Excited pions, $\rho$- and $\omega$- mesons and their decays in a chiral $SU(2) \times SU(2)$ Lagrangian

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Abstract
A chiral $SU(2) \times SU(2)$ Lagrangian containing, besides the usual meson fields, their first radial excitations is considered. The Lagrangian is derived by bosonization of the Nambu–Jona-Lasinio quark model with separable non-local interactions, with form factors corresponding to 3-dimensional ground and excited state wave functions. The spontaneous breaking of chiral symmetry is governed by the NJL gap equation. The first radial excitations of the pions, $\rho$- and $\omega$-mesons are described with the help of two form factors. The weak decay constant $F_{\pi'}$ is calculated. The values for the decay widths of the processes $\rho \to 2\pi$, $\pi' \to \rho \pi$, $\rho' \to 2\pi$, $\rho' \to \omega \pi$ and $\omega' \to \rho \pi$ are obtained in agreement with the experimental data.

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1 Introduction

So far the experimental status of excited light mesons like the $\pi'$ and $K'$ is not yet completely established, requiring further investigations both in experiment and theory [1, 2]. In particular, the theoretical study of radially (orbitally) excited mesons is expected to provide us with a deeper understanding of the internal structure of hadrons and, equivalently, of the underlying effective interquark forces.

In the previous papers [2, 3] of one of the authors (MKV) a simple extension of the NJL-model with nonlocal separable quark interactions for the description of radially excited mesons was proposed. The theoretical foundations for the choice of the polynomial pion-quark form factors were discussed and it was shown that we can choose these form factors in such a way that the mass gap equation conserves the usual form and gives a solution with a constant constituent quark mass. Moreover, the quark condensate does not change after including the excited states in the model, because the tadpoles connected with the excited scalar fields vanish. Thus, in this approach it is possible to describe radially excited mesons above the usual NJL vacuum preserving the usual mechanism of chiral symmetry breaking. Finally, it has been shown that one can derive an effective meson Lagrangian for the ground and excited meson states directly in terms of local fields and their derivatives. The nonlocal separable interaction is defined in Minkowski space in a 3-dimensional (yet covariant) way whereby form factors depend only on the part of the quark- antiquark relative momentum transverse to the meson momentum. This ensures the absence of spurious relative-time excitations [4].

In paper [3] the meson mass spectrum for the ground and excited pions, kaons and the vector meson nonet in the $U(3) \times U(3)$ model of this type was obtained. By fitting the meson mass spectrum all parameters in this model are fixed. This then allows one to describe all the strong, electromagnetic and weak interactions of these mesons without introducing any new additional parameters.

In the papers [2, 3] the weak decay constants $F_{\pi'}$, $F_K$ and $F_{K'}$ were described. In the present work we would like to extend this by demonstrating that this model satisfactorily describes two types of decays. This concerns strong decays like $\rho \to 2\pi, \pi' \to \rho \pi, \rho' \to 2\pi$ associated to divergent quark diagrams as well as the decays $\rho' \to \omega \pi$ and $\omega' \to \rho \pi$ defined by anomalous quark diagrams.

The paper is organized as follows. In section 2, we introduce the effective quark interaction in the separable approximation and describe its bosonization. We discuss the choice of form factors necessary to describe the excited states of the scalar meson, pions, $\rho$, $\omega$ and $a_1$-mesons. In section 3, we derive the effective Lagrangian for the pions and perform the diagonalization leading to the physical pion ground and excited states. In section 4, we perform the diagonalization for the $\rho$ and $\omega$-mesons. In section 5, we fix the parameters of our model and evaluate the masses of the ground and excited states of pions and $\rho$-mesons and the weak decay constants $F_\pi$ and $F_{\pi'}$. In section 6, we evaluate the decay widths of the processes $\rho \to 2\pi, \pi' \to \rho \pi, \rho' \to 2\pi, \rho' \to \omega \pi$ and $\omega' \to \rho \pi$. The obtained results are discussed in section 7.
2 \textit{SU}(2) \times \textit{SU}(2)\ chiral\ Lagrangian\ with\ excited\ meson\ states

In the usual \textit{SU}(2) \times \textit{SU}(2) NJL model a local (current–current) effective quark interaction is used

\begin{equation}
L(\bar{q}, q) = \int d^4x \, \bar{q}(x) \left( i \partial^\mu - m^0 \right) q(x) + L_{\text{int}},
\end{equation}

\begin{equation}
L_{\text{int}} = \sum_{a=1}^{3} \int d^4x \left[ \frac{G_1}{2} (j_\sigma(x) j_\sigma(x) + j_\pi^a(x) j_\pi^a(x)) \\
- \frac{G_2}{2} (j_\rho^a(x) j_\rho^a(x) + j_{\alpha_1}^a(x) j_{\alpha_1}^a(x)) \right],
\end{equation}

where $m^0$ is the current quark mass matrix. We suppose that $m_u^0 \approx m_d^0 = m^0$.

\begin{equation}
\begin{aligned}
  j_\sigma(x) &= \bar{q}(x) q(x), \\
  j_\pi^a(x) &= \bar{q}(x) i \gamma_5 \tau^a q(x), \\
  j_\rho^\mu(x) &= \bar{q}(x) \gamma^\mu \tau^a q(x), \\
  j_{\alpha_1}^a(x) &= \bar{q}(x) \gamma_5 \gamma^\mu \tau^a q(x).
\end{aligned}
\end{equation}

Here $\tau^a$ are the Pauli matrices. The model can be bosonized in the standard way by representing the 4–fermion interaction as a Gaussian functional integral over scalar, pseudoscalar, vector and axial-vector meson fields \[3, 4\]. The effective meson Lagrangian, which is obtained by integration over the quark fields, is expressed in terms of local meson fields. By expanding the quark determinant in derivatives of the local meson fields one then derives the chiral meson Lagrangian.

The Lagrangian \[2\] describes only ground–state mesons. To include excited states, one has to introduce effective quark interactions with a finite range. In general, such interactions require bilocal meson fields for bosonization \[3, 4\]. A possibility to avoid this complication is the use of a separable interaction, which is still of current–current form, eq. \[2\], but allows for non-local vertices (form factors) in the definition of the quark currents, eqs. \[3\],

\begin{equation}
\begin{aligned}
  L_{\text{int}} &= \int d^4x \sum_{i=1}^{N} \sum_{a=1}^{3} \left[ \frac{G_1}{2} (j_{\sigma,i}(x) j_{\sigma,i}(x) + j_{\pi,i}^a(x) j_{\pi,i}^a(x)) \\
  &- \frac{G_2}{2} (j_{\rho,i}^a(x) j_{\rho,i}^a(x) + j_{\alpha_1,i}^a(x) j_{\alpha_1,i}^a(x)) \right],
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
  j_{\sigma,i}(x) &= \int d^4x_1 \int d^4x_2 \, \bar{q}(x_1) F_{\sigma,i}(x; x_1, x_2) q(x_2), \\
  j_{\pi,i}^a(x) &= \int d^4x_1 \int d^4x_2 \, \bar{q}(x_1) F_{\pi,i}^a(x; x_1, x_2) q(x_2), \\
  j_{\rho,i}^a(x) &= \int d^4x_1 \int d^4x_2 \, \bar{q}(x_1) F_{\rho,i}^a(x; x_1, x_2) q(x_2), \\
  j_{\alpha_1,i}^a(x) &= \int d^4x_1 \int d^4x_2 \, \bar{q}(x_1) F_{\alpha_1,i}^a(x; x_1, x_2) q(x_2).
\end{aligned}
\end{equation}

\footnote{The $\omega$-meson will be taken into consideration at the end of the paper.}
Here, \( F_{i,i}^a(x;x_1,x_2) \), \( i = 1, \ldots N \), denote a set of non-local scalar, pseudoscalar, vector and axial-vector quark vertices (in general momentum– and spin–dependent), which will be specified below. Upon bosonization we obtain

\[
L_{\text{bos}}(\bar{q}, q; \sigma, \pi, \rho, a_1) = \int d^4x_1 \int d^4x_2 \bar{q}(x_1)[(i\not\partial_{x_2} - m^0)\delta(x_1 - x_2)
\begin{align*}
&+ \int d^4x \sum_{i=1}^N \sum_{a=1}^3 (\sigma_i(x) F_{i,i}^a(x;x_1,x_2) + \pi_i^a(x) F_{\pi,i}^a(x;x_1,x_2) \\
&+ \rho_i^{a,\mu}(x) F_{\rho,i}^{a,\mu}(x;x_1,x_2) + a_{1,i}^{a,\mu}(x) F_{a_{1,i}}^{a,\mu}(x;x_1,x_2))]q(x_2) \\
&- \int d^4x \sum_{i=1}^N \sum_{a=1}^3 \left[ \frac{1}{2G_1} (\sigma_i^2 (x) + \pi_i^2 (x)) - \frac{1}{2G_2} (\rho_i^{a,\mu} (x) + a_{1,i}^{a,\mu}) \right]. \tag{9}
\end{align*}
\]

This Lagrangian describes a system of local meson fields, \( \sigma_i(x), \pi_i^a(x), \rho_i^{a,\mu}(x), a_{1,i}^{a,\mu}, i = 1, \ldots N \), which interact with the quarks through non-local vertices. These fields are not yet to be associated with physical particles, which will be obtained after determining the vacuum and diagonalizing the effective meson Lagrangian.

In order to describe the first radial excitations of mesons \( N = 2 \), we take the form factors in the form (see \( \text{[2]} \))

\[
\begin{align*}
F_{\pi,2}(k) &= f_\pi(k), & F_{\pi,2}^\prime(k) &= i\gamma_5 \tau^a f_\pi(k), \\
F_{\rho,2}^{a,\mu}(k) &= \gamma^\mu \tau^a f_\rho(k), & F_{a_{1,2}}^{a,\mu}(k) &= \gamma_5 \gamma^\mu \tau^a f_\rho(k), \tag{10}
\end{align*}
\]

\[ f^{U,1}(k) = c^{U,1} (1 + d k^2). \tag{11} \]

We consider here the form factors in the momentum space and in the rest frame of the mesons \( (\text{P}_{\text{meson}} = 0; k \) and \( P \) are the relative and total momentum of the quark-antiquark pair, respectively). For the ground states of mesons one has \( f^{U,1}(k) = 1 \).

After integrating over the quark fields in eq.\( \text{[9]} \), one obtains the effective Lagrangian of the \( \sigma_1, \sigma_2, \pi_1^a, \pi_2^a, \rho_1^{a,\mu}, \rho_2^{a,\mu}, a_{1,1}^{a,\mu} \) and \( a_{1,2}^{a,\mu} \), fields.

\[
L(\sigma', \pi, \rho, a_1, \bar{\sigma}, \bar{\pi}, \bar{\rho}, \bar{a}_1) = \]

\[
- \frac{1}{2G_1} (\sigma'^2 + \pi_1^2 + \pi_2^2) + \frac{1}{2G_2} (\rho_1^2 + a_1^{a,\mu} + \bar{a}_1^2 + \bar{a}_1^2) \\
- i N_\text{c} \text{ Tr } [i\not\partial - m^0 + \sigma' + (i\gamma_5 \pi a + \gamma_\mu \rho_1^\mu + \gamma_5 \gamma_\mu a_1^{a,\mu}) \tau^a \\
+ \bar{\sigma} f^\pi + (i\gamma_5 \bar{\pi}_a f^\pi + \gamma_\mu \bar{\rho}_1^\mu f^\rho + \gamma_5 \gamma_\mu \bar{a}_1^{a,\mu} f^\rho) \tau^aq]. \tag{12}
\]

where we have put \( \sigma_1 = \sigma', \sigma_2 = \bar{\sigma}, \pi_1 = \pi, \pi_2 = \bar{\pi} \) etc. Now let us define the vacuum expectation of the \( \sigma' \) field

\[
< \frac{\delta L}{\delta \sigma'} >_0 = -i N_\text{c} \text{ tr } \int_{\Lambda_3} \frac{d^4k}{(2\pi)^4} \frac{1}{(k - m^0 + < \sigma' >_0)} = - \frac{< \sigma' >_0}{G_1} = 0. \tag{13}
\]

Introduce the new sigma field which vacuum expectation is equal to zero

\[
\sigma = \sigma' - < \sigma' >_0 \tag{14}
\]

and redefine the quark mass

\[
m = m^0 - < \sigma' >. \tag{15}
\]

\[ 4 \]
Then eq. (13) can be rewritten in the form of the usual gap equation

\[ m = m^0 + 8G_1mI_1(m), \]  

(16)

where

\[ I_n(m) = -iN_c \int_{\Lambda_3} \frac{d^4k}{(2\pi)^4} \frac{1}{(m^2 - k^2)^n} \]

(17)

and \( m \) is the constituent quark mass.

### 3 Effective Lagrangian for the ground and excited states of the pions

To describe the first excited states of pions and \( \rho \)-mesons, it is necessary to use form factors \( f^{\pi,\rho}(k) \) (see eq. (11))

\[ f^{\pi,\rho}(k) = c^{\pi,\rho}(1 + dk^2). \]

(18)

Following refs. [2, 3] we can fix the slope parameter \( d \) by using the condition

\[ I_{f..f}^f(m) = 0, \]

(19)

where

\[ I_{f..f}^f(m) = -iN_c \int_{\Lambda_3} \frac{d^4k}{(2\pi)^4} \frac{f^{U}(k)\ldots f^{U}(k)}{(m^2 - k^2)}. \]

(20)

Eq. (19) allows us to conserve the gap equation in the form usual for the NJL model (see eq. (16)), because the tadpole with the excited scalar external field does not contribute to the quark condensate and to the constituent quark mass.

Using eq. (19) and the values of \( m \) and \( \Lambda_3 \) quoted in sec.5 we obtain for the slope parameter \( d \) the value

\[ d = -1.784 \text{ GeV}^{-2}. \]

(21)

Now let us consider the free part of the Lagrangian (12). For the pions we obtain

\[ L^{(2)}(\pi) = \frac{1}{2} \sum_{i,j=1}^{2} \sum_{a=1}^{3} \pi^a_i(P)K_{ij}(P)\pi^a_j(P), \]

(22)

where \( K_{ij}(P) \) given by

\[ K_{ij}(P) = -\delta_{ij} \frac{1}{G_1} - iN_c \text{ tr} \int_{\Lambda_3} \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{k^2 + \frac{i}{2}P - m}i\gamma_5f^\pi_i \frac{1}{k^2 - \frac{i}{2}P - m}i\gamma_5f^\pi_j \right], \]

(23)

\[ f^\pi_1 \equiv 1, \quad f^\pi_2 \equiv f^\pi(k). \]

The integral (23) is evaluated by expanding in the meson field momentum, \( P \). To order \( P^2 \), one obtains

\[ K_{11}(P) = Z_1(P^2 - M_{\pi_1}^2), \quad K_{22}(P) = Z_2(P^2 - M_{\pi_2}^2), \]

\[ K_{12}(P) = K_{21}(P) = \gamma P^2, \]

(24)
where
\[ Z_1 = 4I_2 Z, \quad Z_2 = 4I_2^{ff} \bar{Z}, \quad \gamma = 4I_2^f Z, \] (25)
\[ M_{\pi_1}^2 = (Z_1)^{-1} \left[ \frac{1}{G_1} - 8I_1(m) \right] = \frac{m^0}{4mI_2(m)}, \]
\[ M_{\pi_2}^2 = (Z_2)^{-1} \left[ \frac{1}{G_1} - 8I_1^{ff}(m) \right]. \] (26)

Here, \( Z = 1 - \frac{6m^2}{M_{a_1}^2} \approx 0.7, \bar{Z} = 1 - \frac{\Gamma_\pi^2 6m^2}{M_{a_1}^2} \approx 1, \) \( M_{a_1} \) is the mass of the \( a_1 \) meson and \( \Gamma_\pi \) is given below (see eq. (30) below). \( I_n, I_n^f \) and \( I_n^{ff} \) denote the usual loop integrals arising in the momentum expansion of the NJL quark determinant, but now with zero, one or two factors \( f^U(k), \) eqs. (18), in the numerator (see (20) and below)

\[ I_{n-f}^I(m) = -iN_c \int_{\Lambda^3} \frac{d^4k}{(2\pi)^4} \frac{f^U(k) \cdots f^U(k)}{(m^2 - k^2)^n}. \] (27)

The evaluation of these integrals with a 3–momentum cutoff is described e.g. in ref.[10]. The integral over \( k_0 \) is taken by contour integration, and the remaining 3–dimensional integral is regularized by the cutoff. Only the divergent parts are kept; all finite parts are dropped.

After the renormalization of the pion fields
\[ \pi_i^a = \sqrt{Z_i} \pi_i^0 \] (28)
the Lagrangian (22) takes the form
\[ L_\pi^{(2)} = \frac{1}{2} \left[ (P^2 - M_{\pi_1}^2) \pi_1^2 + 2\Gamma_\pi P^2 \pi_1 \pi_2 + (P^2 - M_{\pi_2}^2) \pi_2^2 \right]. \] (29)
Here
\[ \Gamma_\pi = \frac{\gamma}{\sqrt{Z_1 Z_2}} = \frac{I_2^{ff} \sqrt{Z}}{I_2 I_2^{ff} \bar{Z}}. \] (30)

Using the additional transformation of the pion fields
\[ \pi^a = \cos(\theta_\pi - \theta_0^\pi)\pi_1^a - \cos(\theta_\pi + \theta_0^\pi)\pi_2^a, \]
\[ \pi'^a = \sin(\theta_\pi - \theta_0^\pi)\pi_1^a - \sin(\theta_\pi + \theta_0^\pi)\pi_2^a, \] (31)
where
\[ \sin\theta_0^\pi = \sqrt{\frac{1 + \Gamma_\pi}{2}}, \quad \cos\theta_0^\pi = \sqrt{\frac{1 - \Gamma_\pi}{2}} \] (32)
the Lagrangian (29) takes the diagonal form
\[ L_\pi^{(2)} = \frac{1}{2} (P^2 - M_{\pi}^2) \pi^2 + \frac{1}{2} (P^2 - M_{\pi'}^2) \pi'^2. \] (33)

The factors \( Z \) and \( \bar{Z} \) appear when we take into account the transitions \( \pi_i \rightarrow a_1 \rightarrow \pi_j. \)
Here
\[ M^2_{\pi,\pi'} = \frac{1}{2(1 - \Gamma^2_{\pi})}[M^2_{\pi_1} + M^2_{\pi_2} \pm \sqrt{(M^2_{\pi_1} - M^2_{\pi_2})^2 + (2M_{\pi_1}M_{\pi_2}\Gamma_{\pi})^2}] \] (34)
and
\[ \tan 2\bar{\theta}_\pi = \sqrt{\frac{1}{\Gamma^2_{\pi}} - 1} \left[ \frac{M^2_{\pi_1} - M^2_{\pi_2}}{M^2_{\pi_1} + M^2_{\pi_2}} \right] = -\tan 2\bar{\theta}_0 \left[ \frac{M^2_{\pi_1} - M^2_{\pi_2}}{M^2_{\pi_1} + M^2_{\pi_2}} \right], \quad (2\bar{\theta}_\pi = 2\bar{\theta}_0 + \pi). \] (35)

In the chiral limit \( M_{\pi_1} \to 0, \theta_\pi \to \theta^0_\pi \) (see eqs. (26, 35)) we obtain
\[ M^2_{\pi} = M^2_{\pi_1} + O(M^4_{\pi}) \] (36)
\[ M^2_{\pi'} = \frac{M^2_{\pi_2} + M^2_{\pi_1}\Gamma_{\pi}}{1 - \Gamma^2_{\pi}} + O(M^4_{\pi_1}). \] (37)

Thus, in the chiral limit the effective Lagrangian eq. (29) describes a massless Goldstone pion, \( \pi \), and a massive particle, \( \pi' \).

For the weak decay constants of the pions we obtain (see [2])
\[ F_{\pi} = 2m\sqrt{ZI_2(m)} \cos(\theta_\pi - \theta^0_\pi), \]
\[ F_{\pi'} = 2m\sqrt{ZI_2(m)} \sin(\theta_\pi - \theta^0_\pi). \] (38)

In the chiral limit we have
\[ F_{\pi} = \frac{m}{g_{\pi}}, \quad F_{\pi'} = 0. \] (39)

with \( g_{\pi} = Z^{-1/2}_1 \) which is just the Goldberger-Treimann relation for the coupling constant \( g_{\pi} \). The matrix elements of the divergence of the axial current between meson states and the vacuum equal (PCAC relations)
\[ \langle 0|\partial^\mu A^a_\mu|\pi\rangle = M^2\pi F_{\pi}\delta^{ab}, \] (40)
\[ \langle 0|\partial^\mu A^a_\mu|\pi'\rangle = M^2\pi' F_{\pi'}\delta^{ab}. \] (41)

Then from eqs. (36) and (39) we can see that the axial current is conserved in the chiral limit, because its divergence equals zero, according to the low-energy theorems.

**4 Effective Lagrangian for ground and excited states of the \( \rho(\omega) \)-mesons**

The free part of the effective Lagrangian (12) describing the ground and excited states of the \( \rho \)- and \( \omega \)-mesons has the form
\[ L^{(2)}(\rho, \omega) = -\frac{1}{2} \sum_{i,j=1}^{2} \sum_{\alpha,\beta=0}^{3} \rho^i_{\mu\alpha}(P) R^{\mu\nu}_{ij}(P) \rho^j_{\nu\beta}(P), \] (42)
where
\[ \sum_{a=0}^{3} (p_i^{\mu a})^2 = (\omega_i)^2 + (\rho_i^{0 \mu})^2 + 2 \rho_i^{\pm \mu} \rho_i^{-\mu} \tag{43} \]

and
\[ R_{ij}^{\mu \nu}(P) = \frac{\delta_{ij}}{G_2} g^{\mu \nu} - i N_c \text{ tr} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + \frac{1}{2} P^2 - m_i^2} \gamma^\mu f_{i}^\rho \frac{1}{k^2 + \frac{1}{2} P^2 - m} \gamma^\nu f_{j}^\rho, \]
\[ f_1^\rho \equiv 1, \quad f_2^\rho \equiv f^\rho(k). \tag{44} \]

To order \( P^2 \), one obtains
\[ R_{11}^{\mu \nu} = W_1 [P^2 g^{\mu \nu} - P^\mu P^\nu - g^{\mu \nu} M_{\rho_1}^2], \]
\[ R_{22}^{\mu \nu} = W_2 [P^2 g^{\mu \nu} - P^\mu P^\nu - g^{\mu \nu} M_{\rho_2}^2], \]
\[ R_{12}^{\mu \nu} = R_{21}^{\mu \nu} = \gamma_\rho [P^2 g^{\mu \nu} - P^\mu P^\nu]. \tag{45} \]

Here
\[ W_1 = \frac{8}{3} I_2, \quad W_2 = \frac{8}{3} I_2^f, \quad \gamma_\rho = \frac{8}{3} I_2, \]
\[ M_{\rho_1}^2 = (W_1 G_2)^{-1}, \quad M_{\rho_2}^2 = (W_2 G_2)^{-1}. \tag{46} \]

After renormalization of the \( \rho(\omega) \)-meson fields
\[ \rho_{i}^{\mu \nu} = \sqrt{W_i} \rho_i^{\mu a} \]
we obtain the Lagrangian
\[ L^{(2)}_\rho = -\frac{1}{2} \left[ (g^{\mu \nu} P^2 - P^\mu P^\nu - g^{\mu \nu} M_{\rho_1}^2) \rho_i^{\mu} \rho_i^{\nu} \right. \]
\[ + \left. 2 \Gamma_\rho (g^{\mu \nu} P^2 - P^\mu P^\nu) \rho_1^{\mu} \rho_2^{\nu} + (g^{\mu \nu} P^2 - P^\mu P^\nu - g^{\mu \nu} M_{\rho_2}^2) \rho_2^{\mu} \rho_2^{\nu} \right], \tag{49} \]
where
\[ \Gamma_\rho = \frac{I_2^f(m)}{\sqrt{I_2(m) I_2^f(m)}}. \tag{50} \]

By transforming the \( \rho \)-meson fields similarly to eqs. (31) used for pions, the Lagrangian takes the diagonal form
\[ L^{(2)}_{\rho,\rho'} = -\frac{1}{2} \left[ (g^{\mu \nu} P^2 - P^\mu P^\nu - M_{\rho}^2) \rho^{\mu} \rho^{\nu} + (g^{\mu \nu} P^2 - P^\mu P^\nu - M_{\rho'}^2) \rho'^{\mu} \rho'^{\nu} \right], \tag{51} \]
where \( \rho \) and \( \rho' \) are the physical ground and excited \( \rho \)-meson states and
\[ M_{\rho,\rho'}^2 = \frac{1}{2(1 - \Gamma_\rho^2)} \left[ M_{\rho_1}^2 + M_{\rho_2}^2 \pm \sqrt{(M_{\rho_1}^2 - M_{\rho_2}^2)^2 + (2 M_{\rho_1} M_{\rho_2} \Gamma_\rho)^2} \right]. \tag{52} \]

The same formulae are valid for the \( \omega \)-meson.
5 Numerical estimates

We can now estimate numerically the masses of the pions and ρ-mesons and the weak decay constants $F_\pi$ and $F_\pi'$ in our model.

Because the mass formulae and others equations (for instance, Goldberger – Treimann relation etc.) have new forms in the extended NJL model with excited states of mesons as compared with the usual NJL model, the values of basic parameters ($m$, $\Lambda_3$, $G_1$, $G_2$) could change. However, this does not happen for the parameters $m = 280$ MeV, $\Lambda_3 = 1.03$ GeV and $G_1 = 3.47$ GeV$^{-2}$ (see ref. [10]), because the condition (19) conserves the gap equation in the old form and one can satisfactorily describe the weak decay constant $F_\pi$ and the decay $\rho \to 2\pi$ in the extended model, too, using $m = 280$ MeV and the cutoff parameter $\Lambda_3 = 1.03$ GeV (see below). $G_1$ does not change in the extended model, because $M_\pi \approx M_\pi'$. However, for the coupling constant $G_2$ the new value $G_2 = 12.5$ GeV$^{-2}$ will be used, which differs from the former value $G_2 = 16$ GeV$^{-2}$ (see ref. [10]). It is a consequence of the fact that the mass $M_\rho$ noticeably differs from the physical mass $M_\rho$ of the ground state $\rho$.

Using these basic parameters, the slope parameter $d = -1.784$ GeV$^{-2}$ (see eq. (21)) and choosing the form factor parameters $c^\pi = 1.37$ and $c^\rho = 1.26$, one finds

$$M_\rho = 768.3\text{ MeV}, \quad M_\rho' = 1.49\text{ GeV}, \quad M_\pi = 136\text{ MeV}, \quad M_\pi' = 1.3\text{ GeV}. \quad (53)$$

The experimental values are

$$M_\rho^{\text{exp}} = 768.5 \pm 0.6\text{ MeV}, \quad M_\rho'^{\text{exp}} = 1465 \pm 25\text{ MeV},$$

$$M_\pi^{\text{exp}} = 139.57\text{ MeV}, \quad M_\pi'^{\text{exp}} = 134.98\text{ MeV},$$

$$M_\pi'^{\text{exp}} = 1300 \pm 100\text{ MeV}. \quad (54)$$

$$F_\pi = 93\text{ MeV}, \quad F_\pi' = 0.57\text{ MeV},$$

$$F_\pi' \approx -\cot 2\theta^0 \left( \frac{M_\pi}{M_\pi'} \right)^2 \approx 0.5 \left( \frac{M_\pi}{M_\pi'} \right)^2. \quad (55)$$

6 Decays $\rho \to 2\pi, \pi' \to \rho\pi, \rho' \to 2\pi, \rho' \to \omega\pi$ and $\omega' \to \rho\pi$.

Let us show how the decay widths of the ground and excited states of mesons are calculated in our model. For that we start with the decay $\rho \to 2\pi$. The amplitude describing this decay has the form

$$T_{\rho \to 2\pi} = i \frac{g_\rho}{2} \epsilon_{ijk} (p_j - p_k)\nu \rho^{i}_{\nu} \pi^{j}_{\nu} \pi^{k}_{\nu}, \quad (56)$$

where $p_{j,k}$ are the pion momenta and $\epsilon_{ijk}$ is the antisymmetric tensor. Using the value $\alpha_\rho = \frac{g_\rho^2}{4\pi} \approx 3 \quad (g_\rho \approx 6.1)$ of refs. [4, 3, 7] we obtain for the decay width

$$\Gamma_{\rho \to 2\pi} = \frac{\alpha_\rho}{12 M_\rho^2} (M_\rho^2 - 4 M_\pi^2)^{3/2} = 151.5\text{ MeV}. \quad (57)$$

The experimental value is [4]

$$\Gamma_{\rho \to 2\pi} = 150.7 \pm 1.2\text{ MeV} \quad (58)$$
Now let us calculate this amplitude in our model with the excited states of mesons. For that we rewrite the amplitude $T_{\rho\rightarrow 2\pi}$ in the form

$$T_{\rho\rightarrow 2\pi} = i \epsilon_{ijk} (p_j - p_k)_{\nu} \rho_{ij} \pi^j \pi^k,$$

(59)

and calculate the factor $c_{\rho\rightarrow 2\pi}$ in the new model. Using eqs. (28), (31) and (48) we can find the following expressions for the meson fields $\pi$, $\pi'$ and $\rho$, $\rho'$ ($\alpha = \theta_\pi$, $\beta = \theta_\rho$)

$$\pi_1 = \frac{\sin(\alpha + \alpha_0)\pi - \cos(\alpha + \alpha_0)\pi'}{\sqrt{Z_1 \sin 2\alpha_0}},$$

$$\pi_2 = \frac{\sin(\alpha - \alpha_0)\pi - \cos(\alpha - \alpha_0)\pi'}{\sqrt{Z_2 \sin 2\alpha_0}},$$

(60)

$$\rho_1 = \frac{\sin(\beta + \beta_0)\rho - \cos(\beta + \beta_0)\rho'}{\sin 2\beta_0 \sqrt{8/3 I_2}},$$

$$\rho_2 = \frac{\sin(\beta - \beta_0)\rho - \cos(\beta - \beta_0)\rho'}{\sin 2\beta_0 \sqrt{8/3 I_{2/\rho}}},$$

(61)

or, using the values $I_2 = 0.04, I_{2/\rho}^{ff} = 0.0244, \alpha = 59.5^o, \alpha_0 = 59.15^o, \beta = 79.9^o, \beta_0 = 61.5^o$, we obtain

$$\pi_1 = \frac{0.878\pi + 0.48\pi'}{0.88\sqrt{Z_1}}, \quad \pi_2 = \frac{0.0061\pi - \pi'}{0.88\sqrt{Z_2}},$$

$$\rho_1 = (0.744\rho + 0.931\rho') g_\rho/2, \quad \rho_2 = (0.48 \rho - 1.445 \rho') g_\rho/2.$$

(62)

The decay $\rho \rightarrow 2\pi$ is described by the quark triangle diagrams with the vertices $\rho_1(\pi_1^2 + 2\pi_1 \pi_2 + \pi_2^2)$ and $\rho_2(\pi_1^2 + 2\pi_1 \pi_2 + \pi_2^2)$ (see Fig.1). Using eqs. (60), (61) and (62) the factor $c_{\rho\rightarrow 2\pi}$ is given by

$$c_{\rho\rightarrow 2\pi} = c_{\rho_1\rightarrow 2\pi} + c_{\rho_2\rightarrow 2\pi} = 0.975 g_\rho/2,$$

(63)

$$c_{\rho_1\rightarrow 2\pi} = \frac{\sin(\beta + \beta_0)}{\sin^2 2\alpha_0 \sin 2\beta_0 \sqrt{8/3 I_2}} [ (\sin(\alpha + \alpha_0))^2 + 2\sin(\alpha + \alpha_0)\sin(\alpha - \alpha_0) \Gamma_\pi ]$$

$$+ (\sin(\alpha - \alpha_0))^2 = \frac{\sin^2 2\alpha_0}{\sin 2\beta_0 \sqrt{8/3 I_2}} = 0.745 g_\rho/2,$$

$$c_{\rho_2\rightarrow 2\pi} = \frac{\sin(\beta - \beta_0)}{\sin^2 2\alpha_0 \sin 2\beta_0 \sqrt{8/3 I_{2/\rho}}^{ff}} [ (\sin(\alpha + \alpha_0))^2 \frac{I_2^{ff}}{I_2} ]$$

$$+ 2\sin(\alpha + \alpha_0)\sin(\alpha - \alpha_0) \frac{I_2^{ff}}{\sqrt{I_2 I_{2/\rho}^{ff}}} + (\sin(\alpha - \alpha_0))^2 \frac{I_2^{fff}}{I_2^{ff}}] = 0.227 g_\rho/2.$$

(64)

3Analogous formulae are obtained for the $\omega$-meson.

4Taking into account the $\pi \rightarrow a_1$ transitions on the external pion lines we obtain additional factors $Z$ (\tilde{Z}) in the numerators of our triangle diagrams, which cancel corresponding factors in $Z_i$ (see eqs. (23), (30) and ref. \cite{5}). Therefore, in future we shall ignore the factors $Z$ (\tilde{Z}) in $Z_i$. 

10
Here we used the values $I_2 = 0.04, I_2^f = 0.0185, I_2^{ff} = 0.0289, I_2^{fff} = 0.0224$ and the relation $\Gamma_\pi = -\cos 2\alpha_0$ (see eqs. (32)). Then the decay width $\rho \to 2\pi$ is equal to

$$
\Gamma_{\rho \to 2\pi} = 149 \text{ MeV}.
$$

(65)

In the limit $f = 0$ ($\alpha = \alpha_0, \beta = \beta_0$) from eqs. (64) one finds

$$
c_{\rho \to 2\pi} = c_{\rho_1 \to 2\pi} = g_\rho/2, \quad c_{\rho_2 \to 2\pi} = 0.
$$

(66)

Now let us consider the decay $\pi' \to \rho\pi$. The amplitude of this decay has the form

$$
T_{\pi' \to \rho\pi} = i c_{\pi' \to \rho\pi} \epsilon_{ijk} (p_j + p_k)\nu^i \bar{\rho}_\nu \pi^j \pi^k,
$$

(67)

where

$$
c_{\pi' \to \rho\pi} = c_{\pi' \to \rho_1\pi} + c_{\pi' \to \rho_2\pi}.
$$

(68)

Then for $c_{\pi' \to \rho_1\pi}$ we obtain

$$
c_{\pi' \to \rho_1\pi} = \frac{2}{(\sin 2\alpha_0)^2} \left[ -\sin(\alpha + \alpha_0)\cos(\alpha + \alpha_0) - \sin 2\alpha \Gamma_\pi - \sin(\alpha - \alpha_0)\cos(\alpha - \alpha_0) \right]
$$

$$
= -\sin 2\alpha \cos 2\alpha_0 + \sin 2\alpha \cos 2\alpha_0 = 0 \frac{\sin(\beta + \beta_0)}{\sin 2\beta_0} g_\rho/2 = 0,
$$

(69)

$$
c_{\pi' \to \rho_2\pi} = \frac{2}{(\sin 2\alpha_0)^2} \left[ -\sin(\alpha + \alpha_0)\cos(\alpha + \alpha_0) I_2^f \sqrt{I_2^{ff} I_2^{ff}} - \sin 2\alpha \frac{I_2^{ff}}{\sqrt{I_2^{ff}}} \right]
$$

$$
-\sin(\alpha - \alpha_0)\cos(\alpha - \alpha_0) I_2^{ff} \sin(\beta - \beta_0) \sqrt{\frac{I_2^{ff}}{I_2^{ff}}} g_\rho/2 = -0.573 \frac{g_\rho}{2}.
$$

(70)

For the decay width $\pi' \to \rho\pi$ we obtain

$$
\Gamma_{\pi' \to \rho\pi} = \frac{c_{\pi' \to \rho\pi}^2}{4\pi M_\pi^2 M_\rho^2} \left[ M_\pi^4 + M_\rho^4 + M_\pi^4 - 2(M_\pi^2 M_\rho^2 + M_\pi^2 M_\rho^2 + M_\rho^2 M_\pi^2) \right]^{3/2}
$$

$$
= 220 \text{ MeV}.
$$

(71)

This value is in agreement with the experimental data

$$
\Gamma_{\pi'}^{\text{total}} = 200 - 600 \text{ MeV}.
$$

(72)

The decay $\pi' \to \sigma\pi$ in our model gives only a small contribution to the total decay width of $\pi'$.

For the decay $\rho' \to 2\pi$ we obtain in our model the result

$$
\Gamma_{\rho' \to 2\pi} \approx 22 \text{ MeV}.
$$

(73)

All our results are in agreement with the results of a relativized potential quark model with the $3P_0$-mechanism of meson decays.
In conclusion of this section let us calculate the decay widths of the processes \( \rho' \to \omega \pi \) and \( \omega' \to \rho \pi \). These decays go through anomalous triangle quark loop diagrams. The amplitude of the decay \( \rho' \to \omega \pi \) takes the form

\[
T_{\rho' \to \omega \pi}^{\mu \nu} = \frac{3 \alpha_{\rho} c_{\rho' \to \omega \pi}}{2 \pi F_\pi} \epsilon_{\mu \nu \rho \sigma} q^\rho p^\sigma,
\]

where \( q \) and \( p \) are the momentum of the \( \omega \) and \( \rho' \) meson, respectively. The factor \( c_{\rho' \to \omega \pi} \) is similar to the factors \( c_{\rho \to 2\pi} \) and \( c_{\pi' \to \rho \pi} \) in previous equations and arises from the four triangle quark diagrams with vertices \( \pi_1 (\rho_1 \omega_1 + \rho_2 \omega_1 + \rho_1 \omega_2 + \rho_2 \omega_2) \). Using the estimate

\[
c_{\rho' \to \omega \pi} \approx -0.3,
\]

we obtain for the decay width

\[
\Gamma_{\rho' \to \omega \pi} = \frac{3}{2 \pi M_{\rho'}^2} \left( \frac{\alpha_{\rho} c_{\rho' \to \omega \pi}}{8 \pi F_\pi} \right)^2 \left[ M_{\rho'}^4 + M_\omega^4 + M_\pi^4 - 2(M_\rho^2 M_\omega^2 + M_\rho^2 M_\pi^2 + M_\omega^2 M_\pi^2) \right]^{3/2}
\approx 75 \text{ MeV}.
\]

For the decay \( \omega' \to \rho \pi \) we have the relation

\[
\Gamma_{\omega' \to \rho \pi} \approx 3 \Gamma_{\rho' \to \omega \pi}
\]

leading to the estimate

\[
\Gamma_{\omega' \to \rho \pi} \approx 225 \text{ MeV}.
\]

The experimental values are \([12]\)

\[
\Gamma^{\text{exp}}_{\rho' \to \omega \pi} = 0.21 \Gamma_{\rho' \to \omega \pi}^{\text{tot}} = 65.1 \pm 12.6 \text{ MeV}
\]

and \([1]\)

\[
\Gamma^{\text{exp}}_{\omega' \to \rho \pi} = 174 \pm 60 \text{ MeV}.
\]

Finally, let us quote the ratio of the decay widths \( \rho' \to \omega \pi \) and \( \rho' \to 2\pi \)

\[
\frac{\Gamma_{\rho' \to 2\pi}}{\Gamma_{\rho' \to \omega \pi}} \approx 0.3,
\]

which has to be compared with the experimental value 0.32 (see \([12]\)).

Thus, we can see that all our estimates are in satisfactory agreement with experimental data.

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\(^5\)We shall neglect the diagrams with vertices \( \pi_2 \), because their contribution to the ground state of the pion is very small (see eq. (62)).
7 Summary and conclusions

Our calculations have shown that the main decay of the $\rho$-meson, $\rho \to 2\pi$, changes very weakly after including the excited meson states into the NJL model. The main part of this decay (75%) comes from the $\rho$-vertex without form factor, whereas the remaining 25% of the decay are due to the $\rho$-vertex with form factor. As a result, the new coupling constant $g_\rho$ turns out to be very close to the former value.

For the decay $\pi' \to \rho\pi$ we have the opposite situation. Here the channel connected with the $\rho$-vertex without form factor is closed, because the states $\pi$ and $\pi'$ are orthogonal to each other, and the total decay width of $\pi' \to \rho\pi$ is defined by the channel going through the $\rho$-vertex with form factor. As a result, we obtain the quoted value which satisfies the experimental data [1]. The decay $\pi' \to \sigma\pi$ gives only very small corrections to the total decay width of $\pi'$. Notice that these results are in agreement with the results obtained in the relativized version of the $3P_1$ potential model [1].

For the decay $\rho' \to 2\pi$ we obtain a strong compensation of the contributions of the two channels, connected with $\rho$-vertices with and without form factors, and the corresponding decay width is equal to 22 MeV. Again this value is very close to the result of ref. [1].

It should be emphasized, that the decays $\rho' \to \omega\pi$ and $\omega' \to \rho\pi$ belonging to quite another class of quark loop diagrams (“anomaly diagrams”) are also satisfactorily described by our model. In future applications we are planning to describe the decays of other pseudoscalar, scalar, vector and axial-vector mesons of the $\text{U}(3)$ flavour group, too.

Finally, it is worth mentioning that concerning in particular the bosonization program of QCD, the description of excited mesons has constantly attracted considerable interest by many authors (see e.g. [13, 14] and references therein).

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Fig.1. Triangle diagrams describing the decays:

a) $\rho \rightarrow 2\pi$, when we consider the parts of $\rho_i$ and $\pi_i$ meson fields corresponding only to the ground states $\rho$ and $\pi$;

b) $\pi' \rightarrow \rho\pi$, when we consider the parts of the $\rho_i$ corresponding to the ground state $\rho$ and take for one of the pions $\pi_i$ the part corresponding to the ground state $\pi$ and for the other the excited pion state $\pi'$.

c) $\rho' \rightarrow 2\pi$, when we consider the parts of $\rho_i$ corresponding to the excited $\rho'$ and take for the $\pi_i$ only the ground state pions.
Fig. 1