METRICS OF PINCHED CURVATURE ON HEINTZE SPACES OF CARNOT-TYPE

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ABSTRACT. Rank-one symmetric spaces have a generalization to a larger class of Lie groups that are one-dimensional extensions of nilpotent groups. By examining some metric properties of these symmetric spaces, we motivate and prove the existence of analogous metrics on Heintze spaces of Carnot-type. As an application of this construction, we see that Heintze spaces with H-metrics are of Iwasawa type, fail to have the Einstein condition, and admit a useful symmetric adjoint action. We also introduce the definition of square-pinched algebras, to generalize the quarter-pinched property explored by Eberlein and Heber, and use H-metrics to prove that the Lie algebras corresponding to Heintze spaces of Carnot-type are square-pinched. We show in a special case that this pinching is optimal, appealing to a result of Belegradek and Kapovitch.

1. Introduction

Rank-one symmetric spaces of non-compact type are some of the most interesting and well-studied objects in mathematics. The most famous of these spaces is hyperbolic n-space, which should rightfully be called real hyperbolic space and denoted RH^n. This is to distinguish it from the other manifolds in this class – KH^n for K either the complex, quaternionic, or octave numbers (special care must be taken in this last case – in particular n must equal 1). A unified approach to the three infinite families is available in Chapter 10 of [BH99] and symmetric spaces in general are explored at length in [Hel01].

We prove the following relationship between the metric structure and Lie structure for these spaces in order to motivate a similar result regarding a generalization outside of this restrictive class. The following is a result which can be obtained from section 3 in Eberlein and Heber [EH96], though we give an independent proof. The definition of the terms will come in the next section.

Proposition 3.2 [EH96] Consider the manifold KH^n viewed as an upper half-space with derived subalgebra heis = K^n ⊕ Im(K), normalized so that the supremum of its sectional curvature is −1. This space has the following curvature properties (V, W ∈ T_pKH^n, V vertical):

\[ K(V, W) = -1 \text{ if } W \in (dL_p)_eK^n \]
\[ K(V, W) = -4 \text{ if } W \in (dL_p)_e\text{Im}(K) \]
In particular, vertical tangent planes, which are defined as having one direction orthogonal to the Heisenberg group, define subspaces isometric to real hyperbolic planes with constant curvature $-1$ inside $KH^n$ when the other vector is in the direction of the first element of the lower central series of $H_n^{n-1}$, and rescaled versions of real hyperbolic planes with curvature $-4$ when this vector is in the direction of the second element. It is true that normally the identification of the tangent space with the associated Lie algebra is a priori only defined at the origin. We take advantage of the isometric homogeneity to define the identification everywhere by left-translation (see Definition 4.2).

Although complex hyperbolic space does have constant holomorphic curvature, the failure to be abelian by its associated nilpotent group is the reason it only satisfies pinched negative curvature when seen as a Lie group over the real numbers. Although one may expect that quaternionic hyperbolic space could have a wider range of curvatures, we will see the reason that they do not is exactly because the associated nilpotent Lie group remains 2-step.

In [Hei74b] Heintze answers the question as to whether these symmetric spaces are the only examples of connected manifolds that are homogeneous and have non-positive curvature. He demonstrates that while they are not, all spaces of this form share some key features. We therefore consider those homogeneous, connected manifolds that admit metrics realizing strictly negative sectional curvature which are complete and non-compact, but may not satisfy the restrictive involutive condition of symmetric spaces. The foundational result that we get from [Hei74b] is that all of these spaces can be expressed as Lie groups that admit a semi-direct product decomposition into a nilpotent Lie group with $\mathbb{R}$. A certain class of these, called Heintze spaces of Carnot-type, will admit metrics that satisfyingly generalize the behavior from Proposition 3.2. The definition of terms will come in later sections.

**Theorem 4.10** For any Heintze space of Carnot-type $M$ with stratification $\oplus_i V_i$, there exists a Riemannian metric such that at any point $p \in M$, if $X, Y$ are layered basis tangent vectors in $T_pM$, then the sectional curvature $K(X, Y)$ satisfies the following conditions:

- $K(A, Y) = -i^2$ for vertical planes such that $Y \in V_i$, and
- $-1 \leq K(X, Y) \leq -s^2$ where $s$ is the step nilpotency of the base group.

In fact, we will define an $H$-metric to be a Riemannian metric on a Heintze space of Carnot-type that satisfies those conditions, and hence we conclude by Proposition 3.2 that symmetric spaces are equipped with such an $H$-metric. One should consider this class of metric the correct higher-step analog of the metrics within the intersection $QP \cap AM$ considered in Section 6 of Eberlein and Heber [EH96].
The existence of $H$–metrics allows us to conclude some strong properties about Heintze-spaces of Carnot type. The following observation comes from an examination of the adjoint action by the vertical direction.

**Theorem 4.14** Heintze spaces of Carnot-type have the property that \( \text{ad } A = D_0, S_0 = 0 \) for all vertical vectors \( A \) when the space is equipped with an $H$–metric, and thus are also of Iwasawa-type in the sense of Wol91.

The existence of $H$–metrics for these spaces speak to their role as a generalization of rank-one symmetric spaces. In this way it may be interesting to examine what other metric properties of symmetric spaces as present in Heintze spaces of Carnot-type equipped with $H$–metrics. To this end, we define a Lie algebra to be square-pinched if it admits an inner product such that the sectional curvatures of the Lie group lie in \([-s^2, -1]\) where \( s \) is the step nilpotency of the derived subgroup.

**Theorem 5.3** Any Lie algebra Lie group is a Heintze space of Carnot-type is an SP-algebra.

To prove this statement, we construct an $H$–metric with certain parameters and check the planes not covered in the definition of an $H$–metric to make sure their curvature lies in the right interval. In a special case we can also demonstrate that this level of pinching is optimal.

**Theorem 5.5** Let \( M \) be a Heintze space of Carnot-type whose derived subgroup admits a lattice. Then \( M \) is not $C$–pinched for any $C > \frac{1}{s^2}$, where \( s \) is the step nilpotency of the derived subgroup.

The proof of Theorem 5.5 result comes almost directly from the main theorem of Belegradek and Kapovitch in BK05, in which they describe the pinching bounds on Riemannian manifolds whose fundamental groups have nilpotent subgroups.

Some bounds on the pinched curvature values for Heintze spaces were already known. Work of Pansu in Section 5 of Pan89 can be adapted to demonstrate such bounds. However, values obtained as a consequence of this work are less sharp than the curvature condition \(-s^2 \leq K \leq -1\) we prove in Theorem 5.3. 

From this condition, we will spend a little time exploring some properties of the isometry group of these spaces, which allow us to observe that Heintze spaces of this type do not admit any finite volume or cocompact lattices – a fact originally due to Heintze Hei74a. From this, we may apply a result from AC99 to observe that admitting a lattice is a necessary condition to be Einstein when the vertical adjoint action is symmetric.

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\(^1\) That these bounds can be obtained from the work in Pan89 was communicated to the author by Gabriel Pallier.
Corollary 7.6. Heintze spaces of Carnot-type equipped with $H$-metrics which are not symmetric are not Einstein manifolds

We also define a family of higher step ($s > 2$) Heintze spaces of Carnot-type which arise as Lie subgroups of the special linear group and explicitly describe an $H$-metric for them.

The proof of the main theorem proceeds by considering the ‘upper half-space model’ of a Heintze space by way of the generalized Cayley transform (see [Nis00]). This transformation is a diffeomorphism to the Heintze space expressed as a semidirect product from a manifold which is the base (derived) group cross the positive reals. This upper half-space thus inherits a class of pullback Riemannian metrics as well as a Lie structure. We then construct a metric on this half-space model with the desired properties, using the induced Lie bracket information from the diffeomorphism. Verification of the condition on the vertical tangent planes is done by way of the structure constants for a Lie group, which directly give an expression for sectional curvature; the horizontal planes are examined using machinery developed in [Hei74b] and [EH96]. We continue on to show that this metric can be chosen with a particular parameter in order to conclude pinched curvature properties.

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2. Background

We begin with some definitions to understand the context that these spaces arise from. A reader familiar with Heintze spaces of Carnot-type may safely skip to Section 3. As a convention, we will use capital letters to denote tangent vectors, or elements of the Lie algebra when appropriate. As usual, fraktur letters will denote the Lie (sub)algebras, and $g$ (or $h$ occasionally) will refer to a Riemannian metric.

Definition 2.1. A metric space $X$ is homogeneous if $\text{Isom}(X)$ acts transitively on $X$.

This definition is sometimes referred to as isometrically homogeneous, to distinguish this property from groups acting in different ways. A foundational result in the study of homogeneous spaces is the following theorem of Kobayashi.

Theorem 2.2. [Kob62] A homogeneous Riemannian manifold with nonpositive sectional curvature and negative definite Ricci tensor is simply connected.

From here, significant attention was paid to the case of strictly negative sectional curvature.

Definition 2.3. A connected Riemannian manifold of strictly negative sectional curvature which is homogeneous is called a Heintze space.
The name Heintze space derives from the following theorem of Heintze.

**Theorem 2.4.** [Hei74b] Every Heintze space is isometric to a connected, solvable Lie group with a left-invariant metric. Furthermore, a Heintze space may be represented as a solvable Lie algebra with an inner product.

Heintze recognizes the second part of the above claim as following from the main result in [Hei74b] and Theorem 2.2 from Kobayashi. Because of this identification, we will often move between inner products on Lie algebras and left-invariant metrics on the associated manifold. In [Hei74b], it is further observed that for such a solvable Lie group $g$, it is the case that $[g, g] \perp$ is one dimensional and that $[g, g]$ is nilpotent.

**Definition 2.5.** Let $G$ be a connected, simply connected, solvable Lie group with Lie algebra $g$. The Lie algebra $[g, g]$ is called the derived subalgebra and the Lie group associated to $[g, g]$ is called the base group. A vector in $[g, g] \perp$ is called vertical. A vertical plane is a 2-dimensional subspace of $g$ that contains a vertical vector.

Due to the identification of Heintze spaces with Lie groups, the term Heintze space is often worded as Heintze group. We avoid this wording so as to remember we are considering these manifolds primarily as metric spaces.

Some important properties of these spaces come from Theorem 3 in [Hei74b], which tells us that a solvable Lie algebra is one that comes from a Heintze space exactly when its derived subalgebra has codimension 1 and admits a contracting (equivalently expanding) automorphism; see below for a special kind of such an automorphism. In certain cases of interest to us, this subalgebra will have stronger properties which allows us to pick a natural candidate for a metric on the associated manifold.

**Definition 2.6.** A stratification on a nilpotent Lie algebra $n$ is a decomposition

$$n = V_1 \oplus V_2 \oplus \ldots \oplus V_s$$

with the following properties:

- $[V_i, V_j] = V_{j+1}$
- $[V_i, V_s] = 0$

The subspaces $V_i$ are called the layers and $V_1$ the horizontal layer, and the simply connected, connected Lie group associated to $n$ is called a Carnot group. More information on these groups is available in [LD17].

A stratification as above is also sometimes referred to as a Carnot grading. Observe that a consequence of the above conditions is that $[V_i, V_j] \subset V_{i+j}$. The focus on this specific subclass can be justified by the following result of Pansu.

**Theorem 2.7.** [Pan83] Let $G$ be a nilpotent Lie group equipped with a left-invariant metric. Then the asymptotic cone of $G$ is a Carnot group.

We give some basic examples of these Carnot groups.
Example 2.8. The space $\mathbb{E}^n$ when viewed as a Lie group with associated bracket $[A, B] = 0$ may clearly be given a stratification with $s = 1$.

Example 2.9. The (real) Heisenberg group, $\mathcal{H}$, is a matrix group consisting of $3 \times 3$ matrices of the form, where $a, b, c \in \mathbb{R}$:

\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}
\]

As a Lie group, $\mathcal{H}$ is 3-dimensional and nilpotent of step 2. The associated Lie algebra looks similar, and is the set of matrices of the form:

\[
\begin{pmatrix}
0 & x & z \\
0 & 0 & y \\
0 & 0 & 0
\end{pmatrix}
\]

Together with bilinearity and antisymmetry, the entire Lie bracket is determined by the relations:

$[X, Z] = 0, [X, Y] = 0, [X, Y] = Z$

where the capital letters represent the matrix with a one in the associated place and zeros elsewhere. The stratification on this algebra can be found by allowing $\mathcal{V}_1$ to be the subspace spanned by the vectors $X, Y$ and $\mathcal{V}_2$ spanned by $Z$.

Example 2.10. Higher dimension Heisenberg groups can be obtained by replacing the values $x, y$ by real-valued row and column vectors respectively. In this way we obtain a 2-step nilpotent Lie group consisting of $(k+2) \times (k+2)$ matrices, where $k$ is the size of those vectors.

We should note from this setup that the degree of the stratification matches the step of nilpotency of the Lie group, recalling that $[\mathfrak{g}, \mathfrak{g}]$ will always be nilpotent. However, not every nilpotent Lie group admits a stratification (see Examples 2.7 and 2.8 in [LD17]).

Definition 2.11. A Heintze space of Carnot-type is a connected, simply connected, solvable Lie group such that the derived subalgebra admits a stratification and the adjoint action of vectors in the vertical direction respect the stratification in the sense that each layer is an eigenspace with distinct eigenvalue.

It is important to note here that we have dropped the assumption of any metric in this definition. By equipping them with a left-invariant metric of negative curvature (which is always possible [Hei74b]) we see how Heintze spaces of Carnot-type become (of course!) a subclass of Heintze spaces. A good explanation of the relationship between these classes is available in the introduction of [Xie13]. Note that in that paper, Xie considers more closely the base group rather rather than its associated Heintze space. In
particular, all Carnot groups admit automorphisms which produce Heintze spaces of Carnot-type. More background on Heintze spaces of Carnot-type is available in [CM18] and [Cor18].

3. Sectional Curvature in Symmetric Spaces

It is a well known fact (a good exposition is available in [Luk14]) that both $\mathbb{RH}^n$ and $\mathbb{CH}^n$ can be viewed as Heintze spaces of Carnot-type obtained using a particular left-invariant metric with base group $E_{n-1}$ and $H_{n-1}$ respectively. In order to study quaternionic hyperbolic space and the octave hyperbolic plane, we define the more general framework that the two above examples fit in to.

**Definition 3.1** (Heisenberg group over $K^n$). Let $n \in \mathbb{N}$ and $K$ be one of the following algebras: $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{QU}$ or the octonions $\mathbb{O}$. If $K = \mathbb{O}$ we require that $n = 1$, due to the restrictions on the operation of the normed division algebra $\mathbb{O}$. The imaginary space $\text{Im}(K) \subset K$ is the real vector space consisting of the ‘purely imaginary’ elements of $K$, which has real dimension $\text{dim}(K) - 1 \in \{0, 1, 3, 7\}$.

The Lie algebra $\mathfrak{hei}(K^n)$ is defined to be the central extension of abelian Lie algebras

$$0 \rightarrow \text{Im}(K) \rightarrow \mathfrak{hei}(K^n) \rightarrow K^n \rightarrow 0$$

with underlying real vector space $K^n \oplus \text{Im}(K)$ and the following Lie bracket, recalling that conjugation for $\mathbb{C}, \mathbb{QU}, \mathbb{O}$ consists of flipping the sign of the coefficients for any imaginary element, and is denoted by $a + bj = a - bj$, $a, b \in \mathbb{R}$, $j \in \text{Im}(K)$.

$$[V, W] = \text{Im}\left(\sum_{i=1}^{n} V_i W_i\right) \quad \text{for all } V, W \in K^n.$$  

$$[Z, W] = 0 \quad \text{for all } Z \in \text{Im}(K)$$

We observe that $\mathfrak{hei}$ is a 2-step nilpotent Lie algebra with the following stratification: $V_1 = K^n$ and $V_2 = \text{Im}(K)$. While this is a different view of the Heisenberg group than in Examples 2.9 and 2.10, they are canonically identified, and this view provides the relationship desired to the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{QU}$, $\mathbb{O}$.

We use the properties that sectional curvature has in symmetric spaces to motivate the definition of the metric we will put on Heintze spaces of Carnot-type. In particular we make the following observation. Recall that a vertical vector is one such that $V \in \mathfrak{hei}^\perp$, the space of which must be one-dimensional by [Hei74b]. The following proposition can be found to be a consequence of Proposition 3.16 in [EH96], proven for generic 3-step Carnot solvmanifolds. It should be stated that Eberlein and Heber’s 3-step Carnot

\footnote{The author greatly regrets the lack of a better symbol for this object, given the reservation of $Q$. We will avoid using $H$, which we reserve for hyperbolic space.}
METRICS OF PINCHED CURVATURE ON HEINTZE SPACES OF CARNOT-TYPE

solvmanifold is, to us, a Heintze space of Carnot-type with base group of step 2. Our (independent) proof will take advantage of the special structure present in rank-one symmetric spaces.

Proposition 3.2. [EH96] Consider the manifold $KH^n$ for $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}U\}$ viewed as an upper half-space with derived subalgebra $\mathfrak{heis} = \mathbb{K}^n \oplus \text{Im}(K)$, normalized so that the supremum of its sectional curvature is $-1$. This space has the following curvature properties ($V, W \in T_pKH^n, V$ vertical):

$$K(V, W) = -1 \text{ if } W \in (dL_p)_e \mathbb{K}^n$$

$$K(V, W) = -4 \text{ if } W \in (dL_p)_e \text{Im}(K)$$

Note that in the statement we are taking advantage of the Lie group structure on these spaces. In particular, $(dL_p)_e$ represents the differential of left-translation by any element $p$. This may be immediately forgotten for the purpose of the proof, which we see below, but makes the statement more general. We should also recall that $KH^n$ will have real dimension $kn$ for $k$ the appropriate value in $\{1, 2, 4\}$.

Proof. Because of the isometric homogeneity of symmetric spaces, it suffices to show this property at the origin and may therefore treat with only with vectors that we consider living in $\mathfrak{heis} = \mathbb{K}^n \oplus \text{Im}(K)$, which will be identified with the tangent space at the origin. Due to the fact that real hyperbolic space has constant curvature $-1$ and abelian derived subalgebra, there is nothing to prove. We therefore restrict our attention to $K \neq \mathbb{R}$.

For these cases, we turn to [BH99]. We must first see how the authors define $K$–hyperbolic space. Bridson and Haefliger define the following form on $\mathbb{K}^{n+1}$:

$$Q'(x, y) = -x_1 y_{n+1} - x_{n+1} y_1 + \sum_{i=2}^{n} x_i y_i$$

Equipped with this form, they write the associated space as $\mathbb{K}^{n,1}$, recording the signature of $Q'$. Further, define $\mathbb{K}P^n$ as the projective space over $\mathbb{K}$; the space $\mathbb{K}^{n,1} - \{0\} / \sim$ where

$$(x_1, \ldots, x_{n+1}) \sim (\lambda x_1, \ldots, \lambda x_{n+1}) \text{ for all } \lambda \in \mathbb{K} - \{0\}$$

We now define $KH^n$ as that subset of $\mathbb{K}P^n \ni [x]$ such that $Q'(x, x) < 0$. To choose a representative of each class $[x] \in \mathbb{K}P^n$, we call the homogeneous coordinates of $[x]$ the representative $(x_1, x_2, \ldots, x_n, 1)$, noting that if the last entry is zero then it is impossible for $Q'$ to be negative.

Without loss of generality label the origin as $O := (1, 0, 0, \ldots, 0, 1)$; notice that all entries of $O$ are real and $Q'(O, O) = -1$. We follow the pattern from [BH99] and identify $T_OKH^n$ with the perp space $O^\perp$. The Riemannian structure comes from the following inner product on $O^\perp \ni U, V$:

$$\langle U, V \rangle = \Re(Q'(U, V))$$
The general formula for the inner product at an arbitrary point is slightly more complicated, but reduces to the above at points where \( Q' \) takes on the value of \(-1\). It is important to note that because these tangent vectors live in \( \mathbb{K}^{n,1} \) and not necessarily \( \mathbb{KH}^n \), we will always express them using coordinates that have \( n+1 \) entries (these points are emphatically not represented by homogeneous coordinates). Because \( \mathbb{KH}^n \) is simply connected, the exponential map is bijective, and we may consider for any tangent vector at the origin \( U \in T_O \mathbb{KH}^n \) the element in the space determined by \( \exp_O(U) \), which is given by \( \exp_O(U) = O \cosh(t) + \frac{U}{t} \sinh(t) \) where \( t = ||U|| \).

Now define the bijective map between expressions of tangent spaces as follows:

\[
T : \text{Im}(K) \oplus \mathbb{K}^n \rightarrow T_O \mathbb{KH}^n = O^\perp
\]

\[
T((u_0, u_1, \ldots u_n)) = (u_1, u_2, \ldots u_{n+1})
\]

It is clear that because \( O \) has real first entry and identically zero remaining entries, the image of this map will lie in \( O^\perp \), realizing that \( u_0 \) must be purely imaginary. To complete our description of the entire Lie algebra in this coordinate system, we also need to know how to describe vertical vectors (that one-parameter family of tangent vectors that live within \( (\text{Im}(K) \oplus \mathbb{K}^n)^\perp \)).

As described in Section 10.25 of [BH99], we can represent \( \infty \) in the parabolic model by the point in homogeneous coordinates \((1, 0, \ldots, 0)\), which is clearly not in the space as the last entry is zero. To approach this point from within the space we must scale the first coordinate \( x_1 \) in the positive real direction (a one-parameter family), which by projectivization is equivalent to scaling all other points towards \( 0 \). Thus, we identify a vertical tangent vector \( V \) with a translation in space in the direction from the origin to \((e^{||V||/2}, 0, 0, \ldots, 0, 1)\). Observe that for any such vertical vector \( V \), the first coordinate of this point is always real.

Theorem 10.16 in [BH99] identifies subspaces which are isometric to \( \mathbb{RH}^2 \) as those 2-dimensional subspaces which are totally real; i.e. such that \( Q'(\cdot, \cdot) \) takes on only real values over. It also classifies subspaces isometric to \( \mathbb{RH}^2_{-4} \) – a rescaled copy of the hyperbolic plane which has curvature \(-4\) – as those planes which contain a point \( z \) such that \( Q'(z, z) = -1 \) and are subspaces of \( \mathbb{K} \)-affine ‘lines’. This is defined to be the intersection of \( \mathbb{KH}^n \) with a single factor of the vector space \( \mathbb{K}^{n,1} \).

The proof of \((††)\) comes immediately from our definition of \( T \). We recall that \( O \) is such a point that \( Q'(O, O) = -1 \), and furthermore that the image of \( \text{Im}(K) \) under \( T \) is, by construction, exactly those vectors which are \( \mathbb{K} \)-linearly dependent on vectors in the vertical direction. In other words, we observe that the \( \mathbb{K} \) translates of a vertical vector by the first factor of \( \mathbb{K} \) in \( T_O \mathbb{KH}^n \) coincides with the image \( T(\text{Im}(K)) \).

To prove \((†)\), we compute \( Q'(u, v) \) for \( u = \exp_O(U), v = \exp_O(V) \) such that \( V \) is vertical and \( U \in \mathbb{K}^n \) (the second factor of the Lie algebra). We
recall that \( v \) must be of the form \((r,0,\ldots,0,1)\) for some real number \( r \). We will consider \( U \) a unit vector in the direction of some specified factor of \( \mathbb{K}^n \) – the generic case will follow from linearity. Thus, we see that 
\[
 u = (\cosh(1)/2, \sinh(1)U, 0, \ldots, \cosh(1)) = (1/2, \tanh(1)U, 0, \ldots, 1)
\]
where the final equality is by projectivization. A direct calculation shows that
\[
 Q'(u,v) = -1/2 - r + \tanh(1)U \cdot 0 = -1/2 - r
\]
which is always a real number. \( \square \)

4. Metrics on Heintze spaces of Carnot-type

A Lie group is assumed to come equipped with a topological structure which makes it a manifold and a continuous binary operation that makes it a group. A metric on a Lie group is called \textit{admissible} if it induces the topology already equipped.

**Definition 4.1.** A metric on a Lie group \( G \) is called \textit{left-invariant} if the left group action by \( G \) is an action by isometries.

We will only be interested in left-invariant, admissible metrics. In practice, a left-invariant metric will usually come from a bilinear form on \( \mathfrak{g} \), from which one defines the metric everywhere on \( G \) by declaring the left group action to be by isometries. If \( G \) is a Carnot group, we can make sense of the layers of \( \mathfrak{g} \) at all points by declaring the following:

**Definition 4.2.** Let \( N \) be a stratified nilpotent Lie group with \( n = \bigoplus_i \mathcal{V}_i \), where each \( \mathcal{V}_i \) is equipped with a norm \( || \cdot ||_{\mathcal{V}_i} \). Define the \textit{layers} of \( N \) to be the following sub-tangent bundles, defined at every point \( p \in N \) as:
\[
 \Delta^p_i := (dL_p)e_{\mathcal{V}_i}
\]
where \( dL_p \) represents the differential associated to left translation by \( p \). Equip each of these sub-tangent spaces with the following norm:
\[
 ||(dL_p)e_v|| := ||v||
\]

It is worth noting that in the event that the norms on the tangent space are induced by a Riemannian structure, this layered structure on \( N \) is often called subRiemannian; for some authors this is a requirement for the usage of the term \textit{Carnot group} (see the discussion in Definition 2.3 in [LD17]).

The goal of this section is to pick out a Riemannian structure on an arbitrary Heintze space of Carnot-type that generalizes the properties we observe for rank-one symmetric spaces. Corollary 4.4 below is intended to justify the use of a left-invariant, rather than bi-invariant, metric.

**Theorem 4.3.** [Wol64] Let \( N \) be a nilpotent Lie group which is not abelian. Then the Riemannian manifold obtained by equipping \( n \) with an inner product which is extended to all tangent spaces by left-translation, satisfies the
following condition at every point:

There exist tangent planes $R, S, T$ such that

$$K(R) < 0 = K(S) < K(T)$$

where $K$ represents the sectional curvature.

The following corollary follows from the observation that sectional curvature is always non-negative for Lie groups equipped with bi-invariant metrics.

**Corollary 4.4.** Any nonabelian nilpotent Lie group does not admit a bi-invariant metric.

We know from Theorem 2.4 that any Heintze space is expressible as a semi-direct product of a certain class of Lie groups. Because the information we have about Heintze spaces to this point is topological and algebraic in nature; we do not yet have a preferred way to equip the Lie algebra with an appropriate bilinear form that will allow us to measure distance and therefore curvature, such as is present in $\mathbb{KH}^n$. Instead of metrizing the semidirect product expression, however, we will use the following lemma to define an upper half-space model for a Heintze group, which comes with a pullback Riemannian metric once a left-invariant metric is chosen on the base group.

**Proposition 4.5.** [Nis00] Let $M$ be an Heintze space such that $M = N \rtimes \mathbb{R}$, where $N$ is the base group. Then the map

$$\Phi : N \times \mathbb{R}_+ \ni (n, y) \mapsto (n, \ln(y)) \in M = N \rtimes \mathbb{R}$$

is a diffeomorphism, called the generalized Cayley transform, and so defines a left-invariant pullback Riemannian metric $\Phi^* g$ on $N \times \mathbb{R}_+$. Furthermore, if $M$ is of Carnot-type then

$$\Phi^* g = \frac{1}{y^2}g|_{n_1} + \frac{1}{y^4}g|_{n_2} \ldots \frac{1}{y^{2s}}g|_{n_s} + \frac{dy^2}{y^2}$$

where $\sum g|_{n_i}$ is a left-invariant metric on the derived subalgebra.

We note that, importantly, this diffeomorphism preserves vertical tangent vectors - the differential of this map preserves the property of being orthogonal to $N$ and the eigenspace decomposition of $n$ associated to the adjoint action of the vertical direction (though the eigenvalues change, importantly). Due to this, anything proved of vertical vectors or vertical planes in the half-space representation will be valid when considering the Heintze space as a semidirect product of Lie groups.

In [Nis00], Nishikawa notes in Remark 3.1 that this choice of metric is far from unique, in both that the nilpotent metric is unspecified, and also that the $\ln$ function used could as easily be taken to be, for example, $\log_2$, which induces the denominators above instead as $y, y^2 \ldots y^s, 4y^2$. ‘In general’, he remarks, ‘it is not clear a priori which choice should be canonical.’

We see that when expressed as a matrix the above tensor $\Phi^* g$ is block diagonal, where the size of each block is the rank of the vector space $V_i$ with
the final block being of size one. This is true for any choice of map \( \mathbb{R} \to \mathbb{R}_+ \) or Riemmanian metric on each piece of the derived subalgebra. We will call a Riemnanian metric of this form on the product \( N \times \mathbb{R}_+ \) a horocyclic metric, noting that this represents a metric on the Heintze space itself.

The above half-space model for Heintze spaces agrees with the half-space model for hyperbolic (and K-hyperbolic) spaces. This coordinate structure for a Heintze space will be called its horospherical coordinates. If our space is a Heintze space of Carnot-type, then we get a full description of this coordinate system using the stratification. From here all metrics described will be assumed to be on the upper-half space model of the Heintze space, which is diffeomorphic to the original semi-direct product.

**Definition 4.6.** Suppose \( N \) is a Carnot group with stratification \( n = \bigoplus V_i \). Because \( N \) must be simply connected, the exponential map at the origin is bijective. For any specified basis of \( n \), we give \( N \) exponential coordinates by applying this map to \( n \). Furthermore we define the standard dilation to be a homomorphism of \( (\mathbb{R},+) \) to the automorphisms of \( N \) as follows:

\[
\lambda \mapsto \delta_\lambda, \delta_\lambda : \exp(V_1 + V_2 + \ldots + V_s) \mapsto \exp(\lambda V_1 + \lambda^2 V_2 \ldots + \lambda^s V_s)
\]

where \( V_i \in V_i \).

More information on this map may be found in [DKLL18]. The fact that the standard dilation on a Carnot group represents a group of isometries of common left-invariant metrics is nontrivial and is discussed in, among other places, Definition 2.1 [LD17].

We see that if we can choose a Riemannian metric tensor on the derived subalgebra, Proposition 4.5 now allows us to equip a Heintze space of Carnot-type with a specific admissible, left-invariant metric.

**Definition 4.7.** Let \( N \) be a Carnot group with a stratification of its algebra \( n = \bigoplus V_i \). A layered basis on the tangent bundle of \( N \) is the set left-translates (as in Definition 4.2) of orthogonal bases \( \{e_{ij}\} \) for each \( V_i \). In the event a metric is specified, we assume that a layered basis is orthonormal.

In some parts of the literature, the set of \( \{e_{ij}\} \) is referred to as a basis for \( n \) which is adapted to the stratification. When we consider the group \( N \) as a factor of the half-space model for a Heintze space, the choice of an orthogonal layered basis is made possible by the block-diagonality of \( \Phi^* g \). Left-invariance of horocyclic metrics and homogeneity of our space then guarantees that the translates of these bases will remain orthonormal.

Intersecting the layered basis with \( T_p M \) will give an orthogonal basis for the nilpotent Lie algebra, and if the basis is taken together with \( A \), a (vertical) vector in the positive \( \mathbb{R}_+ \) direction, we get a basis for the whole tangent space at a point in an arbitrary Heintze space of Carnot-type. We will abuse notation in this setting by referring to elements of any tangent space as linear combinations of \( e_{ij} A \) without referencing the translation. Denote by \( e_0 \) the unit vector in the direction of \( A \), recalling that \( K(A, X) = K(e_0, X) \). Later we will make the assumption that \( A = e_0 \).
Proposition 4.8. [Nis00] Let $M$ be a Heintze space of Carnot-type equipped with a horocyclic metric. Then there exists a vertical vector $A$ such that the Lie algebra decomposes $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}A$ and if $\mathfrak{n} = \oplus \mathcal{V}_i$ is the stratification, then the $\mathcal{V}_i$ are eigenspaces of the action by $\text{ad} A$ with eigenvalues $i$ in the Lie structure on $N \times \mathbb{R}_+$ induced by the generalized Cayley transform.

One can check that the transformation $\text{ad} A$ as above is a derivation of $\mathfrak{n}$. The way we will use Proposition 4.8 is to observe that for our chosen coordinates, $\text{ad} A(V_i) = [A, V_i] = iV_i$ for $V_i \in \mathcal{V}_i$. Furthermore, if we choose our horocyclic metric carefully, we may assume $[e_0, V_i] = iV_i$ (or equivalently that $A$ is unit length).

It is important to note that the eigenvalues of this transformation are distinct from the eigenvalues in the linear transformation which is exponentiated in Definition 4.6. This is essentially due to the logarithm present in the second factor of the generalized Cayley transform. For this reason as well, the adjoint action of $A$ acts as derivation of $\mathfrak{n}$, while this is not the case in the for the standard dilation - which can be realized as the action of $\mathbb{R}$ in the semidirect product decomposition of a Heintze space as $N \rtimes \mathbb{R}$ (see 2.G of [Cor18] for more details).

We continue by providing an answer to the question of Nishikawa by defining the following class of Riemannian metric on an arbitrary Heintze space of Carnot-type. In the definition below we use the letter $H$ to remember the motivating example of sectional curvature in Heintze spaces of Carnot-type over Heisenberg groups, as well as the reliance on horospherical coordinates. We also draw inspiration from the use of $H$ to specify special structures on nilpotent Lie groups such as in Kaplan [Kap81].

Definition 4.9. Let $M$ be a Heintze space of Carnot-type with a layered basis $\{e_{i,j}\}$. We say that a Riemannian metric on $M$ is an $H$–metric if it is left-invariant and satisfies the following curvature conditions at any (equivalently every) point:

$$-1 \leq K(e_{i,j}, e_{k,l}) \leq -s^2$$

$$K(e_0, e_{i,j}) = -i^2$$

where the set $\{e_{i,j}\}$ represents a layered basis and $e_0$ is vertical.

Recall that vectors which are vertical in $N \times \mathbb{R}$ remain vertical in the upper half-space model $N \times \mathbb{R}_+$.

Theorem 4.10. All Heintze spaces of Carnot-type admit an $H$–metric.

Observe that $H$–metrics are ‘admissible’ metrics in the sense of Definition 6.5 from Eberlein and Heber [EH96]. We avoid generalizing this notation to avoid clashing with the definition of an admissible metric as one that induces the Lie topology. In particular, $H$–metrics on Heintze spaces of Carnot-type based on 2-step groups will lie in the intersection of $QP \cap AM$, again in the sense of [EH96].
Proposition 4.11. Let $M$ be a Heintze space of Carnot-type with Lie algebra $s$ and $A$ be the vertical vector as in Proposition 4.8. For any left-invariant metric $g$ on the subgroup corresponding to $n = [s, s]$, the horocyclic metric $h$ on $M$ which agrees with $g$ and sets $|A|_h = 1$ has the following curvature property

$$K_h(A, V_i) = -i^2$$

for all $V_i \in \mathcal{V}_i$

To prove this, we recall the following definition.

Definition 4.12. Let $h$ be a Lie algebra with orthonormal basis $\{f_i\}$. Then if $[f_i, f_j] = \Sigma_k \alpha_{ij}^k f_k$, the values $\alpha_{ij}^k$ are called the structure constants of $h$.

Proof. of Proposition 4.11 A result of Milnor from [Mil76] tells us that for such an orthonormal basis $\{f_i\}$

$$K(f_a, f_b) = \sum_k A_{ab}^k$$

where

$$A_{ab}^k = \frac{1}{2} \alpha_{ab}^k (-\alpha_{ab}^k + \alpha_{bk}^a + \alpha_{ka}^b)$$

$$- \frac{1}{4} (\alpha_{ab}^k - \alpha_{bk}^a + \alpha_{ka}^b)(\alpha_{ab}^k + \alpha_{bk}^a - \alpha_{ka}^b - \alpha_{ka}^b \alpha_{kb})$$

Reindex the layered basis $\{e_0, e_{i,j}\}$ to $f_k$ with a function $\psi(k) = i$ if $f_k = e_{i,j}$ for some $j$ ($f_k \in \mathcal{V}_{\psi(k)}$). We can observe the following identities of the structure constants:

$$\alpha_{nm} = 0, \quad \alpha_{0m} = \psi(m)$$

$$\alpha_{nm}^\ell = 0 \text{ if } \psi(\ell) \neq \psi(n) + \psi(m)$$

$$\alpha_{0m}^n = 0, n \neq m$$

To see why these hold, we recognize the following form for the structure constants:

$$\alpha_{nm}^\ell = \langle [e_n, e_m], e_\ell \rangle$$

In this formulation, the first above property follows immediately from the skew-symmetry of the Lie bracket, and the second from the definition of a stratification and the fact that the layers are orthogonal. The last two properties follow from the fact that adjoint action on the layered basis by $\{e_0\}$ is exactly the adjoint action from Proposition 4.8.

To prove the proposition, let $m$ be such that $e_{i,j} = f_m$. Note that $\alpha_{0m}^\ell = 0$ unless $\ell = m$, as the vertical direction is orthogonal to all layers, and each layer is orthogonal to others, but the adjoint action contains $f_m$ as an eigenvector with eigenvalue $\psi(m)$. Thus

$$K(e_0, e_{i,j}) = K(e_0, f_m) = A_{0m}^0 + A_{0m}^m$$
From here we get that $A_{0m}^0 = 0$ when $f_m$ lies in the $i$-th layer $\mathcal{V}_i$, as it will only contain one nonzero term $(\alpha_{0m}^m)$ which is multiplied by a term which is zero. Direct calculation shows that because $\alpha_{0m}^m = \psi(m) = i$, $A_{0m}^m = -i^2$, verifying that $K(e_0, e_{i,j}) = -i^2$. □

Due to this lemma, $|A| = 1$ will be a condition on any metric we put on Heintze spaces of Carnot-type. Furthermore, the above result remains true upon scaling the layered basis, as the zero terms must remain zero by bilinearity of the inner product and Lie bracket. Let us now prove the main theorem.

**Proof.** of Theorem 4.10

Let $M$ be our Heintze space of Carnot-type and $s$ the associated Lie algebra. We will construct an inner product on $s$ which comes from a horocyclic metric $h$ that satisfies certain additional conditions. We begin by assuming that $|A|_h = 1$. We are now guaranteed the vertical plane condition by Proposition 4.11. Thus we must only check the condition on basis planes contained in $n$. Specify a layered basis as in Definition 4.2 and choose the $h$ makes that basis orthonormal (this does not affect the norm of $A$). We recall 1.5 b) from [EH96] the following curvature for such planes, assuming $X,Y$ are orthonormal. Notice the following is valid for any metric on a Heintze space.

$$K(X, Y) = K^n(X, Y) - |\phi(X, Y)|^2$$

where $\phi(X, Y) = \sqrt{D_0}X \wedge \sqrt{D_0}Y$, and $D_0$ is the symmetric part of $\text{ad} A$ - the skew-symmetric part is labelled $S_0$. We claim that $D_0$ has the same eigenvectors and eigenspaces as $\text{ad} A$ in the setting of a horocyclic metric. Indeed this will be true if the transformation $\text{ad} A \sim n$ is symmetric, as of course in this case $S_0 = 0$. We can see this is due to the assumption that the eigenvalues of this transformation are real - by assumption they are exactly the set $\{1, 2, \ldots, s\}$ - and the eigenvectors are mutually orthogonal by the assumption that our metric respects the layered basis. Thus in fact, if the transformation is written in the basis of these elements, it is not just symmetric but even diagonal. We examine the value of $|\phi(V_i, V_j)|^2$ where $V_i, V_j$ are distinct orthonormal vectors in $\mathcal{V}_i, \mathcal{V}_j$ respectively ($i$ possibly equal to $j$).

$$|\phi(V_i, V_j)|^2 = |\sqrt{D_0}V_i \wedge \sqrt{D_0}V_j|^2$$
$$= |\sqrt{i}V_i \wedge \sqrt{j}V_j|^2$$
$$= |\sqrt{ij}(V_i \wedge V_j)|^2$$
$$= (\sqrt{ij})^2 = ij$$

Note that this value is always in the interval $[1, s^2]$. From here, because $h$ is a horocyclic metric, it agrees with some left-invariant Riemannian metric $g$ on $n$. We can now appeal to Gromov in [Gro78] to say that for any $\epsilon$,
\( n = [S, S] \) admits a metric such that \(|K| < 100\epsilon^2\) for all tangent planes. In particular, per 4.4 in [Gro78], this can be done simply by scaling the norm on certain vectors – thus preserving orthogonality of vectors. One should think of this as first fixing lengths of the basis for \( V_1 \), and then scaling down the basis of \( V_2 \) as needed to ensure that

\[
||[e_{1,i}, e_{1,j}]|| \leq \epsilon ||e_{1,i}|| ||e_{1,j}||
\]

This is then done step by step for \( V_2, V_3 \ldots V_3 \), with the assurance that this process terminates due to the fact that a) each \( V_i \) has finite rank and b) \( V_\ell = 0, \ell > s \). We can now insist that the metric on \( N \) induced by the horocyclic metric \( h \) agrees with such a \( g_\epsilon \) for our choice of \( \epsilon > 0 \). Choose \( \epsilon = \frac{1}{20s} \). With this condition in place, we relabel our layered basis to one that is orthonormal, by choosing basis vectors of unit length and parallel to the old ones. We will again label this basis \( \{e_{i,j}\} \). Recall that this does not affect the curvature of vertical planes by Proposition 4.8 - \( A \) remains unit length and the new metric is still a horocyclic metric. We compute the \( N \)-curvature for a plane contained in \( V_1 \), recalling the structure constants and the result of Milnor:

\[
K^n(e_{1,a}, e_{1,b}) = \sum_k A_{ab}^k = \sum_{e_{2,k}} A_{ab}^k
\]

\[
= \sum_{e_{2,k}} \frac{1}{2} \alpha_{ab}^k (-\alpha_{ab}^k + \alpha_{ak}^a + \alpha_{bk}^b)
\]

\[
- \frac{1}{4} \alpha_{ab}^k (\alpha_{ak}^a + \alpha_{bk}^b) (\alpha_{ab}^k + \alpha_{bk}^b - \alpha_{ak}^a) - \alpha_{ka}^a \alpha_{kb}^b
\]

\[
= \sum_{e_{2,k}} \left( \frac{1}{2} \alpha_{ab}^k (-\alpha_{ab}^k) - \frac{1}{4} (\alpha_{ab}^k)^2 \right)
\]

\[
= \sum_{e_{2,k}} \frac{3}{4} (\alpha_{ab}^k)^2 = \sum_{e_{2,k}} \frac{3}{4} \langle [e_{1,a}, e_{1,b}], e_{2,k} \rangle^2
\]

\[
= -\frac{3}{4} ||[e_{1,a}, e_{1,b}]||^2 \leq 0
\]

While of course not all planes will have nonpositive \( N \)-curvature (all left-invariant metrics on nilpotent Lie groups must have planes with both positive and negative curvature), we will see it is important that ones of this form will. In particular, we calculate the curvature of a plane of the form \( \pi = \langle e_{i,j}, e_{k,\ell} \rangle \). First assume it is not the case that both \( i = 1, k = 1 \) or that both \( i = s, k = s \). Then we see

\[
K(e_{i,j}, e_{k,\ell}) = K^N(e_{i,j}, e_{k,\ell}) - |\phi(e_{i,j}, e_{k,\ell})|^2
\]

\[
= K^N(e_{i,j}, e_{k,\ell}) - ik
\]

Now by the assumption that \(|K^N| < \frac{1}{4s^2} < \frac{1}{4} \) because we picked a \( \frac{1}{20s} \)-metric, we are assured this value is in \([-1, -s^2] \). If \( i = 1, k = 1 \), then \( K = K^N - 1 \) which is in \([-1, -s^2] \) because \(-\frac{1}{4s^2} < K^N < 0 \). For the case
that $i = s, k = s$, we see that $e_{s,j}, e_{s,\ell}$ are in the center, and so also we have that $K_N(e_{s,j}, e_{s,\ell}) = 0$. This is also readily seen to be a consequence of the Milnor theorem and the fact that $[e_{s,j}, e_{s,\ell}] = 0$.

We now argue that all the conditions placed on the metric $h$ may be realized simultaneously. Observe that we insisted:

- $h$ restricts to a $\frac{1}{20s}$–metric on $n$ in the sense of Gromov.
- $|A|_h = 1$
- $h$ respects the orthogonal nature of a layered basis $\{e_{i,j}\}$ and is a horocyclic metric.

These conditions are compatible due to the orthogonal decomposition of the Lie algebra $\mathfrak{s} = \mathbb{R}A \oplus \mathcal{V} = \mathbb{R}A \oplus \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_s$, which allows us to scale $\mathbb{R}A$ independently from $\mathcal{V}$, and $\mathcal{V}_i$ independently from $\mathcal{V}_j$. We also see that the construction of an $\epsilon$–flat by Gromov does not change the orthogonality of the chosen basis, only scaling elements of the form $[e, e']$. Therefore we may scale this way for any metric induced on $n$ by $h$.

While the above computation may seem involved, we note that Theorem 2 in the celebrated paper [Hei74b] describes an analogous construction. Indeed, Heintze remarks that only by such a scaling can one even conclude strict negative curvature of such a space. Because our goal will be to conclude pinched negative curvature (and not simply negative curvature at all), we rely on more than just a sole parameter such as the one used in the statement of Theorem 2 of [Hei74b]. We name one of these parameters here which is present in the proof of 4.10.

**Definition 4.13.** Let $(M, g)$ be a Heintze space of Carnot-type with an $H$–metric. If $g$ satisfies the condition

$$||[e_a, e_b]|_g| \leq k \|e_a\|_g \|e_b\|_g$$

we say that $g$ has *Gromov value* $k$.

Observe that all Heintze spaces of Carnot-type have $H$–metrics with Gromov value $k$ for any $k > 0$. We should also observe that as a consequence of the proof of Theorem 4.10 it is certainly not the case that all admissible metrics for these spaces have this pinched negative curvature property. In fact some metrics may have sections of positive curvature, as noted too by Heintze.

One consequence of the previous proof is the symmetric nature of the adjoint action. Due to the importance of the adjoint actions decomposition into symmetric and skew-symmetric parts, we note this result formally.

**Theorem 4.14.** Let $M$ be a Heintze space of Carnot-type equipped with any horocyclic metric. Then the adjoint action $\text{ad} A$ is symmetric and $S_0 = 0$ for all vertical vectors.

**Proof.** Because $M$ admits an $H$–metric, the adjoint action is symmetric with respect to the layered basis, per the above argument. Symmetry of a linear transformation is invariant under change of basis. $\square$
Corollary 4.15. Heintze spaces of Carnot-type equipped with $H$–metrics are of Iwasawa-type in the sense of [Wol91].

One should compare the above result to the discussion at the beginning of section 3.1 in [EH96], where this fact is noted for Heintze spaces of Carnot-type with 2-step base. In some parts of the literature (see [BPR03]) Heintze spaces of Carnot-type are defined to have this property. Note that such spaces in [BPR03] are assumed to be 2-step as well; the authors also use $H$–type to refer to those spaces which are based on Heisenberg groups. Finally, the above result is assumed in [Dru06], however again this is only stated for spaces based on 2-step Carnot groups; Corollary 4.15 holds for Heintze spaces of Carnot-type of any step.

5. Pinched Curvature in Homogeneous Manifolds

We give a definition here of pinched curvature in manifolds and explore the pinching properties of metrics on Heintze spaces.

Definition 5.1. A manifold with all sectional curvatures in the interval $[a, b]$ with $a, b < 0$ is said to be $\frac{b}{a}$–pinched.

Usually, the metric will be normalized to allow $b = -1$ and the pinching constant will be a fraction with numerator 1 - observe that scaling the metric to change the supremum of curvature leaves this fraction invariant. We see from the definition of pinched curvature that if a metric on a manifold is not strictly negatively curved or has the property that the infimum of its sectional curvature is $-\infty$, then it is not $C$–pinched for any $C$.

Eberlein and Heber in [EH96] provide algebraic conditions on a solvable Lie algebra that guarantee it admits an inner product which turns the corresponding Lie group into a manifold with sectional curvature $-\left(2^2\right) = -4 \leq K \leq -1 = -\left(1^2\right)$. Such a manifold is called quarter-pinched. We see $H$–metrics exhibit a very similar type of behavior, saving the fact that the ratio of maximum vertical curvature to minimum vertical curvature is now the reciprocal of an integral square.

Definition 5.2. A solvable Lie algebra $\mathfrak{s}$ is an SP-algebra if it admits an inner product such that the associated manifold is $\frac{1}{s^2}$–pinched where $s$ is the step nilpotency of $[\mathfrak{s}, \mathfrak{s}]$.

Analogously, the manifold will be said to have square-pinched curvature. It is clear not all Lie algebras are square pinched – consider those where the derived subalgebra has codimension greater than one.

Theorem 5.3. Any Lie algebra with inner product $\{\mathfrak{s}, \langle \cdot , \cdot \rangle\}$ whose Lie group $S$ is a Heintze space of Carnot-type is an SP-algebra.

The existence of $H$–metrics gets us most of the way to this result. The following lemma is in service to the proof of Theorem 5.3.
Lemma 5.4. Let \( M \) be a Heintze space of Carnot-type equipped with an \( H \)-metric with Gromov value \( \frac{1}{20s} \). Then for any tangent plane \( \pi \subset s \) there exist orthonormal vectors \( \alpha A + \beta U, V \) generating \( \pi \) such that the following is true for the value \( T = T(U, V) = \langle (\nabla^N_U \text{ad} A)V - (\nabla^N_V \text{ad} A)U, V \rangle \):

- \( |T| < \frac{1}{s} \)
- If \( \pi \cap \mathcal{V}_1 \) is non-trivial, then \( T = 0 \)
- If \( \pi \cap \mathcal{V}_s \) is non-trivial, then \( T = 0 \)

Proof. We break down the definition of \( T = \langle (\nabla^N_U \text{ad} A)V - (\nabla^N_V \text{ad} A)U, V \rangle \).

Recall that for vectors regarded as left-invariant vector fields, we have the following identity (see (7) of [AW76]): 
\[ \nabla_X Y = \frac{1}{2}([X, Y] - (\text{ad}_X)^T Y - (\text{ad}_Y)^T X). \]

In particular it is expressible as a linear combination of terms involving the adjoint action realized by the Lie bracket and its transpose. From this we see that a bound on the size of the Lie bracket given a bound on the inputs allows to bound the values of \( \nabla^N_U \text{ad} A, \nabla^N_V \text{ad} A \) and thus \( T \).

As a left invariant vector field, we can realize \( \text{ad} A \) as the sum of coordinate vector fields using the diagonal expression from the layered basis:
\[ \text{ad} A = \sum_j \psi(j)E_j \text{ where } \psi(j) = i \iff E_j \in \mathcal{V}_i. \]

Then by the Gromov condition that \( ||[X, Y]|| \leq \frac{1}{20s} ||X|| ||Y|| \) and the facts that \( \psi(j) \leq s \) and \( ||U|| \leq ||V|| = 1 \), we know that \( ||(\nabla^N_U \text{ad} A)V - (\nabla^N_V \text{ad} A)U|| \leq \frac{3}{8} \), which means \( |T| < \frac{1}{s} \), as we are projecting the previous vector onto the \( V \) direction.

Now if \( \pi \cap \mathcal{V}_1 \) is nontrivial, we can arrange it so that \( V \in \mathcal{V}_1 \). Notice from the definition of \( T \) that \( (\nabla^N_U \text{ad} A)V - (\nabla^N_V \text{ad} A)U \) lives entirely in \( \oplus_{i \geq 2} \mathcal{V}_i \), which is orthogonal to \( \mathcal{V}_1 \). As such \( \langle (\nabla^N_U \text{ad} A)V - (\nabla^N_V \text{ad} A)U, V \rangle = 0 \).

Finally, if \( \pi \cap \mathcal{V}_s \) is nontrivial, we can use similar reasoning and assume that \( U \in \mathcal{V}_s \). Now see that \( \text{ad} U \) is identically trivial, as \( [U, V_j] \subset \mathcal{V}_{j+s} = 0 \). Similarly we know that \( (\text{ad} U)^T \) is also trivial, which proves the claim. \( \square \)

Proof. of [5.3] Begin by equipping \( S \) with an \( H \)-metric which has a Gromov value \( \frac{1}{20s} \). By the definition of an \( H \)-metric, we already know the sectional curvature of planes determined by layered basis vectors, so we need to consider planes which may be generated by linear combinations of these. We break this argument into two cases for tangent planes \( \pi \).

Case 1) \( \pi \subset n = \mathbb{R}A^4 \)

Let \( U, V \) be orthonormal vectors spanning \( \pi \). Then we recall the formula from [EH96] used in the proof of Theorem 4.10

\[ K(\pi) = K^n(U, V) - |\phi(U, V)|^2 \]

\[ = K^n(U, V) - |\sqrt{D_0}U \wedge \sqrt{D_0}V|^2 \]

Recall that \( \sqrt{D_0}V_i = \sqrt{i}V_i \) when \( V_i \in \mathcal{V}_i \). Now because \( |K^n| < \frac{1}{147} \) the result follows from linearity of \( D_0 \) and the fact that \( K^N \) is strictly negative when constrained to \( \mathcal{V}_1 \) and identically zero on \( \mathcal{V}_s \).

Case 2) \( \pi \) is not orthogonal to \( A \)
Assume \( \pi = \langle \alpha A + \beta U, V \rangle, U, V \in \mathfrak{n} \) for vectors of the form specified in Lemma 5.4; recall they are thus assumed to be orthonormal. In this case we use part c) of 1.5 in Eberlein and Heber [EH96] to find, recalling that \( D_0 = \text{ad} A \):

\[
K(\pi) = -\alpha^2 \langle N_0 V, V \rangle + \beta^2 K(U, V) + 2\alpha\beta \langle (\nabla^N_U \text{ad} A)V - (\nabla^N_V \text{ad} A)U, V \rangle
\]

In that statement, \( N_0 \) is defined to be the transformation defined in Heintze [Hei74b]; i.e. \( N_0 = D_0^2 + [D_0, S_0] \). Thus, in our case, \( N_0 = D_0^2 \).

In the event that \( \pi \) intersects either \( V_1 \) or \( V_s \), the above simplifies by Lemma 5.4 to the following.

\[
K(\pi) = -\alpha^2 \langle N_0 V, V \rangle + \beta^2 K(U, V)
\]

for \( 1 \leq k \leq s \). Because \( K(U, V) \in [-s^2, -1] \), per case 1, and \( \alpha^2 + \beta^2 = 1 \), this completes this case. In the event that \( \pi \) does not intersect either the horizontal or the top layer nontrivially, our bounds for the above terms are instead \( 2 \leq k \leq s-1 \) and \( K(U, V) \in \left[-(s-1)^2, -4\right] \). Now we see that because Lemma 5.4 tells us that \( |T| < \frac{1}{2} \), and so \( |2\alpha\beta T| < \frac{1}{2} \). Therefore \( K \in \left[-(s-1)^2 - 0.5, -3.5\right] \subset \left[-s^2, -1\right] \). \( \square \)

We see that the dependence on \( s \) for the Gromov value was not necessary for verifying the sectional curvature in the definition of an \( H \)-metric. However it is used in the above argument to find an \( H \)-metric which is \( \frac{1}{s^2} \)-pinched. As such we are not guaranteed that all \( H \)-metrics, with arbitrary Gromov value, have this property.

We note above the use of the general plane curvature formula from 1.5 of [EH96]. Indeed one could apply this directly to Proposition 4.11 and prove the statement that way. This however relies on the realization that \( N_0 = D_0^2 \) exactly, which is true because we proved that the adjoint action was symmetric, and thus \( S_0 \) and so \( [D_0, S_0] = 0 \). We may therefore see this sketch of an alternate proof as an independent check on Theorem 4.14.

We may ask about some form of a converse to Theorem 5.3. It is well-known result that for rank-one symmetric spaces of noncompact type, there is no admissible metric of curvature pinched tighter than \( \frac{1}{4}\). This motivates the question: Can a Heintze space of Carnot-type admit a metric with pinching tighter than \( \frac{1}{s^2} \)? We give a partial answer to this question here.

**Theorem 5.5.** Let \( M \) be a Heintze space of Carnot-type whose derived subgroup admits a lattice. Then \( M \) is not \( C \)-pinched for any \( C > \frac{1}{s^2} \), where \( s \) is the step nilpotency of the derived subgroup.

Note that a tighter pinching constant, i.e. one associated to a manifold whose curvature is closer to being constant, is larger. In this way a manifold which is \( C \) pinched is trivially \( C' \) pinched for \( C' < C \).
Proof. Begin by equipping $M$ with a metric. Immediately we see that if this metric does not have strictly negative curvature bounded away from zero, the result is immediate, as it will not be pinched for any value. Therefore we may normalize the metric to assume its largest curvature value is $-1$.

By assumption the derived subgroup is nilpotent; call this subgroup $N$. We know by Theorem 2.1 of [Rag07] that a lattice in a nilpotent Lie group is always cocompact. Let $\Gamma$ represent the lattice assumed to exist in $N$. We can pass to the quotient manifold $M_\Gamma := M/\Gamma$. By construction $\pi_1(M_\Gamma) \cong \Gamma$. Furthermore, by cocompactness and Chapter 2 of [Rag07], we may conclude $\Gamma$ is also $s$-step nilpotent.

Observe that because the action $\Gamma \curvearrowright M$ is by isometries, the sectional curvatures present in $M$ is the same set of curvatures present in $M_\Gamma$; indeed just apply the locally isometric covering map. We may now apply the main result of Belegradek and Kapovitch from [BK05], which states that a manifold whose fundamental group has as a finite index subgroup a nilpotent group of step $s$ must have sectional curvatures, after normalization such that the supremum of its sectional curvature is $-1$, at least as negative as $-s^2$.\[\square\]

In the above way we conclude that an $H$–metric observes the tightest possible constraints on sectional curvature when the Heintze space of Carnot-type admits a lattice of its Carnot group. It is not difficult to see when these lattices exist given the information of the Lie bracket. We recall the following result which is present as Theorem 2.12 in [Rag07] and is originally attributed to Malcev.

Theorem 5.6. [Mal49] Let $N$ be a nilpotent Lie group with Lie algebra $\mathfrak{n}$. Then $N$ admits a lattice if and only if there exists a basis for $\mathfrak{n}$ which makes all structure constants rational.

In fact, Raghunathan in Remark 2.14 of [Rag07] constructs (many) examples of nilpotent Lie algebras which do not admit lattices. A careful reading of this note reveals that in fact those algebras he constructs include some algebras corresponding to Carnot groups of step 2.

6. Special Carnot Groups

Of primary interest when considering Heintze spaces of Carnot-type has been symmetric spaces, or more generally Heintze spaces whose associated nilpotent group is step 2. However, it is useful to have examples of Heintze spaces of Carnot-type which are of arbitrary step.

Definition 6.1. [Nis00] Let $k \in \mathbb{N}$ and consider $SL(k + 1, \mathbb{R})$ the special linear group of degree $k + 1$ with Lie algebra $\mathfrak{sl}(k + 1, \mathbb{R})$, where the bracket operation is matrix multiplication commutation. If $E_{ij}$ is the matrix with a 1 in the $(i, j)$ place and zeroes elsewhere, let

$$\mathfrak{n} = \text{Span}\{E_{ij} \mid j > i\}$$
and let $A$ be the following matrix

$$A = \frac{1}{2} \begin{pmatrix} k & k - 2 & \cdots & -(k - 2) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ -k & -k & \cdots & -k \end{pmatrix}$$

The Lie subalgebra of $\mathfrak{sl}(n+1, \mathbb{R})$ formed by $\mathfrak{sc}(k) = \mathfrak{n} \oplus \mathbb{R}A$ is called the special Carnot algebra and the associated Lie group is the special Carnot group.

The condition that $j$ is strictly larger than $i$ guarantees all of the matrices $E_{ij}$ have trace zero and thus live in $\mathfrak{sl}(k+1)$.

**Proposition 6.2.** For all $k > 2$, $\mathfrak{sc}(k)$ is a Carnot group with grading $\mathcal{V}_\ell = \text{Span}\{E_{ij} \mid j - i = \ell\} = \{X \mid [A, X] = \ell X\}$.

One benefit of this example is that we may describe the $H$–metric significantly more explicitly. To do this, observe that the real rank of $\mathcal{V}_\ell$ is $\text{rk}(\mathcal{V}_\ell) = \#\{(i, j) \mid j - i = \ell, 1 \leq i < j \leq k + 1\} = k - \ell + 1$. The fact that the adjoint action of $A$ matches the form of the Lie bracket in the upper half-space model, rather than the semidirect product, serves as further evidence of the appropriateness of the generalized Cayley transform.

Observe further that the matrices $E_{ij}$ form an orthonormal basis for $\mathfrak{n}$ when considering the inner product induced from being a subalgebra of $\mathfrak{sl}$. Furthermore, elementary matrix calculations guarantee $\langle A, E_{ij} \rangle = 0$, so indeed $\mathbb{R}A$ is orthogonal to $\mathfrak{n}$. However, $|A| \neq 1$.

**Theorem 6.3.** Consider the inner product $\langle \cdot \rangle_H$ on $\mathfrak{sc}(k+1)$ such that:

- $|E_{ij}|_H = \frac{1}{100k(j-i)}$ for all $i, j$
- $|A|_H = 1$
- $\langle E_{ij}, E_{k\ell} \rangle_H = \langle E_{ij}, E_{k\ell} \rangle_{\mathfrak{sl}} = 0$
- $\langle E_{ij}, A \rangle_H = \langle E_{ij}, A \rangle_{\mathfrak{sl}} = 0$

Then $\langle \cdot \rangle_H$ induces an $H$–metric on the Heintze space derived from the special Carnot group.

As these matrices form a basis for $\mathfrak{sc}$, this uniquely determines an inner product.

**Proof.** We first note that $[E_{ij}, E_{k\ell}] = 0$ unless $k = \ell$, in which case $[E_{ij}, E_{j\ell}] = E_{i\ell}$, seeing that it is necessarily the case that $\ell > i$. From this, we see that $H$ has a Gromov value of $\frac{1}{100k}$, and so the induced curvature in $\mathfrak{n}$ is less than $\frac{1}{100k^2}$.

Immediately as a consequence from declaring $|A|_H = 1$ and the orthogonality of the $E_{ij}$, we see from Theorem 4.11 that $K(A, E_{ij}) = -(j-i)^2$, recalling that $E_{ij} \in \mathcal{V}_{j-i}$, and so satisfy the first condition of an $H$–metric.

We get an immense benefit from the fact that the adjoint action of the vertical direction corresponds to a diagonal matrix, as in the construction of an
$H$–metric in the general case. We recall the definition of the transformations $D_0, S_0$ as the symmetric and skew-symmetric parts of $\text{ad} A$ respectively. In particular $D_0 = \frac{1}{2}(\text{ad} A + (\text{ad} A)^T), S_0 = \frac{1}{2}(\text{ad} A - (\text{ad} A)^T), \) (see Section 1.2 of [EH96]). The diagonal nature of $A$ means that $D_0 = \text{ad} A, S_0 = 0$, as in the proof of Theorem 4.10. From this proof also, we use the formula present in [EH96] and Proposition 4.11 to verify that because the nilpotent metric is a $\frac{1}{100}$ metric, $|A| = 1$, and the basis is orthogonal, we get the curvature of basis-planes to be pinched in the correct way. □

7. On the Einstein Condition

Regarded as metric spaces, Heintze spaces of Carnot-type with $H$–metrics generalize rank-one, noncompact symmetric spaces. In this section, all Heintze spaces of Carnot-type are assumed to be equipped with an $H$–metric.

Fact 1. Heintze spaces of Carnot-type are $\delta$–hyperbolic for a uniform $\delta$ when equipped with $H$–metrics.

Proof. By Theorem 5.3, these spaces are CAT$(-1)$. All CAT$(-k)$ spaces are $\delta$–hyperbolic by Proposition 1.2 in Part III of [BH99]. Furthermore, the value of $\delta$ only depends on the value of $k$. □

Recall the following definition for hyperbolic spaces.

Definition 7.1. If $X$ is a proper $\delta$–hyperbolic space, the boundary (or boundary at infinity) of $X$, denoted $\partial X$, is the set of all equivalence classes of geodesic rays in $X$, where two rays $c, c'$ are equivalent if the distance $d(c(t), c'(t)) < K$ for some value $K$ and all $t$.

With this additional piece of information, we can ask questions about the group of isometries for a Heintze space of Carnot-type, $\text{Isom}(M, h)$. To do this, we recall the classification of isometries for these spaces, due to Gromov in [Gro87, §3.1].

Definition 7.2. Let $X$ be a proper $\delta$–hyperbolic space. Isometries $\gamma$ of $X$ come in three mutually exclusive types [GH90, §8.2]:

- elliptic if the limit set of $\gamma$ is empty;
- parabolic if the limit set of $\gamma$ is a single point; and
- loxodromic if the limit set of $\gamma$ is a pair of points.

Furthermore, subgroups of $\text{Isom}(X)$ fall in to exactly one of the following categories.

1. Elementary actions
   (a) elliptic if the limit set of $\Gamma$ is empty;
   (b) parabolic if the limit set of $\Gamma$ is a single point;
   (c) lineal if the limit set of $\Gamma$ consists of a pair of points.

2. Nonelementary actions
   (a) quasi-parabolic if the limit set of $\Gamma$ is uncountable and $\Gamma$ has a unique fixed point $\xi$ in $\partial X$. 
(b) **general type** if the limit set of $\Gamma$ is uncountable and $\Gamma$ has no finite orbits in $\partial X$.

**Proposition 7.3.** Let $\Gamma$ act by isometries on a Heintze space of Carnot-type $(M, h)$ of step $s > 2$, or of step 2 but not a symmetric space. Then this action is either elliptic, parabolic, quasi-parabolic or lineal.

**Proof.** We may begin by identifying the ideal boundary $M(\infty)$ with the visual boundary $\partial M$ of $M$ as a hyperbolic space. Indeed, the geodesic definition and equivalence classes for $M(\infty)$ coincide with $\partial M = \partial_{\infty}M$.

Recall that in the event that $M$ is not isometric to a symmetric space, there exists a unique fixed point $\infty$ common to all elements of $\text{Isom}(M, h)$. In other words, $\text{Stab}_\Gamma(\infty) = \Gamma$ for all subgroups $\Gamma \subset \text{Isom}(M, h)$. The impossibility of general type actions follows immediately. □

Notice that this specific point $\infty$ matches the interpretation of the point $\infty$ as in Proposition 3.2 as the ideal point in the horospherical coordinates, as well as the distinguished ideal point in the generalized Cayley transform for Heintze spaces. This also matches the pointed sphere viewpoint of the boundary for a Heintze space, where the pointed sphere is identified with the one-point compactification of the derived subgroup.

With Proposition 7.3 in hand, we recover a fact about non-symmetric Heintze spaces observed by Heintze in [Hei74a]: that they do not admit any finite-volume quotient. Recall that any lattice in a solvable Lie group must be uniform (see, for example, [Rag07]), and so a finite-volume quotient can only be compact in this setting. Because the Riemannian metric is compatible with the manifold structure from the Lie group, a uniform lattice of Riemannian isometries would induce a uniform Lie group lattice (the same discrete subgroup), so it suffices to contradict the existence of such a metric lattice. A uniform Riemannian lattice however, when seen as acting on a Heintze space of Carnot-type with an $H$–metric, would constitute a general type action by cocompactness.

Intuitively, we can think of the above obstruction as owing to the fact that the stabilizer of the point is the whole group, and so should look like the solvable Lie group itself. However, for discrete (non-elementary) groups hyperbolicity and solvability are incompatible. In fact, Heintze proves something stronger, that the only connected homogeneous Riemannian manifolds of strictly negative curvature that can admit quotients of finite volume are symmetric spaces.

We should note that the fact that non-symmetric Heintze spaces do not admit lattices is well-known and can be seen from multiple different perspectives. In addition to [Hei74a], one can derive the desired result using a Lie theoretic approach by applying [CT11] which completely classifies those unimodular Gromov-hyperbolic Lie groups.

We can use this lack of a lattice for nonsymmetric Heintze spaces to talk about another property of interest for Riemannian manifolds.
Definition 7.4. A Riemannian manifold is an *Einstein manifold* if its Ricci tensor is a constant multiple of its metric tensor.

We recall a result from [AC99].

Proposition 7.5. [AC99] A simply connected, nonflat, Einstein solvmanifold such that the component orthogonal to the derived subalgebra is abelian admits a finite volume quotient if (and only if) the adjoint actions of all vectors in the abelian component are symmetric.

One can quickly verify all of the above hypotheses, including symmetry of the adjoint action by Theorem 4.14 hold for our Heintze spaces of Carnot-type when equipped with $H$–metrics, with the possible exception of the Einstein condition. Therefore we get the following.

Corollary 7.6. Heintze spaces of Carnot-type equipped with $H$–metrics which are not symmetric are not Einstein manifolds.

In fact the above statement holds for any horocyclic metric. This should not be considered surprising - indeed for 2-step Carnot groups, it is demonstrated in 3.22 of [EH96] that the only Heintze spaces of Carnot-type which can admit metrics that make them Einstein manifolds are symmetric.

8. Further Questions

There are a number of natural generalizations to these ideas. We list a few questions that arise from this discussion.

**Question 8.1.** Are all Heintze spaces that arise from square-pinched algebras of Carnot-type when the pinching bound is tight? Does the information of the curvature give a way to put a stratification on the nilpotent base?

If this is true, one expects that a necessary and sufficient condition for a vector to lie in $V_i$ is for the appropriate vertical plane to have curvature $-i^2$.

**Question 8.2.** Is it true that the $\frac{1}{2}$–pinching constant is optimal for all Heintze spaces of Carnot-type, regardless of the existence of lattices of the base group?

The author suspects the answer to the above question is yes.

**Question 8.3.** What does the space of $H$–metrics look like?

Of course, we may consider the properties of other forms of curvature, beyond the failure of the Einstein condition for Ricci curvature.

**Question 8.4.** What can be said about the Ricci and scalar curvatures of an $H$–metric?

The fact that these metrics make the Lie algebra of Iwasawa type should help answer this question; see the formulas present in [Wo91], or [KP10] for a more general context.
Appendix A. A Computation

It is clear that any Riemannian metric on $\mathbb{RH}^n$ that gives constant curvature $-1$ is an $H$–metric. We describe here the next easiest case - $\mathbb{CH}^2$. Recall the Bergman metric on the Siegel domain model of $\mathbb{CH}^2$ is defined as the following metric tensor, with coordinates $(x, y, z, t), t > 0$.

$$
(\mathfrak{g}, \langle \cdot, \cdot \rangle) = \begin{pmatrix}
\frac{4(z+y^2)}{z^2} & \frac{-4xy}{z^2} & 0 & \frac{-2y}{z^2} \\
\frac{-4xy}{z^2} & \frac{4(z+x^2)}{z^2} & 0 & \frac{2x}{z^2} \\
0 & 0 & \frac{1}{z^2} & 0 \\
\frac{-2y}{z^2} & \frac{2x}{z^2} & 0 & \frac{1}{z^2}
\end{pmatrix}
$$

A description of the above metric, and its derivation, are available in [Par03]. The coordinates defining the chart above are valid over all of $\mathbb{CH}^2$, making it generic for the purposes of computation. While this may look similar to a metric arising from Proposition 4.5, it is not of the same form. From this information, however, it is a straightforward computational exercise to generate sectional curvature, by way of Christoffel symbols and the Riemannian curvature tensor. While this may be done by hand under extreme duress, the author recommends the usage of Mathematica. A detailed outline for using the program to generate sectional curvature from a metric tensor is available in [PHO19]. Indeed, this exact example is present in the section regarding applications to topology. Using the referenced tools, or any other method, one finds the following curvatures:

$$
K(\partial_x, \partial_y) = -\frac{x^2 + y^2 + 4z}{x^2 + y^2 + z} \quad K(\partial_x, \partial_z) = -\frac{4y^2 + z}{y^2 + z} \quad K(\partial_x, \partial_t) = -1
$$

$$
K(\partial_y, \partial_z) = -\frac{4y^2 + z}{y^2 + z} \quad K(\partial_y, \partial_t) = -1 \quad K(\partial_z, \partial_t) = -4
$$

Of course, the set $\{\partial_x, \partial_y, \partial_z, \partial_t\}$ does not itself satisfy the conditions of a layered basis - it is neither normal, nor orthogonal. However, it does have pinched negative curvature $[-4, -1]$. Furthermore, if we consider the vertical direction as that subspace generated by $\partial_t$, we see that the vertical planes condition is satisfied by considering $V_1 = \text{Span}(\partial_x, \partial_y), V_2 = \text{Span}(\partial_z)$ (we abuse notation here slightly because we have not specified the Lie structure). Because $\mathbb{CH}^2$ is symmetric and its isometry group acts transitively on the boundary, we lose no generality assuming that $\partial_t$ is vertical. We conclude what is already proven in Proposition 3.2 - that this metric is an $H$–metric.

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