Eliminating leading corrections to scaling in the
3-dimensional $O(N)$-symmetric $\phi^4$ model:

$N = 3$ and $4$

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Abstract

We study corrections to scaling in the $O(3)$- and $O(4)$-symmetric $\phi^4$
model on the three-dimensional simple cubic lattice with nearest neighbour interactions. For this purpose, we use Monte Carlo simulations in connection with a finite size scaling method. We find that there exists a finite value of the coupling $\lambda^*$, for both values of $N$, where leading corrections to scaling vanish. As a first application, we compute the critical exponents $\nu = 0.710(2)$ and $\eta = 0.0380(10)$ for $N = 3$ and $\nu = 0.749(2)$
and $\eta = 0.0365(10)$ for $N = 4$. 
1 Introduction

At a second order phase transition various quantities diverge with power laws. E.g. the magnetic susceptibility behaves as

$\chi \sim C_\pm |t|^{-\gamma}$, \hspace{1cm} (1)

where $t = (T - T_c)/T_c$ is the reduced temperature. The subscripts $+$ and $-$ indicate the high and low temperature phase, respectively. $\gamma$ is the critical exponent of the magnetic susceptibility.

The universality hypothesis says that for all systems within a given universality class the exponent $\gamma$, as other critical exponents, takes exactly the same value. Note that the amplitudes $C_\pm$ depend on the details of the system, while the ratio $C_+/C_-\,$ is universal; i.e. it takes the same value for all systems within a universality class. A universality class is characterized by the spatial dimension of the system, the range of the interaction and the symmetry of the order parameter. See ref. [1].

A central goal of the study of critical phenomena is to obtain precise estimates of universal quantities like the critical exponents or universal amplitude ratios.

Often so called lattice spin models are used to study critical phenomena. The Ising model is the proto-type of such models. In three dimensions, where no exact solution of these models is available, the most precise results are obtained by the analysis of high temperature series or Monte Carlo simulations. For universal quantities, a similar accuracy can be obtained with field-theoretic methods like the $\epsilon$-expansion or perturbation theory in three dimensions. For a detailed discussion see text-books on critical phenomena; e.g. refs. [2, 3].

The accuracy of estimates of universal quantities extracted from high temperature series expansions or Monte Carlo simulations is limited by corrections to the power law (2):

$\chi = C_\pm |t|^{-\gamma} \left(1 + a_\pm |t|^\theta + \ldots\right)$, \hspace{1cm} (2)

where the correction exponent $\theta = \nu \omega \approx 0.5$ is universal. While the exponent $\theta$ is universal, the amplitudes of corrections depend on the parameters of the system. Hence it is a very natural idea to search for parameters at which (non-analytic) corrections to scaling vanish.

In the present work, we study the $O(N)$-invariant $\phi^4$ model on simple cubic lattices with periodic boundary conditions. The lattices have the linear extension $L$ in all three directions. The classical Hamiltonian is given by

$H = -\beta \sum_{<xy>} \vec{\phi}_x \cdot \vec{\phi}_y + \sum_x \vec{\phi}_x^2 + \lambda \sum_x (\vec{\phi}_x^2 - 1)^2$, \hspace{1cm} (3)

where $\vec{\phi}_x \in \mathbb{R}^N$ and $<xy>$ denotes a pair of nearest neighbour sites on the
We study the canonical ensemble; the partition function is given by

\[ Z = \int \mathcal{D}[\phi] \exp(-H), \tag{4} \]

where \( \int \mathcal{D}[\phi] \) is a short hand for the \( N \times L^3 \) dimensional integral over all components of the field at all lattice points.

In the limit \( \lambda = \infty \), the so called \( O(N) \)-invariant non-linear \( \sigma \) model is recovered. In this limit the last term of the Hamiltonian forces the field to unit-length: \( \vec{\phi}_x^2 = 1 \). \( \lambda = 0 \) gives the exactly solvable Gaussian model. Along a critical line \( \beta_c(\lambda) \), the model undergoes a second order phase transition. For all \( \lambda > 0 \), at given \( N \), these transitions belong to the same universality class. Hence the critical exponents, including the correction exponent \( \theta \), are the same for all values of \( \lambda > 0 \). However, in eq. (2) the correction amplitudes \( a_\pm \) become functions of \( \lambda \). Hence, there is a chance to find a zero: \( a_\pm(\lambda^*) = 0 \). Since leading corrections to scaling are due to a unique irrelevant scaling field \( u_3 \), it follows that \( \lambda^* \) is unique for all quantities.

Already in refs. [4, 5] it was suggested that for models that interpolate between the Gaussian and the Ising model, there exists one value of the interpolation parameter, where leading corrections to scaling vanish. These models were studied with high temperature series expansions.

More recently, this idea has been picked up in connection with Monte Carlo simulations and finite size scaling. In refs. [3, 4] a finite size scaling method has been proposed that allows to compute \( \lambda^* \) in a systematic way. While the details of the implementation of ref. [4] and [7] are different, the underlying physical idea is the same.

This programme has been successfully implemented for \( N = 1 \) (Ising universality class) [3, 4, 8] and \( N = 2 \) (XY universality class) [3, 10]. As a result, for these universality classes, the most precise estimates for critical exponents from Monte Carlo simulations were obtained.

In a turn, the results for \( \lambda^* \) obtained from Monte Carlo have been used as input for the analysis of high temperature series of these models. In ref. [11] the Ising universality class and in refs. [11, 12] the XY universality class was studied. As a result, the accuracy of the critical exponents was further improved and in addition, a large set of universal amplitude combinations has been obtained with unprecedented precision.

In the present work we like to answer the question whether this programme can be extended to larger values of \( N \). In ref. [11] it was argued that for \( N > 3 \) there exists no such \( \lambda^* \) for the Hamiltonian eq. (3) on simple cubic lattices. The argument is based on the large \( N \) expansion and the analysis of high temperature series expansions [13].

The paper is organized as follows. In section 2 we briefly review the finite size scaling method that has been applied to eliminate corrections to scaling. The Monte Carlo algorithm is discussed in section 3. Details of the simulations are
given in section 4. In section 5 the numerical results are presented. Next we compare our results for the critical exponents with those given in the literature. Finally we give our conclusions.

2 The finite size scaling method

2.1 Phenomenological couplings

Here, phenomenological couplings are used to locate $\lambda^*$. Nightingale, in his seminal paper on the phenomenological renormalization group [14], proposed to use such a quantity to locate $\beta_c$ and to compute the renormalization group exponents. Phenomenological couplings are invariant under renormalization group transformations. Hence, they are well suited to detect corrections to scaling. The prototype of such a quantity is the so called Binder-cumulant [15]:

$$U_4 = \frac{\langle (\bar{m}^2)^2 \rangle}{\langle \bar{m}^2 \rangle^2},$$

where

$$\bar{m} = \frac{1}{V} \sum_x \tilde{\phi}_x$$

is the magnetisation of the system. Note that also higher moments of the magnetisation could be considered.

Frequently the second moment correlation length divided by the linear extension of the lattice $\xi_{2nd}/L$ has been studied. The second moment correlation length is defined by

$$\xi_{2nd} = \sqrt{\frac{\chi/F - 1}{4 \sin(\pi/L)^2}},$$

where

$$\chi = \frac{1}{V} \left\langle \left( \sum_x \tilde{\phi}_x \right)^2 \right\rangle$$

is the magnetic susceptibility and

$$F = \frac{1}{V} \left\langle \left| \sum_x \exp \left( i \frac{2\pi x_1}{L} \right) \tilde{\phi}_x \right|^2 \right\rangle$$

is the Fourier-transform of the two-point correlation function at the lowest non-vanishing momentum. In order to reduce the statistical error, we averaged the results of all three directions of the lattice. Note that Nightingale [14] studied $\xi/L$, where $\xi$ is the exponential correlation length of a system of the size $L^{D-1} \times \infty$.

The third quantity that we study is the ratio $Z_a/Z_p$ of the partition function $Z_a$ with anti-periodic boundary conditions in one of the three directions and
Z_p with periodic boundary conditions in all directions. Anti-periodic boundary conditions mean that the term $\sum_{xy} \vec{\phi}_x \cdot \vec{\phi}_y$ in the Hamiltonian is multiplied by $-1$ for $x = (L_1, x_2, x_3)$ and $y = (1, x_2, x_3)$. This ratio can be measured with the help of the boundary-flip algorithm, which is a version of the cluster algorithm. The boundary-flip algorithm was introduced in ref. 22 for the Ising model. In [23] the authors have generalized this method to the case of $O(N)$-invariant non-linear $\sigma$-models. As in ref. 8 we use a version of the algorithm that only measures $Z_a/Z_p$ and does not perform the flip to anti-periodic boundary conditions. For a recent discussion of the algorithm see ref. 10.

2.2 The finite size method

Below we shall briefly recall the theoretical basis of the finite size scaling method that has been developed in refs. 6, 7, 8. See also ref. 10.

Let us denote the phenomenological couplings by $R_i$. We define a quantity $\bar{R}$ based on a pair of phenomenological couplings $R_1$ and $R_2$.

First we define $\beta_f$ as the value of $\beta$, where, at given $\lambda$ and $L$, $R_1$ takes the fixed value $R_{1,f}$:

$$R_1(L, \lambda, \beta_f) = R_{1,f}.$$  \hfill (10)

Hence $\beta_f$ is a function of $L$ and $\lambda$. In the language of high energy physics, this is our “renormalization condition”.

Next we define

$$\bar{R}(L, \lambda) \equiv R_2(L, \lambda, \beta_f).$$  \hfill (11)

The behaviour of this quantity gives us direct access to corrections to scaling, as we shall see below. In the following we will frequently refer to $\bar{R}(L, \lambda)$ as “$R_2$ at $R_{1,f}$”.

Let us consider linearized renormalization group transformations in the neighbourhood of the fixed point. In the following, we take explicitly into account the thermal scaling field $u_t$ and the leading irrelevant scaling field $u_3$. In addition, we consider $u_4$ as representative of sub-leading irrelevant scaling fields. For the definition of scaling fields see e.g. the textbook [2].

In order to derive the finite size scaling behaviour of $R$ we block the lattice to a fixed size $L'$. The scaling fields $u'_t$, $u'_3$ and $u'_4$ of the blocked system are given by

$$u'_t = a(\lambda, \beta) L^{1/\nu}$$  \hfill (12)

with $a(\lambda, \beta_c) = 0$ by definition. The irrelevant scaling fields behave as

$$u'_3 = u_3^{(0)}(\lambda) L^{-\omega} + O(\beta - \beta_c(\lambda))$$  \hfill (13)

and

$$u'_4 = u_4^{(0)}(\lambda) L^{-\omega_2} + O(\beta - \beta_c(\lambda)),$$  \hfill (14)
Here we have absorbed the constant factors $L'^{-1/\nu}$, $L'\omega$ and $L'\omega^2$ into $a(\lambda, \beta)$, $u_3^{(0)}(\lambda)$ and $u_4^{(0)}(\lambda)$, respectively. We always simulate at a vanishing external field. Hence also the corresponding scaling field $u_h$ vanishes. While the value of $\omega \approx 0.8$ is well established (see e.g. [16]), the knowledge of $\omega_2$ is rather limited. Using the “scaling field method”, which is a version of Wilsons “exact renormalization group”, the authors of ref. [17] find $\omega_2 = 1.78(11)$ for $N = 3$. Following the figure given in ref. [17] the value of $\omega_2$ for $N = 4$ is similar to that of $N = 3$. In addition, there are corrections due to the breaking of the rotational invariance by the lattice. These corrections have an $\omega_{rot} \approx 2$ [18].

In the case of the Binder-cumulant and the second moment correlation length we have to expect corrections due to the analytic background of the magnetic susceptibility. This amounts to corrections with an exponent $\omega_{back} = 2 - \eta \approx 2$. In our numerical analysis, we will always pessimistically assume $\omega_2 = 1.6$ when we estimate systematic errors due to sub-leading corrections. Since $R$ keeps its value under renormalization group transformations we get

$$R(\beta, \lambda, L) = R'(u'_t, u'_3, u'_4, ...).$$  \hspace{1cm} (15)

Note that we are allowed to skip $L'$ from the list of arguments of $R'$ since $L'$ is fixed. Expanding $R'$ in the scaling fields at the fixed point $\{u'_j = 0\}$ gives

$$R = R^* + D_t u'_t + D_3 u'_3 + D_4 u'_4 + ...$$  \hspace{1cm} (16)

with

$$D_i \equiv \frac{\partial R'}{\partial u'_i} \Bigg|_{\{u'_j = 0\}}.$$  \hspace{1cm} (17)

Inserting eqs. (12,13,14) into eq. (16) yields

$$R = R^* + D_t a(\lambda, \beta_f) L^{1/\nu} + c(\lambda) L^{-\omega} + d(\lambda) L^{-\omega_2} + ... ,$$  \hspace{1cm} (18)

where $c(\lambda) = D_3 u_3^{(0)}(\lambda)$ and $d(\lambda) = D_4 u_4^{(0)}(\lambda)$. Hence

$$a(\lambda, \beta_f) L^{1/\nu} = D_{1, f}^{-1} \left( R_{1, f} - R^*_1 - c_1(\lambda) L^{-\omega} - d_1(\lambda) L^{-\omega_2} \right) + ... .$$  \hspace{1cm} (19)

The subscript 1 has been introduced to indicate that the quantities are related to $R_1$. Taking $R_2$ at $\beta_f$ gives

$$\bar{R}(L, \lambda) = \bar{R}^* + \bar{c}(\lambda) L^{-\omega} + \bar{d}(\lambda) L^{-\omega_2} + ...$$  \hspace{1cm} (20)

with

$$\begin{align*}
R^* &= R^*_2 + \frac{D_{2, t}}{D_{1, t}} (R_{1, f} - R^*_1) , \\
\bar{c}(\lambda) &= c_2(\lambda) - \frac{D_{2, t}}{D_{1, t}} c_1(\lambda)
\end{align*}.$$  \hspace{1cm} (21, 22)
and $\tilde{d}$ is given by an analogous expression.

In eq. (20) we have neglected corrections that are quadratic in the scaling fields. The most important of these is $b \bar{c}(\lambda)^2 L^{-2\omega}$. The exponent $2\omega$ has roughly the same value as $\omega_2$. However, since $\bar{c}(\lambda)$ appears quadratically, we can safely ignore this term in the neighbourhood of $\lambda^*$.

Eliminating leading corrections to scaling means in terms of eq. (20) to find the zero of $\bar{c}(\lambda)$. One can imagine various numerical implementations to find this zero. Here we followed the strategy used in refs. [8, 9]. We have simulated the models close to the critical line for several values of $\lambda$ for various lattices sizes. The results are then fitted by an ansatz motivated by eq. (20). The function $\bar{c}(\lambda)$ is then approximated by interpolation between the $\lambda$-values, where simulations have been performed.

In the following, we will always use either $Z_a/Z_p$ or $\xi_{2nd}/L$ as $R_1$ and $U_4$ as $R_2$. Note that in ref. [8] we have only used $R_1 = Z_a/Z_p$ and in ref. [9] only $R_1 = \xi_{2nd}/L$. Using both quantities gives us better control over systematic errors introduced by sub-leading corrections.

### 3 The Monte Carlo Algorithm

The $O(N)$-invariant non-linear $\sigma$ models were simulated with a cluster algorithm. For finite $\lambda$, following Brower and Tamayo [19], additional updates with a local Metropolis-like algorithm were performed to allow fluctuations of the modulus of the field $\vec{\phi}_x$. Below, we give the details of the cluster algorithm and the local update.

#### 3.1 The wall-cluster update

Several variants of the cluster algorithm have been proposed in the literature. The best known are the (original) Swendsen-Wang algorithm [20] and the single-cluster algorithm of Wolff [21]. Here we have used the wall-cluster algorithm of ref. [7]. In ref. [7], for the 3D Ising model, a small gain in performance compared with the single-cluster algorithm was found. Such a comparison for $N > 1$ remains to be done. The main reason for using the wall-cluster algorithm here is that the wall-cluster update can be combined with the measurement of the ratio of partition functions $Z_a/Z_p$. In the wall-cluster update all clusters are flipped that intersect a plane of the lattice.

The definition of a cluster is the same as in the Swendsen-Wang or the single-cluster algorithm. Following Wolff [21], only the sign of one component $\phi_x(p)$ of the field is changed in a single step of the algorithm. This leads to the probability

$$p_d = \min[1, \exp(-2\beta \phi_x(p) \phi_y(p))] \quad (23)$$
to delete the link $<xy>$. The links that are not deleted are frozen. A cluster is a set of sites that is connected by frozen links.

In ref. [21] the component of the field parallel to a randomly chosen direction is used. Here we choose the $1^{st}$, $2^{nd}$, ... or $N^{th}$ component of $\vec{\phi}$. This simplifies the implementation of the cluster update and the measurement of $Z_a/Z_p$. Also CPU-time is saved since in one update step only one component of the field has to be accessed. In order to compensate the algorithmic disadvantage of this restricted choice, we perform a global rotation of the field after a certain number of cluster updates.

3.2 The local update of the $\phi^4$ model

We sweep through the lattice with a local Metropolis-type updating scheme. A proposal for a new field at the site $x$ is generated by

$$\phi_x^{(i)} = \phi_x^{(i)} + c (r^{(i)} - 0.5)$$

(24)

where the $r^{(i)}$ are random numbers that are uniformly distributed in $[0, 1)$, $i$ runs from 1 to $N$. The proposal is accepted with the probability

$$A = \min[1, \exp(-H' + H)]$$

(25)

where $H'$ is the Hamiltonian of the proposed field $\phi'$ and $H$ the Hamiltonian of the original field. The step-size $c$ is adjusted such that the acceptance rate is about $1/2$. After this Metropolis step, we perform at the same site an over-relaxation step:

$$\vec{\phi}_x = \vec{\phi}_x - 2 \left( \frac{\vec{\phi}_x \cdot \vec{\phi}_n}{\vec{\phi}_n^2} \right) \vec{\phi}_n$$

(26)

Note that this step takes very little CPU-time. Hence it is likely that its benefit out-balances the CPU-cost. For lack of time, we did not carefully check this point.

3.3 The update cycle

Finally let us summarize the complete update cycle:

- local update sweep
  (This step is omitted for the non-linear $\sigma$ models)

- global rotation of the field

- $3 \times N$ wall-cluster updates

The sequence of the $3 \times N$ wall-cluster updates is given by the wall in 1-2, 1-3 and 2-3 plane. In each of the three cases, an update is performed for all $N$ components of the field. For each of the three orientations of the wall we performed a measurement of $Z_a/Z_p$. 
4 The simulations

First we have simulated the $O(3)$-invariant non-linear $\sigma$ model at the best estimate of $\beta_c = 0.693002(12)$ of ref. [24]. We used lattices of size $L = 6, 8, 12, 16, 24, 32$ and $48$. We have performed $25 \times 10^6$ measurements for $L = 6$ up to $L = 32$ and $10^7$ measurements for $L = 48$.

The $N = 3 \phi^4$ model was simulated at $\lambda = 2.0, 4.5$ and $5.0$ on lattices of the linear size $L = 6, 8, 12, 16, 24$ and $32$. For $\lambda = 4.5$ we simulated in addition $L = 48$. In all cases we performed $10^7$ measurements.

We have simulated the $O(4)$-invariant non-linear $\sigma$ model at the estimate of $\beta_c = 0.935861(8)$ of ref. [24]. We studied lattices of size $L = 6, 8, 12, 16, 24, 32$ and $48$. We have performed $25 \times 10^6$ measurements for $L = 6$ up to $L = 16$, $2 \times 10^7$ measurements for $L = 24$, $14 \times 10^6$ measurements for $L = 32$, and $105 \times 10^5$ measurements for $L = 48$.

The $N = 4 \phi^4$ model was simulated at $\lambda = 8.0, 12.0$ and $14.0$ on lattices of the linear size $L = 6, 8, 12, 16, 24$ and $32$. For $\lambda = 12.0$ we simulated in addition $L = 48$. As for $N = 3$, we performed $10^7$ measurements for each parameter set.

A measurement was performed after one update cycle. (See section 3.3.)

In all cases listed above, we performed $10^5$ update cycles before measuring.

In order to reduce the amount of data that is written to disc, we averaged during the simulation the results of 5000 measurements. These averages were saved.

In the case of the $\phi^4$ model we have simulated at estimates of $\beta_f$ from $Z_a/Z_{p.f}$, which were obtained from smaller lattice sizes that have been simulated before and/or short test-simulations.

In our analysis the observables are needed as a function of $\beta$. Given the large statistics we did not use the reweighting technique. Instead we used the Taylor-expansion up to the third order. The coefficients were obtained from the simulation. We always checked that the $\beta_f$ are sufficiently close to the $\beta$ of the simulation such that the error from the truncation of the Taylor-series is well below the statistical error.

As random number generator we have used our own implementation of G05CAF of the NAG-library. The G05CAF is a linear congruential random number generator with modulus $m = 2^{59}$, multiplier $a = 13^{13}$ and increment $c = 0$.

As a check of the correctness of the program and the quality of the random number generator we have implemented the following two non-trivial relations among observables:

$$0 = \frac{1}{2} \beta \sum_{y,nn,x} \langle \vec{\phi}_x \vec{\phi}_y \rangle - \langle \vec{\phi}_y \rangle - 2\lambda \langle (\vec{\phi}_x^2 - 1) \vec{\phi}_x \rangle + \frac{N}{2} \tag{27}$$
and
\[ 0 = \beta \left\langle \left( \sum_{y,nn.x} \left[ \phi_x^1 \phi_y^2 - \phi_x^2 \phi_y^1 \right] \right)^2 \right\rangle - \frac{2}{N} \sum_{y,nn.x} \langle \overrightarrow{\phi}_x \overrightarrow{\phi}_y \rangle, \tag{28} \]
where \( y,nn.x \) indicates that the sum runs over the six nearest neighbours of \( x \).

In order to enhance the statistics, we have summed eqs. (27), (28) over all sites \( x \). We found that for all our simulations these equations are satisfied within the expected statistical errors.

Most of the simulations were performed on 450 MHz Pentium III PCs. For our largest lattice size \( L = 48 \), one update cycle plus a measurement takes 0.67 s, 0.58 s, 0.86 s and 0.75 s for the \( N = 3 \) \( \phi^4 \) model at \( \lambda = 4.5, \beta = 0.68622 \), the \( O(3) \)-invariant non-linear \( \sigma \) model at \( \beta = 0.693002 \), the \( N = 4 \) \( \phi^4 \) model at \( \lambda = 12.0, \beta = 0.90843 \) and the \( O(4) \)-invariant non-linear \( \sigma \) model at \( \beta = 0.935861 \), respectively.

In total, the whole study took about two years on a single 450 MHz Pentium III CPU.

5 Analysis of the data

5.1 Corrections to scaling

In a first step of the analysis we have estimated \( \xi_{2nd}/L^* \) and \( Z_a/Z_p^* \) by fitting the \( O(3) \)- and \( O(4) \)-invariant non-linear \( \sigma \) model data with the ansatz
\[ R(\beta_c) = R^* + c L^{-\omega}, \tag{29} \]
where \( \beta_c, R^* \) and \( c \) are the parameters of the fit. We have fixed \( \omega = 0.8 \). As result we obtain \( Z_a/Z_p^* \approx 0.196 \) and \( \xi_{2nd}/L \approx 0.564 \) for the \( O(3) \) model and \( Z_a/Z_p^* \approx 0.1195 \) and \( \xi_{2nd}/L^* \approx 0.547 \) for the \( O(4) \) model. In the following we shall use these numbers to set \( R_{1,f} \). I.e. \( Z_a/Z_p,f = 0.196 \) and \( \xi_{2nd}/L_f = 0.564 \) for \( N = 3 \) and \( Z_a/Z_p,f = 0.1195 \) and \( \xi_{2nd}/L_f = 0.547 \) for \( N = 4 \).

Next we have analyzed \( \overline{R} \) to study corrections to scaling. To obtain a first impression, we have plotted our results for \( \overline{R} \) with \( Z_a/Z_p,f \) in figure 1 for \( N = 3 \) and in figure 2 for \( N = 4 \). In both cases the range \( 6 \leq L \leq 32 \) is shown.

Let us discuss in detail the \( N = 3 \) case. For the \( O(3) \)-symmetric non-linear \( \sigma \) model we clearly see an increase of \( \overline{R} \) with increasing \( L \) over the whole range of lattices sizes. On the other side, for \( \lambda = 2.0, \overline{R} \) is decreasing. For \( \lambda = 4.5 \) and \( \lambda = 5.0 \) \( \overline{R} \) stays almost constant. This behaviour suggests that leading corrections to scaling vanish at \( \lambda^* \approx 5 \). The behaviour of \( \overline{R} \) for \( N = 4 \) is qualitatively the same as for \( N = 3 \). The figure 2 indicates that \( \lambda^* \approx 13 \).

In the following numerical analysis of the data we demonstrate that the behaviour discussed above is indeed due to leading corrections to scaling and give
an accurate estimate of $\lambda^*$ and its error-bar. For this purpose we fitted our data for $\bar{R}$ with the ansatz

$$
\bar{R} = \bar{R}^* + \bar{c}(\lambda) L^{-\omega}
$$

(30)

with $\bar{R}^*$, $\bar{c}(\lambda)$ for each value of $\lambda$ and $\omega$ as free parameters. For $N = 3$ and $N = 4$, we have performed such fits for three different sets of input data. These sets are given in table 1.

Our results for $\bar{R}^*$ and $\omega$ for $N = 3$ are given in table 2.

The values of $\bar{c}$ from the same fits are summarized in table 3. First of all we notice that our fit results for $\omega$ are consistent with the estimates from field-theoretic methods. (See table 9) This indicates that the corrections are indeed dominated by the leading irrelevant scaling field $u_3$. Given the statistical error and the variation of our result for $\omega$ with the different data sets, we can not provide a more accurate estimate for $\omega$ as the field-theoretic methods.

Having convinced ourselves that we really see leading corrections to scaling we extract an estimate for $\lambda^*$ from the data given in table 3. Therefore, we linearly extrapolate the results of $\bar{c}$ at $\lambda = 4.5$ and $\lambda = 5.0$. Taking the results from the set 1 we arrive at the estimate $\lambda^* = 4.4(7)$, where the result from $U_4$ at $Z_a/Z_{p,f}$ and $U_4$ at $\xi_{2nd}/L_f$ are consistent. The error-bar is computed from the variation of $\lambda^*$ with the data-sets used for the fit. In the case of $U_4$ at $Z_a/Z_{p,f}$ $\lambda^*$ is roughly

Figure 1: $N = 3$. The Binder-cumulant $U$ at $Z_a/Z_{p,f} = 0.196$ as a function of the lattice size $L$ for $\lambda = 2.0, 4.5, 5.0$ and $\infty$. The dotted line should only guide the eye.
Table 1: The sets of input data for fits with eq. (30). For \( N = 3 \) as well as \( N = 4 \) we have fitted with three different sets of input data. In column one the \( \lambda \)-values for \( N = 3 \) and in column two the \( \lambda \)-values for \( N = 4 \) are given. In columns three, four and five the lattice sizes \( L \) that have been included in the fits are listed.

| \( \lambda, N = 3 \) | \( \lambda, N = 4 \) | set 1 | set 2 | set 3 |
|----------------------|----------------------|------|------|------|
| 2.0                  | 8.0                  | 16, 24, 32 | 12, 16, 24 | 8, 12, 16 |
| 4.5                  | 12.0                 | 16, 24, 32, 48 | 12, 16, 24, 32 | 8, 12, 16, 24 |
| 5.0                  | 14.0                 | 16, 24, 32 | 12, 16, 24 | 8, 12, 16 |
| \( \infty \)         | \( \infty \)        | 16, 24, 32, 48 | 12, 16, 24, 32 | 8, 12, 16, 24 |

Table 2: Results for \( \bar{R}^* \) and \( \omega \) for \( N = 3 \) from fits with the ansatz (30). The data-sets that have been used for the fits are given in table 1. The results for \( \bar{c} \) from the same fits are summarized in table 3.

| set | \( \chi^2 \)/d.o.f. | \( \bar{R}^* \) | \( \omega \) |
|-----|----------------------|----------------|-------------|
| \( U_4 \) at \( Z_a/Z_{p,f} \) |
| 1   | 2.32                 | 1.14018(9)    | 0.743(23)   |
| 2   | 1.79                 | 1.14028(9)    | 0.749(18)   |
| 3   | 1.14                 | 1.14021(8)    | 0.799(13)   |
| \( U_4 \) at \( \xi_{2nd}/L_f \) |
| 1   | 2.16                 | 1.13977(10)   | 0.741(22)   |
| 2   | 0.90                 | 1.14000(11)   | 0.732(17)   |
| 3   | 4.06                 | 1.14038(10)   | 0.775(12)   |

Table 3: Results for \( \bar{c} \) for \( N = 3 \) from fits with the ansatz (30). The results for \( \bar{R}^* \) and \( \omega \) are given in table 2.

| set | \( \bar{c}(2.0) \) | \( \bar{c}(4.5) \) | \( \bar{c}(5.0) \) | \( \bar{c}(\infty) \) |
|-----|-------------------|-------------------|-------------------|-------------------|
| \( U_4 \) at \( Z_a/Z_{p,f} \) |
| 1   | 0.0392(33)        | -0.0008(10)       | -0.0042(8)        | -0.0390(24)       |
| 2   | 0.0394(25)        | -0.0011(8)        | -0.0049(7)        | -0.0408(17)       |
| 3   | 0.0451(17)        | -0.0007(6)        | -0.0053(5)        | -0.0461(11)       |
| \( U_4 \) at \( \xi_{2nd}/L_f \) |
| 1   | 0.0464(37)        | -0.0007(11)       | -0.0056(9)        | -0.0464(26)       |
| 2   | 0.0436(27)        | -0.0026(8)        | -0.0074(7)        | -0.0475(18)       |
| 3   | 0.0452(18)        | -0.0064(6)        | -0.0115(5)        | -0.0566(12)       |
Figure 2: $N = 4$. The Binder-cumulant $U$ at $Z_a/Z_{p,f} = 0.1195$ as a function of the lattice size $L$ for $\lambda = 8.0, 12.0, 14.0$ and $\infty$. The dotted line should only guide the eye.

the same for all three sets. However for $U_4$ at $\xi_{2nd}/L_f$ there is a drift to larger results for $\lambda^*$ as the lattice sizes increase. Assuming a convergence proportional $L^{-(\omega+\omega)} \approx L^{-0.8}$ we arrive at our error-estimate.

In tables 4 and 5 we have summarized our results for $\bar{R}^*$ and $\omega$ and $\bar{c}$ for $N = 4$. As for $N = 3$ we see that the values for $\omega$ are consistent with the field-theoretic results.

Clearly, the sign of $\bar{c}(\infty)$ is negative and that of $\bar{c}(8.0)$ is positive. Hence there exists a $\lambda^*$ with $\bar{c}(\lambda^*) = 0$. From linear interpolation of the result for $\lambda = 12.0$ and $\lambda = 14.0$ we arrive at $\lambda^* = 12.5(4.0)$. The error-bar has been computed in the same way as for $N = 3$.

We have also used a slightly different approach to compute $\omega$ from the available data. We have analyzed the difference of $\bar{R}$ at $\lambda = 2.0$ and $\lambda = \infty$. Obviously, $\bar{R}^*$ is cancelled this way. In addition one might expect that corrections that are quadratic in the scaling field $u_3$ cancel since $|\bar{c}(2.0)|$ and $|\bar{c}(\infty)|$ are almost the same. In addition, the analysis of numerical data for $N = 1$ in ref. [8] and $N = 2$ in ref. [9] suggests that sub-leading corrections also cancel to a large extent. Therefore, we have fitted our data with the ansatz

$$\bar{R}(L, \lambda)|_{\lambda=2.0} - \bar{R}(L, \lambda)|_{\lambda=\infty} = \Delta \bar{c} L^{-\omega} .$$

Our results for $N = 3$ and the corresponding results for $N = 4$ are given in table
Table 4: Results for $\bar{R}^*$ and $\omega$ for $N = 4$ from fits with the ansatz (30). The data-sets that have been used for the fits are given in table 1. The results for $\bar{c}$ from the same fits are summarized in table 5.

| set | $\chi^2$/d.o.f. | $\bar{R}^*$ | $\omega$ |
|-----|----------------|-------------|---------|
| $U_4$ at $Z_a/Z_{p,f}$ | | | |
| 1   | 0.75           | 1.09441(6)  | 0.798(52)|
| 2   | 0.84           | 1.09450(7)  | 0.748(41)|
| 3   | 2.07           | 1.09466(6)  | 0.761(27)|

| $U_4$ at $\xi_{2nd}/L$ | | | |
| 1   | 0.77           | 1.09458(7)  | 0.761(50)|
| 2   | 1.08           | 1.09474(8)  | 0.735(37)|
| 3   | 7.51           | 1.09522(8)  | 0.761(25)|

Table 5: Results for $\bar{c}$ for $N = 4$ from fits with the ansatz (30). The results for $\bar{R}^*$ and $\omega$ are given in table 4.

| set | $\bar{c}(8.0)$ | $\bar{c}(12.0)$ | $\bar{c}(14.0)$ | $\bar{c}(\infty)$ |
|-----|----------------|-----------------|-----------------|------------------|
| $U_4$ at $Z_a/Z_{p,f}$ | | | | |
| 1   | 0.0091(19)     | 0.0010(8)       | -0.0023(7)      | -0.0160(22)      |
| 2   | 0.0073(12)     | -0.0001(6)      | -0.0021(5)      | -0.0145(14)      |
| 3   | 0.0061(7)      | -0.0013(4)      | -0.0033(3)      | -0.0165(8)       |
| $U_4$ at $\xi_{2nd}/L$ | | | | |
| 1   | 0.0083(18)     | 0.0011(8)       | -0.0035(6)      | -0.0183(23)      |
| 2   | 0.0065(12)     | -0.0017(6)      | -0.0040(5)      | -0.0184(15)      |
| 3   | 0.0030(7)      | -0.0059(4)      | -0.0080(3)      | -0.0239(10)      |
Table 6: Results for \( \omega \) for \( N = 3 \) and \( N = 4 \) obtained by fitting with the ansatz (31). In the first column we give \( N \) and the phenomenological coupling that has been used to determine \( \beta_f \). In the fit all lattice size with \( L_{\text{min}} \leq L \leq L_{\text{max}} \) have been taken into account.

| \( N = 3, \, U_4 \) at \( Z_a/Z_p = 0.196 \) | \( L_{\text{min}} \) | \( L_{\text{max}} \) | \( \chi^2/\text{d.o.f.} \) | \( \omega \) |
|---|---|---|---|---|
| 6 | 32 | 1.67 | 0.796(7) |
| 8 | 32 | 0.73 | 0.781(10) |

| \( N = 3, \, U_4 \) at \( \xi_2/L = 0.564 \) | \( L_{\text{min}} \) | \( L_{\text{max}} \) | \( \chi^2/\text{d.o.f.} \) | \( \omega \) |
|---|---|---|---|---|
| 6 | 32 | 0.65 | 0.769(6) |
| 8 | 32 | 0.65 | 0.766(9) |

| \( N = 4, \, U_4 \) at \( Z_a/Z_p = 0.1195 \) | \( L_{\text{min}} \) | \( L_{\text{max}} \) | \( \chi^2/\text{d.o.f.} \) | \( \omega \) |
|---|---|---|---|---|
| 6 | 32 | 0.54 | 0.780(15) |
| 8 | 32 | 0.40 | 0.765(22) |

| \( N = 4, \, U_4 \) at \( \xi_2/L = 0.547 \) | \( L_{\text{min}} \) | \( L_{\text{max}} \) | \( \chi^2/\text{d.o.f.} \) | \( \omega \) |
|---|---|---|---|---|
| 6 | 32 | 0.40 | 0.774(14) |
| 8 | 32 | 0.38 | 0.764(20) |

In the case of \( N = 4 \) we have taken the difference of \( \bar{R} \) at \( \lambda = 8.0 \) and \( \lambda = \infty \).

First we notice that a \( \chi^2/\text{d.o.f.} \approx 1 \) is already reached for \( L_{\text{min}} = 6 \). For such a small \( L_{\text{min}} \), fits with ansatz (30) produce \( \chi^2/\text{d.o.f.} = 2.8 \) for \( N = 3, \, U_4 \) at \( Z_a/Z_p = 0.196 \) and \( \chi^2/\text{d.o.f.} = 13.3 \) for \( N = 3, \, U_4 \) at \( \xi_2/L = 0.564 \). This fact indicates that the above mentioned cancellations indeed occur.

The results for \( \omega \) are consistent with those of the field theoretic methods. Certainly our approach is very promising to give competitive results for the correction exponent \( \omega \). However, we would like to have still larger statistics and a larger range of lattice sizes to give a sensible estimate of the systematic error caused by sub-leading corrections.

### 5.2 Critical exponents

We have computed the critical exponents \( \nu \) and \( \eta \) using well established finite size scaling methods. Below we shall only discuss in detail the results for \( N = 3 \). The analysis for \( N = 4 \) has been performed analogously.

The exponent \( \nu \) is computed from the slope of a phenomenological coupling \( R \) at \( \beta_f \) (see eq. (10)):

\[
\left. \frac{\partial R}{\partial \beta} \right|_{\beta_f} = a \, L^{1/\nu} \, .
\]

(32)

As it was pointed out in [24], replacing \( \beta_c \) by \( \beta_f \) simplifies the error-analysis, since the error in \( \beta_c \) needs not to be propagated.

In our study we have considered three different choices of \( R \). Hence in eq. (32) we could in principle consider 9 different combinations. Below we shall restrict ourself to six choices: \( \beta_f \) is fixed either by \( Z_a/Z_{p,f} \) or \( \xi_{2nd}/L_f \). We consider the slope of all three phenomenological couplings \( R \).
Table 7: Results for $\nu$ from fits with the ansatz (32). The range of lattice sizes is always $L_{\text{min}} = 16$ and $L_{\text{max}} = 32$. We have used six different combinations of $R$. These are given by C1: The slope of $U_4$ at $Z_a/Z_{p,f}$; C2: The slope of $Z_a/Z_p$ at $Z_a/Z_{p,f}$; C3: The slope of $\xi_{2\text{nd}}/L$ at $Z_a/Z_{p,f}$; C4: The slope of $U_4$ at $\xi_{2\text{nd}}/L_f$; C5: The slope of $Z_a/Z_p$ at $\xi_{2\text{nd}}/L_f$; C6: The slope of $\xi_{2\text{nd}}/L$ at $\xi_{2\text{nd}}/L_f$.

| $\lambda$ | C1   | C2     | C3     | C4     | C5     | C6     |
|-----------|------|--------|--------|--------|--------|--------|
| 2.0       | 0.7164(23) | 0.7088(10) | 0.7136(13) | 0.7181(24) | 0.7106(12) | 0.7135(12) |
| 4.5       | 0.7071(21) | 0.7083(9)  | 0.7115(12) | 0.7074(21) | 0.7086(10) | 0.7115(12) |
| 5.0       | 0.7078(21) | 0.7085(9)  | 0.7114(11) | 0.7071(21) | 0.7077(10) | 0.7114(11) |
| $\infty$  | 0.7076(12) | 0.7127(5)  | 0.7142(6)  | 0.7056(12) | 0.7136(13) | 0.7141(6)  |

Table 8: Results for $\nu$ from fits with ansatz (32) for $N = 3$ at $\lambda = 4.5$; always $L_{\text{max}} = 48$. The combinations C1,...,C6 are explained in the caption of table 7.

| $L_{\text{min}}$ | C1   | C2     | C3     | C4     | C5     | C6     |
|-------------------|------|--------|--------|--------|--------|--------|
| 8                 | 0.7113(7) | 0.7087(3) | 0.7150(4) | 0.7102(7) | 0.7075(3) | 0.7150(4) |
| 12                | 0.7111(10) | 0.7100(4) | 0.7132(5) | 0.7107(10) | 0.7096(5) | 0.7132(5) |
| 16                | 0.7101(13) | 0.7099(6) | 0.7120(7) | 0.7102(14) | 0.7100(7) | 0.7120(7) |

First we like to study how much the result of $\nu$ from fits with eq. (32) depends on leading corrections to scaling. For this purpose we have fitted for $L_{\text{min}} = 16$ and $L_{\text{max}} = 32$ our data for all available values of $\lambda$ for all six combinations of $R$. The results of these fits are summarized in table 7. The variation of the results with $\lambda$ are rather small. The largest variation we find for the combination C1, where we get $\nu = 0.7164(23)$ for $\lambda = 2.0$ and $\nu = 0.7076(12)$ for $\lambda = \infty$. Hence, we expect that for $\lambda = 4.5$ the effect of corrections to scaling should be smaller than 0.0001.

Next, let us discuss in more detail the results for $\lambda = 4.5$ which is closest to our estimate of $\lambda^*$. Results from fits with ansatz (32) are summarized in table 8. We see that the results approach each other as $L_{\text{min}}$ is increased. In the case of the slope of $U_4$, the results for $\nu$ stay almost constant as $L_{\text{min}}$ is varied. For the slope of $Z_a/Z_p$ we see a slight increase of the estimate of $\nu$. On the other hand for the slope of $\xi_{2\text{nd}}/L$ we see a decrease. Assuming that this behaviour is caused by the sub-leading corrections, we conclude that $\nu = 0.7120$ from $\xi_{2\text{nd}}/L$ is an upper bound. Given the larger stability of the estimate from $Z_a/Z_p$ we take as our final estimate $\nu = 0.710(2)$.

Next we computed the exponent $\eta$ from the finite size behaviour of the magnetic susceptibility

$$
\chi|_{\beta_f} = c \, L^{2-\eta}.
$$

(33)
We restrict the discussion to \( Z_a/Z_{p,f} \) since fixing \( \beta_f \) by \( \xi_{2na}/L_f \) gives very similar numbers.

For the estimate of \( \eta \) we see a much stronger dependence on \( \lambda \) than for \( \nu \). Fitting with ansatz (33) and \( L_{\text{min}} = 16, L_{\text{max}} = 32 \) we obtain \( \eta = 0.03982(33), 0.03614(30), 0.03541(31) \) and \( 0.03269(18) \) for \( \lambda = 2.0, 4.5, 5.0 \) and \( \infty \), respectively.

Fitting the data at \( \lambda = 4.5 \) with \( L_{\text{max}} = 48 \) yields \( \eta = 0.03581(14), 0.03668(19) \) and \( 0.03736(32) \) for \( L_{\text{min}} = 12, 16 \) and \( 24 \), respectively. For \( L_{\text{min}} = 12 \) we get \( \chi^2/d\text{o.f.} = 18.5 \). The strong dependence of the result on \( L_{\text{min}} \) and the large \( \chi^2/d\text{o.f.} \) at \( L_{\text{min}} = 12 \) indicates that there are sizeable sub-leading corrections.

Fitting with an ansatz that includes an analytic background term

\[
\chi_{|\beta_f} = c L^{2-\eta} + b
\]  

(34)
yields \( \chi^2/d\text{o.f.} = 0.38 \) already for \( L_{\text{min}} = 8 \). The results are \( \eta = 0.0384(2), 0.0386(4) \) and \( 0.0381(6) \) for \( L_{\text{min}} = 8, 12 \) and \( 16 \), respectively. To see the effect of leading corrections on this fit we have in addition fitted the data for \( \lambda = 5.0 \) with \( L_{\text{min}} = 8 \) and \( L_{\text{max}} = 32 \). We get \( \eta = 0.0378(4) \).

As our final estimate we quote \( \eta = 0.0380(10) \). The error-bar takes into account statistical errors as well a systematic errors due to leading and sub-leading corrections. These errors are estimated from the spread of the results of the various fits discussed above.

With a similar analysis we arrive at \( \nu = 0.749(2) \) and \( \eta = 0.0365(10) \) for the \( O(4) \) universality class.

6 Comparison with results from the literature

Here we like to compare our results for the critical exponents \( \nu \) and \( \eta \) with previous Monte Carlo studies of the \( O(3) \)- and \( O(4) \)-invariant non-linear \( \sigma \) models. In addition we give selected results from high temperature series, perturbation theory in three dimensions, and the \( \epsilon \)-expansion. The results are summarized in tables 9 and 10 for \( N = 3 \) and \( N = 4 \), respectively. For more references on field theoretic methods and other methods, not discussed here, see e.g. ref. [16].

All Monte Carlo studies listed in the tables 9 and 10 use a simple cubic lattice. In addition, in ref. [29] the body centred cubic lattice is studied. (In table 9 we only give the sc results.) In all studies the lattice sizes are smaller or equal \( L = 48 \), except for ref. [24], where in addition \( L = 64 \) is simulated. The results of the MC studies cited above are extracted from ansätze like eqs. (32,33). (Mostly \( \beta_c \) is used instead of \( \beta_f \).) We see that almost all Monte Carlo results for \( \nu \) are consistent with ours. On the other hand, the results for \( \eta \) are systematically too small, except for ref. [24].

This behaviour can be well understood with our results of section 5.2. The estimates for \( \eta \) from the ansatz (33) are clearly effected by corrections to scaling.
Our results from the $O(3)$- and $O(4)$-invariant non-linear $\sigma$ models for $\eta$ are systematically lower than our final results from the $\phi^4$ models at $\lambda^*$. On the other hand, the results for $\nu$ given in table 7 show only little variation with $\lambda$; i.e. little dependence on leading corrections to scaling.

The authors of ref. [24], who for the first time tried to take into account leading corrections to scaling in the analysis of their data, arrive at rather similar conclusions how leading corrections to scaling affect the estimates of $\eta$ and $\nu$. They extrapolated their results for $\eta$ assuming $L^{-\omega}$ corrections.

However from our analysis of section 5.2 we know that the estimates of $\eta$ obtained from lattices with $L \leq 48$ are also strongly affected by sub-leading corrections with $\omega_2 \approx 2$.

Hence, extrapolating only in $L^{-\omega}$ leads to a wrong amplitude for the $L^{-\omega}$ corrections. As result, the final estimate of ref. [24] for $\eta$ is too large compared with our result.

There exists a large number of refs. on the $\epsilon$-expansion and the perturbative expansion in three dimensions in the literature. As an example, we have chosen the result of a recent analysis by Guida and Zinn-Justin [16]. We notice that these results are consistent with ours for $\eta$ and $\nu$ within the quoted errors. Also note that the error-bars of our estimates for $\nu$ and $\eta$ are smaller than those of the field-theoretic estimates.

There exists also a number of publications on the analysis of high temperature series. In the tables we give the results of a recent analysis [30] using inhomogeneous differential approximants. The coefficients of the high temperature series of $\chi$ and $\mu_2$ are computed up to $\beta^{21}$. In our tables, we only give the results from the unbiased analysis of the simple cubic lattice series. In addition the authors analyze the body centred cubic lattice. The authors also give results obtained from a so called $\theta$-biased analysis, where they make use of the numerical results for $\theta = \omega \nu$ obtained from field-theoretic methods. It is interesting to notice that these biased results (not given in our tables) tend to be less consistent with our results than the un-biased results which we quote in tables 9 and 10.

In tables 9 and 10, we give for $\omega$ the average of the result from $Z_a/Z_{p,f}$ and $\xi_{2nd}/L_f$ with $L_{min} = 8$ taken from table 8. We make no attempt to estimate the systematic errors due to sub-leading corrections. Certainly these errors are larger than that quoted for the field-theoretic estimates of ref. [16]. It is however interesting to note that our results are consistent with those of ref. [16].

Also the Monte Carlo result of ref. [24] for $N = 3$ is consistent with ours. However we cannot confirm their surprising result for $N = 4$.

7 Conclusions

In this study we have demonstrated that the programme of refs. [6, 7] to eliminate leading corrections to scaling in the three dimensional $\phi^4$ model can be extended
Table 9: Results for the critical exponents of the $O(3)$ universality class from various methods. The numbers for $\eta$ that are marked by a $*$ are computed from $\gamma/\nu = 2 - \eta$. Details are discussed in the text.

| method | ref.     | $\nu$    | $\eta$    | $\omega$ |
|--------|----------|----------|-----------|----------|
| IMC    | present work | 0.710(2) | 0.0380(10)| 0.773    |
| MC     | [24]     | 0.7128(14)| 0.0413(15)(1)| 0.78(2)  |
| MC     | [25]     | 0.642(2) | 0.020(1)  |          |
| MC     | [26]     | 0.704(6) | 0.027(2)  |          |
| MC     | [27]     | 0.704(6) | 0.028(2)  |          |
| MC     | [28]     | 0.706(9) | 0.031(7)  |          |
| MC     | [29]     | 0.7036(23)| 0.0250(35)|          |
| HT     | [30]     | 0.7153(35)| 0.0355(25)| 0.782(13)|
| $d = 3$ PT | [16]     | 0.7045(55)| 0.0375(45)| 0.794(18)|

Table 10: Results for the critical exponents of the $O(4)$ universality class from various methods. The numbers for $\eta$ that are marked by a $*$ are computed from $\gamma/\nu = 2 - \eta$. Details are discussed in the text.

| method   | ref.     | $\nu$    | $\eta$    | $\omega$ |
|----------|----------|----------|-----------|----------|
| IMC      | present work | 0.749(2) | 0.0365(10)| 0.765    |
| MC       | [24]     | 0.7525(10)| 0.0384(12)| 1.8(2)   |
| MC       | [31]     | 0.7479(90)| 0.0254(38)|          |
| HT       | [30]     | 0.750(3) | 0.035(9)  |          |
| $d = 3$ PT | [16]     | 0.741(6) | 0.0350(45)| 0.774(20)|
| $\epsilon$-expansion | [16] | 0.737(8) | 0.0360(40)| 0.795(30)|
to $N = 3$ and $N = 4$. In particular, we have found $\lambda^* = 4.4(7)$ for $N = 3$
and $\lambda^* = 12.5(4.0)$ for $N = 4$. Based on this result we have computed the
critical exponents $\nu$ and $\eta$ from finite size scaling. In particular in the case of $\eta$,
the error-bar could be reduced considerably compared with previous Monte Carlo
simulations or field theoretic methods and the analysis of high temperature series.

Since the CPU-time that was used for the present study is still moderate,
further progress can be made by just enlarging the statistics and simulating larger
lattices.

Also, our results for $\lambda^*$ can be used as input for the analysis of high temper-
ature series analogous to refs. [11, 12].

The principle question raised in ref. [11], whether the programme to eliminate
leading corrections is restricted to $N < N_c$, where $N_c$ is finite, remains open.

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