CLASSIFYING $\tau$-TILTING MODULES OVER THE AUSLANDER ALGEBRA OF $K[x]/(x^n)$

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Abstract. We build a bijection between the set $s\tau$-tilt$\Lambda$ of isomorphism classes of basic support $\tau$-tilting modules over the Auslander algebra $\Lambda$ of $K[x]/(x^n)$ and the symmetric group $S_{n+1}$, which is an anti-isomorphism of partially ordered sets with respect to the generation order on $s\tau$-tilt$\Lambda$ and the left order on $S_{n+1}$. This restricts to the bijection between the set tilt$\Lambda$ of isomorphism classes of basic tilting $\Lambda$-modules and the symmetric group $S_n$ due to Brüstle, Hille, Ringel and Röhrle. Regarding the preprojective algebra $\Gamma$ of Dynkin type $A_n$ as a factor algebra of $\Lambda$, we show that the tensor functor $- \otimes_{\Lambda} \Gamma$ induces a bijection between $s\tau$-tilt$\Lambda$ and $s\tau$-tilt$\Gamma$. This recovers Mizuno’s bijection $S_{n+1} \rightarrow s\tau$-tilt$\Gamma$ for type $A_n$.

Contents

1. Introduction 1
2. Preliminaries 3
3. Tilting modules over the Auslander algebra of $K[x]/(x^n)$ 7
4. Support $\tau$-tilting modules over the Auslander algebra of $K[x]/(x^n)$ 15
5. Connection with preprojective algebras of type $A_n$ 21
References 23

1. Introduction

Tilting theory has been central in the representation theory of finite dimensional algebras since the early seventies [BGP, AuPR, B, BrB, HaR]. In this theory, tilting modules play a central role. So it is important to classify tilting modules for a given algebra. There are many algebraists working on this topic which makes the theory fruitful. For more details about classical tilting modules we refer to [AsSS, AnHK].

Recently Adachi, Iyama and Reiten [AIR] introduced $\tau$-tilting theory to generalize the classical tilting theory from viewpoint of mutations. This is very close to the silting theory introduced by [AI] and the cluster tilting theory in the sense of [KR [Y] BMRRT]. The central notion of $\tau$-tilting theory is support $\tau$-tilting modules, and therefore it is important to classify support $\tau$-tilting modules for a given algebra. Recently some authors worked on this topic, e.g. Adachi [A1] classified $\tau$-rigid modules for Nakayama algebras, Adachi [A2] and Zhang [Z] studied $\tau$-rigid modules for algebras with radical square zero, and Mizuno [M] classified support $\tau$-tilting modules for preprojective algebras of Dynkin type. In this context, it is basic to consider algebras with only finitely many support $\tau$-tilting modules, called $\tau$-rigid finite algebras and studied by Demonet, Iyama and Jasso [DLJ]. For more details of $\tau$-tilting theory, we refer to [AAC, AIR, AnMV, HuZ, J, LLY, LRT, W] and so on.

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In this paper we focus on classifying tilting modules and support τ-tilting modules over a class of Auslander algebras. Recall that an algebra Λ is called an Auslander algebra if the global dimension of Λ is less than or equal to 2 and the dominant dimension of Λ is greater than or equal to 2. It is showed by Auslander there is a one-to-one correspondence between Auslander algebras and algebras of finite representation type.

In the rest, let Λ be the Auslander algebra of the algebra $K[x]/(x^n)$. Then Λ is presented by the quiver

\[ 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-2}} n-1 \xrightarrow{a_{n-1}} n \]

with relations $a_1b_2 = 0$ and $a_db_{i+1} = b_da_{i-1}$ for any $2 \leq i \leq n-1$. All modules in this paper are right modules. Denote by $\text{tilt}_\Lambda$ the set of isomorphism classes of basic tilting Λ-modules. We show that each tilting Λ-module is isomorphic to a product of maximal ideals $I_1, \ldots, I_{n-1}$ of Λ. Moreover, we show a strong relationship between basic tilting Λ-modules and the symmetric group $S_n$.

For $w, w' \in S_n$ and $1 \leq i \leq n$, we denote the product $w'w \in S_n$ by $(w'(w))(i) := w'(w(i))$. Denote by $s_i \in S_n$ the transposition $(i, i+1)$ for $1 \leq i \leq n-1$. The length of $w \in S_n$ is defined by $l(w) := \#\{(i, j) \mid 1 \leq i < j \leq n, w(i) > w(j)\}$ and an expression $w = s_i s_j \cdots s_k$ of $w \in S_n$ is called a reduced expression if $l = l(w)$. For elements $w, w' \in S_n$, if $l(w') = l(w) + l(w'w^{-1})$ then we write $w \leq w'$. This gives a partial order on $S_n$ called the left order. The Hasse quiver of $S_n$ has vertices $w$ corresponding to each element $w \in S_n$, and has arrows $w \rightarrow s_i w$ if $l(w) > l(s_i w)$ and $w \leftarrow s_i w$ if $l(w) < l(s_i w)$ for $w \in S_n$ and $1 \leq i \leq n-1$. Now we are in a position to state our first main result.

**Theorem 1.1.** (Theorem 5.3, 5.12) Let Λ be the Auslander algebra of $K[x]/(x^n)$, and $\langle I_1, \ldots, I_{n-1} \rangle$ the ideal semigroup of Λ generated by the maximal ideals $I_1, \ldots, I_{n-1}$.

1. The set $\text{tilt}_\Lambda$ is given by $\langle I_1, \ldots, I_{n-1} \rangle$.
2. There exists a well-defined bijection $I : S_n \cong \langle I_1, \ldots, I_{n-1} \rangle$, which maps $w$ to $I(w) = I_i \cdots I_n$ where $w = s_i \cdots s_k$ is an arbitrary reduced expression.
3. Consequently there exists a bijection $I : S_n \cong \text{tilt}_\Lambda$. In particular $\#\text{tilt}_\Lambda = n!$.
4. The map $I$ in (3) is an anti-isomorphism of posets.

Theorem 1.1(3) has been shown in [BR] by using a combinatorial method. Our method in this paper is rather homological, and we shall modify the method in [BR] for preprojective algebras to the Auslander algebra of $K[x]/(x^n)$ by using basic properties of Auslander algebras in Section 2.

Denote by $sr$-$\text{tilt}_\Lambda$ the set of isomorphism classes of basic support τ-tilting Λ-modules, and by $\mu_i(T)$ the mutation of $T$ with respect to the $i$-th indecomposable direct summand of $T$. The set $sr$-$\text{tilt}_\Lambda$ forms a poset (=partially ordered set) with respect to the generation order. We show the following main result of this paper in Section 4, where the map $I : S_{n+1} \cong sr$-$\text{tilt}_\Lambda$ is an extension of the map $I$ in Theorem 1.1.

**Theorem 1.2.** (Theorems 4.8, 4.10, 4.12) Let Λ be the Auslander algebra of $K[x]/(x^n)$.

1. We have $sr$-$\text{tilt}_\Lambda$ is a disjoint union of $\mu_{i+1} \mu_{i+2} \cdots \mu_n(\text{tilt}_\Lambda)$ for $0 \leq i \leq n$.
2. There exists a bijection $I : S_{n+1} \cong sr$-$\text{tilt}_\Lambda$ which maps $w$ to $I(w) = \mu_1 \mu_2 \cdots \mu_n(\Lambda)$, where $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ is an arbitrary reduced expression. In particular, we have $#sr$-$\text{tilt}_\Lambda = (n+1)!$.
3. The map $I$ in (2) is an anti-isomorphism of posets.

Now let $\Gamma$ be the preprojective algebra of Dynkin type $A_n$. Then there exists a natural surjection $\Lambda \rightarrow \Gamma$, and we get a tensor functor $- \otimes_{\Lambda} \Gamma : \text{mod}_\Lambda \rightarrow \text{mod}_\Gamma$. By using this we get a bijection between $sr$-$\text{tilt}_\Lambda$ and $sr$-$\text{tilt}_\Gamma$. More precisely, we have:

**Theorem 1.3.** (Theorem 5.3) Let $\Lambda$ and $\Gamma$ be as above. Then

1. The map $- \otimes_{\Lambda} \Gamma : sr$-$\text{tilt}_\Lambda \rightarrow sr$-$\text{tilt}_\Gamma$ via $U \rightarrow U \otimes_{\Lambda} \Gamma$ is bijective.
2. The map in (1) is an isomorphism of posets.
As a corollary of Theorems 1.2 and 1.3 we recover Mizuno’s anti-isomorphism \( \mathfrak{S}_{n+1} \rightarrow \text{st-tilt} \Gamma \) since it is the composition of \( - \otimes \Gamma \) in Theorem 1.3 and \( I \) in Theorem 1.2.

**Corollary 1.4.** (Corollary 5.5) Let \( \Lambda \) and \( \Gamma \) be as above. There are isomorphisms between the following posets:

1. The poset \( \text{st-tilt} \Lambda \) with the generation order.
2. The poset \( \text{st-tilt} \Gamma \) with the generation order.
3. The symmetric group \( \mathfrak{S}_{n+1} \) with the left order.
4. The poset \( \text{st-tilt} (\Lambda^{\text{op}}) \) with the opposite of the generation order.
5. The poset \( \text{st-tilt} (\Gamma^{\text{op}}) \) with the opposite of the generation order.
6. The symmetric group \( \mathfrak{S}_{n+1} \) with the opposite of the right order.

The paper is organized as follows: In Section 2, we recall some preliminaries on Auslander algebras, tilting modules and support \( \tau \)-tilting modules for later use. In Section 3, we focus on the tilting modules over the Auslander algebra \( \Lambda \) of \( K[x]/(x^n) \) and we prove Theorem 1.1. In Section 4, we use Theorem 1.1 and some other facts of tilting modules to prove Theorem 1.2. Finally, in Section 5, we apply Theorem 1.2 and Theorem 1.3 to preprojective algebras of Dynkin type \( A_n \) and get Mizuno’s bijection for preprojective algebras of Dynkin type \( A_n \).

Throughout this paper, we denote by \( K \) an arbitrary field, and we consider basic finite dimensional \( K \)-algebras. By a module we mean a right module. For an algebra \( \Lambda \), denote by \( \text{mod} \Lambda \) the category of finitely generated right \( \Lambda \)-modules. For an \( \Lambda \)-module \( M \), we denote by \( \text{add} M \) the full subcategory of \( \text{mod} \Lambda \) whose objects are direct summands of \( M^n \) for some \( n > 0 \). The composition of homomorphisms \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) is denoted by \( gf : X \rightarrow Z \).

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## 2. Preliminaries

In this section we recall some basic properties of Auslander algebras, tilting modules and support \( \tau \)-tilting modules. We begin with the definition of Auslander algebras.

For an algebra \( \Lambda \) and a \( \Lambda \)-module \( M \), denote by \( \text{gl.dim} \Lambda \) the global dimension of \( \Lambda \) and denote by \( \text{proj.dim} M \) (resp. \( \text{inj.dim} M \)) the projective dimension (resp. injective dimension) of \( M \). We recall the following definition.

**Definition 2.1.** An algebra \( \Lambda \) is called an Auslander algebra if \( \text{gl.dim} \Lambda \leq 2 \) and \( E_i(\Lambda) \) is projective for \( i = 0, 1 \), where \( E_i(\Lambda) \) is the \( (i+1) \)-th term in a minimal injective resolution of \( \Lambda \).

Recall that an algebra \( R \) is called representation-finite if \( \text{mod} R \) admits an additive generator \( M \), that is, \( \text{mod} R = \text{add} M \). The following classical result in \cite{AuRS} shows the relationship between representation-finite algebras and Auslander algebras.

**Theorem 2.2.** (1) For an additive generator \( M \) of the category \( \text{mod} R \) over a representation-finite algebra \( R \), the algebra \( \text{End}_R(M) \) is an Auslander algebra.

(2) For an Auslander algebra \( \Lambda \) and an additive generator \( Q \) of the category of projective-injective \( \Lambda \)-module, the algebra \( \text{End}_\Lambda(Q) \) is representation-finite.

(3) The correspondences in (1) and (2) induce mutually inverse bijections between Morita equivalence classes of representation-finite algebras and Morita equivalence classes of Auslander algebras.
We call $\Lambda = \text{End}_R(M)$ in Theorem 2.2 an Auslander algebra of $R$. In this case, for $X \in \text{mod } R$ we denote
\[ P_X = \text{Hom}_R(M, X), \quad P^X = \text{Hom}_R(X, M), \quad S_X = P_X / \text{rad } P_X \quad \text{and} \quad S^X = P^X / \text{rad } P^X. \]
Here $P_-$ is an equivalence between $\text{add } M$ and $\text{add } \Lambda$ and $P^-$ is a duality between $\text{add } M$ and $\text{add } \Lambda^op$. The following statement [AuRS] shows the relationship between almost split sequences of $R$ and projective resolutions of simple $\Lambda$-modules.

**Proposition 2.3.** Let $X$ be an indecomposable $R$-module. Then we have
\begin{enumerate}[1]
  \item $\text{proj.dim } S_X \leq 1$ if and only if $X$ is projective. Then $0 \to P_{\text{rad } X} \to P_X \to S_X \to 0$ is a minimal projective resolution of $S_X$.
  \item $\text{proj.dim } S_X = 2$ if and only if $X$ is nonprojective. Then the almost split sequence $0 \to \tau X \to E \to X \to 0$ gives a minimal projective resolution $0 \to P_{\tau X} \to P_E \to P_X \to S_X \to 0$ of $S_X$.
  \item $\text{proj.dim } S^X \leq 1$ if and only if $X$ is injective. Then $0 \to P^{X/\text{soc } X} \to P^{X} \to S^X \to 0$ is a minimal projective resolution of $S^X$.
  \item $\text{proj.dim } S^X = 2$ if and only if $X$ is noninjective. Then the almost split sequence $0 \to X \to E \to \tau^{-1} X \to 0$ gives a minimal projective resolution $0 \to P^{\tau^{-1} X} \to P^E \to P^{X} \to S^X \to 0$ of $S^X$.
\end{enumerate}

Denote by $(-)^* = \text{Hom}_\Lambda(-, \Lambda)$. We also need the following lemma.

**Lemma 2.4.** Let $R$ and $\Lambda$ be as above and let $X$ be an indecomposable non-projective $R$-module. Then we have
\begin{enumerate}[1]
  \item $\text{Ext}^i_\Lambda(S_X, X) \cong S^{\tau^i X}$ if $i = 2$, and $\text{Ext}^i_\Lambda(S_X, X) = 0$ if $i \neq 2$.
  \item $\text{Ext}^i_\Lambda(S_X, Y) \cong \text{Tor}^i_\Lambda(X, S^{\tau^i X})$ for $Y \in \text{mod } \Lambda$ and $i \in \mathbb{Z}$.
\end{enumerate}

**Proof.** We only prove (2) since the statement (1) follows from (2) immediately.

By Proposition 2.3 there exist projective resolutions
\begin{align*}
0 \to P_{\tau X} \to P_E \to P_X \to S_X \to 0, & \quad (2.1) \\
0 \to P^X \to P^E \to P^{\tau X} \to S^{\tau X} \to 0, & \quad (2.2)
\end{align*}
of $S_X$ and $S^{\tau X}$, respectively. Applying $\text{Hom}_\Lambda(-, Y)$ to (2.1), we obtain a complex
\[ 0 \to \text{Hom}_\Lambda(P_X, Y) \to \text{Hom}_\Lambda(P_E, Y) \to \text{Hom}_\Lambda(P_{\tau X}, Y) \to 0 \]
whose homologies are $\text{Ext}^i_\Lambda(S_X, Y)$. Similarly, applying $Y \otimes_\Lambda -$ to (2.2), we obtain a complex
\[ 0 \to Y \otimes_\Lambda P^X \to Y \otimes_\Lambda P^E \to Y \otimes_\Lambda P^{\tau X} \to 0 \]
whose homologies are $\text{Tor}^i_\Lambda(Y, S^{\tau^i X})$. Because $\text{Hom}_\Lambda(P_{\tau X}, Y) \cong Y \otimes_\Lambda P_{\tau X} \cong Y \otimes_\Lambda P^* = Y \otimes_\Lambda P^{-}$ holds, (2.3) and (2.4) are isomorphic. Thus we obtain the desired isomorphism. \hfill $\square$

The following lemma is useful.

**Lemma 2.5.** Let $\Lambda$ be an Auslander algebra and $X \in \text{mod } \Lambda$. If $\text{Ext}^2_\Lambda(X, \Lambda) \neq 0$, then any composition factor of $\text{Ext}^2_\Lambda(X, \Lambda)$ has projective dimension 2.

**Proof.** We prove the assertion by induction on the length of $X$, which is denoted by $l(X)$.

If $l(X) = 1$, then $\text{Ext}^2_\Lambda(X, \Lambda)$ is simple by Proposition 2.3(1). By Proposition 2.3(4), the projective dimension is 2. Assume that it is true for $l(X) < t$. For the case $l(X) = t$, take an exact sequence $0 \to X' \to X \to X'' \to 0$ such that $l(X') < t$ and $l(X'') < t$ hold. Applying $(-)^*$, one gets an exact sequence $\text{Ext}^2_\Lambda(X'', \Lambda) \to \text{Ext}^2_\Lambda(X, \Lambda) \to \text{Ext}^2_\Lambda(X', \Lambda)$. Since any composition factor of $\text{Ext}^2_\Lambda(X, \Lambda)$ is either the composition factor of $\text{Ext}^2_\Lambda(X', \Lambda)$ or that of $\text{Ext}^2_\Lambda(X'', \Lambda)$, we are done. \hfill $\square$

We also need the following general result on algebras of global dimension 2.

**Lemma 2.6.** Let $\Lambda$ be an algebra with $\text{gl.dim } \Lambda \leq 2$ and $X \in \text{mod } \Lambda$. Then $X^{**}$ is a projective $\Lambda$-module.
Proof. Let \( Q_1 \to Q_0 \to X \to 0 \) be a projective resolution of \( X \). Applying \((-)^*\), we obtain an exact sequence \( 0 \to X^* \to Q_0^* \to Q_1^* \). Hence \( X^* \) is a projective \( \Lambda^{op}\)-module, since \( Q_0^* \) and \( Q_1^* \) are projective \( \Lambda^{op}\)-modules and \( \text{gl.dim} \Lambda \leq 2 \). Thus \( X^{**} \) is a projective \( \Lambda \)-module. \( \square \)

By the lemma above we obtain the following.

**Lemma 2.7.** Let \( \Lambda \) be an Auslander algebra, and let \( X \) be a \( \Lambda \)-module with \( \text{proj.dim} X \leq 1 \). Then the evaluation map \( \varphi_X : X \to X^{**} \) is injective, and the projective dimension of any composition factor of \( X^{**}/X \) is 2.

**Proof.** By \( \text{AusR} \), we get an exact sequence \( 0 \to \text{Ext}^1_{\Lambda^{op}}(\text{Tr} X, \Lambda) \to X \to X^{**} \to \text{Ext}^2_{\Lambda^{op}}(\text{Tr} X, \Lambda) \to 0 \). Then the latter assertion holds by Lemma 2.3.

To prove the former one, it suffices to show that \( \text{Ext}^1_{\Lambda^{op}}(\text{Tr} T, \Lambda) = 0 \). By Lemma 2.4, we only have to show that the projective dimension of any composition factor of \( \text{Tr} T \) is 2, that is, \( \text{Hom}_{\Lambda^{op}}(P, \text{Tr} T) = 0 \) holds for the projective cover \( P \) of any simple \( \Lambda^{op}\)-module \( S \) with \( \text{proj.dim} S \leq 1 \). By Proposition 2.3(3), \( P = P^I \) for some injective \( R \)-module \( I \).

On one hand, take a minimal projective resolution of \( T \):

\[
0 \to P_{X_1} \to P_{X_0} \to T \to 0 \quad (2.5)
\]

Since \( M \) is a generator, then we get an \( R \)-module monomorphism \( f : X_1 \to X_0 \). Applying \( \text{Hom}_R(-, I) \), one has an epimorphism

\[
\text{Hom}_R(X_0, I) \to \text{Hom}_R(X_1, I). \quad (2.6)
\]

On the other hand, applying the functor \((-)^*\) to (2.5), we get an exact sequence \( P^{X_0} \to P^{X_1} \to \text{Tr} T \to 0 \). Then applying the functor \( \text{Hom}_{\Lambda^{op}}(P^I, -) \), one obtains an exact sequence

\[
\text{Hom}_{\Lambda^{op}}(P^I, P^{X_0}) \to \text{Hom}_{\Lambda^{op}}(P^I, P^{X_1}) \to \text{Hom}_{\Lambda^{op}}(P^I, \text{Tr} T) \to 0 \quad (2.7)
\]

So (2.7) can be rewritten as \( \text{Hom}_R(X_0, I) \to \text{Hom}_R(X_1, I) \to \text{Hom}_{\Lambda^{op}}(P^I, \text{Tr} T) \to 0 \). Thus \( \text{Hom}_{\Lambda^{op}}(P^I, \text{Tr} T) = 0 \) by (2.6). \( \square \)

In the rest of this section, \( \Lambda \) is an arbitrary algebra. In the following we recall some basic properties of tilting modules. We begin with the definition of tilting modules.

**Definition 2.8.** We call \( T \in \text{mod}\Lambda \) a **tilting module** if \( T \) satisfies the following conditions

(T1) \( \text{proj.dim}_T \Lambda \leq 1 \).

(T2) \( \text{Ext}^1_\Lambda(T, T) = 0 \).

(T3) There exists a short exact sequence \( 0 \to \Lambda \to T_0 \to T_1 \to 0 \) with \( T_0, T_1 \in \text{add} T \).

The condition (T3) is equivalent to

(T3') The number of non-isomorphic direct summands of \( T \) is equal to that of \( \Lambda \).

Now let us recall some general properties of tilting modules [HaU].

**Lemma 2.9.** Let \( T \) be a tilting \( \Lambda \)-module, and let \( 0 \to Q_1 \to Q_0 \to T \to 0 \) be a minimal projective resolution of \( T \). Then we have the following:

(1) \( (\text{add} Q_1) \cap (\text{add} Q_0) = 0 \) and \( \text{add}(Q_0 \oplus Q_1) = \text{add} \Lambda \) hold.

(2) For a simple \( \Lambda \)-module \( S \), precisely one of \( \text{Hom}_\Lambda(T, S) = 0 \) and \( \text{Ext}^1_\Lambda(T, S) = 0 \) holds.

(3) For a simple \( \Lambda^{op}\)-module \( S \), precisely one of \( T \otimes_\Lambda S = 0 \) and \( \text{Tor}^1_\Lambda(T, S) = 0 \) holds.

We also have the following properties for the tensor products of tilting modules.

**Proposition 2.10.** Let \( T \) be a tilting \( \Lambda \)-module with \( \Gamma = \text{End}_\Lambda(T) \).

(1) Let \( U \) be a tilting \( \Gamma \)-module. If \( \text{Tor}^i_\Gamma(U, U) = 0 \) for any \( i > 0 \) and \( \text{proj.dim}(U \otimes_\Gamma \Gamma) \leq 1 \), then \( U \otimes_\Gamma T \) is a tilting \( \Lambda \)-module with \( \text{End}_\Lambda(U \otimes_\Gamma T) \simeq \text{End}_\Gamma(U) \).

(2) Let \( V \) be a tilting \( \Lambda \)-module. If \( \text{Ext}^i_\Gamma(T, V) = 0 \) for any \( i > 0 \) and \( \text{proj.dim} \text{Hom}_\Lambda(T, V) \leq 1 \), then \( \text{Hom}_\Lambda(T, V) \) is a tilting \( \Gamma \)-module with \( \text{End}_\Gamma(\text{Hom}_\Lambda(T, V)) \simeq \text{End}_\Lambda(V) \).

**Proof.** (1) \( U \otimes_\Gamma T \) is a tilting complex of \( \Lambda \). By our assumption \( U \otimes_\Gamma T \simeq U \otimes_\Lambda T \) holds. Thus the assertion holds. One can show (2) similarly. \( \square \)
Denote by $\tau$ the AR-translation and denote by $|N|$ the number of non-isomorphic indecomposable direct summands of $N$ for a $\Lambda$-module $N$. In the following we recall some basic properties of $\tau$-tilting theory. Firstly, we need the following definition in [AIR].

**Definition 2.11.** (1) We call $N \in \text{mod}\Lambda$ $\tau$-rigid if $\text{Hom}_\Lambda(N, \tau N) = 0$.
(2) We call $N \in \text{mod}\Lambda$ $\tau$-tilting if $N$ is $\tau$-rigid and $|N| = |\Lambda|$.
(3) We call $N \in \text{mod}\Lambda$ support $\tau$-tilting if there exists an idempotent $e$ of $\Lambda$ such that $N$ is a $\tau$-tilting $(\Lambda/(e))$-module.

It is clear that every tilting $\Lambda$-module is a $\tau$-tilting $\Lambda$-module, and hence a support $\tau$-tilting module. Moreover, it is showed in [AIR] tilting $\Lambda$-modules are exactly faithful support $\tau$-tilting modules. The following properties of $\tau$-rigid modules are also needed.

**Lemma 2.12.** ([AIR] Theorem 2.10) Let $\Lambda$ be an algebra. If $N$ is a $\tau$-rigid $\Lambda$-module, then it is a direct summand of a support $\tau$-tilting $\Lambda$-module.

For a $\Lambda$-module $X$, we define a full subcategory of $\text{mod}\Lambda$ by

$$\text{Fac}X := \{Y \in \text{mod}\Lambda \mid \text{There exists an epimorphism } X^n \to Y \text{ for some } n \geq 0\}.$$ 

We define the partial order on $s\tau$-tilt$\Lambda$ called the generation order as follows: For basic support $\tau$-tilting $\Lambda$-modules $T, U$, we write $T \leq U$ if $\text{Fac}T \subseteq \text{Fac}U$. Then the relation $\leq$ gives a partial order on the set $s\tau$-tilt$\Lambda$. Clearly $\Lambda$ is a unique maximal element and 0 is a unique minimal element in $s\tau$-tilt$\Lambda$.

We now recall the Hasse quiver of general posets.

**Definition 2.13.** The Hasse quiver $H(P)$ of a poset $(P, \leq)$ is defined as follows:
(1) The vertices are the elements of the poset $P$.
(2) For $X, Y \in P$, there is an arrow $X \to Y$ if and only if $X > Y$ and there is no $Z \in P$ satisfying $X > Z > Y$.

Then we have the following observation.

**Lemma 2.14.** Two partial orders on a finite set are the same if and only if their Hasse quivers are the same.

Now it is time to recall the mutations of support $\tau$-tilting modules from [AIR].

**Definition 2.15.** Let $T, U \in s\tau$-tilt$\Lambda$, we call $T$ a mutation of $U$ if they have the same indecomposable direct summands except one. Precisely speaking, there are three cases:
(1) $T = V \oplus X$ and $U = V \oplus Y$ with $X \not\cong Y$ indecomposable;
(2) $T = U \oplus X$ with $X$ indecomposable;
(3) $U = T \oplus X$ with $X$ indecomposable.

Moreover, we call $T$ a left mutation (resp. right mutation) of $U$ if $\text{Fac}T \subset \text{Fac}U$ (resp. $\text{Fac}T \supset \text{Fac}U$).

In the following we give a method of calculating left mutations of support $\tau$-tilting modules due to Adachi, Iyama and Reiten [AIR].

**Theorem 2.16.** ([AIR] Theorem 2.30) Let $T = X \oplus V$ be a basic $\tau$-tilting $\Lambda$-module which is the Bongartz completion of $V$, where $X$ is indecomposable. Let $X \xrightarrow{f} V' \xrightarrow{g} Y \to 0$ be an exact sequence, where $f$ is a minimal left (add$V$)-approximation. Then we have the following.
(1) If $Y = 0$, then $V = \mu_X(T)$ holds.
(2) If $Y \neq 0$, then $\mu_X(T) = Y' \oplus V$ holds, where $Y'$ is an indecomposable direct summand of $Y$ such that $Y'^m \simeq Y$ for some integer $m > 0$.

Now let us recall the relationship between mutations and the Hasse quiver, which is given in [HaU, RS] for tilt$\Lambda$ and in [AIR] for $s\tau$-tilt$\Lambda$.
Theorem 2.17. Let \( T, U \in \text{sr-tilt} \Lambda \) (resp. tilt\( \Lambda \)). The following are equivalent.
(1) \( T \) is a left mutation of \( U \).
(2) \( U \) is a right mutation of \( T \).
(3) \( U > T \) and there is no \( V \in \text{sr-tilt} \Lambda \) (resp. tilt\( \Lambda \)) such that \( U > V > T \).
(4) There is an arrow from \( U \) to \( T \) in \( H(\text{sr-tilt} \Lambda) \) (resp. \( H(\text{tilt} \Lambda) \)).

The following result [AIR, Corollary 2.38] gives a method of judging an algebra to be \( \tau \)-rigid finite.

Proposition 3.18. If \( H(\text{sr-tilt} \Lambda) \) admits a finite connected component \( C \), then \( H(\text{sr-tilt} \Lambda) = C \).

3. Tilting modules over the Auslander algebra of \( K[x]/(x^n) \)

Let \( n \geq 1 \) and \( R = K[x]/(x^n) \) a factor algebra of the polynomial ring \( K[x] \). Then the Auslander algebra \( \Lambda \) of \( R \) is presented by the quiver

\[
1 \to a_1 \to a_2 \to a_3 \to \cdots a_{n-1} \to n \to 1
\]

with relations \( a_i b_2 = 0 \) and \( a_i b_{i+1} = b_i a_{i-1} \) for any \( 2 \leq i \leq n-1 \). In this section, we classify all tilting \( \Lambda \)-modules.

Denote by \( \{e_1, \ldots, e_n\} \) a complete set of primitive orthogonal idempotents of \( \Lambda \) and denote by \( P_i = e_i \Lambda \) (resp. \( P^n = \Lambda e_i \)) the indecomposable projective \( \Lambda \)-module (resp. \( \Lambda^{\text{op}} \)-module). It is easy that \( P_1, P_2, \ldots, P_n \) have the following composition series (see \( n = 4 \) for example).

\[
[P_1 | P_2 | P_3 | P_4] = \left[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3 \\
\end{array} \right]
\]

For \( 1 \leq i \leq n \), we define an ideal of \( \Lambda \) by

\[
I_i = P_1 \oplus \cdots \oplus (\text{rad} \, P_i) \oplus \cdots \oplus P_n = P^1 \oplus \cdots \oplus (\text{rad} \, P^n) \oplus \cdots \oplus P^n.
\]

This is a maximal left ideal and also a maximal right ideal. Furthermore, for \( 1 \leq i \leq n \), we define a \((\Lambda, \Lambda)\)-bimodule by \( S_i = \Lambda/I_i \). Clearly we have the following.

Proposition 3.1. Let \( \Lambda \) be the Auslander algebra of \( K[x]/(x^n) \). Then one gets the following
(1) As a \( \Lambda \)-module \( S_i \cong P_i/\text{rad} \, P_i \) is simple. As a \( \Lambda^{\text{op}} \)-module \( S_i \cong P^i/\text{rad} \, P^i \) is simple.
(2) There exists an isomorphism \( P_n \cong \text{DP}^n \) of \( \Lambda \)-modules. Thus \( P_n \) is a projective-injective \( \Lambda \)-module.
(3) For \( 1 \leq i \leq n-1 \), there exist minimal projective resolutions of \( \Lambda \)-modules

\[
0 \to P_i \to P_{i-1} \oplus P_{i+1} \to P_i \to S_i \to 0 \quad \text{and} \quad 0 \to P_i \to P_{i-1} \oplus P_{i+1} \to \text{rad} \, P_i \to 0.
\]

(4) There exist minimal projective resolutions of \( \Lambda \)-modules

\[
0 \to P_{n-1} \to P_n \to S_n \to 0 \quad \text{and} \quad 0 \to P_{n-1} \to \text{rad} \, P_n \to 0.
\]

Now we are in a position to show the following proposition.

Proposition 3.2. For \( 1 \leq i \leq n-1 \), \( I_i \) is a tilting \( \Lambda \)-module and a tilting \( \Lambda^{\text{op}} \)-module.

Proof. We only prove the case of a \( \Lambda \)-module since the case of a \( \Lambda^{\text{op}} \)-module is similar. By definition, we have \( I_i \left(= \bigoplus_{j \neq i} P_j \right) \oplus \text{rad} \, P_i \).

(T1) By Proposition 3.1(3), we have \( \text{proj.dim} \, \text{rad} \, P_i \leq 1 \). Thus \( \text{proj.dim} \, I_i \leq 1 \).

(T2) It suffices to show that \( \text{Ext}^1_\Lambda(\text{rad} \, P_i, I_i) = 0 \). Since there exists an exact sequence \( 0 \to \text{rad} \, P_i \to P_i \to S_i \to 0 \), we have \( \text{Ext}^1_\Lambda(S_i, I_i) = 0 = \text{Ext}^1_\Lambda(\text{rad} \, P_i, I_i) \). By Lemma 2.3, we have \( \text{Ext}^1_\Lambda(S_i, I_i) \cong I_i \otimes_\Lambda S_i \). On the other hand, we have \( P_j \otimes_\Lambda S_i = 0 \) for any \( j \neq i \). By Proposition 3.1(3), there exists an exact sequence \( 0 = (P_{i-1} \oplus P_{i+1}) \otimes_\Lambda S_i \to \text{rad} \, P_i \otimes_\Lambda S_i \to 0 \). Thus we have \( \text{rad} \, P_i \otimes_\Lambda S_i = 0 \) and \( I_i \otimes_\Lambda S_i = 0 \).
Lemma 3.6. show the converse, that is, all basic tilting Λ-modules are included in $T I$ to Proposition 3.4(1), we obtain that $T I$ is a module. The left multiplication

Proof. Theorem 3.5. empty product Λ is also contained in this set. Now we can state the following result.

□

Proposition 3.4. Λ-modules.

Proof. (1) If $j \neq i$, we have $\text{Hom}_\Lambda(I_i, S_j) = 0$. Further, by Proposition 3.1(3)(4), one gets $\text{Hom}_\Lambda(\text{rad} P_i, S_j) = 0$. Thus we have $\text{Hom}_\Lambda(I_i, S_j) = 0$.

(2) Applying $\text{Hom}_\Lambda(-, \Lambda)$ to a short exact sequence

$$0 \to I_i \to \Lambda \to S_i \to 0 \quad (3.1)$$

yields a long exact sequence $0 \to \text{Hom}_\Lambda(S_i, \Lambda) \to \text{Hom}_\Lambda(\Lambda, \Lambda) \to \text{Hom}_\Lambda(I_i, \Lambda) \to \text{Ext}_\Lambda^1(S_i, \Lambda) \to 0$. Then by Lemma 2.3, we have $\text{Hom}_\Lambda(S_i, \Lambda) = \text{Ext}_\Lambda^1(S_i, \Lambda) = 0$, and hence $\text{Hom}_\Lambda(I_i, \Lambda) \cong \text{Hom}_\Lambda(\Lambda, \Lambda) \cong \Lambda$. On the other hand, applying $\text{Hom}_\Lambda(I_i, -)$ to the short exact sequence (3.1), one gets an exact sequence $0 \to \text{Hom}_\Lambda(I_i, I_i) \to \text{Hom}_\Lambda(I_i, \Lambda) \to \text{Hom}_\Lambda(I_i, S_i)$. Using (1), we have $\text{End}_\Lambda(I_i) \cong \text{Hom}_\Lambda(\Lambda, \Lambda) \cong \Lambda$. □

From the argument above, we have the following proposition on the multiplication of tilting Λ-modules.

Proposition 3.4. Let $T$ be a tilting Λ-module and $1 \leq i \leq n - 1$. Then we have the following.

(1) $TI_i$ is a tilting Λ-module, and $\text{End}_\Lambda(TI_i) = \text{End}_\Lambda(T)$.

(2) If $TI_i \neq T$, then $TI_i = T \otimes_\Lambda I_i = T \otimes^L_\Lambda I_i$.

Proof. (2) Since $TI_i \neq T$, then $T \otimes_\Lambda S_i \cong T/TI_i \neq 0$, and we have $\text{Tor}_1^\Lambda(T, S_i) = 0$ by Proposition 2.9(3). Applying $T \otimes_\Lambda -$ to the short exact sequence $0 \to I_i \to \Lambda \to S_i \to 0$, one gets an exact sequence $0 \to \text{Tor}_1^\Lambda(T, S_i) \to T \otimes_\Lambda I_i \to T \otimes_\Lambda \Lambda = T$. Thus the natural map $T \otimes_\Lambda I_i \to T$ is injective and has the image $TI_i$. Thus we obtain $T \otimes_\Lambda I_i \cong TI_i$. Moreover, we have $\text{Tor}_1^\Lambda(T, I_i) \cong \text{Tor}_1^\Lambda(T, S_i) = 0$ for $j \geq 1$ since proj.dim $T \leq 1$. Thus $T \otimes_\Lambda I_i = T \otimes^L_\Lambda I_i$.

(1) If $TI_i = T$, then the assertion is clear. Now assume that $TI_i \neq T$. Since we have $\text{End}_\Lambda(I_i) = \Lambda$ by Proposition 3.3, $T \otimes_\Lambda I_i = TI_i$ is a tilting module with $\text{End}_\Lambda(T) = \text{End}_\Lambda(TI_i)$ by (2) and Proposition 2.10(1). □

Denote by $\langle I_1, \ldots, I_{n-1} \rangle$ the set of ideals of Λ given by products of $I_1, \ldots, I_{n-1}$, where the empty product Λ is also contained in this set. Now we can state the following result.

Theorem 3.5. Any ideal $T$ in $\langle I_1, \ldots, I_{n-1} \rangle$ is a basic tilting Λ-module and a basic tilting $\Lambda^{op}$-module. The left multiplication $\Lambda \to \text{End}_\Lambda(T)$ and the right multiplication $\Lambda^{op} \to \text{End}_{\Lambda^{op}}(T)$ are isomorphisms.

Proof. We only prove the case of a Λ-module since the case of a $\Lambda^{op}$-module is similar.

By Proposition 3.2 each of $I_1, \ldots, I_{n-1}$ is a tilting Λ-module with $\text{End}_\Lambda(I_i) = \Lambda$. If $T = I_iI_{i_2} \cdots I_{i_{k-1}}$ is a tilting Λ-module with $\text{End}_\Lambda(T) = \Lambda$ for $i_1, \ldots, i_k \in \{1, \ldots, n-1\}$, then, according to Proposition 3.3(1), we obtain that $TI_{i_k}$ is a tilting Λ-module with $\text{End}_\Lambda(TI_{i_k}) = \Lambda$. In particular, $TI_i$ is basic. Thus we get the assertion inductively. □

By Theorem 3.5 any element in $\langle I_1, \ldots, I_{n-1} \rangle$ is a basic tilting Λ-module. In the following we show the converse, that is, all basic tilting Λ-modules are included in $\langle I_1, \ldots, I_{n-1} \rangle$. For this aim, we start with the following.

Lemma 3.6. Let $X$ be a Λ-module. For $1 \leq i \leq n - 1$, there exist isomorphisms $\text{Ext}_\Lambda^1(S_i, X) \cong X \otimes_\Lambda S_i$ and $\text{Ext}_\Lambda^1(S_i, X) \cong \text{Tor}_1^\Lambda(X, S_i)$. If $X$ is tilting, then precisely one of them is zero.
Proof. By Lemma 2.4, we have Ext^j_Λ(S_i, X) \cong Tor^j_Λ(X, S_i). The latter statement follows from Proposition 2.9(3).

We need the following properties of tilting Λ-modules.

Proposition 3.7. Let T be a tilting Λ-module, and 1 \leq i \leq n - 1. Then we have the following:

1. Hom_Λ(S_i, T) = 0.
2. proj.dim Hom_Λ(I_i, T) \leq 1.
3. There exist natural inclusions T \subset Hom_Λ(I_i, T) \subset T^{**} = Hom_Λ(I_i, T)^{**}.
4. Hom_Λ(I_i, T)/T \simeq Ext^1_Λ(S_i, T). If T \subseteq Hom_Λ(I_i, T), then Hom_Λ(I_i, T)I_i = T.
5. Hom_Λ(I_i, T) is a tilting Λ-module, and End_Λ(Hom_Λ(I_i, T)) = End_Λ(T) holds.
6. If T is not a projective Λ-module, then there exists 1 \leq i \leq n - 1 such that T \subseteq Hom_Λ(I_i, T).

Proof. We firstly note by Lemma 2.6 that T^{**} is a projective Λ-module. By Lemma 2.4, we have Ext^j_Λ(S_i, T) = 0 = Ext^j_Λ(S_i, T^{**}) for j \neq 2. These facts will be used freely in the later proof.

(1) By Lemma 2.7, we have an exact sequence

0 \rightarrow T^{**} \rightarrow T^{**}/T \rightarrow 0.

(3) Applying Hom_Λ(T^{**}, T) to the exact sequence 0 \rightarrow I_i \rightarrow \Lambda \rightarrow S_i \rightarrow 0 of (Λ, Λ)-bimodules, we obtain an exact sequence

0 \rightarrow Hom_Λ(\Lambda, T) \rightarrow Hom_Λ(I_i, T) \rightarrow Ext^1_Λ(S_i, T) \rightarrow 0 \rightarrow Ext^2_Λ(S_i, T) \rightarrow 0 \rightarrow \cdots

of Λ-modules by (1). Since the Λ^op-module S_i is annihilated by I_i, the Λ-module Ext^1_Λ(S_i, T) is annihilated by I_i and hence isomorphic to S_i^m for some m \geq 0. Hence (3.3) gives an exact sequence 0 \rightarrow T \rightarrow Hom_Λ(I_i, T) \rightarrow S_i^m \rightarrow 0. Applying (−)^* = Hom_Λ(−, Λ), we obtain an exact sequence 0 = (S_i^m)^* \rightarrow Hom_Λ(I_i, T)^* \rightarrow T^* \rightarrow Ext^1_Λ(S_i, T)^* = 0. In particular, we have T^{**} \cong Hom_Λ(I_i, T)^{**} and the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & T \\
\end{array}
\xrightarrow{φ_T}
\begin{array}{ccc}
Hom_Λ(I_i, T) & \rightarrow & S_i^m \\
\end{array}
\xrightarrow{φ_{Hom_Λ(I_i, T)}}
\begin{array}{ccc}
T^{**} & \rightarrow & Hom_Λ(I_i, T)^{**} \\
\end{array}
\]

By (2) and Lemma 2.7, φ_{Hom_Λ(I_i, T)} is a monomorphism and hence (3) follows.

(4) The former assertion is immediate from the exact sequence (3.3). Since Ext^1_Λ(S_i, T) \cong S_i^m is annihilated by I_i, we have T/I_i \subseteq Hom_Λ(I_i, T)I_i \subseteq T. Since T/I_i \cong T \otimes_Λ S_i = 0 holds by Lemma 2.9(3), we obtain Hom_Λ(I_i, T)I_i = T.

(5) If T = Hom_Λ(I_i, T), then it is obvious. Assume that T \neq Hom_Λ(I_i, T). By (2) and Propositions 3.7, 2.10(2), it suffices to prove that Ext^1_Λ(I_i, T) = 0 for any j > 0. We only have to consider the case j = 1 since proj.dim I_i \leq 1. We have Ext^1_Λ(S_i, T) \neq 0 by (4), and hence Ext^1_Λ(I_i, T) \cong Ext^1_Λ(S_i, T) = 0 holds by Lemma 3.6. Thus (5) follows.

(6) By our assumption, T \neq T^{**} holds. By Lemma 2.7 and Proposition 3.1, we can take a simple submodule S_i of T^{**}/T for some 1 \leq i \leq n - 1. Applying Hom_Λ(S_i, −) to the exact sequence (3.2), we get an exact sequence 0 = Hom_Λ(S_i, T^{**}) \rightarrow Hom_Λ(S_i, T^{**}/T) \rightarrow Ext^1_Λ(S_i, T). Thus Ext^1_Λ(S_i, T) \neq 0 by our choice of S_i. Thus Hom_Λ(I_i, T)/T \cong Ext^1_Λ(S_i, T) \neq 0 holds by (4), and we have T \subseteq Hom_Λ(I_i, T).
Lemma 3.8. Let $T \in \langle I_1, \ldots, I_{n-1} \rangle$, and let $f_T : T \to \Lambda$ be a natural inclusion. Then in the following commutative diagram, $\varphi_\Lambda$ and $f_T^{**}$ are isomorphisms.

\[
\begin{array}{c}
T \\
\downarrow f_T \\
\Lambda
\end{array}
\quad \begin{array}{c}
\varphi_\Lambda \\
\downarrow f_T^{**} \\
\Lambda^{**}
\end{array}
\]

Proof. Since $\Lambda$ is projective, it is clear that $\varphi_\Lambda$ is an isomorphism.

Any composition factor of the $\Lambda$-module $\Lambda/T$ has a form $S_i$ for some $1 \leq i \leq n - 1$. By Lemma 2.6, we have $\text{Ext}_i^1(\Lambda/T, \Lambda) = 0$ for $j \neq 2$. Applying $(-)^* = \text{Hom}_\Lambda(-, \Lambda)$ to the exact sequence $0 \to T \xrightarrow{f_T} \Lambda \to \Lambda/T \to 0$, we have an exact sequence $0 = (\Lambda/T)^* \to \Lambda^* \xrightarrow{f_T^*} \Lambda \to \text{Ext}_1^1(\Lambda/T, \Lambda) = 0$. Thus $f_T^*$ is an isomorphism and hence $f_T^{**}$ is an isomorphism.

Now we are in a position to state our first main result in this section.

Theorem 3.9. Let $\Lambda$ be the Auslander algebra of $K[x]/(x^n)$. Then
(1) For any tilting $\Lambda$-module $T$, there exists $U \in \langle I_1, \ldots, I_{n-1} \rangle$ such that $\text{add} T = \text{add} U$.
(2) If two elements $T$ and $U$ in $\langle I_1, \ldots, I_{n-1} \rangle$ are isomorphic as $\Lambda$-modules, then $T = U$.
(3) The set $\text{tilt} \Lambda$ is given by $\langle I_1, \ldots, I_{n-1} \rangle$.
(4) The statements (1), (2) and (3) hold also for $\Lambda^{op}$-modules.

Proof. (1) By Proposition 3.7(4)(5)(6), there exists a finite sequence of tilting $\Lambda$-modules

$T = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_m = T^{**}$

and $i_1, \ldots, i_m \in \{1, \ldots, n - 1\}$ such that $T_{k+1} = \text{Hom}_\Lambda(I_{k+1}, T_k)$ and $T_k = T_{k+1}I_{k+1}$ for any $0 \leq k \leq m - 1$. In particular, we have $T = T_1I_{i_1} = T_2I_{i_1}I_{i_2} = \cdots = T_mI_{i_m} \cdots I_{i_1}$. Because $T^{**}$ is a projective tilting $\Lambda$-module by Lemma 2.4, we have $\text{add} T_m = \text{add} \Lambda$. Thus $\text{add} T = \text{add} U$ holds for $U := I_{i_m} \cdots I_{i_1} \in \langle I_1, \ldots, I_{n-1} \rangle$.

(2) For $T, U \in \langle I_1, \ldots, I_{n-1} \rangle$, assume that there exists a $\Lambda$-module isomorphism $g : T \cong U$.

By Lemma 3.8 there exists a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{g} & U \\
\downarrow f_T & & \downarrow f_U \\
\Lambda & \xrightarrow{\varphi_\Lambda^{**}} & \Lambda^{**}
\end{array}
\]

where $e_T := \varphi_\Lambda^{-1} f_T^{**}$ and $e_U := \varphi_\Lambda^{-1} f_U^{**}$ are isomorphisms. Putting $h = e_Ug^{**}e_T^{-1} : \Lambda \to \Lambda$, we have a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{g} & U \\
\downarrow f_T & & \downarrow f_U \\
\Lambda & \xrightarrow{h} & \Lambda
\end{array}
\]

Since $h$ is given by the left multiplication of an invertible element $x \in \Lambda$, so is $g$. Since $T$ is an ideal of $\Lambda$, we have $U = xT = T$.

(3) This is a consequence of (1), (2) and Theorem 3.5.

(4) One can prove it similarly to (1), (2) and (3).

The mutations of tilting $\Lambda$-modules are described by the following result. Notice that we use the structure of $\Lambda^{op}$-modules when we consider mutations of $\Lambda$-modules.

Proposition 3.10. Let $T \in \langle I_1, \ldots, I_{n-1} \rangle$.
(1) For each $1 \leq i \leq n - 1$, precisely one of the following statements (a) and (b) holds.
(a) If $I_iT \neq T$, then $I_iT = I_i \otimes_A T$ is a left mutation of $T$, and $\text{Hom}_{\Lambda^{op}}(I_i, T) = T$ holds.
(b) If $\text{Hom}_{\Lambda^\varphi}(I_i, T) \neq T$, then $\text{Hom}_{\Lambda^\varphi}(I_i, T)$ is a right mutation of $T$, and $I_i T = T$ holds.

(2) All mutations of $T$ in $\text{tilt}\Lambda$ are given by of the form (1). In particular $T$ has precisely $n - 1$ mutations in $\text{tilt}\Lambda$.

(3) The corresponding statement to (1) and (2) hold for $\Lambda^\varphi$-modules.

Proof. (1) Applying Proposition 3.7 and Proposition 3.10 to the tilting $\Lambda^\varphi$-module $T$, we have that $I_i T$ and $\text{Hom}_{\Lambda^\varphi}(I_i, T)$ are tilting $\Lambda^\varphi$-modules, $\text{End}_{\Lambda^\varphi}(I_i T) = \text{End}_{\Lambda^\varphi}(\text{Hom}_{\Lambda^\varphi}(I_i, T))$. Since $\text{End}_{\Lambda^\varphi}(T) = \Lambda^\varphi$ holds by Theorem 3.5, we have that $I_i T$ and $\text{Hom}_{\Lambda^\varphi}(I_i, T)$ are tilting $\Lambda$-modules. Further we know that $I_i T = \bigoplus_{j=1}^n e_j I_i T$ and $\text{Hom}_{\Lambda^\varphi}(I_i, T) = \bigoplus_{j=1}^n \text{Hom}_{\Lambda^\varphi}(I_i e_j, T)$.

Since $e_j I_i = e_j \Lambda$ and $I_i e_j = \Lambda e_j$ hold for any $j \neq i$. Thus the indecomposable direct summands of $I_i T$ (resp. $\text{Hom}_{\Lambda^\varphi}(I_i, T)$) coincide with those of $T$ except one. By Theorem 2.10, $I_i T$ (resp. $\text{Hom}_{\Lambda^\varphi}(I_i, T)$) is either isomorphic to $T$ or a mutation of $T$. We have $I_i T \cong T \iff S_i \otimes \Lambda T = 0 \iff \text{Hom}_{\Lambda^\varphi}(S_i, T) = 0$,

\[ \text{Hom}_{\Lambda^\varphi}(I_i, T) \cong T \iff \text{Ext}^1_{\Lambda^\varphi}(S_i, T) = 0 \]

by Proposition 3.7. Thus precisely one of $S_i \otimes \Lambda T = 0$ and $\text{Tor}_1^\Lambda(S_i, T) = 0$ holds by Proposition 2.10.

It remains to decide whether the mutation is left or right. We only have to show $\text{Hom}_{\Lambda^\varphi}(I_i, T) \geq T \geq I_i T$. Taking an epimorphism $\Lambda^n \rightarrow I_i$ of $\Lambda$-modules, we have an epimorphism $T^n \rightarrow I_i T$. Thus, we have $T^* \supset (I_i T)^*$ and $T \geq I_i T$. If $U := \text{Hom}_{\Lambda^\varphi}(I_i, T) \geq T$, then we have $I_i U = T$ by Proposition 3.7. Thus we have $\text{Hom}_{\Lambda^\varphi}(I_i, T) = U \geq T$.

(2) Any basic tilting $\Lambda$-module has precisely $n$ indecomposable direct summands. Since $P_i$ is injective by Proposition 3.11, it is a direct summand of any tilting $\Lambda$-module. Therefore the number of mutations of $T$ in $\text{tilt}\Lambda$ is at most $n - 1$, while we have at least $n - 1$ mutations in $\text{tilt}\Lambda$ by (1).

(3) One can prove it similarly to (1) and (2). \hfill $\square$

Immediately we have the following description of the Hasse quiver of tilting $\Lambda$-modules.

**Corollary 3.11.** Let $T \in \langle I_1, \ldots, I_{n-1} \rangle$. Then all arrows in the Hasse quiver of tilting $\Lambda$-modules starting or ending at $T$ are given by the following for $i \in \{1, \ldots, n-1\}$:

\[ \text{Hom}_{\Lambda^\varphi}(I_i, T) \rightarrow T \quad \text{if} \quad T = I_i T, \]

\[ T \rightarrow I_i T \quad \text{if} \quad T \neq I_i T. \]

Thus the number of arrows starting or ending at $T$ is precisely $n - 1$.

We have shown that the set $\text{tilt}\Lambda$ is given by $\langle I_1, \ldots, I_{n-1} \rangle$. In the following we give an explicit description of this set. Let us start with the following elementary observation.

**Proposition 3.12.** Let $\Lambda$ be a basic finite dimensional algebra, $\{e_1, \ldots, e_n\}$ a complete of orthogonal primitive idempotents of $\Lambda$, and $S_1, \ldots, S_n$ the corresponding simple $\Lambda$-modules. For a subset $J$ of $\{1, \ldots, n\}$, we put $e_J = 1 - \sum_{i \in J} e_i$ and $I_J = \Lambda(1 - e_J)\Lambda$.

Then for any $X \in \mod\Lambda$, we have that $XI_J$ is the maximum amongst submodules $Y$ of $X$ satisfying the following condition:

(2) Any composition factor of $X/Y$ has the form $S_i$ for some $i \in J$.

**Proof.** Since $\text{Hom}_{\Lambda}((1 - e_J)\Lambda, X) \cong X(1 - e_J)$, we have

\[ XI_J = X(1 - e_J)\Lambda = \sum_{f \in \text{Hom}_{\Lambda}((1 - e_J)\Lambda, X)} \text{Im} f. \]
The condition (2) holds if and only if \( \text{Hom}_\Lambda((1 - e_j)\Lambda, X/Y) = 0 \) holds if and only if \( \text{Im} f \subset Y \)
holds for any \( f \in \text{Hom}_\Lambda((1 - e_j)\Lambda, X) \) if and only if \( Y \subset XI_j \).

We have the following relations for the multiplication of ideals \( I_1, \ldots, I_{n-1} \).

**Proposition 3.13.** The following relations hold for any \( 1 \leq i, j \leq n - 1 \).

1. \( I_i^2 = I_i \).
2. If \( |i - j| \geq 2 \), then \( I_i I_j = I_j I_i \).
3. If \( |i - j| = 1 \), then \( I_i I_j I_i = I_j I_i I_j \).

**Proof.**

(1) By Propositions 3.12 and 3.1 \( I_i = \Lambda(1 - e_i)\Lambda \) holds. Hence \( I_i^2 = \Lambda(1 - e_i)\Lambda(1 - e_i)\Lambda = \Lambda(1 - e_i)\Lambda = I_i \).

(2) For \( 1 \leq i \neq j \leq n - 1 \), put \( I_{i,j} = \Lambda(1 - e_i - e_j)\Lambda \). Removing all vertices except \( i \) and \( j \) from the quiver with relations of \( \Lambda \), we have the quiver with relations of \( \Lambda/I_{i,j} \). In particular, if \( |i - j| \geq 2 \), then \( \Lambda/I_{i,j} \cong K \times K \). If \( |i - j| = 1 \), then \( \Lambda/I_{i,j} \) is given by the quiver \( \begin{array}{c} i \rightarrow b \rightarrow j \end{array} \) with relations \( ab = ba \) and hence \( \Lambda/I_{i,j} = \left[ \begin{array}{c} i \mid j \end{array} \right] \).

We prove (2). By Proposition 3.12 \( I_{i,j} \supset I_{i,j} \). Since \( \Lambda/I_{i,j} \cong K \times K \), we have \( I_{i,j}/I_{i,j} = 0 \). Hence \( I_{i,j}/I_{i,j} \) holds, and similarly we have \( I_{i,j}/I_{i,j} \). Thus \( I_{i,j}/I_{i,j} = I_{i,j} \).

We prove (3). By Proposition 3.12 \( I_{i,j}I_{j,i} \supset I_{i,j} \). Since \( \Lambda/I_{i,j} = \left[ \begin{array}{c} j \mid i \end{array} \right] \), we have \( I_{i,j}/I_{i,j} = 0 \). Hence \( I_{i,j}/I_{i,j} \) holds, and similarly we have \( I_{i,j}/I_{i,j} = I_{i,j} \). Thus \( I_{i,j}/I_{i,j} = I_{i,j} \).

Now we recall some well-known properties of the symmetric groups. We consider the action of \( S_n \) on \( \mathbb{R}^n \) given by permuting the standard basis \( e_1, \ldots, e_n \). Then \( S_n \) acts on the subspace

\[ V := \{ x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}^n | \sum_{i=1}^n x_i = 0 \}, \]

which has a basis \( \alpha_i := e_i - e_{i+1} \) with \( 1 \leq i \leq n - 1 \). Clearly the action of \( S_n \) on \( V \) is faithful, and we have an injective homomorphism \( S_n \to \text{GL}(V) \) called geometric representation.

The following elementary fact plays an important role in the proof of our main theorem.

**Proposition 3.14.** Let \( S_n \) be the symmetric group of degree \( n \) and \( S_n \supset w \). Then we have the following:

1. \[ \text{[B] Theorem 3.3.1]} \] Any expression \( s_{i_1} s_{i_2} \cdots s_{i_l} \) of \( w \) can be transformed into a reduced expression of \( w \) by applying the following operations (a), (b), (c) repeatedly.
   (a) Remove \( s_i s_i \) in the expression.
   (b) Replace \( s_i s_j \) with \( |i - j| \geq 2 \) by \( s_j s_i \) in the expression.
   (c) Replace \( s_i s_j s_i \) with \( |i - j| = 1 \) by \( s_j s_i s_j \) in the expression.

2. \[ \text{[B] Theorem 3.3.1]} \] Every two reduced expressions of \( w \) can be transformed each other by applying the operations (b) and (c) repeatedly.

3. If \( w = s_{i_1} s_{i_2} \cdots s_{i_l} \) is a reduced expression, then \( s_{i_1} s_{i_2} \cdots s_{i_l} (\alpha_{i_{l+1}}) \) is a positive root for any \( 1 \leq k \leq l - 1 \).

We also need the following proposition.

**Proposition 3.15.** There exists a well-defined surjective map \( S_n \to \langle I_1, \ldots, I_{n-1} \rangle \), which maps \( w \) to \( I(w) = I_{i_1} \cdots I_{i_l} \) where \( w = s_{i_1} \cdots s_{i_l} \) is an arbitrary reduced expression.

**Proof.** First, we show that the map is well-defined. Take two reduced expressions \( w = s_{i_1} \cdots s_{i_l} = s_{j_1} \cdots s_{j_y} \) of \( w \). These two expressions are transformed each other by the operation (b) and (c) in Proposition 3.14. Then by Proposition 3.13 we obtain \( I_{i_1} \cdots I_{i_l} = I_{j_1} \cdots I_{j_y} \).

Next we show that the map is surjective. For any \( I \in \langle I_1, \ldots, I_{n-1} \rangle \), we take a minimal number \( l \) such that \( I = I_{i_1} \cdots I_{i_l} \) holds for some \( i_1, \ldots, i_l \in \{1, \ldots, n-1\} \). Now we put \( w := s_{i_1} \cdots s_{i_l} \). This expression is transformed into a reduced expression of \( w \) by applying (a), (b) and (c) in Proposition 3.14. Since \( k \) is minimal, then (c) would not happen. Therefore \( w = s_{i_1} \cdots s_{i_l} \) is a reduced expression and we have \( I = I(w) \).
Since \( I(w) \) is a tilting \( \Lambda \)-module with \( \text{End}_\Lambda(I(w)) = \Lambda \) for any \( w \in \mathcal{S}_n \) by Proposition 3.15, we have an autoequivalence

\[
- \otimes^L_A I(w) : D^b(\text{mod} \Lambda) \rightarrow D^b(\text{mod} \Lambda)
\]

whose quasi-inverse is given by \( \mathbb{R}\text{Hom}_\Lambda(I(w), -) \). We define a full subcategory \( \mathcal{T} \) of \( D^b(\text{mod} \Lambda) \) by

\[
\mathcal{T} := \{ X \in D^b(\text{mod} \Lambda) \mid \forall i \in \mathbb{Z}, H^i(X)e_i = 0 \}.
\]

The Grothendieck group \( K_0(\mathcal{T}) \) is a free abelian group with basis \([S_1], \ldots, [S_{n-1}]\). We identify \( V \) with \( \mathbb{R} \otimes \mathbb{Z} K_0(\mathcal{T}) \) for any \( 1 \leq i \leq n-1 \).

**Lemma 3.16.** (1) We have an induced autoequivalence \(- \otimes^L_A I(w) : \mathcal{T} \rightarrow \mathcal{T}\).

(2) We have \([ - \otimes^L_A I(w) ] = s_i \) in \( \text{GL}(V) \) for any \( 1 \leq i \leq n-1 \).

**Proof.** (1) We have a triangle \( I(w) \rightarrow \Lambda \rightarrow \Lambda/I(w) \rightarrow I(w)[1] \in D(\text{Mod}\Lambda^{op} \otimes \Lambda) \). Applying \( \text{Proposition 3.17} \), we have a triangle

\[
X \otimes^L_A I(w) \rightarrow X \rightarrow X \otimes^L_A (\Lambda/I(w)) \rightarrow X \otimes^L_A I(w)[1]
\]

in \( D^b(\text{mod} \Lambda) \). Since both \( X \) and \( X \otimes^L_A (\Lambda/I(w)) \) belong to \( \mathcal{T} \), so is \( X \otimes^L_A I(w) \). Thus \( \mathcal{T} \otimes^L_A I(w) \subset \mathcal{T} \) holds. Similarly one can show \( \mathbb{R}\text{Hom}_\Lambda(I(w), \mathcal{T}) \subset \mathcal{T} \). Therefore the assertion follows.

(2) For \( X \in D^b(\text{mod} \Lambda) \) and \( Y \in D^b(\text{mod} \Lambda^{op}) \), let \( \chi(X, Y) := \sum_{k \in \mathbb{Z}} \dim_k H^k(X \otimes_A Y) \).

Then

\[
\chi(S_j, S_i) = \begin{cases} 
2 & i = j \\
-1 & |i - j| = 1 \\
0 & |i - j| \geq 2
\end{cases}
\]

holds for any \( 1 \leq j \leq n-1 \). We have \([S_j] \otimes^L_A I_i = [S_j] - [S_j] \otimes^L_A S_i = [S_j] - \chi(S_j, S_i)[S_i] \) by applying \( 3.4 \) to \( X = S_j \) and \( w = s_i \). Thus the assertion follows easily. \( \square \)

We have the following key observations.

**Proposition 3.17.** Let \( w \in \mathcal{S}_n \) and \( w = s_{i_1} s_{i_2} \cdots s_{i_l} \) a reduced expression.

(1) We have \([ - \otimes^L_A I(w) ] = w^{-1} \) in \( \text{GL}(V) \).

(2) We have \( I_1 \supseteq I_{i_{l-1}} I_i \supseteq \cdots \supseteq I_i \cdots I_1 \) and \( I(w) = I_1 \otimes^L_A \cdots \otimes^L_A I_i \).

(3) Let \( 1 \leq j \leq n-1 \). Then \( l(s_j w) > l(w) \) if and only if \( I(s_j w) < I(w) \).

**Proof.** The assertion (2) implies (1) since Lemma 3.10(2) implies \([ - \otimes^L_A I(w) ] = [ - \otimes^L_A I_i ] \circ \cdots \circ [ - \otimes^L_A I_1 ] \circ [ - \otimes^L_A I_i ] = s_{i_1} \cdots s_{i_l} = w^{-1} \).

(2) Since \( I(w) \supseteq I_{i_{l-1}} I_i \supseteq \cdots \supseteq I_i \cdots I_1 \) and \( I(w) = I_1 \otimes^L_A \cdots \otimes^L_A I_i \).

(3) It suffices to show that \( l(s_j w) > l(w) \) implies that \( I(s_j w) < I(w) \) by replacing \( s_j w \) with \( w \) if necessary. By (2) we have \( I(w) \supseteq I(s_j w) = I_j I(w) \). Then by Proposition 3.10(1)(a), we have \( I(s_j w) < I(w) \).

Now we have the following main result in this section.

**Theorem 3.18.** (1) There exists a well-defined bijection \( \mathcal{S}_n \cong (I_1, \ldots, I_{n-1}) \), which maps \( w \) to \( I(w) = I_1 \cdots I_i \) where \( w = s_{i_1} \cdots s_{i_l} \) is an arbitrary reduced expression.

(2) Consequently, there is a bijection \( I : \mathcal{S}_n \cong \text{tilt} \Lambda \). In particular \( \# \text{tilt} \Lambda = n! \).

(3) The bijection \( I \) in (2) is an anti-isomorphism of posets.

**Proof.** (1) Since the later one is a bijection result of the first one, we only have to show the first one. By Proposition 3.16(1), \( I \) is a well-defined surjective map.

Now we show that the map is injective. If \( I_w = I_{w'} \), then \([ - \otimes^L_A I(w) ] = [ - \otimes^L_A I(w') ] \) in \( \text{GL}(V) \).

By Proposition 3.17(1), the images of \( w \) and \( w' \) in \( \text{GL}(V) \) are the same. Since \( \mathcal{S}_n \rightarrow \text{GL}(V) \) is injective, we have \( w = w' \).

(2) This is immediate from (1) and Theorem 3.9(3).
(3) In the Hasse quiver of the left order on $\mathfrak{S}_n$, arrows ending at $w \in \mathfrak{S}_n$ are given by $w \to s_iw$ with $1 \leq i \leq n - 1$ satisfying $l(s_iw) = \mu_i(I(w))$. By Corollary 3.19, the Hasse quiver of $\text{tilt}\Lambda$ coincides with the opposite of the Hasse quiver of $\mathfrak{S}_n$. Thus $I$ is an anti-isomorphism by Lemma 2.14. □

Immediately we have the following corollary.

**Corollary 3.19.** For any expression $w = s_{i_1} s_{i_2} \cdots s_{i_t} \in \mathfrak{S}_n$, $I(w) = \mu_1 \mu_2 \cdots \mu_t (\Lambda)$ holds.

**Proof.** It suffices to show that, if $l(s_i w) = l(w) + 1$, then $I(s_i w) = \mu_i(I(w))$ holds. Since $I(s_i w) \not\in I(w)$ holds by Proposition 3.16(2), the assertion follows from Theorem 3.10(1)(a). □

To compare with the Hasse quiver of tilting $\Lambda$-modules, we give the Hasse quiver of the left order on the symmetric group $\mathfrak{S}_n$ for $n = 2, 3$.

**Example 3.20.** We describe the Hasse quiver of the left order on $\mathfrak{S}_2$ and $\mathfrak{S}_3$.

1. The Hasse quiver of the left order on $\mathfrak{S}_2$ is the opposite of the following quiver:

![Hasse quiver of $\mathfrak{S}_2$](image)

2. The Hasse quiver of the left order on $\mathfrak{S}_3$ is the opposite of the following quiver:

![Hasse quiver of $\mathfrak{S}_3$](image)

By Corollary 3.19, we can describe the Hasse quiver of tilting modules over the Auslander algebra $\Lambda$ of $K[x]/(x^n)$ for $n = 2, 3$.

**Example 3.21.** Denote by $\Lambda_i$ the Auslander algebra of $K[x]/(x^i)$ for $i = 2, 3$. Then we have

1. The Hasse quiver $H(\text{tilt}\Lambda_2)$ is the following:

$$\Lambda_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow{I_1} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = I_2$$

2. The Hasse quiver $H(\text{tilt}\Lambda_3)$ is the following:

$$\Lambda_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$I_1 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \xrightarrow{I_1I_2} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} = I_2$$

$$I_2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \xrightarrow{I_1I_2} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} = I_2$$

$$I_1I_2I_1 = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} = I_2I_1I_2$$
4. Support τ-tilting modules over the Auslander algebra of $K[x]/(x^n)$

Throughout this section, $\Lambda$ is the Auslander algebra of $K[x]/(x^n)$. In this section, we firstly construct a bijection from the symmetric group $\mathfrak{S}_{n+1}$ to the set $sr$-tilt $\Lambda$ of isomorphism classes of basic support τ-tilting $\Lambda$-modules, and then we show that this is an anti-isomorphism of posets. Recall that $\Lambda$ is presented by the quiver

$$
\begin{array}{cccccccc}
1 & \overset{a_1}{\longrightarrow} & 2 & \overset{a_2}{\longrightarrow} & 3 & \overset{a_3}{\longrightarrow} & \cdots & \overset{a_{n-1}}{\longrightarrow} & n - 1 & \overset{a_n}{\longrightarrow} & n \\
& b_2 & & b_3 & & b_4 & & b_{n-1} & & b_n & \\
\end{array}
$$

Let $M$ be the ideal of $\Lambda$ generated by $e_n$. Then we have $M = \bigoplus_{i=1}^n M_i$. We often use the functor

$$
\bigtriangledown := - \otimes_{\Lambda} (\Lambda/M) : \text{mod} \Lambda \rightarrow \text{mod}(\Lambda/M).
$$

For example, $\Lambda$ and $M$ in the case $n = 4$ are the following.

$$
M = \begin{bmatrix}
4 & 3 & 4 & 1 & 3 & 4 \\
3 & 4 & 1 & 2 & 3 & 4 \\
3 & 4 & 1 & 2 & 3 & 4 \\
4 & 1 & 2 & 3 & 4 & 4
\end{bmatrix} \subset \Lambda = \begin{bmatrix}
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4
\end{bmatrix}.
$$

We start with the some facts on $\mathfrak{S}_{n+1}$. As before we denote by $s_i$ the transposition $(i, i+1)$ in $\mathfrak{S}_{n+1}$ for $1 \leq i \leq n$. Now we prepare the following, which will be used later.

**Lemma 4.1.**

(1) $\mathfrak{S}_{n+1} = \coprod_{i=0}^n \{ s_{i+1} \cdots s_n w | w \in \mathfrak{S}_n \}$.

(2) Let $v \in \mathfrak{S}_n$, $1 \leq i \leq n$ and $w = s_{i+1} \cdots s_nv \in \mathfrak{S}_{n+1}$.

(a) If $j \leq i - 1$, then $s_jw = s_{i+1} \cdots s_n s_j v$.

(b) If $j \geq i + 2$, then $s_jw = s_{i+1} \cdots s_n s_{j-1}v$.

**Proof.** (1) An element $w \in \mathfrak{S}_{n+1}$ belongs to $s_{i+1} \cdots s_n \mathfrak{S}_n$ if and only if $w(n+1) = i + 1$ holds. Thus the assertion follows.

(2) (a) is clear. (b) follows from $s_jw = s_{i+1} \cdots s_{j-2} s_j s_{j-1} s_{j+1} \cdots s_nv = s_{i+1} \cdots s_{j-1} s_j s_{j-1} s_{j+1} \cdots s_nv = s_{i+1} \cdots s_n s_{j-1}v$. \(\square\)

Our first goal is to construct support τ-tilting $\Lambda$-modules by using a similar method of constructing permutations in $\mathfrak{S}_{n+1}$ given by Lemma 4.1. Note that in Theorem 3.18 we have built a bijection between $\mathfrak{S}_n$ and the set of isomorphism classes of basic tilting $\Lambda$-modules. In the following we try to construct support τ-tilting $\Lambda$-modules from tilting $\Lambda$-modules.

We need the following observations on the direct summands of a tilting module $T$.

**Lemma 4.2.** Let $T \in (I_1, \ldots, I_{n-1})$ and $T_i := e_i T$ for $1 \leq i \leq n$. For any $1 \leq i \leq n$, we have

(1) $\text{soc} T_i \cong S_n$.

(2) $T_i$ is either zero or indecomposable with a simple socle $S_{n-i}$.

(3) $T_i$ has no composition factors isomorphic to $S_n$. In particular $\text{Hom}_\Lambda(T_i, T) = 0$.

(4) If $T_i \neq 0$, then one can recover $T_i$ from $T_i$.

**Proof.** (1) Since $M_i \subset T_i \subset P_i$, we have $S_n = \text{soc} M_i \subset \text{soc} T_i \subset \text{soc} P_i = S_n$.

(2) is clear. (3) is immediate from (1). To prove (4), consider the pullback diagram

$$
\begin{array}{ccc}
P_i & \longrightarrow & \overline{P_i} \\
\downarrow & & \downarrow \ \\
T_i & \longrightarrow & \overline{T_i}
\end{array}
$$

Since $\overline{T_i} \subset \overline{P_i}$ is an injective hull as a $(\Lambda/M)$-module and $P_i \rightarrow \overline{P_i}$ is a projective cover as $\Lambda$-module, we have the assertion. \(\square\)

The following results on minimal left approximations are also needed for constructing support τ-tilting $\Lambda$-modules.
Lemma 4.3. Let $T \in (I_1, \ldots, I_{n-1})$ and $T_i := e_iT$ for $1 \leq i \leq n$.
(1) The minimal left add$(\bigoplus_{j=1}^{i-1} T_j)$-approximation of $T_i$ is given by $f_i : T_i \to T_{i-1}$, which is the left multiplication of the arrow $a_{i-1} : i-1 \to i$ in the quiver of $\Lambda$. In this case, $f_i(M_i) = M_{i-1}$.
(2) The minimal left add$(\bigoplus_{j=i+1}^{n-1} T_j)$-approximation of $T_i$ is given by $g_i : T_i \to T_{i+1}$, which is the left multiplication of the arrow $b_{i+1} : i+1 \to i$ in the quiver of $\Lambda$. This is a monomorphism.

Proof. (1) Since the left multiplication gives an isomorphism $\Lambda \cong \text{End}_\Lambda(T)$, we have an equivalence $\text{Hom}_\Lambda(T, -) : \text{add} T \cong \text{add}(\Lambda)$. The minimal left $\text{add}(\bigoplus_{j=1}^{i-1} e_j \Lambda)$-approximation of $e_i \Lambda$ is $e_i \Lambda \to e_{i-1} \Lambda$, which is given by the left multiplication of $a_{i-1}$. Thus the former assertion follows. The latter assertion follows from $f_i(M_i) = a_{i-1}M_i = M_{i-1}$.

(2) One can prove the first assertion similarly to (1). Since the left multiplication of $b_{i+1}$ gives a monomorphism $P_i \to P_{i+1}$, its restriction $g_i$ is also a monomorphism. \[\square\]

For a tilting $\Lambda$-module $T$, we consider the support $\tau$-tilting $\Lambda$-module
$$\mu_{[i+1, n]}(T) := \mu_{i+1} \mu_{i+2} \cdots \mu_{n}(T)$$
obtained by a successive mutation. The following result plays a crucial role.

Proposition 4.4. Let $T \in (I_1, \ldots, I_{n-1})$ and $T_j := e_jT$ for $1 \leq j \leq n$. For $0 \leq i \leq n$, we have
(1) $\mu_{[i+1, n]}(T) = \bigoplus_{j=1}^{i} T_j \oplus \bigoplus_{j=i+1}^{n-1} T_j$.
(2) $T > \mu_{n}(T) > \mu_{[n-1, n]}(T) > \cdots > \mu_{[1, n]}(T)$.

Proof. (1) We prove the assertion by descending induction on $i$. It is clear for $i = n$.

Now we assume that $\mu_{[i+1, n]}(T)$ is $\bigoplus_{j=1}^{i} T_j \oplus \bigoplus_{j=i+1}^{n-1} T_j$. In the following we calculate $\mu_{[i, n]}(T)$ by applying Theorem 2.16.

Firstly, we show that $T_i \not\in \text{Fac}(\bigoplus_{j=1}^{i-1} T_j \oplus \bigoplus_{j=i+1}^{n-1} T_j)$. By Lemma 4.2(3), we have $\text{Hom}_\Lambda(T_j, T_i) = 0$. Thus we only have to show $T_i \not\in \text{Fac}(\bigoplus_{j=1}^{i-1} T_j)$. Otherwise, since $TM = M$ holds as ideals of $\Lambda$, we have $e_i M = T_i M \in \text{Fac}(\bigoplus_{j=1}^{i-1} T_j M) = \text{Fac}(\bigoplus_{j=1}^{i-1} e_j M)$. This is impossible by the explicit form of $M$. Thus the assertion follows.

Next, by Lemma 4.3(1) and the fact that $\pi_i$ is a left $(\text{mod} \Lambda/M)$-approximation of $T_i$, a left $\text{add}(\bigoplus_{j=1}^{i-1} T_j \oplus \bigoplus_{j=i+1}^{n-1} T_j)$-approximation of $T_i$ is given by $f := (f_i) : T_i \to T_{i-1} \oplus T_i$.

Finally, we have a commutative diagram of exact sequences
$$\begin{array}{cccccc}
0 & \longrightarrow & M_i & \longrightarrow & T_i & \longrightarrow & T_{i-1} & \longrightarrow & 0 \\
& & \downarrow{f_i} & & \downarrow{\pi_i} & & \downarrow{\text{Coker} f} & & 0, \\
& & M_i & \longrightarrow & T_{i-1} & \longrightarrow & \text{Coker} f & & 0,
\end{array}$$
we have $\text{Coker} f = T_{i-1}/f_i(M_i) = T_{i-1}$ by Lemma 4.3(1). This is indecomposable by Lemma 4.2(2), and we have $\mu_{[i, n]}(T) = \bigoplus_{j=1}^{i-1} T_j \oplus \bigoplus_{j=i+1}^{n-1} T_j$ by Theorem 2.16. Thus the assertion follows.

(2) By the proof of (1) we get $\mu_{[i, n]}(T)$ is a left mutation of $\mu_{[i+1, n]}(T)$, and hence the assertion holds. \[\square\]

Now we give an example of calculation given in Proposition 4.3.
Example 4.5. Let \( \Lambda \) be the Auslander algebra of \( K[x]/(x^4) \). Taking the trivial tilting module \( \Lambda \), then \( \mu_4(\Lambda) \), \( \mu_3(\Lambda) \), \( \mu_2(\Lambda) \) and \( \mu_1(\Lambda) \) is given as follows.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{bmatrix}
\mu_4
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{bmatrix}
\mu_3
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{bmatrix}
\mu_2
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{bmatrix}
\mu_1
\]

We denote by \( \mu_{i+1,n}(\text{tilt} \Lambda) \) the set of isomorphism classes of support tilting \( \Lambda \)-modules consisting of \( \mu_{i+1,n}(T) \) for any \( T \in \text{tilt} \Lambda \) for \( 0 \leq i \leq n \). Then we have the following proposition.

Lemma 4.6. Let \( \Lambda \) be as above. Then

1. There is a bijection \( \text{tilt} \Lambda \rightarrow \mu_{i+1,n}(\text{tilt} \Lambda) \) via \( T \rightarrow \mu_{i+1,n}(T) \) for any \( 0 \leq i \leq n \).
2. We have \( \mu_{i+1,n}(\text{tilt} \Lambda) \cap \mu_{j+1,n}(\text{tilt} \Lambda) = 0 \) for any \( 0 \leq i \neq j \leq n \).

Proof. (1) This is clear since each \( \mu_i : \text{sr-tilt} \Lambda \rightarrow \text{sr-tilt} \Lambda \) is a bijection.

(2) By Proposition 4.4 and Lemma 4.2(1)(3), the first \( i \) direct summands of \( \mu_{i+1,n}(T) \) have a composition factor \( S_n \), and the other summands do not have a composition factor \( S_n \). Thus the assertion follows.

We have the following relations of mutation in \( \text{sr-tilt} \Lambda \) corresponding to Lemma 4.4(2).

Proposition 4.7. Let \( T \in \{ I_1, \ldots, I_{n-1} \} \), \( 0 \leq i \leq n \) and \( U := \mu_{i+1,n}(T) \).

1. For any \( 1 \leq k \leq i - 1 \), we have \( \mu_k(U) = \mu_{i+1,n}((\mu_k(T))) \). Moreover, \( T > \mu_k(T) \) if and only if \( U > \mu_k(U) \).
2. For any \( i + 2 \leq k \leq n \), we have \( \mu_k(U) = \mu_{i+1,n}((\mu_{k-1}(T))) \). Moreover, \( T > \mu_{k-1}(T) \) if and only if \( U > \mu_k(U) \).
3. We have

\[
\mu_k \mu_{i+1,n}(T) = \begin{cases}
\mu_{i+1,n} & k \leq i - 1, \\
\mu_{i,n} & k = i, \\
\mu_{i+2,n} & k = i + 1, \\
\mu_{i+1,n} & k \geq i + 2.
\end{cases}
\]

Proof. By Proposition 4.4, we have \( U = \bigoplus_{j=1}^{k-1} T_j \oplus \bigoplus_{j=k+1}^{n} T_j \).

(1) Let \( V := \mu_k(T) = \bigoplus_{j=1}^{k-1} T_j \oplus T_k \oplus \bigoplus_{j=k+1}^{n} T_j \). Then \( V \) is a tilting \( \Lambda \)-module with \( T_k \neq T_k \).

(2) Let \( V := \mu_{k-1}(T) = \bigoplus_{j=1}^{k-2} T_j \oplus T_{k-1} \oplus \bigoplus_{j=k}^{n} T_j \). Then \( V \) is a tilting \( \Lambda \)-module with \( T_{k-1} \neq T_{k-1} \).

To prove the later one, it suffices to show that \( T > \mu_k(T) \) implies \( U > \mu_k(U) \). The condition \( T > \mu_k(T) \) is equivalent to \( T_k \notin \text{Fac}(T/T_k) \). Since \( U/U_k \) belongs to \( \text{Fac}(T/T_k) \) by the explicit form in Proposition 4.4, we have \( U_k = T_k \notin \text{Fac}(U/U_k) \). Therefore \( U > \mu_k(U) \).

To show the later one, it suffices to show that \( T < \mu_{k-1}(T) \) implies \( U < \mu_k(U) \). The condition \( T < \mu_{k-1}(T) \) is equivalent to \( T_{k-1} \in \text{Fac}(T/T_{k-1}) \). Since \( T/T_{k-1} \) belongs to \( \text{Fac}(U/U_k) \) by the
explicit form in Proposition 4.4, we have \( U_k = T_{k-1} \in \text{Fac}(T/T_{k-1}) \subset \text{Fac}(U/U_k) \). Therefore \( U < \mu_k(U) \).

(3) Immediate from (1) and (2).

Immediately we have the following complete classification of support \( \tau \)-tilting \( \Lambda \)-modules and indecomposable \( \tau \)-rigid \( \Lambda \)-modules.

**Theorem 4.8.** (1) We have \( sr\mathrm{-tilt} \Lambda = \bigcup_{i=0}^{n} \mu_{[i+1,n]}(\text{tilt} \Lambda) \). In particular, \( \#sr\mathrm{-tilt} \Lambda = (n+1)! \).

(2) Any support \( \tau \)-tilting \( \Lambda \)-module has a form \( T_0 \oplus \cdots \oplus T_i \oplus T_{i+1} \oplus \cdots \oplus T_{n-1} \) for some \( 0 \leq i \leq n \) and \( T \in (I_1, \ldots, I_{n-1}) \) with \( T_j := e_j T \) for \( 1 \leq j \leq n \). Moreover such \( i \) and \( T \) are uniquely determined.

(3) Any indecomposable \( \tau \)-rigid module has a form \( T_i = e_i T \) or \( T_i \) for some \( T \in (I_1, \ldots, I_{n-1}) \) and \( 1 \leq i \leq n \).

(4) The statements (1) and (2) hold for \( \Lambda^{op} \)-modules.

**Proof.** (1) By Lemma 4.6 \( \bigcup_{i=0}^{n} \mu_{[i+1,n]}(\text{tilt} \Lambda) \) is a disjoint union and contains precisely \( (n+1)! \) elements. By Proposition 4.4(3), \( \bigcup_{i=0}^{n} \mu_{[i+1,n]}(\text{tilt} \Lambda) \) is closed under mutation. This is a finite connected component of \( \text{H}(sr\mathrm{-tilt} \Lambda) \) since \( \#\text{tilt} \Lambda = n! \). By Proposition 2.18 we have \( sr\mathrm{-tilt} \Lambda = \bigcup_{i=0}^{n} \mu_{[i+1,n]}(\text{tilt} \Lambda) \).

(2) is clear by (1) and Proposition 4.3 (3) is a straight result of (2) and Lemma 2.12.

The following lemma is also needed.

**Lemma 4.9.** Let \( U \in sr\mathrm{-tilt} \Lambda \) and \( 1 \leq j, k \leq n \).

(1) \( \mu_j \mu_k(U) = U \).

(2) If \( |j - k| \geq 2 \), then \( \mu_j \mu_k(U) = \mu_k \mu_j(U) \).

(3) If \( |j - k| = 1 \), then \( \mu_j \mu_k(U) = \mu_k \mu_j(U) \).

**Proof.** (1) is clear from the definition of mutation.

By Theorem 1.1, we can assume that \( U = \mu_{[i+1,n]}(T) \) for some \( 0 \leq i \leq n \) and \( T \in (I_1, \ldots, I_{n-1}) \). In the both proofs we use Proposition 4.7(3) and Proposition 3.5 frequently.

(2) Without loss of generality, we assume \( k < j \). We divide the proof into seven cases.

(a) If \( k < j \leq i - 1 \), then \( \mu_j \mu_k(U) = \mu_j \mu_k \mu_{[i+1,n]}(T) = \mu_j \mu_{[i+1,n]}(T) = \mu_{[i+1,n]} \mu_j \mu_k(T) = \mu_{[i+1,n]} \mu_k \mu_j(T) = \mu_k \mu_j \mu_{[i+1,n]}(T) = \mu_k \mu_j(T) \).

(b) If \( i + 2 \leq k < j \), then the proof is very similar to (a).

(c) If \( k \leq i - 1 < i + 2 \leq j \), then \( \mu_j \mu_k(U) = \mu_j \mu_k \mu_{[i+1,n]}(T) = \mu_j \mu_{[i+1,n]} \mu_k(T) = \mu_{[i+1,n]} \mu_j \mu_k(T) = \mu_{[i+1,n]} \mu_k \mu_j(T) = \mu_k \mu_j \mu_{[i+1,n]}(T) = \mu_k \mu_j(T) \).

(d) The case \( k = i + 1 < i + 2 = j \), then \( \mu_j \mu_k(U) = \mu_j \mu_k \mu_{[i+1,n]}(T) = \mu_j \mu_{[i+1,n]}(T) = \mu_j \mu_k(T) \).

(e) If \( k \leq i - 2 < i = j \), then the proof is very similar to (d).

(f) If \( k \leq i - 1 < i + 1 = j \), then \( \mu_j \mu_k(U) = \mu_j \mu_k \mu_{[i+1,n]}(T) = \mu_{i+1} \mu_{[i+1,n]}(T) = \mu_k \mu_j \mu_{[i+1,n]}(T) = \mu_k \mu_j(T) \).

(g) If \( k = i + 1 < i + 3 \leq j \), then the proof is very similar to (d).

(3) Without loss of generality, we assume \( k = j + 1 \). We also divide the proof into five cases.

(a) If \( j \leq i - 2 \), then \( \mu_j \mu_k \mu_j \mu_{[i+1,n]}(T) = \mu_j \mu_{[i+1,n]}(T) = \mu_k \mu_j(T) = \mu_k \mu_j \mu_{[i+1,n]}(T) = \mu_k \mu_j(T) \).

(b) If \( j \geq i + 2 \), then the proof is very similar to (a).

(c) If \( j = i - 1 \), then \( \mu_{i-1} \mu_{i-1} \mu_{i-1}(T) = \mu_{i-1} \mu_{i-1} \mu_{[i+1,n]}(T) = \mu_{i-1} \mu_{[i+1,n]} \mu_{i-1}(T) = \mu_{i-1} \mu_{i-1} \mu_{i-1}(T) = \mu_{i-1} \mu_{i-1}(T) \).

(d) If \( j = i \) or \( j = i + 1 \), then the proof is very similar to (c).

Now we are in a position to state one of the main results of this section.

**Theorem 4.10.** Let \( \Lambda \) be the Auslander algebra of \( K[x]/(x^n) \). Then

(1) There exists a bijection \( I: \mathfrak{S}_{n+1} \cong sr\mathrm{-tilt} \Lambda \) which maps \( w \) to \( I(w) = \mu_{i_1} \mu_{i_2} \cdots \mu_{i_t}(\Lambda) \), where \( w = s_{i_1} s_{i_2} \cdots s_{i_t} \) is an arbitrary (not necessarily reduced) expression.
(2) The statements (1) holds for $\Lambda^{op}$-modules.

Proof. (1) Proposition 4.9 and the same argument as in the proof of Theorem 3.18 shows that the map $I$ is well-defined. By Theorem 4.8 we have $\#s\tau$-tilt $\Lambda = (n+1)! = \#\mathfrak{S}_{n+1}$. Thus we only have to show that $I$ is surjective.

By Theorem 4.8 any $U \in s\tau$-tilt $\Lambda$ is written as $\mu_{[i+1,n]}(T)$ for some $T \in \text{tilt} \Lambda$ and $0 \leq i \leq n+1$. By Corollary 3.19, there exists $w \in \mathfrak{S}_n$ such that $T = I(w)$. Then we have $I(s_{i+1} \cdots s_n w) = \mu_{[i+1,n]}(T) = U$. Thus the assertion follows.

(2) We only have to replace $\Lambda$-modules with $\Lambda^{op}$-modules in the proof. □

Our second goal in this section is to show that the map $I$ in Theorem 4.10 is an anti-isomorphism of posets. For this aim, we need the following result.

Proposition 4.11. For $w \in \mathfrak{S}_{n+1}$ and $1 \leq j \leq n$, $l(s_j w) > l(w)$ if and only if $I(s_j w) < I(w)$.

Proof. It suffices to show that $l(s_j w) > l(w)$ implies that $I(s_j w) < I(w)$ by replacing $s_j w$ with $w$ if necessary. Write $w = s_{i+1} \cdots s_n v$ with $0 \leq i \leq n$ and $v \in \mathfrak{S}_n$. Then $l(w) = n - i + l(v)$ and $l(s_j w) = n - i + l(v) + 1$ hold by our assumption. We prove the assertion by comparing $i$ with $j$.

(a) Assume $j \leq i - 1$. By Proposition 4.7(3), we have $I(s_j w) = \mu_j \mu_{i+1,n}(I(v)) = \mu_{i+1,n}\mu_j(I(v)) = \mu_{i+1,n}(I(s_j v))$. Since $s_j w = s_{i+1} \cdots s_n s_j v$ holds, we have $n - i + l(v) + 1 = l(s_j w) \leq n - i + l(s_j v)$ and hence $l(v) + 1 = l(s_j v)$. Then by Theorem 3.18 one has $I(s_j v) < I(v)$, which implies by Proposition 4.7(1) that $I(s_j w) = \mu_{i+1,n}(I(s_j v)) < \mu_{i+1,n}(I(v)) = I(w)$.

(b) Assume $j \geq i + 2$. We have $I(s_j w) = \mu_j \mu_{i+1,n}(I(v)) = \mu_{i+1,n}\mu_j(I(v)) = \mu_{i+1,n}(I(s_j v))$ by Proposition 4.7(3). Since $s_j w = s_{i+1} \cdots s_n s_{j-1} v$ holds by Lemma 4.1(2), we have $n - i + l(v) + 1 = l(s_j w) \leq n - i + l(s_{j-1} v)$ and hence $l(v) + 1 = l(s_{j-1} v)$. Then by Theorem 3.18 one has $I(s_{j-1} v) < I(v)$, which implies by Proposition 4.7(2) that $I(s_j w) = \mu_{i+1,n}(I(s_{j-1} v)) < \mu_{i+1,n}(I(v)) = I(w)$.

(c) Assume $j = i$. By Proposition 4.7(3), we have $I(s_j v) = \mu_{i+1,n}(I(v)) = \mu_i(I(v)) < \mu_{i+1,n}(I(v)) = I(w)$ by Proposition 4.7(2).

(d) The case $j = i + 1$ does not occur. In fact $s_j w = s_{i+2} \cdots s_n v$ implies $l(s_j w) = l(w) - 1$, a contradiction. □

Now we are ready to show the main result on the anti-isomorphisms of posets.

Theorem 4.12. Let $\Lambda$ and $I$ be as in Theorem 4.10. Then $I : \mathfrak{S}_{n+1} \rightarrow s\tau$-tilt $\Lambda$ is an anti-isomorphism of posets, that is, $w_1 \leq w_2$ in $\mathfrak{S}_{n+1}$ if and only if $I(w_1) \geq I(w_2)$ in $s\tau$-tilt $\Lambda$.

Proof. The proof is very similar to the proof of Theorem 3.18(3), we only have to use Proposition 4.11 instead of Proposition 3.17(3). □

To compare with the Hasse quiver of support $\tau$-tilting $\Lambda$-modules, we give the Hasse quiver of the left order on the symmetric group $\mathfrak{S}_n$ for $n = 4$.

Example 4.13. We describe the Hasse quiver of the left order on $\mathfrak{S}_4$. 

CLASSIFYING $\tau$-TILTING MODULES OVER THE AUSLANDER ALGEBRA OF $K[x]/(x^n)$
The Hasse quiver of the left order on $S_4$ is the opposite of the following quiver:

By Theorem 4.14, we now give the Hasse quiver of support $\tau$-tilting modules of the Auslander algebra of $K[x]/(x^n)$ for $n = 2, 3$.

**Example 4.14.** Denote by $\Lambda_i$ the Auslander algebra of $K[x]/(x^i)$ for $i = 2, 3$. Then

1. The Hasse quiver $H(\tau$-tilt$\Lambda_2)$ is of the following form:

2. The Hasse quiver $H(\tau$-tilt$\Lambda_3)$ is of the following form:
CLASSIFYING $\tau$-TILTING MODULES OVER THE AUSLANDER ALGEBRA OF $K[x]/(x^n)$

Let $\Lambda$ be the Auslander algebra of $K[x]/(x^n)$ and $\Gamma$ the preprojective algebra of Dynkin type $A_n$. Thus $\Lambda$ is presented by the quiver

\[
\begin{array}{c}
1 & \overset{a_1}{\rightarrow} & 2 & \overset{a_2}{\rightarrow} & 3 & \cdots & \overset{a_{n-1}}{\rightarrow} & n - 1 & \overset{a_n}{\rightarrow} & n \\
\end{array}
\]

with relations $a_1 b_2 = 0$ and $a_i b_{i+1} = b_i a_{i-1}$ for any $2 \leq i \leq n - 1$, and $\Gamma$ is presented by the same quiver with one additional relation $b_n a_{n-1}$. Thus we have $\Gamma = \Lambda / L$ for the ideal $L$ of $\Lambda$ generated by $b_n a_{n-1}$. Then we have $L = \bigoplus_{i=1}^{n} L_i$ for $L_i := e_i L$. For example, $\Lambda$ and $L$ in the case $n = 4$ is the following.

\[
L = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c}
\begin{array}{c|c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 3 & 4 & 2 & 3 & 4 & 3 \\
\end{array}
\end{array} \right] \subseteq \Lambda = \left[ \begin{array}{c|c|c|c|c|c|c|c|c|c}
\begin{array}{c|c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 3 & 4 & 2 & 3 & 4 & 3 \\
\end{array}
\end{array} \right]
\]

Our aim in this section is to apply Theorems 4.10 and 4.12 to $\Gamma$ and prove that the tensor functor $- \otimes_{\Lambda} \Gamma : \text{mod}\Lambda \to \text{mod}\Gamma$ induces a bijection from $\text{sr-tilt}\Lambda$ to $\text{sr-tilt}\Gamma$. In particular we can get Mizuno’s bijection from the symmetric group $S_{n+1}$ to $\text{sr-tilt}\Gamma$.

Let us start with the following general properties of support $\tau$-tilting modules over an algebra $A$ and its factor algebra $B$ (see [IRRT]).

**Proposition 5.1.** Let $A$ be an algebra and let $B$ be a factor algebra of $A$.

1. If $M$ is a $\tau$-rigid $A$-module, then $M \otimes_{A} B$ is a $\tau$-rigid $B$-module.
2. If $M$ is a support $\tau$-tilting $A$-module, then $M \otimes_{A} B$ is a support $\tau$-tilting $B$-module. Thus we have a map $- \otimes_{A} B : \text{sr-tilt} A \to \text{sr-tilt} B$, which preserves the generation order.
(3) The map in (2) is surjective if $A$ is $\tau$-rigid finite.

Note that $M \otimes \Lambda B$ is not necessarily basic even if $M$ is basic $\tau$-rigid.

We need the following facts.

**Lemma 5.2.** Let $T \in \{I_1, \ldots, I_{n-1}\}$ and $T_i := e_iT$ for $1 \leq i \leq n$. For any $1 \leq i \leq n$, we have

1. $LM = L = ML$ and $T_iL = L_i$.
2. $T_iL_i$ is indecomposable with a simple socle $S_{n-i+1}$.
3. One can recover $T_i$ from $T_iL_i$.

**Proof.** (1) This is clear. (2) Since $M_i \subset T_i \subset P_i$, we have $L_i = M_iL \subset T_iL \subset P_iL = L_i$. The socle of $T_iL_i \leq P_iL_i$ is $S_{n-i+1}$. (3) One can prove in a similar method with Lemma 4.2(4). \qed

Now we can state our main result of this section.

**Theorem 5.3.** Let $\Lambda$ be the Auslander algebra of $K[x]/(x^n)$ and $\Gamma$ the preprojective algebra of Dynkin type $A_n$.

1. The map $- \otimes \Lambda \Gamma : \tau\text{-}\text{tilt}\Lambda \to \tau\text{-}\text{tilt}\Gamma \circ\circ\circ U \to U \otimes \Lambda \Gamma$ is bijective.
2. The map in (1) is an isomorphism of posets.
3. If $X$ is an indecomposable $\tau$-rigid $\Lambda$-module, then $X \otimes \Lambda \Gamma$ is an indecomposable $\Gamma$-module.

**Proof.** (1) For any $U \in \tau\text{-}\text{tilt}\Lambda$, there exists $T \in \{I_1, \ldots, I_{n-1}\}$ and $0 \leq i \leq n$ such that $U = \mu_{[i+1,n]}(T) = T_1 \oplus \cdots \oplus T_i \oplus T_{i+1} \oplus \cdots \oplus T_n$ by Theorem 4.8. In this case, we have

$$U \otimes \Lambda \Gamma = \left\{ \begin{array}{ll} (T_1/L_1) \oplus \cdots \oplus (T_i/L_i) \oplus T_{i+1} \oplus \cdots \oplus T_n & \text{if } i \geq 1, \\ 0 \oplus T_{i+1} \oplus \cdots \oplus T_n & \text{if } i = 0. \end{array} \right.$$ 

For any $1 \leq j \leq n$, $T_j$ does not have $S_n$ a composition factor, and $T_j/L_j$ has $S_n$ as a composition factor. Therefore the integer $i$ can be recovered from $U$ as the number of indecomposable direct summands of $U$ that have $S_n$ as a composition factor. Moreover, by Lemmas 5.2(2) and 4.2(2), the socle of the $j$-th direct summand of $U \otimes \Lambda \Gamma$ is $S_{n-j+1}$ if $1 \leq j \leq i$, and either 0 or $S_{n-j+1}$ if $i+1 \leq j \leq n$.

Now assume that another $U' \in \tau\text{-}\text{tilt}\Lambda$ satisfies $U \otimes \Lambda \Gamma \cong U' \otimes \Lambda \Gamma$, and take $T' \in \{I_1, \ldots, I_{n-1}\}$ and $1 \leq i' \leq n$ such that $U' = \mu_{[i'+1,n]}(T')$. By the argument above, we have $i = i'$. By looking at the socle of each indecomposable direct summand, we have $T_j/L_j \cong T_j'/L_j'$ for any $1 \leq j \leq i$ and $T_j \cong T_j'$ for any $1 \leq j \leq n-1$. They imply $T_j \cong T_j'$ for any $1 \leq j \leq n-1$ by Lemmas 5.2(3) and 4.2(4). Since $T_n = P_n = T'_n$, we have $T \cong T'$ and hence $U = \mu_{[i+1,n]}(T) \cong \mu_{[i'+1,n]}(T') = U'$.

(3) By Theorem 4.8, $X$ has a form $T_1 \otimes \cdots \otimes T_n$ for some $T \in \{I_1, \ldots, I_{n-1}\}$ and $1 \leq i \leq n$. Since $T_i \otimes \Lambda \Gamma = T_i/L_i$ and $T_i \otimes \Lambda \Gamma = T_i$ are indecomposable by Lemmas 5.2(2) and 4.2(2), the assertion follows.

(2) The map $- \otimes \Lambda \Gamma$ preserves the mutation. In fact, if $U = \mu_i(T)$, then $U \otimes \Lambda \Gamma$ and $T \otimes \Lambda \Gamma$ have the same indecomposable direct summands except the $i$-th summand by (3) and the injectivity of $- \otimes \Lambda \Gamma : \tau\text{-}\text{tilt}\Lambda \to \tau\text{-}\text{tilt}\Gamma$. Therefore we have $U \otimes \Lambda \Gamma = \mu_i(T \otimes \Lambda \Gamma)$.

In particular, $- \otimes \Lambda \Gamma$ gives an isomorphism $H(\tau\text{-}\text{tilt}\Lambda) \to H(\tau\text{-}\text{tilt}\Gamma)$ of Hasse quivers. Thus $- \otimes \Lambda \Gamma : \tau\text{-}\text{tilt}\Lambda \to \tau\text{-}\text{tilt}\Gamma$ is an isomorphism of posets by Lemma 2.14 \qed

**Remark 5.4.** Theorem 5.3 gives another proof of Mizuno’s result [M, Theorem 2.21].

On the other hand, we can give another shorter proof by using Mizuno’s result [M, Theorem 2.21]. By Proposition 5.3(3), we have a surjective map $- \otimes \Lambda \Gamma : \tau\text{-}\text{tilt}\Lambda \to \tau\text{-}\text{tilt}\Gamma$. This must be surjective since we know $\#\tau\text{-}\text{tilt}\Lambda = (n+1)! = \#\tau\text{-}\text{tilt}\Gamma$ by Theorem 4.10 and Mizuno’s result.

As a corollary, we get the following.

**Corollary 5.5.** Let $\Lambda$ be the Auslander algebra of $K[x]/(x^n)$ and $\Gamma$ a preprojective algebra of Dynkin type $A_n$. There are isomorphisms between the following posets:

1. The poset $\tau\text{-}\text{tilt}\Lambda$ with the generation order.
The poset $sr$-tilt $\Gamma$ with the generation order.

(3) The symmetric group $\mathfrak{S}_{n+1}$ with the left order.

(4) The poset $sr$-tilt $(\Lambda^{op})$ with the opposite of the generation order.

(5) The poset $sr$-tilt $(\Gamma^{op})$ with the opposite of the generation order.

(6) The symmetric group $\mathfrak{S}_{n+1}$ with the opposite of the right order.

Proof. The isomorphism from (1) to (2) given by $- \otimes A \Gamma$ is showed in Theorem 5.3. The isomorphism from (3) to (1) given by $I$ is showed in Theorems 4.10 and 4.12. The isomorphism between (1) and (4) (resp. (2) and (5)) is given in [AIR].

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