TWISTS OF GENUS THREE JACOBIANS

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1. Introduction

1.0.1. Let $k$ be a field not of characteristic 2. Let $(A, a)$ be a principally polarised Abelian variety of dimension three over $k$.

1.0.2. If $a$ is indecomposable and $k$ is algebraically closed there is a smooth genus three curve $C$ and a $k$ isomorphism of principally polarised Abelian varieties

$$(\text{Jac}(C), \lambda) \cong (A, a).$$

However, if $k$ admits quadratic extensions, then such a $C$ need not exist (see [21] and [22]).

1.0.3. All of the notions in Theorem 1 will be explained within the first 3 sections of this paper, with the exception of part f) which will be explained in section 6.

Let $A_{3,1,4}$ be the fine moduli scheme of principally polarised Abelian varieties with a symplectic level 4 structure; it is a smooth scheme over $\mathbb{Z}[1/2, \sqrt{-1}]$. Let $A'_{3,1,4}$ be the open subscheme corresponding to the principally polarised Abelian varieties whose polarisation is indecomposable. We prove

**Theorem 1.** 1. Let $(A/A_{3,1,4}, A_{3,1,4}, a)$ be the universal principally polarized Abelian scheme with a level 4 structure. There is a symmetric effective principal polar divisor $\Theta$ on $A$ and therefore $\mathcal{O}_A(\Theta)$ has a a global section $s_\Theta$ which is unique up to multiplication by a unit of $\mathbb{Z}[1/2, \sqrt{-1}]$. Let $\chi_{18}$ be the product of the 36 even theta nulls on $A_{3,1,4}$, i.e.

$$\chi_{18} := \bigotimes_{\text{even}} p^* s_\Theta,$$

where the product runs over the even 2 torsion points $p: A_{3,1,4} \rightarrow A$. Then $\chi_{18}$ is unique up to multiplication by an element of $\mathbb{Z}[1/2]^{*36}$. Moreover, $\chi_{18}$ has the following properties:

a) $\chi_{18}$ is a modular form, i.e. there is an isomorphism

$$\bigotimes_{\text{even}} p^* \mathcal{O}_A(\Theta) \cong \omega_{A/A_{3,1,4}}^{18}$$

which is compatible with the classical isomorphism between analytic theta nulls and analytic theta nulls (see 3.3.1 and 3.4.4, cf. Morel-Bailly [12]).

b) $\chi_{18}$ descends to a unique Katz-Siegel modular form, which will also be denoted by $\chi_{18}$, of level 1 away from characteristic 2.

c) Let $H$ be the hyperelliptic locus in $A_{3,1,4}$. The divisor of $\chi_{18}$, on $A'_{3,1,4}$, is equal to $H$.

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This follows from the properness of Torelli morphism and the irreducibility of the moduli space of Abelian varieties of dimension 3 (cf. 2.5.4).
2) Let \((A/S,a)\) be a family (smooth and proper) of principally polarized Abelian varieties over a scheme \(S\). Then there is a Zariski open cover \(U_i\) of \(S\) and local functions \(\Delta_i \in \mathcal{O}_S(U_i)\) with the following properties: for every fibre \((A_x,a_x)\) over a closed point \(x \in S\)
\[ \Delta_i(A_x,a_x) \in k(x)^* \] if and only if \((A_x,a_x)\) is a Jacobian.

The divisor of \(\Delta_i(A_x,a)\) is equal to the hyperelliptic locus.

e) In particular, if \(k \subset \mathbb{C}\), let \(\tau\) denote the element of the Siegel upper half plane corresponding to \(A_C\). Let \(\chi_{18}^\text{an}(\tau)\) denote the product of the 36 analytic even theta nulls at an element \(\tau\). Let \(\xi_1, \xi_2, \xi_3\) be a basis for the Kähler differentials of \(A\) over \(k\). Let \(\Omega_1\) denote the 3 \times 3 matrix of integrals of \(\xi_i\) with respect to the homology classes of \(C^3/\mathbb{Z}^3 + \mathbb{Z}^3\tau\) given by the lattice vectors \((1,0,0), (0,1,0), (0,0,1) \in \mathbb{Z}^3 + \mathbb{Z}^3\tau\). Then
\[ (\frac{\pi}{2})^{54} \frac{\chi_{18}^\text{an}(\tau)}{\det(\Omega_1)^{18}} = \Delta(A,a) \in k, \]
where \(\Delta\) is a function as in part d).

This does not depend on the embedding \(k \subset \mathbb{C}\).

2. There is a Katz-Teichmüller modular form \(\text{Discr}\), defined for families of genus 3 curves over a smooth base, which compatible with the classical discriminant of a smooth quartic. Moreover, \(\text{Discr}\) has the property:

f) \(\text{Discr}\) vanishes on the hyperelliptic locus with multiplicity 1 and its square \(\text{Discr}^2\) descends to a Katz-Siegel modular form equal to \(\chi_{18}\) modulo a power of 2.

1.0.4. Remarks.

1) In a letter to Jaap Top from 2003 ([23]), Jean-Pierre Serre suggested that something like Theorem 1 should be true. In fact he more or less conjectured parts e) and f) of Theorem 1 but without the factor coming from \(\pi\) and \(\Omega_1\). The functions \(\Delta_i\) yield the mysterious square which Serre refers to at the end of the paper [24].

2) The vanishing of \(\chi_{18}(A,a)\) on the hyperelliptic locus is a classically known fact for the corresponding Siegel modular form on the Siegel upper half plane (see [4]). However these classical facts are strictly complex analytic and do not apply to the scheme \(\mathcal{A}_{3,1,4}\) without further techniques. Tsuyumine ([26]) also studied the square root of \(\chi_{18}\) using automorphic theory and observed that its square root exists on the moduli space of curves; which is roughly half of part f) of Theorem 1. The first author to study \(\chi_{18}\) seems to have been Klein (p462 [8]) and he also observed part f) of Theorem 1 was true. He remarks upon an analogous result for \(g = 4\). We think that this is really remarkable - Klein was working in a time when moduli spaces and even the Siegel upperhalf plane were dark and mysterious.

3) In Lemma 2 we show the existence of an effective line bundle \(L\), with a canonical global section \(s\), on the universal Abelian scheme \(A\) with a level 4 structure which gives the universal principal polarisation. We define the even theta null corresponding to an even two torsion point \(p : \mathcal{A}_{3,1,4} \rightarrow A\) as the pullback of \(s\) via the section \(p\). Our reliance upon level 4 structures is the reason we work away from characteristic 2. However, it should be possible to generalise Theorem 1 to characteristic 2 by using Mumford’s theta group and so-called Lagrangian decompositions instead of symplectic level 4 structures.

4) Part e) of Theorem 1 is the only way we know to compute the function in part d). It might be possible to do this computation algebraically. However this would
necessitate making the isomorphism of part a) explicit.

5) Christophe Ritzenthaler and Gilles Lachaud ([20]) have proven a special case of parts e) and f) of Theorem 1 for Jacobians which are \( (2, 2, 2) \) isogenous to a product of 3 elliptic curves; they also proved part c) in this case but for the corresponding Siegel modular form but without multiplicities. Their proof uses explicit calculations of theta functions along with a result of Everett Howe, Franck Leprevost and Bjorn Poonen which relates the coefficients of the elliptic curves to a twisting factor of the Jacobian. It is a remarkable example of how much can be achieved with explicit methods. In particular, part e) of Theorem 1 should be thought of as a generalization of the result of Ritzenthaler and Lachaud. We would not have tried to prove part e) of Theorem 1 had we not been aware of their work.

6) To improve part f) of Theorem 1 to an exact equality it would suffice to give the first few terms of a Taylor expansion of \( \chi_{18} \) and Discr in the neighbourhood of a singular quartic.

1.0.5. The proof of Theorem 1 uses three ingredients: i) the fact that \( \chi_{18} \) exists as a section of line bundle, and that its locus is the hyperelliptic locus; ii) the fact that the Torelli morphism is locally a closed immersion; iii) the fact that the classical isomorphism of the line bundle of theta nulls with the line bundle of modular forms is compatible with the algebraic isomorphism of these line bundles.

We prove fact i) by using David Mumford’s theory of the theta group; this is essentially a descent argument dressed up in the guise of group theory. Fact ii) is due to Frans Oort and Joseph Steenbrink. Fact iii) is due to Laurent Moret-Bailley.

From fact ii) we deduce the existence of the \( \Delta_i \) as in part d) of Theorem 1. From fact i) we deduce that there is a canonical, up to multiplication by a unit in \( \mathbb{Z}[1/2]^* \), map identifying \( \chi_{18} \) with the \( \Delta_i \) on an open cover \( U_i \). From fact iii), we deduce that we may identify the \( \Delta_i \) with the analytic \( \chi_{18} \) modulo an explicit scaling factor.

1.0.6. This paper is structured in the following way: in the first three sections we give the necessary background on moduli spaces of Abelian varieties and curves. We also explain how to construct the effective divisor of Theorem 1 and prove parts a),b) and c) of Theorem 1. In section 4 we study the Torelli morphism and deduce the existence of the functions \( \Delta_i \) of Theorem 1 part d). In section 5 we relate the analytic \( \chi_{18} \) to the algebraic \( \Delta_i \). In section 6 we prove part f) of Theorem 1.

1.0.7. We thank Robert Carls, Everett Howe and Christophe Ritzenthaler for looking at and correcting many points in an earlier version of this paper. We also thank Laurent Moret-Bailly and Bas Edixhoven for helpful remarks regarding the effectiveness of relatively ample line bundles on Abelian schemes.

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Finally, for some time we doubted the validity of the result of Ritzenthaler and Lachaud mentioned in remark 5, primarily because we wrongly believed that \( \chi_{18} \) vanished with multiplicity 2 on the locus of hyperelliptic Jacobians. The author is very much indebted to the Ritzenthaler for the patience that he displayed in explaining his and Lachaud’s result to us. We would not have thought of the
normalization in part e) of Theorem 1 had it not been for the work of Ritzenthaler and Lachaud.

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2. Definitions and conventions

2.1. Basic notation. All rings are commutative with identity. With the exception of this section and section 2.5, all schemes $S$ and rings $R$ considered will be over $\mathbb{Z}[1/2]$. Thus any ring $R$ will necessarily contain $1/2$. We will write $k$ for a field not of characteristic 2 and we write $k^s$ for a separable closure of $k$.

Let $X$ and $T$ be $S$ schemes, we will write $X_T$ for the fibre product

$$X \times_S T.$$ 

A capital $A$ will usually refer to an Abelian variety or an Abelian scheme over some scheme $S$ which should be understood from the context. Likewise a capital $C$ will refer to a smooth curve of a fixed genus $g$, either over a field $k$ or a scheme $S$. Such families of Abelian varieties or curves will always be smooth and proper.

A curly letter $A$ or $M$ will always denote a moduli scheme. The letter $A$ is reserved for moduli of Abelian varieties and the letter $M$ for moduli of curves.

The Jacobian of a curve $C$ will be written as $\text{Jac}(C)$ and its theta divisor, when it exists, as $\Theta$. The principal polarisation induced from $\Theta$ will be written as $\lambda_\Theta$. 
The gothic letter $t$ will refer to the Torelli morphism.

We write $G_m$, $\mu_N$ and $\mathbb{Z}/N$ for the group schemes over $\text{Spec}(\mathbb{Z}[1/2])$ with Hopf algebras

$$\mathbb{Z}[1/2, X, X^{-1}],$$

$$\mathbb{Z}[1/2, X]/(X^N - 1),$$

$$\mathbb{Z}[1/2][\mathbb{Z}/N\mathbb{Z}]$$

respectively. Let $\zeta_N$ be a primitive $N$th root of unity and let $S = \text{Spec}(\mathbb{Z}[\frac{1}{N}, \zeta_N])$. For each $N$ we fix throughout this paper an isomorphism

$$\text{Exp}: (\mathbb{Z}/N) \to (\mu_N),$$

i.e., for each $N$ we choose a primitive $N$th root of unity.

2.2. Abelian schemes and families of curves. Let $S$ be a scheme. An Abelian scheme is a smooth proper morphism

$$\pi: A \to S$$

with a section

$$O: S \to A$$

so that a) $A$ is a group scheme and b) the geometric fibres of $\pi$ are connected projective varieties. This necessarily implies that the geometric fibres of $\pi$ are Abelian varieties and that the group law on $A$ is commutative.

A family of genus $g$ curves is a smooth proper morphism

$$f: C \to S$$

whose geometric fibres are connected genus $g$ curves.

2.3. The Picard scheme.

2.3.1. Let $\pi: X \to S$ be a morphism of schemes and let $T$ be a $S$ scheme. Consider the absolute Picard functor

$$\text{Pic}(X)(T) := \{\text{invertible sheaves } L \text{ over } X_T \}/ \cong_T .$$

Pullback via $\pi$ yields a natural transformation of group valued functors

$$\pi^*: \text{Pic}(S)(T) \to \text{Pic}(X)(T).$$

We define the relative Picard functor as

$$\text{Pic}(X/S)(T) := \text{Pic}(X)(T)/\pi^*\text{Pic}(S)(T).$$

Let $\text{Pic}_{\text{ét}}(X/S)$ denote the sheaf associated to $\text{Pic}(X/S)$ over the étale site.

If $\pi: X \to S$ is a flat morphism of schemes which is locally projective in the Zariski topology and whose geometric fibres are integral then Theorem 4.8 of Kleiman ([7]) tells us that the functor $\text{Pic}_{\text{ét}}(X/S)$ is representable. In particular this theorem applies to the case of $\pi: X \to S$ equal to either an Abelian scheme or a family of genus $g$ curves.
2.3.2. We will write \( \text{Pic}^\tau_{\text{\acute{e}t}}(X/S)(T) \) for the component of \( \text{Pic}(X/S) \) containing the identity (cf. \( [13] \) p23).

Now assume that \( \pi : X \to S \) is either an Abelian scheme or a family of genus \( g \) curves.

In case \( S \) is Noetherian Corollary 6.8 and Proposition 6.9 of Mumford (\( [13] \) p118) tell us that \( \text{Pic}^\tau_{\text{\acute{e}t}}(X/S) \) is an Abelian scheme and Zariski locally projective. We explain how to reduce to the Noetherian case in the following section.

2.3.3. We now explain how to reduce the facts from 2.3.3 about representability of the Picard functor to the Noetherian case. This is required as we define our moduli spaces for Abelian varieties and curves with respect to families of Abelian schemes over a non-Noetherian base.

If \( S \) is non-Noetherian we may reduce to the Noetherian case (see 2.3.3.) in the following way: First note, by the sheaf property, that it is necessary and sufficient to consider affine \( S \).

Then if \( S = \text{Spec}(R) \) and \( \pi : X \to S \) is Zariski locally projective, there is a finitely generated \( R[X_1, \ldots, X_n] \) ideal \( I \) so that

\[
X = \text{Proj}(R[X_1, \ldots, X_n]/I)
\]

Let \( r_1, \ldots, r_m \) be the coefficients of some finite set of generators for \( I \) as an \( R \) module.

Let \( R_{-1} \) be the image of \( \mathbb{Z}[1/2] \) in \( R \) and let \( R_0 = R_{-1}[r_1, \ldots, r_m] \). We note that \( R_0 \) is Noetherian. Let \( I_0 = R_0[X_1, \ldots, X_n] \cap I \). Now put

\[
X_0 = \text{Proj}(R_0[X_1, \ldots, X_n]/I_0).
\]

Then

\[
R[X_1, \ldots, X_n]/I = R \otimes (R_0[X_1, \ldots, X_n]/I_0).
\]

Thus

\[
\text{Pic}^\tau_{\text{\acute{e}t}}(X/S) = \text{Pic}^\tau_{\text{\acute{e}t}}(X_0/\text{Spec}(R_0))/S.
\]

2.3.4. When \( f : C \to S \) is a family of curves we will write \( \text{Jac}(C) \) or \( \text{Jac}(C/S) \) for \( \text{Pic}^\tau_{\text{\acute{e}t}}(C/S) \). When \( \pi : A \to S \) is an Abelian scheme we will write \( A^t \) for \( \text{Pic}^\tau_{\text{\acute{e}t}}(A/S) \).

2.3.5. The definition of \( \text{Pic}(X/S) \) given in section 2.3.1. makes life difficult in certain applications. When \( \pi : X \to S \) has a section \( s : S \to X \), this difficulty can be removed by choosing for each coset of \( \text{Pic}(X)(T)/T^\ast \text{Pic}(T)(T) \) a canonical representative.

To be precise, we define a functor

\[
P_{X/S}(T) := \{(L, \phi) \mid L \in \text{Pic}(X)(T) \text{ and } \phi : s^*T \to \mathcal{O}_T \}/ \cong
\]

where \( \phi \) is an isomorphism. An element of \( P_{X/S}(T) \) is called a line bundle, rigidified or normalized along the section \( s \).

We now explain how to define a natural isomorphism from \( \text{Pic}(X/S) \) to \( P_{X/S} \).

\footnote{This unusual definition is forced on us by our convention that all schemes are over \( \mathbb{Z}[1/2] \).}
Indeed given $L$ and its coset 

$$L \otimes \pi^* \text{Pic}(T),$$

consider the line bundle 

$$M(L) := L \otimes (\pi^* s_T^* L)^{-1}$$

then 

$$M(L) \in L \otimes \pi^* \text{Pic}(T)$$

and there is a canonical isomorphism 

$$\phi_L : \pi^* M(L) \longrightarrow \mathcal{O}_T.$$ 

Moreover, given any other element 

$$L' \in L \otimes \pi^* \text{Pic}(T)$$

we have an isomorphism 

$$\psi : M(L') \longrightarrow M(L)$$

so that 

$$\phi_{L'} = \phi_L \circ \pi^* \psi.$$ 

This yields a natural transformation 

$$\text{Pic}(X/S)(T) \rightarrow P_{X/S}(T).$$

This morphism is surjective since if $(L, \phi)$ is a pair as above then $M(L)$ is isomorphic to $L$ in a way which is compatible with $\phi$.

When $\pi : X \longrightarrow S$ is an Abelian scheme we will use this alternative definition of $\text{Pic}(X/S)$ with $O$ as the section $s$.

2.4. Isogenies, polarisations, the Weil pairing and level structures.

2.4.1. Let $A$ and $B$ be a Abelian schemes over $S$ of relative dimension $n$. An isogeny 

$$\lambda : A \longrightarrow B$$

is a morphism of group schemes over $S$ which is surjective. The kernel of $\lambda$ will be denoted as 

$$A[\lambda];$$

this is a finite group scheme over $S$. For a positive integer $N$ we will write 

$$[N] : A \longrightarrow A$$

for the isogeny which adds a point to itself $N$ times.

2.4.2. A polarisation of an Abelian scheme is an isogeny 

$$a : A \longrightarrow A^t$$

which has the following form at each geometric fibre $A_p$ of $A$: there exists an invertible sheaf $L_p$ on $A_p$ so that 

$$a(x) = t_x^* L_p \otimes L_p^{-1}.$$ 

We call a polarisation principal if it is an isomorphism.
2.4.3. It need not be the case that there is an invertible sheaf \( L \) over \( A \) so that \( p^* L \equiv L_p \mod \text{Pic}''(A_p/k') \) for each geometric point \( p \in S(k') \). However, given a polarisation \( a \) there does exist ([13], p121, Proposition 6.10) an \( L \) over \( A \) so that 
\[
\lambda(L) = 2a,
\]
where \( \lambda(L) \) is given by the obvious formula 
\[
\lambda(L)(x) := t^* x L \otimes L^{-1}.
\]
We are able to define \( \lambda(L) \) in this way because of our definition of Pic as in 2.3.5 (see [13] p120-121 Def 6.2 Prop 6.10 for another definition). In this case we write \( H(L) \) for the kernel of the polarisation \( \lambda(L) \).

2.4.4. The Cartier dual of a finite flat group scheme \( G \) is defined as the group functor \( S \mapsto \text{Hom}(G(S), G_m(S)) \); it is represented by a finite flat group scheme \( G^D \) (see [19]).

Given a principal polarisation \( a : A \rightarrow A' \) the canonical isomorphism between \( A'[N] \) and the Cartier dual of \( A[N] \) (see [19]) yields a pairing 
\[
e_N : A[N] \times A[N] \rightarrow \mu_N.
\]
This is called the Weil \( N \) pairing; it is symplectic and non-degenerate.

2.4.5. A symplectic level \( N \) structure on a principally polarised Abelian scheme \((A, a)\) is an isomorphism of group schemes 
\[
\phi : A[N] \rightarrow (\mathbb{Z}/N)^g_S \times (\mathbb{Z}/N)^g_S
\]
which takes the Weil \( N \) pairing on \( A[N] \) to the standard symplectic pairing on \((\mathbb{Z}/N)^g_S \times (\mathbb{Z}/N)^g_S\) given by the formula 
\[
((x, y), (u, v)) \mapsto \text{Exp}(x \cdot v - y \cdot u).
\]
We will abuse language and also use the phrase ‘level \( N \) structure’ to mean a symplectic level \( N \) structure.

A level \( N \) structure on a family of curves \( \pi : C \rightarrow S \) is a level \( N \) structure on its Jacobian variety.

2.5. Moduli schemes, modular forms, and the Torelli morphism. In this section we permit schemes \( S \) which are not defined over \( \mathbb{Z}[1/2] \). Let \( N \) be a positive integer \( \geq 3 \) and let \( \zeta_N \) be a primitive \( N \)th root of unity.

2.5.1. Consider the functor from the category of schemes over \( \mathbb{Z}[1/N, \zeta_N] \) to the category of sets given by
\[
S \mapsto \{(C, \alpha) \mid C \text{ is a family of genus } g \text{ curves over } S \text{ and } \alpha \text{ is a symplectic level } N \text{ structure on the Jacobian of } C\}/\equiv_S.
\]
Here two families of curves \( f : C \rightarrow S \) and \( h : C' \rightarrow S \) considered isomorphic if there is an isomorphism \( \phi : C \rightarrow C' \) such that \( h \circ \phi = f \).

It turns out, since \( N \geq 3 \), that this functor is representable by a smooth scheme \( \mathcal{M}_{g,N} \) over \( \mathbb{Z}[1/N, \zeta_N] \).
Likewise for $N \geq 3$ the functor

$$S \mapsto \{(A, a, \alpha) \mid A \text{ an Abelian scheme of relative dimension } g \text{ over } S$$

with a principal polarisation $a$ and $\alpha$ is a symplectic level $N$ on $A\}/\sim_S,$

is represented by a smooth scheme $A_{g,1,N}$ over $\mathbb{Z} \left[ \frac{1}{N} \right], \zeta_N].$

2.5.2. One consequence of the representability of the above functors is the existence of universal families of curves and Abelian varieties.

Thus there is a family of genus $g$ curves

$$\pi : C \rightarrow \mathcal{M}_{g,N}$$

with a level $N$ structure

$$\alpha : \text{Jac}(C)[N] \rightarrow (\mathbb{Z}/N)^g_{\mathcal{M}_{g,N}} \times (\mathbb{Z}/N)^g_{\mathcal{M}_{g,N}}.$$  

Moreover given any family of genus $g$ curves

$$f : D \rightarrow S$$

with a level $N$ structure

$$\beta : \text{Jac}(D)[N] \rightarrow (\mathbb{Z}/N)^g_{\mathcal{M}_{g,N}} \times (\mathbb{Z}/N)^g_{\mathcal{M}_{g,N}},$$

there is a unique morphism $h : S \rightarrow \mathcal{M}_{g,N}$ so that $f = h^* \pi$ and $\beta = h^* \alpha$ up to canonical isomorphism. We call the pair $(\pi : C \rightarrow \mathcal{M}_{g,N}, \alpha)$ the universal genus $g$ curve with level $N$ structure.

Similarly, there is a universal principally polarised Abelian scheme of relative dimension $g$ with a level $N$ structure.

2.5.3. Let $C/S$ be a family of genus $g$ curves. The Jacobian has a canonical polarisation (see Prop 6.9 [13] p118)

$$\lambda_\Theta : \text{Jac}(C) \rightarrow \text{Jac}(C)^t.$$  

Moreover if $C/S$ has a section $s : S \rightarrow C$ then $\lambda_\Theta$ is obtained by applying the Picard functor to the morphism

$$j : C(T) \rightarrow \text{Jac}(C)(T) : t \mapsto t - s.$$  

This follows from the proof of Proposition 6.9 in [13].

2.5.4. For $N \geq 3$, we therefore have a morphism of fine moduli schemes

$$t : \mathcal{M}_{g,N} \rightarrow \mathcal{A}_{g,1,N} : C \mapsto (\text{Jac}(C), \lambda_\Theta),$$

which we call the Torelli morphism; it is a proper morphism of degree 2 and is ramified along the hyperelliptic locus. Oort and Steenbrink have show that it is a locally closed immersion (see [13]).

The moduli spaces $\mathcal{M}_{g,N}$ and $\mathcal{A}_{g,N}$ are irreducible. Moreover the locus of decomposably polarised Abelian schemes is closed. Then since the Torelli morphism is proper, for $g = 2$ and 3, we have the Torelli theorem:

Let $(A, a)$ be an indecomposably principally polarised Abelian variety of dimension $g = 2$ or 3 over an algebraically closed field $k$. Then there is a smooth projective irreducible genus $g$ curve $C$ over $k$ such that

$$(\text{Jac}(C), \lambda_\Theta) \cong (A, a).$$
Our Theorem 1 is really just a refined version of the Torelli theorem for genus 3. There is a more precise version of this theorem for non-algebraically closed fields and even for Abelian schemes over a noetherian base (see [21] and [22]).

2.5.5. Let \( \pi : A \to S \) be an Abelian scheme and let \( \Omega_{A/S} \) be the sheaf of relative Kähler differentials on \( A \).

We write

\[
\omega_{A/S} := \det(\pi_! \Omega_{A/S}),
\]
for the determinant of the Hodge bundle \( \pi_! \Omega_{A/S} \).

A Siegel modular form of weight \( \rho \) is defined to be a global section of \( \omega_{A/S}^\rho \).

2.5.6. The definition of 2.5.5 is not “functorial” in that it is defined relative to a fixed Abelian scheme \( A/S \). To remedy this Katz has suggested the following definition of a modular form\(^3\).

Let \( g \) and \( N \) be fixed positive integers, and let \( S \) be a given base scheme. A Katz-Siegel modular form of weight \( \rho \) and level \( N \), relative to \( S \), consists of the following data:

(D) For every \( S \) scheme \( T \) and for every principally polarised Abelian scheme \( \pi : A \to T \) of relative dimension \( g \) with a level \( N \) structure \( \alpha \), there is a global section \( s_{A/T} \) of \( \omega_{A/T}^\rho \) subject to the condition:

(C) Given a principally polarized Abelian scheme \( A'/T' \) with a level \( N \) structure \( \alpha' \), a morphism \( f : T' \to T \) and an isomorphism \( h : A' \to A_{T'} \), compatible with the level structures \( \alpha_{T'} \) and \( \alpha' \), we have

\[
det(dh)^\rho(f^* s_{A/T}) = s_{A'/T'}.\]

We note that if \( N = 1 \) then the condition on level structures is empty.

Thus to give a Katz-Siegel modular form is the same as giving a compatible system of modular forms indexed by Abelian schemes.

This definition can be made completely in terms of stacks, but we leave this as an exercise for the interested reader. (Hint: the first step is to define a category \( \omega \) fibred over the stack \( \mathcal{A}_{g,1} \) which plays the role of the bundle \( \omega_{A/T} \); then consider what a section from \( \mathcal{A}_{g,1} \) to \( \omega \) is.)

2.5.7. Let \( f : C \to S \) be a family of genus \( g \) curves. Let \( \Omega_{C/S} \) be the sheaf of relative Kähler differentials. A Teichmüller modular form of weight \( \rho \) is a section of the bundle

\[
\omega_{C/S}^\rho := \det(f_! \Omega_{C/S})^\rho.
\]

A Katz-Teichmüller modular form \( t_{C/S} \) is defined in exact analogy with the definition of a Katz-Siegel modular form.

2.5.8. Let \( f : C \to S \) be as in 2.5.7, and consider the Jacobian \( \text{Jac}(C)/S \) and its sheaf of relative Kähler differentials \( \Omega_{\text{Jac}(C)/S} \). The Kodaira-Spencer class, gives a canonical isomorphism between

\(^3\)At least in the case of an Abelian scheme of relative dimension 1 ([15]). Our definition excludes twisting by a character, since we do not need the general definition.
and the bundle
\[ \omega_{\text{Jac}(C)/S} \]
defined in section 2.5.5.

Thus for a family of Jacobians, a Siegel modular form defines a Teichmüller modular form and vice versa.

2.5.9. We say that a Katz-Teichmüller modular form \( t_{C/S} \) of genus \( g \) descends to a Katz-Siegel modular form \( s_{A/S} \) of dimension \( g \) if for every family of genus \( g \) curves \( C/S \) we have

\[ t_{C/S} = s_{\text{Jac}(C)/S} \]

using the identification of \( \omega_{C/S} \) and \( \omega_{\text{Jac}(C)/S} \) explained in section 2.5.8.

3. Mumford’s theta group, existence of a principal line bundle and the definition of \( \chi_{18} \)

3.1. Mumford’s theta group as a descent gadget.

3.1.1. Let \( A \) and \( B \) be Abelian schemes over \( S \), let \( \lambda: A \to B \) be an isogeny and let \( L \) be a line bundle over \( A \).

We are interested to know when there exists a line bundle \( M \) over \( B \) so that \( \lambda^*M \cong L \). When such an \( M \) exists we say that \( L \) descends.

Consider the cartesian product

\[
\begin{array}{ccc}
A \times B & \xrightarrow{p_2} & A \\
\downarrow{p_1} & & \downarrow{\lambda} \\
A & \xrightarrow{\lambda} & B.
\end{array}
\]

We will write \( p_{12}, p_{13} \) and \( p_{23} \) for the three distinct projections

\[ p_{13}: A \times_B A \times_B A \to A \times_B A. \]

The following conditions turn out to be necessary for \( L \) to descend

1) there is an isomorphism \( \phi: p_1^*L \to p_2^*L \),
2) up to natural isomorphism we have, \( p_{13}^*\phi = p_{23}^*\phi \circ p_{12}^*\phi \).

We call a pair \( (L, \phi: p_1^*L \to p_2^*L) \), where \( \phi \) is an isomorphism satisfying condition 2, a line bundle with descent datum with respect to \( \lambda \). Line bundles with descent datum form a category in an obvious way. Pullback with via \( \lambda \) gives a functor from the ‘category of line bundles over \( B \)’ to ‘line bundles over \( A \) with descent datum with respect to \( \lambda \).’

A special case of Grothendieck’s descent theory states (II p134 Theorem 4) that the functor \( \lambda^* \) induces an equivalence of categories (in fact this holds for any faithfully flat quasi compact morphism \( \lambda \)). In other words conditions 1) and 2) are necessary and sufficient for \( L \) to descend.
3.1.2. Let \( K \) be the kernel of \( \lambda \). We have isomorphism

\[ i : A \times K \longrightarrow A \times_B A : (x, y) \mapsto (x, x + y). \]

We define two morphisms \( m : A \times K \longrightarrow A \) and \( p : A \times K \longrightarrow A \) by

\[ m(x, y) = x + y \]

and

\[ p(x, y) = x. \]

Then \( i \) gives an isomorphism of \((A \times K, p, m)\) with the fibre product \((A \times_B A, p_1, p_2)\) of \((\lambda : A \longrightarrow B, \lambda : A \longrightarrow B)\).

Similarly the isomorphisms \( i \) and

\[ j : A \times K \times K \longrightarrow A \times_B A \times_B A : (x, y, z) \mapsto (x, x + y, x + y + z), \]

give an isomorphism of \((A \times K \times K, i^{-1} \circ p_{12} \circ j(a, b, c), i^{-1} \circ p_{13} \circ j(a, b, c), i^{-1} \circ p_{23} \circ j(a, b, c))\) with \((A \times_B A \times_B A, p_{12}, p_{13}, p_{23})\).

In other words \( i \) and \( j \) identify \( p_{12}, p_{13} \) and \( p_{23} \) with following morphisms

\[ i^{-1} \circ p_{12} \circ j(a, b, c) = (a, b), \]

\[ i^{-1} \circ p_{13} \circ j(a, b, c) = (a, b + c), \]

\[ i^{-1} \circ p_{23} \circ j(a, b, c) = (a + b, c). \]

The conditions for \( L \) to descend via \( \lambda \) can then be restated as:

1) there is an isomorphism \( \phi : p^* L \longrightarrow m^* L \),

2) up to natural isomorphism we have, \((i^{-1} \circ p_{13} \circ j)^* \phi = (i^{-1} \circ p_{23} \circ j) \phi \circ (i^{-1} \circ p_{12} \circ j)^* \phi; \)

or equivalently, for every \( S \) scheme \( T \) the group scheme \( K \) acts on \( L \), i.e. we have

1) for each \( b \in K(T) \) there is an isomorphism \( \phi_b : L \longrightarrow t_b^* L \),

2) up to natural isomorphism for each \( b, c \in K(T) \) we have \( \phi_{c+b} = (t_b^* \phi_b) \circ \phi_c \).

Thus \( L \) descends to \( B \) if there is an action of \( K \) on \( L \). The Mumford theta group \( G(L) \), which we define in 3.1.3., gives a way of encoding all possible actions of such a \( K \) on \( L \).

3.1.3. Now assume that \( L \) is relatively ample. Consider the isogeny

\[ \lambda(L)(x) := t_x^* L \otimes L^{-1}, \]

where \( x \in A(T) \) for some \( S \) scheme \( T \). Let \( H(L) \) be the kernel of \( \lambda(L) \). If \( x \in H(L)(T) \) then there is an isomorphism

\[ \phi : L_T \longrightarrow t_x^* L_T. \]

We write \( G(L)(T) \) for the set of all pairs \((x, \phi : L_T \longrightarrow t_x^* L_T)\) with \( x \in H(L)(T) \) and \( \phi \) an isomorphism as above. Then \( G(L)(T) \) is endowed with the structure of a group

\[ (x, \phi) \cdot (y, \psi) := (x + y, (t_y^* \phi) \circ \psi). \]

There is a natural projection

\[ G(L)(T) \rightarrow H(L)(T) \]
and the kernel of this map is naturally identified with $\mathbb{G}_m(T)$, which lies in the centre of $G(L)(T)$. Hence we have a short exact sequence

\[ 1 \longrightarrow \mathbb{G}_m(T) \longrightarrow G(L)(T) \longrightarrow H(L)(T) \longrightarrow 1. \]

The functor $T \mapsto G(L)(T)$ is represented by the scheme $\mathbb{G}_m \times H(L)$.

However the group structure of $G(L)(T)$ is not that of the product $\mathbb{G}_m(T) \times H(L)(T)$. Thus as a group scheme $G(L)$ is a non-trivial extension of $H(L)$ by $\mathbb{G}_m$.

We call $G(L)$ the Mumford theta group of $L$ (cf. [14]).

3.1.4. Given a subgroup scheme $K \subset H(L)$ an action of $K$ on $L$ is equivalent to giving a subgroup scheme of $G(L)$ which is isomorphic to $K$ under the canonical projection from $G(L)$ to $H(L)$.

Thus Grothendieck’s descent theorem tells us: for a fixed subgroup scheme $K \subset H(L)$, there is a bijection between lifts $\tilde{K}$ of $K$ and equivalence classes of pairs $(M, \phi)$ where $M$ is a line bundle over $A/S$ and $\phi : L \longrightarrow \pi^*M$ is an isomorphism.

3.1.5. The commutator pairing

\[ e_L(T) : H(L)(T) \times H(L)(T) \longrightarrow \mathbb{G}_m(T) \]

is defined in the following way: let $x, y \in H(L)(T)$ and let $g_x$ and $g_y$ be lifts of $x$ and $y$ respectively in $G(L)(T)$. We then define

\[ e_L(T)(x, y) := g_x g_y g_x^{-1} g_y^{-1}, \]

which is independent of the choice of lifts $g_x$ and $g_y$ as $\mathbb{G}_m(T)$ is in the centre of $G(L)(T)$.

From the definition of multiplication on $G(L)(T)$ we see that this pairing is equal to the Weil $N$ pairing when $H(L) = A[N]$ (e.g. if $N = m^2$ and $L = [m]^*M$ for some principal and ample line bundle $M$). Moreover we note that for a subgroup scheme $K$ of $H(L)$ to lift to $G(L)$ as in section 3.1.4 it is necessary that $e_L(T)$ vanishes on $K(T)$ for all $S$ schemes $T$.

3.2. Existence of an effective symmetric polar divisor for level 4.

3.2.1. Let $\pi : A \longrightarrow S$ be an Abelian scheme with a principal polarisation

\[ a : A \longrightarrow A^t, \]

and a level 4 structure (see section 2.4.5 for the definition)

\[ \phi : A[4] \longrightarrow (\mathbb{Z}/4\mathbb{Z})^g_S \times (\mathbb{Z}/4\mathbb{Z})^g_S. \]

Following 2.4.3 there is an invertible sheaf $L$ over $A$ so that

\[ \lambda(L) = 2a. \]
The invertible sheaf $M := L \otimes [-1]^*L$ satisfies $[-1]^*M \cong M$ and

$$\lambda(M) = 4a.$$ 

Thus $H(M) = A[4]$ and the Mumford theta group of $G(M)$ fits into an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G(M) \longrightarrow A[4] \longrightarrow 0.$$ 

We first show that a lift of $A[2] \subset A[4]$ to $G(M)$ exists. As we will see, this is possible as $M = L \otimes [-1]^*L$.

3.2.2. To lift $A[2]$ to $G(M)$ we proceed as follows. Let $x_1, \ldots, x_{2g}$ be generators of $A[2](S)$. For each $x_i$ we want to find a lift $g_i \in G(M)(S)$ so that

$$g_i^2 = 1 \text{ and } g_ig_j = g_jg_i.$$ 

Then the subgroup of $G(M)$ generated by the $g_i$ is a lift of $A[2]$. The commutativity condition is satisfied for any lifts $g_i$ and $g_j$ of $x_i$ and $x_j$ as

$$g_ig_jg_i^{-1}g_j^{-1} = e_4(x_i, x_j) = (e_4(y_i, y_j))^4 = 1$$

where $2y_i = x_i$ and $2y_j = x_j$ (see 3.1.5). In order to obtain lift $g_i$ of $x_i$ so that $g_i^2 = 1$ we construct a lift $h_i$ so that $h_i^2 = \lambda_i^2$ with $\lambda_i \in \mathbb{G}_m(S)$. We then put $g_i = \frac{h_i}{\lambda_i}$.

We construct such an $h_i$ in the following way. Since $\ker(\lambda(L)) = A[2]$ we have $H(L) = A[2]$. Therefore we may lift $x_i$ to $\gamma_i \in G(L)$. Note that $\gamma_i$ has the form

$$(x_i, \phi_i)$$

for some isomorphism $\phi_i : L \longrightarrow t_{x_i}^*L$. Now define an element $\gamma_i'$ of $G([-1]^*L)$ by the formula

$$(x_i, [-1]^*\phi_i).$$

Consider the following functors from the category of line bundles on $A$ to itself:

$$F(L) := t_{x_i}^*(L \otimes [-1]^*L)$$

and

$$G(L) := (t_{x_i}^*L) \otimes (t_{x_i}^*[-1]^*L).$$

Then $F$ and $G$ are naturally isomorphic, and we will write $\xi_L$ for a natural isomorphism from $F$ to $G$. We also note that

$$t_{x_i}^*F(\phi) = \phi \otimes [-1]^*\phi \text{ and } G(\phi) = t_{x_i}^*\phi \otimes t_{x_i}^*[-1]^*\phi.$$ 

Then $\gamma_i^2 = \gamma_i'^2 = \lambda_i$ for some $\lambda_i \in \mathbb{G}_m(S)$ and

$$h_i = (x_i, \xi_L^{-1} \circ t_{x_i}^*F(\phi_i))$$

is a lift of $x_i$ to $G(M)(S)$ such that $h_i^2 = \lambda_i^2$.

3.2.3. Let $L$ be the line bundle on $A/A[2]$ corresponding to the lift described in section 3.2.2, so that $[2]^*L \cong M$.

Consider the following element of $A^t(S)$:

$$D := [-1]^*L \otimes L^{-1}.$$ 

Then $D^{\otimes 2} = [2]^*D \cong [-1]^*M \otimes M^{-1} \cong \mathcal{O}_A$. Let $D' \in A^t[4](S)$ be such that

$$2D' = D,$$
and put
\[ L' = L \otimes D'. \]
Note that \( L \) and \( L' \) yield the same polarization on \( A \) as \( D' \in \text{Pic}^0(A/S)(S) \).
Now \([-1]^*L' \cong L'\) and \( H(L') = 0 \). Thus \( L' \) yields a principal polarisation of \( A \) and \( L' \) is symmetric in the sense that \([-1]^*L' \cong L'\).

The sheaf \( \pi_*L' \) may have no global section; and indeed if \( L' \) is rigidified along the zero section \( O \), it follows from Theorem 5.1 of [2] that \( \pi_*L' \) is anti-ample. So in this case \( \pi_*L' \) can not have a global section.

However, we can arrange for a twist of \( L' \) to have a global section. Put
\[ L := L' \otimes \pi^*\pi_*L'^{-1}. \]
Then we have a canonical isomorphism \( \psi: \pi_*L \to O_S \) thus \( L \) is effective and has a canonical global section \( s := \psi^{-1}(1) \). Let \( \Theta \) be the divisor of \( s \); then
\[ L \cong O_A(\Theta). \]
Thus \( \Theta \) is an ample relative Cartier divisor which is principal and symmetric.

We have therefore proven

**Lemma 2.** Let \((A/S,a)\) be a principally polarised Abelian variety with a level 4 structure. Then \( A/S \) has a symmetric effective polar divisor \( \Theta \) and \( \Theta \) is unique up to translation by a two torsion point.

3.3. **Symmetric polar divisors, theta nulls and modular forms.** Let \((A/S,a)\) be a principally polarised Abelian scheme of relative dimension \( g \) with a level 4 structure and let \( \Theta \) be an ample effective symmetric polar relative Cartier divisor on \( A/S \) such that \( \lambda(O_A(\Theta)) = a \). Such a \( \Theta \) exists by Lemma 2 and is unique up to translation by a two torsion point. We fix \( \Theta \) throughout the sequel, but we will see that our choice does not matter, except that it might introduce a power of \( 2^{36} \) in our definition of \( \chi_{18} \) - which is what Theorem 1 states.

3.3.1. Following Moret-Bailly ([12]) we define the sheaf of the zeroth theta null
\[ \Theta_g := O^*O_A(\Theta). \]
Now we set
\[ M = O_A(\Theta) \otimes \pi^*O^*O_A(\Theta)^{-1}. \]
Then Theorem 5.1 of [2] yields
\[ \pi_*M^8 \cong \det(\pi_*\Omega_{A/S})^{-4}. \]
Moreover
\[ \pi_*M \cong (O^*O_A(\Theta))^{-1}. \]
Thus
\[ \Theta_g^8 \cong \det(\pi_*\Omega_{A/S})^4. \]
3.4. Even two torsion points.

3.4.1. Let \( p \in A[2](S) \) be a two torsion point; let \( x \) be some geometric point of \( A \) in the image of \( p \). Let \( I = \mathcal{O}_A(-\Theta) \) be the sheaf of ideals corresponding to the symmetric effective line bundle \( \mathcal{O}_A(\Theta) \).

Let \( \mathcal{O}_{A,x} \) be the local ring of \( A \) at \( x \) and let \( m_x \) be the maximal ideal of \( \mathcal{O}_{A,x} \) and let \( I_x \) be the stalk of \( I \) at \( x \). We say that \( p \) is even if

\[
\max\{n \in \mathbb{N} \mid I_x \subset m_x^n\}
\]

is even for every such \( x \). In words this says that a section \( p \) is even if and only if it intersects the divisor of each geometric fibre at a point of even multiplicity of the divisor, and this agrees with the classical notion.

3.4.2. It is not a priori clear from the definition in 3.3.2 that even \( p \) exist. We therefore offer an alternative characterization which proves their existence.

Let \( t_p : A \to A \) be the translation by \( p \) morphism. The line bundle \( t_p^*\mathcal{O}_A(\Theta) \) is effective with a unique non-zero global section \( s \) (up to multiplication by \( \mathcal{O}_S^*(S) \)). Moreover we have an isomorphism

\[
t_p^*L \cong [-1]^*t_p^*L
\]

which induces an involution \( \epsilon_p \in \mu_2(S) \) on \( p^*\mathcal{O}_A(\Theta) \) and therefore on \( t_p^*I \).

We claim that \( p \) is even if and only if \( \epsilon_p = 1 \) on each connected component of \( S \).

Indeed, let \( x \) be a geometric point of \( A \) in the image of \( p \) and let \( f \) be an element of \( \mathcal{O}_{A,x} \) which generates \( (t_p^*I)_x \). Without loss we may assume that \( S \) is connected.

Then

\[
\epsilon_p f = f \circ [-1].
\]

Thus \( f \in m_x^n \) with \( n \) even if and only if \( \epsilon_p = 1 \).

The existence of even \( p \) then follows from the fact that each geometric fibre of \( A/S \) has even two torsion points (see [14] where \( \epsilon_p \) is called \( e_* \): p304 Property iv) of \( e_* \), and p307 Proposition 2), and such points must be the pullback of a global two torsion point if \( A/S \) has a level 2 structure.

3.4.3. Let \( \Theta_{\pi(x)} \) be the divisor of \( A_{\pi(x)} \) given by restricting \( \Theta \) to \( A_{\pi(x)} \). Then \( p \) is even (cf. 3.3.2) precisely if \( \Theta_{\pi(x)} \) contains \( x \) with even multiplicity.

If the relative dimension \( g \) of \( A/S \) is three then for generic \( \pi(x) \) we may identify \( \Theta_{\pi(x)} \) with the theta divisor of the Jacobian \( \text{Jac}(C) \) of a smooth genus three curve \( C \) over the residue field of \( \pi(x) \).

Let \( u \) be a half canonical divisor of \( C \), i.e. \( 2u = K_C \); such a \( u \) is called a theta characteristic. By the theorem of Riemann-Kempf ([6]) the multiplicity of vanishing at \( x \) is equal to the dimension of \( h^0(x + u) \). By Mumford’s results on theta characteristics ([17]) we know there are exactly \( 2^{g-1}(2^g + 1) = 36 \) such \( x \).

3.4.4. We keep the notation of the previous paragraph. The function
is quadratic for the Weil 2 pairing and has Arf invariant 1. Choosing coordinates for $A_x[1](k(x))$, we therefore compute that

$$\sum_{\text{even } x} x = O.$$ 

Thus by the theorem of the cube ([2] p1) we have a canonical isomorphism

$$\otimes_{\text{even } p} p^* \mathcal{O}_A(\Theta) \cong \mathcal{O}_A(\Theta)^{36}. $$

Since we work under the hypothesis that $g = 3$ and $\pi : A \to S$ has a level 4 structure it is possible to improve upon Theorem 5.1 and prove that

$$\text{TH}_3^{36} \cong \det(\pi_* \Omega_{A/S})^{18}. $$

It suffices to prove this over $\mathbb{C}$ and this follows the fact that over $\mathbb{A}_3$ the bundle $\text{TH}_3^{36}$ has a global section $\chi^{an}_{18}$ which transforms as a modular form (cf. [4]). And therefore

$$\det(\pi_* \Omega_{A/S})^{18} \cong \text{TH}_3^{36} \cong O^*(L^{36}) \cong O^*(\otimes_{\text{even } p} p^* L) \cong \otimes_{\text{even } p} p^* L.$$ 

3.5. Definition of $\chi_{18}$, descent to level 1 and vanishing on the hyperelliptic locus.

3.5.1. Let $\pi : A \to A_{3,1,4}$ be the universal Abelian scheme with a level 4 structure and a principal polarisation $a$. Let $\Theta$ be an effective symmetric line bundle as in Lemma 2.

The Satake embedding embeds $A_{3,1,4}$ into projective space, and the projective closure of this image has a boundary of codimension 2. Therefore the global sections of $A_{3,1,4}$ are

$$\mathbb{Z}[1/2, \sqrt{-1}].$$

Let $s_\Theta$ be a global section of $\mathcal{O}_A(\Theta)$. The section $s$ is unique up to multiplication by a unit of $\mathbb{Z}[1/2, \sqrt{-1}]$. At each geometric fibre $A_x$ of $A/S$ the section $s$ cuts out a symmetric polar divisor $\Theta_x$ (note $x$ is now a geometric point of $S$ and not $A$ in contrast to section 3.3) of $A_x$. Such polar divisors $\Theta_x$ are unique up to translation by a two torsion point.

We therefore define

$$\chi_{18} = \prod_{\text{even } p \in A[2](S)} p^* s_\Theta.$$ 

It is a global section of $\det(\pi_* \Omega_{A/S})^{18} \cong \text{TH}_3^{36}$ and thus a level 4 Katz-Siegel modular form of weight 18. It vanishes with multiplicity 1 on the locus of hyperelliptic Jacobians, it also vanishes on the locus of Abelian three folds with a decomposable polarisation (see 3.5.3).

It is natural to ask how $\chi_{18}$ depends on the choice of $\Theta$ and $s_\Theta$. For fixed $\Theta$, we see that $\chi_{18}$ is unique up to multiplication by an element of $\mathbb{Z}[1/2]^{*36}$. 

$$x \mapsto \begin{cases} -1 & \text{if } h^0(x + u) \text{ is not even,} \\ 1 & \text{if } h^0(x + u) \text{ is even.} \end{cases}$$
It is in fact independent of our choice of $\Theta$. Given another choice $\Theta'$, satisfying the hypothesis of Lemma 2, there is a unique two torsion point $p$ so that

$$p^*\Theta = \Theta'.$$

In this case there is a section $s'_{\Theta'}$, such that

$$t^*_p s_{\Theta} = s'_{\Theta'}.$$

3.5.2. Let $\pi: A \to S$ be any principally polarised Abelian scheme over a base $S$ in which 2 is invertible. Without loss, we assume that $S$ is connected. There exists a Galois morphism $f: S' \to S$ so that $A_{S'/S'}$ has a level 4 structure and so that the covering group is $\text{PSp}_6(\mathbb{Z}/4)$ (for example take $S' = A[4]_S$). Using the universal property of $A_{3,1,4}$, this yields a weight 18 modular form $\chi_{18}$ on $A_{S'/S'}$. We now show that $\chi_{18}$ descends to $A/S$.

The group $\text{PSp}_6(\mathbb{Z}/4)$ is a semi-direct product of $\text{PSp}_6(\mathbb{Z}/2)$ by an Abelian group $H$ of exponent 2. Now $H$ acts trivially on two torsion points, and thus $\chi_{18}$ descends to a level 2 Katz-Siegel modular form. Moreover, $\text{PSp}_6(\mathbb{Z}/2)$ is simple and non-Abelian, and therefore has no non-trivial 1 dimensional representations. Thus $\chi_{18}$ descends as a level 2 modular form to a level 1 Katz-Siegel modular form.

In this way we obtain $\chi_{18}$ as a Katz-Siegel modular form of weight 18 and level 1 over $\mathbb{Z}[1/2]$.

3.5.3. We now show that $\chi_{18}$ vanishes on the hyperelliptic locus with multiplicity 1.

Let $A/S$ be as in the preceding paragraph. Let $(A_x, a_x)$ be a fibre of a closed point $x \in S$ with indecomposable polarization $a_x$. Let $k(x)$ be the residue field of $x$ and let $\Theta_x$ be a symmetric polar divisor of $a_x$ over the separable closure $k(x)^s$ of $k(x)$.

We may assume that there is a curve $C$ over $k(x)$ such that $(\text{Jac}(C), \lambda) \otimes k(x)^s \cong (A_x, a_x) \otimes k(x)^s$ and that $\Theta_x$ is a theta divisor of $\text{Jac}(C)$.

Then $\Theta_x$ contains an even $p$ if and only if $C$ is hyperelliptic and in this case $p$ is unique. This is because $\Theta_x = \{a + b - u \in \text{Jac}(C)(k^s) \mid 2u = K_C \text{ and } h^0(u) \text{ even, and so we have that } h^0(p + u) \text{ is even and } p + u = a + b \text{ for some } a, b \in C(k^s)\}$. Hence there is a morphism $f: C \to \mathbb{P}^1$ with polar divisor $a + b$. Thus $f$ is 2 : 1 from $C$ to $\mathbb{P}^1$. Such an $f$ is unique up to an automorphism of $\mathbb{P}^1$ and hence $p$ is also unique. Moreover, it is easy to show that the multiplicity of such an $p$ must be exactly 2, i.e. $p$ is even.

Thus $\chi_{18}$ vanishes with multiplicity 1 on the locus of hyperelliptic Jacobians; it seems it also vanishes on the locus of decomposable principally polarized Abelian varieties, but we will not need this.

4. Existence and properties of the $\Delta_i$.

4.1. Properties of the Torelli morphism.

4.1.1. Let $N$ be an integer $\geq 3$ and let $\mathcal{M}_{g,N}$ be the fine moduli space of smooth genus $g$ curves with a level $N$ structure. Let $\mathcal{A}_{g,1,N}$ be the fine moduli space of principally polarised Abelian varieties of dimension $g$ with level $N$ structure.
Let \( t : \mathcal{M}_{g,N} \rightarrow A_{g,1,N} \) be the Torelli morphism.

A point \( x \in \mathcal{M}_{g,N}(S) \) corresponds to a pair \((C,\beta)\) where \( C \) is a smooth proper family of genus three curves over \( S \) and \( \beta \) is a level \( N \) structure for its Jacobian variety.

If \( C \) is non-hyperelliptic (i.e. a dense set of its fibres are non-hyperelliptic) then the automorphism scheme of \( \text{Jac}(C)/S \) is isomorphic with \( \text{Aut}(C/S) \times \langle [-1] \rangle \) (see [22] and [21]). Otherwise, if \( C \) is hyperelliptic (i.e. all its geometric fibres are hyperelliptic curves), we have \( \text{Aut}(C/S) \cong \text{Aut}(\text{Jac}(C)/S) \) and the hyperelliptic involution is taken to the automorphism \([-1]\).

Thus unless \( C \) is hyperelliptic, there is no automorphism of \( C \) which takes \( \beta \) to \(-\beta\). We therefore have a non-trivial involution

\[
\tau : \mathcal{M}_{g,N} \rightarrow \mathcal{M}_{g,N}
\]
given on \( S \) valued points by the formula

\[
\tau(C,\beta) = (C, -\beta).
\]

Moreover, the fixed locus of \( \tau \) is precisely the hyperelliptic locus.

Let \( V_{g,N} \) be the geometric quotient of \( \mathcal{M}_{g,N} \) under \( \tau \) and let

\[
i : \mathcal{M}_{g,N} \rightarrow V_{g,N}
\]
be the quotient morphism. Since the \([-1]\) automorphism of the Jacobian takes \( \beta \) to \(-\beta\) we see that the Torelli morphism factors through \( i \). We therefore have a morphism

\[
i : V_{g,N} \rightarrow A_{g,1,N};
\]
and the image of \( i \) obviously avoids the locus of Abelian \( g \) folds with a decomposable polarisation.

Oort and Steenbrink ([18]) have shown that \( i \) is a locally closed immersion and that it is \( 2 : 1 \) onto its image. We will therefore identify \( V_{g,N} \) with a locally closed subscheme of \( A_{g,1,N} \).

4.1.2. The moduli spaces \( \mathcal{M}_{g,N} \) and \( A_{g,1,N} \) are known to be smooth, which means they are locally regular.

Let \( \{U_{N,i}\} \) be an affine cover of \( \mathcal{M}_{g,N} \) so that \( U_{N,i} = \text{Spec}(R_{N,i}) \) with \( R_{N,i} \) regular. Assume further that \( \tau(U_{N,i}) = U_{N,i} \).

Let \( R^\tau_{N,i} \) be the ring of invariants of \( R_{N,i} \) under the action of the group \( \langle \tau \rangle \).

Since \( R_{N,i} \) is locally regular, one may show that \( R_{N,i} \) is free of rank 2 as an \( R^\tau_{N,i} \) module. This implies\(^4\) that there is a function \( \Delta_{N,i} \in R^\tau_{N,i} \) so that

---

\(^4\)This definition is equivalent to \( C \) having an involution whose quotient is a family of \( \mathbb{P}^1 \)'s over \( S \) (see [10]).

\(^5\)Here the author is indebted to Marius van der Put for pointing out to him that such a \( \Delta_{N,i} \) existed and that \( \Delta_{N,i} \) and \( \Delta^2_{N,i} \) vanished with the stated multiplicities on the hyperelliptic locus of \( \mathcal{M}_{g,N} \) and \( V_{g,N} \) respectively.
\[ R_{N,i} = R_{N,i}^{\tau}[\sqrt{\Delta_{N,i}}] \]

and moreover \( \tau(\sqrt{\Delta_{N,i}}) = -\sqrt{\Delta_{N,i}}. \) This last condition implies that \( \sqrt{\Delta_{N,i}} \) vanishes with multiplicity 1 on the fixed locus of \( \tau. \) Hence \( \Delta_{N,i} \in R_{N,i}^{\tau} \) vanishes with multiplicity 1 on the image of the fixed locus of \( \tau \) in \( V_{g,N}. \)

But the fixed locus of \( \tau \) is precisely the hyperelliptic locus. Thus the \( \Delta_{N,i} \) cut out the same divisor as \( \chi_{18} \) (see 3.5.3).

**Proposition 3.** Let \( k \) be a field not of characteristic 2. Let \( (A,a,\alpha) \in A_{3,1,4}(k) \) be a principally polarised Abelian threefold defined over \( k \) with a level \( N \) structure \( \alpha. \) Suppose that \( (A,a,\alpha) \in U_{N,i} \) as above for \( g = 3 \) and that the polarisation \( a \) is indecomposable. Then \( \Delta_{N,i}(A,a) \in k^2 \) if and only if \( (A,a) \cong (\text{Jac}(C),\lambda_\Theta) \) for some genus three curve \( C \) over \( k. \)

**Proof:**
There are two cases to consider: i) \( (A,a) \) is a Jacobian and ii) \( (A,a) \) is the \([-1]\) twist of a non-hyperelliptic Jacobian. In the second case there is a unique non-hyperelliptic curve \( C \) over \( k \) and a unique quadratic extension \( k_2 \) of \( k \) (cf. \([21],[22],\) Theorem 3.2) so that

\[ (A,a) \otimes k_2 \cong (\text{Jac}(C),\lambda_\Theta) \otimes k_2. \]

In case i) we have that \( \sqrt{\Delta_{N,i}(C,\alpha)} \in k \) and \( \Delta_{N,i}(A,a) = (\sqrt{\Delta_{N,i}(C,\alpha)})^2 \in k^2. \)

In case ii) \( (\text{Jac}(C),\lambda_\Theta) \) is the \([-1]\) twist of \( (A,a) \) over \( k_2. \) Consider the obvious Galois representation

\[ \rho : \text{Gal}(k_2/k) \rightarrow \text{Aut}(A[N](k_2)). \]

This is trivial as \( A \) has \( k \) rational \( N \) torsion. Let \( \sigma \) be the generator of \( \text{Gal}(k_2/k). \) The Galois representation

\[ \rho^{\text{tw}} : \text{Gal}(k_2/k) \rightarrow \text{Aut}(\text{Jac}(C)[N](k_2)) \]

is then given by the formula

\[ \rho^{\text{tw}}(\sigma) = [-1] \rho(\sigma) = [-1]. \]

Thus \( \text{Jac}(C) \) does not have \( k \) rational \( N \) torsion but \( \text{Jac}(C) \otimes k_2 \) has \( k_2 \) rational \( N \) torsion.

Let \( \alpha^{\text{tw}} \) be the symplectic isomorphism of \( \text{Jac}(C)[N](k_2) \) with \( (\mathbb{Z}/N)^6 \) given by composing the isomorphism of \( \text{Jac}(C)_{k_2} \) with \( A_{k_2} \) with the level \( N \) structure

\[ \alpha : A[4](k_2) \longrightarrow (\mathbb{Z}/N)^6. \]

Then the fibre of the Torelli morphism above \( (A,a) \) is given by the \( \text{Gal}(k_2/k) \) orbit

\[ \{ (C,\alpha^{\text{tw}}), (C,[-1] \circ \alpha^{\text{tw}}) \}. \]

And therefore

\[ [\sqrt{\Delta_{N,i}((C,\alpha^{\text{tw}}), (C,[-1] \circ \alpha^{\text{tw}}))}]^2 \in k_2 - k, \]

hence
This completes the proof of Proposition 3.

4.1.3. Assume now that over the cover $U_{N,i}$, the line bundle $\mathcal{O}_{A_{3,1,N}}(H) \cong \otimes_{\text{even}} p^* \mathcal{O}_A(\Theta)$ is trivial.

Then we have isomorphisms

$$\phi_i : \mathcal{O}_{A_{3,1,N}}(H) \to R_{N,i}^*,$$

such that

$$\phi_i(\chi_{18})|_{U_{N,i} \cap U_{N,j}} = \phi_j(\chi_{18})|_{U_{N,i} \cap U_{N,j}}.$$

Such a system of compatible isomorphisms $(\phi_i)$ is unique up to multiplication by an element of $\Gamma(A_{3,1,N}, \mathcal{O}_{A_{3,1,N}}^*) = \mathbb{Z}[1/2N, \zeta_N]^*$ (cf. 3.5.1).

Assume moreover that over the cover $W_i = \text{Spec}(R_{N,i})$ the bundle $\mathcal{O}_{M_{3,N}}(H)$ is trivial.

Let

$$\psi_i : \mathcal{O}(H) \to R_{N,i},$$

be a trivialization.

Consider the pullback $t^*\chi_{18}$ of $\chi_{18}$ via the torelli morphism $t$.

We may extract a 2cd roots of $t^*\chi_{18}$ over $C$, and hence over $\mathbb{Z}[1/2N, \sqrt{-1}]$, thus

$$\sqrt[4]{t^*\chi_{18}} \in \mathcal{O}_{M_{3,N}}(H).$$

And so

$$\psi_i(\sqrt[4]{t^*\chi_{18}}) = (\sqrt{\Delta_{N,i}})^2$$

up to a unit. And therefore we may choose $\phi_i$ as above so that

$$\phi_i(\sqrt[4]{\chi_{18}}) = \Delta_{N,i}$$

up to a square. We have therefore proven

**Proposition 4.** Given any system of isomorphisms $(\phi'_i)$ such that

$$\phi_i(\chi_{18})|_{U_{N,i} \cap U_{N,j}} = \phi_j(\chi_{18})|_{U_{N,i} \cap U_{N,j}},$$

there is a constant $c \in \mathbb{Z}[1/2N, \zeta_N]^*$, depending only on the system $(\phi'_i)$, so that

$$\phi'_i(\chi_{18}) = c \Delta_{N,i},$$

Moreover, there exists a system of isomorphisms $(\phi_i)$ such that

$$\phi_i(\chi_{18})|_{U_{N,i} \cap U_{N,j}} = \phi_j(\chi_{18})|_{U_{N,i} \cap U_{N,j}},$$

and

$$\phi'_i(\chi_{18}) = \Delta_{N,i}.$$
4.2. Construction of $\Delta_i$.

4.2.1. Let $A/S$ be an Abelian scheme of relative dimension 3 with a principal polarization $a$. We now show the existence of the $\Delta_i$ as in part d) of Theorem 1.

Let $f_N : S' \rightarrow S$ be a Galois cover such that $A_{S'}$ has a level $N$ structure (cf. 3.5.2). Now let $X_{N,i}$ be a Zariski cover of $S$ such that $U_{N,i} := f^{-1}(X_{N,i})$ is affine with a function $\Delta_{N,i}$ coming from the fine moduli space $A_{3,1,N}$ as in 4.1.2.

We first note that since $\chi_{18}$ is a Katz-Siegel modular form, it follows from Proposition 4, that $\Delta_{N,i}$ and is equal to a function $\delta_{N,i}$ on $X_{N,i}$ with locus equal to the hyperelliptic locus.

Moreover, given distinct $N$ and $M$ we have, by Proposition 4, that

$$\delta_{N,i} = \gamma \delta_{M,i},$$

for some unit $\gamma \in \mathbb{Z}[1/2MN, \zeta_{MN}]^*$. Since $\mathbb{Z}[1/2MN, \zeta_{MN}]^*$ is generated by $1/2M, 1/2N, \zeta_M$ and $\zeta_N$, we may choose $\Delta_i$ on $U_{N,i} \cap U_{M,i}$ so that

$$f_M^* \Delta_i = \Delta_{M,i} \text{ and } f_N^* \Delta_i = \Delta_{N,i}.$$  

This choice can be made independently of $M$ and $N$ so that we have, up to multiplication by an element of $\mathbb{Z}[1/2]^2$, a unique choice of $\Delta_i$ with the property that there is a trivialization of $\mathcal{O}_S(H)$ sending $\chi_{18}$ to $\Delta_i$. We have a version of Proposition 3 for the $\Delta_i$.

**Proposition 5.** Let $x$ be a closed point of $U_i \subset S$ with residue field $k$. Let $(A_x, a_x)$ be the fibre of $A$ over $x$. Then

$$\Delta_i(A_x) \in k^2$$

if and only if there is a curve $C$ over $k$ such that $(\text{Jac}(C), \lambda_{\Theta}) \cong (A_x, a_x)$.

**Proof:** From the refined Torelli theorem, we know that there is a curve $C$ over $k$ and an element $D \in k$ such that

$$(\text{Jac}(C), \lambda_{\Theta}) \otimes k(\sqrt{D}) \cong (A_x, a_x) \otimes k(\sqrt{D}).$$

We want to show that $k(\sqrt{D}) = k(\sqrt{\Delta_i(A_x, a_x)})$. From our construction we know that for any two $M, N$

$$\sqrt{\Delta_i(A_x, a_x)} \in k(A[M]) \cap k(A[N]).$$

Hence if $D \in k^2$, then $\Delta_i(A_x, a_x) \in k^2$.

Now suppose $k(\sqrt{D})$ is not contained in $k(A[4])$, but $k(\sqrt{\Delta_i(A_x, a_x)}) \subset k(A[4])$. Now $A$ is the $[-1]$ twist of Jac($C$), and so if $\rho$ is the representation of $\text{Gal}(k(A[4])/k)$ on $A[4]$ then $-\rho$ is the Galois representation of $\text{Gal}(k(A[4])/k)$ on Jac($C$)[4]. Which is a contradiction. So

$$\sqrt{\Delta_i(A_x, a_x)} \notin k(A[4])$$

Consequently,

$$k(A[4], \sqrt{D}) = k(A[4], \sqrt{\Delta_i(A_x, a_x)})$$

and thus $k(\sqrt{D}) = k(\sqrt{\Delta_i(A_x, a_x)})$.

4.2.2. We have a version of Proposition 4 for the $\Delta_i$:
Proposition 6. Given any system of isomorphisms $(\phi'_i)$ such that 
$$\phi_i(\chi_{18})|_{U_i \cap U_j} = \phi_j(\chi_{18})|_{U_i \cap U_j},$$
there is a constant $c \in \mathbb{Z}[1/2]$, depending only on the system $(\phi'_i)$, so that 
$$\phi'_i(\chi_{18}) = c\Delta_i.$$ 
Moreover, there exists a system of isomorphisms $(\phi_i)$ such that 
$$\phi_i(\chi_{18})|_{U_i \cap U_j} = \phi_j(\chi_{18})|_{U_i \cap U_j},$$
and 
$$\phi'_i(\chi_{18}) = \Delta_i.$$

5. Theorem 1e

5.1. Definition of the analytic $\chi^\text{an}_{18}$.

5.1.1. Let $\mathfrak{h}_3$ be the Siegel upper half space of degree 3: that is the complex domain of $3 \times 3$ symmetric complex matrices with positive definite imaginary part. Let $\pi^\text{an}: A^\text{an} \longrightarrow \mathfrak{h}_3$ be the universal complex analytic Abelian threefold over $\mathfrak{h}_3$.

Analytic uniformization provides a family $f: \mathbb{C}^3 \times \mathfrak{h}_3 \longrightarrow \mathfrak{h}_3$ of complex vector spaces, and a family $h: \Lambda \longrightarrow \mathfrak{h}_3$ of lattices of rank 6 whose fibre above a point $\tau$ is 
$$h^{-1}(\tau) := \mathbb{Z}^3 + \tau \mathbb{Z}^3 \subset f^{-1}(\tau) = \mathbb{C}^3 \times \{\tau\}$$
so that 
$$A^\text{an} \cong \mathbb{C}^3 \times \mathfrak{h}_3/\Lambda$$
as analytic manifolds above $\mathfrak{h}_3$.

Let $z = (z_1(\tau), z_2(\tau), z_3(\tau))$ denote the standard coordinates of $\mathbb{C}^3 \times \{\tau\} \subset \mathbb{C}^3 \times \mathfrak{h}_3$.

The Riemann theta function is a holomorphic function on $\mathbb{C}^3 \times \mathfrak{h}_3$ defined by the formula 
$$\vartheta(z; \tau) := \sum_{n \in \mathbb{Z}^3} e^{\pi \sqrt{-1} n' \tau n + 2\pi \sqrt{-1} n \cdot z}.$$ 
We have a canonical level 2 structure on $\pi^\text{an}: A^\text{an} \longrightarrow \mathfrak{h}_3$ given by the standard coordinates on $\mathbb{Z}^3 + \tau \mathbb{Z}^3$. Using this we identify a two torsion point $p \in A^\text{an}[2](\mathfrak{h}_3)$ with a unique element $m + \tau m' \in 1/2(\mathbb{Z}^3 + \tau \mathbb{Z}^3)$ such that $m$ and $m'$ have entries equal to either 0 or 1/2.

The analytic theta nulls are then holomorphic functions on $\mathfrak{h}_3$ given by the formulas 
$$\vartheta\left[\begin{array}{c} m \\ m' \end{array}\right](0; \tau) := \vartheta(m + \tau m'; \tau).$$ 
The function $p \mapsto 4m \cdot m'$ mod 2 defines a quadratic form with values in $\mathbb{Z}/2$ which is zero if and only if $p$ is a point in the divisor of $\vartheta(z; \tau)$ of even multiplicity. Thus an analytic theta null 
$$\vartheta\left[\begin{array}{c} m \\ m' \end{array}\right](0; \tau)$$
is called even if $4m \cdot m'$ is even.

We therefore define
\[
\chi_{18}^{an}(\tau) := \prod_{4m, m' \in \mathbb{Z}} \vartheta \left[ \frac{m}{m'} \right](0; \tau).
\]

5.2. Moret-Bailley’s isomorphism.

5.2.1. Consider the analytic isomorphism on \( h_3 \)

\[
\omega_{A/\mathfrak{h}_3}^{18} \otimes \mathcal{O}_{\mathfrak{h}_3}(-H) \cong \mathcal{O}_{\mathfrak{h}_3}
\]
given by sending

\[
(2\pi \sqrt{-1})^{54} \chi_{18}^{an}(\tau)(dz_1 \wedge dz_2 \wedge dz_3)^{18}
\]
to the constant function 1. Combing section 3.3.1 and 3.3.4 with [12] we deduce that this isomorphism is compatible with the isomorphism

\[
\omega_{A_{A_{3,1, N}}}^{18} \otimes \mathcal{O}_{A_{3,1, N}}(-H) \cong \mathcal{O}_{A_{3,1, N}},
\]
over \( \mathbb{Z}[\frac{1}{N}, \zeta_N] \) of section 3.3.1.

Let \( A/S \) be an Abelian scheme of relative dimension 3 with a principal polarisation \( a \). Let \( U_i \) be an open cover of \( S \) over which the functions \( \Delta_i \) of 4.2 exist, and over which the sheaf of Kähler differentials is free with basis \( \xi_1, \xi_2, \xi_3 \). Then Proposition 6 guarantees that there is a constant \( c \in \mathbb{Z}[1/2]^* \), independent of \( S \) and \( i \), so that

\[
c\Delta_i(\xi_1 \wedge \xi_2 \wedge \xi_3)^{18}
\]
is mapped to the constant function 1 by the isomorphism of section 3.3.1.

Thus, tensoring \( S \) with \( C \) and embedding \( S \otimes C \) into \( \mathfrak{h}_3 \), we have

\[
c\Delta_i(\xi_1 \wedge \xi_2 \wedge \xi_3)^{18} = (2\pi \sqrt{-1})^{54} \chi_{18}^{an}(\tau)(dz_1 \wedge dz_2 \wedge dz_3)^{18}.
\]

5.2.2. Consider the relative differential forms \( dz_1, dz_2 \) and \( dz_3 \) on \( A^{an} \). They can be expressed in terms of algebraic invariant differential forms \( \xi_1, \xi_2, \xi_3 \) by means of the period matrix \( \Omega_1 \)

\[
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix} = \Omega_1
\begin{pmatrix}
dz_1 \\
dz_2 \\
dz_3
\end{pmatrix}.
\]

The matrix \( \Omega_1 \) is obtained in the following way: let \( \gamma_1, \gamma_2, \gamma_3 \) denote the homology classes of \( A = C^3/\mathbb{Z}^3 + \tau \mathbb{Z}^3 \) corresponding to the lattice vectors \((1, 0, 0), (0, 1, 0), (0, 0, 1)\).

Then the entries of \( \Omega_1 \) are obtained by integrating the \( \xi_i \) along the \( \gamma_j \).

5.2.3. The final formula of 5.2.1 yields

\[
c\Delta_i = \frac{(2\pi \sqrt{-1})^{54} \chi_{18}^{an}}{\det(\Omega_1)^{18}}.
\]

We note the the quantity on the right hand side is invariant under symplectic transformations, and it depends on \( A/S \), as \( \Omega_1 \) depends on \( A/S \). In particular if \( S \) is the spectrum of a field \( k \), and \( A^{tw} \) is the \([-1]\) twist of \( A \) over the field \( k(\sqrt{D}) \) then \( \det(\Omega_1)^{18}_A \) and \( \det(\Omega_1)^{18}_{A^{tw}} \) differ by \( D^{27} \).
Explicit numerical calculation\footnote{Christophe Ritzenthaler showed the author Maple code which does this: given an affine quartic, Maple computes the period matrix in \( \eta_{3} \). Then computation with the analytic theta nulls and the algebraic invariant differentials shows in specific examples that the right hand side of the above formula is a rational square when multiplied by \(-2^{-108}\).} with the analytic theta nulls determines that this is the case for \( c = -2^{108} \). And this shows Theorem 1 part e).

6. Theorem 1F

The definition of \( \text{Discr} \) is made in two steps. It is first defined in the classical setting; then via pulling back, it is defined as a Katz-Teichmüller modular form.

6.1. Construction of the Discriminant.

6.1.1. We work over the base \( \mathbb{Z}[1/2] \). Give \( \mathbb{P}^{15} \) coordinates \( a_{ijl} \) with \( i,j,l \in \{0,\cdots,4\} \) and \( i + j + l = 4 \). Consider the classical family of plane quartics:

\[
\gamma F : C \longrightarrow \mathbb{P}^{15}\n\]

where \( C \) is the subscheme of \( \mathbb{P}^{15}_{\mathbb{R}} \) defined by the equation

\[
(F : \sum_{i,j,l} a_{ijl}X^{i}Y^{j}Z^{l}) = 0.
\]

Let \( F_{X}, F_{Y} \) and \( F_{Z} \) be the derivatives of \( F \) with respect to \( X, Y \) and \( Z \).

The Discriminant of \( \gamma F : C \longrightarrow \mathbb{P}^{15} \) is defined to be the resultant of \( F_{X}, F_{Y} \) and \( F_{Z} \). We will now explain what this means.

6.1.2. Let \( U \) be an affine open subset of \( \mathbb{P}^{15} \) with coordinate ring \( R \) which has the unique factorization property. Consider the restriction \( C_{U} \) of \( C \) to \( U = \text{Spec}(R) \). The scheme \( \mathbb{P}^{15}_{U} \) is then obtained as the Proj of the ring \( R[X,Y,Z] \). For each positive integer \( m \), let \( I_{m} \) denote the \( R \) module of degree \( m \) homogeneous forms in \( X, Y \) and \( Z \) with coefficients in \( R \). Then \( I_{m} \) is free of rank \( (m+2)(m+1)/2 \) over \( R \).

The polynomials \( F_{X}, F_{Y} \) and \( F_{Z} \) have a common zero at a closed point \( (a_{ijl}) \in \mathbb{P}^{15} \) if and only if the corresponding quartic \( F_{(a_{ijl})} \) is singular. Write \( \gamma \) for an element of \( I_{4} \). Then the polynomials \( F_{X}, F_{Y} \) and \( F_{Z} \) have a common zero at a closed point \( (a_{ijl}) \) if and only if for all \( \gamma \in I_{4} \) the polynomials

\[
\gamma F_{X} + \gamma F_{Y} + \gamma F_{Z}
\]

have a common zero.

We have an \( R \) matrix \( M \) taking \( I_{4}^{3} \) to \( I_{7} \) which is given by multiplying the monomial elements of \( I_{4} \) by \( F_{X}, F_{Y} \) and \( F_{Z} \). Evaluating \( M \) at a closed point \( (a_{ijl}) \in \mathbb{P}^{15} \), we find that \( F_{(a_{ijl})} \) is smooth if and only if the maximal minors of \( M_{(a_{ijl})} \) are all non-singular. The resultant (cf. [\textit{\text{Ill}}] p3, cf. [\textit{\text{I}}] pp388-404) of \( F_{X}, F_{Y} \) and \( F_{Z} \) is then the highest common factor of the maximal minors of \( M \). It is thus zero at precisely the singular fibres of \( C \) (cf. [\textit{\text{Ill}}] p13).

Let \( B \) be an invertible \( 3 \times 3 \) matrix with entries in \( R \); then \( \text{Res}(F_{X} \circ B^{-1}, F_{Y} \circ B^{-1}, F_{Z} \circ B^{-1}) = \det(B)^{3} \text{Res}(F_{X}, F_{Y}, F_{Z}) \) (cf. [\textit{\text{Ill}}] pp8-9).

Let \( V \) be the open subscheme of \( \mathbb{P}^{15} \) where the resultant of \( F_{X}, F_{Y} \) and \( F_{Z} \) does not vanish.

Since smooth plane quartics are embedded via their canonical bundle, we have defined a weight 9 Teichmüller modular form \( \text{Discr}_{C/V} \).
6.1.3. We define $\text{Discr}_{C/M_{3,4}}$ by mapping $C$ to $\text{Proj}(\Omega_{C/M_{3,4}})$ and using the above procedure.

Comparing $\text{Discr}_{C/M_{3,4}}$ with $\text{Discr}_{C/V}$ we see that it is independent of the level 4 structure, and thus descends to a weight 9 Katz-Teichmüller modular form.

A one dimensional family of quartics with equation

$$F^2 + t^2 H = 0$$

with $F$ a conic and $H$ a quartic, can be normalized to obtain a smooth curve $C$ over $\mathbb{A}^1 = \text{Spec}(k[t])$ whose fibre above zero is hyperelliptic. Thus $\text{Discr}_{C/\mathbb{A}^1}$ vanishes on the hyperelliptic locus. And hence $\text{Discr}_{C/S}$ vanishes on the hyperelliptic locus of $S$ for any family of genus three curves $C$.

Considered as a Katz-Teichmüller modular form, $\chi_{18}$ vanishes with multiplicity 2 on the hyperelliptic locus and has weight 18. Thus up to a unit of $\mathbb{Z}[1/2]$ it is equal to $\text{Discr}^2$ over $M_{3,4}$ the modular form $\chi_{18}$ is equal to $\text{Discr}^2$ modulo a unit in $\mathbb{Z}[1/2, \sqrt{-1}]$ whereas over $M_{3,3}$ it is equal to $\text{Discr}^2$ modulo a unit in $\mathbb{Z}[1/2, 1/3\zeta_3]$. Thus over $M_{3,12}$ we see that $\text{Discr}^2$ is equal to $\chi_{18}$ modulo a unit of $\mathbb{Z}[1/2]$.

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