COMPUTATION OF THE FIRST STIEFEL–WHITNEY CLASS FOR THE VARIETY $\mathcal{M}_{0,n}^\mathbb{R}$

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Abstract. We compute the class $W_{n-4}(\mathcal{M}_{0,n}^\mathbb{R})$, which is Poincaré dual to the first Stiefel–Whitney class for the variety $\mathcal{M}_{0,n}^\mathbb{R}$ in terms of the natural cell decomposition of $\mathcal{M}_{0,n}^\mathbb{R}$.

1. Introduction

Let $\mathcal{M}_{0,n}^\mathbb{R}$ be the Deligne–Mumford compactification of the moduli space of algebraic curves of genus 0 with $n$ marked and numbered points. In [3, 4, 6], topology of this variety is studied. Many topological and algebraic characteristics of this variety are investigated. In particular, the structure of the cell decomposition is introduced. By means of this cell decomposition, Euler characteristics of the variety are computed, and Betti numbers are found.

This paper is devoted to the investigations of the first Stiefel–Whitney class for the variety $\mathcal{M}_{0,n}^\mathbb{R}$. More precisely, we provide a natural geometric interpretation of the class $W_{n-4}(\mathcal{M}_{0,n}^\mathbb{R})$, which is Poincaré dual to the first Stiefel–Whitney class of $\mathcal{M}_{0,n}^\mathbb{R}$.

The main result of the paper is the proof of the following two statements.

Theorem 1.1. Let $n \geq 5$, $\mathcal{M}_{0,n}^\mathbb{R}$ be the real moduli space of algebraic curves of genus 0 with $n$ marked and numbered points, say $\{1, 2, \ldots, n\}$. Let $\mathcal{M}_{0,n}^\mathbb{R}$ be its Deligne–Mumford compactification. We consider the cell decomposition of $\mathcal{M}_{0,n}^\mathbb{R}$ defined in Construction 2.6. Then the class $W_{n-4}(\mathcal{M}_{0,n}^\mathbb{R})$ (which is Poincaré dual to the first Stiefel–Whitney class of $\mathcal{M}_{0,n}^\mathbb{R}$) consists exactly of those cells of co-dimension 1 that satisfy the following condition: an irreducible component of the curve that contains at most one point from the set $\{1, 2, 3\}$ contains an odd number of points from the set $\{1, 2, \ldots, n\}$.

Corollary 1.2. Let $n \geq 6$ be even, and let $\mathcal{M}_{0,n}^\mathbb{R}$ be the Deligne–Mumford compactification of the moduli space of algebraic curves of genus 0 with $n$ marked and numbered points. We consider the cell decomposition of $\mathcal{M}_{0,n}^\mathbb{R}$ defined in Construction 2.6. Then the class $W_{n-4}(\mathcal{M}_{0,n}^\mathbb{R})$ consists exactly of the cells of co-dimension 1 such that each irreducible component of the curve contains an odd number of marked points.

The first Stiefel–Whitney class of this variety was first computed in [1]. In this work, we point out another representative for the class, which is Poincaré dual to the first Stiefel–Whitney class of $\mathcal{M}_{0,n}^\mathbb{R}$. The advantage of our method is the direct application of the corresponding cell decomposition structure. We provide an easy combinatorial and geometric characterization of cells that provides the possibility to determine whether a cell lies in the class under consideration or not.

Further we plan to apply the obtained characterization for geometric interpretation of generators and relations in the cohomology algebra $H^\ast(\mathcal{M}_{0,n}^\mathbb{R}, \mathbb{Q})$. We are going to prove their close connections with those that were found by Keel [7] in the complex case and to point out the principal difference with the complex case.

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Our work is organized as follows. Section 2 contains necessary definitions and a detailed description of the cell decomposition of the variety $\mathcal{M}_{0,n}^\mathbb{R}$. This theory is illustrated by the description of the cell decomposition of $\mathcal{M}_{0,5}^\mathbb{R}$, which includes the number of cells, adjacency types, and graphic illustration of the adjacency. Section 3 is devoted to the computation of the first Stiefel–Whitney class separately for $n = 5$ and $n \geq 6$.

2. The Deligne–Mumford Compactification of the Space $\mathcal{M}_{0,n}^\mathbb{R}$

2.1. Stable Curves. Following [4], we define real stable curves as “cacti-like” structures, i.e., 3-dimensional “trees” (in the graph theoretical sense), consisting of flat circles with the points $\{1, 2, \ldots, n\}$ on them.

Definition 2.1 ([2]). A stable curve of genus 0 with $n$ marked points over the field of real numbers $\mathbb{R}$ is a finite union of real projective lines $C = C_1 \cup C_2 \cup \cdots \cup C_p$ with $n$ different marked points $z_1, z_2, \ldots, z_n \in C$ if the following conditions hold.

1. For each point $z_i$, there exists a unique line $C_j$ such that $z_i \in C_j$.
2. For any pair of lines, $C_i \cap C_j$ is either empty or consists of one point, and in the latter case the intersection is transversal.
3. The graph corresponding to $C$ (the lines $C_1, C_2, \ldots, C_p$ correspond to the vertices; two vertices are incident to the same edge if and only if the corresponding lines have nonempty intersection) is a tree.
4. The total number of special points (i.e., marked points or intersection points) that belong to a given component $C_j$ is at least 3 for each $j = 1, \ldots, p$.

We say that $p$ is the number of components of the stable curve.

![Fig. 1. A stable curve over $\mathbb{R}$ of genus 0 with 9 marked points.](image)

Definition 2.2. Let $C = (C_1, C_2, \ldots, C_p, z_1, z_2, \ldots, z_n)$ and $C' = (C'_1, C'_2, \ldots, C'_p, z'_1, z'_2, \ldots, z'_n)$ be stable curves of genus 0 with $n$ marked and numbered points. The curves $C$ and $C'$ are called equivalent if there exists an isomorphism of algebraic curves $f: C \to C'$ such that $f(z_i) = z'_i$ for all $i = 1, \ldots, n$.

2.2. Moduli Space $\mathcal{M}_{0,n}^\mathbb{R}$.

Definition 2.3. Let $n \geq 3$. The Deligne–Mumford compactification of the moduli space of genus 0 real algebraic curves with $n$ marked points $\mathcal{M}_{0,n}^\mathbb{R}$ is the set of equivalence classes of genus 0 stable curves with $n$ marked points defined over $\mathbb{R}$.
Theorem 2.4 ([3]). Let $n > 4$. Then the space $\overline{M}_{0,n}^\mathbb{R}$ is a nonorientable compact variety of real dimension $\dim(\overline{M}_{0,n}^\mathbb{R}) = n - 3$.

2.3. The Cell Decomposition of $\overline{M}_{0,n}^\mathbb{R}$.

Remark 2.5. There exists a natural structure of cell decomposition for the space $\overline{M}_{0,n}^\mathbb{R}$. This structure is described, e.g., in [3, 6].

Construction 2.6 (description of the cell decomposition of the compactified space $\overline{M}_{0,n}^\mathbb{R}$ [3, 6]). Following [3], let us consider a right $n$-gon, possibly with several nonintersecting diagonals. We label different cells of the moduli space $\overline{M}_{0,n}^\mathbb{R}$ by such $n$-gons whose sides correspond to the marked points and are labeled by $z_1, \ldots, z_n$. Here the polygons that can be transformed to each other by the dihedral group action label the same cell. The cells of maximal dimension are labeled by $n$-gons without diagonals. The cells of codimension 1 are labeled by $n$-gons with one diagonal. Note that these cells consist exactly of 2-component stable curves. The cells of codimension 3, i.e., corresponding to 3-component curves, are labeled by $n$-gons with two diagonals. In general, a cell of codimension $k$ is labeled by an $n$-gon $M$ with $k$ diagonals. These diagonals divide $M$ into $k+1$ polygons $M_1, \ldots, M_{k+1}$. The edges of $M_1, \ldots, M_{k+1}$ that are edges of $M$ are labeled by the points marking the components of the curve. Note that condition (3) guarantees that different diagonals do not intersect outside the vertices of $M$. Condition (4) guarantees that each of the polygons $M_1, \ldots, M_{k+1}$ has at least three sides, i.e., it is a polygon.

Following [3], we denote by $G^L(n, k)$ the set of $n$-gons with $k$ nonintersecting diagonals and labels $z_1, \ldots, z_n$ on the edges.

Definition 2.7. A twist of $M \in G^L(n, k)$ along a diagonal $d$ is the $n$-gon $M' \in G^L(n, k)$ obtained from $M$ by cutting $M$ along $d$, then a $180^\circ$ rotation of one (any one) of the parts relative to the axis orthogonal to $d$ in the plane of the $n$-gon, and gluing the obtained two parts along $d$.

Remark 2.8. Let $M'$ be the twist of $M$, let labels of the sides of $M$ be ordered as $z_1, \ldots, z_n$, and let the sides marked by $z_1, \ldots, z_k$ be separated by $d$ from the sides marked by $z_{k+1}, \ldots, z_n$. Then the sides of $M'$ have ordered labels $z_1, \ldots, z_k, z_n, z_{n-1}, \ldots, z_{k+1}$.

Construction 2.9 (description of the cell decomposition of the compactified space $\overline{M}_{0,n}^\mathbb{R}$. Prolongation [3, 6]). Assume that polygons $M$ and $M' \in G^L(n, k)$ can be transformed to each other by a series of twists. Then $M$ and $M'$ mark the same cell.

Remark 2.10. A special charm of this construction is that marked points and singular points (the points of intersection of different components of a curve) do not have fundamental differences. Namely, both of them are denoted by edges of a polygon. Moreover, each component $C_i$ of the curve (as well as a connected union of several components) is denoted by a polygon. This polygon marks the cell of the cell decomposition (for the moduli space of a smaller dimension) that contains $C_i$.

Example 2.11. The cell decomposition of $\overline{M}_{0,n}^\mathbb{R}$ contains $(n - 1)!/2$ cells of maximal dimension $n - 3$.

Example 2.12. $\overline{M}_{0,4}^\mathbb{R}$ is a circle consisting of three cells of dimension 1 and three cells of dimension 0. Figure 2 represents the cell decomposition of $\overline{M}_{0,4}^\mathbb{R}$. Near each cell we provide its “typical” representative, i.e., one of the stable curves that constitute this cell.
2.4. The Cell Decomposition of the Variety $\overline{M^5_{0,5}}$. By Example 2.11, the found cell decomposition of $\overline{M^5_{0,5}}$ consists of 12 cells of maximal dimension. Each cell is labeled by a pentagon. The cell labeled by the pentagon with sides marked by the symbols 1, 2, 3, 4, and 5 is represented in Fig. 3. All curves that are in this cell have the form shown in Fig. 3 inside the pentagon. Outside the pentagon the stable curves are shown that are obtained by moving to each of the boundaries. In Fig. 4, two cells of codimension 1 are shown. The cell marked by the letter $A$ corresponds to the lower edge of the pentagon drawn in Fig. 3. The cell marked by $B$ corresponds to the next edge in the counterclockwise order.
Fig. 4. Forms of the boundary cells.

Fig. 5. Two adjoint cells of $\mathcal{M}_{0,5}^{\text{reg}}$.

Fig. 6. $\overline{\mathcal{M}}_{0,5}$.

Two adjoint cells are shown in Fig. 5. Adjoining between all 12 cells is shown in Fig. 6.
3. Stiefel–Whitney Class of $\overline{M}_{0,n}$

3.1. Some General Remarks.

Remark 3.1. We use the following theorem in order to compute the homological class $W_{n-4}$, which is Poincaré dual to the first Stiefel–Whitney class of the variety $\overline{M}_{0,n}$. For the computations, we use the introduced structure of the natural cell decomposition of $\overline{M}_{0,n}$.

Theorem 3.2 ([8, p. 119], [5], [1]). Let $M$ be a smooth compact variety without a boundary, $K$ be a cell decomposition of $M$, $k_j \subset K$ denote the cells of maximal dimension $d$. Let us fix the orientation on the cells of maximal dimension $k_j$. Then homological class dual to the first Stiefel–Whitney class of $M$ can be represented in the form

$$W_{d-1}(M) = \left( \frac{1}{2} \sum \partial k_j \right) \mod 2. \tag{3.1}$$

The main idea of the proof of Theorem 1.1 is to introduce global coordinates on the moduli space $\mathcal{M}_{0,n}$, which determine the orientation in each cell. Certainly, these coordinates can be prolonged up to the boundary for some cells, but not all cells. If the introduced coordinates can not be prolonged up to the boundary, we introduce some other coordinates that can be prolonged up to that boundary, and compute the Jacobian of the transition functions between different coordinates. Then we can determine whether two adjacent cells induce the same orientations on their common boundary, or the opposite ones. If two adjacent cells induce opposite orientations on their common boundary, then their influences in formula (3.1) annihilate each other. Otherwise, we add the influences. Then, after dividing by 2 and considering the result modulus 2, we get that the cell is in the class $W_{n-4}(\mathcal{M}_{0,n})$ with the coefficient 1. So, the cell is in $W_{n-4}(\mathcal{M}_{0,n})$ if and only if it is a common boundary of two cells of maximal dimension that have the same orientations (in the global coordinates determined in the open part of the moduli space).

We are going to realize this program.

In order to find the first Stiefel–Whitney class of the variety $\overline{M}_{0,n}$, we fix one of the possible orientations for the maximal dimension cells of the space $\mathcal{M}_{0,n}$.

Definition 3.3. A coordinate map on the space $\mathcal{M}_{0,n}$ is the map $\varphi: \mathcal{M}_{0,n} \rightarrow \mathbb{R}^{n-3}$.

Let $(P_1(\mathbb{R}), z_1, \ldots, z_n) \in \mathcal{M}_{0,n}$, $z_i \in P_1(\mathbb{R})$, i.e., we consider a 1-component curve, which is a projective line, with $n$ marked points on it. Since the points $z_1, \ldots, z_n \in P_1(\mathbb{R})$ are determined up to the linear-fractional transformation, we can and we do fix $z_1 = 0$, $z_2 = 1$, and $z_3 = \infty$. We define

$$\varphi(P_1(\mathbb{R}), z_1, \ldots, z_n) = (z_4, \ldots, z_n).$$
The chosen system of coordinates \((z_1, \ldots, z_n)\) in a cell or several cells of the space \(\mathcal{M}_{0,n}^\mathbb{R}\) will be called the \textit{parametrization} of these cells.

We fix the standard orientation of the space \(\mathbb{R}^{n-3}\). This determines the orientation on the cells of maximal dimension. In the chosen parametrization, we can easily see the boundaries of the cells shown in Fig. 8, which correspond to the gluing of the points marked by \(i\) and \(j\), where \(1 \leq i \leq n, 4 \leq j \leq n\), i.e., the boundaries drawn in Fig. 9.

**Lemma 3.4.** For all \(n \geq 5\) and for all \(i, j, 1 \leq i \leq n, 4 \leq j \leq n\), the cells \(K_{ij|l_1\ldots l_{n-2}}\) drawn in Fig. 9 are not in the class \(W_{n-4}(\mathcal{M}_{0,n}^\mathbb{R})\).

**Proof.** For all \(i, j, 1 \leq i \leq n, 4 \leq j \leq n\), the cells of maximal dimension having the cell in Fig. 9 as their common boundary induce opposite orientations on this cell. Hence, by Theorem 3.2 the common boundary \(K_{ij|l_1\ldots l_{n-2}}\) is included in the sum (3.1) twice with opposite signs, whence it is not in the class \(W_{n-4}(\mathcal{M}_{0,n}^\mathbb{R})\).

\[\square\]

**3.2. Computation of \(W_1(\overline{\mathcal{M}_{0,5}^\mathbb{R}})\).** By Lemma 3.4, it remains to consider the cells of codimension 1 of the form

where \(1 \leq i, j \leq 3\).

We start with \(i = 1, j = 2\).
Lemma 3.5. The boundary cell labeled by the pentagon

\[ K_{12|345} \]

is in the class \( W_1(M_{0,5}^\mathbb{R}) \).

Proof. (1) Let us consider the coordinates on \( M_{0,5}^\mathbb{R} \) that can be prolonged to this boundary. To do this, similarly to Definition 3.3, we construct an appropriate coordinate map \( \varphi_1 : M_{0,5}^\mathbb{R} \to \mathbb{R}^2 \). Let \((P_1(\mathbb{R}), y_1, \ldots, y_5) \in M_{0,5}^\mathbb{R}\) be a point of the moduli space \( M_{0,5}^\mathbb{R} \). Let us consider a parametrization of the curve \( P_1(\mathbb{R}) \) such that \( y_3 = \infty, y_4 = 0 \), and \( y_5 = 1 \). We set \( \varphi_1(P_1(\mathbb{R}), y_1, \ldots, y_5) = (y_1, y_2) \).

Now we find the transition function from the coordinates \((z_4, z_5)\) to the coordinates \((y_1, y_2)\). To do this, we write \( z\)- and \( y\)-coordinates on \( P_1(\mathbb{R}) \) for each of the five marked points and find the rational-fractional transformation that maps the coordinates to each other. The condition of the coordinate transformation has the form

\[
\begin{array}{cccccc}
& 1 & 2 & 3 & 4 & 5 \\
z\text{-coordinates} & 0 & 1 & \infty & z_4 & z_5 \\
y\text{-coordinates} & y_1 & y_2 & \infty & 0 & 1 \\
\end{array}
\]

We are looking for a linear-fractional transformation of the form

\[
f(t) = \frac{at + b}{ct + d}.
\]

We have \( f(z_3) = f(\infty) = \infty \), which implies that \( c = 0 \) and \( d = 1 \). Further, substituting the known values we get

\[
f(z_4) = 0 = \frac{az_4 + b}{cz_4 + d} = az_4 + b, \quad f(z_5) = 1 = \frac{az_5 + b}{cz_5 + d} = az_5 + b.
\]

Then

\[
a = \frac{1}{z_5 - z_4}, \quad b = \frac{z_4}{z_4 - z_5}.
\]

We obtain that

\[
f(t) = \frac{z_4 - t}{z_4 - z_5}.
\]

Therefore,

\[
y_1 = f(0) = \frac{z_4}{z_4 - z_5}, \quad y_2 = f(1) = \frac{z_4 - 1}{z_4 - z_5}.
\]

(2) Let us consider two cells of maximal dimension such that the cell \( K_{12|345} \) is their common boundary. Orientations of these maximal dimension cells provided by the standard orientation of the plane \((y_1, y_2)\) are opposite and thus induce opposite orientations on \( K_{12|345} \). We must understand whether the orientations of these two cells provided by the orientation of the plane \((z_1, z_2)\) are the same or opposite. Thus, we must check whether the orientation of the maximal dimension cells provided by the orientation of \((y_1, y_2)\) coincides with the orientation provided by the orientation of \((z_4, z_5)\).

We compute the Jacobian of coordinate change from \((z_4, z_5)\) to \((y_1, y_2)\):

\[
J = \det \begin{pmatrix}
\frac{\partial y_1}{\partial z_4} & \frac{\partial y_1}{\partial z_5} \\
\frac{\partial y_2}{\partial z_4} & \frac{\partial y_2}{\partial z_5}
\end{pmatrix}.
\]
Direct computations, which we omit to shorten the text, show that

\[
J = \det \begin{pmatrix}
-z_5 & z_4 \\
(z_4 - z_5)^2 & (z_4 - z_5)^2 \\
1 - z_5 & 1 + z_4 \\
(z_4 - z_5)^2 & (z_4 - z_5)^2
\end{pmatrix} = \frac{1}{(z_4 - z_5)^4}(z_5 - z_4).
\]

(3) The cell \(K_{12|345}\) is the common boundary of the following two cells:

These cells correspond to the following orders of the marked points:

\[
\text{left cell: } z_4, z_5, 0, 1, \infty, \quad \text{right cell: } 0, 1, z_5, z_4, \infty.
\]

respectively.

(4) So, we have that \(z_4 < z_5 < 0\) for the cell marked by the left pentagon, and \(1 < z_5 < z_4 < \infty\) for the cell marked by the right pentagon. Then for the left cell we get \(J > 0\), and for the right one, \(J < 0\). Since in the parametrization \((y_1, y_2)\) these cells have opposite orientations and the Jacobians have opposite signs, we get that in the parametrization \((z_4, z_5)\) these cells have the same orientation. Hence, the cell \(K_{12|345}\) is included twice in the expression for \(W_1(M_{0,5}^{\mathbb{R}})\). Multiplying the sum of boundary cells by \(1/2\), we get that the cell \(K_{12|345}\) goes with the coefficient 1, which remains fixed modulus 2. Therefore, the class \(W_1(M_{0,5}^{\mathbb{R}})\) contains the cell labeled by the pentagon \(K_{12|345}\).

Lemma 3.6. Boundary cells labeled by the pentagons

\[
K_{12|435} \quad K_{12|354}
\]

are in the class \(W_1(M_{0,5}^{\mathbb{R}})\), which is Poincaré dual to the first Stiefel–Whitney class of \(M_{0,5}^{\mathbb{R}}\).

Proof. (1) Our proof is similar to the proof of Lemma 3.5, and we use the coordinates \((y_1, y_2)\) on \(M_{0,5}^{\mathbb{R}}\) introduced in that proof. The cell labeled by \(K_{12|435}\) is the common boundary of the cells labeled by the pentagons

\[
\text{left cell: } z_4, z_5, 0, 1, \infty, \quad \text{right cell: } 0, 1, z_5, z_4, \infty.
\]

The value of the Jacobian for the change of coordinates from \((z_4, z_5)\) to \((y_1, y_2)\) by Lemma 3.5 is equal to

\[
J = \frac{z_5 - z_4}{(z_4 - z_5)^4}.
\]
For all elements of these two cells we have the following order of points on curves:

\[ z_5 \, z_4 \, 0 \, 1 \, \infty , \quad 0 \, 1 \, z_4 \, z_5 \, \infty . \]

So, for the cell marked by the left pentagon we get \( J < 0 \), and for the right one, \( J > 0 \). Since in the coordinates \((y_1, y_2)\) these cells have opposite orientations, in the coordinates \((z_4, z_5)\) they have the same orientation. Hence their common boundary, i.e., the cell labeled by \( K_{12|35} \), is in the class \( W_1(M_{0,5}^\mathbb{R}) \).

(2) Similarly, for the cell labeled by \( K_{12|354} \), the Jacobian value is the same; however the points on the curves in the corresponding cells of maximal dimension are ordered as follows:

\[ z_5 \, z_4 \, 0 \, 1 \, \infty , \quad 0 \, 1 \, z_4 \, z_5 \, \infty . \]

Thus, for one of them we get \( J < 0 \), and for the other one, \( J > 0 \). Since in the coordinates \((y_1, y_2)\) these cells have opposite orientations, we get that in the coordinates \((z_4, z_5)\) they have the same orientation. Hence their common boundary, i.e., the cell labeled by \( K_{12|354} \), is in the class \( W_1(M_{0,5}^\mathbb{R}) \).

**Lemma 3.7.** The boundary cells labeled by the pentagons

\[
\begin{array}{c}
K_{13|542} \\
\begin{array}{cc}
1 & 3 \\
2 & 5 \\
4 & \end{array}
\end{array}
\quad
\begin{array}{c}
K_{13|452} \\
\begin{array}{cc}
1 & 3 \\
2 & 5 \\
4 & \end{array}
\end{array}
\quad
\begin{array}{c}
K_{13|524} \\
\begin{array}{cc}
1 & 3 \\
2 & 5 \\
4 & \end{array}
\end{array}
\]

are in the class \( W_1(M_{0,5}^\mathbb{R}) \).

**Proof.** (1) For the investigation of these three cells, we consider new coordinates on \( M_{0,5}^\mathbb{R} \). As in Definition 3.3, we construct the coordinatization map \( \varphi_2: M_{0,5}^\mathbb{R} \to \mathbb{R}^2 \). Let \((P_1(\mathbb{R}), x_1, \ldots, x_5) \in M_{0,5}^\mathbb{R}\) be a point of the moduli space \( M_{0,5}^\mathbb{R} \). We choose the parametrization \( x \) of \( P_1(\mathbb{R}) \) such that \( x_2 = 0 \), \( x_4 = 1 \), and \( x_5 = \infty \) and set \( \varphi_2(P_1(\mathbb{R}), x_1, \ldots, x_5) = (x_1, x_3) \).

Let us find the expression for the coordinates \((x_1, x_3)\) via the coordinates \((z_4, z_5)\). To do this, we set to each of the marked points \( i \) its \( x \)-coordinates and \( z \)-coordinates on \( P_1(\mathbb{R}) \). Then we will find the rational-fractional function that maps \( z \)-coordinates to \( x \)-coordinates. The condition is

\[
\begin{array}{cc}
i & 1 \, 2 \, 3 \, 4 \, 5 \\
x \text{-coordinate} & x_1 \, 0 \, x_3 \, 1 \, \infty \\
z \text{-coordinate} & 0 \, 1 \, \infty \, z_4 \, z_5 \\
\end{array}
\]

We find rational-fractional transformation \( f(t) \) by the conditions \( f(1) = 0 \), \( f(z_4) = 1 \), and \( f(z_5) = \infty \). Then

\[
f(t) = \frac{z_4 - z_5}{z_4 - 1} \, \frac{t - 1}{t - z_5}.
\]

Hence in this case the Jacobian of the coordinate change is equal to

\[
J = \det \begin{pmatrix}
\frac{z_5 - 1}{z_5(z_4 - 1)^2} & -\frac{z_4}{z_5(z_4 - 1)^2} \\
\frac{z_5 - 1}{z_5(z_4 - 1)^2} & \frac{1}{z_5(z_4 - 1)^2}
\end{pmatrix} = \frac{(1 - z_5)(z_5 - z_4)}{(z_4 - 1)^3 z_5^2}.
\]
(2) The cell labeled by $K_{13|542}$ is the common boundary of the cells labeled by

![Diagram of cells labeled by $K_{13|542}$]

The following variants of marked point order correspond to these cells:

\[
0 \quad 1 \quad z_4 \quad z_5 \quad \infty , \\
0 \quad z_5 \quad z_4 \quad 1 \quad \infty .
\]

Then $0 < 1 < z_4 < z_5 < \infty$, whence $J < 0$ for the cell labeled by the left pentagon. Also $0 < z_5 < z_4 < 1 < \infty$ and $J > 0$ for the cell labeled by the right pentagon. As in the previous lemmas, we get that the cell labeled by $K_{13|542}$ is in the class $W_1(M_{0,5}^\mathbb{R})$.

(3) Similarly, the cell labeled by $K_{13|452}$ is the common boundary of the cells labeled by

![Diagram of cells labeled by $K_{13|452}$]

i.e., the cells containing curves with the following order of the marked points

\[
0 \quad 1 \quad z_5 \quad z_4 \quad \infty , \\
0 \quad z_4 \quad z_5 \quad 1 \quad \infty ,
\]

respectively.

Then $J > 0$ for the cell labeled by the left pentagon, and $J < 0$ for the cell labeled by the right pentagon. Hence, the cell marked by $K_{13|452}$ is in the class $W_1(M_{0,5}^\mathbb{R})$.

(4) Finally, the cell labeled by $K_{13|524}$ is the common boundary of two cells, each of them consisting of the curves with the following order of marked points:

\[
0 \quad z_4 \quad 1 \quad z_5 \quad \infty , \\
0 \quad z_5 \quad 1 \quad z_4 \quad \infty ,
\]

respectively.

Then $J > 0$ for the cell represented on the left-hand side, and $J < 0$ for the cell represented on the right-hand side. Therefore, the cell labeled by $K_{13|524}$ is in the class $W_1(M_{0,5}^\mathbb{R})$.

**Lemma 3.8.** All three boundary cells labeled by the pentagons with the diagonal cutting the sides marked by 2 and 3, i.e., labeled by the pentagons of the form

\[
K_{23|451} \quad K_{23|541} \quad K_{23|514}
\]

are in the class $W_1(M_{0,5}^\mathbb{R})$.  

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Proof. (1) Points of these three cells are stable curves such that the marked points with labels 2 and 3 are on the separate component. Thus, to investigate these cells, we must consider the coordinates on $\mathcal{M}_{0,5}^R$ in which gluing of the points $z_2$ and $z_3$ can be seen. For this, similarly to Definition 3.3, we construct the coordinatization map $\varphi_3: \mathcal{M}_{0,5}^R \rightarrow \mathbb{R}^2$. Let $(P_1(\mathbb{R}), u_1, \ldots, u_5) \in \mathcal{M}_{0,5}^R$ be an element of the moduli space $\mathcal{M}_{0,5}^R$. We choose the parametrization $u$ on the curve $P_1(\mathbb{R})$ such that $u_1 = 0$, $u_4 = 1$, and $u_5 = \infty$. Let us set $\varphi_3(P_1(\mathbb{R}), u_1, \ldots, u_5) = (u_2, u_3)$.

To find the transition functions from $(z_4, z_5)$ to $(u_2, u_3)$, we write the correspondence between different coordinates of the points:

| point $i$ | 1 | 2 | 3 | 4 | 5 |
|-----------|---|---|---|---|---|
| $u$-coordinate | $u_2$ | $u_3$ | 1 | $\infty$ |  |
| $z$-coordinate | 0 | 1 | $\infty$ | $z_4$ | $z_5$ |

Then the corresponding linear-fractional transformation has the form

$$f(t) = \frac{z_4 - z_5}{z_4} \frac{t}{t - z_5},$$

and one can easily write the explicit formulas for $u_2$ and $u_3$ as the functions of $z_4$, $z_5$.

Therefore, the Jacobian of this change of coordinates has the form

$$J = \det \begin{pmatrix} \frac{z_5}{z_4} & \frac{z_4 - 1}{z_4} \\ \frac{1 - z_5}{z_5} & \frac{1}{z_4} \end{pmatrix} = \frac{z_5(z_5 - z_4)}{z_4^2(1 - z_5)^2}.$$ 

(2) The cell labeled by $K_{23|451}$ is the common boundary of the two cells labeled by

These cells correspond to the following orders of marked points on the curves:

$$z_4 \quad z_5 \quad 0 \quad 1 \quad \infty,$$

Then $z_4 < z_5 < 0 < 1 < \infty$, whence $J < 0$ for the cell labeled by the left pentagon, and $0 < z_5 < z_4 < 1 < \infty$, $J > 0$ for the cell labeled by the right pentagon. As in the previous cases we get that the cell labeled by $K_{23|451}$ is in the class $W_1(\mathcal{M}_{0,5}^R)$.

(3) Similarly, the cell labeled by $K_{23|541}$ is the common boundary of the cells labeled by

These cells correspond to the following orders of marked points on the curves:

$$z_5 \quad z_4 \quad 0 \quad 1 \quad \infty.$$
Then $J < 0$ for the cell labeled by the left pentagon, and $J > 0$ for the cell labeled by the right pentagon. Thus, the cell labeled by $K_{23|541}$ is in the class $W_1(\mathcal{M}_{0,5}^\mathbb{R})$.

(4) Finally, the cell labeled by $K_{23|514}$ is the common boundary of the cells that consist of the curves with the following order of marked points:

\[ z_5 \ 0 \ 0 \ z_4 \ 1 \ \infty \ , \quad z_4 \ 0 \ 0 \ z_5 \ 1 \ \infty . \]

Then $J > 0$ for the cell represented on the left figure, and $J < 0$ for the cell represented on the right figure. Hence, the cell labeled by $K_{23|514}$ is in the class $W_1(\mathcal{M}_{0,5}^\mathbb{R})$.

**Corollary 3.9.** Poincaré dual to the first Stiefel–Whitney class of $\mathcal{M}_{0,5}^\mathbb{R}$, the class $W_1(\mathcal{M}_{0,5}^\mathbb{R})$ contains all cells labeled by the pentagons of the form

\[ \begin{array}{c}
  i \\
  j \\
\end{array} \]

where $1 \leq i, j \leq 3$, and only these cells.

**3.3. Computation of $W_{n-4}(\mathcal{M}_{0,n}^\mathbb{R})$ for $n \geq 6$.** In this section, we introduce special coordinates that provide the possibility of investigating the situation near some boundaries that are of interest. In order to do this, we draw the curve $P_1(\mathbb{R})$ in the form of hyperbola $xy = \varepsilon$. Approaching the boundary corresponds to taking the limit $\varepsilon \to 0$ under the fixed coordinates $x$ or $y$ of marked points on the curve. Approaching the boundary transforms the hyperbola to a pair of intersecting lines. As local coordinates in a neighborhood of the boundary we use the parameter $\varepsilon$, $x$-coordinates of the points with positive $x$-coordinates, and $y$-coordinates for the points with negative $y$-coordinates. We fix coordinates of some 4 points. Then the rest of the $n - 4$ points and $\varepsilon$ provide exactly the required $n - 3$ coordinates. Below (see formulas (3.2) and (3.3)), we show that for sufficiently small $\varepsilon \neq 0$ the Jacobian of the corresponding transition function cannot be zero; thus, this parametrization indeed provides local coordinates on $\mathcal{M}_{0,n}^\mathbb{R}$.

**Definition 3.10.** Let us define the coordinates in the following way. Let the boundary under consideration be labeled by the polygon $K_{l_1,\ldots,l_m|k_1,\ldots,k_{n-m}}$, which is drawn at the middle part of Fig. 10. This boundary is a common boundary of the cells labeled by the polygons $K_{l_1,\ldots,l_m,k_1,\ldots,k_{n-m}}$ and $K_{l_1,\ldots,l_m,k_{n-m},\ldots,k_1}$ drawn on the left side and the right side in Fig. 10, respectively. The cell labeled by $K_{l_1,\ldots,l_m,k_1,\ldots,k_{n-m}}$ is parametrized by $\varepsilon > 0$ and corresponds to the hyperbola lying in the intersection of right and upper semi-planes and in the intersection of left and lower semi-planes in Fig. 12. Similarly, the cell labeled by

\[ xy = \varepsilon \]

\[ \begin{array}{c}
  l_1 \\
  l_2 \\
  l_3 \\
  l_4 \\
  l_5 \\
\end{array} \quad \begin{array}{c}
  k_1 \\
  k_2 \\
  k_3 \\
  k_4 \\
  k_5 \\
\end{array} \]

\[ \begin{array}{c}
  i_1 \\
  i_2 \\
  i_3 \\
  i_4 \\
  i_5 \\
\end{array} \quad \begin{array}{c}
  j_1 \\
  j_2 \\
  j_3 \\
  j_4 \\
  j_5 \\
\end{array} \]

**Fig. 10.** $xy = \varepsilon$. 204
Lemma 3.12. Let \( \sigma \in S_3 \) be a permutation of the points 0, 1, and \( \infty \). Let \( \varphi: \mathcal{M}_{0,n}^\mathbb{R} \to \mathbb{R}^{n-3} \) be a coordinatization map, constructed similarly to Definition 3.3 in the following way. If \( (\mathbb{P}_1(\mathbb{R}), w_1, \ldots, w_n) \in \mathcal{M}_{0,n}^\mathbb{R} \) is a point of the moduli space \( \mathcal{M}_{0,n}^\mathbb{R} \), and parametrization on the curve \( \mathbb{P}_1(\mathbb{R}) \) is chosen in such a way that \( w_3 = \sigma(\infty) \), \( w_2 = \sigma(1) \), and \( w_1 = \sigma(0) \), then \( \varphi(\mathbb{P}_1(\mathbb{R}), w_1, \ldots, w_n) = (w_4, \ldots, w_n) \). Let the orientation of all cells be provided by the standard orientation of the space \( \mathbb{R}^{n-3} = \{ (w_4, \ldots, w_n) \} \). Then either the orientation of all cells induced by the coordinates \( \varphi_{\sigma} \) coincides with the orientation of these cells induced by the coordinates \( \varphi \) or for all cells these orientations are opposite.

Proof. All permutations of the values 0, 1, and \( \infty \) on the projective line \( \mathbb{P}_1(\mathbb{R}) \) are compositions of the transformations \( f_1(t) = 1 - t \) and \( f_2(t) = 1/t \). Hence the change from the coordinates \( z \) to the coordinates \( w \) is given by a composition of the maps \( z_i \to 1 - w_i \) for all \( i = 4, \ldots, n \) and \( z_i \to 1/w_i \) for all \( i = 4, \ldots, n \). Since these transformations are diagonal, the corresponding Jacobians are

\[
J_1 = \prod_{k=4}^n \frac{\partial(1 - z_k)}{\partial z_k} = (-1)^{n-3}
\]

and

\[
J_2 = \prod_{k=4}^n \frac{\partial\left( \frac{1}{z_k} \right)}{\partial z_k} = \prod_{k=4}^n \frac{-1}{z_k^2} = (-1)^{n-3} \prod_{k=4}^n \frac{1}{z_k^2}.
\]

So, in both cases orientation does not depend on the cell, but depends on the oddity of \( n \). If \( n \) is even, then the change of coordinates converts the orientation of any cell to the opposite one. If \( n \) is odd, then the orientation of any cell remains the same.

Composition of transformations corresponds to the multiplication of the Jacobians; hence in all the cases either the change of coordinates changes orientation of all cells or it leaves orientations of all cells fixed.

3.3.1. The case where the points 1, 2, and 3 are in the same component of the boundary. In this case, corresponding cells and coordinates on them are shown in Figs. 11, 12, and 13.

Lemma 3.13. Let \( n \geq 6 \) and all three points 1, 2, and 3 be in one component of the boundary. Then the corresponding cell is in the class \( W_{n-4}(\overline{\mathcal{M}}_{0,n}^\mathbb{R}) \) if and only if the component of the boundary that does not contain the points 1, 2, and 3 contains an odd number of marked points.

Proof. Up to renaming the marked points, we can and do assume that \( l_{i_1}, l_{i_2} \in \{1, 2, 3\} \). Applying Lemma 3.12, we can set \( l_{i_1} = 1 \) and \( l_{i_2} = 2 \). Then we denote \( l_{i_3} = 3 \) for some \( i_3 \). Hence we obtain that the abscissas of the points 1 and 2 are equal to 1 and 2, respectively. We denote the abscissa of the point 3 by \( x_3 \) and the abscissas of the points \( l_i \in \{l_1, \ldots, l_m\} \setminus \{1, 2, 3\} \) by \( x_l \). Here \( m \geq 3 \) by the assumptions of the lemma asserting that the points 1, 2, and 3 lie in one component.

Also we denote the ordinates of the points \( k_j \in \{k_1, \ldots, k_{n-m}\} \setminus \{k_1, k_2\} \) by \( y_{k_j} \).

Let us find the transition functions from the chosen coordinates \( x \), \( y \), and \( \varepsilon \) to the coordinates \( z \) introduced in Definition 3.3.
Point-wise the change of coordinates looks as follows:

\[
\begin{array}{c}
\begin{array}{cccccccc}
i & 1 & 2 & 3 & \ldots & l_i & \ldots & k_j & \ldots & k_{j_1} & k_{j_2} \\
z\text{-coordinates} & 0 & 1 & \infty & \ldots & z_{l_i} & \ldots & z_{k_j} & \ldots & z_{k_{j_1}} & z_{k_{j_2}} \\
x\text{-coordinates} & 1 & 2 & x_3 & \ldots & x_{l_i} & \ldots & \varepsilon/y_{k_j} & \ldots & -\varepsilon & -\varepsilon/2 \\
y\text{-coordinates} & \varepsilon & \varepsilon/2 & \varepsilon/x_3 & \ldots & \varepsilon/x_{l_i} & \ldots & y_{k_j} & \ldots & -1 & -2 \\
\end{array}
\end{array}
\]

We are looking for the linear-fractional transformation

\[
f(t) = \frac{at + b}{ct + d}
\]
that maps the $x$-coordinates of the points 1, 2, and 3 to their $z$-coordinates. Then

$$f(t) = (2 - x_3)rac{t - 1}{t - x_3}.$$ 

We compute the Jacobian of the change of coordinates $\varepsilon, x_{l_1}, y_{k_1}$ by the coordinates $z_4, \ldots, z_n$, writing rows and columns of the matrix of partial derivatives in the following order. The rows:

$$z_{k_1}, z_{k_2}, z_{l_1}, \ldots, z_{l_m}, z_{k_1}, \ldots, z_{k_{n-m}}$$

(without $z_{l_{t_1}}, z_{l_{t_2}}, z_{l_{t_3}}, z_{k_{j_1}}, z_{k_{j_2}}$). The columns:

$$\varepsilon, x_3, x_{l_1}, \ldots, x_{l_m}, y_{k_1}, \ldots, y_{k_{n-m}}$$

(without $x_{l_{t_1}}, x_{l_{t_2}}, y_{k_{j_1}}, y_{k_{j_2}}$, which are fixed, and $x_{l_{t_3}}$, which is in the second column). Note that

$$\frac{\partial z_{l_s}}{\partial x_{l_s}} = 0 \quad \text{if } s \neq i,$$

$$\frac{\partial z_{l_t}}{\partial y_{k_t}} = 0 \quad \text{for all } i \text{ and } t,$$

$$\frac{\partial z_{k_i}}{\partial x_{l_s}} = 0 \quad \text{for all } j \text{ and } s,$$

$$\frac{\partial z_{k_i}}{\partial y_{k_t}} = 0 \quad \text{if } t \neq j.$$

Then the Jacobi matrix of the change of coordinates has the following block-triangular form:

$$F = \begin{pmatrix} F_1 & O_1 & O_2 \\ X_1 & D_1 & O_3 \\ X_2 & O_4 & D_2 \end{pmatrix},$$

where

$$F_1 = \begin{pmatrix} \frac{\partial z_{k_{j_1}}}{\partial \varepsilon} & \frac{\partial z_{k_{i_1}}}{\partial x_3} \\ \frac{\partial z_{k_{j_2}}}{\partial \varepsilon} & \frac{\partial z_{k_{i_2}}}{\partial x_3} \end{pmatrix}.$$

$O_1, O_2, O_3,$ and $O_4$ are the zero matrices of the sizes $2 \times (n-m-2)$, $2 \times (n-m-2)$, $(m-3) \times (n-m-2)$, and $(n-m-2) \times (m-3)$, respectively, $X_1$ and $X_2$ are unknown matrices of sizes $(m-3) \times 2$ and $(n-m-2) \times 2$, respectively, $D_1$ is a diagonal $((m-3) \times (m-3))$-matrix of partial derivatives $\partial z_{l_i}/\partial x_{l_i}$, and $D_2$ is a diagonal $((n-m-2) \times (n-m-2))$-matrix of partial derivatives $\partial z_{k_j}/\partial y_{k_j}$. Then $J = \det F = \det F_1 \cdot \det D_1 \cdot \det D_2$. Let us compute each factor separately:

(1) The determinant of the matrix $F_1$.

$$J_1 = \det F_1 = \det \begin{pmatrix} -(2 - x_3)(1 - x_3) & -(\varepsilon + 1)(\varepsilon + 2) \\ -(\varepsilon + x_3)^2 & -(\varepsilon + 3)(\varepsilon + 2) \\ -(2 - x_3)(2 - 2x_3) & -(\varepsilon + 2)(\varepsilon + 4) \\ (\varepsilon + 2x_3)^2 & (\varepsilon + 4x_3)^2 \end{pmatrix} = \frac{(2 - x_3)(1 - x_3)(4 - \varepsilon)^2}{(\varepsilon + x_3)^2} (\varepsilon + 2x_3)^2.$$

(2) Diagonal entries of $D_1$ and $D_2$. Since

$$z_{l_i} = (2 - x_3) \cdot \frac{x_{l_i} - 1}{x_{l_i} - x_3},$$

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we get
\[ \frac{\partial z_{i_l}}{\partial x_{i_l}} = (2 - x_3)(x_3 - 1) \cdot \frac{-1}{(x_{i_l} - x_3)^2}. \]
Similarly,
\[ z_{k_j} = (2 - x_3) \cdot \frac{\varepsilon/y_{k_j} - 1}{\varepsilon/y_{k_j} - x_3}; \]
hence,
\[ \frac{\partial z_{k_j}}{\partial y_{k_j}} = \varepsilon(2 - x_3)(x_3 - 1) \cdot \frac{1}{(\varepsilon - y_{k_j}x_3)^2}. \]
The determinant of the product of \( D_1 \) and \( D_2 \) is the product of all the found values. Then
\[ J = J_1 \det D_1 \det D_2 = (2 - x_3)^n - 2(x_3 - 1)^n - 2(-1)^{m+1} \varepsilon^{n-m-2} \cdot J_2, \]
where
\[ J_2 = \frac{4 - \varepsilon^2}{(\varepsilon + x_3)^2(\varepsilon + 2x_3)^2} \prod_{i=4}^{m} \frac{1}{(x_{i_l} - x_3)^2} \prod_{j \in \{1, \ldots, n-m\} \setminus \{j_1, j_2\}} \frac{1}{(\varepsilon - y_{k_j}x_3)^2}. \]
Note that in the neighborhood of \( \varepsilon = 0 \) it holds that \( J_2 > 0 \), so this factor does not influence the sign of the Jacobian.

It follows from formula (3.2) that if \( n - m \) is even, then the Jacobian \( J \) does not change sign while \( \varepsilon \) goes from \( \varepsilon < 0 \) to \( \varepsilon > 0 \). If \( n - m \) is odd, then it changes sign.

Let \( n - m \) be even, i.e., let there be an even number of marked points in the component of the boundary that does not contain the points 1, 2, and 3. The cells of maximal dimension corresponding to \( \varepsilon > 0 \) and \( \varepsilon < 0 \) have opposite orientations. Hence, arguing as in the proof of Lemma 3.5, we obtain that in the coordinates \((z_4, \ldots, z_n)\) these cells have opposite orientations. Therefore, their common boundary is included twice in the expression for \( W_{n-4}(M_{0,n}) \) and with opposite signs, hence its influence is zero.

Now let \( n - m \) be odd, i.e., there are an odd number of marked points in the component of the boundary that does not contain the points 1, 2, and 3. The cells of maximal dimension corresponding to \( \varepsilon > 0 \) and \( \varepsilon < 0 \) have opposite orientations. Hence we obtain that in the coordinates \((z_4, \ldots, z_n)\) these cells have the same orientations, since the sign of \( J \) is changed at the point \( \varepsilon = 0 \). Therefore, their common boundary is included twice with the same sign in the expression for \( W_{n-4}(M_{0,n}) \). After dividing by 2 and taking the result modulus 2, we get that the common boundary of these two cells is in the class \( W_{n-4}(M_{0,n}) \). This concludes the proof of the lemma.

3.3.2. The case where the points 1, 2, and 3 are in the different components of the boundary. The case under consideration is presented in Fig. 14.

Lemma 3.14. Let \( n \geq 6 \) and let only two of the points 1, 2, and 3 be in the same component of the boundary \( K \). Then \( K \) is in the class \( W_{n-4}(M_{0,n}) \) if and only if there are an odd number of marked points on the component of \( K \) that contains the third point from the set \( \{1, 2, 3\} \).

Proof. Applying Lemma 3.12, up to renaming the marked points, we can and we do set \( l_{i_1} = 1, l_{i_2} = 2, \) and \( k_{j_1} = 3. \) Then we obtain that the abscissas of the points 1 and 2 are 1 and 2, respectively, and the ordinate of the point 3 is \(-1\). We denote the abscissas of the points \( l_i \in \{l_1, \ldots, l_m\} \setminus \{1, 2\} \) by \( x_{i_l} \); here \( m \geq 2 \). Let us denote ordinates of the points \( k_j \in \{k_1, \ldots, k_{n-m}\} \setminus \{k_{j_1}, k_{j_2}\} \) by \( y_{k_j} \).

We find the transition functions from the chosen coordinates \( x, y, \) and \( \varepsilon \) to the coordinates \( z \) introduced in Definition 3.3.

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Point-wise the change of coordinates looks as follows:

\[
\begin{array}{cccccccc}
i & 1 & 2 & 3 & \ldots & i_l & \ldots & k_j & \ldots & k_{j_2} \\
z\text{-coordinates} & 0 & 1 & \infty & \ldots & z_{i_l} & \ldots & z_{k_j} & \ldots & z_{k_{j_2}} \\
x\text{-coordinates} & 1 & 2 & -\epsilon & \ldots & x_{i_l} & \ldots & \epsilon/y_{k_j} & \ldots & -\epsilon/2 \\
y\text{-coordinates} & \epsilon & \epsilon/2 & -1 & \ldots & \epsilon/x_{i_l} & \ldots & y_{k_j} & \ldots & -2
\end{array}
\]

We are looking for the linear-fractional transformation

\[
f(t) = \frac{at + b}{ct + d}
\]

that maps coordinates \(x, y, \epsilon\) of the points 1, 2, and 3 to their \(z\)-coordinates. Then

\[
f(t) = (2 + \epsilon) \frac{t - 1}{t + \epsilon}.
\]

We compute the Jacobian of the change of coordinates \(\epsilon, x_{i_l}, y_{k_j}\) by the coordinates \(z_4, \ldots, z_n\), writing rows and columns of the matrix of partial derivatives in the following order. The rows:

\[
z_{k_{j_2}}, z_{i_1}, \ldots, z_{i_m}, z_{k_1}, \ldots, z_{k_{n-m}}
\]

(without \(z_{i_1}, z_{i_2}, z_{k_1}, z_{k_{j_2}}\)). The columns:

\[
\epsilon, x_{i_1}, \ldots, x_{l_m}, y_{k_1}, \ldots, y_{k_{n-m}}
\]
(without \(x_{t_1}, x_{t_2}, y_{k_{j_1}}, y_{k_{j_2}}\), which are fixed). As in the previous lemma we note that

\[
\frac{\partial z_{l_i}}{\partial x_{l_s}} = 0 \quad \text{if } s \neq i,
\]

\[
\frac{\partial z_{l_i}}{\partial y_{k_t}} = 0 \quad \text{for all } i \text{ and } t,
\]

\[
\frac{\partial z_{k_j}}{\partial x_{l_s}} = 0 \quad \text{for all } j \text{ and } s,
\]

\[
\frac{\partial z_{k_j}}{\partial y_{k_t}} = 0 \quad \text{if } t \neq j.
\]

Then the Jacobi matrix of this change of coordinates is triangular with zeros above the diagonal:

\[
F = \begin{pmatrix}
\frac{\partial z_{k_{j_2}}}{\partial \varepsilon} & 0 \\
\ast & D
\end{pmatrix},
\]

where \(\ast\) denotes unknown elements of the first column, the first row is zero, except the entry in the position \((1, 1)\), and \(D\) is a diagonal \(((n - 4) \times (n - 4))\) matrix. The first \(m - 2\) diagonal entries of \(D\) have the form \(\frac{\partial z_{l_i}}{\partial x_{l_i}}\); the next \(n - m - 2\) entries have the form \(\frac{\partial z_{k_j}}{\partial y_{k_j}}\).

Let us compute all the factors separately:

\[
\frac{\partial z_{k_{j_2}}}{\partial \varepsilon} = \frac{4 - \varepsilon^2}{\varepsilon^2}
\]

\[
\frac{\partial z_{l_i}}{\partial x_{l_i}} = \frac{(2 + \varepsilon)(1 + \varepsilon)}{(x_{l_i} + \varepsilon)^2}
\]

\[
\frac{\partial z_{k_j}}{\partial y_{k_j}} = -\frac{(2 + \varepsilon)(1 + \varepsilon)}{\varepsilon(y_{k_j} + 1)^2}.
\]

Then

\[
J = (-1)^{n-m-1} \frac{1}{\varepsilon^{n-m}} (2 + \varepsilon)^{n-4} (1 + \varepsilon)^{n-4} \prod_{i=3}^{m} \frac{1}{(x_{l_i} - \varepsilon)^2} \cdot \prod_{j \in \{3, \ldots, n-m\} \setminus \{3, j_2\}} \frac{1}{(1 + y_{k_j})^2}.
\]

(3.3)

Note that in the neighborhood of \(\varepsilon = 0\) the sign of the Jacobian is determined by the factor \(1/\varepsilon^{n-m}\) only.

Formula (3.3) implies that if \(n - m\) is even, then the Jacobian \(J\) does not change sign going from \(\varepsilon < 0\) to \(\varepsilon > 0\), and if \(n - m\) is odd, then \(J\) changes sign. Thus, repeating the arguments at the end of Lemma 3.13, we obtain that if \(n - m\) is even, then \(K\) is not in the class \(W_{n-4}(\overline{M}_{0,n}^R)\), and if \(n - m\) is odd, then \(K\) is in this class, as required. \(\square\)

Consequent application of these lemmas concludes the proof of Theorem 1.1.

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