ABSTRACT

We give an expanded discussion of the proposal that spacetime super-symmetry representations may be viewed as having their origins in 1D theories that involve a special class of real Clifford algebras. These 1D theories reproduce the supersymmetric structures of spacetime supersymmetric theories after the latter are reduced on a 0-brane. This leads us to propose that spacetime appears as a bundle in KO-theory.

PACS: 04.65.+e, 11.15.-q, 11.25.-w, 12.60.J

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1 Introduction

In previous work on the subject of 1D, arbitrarily large $N$-extended supersymmetry [1, 2], it was shown how to reformulate the usual notion of supersymmetry transformations on an NSR $D0$-brane, defined through the action of a set of derivatives, in terms of an algebraic structure dubbed the “general real algebra of extension $N$ and dimension $d$,” or $\mathcal{GR}(d, N)$ for short. In many ways the matrices that are used to realize these constructs behave as real-valued Euclidean space analogues of the usual Pauli matrices and hence our assigned moniker. The spinning particle is a unique supersymmetrical system at this time, in that we have complete knowledge of its off-shell representation theory, i.e., the auxiliary fields are known for all $N$. The problem of finding the off-shell linear representations of all supersymmetrical systems has remained unsolved for almost a quarter of a century [3].

Insight into this general problem may arise from a more careful investigation of the spinning particle system. In previous work, this same algebraic structure was found to occur in all on-shell $N$-extended scalar (and their dual vector) multiplets in three dimensional superspaces. Since this is the case, the way is open to study the problem of the arbitrarily $N$-extended supersymmetric BF theories. This class of theories naturally leads to “zero-curvature” conditions on field strength and this means there is the possibility of a link to integrable systems.

Our main motivation is to understand these two problems. In the following we will present the $\mathcal{GR}(d, N)$ algebras more explicitly than in our previous work. Most importantly, we will investigate how these algebras organize the problem of component fields in conventional superspace and in superfields. Further, using 1D models constructed from these algebras we will show that these algebras can encode the structure of supermultiplets from higher dimensions. This letter begins with an abstract definition of $\mathcal{GR}(d, N)$, followed by some useful derivations using an explicit realization of these algebras. We then focus our attention on writing field theories using $\mathcal{GR}(d, N)$.

The form of this paper is as follows. Formal arguments are presented in Section 2. These include geometrical motivations for the $\mathcal{GR}$ algebra, and conventions for the representation of these algebras. In section 3, we show how these algebras encode the supersymmetric structure of higher dimensional supersymmetric theories. Finally, section 4 shows how we apply the $\mathcal{GR}$ algebras to write off-shell $N$-extended spinning particle actions.
2 Formal Arguments

In this section we present a geometrical description for the definition of the $\mathcal{GR}(d, N)$ algebra. We also show the explicit connection between these algebras and real Clifford algebras. The connection to real Clifford algebras is then exploited in order to understand the structure of the enveloping algebras of $\mathcal{GR}(d, N)$.

2.1 Geometric Preliminaries

Let $d \geq 1$ be some fixed natural number and consider a collection of $d + d$ diffeomorphisms $\phi_i, \psi_l : \mathbb{R} \to \mathbb{R}^d$, with $i = 1, \ldots, d$ and $l = 1, \ldots, d$. Define $V_L$ and $V_R$ be the free vector spaces generated by $\{\phi_i\}_{i=1}^d$ and $\{\psi_l\}_{l=1}^d$, respectively. Note that $V_L \cong \mathbb{R}^d \cong V_R$, however, we do not wish to identify the two.

Next we consider the set of linear transformations acting on these spaces in the following manner. Let $\{\mathcal{M}_L\}$ denote all linear maps that send elements of $V_L$ into elements of $V_R$, $\{\mathcal{M}_R\}$ denote all linear maps that send elements of $V_R$ into elements of $V_L$, $\{\mathcal{U}_L\}$ denote all linear maps that send elements of $V_L$ into elements of $V_L$ and $\{\mathcal{U}_R\}$ denote all linear maps that send elements of $V_L$ into elements of $V_L$. Thus we have\(^5\),

$$
\begin{align*}
\mathcal{M}_L : & V_L \to V_R, \\
\mathcal{M}_R : & V_R \to V_L, \\
\mathcal{U}_L : & V_L \to V_L, \\
\mathcal{U}_R : & V_R \to V_R.
\end{align*}
$$

Since the dimension of the vector spaces is $d$, it follows that $\dim \{\mathcal{M}_L\} = \dim \{\mathcal{M}_R\} = \dim \{\mathcal{U}_L\} = \dim \{\mathcal{U}_R\} = d^2$.

The definition of these maps implies that the compositions $\mathcal{M}_R \circ \mathcal{M}_L$ and $\mathcal{M}_L \circ \mathcal{M}_R$ have the properties

$$
\begin{align*}
\mathcal{M}_R \circ \mathcal{M}_L : & V_L \to V_L, \\
\mathcal{M}_L \circ \mathcal{M}_R : & V_R \to V_R,
\end{align*}
$$

and are thus elements of $\{\mathcal{U}_L\}$ and $\{\mathcal{U}_R\}$ respectively.

All of these structures may be illustrated by means of a “Placement-putting Graph” that is presented on the following page. The $d$-dimensional vector spaces are represented in a Venn diagram containing two disjoint sets and the linear transformations are represented by directed line segments that act between the sets appropriately.

\(^5\)Here it understood that $\mathcal{M}_L$ is an element of $\{\mathcal{M}_L\}$. 
The line segments may be regarded as being composed of $d^2$ “fibers” representing the $d^2$-linearly independent linear maps acting between the vector spaces.

Two conditions are necessary to define the $\mathcal{GR}(d,N)$ algebra. We will show how these conditions yield a representation of supersymmetry in 1D. The first condition is a definition of the fundamental generators of $\mathcal{GR}(d,N)$. The second is a statement of how the generators behave when an inner product is defined on the vector spaces $\mathcal{V}_L$ and $\mathcal{V}_R$. Consider the family of $N + N$ (for any natural number $N$) linear maps $L_i \in \{\mathcal{M}_L\}$ and $R_i \in \{\mathcal{M}_R\}, i = 1, \cdots, N$ such that

\[
\begin{align*}
L_i \circ R_k + L_k \circ R_i &= -2\delta_{i,k} I_{\mathcal{V}_L}, \\
R_i \circ L_k + R_k \circ L_i &= -2\delta_{i,k} I_{\mathcal{V}_R}, \quad \forall \ i, k = 1, \cdots, N.
\end{align*}
\]

Here $I_{\mathcal{V}_L}$ and $I_{\mathcal{V}_R}$ denote the identity maps on $\mathcal{V}_L$ and $\mathcal{V}_R$ respectively. Next we equip $\mathcal{V}_L$ and $\mathcal{V}_R$ with Euclidean inner products $\langle \cdot, \cdot \rangle_{\mathcal{V}_L}$ and $\langle \cdot, \cdot \rangle_{\mathcal{V}_R}$ respectively. The second
defining relation of the $\mathcal{GR}(d, N)$ algebra is the condition that for any $I = 1, \ldots, N$

$$
\langle \phi, L_i(\psi) \rangle_{V_L} = -\langle R_i(\phi), \psi \rangle_{V_R} \quad \forall (\phi, \psi) \in V_L \oplus V_R.
$$

This equation implies that the objects $R_i$ equal to minus the adjoints of $L_i$ and the definitions in (3) can be solely expressed in term of the sets of $L_i$ and their adjoints $L_i^*$. In all, then, we define the general real algebra of level $N$ and dimension $d$ to be the sub-algebra of $\{M_R\} \oplus \{M_L\}$ generated by the relations (3) and (4). We will occasionally resort to the notation $\mathcal{GR}(N)$ when referring to the abstract algebra or when the dimension $d$ is otherwise unimportant.

We will now motivate the definition of the $\mathcal{GR}(N)$ algebra by using it to write a supersymmetric theory in 1D as follows. Let $\{\phi_k\} \subset V_L$ and $\{\psi^\ell\} \subset V_R$ denote a set of $d + d$ real-valued fields, henceforth to be known as bosonic and fermionic respectively. We define an algebraic derivation $\delta : V_L \oplus V_R \to V_L \oplus V_R$ by the relation

$$
\delta_\alpha : (\phi, \psi) \mapsto (i\alpha \cdot L(\psi), -i\alpha \cdot R(D\phi)),
$$

where $D := i\partial$ is the 1D translation generator and $\alpha \cdot L = \alpha^i L_i$ and $\alpha \cdot R = \alpha^i R_i$ are elements of the $\mathcal{GR}(N)$ algebra described above. Owing to this we may easily derive (using (3) and the facts that the parameter and fermion are classical anti-commuting parameters)

$$
[\delta_\alpha, \delta_\beta] = -2\alpha \cdot \beta D,
$$

on $V_L \oplus V_R$, which is nothing but the statement that these algebraic derivations are, in fact, generators of supersymmetry. Next, consider the standard free lagrangian

$$
\mathcal{L} = -\frac{1}{2}(D\phi) \cdot (D\phi) + \psi \cdot D\psi = \frac{1}{2} \langle \phi, D^2 \phi \rangle_{V_L} + \frac{1}{2} \langle \psi, D\psi \rangle_{V_R},
$$

up to total derivatives, which we neglect throughout this work. It is easy to check that owing to the relation (4), the action constructed from this lagrangian commutes with the derivation defined in (5):

$$
\delta_\alpha \mathcal{L} = -i\alpha \cdot \left\{ \langle D\phi, L(D\psi) \rangle_{V_L} + \langle R(D\phi), D\psi \rangle_{V_R} \right\} = 0.
$$

Note that the relations (3) above are not consistent for all values of $N$ and $d$. In particular, for any $N \in \mathbb{N}$, there is a minimum value of $d$, which we will denote by $d_N$, such that $N$ linearly independent quantities $L_i$ and $N$ linearly independent quantities $R_i$ can be constructed to satisfy (3) and (4). If we write $N$ as related to $n$ and $m$ through the equations $N = n + 8m$ where $1 \leq n \leq 8$ and further, if we use
the rule that if \( N = 8k \Rightarrow n = 8, \& m = k - 1 \) for \( k = 1, 2, ..., \infty \). then the minimal value of \( d_N \) can be expressed in the form of a functional relationship

\[
d_N = 2^{4m} F_{RH}(N) = 16^m F_{RH}(n)
\]

where \( F_{RH}(N) \) is the Radon-Hurwitz function as was noted in [4]. In writing the second form of this relation, we have used the period eight property of this function. In our previous work [1, 2], the Radon-Hurwitz function appeared in tabular form.

Our previous work also presented an algorithm for the construction of an explicit matrix representation for \( L_i \) and \( R_i \) for all values of \( N \) and \( d_N \).

In what follows we will be interested not only in the \( GR(d, N) \) algebra but two other related algebras. The first is the enveloping algebra of \( GR(d, N) \) denoted \( EGR(d, N) \). This \( (EGR(d, N)) \) denotes the set of all linear maps on and between \( V_L \) and \( V_R \). More simply stated:

\[
EGR := \{ M_L \} \oplus \{ M_R \} \oplus \{ U_L \} \oplus \{ U_R \}
\]

This new algebraic structure contains the \( GR(d, N) \) sub-algebra. Because of the structure of (3) it is natural to work with objects that are antisymmetric combinations of \( L_i \) and \( R_i \). We therefore make the following definitions. Let \( f_{[n]} \) denote the \( n \)-th antisymmetric combination beginning with \( L_i \), and \( \hat{f}_{[n]} \) the \( n \)-th antisymmetric combination beginning with \( R_i \). For example:

\[
\begin{align*}
f_{[1]} & := L_i \\
\hat{f}_{[1]} & := R_i \\
f_{[2]} & := L_{[i} \circ R_{j]} \\
\hat{f}_{[2]} & := R_{[i} \circ L_{j]}
\end{align*}
\]

where \([IJ]\) denotes antisymmetrization. The algebra formed by wedging \( L \)'s and \( R \)'s in this manner will be denoted by \( \Lambda GR(d, N) \). Although it sometimes happens that \( EGR(d, N) \cong \Lambda GR(d, N) \), the generic situation is that \( \Lambda GR(d, N) \) forms a proper sub-algebra of \( EGR(d, N) \). When this happens, \( \Lambda GR(d, N) \) does not span the space of all linear mapping on and between \( V_L \) and \( V_R \). It is in this sense that \( EGR(d, N) \) completes \( GR(d, N) \).

It is hoped that the discussion above has convinced the reader that by allowing the elements of \( V_L \) and \( V_R \) to depend upon a real parameter for which the derivation \( D \) is well defined, a natural representation of 1D supersymmetry is induced among the elements. However, this should also suggest to the reader that by permitting the elements of \( EGR(N) \) to also depend on such a parameter implies that the elements of \( EGR(N) \) must also carry a representation of 1D supersymmetry. We will see that this is so in the next section.
With these preliminaries out of the way, we turn in the next section to the representation theory of these $\mathcal{G}\mathcal{R}(N)$ algebras. As explained there, the representations of this algebra can be derived directly from the known real representation theory of ordinary Clifford algebras. What may, perhaps, come as a surprise to some readers is the fact that the correspondence between the representations of the two algebras is not entirely trivial. In fact, as will be explained below, the condition (4) above amounts in the use of Clifford algebras to requiring the existence of an extra matrix which anti-commutes with the algebra elements and squares to the identity.

2.2 Enveloping Algebra Representation Theory

As mentioned in the previous section, it turns out that $\mathcal{G}\mathcal{R}(d, N)$ fits naturally into the context of real Clifford algebras. We note first that we need only consider $C(N, 1)$ Clifford algebras, that is (with $I = 1, \ldots, N + 1$):

$$\gamma^{(t\gamma^j)} = -2\eta^{tj}\mathbf{1}$$  \hspace{1cm} (12)

Where $\eta^{tj} = \text{diag}(1 \ldots 1, -1)$.$^6$ We find the irreducible representations of $\mathcal{G}\mathcal{R}(d,N)$ within the context of this family of Clifford algebras. It turns out that $C(N, 1)$ Clifford algebras naturally contain projection operators that lead to $\mathcal{G}\mathcal{R}(d,N)$. We can see this by constructing projection operators as follows:

$$P_\pm = \frac{1}{2}(\mathbf{1} \pm \gamma^{N+1})$$  \hspace{1cm} (13)

where $\gamma^{N+1}$ is the $(N + 1)$-th $\gamma$-matrix in (12). These operators satisfy the usual projection algebra, $P_i P_j = \delta_{ij} P_i$, and have the following property when acting on $\gamma^I$:

$$P_\pm \gamma^I = \gamma^I P_\mp, \hspace{0.5cm} P_\pm \gamma^{N+1} = \pm P_\pm$$  \hspace{1cm} (14)

With these projectors we make the following identifications:

$$L_i \equiv P_+ \gamma_i P_-, \hspace{0.5cm} R_i \equiv P_- \gamma_i P_+,$$  \hspace{1cm} (15)

and we can now see that $\mathcal{G}\mathcal{R}(d, N)$ arises naturally:

$$L_{(i} R_{j)} = P_+ \gamma_{(i} P_- \gamma_{j)} P_+ = -2\delta_{ij} P_+ \equiv -2\delta_{ij} \mathbf{1}_+,$$

$$R_{(i} L_{j)} = P_- \gamma_{(i} P_+ \gamma_{j)} P_- = -2\delta_{ij} P_- \equiv -2\delta_{ij} \mathbf{1}_-. $$  \hspace{1cm} (16)

$^6$Please note the different indices. $I$ corresponds to the adjoint Clifford algebra index that runs from 1 to $N + 1$, whereas $i$ is the adjoint index corresponding to $\mathcal{G}\mathcal{R}(d,N)$ taking values 1 to $N$. 

7
We can also exhibit the skew symmetry relation of the L’s to the R’s. For the \( C(\mathcal{N}, 1) \) Clifford algebras we can always find a basis such that:

\[
(\gamma^{N+1})^T = \gamma^{N+1} \\
(\gamma^i)^T = -\gamma^i
\]  

then it follows that:

\[
L^T_i = (P_+ \gamma_i P_-)^T = P_-^T \gamma_i^T P_+^T = -P_+ \gamma_i P_- = -R_i
\]  

Thus, we have shown how all of the defining properties of \( \mathcal{G}\mathcal{R}(d, \mathcal{N}) \) can be realized by considering only the real Clifford algebras \( \mathcal{C}(\mathcal{N}, 1) \).

A natural expectation\(^7\) is that \( \mathcal{E}\mathcal{G}\mathcal{R}(\mathcal{N}) = \wedge \mathcal{G}\mathcal{R}(\mathcal{N}) \) i.e. represented:

\[
\{\mathcal{U}_L\} = \{P_+, P_+ \gamma_1 P_+, \ldots, P_+ \gamma_{[N]} P_+\} \\
\{\mathcal{M}_L\} = \{P_+ \gamma^i P_-, \ldots, P_+ \gamma_{[N-1]} P_-\} \\
\{\mathcal{U}_R\} = \{P_-, P_- \gamma^i P_+, \ldots, P_- \gamma_{[N]} P_+\} \\
\{\mathcal{M}_R\} = \{P_- \gamma^i P_-, \ldots, P_- \gamma_{[N-1]} P_+\}
\]  

Where \([N]\) means antisymmetrization over \( N \) indices. This procedure realizes the four transformations between the two spaces and automatically obeys the left/right composition rule since \( P_\pm P_\mp = 0 \). We can also see why each space of \( \wedge \mathcal{G}\mathcal{R}(N) \) contains half of the elements that \( \wedge C(N, 1) \) contains. The Clifford algebra element \( \gamma^{N+1} \) does not appear in (19) because of (14).

When dealing with Minkowski-space Dirac matrices, it is customary to use the analogs of L and R (i.e. the usual \( \sigma \)-matrices) to generate the analogs of \( \{\mathcal{M}_L\} \), \( \{\mathcal{M}_R\} \), \( \{\mathcal{U}_L\} \) and \( \{\mathcal{U}_R\} \) by repeatedly taking higher and higher order nested commutators. The procedure generally fails for the \( \mathcal{G}\mathcal{R}(d, \mathcal{N}) \) algebras as implied by a work of Okubo [7] who also gave the proper way to build the complete enveloping algebras. His classification of the enveloping algebras point to the existence of three distinct types of real Clifford algebras\(^8\):

(a.) N-type (normal)  
(b.) AC-type (almost complex)  
(c.) Q-type (quaternionic)  

\(^7\)However, this natural expectation is wrong as will be discussed below.  
\(^8\)In the mathematical literature, these are simply referred to as irreducible Clifford modules that are Real, Complex, or Quaternionic, respectively.
In the case of the normal enveloping algebras (a.), we can begin with the $N+1$ representations $\gamma^I$ and use their wedge products to form a set $\{\Gamma\}$. This set is equivalent to the enveloping algebra

$$\{\Gamma\} = \{\mathbf{I}, \gamma^I, \gamma^I\gamma^J, \ldots, \gamma^{[N+1]}\}$$

(20)

In turn, this set can be split according to whether elements contain even or odd powers of the $\gamma^I$-matrices.

$$\{\Gamma_e\} = \{\mathbf{I}, \gamma^{N+1}, \gamma^{ij}, \gamma^{ij}\gamma^{N+1}, \gamma^{ijkl}, \ldots\}$$

(21)

$$\{\Gamma_o\} = \{\gamma^I, \gamma^I\gamma^{N+1}, \gamma^{ijkl}, \gamma^{ijkl}\gamma^{N+1}, \ldots\}$$

(22)

Finally, the elements of $\{\Gamma_e\}$ and $\{\Gamma_o\}$ are such that they can be put into a two-to-one correspondence with $\{M_L\}$, $\{M_R\}$, $\{U_L\}$ and $\{U_R\}$ according to (19).

For the almost complex enveloping algebra (b.), the wedge products span only one half of the enveloping algebra. These algebras contain one element $J$ that commutes with $\gamma^I$ and squares to $-\mathbf{I}$. In order to completely identify all of the elements, we must introduce one additional quantity (denoted by Okubo as $D$) that anti-commutes with $\gamma^I$. $D$ is used to double the number of wedge products. Thus, the enveloping algebra of the almost complex representations takes the form:

$$\{\Gamma\} = \{\mathbf{I}, D, \gamma^I, \gamma^I D, \ldots, \gamma^{[N+1]}, \gamma^{[N+1]} D\}$$

(23)

This enveloping algebra can now be split into even and odd parts relative to $\gamma^I$. These even and odd algebras can then be put into a two-to-one correspondence with $\mathcal{EGR}(N)$ in the same manner as the normal case above.

Finally, in the case of quaternionic enveloping algebras, (c.), the wedge products yield one quarter of the enveloping algebra. This is remedied by introducing three objects that commute with the elements $\gamma^I$ [7]. We denote these three objects by $\mathcal{E}^{\hat{a}}$ with $\hat{a} = (1, 2, 3)$ and they satisfy

$$[\mathcal{E}^{\hat{a}}, \mathcal{E}^{\hat{b}}] = 2\epsilon^{\hat{a}\hat{b}\hat{c}} \mathcal{E}^{\hat{c}}, \quad \mathcal{E}^{\hat{a}} \mathcal{E}^{\hat{b}} = -\delta^{\hat{a}\hat{b}} \mathbf{I}.$$ 

(24)

The enveloping algebra now takes the form:

$$\{\Gamma\} = \{\mathbf{I}, \mathcal{E}^{\hat{a}}, \gamma^I, \gamma^I \mathcal{E}^{\hat{a}}, \ldots, \gamma^{[N+1]}, \gamma^{[N+1]} \mathcal{E}^{\hat{a}}\}$$

(25)

Splitting into odd and even as before, we again arrive at a two-to-one correspondence between the elements of this enveloping algebra and $\mathcal{EGR}(N)$.

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9Actually $J = \pm \gamma^1 \gamma^2 \ldots \gamma^{N+1}$
The fact that the enveloping algebras have these three distinct structures makes a great difference when considering either the left or right multiplication of the enveloping algebras by single $\gamma$-matrices. In the normal case such a multiplication will cause an orbit to move through the entirety of $\{M_L\}, \{M_R\}, \{U_L\}$ and $\{U_R\}$. In the almost complex and quaternionic cases, the results of such a multiplication are very different. In the almost complex case such a multiplication will only move through half of $\{M_L\}, \{M_R\}, \{U_L\}$ and $\{U_R\}$ since such multiplications can never produce objects in the “$\mathcal{D}$-sector.” Similarly, in the quaternionic case such a multiplication will only move through one-fourth of $\{M_L\}, \{M_R\}, \{U_L\}$ and $\{U_R\}$ since such multiplications can never produce objects in the “$\mathcal{E}^{\alpha}$-sector.” We will refer to the part of the enveloping algebras that is independent of either $\mathcal{D}$ or $\mathcal{E}^{\alpha}$ as “the normal part (NP) of the enveloping algebra.” Thus the dimension of the normal part of $\mathcal{E}\mathcal{G}\mathcal{R}(N)$ (which is $\wedge^{}\mathcal{G}\mathcal{R}(N)$) is $2^N$.

In our previous work [1, 2], we constructed explicit representations of the objects in (15) and also developed an algorithm for constructing arbitrarily large $N$ representations. The results of this analysis are presented in the following table for the cases $1 \leq N \leq 8$.

| $\mathcal{G}\mathcal{R}(d, N)$ | $\dim C(N, 1)$ | $\mathcal{E}\mathcal{G}\mathcal{R} (d, N)$ generators | Type |
|-----------------------------|----------------|---------------------------------|------|
| $\mathcal{G}\mathcal{R}(8, 8)$ | 16             | $f_1$                           | $N$  |
| $\mathcal{G}\mathcal{R}(8, 7)$ | 16             | $f_1$                           | $N$  |
| $\mathcal{G}\mathcal{R}(8, 6)$ | 16             | $f_1, \mathcal{D}$             | AC   |
| $\mathcal{G}\mathcal{R}(8, 5)$ | 16             | $f_1, \mathcal{E}^{\alpha}$    | Q    |
| $\mathcal{G}\mathcal{R}(4, 4)$ | 8              | $f_1, \mathcal{E}^{\alpha}$    | Q    |
| $\mathcal{G}\mathcal{R}(4, 3)$ | 8              | $f_1, \mathcal{E}^{\alpha}$    | Q    |
| $\mathcal{G}\mathcal{R}(2, 2)$ | 4              | $f_1, \mathcal{D}$             | AC   |
| $\mathcal{G}\mathcal{R}(1, 1)$ | 2              | $f$                             | $N$  |

Table I

By using Bott periodicity this table can be extended to all values of $N$. We also see that the dimension of the irreducible representations of $C(N, 1)$ are exactly twice that of $\mathcal{G}\mathcal{R}(d, N)$. Therefore, it makes sense that $\mathcal{G}\mathcal{R}(d, N)$ can be obtained by projection from $C(N, 1)$.

---

10By dimension in the following table we are referring the size of the matrices that comprise $C(N, 1)$, or the dimension of the spin representation it acts upon.

10
This concludes our discussion of the overall mathematical structure of \( GR(N) \). In what follows we provide a more explicit representation of \( \Lambda GR(d, N) \) and construction of \( \mathcal{E}GR(d, N) \) for \( N \leq 8 \). We will use \( GR(d, N) \) notation and when necessary we will revert to the Clifford algebra representations.

### 2.3 \( GR(8, 8) \) and Dimensional Reductions

The Clifford algebra \( C(8, 1) \) is 16 dimensional. This means that the associated \( GR \)-algebra is \( GR(8,8) \). Since \( C(8, 1) \) is normal we will not have to introduce a new matrix to complete the enveloping algebra. Therefore, we can write \( \mathcal{E}GR(8) \) using the elements of \( \Lambda GR(8,8) \):

\[
\mathcal{E}GR(8)_{ij} \cong \{U_L^8\} = \{I, f_{1j}, f_{1jk}\}
\]

\[
64 = \{1, 28, 35\}
\]

\[
\mathcal{E}GR(8)_{ij} \cong \{M_L^8\} = \{f_1, f_{1jk}\}
\]

\[
64 = \{8, 56\}
\]

Naively, we would expect 5, 6, 7 and 8-forms in \( \mathcal{E}GR(8) \) and that the 4-form is a 70. However, if we write these forms using the Clifford algebra basis we see that these forms are the duals of the 3, 2, 1, and 0-forms respectively and that the 4-form has a definite duality. Taking:

\[
\gamma^{[8]} = \ell \epsilon^{[8]} \gamma^9,
\]

where \( \epsilon^{[8]} \) is the totally antisymmetric tensor with eight indices and \( \ell = \pm 1 \), we have:

\[
f_{[8]} = P_+ \gamma^{[8]} P_+ = \ell \epsilon^{[8]} P_+ \gamma^9 P_+ = \ell \epsilon^{[8]} P_+ \equiv \ell \epsilon^{[8]} I
\]

\[
f_{[7]} = P_+ \gamma^{[7]} P_- = -P_+ \gamma^{[7]} \gamma^1 P_+ \gamma^1 = -\ell \epsilon^{[7]} P_+ \gamma^9 P_+ \gamma^1
\]

\[
= -\ell \epsilon^{[7]} P_+ \gamma^1 P_- = -\ell \epsilon^{[7]} f_{i1} \quad I \notin [7]
\]

\[
f_{[6]} = P_+ \gamma^{[6]} P_+ = -\frac{1}{2!} P_+ \gamma^{[6]} \gamma^{ijkl} \gamma^{ijkl} P_+
\]

\[
= -\ell \epsilon^{[6]} P_+ \gamma^9 P_+ \gamma^{1j} = -\ell \epsilon^{[6]} f_{ij} \quad I, J \notin [6]
\]

\[
f_{[5]} = P_+ \gamma^{[5]} P_- = \frac{1}{3!} P_+ \gamma^{[5]} \gamma^{1jk} \gamma^{1jk} P_-
\]

\[
= \ell \epsilon^{[5]} \gamma^{1jk} P_+ \gamma^9 P_+ \gamma^{1jk} = \ell \epsilon^{[5]} f_{1jk} \quad I, J, K \notin [5]
\]

11
and:

\[
f_{IJKL} = P_+ \gamma^{IJKL} P_+ = \frac{1}{4!} P_+ \gamma^{IJKL} \gamma^{MNOP} \gamma^{MNOP} P_+ = \frac{\ell}{4!} \epsilon^{IJKLMNOP} f_{MNOP}
\]

This explains the economy of \(EGR(8)\) written in terms of \(GR(8,8)\). Note that the “hatted” objects may have different minus signs since, \(P_+ \gamma^9 P_+ = -P_+\).

It is a fact that the irreducible representations of \(C(7,1), C(6,1),\) and \(C(5,1)\) are 16 dimensional. Because of this, we can obtain the enveloping algebras of \(GR(8,7), GR(8,6),\) and \(GR(8,5)\) by dimensional reduction from \(EGR(8,8)\). From this perspective we will see why \(EGR(8,7)\) is a normal algebra. We will also see how the almost complex and quaternionic structures appear in \(EGR(8,6)\) and \(EGR(8,5)\), respectively.

\(EGR(8,7)\) takes the form:

\[
EGR(8,7)_{ij} \cong \{U^i_j\} = \{I, f_{13}, f_{1JKL}, f_{[6]}\}
\]

\(64 = \{1, 21, 35, 7\}\)

\[
EGR(8,7)_{ij} \cong \{M^i_j\} = \{f_{1}, f_{LJK}, f_{[6]}, f_{[7]}\}
\]

\(64 = \{7, 35, 21, 1\}\) \(33\)

One may ask “Where did the duality go?” The answer to this question is that the dimensional reduction of \(C(8,1)\) to \(C(7,1)\) uses the duality to transform all of the “eights” away. For example:

\[
f_{[6]} = -\frac{\ell}{24!} \epsilon^{[6]}_{[8]} f_{18}\)

From these arguments, we see that \(EGR(8,7)\) is normal. Because \(EGR(8,7)\) is normal we can use it, instead of \(EGR(8,8)\), to obtain \(EGR(8,6)\) and \(EGR(5,8)\) via dimensional reduction. \(EGR(6,8)\) looks like:

\[
EGR(8,6)_{ij} = \{I, f_{17}, f_{13}, f_{1JK7}, f_{1JKL}, f_{[5]}_{[7]}, f_{[6]}\}
\]

\(64 = \{1, 6, 15, 20, 15, 6, 1\}\)

\[
EGR(8,6)_{ij} = \{f_{7}, f_{1}, f_{137}, f_{LJK}, f_{[4]}_{[7]}, f_{[5]}_{[7]}, f_{[6]}\}
\]

\(64 = \{1, 6, 15, 20, 15, 6, 1\}\) \(35\)

Here we see that \(f_{7}\) plays the role of the projected \(\mathcal{D}\). \(\mathcal{D}\) should be in the mixed space since it anti-commutes with \(\gamma^I\). Furthermore, \(\mathcal{D} \mathcal{D} = -I\) necessarily. Reducing once more we have:

\[
EGR(8,5)_{ij} = \{I, f_{67}, f_{16}, f_{17}, f_{13}, f_{1JK7}, f_{1JKL}, f_{1JKL67}, f_{[5]}_{[7]}, f_{[5]}_{[6]}\}
\]
Here we see the quaternionic structure:

\[ E^\alpha = \{ f_{67}, f_{[5]6}, f_{[5]7} \} \]

In the following sections we will see representations of a different quaternionic and almost complex enveloping algebras. These cases can not be understood as dimensional reductions of larger Clifford algebras of the form \( C(N + A, 1) \).

### 2.4 \( GR(4, 4) \) and Dimensional Reduction to \( GR(3, 4) \)

The irreducible representation of \( C(4, 1) \) is 8 dimensional and quaternionic. To construct the enveloping algebra we take the wedge products and the quaternions. Thus \( EGR(4, 4) \) takes the form:

\[
EGR(4)_{ij} = \{ I, f_{ij}, E^\alpha, f_{i}E^\alpha \}
\]

\[
EGR(4)_{i}^\alpha = \{ f_{i}, f_{i}E^\alpha \}
\]

As in the case of \( EGR(8) \), there is some duality that must be taken care of in order to write the enveloping algebra in this form. In this case we have

\[
\gamma^{IJKL} = \ell \epsilon^{IJKL} \gamma^{5},
\]

and upon projection

\[
f^{IJKL} \equiv P_+ \gamma^{IJKL} P_+ = \ell \epsilon^{IJKL} P_+ \gamma^{5} P_+ \equiv \ell \epsilon^{IJKL} I
\]

\[
f^{IJK} \equiv P_+ \gamma^{IJK} P_+ = -P_+ \gamma^{IJK} \gamma^{L} P_+ \gamma^{L} \equiv -\ell \epsilon^{IJKL} f_{L} \quad L \neq I, J, K
\]

\[
f^{IJ} \equiv P_+ \gamma^{IJ} P_+ = -\frac{\ell}{2} \epsilon^{IJKL} P_+ \gamma^{5} P_+ \gamma^{KL} \equiv -\frac{\ell}{2} \epsilon^{IJKL} f_{KL}
\]

Here we see that the two-form has definite duality, and the 3 and 4 forms are the duals of the 1 and 0 forms. We have also projected the quaternionic generators:

\[
E^\alpha := P_+ E^\alpha P_+ , \quad \tilde{E}^\alpha := P_- E^\alpha P_-
\]
The Clifford algebra $C(3,1)$ is 8 dimensional and quaternionic just like $C(4,1)$. This similarity means that we can just dimensionally reduce $\mathcal{EGR}(4,4)$ to get $\mathcal{EGR}(4,3)$:

$$
\mathcal{EGR}(3)_{ij} = \{ I, f_{ij}, \mathcal{E}^\alpha, f_{ij} \mathcal{E}^\alpha \}
$$

$$
16 = \{ 1, 3, 3, 9 \}
$$

$$
\mathcal{EGR}(3)_{ij} = \{ f_i, f_{ijk}, f_{ijk} \mathcal{E}^\alpha, f_i \mathcal{E}^\alpha \}
$$

$$
16 = \{ 3, 1, 3, 9 \}\quad (44)
$$

Note, as in the case of $\mathcal{GR}(7,8)$, we have used the duality to transform all of the “fours” away.

### 2.5 $\mathcal{GR}(2,2)$ and $\mathcal{GR}(1,1)$

$C(2,1)$ is an almost complex, 4 dimensional Clifford algebra. So the enveloping algebra of $\mathcal{GR}(2,2)$ looks like:

$$
\mathcal{EGR}(2,2)_{ij} = \{ I, f_{ij}, D, f_{ij} D \}
$$

$$
4 = \{ 1, 1, 1, 1 \}
$$

$$
\mathcal{EGR}(2,2)_{ij} = \{ f_i, f_i D \}
$$

$$
4 = \{ 2, 2 \}\quad (45)
$$

In this case $D\hat{D} = I$. The final algebra to consider is $\mathcal{GR}(1,1)$. The Clifford algebra $C(1,1)$ is two dimensional and normal. A one dimensional algebra may not be worth noting, but these structures obey Bott periodicity. So even though $\mathcal{GR}(1,1)$ is trivial, the form of $\mathcal{EGR}(1,1)$ will be the same for $N = 9, 17 \ldots$. It is just the wedge products:

$$
\mathcal{EGR}(1,1)_{ij} = \{ I \}
$$

$$
1 = \{ 1 \}
$$

$$
\mathcal{EGR}(1,1)_{ij} = \{ f \}
$$

$$
1 = \{ 1 \}\quad (46)
$$

We note that there is no duality in these cases. This is because $N + 1 = 2n$ in this case. If $N + 1$ had been odd, as in the case $N = 8$ above, there would have been duality. This concludes the mathematical discussion of $\mathcal{GR}$-algebras. In what follows we will show how the $\mathcal{GR}$-algebras can lead to a better understanding of supersymmetric theories in higher dimensions.
3 Connections to Higher Dimensional Supersymmetric Systems

The set of objects $\phi_i$ and $\psi_l$ with which we began our discussion in the introduction provide a purely algebraic definition of a “superfield$^{11}$.” So we may call the set $(\phi_i, \psi_l)$ a superfield. In particular, the elements of $\mathcal{V}_L$ may be taken to be a set of bosonic commuting 1D fields and the elements of $\mathcal{V}_R$ may be taken to be a set of fermionic anti-commuting 1D fields. When the parameters $\alpha^i$ that first appear below (5) are also considered to be anti-commuting, we obtain a description of a system that realizes $N$-extended supersymmetry. The simplest invariant action for this system appears in (7).

However, as already pointed out, the superfields defined above are not sufficient to describe spinning particle systems in Minkowski space. The basic problem is that the bosonic components that correspond to the “$x$-space” coordinate of the spinning particle within the framework of these representations necessarily increase with $N$. Thus, the superfields defined above are appropriate for “iso-spinning particle” systems but not spinning particle systems. To overcome this, it was suggested that there is another$^{12}$ Clifford-algebraic definition of a superfield. It was observed that the elements of $\{U_L\} \oplus \{M_R\}$ (as well as $\{U_R\} \oplus \{M_L\}$) can be used to define superfields in precisely the same way as the elements $\mathcal{V}_L$ and $\mathcal{V}_R$. This second Clifford-algebraic definition of a superfield appears to be compatible with the structure of spinning particle models in Minkowski space.

To this end, introduce a set of maps $(\Phi_i^j, \Psi_i^l): \mathbb{R} \rightarrow \{U_L\} \oplus \{M_R\}$ and define an algebraic derivation $\delta_\alpha : \{U_L\} \oplus \{M_R\} \rightarrow \{U_L\} \oplus \{M_R\}$ by the relations

$$\delta_\alpha : (\Phi, \Psi) \mapsto (i\alpha \cdot L(\Psi), -i\alpha \cdot R(D\Phi)) \quad . \quad (47)$$

Above we mean that $L$ and $R$ act upon $\Phi$ and $\Psi$ by left composition. Acting again with $\delta_\beta$ we find that the condition in (6) is satisfied on this representation of fields.

We will see that this representation of $\mathcal{GR}(d, N)$ plays a key role in connection with higher dimensional theories.

The constructions above also naturally engender the realization of “twisted” superfield representations. If we concentrate only on the field representations and neglect the question of an invariant action, then there exists a potentially interesting ambiguity. Let us focus upon two sets of objects $(\phi_i^{(1)}, \psi_i^{(1)})$ and $(\phi_i^{(2)}, \psi_i^{(2)})$ where it should

$^{11}$As we will see this definition is equivalent to the usual Salam-Strathdee definition of a superfield.

$^{12}$In fact, there are an infinite number of such definitions.
be noted that these two sets of fields possess different markings of their index types. Both of these sets will necessarily define supermultiplets that satisfy a supersymmetry algebra. If they are inequivalent, we may call one the twisted version of the other. There is also a second potential source of twisted multiplets. If there exist several inequivalent representations of a given $\mathcal{GR}(N)$ algebra\textsuperscript{13}, then the fields defined with respect to these inequivalent representations may also be referred to as the twisted versions of one another. Finally, one can consider a set of mapping operations whose definition is given by the exchange among pairs of twisted multiplets. We have long called such maps, “mirror maps.”

The concepts of twisted multiplets and mirror maps can also be defined to act upon sets of objects that are defined to lie in $\{\mathcal{M}_R\} \oplus \{\mathcal{U}_L\}$ (as well as $\{\mathcal{M}_L\} \oplus \{\mathcal{U}_R\}$). So for these larger superfield representations very similar ambiguities may also be realized. In particular, the fact that normal, almost complex and quaternionic cases exist leads directly the occurrence of mirror representations here. Consider a set of objects $(\hat{\Phi}_i^j, \hat{\Psi}_i^j)$ with $\hat{\Phi}_i^j \in \{\mathcal{U}_R\}$ and $\hat{\Psi}_i^j \in \{\mathcal{M}_L\}$. Next choose only the cases where the enveloping algebra contains an almost complex structure $J$. Finally, we define define an algebraic derivation $\delta_\alpha : \{\mathcal{U}_R\} \oplus \{\mathcal{M}_L\} \rightarrow \{\mathcal{U}_R\} \oplus \{\mathcal{M}_L\}$ by the relations

$$\delta_\alpha : (\hat{\Phi}, \hat{\Psi}) \mapsto (i\alpha \cdot J_L(\hat{\Psi}), i\alpha \cdot J_R(D\hat{\Phi})) \quad ,$$

(48)

The set $(\hat{\Phi}, \hat{\Psi})$ is a “twisted multiplet” relative to the set $(\Phi, \Psi)$. Similar generalizations can occur for quaternionic representations\textsuperscript{14}.

Due to the structure of the enveloping algebras discussed previously, the superfields that appear in (47) are in general reducible representations. Only in the cases where the enveloping algebra is normal, do these superfields form irreducible representations and this only occurs for the cases of $N = 1, 7$ and $8 \text{ mod}(8)$. For all other values of $N$, only the normal parts of the almost complex and quaternionic enveloping algebras form irreducible superfields.

However, even in the case of either normal enveloping algebras or restricting to the normal part of enveloping algebras, there is a remarkable property of these irreducible representations. In these cases there is a freedom to transmute “auxiliary fields” into “physical fields” and vice-versa. In the following, we display this property and show some connections between $\mathcal{GR}(d, N)$ and higher dimensional theories.

\textsuperscript{13}From the mathematical literature, the is known to occur for quaternionic case.  
\textsuperscript{14}It may be the case that there occur a full $S^2$ of such twistings resulting from the distinct ways of choosing a complex structure on the quaternions.
3.1 Auxiliary/Physical Field Duality in $\mathcal{GR}(d, N)$

Since reduction on a cylinder is a universal procedure, it can be applied to any field theory in any dimension greater than one. Let us begin with the scalar and spinor multiplets of the well known heterotic string. The starting point includes the (1,0) “scalar” superfields and (1,0) “heterotic fermion” superfields,

$$X_\mu(\zeta^+, \tau, \sigma) = X_\mu(\tau, \sigma) + \zeta^+ \psi_+^\mu(\tau, \sigma) \quad ,$$  \hspace{1cm} \lambda_-(\zeta^+, \tau, \sigma) = \eta_-(\tau, \sigma) + i \zeta^+ F^\tau(\tau, \sigma) \ .$$  

(49)

The well-known supersymmetrically invariant actions for these take the forms

$$S_{\text{scalar}} = \int d^2 \sigma d\zeta^+ \left[ i \frac{1}{2} \eta_\mu \xi^\mu \partial_\tau D_{\zeta} X_\mu \right] ,$$

$$S_{\text{spinor}} = \int d^2 \sigma d\zeta^+ \left[ - \frac{1}{2} \eta_\mu \xi^\mu \partial_\tau \eta_\mu + i \frac{1}{2} \eta_\mu \eta_\mu \partial_\tau \psi_+^\mu \right] .$$

(50)

The 2D actions in (50), under the action of the reduction, become

$$S_{\text{scalar}} = \int d\tau d\zeta \left[ i \frac{1}{2} X \partial_\tau D_\zeta X \right] = \int d\tau \left[ - \frac{1}{2} X \partial_\tau^2 X - i \frac{1}{2} \psi \partial_\tau \psi \right] ,$$

$$S_{\text{spinor}} = \int d\tau d\zeta \left[ \frac{1}{2} \lambda \partial_\zeta \lambda \right] ,$$

(53)

$$= \int d^2 \sigma \left[ - i \frac{1}{2} \eta \partial_\tau \eta + \frac{1}{2} F F \right] .$$

(54)
The \((X, \psi)\) multiplet and the \((\eta, F)\) multiplet are both representations of \(\mathcal{GR}(1, 1)\). For the first of these we may make the identifications \(X \in \mathcal{V}_L\) and \(\psi \in \mathcal{V}_R\) and for the second \(\eta \in \mathcal{V}_L\) and \(F \in \mathcal{V}_R\).

There is a transformation that maps one of these representation into the other. This can be seen by simply making the following field re-definitions

\[
(X, \psi) \leftrightarrow (\partial^{-1}_\tau F, \eta),
\]

which has the effect of transforming the two actions one into the other, \(S_{\text{scalar}} \leftrightarrow S_{\text{spinor}}\). An interesting feature of the re-definition in (55) is that although it involves a formally non-local transformation, there is no sign of the non-locality (after implementing the map) in either the transformation laws or the actions. A more remarkable feature of the map defined by (55) is that it acts to map the “physical field” \(X\) in the 1D scalar multiplet into the “auxiliary field” \(F\) of the 1D spinor multiplet and vice-versa. So within the representation theory of \(\mathcal{GR}(d, N)\) bosonic fields which in higher dimensions correspond to propagating and auxiliary fields are accorded a unified treatment and a “duality” map (55) exists between them.

One other fact to note about the map in (55) is that from the point of view of the superfields it corresponds to

\[
\Lambda \leftrightarrow -i D_\zeta X.
\]

Thus implementing the map also has the effect upon a superfield construction of “changing” where the component fields appear in the \(\zeta\)-expansion of superfields. It should noted that the equation in (56) is particular to 1D. In the original 2D theory (49), (50) the 2D Lorentz invariance forbids the possibility to write (56). It should also be clear that only in the case of equal numbers of scalar multiplets and spinor multiplets can such a transformation be implemented.

### 3.2 “Root Superfields” in \(\mathcal{GR}(d, N)\)

The observation that such non-local transformations exist for \(\mathcal{GR}(d, N)\) allows for an interesting generalization for higher values of \(d\) and \(N\) when we start with superfields that are defined by Clifford algebra expansions as described in (47). Due to their definitions, it follows that we may write

\[
\Phi = \phi(\tau) I + \phi_{11} f_{12} \phi_{12} + \phi_{11} f_{12} f_{13} \phi_{13} + \ldots, \\
\Psi = \psi_{11} \hat{f}_{12} + \psi_{12} \hat{f}_{12} \hat{f}_{13} + \ldots.
\]

18
Written with these definitions, the component fields are analogous to components of $\mathbf{X}$. However, we observe that there exist many distinct “dualities” like that defined in (55) that may be implemented on these components. The “dualized” superfields have the forms

$$\tilde{\Phi} = [(\partial_\tau)^{a_0} \Phi] \mathbf{I} + [(\partial_\tau)^{a_2} \phi_{1_1 1_2}] f_{1_1 1_2} + [(\partial_\tau)^{a_4} \phi_{1_1 1_2 1_3 1_4}] f_{1_1 1_2 1_3 1_4} + \ldots ,$$

$$\tilde{\Psi} = [(\partial_\tau)^{a_1} \psi_{1_1}] \hat{f}_{1_1} + [(\partial_\tau)^{a_3} \psi_{1_1 1_2 1_3}] \hat{f}_{1_1 1_2 1_3} + \ldots .$$

(58)

There are many different choices for the non-positive integer exponents $a_0, a_1, \ldots$ such that using (47) leads to purely local transformation laws among the component fields. However, just as in our “toy” example of looking at the effect of (55) on the $\zeta$-expansion of the equivalent superfields, here the different choices of the exponents “shift” the various component fields among the different $\zeta$-levels of Salam-Strathdee superfields. In the previous work of [1, 2] this freedom was exploited to define a set of component fields that have been suggested to provide an off-shell representation for the arbitrary $N$-extended spinning particle. In the chapter on applications, we will return to this proposed description.

It might at first appear that the 1D duality between physical bosons and auxiliary bosons is peculiar to the case of the heterotic starting point of our discussion. The next point we wish to make is that this is not the case and the simplest context for giving this demonstration is to consider the 2D, $N = 1$ scalar superfield

$$\mathbf{X}(\zeta^+, \zeta^-, \tau, \sigma) = X(\tau, \sigma) + \zeta^+ \psi^+ (\tau, \sigma) + \zeta^- \psi^- (\tau, \sigma) + i \zeta^+ \zeta^- F(\tau, \sigma).$$

(59)

After reduction to 1D on a cylinder this takes the form

$$\mathbf{X}(\zeta^+, \zeta^-, \tau) = X(\tau) + \zeta^+ \psi^+ (\tau) + \zeta^- \psi^- (\tau) + i \zeta^+ \zeta^- F(\tau).$$

(60)

Note that since the two spinors $\psi^+$ and $\psi^-$ as well as $\epsilon^+$ and $\epsilon^-$ are independent quantities, we retain the indices from 2D even though they no longer carry a representation of helicity.

The presence of the indices in 1D allows us to use superspace conjugation so that

$$X = [X]^*, \quad \psi^+ = -[\psi^+]^*, \quad \psi^- = -[\psi^-]^*, \quad F = [F]^*.$$

(61)

The supersymmetry variation of the superfield in (59) can also be reduced on a
cylinder and afterward takes the form

\[
\begin{align*}
\delta Q X &= \epsilon^+ \psi_+ + \epsilon^- \psi_- , \\
\delta Q \psi_+ &= i \epsilon^+ \partial_r X + i \epsilon^- F , \\
\delta Q \psi_- &= i \epsilon^- \partial_r X - i \epsilon^+ F , \\
\delta Q F &= - \left[ \epsilon^+ \partial_r \psi_- - \epsilon^- \partial_r \psi_+ \right].
\end{align*}
\]

(62)

so that when evaluated on any of the fields we have

\[
[\delta Q_1 , \delta Q_1] = - i 2 \left[ \epsilon_1^+ \epsilon_2^+ \partial_r + \epsilon_1^- \epsilon_2^- \partial_r \right].
\]

(63)

We next take a Clifford algebraic superfield of the form given in (58) expanded over the normal part of \( U_L(2, 2) \oplus M_L(2, 2) \)

\[
\begin{align*}
\Phi &= \left[ (\partial_r)^{a_0} \phi \right] I + \left[ (\partial_r)^{a_2} \phi_{i_1 i_2} \right] f^{i_1 i_2} , \\
\Psi &= \left[ (\partial_r)^{a_1} \psi_{i_1} \right] \hat{f}^{i_1} .
\end{align*}
\]

(64)

Upon making the identifications \( a_0 = a_1 = 0, \ a_2 = -1 \) and

\[
\alpha^i = (\epsilon^+, \epsilon^-) , \ \psi^i = (\psi_+, \psi_-) , \ \phi = X , \ \phi_{i_1 i_2} = \epsilon_{i_1 i_2} F .
\]

(65)

it is seen that the transformation law in (47) exactly reproduces the results in (62). An interesting feature to note is that the exponent for \( F \) takes on a different value from those of \( X \) and \( \psi^i \). Since the former is known to be an “auxiliary” field while the latter two are “physical” fields, this suggest that the presence of the operator \( \partial^{-1} \) is associated with this distinction.

We have seen that there is a sense in which the 1D Clifford-algebraic superfield in (64) “encodes” the structure of the 2D, \( N = 1 \) superfield in (59). We will refer to the former of these superfields as the “root” superfield for the latter. This brings us to a conjecture

\[
All \ superfields \ that \ provide \ a \ linear \ representation \ of \ spacetime \ super-symmetry \ in \ all \ dimensions \ can \ be \ represented \ as \ Clifford-algebraic \ root \ superfields.
\]

It is our eventual hope that the root superfield concept will prove useful in understanding off-shell representation theory in higher dimensions as the concepts of “roots and weights” play a similar role in Lie algebra theory.
3.3 Higher Dimensional Off-shell v.s. On-shell SUSY and Embedding in $\mathcal{GR}(d, N)$ Representations

The fact that higher dimensional superfields can be related to 1D root superfields has other interesting implications for making a Clifford-algebraic distinction between on-shell superfields and off-shell superfields. This can be seen by re-considering the 2D, $N = 1$ scalar multiplet in (59). The 2D, $N = 1$ off-shell supersymmetry variations take the form

$$
\delta_Q X = \epsilon^+ \psi_+ + \epsilon^- \psi_- ,
$$
$$
\delta_Q \psi_+ = i \epsilon^+ \partial_+ X + i \epsilon^- F ,
$$
$$
\delta_Q \psi_- = i \epsilon^- \partial_- X - i \epsilon^+ F ,
$$
$$
\delta_Q F = - \left[ \epsilon^+ \partial_\pm \psi_+ - \epsilon^- \partial_\pm \psi_- \right] .
$$

so that when evaluated on any of the fields we have

$$
[\delta_{Q_1}, \delta_{Q_2}] = - i 2 \left[ \epsilon_1^+ \epsilon_2^+ \partial_+ + \epsilon_1^- \epsilon_2^- \partial_- \right] .
$$

In the previous section, we established that the reduction of these off-shell variations yields the 1D variations that appear in (62). In turn these variations could be embedded into the transformation law of (47) if the Clifford-algebraic superfield was expanded over the normal part of $U_L(2, 2) \oplus M_L(2, 2)$ embedded in $EGR(2, 2)$.

Let us consider the on-shell massless limit of (66) which begins by imposing the condition that the usual auxiliary component field $F$ should be subjected to the algebraic condition $F = 0$. From the last line in (66), if $F = 0$, it follows that

$$
\partial_\pm \psi_\pm = 0 .
$$

The solutions to these equations are very simple, namely

$$
\psi_- = \psi_- (\sigma^-) , \quad \psi_+ = \psi_+ (\sigma^+) .
$$

Subject to these restrictions and as well using the restriction $F = 0$, we next apply $\partial_- \partial_-$ to the second line of (66) (or alternately applying $\partial_\pm$ to the third line of (66)). We find

$$
\partial_\pm \partial_\pm X = 0 ,
$$

whose solution is given by

$$
X = X_L(\sigma^+) + X_R(\sigma^-) .
$$
The solutions in (69) and (71) can be substituted back into (66) and since the 2D light-cone coordinates $\sigma^+$ and $\sigma^-$ are independent, it follows that the equations in (71) can actually be separated into two distinct sets of equations so that we can write,

$$
\delta_Q [X_L(\sigma^+)] = \epsilon^+ \psi_+(\sigma^+), \quad \delta_Q [\psi_+(\sigma^+)] = i\epsilon^+ \partial_+ X_L(\sigma^+),
$$
$$
\delta_Q [X_R(\sigma^-)] = \epsilon^- \psi_-(\sigma^-), \quad \delta_Q [\psi_-(\sigma^-)] = i\epsilon^- \partial_- X_R(\sigma^-).
$$

(72)

Now we can reduce these results on a cylinder and then ask, “What $\mathcal{GR}(d, N)$ Clifford-algebraic superfields can re-produce these results?” The answer turns out to be rather interesting. As can easily be seen in (72), the component fields there actually form two distinct representations; $X_L$ and $\psi_+$ form one representation and $X_R$ and $\psi_-$ form another. The Clifford-algebraic superfields that produces these transformation laws are valued in $\mathcal{EGR}(1, 1)$. Stated another way, the results in (72) constitute a reducible representation of $\mathcal{EGR}(1, 1)$.

The lesson from the example is starkly clear. The $\mathcal{GR}(d, N)$ characterization of the same multiplet changes depending on whether the multiplet is on-shell or off-shell. Stated another way, for a given supermultiplet, there is an algebraic way to distinguish between its on-shell versus off-shell representation. From this explicit example, we are led to a second conjecture regarding the off-shell versus on-shell distinction for spacetime supersymmetric representations when viewed from their embedding into $\mathcal{GR}(d, N)$

If an on-shell supermultiplet is embedded into a representation of $\mathcal{EGR}(d_N, N)$, then an off-shell representation of this supermultiplet is embedded into $\mathcal{EGR}(d_{2N}, 2N)$.

3.4 4D Chiral Superfield on a D0-Brane

In light of our presentation in the previous chapter, it seems plausible that we might gain insight into the possibility of embedding the superfield representations of 4D spacetime supersymmetry. Let us begin with a 1D representation that bears a striking resemblance to the 4D, $N = 1$ chiral superfield. We introduce a multiplet $(\mathcal{Z}(\tau), \varphi_1(\tau), F(\tau))$ that forms an 1D, $N = 4$ scalar multiplet. The global supersymmetry variations of these fields when reduced to 1D read

$$
\delta Q \mathcal{Z} = \epsilon^I \varphi_I, \quad \delta Q \varphi_I = i\tau_I (\partial_\tau \mathcal{Z}) + i\epsilon^K \epsilon_{K I} F, \quad \delta Q F = \tau_I (\partial_\tau \varphi_K) \epsilon^{I K},
$$

(73)
A few words about notation are in order. These results are directly obtained from the reduction. The fields \((Z(\tau), \varphi_1(\tau), F(\tau))\) are all complex and functions of \(\tau\). The indices \(I\) on the supersymmetry parameter \(\epsilon\) and the fermion field \(\varphi\) in 1D correspond to an isospin index that takes on two values. This is not to be identified with the index \(I\) which takes on four values.

When evaluated on the fields that appear in (73), the commutator algebra takes the form

\[
[\delta_{Q_1}, \delta_{Q_2}] = -i (\epsilon_1^I \bar{\epsilon}_2^I + \bar{\epsilon}_1^I \epsilon_2^I) \partial_\tau ,
\]  

(74)

but this is not the basis for extracting the \(G\mathcal{R}(d, N)\) structure of the theory. For this purpose it is necessary to express the supersymmetry parameter and fields in terms of real quantities by writing

\[
\epsilon^I \equiv (\epsilon^{1(1)} + i\epsilon^{1(2)}, \epsilon^{2(1)} + i\epsilon^{2(2)}) ,
\]

\[
Z \equiv A + iB , \quad F \equiv F + iG ,
\]

\[
\varphi_I \equiv (\varphi_{1(1)} + i\varphi_{1(2)}, \varphi_{2(1)} + i\varphi_{2(2)}) ,
\]

(75)

Once these definitions are made, then it is possible to introduce four component quantities \(\alpha^I\) and \(\psi^I\) via the definitions

\[
\alpha^I = (\epsilon^{1(1)}, \epsilon^{1(2)}, \epsilon^{2(1)}, \epsilon^{2(2)}) ,
\]

\[
\psi^I = -i (\varphi_{1(1)}, \varphi_{1(2)}, \varphi_{2(1)}, \varphi_{2(2)}) .
\]

(76)

It is a simple matter to verify that (74) using the definitions in (75) and (76) takes the form of (6).

Finally, the transformation laws in (73) must be expressed in terms of \(A, B, F, G, \psi^I\) and \(\epsilon^I\) in order to uncover how the 4D, \(N = 1\) chiral multiplet is embedded into representations of \(G\mathcal{R}(d, N)\). In terms of these, we find

\[
\delta_{Q\alpha} = i [\alpha^1 \psi^2 + \alpha^2 \psi^1 + \alpha^3 \psi^4 + \alpha^4 \psi^3 ] ,
\]

\[
\delta_{Q\beta} = i [ -\alpha^1 \psi^1 + \alpha^2 \psi^2 - \alpha^3 \psi^3 + \alpha^4 \psi^4 ] ,
\]

\[
\delta_{Q\psi^1} = [ -\alpha^1 (\partial_\tau B) + \alpha^2 (\partial_\tau A) + \alpha^3 G + \alpha^4 F ] ,
\]

\[
\delta_{Q\psi^2} = [ \alpha^1 (\partial_\tau A) + \alpha^2 (\partial_\tau B) - \alpha^3 F + \alpha^4 G ] ,
\]

\[
\delta_{Q\psi^3} = [ -\alpha^1 G - \alpha^2 F - \alpha^3 (\partial_\tau B) + \alpha^4 (\partial_\tau A) ] ,
\]

\[
\delta_{Q\psi^4} = [ \alpha^1 F - \alpha^2 G + \alpha^3 (\partial_\tau A) + \alpha^4 (\partial_\tau B) ] ,
\]

\[
\delta_{QF} = i \partial_\tau [\alpha^1 \psi^4 - \alpha^2 \psi^3 - \alpha^3 \psi^2 + \alpha^4 \psi^1 ] ,
\]

\[
\delta_{QG} = i \partial_\tau [-\alpha^1 \psi^3 - \alpha^2 \psi^4 + \alpha^3 \psi^1 + \alpha^4 \psi^2 ] .
\]

(77)
It is clearly the case that these variations can be written in the forms
\[\delta_Q A = i \alpha^I L^{(A)}_{i k} \psi^K, \quad \delta_Q B = i \alpha^I L^{(B)}_{i k} \psi^K,\]
\[\delta_Q \psi^K = - \left[ R^{(A)KL} (\partial_\tau A) + R^{(B)KL} (\partial_\tau B) + R^{(F)KL} F + R^{(G)KL} G \right] \alpha^L,\]
\[\delta_Q F = i \alpha^I L^{(F)}_{i k} \partial_\tau \psi^K, \quad \delta_Q G = i \alpha^I L^{(G)}_{i k} \partial_\tau \psi^K,\]
(78)
in terms of some constant coefficients (the L’s and R’s). These variations satisfy (6) when evaluated on any of the real fields. The explicit forms of the L and R quantities are given by
\[\left( L^{(A)}, L^{(B)}, L^{(F)}, L^{(G)} \right) = \left( I \otimes \sigma^1, -I \otimes \sigma^3, -\sigma^2 \otimes \sigma^2, -i\sigma^2 \otimes I \right),\]
\[\left( - R^{(A)}, - R^{(B)}, - R^{(F)}, R^{(G)} \right).\]
(79)

If we use the symbol \(\mathcal{F}\) where \(\mathcal{F} = \{(A), (B), (F), (G)\}\) to denote the different bosonic fields then the L’s and R’s satisfy the algebra of \(\mathcal{G}\mathcal{R}(4)\):
\[L^F_{i k} R^F_{k l} + L^F_{i k} R^F_{k l} = -2 \delta^F_{i l} \delta^F_{k L}.\]
(80)
Consequently, we are forced to conclude that the bosonic fields A, B, F and G constitute the same representation of \(\mathcal{G}\mathcal{R}(d, N)\) as the four components of \(\psi^K\) or \(\alpha^K\). This is true in spite of the fact that the engineering dimensions of the first two bosonic fields are different from that of the later two bosonic fields. The condition (80) insures that the usual supersymmetry algebra is obeyed on all the bosonic fields.

The closure of the algebra on the fermionic fields, however, requires something quite different. In particular, the Fierz identity
\[\sum_{F} \left( L^F_{i k} R^F_{l m} + L^F_{m k} R^F_{l i} \right) = -2 \delta^F_{i m} \delta^F_{j k}.\]
(81)
must be satisfied. Direct calculation using the representation in (79) shows that it is. We emphasize that not all L’s and R’s that provide representations of \(\mathcal{G}\mathcal{R}(d, N)\) satisfy this Fierz condition.

### 3.5 4D Chiral Superfield Alternate Embedding

In our just concluded discussion, we have seen that the fundamental representation of 4D, \(N = 1\) supersymmetry, the chiral multiplet, does indeed provide a realization of structures associated with the geometry indicated in figure one for \(\mathcal{G}\mathcal{R}(4, 4)\). There is also a second way to interpret the chiral multiplet, namely it is also a representation of
\( \mathcal{EGR}(2, 2) \). To begin this demonstration we first introduce four \( \tau \)-dependent “fields” \( \hat{\Psi}_{k l}, \Psi_{k l}, \Phi_{k l} \) and \( \Phi_{k l} \) so that

\[
\hat{\Psi}_{k l} \in \{\mathcal{M}_L\} \quad , \quad \Psi_{k l} \in \{\mathcal{M}_R\} \quad , \\
\Phi_{k l} \in \{\mathcal{U}_L\} \quad , \quad \Phi_{k l} \in \{\mathcal{U}_R\} .
\]  

(82)

so that collectively\(^{15}\) these are valued in the entirety of \( \mathcal{EGR}(2, 2) \) as indicated in (10).

We next propose as their transformation laws,

\[
\delta Q \Phi_{k l} = i \alpha^I (L^I)_k^\ell \hat{\Psi}_{\ell l} + i \bar{\alpha}^I (L^I)_l^\ell \hat{\Phi}_{k \ell} , \\
\delta Q \Psi_{k l} = - \alpha^I (R^I)_k^\ell \partial_\tau \Phi_{\ell l} + \bar{\alpha}^I (L^I)_l^\ell \partial_\tau \hat{\Phi}_{k \ell} , \\
\delta Q \hat{\Psi}_{k l} = - \alpha^I (L^I)_k^\ell \partial_\tau \hat{\Phi}_{\ell l} - \bar{\alpha}^I (R^I)_l^\ell \partial_\tau \Phi_{k \ell} , \\
\delta Q \hat{\Phi}_{k l} = i \alpha^I (R^I)_k^\ell \hat{\Psi}_{\ell l} - i \bar{\alpha}^I (R^I)_l^\ell \Psi_{k \ell} .
\]  

(83)

These variations close under commutation to:

\[
[\delta_1, \delta_2] = - i 2 (\alpha^1 \alpha^2 + \bar{\alpha}^1 \bar{\alpha}^2) \partial_\tau ,
\]  

(84)

In \( \mathcal{G}R(2, 2) \) we have the following conventions and identities:

\[
X^{[IJ]} = \epsilon^{[IJ]} \epsilon^{MN} X^{MN} \equiv \epsilon^{[IJ]} \epsilon \cdot X , \\
f^{[IJ]} \equiv \frac{1}{2} L^{[I} R^{J]} = \frac{1}{2} \epsilon^{[IJ]} \epsilon \cdot f , \\
L^I R^J = - \delta^{IJ} + \frac{1}{2} \epsilon^{IJ} \epsilon \cdot f , \\
Tr[\epsilon \cdot f] = 0 , \quad (\epsilon \cdot f)^2 = - 4 \mathbf{I} .
\]  

(85)

We expand the \( \mathcal{EGR}(2, 2) \) fields in the following manner:

\[
\Phi_{k l} = \frac{1}{2} \delta_{k l} B + \frac{1}{4} (\epsilon \cdot f)_{k l} (\partial_\tau)^{-1} G , \\
\hat{\Phi}_{k l} = \frac{1}{2} \delta_{k l} A + \frac{1}{4} (\epsilon \cdot \hat{f})_{k l} (\partial_\tau)^{-1} F , \\
\Psi_{k l} = - \frac{1}{2} R^I_{k l} \psi^I , \\
\hat{\Psi}_{k l} = - \frac{1}{2} L^I_{k l} \hat{\psi}^I .
\]  

(86)

We propose that these Clifford algebraic superfields may be regarded as “root” superfields for the 4D, \( N = 1 \) chiral multiplet.

Using \( L^I = -(R^I)^T \) to do proper matrix multiplication when necessary, we can

\(^{15}\)The fields do not, however, completely saturate these spaces.
extract the “component” results from the above variations.

\[
\begin{align*}
\delta Q A &= i\alpha^1\tilde{\psi}^1 + i\bar{\alpha}^1\psi^1 \\
\delta Q B &= i\alpha^1\psi^1 - i\bar{\alpha}^1\tilde{\psi}^1 \\
\delta Q F &= -i\alpha^1\epsilon^{1ij}\partial_r\tilde{\psi}^j + i\bar{\alpha}^1\epsilon^{1ij}\partial_r\psi^j \\
\delta Q G &= -i\alpha^1\epsilon^{1ij}\partial_r\psi^j - i\bar{\alpha}^1\epsilon^{1ij}\partial_r\tilde{\psi}^j \\
\delta Q \psi^1 &= \alpha^1\partial_r B + \bar{\alpha}^1\partial_r A - \alpha^1\epsilon^{3ij}G + \bar{\alpha}^1\epsilon^{3ij}F \\
\delta Q \tilde{\psi}^1 &= \alpha^1\partial_r A - \bar{\alpha}^1\partial_r B - \alpha^1\epsilon^{3ij}F - \bar{\alpha}^1\epsilon^{3ij}G
\end{align*}
\] (87)

These variations are exactly those that appear in (77) after the following redefinitions:

\[
\begin{align*}
\alpha^1 &\to \tilde{\alpha}^1, \quad \alpha^2 \to \alpha^1, \quad \alpha^3 \to \bar{\alpha}^2, \quad \alpha^4 \to \alpha^2, \\
\psi^1 &\to \tilde{\psi}^1, \quad \psi^2 \to \psi^1, \quad \psi^3 \to \tilde{\psi}^2, \quad \psi^4 \to \psi^2.
\end{align*}
\] (88)

One of the interesting points about this embedding of the transformations in (77) into representations of EGR(2, 2) is that it suggests that a non-trivial role may be played by the \(D\)-element associated with this algebra. In particular, the expansions in (86) are not not unique. Referring back to the results in Table I, we see that EGR(2, 2) is one of the almost complex cases. This implies that we can utilize the alternate expansion given by

\[
\begin{align*}
\Phi'_{kl} &= \frac{1}{2}D_{kl} B' + \frac{1}{4}(\epsilon \cdot fD)_{kl}(\partial_r)^{-1}G', \\
\tilde{\Phi}'_{kl} &= \frac{1}{2}\tilde{D}_{kl} A' + \frac{1}{4}(\epsilon \cdot f\tilde{D})_{kl}(\partial_r)^{-1}F', \\
\Psi'_{kl} &= -\frac{1}{2}(L^1\tilde{D})_{kl}\psi'^1, \\
\tilde{\Psi}'_{kl} &= -\frac{1}{2}(R^1D)_{kl}\tilde{\psi}'^1,
\end{align*}
\] (89)

where we use the prime superscript to distinguish these fields from those that appear in (86).

In order to find the form of the supersymmetry transformation laws for this representation, we propose the ansatz below.

\[
\begin{align*}
\delta Q \Phi'_{k\ell} &= i\alpha^1(L^1)_{k}^{\ell} \tilde{\Psi}'_{\ell i} + i\bar{\alpha}^1(R^1)_{k}^{\ell} \Psi'_{k\ell}, \\
\delta Q \Psi'_{k\ell} &= \alpha^1(L^1)_{k}^{\ell} \partial_r\tilde{\Phi}'_{\ell i} - \bar{\alpha}^1(R^1)_{k}^{\ell} \partial_r\Phi'_{k\ell}, \\
\delta Q \tilde{\Psi}'_{k\ell} &= -\alpha^1(R^1)_{k}^{\ell} \partial_r\tilde{\Phi}'_{\ell i} - \bar{\alpha}^1(L^1)_{k}^{\ell} \partial_r\Phi'_{k\ell}, \\
\delta Q \tilde{\Phi}'_{k\ell} &= -i\alpha^1(R^1)_{k}^{\ell} \Psi'_{\ell i} + i\bar{\alpha}^1(L^1)_{k}^{\ell} \tilde{\Psi}'_{k\ell}.
\end{align*}
\] (90)

Now it is a fact that when the expansions in (89) are substituted into the variations in (90), the component fields in (89) are found to obey exactly the same transformation
laws as appear in (87). In other words, this is an alternate description of the same multiplet.

This leads to a minor dilemma. If the descriptions obtained previously describe the chiral multiplet, how then is the “anti-chiral multiplet” to be described? A simple solution is provided by using local and non-local re-definitions as appear in (55). In particular we introduce new expansions

\[ \Phi_{kl}^c = \frac{1}{2} \delta_{kl} (\partial_\tau)^{-1} G^c + \frac{1}{4} (\epsilon \cdot f)_{kl} B^c, \]

\[ \hat{\Phi}_{kl}^c = \frac{1}{2} \delta_{kl} (\partial_\tau)^{-1} F^c + \frac{1}{4} (\epsilon \cdot \hat{f})_{kl} A^c, \]

\[ \Psi_{kl}^c = -\frac{1}{2} R^{1}_{kl} \psi^e, \]

\[ \hat{\Psi}_{ik}^c = -\frac{1}{2} R^{1}_{ik} \hat{\psi}^e. \]

and use the variations from (83) The motivation for this can be seen by a review of the relation between chiral and anti-chiral multiplets in 4D.

In 4D, \( N = 1 \) superspace, the component fields \( (A, B, \psi_\alpha, F \) and \( G \) of a chiral multiplet may be defined by the \( D \)-expansion of a chiral superfield

\[ A(x) + iB(x) \equiv \Phi|, \psi_\alpha(x) \equiv D_\alpha \Phi|, F(x) + iG(x) \equiv D^2 \Phi|. \]

But the component fields \( (A^c, B^c, \psi^e_\alpha, F^c \) and \( G^c \) of an \( \) anti-chiral multiplet can also be defined from the same superfield

\[ A^c(x) + iB^c(x) \equiv D^2 \Phi|, \psi^e_\alpha(x) \equiv i \partial_\alpha D^a \Phi|, F^c(x) + iG^c(x) \equiv \Box \Phi|\]

In particular, it can be seen that the spinor in the anti-chiral multiplet is related to the spinor in the chiral multiplet by

\[ \overline{\psi}_\alpha^c(x) = i \partial_\alpha \psi^e_\alpha(x). \]

Using the expansions in (91) and the variations in (84) leads to the component results

\[ \delta A^c = -i \alpha^I \epsilon^{IJ} \hat{\psi}^c J + i \bar{\alpha}^I \epsilon^{IJ} \hat{\psi}^c J, \]

\[ \delta B^c = -i \alpha^I \epsilon^{IJ} \psi^e J - i \bar{\alpha}^I \epsilon^{IJ} \hat{\psi}^c J, \]

\[ \delta F^c = i \alpha^I \partial_\tau \hat{\psi}^e 1 + i \bar{\alpha}^I \partial_\tau \psi^e 1, \]

\[ \delta G^c = i \alpha^I \partial_\tau \psi^e 1 - i \alpha^I \partial_\tau \hat{\psi}^c 1, \]

\[ \delta \psi^e 1 = -\alpha^J \epsilon^{JI} \partial_\tau B^c + \bar{\alpha}^J \epsilon^{JI} \partial_\tau A^c + \alpha^I G^c + \bar{\alpha}^I F^c, \]

\[ \delta \hat{\psi}^c 1 = -\alpha^J \epsilon^{JI} \partial_\tau A^c - \bar{\alpha}^J \epsilon^{JI} \partial_\tau B^c + \alpha^I F^c - \bar{\alpha}^I G^c. \]
Now the transformation laws in (95) are related to those in (83) via the following field re-definitions.

\[
\begin{align*}
\partial_\tau A &= F^c, \quad \partial_\tau \psi^I = \psi^{cI}, \quad F = \partial_\tau A^c, \\
\partial_\tau B &= G^c, \quad \partial_\tau \hat{\psi}^I = \hat{\psi}^{cI}, \quad G = \partial_\tau B^c.
\end{align*}
\] (96)

It is obvious now that spinors in our proposed 1D, \(N = 4\) “chiral multiplet” are related to the spinors in our proposed 1D, \(N = 4\) “anti-chiral multiplet” in a manner similar to the 4D, \(N = 1\) theory. All of this suggests that the representation in (91) is the root superfield representation of the usual anti-chiral multiplet familiar from 4D, \(N = 1\) supersymmetry.

### 3.6 Higher Rank \(\mathcal{EGR}(2, 2)\) Representations

Let us call the quartet of fields

\[
\begin{align*}
\Phi_{k_1 k_2} \\
\Psi_{\hat{k}_1 \hat{k}_2} \\
\hat{\Psi}_{k_1 \hat{k}_2} \\
\hat{\Phi}_{\hat{k}_1 \hat{k}_2}
\end{align*}
\] (97)

a rank two representation. Having seen evidence this quartet of fields forms a 1D, \(N = 4\) representation that is closely related to the 4D, \(N = 1\) chiral multiplet suggests that there should exist even more complicated 1D, \(N = 4\) \(\mathcal{EGR}(2, 2)\) representations related to other 4D, \(N = 1\) supermultiplets. Let us discuss the general outline of this possible relation. As a first step let us introduce an alternate geometrical interpretation of the quartet above. We can consider a cartesian product space of the form

\[
\mathcal{P}_2 \equiv \mathcal{V}_L^{(1)} \times \mathcal{V}_L^{(2)} \times \mathcal{V}_R^{(1)} \times \mathcal{V}_R^{(2)}.
\] (98)

Next we introduce a set of maps that act on rank-two projections of this space into the real numbers defined by

\[
\begin{align*}
\Phi_{k_1 k_2} : \mathcal{V}_L^{(1)} \times \mathcal{V}_L^{(2)} &\rightarrow \mathbb{R}^1, \\
\Psi_{\hat{k}_1 \hat{k}_2} : \mathcal{V}_R^{(1)} \times \mathcal{V}_L^{(2)} &\rightarrow \mathbb{R}^1, \\
\hat{\Psi}_{k_1 \hat{k}_2} : \mathcal{V}_L^{(1)} \times \mathcal{V}_R^{(2)} &\rightarrow \mathbb{R}^1, \\
\hat{\Phi}_{\hat{k}_1 \hat{k}_2} : \mathcal{V}_R^{(1)} \times \mathcal{V}_R^{(2)} &\rightarrow \mathbb{R}^1.
\end{align*}
\] (99)

These coefficients are precisely the quartet that appears in (97).
The generalization of this construction is obvious. These begin by the introduction of

\[ P_n \equiv V^{(1)}_L \times \ldots \times V^{(n)}_L \times V^{(1)}_R \times \ldots \times V^{(n)}_R. \]  

Next one can consider all rank \( n \) tensors that map rank \( n \) projections of \( P_n \) into \( \mathbb{R}^1 \).

For \( n = 3 \), this leads to the fields

\[
\begin{align*}
\Phi_{k_1 k_2 k_3} \\
\psi_{\bar{k}_1 k_2 k_3} & \quad \psi_{k_1 k_2 k_3} & \quad \psi_{k_1 k_2 \hat{k}_3} \\
\psi_{k_1 \hat{k}_2 k_3} & \quad \psi_{\bar{k}_1 k_2 k_3} & \quad \psi_{k_1 k_2 k_3} \\
\Phi_{\bar{k}_1 \bar{k}_2 \bar{k}_3} & 
\end{align*}
\]

or the \( n = 4 \) representation of the form

\[
\begin{align*}
\Phi_{k_1 k_2 k_3 k_4} \\
\psi_{\bar{k}_1 k_2 k_3 k_4} & \quad \psi_{k_1 k_2 k_3 k_4} & \quad \psi_{k_1 k_2 k_3 k_4} & \quad \psi_{k_1 k_2 k_3 \hat{k}_4} \\
\Phi_{k_1 \bar{k}_2 k_3 k_4} & \quad \Phi_{k_1 k_2 \bar{k}_3 k_4} & \quad \Phi_{k_1 k_2 k_3 \hat{k}_4} & \quad \Phi_{k_1 k_2 k_3 k_4} & \quad \Phi_{k_1 k_2 k_3 \hat{k}_4} \\
\psi_{\bar{k}_1 \bar{k}_2 \bar{k}_3 k_4} & \quad \psi_{k_1 \bar{k}_2 k_3 k_4} & \quad \psi_{k_1 k_2 \bar{k}_3 k_4} & \quad \psi_{k_1 k_2 k_3 \hat{k}_4} & \quad \psi_{k_1 k_2 k_3 k_4} \\
\Phi_{\bar{k}_1 \bar{k}_2 \bar{k}_3 \bar{k}_4} &
\end{align*}
\]

(On each row, the number of fields is simply determined by the binomial coefficients.)

These simple considerations lead us to suggest that only the even rank representations can correspond to 4D, \( N = 1 \) superfields. The next question becomes whether there exist for (101) and (102) a set of variations analogous to (83) that satisfy, for these representations, the condition in (84)? We will not attempt to prove this but are confident that this is the case.

Our arguments suggest that the vector multiplet appears as a root superfield within the rank four structure associated with various representations of \( EGR(2, 2) \). The next highest even-rank structures, the rank six tensors, would therefore be associated with matter gravitino multiplets [8]. Finally the rank eight structures must contain the root representation of the 4D, \( N = 1 \) supergravity multiplet. However, the supersymmetry variations of the root superfield representation of both the matter gravitino multiplet and super gravity multiplet must be of the type that was suggested for the spinning particle models of [1, 2].

Since the representation in (97) seems to correspond to a chiral multiplet, the representation in (102) seems appropriate to describe the usual 4D, \( N = 1 \) vector multiplet.
One possible use for these higher rank root superfields is for counting the degrees of freedom of higher dimensional supersymmetric irreducible multiplets. Since dimensional reduction on a cylinder is well defined, any higher dimensional multiplet must be related to some $\mathcal{GR}$ representation. In order to do this counting, we must know the total number of components of the supersymmetry parameters. For example, in $D = 4, N = 1$ there are four components in the supersymmetry parameter. This means that we are looking for a $N = 4$ one dimensional representation. This can be obtained by using $\mathcal{GR}(4, 4)$, or by using $\mathcal{EGR}(2, 2)$. Both of these descriptions had four supersymmetry parameters. This also tells us that there are no smaller representations then the chiral field in four dimensions.

Could we do this for $N = 2$ in 4D or even higher 4D, $N$-extended representations? At present this is not clear. However, a hopeful starting point would be to begin with structures associated with $\mathcal{EGR}(2N, 2N)$.

4 Other Applications

In the following sections, we will present some concrete applications of the results discussed in this work. We should point out that these examples have appeared before in very brief discussions. So part of the benefit of our return to these is to give a much fuller description.

4.1 Off-shell 1D, $N$-extended Spinning Particles

We turn now to the discussion of free spinning particle (which was our original motivation for introducing these algebraic structures). Utilizing the concise $\mathcal{GR}(d, N)$ notation this action takes the form

$$S_{\text{Spng Part.}} = \int d\tau \left[ -i\frac{1}{2}d^{-1}(\mathcal{X}^i)^{\hat{k}}_{\hat{k}} \partial_\tau \mathcal{X}^i_{\hat{k}} - i\frac{1}{2} \pi_1 \partial_\tau \pi_1 - \frac{1}{2}P^2 - \frac{1}{2}d^{-1}((\mathcal{G}^i)^{\hat{j}}_{\hat{j}} \mathcal{G}^j_{\hat{j}}) - i \Psi_1 (\partial_\tau \pi_1) + P(\partial_\tau \mathcal{X}^i) + d^{-1}\mathcal{G}^j_{\hat{j}} F^i_{\hat{i}} + id^{-1}\mathcal{X}^i_{\hat{k}} \Lambda^i_{\hat{k}} \right].$$

As described in [1, 2], this action is actually constructed from two distinct multiplets; the USPM (the Universal Spinning Particle Multiplet; $X, \Psi_1, F^i_{\hat{j}}, \Lambda^i_{\hat{k}}$) and its canonically conjugate momentum multiplet ($\pi_1, \mathcal{X}^i_{\hat{k}}, P, \mathcal{G}^i_{\hat{j}}$). The fields $(X, F^i_{\hat{j}}) \oplus (\Psi_1, \Lambda^i_{\hat{k}})$ form a representation of the normal part of $\mathcal{U}_L \oplus \mathcal{M}_R$ and the fields $(P, \mathcal{G}^i_{\hat{j}}) \oplus (\pi_1, \mathcal{X}^i_{\hat{k}})$ form a representation of the normal part of $\mathcal{U}_R \oplus \mathcal{M}_L$. 

30
We have also used the result described in the fifth chapter. In our original presentation of these results [1, 2] we described how it was necessary to begin with a Cartesian product of the "iso-spinning particle" with a space isomorphic to $\mathcal{V}_L$. The iso-spinning particle is geometrically described by the superfield and transformation laws that appear in (5). After taking the Cartesian product, we arrive at the superfield and and transformation laws that appear in (47). Finally in our earlier works [1, 2], we wrote a theory that is described by a the superfield of the form that appears in (58). The exponents were chosen in such a way that only the 0-brane coordinate $X(\tau)$ and NSR fermions $\Psi_i(\tau)$ are propagating degrees of freedom.

The supersymmetry transformation laws of the spinning particle supermultiplets that leave the action invariant are given by

$$
\delta_Q X = i\alpha^1 \Psi_1,
\delta_Q \Psi_i = -\left[ \alpha_i (\partial_\tau X) + d^{-1} \alpha^j (f_{1j})^i \mathcal{F}^j \right],
\delta_Q \mathcal{F}^j = i\alpha^i (f_{1k})^j (\partial_\tau \Psi_k) + i\alpha^k (L_k)^i \Lambda_k^j,
\delta_Q \Lambda_k^j = \alpha^k \partial_\tau \left[ (R_k)_k^i \mathcal{F}^j + d^{-1} (R_1)_k^j (f_{1j})_1^i \mathcal{F}^i \right],
\delta_Q \pi^i = \alpha_i P + d^{-1} \alpha^k (f_{1k})_j \mathcal{G}^j,
\delta_Q \mathcal{G}^j = -\alpha^k (L_k)^i \mathcal{G}^j + d^{-1} \alpha^k (L_1^k)^i \mathcal{F}^i,
\delta_Q P = -i \alpha_i \partial_\tau \pi^i,
\delta_Q \mathcal{X}^i = -i \partial_\tau \left[ \alpha^j (f_{1j})^i \pi^j + \alpha^k (R_k)^i \mathcal{X}^j \right],
$$

(104)

and where $\mathcal{F}^j = (L^1)_j^k \Lambda_k^j = \mathcal{G}^j = (R_1)_k^i \mathcal{X}^j = 0$. These are equivalent to the transformation laws given in (47). A more geometrical way to understand these transformation laws is to note that the quantities $\Phi_1$, $\Psi_1$, $\Phi_2$, $\Psi_2$ defined by

$$
(\Phi_1)_ik \equiv X_{Ik} + \left[ (\partial_\tau)^{-1} \mathcal{F}^i k \right], \quad (\Psi_1)_{ki} \equiv \Psi_1 (\hat{f}_1)_{ki} + \left[ (\partial_\tau)^{-1} \Lambda_{ki} \right],
(\Phi_2)_{ik} \equiv P_{I} + G_{ik}, \quad (\Psi_2)_{ik} \equiv \pi_1 (f_{1ik}) + \mathcal{X}_{ik},
$$

(105)

are such that $\Phi_2$ and $\Psi_2$ have exactly the transformation law given in (47). Similarly $\Phi_1$ and $(\Psi_1)^I$ also have exactly the transformation law given in (47).

The action in (103) can be somewhat simplified by making three field redefinitions

$$
\mathcal{F}^j \to \mathcal{F}^j + \frac{1}{2} \mathcal{G}^j, \quad (\Lambda^i)_1^k \to (\Lambda^i)_1^k + \partial_\tau \mathcal{X}^k, \quad \Psi_i \to \Psi_i - \pi_i,
$$

(106)

so that the redefined action is

$$
S_{Spng Part.} = \int d\tau \left[ -\frac{1}{2} P^2 + P(\partial_\tau X) - i \Psi_i (\partial_\tau \pi_i) + d^{-1} \mathcal{G}^i \mathcal{F}^j + id^{-1} \mathcal{X}^k \Lambda_k^j \right].
$$

(107)
The bosonic fields \((F^i_j, G^i_j)\) and fermionic fields \((\Lambda^i_k, \chi^i_k)\) are to be expanded as elements only of the normal parts of the enveloping algebras\(^{17}\), subject to the algebraic constraints given below (104).

\[
F^i_j = \frac{1}{2} (f^{i_1i_2})^j_i F_{i_1i_2} + \frac{1}{4!} (f^{i_1i_2i_3i_4})^j_i F_{i_1i_2i_3i_4} + \cdots + \frac{1}{(2p)!} (f^{i_1\ldots i_{2p}})^j_i F_{i_1\ldots i_{2p}} , \\
\Lambda^i_k = \frac{1}{3!} (f^{i_1i_2i_3})^k_j \Lambda_{i_1i_2i_3} + \cdots + \frac{1}{(2q+1)!} (f^{i_1\ldots i_{2q+1}})^k_j \Lambda_{i_1\ldots i_{2q+1}} , \\
G^i_j = \frac{1}{2} (f^{i_1i_2})^j_i G_{i_1i_2} + \frac{1}{4!} (f^{i_1i_2i_3i_4})^j_i G_{i_1i_2i_3i_4} + \cdots + \frac{1}{(2p)!} (f^{i_1\ldots i_{2p}})^j_i G_{i_1\ldots i_{2p}} , \\
\chi^i_k = \frac{1}{3!} (f^{i_1i_2i_3})^k_j \chi_{i_1i_2i_3} + \cdots + \frac{1}{(2q+1)!} (f^{i_1\ldots i_{2q+1}})^k_j \chi_{i_1\ldots i_{2q+1}} . \tag{108}
\]

When these expansions are inserted into the spinning particle action it is seen to lead to

\[
S_{Spng \ Part.} = \int d\tau \left[ -\frac{1}{2} P^2 + P(\partial_\tau X) - i \Psi_I (\partial_\tau \pi_I) + \sum_\ell c_{2\ell} G_{i_1\ldots i_{2\ell}} F_{i_1\ldots i_{2\ell}} + i \sum_\ell c_{2\ell+1} \chi_{i_1\ldots i_{2\ell+1}} \Lambda_{i_1\ldots i_{2\ell+1}} \right] , \tag{109}
\]

here the values of the coefficients \(c_{2\ell}\) and \(c_{2\ell+1}\) can be found by using the trace relations (143) given in the appendix. Likewise, the expansion of (108) can be substituted into (104) in order to derive the explicit transformation laws of all of the component fields. For a fixed value of \(N\) there are \(2^N\) bosons and \(2^N\) fermions in the action of (109).

For the values \(N=1,2\) and 4, the action and theory described above is reducible. In these cases there is a truncation that may be performed to obtain a smaller representation. The reason for the exceptional nature of these cases can be traced back to the representation theory of the \(\mathcal{EGR}\) algebras. For these cases, the condition

\[
\dim(\{f_I\}) = 1 + \sum_{p=1}^{p_{max}} \dim(f_{i_1\ldots i_{2p}}) \tag{110}
\]

is satisfied for some integer \(p_{max}\). As a consequence the number of NSR fermions is equal to the number of propagating and auxiliary bosons. Whenever this condition is satisfied, the theory described above is reducible and the only cases of which we are aware are precisely \(N=1,2,\) and 4 which we discuss below.

For the exceptional \(N=4\) case, the action for the spinning particle is of the form

\[
S_{Spng \ Part.}^{N=4\text{ex}} = \int d\tau \left[ \frac{1}{2} (\partial_\tau X)(\partial_\tau X) + i \frac{1}{2} \Psi_I \partial_\tau \Psi_I + \frac{1}{4} F_{ij} F_{ij} \right] , \tag{111}
\]

\(^{17}\)This restriction to the normal parts of the enveloping algebra has not been stated in our previous works on these models.
where the auxiliary field satisfies \( F_{ij} = \frac{1}{2} \xi \epsilon_{ijkl} F_{kl} \) for \( \xi = \pm 1 \). The \( N = 4 \) action is invariant under the supersymmetry variations

\[
\begin{align*}
\delta_Q X &= i \alpha_i \Psi^i, \quad \delta_Q \Psi^i = -\alpha_i (\partial_\tau X) + \alpha_k F_{ki}, \\
\delta_Q F_{ij} &= -i \frac{1}{2} \left[ \alpha_i \partial_\tau \Psi^i - \alpha_j \partial_\tau \Psi^j + \xi \alpha_k \epsilon_{ijkl} (\partial_\tau \Psi^l) \right].
\end{align*}
\] (112)

The \( N = 2 \) exceptional truncation of this is given by

\[
S_{\text{Spng Part.}}^{N=2\text{ex}} = \int d\tau \left[ \frac{1}{2} (\partial_\tau X)(\partial_\tau X) + \frac{1}{2} F F \right],
\] (113)

with transformation laws given by

\[
\begin{align*}
\delta_Q X &= i \alpha \Psi, \\
\delta_Q \Psi &= -\alpha (\partial_\tau X).
\end{align*}
\] (114)

Finally there is the \( N = 1 \) theory

\[
S_{\text{Spng Part.}}^{N=1\text{ex}} = \int d\tau \left[ \frac{1}{2} (\partial_\tau X)(\partial_\tau X) + \frac{1}{2} F F \right],
\] (115)

with transformation laws given by

\[
\begin{align*}
\delta_Q X &= i \alpha \Psi, \\
\delta_Q \Psi &= -\alpha (\partial_\tau X).
\end{align*}
\] (116)

Up to this point in our discussion, all dynamical variables under consideration were ordinary functions of \( \tau \), our time-like variable. We now wish to consider a superspace formalism that is compatible with our previous discussion. We extend our one dimensional world-line parametrized by \( \tau \) to a one dimensional super world-line parametrized by \( (\tau, \zeta^I) \) where \( \zeta^I \) is a Grassmann coordinate\(^{18}\) with \( I = 1, \ldots, N \).

A standard rule for identifying the Salam-Strathdee superfields associated with a component-level description is that the component fields with the highest engineering dimension and which transform solely into derivative terms under a supersymmetry transformation must occur as the last field in the Grassmann coordinate expansion of the superfield. This rule is applied in the following discussion.

The analysis of the exceptional irreducible cases of \( N = 1, 2 \) and 4 is simplest, so we carry this out first. Beginning our considerations in the \( N = 4 \) case, we see that there is only a bosonic prepotential \( U^{ij}(\zeta, \tau) \), \( U_{ij} = \frac{1}{2} \epsilon_{ijkl} U^{kl} \) which describes the entire theory. The manner in which the component fields appear is rather obvious. We can define these through projection. For the \( N = 4 \) case these are

\[
X = D_k D^l U^{kl},
\]

\(^{18}\)It has long been our convention to reserve the symbol \( \theta \) for the Grassmann coordinate associated with the target manifold.
\[ \Psi_I = i D_i D_K D_L U^{KL} , \]
\[ F^{KL} = \frac{1}{4!} \epsilon_{IJKLMN} D_i D_j D_K D_L D_M D_N U^{KL} . \] (117)

and the superfield action for the multiplet is simply
\[ S_{N=4}^{ex} = \int d\tau d^4\zeta \left[ \frac{1}{2} U^{IJ} D_I D_J U^{KL} \right] = \int d\tau d^4\zeta \left[ \frac{1}{2} U^{KL} F^{KL} \right] . \] (118)

In the remaining exceptional cases, both theories are described by a scalar prepotential superfield \( U(\zeta, \tau) \). While in the \( N = 2 \) case the components can be defined by
\[ X = U , \quad \Psi_1 = i D_i U , \quad F = i \frac{1}{2} \epsilon^{1j} D_i D_j U . \] (119)

and the superfield action for the multiplet is simply
\[ S_{N=2}^{ex} = \int d\tau d^2\zeta \left[ i \frac{1}{4} U \epsilon^{1j} D_i D_j U \right] = \int d\tau d^2\zeta \left[ \frac{1}{2} U F \right] . \] (120)

Finally in the \( N = 1 \) case the components can be defined by
\[ X = U , \quad \Psi = i D U . \] (121)

and the superfield action for the multiplet is simply
\[ S_{N=1}^{ex} = \int d\tau d\zeta \left[ i \frac{1}{2} U D U \right] = \int d\tau d\zeta \left[ \frac{1}{2} U \Psi \right] . \] (122)

For \( N = 3 \), we introduce the prepotential superfield
\[ U^{11213}(\zeta, \tau) , \] (123)
along with two other prepotential superfields,
\[ (V(\zeta, \tau), V^{112}(\zeta, \tau)) , \] (124)

For \( N > 4 \), we introduce the \( 2^{N-1} - 1 \) prepotential superfields
\[ (U^{11213}(\zeta, \tau), \ldots, U^{112^{1-2}p+1}(\zeta, \tau)) , \] (125)
along with another set of \( 2^{N-1} \) prepotential superfields,
\[ (V(\zeta, \tau), V^{112}(\zeta, \tau), \ldots, V^{11^{1-2}p}(\zeta, \tau)) , \] (126)
whose Grassmann coordinate expansions contain all of the component fields in (104) plus many more.
The number of component fields contained in (125) and (126) is $2^N(2^N - 1)$. This should be compared with the $2^{N+1}$ total number of Wess-Zumino gauge component fields in (104). The results in (125) and (126) follow from the fact that highest engineering dimension component fields of the USPM correspond to spinorial auxiliary fields $\Lambda_k^i$ and for the conjugate momentum multiplet these are the bosonic fields $P$ and $G^j$.

In other words, the superfields in (125) and (126) are the unconstrained pre-potentials for the off-shell spinning particle with $N$-extended supersymmetries on its worldline. The nature of the superfields changes drastically according to the value of $N$. If $N$ is even, then these superfields are bosonic quantities. If $N$ is odd, then these superfields are fermionic. The position coordinate is contained in the “smallest” $U$ pre-potential. In the ordinary member of the generic sequence, the 0-brane coordinate, canonical conjugate momentum, NSR fermion and spinorial momentum can be defined by

\begin{align*}
X(\tau) &\propto \frac{1}{(N-3)!3!} \epsilon^{I_1 \cdots I_N} D^{I_1} \cdots D^{I_{N-3}} U^{I_{N-2}I_{N-1}I_N} |, \\
P(\tau) &\propto \frac{1}{N!} \epsilon^{I_1 \cdots I_N} D^{I_1} \cdots D^{I_N} V |, \\
\Psi^I(\tau) &\propto \frac{1}{N!} \epsilon^{I_1 \cdots I_N} D^{I_1} \cdots D^{I_{N-3}} U^{I_{N-2}I_{N-1}I_N} |, \\
\pi^I(\tau) &\propto \frac{1}{(N-1)!} \epsilon^{I_2 \cdots I_N} D^{I_2} \cdots D^{I_N} V |,
\end{align*}

and the remaining component fields in (104) can easily be assigned as the $\zeta \to 0$ limit of the $D$’s acting on the superfields in (125) and (126). Note that since the engineering dimensions of $X$, $P$, $\psi_I$ and $\pi_I$ are fixed, as $N$ increases the engineering dimensions of $U^{I_1I_2I_3}$, $\ldots$, $U^{I_1 \cdots I_{2p+1}}$, $V$, $\ldots$, $V^{I_1 \cdots I_{2p}}$ become increasingly negative in order to compensate for the numbers of spinorial superderivatives.

From the expression in (127) it can be seen that $X$, $P$, $\Psi^I$ and $\pi^I$ are expressed in terms of the unconstrained pre-potential superfields. This implies that the superfields $X$, $P$, $\Psi^I$ and $\pi^I$ must satisfy some set of differential equations. Such differential equation on superfields are called “constraints.” This observation leads us to our third conjecture in this work.

The constraints to which all irreducible superfields in all dimensions are subjected insure that irreducible supermultiplets are also irreducible representations of the $\mathcal{GR}(d,N)$ algebra.

In passing, it is worth mentioning that after all the component fields in (104) are expressed in terms of the pre-potentials in (125) and (126), the action (103) can be
written as the integral of a superfield Lagrangian

\[
S_{\text{Spng Part.}} = \int d\tau \int d^N \zeta \mathcal{L}_{\text{Spng Part.}}(U^{11,2,13}, \ldots, U^{1,\cdots,2p+1}; V, \ldots, V^{1,\cdots,2p}) ,
\]

\[
\int d^N \zeta \equiv \frac{1}{N!} \epsilon^{1,\cdots,1_N} D^{1} \cdots D^{1_N} ,
\]

(128)

which describes the spinning particle for arbitrary values of \( N \) (except 1,2 and 4). For \( N \geq 3 \), all the actions suggested in this chapter to describe spinning particles are superfield gauge theories. The existence of these superfield actions is a direct consequence of the existence of the pre-potential superfields \( U^{11,2,13}, \ldots, U^{1,\cdots,2p+1}, V, \ldots, V^{1,\cdots,2p} \).

We end this section by pointing out that the \( \mathcal{GR}(d, N) \) Clifford algebra approach has gone well beyond the naive use of a Salam-Strathdee superspace. By its use we have; (a.) identified a set of superfield prepotentials (125), (126) and (b.) identified the component fields that remain in the WZ gauge of the pre-potentials (104). It is now also obvious that had we simply begun with the action in (109) together with the transformation laws after substitution of (108) into (104), the “spinorial” indices of \( \mathcal{GR}(d, N) \) do not explicitly appear anywhere in the formulation! In other words, the \( \mathcal{GR}(d, N) \) origin of the multiplets of pre-potentials is totally hidden.

4.2 On-shell 3D, \( N \)-extended Vector Multiplets

Simultaneous with our initial exposition on the role of the \( \mathcal{GR}(d, N) \) algebras in 1D \( N \)-extended systems, it was also noted that this same algebraic structure plays a role in the construction of on-shell \( N \)-extended supersymmetrical vector multiplets in three dimensions.

The following supersymmetry variations close up to terms involving the Dirac equations of the spinor fields

\[
\delta_Q B_i^j = \epsilon^{a1} (L_1)_k^j \left[ \delta_i^k \lambda_{\alpha_k}^j - d^{-1} \delta_i^j \lambda_{\alpha_k}^k \right] ,
\]

\[
\delta_Q \lambda_{\alpha_k}^k = i \epsilon^{a1} (R_1)_k^j (\gamma^d)_{\alpha\beta} \left[ \partial_{\alpha} B_j^k + \frac{1}{4} d^{-1} \delta_j^k \epsilon^{bc} F_{bc} \right] ,
\]

\[
\delta_Q A_{\alpha} = i \epsilon^{a1} (L_1)_k^j (\gamma^d)_{\alpha\beta} \lambda_{\beta}^k .
\]

(129)

The conventions used here to describe 3D Lorentz spinors and vectors are given by

\[
\eta_{ab} = \text{diag}(1,-1,-1) , \quad \epsilon_{abc} \epsilon^{def} = \delta_{ab} \delta^{ef} - \delta_{af} \delta^{be} , \quad \epsilon^{012} = +1 ,
\]

\[
(\gamma^a)^\alpha (\gamma^b)_{\gamma}^\beta = \eta_{ab} \delta_{\alpha}^\beta + i \epsilon_{abc} (\gamma^c)^{\alpha\beta} .
\]

(130)
In these expressions, $\epsilon_{abc}$ is the Levi-Civita tensor. Some useful Fierz identities are:

$$
(\gamma^a)_{\alpha\beta} (\gamma_a)^{\delta} = -\delta_{\alpha\gamma} \delta_{\beta\delta},
$$

$$
\epsilon^{abc} (\gamma_b)_{\alpha\beta} (\gamma_c)^{\gamma} = -i C_{\alpha\gamma} (\gamma^b)_{\beta} - i (\gamma^b)_{\alpha\gamma} \delta_{\beta\delta}.
$$

(131)

where $C_{\alpha\beta} = i \epsilon_{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ is also a Levi-Civita tensor.

The fields $B^i_j$ and $\lambda^{\alpha_k}_k$ may be regarded as DKP fields in the normal part of the enveloping algebra (as in the case of the spinning particle). Consequently, each of these field operators can be expanded as

$$
B^i_j(x) = \frac{1}{2} (f^{112})_i^j \varphi_{112}^j (x) + \frac{1}{4!} (f^{11214}_i)^j \varphi_{11214}^j (x) + \cdots
$$

$$
+ \frac{1}{(2p)!} (f^{11 \cdots 12p})_i^j \varphi_{11 \cdots 12p}^j (x),
$$

$$
\lambda^{\alpha_k}_k(x) = (R^1)^j_k \lambda^{\alpha_k}_k (x) + \frac{1}{3!} (f^{1121}_j)_k \lambda^{\alpha_k}_j (x) + \cdots
$$

$$
+ \frac{1}{(2q+1)!} (f^{11 \cdots 12q+1})_j^k \lambda^{\alpha_k}_j (x).
$$

(132)

All of the results above are cast in the form of component formulations. We now switch to the superfield viewpoint by the introduction of connection superfields $\Gamma_{\dot{a}}$. These superfields are expanded over the three bosonic coordinates and $N$ (three dimensional) spinor coordinates and are used to introduce superspace gauge covariant derivatives

$$
\nabla_{\dot{a}} \equiv (\nabla_{\alpha_1}, \nabla_{\dot{a}}),
$$

$$
\nabla_{\alpha_1} \equiv D_{\alpha_1} + i g \Gamma_{\alpha_1 \dot{a}} t_{\dot{a}}, \quad \nabla_{\dot{a}} \equiv \partial_{\dot{a}} + i g \Gamma_{\dot{a}} t_{\dot{a}}
$$

(133)

where $t_{\dot{a}}$ denote a set of generators for a set of Abelian symmetries and satisfies $(t_{\dot{a}})^* = -(t_{\dot{a}})$. The supersymmetry transformation laws above are consistent with a superspace covariant derivative $\nabla_{\alpha_1}$ associated with an Abelian group and which satisfies the restrictions

$$
[\nabla_{\alpha_1}, \nabla_{\beta_j}] = i 2 \delta_{1j} (\gamma^\alpha)_{\alpha\beta} \nabla_{\dot{a}} + i 4 g C_{\alpha\beta} (f_{1j})_i^j B_{i \dot{a}} t_{\dot{a}},
$$

$$
[\nabla_{\alpha_1}, \nabla_{\dot{a_j}}] = 2 g (\gamma^\alpha)_{\alpha\beta} (L^1)_j^k \lambda^{\beta_j \dot{a}} t_{\dot{a}},
$$

$$
[\nabla_{\dot{a}}, \nabla_{\dot{b}}] = i g F_{\dot{a} \dot{b}} t_{\dot{a}}.
$$

(134)

In order to show that these constraints satisfy the usual superspace Bianchi identities, it is required to note the identities

$$
(f_{1j})_p^q (L^1)_p \dot{q} = - \delta_{jK} (L^1)_p \dot{q} + \delta_{1K} (L^1)_p \dot{q} + (f_{1jK})_p \dot{q},
$$

$$
(f_{1j})_i^k = \frac{1}{2} [ (L^1)_i \dot{q} (R^1)_q^k - (L^1)_i \dot{q} (R^1)_q^k ]
$$

(135)

where $(f_{1jK})$ is a 3-form element as discussed in section (2.2).
An interesting point about the equations (134) is that they demonstrate that only \( \varphi_{IJ} \) and \( \lambda_{\alpha I} \) appear as components of the field strength superfield. The remaining fields in (132) appear via derivatives at higher orders in the \( \theta \)-expansions of the superfields \( \varphi_{IJ}(x) \) and \( \lambda_{\alpha I}(x) \). The component fields \( \varphi_{i_1i_2i_3i_4}(x), \ldots, \varphi_{i_1\cdots i_{2p}}(x) \) and \( \lambda_{\alpha_1i_1i_2i_3}(x), \ldots, \lambda_{\alpha_1\cdots i_{2q+1}}(x) \) provide examples of a phenomenon that has not been observed in superspace formulations previously. We may call these “exo-field strength components” because they play the same roles at the usual field strength components \( \varphi_{IJ}(x) \) and \( \lambda_{\alpha I}(x) \) but they do not occur within the set of conventional field strength superfields common to all known Salam-Strathdee superspace constructions.

There is one other interesting implication of the presence of the exo-field strength components. Due to the fact that they only occur via their spacetime derivatives in (134) implies that there is a huge space of both bosonic and fermionic moduli associated with these theories in general. This follows from the invariance of the results in (134) with respect to transformations of the forms

\[
\delta_m \varphi_{i_1i_2i_3i_4}(x) = (c_0)_{i_1i_2i_3i_4}, \ldots, \delta_m \varphi_{i_1\cdots i_{2p}}(x) = (c_0)_{i_1i_2i_3i_{2p}} \tag{136}
\]

where the quantities \((c_0)_{i_1i_2i_3i_4}, \ldots, (c_0)_{i_1i_2i_3i_{2p}}\) are bosonic constants and \((\alpha_0)_{i_1i_2i_3}, \ldots, (\alpha_0)_{i_1i_2i_3i_{2q+1}}\) are fermionic constants. The dimensions of the bosonic and fermionic space of moduli are respectively given by

\[
\dim\left(\mathcal{U}_L / \mathbf{I} \oplus f_{IJ}\right) \quad & \quad \dim\left(\mathcal{M}_L / f_{1}\right) \tag{137}
\]

and these formulae are only evaluated over the normal parts of the spaces.

There are two obvious actions that are invariant under the supersymmetry variations in (129). The first of these is the usual kinetic energy of a vector supermultiplet

\[
S_{3DVM} = \int d^3x \left[ -\frac{1}{4} F_{\alpha \beta} F_{\alpha \beta} + i d^{-1} \lambda_{\alpha k} (\gamma^a)_{\alpha \beta} \partial_a \lambda_{\beta k} + d^{-1} (\partial^a B_j^i) (\partial_a B_j^i) \right]. \tag{138}
\]

From its form, the space of moduli described in (136) is seen to leave this action invariant.

As well, it is possible to construct the supersymmetric BF-theory. In order to do this, it is first necessary to introduce the supersymmetrical dual multiplet. The fields of this supermultiplet can be defined by that fact that their transformation laws are such that the BF-action

\[
S_{3DBF} = \int d^3x \left[ \frac{1}{2} \epsilon_{\alpha \beta \gamma} B_\alpha F_{\beta \gamma}(A) + \lambda_{\alpha k} i k \lambda_{\alpha k} + \mathcal{H}^j i B_j^i \right]. \tag{139}
\]

38
is invariant under the supersymmetry variations. The variations of the dual multiplet components that accomplish this are given by

\[ \delta Q_{\alpha } B_{k}^{a} = i d^{-1} e^{\alpha 1} (\gamma _{\alpha }^{a})_{\alpha \beta } \chi _{k}^{\beta } (R_{i})_{k}^{i} , \]
\[ \delta Q_{\alpha } X_{\alpha i}^{a} = e^{\alpha 1} \left[ C_{\alpha \beta } H_{i}^{l} + i \frac{1}{2} \delta _{i}^{j} (\gamma _{\alpha }^{a})_{\alpha \beta } e_{a}^{k} F_{\beta c} (B) \right] (L_{i})_{k}^{l} , \]
\[ \delta Q_{i} H_{i,j}^{j} = - i e^{\alpha 1} (\gamma _{\alpha }^{a})_{\alpha \beta } \partial _{a} \left[ \chi _{i}^{\beta } k \delta _{k}^{j} - d^{-1} \delta _{i}^{j} \chi _{k}^{\beta k} \right] (R_{i})_{k}^{i} . \] (140)

Finally it is interesting to note that the action in (139) is a superconformal action. Via the AdS/CFT correspondence there should exist a 4D, \( N \)-extended AdS supergravity theory that is closely related to this action.

In closing this section, it should pointedly be noted that we have not attempted to construct the non-Abelian extensions of the models discussed. The non-linearities due to the presence of non-trivial commutators can be expected to place stringent restrictions on \( N \). This is a topic for possible future study.

5 The \( N = 8 \) Spinning Particle - Supergravity

Surprise

Before ending our recitation on the relation between \( \mathcal{G}\mathcal{R}(d, N) \) Pauli algebras and supersymmetric algebras, there is a surprising observation to be made. We return to the case of the \( \mathcal{G}\mathcal{R}(8, 8) \) enveloping algebras where we found

\[ \mathcal{U}_{L} = \{ \mathbf{I}, f_{1}, f_{12}, f_{13}, f_{14}, f_{1234} \} , \quad \mathcal{U}_{R} = \{ \mathbf{I}, \tilde{f}_{1}, \tilde{f}_{12}, \tilde{f}_{13} \} \]
\[ \mathcal{M}_{L} = \{ f_{1}, f_{12}, f_{13}, f_{14} \} , \quad \mathcal{M}_{R} = \{ \tilde{f}_{1}, \tilde{f}_{12} \} . \] (141)

One of our other on-going lines of investigation [6] has been a model-independent formulation of super Virasoro algebras. There we have seen that the co-adjoint of the totality of generators required to form a closed super Virasoro algebra naturally leads to the appearance of fields that bare a striking resemblance to the spectrum of supergravity theories. In particular, the spin of the co-adjoint fields is determined by the relation

\[ s \equiv \left( 2 - \frac{1}{2} p \right) , \] (142)

where \( p \) is the rank of the generator (all such generators are forms in this approach) associated with the co-adjoint field. If we now simply apply this observation to the forms that appear in the enveloping algebra above, we are led to the spins and degeneracies indicated in the following table.
These spins and degeneracies are exactly those of on-shell 4D, $N = 8$ supergravity. We thus assert that each state of the on-shell supergravity theory is in one-to-one correspondence with the elements of $EGR(8,8)$. It is our suspicion that the relationship we have elucidated here is no accident but instead is hinting at a new deep relationship between the $GR(d, N)$ Pauli-Clifford algebras and $GR$ super Virasoro algebras on one side and supergravity and superstring/M-theory on the other. Stated another way, upon choosing $N = 8$, each of the “fibers” of the directed links in PpG diagram can apparently be associated with one of the states in $N = 8$ supergravity.

6 Conclusion

With this paper, we hope to have provided the reader with convincing arguments that show that there exists a deep and largely overlooked connection between supersymmetrical theories and the theory of a special class of real Clifford algebras. Evidence for these connections first began to emerge from investigations of spinning particles [1, 2]. It appears within the context of the 1D spinning particle theories that the component fields of superfields have two simultaneous interpretations. First, as is widely known, the component fields are the coefficients of superfields when expanded over a set of Grassmann coordinates. Our work shows that these components may also be interpreted as the coefficients of the expansions of a set of linear operators acting on the vector space $V_L \oplus V_R$. In the former approach the field operators are
characterized by a Grassmann algebra while in the latter they are defined through a Clifford algebra.

One other intriguing possible application of our work is to strengthen the relation of KO-theory to supersymmetrical representation theory. Some years ago [9], it was shown that topological indices can be related to supersymmetrical quantum mechanical models closely related to the action in (7). In turn, the present work suggests how the representation theory of the supersymmetrical model is itself related to the $\mathcal{GR}(d, N)$ Pauli-Clifford algebras. It is also well known that KO-theory is related to real Clifford algebras. So we see a nexus involving supersymmetry, $\mathcal{GR}$ algebras and KO-theory. The results in section (3.5) show that the usual component fields of spacetime supersymmetrical theory seem to possess root superfield representations in 1D, $N = 4$ theories and that these component fields correspond in a definite way to geometrical structures associated with real Clifford algebras.

It has been observed by Landweber [10] that our construction may be linked to KO-theory by noting that we construct $KO^{-k}(X)$ by looking at $\mathbb{Z}_2$-graded $Cl(k)$ bundles over $X$, and identifying ones which admit a $Cl(k+1)$ action. Our present work takes $X$ to simply be $R^1$ parametrized here by $\tau$. Upon the imposition of boundary conditions, in other words deal with K-theory with compact supports or take the one point compactification, $KO(R)$ shifts degree by one. So it may be possible to view space-time locally as a bundle over the timelike direction.

We wish to comment based upon an accumulating amount of evidence that it is conceivable that all aspects of supersymmetrical theories in all dimensions are encoded in some manner in 1D, $N$-extended theories. This thought has been in the background of this line of research every since we began it some time ago [1, 2]. Two of the main results in this present paper add more such evidence. One of these is the discussion that was given in sections (3.5-3.6). There it was seen in a rather precise way, how the fields of the 4D, $N = 1$ chiral multiplet seem possess an alternate interpretation of being associated with certain irreducible tensor operators that act on the spaces $\mathcal{V}_L$ and $\mathcal{V}_R$ (as well as Cartesian products of these spaces). In equation (86) we have seen that the 4D, $N = 1$ chiral multiplet seems to possess a “root superfield” representation. We have also outlined how other representations of 4D, $N = 1$ supersymmetry likely also possess such representations among the higher rank representations analogous to that used for the chiral multiplet. A second striking piece of evidence is the unexpected apparent link between some representations of the $\mathcal{EGR}(8, 8)$ algebras and the states that appear in 4D, $N = 8$ supergravity.

All of this suggests that there exist some new type of “holography” at work here.
Finally, we have seen that a Clifford-algebraic based construction of supersymmetrical theories exist independent of the more traditional Salam-Strathdee superspace based constructions. However, as explicitly seen in sections (4.1-4.2), the Clifford-algebraic based construction is perfectly compatible with off-shell and on-shell Salam-Strathdee superspace based constructions. We ultimately expect this to be universally true. Exploring these relations further will be a primary purpose of future studies along these lines.

“We share a philosophy about Linear Algebra: We think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury.” – I. Kaplansky.

Acknowledgments

The authors would like to acknowledge, John H. Schwarz and H. Tuck for the hospitality extended during their visit to the California Institute of Technology, where some of this research was undertaken. Additionally, S.J.G. wishes to recognize the support rendered by the Caltech administration during this visit. We also thank Prof. V. G. J. Rodgers for assistance with this paper. Finally, we wish to thank Prof. Gregory Landweber for his critical reading of an earlier version of this work and making numbers of valuable comments.
7 Appendix: Trace Relations and Inner Product Structure on $EGR(d, N)$

In this section we present a symmetric bilinear form on the enveloping algebra $EGR(8)$. This will induce, via the reduction procedure described above, an inner product on the algebras $EGR(N)$ for $5 \leq N \leq 8$. In order to define the inner product on the enveloping algebra, it is necessary to investigate the traces which may be defined on the generators. Due to the mixed markings on the elements of $EGR(8) \cong \{M_L\} \oplus \{M_R\} \oplus \{U_L\} \oplus \{U_R\}$, only the traces on the following subspaces may be defined properly: $\{U_R\} \oplus \{U_L\}$, $\{U_L\} \oplus \{U_R\}$, and $\{M_L\} \oplus \{M_R\}$. The relevant traces are given by

$$\text{tr} \left[ \tilde{f}_{i_1 \ldots i_p} \tilde{f}^{j_1 \ldots j_q} \right] = (-)^q \cdot d \cdot \text{sgn} \left( \frac{12 \cdots q}{q \cdots 21} \right) \cdot \delta_{p,q} \cdot \delta_{i_1 \ldots i_q}^{j_1 \ldots j_q}$$

with all other traces vanishing. Here $\tilde{f}_{[p]} := f_{i_1 \ldots i_p}$ if $p$ is even and $\tilde{f}_{[p]} \in \{M_L\}$ or $\{M_R\}$ if $p$ is odd. Also, $d$ here is the dimension of the representation, $\ell$ gives the duality of the $\frac{N}{2}$-form and $\text{sgn} \left( \frac{12 \cdots q}{q \cdots 21} \right) = (-) \sum_{n=1}^{q-1} n = \left( - \frac{q(q-1)}{2} \right)$ denotes the sign of the permutation reversing the order of the $q$ indices.

Using these traces we may proceed to define the following inner product. Let

$$\phi_i^j = \sum_{p=0}^{N/2} \phi_{i_1 \ldots i_{2p}} (f_{i_1 \ldots i_{2p}})_{i}^j \in \{U_L\}$$

$$\hat{\phi}_k^i = \sum_{p=0}^{N/2} \hat{\phi}_{i_1 \ldots i_{2p}} (\tilde{f}_{i_1 \ldots i_{2p}})_{k}^i \in \{U_R\}$$

$$\psi_k^i = \sum_{p=0}^{N/2-1} \psi_{i_1 \ldots i_{(2p+1)}} (f_{i_1 \ldots i_{(2p+1)}})_{k}^i \in \{M_L\}$$

$$\hat{\psi}_k^i = \sum_{p=0}^{N/2-1} \hat{\psi}_{i_1 \ldots i_{(2p+1)}} (\tilde{f}_{i_1 \ldots i_{(2p+1)}})_{k}^i \in \{M_R\}.$$ (144)

We define $\langle \cdot, \cdot \rangle$ by

$$\langle \Phi^{(1)}, \Phi^{(2)} \rangle \mapsto \frac{1}{d} \sum_{p,q=0}^{N/2} \frac{1}{(p+q)!} \cdot \phi^{(1)}_{i_1 \ldots i_{2p}} \phi^{(2)}_{j_1 \ldots j_{2q}} \cdot \text{tr} \left[ f_{i_1 \ldots i_{2p}} f_{j_2 \ldots j_{1}} \right]$$

$$\langle \hat{\Phi}^{(1)}, \hat{\Phi}^{(2)} \rangle \mapsto \frac{1}{d} \sum_{p,q=0}^{N/2} \frac{1}{(p+q)!} \cdot \hat{\phi}^{(1)}_{i_1 \ldots i_{2p}} \hat{\phi}^{(2)}_{j_1 \ldots j_{2q}} \cdot \text{tr} \left[ \tilde{f}_{i_1 \ldots i_{2p}} \tilde{f}_{j_2 \ldots j_{1}} \right].$$
\[
(\Psi^{(1)}, \Psi^{(2)}) \mapsto -\frac{1}{d} \sum_{p,q=0}^{N/2-1} \frac{1}{(p+q+1)!} \cdot \psi^{(1)}_{1^{[1]} \cdots 1^{(2p+1)}} \psi^{(2)}_{2^{[1]} \cdots 2^{(2q+1)}} \cdot \text{tr} \left[ f_{1^{[1]} \cdots 1^{(2p+1)}} f_{1^{[1]} \cdots 1^{(2q+1)}} \right],
\]

\[
(\hat{\Psi}^{(1)}, \hat{\Psi}^{(2)}) \mapsto -\frac{1}{d} \sum_{p,q=0}^{N/2-1} \frac{1}{(p+q+1)!} \cdot \hat{\psi}^{(1)}_{1^{[1]} \cdots 1^{(2p+1)}} \hat{\psi}^{(2)}_{2^{[1]} \cdots 2^{(2q+1)}} \cdot \text{tr} \left[ \hat{f}_{1^{[1]} \cdots 1^{(2p+1)}} \hat{f}_{1^{[1]} \cdots 1^{(2q+1)}} \right].
\]

(145)

Note that the adjoint indices on the second generator under the trace stand in reverse order w.r.t. the indices on its accompanying component field.

The following property of this inner product is crucial to the construction of \(N\)-extended supersymmetry Lagrangians: The non-vanishing piece of these traces which are not proportional to the \(\epsilon\) tensor are positive definite. e.g.

\[
\langle \Phi^{(1)}, \Phi^{(2)} \rangle = + \frac{1}{d} \sum_{p,q=0}^{N/2} \frac{1}{(p+q)!} \cdot \phi^{(1)}_{1^{[1]} \cdots 1^{2p}} \phi^{(2)}_{1^{[1]} \cdots 1^{2q}} \cdot \text{tr} \left[ f_{1^{[1]} \cdots 1^{2p}} f_{1^{[1]} \cdots 1^{2q}} \right]
\]

\[
= + \frac{1}{d} \sum_{p,q=0}^{N/2} \frac{1}{(p+q)!} \cdot \phi^{(1)}_{1^{[1]} \cdots 1^{2p}} \phi^{(2)}_{1^{[1]} \cdots 1^{2q}} \cdot (-)^{2q} \cdot d \cdot \left[ \text{sgn} \left( \frac{12 \cdots q}{q \cdot 21} \right) \right]^2
\]

\[
\cdot \delta_{p-q} \cdot \delta_{1^{[1]} \cdots 1^{2p}} \cdot \delta_{1^{[1]} \cdots 1^{2q}} + \epsilon - \text{terms}
\]

\[
= + \sum_{p=0}^{N/2} \phi^{(1)}_{1^{[1]} \cdots 1^{2p}} \phi^{(2)}_{1^{[1]} \cdots 1^{2p}} + \epsilon - \text{terms.}
\]

(146)

Without this result, the would-be kinetic terms in a generic lagrangian would introduce classical ghosts into the particle spectrum.

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