Self-generated interior blow-up solutions in fractional elliptic equation with absorption

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Abstract

In this paper we study positive solutions to problem involving the fractional Laplacian

\[
\begin{cases}
(-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, & x \in \Omega \setminus \mathcal{C}, \\
u(x) = 0, & x \in \Omega^c, \\
\lim_{x \in \Omega \setminus \mathcal{C}, x \to \mathcal{C}} u(x) = +\infty,
\end{cases}
\]

(0.1)

where \( p > 1 \) and \( \Omega \) is an open bounded \( C^2 \) domain in \( \mathbb{R}^N \), \( \mathcal{C} \subset \Omega \) is a compact \( C^2 \) manifold with \( N - 1 \) multiples dimensions and without boundary, the operator \((-\Delta)^\alpha\) with \( \alpha \in (0, 1) \) is the fractional Laplacian.

We consider the existence of positive solutions for problem (0.1). Moreover, we further analyze uniqueness, asymptotic behaviour and nonexistence.

Key words: Fractional Laplacian, Existence, Uniqueness, Asymptotic behavior, Blow-up solution.
1 Introduction

In 1957, a fundamental contribution due to Keller in [11] and Osserman in [19] is the study of boundary blow-up solutions for the non-linear elliptic equation

\[
-\Delta u + h(u) = 0 \text{ in } \Omega, \\
\lim_{x \in \Omega, x \to \partial \Omega} u(x) = +\infty. \tag{1.1}
\]

They proved the existence of solutions to (1.1) when \( h : \mathbb{R} \to [0, +\infty) \) is a locally Lipschitz continuous function which is nondecreasing and satisfies the so called Keller-Osserman condition. From then on, the result of Keller and Osserman has been extended by numerous mathematicians in various ways, weakening the assumptions on the domain, generalizing the differential operator and the nonlinear term for equations and systems. The case of \( h(u) = u^p \) with \( p = \frac{N+2}{N-2} \) is studied by Loewner and Nirenberg [15], where in particular uniqueness and asymptotic behavior were obtained. After that, Bandle and Marcus [2] obtained uniqueness and asymptotic for more general non-linearities \( h \). Later, Le Gall in [9] established a uniqueness result of problem (1.1) in the domain whose boundary is non-smooth when \( h(u) = u^2 \). Marcus and Véron [16, 18] extended the uniqueness of blow-up solution for (1.1) in general domains whose boundary is locally represented as a graph of a continuous function when \( h(u) = u^p \) for \( p > 1 \). Under this special assumption on \( h \), Kim [12] studied the existence and uniqueness of boundary blow-up solution to (1.1) in bounded domains \( \Omega \) satisfying \( \partial \Omega = \partial \bar{\Omega} \). For another interesting contributions to boundary blow-up solutions see for example Kondratev, Nikishkin [13], Lazer, McKenna [14], Arrieta and Rodríguez-Bernal [1], Chuaqui, Cortázar, Elgueta and J. García-Melián [4], del Pino and Letelier [5], Díaz and Letelier [6], Du and Huang [7], García-Melián [10], Véron [20], and the reference therein.

In a recent work, Felmer and Quaas [8] considered a version of Keller and Osserman problem for a class of non-local operator. Being more precise, they considered as a particular case the fractional elliptic problem

\[
\begin{aligned}
(-\Delta)^\alpha u(x) + |u|^{p-1}u(x) &= f(x), & x &\in \Omega, \\
u(x) &= g(x), & x &\in \bar{\Omega}^c, \\
\lim_{x \in \Omega, x \to \partial \Omega} u(x) &= +\infty,
\end{aligned} \tag{1.2}
\]

where \( p > 1 \), \( f \) and \( g \) are appropriate functions and \( \Omega \) is a bounded domain with \( C^2 \) boundary. The operator \((-\Delta)^\alpha\) is the fractional Laplacian which is defined as

\[
(-\Delta)^\alpha u(x) = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{\delta(u, x, y)}{|y|^{N+2\alpha}} dy, \quad x \in \Omega, \tag{1.3}
\]
with $\alpha \in (0, 1)$ and $\delta(u, x, y) = u(x + y) + u(x - y) - 2u(x)$.

In [8] the authors proved the existence of a solution to (1.2) provided that $g$ explodes at the boundary and satisfies other technical conditions. In case the function $g$ blows up with an explosion rate as $d(x)^\beta$, with $\beta \in [-\frac{2\alpha}{p-1}, 0)$ and $d(x) = \text{dist}(x, \partial \Omega)$, it is shown that the solution satisfies

$$0 < \liminf_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{-\beta} \leq \limsup_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty.$$ 

Here the explosion is driven by the external value $g$ and the external source $f$ has a secondary role, not intervening in the explosive character of the solution.

More recently, Chen, Felmer and Quaas [3] extended the results in [8] studying existence, uniqueness and non-existence of boundary blow-up solutions when the function $g$ vanishes and the explosion on the boundary is driven by the external source $f$, with weak or strong explosion rate. Moreover, the results are extended even to the case where the boundary blow-up solutions in driven internally, when the external source and value, $f$ and $g$, vanish. Existence, uniqueness, asymptotic behavior and non-existence results for blow-up solutions of (1.2) are considered in [3]. In the analysis developed in [3], a key role is played by the function $C : (-1, 0] \to \mathbb{R}$, that governs the behavior of the solution near the boundary. The function $C$ is defined as

$$C(\tau) = \int_{0}^{+\infty} \frac{\chi_{(0,1)}(t)|1 - t|^{\tau} + (1 + t)^{\tau} - 2}{t^{1+2\alpha}} dt$$ (1.4)

and it possess exactly one zero in $(-1, 0)$ and we call it $\tau_0(\alpha)$. In what follows we explain with more details the results in the case of vanishing external source and values, that is $f = 0$ in $\Omega$ and $g = 0$ in $\Omega^c$, which is the case we will consider in this paper. In Theorem 1.1 in [3], we proved that whenever

$$1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)},$$

then problem (1.2) admits a unique positive solution $u$ such that

$$0 < \liminf_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} \leq \limsup_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty.$$ 

On the other hand, we proved that when $p \geq 1$, then problem (1.2) does not admit any solution $u$ such that

$$0 < \liminf_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{-\tau} \leq \limsup_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{-\tau} < +\infty,$$ (1.5)
for any $\tau \in (-1, 0) \setminus \{\tau_0(\alpha), -\frac{2\alpha}{p-1}\}$. We observe that the non-existence result does not include the case when $u$ has an asymptotic behavior of the form $d(x)^{\tau_0(\alpha)}$, where $\tau_0(\alpha)$ is precisely where $C$ vanishes. We have a special existence result in this case, precisely if

$$\max\{1 - \frac{2\alpha}{\tau_0(\alpha)} + \frac{\tau_0(\alpha) + 1}{\tau_0(\alpha)}, 1\} < p < 1 - \frac{2\alpha}{\tau_0(\alpha)},$$

then, for any $t > 0$, problem (1.2) admits a positive solution $u$ such that

$$\lim_{x \in \Omega, x \to \partial\Omega} u(x)d(x)^{-\tau_0(\alpha)} = t.$$

Motivated by these results and in view of the non-local character of the fractional Laplacian we are interested in another class of blow-up solutions. We want to study solutions that vanish at the boundary of the domain $\Omega$ but that explodes at the interior of the domain, near a prescribed embedded manifold. From now on, we assume that $\Omega$ is an open bounded domain in $\mathbb{R}^N$ with $C^2$ boundary, and that there is a $C^2$, $(N - 1)$-dimensional manifold $C$ without boundary, embedded in $\Omega$, such that, it separates $\Omega \setminus C$ in exactly two connected components. We denote by $\Omega_1$ the inner component and by $\Omega_2$ the external component, that is $\bar{\Omega}_1 \cap \partial\Omega = \emptyset$ and $\bar{\Omega}_2 \cap \partial\Omega = \partial\Omega$. Throughout the paper we will consider the distance function

$$D : \Omega \setminus C \to \mathbb{R}_+, \quad D(x) = \text{dist}(x, C). \quad (1.6)$$

Let us consider the equations, for $i = 1, 2$,

\[
\begin{cases}
(-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, & x \in \Omega_i, \\
u(x) = 0, & x \in \bar{\Omega}_i^c, \\
\lim_{x \in \Omega_i, x \to \partial\Omega_i} u(x) = +\infty,
\end{cases}
\quad (1.7)
\]

which have solutions $u_1$ and $u_2$, for $i = 1, 2$ respectively, in the appropriate range of the parameters. In the local case, that is, $\alpha = 1$, these two solutions certainly do not interact among each other, but when $\alpha \in (0, 1)$, due to the non-local character of the fractional Laplacian and the non-linear character of the equation the solutions on each side of $\Omega$ interact and it is precisely the purpose of this paper to study their existence, uniqueness and non-existence.

In precise terms we consider the equation

\[
\begin{cases}
(-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, & x \in \Omega \setminus C, \\
u(x) = 0, & x \in \Omega^c, \\
\lim_{x \in \Omega \setminus C, x \to C} u(x) = +\infty,
\end{cases}
\quad (1.8)
\]
where \( p > 1 \), \( \Omega \) and \( C \subset \Omega \) are as described above. The explosion of the solution near \( C \) is governed by a function \( c : (-1, 0] \to \mathbb{R} \), defined as

\[
c(\tau) = \int_0^{+\infty} \frac{|1-t|^\tau + (1+t)^\tau - 2}{t^{1+2\alpha}} dt.
\]

This function plays the role of the function \( C \) used in [3], but it has certain differences. In Section §2 we prove the existence of a number \( \alpha_0 \in (0, 1) \) such that \( \alpha \in [\alpha_0, 1) \) the function \( c \) is always positive in \((-1, 0)\), while if \( \alpha \in (0, \alpha_0) \) then there exists a unique \( \tau_1(\alpha) \in (-1, 0) \) such that \( c(\tau_1(\alpha)) = 0 \) and \( c(\tau) > 0 \) in \((-1, \tau_1(\alpha))\) and \( c(\tau) < 0 \) in \((\tau_1(\alpha), 0)\), see Proposition 2.1. We notice here that \( \tau_1(\alpha) > \tau_0(\alpha) \) if \( \alpha \in (0, \alpha_0) \).

Now we are ready to state our main theorems on the existence uniqueness and asymptotic behavior of interior blow-up solutions to equation (1.8). These theorems deal separately the case \( \alpha \in (0, \alpha_0) \) and \( \alpha \in [\alpha_0, 1) \).

**Theorem 1.1** Assume that \( \alpha \in (0, \alpha_0) \) and the assumptions on \( \Omega \) and \( C \).

Then we have:

(i) If

\[
1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_1(\alpha)},
\]

then problem (1.8) admits a unique positive solution \( u \) satisfying

\[
0 < \liminf_{x \in \Omega \backslash C, x \to C} u(x)D(x)^{\frac{2\alpha}{\tau_1(\alpha)}} \leq \limsup_{x \in \Omega \backslash C, x \to C} u(x)D(x)^{\frac{2\alpha}{\tau_1(\alpha)}} < +\infty.
\]

(ii) If

\[
\max\{1 - \frac{2\alpha}{\tau_1(\alpha)} + \frac{\tau_1(\alpha) + 1}{\tau_1(\alpha)}, 1\} < p < 1 - \frac{2\alpha}{\tau_1(\alpha)}.
\]

Then, for any \( t > 0 \), there is a positive solution \( u \) of problem (1.8) satisfying

\[
\lim_{x \in \Omega \backslash C, x \to C} u(x)D(x)^{-\tau_1(\alpha)} = t.
\]

(iii) If one of the following three conditions holds

a) \( 1 < p \leq 1 + 2\alpha \) and \( \tau \in (-1, 0) \backslash \{\tau_1(\alpha)\} \),

b) \( 1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_1(\alpha)} \) and \( \tau \in (-1, 0) \backslash \{\tau_1(\alpha), -\frac{2\alpha}{p-1}\} \) or

c) \( p \geq 1 - \frac{2\alpha}{\tau_1(\alpha)} \) and \( \tau \in (-1, 0) \),

then problem (1.8) does not admit any solution \( u \) satisfying

\[
0 < \liminf_{x \in \Omega \backslash C, x \to C} u(x)D(x)^{-\tau} \leq \limsup_{x \in \Omega \backslash C, x \to C} u(x)D(x)^{-\tau} < +\infty.
\]
We observe that this theorem is similar to Theorem 1.1 in [3], where the role of \( \tau_0(\alpha) \) is played here by \( \tau_1(\alpha) \). A quite different situation occurs when \( \alpha \in [\alpha_0, 1) \) and the function \( c \) never vanishes in \((-1, 0)\). Precisely, we have

**Theorem 1.2** Assume that \( \alpha \in [\alpha_0, 1) \) and the assumptions on \( \Omega \) and \( C \). Then we have:

(i) If \( p > 1 + 2\alpha \), then problem \((1.8)\) admits a unique positive solution \( u \) satisfying \((1.11)\).

(ii) If \( p > 1 \), then problem \((1.8)\) does not admit any solution \( u \) satisfying \((1.14)\) for any \( \tau \in (-1, 0) \setminus \{-\frac{2\alpha}{p-1}\} \).

Comparing Theorem 1.1 with Theorem 1.2 we see that the range of existence for the absorption term is quite larger for the second one, no constraint from above. The main difference with the results in [3], Theorem 1.1, with vanishing \( f \) and \( g \) occurs when \( \alpha \) is large and the function \( c \) does not vanish, allowing thus for existence for all \( p \) large. This difference comes from the fact that the fractional Laplacian is a non-local operator so that in the interior blow-up, in each of the domains \( \Omega_1 \) and \( \Omega_2 \) there is a non-zero external value, the solutions itself acting on the other side of \( C \).

The proof of our theorems is obtained through the use of super and sub-solutions as in [3]. The main difficulty here is to find the appropriate super and sub-solutions to apply the iteration technique to fractional elliptic problem \((1.8)\). Here we make use of some precise estimates based on the function \( c \) and the distance function \( D \) near \( C \).

This article is organized as follows. In section \( \S \) 2, we introduce some preliminaries and we prove the main estimates of the behavior of the fractional Laplacian when applied to suitable powers of the function \( D \). In section \( \S \) 3 we prove the existence of solution to problem \((1.8)\) as given in Theorem 1.1 and Theorem 1.2. Finally, in Section \( \S \) 4 we prove the uniqueness and nonexistence statements of these theorems.

## 2 Preliminaries

In this section, we recall some basic results from [3] and obtain some useful estimate, which will be used in constructing super and sub-solutions of problem \((1.8)\). The first result states as:

**Theorem 2.1** Assume that \( p > 1 \) and there are super-solution \( \bar{U} \) and sub-solution \( U \) of problem \((1.8)\) such that

\[
\bar{U} \geq U \quad \text{in} \quad \Omega \setminus C, \quad \liminf_{x \in \Omega \setminus C, x \to C} U(x) = +\infty, \quad \bar{U} = U = 0 \quad \text{in} \quad \Omega^c.
\]
Then problem (1.8) admits at least one positive solution $u$ such that

$$U \leq u \leq \bar{U} \quad \text{in } \Omega \setminus C.$$  

**Proof.** The procedure is similar to the proof of Theorem 2.6 in [3], here we give the main differences.

Let us define $\Omega_n := \{ x \in \Omega \mid D(x) > 1/n \}$ then we solve

$$\begin{cases}
(-\Delta)\alpha u_n(x) + |u_n|^{p-1}u_n(x) = 0, & x \in \Omega_n, \\
u_n(x) = U, & x \in \Omega_n^c.
\end{cases} \quad (2.1)$$

To find these solutions of (2.1) we observe that for fix $n$ the method of section 3 of [8] applies even if the domain is not connected since the estimate of Lemma 3.2 holds with $\delta < 1/2n$ (see also Proposition 3.2 part ii) in [3], form here sub and super-solution can be construct for the Dirichlet problem and then existence holds for (2.1) by an iteration technique (see also section 2 of [3] for that procedure). Then as in Theorem 2.6 in [3] we have

$$U \leq u_n \leq u_{n+1} \leq \bar{U} \quad \text{in } \Omega.$$  

By monotonicity of $u_n$, we can define

$$u(x) := \lim_{n \to +\infty} u_n(x), \quad x \in \Omega \quad \text{and} \quad u(x) := 0, \quad x \in \Omega^c.$$  

Which, by a stability property, is a solution of problem (1.8) with the desired properties. \hfill \Box

In order to prove our existence result, it is crucial to have available super and sub-solutions to problem (1.8). To this end, we start describing the properties of $c(\tau)$ defined in (1.9), which is a $C^2$ function in $(-1,0)$.

**Proposition 2.1** There exists a unique $\alpha_0 \in (0,1)$ such that

(i) For $\alpha \in [\alpha_0,1)$, we have $c(\tau) > 0$, for all $\tau \in (-1,0)$;

(ii) For any $\alpha \in (0,\alpha_0)$, there exists unique $\tau_1(\alpha) \in (-1,0)$ satisfying

$$c(\tau) \begin{cases}
> 0, & \text{if } \tau \in (-1,\tau_1(\alpha)), \\
= 0, & \text{if } \tau = \tau_1(\alpha), \\
< 0, & \text{if } \tau \in (\tau_1(\alpha),0)
\end{cases} \quad (2.2)$$

and

$$\lim_{\alpha \to \alpha_0^-} \tau_1(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to 0^+} \tau_1(\alpha) = -1. \quad (2.3)$$

Moreover, $\tau_1(\alpha) > \tau_0(\alpha)$, for all $\alpha \in (0,\alpha_0)$, where $\tau_0(\alpha) \in (-1,0)$ is the unique zero of $C(\tau)$, defined in (1.4).
Proof. From (1.9), differentiating twice we find that

$$c''(\tau) = \int_{0}^{+\infty} \frac{|1 - t|^\tau (\log |1 - t|)^2 + (1 + t)^\tau (\log(1 + t))^2}{t^{1+2\alpha}} dt > 0, \quad (2.4)$$

so that $c$ is strictly convex in $(-1, 0)$. We also see easily that

$$c(0) = 0 \quad \text{and} \quad \lim_{\tau \to -1^+} c(\tau) = \infty. \quad (2.5)$$

Thus, if $c'(0) \leq 0$ then $c(\tau) > 0$ for $\tau \in (-1, 0)$ and if $c'(0) > 0$, then there exists $\tau_1(\alpha) \in (-1, 0)$ such that $c(\tau) > 0$ for $\tau \in (-1, \tau_1(\alpha))$, $c(\tau) < 0$ for $\tau \in (\tau_1(\alpha), 0)$ and $c(\tau_1(\alpha)) = 0$. In order to complete our proof, we have to analyze the sign of $c'(0)$, which depends on $\alpha$ and to make this dependence explicit, we write $c'(0) = T(\alpha)$. We compute $T(\alpha)$ from (1.9), differentiating and evaluating in $\tau = 0$

$$T(\alpha) = \int_{0}^{+\infty} \frac{\log |1 - t^2|}{t^{1+2\alpha}} dt. \quad (2.6)$$

We have to prove that $T$ possesses a unique zero in the interval $(0, 1)$. For this purpose we start proving that

$$\lim_{\alpha \to 1^-} T(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha \to 0^+} T(\alpha) = +\infty. \quad (2.7)$$

The first limit follows from the fact that $\log(1 - s) \leq -s$, for all $s \in [0, 1/4]$, and so

$$\lim_{\alpha \to 1^-} \frac{1}{\alpha} \int_{0}^{1/2} \frac{\log(1 - t^2)}{t^{1+2\alpha}} dt \leq - \lim_{\alpha \to 1^-} \frac{1}{\alpha} \int_{0}^{1/2} t^{-1+2\alpha} dt = -\infty$$

and the fact that exists a constant $t_0$ such that

$$\int_{1/2}^{+\infty} \frac{\log |1 - t^2|}{t^{1+2\alpha}} dt \leq t_0, \quad \text{for all } \alpha \in (1/2, 1).$$

The second limit in (2.7) follows from

$$\lim_{\alpha \to 0^+} \frac{1}{\alpha} \int_{1/2}^{+\infty} \frac{\log |1 - t^2|}{t^{1+2\alpha}} dt \geq \log 3 \lim_{\alpha \to 0^+} \frac{1}{\alpha} \int_{1/2}^{+\infty} t^{-1-2\alpha} dt = +\infty$$

and the fact that there exists a constant $t_1$ such that

$$\int_{0}^{1/4} \frac{\log |1 - t^2|}{t^{1+2\alpha}} dt \leq t_1, \quad \text{for all } \alpha \in (0, 1/2).$$

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On the other hand we claim that
\[ T'(\alpha) = -2 \int_0^{+\infty} \frac{\log |1-t^2|}{t^{1+2\alpha}} \log t \, dt < 0, \quad \alpha \in (0, 1). \tag{2.8} \]

In fact, since \( \log |1-t^2| \log t \) is negative only for \( t \in (1, \sqrt{2}) \), we have
\[
\int_0^{+\infty} \frac{\log |1-t^2|}{t^{1+2\alpha}} \log t \, dt > \int_0^{\sqrt{2}-1} \frac{-t^2}{t^{1+2\alpha}} \log t \, dt + \int_1^{\sqrt{2}} \log(t-1) \log t \, dt \\
\geq \int_0^{\sqrt{2}-1} \frac{-t^2}{t^{1+2\alpha}} \log t \, dt + \int_1^{\sqrt{2}-1} \log(1+t) \log t \, dt \\
= -\int_0^{\sqrt{2}-1} \frac{t^{-2\alpha}}{t^{1+2\alpha}} \log t \, dt + \int_0^{\sqrt{2}-1} \log(1+t) \log t \, dt \\
\geq -\int_0^{\sqrt{2}-1} \frac{t^{-2\alpha}}{t^{1+2\alpha}} \log t \, dt + \int_0^{\sqrt{2}-1} t \log t \, dt > 0.
\]

Then, (2.7) and (2.8) the existence of the desired \( \alpha_0 \in (0, 1) \) with the required properties follows, completing \((i)\) and (2.2) in \((ii)\).

To continue with the proof of our proposition, we study the first limit in (2.3). We assume that there exist a sequence \( \alpha_n \in (0, \alpha_0) \) and \( \tilde{\tau} \in (-1, 0) \) such that
\[
\lim_{n \to +\infty} \alpha_n = \alpha_0 \quad \text{and} \quad \lim_{n \to +\infty} \tau_1(\alpha_n) = \tilde{\tau}
\]
and so \( c(\tilde{\tau}) = 0 \). Moreover \( c(0) = 0 \) and \( c'(0) = T(\alpha_0) = 0 \), contradicting the strict convexity of \( c \) given by (2.4). Next we prove the second limit in (2.3). We proceed by contradiction, assuming that there exist a sequence \( \{\alpha_n\} \subset (0, 1) \) and \( \tilde{\tau} \in (-1, 0) \) such that
\[
\lim_{n \to +\infty} \alpha_n = 0 \quad \text{and} \quad \tau_1(\alpha_n) \geq \tilde{\tau} > -1, \quad \text{for all} \ n \in \mathbb{N}.
\]

Then there exist \( C_1, C_2 > 0 \), depending on \( \tilde{\tau} \), such that
\[
\int_0^2 \frac{|1-t|\tau_1(\alpha_n) + (1+t)^{\tau_1(\alpha_n)} - 2}{t^{1+2\alpha_n}} \, dt \leq C_1
\]
and
\[
\lim_{n \to +\infty} \int_0^{+\infty} \frac{|1-t|\tau_1(\alpha_n) + (1+t)^{\tau_1(\alpha_n)} - 2}{t^{1+2\alpha_n}} \, dt \leq -C_2 \lim_{n \to +\infty} \int_0^{+\infty} \frac{1}{t^{1+2\alpha_n}} \, dt = -\infty.
\]

Then \( c(\tau_1(\alpha_n)) \to -\infty \) as \( n \to +\infty \), which is impossible since \( c(\tau_1(\alpha_n)) = 0 \).
We finally prove the last statement of the proposition. Since \( \tau_0(\alpha) \in (-1,0) \) is such that \( C(\tau_0(\alpha)) = 0 \) and we have, by definition, that

\[
c(\tau) = C(\tau) + \int_{1}^{+\infty} \frac{(t - 1)^{\tau}}{t^{1+2\alpha}} dt,
\]

we find that \( c(\tau_0(\alpha)) > 0 \), which together with \( \tau \in [2,2] \), implies that \( \tau_0(\alpha) \in (-1, \tau_1(\alpha)) \).

Next we prove the main proposition in this section, which is on the basis of the construction of super and sub-solutions. By hypothesis on the domain \( \Omega \) and the manifold \( C \) of the construction of super and sub-solutions. By hypothesis on the domain \( \delta \), there exists \( \delta > 0 \) such that the distance functions \( d(\cdot) \), to \( \partial \Omega \), and \( D(\cdot) \), to \( C \), are of class \( C^2 \) in \( B_\delta \) and \( A_\delta \), respectively, and \( \text{dist}(A_\delta, B_\delta) > 0 \), where \( A_\delta = \{ x \in \Omega \mid D(x) < \delta \} \) and \( B_\delta = \{ x \in \Omega \mid d(x) < \delta \} \). Now we define the basic function \( V_\tau \) as follows

\[
V_\tau(x) := \begin{cases} 
D(x)^\tau, & x \in A_\delta \setminus C, \\
\frac{1}{C} D(x)^{\tau-2\alpha}, & x \in B_\delta, \\
l(x), & x \in \Omega \setminus (A_\delta \cup B_\delta), \\
0, & x \in \Omega^c, 
\end{cases}
\]

(2.9)

where \( \tau \) is a parameter in \((-1,0)\) and the function \( l \) is positive such that \( V_\tau \) is of class \( C^2 \) in \( \mathbb{R}^N \setminus C \).

**Proposition 2.2** Let \( \alpha_0 \) and \( \tau_1(\alpha) \) be as in Proposition 2.1.

(i) If \( (\alpha, \tau) \in [\alpha_0, 1) \times (-1,0) \) or \( (\alpha, \tau) \in (0, \alpha_0) \times (-1, \tau_1(\alpha)) \), then there exist \( \delta_1 \in (0, \delta] \) and \( C > 1 \) such that

\[
\frac{1}{C} D(x)^{\tau-2\alpha} \leq -(-\Delta)^\alpha V_\tau(x) \leq C D(x)^{\tau-2\alpha}, \quad x \in A_{\delta_1} \setminus C.
\]

(ii) If \( (\alpha, \tau) \in (0, \alpha_0) \times (\tau_1(\alpha), 0) \), then there exist \( \delta_1 \in (0, \delta] \) and \( C > 1 \) such that

\[
\frac{1}{C} D(x)^{\tau-2\alpha} \leq -(-\Delta)^\alpha V_\tau(x) \leq C D(x)^{\tau-2\alpha}, \quad x \in A_{\delta_1} \setminus C.
\]

(iii) If \( (\alpha, \tau) \in (0, \alpha_0) \times \{\tau_1(\alpha)\} \), then there exist \( \delta_1 \in (0, \delta] \) and \( C > 1 \) such that

\[
|(-\Delta)^\alpha V_\tau(x)| \leq C D(x)^{\min(\tau, 2\tau-2\alpha+1)}, \quad x \in A_{\delta_1} \setminus C.
\]

This proposition and its proof has many similarities with Proposition 3.2 in [3], but it has also important differences so we give a complete proof of it.
Proof. By compactness of $\mathcal{C}$, we just need to prove that the corresponding inequality holds in a neighborhood of any point $\bar{x} \in \mathcal{C}$ and, without loss of generality, we may assume $\bar{x} = 0$. For a given $0 < \eta \leq \delta$, we define

$$Q_\eta = (-\eta, \eta) \times B_\eta \subset \mathbb{R} \times \mathbb{R}^{N-1},$$

where $B_\eta$ denotes the ball centered at the origin and with radius $\eta$ in $\mathbb{R}^{N-1}$. We observe that $Q_\eta \subset \Omega$. Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a $C^2$ function such that $(z_1, z') \in \mathcal{C} \cap Q_\delta$ if and only if $z_1 = \varphi(z')$. We further assume that $e_1$ is normal to $C$ at $\bar{x}$ and then there exists $C > 0$ such that $|\varphi(z')| \leq C|z'|^2$ for $|z'| \leq \delta$. Thus, choosing $\eta > 0$ smaller if necessary we may assume that $|\varphi(z')| < \frac{\eta}{2}$ for $|z'| \leq \eta$. In the proof of our inequalities, we will consider a generic point along the normal $x = (x_1, 0) \in A_{\eta/4}$, with $0 < |x_1| < \eta/4$. We observe that $|x - \bar{x}| = D(x) = |x_1|$. By definition we have

$$- (-\Delta)^\alpha V_\tau(x) = \frac{1}{2} \int_{Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy + \frac{1}{2} \int_{\mathbb{R}^N \setminus Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy. \quad (2.10)$$

It is not difficult to see that the second integral is bounded by $C x_1^2$, for an appropriate constant $C > 0$, so that we only need to study the first integral, that from now on we denote by $\frac{1}{2}E(x_1)$.

Our first goal is to obtain positive constants $c_1, c_2$ so that lower bound for $E(x_1)$

$$E(x_1) \geq c_1 c(\tau)|x_1|^{\tau-2\alpha} - c_2 |x_1|^{\min\{\tau, 2\tau-2\alpha+1\}} \quad (2.11)$$

holds, for all $|x_1| \leq \eta/4$. For this purpose we assume that $0 < \eta \leq \delta/2$, then for all $y = (y_1, y') \in Q_\eta$ we have that $x \pm y \in Q_\delta$, so that

$$D(x \pm y) \leq |x_1 \pm y_1 - \varphi(\pm y')|, \quad \text{for all} \ y \in Q_\eta.$$ 

From here and the fact that $\tau \in (-1, 0)$, we have that

$$E(x_1) = \int_{Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy \geq \int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy + \int_{Q_\eta} \frac{J(y) + J(-y)}{|y|^{N+2\alpha}} dy, \quad (2.12)$$

where the functions $I$ and $J$ are defined, for $y \in Q_\eta$, as

$$I(y) = |x_1 - y_1|^\tau + |x_1 + y_1|^\tau - 2 x_1^\tau \quad (2.13)$$

and

$$J(y) = |x_1 + y_1 - \varphi(y')|^\tau - |x_1 + y_1|^\tau. \quad (2.14)$$

In what follows we assume $x_1 > 0$ (the case $x_1 < 0$ is similar). For the first term of the right hand side in $(2.12)$, we have

$$\int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy = x_1^{\tau-2\alpha} \int_{Q_{\frac{\eta}{1+x}}} \frac{|1 - z_1|^\tau + |1 + z_1|^\tau - 2}{|z|^{N+2\alpha}} dz.$$
On one hand we have that, for a constant $c_1$, we have
\[
\int_{\mathbb{R}^N} \frac{|1 - z_1|^\tau + |1 + z_1|^\tau - 2}{|z|^{N+2\alpha}} \, dz = 2c(\tau) \int_{\mathbb{R}^{N-1}} \frac{1}{(|z'|^2 + 1)^{\alpha/2}} \, dz' = c_1 c(\tau),
\]
and, on the other hand, for constants $C_2$ and $C_3$ we have
\[
\left| \int_{z_N \geq \frac{N}{z_1^2}} \frac{|1 - z_1|^\tau + |1 + z_1|^\tau - 2}{|z|^{N+2\alpha}} \, dz \right|
\leq \int_{z_1 \geq \frac{N}{z_1^2}} (|1 - z_1|^\tau + |1 + z_1|^\tau + 2) \, dz_1 \int_{|z'| \geq \frac{N}{z_1^2}} \frac{dz'}{|z'|^{N+2\alpha}} \leq C_2 x_1^{2\alpha}
\]
and
\[
\left| \int_{z_1 \geq \frac{N}{z_1^2}} \int_{\mathbb{R}^{N-1}} \frac{|1 - z_1|^\tau + |1 + z_1|^\tau - 2}{|z|^{N+2\alpha}} \, dz \right|
\leq 2 \int_{z_1 \geq \frac{N}{z_1^2}} \frac{|1 - z_1|^\tau + |1 + z_1|^\tau + 2}{z_1^{1+2\alpha}} \, dz_1 \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |z'|^2)^{\alpha/2}} \, dz' \leq C_3 x_1^{2\alpha}.
\]
Consequently, for an appropriate constant $c_2$
\[
\left| \int_{Q_n} \frac{I(y)}{|y|^{N+2\alpha}} \, dy - c_1 c(\tau)x_1^{\tau-2\alpha} \right| \leq c_2 x_1^\tau. \quad (2.15)
\]
Next we estimate the second term of the right hand side in (2.12). Since
\[
\int_{Q_n} \frac{J(-y)}{|y|^{N+2\alpha}} \, dy = \int_{Q_n} \frac{J(y)}{|y|^{N+2\alpha}} \, dy,
\]
we only need to estimate
\[
\int_{Q_n} \frac{J(y)}{|y|^{N+2\alpha}} \, dy = \int_{B_n} \int_{-\eta}^{\eta} \frac{|x_1 + y_1 - \varphi(y')|^\tau - |x_1 + y_1|^\tau}{(y_1^2 + |y'|^2)^{\alpha/2}} \, dy_1 \, dy'. \quad (2.16)
\]
We notice that $|x_1 + y_1 - \varphi(y')| \geq |x_1 + y_1|$ if and only if
\[
\varphi(y')(x_1 + y_1 - \frac{\varphi(y')}{2}) \leq 0.
\]
From here and (2.16), we have
\[
\int_{Q_n} \frac{J(y)}{|y|^{N+2\alpha}} \, dy \geq \int_{B_n} \int_{-\eta}^{\eta} \frac{|x_1 + y_1 - \varphi(y')|^\tau - |x_1 + y_1|^\tau}{(y_1^2 + |y'|^2)^{\alpha/2}} \, dy_1 \, dy'
+ \int_{B_n} \int_{-\eta}^{\eta} \frac{|x_1 + y_1 - \varphi(y')|^\tau - |x_1 + y_1|^\tau}{(y_1^2 + |y'|^2)^{\alpha/2}} \, dy_1 \, dy'
= E_1(x_1) + E_2(x_1),
\]

where \( \varphi_+(y') = \max\{\varphi(y'), 0\} \) and \( \varphi_-(y') = \min\{\varphi(y'), 0\} \). We only estimate \( E_1(x_1) \) (\( E_2(x_1) \) is similar). Using integration by parts, we obtain

\[
E_1(x_1) = \int_{B_n} \int_{x_1 - \eta}^{\varphi_+(y')} (y_1 - \varphi_+(y'))^\tau - |y_1|\tau \ dy_1 dy' \\
= \int_{B_n} \int_{x_1 - \eta}^{0} (\varphi_+(y') - y_1)^\tau - (-y_1)^\tau \ dy_1 dy' \\
\quad + \int_{B_n} \int_{0}^{\varphi_+(y')} \frac{\varphi_+(y') - y_1^\tau - y_1^\tau}{(y_1 - x_1)^2 + |y'|^2} \ dy_1 dy' \\
= \frac{1}{\tau + 1} \int_{B_n} \left[ \frac{-\varphi_+(y')^\tau + 1}{\eta - x_1 + \varphi_+(y')^\tau + 1} \right] d\eta \\
- \frac{N + 2\alpha}{\tau + 1} \int_{B_n} \int_{x_1 - \eta}^{0} \frac{\varphi_+(y') - y_1^\tau + 1}{(y_1 - x_1)^2 + |y'|^2} \ dy_1 dy' \\
+ \frac{1}{\tau + 1} \int_{B_n} \left[ \frac{-2^{-\tau} \varphi_+(y')^\tau + 1}{(\varphi_+(y')^2 - x_1)^2 + |y'|^2} \right] d\eta \\
+ \frac{N + 2\alpha}{\tau + 1} \int_{B_n} \int_{0}^{\varphi_+(y')} \frac{\varphi_+(y') - y_1^\tau + 1}{(y_1 - x_1)^2 + |y'|^2} \ dy_1 dy' \\
\geq \frac{-2^{-\tau}}{\tau + 1} \int_{B_n} \left[ \frac{\varphi_+(y')^\tau + 1}{(\varphi_+(y')^2 - x_1)^2 + |y'|^2} \right] d\eta \\
+ \frac{N + 2\alpha}{\tau + 1} \int_{B_n} \int_{0}^{\min(\varphi_+(y'), x_1)} \frac{\varphi_+(y') - y_1^\tau + 1}{(y_1 - x_1)^2 + |y'|^2} \ dy_1 dy' \\
= A_1(x_1) + A_2(x_1). \tag{2.17}
\]

In order to estimate \( A(x_1) \), we split \( B_n \) in \( O = \{ y' \in B_n : \varphi_+(y') - x_1 \geq \frac{x_1}{2} \} \) and \( B_n \setminus O \). On one hand we have

\[
\int_{O} \frac{|y'|^{2\tau + 2}}{(\varphi_+(y')^2 - x_1)^2 + |y'|^2} \ dy' \leq x_1^{2\tau - 2\alpha + 1} \int_{B_n \setminus x_1} \frac{|z'|^{2\tau + 2}}{(1/4 + |z'|^2)^{N + 2\alpha}} \ dz' \\
\leq C(x_1^{2\tau - 2\alpha + 1} + x_1). 
\]

On the other hand, for \( y' \in B_n \setminus O \) we have that \( |y'| \geq c_1 \sqrt{x_1} \), for some constant \( c_1 \), and then

\[
\int_{B_n \setminus O} \frac{|y'|^{2\tau + 2}}{(\varphi_+(y')^2 - x_1)^2 + |y'|^2} \ dy' \leq \int_{B_n \setminus B_n \setminus \sqrt{x_1}} |y'|^{2\tau + 2 - N - 2\alpha} \ dy'.
\]
Thus, for some \( C > 0 \),
\[
A_1(x_1) \geq -C x_1^{\min\{\tau, 2\tau - 2\alpha + 1\}}. \tag{2.18}
\]

Next we estimate \( A_2(x_1) \):
\[
A_2(x_1) \geq \frac{2(N + 2\alpha)}{\tau + 1} \int_{B_n} \int_{0}^{x_1} \varphi_+ (y')^{\tau + 1} (y_1 - x_1) \left( (y_1 - x_1)^2 + |y'|^{N + 2\alpha + 1} \right) dy_1 dy'
\]
\[
\geq C \int_{B_n} \int_{0}^{x_1} \frac{|y'|^{2\tau + 2} (y_1 - x_1)}{(y_1 - x_1)^2 + |y'|^{N + 2\alpha + 1}} dy_1 dy'
\]
\[
\geq C x_1^{2\tau - 2\alpha + 1} \int_{B_n/x_1} \int_{0}^{1} \frac{|z'|^{2\tau + 2} (z_1 - 1)}{(z_1 - 1)^2 + |z'|^{N + 2\alpha + 1}} dz_1 dz'
\]
\[
\geq -C_1 x_1^{\min\{\tau, 2\tau - 2\alpha + 1\}},
\]
for some \( C, C_1 > 0 \). From here, (2.17) and (2.18) we obtain, for some \( C > 0 \)
\[
E_1(x_1) \geq -C x_1^{\min\{\tau, 2\tau - 2\alpha + 1\}}. \tag{2.19}
\]

Using the similar estimate for \( E_2(x_1) \), we obtain
\[
\int_{Q_\eta} \frac{J(y) + J(-y)}{|y|^{N + 2\alpha}} dy \geq -C x_1^{\min\{\tau, 2\tau - 2\alpha + 1\}}. \tag{2.20}
\]

Thus, from (2.12), (2.15), (2.19) and noticing that these inequalities also hold with \( x_1 < 0 \) with the obvious changes, we conclude the lower bound for \( E(x_1) \) we gave in (2.11). Our second goal is to get an upper bound for \( E(x_1) \) and for this, we first recall Lemma 3.1 in [3] to obtain
\[
D(x \pm y)^\tau \leq (x_1 \pm y_1 - \varphi(y'))^\tau (1 + C|y'|^2), \text{ for all } |x_1| \leq \eta/4, y = (y_1, y') \in Q_\eta.
\]

From here we see that
\[
E(x_1) \leq \int_{Q_\eta} \frac{I(y)}{|y|^{N + 2\alpha}} dy + \int_{Q_\eta} \frac{J(y) + J(-y)}{|y|^{N + 2\alpha}} dy
\]
\[
+ C \int_{Q_\eta} \frac{I(y) + J(y) + J(-y)}{|y|^{N + 2\alpha}} |y'|^2 dy. \tag{2.20}
\]

We denote by \( E_3(x_1) \) the third integral above. The first integral was studied in (2.15), so we study the second integral and that we only need to consider
the term $J(y)$, since the other is completely analogous. We see that $|x_1 + y_1 - \varphi(y')| \leq |x_1 + y_1|$ if and only if

$$\varphi(y')(x_1 + y_1 - \varphi(y') \frac{1}{2}) \geq 0.$$ 

As before, we will consider only the case $x_1 > 0$, since the other one is analogous. From (2.16) we have

$$\int_{Q_n} \frac{J(y)}{|y|^{N+2\alpha}} dy \leq \int_{B_n} \int_{-\eta}^{-x_1 + \frac{\varphi(y')}{2}} \frac{|x_1 + y_1 - \varphi(y')|^\tau - |x_1 + y_1|^\tau}{(y_1^2 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy_1 dy'$$

$$+ \int_{B_n} \int_{-x_1 + \frac{\varphi(y')}{2}}^{x_1 - \frac{\varphi(y')}{2}} \frac{|x_1 + y_1 - \varphi(y')|^\tau - |x_1 + y_1|^\tau}{(y_1^2 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy_1 dy'$$

$$= F_1(x_1) + F_2(x_1).$$

Next we estimate $F_1(x_1)$ ($F_2(x_1)$ is similar), using integration by parts

$$F_1(x_1)$$

$$= \int_{B_n} \int_{x_1 - \eta}^{x_1 - \frac{\varphi(y')}{2}} \frac{|y_1 - \varphi(y')|^\tau - |y_1|^\tau}{((y_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy_1 dy'$$

$$+ \int_{B_n} \int_{x_1 - \eta}^{x_1 - \frac{\varphi(y')}{2}} \frac{\varphi(y') - y_1)^\tau - (y_1)^\tau}{((y_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy_1 dy'$$

$$= \frac{1}{\tau + 1} \int_{B_n} \int_{x_1 - \eta}^{x_1 - \frac{\varphi(y')}{2}} \frac{(-\varphi(y')^\tau + 1)}{(y_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy_1 dy'$$

$$- \frac{N + 2\alpha}{\tau + 1} \int_{B_n} \int_{x_1 - \eta}^{x_1 - \frac{\varphi(y')}{2}} \frac{\varphi(y') - y_1)^\tau + 1 - (y_1)^\tau + 1}{((y_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy_1 dy'$$

$$+ \frac{1}{\tau + 1} \int_{B_n} \int_{x_1 - \eta}^{x_1 - \frac{\varphi(y')}{2}} \frac{2\tau - (\varphi(y')^\tau + 1)}{((y_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy_1 dy'$$

$$+ \frac{N + 2\alpha}{\tau + 1} \int_{B_n} \int_{x_1 - \eta}^{x_1 - \frac{\varphi(y')}{2}} \frac{(y_1 - \varphi(y'))^\tau + 1 - (y_1)^\tau + 1}{((y_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy_1 dy'$$

$$\leq \frac{1}{\tau + 1} \int_{B_n} \int_{x_1 - \eta}^{x_1 - \frac{\varphi(y')}{2}} \frac{2\tau - (\varphi(y')^\tau + 1)}{((y_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy_1 dy' = B(x_1).$$

Since $(x_1 - \frac{\varphi(y')}{2} \geq x_1^2$, we have

$$B(x_1) \leq \frac{2\tau - (\varphi(y')^\tau + 1)}{\tau + 1} \int_{B_n} \frac{(\varphi(y')^\tau + 1)}{(x_1 + |y'|^2)^{\frac{N+2\alpha}{4}}} dy'.$$
\[ \leq C \int_{B_n} \frac{|y'|^{2\tau + 2}}{(x_1^2 + |y'|^2)^{\frac{N + 2\alpha}{2}}} \, dy' \leq C x_1^{\min\{\tau, 2\tau - 2\alpha + 1\}}, \]

for some \( C > 0 \) independent of \( x_1 \). Thus we have obtained that

\[ F_1(x_1) \leq C x_1^{\min\{\tau, 2\tau - 2\alpha + 1\}}. \quad (2.21) \]

Similarly, we can get an analogous estimate for \( F_2(x_1) \) and these two estimates imply

\[ \int_{Q_n} \frac{J(y) + J(-y)}{|y|^{N + 2\alpha}} \, dy \leq C x_1^{\min\{\tau, 2\tau - 2\alpha + 1\}}. \quad (2.22) \]

Finally we obtain

\[ \int_{Q_n} \frac{I(y)}{|y|^{N + 2\alpha}} |y'|^2 \, dy = x_1^{\tau - 2\alpha + 2} \int_{Q_n} \frac{|1 - z_1|^\tau + |1 + z_1|^\tau - 2}{|z|^{N + 2\alpha}} |z'|^2 \, dz \leq C x_1^{\min\{\tau, \tau - 2\alpha + 2\}} \]

and, in a similar way,

\[ \int_{Q_n} \frac{J(y)|y'|^2}{|y|^{N + 2\alpha}} \, dy \leq C x_1^{\min\{\tau, 2\tau - 2\alpha + 1\}}. \]

From the last two inequalities we obtain

\[ E_3(x_1) \leq C x_1^{\min\{\tau, 2\tau - 2\alpha + 1\}}. \quad (2.23) \]

Then, taking into account \( (2.20), (2.15), (2.22), (2.23) \) and considering also the case \( x_1 < 0 \), we obtain

\[ E(x_1) \leq c_1 c(\tau)|x_1|^{\tau - 2\alpha} + c_2 |x_1|^{\min\{\tau, 2\tau - 2\alpha + 1\}}. \quad (2.24) \]

From inequalities \( (2.11), (2.24) \) and Proposition \( 2.1 \) the result follows. \( \square \)

### 3 Existence of large solution

This section is devoted to use Proposition \( 2.2 \) to prove the existence of solution of problem \( (1.8) \). To this purpose, our main goal is to construct appropriate sub-solution and super-solution of problem \( (1.8) \) under the hypotheses of Theorem \( 1.1 \) (i), (ii) and Theorem \( 1.2 \) (i).

We begin with a simple lemma that reduces the problem to find them only in \( A_\delta \setminus C \).
Lemma 3.1 Let $U$ and $W$ be classical ordered super and sub-solution of (1.8) in the sub-domain $A_\delta \setminus C$. Then there exists $\lambda$ large such that $U_\lambda = U + \lambda \bar{V}$ and $W_\lambda = W - \lambda \bar{V}$, are ordered super and sub-solution of (1.8), where $\bar{V}$ is the solution of
\[
\begin{align}
(-\Delta)^{\alpha} \bar{V}(x) &= 1, \quad x \in \Omega, \\
\bar{V}(x) &= 0, \quad x \in \Omega^c.
\end{align}
\] (3.1)

Remark 3.1 Here $U, W : \mathbb{R}^N \rightarrow \mathbb{R}$ are classical ordered of super and sub-solution of (1.8) in the sub-domain $A_\delta \setminus C$ if $U$ satisfies
\[
(-\Delta)^{\alpha} U + |U|^{p-1} U \geq 0 \quad \text{in} \quad A_\delta \setminus C
\]
and $W$ satisfies the reverse inequality. Moreover, they satisfy
\[
U \geq W \quad \text{in} \quad \Omega \setminus C, \quad \liminf_{x \in \Omega \setminus C, x \rightarrow C} W(x) = +\infty, \quad U = W = 0 \quad \text{in} \quad \Omega^c.
\]

Proof. Notice that by the maximum principle $\bar{V}$ is nonnegative in $\Omega$, therefore $U_\lambda \geq U$ and $W_\lambda \leq W$, so they are still ordered. In addition $U_\lambda$ satisfies
\[
(-\Delta)^{\alpha} U_\lambda + |U_\lambda|^{p-1} U_\lambda \geq (-\Delta)^{\alpha} U + |U|^{p-1} U + \lambda > 0 \quad \text{in} \quad \Omega \setminus C.
\]
This inequality holds because of our assumption in $A_\delta \setminus C$ and the fact that $(-\Delta)^{\alpha} U + |U|^{p-1} U$ is continuous in $\Omega \setminus A_\delta$ and by taking $\lambda$ large enough.

By the same type of arguments we find that $W_\lambda$ is a sub-solution. \(\square\)

Proof of existence results in Theorem 1.1 (i) and Theorem 1.2 (i). We define
\[
U_\mu(x) = \mu V_\tau(x) \quad \text{and} \quad W_\mu(x) = \mu V_\tau(x), \quad x \in \mathbb{R}^N \setminus C,
\] (3.2)
where $V_\tau$ is defined in (2.9) with $\tau = -\frac{2\alpha}{q-1}$

1. $U_\mu$ is Super-solution. By hypotheses of Theorem 1.1 (i) and Theorem 1.2 (i), we notice that
\[
\tau \in (-1, 0), \quad \text{for} \quad \alpha \in [\alpha_0, 1),
\]
\[
\tau \in (-1, \tau_1(\alpha)), \quad \text{for} \quad \alpha \in (0, \alpha_0)
\]
and $\tau p = \tau - 2\alpha$, then we use Proposition 2.2 part (i) to obtain that there exist $\delta_1 \in (0, \delta]$ and $C > 1$ such that
\[
(-\Delta)^p U_\mu(x) + U_\mu^p(x) \geq -C\mu D(x)^{\tau - 2\alpha} + \mu^p D(x)^{\tau p}, \quad x \in A_{\delta_1} \setminus C.
\]
Then there exist $\mu_1 > 1$ such that for $\mu \geq \mu_1$, we have
\[ (-\Delta)^\alpha U_\mu(x) + U_\mu^p(x) \geq 0, \ x \in A_{\delta_1} \setminus C. \]

2. $W_\mu$ is Sub-solution. We use Proposition 2.2 part (i) to obtain that there exist $\delta_1 \in (0, \delta]$ and $C > 1$ such that for $x \in A_{\delta_1} \setminus C$, we have
\[ (-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \leq -\frac{\mu}{C} D(x)^{\tau - 2\alpha} + \mu^p D(x)^{\alpha p} \]
\[ \leq \left( \frac{-\mu}{C} + \mu^p \right) D(x)^{\tau - 2\alpha}. \]

Then there exists $\mu_3 \in (0, 1)$ such that for all $\mu \in (0, \mu_3)$, it has
\[ (-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \leq 0, \ x \in A_{\delta_1} \setminus C. \]

To conclude the proof we use Lemma 3.1 and Proposition 2.2.

Prove of Theorem 1.1 (ii). For any given $t > 0$, we denote
\[ U(x) = tV_{\tau_1(\alpha)}(x), \ x \in \mathbb{R}^N \setminus C, \]
\[ W_\mu(x) = tV_{\tau_1(\alpha)}(x) - \mu V_{\bar{\tau}}(x), \ x \in \mathbb{R}^N \setminus C \]
where $\bar{\tau} = \min\{\tau_1(\alpha)p + 2\alpha, \frac{1}{2}\tau_1(\alpha)\} < 0$. By (1.12), we have
\[ \bar{\tau} \in (\tau_1(\alpha), 0), \ \bar{\tau} - 2\alpha < \min\{\tau_1(\alpha), 2\tau_1(\alpha) - 2\alpha + 1\} \text{ and } \bar{\tau} - 2\alpha < \tau_1(\alpha)p. \] (3.3)

1. $U$ is Super-solution. We use Proposition 2.2 (iii) to obtain that for any $x \in A_{\delta_1} \setminus C$,
\[ (-\Delta)^\alpha U(x) + U^p(x) \geq -Ct D(x)^{\min\{\tau_1(\alpha), 2\tau_1(\alpha) - 2\alpha + 1\}} + t^p D(x)^{\tau_1(\alpha)p}, \]

then there exists $\delta_2 \in (0, \delta_1]$ such that
\[ (-\Delta)^\alpha U(x) + U^p(x) \geq 0, \ x \in A_{\delta_2} \setminus C. \]

2. $W_\mu$ is Sub-solution. We use Proposition 2.2 (ii) and (iii) to obtain that for $x \in A_{\delta_1} \setminus C$,
\[ (-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \leq Ct D(x)^{\min\{\tau_1(\alpha), 2\tau_1(\alpha) - 2\alpha + 1\}} \]
\[ -\frac{\mu}{C} D(x)^{\bar{\tau} - 2\alpha} + t^p D(x)^{\tau_1(\alpha)p}. \]

Then there exists $\delta_2 \in (0, \delta_1]$ such that for any $\mu \geq 1$, we have
\[ (-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \leq 0, \ x \in A_{\delta_2} \setminus C. \]

To conclude the proof we use Lemma 3.1 and Proposition 2.2.
4 Uniqueness and nonexistence

We prove the uniqueness statement by contradiction. Assume that \( u \) and \( v \) are solutions of problem (1.8) satisfying (1.11). Then there exist \( C_0 \geq 1 \) and \( \bar{\delta} \in (0, \delta) \) such that

\[
\frac{1}{C_0} \leq v(x)D(x)^{-\tau}, \quad u(x)D(x)^{-\tau} \leq C_0, \quad \forall x \in A_{\bar{\delta}} \setminus C,
\]  

(4.4)

where \( \tau = -\frac{2\alpha}{p-1} \). We denote

\[
A = \{ x \in \Omega \setminus C \mid u(x) > v(x) \}. \quad (4.5)
\]

Then \( A \) is open and \( A \subset \Omega \). Then the uniqueness in Theorem 1.2 (i) and Theorem 1.1 (i) is a consequence of the following result:

**Proposition 4.1** Under the hypotheses of Theorem 1.2 (i) and Theorem 1.1 (i), we have

\[
A = \emptyset.
\]

**Proof.** The procedure of proof is similar as Section §5 in [3], noting that we need to replace \( d(x) \) by \( D(x) \) and \( \partial \Omega \) by \( C \). \hfill \Box

From Proposition 4.1 we can prove uniqueness part in Theorem 1.1 (i) and Theorem 1.2 (i).

The final goal in this note is to consider the nonexistence of solutions \( s \) of problem (1.8) under the hypotheses of Theorem 1.1 (iii) and Theorem 1.2 (ii).

**Proposition 4.2** Under the hypotheses of Theorem 1.1 (iii) and Theorem 1.2 (ii), we assume that \( U_1 \) and \( U_2 \) are both sub-solutions (or both super-solutions) of (1.8) satisfying that \( U_1 = U_2 = 0 \) in \( \Omega^c \) and

\[
0 < \liminf_{x \in \Omega \setminus C, \ x \to C} U_1(x)D(x)^{-\tau} \leq \limsup_{x \in \Omega \setminus C, \ x \to C} U_1(x)D(x)^{-\tau}
\]

\[
< \liminf_{x \in \Omega \setminus C, \ x \to C} U_2(x)D(x)^{-\tau} \leq \limsup_{x \in \Omega \setminus C, \ x \to C} U_2(x)D(x)^{-\tau} < +\infty,
\]

for \( \tau \in (-1, 0) \). For the case \( \tau p > \tau - 2\alpha \), we further assume that

(i) if \( U_1, U_2 \) are sub-solutions, there exist \( C > 0 \) and \( \tilde{\delta} > 0 \),

\[
(-\Delta)^\alpha U_2(x) \leq -CD(x)^{\tau-2\alpha}, \quad x \in A_{\tilde{\delta}} \setminus C; \quad (4.6)
\]

or

(ii) if \( U_1, U_2 \) are super-solutions, there exist \( C > 0 \) and \( \tilde{\delta} > 0 \),

\[
(-\Delta)^\alpha U_1(x) \geq CD(x)^{\tau-2\alpha}, \quad x \in A_{\tilde{\delta}} \setminus C. \quad (4.7)
\]
Then there doesn’t exist any solution $u$ of (1.8) such that
\[
\limsup_{x \in \Omega \setminus C, x \to C} \frac{U_1(x)}{u(x)} < 1 < \liminf_{x \in \Omega \setminus C, x \to C} \frac{U_2(x)}{u(x)}. \tag{4.8}
\]

**Proof.** The proof is similar as Proposition 6.1 in [3], noting again that we need to replace $d(x)$ by $D(x)$ and $\partial \Omega$ by $C$.

With the help of Proposition 2.2 for given $t_1 > t_2 > 0$, we construct two sub-solutions (or both super-solutions) $U_1$ and $U_2$ of (1.8) such that
\[
\lim_{x \in \Omega \setminus C, x \to C} U_1(x) D(x)^{-\tau} = t_1, \quad \lim_{x \in \Omega \setminus C, x \to C} U_2(x) D(x)^{-\tau} = t_2.
\]
So what we have to do is to prove that for any $t > 0$, we can construct super-solution (sub-solution) of problem (1.8).

**Proof of Theorem 1.1 (iii) and Theorem 1.2 (ii).** We divide our proof of the nonexistence results into several cases under the assumption $p > 1$.

**Zone 1:** We consider $\tau \in (\tau_1(\alpha), 0)$ and $\alpha \in (0, \alpha_0)$. By Proposition 2.2 (ii), there exists $\delta_1 > 0$ such that
\[
(-\Delta)^\alpha V_\tau(x) \geq \frac{1}{C} D(x)^{r-2\alpha}, \quad x \in A_{\delta_1} \setminus C. \tag{4.9}
\]
Since $V_\tau$ is $C^2$ in $\Omega \setminus C$, then there exists $C > 0$ such that
\[
|(-\Delta)^\alpha V_\tau(x)| \leq C, \quad x \in \Omega \setminus A_{\delta_1}. \tag{4.10}
\]
Let $\bar{U} := V_\tau + CV$ in $\mathbb{R}^N \setminus C$, then we have $\bar{U} > 0$ in $\Omega \setminus C$,
\[
(-\Delta)^\alpha \bar{U} \geq 0 \quad \text{in} \quad \Omega \setminus C \quad \text{and} \quad (-\Delta)^\alpha \bar{U}(x) \geq \frac{1}{C} D(x)^{r-2\alpha}, \quad x \in A_{\delta_1} \setminus C.
\]
Then, we have that $t\bar{U}$ is super-solution of (1.8) for any $t > 0$. Using Proposition 2.2 we see that there is no solution of (1.8) satisfying (1.14).

**Zone 2:** We consider $\tau - 2\alpha < \tau p$ and
\[
\tau \in \begin{cases} 
(-1, 0), & \alpha \in [\alpha_0, 1), \\
(-1, \tau_1(\alpha)), & \alpha \in (0, \alpha_0).
\end{cases}
\]
Let us define
\[
W_{\mu,t} = tV_\tau - \mu \bar{U} \quad \text{in} \quad \mathbb{R}^N \setminus C,
\]
where $t, \mu > 0$. By Proposition 2.2 (i), for $x \in A_{\delta_1} \setminus C$,
\[
(-\Delta)^\alpha W_{\mu,t}(x) + |W_{\mu,t}|^{p-1} W_{\mu,t}(x) \leq -\frac{t}{C} D(x)^{r-2\alpha} + t^p D(x)^{\tau p}.
\]

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For any fixed $t > 0$, there exists $\delta_2 \in (0, \delta_1]$, for all $\mu \geq 0$,

\[
(-\Delta)^{\alpha} W_{\mu, t}(x) + |W_{\mu, t}|^{p-1} W_{\mu, t}(x) \leq 0, \quad A_{\delta_2} \setminus \mathcal{C}.
\]  

(4.11)

To consider $x \in \Omega \setminus A_{\delta_2}$, in fact, there exists $C_1 > 0$ such that

\[
t|(-\Delta)^{\alpha} V_{\tau}(x)| + t^p V_{\tau}^p(x) \leq C_1, \quad x \in \Omega \setminus A_{\delta_2}
\]

and

\[
(-\Delta)^{\alpha} W_{\mu, t}(x) + |W_{\mu, t}|^{p-1} W_{\mu, t}(x) \leq C_1 t - \mu, \quad x \in \Omega \setminus A_{\delta_2}
\]

For given $t > 0$, there exists $\mu(t) > 0$ such that

\[
(-\Delta)^{\alpha} W_{\mu(t), t}(x) + |W_{\mu(t), t}|^{p-1} W_{\mu(t), t}(x) \leq 0, \quad x \in \Omega \setminus A_{\delta_2}.
\]  

(4.12)

Therefore, together with (4.11) and (4.12), for any given $t > 0$, there subsolutions $W_{\mu(t), t}$ of problem (1.8) and by Proposition 4.2, we see that there is no solution $u$ of (1.8) satisfying (1.14).

**Zone 3:** We consider $\tau - 2\alpha > \tau p$ and

\[
\tau \in \begin{cases} (-1, 0), & \alpha \in [\alpha_0, 1), \\ (-1, \tau_1(\alpha)), & \alpha \in (0, \alpha_0). \end{cases}
\]

We denote that

\[
U_{\mu, t} = t V_{\tau} + \mu \tilde{V} \quad \text{in} \quad \mathbb{R}^N \setminus \mathcal{C},
\]

where $t, \mu > 0$. Here $U_{\mu, t} > 0$ in $\Omega \setminus \mathcal{C}$. By Proposition 2.2 (i),

\[
(-\Delta)^{\alpha} U_{\mu, t}(x) + U_{\mu, t}^p(x) \geq -C t D(x)^{\tau - 2\alpha} + t^p D(x)^{\tau p}, \quad x \in A_{\delta_1} \setminus \mathcal{C}.
\]

For any fixed $t > 0$, there exists $\delta_2 \in (0, \delta_1]$, for all $\mu \geq 0$,

\[
(-\Delta)^{\alpha} U_{\mu, t}(x) + U_{\mu, t}^p(x) \geq 0, \quad x \in A_{\delta_2} \setminus \mathcal{C}.
\]  

(4.13)

For $x \in \Omega \setminus A_{\delta_2}$, we see that $(-\Delta)^{\alpha} V_{\tau}$ is bounded and

\[
(-\Delta)^{\alpha} U_{\mu, t}(x) + U_{\mu, t}^p(x) \geq -C t + \mu.
\]

For given $t > 0$, there exists $\mu(t) > 0$ such that

\[
(-\Delta)^{\alpha} U_{\mu(t), t}(x) + U_{\mu(t), t}^p(x) \geq 0, \quad x \in \Omega \setminus A_{\delta_2}.
\]  

(4.14)
Combining with (4.13) and (4.14), we have that for any $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)\alpha U_{\mu(t),t}(x) + U_{\mu(t),t}^p(x) \geq 0, \quad x \in \Omega \setminus C.$$ 

Therefore, for any given $t > 0$, there is a super-solution $U_{\mu(t),t}$ of problem (1.8) and by Proposition 4.2, we see that there is no solution of (1.8) satisfying (1.14).

We see that Zones 1 and 2 cover Theorem 1.1 part (iii) a) since $\tau > -2\alpha/(p-1)$. From Zones 1, 2 and 3 we cover Theorem 1.1 part (iii) b) since $\tau_1(\alpha) > 2\alpha/(p-1)$. Moreover, from Zone 1 to Zone 3, we cover the parameters in part (iii) c) of Theorem 1.1 since $\tau_1(\alpha) < 2\alpha/(p-1)$. Finally, Theorem 1.2 part ii) can be obtained from Zone 2 and Zone 3. This completes the proof.

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References

[1] J. M. Arrieta and A. Rodríguez-Bernal, Localization on the boundary of blow-up for reaction-diffusion equations with nonlinear boundary conditions, *Comm. Partial Diff. Eqns.*, 29, 1127-1148, 2004.

[2] C. Bandle and M. Marcus, Large solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behaviour, *J. Anal. Math.*, 58, 9-24, 1992.

[3] H. Chen, P. Felmer and A. Quaas, Large solution to elliptic equations involving fractional Laplacian, *Preprint*.

[4] M. Chuaqui, C. Cortázar, M. Elgueta and J. García-Melián, Uniqueness and boundary behaviour of large solutions to elliptic problems with singular weights, *Comm. Pure Appl. Anal.*, 3, 653-662, 2004.

[5] M. del Pino and R. Letelier, The influence of domain geometry in boundary blow-up elliptic problems, *Nonlinear Analysis: Theory, Methods & Applications*, 48(6), 897-904, 2002.
[6] G. Díaz and R. Letelier, Explosive solutions of quasilinear elliptic equations: existence and uniqueness, *Nonlinear Analysis: Theory, Methods & Applications*, 20(2), 97-125, 1993.

[7] Y. Du and Q. Huang, Blow-up solutions for a class of semilinear elliptic and parabolic equations, *SIAM J. Math. Anal.*, 31, 1-18, 1999.

[8] P. Felmer and A. Quaas, Boundary blow up solutions for fractional elliptic equations. Asymptotic Analysis, Volume 78 (3), 123-144, 2012.

[9] J. F. Le Gall, A path-valued Markov process and its connections with parital differential equations. In Proc. First European Congress of Mathematics, Vol. II (A. Joseph, F. Mignot, F. Murat, B. Prum and R. Rentschler, eds.) 185-212, 1994. Birkhäuser, Boston.

[10] J. García-Melián, Nondegeneracy and uniqueness for boundary blow-up elliptic problems, *J. Diff. Eqns.*, 223(1), 208-227, 2006.

[11] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.*, 10, 503-510, 1957.

[12] S. Kim, A note on boundary blow-up problem of $\Delta u = u^p$, *IMA preprint No.*, 18-20, 2002.

[13] V. A. Kondratev, V. A. Nikishkin, Asymptotics near the boundary, of a solution of a singular boundary value problem for a semilinear elliptic equation, *Differential Equations* 26 (1990), 345-348.

[14] A. C. Lazer, P. J. McKenna, Asymptotic behaviour of solutions of boundary blow-up problems, *Differential Integral Equations* 7 (1994), 1001-1019.

[15] C. Loewner and L. Nirenberg, Parital differential equations invariant under conformal or projective transformations, *In Contributions to analysis*, Academic Press, New York, 245-272, 1974.

[16] M. Marcus and L. Véron, Uniqueness of solutions with blow up at the boundary for a class of nonlinear elliptic equations, *C. R. Acad. Sci. Paris sér. I Math.* 317(6), 559-563, 1993.

[17] M. Marcus and L. Véron, Existence and uniqueness results for large solutions of general nonlinear elliptic equation, *J. Evol. Equ.* 3, 637-652, 2003.
[18] M. Marcus and L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, *Ann. Inst. H. Poincaré* 14(2), 237-274, 1997.

[19] R. Osserman, On the inequality $\Delta u = f(u)$, *Pac. J. Math.* 7, 1641-1647, 1957.

[20] L. Véron, Semilinear elliptic equations with uniform blow-up on the boundary, *J. Anal. Math.* 59(1), 231-250, 1992.