LOW REGULARITY FOR A QUADRATIC SCHRÖDINGER EQUATION ON $\mathbb{T}$

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ABSTRACT. — In this paper we consider a Schrödinger equation on the circle with a quadratic nonlinearity. Thanks to an explicit computation of the first Picard iterate, we give a precision on the dynamic of the solution, whose existence was proved by C. E. Kenig, G. Ponce and L. Vega \[15\]. We also show that the equation is well-posed in a space $H^{s,p}(\mathbb{T})$ which contains the Sobolev space $H^s(\mathbb{T})$ when $p \geq 2$.

RÉSUMÉ. — Dans cet article on s'intéresse à une équation de Schrödinger sur le cercle avec une non-linéarité quadratique. Un calcul explicite de la première itérée de Picard permet de donner une précision sur la dynamique de la solution, dont l’existence a été démontrée par C. E. Kenig, G. Ponce et L. Vega \[15\]. On montre également que l’équation est bien posée dans un espace $H^{s,p}(\mathbb{T})$ qui contient l’espace de Sobolev $H^s(\mathbb{T})$ lorsque $p \geq 2$.

1. Introduction

Denote by $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the unidimensional torus. In this paper we consider the following nonlinear Schrödinger equation

\[
\begin{cases}
i\partial_t u + \Delta u = \kappa u^2, & (t, x) \in \mathbb{R} \times \mathbb{T}, \\
u(0, x) = f(x) \in X,
\end{cases}
\]

\[\text{(1.1)}\]

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where $X$ is a Banach space (the space of the initial conditions).
This equation has been intensively studied in the case $x \in M$ where $M$ is a Riemannian manifold and for different nonlinearities, usually of the form
\[ F(u, \overline{u}) = \pm u^{p_1} \overline{u}^{p_2}, \quad \text{where} \quad p_1, p_2 \in \mathbb{N}. \]

Here we mainly discuss the results in one dimension for quadratic nonlinearities. For the other cases see [7], [15], [3], and references therein.

1.1. Previous results on the real line. —
In the case $x \in \mathbb{R}$, J. Ginibre and G. Velo [10], Y. Tsutsumi [17], T. Cazenave and F. B. Weissler [7] showed that the Cauchy problem is well posed for $f \in L^2(\mathbb{R})$, for every nonlinearity of the type (1.2) with $p_1 + q_1 \leq 5$. The proof relies on the use of Strichartz inequalities, which are of the form
\[ \| e^{it \Delta} f \|_{L^p(\mathbb{R}, L^q(\mathbb{R}))} \leq C \| f \|_{L^2(\mathbb{R})}, \quad \text{with} \quad \frac{1}{p} + \frac{2}{q} = \frac{1}{2}. \]

In [15], C. E. Kenig, G. Ponce, and L. Vega show that (1.1) is well posed in $X = H^s(\mathbb{R})$:
- for $s > -3/4$ in the case $F(u, \overline{u}) = \pm u^2$ or $F(u, \overline{u}) = \pm \overline{u}^2$;
- for $s > -1/4$ in the case $F(u, \overline{u}) = \pm |u|^2$.

To obtain these results, the authors prove some bilinear estimates in the conormal spaces $X^{s,b}$ (see Definition 1.2), and they also show that these estimates are optimal, and as a consequence it is impossible to perform a usual fixed point argument in these spaces, below the threshold $s = -3/4$ (resp. $s = -1/4$). Notice that the $X^{s,b}$ spaces distinguish the structure of the nonlinearity, which was not the case for the Strichartz spaces.

In [1], I. Bejenaru and T. Tao extend the well-posedness results to $s \leq -1$ in the case $F(u, \overline{u}) = u^2$, and show that the equation (1.1) is ill-posed in $H^s(\mathbb{R})$ when $s < -1$.

1.2. Previous results on the torus. —
In the case $x \in \mathbb{T}$, J. Bourgain [2] established the embedding $X^{0,3/8} \subset L^4_{x,t}$, which permitted to show that the problem (1.1) is locally well posed in $L^2(\mathbb{T})$, for every nonlinearity (1.2) with $p_1 + p_2 \leq 3.$

Then, C. E. Kenig, G. Ponce, and L. Vega [15], thanks to bilinear estimates in $X^{s,b}$ (see Theorem 1.4 below), obtained the well posedness of (1.1) in $H^s(\mathbb{T})$ for $s > -1/2$ in the case $F(u, \overline{u}) = \pm u^2$ or $F(u, \overline{u}) = \pm \overline{u}^2$. Again, these estimates fail if $s < -1/2.$
1.3. The $\mathcal{H}^{s,p}(\mathbb{T})$ and $X^{s,b}$ spaces. —

Now we introduce the $\mathcal{H}^{s,p}(\mathbb{T})$ spaces

**Definition 1.1.** — ($\mathcal{H}^{s,p}$ spaces)

For $s \in \mathbb{R}$ and $p \geq 1$, denote by $\mathcal{H}^{s,p}(\mathbb{T})$ the completion of $\mathcal{C}^\infty(\mathbb{T})$ with respect to the norm

$$
\|f\|_{\mathcal{H}^{s,p}} = \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{sp} |\hat{f}(n)|^p \right)^{\frac{1}{p}}.
$$

Here $\hat{f}(n)$ denotes the Fourier coefficient of $f$ (see (2.6)).

These spaces were introduced by L. Hörmander (see [14], Section 10.1).

There are several motivations to introduce these spaces

- First notice that $\mathcal{H}^{s,2}(\mathbb{T}) = \mathcal{H}^{s}(\mathbb{T})$, and for $p > 2$ we have the (strict) inclusion $\mathcal{H}^{s}(\mathbb{T}) \subset \mathcal{H}^{s,p}(\mathbb{T})$.

- Then, the space $\mathcal{H}^{s,p}$ scales like $\mathcal{H}^{s(p)}$ where $s(p) = -\frac{1}{2} + s + \frac{1}{p}$. Hence, if $s(p) < -\frac{1}{2}$, the space $\mathcal{H}^{s,p}$ contains elements $f$ such that $|\hat{f}(n)| \to +\infty$ when $n \to +\infty$. Therefore we can go closer to the scaling of the equation (1.1) which is $-\frac{3}{2}$.

- T. Cazenave, L. Vega and M. C. Vilela [6] where the first authors to study nonlinear Schrödinger equations in $\mathcal{H}^{s,p}$-like spaces. In fact they show that a class of NLS equations on $\mathbb{R}^N$ is well-posed if the linear flow belongs to some weak $L^p$ space. Moreover they prove that this condition can be ensured if the initial data $f$ satisfies $\hat{f} \in L^{p,\infty}(\mathbb{R}^N)$ for some $p \geq 1$. This latter space is a continuous version of the space $\mathcal{H}^{s,p}$.

- In [12] A. Grünrock establishes bilinear and trilinear estimates in conormal spaces $X^{s,b}_{p,q}$ (see definition below) based on $L^r$. This permits him to show that the cubic Schrödinger equation

$$
i \partial_t u + \Delta u = \pm |u|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

is well-posed for initial conditions in the corresponding continuous version of the space $\mathcal{H}^{s,p}$. He obtains analogous results for the DNLS equation [12] and for the mKdV equation [11].

In [8], M. Christ shows that the modified cubic problem

$$
\begin{cases}
  i \partial_t u + \Delta u = \pm (|u|^2 - 2\mu(|u|^2))u, \quad \mu(|v|^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |v(x)|^2 dx,
  \\
  u(0, x) = f(x) \in \mathcal{H}^{s,p}((\mathbb{T}),
\end{cases}
$$

is well posed in $\mathcal{H}^{s,p}((\mathbb{T})$ for any $s \geq 0$ and $p \geq 1$. See [8] for precise statements.

Recently, A. Grünrock and S. Herr [13] have shown the well-posedness in $\mathcal{H}^{s,p}$ spaces of the DNLS equation on the torus, thanks to multilinear estimates.
See [8, 11, 12, 13] for other features of the spaces $H^{s,p}$ and more references.

• Notice that the $H^{s,p}$ is preserved by the linear Schrödinger flow. Write

$$f(x) = \sum_{n \in \mathbb{Z}} \alpha_n e^{in x},$$

then $e^{it \Delta} f(x) = \sum_{n \in \mathbb{Z}} \alpha_n e^{-in^2 t} e^{in x},$

and for all $t \in \mathbb{R}$, $\|e^{it \Delta} f\|_{H^{s,p}} = \|f\|_{H^{s,p}}$.

We now define the $X^{s,b}$ spaces

**Definition 1.2.** — ($X^{s,b}$ spaces)

(i) For $s, b \in \mathbb{R}$, denote by $X^{s,b} = X^{s,b}(\mathbb{R} \times T)$ the completion of $C^\infty(T, S(\mathbb{R}))$ with respect to the norm

$$\|F\|_{X^{s,b}} = \left( \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \langle \tau + n^2 \rangle^{2b} \langle n \rangle^{2s} |\hat{F}(\tau, n)|^2 d\tau \right)^{\frac{1}{2}}.$$

(ii) Let $T > 0$, we define the restriction spaces $X^{s,b}_T = X^{s,b}([-T, T] \times T)$ by

$$(1.3) \|F\|_{X^{s,b}_T} = \inf \left\{ \|\psi(\frac{t}{T}) F\|_{X^{s,b}}, F \in X^{s,b} \text{ with } \psi \in S(\mathbb{R}) \text{ s.t. } \psi|_{[-1,1]} = 1 \right\}.$$ 

Here $\hat{F}$ stands for the space-time Fourier transform (see (2.7)).

In the following, we will mainly use the space $X^{s,b}_1 = X^{s,b}([-1,1] \times T)$.

We recall the key estimates which permit to perform a fixed point argument in the $X^{s,b}$ spaces, and to deduce that the equation (1.1) is well posed in $H^s$ for $s > -\frac{1}{2}$.

**Proposition 1.3.** — Let $s \leq 0$ and $\frac{1}{2} < b \leq 1$. Then for all $F \in X^{s,b-1}_1$, we have

$$\left\| \int_0^t e^{i(t-t')\Delta} F(t', \cdot) dt' \right\|_{X^{s,b}_1} \leq C \|F\|_{X^{s,b-1}_1}.$$ 

See [9] for a proof. Notice this estimate holds in the general case of a riemannian manifold, indeed the proof reduces to time integrations. Notice also that we always have the estimate

$$\|e^{it \Delta} f\|_{X^{s,b}_1} \leq C \|f\|_{H^s},$$

but we won’t use it in this paper.

The following theorem is one of the main results of [15] (see Theorem 1.9. in [13])

**Theorem 1.4.** — (Kenig-Ponce-Vega [15]) Let $-\frac{1}{2} < s \leq 0$, then there exists $b_0 > \frac{1}{2}$ such that for all $\frac{1}{2} < b \leq b_0$ and all $v, w \in X^{s,b}(\mathbb{R} \times T)$

$$(1.4) \|\nabla v\|_{X^{s,b-1}} \lesssim \|v\|_{X^{s,b}} \|w\|_{X^{s,b}}.$$
Moreover, for any $s < -\frac{1}{2}$ and $b \in \mathbb{R}$, an estimate of the form (1.4) fails.

We can deduce the following

**Corollary 1.5.** — Let $-\frac{1}{2} < s \leq 0$, then there exists $b_0 > \frac{1}{2}$ such that for all $\frac{1}{2} < b \leq b_0$ and all $v, w \in X^{s,b}([-1, 1] \times T)$

\[
\|v w\|_{X^{s,b}} \lesssim \|v\|_{X^{s,b}} \|w\|_{X^{s,b}}.
\]

**Proof.** — Let $\psi_1, \psi_2 \in C^\infty_0(\mathbb{R})$ be so that $\psi_1, \psi_2 = 1$ on $[-1, 1]$ and $\text{supp } \psi_1, \psi_2 \subset [-2, 2]$. Then by (1.4) applied to $\psi_1(t)v$ and $\psi_2(t)w$, we obtain

\[
\|v w\|_{X^{s,b}} \leq \|\psi_1(t)\overline{v}\psi_2(t)w\|_{X^{s,b}} \lesssim \|\psi_1 v\|_{X^{s,b}} \|\psi_2 w\|_{X^{s,b}},
\]

and the result follows, by choosing $\psi_1$ and $\psi_2$ which realise the infimum for the $X^{s,b}([-1, 1] \times T)$ norm.

### 2. Main results of this paper

#### 2.1. Local well posedness in the Sobolev scale. —

Our first result is a precision on the dynamic of the solution of (1.1) when the initial condition $f$ is in $H^{s_0}(T)$ with $-\frac{1}{2} < s_0 \leq 0$.

Let $f \in \mathcal{D}'(T)$. Then define

\[
u_0(t, x) = e^{it\Delta} f(x) = \sum_{n \in \mathbb{Z}} \tilde{f}(n) e^{-in^2 t} e^{inx},
\]

the free Schrödinger evolution and

\[
u_1(t, x) = -i \int_0^t e^{i(t-t')\Delta}(\overline{\nu_0})'(t', x) dt',
\]

the first Picard iterate of the equation (1.1). Then we will show that there exists $b > \frac{1}{2}$ so that

\[
\|\nu_1\|_{X^{0,b}([-1, 1] \times T)} \lesssim \|f\|_{H^{s_0}(T)}^2.
\]

Hence, $\nu_1$ is more regular than $f$: there is a gain of $|s_0|$ derivative. We will take profit of this phenomenon to prove that it is also the case for $u - e^{it\Delta} f$, where $u$ is the solution of (1.1).

**Theorem 2.1.** — Let $\kappa = \pm 1$. Let $-\frac{1}{2} < s_0 \leq 0$ and $f \in H^{s_0}(T)$. Then there exist $b > \frac{1}{2}$ and $T > 0$ such that there exists a unique solution $u$ to (1.1) in the space

\[
Y_T^{0,b} = \left( e^{it\Delta} f + X^{0,b}([-T,T] \times \mathbb{T}) \right).
\]
Moreover, given $0 < T' < T$ there exist $R = R(T') > 0$ such that the map $	ilde{f} \mapsto \tilde{u}(t)$ from $\{ \tilde{f} \in H^{s_0}(T) : \| \tilde{f} - f \|_{H^{s_0}} < R \}$ into the class \( (2.2) \) with $T'$ instead of $T$ is Lipschitz.

This result will be obtained with a contraction argument in the space $X^{0,b}$ (thanks to the gain of regularity), and therefore we will only need the estimate \( (1.4) \) with $s = 0$.

**2.2. Local well posedness in the $\mathcal{H}^{s,p}$ scale. —**

We can use the gain of regularity of the first Picard iterate to solve the Cauchy problem \( (1.1) \) for data $f \in \mathcal{H}^{s,p}(T)$, and this will improve slightly the result of [15], as we have the inclusion $H^{s_0}(T) \subset H^{s_0,p}(T)$ for $p > 0$.

The following condition on the real numbers $s_0$ and $p$ will be needed for our result

\[ \frac{3}{p} + s_0 > \frac{5}{6}. \]

**Theorem 2.2. —** Let $\kappa = \pm 1$. Let $s_0 > -\frac{1}{2}$ and let $p > 2$ be so that the condition \( (2.3) \) is satisfied. Let $f \in \mathcal{H}^{s_0,p}(T)$. Then for all $s_1 < -1 + \frac{2}{p}$ there exist $b > \frac{1}{2}$, $s_1 < s < -1 + \frac{2}{p}$, and $T > 0$ such that there exists a unique solution $u$ to \( (1.1) \) in the space

\[ Y^{s,b}_{T} = \left( e^{it\Delta} f + X^{s,b}([-T,T] \times \mathbb{T}) \right). \]

Moreover, given $0 < T' < T$ there exist $R = R(T') > 0$ such that the map $	ilde{f} \mapsto \tilde{u}(t)$ from $\{ \tilde{f} \in \mathcal{H}^{s_0,p}(T) : \| \tilde{f} - f \|_{\mathcal{H}^{s_0,p}} < R \}$ into the class \( (2.2) \) with $T'$ instead of $T$ is Lipschitz.

To prove Theorem 2.2 we will use the estimate \( (1.4) \) in its full strength.

From the previous result, we can immediately deduce

**Corollary 2.3. —** Let $\alpha < \frac{1}{18}$ and let $f \in \mathcal{D}'(T)$ be such that $|\tilde{f}(n)| \lesssim (n)^{\alpha}$. Then there exist $s > -\frac{1}{2}$, $b > \frac{1}{2}$ and $T > 0$ such that there exists a unique solution to \( (1.1) \) in the space

\[ Y^{s,b}_{T} = \left( e^{it\Delta} f + X^{s,b}([-T,T] \times \mathbb{T}) \right). \]

For instance: Let $0 < \varepsilon < 1$ be small and $\alpha = \frac{1}{18} - \varepsilon$. Define $f \in \mathcal{D}'(T)$ by $\tilde{f}(n) = (n)^{\alpha}$. Then $f \in \mathcal{H}^{s}(T)$ for $s < -\frac{1}{2} - \frac{1}{18} + \varepsilon < -\frac{1}{2}$, but $f \in \mathcal{H}^{s_0,p}(T)$ for some $(s_0, p)$ which satisfies the assumptions of Theorem 2.2.
Remark 2.4. — The result of Theorem 2.2 is interesting when $s_0$ is close to $-\frac{1}{2}$, and $p$ as big as possible, under the assumption (2.3). Let $0 < \varepsilon < 1$ be small and set $s_0 = -\frac{1}{2} + \varepsilon$. Then $p > 2$ satisfies (2.3) iff
\[
\frac{4}{9} - \frac{1}{3}\varepsilon < \frac{1}{p} < \frac{1}{2}.
\]
Hence, the parameter $s$ in Theorem 2.2 can be chosen close to $-\frac{1}{9}$. In other words there is a gain of $\sim \frac{1}{2} - \frac{1}{9} = \frac{7}{18}$ derivative.

Remark 2.5. — The conclusions of Theorem 2.1 and 2.2 are likely to hold with the nonlinearities $F(u) = \pm u^2$, but we do not pursue this here.

2.3. Notations and plan of the paper. —

For $F \in \mathcal{S}(\mathbb{R})$ we define the time-Fourier transform by
\[
\hat{F}(\tau) = \int_{\mathbb{R}} e^{-i\tau t} F(t) dt,
\]
which has the following properties
\[
(2.5) \quad \hat{F}(\tau) = \overline{F(-\tau)} \quad \text{and} \quad \hat{F}(e^{i\theta}(\tau)) = \hat{F}(\tau - \theta) \quad \text{for all} \; \theta \in \mathbb{R}.
\]
Each $F \in C^\infty(\mathbb{T}, \mathcal{S}(\mathbb{R}))$ admits the Fourier expansion
\[
(2.6) \quad F(t,x) = \sum_{n \in \mathbb{Z}} \hat{F}(t,n)e^{inx}, \quad \text{where} \quad \hat{F}(\tau,n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} F(t,x) dx,
\]
is the periodic Fourier coefficient of $F$.
Finally, we denote by
\[
(2.7) \quad \tilde{F}(\tau,n) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} e^{-i(\tau t + nx)} F(t,x) dt dx,
\]
the space-time Fourier transform.

Notations. — In this paper $c, C$ denote constants the value of which may change from line to line. These constants will always be universal, or depending only on fixed quantities. We use the notations $a \sim b$, $a \lesssim b$ if $\frac{1}{C}b \leq a \leq Cb$, $a \leq Cb$ respectively.

In Section 3 we make explicit computations to estimate the first Picard iteration in $X^{s,b}$ spaces.
Then, in Section 4 we establish a bilinear estimate in $X^{s,b}$ spaces.
In Section 5 we follow an idea of N. Burq and N. Tzvetkov [4, 5] and look for a solution of (1.1) of the form $u = e^{it\Delta} f + v$. The existence and uniqueness of $v$
is then proved with a fixed point argument, using the estimates of the previous sections.

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3. The first Picard iteration

**Lemma 3.1.** Let \( \varphi \in \mathcal{S}(\mathbb{R}) \). Then

\[
\int_{\mathbb{R}} \frac{1}{\langle \tau + A \rangle} |\varphi(\tau)| d\tau \lesssim \frac{1}{\langle A \rangle},
\]

uniformly in \( A \in \mathbb{R} \).

**Proof.** As \( \varphi \) is in the Schwartz class \( |\varphi(\tau)| \lesssim \langle \tau \rangle^{-3} \).

Then notice that \( \langle \tau \rangle \langle \tau + A \rangle \gtrsim \langle A \rangle \), therefore

\[
\int_{\mathbb{R}} \frac{\langle A \rangle}{\langle \tau + A \rangle} |\varphi(\tau)| d\tau \lesssim \int_{\mathbb{R}} \frac{\langle A \rangle}{\langle \tau \rangle \langle \tau + A \rangle} d\tau \lesssim 1,
\]

hence the result.

Let \( f \in \mathcal{D}'(\mathbb{T}) \), denote by \( \alpha_n = \tilde{f}(n) \). Then define

\[
(3.1) \quad u_0(t, x) = e^{it\Delta} f(x) = \sum_{n \in \mathbb{Z}} \alpha_n e^{-in^2t} e^{inx},
\]

the free Schrödinger evolution and

\[
(3.2) \quad u_1(t, x) = -i \int_0^t e^{i(t-t')\Delta}(t') \, dt',
\]

which is the first Picard iterate of the equation (1.1).

**Proposition 3.2.** Let \(- \frac{1}{2} < s_0 \leq 0 \) and \( p \geq 2 \). Then there exists \( b_1 > \frac{1}{2} \) such that for all \( \frac{1}{2} < b < b_1 \), all \( f \in \mathcal{H}^{s_0-p}(\mathbb{T}) \) and all \( s < -1 + 2/p \) we have

\[
(3.3) \quad \| u_1 \|_{X^{s, s}([-1, 1] \times \mathbb{T})} \lesssim \| f \|_{\mathcal{H}^{s_0-p}(\mathbb{T})}^2.
\]

Moreover, in the case \( p = 2 \), the estimate (3.3) holds for \( s = 0 \).

**Remark 3.3.** — The result of Proposition 3.2 shows that the first Picard iterate is more regular than the initial condition, when \( s_0 \) is close to \(- \frac{1}{2}\) and \( p < 4 \). In this case, we can take \( s > s_0 \).

The result we stated is not optimal when \( s_0 \) is far from \(- \frac{1}{2}\).
Proof. — Let $b > \frac{1}{2}$ to be chosen later. Denote by $\beta = 2(1-b) < 1$ and $\sigma = -s \geq 0$.

Let $\psi_0 \in \mathcal{C}_0^\infty(\mathbb{R})$ s.t. $\psi_0 = 1$ on $[-1, 1]$, and $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ s.t. $\psi_0 \psi = \psi_0$. Then by Definition 1.2 and Proposition 1.3 we have

$$
\|u_1\|_{X^{s,b}([-1,1] \times \mathbb{T})} \leq \|\psi_0(t) u_1\|_{X^{s,b}(\mathbb{R} \times \mathbb{T})} \leq \|\psi(t) u_0\|_{X^{s,b-1}(\mathbb{R} \times \mathbb{T})}.
$$

(3.4)

Now by the expression (3.1), we have (with the change of variables uniformly in $(\tau, p)$)

$$
\psi(t)(\overline{u_0}^2) = \psi(t) \sum_{(n,m) \in \mathbb{Z}^2} \overline{\alpha_n \alpha_m} e^{i(n^2 + m^2)t} e^{-(n+m)x}.
$$

Hence we deduce the Fourier coefficients of $\psi(t)(\overline{u_0}^2)$:

$$
c_p(t) := \sum_{n \in \mathbb{Z}} \overline{\alpha_n \alpha_{-n-p}} e^{i(n^2 + (n+p)^2)t} = \psi(t)(\overline{u_0}^2)(p).
$$

From the properties (2.5) of the time-Fourier transform, we deduce

$$
\hat{c}_p(\tau) = \sum_{n \in \mathbb{Z}} \overline{\alpha_n \alpha_{-n-p}} \hat{\psi}(\tau - n^2 - (n+p)^2),
$$

and by Definition 1.2 we have

$$
I := \|\psi(t)(\overline{u_0}^2)\|^2_{X^{s,b-1}(\mathbb{R} \times \mathbb{T})} = \sum_{p \in \mathbb{Z}} \int_{\mathbb{R}} \langle \tau + p^2 \rangle^{-\beta} \langle p \rangle^{2s} |\hat{c}_p(\tau)|^2 d\tau,
$$

with $\beta = 2(1-b)$. Now, by Lemma 3.4 (see below for the statement and proof) we have

$$
|\hat{c}_p(\tau)|^2 \lesssim \sum_{n \in \mathbb{Z}} |\alpha_n|^2 |\alpha_{-n-p}|^2 |\hat{\psi}|(\tau - n^2 - (n+p)^2),
$$

uniformly in $(\tau, p) \in \mathbb{R} \times \mathbb{Z}$. With the change of variables $m = -n - p$ and $\tau' = \tau - n^2 - m^2$, we deduce

$$
I \lesssim \sum_{n \in \mathbb{Z}} \sum_{p \in \mathbb{Z}} \int_{\mathbb{R}} \frac{(p)^{2s}}{\langle \tau + p^2 \rangle^{\beta}} |\alpha_n|^2 |\alpha_{-n-p}|^2 |\hat{\psi}|(\tau - n^2 - (n+p)^2) d\tau
$$

$$
= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} \frac{(n+m)^{2s}}{\langle \tau + (n+m)^2 \rangle^{\beta}} |\alpha_n|^2 |\alpha_m|^2 |\hat{\psi}|(\tau - n^2 - m^2) d\tau
$$

$$
= \sum_{(n,m) \in \mathbb{Z}^2} \int_{\mathbb{R}} \frac{(n+m)^{2s}}{\langle \tau + (n+m)^2 + n^2 + m^2 \rangle^{\beta}} |\alpha_n|^2 |\alpha_m|^2 |\hat{\psi}|(\tau) d\tau.
$$

(3.7)

Apply Lemma 3.4 with $A = (n+m)^2 + n^2 + m^2$. Denote by $\sigma = -s \geq 0$. Then from (3.7) we deduce
From here we assume that $\sigma > 0$. For $m \in \mathbb{Z}$, denote by

$$
g_m = \sum_{n \in \mathbb{Z}} \frac{|\alpha_n|^2}{\langle n + m \rangle^{2\sigma} \langle n \rangle^{\beta}}.
$$

thanks to the inequality $\langle n^2 + m^2 \rangle \geq \langle n \rangle \langle m \rangle$, from (3.8) we deduce

$$
I \lesssim \sum_{m \in \mathbb{Z}} \gamma_m |\alpha_m|^{2 - \frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} \frac{|\alpha_n|^2}{\langle n + m \rangle^{2\sigma} \langle n \rangle^{\beta}} \right) \left( \sum_{m \in \mathbb{Z}} \gamma_m \right)^{\frac{1}{q_1}},
$$

with

$$
\frac{1}{q_1} = 1 - \frac{2}{p}.
$$

To estimate the last term in (3.10), we observe that

$$
\gamma_m = \left( \frac{|\alpha_k|^2}{\langle k \rangle^{\beta} \langle j \rangle^{2\sigma}} \right)(m),
$$

then by Young’s inequality, for all $p_1, r_1 \geq 1$ so that

$$
\frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{r_1} - 1,
$$

and so that for $2\sigma r_1 > 1$, we have

$$
\left( \sum_{m \in \mathbb{Z}} \gamma_m \sum_{k \in \mathbb{Z}} \frac{|\alpha_k|^2 p_1}{\langle k \rangle^{3\beta p_1}} \right)^{\frac{1}{q_1}} \left( \sum_{m \in \mathbb{Z}} \gamma_m \sum_{j \in \mathbb{Z}} \frac{1}{\langle j \rangle^{2\sigma r_1}} \right)^{\frac{1}{r_1}}.
$$

We take $p_1 = p/2$. This choice together with the conditions (3.11), (3.12) and $2\sigma r_1 > 1$ yields

$$\sigma > \frac{1}{2r_1} = 1 - \frac{2}{p},$$

and thus by (3.9), (3.10) and (3.13) we obtain

$$I \lesssim \|f\|_{H^{-\beta/2, p}}^4.$$

Now we choose $b > \frac{1}{2}$ such that $\beta = -2s_0$, i.e. $b = 2(1 - \beta) = 1 + s_0$, and thus $\frac{1}{2} < b \leq 1$, as we assumed that $-\frac{1}{2} < s_0 \leq 0$. 

Together with (3.4), this concludes the proof of the first statement of Proposition 3.2.

Now we deal with the case \( p = 2 \) and \( \sigma = 0 \).

By 3.8 we only have to bound the term

\[
J := \sum_{(n,m) \in \mathbb{Z}^2} |\alpha_n|^2 |\alpha_m|^2 \langle n^2 + m^2 \rangle^\beta.
\]

Thanks to the inequality \( \langle n^2 + m^2 \rangle \geq \langle n \rangle \langle m \rangle \), we get

\[
J \leq \sum_{(n,m) \in \mathbb{Z}^2} |\alpha_n|^2 |\alpha_m|^2 \langle \tau \rangle^\beta \langle \tau \rangle^\beta = \|f\|_{H^0}^4,
\]

which was the claim. \( \square \)

**Lemma 3.4.** — Let \( \hat{c}_p(\tau) \) be defined by (3.6). Then there exists \( C > 0 \), which only depends on \( \psi \), so that

\[
|\hat{c}_p(\tau)|^2 \leq C \sum_{n \in \mathbb{Z}} |\alpha_n|^2 |\alpha_{n-p}|^2 |\hat{\psi}(\tau - n^2 - (n + p)^2)|,
\]

for all \( (\tau, p) \in \mathbb{R} \times \mathbb{Z} \).

**Proof.** — Denote by

\[
\hat{\psi}_1(\tau, n, p) = \hat{\psi}(\tau - n^2 - (n + p)^2),
\]

then

\[
|\hat{c}_p(\tau)|^2 = \sum_{(n,m) \in \mathbb{Z}^2} \alpha_n \alpha_{n-p} \alpha_m \alpha_{m-p} \hat{\psi}_1(\tau, n, p) \hat{\psi}_2(\tau, m, p),
\]

and with the change of variables \( m = n + k \) we obtain

\[
|\hat{c}_p(\tau)|^2 = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \alpha_n \alpha_{n-p} \alpha_{n+k} \alpha_{n+k-p} \hat{\psi}_1(\tau, n, p) \hat{\psi}_2(\tau, n + k, p).
\]

As \( \hat{\psi} \in \mathcal{S}(\mathbb{R}) \), for all \( N \in \mathbb{N} \), \( |\hat{\psi}| \lesssim \langle \tau \rangle^{-N} \). In the remaining of the proof, the constant \( N \) may change from line to line. By the inequality \( \langle A + B \rangle \lesssim \langle A \rangle \langle B \rangle \), we have

\[
|\hat{\psi}_1(\tau, n, p)\hat{\psi}_2(\tau, n + k, p)| \lesssim \frac{|\hat{\psi}_1(\tau, n, p)|^{\frac{1}{2}} |\hat{\psi}_2(\tau, n + k, p)|^{\frac{1}{2}}}{\langle \tau - n^2 - (n + p)^2 \rangle^N \langle \tau - (n + k)^2 - (n + k + p)^2 \rangle^N}
\]

\[
\lesssim \frac{\langle \hat{\psi}_1(\tau, n, p) \rangle^{\frac{1}{2}} \langle \hat{\psi}_1(\tau, n + k, p) \rangle^{\frac{1}{2}}}{(2k(2n + k + p))^{N}}.
\]
• If $k = 0$ or $k = -2n - p$, in the sum (3.16), we immediately get the bound (3.14).
• Denote by
  \[
  I_p(\tau) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_p^*} \alpha_n \alpha_{-n-p} \alpha_{n+k} \alpha_{-n-k-p} \hat{\psi}_1(\tau, n, p) \hat{\psi}_2(\tau, n + k, p).
  \]
  where $\mathbb{Z}_p^* = \mathbb{Z}\setminus\{0, -2n - p\}$.
If $k \neq 0$ and $k \neq -2n - p$, observe that
  \[
  \langle 2k (2n + k + p) \rangle^2 \gtrsim \langle k \rangle^2 \langle 2n + k + p \rangle,
  \]
thus by (3.16)
  \[
  |\hat{\psi}_1(\tau, n, p) \hat{\psi}_1(\tau, n + k, p)| \lesssim \langle \hat{\psi}_1(\tau, n, p) \rangle^{1/2} \langle \hat{\psi}_1(\tau, n + k, p) \rangle^{1/2},
  \]
and
  \[
  I_p(\tau) \lesssim \sum_{n \in \mathbb{Z}} |\alpha_n| |\alpha_{-n-p}| \langle \hat{\psi}_1(\tau, n, p) \rangle^{1/2} \left( \sum_{k \in \mathbb{Z}} |\alpha_{n+k}| |\alpha_{-n-k-p}| \langle \hat{\psi}_1(\tau, n + k, p) \rangle^{1/2} \right)^{1/2} \langle \hat{\psi}_1(\tau, n, p) \rangle^{1/2} \langle \hat{\psi}_1(\tau, n + k, p) \rangle^{1/2} \langle \hat{\psi}_1(\tau, n + k, p) \rangle^{1/2} \left( \sum_{j \in \mathbb{Z}} |\alpha_j| |\alpha_{j-p}| \langle \hat{\psi}_1(\tau, j, p) \rangle^{1/2} \langle \hat{\psi}_1(\tau, j, p) \rangle^{1/2} \langle \hat{\psi}_1(\tau, j, p) \rangle^{1/2} \langle \hat{\psi}_1(\tau, j, p) \rangle^{1/2} \right)^{1/2},
  \]
after the change of variables $j = k + n$ in the second sum.
Now by Cauchy-Schwarz
  \[
  \sum_{j \in \mathbb{Z}} |\alpha_j| |\alpha_{j-p}| \langle \hat{\psi}_1(\tau, j, p) \rangle^{1/2} \langle \hat{\psi}_1(\tau, j, p) \rangle^{1/2} \langle \hat{\psi}_1(\tau, j, p) \rangle^{1/2} \langle \hat{\psi}_1(\tau, j, p) \rangle^{1/2} \left( \sum_{l \in \mathbb{Z}} \frac{1}{(n-l)^N} \frac{1}{(n + l + p)^N} \right)^{1/2},
  \]
where
  \[
  d(\tau, p) = \sum_{j \in \mathbb{Z}} |\alpha_j|^2 |\alpha_{j-p}|^2 |\psi_1(\tau, j, p)|,
  \]
and as $\langle n - l \rangle \langle n + l + p \rangle \gtrsim \langle 2n + p \rangle$,
  \[
  \sum_{l \in \mathbb{Z}} \frac{1}{(n-l)^N} \frac{1}{(n + l + p)^N} \lesssim \frac{1}{(2n + p)^N} \sum_{l \in \mathbb{Z}} \frac{1}{(n-l)^N} \frac{1}{(n + l + p)^N} \lesssim \frac{1}{(2n + p)^N}.
  \]
by Cauchy-Schwarz. Thus

\[ I_p(\tau) \lesssim d(\tau, p)^{\frac{1}{2}} \sum_{n \in \mathbb{Z}} |\alpha_n| |\alpha_{-n}\psi_1(\tau, n, p)|^{\frac{1}{2}} \frac{1}{(2n + p)^N} \]

\[ \lesssim d(\tau, p)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z}} |\alpha_n|^2 |\alpha_{-n}\psi_1(\tau, n, p)| \right)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z}} \frac{1}{(2n + p)^N} \right)^{\frac{1}{2}} \]

\[ \lesssim d(\tau, p), \]

which completes the proof. \( \square \)

4. The bilinear estimate

This section is devoted to the proof of the following result

**Proposition 4.1.** — Let \(-\frac{1}{2} < s_0 \leq 0\) and \(p \geq 2\). Then for all

\[ -\frac{1}{6} - s_0 - \frac{1}{p} < s \leq 0, \]

there exists \(b_2 > \frac{1}{2}\) such that for all \(\frac{1}{2} < b < b_2\), all \(f \in \mathcal{H}^{s_0,p}(\mathbb{T})\) and all \(v \in X^{s,b}_1(\mathbb{R} \times \mathbb{T})\)

\[ \left\| \int_0^t e^{i(t-t')\Delta} \frac{u_0}{w_0} v(t', \cdot) dt' \right\|_{X^{s,b}_1([-1,1] \times \mathbb{T})} \lesssim \|f\|_{\mathcal{H}^{s_0,p}} \|v\|_{X^{s,b}_1([-1,1] \times \mathbb{T})}, \]

where \(u_0(t) = e^{it\Delta} f\).

Proposition 4.1 shows that, under condition (4.1), the term

\[ \int_0^t e^{i(t-t')\Delta} \frac{u_0}{w_0} v(t', \cdot) dt', \]

has the regularity of \(v\), even if \(f\) is less regular. For instance, with \(p = 2\) and \(s = 0\), we obtain

\[ \left\| \int_0^t e^{i(t-t')\Delta} \frac{u_0}{w_0} v(t', \cdot) dt' \right\|_{X^{0,b}_1([-1,1] \times \mathbb{T})} \lesssim \|f\|_{\mathcal{H}^{0,p}} \|v\|_{X^{0,b}_1([-1,1] \times \mathbb{T})}, \]

whenever \(s_0 > -\frac{1}{2} - \frac{1}{6}\).

We now state a few technical results.

We will need the following lemma which is proved in \([15]\).

**Lemma 4.2.** — If \(\gamma > \frac{1}{2}\), then we have

\[ \sup_{y \in \mathbb{R}} \sum_{n \in \mathbb{Z}} \frac{1}{(n-y)^{2\gamma}} < \infty, \]
and
\begin{equation}
(4.4) \quad \sup_{(y,z)\in\mathbb{R}^2} \sum_{n\in\mathbb{Z}} \frac{1}{(z+n(n-y))^{\gamma}} < \infty.
\end{equation}

**Proof.** — • Let $y \in \mathbb{R}$. Up to a shift in $n$, we can assume that $y \in [0,1]$. Then $\langle n-y \rangle \geq \frac{1}{2} \langle n \rangle$, hence the estimate (4.3).

• Denote by $r_1 = r_1(y,z)$ and $r_2 = r_2(y,z)$ the complex roots of the polynomial $z + X(X-y)$. Then

$$z + n(n-y) = (n-r_1)(n-r_2).$$

There are at most 10 indexes $n$ such that $|n-r_1| \leq 2$ or $|n-r_2| \leq 2$. The remaining $n$'s satisfy

$$\langle (n-r_1)(n-r_2) \rangle \geq \frac{1}{2} \langle n-r_1 \rangle \langle n-r_2 \rangle.$$

Hence by the Cauchy-Schwarz inequality

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n(n-y))^{\gamma}} \lesssim \left( \sum_{n\in\mathbb{Z}} \frac{1}{\langle n-r_1 \rangle^{2\gamma}} \right)^{\frac{1}{2}} \left( \sum_{n\in\mathbb{Z}} \frac{1}{\langle n-r_2 \rangle^{2\gamma}} \right)^{\frac{1}{2}},$$

which yields the result by (4.3). \hfill \Box

**Corollary 4.3.** — If $\gamma_1, \gamma_2 > \frac{1}{2}$, then

\begin{equation}
(4.5) \quad \sup_{(k,\tau)\in\mathbb{Z}^2\times\mathbb{R}} \sum_{n\in\mathbb{Z}} \frac{1}{\langle -\tau + (n+k)^2 + n^2 \rangle^{\gamma_1}} < \infty,
\end{equation}

and

\begin{equation}
(4.6) \quad \sup_{(m,k,\tau)\in\mathbb{Z}^2 \times \mathbb{R}} \sum_{n\in\mathbb{Z}} \frac{1}{\langle \tau - (n+k)^2 + (n+m)^2 + m^2 \rangle^{2\gamma_2}} < \infty,
\end{equation}

where $\mathbb{Z}^2_* = \{ (m,k) \in \mathbb{Z}^2, \text{ s.t. } m \neq k \}$.

**Proof.** — • We first prove the estimate (4.5). For all $\tau, n, k$ we have

$$\langle -\tau + (n+k)^2 + n^2 \rangle = \langle -\tau + k^2 + 2n(n+k) \rangle \geq \langle -\tau + k^2 + n(n+k) \rangle.$$

The estimate then follows from (4.4) with $\gamma_1 > \frac{1}{2}$, $y = -k$ and $z = (-\tau + k^2)/2$.

• We now turn to the proof of (4.6). If $m \neq k$ are integers, then $|m-k| \geq 1$ and thus

$$|\tau - (n+k)^2 + (n+m)^2 + m^2| = 2|m-k||\frac{\tau-k^2 + 2m^2}{2(m-k)} + n|$$

$$\geq |C+n|,$$
with \( C = (\tau - k^2 + 2m^2)/(2(m - k)) \). Therefore
\[
\langle \tau - (n + k)^2 + (n + m)^2 + m^2 \rangle \geq \langle n + C \rangle,
\]
and the estimate follows from an application of (4.3).

**Lemma 4.4.** — If \( \gamma > \frac{1}{2} \), then
\[
\sum_{n \in \mathbb{Z}} \frac{1}{\langle n^2 + y^2 \rangle^\gamma} \lesssim \frac{1}{\langle y \rangle^{2\gamma - 1}}.
\]

**Proof.** — We can assume that \( y > 0 \). We compare the sum with an integral, and with the change of variables \( x = yt \) we obtain
\[
\sum_{n \in \mathbb{Z}} \frac{1}{\langle n^2 + y^2 \rangle^\gamma} \lesssim \sum_{n \in \mathbb{N}} \frac{1}{\langle n^2 + y^2 \rangle^\gamma} \lesssim \int_0^{+\infty} \frac{dx}{\langle x^2 + y^2 \rangle^\gamma}
\]
\[
\lesssim \frac{1}{\langle y \rangle^{2\gamma - 1}} \int_0^{+\infty} \frac{dt}{(t^2 + 1)^\gamma} \lesssim \frac{1}{\langle y \rangle^{2\gamma - 1}},
\]
which was the claim.

**Proof of Proposition 4.1.** — Let \( f \in H_{s,0}^s(T) \) and write
\[
f(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}.
\]
Denote by \( u_0(t) = e^{it\Delta} f \) the free Schrödinger evolution of \( f \). Then
\[
u_0(t, x) = e^{it\Delta} f(x) = \sum_{n \in \mathbb{Z}} a_n e^{-in^2t} e^{inx}.
\]

Let \( v \in X_{1,s,b}^s(\mathbb{R} \times T) \), and let \( \psi_0 \in C_0^\infty(\mathbb{R}) \) be so that \( \psi_0 = 1 \) on \([-1, 1]\) and \( \text{supp } \psi_0 \subset [-2, 2] \). Moreover, we choose \( \psi_0 \) such that
\[
\|v\|_{X_{1,s,b}^s}^2 = \|\psi_0(t) v\|_{X_{1,s,b}^s}^2.
\]
Then we consider the following Fourier expansion
\[
\psi_0(t) v(t, x) = \sum_{n \in \mathbb{Z}} b_n(t) e^{inx}.
\]
Thus by Definition 1.2 and (4.8) we have
\[
\|v\|_{X_{1,s,b}^s}^2 = \|\psi_0(t) v\|_{X_{1,s,b}^s}^2 = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \langle \tau + n^2 \rangle^{2b} \langle (n) \rangle^{2s} |\hat{b}_n(\tau)|^2 d\tau.
\]
Now, use the expressions (4.7) and (4.9) to compute
\[
\psi_0(t) u_0 v(t, x) = \sum_{(j,k) \in \mathbb{Z}^2} a_j b_k(t) e^{-itj^2} e^{i(j+k)x}
\]
therefore
\[
\psi_0(t) u_0 v(t, x) = \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} a_{-n-k} b_k(t) e^{-it(n+k)^2} \right) e^{-inx},
\]

with
\[
c_n(t) = \sum_{k \in \mathbb{Z}} a_{-n-k} b_k(t) e^{it(n+k)^2}.
\]

Now from the properties (2.5) of the time-Fourier transform, we deduce
\[
\hat{c}_n(\tau) = \sum_{k \in \mathbb{Z}} a_{-n-k} b_k(t) e^{-it(n+k)^2}(\tau) = \sum_{k \in \mathbb{Z}} a_{-n-k} b_k(t) e^{-it(n+k)^2}(-\tau)
\]
\[
= \sum_{k \in \mathbb{Z}} a_{-n-k} \hat{b}_k(-\tau + (n + k)^2).
\]

Now write
\[
\hat{c}_n(\tau) = \sum_{k \in \mathbb{Z}} a_{-n-k} \hat{b}_k\left(\frac{k}{(\tau + (n + k)^2 + k^2)^b}\right)\langle k \rangle^2 \langle \tau + (n + k)^2 + k^2 \rangle^b \hat{b}_k(\tau + (n + k)^2),
\]
and by the Cauchy-Schwarz inequality we obtain
\[
|\hat{c}_n(\tau)|^2 \leq \left( \sum_{j \in \mathbb{Z}} A_{j,n}(\tau) \right) \left( \sum_{k \in \mathbb{Z}} B_{k,n}(\tau) \right),
\]
where
\[
A_{j,n}(\tau) = \frac{|a_{-n-j}|^2}{\langle j \rangle^{2s} \langle \tau + (n + j)^2 + j^2 \rangle^{2b}},
\]
and
\[
B_{k,n}(\tau) = \langle k \rangle^{2s} \langle \tau + (n + k)^2 + k^2 \rangle^{2b} |\hat{b}_k|^2 (\tau + (n + k)^2).
\]

Now by Proposition 1.3 for \( \frac{1}{2} < b < 1 \) and \( s \in \mathbb{R} \)
\[
\| \int_0^t e^{i(t-t') \Delta} u_0 v(t', \cdot) dt' \|_{X_1^{s,b}} \lesssim \| u_0 v \|_{X_1^{s,b-1}} \leq \| \psi_0(t) u_0 v \|_{X_1^{s,b-1}},
\]
where the second inequality is a consequence of Definition 1.2.
Then by (4.11) and (4.12) we obtain

\[
\|\psi_0(t)\|_{X_{n,b-1}}^2 = \sum_{n \in \mathbb{Z}} \int_\mathbb{R} (\tau + n^2)^{2(b-1)} \langle n \rangle^{2s} \hat{\psi}_n(\tau)^2 \, d\tau
\leq \sum_{n \in \mathbb{Z}} \int_\mathbb{R} \frac{\langle n \rangle^{2s}}{(\tau + n^2)^{2(1-b)}} \left( \sum_{j \in \mathbb{Z}} A_{j,n}(\tau) \right) \left( \sum_{k \in \mathbb{Z}} B_{k,n}(\tau) \right) \, d\tau
\]
\[
= \sum_{k \in \mathbb{Z}} \sum_{n,m} \int_\mathbb{R} \left( \sum_{j \in \mathbb{Z}} \frac{\langle n \rangle^{2s} A_{j,n}(\tau)}{\langle \tau + n^2 \rangle^{2(1-b)}} \right) B_{k,n}(\tau) \, d\tau.
\]
Now, thanks to the change of variables \( \tau' = -\tau + (n+k)^2 \) and (4.14) we deduce

\[
\|\psi(t)\|_{X_{n,b-1}}^2 \leq \sum_{k \in \mathbb{Z}} \sum_{n,m} \int_\mathbb{R} \left( \sum_{j \in \mathbb{Z}} \frac{\langle n \rangle^{2s} A_{j,n}(-n') \langle -\tau' + (n+k)^2 \rangle}{\langle -\tau' + (n+k)^2 \rangle^{2(1-b)}} \right) B_{k,n}(\tau) \, d\tau'
\]
\[
= \sum_{k \in \mathbb{Z}} \int_\mathbb{R} \left( \sum_{(n,j) \in \mathbb{Z}^2} \frac{\langle n \rangle^{2s} A_{j,n}(-\tau')}{\langle -\tau' + (n+k)^2 \rangle^{2(1-b)}} \right) \langle k \rangle^{2s} \langle -\tau' + k^2 \rangle^{2b} |\hat{b}_{k-n} |^2 (\tau') \, d\tau'
\]
\[
\leq \sup_{(k,\tau) \in \mathbb{Z} \times \mathbb{R}} \sum_{(n,j) \in \mathbb{Z}^2} \frac{\langle n \rangle^{2s} A_{j,n}(-\tau')}{\langle -\tau + (n+k)^2 \rangle} \sum_{k \in \mathbb{Z}} \int_\mathbb{R} \langle k \rangle^{2s} \langle -\tau' + k^2 \rangle^{2b} |\hat{b}_{k-n} |^2 (\tau') \, d\tau'
\]
\[
= \|v\|_{X_{n,b-1}}^2 \sup_{(k,\tau) \in \mathbb{Z} \times \mathbb{R}} \left[ \sum_{(n,j) \in \mathbb{Z}^2} \frac{\langle n \rangle^{2s} A_{j,n}(-\tau')}{\langle -\tau + (n+k)^2 \rangle} \right],
\]
by (4.10).

It remains to estimate the term

\[
I(k,\tau) := \sup_{(k,\tau) \in \mathbb{Z} \times \mathbb{R}} \left[ \sum_{(n,j) \in \mathbb{Z}^2} \frac{\langle n \rangle^{2s} A_{j,n}(-\tau')}{\langle -\tau + (n+k)^2 \rangle} \right],
\]
uniformly in \((k,\tau) \in \mathbb{Z} \times \mathbb{R} \).

By the definition (4.13) of \( A_{j,n} \) and the change of indexes \( m = -n - j \), we have

(4.15)

\[
I(k,\tau) = \sum_{(n,j) \in \mathbb{Z}^2} \frac{\langle n \rangle^{2s} |a_{-n-j}|^2}{(j)^{2s} \langle -\tau + (n+k)^2 \rangle^{2(1-b)} \langle \tau - (n+k)^2 + (n+j)^2 + j^2 \rangle^{2b}}
\]
\[
= \sum_{(n,m) \in \mathbb{Z}^2} \frac{\langle n \rangle^{2s} |a_m|^2}{(n+m)^{2s} \langle -\tau + (n+k)^2 \rangle^{2(1-b)} \langle \tau - (n+k)^2 + m^2 + (n+m)^2 \rangle^{2b}}
\]
\[
:= \sum_{(n,m) \in \mathbb{Z}^2} I_{n,m}(k,\tau).
\]
Denote by
\[ R_1 = R_1(\tau, n, k) = -\tau + (n + k)^2 + n^2, \]
\[ R_2 = R_2(\tau, n, k, m) = \tau - (n + k)^2 + m^2 + (n + m)^2. \]
Denote by \( \sigma = -s > 0 \) and \( \sigma_0 = -s_0 \geq 0 \). Write \( b = \frac{1}{2} + \varepsilon \). Then introduce
\[ \beta_1 = 2(1 - b) = 1 - 2\varepsilon < 1 \quad \text{and} \quad \beta_2 = 2b = 1 + 2\varepsilon > 1. \]
Therefore, \( I_{n,m} \) can be rewritten
\[
(4.16) \quad I_{n,m}(k, \tau) = \frac{(n + m)^{2\sigma}}{(n)^{2\sigma}} \frac{|a_m|^2}{\langle R_1 \rangle^{\beta_1} \langle R_2 \rangle^{\beta_2}}.
\]
• Observe that \( \beta_1 \leq \beta_2 \). Thus by \((4.16)\), for all \( m \neq k \) and \( 0 \leq \theta \leq 1 \)
\[
\sum_{n \in \mathbb{Z}} I_{n,m}(k, \tau) \leq |a_m|^2 \sum_{n \in \mathbb{Z}} \frac{(n + m)^{2\sigma}}{(n)^{2\sigma}} \frac{1}{\langle R_1 \rangle^{\beta_1}} \frac{1}{\langle R_2 \rangle^{\beta_2}}
\]
\[
(4.17) \quad \leq |a_m|^2 \sup_{n \in \mathbb{Z}} \left[ \frac{(n + m)^{2\sigma}}{(n)^{2\sigma}} \frac{1}{\langle R_1 \rangle^{(1-\theta)\beta_1}} \frac{1}{\langle R_2 \rangle^{(1-\theta)\beta_2}} \sum_{n \in \mathbb{Z}} \frac{1}{\langle R_1 \rangle^{\theta \beta_1}} \frac{1}{\langle R_2 \rangle^{\theta \beta_2}}. \right]
\]
For \( p, q \geq 1 \) such that \( 1/p + 1/q = 1 \) we have the Hölder inequality
\[
(4.18) \quad \sum_{n \in \mathbb{Z}} \frac{1}{\langle R_1 \rangle^{\theta \beta_1}} \frac{1}{\langle R_2 \rangle^{\theta \beta_2}} \leq \left( \sum_{n \in \mathbb{Z}} \frac{1}{\langle R_1 \rangle^{\theta \beta_1 p}} \right)^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} \frac{1}{\langle R_2 \rangle^{\theta \beta_2 q}} \right)^{\frac{1}{q}}.
\]
Now choose \( p, q \) such that \( \theta \beta p = \frac{1}{2} + \varepsilon \) and \( \theta \beta q = 1 + 2\varepsilon \), i.e.
\[ p = \frac{3}{2}, \quad q = 3, \quad \text{and thus} \quad \theta = \frac{1 + 2\varepsilon}{3(1 - 2\varepsilon)}. \]
(Notice that \( 0 \leq \theta \leq 1 \) if \( \varepsilon > 0 \) is small enough). With these choices, by Corollary \((4.3)\) all the sums in \((4.18)\) are uniformly bounded with respect to \( (m, k, \tau) \in \mathbb{Z}_+ \times \mathbb{R} \). Therefore, for \( m \neq k \) we have
\[
(4.19) \quad \sum_{n \in \mathbb{Z}} I_{n,m}(k, \tau) \lesssim |a_m|^2 \sup_{n \in \mathbb{Z}} \left[ \frac{(n + m)^{2\sigma}}{(n)^{2\sigma}} \frac{1}{\langle R_1 \rangle^{(1-\theta)\beta_1}} \frac{1}{\langle R_2 \rangle^{(1-\theta)\beta_2}} \right].
\]
Now we bound the \( \sup_{n \in \mathbb{Z}} \) in \((4.19)\). Notice that we have the inequalities
\[ \frac{1}{\langle R_1 \rangle \langle R_2 \rangle} \leq \frac{1}{\langle R_1 + R_2 \rangle} = \frac{1}{\langle n^2 + m^2 + (n + m)^2 \rangle} \lesssim \frac{1}{\langle m \rangle^2}, \]
and \( \langle n + m \rangle \lesssim \langle n \rangle \langle m \rangle \). Hence
\[
(4.20) \quad \sup_{n \in \mathbb{Z}} \left[ \frac{(n + m)^{2\sigma}}{(n)^{2\sigma}} \frac{1}{\langle R_1 \rangle^{(1-\theta)\beta_1}} \frac{1}{\langle R_2 \rangle^{(1-\theta)\beta_2}} \right] \lesssim \frac{1}{\langle m \rangle^{2(1-\theta)\beta_1 - 2\sigma}}.
\]
Then thanks to \((4.20)\), for \( m \neq k \), \((4.19)\) becomes
\[
\sum_{n \in \mathbb{Z}} I_{n,m}(k, \tau) \lesssim \frac{|a_m|^2}{\langle m \rangle^{2(1-\theta)\beta_1 - 2\sigma}} = \frac{|a_m|^2}{\langle m \rangle^{\frac{2}{3}(1-4\varepsilon)-2\sigma}}.
\]
and by summing up, we obtain

\[(4.21) \quad \sum_{(n,m) \in \mathbb{Z}^2, m \neq k} I_{n,m}(k, \tau) \lesssim \sum_{m \in \mathbb{Z}} \frac{|a_m|^2}{\langle m \rangle^{\frac{1}{2}(1-4\varepsilon)-2\sigma}} = \sum_{m \in \mathbb{Z}} \frac{|a_m|^2}{\langle m \rangle^{2\sigma_0} \langle m \rangle^\gamma},\]

with

\[(4.22) \quad \eta = \frac{4}{3}(1-4\varepsilon) - 2\sigma_0 - 2\sigma.\]

Now apply Hölder to (4.21): For all \(p \geq 2\) and \(1/q = 1 - 2/p\) so that \(q\eta > 1\), we can write

\[
\sum_{(n,m) \in \mathbb{Z}^2, m \neq k} I_{n,m}(k, \tau) \lesssim \left( \sum_{m \in \mathbb{Z}} \frac{|a_m|^p}{\langle m \rangle^{\sigma_0 p}} \right)^{\frac{1}{p}} \left( \sum_{j \in \mathbb{Z}} \frac{1}{\langle j \rangle^{q\eta}} \right)^{\frac{1}{q}}.
\]

By (4.22), the condition \(q\eta > 1\) is equivalent to

\[
\frac{4}{3}(1-4\varepsilon) - 2\sigma_0 - 2\sigma = \eta > \frac{1}{q} = 1 - \frac{2}{p},
\]

or

\[(4.23) \quad \sigma < \frac{1}{6} - \sigma_0 + \frac{1}{p} - \frac{8}{3}\varepsilon.\]

Assume that (4.1) is satisfied. Then for \(0 < \varepsilon \leq \varepsilon_1\) (for \(\varepsilon_1\) small enough), the condition (4.23) is also satisfied and we have

\[
\sum_{(n,m) \in \mathbb{Z}^2, m \neq k} I_{n,m}(k, \tau) \lesssim \|f\|_{H^{\sigma_0,p}}^2.
\]

• We now consider the case \(m = k\).

By (4.15), we have to bound, uniformly in \((k, \tau) \in \mathbb{Z} \times \mathbb{R}\), the term

\[
\sum_{n \in \mathbb{Z}} I_{n,k}(k, \tau) = |a_k|^2 \sum_{n \in \mathbb{Z}} \frac{(n+k)^{2\sigma}}{(n)^{2\sigma}} \frac{1}{(-\tau + (n+k)^2 + n^2)^{\beta_1} (\tau + k^2)^{\beta_2}}.
\]

By the inequality \(|a + b| \leq |a|(|b|\beta_1)\beta_1\) and Lemma 4.4, we obtain (recall that \(\beta_1 = 1 - 2\varepsilon\))

\[
\sum_{n \in \mathbb{Z}} I_{n,k}(k, \tau) \leq |a_k|^2 \sum_{n \in \mathbb{N}} \frac{(n+k)^{2\sigma}}{(n)^{2\sigma}} \frac{1}{(k^2 + n^2)^{1-2\varepsilon} (\tau + k^2)^{\beta_2}} \\
\leq |a_k|^2 \sum_{n \in \mathbb{N}} \frac{(n+k)^{2\sigma}}{(n^{2\sigma}(k^2 + n^2)^{1-2\varepsilon} (\tau + k^2)^{\beta_2}} \\
\lesssim |a_k|^2 \sum_{n \in \mathbb{N}} \frac{1}{(n^{2\sigma}(k^2 + n^2)^{1-\sigma-2\varepsilon} \\
\lesssim |a_k|^2 \sum_{n \in \mathbb{N}} \langle n \rangle^{2\sigma (k^2 + n^2)^{1-\sigma-2\varepsilon}}
\]
Now we compare this sums with an integral: Thanks to the change of variables $x = |k| y$ we obtain, as $\sigma < \frac{1}{2}$

$$
\sum_{n \in \mathbb{Z}} I_{n,k}(k, \tau) \lesssim |a_k|^2 \int_0^{+\infty} \frac{dx}{\langle x \rangle^{2\sigma} (k^2 + x^2)^{1-\sigma-2\varepsilon}} \\
\lesssim \frac{|a_k|^2}{\langle k \rangle^{1-4\varepsilon}} \int_0^{+\infty} \frac{dy}{y^{2\sigma} (1 + y^2)^{1-\sigma-2\varepsilon}} \\
\lesssim \frac{|a_k|^2}{\langle k \rangle^{1-4\varepsilon}} \lesssim \|f\|_{H^{s_0,p}}^2 ,
$$

whenever $1 - 4\varepsilon \geq 2\sigma_0 = -2s_0$, i.e. for $0 < \varepsilon \leq \varepsilon_2$.

Finally, set $b_2 = \frac{1}{2} + \varepsilon$, with $\varepsilon = \min (\varepsilon_1, \varepsilon_2)$. This concludes the proof.

---

5. Proof of the main theorem

We now have all the ingredients to prove Theorem 2.2 (observe that Theorem 2.1 is a particular case of the latter).

Proof of Theorem 2.2 — To take profit of the gain of regularity of the first Picard iterate (Proposition 3.2) we write $u = e^{it\Delta} f + v$ and where $v$ lives in a smaller space than $u$. This idea was used by N. Burq and N. Tzvetkov in the context of supercritical wave equations.

We plug this expression in the integral equation

$$
u = e^{it\Delta} f - ik \int_0^t e^{i(t-t')\Delta (\overline{u^2})}(t', x) dt',
$$

then we will show that the map $K$ defined by

$$
K(v) = -ik \int_0^t e^{i(t-t')\Delta (\overline{u^2})}(t', x) dt' - 2ik \int_0^t e^{i(t-t')\Delta (\overline{v^2})}(t', x') dt' - ik \int_0^t e^{i(t-t')\Delta (\overline{v^2})}(t', x) dt',
$$

is a contraction.

Let $p \geq 2$ and $s_0 > -\frac{1}{2}$ satisfy the condition (2.3), i.e.

$$
\frac{3}{p} + s_0 > \frac{5}{6},
$$

then there exists $s > -\frac{1}{2}$ so that

$$
-\frac{1}{6} - s_0 - \frac{1}{p} < s < -1 + \frac{2}{p},
$$
and we can use the estimates (1.4), (3.3) and (4.2) to obtain: There exist $b > \frac{1}{2}$ and $C \geq 1$ such that
\begin{align*}
\|K(v)\|_{X^{s,b}_1} &\leq C \left( \|f\|_{H^{s,0,p}}^2 + \|f\|_{H^{s,0,p}} \|v\|_{X^{s,b}_1} + \|v\|_{X^{s,b}_1}^2 \right), \\
\text{and} \\
\|K(v_1) - K(v_2)\|_{X^{s,b}_1} &\leq C \left( \|f\|_{H^{s,0,p}} + \|v_1 + v_2\|_{X^{s,b}_1} \right) \|v_1 - v_2\|_{X^{s,b}_1}.
\end{align*}

- The case of small initial data. We assume that $\|f\|_{H^{s,0,p}} = \mu \ll 1$. Then we show that $K$ is a contraction on the ball of radius $C\mu$ in $X^{s,b}$, for $\mu$ small enough. For $\|v_1\|_{X^{s,b}}, \|v_2\|_{X^{s,b+1}} \leq C\mu$, we deduce from (5.1) and (5.2) that
\begin{align*}
\|K(v)\|_{X^{s,b}_1} &\leq C \left( \mu^2 + \mu \|v\|_{X^{s,b}_1} + \|v\|_{X^{s,b}_1}^2 \right) \leq 3C^2 \mu^2, \\
\text{and} \\
\|K(v_1) - K(v_2)\|_{X^{s,b}_1} &\leq C \left( \mu + \|v_1 + v_2\|_{X^{s,b}_1} \right) \|v_1 - v_2\|_{X^{s,b}_1} \leq 3C^2 \mu \|v_1 - v_2\|_{X^{s,b}_1},
\end{align*}
and the result follows if we choose $\mu$ so that $3C^2 \mu < 1$.

The argument to show the uniqueness of the solution in the whole space is similar to the argument given in [15], we do not give more details here.

- The general case. Let $u$ be a solution of (1.1), then for all $\lambda > 0$, $u_\lambda$ defined by $u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x)$ in also a solution of the equation, but on a torus of period $2\pi/\lambda$. It is easy to check that the estimates (1.4), (3.3) and (4.2) still hold uniformly w.r.t $\lambda > 0$, if we replace $\mathbb{R}/(2\pi \mathbb{Z})$ with $\mathbb{R}/(\frac{2\pi}{\lambda} \mathbb{Z})$ (see Molinet [16] for more details). Now as
\[ \|f\|_{H^{s,0,p}} = \|u_\lambda(0,\cdot)\|_{H^{s,0,p}} \sim \lambda^{1+s+\frac{1}{p}}, \]
which tends to 0, we can apply the result of the previous case, and find a unique solution $u \in X^{s,b}([-\lambda^2, \lambda^2] \times \mathbb{T})$, for $\lambda$ small enough.

- The argument showing the regularity of the flow map is exactly the same as in [15], hence we omit the proof here.

\begin{remark}
We may compute the following Picard iterates of $u$. Therefore we could look for a solution to (1.1) of the form $u = u_0 + u_1 + \cdots + u_n + v$, where the $u_j$'s are known explicitly and where the unknown $v$ in more regular than $u_n$. A fixed point argument on $v$ would improve a bit the range (2.3). However we do not pursue this strategy as we do not think this will give an optimal result.
\end{remark}

\begin{remark}
The conclusion of Theorem 2.2 may be improved using estimates in $X^{s,b}_{p,q}$ space, i.e. $X^{s,b}$ spaces based on $L^p$ in the space frequency variable and $L^q$ in the variable $\tau$. See [13] for such a strategy for the DNLS equation.
\end{remark}
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