PARTITIONS OF CYCLIC WORDS AND GOLDMAN-TURAEV LIE BIALGEBRA

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ABSTRACT. The free \( \mathbb{Z} \)-module generated from the set of non-trivial homotopy classes of closed curves on an oriented surface has the structure of Lie bialgebra by two operations, Goldman bracket and Turaev cobracket. M. Chas gave a combinatorial redefinition of these operations through the identification of the homotopy classes of closed curves on the surface with the cyclic words associated with the word representation of elements of the fundamental group of the surface. We present a new approach to give a combinatorial definition of the bracket and cobracket, focusing on the information given by the partitions of cyclic words.

1. Introduction

Let \( \Sigma = \Sigma_{g,1} \) be a compact oriented surface of genus \( g \geq 1 \) with a connected boundary, and \( \hat{\pi} \) the set of (free) homotopy classes of oriented closed curves on \( \Sigma \). The free \( \mathbb{Z} \)-module generated from \( \hat{\pi} \) has a structure of Lie algebra by an operation called the Goldman bracket \( [3] \). Let \( 1 \in \hat{\pi} \) be the homotopy class of the trivial loop. The quotient Lie algebra \( \mathbb{Z}\hat{\pi}/\mathbb{Z}1 \) has an operation called the Turaev cobracket which gives a structure of Lie bialgebra on \( \mathbb{Z}\hat{\pi}/\mathbb{Z}1 \) \( [4] \).

These two operations are deeply related to the geometric intersection number of loops on a surface, since they both are defined from the intersections of representatives of homotopy classes of loops and are independent of the choice of the representatives. In fact, these operations determine the geometric intersection number of certain types of loops, as shown in \([1], [2]\). In these works a “combinatorial” definitions of the bracket and cobracket introduced by M. Chas are used. “Combinatorial” means as follows: The elements of \( \hat{\pi} \) can be represented as reduced cyclic words (we will recall in the next section). Chas redefined the bracket and cobracket on the reduced cyclic words using information of the letters of the words.

In this paper, we will discuss a new approach of the combinatorial definition of the Goldman bracket and the Turaev cobracket. We will focus on the partitions of cyclic words and introduce the notion of linking number for a pair of partitions of words. We then define an operation on (not necessary reduced) cyclic words which yields a sum of elements in \( \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi} \) such that each element is given from a pair of partitions of a word with the linking number of the partitions as coefficient. We will see that this operation gives an new definition of the Turaev cobracket. The Goldman bracket is redefined in a similar manner.

This paper is organised as follows: In §2 we will recall the original definitions of the Goldman bracket and the Turaev cobracket, and prepare some notions about cyclic words representing the homotopy classes of free loops on \( \Sigma \). In §3 we organise the connection between the letters and the partitions of a cyclic word in the form of a chain complex, and introduce the notion of linking number of a pair of partitions of words.
of a cyclic word (or cyclic words) which gives information of the self-intersections of a loop on \( \Sigma \) corresponding to the word (or the intersections of loops on \( \Sigma \) corresponding to the words). Finally in \( \S 4 \) we give our combinatorial definitions of the Turaev cobracket and the Goldman bracket. In addition, we will see that one can confirm the well-definedness of the operations as a map on \( \mathbb{Z}\pi/\mathbb{Z}1 \) directly from our definitions.

2. Preliminaries

In this section we first recall the original definitions of two operations of Goldman-Turaev Lie bialgebra, and then review the notion of the cyclic words corresponding to the free homotopy classes of loops on \( \Sigma \).

2.1. Goldman bracket. For two homotopy classes of oriented loops \( \alpha \) and \( \beta \) in \( \hat{\pi} \), take representatives \( \tilde{\alpha} \) and \( \tilde{\beta} \) respectively so that \( \tilde{\alpha} \) and \( \tilde{\beta} \) are in general position, i.e., their intersection points are all transversal double points. Let \( \Gamma(\tilde{\alpha}, \tilde{\beta}) \) be the set of all intersection points of \( \tilde{\alpha} \) with \( \tilde{\beta} \), and for each \( p \in \Gamma(\tilde{\alpha}, \tilde{\beta}) \), let \( \tilde{\alpha}_p \) (\( \tilde{\beta}_p \)) the loop \( \tilde{\alpha} \) (\( \tilde{\beta} \) respectively) viewed as an oriented loop based at \( p \). Then the bracket of \( \alpha \) and \( \beta \) is defined as follows;

\[
[\alpha, \beta] := \sum_{p \in \Gamma(\tilde{\alpha}, \tilde{\beta})} \text{sign}(\tilde{\alpha}_p, \tilde{\beta}_p)|\tilde{\alpha}_p\tilde{\beta}_p|,
\]

where \( \text{sign}(\tilde{\alpha}_p, \tilde{\beta}_p) \) is +1 if the pair of the tangent vector of \( \tilde{\alpha} \) and of \( \tilde{\beta} \) at \( p \) form the positive basis of the tangent plane of \( \Sigma \) at \( p \) and is −1 otherwise, and \( |\tilde{\alpha}_p\tilde{\beta}_p| \) denotes the (free) homotopy class of the based loop \( \tilde{\alpha}_p\tilde{\beta}_p \). In \( [3] \), Goldman proved well-definedness of this operation. Linearly expanding this bracket onto \( \mathbb{Z}\hat{\pi} \), we obtain the Goldman bracket \([,] : \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi} \rightarrow \mathbb{Z}\hat{\pi} \).

2.2. Turaev cobracket. For a homotopy classes of oriented loops \( \alpha \) in \( \hat{\pi} \), take a representative \( \tilde{\alpha} \) in general position. Let \( \Gamma(\tilde{\alpha}) \) be the set of all self-intersection points of \( \tilde{\alpha} \), and for each \( p \in \Gamma(\tilde{\alpha}) \), let \( \tilde{\alpha}^1_p \) and \( \tilde{\alpha}^2_p \) the two oriented loops based at \( p \) obtained by dividing \( \tilde{\alpha} \) at \( p \). (Either one can be \( \tilde{\alpha}^1_p \).) Then we may define the following operation;

\[
\Delta(\alpha) := \sum_{p \in \Gamma(\tilde{\alpha})} \text{sign}(\tilde{\alpha}^1_p, \tilde{\alpha}^2_p) \left( |\tilde{\alpha}^1_p| \otimes |\tilde{\alpha}^2_p| - |\tilde{\alpha}^2_p| \otimes |\tilde{\alpha}^1_p| \right),
\]

where \( \text{sign}(\tilde{\alpha}^1_p, \tilde{\alpha}^2_p) \) is the same as above, and \( |\tilde{\alpha}^1_p| \) (\( |\tilde{\alpha}^2_p| \)) also denotes the (free) homotopy class of the based loop \( \tilde{\alpha}^1_p \) (\( \tilde{\alpha}^2_p \) respectively). Since the value \( \Delta(\alpha) \in \mathbb{Z}\hat{\pi} \) has an ambiguity by \( 1 \in \mathbb{Z}\hat{\pi} \), we need to take the quotient \( \mathbb{Z}\hat{\pi}/\mathbb{Z}1 \) to have well-definedness of the operation. Linearly expanding this operation onto \( \mathbb{Z}\hat{\pi}/\mathbb{Z}1 \), we obtain the Turaev cobracket \([,] : \mathbb{Z}\hat{\pi}/\mathbb{Z}1 \rightarrow \mathbb{Z}\hat{\pi}/\mathbb{Z}1 \otimes \mathbb{Z}\hat{\pi}/\mathbb{Z}1 \).

2.3. Words and cyclic words. Let \( \{a_k, b_k\}_{k=1,2,...,g} \) be a set of \( 2g \) based loops on \( \Sigma \) such that they are mutually disjoint except for the base-point, mutually non-parallel and each pair \( (a_k, b_k) \) (\( 1 \geq k \geq g \)) gives a one-handle part of a handle decomposition of \( \Sigma \). We will also denote by \( a_k \) (resp. \( b_k \)) the based homotopy class of the based loop \( a_k \) (resp. \( b_k \)). The fundamental group of the surface \( \Sigma \), we will denote it by \( \pi \), is identified with the free group of rank \( 2g \) with respect to the generators \( \{a_k, b_k\}_{k=1,...,g} \).
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On the free group \( \pi \), the alphabet is the set \( \mathcal{A} = \{a_k, a_k^{-1}, b_k, b_k^{-1}\}_{k=1}^{g} \), and the letters are the elements of \( \mathcal{A} \). We denote by \( W(\mathcal{A}) \) the set of words on \( \mathcal{A} \). The contraction (resp. the cyclic contraction) on a word in \( W(\mathcal{A}) \) is an operation that removing successive two letters (resp. the first and the last letter) if they are inverse each other. The cyclic words on \( \mathcal{A} \) are the equivalent classes of \( W(\mathcal{A}) \) with respect to the equivalence relation “becomes the same word by contractions and cyclic contractions”. Throughout this paper we will regard that the words in \( W(\mathcal{A}) \) as representatives of cyclic words by considering that the last letter of a word is followed by the first letter.

3. PARTITIONS OF WORDS AND THEIR LINKING NUMBERS

In this section we discuss a correspondence between sub-arcs of a free loop and partitions of the word associated with the loop. We then introduce the notion of linking numbers of a pair of petitions of words. In the rest of this paper we take the base-point on \( \partial \Sigma \) and generator loops \( \{a_1, b_1, \ldots, a_g, b_g\} \) of \( \pi = \pi_1(\Sigma) \) as shown in Figure 1. Recall that the based loops on \( \Sigma \) are represented by the words in \( W(\mathcal{A}) \), which consist of the letters of the alphabet \( \mathcal{A} = \{a_k, a_k^{-1}, b_k, b_k^{-1}\}_{1 \leq k \leq g} \).

3.1. Partitions of a word. We consider the partitions of a word each of which splits the word in two. For a word \( w = x_1x_2\ldots x_p \) in \( W(\mathcal{A}) \), we call the partition between \( x_i \) and \( x_{i+1} \) the \( i \)-th partition of \( w \), and denote it by \( \phi_i w \). We also consider the \( p \)-th partition \( \phi_p w \) between \( x_p \) and \( x_1 \) so that we may regard the word as representative of a cyclic word.

3.2. Arc diagram of a word. We will introduce “arc diagram” of a word in \( W(\mathcal{A}) \) to associate the partitions of the word with the self-intersections of the corresponding loop on \( \Sigma \).

Let \( D \) be a small disk neighbourhood of the base-point on \( \Sigma \), and \( \partial_0 D \) the (open) subarc of \( \partial D \) which is the intersection of \( \partial D \) with the interior of \( \Sigma \). We assume that any of the generator loops \( a_1, b_1, \ldots, a_g, b_g \) intersects just twice with \( \partial_0 D \). We call the first intersection point at which a generator loop crosses \( \partial_0 D \) while following the loop from the base-point along its direction, the gate of the loop, and consider that the another intersection point of the generator loop with \( \partial_0 D \) is the gate of the inverse of the generator loop. We give the orientation on \( \partial_0 D \) that matches with the direction from the gate of \( a_1 \) to the gate of \( b_1 \). For two distinct gates \( x \) and \( y \) on \( \partial_0 D \), we express as \( x < y \) if the direction from \( x \) to \( y \) along \( \partial_0 D \) coincides the orientation of \( \partial_0 D \), and as \( x > y \) otherwise. Relating to the order of the gates of

![Figure 1. generators of \( \pi \)](image-url)
the generator loops and their inverses, we give the following order on the alphabet $\mathcal{A}$:

$$a_1 < b_1 < a_1^{-1} < b_1^{-1} < a_2 < b_2 < a_2^{-1} < b_2^{-1} < \cdots < a_q < b_q < a_q^{-1} < b_q^{-1}.$$  

For each word $w = x_1 x_2 \ldots x_p \in W(\mathcal{A})$, we will fix a way of choosing a based loop on $\Sigma$ represented by $w$ as follows, and denote the chosen loop by $\ell_b(w)$:

i) The based loop $\ell_b(w)$ is the union of based loops mutually disjoint except for the base-point each of which corresponds to each of the letters of $w$.

ii) Let the letter $x_i$ denote the gate of the sub-loop of $\ell_b(w)$ corresponding $x_i$, and $x_i^{-1}$ the gate of the inverse of the sub-loop. The gates $x_1, \ldots, x_p$ and the gates $x_1^{-1}, \ldots, x_p^{-1}$ are lined up on $\partial_0D$ by the following rule: Two gates $x_i^\varepsilon$ and $x_j^\delta$ ($i < j$, $\varepsilon, \delta \in \{-1, 1\}$) are in the order of $\mathcal{A}$ if these letters are distinct. In the case where these letters are the same, they are in order of $x_i^\varepsilon < x_j^\delta$ if the letter is $a_k$ or $b_k$, and $x_i^\varepsilon > x_j^\delta$ otherwise.

By altering $\ell_b(w)$ within $D$ so that each proper arc connects gates directly, without touching the base-point, we now obtain a free loop on $\Sigma$, which is (freely) homotopic to $\ell_b(w)$. We denote it by $\ell(w)$.

We call the disk $D$ with proper arcs $\ell(w) \cap D$ the arc diagram of $w$ (See Figure 2 (left)). Because of obvious one-to-one correspondence between the partitions of $w$ and the proper arcs on the arc diagram of $w$, we also denote by $\phi_w$ the simple oriented proper arc from the gate $x_i^{-1}$ to the gate $x_{i+1}$. ($\phi_w$ is the arc connecting $x_p^{-1}$ and $x_1$.)

We also define the arc diagram for the pair of words to use for redefinition of the Goldman bracket in §4. The arc diagram of a pair of words $v = x_1 x_2 \ldots x_p$ and $w = y_1 y_2 \ldots y_q$ is the arc diagram of $v$ overlapping with the arc diagram of $w$ such that $\partial D$, $\partial_0 D$ and the base-point of two diagrams are identified each, and the gates of $v$ and the gates of $w$ are lined up on $\partial_0 D$ in the following order: A gate $x_i^\varepsilon$ of $v$ and a gate $y_j^\delta$ of $w$ ($1 \leq i \leq p$, $1 \leq j \leq q$, $\varepsilon, \delta \in \{-1, 1\}$) are in the order of $\mathcal{A}$ if their letters are distinct. In the case where these letters are the same, they are in order of $x_i^\varepsilon < y_j^\delta$ if the letter is $a_k$ or $b_k$, and $x_i^\varepsilon > y_j^\delta$ otherwise (See Figure 2 (right)). Note that the order between a gate of $v$ and a gate of $w$ does not depend on their positions on each word.

![Figure 2. The arc diagram of a word $a_1 b_1^{-1} a_2 a_1$ (left) and the arc diagram of a pair of words $a_1 a_2^{-1}$ and $a_1 b_1^{-1} a_2$ (right).](image)

## 3.3. Linking number of partitions of words

Let $D$ be an arc diagram (of a word or a pair of words). We denote the set of gates of $D$ by $G(D)$, the oriented proper arcs on $D$ by $A(D)$, and the arc in $A(D)$ from a gate $x$ to a gate $y$ by $(xy)$.
We consider a chain complex obtained from $D$ as follows: The degree 0 chain group $C_0(D)$ is a free $\mathbb{Z}$-module generated from $G(D)$ and the degree 1 chain group $C_1(D)$ is a free $\mathbb{Z}$-module generated from $A(D)$. The chain groups of degree more than 2 are all 0. The boundary map $\partial = \partial_1 : C_1(D) \rightarrow C_0(D)$ is defined by $\partial(xy) = y - x$ (with linearly expanding on $C_1(D)$), and the boundary maps of other degree are all zero map.

On $C_0(D)$ we give a bi-linear form directly associated with the order of the gates on $\partial D$.

**Definition 3.1.** A bi-linear alternating form $\cdot : C_0(D) \times C_0(D) \rightarrow \mathbb{Z}$ is defined by

$$x \cdot y := \begin{cases} +1 & (x < y) \\ 0 & (x = y), \forall x, y \in G(D). \\ -1 & (x > y) \end{cases}$$

**Remark 3.2.**

i) We may apply the operation $\cdot$ to the letters of a word $w$ (and their inverses) directly without bringing up the arc diagram of the word. Namely we may define the operation $\cdot$ as follows: For the $i$-th letter $x_i$ and the $j$-th letter $x_j$ $(1 \leq i < j \leq p)$ of a word $w = x_1 x_2 \ldots x_p$, we define

$$x_i^\epsilon \cdot x_j^\delta = -x_j^\delta \cdot x_i^\epsilon = \begin{cases} +1 & (x_i^\epsilon < x_j^\delta \text{ or } x_i^\epsilon = x_j^\delta \in \{a_k, b_k \} \text{ for } 1 \leq k \leq g) \\ -1 & (x_i^\epsilon > x_j^\delta \text{ or } x_i^\epsilon = x_j^\delta \in \{a_k^{-1}, b_k^{-1} \} \text{ for } 1 \leq k \leq g) \end{cases},$$

where $\epsilon$ and $\delta$ are $\pm 1$, and the inequalities are of the order of $A$.

ii) On the definition of the operation $\cdot$ we may assign any value to $x \cdot x$, since the operation $\cdot$ is not performed on the same two gates throughout this paper.

We now introduce the notion of the linking number of a pair of proper arcs on $D$, which define a bi-linear alternating form on $C_1(D)$.

**Definition 3.3.** The linking number $\text{lk}(\alpha, \beta)$ of $\alpha$ and $\beta$ in $A(D)$ is defined as

$$\text{lk}(\alpha, \beta) = \frac{1}{2} \partial \alpha \cdot \partial \beta.$$ 

This defines the linking number of a pair of partitions of a word through the identification between the proper arcs of the arc diagram and the partitions of the word. Namely for the $i$-th partition $\phi_i w$ and the $j$-th partition $\phi_j w$ of a word $w = x_1 \ldots x_p$ we may define the linking number $\text{lk}(\phi_i w, \phi_j w)$ as

$$\text{lk}(\phi_i w, \phi_j w) = \frac{1}{2} \partial \phi_i w \cdot \partial \phi_j w = \frac{1}{2} (x_{i+1} - x_i^{-1}) \cdot (x_{j+1} - x_j^{-1}).$$

**Example 3.4.** For a word $w = a_1 b_1^{-1} a_2 a_1$,

$$\text{lk}(\phi_1 w, \phi_2 w) = \frac{1}{2} (b_1^{-1} - a_1^{-1}) \cdot (a_1 - a_2^{-1})$$

$$= \frac{1}{2} (b_1^{-1} \cdot a_1 - b_1^{-1} \cdot a_2^{-1} - a_1^{-1} \cdot a_1 + a_1^{-1} \cdot a_2^{-1})$$

$$= \frac{1}{2} ((-1) - (+1) - (-1) + (+1)) = 0,$$
We will show that the value $1/2(a_1 - a_2^{-1}) \cdot (a_1 - a_1^{-1})$
and Goldman bracket. For $w$

\[ \text{Definition 4.1.} \]

\[ \square \]

Note that all self-intersection points of the loop $\phi(w, \phi(w))$

\[ \text{Proposition 3.1.} \]

\[ \text{Proof.} \]

\[ \text{Note that all self-intersection points of the loop } \phi(w, \phi(w)) \text{ is equal to the sign of the intersection number of the pair.} \]

In the case where $w < x < z$ and $w < y < z$, the linking number $\frac{1}{2} \partial(xy) \cdot \partial(zw)$

\[ \text{Note that} \]

\[ X = \begin{cases} -2 & (w < x < z) \\
+2 & (z < x < w) \\
0 & (\text{otherwise}) \end{cases}, \quad
Y = \begin{cases} -2 & (w < y < z) \\
+2 & (z < y < w) \\
0 & (\text{otherwise}) \end{cases}. \]

4. Combinatorial Goldman-Turaev Lie bialgebra

In this section we give new combinatorial definitions of the Turaev cobracket
and Goldman bracket. For $w \in W(A)$, we denote by $|w|$ the conjugacy class of
the element in $\pi = W(A)/\text{contraction represented by } w$, which is identified with
the element of $\hat{\pi}$ represented by the (free) loop $\ell(w)$ on $\Sigma$ mentioned in $\mathbb{3}$

4.1. Turaev cobracket. For $w \in W(A)$ of length $p$, let $w_{s,t}$ denote its sub-word from the $s$-th letter to the $t$-th letter if $s \leq t$, and define $w_{s,t}$ as $w_{s,p}w_{1,t}$ if $s > t$. The index numbers are considered in modulo $p$.

\[ \text{Definition 4.1.} \]

\[ \delta(w) = \sum_{1 \leq i < j \leq p} \text{lk}(\phi_i w, \phi_j w) (|w_{i+1,j}| \otimes |w_{j+1,i}| - |w_{j+1,i}| \otimes |w_{i+1,j}|). \]

\[ \text{Theorem 4.1.} \]

\[ \text{For any word } w \in W(A), \delta(w) = \Delta(|\ell(w)|). \]

\[ \text{Proof.} \]

Note that all self-intersection points of the loop $\ell(w)$ of a word $w \in W(A)$ are displayed in the arc diagram of $w$. Let $\ell(w)_1$ and $\ell(w)_2$ be the oriented loops based
for the pair of partitions \((\phi_x, \ell)\) at \(x\). We see that the free homotopy class of \(\ell\) is immediate from Proposition \(3.1\) that by the following simple observation. Let \(w\) of \(\delta\) of terms in \(\delta\) of \(\delta\) as a map from the set of cyclic words on \(A\) completes the proof.

\[
\Delta([\ell(w)]) = \sum_{x \in \Gamma(\ell(w))} \text{sign}(\ell(w)^1_x, \ell(w)^2_x) (|\ell(w)^1_x| \otimes |\ell(w)^2_x| - |\ell(w)^2_x| \otimes |\ell(w)^1_x|).
\]

It is immediate from Proposition \(3.1\) that

\[
\text{sign}(\ell(w)^1_x, \ell(w)^2_x) = \text{lk}(\phi_1 w, \phi_2 w)
\]

for the pair of partitions \((\phi_1 w, \phi_2 w)\) such that they intersect at \(x\), and one can see that the (free) homotopy class of \(\ell(w)^1_x\) and \(\ell(w)^2_x\) are respectively identified with \(|w_{i+1,j}^1|\) and \(|w_{j+1,i}^2|\). We also know from Proposition \(3.1\) that the value \(\text{lk}(\phi_1 w, \phi_2 w) = 0\) for the pairs of partitions of \(w\) which has no intersections. This completes the proof. \(\square\)

It follows from Theorem \(4.1\) that the map \(\delta\) induces a map from the set of cyclic words on \(A\) to \(\mathbb{Z}\pi/\mathbb{Z}1 \otimes \mathbb{Z}\pi/\mathbb{Z}1\) and gives a new combinatorial definition of Turaev cobracket through identifying the set of cyclic words with \(\pi\) and linearly expanding \(\delta\) onto \(\mathbb{Z}\pi\).

**Remark 4.2.** From our definition of \(\delta\) we may directly show the well-definedness of \(\delta\) as a map from the set of cyclic words on \(A\) to \(\mathbb{Z}\pi/\mathbb{Z}1 \otimes \mathbb{Z}\pi/\mathbb{Z}1\) as follows:

We first note that one can see the invariance of \(\delta(w)\) under the cyclic permutation by the following simple observation. Let \(w'\) be a word given from a word \(w = x_1 x_2 \ldots x_p\) by one cyclic permutation, i.e., \(w' = x_2 \ldots x_p x_1\). Then the partition \(\phi_1 w\) of \(w\) turns into the partition \(\phi_p w'\) of \(w'\). This causes the inversion of coefficients of terms in \(\delta(w)\) which are given by \(\phi_1 w\) with other partitions i.e., \(\partial \phi_1 w \cdot \partial \phi_1 w = (-1) \partial \phi_{i-1} w' \cdot \partial \phi_p w\), while the wedge part \(w_{1i} \wedge w_{1i}\) of the terms turn into \(w'_{i-1,p} \wedge w'_{p,i-1} = -w_{1i} \wedge w_{1i}\). Therefore \(\partial (w') = \delta(w)\).

Now all that remains is to show \(\delta(x^{-1}wx) = \delta(w)\) for \(\forall x \in A\) and \(\forall w = x_1 x_2 \ldots x_p \in W(A)\). We denote \(x^{-1}wx\) by \(\overline{x} w\) and identify \(|u| \otimes |v| - |v| \otimes |u|\) with \(|u| \wedge |v|\) for short. Note the following three simple facts in advance:

i) \(\partial \phi_{p+1}(\overline{x} w) = x^{-1} - x^{-1} = 0\),

ii) \(|u| \wedge |\overline{x} x^{-1}| = 0\) in \(\mathbb{Z}\pi/\mathbb{Z}1\), because of \(|\overline{x} x^{-1}| = 1\),

iii) \(\partial \phi_1(x^{-1}w) + \partial \phi_{p+1}(\overline{x} w) = (x^{-1} - x^{-1}) + (x^{-1} - x_p) = \partial \phi_p w\).
Using these facts we have
\[
\delta(\tau w) = \sum_{2 \leq j \leq p} \text{lk}(\phi_1(\tau w), \phi_{j+1}(\tau w)) |w_{1,j-1}| \land |w_{1,p}x| + \sum_{2 \leq i \leq p} \text{lk}(\phi_i(\tau w), \phi_{i+1}(\tau w)) |w_{i,p}| \land |w_{1,i-1}x^{-1}| + \sum_{2 \leq i < j \leq p} \frac{\partial \phi_{i-1} w \cdot (\partial \phi_i(\tau w) + \partial \phi_{i+1}(\tau w))}{2} |w_{i,p}| \land |w_{1,i-1}x| + \sum_{2 \leq i < j \leq p} \text{lk}(\phi_i w, \phi_{j+1} w) |w_{i,j-1}| \land |w_{j,i-1}|
\]
\[
= \delta(w).
\]

4.2. Goldman bracket. Let \( \nu \) be the cyclic permutation on the words in \( W(\mathcal{A}) \) given by \( \nu(x_1x_2 \ldots x_p) = x_2 \ldots x_p x_1 \ (x_i \in \mathcal{A}) \).

**Definition 4.3.** We define a map \( \langle \rangle : W(\mathcal{A}) \times W(\mathcal{A}) \rightarrow \mathbb{Z}\pi \) as follows: For any words \( v \) and \( w \) in \( W(\mathcal{A}) \), we set
\[
\langle v, w \rangle = \sum_{1 \leq i \leq p} \sum_{1 \leq j \leq q} \text{lk}(\phi_i v, \phi_j w) |\nu^i(v)\nu^j(w)|.
\]

**Theorem 4.2.** For any two words \( v \) and \( w \) in \( W(\mathcal{A}) \), \( \langle v, w \rangle = ||\ell(v)||, ||\ell(w)|| \).

**Proof.** The all intersection points of \( \ell(v) \) with \( \ell(w) \) are displayed in the arc diagram of the pair \( v \) and \( w \) as the intersection of proper arcs in \( \{\phi_i(v)\}_{1 \leq i \leq p} \) with proper arcs in \( \{\phi_j(w)\}_{1 \leq j \leq q} \). The rest of this proof is the same as the proof of Theorem 4.1. \( \square \)

In the same way as in Remark 4.2, we may see the well-definedness of \( \langle \rangle \) as the map from the direct product of the set of cyclic words with itself to \( \mathbb{Z}\pi \) directly from our definition.

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