RADON MEASURE SOLUTIONS FOR STEADY COMPRESSIBLE HYPERSONIC-LIMIT EULER FLOWS PASSING CYLINDRICALLY SYMMETRIC CONICAL BODIES

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Abstract. We study steady uniform hypersonic-limit Euler flows passing a finite cylindrically symmetric conical body in the Euclidean space $\mathbb{R}^3$, and its interaction with downstream static gas lying behind the tail of the body. Motivated by Newton’s theory of infinite-thin shock layers, we propose and construct Radon measure solutions with density containing Dirac measures supported on surfaces and prove the Newton-Busemann pressure law of hypersonic aerodynamics. It happens that if the pressure of the downstream static gas is quite large, the Radon measure solution terminates at a finite distance from the tail of the body. The main difficulty of the analysis is a correct definition of Radon measure solutions. The results are helpful to understand mathematically some physical phenomena and formulas about hypersonic inviscid flows.

1. Introduction. Considering steady inviscid non-isentropic compressible gas flow in the three-dimensional Euclidean space $\mathbb{R}^3$ with Cartesian coordinates $(x, y, z)$, the governing Euler system reads

$$F(U)_x + G(U)_y + H(U)_z = 0, \quad U = (\rho, u, v, w, E)^	op,$$

where

$$F(U) = (\rho u, \rho u^2 + p, \rho u v, \rho u w, \rho u E)^	op,$$

$$G(U) = (\rho v, \rho u v, \rho v^2 + p, \rho v w, \rho v E)^	op.$$
and the unknowns $\rho, p, u, v, w$ represent respectively the density, pressure and velocity component along $x$, $y$, $z$-axis of the flow. Furthermore, $E$ is the total enthalpy per unit mass. For polytropic gas, $p$ and $E$ are supposed to satisfy the following constitutive relation

$$p = \frac{\gamma - 1}{\gamma} \rho \cdot \left( E - \frac{q^2}{2} \right)$$

with $q^2 = u^2 + v^2 + w^2$, and $\gamma > 1$ the adiabatic exponent. Equivalent but more usual expressions of (1.2) are

$$E = \frac{q^2}{2} + \frac{\gamma - 1}{\gamma} \rho \quad \text{and} \quad p = \kappa \rho \gamma \exp \left( \frac{S}{c_v} \right).$$

In the latter, $S$ denotes the specific entropy, and $\kappa, c_v$ are some positive constants. The sonic speed is $c = \sqrt{\gamma p / \rho}$, and Mach number is $M = q / c$. According to [26, Proposition 1, p. 4], hypersonic-limit Euler flows are actually pressureless inviscid flows, since for given (scaled) uniform upstream supersonic flow $U_0$, the limit that the Mach number $M_0$ increases to $+\infty$ is equivalent to the limit that the adiabatic exponent $\gamma$ decreases to 1. Hence from (1.2) one has $p_0 = 0$. However, when particles in the pressureless flow impinge on a solid surface, they were forced to move on the surface, thus produce nontrivial pressure distributions, as well as a layer of concentrated mass. This is a very fundamental issue for hypersonic aerodynamics [2, section 1.3.1]. On the other hand, it had been noticed for a long time that for $\gamma = 1$, the system (1.1), namely the pressureless Euler system, is weakly hyperbolic in the sense that the eigenvalues are real and identical, the associated eigenvectors are not complete, and linearly degenerate (cf. [13, p. 325]). Hence some Riemann problems can only be solved in the class of Radon measures that are singular to the Lebesgue measure $\mathcal{L}^3$ of $\mathbb{R}^3$. Therefore one needs a proper interpretation of (1.1) when the unknowns are Radon measures, instead of locally integrable functions with respect to $\mathcal{L}^3$. It is preferable if one could characterize the concentration layers by mathematical analysis of Radon measure solutions, and derive classical physical formulas on pressure distributions of bodies in hypersonic flows, namely the celebrated Newton’s sine-squared pressure law and its extensions by Busemann when the surface is curved [14, chapter 3, section 4, p. 120, p. 123]. See [1, 2, 14, 21] for nice introductions on hypersonic flows.

In this paper, we are concerned with the uniform hypersonic-limit Euler flow passing a finite cylindrically symmetric conical body \{$(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq x_*, \, y^2 + z^2 \leq f(x)^2$\} without attacking angle, while with downstream static gas lying behind the tail cross-section of the cone that is denoted by $\Gamma^s_b \doteq \{(x, y, z) \in \mathbb{R}^3 : x = x_*, \, y^2 + z^2 = f(x)^2\}$, see Figure 1. Here $f(x) \in C^2([0, x_*])$ is given and assumed to satisfy $f(0) = 0$, $f'(0^+) > 0$, and $f(x) > 0$, $f'(x) \geq 0$ for $0 < x \leq x_*$. Hence the domain is $\Omega^s \doteq \{(x, y, z) \in \mathbb{R}^3 : 0 < x \leq x_*, \, y^2 + z^2 > f(x)^2\} \cup \{x > x_*\}$. The solid lateral boundary of the cone is denoted by

$$\Gamma^s_j \doteq \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq x_*, \, y^2 + z^2 = f(x)^2\}.$$  

It is assumed that the flow satisfies the slip condition

$$uf(x)f'(x) = vy + wz \quad \text{on} \quad \Gamma^s_j.$$  

(1.3)
Without loss of generality (cf. [26, p. 3] for a non-dimensional scaling), the uniform upstream incoming flow can be taken as
\[ U = U_0 = (\rho_0, u_0, v_0, w_0, E_0)^\top = (1, 1, 0, 0, E_0)^\top \] on \{x = 0\}. \hspace{1cm} (1.4)

Here \( E_0 > \frac{1}{2} \) is a given constant, while \( p_0 = 0 \), by recalling (1.2) as well as \( \gamma = 1 \) for limiting hypersonic flow. The downstream uniform static gas is denoted by
\[ U = (\rho, 0, 0, 0, E)^\top. \hspace{1cm} (1.5) \]

**Figure 1.** The upstream hypersonic-limit flow is separated from the downstream static gas by an axially-symmetric free concentration interface.

The problems of supersonic flows past conical bodies have attracted many mathematicians for intensive studies in the past decades, due to its fundamental applications in aerodynamics (cf. [1, 2]), and various difficulties in analysis (cf. [4, 3, 8, 25] and references therein). Previous studies had concerned local solutions, global solutions, piecewise smooth solutions, general integrable weak solutions, isentropic ir-rotational flows, supersonic shocks, transonic shocks, polytropic gases etc. (cf. [4, 3, 10, 5, 7, 6, 9, 15, 27] and references therein). Now we have many impressive theorems and systematic methods to study supersonic flows passing bodies, thanks to the significant contributions made by Prof. Shuxing Chen and his collaborators. Motivated by their research, we continue to investigate hypersonic flows. Hu and Zhang [17] studied global solutions for isentropic ir-rotational flows past symmetric cones with quite large upstream Mach number. Kuang, Xiang, and Zhang [20] proved hypersonic similarity laws for potential flows past wedges. However, to our knowledge, there is no results on limiting hypersonic flows, except Qu, Yuan for conical flows [25], in which the authors presented an analytical proof of Newton’s sine-squared pressure law for a circular cone with zero attacking angle, by solving Radon measure solutions of the conical Euler equations on the unit sphere in \( \mathbb{R}^3 \).

The main difficulty of studying limiting hypersonic flows is a proper definition of Radon measure solutions of initial-boundary value problems of compressible Euler equations. We note that there are many works on the Cauchy problems of unsteady Euler equations of pressureless flow or Chaplygin gas, and other singular hyperbolic conservation laws for initial value problems, mainly motivated by mathematical curiosity (see, for example, [11, 12, 23, 22, 28] and references therein). These works present different definitions of measure solutions of hyperbolic conservation laws.
For example, Huang and Wang’s method of generalized potentials and they proved well-posedness for Cauchy problem of unsteady one-space dimensional pressureless Euler equations with Radon measures being initial data [18]. Nedeljkov introduced shadow wave solution in [22] to deal with the delta shocks to certain systems of conservation laws, and studied interactions of shadow waves and their entropy admissible conditions. Paiva [23] applied Sarrico’s $\alpha$-multiplication of distributions to construct delta waves for a modified Euler system of isentropic gases. Our studies demonstrate that these works are not only mathematically interesting, but also relevant to prominent physical phenomena.

As an indispensable step toward solvability of the general three-dimensional problem of hypersonic-limit flow passing conical bodies, for which there would involve complicated geometry and nonlinear partial differential equations on surfaces, in this paper we study the problems by assuming cylindrical symmetry, as indicated above. This enables us to employ the standard cylindrical coordinates $(x, r = \sqrt{y^2 + z^2}, \theta)$ of $\mathbb{R}^3$ (with $\theta \in [-\pi, \pi]$) so that $\tan \theta = z/y$, and ignore swirl, and reduce the problem to the case of two-independent variables. The price is the fluxes depend on the radius $r$, hence they are no longer homogeneous. One should be careful to define Radon measure solutions to take into account of the geometric effects. This makes the analysis more involved than the planar case studied in [19]. Mathematically, we are directed to study a nonlinear integral-differential equation. The result for the case that the cone is straight and infinitely long, with zero attacking angle, follows directly from the present work (by taking $x^* = +\infty$). It is consistent with [25]. The difference is that the surface of the cone considered here could be curved and we can prove the more general Newton-Busemann formula.

The main results of this paper are summarized as the following two theorems.

**Theorem 1.1.** For uniform hypersonic-limit Euler flow passing a finite cylindrically symmetric conical surface $\Gamma_f \doteq \{(x, r, \theta): 0 \leq x \leq x_*, r = f(x), \theta \in [-\pi, \pi]\}$, suppose that for $0 \leq x < x_*$, there hold

$$f(0) = 0, \quad f'(0+) > 0, \quad f'(x) \geq 0, \quad f''(x)K(x) + f(x)f'(x)^2\sqrt{1 + f'(x)^2} > 0,$$

where

$$K(x) \doteq \int_0^x \frac{f(t)f'(t)}{\sqrt{1 + f'(t)^2}} \, dt,$$

then there always exists a Radon measure solution to problem (1.1)-(1.4) in the sense of Definition 2.3, which is given by (3.17)-(3.19), with the density containing a weighted Dirac measure supported on $\Gamma_f \doteq \{(x, r): 0 \leq x \leq x_*, r = f(x)\}$.

In particular, there holds the Newton-Busemann formula of drags/lifts on cylindrically symmetric objects in hypersonic flows:

$$F \doteq 2\pi \int_0^{x_*} w_p(x)f(x)(f'(x), 0, 0) \, dx,$$

where

$$w_p(x) \doteq \frac{f''(x)K(x) + f(x)f'(x)^2\sqrt{1 + f'(x)^2}}{f(x)(1 + f'(x)^2)^{\frac{3}{2}}}.$$

**Remark 1.** The condition (1.6) implies $w_p > 0$. The celebrated Newton-Busemann formula for pressure distribution on cylindrically symmetric obstacles in hypersonic flows was given by [2, (3.30)] in terms of $C_p$, which is exactly $2w_p$ here. Notice that
$f'(x) = \tan \theta$ for $\theta$ appearing there, and $dr = f'(x)dx$, with $r$ corresponding to $y$, one verifies easily the equivalence of $2u_p$ here with $C_p$ in [2, (3.30)]. There is an extra factor 2, which comes from the scaling $(3.16)$, namely $C = (p - p_0)/(\frac{1}{2} \rho_0 u_0^2)$, used in [2, section 3.3]. We used however the scaling $(p - p_0)/(\rho_0 u_0^2)$, see the last line in [26, p. 3].

Particulary, for a straight cone with half-open angle $\theta_0$, namely $f(x) = x \tan \theta_0$ for $x \in [0, x_\star]$, we have $\mathbf{F} = (\rho_\infty u_\infty^2 \pi (R \sin \theta_0)^2, 0, 0)$, where $R = x, \tan \theta_0$. For a ball with radius $R$, i.e. $f(x) = R \sqrt{1 - (\frac{R-x}{R})^2}$ for $x \in [0, R]$, we have $\mathbf{F} = \rho_\infty u_\infty^2 (\frac{1}{4} \pi R^2, 0, 0)$.

**Theorem 1.2.** For uniform hypersonic-limit Euler flow passing a finite cylindrically symmetric conical surface $\Gamma_f^\circ$ with $f$ satisfying $(1.6)$, suppose that the hypersonic-limit flow behind $\{x = x_\star\}$ is separated from the static gas with constant pressure $p$ by a free concentration interface. Then there exists a global Radon measure solution $(4.7)-(4.10)$ if $p \in (0, 1]$. If instead $p > 1$, there is only a local Radon measure solution $(4.7)-(4.10)$, defined for $x \in [0, x_\star]$ and the free concentration interface terminates on the circumference centered at $\{x = x_\star, r = 0\}$, with radius $r = l(x_\star)$ satisfying $(4.26)$.

**Remark 2.** In particular, we have the following relations between radius of the free interface and the pressure of the static gas:

1) if $p = 1$, the radius of the free interface is given by an exponential function $(4.23)$;
2) if $\rho = 0$, the radius of the free interface is given by a real root of a cubic algebraic equation $(4.24)$;
3) if $0 < p < 1$, the radius of the free interface is given implicitly by $(4.25)$.

The rest of the paper contains three sections. In Section 2, we reformulate the problem in the cylindrical coordinates and prove two lemmas on the generalized Rankine-Hugoniot (RH) conditions for concentration layers which are special piecewise-smooth Radon measure solutions of linear conservation laws. A rigorous definition of Radon measure solutions for boundary value problems of compressible Euler equations is presented, with the nonlinearity decoded in some nonlinear relations of Radon-Nikodym derivatives of measures. This strategy avoids the unnecessary confusion on multiplications of measures and discontinuous functions in the literature. Another lemma is about solutions of an integral-differential equation that the free concentration layer satisfies. In Section 3, we present the proof of Theorem 1.1. Theorem 1.2 is proved in Section 4.

2. Preliminaries.

2.1. **Formulation of problems in cylindrical coordinates.** We firstly reformulate the problems in the cylindrical coordinates. By the transformation $x = x, y = r \cos \theta, z = r \sin \theta$, and set $\tilde{v} = v \cos \theta + w \sin \theta, \tilde{w} = -v \sin \theta + w \cos \theta$, the Euler system $(1.1)$ takes the following form in the cylindrical coordinates $(x, r, \theta)$ of $\mathbb{R}^3$:

\[
\begin{align*}
(rpu)_x + (rpuv)_r + (r\tilde{w})_\theta = 0, \\
\left[ r(pu^2 + p) \right]_x + (rpuv)_r + (rpu\tilde{w})_\theta = 0, \\
\left[ r(pu\tilde{v}) + r(p\tilde{v}^2 + p) \right]_x + (rpu\tilde{w})_r + (r\tilde{w}^2 + p)_\theta = 0, \\
\left[ r(pu\tilde{w}) + (r\tilde{v}^2 + p) \right]_x + (rpu\tilde{w})_r + (r\tilde{w}^2 + p)_\theta = 0, \\
\left[ rpuE \right]_x + (rpuE)_r + (r\tilde{w}E)_\theta = 0.
\end{align*}
\]
We are looking for cylindrically symmetric solutions, namely all the unknowns are independent of $\theta$. By dropping the tilde “$\tilde{\ }$” of $\tilde{v}$ in the sequel, and further assuming that $\tilde{w} \equiv 0$ — which is natural, as the upcoming flow is uniform and without swirl, the equations (2.1) are simplified to

$$
\begin{align*}
(rpu)_x + (rpv)_r &= 0, \\
[r(pu^2 + p)]_x + (rpuv)_r &= 0, \\
(rpuv)_x + [r(pu^2 + p)]_r - p &= 0, \\
(rpuE)_x + (rpvE)_r &= 0.
\end{align*}
$$

(2.2)

Notice that in the state equation (1.2), one shall take $q^2 = u^2 + v^2$ now.

It is easily seen that the lateral boundary surface of the cylindrically symmetric cone is $\Gamma^*_f = \{(x, r, \theta) : 0 \leq x \leq x_*, r = f(x), \theta \in [-\pi, \pi]\}$, and the domain under consideration is $\Omega^* = \{(x, r, \theta) : 0 < x \leq x_*, r > f(x), \text{ or } x_* < x, \theta \in [-\pi, \pi]\}$. To avoid confusion, we will use hereafter notations with superscript * to represent sets in $\mathbb{R}^3$, while notations without * to represent sets on the $(x, r)$ half-plane. Hence the domain considered on the $(x, r)$ half-plane is

$$
\Omega \doteq \Omega_1 \cup \Omega_2 \doteq \{(x, r) : 0 < x \leq x_*, r > f(x)\} \cup \{(x, r) : x > x_*, r \geq 0\},
$$

(2.3)

and the lateral surface of the cone is $\Gamma_f \doteq \{(x, r) : 0 \leq x \leq x_*, r = f(x)\}$. The slip condition (1.3) becomes

$$
v = uf^f(x) \quad \text{on } \Gamma_f.
$$

(2.4)

Correspondingly, the initial condition is

$$
U = U_0 = (\rho_0, u_0, v_0, E_0)^\top = (1, 1, 0, E_0)^\top \quad \text{on } \{x = 0, r \geq 0\}.
$$

(2.5)

The static gas behind $\Gamma_b \doteq \{(x_*, r) : 0 \leq r \leq f(x_*)\}$ is

$$
\underline{U} = (\underline{\rho}, 0, 0, \underline{E})^\top, \quad \text{and } \underline{p} = \frac{\gamma}{\gamma - 1} \underline{p}E.
$$

(2.6)

Notice that the static gas is polytropic gas with $\gamma > 1$. Thus the case $\underline{p} = 0$ could also represent vacuum.

Therefore the first problem (for Theorem 1.1) is reduced to seek a solution of the initial-boundary value problem (2.2), (1.2), (2.4)-(2.5) in the domain $\Omega_1$. The second problem (for Theorem 1.2) is to seek a free concentration layer in $\Omega_2$ with radius $r = l(x)$ such that $U = \underline{U}$ if $x > x_*$, $0 \leq r < l(x)$.

2.2. Radon measures and weighted Dirac measures. Since previous physical and mathematical studies have shown that mass may concentrate on the lateral boundary of the cone, the density is no longer a measurable function with respect to the standard Lebesgue measure. To establish a proper mathematical theory for these problems, we need study their measure solutions. We now review some basic facts about Radon measures.

Let $\mathcal{B}$ be the Borel $\sigma$-algebra of a locally-compact Hausdorff topological space $\mathcal{X}$, and $m$ a Radon measure on $\mathcal{B}$. By Riesz Representation Theorem [16, section 1.8, p. 59], there is a pairing between $m$ and a test function $\phi \in C_0(\mathcal{X})$ (the set of continuous functions on $\mathcal{X}$ with compact supports), which is given by

$$
\langle m, \phi \rangle = \int_{\mathcal{X}} \phi \, dm.
$$

(2.7)
Except the Lebesgue measure \( \mathcal{L}^n \) of the Euclidean space \( \mathbb{R}^n \), the most important example we need in this paper is the Dirac measure supported on a lower-dimensional manifold \( S \) of \( \mathbb{R}^n \). Suppose \( S \) has Hausdorff dimension \( k \), and write the \( k \)-dimensional Hausdorff measure in \( \mathbb{R}^n \) as \( \mathcal{H}^k \), we could define a weighted Dirac measure \( w_S \delta_S \) by

\[
\langle w_S \delta_S, \phi \rangle \doteq \int_S w_S(\xi)\phi(\xi)\,d\mathcal{H}^k(\xi), \quad \forall \phi \in C_0(\mathbb{R}^n),
\]

where the weight \( w_S : \xi \in S \mapsto \mathbb{R} \) is an integrable function with respect to the measure \( \mathcal{H}^k \). The value of \( w_S \delta_S \) on an open set \( A \subset \mathbb{R}^n \) is given by

\[
w_S \delta_S(A) = \int_{A \cap S} w_S(\xi)\,d\mathcal{H}^k(\xi).
\]

Readers familiar with geometric measure theory recognize easily that the weighted Dirac measure we presented here is a special case of the more general concept of varifold. Sometimes we also use the standard notation \( (m|A) \) to denote restricting the measure \( m \) on an \( m \)-measurable set \( A \), defined by \( (m|A)(C) \doteq m(A \cap C) \) for any \( m \)-measurable set \( C \).

Now let \( \mathcal{T} \) be the upper half plane \( \mathbb{R}^2_+ \doteq \{(x, r) : x \in \mathbb{R}, \ r \geq 0\} \), which is considered as a topological subspace of the Euclidean plane \( \mathbb{R}^2 \). We may define a Radon measure \( \mathcal{L}^2 \) as follows:

\[
\langle \mathcal{L}^2, \phi \rangle \doteq \int_{\mathbb{R}^2_+} \phi(x, r)\, r\,dx\,dr, \quad \forall \phi \in C_0(\mathbb{R}^2_+).
\]

Suppose that \( L \) is a Lipschitz curve on \( \mathbb{R}^2_+ \), with a parameter representation \( (x(t), r(t)) \) for \( t \in [0, T) \), and \( w_L(t) \) a continuous function, we could define the following circular Dirac measure

\[
\langle w_L \delta_L, \phi \rangle \doteq \int_0^T w_L(t)\phi(x(t), r(t))r(t)\sqrt{(x'(t))^2 + (r'(t))^2}\,dt, \quad \forall \phi \in C_0(\mathbb{R}^2_+).
\]

We may also define the standard weighted Dirac measure supported on \( L \) by

\[
\langle w_L \delta_L, \phi \rangle \doteq \int_0^T w_L(t)\phi(x(t), r(t))\sqrt{(x'(t))^2 + (r'(t))^2}\,dt, \quad \forall \phi \in C_0(\mathbb{R}^2_+),
\]

and particularly, the Dirac measure supported on a point \( \xi \) by

\[
\langle w_\xi \delta_\xi, \phi \rangle \doteq w_\xi \phi(\xi), \quad \forall \phi \in C_0(\mathbb{R}^2_+),
\]

where \( w_\xi \) is a constant. We remark that \( \mathcal{L}^2 \) represents the Lebesgue measure \( \mathcal{L}^2 \) for the cylindrically symmetric case (up to a factor \( 2\pi \)). Similarly the circular Dirac measure \( \delta_L \) actually supports on the surface \( S \) obtained by rotating \( L \) along \( x \)-axis in \( \mathbb{R}^3 \). If a measure \( \lambda \) is absolutely continuous with respect to a nonnegative measure \( \mu \), it is written as \( \lambda \ll \mu \), and the Radon-Nikodym derivative is denoted by \( d\lambda/d\mu \) (cf. [16, section 1.6.2, p. 50]). For example, we have

\[
\frac{d\mathcal{L}^2}{d\mathcal{L}^2} = r, \quad \frac{d\delta_L}{d\delta_L} = r.
\]

These measures are what we need to construct specific Radon measure solutions of the limiting hypersonic Euler equations. The former (with tildes) are physically meaningful and independent of coordinates, while the latter (without tildes) are easier to calculate for our purpose by applying directly the generalized Rankine-Hugoniot conditions introduced below.
In this paper, the support $S$ of the weighted Dirac measure is also called delta shock, or concentration interface. We usually choose $t = x$, then $\sqrt{1 + (r'(x))^2}w_L(x)$ characterizes the strength of a delta shock in the $(x, r)$ half-plane [24, Lemma 3.6].

2.3. Generalized Rankine-Hugoniot conditions. Before considering the more complicated Euler system (2.2) with geometric effects, we first show how to define Radon measure solutions of linear equations and derive the generalized Rankine-Hugoniot conditions across delta shocks.

In this subsection we write a generic point in $\mathbb{R}^2$ as $(t, x)$. We say a triple of Radon measures $(m, n, \varphi)$ solves the balance law

$$m_t + n_x = \varphi, \quad t > 0, \quad x \in \mathbb{R}$$

with initial data $m(0, x)$, which is a Radon measure on the real line $\{t = 0\}$, if there holds the following identity:

$$\langle m, \phi_t \rangle + \langle n, \phi_x \rangle + \langle \varphi, \phi \rangle + \langle m(0, x), \phi(0, x) \rangle = 0, \quad \text{for all } \phi \in C^1_c(\mathbb{R}^2),$$

where $C^1_c(\mathbb{R}^2)$ is the set of continuously differentiable functions in the Euclidean plane with compact supports.

An important case is to consider the following Riemann initial data

$$m(0, x) = m_l I_{\{x < 0\}} L^1 + m_r I_{\{x > 0\}} L^1 + w^0_n \delta_{(t = 0, x = 0)},$$

$$n(0, x) = n_l I_{\{x < 0\}} L^1 + n_r I_{\{x > 0\}} L^1 + w^0_m \delta_{(t = 0, x = 0)},$$

where all $m_l, m_r, n_l, n_r$ are functions, $w^0_m, w^0_n$ are constants, and $I_A$ is the indicator function of a set $A$, namely $I_A(\xi) = 1$ if $\xi \in A$ and $I_A(\xi) = 0$ otherwise. We call

$$m = m_l(t, x) I_{\{x < x(t)\}} L^2 + m_r(t, x) I_{\{x > x(t)\}} L^2 + w_m(t) \delta_{\{x = x(t)\}},$$

$$n = n_l(t, x) I_{\{x < x(t)\}} L^2 + n_r(t, x) I_{\{x > x(t)\}} L^2 + w_n(t) \delta_{\{x = x(t)\}},$$

$$\varphi = p_l(t, x) I_{\{x < x(t)\}} L^2 + p_r(t, x) I_{\{x > x(t)\}} L^2$$

a piecewise smooth Radon measure solution of (2.16)-(2.17), if (2.18)-(2.20) is a Radon measure solution of (2.16)-(2.17), and the functions

$$(m_l(t, x), n_l(t, x), p_l(t, x)), \quad (m_r(t, x), n_r(t, x), p_r(t, x))$$

are Lipschitz continuous, satisfy their initial values on $\{t = 0\}$ respectively in a continuous way, and $L^2$ a.e. hold

$$\partial_t m_l(t, x) + \partial_x m_l(t, x) = p_l(t, x), \quad t > 0, \quad x < x(t),$$

$$\partial_t m_r(t, x) + \partial_x m_r(t, x) = p_r(t, x), \quad t > 0, \quad x > x(t).$$

Lemma 2.1 (Generalized Rankine-Hugoniot conditions). Suppose that $(m, n, \varphi)$ is a piecewise smooth Radon measure solution given by (2.18)-(2.20). Then there hold

$$\begin{cases} w_m(0) \sqrt{1 + x'(0)^2} = w^0_m, \\ \frac{d}{dt} w_m(t) \sqrt{1 + x'(t)^2} = [m] x'(t) - [n], \end{cases}$$

where $[m] = m_r(t, x(t)+) - m_l(t, x(t)-), \quad [n] = n_r(t, x(t)+) - n_l(t, x(t)-)$.

Proof. Substituting (2.18)-(2.20) into (2.16), it follows that

$$\begin{align*}
\int_{x < x(t)} m_l(t, x) \phi_l(t, x) \, dx \, dt + \int_0^\infty w_m(t) \phi_l(t, x) \sqrt{1 + x'(t)^2} \, dt \\
+ \int_{x > x(t)} m_r(t, x) \phi_l(t, x) \, dx \, dt + \int_{x < x(t)} n_l(t, x) \phi_r(t, x) \, dx \, dt
\end{align*}$$
\[
+ \int_0^\infty w_m(t)\phi_x(t,x)\sqrt{1+x'(t)^2}\,dt + \iint_{x>x(t)} n_r(t,x)\phi_x(t,x)\,dx\,dt
+ \iint_{x<x(t)} p_l(t,x)\phi(t,x)\,dx\,dt + \iint_{x>x(t)} p_r(t,x)\phi(t,x)\,dx\,dt
+ \int_{-\infty}^0 m_l\phi(0,x)\,dx + \int_0^\infty m_r\phi(0,x)\,dx + w_m^0\phi(0,0) = 0.
\]

By Green’s formula and (2.21)-(2.22), one has
\[
\int_0^\infty \left[ (m_r(t,x(t)+)-m_l(t,x(t)-))x'(t) -(n_r(t,x(t)+)-n_l(t,x(t)-)) \right] \phi(t,x(t))\,dt
+ (w_m^0-w_m(0)\sqrt{1+x'(0)^2})\phi(0,0) - \int_0^\infty \frac{d(w_m(t)\sqrt{1+x'(t)^2})}{dt}\phi(t,x(t))\,dt
+ \int_0^\infty [w_n(t)-w_m(t)x'(t)]\sqrt{1+x'(t)^2}\phi_x(t,x(t))\,dt = 0.
\]
Then (2.23) follows from the arbitrariness of \(\phi(0,0)\), \(\phi(t,x(t))\) and \(\phi_x(t,x(t))\).

The equations in (2.23) are called the generalized Rankine-Hugoniot (RH) conditions, since if there is no concentration, namely \(w_m(t) \equiv 0\), we get the classical RH jump condition for discontinuities in piecewise smooth integral weak solutions of balance laws. Note that once concentration appears, besides the information on initial value, there are two RH conditions from a single balance law.

In applications, it may happen that the mass concentrates on a known \(C^2\) curve \(x = x(t)\) which is subjected to some extra boundary conditions (like the slip condition on solid boundaries), so we also present a unilateral RH condition formulated as follows. Consider the equation
\[
m_t + n_x = \varphi + w_p(t)\delta_{\{x=x(t)\}}, \quad t > 0, \ x \leq x(t),
\]
with the initial data
\[
\begin{align*}
m(0,x) &= m_l\{x<0\}\mathcal{L}^1 + w_m^0\delta_{\{t=0,x=0\}}, \\
n(0,x) &= n_l\{x<0\}\mathcal{L}^1 + w_n^0\delta_{\{t=0,x=0\}}, \quad w_p(0) = w_p^0
\end{align*}
\]
and slip boundary condition
\[
n\{\{x = x(t)\} = x'(t) \cdot m\{\{x = x(t)\}, \quad t > 0.
\]
This problem is understood in the sense that the Radon measures \(m = m\{\{x \leq x(t)\}, n = n\{\{x \leq x(t)\}, \text{ and function } w_p(t)\) satisfy
\[
\langle m, \phi_t \rangle + \langle n, \phi_x \rangle + \langle \varphi, \phi \rangle + \langle m(0,x), \phi(0,x) \rangle + \langle w_p(t)\delta_{\{x=x(t)\}}, \phi \rangle = 0 \quad (2.27)
\]
for all \(\phi \in C^1_0(\mathbb{R}^2)\). A piecewise smooth Radon measure solution could be defined similarly. We remark that with the applications to Euler equations in mind, here \(m, n\) are momentum flux, while the strange-looking term \(w_p\delta_{\{x=x(t)\}}\) represents the impact on the boundary by the gas, which does not appear in the studies of Cauchy problems, see also explanations in [26, p. 9].

**Lemma 2.2** (Unilateral generalized Rankine-Hugoniot conditions). Assume that
\[
m = m_l(t,x)\mathcal{L}^2 + w_m(t)\delta_{\{x=x(t)\}}, \quad (2.28)
\]
\[
n = n_l(t,x)\mathcal{L}^2 + w_n(t)\delta_{\{x=x(t)\}}, \quad (2.29)
\]
\[
\varphi = p_l(t,x)\mathcal{L}^2, \quad w_p = w_p(t) \quad (2.30)
\]
is a piecewise smooth Radon measure solution to the initial-boundary value problem (2.42)-(2.26). Then it holds on \( x = x(t) \) that

\[
\begin{aligned}
\langle w_m(0) \rangle \sqrt{1 + x'(t)^2} = w_m(0), \\
\langle w_n(t) \rangle = w_m(t)x'(t), \\
\langle w_p(0) \rangle = w_p(0), \\
\frac{d\langle w_m(t) \rangle \sqrt{1 + x'(t)^2}}{dt} = n_m(x(t) - m_I(t)), \quad \gamma(t) - x(t) + w_p(t)\sqrt{1 + x'(t)^2}.
\end{aligned}
\tag{2.31}
\]

**Proof.** Substituting (2.28)-(2.30) into (2.27), one finds

\[
\begin{aligned}
\int_{x < x(t)} m_I(t, x)\phi_x(t, x) \, dx \, dt + \int_0^\infty w_m(t)\phi_t(t, x)\sqrt{1 + x'(t)^2} \, dt \\
+ \int_{x < x(t)} n_I(t, x)\phi_x(t, x) \, dx \, dt + \int_0^\infty w_n(t)\phi_x(t, x)\sqrt{1 + x'(t)^2} \, dt \\
+ \int_0^\infty w_p(t)\phi(t, x(t))\sqrt{1 + x'(t)^2} \, dt + \int_{-\infty}^0 n_m(0, x) \, dx + w_m(0)\phi(0, 0) = 0.
\end{aligned}
\tag{2.32}
\]

Green’s formula and (2.21) satisfied by piecewise smooth solutions yield that

\[
\begin{aligned}
\int_0^\infty \left( n_I(t, x(t) - m_I(t, x(t)) - x'(t) + w_p(t)\sqrt{1 + x'(t)^2} - \frac{d\langle w_m(t) \rangle \sqrt{1 + x'(t)^2}}{dt} \right) \\
\cdot \phi(t, x(t)) \, dt - \int_0^\infty \left( w_n(t) - w_m(t)\sqrt{1 + x'(t)^2} \right) \phi_x(t, x(t)) \, dt \\
+ (w_m^0 - w_m(0)\sqrt{1 + x'(0)^2})\phi(0, 0) = 0.
\end{aligned}
\tag{2.33}
\]

Hence (2.31) follows from arbitrariness of \( \phi(0, 0) \), \( \phi(t, x(t)) \) and \( \phi_x(t, x(t)) \).

**2.4. Radon measure solutions of Euler equations.** The following definition of Radon measure solutions utilized the special nonlinear structure of the compressible Euler equations, while notice that it works for general polytropic gases, which enables us to consider multi-phase flows, such as gases with different constitutive relations in different regions separated by delta shocks, which is a special case of systems of conservation laws with discontinuous fluxes.

**Definition 2.3.** For fixed \( \gamma \geq 1 \), let \( m^0_j, m^1_j \) (\( j = 0, 1, 2, 3 \)), \( \nu, \tilde{\nu}, \bar{\nu} \) be Radon measures on \( \Omega \) (cf. (2.3)), and \( w_p \in L^1_{loc}(\mathbb{R}^+ \cup \{0\}, \mathcal{L}^1) \). We call \( (\rho, u, \nu, E) \) a Radon measure solution to the initial-boundary value problem (2.2), (1.2), (2.4)-(2.6) in the domain \( \Omega \), provided that:

1) [Linear relaxation] for any \( \phi \in C^0_0(\mathbb{R}^2) \), there hold

\[
\begin{aligned}
\langle m^0_a, \phi_x \rangle + \langle m^0_b, \phi_r \rangle + \int_0^\infty r\rho_0u_0\phi(0, r) \, dr = 0, \\
\langle m^1_a + \nu, \phi_x \rangle + \langle m^1_b, \phi_r \rangle + \langle w_p\nu_1\tilde{\nu}_r, \phi \rangle + \int_0^\infty r(\rho_0u_0^2 + p_0)\phi(0, r) \, dr \\
+ \int_{f(x_\star)}^f(x_\star) r\phi(x_\star, r) \, dr = 0,
\end{aligned}
\tag{2.34}
\]

\[
\begin{aligned}
\langle m^2_a, \phi_x \rangle + \langle m^2_b + \nu, \phi_r \rangle + \langle \tilde{\nu}, \phi \rangle + \langle w_p\nu_2\tilde{\nu}_r, \phi \rangle + \int_0^\infty r\rho_0u_0v_0\phi(0, r) \, dr = 0, \\
\langle m^3_a, \phi_x \rangle + \langle m^3_b, \phi_r \rangle + \int_0^\infty r\rho_0u_0E_0\phi(0, r) \, dr = 0,
\end{aligned}
\tag{2.35}
\]

where \( \mathbf{n} = (n_1, n_2) \) denotes the unit normal on \( \Gamma_f \), pointing into \( \Omega_1 \);
Proof. By the definition of Radon measure solution, we have condition to select physically relevant solution, since our construction leads to a non-negative Radon measure, such that \( \phi \ll \rho \), \( \tilde{\phi} \ll \rho \), \( (m^a_1, m^b_3, m^a_1) \ll \rho \) \( (j = 0, 1, 2, 3) \), and the corresponding Radon-Nikodym derivatives satisfy \( \rho \)-a.e.

\[
\begin{align*}
  u &= \frac{dm^0_a}{d\rho} \quad \text{and} \quad v = \frac{dm^0_b}{d\rho}, \\
  u &= \frac{dm^1_a/d\rho}{dm^0_a/d\rho} = \frac{dm^1_b/d\rho}{dm^0_b/d\rho}, \quad v = \frac{dm^2_a/d\rho}{dm^0_a/d\rho} = \frac{dm^2_b/d\rho}{dm^0_b/d\rho},
\end{align*}
\]

and there is a \( \rho \)-a.e. function \( E \), so that

\[
E = \frac{dm^3_a/d\rho}{dm^0_a/d\rho} = \frac{dm^3_b/d\rho}{dm^0_b/d\rho};
\]

iii) [state function of gas] if \( \rho \ll \tilde{\mathcal{L}}^2 \) with Radon-Nikodym derivative \( \rho(\mathbf{r}, t) \), \( \phi \ll \tilde{\mathcal{L}}^2 \) with Radon-Nikodym derivative \( \tilde{\rho}(\mathbf{r}, t) \), in a neighborhood \( \mathcal{N} \) of \((\mathbf{r}, t) \in \Omega \), then \( \tilde{\mathcal{L}}^2 \)-a.e. there hold

\[
p = \frac{\gamma - 1}{\gamma} \rho \left( E - \frac{u^2 + v^2}{2} \right), \quad p = \bar{\rho} \quad \text{in} \quad \mathcal{N}. \tag{2.41}
\]

It is easy to see that classical solutions and integral weak solutions could be identified as Radon measure solutions. In the present work, we do not need entropy condition to select physically relevant solution, since our construction leads to a unique solution that complies with physical results.

\[
\begin{align*}
\langle m, \phi_t \rangle + \langle n, \phi_x \rangle &= 0. \tag{2.42}
\end{align*}
\]

Substituting (2.18), (2.19) into (2.41), one finds

\[
\begin{align*}
  m_1(t, x) \int \int_{x < x(t)} \phi(t, x) \, dt \, dx + \int_0^\infty w_m(t) \phi(t, x(t)) \sqrt{1 + x'(t)^2} \, dt \\
  + m_r \int \int_{x > x(t)} \phi(t, x) \, dt \, dx + n_1(t, x) \int \int_{x < x(t)} \phi(t, x) \, dx \, dt \\
  + \int_0^\infty w_n(t) \phi(t, x) \sqrt{1 + x'(t)^2} \, dt + n_r \int \int_{x > x(t)} \phi(t, x) \, dx \, dt = 0. \tag{2.43}
\end{align*}
\]

Using Green’s formula, we obtain

\[
\begin{align*}
  m_1(t, x) \int_0^\infty \phi(t, x) x'(t) \, dt \, dx - w_m(0) \sqrt{1 + x'(0)^2} \\
  - \int_0^\infty \frac{d(w_n(t) \sqrt{1 + x'(t)^2})}{dt} \phi(t, x) \, dt - \int_0^\infty w_m(t) \sqrt{1 + x'(t)^2} \phi(t, x) x'(t) \, dt \\
  - n_1(t, x) \int_0^\infty \phi(t, x) \, dt \, dx + \int_0^\infty w_n(t) \phi(t, x) \sqrt{1 + x'(t)^2} \, dt = 0. \tag{2.44}
\end{align*}
\]

By arbitrariness of \( \phi \), this means

\[
\begin{align*}
  w_m(0) &= 0, \quad w_n(t) = w_m(t) x'(t), \\
  \frac{d(w_m(t) \sqrt{1 + x'(t)^2})}{dt} &= m_1(t, x) x'(t) - n_1(t, x).
\end{align*}
\]

\( \square \)
2.5. A nonlinear integral-differential equation. The following nonlinear integral-differential equation appears naturally when we use generalized RH conditions to construct delta shocks. We just list the results to be used later, although we can solve it for more general cases.

Lemma 2.4. Suppose that \( l = l(x) \) solves on \( x \geq x_* \) that

\[
l'(x) \left[ \alpha_0 l^2(x) + \hat{\beta} \right] = \lambda_0 \int_{x_*}^x l(t) \, dt + \hat{\gamma}, \quad l(x_*) = \gamma_1, \tag{2.45}
\]

where \( \alpha_0, \hat{\beta}, \lambda_0 \geq 0, \hat{\gamma} \geq 0, \gamma_1 > 0 \) are constants, and satisfies \( \alpha_0 \gamma_1^2 + \hat{\beta} > 0 \). Set

\[
\gamma_2 = \frac{\hat{\gamma}}{(\alpha_0 \gamma_1^2 + \hat{\beta})}. \tag{2.49}
\]

Then

1) if \( \alpha_0 = 0, \lambda_0 > 0, \hat{\beta} > 0 \), the only global solution of (2.45) is

\[
l(x) = \frac{\gamma_1 + \gamma_2 \sqrt{\frac{2}{\lambda_0}}}{2} \exp \left( \sqrt{\frac{\lambda_0}{\alpha_0}} (x-x_*) \right) \left[ 1 - \frac{\gamma_2 \sqrt{\frac{2}{\lambda_0}}}{\alpha_0} \exp \left( \sqrt{\frac{\lambda_0}{\alpha_0}} (x-x_*) \right) \right]; \tag{2.46}
\]

2) if \( \alpha_0 > 0, \lambda_0 = 0, \) there exists a unique global solution of (2.45) that is increasing, and given by a real root of the cubic equation

\[
\alpha_0 \left( l^2(x) - \frac{\gamma_1^3}{3} \right) + \hat{\beta} l(x) - \gamma_1 = \hat{\gamma} (x-x_*), \quad x \geq x_*; \tag{2.47}
\]

3) if \( \alpha_0 > 0, \lambda_0 > 0, \) there exists a unique global solution of (2.45), which is increasing and satisfies

\[
\int_{\gamma_1}^{l(x)} \sqrt{\frac{2\alpha_0 (\alpha_0 t^2 + \hat{\beta})^2}{\lambda_0}} \, dt = x - x_*, \tag{2.48}
\]

where \( C_* = (\lambda_0 - 2\alpha_0 \gamma_1^2)(\alpha_0 \gamma_1^2 + \hat{\beta})^2 \);

4) if \( \alpha_0 < 0, \lambda_0 > 0, \) then (2.45) admits only a local solution that is given implicitly by (2.48) for \( x \in [x_*, x^2] \), and \( x^2 \) is a finite number so that \( l'(x^2) = +\infty \).

Proof. 1. If \( \alpha_0 = 0, \lambda_0 > 0, \hat{\beta} > 0 \), then (2.45), becomes \( \hat{\beta} l'(x) = \lambda_0 \int_{x_*}^x l(t) \, dt + \hat{\gamma} \), which implies that \( \hat{\beta} l''(x) = \lambda_0 l(x) \). Thus the global solution of (2.45) is unique and given by (2.46).

2. If \( \alpha_0 > 0, \lambda_0 = 0, \) then (2.45) is reduced to

\[
(\alpha_0 l^2(x) + \hat{\beta}) l'(x) = \hat{\gamma}, \quad l(x_*) = \gamma_1. \tag{2.49}
\]

The assumption \( \hat{\gamma} \geq 0 \) and \( \alpha_0 \gamma_1^2 + \hat{\beta} > 0 \) indicate that \( l'(x) \geq 0, l(x) \geq \gamma_1 \) and \( \alpha_0 l(x)^2 + \hat{\beta} \geq \alpha_0 \gamma_1^2 + \hat{\beta} > 0 \) for all \( x \geq x_* \). Hence (2.49) has a global solution. Direct integration yields the solution is given implicitly by a real root of the cubic equation

\[
l^3(x) + \frac{3\hat{\beta}}{\alpha_0} l(x) - \gamma_1^3 - \frac{3\hat{\beta}}{\alpha_0} \gamma_1 - \frac{3\hat{\gamma}}{\alpha_0} (x-x_*) = 0. \tag{2.50}
\]

\[
l_1(x) = \omega \left\{ - C_3 \left[ \left( \frac{C_3}{2} \right)^2 + \left( \frac{\hat{\beta}}{\alpha_0} \right)^3 \right] \right\}^{\frac{1}{4}} + \omega^2 \left\{ - C_3 \left[ \left( \frac{C_3}{2} \right)^2 + \left( \frac{\hat{\beta}}{\alpha_0} \right)^3 \right] \right\}^{\frac{1}{4}}, \tag{2.51}
\]

\[
l_2(x) = \omega \left\{ - C_3 \left[ \left( \frac{C_3}{2} \right)^2 + \left( \frac{\hat{\beta}}{\alpha_0} \right)^3 \right] \right\}^{\frac{1}{4}} + \omega^2 \left\{ - C_3 \left[ \left( \frac{C_3}{2} \right)^2 + \left( \frac{\hat{\beta}}{\alpha_0} \right)^3 \right] \right\}^{\frac{1}{4}}, \tag{2.52}
\]

\[
l_3(x) = \omega^2 \left\{ - C_3 \left[ \left( \frac{C_3}{2} \right)^2 + \left( \frac{\hat{\beta}}{\alpha_0} \right)^3 \right] \right\}^{\frac{1}{4}} + \omega \left\{ - C_3 \left[ \left( \frac{C_3}{2} \right)^2 + \left( \frac{\hat{\beta}}{\alpha_0} \right)^3 \right] \right\}^{\frac{1}{4}}, \tag{2.53}
\]
3. For $\alpha_0 > 0$, $\lambda_0 > 0$, taking derivative in both sides of (2.45) yields
\[ (\alpha_0 l''(x) + \beta)l''(x) = [\lambda_0 - 2\alpha_0(\lambda_1'(x))^2]l(x). \quad (2.52) \]
Since $l(x_*) = \gamma_1 > 0$, $l'(x_*) = \gamma_2 \geq 0$ and $\alpha_0 \gamma_2^2 + \beta > 0$, it is clear that $\alpha_0 l''(x) + \beta > 0$ as we assumed that $l(x)$ to be monotonically increasing. Thus by multiplying $-2\alpha_0 l''(x)/[(\alpha_0 l''(x) + \beta)(\lambda_0 - 2\alpha_0(\lambda_1'(x))^2)]$ to both sides of (2.52) and integration, we have
\[ \lambda_0 - 2\alpha_0(\lambda_1'(x))^2 = C_0(\alpha_0 l''(x) + \beta)^{-2}, \quad (2.53) \]
where $C_0 = (\lambda_0 - 2\alpha_0 \gamma_0^2)(\alpha_0 \gamma_0^2 + \beta)^2$. Consequently,
\[ l'(x) = \sqrt{\frac{\lambda_0(\alpha_0 l''(x) + \beta)^2 - C_0}{2\alpha_0(\alpha_0 l''(x) + \beta)^2}} \geq 0 \quad (2.54) \]
with the right-hand side equals zero only at $x = x_*$ when $\gamma_2 = 0$, and the expression under the radical sign $\sqrt{\cdot}$ is automatically positive otherwise. Integrating both sides of (2.54) with respect to $x$ yields
\[ \int_{\gamma_1}^{l(x)} \sqrt{\frac{2\alpha_0(\alpha_0 l''(x) + \beta)^2 - C_0}{\lambda_0(\alpha_0 l''(x) + \beta)^2}} \, dx = x - x_*, \quad (2.55) \]
which gives implicitly the solution $l = l(x)$. For the exceptional case that $\gamma_2 = 0$, the resultant improper integral still converges at $x = x_*$, as the singularity of the integrand is $O(1/\sqrt{x - x_*})$ near $x = x_*$. 

4. However, for $\alpha_0 < 0$, $\lambda_0 > 0$, from (2.54) we observe that the solvability of (2.55) requires $\alpha_0 l''(x) + \beta \neq 0$, which generally fails now. Indeed, since $\alpha_0 \gamma_2^2 + \beta > 0$ and $l'(x) > 0$, there exists a finite $x^* > x_*$, such that $\alpha_0 l''(x^*) + \beta = 0$. (Notice that the integral in (2.55) with lower limit $\gamma_1$ and upper limit $\sqrt{\beta/\alpha_0}$ is finite.) Hence we infer from (2.54) that $l'(x^*) = \infty$, that is to say, the curve $l(x)$ terminates at $x = x^*$. So (2.55) is solvable only in $[x_*, x^*]$. We conclude that (2.45) has just a local solution in $[x_*, x^*]$ if $\alpha_0 < 0$, $\lambda_0 > 0$. 

In this section, we construct a Radon measure solution of the hypersonic-limit Euler flow passing a finite cylindrically symmetric cone by the unilateral Rankine-Hugoniot conditions, and then solve the interaction between the hypersonic-limit flow and the downstream pressureless jet flow behind the finite cone, by applying the generalized Rankine-Hugoniot conditions.

**Lemma 2.5. (unilateral generalized Rankine-Hugoniot conditions)** Consider a scalar equation
\[ m_t + n_x = 0, \quad t > 0, \quad x \in \mathbb{R}. \quad (2.56) \]
It is assumed that there is a uniform gas on one side of the solid boundary, namely
\[ m = m_1(t, x) \mathbb{1}_{(x < x(t))} \mathcal{L}^2 + w_m(t) \delta_{(x=x(t))}, \quad (2.57) \]
\[ n = n_1(t, x) \mathbb{1}_{(x < x(t))} \mathcal{L}^2 + w_n(t) \delta_{(x=x(t))}, \quad (2.58) \]
then the following unilateral generalized Rankine-Hugoniot conditions
\[ \begin{cases} \frac{d(w_m(t)\sqrt{1 + x'(t)^2})}{dt} = m_1(t, x)x'(t) - n_1(t, x), \\ w_n(t) = w_m(t)x'(t), \end{cases} \quad (2.59) \]
should be satisfied on the discontinuity line $x = x(t)$.
\textbf{Proof.} From the definition of Radon measure solution, we have
\[ \langle m, \phi_t \rangle + \langle n, \phi_x \rangle = 0. \tag{2.60} \]
Substituting (2.57), (2.58) into (2.60), one finds
\[
m_l(t, x) \int_{x < x(t)} \phi(t, x) x' dt dx + \int_0^\infty w_m(t) \phi(t, x) \sqrt{1 + x'(t)^2} dt
\]
\[+ n_l(t, x) \int_{x < x(t)} \phi(t, x) dx + \int_0^\infty w_n(t) \phi(t, x) x' dt = 0. \tag{2.61} \]
Using Green’s formula, we obtain
\[
m_l(t, x) \int_0^\infty \phi(t, x) x'(t) dt dx - w_m(0) \sqrt{1 + x'(0)^2}
\]
\[- \int_0^\infty \frac{d(w_m(t) \sqrt{1 + x'(t)^2})}{dt} \phi(t, x) dt - \int_0^\infty w_m(t) \sqrt{1 + x'(t)^2} \phi(t, x) x' dt
\]
\[- n_l(t, x) \int_0^\infty \phi(t, x) dt dx + \int_0^\infty w_n(t) \phi(t, x) x' dt = 0. \tag{2.62} \]
By arbitrariness of \( \phi \), this means
\[ w_m(0) = 0, \quad w_n(t) = w_m(t) x'(t), \tag{2.63} \]
\[ \frac{d(x(t) w_m(t) \sqrt{1 + x'(t)^2})}{dt} = m_l(t, x) x'(t) - n_l(t, x). \]

3. Infinite-thin shock layers and Newton-Busemann pressure law. Due to hyperbolicity, the flow ahead of \( \{ x = x_* \} \) is totally determined by upstream hypersonic-limit flow at \( \{ x = 0 \} \) and the cone. Newton had calculated the drag of a moving hard ball in pressureless flows in 1687 (\textit{Mathematical Principles of Natural Philosophy}, Proposition 34), for which he used the now called Newton’s sine-squared law, which turned out to be the fundamental formula for hypersonic aerodynamics, and Busemann generalized Newton’s formula to curved conical bodies in 1933, by taking into account of the centrifugal force caused by bending of the cone’s boundary. It was observed that the most prominent feature of hypersonic flow is that the conical shock-front ahead of the cone approaches the cone’s surface as the upstream Mach number increases to infinity, hence an infinite-thin shock layer appears and mass concentrates on the cone’s surface [1, section 15.4].

We thus suppose that a solution of the initial-boundary value problem (2.2), (1.2), (2.4), (2.5) in the domain \( \Omega_1 \) (cf. (2.3)) is given by the measures
\[
m_a^0 = \rho_0 u_0 \hat{L}^2 |\Omega_1 + w_0^3(x) \delta_{\Gamma_f}, \quad m_0^0 = w_0^3(x) \delta_{\Gamma_f}, \tag{3.1}
\]
\[m_a^1 = \hat{L}^2 |\Omega_1 + w_a^3(x) \delta_{\Gamma_f}, \quad m_0^1 = w_a^3(x) \delta_{\Gamma_f}, \quad \varphi = 0, \quad \hat{\varphi} = 0, \tag{3.2}
\]
\[m_a^2 = w_a^2(x) \delta_{\Gamma_f}, \quad m_0^2 = w_0^2(x) \delta_{\Gamma_f}, \tag{3.3}
\]
\[m_a^3 = E_0 \hat{L}^2 |\Omega_1 + w_a^3(x) \delta_{\Gamma_f}, \quad m_0^3 = w_0^3(x) \delta_{\Gamma_f}, \tag{3.4}
\]
where \( w_j^a(x) \) (\( j = 0, 1, 2, 3 \)) \( (0 \leq x \leq x_*) \) are functions to be determined. Notice that constants are solutions to the Euler equations, by (2.14) and Lemma
2.2, we only need to solve the unilateral generalized RH conditions

\[
\begin{align*}
\frac{d(f(x)w_0^0(x))}{dx}\sqrt{1+f'(x)^2} &= \rho_0 u_0 f(x)f'(x) - \rho_0 v_0 f(x), \\
\frac{d(f(x)w_1^1(x))}{dx}\sqrt{1+f'(x)^2} &= (\rho_0 u_0^2 + \rho_0 - w_p(x))f(x)f'(x) - \rho_0 u_0 v_0 f(x), \\
\frac{d(f(x)w_2^2(x))}{dx}\sqrt{1+f'(x)^2} &= \rho_0 u_0 v_0 f(x) - (\rho_0 v_0^2 + \rho_0 + w_p(x))f(x), \\
\frac{d(f(x)w_3^3(x))}{dx}\sqrt{1+f'(x)^2} &= \rho_0 u_0 E_0 f(x)f'(x) - \rho_0 v_0 E_0 f(x),
\end{align*}
\]

(3.5)

together with

\[
f(0)w_0^j(0) = 0, \quad w_0^j(x) = f'(x)w_0^j(x), \quad j = 0, 1, 2, 3. \tag{3.6}
\]

By (3.1), (3.6), and (3.5)_1, we deduce

\[
w_0^0(0)f(0) = 0, \quad w_0^1(x) = f'(x)w_0^0(x), \\
\frac{d(f(x)w_0^0(x))}{dx}\sqrt{1+f'(x)^2} = f'(x)f(x), \quad 0 \leq x \leq x_*,
\]

(3.7)

which yield that

\[
w_0^0(x)\sqrt{1+f'(x)^2} = \frac{f(x)}{2}, \quad w_0^0(x) = f'(x)w_0^0(x). \tag{3.8}
\]

By continuity of \(w_0^0(x)\) and \(f(0) = 0\), we see \(w_0^0(0) = \lim_{x \to 0^+} \frac{f(x)}{\sqrt{1+f'(x)^2}} = 0\).

Similarly, we get

\[
w_0^3(x)\sqrt{1+f'(x)^2} = \frac{E_0 f(x)}{2}, \quad w_0^3(x) = f'(x)w_0^3(x), \quad 0 \leq x \leq x_*. \tag{3.9}
\]

Owing to (3.2), (3.6), and (3.5)_2, we derive

\[
w_1^1(0)f(0) = 0, \quad w_1^1(x) = f'(x)w_1^1(x), \\
\frac{d(f(x)w_1^1(x))}{dx}\sqrt{1+f'(x)^2} = f'(x)f(x)(1-w_p(x)), \quad 0 \leq x \leq x_*. \tag{3.10}
\]

In view of (3.3), (3.6) and (3.5)_3, we have

\[
w_2^2(0)f(0) = 0, \quad w_2^2(x) = f'(x)w_2^2(x), \\
\frac{d(f(x)w_2^2(x))}{dx}\sqrt{1+f'(x)^2} = f(x)w_p(x), \quad 0 \leq x \leq x_*. \tag{3.11}
\]

By requirements (2.38) and (2.39), there holds

\[
w_2^a(x) = f'(x)w_2^a(x). \tag{3.12}
\]

Combining (3.10), (3.11) and (3.12), one has

\[
f(x)w_1^1(x) = \frac{K(x)}{1+f'(x)^2}, \tag{3.13}
\]

\[
f(x)w_p(x) = \frac{f''(x)K(x) + f(x)f'(x)^2\sqrt{1+f'(x)^2}}{(1+f'(x)^2)^{\frac{3}{2}}}, \tag{3.14}
\]

where

\[
K(x) = \int_0^x \frac{f(t)f'(t)}{\sqrt{1+f'(t)^2}} \, dt \geq 0, \quad 0 \leq x \leq x_*. \tag{3.15}
\]

\]
The continuity of \( w_1^1(x) \) implies that

\[
 w_1^1(0) = \lim_{x \to 0^+} w_1^1(x) = \lim_{x \to 0^+} \frac{K(x)}{f(x)(1 + f'(x)^2)} = 0.
\]

In the same way, one has \( w_2^1(0) = 0 \) and

\[
 w_p(0) = \lim_{x \to 0^+} w_p(x) = \lim_{x \to 0^+} \frac{f''(x)K(x) + f(x)f'(x)^2\sqrt{1 + f'(x)^2}}{f(x)(1 + f'(x)^2)^{\frac{3}{2}}} = \frac{f'(0_+)^2}{1 + f'(0_+)^2} > 0.
\]

For \( x \in (0, x^*) \), by (3.14) and the assumption (1.6), we have

\[
 w_p(x) = \frac{f''(x)K(x) + f(x)f'(x)^2\sqrt{1 + f'(x)^2}}{f(x)(1 + f'(x)^2)^{\frac{3}{2}}} > 0. \tag{3.16}
\]

Therefore, the force exerted on \( \Gamma_f^y \) by the gas is given by

\[
 F = -\int_{\Gamma_f^y} w_p \hat{n} \, dH^2 = 2\pi \int_0^{x^*} w_p(x)f(x)(f'(x), 0, 0) \, dx.
\]

(Notice that \( \hat{n} \) here is the normal on \( \Gamma_f^y \), which is a surface in \( \mathbb{R}^3 \).) This proves Newton-Busemann formula.

Finally we write explicitly the Radon measure solution constructed. Returning to (2.39), we have

\[
 u|_{\Gamma_f^y} = \frac{2K(x)}{f^2(x)\sqrt{1 + f'(x)^2}}, \quad v|_{\Gamma_f^y} = \frac{2f'(x)K(x)}{f^2(x)\sqrt{1 + f'(x)^2}},
\]

hence for \( 0 < x \leq x^* \),

\[
 u = \mathbb{I}_{\Omega_1} + \frac{2K(x)}{f^2(x)\sqrt{1 + f'(x)^2}} \delta_{\Gamma_f^y}, \quad v = \frac{2f'(x)K(x)}{f^2(x)\sqrt{1 + f'(x)^2}} \delta_{\Gamma_f^y}, \tag{3.17}
\]

where

\[
 u(0+, 0) = \lim_{x \to 0^+} \frac{2K(x)}{f(x)^2\sqrt{1 + f'(x)^2}} = \frac{1}{1 + f'(0_+)^2},
\]

\[
 v(0+, 0) = \lim_{x \to 0^+} \frac{2f'(x)K(x)}{f(x)^2\sqrt{1 + f'(x)^2}} = \frac{f'(0_+)}{1 + f'(0_+)^2}.
\]

By (2.38), (3.8), and (3.17), we determine that

\[
 \varrho = \mathcal{L}^2[\Omega_1 + \frac{f^3(x)}{4K(x)} \delta_{\Gamma_f^y}, \ 0 < x \leq x^*]. \tag{3.18}
\]

Furthermore, from (2.40), (3.8), and (3.9), we take

\[
 E = E_0\mathbb{I}_{\Omega_1} + E_0\delta_{\Gamma_f^y}. \tag{3.19}
\]

Therefore, (3.17)-(3.19) give the Radon measure solution modeling infinite-thin shock layers for the hypersonic-limit flow passing a cylindrically symmetric cone. Theorem 1.1 is proved.

**Remark 3.** Taking \( f(x) = \sqrt{x} \), direct calculations show that (1.6), namely \( f''(x) > -\frac{f(x)f'(x)^2\sqrt{1 + f'(x)^2}}{K(x)} \), always holds for \( x \in (0, +\infty) \). The solution that a hypersonic-limit flow passing an infinite cylindrically symmetric cone follows immediately by taking \( x^* = +\infty \).
Remark 4. If we wish to design a cylindrically symmetric conical profile which bears uniform force per unit area from the upcoming hypersonic-limit flow, i.e., \( w_p \) is a constant \( p \), then the radius \( f(x) \) shall be a solution of the following integral-differential equation

\[
f''(x)K(x) + f(x)f'(x)^2\sqrt{1+f'(x)^2} = pf(x)(1+f'(x)^2)^{3/2},
\]

where \( K(x) \) is defined by (3.15). If the cone is straight, namely \( f''(x) = 0 \), then \( f'(x) = k_0 \), with \( k_0 \) a constant, and it follows from (3.20) that \( p = \frac{k_0^2}{1+k_0^2} \). This is the classical Newton’s sine-squared law. On the contrary, some computation shows that \( (KK'/f)' = pf \), which implies \( K(x) = \sqrt{p} \int_0^x f(t) \, dt \), and from this we infer that circular cone is the only pointed body (i.e. \( f(0) = 0 \)) with cylindrical symmetry that bears uniform pressure in the hypersonic-limit flow.

Remark 5. Notice that \( w_p \) is the force of lift/drag per unit area exert by the fluid onto the cone. It results from the changes of momentum when particles hitting the cone. The force on the surface of the cone is nonzero indicates that there is a mass concentration layer on the surface of the cone. If \( w_p = 0 \) somewhere, it is possible that there is no concentration, or the mass concentration layer leaves the surface of the cone, thus becomes a free interface called delta shock, and form a vacuum between it and the cone. Once vacuum appears between the gas and the cone, we need to study interactions of the hypersonic-limit flow with vacuum, which is a special case of interactions with pressureless static gas, a case to be discussed in the next section.

4. Interactions of hypersonic-limit flow and static gas. We study the existence of a Radon measure solution of the interactions between the hypersonic-limit flow passing a finite cylindrically symmetric cone and a uniform static gas which is given and lying behind the cut-off plane. Now the domain we consider is \( \Omega = \Omega_c \cup \Omega_t \cup \Gamma_f \cup \Gamma_l \), where

\[
\Gamma_l = \{(x, r) : x_* < x \leq x_o, \ r = l(x)\},
\]

and

\[
\Omega_c = \{(x, r) : 0 < x \leq x_* , \ r > f(x); \ x_* < x \leq x_o, \ r > l(x)\},
\]

\[
\Omega_t = \{(x, r) : x_* < x \leq x_o, \ 0 \leq r < l(x)\}
\]

are respectively the region occupied by the uniform upcoming hypersonic-limit flow above \( \Gamma_l \), and the region of uniform static gas below \( \Gamma_l \) (see Figure 1). The unknown function \( r = l(x) \) defined on \( [x_*, x_o] \) (with \( x_o \) possibly being \(+\infty\)) represents the free concentration layer (delta shock).

Suppose that

\[
m^0_a = \rho_0 u_0 L^2 [w^0_a(x) \delta_{\Gamma_f} + \delta^0_a(x) \delta_{\Gamma_l}],
\]

\[
m^0_o = w^0_o(x) \delta_{\Gamma_l},
\]

\[
m^1_a = \delta^1_a \Omega_c + w^1_a(x) \delta_{\Gamma_f} + \delta^1_a \Gamma_l,
\]

\[
m^1_o = w^1_o(x) \delta_{\Gamma_f} + \delta^1_o \Gamma_l,
\]

\[
m^2_a = w^2_a(x) \delta_{\Gamma_f} + \delta^2_a \Gamma_l,
\]

\[
m^2_o = w^2_o(x) \delta_{\Gamma_f} + \delta^2_o \Gamma_l,
\]

\[
m^3_a = \delta_{\Gamma_c} L^3 [w^3_a(x) \delta_{\Gamma_f} + \delta^3_a \Gamma_l],
\]

\[
m^3_o = w^3_o(x) \delta_{\Gamma_f} + \delta^3_o \Gamma_l.
\]
Here $\bar{w}_0^j(x), \bar{w}_b^j(x)$ $(j = 0, 1, 2, 3)$ are functions to be determined. We do not need to calculate all the weights by Definition 2.3, since the problem is hyperbolic in the $x$-direction. We just need to consider a singular Riemann problem of the Euler equations with the initial data given by Theorem 1.1 as follows:

$$\begin{align*}
\rho(x_*, r) &= L^1\{ r > f(x_*) \} + \frac{f'(x_*)}{4K(x_*)} \delta_{(r = f(x_*) \land 0 \leq r < f(x_*))}, \\
u(x_*, r) &= I_{(r > f(x_*))} + \frac{f'(x_*)}{2K(x_*)} L^1\{ r = f(x_*) \}, \\
v(x_*, r) &= \frac{2f'(x_*)K(x_*)}{f^2(x_*) \sqrt{1 + f'(x_*)^2}} I_{(r = f(x_*) \land 0 \leq r < f(x_*))}, \\
E(x_*, r) &= E_0 I_{(r > f(x_*))} + E_1 I_{(r = f(x_*))} + E_2 I_{(0 \leq r < f(x_*))},
\end{align*}$$

hence Lemma 2.1 leads us to solve the generalized RH conditions

$$\begin{align*}
dx{d(l(x)\bar{w}_0^0(x))\sqrt{1 + l'(x)^2}} &= [l(x)\rho u][l'(x) - [l(x)\rho v]], \\
dx{d(l(x)\bar{w}_0^j(x))\sqrt{1 + l'(x)^2}} &= [l(x)(\rho u^2 + p)][l'(x) - [l(x)\rho v]], \\
dx{d(l(x)\bar{w}_0^3(x))\sqrt{1 + l'(x)^2}} &= [l(x)\rho u E][l'(x) - [l(x)\rho v E]], \\
dx{d(l(x)\bar{w}_0^3(x))\sqrt{1 + l'(x)^2}} &= [l(x)\rho v E][l'(x) - [l(x)\rho u E]] - [l(x)\rho u][l'(x) - [l(x)\rho v]], \\
\bar{w}_0^j(x) &= l'(x)\bar{w}_0^j(x), \quad j = 0, 1, 2, 3, \\
l(x_*)\bar{w}_0^j(x_*)\sqrt{1 + l'(x_*)^2} &= f(x_*) \bar{w}_0^j(x_*) \sqrt{1 + f'(x_*)^2}
\end{align*}$$

for $x \in (x_*, x_0]$.

**Lemma 4.1.** The singular Riemann problem (2.2), (4.5) admits a Radon measure solution given by

$$\begin{align*}
u &= l\Omega_0 + \frac{2K(x)}{f^2(x) \sqrt{1 + f'(x)^2}} l\Gamma_j + u|\Gamma_l| l\Gamma_j, \\
v &= \frac{2f'(x)K(x)}{f^2(x) \sqrt{1 + f'(x)^2}} l\Gamma_j + v|\Gamma_l| l\Gamma_j, \\
\rho &= L^2|\Omega_0 + \frac{\rho L^2}{4K(x)} + \frac{f'(x)}{4K(x)} \delta_{\Gamma_j} + \rho|\Gamma_l, \\
E &= E_0 l\Omega_0 + E_1 l\Omega_1 + E_2 l\Gamma_j + E_3 l\Gamma_l,
\end{align*}$$

where

$$\begin{align*}
u|\Gamma_j &= \frac{2 l^3(x) \int_{l(x)}^{x} l(t) dt}{L^2(x)}, \\
u|\Gamma_l &= \frac{2 l^3(x)}{L^2(x)} \left[ (1 - \rho) (l^2(x) - f^2(x_*)) + \frac{2K(x)}{\sqrt{1 + f'(x)^2}} \right], \\
\rho|\Gamma_l &= \frac{2 \sqrt{1 + l'(x)^2} \left[ (1 - \rho) (l^2(x) - f^2(x_*)) + \frac{2K(x)}{\sqrt{1 + f'(x)^2}} \right]}{2L^2(x)},
\end{align*}$$

provided that $r = l(x)$ exists for $x > x_*$, which is to be determined by Propositions 1-4 below.
Proof. We are to find a solution of the ODE (4.6). According to (4.6) and (3.8),
\[ l(x)\bar{w}_a^0(x_0)\sqrt{1 + l'(x_0)^2} = f(x_0)\bar{w}_a^0(x_0)\sqrt{1 + f'(x_0)^2} = \frac{f^2(x_0)}{2}. \] (4.13)
Since \( v_0 = v = 0 \), it is seen from (4.6) and (4.5) that
\[ \bar{w}_a^0(x)\sqrt{1 + l'(x)^2} = \frac{l(x)}{2}. \] (4.14)
Similarly, we have
\[ \bar{w}_a^1(x)\sqrt{1 + l'(x)^2} = \frac{E_0l(x)}{2}. \] (4.15)
Owing to (4.6), (3.13), (4.6) and (4.5), there hold
\[ l(x)\bar{w}_a^1(x)\sqrt{1 + l'(x)^2} = f(x_0)\bar{w}_a^1(x_0)\sqrt{1 + f'(x_0)^2} = \frac{K(x_0)}{\sqrt{1 + f'(x_0)^2}}, \] (4.16)
\[ \frac{d}{dx}(l(x)\bar{w}_a^1(x))\sqrt{1 + l'(x)^2} = l(x)l'(x)(1 - p). \] (4.17)
We thus get
\[ l(x)\bar{w}_a^1(x)\sqrt{1 + l'(x)^2} = \frac{(1 - p)[l^2(x) - f^2(x_0)]}{2} + \frac{K(x_0)}{\sqrt{1 + f'(x_0)^2}}. \] (4.18)
From (4.6), (3.12), (4.6), and (4.5), one has
\[ l(x_0)\bar{w}_a^2(x_0)\sqrt{1 + l'(x_0)^2} = \frac{f'(x_0)K(x_0)}{\sqrt{1 + f'(x_0)^2}}, \] \[ l(x)\bar{w}_a^2(x)\sqrt{1 + l'(x)^2} = \frac{p}{2} \int_{x_0}^{x} l(t) dt + \frac{f'(x_0)K(x_0)}{\sqrt{1 + f'(x_0)^2}}. \] (4.19)
Notice (2.39), we obtain by (4.14), (4.18), and (4.19) the desired (4.11). By (2.40), (4.14) and (4.15), it follows that
\[ E|_{x_1} = E_0. \] (4.20)
Moreover, thanks to (2.38), (4.14), and (4.11), we have (4.12).

Combining with (3.17)-(3.19), we derive the Radon measure solution (4.7)-(4.10) of the singular Riemann problem (2.2), (4.5). The last term of (4.10) means that the static gas does not flow into delta wave. Lemma 4.1 is proved.

We now solve the delta shock, i.e., \( r = l(x) \). From (4.7)-(4.10), if we can solve \( l(x) \), a Radon measure solution of the interaction between the hypersonic-limit flow and a uniform static gas is determined. By (4.6), we recover the slip condition \( v = l'(x)u \), which leads to a nonlinear integral-differential equation for \( l(x) \):
\[ \begin{align*}
\left\{ \begin{array}{l}
l'(x)\left[(1 - p)(l^2(x) - f^2(x_0)) + \frac{2K(x_0)}{\sqrt{1 + f'(x_0)^2}}\right] = 2p \int_{x_0}^{x} l(t) dt + \frac{f'(x_0)K(x_0)}{\sqrt{1 + f'(x_0)^2}}, \\
l(x_0) = f(x_0),
\end{array} \right. \tag{4.21}
\end{align*} \]
Comparing to notations used in Lemma 2.4, we see
\[ \alpha_0 = 1 - \frac{p}{2}, \quad \beta = 2\alpha_1 - (1 - p)f^2(x_0), \quad \lambda_0 = 2\beta, \quad \gamma = 2\alpha_1f'(x_0) \geq 0, \] (4.22)
where \( \alpha_1 = \frac{K(x_0)}{\sqrt{1 + f'(x_0)^2}} > 0 \). Hence \( (1 - p)f^2(x_0) + \beta = 2\alpha_1 > 0 \).

Proposition 1. If \( p = 1 \), then the nonlinear integral-differential equations (4.21) has a global solution given by exponential functions.
Proof. If \( p = 1 \), then (4.22) implies \( \alpha_0 = 0 \), \( \lambda_0 = 2 > 0 \), \( \beta = 2\alpha_1 > 0 \). By Lemma 2.4 1), we deduce that (4.21) has a global solution
\[
l(x) = \frac{f(x_0) + f'(x_0)\sqrt{\alpha_1}}{2} \exp \left( \frac{x - x_0}{\sqrt{\alpha_1}} \right) + \frac{f(x_0) - f'(x_0)\sqrt{\alpha_1}}{2} \exp \left( \frac{x - x_0}{\sqrt{\alpha_1}} \right).
\]
(4.23)

Proposition 2. If \( p = 0 \), then (4.21) has a solution given by a real root of a cubic algebraic equation.

Proof. If \( p = 0 \), then (4.22) indicates \( \alpha_0 = 1 > 0 \), \( \lambda_0 = 0 \). According to Lemma 2.4 2), we deduce that (4.21) has a unique global solution satisfying
\[
\frac{\ell^3(x) - f^3(x_*)}{3} + \tilde{\beta}(l(x) - f(x_*)) = 2\alpha_1 f'(x_*)(x - x_*).
\]
(4.24)

We note that if \( f'(x_*) > 0 \), then \( l(x) \) is monotonically increasing to \( \infty \) as \( x \to \infty \), and the delta shock touches the point \( x = x_* \) in a continuously differentiable way, namely \( l''(x) = f'(x_*) \) holds automatically. If instead \( f'(x_*) = 0 \), then \( l(x) \equiv f(x_*) \).

Remark 6. The shape of the free interface separating hypersonic-limit flow and static pressureless gas is the same as the case for vacuum.

Proposition 3. If \( 0 < p < 1 \), then (4.21) has a unique solution \( l(x) \) satisfying
\[
\int_{f(x_*)}^{l(x)} \sqrt{\frac{[\alpha_1 + (1 - p)(t^2 - f^2(x_*))]^2}{p(1 - p)(t^2 - f^2(x_*))^2 + 4\alpha_1 (t^2 - f^2(x_*)) + 4\alpha_1^2 f'(x_*)^2}} \, dt = x - x_*
\]
(4.25)
for all \( x \geq x_* \).

Proof. If \( 0 < p < 1 \), then (4.22) implies \( \alpha_0 = 1 - p > 0 \), \( \lambda_0 = 2p > 0 \). Thanks to Lemma 2.4 3), we infer that (4.21) admits a unique global solution solving (4.25).

Proposition 4. If \( p > 1 \), then (4.21) has a local solution implicitly given by (4.25).

Proof. If \( p > 1 \), then \( \alpha_0 = 1 - p < 0 \), \( \lambda_0 > 0 \). By Lemma 2.4 4), there exists only a local solution of (4.21) satisfying (4.25). Let \( x^\dagger \) be the number so that
\[
\alpha_0^2 (x^\dagger) + \beta = (1 - p)(t^2 - f^2(x_*)) + 2\alpha_1 = 0.
\]
(4.26)
This means \( l'(x^\dagger) = \infty \), \( u(x^\dagger) = 0 \), thus the delta shock can not be prolonged downstream further, as it becomes “space-like” at \( x = x^\dagger \). Therefore, piecewise constant Radon measure solution satisfying (4.25) only exists in \([x_*, x^\dagger] \) as \( p > 1 \).

Theorem 1.2 is proved by Lemma 4.1 and Propositions 1-4.

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