SHEAR AND VORTICITY OF PERFECT-FLUID SPACETIMES
AND THE SHEAR-FREE CONJECTURE

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Abstract. We obtain expressions for the shear and the vorticity tensor \( \sigma \) of perfect-fluid spacetimes, in terms of the divergence of the Weyl tensor. For such spacetimes, we prove that if the gradient of the energy density is parallel to the velocity, then either the expansion rate is zero, or the vorticity vanishes. This statement recalls the “shear-free conjecture” for a perfect barotropic fluid: vanishing shear implies either vanishing expansion rate or vanishing vorticity. Finally, we give a new condition for a perfect fluid to be a Generalized Robinson-Walker spacetime.

1. Introduction

In a Lorentzian manifold, a velocity field \( u^k \) is a time-like unit vector field, \( u_k u^k = -1 \). Its covariant gradient has a standard decomposition into terms with geometric meaning [7]:

\[
\nabla_k u_l = \frac{\theta}{n-1} h_{kl} - u_k \dot{u}_l + \sigma_{kl} + \omega_{kl}
\]

Namely, \( \theta = \nabla_k u^k \) is the expansion scalar, \( h_{kl} = g_{kl} + u_k u_l \) is an orthogonal projector, \( \dot{u}_k = u^m \nabla_m u_k \) is the acceleration (orthogonal to the velocity),

\[
\omega_{kl} = \frac{1}{2} h_{kl} (\nabla_i u_j - \nabla_j u_i)
\]

\[
= \frac{1}{2} (\nabla_k u_l - \nabla_l u_k) + \frac{1}{2} (u_k \dot{u}_l - u_l \dot{u}_k)
\]

is the (antisymmetric) vorticity tensor, with \( \omega_{jk} u^k = 0 \), and

\[
\sigma_{kl} = \frac{1}{2} h_{kl} (\nabla_i u_j + \nabla_j u_i) - \frac{\theta}{n-1} h_{kl}
\]

\[
= \frac{1}{2} (\nabla_k u_l + \nabla_l u_k - \frac{2n}{n-1} h_{kl}) + \frac{1}{2} (u_k \dot{u}_l - u_l \dot{u}_k)
\]

is the (traceless, symmetric) shear tensor, with \( \sigma_{jk} u^k = 0 \). Their expressions depend on the connection of the spacetime. In this paper we consider perfect-fluid spacetimes, i.e. Lorentzian manifolds whose Ricci tensor takes the form

\[
R_{ij} = A g_{ij} + B u_i u_j
\]

where \( u_k \) is a velocity field and \( A, B \) are scalar fields, with \( B \neq 0 \). Geometers identify the special form (4) of the Ricci tensor as the defining property of “quasi Einstein manifolds” (with arbitrary metric signature). The case \( A = 0 \) defines

\[
\text{Shear and vorticity of perfect-fluid spacetimes and the shear-free conjecture.}
\]

Date: 11 sept 2017.

2010 Mathematics Subject Classification. Primary 53B30, 53B50, Secondary 53C80, 83C15.

Key words and phrases. shear, vorticity, perfect fluid, shear-free conjecture, torse-forming vector, Generalized Robertson-Walker spacetimes.
“Ricci-simple” manifolds [14]. For perfect-fluid spacetimes of dimension $n$ we obtain the following expressions of the terms in (1) (Theorem 2.1):

$$\theta = -\frac{1}{2B} u^k \nabla_k [(n-2)A + B]$$ (5)

$$\dot{u}_j = \frac{1}{2B} h_j^k \nabla_k [(n-2)A - B]$$ (6)

$$\sigma_{kl} = -\frac{1}{2B} \frac{n-2}{n-3} u^j (h_k^q \nabla^m C_{jqlm} + h_l^q \nabla^m C_{jkqm})$$ (7)

$$\omega_{kl} = \frac{1}{2B} \frac{n-2}{n-3} u^j (h_k^q \nabla^m C_{jqlm} - h_l^q \nabla^m C_{jkqm})$$ (8)

where $C_{jklm}$ is Weyl’s curvature tensor. This new result enables us to prove an interesting statement (Theorem 3.3):

If the velocity and the Weyl tensor of a perfect-fluid spacetime satisfy

$$u^j u^l \nabla_m C_{jklm} = 0$$ (9)

then either the expansion rate $\theta$ or the vorticity $\omega_{kl}$ vanish. The condition is equivalent to $h_j^k \nabla_j [(n-2)A + B] = 0$. In general relativity it corresponds to the gradient of the energy density being parallel to the velocity.

The same twofold conclusion appears in the “shear-free conjecture” standing for many years: If the velocity of a barotropic perfect fluid is shear-free ($\sigma_{ij} = 0$) then either the expansion $\theta$ or the vorticity $\omega_{ij}$ of the fluid vanishes. Here, the shear-free and barotropic hypotheses are replaced by (9).

For the rich history of the shear-free conjecture, with its several variants of hypotheses, we refer to the review by Van den Bergh and Slobodeanu [21], the recent work [18], or the book [20] on exact solutions of Einstein’s equations (Ch. 6.2).

We mention some achievements. In 1950 Gödel claimed that a spatially homogeneous shear-free dust model could either expand or rotate, but not both [9]. The condition of homogeneity was relaxed in the “dust shear-free theorem” proved by Ellis with tetrads: if a dust solution of the Einstein field equations is shear-free in a domain, it cannot both expand and rotate in the domain [6]. Dust matter has zero pressure. In barotropic perfect fluids, pressure and energy density are related by an equation of state $p = p(\rho)$ with $p + \rho \neq 0$. Collins classified the barotropic shear-free perfect fluids with $\theta \neq 0$ and null vorticity [4].

Some cases where the conjecture was proven: Lang and Collins [10] with the assumption that the expansion $\theta$ and the energy density are functionally dependent, Senovilla et al. [17] with the condition that the acceleration is proportional to the vorticity vector (including the case $\dot{u}_k = 0$), Sopuerta [19] when $\theta$ and the rotation scalar are functionally dependent.

In Einstein spaces, if the Weyl tensor is purely electric or magnetic, a shear-free velocity field is necessarily irrotational unless the space-time has constant curvature [1]. The result was extended to dissipative axially-symmetric fluids [8], with the finding that it is irrotational if and only if the Weyl tensor is purely electric.

For rotating shear-free barotropic perfect fluids, if the Weyl tensor is purely electric, then $\theta = 0$ [3]. Collins and Wainwright [5] proved that any barotropic perfect fluid solution of the field equations of general relativity, which is shear-free, irrotational with $\theta \neq 0$, is either a Friedmann-Robertson-Walker model, or spherically symmetric Wyman solutions, or a special class of plane-symmetric models. In particular,
if \( p = (\gamma - 1) \rho \), then only the FRW model remains.

In Section 2, for perfect fluid spacetimes, we express the components of the gradients of the velocity in terms of the Weyl tensor. In Section 3 we prove our main statement, that replaces the shear-free and barotropic hypotheses with the condition \( (3) \) on the Weyl tensor. In Section 4 we discuss the structure of spacetimes admitting a velocity field that is both shear-free and vorticity-free. In particular, in Theorem 1.2 we significantly weaken the hypotheses given in ref. [11], for a perfect fluid to be a Generalized Robinson-Walker space-time.

2. Velocity terms for perfect-fluid spacetimes

**Theorem 2.1.** For a perfect-fluid spacetime with velocity field \( u_j \), the expansion rate, the acceleration, the shear and the vorticity tensors are given by the expressions [5]–[8].

**Proof.** The covariant divergence of eq. (11) is \( \nabla_j [(n-2)A - B] = 2u_j \dot{B} + 2B(\dot{u}_j + \theta u_j) \), where \( \dot{B} = u^k \nabla_k B \); contraction with \( u^j \) gives the equation for \( \theta \): \( (n-2) \dot{A} + \dot{B} = -2\theta B \). Elimination or \( \theta \) gives the acceleration \( \dot{u}_j \):

\[
\nabla_j [(n-2)A - B] + u_j [(n-2) \dot{A} - \dot{B}] = 2B \dot{u}_j
\]

In the general expression for the divergence of the conformal tensor,

\[
\nabla^m C_{jklm} = \frac{n-3}{n-2} \left[ \nabla_k R_{jl} - \nabla_j R_{kl} - \frac{1}{2(n-1)} (g_{jl} \nabla_k R - g_{kl} \nabla_j R) \right]
\]

the expressions \( (3) \) of \( R_{ij} \) and the curvature scalar \( R = nA - B \) are inserted:

\[
\frac{n-2}{n-3} \nabla^m C_{jklm} = \nabla_k (Bu_j u_l) - \nabla_j (Bu_k u_l) + (g_{jl} \nabla_k - g_{kl} \nabla_j) \frac{(n-2)A + B}{2(n-1)}
\]

The expression is contracted with \( u^j \):

\[
\frac{n-2}{n-3} u^j \nabla^m C_{jklm} = -B(\nabla_k u_l + u_k \dot{u}_l + u_l \dot{u}_k) - u_l (\nabla_k B + u_k \dot{B}) + (u_l \nabla_k - g_{kl} u^j \nabla_j) \frac{(n-2)A + B}{2(n-1)}
\]

The expression \( (11) \) for \( \nabla_k u_l \) is used:

\[
\frac{n-2}{n-3} u^j \nabla^m C_{jklm} = \frac{(n-2)A + B}{2(n-1)} h_{kl} - B(\sigma_{kl} + \omega_{kl} - u_k \dot{u}_k) - u_l (\nabla_k B + u_k \dot{B}) + (u_l \nabla_k - g_{kl} u^j \nabla_j) \frac{(n-2)A + B}{2(n-1)}
\]

\[
= \frac{(n-2)A - (2n-3)B}{2(n-1)} u_k u_l - B(\sigma_{kl} + \omega_{kl} - u_k \dot{u}_k) + u_l (\nabla_k [n-2)A + B])
\]

The expression for the acceleration is used:

\[
B(\sigma_{kl} + \omega_{kl}) = -\frac{n-2}{n-3} u^j \nabla^m C_{jklm} - \frac{n-2}{2(n-1)} u_j h_{km} \nabla^m [(n-2)A + B]
\]

Contraction with \( u^j \) gives an interesting relation:

\[
\frac{n-2}{2(n-1)} h_{km} \nabla^m [(n-2)A + B] = -\frac{n-2}{n-3} u^j \nabla^m C_{jklm}
\]

Therefore:

\[
B(\sigma_{kl} + \omega_{kl}) = -\frac{n-2}{n-3} u^j h_{kl} \nabla^m C_{jklm}
\]

The symmetric and antisymmetric parts are separated.
Lemma 3.1. on the following two lemma by Senovilla et al. [17]:

vorticity is zero. We now obtain a different condition for zero vorticity. It is based on

\[
\begin{align*}
\sigma_{kl} &= -\frac{n-2}{2(n-3)} u^j \nabla^m (C_{jklm} + C_{jikm}) - \frac{n-2}{2(n-3)} u_l u_k [(n-2)A + B] \\
&\quad - \frac{n-2}{4(n-1)} (u_l \nabla_k + u_k \nabla_l) [(n-2)A + B]
\end{align*}
\]

\[
\omega_{kl} = \frac{n-2}{2(n-3)} u^j \nabla^m C_{klijm} - \frac{n-2}{4(n-1)} (u_l \nabla_k + u_k \nabla_l) [(n-2)A + B]
\]

Proof. By the identity

\[
\nabla^m (C_{jklm} + C_{jikm}) = \frac{n-2}{2(n-3)} u_l u_k [(n-2)A + B]
\]

\[
\frac{n-2}{2(n-3)} u^j \nabla^m C_{klijm} - \frac{n-2}{4(n-1)} (u_l \nabla_k + u_k \nabla_l) [(n-2)A + B]
\]

If Einstein’s equations hold, \( R_{ij} - \frac{1}{2} R g_{ij} = 8 \pi T_{ij} \), the geodesic tensor of a perfect fluid spacetime corresponds to the stress-energy tensor of a perfect fluid with same viscosity. If \( u^k \) is the velocity field and \( \rho \) is the energy density, then either \( \omega_{kl} = 0 \) means \( u^k \nabla_k \rho = 0 \), or

\[
\theta = \frac{u^k \nabla_k \rho}{\rho + p}
\]

\[
\dot{u}_j = -\frac{1}{\rho + p} h_j k \nabla_k \rho
\]

\[
(\rho + p) \sigma_{kl} = -\frac{n-2}{16(n-3)} u^j \nabla^m (C_{jklm} + C_{jikm}) - \frac{n-2}{n-3} u_l u_k \dot{\rho}
\]

\[
- \frac{n-2}{2(n-1)} (u_l \nabla_k + u_k \nabla_l) \rho
\]

\[
(\rho + p) \omega_{kl} = \frac{n-2}{16(n-3)} u^j \nabla^m C_{klijm} - \frac{n-2}{2(n-1)} (u_l \nabla_k + u_k \nabla_l) \rho
\]

The requirement \( B \neq 0 \) for perfect-fluid spacetimes means \( p + \rho \neq 0 \).

3. A CONDITION FOR VANISHING VORTICITY

If the velocity field is closed (\( \nabla_i u_j = \nabla_j u_i \)), then it is geodesic (\( \dot{u}_k = 0 \)) and the vorticity is zero. We now obtain a different condition for zero vorticity. It is based on the following two lemma by Senovilla et al. [17]:

**Lemma 3.1.** Let \( A_{jkl} = \frac{1}{4} [u_j (\nabla_k u_l - \nabla_l u_k) + u_k (\nabla_l u_j - \nabla_j u_k) + u_l (\nabla_j u_k - \nabla_k u_j)] \).

1) \( \omega_{kl} = 0 \) if and only if \( A_{jkl} = 0 \).

2) If there are scalar functions \( \lambda \), \( f \) such that \( u_j = \lambda \nabla_j f \) then \( \omega_{kl} = 0 \).

**Proof.** By the identity \( u_j \omega_{kl} + u_k \omega_{lj} + u_l \omega_{jk} = A_{jkl} \) it follows that \( \omega_{kl} = 0 \) means \( A_{jkl} = 0 \). The other way, if \( A_{jkl} = 0 \) then one evaluates \( 0 = u^l A_{jkl} = \frac{1}{4} (-\nabla_k u_l + \nabla_l u_k - \nabla_l u_k + u_k) = -\omega_{kl} \).

If \( u_j = \lambda \nabla_j f \) then direct substitution gives \( A_{jkl} = 0 \), i.e. \( \omega_{kl} = 0 \).

**Lemma 3.2.** If \( h_k^m \nabla_m f = 0 \) then, either \( f \) is constant, or \( \omega_{ij} = 0 \).

**Proof.** Being \( \nabla_k f + u_k \dot{f} = 0 \), if \( \dot{f} = 0 \) it follows that \( \nabla_k f = 0 \) i.e. \( f \) is constant. If \( \dot{f} \neq 0 \) then \( u_k = -(\nabla_k f)/\dot{f} \). Then, by Lemma 3.1 \( \omega_{kl} = 0 \).

Now, thanks to the expressions (13)-(16), the theorem follows:

**Theorem 3.3.** In a perfect fluid, if \( u^q u^p \nabla^m C_{qkpm} = 0 \) then either \( \theta = 0 \) or \( \omega_{kl} = 0 \).

**Proof.** From the identity (12) it follows that \( h_k^m \nabla_m [(n-2)A + B] = 0 \). Then Lemma 3.2 implies that either \( (n-2)A + B \) is constant in spacetime, i.e. \( \theta = 0 \), or \( \omega_{kl} = 0 \).
Remark 3.4. In general relativity the condition in the theorem is $h^m_k \nabla_m \rho = 0$, meaning that the gradient of the energy density is parallel to the velocity. The possible outcome of the theorem $\nabla_k [(n-2)A + B] = 0$ means that the energy density $\rho$ is uniform in spacetime.

4. Shear and vorticity-free perfect-fluid spacetimes

The vanishing of terms in the decomposition of a velocity field, determines a hierarchy of spacetimes. If a spacetime admits a shear-free and vorticity-free velocity field, $\nabla_i u_j = \theta h_{ij} - u_i u_j$, then it admits a line element of the form \cite{16}:

$$ds^2 = -\varphi(t, \vec{x})^2 dt^2 + f(t, \vec{x})^2 g^*_{\mu\nu}(\vec{x}) dx^\mu dx^\nu$$

In the geometric literature, \cite{21} represents a doubly twisted product. If the velocity field is also geodesic, $\dot{u}_k = 0$, then it satisfies the torse-forming condition $\nabla_i u_j = \frac{\theta}{n-1} h_{ij}$, which is necessary and sufficient for the spacetime to be twisted \cite{13}, \cite{2}, with metric

$$ds^2 = -dt^2 + f(t, \vec{x})^2 g^*_{\mu\nu}(\vec{x}) dx^\mu dx^\nu$$

If the velocity is also an eigenvector of the Ricci tensor, the spacetime is a Generalized Robertson-Walker (GRW) spacetime \cite{12}, with metric

$$ds^2 = -dt^2 + f(t)^2 g^*_{\mu\nu}(\vec{x}) dx^\mu dx^\nu$$

(if the Weyl tensor is zero, it is a Robertson-Walker space-time). Finally, if also $\theta = 0$ the space is the product of disjoint manifolds: $ds^2 = -dt^2 + g^*_{\mu\nu}(\vec{x}) dx^\mu dx^\nu$.

As a consequence of Theorem 2.1 we have:

Corollary 4.1. The velocity of a perfect-fluid space-time is shear-free and vorticity-free if and only if the Weyl tensor satisfies $h^m_k \nabla_m C_{jklm} = 0$ i.e.

$$u^j (u_i u^l \nabla^m C_{jklm} + \nabla^m C_{jklm}) = 0.$$ \hspace{1cm} \text{(21)}$$

The simplest case $\nabla^m C_{jklm} = 0$ includes locally symmetric, conformally flat and conformally symmetric perfect-fluid spacetimes. In \cite{11} we proved that a perfect-fluid spacetime with $\nabla_i u_j = \nabla_j u_i$ and $\nabla_m C_{jklm} = 0$ is a GRW spacetime. However, both hypotheses of closedness and conformal harmonicity can now be weakened:

Theorem 4.2. Let $M$ be a perfect-fluid spacetime. If $u^k$ is geodesic ($\dot{u}_k = 0$) and if $h_i^l u^j \nabla_m C_{jklm} = 0$, then $M$ is a GRW spacetime.

Proof. The condition on the Weyl tensor implies that the velocity is both shear-free and vorticity-free. Being geodesic, the velocity is torse-forming: $\nabla_i u_j = \theta h_{ij} / (n-1)$. For a perfect-fluid spacetime, the velocity is an eigenvector of the Ricci tensor. Then $M$ is a GRW spacetime. \hfill \Box

Now we give an example of perfect-fluid space-time that is shear-free and vorticity-free \cite{15}. The Ricci tensor has the form \cite{14}, and is also recurrent with 1-form $\eta_k \neq 0$:

\hspace{2cm}$$\nabla_k R_{jl} - \nabla_j R_{kl} = \eta_k R_{jl} - \eta_j R_{kl}$$ \hspace{1cm} \text{(22)}$$

\begin{thebibliography}{99}

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A contraction with the metric tensor gives: $\nabla_k R = 2\eta_k [(n-1)A - B] - 2Bu_k u^j \eta_j$. Contraction with $u^k$ gives: $u^k \nabla_k R = 2(n-1)Au^j \eta_j$. Next evaluate

$$u^j \nabla_m C_{jklm} = \frac{n-3}{n-2} \left[ \eta_k R_{jl} u^j - u^j \eta_j R_{kl} - \frac{1}{2(n-1)} (u_l \nabla_k R - g_{kl} u^j \nabla_j R) \right]$$

$$= - \frac{n-3}{n-1} Bu_l h_{km} \eta^m$$

It is clear that $h_i^l u^j \nabla_m C_{jklm} = 0$. Then $\sigma_{kl} = 0$ and $\omega_{kl} = 0$.

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