ON THE SEMI-RELATIVE CONDITION FOR
CLOSED (TOPOLOGICAL) STRINGS

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ABSTRACT

We provide a simple lagrangian interpretation of the meaning of the $b^-_0$ semi-relative condition in closed string theory. Namely, we show how the semi-relative condition is equivalent to the requirement that physical operators be cohomology classes of the BRS operators acting on the space of local fields covariant under world-sheet reparametrizations. States trivial in the absolute BRS cohomology but not in the semi-relative one are explicitly seen to correspond to BRS variations of operators which are not globally defined world-sheet tensors. We derive the covariant expressions for the observables of topological gravity. We use them to prove a formula that equates the expectation value of the gravitational descendant of ghost number 4 to the integral over the moduli space of the Weil-Peterson Kähler form.
1. Introduction

Physical states in closed string theory are identified with classes of the semi-relative BRS cohomology. The semi-relative BRS cohomology is given by equivalent classes of elements in the state space $\mathcal{H}$ of the string quantized on the infinite cylinder in the conformal gauge. A given class in this cohomology is identified by a state $|\psi\rangle$ satisfying the $b_0^-$ condition

$$(b_0 - \bar{b}_0)|\psi\rangle = 0,$$

and annihilated by the BRS operator $Q_{BRS}$ modulo $Q_{BRS}|\lambda\rangle$, for any state $|\lambda\rangle$ annihilated by $b_0^-$ $\equiv b_0 - \bar{b}_0$. In the above formulae, $b_0$ and $\bar{b}_0$ are the operators that correspond to the zero modes of the antighost fields $b(z)$ and $\bar{b}(\bar{z})$. The usual BRS cohomology, consisting of $Q_{BRS}$-invariant states modulo $Q_{BRS}|\lambda\rangle$ for arbitrary $|\lambda\rangle$, is known as the absolute string cohomology. Therefore the relative cohomology is the cohomology of the BRS operator acting on the subspace of the states annihilated by both $b_0$ and $\bar{b}_0$.

The validity of the condition (1) has been argued in [1] and [2], in the context of the so-called “operator formalism” [3]. Let us briefly outline this argument.

The central idea of the operator formalism is that, given a conformal field theory with $c = 0$, (e.g. a string background), one can associate to each Riemann surface $\Sigma_g$ of genus $g$, with a marked point $P \in \Sigma_g$ and a local coordinate system $z_P$ centered around $P$, a state $|\Sigma_g, P, z_P\rangle$ in the canonical state space $\mathcal{H}$. $|\Sigma_g, P, z_P\rangle$ is (formally) identified with the vacuum wave functional of the string action on $\Sigma_g/D$, where $D$ is the disk centered in $P$ with boundary at $z_P = 1$. The triple $X \equiv (\Sigma_g, P, z_P)$ defines a point in $\mathcal{P}_{g,1}$, the space of Riemann surfaces with a single marked point and a given local complex coordinate system around $P$. One similarly defines the spaces $\mathcal{P}_{g,n}$. In this case, the vacuum functional on $\Sigma_g$ with $n$ disks deleted defines a state in the tensor product $\mathcal{H}^\otimes n$. It is convenient to think of $\mathcal{P}_{g,1}$ as an infinite-dimensional fiber bundle over $\mathcal{M}_{g,1}$, the (finite-dimensional) moduli space of Riemann surfaces of genus $g$ with one puncture. Thus, the string functional integral giving the vacuum functional defines a (formal) map from $\mathcal{P}_{g,1}$ to $\mathcal{H}$ (and analogously defines maps from $\mathcal{P}_{g,n}$ to $\mathcal{H}^\otimes n$).

Given this map, one can define the “correlation functions” on $\Sigma_g$ of the “operators” $\psi_1, ..., \psi_n$ corresponding to the states $|\psi_1\rangle, ..., |\psi_n\rangle$ in $\mathcal{H}$, inserted at the points $P_1, ..., P_n$ of $\Sigma_g$, as the numbers

$$<\psi_1(P_1,z_1)\psi_1(P_n,z_n)\rangle = <\psi_1| \otimes ... \otimes <\psi_n| \Sigma_g, P_1, z_1, ..., P_n, z_n >.$$
In string theory, however, the objects of interest are not the functions over $M_{g,n}$ defined in Eq. (2), but top forms on $P_{g,n}$ which can be integrated over the moduli space to give string amplitudes. One can easily construct a $(3g-3+n)$-form $\tilde{\mu}_{\psi_1,\ldots,\psi_n}$ on $P_{g,n}$ in terms of the map from $P_{g,n}$ to $H^{\otimes n}$ discussed above, by using the Beilinson-Konsevitch action of the Virasoro algebra on the augmented moduli space $P_{g,n}$ [4], [5]:

$$\tilde{\mu}_{\psi_1,\ldots,\psi_n}(V_1,\ldots,V_{3g-3+n};\bar{V}_1,\ldots,\bar{V}_{3g-3+n}) = (\langle \psi_1| \otimes \ldots \otimes \langle \psi_n|) b(v_1)b(\bar{v}_1) \ldots $$

$$\ldots b(v_{3g-3+n})b(\bar{v}_{3g-3+n})|\Sigma_g,P_1,z_1,\ldots,P_n,z_n>,$$

(3)

where $V_i$ ($\bar{V}_i$), with $i = 1,\ldots,3g - 3 + n$, are tangent (anti)holomorphic vectors to $P_{g,n}$ at the point $(\Sigma_g,P_1,z_1,\ldots,P_n,z_n)$; $v_i$ ($\bar{v}_i$) are corresponding vector fields on $\Sigma$, (anti)holomorphic on the unit disk $D$ minus the point $P$ and meromorphic at $P$; $b(v_i) \equiv \oint_P dz P b_{zz} v_i^z$ are the associated anti-ghost insertions.

The problem with $\tilde{\mu}_{\psi_1,\ldots,\psi_n}$ is that it is a form on $P_{g,n}$, rather than on $M_{g,n}$. By selecting a section $\sigma$ of $P_{g,n}$ over $M_{g,n}$ - i.e. by choosing local coordinate systems $(z_{P_1},\ldots,z_{P_n})$ that vary smoothly as $(\Sigma_g,P_1,\ldots,P_n)$ varies over $M_{g,n}$ - one obtains a $3g - 3 + n$ form $\mu_{\psi_1,\ldots,\psi_n}$ on $M_{g,n}$ by pulling back $\tilde{\mu}_{\psi_1,\ldots,\psi_n}$ via $\sigma$:

$$\mu_{\psi_1,\ldots,\psi_n} = \sigma^* \tilde{\mu}_{\psi_1,\ldots,\psi_n}. $$

(4)

$\mu_{\psi_1,\ldots,\psi_n}$ depends on $\sigma$, that is on the chosen family of local coordinate systems $z_{P_i}$. However, standard arguments show that if the states $|\psi_i>$ are BRS closed then $\tilde{\mu}_{\psi_1,\ldots,\psi_n}$ is a closed form on $P_{g,n}$, and the cohomology class of the pulled back $\mu_{\psi_1,\ldots,\psi_n}$ is independent on $\sigma$ [1].

As pointed out in [1], the problem with Eq. (4) is that a global section $\sigma$ does not exist (for generic genus $g$). The best that one can do is to choose $\sigma$ to be a section continuous up to a phase. For the corresponding $\mu_{\psi_1,\ldots,\psi_n}$ to be continuous on $M_{g,n}$ it is necessary that $\tilde{\mu}_{\psi_1,\ldots,\psi_n}$ be invariant under the transformation corresponding to $L_0 - \tilde{L}_0 \equiv L_0^-$ and that $\tilde{\mu}_{\psi_1,\ldots,\psi_n}$ annihilates tangent vectors in the direction of this Virasoro generator. This is the case precisely when the physical states $|\psi_i>$ satisfy the $b_0^-$ condition of Eq. (1).

To summarize, the derivation of the semi-relative condition (1) in the operator formalism is strictly Hamiltonian. “Correlators” of operators on higher genus Riemann surfaces are defined via scalar products in the canonical state space $H$. Such definition involves the formal map from the infinite-dimensional bundle $P_{g,1}$ to the canonical space of states. It
relies on some geometrical properties of this bundle – namely, on the Beilinson-Konsevitch Virasoro action upon it and on the absence of a global section. The fact that states trivial in the absolute BRS cohomology but non-trivial in the semi-relative cohomology do not in general decouple is, ultimately, a self-consistency requirement of the operator formalism: in this approach a consistent definition of string amplitudes would not even be possible, were the $b_0^-$ condition not fulfilled.

However, it should be clear that in a lagrangian field theoretical approach to first quantized closed strings, the relevant BRS cohomology is neither a matter of choice nor of definition. Given a reparametrization invariant lagrangian on a two-dimensional arbitrary closed surface, the physical states of the corresponding closed string are automatically determined by the gauge invariance of the theory.

Here, in fact, we show that in the covariant lagrangian approach to first quantized closed bosonic strings and topological strings, the semi-relative condition (1) is identical to the requirement that physical observables be covariant under coordinate reparametrizations of the world-sheet. In other words, we prove that the physical states in the semi-relative cohomology are in one-to-one correspondence with non-trivial classes of the BRS operator acting on the space of covariant local field operators. The fact that operators corresponding to states trivial in the absolute cohomology but not trivial in the semi-relative one do not decouple in physical amplitudes is the direct consequence of the non-vanishing of world-sheet integrals of two-forms which are locally but not globally exact. Therefore, the “equivariance” principle of closed string theory is simply that of covariance under general coordinate transformations. This in turn is dictated by the fact that string amplitudes are integral over the world-sheet of correlators of field operators.

Beyond the virtue of simplicity, our novel derivation of the well known condition (1) also has the advantage of leading to covariant expressions for the operatorial observables in question. We expect that the covariant expressions we derive for the observables of topological strings will be important for the explicit field theoretical calculation of their correlators, still lacking in the literature. We intend to report on this in a future work.
2. Closed bosonic critical string

Let $g_{\mu\nu}$ be the two-dimensional metric on the world-sheet, $c^\mu$ the ghost fields, $X$ the matter fields. We will denote by $s$ the BRS operator acting on the fields. Since in critical strings the Liouville field is not dynamical, gauge transformations are two-dimensional diffeomorphisms accompanied with a compensating Weyl transformation. Therefore the action of $s$ is:

$$sg_{\mu\nu} = \delta_c g_{\mu\nu} - (Dc) g_{\mu\nu}$$
$$sc^\mu = \frac{1}{2} \delta_c c^\mu = c^\nu \partial_\nu c^\mu$$
$$sX = \delta_c X,$$

where $\delta_c$ denotes the action of the diffeomorphisms with parameters $c^\mu$, and $Dc \equiv D_\alpha c^\alpha$, with $D_\alpha$ being the covariant derivative. Let $d$ be the exterior derivative on forms; $s$ and $d$ commute among themselves. To each two-form $\Omega^{(2)}$, $s$-closed modulo $d$, one can associate a one-form $\Omega^{(1)}$ and a zero-form $\Omega^{(0)}$ satisfying the famous descent equations:

$$s\Omega^{(2)} = d\Omega^{(1)}$$
$$s\Omega^{(1)} = d\Omega^{(0)}$$
$$s\Omega^{(0)} = 0.$$  \hspace{1cm} (6)

Furthermore, if $\Omega^{(0)}$ is a local observable, $\Omega^{(2)}$ belongs to the cohomology of $s$ modulo $d$. Noting that the cohomology of $s$ in the space of the two and one-forms is trivial, we can revert the statement above: if $\Omega^{(2)}$ belongs to the cohomology of $s$ modulo $d$, $\Omega^{(0)}$ is necessarily in the cohomology of $s$.

Now take

$$\Omega^{(2)} = \frac{1}{2} \sqrt{g} R \epsilon_{\mu\nu} dx^\mu \wedge dx^\nu \equiv \mathcal{R}^{(2)}$$  \hspace{1cm} (7)

where $g = det(g_{\mu\nu})$, $R$ is the scalar two-dimensional curvature and $\epsilon_{\mu\nu}$ is the antisymmetric numeric tensor defined by $\epsilon_{12} = 1$. One can verify that $\Omega^{(2)}$ satisfies the descent equations (6) with $\Omega^{(0)}$ and $\Omega^{(1)}$ given by:

$$\Omega^{(1)} = \sqrt{g} \epsilon_{\mu\nu} (c^\nu R + g^{\nu\rho} \partial_\rho(Dc)) dx^\mu$$
$$\Omega^{(0)} = \sqrt{g} \epsilon_{\mu\nu} \left( \frac{1}{2} c^\mu c^\nu R + c^\mu g^{\nu\rho} \partial_\rho(Dc) \right).$$  \hspace{1cm} (8)

$\Omega^{(0)}$ is the local operator which corresponds to the dilaton state at zero momentum $|D> = (c_{-1} c_1 - \bar{c}_{-1} \bar{c}_1)|0>$, as it is apparent by considering $\Omega^{(0)}$ in the conformal gauge $g_{\mu\nu} = e^\phi \delta_{\mu\nu}$ and applying it to the $SL(2,C)$ invariant vacuum $|0>$. $|D>$ is the unique example.
in critical string of a state which is trivial in the absolute $Q_{BRS}$ cohomology (since $|D > = Q_{BRS}c_0|0 >$) but non-trivial in the semi-relative one (since $b_0^-(c_0^-|0 >) \neq 0$), and which therefore does not decouple in generic amplitudes.

It is now clear how this translates into our covariant field theoretical framework. If $\Omega^{(2)} = d\omega^{(1)}$, then it follows easily from the descent equations that

$$\Omega^{(1)} = s\omega^{(1)} + d\omega^{(0)}$$

(9)

for some $\omega^{(0)}$, and

$$\Omega^{(0)} = s\omega^{(0)}.$$  

(10)

However if $\omega^{(1)}$ cannot be chosen to be a covariant one-form, $\omega^{(0)}$ will not be a scalar under reparametrizations, as it is implied by Eq. (9). This is precisely the case for the operators corresponding to the dilaton state. $\Omega^{(2)}$ in Eq. (7) is locally but not globally exact, that is $R^{(2)} = d\omega^{(1)}$ but $\omega^{(1)}$ is not a globally defined one-form. Therefore one can write $\Omega^{(0)}$ in Eq. (8) as the $s$ variation of something, $\Omega^{(0)} = s\omega^{(0)}$, but $\omega^{(0)}$ will not be coordinate independent. Moreover, the triviality of the $s$-cohomology in the space of the two and one-forms guarantees the absence of any globally defined $\omega^{(0)}$ satisfying Eq. (10).

The facts that the dilaton state does not decouple in generic physical amplitudes and that its expectation value is topological follow directly from the covariant two-form representation in Eq. (7). In contrast, the calculation of the dilaton correlation functions in the conformal operator formalism requires a careful consideration of the transformation properties of non-covariant correlators under changes of coordinate patches [6], [1].

One can calculate explicitly $\omega^{(0)}$ and $\omega^{(1)}$ by choosing a particular coordinate system. In order to compare with the conformal field theory formalism, let us choose a system of holomorphic coordinates with metric given by $ds^2 = |\lambda(dz + \mu d\bar{z})|^2$, where $\mu$ is a Beltrami differential. Then

$$\omega^{(1)} = \frac{2}{\Theta} \left( \partial\mu - \mu \bar{\partial}\bar{\mu} + \frac{1}{2} (\bar{\nabla} - \mu \nabla) ln\Theta \right) d\bar{z} - c.c.,$$

(11)

where we introduce the symbol $\Theta \equiv 1 - \mu \bar{\mu}$ and the derivative $\nabla \equiv \partial - \bar{\mu} \bar{\partial}$, and where $c.c.$ denotes the conjugate expression in which all quantities are substituted with their barred expressions. For $\omega^{(0)}$ one obtains the result:

$$\omega^{(0)} = \nabla c + \bar{c}(\omega^{(1)} \bar{z} - c.c..$$

(12)

Going to a conformal frame with $\mu = 0$, $\omega^{(0)}$ reduces to $\partial c - \bar{\partial}\bar{c}$, the conformal operator creating the state $c_0^-|0 >$ which does not satisfy the semi-relative condition (4).
3. Topological gravity

The basic fields of topological gravity \([7]\) are the metric \(g_{\mu\nu}\) and its gravitino superpartner \(\psi_{\mu\nu}\), together with the anticommuting ghost \(c^\mu\) and their commuting superpartners \(\gamma^\mu\).

Note that in this case the Liouville field is a dynamical field and thus is not inert under BRS transformations. The nilpotent BRS transformations are:

\[
\begin{align*}
sg_{\mu\nu} &= \delta_c g_{\mu\nu} + \psi_{\mu\nu} \\
sp_{\mu\nu} &= \delta_c \psi_{\mu\nu} - \delta_\gamma g_{\mu\nu} \\
sc^\mu &= \frac{1}{2} \delta_c c^\mu + \gamma^\mu = c^\nu \partial_\nu c^\mu + \gamma^\mu \\
sg^\mu &= \frac{1}{2} \delta_\gamma c^\mu - \frac{1}{2} \delta_c \gamma^\mu = c^\nu \partial_\nu \gamma^\mu - \partial_\nu c^\mu \gamma^\nu,
\end{align*}
\]  

(13)

where \(\delta_c\) and \(\delta_\gamma\) are the variations under reparametrizations with parameters \(c^\mu\) and \(\gamma^\mu\) respectively.

Let us solve again the descent equations (6) with \(\Omega^{(2)}\) given by the Euler two-form as in Eq. (7) and the BRS transformations defined by Eq.(13). For the one and zero-forms one obtains the following expressions:

\[
\begin{align*}
\Omega^{(1)} &= \sqrt{g} \epsilon_{\mu\nu} (c^\nu R + D_\rho (\bar{\psi}^\rho - \frac{1}{2} g^\rho_\sigma \psi^\sigma)) dx^\mu \\
\Omega^{(0)} &= \sqrt{g} \epsilon_{\mu\nu} \left(\frac{1}{2} c^\mu c^\nu R + c^\mu D_\rho (\bar{\psi}^\rho - \frac{1}{2} g^\rho_\sigma \psi^\sigma) + D^\mu \gamma^\nu - \frac{1}{4} \bar{\psi}_\rho \bar{\psi}^\rho\right),
\end{align*}
\]

(14)

where \(\bar{\psi}_{\mu\nu} \equiv \psi_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \psi^\sigma\) is the traceless part of the gravitino. These formulas for \(\Omega^{(1)}\) and \(\Omega^{(0)}\) are the covariant generalizations of the expressions which hold in the conformal gauge and which are given in \([8]\).

As before, since \(\Omega^{(2)} = d\omega^{(1)}\) is locally exact, it follows from the descent equation that \(\Omega^{(0)} = s \omega^{(0)}\), but with \(\omega^{(0)}\) not being a reparametrization invariant scalar. Using a holomorphic coordinate system parametrized by the Beltrami differential \(\mu\), one can compute \(\omega^{(0)}\) to be:

\[
\omega^{(0)} = \nabla c + \bar{c}(\omega^{(1)}) \bar{\zeta} + \frac{\bar{\mu} \psi}{\Theta} - c.c.,
\]

(15)

where \(\psi = s \mu\) and \(\bar{\psi} = s \bar{\mu}\) are the components of the traceless part of the gravitino field \(\bar{\psi}_{\mu\nu}\) in the holomorphic coordinate system. In a conformal frame in which \(\mu = 0\), one can again check that \(\omega^{(0)}\) reduces to \(\partial c - \bar{\partial} \bar{c}\), the conformal operator creating the state \(c_0^0 |0\rangle\) which violates the semi-relative condition in Eq.(1).
Thanks to the commutative nature of the superghosts $\gamma^\mu$, of topological gravity, one can construct an infinite set of non-vanishing observables $\Omega_n^{(0)} = (\Omega^{(0)})^n$ with $n = 0, 1, \ldots$ by taking positive powers of the “dilaton” operator $\Omega^{(0)}$. The operators $\Omega_n^{(0)}$ are believed to exhaust the BRS local cohomology. They all create states which are trivial in the absolute BRS state cohomology but non-trivial in the semi-relative one. The one-form and two-form expressions associated to $\Omega_n^{(0)}$ can be expressed in terms of the operators $\Omega^{(0)}$, $\Omega^{(1)}$ and $\Omega^{(2)}$: 

\[ \Omega_n^{(1)} = n \Omega_n^{(0)n-1}\Omega^{(1)} \]
\[ \Omega_n^{(2)} = n \Omega_n^{(0)n-1}\Omega^{(2)} + \frac{1}{2} n(n-1)\Omega_n^{(0)n-2}\Omega^{(1)} \wedge \Omega^{(1)}. \]  

It is expected that the word-sheet integral of the matrix elements of the two-form $\Omega_n^{(2)}$ be a closed $2n-2$-form on the moduli space $\mathcal{M}_g$. In particular, $<\int_{\Sigma_g} \Omega_2^{(2)}>$ should correspond to a closed (1,1) form on $\mathcal{M}_g$.

We intend to present a more detailed analysis of the correlation functions of the integrated observables $\Omega_n^{(2)}$ in a covariant lagrangian approach in a forthcoming paper. Here we limit ourselves to the following observation.

When considering vacuum expectation values of the integrated observables $\Omega_n^{(2)}$, all the terms in Eq. (14) but the term bilinear in the gravitino field average to zero. The gravitino field $\tilde{\psi}^{\mu\nu}$ is naturally identified with a cotangent vector on $\mathcal{M}_g$, and one can show that after performing the fermionic functional integration, it can be substituted with following expression:

\[ \tilde{\psi}^{\mu\nu} = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{\mu\nu}) dm^i \equiv \frac{1}{\sqrt{g}} dm (\sqrt{g} g^{\mu\nu}), \]  

(17)

where $m_i$ are coordinates on $\mathcal{M}_g$, $\partial_i$ and $dm^i$ are the corresponding derivatives and one-form elements. $g^{\mu\nu} = g^{\mu\nu}(x; m^i)$ is a two-dimensional metric which represents the gauge equivalence class of metrics corresponding to the point of $\mathcal{M}_g$ with coordinates $\{m^i\}$. On account of the explicit formulas given in Eq. (14) above, the vacuum expectation value of $\Omega_2^{(2)}$ is reduced therefore to the following closed (1,1) form on the moduli space $\mathcal{M}_g$:

\[ \mu_{\mathcal{M}_g} \equiv <\int_{\Sigma} \Omega_2^{(2)}> = \int_{\Sigma} d^2 x g_{\alpha\beta} \left( -\frac{1}{2} R\tilde{\psi}^{\alpha\mu}\tilde{\psi}_\mu^\beta + D_\mu \tilde{\psi}^{\mu\alpha} D_\nu \tilde{\psi}^{\nu\beta} \right). \]  

(18)

One can verify that the two-form in Eq. (18), when evaluated using a constant curvature section $g^{\mu\nu}(x; m^i)$, coincides with the Weil-Peterson Kähler two-form on $\mathcal{M}_g$. 

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Hence the expectation value of $3g - 3$ integrated $\Omega_2^{(2)}$'s should formally be given by the volume of $\mathcal{M}_g$ computed via the Weil-Peterson symplectic form. To be able to integrate over $\mathcal{M}_g$ powers of the $\mu_{\mathcal{M}_g}$ given in Eq. (18), one needs to study the problem of extending $\mu_{\mathcal{M}_g}$ to $\overline{\mathcal{M}}_g$, the Deligne-Mumford compactification of $\mathcal{M}_g$. Because of the divergencies of the Weil-Peterson form at the boundary of $\mathcal{M}_g$ [12], this is a delicate issue and corresponds in the field theoretical language to the problem of possible BRS anomalies induced by contributions at the boundary of $\mathcal{M}_g$ [13].
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