On the chromatic numbers of spheres in $\mathbb{R}^n$*

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1 Introduction

In this paper, we study a classical problem going back to H. Hadwiger, E. Nelson, and P. Erdős. Let $(X, \rho)$ be a metric space. Consider a set $\mathcal{A}$ of distinct positive reals. We call the value

$$\chi((X, \rho); \mathcal{A}) = \min \{ \chi : X = X_1 \cup \ldots \cup X_\chi, \forall i \forall x, y \in X_i \rho(x, y) \notin \mathcal{A} \}$$

the chromatic number of the space $(X, \rho)$ with the set of forbidden distances $\mathcal{A}$. In other words, $\chi((X, \rho); \mathcal{A})$ is the minimum number of colours needed to paint all the points in $X$ so that any two points at a distance from $\mathcal{A}$ apart receive different colours.

Various metric spaces and sets of forbidden distances have been considered by many authors. Let us briefly review the most important cases.

1. $(X, \rho) = (\mathbb{R}^n, \ell_2^2)$, $\mathcal{A} = \{1\}$. Here

$$\ell_2^2(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2},$$

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).$$

This is the classical case, which is deeply investigated. We use a simpler standard notation $\chi(\mathbb{R}^n)$ for the corresponding chromatic number. Numerous results concerning $\chi(\mathbb{R}^n)$ can be found in the books [1], [2] and surveys [3], [4]. For our further

*This work is done under the financial support of the following grants: the grant 09-01-00294 of Russian Foundation for Basic Research, the grant MD-8390.2010.1 of the Russian President, the grant NSh-691.2008.1 supporting Leading scientific schools of Russia, a grant of Dynastia foundation.
purposes, only the following bounds will be useful:

\[ \chi(\mathbb{R}^n) \geq (\zeta_1 + o(1))^n, \quad \zeta_1 = \frac{1 + \sqrt{2}}{2} = 1.207... \quad \text{(see [3])}, \]

\[ \chi(\mathbb{R}^n) \geq (\zeta_2 + o(1))^n, \quad \zeta_2 = 1.239... \quad \text{(see [6])}, \]

\[ \chi(\mathbb{R}^n) \leq (3 + o(1))^n \quad \text{(see [7])}. \]

2. \((X, \rho) = (\mathbb{R}^n, l_2), |A| = k, k \in \mathbb{N} \). Here the best known results are given in the paper [8].

3. \((X, \rho) = (\mathbb{R}^n, l_2), |A| = \infty \). Here the paper [9] should be cited.

4. \((X, \rho) = (\mathbb{R}^n, l_p), |A| = k, k \in \mathbb{N} \), where

\[ l_p(x, y) = \sqrt[p]{|x_1 - y_1|^p + \ldots + |x_n - y_n|^p}, \quad p \in [1, \infty), \]

\[ l_\infty(x, y) = \max_{i=1,\ldots,n} |x_i - y_i|. \]

These cases were studied in [10], [11], [12], [13].

5. \((X, \rho) = (\mathbb{Q}^n, l_p), |A| = k, k \in \mathbb{N} \). See [4], [13], [14], [15] for multiple references.

Another interesting series of metric spaces is generated by spheres \(S_{r-1}^n\) of radii \(r \geq \frac{1}{2}\) in \(\mathbb{R}^n\): \((X, \rho) = (S_{r-1}^n, l_2), A = \{1\}\). Studying

\[ \chi(S_{r-1}^n) = \chi((S_{r-1}^n, l_2); \{1\}) \]

was proposed by Erdős who conjectured in [16] that \(\chi(S_{r-1}^n) \to \infty\) for any fixed value of \(r > \frac{1}{2}\). It is obvious that \(\chi(S_{1/2}^{n-1}) = 2\), and L. Lovász proved Erdős’ conjecture in [17] using topological tools (see also [18]). The exact assertion of Lovász is as follows:

for any \(r > \frac{1}{2}\) and \(n \in \mathbb{N}\), the inequality holds \(\chi(S_{r}^{n-1}) \geq n\); if \(r < \sqrt{\frac{n}{2n+2}} \sim \frac{1}{\sqrt{2}}\), i.e.,

the length of any side of a regular \(n\)-simplex inscribed into \(S_{r}^{n-1}\) is smaller than 1, then \(\chi(S_{r}^{n-1}) \leq n + 1\). Although this result is widely cited (see, e.g., [3]), its second part is completely wrong (see Section 5). Actually, for every \(r > \frac{1}{2}\), the quantity \(\chi(S_{r}^{n-1})\) grows exponentially, not linearly.
In this paper, we will do a careful analysis of the asymptotic behaviour of the value $\chi(S^{n-1}_r)$. We will study even some cases when $r$ may depend on $n$.

Before proceeding to formulating our main results, let us mention some more papers on the chromatic numbers of spheres: [19], [20].

### 2 Statements of the main results

The starting point for our investigation is the following assertion.

**Theorem 1.** For any $r > \frac{1}{2}$, there exist a constant $\gamma = \gamma(r) > 1$ and a function $\varphi(n) = \varphi(n, r) = o(1)$, $n \to \infty$, such that for every $n \in \mathbb{N}$, the inequality holds

$$\chi(S^{n-1}_r) \geq (\gamma + \varphi(n))^n.$$

Theorem 1 sais that, for any fixed radius, the chromatic number grows exponentially in the dimension. Of course it is possible to make the value of $\gamma(r)$ a bit more concrete. The first step in this direction is given in Theorem 2.

**Theorem 2.** For any $r \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, there exists a function $\delta(n) = \delta(n, r) = o(1)$, $n \to \infty$, such that for every $n \in \mathbb{N}$, the inequality holds

$$\chi(S^{n-1}_r) \geq 2 \left(\frac{1}{8r^2}\right)^\frac{1}{8r^2} \left(1 \frac{1}{8r^2}\right)^{1-\frac{1}{8r^2}} + \delta(n)^n.$$

Looking at Theorem 2, we see that if $r$ becomes closer and closer to $\frac{1}{\sqrt{2}}$, then the constant

$$\gamma = 2 \left(\frac{1}{8r^2}\right)^\frac{1}{8r^2} \left(1 \frac{1}{8r^2}\right)^{1-\frac{1}{8r^2}}$$

approaches the value $\zeta_3 = 1.139...$. Since $S^{n-1}_r \subset \mathbb{R}^n$ leading to $\chi(S^{n-1}_r) \leq \chi(\mathbb{R}^n)$, one may not expect that $\zeta_3$ could be somehow replaced by anything greater than $\zeta_2$ (cf. Introduction). However, there is some room to spare here, and in Section 7 we will exhibit a further optimization process providing even larger constants.
At the same time, if \( r \geq \frac{1}{\sqrt{2}} \), then we certainly have

\[
\chi(S^n_r) \geq \chi(S^{n-1}_{r'}) \geq (1.139 + o(1))^n, \quad r' < \frac{1}{\sqrt{2}} \leq r.
\]

So, once again, for any fixed value of radius, the chromatic number is essentially exponential in \( n \). Comparing our results with those due to Lovász, we get the following assertion.

**Theorem 3.** For any \( r > \frac{1}{2} \), there exists an \( n_0 \) such that for every \( n \geq n_0 \), \( \chi(S^n_{r-1}) > n + 1 \).

On the one hand, Theorem 3 shows that the bound \( \chi(S^n_{r-1}) \leq n + 1 \) is false, provided we fix \( r \) and let \( n \) go to infinity. On the other hand, the result of Theorem 3 is much stronger than that of Lovász only for the values of \( n \) which are big enough. So in small dimensions, the lower estimate \( \chi(S^n_{r-1}) \geq n + 1 \) is still the best known (and true).

The gap between exponents and linear functions is quite large. Thus, one may expect that superlinear lower bounds for \( \chi(S^n_{r-1}) \) would be possible not only for a constant \( r > \frac{1}{2} \), but also for some sequences \( r_n \to \frac{1}{2} \). The most general assertion of this kind is in Theorem 4.

**Theorem 4.** Let \( \mathbb{P} \) be the set of prime numbers. Let \( f(x) \) be such a function that for any \( x \in \mathbb{R}, x \geq 0 \),

\[
x + f(x) = \min\{p \in \mathbb{P}: p > x\}.
\]

Let

\[
m(x) = \max\{m < x: m \equiv 0 \pmod{4}\}.
\]

Consider a sequence \( \{r_n\}_{n=1}^{\infty} \), where \( r_n > \frac{1}{2} \) for each \( n \in \mathbb{N} \). Set

\[
p(n) = \frac{m(n)}{8r^2_n} + f\left(\frac{m(n)}{8r^2_n}\right).
\]

If

\[
\frac{m(n)}{4} < p(n) \leq \frac{m(n)}{2}, \quad n \in \mathbb{N},
\]

then,

\[
\chi(S^n_{r_n-1}) \geq \frac{C^{m(n)/2}_{m(n)}}{C^{p(n)}_{m(n)}}.
\]
Translating Theorem 4 into a form of Theorem 3, we get

**Theorem 5.** Consider a sequence \( \{r_n\}_{n=1}^{\infty} \), where \( r_n > \frac{1}{2} \) for each \( n \in \mathbb{N} \). Let \( p(n) \) be the same as in Theorem 4. If
\[
\frac{m(n)}{4} < p(n) < \frac{m(n)}{2} - \sqrt{\frac{m(n) \ln m(n)}{\kappa}}, \quad \kappa < 2, \quad n \in \mathbb{N},
\]
then,
\[
\chi(S_{r_n}^{n-1}) > n + 1, \quad \forall \ n \geq n_0.
\]

The quality of Theorem 5 depends on the estimates for the function \( f(x) \). Determining the exact asymptotic behaviour of \( f(x) \) is a very hard problem of analytical number theory (see [21]). As far as we know, the best upper estimate is \( f(x) = O \left( x^{0.525-\varepsilon} \right) \) with a so small \( \varepsilon > 0 \) that the authors did not care of it (see [22]). However, it is conjectured that \( f(x) = O(\ln x) \) (see [23]). The tightest lower bound is given in [24] and [25], but it is sublogarithmic and apparently far enough from the truth. Using this information, we may derive

**Theorem 6.** Assume that \( c_0 > 0 \) is such that \( f(x) \leq c_0 x^{0.525} \) for every \( x \). Then, there exists a constant \( c'_0 > 0 \) such that for any sequence of radii \( r_n \) satisfying the inequality
\[
r_n \geq \frac{1}{2} + \frac{c'_0}{n^{0.475}},
\]
we have the bound
\[
\chi(S_{r_n}^{n-1}) > n + 1, \quad \forall \ n \geq n_0.
\]

**Theorem 7.** Assume that \( c_1 > 0 \) is such that \( f(x) \leq c_1 \ln x \) for every \( x \). Then, there exists a constant \( c'_1 > 0 \) such that for any sequence of radii \( r_n \) satisfying the inequality
\[
r_n \geq \frac{1}{2} + c'_1 \sqrt{\frac{\ln n}{n}},
\]
we have the bound
\[
\chi(S_{r_n}^{n-1}) > n + 1, \quad \forall \ n \geq n_0.
\]
So \( r_n > \frac{1}{2} \) may be quite close to the value \( \frac{1}{2} \), and, nevertheless, the chromatic numbers will exceed the Lovász upper estimate. Finally, it is of interest for which sequences of \( r_n \), we do really have the bound \( \chi(S^{n-1}_{r_n}) \leq n + 1 \).

**Theorem 8.** There exists a constant \( c_2 > 0 \) such that for any sequence of radii \( r_n \) satisfying the inequality

\[
r_n \leq \frac{1}{2} + \frac{c_2}{n},
\]

we have the bound

\[
\chi(S^{n-1}_{r_n}) \leq n + 1, \quad \forall \ n \geq n_0.
\]

Further structure of the paper is as follows. In Section 3, we shall give proofs for Theorems 1 – 4. Section 4 will be devoted to proving Theorems 5 – 7. In Section 5, we shall discuss Theorem 8. In Section 6, some more comments and suggestions will be given. In particular, we shall exhibit more general upper estimates for \( \chi(S^{n-1}_{r_n}) \) than those in Theorem 8. In Section 7, we shall present a general scheme for obtaining better (and, in some sense, optimal) constants \( \gamma \) than those appearing in Theorems 1 and 2.

### 3 Proofs of Theorems 1 – 4

Among Theorems 1 – 3, Theorem 2 covers both Theorem 1 and Theorem 3. So we start by proving Theorem 2.

Fix an \( r \in \left( \frac{1}{2}, \frac{1}{\sqrt{2}} \right) \) and an \( n \in \mathbb{N} \). Let \( m < n \) be the maximum natural number which is divisible by 4. Let us find \( a' \) from the relation

\[
\frac{\sqrt{m}}{\sqrt{2m-2a'}} = r, \quad \text{i.e.,} \quad a' = \frac{m(2r^2 - 1)}{2r^2}.
\]

Let \( p \) be the smallest prime number satisfying the inequality

\[
p > \frac{m - a'}{4} = \frac{m}{8r^2}.
\]
Set 

\[ a = m - 4p < a'. \]

Consider the following graph \( G = (V, E) \):

\[
V = \left\{ \mathbf{x} = (x_1, \ldots, x_m) : x_i \in \left\{ \frac{1}{\sqrt{2m - 2a}}, \frac{1}{\sqrt{2m - 2a}} \right\}, \ x_1 + \ldots + x_m = 0 \right\},
\]

\[
E = \left\{ \{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in V, \ l_2(\mathbf{x}, \mathbf{y}) = 1 \right\}.
\]

Obviously \( V \subset S_{r'}^{m-1} \), where

\[
r' = \frac{\sqrt{m}}{\sqrt{2m - 2a}} < \frac{\sqrt{m}}{\sqrt{2m - 2a'}} = r.
\]

If we use the standard notation \( \chi(G) \) for the chromatic number of \( G \) and \( \alpha(G) \) for its independence number, then we get

\[
\chi(S_{r'}^{m-1}) \geq \chi(S_{r}^{m-1}) \geq \chi(G) \geq \frac{|V|}{\alpha(G)} = \frac{C_m^{m/2}}{\alpha(G)}.
\]

So we are led to estimate \( \alpha(G) \) from above. It is convenient to transform \( G = (V, E) \) into an \( H = (W, F) \):

\[
W = \left\{ \mathbf{x} \cdot \sqrt{2m - 2a} : \mathbf{x} \in V \right\}, \quad F = \left\{ \{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in W, \ l_2(\mathbf{x}, \mathbf{y}) = \sqrt{2m - 2a} \right\}.
\]

Let us denote by \((\mathbf{x}, \mathbf{y})\) the Euclidean scalar product of \(\mathbf{x}\) and \(\mathbf{y}\). Since for any \(\mathbf{x} \in W\), 
\((\mathbf{x}, \mathbf{x}) = m\), we may rewrite \(F\) as follows:

\[
F = \left\{ \{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in W, \ (\mathbf{x}, \mathbf{y}) = a \right\}.
\]

Notice that for \(\mathbf{x}, \mathbf{y} \in W\), the quantity \((\mathbf{x}, \mathbf{y})\) lies in the interval \([-m, m]\) and is congruent to zero modulo 4. The last observation is due to the fact that \(m \equiv 0 \pmod{4}\) and every vector \(\mathbf{x} \in W\) contains an even number of negative coordinates. Also,

\[
m - 8p < m - \frac{m}{r^2} < -m.
\]

Thus, for every \(\mathbf{x}, \mathbf{y} \in W\),

\[
(\mathbf{x}, \mathbf{y}) \equiv m \pmod{p} \iff (\mathbf{x}, \mathbf{y}) = m \text{ or } (\mathbf{x}, \mathbf{y}) = a. \tag{1}
\]
Now, we are about to prove that $\alpha(G) = \alpha(H) \leq C_m^p$. Take an arbitrary
\[ Q = \{ x_1, \ldots, x_s \} \subset W, \quad \forall i \forall j, \quad (x_i, x_j) \neq a. \quad (2) \]

In other words, $Q$ is an independent set in $H$. We have to show that $s \leq C_m^p$. For this purpose, we use the linear algebra method (see [5], [26], [27], [28]).

To each vector $x \in W$ we assign a polynomial $P_x \in \mathbb{Z}/p\mathbb{Z}[y_1, \ldots, y_m]$. First, we take
\[
P_x'(y) = \prod_{i \in I} (i - (x, y)),
\]
where
\[ I = \{0, 1, \ldots, p - 1\} \setminus \{m \pmod{p}\}, \quad y = (y_1, \ldots, y_m), \]
and so $P_x' \in \mathbb{Z}/p\mathbb{Z}[y_1, \ldots, y_m]$. Obviously,
\[
\forall x, y \in W \quad P_x'(y) \equiv 0 \pmod{p} \iff (x, y) \neq m \pmod{p}. \quad (3)
\]

Second, we represent $P_x'$ as a sum of monomials. If a monomial has the form
\[
y_{i_1}^{\alpha_{i_1}} \cdots y_{i_q}^{\alpha_{i_q}}, \quad \alpha_{i_1} > 0, \ldots, \alpha_{i_q} > 0,
\]
then we replace it by
\[
y_{i_1}^{\beta_{i_1}} \cdots y_{i_q}^{\beta_{i_q}},
\]
where $\beta_{i_\nu} = 1$, provided $\alpha_{i_\nu}$ is odd, and $\beta_{i_\nu} = 0$, provided $\alpha_{i_\nu}$ is even. Eventually, we get a polynomial $P_x$. It is worth noting that this polynomial does also satisfy property (3).

It follows from properties (1), (2), and (3) that the polynomials
\[ P_{x_1}, \ldots, P_{x_s} \]
assigned to the vectors of the set $Q$ are linearly independent over $\mathbb{Z}/p\mathbb{Z}$. It is also easy to see that the dimension of the space generated by
\[ P_{x_1}, \ldots, P_{x_s} \]
does not exceed $C_m^p$. Thus, $s = |Q| \leq C_m^p$ and, therefore,
\[ \chi(S_r^{m-1}) \geq \frac{C_m^{m/2}}{C_p^m}. \]
Standard analytical tools (like Stirling’s formula) together with \( p \sim \frac{m}{8r^2} \) give us, finally, the expected bound

\[
\chi(S_r^{n-1}) \geq \left( 2 \left( \frac{1}{8r^2} \right)^{\frac{1}{8r^2}} \left( 1 - \frac{1}{8r^2} \right)^{1 - \frac{1}{8r^2}} + \delta(n) \right)^n,
\]

which completes the proof of Theorems 1 – 3.

The proof of Theorem 4 is now clear. We just reproduce the above argument with \( r_n \) instead of \( r \). The only thing one has to explain here is why we impose additional conditions on the value of a prime. Indeed, the inequality \( p(n) > \frac{m(n)}{4} \) is quite important, since property (1) becomes false without it. As for the inequality \( p(n) \leq \frac{m(n)}{2} \), it is necessary to correctly estimate the independence number of our graph \( G \) by the quantity \( C_m^p \). Moreover, \( \chi(G) = 1 \), provided \( p(n) > \frac{m(n)}{2} \), and the result is trivial. Theorem 4 is proved.

4 Proofs of Theorems 5 – 7

4.1 Proof of Theorem 5

Set \( m = m(n) \), \( p = p(n) \). Since the function \( \frac{C_m^{m/2}}{C_m^p} \) is decreasing in \( p \), we just have to show that for

\[
p = \left[ \frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}} \right],
\]

the inequality \( \frac{C_m^{m/2}}{C_m^p} > n + 1 \) is true for large values of \( n \). We have

\[
\frac{C_m^{m/2}}{C_m^p} = \left( \frac{m}{2} + 1 \right) \cdot \left( \frac{m}{2} + 2 \right) \cdot \ldots \cdot \left( \frac{m}{2} + \left( \frac{m}{2} - p \right) \right) = \left( 1 + \frac{2}{m} \right) \cdot \left( 1 + \frac{4}{m} \right) \cdot \ldots \cdot \left( 1 + \frac{m-2p}{m} \right) \sim e^{(m-2p)^2/2m} \geq e^{2\ln m/\kappa} = m^{2\kappa}.
\]

By a condition of Theorem 5, \( \kappa < 2 \). Thus,

\[
m^{2\kappa}(1 + o(1)) > n + 1, \quad \forall n \geq n_0.
\]

Theorem 5 is proved.
We just have to show that for our choice of \( r_n \),

\[
p = \frac{m}{8r_n^2} + f \left( \frac{m}{8r_n^2} \right) < \frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}},
\]

provided \( \kappa < 2 \) is a constant and \( n \) is large enough.

Indeed, assume that \( c'_0 \) is large (say, \( c'_0 > c_0 \)). Then,

\[
p \leq \frac{m}{8 \left( \frac{1}{2} + \frac{c'_0}{n^{0.475}} \right)^2} + c_0 \left( \frac{m}{8 \left( \frac{1}{2} + \frac{c'_0}{n^{0.475}} \right)^2} \right)^{0.525} < \frac{m}{8 \left( \frac{1}{4} + \frac{c'_0}{n^{0.475}} \right)} + c_0 \left( \frac{m}{8 \left( \frac{1}{4} + \frac{c'_0}{n^{0.475}} \right)} \right)^{0.525} = \frac{m}{2} \left( 1 - \frac{4c'_0}{n^{0.475}} + o \left( \frac{1}{\sqrt{n}} \right) \right) + c_0 \left( \frac{m}{2} \left( 1 - \frac{4c'_0}{n^{0.475}} + o \left( \frac{1}{\sqrt{n}} \right) \right) \right)^{0.525}.
\]

For any sufficiently large value of \( n \), the last quantity is bounded from above by

\[
\frac{m}{2} - c'_0 m^{0.525} + c_0 m^{0.525} = \frac{m}{2} - c''_0 m^{0.525}, \quad c''_0 > 0.
\]

Obviously, for any \( n \geq n_0 \),

\[
\frac{m}{2} - c''_0 m^{0.525} < \frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}}.
\]

Theorem 6 is proved.

### 4.3 Proof of Theorem 7

Let us briefly write down a series of inequalities similar to those in 4.2:

\[
p \leq \frac{m}{8 \left( \frac{1}{2} + c'_1 \sqrt{\ln n / n} \right)^2} + c_1 \ln \left( \frac{m}{8 \left( \frac{1}{2} + c'_1 \sqrt{\ln n / n} \right)^2} \right) < \frac{m}{2} \left( 1 - 4c'_1 \sqrt{\ln n / n} + o \left( \frac{1}{n^{3/2}} \right) \right) + c_1 \ln \left( \frac{m}{2} \left( 1 - 4c'_1 \sqrt{\ln n / n} + o \left( \frac{1}{n^{3/2}} \right) \right) \right) < \frac{m}{2} - c'_1 \sqrt{m \ln m},
\]

and we are done.
5 Proof of Theorem 8

Let us take $S_{1/2}^{n-1}$ and divide it into $n+1$ parts of smallest possible diameters. To this end, we inscribe a regular $n$-simplex $\Delta^n$ into $S_{1/2}^{n-1}$ and consider multidimensional polyhedral cones $C_1, \ldots, C_{n+1}$ with common vertex at the center of $S_{1/2}^{n-1}$ and coming through the $(n-1)$-faces of $\Delta^n$. Obviously,

$$S_{1/2}^{n-1} = (S_{1/2}^{n-1} \cap C_1) \cup \ldots \cup (S_{1/2}^{n-1} \cap C_{n+1}). \quad (4)$$

In principle, it is a good exercise in multidimensional geometry to prove that for any $i$,

$$\operatorname{diam} (S_{1/2}^{n-1} \cap C_i) = 1 - \Theta \left( \frac{1}{n} \right).$$

It follows immediately from this observation that we may inflate $S_{1/2}^{n-1}$ at most

$$\frac{1}{1 - \Theta \left( \frac{1}{n} \right)} = 1 + \Theta \left( \frac{1}{n} \right)$$

times in order to get a partition of the resulting sphere into parts of diameter not exceeding 1. Thus, for a constant $c_2 > 0$, we have an appropriate coloring of $S_{1/2+c_2/n}^{n-1}$, which completes the proof of Theorem 8.

Apparently, in [17], the same construction was proposed. However, the author assumed that the diameter of any part in the corresponding partition is attained on the sides of a regular $n$-simplex $\Delta^n$. This is true only for $n=2$. Already in $\mathbb{R}^3$, the diameter of a part is $\sqrt{\frac{3+\sqrt{3}}{6}} = 0.888\ldots$, which is not the length of a side of a tetrahedron inscribed into $S_{1/2}^2$.

6 Comments and upper bounds

First of all, it is worth noting that there is still a certain gap between the estimates

$$r_n \geq \frac{1}{2} + c'_1 \sqrt{\frac{\ln n}{n}} \quad (5)$$

and

$$r_n \leq \frac{1}{2} + \frac{c_2}{n}. \quad (6)$$
Removing this gap could be a good problem. As for (6), it cannot be enlarged by any refinement of the techniques of the previous section. The point is that the partition (4) is best possible: for any other decomposition of $S_{1/2}^{n-1}$ into $n + 1$ parts, there exists a part whose diameter is not less than each of the diameters $\text{diam} (S_{1/2}^{n-1} \cap C_i)$. Of course it is not necessary to divide a sphere into parts with diameters strictly smaller than 1; we just need to cut it in such a way that no part would contain a pair of points at the unit distance. However, we do not know such a partition. Perhaps it is easier to improve (5). One should combine linear algebra of Section 3 with some additional ideas.

Let us say a few words about general upper estimates for $\chi(S_{r_n}^{n-1})$. The simplest observation here is that

$$\chi(S_{r_n}^{n-1}) \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n \quad (\text{cf. Introduction}).$$

Thus, for constant values $r > \frac{1}{2}$ (as in Theorems 1 – 3), we already get the order of magnitude for any quantity $\log \chi(S_{r_n}^{n-1})$.

In [29], C.A. Rogers proved that any sphere of radius $r$ in $\mathbb{R}^n$ can be covered by

$$\left(\frac{r}{\rho} + o(1)\right)^n$$

spheres of radius $\rho < r$. In our case, this means that

$$\chi(S_{r_n}^{n-1}) \leq (2r_n + o(1))^n.$$

If $r_n < 3/2$, then this bound is better than that in (7).

More precisely, Rogers’ estimate is as follows: there is an absolute constant $c > 0$ such that, if $r > \frac{1}{2}$ and $n \geq 9$, any $n$-dimensional spheres of radius $r$ can be covered by less than $cn^{5/2}(2r)^n$ spheres of radius $\frac{1}{2}$. A so precise formulation is useless when $r$ is a constant, but coming again to $r_n \to \frac{1}{2}$ we may carefully apply this statement in order to obtain upper bounds like

$$\chi(S_{r_n}^{n-1}) \leq 2cn^{5/2}(2r_n)^n = \Theta\left(n^{5/2}(2r_n)^n\right).$$

Here the factor 2 is due to the fact that $\chi(S_{1/2}^{n-1}) = 2$. One should not forget that if, for example, $r_n = \frac{1}{2} + \Theta\left(\frac{1}{n}\right)$, then $(2r_n)^n = \Theta(1)$, so that estimate (8) is very good.

It is possible to evaluate even more sophisticated bounds for $\chi(S_{r_n}^{n-1})$, but this is not so interesting.
7 A possible way for improving Theorem 2

7.1 Statements of the results

Fix again an \( r > \frac{1}{2} \). Let \( m = m(n) < n \) for every \( n \) and \( m \sim n \) for \( n \to \infty \). Assume that \( t = t(n) \in \mathbb{N} \),

\[
b_1 = b_1(n) \in \mathbb{Z}, \ldots, \ b_t = b_t(n) \in \mathbb{Z},
\]

\[
l_1 = l_1(n) \in \mathbb{N}, \ldots, \ l_t = l_t(n) \in \mathbb{N}, \ l_1 + \ldots + l_t = m.
\]

Consider

\[
V = V(n) = \{x = (x_1, \ldots, x_m): x_i \in \{b_1, \ldots, b_t\}, |\{i: x_i = b_j\}| = l_j, \ j = 1, \ldots, t\}.
\]

Let \( d = d(n) \) be the maximum natural number such that for any \( x, y \in V \), we have \( (x, y) \equiv 0 \pmod{d} \). Note that \( V \) is an obvious analog of the set \( W \) from Section 3, where \( d \) was equal to 4. Set

\[
\bar{s} = \bar{s}(n) = \max_{x, y \in V} (x, y), \ s = s(n) = \min_{x, y \in V} (x, y).
\]

Find \( a' = a'(n) \) from the relation

\[
\frac{\sqrt{s}}{\sqrt{2\bar{s} - 2a'}} = r.
\]

Define \( p = p(n) \) as the minimum prime number satisfying the inequality

\[
p > \frac{s - a'}{d}.
\]

Finally, we choose \( a = a(n) \) from the condition

\[
p = \frac{\bar{s} - a}{d}, \ \text{i.e.}, \ a = \bar{s} - dp < a'.
\]

We get the following theorem.

**Theorem 9.** If \( a > s \) and \( \bar{s} - 2dp < s \), then

\[
\chi(S_r^{m-1}) \geq \frac{L}{M},
\]

where

\[
L = \frac{m!}{l_1! \cdot \ldots \cdot l_t!}, \ M = \sum_{(s_1, \ldots, s_t) \in A} \frac{m!}{s_1! \cdot \ldots \cdot s_t!}.
\]
In Theorem 9 we optimize over the parameters $t, b_1, \ldots, b_t$, and $l_1, \ldots, l_t$. This optimization can be a bit simpler, provided we suppose that $l_i \sim l_i^0 n$, where $l_i^0 \in (0, 1)$. Actually this does not substantially change results. In our case, we get

**Corollary.** The estimate holds

$$
\chi(S_r^{m-1}) \geq \left( \frac{L_0}{M_0} + o(1) \right)^n,
$$

where

$$
L_0 = e^{-l_1^0 \ln l_1^0 - \ldots - l_t^0 \ln l_t^0}, \quad M_0 = \max_{(s_1^0, \ldots, s_t^0) \in A_0} e^{-s_1^0 \ln s_1^0 - \ldots - s_t^0 \ln s_t^0},
$$

$$
A_0 = \left\{ (s_1^0, \ldots, s_t^0) : s_i^0 \in (0, 1), s_1^0 + \ldots + s_t^0 = 1, s_1^0 + 2s_2^0 + \ldots + (t - 1)s_{t-1}^0 \leq \frac{p}{n} \right\}.
$$

We shall prove Theorem 9 in §7.2. Corollary can be easily derived from Theorem 9 using Stirling’s formula and other standard tools of analysis.

In this paper, we shall not evaluate optimization from Corollary. Here we only cite the papers [8], [13], in which similar optimization procedures were carefully realized.

### 7.2 Proof of Theorem 9

Let us start by noting that all the parameters in Theorem 9 are chosen to generalize the approach that we used in Section 3. We have already mentioned that the quantity $d$ plays the role of the number 4 in the corresponding argument. Almost all the other notations are also completely parallel to those appearing in Section 3. Here only $m$ should be replaced by $\bar{s}$, and we just consider $V$ as an analog to $W$, without introducing two similar sets $V$ and $W$ as it was done in Section 3.

Set $G = (V, E)$ with

$$
E = \left\{ \{x, y\} : x, y \in V, (x, y) = a \right\}.
$$
We think it is now obvious that
\[
\chi(S_r^{n-1}) \geq \chi(G) \geq \frac{|V|}{\alpha(G)} = \frac{L}{\alpha(G)}.
\]
So it remains to prove that \(\alpha(G) \leq M\). This is done by the same linear algebra method as in Section 3.

Indeed, by the conditions of Theorem 9, we have, for every \(x, y \in V\),
\[
(x, y) \equiv s \pmod{p} \iff (x, y) = s \text{ or } (x, y) = a.
\]
(1')

Take an arbitrary
\[
Q = \{x_1, \ldots, x_s\} \subset V, \quad \forall \; i \forall \; j, \; (x_i, x_j) \neq a.
\]
(2')

We are about to show that \(s \leq M\).

To each vector \(x \in V\) we assign a polynomial \(P_x \in \mathbb{Z}/p\mathbb{Z}[y_1, \ldots, y_m]\). First, we take
\[
P'_x(y) = \prod_{i \in I} (i - (x, y)),
\]
where
\[
I = \{0, 1, \ldots, p - 1\} \setminus \{s \pmod{p}\}, \quad y = (y_1, \ldots, y_m),
\]
and so \(P'_x \in \mathbb{Z}/p\mathbb{Z}[y_1, \ldots, y_m]\). Obviously,
\[
\forall \; x, y \in W \quad P'_x(y) \equiv 0 \pmod{p} \iff (x, y) \neq s \pmod{p}.
\]
(3')

Second, we represent \(P'_x\) as a sum of monomials. We use the fact that
\[
(y_i - b_1) \cdot (y_i - b_2) \cdot \ldots \cdot (y_i - b_t) = 0,
\]
for any \(y \in V\). So we get a polynomial \(P_x\) of degree \(< t\). It is worth noting that this polynomial does also satisfy property (3').

It follows from properties (1'), (2'), and (3') that the polynomials
\[
P_{x_1}, \ldots, P_{x_s}
\]
assigned to the vectors of the set \(Q\) are linearly independent over \(\mathbb{Z}/p\mathbb{Z}\). Now it is easy to see that the dimension of the space generated by
\[
P_{x_1}, \ldots, P_{x_s}
\]
does not exceed \(M\). Thus, \(s = |Q| \leq M\) and, therefore, Theorem 9 is proved.
References

[1] A. Soifer, *The Mathematical Coloring Book*, Springer, 2009.

[2] P. Brass, W. Moser, J. Pach, *Research problems in discrete geometry*, Springer, 2005.

[3] L.A. Székely, *Erdős on unit distances and the Szemerédi - Trotter theorems*, Paul Erdős and his Mathematics, Bolyai Series Budapest, J. Bolyai Math. Soc., Springer, 11 (2002), 649 - 666.

[4] A.M. Raigorodskii, *The Borsuk problem and the chromatic numbers of some metric spaces*, Uspekhi Mat. Nauk, 56 (2001), N1, 107 - 146; English transl. in Russian Math. Surveys, 56 (2001), N1, 103 - 139.

[5] P. Frankl, R.M. Wilson, *Intersection theorems with geometric consequences*, Combinatorica, 1 (1981), 357 - 368.

[6] A.M. Raigorodskii, *On the chromatic number of a space*, Uspekhi Mat. Nauk, 55 (2000), N2, 147 - 148; English transl. in Russian Math. Surveys, 55 (2000), N2, 351 - 352.

[7] D.G. Larman, C.A. Rogers, *The realization of distances within sets in Euclidean space*, Mathematika, 19 (1972), 1 - 24.

[8] E.S. Gorskaya, I.M. Mitricheva, V.Yu. Protasov, A.M. Raigorodskii, *Estimating the chromatic numbers of Euclidean spaces by methods of convex minimization*, Mat. Sbornik, 200 (2009), N6, 3 - 22; English transl. in Sbornik Math., 200 (2009), N6, 783 - 801.

[9] N.G. Moshchevitin, A.M. Raigorodskii, *On colouring the space $\mathbb{R}^n$ with several forbidden distances*, Mat. Zametki, 81 (2007), N5, 733 - 744; English transl. in Math. Notes, 81 (2007), N5, 656 - 664.

[10] J.-H. Kang, Z. Füredi, *Distance graphs on $\mathbb{Z}^n$ with $l_1$-norm*, Theoretical Comp. Sci. 319 (2004), N1 - 3, 357 - 366.
[11] A.M. Raigorodskii, *On the chromatic number of a space with l_q - norm*, Uspekhi Mat. Nauk, 59 (2004), N5, 161 - 162; English transl. in Russian Math. Surveys, 59 (2004), N5, 973 - 975.

[12] A.M. Raigorodskii, M.I. Absalyamova, *A lower bound for the chromatic number of the space R^n with k forbidden distances and metric l_1*, Chebyshev Sbornik, 7 (2006), N4 (20), 105 - 113 (in Russian).

[13] A.M. Raigorodskii, I.M. Shitova, *On the chromatic numbers of real and rational spaces with several real or rational forbidden distances*, Mat. Sbornik, 199 (2008), N4, 107 - 142; English transl. in Sbornik Math., 199 (2008), N 4, 579 - 612.

[14] D.R. Woodall, *Distances realized by sets covering the plane*, J. Combin. Th. (A), 14 (1973), 187 - 200.

[15] M. Benda, M. Perles, *Colorings of metric spaces*, Geombinatorics, 9 (2000), 113 - 126.

[16] P. Erdős, R.L. Graham, *Problem proposed at the 6th Hungarian combinatorial conference*, Eger, July 1981.

[17] L. Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math., 45 (1983), 317 - 323.

[18] J. Matousek, *Using Borsuk – Ulam Theorem*, Springer, 2003.

[19] G.J. Simmons, *On a problem of Erdős concerning 3-colouring of the unit sphere*, Discrete Math, 8 (1974), 81 - 84.

[20] G.J. Simmons, *The chromatic number of the sphere*, J. Austral. Math. Soc. Ser, 21 (1976), 473 - 480.

[21] P. Erdős, *Some unsolved problems*, Magyar Tud. Akad. Mat. Kutató Int. Közl., 6 (1961), 221 - 254.
[22] R.C. Baker, G. Harman, J. Pintz, *The difference between consecutive primes, II*, Proceedings of the London Mathematical Society, 83 (2001), 532 - 562.

[23] H. Cramér, *On the order of magnitude of the difference between consecutive prime numbers*, Acta Arithmetica, 2 (1936), 23 - 46.

[24] A. Schönhage, *Eine Bemerkung zur Konstruktion grosser Primzahllucken*, Archiv der Math., 14 (1963), 29 - 30.

[25] R.A. Rankin, *The difference between consecutive prime numbers*, V Proc. Edinburgh. Math. Soc, 13 (1962-1963), 331 - 332.

[26] N. Alon, L. Babai, H. Suzuki, *Multilinear polynomials and Frankl - Ray-Chaudhuri - Wilson type intersection theorems*, J. Comb. Th., Ser. A, 58 (1991), 165 - 180.

[27] L. Babai, P. Frankl, *Linear algebra methods in combinatorics*, Part 1, Department of Computer Science, The University of Chicago, Preliminary version 2, September 1992.

[28] A.M. Raigorodskii, *The linear algebra method in combinatorics*, Moscow Centre for Continuous Mathematical Education (MCCME), Moscow, Russia, 2007 (book in Russian).

[29] C.A. Rogers, *Covering a sphere with spheres*, Mathematika, 10 (1963), 157 - 164.