Radiation Reaction of Charged Particles Orbiting a Magnetized Schwarzschild Black Hole

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Abstract

In many astrophysically relevant situations, radiation-reaction forces acting upon a charge cannot be ignored, and the question of the location and stability of circular orbits in such a regime arises. The motion of a point charge with radiation reaction in flat spacetime is described by the Lorenz–Dirac (LD) equation, while in curved spacetime it is described by the DeWitt–Brehme (DWB) equation containing the Ricci term and a tail term. We show that for the motion of elementary particles in vacuum metrics, the DWB equation can be reduced to the covariant form of the LD equation, which we use here. Generically, the LD equation is plagued by runaway solutions, so we discuss computational ways of avoiding this problem when constructing numerical solutions. We also use the first iteration of the covariant LD equation, which is the covariant Landau–Lifshitz equation, comparing the results of these two approaches and showing the smallness of the third-order Schott term in the ultrarelativistic case. We calculate the corresponding energy and angular momentum loss of a particle and study the damping of charged particle oscillations around an equilibrium radius. We find that, depending on the orientation of the Lorentz force, the oscillating charged particle either spirals down to the black hole or stabilizes the circular orbit by decaying its oscillations. The latter case leads to the interesting new result of the particle orbit shifting outwards from the black hole. We also discuss the astrophysical relevance of the presented approach and provide estimates of the main parameters of the model.

Key words: accretion, accretion disks – black hole physics – magnetic fields – radiation mechanisms: non-thermal – relativistic processes

1. Introduction

The synchrotron radiation (SR) emitted by a charged particle leads to the appearance of a back-reaction force, which can significantly affect the particle’s motion. The purpose of this paper is to study the motion of charged particles in the combined magnetic and gravitational fields, taking into account the radiation-reaction force. There is convincing evidence that magnetic fields are indeed present in the vicinity of black holes. Observations of the Galactic center (Eatough et al. 2013) have demonstrated the existence of strong magnetic fields of hundreds of Gauss in the vicinity of the supermassive black hole (SMBH) at the Galactic center. Recent studies of quasiperiodic oscillations (QPOs) observed in black hole microquasars have shown possible signatures of Galactic magnetic fields in the vicinity of three microquasars (Kološ et al. 2017), if “magnetic” generalizations of the geodesic models of twin high-frequency (HF) QPOs (Stuchlík et al. 2013) are applied to the data. HF QPOs and relativistic jets observed in microquasars indicate the presence of an external magnetic field, influencing both oscillations in the accretion disk and the creation of jets. The presence of a radiation-reaction force and its relevance in the vicinity of black holes can make sufficient contributions to the shifts in QPO frequencies, which are usually observed. Moreover, the radiation reaction can support the accretion of charged particles from the accretion disk towards the black hole. Depending on the magnitude of the external magnetic field, the radiation-reaction force can considerably shift the stable orbits of a particle, which can sufficiently influence the predictions of black hole parameters.

The weakness of the magnetic field in our study is understood in the sense that it does not perturb the spacetime metric; the corresponding condition on the field magnitude $B$ in the vicinity of a Schwarzschild black hole of mass $M$ reads (Gal’tsov & Petukhov 1978)

$$B \ll B_G = \frac{e^4}{G^{3/2}M_\odot \left( \frac{M}{M_\odot} \right)} \sim 10^{13} \frac{M_\odot}{M} \text{ Gauss.}$$

(1)

This weakness is compensated by the large ratio $e/m$ for electrons and protons, whose motion will be essentially affected by magnetic fields already of the order of a few Gauss.

The study of particle motion and electrodynamics in magnetized black holes began long ago—see, e.g., Gal’tsov & Petukhov (1978) and Blandford & Znajek (1977); for a review of early results, see Aliev & Gal’tsov (1989). Equatorial orbits are of primary interest in the theory of accretion disks. It has been shown that the innermost stable circular orbits (ISCOs) in the field of a magnetized black hole are shifted towards the horizon (Gal’tsov & Petukhov 1978) for a suitable direction of rotation. Recently, detailed studies of the equatorial motion of charged particles in weakly magnetized Schwarzschild and Kerr black holes were performed in Frolov & Shoom (2010), Kološ et al. (2015), and Tursunov et al. (2016), revealing a number of new types of motion.

Estimates show that in many physically interesting cases, the radiation-reaction force is non-negligible, and this has prompted us to consider what happens once the reaction force is taken into account. The general problem of SR and the treatment of the radiation-reaction force in curved spacetime have been widely studied in the literature. Below, we recall a few of them. The electromagnetic radiation emitted by
relativistic particles in the presence of strong gravity has been studied, e.g., in Poisson (2004), Johnston et al. (1973), Zerilli (2000), and Price et al. (2013). The problem of SR in curved spacetime has been studied in Sokolov et al. (1983, 1978) using covariantization of the flat-space results. Actually, we will use here the same approach to describe the reaction force. For the recent treatment of radiation reaction in flat space, see Gal’tsov & Spirin (2006), where the derivation of the Shott term was given explicitly along the lines of the Teitelboim idea to associate it with the Coulomb part of the electromagnetic field of a point charge. It is worth noting that the problem of the Shott contribution to the reaction force (the third derivative term) still remains a source of discussion (see, e.g., Poisson 2004), so we will touch on it in this paper, too.

The generalization of the radiation-reaction equation to curved space was given by DeWitt & Brehme (1960). Their result (corrected by Hobbs 1968) shows that radiation reaction in curved spacetime is essentially non-local and appears to solve the integral equation. But, contrary to the gravitational radiation reaction in collisions of black holes, electromagnetic radiation from point particles can still be simplified to a local theory.

Recently, the generalization of the radiation-reaction equation to higher dimensions was discussed in Gal’tsov (2002) and Gal’tsov & Spirin (2007), which may be relevant to extra-dimensional models. Radiation from hypothetical massless charges was considered in Gal’tsov (2015), shedding new light on the relationship between the ultrarelativistic and the massless limits in SR and the radiation-reaction equation. The motion of point particles with scalar, electric, and mass charges has been reviewed in Poisson (2004). The problem of radiation reaction for extended charged particles has been studied in Cremonini & Tessarotto (2011); its quantum aspects have been discussed in Cremonini & Tessarotto (2015). A statistical treatment of the radiation-reaction problem is given in Cremonini & Tessarotto (2013). The problem of gravitational self-force of test particles is summarized in Barack (2014). Nevertheless, despite the active interest in this topic and a comprehensive literature review, successful attempts to integrate the equations of motion in the curved background combined with external electromagnetic fields are quite rare.

In spite of the weakness of the radiation-reaction force in ordinary stellar astrophysical situations, it was recently found that in high-energy plasma processes in pulsar magnetospheres, black hole accretion disks, hot accretion and active galactic nuclei, relativistic jets, and so on, which are dominated by magnetic reconnection in plasma, radiation reaction may play a crucial role. The new theory, named radiative magnetic reconnection (for a recent review, see Uzdensky 2016), predicts noticeable radiative effects on astrophysical processes such as reconnection-reaction limits on particle acceleration, radiative cooling, radiative resistivity, the braking of reconnection outflows by radiation drag, radiation pressure, viscosity, and even pair creation at highest energy densities. Radiative reconnection theory is based on the flat-space description of radiation reaction adapted to phenomena in strong gravitational fields, which is well justified in dense plasmas.

However, in the case of diluted media, when individual particle motion is relevant, it seems more adequate to use a description of radiation reaction in curved spacetime from the very beginning. This is the main goal of the present paper. Here, we would like both to discuss conceptual problems of such a description and to analyze numerically the modification of particle motion within a simple model of a weakly magnetized Schwarzschild black hole. Classifying the motion of charged particles by the given set of parameters, we find explicit trajectories for each class of orbits and study the evolutions of the particle energy, angular momentum, etc. The motion of charged particles in the combined magnetic and gravitational fields without radiation reaction reveals interesting new types of trajectories. We intend to apply the full machinery of the curved spacetime radiation-reaction theory to these problems. We will also discuss possible applications of our results in some astrophysical scenarios.

This paper is organized as follows. In Section 2, we test the dynamical equations with radiation-reaction force in flat spacetime using two main approaches and compare the results. In Section 3, we present the properties of the circular motion of non-radiating charged particles in a weakly magnetized Schwarzschild black hole, focusing our attention on the bounded orbits and charged particle oscillations. In Section 4, we give the general relativistic treatment of the radiation-reaction force in curved spacetime, and under reasonable assumptions we give the explicit form of the equations of motion of a charged particle in the vicinity of a Schwarzschild black hole immersed in an external asymptotically uniform magnetic field. In Section 5, we analyze the trajectories of charged particles for different classes of orbits and find the evolutions of the relevant parameters of the system during and after the decay of particle oscillations. Switching to Gaussian units in Section 6, we estimate the characteristic decay times of the charged particle oscillations and discuss the relevance of the model in astrophysical situations. We summarize our results in Section 7.

Throughout the paper, we use the spacetime signature (−, +, +, +) and the system of geometric units in which $G = 1 = c$. However, for expressions having astrophysical relevance, we use the constants explicitly. Greek indices are taken to run from 0 to 3.

2. Radiation Reaction in Flat Spacetime

For completeness, we first summarize the results of the motion of a charged particle in flat spacetime. The equation that describes the motion of a charged particle in a magnetic field in general contains two forces,

$$\frac{du^\mu}{d\tau} = f_L^\mu + f_R^\mu,$$  \hspace{1cm} (2)

where $f_L^\mu = (q/m)F^{\mu\nu}u_\nu$ is the Lorentz force, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the tensor of the electromagnetic field, and $u^\nu(\tau) = dx^\nu/\,d\tau$ is a four-velocity of the particle. The last term $f_R^\mu$ is the radiation-reaction force, which in the non-relativistic case has to lead to the expression $3q^2d^2u^\mu/2m\,d\tau^2$. Moreover, any force vector has to satisfy the condition $f_R^\mu u_\mu = 0$. This implies that the correct covariant form of the expression for the radiation-reaction force is

$$f_R^\mu = \frac{2q^2}{3m}\left(\frac{d^2u^\mu}{d\tau^2} + u_\nu u^\nu \frac{d^2u^\mu}{d\tau^2}\right).$$  \hspace{1cm} (3)

This expression was found by Dirac, and the motion equation is sometimes called the Lorentz–Abraham–Dirac or Lorentz–Dirac (LD) equation. The first term in the parentheses, also
known as the Schott term, arises from the particle electromagnetic momentum (Gal’tsov & Spirin 2006). The second term in parentheses is the radiation recoil term, which corresponds to the relativistic correction of the radiation-reaction force. The four-velocity of the charged particle satisfies the following equations:

\[ u_\alpha u^\alpha = -1, \quad u_\alpha \dot{u}^\alpha = 0, \quad u_\alpha \ddot{u}^\alpha = -\ddot{u}_\alpha. \]  

(4)

Thus, the motion of a radiating charge is described by a third-order differential equation in coordinates, rather than the habitual second order. This may lead to unphysical solutions—

the existence of pre-acceleration in the absence of external forces. Even though the unphysical solutions can be removed by properly chosen initial conditions, the integration of the exact form of the equations of motion, (2) and (3), is inconvenient, due to the exponential increase of the computational error in practical calculations.

However, one can reduce the order of the LD equation using the method proposed in Landau & Lifshitz (1975), i.e., by rewriting the self-force in terms of the external force and the four-velocity of a particle. Substituting the higher-order terms in Equation (3) with the derivatives of the Lorentz force, we get the equation in the following form:

\[
\frac{du^\alpha}{d\tau} = f^\mu_L + \frac{2\alpha^2}{3m}(\delta^\alpha_\mu + u^\alpha u_\mu) \frac{df^\alpha_L}{d\tau}. \]  

(5)

This equation, usually referred to as the Landau–Lifshitz (LL) equation, has important consequences: it is of the second order, does not violate the principle of inertia, and the self-force vanishes in the absence of the external (Lorentz) force (Rohrlich 2001; Poisson 1999). The self-contained derivation of Equation (5) in terms of retarded potentials is given in Poisson et al. (2011). Equation (5) can be applied to cases with any external forces acting on a charged particle instead of the Lorentz force. In the case where \( f^\mu_L = \frac{2}{m} F^\mu_\nu u_\nu \), the radiation-reaction force can be rewritten in the form

\[
f^\mu_R = k\tilde{q} \{ F^\mu_{\nu\rho} u^\nu u^\rho + \tilde{q}(F^\mu_{\nu\rho} F^\rho_{\mu\nu} - F^\nu_{\rho\sigma} F^\rho_{\mu\sigma} u^\mu u^\nu) u^\rho \}, \]  

(6)

where \( \tilde{q} = q/m \) is the specific charge of the particle, \( k = (2/3) \tilde{q} q \), and the comma in the first term denotes the partial derivative with respect to the coordinate \( t^\alpha \). Spohn (2000) concluded that using the LL equation is identical to imposing Dirac’s asymptotic condition \( \lim_{t\to\infty} \dot{u}^\alpha = 0 \) on the LD equation. It was later confirmed by Rohrlich (2001) that the reduced form of the equation of motion is exact, rather than approximative, though the LL equation was proposed in Landau & Lifshitz (1975) as an approximative solution to the third-order LD equation. More details on the treatment of the radiation reaction of charged particles in flat spacetime can be found in the book by Spohn (2004). In our numerical study, we found that the LL approximation is perfectly applicable if the Schott term is small with respect to the radiation recoil term, which is the case we consider here. Below we show a representative example of charged particle motion in an external uniform magnetic field, integrating both LD and LL equations. The results of the numerical studies of LD and LL equations for the motion of a charged particle in a uniform magnetic field in flat spacetime are in accord with the analytical treatment of the radiation-reaction force performed in Spohn (2000).

2.1. Charged Particle in a Uniform Magnetic Field

Let us consider the motion of a charged particle in a homogeneous magnetic field aligned along the z-axis, such that the independent nonvanishing component of the electromagnetic tensor is \( F_{\nu\gamma} = B \). We introduce the new parameter in the form \( B = qB/(2m) \), where the factor 1/2 is added in order to correspond to the similar parameter introduced in the curved spacetime case.

Both LD and LL equations lead to equivalent results, albeit they differ in the number of initial conditions. In the case of the LD equation, one needs to set the values of nine constants—
arbitrary independent components of the initial position, velocity, and acceleration of the charged particle. The other three constants are given by the normalization condition, Equation (4). However, the direct integration of higher-order equations leads to the exponential increase of the computational error in a very short time. It is interesting to note that the problem of the time-dispersion error can be greatly reduced by integrating equations of motion backwards in time. A similar method of solving LD equations has been proposed in the past by Huschilt & Baylis (1976).

On the other hand, the reduced-order equations of motion (5) can be written explicitly in the form

\[
\frac{du^\alpha}{d\tau} = 2Bu^\gamma - 4kB^2(1 + u_\gamma^2)u^\gamma, \]  

(7)

\[
\frac{du^\gamma}{d\tau} = -2Bu_\gamma - 4kB^2(1 + u_\gamma^2)u_\gamma, \]  

(8)

\[
\frac{du^\gamma}{d\tau} = -4kB^2u^2 u^\gamma, \]  

(9)

\[
\frac{du^\gamma}{d\tau} = -4kB^2u^2 u^\gamma. \]  

(10)

Here, \( u_\gamma^2 = (u^\gamma)^2 + (u^\nu)^2 \) is square of the particle velocity in the plane orthogonal to the magnetic field and the \( z \)-axis. The representative trajectory of a radiating charged particle corresponding to both LD and LL cases is shown in Figure 1. The initial conditions are chosen as follows: the initial plane velocity is \( u_\gamma = 0.8c, \) the velocity in the vertical direction is \( u_0^2 = 0.5c, \) the magnetic field is aligned along the \( z \)-axis, the magnetic parameter is chosen to be \( B = 1, \) and the radiation parameter \( k = 0.01. \) The estimations of the parameters in realistic situations are given in Section 6. Energy and angular momentum loss lead to the decay of the plane velocity of the particle, while the vertical component remains constant. An apparent deceleration along the \( z \)-axis, shown in Figure 1 (solid thick curve in the middle plot), appears only in the frame moving with the particle, while no forces are applied in the \( z \)-direction. The velocity with respect to the static observer, \( v^{\gamma} = dz/dt \equiv u^{\gamma}/u^t, \) remains constant.

2.2. Energy and Momentum Loss

The rate of energy loss of the particle can be evaluated from Equation (10). Modifying it for a static observer, we get

\[
\frac{dE}{dt} = -kB^2\tilde{q}^2u_\gamma(t)^2. \]  

(11)
Integrating this equation, one can obtain the energy of the particle in a given moment of time. The evolution of the energy in time is shown in Figure 1 for the given trajectory. Thus, energy loss will be given only by the change of the plane velocity \( u_\perp \) in time, while the kinetic energy associated with the motion in the \( z \) direction will be conserved. The plane velocity \( u_\perp \) of the particle decreases in time and asymptotically tends to zero, as represented by the dotted curve of the middle plot of Figure 1. This implies that there exists an irreducible (specific) energy of the radiating charged particle, which corresponds to the final state of the particle, having the following simple form:

\[
\mathcal{E}_0 = (1 - (v_0^2)^{\frac{1}{2}}),
\]  

(12)

where \( v_0 \) is the vertical velocity of the particle along the \( z \)-axis, measured by the static observers and which is constant during the radiation process.

In order to find the rate of angular momentum loss, one can fix the motion in a plane by taking \( u^z = 0 \). In general, the specific angular momentum for the motion in a plane is defined by the formula

\[
\mathcal{L} = \rho^2 \frac{d\phi}{d\tau} + \tilde{q} A_\varphi,
\]

where \( \rho \) is the gyroradius of the particle trajectory. Recalling that \( \frac{d\phi}{d\tau} = u_\perp \gamma / \rho \) and \( \rho = u_\perp / \omega_L \), where \( \omega_L = qB/m \equiv 2B \) is the Larmor frequency, one can write the specific angular momentum of the radiating charged particle in a given moment of time in the form

\[
\mathcal{L} = \frac{u_\perp (\tau)^2}{4B} [2\gamma(\tau) + 1], \quad \gamma = (1 - u_\perp^2)^{\frac{1}{2}}.
\]  

(13)

Solving first two equations of motion, Equations (7) and (8), and substituting into Equation (13), we get the evolution of the angular momentum in time, which is represented in Figure 1. Unlike the energy of the particle, the angular momentum asymptotically tends to zero for large \( \tau \). This occurs because the gyroradius \( \rho \) of the charged particle tends to zero as well, while \( \mathcal{L} \) is proportional to \( \rho \).

One can find the ratio between the angular momentum loss \( \dot{\mathcal{L}} = d\mathcal{L}/dt \) and the energy loss \( \dot{\mathcal{E}} = d\mathcal{E}/dt \) in the form

\[
\frac{\dot{\mathcal{L}}}{\dot{\mathcal{E}}} = \frac{r^2 \Omega u_\perp^2 + 1}{u_\perp^2}.
\]  

(14)
where $\Omega$ is the angular frequency of the charged particle measured by the observers at rest. In Cartesian coordinates, $\Omega$ takes the form
\[ \Omega = \frac{xy - yx}{x^2 + y^2}, \tag{15} \]
where overdots denote the derivative with respect to the coordinate time $\tau$.

### 2.3. Decay Time

One can find the characteristic time required to decay the energy of a radiating charged particle in the following way. Since the velocity in the magnetic field direction is constant, one can consider only the planar motion of the particle by taking $u_z = 0$. This implies that according to the condition $u^\mu u_\mu = -1$, we get $u_\tau^2 = (u^\tau)^2 - 1$. Thus, Equation (10) can be rewritten in the form
\[ \frac{dE}{d\tau} = -\mathcal{K}(E^3 - E), \quad \mathcal{K} = 4kB^2. \tag{16} \]
Integrating this equation, we get the particle energy in a given moment of time,
\[ E(\tau) = \frac{E_0 e^{\mathcal{K} \tau}}{\sqrt{1 + E_0^2 e^{2\mathcal{K} \tau} - 1}}, \tag{17} \]
where the integration constant $E_0$ is the initial specific energy of the particle. Asymptotically in time, the specific energy tends to the particle rest energy, being equal to 1. Thus, the decay time during which the specific energy will be lowered from $E_0$ to $E_f$ due to radiation takes the following form:
\[ T = \frac{1}{2\mathcal{K}} \ln \left( \frac{E_f^2 (E_f^2 - 1)}{E_0^2 (E_0^2 - 1)} \right), \tag{18} \]
where $\mathcal{K} = 4kB^2$. Since the energy in Equation (17) is an exponentially decreasing function, the particle energy cannot be reduced to 1 in practical calculations.

### 3. Charged Particle Orbiting a Black Hole without Radiation Reaction

In this section, we briefly summarize previous results related to the particle motion in the field of magnetized black holes presented in Kološ et al. (2015), Stuchlík & Kološ (2016), and Tursunov et al. (2016). The interval in the Schwarzschild black hole spacetime in spherical coordinates $(t, r, \theta, \phi)$ reads
\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{19} \]
where $M$ is the black hole mass, and the function $f(r)$ is the lapse function given by
\[ f(r) = 1 - \frac{2M}{r}. \tag{20} \]
Hereafter, without loss of generality, we put the mass of the black hole to be equal to unity, $M = 1$. Let us consider a black hole immersed in an external asymptotically uniform magnetic field. In the case when this field is weak, implying that the metric of the Schwarzschild black hole is not violated (see Equation (1)), one can write the solution of the Maxwell equation for the four-vector potential of the electromagnetic field $A^\mu$ in the following form (Wald 1984):
\[ A_\phi = \frac{B}{2}, \quad g_{\phi\phi} = \frac{B}{2} r^2 \sin^2 \theta, \tag{21} \]
which is the only nonzero component of the four-vector potential $A^\mu$, and $B$ is the strength of the magnetic field at spatial infinity, which is taken to be constant. The antisymmetric tensor of the electromagnetic field $F_{\mu\nu} = A_{\nu\mu} - A_{\mu\nu}$ in this case has only two independent nonzero components,
\[ F_{\tilde{r}\phi} = Br \sin^2 \theta, \quad F_{\tilde{r}\phi} = Br^2 \sin \theta \cos \theta. \tag{22} \]

#### 3.1. Bounded Motion around a Black Hole

The motion of charged particles is described by the Lorentz equation in curved spacetime. In the case when radiative processes can be ignored, one can write for the particle of mass $m$ and charge $q$ the Lorentz equation of motion
\[ \frac{Du^\mu}{d\tau} = \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta + \frac{q}{m} F^\mu_{\nu} u^\nu, \]
where $u^\mu$ is the four-velocity of the particle, normalized by the condition $u^\alpha u_\alpha = -1$, $\tau$ is the proper time of the particle, and the components of $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols.

The symmetry of the Schwarzschild black hole, Equation (19), and the external magnetic field gives us a right to find the conserved quantities associated with the time and space components of the generalized four-momentum $p_\alpha = p_\alpha + A_\alpha$. Thus, the energy and angular momentum of the charged particle in the presence of an external uniform magnetic field take the following form:
\[ E = -p_t = mf(r) \frac{dt}{d\tau}, \tag{24} \]
\[ L = \pi_\phi = m r^2 \sin^2 \theta \left( \frac{d\phi}{d\tau} + \frac{qB}{m} \right). \tag{25} \]
In this case, the circular motion of charged particles is always bounded in the plane orthogonal to the magnetic field lines, which corresponds to the equatorial plane $\theta = \pi/2$. The boundary of the motion is governed by the shape of the effective potential, determined by the equation
\[ E^2 = V_{eff}(r, \theta; \mathcal{L}, B), \tag{26} \]
Using the notation
\[ E = \frac{m}{f(r)} \mathcal{L} = \frac{L}{m}, \quad B = \frac{qB}{2m}, \tag{27} \]
we can write the effective potential in the form
\[ V_{eff}(r, \theta) = f(r) \left[ 1 + \left( \frac{\mathcal{L}}{r \sin \theta} - Br \sin \theta \right)^2 \right]. \tag{28} \]
The effective potential (28) shows a clear symmetry $(\mathcal{L}, B) \rightarrow (-\mathcal{L}, -B)$, which allows the following two situations to be distinguished:
- minus configuration, with $\mathcal{L} > 0, B < 0$ (equivalent to $\mathcal{L} < 0, B > 0$): magnetic field and angular momentum parameters have opposite signs, and the Lorentz force is attracting the charge towards the black hole.
3.2. Charged Particle Oscillations

Charged particles can undergo stable quasiharmonic oscillations in radial and vertical directions in the vicinity of a magnetized black hole. In the case where the oscillations are small enough in comparison with the radius of the corresponding stable circular orbit, one can find the locally measured frequencies of the vertical $\omega_0$ and radial $\omega_\ell$ oscillations to be equal to

$$\omega_0^2 = \frac{L_c^2}{r^4} - B^2,$$

$$\omega_\ell^2 = \frac{1}{(r-2)^2} \left\{ (r-2)^2 (B^2 r^4 + 3L_c^2) - 2r(B^2 - L_c^2)^2 - 2r^4 \right\},$$

where $L_c$ is the specific angular momentum at the circular orbit; for details, see Kološ et al. (2015). In addition to the frequencies given above, one can also find the Keplerian axial frequency $\omega_\phi$ and the Larmor angular frequency $\omega_L$, which are given by

$$\omega_\phi = \frac{d\phi}{d\tau} = U_\phi = \frac{L_c}{g_{\phi\phi}} - B, \quad \omega_L = \frac{qB}{m} = 2B.$$

As one can see, the frequency $\omega_L$ does not depend on the $r$ coordinate and plays a crucial role in the regions detached from the black hole. The characteristic oscillations of charged particles and representative plots of charged particle trajectories around a Schwarzschild black hole can be found in Figure 7 of Kološ et al. (2015). For the particle to oscillate in the vicinity of a black hole, its energy has to be larger than the minimum of the effective potential but keeping the finite type of the motion. Thus, the minimum energy of a charged particle in trapped states corresponds to the stable circular orbits given by the minimum of the effective potential (28). The maximum energy at the trapped states is given by the unstable circular orbit with the corresponding angular momentum. The energy of the charged particle in a circular orbit is given by

$$\mathcal{E}_\pm^2 = \frac{r_\pm^2}{(r-3)^2} (r - 3 + 2B^2 r^3 \pm 2Br \sqrt{r-3 + B^2 r^4 f^2}),$$

where the signs correspond to the maximum and minimum energies of a particle in a circular orbit, and $f = 1 - 2/r$ is the lapse function. The difference between $\mathcal{E}_+$ and $\mathcal{E}_-$ can be very large for large values of the magnetic parameter $B$, representing the charged particles accelerated up to ultrarelativistic velocities (Kološ et al. 2015; Stuchlík & Kološ 2016).

Some types of quasiharmonic epicyclic motion are possible only in the presence of a magnetic field. One interesting and illustrative example of the effect of the magnetic field on charged particle motion in the black hole background is the appearance of curled trajectories, demonstrated in Figure 4, which corresponds to the plus configuration with a repulsive Lorentz force. The conservation of angular momentum, Equation (25), in the absence of radiation leads to the equation

$$\dot{\phi} = \frac{L}{r^2} - B. \tag{33}$$

In the non-magnetic case, the right-hand side of the equation above is positive for positive $L$. The presence of a magnetic field with $B > 0$ decreases the velocity in the $\phi$-direction, which can become negative if

$$L > L_{cs}(r; B) \equiv Br^2. \tag{34}$$

For the energy, this condition means

$$\mathcal{E} > \mathcal{E}_s(L; B) \equiv \sqrt{1 - 2B/L}. \tag{35}$$

Thus, decreasing the azimuthal velocity of a particle through the influence of the magnetic field and keeping the above given conditions, we get so-called curled trajectories, which are not possible in the absence of a magnetic field. This type of orbit does not appear in the case of an attractive Lorentz force, which corresponds to the minus configurations. Dependence of angular momentum, energy, and azimuthal velocity of a charged particle on the location of the circular orbit for different values of the magnetic parameter is shown in Figure 2.

In the next sections, we show that the oscillations of charged particles in a magnetic field near a Schwarzschild black hole will be damped due to SR, and in particular cases, the charged particle orbits will not be able to stay stable, bringing the particle to the black hole due to the effect of the radiation-reaction force.

4. Radiation Reaction in the Field of Magnetized Black Holes

4.1. Radiation-reaction force

The motion of a relativistic charged particle is governed by the LD equation, which includes the influence of external electromagnetic fields and the corresponding radiation-reaction force. The last force arises from the radiative field of the charged particle, and the equations of motion in general can be written in the form

$$\frac{D\mu^\nu}{d\tau} = \tilde{q} F^\nu_\mu u^\nu + \tilde{q} F^\mu_\nu u^\nu, \tag{36}$$

where the first term on the right-hand side of Equation (36) corresponds to the Lorentz force with electromagnetic tensor $F_{\mu\nu}$ given by Equation (22), while the second term is the self-force of charged particles with the radiative field $F_{\mu\nu} = A^\mu_{,\nu} - A^\mu_{,\nu}$. The vector potential of the self-electromagnetic field of the charged particle satisfies the wave equation

$$\square A^\mu - R^\mu_\nu A^\nu = -4\pi j^\mu, \tag{37}$$

where $j^\mu$ is the four-current. In order to make use of the Lorentz force, we have to solve the wave equation for the potential.
where $\Box = g^{\mu\nu}D_\mu D_\nu$, $D_\mu$ is the covariant differentiation, and $R^\mu_\nu$ is the Ricci tensor. The retarded solution to Equation (37) for the vector potential takes the form

$$A^\mu(x) = q \int G^\mu_{\tau \lambda}(x, z(\tau)) u^\lambda d\tau,$$

(38)

where $G^\mu_{\tau \lambda}$ is the retarded Green function, and the integration is taken along the worldline of the particle $z$, i.e., $u^\mu(\tau) = dz^\mu(\tau)/d\tau$. For details, see, e.g., Poisson (2004). The covariant generalization of the dynamics of a radiating charged particle in curved spacetime has been derived in DeWitt & Brehme (1960) and completed in Hobbs (1968) using the tetrad formalism. The explicit form of Equation (36) for the motion of a charged particle experiencing radiation-reaction force in curved spacetime reads

$$\frac{Du^\mu}{d\tau} = q F^\mu_\nu u^\nu + \frac{q^2}{3m} \left( 2D^2u^\mu + u^\mu u_\nu D^2u^\nu \right) + \frac{q^2}{3m} (R^\mu_\lambda u^\lambda + R^\nu_\lambda u^\lambda u^\nu + 2u^\lambda D_{\tau} u_\nu),$$

(39)

where the last term of Equation (39) is the tail integral

$$f^{\mu}_{\text{tail}} = \int_{-\infty}^{\tau - 0^+} D^\lambda G^{\mu}_{\lambda \beta}(z(\tau), z(\tau')) u^\beta d\tau'.$$

(40)

A detailed derivation of the equations of motion of radiating charged particles can be found in Hobbs (1968) and Poisson (2004). The integral in the tail term is evaluated over the past history of the charged particle, with primes indicating its prior positions. All other quantities are evaluated at the current position of the particle $z(\tau)$. The term containing the Ricci tensor vanishes in the vacuum metrics, so this term is irrelevant in our case. The existence of the “tail” integral in Equation (39) implies that the reaction force in curved spacetime has a non-local nature, because the motion of the charged particle depends on its entire history and not only on its current state. The radiation field $F_{\mu\nu}$ in Equation (36), emitted by the charged particle, interacts with the curvature of the background spacetime and returns to the particle with a delay corresponding to the tail integral in Equation (39). In this sense, the electromagnetic field radiated by the charged particle carries the information about the history of the particle. Even in the absence of external forces, such as the Lorentz force, the free trajectory of the charged particle does not follow the geodesics, which is one of the most important consequences of Equation (39).

However, for the purposes of the present paper, the tail term can be ignored, as we show below. The tail terms can be estimated based on the results of Dewitt & Dewitt (1964) and Smith & Will (1980), as well as multiple subsequent papers. For a particle with the charge $q$ and mass $m$, the ratio of the tail force $F_{\text{tail}} \sim GMq^2/(r^2c^2)$ to the Newton force $F_N \sim GMm/r^2$ in the vicinity of a black hole ($r \sim r_H = 2GM/c^2$) of stellar mass $M \sim 10M_\odot$ is

$$\frac{F_{\text{tail}}}{F_N} \sim \frac{q^2}{mMG} \sim 10^{-19} \left( \frac{q}{e} \right)^2 \left( \frac{m_e}{m} \right) \left( \frac{10M_\odot}{M} \right).$$

(41)

where $e$ and $m_e$ are the charge and mass of an electron. For SMBHs with mass $M \sim 10^6M_\odot$, this ratio is eight orders lower. On the other hand, the radiation-reaction force (second term on the right-hand side of Equation (39)) depends on the presence of an external force, arising in our case from the external magnetic field. According to Piotrovich et al. (2011) and Baczko et al. (2016), the characteristic values of the magnetic fields near stellar-mass black holes and SMBHs are $B \sim 10^3G$ for $M = 10^6M_\odot$ and $B \sim 10^4G$ for $M = 10^9M_\odot$. Thus, for a particle with velocity comparable to the speed of light, $v \sim c$, the ratio of the radiation-reaction force $F_{\text{RR}} \sim q^4B^2/(m^2c^4)$ to the Newton force gives an order

$$\frac{F_{\text{RR}}}{F_N} \sim \frac{q^4B^2MG}{m^3c^5} \sim 10^3 \left( \frac{q}{e} \right)^4 \left( \frac{m_e}{m} \right)^3 \left( \frac{B}{10^3G} \right) \left( \frac{M}{10M_\odot} \right).$$

(42)

The ratio of $F_{\text{RR}}$ to $F_N$ for SMBHs with $M = 10^6M_\odot$ and magnetic field $B \sim 10^4G$ gives the same order of magnitude. The above estimations of the tail term apply in the non-relativistic, or moderately relativistic, case, when the Lorentz factor is of the order of unity. A more general argument is based on the treatment of the gravitational self-force in the Schwarzschild and Kerr spacetimes. As was shown by Gal’tsov (1982), the radiative part of the self-force (based on the half-difference of the retarded and advanced Green’s function) in the Kerr metric satisfies the (averaged over time) balance equations for the energy and angular momentum of radiation of all spins $s = 0, 1, 2$, in the sense that local work of the self-force is equal to radiated fluxes at infinity and at the black hole horizon. This is valid independently of the particle velocity and extends to the ultrarelativistic case. Later, it was shown that a similar balance is valid for Carter’s constant (Mino 2003), which completes the set of quantities determining geodesic motion in the Kerr field. The conservative part of the self-force (half-sum of the retarded and advanced potentials) is more difficult to compute since this demands a proper elimination of divergences. This caused a vivid discussion in the literature from the mid-1990s to early 2000s, which is nicely reviewed in Tanaka (2006); for more recent work on the same subject containing further references, see Fujita et al. (2017), resulting in the consensus opinion that this contribution is small with respect to radiative self-force, especially with a growing Lorentz factor (Pound et al. 2005; Sago et al. 2005). Therefore, if one is interested in estimating the gravitational and electromagnetic tail terms in the ultrarelativistic limit, using the above results, one can revoke the gravitational synchrotron radiation (GSR), which was computed earlier for spins $s = 0, 1, 2$, most notably in Chrzanski & Misner (1974) and in Breuer (1975), by comparing to flat-space SR. From these results, one can see that GSR in the ultrarelativistic limit is suppressed by the square of the Lorentz factor with respect to SR. These arguments lead us to conclude that for elementary particles moving with any velocity in both gravitational and electromagnetic fields, the purely gravitational tail term can be ignored in comparison with the electromagnetic (properly covariantized) radiation-reaction force. Thus, for purposes of the present paper, the equation of motion (39) can be simplified to the following covariant form of the LD equation:

$$\frac{Du^\mu}{d\tau} = q F^\mu_\nu u^\nu + f^\mu_R,$$

(43)

with the radiation-reaction force given by

$$f^\mu_R = \frac{2q^2}{3m} \left( \frac{D^2u^\mu}{dt^2} + u^\mu u_\nu \frac{D^2u^\nu}{dt^2} \right).$$

(44)

Introducing the four-acceleration as a covariant derivative of the four-velocity, $a^\mu = Du^\mu/d\tau$, one can rewrite the term
$D^2 u^\alpha / d\tau^2$ as follows

\[
\frac{D^2 u^\alpha}{d\tau^2} = \frac{D a^\mu}{d\tau} = \frac{da^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha a^\beta
\]

\[
= \frac{d}{d\tau} \left( u^\mu + \Gamma^\mu_{\alpha\beta} u^\alpha a^\beta \right) + \Gamma^\mu_{\alpha\beta} \left( \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta a^\gamma \right)
\]

\[
= \frac{d^2 u^\alpha}{d\tau^2} + \left( \frac{\partial \Gamma^\mu_{\alpha\beta}}{\partial \zeta^2} u^\mu u^\beta + 3 \Gamma^\mu_{\alpha\beta} \frac{du^\beta}{d\tau} + \Gamma^\alpha_{\beta\gamma} \Gamma^\mu_{\alpha\gamma} u^\beta a^\gamma \right) u^\alpha.
\]

Thus, the general relativistic equations of radiating charged particle motion are given by Equations (44) and (45). However, the full form of the equations is plagued by runaway solutions. One can avoid this problem in a similar way, as in the flat spacetime case, namely by reducing the order of differential equations. In the absence of the radiation-reaction force, the motion of the charged particle in an external asymptotically uniform magnetic field is governed by Equation (23). Taking the covariant derivative with respect to the proper time from both sides of Equation (23), we get

\[
\frac{D^2 u^\alpha}{d\tau^2} = \frac{d}{dx^\nu} \left( F^\alpha_{\beta} u^\beta u^\mu + \tilde{q} F^\alpha_{\beta} F^\beta_{\nu} u^\mu u^\nu \right).
\]

Substituting Equation (46) into Equation (44), we get the radiation-reaction force in the form

\[
f^\alpha_R = k \tilde{q} \left( D F^\alpha_{\beta} \frac{dx^\beta}{du} u^\mu + \tilde{q} \left( F^\alpha_{\beta} F^\beta_{\nu} + F_{\mu\nu} F^\nu_{\rho} u^\rho u^\alpha \right) u^\lambda \right),
\]

where the covariant derivative from the second rank tensor reads

\[
\frac{D F^\alpha_{\beta}}{dx^\mu} = \frac{\partial F^\alpha_{\beta}}{\partial x^\mu} + \Gamma^\alpha_{\beta\gamma} F^\gamma_{\mu} - \Gamma^\mu_{\gamma\beta} F^\alpha_{\gamma}.
\]

Equations (43) and (47) give a covariant form of the LL equations. Below we test these equations in the particular case of the motion of charged particles around a Schwarzschild black hole immersed in an external asymptotically uniform magnetic field.

The motion of charged particles in the vicinity of a magnetized Schwarzschild black hole is generally chaotic (Kopáček & Karas 2014; Kopáček et al. 2010). However, close to the minimum of the effective potential, which corresponds to the stable circular orbit, the motion of the charged particle is regular, being of harmonic character. The motion is also regular, if the particle is moving entirely in the equatorial plane and the chaotic behavior appears with increasing inclination angle. Here, we focus our attention on the regular motion only. In order to represent the equations of motion explicitly, we fix the plane of the motion at the equatorial plane, $\theta = \pi/2$, of a magnetized black hole. However, to explore the trajectories of particles, we use the full set of equations of motion and solve them numerically. Without loss of generality, one can again equalize the mass of a black hole to unity, $M = 1$. The nonvanishing components of equations of motion of radiating charged particles moving around a Schwarzschild black hole immersed in an external asymptotically uniform magnetic field take the following form:

\[
\frac{du^\alpha}{d\tau} = \frac{2u^\alpha u^\mu}{r(2 - r)} - \frac{2kB u^\mu}{r} \left( 2Brf \left[ f(u^\beta)^2 - 1 \right] - u^\mu \right),
\]

\[
\frac{du^\alpha}{d\tau} + \frac{k}{2} B f^2 (u^\beta)^2 - u^\alpha \left( 2Brf \left[ f(u^\beta)^2 - 1 \right] - u^\mu \right)
\]

where $f$ is the lapse function given by Equation (20), $B = qB/(2m)$ is the magnetic parameter, and $k = 2q^2/(3m)$ is the radiation parameter.

### 4.2. Energy and Angular Momentum Loss

The total energy–momentum radiated by the charged particle is equal to the integral of the radiation-reaction force taken along the worldline of the particle. In flat spacetime, the radiated four-moment of a particle with charge $q$ is given by $dP^\mu / d\tau = \tilde{q}^2 u^\alpha a^\alpha u^\mu$. Synchrotron radiation in curved spacetime has been studied in Sokolov et al. (1983, 1978) using covariantization of the flat-space results. The problem has been revisited more recently in Shoom (2015), where, however, the radiation-reaction force is not taken into account. The evolutions of the particular components of the four-momentum of the particle with the radiation-reaction force can be found from Equations (43) and (47). For the motion at the equatorial plane, the energy loss is given by

\[
\frac{dE}{d\tau} = -2kB \left[ 2Bf^3 - E \right] \left( 2Bf + u^\theta \right)
\]

For an ultrarelativistic particle with $E \gg 1$, the leading contribution to the energy loss is given by the first term in square brackets of Equation (52). However, for small velocities close to the stable circular orbits, the last two terms can play significant roles. Both situations will be studied in detail in the following section. Similarly, one can find the rate of angular momentum loss as

\[
\frac{dL}{d\tau} = 4B^2 k u^\theta (f^2 (u^\beta)^2 - f) - 2u^\alpha u^\beta \left( r - 4B^2 k^2 \right) + 2rB u^\alpha.
\]

In the next section, we will test the equations of motion numerically for the general bounded motion of a radiating charged particle not restricted to the equatorial plane.

### 5. Damping of Oscillations

In this section, we study the damping of charged particle oscillations due to the radiation-reaction force for both “+” and “−” configurations. In particular, we demonstrate that the particles at “−” configurations spiral down to the black hole, while for those at “+” configurations, the motion remains stable. In Section 5.3, we study the evolution of circular orbits under the influence of self-force. We show that in such a case, the circular orbits can be shifted outwards from the black hole. As pointed out in Section 3, depending on the direction of the Lorentz force, one can distinguish two qualitatively different types of motion. In a radiating case, one can see such differences as well. The motion along a general worldline highly depends on the initial energy and the position of the particle. Radiation effect on the charged particle motion also
has different timescales depending on whether the particle is initially oscillating or not. A representative comparison of trajectories of oscillating charged particles in the presence and absence of the radiation-reaction force is illustrated in Figure 3 for the attractive Lorentz force and in Figure 4 for the repulsive one. Note that the trajectories represented in Figures 3 and 4 are

**Figure 3.** Representative comparison of trajectories of non-radiating (first line) and radiating (second line) charged particles in the minus configuration, corresponding to the attractive Lorentz force from different view angles around the black hole in a magnetic field. The initial conditions in both cases are chosen to be the same and shown in the second row plots. The starting point is indicated by the black dot. A radiating charged particle escapes the initial boundary of the motion (dashed contours in the third row) governed by an effective potential, Equation (28), and collapses to the black hole due to the loss of angular momentum. The magnetic field is aligned with the z-axis. The trajectory of the radiating charged particle corresponds to the integration of the full set of equations of motion, Equations (43) and (47), without fixing the plane of motion.

**Figure 4.** Similar to Figure 3; the comparison of trajectories of non-radiating and radiating charged particles in the case of a repulsive Lorentz force, i.e., the plus configuration. A radiating particle stays in the region of the initial boundary of the motion and radiates the oscillatory part of its energy–momentum, tending to a stable circular orbit.
plotted by integration of the full form of the equations of motion (43) and (47) without restriction to the equatorial plane. In both cases, the motion of the charge is initially bounded. The charged particle starts its motion slightly above the equatorial plane (keeping the regular character of its motion) in the vicinity of a weakly magnetized Schwarzschild black hole. The initial energy and angular momentum of the particle correspond to the state close to, but above the minimum of, the effective potential, which generates the barrier from which the particle bounces. Due to the action of the radiation-reaction force, the charged particle changes the oscillatory character of its motion because of the loss of energy and angular momentum. The final state of the charged particle depends on the direction of the Lorentz force. For the Lorentz force directed towards the horizon of the black hole, SR leads to the collapse of the initially stable particle into the black hole, the gyrocenter of the oscillating charged particle is located near its orbit (center of “curls”), while in the case of the minus configuration, the gyrocenter of the orbit is located inside the horizon (it coincides with a black hole singularity for the circular motion). Indeed, from the analogy with the flat spacetime formula (5), the sign of the radiation-reaction force depends on the alignment of the Lorentz force due to the reduction of order procedure performed in the beginning. Thus, the “Lorentz-repulsive” case with radiation reaction represents the damping of oscillations, turning the oscillatory epicyclic type of motion to nearly circular orbit. A strong nonlinearity of the equations of motion shifts the location of the stable circular orbits as well; however, as we will see below, this effect is relatively slow in comparison to the process of radiative damping of oscillations. One can conclude that for relatively short periods of time, an oscillating charged particle experiencing a repulsive Lorentz force and radiation-reaction force will damp its oscillations, thus settling down to the circular orbit. This result is represented in Figure 7, which is in accordance with the qualitative predictions given in Shoom (2013).

5.1. Evolution of Energy and Angular Momentum

In fact, SR carries away the kinetic energy of the particle. One can see in the \(E-\tau\) plots of Figure 5 and Figure 6 that the energy slightly decreases in both cases. However, the azimuthal angular momentum of the particle changes differently depending on the direction of the Lorentz force. While in the minus configurations the charged particle decreases its angular momentum due to radiation, in the plus configurations, the angular momentum quasiperiodically increases. Quasiperiodicity is connected with the quasiharmonic oscillations of the particle in the motion with “curls”. The increase of the angular momentum of the particles in plus configurations becomes clear when one compares the angular momentum of the charged particles in the motion with curls, with \(\mathcal{L}_p\) given by Equation (34), with that of the charged particles in circular orbits given by Kološ et al. (2015),

\[
\mathcal{L}_c = \frac{-Br^2 + r\sqrt{B^2r^2(r^2 - 2)^2 + r^3}}{r - 3}.
\] (54)

Let us now assume that the initial angular momentum of the particle is \(\approx \mathcal{L}_p\), while the angular momentum at the final state is nearly equal to that in the stable circular orbit, \(\approx \mathcal{L}_c\). From the fact that \(\mathcal{L}_c > \mathcal{L}_p\) (see details in Kološ et al. 2015), one can conclude that the particle is actually gaining angular momentum by reducing radial oscillations due to the radiation-reaction force. In the minus configurations, the trajectories with curls, i.e., the regions with a negative angular velocity, are not observed. Therefore, a charged particle initially located at a
bounded orbit around a black hole continuously loses its angular momentum due to the radiation-reaction force. When the angular momentum of the particle becomes lower than the one corresponding to the ISCO, $L < L_{\text{ISCO}}$, the particle collapses into the black hole. The value of the angular momentum at the ISCO is given by Kološ et al. (2015),

$$L_{\text{ISCO}} = -2Br_l + \sqrt{B^2r_l^2(5r_l^2 - 4r_l + 4) + 2r_l},$$

where $r_l$ is the ISCO radius. A detailed analysis of the boundaries of the angular momentum separating different types of orbits in the vicinity of magnetized black holes can be found in Kološ et al. (2015) for non-rotating black holes and in Tursunov et al. (2016) for magnetized Kerr black holes.

### 5.2. Lifetime of Oscillations

One can calculate the relaxation time $\tau$ of a charged particle required for the decay of the radial oscillations due to the radiation-reaction force. The calculation of the relaxation time makes sense in plus configurations only, when the radiation reaction does not cause the particle to collapse to the black hole. The rate of energy loss can be written as

$$\dot{E} = \frac{E_f - E_i}{\tau},$$

where $E_i$ and $E_f$ are the initial and final energies of the particle. For the particle with velocity close to the speed of light, $v \sim c$, the leading contributor to energy loss is given by the first term of expression (52), which can be solved analytically, giving

$$\frac{dE}{d\tau} = -4B^2kE^3, \quad E(\tau) = \frac{E_i}{\sqrt{1 + 8B^2kE_i^2\tau}}.$$  \hspace{1cm} (57)

Extracting $\tau$ from Equation (57) at the final energy, we get

$$\tau = \frac{1}{4kB^2} \frac{E_i^2 - E_f^2}{E_i^2 E_f^2}. \hspace{1cm} (58)$$

One can find the upper limit of the relaxation time required to lower the maximum energy at the bounded orbit to the lowest energy corresponding to the circular orbit, where no oscillations exist. Thus, identifying $E_i = E_+$ and $E_i = E_-$, given by Equation (32), and assuming locations of extrema $E_{\text{max}} \approx E_{\text{min}} = r$ (indeed, according to Kološ et al. 2015), for large values of $B$ the locations of extrema are very close, we get

$$\tau_{\text{max}} = \frac{\sqrt{r - 3 + B^2f^2r^4}}{kB(1 + 4B^2f^2)}.$$  \hspace{1cm} (59)

Note that this equation corresponds to particles with ultra-relativistic velocities. This implies that for large values of the magnetic parameter $B$, one can write Equation (59) in the following simple form:

$$\tau_{\text{max}} \approx \frac{1}{kBf^2(\tau)}, \quad B \gg 1,$$  \hspace{1cm} (60)

where $f(\tau) = 1 - 2M/\tau$ is the lapse function. In particular, this equation shows that closer to the black hole, the decay of the oscillations of the charged particle is slower.

It is useful to compare the maximum relaxation time $\tau_{\text{max}}$ of the charged particle’s oscillations with the orbital time of the charged particle around the black hole. The orbital frequency $\omega_0$ is defined in Equation (31). Considering the stable circular orbits at the equatorial plane, we get the time of one revolution around the black hole in the form

$$\tau_\omega = \frac{2\pi r^2}{L_c - Br^2},$$  \hspace{1cm} (61)

where $L_c$ is given by Equation (54). Dividing Equation (59) by Equation (61), we get the maximum number of revolutions around the black hole during which the radial oscillations decay:

$$N_{\text{max}} \equiv \frac{\tau_\max}{\tau_\omega} = \frac{BrA}{\pi k(r - 2)^2(4B^2r^2 + 1)(Br(r - 2) + A)},$$  \hspace{1cm} (62)

where $A = \sqrt{r - 3 + B^2r^2(r - 2)^2}$. The dependence of $N_{\text{max}}$ on the magnetic parameter $B$ is given in Figure 8 for different values of the radiation parameter $k$. Since both the parameters $B$ and $k$ depend on the specific charge of the charged particle, the parameters are not really independent. This implies that the parameter $k$ can be expressed as a fraction of the parameter $B$. 

Figure 6. Decay of oscillations due to the radiation reaction of a plus-configuration particle and the corresponding evolution in time $\tau$ of the radius of orbit $r$, angular momentum $L$, energy $E$, radial velocity $u^r$, angular azimuthal velocity $u^\phi$, and gamma factor $u^\gamma = df/d\tau \equiv \gamma$. The angular momentum and energy are measured with respect to an observer at rest at infinity. The starting point of the particle is indicated by the black dot.
As one can see from Figure 8, the radiation reaction of the charged particle cannot be ignored even for small values of the parameter $k$. For large values of $\mathcal{B}$, the radial oscillations of an initially oscillating charged particle may decay, thus the charged particle may end up in a circular orbit within only a fraction of the total orbit around the black hole. The estimations of both parameters $\mathcal{B}$ and $k$, and the lifetime of the charged particle’s oscillations in realistic scenarios for electrons and ions will be given in Section 6.

The equations of motion (43) and (47) allow us to repeat the calculations of the damping of vertical oscillations (in $\theta$) of charged particles as well. In the case when the initial velocity of a particle is ultrarelativistic, the leading term responsible for the radiation reaction for all components of the velocity is the last term of Equation (47). This implies that the decay rate of the vertical oscillations of the charged particle is similar to the decay rate of the radial oscillations. The representative plots of the relevant parameters of the charged particles during the decay of the vertical oscillations is presented in Figure 9.

### 5.3. Circular Orbit Widening

Another new consequence of the radiation-reaction force in the vicinity of a black hole and in the presence of an external magnetic field is the evolution of the circular orbits of charged particles during the radiation process and the shifting of the radius of the circular orbit outwards from the black hole. Obviously, the last case can be realized for the repulsive Lorentz force only, while in the attractive case, as we have already seen, the particle spirals down into the black hole, and no closed circular orbits can be formed. Let us consider a purely circular motion of a charged particle revolving around a black hole at the equatorial plane in the presence of a uniform magnetic field (see Figure 10). While small oscillations can appear, they will be damped relatively quickly, and the energy of the particle can be considered to be always located near the minimum of the effective potential corresponding to the circular orbits. The radiation-reaction force in the frame moving with the particle vanishes, however, in the local geodesic frame of reference, the self-force is directed antiparallel to the particle velocity and concentrated in the narrow cone along the motion. This implies that, in fact, the radiation reaction reduces the angular velocity of the particle. Reducing the angular velocity while keeping the circular character of the motion causes the increase of the radius of the circular orbit. This, in turn, increases the particle energy and angular momentum with respect to the observer at rest at infinity. Indeed, the energy and angular momentum of the charged particle for an observer at infinity are given as

$$ E = \left(1 - \frac{2}{r}\right)u', $$

and

$$ \mathcal{L} = r^2 (u^\phi + \mathcal{B}), $$

which are not conserved for radiating particles. When the radius of the orbit $r$ increases faster than the deceleration of the velocities $u'$ and $u^\phi$, then the energy and angular momentum of the charged particle will increase accordingly. An example of the widening of the circular orbit of the radiating charged particle is illustrated in Figure 10. Note that the parameter $E$ is represented as in previous plots. The angular momentum and energy are measured with respect to an observer at rest at infinity. The starting point of the particle is indicated by the black dot.

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responsible for the radiation reaction \( k \) is taken to be unrealistically large in order to obtain a representative plot, since for small values of the parameter \( k \), due to the slowness of the process, the shift of the orbit would be barely seen.

The expansion of the circular orbits is much slower in comparison to the decay of the oscillations. Representative changes of the positions of the charged particles over time for attractive and repulsive Lorentz forces are demonstrated in Figure 11. In the case of the attractive Lorentz force \( (B < 0) \), the charged particle plunges into the black hole, as described in the previous section. The wiggles in the \( B < 0 \) case appear due to the precession of the particle around the circular orbit, as can be also seen in the \( r - \tau \) plot of Figure 5. In the repulsive case, the particle starts to oscillate inside a barrier given by the effective potential at a given moment of time. Due to the radiation-reaction force, its oscillations decay, ending up at a circular orbit with a larger radius than the initial position. At the circular orbit, the particle continues to slow down its orbital velocity, keeping a stable motion, which affects the location of the circular orbit, shifting it towards infinity. Note that the timescale of the plot is logarithmic, which implies that the shifting of the circular orbit of the particle is relatively slow.

The energy loss of this process is given by Equation (52), where all terms are important since the motion is not necessarily ultrarelativistic. Since all quantities in Equation (52) are implicit functions of time, one can rewrite this equation in the form

\[
\dot{E} = -A_1 \dot{x}^3 + A_2 \dot{x} x(\tau), \tag{65}
\]

where the overdot denotes the derivative with respect to the proper time \( \tau \), \( A_1 = 4kB^2 \) and \( A_2 = 2kB \) are constants, and \( x(\tau) = 2kBf(\tau) + u^\theta(\tau)/r(\tau) \). The analytical form of the
solution is found to be
\[
\mathcal{E}(\tau) = \frac{\mathcal{E} e^{A X(\tau)}}{\left(1 + 2A \mathcal{E}^2 \int_0^\infty e^{2A X(\tau')} d\tau'\right)^2},
\]
where \(X(\tau) = \int_0^\tau x(\tau) d\tau\). Equation (65) can be solved numerically for the given set of equations of motion, and it is performed for all \(\mathcal{E} - \tau\) dependence plots of the present paper. The detailed analysis of the effect of the radiative widening of orbits is left for our future studies.

6. Relevance to Astrophysics

In order to relate our results to realistic astrophysical scenarios, we perform estimations of the relevant parameters of the discussed model. Even though the magnetic field does not violate the geometry of the background spacetime, satisfying Equation (1), one cannot ignore its effect on the motion of charged particles due to the large values of the specific charge (charge per mass ratio) for elementary particles. Restoring the world constants, the dimensionless parameter \(\mathcal{B}\), widely used in the present paper, takes the form
\[
\mathcal{B} = \frac{|q| B GM}{2 m_e c^4},
\]
reflecting the relative influence of the gravitational and magnetic fields on the charged particle motion. According to Piotrovich et al. (2011) and Baczko et al. (2016), the characteristic values of the magnetic fields near the stellar-mass and SMBHs are \(B \sim 10^8 G\) for \(M = 10 M_\odot\) and \(B \sim 10^9 G\) for \(M = 10^9 M_\odot\). For electrons, one can estimate the parameter \(\mathcal{B}\) as
\[
\mathcal{B}_{BH} \approx 4.32 \times 10^{10} \quad \text{for} \quad M = 10 M_\odot, \quad (68)
\]
\[
\mathcal{B}_{SMBH} \approx 4.32 \times 10^{14} \quad \text{for} \quad M = 10^9 M_\odot. \quad (69)
\]
As a representative example, one can estimate the parameter \(\mathcal{B}\) for an electron in the vicinity of the SMBH at the center of the Milky Way, which is currently the best-studied SMBH candidate. The equipartition strength of the magnetic field near the Galactic center is usually considered to be of tens of Gauss (Johnson et al. 2015). Moreover, multifrequency measurements of the pulsar near the Galactic center by Eatough et al. (2013) demonstrate the existence of a strong magnetic field of a few hundred Gauss near the event horizon of the SMBH. The mass of the black hole candidate is measured to be \(4.3 \times 10^6 M_\odot\) (for more details about Sgr A*, see the recent review by Eckart et al. 2017). Thus, the parameter \(\mathcal{B}\) for the electrons surrounding the SMBH at the center of our Galaxy can be estimated as
\[
\mathcal{B}_{SgrA^*} \approx \frac{|e| B GM}{2 m_e c^4} \approx 1.86 \times 10^{10}, \quad (70)
\]
For protons, the values of \(\mathcal{B}\) in Equations (68)–(70) are lower by a factor \(m_p/m_e \approx 1836\). Large values of the parameter \(\mathcal{B}\) in realistic scenarios imply that the effects of the magnetic field on the dynamics of charged particles play one of the essential roles.

The radiation-reaction force acting on a charged particle is represented by the dimensionless parameter \(k\), which has the form
\[
k = \frac{2 q^2}{3 m G M}. \quad (71)
\]
The value of the parameter \(k\) is much lower than that of \(\mathcal{B}\). For electrons orbiting stellar-mass and SMBHs, we have, respectively,
\[
k_{BH} \sim 10^{-19} \quad \text{for} \quad M = 10 M_\odot, \quad (72)
\]
\[
k_{SMBH} \sim 10^{-27} \quad \text{for} \quad M = 10^9 M_\odot. \quad (73)
\]
For electrons around Sgr A*, we have \(k_{SgrA^*} \sim 10^{-25}\). For protons, the values of the \(k\) parameter is lower by a factor \(m_p/m_e \approx 1836\), as in the case with \(\mathcal{B}\). Despite the weakness of the parameter \(k\) as compared to \(\mathcal{B}\), it enters into the equations for ultrarelativistic particles as \(kB^2\), which can make the effect of the radiation-reaction force considerably large.

One can estimate the timescale of the decay of the charged particle oscillations. Typical orders of magnitude of the oscillation decay time of an electron and proton orbiting a black hole are given in Table 1. One can compare these values with the timescale of one orbit of a particle, \(\tau_o\), around stellar-mass and SMBHs. For the orbit at ISCO, we have
\[
\tau_e \sim 10^{-3} \quad \text{s}, \quad \text{for} \quad M = 10 M_\odot, \quad (74)
\]
\[
\tau_e \sim 10^4 \quad \text{s}, \quad \text{for} \quad M = 10^9 M_\odot. \quad (75)
\]
For Sgr A*, the electron decay time is about \(10^4\) s, while the orbiting time at ISCO is \(\sim 10^3\) s. It is interesting to note that for ions, the decay time of oscillations is much less than that for electrons, namely, for the factor of \((m_p/m_e)^3 \sim 10^{10}\). Thus, one can conclude that the radiation reaction of electrons can be quite relevant for astrophysically plausible magnetic fields providing reasonable decay times, while the radiation reaction of ions can be relevant only in the presence of magnetic fields much stronger than those corresponding to black holes, e.g., in the vicinity of neutron stars.

Estimations of the timescale of the radiative widening of the charged particle orbit show that this process is slower than the damping of oscillations due to radiation reaction by about \(10^{10}\) times in average for the values of the parameters \(\mathcal{B}\) and \(k\) given by Equations (72) and (68). However, in some astrophysical scenarios with large magnetic fields, this process can be potentially measured.

The radiation reaction and related decay time of particle oscillations and can have a significant effect on the plasma surrounding the black hole, when the decay time of the charged particle becomes smaller than the average time between collisions of particles in plasma.
7. Summary

We have studied the radiation reaction of charged particles moving around a Schwarzschild black hole immersed in an external asymptotically uniform magnetic field. We started our analysis from the study of the motion in flat spacetime, described by the LD equation and by its reduced form—the LL equation. We have tested both equations numerically and found that in the simplified framework of the asymptotically uniform magnetic field, both approaches lead to the same result. The LL equation is more convenient as it is a second-order differential equation, which can be solved in the usual manner, while for the LD equation, we have to perform the integration of the dynamical equations backwards in time in order to avoid the exponential increase of computational error.

We have shown that the tail term appearing in the general formalism for the motion in curved spacetime can be ignored in the presence of the magnetic field. Ignoring the Ricci term for the Schwarzschild black hole case enables the covariant form of the LD equation or its reduced form—the covariant LL equation—to be used. The motion of charged particles around a non-rotating black hole in the presence of an external uniform magnetic field can be classified into two different types, differing by the orientation of the Lorentz force directed towards and outwards from the black hole, which we called the minus and plus configurations, respectively. We concentrated our attention on the quasibounded orbits characterized by the presence of maxima and minima of the effective potential. In such orbits, the charged particles can undergo stable quasi-harmonic oscillations in the radial and vertical directions. We have shown that the presence of the radiation-reaction force leads to the decay of the particle oscillations, while the decay time increases as the orbit approaches the black hole. The final state of the particle depends on the orientation of the Lorentz force with respect to the black hole. In the case when the Lorentz force is directed toward the black hole, the radiation reaction leads to the fall of the charged particle from an initially stable orbit into the black hole. Inversely, when the Lorentz force is repulsive, the orbit of the charged particle remains bounded while oscillations decay.

We have shown that in the absence of oscillations, the radiation-reaction force can shift the circular orbits outwards from the black hole. This can occur only in the case where the Lorentz force is repulsive. While the stability of the circular orbit is conserved, the radiation reaction acting against the particle motion leads to the decrease of the linear velocity of the particle, which in turn leads to the widening of the orbit. Even though the kinetic energy of a particle decreases due to the SR, its potential energy increases faster due to the widening of the orbit. One can find the efficiency of this process, defined as the ratio between the gain energy required to shift the particle from ISCO to infinity to its final energy. The value of the efficiency depends on the initial position of the particle and the parameters $B$ and $k$, which governs the magnetic field and radiation-reaction force, respectively. The efficiency can reach the values, close to the maximal 100%. The maximal efficiency, however, can never be achieved, since it would require $B \to \infty$, and the ISCO would coincide with the black hole horizon. In the absence of a magnetic field, the formal efficiency of the mechanical process of shifting the particle orbit from infinity to the ISCO is 5.7%. One needs to note that the process of widening of orbits is many orders of magnitude slower than the process of cooling of the particle oscillations.

The physical mechanism causing the gain in energy requires further investigations.

We have estimated the relevant parameters of the model as well as the decay times of the particle oscillations for typical values of the magnetic field for stellar-mass and SMBHs. Assuming the test particle to be an electron, we have found that the values of the decay times are reasonable and can be measured in most astrophysical scenarios. Despite the slowness of the process of the radiative widening of orbits, the estimations show that for stellar-mass black holes in the presence of magnetic fields of order $10^8$ G and larger, the widening of orbits can be potentially measured.

We expect that our results can be relevant in treating a variety of astrophysical phenomena observed in microquasars or active galactic nuclei containing stellar-mass or SMBHs. These results can put relevant limits on the validity of recent models of high-frequency quasi-periodic oscillations (HF QPOs) or the motion of jets observed in microquasars. The limits implied by the radiation-reaction forces can be clearly relevant for both the magnetized geodesic models of HF QPOs discussed in Kološ et al. (2017; see also Stuchlík et al. 2013), or the string loop models of jets or HF QPOs discussed in Jacobson & Sotiriou (2009), Stuchlík & Kološ (2012), Kološ & Stuchlík (2013), and Tursunov et al. (2014).

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15

THE ASTROPHYSICAL JOURNAL, 861:2 (16pp), 2018 July 1

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