On the convergence of double Fourier series of functions of bounded partial generalized variation.

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ABSTRACT. The convergence of double Fourier series of functions of bounded partial \( \Lambda \)-variation is investigated. The sufficient and necessary conditions on the sequence \( \Lambda = \{\lambda_n\} \) are found for the convergence of Fourier series of functions of bounded partial \( \Lambda \)-variation.

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On the convergence of double Fourier series
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1. CLASSES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

In 1881 Jordan [1] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation etc., see [2]-[5]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [6].

Let $f$ be a real function of two variable of period $2\pi$ with respect to each variable. Given intervals $I = (a, b)$, $J = (c, d)$ and points $x, y$ from $T := [0, 2\pi]$ we denote $f(I, y) := f(b, y) - f(a, y)$, $f(x, J) = f(x, d) - f(x, c)$ and $f(I, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d)$.

Let $E = \{I_i\}$ be a collection of nonoverlapping intervals from $T$ ordered in arbitrary way and let $\Omega$ be the set of all such collections $E$. Denote by $\Omega_n$ set of all collections of $n$ nonoverlapping intervals $I_k \subset T$.

For the sequence of positive numbers $\Lambda = \{\lambda_n\}_{n=1}^\infty$ we denote

$$\Lambda V_1(f) = \sup_y \sup_{E \in \Omega} \sum_n \frac{|f(I_i, y)|}{\lambda_i} \quad (E = \{I_i\}),$$

$$\Lambda V_2(f) = \sup_x \sup_{F \in \Omega} \sum_m \frac{|f(x, J_j)|}{\lambda_j} \quad (F = \{J_j\}),$$

$$\Lambda V_{1,2}(f) = \sup_{F, E \in \Omega} \sum_i \sum_j \frac{|f(I_i, J_j)|}{\lambda_i \lambda_j}.$$

**Definition 1.** We say that the function $f$ has Bounded $\Lambda$-variation on $T = [0, 2\pi]^2$ and write $f \in \Lambda BV$, if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda V_{1,2}(f) < \infty.$$

We say that the function $f$ has Bounded Partial $\Lambda$-variation and write $f \in P\Lambda BV$ if

$$P\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \ldots$) the classes $\Lambda BV$ and $P\Lambda BV$ coincide with the Hardy class $BV$ and PBV respectively. Hence it is reasonable to assume that $\lambda_n \to \infty$ and since the intervals in $E = \{I_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$1 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{n \to \infty} \lambda_n = \infty.$$  \hspace{1cm} (1.1)

In the case when $\lambda_n = n$, $n = 1, 2\ldots$ we say Harmonic Variation instead of $\Lambda$-variation and write $H$ instead of $\Lambda$ (HBV, PHBV, HV(f), etc).

The notion of $\Lambda$-variation was introduced by D. Waterman [4] in one dimensional case and A. Sahakian [10] in two dimensional case.
Definition 2. Let $\Phi$ be a strictly increasing continuous function on $[0, +\infty)$ with $\Phi(0) = 0$. We say that the function $f$ has bounded partial $\Phi$-variation on $T^2$ and write $f \in PBV_\Phi$, if

$$V^{(1)}_\Phi(f) := \sup_{y} \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^{n} \Phi (|f (I_i, y)|) < \infty, \quad n = 1, 2, ... ,$$

$$V^{(2)}_\Phi(f) := \sup_{x} \sup_{\{J_j\} \in \Omega_m} \sum_{j=1}^{m} \Phi (|f (x, J_j)|) < \infty, \quad m = 1, 2, ....$$

In the case when $\Phi(u) = u^p$, $p \geq 1$, the notion of bounded partial $p$-variation (class $PBV_p$) was introduced in [8].

Theorem 1. Let $\Lambda = \{\lambda_n = n \gamma_n\}$ and $\gamma_n \geq \gamma_{n+1} > 0$, $n = 1, 2, ...$.

1) If

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty,$$

then $PABV \subset HBV$.

2) If, in addition, for some $\delta > 0$

$$\gamma_n = O(\gamma_n^{1+\delta}) \quad as \quad n \to \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty,$$

then $PABV \notin HBV$.

Proof. 1) Let $f \in PABV$ and

$$\sum_{i,j=1}^{\infty} \frac{|f (I_i, J_j)|}{i j} = \sum_{i \leq j} \frac{|f (I_i, J_j)|}{i j} + \sum_{i > j} \frac{|f (I_i, J_j)|}{i j} := I_1 + I_2.$$

Then according to (1.2),

$$I_1 = \sum_{i=1}^{\infty} \frac{1}{i} \sum_{j=i}^{\infty} \frac{|f (I_i, J_j)|}{j}$$

$$\leq 2 \sum_{i=1}^{\infty} \frac{1}{i} \sup_{x} \sum_{j=i}^{\infty} \frac{|f (x, J_j)| \lambda_j}{\lambda_j j}$$

$$\leq 2 \sum_{i=1}^{\infty} \frac{\lambda_i}{i^2} \sup_{x} \sum_{j=i}^{\infty} \frac{|f (x, J_j)|}{\lambda_j}$$

$$\leq 2 \Lambda V_2 (f) \sum_{i=1}^{\infty} \frac{\lambda_i}{i^2} \leq c \Lambda V_2 (f) < \infty.$$

Similarly, $I_2 \leq c \Lambda V_1 (f) < \infty$. 


2) In the proof of the second statement of Theorem 1 we use the following well known lemma.

Lemma 1. Let \( u_i \) and \( v_i, i = 1, 2, \ldots, j \) be two increasing (decreasing) sequences of positive numbers. Then for any rearrangement \( \{\sigma(i)\} \) of the set \( \{1, 2, \ldots, j\} \)

\[
\sum_{i=1}^{j} u_i v_{i-1} \leq \sum_{i=1}^{j} u_i v_{\sigma(i)} \leq \sum_{i=1}^{j} u_i v_i.
\]

Let (1.3) and (1.4) be fulfilled and define

\[
f(x, y) := \begin{cases} t_j, & x = \frac{1}{i}, \ y = \frac{1}{j}, \ j < i \leq j + m_j, \ i, j = 1, 2, \ldots, \\ 0, & \text{otherwise}
\end{cases},
\]

where

\[
t_j := \left( \sum_{i=1}^{m_j} \frac{1}{\lambda_j} \right)^{-1}, \quad m_j = \left\lfloor j^{1+\delta} \right\rfloor, \quad j = 1, 2, \ldots
\]

Let \( x = 1/i \) and let \( j(i) \) be the smallest integer satisfying

\[
j(i) + m_{j(i)} \geq i.
\]

Since \( t_j \) is decreasing and \( \lambda_j \) is increasing, using Lemma 1 we can write

\[
\sup_{E \in \Omega} \sum_{j=1}^{\infty} \frac{|f(1/i, J_j)|}{\lambda_j} = \sum_{j=j(i)}^{i-1} t_j \frac{1}{\lambda_j} \leq t_j(i) \sum_{j=1}^{m_j} \frac{1}{\lambda_j} = 1.
\]

Hence

\[
\Lambda V_2(f) \leq 1.
\]

For \( y = 1/j \) we have

\[
\sup_{E \in \Omega} \sum_{i=1}^{\infty} \frac{|f(i, 1/j)|}{\lambda_i} = t_j \sum_{i=1}^{m_j} \frac{1}{\lambda_i} = 1.
\]

Consequently,

\[
\Lambda V_1(f) \leq 1.
\]

Combining (1.7) and (1.8) we conclude that \( f \in PABV \).

Now we prove that \( f \notin HBV \). From (1.3) and (1.5) follows that

\[
\sum_{i=1}^{m_j} \frac{1}{\lambda_i} = \sum_{i=1}^{m_j} \frac{1}{i \gamma_i} \leq C \log m_j \gamma_mj \leq C \frac{\log j}{\gamma_j}.
\]

Hence

\[
t_j \cdot \log j \geq c \gamma_j, \quad j = 2, 3, \ldots
\]
and from the definition of $f$, (1.5) and (1.4) we obtain

$$\sup_{E,F \in \Omega} \sum_{i,j} |f(I_i, J_j)|_{ij} \geq \sum_{j=1}^{\infty} \frac{t_j}{j} \sum_{i=j+1}^{j+m_j} \frac{1}{i} \geq c \sum_{j=1}^{\infty} \frac{t_j}{j} \log(j + m_j) \geq c \sum_{j=1}^{\infty} \frac{\gamma_j}{j} = \infty.$$ 

Theorem 1 is proved.

Taking $\lambda_n \equiv 1$ and $\lambda_n = n$ in Theorem 1, we get

**Corollary 1.** $PBV \subset HBV$ and $PHBV \not\subset HBV$.

**Corollary 2.** Let $\Phi$ and $\Psi$ are conjugate functions in the sense of Yung ($ab \leq \Phi(a) + \Psi(b)$) and let for some $\{\lambda_n\}$ satisfying (1)

\begin{equation}
(1.10) \quad \sum_{n=1}^{\infty} \Psi \left( \frac{1}{\lambda_n} \right) < \infty.
\end{equation}

Then $PBV_{\Phi} \subset HBV$. In particular, $PBV_{\Phi} \subset HBV$ for any $p > 1$.

Indeed, from the inequality $\frac{a}{b} \leq \Phi(a) + \Psi\left( \frac{1}{b} \right)$ follows that $PBV_{\Phi} \subset PABV$ under assumption (1.10), and $PABV \subset HBV$ if (1.1) holds.

**Definition 3** (see [9]). The partial modulus of variation of a function $f$ are the functions $v_1(n, f)$ and $v_2(m, f)$ defined by

$$v_1(n, f) := \sup_y \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^{n} |f(I_i, y)|, \quad n = 1, 2, \ldots,$$

$$v_2(m, f) := \sup_x \sup_{\{J_k\} \in \Omega_m} \sum_{k=1}^{m} |f(x, J_k)|, \quad m = 1, 2, \ldots.$$

For functions of one variable the concept of the modulus variation was introduced by Chanturia [5].

**Theorem 2.** If $f \in B$ is bounded on $T^2$ and

$$\sum_{n=1}^{\infty} \frac{\sqrt[n]{v_j(n, f)}}{n^{3/2}} < \infty, \quad j = 1, 2,$$

then $f \in HBV$. 
Proof. Using Abel transformation we can write

\[ \sum_{k=1}^{m} \left| \frac{f(x, J_k)}{k} \right| = \sum_{k=1}^{m-1} \left( \frac{1}{k} - \frac{1}{k+1} \right) \sum_{l=1}^{k} \left| f(x, J_l) \right| + \frac{1}{m} \sum_{k=1}^{m} \left| f(x, J_k) \right| \]

\[ \leq \sum_{k=1}^{m-1} \frac{1}{k^2} \left( \sum_{l=1}^{k} \left| f(x, J_l) \right| \right)^{1/2} \left( \sum_{l=1}^{k} \left| f(x, J_l) \right| \right)^{1/2} + c \]

\[ \leq c \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2} \left( \sum_{l=1}^{k} \left| f(x, J_l) \right| \right)^{1/2} + c \]

\[ \leq c \sum_{k=1}^{\infty} \frac{\sqrt{v_2(k, f)}}{k^{3/2}} + c \leq c < \infty. \]

Consequently,

(1.11) \quad HV_2(f) < \infty.

Analogously, we can prove that

(1.12) \quad HV_1(f) < \infty.

Using Hardy transformation we obtain

\[ \sum_{i=1}^{n} \sum_{j=1}^{m} \begin{vmatrix} \frac{f(I_i, J_j)}{ij} \\ \end{vmatrix} \]

\[ = \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \left( \frac{1}{i} - \frac{1}{i+1} \right) \left( \frac{1}{j} - \frac{1}{j+1} \right) \sum_{l=1}^{i} \sum_{s=1}^{j} \left| f(I_i, J_s) \right| \]

\[ + \frac{1}{n} \sum_{j=1}^{m-1} \left( \frac{1}{j} - \frac{1}{j+1} \right) \sum_{l=1}^{n} \sum_{s=1}^{j} \left| f(I_l, J_s) \right| \]

\[ + \frac{1}{m} \sum_{i=1}^{n-1} \left( \frac{1}{i} - \frac{1}{i+1} \right) \sum_{l=1}^{m} \sum_{s=1}^{i} \left| f(I_l, J_s) \right| \]

\[ + \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \left| f(I_i, J_j) \right| \]

\[ = I + II + III + IV. \]

Since

\[ \sum_{l=1}^{i} \sum_{s=1}^{j} \left| f(I_l, J_s) \right| \leq 2i \sup_{x} \sum_{s=1}^{j} \left| f(x, J_s) \right| \leq 2i v_2(j, f) \]

and

\[ \sum_{l=1}^{i} \sum_{s=1}^{j} \left| f(I_l, J_s) \right| \leq 2j \sup_{y} \sum_{l=1}^{i} \left| f(I_l, y) \right| \leq 2j v_1(i, f) \]
we can write

\[
I \leq \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \frac{1}{j^2} \left( \sum_{l=1}^{i} \sum_{s=1}^{j} |f(I_l, J_s)| \right)^{1/2} \left( \sum_{l=1}^{i} \sum_{s=1}^{j} |f(I_l, J_s)| \right)^{1/2}
\]

(1.14) \leq 2 \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} \frac{\sqrt{ij}v_2(j, f) v_1(i, f)}{j^2} \leq \infty,

\[
II \leq \frac{1}{n} \sum_{j=1}^{m-1} \frac{1}{j^2} \left( \sum_{l=1}^{n} \sum_{s=1}^{j} |f(I_l, J_s)| \right)^{1/2} \left( \sum_{l=1}^{n} \sum_{s=1}^{j} |f(I_l, J_s)| \right)^{1/2}
\]

(1.15) \leq \frac{1}{n} \sum_{j=1}^{m-1} \frac{\sqrt{n}v_2(j, f)}{j^2}

\leq \frac{\sqrt{v_1(n, f)}}{\sqrt{n}} \sum_{j=1}^{\infty} \frac{\sqrt{v_2(j, f)}}{j^{3/2}} \leq c < \infty, n = 1, 2, ...

Analogously, we can prove that

(1.16)

\[III \leq c < \infty,\]

(1.17)

\[IV \leq 2 \sqrt{\frac{v_1(n, f) v_2(m, f)}{n m}} \leq c < \infty, n, m = 1, 2,....\]

Combining (1.11)-(1.17), we conclude that \( f \in HBV. \) Theorem 2 is proved.

2. CONVERGENCE OF TWO-DIMENSIONAL TRIGONOMETRIC FOURIER SERIES

Let \( f \in L^1(T^2), \ T^2 := [0, 2\pi]^2. \) The Fourier series of \( f \) with respect to the trigonometric system is the series

\[
S[f] := \sum_{m,n=-\infty}^{+\infty} \hat{f}(m,n) e^{imx}e^{iny},
\]

where

\[
\hat{f}(m,n) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} f(x,y) e^{-imx}e^{-iny} dx dy
\]

are the Fourier coefficients of the function \( f. \) The rectangular partial sums are defined as follows:

\[
S_{M,N}[f(x,y)] := \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{f}(m,n) e^{imx}e^{iny},
\]
In this paper we consider convergence of only rectangular partial sums (convergence in the sense of Pringsheim) of double Fourier series.

We denote by $C(T^2)$ the space of continuous functions which are $2\pi$-periodic with respect to each variable with the norm
\[
\|f\|_C := \sup_{x,y \in T^2} |f(x,y)|.
\]

For the function $f$ defined on $T^2$ we denote by $f(x \pm 0, y \pm 0)$ the open co-ordinate quadrant limits (if exist) at the point $(x, y)$ and let $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$ be the arithmetic mean
\[
(2.1) \quad \frac{1}{4}\{f(x + 0, y + 0) + f(x + 0, y - 0) + f(x - 0, y + 0) + f(x - 0, y - 0)\}.
\]

The well known Dirichlet-Jordan theorem (see [7]) states that the Fourier series of a function $f(x)$, $x \in T$ of bounded variation converges at every point $x$ to the value \( \frac{f(x + 0) + f(x - 0)}{2} \). If $f$ is in addition continuous on $T$ the Fourier series converges uniformly on $T$.

Hardy [6] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if function $f(x, y)$ has bounded variation in the sense of Hardy ($f \in BV$), then $S[f]$ converges at any point $(x, y)$ to the value $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$. If $f$ is in addition continuous on $T^2$ then $S[f]$ converges uniformly on $T^2$.

**Theorem S** (Sahakian [10]). The Fourier series of a function $f(x, y) \in HBV$ converges to $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$ at any point $(x, y)$, where the quadrant limits (2.1) exist. The convergence is uniformly on any compact $K$, where the function $f$ is continuous.

Theorem S was proved in [10] under assumption that the function is continuous on some open set containing $K$ while Sargsyan noticed in [11], that the continuity of $f$ on the compact $K$ is sufficient.

Analogs of Theorem S for higher dimensions can be found in [12] and [13]. Convergence of spherical and other partial sums of double Fourier series of functions of bounded $\Lambda$-variation was investigated in details by Dyachenko (see [14], [15] and references therein).

The first author [9] has proved that if $f$ is continuous function and has bounded partial $p$-variation ($f \in PBV_p$) for some $p \in [1, +\infty)$ then $S[f]$ converges uniformly on $T^2$. Moreover, the following is true

**Theorem G** (Goginava [9]). Let $f \in C(T^2)$ and
\[
\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \ j = 1, 2.
\]

Then $S[f]$ converges uniformly on $T^2$.

Theorems 1, 2, Corollary 2 and Theorem S imply
Theorem 3. Let \( f \in PAV \) with
\[
\sum_{j=1}^{\infty} \frac{\lambda_j}{j^2} < \infty, \quad \frac{\lambda_j}{j} \downarrow 0.
\]
Then \( S[f] \) converges to \( \sum f(x, y) \) in any point \((x, y)\), where the quadrant limits \((2.1)\) exist. The convergence is uniformly on any compact \( K \), where the function \( f \) is continuous.

Theorem 4. Let \( f \in B \) and
\[
\sum_{n=1}^{\infty} \sqrt{v_j(n, f)} \frac{1}{n^{3/2}} < \infty, \quad j = 1, 2.
\]
Then \( S[f] \) converges to \( \frac{1}{4} \sum f(x \pm 0, y \pm 0) \) in any point \((x, y)\), where the quadrant limits \((2.1)\) exist. The convergence is uniformly on any compact \( K \), where the function \( f \) is continuous.

Corollary 3. Let \( f \in B \) and \( v_1(k, f) = O(k^\alpha), v_2(k, f) = O(k^\beta), 0 < \alpha, \beta < 1 \). Then \( S[f] \) converges to \( \frac{1}{4} \sum f(x \pm 0, y \pm 0) \) in any point \((x, y)\), where the quadrant limits \((2.1)\) exist. The convergence is uniformly on any compact \( K \), where the function \( f \) is continuous.

Theorem 5. Let \( f \in PBV_p, p \geq 1 \). Then \( S[f] \) converges to \( \frac{1}{4} \sum f(x \pm 0, y \pm 0) \) in any point \((x, y)\), where the quadrant limits in \((2.1)\) exist. The convergence is uniformly on any compact \( K \), where the function \( f \) is continuous.

From Theorem 3 follows that for any \( \delta > 0 \) the Fourier series of the function \( f \in P\{n \log n\} BV \) converges to \( \frac{1}{4} \sum f(x \pm 0, y \pm 0) \) in any point \((x, y)\), where the quadrant limits \((2.1)\) exist. Moreover, one can not take here \( \delta = 0 \) (see Theorem 6). It is interesting to compare this result with that obtained by M. Dyachenko and D. Waterman in [16].

Dyachenko and Waterman [16] introduced another class of functions of generalized bounded variation. Denoting by \( \Gamma \) the set of finite collections of nonoverlapping rectangles \( A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset T^2 \) we define
\[
\Lambda^*V(f) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.
\]

Definition 4 (Dyachenko, Waterman). Let \( f \) be a real function on \( T^2 := [0, 2\pi] \times [0, 2\pi] \). We say that \( f \in \Lambda^*BV \) if
\[
\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^*V(f) < \infty.
\]

Theorem DW ([16]). If \( f \in \left\{ \frac{n}{\log n} \right\}^* BV \), then in any point \((x, y)\) the quadrant limits \((2.1)\) exist and the double Fourier series of \( f \) converges to \( \frac{1}{4} \sum f(x \pm 0, y \pm 0) \).

Moreover, the sequence \( \left\{ \frac{n}{\log n} \right\} \) can not be replaced with any sequence \( \left\{ \frac{\alpha_n}{\log n} \right\} \), where \( \alpha_n \to \infty \).
It is easy to show (see [16]), that \( \left\{ \frac{n}{\log n} \right\}^* BV \subset HBV \), hence the convergence part of Theorem DW follows from Theorem S. It is essential that the condition \( f \in \left\{ \frac{n}{\log n} \right\}^* BV \) guarantees the existence of quadrant limits.

The next theorem in particular shows that in Theorem S the condition \( HV_{1,2}(f) < \infty \) is necessary, i.e the boundedness of partial harmonic variation is not sufficient for the convergence of Fourier series of continuous function.

**Theorem 6.** Let \( \Lambda = \{ \lambda_n = n\gamma_n \} \) where \( \gamma_n \) is a decreasing sequence satisfying (1.3) and (1.4). Then there exists a continuous function \( f \in \text{P} \Lambda BV \) with Fourier series that diverges at \((0,0)\).

We need the following simple lemma that easily follows from Lemma 1.

**Lemma 2.** Let the function \( g(t) \) be defined on \( T \) and

\[ 0 = t_1 < t_2 < \ldots < t_{2m} = 2\pi. \]

Suppose \( g \) is increasing on \([t_i, t_{i+1}]\) if \( i \) is odd and is decreasing, if \( i \) is even.

If

\[ |g(t_{i+1}) - g(t_i)| > |g(t_{i+2}) - g(t_{i+1})|, \quad i = 1, 2, \ldots, 2m - 2, \]

then

\[ \lambda BV(g) = \sum_{i=1}^{2m-1} \frac{|g(t_{i+1}) - g(t_i)|}{\lambda_i}, \]

for any sequence \( \Lambda = \{ \lambda_n \} \) satisfying (1.7).

**Proof of Theorem 6.** It is not hard to see, that for any sequence \( \Lambda = \{ \lambda_n \} \) satisfying (1.1) the class \( C(T^2) \cap \text{P} \Lambda BV \) is a Banach space with the norm \( \|f\|_{\text{P} \Lambda BV} := \|f\|_C + \lambda BV(f) \).

Let \( \Lambda = \{ \lambda_n \} \) be as in Theorem 6 and denote

\[ A_{i,j} = \left[ \frac{\pi i}{N + 1/2}, \frac{\pi (i+1)}{N + 1/2} \right) \times \left[ \frac{\pi j}{N + 1/2}, \frac{\pi (j+1)}{N + 1/2} \right). \]

Define \( t_j \) and \( m_j \) as in (1.5) and consider the function

\[ f_N(x,y) = \sum_{(i,j) \in W} t_j \chi_{A_{i,j}}(x,y) \sin \left( N + \frac{1}{2} \right) x \cdot \sin \left( N + \frac{1}{2} \right) y, \]

where \( \chi_A(x,y) \) is the characteristic function of the set \( A \subset T^2 \) and

\[ W := \{(i,j) : j < i < j + m_j, \quad 1 \leq j < N_\delta \}, \quad N_\delta = \left( \frac{N}{2} \right)^{1+\delta}. \]

Each summand in the sum (2.2) is continuous on the rectangle \( A_{i,j} \) and vanishes on its boundary, hence \( f_N \in C(T^2) \).

Next, in view of Lemma 2, using the same arguments as in the proof of (1.7) and (1.8), we get

\[ AV_1(f_N) \leq 1, \quad AV_2(f_N) \leq 1. \]
Hence $f_N \in \text{PABV}$ and

\begin{equation}
\|f_N\|_{\text{PABV}} \leq 3, \quad N = 1, 2, \ldots
\end{equation}

Observe that $N_\delta < N$ and $j + m_j < N$, if $j < N_\delta$, hence $A_{i,j} \subset T^2$, if $(i,j) \in W$. Taking into account (1.5) and (1.9), for the square partial sum of the Fourier series of $f_N$ at $(0,0)$ we get

\begin{align}
\pi \cdot S_{N,N}[f_N,(0,0)] &= \int_{T^2} f_N(x,y)D_N(x)D_N(y)dxdy \\
&= \sum_{(i,j) \in W} t_j \int_{A_{i,j}} \sin^2 \left( N + \frac{1}{2} \right) x \cdot \sin^2 \left( N + \frac{1}{2} \right) y \frac{dx \cdot dy}{4\sin^2 \frac{x}{2} \sin^2 \frac{y}{2}} \\
&\geq c \sum_{j=1}^{N_\delta} t_j \sum_{j=m_j+1}^{N_\delta} \frac{1}{j} \geq c \sum_{j=1}^{N_\delta} t_j \log(j + m_j) \geq c \sum_{j=1}^{N_\delta} \frac{\gamma_j}{j} \to \infty.
\end{align}

as $N \to \infty$, where $c$ is an absolute constant.

Applying the Banach-Steinhaus Theorem, from (2.3) and (2.4) we obtain that there exists a continuous function $f \in \text{PABV}$ such that

$$\sup_N |S_{N,N}[f,(0,0)]| = \infty.$$
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