THE DOUGLAS LEMMA FOR VON NEUMANN ALGEBRAS AND SOME APPLICATIONS

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ABSTRACT. In this article, we discuss some applications of the well-known Douglas factorization lemma in the context of von Neumann algebras. Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on a complex Hilbert space $\mathcal{H}$, and $\mathcal{R}$ be a von Neumann algebra acting on $\mathcal{H}$. We prove some new results about left (or, one-sided) ideals of von Neumann algebras; for instance, we show that every left ideal of $\mathcal{R}$ can be realized as the intersection of a left ideal of $\mathcal{B}(\mathcal{H})$ with $\mathcal{R}$. We also generalize a result by Loebl and Paulsen (Linear Algebra Appl. 35 (1981), 63–78) pertaining to $C^*$-convex subsets of $\mathcal{B}(\mathcal{H})$ to the context of $\mathcal{R}$-bimodules.

Keywords: Douglas lemma, Left ideals of von Neumann algebras, $C^*$-convexity

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1. Introduction

In [4], Douglas notes that the notions of majorization, factorization, and range inclusion, for operators on a Hilbert space are intimately connected. We mention the main result of [4] below which is referred to as the Douglas lemma or the Douglas factorization theorem in the literature.

**Theorem 1.1** (Douglas lemma). For bounded operators $A, B$ on a Hilbert space $\mathcal{H}$, the following statements are equivalent:

(i) (majorization) $A^*A \leq \lambda^2 B^*B$ for some $\lambda \geq 0$;
(ii) (factorization) $A = CB$ for some bounded operator $C$ on $\mathcal{H}$;
(iii) (range inclusion) $\text{range}(A^*) \subseteq \text{range}(B^*)$.

It naturally appears in many contexts, and as Douglas observed, “... fragments of these results are to be found scattered throughout the literature (usually buried in proofs) ....”

In this article, we give a constructive proof of the Douglas lemma for von Neumann algebras. In §3, we discuss some applications of this result to the structure of left (or, one-sided) ideals of von Neumann algebras, and to the notion of $C^*$-convexity in the context of bimodules over a von Neumann algebra.

For the convenience of the reader, we briefly recall some basic notions in operator algebras and set up the notation. We shall denote a complex Hilbert space by $\mathcal{H}$ and the set of bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. The family $\mathcal{B}(\mathcal{H})$ is an algebra relative to the usual addition and multiplication (composition) of operators. Let $\| \cdot \|$ denote the usual operator norm. Provided with this norm, $\mathcal{B}(\mathcal{H})$ becomes a Banach algebra. A family $\Gamma$ of operators...
on $\mathcal{H}$ is said to be “self-adjoint” when $A^*$, the adjoint-operator of $A$, is in $\Gamma$ if $A$ is in $\Gamma$. The norm-closed self-adjoint subalgebras of $\mathcal{B}(\mathcal{H})$ are called “C*-algebras” and those closed in the weak-operator topology on $\mathcal{B}(\mathcal{H})$ are the “von Neumann algebras”. Our von Neumann algebras are required to contain the identity operator $I$ on $\mathcal{H}$, that is, $Ix = x$ for each $x$ in $\mathcal{H}$. We assume that our C*-algebras are unital. We often denote a C*-algebra by “$\mathfrak{A}$” and a von Neumann algebra by “$\mathcal{R}$.” In the rest of this section, we discuss the two contexts in which we apply the Douglas lemma - (i) one-sided ideal structure of von Neumann algebras, (ii) $C^*$-convexity in bimodules over a C*-algebra.

1.1. Ideals of $C^*$-algebras and von Neumann algebras. The structures of the left ideals and right ideals in a $C^*$-algebra, $\mathfrak{A}$, are very closely tied to the representation theory of $\mathfrak{A}$; for instance, via the GNS construction. In particular, the representation theory of $\mathcal{B}(\mathcal{H})$ is very much a part of this. It is well-known that the weak-operator closed left ideals in a von Neumann algebra $\mathcal{R}$ are left principal ideals of the form $\mathcal{R}E$ for a projection $E$ in $\mathcal{R}$. In Lemma 3.2 we prove that for a positive self-adjoint operator $A$ in $\mathcal{R}$, the left principal ideal $\mathcal{R}A$ is weak-operator closed if and only if 0 is an isolated point in the spectrum of $A$.

For the discussion in this paragraph, we assume that the von Neumann algebra $\mathcal{R}$ acting on $\mathcal{H}$ is infinite-dimensional, so as to avoid making vacuous statements. In Theorem 3.7 we show that a norm-closed left ideal in $\mathcal{R}$ which is not weak-operator closed must be (algebraically) generated by uncountably many operators in $\mathcal{R}$. We include a proof of the fact that the lattice of norm-closed left ideals of a $C^*$-algebra acting on $\mathcal{H}$ may be derived from the lattice of norm-closed left ideals of $\mathcal{B}(\mathcal{H})$, via intersection with the $C^*$-algebra. On a similar note, in Corollary 3.10 we show that the lattice of left ideals of $\mathcal{R}$ may be derived from the lattice of left ideals of $\mathcal{B}(\mathcal{H})$, via intersection with $\mathcal{R}$. Although it is straightforward to see that the intersection of a left ideal of $\mathcal{B}(\mathcal{H})$ with $\mathcal{R}$ is a left ideal of $\mathcal{R}$, what we show is that every left ideal of $\mathcal{R}$ can be obtained in such a manner.

1.2. $C^*$-convexity. The numerical range of an operator $T$ in $\mathcal{B}(\mathcal{H})$ is defined as

$$W(T) := \{(Tx, x) : x \in \mathcal{H}, \|x\| = 1\}.$$ 

The Toeplitz-Hausdorff theorem (cf. [12], [3]) states that the numerical range of a bounded operator is a convex set. This subset of the complex plane $\mathbb{C}$ succinctly captures information about the eigenvalues, algebraic, analytic structure of $T$ in the geometry of its boundary. In [1], [2], Arveson defines the notion of the $n$th-matrix range (a generalized non-commutative numerical range) of an operator $T$ as

$$W_n(T) := \{\Phi(T) : \Phi \text{ is a unital completely positive map from } C^*(T) \text{ to } M_n(\mathbb{C})\}.$$ 

The description of completely positive maps given by Stinespring’s theorem (cf. [11]), and Choi’s theorem (cf. [3]) for completely positive maps between finite-dimensional $C^*$-algebras involving Kraus operators suggest the importance of studying a non-commutative version of convexity called $C^*$-convexity which we define below in the context of bimodules over a $C^*$-algebra.

\(^1\)Note that $W_1(T)$ is not necessarily the same as the numerical range $W(T)$. But they have the same closure in $\mathbb{C}$.
Definition 1.2. Let \( \mathfrak{A} \) be a \( C^* \)-algebra with identity \( I \) and \( \mathfrak{H} \) be a \( \mathfrak{A} \)-bimodule. For vectors \( A_1, \ldots, A_n \) in \( \mathfrak{H} \) \((n \in \mathbb{N})\), and operators \( T_1, \ldots, T_n \in \mathfrak{A} \) satisfying \( T_1^*T_1 + \cdots + T_n^*T_n = I \), the vector \( T_1^*A_1T_1 + \cdots + T_n^*A_nT_n \) in \( \mathfrak{H} \) is called a \( C^* \)-convex (or \( \mathfrak{A} \)-convex) combination of the \( A_i \)'s.

A subset \( \mathcal{J} \) of \( \mathfrak{H} \) is said to be \( C^* \)-convex in \( \mathfrak{H} \) (or \( \mathfrak{A} \)-convex) if for vectors \( A_1, \ldots, A_n \) in \( \mathcal{J} \) \((n \in \mathbb{N})\), every \( \mathfrak{A} \)-convex combination of the \( A_i \)'s is also in \( \mathcal{J} \).

Definition 1.3. For vectors \( A_1, A_2, \ldots, A_n \) in the \( \mathfrak{A} \)-bimodule \( \mathfrak{H} \), the set

\[
\{ \sum_{i=1}^{n} T_i^*A_iT_i : T_1, T_2, \ldots, T_n \in \mathfrak{A}, \sum_{i=1}^{n} T_i^*T_i = I \} \subseteq \mathfrak{H}
\]

is said to be the \( C^* \)-polypode (or \( \mathfrak{A} \)-polypode) generated by the \( n \)-tuple \( \mathbf{A} := (A_1, A_2, \ldots, A_n) \). The \( \mathfrak{A} \)-polypode generated by a 2-tuple \( (A_1, A_2) \) is called the \( C^* \)-segment (or \( \mathfrak{A} \)-segment) joining the elements \( A_1, A_2 \) in \( \mathfrak{H} \).

We will use the terms “\( C^* \)-convex set”, “\( C^* \)-polypode”, “\( C^* \)-segment” in the context of a bimodule over a general \( C^* \)-algebra. In the setting of a specific \( C^* \)-algebra \( \mathfrak{A} \), we prefer to use the terms “\( \mathfrak{A} \)-convex set”, “\( \mathfrak{A} \)-polypode”, “\( \mathfrak{A} \)-segment”.

Example 1.4. (i) A \( C^* \)-algebra may be viewed as a \( \mathbb{C} \)-bimodule with the left and right action both given by the usual scaling. In this case, \( C^* \)-convexity reduces to the usual notion of convexity.

(ii) A \( C^* \)-algebra \( \mathfrak{A} \) may be viewed as a \( \mathfrak{A} \)-bimodule with the left action (right action, respectively) given by multiplication on the left (on the right, respectively). For \( \mathfrak{A} = \mathcal{B}(\mathcal{H}) \) this is the context in which \( C^* \)-convexity is discussed by Loebl and Paulsen in [10].

Example 1.5. Consider \( M_n(\mathbb{C}) \) as an \( M_n(\mathbb{C}) \)-bimodule in the sense of Example 1.4 (ii). The \( n \)-th-matricial ranges \( W_n(T) \subset M_n(\mathbb{C}) \) are not only convex but also \( M_n(\mathbb{C}) \)-convex.

At this point, we direct the interested reader to [13] for an exposition on some basic results in the theory of \( C^* \)-convexity. Line segments in complex (or real) vector spaces are the most basic of convex sets. But although every \( C^* \)-convex set is convex, as a consequence of the non-commutativity of the “coefficients”, the \( C^* \)-segments in bimodules over a \( C^* \)-algebra need not be \( C^* \)-convex or even convex. For instance, in \( M_2(\mathbb{C}) \) (viewed as an \( M_2(\mathbb{C}) \)-bimodule) consider

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]

As \( A \) is unitarily equivalent to \( C \), the \( M_2(\mathbb{C}) \)-segment \( S(A, B) \) contains \( C \). The mean of \( A \) and \( C \) is of full-rank but the elements in \( S(A, B) \) have rank 0 or 1. Hence \( \frac{A+C}{2} \) is not in \( S(A, B) \) although both \( A, C \) are in \( S(A, B) \).

A subset \( S \) of a complex vector space \( V \) is said to be convex if it contains the line segment joining any two points in it. Equivalently, the subset \( S \) is said to be convex if it contains all convex combinations of its elements.
Definition 1.6. Let $\mathfrak{A}$ be a unital $C^*$-algebra. We say that a set $\mathcal{I}$ in a $\mathfrak{A}$-bimodule $\mathcal{H}$ is $C^*$-2-convex in $\mathcal{H}$ (or $\mathfrak{A}$-2-convex) if the $\mathfrak{A}$-segment joining any two elements in $\mathcal{I}$ is contained in $\mathcal{I}$.

For complex vector spaces (thought of as $\mathbb{C}$-bimodules in the sense of Example 1.3 (i)), the notions of $C^*$-convexity and $C^*$-2-convexity coincide. Let $\mathfrak{A}$ be a $C^*$-algebra. In a $\mathfrak{A}$-bimodule $\mathcal{H}$, clearly a $\mathfrak{A}$-convex set is $\mathfrak{A}$-2-convex. Because of the (generally) non-convex nature of $C^*$-segments, it is not readily apparent whether every $\mathfrak{A}$-2-convex set is also $\mathfrak{A}$-convex. As determining $C^*$-2-convexity of a set is a more direct affair involving pairs of elements, a result in the affirmative would be of practical utility in determining $C^*$-convexity of subsets of $\mathcal{H}$.

The results in [10] Theorem 15, 16] proved by Loebl and Paulsen may be more generally viewed as having shown that $B(\mathcal{H})$-convexity and $B(\mathcal{H})$-2-convexity are equivalent concepts in the context of $B(\mathcal{H})$-bimodules. In Theorem 3.13 for a finite von Neumann algebra $\mathcal{R}$, we show that a subset $\mathcal{I}$ of a $\mathcal{R}$-bimodule is $\mathcal{R}$-convex if and only if $\mathcal{I}$ is $\mathcal{R}$-2-convex. In Theorem 3.15 we prove a similar equivalence for properly infinite von Neumann algebras. Using the type decomposition of von Neumann algebras, we obtain the result for any von Neumann algebra.

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2. The Douglas lemma for von Neumann algebras

Theorem 2.1 (Douglas factorization lemma). Let $\mathcal{R}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. For $A, B$ in $\mathcal{R}$ the following are equivalent:

(i) $A^*A \leq \lambda^2B^*B$ for some $\lambda \geq 0$;
(ii) $A = CB$ for some operator $C$ in $\mathcal{R}$.

In addition, if $A^*A = B^*B$, then $C$ can be chosen to be a partial isometry with initial projection the range projection of $B$, and final projection as the range projection of $A$.

Proof. (i) $\implies$ (ii)

For any vector $x$ in the Hilbert space $\mathcal{H}$, we have that $\|Ax\|^2 = \langle A^*Ax, x \rangle \leq \lambda^2 \langle B^*Bx, x \rangle = \lambda^2\|Bx\|^2$ which implies $\|Ax\| \leq \lambda\|Bx\|$. Thus if $Bx = 0$, it follows that $Ax = 0$ and the linear map $C$ defined on the range of $B$ by $C(Bx) = Ax$ is well-defined and also bounded (with norm less than or equal to $\lambda$). Thus we may extend the domain of definition of $C$ to $\overline{\text{ran}(B)}$ the closure of the range of $B$. If $z$ is a vector in $\overline{\text{ran}(B)}$, we define $Cz = 0$. Thus $C$ is a bounded operator on $\mathcal{H}$ such that $A = CB$ with $\|C\| \leq \lambda$.

Let $R$ be a self-adjoint operator in the commutant $\mathcal{R}'$ of $\mathcal{R}$. Then $RA = AR, RB = BR$ and the linear subspace $\overline{\text{ran}(B)}$ is invariant under $R$ and so is the closed subspace $\overline{\text{ran}(B)}$ (as $R$ is self-adjoint). For vectors $x_1$ in $\mathcal{H}$ and $x_2$ in $\overline{\text{ran}(B)}$, we have that $CR(Bx_1 + x_2) = CRBx_1 + C(Rx_2) = CB(Rx_1) + 0 = A(Rx_1) = R(Ax_1) = RCBx_1 = RC(Bx_1 + x_2)$. Thus
If $A = CB$ for some operator $C \in \mathcal{R}$, then $A^*A = B^*C^*CB \leq \|C\|^2B^*B$. Thus, we may pick $\lambda = \|C\|$.

If $A^*A = B^*B$, then $\|Af\| = \|Bf\|$ for any vector $f$ in $\mathcal{H}$. Thus, the second part follows from the explicit definition of the operator $C$ earlier in the proof. □

The polar decomposition theorem for von Neumann algebras is a direct consequence of the Douglas lemma.

**Corollary 2.2 (Polar decomposition theorem).** Let $\mathcal{R}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. For an operator $A$ in $\mathcal{R}$, there is a partial isometry $V$ with initial projection the range projection of $(A^*A)\frac{1}{2}$, and final projection as the range projection of $A$ such that $A = V(A^*A)\frac{1}{2}$.

*Proof.* Let $B$ denote the operator $(A^*A)\frac{1}{2}$. Clearly $A^*A = B^*B$ and thus from the second part of the Douglas lemma, the corollary follows. □

### 3. Applications

#### 3.1. Left ideals of von Neumann algebras.

**Lemma 3.1.** Let $A, B$ be operators in a von Neumann algebra $\mathcal{R}$. The left ideal $\mathcal{R}A$ in $\mathcal{R}$ is contained in the left ideal $\mathcal{R}B$ if and only if $A^*A \leq \lambda^2B^*B$ for some $\lambda \geq 0$. As a consequence, for any $A$ in $\mathcal{R}$, we have that $\mathcal{R}A = \mathcal{R}\sqrt{A^*A}$.

*Proof.* Its straightforward to see that $A$ is in $\mathcal{R}B$ if and only if $\mathcal{R}A \subseteq \mathcal{R}B$. And from Theorem 2.1, we have that $A$ is in $\mathcal{R}B$ if and only if $A^*A \leq \lambda^2B^*B$ for some $\lambda \geq 0$.

Further, $\mathcal{R}A = \mathcal{R}B$ if and only if $B^*B \leq \lambda^2A^*A$ and $A^*A \leq \mu^2B^*B$ for some $\lambda, \mu \geq 0$. In particular, if $A^*A = B^*B$, then $\mathcal{R}A = \mathcal{R}B$. Noting that $A^*A = \sqrt{A^*A}\sqrt{A^*A}$, we conclude that $\mathcal{R}A = \mathcal{R}\sqrt{A^*A}$. □

**Lemma 3.2.** Let $A$ be an operator in a von Neumann algebra $\mathcal{R}$. Then the left ideal $\mathcal{R}A$ is weak-operator closed if and only if 0 is an isolated point in the spectrum of $A^*A$.

*Proof.* If $A = 0$, the conclusion is straightforward. So we may assume that $A \neq 0$.

If $\mathcal{R}A$ is weak-operator closed, there is a unique projection $E$ in $\mathcal{R}$ such that $\mathcal{R}A = \mathcal{R}E$. From Lemma 3.1, there are $\mu, \lambda > 0$ such that $\mu^2E \leq A^*A \leq \lambda^2E$. This tells us that the spectrum of $A^*A$ is contained in $\{0\} \cup [\mu, \lambda]$ which implies that 0 is an isolated point in the spectrum of $A^*A$.

For the converse, let 0 be an isolated point in the spectrum of $A^*A$. By the spectral mapping theorem, 0 is also an isolated point in the spectrum of $\sqrt{A^*A}$. Let the distance of 0 from $\text{sp}(\sqrt{A^*A}) - \{0\}$ (which is compact as 0 is isolated) be $\mu > 0$ and $\lambda = \|A\|$. Let $F$
be the projection onto the kernel of $\sqrt{A^*A}$, which is the largest projection in $\mathcal{R}$ such that $\sqrt{A^*A}F = 0$. We have that $\mu^2(I - F) \leq A^*A \leq \lambda^2(I - F)$. Thus $\mathcal{R}A = \mathcal{R}(I - F)$ which is weak-operator closed.

**Proposition 3.3.** Let $A$ be an operator in a von Neumann algebra $\mathcal{R}$ acting on the Hilbert space $\mathcal{H}$. Then the left ideal $\mathcal{R}A$ is norm-closed if and only if $\mathcal{R}A$ is weak-operator closed.

**Proof.** Let $\mathcal{R}A$ be norm-closed. Without loss of generality, we may assume that $A$ is positive (since $\mathcal{R}A = \mathcal{R}\sqrt{A^*A}$). By the Stone-Weierstrass theorem, for a continuous function $f$ on $\text{sp}(A)$ vanishing at $0$, $f(A)$ is in $\mathcal{R}A$. In particular, $\sqrt{A}$ is in $\mathcal{R}A$. By Lemma 3.1, there is a $\lambda > 0$ such that $(\sqrt{A})^2 = A \leq \lambda^2 A^2$. The operator $\lambda^2 A^2 - A$ is positive and by the spectral mapping theorem,

$$\text{sp}(\lambda^2 A^2 - A) := \{\lambda^2 \mu^2 - \mu : \mu \in \text{sp}(A)\}.$$  

For a non-zero element $\mu$ in the spectrum of $A$, $\lambda^2 \mu^2 - \mu \geq 0 \Rightarrow \mu \geq \frac{1}{\sqrt{\lambda}}$. This tells us that $0$ is an isolated point in the spectrum of $A$ and hence by Lemma 3.2, $\mathcal{R}A$ is weak-operator closed.

The converse is straightforward as the weak-operator topology on $\mathcal{R}$ is coarser than the norm topology. \hfill \Box

Let $\mathcal{R}$ be a von Neumann algebra. We use the notation $\langle V \rangle$, to denote the linear span of a subset $V$ of $\mathcal{R}$.

**Definition 3.4.** Let $S$ be a family of operators in the von Neumann algebra $\mathcal{R}$. The smallest left ideal of $\mathcal{R}$ containing $S$ is denoted by $\langle \mathcal{R}S \rangle$ and said to be generated by $S$. A left ideal $\mathcal{I}$ is said to be finitely generated (countably generated, respectively) if $\mathcal{I} = \langle \mathcal{R}S \rangle$ for a finite (countable, respectively) subset $S$ of $\mathcal{R}$. Here we take a moment to stress that the set of generators is considered in a purely algebraic sense.

**Proposition 3.5.** Let $A_1, A_2$ be operators in a von Neumann algebra $\mathcal{R}$. Then $\mathcal{R}A_1 + \mathcal{R}A_2 = \mathcal{R}\sqrt{A_1^*A_1 + A_2^*A_2}$. Thus, every finitely generated left ideal of $\mathcal{R}$ is a principal ideal.

**Proof.** Consider the operators $A, \tilde{A}$ in $M_2(\mathcal{R})$ represented by,

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \sqrt{A_1^*A_1 + A_2^*A_2} & 0 \\ 0 & 0 \end{bmatrix}$$

It is easy to see that $A^*A = \tilde{A}^*\tilde{A}$. By Lemma 3.1, $M_2(\mathcal{R})A = M_2(\mathcal{R})\tilde{A}$ and comparing the $(1, 1)$ entry on both sides, our result follows. Inductively, for operators $A_1, \ldots, A_n$ in $\mathcal{R}$, we see that

$$\mathcal{R}A_1 + \cdots + \mathcal{R}A_n = \mathcal{R}\sqrt{A_1^*A_1 + \cdots + A_n^*A_n}.$$  

In conclusion, every finitely generated left ideal of $\mathcal{R}$ is singly generated. \hfill \Box

**Corollary 3.6.** If $\mathcal{I}$ is a norm-closed left ideal of $\mathcal{R}$ which is finitely generated, then $\mathcal{I}$ is weak-operator closed.

**Proof.** A straightforward consequence of Proposition 3.3, 3.5. \hfill \Box

**Theorem 3.7.** If $\mathcal{I}$ is a norm-closed left ideal of $\mathcal{R}$ which is countably generated, then $\mathcal{I}$ is weak-operator closed (and thus, a left principal ideal).
Proof. Let \( \mathcal{I} \) be a countably generated norm-closed left ideal of \( \mathcal{B} \) with generating set \( \mathcal{S} := \{ A_i : i \in \mathbb{N} \} \). We prove that it must be weak-operator closed. Noting that \( \mathcal{B} A_i = \mathcal{B} \sqrt{A_i^* A_i} \) and after appropriate scaling, we may assume that the \( A_i \)'s are positive and \( \| A_i \| \leq 1 \). For \( n \in \mathbb{N} \), define \( B_n := \sqrt{\sum_{i=1}^{n} A_i^2} \). Thus the sequence \( \{ B_n^2 \}_{i=1}^{\infty} \) is an increasing Cauchy sequence of positive operators in \( \mathcal{I} \), and \( \lim_{n \to \infty} B_n^2 = B^2 \) exists. As \( \mathcal{I} \) is norm-closed, the positive operator \( B := \sqrt{\lim_{n \to \infty} B_n^2} \) is in \( \mathcal{I} \) and thus \( \mathcal{B} B \subseteq \mathcal{I} \). Also for each \( n \in \mathbb{N} \) as \( A_i^2 \leq 2^n B_n^2 \leq 2^n B^2 \), by Lemma 3.1, we have that \( \mathcal{A} A_i \subseteq \mathcal{R} B \). Thus \( \mathcal{I} \subseteq \mathcal{R} B \) and combined with the previous conclusion, \( \mathcal{I} = \mathcal{R} B \). By Corollary 3.6, being norm-closed, \( \mathcal{I} = \mathcal{R} B \) is also weak-operator closed. \( \square \)

Below we note a result about norm-closed left ideals of represented C*-algebras. In the results that follow after, we will see how a similar conclusion holds for left ideals in von Neumann algebras.

**Proposition 3.8.** Let \( \mathcal{A} \) be a C*-algebra acting on the Hilbert space \( \mathcal{H} \) and let \( \mathcal{I} \) be a norm-closed left ideal of \( \mathcal{A} \). Then there is a norm-closed left ideal \( \mathcal{J} \) of \( \mathcal{B}(\mathcal{H}) \) such that \( \mathcal{I} = \mathcal{J} \cap \mathcal{A} \).

Proof. For a state \( \rho \) on a C*-algebra, we denote its left kernel, as defined in [8, §4.5.2], by \( \mathcal{L}_\rho \). Let \( \mathcal{P}^I \) denote the set of pure states on \( \mathcal{A} \) whose left kernels contain \( \mathcal{I} \). Then from Theorem 3.2 in [8], we have that

\[
\mathcal{I} = \bigcap_{\rho \in \mathcal{P}^I} \mathcal{L}_\rho
\]

A pure state \( \rho \) on \( \mathcal{A} \) can be extended to a pure state \( \overline{\rho} \) on \( \mathcal{B}(\mathcal{H}) \). We denote the set of all such extensions of the states in \( \mathcal{P}^I \) by \( \mathcal{P}^J \). Being an intersection of norm-closed left ideals, the set

\[
\mathcal{J} := \bigcap_{\overline{\rho} \in \mathcal{P}^J} \mathcal{L}_{\overline{\rho}}
\]

is also a norm-closed left ideal of \( \mathcal{B}(\mathcal{H}) \). Clearly if \( \overline{\rho} \) in \( \mathcal{P}^J \) is an extension of a state \( \rho \) in \( \mathcal{P}^I \), we have that \( \mathcal{L}_{\overline{\rho}} \cap \mathcal{A} = \mathcal{L}_\rho \). Thus we conclude that \( \mathcal{I} = \mathcal{J} \cap \mathcal{A} \). \( \square \)

**Proposition 3.9.** Let \( \mathcal{R}_1, \mathcal{R}_2 \) be von Neumann algebras acting on the Hilbert space \( \mathcal{H} \). Let \( A \) be an operator in \( \mathcal{R}_1 \cap \mathcal{R}_2 \). Then \( \mathcal{R}_1 A \cap \mathcal{R}_2 = \mathcal{R}_1 A \cap \mathcal{R}_2 A = (\mathcal{R}_1 \cap \mathcal{R}_2) A \).

Proof. Let \( B \) be an operator in \( \mathcal{R}_1 A \cap \mathcal{R}_2 \). As \( B \in \mathcal{R}_1 A \), we have that \( B^* B \leq \lambda^2 A^* A \) for some \( \lambda \geq 0 \). As \( B, A \) are both in \( \mathcal{R}_2 \), we conclude from Lemma 3.1 that \( B \) is also in \( \mathcal{R}_2 \). Thus \( B \in \mathcal{R}_1 A \cap \mathcal{R}_2 A \). This proves that \( \mathcal{R}_1 A \cap \mathcal{R}_2 \subseteq \mathcal{R}_1 A \cap \mathcal{R}_2 A \). The reverse inclusion is obvious. Thus \( \mathcal{R}_1 A \cap \mathcal{R}_2 = \mathcal{R}_1 A \cap \mathcal{R}_2 A \).

By considering the von Neumann algebra \( \mathcal{R}_1 \cap \mathcal{R}_2 \) (in place of \( \mathcal{R}_2 \)), we have from the above that \( \mathcal{R}_1 A \cap \mathcal{R}_2 = \mathcal{R}_1 A \cap ((\mathcal{R}_1 \cap \mathcal{R}_2) A) = \mathcal{R}_1 A \cap (\mathcal{R}_1 \cap \mathcal{R}_2) A = (\mathcal{R}_1 \cap \mathcal{R}_2) A \). \( \square \)

**Corollary 3.10.** Let \( \mathcal{R}_1, \mathcal{R}_2 \) be von Neumann algebras acting on the Hilbert space \( \mathcal{H} \). Let \( \mathcal{S} \) be a family of operators in \( \mathcal{R}_1 \cap \mathcal{R}_2 \). Then \( (\mathcal{R}_1 \mathcal{S}) \cap \mathcal{R}_2 = ((\mathcal{R}_1 \cap \mathcal{R}_2) \mathcal{S}) \).
Proof. Let $A, B$ be operators in $\mathcal{S}$. From Proposition 3.5 [3.10], we have that $\langle \mathcal{R}_1 \{ A, B \} \rangle \cap \mathcal{R}_2 = (\mathcal{R}_1 A + \mathcal{R}_1 B) \cap \mathcal{R}_2 = \mathcal{R}_1 A^* A + B^* B \cap \mathcal{R}_2 = (\mathcal{R}_1 \mathcal{R}_2) A^* A + (\mathcal{R}_1 \mathcal{R}_2) B = ((\mathcal{R}_1 \mathcal{R}_2) \{ A, B \})$. Thus $\langle \mathcal{R}_1 \mathcal{S} \rangle \cap \mathcal{R}_2 = ((\mathcal{R}_1 \mathcal{R}_2) \mathcal{S})$. \hfill $\square$

The corollary below is in the same vein as Proposition 3.8. In effect, it says that every left ideal of a represented von Neumann algebra may be viewed as the intersection of a left ideal of the full algebra of bounded operators on the underlying Hilbert space with the von Neumann algebra.

**Corollary 3.11.** Let $\mathcal{R}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Let $\mathcal{I}$ be a left ideal of $\mathcal{R}$. Then there is a left ideal $\mathcal{J}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{I} = \mathcal{J} \cap \mathcal{R}$.

**Proof.** By choosing $\mathcal{R}_1 = \mathcal{B}(\mathcal{H}), \mathcal{R}_2 = \mathcal{R}$ and $\mathcal{S} = \mathcal{I}$ and using Corollary 3.10 we see that for $\mathcal{J} := \langle \mathcal{B}(\mathcal{H}) \mathcal{I} \rangle$, we have that, $\mathcal{I} = \langle \mathcal{R} \mathcal{I} \rangle = \langle (\mathcal{B}(\mathcal{H}) \cap \mathcal{R}) \mathcal{I} \rangle = \mathcal{J} \cap \mathcal{R}$ and $\mathcal{J}$ is a left ideal of $\mathcal{B}(\mathcal{H})$. \hfill $\square$

### 3.2. Results on $C^*$-2-convex sets

In this subsection, we prove that $C^*$-2-convex subsets of bimodules over a von Neumann algebra are $C^*$-convex.

**Lemma 3.12.** Let $\mathcal{R}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$, with identity $I$. If $T_1, \cdots, T_n$ are operators in $\mathcal{R}$ such that $T_1^* T_1 + \cdots + T_n^* T_n = I$, then there are operators $S_1, \cdots, S_{n-1}$ in $\mathcal{R}$ such that $T_i = S_i \sqrt{I - T_n^* T_n}$, $1 \leq i \leq n - 1$ and $S_1^* S_1 + \cdots + S_{n-1}^* S_{n-1}$ is the range projection of $\sqrt{I - T_n^* T_n}$.

**Proof.** As $T_1^* T_1 + \cdots + T_n^* T_n - I = T_n^* T_n$, we have that

$$T_i^* T_i \leq \sqrt{I - T_n^* T_n} \sqrt{I - T_n^* T_n}, 1 \leq i \leq n - 1.$$

Since $\sqrt{I - T_n^* T_n}$ is self-adjoint, the orthogonal complement of the range of $\sqrt{I - T_n^* T_n}$ is equal to the kernel of $\sqrt{I - T_n^* T_n}$. From the proof of the Douglas lemma for von Neumann algebras in Theorem 2.1, we have for each $i \in \{1, 2, \cdots, n - 1\}$, an operator $S_i$ in $\mathcal{R}$ such that $T_i = S_i \sqrt{I - T_n^* T_n}$, and $S_i z = 0$ for any vector $z$ in ker($\sqrt{I - T_n^* T_n}$). Note that,

$$I - T_n^* T_n = T_1^* T_1 + \cdots + T_{n-1}^* T_{n-1} = \sqrt{I - T_n^* T_n} (S_1^* S_1 + \cdots + S_{n-1}^* S_{n-1}) \sqrt{I - T_n^* T_n}.$$ 

As a result, for every vector $x$ in the range of $\sqrt{I - T_n^* T_n}$, we have $\langle x, x \rangle = \langle (S_1^* S_1 + \cdots + S_{n-1}^* S_{n-1}) x, x \rangle$. In addition, for every vector $y$ in the kernel of $\sqrt{I - T_n^* T_n}$, we have $\langle (S_1^* S_1 + \cdots + S_{n-1}^* S_{n-1}) y, y \rangle = 0$. Thus the operator $S_1^* S_1 + \cdots + S_{n-1}^* S_{n-1}$ must be the range projection of $\sqrt{I - T_n^* T_n}$. \hfill $\square$

**Proposition 3.13.** For a finite von Neumann algebra $\mathcal{R}$ and a Hilbert $\mathcal{R}$-bimodule $\mathcal{S}$, a subset $\mathcal{I}$ of $\mathcal{S}$ is $\mathcal{R}$-convex if and only if $\mathcal{I}$ is $\mathcal{R}$-2-convex.

**Proof.** If $\mathcal{I}$ is $\mathcal{R}$-convex, the $\mathcal{R}$-segment $S(A_1, A_2)$ is clearly in $\mathcal{I}$ for any $A_1, A_2 \in \mathcal{I}$ as it consists of $\mathcal{R}$-convex combinations of $A_1$ and $A_2$. For the other direction, we inductively prove that for $(A_1, \cdots, A_n)$, an $n$-tuple of vectors from $\mathcal{I}$ and $T_1, \cdots, T_n \in \mathcal{R}$ satisfying $T_1^* T_1 + \cdots + T_n^* T_n = I$, the $\mathcal{R}$-convex combination $T_1^* A_1 T_1 + \cdots T_n^* A_n T_n$ is in $\mathcal{I}$. For $n = 1, 2$, the above is clearly true from the $\mathcal{R}$-2-convexity of $\mathcal{I}$.

As $\mathcal{R}$ is finite, by [7, Exercise 6.9.10(ii)], each of the $T_i$’s has a unitary polar decomposition, that is, there are unitary operators $U_i$ and positive operators $P_i$ such that $T_i = U_i P_i$, for $1 \leq
Proof. It is straightforward from the definitions that every $S_i$ subset $T$ words, for operators $R$. For an infinite von Neumann algebra Proposition 3.15. Let $F$ denote the projection onto the kernel of $\sqrt{I - P_n^2}$. As $F = I - E$, clearly $F$ is in $\mathcal{R}$. For $i \in \{1, \ldots, n - 1\}$ as $\ker(\sqrt{I - P_n^2}) \subseteq \ker(P_i)$, we have that $PF = FP_i = 0$ and as a result $FS_i = 0$. Now define $S_i := S_i + \frac{P}{\sqrt{n-1}}, 1 \leq i \leq n - 1$. We see that $S_i S_1 + \cdots + S_{n-1} S_n = (S_i S_1 + \frac{P}{\sqrt{n-1}}) + \cdots + (S_{n-1} S_{n-1} + \frac{P}{\sqrt{n-1}}) = E + F = I$ and $P_i = S_i \sqrt{I - P_n^2}$. By the induction hypothesis, $A' := S_i A_1 S_1 + \cdots S_{n-1} A'_{n-1} S_{n-1}$ is in $\mathcal{S}$. As $T_n A_1 T_1 + \cdots T_{n-1} A_n T_n = \sqrt{I - P_n^2} A' \sqrt{I - P_n^2} + P_n A'_n P_n$, being a $\mathcal{R}$-convex combination of $A'$ and $A'_n$, it must be in $\mathcal{S}$. □

At this point, we refer the reader to Chapter 6 of [9] for a detailed account of the comparison theory of projections in von Neumann algebras. We denote the Murray-von Neumann equivalence relation for projections in a von Neumann algebra by $\sim$ and the partial order it begets by $\leq$. Below we mention (without proof) the halving lemma (see [9, Lemma 6.3.3]) for properly infinite projections in a von Neumann algebra as it will be repeatedly used in Proposition 3.15.

Lemma 3.14 (Halving Lemma). Let $E$ be a properly infinite projection in a von Neumann algebra $\mathcal{R}$. There is a projection $F$ in $\mathcal{R}$ such that $F \leq E$ and $F \sim (E - F) \sim E$.

Proposition 3.15. For an infinite von Neumann algebra $\mathcal{R}$ and a Hilbert $\mathcal{R}$-bimodule $\mathcal{S}$, a subset $\mathcal{S}$ of $\mathcal{S}$ is $\mathcal{R}$-convex if and only if $\mathcal{S}$ is $\mathcal{R}$-$2$-convex.

Proof. It is straightforward from the definitions that every $\mathcal{R}$-convex subset is $\mathcal{R}$-$2$-convex. We prove the other direction inductively. For $n \in \mathbb{N}$, let the $\mathcal{R}$-polytope generated by any $(n - 1)$-tuple of elements from $\mathcal{S}$ be contained in $\mathcal{S}$. Below we prove that the $\mathcal{R}$-polytope generated by an $n$-tuple $(A_1, A_2, \ldots, A_n)$ of elements from $\mathcal{S}$ is contained in $\mathcal{S}$. In other words, for operators $T_1, \ldots, T_n$ in $\mathcal{R}$ such that $\sum_{i=1}^n T_i = I$, we prove that $\sum_{i=1}^n T_i A_i T_i$ is in $\mathcal{S}$. Note that for $n = 1, 2$, the above is clearly true from the hypothesis of $\mathcal{R}$-$2$-convexity of $\mathcal{S}$.

Repeatedly using Lemma 3.14 consider mutually orthogonal projections $E_1, E_2, \ldots, E_n$ and projections $E_{n1}, E_{n2}, \ldots, E_{nn}$ in $\mathcal{R}$ such that

$$E_1 + E_2 + \cdots + E_n = I,$$
$$E_n = E_{n1} + E_{n2} + \cdots + E_{nn},$$
$$E_1 \sim E_2 \sim \cdots \sim E_n \sim E_{n1} \sim E_{n2} \sim \cdots \sim E_{nn} \sim I.$$

For $i \in \{1, 2, \ldots, n - 1\}$, define $F_i := E_i + E_{ni}$. s $I \sim E_i \leq F_i \leq I$, from the reflexivity of $\leq$ we have that $E_i \sim F_i \sim I$. For $i \in \{1, 2, \ldots, n - 1\}$, let $W_i$ be a partial isometry with initial projection $F_i$ and final projection $I$, and $W'_i$ be a partial isometry with initial projection $E_i$ and final projection $F_i$. Define $W := \sum_{i=1}^{n-1} W'_i$. Note that $W$ itself is a partial isometry with initial projection $\sum_{i=1}^{n-1} E_i = I - E_n$ and final projection $\sum_{i=1}^{n-1} F_i = I$. Further let $W_n$ be a partial isometry with initial projection $E_n$ and final projection $I$. For $i \in \{1, \ldots, n - 1\}$, let $V_i := W_i W$ and define $V_n := W_n$. To assist the reader in navigating the maze of partial
isometries we have defined, we tabulate the partial isometries and their initial and final projections in Table 11. Recall that for a partial isometry $V$ with initial projection $E$ and final projection $F$, we have that $V^*V = E$ and $VV^* = F$.

| Partial Isometry | Initial Projection | Final Projection |
|------------------|--------------------|------------------|
| $W_i$            | $F_i$              | $I$              |
| $W'_i$           | $E_i$              | $F_i$            |
| $W(= \sum_{j=1}^{n-1} W'_j)$ | $I - E_n$          | $I$              |
| $W_n$            | $E_n$              | $I$              |
| $V_i$            | $E_i$              | $I$              |

Table 1. Reference table for the partial isometries. The index $i$ ranges from 1 to $n - 1$.

By the induction hypothesis, the vector $A := W_1^*A_1W_1 + \cdots + W_{n-1}^*A_{n-1}W_{n-1}$ is in $\mathcal{S}$ as $\sum_{i=1}^{n-1} W_i^*W_i = \sum_{i=1}^{n-1} F_i = I$. Note that $W^*AW = \sum_{i=1}^{n-1} V_i^*A_iV_i$ and the vector $W^*AW + W_1^*A_1W_n(= \sum_{i=1}^n V_i^*A_iV_i)$ is in the $\mathcal{R}$-segment joining $A$ and $A_n$ as $W^*W + W_n^*W_n = (I - E_n) + E_n = I$, and thus $\sum_{i=1}^n V_i^*A_iV_i$ is in $\mathcal{S}$. We further have that $\sum_{i=1}^n V_i^*V_i = (\sum_{i=1}^n W^*W_i^*W_iW) + W_n^*W_n = (\sum_{i=1}^n W^*F_iW) + E_n = (\sum_{i=1}^n E_i) + E_n = I$. Consider the operator $\tilde{V} := V_1^*T_1 + \cdots + V_n^*T_n$ in $\mathcal{R}$ and the vector $\tilde{A} := V_1^*A_1V_1 + \cdots + V_n^*A_nV_n$ in $\mathcal{S}$. As $V_iV_j^* = \delta_{ij}I$ for $1 \leq i, j \leq n$, note that $\tilde{V}^*\tilde{V} = \sum_{i=1}^n \sum_{j=1}^n T_i^*V_jV_j^*T_i = \sum_{i=1}^n T_i^*T_i = I$ and $\tilde{V}^*\tilde{A}\tilde{V} = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n T_i^*V_iV_jV_j^*A_jV_j^*T_k = \sum_{i=1}^n T_i^*A_iT_i$. We have already proved that $\tilde{A}$ is in $\mathcal{S}$. Thus we have that $\tilde{V}^*\tilde{A}\tilde{V} = \sum_{i=1}^n T_i^*A_iT_i$ is in $\mathcal{S}$. This finishes the proof. \hfill \Box

From the type decomposition of von Neumann algebras (see [9, Theorem 6.5.2.]), for a von Neumann algebra $\mathcal{R}$, we have central projections $E, F$ in $\mathcal{R}$ such that $E + F = I$, $\mathcal{R}E$ is a finite von Neumann algebra acting on $E(\mathcal{H})$, and $\mathcal{R}F$ is a properly infinite von Neumann algebra acting on $F(\mathcal{H})$. Thus combining Proposition 3.13 and Proposition 3.15, we have the following theorem.

**Theorem 3.16.** For a von Neumann algebra $\mathcal{R}$ and an $\mathcal{R}$-bimodule $\mathcal{H}$, a subset $\mathcal{I}$ of $\mathcal{H}$ is $\mathcal{R}$-convex if and only if $\mathcal{I}$ is $\mathcal{R}$-2-convex.

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