also, by applying the work of [36] and [26] to our weighted stationary \( k \)-Bratteli diagrams \( \mathcal{B}_\Lambda \), we can obtain an explicit formula for \( \Delta_s \) on characteristic functions as follows.

For a finite path \( \eta = (\eta_i)_{i=1}^n \) (where each \( \eta_i \) is an edge) in \( \mathcal{B}_\Lambda \), we write \( \chi_\eta \) for the characteristic function of the set \( \{\eta\} \subseteq X_{\Lambda} \) of infinite paths of \( \mathcal{B}_\Lambda \) whose initial segment is \( \eta \), and \( \eta(0, i) \) for \( \eta_1 \cdots \eta_i \). We denote by \( \eta(0, 0) \) the vertex \( r(\eta) \). Also, for \( \gamma \in F\mathcal{B}_\Lambda \), we set

\[
\frac{1}{F_{\gamma}} = \sum_{(e,e') \in \text{ext}_1(\gamma)} \mu([\gamma e])\mu([\gamma e']),
\]

where \( \text{ext}_1(\gamma) \) is the set of pairs \((e, e')\) of edges in \( \mathcal{B}_\Lambda \) with \( e \neq e' \) and \( r(e) = r(e') = s(\gamma) \).

From Remark 4.6 we know that if Equation (8) holds for the weighted stationary \( k \)-Bratteli diagram \((\mathcal{B}_\Lambda, \omega_\delta)\) associated to a higher-rank graph \( \Lambda \), then \( \text{ext}_1(\gamma) \) is nonempty for all \( \gamma \in F\mathcal{B}_\Lambda \). Since we need to invoke Equation (8) in order to guarantee that the measures \( \mu_\delta \) and \( M \) agree on \( X_{\Lambda} \cong \Lambda^\infty \), we will also assume without loss of generality that \( \text{ext}_1(\gamma) \) is always nonempty; equivalently, that \( F_{\gamma} < \infty \). Then, as in Section 4 of [26], for each \( s \in \mathbb{R} \), we have
\[ \Delta_s(x_{[\eta]}) = -\sum_{i=0}^{\ell-1} 2F_{\eta(0,i)} w(\eta(0,i))^{s-2} \left( \mu([\eta(0,i)] \setminus [\eta(0,i+1)]) \right) \chi_{[\eta]} \]

Of more relevance to the current paper is Theorem 4.3 of [26], which tells us that each finite path \( \gamma \) in \( \mathcal{B}_\Lambda \) determines an eigenspace \( E_\gamma \) for \( \Delta_s \):

\[ E_\gamma = \text{span} \left\{ \frac{1}{\mu[\gamma e]} \chi_{[\gamma e]} - \frac{1}{\mu[\gamma e']} \chi_{[\gamma e']} : (e, e') \in \text{ext}_1(\gamma) \right\}. \tag{30} \]

The other eigenspaces are \( E_{-1} = \text{span}\{\chi_{\Lambda^\infty}\} \) and \( E_0 = \text{span} \left\{ \frac{1}{\mu([v])} \chi_{[v]} - \frac{1}{\mu([v'])} \chi_{[v']} : v \neq v' \in \mathcal{V}_0 \right\} \).

Observe that if \( \gamma \neq \eta \), then \( E_\gamma \perp E_\eta \). Also, the eigenspaces of \( \Delta_s \) are independent of \( s \), although the eigenvalues (as described in [36, 26]) depend on the choice of \( s \in \mathbb{R} \).

In Theorem 4.3 below, we compare the eigenspaces \( E_\gamma \) with the wavelet decomposition of \( L^2(\Lambda^\infty, M) \) which was constructed in [17] out of a representation of the \( C^* \)-algebra \( C^*(\Lambda) \) on \( L^2(\Lambda^\infty, M) \). Before recalling this wavelet decomposition, we first review the construction of the \( C^* \)-algebra \( C^*(\Lambda) \) associated to a higher-rank graph.

**Definition 4.1.** [32] Let \( \Lambda \) be a finite \( k \)-graph with no sources. \( C^*(\Lambda) \) is the universal \( C^* \)-algebra generated by a collection of partial isometries \( \{s_\lambda\}_{\lambda \in \Lambda} \) satisfying the Cuntz-Krieger conditions:

\begin{enumerate}
  \item[(CK1)] \( \{s_v : v \in \Lambda^0\} \) is a family of mutually orthogonal projections;
  \item[(CK2)] Whenever \( s(\lambda) = r(\eta) \) we have \( s_\lambda s_\eta = s_{\lambda\eta} \);
  \item[(CK3)] For any \( \lambda \in \Lambda \), \( s_\lambda^* s_\lambda = s_{s(\lambda)} \);
  \item[(CK4)] For all \( v \in \Lambda^0 \) and all \( n \in \mathbb{N}^k \), \( \sum_{\lambda \in vE\Lambda} s_\lambda s_\lambda^* = s_v \).
\end{enumerate}

We now review the “standard representation” of \( C^*(\Lambda) \) on \( L^2(\Lambda^\infty, M) \), which we denote by \( \pi \). It is this representation, first described in Theorem 3.5 of [17], which gives the wavelets that will be used in the sequel. For \( p \in \mathbb{N}^k \) and \( \lambda \in \Lambda \), let \( \sigma^p \) and \( \sigma_\lambda \) be the shift map and prefixing map given in Remark 2.7.9(b). If we let \( S_\lambda := \pi(s_\lambda) \), the image of the standard generator \( s_\lambda \) of \( C^*(\Lambda) \), then Theorem 3.5 of [17] tells us that \( S_\lambda \) is given on characteristic functions of cylinder sets by

\[ S_\lambda \chi_{[\eta]}(x) = \chi_{[\eta]}(x) \rho(\Lambda)^{d(\lambda)/2} \chi_{[\eta]}(\sigma^{d(\lambda)}(x)) \]

\[ = \begin{cases} 
  \rho(\Lambda)^{d(\lambda)/2} & \text{if } x = \lambda \eta y \text{ for some } y \in \Lambda^\infty \\
  0 & \text{otherwise}
\end{cases} \]

\[ = \rho(\Lambda)^{d(\lambda)/2} \chi_{[\lambda \eta]}(x). \tag{31} \]

We can think of the operators \( S_\lambda \) as combined “scaling and translation” operators, since they change both the size and the range of a cylinder set \([\eta]\), and are intimately tied to the geometry of the \( k \)-graph \( \Lambda \).
Theorem 4.3 below shows that when the measure $\mu_{\Lambda}$ induced by the Dixmier trace agrees with the measure $M$, the eigenspaces of the Laplace-Beltrami operators refine the wavelet decomposition of $[17]$ which arises from the standard representation $\pi$. In order to state and prove this Theorem, we first review this wavelet decomposition.

For each $n \in \mathbb{N}$, write

$$\mathcal{Y}_n = \text{span}\{\chi_\lambda : d(\lambda) = (n, \ldots, n)\}, \quad \text{and} \quad \mathcal{W}_n = \mathcal{Y}_{n+1} \cap \mathcal{Y}_n^\perp.$$  

We know from Lemma 4.1 of [17] that $\{\chi_\lambda : d(\lambda) = (n, \ldots, n)\}$ for some $n \in \mathbb{N}$ densely spans $L^2(\Lambda^\infty, M)$. Consequently,

$$L^2(\Lambda^\infty, M) = \mathcal{W}_0 \oplus \bigoplus_{n \in \mathbb{N}} \mathcal{W}_n. \quad (32)$$

Proposition 4.2 below establishes that the subspaces $\mathcal{W}_n := \mathcal{Y}_{n+1} \cap \mathcal{Y}_n^\perp$ are precisely the wavelet subspaces which were denoted $\mathcal{W}_{n,\Lambda}$ in Theorem 4.2 of [17]. Indeed, one can think of the subspaces $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$ as a “multiresolution analysis” for $L^2(\Lambda^\infty, M)$. With this perspective, researchers familiar with wavelet theory will find it natural that the wavelet spaces $\mathcal{W}_{n,\Lambda}$ of [17] arise in this fashion from a multiresolution analysis.

For the proof of our main result, Theorem 4.3, as well as for the proof of Proposition 4.2, it will be convenient to work with a specific basis for $\mathcal{W}_0$. For each vertex $v$ in $\Lambda$, let

$$D_v = v\Lambda^{(1,\ldots,1)}.$$  

One can show (cf. [23] Lemma 2.1(a)) that $D_v$ is always nonempty.

Enumerate the elements of $D_v$ as $D_v = \{\lambda_0, \ldots, \lambda_{\#(D_v)-1}\}$. Observe that if $D_v = \{\lambda\}$ is a 1-element set, then $[v] = [\lambda]$. If $\#(D_v) > 1$, then for each $1 \leq i \leq \#(D_v) - 1$, we define

$$f^{i,v} = \frac{1}{M[\lambda_0]} \chi_{[\lambda_0]} - \frac{1}{M[\lambda_i]} \chi_{[\lambda_i]} \quad (33)$$

One easily checks that in $L^2(\Lambda^\infty, M)$, $\langle f^{i,v}, \chi_{[w]} \rangle = 0$ for all $i$ and all vertices $v, w$, and that

$$\{f^{i,v} : v \in \Lambda^0, 1 \leq i \leq \#(D_v) - 1\}$$

is an orthogonal basis for $\mathcal{W}_0 = \mathcal{Y}_1 \cap \mathcal{Y}_0^\perp \subseteq L^2(\Lambda^\infty, M)$.

The following Proposition justifies the labeling of the orthogonal decomposition of $L^2(\Lambda^\infty, M)$ given in Equation (32) as a wavelet decomposition; it is generated by applying our “scaling and translation” operators $S_\Lambda$ to a finite family $\{f^{i,v}\}_{i,v}$ of “mother functions.”

**Proposition 4.2.** For any $n \in \mathbb{N}$, the set

$$S_n = \{S_\Lambda f^{i,s(\lambda)} : d(\lambda) = (n, \ldots, n), 1 \leq i \leq \#(D_{s(\lambda)}) - 1\}$$

is a basis for $\mathcal{W}_n = \mathcal{Y}_{n+1} \cap \mathcal{Y}_n^\perp$.

**Proof.** The formulas (31) and (33) show that if $d(\lambda) = (n, \ldots, n)$, then $S_\Lambda f^{i,s(\lambda)}$ is a linear combination of characteristic functions of cylinder sets of degree $(n+1, \ldots, n+1)$. Thus, to see that $S_\Lambda f^{i,s(\lambda)} \in \mathcal{W}_n$ for each such $\lambda$ and each $1 \leq i \leq \#(D_{s(\lambda)}) - 1$, we must check that $\langle S_\Lambda f^{i,s(\lambda)}, \chi_{[\eta]} \rangle = 0$ whenever $d(\eta) = (n, \ldots, n)$. We compute:

$$\frac{1}{\rho(\Lambda)d(\lambda)\Lambda^\infty/2} \langle S_\Lambda f^{i,s(\lambda)}, \chi_{[\eta]} \rangle = \frac{1}{M[\lambda_0]} \int_{\Lambda^\infty} \chi_{[\eta]} \chi_{[\lambda_0]} \, dM - \frac{1}{M[\lambda_i]} \int_{\Lambda^\infty} \chi_{[\eta]} \chi_{[\lambda_i]} \, dM$$

$$= \begin{cases} 0, & \eta \neq \lambda \\ \frac{M[\lambda_0]}{M[\lambda_0]} - \frac{M[\lambda_i]}{M[\lambda_i]}, & \lambda = \eta. \end{cases}$$

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Using the formula for $M$ given in Equation (33), we see that

$$\frac{M[\lambda \lambda_0]}{M[\lambda_0]} - \frac{M[\lambda \lambda_i]}{M[\lambda_i]} = \rho(\Lambda)^{-d(\lambda)} - \rho(\Lambda)^{-d(\lambda)} = 0.$$  

In other words, $\langle S \delta^{i,s}(\lambda), \chi_{[n]} \rangle_M = 0$ always, so $S \delta^{i,s}(\lambda) \perp V_n$, and hence $S \delta^{i,s}(\lambda) \in W_n$ for all $\lambda$ and for all $i$. Moreover, $S_n$ is easily seen to be a linearly independent set: if $d(\lambda) = d(\lambda') = (n, \ldots, n)$ and $d(\lambda_i) = d(\lambda'_i) = (1, \ldots, 1)$,

$$[\lambda \lambda_i] \cap [\lambda' \lambda_i'] = \delta_{\lambda, \lambda'} \delta_{\lambda_i, \lambda_i'}. $$

Since $\dim W_n = \dim V_{n+1} - \dim V_n = \#(\Lambda^{n+1,...,n+1}) - \#(\Lambda^{n,...,n})$ and

$$\#(S_n) = \sum_{\lambda \in \Lambda^{n,...,n}} (\#(D_{\delta}(\lambda)) - 1) = \#(\Lambda^{n+1,...,n+1}) - \#(\Lambda^{n,...,n})$$

we have $W_n = \text{span} S_n$ as claimed. \hfill $\square$

### 4.1 Wavelets and eigenspaces for $\Delta_s$

In this section, we prove our Theorem relating the wavelet decomposition (32) with the eigenspaces $V$ of the Laplace-Beltrami operators $\Delta_s$ in the case when $A := A_1 \cdots A_k$ is irreducible. Recall from Corollary 3.10 that in this case, the measure $\mu = \mu_\delta$ used in the following theorem agrees with the measure $M$ from Proposition 8.1 of [23], which was described in Equation (33) above.

**Theorem 4.3.** Let $\Lambda$ be a finite, strongly connected $k$-graph with adjacency matrices $A_i$. Suppose that $A = A_1 \cdots A_k$ is irreducible. For any weight $w_\delta$ on the associated Bratteli diagram $B_\Lambda$ as in Proposition 2.7.2, such that Equation (33) is valid for the ultrametric Cantor set ($\Lambda^\infty, d_{w_\delta}$), the eigenspaces of the associated Laplace-Beltrami operators $\Delta_s$ refine the wavelet decomposition of (32):

$$V_0 = E_{-1} \oplus E_0 \quad \text{and} \quad W_n = \text{span} \{ E_\gamma : |\gamma| = nk + t, 0 \leq t \leq k - 1 \}.$$  

**Proof.** First observe that under the identification of $\Lambda^0 \subseteq \Lambda$ with $V_0 \subseteq B_\Lambda$, we have $E_0 \subseteq V_0$ and $E_{-1} \subseteq V_0$, since the spanning vectors of both $E_0$ and $E_{-1}$ are linear combinations of $\chi_{[n]}$ for vertices $v$. Thus $E_{-1} \oplus E_0 \subseteq V_0$. For the other inclusion, writing $\mu := \mu_\delta$ for the Dixmier trace measure, we compute

$$\left(1 + \sum_{w \neq v \in \Lambda^0} \frac{\mu[w]}{\mu[v]} \right) \chi_{[v]} = \chi_{\Lambda^\infty} = \sum_{w \neq v \in \Lambda^0} \chi_{[w]} + \sum_{w \neq v} \frac{\mu[w]}{\mu[v]} \chi_{[v]}$$

$$= \chi_{\Lambda^\infty} - \sum_{w \neq v} \mu[w] \left( \frac{1}{\mu[w]} \chi_{[w]} - \frac{1}{\mu[v]} \chi_{[v]} \right).$$  

By rescaling, we see that $\chi_{[v]} \in E_{-1} \oplus E_0$, and hence $V_0 = E_{-1} \oplus E_0$ as claimed.

To examine the claim about $W_n$, let $\eta \in FB_\Lambda$ with $|\eta| = nk + t$. In other words, $\eta$ represents an element of degree $(n+1, \ldots, n+1, n, \ldots, n)$ in the associated $k$-graph. Choose a typical generating element $f_\eta$ of $E_\eta$ as in Equation (33),

$$f_\eta = \frac{1}{\mu[\eta e]} \chi_{[\eta e]} - \frac{1}{\mu[\eta e']} \chi_{[\eta e']},$$  

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where \((e, e') \in \text{ext}_1(\eta)\). Write \(\eta = \eta_0 \eta_1\), where \(d(\eta_0) = (n, \ldots, n)\) and \(d(\eta_1) = (1, \ldots, 1, 0, \ldots, 0)\). Enumerate the paths in \(r(\eta_0) \Lambda^{(1, \ldots, 1)}\) as

\[
\{\lambda_0, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+\ell}, \lambda_{m+\ell+1}, \ldots, \lambda_{m+\ell+p}\}
\]

where the paths \(\lambda_i\) for \(0 \leq i \leq m\) are the extensions of \(\eta e\) and the paths \(\lambda_i\) for \(m + 1 \leq i \leq m + \ell\) are the extensions of \(\eta e'\). Then

\[
f_\eta = \frac{1}{\mu(\eta e')} \sum_{i=0}^{m} \chi[\eta_0 \lambda_i] - \frac{1}{\mu(\eta e')} \sum_{i=m+1}^{m+\ell} \chi[\eta_0 \lambda_i]. \tag{34}
\]

Using Equations (31) and (33), and the fact that \(\mu = M\), we obtain

\[
S_{\eta_0} f_{i,r(\eta)} = \rho(\Lambda)^{(n/2, \ldots, n/2)} \left( \frac{1}{\mu(\eta_0)} \chi[\eta_0 \lambda_0] - \frac{1}{\mu(\eta_0)} \chi[\eta_0 \lambda_1]\right),
\]

and hence

\[
S_{\eta_0} \left( \sum_{i=1}^{m} -\frac{\mu[\lambda_i]}{\mu(\eta e')} f_{i,r(\eta)} + \sum_{i=m+1}^{m+\ell} \frac{\mu[\lambda_i]}{\mu(\eta e')} f_{i,r(\eta)} \right)
\]

\[
= \rho(\Lambda)^{(n/2, \ldots, n/2)} \left( \frac{1}{\mu(\eta e')} \sum_{i=1}^{m} \chi[\eta_0 \lambda_i] - \frac{1}{\mu(\eta e')} \sum_{i=m+1}^{m+\ell} \chi[\eta_0 \lambda_i]\right)
\]

\[
+ \frac{1}{\mu(\lambda_0)} \chi[\eta_0 \lambda_0] \left( \sum_{i=1}^{m} -\frac{\mu[\lambda_i]}{\mu(\eta e')} + \sum_{i=m+1}^{m+\ell} \frac{\mu[\lambda_i]}{\mu(\eta e')} \right)
\]

\[
= \rho(\Lambda)^{(n/2, \ldots, n/2)} \left( f_\eta + \frac{1}{\mu(\lambda_0)} \chi[\eta_0 \lambda_0] \left( \sum_{i=0}^{m} -\frac{\mu[\lambda_i]}{\mu(\eta e')} + \sum_{i=m+1}^{m+\ell} \frac{\mu[\lambda_i]}{\mu(\eta e')} \right) \right). \tag{35}
\]

Since the paths \(\lambda_i\), for \(0 \leq i \leq m\), constitute the extensions of \(\eta e\) with the same degree \((1, \ldots, 1)\), we have \(\sum_{i=0}^{m} \mu[\lambda_i] = \mu(\eta e)\). Similarly, \(\sum_{j=m+1}^{m+\ell} \mu[\lambda_i] = \mu(\eta e')\). Moreover,

\[
\frac{\mu[\eta e]}{\mu(\eta e')} = \rho(\Lambda)^{d(\eta e) - d(\eta e')} = \rho(\Lambda)^{d(\eta_0)} = \frac{\mu[\eta e]}{\mu(\eta e')}.
\]

In other words, the coefficient of \(\chi[\eta_0 \lambda_0]\) in Equation (33) is zero, and so \(f_\eta \in \mathcal{W}'\).

If our “preferred path” \(\lambda_0\) is not an extension of either \(e\) or \(e'\), Equations (34) and (35) hold in a modified form without the zeroth term, and we again have \(f_\eta \in \mathcal{W}'\). In other words,

\[
E_\eta \subseteq \mathcal{W}' \quad \text{whenever} \quad |\eta| = nk + t.
\]

To see that \(\mathcal{W}' = \bigoplus_{t=0}^{k-1} \bigoplus_{|\eta|=nk+t} E_\eta\), we again use a dimension argument. If \(|\eta| = nk + t\), we know from [26] Theorem 4.3 that \(\dim E_\eta = \#(s(\eta) \Lambda^{\epsilon_{t+1}}) - 1\). Since we have a bijection between

\[
\bigcup_{|\eta|=nk+t} s(\eta) \Lambda^{\epsilon_{t+1}} \quad \text{and} \quad \Lambda^{d(\eta) + \epsilon_{t+1}},
\]

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\[
\dim \left( \bigoplus_{t=0}^{k-1} \bigoplus_{|\eta|=nk+t} E_{\eta} \right) = \sum_{t=1}^{k} \#(\Lambda^{(n+1, \ldots, n+1, n, \ldots, n)}) - \sum_{t=0}^{k-1} \#(\Lambda^{(n+1, \ldots, n+1)}) \\
= \#(\Lambda^{(n+1, \ldots, n+1)}) - \#(\Lambda^{(n, \ldots, n)}) \\
= \dim W_n.
\]

Remark 4.4. Recall that a directed graph with adjacency matrix \( A \) gives rise to both a stationary Bratteli diagram with adjacency matrix \( A \), and a 1-graph – namely, the category of its finite paths. Moreover, for many 1-graphs the wavelets of [17, Section 4] agree with the wavelets of [34, Section 3]. (Marcolli and Paolucci only considered in [34] strongly connected directed graphs whose adjacency matrix \( A \) has entries from \( \{0, 1\} \); but for all such directed graphs, the wavelets of [17, Section 4] agree with the wavelets of [34, Section 3].) Thus, in this situation, Theorem 4.3 above implies that the eigenspaces of the Laplace-Beltrami operators \( \Delta_s \) associated to the stationary Bratteli diagram with adjacency matrix \( A \), as in [26] Section 4, refine the graph wavelets from Section 3 of [34].

Remark 4.5. In [16], four of the authors of the current paper introduced for any \( k \)-tuple \( J = (J_1, J_2, \ldots, J_k) \in \mathbb{N}^k \) the so-called \( J \)-shaped wavelet decomposition of the Hilbert space \( L^2(\Lambda^\infty, M) : \)

\[
L^2(\Lambda^\infty, M) = \mathcal{V}_0 \oplus \bigoplus_{q \in \mathbb{N}} \mathcal{W}^{J_q}_{q,\ell}.
\]

It is not difficult to modify our definition of the \( k \)-stationary Bratteli diagram associated to \( \Lambda \) and obtain a new Bratteli diagram using \( J : \)

\[
\mathcal{B}^J_\Lambda = ((\mathcal{V}^J_\Lambda)^n, (\mathcal{E}^J_\Lambda)^n),
\]

where \( (\mathcal{V}^J_\Lambda)^n = \mathcal{V}_0 = \Lambda^0 \) for all \( n \), and if \( n = q(J_1 + \cdots + J_k) + (J_1 + \cdots + J_\ell) + t \) for some \( 0 \leq t < J_{\ell+1} \), then \( (\mathcal{E}^J_\Lambda)^n \) has adjacency matrix

\[
(A_1^{J_1} A_2^{J_2} \cdots A_k^{J_k})^q (A_1^{J_1} \cdots A_\ell^{J_\ell}) A_\ell^{J_{\ell+1}}.
\]

Analogously, one can modify the definition of the weight \( w_\delta \) from Equation (5) to obtain a weight, and hence an ultrametric, on \( \mathcal{B}_\Lambda \) whenever \( 0 < \delta < 1 \). Assuming that Hypothesis 3.5 holds in this setting, we thus obtain a Pearson-Bellissard type spectral triple for \( X_{\mathcal{B}^J_\Lambda} \cong \Lambda^\infty \), whose associated Dixmier trace agrees with the measure \( M \) given in Equation (3) on \( \Lambda^\infty \) if \( A_1^{J_1} \cdots A_k^{J_k} \) is irreducible, as in Corollary 3.10. Then, constructing the associated Laplace-Beltrami operators, an easy modification of the proof of Theorem 4.3 shows that

\[
\mathcal{W}^{J_q}_{q,\ell} = \text{span}\{E_\gamma : q(J_1 + \cdots + J_k) \leq |\gamma| < (q + 1)(J_1 + \cdots + J_k)\}
\]

in this more general case, as well.

References

[1] M. Amini, Elliott. G.A., and N. Golestani, The category of ordered Bratteli diagrams, arXiv:1509:07246, 2015.
[2] G. Battle, P. Federbush, and P. Uhlig, *Wavelets for quantum gravity and divergence-free wavelets*, Appl. Comput. Harmon. Anal. 1 (1994), 295–297.

[3] A. Berman and R.J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Classics in Applied Mathematics, vol. 9, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994, Revised reprint of the 1979 original.

[4] S. Bezuglyi and P.E.T. Jorgensen, *Representations of Cuntz-Krieger relations, dynamics on Bratteli diagrams, and path-space measures*, Trends in harmonic analysis and its applications, Contemp. Math., vol. 650, Amer. Math. Soc., Providence, RI, 2015, pp. 57–88.

[5] J.H. Brown, L.O. Clark, C. Farthing, and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum 88 (2014), 433–452.

[6] J.H. Brown, G. Nagy, S. Reznikoff, A. Sims, and D.P. Williams, *Cartan subalgebras in C*-algebras of Hausdorff étale groupoids*, Integral Equations Operator Theory 85 (2016), 109–126.

[7] T.M. Carlsen, S. Kang, J. Shotwell, and A. Sims, *The primitive ideals of the Cuntz-Krieger algebra of a row-finite higher-rank graph with no sources*, J. Funct. Anal. 266 (2014), 2570–2589.

[8] E. Christensen, C. Ivan, and M.L. Lapidus, *Dirac operators and spectral triples for some fractal sets built on curves*, Adv. Math. 217 (2008), 42–78.

[9] L.O. Clark, A. an Huef, and A. Sims, *AF-embeddability of 2-graph algebras and quasidiagonality of k-graph algebras*, J. Funct. Anal. 271 (2016), 958–991.

[10] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.

[11] , *On the spectral characterization of manifolds*, J. Noncommut. Geom. 7 (2013), 1–82.

[12] A. Connes and J. Lott, *Particle models and noncommutative geometry*, Nuclear Phys. B Proc. Suppl. 18B (1990), 29–47 (1991).

[13] K.R. Davidson and D. Yang, *Periodicity in rank 2 graph algebras*, Canad. J. Math. 61 (2009), 1239–1261.

[14] J. Ellis, N.E. Marvomatos, D.V. Nanopoulos, and A.S. Sakharov, *Quantum-gravity analysis of gamma-ray bursts using wavelets*, A&A 402 (2003), 409–424.

[15] C. Farsi, E. Gillaspy, A. Julien, S. Kang, and J. Packer, *Wavelets and spectral triples for fractal representations of Cuntz algebras*, to appear, Contemporary Mathematics.

[16] C. Farsi, E. Gillaspy, S. Kang, and J. Packer, *Wavelets and graph C*-algebras*, to appear Excursions in Harmonic Analysis, vol. 5, arXiv:1601.00061.

[17] , *Separable representations, KMS states, and wavelets for higher-rank graphs*, J. Math. Anal. Appl. 434 (2016), 241–270.

[18] D. Guido and T. Isola, *Dimensions and singular traces for spectral triples, with applications to fractals*, J. Funct. Anal. 203 (2003), 362–400.
[19] G.H. Hardy and M. Riesz, *The general theory of Dirichlet’s series.*, Cambridge University Press, 1915.

[20] R.A. Horn and C.R. Johnson, *Matrix analysis*, second ed., Cambridge University Press, Cambridge, 2013.

[21] A. an Huef, S. Kang, and I. Raeburn, *Spatial realisations of KMS states on the C*-algebras of higher-rank graphs*, J. Math. Anal. Appl. 427, 977–1003.

[22] A. an Huef, M. Laca, I. Raeburn, and A. Sims, *KMS states on C*-algebras associated to higher-rank graphs*, J. Funct. Anal. 266 (2014), 265–283.

[23] ______, *KMS states on the C*-algebra of a higher-rank graph and periodicity in the path space*, J. Funct. Anal. 268 (2015), 1840–1875.

[24] J.Bellissard, R.Benedetti, and J.-M.Gambaudo, *Spaces of tilings, finite telescopic approximations and gap-labeling*, Comm. Math. Phys.

[25] A. Jonsson, *Wavelets on fractals and Besov spaces*, J. Fourier Anal. Appl. 4 (1998), 329–340.

[26] A. Julien and J. Savinien, *Transverse Laplacians for substitution tilings*, Comm. Math. Phys. 301 (2011), 285–318.

[27] W. Kalau, *Hamilton formalism in non-commutative geometry*, J. Geom. Phys. 18 (1996), 349–380.

[28] S. Kang and D. Pask, *Aperiodicity and primitive ideals of row-finite k-graphs*, Internat. J. Math. 25 (2014), 1450022, (25 pages).

[29] A. Yu. Khrennikov and S.V. Kozyrev, *Pseudodifferential operators on ultrametric spaces and ultrametric wavelets*, Izv. Math. 69 (2005), 989–1003.

[30] ______, *Wavelets on ultrametric spaces*, Appl. Comput. Harmon. Anal. 19 (2005), 61–76.

[31] S.V. Kozyrev, *Wavelet theory as p-adic spectral analysis*, Izv. Ross. Akad. Nauk Ser. Mt. 66 (2002), 149–158.

[32] A. Kumjian and D. Pask, *Higher rank graph C*-algebras*, New York J. Math. 6 (2000), 1–20.

[33] M.L. Lapidus, *Towards a noncommutative fractal geometry? Laplacians and volume measures on fractals*, Harmonic analysis and nonlinear differential equations (Riverside, CA, 1995), Contemp. Math., vol. 208, Amer. Math. Soc., Providence, RI, 1997, pp. 211–252.

[34] M. Marcolli and A.M. Paolucci, *Cuntz-Krieger algebras and wavelets on fractals*, Complex Analysis and Operator Theory 5 (2011), 41–81.

[35] D. Pask, I. Raeburn, M. Rørdam, and A. Sims, *Rank-two graphs whose C*-algebras are direct limits of circle algebras*, J. Funct. Anal. 239 (2006), 137–178.

[36] J. Pearson and J. Bellissard, *Noncommutative Riemannian geometry and diffusion on ultrametric Cantor sets*, J. Noncommut. Geom. 3 (2009), 447–480.
[37] I. Raeburn, A. Sims, and T. Yeend, *The C*-algebras of finitely aligned higher-rank graphs*, J. Funct. Anal. 213 (2004), 206–240.

[38] D.I. Robertson and A. Sims, *Simplicity of C*-algebras associated to higher-rank graphs*, Bull. Lond. Math. Soc. 39 (2007), 337–344.

[39] U.G. Rothblum, *Expansions of sums of matrix powers*, SIAM Rev. 23 (1981), 143–164.

[40] J. Spielberg, *Graph-based models for Kirchberg algebras*, Journal of Operator Theory 57 (2007), 347–374.

[41] R. Strichartz, *Construction of orthonormal wavelets*, Wavelets: mathematics and applications, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1994, pp. 23–50.

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