On forward-order law for core inverse in rings

Amit Kumar and Debasish Mishra

Abstract. This article establishes a few sufficient conditions of the forward-order law for the core inverse of elements in rings with involution. It also presents the forward-order law for the weighted core inverse and the triple forward-order law for the core inverse. Additionally, we discuss the hybrid forward-order law involving different generalized inverses like the Moore–Penrose inverse, the group inverse, and the core inverse.

Mathematics Subject Classification. 15A09, 16W10, 16B99.

Keywords. Core inverse, Weighted core inverse, Forward-order law, Hybrid forward-order law.

1. Introduction

Inverses/generalized inverses of a product of two or three elements over a ring have been investigated by many researchers. For instance, if \( a \) and \( b \) are a pair of invertible elements, then \( ab \) is also invertible, and the inverse of the product \( ab \) satisfying
\[
(ab)^{-1} = b^{-1}a^{-1},
\]
is known as the reverse-order law. On the other hand,
\[
(ab)^{-1} = a^{-1}b^{-1}
\]
is known as the forward-order law. While the reverse-order law does not hold for different generalized inverses (see the next section for its definition), the forward-order law is not valid even for invertible elements. One of the fundamental topics in the theory of generalized inverses is investigating various reverse-order laws and forward-order laws. For example, Mosić and Djordjević [12] expanded the reverse-order law for the Moore–Penrose inverse of a matrix to the reverse-order law for the Moore–Penrose inverse of an element over a ring. In 2012, Mosić and Djordjević [10] extended the reverse-order law for the
group inverse in Hilbert spaces to rings. In 2017, Zhu and Chen [19] obtained the forward-order law for the Drazin inverse in a ring. Zhu [20,22,23] conferred several results on additive properties, the reverse-order law and the forward-order law. In 2012, Mosić and Djordjević [13] provided the hybrid reverse-order law between the group inverse and the Moore–Penrose inverse. In 2018, Zou et al. [21] provided the reverse-order law for the generalized core inverse. In 2022, Sahoo et al. [17] gave the reverse-order law for the weighted core inverse. In 2021, Gao et al. [5] provided the reverse-order law for the generalized pseudo core inverse. In 2021, Li et al. [9] studied the forward-order law for the core inverse and the hybrid forward-order law among the core inverse, the Moore–Penrose inverse and the group inverse in a matrix setting. The vast literature on the core inverse and the weighted core inverse with its multifarious extensions in different areas of mathematics motivate us to study the following two problems.

(i) When does the forward-order law for the core inverse and the weighted core inverse hold?

(ii) When does the triple forward-order law hold?

In applications, the core inverse is used to find the Bott–Duffin inverse [8]. Also, one can compute the Moore–Penrose inverse and the group inverse by using the forward-order law for the core inverse. The theory of generalized inverses over rings is also used in cryptography [6]. For example: To find the solution of \( y = ax + b \) (mod 26) which is \( x = a^{-1}(y - b) \) (mod 26), where \( a^{-1} \) is a generalized inverse of \( a \) [16]. The original idea of introducing matrix partial orders comes from partial orders that were defined in the context of semigroups. It is thus interesting to study the above-stated problems in rings.

The main goal of this article is to study the forward-order law for the core inverse and the weighted core inverse. Furthermore, examples that stress their importance are presented, and counterexamples are also illustrated. Before we begin, we organize this article as follows. In Sect. 2, we recall some preliminaries. Section 3 presents a few conditions for the forward-order law for the core inverse. Some conditions for the forward-order law for the weighted core inverse are also established. Finally, the triple forward-order law is discussed for the core inverse. Section 4 provides some necessary and sufficient conditions for the hybrid forward-order law.

2. Preliminaries

Throughout this article, \( R \) denotes a unital \( * \)-ring, that is, a ring with unity 1 and an involution \( * \). A ring \( R \) is called a proper ring if \( a^*a = 0 \) \( \Rightarrow \) \( a = 0 \) for all \( a \in R \). An involution \( * \) is an anti-isomorphism of order 2 that satisfies the
conditions 
\[(a + b)^* = a^* + b^*, \ (ab)^* = b^*a^*, \text{ and } (a^*)^* = a, \text{ for all } a, b \in R.\]

An element \(a\) is said to be Hermitian if \(a^* = a\), and is called idempotent if \(a^2 = a\). The left annihilator of \(a \in R\) is given by \(\mathcal{L}(a) = \{x \in R : xa = 0\}\) and the right annihilator of \(a\) is given by \((a)\mathcal{R} = \{x \in R : ax = 0\}\). An element \(a \in R\) is Moore–Penrose invertible if there exists a unique element \(x \in R\) that satisfies the equations:

\[
(1.) \ axa = a, \ (2.) \ xax = x, \ (3.) \ (ax)^* = ax, \text{ and } (4.) \ (xa)^* = xa.
\]

Then \(x\) is called the Moore–Penrose inverse \([14]\) of \(a\), and is denoted as \(x = a^\dagger\).

By \(R^\dagger\), we denote the set of all Moore–Penrose invertible elements of \(R\). The set of all elements which satisfy any of the combinations of the above four equations is denoted as \(a\{i, j, k, l\}\), where \(i, j, k, l \in \{1, 2, 3, 4\}\), and is called an \{\(i, j, k, l\)\} inverse of \(a\). A first and third inverse of \(a\) is denoted as \(a^{(1,3)}\). The set of first and third invertible elements of \(R\), is denoted by \(R^{(1,3)}\).

An element \(a\) is called Drazin invertible \([4]\) if there exists a unique element \(x \in R\) such that \(xa^{k+1} = a^k, ax = xa, \text{ and } ax^2 = x\), for some positive integer \(k\). If the Drazin inverse of \(a\) exists, then it is denoted by \(a^d\). The smallest positive integer \(k\) is called the Drazin index, denoted \(i(a)\). The set of all Drazin invertible elements of \(R\) will be denoted by \(R^d\). If \(i(a) = 1\), then the Drazin inverse of \(a\) is called the group inverse of \(a\), and is denoted by \(a^\#\). The set of group invertible elements of \(R\) will be denoted by \(R^\#\).

The perusal of the core inverse is one of the areas of generalized inverses that has caught the interest of numerous researchers in the past few decades. Firstly, Baksalary and Trenkler \([1]\) introduced the core inverse for complex matrices. Let \(A \in \mathbb{C}^{n \times n}\). A matrix \(A^@ \in \mathbb{C}^{n \times n}\) is called the core inverse of \(A\) if

\[
AA^@ = P_A \text{ and } R(A^@) \subseteq R(A),
\]

where \(P_A\) is the orthogonal projector onto \(R(A)\), and \(R(A)\) is the column space of \(A\). Motivated by this work, Rakić et al. \([15]\) introduced the core inverse in rings. An element \(a^@ \in R\) satisfying

\[
aa^@a = a, a^@R = aR, \text{ and } Ra^@ = Ra^*,
\]

is called the core inverse of \(a\). The authors proved that an element \(a \in R\) is core invertible if and only if there exists \(x \in R\) which satisfies

\[
axa = a, xax = x, (ax)^* = ax, xa^2 = a, \text{ and } ax^2 = x.
\]

The element \(x\) is known as the core inverse of \(a\). And it is unique (if it exists).

In 2017, Xu et al. \([18]\) proved that if \(a\) satisfies these three conditions

\[
(ax)^* = ax, \ ax^2 = x, \text{ and } xa^2 = a,
\]
then $a$ is core invertible. The set of all core invertible elements of $R$ will be denoted by $R^\Diamond$. In 2018, Mosić et al. [11] extended the notion of the core inverse to the weighted core inverse in a ring with involution. Let $a \in R$, and $e \in R$ be an invertible element with $e^* = e$. Then a unique element $x \in R$ is said to be an $e$-weighted core inverse if

$$ax^2 = x, xa^2 = a, \text{ and } (eax)^* = eax.$$ 

The $e$-weighted core inverse of an element $a \in R$ is denoted by $a^e(\Diamond)$ (if it exists). The set of all $e$-weighted core invertible elements of $R$ is denoted by $R^{e(\Diamond)}$. These results will often be used later in this article.

**Lemma 2.1.** (Corollary 3.4, [3]) Let $a, x \in R$ with $xa = ax$ and $xa^* = a^*x$. If $a \in R^\Diamond$, then $xa^\Diamond = a^\Diamond x$.

**Theorem 2.2.** (Theorem 2.3, [11]) Let $e \in R$ be invertible Hermitian element. If $a \in R^# \cap R^{e(\Diamond)}$, then $a^# = (a^e(\Diamond))^2a$.

**Theorem 2.3.** (Theorem 3.1, [21]) Let $a, x \in R$. Then the following are equivalent:

(i) $a \in R^\Diamond$, and $x = a^\Diamond$;
(ii) $axa = a, xR = aR, \text{ and } Rx \subseteq Ra^*$;
(iii) $axa = a, ^o x = ^o a, \text{ and } (a^*)^o \subseteq x^o$;
(iv) $xax = x, xR = aR, \text{ and } Ra^* \subseteq Rx$;
(v) $xax = x, xR = aR, \text{ and } Ra^* \subseteq Rx$;
(vi) $xax = x, ^o x = ^o a, \text{ and } x^o \subseteq (a^*)^o$.

3. **Forward-order law for the core inverse**

This section provides some sufficient conditions under which the forward-order law holds for the core inverse. First, we prove some results for the core inverse. We discuss the forward-order law for the core inverse and the weighted core inverse. We then present some characterizations of the forward-order law for the core inverse. This section begins with the following lemma.

**Lemma 3.1.** Let $a \in R^\Diamond$. Then the following conditions hold:

(i) $(a^*)^o = (a^\Diamond)^o$;
(ii) $^o[(a^\Diamond)^*] = ^o(a) = ^o(a^\Diamond)$;
(iii) $a^\Diamond b = a^\Diamond c$ if and only if $a^*b = a^*c$, where $b, c \in R$;
(iv) $ba^\Diamond = ca^\Diamond$ if and only if $ba = ca$, where $b, c \in R$;
(v) $^o(b(a^\Diamond)^*) = ^o(ba^\Diamond)$, where $b \in R$. 

Proof. (i) From Theorem 2.3 parts (iii) and (vi), we get $(a^*)^\circ = (a^\otimes)^\circ$.

(ii) From Theorem 2.3 part (iii), we get $\circ(a) = \circ(a^\otimes)$. Let $x \in \circ([a^\otimes]^*)$. So, $x(a^\otimes)^* = 0$. Post-multiplying $x(a^\otimes)^* = 0$ by $a^*a$, we get $x(aa^\otimes)^*a = 0$, i.e., $aaxa^\otimes = 0$, i.e., $xa = 0$. Hence,

$$\circ[(a^\otimes)^*] \subseteq \circ(a). \tag{3.1}$$

Conversely, if $x \in \circ(a)$, then $xa = 0$. So, $xaxa^\otimes = 0$, i.e., $x(a^\otimes)^*a^* = 0$. This implies $x(a^\otimes)^*a^*(a^\otimes)^* = 0$, i.e., $x(a^\otimes)^* = 0$. Thus,

$$\circ(a) \subseteq \circ[(a^\otimes)^*]. \tag{3.2}$$

From (3.1) and (3.2), we find $\circ[(a^\otimes)^*] = \circ(a)$. Hence, $\circ[(a^\otimes)^*] = \circ(a) = \circ(a^\otimes)$.

(iii) We know that $(a^\otimes)^* = aa^\otimes$. We have $a^\otimes b = a^\otimes c$. Pre-multiplying $a^\otimes b = a^\otimes c$ by $a^*a$, we get $a^*(a^\otimes)^*a^*b = a^*(a^\otimes)^*a^*c$, i.e., $a^*b = a^*c$. Conversely, we assume that $a^*b = a^*c$. Pre-multiplying $a^*b = a^*c$ by $a^\otimes(a^\otimes)^*$, we get $a^\otimes b = a^\otimes c$.

(iv) We have $ba^\otimes = ca^\otimes$. Post-multiplying $ba^\otimes = ca^\otimes$ by $a^2$, we obtain $ba^\otimes a^2 = ca^\otimes a^2$, i.e., $ba = ca$. Conversely, we suppose that $ba = ca$. Now, post-multiplying $ba = ca$ by $(a^\otimes)^2$, we get $ba^\otimes = ca^\otimes$.

(v) Let $x \in \circ(b(a^\otimes)^*)$. Then we have $xb(a^\otimes)^* = 0$ implies that $xb(a^\otimes)^*a^* = 0$.

So, $xb(aa^\otimes)^* = 0$, i.e., $xbaa^\otimes = 0$. Post-multiplying $xbaa^\otimes = 0$ by $a^2$, and from $aa^\otimes a = a$, we get $xba = 0$. So, $x \in \circ(ba)$, which implies $\circ(b(a^\otimes)^*) \subseteq \circ(ba)$. Conversely, $x \in \circ(ba)$ yields $xba = 0$. Post-multiplying $xba = 0$ by $a^\otimes$, and from $(aa^\otimes)^* = (a^\otimes)^*a^*$, we get $xb(a^\otimes)^*a^* = 0$. Again, post-multiplying $xb(a^\otimes)^*a^* = 0$ by $(a^\otimes)^*$, and using the identity $(a^\otimes)^*a^*(a^\otimes)^* = (a^\otimes)^*$, we get $xb(a^\otimes)^* = 0$. So, $x \in \circ(b(a^\otimes)^*)$, which implies that $\circ(ba) \subseteq \circ(b(a^\otimes)^*)$. Hence, $\circ(b(a^\otimes)^*) = \circ(ba)$.

An example that shows that the forward-order law does not always hold for the core inverse is produced next.

Example 3.1. Let $R = \{a_0 + a_1i + a_2j + a_3k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k, jk = -k = i, ki = -ik = j$, be a ring (quaternion polynomial ring) with conjugate involution. So, $\overline{-1}a_i$ is the core inverse of $a_1i$ and $\overline{-1}a_j$ is the core inverse of $a_2j$, where $a_1, a_2 \neq 0$.

Then $(a_1ia_2j)^\otimes = (a_1a_2j)^\otimes = (a_1a_2k)^\otimes = \frac{-1}{a_1a_2^2}k$, and $(a_1i)^\otimes(a_2j)^\otimes = \frac{(-1)}{a_1a_2^2}j = \frac{1}{a_1a_2}ij = \frac{1}{a_1a_2^2}k$. Hence, $(a_1ia_2j)^\otimes \neq (a_1i)^\otimes(a_2j)^\otimes$.

Now, we present the forward-order law for the core inverse under the assumption of a few conditions.

Theorem 3.2. Let $a, b \in R^\otimes$ with $aba = ba^2 = a^2b$. If $aba^\otimes = aa^\otimes b$ and $abb^\otimes = bb^\otimes a$, then $(ab)^\otimes = a^\otimes b^\otimes$. 
Proof. Taking the involution on $abb^\circ = bb^\circ a$, we have $a^*bb^\circ = bb^\circ a^*$. By Lemma 2.1, we obtain $a^\circ bb^\circ = bb^\circ a^\circ$. Now, we will prove that $(ab)^\circ = a^\circ b^\circ$ by using the definition of the core inverse.

$$aba^\circ b^\circ a^\circ b^\circ = aa^\circ bb^\circ a^\circ b^\circ (\text{since } aba^\circ = aa^\circ b)$$

$$= aa^\circ a^\circ bb^\circ b^\circ (\text{using } a^\circ bb^\circ = bb^\circ a^\circ)$$

$$= a^\circ b^\circ,$$

$$a^\circ b^\circ abab = a^\circ b^\circ ba^2b (\text{since } aba = ba^2)$$

$$= a^\circ b^\circ b^2a^2$$

$$= a^\circ ba^2$$

$$= a^\circ a^2b$$

$$= ab,$$

and

$$(aba^\circ b^\circ)^\circ = (aa^\circ bb^\circ)^\circ (\text{since } aba^\circ = aa^\circ b)$$

$$= (abb^\circ a^\circ)^\circ (\text{using } a^\circ bb^\circ = bb^\circ a^\circ)$$

$$= (bb^\circ aa^\circ)^\circ$$

$$= (aa^\circ)^\circ (bb^\circ)^\circ$$

$$= aa^\circ bb^\circ$$

$$= aba^\circ b^\circ.$$

Hence, $(ab)^\circ = a^\circ b^\circ$. \hfill \Box

Note that the condition $aba = ba^2 = a^2b$ in the above result can be replaced by $ab = ba$. The next result provides another set of sufficient conditions for the forward-order law.

**Theorem 3.3.** Let $a, b \in R^\circ$. If $a^*b = a^*a(ab)^\circ b^2$ and $bb^\circ a = abb^\circ$, then $(ab)^\circ = a^\circ b^\circ$.

**Proof.** From Lemma 3.1 (iii), $a^*b = a^*a(ab)^\circ b^2$ yields $a^\circ b = a^\circ a(ab)^\circ b^2$. From $ab = abb^\circ b$, and $bb^\circ a = abb^\circ$, we get $ab = bb^\circ ab$. Taking the involution on $ab = bb^\circ ab$, we obtain $(ab)^\circ = (ab)^\circ b^\circ$. From Lemma 3.1 (iii), $(ab)^\circ = (ab)^\circ b^\circ$. Now,

$$a^\circ b^\circ = a^\circ aa^\circ b(b^\circ)^2 = a^\circ (aa^\circ)^\circ b(b^\circ)^2 = a^\circ (a^\circ)^\circ a^*b(b^\circ)^2$$

$$= a^\circ (a^\circ)^\circ a^*a(ab)^\circ b^2(b^\circ)^2 = a^\circ a(ab)^\circ bb^\circ = a^\circ a(ab)^\circ$$

$$= a^\circ a(ab)^\circ (ab)^\circ = ab(ab)^\circ (ab)^\circ = (ab)^\circ.$$

Hence, $(ab)^\circ = a^\circ b^\circ$. \hfill \Box

A characterization of the forward-order law for a class of elements satisfying the condition $bb^\circ a = abb^\circ$ is established below.
Theorem 3.4. Let $a, b \in R^\oplus$ with $bb^\oplus a = ab^\oplus$. Then the following are equivalent:

(i) $ab \in R^\oplus$ and $(ab)^\oplus = a^\oplus b^\oplus$;

(ii) $ab \in R^\#, \, ab^\oplus R \subseteq a^\oplus b^\oplus R$, and $a^*b^\oplus(1 - (aba^\oplus b^\oplus)^*)b = 0$.

Proof. (i)$\Rightarrow$(ii): From Theorem 2.3 (viii), $ab \in R^\#$. Now, $ab^\oplus R = ab(b^\oplus)^2R = (ab)^\oplus(ab)^2(b^\oplus)^2R \subseteq (ab)^\oplus R = a^\oplus b^\oplus R$. So, $ab^\oplus R \subseteq a^\oplus b^\oplus R$. Further, we obtain

\[ a^*b^\oplus b = (aa^\oplus a)^*b^\oplus b = a^*aa^\oplus b^\oplus b = a^*a(ab)^\oplus b = a^*(aa^\oplus)^*b^\oplus aba^\oplus b^\oplus b = a^*(a^\oplus)^*a^*b^\oplus aba^\oplus b^\oplus b = a^*b^\oplus aba^\oplus b^\oplus b = a^*b^\oplus ab(ab)^\oplus b = a^*b^\oplus(ab(ab)^\oplus)^*b = a^*b^\oplus(ab(ab)^\oplus)^*b. \]  

Equation (3.3) implies that $a^*b^\oplus(1 - (aba^\oplus b^\oplus)^*)b = 0$.

(ii)$\Rightarrow$(i): We will show that $(ab)^\oplus = a^\oplus b^\oplus$ by using Theorem 2.3 (viii). Set $x = a^\oplus b^\oplus$. From part (ii), we have $a^*b^\oplus(1 - (aba^\oplus b^\oplus)^*)b = 0$, i.e., $a^*b^\oplus b = a^*b^\oplus(aba^\oplus b^\oplus)^*b$. We can write, $a^*b^\oplus = (a^*b^\oplus b)b^\oplus$, which implies that $a^*b^\oplus = a^*b^\oplus(ab(ab)^\oplus)^*bb^\oplus$. Taking the involution on both sides of the last equality $a^*b^\oplus = a^*b^\oplus(ab(ab)^\oplus)^*bb^\oplus$, we get

\[ (b^\oplus)^*a = (bb^\oplus)^*aba^\oplus b^\oplus(b^\oplus)^*a. \]  

Post-multiplying (3.4) by $a^\oplus(a^\oplus)^*$, we obtain

\[ (b^\oplus)^*(a^\oplus)^* = (bb^\oplus)^*aba^\oplus b^\oplus(b^\oplus)^*(a^\oplus)^*, \]

i.e.,

\[ (a^\oplus b^\oplus)^* = (bb^\oplus)^*aba^\oplus b^\oplus(a^\oplus b^\oplus)^*. \]  

Again, taking the involution on both sides of (3.5), we get

\[ a^\oplus b^\oplus = a^\oplus b^\oplus(ab(ab)^\oplus)^*(bb^\oplus)^* = a^\oplus b^\oplus(bb^\oplus aba^\oplus b^\oplus)^* = a^\oplus b^\oplus(abb^\oplus ab(ab)^\oplus)^*(since \, bb^\oplus a = abb^\oplus) = a^\oplus b^\oplus(ab(ab)^\oplus)^*. \]
Pre-multiplying (3.6) by \(ab\), we obtain
\[
ab @ b @ = ab @ b @ (ab @ b @)^*.
\] (3.7)

By equation (3.7), we have \((ab @ b @)^* = ab @ b @\), i.e., \((abx)^* = abx\). The previous equality \((ab @ b @)^* = ab @ b @\) and equation (3.6) imply that \(a @ b @ a @ b @ = a @ b @\), i.e., \(xabx = x\). From \(ab @ R \subseteq a @ b @ R = xR\), we get \(ab @ = a @ b @ y\), where \(y \in R\). Then \(ab = ab @ b^2 = a @ b @ yb^2\). And \(abR = a @ b @ yb^2 R \subseteq a @ b @ R\), i.e., \(abR \subseteq a @ b @ R = xR\). By Theorem 2.3 (viii), we thus have \(ab \in R @\) and \((ab)^* = x = a @ b @\).

\(\square\)

The next result provides sufficient conditions for which the set of core invertible elements satisfies the commutative property.

**Theorem 3.5.** Let \(a, b \in R @\) with \(a @ b = b @ a^*\) and \(bab = ab^2\). If \(a @ b = b @ a\) and \(a @ b = ba @\), then \((ab @) = a @ b @ = b @ a @\).

**Proof.** First we will show that \((ab @) = a @ b @\) by using Theorem 2.3 (ii). Set \(x = a @ b @\), we obtain
\[
abxab = ab @ b @ ab = aa @ bb @ ab
\]
\[
= a a @ b @ b @ b
\]
\[
= aa @ bab(b @)^2 b
\]
\[
= aa @ ab^2 (b @)^2 b
\]
\[
= ab b @ b
\]
\[
= ab.
\]

So, the first condition of Theorem 2.3 (ii) is satisfied. Next, we show that \(xR = abR\) which is the second condition of Theorem 2.3 (ii). By hypothesis, we have \(a @ b = ba @\). Thus
\[
a @ b @ R = a(a @)^2 b(b @)^2 R = ab(a @)^2 b^2 R = a @ b @ a @ b^2 R = a @ b @ R,\]

i.e., \(a @ b @ R \subseteq abR\). Hence, \(xR \subseteq abR\). Conversely, we have \(b @ a = ab @\). So, \(abR = a @ a @ b @ b^2 R = a @ b @ a @ b^2 R \subseteq a @ b @ R\), i.e., \(abR \subseteq a @ b @ R\). So, \(abR \subseteq xR\). Hence, \(abR = xR\). One can now apply Theorem 2.3 (ii) if the third condition \(Rx \subseteq R(ab)^*\) holds. We have \(Rx = Ra @ b @ = Ra @ aa @ b @ \subseteq R(a a @)^* b @ = R(a @)^* a @ b @ \subseteq Ra @ b @ = Rb @ a^* = Rb @ bb @ a^* \subseteq R(b b @)^* a^* = R(b @)^* b @ a^* \subseteq R(ab)^*\), i.e., \(Rx = Ra @ b @ \subseteq R(ab)^*\). Hence, Theorem 2.3 (ii) yields
\[(ab)^* = a @ b @.\]

We also have \(a @ b = b @ a^*\) and \(ab @ = b @ a\). By Lemma 2.1, we obtain \(a @ b @ = b @ a @\). Thus,
\[(ab)^* = a @ b @ = b @ a @.\]

\(\square\)
The following example demonstrates the above theorem.

**Example 3.2.** Let \( R = \mathbb{R}^{3 \times 3}[x] \) with conjugate involution. Suppose \( A = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} \)

+ \( x^2 - 1 \) and \( B = P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} + < x^2 - 1 > \in R. \) Then \( AB = P \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} + < x^2 - 1 > \) and we have \( A\# = P \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} + < x^2 - 1 > \). Hence, \( (AB)\# = A\# B\# = B\# A\# \).

Kumar and Mishra [7] proposed the following result for idempotent elements.

**Theorem 3.6.** (Theorem 2.11, [7]) Let \( a, b \in R\# \). Then the following hold:

(i) \((1 - x) a = b\) if and only if \( x b = 0\) and \( x (x) \subseteq x (a - b)\);

(ii) \( a (1 - y) = b\) if and only if \( y b = 0\) and \( y (y) \subseteq y (a - b)\).

If \( b \in R\# \), then \( b\#\) is an idempotent element. By Theorem 3.6, we thus have the following remark.

**Remark 3.2.** Let \( a, b \in R\# \) with \( ab = b\#\) \( ab\). Then \( x (bb\#) \subseteq x (ab)\).

A characterization of the forward-order law is presented below.

**Theorem 3.7.** Let \( a, b, ab \in R\#. \) Then \( (ab)\# = a\# b\# \) if and only if \( a\# a (ab)\# = a^* b\# \).
Proof. We know that \((aa^\oplus)^* = (a^\oplus)^*a^*\). Now, pre-multiplying \((ab)^\oplus = a^\oplus b^\oplus\) by \(a^*a\), we obtain \(a^*a(ab)^\oplus = a^*(a^\oplus)^*a^*b^\oplus\) which implies \(a^*a(ab)^\oplus = a^*b^\oplus\). Conversely, let \(a^*a(ab)^\oplus = a^*b^\oplus\). Pre-multiplying \(a^*a(ab)^\oplus = a^*b^\oplus\) by \((a^\oplus)^*\), we get \((ab)^\oplus = aa^\oplus b^\oplus\). Again, pre-multiplying \((ab)^\oplus = aa^\oplus b^\oplus\) by \(a^\oplus\), we obtain \(a^\oplus a(ab)^\oplus = a^\oplus b^\oplus\). Further, we have
\[a^\oplus a(ab)^\oplus = a^\oplus a(ab)(ab)^\oplus = ab(ab)^\oplus(ab)^\oplus = (ab)^\oplus.\]
Hence, \((ab)^\oplus = a^\oplus b^\oplus\). \(\square\)

The following two results provide some necessary conditions of the forward-order law for the core inverse in a ring.

**Theorem 3.8.** Let \(a, b \in R^\oplus\). If \((ab)^\oplus = a^\oplus b^\oplus\), then the following statements are equivalent:

(i) \(a^*b = a^*a(ab)^\oplus b^2\);
(ii) \(aa^\oplus b = a(ab)^\oplus b^2\);
(iii) \(a^\oplus b = (ab)^\oplus b^2\).

**Proof.** (i) \(\Rightarrow\) (ii): We have \(a^*b = a^*a(ab)^\oplus b^2\). Now, pre-multiplying \(a^*b = a^*a(ab)^\oplus b^2\) by \((a^\oplus)^*\), we get \((aa^\oplus)^*b = (aa^\oplus)^*a(ab)^\oplus b^2\), i.e., \(aa^\oplus b = aa^\oplus a(ab)^\oplus b^2\), i.e., \((aa^\oplus b)^2 = a(ab)^\oplus b^2\).

(ii) \(\Rightarrow\) (iii): We have \(aa^\oplus b = a(ab)^\oplus b^2\). Now, pre-multiplying \(aa^\oplus b = a(ab)^\oplus b^2\) by \(a^\oplus\), and from \((ab)^\oplus = a^\oplus b^\oplus\), we get \(a^\oplus aa^\oplus b = a^\oplus aa^\oplus b^\oplus b^2\) implies \(a^\oplus b = (ab)^\oplus b^2\).

(iii) \(\Rightarrow\) (i): Pre-multiplying (iii) by \(a^*a\), we get (i). \(\square\)

**Theorem 3.9.** Let \(a, b \in R^\oplus\). If \((ab)^\oplus = a^\oplus b^\oplus\), then the following conditions hold:

(i) \((b^\oplus)^*aR = abR, and a^\oplus b^\oplus(ab)^2 = ab\);
(ii) \((a^\oplus b^\oplus)^*R = abR, and a^*b^\oplus(ab)^2 = a^*a^2b\).

**Proof.** (i) From the properties of the core inverse, we obtain \((b^\oplus)^*aR = (b^\oplus)^*a^\oplus aR = (b^\oplus)^*(aa^\oplus)^*aR = (aa^\oplus b^\oplus)^*aR = (a^\oplus b^\oplus)^*a\) \(\subseteq ((ab)^\oplus)^*R = ((ab)^\oplus(ab)^\oplus)^*R = (ab(ab)^\oplus)^*R \subseteq ab(ab)^\oplus R \subseteq abR\). Hence, \((b^\oplus)^*aR \subseteq abR\). Again, \(axa = a\) and \((ax)^* = ax\), yield \(abR = ab(ab)^\oplus abR \subseteq ab(ab)^\oplus R = (b^\oplus)^*(a^\oplus)^*R \subseteq (b^\oplus)^*(a^\oplus a^\oplus)^*R = (b^\oplus)^*(aa^\oplus)^*a^*R \subseteq (b^\oplus)^*a^\oplus R \subseteq (b^\oplus)^*a\) \(\subseteq abR\). Hence, \((b^\oplus)^*aR = abR\). Since \((ab)^\oplus = a^\oplus b^\oplus\), we thus have \(ab = (ab)^\oplus(ab)^2 = a^\oplus b^\oplus(ab)^2\).

(ii) From part (i), we have \((b^\oplus)^*aR = abR\). Now, \((b^\oplus)^*aR = (b^\oplus)^*a^\oplus aR = (b^\oplus)^*(aa^\oplus)^*aR = (aa^\oplus b^\oplus)^*aR \subseteq (a^\oplus b^\oplus)^*R = (b^\oplus)^*(a^\oplus)^*R \subseteq (b^\oplus)^*a^\oplus R \subseteq (b^\oplus)^*aR\). So, \((b^\oplus)^*R = (a^\oplus b^\oplus)^*R\). Hence, \(abR = (a^\oplus b^\oplus)^*R\). Pre-multiplying \(a^\oplus b^\oplus(ab)^2 = ab\) by \(a^*a\), we get \(a^*b^\oplus(ab)^2 = a^*a^2b\). \(\square\)
The next result provides a relation among the weighted core inverse of $ab$, the weighted core inverse of $a$, and the core inverse of $b$.

**Theorem 3.10.** Let $a \in R^\# \cap R^{e,[]}$ be such that $\circ(a) = \{0\}$, and $b \in R^\#$ with $aba = ba^2 = a^2b$. If $eabb = bb^eab$, $aba^{e,[]} = aae^{e,}b$, and $abb^e = bb^a$, then $(ab)^{e,[]} = a^{e,}b^e$.

**Proof.** Taking the involution on $abb^e = bb^ea$, we get $bb^ea^* = a^*bb^e$. Post-multiplying $abb^e = bb^ea$ by $(a^{e,[]})^2a$, we get $abb^e(a^{e,[]})^2a = bb^ea(a^{e,[]})^2a$, i.e., $abb^e(a^{e,[]})^2a = bb^ea^{e,[]}a$. Using $a^# = (a^{e,[]})^2a$ by the Theorem 2.2, we get $abb^ea^# = bb^ea^{e,[]}a$. We know that if $ab = ba$, then $a^#b = ba^#$. Therefore, $aa^#bb^e = bb^ea^{e,[]}a$, i.e., $a(a^{e,[]})^2abb^e = bb^ea^{e,[]}a$, i.e., $a^{e,[]}babb^e = bb^ea^{e,[]}a$, i.e., $(a^{e,[]}bb^e - bb^ea^{e,[]})a = 0$, i.e., $(a^{e,[]}bb^e - bb^ea^{e,[]}) \in \circ(a) = \{0\}$. Therefore, $a^{e,[]}bb^e = bb^ea^{e,[]}$.

Now, we will prove that $(ab)^{e,[]} = a^{e,}b^e$ by using the definition of the weighted core inverse.

\[
aba^{e,[]}b^eab = aab^{e,[]}b^eab (\text{using } aba^{e,[]} = aab^{e,[]})
\]
\[
= aab^{e,[]}b^eab (\text{using } aba^{e,[]} = aab^{e,[]})
\]
\[
= a^{e,}b^e,
\]
\[
abab = a^{e,}ab^2b
\]
\[
= a^{e,}b^2a^2
\]
\[
= a^{e,}a^2b
\]
\[
= ab,
\]

and

\[(eaba^{e,[]}b^e)^* = (eaa^{e,[]}bb^e)^*
\]
\[
= (eabb^ea^{e,[]})^*
\]
\[
= (ebba^{e,[]}aa^e)^*
\]
\[
= (bb^eaa^e)^*
\]
\[
= (eaa^e)(bb^e)^*
\]
\[
= ea^{e,[]}bb^e
\]
\[
= eaba^{e,[]}b^e.
\]

Hence, $(ab)^{e,[]} = a^{e,}b^e$. 

The following result presents some sufficient conditions of the forward-order law for the weighted core inverse.

**Theorem 3.11.** Let $a, b \in R^{e,[]}$. If $a^{e,[]}b = b^{e,[]}a$ and $bb^{e,[]} = ab^{e,[]}$, then $(ab)^{e,[]} = a^{e,}b^{e,[]} = (b^{e,[]})^2$. 

Proof. \( a^{e,\otimes} b^{e,\otimes} = a^{e,\otimes} b(b^{e,\otimes})^2 = b^{e,\otimes} ab^{e,\otimes} = b^{e,\otimes} bb^{e,\otimes} b^{e,\otimes} = b^{e,\otimes} b^{e,\otimes} = (b^{e,\otimes})^2 \). Now, check that \( a^{e,\otimes} b^{e,\otimes} \) is the weighted core inverse of \( ab \) by the definition.

\[
eaba^{e,\otimes} b^{e,\otimes} = eab^{e,\otimes} b^2 a^{e,\otimes} b^{e,\otimes}
= ebb^{e,\otimes} b^2 (b^{e,\otimes})^2 (\text{using } (b^{e,\otimes})^2 = a^{e,\otimes} b^{e,\otimes})
= ebb^{e,\otimes} bb^{e,\otimes}
= ebb^{e,\otimes}.
\]

(3.8)

From equation (3.8), \((eaba^{e,\otimes} b^{e,\otimes})^* = eaba^{e,\otimes} b^{e,\otimes} \). Further,

\[
aba^{e,\otimes} b^{e,\otimes} a^{e,\otimes} b^{e,\otimes} = ab^{e,\otimes} b^2 (b^{e,\otimes})^2 (\text{using } (b^{e,\otimes})^2 = a^{e,\otimes} b^{e,\otimes})
= b^2 (b^{e,\otimes})^2 (b^{e,\otimes})^2 (\text{using } bb^{e,\otimes} = ab^{e,\otimes})
= (b^{e,\otimes})^2
= a^{e,\otimes} b^{e,\otimes},
\]

and

\[
a^{e,\otimes} b^{e,\otimes} abab = a^{e,\otimes} b^{e,\otimes} ab^{e,\otimes} b^2 a^{e,\otimes} b^2
= (b^{e,\otimes})^2 b^{e,\otimes} b^2 bb^{e,\otimes} b^2
= (b^{e,\otimes})^2 b^2 b^2
= b^2
= bb
= bb^{e,\otimes} b^2
= ab^{e,\otimes} b^2
= ab.
\]

Hence, \((ab)^{e,\otimes} = a^{e,\otimes} b^{e,\otimes} = (b^{e,\otimes})^2 \). \(\square\)

We now show that the forward-order law for the weighted core inverse holds under the assumption \(ab = b^2\).

**Theorem 3.12.** Let \( a \in R^{e,\otimes} \) and \( b \in R^{#} \cap R^{e,\otimes} \). If \( ab = b^2 \), then

(i) \( ab \in R^{e,\otimes} \) and \( (ab)^{e,\otimes} = a^{e,\otimes} b^{e,\otimes} \);

(ii) \( a a^{e,\otimes} b^{e,\otimes} (baa^{e,\otimes})^2 = baa^{e,\otimes}, \) \( ebba^{e,\otimes} (aa^{e,\otimes} b^{e,\otimes})^2 = eb^{e,\otimes} \) and \( (eaba^{e,\otimes} b^{e,\otimes})^* = ebba^{e,\otimes} b^{e,\otimes} \).

**Proof.** (i) By the definition of group inverse, we can write \( b = b^2 b^# \). The hypotheses \( ab = b^2 \) and \( b = b^2 b^# \) imply that \( b = abb^# \), i.e., \( b = a^{e,\otimes} a^2 b^2 b^# \), i.e., \( b = a^{e,\otimes} b^3 b^# \). So, \( b = a^{e,\otimes} b^2 \). Again, by \( ab = b^2 \), we get \( b^{e,\otimes} b = (b^{e,\otimes})^2 b^2 = (b^{e,\otimes})^2 ab \). Similarly, \( bb^{e,\otimes} = b^2 (b^{e,\otimes})^2 = ab(b^{e,\otimes})^2 = ab^{e,\otimes} \). Further, we get

\[
a^{e,\otimes} b^{e,\otimes} abab = a^{e,\otimes} b^{e,\otimes} b^2 b^2 (\text{using } ab = b^2)
\]
\[= a^{e\otimes} b b^2\]
\[= b^2 (\text{using } a^{e\otimes} b^2 = b)\]
\[= ab,\]
\[aba^{e\otimes} b^{e\otimes} a^{e\otimes} b^{e\otimes} = aba^{e\otimes} b^{e\otimes} a^{e\otimes} b^2 (b^{e\otimes})^3 (\text{using } b^{e\otimes} = b(b^{e\otimes})^2)\]
\[= aba^{e\otimes} b^{e\otimes} (b^{e\otimes})^3 (\text{using } a^{e\otimes} b^2 = b)\]
\[= aba^{e\otimes} (b^{e\otimes})^3\]
\[= a^{e\otimes} a^2 b a^{e\otimes} (b^{e\otimes})^3\]
\[= a^{e\otimes} b^3 a^{e\otimes} (b^{e\otimes})^3\]
\[= a^{e\otimes} b^3 a^{e\otimes} b^2 (b^{e\otimes})^5\]
\[= a^{e\otimes} b^4 (b^{e\otimes})^5\]
\[= a^{e\otimes} b^{e\otimes},\]
and
\[eaba^{e\otimes} b^{e\otimes} = eb^2 a^{e\otimes} b^2 (b^{e\otimes})^3 (\text{using } b^{e\otimes} = b(b^{e\otimes})^2)\]
\[= eb^3 (b^{e\otimes})^3\]
\[= ebb^{e\otimes}. \tag{3.9}\]

From (3.9), \((eaba^{e\otimes} b^{e\otimes})^* = eaba^{e\otimes} b^{e\otimes}\). Hence, \((ab)^{e\otimes} = a^{e\otimes} b^{e\otimes}\).

(ii) Putting \(a^{e\otimes} b^{e\otimes}\) directly in the equations, we have
\[ebaa^{e\otimes} b^{e\otimes} = ebaa^{e\otimes} b^2 (b^{e\otimes})^3\]
\[= ebaa^{e\otimes} ab(b^{e\otimes})^3\]
\[= ebab(b^{e\otimes})^3\]
\[= eb^3 (b^{e\otimes})^3\]
\[= ebb^{e\otimes}, \tag{3.10}\]

which implies \((eaba^{e\otimes} b^{e\otimes})^* = eaba^{e\otimes} b^{e\otimes}\). Further, we obtain
\[aa^{e\otimes} b^{e\otimes} (baa^{e\otimes})^2 = aa^{e\otimes} b^{e\otimes} baa^{e\otimes} baa^{e\otimes}\]
\[= aa^{e\otimes} b^2 (b^{e\otimes})^3 baa^{e\otimes} baa^{e\otimes}\]
\[= aa^{e\otimes} ab(b^{e\otimes})^3 baa^{e\otimes} baa^{e\otimes}\]
\[= ab(b^{e\otimes})^3 baa^{e\otimes} baa^{e\otimes}\]
\[= b^2 (b^{e\otimes})^3 baa^{e\otimes} baa^{e\otimes}\]
\[= b^{e\otimes} baa^{e\otimes} b^{e\otimes} b^2 aa^{e\otimes}\]
\[= b^{e\otimes} baa^{e\otimes} b^2 (b^{e\otimes})^3 b^2 aa^{e\otimes}\]
\[= b^{e\otimes} baa^{e\otimes} ab(b^{e\otimes})^3 b^2 aa^{e\otimes}\]
\[= b^{e\otimes} bab(b^{e\otimes})^3 b^2 aa^{e\otimes}\]
\[
\begin{align*}
&= b^e \cdot b b^2 (b^e) 3 b^2 a a^e \cdot e \\
&= b^2 (b^e) 3 b^2 a a^e \cdot e \\
&= b(b^e) 2 b^2 a a^e \cdot e \\
&= baa^e \cdot e ,
\end{align*}
\]
and
\[
\begin{align*}
\text{ebaa}^e (aa^e \cdot b^e) 2 &= \text{ebaa}^e (aa^e \cdot b^e) (aa^e \cdot b^e) \\
&= \text{ebaa}^e b^e (aa^e \cdot b^e) \\
&= \text{ebb}^e (aa^e \cdot b^e) (\text{by equation 3.10}) \\
&= \text{ebb}^e (aa^e \cdot b^e)^3 \\
&= \text{ebb}^e (aa^e \cdot ab(b^e))^3 \\
&= \text{ebb}^e b^2 (b^e)^3 \\
&= \text{ebb}^e \\
\end{align*}
\]

\[\square\]

An immediate consequence of the above result is shown next as a corollary.

**Corollary 3.13.** Let \(a, b \in R^e \cdot e\). If \(ab = b^2\) and \(e = 1\), then
1. \(ab \in R^e\) and \((ab)^e = a^e b^e\);
2. \(aa^e b^e (baa^e)^2 = baa^e\), \(baa^e (aa^e b^e)^2 = b^e\) and \((baa^e b^e)^* = baa^e b^e\).

An element \(a\) is said to be a *projection* if \(a^2 = a = a^*\). The next result is about the triple forward-order law.

**Theorem 3.14.** Let \(a, b, c \in R^e\) with a projection element \(a\). If \(abc = bca\) and \((bc)^e = b^e c^e\), then \((abc)^e = a^ e b^e c^e\).

**Proof.** The conditions \(abc = bca\) and \(a^* = a\) imply that \(a^* b c = b c a^*\). By Lemma 2.1, \(a^e b c = b c a^e\). Pre-multiplying \(a^e b c = b c a^e\) by \(a\), we have \(a a^e b c = abca^e\), i.e., \(a a^e b c = b c a a^e\). Taking the involution on both sides of the equation \(a a^e b c = b c a a^e\), we get \((bc)^ e a a^e = a a^e (bc)^*\). Again, by Lemma 2.1, we have \((bc)^ e a a^e = a a^e (bc)^\). Now, we will prove that \((abc)^e = a^e b^e c^e\) by using the definition of the core inverse.

\[
\begin{align*}
abc(a^e b^e c^e) 2 &= abca^e b^e c^e a^e b^e c^e \\
&= bcaa^e (bc)^ e a^e (bc)^ e \\
&= bc(b)^ e a a^e (bc)^ e \\
&= bc(b)^ e a^2 (a^ e)^2 (bc)^ (\text{since } a^2 = a) \\
&= bc(b)^ e a a^e (bc)^ \\
\end{align*}
\]
\[ = bc(bc)^\#(bc)^\#aa^\#
\]
\[ = (bc)^\#aa^\#
\]
\[ = aa^\#(bc)^\#
\]
\[ = a^2(a^\#)^2b^\#c^\# (\text{since } a^2 = a)
\]
\[ = a(a^\#)^2b^\#c^\#
\]
\[ = a^\#b^\#c^\#.
\]
\[ a^\#b^\#c^\#(abc)^2 = a^\#(bc)^\#abcabc
\]
\[ = a^\#(bc)^\#(bc)^2a^2
\]
\[ = a^\#bca^2
\]
\[ = a^\#a^2bc
\]
\[ = abc,
\]
and
\[ (abca^\#b^\#c^\#)^* = (aa^\#bcb^\#c^\#)^*
\]
\[ = (aa^\#bc(bc)^\#)^*
\]
\[ = (bc(bc)^\#)^*(aa^\#)^*
\]
\[ = bc(bc)^\#aa^\#
\]
\[ = bcaaa^\#(bc)^\#
\]
\[ = abca^\#b^\#c^\#.
\]

Hence, \((abc)^\# = a^\#b^\#c^\#\). \qed

Note that if \(c\) is the unity, then the above theorem coincides with Remark 3.1. Next, if \(p = a^\#abca^\#\), and \(q = aa^\#b^\#c^\#a\), then \(p\) and \(q\) have the following properties.

**Theorem 3.15.** Let \(a, b, c \in R^\#\). If \((abc)^\# = a^\#b^\#c^\#\), then

\((i)\) \(q \in p\{1, 2\}\) and \(p \in q\{1, 2\};
\((ii)\) \(a^*apq\) and \(pqa^\#(a^\#)^*\) are both Hermitian;
\((iii)\) \(pq\) and \(qp\) are both idempotent elements.

**Proof.**

\((i)\)

\[ pqp = a^\#abca^\#aa^\#b^\#c^\#aa^\#abca^\#
\]
\[ = a^\#abca^\#b^\#c^\#abca^\#
\]
\[ = a^\#(abc)(abc)^\#(abc)a^\#
\]
\[ = a^\#abca^\# = p.
\]
And

\[ qpq = a a @ b @ c @ a a @ b @ c @ a \]
\[ = a a @ b @ c @ a a b @ c @ a \]
\[ = a a @ b @ c @ abca @ (abc) @ a \]
\[ = a a @ b @ c @ a \]
\[ = a a @ b @ c @ a = q. \]

Hence, \( q \in p\{1, 2\} \) and \( p \in q\{1, 2\}. \)

(ii)

\[ (a^* apq)^* = (a^* a a @ abca @ a a @ b @ c @ a)^* \]
\[ = (a^* abca @ b @ c @ a)^* \]
\[ = (a^* abc(abc)^@)^* \]
\[ = a^* (abc(abc)^@)^*(a^*)^* \]
\[ = a^* abca @ abca @ a a @ b @ c @ a \]
\[ = a^* apq. \]

And

\[ (pqa @ (a^@))^* = (a @ abca @ a a @ b @ c @ a a @ (a^@))^* \]
\[ = (a @ abca @ b @ c @ (aa)^@)^*(a^@)^* \]
\[ = (a @ abca @ b @ c @ (a^@)^* a @ (a^@))^* \]
\[ = (a @ abca @ b @ c @ (a^@)^* )^* \]
\[ = (a @ abc(abc)^@ @ (a^@))^* \]
\[ = ((a^@)^* ((abc)(abc)^@)^*(a^@)^* \]
\[ = a @ abc @ abc @ (a^@)^* \]
\[ = pqa @ (a^@)^*. \]

Hence, \( a^* apq \) and \( pqa @ (a^@)^* \) are both Hermitian.

(iii) From part (i), \( qpq = p \) and \( qpq = q \). So, \( (pq)^2 = pq \) and \( (qp)^2 = qp \).

This example shows that the additive property of the core inverse (i.e., \( (a + b)^@ = a^@ + b^@ \)) does not hold always.

**Example 3.3.** Let \( 2^+ < x^2 >, 1^+ < x^2 > \in \frac{\mathbb{Z}_5[x]}{<x^2>} \) with conjugate involution.

Let \( 3^+ < x^2 > \) be the core inverse of \( 2^+ < x^2 > \) and \( 1^+ < x^2 > \) be the core inverse of \( 1^+ < x^2 > \). Then \( (2^+ < x^2 > + 1^+ < x^2 >)^@ = (3^+ < x^2 >)^@ = 2^+ < x^2 > \).

But

\[ (2^+ < x^2 >)^@ + (1^+ < x^2 >)^@ = 3^+ < x^2 > + 1^+ < x^2 > = 4^+ < x^2 >. \]
In 2022, Baksalary et al. [2] provided certain sufficient conditions under which the Moore–Penrose inverse is additive, i.e., $(A + B)\dagger = A\dagger + B\dagger$. Next, we establish a result for the additive property of the core inverse in rings.

**Theorem 3.16.** Let $a, b \in R^\circ$. If $ab = ba = 0$ and $ab^* = b^*a$, then $(a + b)^\circ = a^\circ + b^\circ$.

**Proof.** We have $ab = ba = 0$, which yields $(a + b)^k = a^k + b^k$ for every positive integer $k$, by the binomial expansion. From Remark 3.1, we have $(ab)^\circ = a^\circ b^\circ = b^\circ a^\circ$. If $ab = ba = 0$, then $b^\circ a^\circ = a^\circ b^\circ = 0$. Again, by the binomial expansion, we have $(a^\circ + b^\circ)^k = (a^\circ)^k + (b^\circ)^k$ for every positive integer $k$. Further, we obtain

$$
(a + b)(a^\circ + b^\circ)^2 = (a + b)((a^\circ)^2 + (b^\circ)^2)
= a(a^\circ)^2 + a(b^\circ)^2 + b(a^\circ)^2 + b(b^\circ)^2
= a^\circ + a(b^\circ)^2 + b(a^\circ)^2 + b^\circ
= a^\circ + ab(b^\circ)^3 + ba(a^\circ)^3 + b^\circ
= a^\circ + b^\circ,
$$

(3.11)

and

$$
(a^\circ + b^\circ)(a + b)^2 = (a^\circ + b^\circ)(a^2 + b^2)
= a^\circ a^2 + a^\circ b^2 + b^\circ a^2 + b^\circ b^2
= a + a^\circ b^\circ b^3 + b^\circ a^\circ a^3 + b
= a + b
$$

(3.12)

and

$$
((a + b)(a^\circ + b^\circ))^* = (aa^\circ + ab^\circ + ba^\circ + bb^\circ)^*
= (aa^\circ + ab(b^\circ)^2 + ba(a^\circ)^2 + bb^\circ)^*
= (aa^\circ)^* + (bb^\circ)^*
= (a + b)(a^\circ + b^\circ).
$$

(3.13)

From (3.11), (3.12) and (3.13), we get $(a + b)^\circ = a^\circ + b^\circ$. □

4. **Hybrid forward-order law**

In this section, we present the hybrid forward-order law which says the Moore–Penrose inverse of the product of two elements is equal to the product of the core inverses of each element in the same order. We also discuss the other hybrid forward-order law. We start this section with a result which will be helpful in proving the main results of this section.

**Theorem 4.1.** Let $a \in R$ and $b \in R^\circ$. Then $(b^\circ)^*aR = abR$ if and only if $abR \subseteq bR$ and $b^*aR = (b^*)^2 abR$. 
Proof. Suppose \((b^\oplus)^*aR = abR\). Therefore, \(abR = (b^\oplus)^*aR = (b^\oplus)^{(b^\oplus)^*}aR = bb^\oplus(b^\oplus)^*aR \subseteq bR\). So, \(abR \subseteq bR\). And \((b^\oplus)^*aR = abR\) implies \(ab \in (b^\oplus)^*aR\), i.e., there exists \(x \in R\) such that \(ab = (b^\oplus)^*ax\). Pre-multiplying \(ab = (b^\oplus)^*ax\) by \((b^\oplus)^2\), we get \((b^\oplus)^2ab = b^*ax\). Hence, \((b^\oplus)^2abR \subseteq b^*aR\). Again, \((b^\oplus)^*aR = abR\) implies \((b^\oplus)^*a \in abR\), i.e., there exists \(y \in R\) such that \((b^\oplus)^*a = aby\). Pre-multiplying \((b^\oplus)^*a = aby\) by \((b^\oplus)^2\), we get \(ba = (b^\oplus)^2aby\). So, \(baR \subseteq (b^\oplus)^2abR\). We conclude that \(b^*aR = (b^\oplus)^2abR\).

Conversely, we assume that \(abR \subseteq bR\) and \(b^*aR = (b^\oplus)^2abR\). Now, \(abR \subseteq bR\) implies \(ab = bz\), where \(z \in R\). Pre-multiplying \(ab = bz\) by \(bb^\oplus\), we get \(bb^\oplus ab = ab\). So, \(abR = bb^\oplus abR\). Again, pre-multiplying \(b^*aR = (b^\oplus)^2abR\) by \(((b^\oplus)^2)^*\), we conclude \((b(b^\oplus)^2)^*aR = (b^2(b^\oplus)^2)^*abR\) yields \((b^\oplus)^*aR = bb^\oplus abR\). Hence, \((b^\oplus)^*aR = abR\).

\(\square\)

**Theorem 4.2.** Let \(a, b \in R^\oplus\). If \(abb^\oplus = bb^\oplus a\), then \((ab)^{(1,2,3)}\) exists and \((ab)^{(1,2,3)} = b^\oplus a^\oplus\).

**Proof.** Applying the involution on \(abb^\oplus = bb^\oplus a\), we get \(a^*bb^\oplus = bb^\oplus a^*\). Then by Lemma 2.1, we obtain \(a^\oplus bb^\oplus = bb^\oplus a^\oplus\). Now,

\[
\begin{align*}
abb^\oplus a^\oplus ab &= aab^\oplus ab \\
&= aab^\oplus b \\
&= ab,
\end{align*}
\]

\[
\begin{align*}
bb^\oplus a^\oplus abb^\oplus a^\oplus &= bb^\oplus a^\oplus bb^\oplus a^\oplus \\
&= bb^\oplus a^\oplus a^\oplus bb^\oplus a^\oplus \\
&= bb^\oplus a^\oplus ,
\end{align*}
\]

and

\[
\begin{align*}
(ab^\oplus a^\oplus)^* &= (bb^\oplus a^\oplus)^* \\
&= (aa^\oplus)^*(bb^\oplus)^* \\
&= aa^\oplus bb^\oplus \\
&= abb^\oplus a^\oplus .
\end{align*}
\]

Hence, \((ab)^{(1,2,3)} = b^\oplus a^\oplus\). \(\square\)

**Lemma 4.3.** Let \(a \in R^\oplus\) and \(a, ab \in R^\uparrow\). Then the following are equivalent:

\((i)\) \(a^*b = a^*(ab)^\uparrow b^2\); 
\((ii)\) \(aa^\oplus b = (a(ab)^\uparrow b^2\); 
\((iii)\) \(aa^\uparrow b = a(ab)^\uparrow b^2\).

**Proof.** (i) \(\Rightarrow\) (ii) Pre-multiplying \(a^*b = a^*(ab)^\uparrow b^2\) by \((a^\oplus)^*\), we obtain \((aa^\oplus)^*b = (aa^\oplus)^*(a(ab)^\uparrow b^2\), i.e., \(aa^\oplus b = a(ab)^\uparrow b^2\).

(ii) \(\Rightarrow\) (iii) Pre-multiplying \(aa^\oplus b = a(ab)^\uparrow b^2\) by \(a^*\), we have \(a^*aa^\oplus b = a^*(a(ab)^\uparrow b^2\) yields \(a^*b = a^*(a(ab)^\uparrow b^2\). Again, pre-multiplying \(a^*b = a^*(a(ab)^\uparrow b^2\) by \((a^\uparrow)^*\), we get \((aa^\uparrow)^*b = (aa^\uparrow)^*(a(ab)^\uparrow b^2\), i.e., \(aa^\uparrow b = a(ab)^\uparrow b^2\).
(iii) \(\Rightarrow\) (i) Pre-multiplying part (iii) by \(a^*\), we have part (i).

\[\square\]

The hybrid forward-order law involving the Moore–Penrose inverse and the core inverse is presented below.

**Theorem 4.4.** Let \(a, b \in R^\ddagger\) and \(ab \in R^\dagger\). Then \((ab)^\dagger = a^\ddagger b^\ddagger\) if and only if \(abR \subseteq bR\), \(Rab \subseteq Ra^*\) and \(a^*b = a^*a(ab)^\dagger b^2\).

**Proof.** Now, assume \((ab)^\dagger = a^\ddagger b^\ddagger\). Then

\[a^*b = (aa^\ddagger a)b^\ddagger b^2 = a^*aa^\ddagger b^\ddagger b^2 = a^*a(ab)^\dagger b^2.\]

Now, from \(ab = ab(ab)^\dagger ab\) and \(b^\ddagger = b^\ddagger bb^\ddagger\), we get

\[abR = ab(ab)^\dagger abR\]
\[= [(ab)(ab)^\dagger]*abR\]
\[= ((ab)^\dagger)^*(ab)^*abR\]
\[\subseteq ((ab)^\dagger)^* R\]
\[= (a^\ddagger b^\ddagger)^* R\]
\[= (b^\ddagger)^*(a^\ddagger)^* R\]
\[\subseteq (b^\ddagger)^* R\]
\[= (b^\ddagger bb^\ddagger)^* R\]
\[\subseteq bb^\ddagger R\]
\[\subseteq bR.\]

So, \(abR \subseteq bR\). Furthermore,

\[Rab = Rab(ab)^\dagger ab\]
\[= Rab((ab)^\dagger ab)^*\]
\[= Rab(ab)^*((ab)^\dagger)^*\]
\[\subseteq R((ab)^\dagger)^*\]
\[= R(a^\ddagger b^\ddagger)^*\]
\[\subseteq R(a^\ddagger)^*\]
\[= R(a(a^\ddagger)^2)^*\]
\[\subseteq Ra^*.\]

Hence, \(Rab \subseteq Ra^*\).

Conversely, the hypothesis are \(abR \subseteq bR\), \(Rab \subseteq Ra^*\) and \(a^*b = a^*a(ab)^\dagger b^2\). Since, \(abR \subseteq bR\) implies \(ab = bz\), where \(z \in R\). Pre-multiplying \(ab = bz\) by \(bb^\ddagger\), we get \(bb^\ddagger ab = ab\). Taking the involution on both sides of \(bb^\ddagger ab = ab\), we get \((ab)^* = (ab)^*bb^\ddagger\). Now, pre-multiplying \((ab)^* = (ab)^*bb^\ddagger\) by \((ab)^((ab)^\dagger)^*\), we
obtain \((ab)\dagger = (ab)^\dagger b b^\otimes\). And \(Rab \subseteq Ra^\ast\) gives \(ab \in Ra^\ast\), i.e., \(ab = ya^\ast\), where \(y \in R\). Post-multiplying \(ab = ya^\ast\) by \((a^\otimes a)^\ast\), we get \(ab = ab(a^\otimes a)^\ast\). Taking the involution on both sides of \(ab = ab(a^\otimes a)^\ast\), we have \((ab)^\ast = a^\otimes a(ab)^\ast\). Now, post-multiplying \((ab)^\ast = a^\otimes a(ab)^\ast\) by \(((ab)\dagger)^\ast (ab)\dagger\), we get \((ab)\dagger = a^\otimes a(ab)\dagger\). Therefore,

\[
a^\otimes a(ab)\dagger b b^\otimes = (ab)\dagger b b^\otimes = (ab)^\dagger.
\]

Now, pre- and post-multiplying \(a^\ast b = a^\ast a(ab)^\dagger b^2\) by \(a^\otimes (a^\otimes)^\ast\) and \((b^\otimes)^2\), respectively, we have \(a^\otimes b^\otimes = a^\otimes a(ab)^\dagger b^\otimes\). Hence, \((ab)\dagger = a^\otimes b^\otimes\). \(\Box\)

In the above theorem, we can replace \(a^\ast b = a^\ast a(ab)^\dagger b^2\) by \(aa^\otimes b = (a^\dagger b)^2\) or \(aa^\ast b = (a^\dagger b)^2\) by Lemma 4.3. Next, we establish a result for the hybrid forward-order law involving the Moore–Penrose inverse and the core inverse.

**Theorem 4.5.** Let \(a, b \in R^\otimes\) and \(ab \in R^\dagger\). Then

(i) \((ab)\dagger = a^\otimes b^\otimes\) if and only if \(abR \subseteq bR\) and \(a^\otimes b = (ab)^\dagger b^2\);

(ii) \((ab)\dagger = a^\otimes b^\otimes\) if and only if \(ab \subseteq Ra^\ast\) and \(a^\ast b^\otimes = a^\ast a(ab)\dagger\);

(iii) \((ab)\dagger = a^\otimes b^\otimes\) if and only if \((b^\otimes)^\ast aR \subseteq abR\) and \((ab)\dagger = a^\otimes b^\otimes ab(ab)^\ast\).

**Proof.**

(i) By Theorem 4.4, \((ab)\dagger = a^\otimes b^\otimes\) yields \(abR \subseteq bR\). Further, \((ab)\dagger b^2 = a^\otimes b^\otimes b^2 = a^\otimes b\). Conversely, \(abR \subseteq bR\) yields \(ab = bz\), where \(z \in R\). Pre-multiplying \(ab = bz\) by \(b b^\otimes\), we get \(b b^\otimes ab = ab\). Taking the involution on \(b b^\otimes ab = ab\), we get \((ab)^\ast b b^\otimes = (ab)^\ast\). Pre-multiplying \((ab)^\ast b b^\otimes = (ab)^\ast\) by \(((ab)\dagger)^\ast\), we have \((ab)\dagger b b^\otimes = (ab)\dagger\). Post-multiplying \(a^\otimes b = (ab)\dagger b^2\) by \((b^\otimes)^2\), we get \(a^\otimes b^\otimes = (ab)\dagger b b^\otimes\). Hence, \((ab)\dagger = a^\otimes b^\otimes\).

(ii) If \((ab)\dagger = a^\otimes b^\otimes\), then \(Rab \subseteq Ra^\ast\) by Theorem 4.4. Further, \(a^\ast a(ab)^\dagger = a^\ast a(ab)^\dagger a^\otimes b^\otimes = a^\ast b^\otimes = a^\ast a(ab)^\dagger\). Conversely, if \(Rab \subseteq Ra^\ast\), then \(ab = za^\ast\), where \(z \in R\). Post-multiplying \(ab = za^\ast\) by \((a^\otimes)^\ast\), we get \(ab = ab(a^\otimes a)^\ast\). Applying the involution on \(ab = ab(a^\otimes a)^\ast\), we get \((ab)^\ast = a^\otimes a(ab)^\ast\). Again, post-multiplying \((ab)^\ast = a^\otimes a(ab)^\ast\) by \(((ab)\dagger)^\ast\), we obtain \((ab)\dagger = a^\otimes a(ab)\dagger\). Pre-multiplying \(a^\otimes b^\otimes = a^\ast a(ab)\dagger\) by \((a^\otimes)^\ast\), we get \(a^\otimes b^\otimes = a^\otimes a(ab)\dagger\). Hence, \((ab)\dagger = a^\otimes b^\otimes\).

(iii) Suppose \((ab)\dagger = a^\otimes b^\otimes\). Then \((b^\otimes)^\ast aR \subseteq abR\) and \((ab)\ast = a^\otimes b^\otimes ab(ab)^\ast\) are obvious. Conversely, if \((ab)\ast = a^\otimes b^\otimes ab(ab)^\ast\), then \((ab)\dagger = a^\otimes b^\otimes ab(ab)^\dagger\). And \((b^\otimes)^\ast aR \subseteq abR\) gives \((b^\otimes)^\ast a \in abR\), i.e., \((b^\otimes)^\ast a = abx\), where \(x \in R\). Taking the involution on both sides of \((b^\otimes)^\ast a = abx\), we get \(a^\ast b^\otimes = x^\ast (ab)^\ast\), i.e., \(a^\ast b^\otimes = x^\ast (ab)^\ast ab(ab)^\dagger\), i.e., \(a^\ast b^\otimes = (abx)^\ast ab(ab)^\dagger\).

So, \(a^\ast b^\otimes = a^\ast b^\otimes ab(ab)^\dagger\). By Lemma 3.1, \(a^\otimes b^\otimes = a^\otimes b^\otimes ab(ab)^\dagger\). Hence, \((ab)\dagger = a^\otimes b^\otimes\). \(\Box\)

Some equivalent conditions for two core invertible elements are obtained next.

**Theorem 4.6.** Let \(a, b \in R^\otimes\). Then the following are equivalent:

[Theorem statement and proof are not provided]
\begin{align*}
(i) \quad (ab)^* &= a^@b^@ab(ab)^*; \\
(ii) \quad Rab \subseteq Ra^* \text{ and } a^*a(ab)^* &= a^*b^@ab(ab)^*.
\end{align*}

**Proof.** (i) ⇒ (ii): Pre-multiplying \((ab)^* = a^@b^@ab(ab)^*\) by \(a^*a\), we obtain \(a^*a\ (ab)^* = a^*b^@ab(ab)^*\). We take the involution on both sides of \((ab)^* = a^@b^@ab(ab)^*\), and we obtain \(ab = ab(ab)^*(b^@)^*(a^@)^*, \text{ i.e., } ab = ab(ab)^*(b^@)^*((a^@)^2)^*a^*\). Hence, \(Rab \subseteq Ra^*\).

(ii) ⇒ (i): \(Rab \subseteq Ra^*\) implies \(ab \in Ra^*\), i.e., \(ab = xa^*\). Taking the involution on both sides of \(ab = xa^*\), we have \((ab)^* = ax^*\). Pre-multiplying \((ab)^* = ax^*\) by \(a^@a\), we get \(a^@a(ab)^* = ax^*\), i.e., \(a^@a(ab)^* = (ab)^*\). Again, pre-multiplying \(a^*a(ab)^* = a^*b^@ab(ab)^*\) by \(a^@a(ab)^*\), we obtain \(a^@a(ab)^* = a^@b^@ab(ab)^*\). Hence, \((ab)^* = a^@b^@ab(ab)^*\).

\[\Box\]

**Theorem 4.7.** Let \(a, b \in R^\oplus\) and \(ab \in R^\dagger\). If \((ab)^\dagger = a^@b^@\), then \((b^\oplus)^*aR = abR\) and \((ab)^* = a^@b^@ab(ab)^*)\).

**Proof.** First we prove \((b^\oplus)^*aR = abR\). Now, by Theorem 4.5 (iii),
\[(b^\oplus)^*aR \subseteq abR,\]

and
\[abR = ab(ab)^\dagger abR = ((ab)^\dagger)^*(ab)^*abR \subseteq ((ab)^\dagger)^*R = (a^@b^@)^*R = (b^\oplus)^*(a^@)^*R = (b^\oplus)^*(a^@a^@)^*R = (b^\oplus)^*aa^@a^@\cdot(a^@)^*R \subseteq (b^\oplus)^*aR.\]

So, \((b^\oplus)^*aR = abR\). We take the involution on \(ab = ab(ab)^\dagger ab\), and we obtain \((ab)^* = (ab)^\dagger ab(ab)^*\) which implies \((ab)^\dagger = a^@b^@ab(ab)^*\) as \((ab)^\dagger = a^@b^@\).

\[\Box\]

A result for the hybrid forward-order law and the hybrid reverse-order law involving the Moore–Penrose inverse and the core inverse is presented next.

**Theorem 4.8.** Let \(a, b \in R^\dagger\) and \(ab \in R^\oplus\). Then
\[(i) \quad (ab)^@ = a^@b^\dagger \text{ if and only if } abR \subseteq a^*R \text{ and } a^@a(ab)^@ = a^\dagger b^\dagger; \]
\[(ii) \quad (ab)^@ = b^\dagger a^\dagger \text{ if and only if } abR \subseteq b^*R \text{ and } b^\dagger b(ab)^@ = b^\dagger a^\dagger.\]

**Proof.** (i) Let \((ab)^@ = a^\dagger b^\dagger\). Then \(abR = (ab)^@((ab)^\dagger)^2R \subseteq (ab)^@R = a^\dagger b^\dagger R \subseteq a^1R = a^\dagger aa^\dagger R = a^*(a^\dagger)^*a^\dagger R \subseteq a^*R\), i.e., \(abR \subseteq a^*R\). And \(a^\dagger a(ab)^@ = a^\dagger aa^\dagger b^\dagger = a^\dagger b^\dagger.\)
Conversely, the hypothesis $abR \subseteq a^*R$ yields $ab = a^*y$, where $y \in R$. Pre-multiplying $ab = a^*y$ by $a^\dagger a$, we obtain $a^\dagger aab = a^\dagger aa^*y$ which implies $a^\dagger aab = a^\dagger (a^\dagger)^*a^*y$, i.e., $a^\dagger aab = a^*y$. So, $a^\dagger aab = ab$. Post-multiplying $a^\dagger aab = ab$ by $(ab)^\odot^2$, we have $a^\dagger a(ab)^\odot = (ab)^\odot$. From the hypothesis $a^\dagger a(ab)^\odot = a^\dagger b^\dagger$ and the previous equality $a^\dagger a(ab)^\odot = (ab)^\odot$, we arrive at

$$(ab)^\odot = a^\dagger b^\dagger.$$ (ii) Assume that $(ab)^\odot = b^\dagger a^\dagger$. Then $abR = (ab)^\odot(ab)^2R \subseteq (ab)^\odot R = b^\dagger a^\dagger R \subseteq b^\dagger bb^\dagger R = b^\star (b^\dagger)b^\dagger R \subseteq b^\star R$, i.e., $abR \subseteq b^\star R$. And $b^\dagger b(ab)^\odot = b^\dagger bb^\dagger a^\dagger = b^\dagger a^\dagger$.

Conversely, from the assumption $abR \subseteq b^\star R$, we get $ab = b^\star y$, where $y \in R$. Pre-multiplying $ab = b^\star y$ by $b^\dagger b$, we obtain $b^\dagger bab = b^\dagger bb^\dagger y$ which implies $b^\dagger bab = b^\star (b^\dagger)b^\star y$, i.e., $b^\dagger bab = b^\star y$. So, $b^\dagger bab = ab$. Post-multiplying $b^\dagger bab = ab$ by $(ab)^\odot^2$, we have $b^\dagger b(ab)^\odot = (ab)^\odot$. From the hypothesis $b^\dagger b(ab)^\odot = b^\dagger a^\dagger$ and the previous equality $b^\dagger b(ab)^\odot = (ab)^\odot$, we obtain

$$(ab)^\odot = b^\dagger a^\dagger.$$

Pre-multiplying $a^\star b = a^\star a(ab)^\#b^2$ by $(ab)^\odot^\star$, we get $aa^\odot b = a(ab)^\#b^2$.

Pre-multiplying $aa^\odot b = a(ab)^\#b^2$ by $a^\odot$, we have $a^\odot b = (ab)^\#b^2$. Again, pre-multiplying $a^\odot b = (ab)^\#b^2$ by $a^\star a$, we get $a^\star b = a^\star a(ab)^\#b^2$. This leads to the following lemma.

**Lemma 4.9.** Let $a, ab \in R^\odot$. Then the following are equivalent:

(i) $a^\star b = a^\star a(ab)^\#b^2$;
(ii) $aa^\odot b = a(ab)^\#b^2$;
(iii) $a^\odot b = (ab)^\#b^2$.

Next, we establish a result for the hybrid forward-order law involving the group inverse and the core inverse.

**Theorem 4.10.** Let $a, b, ab \in R^\odot$. Then

(i) $(ab)^\# = a^\odot b^\odot$ if and only if $Rab \subseteq R^a$ and $a^\odot b = (ab)^\#b^2$;
(ii) $(ab)^\# = a^\odot b^\odot$ if and only if $a^\odot bR = abR$ and $ab = (ab)^2a^\odot b^\odot$;
(iii) $(ab)^\# = a^\odot b^\odot$ if and only if $Ra^\star b^\odot = Rab = Ra^\odot b^\odot$ and $ab = a^\odot b^\odot(ab)^2$;
(iv) $(ab)^\# = a^\odot b^\odot$ if and only if $a^\star a(ab)^\# = a^\star b^\odot$.

**Proof.** (i) Assume $(ab)^\# = a^\odot b^\odot$. Then $a^\star a(ab)^\#b^2 = a^\star aa^\odot b^\odot b^2 = a^\star (a^\odot)^\#a^\star b = a^\star b$, i.e., $a^\star b = a^\star a(ab)^\#b^2$. By Lemma 4.9, $a^\star b = a^\star a(ab)^\#b^2$ implies $a^\odot b = (ab)^\#b^2$. Further,

$$Rab = R(ab)^2(ab)^\# \subseteq R(ab)^\# = Ra^\odot b^\odot \subseteq Rb^\odot(b^\odot)^\# \subseteq Rb^\star.$$
Conversely, $Rab \subseteq Rb^*$ and $a^{\#}b = (ab)^{\#}b^2$ imply $abbb^{\#} = ab$ and $a^{\#}b^{\#} = (ab)^{\#}bb^{\#}$, respectively. And $abbb^{\#} = ab$ is equivalent to $(ab)^{\#}bb^{\#} = (ab)^{\#}$. So, $(ab)^{\#} = a^{\#}b^{\#}$.

(ii) We know that $ab = (ab)^2(ab)^{\#}$. So, $ab = (ab)^2a^{\#}b^{\#}$. Further, $abR = (ab)^{\#}(ab)^2R \subseteq a^{\#}b^R \subseteq a^{\#}b^R \subseteq (ab)^{\#}R \subseteq abR$.

So, $a^{\#}b = abR$.

Conversely, the hypothesis $a^{\#}bR = abR$ implies $a^{\#}bR \subseteq abR$. Also, we have $ab = (ab)^2a^{\#}b^{\#}$. The first one $a^{\#}bR \subseteq abR$ yields $a^{\#}b \in abR$. So, we have $a^{\#}b = aby$, where $y \in R$. Pre-multiplying $a^{\#}b = aby$ by $ab(ab)^{\#}$, we get $ab(ab)^{\#}a^{\#}b = a^{\#}b$. Post-multiplying $ab(ab)^{\#}a^{\#}b = a^{\#}b$ by $(b^{\#})^2$, we have $ab(ab)^{\#}a^{\#}b^{\#} = a^{\#}b^{\#}$. The last equality $ab = (ab)^2a^{\#}b^{\#}$ is equivalent to $(ab)^{\#} = ab(ab)^{\#}a^{\#}b^{\#}$. Therefore, $(ab)^{\#} = a^{\#}b^{\#}$.

(iii) We know that $ab = (ab)^{\#}(ab)^2$. Therefore, $ab = a^{\#}b^{\#}(ab)^2$. Further, $Rab = R(ab)^2(ab)^{\#} \subseteq R(ab)^{\#} \subseteq Ra^{\#}b^{\#} \subseteq Ra^{\#}b^{\#} \subseteq R(ab)^{\#} \subseteq Rab$.

So, $Ra^{\#}b^{\#} = Rab = Ra^{\#}b^{\#}$.

Conversely, the hypothesis $Ra^{\#}b^{\#} = Rab$ implies that $Ra^{\#}b^{\#} \subseteq Rab$. Now, $Ra^{\#}b^{\#} \subseteq Rab$ gives $a^{\#}b^{\#} \in Rab$, i.e., $a^{\#}b^{\#} = xab$, where $x \in R$. Post-multiplying $a^{\#}b^{\#} = xab$ by $(ab)^{\#}ab$, we get $a^{\#}b^{\#}(ab)^{\#}ab = a^{\#}b^{\#}$. Now, $a^{\#}b^{\#}(ab)^{\#}ab = a^{\#}b^{\#}$ yields $a^{\#}b^{\#}(ab)^{\#}ab = a^{\#}b^{\#}$, by Lemma 3.1. And $ab = a^{\#}b^{\#}(ab)^2$ is equivalent to $(ab)^{\#} = a^{\#}b^{\#}(ab)^{\#}ab$. Thus, $(ab)^{\#} = a^{\#}b^{\#}$.

(iv) If $(ab)^{\#} = a^{\#}b^{\#}$, then $a^{\#}a(ab)^{\#} = a^{\#}aa^{\#}b^{\#}$ yields $a^{\#}a(ab)^{\#} = a^{\#}b^{\#}$. Conversely, if we pre-multiplying $a^{\#}a(ab)^{\#} = a^{\#}b^{\#}$ by $a^{\#}(a^{\#})^*$, then we get $a^{\#}a(ab)^{\#} = a^{\#}b^{\#}$. And $a^{\#}a(ab)^{\#} = a^{\#}a(ab)^{\#}(ab)^2 = ab((ab)^{\#})^2 = (ab)^{\#}$. Therefore, $(ab)^{\#} = a^{\#}b^{\#}$.

The following lemma combines Lemma 2.5 and Lemma 2.6 of [15].

**Lemma 4.11.** Let $a, b \in R$.

(i) If $aR \subseteq bR$, then $^\circ(b) \subseteq ^\circ(a)$;

(ii) If $b$ is regular and $^\circ(b) \subseteq ^\circ(a)$, then $aR \subseteq bR$;

(iii) If $Ra \subseteq Rb$, then $(b)^\circ \subseteq (a)^\circ$;

(iv) If $b$ is regular and $(b)^\circ \subseteq (a)^\circ$, then $Ra \subseteq Rb$.

An element $a$ is said to be *regular* if $a \in R^{(1)}$. And every core invertible element is a regular element. We conclude the article with the following three results without their proofs. These can be proved using Theorem 4.1, Theorem 4.4, Theorem 4.5, and Lemma 4.11.

**Theorem 4.12.** Let $a \in R$ and $b \in R^{\#}$. Then $(b^{\#})^\circ aR = abR$ if and only if $^\circ(b) \subseteq ^\circ(ab)$ and $b^\circ aR = (b^\circ)^2abR$. 
Theorem 4.13. Let $a, b \in R^{\circ}$ and $ab \in R^{\dagger}$. Then $(ab)^\dagger = a^{\circ}b^{\circ}$ if and only if $\circ(b) \subseteq \circ(ab)$, $(a^*)^\circ \subseteq (ab)^\circ$ and $a^*b = a^*a(ab)^\dagger b^2$.

Theorem 4.14. Let $a, b \in R^{\circ}$ and $ab \in R^{\dagger}$. Then

(i) $(ab)^\dagger = a^{\circ}b^{\circ}$ if and only if $\circ(b) \subseteq \circ(ab)$ and $a^{\circ}b = (ab)^\dagger b^2$;

(ii) $(ab)^\dagger = a^{\circ}b^{\circ}$ if and only if $(a^*)^\circ \subseteq (ab)^\circ$ and $a^*b^{\circ} = a^*a(ab)^\dagger$.

5. Conclusion

The important findings are summarized as follows:

- The forward-order laws for the core inverse and the weighted core inverse have been introduced in rings.
- The triple forward-order law for the core inverse has been established.
- Finally, we have presented a few necessary and sufficient conditions of the hybrid forward-order law.

Acknowledgements

The authors thank the anonymous referee for carefully reading the earlier draft and for suggestions that improved the article’s presentation. The first author acknowledges the support of the Council of Scientific and Industrial Research-University Grants Commission, India. We thank Aaisha Be and Vaibhav Shekhar for their helpful suggestions on some parts of this article.

Data Availability Statements Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.
References

[1] Baksalary, O.M., Trenkler, G.: Core inverse of matrices. Linear Multilinear Algebra 58(6), 681–697 (2010)
[2] Baksalary, O. M., Sivakumar, K. C., Trenkler, G.: On the Moore-Penrose inverse of a sum of matrices. Linear Multilinear Algebra (2022). https://doi.org/10.1080/03081087.2021.2021132
[3] Chen, J.L., Zhu, H.H., Patrício, P., Zhang, Y.L.: Characterizations and representations of core and dual core inverses. Can. Math. Bull. 60, 269–282 (2017)
[4] Drazin, M.P.: Commuting properties of generalized inverses. Linear Multilinear Algebra 61(12), 1675–1681 (2013)
[5] Gao, Y.F., Chen, J.L., Wang, L., Zou, H.L.: Absorption laws and reverse order laws for generalized core inverses. Commun. Algebra 49(8), 3241–3254 (2021)
[6] Hartwig, R.E., Levine, J.: Applications of the Drazin inverse to the Hill cryptographic system. Part III, Cryptologia 5(2), 67–77 (1981)
[7] Kumar, A., Mishra, D.: On WD and WDMP generalized inverses in rings submitted (2022)
[8] Kyrchei, I.I.: Determinantal representations of the core inverse and its generalizations with applications. Hindawi (2019). https://doi.org/10.1155/2019/1631979
[9] Li, T., Mosić, D., Chen, J.L.: The forward order laws for the core inverse. Aequ. Math. 95, 415–431 (2021)
[10] Mosić, D., Djordjević, D.S.: Reverse order law for the group inverse in rings. Appl. Math. Comput. 219, 2526–2534 (2012)
[11] Mosić, D., Deng, C., Ma, H.: On a weighted core inverse in a ring with involution. Commun. Algebra 46(6), 2332–2345 (2018)
[12] Mosić, D., Djordjević, D.S.: Reverse order law for Moore-Penrose inverse in C* algebras. Electron. J. Linear Algebra 22, 92–111 (2011)
[13] Mosić, D., Djordjević, D.S.: Some results on the reverse order law in rings with involution. Aequ. Math. 83(3), 271–282 (2012)
[14] Penrose, R.: A generalized inverse for matrices. Camb. Philos. Soc. 51, 406–413 (1955)
[15] Rakić, D.S., Dincic, N.C., Djordjević, D.S.: Group, Moore-Penrose, core and dual core inverse in rings with involution. Linear Algebra Appl. 463, 115–133 (2014)
[16] Rao, K.B.: The Theory of Generalized Inverses Over Commutative Rings, vol. 17. CRC Press, Boca Raton (2002)
[17] Das, S., Sahoo, J.K., Behera, R.: Further results on weighted core inverse in a ring. Linear Multilinear Algebra (2022). https://doi.org/10.1080/03081087.2022.2128023
[18] Xu, S., Chen, J.L., Zhang, X.X.: New characterizations for core inverses in rings with involution. Front. Math. China 12(1), 231–246 (2017)
[19] Zhu, H.H., Chen, J.L.: Additive and product properties of Drazin inverses of elements in a ring. Bull. Malays. Math. Sci. Soc. 40(1), 259–278 (2017)
[20] Zhu, H.H., Peng, F.: Projections generated by Moore-Penrose inverses and core inverses. J. Algebra Appl. 20(3), 2150027 (2021)
[21] Zou, H.H., Chen, J.L., Patrício, P.: Reverse order law for core inverse in rings. Mediterr. J. Math. 15(3), 1–17 (2018)
[22] Zhu, H.H., Chen, J.L., Patrício, P.: Reverse order law for inverse along an element. Linear Multilinear Algebra 65(1), 166–177 (2017)
[23] Zhu, H.H., Chen, J.L., Patrício, P., Mary, X.: Centralizer’s applications to the inverse along an element. Appl. Math. Comput. 315, 27–33 (2017)
Amit Kumar and Debasisha Mishra
Department of Mathematics
National Institute of Technology Raipur
Raipur
India
e-mail, dmishra@nitrr.ac.in:

Amit Kumar
e-mail, amitdhull513@gmail.com:

Received: June 21, 2022
Revised: December 2, 2022
Accepted: December 8, 2022