POSITIVE SOLUTION TO EXTREMAL PUCCI’S EQUATIONS WITH SINGULAR AND GRADIENT NONLINEARITY

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ABSTRACT. In this paper, we establish the existence of a positive solution to
\[
\begin{align*}
- \mathcal{M}_{\Lambda, \lambda}^{+}(D^2u) + H(x, Du) &= \frac{k(x)f(u)}{u^\alpha} \quad \text{in } \Omega, \\
u > 0 &\quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
under certain conditions on \(k, f, H\), using viscosity sub-and supersolution method. The main feature of this problem is that it has singularity as well as a superlinear growth in the gradient term. We use Hopf-Cole transformation to handle the superlinear gradient term and an approximation method combined with suitable stability result for viscosity solution to outfit the singular nonlinearity. This work extends and complements the recent works on elliptic equations involving singular as well as superlinear gradient nonlinearities.

1. Introduction. The goal of this paper is to establish the existence of a positive solution to
\[
\begin{align*}
- \mathcal{M}_{\Lambda, \lambda}^{+}(D^2u) + H(x, Du) &= \frac{k(x)f(u)}{u^\alpha} \quad \text{in } \Omega, \\
u > 0 &\quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where \(\Omega \subset \mathbb{R}^n\) is a smooth bounded domain, \(0 < \alpha < 1\), and \(\mathcal{M}_{\Lambda, \lambda}^{+}\) is Pucci’s extremal operator and \(H\) has a superlinear growth in the gradient. We specify the conditions on \(k, f, H\) later. For given \(\lambda, \Lambda\) satisfying \(0 < \lambda \leq \Lambda < \infty\), Pucci’s extremal operators are defined as follows:
\[
\mathcal{M}_{\Lambda, \lambda}^{\pm}(M) = \Lambda \sum_{\pm e_i > 0} e_i + \lambda \sum_{\pm e_i < 0} e_i,
\]
where \(e_i\)’s are the eigenvalues of \(M\) and \(M \in S(n)\), where \(S(n)\) is the set of all \(n \times n\) real symmetric matrices. In case when \(\lambda = \Lambda = 1\), it is easy to see that
\[
\mathcal{M}_{\Lambda, \lambda}^{+}(D^2u) = \Delta u.
\]

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Thus the quasilinear version of (1) is the following problem
\[
\begin{aligned}
- \Delta u + H(x, Du) &= \frac{k(x)f(u)}{u^{\alpha}} \quad \text{in } \Omega, \\
u > 0 &\quad \text{in } \Omega, \\
u = 0 &\quad \text{on } \partial \Omega.
\end{aligned}
\]

(3)

In case when either \( H = 0 \), or \( H \) has a linear growth in the gradient, (3) originates from many branches of science and engineering and has been attracting continuous attention since last three decades. These kind of singular elliptic problems arise in the study of singular minimal surface \([10]\), non Newtonian fluids \([12]\) and boundary layer phenomenon for viscus fluids \([11]\), steady state of thin films \([5, 6]\), modeling of MEMS devices \([39]\). The main progress in the study of singular elliptic equations was made by M. G. Crandall et al. in \([19]\), see also \([15, 16, 25, 35, 36]\). In case, when \( f \equiv 1 \) and \( H \equiv 0 \), (3) was studied in \([29, 37]\). Then, subsequently, its generalizations has been studied by many authors. For the complete list of references in this direction, we refer to survey \([31]\). In the context of fully nonlinear elliptic equations such problems have been considered in \([23]\), where authors studied the existence and regularity properties of the solutions.

Recently, the existence of positive solution to Pucci’s extremal equation involving singular and sublinear nonlinearity has been established in \([46]\). The work \([46]\) generalizes the existence results for semilinear elliptic equations with singular and sublinear nonlinearity to the equations involving Pucci’s extremal operator.

In case \( \alpha = 0 \), (1) has been of independent interest. Such type of problems first of all appear in the thesis \([24]\) and later have been studied by many authors, see \([32, 33, 34, 44]\). We refer to \([47]\) for the Lyapunov type inequality of the nontrivial solution of (1) in the special case of \( H, f \) and with \( \alpha = 0 \). We mention that the maximum and comparison principles for fully nonlinear elliptic equations with superlinear gradient terms have also been discussed in \([1, 13, 26]\). For more references in this direction, we refer to the survey \([45]\). We would like to mention that in \([44]\), Sirakov studied more general problems than (1) but with \( \alpha = 0 \).

Let us recall some of the previous works in the context of (3) with superlinear term \( H(x, Du) \) as well as \( \alpha > 0 \). There is a lot of references dealing with existence of solution of (3). In \([43]\), Cui studied the existence of positive solution to equation containing quadratic gradient term as well as singular nonlinearity. Zhang \([49]\) considered the singular elliptic equation of the form (3) and established some nonexistence results concerning the classical solution. Further, Zhang \([48]\) considered the problem (3) with \( H(x, Du) = c|Du|^q + \sigma \), and \( k(x)f(u) \equiv 1 \), and defined the range for \( c \) and \( \sigma \), to guarantee the existence of classical solution as well as weak solution. Furthermore, they have also shown that smoothness of the solution depends upon \( \alpha \). While in \([27]\), Ghergu and Rădulescu considered the same problem with \( q = 2 \) and studied the bifurcation problem to establish the precise range for the existence of classical solution. Ghergu et al. \([28]\) considered (3) with \( f \equiv 1 \) and an additional sublinear term containing a parameter. They proved that if \( k > 0 \), then there is always a classical solution. On the other hand, they also have given the condition of existence as well as nonexistence in case \( k < 0 \). For precise conditions, we refer to Theorems 1.2 and 1.3 there. Now, we mention the work \([30]\), where Giarrusso and Porru considered problem (3) with \( H \) also depending on \( u \) variable, more precisely, increasing in \( u \) and studied the existence as well as asymptotic behaviour of the classical solution. In \([40]\), Porru and Vitolo considered the same problem as in
[30], but here gradient term is quadratic only and studied the existence of positive solution and its asymptotic behaviour. Finally, we would like to mention the work [20], where the authors considered the problem (3) with \( k \) as some function of the distance function as well as equation contains sublinear term. Here the authors describe the conditions which ensure the existence as well as nonexistence of classical solution. Recently, Faraci et al. [22] considered the problem of type (3), and proved the existence of classical as well as weak solution through fixed point argument. Let us also mention the works, where superlinear gradient as well as singularity occur in a single term as a multiplier of each other, see [2, 3, 7, 8] and references therein. Finally, we mention the works [38], where Benrouma dealt with system of equations of the type considered in the previous works. Despite being available numerous literature on the existence questions to semilinear/quasilinear version of (1), in the best our knowledge, we are not aware of the existence results to fully nonlinear elliptic equations with superlinear gradient and singular term.

The aim of this paper is to establish the existence of a solution to fully nonlinear elliptic equations with superlinear gradient and singular term.

More precisely, we establish the existence of positive solution to (1). To establish the existence of a solution to (1) is challenging because of the following facts:

(i) Due to extremal Pucci operator involved in equation, there is no variational structure of the problem and therefore one cannot use the standard techniques which are applicable for semilinear/quasilinear elliptic equations.

(ii) The problem includes the superlinear growth in gradient term which poses an extra difficulty because these problems, in general, do not satisfy the maximum principle, see Section 3 [34]. Here, superlinear growth in gradient term is handled by a suitable transformation.

(iii) The problem under consideration includes a singular term which adds an additional challenge. We use an approximation method combined with stability result for viscosity solution to handle the singular nonlinearity.

At this point, we remark that since our operator is of the nondivergence form and measurable in \( x \) variable so the appropriate notion of solution is the so-called \( L^n \)-viscosity solution, see Definition 2.1 for the details.

This works extend and complement the earlier research works on singular elliptic equations due to the following:

(a) In case \( q = 1 \), where \( q \) is given next in (H2), and \( f(t) \equiv 1 \), the existence and regularity properties of positive solution to even more general problem than (1) has been studied in [23]. Thus, our results extend the work of [23] because here, we deal with the problem which has superlinear growth in the gradient as well as we allow an additional nonlinear term \( f \).

(b) In case \( \alpha = 0 \) and either \( q = 1 \) or \( H \equiv 0 \), our operator reduces to proper operator. The existence of solutions to these type of problems been studied in [18, 17]. In addition to this, when \( \alpha = 0 \), the existence of solution to (1) has been established in [44]. So our results extend the results there in the sense that we are considering singular nonlinearity as well as superlinear growth in gradient simultaneously.

(c) In case of \( M_{\lambda,\Lambda}(D^2u) = \Delta u \), the existence of positive classical solution to (1) has been established in [22, 28, 30, 48]. In this continuation, we also mention the work [14], where authors establish the existence of positive solution to (1) with \( M_{\lambda,\Lambda}(D^2u) = \Delta u, H \equiv 0 \) and \( f(0) < 0 \). So our works generalize these works to fully nonlinear elliptic equations in the framework of viscosity solution.
Below, we list the conditions on \( f, k \) and \( H \) and state the main theorems of this paper. In (1), we assume that \( H : \Omega \times \mathbb{R}^n \rightarrow [0, \infty) \) is measurable in \( x \), continuous in \( p \). Moreover, we also assume the following hypotheses on \( H \):

(H1) \( H(x, 0) = 0, \forall x \in \Omega \).

(H2) \( |H(x, p) - H(x, p')| \leq b(x)(|p|^{q-1} + |p'|^{q-1})|p - p'| \) for \( 1 < q \leq 2 \),

\((x, p), (x, p') \in \Omega \times \mathbb{R}^n \) and \( 0 \leq b \in L^\infty(\Omega) \).

Let us state the main theorems of this paper which we prove in ensuing sections.

**Theorem 1.1.** Let \( f \) be a nonnegative, nonincreasing, locally Lipschitz continuous function with \( f(0) > 0 \), and \( k \) be a nonnegative bounded continuous function on \( \Omega \) with \( \inf_{\Omega} k > 0 \). Suppose also that (H1)-(H2) hold, then there exists a positive solution to (1) in case \( f \) is motivated by the study of similar problems but in the context of Laplace equation, see [14].

The article is organized as follows. In Section 2, we recall important auxiliary results which are used in this article. Section 3 deals with the existence of positive solution to (1) with \( f(0) > 0 \) and a remark on the case \( f(0) = 0 \). In Section 4, we establish the existence of a positive solution to (1) with \( f(0) < 0 \). An appendix is given in the last section.

2. **Auxiliary results.** In this section, we recall a few auxiliary results which are used throughout this article. Let us start with simple inequalities which have been used repeatedly. The first inequality says that if \( 1 \leq q \leq 2 \), then for any \( p \in \mathbb{R}^n \), we have

\[ |p|^q \leq (2 - q)|p| + (q - 1)|p|^2. \tag{4} \]

The second inequality says that for any two nonnegative real numbers \( x, y \geq 0 \), the following holds:

\[ |x^q - y^q| \leq \begin{cases} |x - y|^q, & \text{if } 0 < q < 1, \\ q(x^{q-1} + y^{q-1})|x - y|, & \text{if } 1 \leq q < \infty, \end{cases} \tag{5} \]

for the details, see Chapter 3, Exercise 24 [42].

By Definition of Pucci’s extremal operator (2), one can easily see that the following relations hold:

\[ \mathcal{M}_{\lambda, \Lambda}^+(\theta M) = \theta \mathcal{M}_{\lambda, \Lambda}^+(M) \text{ for } \theta \geq 0. \tag{6} \]

\[ \mathcal{M}_{\lambda, \Lambda}^-(M) = -\mathcal{M}_{\lambda, \Lambda}^-(M). \tag{7} \]

\[ \mathcal{M}_{\lambda, \Lambda}^+(M) + \mathcal{M}_{\lambda, \Lambda}(N) \leq \mathcal{M}_{\lambda, \Lambda}(M + N) \leq \mathcal{M}_{\lambda, \Lambda}^+(M) + \mathcal{M}_{\lambda, \Lambda}^-(N), \tag{8} \]

\( \forall M, N \in S(n) \), where \( S(n) \) is the set of all \( n \times n \) real symmetric matrices. Let us recall the definition of \( L^n \)-viscosity solution.
Definition 2.1 ([17]). A function \( u \in C(\bar{\Omega}) \) is called \( L^n \)-viscosity subsolution (resp., supersolution) of (1) in \( \Omega \) if for all \( \phi \in W^{2,n}_{loc}(\Omega) \) and \( x \in \Omega \) at which \( u - \phi \) has local maximum (resp., minimum), we have

\[
\begin{align*}
\text{ess lim inf}_{y \to x} (-M^+_{\lambda,A}(D^2\phi) + H(y, D\phi) - \frac{k(y)f(u)}{u^\alpha} & \leq 0), \\
\text{ess lim sup}_{y \to x} (-M^+_{\lambda,A}(D^2\phi) + H(y, D\phi) - \frac{k(y)f(u)}{u^\alpha} & \geq 0).
\end{align*}
\]

Further, \( u \in C(\bar{\Omega}) \) is called \( L^n \)-viscosity solution of (1) if it is a subsolution and supersolution of (1). Also, when a function satisfies an equation or inequality in the \( L^n \)-viscosity sense, we simply say that equation or inequality holds in viscosity sense. We also recall another notion of the solution so-called \( L^n \)-strong solution. A function \( u \in W^{2,n}_{loc}(\Omega) \cap C(\bar{\Omega}) \) is called \( L^n \)-strong solution of (1) if it satisfies (1) almost everywhere in \( \Omega \). Similarly, \( L^n \)-strong sub and supersolution are also defined.

Next, we state the existence of eigenvalue and eigenfunction to the following eigenvalue problem

\[
\begin{align*}
M^+_{\lambda,A}(D^2\phi) + \gamma |D\phi| - \mu^+ \phi & = 0 & \text{in } \Omega, \\
\phi & = 0 \text{ on } \partial \Omega,
\end{align*}
\]

see [41] or [4, 9] for the details.

Theorem 2.2 (Theorem 1.1 [41]). Let us consider the eigenvalue problem (10) for some \( \gamma \geq 0 \). There exists \( \phi \in W^{2,p}_{loc}(\Omega) \cap C(\bar{\Omega}) \) for all \( 1 < p < \infty \) and \( \mu_1^+ > 0 \) such that \( (\mu_1^+, \phi) \) is a solution of (10). Furthermore, any other positive solution to (10) is of the form \( (\mu_1^+, k\phi) \) for some \( k > 0 \).

Next, we present Höpf type lemma in the context of viscosity solution.

Theorem 2.3 (Theorem 4 [23]). Suppose \( u \in C(\bar{\Omega}) \) is a nonnegative viscosity solution of

\[
\begin{align*}
M^+_{\lambda,A}(D^2\phi) - \gamma |D\phi| - \delta u & \leq 0 \text{ in } \Omega,
\end{align*}
\]

where \( \gamma, \delta \geq 0 \). Then either \( u \equiv 0 \) or \( u > 0 \) in \( \Omega \) and at any point \( y \in \partial \Omega \) at which \( u(y) = 0 \), we have \( \liminf_{t \to 0^+} \frac{u(y + \tilde{N}t) - u(y)}{t} > 0 \), where \( \tilde{N} \) is the interior normal to \( \partial \Omega \) at \( y \).

As a consequence of Theorem 2.3, we have the following remark.

Remark 1. There exists a neighbourhood \( G \) of \( \partial \Omega \) such that eigenfunction \( \phi \) from Theorem 2.2 satisfies \( |D\phi| \geq C > 0 \) for some constant \( C \).

We borrow the following lemma from [44], which is used to handle the superlinear gradient term.

Lemma 2.4 (Lemma 2.3 [44]). Let \( u \in W^{2,n}_{loc}(\Omega) \) and \( \mu > 0 \). Set

\[
v = \frac{e^{\mu u} - 1}{\mu}, \quad w = \frac{1 - e^{-\mu u}}{\mu}.
\]
Then almost everywhere in $\Omega$, we have
\[
Dv = (1 + \mu v)Du, \quad Dw = (1 - \mu v)Du.
\]  
(12)

\[
\mu|Du|^2 + M_{\lambda,A}^+(D^2u) \leq \frac{M_{\lambda,A}^+(D^2v)}{1 + \mu v} \leq \mu|Du|^2 + M_{\lambda,A}^+(D^2u)
\]  
(13)

\[
-\mu|Du|^2 + M_{\lambda,A}^-(D^2u) \leq \frac{M_{\lambda,A}^-(D^2w)}{1 - \mu w} \leq -\mu|Du|^2 + M_{\lambda,A}^-(D^2u),
\]  
(14)

and $u = 0$ (resp., $u > 0$) is equivalent to $v = 0$ (resp., $v > 0$) and to $w = 0$ (resp., $w > 0$). The same inequalities hold in the viscosity sense.

3. **Proof of Theorem 1.1.** Problem (1) under consideration is singular in nature. So we use an approximation method to establish the existence of positive solution to (1). More precisely, we approximate (1) with a sequence of regular problems $(R_3)$. For each fixed $\delta > 0$, we first establish the existence of positive solution of $(R_3)$ by using monotone iteration method. Finally, we use interior Hölder estimate (Theorem 2 [44]) and the stability result (Theorem 4 [44]) for the viscosity solution to establish the existence of positive solution of (1). Let us consider the following regularized problem $(R_3)$ corresponding to (1) as follows:

\[
(R_3) \quad \begin{cases} 
-M_{\lambda,A}^+(D^2u) + H(x, Du) = \frac{ kf(u) }{ (u + \delta)^{\alpha} } & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

3.1. **Comparison principle.**

**Proposition 1.** Suppose that assumptions of Theorem 1.1 hold and $u$, $v$ be respectively, viscosity sub-and supersolution to $(R_3)$ with $u \leq v$ on $\partial \Omega$. If either $u$ or $v$ $\in W^{2,\alpha}_{\text{loc}}(\Omega)$, then $u \leq v$ in $\Omega$.

**Proof.** We prove this proposition by the method of contradiction. For that, let us set

\[ A = \{ x \in \Omega \mid u(x) > v(x) \}. \]

If $A = \emptyset$, then we are done, otherwise, for each $0 \leq \delta < 1$, we have

\[ u + \delta > v + \delta \text{ on } A \text{ and } u = v \text{ on } \partial A. \]

Moreover, by noting that $k \geq 0$ and $f$ is nonincreasing, we also have

\[
\frac{ kf(u) }{ (u + \delta)^{\alpha} } \leq \frac{ kf(v) }{ (v + \delta)^{\alpha} } \text{ in } A.
\]  
(15)

Thus, in view of (15) and the fact that $u$ and $v$ are respectively, sub and supersolution of $(R_3)$, we find that

\[
-M_{\lambda,A}^+(D^2u) + H(x, Du) \leq \frac{ kf(u) }{ (u + \delta)^{\alpha} } \leq \frac{ kf(v) }{ (v + \delta)^{\alpha} } \leq -M_{\lambda,A}^+(D^2v) + H(x, Dv),
\]  
(16)

holds in $A$, in the viscosity sense. Now, since we are assuming that one of $u$ or $v$ is in $W^{2,\alpha}_{\text{loc}}(\Omega)$, so we can compute as if both $u$ and $v$ were strong solution, see [44]. Without loss of generality, we assume that $v \in W^{2,\alpha}_{\text{loc}}(\Omega)$. So, the following holds in $A$:

\[
M_{\lambda,A}^+(D^2u) - M_{\lambda,A}^+(D^2v) + H(x, Dv) - H(x, Du) \geq 0.
\]
Using (7), (8) and (H2), we get
\[ M_{\lambda,\Lambda}^+(D^2(u - v)) + b(x)(|Du|^q - 1 + |Du|^q)|D(u - v)| \geq 0 \]
or
\[ M_{\lambda,\Lambda}^+(D^2(u - v)) + b(x)|D(u - v)|(|Du|^q - 1 + 2|Du|^q - 1) \geq 0. \] (17)

Let us set \( w = u - v \), then in view of the following inequality
\[ |Du|^q - 1 - |Du|^q \leq \max\{ |Du|^q - 1 - |Du|^q \} \leq \sup\{ |Du|^q - 1 - |Du|^q \} \leq |D(u - v)|^q - 1, \] (18)
which is a consequence of (5) and \( 0 < q - 1 \leq 1 \), we find that \( w \) satisfies the following inequality
\[ M_{\lambda,\Lambda}^+(D^2w) + b(x)|Dw|^q + 2b(x)|Dv|q - 1|Dw| \geq 0 \text{ in } A. \] (19)

Using (4), we obtain the following:
\[ M_{\lambda,\Lambda}^+(D^2w) + \|b\|_{L^\infty(\Omega)}(q - 1)|Dw|^2 + b(x)((2q - q) + 2|Dv|^q - 1)|Dw| \geq 0, \text{ in } A. \] (20)

By setting \( l = \frac{\|b\|_{L^\infty(\Omega)}(q - 1)}{A} \) and \( \tilde{b} = b(x)((2q - q) + 2|Dv|^q - 1) \), (20) can be rewritten as follows:
\[ M_{\lambda,\Lambda}^+(D^2w) + l|Dw|^2 + \tilde{b}(x)|Dw| \geq 0 \text{ in } A. \] (21)

Now, let us define a function \( \tilde{w} = \frac{e^{\mu w} - 1}{\mu} \), which in view of Lemma 2.4, satisfies the following inequality
\[ M_{\lambda,\Lambda}^+(D^2\tilde{w}) + \tilde{b}(x)|D\tilde{w}| \geq 0 \text{ in } A. \] (22)

Let us take an arbitrary but fix \( \epsilon > 0 \), and define \( A_\epsilon = \{ x \in A \mid \tilde{w}(x) > \epsilon \} \subset \subset \Omega. \) Then, it is clear that \( \tilde{b} \) satisfies all the assumptions of Theorem 3 [44] in \( A_\epsilon \). Consequently, by Theorem 3 [44], we get \( \tilde{w} \leq \epsilon \) in \( A_\epsilon \), which contradicts the definition of \( A_\epsilon \). So for each \( \epsilon > 0 \), \( A_\epsilon = \emptyset \), consequently \( \tilde{w} \leq 0 \) in \( A \). Thus, by Lemma 2.4, \( w \leq 0 \) in \( A \), which contradicts the definition of \( A \) hence \( A = \emptyset \) and the proposition is proved.

**Remark 2.** From the proof of above proposition, it is clear that the same assertion holds for the following equation
\[-M_{\lambda,\Lambda}^+(D^2u) + H(x, Du) + Cu = f(x) \text{ in } \Omega,\] (23)
where \( C > 0 \). That is, if \( u \) and \( v \) are respectively, sub and supersolution of (23) and either \( u \) or \( v \) is in \( W^{2,n}_{loc}(\Omega) \). Then \( u \leq v \) on \( \partial\Omega \), implies \( u \leq v \) in \( \Omega \).

The next proposition deals with existence of positive solution to the regularized problem \((R_\delta)\).

**Proposition 2.** Under the assumptions of Theorem 1.1, for each \( 0 < \delta < 1 \), there exists a positive solution to \((R_\delta)\). Furthermore,
\[ u \leq w_\delta \leq v. \]

For the definitions of \( u \) and \( v \), see Lemma 3.2 and (24), respectively. We prove Proposition 2 by the method of monotone iteration. In order to apply the monotone iteration, we first construct an appropriately ordered sub and supersolution of \((R_\delta)\) in next lemma. We mention that the sub and supersolution of \((R_\delta)\) constructed below, is independent of \( \delta \). More precisely, we consider the following function
\[ v = \frac{1}{l} \ln(1 + Ml\phi^\delta), \] (24)
where $l = \frac{\|b\|_{L^\infty(\Omega)}(q-1)}{\lambda}$, $0 < \beta < 1$ and $\phi$ is the eigenfunction from Theorem 2.2 with $\gamma = \|b\|_{L^\infty(2-q)}$, and prove Lemma 3.1. Similarly, by considering the function $u = c\phi$, where $c$ is a small positive constant and $\phi$ is again the eigenfunction as in the definition of $v$, we prove Lemma 3.2, see next.

**Lemma 3.1.** Under the assumptions of Proposition 2, there is a constant $M > 0$ (large enough) such that the function $v$ defined by (24) is a supersolution of $(R_\delta)$ for any $0 < \delta < 1$. Moreover, it remains a supersolution of $(R_\delta)$ if $M$ is replaced by any $M'$ satisfying $M' \geq M$.

**Proof.** We construct a positive supersolution of $(R_\delta)$, i.e., to construct a positive $v$ satisfying

$$-\mathcal{M}_{\lambda,\Delta}^+(D^2v) + H(x, Dv) \geq \frac{k(x)f(v)}{(v + \delta)\alpha},$$

(25)

Since, by (H1), (H2) and (5), $H$ satisfies the following inequality

$$H(x, Du) \geq -\|b\|_{L^\infty(\Omega)}(q-1)|Du|^2 - \|b\|_{L^\infty(\Omega)}(2-q)|Du|.$$  

(26)

Therefore, in view of (26) and hypothesis on $k$, it is sufficient to construct a positive $v$ satisfying the following inequality

$$-\mathcal{M}_{\lambda,\Delta}^+(D^2v) - \|b\|_{L^\infty(\Omega)}(q-1)|Du|^2 - \|b\|_{L^\infty(\Omega)}(2-q)|Du| \geq \frac{f(v)|k|_{L^\infty(\Omega)}}{(v + \delta)\alpha},$$

or equivalently

$$\mathcal{M}_{\lambda,\Delta}^+(D^2v) + \frac{\|b\|_{L^\infty(\Omega)}(q-1)}{\lambda}|Du|^2 + \|b\|_{L^\infty(\Omega)}(2-q)|Du| + \frac{f(v)|k|_{L^\infty(\Omega)}}{(v + \delta)\alpha} \leq 0.$$  

(27)

Let us define a function $w = \frac{e^{lv} - 1}{l}$, where $l = \frac{\|b\|_{L^\infty(\Omega)}(q-1)}{\lambda}$. So in view of Lemma 2.4 (first inequality in (13)), if we find a positive function $w$ satisfying

$$\mathcal{M}_{\lambda,\Delta}^+(D^2w) + \|b\|_{L^\infty(\Omega)}(2-q)|Dw| \leq \frac{f(l\ln(1 + lw))|k|_{L^\infty(\Omega)}}{(1 + lw)^\alpha} \leq 0,$$  

(28)

or

$$\mathcal{M}_{\lambda,\Delta}^+(D^2w) + \|b\|_{L^\infty(\Omega)}(2-q)|Dw| \leq \frac{f(l\ln(1 + lw))|k|_{L^\infty(\Omega)}}{(1 + lw)^\alpha} \leq 0,$$  

(29)

then $v = \frac{1}{l}\log(1 + lw)$ satisfies (27). Thus in view of this, our aim is to find a positive $w$ satisfying (29). Let us consider $w = M\phi^\beta$, where $\phi$ is the eigenfunction from (10) with $\gamma = \|b\|_{L^\infty(\Omega)}(2-q)$, $0 < \beta < 1$, and compute

$$\begin{cases} Dw = M\beta\phi^{\beta-1}D\phi, \\ D^2w = M\beta(\beta-1)\phi^{\beta-2}D\phi \otimes D\phi + M\beta\phi^{\beta-1}D^2\phi. \end{cases}$$  

(30)
Now, putting the values of $w$, $Dw$ and $D^2w$ in (29), we get

\[
\mathcal{M}^+_{\Lambda} (D^2 w) + ||\nabla||_{L^\infty(\Omega)} (2 - q) |Dw| + \left( \frac{1 + lw}{1 + \ln (1 + lw)} \right)^\alpha ||\nabla L^\infty(\Omega) ||_{L^\infty(\Omega)}
\]

\[
= \mathcal{M}^+_{\Lambda}(M \phi^\alpha - 1) \phi^{\alpha - 2} D\phi \otimes D\phi + M \phi^\alpha - 1 D^2 \phi + ||\nabla||_{L^\infty(\Omega)} (2 - q) M \phi^\alpha - 1 |D\phi|
\]

\[
+ \left( \frac{1 + M \phi^\beta}{1 + \ln (1 + M \phi^\beta)} \right)^\alpha ||\nabla L^\infty(\Omega) ||_{L^\infty(\Omega)}
\]

\[
\leq \mathcal{M}^+_{\Lambda}(M \phi^\alpha - 1) \phi^{\alpha - 2} D\phi \otimes D\phi + M \phi^\alpha - 1 \left[ \mathcal{M}^+_{\Lambda}(D^2 \phi) + ||\nabla||_{L^\infty(\Omega)} (2 - q) |D\phi| \right]
\]

\[
+ \left( \frac{1 + M \phi^\beta}{1 + \ln (1 + M \phi^\beta)} \right)^\alpha \quad \text{(by using inequality (8) and (6))}
\]

\[
= -M \phi^\alpha - 1 \lambda |D\phi|^2 \phi - 2 - M \mu^+ \beta \phi^\beta + \left( \frac{1 + M \phi^\beta}{1 + \ln (1 + M \phi^\beta)} \right)^\alpha, \quad \text{(31)}
\]

where in the last equality, we have used the fact that $\phi$ is the eigenfunction from (10) with $\gamma = ||\nabla||_{L^\infty(\Omega)} (2 - q)$ and

\[
\mathcal{M}^+_{\Lambda}(M \phi^\alpha - 1) \phi^{\alpha - 2} D\phi \otimes D\phi = -M \phi^\alpha - 1 \lambda |D\phi|^2 \phi - 2 - M \mu^+ \beta \phi^\beta.
\]

Equation (32) is a consequence of the facts $0 < \beta < 1$, (7) and $D\phi \otimes D\phi$ has the only nontrivial eigenvalue $|D\phi|^2$, combined with the definition of Pucci’s extremal operator. Thus, in view of (31), in order to show that $w$ satisfies (29), it is sufficient to choose $M > 0$ large enough such that

\[
-M \phi^\alpha - 1 \lambda |D\phi|^2 \phi - 2 - M \mu^+ \beta \phi^\beta + \left( \frac{1 + M \phi^\beta}{1 + \ln (1 + M \phi^\beta)} \right)^\alpha \leq 0,
\]

holds in $\Omega$. The choice of large $M > 0$ is accomplished in two steps.

**Step (i).** In this step, we will find a neighbourhood, say, $N$ of $\partial \Omega$ and a large value of $M$, say $M_1$, such that (33) holds in $\Omega \cap N$ for any $M \geq M_1$. In view of Remark 1, there exists a neighbourhood, say $N_1$, of $\partial \Omega$ and a positive constant $L$ such that $|D\phi| \geq L$ in $N_1$. Thus the left hand side of (33) in $N_1$ satisfies the following inequality:

\[
\left[ -M \phi^\alpha - 1 \lambda |D\phi|^2 - M \mu^+ \beta \phi^\beta + \left( \frac{1 + M \phi^\beta}{1 + \ln (1 + M \phi^\beta)} \right)^\alpha \right] \phi^{\beta - 2}
\]

\[
\leq \left[ -M \phi^\alpha - 1 \lambda L^2 - M \mu^+ \beta \phi^\beta + \left( \frac{1 + M \phi^\beta}{1 + \ln (1 + M \phi^\beta)} \right)^\alpha \right] \phi^{\beta - 2},
\]

where $\sigma = 2 - \alpha \beta - \beta > 0$ because $0 < \alpha, \beta < 1$. Now, choose $M_3$ such that for any $M \geq M_3$, the following holds in $N_1$:

\[
-M \phi^\alpha - 1 \lambda L^2 - M \mu^+ \beta \phi^\beta + 2^\alpha \left( \frac{1}{M^\alpha} + M \phi^\beta \right) \left( \frac{1}{1 + \ln (1 + M \phi^\beta)} \right)^\alpha \phi^{\beta - 2} \leq 0,
\]

which is always possible because of the first term and $f$ is nonincreasing. Furthermore, we can also choose an $\epsilon_1 > 0$ and another neighbourhood $N_2$ of $\partial \Omega$ such that
Let us choose \( \phi \) the following holds
\[
\frac{1}{2} < \frac{\ln(1 + x)}{x} < \frac{3}{2} \quad \text{if } 0 \leq x < \delta_1,
\]
and
\[
[f(0)]_L^\alpha (1 + l\phi^\beta)\phi^{2 - \beta} < \beta(1 - \beta)\lambda L^2[\ln(1 + \epsilon_1)]^\alpha \text{ in } N_2, \tag{38}
\]
where (38) possible since \( \phi = 0 \) on \( \partial\Omega \) and righthand side is a positive constant.

Now, we claim that for any \( M \geq M_1 = \max\{2, M_3\} \), (33) is satisfied in \( N = N_1 \cap N_2 \).

Let us take an arbitrary but fixed \( x \in N \), and for any \( M \), in view of (33), we find that (33) holds.

In the case (a), by (37), we have
\[
\frac{1}{2} \leq (1 + M\phi^\beta)\phi^{2 - \beta} < \beta(1 - \beta)\lambda L^2[\ln(1 + \epsilon_1)]^\alpha \quad \text{and} \quad a, A > 0.
\]

So as a consequence of (36) and (37), we find that (33) holds. In the case (b), (38) takes the following form:
\[
[f(0)]_L^\alpha (1 + l\phi^\beta)\phi^{2 - \beta} < \beta(1 - \beta)\lambda L^2[\ln(1 + M\phi^\beta)]^\alpha,
\]
\[
[f(0)]_L^\alpha \frac{(1 + M\phi^\beta)}{\ln(1 + M\phi^\beta)} < \beta(1 - \beta)\lambda L^2[\ln(1 + M\phi^\beta)]^\alpha \quad \text{(since } M \geq 2). \tag{39}
\]

So by noting that \( f \) is nonincreasing and \(|D\phi| \geq L\), we have:
\[
-M_1^+ \beta a^2 - \beta(1 - \beta)\lambda|D\phi|^2 + \frac{(1 + M\phi^\beta)f(\frac{1}{2} \ln(1 + M\phi^\beta))\|k\|_{L^\infty(\Omega)}\phi^{2 - \beta}}{[\frac{1}{2} \ln(1 + M\phi^\beta)]^\alpha} \leq 0. \tag{40}
\]

Thus, (33) again follows from (40) combined with (34).

**Step (ii).** In this step, our aim is to find a large value of \( M \), say \( M_2 \), such that for any \( M \geq M_2 \) (33) is satisfied in \( \Omega \setminus N \). For this, let us set
\[
a = \inf_{\Omega \setminus N} \phi, \quad A = \sup_{\Omega \setminus N} \phi. \tag{41}
\]

Note that \( a, A > 0 \) and the following inequality holds in \( \Omega \setminus N \)
\[
-M_1^+ \beta a^2 - \beta(1 - \beta)\lambda|D\phi|^2 + \frac{(1 + M\phi^\beta)f(\frac{1}{2} \ln(1 + M\phi^\beta))\|k\|_{L^\infty(\Omega)}\phi^{2 - \beta}}{[\frac{1}{2} \ln(1 + M\phi^\beta)]^\alpha} \leq -M_1^+ \beta a^2 - \beta(1 - \beta)\lambda|D\phi|^2 + \frac{(1 + M\lambda a^\beta)f(\frac{1}{2} \ln(1 + M\lambda a^\beta))\|k\|_{L^\infty(\Omega)}A^{2 - \beta}}{[\ln(1 + M\lambda a^\beta)]^\alpha}. \tag{42}
\]

Let us choose \( M_2 \) sufficiently large such that for any \( M \geq M_2 \), we have
\[
f(0)\frac{1}{M} (1 + lA^\beta)\|k\|_{L^\infty(\Omega)}^{\alpha} A^{2 - \beta} \leq \mu_1^+ \beta a^2 [\ln(1 + M\lambda a^\beta)]^\alpha, \tag{43}
\]
which is always possible. For any such value of \( M \), multiplying the inequality (43) by \( M[\ln(1 + M\lambda a^\beta)]^{-\alpha} \), rearranging the terms and using the fact that \( f \) is nonincreasing, we find that the right hand side of (42) is nonpositive. Thus, in view of (34), for any \( M \geq M_2 \), (33) holds also in \( \Omega \setminus N \).

Now, if we set \( M = 1 + \max\{M_1, M_2\} \), where \( M_1 \) and \( M_2 \) are as in steps (i) and (ii), respectively, then it is clear that for this value of \( M \), \( w = M\phi \) satisfies (29). Consequently, \( v = \frac{1}{2} \ln(1 + lM\phi^\beta) \) satisfies (25) and hence is a supersolution of (R₆). Furthermore, from the proof it is clear that, for any \( M' \geq M \), \( \frac{1}{2} \ln(1 + lM'\phi^\beta) \) is also a supersolution of (R₆). □
Lemma 3.2. Under the assumptions of the Proposition 2, \( u = c\phi \) is a subsolution to \((R_3)\) for sufficiently small value \( c \). Furthermore, for any \( 0 < c' < c, c\phi \) is again a subsolution of \((R_3)\).

Proof. Here, we use \( \inf_{\Omega} k > 0 \) and \( f(0) > 0 \). Since \( f(0) > 0 \) so there exists a \( \epsilon_2 > 0 \) such that

\[
f(t) > 0, \text{ for all } t \in [0, \epsilon_2].
\]

Let us set \( \inf_{[0, \epsilon_2]} f = f(\epsilon_2) > 0 \). In order to find the subsolution of \((R_3)\) for any \( 0 < \delta < 1 \), it is sufficient to find a positive function \( u \) satisfying

\[
- \mathcal{M}^+_{\lambda, \Lambda}(D^2u) + H(x, Du) \leq \frac{f(u)k(x)}{(u + 1)^\alpha}.
\]

Again, in view of \((H1)\) and \((H2)\), it is sufficient to find a positive \( u \) satisfying

\[
- \mathcal{M}^+_{\lambda, \Lambda}(D^2u) + b(x)|Du|^q \leq \frac{f(u)k(x)}{(u + 1)^\alpha} \text{ in } \Omega.
\]

Let us consider \( u = c\phi \), where \( \phi \) is the same as above in the construction of super-solution, and choose \( c > 0 \) sufficiently small such that

\[
c\mu_k^+ + c\|b\|_{L^\infty(\Omega)}(2 - q)|D\phi| + c^\alpha\|b\|_{L^\infty(\Omega)}|D\phi|^q - \frac{f(\epsilon_2)\inf_{\Omega} k}{(c\phi + 1)^\alpha} \leq 0,
\]

and \( c\phi \in [0, \epsilon_2] \), which is always possible since \( f(\epsilon_2)\inf_{\Omega} k > 0 \). Thus, with the above choice of \( c, u = c\phi \) is a subsolution to \((44)\) and therefore a subsolution of \((R_3)\) for any \( 0 < \delta < 1 \). Furthermore, it is also clear that for any choice of \( c' < c, c\phi \) is also a subsolution to \((R_3)\).

Thus, from Lemmas 3.1 and 3.2, we have a family of super and subsolution of \((R_3)\). Now, we extract a sub and supersolution which are ordered. For this, let us choose \( c \) sufficiently small such that \( 2lc\phi^\beta < \epsilon_1 \), where \( \epsilon_1 \) is defined in \((38)\), then

\[
c\phi \leq c\phi^\beta < \frac{1}{l} \ln(1 + 2lc\phi^\beta) \leq \frac{1}{l} \ln(1 + M\phi^\beta),
\]

where the first inequality holds because \( \|\phi\|_{L^\infty(\Omega)} = 1 \) and \( 0 < \beta < 1 \). Thus with the above values of \( c \) and \( M \), let us denote corresponding sub and supersolution by \( u = c\phi \) and \( v = \frac{1}{l} \ln(1 + M\phi^\beta) \), respectively, then \( u \leq v \). We will use these values of \( u \) and \( v \) in the proof of the following proposition.

Now, we are ready to give a proof of Proposition 2.

3.2. Proof of Proposition 2.

Proof. Let us set \( \Gamma = \sup_{\Omega} v \). Now, for any fixed \( \delta \in (0, 1) \), by using Lipschitz continuity of \( f \) and boundedness of \( k \), we can find a positive constant, say \( C \), such that

\[
k(x)f(t_1) \frac{(t_1 + \delta)^\alpha}{(t_1 + \delta)^\alpha} + C t_1 \leq k(x)f(t_2) \frac{(t_2 + \delta)^\alpha}{(t_2 + \delta)^\alpha} + C t_2 \text{ for } t_1, t_2 \in [0,\Gamma],
\]

with \( t_1 \leq t_2 \), see Proposition 3(see appendix for details). Let us also note that finding a solution \( w_\delta \) to \((R_3)\) is equivalent to finding the solution of the following problem:

\[
\begin{cases}
- \mathcal{M}^+_{\lambda, \Lambda}(D^2w_\delta) + H(x, Dw_\delta) + Cw_\delta = \frac{k(x)f(w_\delta)}{(w_\delta + \delta)^\alpha} + Cw_\delta \text{ in } \Omega, \\
w_\delta = 0 \text{ on } \partial\Omega.
\end{cases}
\]
It is also easy to observe that supersolution and subsolution of \((R_\delta)\) constructed in Lemmas 3.1 and 3.2 are also a supersolution and subsolution of \((48)\), respectively. Now, let us consider the solution \(w_{\delta,1}\) to the following problem

\[
\begin{cases}
-M_\lambda + \lambda, \Lambda(D^2 w_{\delta,1}) + H(x, Dw_{\delta,1}) + Cw_{\delta,1} = \frac{k(x)f(u)}{(u + \delta)^\alpha} + Cu & \text{in } \Omega, \\
w_{\delta,1} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(u = c\phi\) is the subsolution constructed above. For the existence of solution to \((49)\), we refer Theorem 1(i) \([44]\). Since \(u = c\phi\) is also a subsolution to \((48)\) and \(u \in W^{2,n}_{loc}(\Omega)\), so by comparison principle (Remark 2), we find

\[u \leq w_{\delta,1} \leq v.\]

Proceeding inductively, we find a sequence of solutions \(w_{\delta,m}\) to the following problem

\[
\begin{cases}
-M_\lambda + \lambda, \Lambda(D^2 w_{\delta,m}) + H(x, Dw_{\delta,m}) + Cw_{\delta,m} = \frac{k(x)f(w_{\delta,m-1})}{(w_{\delta,m-1} + \delta)^\alpha} + Cw_{\delta,m-1} & \text{in } \Omega, \\
w_{\delta,m} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

satisfying \(u \leq w_{\delta,m} \leq v\). Now, by using the \(C^\alpha\) estimate (Theorem 2 \([44]\)), we find a subsequence, say, \(w_{\delta,m_k}\) of \(w_{\delta,m}\) converges uniformly to \(w_\delta\), which in view of the stability result (Theorem 4 \([44]\)) for the viscosity solution satisfies \((48)\) or equivalently \((R_\delta)\). Furthermore, we also have

\[u \leq w_\delta \leq v.\]

**Remark 3.** Proposition 2 gives the existence of a positive solution \(w_\delta\) of \((R_\delta)\) for each \(0 < \delta < 1\). In particular, for \((R_{\frac{1}{m}})\) for each \(m \in \mathbb{N}\). Below, we denote the solution of \((R_{\frac{1}{m}})\) by \(w_m\) instead of \(w_{\frac{1}{m}}\). So in view of this notation, we have

\[u \leq w_m \leq v.\]

Now, we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Choose \(\delta_m = \frac{1}{m}\) and corresponding sequence \(\{w_m\}\) of solution of \((R_{\delta_m})\) satisfying \((51)\). Now, using Theorem 2 \([44]\) (interior estimate) and stability result (Theorem 4 \([44]\)) for viscosity solution and standard diagonal process, we find a subsequence of \(\{w_m\}\) converges uniformly to a \(w \in C(\Omega)\), which is the positive solution of \((1)\). \(\Box\)

**Remark 4.** Due to the lack of regularity of the solution, we cannot say about the uniqueness of the solution of \((1)\).

**Remark 5.** Note that when \(f(0) = 0\) and \(f\) nonincreasing, so \(f\) reduces to identically 0. So in this case, \((1)\) takes the following form

\[
\begin{cases}
-M_\lambda + \lambda, \Lambda(D^2 u) + H(x, Du) = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

\[52\]
Here, by using \( H(x,0) = 0 \), it is clear that \( u \equiv 0 \) is a solution to (52), further, \( 0 \in W_{2,n}^{2,r}(\Omega) \), so we apply the comparison principle to conclude that any nonnegative solution to (52) is \( u \equiv 0 \).

4. Case \( f(0) < 0 \). In this section, we consider the case \( f(0) < 0 \) and establish the existence of a positive solution to (1). Here, we assume that \( H(x, Du) = b(x)|Du|^q \), where \( 1 < q \leq 2 \), so problem (1) reduces to the following problem:

\[
-\mathcal{M}_{\lambda,\Lambda}^+(D^2u) + b(x)|Du|^q = \frac{k(x)f(u)}{u^\alpha} \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\]

Since we assume that \( f(0) < 0 \) so as long as \( k(x) \geq 0 \), problem (53) is called semipositone problem. In this section, we show that there exists a positive solution to (53). The approach of the proof is to truncate the problem around the origin. First, we establish the existence of positive solution to the truncated problem by constructing a sub and supersolution of the truncated problem and finally pass the limit to the truncated problem (see (84)). We see that in contrast to the case \( f(0) > 0 \), the construction of a subsolution is more difficult than the construction of a supersolution to the truncated problem. Let us proceed to prove a Theorem 4.1 which is needed in the construction of a subsolution of (84), see next.

**Theorem 4.1.** Suppose that \( k \in L^p(\Omega) \) for \( p > n \) and

\[
-\mathcal{M}_{\lambda,\Lambda}^+(D^2u) + b(x)\eta^{q-1}|u|(q-1)|Du|^q = k(x) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\]

where \( 1 < \eta < 2 \), \( 1 < q \leq 2 \) and \( b \in L^\infty(\Omega) \). Suppose also that the following condition holds:

\[
\|b\|_{L^\infty} \|k\|_{L^p(\Omega)}^{q(q-1)} < \frac{1}{\eta^{q-1}q^2(q-1)\gamma},
\]

where \( \rho > 0 \) (which is defined by (70)). Then there exists an \( L^p \)-strong solution \( u \in W^{2,r}(\Omega) \) of (54). Furthermore,

\[
\|u\|_{W^{2,r}(\Omega)} \leq \tilde{C}\|k\|_{L^p(\Omega)}.
\]

**Proof.** We prove this theorem by fixed point argument. Let us define a number \( r \) by the following equation:

\[
\frac{1}{p} = \frac{(\eta - 1)(q - 1) + q}{r}.
\]

That is, \( r = p[(\eta - 1)(q - 1) + q] \). In view of \( (\eta - 1)(q - 1) > 0 \) and \( q > 1, r \geq p \). Let us define a map \( S : W^{1,r}(\Omega) \longrightarrow W^{1,r}(\Omega) \) by

\[
Su = u,
\]

where \( u \) is the unique \( L^p \)-strong solution of the following problem

\[
-\mathcal{M}_{\lambda,\Lambda}^+(D^2u) = k(x) - \eta^{q-1}b(x)|v|(q-1)|Dv|^q \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\]

The map \( S \) is well defined. For, \( v \in W^{1,r}(\Omega) \), \( |v|(q-1) \in L^{(\eta-1)(q-1)}(\Omega) \) and \( |Dv|^q \in L^\frac{p}{r}(\Omega) \). So as a consequence of (57) and Hölder’s inequality, we have
where we have used the following inequality:
\[ |v|^{(q-1)(q-1)}|Dv|^q \in L^p(\Omega). \]

Now, by Proposition 2.4 [32], Equation (58) has unique \( L^p \)-strong solution. In fact, \( u \in W^{2,p}(\Omega) \) and also satisfies the following estimates
\[ \|u\|_{L^\infty(\Omega)} \leq C(\|k\|_{L^p(\Omega)} + \|b\|_{L^\infty(\Omega)}\eta^{q-1}\|v\|_{L^{q(1)}}^{(q-1)}\|Dv\|_{L^q(\Omega)}), \]
and
\[ \|u\|_{W^{2,p}(\Omega)} \leq \tilde{C}(\|k\|_{L^p(\Omega)} + \|b\|_{L^\infty(\Omega)}\eta^{q-1}\|v\|_{L^{q(1)}}^{(q-1)}\|Dv\|_{L^q(\Omega)}), \]
where we have used the following inequality:
\[ \|v|^{(q-1)(q-1)}|Dv|^q \|_{L^p(\Omega)} \leq \|v|^{(q-1)(q-1)}\|Dv\|_{L^q(\Omega)}^q, \]

where \( \|v\|_{L^\infty(\Omega)} \leq \|v\|_{L^{q(1)}} \|Dv\|_{L^q(\Omega)} \|Dv\|_{L^q(\Omega)}. \)

Thus \( S \) is well defined. Moreover, in view of (60) and compact embedding \( W^{2,p}(\Omega) \hookrightarrow W^{1,r}(\Omega) \), \( S \) is a compact operator. Now, we claim that \( S \) is also continuous. Let \( \{u_k\} \subset W^{1,r}(\Omega) \) be a sequence such that \( u_k \rightarrow v \) in \( W^{1,r}(\Omega) \) and set \( Sv_k = u_k \). Thus by (60), we have
\[ \|Sv_k\|_{W^{2,p}(\Omega)} \leq C_1 \text{ for } k = 1, 2, \ldots. \]

So, by the reflexivity of \( W^{2,p}(\Omega) \), there exists a subsequence \( \{u_{k_j}\} \) of \( \{u_k\} \) and a function \( u \in W^{2,p}(\Omega) \) such that \( Sv_k \rightharpoonup u \) in \( W^{2,p}(\Omega) \) and by compactness of the embedding \( W^{2,p}(\Omega) \hookrightarrow W^{1,r}(\Omega) \)
\[ Sv_k \rightarrow u \text{ in } W^{1,r}(\Omega). \]

Next, we show that \( Sv = u \) and consequently the continuity of \( S \) follows, see, pp.542 [21]. Let us set
\[ h_k(x) = k(x) - b(x)\eta^{q-1}|v_k(x)|^{(q-1)(q-1)}|Dv_k(x)|^q, \]
and consider
\[ \|h - h_k\|_{L^p(\Omega)} = \|b(x)\eta^{q-1}(|v_k(x)|^{(q-1)(q-1)}|Dv_k|^q - |v|^{(q-1)(q-1)}|Dv|^q)|L^p(\Omega) \]
\[ \leq \eta^{q-1}\|b\|_{L^\infty(\Omega)}\|v_k|^{q-1}|Dv_k|^q - |v|^{(q-1)(q-1)}|Dv|^q\|_{L^p(\Omega)} \]
\[ \leq \eta^{q-1}\|b\|_{L^\infty(\Omega)} \left( \|v|^{(q-1)(q-1)}|Dv|^q\|_{L^p(\Omega)} + I_1 \right) + \eta^{q-1}\|b\|_{L^\infty(\Omega)} \left( \|Dv_k|^{q(1)} - |Dv|^q\|_{L^p(\Omega)} \right), \]
where the last inequality is obtained by adding and subtracting \( |v|^{(q-1)(q-1)}|Dv|^q \).

Now, by using inequality (5) and noting the fact that \( 0 < (q - 1)(\eta - 1) \leq 1 \), we get
\[ I_1 = \|v|^{(q-1)(q-1)} - |v|^{(q-1)(q-1)}|Dv_k|^q\|_{L^p(\Omega)} \]
\[ \leq \|v_k|^{(q-1)(q-1)} - |v|^{(q-1)(q-1)}\|_{L^{q(1)(q-1)}} \|Dv_k|^q\|_{L^q(\Omega)} \]
\[ \leq \|v_k - v\|_{L^{q(1)(q-1)}} \|Dv_k|^q\|_{L^q(\Omega)}. \]
Further, since
and thus in view of (63),
By using (5), we find
Now, consider
the right hand side
Next, we show that
Now, as
so by using Hölder’s and triangle inequality, we obtain
and thus in view of (63), \( I_2 \) satisfies the following inequality:
Now, as \( v_k \to v \) in \( W^{1,r}(\Omega) \), so by (62) and (66), we get
and consequently, \( \| h_k - h \|_{L^p(\Omega)} \to 0 \) as \( k \to \infty \). Since \( u_k \) and \( Sv \) satisfy (58) with the right hand side \( h_k \) and \( h \), respectively, so \( w_k = u_k - Sv \) satisfies
Next, we show that \( S : B(0,R) \subset W^{1,r}(\Omega) \to B(0,R) \), for some \( R > 0 \). Let \( \| v \|_{W^{1,r}(\Omega)} \leq R \), where \( R \) will be chosen below, so
Thus as \( k \to \infty \), we find \( u_k \to Sv \), that is, \( Sv_k \to Sv \). Since the limit is unique so \( Sv = u \).
In the above calculation, \( D \) is the embedding constant and \( \bar{C} \) is the constant from (60). Now, set
\[
2D\bar{C} = \rho \quad (70)
\]
and on choosing \( R = \rho \|k\|_{L^p(\Omega)} \) yields that
\[
\|Sv\|_{W^{1,r}(\Omega)} \leq \frac{R}{2} \left( 1 + \rho \eta^{q-1} \|b\|_{L^\infty(\Omega)} R^{(q-1)\eta} \right). \quad (71)
\]
From the definition of \( R \), we have
\[
\rho \eta^{q-1} \|b\|_{L^\infty(\Omega)} R^{(q-1)\eta} = \eta^{q-1} \rho^{(q-1)\eta+1} \|b\|_{L^\infty(\Omega)} \|k\|_{L^p(\Omega)}^{(q-1)\eta}. \quad (72)
\]
So from (71), (72) and (55), we get
\[
\|Sv\|_{W^{1,r}(\Omega)} \leq R.
\]
Now, since the operator \( S \) is compact, so \( S(B(0,R)) \) is precompact in \( W^{1,r}(\Omega) \), consequently, \( S : B(0,R) \to B(0,R) \) has a fixed point which is an \( L^p \)-strong solution of (54). (60) and (56) imply (56) and the proof is complete. \( \square \)

**Remark 6.** Before proceeding further, we remark that here and afterwards, we assume that \( k \) and \( b \) satisfy (55).

Next, we prove that (1) has at least one positive solution in case \( f(0) < 0 \). More precisely, we have the following theorem:

**Theorem 4.2.** Let \( f : [0,\infty) \to \mathbb{R} \) be a bounded, locally Lipschitz continuous function such that \( f(0) < 0 \) and \( H(x,Du) = b(x)|Du|^q \), where \( 0 \leq b(x) \in L^\infty(\Omega) \), \( 1 < q \leq 2 \). Let us set \( \eta = \frac{2}{q-1} \), where \( 0 < \alpha < 1 \). If \( f(t^n) \geq -\frac{\alpha(\eta-1)\rho^2}{2M} := -M \) on \([0,s]\), and \( f(t^n) \geq \eta t^n \) on \([s,S]\), where \( s, S \) and \( \theta \) are given by (82) and (75), respectively. Then (1) has at least one positive solution provided \( b \) and \( k \) satisfy (55).

In order to prove Theorem 4.2, we need some background. Let us introduce the following notation:
\[
\Omega_\tau = \{x \in \Omega : d(x,\partial\Omega) > \tau\},
\]
where \( d(x,\partial\Omega) \) represents the distance of the point \( x \) from the boundary \( \partial\Omega \). For each \( \tau > 0 \), we consider the following problem:
\[
\begin{cases}
-\mathcal{M}_A^{I}(D^2v_\tau) + b(x)\eta^{q-1}|v_\tau|^{(q-1)(\eta-1)}|Dv_\tau|^q = k(x)\chi_{\Omega_\tau} & \text{in } \Omega, \\
v_\tau = 0 & \text{on } \partial\Omega,
\end{cases} \quad (73)
\]
where \( \chi_{\Omega_\tau} \) denotes the characteristic function of the set \( \Omega_\tau \). By Theorem 4.1, there exists \( v_\tau \in W^{2,p}(\Omega) \), a solution of (73).

**Lemma 4.3.** Let \( v_\tau \) be a solution of (73) for \( \tau > 0 \). Then \( v_\tau \geq 0 \) and \( v_\tau \to v \) in \( C^1(\Omega) \), where \( v \) satisfies (73) with righthand side \( k \). Also, there exists some \( \tau > 0 \) such that the following holds
\[
\min \{|Dv_\tau(x)| : x \in \Omega \setminus \Omega_\tau\} > \frac{\theta}{2}, \quad (74)
\]
where \( \theta \) is defined as follows:
\[
\theta := \min\{|Dv(x)| : x \in \partial\Omega\} > 0. \quad (75)
\]
Proof. We claim that $v_\tau \geq 0$. In order to prove the claim, we take arbitrary but fixed $\tau$ and set $\bar{b}(x) = b(x)\eta^{q-1}|v_\tau|^{(q-1)(q-1)}$. By an easy observation that $\bar{b} \in L^\infty(\Omega)$, we find that the following holds:

\begin{align*}
-\mathcal{A}_A(D^2v_\tau) + \bar{b}(x)Dv_\tau|^q &= k(x)\chi_\Omega, \\
-\mathcal{A}_A(D^2v_\tau) + |\bar{b}|_{L^\infty(\Omega)}Dv_\tau|^q &\geq k(x)\chi_\Omega, \\
-\mathcal{A}_A(D^2v_\tau) + |\bar{b}|_{L^\infty(\Omega)}(q-1)|Dv_\tau|^2 + |\bar{b}|_{L^\infty(\Omega)}(2-q)|Dv_\tau| &\geq k(x)\chi_\Omega \geq 0,
\end{align*}

where $p > n$.

Following the similar calculation as earlier, we get

\begin{align*}
\mathcal{A}_A(D^2v_\tau) - |\bar{b}|_{L^\infty(\Omega)}(q-1)|Dv_\tau|^2 - |\bar{b}|_{L^\infty(\Omega)}(2-q)|Dv_\tau| &\leq 0.
\end{align*}

(76)

Let us set $\bar{\tau} = \frac{|\bar{b}|_{L^\infty(\Omega)}(q-1)}{\lambda}$ and define the following function

\begin{equation}
\w_\tau = \frac{1 - e^{-l\tau}}{l}.
\end{equation}

By Lemma 2.4, $\w_\tau$ satisfies the following inequality:

\begin{align*}
\mathcal{A}_A(D^2\w_\tau) - |\bar{b}|_{L^\infty(\Omega)}(2-q)|D\w_\tau| &\leq 0,
\end{align*}

(77)

and so by Theorem 3 [44], $\w_\tau \geq 0$ in $\Omega$, and consequently, $v_\tau \geq 0$.

Again, from (56), we have

\begin{equation}
|v_\tau|_{W^{2,p}(\Omega)} \leq C\|k\|_{L^p}.
\end{equation}

By noting that $p > n$, we find that $\{v_\tau\}$ is bounded in $C^{1,\alpha}(\Omega)$, so as a consequence of the compact embedding $C^{1,\alpha}(\Omega) \subset C^1(\Omega)$, there is a $v \in C^1(\Omega)$ and a subsequence of $\{v_\tau\}$ which we still denote by $v_\tau$ such that $v_\tau \to v$ in $C^1(\Omega)$, moreover $v \geq 0$. We claim that $v$ satisfies (73) with right hand side $k$. For this, let us set

\begin{equation}
g_\tau(x) = |v_\tau(x)|^{(q-1)(q-1)}|Dv_\tau|^q
\end{equation}

and $g(x) = |v(x)|^{(q-1)(q-1)}|Dv|^q$.

Following the similar calculation as earlier, we get

\begin{equation}
|g_\tau(x) - g(x)| \leq q|v_\tau(x)|^{(q-1)(q-1)}(|Dv_\tau(x)|^{q-1} + |Dv(x)|^{q-1})|Dv_\tau(x) - Dv(x)|
\end{equation}

and so $g_\tau(x) \to g(x)$ uniformly in $\bar{\Omega}$. Thus $g_\tau \to g$ in $L^p(\Omega)$ for all $p \in [1, \infty)$. Since $v_\tau$ satisfies (73) so by an application of the stability result (Theorem 5 [23]) for viscosity solution, we find that $v$ satisfies the following equation

\begin{equation}
\begin{cases}
-\mathcal{A}_A(D^2v) + b(x)\eta^{q-1}|v|^{(q-1)(q-1)}Dv|^q &= k(x) \text{ in } \Omega, \\
v &= 0 \text{ on } \partial\Omega.
\end{cases}
\end{equation}

(79)

Note that $\bar{b} = \eta b\eta^{q-1}|v|^{(q-1)(q-1)} \in L^\infty(\Omega)$ and define the function $w = \frac{1 - e^{-l\tau}}{l}$, where $l = \frac{|\bar{b}|_{L^\infty(\Omega)}(q-1)}{\lambda}$. Notice also that $w$ is non negative, $w \neq 0$ and $w = 0$ for all $x \in \partial\Omega$. Following the similar steps as above, $w$ satisfies (77) with $|\bar{b}|_{L^\infty(\Omega)}$ replaced by $|\bar{b}|_{L^\infty(\Omega)}$. So by Höpfl type lemma, $Dw(x) \neq 0$ for all $x \in \partial\Omega$. Let us consider

\begin{equation}
\theta := \min\{|Dv(x)| : x \in \partial\Omega\} > 0
\end{equation}

and choose $\tau > 0$ such that

\begin{equation}
\min\{|Dv_\tau(x)| : x \in \bar{\Omega} \setminus \Omega_\tau\} > \frac{\theta}{2}.
\end{equation}

(81)

This completes the proof. \qed
Let us set
\[ s = \min\{v_\tau(x) : x \in \bar{\Omega}_\tau\} \text{ and } S = \max\{v_\tau(x) : x \in \bar{\Omega}_\tau\} \]  
(82)
corresponding to a suitably chosen \( \tau \) in Lemma 4.3. Since \( v_\tau \geq \frac{\theta}{2} \tau \) on \( \partial \Omega_\tau \), so by maximum principle as above \( v_\tau \geq \frac{\theta}{2} \tau \) in \( \bar{\Omega}_\tau \). Thus \( s \geq \frac{\theta}{2} \tau \).

Let us construct an easy example of \( f \) satisfying the conditions of Theorem 4.2.

**Example 1.** Consider the following function
\[ f(t) = \frac{lt}{1+t} - M \text{ for } t \in [0, \infty), \]
where \( l \) is a parameter and \( M > 0 \) is a constant. Note that \( f(0) = -M < 0 \), and nondecreasing. Further, by choosing \( l \) such that
\[ \frac{ls^\eta}{1+s^\eta} \geq \eta S + M, \]
where \( s < S \) as in Theorem 4.2, we find that
\[ f(t^\eta) = \frac{lt^\eta}{1+t^\eta} - M \geq \eta S \geq \eta t \text{ for any } t \in [s, S]. \]

Now, we are prepared to present the proof of Theorem 4.2 in case \( f(0) < 0 \). In this case, we assume that \( H(x, Du) = b(x)|Du|^q \) for some nonnegative function \( b \in L^\infty(\Omega) \) and \( 1 < q \leq 2 \), and prove the following existence theorem.

Now, we prove Theorem 4.2.

**Proof.** The idea of the proof is to truncate the function around origin. We set \( g(t) = f(t)t^{-\alpha} \) and for each \( m \in \mathbb{N} \), define
\[ g_m(t) = \begin{cases} g(t), & \text{if } t \geq \rho_m, \\ \max\{g(\rho_m), g(t)\}, & \text{if } 0 < t \leq \rho_m, \end{cases} \]
(83)
where \( \{\rho_m\} \) is a decreasing sequence of positive real numbers such that \( \rho_m \to 0 \) as \( m \to \infty \). As \( \lim_{t \to 0^+} g(t) = -\infty \), so for each \( m \), there is some \( \rho'_m \leq \rho_m \) such that \( g_m = g(\rho_m) \) on \( (0, \rho'_m] \), consequently, \( g_m \) can be extended to have value \( g(\rho_m) \) at \( 0 \).

Now, we will consider the following truncated boundary value problem
\[ \begin{cases} -M^+_{\lambda, \Lambda}(D^2 u) + b(x)|Du|^q = k(x)g_m(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial \Omega, \end{cases} \]
(84)
and show that it has a positive solution by constructing a positive subsolution and an appropriately ordered supersolution. Let us take \( u = u^0_\tau \), where \( \tau > 0 \) fixed as in Lemma 4.3, and differentiate it twice to get
\[ Du = \eta v_\tau^{\eta-1}Dv_\tau, \]
\[ D^2 u = \eta(\eta - 1)v_\tau^{\eta-2}Dv_\tau \otimes Dv_\tau + \eta v_\tau^{\eta-1}D^2 v_\tau, \]
(85)
where again \( x \otimes x \) is an \( n \times n \) matrix with \( ij \)-entry \( (x_ix_j) \) and \( x = (x_1, x_2, \cdots, x_n) \). In the following, we use the fact that \( v_\tau \) is nonnegative and \( Dv_\tau \otimes Dv_\tau \) has \( |Dv_\tau|^2 \) as the only nontrivial eigenvalue. Let us consider
\[ -M^+_{\lambda, \Lambda}(D^2 u) + b(x)|Du|^q \]
\[ = -M^+_{\lambda, \Lambda}(\eta(\eta - 1)v_\tau^{\eta-2}Dv_\tau \otimes Dv_\tau + \eta v_\tau^{\eta-1}D^2 v_\tau) \]
\[ + b(x)\eta^q v_t^{(q-1)}|Dv_t| \]
\[ \leq -M_{\lambda, A}(\eta(n-1)v_t^{-2}Dv_t \otimes Dv_t) - M_{\lambda, A}(\eta v_t^{q-1}D^2v_t) + b(x)\eta^q v_t^{(q-1)}|Dv_t| \]
\[ = -\lambda \eta(n-1)v_t^{-2}|Dv_t|^2 + \eta v_t^{q-1}[\lambda x + b(x)]v_t^{q-1}Dv_t \]
\[ = -\lambda \eta(n-1)v_t^{-2}|Dv_t|^2 + \eta v_t^{q-1}kx_\Omega. \]

We also have \( \eta - 2 = -\eta_m \), so
\[ -M_{\lambda, A}(D^2u) + b(x)|Du|^q \]
\[ \leq -\lambda \eta(n-1)\frac{\theta^2}{4} \frac{k}{||b||L^\infty(\Omega)} (v_0^2)^{-\alpha} \chi_{\{x \in \Omega : 0 \leq v_\tau \leq x\}} + \eta kv_\tau (v_0^2)^{-\alpha} \chi_{\{x \in \Omega : s \leq v_\tau \leq s\}} \]
\[ = k\left[ -\eta(n-1)\frac{\theta^2}{4} \frac{1}{||b||L^\infty(\Omega)} \chi_{\{x \in \Omega : 0 \leq v_\tau \leq x\}} + \eta v_\tau \chi_{\{x \in \Omega : s \leq v_\tau \leq s\}} \right] (v_0^2)^{-\alpha} \]
\[ \leq kf(v_0^2)(v_0^2)^{-\alpha} \]
\[ \leq k \eta_m(v_0^2) \]

for all \( m \in \mathbb{N} \), where in the first inequality, we have used (74) and second follows from the definition of \( f \). Thus \( u = v_0^2 \) is a subsolution of (84) for each \( m \). In order to construct a positive supersolution to (84), we proceed by observing that we can find a positive constant \( C \) satisfying \( C \geq \max_{t \geq 0} f(t)^{-\alpha} \). It is possible because \( f \) is a bounded continuous and \( f(0) < 0 \). From the definition of \( g_m \), it is clear that \( g_m(t) \leq C \) for all \( t \geq 0 \). In view of the above observation, it is sufficient to construct a positive supersolution to the following problem:

\[
\begin{cases}
-\lambda M_{\lambda, A}(D^2v) + b(x)|Dv|^q \geq k(x)C & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega. 
\end{cases}
\]

Let us consider the function \( v = M\phi^\nu \), where \( 0 < \nu < 1 \) and \( \phi \) is the eigenfunction from Theorem 2.2 with \( \gamma = 0 \). Differentiating \( v \) twice, we get
\[ Dv = M\nu \phi^{\nu-1}D\phi, \]
\[ D^2v = M\nu(\nu - 1)\phi^{\nu-2}D\phi \otimes D\phi + M\nu^2 \phi^{\nu-1}D^2\phi. \]

On putting these values in (86), we find
\[ -\lambda M_{\lambda, A}(D^2v) + b(x)|Dv|^q \]
\[ = -\lambda M_{\lambda, A}(M\nu(\nu - 1)\phi^{\nu-2}D\phi \otimes D\phi + M\nu^2 \phi^{\nu-1}D^2\phi) + b(x)M^q \phi^{\theta \theta (\nu-1)}|D\phi|^q \]
\[ \geq -\lambda M_{\lambda, A}(M\nu(\nu - 1)\phi^{\nu-2}D\phi \otimes D\phi) - M_{\lambda, A}(M\nu^2 \phi^{\nu-1}D^2\phi) + b(x)M^q \phi^{\theta \theta (\nu-1)}|D\phi|^q \]
\[ = -\lambda M_{\lambda, A}(-M\nu(\nu - 1)\phi^{\nu-2}D\phi \otimes D\phi + M\nu^2 \phi^{\nu-1}[-M_{\lambda, A}(D^2\phi)]) + b(x)M^q \phi^{\theta \theta (\nu-1)}|D\phi|^q \]
\[ = M\nu(1 - \nu)\lambda \phi^{\nu-2}|D\phi|^2 + M\nu^2 \phi^{\nu-1}b(x)M^q \phi^{\theta \theta (\nu-1)}|D\phi|^q. \]

Now, using the same method as in the proof of Proposition 2, we can find sufficiently large value of \( M > 0 \) so that
Thus \( v = M\phi^s \) satisfies (86) and consequently \( v \) works as a supersolution to (84) for all \( m \in \mathbb{N} \). Now, by the choice of \( C \) above, we have
\[
g_m(u) \leq Ck(x),
\]
consequently,
\[
-M_{\lambda,\Lambda}(D^2u) + b(x)|Du|^q \leq k(x)g_m(u) \leq Ck(x) \leq -M_{\lambda,\Lambda}(D^2v) + b(x)|Dv|^q. (89)
\]
So by the comparison principle, we have \( u \leq v \). Proceeding as in the proof of Proposition 2, by using Lemma 5.1 (see appendix for details), for each \( m \in \mathbb{N} \), we find a solution, say, \( w_m \) to (84) satisfying
\[
u \leq w_m \leq v. \tag{90}
\]
Consequently, on each compact subdomains of \( \Omega \), we have an \( L^\infty \)-bound and a strictly positive lower bound for \( w_m \). Thus, by using the interior estimate (Theorem 2 [44]) and the stability properties for the viscosity solution (Theorem 4 [44]), \( w_m \) converges locally uniformly to a positive solution of (53) and this completes the proof. \( \square \)

5. Appendix.

**Proposition 3.** Assume that \( f \) satisfies the conditions of Theorem 1.1, then for any fixed \( \delta > 0 \), there is a positive constant \( C \) such that for any \( t_1, t_2 \in [0, \Gamma] \) with \( t_1 < t_2 \), we have
\[
\frac{k(x)f(t_1)}{(t_1 + \delta)^\alpha} + Ct_1 \leq \frac{k(x)f(t_2)}{(t_2 + \delta)^\alpha} + Ct_2.
\]

**Proof.** Let us consider
\[
\left| \frac{k(x)f(t_1)}{(t_1 + \delta)^\alpha} - \frac{k(x)f(t_2)}{(t_2 + \delta)^\alpha} \right| \leq \sup_{\Omega} k \left[ \frac{f(t_1)}{(t_1 + \delta)^\alpha} - \frac{f(t_2)}{(t_2 + \delta)^\alpha} \right] \\
\leq \sup_{\Omega} k \left[ \frac{f(t_1)}{(t_1 + \delta)^\alpha} - \frac{f(t_2)}{(t_2 + \delta)^\alpha} \right] + \frac{1}{(t_2 + \delta)^\alpha} \left| f(t_1) - f(t_2) \right| \\
\leq \sup_{\Omega} k \left[ f(0) \alpha \frac{\delta^{\alpha+1}}{t_1 - t_2} + \frac{1}{\delta^\alpha} L(f, \Gamma) \right] \left| t_1 - t_2 \right| \\
= C \left| t_1 - t_2 \right|,
\]
where we have used \( f(t_1) \leq f(0), C = \sup_{\Omega} k \left[ f(0) \alpha \frac{\delta^{\alpha+1}}{t_1 - t_2} + \frac{1}{\delta^\alpha} L(f, \Gamma) \right] \) and \( L(f, \Gamma) \) is the Lipschitz constant of \( f \) on \([0, \Gamma]\). Now, since \( t_1 < t_2 \), so
\[
\frac{k(x)f(t_1)}{(t_1 + \delta)^\alpha} - \frac{k(x)f(t_2)}{(t_2 + \delta)^\alpha} \leq C(t_2 - t_1),
\]
that is,
\[
\frac{k(x)f(t_1)}{(t_1 + \delta)^\alpha} + Ct_1 \leq \frac{k(x)f(t_2)}{(t_2 + \delta)^\alpha} + Ct_2.
\]
\( \square \)
Lemma 5.1. For any fixed \( m \in \mathbb{N} \), we can find a positive constant \( C = C(m, \rho_m, \rho'_m, \alpha) \) such that

\[
g_m(t) + Ct \text{ is increasing in } t \in [0, \Gamma],
\]

that is, for any \( t_1, t_2 \in [0, \Gamma] \) with \( t_1 < t_2 \), we have

\[
g_m(t_1) + Ct_1 \leq g_m(t_2) + Ct_2.
\]

Proof. For any fixed \( m \), \( g_m \) is Lipschitz continuous in \([\rho'_m, \Gamma]\) because \( 1/t^\alpha \) and \( f \) are Lipschitz continuous in \([\rho'_m, \Gamma]\). So there is a constant \( C_1 \) depending on \( \alpha, \rho'_m \) and Lipschitz constant of \( f \) in \([\rho'_m, \Gamma]\) such that for any \( t_1, t_2 \in [\rho'_m, \Gamma] \) with \( t_1 < t_2 \), the following holds:

\[
g_m(t_1) + C_1 t_1 \leq g_m(t_2) + C_1 t_2.
\]

Now, for any \( t_1, t_2 \in [0, \Gamma] \) with \( t_1 < t_2 \) there are three possibilities:

(i) \( t_1, t_2 \in [0, \rho'_m] \)
(ii) \( t_1, t_2 \in [\rho'_m, \Gamma] \)
(iii) \( t_1 \in [0, \rho'_m] \) and \( t_2 \in [\rho'_m, \Gamma] \).

In the first case, we have

\[
g_m(t_1) = g(\rho_m) = g_m(t_2),
\]

so for any \( C_2 > 0 \), we have

\[
g_m(t_1) + C_2 t_1 \leq g_m(t_2) + C_2 t_2.
\]

In the second case, we have positive constant \( C_1 \) from (93) so that (92) holds. In the case (iii), we further consider the following two subcases:

(iii.a) \( t_2 \in [\rho'_m, \rho_m] \)
(iii.b) \( t_2 \in [\rho_m, \Gamma] \).

In the case (iii)(a), by definition of \( g_m \), we have

\[
g(m)(t_2) \geq g_m(\rho_m) = g_m(t_1),
\]

so for any positive constant say \( C_3 \) (92) holds. In the case (iii)(b), we have

\[
|g_m(t_1) - g_m(t_2)| = |g_m(t_1) - g_m(\rho_m) + g_m(\rho_m) - g_m(t_2)|
\leq |g_m(t_1) - g_m(\rho_m)| + |g_m(\rho_m) - g_m(t_2)|
= |g_m(\rho_m) - g_m(t_2)| \quad \text{(since } g_m(t_1) = g_m(\rho_m))
\leq C_4 |\rho_m - t_2| \quad \text{(by the Lipschitz continuity of } g_m \text{ on } [\rho'_m, \Gamma])
= C_4 (t_2 - \rho_m) \quad \text{(since } t_2 \geq \rho_m)
\leq C_4 (t_2 - t_1) \quad \text{(since } t_1 \leq \rho_m).
\]

Thus in this case also, we have (92) with the constant \( C_4 \). So by taking \( C = \max\{C_1, C_2, C_3, C_4\} \), we have (92).

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