LARGE N QUANTUM CRYPTOGRAPHY

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In quantum cryptography, the level of security attainable by a protocol which implements a particular task $N$ times bears no simple relation to the level of security attainable by a protocol implementing the task once. Useful partial security, and even near-perfect security in an appropriate sense, can be obtained for $N$ copies of a task which itself cannot be securely implemented. We illustrate this with protocols for quantum bit string commitment and quantum random number generation between mistrustful parties.

1 Introduction

It is now well known that quantum information can guarantee classically unattainable security in a variety of important cryptographic tasks. We know too from no-go results that quantum cryptography cannot guarantee perfect security for every task. We cannot presently characterise precisely the tasks for which perfectly secure quantum protocols exist, or even the range of cryptographic tasks for which perfectly secure quantum protocols might possibly exist, because quantum cryptography involves more than devising quantum protocols for tasks known to be useful in classical cryptography. The properties of quantum information allow new and cryptographically useful tasks, which have no classical counterpart. Also, reductions and relations between classical cryptographic tasks need not necessarily apply to their quantum equivalents. This means that there is a wider range of tasks to consider, and that no-go theorems may not necessarily be quite as powerful as classical reasoning would suggest.

These remarks apply in particular to bit commitment and coin tossing, important cryptographic protocols whose potential for physically secure implementation has been extensively investigated. It is known that unconditionally secure quantum bit commitment is impossible for non-relativistic protocols: that is, protocols in which the two parties are restricted to single pointlike sites, or more generally, in which the signalling constraints of special relativity are ignored. No unconditionally secure non-relativistic coin tossing protocol has been found; no proof that no such protocols exist has yet been published either.

Unconditionally secure bit commitment is conjectured to be possible between parties controlling appropriately separated pairs of sites, when the impossibility of superluminal signalling is taken into account. Unconditionally secure coin tossing is simple to implement under these conditions. However, we restrict attention to non-relativistic protocols in the rest of this paper,
taking this as understood rather than inserting “non-relativistic” throughout.

Some variants of bit commitment, for which non-relativistic protocols are not known to be impossible, have previously been studied. We consider here a different generalisation, bit string commitment, in which one party commits many bits to another in a single protocol. Two non-relativistic bit string commitment protocols, which offer classically unattainable levels of security against cheating, are described.

2 Bit string commitment

Consider the following classical cryptographic problem. Two mistrustful parties, A and B, need a protocol which will (i) allow A to commit a string \(a_1a_2\ldots a_n\) of bits to B, and then, (ii) at any later time of her choice, reveal the committed bits. The protocol should prevent A from cheating, in the sense that she should have little or no chance of unveiling bits \(a'_i\) different from the \(a_i\) without B being able to detect the attempted detection. In other words, A should be genuinely committed after the first stage. The protocol should also prevent B from being able to completely determine the bit string. More precisely, it must guarantee that, before revelation, B has little or no chance of obtaining more than \(m\) bits of information about the committed string, for some fixed integer \(m < n\).

This \((m,n)\) bit string commitment problem is a generalisation of the standard bit commitment problem, in which \(n = 1\) and \(m = 0\). Clearly, a protocol for bit commitment would solve this generalised problem, since the protocol could be repeated \(n\) times to commit each of the \(a_i\), and B would be able to obtain no information about the committed string. Conversely, classical reasoning implies that a protocol for the generalised problem, for any integers \(m\) and \(n\) with \(m < n\), could be used as a protocol for standard bit commitment. For \(A\) and \(B\) can use any coding of a single bit \(a\) by the \(n\) bit string such that none of the \(m\) bits available to \(B\) give information about \(a\), and then use the protocol to commit \(A\) to \(a\).

Classically, then, \((m,n)\) bit string commitment is essentially equivalent to bit commitment. However, there is no obvious equivalence between quantum \((m,n)\) bit string commitment and quantum bit commitment. The impossibility of unconditionally secure quantum bit commitment does not necessarily imply that, with an analogous definition of security, unconditionally secure quantum bit string commitment is impossible. In fact, the next sections show it can be achieved.

3 Protocol 1

Define qubit states \(\psi_0 = |0\rangle\) and \(\psi_1 = \sin \theta |0\rangle + \cos \theta |1\rangle\), where \(\sin^2 \theta = \delta\). We take \(\theta > 0\) and \(r = n - m\) to be security parameters for the protocol.
Commitment: To commit a string \( a_1 \ldots a_n \) of bits to \( B \), \( A \) sends the qubits \( \psi_{a_1}, \ldots, \psi_{a_n} \), sequentially.

Unveiling: To unveil, \( A \) simply declares the values of the string bits, and hence the qubits sent. Assuming that \( B \) has not disturbed the qubits, he can test the bit values \( a_i' \) claimed by \( A \) at unveiling by measuring the projection onto \( \psi_{a_i'} \) on qubit \( i \), for each \( i \). If he obtains eigenvalue 1 in each case, he accepts the unveiling as an honest revelation of a genuine commitment. If he obtains eigenvalue 0 in any case, he concludes (assuming that noise is negligible) that \( A \) has cheated.

Security against \( A \): Whatever strategy \( A \) follows, once she transmits the qubits to \( B \), their respective density matrices \( \rho_i \) are fixed. Let \( p_j^i = \langle \psi_j | \rho_i | \psi_j \rangle \) be the probability of \( B \) accepting a revelation of \( j \) for the \( i \)-th bit. We have
\[
p_0^i + p_1^i \leq \cos^2((\pi/4) - (\theta/2)) + \sin((\pi/4) + (\theta/2)),
\]
which is \( \leq 1 + \theta \) for small \( \theta \). This is the standard definition of security against \( A \) for an individual bit commitment, with security parameter \( \theta \). In other words, \( A \)'s scope for cheating on any bit of the string is limited to slightly increasing the probability of revealing a 0 or 1, by an amount \( \leq \theta \), which can be made arbitrarily small by choosing the security parameters appropriately.

Security against \( B \): We assume that, prior to the commitment, \( B \) has no information about the bit string and regards every possible value as equiprobable. From \( B \)'s perspective, then, he has to obtain information about a density matrix of the form
\[
\rho = (1/2^n) \sum_{a_1 \ldots a_n} |\psi_{a_1} \ldots \psi_{a_n}\rangle\langle \psi_{a_1} \ldots \psi_{a_n}|. \tag{2}
\]
Holevo's theorem\(^{10}\) tells us that the accessible information available to \( B \) by any measurement on \( \rho \) is bounded by the entropy
\[
S(\rho) = \frac{(1 + \sin \theta)}{2} \log_2((1 + \sin \theta)/2) + \\
\frac{(1 - \sin \theta)}{2} \log_2((1 - \sin \theta)/2))^n. \tag{3}
\]
Now, for any fixed \( \theta > 0 \), we have \( S(\rho) < n \). For any fixed \( r \), by taking \( n \) sufficiently large, we can ensure \( n - S(\rho) > r \). In other words we can ensure that, however \( B \) proceeds, an average of at least \( r \) bits of information about the string will remain inaccessible to him. By choosing \( n \) suitably large, we can also ensure that the probability of his obtaining more than \( n - r \) bits of information about the string is smaller than \( \epsilon \), for any given \( \epsilon > 0 \).

A more efficient version of this protocol can be devised using qudit states\(^{11}\) — an observation I owe to Rob Spekkens.
4 Protocol 2

Protocol 1 ensures bit-wise security against $A$, but uses a rather inefficient bit string coding which allows $B$ to obtain almost all of the bit string before revelation. For large $n$, more efficient codings allow the security against $B$ to be greatly enhanced, though with a weakened notion of security against $A$.

We again take $\theta > 0$ to be a security parameter and write $\epsilon = \sin \theta$. Now, for any $\theta > 0$ and large $n$, explicit constructions are known for sets $v_1, \ldots, v_{f(n)}$ of vectors in $H^n$ such that $|\langle v_i | v_j \rangle| < \sin \theta$ for all $i \neq j$, with the property that $f(n) = O(\exp(Cn))$, where $C$ is a positive constant that depends on $\theta$. (The use of these constructions for efficient quantum coding of classical information has previously been noted by Buhrman et al.) A string of $O(Cn)$ bits can thus be encoded by vectors in $H^n$, such that the overlap between the code vectors for two distinct strings is always less than $\sin \theta$, suggesting the following bit string commitment protocol.

Commitment: Let $N$ be the number of bits that can be encoded in $H^n$ by the above construction. To commit a string $a_1 \ldots a_N$ of bits to $B$, $A$ sends the state $v_{a_1 \ldots a_N}$, treating the index as a binary number.

Unveiling: To unveil, $A$ simply declares the values of the string bits, and hence the state sent. Assuming that $B$ has not disturbed the qubits, he can test $A$’s claim at unveiling by measuring the projection onto $v_{a_1 \ldots a_N}$. If he obtains eigenvalue 1, he accepts the unveiling as an honest revelation of a genuine commitment. If he obtains eigenvalue 0, he concludes that $A$ has cheated.

Security against $A$: As before, once $A$ transmits a quantum state to $B$, its density matrix $\rho$ is fixed. Consider some set $i_1, \ldots, i_r$ of bit strings which $A$ might wish to maintain the option of revealing after commitment. Let $P_i$ be the projection onto $v_i$, let $p_i = \text{Tr}(\rho P_i)$ be the probability of $A$ successfully revealing string $i$, and write

$$Q = P_{i_1} + \ldots + P_{i_r}. \quad (4)$$

It is not too hard to verify that

$$\text{Tr}(\rho Q) \leq 1 + (r - 1)\epsilon \quad (5)$$

In other words,

$$p_{i_1} + \ldots + p_{i_r} \leq 1 + f(\epsilon, r), \quad (6)$$

where, for any fixed $r$, $f$ can be made as small as desired by choosing $\theta$ suitably small.

So, given that $A$ is determined to reveal a bit string from some finite set of size $r$, her scope for cheating is limited to increasing the probability of
revealing any given element of the set by a fixed amount. For any fixed \( r \), that amount can be made arbitrarily small by choosing the security parameters appropriately. If \( B \)'s concern is to prevent cheating of this type, for some predetermined \( r \), the protocol can guarantee him security.

**Security against B:** Holevo's theorem implies that the information about the \( N \approx Cn \) bit string accessible to \( B \) is at most \( \log n \) bits.

5 **Asymptotically secure coin tossing**

Consider the following non-relativistic protocol for generating a string of \( N \) random bits between mistrustful parties. We assume that \( N \) is large, and take \( M \) also to be large, with \( \log M \ll N \). A prepares \( M \) batches of \( N \) Bell singlet states, and sends one particle from each of the \( MN \) singlets to \( B \). \( B \) chooses \( (M-1) \) of the batches, and asks \( A \) to send the second particle from each of the \((M-1)N\) pairs of particles are indeed singlets. If not, he concludes that \( A \) is cheating, and the protocol ends. If so, he accepts that \( A \) is honest. A and \( B \) then use the last batch of singlets to generate \( N \) random bits, by carrying out correlated measurements (say of \( \sigma_z \)) and converting the results to a bit string using a previously agreed protocol.

**Security against A:** A can only cheat by preparing non-singlet states which bias the outcomes towards those she would prefer. Her scope for cheating is limited by the cut-and-choose step of the protocol, which ensures that, if any batch has low fidelity to \( N \) singlet states, her cheating will almost surely be detected.

**Security against B:** \( B \) can cheat by carrying out measurements on every particle from every batch sent to him, deciding which batch gives the bit string most favourable for his purposes, and choosing the other \((M-1)\) for the test. However, this will allow him to fix only \( \approx \log M \) bits of information about the \( N \) bit string. With suitable \( M, N \) this is an insignificant fraction.

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