**Abstract.** Consider an \( m \times n \) table \( T \) and lattices paths \( \nu_1, \ldots, \nu_k \) in \( T \) such that each step \( \nu_{i+1} - \nu_i \in \{(1, 1), (1, 0), (1, -1)\} \). The number of paths from the \((1, i)\)-cell (resp. first column) to the \((s, t)\)-cell is denoted by \( C^i(s, t) \) (resp. \( C(s, t) \)). Also, the number of all paths from the first column to the last column is denoted by \( I_m(n) \). We give explicit formulas for the numbers \( C^1(s, t) \) and \( C(s, t) \).

1. Introduction

A lattice path in \( \mathbb{Z}^2 \) is the drawing in \( \mathbb{Z}^2 \) of a sum of vectors from a fixed finite subset \( S \) of \( \mathbb{Z}^2 \), starting from a given point, say \((0, 0)\) of \( \mathbb{Z}^2 \). A typical problem in lattice paths is the enumeration of all \( S \)-lattice paths (lattice paths with respect to the set \( S \)) with a given initial and terminal point satisfying possibly some further constraints. A nontrivial simple case is the problem of finding the number of lattice paths starting from the origin \((0, 0)\) and ending at a point \((m, n)\) using only right step \((1, 0)\) and up step \((0, 1)\) (i.e., \( S = \{(1, 0), (0, 1)\} \)). The number of such paths are known to be the binomial coefficient \( \binom{m+n}{n} \). Yet another example, known as the ballot problem, is to find the number of lattice paths from \((1, 0)\) to \((m, n)\) with \( m > n \), using the same steps as above, that never touch the line \( y = x \). The number of such paths, known as ballot number, equals \( \frac{m-n}{m+n} \binom{m+n}{n} \). In the special case where \( m = n + 1 \), the ballot number is indeed the Catalan number \( C_n \).

Let \( T_{m,n} \) denote the \( m \times n \) table in the plane and \((x, y)\) be the cell in the columns \( x \) and row \( y \) (and refer to it as the \((x, y)\)-cell). The set of lattice paths from the \((i, j)\)-cell to the \((s, t)\)-cell, with steps belonging to a finite set \( S \), is denoted by \( L((i, j) \to (s, t); S) \), and the number of those paths is denoted by \( L((i, j) \to (s, t); S) \), where \( 1 \leq i, s \leq m \)

\[ \text{2010 Mathematics Subject Classification. Primary 05A15; Secondary 11B37, 11B39.} \]

\[ \text{Key words and phrases. Direct animals, Lattice paths, Dyck paths, Perfect lattice paths, Ballot numbers, Motzkin numbers.} \]
and \(1 \leq j, t \leq n\). We put \(|L((i, j) \rightarrow (s, t); S)| = l((i, j) \rightarrow (s, t); S)|\) which means the number of all lattice paths from the \((i, j)\)-cell to the \((s, t)\)-cell.

Throughout this paper, for the table \(T_{m,n}\), we set \(S = \{(1, 1), (1, 0), (1, -1)\}\), and the corresponding lattice paths starting from the first column and ending at the last column are called \textit{perfect lattice paths}. The number of all perfect lattice paths is denoted by \(I_m(n)\), that is,

\[
I_m(n) = \sum_{i,j=1}^{m} l((i, j) \rightarrow (s, t); S).
\]

The values of \(I_m(n)\) is OEIS sequence \texttt{A081113} and \texttt{A296449}.

Sometimes it is more convenient to name each step of lattice paths by a letter, and hence every lattice path will be encoded as a \textit{lattice word}. We label the steps of the set \(S = \{(1, 1), (1, 0), (1, -1)\}\) by letters \(u = (1, 1), r = (1, 0),\) and \(d = (1, -1)\); also if \(h\) is a letter of the word \(W\), order or size of \(h\) in \(W\) is the number of times the letter \(h\) appears in the word \(W\) and it is denoted by \(|h| = |h|_W\).

In this paper, by using ballot numbers, we give explicit formulas for the numbers \(C^1(s, t)\) and \(C(s, t)\) where are defined in the section 2. We closed this paper by calculating the number of perfect lattice paths without restrictions in the table \(T\).

### 2. Computing \(I_m(n)\) in Special Cases

In this section, we give formulas for the number \(I_m(n)\) in the cases where \(n + 1 \leq m \leq 2n\) and \(2n \leq m\). To achieve this goal, we must to recall some further notations from [3]. Let \(T\) be the \(m \times n\) table and \(S = \{(1, 0), (1, 1), (1, -1)\}\). The number of lattice paths from the \((1, i)\)-cell to the \((s, t)\)-cell is denoted by \(C^i(s, t)\). Indeed, \(C^i(s, t) = l((1, i) \rightarrow (s, t); S)\). Table 1 illustrates the values of \(C^8(s, t)\) for \(1 \leq s, t \leq 8\). Also, the number of lattice paths from the first column to the \((s, t)\)-cell is denoted by \(C_{m,n}(s, t)\), that is,

\[
C_{m,n}(s, t) = \sum_{i=1}^{m} C^i(s, t).
\]

To avoid confusion we may use simply notation \(C(s, t)\) for \(C_{m,n}(s, t)\). For the square table \(T_{n,n}\), some values of \(C(n, n)\) is 1, 1, 2, 5, 13, 35, 96, 267, 750, ... are given in the OEIS sequence \texttt{A005773}. D. Gouyou-Beauchamps, G. Viennot show that \(C(n, n)\) enumerate the number of directed animals of size \(n\) (or directed \(n\)-ominoes in standard position) [1]. Clearly, \(I_m(n)\) is the number of words \(a_1a_2...a_{n-1}a_n\) \((a_i \in \{1, \ldots, m\}\) such that \(|a_{i+1} - a_i| \leq 1\) for all \(i = 1, \ldots, n - 1\). Figure 1 shows perfect lattice
paths in $T_{2,3}$ and the corresponding words, where the $i^{th}$ letter indicates the rows whose $i^{th}$ point of the paths belongs to.

In what follows, the number of lattice paths from $(1,1)$-cell to $(s,t)$-cell ($1 \leq s \leq n$ and $1 \leq t \leq m$), using just the two steps $(1,1)$ and $(1,-1)$, is denoted by $A(s,t)$. In other words, $A(s,t) = l(1,1; s,t : S')$, where $S' = \{(1,1), (1,-1)\}$. Table 1 illustrates the values of $A(s,t)$ for $1 \leq s,t \leq 8$. Clearly $A(s,t) = 0$ for $s < t$, and that $A(s,t)$ is the number lattice paths from the $(1,1)$-cell to $(s,t)$-cell not sliding above the line $y = x$. One observe that $A(s,t) = 0$ if $s,t$ have distinct parities as the paths counted by $A(s,t)$ begins from $(1,1)$ and every step in $S'$ keeps the parities of entries so that such paths never meet $(s,t)$-cells with $(s,t)$ having distinct parities. Using the symbols $u$ and $d$, the number $A(s,t)$ counts the words of length $s-1$ on $\{u,d\}$ whose all initial subwords have more or equal $u$ than $d$. For example, Table 1 tells us $A(6,1) = 5$ and the corresponding five words are $uuudd, uudud, uuddu, uuddu, uddud$.

Analogous to $A(s,t)$, the number $C^1(s,t)$ counts the words $a_1a_2 \ldots a_i$ with $1 \leq a_i \leq t$ such that $|a_{i+1} - a_i| \leq 1$ for all $1 \leq i \leq s-1$. In other words, $C^1(s,t)$ counts the number of words of length $s-1$ on $\{u,r,d\}$ whose all initial subwords have more or equal $u$ than $d$. For example, Table 1 shows that $C^1(4,1) = 5$, and the corresponding five words are $uud, urr, rru, udu, rur$. 
Table 1. Values of $C^1(s, t)$ (left), and values of $A(s, t)$ (right)

\[
\begin{array}{cccc|cccc}
1 & 7 & 1 & 6 & 27 & 1 & 0 & 0 \\
1 & 5 & 20 & 70 & 1 & 0 & 5 & 0 \\
1 & 4 & 14 & 44 & 133 & 1 & 0 & 4 & 0 \\
1 & 3 & 9 & 25 & 69 & 189 & 1 & 0 & 3 & 0 \\
1 & 2 & 5 & 12 & 30 & 76 & 196 & 1 & 0 & 2 & 5 \\
1 & 1 & 2 & 4 & 9 & 21 & 51 & 127 & 1 & 0 & 1 & 0 \\
\end{array}
\]

**Theorem 2.1.** For all $1 \leq s, t \leq m$, we have

\[
C^1(s, t) = \sum_{i=0}^{\left\lfloor \frac{s-t}{2} \right\rfloor} \binom{s-1}{s-t-2i} A(t+2i, t).
\]

**Proof.** Let $P$ be a lattice path starting from the $(1,1)$-cell and ending at the $(s,t)$ with three steps $u, r, d$ that never slides above the line $y = x$. Clearly, $|u|_P - |d|_P = t - 1$. Since the number of steps to reach the $(s,t)$-cell is $s - 1$ steps, we must have

\[
|r|_P = s - 1 - |u|_P - |d|_P = s - t - 2|d|_P.
\]

Omitting all the $r$ steps from $P$ yields a path $P'$ from $(1,1)$-cell to the $(t + 2|d|_P, t)$-cell using only $u$ and $d$ steps. For any such a path $P'$ one can recover

\[
\left( |u|_{P'} + |d|_{P'} + s - t - 2|d|_{P'} \right) = \binom{s-1}{s-t-2|d|_{P'}}
\]

paths $P$ by inserting right steps $r$ among $u$ and $d$ steps of $P'$, from which the result follows. \hfill \Box

**Example 2.2.** From Table 1 we read $C^1(8, 4) = 133$. Using theorem 2.1, we can compute $C^1(8, 4)$ alternately as

\[
C^1(8, 4) = \sum_{i=0}^{\left\lfloor \frac{8-4}{2} \right\rfloor} \binom{8-1}{8-4-2i} A(2i + 4, 4)
= \cdot Choose 4A(4, 4) + 7Choose 2A(6, 4) + 7Choose 0A(8, 4)
= 35 \times 1 + 21 \times 4 + 1 \times 14 = 133.
\]

The numbers $A(s, t)$ are indeed computed as in the ballot problem were the paths can touch the $y = x$ line but never go above it. The
number of such ballot paths from \((1, 0)\) to \((m, n)\) is \(\frac{m-n+1}{m+1} \binom{m+n}{m}\). Recall that \(A(s, t)\) is the number of words \(W\) of length \(s-1\) on \(\{u, d\}\) with more or equal \(u\) than \(d\) in any initial subword, hence \(A(s, t)\) is equal to the above number with \(m := |u|_W\) and \(n := |d|_W\). Now since \(|u|_W + |d|_W = s - 1\) and \(|u|_W - |d|_W = t - 1\), it follows that \(m = (s + t)/2 - 1\) and \(n = (s - t)/2\). Hence we obtain the following

**Lemma 2.3.** Inside the \(n \times n\) table, we have

\[
A(s, t) = \frac{2t}{s + t} \left( \binom{s - 1}{\frac{s - t}{2}} \right).
\]

for all \(1 \leq s, t \leq n\).

**Corollary 2.4.** Inside the \(n \times n\) table, we have

\[
C^1(s, t) = \sum_{i=0}^{\left\lfloor \frac{s-1}{2} \right\rfloor} \frac{t}{t + i} \left( \binom{s - 1}{s - t - 2i} \binom{t + 2i - 1}{i} \right).
\]

for all \(1 \leq s, t \leq n\).

In [3], we have computed the number \(I_n(n)\) for all \(n \geq 1\). In what follows, we shall give formulas for \(I_m(n)\), where \(n + 1 \leq m \leq 2n\). To achieve this goal, we use the numbers \(H(s, t)\) inside the \(m \times n\) table defined as

\[
H(s, t) = \sum_{i=1}^{t} C^1(s, i),
\]

where \(1 \leq s \leq n\) and \(1 \leq t \leq m\). Table 2 illustrates some values of \(H(s, s)\). One observe that \(H(s, s) = C(s, s)\) for all \(s \leq m = 5\).

| \(s\) | 1 | 2 | 3 | 4 | 5 |
|------|---|---|---|---|---|
| \(H(s, s)\) | 1 | 2 | 5 | 13 | 36 |

**Table 2.** Some values of \(H(s, s)\) for \(T_{5,10}\)

**Lemma 2.5.** Inside the \(m \times n\) table with \(m \leq n \leq 2m\), we have

\[
H(n, m) = C(n, n) - \sum_{i=m}^{n-1} 3^{n-i-1} C^1(i, m).
\]


Proof. Consider the $m \times n$ table $T$ as the subtable of the $n \times n$ table $T'$ with $T$ in the bottom. We know that $C(n, n) = H(n, n)$ is the number of all perfect lattice paths from the $(1, 1)$-cell to the last columns. However, some lattice paths leave $T$ in rows $m + 1, m + 2, \ldots n$ of $T'$. We shall count the number of such paths. Consider the column $i$ a path starting from $(1, 1)$ left the table $T$ for the first times. Clearly, $m + 1 \leq i \leq n$. The number of such paths is $C_1(i - 1, m)$ and the number of paths starting from $(i, m + 1)$-cell and ending at the last column in simply $3^{n-i}$. Thus the number of such paths leaving $T$ is equal to

$$\sum_{i=m+1}^{n} 3^{n-i}C_1(i-1, m) = \sum_{i=m}^{n-1} 3^{n-i-1}C_1(i, m),$$

from which the result follows. □

Example 2.6. Using Lemma 2.5, we can calculate $H(9, 5)$ as

$$H(9, 5) = C(9, 9) - \sum_{i=5}^{8} 3^{8-i}C_1(i, 5)$$

$$= 2123 - (3^{8-5}C_1(5, 5) + 3^{8-6}C_1(6, 5) + 3^{8-7}C_1(7, 5) + 3^{8-8}C_1(8, 5))$$

$$= 2123 - (27 \times 1 + 9 \times 5 + 3 \times 19 + 1 \times 63) = 1931.$$

Lemma 2.7. Inside the $m \times n$ table, we have

$$C_1(n, m) = \sum_{i=1}^{m} C_1(s, i) \times C_1(n - s + 1, m - i + 1).$$

for all $1 \leq s \leq n$.

Proof. Every path from the $(1, 1)$-cell to $(n, m)$-cell crosses the $s^{th}$ column at some row, say $i$. The number of such paths equals the number $C_1(s, i)$ of paths from the $(1, 1)$-cell to $(s, i)$-cell multiplied by the number $C_1(n - s + 1, m - i + 1)$ of paths from the $(n, m)$-cell to $(s, i)$-cell (in reversed direction). The result follows. □
Example 2.8. Table 1 shows that $C^1(9, 5) = 195$. Lemma 2.7 gives an alternate way to compute $C^1(9, 5)$ as in the following:

$$C^1(9, 5) = \sum_{i=1}^{5} C^1(5, i)C^1(9 - 5 + 1, 5 - i + 1)$$

$$= C^1(5, 1)C^1(5, 5) + C^1(5, 2)C^1(5, 4) + C^1(5, 3)C^1(5, 3)$$

$$+ C^1(5, 4)C^1(5, 2) + C^1(5, 5)C^1(5, 1)$$

$$= 9 \times 1 + 12 \times 4 + 9 \times 9 + 4 \times 12 + 1 \times 9 = 195.$$

Lemma 2.9. Inside the $m \times n$ table, we have

$$C(s, t) = 3^{s-1} - \sum_{i=t+1}^{s-1} 3^{s-i-1}C^1(i, t) - \sum_{i=m+2-t}^{s-1} 3^{s-i-1}C^1(i, m+1-t)$$

for all $s \leq n + 2$.

Proof. Allowing the paths leak out of the $m \times n$ table $T$, the number of all perfect lattice paths from the first column to the $(s, t)$-cell is $3^{s-1}$ minus those paths leaving $T$ at some step. Suppose a path leaves $T$ for the last time at $(0, i)$-cell. Then $i = 1, \ldots, s - t - 1$ and the number of such paths equals $3^{i-1}C^1(s - i, t)$. Analogously, the number of paths leaving $T$ for the last time at $(i, m + 1)$-cell is $3^{i-1}C^1(s - i, m + 1 - t)$, and that $0 \leq i \leq s - (m + 1 - t)$. Hence the result follows by changing $i$ to $s - i$. \qed

In the [3] the authors obtained the following relation for $C_n$ in terms of Motzkin numbers $M_n$.

Lemma 2.10. Inside the square $n \times n$ table we have

$$C_n = 3C_{n-1} - M_{n-2}.$$

Utilizing the above recurrence relation, we prove the following theorem.

Theorem 2.11. Let $T$ be a $n \times n$ table. Then

$$C_{n+1} = \sum_{i=3}^{n+1} M_{i-3}C_{n-i+1},$$

where $M_i$ is the $i^{th}$ Motzkin number.

Proof. $C_n$ is the number of perfect lattice paths from the cell $(1, 1)$ to $(n, n)$. Consider $P_n \in C_n$ is the paths of length $n$ in the following table. If the first step of $P_n$ is $(1, 0)$, the number of perfect lattice paths from this cell to the cell $(n - 1, n - 1)$ is $C_{n-1}$. Now, let the first step of $P_n$
is \((1, 1)\). There are two cases for the number of lattice paths from the cell \((1, 1)\) to the cell \((n, n)\). First, for the next steps, the lattice path \(P_n\) never back to the first row, the number of such lattice paths is \(C_{n-1}\).

Let \(P_n\) back to the first row in the \(i^{th}\) step. So, the number of perfect lattice paths from the cell \((1, 1)\) to the cell \((i−1, 2)\) which staying weakly upper the line \(y = 1\) is \(M_{i−3}\). It is remind to calculate the lattice paths from the cell \((i, 0)\) to the cell \((n, n)\) of the path \(P_n\) that is equal \(C_{n−i+1}\). We have

\[
C_n = C_{n−1} + C_{n−1} + \sum_{i=3}^{n} M_{i−3}C_{n−i+1}
\]

\[
= 2C_{n−1} + \sum_{i=3}^{n} M_{i−3}C_{n−i+1},
\]

Now, by using of 2.10 we can write

\[
C_{n−1} = M_{n−2} + \sum_{i=3}^{n} M_{i−3}C_{n−i+1},
\]

Then

\[
C_{n−1} = \sum_{i=3}^{n+} M_{i−3}C_{n−i+1}.
\]

\[\Box\]

**Theorem 2.12.** Inside the \(m \times n\) table, we have

\[
\mathcal{I}_m(n) = \sum_{i=1}^{m} C(a, i)C(b, i)
\]

for all \(a, b \geq 1\) such that \(a + b = n + 1\). In other words, the inner product of columns \(a\) and \(b\) equals \(\mathcal{I}_m(n)\). In particular, if \(n = 2k − 1\) is odd, then

\[
\mathcal{I}_m(n) = \sum_{i=1}^{m} C_{2i,k,i}^2.
\]

**Proof.** Every perfect lattice path crosses the column \(a\) at some row, say \(i\). The number of such paths equals the number \(C(a, i)\) of paths from the first column to the \((a, i)\)-cell multiplied by the number \(C(n−(a−1)a, i) = C(b, i)\) of paths from the last column to that cell, from which the result follows. \[\Box\]
3. Perfect lattice paths in the plane

In this section, we calculate the number of all perfect lattice paths from (1,1)-cell to (x,y)-cell in the whole space (not restricted to a table). We denote these lattice paths with \( S(x,y) \).

**Theorem 3.1.** The number \( S(x,y) \) is given by

\[
S(x,y) = \sum_{r=0}^{x-1} \binom{x-1}{r} \binom{x-r-1}{\frac{x-y-r}{2}} = \sum_{d=0}^{\lfloor \frac{x-y}{2} \rfloor} \binom{x-1}{d} \binom{x-d-1}{x-y-2d}.
\]

**Proof.** Let \( P \) be a lattice path starting from the (1,1)-cell and ending at the (s,t)-cell which uses three steps \( u, r, d \). Then \( P \) is equivalent to a word on \( u, r, d \) satisfying

\[
|u|_P + |r|_P + |d|_P = x - 1 \quad \text{and} \quad |u|_P - |d|_P = y - 1,
\]

which implies that \( |r|_P + 2|d|_P = x - 1 \). The first equality follows from choosing first \( r \) letter \( r \) among \( x - 1 \) letters and then choosing \( d = (x - y - r)/2 \) letter \( d \) from \( x - r - 1 \) remaindered letters, while the second equality follows from choosing first \( d \) letter \( d \) among \( x - 1 \) letter and then choosing \( r = x - y - 2d \) letter \( r \) from \( x - d - 1 \) remaindered letters. \( \square \)

Let \( T = T_{m,n} \) and \( S(a,b)(x,y) \) denote the number of all perfect lattice path from \((a,b)\)-cell to \((x,y)\)-cell without leaving the table \( T \).

|   |   |   |   |
|---|---|---|---|
| 6 |   |   | 1 |
| 5 |   | 1 | 5 |
| 4 | 1 | 4 | 15|
| 3 | 1 | 3 | 10| 30|
| 2 | 1 | 2 | 6 | 16 | 45|
| 1 | 1 | 3 | 7 | 19 | 51|
| 0 | 1 | 2 | 6 | 16 | 45|
| -1| 1 | 3 | 10| 30 |
| -2|   | 1 | 4 | 15 |
| -3|   |   | 1 | 5 |
| -4|   |   |   | 1 |

**Table 3.** Some values of \( S(x,y) \)

As before one can compute \( S(a,b)(x,y) \) by subtracting the number of all those paths starting from \((a,b)\)-cell and ending at \((x,y)\)-cell and leave the table from the total number of such paths. We have
Theorem 3.2. Inside the \( m \times n \) table, we have
\[
S_{a,b}(x, y) = S(x - a + 1, y - b + 1) - \sum_{x' = a + b}^{x - y} C^1(x' - a, b)S(x - x' + 1, y + 1) - \sum_{x' = m + a - b + 1}^{x + y - m - 1} C^1(x' - a, m - b + 1)S(x - x' + 1, m - y),
\]
where \( a \leq x \) and \( b \leq y \).

Proof. Let \( T := T_{m,n} \). Clearly, the number of \( \{u, r, C\} \)-paths starting from \((a, b)\)-cell and ending at \((x, y)\)-cell equals \( S(x - a + 1, y - b + 1) \) after a suitable shift. Now we count the number of those paths leaving \( T \). Every such path leaves \( T \) from the first row or the last row. Suppose a path leaves \( T \) at \((x', 0)\)-cell for the first times. Clearly, \( a + b \leq x' \leq x - y \), and the number of such paths equals the number of paths inside \( T \) (in reverse direction) starting from \((x' - 1, 1)\)-cell and ending at \((a, b)\)-cell (namely, \( C^1(x' - a, b) \)) multiplied by the number of paths starting at \((x', 0)\) and ending at \((x, y)\)-cell (namely, \( S(x - x' + 1, y + 1) \)). A similar argument shows that if a path \( P \) leaves \( T \) at \((x', m + 1)\) for the first times, then \( m + a - b + 1 \leq x' \leq x - y - m - 1 \) and the number of those paths equals \( C^1(x' - a, m - b + 1)S(x - x' + 1, m - y) \), as required.

Example 3.3. Utilizing Table 3 and Theorems 3.1 and 3.2, we observe that, inside the \( 8 \times 8 \) table,
\[
S_{(2,1)}(7, 3) = S(6, 3) - \sum_{x' = 3}^{4} C^1(x' - 2, 1)S(8 - x', 4) - \sum_{x' = 10}^{1} C^1(x' - 2, 8)S(8 - x', 5) = S(6, 3) - C^1(1, 1)S(5, 4) - C^1(2, 1)S(4, 4) = 30 - 1 \times 4 - 1 \times 1 = 25.
\]

Using Theorem 3.2, we can obtain a formula for \( I_m(n) \) in the case where \( m + 3 \leq n \leq 2m + 5 \). In the following, we obtain \( I_m(n) \) in the more general case that \( n \geq 2m \).

Theorem 3.4. Inside table \( T_{m,km+j} \), where \( 0 \leq j \leq m - 1 \) and \( k \geq 2 \), we have
\[
I_m(km + j) = \sum_{1 \leq h_1, h_2, \ldots, h_k \leq m} \Pi_{l=1}^{k-1} S_{(t,h_l)}(m + 1, h_{l+1})C(m, h_1)C(j + 1, h_k),
\]
where \( k \leq \sum_{i=1}^{k} h_i \leq km; \ 1 \leq h_i \leq m \).
Proof. For positive integers $k$ and $0 \leq j \leq m - 1$, let $T$ be a table with $m$ rows and $n = km + j$ columns. For $0 \leq i \leq m$, let $h_i$ be a cells in column $m$. The number of all perfect lattice paths from the first columns to the cell $(i, m)$ is equal $C(m, h_1)$ and the number of perfect lattice paths from the cells $(h_k, mk)$ to the last column is equal $C(j + 1, h_k)$, for positive integers $k$ and $0 \leq j \leq m - 1$. Now, the number of ways to arrive from the cells $(h_i, i \times m)$ to the cells $(h_{i+1}, (i + 1)m)$ is $S_{(h_i)}(m + 1, h_{i+1})$. So, the number of all perfect lattice paths from the first column to the last columns in the $T$ with $m$ rows and $n = km + j$ columns is equal

$$I_m(km + j) = \sum_{1 \leq h_1, ..., h_k \leq m} C(m, h_1)C(j + 1, h_k) \prod_{t=1}^{k-1} S_{(h_t)}(m + 1, h_{t+1}).$$

□

References

[1] Gouyou-Beauchamps, D., and Gérard Viennot. Equivalence of the two-dimensional directed animal problem to a one-dimensional path problem. Advances in Applied Mathematics 9.3 (1988): 334-357.

[2] Sequence A081113, On-Line Encyclopaedia of Integer Sequences, published electronically at http://oeis.org/.

[3] D. Yaqubi, M. Farrokhi D. G., and H. Ghasemian Zoeram, Lattice paths inside a table I, Submitted.

Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran
Email address: daniel_yaqubi@yahoo.es

Institute for Advanced Studies in Basic Sciences (IASBS), and the Center for Research in Basic Sciences and Contemporary Technologies, IASBS, P.O.Box 45195-1159, Zanjan 66731-45137, Iran
Email address: m.farrokhi.d.g@gmail.com, farrokhi@iasbs.ac.ir

Amirkabir University of Technology (Tehran Polytechnique), Tehran, Iran
Email address: m.zamani28@aut.ac.ir