**Abstract**

It is shown that time-independent solutions to the (2+1)-dimensional non-linear O(3) sigma model may be placed in correspondence with surfaces of constant mean curvature in three-dimensional Euclidean space. The tools required to establish this correspondence are provided by the classical differential geometry of surfaces. A constant-mean-curvature surface induces a solution to the O(3) model through the identification of the Gauss map, or normal vector, of the surface with the field vector of the sigma model. Some explicit solutions, including the solitons and antisolitons discovered by Belavin and Polyakov, and a more general solution due to Purkait and Ray, are considered and the surfaces giving rise to them are found explicitly. It is seen, for example, that the Belavin-Polyakov solutions are induced by the Gauss maps of surfaces which are conformal to their spherical images, i.e. spheres and minimal surfaces, and that the Purkait-Ray solution corresponds to the family of constant-mean-curvature helicoids first studied by do Carmo and Dajczer in 1982. A generalisation of this method to include time-dependence may shed new light on the rôle of the Hopf invariant in this model.
1 Introduction

It has been known for a long time that there is a close link between certain non-linear partial differential equations, or ‘evolution equations’, and the classical differential geometry of surfaces in three-dimensional Euclidean space. These non-linear equations arise from ‘compatibility conditions’ imposed on the linear Gauss-Weingarten equations which describe the surface. For example, the sine-Gordon equation is naturally associated with pseudospherical surfaces, and the Liouville equation with spherical and minimal surfaces. This situation, where a non-linear compatibility condition follows from an over-determined linear system, is exactly mirrored in the ‘inverse scattering method’ due to Gardner et al.  It was shown by Lund  building on earlier results, that this method may be combined with classical surface theory to give a geometric interpretation to the procedure of finding the linear system of which a given evolution equation is the compatibility condition. A general class of such equations — the ‘AKNS system’, which contains the sine-Gordon, Korteweg-de Vries (KdV), modified KdV, and non-linear Schrödinger equations as special cases — has been considered in this geometric light.

In this paper, we apply the tools of classical surface theory to a popular non-linear sigma model. The O(3) model in 2+1 dimensions has received much attention as a result of its resemblance to (3+1)-dimensional non-Abelian gauge theories (see, for example, refs. and ). The approach adopted here is somewhat different from that used by Lund in that we regard the (non-linear) field equation of the O(3) model not as the compatibility condition of a surface but rather as a constraint on the normal vector to the surface. The normal, or, more correctly, the Gauss map (described in Section ), is identified with the field vector of the O(3) model. In this way a correspondence is defined between surfaces in three dimensions and solutions to the field equation of the model. Similar techniques have been applied in the past to the non-linear O(4) model in two dimensions and to the O(3) model in 1+1 dimensions. However, there are differences between the approaches used in these papers and the work presented here. In ref. the field vector of the O(4) model was identified with the position vector, rather than the normal vector, of a surface and the problem led back to the sine-Gordon case of the AKNS system. In ref. the field vector was equated with the surface normal, as is done here, but a parametrisation of the surface — the so-called ‘asymptotic’ coordinates — was adopted which obscured the variety and interrelationships of the model’s solutions. Namely, it was found that the surfaces to which the field vector is
normal were all pseudospheres. By contrast, it will be shown below that the use of isothermal coordinates leads to a much more faithful correspondence between the solutions to the O(3) model and surfaces in $\mathbb{R}^3$. An expanded account of this work is given by Ody.\textsuperscript{13}

We begin in Section 2 by summarising the salient features of the O(3) model and a number of its time-independent solutions. The relevant differential geometry is given in Section 3 and then combined with the mathematics of the O(3) model in Section 4. In addition to finding the surfaces associated with particular solutions, it is shown that quantities in the sigma model, such as topological charge, energy density, and total energy, can be directly identified with surface-geometric quantities. Section 5 contains a summary and concluding remarks.
2 The non-linear O(3) model in two dimensions

Here we briefly describe the time-independent O(3) sigma model and a number of solutions to its field equation. For further details, see refs. 10 or 14.

The Lagrangian of the model is simply

\[ \mathcal{L} = \frac{1}{2} \partial_i n_a, \partial_i n_a \equiv \frac{1}{2} n_i \cdot n_i \]  

(2.1)

where there is a summation over both the internal index \( a = 1, 2, 3 \) and the space index \( i = 1, 2 \), and the notation \( n_i \) is shorthand for \( \partial_i n \). The field vector \( n(x_1, x_2) = (n_1, n_2, n_3) \) of the model is a unit vector in internal space as a result of the non-linear constraint

\[ n \cdot n \equiv n_1^2 + n_2^2 + n_3^2 = 1 \quad \text{for all } x_1, x_2. \]  

(2.2)

It is therefore sensible to adopt a polar coordinate system in which

\[ n = \begin{pmatrix} \sin \Theta \cos \Phi \\ \sin \Theta \sin \Phi \\ \cos \Theta \end{pmatrix} \]  

(2.3)

where \( \Theta \) and \( \Phi \) are functions of \( x_1 \) and \( x_2 \). The field manifold is clearly the unit sphere \( S^2 \) centred on the origin of internal space \( \mathbb{R}^3 \). It follows from (2.1) and (2.2) that the Euler-Lagrange equation, or field equation, of the model is

\[ \nabla^2 n - (n \cdot \nabla^2 n)n = 0. \]  

(2.4)

Some solutions to this non-linear equation are described below.

It proves convenient to map the field manifold \( S^2 \) onto a plane via a stereographic projection from the ‘south pole’ \( n = (0, 0, -1) \). We define

\[ W_1 \equiv \frac{n_1}{1 + n_3} = \tan \frac{1}{2} \Theta \cos \Phi, \quad W_2 \equiv \frac{n_2}{1 + n_3} = \tan \frac{1}{2} \Theta \sin \Phi \]  

(2.5)

where \( (W_1, W_2) \) are Cartesian coordinates in the \( n_1n_2 \)-plane of internal space. We can define a complex function \( W(z, z^*) \) by

\[ W \equiv W_1 + iW_2 = \frac{n_1 + in_2}{1 + n_3} = \tan \frac{1}{2} \Theta e^{i\Phi} \]  

(2.6)

where \( z \equiv x_1 + ix_2 = re^{i\phi} \). The inverse projection is given by

\[ n = \frac{1}{1 + |W|^2} \begin{pmatrix} 2 \text{Re} W \\ 2 \text{Im} W \\ 1 - |W|^2 \end{pmatrix} \]  

(2.7)
or, equivalently, by substituting

\[ \Theta = 2 \tan^{-1} |W|, \quad \Phi = \arg W \]  

(2.8)

into (2.3). In terms of \( W \), the field equation (2.4) becomes

\[ \partial \partial^* W - \frac{2 W^* \partial W \partial* W}{1 + |W|^2} = 0 \]  

(2.9)

where \( \partial \) and \( \partial^* \) are shorthand for \( \partial / \partial z \) and \( \partial / \partial z^* \) respectively. The function \( W \), instead of \( n \), may be regarded as the fundamental object in the O(3) model.

The total energy \( E_{\text{tot}} \) of a time-independent solution \( n(x_1, x_2) \) is given by

\[ E_{\text{tot}} = \int_{\mathbb{R}^2} \mathcal{E} \, dx_1 \wedge dx_2 \]  

(2.10)

where the energy density \( \mathcal{E}(x_1, x_2) \) is

\[ \mathcal{E} = \frac{1}{2} n_{,i} \cdot n_{,i} = -\frac{1}{2} n \cdot \nabla^2 n. \]  

(2.11)

The integral in (2.10) will be finite if \( n \) tends to the same constant value \( n_0 \) at large distances from the origin of the \( x_1x_2 \)-plane, i.e. if

\[ \lim_{r \to \infty} n = n_0 \quad \text{for all } \phi. \]  

(2.12)

This behaviour of the field at infinity endows the coordinate space \( \mathbb{R}^2 \) with the topology of the sphere \( S^2 \). A field \( n(x_1, x_2) \) which obeys (2.12) is then a mapping from this ‘compactified’ coordinate space \( S^2 \) to the field manifold \( S^2 \). Such maps fall into homotopy classes as a result of the homotopy

\[ \pi_2(S^2) \sim \mathbb{Z}. \]  

(2.13)

The degree \( Q \) of the map \( n : S^2 \to S^2 \) is an integer, positive or negative, which is the same for all maps in the same homotopy class and is interpreted physically as a topological charge. An expression for \( Q \) can be found by integrating the pullback \( n^* \beta \) of the normalised area element \( \beta \) on the field manifold \( S^2 \):

\[ \beta = \frac{1}{4\pi} \star n \cdot d\mathbf{n} = \frac{1}{8\pi} \mathbf{n} \cdot d\mathbf{n} \times d\mathbf{n}. \]  

(2.14)

(Here, \( \star \) is the Hodge ‘star’ operator.) If \( n \) obeys the boundary condition (2.12) then we can write

\[ Q = \int_{\mathbb{R}^2} n^* \beta \equiv \int_{\mathbb{R}^2} J \, dx_1 \wedge dx_2 \]  

(2.15)
where \( J(x_1, x_2) \) is the charge density given by

\[
J = \frac{1}{8\pi} \epsilon_{ij} \mathbf{n} \cdot \mathbf{n}_i \times \mathbf{n}_j = \frac{1}{4\pi} \mathbf{n} \cdot \mathbf{n}_1 \times \mathbf{n}_2.
\]  

(2.16)

In terms of the complex function \( W \), we have

\[
\mathcal{E} = \frac{4(|\partial W|^2 + |\partial^* W|^2)}{(1 + |W|^2)^2} \]

(2.17)

\[
J = \frac{|\partial W|^2 - |\partial^* W|^2}{\pi(1 + |W|^2)^2}.
\]

(2.18)

The homotopy classification (2.13) applies only to finite-energy solutions, i.e. those which obey (2.12). It was shown by Belavin and Polyakov\textsuperscript{16} that such solutions satisfy, in addition to the field equation (2.4), the so-called ‘duality equations’

\[
\mathbf{n}_i = \pm \epsilon_{ij} \mathbf{n} \times \mathbf{n}_j.
\]

(2.19)

The two equations \((i = 1, 2)\) contained in (2.19) are not independent. We will refer to the \( i = 1 \) case as ‘the’ duality equation:

\[
\mathbf{n}_1 = \pm \mathbf{n} \times \mathbf{n}_2.
\]

(2.20)

The total energy \( E_{\text{tot}} \) of a field \( \mathbf{n} \) which satisfies (2.20) is related to its topological charge \( Q \) by

\[
E_{\text{tot}} = 4\pi |Q|.
\]

(2.21)

In terms of \( W \), the duality equation becomes the Cauchy-Riemann conditions

\[
W_{1,1} = \mp W_{2,2}
\]

(2.22a)

\[
W_{2,1} = \pm W_{1,2}
\]

(2.22b)

i.e.

\[
W = \omega
\]

(2.23)

where \( \omega \) is a complex analytic function of either \( z \) (lower signs) or \( z^* \) (upper signs). This solution (2.23) is the Belavin-Polyakov (BP) solution. It was shown by Garber et al.\textsuperscript{17} that this is the only finite-energy solution to the field equation of the O(3) model. The function \( \omega \) may in fact be meromorphic, but usually the following entire function is used:

\[
\omega(z) = \prod_n \left( \frac{z - z_{0n}}{\rho_n} \right)^{q_n}
\]

(2.24)
where the \( \rho \)'s are real constants, the \( z_0 \)'s are complex constants, and the \( q \)'s are positive integers. Each of the factors in the product represents a ‘soliton’ centred on the point \((x_{1n}, x_{2n}) = (\text{Re} \, z_{0n}, \text{Im} \, z_{0n})\) and having topological charge \( q_n \). The quantity \( \rho_n \) is a scale factor which determines the size of the soliton. Replacing \( z \) with \( z^\ast \) in (2.24) gives the corresponding ‘antisoliton’ solution \( \omega(z^\ast) \). The total topological charge \( Q \) is independent of the \( z_0 \)'s and \( \rho \)'s:

\[
Q = \mp \sum_n q_n. \tag{2.25}
\]

The solution which represents a single soliton of charge \( q \), or a single antisoliton of charge \(-q\), may always be brought to the form

\[
W = \begin{cases} 
  z^{*q} & \text{(antisoliton)} \\
  z^q & \text{(soliton)}
\end{cases} \tag{2.26}
\]

by rescaling the Cartesian coordinates \( x_1 \) and \( x_2 \).

A number of other solutions to the field equation (2.9) have come to light since Belavin and Polyakov’s work. There is the so-called ‘meron’ solution

\[
W = \frac{\omega}{|\omega|} \tag{2.27}
\]

which is analogous to the meron solution in four-dimensional Yang-Mills theory. The function \( \omega \) in (2.27) is still the analytic BP solution but the total energy of a meron is infinite, which is expected in the light of the uniqueness proof of Garber et al. Then a solution was found which contains both (2.23) and (2.27) as special cases, and which in fact interpolates between them. This solution can be written as

\[
W = \text{Re}^{i\Phi} \tag{2.28}
\]

where

\[
R^2 = \frac{1 + D \text{sn}(\eta|D)}{1 - D \text{sn}(\eta|D)} \tag{2.29a}
\]

and

\[
\eta \equiv \ln |\omega|, \quad \Phi = -i \ln \left( \frac{\omega}{|\omega|} \right). \tag{2.29b}
\]

In these expressions, \( \omega \) is the BP solution, \( D \) is a real parameter lying the range \( 0 \leq D \leq 1 \), and \( \text{sn}(\eta|D) \) is a Jacobi elliptic function of the quantity \( \eta \) with parameter \( D \). When \( D = 1 \) the solution (2.29) reduces to (2.23), and when \( D = 0 \) it reduces to (2.27). If the function \( \omega \)
is analytic in $z$ then (2.29) interpolates between solitons and merons, whereas if $\omega$ is analytic in $z^*$ then the interpolation is between antisolitons and antimeron. Some special cases of (2.29) where the function $\omega$ took certain forms, such as (2.24), have been examined.\textsuperscript{24, 25, 26}

A yet more general solution to (2.9) was discovered by Purkait and Ray.\textsuperscript{27} They took the modulus $R = R(U)$ and argument $\Phi = \Phi(U,V)$ of $W$ to be functions of the real and imaginary parts $U(x_1,x_2)$ and $V(x_1,x_2)$ of an analytic function $T = U + iV$. By direct substitution into the field equation, they found that $\Phi$ is given by

$$\Phi(U,V) = c_2 V + c_3 \int \frac{(R^2 + 1)^2}{R^2} dU$$

(2.30a)

and that $R(U)$ is given by inverting the elliptic integral

$$U(R) = \frac{1}{2} \int [Q(x)]^{-\frac{1}{4}} dx$$

(2.30b)

where $x \equiv R^2$ and

$$Q(x) \equiv -c_3^2 x^4 + (c_1 - 2c_2^2) x^3 + (2c_1 + c_2^2 - 2c_3^2) x^2 + (c_1 - 2c_3^2) x - c_3^2$$

(2.30c)

with $c_1$, $c_2$, $c_3$ being arbitrary real constants. There are two special cases of this solution which will concern us here. One is the interpolating solution (2.29) encountered above: the Purkait-Ray solution (2.30) reduces to (2.29) if

$$c_1 \neq 0, \quad c_2 \neq 0, \quad c_3 = 0, \quad D^2 \equiv 4c_1 c_2^{-2} + 1 > 0$$

(2.31)

and

$$T \equiv \frac{1}{c_2} \ln \omega, \quad i.e. \quad \eta \equiv c_2 U.$$  

(2.32)

The second case of interest is that where $c_1 \neq 0$, $c_2 \neq 0$, $c_3 \neq 0$. In this case we find

$$R^2 = \frac{1 + \beta \text{sn}(\alpha|\tilde{D})}{1 - \beta \text{sn}(\alpha|\tilde{D})}$$

(2.33a)

$$\Phi = c_2 V + \frac{4c_3}{c_2} \int \frac{d\eta}{1 - \beta^2 \text{sn}^2(\alpha|\tilde{D})}$$

(2.33b)

where $\eta \equiv c_2 U$ and

$$\tilde{D} \equiv \frac{\beta}{\alpha}$$

(2.34)

$$\alpha^2 \equiv \frac{c_1 - 6c_3^2 + \delta}{c_1 + 2c_3^2 + \delta}, \quad \beta^2 \equiv \frac{c_1 - 6c_3^2 - \delta}{c_1 + 2c_3^2 - \delta}$$

(2.35)
\[ \delta^2 \equiv (c_1 + 2c_3^2)^2 + 4c_2^2c_3^2 \]  

where the constants \( c_1, c_2, c_3 \) are such that \( \alpha^2 > 0 \) and \( \beta^2 > 0 \). This second case of interest will be referred to as the ‘helicoidal solution’, for reasons to be seen below. Note that this solution is separate from the interpolating solution (2.29), i.e. (2.33) does not reduce to (2.29) if \( c_3 \) is set to zero. This is because if \( c_3 = 0 \) then \( \delta = \pm c_1 \), and therefore either \( \alpha \) or \( \beta \) is indeterminate.
3 Surfaces of constant mean curvature

In this section we summarise some features of the classical differential geometry of surfaces and, in particular, of surfaces of constant mean curvature. For further details, see refs. [1] or [2], for example.

A surface $S$ in $\mathbb{R}^3$ is uniquely specified, up to its overall position in space, by giving its first and second fundamental forms $I$ and $II$:

\begin{align*}
I &= E(du^2 + dv^2) \\
II &= L\,du^2 + 2M\,du\,dv + N\,dv^2
\end{align*}

(3.1a) (3.1b)

where $(u, v)$ are isothermal surface parameters and $E, L, M, N$ are functions of $u$ and $v$, subject to certain constraints described below. The form $I$ is just the metric on the surface $S$. Conversely, if the position vector $X(u, v)$ of $S$ is known then we can find the four functions from

\begin{align*}
E &= |X_u|^2 = |X_v|^2 \\
L &= -X_u \cdot n_u, \quad M = -X_u \cdot n_v, \quad N = -X_v \cdot n_v
\end{align*}

(3.2a) (3.2b)

where $X_u \equiv X_u \equiv \partial_u X$ and $X_v \equiv X_v \equiv \partial_v X$ are the two tangent vectors to the surface and

\[ n \equiv E^{-1}(X_u \times X_v) \]

(3.3)

is the unit normal vector. We now introduce the mean curvature $H(u, v)$ and Hopf function $\Psi(\zeta, \zeta^*)$ of the surface:

\begin{align*}
H &= \frac{1}{2} e^{-2\Sigma}(L + N) \\
\Psi &= -2 \frac{\partial X}{\partial \zeta} \cdot \frac{\partial n}{\partial \zeta} = \frac{1}{2}(L - N) - iM,
\end{align*}

(3.4) (3.5)

where

\[ E \equiv e^{2\Sigma} \]

(3.6)

and

\[ \zeta \equiv u + iv. \]

(3.7)
These functions $H$ and $\Psi$ encode many properties of the surface $S$. In terms of these quantities, the forms $I$ and $\Pi$ may be written

\begin{align}
I &= e^{2\Sigma}|d\zeta|^2 \\
\Pi &= \frac{1}{2}(\Psi d\zeta^2 + \Psi^*d\zeta^*^2) + He^{2\Sigma}|d\zeta|^2
\end{align}

i.e.

\begin{align}
L &= HE + \text{Re } \Psi \\
M &= -\text{Im } \Psi \\
N &= HE - \text{Re } \Psi.
\end{align}

If $E$, $L$, $M$, $N$ are given functions then the position vector $X(u,v)$ may in principle be found by solving the Gauss-Weingarten equations. In isothermal coordinates these equations take the form

\begin{align}
X_{u,u} &= \Sigma_{,u}X_u - \Sigma_{,v}X_v + Ln \\
X_{u,v} &= \Sigma_{,v}X_u + \Sigma_{,u}X_v + Mn \\
X_{v,v} &= -\Sigma_{,u}X_u + \Sigma_{,v}X_v + Nn \\
n_{,u} &= -e^{-2\Sigma}(LX_u + MX_v) \\
n_{,v} &= -e^{-2\Sigma}(MX_u + NX_v)
\end{align}

These over-determined linear equations are subject to certain non-linear compatibility conditions which follow from the requirement that mixed second-order derivatives of $X_u$, $X_v$ and $n$ commute. These are the Gauss and the Codazzi-Mainardi conditions, and in isothermal coordinates they take the respective forms

\begin{align}
LN - M^2 &= -e^{2\Sigma}\nabla^2\Sigma \\
\frac{\partial \Psi}{\partial \zeta^*} - e^{2\Sigma}\frac{\partial H}{\partial \zeta} &= 0.
\end{align}

Another important quantity is the Gaussian curvature $K(u,v)$:

\begin{align}
K = \frac{LN - M^2}{E^2} &= -e^{-2\Sigma}\nabla^2\Sigma.
\end{align}

If $K$ is integrated over the whole surface area $A$ of $S$ then the result is the total curvature, denoted $K_{\text{tot}}$:

\begin{align}
K_{\text{tot}} \equiv \int_S K \, dA = \int_S KE \, du \wedge dv = -\int_S \nabla^2\Sigma \, du \wedge dv,
\end{align}
and if $S$ is compact then it follows from the Gauss-Bonnet theorem that

$$K_{\text{tot}} = 2\pi \chi$$  \hspace{1cm} (3.16)

where $\chi$ is the Euler characteristic of $S$. These relations will be needed in the remainder of this paper.

The Gauss map of a surface $S$ is constructed by parallel-transporting the normal vector $n$ at each point of $S$ to some origin $O$ in $\mathbb{R}^3$. Since $n$ is a unit vector, the locus of the tips of these transported vectors will be some region $\Omega$ of the unit sphere $S^2$. The region $\Omega$ is called the spherical image of $S$. The mapping induced by $n$ from the surface $S$ to its spherical image is the Gauss map:

$$n : S \to S^2.$$  \hspace{1cm} (3.17)

The normal vector itself is sometimes referred to as ‘the Gauss map’. It is straightforward to find the degree $\delta$ of the mapping (3.17): if $n^* \beta$ is the pullback to $S$ of the normalised element of area $\beta$ on $S^2$ then

$$\delta = \int_S n^* \beta$$

$$= \frac{1}{4\pi} \int_S n \cdot n_u \times n_v \, du \wedge dv$$

$$= \frac{1}{4\pi} \int_S KE \, du \wedge dv$$

$$= \frac{K_{\text{tot}}}{4\pi}$$

$$= \frac{\chi}{2}$$  \hspace{1cm} (3.18)

where (3.13) holds if the surface is compact. Two kinds of surface which are important to this work are spheres and minimal surfaces. These surfaces have the property that they are conformal to their spherical images.

We shall now describe these, and other, surfaces of constant mean curvature in more detail. In Section 4 it will be seen how they correspond to the various time-independent solutions to the $O(3)$ sigma model described in Section 3. Note first that if the mean curvature $H$ of a surface $S$ is a constant then the Codazzi-Mainardi condition (3.13) implies that the Hopf function $\Psi$ is a function of $\zeta$ only:

$$\Psi = \Psi(\zeta) \iff S \text{ is a surface of constant mean curvature (CMC)}.$$  \hspace{1cm} (3.20)
The sphere

Consider first a CMC surface having $\Psi = 0$, $H \neq 0$. In this case, the first and second fundamental forms (3.8) are related by

$$II = HI,$$

(3.21)

from which it follows that $S$ is a sphere of radius $H^{-1}$. Since $H$ is therefore simply an overall scale factor for the surface, we may set $H = 1$ without loss of generality.

$$\Psi = 0, \ H = 1 \implies S \text{ is a sphere of unit radius.} \ (3.22)$$

Minimal surfaces

Now let the mean curvature $H$ be zero at all points $(u,v)$ of the surface $S$. This is the defining property of a minimal surface. The Hopf function $\Psi(\zeta)$ determines a minimal surface via the Weierstrass-Enneper representation formula for the position vector $X(u,v)$:

$$X(u,v) = \frac{1}{2} \text{Re} \int V(\zeta)\Psi(\zeta) \, d\zeta \ (3.23a)$$

where the vector $V(\zeta)$ is

$$V(\zeta) \equiv \begin{pmatrix} \zeta^2 - 1 \\ i(\zeta^2 + 1) \\ 2\zeta \end{pmatrix}. \ (3.23b)$$

To each analytic function $\Psi(\zeta)$ there corresponds a minimal surface $S$ given by (3.23). From (3.23) it is straightforward to derive the forms $I$ and $II$ for a minimal surface:

$$I = \frac{1}{4} |\Psi|^2 (1 + |\zeta|^2)^2 |d\zeta|^2 \ (3.24a)$$

$$II = \frac{1}{2} (\Psi d\zeta^* + \Psi^* d\zeta^{*2}) \ (3.24b)$$

i.e. $L = -N = \text{Re} \Psi$, $M = -\text{Im} \Psi$. \ (3.25)

The normal vector $n(u,v)$ is found to be

$$n = \frac{1}{1 + |\zeta|^2} \begin{pmatrix} 2 \text{Re} \zeta \\ -2 \text{Im} \zeta \\ 1 - |\zeta|^2 \end{pmatrix} \ (3.26)$$

which is notable because it does not depend on the Hopf function $\Psi(\zeta)$. Therefore, locally, any two minimal surfaces have the same Gauss map. In this sense minimal surfaces are...
exceptional because in general a surface is essentially determined by its Gauss map. The form (3.21) of $n$ indicates that $\zeta$ can be regarded as a stereographic projection of the Gauss map. The complex variable $\zeta = u + iv$ itself will therefore sometimes be referred to as the Gauss map of the surface.

The metric (3.24a) on a minimal surface depends on the modulus of the Hopf function $\Psi(\zeta)$. Therefore a one-parameter family of isometric minimal surfaces may be obtained by subjecting $\Psi$ to the Bonnet transformation

$$\Psi \rightarrow e^{i\alpha}\Psi, \quad 0 \leq \alpha < 2\pi$$

(3.27)

where $\alpha$ is the ‘Bonnet angle’. The surfaces obtained in this way from a given minimal surface $S$ are said to be associate to $S$. On the other hand, if $\Psi$ is multiplied by a real constant $c$ then the surface $S$ is simply scaled in size by a factor $c$. Such a transformation does not give rise to an essentially distinct surface and is therefore trivial.

The simplest choice for $\Psi$, namely

$$\Psi = 1,$$  

(3.28)

gives rise to Enneper’s minimal surface. This surface is an example of a Bour surface: a minimal surface which is applicable ($i.e.$ isometric via a one-parameter transformation) to a surface of revolution. The Hopf function $\Psi(\zeta)$ of an arbitrary Bour surface is given by

$$\Psi = ce^{i\alpha}\zeta^{b-2}$$

(3.29)

where $ce^{i\alpha}$ is an arbitrary complex number incorporating the scale factor $c$ and Bonnet angle $\alpha$, and $b$ is an arbitrary real constant which determines the Bour surface. Specifying $b$ fixes the ‘Bonnet class’ of the surface, $i.e.$ the family of associate minimal surfaces to which it belongs, and the angle $\alpha$ determines a unique surface in that class.

If a coordinate transformation

$$u = u(x_1, x_2), \quad v = v(x_1, x_2)$$

(3.30)

to new independent parameters $(x_1, x_2)$ is made then the Gauss map $\zeta$ will be regarded as a complex function of the variables $z \equiv x_1 + ix_2$ and $z^*$:

$$\zeta = \zeta(z, z^*)$$

(3.31)

If $\zeta$ is the Gauss map of a minimal surface or a sphere then it is an analytic function of $z$ or $z^*$ respectively. Similarly, if $\zeta(z, z^*)$ is the Gauss map of a CMC surface (with $H \neq 0$) then
it is harmonic. If, in addition, both coordinate systems \((u,v)\) and \((x_1,x_2)\) are isothermal then the Hopf function \(\psi\) in the new parametrisation is related to \(\Psi\) by
\[
\Psi \, d\zeta^2 = \psi \, dz^2.
\] (3.32)

Our use of lower case to denote the Hopf function in the \((x_1,x_2)\)-parametrisation will be applied to the other quantities of surface theory. When referred to the parameters \(u\) and \(v\) we will write, for example, \(E, L, M, N, \Sigma, \Psi\), etc. The corresponding quantities in the \((x_1,x_2)\)-parametrisation will be written \(e, l, m, n, \sigma, \psi\), etc. The notation of quantities which are invariant under a transformation of the form (3.30), such as \(K, H, n\), etc., will not be changed.

Many general results pertaining to arbitrary CMC surfaces, and, indeed, to surfaces in general may be obtained from the Kenmotsu representation formula\footnote{\(H\) and \(n\), for example, are invariant up to a sign. If the parameter transformation preserves the orientation of the surface then the sign of these quantities does not change.}, which gives the position vector \(X(x_1,x_2)\) of a surface in terms of its mean curvature \(H(x_1,x_2)\) and Gauss map \(\zeta(z,z^*)\).

For example, we find for a CMC surface the following relations:
\[
I = e^{2\sigma}|dz|^2 \quad \text{where} \quad e^{2\sigma} = \frac{4|\partial^* \zeta|^2}{H^2(1 + |\zeta|^2)^2} \tag{3.33}
\]
\[
\psi(z) = e^{2\sigma} H \frac{\partial \zeta}{\partial^* \zeta} \tag{3.34}
\]
\[
K_{\text{tot}} = 4 \int_S \frac{|\partial^* \zeta|^2 - |\zeta|^2}{(1 + |\zeta|^2)^2} \, dx_1 \wedge dx_2. \tag{3.35}
\]

The Gauss map \(\zeta(z,z^*)\) is harmonic, i.e. it satisfies
\[
\partial \partial^* \zeta - \frac{2\zeta^* \partial \zeta \partial^* \zeta}{1 + |\zeta|^2} = 0, \tag{3.36}
\]
and the normal vector \(n(x_1,x_2)\) is related to \(\zeta\) by
\[
\mathbf{n} = \frac{1}{1 + |\zeta|^2} \begin{pmatrix} 2 \text{Re} \zeta \\ -2 \text{Im} \zeta \\ 1 - |\zeta|^2 \end{pmatrix}. \tag{3.37}
\]

An important result in the theory of CMC surfaces is the following: the directions of principle curvature on a CMC surface form an isothermal coordinate system.\footnote{\textit{An immediate}}
consequence is that, in a parametrisation by these lines of curvature, the second fundamental form $II$ is diagonal and therefore that $\psi(z)$ is purely real:

$$m = -\text{Im} \psi = 0.$$  \hspace{1cm} (3.38)

Moreover, it follows from the analyticity of $\psi$ — see (3.20) — that $\psi$ is a real constant.

There are two further kinds of CMC surfaces which are relevant to the O(3) sigma model and these are described next.

### Delaunay surfaces

The sphere, for example, is a surface of revolution of constant mean curvature, or Delaunay surface. Delaunay surfaces form a one-parameter family for which a general expression for the position vector $X$ exists. Let $C = \{(X(s), Z(s))\}$ be a curve in the $(xz)$-plane of $\mathbb{R}^3$, parametrised by arc-length $s$. Revolving the curve $C$ around the $z$-axis leads to a surface of revolution $S$ with position vector

$$X(s, \theta) = \begin{pmatrix} X(s) \cos \theta \\ X(s) \sin \theta \\ Z(s) \end{pmatrix}$$  \hspace{1cm} (3.39)

where $0 \leq \theta < 2\pi$. From (3.39) we find

$$I = ds^2 + X^2 d\theta^2$$  \hspace{1cm} (3.40a)

$$II = (X'Z'' - X''Z')ds^2 + XZ'd\theta^2$$  \hspace{1cm} (3.40b)

which shows that the parametric lines of $S$, i.e. its meridians and parallels, are also its lines of principal curvature (because $II$ is diagonal). However, the parameters $(s, \theta)$ are not isothermal; an isothermal coordinate system will be introduced below. The functions $X(s)$ and $Z(s)$ can be found explicitly once the mean curvature $H(s, \theta)$ is given. In the present case we require $H = \text{constant}$, and it proves convenient to choose $H = \frac{1}{2}$. Then it can be shown that

$$X = (1 + D^2 + 2D \sin s)^{\frac{1}{2}}$$  \hspace{1cm} (3.41)

$$Z = \int \frac{1 + D \sin s}{X} \, ds$$  \hspace{1cm} (3.42)

where $D \geq 0$ is a real parameter. The surface corresponding to $D = 0$ is the right-circular cylinder; when $0 < D < 1$ the Delaunay surface is an unduloid; when $D = 1$ it is a sphere; and for $D > 1$ the surface is called a nodoid.
To make use of the general results for CMC surfaces described above an isothermal parametrisation of the Delaunay surfaces must be introduced. Defining the new variable $\eta(s)$ by

$$\eta \equiv \int \frac{ds}{X}$$

(3.43)

causes the first fundamental form (3.40a) to become

$$I = X^2(d\eta^2 + d\theta^2).$$

(3.44)

Therefore the parameters $(\eta, \theta)$ are isothermal. The problem now is to find the Gauss map $\zeta$ of a Delaunay surface, and it is shown in the Appendix that

$$\zeta = Re^{-i\Phi}$$

(3.45)

where

$$R^2 = \frac{1 + D \text{sn}(\eta|D)}{1 - D \text{sn}(\eta|D)}$$

(3.46a)

$$\Phi = \theta + \pi.$$  

(3.46b)

Now, it is a fact that two isothermal parametrisations $(\eta, \theta)$ and $(x_1, x_2)$, say, of a surface $S$ are related by

$$\eta + i\theta = \tau(x_1 \pm ix_2)$$

(3.47)

where $\tau$ is a complex analytic function of either $z \equiv x_1 + ix_2$ or $z^*$. In the present case, where $S$ is a Delaunay surface, the function $\tau$ must be analytic in $z$. To see why this is so, consider the special case $D = 1$. The surface is then a sphere. When $D = 1$ the elliptic function $\text{sn}(\eta|D)$ becomes

$$\text{sn}(\eta|1) = \tanh \eta$$

(3.48)

and therefore the Gauss map (3.46) reduces to

$$\zeta = e^{\eta - i\Phi} = -e^{\tau^*}.$$  

(3.49)

But the Gauss map of a sphere is analytic in $z^*$. Consequently, (3.43) implies that

$$\tau = \tau(z).$$

(3.50)

It follows from Kenmotsu's representation theorem for CMC surfaces that if $\eta(x_1, x_2)$ and $\theta(x_1, x_2)$ are such that (3.50) holds then the function $\zeta(z, z^*)$ given by (3.45) and (3.46) is harmonic. It will be seen presently that this is indeed the case.
Helicoids of constant mean curvature

A natural generalisation of a surface of revolution is the surface formed when the generating curve $C$ is simultaneously revolved around the $z$-axis and translated, at constant speed, in a direction parallel to the $z$-axis. Such a surface is called a generalised helicoid. Let $2\pi h$, where $h$ is a constant, be the distance in the $z$-direction by which a given point on $C$ moves after one revolution of $C$ about the $z$-axis. Then the position vector of a generalised helicoid is given by

$$X(s,\theta) = \begin{pmatrix} X(s) \cos \Delta(s, \theta) \\ X(s) \sin \Delta(s, \theta) \\ Z(s) + h\Delta(s, \theta) \end{pmatrix}$$ (3.51)

where $X(s)$ and $Z(s)$ define the curve $C$ and $\Delta(s, \theta)$ is a function to be found. Generalised helicoids of constant mean curvature were first studied by do Carmo and Dajczer. If the mean curvature is chosen to be $H = \frac{1}{2}$ then their expressions for the functions $X$, $Z$ and $\Delta$ take the forms

$$X = (1 + D^2 + 2D \sin s)^{\frac{3}{2}}$$ (3.52)

$$Z = \int Y(1 + D \sin s) \frac{1}{X^2} \, ds$$ (3.53)

$$\Delta = \theta - h \int \frac{1 + D \sin s}{X^2 Y} \, ds$$ (3.54)

where

$$Y^2 \equiv X^2 + h^2.$$ (3.55)

Clearly, the Delaunay surfaces are obtained if $h$ is set to zero.

The first fundamental form of a CMC helicoid is

$$I = ds^2 + Y^2 \, d\theta^2,$$ (3.56)

which may be brought to the isothermal form

$$I = Y^2(d\eta^2 + d\theta^2)$$ (3.57)

by the introduction of the new variable

$$\eta \equiv \int \frac{ds}{Y}.$$ (3.58)
Again the quantity of interest is the Gauss map $\zeta$. It is shown in the Appendix that $\zeta$ is given by (3.45) where now

$$R^2 = \frac{1 + \beta \text{sn}(\alpha \eta \tilde{D})}{1 - \beta \text{sn}(\alpha \eta \tilde{D})}$$  \hspace{1cm} (3.59a)$$

$$\Phi = \theta + \pi - h \int \frac{d\eta}{1 - \beta^2 \text{sn}^2(\alpha \eta \tilde{D})}$$  \hspace{1cm} (3.59b)$$

with

$$\tilde{D} = \frac{\beta}{\alpha}$$  \hspace{1cm} (3.60a)$$

$$\alpha \equiv \frac{1}{2}(A + B), \quad \beta \equiv \frac{1}{2}(A - B)$$  \hspace{1cm} (3.60b)$$

$$A^2 \equiv (1 + D)^2 + h^2, \quad B^2 \equiv (1 - D)^2 + h^2.$$  \hspace{1cm} (3.60c)$$

Just as for Delaunay surfaces, if the isothermal parameters $(\eta, \theta)$ are related to other isothermal coordinates $(x_1, x_2)$ by (3.50) then the Gauss map $\zeta(z, z^*)$ will be harmonic.
4 CMC surfaces and the O(3) model

Now we combine the material of the previous two sections to give an explicit correspondence between static solutions to the non-linear O(3) sigma model and surfaces of constant mean curvature in $\mathbb{R}^3$.

The correspondence is founded upon the identification of the field vector $n$ of the O(3) model with the normal vector, or, more correctly, the Gauss map, of a surface. Both objects are three-component two-parameter unit vectors taking values in the subset $S^2$ of $\mathbb{R}^3$. The explicit functional form of the relation (3.30) between the spatial coordinates $(x_1, x_2)$ and the general surface parameters $(u, v)$ is to be found.

The starting point is to combine the duality or field equation of the O(3) model with the differential equations of surface theory. If the field vector and normal vector are to be identified then both sets of equations must be satisfied simultaneously. By adhering to the notational convention described in Section 3 the fundamental forms $I$ and $II$ of the surfaces to be found will be denoted

$$I = E(dx_1^2 + dx_2^2) = e^{2\sigma}|dz|^2$$

$$II = l dx_1^2 + 2m dx_1 dx_2 + n dx_2^2 = \frac{1}{2}(\psi dz^2 + \psi^* dz^{*2}) + He^{2\sigma}|dz|^2$$

where $H(x_1, x_2)$ and $\psi(z, z^*)$ are the mean curvature and Hopf function, respectively, in the $(x_1, x_2)$-parametrisation:

$$H = \frac{1}{2}e^{-2\sigma}(l + n)$$

$$\psi = \frac{1}{2}(l - n) - im.$$ 

In terms of $x_1$ and $x_2$ the Weingarten equations (3.11) are

$$n_1 = -e^{-2\sigma}(lX_1 + mX_2)$$

$$n_2 = -e^{-2\sigma}(mX_1 + nX_2)$$

where $X_i \equiv \partial_i X$. By differentiating equations (4.4) again it is found that

$$\nabla^2 n = -e^{-2\sigma}\{[l, 1 + m, 2 - (l + n)\sigma, 1]X_1 + [m, 1 + n, 2 - (l + n)\sigma, 2]X_2 + (l^2 + 2m^2 + n^2)n\}.$$ 

We can now relate some surface-geometric quantities to the energy density $\mathcal{E}(x_1, x_2)$ of the O(3) model. If either (4.4) or (4.5) is used in the expression (2.11) for $\mathcal{E}$ then we find

$$\mathcal{E} = -e^{-2\sigma}(l^2 + 2m^2 + n^2)$$
The Gaussian curvature \( K(x_1, x_2) \) may be expressed in terms of \( \mathcal{E} \). We find from (4.1) and (4.6) that

\[
K = e^{-4\sigma} (ln - m^2)
\]

\[
= e^{-2\sigma} \mathcal{E} - 2e^{-4\sigma} |\psi|^2
\]

\[
= 2H^2 - e^{-2\sigma} \mathcal{E}.
\]

These general relations will be applied to the special case of CMC surfaces below.

We consider the two classes of solution to the O(3) model separately: those which satisfy the duality equation (2.20), and those which do not.

**Surfaces arising from the duality equation**

In the duality equation (2.20) we substitute for \( n_1 \) and \( n_2 \) from the Weingarten equations (4.4), and for \( n \) from

\[
n = e^{-2\sigma} (X_1 \times X_2).
\]

Then, on equating coefficients of \( X_1 \) and \( X_2 \), we find that the duality equation implies the following constraints on the second fundamental form:

\[
n = \mp l, \quad m = \pm m.
\]

We consider the two choices of sign separately.

The solutions to the O(3) model which follow from choosing the lower sign in the duality equation, and hence in (4.10), are the Belavin-Polyakov solitons given by (2.23) and (2.24). The surfaces corresponding to these solutions are to be found and will be denoted \( S_+ \). From (4.10) we find that the Hopf function (4.3) of \( S_+ \) is identically zero,

\[
\psi = 0,
\]

and therefore from (4.1) that

\[
\| = He^{2\sigma} |dz|^2 = HI
\]

where

\[
H = e^{-2\sigma} l.
\]
The Codazzi-Mainardi condition (3.13), which in the present notation reads
\[ \partial^* \psi - e^{2\sigma} \partial H = 0, \] (4.14)
therefore implies that \( H \) is a constant, as a result of (4.11). Note that \( H \) cannot be identically zero, for it would follow from (4.12) that \( II = 0 \) and therefore that the surface \( S_+ \) is a plane.
A plane has a trivial Gauss map—a single point on \( S^2 \)—which corresponds to the ground state \( n = n_0 \) of the O(3) model. Consequently, we shall assume here that \( H = \text{constant} \neq 0 \).
We can now use results from the theory of CMC surfaces. The mean curvature \( H \) is just a scale factor for such a surface and may be set to unity, for convenience. Then we see from (4.12) that \( I = II \). Thus, from (3.22), the surface \( S_+ \) corresponding to soliton solutions of the O(3) model is the unit sphere \( S^2 \). This result is somewhat surprising when one considers the multitude of possible functional forms of the field vector \( n(x_1, x_2) \) which can follow from the general soliton configuration (2.24). Yet the surface to which \( n \) is normal is always the unit sphere. But, clearly, different functions \( \omega(z) \) will lead to different parametrisations of the sphere \( S_+ \). This can be made explicit by showing that \( I \) can always be brought to the standard form of the metric on a sphere. To see this, consider the expression (4.6) for the energy density \( \mathcal{E} \). We find immediately that
\[ e^{2\sigma} = \mathcal{E}. \] (4.15)
But the energy density \( \mathcal{E} \) of a soliton configuration \( W = \omega(z) \) is found from (2.17) to be
\[ \mathcal{E} = \frac{4}{(1 + |\omega|^2)^2} \left| \frac{d\omega}{dz} \right|^2 \] (4.16)
\[ = \frac{\sin^2\Theta}{|\omega|^2} \frac{d|\omega|^2}{|dz|^2} + |\omega|^2 \frac{d\Phi^2}{|dz|^2} \] (4.17)
where \( \Theta \) and \( \Phi \) are defined by (2.8). Noting that
\[ \frac{d\Theta}{d|\omega|^2} = \frac{\sin \Theta}{|\omega|^2} \] (4.18)
enables (4.15) and (4.17) to be combined to give
\[ I = \mathcal{E}|dz|^2 = d\Theta^2 + \sin^2\Theta d\Phi^2, \] (4.19)
as claimed. The dependence of \( \Theta \) and \( \Phi \) on the variables \( x_1 \) and \( x_2 \) will of course depend on the functional form of \( \omega(z) \).
We may find the explicit form of the relation (3.30) between \((x_1, x_2)\) and \((u, v)\). We equate expression (3.37) for the normal vector of a CMC surface with (2.7) for the field vector of the O(3) model. This yields
\[
u(x_1, x_2) = \Re \omega(z), \quad v(x_1, x_2) = -\Im \omega(z),
\]
i.e. the Gauss map \(\zeta\) of \(S_+\) is given by
\[
\zeta(z^*) = \omega^*(z^*).
\](4.20)

Now consider the Gaussian curvature \(K\) of the surface \(S_+\). We see immediately from either (4.7) or (4.8) that
\[
K = 1,
\](4.21)
as expected for a unit sphere. However, the total curvature \(K_{\text{tot}}\) is not always \(4\pi\). It is in fact equal to the total energy \(E_{\text{tot}}\) of the soliton configuration \(\omega(z)\):
\[
K_{\text{tot}} = \int K e^{2\sigma} \, dx_1 \wedge dx_2 = \int \mathcal{E} \, dx_1 \wedge dx_2 = E_{\text{tot}} = 4\pi Q
\](4.22)
where \(Q > 0\). Equation (3.18) then confirms that \(Q\) can be identified with the degree \(\delta\) of the Gauss map of the surface. A sphere is a compact surface and therefore the Euler characteristic \(\chi\) is defined. Invoking (3.16) gives
\[
\chi = 2Q.
\](4.23)
These facts show that when \(Q > 1\) the surface \(S_+\) is actually a multiple covering of the unit sphere \(S^2\).

Now consider the upper sign in the duality equation (2.20). This choice leads to the Belavin-Polyakov antisolitons given by \(W = \omega(z^*)\). The surfaces to be found in this case will be denoted \(S_-\). Taking the upper sign in (4.10) gives
\[
n = -l
\](4.24)
which shows straight away that the mean curvature (4.2) is identically zero. Therefore the surfaces \(S_-\) corresponding to antisoliton solutions of the O(3) model are minimal surfaces. The position vector of a minimal surface is given by the Weierstrass-Enneper representation formula (3.23). The task is to find the functional form of the Hopf function \(\psi\) which follows from the antisoliton configuration \(\omega(z^*)\). Then the minimal surfaces \(S_-\) may be constructed.
explicitly. To begin with, the expressions for the field vector and normal vector are equated to give

\[ \zeta(z) = \omega^*(z). \tag{4.25} \]

The metric factor \( e^{2\sigma} \) is found from (4.6) to be related to the energy density \( \mathcal{E} \) by

\[ e^{2\sigma} = |\psi|^2 \mathcal{E}^{-1}. \tag{4.26} \]

In order to make the results obtained in this section directly comparable with those found in the case of the lower sign, we shall take \( x_1 \) and \( x_2 \) to be parameters along the lines of curvature of \( S_- \). The Hopf function \( \psi \) will then be a real constant because \( S_- \) is a CMC surface. So although taking the upper sign in (4.10) gives no restriction on the coefficient \( m \), the parametrisation of \( S_- \) may nevertheless be chosen to make \( m \) vanish. For convenience we will set

\[ \psi = 1. \tag{4.27} \]

Then we find the fundamental forms (4.1) of \( S_- \) to be

\[
I = \mathcal{E}^{-1}|dz|^2 = \frac{1}{4}(1 + |\zeta|^2)^2 \left( \frac{dz}{d\zeta} \right)^2 |dz|^2 \tag{4.28a}
\]

\[
II = \frac{1}{2}(dz^2 + dz^*{}^2) \tag{4.28b}
\]

where (4.23) has been used in (4.28a). The function \( z(\zeta) \) is the inverse of \( \zeta(z) \). Now, rearranging (3.32) gives the form of the Hopf function \( \Psi(\zeta) \) in the \((u, v)\) parametrisation:

\[ \Psi = \left( \frac{dz}{d\zeta} \right)^2 \tag{4.29} \]

The Weierstrass-Enneper formula (3.23) then gives the position vector \( \mathbf{X}(u, v) \) of the minimal surface \( S_- \) corresponding to the analytic function \( \Psi(\zeta) \). Let us consider some examples. A single antisoliton of unit topological charge is represented by the analytic function \( \omega(z^*) = z^* \).

We find, then, that \( \zeta = \omega^* = z \) and therefore from (4.29) that \( \Psi = 1 \). The minimal surface defined by this Hopf function is Enneper’s surface. Next, consider a single antisoliton of arbitrary charge \( q \), represented by the function (2.26):

\[ \omega(z^*) = z^*q. \tag{4.30} \]

The inverse function \( z(\zeta) \) is clearly

\[ z = e^{2\pi is/q} \zeta^{1/q} \tag{4.31} \]
where $s$ is any integer. The Hopf function \( \Psi \) is therefore

\[
\Psi = \frac{1}{q^2} e^{4\pi i s/q} \zeta^{2/q-2}.
\]

(4.32)

This is the Hopf function of a Bour surface: a comparison with (3.29) yields

\[
b = \frac{2}{q}, \quad \alpha = \frac{4\pi s}{q}, \quad c = \frac{1}{q^2}.
\]

(4.33)

The scale factor $c$ may be neglected; the Bour surface is determined essentially by the exponent $b$ and the Bonnet angle $\alpha$. But note that fixing the value of $q$ does not specify $\alpha$ uniquely.

A moment’s thought shows that, for a given value of $q$, distinct surfaces are generated if the integer $s$ takes the values

\[
s = 1, 2, \ldots, d
\]

(4.34)

where the ‘multiplicity’ $d$ is given by

\[
d = \begin{cases} 
q, & \text{if } q \text{ is odd}, \\
\frac{q}{2}, & \text{if } q \text{ is even}.
\end{cases}
\]

(4.35)

Therefore there are $d$ Bour surfaces, associate to each other, corresponding to an antisoliton of charge $q$.

Calculations analogous to those in the case of the lower sign may be carried out. The Gaussian curvature is found from (4.8) to be

\[
K = -e^{-4\sigma}
\]

(4.36)

and the total curvature $K_{\text{tot}}$ to be

\[
K_{\text{tot}} = \int Ke^{2\sigma} \, dx_1 \wedge dx_2 = -\int E \, dx_1 \wedge dx_2 = -E_{\text{tot}} = 4\pi Q
\]

(4.37)

where $Q < 0$. The integer $Q$ is again identified with the degree $\delta$ of the Gauss map but since there are no compact minimal surfaces the Euler characteristic $\chi$ is not defined.

To summarise, finite-energy solutions to the two-dimensional O(3) sigma model are induced by the Gauss maps of surfaces in $\mathbb{R}^3$ which are conformal to their spherical images. Explicitly, the soliton solutions of the O(3) model are induced by the Gauss maps $\zeta(z^*)$ of multiple coverings of the unit sphere $S^2$, for which the metric is given by (4.19) and the mean curvature and Hopf function are

\[
H = 1, \quad \psi = 0.
\]

(4.38)
Antisoliton solutions are induced by the Gauss maps $\zeta(z)$ of certain minimal surfaces, for which
\[ H = 0, \quad \psi = 1. \tag{4.39} \]
The minimal surfaces can be found explicitly by going to the $(u,v)$-parametrisation, in which the Hopf function $\Psi$ has the form \([\textbf{1.29}]\), and by using the Weierstrass-Enneper representation formula \([\textbf{3.23}]\). In both cases, the functional form of the Gauss map $\zeta$ must be that of a polynomial in its argument. Then a solution of the form \([\textbf{2.24}]\) will be induced.

**Surfaces arising from the field equation**

We now turn to the general problem of determining which surfaces $S$ have normal vectors $\mathbf{n}$ which obey the field equation, but not the duality equation, of the O(3) sigma model.

The field equation \([\textbf{2.4}]\) implies simply that the coefficients of $X_1$ and $X_2$ in the expression \([\textbf{1.5}]\) for $\nabla^2 \mathbf{n}$ must vanish. By introducing the mean curvature $H$, given by \([\textbf{4.2}]\), these coefficients become
\[
\frac{1}{2}(l-n)_1 + m_2 + e^{2\sigma} H_1 = 0 \tag{4.40a}
\]
\[
\frac{1}{2}(l-n)_2 - m_1 - e^{2\sigma} H_2 = 0. \tag{4.40b}
\]
Multiplying the second of these equations by $i$ and adding it to the first gives
\[
\partial^* \psi + e^{2\sigma} \partial H = 0 \tag{4.41}
\]
which may be combined with the Codazzi-Mainardi condition \([\textbf{4.14}]\) to give
\[
\partial^* \psi = 0, \quad \partial H = 0. \tag{4.42}
\]
These relations imply that the field equation of the O(3) model, when regarded as a constraint on the normal vector to a surface $S$, is the condition that $S$ be of constant mean curvature.

Now the two expressions \([\textbf{2.7}]\) and \([\textbf{3.37}]\) for the vector $\mathbf{n}$ may be equated. The result is that the stereographic projection $\zeta(z,z^*)$ of the normal vector can be identified in all cases with the complex conjugate of the stereographic projection $W(z,z^*)$ of the field vector:
\[
\zeta(z,z^*) = W^*(z,z^*). \tag{4.43}
\]
This result encompasses \([\textbf{1.20}]\) and \([\textbf{4.25}]\). Also, the expressions \([\textbf{4.22}]\) and \([\textbf{4.37}]\) for the total curvature $K_{\text{tot}}$ are special cases of the following result, obtained by comparing the integrand in \([\textbf{3.35}]\) with the charge density $J(x_1, x_2)$ given by \([\textbf{2.18}]\):
\[
K_{\text{tot}} = 4\pi \int J \, dx_1 \wedge dx_2 = 4\pi Q. \tag{4.44}
\]
The relation $K_{\text{tot}} = \pm E_{\text{tot}}$ follows if the duality equation holds, *i.e.* in the case of finite total energy.

Upon the identification of $\zeta$ with $W^*$, the Kenmotsu equation (3.36) becomes the complex conjugate of the field equation (2.9). Therefore any time-independent solution $W(z, z^*)$ of the O(3) model may be regarded as (the conjugate of) the Gauss map of a CMC surface $S$ in $\mathbb{R}^3$. In principle, $S$ may be constructed explicitly by inserting $\zeta = W^*$ into the Kenmotsu representation formula. Conversely, every CMC surface $S$ will induce via its Gauss map $\zeta$ a solution to the O(3) model. If $S$ is minimal or a sphere then the proviso given above—that the Gauss map $\zeta$ must be a polynomial—must hold. Examples of solutions induced by non-minimal, non-spherical CMC surfaces are given next.

**The Delaunay solution**

The following result is an immediate consequence of identifying (2.29) as the Gauss map (3.46): the Gauss map of a Delaunay surface induces the elliptic interpolating solution of the O(3) sigma model in the case where the function $\omega$ is analytic in $z$. The constraint on $\omega$ is necessary for consistency with the special case $D = 1$, for in that case the Delaunay surface is a sphere and it was found above that the sphere $S^2$ corresponds to soliton solutions, for which $W = \omega(z)$. Therefore, for example, the interpolation between a meron and a soliton is realised by the transformation between a cylinder ($D = 0$) and a sphere ($D = 1$). In this case, then, the interpolating solution may be dubbed the ‘Delaunay solution’. On the other hand, if the function $\omega$ is analytic in $z^*$ then when $D = 1$ the interpolating solution (2.29) reduces to the antisoliton configuration $W = \omega(z^*)$. The surfaces which correspond to this solution are the minimal surfaces found above, none of which are contained in the one-parameter family of equations (3.45) and (3.46). So when $\omega = \omega(z^*)$ there is an obstruction to the interpretation of (2.29) in terms of the Gauss map of Delaunay surfaces. Instead, the interpolation between an antimeron and an antisoliton corresponds to a transformation between a cylinder and a minimal surface, but we know of no family of CMC surfaces which realise this.

If it is assumed that $\omega$ is analytic in $z$ then the following relation between the isothermal parameters ($\eta, \theta$) on the Delaunay surface and the complex function $\omega(z)$ may be deduced from (2.32):

$$\omega = -e^\tau$$

where $\tau \equiv \eta + i\theta$. Equation (4.45) is in agreement with (3.49), which holds when $D = 1$. 

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The helicoidal solution

Compare the Gauss map (3.59) of a CMC helicoid with the Purkait-Ray solution (2.33) of the O(3) model. The correspondence is obvious and we may relate the two parameters $\tilde{D}$ and $h$ of the CMC helicoid to the constants $c_1, c_2, c_3$ appearing in (2.33):

$$\tilde{D}^2 = \frac{\beta^2}{\alpha^2} = \frac{2c_1 + c_2^2 + 4c_3^2 + 2\delta}{2c_1 + c_2^2 + 4c_3^2 - 2\delta}$$

(4.46)

where $\delta$ is given by (2.36), and

$$h = -4c_2^{-1}c_3.$$  \hspace{1cm} (4.47)

Therefore, when the pitch $h$ of the helicoid is non-zero, the Gauss map of a helicoid of constant mean curvature induces the Purkait-Ray solution of the O(3) sigma model. This is the justification for calling (2.33) the ‘helicoidal solution’.
5 Conclusion

In this paper we have shown that the Belavin-Polyakov soliton solutions of the non-linear O(3) sigma model in two dimensions are induced by the Gauss maps of multiple coverings of the unit sphere. These surfaces have a total curvature $K_{\text{tot}}$ equal to the total energy $E_{\text{tot}} = 4\pi Q$, $Q > 0$, of the soliton configuration, and an Euler characteristic $\chi$ equal to twice the total topological charge $Q$. We have shown that antisoliton solutions are the Gauss maps of certain minimal surfaces. The total curvature $K_{\text{tot}}$ of these surfaces is the negative of the total energy: $K_{\text{tot}} = -E_{\text{tot}} = 4\pi Q$, $Q < 0$.

Other solutions, for which, in general, the total energy $E_{\text{tot}}$ is infinite, were found to be the Gauss maps of surfaces of constant mean curvature. The differential equation which is satisfied by the Gauss map of such a surface is the field equation of the O(3) model. The total curvature $K_{\text{tot}}$ is still equal to $4\pi Q$ but now the topological charge $Q$ need be neither integral nor finite. The elliptic interpolating solution, in the case of interpolation between merons and solitons, is induced by the Gauss map of the one-parameter family of Delaunay surfaces. A more general solution, due to Purkait and Ray, was found to be the Gauss map of the two-parameter family of CMC helicoids discovered by do Carmo and Dajczer.

There exist other CMC surfaces in $R^3$, known as ‘Wente tori’, but there are few explicit expressions for the position vector of such surfaces. A number are given in a paper by Walter, but their complicated algebraic form prohibits easy use. If the Gauss map of a Wente torus could be deduced then it would correspond to a new solution to the O(3) model. Also, it would be satisfying if a family of CMC surfaces could be found which corresponds to the interpolation between antimerons and antisolitons.

The obvious generalisation of the present work is to include time-dependence in the sigma model. The field vector $\mathbf{n}(x_1, x_2, t)$ could then be regarded in one of two ways, either as the normal to a moving two-dimensional surface in $R^3$ or as the normal to a three-dimensional hypersurface in $R^4$. Work in this direction could well prove useful, for the following reason: time-dependence in the O(3) model causes new topological features to arise which could benefit from a surface-geometric interpretation. For example, the charge density $J(x_1, x_2)$ becomes one component of a topological current $J_\mu(x_1, x_2, t)$. Further, field configurations $\mathbf{n}(x_1, x_2, t)$ which tend to the same constant value at both spatial and temporal infinity cause spacetime to be compactified to $S^3$. The field is then a mapping from $S^3$ to $S^2$, and such
mappings fall into homotopy classes according to

\[ \pi_3(S^2) \sim \mathbb{Z}. \]

The integer which labels these classes is the Hopf invariant \( H \) which, in the O(3) model, has the effect of imparting fractional spin and statistics to the Belavin-Polyakov soliton after quantisation. But the literature on the subject of the Hopf invariant in the O(3) sigma model is rife with inaccuracies and seeming inconsistencies, although some papers are remedying this situation.

It may be that if the results presented here can be generalised to include time-dependence then a surface-geometric interpretation of quantities such as \( J_\mu \) and \( H \) can be found. A similar idea involving the geometry of space curves has been used recently by Balakrishnan et al. These authors found that the current \( J_\mu \) arises from anholonomy effects when a point \( (x_1, x_2, t) \equiv (x, y, z) \) on a three-parameter space curve moves to \( (x + \Delta x, y + \Delta y, z + \Delta z) \). The field vector \( \mathbf{n} \) is identified with the tangent to the curve. An analogous treatment in terms of surfaces would complement the work of Balakrishnan et al. and provide an alternative, and perhaps richer, geometric view of the structure of the time-dependent O(3) model.

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Appendix: The Gauss map of a CMC helicoid

This appendix contains a derivation of equations (3.59) which, when combined with (3.45), give the Gauss map \( \zeta \) of a helicoid of constant mean curvature. If the parameter \( h \) is set to zero in the derivation then the Gauss map of a Delaunay surface, equations (3.46), is obtained. We work initially in the \((s, \theta)\)-parametrisation and later change to the isothermal coordinated \((\eta, \theta)\), where \( \eta(s) \) is defined by (3.58).

From equations (3.51)–(3.55) it is seen firstly that the tangent vectors to a CMC helicoid are

\[
X_s = \frac{1}{XY} \begin{pmatrix} YD \cos s \cos \Delta + h(1 + D \sin s) \sin \Delta \\ YD \cos s \sin \Delta - h(1 + D \sin s) \cos \Delta \\ X(1 + D \sin s) \end{pmatrix}, \quad X_\theta = \begin{pmatrix} -X \sin \Delta \\ X \cos \Delta \\ h \end{pmatrix}
\]  

and therefore that the unit normal vector is

\[
n = \frac{-1}{XY} \begin{pmatrix} Y(1 + D \sin s) \cos \Delta - hD \cos s \sin \Delta \\ Y(1 + D \sin s) \sin \Delta + hD \cos s \cos \Delta \\ -XD \cos s \end{pmatrix}.
\]

To ensure consistency with the results obtained for spheres and minimal surfaces we take the form of the stereographic projection to be

\[
\zeta = \frac{n_1 - in_2}{1 + n_3} = e^{-i(\Delta + \pi)} \left( \frac{Y(1 + D \sin s) - ihD \cos s}{X(Y + D \cos s)} \right).
\]  

The numerator of the fraction in the bracket is converted to the modulus-argument form

\[
Y(1 + D \sin s) - ihD \cos s \equiv \rho e^{-i\gamma}
\]

where

\[
\rho^2 \equiv Y^2(1 + D \sin s)^2 + h^2D^2 \cos^2 s = X^2(Y^2 - D^2 \cos^2 s)
\]

\[
\gamma \equiv \tan^{-1} \left( \frac{hD \cos s}{Y(1 + D \sin s)} \right)
\]

\[
= -hD \int \frac{Y^2(D + \sin s) + D(1 + D \sin s) \cos^2 s}{X^2Y(Y^2 - D^2 \cos^2 s)} ds.
\]

Then the Gauss map (5.3) becomes

\[
\zeta = e^{-i\Phi} \left( \frac{1 - f}{1 + f} \right)^{1/3}
\]  

(5.6)
where

\[ \Phi = \Delta + \pi + \gamma \]
\[ = \theta + \pi - h \int \frac{ds}{Y(1 - f^2)} \]  \hspace{1cm} (5.7a)

and

\[ f \equiv D Y^{-1} \cos s. \] \hspace{1cm} (5.7b)

Next, we change variables temporarily in order to simplify equation (5.4) for \( \zeta \). The change of variables amounts essentially to inverting (3.58), i.e. to finding \( s \) in terms of \( \eta \). The integral in (3.58) is elliptic and may be written in terms of the standard elliptic integral

\[ t(s) \equiv \int_{0}^{\phi(s)} \frac{dx}{(1 - k^2 \sin^2 x)^{1/2}}. \] \hspace{1cm} (5.8)

In fact we have

\[ \eta = -2A^{-1} t \] \hspace{1cm} (5.9)

\[ \phi(s) \equiv \sin^{-1} \left( \frac{1 - \sin s}{2} \right)^{1/2} \] \hspace{1cm} (5.10)

\[ k^2 \equiv \frac{4 \tilde{D}}{(1 + \tilde{D})^2} \] \hspace{1cm} (5.11)

where \( A \) and \( \tilde{D} \) are defined by equations (3.60). Now, by definition of the Jacobi elliptic functions, we have

\[ \text{sn}(t|k) = \sin \phi \] \hspace{1cm} (5.12)

and therefore (5.10) gives

\[ \sin s = 1 - 2 \text{sn}^2(t|k) \] \hspace{1cm} (5.13a)

\[ \cos s = 2 \text{sn}(t|k) \text{cn}(t|k). \] \hspace{1cm} (5.13b)

Since, by (5.9), \( t \) is a function of \( \eta \), either of these relations (5.13) may be taken as defining the function \( s(\eta) \) which is the inverse of \( \eta(s) \). We now change temporarily to the variables \( (t, \theta) \). It follows from (5.11), (3.60), and the relation

\[ \text{dn}^2(t|k) = 1 - k^2 \text{sn}^2(t|k) \] \hspace{1cm} (5.14)

that

\[ Y = A \text{dn}(t|k). \] \hspace{1cm} (5.15)
Therefore equations (5.13) yield

\[ ds = -2 \, dn(t|k) \, dt = -2A^{-1}Y \, dt. \]  

(5.16)

Consequently, equations (5.7) for the quantities \( \Phi \) and \( f \) become

\[ \Phi = \theta + \pi + 2h \int \frac{dt}{1 - f^2} \]  

(5.17a)

\[ f = \left( \frac{2D}{A} \right) \frac{\text{sn}(t|k) \, \text{cn}(t|k)}{\text{dn}(t|k)}. \]  

(5.17b)

We now invoke the ‘ascending Landen transformation’ for the Jacobi function \( \text{sn} \):

\[ \text{sn}(x|m) = \left( \frac{2}{1 + m} \right) \frac{\text{sn}(y|n) \, \text{cn}(y|n)}{\text{dn}(y|n)} \]  

(5.18)

where

\[ y \equiv \left( \frac{1 + m}{2} \right) x \]  

(5.19a)

\[ n^2 \equiv \frac{4m}{(1 + m)^2}. \]  

(5.19b)

If the identifications

\[ y = t, \quad n = k, \quad m = \tilde{D} \]  

(5.20)

are made then equation (5.19b) coincides with (5.11) and (5.18) becomes, on using (3.60),

\[ \text{sn}(x|\tilde{D}) = \left( \frac{A + B}{A} \right) \frac{\text{sn}(t|k) \, \text{cn}(t|k)}{\text{dn}(t|k)}. \]  

(5.21)

We use (5.21) and (3.60) to bring (5.17b) to the form

\[ f = \beta \, \text{sn}(x|\tilde{D}). \]  

(5.22)

Now we revert to the variables \((\eta, \theta)\) by finding \(x\) in terms of \(\eta\) from (5.19a) and (3.60):

\[ x = \left( \frac{2}{1 + \tilde{D}} \right) t = -\alpha \eta. \]  

(5.23)

Therefore the quantity \(f\) becomes

\[ f = -\beta \, \text{sn}(\alpha \eta|\tilde{D}). \]  

(5.24)

Finally, we bring together equations (5.4), (5.17a) and (5.24) to find that the Gauss map \(\zeta\) of a CMC helicoid takes the form (3.45) with \(R\) and \(\Phi\) given by (3.59), as claimed.
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