Unilateral Small Deviations for the Integral of Fractional Brownian Motion

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Abstract. We consider the paths of a Gaussian random process $x(t)$, $x(0) = 0$ not exceeding a fixed positive level over a large time interval $(0, T)$, $T \gg 1$. The probability $p(T)$ of such event is frequently a regularly varying function at $\infty$ with exponent $\theta$. In applications this parameter can provide information on fractal properties of processes that are subordinate to $x(\cdot)$. For this reason the estimation of $\theta$ is an important theoretical problem. Here, we consider the process $x(t)$ whose derivative is fractional Brownian motion with self-similarity parameter $0 < H < 1$. For this case we produce new computational evidence in favor of the relations $\log p(T) = -\theta \log T (1 + o(1))$ and $\theta = H (1 - H)$. The estimates of $\theta$ are to within 0.01 in the range $0.1 \leq H \leq 0.9$. An analytical result for the problem in hand is known for the markovian case alone, i.e., for $H = 1/2$. We point out other statistics of $x(t)$ whose small values have probabilities of the same order as $p(T)$ in the log scale.

Key words: fractional Brownian motion, fractality, long excursions, small deviations, Monte Carlo methods

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1 Introduction

The asymptotics of tail probabilities $P(\max \Delta x(s) > u), u \to \infty$ of a Gaussian random process $x(t)$ is a problem that has received a sufficiently complete solution (see, e.g., [5], [12]). Considerable progress also occurs in the study of probabilities of small deviations, that is, of events of the form $\{\max_{\Delta} |x(s)| < \varepsilon\}$ as $\varepsilon \downarrow 0$ (see, e.g., [7], [6]). However, there is practically a complete absence of general approaches to the analysis of probabilities of unilateral small deviations, i.e., of probabilities of the form

$$p(u|T) = P(\max_{\Delta T} (x(s) - x(0)) < u)$$

(1)

for small $u$ or large $T$. Here, $T$ is the similarity ratio and $\Delta$ a finite closed interval that contains 0. If $x(\cdot)$ is a self-similar process, that is, finite dimensional distributions of $\{x(t\Lambda)\}$ and $\{|\Lambda|^h x(t)\}$ are identical for any $\Lambda \neq 0$, then $p(u|T) = p(uT^{-h}|1)$, so that the problems on the asymptotics of (1) for the $u$ and $T$ indicated above are equivalent.

Exponential asymptotics are typical of large and small deviations of $x(\cdot)$, while power law asymptotics are typical of unilateral small deviations. Power law asymptotics are rather popular in physics, since they frequently provide information on fractal properties of physical processes (see the examples later in this paper). In this connection one is faced with the task of calculating the exponent

$$\theta = -\lim \log p(1|T)/\log T, \quad T \to \infty.$$  

(2)

We shall refer to the exponent $\theta$ for a sequence of events $A_T$ in what follows, when $\log P(A_T) = -\theta \log T(1 + o(1))$ as $T \to \infty$.

There are few explicit estimates of [12]. Molchan ([11]) has found the exponent $\theta$ for fractional Brownian motion (FBM), $b_H(s)$. That one-parameter family of processes is specified by the requirement $b_H(0) = 0$ and by the structure function

$$E|b_H(t) - b_H(s)|^2 = \sigma |t - s|^{2H}, \quad 0 < H < 1,$$
where $H$ is the self-similarity parameter of the process. When $H = 1/2$, $b_H(s)$ becomes Brownian motion.

According to [9], $\theta = 1 - H$ when $\Delta \cdot T = (0, T)$ and $\theta = 1$ when $\Delta \cdot T = (-T, T)$. The last estimate is of interest in that it is independent of $H$. The property in question also remains valid for FBM with multidimensional time, $t \in R^d$. In that case one has $\theta = d$, if $\Delta \cdot T = \{-T, T\}$.

In mathematical physics one is interested in the parameter $\theta$ for the integral of fractional Brownian motion (IFBM), that is, for the process $x(t) = \int_0^t b_H(s)ds$, $|t| < \infty$ [14], [15]. By now, the problem of $\theta$ has been solved for $H = 1/2$ only, when $(x(t), x'(t))$ is a Markov process. Sinai [13] showed that $\theta = 1/4$ for $\Delta \cdot T = (0, T)$. Since paths of $x(t)$ in $(-T, 0)$ and $(0, T)$ are independent, it follows that $\theta = 1/2$ for $\Delta \cdot T = (-T, T)$. The general case $0 < H < 1$ was studied by the present authors ([10]) both analytically and numerically. Our analysis suggests the following hypothesis:

$$
\theta = \begin{cases} 
1 - H, & \Delta \cdot T = (-T, T) \\
(1 - H)H, & \Delta \cdot T = (0, T)
\end{cases} 
$$

(3)

The hypothesis (3a) is corroborated by a related result in [3] and by numerical calculations. As to (3b), it has been confirmed numerically only for the interval $0.1 \leq H \leq 0.6$ to a low accuracy ($\delta = 0.03$). Here we continue the numerical analysis of our hypothesis (3b) by making use of analytical results derived in [10], but radically modify the evaluation strategy. This allows corroboration of (3b) in the entire range of $H$ to within $\delta < 0.01$. One by-product is to provide support in favor of the following asymptotics:

$$
P(x(s)x(1) > 0, 1 \leq s \leq T) = O((\log T)^{\alpha(H)}T^{-H(1-H)}), \quad T \to \infty,
$$

where $\alpha(H)$ may have the form $H - 1/2$.

2 Examples

Consider a few examples where exponents like (2) are used.

Let $M(t) = \max_{(0,t)}(x(s) - x(0))$ be the record function of the process $x(\cdot)$. Levy has shown (see [11]) that the record function of Brownian motion is
similar to Cantor’s staircase. Its points of growth make a set $S$ of Hausdorff dimension $\dim S = 1/2$. The situation for the FBM family is analogous: $\dim S = H$. This result is based on two facts for FBM in $\Delta \cdot T = (0, T)$: the exponent is $\theta = 1 - H$, and

$$\log[P(G(T) < 1)/P(M(T) < 1)] = o(\log T),$$

(4)

where $G(T)$ is the position of the maximum $M(T)$ in the interval $\Delta \cdot T$ \cite{9}.

Denote by $Z_+$ the first zero of $b_H(t)$ after the time $t = 1$; then, similarly to (4),

$$\log[P(Z_+ > T)/P(M(T) < 1)] = o(\log T).$$

Consequently, the log asymptotics of $P(M(T) < 1)$ also determines the log asymptotics of long excursions of FBM. The interest in such asymptotics is rather broad in physics and engineering (see review \cite{5}).

There is a similar problem in geology bearing on the dimension of fractal sets. Molchan and Turcotte \cite{11} consider a simple one-dimensional model of sedimentation in shallow seas. The model involves two mechanisms that produce the sediment: tectonic forces which cause relative rise/fall of sea level on the one hand and erosion on the other. A sea level fall brings the upper sedimentary layers above the water and causes a fast (on the geological time scale) erosion of these. The erosion process causes time gaps (unconformities) in datings of the sedimentary layers. A sea level rise provokes sedimentation. The latter process is considered to be rapid, so that the sediment follows sea level variations practically continuously. Supposing the sea level history to be described by a process $x(t)$, the drilling data (layer depth and date of formation) must be described by the function

$$y(t) = \min_{s > t} x(s).$$

Points of growth of $y(t)$ (the set $S$) correspond to the dates of those layers which have been preserved in the sedimentary sequence during the entire history of sedimentation. The common assumption makes $x(s)$ a process with stationary increments; the self-similarity of $S$ in an extensive range of scales suggests self-similarity for the stochastic component of $x(\cdot)$. When $x(s)$
is also assumed to be Gaussian, one arrives at the model \(x(s) = as + b_H(s), a > 0\) (see e.g. [2]). The dimension of the support of \(y(t)\) for the above model of \(x(\cdot)\) can be found similarly to the preceding problem, giving the result \(\dim S = H\) ([11]).

The second example is concerned with the fractal nature of solutions to the Burgers equation with random initial data:

\[
\begin{align*}
  u_t + uu_x &= \nu u_{xx} \\
  u(0, x) &= v(x), \quad |x| < \infty
\end{align*}
\]

where \(\int x v(a) \, da = o(x^2), x \gg 1\) (see, e.g., [16]).

When the viscosity \(\nu\) is infinitely small, equation (5) when considered in the one-dimensional case describes the following dynamics of adhesive particles. A particle at the initial time at position \(a\) has the mass \(da\) and the momentum \(v(a)da\). The particle starts moving at velocity \(v(a)\) and preserves that velocity until the first collision with its neighbors. The colliding particles stick together and continue their motion by following the laws of preservation of mass and momentum, and so on. The question that arises is what is the dimension of the initial positions of the particles \(S\) that have not collided until time \(t_0\). Such positions are called Lagrangian regular points.

The question is amenable to a purely geometrical interpretation. Let \(U(x)\) be a convex minorant of the curve \(\xi(x) = \int_0^x v(a) \, da + x^2/2\). Then the right-hand derivative \(U'(x)\) is nondecreasing function, and its points of growth correspond to Lagrangian regular points \(S\). The dimensionality problem of \(S\) in the case \(v(a) = b_{1/2}(a)\) has been solved by Sinai [14]: \(\dim S = 1/2\). The solution is based on estimating the exponent (2) for the integral of Brownian motion in the interval \((0, T)\). For the case \(v(a) = b_H(a)\) it can be asserted that \(\dim S = H\), if \(\theta\) for IFBM in \(\Delta \cdot T = (-T, T)\) admits of the bound \(\theta \geq (1 - H)\) ([10]).

The exponents (2) for the maximum of IFBM are closely related to the exponents of other statistics. To be more precise we introduce the following notation: \(M(T) = \max_{\Delta T} x(t)\), \(G(T)\) is the position of \(M(T)\) in \(\Delta \cdot T\), and \(A(T) = \int_{\Delta T} 1_{x(s) > 0} \, ds\) is the occupation time of \(x(t)\) above 0 in \(\Delta \cdot T\).

**Statement 1.** (Molchan & Khokhlov, [10]). Let \(\Delta \cdot T = (0, T)\) or \((-T, T)\). For \(x(t) = \text{IFBM}(t), t \in \Delta \cdot T\) the exponents of the events \(\{M(T) < 1\}, \{|G(T)| < 1\}, \{A(T) < 1, |G(T)| < T\}, \{x(t) < 0, t \in \Delta \cdot T, |t| > 1\},\)
(a) are identical, when they exist;
(b) exist or do not exist simultaneously.

It is a known fact that $G(1)$, $A(1)$ and the last zero, $Z$, for Brownian motion in $(0,1)$ have identical distributions, namely, the arcsine law. Statement 1 is also true for FBM ([9]). Therefore we can consider this result as a weak version of the arcsine law for FBM and IFBM. Statement 1 provides a certain degree of freedom in numerical analyses of [2].

3 Evaluation of $\theta$

We are going to evaluate the exponent of the event \{$M(T) < 1$\} for the IFBM process in the interval $(0,T), T \gg 1$ using the Monte Carlo method. To select a suitable strategy note the following. Statement 1 gives some information on the possible structure of a typical IFBM path with a low maximum, $M(T) < 1$. Namely, the position of the maximum $G(T)$ and the total time $A(T)$ where IFBM $> 0$ do not practically grow (the growth is most likely to be a logarithmic one). In most of the cases the path goes to the lower half-plane after the lapse of a fixed time, because there are no limitations on the amplitude from below. Consequently, the essential information concerning the low maximum is available around the initial point $t = 0$.

Suppose we generate IFBM on a uniform lattice $(0, \delta, \ldots, L\delta)$ using triangular factorization of the correlation matrix. We note that the correlation structure of the process should be reproduced exactly in the case under consideration, since we are dealing with rare events \{$M(T) < 1$\}. Hence we shall need a memory of order $L^2$ for generating a large number of samples. However, the second-degree growth in memory leads to limitations on lattice width, hence on the information concerning the low maximum. This heuristic argument is borne out in [10]. Following the above strategy, we have not succeeded in evaluating $\theta$ for $0.6 \leq H \leq 1$.

The way out consists in considering the IFBM in a log time scale. To do this, we consider the Lamperti transformation which converts \{IFBM$(t), t > 0$\} into a stationary process $x(t)$:

$$x(t) = c \exp(-(1 + H)t)\text{IFBM}(e^t), \quad |t| < \infty.$$  \hspace{1cm} (6)

We have normalized the process so as to make $Ex^2(t) = 1$, hence $c = (1 + 2H)^{-1/2}$. The correlation function $r(t)$ for $x(t)$ is
\[ r(t) = 2c^2(1 + H) \cosh(HT) - c^2 \cosh((1 + H)t) + 0.5c^2|\sinh(t/2)|^{2H+2} \]

and

\[ r(t) = \begin{cases} 1 - 0.5(1 - H^2)t^2 + 0.5c^2t^2 + O(t^4), & t \to 0 \\ c(H) \cdot \exp(-\rho t) \cdot (1 + o(1)), & t \to \infty, \end{cases} \]

where \( \rho = \min(H, 1 - H) \).

We can see that \( x(t) \) has a smoothness of order \( 1 + H - \varepsilon \) (\( \varepsilon > 0 \)), the number of zeroes of \( x(\cdot) \) is locally finite, and the mean interzero distance is, according to Rice, given by

\[ \Delta_0 = \pi(1 - H^2)^{-1/2}. \] (7)

The transformation \( [6] \) does not preserve the point \( (G(T), M(T)) \) as an extreme one in a sample of \( x(\cdot) \). However, the event

\[ \{ \text{IFBM}(t) < 0, \ 1 < t < T \} = A_T \]

is easily rewritten in terms of \( x(\cdot) \) on a finite interval:

\[ A_T = \{ x(\tau) < 0, \ 0 \leq \tau < T' = \ln T \} := \tilde{A}_T. \]

Consequently, the desired exponent

\[ \theta = - \lim_{T' \to \infty} \ln P(\tilde{A}_{T'})/T' \] (8)

is converted to exponential from a power-law one. As a result, the evaluation of \( \theta \) splits into three steps:

- generating a stationary process \( x(t), t > 0 \);
- evaluating the distribution of the first zero, \( Z \), for \( x(t) \);
- finding \( [8] \) for the tail of the distribution of \( Z \).
The Generation of $x(t)$. The process $x(t)$ was generated as a stationary sequence $x(\delta \cdot k)$ with the exact correlation function $r(\delta k)$, $k = 0, 1, \ldots, L$. This was done using the triangular representation

$$x(k\delta) = \sum_{i=0}^{k} a(i|k)\varepsilon_i \quad (9)$$

in terms of the standard white noise $\{\varepsilon_i\}$. The representation is implemented by using the progressive Schur algorithm (see [1]). The discretization step $\delta$ is specified by the number of points $n_0$ per mean period $\Delta_0$ for the zeroes of $x(t)$ (see (7)). Since $x(\cdot)$ has a smoothness of order $\sim 1 + H$, and we are interested in long excursions of $x(t)$, we can well use moderate numbers for $n_0$; we had $n_0 = 50$ in the calculations. The length $L$ can be found from the requirement $P(Z > L\delta) = \varepsilon$ where $\varepsilon$ is small.

As is shown by some preliminary evaluations of the distribution of $Z$ (see Fig. 1), the function $P(Z > t)$ for $H = 0.1 \div 0.4$ (left) and $H = 0.5 \div 0.9$ (right). Straight lines correspond to $y = H(1 - H)x$.

$Z_\varepsilon = -\ln \varepsilon/[H(1 - H)]$. \quad (10)
When $\varepsilon = 10^{-4}$ and $H = 0.2 - 0.8$, one has $L\delta \simeq 60$, and $L\delta = 100$ when $H = 0.1, 0.9$. The Schur algorithm is a recursive one, so it may become unstable as $L$ increases. The instability manifests itself in parasitic oscillations of the $a(i|k)$ at large $i$ (see (9)). These effects are typical of $i \cdot \delta \geq 40$ and hardly can always be overcome by using available accuracy (for instance long double in C). Hence it follows that

- there are computational difficulties in the way of analyzing the distribution of $Z$ when $H$ is close to either 0 or 1; more exactly, when $|H - 0.5| > 0.4$;
- evaluation of $\theta$ from the values $Z > Z_\varepsilon$, $\varepsilon < 10^{-4}$ calls for higher computation accuracy.

The last conclusion is important, because we do not know when the log linear asymptotics for the tail of the distribution of $Z$ becomes valid.

**Evaluation of $\theta$.** The parameter $\theta$ was evaluated in the series of intervals $\Delta_\varepsilon = (Z_\varepsilon, Z_{\varepsilon/10})$ with $\varepsilon = 0.01; 0.003; 0.001$ for the range $H = 0.1 - 0.9$. Figure 1 does not contradict the assumption of linearity for the plot of $(t, \ln P(Z > t))$, $t \in \Delta_\varepsilon$. For this reason we use for the slope of the plot the maximum likelihood (ML) estimate $\widehat{\theta}$ corresponding to the distribution $\{c(\theta)e^{-\theta t}, t \in \Delta_\varepsilon\}$. Namely, $\widehat{\theta} = x/|\Delta_\varepsilon|$ where $|\Delta_\varepsilon| = \frac{1}{\theta_0} \ln 10$ is the length of $\Delta_\varepsilon$, $\theta_0 = H(1 - H)$, and $x$ is the root of

$$x^{-1} - (e^x - 1)^{-1} = [Z > \varepsilon - Z_\varepsilon]/|\Delta_\varepsilon|.$$  \hfill (11)

Here, $< Z >_\varepsilon$ is the empirical mean of all $Z$ observed in the interval $\Delta_\varepsilon$.

ML estimates of $\theta$ based on $N = 16 \times 300,000$ paths of $x(\cdot)$ are shown in Fig. 2. They demonstrate that the slopes $\hat{\theta}_\varepsilon$ in the intervals $\Delta_\varepsilon, \varepsilon = 0.01, 0.003, 0.001$ are well consistent among themselves and are identical with the hypothetical values $\theta_0 = H(1 - H)$ to within 0.01. Note that the left-hand endpoint of $\Delta_\varepsilon$ is approximately identical with the $(1 - \varepsilon)$ quantile of the distribution of $Z$. Consequently, the number of observations $N_\varepsilon$ used to estimate $\theta_\varepsilon$ is approximately equal to $N \cdot \varepsilon$.

The ML estimate of $\theta$ for the truncated exponential distribution $\{c(\theta)e^{-\theta t}, t \in (\ln \varepsilon^{-1}, \ln 10/\varepsilon)\theta^{-1}\}$ has the standard deviation $\sigma_\varepsilon \simeq 1.7014 \theta N_\varepsilon^{-1/2}$, where $N_\varepsilon \gg 1$ is the number of observations. In our case the slope estimates are close to $\theta_0 = H(1 - H)$, while $N_\varepsilon$ is large, hence $\tilde{\sigma}_\varepsilon = 1.7 \theta_0 N_\varepsilon^{-1/2}$ can serve as a satisfactory theoretical estimate of the standard deviation for $\hat{\theta}_\varepsilon$. This statement is corroborated by our experiments carried out to check the operation of the random digit generator and the Schur
algorithm. The estimates $\hat{\theta}_\varepsilon$ were derived above by averaging over 16 serial estimates $\hat{\theta}_{\varepsilon}^{ser}$. Each series consists of 300,000 paths of $x(\cdot)$. The empirical variance of the averaged estimate $\hat{\theta}_\varepsilon$ is in good agreement with the theoretical value. Examples are given below in Table 1.

Figure 3 shows interval estimates $\hat{\theta}_\varepsilon \pm \tilde{\sigma}_\varepsilon$ of the slope $\hat{\theta}_\varepsilon$ for $\varepsilon = 0.01$. Even though we have seen above that $|\hat{\theta}_\varepsilon - \theta_0| < 0.01$, Fig. 3 provides evidence of a significant discrepancy between empirical and hypothetical estimates of $\theta$. Furthermore we can see that $\hat{\theta}_\varepsilon(H) > \theta_0(H)$ when $H < 0.5$ and $\hat{\theta}_\varepsilon(H) < \theta_0(H)$ when $H > 0.5$ for all $\varepsilon = 0.01, 0.003$ and 0.001 (Fig. 2).

The above inference cannot be ascribed to the effect of discretization. This is confirmed by the estimates of $\theta$ for $H = 0.7$ with different values of the discretization parameter: $n_0 = 30$ and 100 steps per period $\Delta_0$ (see Table 1). The estimates of $\theta_\varepsilon$ correspond to the intervals $\Delta_\varepsilon = (\ln \frac{1}{\varepsilon}, \ln \frac{10}{\varepsilon}) \cdot \theta_0^{-1}$ with $\varepsilon = 0.1, 0.01, 0.001$ and were derived by averaging over $r$ serial estimates $\hat{\theta}_{\varepsilon}^{ser}$. Each series consists of $10^6$ paths of $x(\cdot)$, $r = 120$ in the case $n_0 = 30$.
Table 1: Estimates $\hat{\theta}$ in intervals $\Delta_\varepsilon$ for $H = 0.7$ and discretization parameter $n_0 = 30$ (top) and $n_0 = 100$ (bottom). The standard deviations of $\hat{\theta}$ are $\hat{\sigma}_{\hat{\theta}}$ (empirical) and $\tilde{\sigma}_{\hat{\theta}}$ (theoretical).

| $n_0$ | $\varepsilon$ | $\hat{\theta}$ | $\hat{\sigma}_{\hat{\theta}}$ | $\tilde{\sigma}_{\hat{\theta}}$ |
|-------|----------------|-----------------|-----------------|-----------------|
| 30    | 0.010          | 1.998           | 0.0031          | 0.00032          |
|       | 0.001          | 1.997           | 0.0089          | 0.00096          |
| 100   | 0.010          | 2.001           | 0.00111         | 0.0012           |
|       | 0.001          | 1.989           | 0.00107         | 0.00112          |

and $r = 90$ in the case $n_0 = 100$. Table 1 also lists empirical and theoretical standard deviations of the estimates of $\theta_\varepsilon$. It appears from Table 1 that the estimates $\hat{\theta}_\varepsilon \approx 0.200$ are independent of the discretization step and there are significant deviation of $\hat{\theta}_\varepsilon$ from the hypothetical value $\theta_0 = 0.21$.

We thus have to reject the hypothesis that the tail of the distribution of $Z$ has a purely exponential asymptotics with parameter $\theta = \theta_0$. However, this does not rule out the hypothesis proper of the exponent $\theta = \theta_0$ for the events $\{Z > T\}$, $T \gg 1$. We are going to show that the distribution

$$P(Z > t) = ct^{\alpha}e^{-\theta_0 t} (1 + o(1)), \quad t \to \infty \quad (12)$$

is consistent with our estimates of $\theta_\varepsilon$.

To do this, let us replace the empirical mean $<Z>_{\varepsilon}$ in (11) with $EZ1_{z \in \Delta_\varepsilon}$ assuming (12) and $o(1) = 0$. Solving the equation yields the expected value of $\hat{\theta}_\varepsilon$ under (12). These estimates $\hat{\theta}_\varepsilon$ are shown in Fig. 3 for $\alpha = H - 0.5$. It is seen that the $\hat{\theta}_\varepsilon$ are in very good agreement with the empirical estimates $\hat{\theta}_\varepsilon$ for all $H = 0.1 - 0.9$. Figure 4 provides a more detailed view of the residuals $R_\varepsilon = \hat{\theta}_\varepsilon - \hat{\theta}_\varepsilon$.

Both Figs. 3 and 4 show that the empirical estimates $\hat{\theta}_\varepsilon$ can be well fitted using the extra parameter $\alpha$. However, it is very difficult to get $\alpha$ with a suitable resolution $\delta$, say $\delta = 0.02 - 0.03$, if $\alpha$ is small, as is the case for the model $\alpha = H - 0.5$. Indeed, suppose the parameter $\theta$ is known and $\theta = \theta_0$. The Cramer-Rao inequality yields the optimal variance $\sigma^2_{\text{opt}}$ for $\alpha$:
Figure 3: Hypothetical values of $\theta_0 = H(1 - H)$ and interval estimates $\hat{\theta}_\varepsilon \pm \tilde{\sigma}_\varepsilon$ of the exponent $\theta$ for the interval $\Delta_\varepsilon$, $\varepsilon = 0.01$. Number of paths is $N = 25 \times 300,000$. Boxes correspond to the expected values $\tilde{\theta}_\varepsilon$ of $\hat{\theta}$ under condition 12 and $\alpha = H - 0.5$.

$$\sigma_{\text{opt}} \simeq N_\Delta^{-1/2} \sigma^{-1}(\ln Z_\Delta)$$

where $\sigma^2(\ln Z_\Delta)$ is the variance of $\ln Z$ for observations of $Z$ in the interval $\Delta$, $N_\Delta$ being the number of the observations. Hence

$$N_\Delta \simeq [\sigma_{\text{opt}} \cdot \sigma(\ln Z_\Delta)]^{-2}.$$  

If $\Delta = (Z_\varepsilon, A/\theta_0)$ and $Z_\varepsilon$ is given by (10), then the total number of paths of $x(\cdot)$ is

$$N \simeq [\sigma_{\text{opt}} \cdot \sigma(\ln Z_\Delta)]^{-2}\varepsilon^{-1}.$$
Figure 4: Residuals $R = \tilde{\theta}_\varepsilon - \hat{\theta}_\varepsilon$, $\varepsilon = 0.01$ (see Fig. 3). Vertical lines correspond to two levels of $\hat{\theta}_\varepsilon$ deviations: a) $\pm \hat{\sigma}_\varepsilon$ (bold) and b) $\pm 2\hat{\sigma}_\varepsilon$, (thin) where $\hat{\sigma}_\varepsilon$ is empirical standard deviation of the estimate $\hat{\theta}$.

One has $|\alpha| < 0.5$ in the model $\alpha = H - 0.5$. For this range of $\alpha$ with $\varepsilon = 0.01$ and 0.001 for $A = 20$ one has $\sigma(\ln Z_\Delta) \simeq 0.15883$ and 0.11536, respectively. The requirement $\sigma_{opt} = 0.01$ makes the number $N$ large enough: $40 \cdot 10^6$ if $\varepsilon = 0.01$ and $750 \cdot 10^6$ if $\varepsilon = 0.001$.

4 Conclusion

We have shown that the exponent hypothesis $\theta_0 = H(1 - H)$ for a series of statistics related to IFBM (see Statement 1) is well corroborated by our computations. Further refinement of the tail probabilities for these statistics faces considerable computational difficulties in view of the amount of computation required, the computation accuracy, and checks on the random digit generator.

The symmetry of the exponent: $\theta(H) = \theta(1 - H)$ which $\theta_0$ has by definition is not obvious in our problem because of quite different properties of
FBM for $H < 1/2$ and $H > 1/2$. Our analysis suggests that this difference can manifest itself in more sophisticated asymptotics, e.g.

$$P(\text{IFBM}(t) < 1, \ 0 < t < T) = O(T^{-\theta_0}(\log T)^{\alpha(H)})$$

where $\alpha(1/2) = 0$. Of course this relation, as well as $\theta_0 = H(1 - H)$, are needed in analytical corroboration.

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