Non-Abelian Stokes Theorem and Quark Confinement in $SU(N)$ Yang-Mills Gauge Theory

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We derive a new version of the non-Abelian Stokes theorem for the Wilson loop in the $SU(N)$ case by making use of the coherent state representation on the coset space $SU(N)/U(1)^{N-1}$, the flag space. We consider the $SU(N)$ Yang-Mills theory in the maximal Abelian gauge in which $SU(N)$ is broken down to $U(1)^{N-1}$. First, we show that the Abelian dominance in the string tension follows from this theorem and the Abelian-projected effective gauge theory that was derived by one of the authors. Next (but independently), combining the non-Abelian Stokes theorem with a novel reformulation of the Yang-Mills theory recently proposed by one of the authors, we proceed to derive the area law of the Wilson loop in four-dimensional $SU(N)$ Yang-Mills theory in the maximal Abelian gauge. Owing to dimensional reduction, the planar Wilson loop at least for the fundamental representation in four-dimensional $SU(N)$ Yang-Mills theory can be estimated by the diagonal (Abelian) Wilson loop defined in the two-dimensional $CP^{N-1}$ model. This derivation shows that the fundamental quarks are confined by a single species of magnetic monopole. The origin of the area law is related to the geometric phase of the Wilczek-Zee holonomy for $U(N−1)$. The calculations are performed using the instanton calculus (in the dilute instanton-gas approximation) and using the large $N$ expansion (in the leading order).

§1. Introduction

Gell-Mann and Zweig$^1$ predicted in the mid-1960s that all hadrons (i.e., baryons and mesons) are composed of the fundamental constituents having fractional charges, $\pm \frac{2}{3} e$ or $\pm \frac{1}{3} e$, with $e$ being the elementary charge. Now, the fundamental constituent is called the quark and the proposed theory is called the quark model. The predictions of this model are consistent with the results of experiments performed over the past thirty years. The strong interaction among quarks and anti-quarks is mediated by the gluon, which is described by the quantized Yang-Mills gauge field theory.$^2$ The present fundamental theory describing the quark and the gluon is provided by quantum chromodynamics (QCD), which is a non-Abelian gauge theory with the gauge group $G = SU(3)$ corresponding to three colors. However, neither an isolated quark nor an isolated anti-quark has ever been observed experimentally. In the present understanding, they are believed to be confined in hadrons. This is the hypothesis of quark confinement. Quark confinement could be explained theoretically within the framework of QCD, although no one has achieved a rigorous proof of quark confinement. This is one of the most important problems to be solved in theoretical physics.
QCD has a remarkable property, called asymptotic freedom, which was discovered by Gross, Wilczek and Politzer and independently by 't Hooft. Asymptotic freedom does not appear in the most successful quantized field theory, quantum electrodynamics (QED). As is well known, QED is the Abelian gauge theory for the electron and the photon in which the electromagnetic interaction is described by the quantized Maxwell gauge field theory with the gauge group $G = U(1)$. Asymptotic freedom is a consequence of gluon self-interactions. Therefore, this is a very characteristic feature of non-Abelian gauge theory.

The purpose of this article is to demonstrate quark confinement within QCD based on the Wilson criterion for quark confinement, i.e., the area law of the Wilson loop. The Wilson loop is a gauge invariant quantity and hence the Wilson criterion is also gauge invariant. The formulation of lattice gauge theory proposed by Wilson is manifestly gauge invariant and does not need the gauge fixing. It is easy to show that the strong coupling expansion in lattice gauge theory leads to the area law of the Wilson loop. However, this result has not yet been continued to the weak coupling region, where the string tension is expected to obey the scaling law suggested from the result of the renormalization group based on loop calculations. The first indication of the area law of the Wilson loop for arbitrary coupling constant was found in a study based on numerical simulations within the lattice gauge theory by Creutz for $G = SU(2)$ and $SU(3)$. Although the numerical evidence of quark confinement was indeed a great progress toward a complete understanding of quark confinement, the analytical proof is still lacking.

This work was initiated to justify the dual superconductor picture of the QCD vacuum proposed in the mid-1970s within the framework of continuum quantum field theory. For dual superconductivity to occur, magnetic monopoles must be condensed, just as ordinary superconductivity requires condensation of Cooper pairs. In fact, the importance and the validity of taking into account magnetic monopoles in quark confinement has been demonstrated, at least for simplified four-dimensional and lower-dimensional models, especially, by Polyakov, and recently for the four-dimensional Yang-Mills theory and QCD with extended supersymmetries by Seiberg and Witten. In this scenario, quark confinement is realized due to condensation of magnetic monopoles. Recent developments in numerical simulations on the lattice confirm the existence of dual superconductivity in QCD, at least, under a specific gauge fixing called the Abelian gauge.

This article gives a detailed exposition of the results on quark confinement that were announced in a previous article for $G = SU(3)$ together with new results for $G = SU(N)$ with arbitrary $N$. They are extensions of the analyses of the Yang-Mills theory in the maximal Abelian (MA) gauge given in a series of articles, where the case of $SU(2)$ was explicitly worked out. In this process, we have found that the extension from $SU(2)$ to $SU(3)$ is non-trivial, but the extension from $SU(3)$ to $SU(N)$, $N > 3$, is rather straightforward. New features come out when we begin to analyze the $SU(N)$ case with $N \geq 3$. It seems that they have been overlooked to this time in the conventional approach based on the maximal Abelian gauge.

The MA gauge is a partial gauge fixing from the original non-Abelian gauge group $G$ to its subgroup $H$ in which the gauge degrees of freedom of the coset
$G/H$ are fixed. Even after the MA gauge, there is a residual gauge group $H$ which is taken to be the maximal torus subgroup $H = U(1)^{N-1}$. After the MA gauge, the magnetic monopole is expected to appear, since the Homotopy group $\pi_2(G/H)$ is non-trivial, i.e.,

$$\pi_2(SU(N)/U(1)^{N-1}) = \pi_1(U(1)^{N-1}) = \mathbb{Z}^{N-1}. \quad (1.1)$$

This implies that the breaking of gauge group $G \to H$ by partial gauge fixing leads to $(N - 1)$ species of magnetic monopoles. However, we do not necessarily need to consider the maximal breaking $SU(N) \to U(1)^{N-1}$, although the maximal torus group is desirable as a gauge group of the low-energy effective Abelian gauge theory.\(^{12}\) Actually, even if we restrict $H$ to a continuous subgroup $\ast$ of $G$, there are other possibilities for choosing $H$, e.g., we can choose a subgroup $\tilde{H}$ such that

$$G \supset \tilde{H} \supset H := U(1)^{N-1}. \quad (1.2)$$

The possible number of cases for choosing $\tilde{H}$ increases as $N$ increases. We have found\(^{11}\) that the group $\tilde{H}$ may depend on the representation to which the quark belongs when $N \geq 3$ and that it suffices to take $\tilde{H} = U(N - 1)$ for the fundamental quark to be confined in the sense of the area law of the Wilson loop under the partial gauge fixing. Here $\tilde{H}$ is equal to the maximal stability group specified by the highest-weight state of the representation of the quark in the Wilson loop. This is a new feature which does not show up in the $SU(2)$ case. Nevertheless, this does not mean that the choice of the maximal torus does not lead to quark confinement. In fact, even if we choose the maximal torus, the area law can be derived. This is because the coset $G/\tilde{H}$ is contained in $G/H$, i.e., $G/\tilde{H} \subset G/H$, so that the Wilson loop does not feel the whole of $G/H$, but only feels the components of $G/\tilde{H}$ that are contained in $G/H$. In other words, the variables belonging to $G/H - G/\tilde{H}$ are irrelevant for the expectation value of the Wilson loop, as can be seen from the non-Abelian Stokes theorem (NAST) that was presented in Ref.\,11) and is derived in this article. Therefore, a single kind of magnetic monopole is sufficient for confining a fundamental quark, since

$$\pi_2(SU(N)/U(N - 1)) = \pi_1(U(1)) = \mathbb{Z}. \quad (1.3)$$

Our results show that two partial gauge fixings $SU(3) \to U(2)$ and $SU(3) \to U(1) \times U(1)$ lead to the same result for confinement as far as the fundamental quarks are concerned.\(^{\ast\ast}\)

The NAST plays the crucial role in this article. The NAST has a number of versions which have been derived by many authors.\(^{20}\) The version of NAST derived in this article is based on the idea of Dyakonov and Petrov,\(^{21}\) who derived an $SU(2)$ version and suggested a method of generalization. We derive a version of NAST based on the coherent state representation\(^{22}\) on the flag space,\(^{23},24\) not on the method \(^{\ast}\) The possibility of a discrete subgroup has been extensively investigated recently from the viewpoint of the Abelian gauge, e.g., the center $\mathbb{Z}_N$ for $SU(N)$ (see, e.g., Ref.\,18)).

\(^{\ast\ast}\) See Ref.\,19) for a result of the simulation on a lattice.
suggested by them. The coherent state representation is used in a different fashion to
derive an SU(2) version of the NAST in Ref. 15), but the extension to SU(N), N ≥ 3,
was a non-trivial issue which prevented us from presenting immediate publication
of general SU(N) results. The NAST is not only mathematically (or technically)
important but also physically interesting as we now discuss.

First, the NAST enables us to write the non-Abelian Wilson loop,
\[ W_C[A] := \frac{1}{N} \text{tr} \left( \mathcal{P} \exp \left( ig \oint_C A \right) \right), \quad \text{(1.4)} \]
in terms of the Abelian-field strength (curvature two-form) \( f = da \) with the Abelian
gauge potential (connection one-form) \( a \):
\[ W_C[A] = \int [d\mu(V)]_C \exp \left( ig \oint_C a \right) = \int [d\mu(V)]_C \exp \left( ig \int_S f \right). \quad \text{(1.5)} \]
Combining this fact with the Abelian-projected effective gauge theory (APEGT)
derived by one of the authors,\(^12\) we can explain the Abelian dominance\(^{25,26}\) in the
Wilson loop. The APEGT is an Abelian gauge theory obtained by integrating out
the massive degrees of freedom, i.e., the off-diagonal gluon gauge field \( A^a_\mu \) with mass \( m_A \). Hence, the APEGT can be written in terms of the diagonal massless gauge field \( a^i_\mu \) and the anti-symmetric (Abelian) tensor field \( B^i_{\mu\nu} \) together with the ghost and
anti-ghost fields \( C^a \) and \( \bar{C}^a \), where the index \( i \) denotes the diagonal components and
\( a \) the off-diagonal ones. Therefore the APEGT is regarded as a low-energy effective
theory (LEET) which is valid in the long-distance (or low-energy) region \( R > m_A^{-1} \).
The Abelian gauge field \( b^i_\mu \) obtained after the Hodge decomposition of \( B^i_{\mu\nu} \) can be
identified with the Abelian gauge field dual to \( a^i_\mu \). In fact, we can obtain the theory
with an action \( S[b] \) written in terms of \( b^i_\mu \) alone by integrating out all the fields
other than \( b^i_\mu \) in APEGT, and the theory can be rewritten into the dual Ginzburg-
Landau theory, i.e., the dual Abelian Higgs theory, provided that magnetic monopole
condensation occurs, i.e., \( \langle k_\mu k_\mu \rangle \neq 0 \). In the dual Ginzburg-Landau theory, the
coupling constant \( g \) in the original theory is replaced by the inverse coupling constant
\( 1/g \) which is proportional to the magnetic charge. Therefore, the dual theory can
be identified with the magnetic theory. On the other hand, the theory with an
action \( S[a] \) written in terms of \( a^i_\mu \) alone is an Abelian gauge theory, but the scale
dependence of the coupling constant \( g(\mu) \) is the same as that in the original Yang-
Mills theory. Thus the low-energy effective Abelian gauge theory exhibits asymptotic
freedom reproducing the original renormalization-group beta function \( \beta(g) \). This is a
manifestation of an approximate weak-strong or electro-magnetic duality between
two low-energy effective theories described by \( S[a] \) and \( S[b] \).

Next, the NAST is able to separate the piece \( \omega \) which corresponds to the mag-
netic monopole in the Abelian field \( a = C + \omega \). Indeed, we can write the \( SU(N) \)
version of the \`t Hooft-Polyakov tensor describing the magnetic monopole.\(^{27}\) Hence
we can separate the contribution of the magnetic monopole in the area law of the
Wilson loop and explain the magnetic monopole dominance in quark confinement.
In fact, our derivation of the area law estimates only the monopole contribution,
\( \Omega_K = d\omega \). Moreover, the NAST tells us that the essential ingredient in the area
law lies in the geometric phase, which is concerned with the holonomy group of \( U(N-1) \). Thus we see that quark confinement is intimately related to the geometry of Yang-Mills gauge theory, in sharp contrast with the conventional wisdom.

We present two methods to derive the area law of the Wilson loop by making use of the NAST. The first method is to use the APEGT for estimating the diagonal Wilson loop; for a sufficiently large Wilson loop \( (|C| \gg m^{-1}_A) \), the expectation value of the non-Abelian Wilson loop in Yang-Mills theory is reduced to that of the Abelian Wilson loop in APEGT:

\[
\left\langle W^C[A] \right\rangle_{YM} = \left\langle \exp \left( ig \oint_C a \right) \right\rangle_{YM} \rightarrow \left\langle \exp \left( ig \oint_C a \right) \right\rangle_{APEGT} \quad (1.6)
\]

Then we can apply the result of Ref. 14), confinement in the Abelian gauge theory, to show quark confinement in Yang-Mills theory.

The second method is to treat the non-Abelian gauge theory directly, without going through the effective Abelian gauge theory, based on the novel reformulation of Yang-Mills theory in the MA gauge which was proposed by one of the authors.\(^{13}\) The novel reformulation regards the Yang-Mills theory as the perturbative deformation of a topological quantum field theory (TQFT). An advantage of this reformulation in the MA gauge is that the derivation of the area law of the non-Abelian Wilson loop in the four-dimensional \( SU(N) \) Yang-Mills theory is reduced to that of the diagonal (Abelian) Wilson loop in the two-dimensional coset \((G/H)\) non-linear sigma (NLS) model, at least when the Wilson loop is planar. Therefore the four-dimensional problem is reduced to a two-dimensional problem. This dimensional reduction is a remarkable feature of the modified MA gauge\(^*\) caused by hidden supersymmetry. The Yang-Mills coupling constant \( g \) of the four-dimensional Yang-Mills theory is mapped into the coupling constant in the two-dimensional NLS model. Hence the coupling constant is expected to run in the same way as in the original Yang-Mills theory, since the perturbative deformation part provides the necessary running, as is well known from the loop calculation. For the fundamental quark, we are allowed to restrict the flag space \( F_{N-1} := SU(N)/U(1)^{N-1} \) to the complex projective space \( CP^{N-1} := SU(N)/U(N-1) \). This greatly simplifies the actual treatment.

Another advantage of this reduction is that the magnetic monopole contribution to the Wilson loop in the four-dimensional Yang-Mills theory in the MA gauge is equal to the instanton contribution in the corresponding two-dimensional NLS model. Indeed, the diagonal Wilson loop can be written as the area integral of the instanton density \( \Omega_K \) over the area \( S \) bounded by the loop \( C \). This correspondence may shed more light on the strong correlation between magnetic monopoles and instantons observed in the Monte Carlo simulations, since the two-dimensional instanton is identified as a subclass of the four-dimensional Yang-Mills instanton (see e.g. Ref. 13)).

In this article the expectation value of the Wilson loop is estimated by combining the instanton calculus and the large \( N \) expansion. (See Refs. 28)–31) for reviews of the large \( N \) expansion.) We focus on the \( CP^{N-1} \) model corresponding to the

\(^*\) We must modify the MA gauge slightly in order to keep the supersymmetry, where the supersymmetry is expressed by the orthosymplectic group \( OSp(4|2) \).
fundamental quark. First, the instanton calculus is performed within the dilute gas approximation. It is shown that the calculation in the $SU(N)$ case reduces to that in the $SU(2)$ case. It is well known that the large $N$ expansion is a non-perturbative technique which can be systematically improved. We derive the area law to leading order in the large $N$ expansion, namely, in the region of large $N$ and weak coupling $g$. We hope that our derivation of quark confinement based on the dimensional reduction and the large $N$ expansion may shed more light on the relationship between QCD and string theory, as first suggested by ’t Hooft. \(32\)

This article is organized as follows. In the first half, we give a derivation of the NAST and discuss its implications. Sections 2 and 3 are preparations for §4. In §2 we review the construct of the coherent state on the flag space for the general compact semi-simple group $G$. In §3, we present the explicit form of the coherent state on the flag space for $G = SU(N)$. We define the maximal stability group $\tilde{H}$, which is very important in the following discussion. In §4, making use of the results of §§2 and 3, we derive a new version of the non-Abelian Stokes theorem for $G = SU(N)$. Although we discuss only the case of $SU(N)$ explicitly, it is straightforward to extend this theorem to an arbitrary compact semi-simple group $G$. This version of the non-Abelian Stokes theorem is very interesting not only from the mathematical but also from the physical point of view, since the non-Abelian Wilson loop is expressed as the surface integral of the two-form (i.e., the generalized ’t Hooft-Polyakov tensor), which leads to the magnetic monopole. This fact is intimately related with the Abelian and magnetic monopole dominance in quark confinement, as discussed in subsequent sections.

In the second half, we derive the area law of the Wilson loop. In §5 we discuss the magnetic monopole in $SU(N)$ Yang-Mills theory. In order to specify the type of possible magnetic monopoles, it turns out that the maximal stability group is more important than the maximal torus group $H$. In §6 Abelian dominance in the Wilson loop is shown in the $SU(N)$ Yang-Mills theory in the maximal Abelian gauge based on the Abelian-projected effective gauge theory and the non-Abelian Stokes theorem. In §7 we briefly review a novel reformulation of the Yang-Mills theory which has been proposed by one of the authors\(^{13}\) to derive quark confinement. This reformulation is called the (perturbative) deformation of the topological quantum field theory. We apply this reformulation to derive the area law of the Wilson loop in $SU(N)$ Yang-Mills theory in §§8 and 9. In §9 we show within this reformulation that the area law of the Abelian Wilson loop in the two-dimensional nonlinear sigma model for the flag space $G/\tilde{H}$ is sufficient to derive the area law of the four-dimensional Yang-Mills theory in the maximal Abelian gauge. At the same time, this derivation leads to the magnetic monopole dominance in the area law. In §8 we demonstrate the area law of the Wilson loop in the nonlinear sigma model in an approach based on naive instanton calculus. For the fundamental quark, we have only to deal with the $CP^{N-1}$ model. In §9 we derive the area law based on the large $N$ expansion. These results imply the area law of the non-Abelian Wilson loop in the four-dimensional $SU(N)$ Yang-Mills theory. The final section contains the conclusion of this article.

In Appendix A, we give derivations of the inner product of the coherent states and the invariant measure on the flag space, which are presented in §3. In Appendix
B we explain the method of obtaining $CP^1$ and $CP^2$ by gluing the complex planes. In Appendix C, we explain two ways to characterize the element of the flag space and the manner of formulating the NLS model using these parameterizations. In Appendix D we summarize the large $N$ expansion of $CP^{N-1}$. In Appendix E supplementary material on the $1/N$ expansion is presented.

§2. Coherent state and maximal stability group

First, we construct the coherent state $|\xi, \Lambda\rangle$ corresponding to the coset representatives $\xi \in G/\tilde{H}$. We follow the method of Feng, Gilmore and Zhang. For inputs, we prepare the following:

(a) the gauge group $G$ and the Lie algebra $\mathcal{G}$ of $G$ with the generators $\{T^A\}$, which obey the commutation relations

$$[T^A, T^B] = if^{AB}_C T^C,$$

where the $f^{AB}_C$ are the structure constants of the Lie algebra. If the Lie algebra is semi-simple, it is more convenient to rewrite the Lie algebra in terms of the Cartan basis $\{H_i, E_\alpha, E_{-\alpha}\}$. There are two types of basic operators in the Cartan basis, $H_i$ and $E_\alpha$. The operators $H_i$ may be taken as diagonal, while $E_\alpha$ are the off-diagonal shift operators. They obey the commutation relations

$$[H_i, H_j] = 0,$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha,$$

$$[E_\alpha, E_{-\alpha}] = \alpha_i H_i,$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha;\beta} E_{\alpha+\beta}, & (\alpha + \beta \in R) \\ 0, & (\alpha + \beta \notin R, \alpha + \beta \neq 0) \end{cases}$$

where $R$ is the root system, i.e., a set of root vectors $\{\alpha_1, \cdots, \alpha_r\}$, with $r$ the rank of $G$.

(b) The Hilbert space $V^A$ is a carrier (the representation space) of the unitary irreducible representation $\Gamma^A$ of $G$.

(c) We use a reference state $|\Lambda\rangle$ within the Hilbert space $V^A$, which can be normalized to unity: $\langle\Lambda|\Lambda\rangle = 1$.

We define the maximal stability subgroup (isotropy subgroup) $\bar{H}$ as a subgroup of $G$ that consists of all the group elements $h$ that leave the reference state $|\Lambda\rangle$ invariant up to a phase factor:

$$h|\Lambda\rangle = |\Lambda\rangle e^{i\phi(h)}, h \in \bar{H}.$$

* Note that any compact semi-simple Lie group is a direct product of compact simple Lie group. Therefore, it is sufficient to consider the case of a compact simple Lie group. In the following we assume that $G$ is a compact simple Lie group, i.e., a compact Lie group with no closed connected invariant subgroup.
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The phase factor is unimportant in the following discussion because we consider the expectation value of operators in the coherent state. Let $H$ be the Cartan subgroup of $G$, i.e., the maximal commutative semi-simple subgroup in $G$, and let $\mathcal{H}$ be the Cartan subalgebra in $\mathcal{G}$, i.e., the Lie algebra for the group $H$. The maximal stability subgroup $\tilde{H}$ includes the Cartan subgroup $H = U(1)^r$, i.e., $H \subset \tilde{H}$.

For every element $g \in G$, there is a unique decomposition of $g$ into a product of two group elements,

$$g = \xi h, \quad \xi \in G/\tilde{H}, \quad h \in \tilde{H},$$

for $g \in G$. We can obtain a unique coset space $G/\tilde{H}$ for a given $|\Lambda\rangle$. The action of arbitrary group element $g \in G$ on $|\Lambda\rangle$ is given by

$$g|\Lambda\rangle = \xi h|\Lambda\rangle = \xi|\Lambda\rangle e^{i\phi(h)}.$$  

The coherent state is constructed as $|\xi, \Lambda\rangle = \xi|\Lambda\rangle$. This definition of the coherent state is in one-to-one correspondence with the coset space $G/\tilde{H}$ and the coherent states preserve all the algebraic and topological properties of the coset space $G/\tilde{H}$.

If $\Gamma(\mathcal{G})$ is Hermitian, then $H_i^\dagger = H_i$ and $E_\alpha^\dagger = E_{-\alpha}$. Every group element $g \in G$ can be written as the exponential of a complex linear combination of diagonal operators $H_i$ and off-diagonal shift operators $E_\alpha$. Let $|\Lambda\rangle$ be the highest-weight state, i.e., $H_j|\Lambda\rangle = \Lambda_j|\Lambda\rangle$, $E_\alpha|\Lambda\rangle = 0$ for $\alpha \in R_+$, where $R_+(R_-)$ is a subsystem of positive (negative) roots. Then the coherent state is given by

$$|\xi, \Lambda\rangle = \exp \left[ \sum_{\beta \in R_-} \left( \eta_\beta E_\beta - \bar{\eta}_\beta E_\beta^\dagger \right) \right] |\Lambda\rangle, \quad \eta_\beta \in C,$$

such that the following hold:

(i) $|\Lambda\rangle$ is annihilated by all the (off-diagonal) shift-up operators $E_\alpha$ with $\alpha \in R_+$, $E_\alpha|\Lambda\rangle = 0(\alpha \in R_+)$;

(ii) $|\Lambda\rangle$ is mapped into itself by all diagonal operators $H_i$, $H_i|\Lambda\rangle = \Lambda_i|\Lambda\rangle$;

(iii) $|\Lambda\rangle$ is annihilated by some shift-down operators $E_\alpha$ with $\alpha \in R_-$, not by other $E_\beta$ with $\beta \in R_-$: $E_\alpha|\Lambda\rangle = 0$(some $\alpha \in R_-$); $E_\beta|\Lambda\rangle = |\Lambda + \beta\rangle$(some $\beta \in R_-$);

and the sum $\sum_\beta$ is restricted to those shift operators $E_\beta$ which obey (iii).

The coherent states are normalized to unity:

$$\langle \xi, \Lambda | \xi, \Lambda \rangle = 1.$$  

The coherent state spans the entire space $V^\Lambda$. However, the coherent states are non-orthogonal:

$$\langle \xi', \Lambda | \xi, \Lambda \rangle \neq 0.$$  

By making use of the the group-invariant measure $d\mu(\xi)$ of $G$ which is appropriately normalized, we obtain

$$\int |\xi, \Lambda\rangle d\mu(\xi) \langle \xi, \Lambda | = I,$$

which shows that the coherent states are complete, but in fact over-complete. This resolution of identity is very important to obtain the path integral formula of the Wilson loop given in §4.

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The coherent states $|\xi, \Lambda\rangle$ are in one-to-one correspondence with the coset representatives $\xi \in G/\tilde{H}$:

$$|\xi, \Lambda\rangle \leftrightarrow G/\tilde{H}. \quad (2.13)$$

In other words, $|\xi, \Lambda\rangle$ and $\xi \in G/\tilde{H}$ are topologically equivalent.

§3. **Flag space and coherent state for $SU(N)$**

3.1. **$SU(2)$ coherent state**

In the case of $SU(2)$, the maximal stability group agrees with the maximal torus group $U(1)$ irrespective of the representation. The $SU(2)$ case is well known (see, e.g., Ref. 15). The weight and root diagrams are given in Fig. 1.

The coherent state for $F_1 := SU(2)/U(1)$ is obtained as

$$|j, w\rangle = \xi(w)|j, -j\rangle = e^{\xi J_+ - \bar{\xi} J_-} |j, -j\rangle = \frac{1}{(1 + |w|^2)^j} e^{w J_+} |j, -j\rangle, \quad (3.1)$$

where $|j, -j\rangle$ is the lowest state, $|j, m = -j\rangle$, of $|j, m\rangle$, and

$$J_+ = J_1 + i J_2, \quad J_- = J_+^\dagger, \quad w = \frac{\zeta \sin |\zeta|}{|\zeta| \cos |\zeta|}. \quad (3.2)$$

Note that $(1 + |w|^2)^{-j}$ is a normalization factor that ensures $\langle j, w|j, w\rangle = 1$, which is obtained from the Baker-Campbell-Hausdorff (BCH) formulas. The invariant measure is given by

$$d\mu = \frac{2j + 1}{4\pi} \frac{dwd\bar{w}}{(1 + |w|^2)^2}. \quad (3.3)$$

For $J_A = \frac{1}{2}\sigma^A (A = 1, 2, 3)$ with Pauli matrices $\sigma^A$, we obtain $J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and

$$e^{w J_+} = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in F_1 = CP^1 = SU(2)/U(1) \cong S^2. \quad (3.4)$$

Fig. 1. Root diagram and weight diagram of the fundamental representation of $SU(2)$ where $\Lambda$ is the highest weight of the fundamental representation.
The complex variable $w$ is a $\mathbb{CP}^1$ variable written as $w = e^{i\phi} \cot \frac{\theta}{2}$, in terms of the polar coordinate on $S^2$ or Euler angles, see Ref. 13). We introduce the $O(3)$ vector $\mathbf{n}$ as

$$n^1 := \sin \theta \cos \phi, \quad n^2 := \sin \theta \sin \phi, \quad n^3 := \cos \phi. \quad (3.5)$$

The relation

$$n^A(x) = \tilde{\phi}_a(x) \sigma^A_{ab} \phi_b(x) \quad (a, b = 1, 2) \quad (3.6)$$

is equivalent to

$$n_1 = 2 \Re(\phi_1 \bar{\phi}_2), \quad n_2 = 2 \Im(\phi_1 \bar{\phi}_2), \quad n_3 = |\phi_1|^2 - |\phi_2|^2. \quad (3.7)$$

The complex coordinate $w$ obtained by the stereographic projection from the north pole is identical to the inhomogeneous local coordinates of $\mathbb{CP}^1$ when $\phi_2 \neq 0$,

$$w = w^{(1)} + iw^{(2)} = \frac{n_1 + in_2}{1 - n_3} = \frac{2\phi_1 \bar{\phi}_2}{(|\phi_1|^2 + |\phi_2|^2) - (|\phi_1|^2 - |\phi_2|^2)} = \frac{\phi_1}{\phi_2}. \quad (3.8)$$

The stereographic projection from the south pole leads to

$$w = \frac{n_1 + in_2}{1 + n_3} = \left( \frac{\phi_2}{\phi_1} \right)^* \quad (3.9)$$

if $\phi_1 \neq 0$. The variable $w$ is $U(1)$ gauge invariant. Another representation of $\mathbf{n}$ is obtained by using the parameterization (3.4) of the $F_1$ variable $\xi$:

$$n^A = \langle A| \xi(w) \dagger \sigma^A \xi(w) | A \rangle = (\tilde{\phi}_1 \quad 0) \begin{pmatrix} 1 & \bar{\omega} \\ 0 & 1 \end{pmatrix} \sigma^A \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}. \quad (3.10)$$

This leads to

$$n_1 = |\phi_1|^2 (w + \bar{\omega}), \quad n_2 = -i|\phi_1|^2 (w - \bar{\omega}), \quad n_3 = |\phi_1|^2 (1 - w\bar{\omega}). \quad (3.11)$$

Indeed, this agrees with (3.7) if $w = (\phi_2/\phi_1)^*$. The entire space of $F_1$ is covered by two charts,

$$\mathbb{CP}^1 = U_1 \cup U_2, \quad U_a = \{(\phi_1, \phi_2) \in \mathbb{CP}^1; \phi_a \neq 0\}. \quad (3.12)$$

### 3.2. $SU(3)$ coherent state

For concreteness, we first focus on the $SU(3)$ case. The general $SU(N)$ case will be discussed in the final part of this section. The highest weight $\Lambda$ of the representation specified by the Dynkin index $[m, n]$ can be written as

$$\Lambda = m\vec{h}_1 + n\vec{h}_2, \quad (3.13)$$

where $m$ and $n$ are non-negative integers for the highest weight and $\vec{h}_1$ and $\vec{h}_2$ are the highest weights of the two fundamental representations of $SU(3)$ corresponding to $[1, 0]$ and $[0, 1]$, respectively (see Fig. 2)

$$\vec{h}_1 = \left( \frac{1}{2}, \frac{1}{2\sqrt{3}} \right), \quad \vec{h}_2 = \left( 0, \frac{1}{\sqrt{3}} \right). \quad (3.14)$$
Non-Abelian Stokes Theorem and Quark Confinement

The weight diagram and root vectors required to define the coherent state in the fundamental representations $[1, 0] = 3$, $[0, 1] = 3^*$ of $SU(3)$ where $\vec{\Lambda} = \vec{h}_1 = \nu^1 := (\frac{1}{2}, \frac{1}{2\sqrt{3}})$ is the highest weight of the fundamental representation, and the other weights are $\nu^2 := (-\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and $\nu^3 := (0, -\frac{1}{\sqrt{3}})$.

Therefore, we obtain

$$\vec{\Lambda} = \left(\frac{m}{2}, \frac{m + 2n}{2\sqrt{3}}\right). \tag{3.15}$$

The generators of $SU(3)$ in the Cartan basis are written as \{\(H_1, H_2, E_\alpha, E_\beta, E_{\alpha + \beta}, E_{-\alpha}, E_{-\beta}, E_{-\alpha - \beta}\)\}, where $\alpha$ and $\beta$ are the two simple roots. (See Fig. 3 for the explicit choice.)

If $mn = 0$, ($m = 0$ or $n = 0$), the maximal stability group $\tilde{H}$ is given by $\tilde{H} = U(2)$ with generators \{\(H_1, H_2, E_\beta, E_{-\beta}\)\} (case (I)). Such a degenerate case occurs when the highest-weight vector $\vec{\Lambda}$ is orthogonal to some root vectors (see Fig. 2). If $mn \neq 0$ ($m \neq 0$ and $n \neq 0$), $H$ is the maximal torus group $H = U(1) \times U(1)$ with generators \{\(H_1, H_2\)\} (case (II)). This is a non-degenerate case (see Fig. 4). Therefore, for the highest weight $\Lambda$ in case (I), the coset $G/\tilde{H}$ is given by

$$SU(3)/U(2) = SU(3)/(SU(2) \times U(1)) = CP^2, \tag{3.16}$$

whereas in case (II),

$$SU(3)/(U(1) \times U(1)) = F_2. \tag{3.17}$$

Here, $CP^n$ is the complex projective space and $F_n$ is the flag space. Therefore, the two fundamental representations belong to case (I), and hence the maximal stability group is $U(2)$, rather than the maximal torus group $U(1) \times U(1)$. The implications of this fact for the mechanism of quark confinement is discussed in subsequent sections.

The coherent state for $F_2 = SU(3)/U(1)^2$ is given by

$$|\xi, \Lambda\rangle = \xi(w)|\Lambda\rangle := V^\dagger(w)|\Lambda\rangle, \tag{3.18}$$

This choice of $h_2$ is different from that in Ref. 11. It is adopted so as to obtain the $SU(3)$ case when considering the $N = 3$ case of $SU(N)$ case studied in the next subsection.
with the highest- (lowest-) weight state $|\Lambda\rangle$, i.e.,

$$|\xi, \Lambda\rangle = \exp \left[ \sum_{\alpha \in R^+} (\zeta_{\alpha} E_{-\alpha} - \bar{\zeta}_{\alpha} E_{-\alpha}^\dagger) \right] |\Lambda\rangle = e^{-\frac{1}{2}K(w,\bar{w})} \exp \left[ \sum_{\alpha \in R^+} \tau_{\alpha} E_{-\alpha} \right] |\Lambda\rangle,$$

(3.19)

where $e^{-\frac{1}{2}K}$ is the normalization factor obtained from the Kähler potential (explained below):

$$K(w,\bar{w}) := \ln[(\Delta_1(w,\bar{w}))^m(\Delta_2(w,\bar{w}))^n],$$

(3.20)

$$\Delta_1(w,\bar{w}) := 1 + |w_1|^2 + |w_2|^2, \Delta_2(w,\bar{w}) := 1 + |w_3|^2 + |w_2 - w_1w_3|^2.$$  (3.21)

The coherent state $|\xi, \Lambda\rangle$ is normalized, so that $\langle \xi, \Lambda | \xi, \Lambda \rangle = 1$. We show in Appendix A that the inner product is given by

$$\langle \xi', \Lambda | \xi, \Lambda \rangle = e^{K(w',\bar{w'})} e^{-\frac{1}{2}[K(w',\bar{w'}) + K(w,\bar{w})]},$$

(3.22)

where

$$K(w,\bar{w'}) := \ln[1 + \bar{w}_1'w_1 + \bar{w}_2'w_2]^m[1 + \bar{w}_3'w_3 + (\bar{w}_2' - \bar{w}_1')\bar{w}_3'](w_2 - w_1w_3)^n.$$  (3.23)

Note that $K(w,\bar{w'})$ reduces to the Kähler potential $K(w,\bar{w})$ when $w' = w$, in agreement with the normalization. It follows from the general formula (see discussion of
the $SU(N)$ case) that the $SU(3)$ invariant measure is given (up to a constant factor) by

$$d\mu(\xi) = d\mu(w, \bar{w}) = D(m, n)[(\Delta_1)^m(\Delta_2)^n]^{-2}\prod_{\alpha=1}^{3}dw_\alpha d\bar{w}_\alpha,$$

(3.24)

where $D(m, n) = \frac{1}{2}(m+1)(n+1)(m+n+2)$ is the dimension of the representation. For the choice of shift-up $(E_{+1})$ or shift-down $(E_{-1})$ operators

$$E_{\pm 1} := \frac{\lambda_1 \pm i\lambda_2}{2\sqrt{2}}, \quad E_{\pm 2} := \frac{\lambda_4 \pm i\lambda_5}{2\sqrt{2}}, \quad E_{\pm 3} := \frac{\lambda_6 \pm i\lambda_7}{2\sqrt{2}},$$

(3.25)

with the Gell-Mann matrices $\lambda_A (A = 1, \ldots, 8)$, we obtain

$$\exp\left[\sum_{i=1}^{3}\tau_i E_{-i}\right] = \begin{pmatrix} 1 & w_1 & w_2 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{pmatrix}^t \in F_2 = SU(3)/U(1)^2,$$

(3.26)

where we have used the abbreviation $E_{\pm i} \equiv E_{\pm\alpha(i)} (i = 1, 2, 3)$. These two sets of three complex variables are related as (see Appendix A)

$$w_1 = \frac{\tau_1}{\sqrt{2}}, \quad w_2 = \frac{\tau_2}{\sqrt{2}} + \frac{\tau_1 \tau_3}{4}, \quad w_3 = \frac{\tau_3}{\sqrt{2}},$$

(3.27)

or conversely

$$\tau_1 = \sqrt{2}w_1, \quad \tau_2 = \sqrt{2}\left(w_2 - \frac{w_1 w_3}{2}\right), \quad \tau_3 = \sqrt{2}w_3.$$

(3.28)

The complex projective space $CP^2$ is covered by three complex planes $C$ through holomorphic maps $(33)$ (see Appendix B). The parameterization of $SU(3)$ in terms of eight angles is also possible in $SU(3)$, just as $SU(2)$ is parameterized by three Euler angles (see Ref. 34)).

### 3.3. $SU(N)$ case

For $SU(N) = SU(n+1)$, the flag space $F_n\equiv SU(n+1)/U(1)^n \ni V$. $F_n$ is a compact Kähler manifold, $(35),(36)$ which is a homogeneous but nonsymmetric manifold of dimension $\text{dim} F_n = n(n+1)/2$.

Since the flag manifold $F_n$ is a Kähler manifold, $(35),(36)$ it possesses complex local coordinates $w_\alpha$, a Hermitian Riemannian metric,

$$ds^2 = g_{\alpha\beta}dw^\alpha d\bar{w}^\beta,$$

(3.30)

and a corresponding two-form, called the Kähler form,$^\ast$)

$$\Omega_K = ig_{\alpha\beta}dw^\alpha \wedge d\bar{w}^\beta,$$

(3.31)

$^\ast$) The imaginary unit $i$ is needed to make the Kähler two-form real, since $g_{\alpha\beta} = g_{\alpha\beta} = g_{\beta\alpha}$.
which is closed, i.e.,

\[ d\Omega_K = 0. \quad (3.32) \]

Any closed form \( \Omega_K \) is locally exact (\( \Omega_K = d\omega \)), due to Poincaré’s lemma. The condition (3.32) is equivalent to

\[ \frac{\partial g_{\alpha\bar{\beta}}}{\partial w^\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial w^\alpha}, \quad \text{or} \quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial w^\gamma} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial w^\beta}. \quad (3.33) \]

This holds if and only if the metric \( g_{\alpha\beta} \) can be obtained from a real scalar function \( K \) as

\[ g_{\alpha\bar{\beta}} = \frac{\partial}{\partial w^\alpha} \frac{\partial}{\partial w^\beta} K, \quad (3.34) \]

where \( K = K(w, \bar{w}) \) is called the Kähler potential. Then the Kähler two-form is obtained from (3.31) as

\[ \Omega_K = i\bar{\partial}\partial K. \quad (3.35) \]

On the flag space, there transitively act two groups, \( G = SU(n+1) \) and its complexification \( G^c = SL(n+1, \mathbb{C}) \). Any element of \( F_n \) can be written as an upper triangular \((n+1) \times (n+1)\) matrix, whose main diagonal elements are all 1 and whose upper \( n(n+1)/2 \) elements are complex numbers, \( w_\alpha \in \mathbb{C} \):

\[ \xi = \begin{pmatrix} 1 & w_1 & w_2 & \cdots & \cdots & w_n \\ 0 & 1 & w_{n+1} & \cdots & \cdots & w_{2n-1} \\ 0 & 0 & 1 & w_{2n} & \cdots & w_{3n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & w_{n(n+1)/2} \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \in F_n. \quad (3.36) \]

Therefore, we can write

\[ F_n = SL(n+1, \mathbb{C})/B_- \ni \xi, \quad (3.37) \]

where \( B_- (B_+) \) is the Borel subgroup, i.e., the group of lower (upper) triangular matrices with determinant equal to 1 (Iwasawa decomposition). This definition (3.37)* should be compared with the first definition (3.29). The mapping \( G/H \to G^c/B_- \) is a generalization of the stereographic projection in the \( G = SU(2) \) case.\(^{13}\)

The action of the group \( SL(n+1, \mathbb{C}) \) on \( F_n \), \( g : V \to V_g \) can be found through the Gauss decomposition,

\[ V \cdot g = T DV_g, \quad g \in SL(n+1, \mathbb{C}), \quad T \in Z_-(n+1), \quad V_g \in Z_+(n+1), \quad (3.38) \]

where \( Z_+(n+1)(Z_-(n+1)) \) is the set of upper (lower) triangular matrices whose main diagonal elements are all 1 and \( D \) is a diagonal matrix with determinant equal

*\(^1\) Note that \( \xi \) is not necessarily unitary as a matrix under this definition.
to 1. The elements of the factors $T, D$ and $V_g$ are rational functions of the elements of $g$.

The group $G = SU(N)$ has rank $N - 1$, and the Cartan subalgebra is constructed from $(N - 1)$ diagonal generators $H_i$. Hence, there are $N(N - 1)$ off-diagonal shift operators $E_\alpha$, since $\text{dim}SU(N) := N^2 - 1 = (N - 1) + N(N - 1)$. Therefore, the total number of roots is $N(N - 1)$, of which there are $N - 1$ simple roots. Other roots are constructed as linear combination of the simple roots. Also, there are $N$ weight vectors. An element of $SU(N)$ is represented by the $N \times N$ unitary matrices with determinant 1 that are generated by traceless Hermitian matrices, $N^2 - 1$ linearly independent generators $T^A (A = 1, \ldots, N^2 - 1)$. The generators are normalized as

$$\text{tr}(T^AT^B) = \frac{1}{2} \delta_{AB}. \quad (3.40)$$

Each off-diagonal generator $E_\alpha$ has a single non-zero element $1/\sqrt{2}$. The diagonal generator $H_m$ is defined by

$$(H_m)_{ab} = \frac{1}{\sqrt{2m(m+1)}} \left( \sum_{k=1}^{m} \delta_{ak}\delta_{bk} - m\delta_{a,m+1}\delta_{b,m+1} \right) \quad (3.41)$$

$$= \frac{1}{\sqrt{2m(m+1)}} \text{diag}(1, \ldots, 1, -m, 0, \ldots, 0). \quad (3.42)$$

For $m = 1$ to $N - 1$, the first $m$ diagonal elements (beginning from the upper left-hand corner) of $H_m$ are 1, the next one is $-m$, and the rest of the diagonal elements are 0. Thus $H_m$ is traceless. The weight vectors (eigenvectors of all $H_i$: $H_j|\nu\rangle = \nu^j|\nu\rangle$) of the fundamental representation $N$ ($N$-dimensional irreducible representation of $SU(N)$) are given by

$$\begin{align*}
\nu^1 &= \left( \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{3}}, \ldots, \frac{1}{\sqrt{2m(m+1)}}, \ldots, \frac{1}{\sqrt{2(N-1)N}} \right), \\
\nu^2 &= \left( -\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{3}}, \ldots, \frac{1}{\sqrt{2m(m+1)}}, \ldots, \frac{1}{\sqrt{2(N-1)N}} \right), \\
\nu^3 &= \left( 0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}, \ldots, \frac{1}{\sqrt{2(N-1)N}} \right), \\
&\vdots \\
\nu^{m+1} &= \left( 0, 0, \ldots, 0, -\frac{m}{\sqrt{2m(m+1)}}, \ldots, \frac{1}{\sqrt{2(N-1)N}} \right), \\
&\vdots \\
\nu^N &= \left( 0, 0, \ldots, 0, -\frac{N+1}{\sqrt{2(N-1)N}} \right). \quad (3.43)
\end{align*}$$

*) For $n = 1$,

$$V = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad V_g = \begin{pmatrix} 1 & \frac{aw+b}{cw+d} \\ 0 & 1 \end{pmatrix}. \quad (3.39)$$

Hence $w$ is the complex one-dimensional representation of $SL(2, C)$. 

---

*For $n = 1$,
All the weight vectors have the same length, and the angles between different weights are the same:

$$\nu^i \cdot \nu^i = \frac{N-1}{2N}, \quad \nu^i \cdot \nu^j = -\frac{1}{2N}, \quad (\text{for } i \neq j) \quad (3.44)$$

The weights constitute a polygon in the $N-1$ dimensional space. This implies that any weight can be used as the highest weight. A weight will be called positive if its last non-zero component is positive. With this definition, the weights satisfy

$$\nu^1 > \nu^2 > \cdots > \nu^N. \quad (3.45)$$

The simple roots are given by

$$\alpha^i = \nu^i - \nu^{i+1}. \quad (i = 1, \cdots, N-1) \quad (3.46)$$

Explicitly, we have

$$\alpha^1 = (1, 0, \cdots, 0),$$
$$\alpha^2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, \cdots, 0\right),$$
$$\alpha^3 = \left(0, -\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{3}, 0, \cdots, 0\right),$$
$$\vdots$$
$$\alpha^m = \left(0, 0, \cdots, -\sqrt{\frac{m-1}{2m}}, \sqrt{\frac{m+1}{2m}}, 0, \cdots, 0\right),$$
$$\vdots$$
$$\alpha^{N-1} = \left(0, 0, \cdots, -\sqrt{\frac{N-2}{2(N-1)}}, \sqrt{\frac{N}{2(N-1)}}\right). \quad (3.47)$$

As can be shown from (3.44), all these roots have length 1, the angles between successive roots are the same, $2\pi/3$, and other roots are orthogonal:

$$\alpha^j \cdot \alpha^j = 1, \quad \alpha^i \cdot \alpha^j = -\frac{1}{2}, \quad (j = i \pm 1) \quad (3.48)$$
$$\alpha^i \cdot \alpha^j = 0. \quad (j \neq i, i \pm 1) \quad (3.49)$$

This fact is usually expressed by a Dynkin diagram (see Fig. 5).

If we choose $\nu^1$ as the highest-weight $\bar{\Lambda}$ of the fundamental representation $\mathbf{N}$, some of the roots are orthogonal to $\nu^1$. From the above construction, it is easy to see

\[\begin{array}{c}
\alpha_1 \\
- \\
\alpha_2 \\
- \\
\alpha_{N-2} \\
\alpha_{N-1}
\end{array}\]

Fig. 5. The Dynkin diagram of $SU(N)$. 
that only one simple root $\alpha_1$ is not-orthogonal to $\nu^1$, and that all the other simple roots are orthogonal:

$$\nu^1 \cdot \alpha^1 \neq 0, \quad \nu^1 \cdot \alpha^2 = \nu^1 \cdot \alpha^3 = \cdots = \nu^1 \cdot \alpha^{N-1} = 0. \quad (3.50)$$

Therefore, all the linear combinations constructed from $\alpha^2, \cdots, \alpha^{N-1}$ are also orthogonal to $\nu^1$. Non-orthogonal roots are obtained only when $\alpha_1$ is included in the linear combinations. It is not difficult to show that the total number of non-orthogonal roots is $2(N-1)$, and hence there are $N(N-1) - 2(N-1) = (N-2)(N-1)$ orthogonal roots. The $(N-2)(N-1)$ shift operators $E_\alpha$ corresponding to these orthogonal roots together with the $(N-1)$ Cartan subalgebra $H_i$ constitute the maximal stability subgroup $\tilde{H} = U(N-1)$, since $2(N-1) = \dim U(N-1)$.

Thus, for the fundamental representation, the stability subgroup $\tilde{H}$ of $SU(N)$ is given by $\tilde{H} = U(N-1)$. In order to describe the coset space $G/\tilde{H}$, we need only $(N-1)$ complex numbers, since

$$G/\tilde{H} = SU(N)/U(N-1), \quad (3.51)$$

and $\dim G/\tilde{H} = 2(N-1)$. We conclude that $G/\tilde{H} = CP^{N-1}$, where $CP^{N-1}$ is the $(N-1)$-dimensional complex projective space, which is a submanifold of the flag manifold $F_{N-1}$.

The complex projective space $CP^n$ is the compact Kähler symmetric space with $\dim CP^n = n$. The $SU(n+1)$ group can act transitively on this manifold, and this manifold can be considered as a factor space:

$$CP^n = SU(n+1)/(SU(n) \times U(1)). \quad (3.52)$$

An element of $CP^n$ can be expressed using the $n$ complex variables $w_1, \cdots, w_n$ as

$$\begin{pmatrix}
1 & w_1 & w_2 & \cdots & \cdots & w_n \\
0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 1
\end{pmatrix} \in CP^n. \quad (3.53)$$

The Kähler potential of $F_n$ ($n = N-1$) is obtained as follows. Let $H$ be the Hermitian matrix defined by $H = VV^\dagger, V \in F_n$. Consider the Gauss decomposition

$$H = VV^\dagger = TDV, \quad (3.54)$$

where $T \in Z_-(n+1)$ and $D$ is a diagonal matrix with determinant equal to $1$. $H$ has the upper principal minors $\Delta_\ell(VV^\dagger)(\ell = 1, \cdots, n+1)$, which are equal to the following products of the elements $\delta_\ell$ of the diagonal matrix $D = \text{diag}(\delta_1, \cdots, \delta_n)$:

$$\Delta_1 = \delta_1, \quad \Delta_2 = \delta_1 \delta_2, \quad \cdots, \quad \Delta_n = \delta_1 \delta_2 \cdots \delta_n, \quad \Delta_{n+1} = 1. \quad (3.55)$$
Let $d_\ell (\ell = 1, \ldots, n)$ be the Dynkin index of $SU(n+1)$. Then the Kähler potential of $F_n$ is given in the form,

$$K(w, \bar{w}) = \sum_{\ell=1}^{n} d_\ell K_\ell(w, \bar{w}) = \sum_{\ell=1}^{n} d_\ell \ln \Delta_\ell(w, \bar{w}) = \ln \left[ \prod_{\ell=1}^{n} (\Delta_\ell(w, \bar{w}))^{d_\ell} \right].$$

(3.56)

The function $K(w, \bar{w}')$ can also be obtained from the Gauss decomposition of $VV'$:

$$K(w, \bar{w}') = \ln \left[ \frac{\prod_{\ell=1}^{n} \Delta_\ell(w, \bar{w}')} {d_\ell} \right].$$

(3.57)

The $SU(n+1)$ invariant measure on $F_n$ is written, up to a multiplicative factor, as

$$d\mu(V, \bar{V}) = \rho(V, \bar{V}) \frac{n(n+1)}{2} \prod_{\alpha=1}^{n} dw_\alpha d\bar{w}_\alpha,$$

(3.58)

where $d_\ell = 1$ for all $\ell$. The density $\rho$ of the invariant measure is calculated from

$$\rho = \det(g_{\alpha\bar{\beta}}) = \det \left( \frac{\partial^2 K}{\partial w^\alpha \partial \bar{w}^\beta} \right).$$

(3.60)

The Kähler potential of the $CP^{N-1}$ manifold is given by

$$K = m \ln \left( 1 + \sum_{\alpha=1}^{N-1} |w_\alpha|^2 \right) = m \ln \left( 1 + ||w||^2 \right),$$

(3.61)

where

$$||w||^2 := \sum_{a=1}^{N-1} |w_a|^2.$$  

(3.62)

Hence, the metric reads

$$g_{\alpha\bar{\beta}} = m \frac{(1 + ||w||^2)\delta_{\alpha\beta} - \bar{w}_\alpha w_\beta} {(1 + ||w||^2)^2}.$$  

(3.63)

The above construction of the coherent state can be extended to an arbitrary compact semi-simple Lie group (see Chapter 11 of Perelomov).

§4. Non-Abelian Stokes theorem

In this section we derive a new version of the non-Abelian Stokes theorem (NAST) based on the coherent state obtained in the previous section. An advantage of this version of NAST is that it is possible to separate the magnetic monopole contribution in the Wilson loop and that it is very helpful to understand the dual superconductor picture of quark confinement in QCD.
4.1. Non-Abelian Stokes theorem for \( SU(N) \)

We consider the infinitesimal deviation \( \xi' = \xi + d\xi \) (which is sufficient to derive the path integral formula). From (3.19),

\[
|\xi', \Lambda \rangle = e^{-\frac{i}{2}K(w, \bar{w})} \exp \left[ \sum_{\alpha} \tau_{\alpha}(w)E_{-\alpha} \right] |\Lambda \rangle,
\]
we find

\[
\langle \xi + d\xi, \Lambda | \xi, \Lambda \rangle = \exp \left[ i\omega + O((dw)^2) \right],
\]

\[
\omega(x) := \langle \Lambda | i\xi'^\dagger(x) \partial \xi(x)|\Lambda \rangle,
\]

where \( d \) denotes the exterior derivative

\[
d := dx^\mu \frac{\partial}{\partial x^\mu} := dx^\mu \partial_{\mu}.
\]

Then \( \omega \) is the one-form

\[
\omega = dx^\mu \omega_{\mu}, \quad \omega_{\mu} = \langle \Lambda | i\xi'^\dagger(x) \partial_{\mu} \xi(x)|\Lambda \rangle.
\]

Here the \( x \) dependence of \( \xi \) comes from that of \( w(x) \) (the local field variable \( w(x) \)), i.e., \( \xi(x) = \xi(w(x)) \).

The exterior derivative is regarded as the operator

\[
d = \partial + \bar{\partial} = dw_\alpha \frac{\partial}{\partial w_\alpha} + d\bar{w}_\beta \frac{\partial}{\partial \bar{w}_\beta},
\]

where the operators \( \partial \) and \( \bar{\partial} \) are called the Dolbeault operators.\(^{35} \) From the inner product (3.22),

\[
\langle \xi', \Lambda | \xi, \Lambda \rangle = e^{K(w, \bar{w}')} e^{-\frac{1}{2}K(w', \bar{w}'+K(w, \bar{w})},
\]

we obtain another expression for \( \omega \) using the Kähler potential \( K \):

\[
\omega = \frac{i}{2} (\partial - \bar{\partial}) K = \frac{i}{2} \left( dw_\alpha \frac{\partial}{\partial w_\alpha} - d\bar{w}_\beta \frac{\partial}{\partial \bar{w}_\beta} \right) K.
\]

The Wilson loop operator \( W^C[A] \) is defined as the trace of the path-ordered exponent along the closed loop \( C \) as

\[
W^C[A] := \frac{1}{N} \text{tr} \left[ P \exp \left( ig \oint_C A \right) \right],
\]

where \( N \) is the dimension of the representation \( (N = \text{dim}(1_R) = \text{tr}(1_R)) \), and \( A \) is the (Lie-algebra valued) connection one-form

\[
A(x) = A^A_\mu(x) T^A dx^\mu = A^A(x) T^A.
\]
Consider a curve starting from $x_0$ and ending at $x$. We parameterize this curve by the parameter $t$ and define

$$W_{ab}(t, t_0) := \left[ \mathcal{P} \exp \left( ig \int_{x_0(t_0)}^{x(t)} dx^\mu A_\mu(x) \right) \right]_{ab} = \left[ \mathcal{P} \exp \left( ig \int_{t_0}^{t} dt A(t) \right) \right]_{ab},$$

(4.11)

where

$$A(t) := A_\mu(x) dx^\mu / dt.$$  

(4.12)

Then the wavefunction defined by

$$\psi_a(t) = W_{ab}(t, t_0) \psi_b(t_0)$$

(4.13)

is a solution of the Schrödinger equation

$$\left[ i \frac{d}{dt} + gA(t) \right]_{ab} \psi_b(t) = 0.$$  

(4.14)

Note that the Wilson loop operator is obtained by taking the trace of (4.11) for the closed loop, say $C$:

$$W^C[A] = \frac{1}{\mathcal{N}} \text{tr}(W(t, 0)) = \frac{1}{\mathcal{N}} \sum_{a=1}^{\text{dim. of rep.}} W_{aa}(t, 0).$$

(4.15)

This implies that it is possible to write the path integral representation of the Wilson loop operator if we identify $A(t)$ with the Hamiltonian as

$$H(t) = -gA(t) = -gA_\mu(x) dx^\mu / dt.$$  

(4.16)

First, we define the path-ordered exponent by discretizing the time interval $t$ into $N$ infinitesimal steps and subsequently taking the limit $N \to \infty, \epsilon \to 0$ with $N \epsilon = t$ fixed as

$$\text{tr} \left\{ \mathcal{P} \exp \left[ ig \int_{0}^{t} dt A(t) \right] \right\} = \lim_{N \to \infty, \epsilon \to 0} \text{tr} \left\{ \mathcal{P} \prod_{n=0}^{N-1} [1 + i \epsilon g A(t_n)] \right\},$$

(4.17)

where $t_n := n \epsilon, \epsilon := t / N$. For simplicity, we set $t_0 = 0$ and $t_N = t$. Next, we use the coherent state $|\xi, A\rangle$, following Ref. 15). On the right-hand side of (4.17), we replace the trace with

$$\frac{1}{\mathcal{N}} \text{tr}(\cdots) = \int d\mu(\xi_N) \langle \xi_N, A | (\cdots) | \xi_N, A \rangle,$$

(4.18)

and insert the complete set (resolution of unity)

$$I = \int |\xi_n, A\rangle d\mu(\xi_n) \langle \xi_n, A |.$$  

(4.19)

Then we obtain

$$\frac{1}{\mathcal{N}} \text{tr} \left\{ \mathcal{P} \exp \left[ ig \int_{0}^{t} dt A(t) \right] \right\}$$
We obtain the path integral representation of the Wilson loop,
\[ W^C[A] = \lim_{N \to \infty, \epsilon \to 0} \prod_{n=1}^{N-1} \int d\mu(\xi_n) \langle \xi_n, A|1 + igA(t_{n-1})|\xi_{n-1}, A \rangle d\mu(\xi_{n-1}) \times \langle \xi_{n-1}, A|1 + igA(t_{n-2})|\xi_{n-2}, A \rangle d\mu(\xi_{n-2}) \times \cdots \times d\mu(\xi_1) \langle \xi_1, A|1 + igA(t_0)|\xi_0, A \rangle \]
\[ = \lim_{N \to \infty, \epsilon \to 0} \prod_{n=1}^{N} \int d\mu(\xi_n) \prod_{n=0}^{N-1} \langle \xi_{n+1}, A|[1 + igA(t_n)]|\xi_n, A \rangle \]
\[ = \lim_{N \to \infty, \epsilon \to 0} \prod_{n=1}^{N} \int d\mu(\xi_n) \prod_{n=0}^{N-1} [1 + ig\bar{A}(t_n)] \prod_{n=0}^{N-1} \langle \xi_{n+1}, A|\xi_n, A \rangle \]
\[ = \lim_{N \to \infty, \epsilon \to 0} \prod_{n=1}^{N} \int d\mu(\xi_n) \exp \left[ i \sum_{n=0}^{N-1} g\bar{A}(t_n) \right] \prod_{n=0}^{N-1} \langle \xi_{n+1}, A|\xi_n, A \rangle, \tag{4.20} \]
where we have used \( \xi_0 = \xi_N \) and have defined
\[ \bar{A}(t_n) := \frac{\langle \xi_{n+1}, A|A(t_n)|\xi_n, A \rangle}{\langle \xi_{n+1}, A|\xi_n, A \rangle}. \tag{4.21} \]
Up to \( O(\epsilon^2) \), we find
\[ \bar{A}(t_n) := \langle \xi_n, A|A(t_n)|\xi_n, A \rangle + O(\epsilon^2) = \langle A|\xi(t_n)^\dagger A(t_n)\xi(t_n)|A \rangle + O(\epsilon^2), \tag{4.22} \]
and
\[ \langle \xi_{n+1}, A|\xi_n, A \rangle = \langle \xi(t_n), A|\xi(t_n), A \rangle + \epsilon \langle \xi(t_n), A|\epsilon\xi(t_n), A \rangle + O(\epsilon^2) \]
\[ = \exp[\epsilon \langle \xi(t_n), A|\xi(t_n), A \rangle + O(\epsilon^2)] \]
\[ = \exp[-i\epsilon A\xi(t_n)^\dagger A \xi(t_n)|A] + O(\epsilon^2)] \]
\[ = \exp[i\epsilon A\xi(t_n)^\dagger \xi(t_n)|A] + O(\epsilon^2)], \tag{4.23} \]
where we have used \( \langle \xi(t_n), A|\xi(t_n), A \rangle = 1 \). Therefore we arrive at the expression
\[ W^C[A] = \lim_{N \to \infty, \epsilon \to 0} \prod_{n=1}^{N} \int d\mu(\xi(t_n)) \]
\[ \times \exp \left\{ i\epsilon \sum_{n=0}^{N-1} \langle A|[\xi(t_n)^\dagger A(t_n)\xi(t_n) + ig^{-1}\xi(t_n)^\dagger \xi(t_n)]A \rangle \right\}. \tag{4.24} \]
Thus we obtain the path integral representation of the Wilson loop,
\[ W^C[A] = \int [d\mu(\xi)]_C \exp \left( ig \oint_C \langle A|VAV^\dagger + \frac{i}{g}VdV^\dagger \rangle A \rangle \tag{4.25} \]
where \([d\mu(\xi)]_C \) is the product measure of \( d\mu(w(x), \bar{w}(x)) \) along the loop \( C \):
\[ [d\mu(\xi)]_C := \lim_{N \to \infty, \epsilon \to 0} \prod_{n=1}^{N} \int d\mu(\xi_n). \tag{4.26} \]
Using the (usual) Stokes theorem, \( \oint_{\partial S} \omega = \int_S d\omega \), we obtain the non-Abelian Stokes theorem (NAST):

\[
W^C[A] = \int [d\mu(\xi)]_C \exp \left( ig \oint_C \left[ n^A A^A + \frac{1}{g} \Omega_K \right] \right).
\]

Here we have defined

\[
n^A(x) := \langle A| \xi^\dagger(\xi) T^A \xi(x) |A\rangle,
\]

\[
\omega(x) := \langle A| i\xi^\dagger(x) d\xi(x) |A\rangle,
\]

and

\[
\Omega_K := d\omega.
\]

Taking into account (4.8), we find that this \( \Omega_K \) is identical to the Kähler two-form, in agreement with the general statement (3.35), i.e.,

\[
\Omega_K = d\omega = (\partial + \bar{\partial}) \frac{i}{2} (\partial - \bar{\partial}) K = i\partial\bar{\partial} K,
\]

since the identity \( d^2 = 0 \) leads to \( \partial^2 = 0 = \bar{\partial}^2, \partial\bar{\partial} + \bar{\partial}\partial = 0 \). Therefore, the second term, \( \omega \) or \( \Omega_K \), in the exponent of the NAST is entirely determined from the Kähler potential of the flag manifold.

For \( SU(N) \), the topological part,

\[
\gamma := \oint_C \omega = \int_S \Omega_K,
\]

corresponding to the residual \( U(N-1) \) invariance is interpreted as the geometric phase of the Wilczek-Zee holonomy,\(^{41}\) just as in the \( SU(2) \) case it is interpreted as the Berry-Aharonov-Anandan phase for the residual \( U(1) \) invariance. The details of this point will be given in a subsequent article.\(^{42}\)

### 4.2. Fundamental representation of \( SU(N) \) and \( CP^{N-1} \) variable

Defining

\[
\omega_a(x) := \langle V^\dagger(x)|A\rangle_a, \quad (a = 1, \cdots, N)
\]

we can write

\[
n^A(x) := \langle A| V(x) T^A V^\dagger(x) |A\rangle = \bar{\omega}_a(x)(T^A)_{ab}\omega_b(x).
\]

In the \( CP^{N-1} \) case, in particular, the highest-weight state is given by a column vector,

\[
|A\rangle = \begin{pmatrix} 1 \\
0 \\
\vdots \\
0
\end{pmatrix},
\]

\[
\omega_a(x) := \langle V^\dagger(x)|A\rangle_a, \quad (a = 1, \cdots, N)
\]

we can write

\[
n^A(x) := \langle A| V(x) T^A V^\dagger(x) |A\rangle = \bar{\omega}_a(x)(T^A)_{ab}\omega_b(x).
\]

In the \( CP^{N-1} \) case, in particular, the highest-weight state is given by a column vector,
and then we can write
\[ n^A(x) := \langle A|U(x)T^A U^\dagger(x)|A \rangle = \bar{\phi}_a(x)(T^A)_{ab}\phi_b(x), \]
(4.36)
where \( U \in SU(N) \) and
\[ \phi_a(x) := (U^\dagger(x)|A) = \bar{U}_{1a}(x). \]
(4.37)
Then \( n^A(x) \) can be rewritten as
\[ n^A(x) = (U(x)T^A U^\dagger(x))_{11}. \]
(4.38)
Note that the \( CP^{N-1} \) variables \( \phi_a(a = 1, \cdots, N) \) are subject to the constraint
\[ \sum_{a=1}^{N} |\phi_a(x)|^2 = 1. \]
(4.39)
This is clearly satisfied by the unitarity of \( U \), \( \sum_{a=1}^{N} U_{1a} U_{1a}^\ast = (U(x)U^\dagger(x))_{11} = 1. \)

Now, we examine another expression (the adjoint orbit representation),
\[ n^A(x) = 2\text{tr}(U^\dagger(x)\mathcal{H} U(x)T^A), \]
(4.40)
or equivalently,
\[ n(x) := n^A(x)T^A = U^\dagger(x)\mathcal{H} U(x). \]
(4.41)
Here \( \mathcal{H} \) is defined by
\[ \mathcal{H} = \bar{A} \cdot (H^1, \cdots, H^{N-1}) = \sum_{i=1}^{N-1} A^i H^i = \frac{1}{2} \text{diag} \left( \frac{N-1}{N}, -\frac{1}{N}, \cdots, -\frac{1}{N} \right), \]
(4.42)
where we have used (3.42) and
\[ A^i = \frac{1}{\sqrt{2i(i+1)}}. \]
(4.43)
For \( SU(3) \), when the Dynkin index \([m,n] = [1,0] \) or \([0,1] \), (4.42) reduces to
\[ \mathcal{H} = \bar{A} \cdot \left( \frac{\lambda^3}{2}, \frac{\lambda^8}{2} \right) = \frac{1}{2} \text{diag} \left( \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right), \text{ or } \frac{1}{2} \text{diag} \left( \frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right). \]
(4.44)
Note that two elements agree with each other. Hence the adjoint orbit cannot cover all the flag space \( F_2 \). This is the \( CP^2 \) case. It is easy to see that the two definitions (4.36) and (4.40) are equivalent,
\[ n^A(x) = (U(x)T^A U^\dagger(x))_{11} = 2\text{tr}(\mathcal{H} U(x)T^A U^\dagger(x)), \]
(4.45)
since \( UT^A U^\dagger \) is traceless.\(^*\)

\(^*\) When \([m,n] = [1,1] \), on the other hand, \( \mathcal{H} = \text{diag}(1,-1,0) \), and all the diagonal elements are different. Therefore \( n(x) \) moves on the whole flag space, \( F_2 \).
The correspondence between the $F_{N-1}$ variables $w_a$ and the $CP^{N-1}$ variables $\phi_a$ is given, e.g., by

$$
\phi_1 = w_1, \quad \phi_2 = w_2, \quad \ldots, \quad \phi_{N-1} = w_{N-1}, \quad \phi_N = 1.
$$

(4.46)

Thus $\omega$ is an inhomogeneous coordinate, $\omega_a = \phi_a/\phi_N = w_a$ ($a = 1, \ldots, N-1$), by definition. In the $CP^{N-1}$ case (for the fundamental representation), we can perform the replacement

$$
\langle \Lambda | f(V) | \Lambda \rangle = 2\text{tr}[Hf(U)]
$$

(4.47)

if $f(V)$ belongs to the Lie algebra of $G$ (and hence $f(V)$ is traceless, i.e., $\text{tr}f(V) = 0$). Thus, we obtain another expression for $\omega$,

$$
\omega(x) := 2\text{tr}[HiU(x)dU(x)] = -i2\text{tr}[HdU(x)U^\dagger(x)],
$$

(4.48)

which is a diagonal piece of the Maurer-Cartan one-form,

$$
\vartheta := dUU^{-1}.
$$

(4.49)

It turns out that the two-form $\Omega_K$ is the symplectic two-form,

$$
\Omega_K = d\omega = 2\text{tr}(H[U^{-1}dU,U^{-1}dU]) = 2\text{tr}(n[dn,dn]).
$$

(4.50)

Our choice of $H$ is the most economical one (see Ref. 44 for different choices and more discussion of the related issues).

From the Kähler potential of $CP^{N-1}$ (3.61) and the relation (4.8), the connection one-form $\omega$ reads

$$
\omega := \frac{i}{2} \frac{\bar{w}_\alpha dw_\alpha - d\bar{w}_\alpha w_\alpha}{1 + \bar{w}_\alpha w_\alpha},
$$

(4.51)

which is equal to

$$
\omega := i \frac{\bar{w}_\alpha dw_\alpha}{1 + \bar{w}_\alpha w_\alpha},
$$

(4.52)

up to the total derivative. By taking the exterior derivative, we obtain

$$
\Omega_K = d\omega = i \frac{(1 + ||w||^2)\delta_{\alpha\beta} - \bar{w}_\alpha w_\beta}{(1 + ||w||^2)^2}dw^\alpha \wedge d\bar{w}^\beta,
$$

(4.53)

which agrees with the metric (3.63).

4.3. An implication of the NAST

The NAST (4.27) implies that the Wilson loop operator is given by

$$
W^C[A] = \int [d\mu(\xi)]_C \exp \left( ig \oint_C a \right) = \int [d\mu(\xi)]_C \exp \left( ig \int_{S,C=\partial S} f \right).
$$

(4.54)

First, $a$ is the connection one-form

$$
a := n^A A^A + \frac{1}{g}\omega = \langle \Lambda | A^V | \Lambda \rangle,
$$

(4.55)
where $A^V$ is obtained as the gauge transformation of $A$ by $V \in F_{N-1}$:

$$A^V := V A V^\dagger + \frac{i}{g} V dV^\dagger = \xi^\dagger A \xi + \frac{i}{g} \xi^\dagger d\xi.$$  \hspace{1cm} (4.56)

For a quark in the fundamental representation, we can write

$$a = 2 \text{tr}(HA^V).$$  \hspace{1cm} (4.57)

Therefore, the one-form $a$ is equal to the diagonal piece of the gauge-transformed potential $A^V$. This fact is very useful to derive the Abelian dominance in the low-energy physics of QCD (see §6).

Next, $f$ is the curvature two-form,

$$f := da = dC + \frac{1}{g} d\omega = dC + \frac{1}{g} \Omega_K,$$  \hspace{1cm} (4.58)

where we have defined the one-form

$$C := n^A A^A.$$  \hspace{1cm} (4.59)

The anti-symmetric tensor $f_{\mu\nu}$ can be called the generalized 't Hooft-Polyakov tensor for the following reasons: (1) it gives a non-vanishing magnetic monopole (current), where only the second term $\Omega_K$ gives a non-trivial contribution; (2) it is invariant under the full gauge transformation, although it is an Abelian field strength. These facts are demonstrated as follows.

First, we characterize the flag space in complex coordinates. More precisely, the target space at each space-time point $x \in \mathbb{R}^D$ is parameterized by the complex variables, $w^\alpha = w^\alpha(x)$. The Kähler two-form is rewritten as

$$\Omega_K = ig_{\alpha\bar{\beta}} \partial_\mu w^\alpha \partial_\nu \bar{w}^\beta dx^\mu \wedge dx^\nu.$$  \hspace{1cm} (4.60)

On the other hand,

$$\Omega_K := d\omega = \frac{1}{2} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\mu \wedge dx^\nu := \frac{g}{2} \Omega^\mu_\nu dx^\mu \wedge dx^\nu.$$  \hspace{1cm} (4.61)

Then the second piece $g^{-1} \Omega_K$ of $f$ can be written as

$$f^\Omega_{\mu\nu} = \frac{1}{g} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) = \frac{i}{g} g_{\alpha\bar{\beta}} \partial_\mu w^\alpha \partial_\nu \bar{w}^\beta,$$  \hspace{1cm} (4.62)

and hence

$$f_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + f^\Omega_{\mu\nu}. $$  \hspace{1cm} (4.63)

The magnetic monopole current $k_\mu$ is obtained as the divergence of the dual tensor $\ast f^\Omega_{\mu\nu}$:

$$k_\mu := \partial_\nu \ast f_{\mu\nu},$$  \hspace{1cm} (4.64)

where the Hodge dual of $f_{\mu\nu}$ in four dimensions is defined by

$$\ast f_{\mu\nu} := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}.$$  \hspace{1cm} (4.65)
The first piece \( dC \) in \( f \) does not contribute to the magnetic current, due to the Bianchi identity. On the other hand, the second term \( \Omega_K \) in \( f \) can lead to a non-vanishing magnetic current, as shown shortly. Here it should be remarked that the expression for \( \Omega_K \) given in terms of the local coordinate \( w_\alpha \) leads to a vanishing magnetic current. In fact, if the metric is given by the Kähler potential we have

\[
k_\mu = \frac{i}{2g} \epsilon_{\mu\rho\sigma\sigma} \partial_\nu \left( g_{\alpha\beta} \partial_\rho w^\alpha \partial_\sigma \bar{w}^\beta \right)
\]

\[
= \frac{i}{2g} \epsilon_{\mu\rho\sigma\sigma} \partial_\nu g_{\alpha\beta} \partial_\rho w^\alpha \partial_\sigma \bar{w}^\beta
\]

\[
= \frac{i}{2g} \epsilon_{\mu\rho\sigma\sigma} \left( \frac{\partial g_{\alpha\beta}}{\partial w^\gamma} \partial_\nu w^\gamma + \frac{\partial g_{\alpha\beta}}{\partial \bar{w}^\gamma} \partial_\nu \bar{w}^\gamma \right) \partial_\rho w^\alpha \partial_\sigma \bar{w}^\beta
\]

\[
= \frac{i}{2g} \epsilon_{\mu\rho\sigma\sigma} \left( \frac{\partial K}{\partial w^\gamma} \partial_\nu w^\gamma \partial_\rho w^\alpha \partial_\sigma \bar{w}^\beta + \frac{\partial K}{\partial \bar{w}^\gamma} \partial_\nu \bar{w}^\gamma \partial_\rho w^\alpha \partial_\sigma \bar{w}^\beta \right) = 0, \tag{4.66}
\]

where we have used the antisymmetric property of \( \epsilon_{\mu\rho\sigma\sigma} \) under the exchange of \( \nu \) and \( \rho \), and \( \nu \) and \( \sigma \). However, this does not imply the vanishing total magnetic flux or magnetic charge.

We recall that this situation is similar to that of the Wu-Yang monopole\(^{46}\) compared with the original Dirac monopole.\(^{45}\) There are two ways to describe the Dirac magnetic monopole. One is to use a single vector potential with (line) singularities, called the Dirac string, where the singularities are distributed on a semi-infinite line going through the origin of the space coordinates. In the absence of singularities, the vector potential gives a vanishing magnetic charge, due to the Bianchi identity,

\[
\Phi = \oint B \cdot dS = \oint \text{curl} A \cdot dS = \int_{D^3} \text{divcurl} A dV = 0. \tag{4.67}
\]

Therefore, the singularities must produce a magnetic charge which has the same magnitude as the magnetic charge at the origin, but opposite sign. Another way is to introduce more than one vector potential to avoid singularities. Each vector potential \( A^\alpha \) is defined in a sub-region \( U_\alpha \) of the sphere \( S^2 \) such that \( A^\alpha \) is regular in each region \( U_\alpha \) and the union of the sub-regions covers the whole sphere. Thus, the Bianchi identity leads to zero magnetic flux in each sub-region. Note that we cannot apply the Gauss theorem \( \text{divcurl} A = 0 \), since \( U_\alpha \) is not a closed surface. The total magnetic flux is recovered by summing up all the contributions of the differences of vector potentials on the boundary \( B_{\alpha,\beta} \) between two regions \( U_\alpha \) and \( U_\beta \)

\[
\Phi = \sum_\alpha \oint_{U_\alpha} \text{curl} A^\alpha \cdot dS = \sum_\alpha \oint_{B_{\alpha,\beta}} (A^\alpha - A^\beta) d\ell, \tag{4.68}
\]

where the minus sign follows from the fact that the orientation of the boundary is opposite for neighboring regions. The difference is given by the gauge transformation, \( A^\alpha - A^\beta = \nabla \Lambda_{\alpha,\beta} \). This recovers the same magnetic flux as in the former case.

The variable \( w_\alpha \) corresponds to \( A^\alpha \) in the case of the Wu-Yang monopole. Therefore, to show the existence of a non-zero magnetic flux, we must specify the method of gluing different coordinate patches on the boundary. These subtleties are avoided
by using a different parameterization. This generalizes the argument given by 't Hooft and Polyakov for the SU(2) magnetic monopole. The antisymmetric tensor \( f_{\mu\nu} \) given by (4.58) is the SU(\(N\)) generalization of the 't Hooft-Polyakov tensor for SU(2). In the SU(2) case, \( a = 2\text{tr}(T^3\mathcal{A}^V) \) for any representation, and the two-form \( f := da \) is the Abelian field strength, which is invariant under the SU(2) transformation. Hence the two-form \( f \) is identically the 't Hooft-Polyakov tensor,

\[
f_{\mu\nu}(x) := \partial_\mu(n^A(x)A^A_\nu(x)) - \partial_\nu(n^A(x)A^A_\mu(x)) - \frac{1}{g} n(x) \cdot (\partial_\mu n(x) \times \partial_\nu n(x)), \tag{4.69}
\]

describing the magnetic flux emanating from the magnetic monopole, if we identify \( n^A \) with the direction of the Higgs field:

\[
\hat{\phi}^A := \phi^A/|\phi|, \quad |\phi| := \sqrt{\phi^A\phi^A}. \tag{4.70}
\]

The complex coordinate representation reads

\[
f^\Omega_{\mu\nu}(x) = \frac{1}{g} \frac{1}{(1 + |w(x)|^2)^2} \partial_\mu w(x) \partial_\nu \bar{w}(x). \tag{4.71}
\]

In general, the (curvature) two-form \( f = d(n^A\mathcal{A}^A) + \Omega_K \) in the NAST is the Abelian field strength, which is invariant under the full non-Abelian gauge transformation of \( G = SU(N) \):*

\[
f_{\mu\nu}(x) := \partial_\mu(n^A(x)A^A_\nu(x)) - \partial_\nu(n^A(x)A^A_\mu(x)) + \frac{i}{g} n(x) \cdot [\partial_\mu n(x), \partial_\nu n(x)]. \tag{4.75}
\]

The invariance of \( f \) is obvious from the NAST (4.54), since \( W^C[\mathcal{A}] \) is gauge invariant and the measure \( [d\mu(\xi)]_C \) is also invariant under the \( G \) gauge transformation. In the case of the fundamental representation, the invariance is easily seen, because it is possible to rewrite (4.58) or (4.75) into the manifestly gauge-invariant form**

\[
f_{\mu\nu}(x) := 2\text{tr}\left(n(x)\mathcal{F}_{\mu\nu}(x) + \frac{i}{g} n(x)[D_\mu n(x), D_\nu n(x)]\right), \tag{4.76}
\]

*) The normalization

\[
\text{tr}(T^A T^B) = \frac{1}{2}\delta_{AB} \tag{4.72}
\]

holds for any group. For SU(2), \( \text{tr}(T^A T^B T^C) = \frac{1}{4}i\epsilon_{ABC} \). For SU(3), \( \text{tr}(T^A[T^B, T^C]) = \frac{1}{2}if_{ABC} \), while

\[
\text{tr}(T^A T^B T^C) = \frac{1}{4}(if_{ABC} + d_{ABC}). \tag{4.73}
\]

Here we have used \( T^B T^C = \frac{1}{2}[T^B, T^C] + \frac{1}{2}\{T^B, T^C\}, [T^B, T^C] = if_{BCD}T^D, \) and \( \{T^B, T^C\} = \frac{1}{2}\delta_{AB}I + d_{BCD}T^D \), where \( d_{ABC} \) is completely symmetric in the indices. Furthermore, we find

\[
\text{tr}(T^A T^B T^C T^D) = \frac{1}{12}\delta_{AB}\delta_{CD} - \frac{1}{8}f_{ABE}f_{CDE} + \frac{1}{8}d_{ABEDCDE} + \frac{i}{8}(f_{ABE}d_{CDE} + f_{CDE}d_{ABE}). \tag{4.74}
\]

**) An explicit derivation of this form is given by Hirayama and Ueno in Ref. 47), where the corresponding expression in the adjoint representation is also given.
where
\[ F_{\mu\nu}(x) := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)] \] (4.77)
and
\[ D_\mu n(x) := \partial_\mu n(x) - ig[A_\mu(x), n(x)]. \] (4.78)

In fact, we obtain the magnetic charge
\[ g_m = \int_{D^3} d^3 x k_0 \] (4.79)
\[ = \int_{D^3} d^3 x \frac{1}{2} \epsilon_{ijk} \partial_i f_{jk} \quad (i, j, k = 1, 2, 3) \]
\[ = \int_{D^3} d^3 x \frac{1}{2} \epsilon_{ijk} \partial_i i g \cdot [\partial_j n, \partial_k n] \]
\[ = \int d^2 x \frac{1}{2} \epsilon_{ijk} \frac{i}{g} n \cdot [\partial_j n, \partial_k n] \]
\[ = \int_{S^2} d^2 x \frac{1}{2} \epsilon_{ab} \frac{i}{g} n \cdot [\partial_a n, \partial_b n] \quad (a, b = 1, 2) \]
\[ = \frac{2}{g} \int \Omega_K = \frac{2}{g} 2\pi Q, \] (4.80)
where we have used (4.50) in the last step. Here \( Q \) is the integer-valued instanton charge in the NLS model in two-dimensional space, \( S^2 = R^2 \cup \{ \infty \} \) (see §8).

The contribution from the magnetic monopole is replaced by the instanton in the two-dimensional NLS model. The magnetic charge satisfies the Dirac quantization condition,
\[ g_m g = 2\pi Q = 2\pi n. \quad (n = 0, \pm 1, \pm 2, \cdots) \] (4.81)

4.4. Explicit forms of \( \omega \) and \( \Omega_K \) for \( SU(3) \) and \( SU(2) \)

For \( SU(3) \), we find that \( \omega \) is given by
\[ \omega = im\frac{w_1 d\bar{w}_1 + w_2 d\bar{w}_2}{\Delta_1(w, \bar{w})} \]
\[ + in\frac{w_3 d\bar{w}_3 + (w_2 - w_1 w_3)(d\bar{w}_2 - \bar{w}_1 d\bar{w}_3 - \bar{w}_3 d\bar{w}_1)}{\Delta_2(w, \bar{w})}, \] (4.82)
up to the total derivative. Hence, we obtain
\[ \Omega_K = d\omega = im(\Delta_1)^{-2}[(1 + |w_1|^2)dw_2 \wedge d\bar{w}_2 - \bar{w}_2 w_1 dw_2 \wedge d\bar{w}_1 \]
\[ -w_2 \bar{w}_1 dw_1 \wedge d\bar{w}_2 + (1 + |w_2|^2)dw_1 \wedge d\bar{w}_1 \]
\[ + in(\Delta_2)^{-2}[\Delta_1 dw_3 \wedge d\bar{w}_3 - (w_1 + \bar{w}_3 w_2)dw_3 \wedge (d\bar{w}_2 - \bar{w}_3 d\bar{w}_1) \]
\[ - (\bar{w}_1 + w_3 \bar{w}_2) (dw_2 - w_3 dw_1) \wedge d\bar{w}_3 \]
\[ + (1 + |w_3|^2) (dw_2 - w_3 dw_1) (d\bar{w}_2 - \bar{w}_3 d\bar{w}_1)]. \] (4.83)

The Kähler potential for \( F_2 \) is given by
\[ K(w, \bar{w}) = \ln[(\Delta_1)^m(\Delta_2)^n]. \] (4.84)
For $CP^2$, it reads
\[ K(w, \bar{w}) = \ln[(\Delta_1)^m], \] (4.85)
which is obtained as a special case of $F_2$ by setting $w_3 = 0$ and $n = 0$. Hence, we obtain
\[ \omega = im \frac{w_1d\bar{w}_1 + w_2d\bar{w}_2}{\Delta_1(w, \bar{w})}, \] (4.86)
up to the total derivative, and
\[ \Omega_K = d\omega = im(\Delta_1)^{-2}[(1 + |w_1|^2)dw_2 \wedge d\bar{w}_2 - w_2w_1dw_2 \wedge d\bar{w}_1 \nonumber \\
- w_2\bar{w}_1dw_1 \wedge d\bar{w}_2 + (1 + |w_2|^2)dw_1 \wedge d\bar{w}_1]. \] (4.87)
This should be compared with the case $F_1 = CP^1$,
\[ K(w, \bar{w}) = m \ln[(1 + |w|^2)], \quad m = 2j. \] (4.88)
For $SU(2)$, we reproduce the well-known results,
\[ \omega = im \frac{wd\bar{w}}{1 + |w|^2}, \] (4.89)
and
\[ \Omega_K = igw_1dw \wedge d\bar{w} = im \frac{dw \wedge d\bar{w}}{(1 + |w|^2)^2}. \] (4.90)
The explicit form of $\Omega_K$ is necessary to carry out the instanton calculus in the following.

§5. Magnetic monopole in $SU(N)$ Yang-Mills theory

In the dual superconductor picture of quark confinement, the magnetic monopoles give the dominant contribution to the area law of the Wilson loop or the string tension. Following the ’t Hooft argument,\(^{10}\) the partial gauge fixing $G \rightarrow H$ realizes the magnetic monopole in Yang-Mills gauge theory, even in the absence of an elementary scalar field. In the conventional approach based on the MA gauge, the residual gauge group was chosen to be the maximal torus subgroup $H = U(1)^N - 1$ for $G = SU(N)$. This choice immediately determines the type of magnetic monopoles. We now re-examine this issue.

We have learned that the magnetic monopole which is responsible for the area law of the Wilson loop is determined by the maximal stability group $\tilde{H}$ rather than the residual gauge group $H$. This is a new feature that appears in $SU(N)$ for $N \geq 3$. It seems that this possibility has been overlooked in the lattice community, as far as we know. Indeed, this situation occurs only for $SU(N)$ with $N \geq 3$. Therefore, we must distinguish the maximal stability group $\tilde{H}$ from the residual gauge group $H$. In general, the maximal stability group $\tilde{H}$ is larger than the maximal torus subgroup: $H = U(1)^{N-1} \subset \tilde{H}$. Hence, the coset space is smaller than in the maximal torus case, i.e., $G/\tilde{H} \subset G/H$. 

\[ \]
The existence of magnetic monopoles is suggested by the non-trivial Homotopy groups $\pi_2(G/H)$. In case (II), we have

$$\pi_2(F_2) = \pi_2(SU(3)/(U(1) \times U(1))) = \pi_1(U(1) \times U(1)) = Z + Z. \quad (5.1)$$

On the other hand, in case (I), i.e., $[m, 0]$ or $[0, n]$, we have

$$\pi_2(CP^2) = \pi_2(SU(3)/U(2)) = \pi_1(U(2)) = \pi_1(SU(2) \times U(1)) = \pi_1(U(1)) = Z. \quad (5.2)$$

Note that the $CP^{N-1}$ model possesses only local $U(1)$ invariance for any $N \geq 2$. It is this $U(1)$ invariance that corresponds to a single kind of Abelian magnetic monopole appearing in case (I). This magnetic monopole may be related to the non-Abelian magnetic monopole proposed by Weinberg et al. $^{48}$ The explicit solution for the magnetic monopole in $SU(3)$ gauge theories are discussed in Ref. 49).

This situation should be compared with the $SU(2)$ case, where the maximal stability group is always given by the maximal torus $H = U(1)$, irrespective of the representation. Therefore, the coset is given by

$$G/H = SU(2)/U(1) = F_1 = CP^1 \cong S^2 \quad (5.3)$$

and

$$\pi_2(SU(2)/U(1)) = \pi_2(F_1) = \pi_2(CP^1) = Z, \quad (5.4)$$

for arbitrary representation.

For $SU(N)$, our results suggest that the fundamental quarks are to be confined when the maximal stability group $\tilde{H}$ is given by $\tilde{H} = U(N-1)$ and

$$\pi_2(G/\tilde{H}) = \pi_2(SU(N)/U(N-1)) = \pi_2(CP^{N-1}) = Z, \quad (5.5)$$

while the adjoint quark is related to the maximal torus $\tilde{H} = U(1)^{N-1}$ and

$$\pi_2(G/H) = \pi_2(SU(N)/U(1)^{N-1}) = \pi_2(F_{N-1}) = Z^{N-1}. \quad (5.6)$$

This observation is in sharp contrast with the conventional treatment, in which the $(N-1)$ species of magnetic monopoles corresponding to the residual maximal torus group $U(1)^{N-1}$ of $G = SU(N)$ are taken into account on equal footing. In fact, the NAST derived in this article shows that the fundamental quark feels only the $U(1)$ that is embedded in the maximal stability group $U(N-1)$ as a magnetic monopole. This is a component along the highest weight.

§6. Abelian dominance in $SU(N)$ gauge theory

6.1. APEGT as a low-energy effective theory

The Abelian dominance in $SU(N)$ Yang-Mills theory can be explained as follows. First, we adopt the maximal Abelian (MA) gauge. The MA gauge for $SU(N)$ is defined as follows. Consider the Cartan decomposition of $A$ into diagonal ($H$) and off-diagonal ($G/H$) pieces,

$$A(x) = A^A(x)T^A = a^i(x)H^i + A^a(x)T^a. \quad (A = 1, \ldots, N^2 - 1) \quad (6.1)$$
In particular, for $G = SU(3)$,
\[
H^1 = \frac{\lambda_3}{2}, \quad H^2 = \frac{\lambda_8}{2}, \quad T^a = \frac{\lambda_a}{2}, \quad (a = 1, 2, 4, 5, 6, 7) \tag{6.2}
\]

The MA gauge is obtained by minimizing the functional of off-diagonal fields,
\[
\mathcal{R} := \int d^4x \frac{1}{2} A_\mu^a(x) A^{\mu a}(x) := \int d^4x \text{tr}_{G/H}(A_\mu(x) A^\mu(x)), \tag{6.3}
\]
under the gauge transformation. Under the infinitesimal gauge transformation $\Lambda$, $\mathcal{R}$ transforms as
\[
\delta_\Lambda \mathcal{R} = \int d^4x A^{\mu a} \delta_\Lambda A_\mu^a \\
= \int d^4x A^{\mu a} (\partial_\mu A^a + gf^{aij} a_\mu^i A^j + g f^{ab} A^b A^C) \\
= - \int d^4x (\partial_\mu A^{\mu a} + g f^{abi} a_\mu^i A^{\mu b}) A^a, \tag{6.4}
\]
since $f^{ABC}$ is completely antisymmetric in the indices and $f^{aij} = 0$ ($T^i$ and $T^j$ commute). Therefore, the condition $\delta_\Lambda \mathcal{R} = 0$ for arbitrary $\Lambda$ leads to the differential MA gauge given by
\[
\partial_\mu A^{\mu a}(x) - g f^{abi} a_\mu^i(x) A^{\mu b}(x) := (D_\mu [a] A^\mu)^a = 0. \tag{6.5}
\]

The $SU(N)$ Yang-Mills theory in the MA gauge\(^*\) is given by
\[
S_{\text{YM}}^{\text{total}} = S_{\text{YM}} + S_{\text{GF+FP}}, \tag{6.6}
\]
\[
S_{\text{YM}} = \int d^4x -\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu A}, \tag{6.7}
\]
\[
S_{\text{GF+FP}} = - \int d^4x i \delta_B \left[ \bar{C}^a \left( D_\mu [a] A^\mu + \frac{a}{2} B \right)^a \right], \tag{6.8}
\]
where $\delta_B$ is the Becchi-Rouet-Stora-Tyupin (BRST) transformation,
\[
\delta_B A_\mu(x) = D_\mu [A] C(x) := \partial_\mu C(x) - ig [A_\mu(x), C(x)], \\
\delta_B C(x) = \frac{i}{2} g [C(x), C(x)], \\
\delta_B \bar{C}(x) = i B(x), \\
\delta_B B(x) = 0, \tag{6.9}
\]
which is nilpotent, i.e., $\delta_B^2 \equiv 0$. The generating functional is given by
\[
Z_{\text{YM}}[J] = \int [dA][dC][d\bar{C}][dB] \exp(iS_{\text{YM}}^{\text{total}} + iS_J). \tag{6.10}
\]
\(^*\) In order to fix the residual Abelian gauge group $H$, we add an additional GF + FP term, e.g.,
\[
- \int d^4x i \delta_B \left[ \bar{C}^a \left( \partial_\mu a^\mu + \frac{a}{2} B \right)^a \right].
\]
Now we proceed to derive the effective Abelian gauge theory in the MA gauge by integrating out the off-diagonal gauge fields (together with the ghost and anti-ghost fields), $A^a, C^a, \bar{C}^a$ and $B^a$, as done in Refs. 50) and 12). Then the $SU(N)$ Yang-Mills theory can be reduced to the $U(1)^{N-1}$ Abelian gauge theory, which is written in terms of the diagonal fields, $a^i, C^i, \bar{C}^i$ and $B^i$, alone:

$$Z_{YM}[J] = \int [da^i][dC^i][d\bar{C}^i][dB^i] \exp(iS_{\text{APEGT}}^\text{total} + i\tilde{S}_J),$$ (6.11)

where

$$\exp(iS_{\text{APEGT}}^\text{total} + i\tilde{S}_J) = \int [dA^a][dC^a][d\bar{C}^a][dB^a] \exp(iS_{\text{YM}}^\text{total} + iS_J).$$ (6.12)

In particular, the partition function reads

$$Z_{YM}[0] = \int [da^i][dC^i][d\bar{C}^i][dB^i] \exp(iS_{\text{APEGT}}^\text{total}) := Z_{\text{APEGT}}.$$ (6.13)

The Abelian gauge theory obtained in this way is called the “Abelian-projected effective gauge theory” (APEGT). It has been shown that the APEGT is an Abelian gauge theory whose gauge coupling constant $g$ runs according to the same renormalization-group beta function as in the original $SU(N)$ Yang-Mills theory,

$$S_{\text{APEGT}}^\text{total}[a^i, C^i, \bar{C}^i, B^i] = \int d^4x \frac{-1}{4g^2(\mu)}(da^i, da^i) + S_{\text{GF}},$$ (6.14)

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \frac{b_0}{8\pi^2} \ln \frac{\mu}{\mu_0}, \quad b_0 = \frac{11N}{3} > 0,$$ (6.15)

up to the one-loop level.\(^*\) Hence

$$\left\langle f[a^j] \right\rangle_{\text{YM}} = Z_{YM}^{-1} \int [dA][dC][d\bar{C}][dB] \exp(iS_{\text{YM}}^\text{total}) f[a^j]$$ (6.16)

$$\cong Z_{\text{APEGT}}^{-1} \int [da^i][dC^i][d\bar{C}^i][dB^i] \exp(iS_{\text{APEGT}}^\text{total}) f[a^j]$$ (6.17)

$$= \left\langle f[a^j] \right\rangle_{\text{APEGT}}.$$ (6.18)

6.2. Modified MA gauge

In the MA gauge, it is expected\(^{25}\) that the off-diagonal gauge fields $A^a_\mu$ have non-zero mass $m_A(m_A \neq 0)$, whereas the diagonal gauge fields $a^i_\mu$ remain massless ($m_a = 0$). Therefore, the APEGT obtained in this way can be regarded as the low-energy effective gauge theory of Yang-Mills theory. In the framework of lattice gauge theory, this prediction was confirmed in numerical calculations by Amemiya and Suganuma\(^{52}\) for $G = SU(2)$ and $SU(3)$. In the framework of the continuum gauge field theory, on the other hand, it is more efficient to modify the MA gauge

\(^{*}\) See Ref. 12) for details. The result of Refs. 50) and 12) for $SU(2)$ can be generalized to $SU(N)$ in straightforward way, at least at the one-loop level.\(^{51}\) At the two-loop level, this is not trivial. The two-loop result will be given in Ref. 51).
into the $OSp(D|2)$ invariant form by making use of the BRST $\delta_B$ and anti-BRST $\bar{\delta}_B$ transformations, as proposed by one of the authors: \(^{13)}\)

$$S_{GF+FP}[A, C, \bar{C}, B] := \int d^4 x \ i\delta_B \bar{\delta}_B \text{tr}_{G/H} \left[ \frac{1}{2} A_\mu A^\mu - \frac{\alpha}{2} i C \bar{C} \right], \quad (6.19)$$

where $\alpha$ corresponds to the gauge fixing parameter. Note that (6.19) is obtained from (6.8) by adding the ghost self-interaction terms and by adjusting the parameter for the ghost self-interaction term, since

$$-\bar{\delta}_B \left[ \frac{1}{2} A^a_\mu A^{\mu a} - \frac{\alpha}{2} i C^a \bar{C}^a \right] = \bar{C}^a \left( D_\mu [a] A^\mu + \frac{\alpha}{2} B \right)^a - i \frac{\alpha}{2} f^{abi} C^a C^b C^i - i \frac{\alpha}{4} f^{abc} C^a \bar{C}^b \bar{C}^c. \quad (6.20)$$

The special case $\alpha = -2$ is discussed in previous articles, \(^{13)},^{15)}\) For $G = SU(2)$, the last two terms reduce to $2i \bar{C}^1 C^2 C^3 = -2C^3 \bar{C}^1 - \bar{C}^1$ in agreement with the previous result \(^{13)}\) (See Ref. 13) for details).

In the modified MA gauge (6-19), the non-zero mass generation of off-diagonal components was demonstrated analytically (at least in the topological sector), using dimensional reduction to the two-dimensional coset non-linear (NLS) sigma model (see section IV.C of Ref. 13)). Quite recently, dynamical mass generation of the off-diagonal gluons has been shown to take place due to ghost–anti-ghost condensation caused by the attractive quartic ghost interaction (contained in the gauge fixing term of the modified MA gauge), which is necessary to maintain renormalizability in four dimensions. The off-diagonal gluon mass obtained in this way can be written in terms of the intrinsic scale of the gluodynamics, $\Lambda_{QCD}$, and hence it is a renormalization-group invariant quantity. (For more details, see Ref. 42). Therefore, integration of the massive off-diagonal gauge fields is interpreted as a step of the Wilsonian renormalization group,\(^{*} \) and the APEGT can describe low-energy physics on the length scale $R > m_A^{-1}$. In this sense, the APEGT can be regarded as the low-energy effective gauge theory of Yang-Mills theory.

6.3. Abelian dominance

With the NAST for $SU(N)$ just derived, Abelian dominance in $SU(N)$ Yang-Mills theory is explained as follows, in a manner based on the same argument as that in the $SU(2)$ case.\(^{**})$

The NAST (4.27) implies that the expectation value of the Wilson loop in the

\(^{*})\) Rigorously speaking, all high-energy modes should be integrated out in the Wilsonian renormalization group. Hence, we must integrate out the high-energy mode of the diagonal fields, $a_\mu$, too. The result is the same as the above, at least at the one-loop level. At the two-loop level, we must be more careful in dealing with the high-energy mode (see Ref. 51)).

\(^{**})\) Abelian dominance in the low-energy region of $SU(2)$ QCD is demonstrated in Ref. 15) using the result in Ref. 12) combined with the $SU(2)$ NAST.\(^{21)},^{15}\) The monopole dominance is derived for $SU(2)$ in Ref. 15) by showing that the dominant contribution to the area law comes from the monopole piece alone, $\Omega_K = d \omega = \text{tr}(n|dn, dn|)$. 

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SU(N) Yang-Mills theory is given by

\[
\langle W^C[A] \rangle_{YM} = \int [d\mu(V)]_C \left\langle \exp \left( ig \oint_C a \right) \right\rangle_{YM},
\]

where the one-form \(a\) is written as

\[
a = dx^\mu a_\mu = 2\text{tr}(HA^V)
\]

for a quark in the fundamental representation. Thus, the one-form \(a\) is equal to the diagonal piece of the gauge-transformed potential \(A^V\), i.e., a component along the highest-weight vector of the fundamental representation of \(SU(N)\). \(^*\) By applying the above result (6.18) to (6.21), we obtain

\[
\langle W^C[A] \rangle_{YM} \approx \langle \exp \left( ig \oint_C a \right) \rangle_{APEGT}
\]

\[
\approx \langle \exp \left( i \oint_C \omega \right) \rangle_{APEGT},
\]

where we have used in the first equality the fact that the integration over \(V\) is redundant after having taken the expectation value \(\langle \cdot \rangle_{YM}\). This implies the Abelian dominance for the large Wilson loop in \(SU(N)\) gauge theory in the sense that the expectation value of the non-Abelian Wilson loop in Yang-Mills theory is nearly equal to (or dominated by) that of the Abelian Wilson loop in the APEGT, where the Wilson loop \(C\) is so large that the APEGT is valid in that region, i.e., \(R, T > m_A^{-1}\) for a rectangular Wilson loop with sides \(R\) and \(T\). The APEGT is valid in the range \(\Lambda_{QCD} < \mu \sim R^{-1} < m_A\).

6.4. Monopole dominance and the area law

In our framework, the Abelian dominance and the monopole dominance are understood as implying the first and the second relations below, respectively:

\[
\langle W^C[A] \rangle_{YM} \approx \langle \exp \left( ig \oint_C a \right) \rangle_{APEGT}
\]

\[
\approx \langle \exp \left( i \oint_C \omega \right) \rangle_{APEGT},
\]

Numerical simulations show that the monopole part obeys the area law and \(\sigma_{\text{Abel}}\) exhausts the full string tension obtained from the non-Abelian Wilson loop (i.e., monopole dominance in the string tension or area law),

\[
\langle \exp \left( i \oint_C \omega \right) \rangle_{APEGT} \sim \exp(-\sigma_{\text{Abel}}|S|),
\]

\(^*\) Therefore, the component \(a\) along the highest-weight is obtained as an appropriate linear combination of \(a^i\). Other components orthogonal to the highest-weight do not contribute to the expectation value of the Wilson loop, since the APEGT (6.14) is an Abelian gauge theory without self-interaction among the gauge fields.
while \( \langle \exp (ig \oint_C a - i \oint_C \omega) \rangle_{\text{APEGT}} \) does not obey the area law. This result implies the area law of the original non-Abelian Wilson loop,

\[
\langle W^C[A] \rangle_{\text{YM}} \sim \exp(-\sigma|S|) \quad \sigma \approx \sigma_{\text{Abel}}.
\]

(6.29)

In Ref. 16), the monopole dominance and the area law of the Wilson loop have been demonstrated using the APEGT for \( G = SU(2) \). Now this scenario can be extended to \( G = SU(N) \). The important remarks are in order.

1. The APEGT has a running coupling constant which increases as the relevant energy decreases (asymptotic freedom), and thus the APEGT is in the strong coupling region in the low-energy regime (see Fig. 6).

2. The Abelian gauge group in APEGT is a compact group embedded in the compact non-Abelian gauge group \( SU(N) \). It is the compactness that causes the phase transition in the APEGT at \( \alpha_c = \frac{\pi}{4} \), which separates the Coulomb (conformal) phase (\( \alpha < \alpha_c \)) from the strong coupling phase (\( \alpha > \alpha_c \)). This follows from the Berezinski-Kosterlitz-Thouless (BKT) phase transition in the two-dimensional \( O(2) \) NLS model obtained through dimensional reduction.

3. In the low-energy region such that \( \alpha(\mu) > \alpha_c \), the APEGT is in the strong coupling phase which confines the quark due to vortex condensation. The above described strategy is schematically depicted below.
§7. Reformulation of Yang-Mills theory

In this section we summarize the novel reformulation of the Yang-Mills theory proposed in Ref. 13) and elaborated in Ref. 17). This material is necessary for subsequent sections.

7.1. Deformation of topological field theory

We consider the quantization of Yang-Mills theory on the topological background field. We decompose the connection $A$ as

$$A_\mu(x) = \Omega_\mu(x) + Q_\mu(x)$$

and identify $Q$ with the quantum fluctuation field on the background field $\Omega$. For arbitrary but fixed background field $\Omega$, the generating functional is given by

$$\tilde{Z}[J,\Omega] := \int [dQ][d\tilde{C}][d\tilde{\tilde{C}}][d\tilde{B}] \exp\{iS_{YM}[\Omega + Q]$$

$$+ i\tilde{S}_{GF}[Q, \tilde{C}, \tilde{\tilde{C}}, \tilde{B}] + i(J_\mu \cdot Q_\mu)\},$$

where $S_{YM}[A]$ is the usual Yang-Mills action,

$$S_{YM}[A] = -\int d^Dx \frac{1}{4} F^A_{\mu\nu}[A] F^{\mu\nu A}[A],$$

and $\tilde{S}_{GF}$ is the gauge-fixing and FP ghost term for the quantum fluctuation field $Q$:

$$\tilde{S}_{GF}[Q, \tilde{C}, \tilde{\tilde{C}}, \tilde{B}] := -\int d^Dx i\tilde{\delta}_B \text{tr}_G \left[ \tilde{C} \left( \tilde{F}[Q] + \frac{\tilde{\alpha}}{2} \tilde{B} \right) \right].$$

We wish to retain gauge invariance for the background field $\Omega$ even after the gauge fixing for $Q$. This is realized by choosing the background field (BGF) gauge fixing condition

$$\tilde{F}^A[Q] := D_\mu^{AB}[\Omega] Q^{\mu B} = 0.$$ (7.5)

In fact, in the BGF gauge, $\tilde{Z}[J,\Omega]$ is invariant under the gauge transformation of the background field; the infinitesimal version is given by $\delta \Omega_\mu = D_\mu[\Omega] \omega$. Hence, the theory with the action

$$\tilde{S}_{eff}[J,\Omega] := -i \ln \tilde{Z}[J,\Omega],$$

is defined only on the space of the gauge orbit. Suppose that the background field satisfies the equation $F^A[\Omega] = 0$. In order to consider the quantized Yang-Mills theory on all possible background fields satisfying the equations $F^A[\Omega] = 0$, we define the total generating functional

$$Z[J] = \int [d\Omega_\mu][dC][d\tilde{C}][dB] \tilde{Z}[J,\Omega] \exp(iS_{TQFT}[\Omega, C, \tilde{C}, B]) \exp[i(J_\mu \cdot \Omega^\mu)]$$

$$= \int [d\Omega_\mu][dC][d\tilde{C}][dB] \exp\{i\tilde{S}_{eff}[J,\Omega] + iS_{TQFT}[\Omega, C, \tilde{C}, B] + i(J_\mu \cdot \Omega^\mu)\},$$

(7.7)
where $S_{\text{TQFT}}[\Omega, C, \bar{C}, B]$ corresponds to the gauge-fixing term for the background field $\Omega_\mu$. In order to describe the magnetic monopole as a topological background field in Yang-Mills theory, we choose the MA gauge for $\Omega_\mu$,

$$S_{\text{TQFT}}[\Omega, C, \bar{C}, B] := - \int d^Dx \ i \delta_B \ tr_{G/H} \left[ \bar{C} \left( F[\Omega] + \frac{\alpha}{2} B \right) \right], \quad (7.8)$$

where

$$F^a[\Omega] := D_\mu [\Omega^i] \Omega^{\mu b} := (\partial_\mu \delta^{ab} - g f^{abi} \Omega_\mu^i) \Omega^{\mu b}. \quad (i = 1, \cdots, N - 1) \quad (7.9)$$

Note that the trace is taken on the coset $G/H$, not on the entire $G$, in the MA gauge.

Under the identification

$$\Omega_\mu(x) := \frac{i}{g} U(x) \partial_\mu U^\dagger(x), \quad Q_\mu(x) := U(x) \mathcal{V}_\mu(x) U^\dagger(x), \quad (7.10)$$

we assume that all the topologically non-trivial configurations come from $\Omega$, whereas $\mathcal{V}$ denotes topologically trivial configurations. Therefore, $\mathcal{V}$ changes under the small gauge transformation, while $\Omega$ includes the effect of large gauge transformations. Therefore, we must take into account the finite gauge rotation $U$, without restriction to the infinitesimal gauge transformation. Under the identification (7.10), the Yang-Mills action is invariant,

$$S_{\text{YM}}[A] = S_{\text{YM}}[\Omega + Q] = S_{\text{YM}}[\mathcal{V}] = \int d^Dx \ -\frac{1}{4} \mathcal{F}_{\mu\nu}^{A}[\mathcal{V}] \mathcal{F}^{\mu\nu A}[\mathcal{V}], \quad (7.11)$$

while the gauge-fixing term (7.4) is changed into

$$\tilde{S}_{\text{GF}}[\mathcal{V}, \gamma, \bar{\gamma}, \beta] := - \int d^Dx \ i \delta_B \ tr_{G} \left[ \bar{\gamma} \left( \partial^{\mu} \mathcal{V}_\mu + \frac{\alpha}{2} \beta \right) \right]. \quad (7.12)$$

This implies that the background gauge for $Q$ is changed into the Lorentz gauge for $\mathcal{V}$, $\partial_\mu \mathcal{V}_\mu = 0$. Then the generating functional in the background field $\Omega$ is cast into

$$\tilde{Z}[J, \Omega] = \int [d\mathcal{V}][d\gamma][d\bar{\gamma}][d\beta] \exp \left\{ i S_{\text{YM}}[\mathcal{V}] + i \tilde{S}_{\text{GF}}[\mathcal{V}, \gamma, \bar{\gamma}, \beta] + i (J_\mu \cdot \mathcal{V}^{\mu} U^\dagger) \right\}, \quad (7.13)$$

where $\mathcal{V}_\mu, \gamma, \bar{\gamma}$ and $\beta$ are defined by the adjoint rotations of $Q_\mu, \bar{C}, \bar{\bar{C}}$ and $\bar{B}$ respectively:

$$\mathcal{V}_\mu := U^\dagger Q_\mu U, \quad \gamma := U^\dagger \bar{C} U, \quad \bar{\gamma} := U^\dagger \bar{\bar{C}} U, \quad \beta := U^\dagger \bar{B} U. \quad (7.14)$$

Thus the total generating functional reads

$$Z[J] = \int [dU][dC][d\bar{C}][dB] \tilde{Z}[J, \Omega] \exp(i S_{\text{GF}}[\Omega, C, \bar{C}, B]) \exp[i (J_\mu \cdot \Omega^\mu)]$$

$$= \int [dU][dC][d\bar{C}][dB] \exp \left\{ i \tilde{S}_{\text{eff}}[J, \Omega] + i S_{\text{TQFT}}[\Omega, C, \bar{C}, B] + i (J_\mu \cdot \Omega^\mu) \right\}, \quad (7.15)$$
where we have made a change of variable from $\Omega$ to $U$. The measure $[dU]$ is invariant under the multiplication

$$U(x) \to \tilde{U}(x)U(x), \quad (7.16)$$

which leads to the finite gauge transformation of the background field,

$$\Omega(x) \to \tilde{U}(x)\Omega(x)\tilde{U}^\dagger(x) + \frac{i}{g} \tilde{U}(x)d\tilde{U}^\dagger(x), \quad (7.17)$$

The modified MA gauge proposed in Ref. 13) leads to the TQFT with the action

$$S_{\text{TQFT}}[\Omega, C, \bar{C}, B] := \int_{R^D} d^Dx \ i \delta_B \delta_B \text{tr}_{G/H} \left[ \frac{1}{2} \Omega^\mu \Omega_\mu - \frac{\alpha}{2} iC\bar{C} \right]. \quad (7.18)$$

The expectation value of the functional $f(A)$ of $A$ is calculated as follows. If the functional is of the form $f(A) = g(V_\mu, U)h(U)$, then the expectation value is calculated according to

$$\langle f(A) \rangle_{\text{YM}} = \langle g(V_\mu, U)h(U) \rangle_{\text{TQFT}}, \quad (7.19)$$

where the sector of the perturbative deformation is defined by

$$\langle (\cdots) \rangle_{\text{YM}} = Z_{\text{YM}}^{-1} \int [dV][d\gamma][d\bar{\gamma}][d\beta] \exp \left\{ iS_{\text{YM}}[V] + i\tilde{S}_{\text{GF}}[V, \gamma, \bar{\gamma}, \beta] \right\} (\cdots),$$

$$Z_{\text{YM}} := \int [dV][d\gamma][d\bar{\gamma}][d\beta] \exp \left\{ iS_{\text{YM}}[V] + i\tilde{S}_{\text{GF}}[V, \gamma, \bar{\gamma}, \beta] \right\}, \quad (7.20)$$

and the sector topological field theory is defined by

$$\langle (\cdots) \rangle_{\text{TQFT}} := Z_{\text{TQFT}}^{-1} \int [dU][dC][d\bar{C}][dB] \exp \{ iS_{\text{TQFT}}[\Omega, C, \bar{C}, B] \} (\cdots),$$

$$Z_{\text{TQFT}} := \int [dU][dC][d\bar{C}][dB] \exp \{ iS_{\text{TQFT}}[\Omega, C, \bar{C}, B] \}. \quad (7.21)$$

This reformulation of the Yang-Mills theory is called the “perturbative deformation of a topological quantum field theory”. The expectation value $\langle (\cdots) \rangle_{\text{YM}}$ for the field $V$ is calculated using a perturbation theory in terms of the coupling constant $g$. On the other hand, the expectation value $\langle (\cdots) \rangle_{\text{TQFT}}$ should be calculated in a non-perturbative way to incorporate the topological contribution. Here $U$ is a compact gauge variable corresponding to a finite gauge transformation. In the instanton calculus, the integration measure $[dU]$ is replaced with the (finite-dimensional) integration over the collective coordinates of the instanton.

### 7.2. Dimensional reduction to the NLS model in the MA gauge

It has been shown$^{13}$ that, due to the hidden supersymmetry $OSp(D|2)$, the TQFT part (7.18) is reduced to the $(D - 2)$-dimensional coset $(G/H)$ nonlinear sigma (NLS) model,

$$S_{\text{NLSM}}[U, C, \bar{C}] := \alpha \pi \int_{R^{D-2}} d^{D-2}x \ \text{tr}_{G/H} \left[ \frac{1}{2} \Omega^\mu \Omega_\mu - \frac{\alpha}{2} iC\bar{C} \right], \quad (7.22)$$

$^1$ See Remark 10.1.
where, for the matrix element $\Omega_{ab}$ (see Appendix C),

$$\text{tr}_{G/H} \left[ \frac{1}{2} \Omega_{\mu}(x) \Omega_{\mu}(x) \right] = \sum_{a,b:a<b} (\Omega_{\mu}(x))_{ab} (\Omega_{\mu}(x))_{ab}. \quad (7.23)$$

By making use of the complex coordinates in the flag space $G/H$, the action can be rewritten as (see Appendix C)

$$S_{\text{NLSM}} = \frac{\alpha \pi}{2g^2} \int_{\mathbb{R}^{D-2}} d^{D-2}x g_{\alpha\bar{\beta}} \frac{\partial w^\alpha}{\partial x_a} \frac{\partial \bar{w}^{\bar{\beta}}}{\partial x_a}, \quad (a = 1, \cdots, D - 2) \quad (7.24)$$

where we have omitted to write the decoupled ghost term, $C(x)\bar{C}(x)$. In particular, for $D = 4$,

$$S_{\text{NLSM}} = \frac{\alpha \pi}{2g^2} \int_{C} dzd\bar{z} \ g_{\alpha\bar{\beta}} \left( \frac{\partial w^\alpha}{\partial z} \frac{\partial \bar{w}^{\bar{\beta}}}{\partial z} + \frac{\partial w^\alpha}{\partial \bar{z}} \frac{\partial \bar{w}^{\bar{\beta}}}{\partial \bar{z}} \right), \quad (7.25)$$

where $z = x + iy = x_1 + ix_2 \in \mathbb{C} \cong \mathbb{R}^2$ and $dxdy = dx_1dx_2 = \frac{i}{2}dzd\bar{z}$. The $G = SU(2)$ case is analyzed in Ref. 13), and in that case we have

$$S_{\text{NLSM}} = \frac{\alpha \pi}{2g^2} \int_{C} dzd\bar{z} \ \frac{1}{(1 + |w|^2)^2} \left( \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} + \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right). \quad (7.26)$$

The above described strategy is schematically depicted below.\(^\ast\)

\[\text{D-dim. QCD with a gauge group } G\]
\[\downarrow \text{MA gauge}\]
\[\text{D-dim. Perturbative QCD} \quad \otimes \text{deform} \quad \text{D-dim. TQFT}\]
\[\downarrow \text{Dimensional reduction}\]
\[\text{D-dim. Perturbative QCD} \quad \otimes \text{deform} \quad (\text{D-2})\text{-dim. G/H NLSM}\]

\section{Area law of the Wilson loop (I)}

In this section we derive the area law of the Wilson loop based on the instanton calculus.\(^{53-55}\) A more systematic estimation is given in the next section based on the large $N$ expansion.

\(^{\ast}\) When we encounter the NLS model obtained through dimensional reduction in the following sections, the coupling constant $g$ should be replaced as

$$g^2 \rightarrow \frac{2}{\alpha} g^2, \quad (7.27)$$

since we do not express the $\alpha$ dependence explicitly.
The static potential \( V(R) \) is evaluated from the rectangular Wilson loop \( C \) with sides \( T \) and \( R \) according to
\[
V(R) := -\lim_{T \to \infty} \frac{1}{T} \ln \langle W^C[A] \rangle_{\text{YM4}}.
\]
(8.1)

The (full) string tension \( \sigma \) is defined by
\[
\sigma := -\lim_{A(C) \to \infty} \frac{1}{A(C)} \ln \langle W^C[A] \rangle_{\text{YM4}},
\]
(8.2)

where \( A(C) \) is the minimal area of the surface spanned by the Wilson loop \( C \). Of course, the rectangular loop has minimal area: \( A(C) = TR \).

Using the NAST for the Wilson loop operator,
\[
W^C[A] = \int [d\mu(\xi)]_C \exp \left( ig \oint_C n^A A^A + i \oint_C \omega \right),
\]
(8.3)

we can write its expectation value in the Yang-Mills theory as
\[
\langle W^C[A] \rangle_{\text{YM4}} = \left\langle \exp \left[ ig \oint_C dx^\mu n^A(x) V^A_\mu(x) \right] \right\rangle_{\text{pYM4}} \exp \left[ i \oint_C \omega \right]_{\text{TQFT4}},
\]
(8.4)

where the measure \( \int [d\mu(\xi)]_C \) gives a redundant factor which does not affect the string tension in the area law, as shown in Section IV.C of Ref. 15) and hence it is omitted in the following.

### 8.1. Perturbative expansion and dimensional reduction

In our reformulation, the field \( V^A_\mu(x) \) is identified with the perturbative deformation. Thus we expand the first exponential of (8.4) in powers of the coupling constant \( g \):
\[
\left\langle \exp \left[ ig \oint_C dx^\mu n^A(x) V^A_\mu(x) \right] \right\rangle_{\text{pYM4}} = 1 + \sum_{n=1}^{\infty} \frac{(ig)^n}{n!} \oint_C dx_1^{\mu_1} \oint_C dx_2^{\mu_2} \cdots \oint_C dx_n^{\mu_n} n^{A_1}(x_1)n^{A_2}(x_2) \cdots n^{A_n}(x_n)
\times \langle V^{A_1}_{\mu_1}(x_1)V^{A_2}_{\mu_2}(x_2) \cdots V^{A_n}_{\mu_n}(x_n) \rangle_{\text{pYM4}}.
\]
(8.5)

Hence, we obtain
\[
\langle W^C[A] \rangle_{\text{YM4}} = \left\langle \exp \left[ i \oint_C \omega \right] \right\rangle_{\text{TQFT4}} + \sum_{n=1}^{\infty} \frac{(ig)^n}{n!} \oint_C dx_1^{\mu_1} \oint_C dx_2^{\mu_2} \cdots \oint_C dx_n^{\mu_n} \times \langle n^{A_1}(x_1)n^{A_2}(x_2) \cdots n^{A_n}(x_n) \exp \left[ i \oint_C \omega \right] \rangle_{\text{TQFT4}},
\]

\[
\times \langle n^{A_1}(x_1)n^{A_2}(x_2) \cdots n^{A_n}(x_n) \exp \left[ i \oint_C \omega \right] \rangle_{\text{TQFT4}}.
\]
\[ \times (\nabla^{A_1}_{\mu_1}(x_1) \nabla^{A_2}_{\mu_2}(x_2) \cdots \nabla^{A_n}_{\mu_n}(x_n))_{\text{pYM}_4}. \]  

(8.6)

\[ = \left\langle \exp \left[ i \oint_C \omega \right] \right\rangle_{\text{TQFT}_4} \times \left[ 1 + \sum_{n=1}^{\infty} \frac{(ig)^n}{n!} \int_C dx_1^{\mu_1} \int_C dx_2^{\mu_2} \cdots \int_C dx_n^{\mu_n} (\nabla^{A_1}_{\mu_1}(x_1) \nabla^{A_2}_{\mu_2}(x_2) \cdots \nabla^{A_n}_{\mu_n}(x_n))_{\text{pYM}_4} \right. \]

\[ \left. \times \langle n^{A_1}(x_1)n^{A_2}(x_2) \cdots n^{A_n}(x_n) \exp \left[ i \oint_C \omega \right] \rangle_{\text{TQFT}_4} \right]. \]

(8.7)

We restrict the Wilson loop to a planar loop. This choice has the following advantage. For the planar Wilson loop \( C \), the Parisi-Sourlas dimensional reduction leads to the following identities:

\[ \langle \exp \left[ i \oint_C \omega \right] \rangle_{\text{TQFT}_4} = \langle \exp \left[ i \oint_C \omega \right] \rangle_{\text{NLSM}_2}, \]

(8.8)

and

\[ \langle n^{A_1}(x_1) \cdots n^{A_n}(x_n) \exp \left[ i \oint_C \omega \right] \rangle_{\text{TQFT}_4} = \langle n^{A_1}(x_1) \cdots n^{A_n}(x_n) \exp \left[ i \oint_C \omega \right] \rangle_{\text{NLSM}_2}, \]

(8.9)

where \( x_1, \cdots, x_n \in C \subset \mathbb{R}^2 \). This is because, in the case of the fundamental representation of \( SU(N) \), \( n^A \) and \( \omega \) can be written in terms of \( U \) as (see (4.28) and (4.29))

\[ n^A(x) = U_{1a}(x) (T^A)_{ab} U_{1b}(x) = \bar{\phi}_a(x) (T^A)_{ab} \phi_b(x), \]

(8.10)

\[ \omega(x) = \frac{i}{2} [U_{1a}(x) d\bar{U}_{1a}(x) - dU_{1a}(x) \bar{U}_{1a}(x)] \]

\[ = \frac{i}{2} [\bar{\phi}_a(x) d\phi_a(x) - d\bar{\phi}_a(x) \phi_a(x)]. \]

(8.11)

Taking the logarithm of the Wilson loop (8.7) and expanding it in powers of the coupling constant, we obtain

\[ \ln\langle W^C[A]\rangle_{\text{YM}_4} \]

(8.12)

\[ = \ln\left\langle \exp \left[ i \oint_C \omega \right] \right\rangle_{\text{NLSM}_2} \]

\[ + \ln \left[ 1 - \frac{g^2}{2} \int_C dx^\mu \int_C dy^\nu G^{AB}_{\mu\nu}(x,y) \frac{\langle n^A(x)n^B(y) \exp \left[ i \oint_S \Omega_K \right] \rangle_{\text{NLSM}_2}}{\langle \exp \left[ i \oint_C \omega \right] \rangle_{\text{NLSM}_2}} + O(g^4) \right] \]

\[ = \ln\left\langle \exp \left[ i \oint_C \omega \right] \right\rangle_{\text{NLSM}_2}. \]
\[ -\frac{g^2}{2} \int_C \, \text{d}x^\mu \int_C \, \text{d}y^\nu C^{AB}_{\mu\nu}(x, y) \frac{\langle n^A(x)n^B(y) \exp [i \oint_C \omega] \rangle_{\text{NLSM}_2}}{\langle \exp [i \oint_C \omega] \rangle_{\text{NLSM}_2}} + O(g^4), \quad (8.13) \]

where we have defined the two-point function

\[ G_{\mu\nu}^{AB}(x, y) := \langle V^A_{\mu}(x) V^B_{\nu}(y) \rangle_{\text{pYM}_4}. \quad (8.14) \]

In the rest of this section we focus on the first term in (8.13). The remaining terms will be estimated in the next section.

### 8.2. The instanton in $F_{N-1}$ and $\mathbb{CP}^{N-1}$ models

We wish to demonstrate the area law for the expectation value,

\[ \langle \exp \left( i \oint C \omega \right) \rangle_{\text{NLSM}_2} \equiv \langle \exp \left( i \int_S \Omega_K \right) \rangle_{\text{NLSM}_2} \quad (8.15) \]

\[ = Z_{\text{NLSM}_2}^{-1} \int [d\mu(w, \bar{w})] \exp(-S_{\text{NLSM}_2}[w, \bar{w}]) \exp \left( i \int_S \Omega_K \right), \quad (8.16) \]

where the two-dimensional NLS model is defined by the action

\[ S_{\text{NLSM}_2} = \frac{\pi}{g^2(\mu)} \int_{\mathbb{R}^2} d^2x g_{\alpha\beta} \frac{\partial w^\alpha}{\partial x_a} \frac{\partial \bar{w}^\beta}{\partial x_a} \quad (8.17) \]

\[ = \frac{\pi}{g^2(\mu)} \int_{\mathbb{C}} dz d\bar{z} \gamma_{\alpha\beta} \left( \frac{\partial w^\alpha}{\partial \bar{z}} \frac{\partial \bar{w}^\beta}{\partial z} + \frac{\partial w^\alpha}{\partial z} \frac{\partial \bar{w}^\beta}{\partial \bar{z}} \right), \quad (8.18) \]

where

\[ z = x + iy = x_1 + ix_2 \in \mathbb{C} \cong \mathbb{R}^2, \quad dx dy = dx_1 dx_2 = \frac{i}{2} dz d\bar{z}, \quad (8.19) \]

and $g(\mu)$ is the Yang-Mills coupling constant.\(^*)\)

Note that the action satisfies the inequality,

\[ S_{\text{NLSM}} = \frac{\pi}{2g^2(\mu)} \int_{\mathbb{R}^2} d^2x g_{\alpha\beta}(\partial_a w^\alpha \pm i\epsilon_{ab} \partial_b w^\alpha)(\partial_a w^\beta \pm i\epsilon_{ac} \partial_c w^\beta)^* \]

\[ \pm i \frac{\pi}{g^2(\mu)} \int_{\mathbb{R}^2} d^2x \epsilon_{ab} g_{\alpha\beta} \frac{\partial w^\alpha}{\partial x_a} \frac{\partial \bar{w}^\beta}{\partial x_b} \quad (8.20) \]

\[ \geq \pm i \frac{\pi}{g^2(\mu)} \int_{\mathbb{R}^2} d^2x \epsilon_{ab} g_{\alpha\beta} \frac{\partial w^\alpha}{\partial x_a} \frac{\partial \bar{w}^\beta}{\partial x_b}. \quad (8.21) \]

This inequality is saturated when $w^\alpha$ satisfies the equation $\partial_a w^\alpha \pm i\epsilon_{ab} \partial_b w^\alpha = 0$, which is equivalent to the Cauchy-Riemann equation,

\[ \partial_z w^\alpha := (\partial_1 + i\partial_2) w^\alpha = 0. \quad (8.22) \]

The solution $w^\alpha = f^\alpha(z)$ is an arbitrary rational function of $z$.

\(^*)\) The running of the coupling constant is given by the perturbative deformation in four-dimensional Yang-Mills theory, as in (6.15).
The finite action configuration of the coset NLS model is provided by the instanton solution, which is a solution of the Cauchy-Riemann equation (8.22). It is known\textsuperscript{23} that the integer-valued topological charge $Q$ of the instanton in the $F_{N-1}$ NLS model is given by the integral of the Kähler 2-form over $R^2$:

$$Q = \frac{1}{\pi} \int_{R^2} \Omega_K = \int_{R^2} \frac{d^2 x}{\pi} \epsilon_{abc} \partial w^a \partial \bar{w}^b \frac{\partial w^a}{\partial x^a} \frac{\partial w^b}{\partial x^b} = \int_{C} \frac{dz \bar{z}}{\pi} g_{\alpha \bar{\beta}} \left( \frac{\partial w^\alpha}{\partial z} \frac{\partial \bar{w}^\beta}{\partial \bar{z}} - \frac{\partial w^\alpha}{\partial \bar{z}} \frac{\partial \bar{w}^\beta}{\partial z} \right).$$

This is a generalization of the $F_1$ case ($N = 2$), where

$$Q = \frac{i}{2\pi} \int_{C} \frac{dwd\bar{w}}{(1 + |w|^2)^2} = \frac{i}{2\pi} \int_{S^2} \frac{dzd\bar{z}}{(1 + |w|^2)^2} \left( \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} - \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right).$$

Thus the instanton solution is characterized by the integral topological charge $Q \in \mathbb{Z}$. For instanton ($Q > 0$) and anti-instanton ($Q < 0$) configurations with a topological charge $Q$, the action has

$$S_{\text{NLSM}} = \frac{\pi^2}{g^2} |Q|.$$ (8.25)

The metric in the Kähler manifold $F_{N-1}$ is obtained in accordance with the relation $g_{\alpha \bar{\beta}} = \frac{\partial}{\partial w^\alpha} \frac{\partial}{\partial \bar{w}^\beta} K$ from the Kähler potential,

$$K(w, \bar{w}) = \sum_{\ell=1}^{N-1} d_\ell K_\ell(w, \bar{w}) = \sum_{\ell=1}^{N-1} d_\ell \ln \Delta_\ell(w, \bar{w}),$$

with the Dynkin indices $d_\ell (\ell = 1, \ldots, N-1)$. Then the integral $\int \Omega_K$ over the whole two-dimensional space reads

$$\int_{R^2} \Omega_K = \pi Q = \pi \sum_{\ell=1}^{N-1} d_\ell Q_\ell,$$ (8.27)

where $Q_\ell$ are integer-valued topological charges. Hence, $\Omega_K(x)/\pi$ is identified with the density of the topological charge (up to the weight due to the index $d_\ell$).

Now we consider the $CP^{N-1}$ model. If we identify $w_\alpha$ with the inhomogeneous coordinates, e.g., $w_\alpha := \frac{\phi_\alpha}{\phi_N}$, the metric (3.63) can be rewritten as

$$g_{\alpha \bar{\beta}}(\phi) = \frac{(||\phi||^2)\delta_{\alpha \bar{\beta}} - \bar{\phi}_\alpha \phi_\beta}{(||\phi||^2)^2},$$ (8.28)

where

$$||\phi||^2 := \sum_{\alpha=1}^{N} |\phi_\alpha|^2 = |\phi_N|^2 \left( 1 + ||w||^2 \right).$$ (8.29)

The action of the $CP^{N-1}$ model is given by

$$S_{CP^{N-1}} = \frac{\pi}{g^2} \int d^d x g_{\alpha \bar{\beta}}(w) \partial_\mu w^\alpha \partial_\mu \bar{w}^\beta,$$ (8.30)
or equivalently,

\[
S_{CPN-1} = \frac{\pi}{g^2} \int d^4x g_{\alpha\beta}(\phi) \partial_\mu \phi^\alpha \partial_\mu \bar{\phi}^\beta. \tag{8.31}
\]

Under the constraint \(||\phi||^2 = 1\), the action can be written as

\[
S_{CPN-1} = \frac{\beta g}{2} \int d^4x (\delta_{\alpha\beta} - \bar{\phi}_\alpha \phi_\beta) \partial_\mu \phi^\alpha \partial_\mu \bar{\phi}^\beta
\]

\[
= \frac{\beta g}{2} \int d^4x [\partial_\mu \bar{\phi}^\alpha \partial_\mu \phi^\alpha + (\bar{\phi}^\alpha \partial_\mu \phi^\alpha)(\bar{\phi}^\beta \partial_\mu \phi^\beta)]. \tag{8.32}
\]

This agrees with the action of the \(CP^{N-1}\) model presented in Ref. 13). (See Appendix C for more details).

8.3. Area law in the dilute instanton-gas approximation

If the Wilson loop is large compared with the typical size of the instanton, \(\int_S \Omega_K(x)/\pi\) in (8.16) counts the number of instantons \(n_+\) minus anti-instantons \(n_-\) that are contained inside the area \(S \subset \mathbb{R}^2\) bounded by the loop \(C\):

\[
\int_S \Omega_K = \pi (n_+ - n_-). \tag{8.33}
\]

(See Fig. 7.) Thus, the expectation value \(\langle \exp(i \int_S \Omega_K) \rangle_{NLSM}\) is calculated by summing over all the possible cases of instanton and anti-instanton configurations (i.e., by the integration over the instanton moduli). In this calculation, we use

\[
S_{NLSM} = \frac{\pi^2}{g^2} |Q| = \frac{\pi^2}{g^2} (n_+ + n_-), \quad n_\pm = n_{\pm}^{in} + n_{\pm}^{out}, \tag{8.34}
\]

where \(n_+^{out}\) \((n_-^{out})\) is the number of instantons (anti-instantons) outside \(S\), and \(n_+\) \((n_-)\) is the total number of instantons (anti-instantons). For the quark in the fundamental representation \(N\) \((d_1 = 1, d_2 = d_3 = \cdots = d_{N-1} = 0)\), this is easily carried out as follows.

For the \(SU(3)\) case with \([1,0]\), an element \(\xi \in F_2\) is independent of \(w_3\), so that \(w_3\) is redundant in this case. Hence, it suffices to consider the \(CP^2\) model for the fundamental quark (up to Weyl symmetry). For \(CP^2\), the Kähler two-form is given by (4.87)

\[
\Omega_K = im(\Delta_1)^{-2}[(1 + |w_1|^2)dw_2 \wedge d\bar{w}_2 - \bar{w}_2 w_1 dw_2 \wedge d\bar{w}_1
- w_2 \bar{w}_1 dw_1 \wedge d\bar{w}_2 + (1 + |w_2|^2)dw_1 \wedge d\bar{w}_1]. \tag{8.35}
\]
When \( w_2 = 0 \), \( \Omega_K \) reduces to
\[
\Omega_K = i(1 + |w_1|^2)^{-2} dw_1 \wedge d\bar{w}_1.
\]
(8.36)

Similarly, when \( w_1 = 0 \), we have
\[
\Omega_K = i(1 + |w_2|^2)^{-2} dw_2 \wedge d\bar{w}_2.
\]
(8.37)

For a polynomial \( w_\alpha = w_\alpha(z) \) in \( z = x + iy \) of order \( n \), we find the instanton charge
\[
\int \Omega_K = \pi Q, \quad Q \in \mathbb{Z}.
\]
(8.38)

This is the same situation as that encountered in \( SU(2) \), in which case
\[
\int \Omega_K = 2j\pi Q,
\]
(8.39)

where \( j = 1/2 \) corresponds to the fundamental representation.\(^{13,15}\) Thus the Wilson loop can be estimated by the naive instanton calculus. In fact, the dilute instanton gas approximation leads to the area law for the Wilson loop (see Ref. 13)). Here the factor \( \pi \) is very important. The integral of the Kähler two-form \( \Omega_K \) is a multiple of \( \pi \). If we had a factor \( 2\pi \), the area law would not hold, just as in the \( j = 1 \) case of \( SU(2) \).

For the \( SU(N) \) case with Dynkin index, \([1, 0, \cdots, 0]\), it suffices to consider the \( CP^{N-1} \) model. When \( w_a \neq 0 \) and \( w_b = 0 \) for all \( b \neq a \), the Kähler two-form (4.53) for \( CP^{N-1} \) reduces to
\[
\Omega_K = i(1 + |w_a|^2)^{-2} dw_a \wedge d\bar{w}_a,
\]
(8.40)

with no summation over \( a \). Therefore the above argument can be applied to \( SU(N) \) for any \( N \). This implies the confinement of fundamental quarks in \( SU(N) \) Yang-Mills theory within the approximation of a dilute instanton gas. This naive instanton calculation can be improved by including fluctuations from the instanton solutions following Ref. 55), and this issue will be discussed in detail in a subsequent article.\(^{43}\)

\section{Area law of the Wilson loop (II)}

The derivation of the area law of the Wilson loop in the four-dimensional Yang-Mills theory in the MA gauge is reduced to demonstrating the area law of the diagonal Wilson loop in the two-dimensional coset NLS model. In this section we complete a derivation of the area law of the Wilson loop in the fundamental representation. Here we use the large \( N \) expansion\(^{56)-60}\) for the coset NLS model. (See, e.g., Refs. 28)-31) for reviews of the large \( N \) expansion.)

To perform the large \( N \) expansion, it is convenient to introduce the new variables
\[
P_{ab}(x) := \bar{\phi}_a(x)\phi_b(x) = U_{1a}(x)\bar{U}_{1b}(x),
\]
(9.1)
\[
\gamma^a_{\mu}(x) := \gamma^a_{\mu}(x)(T^A)_{ab},
\]
(9.2)
which are used to rewrite
\[ C_{\mu}(x) := n^{A}(x)\nu^{A}_{\mu}(x) = P_{ab}(x)\nu^{ab}_{\mu}(x), \]  
(9.3)

where \( A = 1, \cdots, N^2 - 1 \) and \( a, b = 1, \cdots N \) for \( SU(N) \).

In terms of the above variables, the expansion (8.7) of the expectation value of the Wilson loop operator in powers of the coupling constant \( g \) is rewritten as
\[ \langle W^{C}[A] \rangle_{YM4} = \left\langle \exp \left[ i \int_{C} \omega \right] \right\rangle_{TQFT4} + \sum_{n=1}^{\infty} \frac{(ig)^{n}}{n!} \int_{C} dx_{1}^{\mu_{1}} \int_{C} dx_{2}^{\mu_{2}} \cdots \int_{C} dx_{n}^{\mu_{n}} \times \langle \nu^{a_{1}b_{1}}_{\mu_{1}}(x_{1})\nu^{a_{2}b_{2}}_{\mu_{2}}(x_{2}) \cdots \nu^{a_{n}b_{n}}_{\mu_{n}}(x_{n}) \rangle_{pYM4} \times \langle P_{a_{1}b_{1}}(x_{1})P_{a_{2}b_{2}}(x_{2}) \cdots P_{a_{n}b_{n}}(x_{n}) \exp \left[ i \int_{C} \omega \right] \rangle \left\rangle_{TQFT4} \right. \]  
(9.4)

\[ = \left\langle \exp \left[ i \int_{C} \omega \right] \right\rangle_{TQFT4} \times \left[ 1 + \sum_{n=1}^{\infty} \frac{(ig)^{n}}{n!} \int_{C} dx_{1}^{\mu_{1}} \int_{C} dx_{2}^{\mu_{2}} \cdots \int_{C} dx_{n}^{\mu_{n}} \times \langle \nu^{a_{1}b_{1}}_{\mu_{1}}(x_{1})\nu^{a_{2}b_{2}}_{\mu_{2}}(x_{2}) \cdots \nu^{a_{n}b_{n}}_{\mu_{n}}(x_{n}) \rangle_{pYM4} \times \langle P_{a_{1}b_{1}}(x_{1})P_{a_{2}b_{2}}(x_{2}) \cdots P_{a_{n}b_{n}}(x_{n}) \exp \left[ i \int_{C} \omega \right] \rangle \right\rangle_{TQFT4} \right]. \]  
(9.5)

The diagrams needed to calculate this expectation value are drawn in Fig. 9 based on the Feynmann rule given in Fig. 8. Here it should be remarked that the definition of the Wilson loop operator
\[ \langle W^{C}[A] \rangle := \frac{1}{N} \left\langle \text{tr} \left[ \mathcal{P} \exp \left( ig \int_{C} dx^{\mu} A_{\mu}(x) \right) \right] \right\rangle_{YM4}, \]  
(9.6)

includes the normalization factor \( N^{-1} \) and that the expectation value (9.6) of the Wilson loop may have a well-defined large \( N \) limit. In particular, in the zero coupling limit, the expectation value reduces to one.

9.1. Large \( N \) expansion and dimensional reduction

It is known\(^{32}\) that only the planar diagrams contribute to the expectation value
\[ \langle A_{\mu_{1}}^{a_{1}b_{1}}(x_{1})A_{\mu_{2}}^{a_{2}b_{2}}(x_{2}) \cdots A_{\mu_{n}}^{a_{n}b_{n}}(x_{n}) \rangle_{YM4} \]  
(9.7)
in the leading order of the large \( N \) expansion. See Fig. 9. However, it is extremely difficult to sum up the infinite number of terms belonging to the leading order of the large \( N \) expansion and to obtain a closed expression in the four-dimensional case. Of course, this does not exclude the possibility that the closed expression obtained by summing up all the leading diagrams may exhibit the area law. In fact, this strategy
Fig. 8. Feynmann rule and the corresponding large $N$ rule (double line notation due to 't Hooft) in QCD. Propagators: (a) quark propagator, (b) gluon propagator. Vertices: (c) quark-gluon vertex ($g_{YM}$), (d) three-gluon vertex ($g_{YM}^2$), (e) four-gluon vertex ($g_{YM}^4$).

has been applied in the two-dimensional case and has successfully lead to the area law (see, e.g., Ref. 61)).

For the planar Wilson loop $C$, we have already shown that the Parisi-Sourlas dimensional reduction occurs and that the TQFT sector reduces to the two-dimensional coset NLS model, i.e. the NLS model on the flag space $F_{N-1}$. Hence, we obtain

$$\langle e^{i \oint C \omega} \rangle_{\text{TQFT}_4} = \langle e^{i \oint C \omega} \rangle_{\text{NLSM}_2}, \quad (9.8)$$

and

$$\langle P_{a_1b_1}(x_1) \cdots P_{a_nb_n}(x_n)e^{i \oint C \omega} \rangle_{\text{TQFT}_4} = \langle P_{a_1b_1}(x_1) \cdots P_{a_nb_n}(x_n)e^{i \oint C \omega} \rangle_{\text{NLSM}_2}, \quad (9.9)$$

where $x_1, \ldots, x_n \in C \subset \mathbb{R}^2$. For the quark in the fundamental representation of $SU(N)$, the relevant NLS model can be restricted to the $CP^{N-1}$ model.

We now return to the expression (9.5) obtained by way of the NAST and apply the large $N$ expansion to the (perturbative) deformation sector and the TQFT sector simultaneously. Taking the logarithm of the Wilson loop, therefore, we obtain

$$\ln \langle W^C[A] \rangle_{YM_4}$$

$$= \ln \left\langle \exp \left[ i \oint C \omega \right] \right\rangle_{CP^{N-1}} + \ln \left[ 1 - f[C] + O(\lambda^2/N^2) \right]$$
Fig. 9. Examples of Feynmann diagrams and the corresponding double line notations that appear in calculating the expectation value of the Wilson loop operator. The order of each diagram is estimated using the rule given in Fig. 8 as (a) \( g_* N^2 = \lambda N \), (b) \( g_* N^3 = \lambda^2 N \), (c) \( g_* N^3 = \lambda^2 N \), (d) \( g_* N = \lambda^2 / N \), (e) \( g_* N^4 = \lambda^3 N \), (f) \( g_* N^3 = \lambda^2 N \), and (g) \( g_* N^2 = \lambda^2 N^0 \).

[Here note that each contribution should be divided by the normalization factor \( N \) for the fundamental quark, in agreement with the definition (9.6).] In the leading order of large \( N \) expansion, the leading contributions come from the planar diagrams, e.g., (a), (b), (c), (e) and (f), which are furthermore classified by the order of \( \lambda \). Note that the contribution from a nonplanar diagram (d) is suppressed for large \( N \). The diagram (g) is the vacuum polarization diagram due to quark–anti-quark pair creation and annihilation, which is neglected in the pure Yang-Mills theory without dynamical quarks.

\[
\ln \left\langle \exp \left[ i \oint_C \omega \right] \right\rangle_{CP_2^{N-1}} - f[C] + O(\lambda^2 / N^2),
\]

where we have defined the two-point correlation function

\[
G_{\mu\nu}^{ab,cd}(x,y) := \langle V_{\mu}^{ab}(x) V_{\nu}^{cd}(y) \rangle_{YM_4},
\]

and

\[
f[C] := \frac{g^2}{2} \int_C dx^\mu \int_C dy^\nu G_{\mu\nu}^{ab,cd}(x,y) \frac{\langle P_{ab}(x) P_{cd}(y) \exp \left[ i \oint_C \omega \right] \rangle_{CP_2^{N-1}}}{\langle \exp \left[ i \oint_C \omega \right] \rangle_{CP_2^{N-1}}}. \]

\[ \text{(9.10)} \]

\[ \text{(9.11)} \]

\[ \text{(9.12)} \]
It should be remarked that the $f[C]$ is $O(\lambda/N)$. This is different from the result of the usual large $N$ expansion of the Yang-Mills theory, i.e., diagram (a) of Fig. 9, which is $O(\lambda)$. This fact is shown as follows. In a previous article,$^{17}$ it is shown that the perturbative sector obeys the Lorentz-type gauge fixing, $\partial_\mu \mathcal{V}_\mu = 0$, by virtue of the background gauge. Here we adopt the Feynman gauge to simplify the calculation.

Then the propagator for $\mathcal{V}_\mu$ reads

$$\langle \mathcal{V}_\mu^A(x) \mathcal{V}_\nu^B(y) \rangle_{pYM_4} = \delta^{AB}\delta_{\mu\nu}D(x, y), \quad D(x, y) := \frac{1}{4\pi^2|x - y|^2}. \quad (9.13)$$

Then we find

$$G^{ab,cd}_{\mu\nu}(x, y) := \langle \mathcal{V}_\mu^a(x) \mathcal{V}_\nu^c(y) \rangle_{pYM_4}$$

$$= \langle \mathcal{V}_\mu^A(x) \mathcal{V}_\nu^B(y) \rangle_{pYM_4}(T^A)_{ab}(T^B)_{cd} \quad (9.15)$$

$$= \delta_{\mu\nu}D(x, y) \sum_{A=1}^{N^2-1} (T^A)_{ab}(T^A)_{cd}, \quad (9.16)$$

where for $G = SU(N)$,

$$\sum_{A=1}^{N^2-1} (T^A)_{ab}(T^A)_{cd} = \frac{N^2 - 1}{2N} \delta_{ab} = C_2(R)\delta_{ab}, \quad (9.17)$$

This implies that

$$\sum_{A=1}^{N^2-1} (T^A T^A)_{ab} = \frac{N^2 - 1}{2N} \delta_{ab} = C_2(R)\delta_{ab}, \quad (9.18)$$

where $C_2(R)$ is the quadratic (second order) Casimir invariant of the fundamental representation. If $G = U(N)$, the relation is simplified as $\sum_{A=1}^{N^2}(T^A)_{ab}(T^A)_{cd} = \frac{1}{2}\delta_{ab}\delta_{bc}$. The difference between $SU(N)$ and $U(N)$ disappears in the large $N$ limit.

To leading order, we can set (see Appendix E)

$$\langle P_{ab}(x)P_{cd}(y) \exp[i \oint C \omega] \rangle_{CP^{N-1}} \approx \langle P_{ab}(x)P_{cd}(y) \rangle_{CP^{N-1}}. \quad (9.19)$$

Thanks to the $SU(N)$ invariance, it is easy to see that

$$\langle P_{ab}(x)P_{cd}(y) \rangle_{CP^{N-1}} = \left( \delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd} \right) Q(x, y), \quad (9.20)$$

where

$$Q(x, y) = \frac{\langle \bar{\phi}_a(x)\phi_b(x)\bar{\phi}_b(y)\phi_a(y) \rangle_{CP^{N-1}} - \frac{1}{N^2} \langle \bar{\phi}_a(x)\phi_a(x)\bar{\phi}_b(y)\phi_b(y) \rangle_{CP^{N-1}}}{N^2 - 1}. \quad (9.21)$$
This leads to
\[ g^2 G^{ab,cd}_{\mu\nu}(x, y) \frac{\langle P_{ab}(x)P_{cd}(y) \exp [i f_C \omega] \rangle_{CP_{N-1}^2}}{\langle \exp [i f_C \omega] \rangle_{CP_{N-1}^2}} = \delta_{\mu\nu} D(x, y) g^2 \sum_{A=1}^{N^2-1} (T^A)_{ab}(T^A)_{cd} \left( \delta_{ad} \delta_{bc} - \frac{1}{N} \delta_{ab} \delta_{cd} \right) Q(x, y) = \delta_{\mu\nu} D(x, y) g^2 \sum_{A=1}^{N^2-1} \left[ \text{tr}(T^A T^A) - \frac{1}{N} \text{tr}(T^A) \text{tr}(T^A) \right] Q(x, y) = \delta_{\mu\nu} D(x, y) g^2 \frac{N^2 - 1}{2} Q(x, y). \] (9.22)

Note that \( Q(x, y) \) is \( O(N^{-2}) \), and hence \( f[C] \) is \( O(\lambda/N) \). This is because we consider the large \( N \) expansion of the \( CP^{N-1} \) model (see Appendix D) with the Lagrangian (see (C.23))
\[ \mathcal{L}_{CP^{N-1}} = \frac{N}{g_0^2} |(\partial_\mu + iV_\mu(x))\phi^\alpha(x)|^2, \quad g_0^2 := \frac{2g_{YM}^2 N}{\pi}, \] (9.23)
where \( V_\mu(x) \) is the composite gauge field,
\[ V_\mu(x) = \frac{i}{2} (\bar{\phi}^\alpha(x) \partial_\mu \phi^\alpha(x) - \partial_\mu \bar{\phi}^\alpha(x) \phi^\alpha(x)), \] (9.24)
under the constraint
\[ \phi^\dagger(x) \phi(x) := \bar{\phi}^\alpha(x) \phi^\alpha(x) = 1. \] (9.25)

It is not difficult to show that the above estimation gives the correct order for the higher-order terms, e.g., (b) and (e) in Fig. 9, by making use of the relations (4.73) and (4.74). For example, \( g^3 \langle V_{\mu_1}^A(x_1) V_{\mu_2}^B(x_2) V_{\mu_3}^C(x_n) \rangle_{pYM} \), is proportional to \( ig^4 f^{ABC} \), and \( ig^4 f^{ABC} \text{tr}[T^A T^B T^C]/N^3 = -g^4 f^{ABC} f^{ABC} / (4N^3) = O(g^4) \), since \( f^{ABC} f^{ABD} = C_2(\text{Adj}) \delta^{CD} \) and \( C_2(\text{Adj}) = N \).

Another way to understand this result is based on the idea of the reduction of degrees of freedom that are responsible to the Wilson loop. The flag space has dimension \( \dim F_{N-1} = N(N - 1) \), whereas \( CP^{N-1} \) has dimension \( \dim CP^{N-1} = 2(N - 1) \). Therefore, the number of relevant degrees of freedom is reduced for the fundamental quark for large \( N \), since \( \dim CP^{N-1} \cong 2\dim F_{N-1}/N \) for large \( N \). Indeed, this result is expected from the NAST given by (4.54),
\[ W^C[A] = \int [d\mu(\xi)]_C \exp \left( ig \int_C a \right). \] (9.26)

The Abelian gauge field \( a = \text{tr}(\mathcal{H}A) \) has only two physical degrees of freedom, while the non-Abelian gauge field \( A = A^A T^A \) in the Wilson loop (9.6) has \( 2(N^2 - 1) \) components. Thus the large \( N \) expansion is reduced to a perturbative expansion in the coupling constant \( g \). In this sense, the large \( N \) expansion combined with the NAST justifies the identification of the deformation part with the perturbative part.
Then we find that the $f[C]$ is of order $O(\lambda/N)$. Therefore, to leading order in the large $N$ expansion, the static potential and the string tension are given by

$$V(R) = -\lim_{T \to \infty} \frac{1}{T} \ln \exp \left[ i \oint_C \omega \right]_{CP^{N-1}} + \lim_{T \to \infty} \frac{f[C]}{T} + O\left(\frac{\lambda^2}{N^2}\right), \quad (9.27)$$

$$\sigma = -\lim_{R,T \to \infty} \frac{1}{RT} \left( \ln \exp \left[ i \oint_C \omega \right]_{CP^{N-1}} + f[C] \right) + O\left(\frac{\lambda^2}{N^2}\right). \quad (9.28)$$

Now we proceed to estimate the second term, $f[C]$. We will show that the second term gives at most the perimeter law, so that the area law (if it exists) is provided by the first term. Because of the factor $\delta_{\mu\nu}$ in $\langle V_A^a(x) V_B^b(y) \rangle_{PYM_4}$, only integration between parallel sides $dx$ and $dy$ gives a contribution to $f[C]$. Thus $f[C]$ is reduced to

$$f[C] = \frac{g^2}{4} \oint_C dx^\mu \oint_C dy^\mu D(x,y) G_C(x,y), \quad (9.29)$$

where we have defined the correlation function for the composite operators as

$$G_C(x,y) := 2(T^A)_{ab}(T^A)_{cd} \langle P_{ab}(x) P_{cd}(y) \exp \left[ i \oint_C \omega \right] \rangle_{CP^{N-1}}$$

$$\approx 2(T^A)_{ab}(T^A)_{cd} \left( \delta_{ad} \delta_{bc} - \frac{1}{N} \delta_{ab} \delta_{cd} \right) Q(x,y) \quad (9.30)$$

$$= (N^2 - 1)Q(x,y). \quad (9.31)$$

First, if we restrict our consideration to topologically trivial configurations, i.e.,

$$n^A(x)n^A(y) \equiv P_{ab}(x) P_{cd}(y) (T^A)_{ab}(T^A)_{cd} \approx n^A(\infty)n^A(\infty) \equiv \frac{1}{2}(1 - N^{-1}), \quad (9.32)$$

then we obtain $G_C(x,y) \approx 1 + O(N^{-1})$ and

$$f[C] \approx \frac{\lambda}{2N} \left(1 + O(N^{-1})\right) h[C], \quad h[C] := \frac{1}{2} \oint_C dx^\mu \oint_C dy^\mu D(x,y). \quad (9.33)$$

By taking into account all the contributions from parallel sides $dx$ and $dy$ (see Fig. 10), $h[C]$ is calculated as, for $T > R \gg 1$,

$$h[C] = -\frac{1}{4\pi} \frac{T}{R} + \frac{1}{2\pi^2} \frac{T + R}{\epsilon} + \frac{1}{2\pi^2} \ln \frac{R}{\epsilon}, \quad (9.34)$$

where $\epsilon$ is the ultraviolet cutoff included to avoid the coincidence of $x$ and $y$ (see Appendix of Ref. 15)). In $h[C]$, the first term corresponds to the Coulomb potential in four dimensions,

$$V_C(R) = -\frac{g^2}{4\pi} \frac{1}{R} + \text{const} + O(\lambda^2/N^2), \quad (9.35)$$

$$\approx \frac{\lambda}{2N} \left(1 + O(N^{-1})\right) h[C],$$

$$f[C] \approx \frac{\lambda}{2N} \left(1 + O(N^{-1})\right) h[C],$$

$$h[C] := \frac{1}{2} \oint_C dx^\mu \oint_C dy^\mu D(x,y). \quad (9.36)$$
Fig. 10. A rectangular Wilson loop and the contribution to the Wilson integral in which \(x\) and \(y\) run over (a) opposite sides, and (b) same sides.

Fig. 11. A circular Wilson loop.

and the second term in \(h[C]\) corresponds to the self-energy of quark and anti-quark. Furthermore, if we take into account the \(O(g^4)\) correction, the coupling constant begins to run and the bare coupling \(g\) in (9.36) is replaced by the running coupling constant \(g = g(\mu)\) (see, e.g., Kogut\(^{62}\)).\(^{*)}\) In the topologically trivial case, therefore, the second term \(f[C]\) cannot give a non-vanishing string tension.

Next, we consider the topologically non-trivial case. We begin to estimate (9.29) for a circular Wilson loop \(C\) with diameter \(R\) (see Fig. 11). If we avoid the coinciding case, \(x = y\), \(f[C]\) has a contribution only when \(x\) and \(y\) are at opposite ends of a diameter, i.e., \(|x - y| = R\). Therefore, \(D(x, y)\) and \(G(x, y)\) are functions of \(R\), due to translational invariance. For any \(x, y \in C\), \(|x - y| = R\) and

\[
\oint_C dx^\mu \oint_C dy^\mu D(R) = \frac{\pi^2 R^2}{4\pi^2 R^2} = \frac{1}{4}.
\]

Hence we obtain

\[
f[C] = \frac{\lambda}{8N} G_C(x, y).
\]

It is clear that \(f[C]\) is not sufficient to give a non-vanishing string tension, since \(G_C(x, y)\) exhibits exponential decay for large \(R := |x - y|\). Note that \(P_{ab}(x)\) and \(\oint_C \omega = \int_S d\omega\) are \(U(1)\) gauge invariant quantities, so that \(G_C(x, y)\) is also \(U(1)\) gauge invariant. In the large \(N\) expansion, we can give a more precise estimate of the second term (see Appendix E).

\(^{*)}\) The contribution up to \(O(\lambda^2)\) in the leading order diagrams (planar diagram) in the large \(N\) expansion leads to a running coupling that differs from that in the usual perturbative calculation in the coupling constant \(g\).
Finally, we consider the topologically nontrivial case of a rectangular Wilson loop $C$ with side lengths $R$ and $T$ (see Fig. 10). In this case, we cannot give a precise estimate of the second term, since we cannot perform the integration exactly. To leading order in the $1/N$ expansion, it turns out that

$$ G_C(x,y) \approx \tilde{G}(x,y) := 2(T^A)_{ab}(T^A)_{cd} \langle P_{ab}(x)P_{cd}(y) \rangle_{CP^{N-1}}, \quad (9.39) $$

and that $\tilde{G}(x,y)$ decays exponentially for sufficiently large $|x - y|$ (see Appendix E). Note that $x$ and $y$ are located on the opposite sides of the rectangular Wilson loop. Therefore, there exists an uniform upper bound,

$$ |\tilde{G}(x,y)| \leq \tilde{G}(R). \quad (9.40) $$

Hence, there exists an upper bound on $f[C]$ for a sufficiently large Wilson loop such that $T > R \gg 1$:

$$ |f[C]| \leq \frac{\lambda}{2N} \tilde{G}(R) h[C]. \quad (9.41) $$

Here $h[C]$ is calculated in the same way as in (9.35) by taking into account all the contributions from parallel sides $dx$ and $dy$ (see Fig. 10). Therefore the second term $f[C]$ cannot give non-vanishing string tension in the topologically nontrivial case.

Thus, within this reformulation, the area law of the non-Abelian Wilson loop and the linear static potential in four-dimensional $SU(N)$ Yang-Mills theory is realized if and only if the diagonal Wilson loop $\langle \exp \left( i \oint_C \omega \right) \rangle_{CP^{N-1}}$ in the two-dimensional NLS model obeys the area law

$$ \langle \exp \left( i \oint_C \omega \right) \rangle_{CP^{N-1}} \equiv \langle \exp \left( i \int_S \Omega_K \right) \rangle_{CP^{N-1}} \sim \exp(-\sigma_0 TR). \quad (9.42) $$

In other words, the area law or the linear potential between the fundamental quark and anti-quark is obtained from the topological $TQFT_4$ piece alone:

$$ \langle \exp \left( i \oint_C \omega \right) \rangle_{TQFT_4} = \langle \exp \left( i \oint_C \omega \right) \rangle_{CP^{N-1}} \sim \exp(-\sigma_0 TR). \quad (9.43) $$

In any case, the derivation of the area law is reduced to a two-dimensional problem.

It should be remarked that only the total static potential,

$$ V(R) = \sigma_0 R - \frac{g^2(\mu)}{4\pi} \frac{1}{R} + \text{const}, \quad (9.44) $$

is gauge invariant. Thus the linear potential piece alone is not gauge invariant. However, in the large $R$ limit, $R \to \infty$, the linear potential is dominant in $V(R)$ so that the linear potential piece becomes essentially gauge invariant.

### 9.2. Area law to the leading order in the large $N$ expansion

By the rescaling of the field $\phi$ in the Lagrangian (9.23), another form of the Lagrangian of the $CP^{N-1}$ model is obtained as

$$ \mathcal{L}_{CP^{N-1}} = \partial_\mu \bar{\phi}^\alpha(x)\partial_\mu \phi^\alpha(x) + \frac{\theta^2}{4N} (\bar{\phi}^\alpha(x)\partial_\mu \phi^\alpha(x) - \partial_\mu \bar{\phi}^\alpha(x)\phi^\alpha(x))^2. \quad (9.45) $$
with the constraint
\[ \phi^\dagger(x)\phi(x) := \bar{\phi}^a(x)\phi^a(x) = \frac{N}{g_0^2}, \quad g_0^2 := \frac{g_{YM}^2 N}{\pi}. \] (9.46)

It is useful to consider the Schwinger parameterization,\(^{63)}\)
\[ \phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} = e^{i\varphi} \left( 1 + \frac{g_0^2}{4N} u^\dagger u \right)^{-1} \left( \sqrt{\frac{N}{g_0}} \left( 1 - \frac{g_0^2}{4N} u^\dagger u \right) \right), \] (9.47)
where
\[ u = \begin{pmatrix} u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}, \quad u^\dagger u := \bar{u}_\alpha u_\alpha. \quad (\alpha = 1, \cdots, N-1) \] (9.48)

Note that there is no constraint for the variable \( u \), since the Schwinger parameterization automatically satisfies the constraint (9.46). We can rewrite various quantities in terms of \( u \) without the constraint, e.g.,
\[ \phi^\dagger d\phi - d\phi^\dagger \phi = \frac{u^\dagger du - du^\dagger u}{(1 + \frac{g_0^2}{4N} u^\dagger u)^2} + 2i \frac{N}{g_0} d\varphi. \] (9.49)

Then the \( U(1) \) gauge field \( V = V_\mu dx^\mu \) in the \( CP^{N-1} \) model is written as
\[ V := \frac{i}{2N} (\phi^\dagger d\phi - d\phi^\dagger \phi) = \frac{i}{2N} \left( \frac{u^\dagger du - du^\dagger u}{(1 + \frac{g_0^2}{4N} u^\dagger u)^2} - d\varphi. \right. \] (9.50)

We identify the complex coordinate \( w \) in the Kähler manifold with the Schwinger variable as
\[ w_\alpha = \sqrt{\frac{g_0^2}{N}} \frac{u_\alpha}{1 - \frac{g_0^2}{4N} u^\dagger u}. \quad (\alpha = 1, \cdots, N-1) \] (9.51)

This leads to
\[ \bar{w}_\alpha dw_\alpha - d\bar{w}_\alpha w_\alpha = \frac{1}{(1 + \frac{g_0^2}{4N} u^\dagger u)^2} \frac{g_0^2}{N} (\bar{u}_\alpha du_\alpha - d\bar{u}_\alpha u_\alpha), \] (9.52)
and
\[ 1 + \bar{w}_\alpha w_\alpha = \frac{(1 + \frac{g_0^2}{4N} u^\dagger u)^2}{(1 - \frac{g_0^2}{4N} u^\dagger u)^2}. \] (9.53)

Then we find the following expression for \( \omega \) in terms of \( u \):
\[ \omega := \frac{i}{2} \frac{\bar{w}_\alpha dw_\alpha - d\bar{w}_\alpha w_\alpha}{1 + \bar{w}_\alpha w_\alpha} = \frac{i}{2N} \frac{u^\dagger du - du^\dagger u}{(1 + \frac{g_0^2}{4N} u^\dagger u)^2}. \] (9.54)
Thus the connection one-form is given by

\[ V = \frac{i}{2} \bar{w}_\alpha dw_\alpha - \frac{d\bar{w}_\alpha w_\alpha}{1 + \bar{w}_\alpha w_\alpha} - d\varphi = \omega - d\varphi, \tag{9.55} \]

and the Abelian curvature two-form is equal to the Kähler two-form:

\[ dV = d\omega = \Omega_K = ig_{\alpha\bar{\beta}} dw_\alpha \wedge d\bar{w}_\beta, \tag{9.56} \]

\[ g_{\alpha\bar{\beta}} = \frac{\Delta \delta_{\alpha\bar{\beta}} - \bar{w}_\alpha w_\beta}{\Delta}, \quad \Delta := 1 + ||w||^2. \tag{9.57} \]

By way of the variable \( u \), we have found that the connection one-form \( \omega \) appearing in the NAST is equal to the gauge-invariant part of \( V \). Therefore the diagonal Wilson loop for \( CP^{N-1} \) model is equal to

\[ \langle \exp \left( i \oint_C \omega \right) \rangle_{CP^{N-1}} = \langle \exp \left( i \oint_C V \right) \rangle_{CP^{N-1}}. \tag{9.58} \]

For the \( CP^1 \) model, this reduces to \( V \) (6.51) in Ref. 13) for \( G = SU(2) \).

The expectation value \( \langle \exp (i \oint_C V) \rangle_{CP^{N-1}} \) is calculated in Appendix D in the large \( N \) expansion in a manner based on pioneering works. The result agrees with the result of Campostrini and Rossi. To leading order in the \( 1/N \) expansion, the Wilson loop obeys the area law for all non-self-intersecting loops:

\[ \langle \exp \left( i \oint_C V \right) \rangle_{CP^{N-1}} = \exp \left[ -\frac{6\pi}{N} m_{\phi}^2 \text{Area}(C) \right], \tag{9.59} \]

where

\[ m_{\phi}^2 = \mu^2 \exp \left[ -\frac{2\pi^2}{g^2(\mu)} \right]. \tag{9.60} \]

Thus the string tension is obtained as

\[ \sigma_0 = \frac{6\pi}{N} m_{\phi}^2. \tag{9.61} \]

Here \( m_{\phi} \) is the mass of the \( CP^{N-1} \) field \( \phi \). The \( m_{\phi}^2 \) is equal to the vacuum expectation value \( \langle \sigma(x) \rangle \) of the Lagrange multiplier field \( \sigma \) from the correspondence \( \sigma(x)\bar{\phi}(x)\phi(x) \rightarrow m_{\phi}^2 \bar{\phi}(x)\phi(x) \). In the propagator of the vector field \( V \), a massless pole appears. Hence, the auxiliary vector field \( V \) becomes a dynamical gauge field, giving rise to a linear confining potential between \( \phi \) and \( \bar{\phi} \). On the other hand, the Lagrange multiplier field \( \sigma \) for the constraint (9.46) becomes massive, so that it does not contribute to the confining potential between \( \phi \) and \( \bar{\phi} \). Thus we have completed a proof of quark confinement in four-dimensional \( SU(N) \) Yang-Mills theory based on the Wilson criterion to leading order in the large \( N \) expansion within our reformulation of the Yang-Mills theory.

§10. Remarks

Some remarks are in order to avoid confusion.
10.1. Calculating gauge invariant quantity in the gauge non-invariant theory

The Wilson loop operator $W_C[A]$ is invariant under the gauge transformation denoted by $U(x)$, which is defined by the decomposition of the variable in (7.1) or (7.10), since it is a gauge-invariant quantity. Therefore, the expression of the Wilson loop operator given by the non-Abelian Stokes theorem derived in this paper is also invariant under arbitrary (i.e., infinitesimal and finite) gauge transformation. Therefore, the variable $U(x)$ does not contribute to the Wilson loop operator, and we can write

$$W_C[A] = W_C[V], \quad (10.1)$$

which is a mathematical identity. This is indeed the situation before taking the expectation value $\langle W_C[A] \rangle_{YM}$ in terms of the Yang-Mills theory. However, this seems to contradict the claim of this paper; the area law can be derived from the contribution of the topologically non-trivial degrees of freedom expressed by the variable $U(x)$, which is described by the TQFT. In fact, we have used the formula (7.19) to calculate the expectation value of the Wilson loop operator in § 8. Therefore, if we set

$$g(V, U) = g(V) = W_C[V], \quad h(U) = 1, \quad (10.2)$$

in the expectation value $\langle \cdot \rangle_{YM}$, we would obtain the inconsistent result

$$\langle W_C[A] \rangle_{YM} = \langle W_C[V] \rangle_{pYM}(1)_{TQFT} = \langle W_C[V] \rangle_{pYM}? \quad (10.3)$$

since this implies that there is no contribution from the topological part expressed by $U$ and that the Wilson loop can be calculated by the perturbative part only.\(^\ast\)

This apparent contradiction can be understood as follows. Elimination of the $U$-dependence is possible only when the gauge theory itself is formulated in a gauge invariant way, just as the lattice gauge theory can be written in a manifestly gauge invariant manner without gauge fixing. In such a case, we can perform the gauge transformation so that a gauge invariant quantity, e.g., the Wilson loop, has no dependence on $U(x)$ also in the expectation value. However, our strategy is totally different from this case. That is to say, we calculate the gauge-invariant quantity using the gauge fixed (i.e., gauge non-invariant) theory, where the off-diagonal components and the diagonal components are subject to different gauge fixing conditions. Therefore, the total action $S_{YM}^{\text{tot}}[A, \cdots] = S_{YM}[A] + S_{GF+FP}[A, \cdots]$ obtained by adding the GF+FP term $S_{GF+FP}[A, \cdots]$ to the Yang-Mills action $S_{YM}[A]$ does not have the same form as the gauge-transformed total action $S_{YM}^{\text{tot}}[V, B, C, \bar{C}]$ obtained by the gauge rotation $U$, although $S_{YM}[A] = S_{YM}[V]$. In this sense,

$$\langle W_C[A] \rangle_{YM} \neq \langle W_C[V] \rangle_{YM}, \quad (10.4)$$

where

$$\langle W_C[A] \rangle_{YM} := Z_{YM}^{-1} \int \mathcal{D}A_\mu \cdots \exp\{iS_{YM}^{\text{tot}}[A, \cdots]\} W_C[A]. \quad (10.5)$$

\(^\ast\) The author would like to thank Giovanni Prosperi and a referee for pointing out this issue.
Thus it should be remarked that the formula (7.19) holds only when the field variable $A$ is separated into $V$ and $U$ and at the same time the respective component is identified with the topologically trivial (perturbative) and non-trivial (non-perturbative) components, respectively, where the $V$-dependent part is calculated using perturbative Yang-Mills (pYM) theory. This is an assumption of our approach. In this sense, Eq. (7.19) holds under this assumption and it is not a mathematical identity. In our approach, an arbitrary gauge transformation is not allowed under the functional of the expectation value, i.e., $\langle \cdot \rangle_{YM}$, once the respective component is identified with the relevant degrees of freedom in Yang-Mills theory. (Thus, if we wish to change the integration variable as in (10.2), the total Yang-Mills theory should be modified simultaneously. This leads to a more complicated theory.) In this paper we have fixed the gauge degrees of freedom for the variable $U(x)$ using the modified MA gauge. In other words, we have chosen a specific field configuration of $\{U(x)\}$ to calculate the physical quantity in such a way that the $U$-dependence is controlled by the MA gauge fixing term. This is an essential point of our strategy. In this sense, the physics enters in when we consider the expectation value.

10.2. The Gribov problem and dimensional reduction

Dimensional reduction of the Parisi-Sourlas type was applied to the random field Ising model$^{73}$ and scalar field theories with random external sources$^{74)-76}$ based on non-perturbative$^{74),75}$ and rigorous methods.$^{73),76}$ In these models it has been recognized$^{77),78}$ that the Parisi-Sourlas correspondence (between random systems in $d$ dimensions and the corresponding pure systems in $d - 2$ dimensions, or supersymmetric theory in $d$ dimensions and the corresponding bosonic theory in $d - 2$ dimensions) is exact only in the case of unique solutions to the classical equation of motion. In fact, it was rigorously shown$^{73}$ that the three-dimensional Ising model in a random magnetic field exhibits long-range order at zero temperature and small disorder. This implies that the lower critical dimension $d_\ell$ for this model is 2 (ruling out $d_\ell = 3$), where the lower-critical dimension is the dimension above which $(d > d_\ell)$ long-range ferromagnetic order can exist. Therefore, this result contradicts the naive prediction$^{72}$ between the random field Ising model in $d$ dimensions and the pure Ising model in $d - 2$ dimensions.

In gauge theories, a similar problem can in principle arise due to the existence of Gribov copies, although much is not known on this issue. Indeed, it is known$^{80}$ that Gribov copies exist for the naive MA gauge. On the other hand, it is not known whether the modified MA gauge with $OSp(D|2)$ symmetry possesses the Gribov copies or not. If Gribov copies exist, first of all, the naive BRST formulation breaks down. Even if a modified BRST formulation can exist, the BRST symmetry can be spontaneously broken,$^{79}$ at least in the topological sector (described by the modified MA gauge action), where the large (or finite) gauge transformation with non-trivial winding number plays a crucial role. In this case, the vacuum in the topological sector is not annihilated by the BRST charge $Q_B$ (i.e., $Q_B|0\rangle_{TQFT} \neq 0$), whereas the perturbative sector (in which there exists small quantum fluctuation around an arbitrary but fixed topological background) is characterized by the unbroken BRST charge, i.e., $Q_B|0\rangle_{pYM} = 0$. Even in this case, dimensional reduction should take
place, since we have used neither the field equations nor the above property of the BRST charge acting on the state. In other words, the dimensional reduction is an origin of the spontaneous breaking of the BRST symmetry (see Refs. 17 and 42). Moreover, the BRST symmetry can also be broken by other mechanisms, e.g., radiative corrections.

In any case, the spontaneous breaking of the BRST symmetry in the topological sector implies that the expectation value of the gauge invariant operator depends on the gauge fixing parameter $\alpha$ for the topological sector. In this case, we cannot conclude the $\alpha$-independence of the physical (gauge-invariant) quantities. Rather, the value of $\alpha$ should be determined as a physical parameter by the theory or experiments. This is reasonable if we believe that the real world is described by a unique quantum theory and that nature is realized at an appropriate gauge fixing, since the classical theory does not really exist in nature, and hence ambiguities arising in the quantization are absent from the beginning. It is a challenge to give a definite answer to these questions. The author would like to thank the referee for pointing out the Gribov problem as an obstacle to the dimensional reduction.

§11. Conclusion and discussion

We have given a new version of the non-Abelian Stokes theorem for $G = SU(N)$ with $N \geq 2$ which reduces to the previous result for $SU(2)$. This version of the non-Abelian Wilson loop is very helpful to see the role played by the magnetic monopole in the calculation of the expectation value of the non-Abelian Wilson loop. Combining this non-Abelian Stokes theorem with the Abelian-projected effective gauge theory for $SU(N)$, we have explained the Abelian dominance for the Wilson loop in $SU(N)$ Yang-Mills gauge theory. For $SU(N)$ with $N \geq 3$, we must distinguish the maximal stability group $\tilde{H}$ and the residual gauge group $H$, which is taken to be the maximal torus group $H = U(1)^{N-1}$.

In order to demonstrate the magnetic monopole dominance and the area law of the Wilson loop, we have used a novel reformulation of the Yang-Mills theory which has been proposed by one of the authors. This reformulation is based on the identification of the Yang-Mills theory with the perturbative deformation of a topological quantum field theory. This framework deals with the gauge action $S_{YM}$ and the gauge-fixing action $S_{GF}$ on an equal footing. The gauge action is characterized from the viewpoint of the geometry of connections. On the other hand, the gauge-fixing part is related to the topological invariant (Euler characteristic) determined by a global topology. Therefore, the gauge-fixing part can have a geometric meaning from a global viewpoint. This point has not been emphasized in textbooks on quantum field theory. (See Ref. 17 for more details.)

Our approach relies heavily on a specific gauge, the MA gauge. In spite of the absence of an elementary scalar field in Yang-Mills theory, the MA gauge allows the existence of the magnetic monopole. The magnetic monopole is considered as a topological soliton composed of gauge degrees of freedom, providing the composite scalar field. At least in this gauge, (Parisi-Sourlas) dimensional reduction occurs due to the supersymmetry hidden in the gauge fixing part in the MA gauge. By
making use of the non-Abelian Stokes theorem within this reformulation of the Yang-Mills theory, the derivation of the area law of the non-Abelian Wilson loop in four-dimensional Yang-Mills theory has been reduced to the two-dimensional problem of calculating the expectation value of the Abelian Wilson loop in the coset $G/H$ non-linear sigma model. This is the main result of this article.

In particular, in order to demonstrate confinement of the fundamental quark in the four-dimensional $SU(N)$ Yang-Mills theory in the MA gauge, we have only to consider the two-dimensional $CP^{N-1}$ model. From the topological point of view, the Abelian Wilson loop is equivalent to the area integral (enclosed by the Wilson loop) of the instanton density in the two-dimensional NLS model. This implies that the calculation of the magnetic monopole contribution to the Wilson loop in four-dimensional Yang-Mills theory was translated into that of the instanton contribution in the two-dimensional NLS model when the Wilson loop is contained in the two-dimensional plane, i.e. when the Wilson loop is planar. In addition, the two-dimensional instanton is considered as a subclass of the four-dimensional Yang-Mills instanton (see Ref. 13)). This suggests that the area law of the Wilson loop can be derived by taking into account the contributions of a restricted class of Yang-Mills instantons. A Monte Carlo simulation on a lattice will be efficient in confirming this dimensionally reduced picture of quark confinement.64)

Moreover, naive instanton calculus (the dilute gas approximation) in the coset NLS model leads to the area law of the Wilson loop of the original four-dimensional Yang-Mills theory. Improvements of the dilute gas approximation are necessary to confirm the area law based on the above picture. However, this is rather difficult, as found more than twenty years ago.55) An advantage of the extension of the above strategy to $SU(N)$ with arbitrary $N$ is that the large $N$ systematic expansion can be used in the calculation of the Wilson loop, and that the area law of the Wilson loop has been demonstrated. These results confirm the area law of the Wilson loop. It is known that the large $N$ Yang-Mills theory is related to the string theory. The correspondence between instanton calculus and the large $N$ expansion and a related issue in the large $N$ expansion in Yang-Mills theory will also be discussed in a subsequent article.43) These considerations should shed more light on the confining string picture.65)

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Appendix A

Normalization of the Coherent State

In this appendix we derive the normalization factor $N$ of coherent state $|\xi, \Lambda\rangle$ which parametrizes $G/\tilde{H}$,

$$N := \langle \Lambda | \exp [\bar{\tau}_\alpha E_\alpha] \exp [\tau_\beta E_{-\beta}] | \Lambda \rangle \tag{A.1}$$

$$= \sum_{K, L=0}^{\infty} \frac{1}{K! L!} \bar{\tau}_{\alpha_1} \cdots \bar{\tau}_{\alpha_K} \tau_{\beta_1} \cdots \tau_{\beta_L} \langle \Lambda | E_{\alpha_1} \cdots E_{\alpha_K} E_{-\beta_1} \cdots E_{-\beta_L} | \Lambda \rangle. \tag{A.2}$$

A.1. $SU(3)$ coherent state

We use the positive root $\alpha^{(i)}$ and Dynkin index $[m, n]$ in $SU(3)$, which are defined by Fig. 3 and (3.13). This definition may be rewritten as

$$\alpha^{(1)} \cdot \vec{A} = \frac{m}{2}, \quad \alpha^{(3)} \cdot \vec{A} = \frac{n}{2}. \tag{A.3}$$

For $[m, n] = [m, 0]$ (resp, $[0, n]$), $\alpha$ and $\beta$ run over 1, 2 (resp, 2, 3). Then the corresponding coherent state parametrizes $CP^2$. From the orthogonality of the states which span the representation space, the terms contributing to $N$ in (A.2) must satisfy the condition

$$\alpha^1 + \ldots + \alpha^K - \beta^1 - \ldots - \beta^L = 0. \tag{A.4}$$

In the $[m, 0]$ case, this condition implies that the number of $\alpha^{(1)}$ and $\alpha^{(2)}$ in $K$ positive roots is equal to that of $\alpha^{(1)}$ and $\alpha^{(2)}$ in $L$ negative roots. Since $[E_1, E_2] = 0$, we have only to estimate the terms

$$N_{k, l} := \langle \Lambda | E_{\alpha_1} \cdots E_{\alpha_k} E_{-\beta_1} \cdots E_{-\beta_l} | \Lambda \rangle. \tag{A.5}$$

We begin with the term

$$N_{0, l} = \langle \Lambda | E_{-\beta_1} \cdots E_{-\beta_l} | \Lambda \rangle = N_{0, l} \langle \Lambda - \alpha^{(2)} | \Lambda - \alpha^{(2)} \rangle, \tag{A.6}$$

where we have used in the last equality the fact that $E_{-\beta_1} \cdots E_{-\beta_l} | \Lambda \rangle$ has a weight $\Lambda - \alpha^{(2)}$ and is proportional to the state $| \Lambda - \alpha^{(2)} \rangle$, which is normalized as

$$\langle \Lambda - \alpha^{(2)} | \Lambda - \alpha^{(2)} \rangle = 1. \tag{A.7}$$

Exchanging the rightmost $E_2$ with $E_{-2}$ and using

$$[E_2, E_{-2}] = \alpha^{(2)} \cdot H, \tag{A.8}$$

$$\alpha^{(2)} \cdot H \langle \Lambda - j \alpha^{(2)} | \Lambda - j \alpha^{(2)} \rangle = (\frac{m}{2} - j) \langle \Lambda - j \alpha^{(2)} | \Lambda - j \alpha^{(2)} \rangle, \quad (0 < j < m) \tag{A.9}$$
we obtain the recursion relation for $N_{0,l}$,

$$
N_{0,l} = \left\{ \frac{m}{2} - (l - 1) \right\} N_{0,l-1} + \langle A| E_2 \cdots E_{-2} E_{-2} \cdots E_{-2} |A \rangle
$$

$$
= \left( \frac{m}{2} - (l - 1) + \frac{m}{2} - (l - 2) + \cdots + \frac{m}{2} \right) N_{0,l-1}
$$

$$
= l(m - l + 1) N_{0,l-1},
$$

(A.10)

where we have used $E_2 |A \rangle = 0$. From this relation and from $N_{0,0} = \langle A|A \rangle = 1$, we obtain

$$
N_{0,l} = \frac{(ll)^2(m)}{2^l}.
$$

(A.11)

Similarly,

$$
N_{k,0} = \frac{(kk)^2(m)}{2^k}.
$$

(A.12)

For the general terms (A.5), we have

$$
N_{k,l} = N_{k,0} \langle A - k\alpha^{(1)}| E_2 \cdots E_{-2} E_{-2} \cdots E_{-2} |A - k\alpha^{(1)} \rangle
$$

$$
= N_{k,0} \frac{k(m - k - l + 1)}{2} \langle A - k\alpha^{(1)}| E_2 \cdots E_{-2} E_{-2} \cdots E_{-2} |A - k\alpha^{(1)} \rangle
$$

$$
= \frac{(kk)^2(m)}{2^k} \frac{(ll)^2(m-k)}{2^l}.
$$

(A.13)

Finally, the normalization factor $N$ is given by

$$
N = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(k+l)!^2} \left( \begin{array}{c} k + l \\ l \end{array} \right) \prod_{\alpha} \tau_\alpha \cdots \prod_{\beta} \bar{\tau}_\beta \prod_{\gamma} \tau_\gamma \cdots \prod_{\delta} \bar{\tau}_\delta
$$

$$
= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \prod_{\alpha} \tau_\alpha \prod_{\beta} \bar{\tau}_\beta \prod_{\gamma} \tau_\gamma \cdots \prod_{\delta} \bar{\tau}_\delta
$$

$$
= \left( 1 + \left( \frac{\tau_1 \bar{\tau}_1}{2} \right)^m \right) = \exp K_{CP^2} (\bar{w}, w)
$$

(A.14)

where $w_1 := \tau_1 / \sqrt{2}$ and $w_2 := \tau_2 / \sqrt{2}$, and $K_{CP^2} (w)$ is the Kähler potential of $CP^2$.

For $F_2$, i.e., $mn \neq 0$, $\alpha$ and $\beta$ in (A.2) run over 1, 2, 3. Here we apply almost the same steps as in the case of $CP^2$. The result is

$$
N := \langle A| \exp \sum_{\alpha=1}^{3} \tau_\alpha E_\alpha \exp \sum_{\beta=1}^{3} \bar{\tau}_\beta E_{-\beta} |A \rangle
$$
\[
E \frac{1}{\sqrt{2}} + \frac{\tau_3 / 4}{\sqrt{2}} = \left(1 + \frac{\tau_1^2}{\sqrt{2}} + \frac{\tau_2^2}{\sqrt{2}} + \frac{\tau_1 \tau_3}{4}\right) \left(1 + \frac{\tau_3^2}{\sqrt{2}} + \frac{\tau_2^2}{\sqrt{2}} - \frac{\tau_1 \tau_3}{4}\right)^n
\]

where \(w_1 := \tau_1 / \sqrt{2}, \ w_2 := \tau_2 / \sqrt{2} + \tau_1 \tau_3 / 4, \ w_3 := \tau_3 / \sqrt{2}\).

**A.2. \(SU(N)\) coherent state**

In \(SU(N)\), the simple root \(\alpha^i\) and the Dynkin index \([m_1, \cdots m_{N-1}]\) are defined by (3.47) and \(\alpha^i \cdot \Lambda := m_i / 2\). In general, the roots \(E_{\pm \alpha}\) belonging to the coset group \(G/H\) are not orthogonal to the highest weight \(\Lambda\).\(^*)\) We restrict our consideration to only the case of \([m, 0, \cdots, 0]\), in which case only \(\alpha^1\) is not orthogonal to \(\Lambda\). Any positive root \(\tilde{\alpha}\) is given by a linear combination of simple roots \(\alpha^i\). The positive roots \(\tilde{\alpha}_i\) in (A.2) are not orthogonal to \(\Lambda\), so they must include \(\alpha^1\). It turns out that such roots are given by \(\tilde{\alpha}_i := \alpha^1 + \cdots + \alpha^i (i = 1, \cdots N - 1)\), which are realized by \((E_{\tilde{\alpha}_i})_{kl} = \delta_{i,k} \delta_{i+1,l} / \sqrt{2}\) in the \(N\) representation. The corresponding coherent state parametrizes \(CP^{N-1}\). We can easily extend the \(CP^2\) case to that of \(CP^{N-1}\) using the fact that \([E_{\tilde{\alpha}_i}, E_{\tilde{\alpha}_j}] = 0\) and \(\tilde{\alpha}_i \cdot \tilde{\Lambda} = m/2\). (This situation is same as in the \(SU(3)\) case, i.e., \([E_1, E_2] = 0\), and \(\alpha^{(1)} \cdot \tilde{\Lambda} = \alpha^{(2)} \cdot \tilde{\Lambda} = m/2\).) Thus we obtain

\[
N := \langle \Lambda | \exp \sum_{i=1}^{N-1} [\tilde{\tau}_i E_{\tilde{\alpha}_i}] \exp \sum_{j=1}^{N-1} [\tilde{\tau}_j E_{-\tilde{\alpha}_j}] | \Lambda \rangle
\]

\[= \left(1 + \sum_{i=1}^{N-1} (\tilde{\tau}_i \tilde{\tau}_i / 2)\right)^m =: \left(1 + \sum_{i=1}^{N-1} (\bar{w}_i w_i)\right)^m
\]

\[= \exp K_{CP^{N-1}}(\bar{w}, w).
\]

**Appendix B**

**From \(CP^1\) to \(CP^2\)**

For an arbitrary element \((\phi_1, \phi_2, \phi_3)\) of \(W\),

\[W = C^3 - (0, 0, 0),\]

the entire set of ratios \(\phi_1 : \phi_2 : \phi_3\) is called the complex projective plane and is denoted by \(P^2(C)\) or \(CP^2\):

\[(\alpha \phi_1 : \alpha \phi_2 : \alpha \phi_3) = (\phi_1 : \phi_2 : \phi_3), \quad \alpha \in C, \quad \alpha \neq 0.\]

Defining the subset \(U_a(a = 1, 2, 3)\) of \(CP^2\) as

\[U_a = \{ (\phi_1 : \phi_2 : \phi_3) \in CP^2; \phi_a \neq 0 \} \subset CP^2,\]

we observe that

\[(\phi_1 : \phi_2 : \phi_3) = \left(1 : \frac{\phi_2}{\phi_1} : \frac{\phi_3}{\phi_1}\right) \in U_1.\]

\(^*)\) From the definition of the coherent state (2.9), we have \(E_{\pm \alpha} |\Lambda\rangle \neq 0\) and \([E_{\alpha}, E_{-\alpha}] |\Lambda\rangle = \alpha \cdot \Lambda |\Lambda\rangle \neq 0.\)
The mapping \( \varphi_1 \) from \( U_1 \) to \( C^2 \),
\[
\varphi_1 : (\phi_1 : \phi_2 : \phi_3) \in U_1 \rightarrow \left( \frac{\phi_2}{\phi_1} : \frac{\phi_3}{\phi_1} \right) \in C^2,
\]
is a bijection, i.e., a surjection (onto-mapping) and injection (one-to-one mapping). The inverse mapping is given by
\[
\varphi_1^{-1} : (x, y) \in C^2 \rightarrow (1 : x : y) \in U_1.
\]
Similarly, the following maps \( \varphi_2 \) and \( \varphi_3 \) are also bijections from \( U_2 \) and \( U_3 \) to \( C^2 \):
\[
\varphi_2 : (\phi_1 : \phi_2 : \phi_3) \in U_2 \rightarrow \left( \frac{\phi_1}{\phi_2} : \frac{\phi_3}{\phi_2} \right) \in C^2,
\]
\[
\varphi_3 : (\phi_1 : \phi_2 : \phi_3) \in U_3 \rightarrow \left( \frac{\phi_1}{\phi_3} : \frac{\phi_2}{\phi_3} \right) \in C^2.
\]
Since
\[
CP^2 = U_1 \cup \{(0 : \phi_2 : \phi_3)\}, \quad (\phi_2 : \phi_3) \neq 0,
\]
(\( \phi_2 : \phi_3 \)) determines a point in \( CP^2 \). Conversely, for a point \( (b_1 : b_2) \) in \( CP^1 \),
\( (0 : b_1 : b_2) \) defines a point in \( CP^2 - U_1 \). Thus the map
\[
(0 : \phi_2 : \phi_3) \in CP^2 - U_1 \rightarrow (\phi_2 : \phi_3) \in CP^1
\]
is one-to-one and onto. Then we can identify \( CP^2 - U_1 \) with \( CP^1 \) as \( CP^2 - U_1 \cong CP^1 \).

On the other hand, the identification of \( U_1 \) and \( C^2 \), \( U_1 \cong C^2 \), by the mapping \( \varphi_1 \) leads to
\[
CP^2 = U_1 \cup CP^1 \cong C^2 \cup CP^1.
\]
Using \( CP^1 = C^1 \cup \{(0 : 1)\} \), we can write \( CP^2 = C^2 \cup C^1 \cup \{(0 : 1)\} \equiv C^2 \cup C^1 \cup C^0 \).
\( CP^2 - U_1 \) is called the “line at infinity” and is denoted by \( \ell_\infty \), \( CP^2 - U_1 = \ell_\infty \).

Thus we can also write
\[
CP^2 = U_1 \cup \ell_\infty, \quad U_1 \cong C^2, \quad \ell_\infty \cong CP^1,
\]
where \( C^2 \) is called the “complex affine plane”. The homogeneous coordinates \( (\phi_1 : \phi_2 : \phi_3) \) are related to an element \( (x, y) \) in the affine plane as
\[
x = \frac{\phi_2}{\phi_1}, \quad y = \frac{\phi_3}{\phi_1}, \quad \ell_\infty \cong \{\phi_1 = 0\}.
\]

\( CP^2 \) is obtained as \( CP^2 = U \cup V \cup W \) by gluing the three affine planes \( U, V \) and \( W \) using the relations
\[
u_1 = \frac{1}{\nu_1}, \quad u_2 = \frac{v_2}{v_1}, \quad v_1 = \frac{w_1}{w_2}, \quad v_2 = \frac{1}{w_2}, \quad w_1 = \frac{1}{u_2}, \quad w_2 = \frac{u_1}{u_2},
\]
where \( (u_1, u_2), (v_1, v_2) \) and \( (w_1, w_2) \) are the coordinates of \( U, V \) and \( W \) defined in terms of homogeneous coordinates of \( CP^2 \) as
\[
(u_1, u_2) = \left( \frac{\phi_2}{\phi_1}, \frac{\phi_3}{\phi_1} \right), \quad (v_1, v_2) = \left( \frac{\phi_1}{\phi_2}, \frac{\phi_3}{\phi_2} \right), \quad (w_1, w_2) = \left( \frac{\phi_1}{\phi_3}, \frac{\phi_2}{\phi_3} \right).
\]
Note that the method of gluing affine planes is not unique. In fact, \( CP^2 \) can be obtained from the following gluing:

\[
\begin{align*}
  u_1 &= \frac{w_2}{w_1}, & u_2 &= \frac{1}{w_1}, & v_1 &= \frac{1}{u_1}, & v_2 &= \frac{u_2}{u_1}, & w_1 &= v_1/v_2, & w_2 &= \frac{1}{v_2}. \\
\end{align*}
\]

(B.16)

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### Appendix C

**Nonlinear Sigma Model of the Flag Space**

#### C.1. Correspondence between \( SU(N)/T \) and \( SL(N,C)/B \)

When an element \( \xi \) in \( F_{N-1} \) is expressed by the complex coordinate,

\[
\xi = \begin{pmatrix}
1 & w_1 & w_2 & \cdots & w_n \\
0 & 1 & w_{n+1} & \cdots & w_{2n-1} \\
0 & 0 & 1 & \cdots & w_{3n-3} \\
0 & 0 & \cdots & 0 & w_{n(n+1)/2} \\
0 & 0 & \cdots & \cdots & 1
\end{pmatrix}^T \in F_n, \tag{C.1}
\]

we find \( \det \xi = 1 \). Hence \( \xi \) is an element of \( SL(N,C) \). There is the isomorphism \( SU(N)/T \cong SL(N,C)/B \). However, \( \xi \) in this form is not necessarily unitary. The corresponding unitary matrix \( V \in SU(N) \) is obtained as follows. First, \( \xi \) as an element of \( SL(N,C) \) is expressed in terms of the column vectors:

\[
\xi = (E_1, E_2, \cdots, E_N) \in SL(N,C) = SU(N)^C. \tag{C.2}
\]

By applying the Gramm-Schmidt orthogonalization, we can obtain a set of mutually orthogonal vectors \( (E'_1, E'_2, \cdots, E'_N) \) from \( (E_1, E_2, \cdots, E_N) \) as

\[
E'_1 := E_1, \\
E'_2 := E_2 - \frac{(E_2, E'_1)}{(E'_1, E'_1)} E'_1, \\
E'_N := E_N - \frac{(E_N, E'_{N-1})}{(E'_{N-1}, E'_{N-1})} E'_{N-1} - \cdots - \frac{(E_N, E'_1)}{(E'_1, E'_1)} E'_1, \tag{C.3}
\]

where the inner product is defined by \( (E_i, E_j) := E_i^T \cdot E_j \). Using the normalized vectors \( e_j := E'_j/||E'_j|| \), we obtain an element in \( SU(N) \),

\[
V = (e_1, e_2, \cdots, e_N) \in SU(N). \tag{C.4}
\]

In fact, \( \det V = 1 \) and \( V \) is unitary, since

\[
V^\dagger V = \begin{pmatrix}
\bar{e}_1^T \\
\vdots \\
\bar{e}_N^T
\end{pmatrix} \cdot (e_1, e_2, \cdots, e_N), \quad (V^\dagger V)_{ij} = \bar{e}_i^T \cdot e_j = (e_i, e_j) = \delta_{ij}. \tag{C.5}
\]
In general, the unitary matrix $V$ is related to its complexification $V^C$ by

$$V = V^C B, \quad V \in SU(N), \quad V^C \in SL(N, C),$$

(C.6)

where $B$ is an upper triangular matrix. This is nothing but the Iwasawa decomposition.\(^\ast\) Since the upper triangular matrices form a group, we have

$$V^C = V B^{-1} = V B',$$

(C.8)

where $B' = B^{-1}$ is also upper triangular. This implies $\xi = V B$ by the above construction. Therefore, $V$ is indeed the element of $SU(N)$ corresponding to $\xi$. Note that the multiplication by the matrix $B$ leaves the highest-weight state $|\Lambda\rangle$ invariant, so that

$$\xi |\Lambda\rangle = VB |\Lambda\rangle = V |\Lambda\rangle.$$

(C.9)

For example,

$$e_1 = E_1' / ||E_1'|| = E_1 / ||E_1||,$$

i.e.,

$$e_1 = \frac{1}{\Delta^{1/2}} \begin{pmatrix} 1 \\ -w_1 \\ \vdots \\ -w_N \end{pmatrix}, \quad \Delta := 1 + ||w||^2 = 1 + \sum_{a=1}^N |w_a|^2.$$

(C.10)

The Mauer-Cartan form is

$$V^{-1} dV = V^\dagger dV = \begin{pmatrix} \bar{e}_1^T \\ \vdots \\ \bar{e}_N^T \end{pmatrix} (de_1, \ldots, de_N) = \begin{pmatrix} \bar{e}_1^T de_1 & \bar{e}_1^T de_2 & \cdots & \bar{e}_1^T de_N \\ \bar{e}_2^T de_1 & \bar{e}_2^T de_2 & \cdots & \bar{e}_2^T de_N \\ \vdots & \vdots & \ddots & \vdots \\ \bar{e}_N^T de_1 & \bar{e}_N^T de_2 & \cdots & \bar{e}_N^T de_N \end{pmatrix},$$

(C.11)

which can be decomposed into diagonal and off-diagonal parts as

$$V^{-1} dV = \sum_{i=1}^N \bar{e}_i^T de_i I_{ii} + \sum_{a \neq b} \bar{e}_a^T de_b E_{ab}.$$  

(C.12)

In the fundamental representation, we have

$$\omega = \langle \Lambda | i V^{-1} dV |\Lambda\rangle = i(V^{-1} dV)_{11} = i\bar{e}_1^T de_1,$$

(C.13)

$$\eta^A = \langle \Lambda | V^{-1} T^A V |\Lambda\rangle = \bar{e}_1^T (T^A) e_1.$$  

(C.14)

Hence, substituting (C.10) into (C.13), we obtain

$$\omega = i \frac{\bar{w}_a dw_a - d\bar{w}_a w_a}{\Delta}.$$  

(C.15)

\(^\ast\) Any element $g_c \in G^C$ may be factorized as

$$g_c = gb, \quad g \in G, \quad b \in B$$

(C.7)

in a unique fashion, up to torus elements that are common to $G$ and $B$.\(^\ast\)
The Lagrangian density of the coset $G/H$, i.e. the flag NLS model,
\[
L_{\text{NLSM}} = \frac{\beta g}{2} \text{tr}_{G/H}(iV^{-1}\partial_{\mu}V iV^{-1}\partial_{\mu}V),
\]  
(C.16)
can be written in terms of the off-diagonal elements as
\[
L_{\text{NLSM}} = \beta g \sum_{a,b:a<b} (\Omega_{\mu})_{ab}(\Omega_{\mu})_{ab} = \beta g \sum_{a,b:a<b} (e_a, \partial_{\mu}e_b)(e_a, \partial_{\mu}e_b),
\]  
(C.17)
where we have used $(e_i, e_j) := \bar{e}_i^T e_j = \delta_{ij}$.

Especially, the Lagrangian of the $CP^{N-1}$ model is obtained as a special case of (C.17) as follows. Using the definition (4.37),
\[
n^A = (UT^A U^\dagger)_{11} = U_{1a} (T^A)_{ab} U_{1b} = U_{1a} (T^A)_{ab} U^\dagger_{b1},
\]  
(C.18)
and $UU^\dagger = 1$, we find
\[
\partial_{\mu} n^A \partial_{\mu} n^A = \partial_{\mu} U_{1a} \partial_{\mu} U^\dagger_{a1} U_{1b} U^\dagger_{b1} + U_{1a} \partial_{\mu} U^\dagger_{a1} U_{1b} \partial_{\mu} U^\dagger_{b1}
\]  
\[
= \sum_{b=2}^N (iU \partial_{\mu} U^\dagger)_{1b} (iU \partial_{\mu} U^\dagger)_{1b} = \sum_{b=2}^N (e_1, \partial_{\mu} e_b)^2.
\]  
(C.19)
If we use (4.36)$(\phi_a = \bar{U}_{1a} = U^\dagger_{a1})$, we obtain another expression,
\[
\partial_{\mu} n^A \partial_{\mu} n^A = \partial_{\mu} \bar{\phi} \cdot \partial_{\mu} \phi + (\bar{\phi} \cdot \partial_{\mu} \bar{\phi})(\bar{\phi} \cdot \partial_{\mu} \phi),
\]  
(C.20)
where we have used $\phi^\dagger \cdot \phi = 1$. Thus the Lagrangian of $CP^{N-1}$ model is obtained as
\[
L_{CP^{N-1}} = \frac{\beta g}{2} \partial_{\mu} n \cdot \partial_{\mu} n \quad (\mu = 1, \ldots, d)
\]  
(C.21)
\[
= \frac{\beta g}{2} \sum_{b=2}^N (e_1, \partial_{\mu} e_b)^2
\]  
(C.22)
\[
= \frac{\beta g}{2} g_{\alpha\beta}(\phi) \partial_{x^\alpha} \partial_{x^\beta},
\]  
(C.23)
where
\[
g_{\alpha\beta}(\phi) := \delta_{\alpha\beta} - \bar{\phi}_\alpha \phi_\beta.
\]  
(C.24)
This agrees with the Lagrangian obtained from the Kähler potential,
\[
L_{CP^{N-1}} = \frac{\beta g}{2} g_{\alpha\beta}(w) \partial_{\mu} w^\alpha \partial_{\mu} \bar{w}^\beta,
\]  
(C.25)
with
\[
g_{\alpha\beta}(w) = \frac{(1 + ||w||^2) \delta_{\alpha\beta} - \bar{w}_\alpha w_\beta}{(1 + ||w||^2)^2}.
\]  
(C.26)

The explicit construction of $V$ is given in the following section.
C.2. \(SU(2)\)

For \(G = SU(2)\), we have

\[
\xi = \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} = (E_1, E_2), \quad E_1 = \begin{pmatrix} 1 \\ -w \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(C.27)

It is easy to see that

\[
(E_1, E_1) = 1 + w\bar{w}, \quad (E_2, E_1) = -\bar{w}.
\]

(C.28)

Hence we obtain

\[
e_1 = \Delta^{-1/2} \begin{pmatrix} 1 \\ -w \end{pmatrix}, \quad e_2 = \Delta^{-1/2} \begin{pmatrix} \bar{w} \\ 1 \end{pmatrix}, \quad \Delta := 1 + |w|^2,
\]

(C.29)

and

\[
V = (e_1, e_2) = \Delta^{-1/2} \begin{pmatrix} 1 & \bar{w} \\ -w & 1 \end{pmatrix}.
\]

(C.30)

The elements of the one-form \(V^{-1}dV\) are

\[
e_1^T d\bar{e}_1 = \frac{1}{2} \Delta^{-1} (wd\bar{w} - \bar{w}dw),
\]

(C.31)

\[
e_2^T d\bar{e}_1 = -\Delta^{-1} d\bar{w}.
\]

(C.32)

The Lagrangian of the \(F_1 = CP^1\) model reads

\[
\mathcal{L}_{\text{NLSM}} = -\beta_g \Delta^{-2} \partial_\mu w \partial_\mu \bar{w} = -\beta_g \frac{1}{(1 + |w|^2)^2} \partial_\mu w \partial_\mu \bar{w}.
\]

(C.33)

C.3. \(SU(3)\)

For \(G = SU(3)\), we have

\[
\xi = \begin{pmatrix} 1 & 0 & 0 \\ -w_1 & 1 & 0 \\ -w_2 & w_3 & 1 \end{pmatrix},
\]

(C.34)

or,

\[
E_1 = \begin{pmatrix} 1 \\ -w_1 \\ -w_2 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 1 \\ w_3 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

(C.35)

Then we have

\[
(E_1, E_1) = 1 + w_1 \bar{w}_1 + w_2 \bar{w}_2, \quad (E_2, E_1) = -\bar{w}_1 - w_3 \bar{w}_2, \ldots.
\]

(C.36)

A straightforward calculation leads to \(V = (e_1, e_2, e_3) \in SU(3)\), with

\[
e_1 = (\Delta_1)^{-1/2} \begin{pmatrix} 1 \\ -w_1 \\ -w_2 \end{pmatrix},
\]
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\[ e_2 = (\Delta_1 \Delta_2)^{-1/2} \begin{pmatrix} \bar{w}_1 + w_3 \bar{w}_2 \\ 1 + |w_2|^2 - w_1 \bar{w}_2 w_3 \\ -w_2 \bar{w}_1 + w_3 + w_3 |w_1|^2 \end{pmatrix}, \]

\[ e_3 = (\Delta_2)^{-1/2} \begin{pmatrix} \bar{w}_2 - w_1 \bar{w}_3 \\ -w_3 \\ 1 \end{pmatrix}, \]  \hspace{1cm} (C.37)

where

\[ \Delta_1 := 1 + |w_1|^2 + |w_2|^2, \quad \Delta_2 := 1 + |w_2 - w_1 w_3|^2 + |w_3|^2. \]  \hspace{1cm} (C.38)

The off-diagonal elements of the one-form \( V^{-1}dV \) are

\[ e_2^T \bar{d}e_1 = (\Delta_1)^{-1}(\Delta_2)^{-1/2}[(1 + |w_2|^2 - w_1 \bar{w}_2 w_3) d\bar{w}_1 + (-w_2 \bar{w}_1 + w_3 + w_3 |w_1|^2) d\bar{w}_2], \]

\[ e_3^T \bar{d}e_1 = (\Delta_1)^{-1/2}(\Delta_2)^{-1/2}[d\bar{w}_2 - \bar{w}_3 d\bar{w}_1], \]

\[ e_3^T \bar{d}e_2 = (\Delta_1)^{-1/2}(\Delta_2)^{-1}[(w_1 + \bar{w}_3 w_2)(d\bar{w}_2 - \bar{w}_3 d\bar{w}_1) - \Delta_1 d\bar{w}_3]. \]  \hspace{1cm} (C.39)

For \( CP^2 \), we have

\[ e_1 = (\Delta_1)^{-1/2} \begin{pmatrix} 1 \\ -w_1 \\ -w_2 \end{pmatrix}, \]  \hspace{1cm} (C.40)

\[ e_2 = (\Delta_1 \Delta_2)^{-1/2} \begin{pmatrix} \bar{w}_1 \\ 1 + |w_2|^2 \\ -w_2 \bar{w}_1 \end{pmatrix}, \]  \hspace{1cm} (C.41)

\[ e_3 = (\Delta_2)^{-1/2} \begin{pmatrix} \bar{w}_2 \\ 0 \\ 1 \end{pmatrix}, \]  \hspace{1cm} (C.42)

where

\[ \Delta_1 := 1 + |w_1|^2 + |w_2|^2 = 1 + ||w||^2, \quad \Delta_2 := 1 + |w_2|^2. \]  \hspace{1cm} (C.43)

The off-diagonal elements of the one-form \( V^{-1}dV \) are

\[ e_2^T \bar{d}e_1 = (\Delta_1)^{-1}(\Delta_2)^{-1/2}[(1 + |w_2|^2) d\bar{w}_1 - w_2 \bar{w}_1 d\bar{w}_2], \]

\[ e_3^T \bar{d}e_1 = (\Delta_1)^{-1/2}(\Delta_2)^{-1/2}[d\bar{w}_2], \]

\[ e_3^T \bar{d}e_2 = (\Delta_1)^{-1/2}(\Delta_2)^{-1}[w_1 d\bar{w}_2]. \]  \hspace{1cm} (C.44)

The Lagrangian of \( CP^2 \) model is given by

\[ \mathcal{L}_{CP^2} = \frac{\beta_g}{2} \sum_{b=2}^{3} |(e_1, \partial_{\mu} e_b)|^2 \]

\[ = \frac{\beta_g}{2} \frac{1}{(1 + ||w||^2)^2} [(1 + |w_2|^2) \partial_{\mu} w_1 \partial_{\mu} \bar{w}_1 + (1 + |w_1|^2) \partial_{\mu} w_2 \partial_{\mu} \bar{w}_2 - \bar{w}_1 w_2 \partial_{\mu} w_1 \partial_{\mu} \bar{w}_2 - w_1 \bar{w}_2 \partial_{\mu} w_2 \partial_{\mu} \bar{w}_1]. \]  \hspace{1cm} (C.46)
Appendix D

Large $N$ Expansion of the $\mathbb{CP}^{N-1}$ Model

The generating function of the $\mathbb{CP}^{N-1}$ model is defined by

$$Z[J, \bar{J}, J_\mu] := \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \prod_x \delta \left( |\phi(x)|^2 - \frac{N}{g_0^2} \right) \times \exp \left\{ -S + \int d^2x [\bar{J} \cdot \phi + \bar{\phi} \cdot J + J_\mu V_\mu] \right\},$$  \hspace{1cm} (D.1)

where $S$ is the action of the $\mathbb{CP}^{N-1}$ model,

$$S := \int d^2x \left[ \partial_\mu \bar{\phi} \cdot \partial_\mu \phi + \frac{g_0^2}{4N} (\bar{\phi} \cdot \partial_\mu \phi) (\bar{\phi} \cdot \partial_\mu \phi) \right]$$  \hspace{1cm} (D.2)

$$= \int d^2x \left[ \partial_\mu \bar{\phi} \cdot \partial_\mu \phi - \frac{N}{g_0^2} V_\mu V_\mu \right],$$  \hspace{1cm} (D.3)

and the auxiliary vector field $V_\mu$ is defined by

$$V_\mu(x) := \frac{g_0^2}{2N} i(\bar{\phi}(x) \cdot \partial_\mu \phi(x)) = \frac{g_0^2}{2N} i(\bar{\phi}(x) \cdot \partial_\mu \phi(x) - \partial_\mu \bar{\phi}(x) \cdot \phi(x)).$$  \hspace{1cm} (D.4)

Introducing the Lagrange multiplier fields $\sigma(x)$ and $A_\mu(x)$, we can rewrite as

$$\prod_x \delta \left( |\phi(x)|^2 - \frac{N}{g_0^2} \right) \exp \left\{ \int d^2x \left[ \frac{N}{g_0^2} V_\mu(x)V_\mu(x) + J_\mu(x)V_\mu(x) \right] \right\},$$

$$= \int \mathcal{D}\sigma \int \mathcal{D}A_\mu \exp \left\{ \int d^2x \left\{ \frac{i}{\sqrt{N}} \sigma \left( |\phi|^2 - \frac{N}{2g_0^2} \right) - \frac{1}{N} A_\mu A_\mu |\phi|^2 - m^2 |\phi|^2 

+ \frac{1}{\sqrt{N}} A_\mu [i(\bar{\phi} \cdot \partial_\mu \phi) + J_\mu] - \frac{g_0^2}{4N} J_\mu J_\mu \right\} \right\},$$  \hspace{1cm} (D.5)

where we have inserted the mass term $m^2|\phi|^2$ for later convenience and have chosen a specific normalization for the field $\sigma$. Then we obtain

$$Z[J, \bar{J}, J_\mu] := \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \int \mathcal{D}\sigma \int \mathcal{D}A_\mu \exp \left\{ -\int d^2x \left[ \bar{\phi} \cdot \Delta_B \phi + \frac{i}{\sqrt{N}} \frac{\sqrt{N}}{g_0^2} \sigma \right] 

+ \int d^2x \left[ \bar{\phi} \cdot \phi + \bar{\phi} \cdot J + \frac{1}{\sqrt{N}} A_\mu J_\mu - \frac{g_0^2}{4N} J_\mu J_\mu \right] \right\},$$  \hspace{1cm} (D.6)

where

$$\Delta_B := -D_\mu D_\mu + m^2 - \frac{i}{\sqrt{N}} \sigma(x), \quad D_\mu := \partial_\mu + \frac{i}{\sqrt{N}} A_\mu(x).$$  \hspace{1cm} (D.7)
This theory has global $SU(N)$ invariance corresponding to rotations of $\phi_a$. Moreover, it has local $U(1)$ gauge invariance under the transformation
\begin{align*}
\phi'_a(x) &= e^{iA(x)}\phi_a(x), \quad (a = 1, \cdots, N) \\
A'_\mu(x) &= A_\mu(x) - \sqrt{N}\partial_\mu A(x), \\
\sigma'(x) &= \sigma(x).
\end{align*}

We can perform the integration over $\phi$ and $\bar{\phi}$ to obtain
\begin{equation}
Z[J, \bar{J}, J_\mu] := \int D\sigma \int DA_\mu \exp \left\{ -S_{\text{eff}} \right. \\
& \left. + \int d^2x \left[ \bar{J}\Delta_B^{-1}J + \frac{1}{\sqrt{N}}A_\mu J_\mu - \frac{g_0^2}{4N}J_\mu J_\mu \right] \right\},
\end{equation}

where
\begin{equation}
S_{\text{eff}} := N\text{Tr} \ln \Delta_B + \frac{i\sqrt{N}}{g_0^2} \int d^2x \sigma(x).
\end{equation}

The effective action can be expanded in a power series of $1/N$:
\begin{equation}
S_{\text{eff}} = \sum_{n=1}^{\infty} N^{1-n/2}g^{(n)} = \sqrt{N}S^{(1)} + N^0S^{(2)} + N^{-1/2}S^{(3)} + \cdots.
\end{equation}

The diagrammatic representation of this expansion is given in Fig. 13 using the rule given in Fig. 12.

First, the order $N^{1/2}$ term corresponds to the diagrams (a) and (b) in Fig. 13:
\begin{equation}
S^{(1)} = \frac{i}{g_0^2} \int d^2x \sigma(x) - i\text{Tr}[(-\partial^2 + m^2)^{-1}\sigma]
\end{equation}
\begin{equation}
= i\tilde{\sigma}(0) \left[ \frac{1}{g_0^2} - \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + m^2} \right],
\end{equation}

where we have used the Fourier transformation
\begin{equation}
\tilde{\sigma}(p) = \int d^2xe^{-ipx}\sigma(x).
\end{equation}

The integral in (D.13) is ultraviolet divergent. It can be regularized by introducing the cutoff $\Lambda$. The saddle point condition $S^{(1)} = 0$ requires the bare coupling constant $g_0$ to vary with the cutoff $\Lambda$ according to
\begin{equation}
\frac{1}{g_0^2(\Lambda)} = \frac{1}{4\pi} \ln \frac{\Lambda^2}{m^2}.
\end{equation}

In other words, if the bare coupling $g_0$ varies with respect to the cutoff according to
\begin{equation}
\frac{1}{g_0^2(\Lambda)} - \frac{1}{g_0^2(\mu)} = \frac{1}{4\pi} \ln \frac{\Lambda^2}{\mu^2},
\end{equation}

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the divergences cancel in $S^{(1)}$. This implies the asymptotic freedom of the $CP^{N-1}$ model. Imposing the condition $S^{(1)} = 0$, therefore, we obtain

$$m^2 = \mu^2 \exp \left[-\frac{4\pi}{g^2_R(\mu)}\right].$$ \hspace{1cm} (D.17)

Next, the order $N^0$ term corresponds to the diagrams (c),(d) and (e) in Fig. 13:

$$S^{(2)} = \frac{1}{2} \int d^2 x \int d^2 y [\sigma(x) \Gamma(x,y) \sigma(y) + A_\mu(x) \Gamma_{\mu\nu}(x,y) A_\nu(y)],$$ \hspace{1cm} (D.18)

where the Fourier transformation of $\Gamma(x,y)$ and $\Gamma_{\mu\nu}(x,y)$ are respectively given by (see Fig. 13)

$$\tilde{\Gamma}(p) = \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(p^2 + m^2)((p + q)^2 + m^2)}$$

$$= \frac{1}{2\pi} \frac{1}{\sqrt{p^2(4m^2 + \sqrt{p^2})}} \ln \frac{\sqrt{p^2 + 4m^2 + \sqrt{p^2}}}{\sqrt{p^2 + 4m^2 - \sqrt{p^2}}}.$$ \hspace{1cm} (D.19)

(a) \hspace{2cm} (d) \hspace{2cm} (b) \hspace{2cm} (e) \hspace{2cm} (c) \hspace{2cm} (f)

\begin{align*}
\text{Fig. 12.} & \quad \text{Graphical representation of the large $N$ expansion in the $CP^{N-1}$ model. The propagators are as follows: (a) $\phi$ propagator, $\delta_{ab}(p^2 + m^2)^{-1}$; (b) $\sigma$ propagator, $\Gamma(p)^{-1}$; (c) $A_\mu$ propagator,} \\
& \quad \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) [((p^2 + m^2)\tilde{\Gamma}(p) - \frac{1}{\pi})]^{-1}. \quad \text{The vertices are as follows: (d) $\sigma\phi_a\bar{\phi}_b$ vertex ($\frac{1}{\sqrt{N}}\delta_{ab}$); (e) $A_\mu\phi_a\phi_b$ vertex ($\frac{1}{\sqrt{N}}\delta_{ab}(p_\mu + p'_\mu)$); (f) $A_\mu A_\nu\phi_a\bar{\phi}_b$ vertex ($\frac{1}{N}\delta_{ab}\delta_{\mu\nu}$).}
\end{align*}
Fig. 13. Examples of Feynmann diagrams. (a) and (b) are tadpole diagrams of order $N^{1/2}$. (c),(d) and (e) are vacuum polarization diagrams of order $N^0$. (f) and (g) are order $N^{-1/2}$ diagrams.

and

$$\tilde{\Gamma}_{\mu\nu}(p) = 2\delta_{\mu\nu} \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + m^2} - \int \frac{d^2q}{(2\pi)^2} \frac{(p_{\mu} + 2q_{\mu})(p_{\nu} + 2q_{\nu})}{(p^2 + m^2)((p + q)^2 + m^2)}$$

$$= \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \left[\frac{1}{(p^2 + 4m^2)}\tilde{\Gamma}(p) - \frac{1}{\pi}\right]. \quad \text{(D.20)}$$

In the neighbourhood of $p^2 = 0$, we have

$$\tilde{\Gamma}(p) = \frac{1}{24\pi m^4} (6m^2 - p^2) + O(p^4), \quad \tilde{\Gamma}_{\mu\nu}(p) = \left[\frac{p^2}{12\pi m^2} + O(p^4)\right] \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right). \quad \text{(D.21)}$$

Thus we obtain the low-energy effective action,

$$S^{(2)} \cong \int d^2x \frac{1}{24\pi m^4} \sigma(x) \left(\partial^2 + 6m^2\right) \sigma(x) + \int d^2x \frac{1}{48\pi m^2} (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))^2. \quad \text{(D.22)}$$
It is important to note that the kinetic terms for the Lagrangian multiplier fields are generated. The field $\sigma$ becomes massive, while the field $A_\mu (= \sqrt{N} V_\mu)$ is massless.

The line integral in the Wilson loop can be rewritten as

$$\oint_C d\xi \mu V_\mu(\xi) = \int d^2z V(z) J_\mu(z) =: (V, J_\mu),$$

where

$$J_\mu(z) = \oint_C d\xi \mu \delta^2(z - \xi) = \epsilon_{\mu\nu} \partial_\nu \Phi(z), \quad \Phi(z) = \begin{cases} 1 & (z \in S_C) \\ 0 & (z \notin S_C) \end{cases}. \quad (D.24)$$

Here $S_C$ is the area bounded by the loop $C$. Up to leading order, we can perform the Gaussian integration to obtain

$$\langle \exp \left[ i \oint_C d\xi \mu V_\mu(\xi) \right] \rangle_{CP^{N-1}} \simeq Z^{-1}[0, 0, 0] \int D\sigma DA_\mu e^{-S(2)} \exp \left( \frac{i}{\sqrt{N}} (A_\mu, J_\mu) \right) = \text{const} \exp \left[ -\frac{1}{2} \frac{12 \pi m^2}{N} (J_\mu, \Delta^{-1} J_\mu) \right]. \quad (D.25)$$

Thus we obtain the area law,

$$\langle \exp \left[ i \oint_C d\xi \mu V_\mu(\xi) \right] \rangle_{CP^{N-1}} = \text{const} \exp \left[ -\frac{6 \pi m^2}{N} |S_C| \right], \quad (D.26)$$

since $J = *d\Phi$ and $\Delta := d\delta + \delta d$, and hence

$$(J_\mu, \Delta^{-1} J_\mu) := \int d^2x \int d^2y J_\mu(x) \Delta^{-1}(x, y) J_\mu(y) = (\Phi, \Phi) = |S_C|. \quad (D.27)$$

The dynamically generated gauge field $A_\mu$ produces a long-range force with a linear potential that confines the $\phi$s. Both global $SU(N)$ and local $U(1)$ symmetries are unbroken in two dimensions due to the Coleman theorem. In dimensions $D > 2$, it has been shown $^{70}$ that there is a critical point $g_c$ such that for $g < g_c$ $SU(N)$ and $U(1)$ symmetries are broken, $\langle \phi_a \rangle \neq 0$, while $\langle \sigma \rangle = 0$. In this phase $\phi$ is regarded as the Nambu-Goldstone particle, since $m_\phi = 0$. A massless vector pole exists in the propagator $\langle A_\mu(x)A_\nu(0) \rangle$. $^{71}$ For $g > g_c$, on the other hand, the $SU(N)$ and $U(1)$ symmetries are exact, implying $\langle \phi_a \rangle = 0$, and $\langle \sigma \rangle \neq 0$. In this phase, $\phi$ is massive, $m_\phi = \langle \sigma \rangle$. For $D = 2$, $g_c = 0$. For more details on large $N$ results, see Refs. 66–69).

**Appendix E**

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**Large N Estimation**

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If we write $n^A(x)$ in terms of the $CP^{N-1}$ variable $\phi_a(x)$ as

$$n^A(x) = \bar{\phi}_a(x) (T^A)_{ab} \phi_b(x),$$

we obtain

$$n^A(x)n^A(y) = \bar{\phi}_a(x) (T^A)_{ab} \phi_b(x) \bar{\phi}_c(y) (T^A)_{cd} \phi_d(y)$$
\[ n(x) \cdot n(x) = n^A(x)n^A(x) = \frac{1}{2} \left[ 1 - \frac{1}{N} \right] (\bar{\phi}(x) \cdot \phi(x))^2. \]  
(E.4)

The constraint \( \bar{\phi}(x) \cdot \phi(x) = 1 \) leads to \( n(x)n(x) = \frac{1}{2} \left[ 1 - \frac{1}{N} \right] \) . The expectation value reads
\[ 2\langle n^A(x)n^A(y) \rangle = \langle (\bar{\phi}(x) \cdot \phi(y))(\phi(x) \cdot \bar{\phi}(y)) \rangle - \frac{1}{N} \langle |\phi(x)|^2 |\phi(y)|^2 \rangle. \]  
(E.5)

The factorization in the large \( N \) expansion leads to
\[ 2\langle n^A(x)n^A(y) \rangle \cong \langle \bar{\phi}(x) \cdot \phi(y) \rangle \langle \phi(x) \cdot \bar{\phi}(y) \rangle - \frac{1}{N} \langle |\phi(x)|^2 |\phi(y)|^2 \rangle, \]
\[ = |\langle \bar{\phi}(x) \cdot \phi(y) \rangle|^2 - \frac{1}{N} \langle |\phi(x)|^2 |\phi(y)|^2 \rangle. \]  
(E.6)

The large \( N \) expansion shows that the field \( \phi \) becomes massive, so that the two-point function \( \langle \bar{\phi}(x) \cdot \phi(y) \rangle \) exhibits exponential decay.

To leading order in the large \( N \) expansion, the correlation function \( G_C(x,y) \) defined by (9.32) in the \( CP_2^{N-1} \) model reads
\[ G_C(x,y) \cong \bar{G}(x,y) := \left( \delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd} \right) \langle P_{ab}(x)P_{ba}(y) \rangle_{CP_2^{N-1}}. \]  
(E.7)

Note that \( P \) is a projection operator with the properties
\[ P^2(x) = P(x), \quad \text{tr} P(x) = 1. \]  
(E.8)

The composite operator \( P \) is \( U(1) \) gauge invariant. \( SU(N) \) invariance implies that
\[ \langle P_{ab}(x) \rangle = \langle \bar{\phi}_a(x)\phi_b(x) \rangle = \frac{1}{N} \delta_{ab}. \]  
(E.9)

It is easy to show that
\[ \bar{G}(x,y) = \langle \bar{\phi}_a(x)\phi_b(x)\bar{\phi}_b(y)\phi_a(y) \rangle - \frac{1}{N} \]
\[ = \langle D^2(x,y) \rangle + \frac{1}{N} \langle D(x,x)D(y,y) \rangle - \frac{1}{N} \]
\[ = \langle D^2(x,y) \rangle + \frac{1}{N} \langle D(x,x)D(y,y) \rangle_{\text{conn}}, \]  
(E.10)

where \( \langle D(x,x) \rangle = 1 \).
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