Simply Generated Trees, B-Series
and Wigner Processes

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Abstract
We consider simply generated trees, like rooted plane trees, and consider the problem of computing generating functions of so-called bare functionals, like the tree factorial, using B-series from Butcher’s theory. We exhibit a special class of functionals from probability theory: the associated generating functions can be seen as limiting traces of product of semi-circular elements.

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1 Introduction

Let $F_n$ denote the set of rooted plane trees of size $n$. Simply generated trees are families of trees obtained by assigning weights $\omega(t)$ to the elements $t \in F = \cup_n F_n$ using a degree function $\psi(z) = 1 + \sum_{k \geq 1} \psi_k z^k$ (see [20]). Basically, the weight $\omega(t)$ of some $t \in F$ is obtained by multiplying the factors $\psi_{d(v)}$ over the nodes $v$ of $t$, where $d(v)$ denotes the outdegree of $v$. Our main topic is the study of generating functions

$$Y(z) = \sum_{t \in F} \omega(t) B(t) z^{|t|},$$

associated with multiplicative functions $B : F \rightarrow \mathbb{R}$ defined recursively by using a sequence of real numbers $\{B_k\}_{k \in \mathbb{N}^+}$. We call such multiplicative functions bare Green functions: $\sum_{t \in F_n} B(t) \omega(t)$ represents the sum of the
Feynman amplitudes associated to the relevant diagrams of size $n$ in some field theory, and the generating function is then a part of the perturbative expansion of the solution of some equation describing the system (see $[3, 6, 8, 15]$).

In Section 4, we give an equation satisfied by $Y$ when the weights $B_k$ come from some master function $L(z) = \sum_{m \geq 0} L_m z^m$, with $B_k \equiv L(k)/k$, $\forall k \in \mathbb{N}^+$. We use series indexed by trees, the so-called B-series, as defined in $[13, 14]$, to show in Theorem 1 that $Y$ solves

$$Y' = L(1 + \theta)\Psi(Y),$$

where $\theta$ is the differential operator $\theta = zd/dz$. $[1]$ considers a similar problem for additive tree functionals $s(t)$ defined on varieties of increasing trees, like $s(t) = \ln(B(t))$. Assuming some constraints on the degree function $\Psi(z)$, it is proven that the exponential generating function

$$S(z) = \sum_{t \in \mathcal{F}} \omega(t)s(t)z^{|t|}/|t|!,$$

is given by the formula

$$S(z) = W'(z) \int_0^z \frac{F'(u)/W'(u)}{u} du,$$

where $F(u) = \sum_{m \geq 0} \ln(B_m)W_m u^m/m!$ and $W(z) = \sum_{m \geq 0} W_m z^m/m!$ solves $W' = \Psi(W)$. We also consider a central functional called the tree factorial, denoted by $t!$ in the sequel, which is relevant in various fields, like algorithmics $[9, 18]$, stochastics $[11, 21]$, numerical analysis (see example $[5, 14]$), and physics $[6, 15]$. We focus on its negative powers $1/(t!)^{l+1}$, $l \in \mathbb{N}$, which do not admit a master function when $l \geq 1$. $[6]$ solved the case $l = 1$ by using the so-called Butcher’s group (see for example $[13, 14]$). We provide in Theorem 2 a differential equation for the associated generating function, $\forall l \in \mathbb{N}$.

In Section 5, we define special multiplicative functionals for which the weights $B_k$ are related to the covariance function $r$ of some gaussian process, as $B_k = \beta^2 r(2k - 1)$, for some positive constant $\beta > 0$. We show that the generating function $Y$ is related to the mean normalized trace of products of large symmetric random matrices having independent and identically distributed versions of the process as entries. Theorem 3 gives then a differential equation for the evolution of the trace of a stationary Wigner processes. It follows that most of the examples given in $[3, 15]$ can be expressed in terms of traces of large random matrices. In Section 6, we show how B-series can be useful for studying traces of triangular operators appearing in free probability.
2 Basic notions

A rooted tree \( t \in \mathcal{R} \) is a triple \( t = (r, V, E) \) such that i) \((V, E)\) is a non-empty directed tree with node set \( V \) and edge set \( E \), ii) all edges are directed away from the root \( r \in V \). The set of rooted trees of order \( n \) is denoted by \( \mathcal{R}_n \), and the set of rooted trees is \( \mathcal{R} = \bigcup_n \mathcal{R}_n \). A rooted plane tree \( t \in \mathcal{F} \) is a quadruple \( t = (r, V, E, L) \) satisfying i) and ii) and iii) \( L := \{ (\{w : vw \in E\}, L_v) : v \in V \} \) is a collection of \(|V|\) linear orders. Given \( v \in V \), let \( \text{ch}(v) := \{ w : vw \in E \} \) be the set of children of \( v \). \( d(v) := |\text{ch}(v)| \) is the outdegree of \( v \). A rooted planar tree can be seen in the plane with the root in the lowest position, such that the orders \( L_v \) coincide with the left-right order. Next consider the partial ordering \((V, \leq)\) defined by \( u \leq v \) if and only if \( u \) lies on the path linking \( r \) and \( v \). Given \( v \in V \) and \( t \in \mathcal{R} \) let \( t_v \) be the subtree of \( t \) rooted at \( v \) spanned by the subset \( \{ w : v \leq w \} \). A rooted labelled tree is a quadruple \( t = (r, V, E, l) \) satisfying i) and ii), with a labelling \( l : V \setminus \{ r \} \to [|V|] := \{1, \ldots ,|V|\} \) such that \( l(u) < l(v) \) when \( u < v \). The set of rooted labelled trees of order \( n \) is denoted by \( \mathcal{L}_n \). Let \( \mathcal{L} = \bigcup_n \mathcal{L}_n \). This family is a special variety of increasing trees, as defined in [1, 12]. We next assign weights to the elements of \( \mathcal{F}_n \), the set of rooted planar trees of order \( n \): the resulting family of trees is said to be simply generated (see [21]). Given a sequence \( \psi = \{ \psi_k \}_{k \in \mathbb{N}} \) of real numbers with \( \psi_0 = 1 \), define recursively the weight \( \omega(t) \) of \( t \in \mathcal{F} \) as

\[
\omega(t) = \psi_k \prod_{i=1}^k \omega(t_i), \quad k = d(r), \quad \omega(t) = \prod_{v \in V} \psi_{d(v)}.
\]

where \( t_1, \ldots , t_k \) are the \( d(r) \) subtrees of \( t \) rooted at \( \text{ch}(r) \). Let \( \psi(z) := 1 + \sum_{k=1}^{\infty} \psi_k z^k \) be the generating function of the weight sequence \( \psi \). Our favourite example is \( \psi(z) = 1/(1 - z) \), with \( \omega(t) = 1, \forall t \) (see [11, 20] for various interesting choices).

We will be concerned with functionals \( B : \mathcal{F} \to \mathbb{R} \), where \( \mathcal{F} = \bigcup_n \mathcal{F}_n \), called bare Green functions. This terminology is taken from quantum field theory where bare Green functions occur during the action of the renormalization group (see for example [3], § 4.2 or [4], § 6.1). Let \( \mathcal{B} \) denote the set of bare Green functions. Any element \( B \in \mathcal{B} \) is given through a sequence of functions \( B_k : \mathbb{R} \to \mathbb{R} \), \( k \in \mathbb{N}^+ \), which are usually Laurent series in some variable \( x \) (see for example [3]). In what follows, we simply write the sequence as \( \{ B_k \}_{k \in \mathbb{N}^+} \).

**Definition 1** The bare Green function \( B \in \mathcal{B} \), \( B : \mathcal{F} \to \mathbb{R} \), associated
with the sequence of functions $\{B_k\}_{k \in \mathbb{N}_+}$ is defined recursively as

$$B(t) = B_{|t|} \prod_{i=1}^{k} B(t_i),$$

where $t_1, \cdots, t_k$ are the $d(r)$ subtrees of $t$ rooted at $\text{ch}(r)$, and where $|t|$ denotes the number of nodes of $t$.

Notice that the value of $B$ at $t \in \mathcal{F}$ does not depend on the linear orders and is independent of the labellings. When dealing with rooted trees, we will adopt the notation $t = B_+(t_1, \cdots, t_k)$ for the operation of grafting the rooted trees $t_1, \cdots, t_k$, that is by considering the tree $t$ obtained by the creation of a new node $r$ (the root) and then joining the roots of $t_1, \cdots, t_k$ to $r$. Bare Green functions appeared also in the probabilistic literature in specific situations. The basic example, in algorithmics [9, 18], in numerical analysis (see [4, 5, 14]), in stochastics [11, 20] and in physics (see for example [6]) is the tree factorial, defined by

**Definition 2** Let $t \in \mathcal{R}$ with $t = B_+(t_1, \cdots, t_k)$. Then the tree factorial $t!$ is the functional $B \in \mathcal{B}$ defined by $t! = |t| \prod_{i=1}^{k} t_i!$, associated with the sequence $\{B_k\}$ given by $B_k \equiv k$.

**Remark 3** It should be pointed out that the functional acting on trees, given as $s(t) = \ln(B(t))$, for $B \in \mathcal{B}$ with $B_k > 0$, $\forall k \geq 1$, is an inductive map or an additive tree functional, as defined in [17]. Interestingly, $B(t) = 1/t!$ is used in [18] to define a probability measure on random search binary trees, and [9, 11] provide precise asymptotics for $\ln(t!)$.}

### 3 Generating functions

We first give some basic results on tree factorials, symmetry factors, and generating functions associated with bare Green functions.

**Definition 4** Let $t \in \mathcal{R}$. Then $\alpha(t)$ is the number of rooted labelled trees $t' \in \mathcal{L}$ of shape $t \in \mathcal{R}$, where the shape of a labelled tree $(r, V, E, l)$ is $(r, V, E)$, $\kappa(t)$ is the number of rooted plane trees of shape $t$, and $\sigma(t)$ is the symmetry factor of the tree, to be defined later. Moreover, let $\omega_\mathcal{L}$ be the weight function associated with elements of $\mathcal{L}$, with weights given by $\psi_k \equiv 1/k!$.

Notice that $\alpha(t)$ is the Connes-Moscovici weight in quantum field theory (see [3, 7]). The symmetry factor satisfies the recursive definition:

$$\sigma(\{r\}) = 1,$$
where the indices $n_i$ means that $t$ is obtained by grafting $n_1$ times the tree $t_1$, and so on, where we assume that the $t_i$ are all different as rooted trees.

**Lemma 5** Let $t \in \mathcal{R}$. Then

$$\alpha(t)\sigma(t) = \frac{|t|!}{t!},$$

(1)

and

$$\alpha(t)t! = |t|\omega_L(t)\kappa(t).$$

(2)

**Proof:** (1) is well known (see for example [5]). Suppose that $t \in \mathcal{R}$ is such that $t = B_+(t_{n_1}^{n_1}, \ldots, t_{n_k}^{n_k})$, the grafting of $n_1$ times the tree $t_1$, and so on, where we set that the trees $t_1, \ldots, t_k$ are different as rooted trees. Then

$$\kappa(t) = \left(\frac{n_1 + \cdots + n_k}{n_1! \cdots n_k!}\right)^{n_1} \cdots \kappa(t_1)^{n_1} \cdots \kappa(t_k)^{n_k}.$$ 

Using the recursive definition of $\omega(t)$ and the definition of $\omega_L$, we have

$$\omega_L(t) = \frac{1}{(n_1 + \cdots + n_k)!^{n_1} \cdots \omega_L(t_1)^{n_1} \cdots \omega_L(t_k)^{n_k}}.$$ 

Therefore

$$\frac{1}{\omega_L(t)\kappa(t)} = n_1! \cdots n_k!(\frac{1}{\omega_L(t_1)\kappa(t_1)})^{n_1} \cdots (\frac{1}{\omega_L(t_k)\kappa(t_k)})^{n_k},$$

and the results follows from the recursive definition of the symmetry factor. □

Then

$$\sum_{t \in \mathcal{F}_n} B(t)\omega(t) = \sum_{t \in \mathcal{R}_n} B(t)\frac{\omega(t)}{\omega_L(t)}\alpha(t)\frac{t!}{|t|!}\frac{t!}{|t|!},$$

(3)

where we have used (2) of Lemma 5.

Consider the generating function

$$Y(z) = \sum_{n \in \mathbb{N}^+} \sum_{t \in \mathcal{R}_n} z^n \sum_{i \in \mathcal{R}_n} \alpha(i)B(i)!\omega(i)/\omega_L(i).$$

(4)

Given $t \in \mathcal{R}$, the ratio $\omega/\omega_L$ is associated with the weight sequence $\psi_k \equiv \psi_k k!$; using the expansion $\psi(z) = 1 + \sum_{k \geq 1} \psi_k z^k = 1 + \sum_{k \geq 1} (\psi_k/k!) z^k$, we see that $\psi_k \equiv \psi^{(k)}(0)$. Consider the elementary differentials $\delta$ (see Section 4) defined by
Definition 6

\[ \delta_{\{s\}} = 1, \quad \delta_t = \psi^{(k)}(0) \prod_{i=1}^{k} \delta_t_i, \quad \frac{\omega}{\omega_L} = \delta, \]

when \( t = B_+(t_1, \ldots, t_k) \), where * denotes the tree of a single node. For a map \( a : \mathcal{R} \cup \{\emptyset\} \rightarrow \mathbb{R} \), a formal power series of the form \( Y(z) = a(\emptyset) y_0 + \sum_{t \in \mathcal{R}} z^{\|t\|} a(t) \alpha(t) / |t|! \) is called a B-series \([13, 14]\).

Remark 7 When \( B(t) = t! \), the series \( Y \) is given by

\[ Y(z) = \sum_{t \in \mathcal{L}} (\omega(t)/\omega_L(t)) z^{|t|}/|t|!. \]

Set \( \phi_k = \psi_k k!, \forall k \), and consider the degree function \( \phi(z) = 1 + \sum_{k \geq 1} (\phi_k/k!) z^k \).

Following \([13]\), \( Y \) solves \( Y' = \phi(Y) \) (see also \([20]\)). We shall see in the next section that it is a natural consequence of B-series expansions of solutions of ordinary differential equations.

4 Runge-Kutta methods for functionals over trees

Consider a dynamical system on \( \mathbb{R} \)

\[ \frac{d}{ds} X(s) = F(X(s)), \quad X(s_0) = X_0, \]

for some smooth \( F : \mathbb{R} \rightarrow \mathbb{R} \). The solution of this dynamical system has a B-series expansion of the form

\[ X(s) = X_0 + \sum_{t \in \mathcal{R}} \frac{(s - s_0)^{|t|}}{|t|!} \alpha(t) \delta_t(s_0), \]

where the elementary differentials \( \delta \) is defined recursively by

\[ \delta_{\{s\}} = f(s_0), \quad \delta_t = \frac{\partial^k F}{\partial s^{k_1} \delta t_1 \cdots \delta t_k}, \]

when \( t = B_+(t_1, \ldots, t_k) \). These kinds of expansions have been treated in great detail in \([4]\) and \([5]\) and developed independently in combinatorics (see for example \([16, 17]\)). Suppose that \( s_0 = 0 \) for simplicity. Butcher considered what happens with numerical approximations of the exact solution, the Runge-Kutta methods, which are themselves B-series \([13, 14]\); here we focus on specific B-series, which are associated to bare Green functions. Let \( B \in \mathcal{B} \) be such that there exists a power series

\[ L(z) = \sum_{m \geq 0} L_m z^m, \]
Given a power series $Y(z) = \sum_{m=0}^{\infty} a_m z^m$ converging for $|z| \leq 1$, we can define $L(\theta + 1)(Y)(z) := \sum_{m=0}^{\infty} a_m L(m + 1)z^m$, which converges for $|z| \leq 1$ when the sequence $(L(k))_{k \geq 1}$ grows subexponentially. We will not focus on convergence questions here, and work at the formal level. Let $L$ be a bare Green function with weights $(B_k)_{k \geq 1}$, such that \[(5)\] holds for some power series $L$. It should be pointed out that [3, 6, 15] deal with the master function $L$, but do not give explicitly an equation for $Y$. The next Theorem provides an equation; its proof uses explicitly B-series.

**Theorem 1** The formal power series

$$Y(z) = \sum_{t \in R} \frac{z^{\left| t \right|}}{\left| t \right|!} \alpha(t)! B(t) \delta_t,$$

solves $Y' = L(1 + \theta)\psi(Y)$.

**Proof:**

$$\psi(Y(z)) = \sum_{k \geq 0} \frac{\psi^{(k)}(0)}{k!} \sum_{(t_1, \ldots, t_k) \in R^k} \frac{z^{\sum_{i=1}^{k} \left| t_i \right|}}{\left| t_1 \right|! \cdots \left| t_k \right|!} \prod_{i=1}^{k} \alpha(t_i) B(t_i) \delta_{t_i}.$$  

For given $(t_1, \ldots, t_k) \in R^k$, set $t = B_{+}(t_1, \ldots, t_k)$. Then $\sum_i |t_i| = |t| - 1$, $\psi^{(k)}(0) \delta_{t_1} \cdots \delta_{t_k} = \delta_t$, and $B(t_1) \cdots B(t_k) = B(t)/B_{|t|}$. The associated term becomes

$$z^{\left| t \right| - 1} \frac{B(t)}{B_{|t|}} \delta_t \frac{\alpha(t_1) \cdots \alpha(t_k) \ t!}{\left| t_1 \right|! \cdots \left| t_k \right|! |t|}.$$  

Next, every rooted tree $t \in R$ can be decomposed uniquely as $t = B_{+}(t_1^{n_1}, \ldots, t_m^{n_m})$, meaning that $t$ is obtained by grafting $n_1$ times $t_1$ and so on, where the $t_i$ are different as rooted trees, with $k = n_1 + \cdots + n_m$. Collecting the terms associated with $t$, we get the contribution

$$\frac{z^{\left| t \right| - 1}}{k!} \frac{B(t)}{B_{|t|}} \delta_t \frac{\alpha(t_1) \cdots \alpha(t_k) \ t!}{|t|} \sum_{(t'_1, \ldots, t'_k) \in R^k} \frac{\alpha(t'_1) \cdots \alpha(t'_k)}{|t'_1|! \cdots |t'_k|!}.$$

with

$$B_k = \frac{L(k)}{k!}, \quad \forall k \in \mathbb{N}^+.$$
where * means that the sum is taken over all the collections \((t'_1 \cdots t'_k) \in \mathcal{R}^k\) such that \(t = B_+(t_1', \ldots, t_k')\). The above sum reduced then to
\[
\frac{(n_1 + \cdots + n_m)! \alpha(t_1)^{n_1} \cdots \alpha(t_m)^{n_m}}{n_1! \cdots n_m!} \cdot \frac{k! |t_1|!^{n_1} \cdots |t_m|!^{n_m}}{\sigma(t_1)t_1!^{n_1} \cdots \sigma(t_m)t_m!^{n_m}} = \frac{1}{n_1!} \left( \frac{1}{\sigma(t_1)t_1!^{n_1}} \right)^{n_1} \cdots \frac{1}{n_m!} \left( \frac{1}{\sigma(t_m)t_m!^{n_m}} \right)^{n_m},
\]
where we use the first identity of Lemma \(\text{E}\). Using the recursive definition of the symmetry factor \(\sigma\), we obtain
\[
\sum_{(t'_1 \cdots t'_k) \in \mathcal{R}^k} \frac{\alpha(t'_1) \cdots \alpha(t'_k)}{k! |t'_1|! \cdots |t'_k|!} = \frac{1}{t_1!^{n_1} \cdots t_m!^{n_m} \sigma(B_+(t_1, n_1, \ldots, t_m, n_m))} = \frac{|t| \alpha(t)}{t! \sigma(t)} = \frac{|t| \alpha(t)}{|t|!}.
\]
We thus get that the contribution associated with \(t \in \mathcal{R}\) is given by
\[
\frac{z^{|t| - 1} B(t)}{k! B_{|t|}} \delta_t \frac{1}{|t|!} \frac{|t| \alpha(t)}{|t|!} = \frac{z^{|t| - 1} B(t)}{|t|!} \alpha(t) \delta_t !.
\]
Therefore
\[
L(\theta + 1) \psi(Y) = \sum_{t \in \mathcal{R}} \frac{B(t)}{B_{|t|}} \alpha(t) t! \delta_t \frac{L(\theta + 1)(z^{|t| - 1})}{|t|!} = \sum_{t \in \mathcal{R}} \frac{B(t)}{B_{|t|}} \alpha(t) t! \delta_t \frac{L(|t|)z^{|t| - 1}}{|t|!} = \sum_{t \in \mathcal{R}} \frac{B(t)}{B_{|t|}} \alpha(t) t! \delta_t \frac{B_{|t|} |t| z^{|t| - 1}}{|t|!} = \sum_{t \in \mathcal{R}} \frac{z^{|t| - 1}}{(|t| - 1)!} B(t) t! \delta_t = \frac{dY}{dz}.
\]

\(\Box\)

**Remark 8** As we have observed in Remark \(\text{E}\), \(s(t) = \ln(B(t))\) is an inductive map when the weights \(B_k\) are positive. It turns out that the exponential generating function associated with \(s\) can be given as an integral transform for varieties of increasing trees (see for example Section \(\text{I}\)). This is the topic of \(\text{I}\).

**Example 9**

When \(L(z) = z\), with \(B_k \equiv 1\), and \(\psi(z) = 1/(1 - z)\), one has \(\sum_{t \in \mathcal{R}_{n+1}} B(t) = C_n\), the Catalan number of order \(n\), with \(C_n = \binom{2n}{n}/(n + 1)\). Then \(Y(z) = \ldots\)
$z \sum_{n \geq 0} z^n C_n$ is solution of the differential equation $Y''(z) = L(1 + \theta)(1/(1 - Y(z)))$, that is $Y''(z) = (z/(1 - Y(z)))'$. The unique solution with $Y(0) = 0$ satisfies $Y(z) = z/(1 - Y(z))$, or $Y(z) = (1 - \sqrt{1 - 4z})/2$, corresponding to a well known result.

\[\square\]

\[6\], § 5.3, considers the case where $B(t) = (1/t!)^2$, which is not of the form given in \[5\]; in this situation, $B_k = 1/k^2$, with $L(z) = 1/z$. The solution is obtained by using the structure of the so-called Butcher’s group of B-series (that is series of the form \[6\]), where the group structure is given in \[13\] \[14\] by tensoring known B-series:

**Example 10**

Consider the bare functional given by $B_k \equiv 1/k^2$, with $B(t) = 1/t^2$. Following Brouder, the associated B-series, as given in \[6\], is solution of the second order differential equation

$$zY'' + Y' = \psi(Y).$$

When $\psi(z) = \exp(z)$, the solution is given by

$$Y(z) = -2 \ln(1 - z/2) = \sum_{n=1}^{\infty} z^n \frac{1}{n2^{n-1}},$$

giving

$$\sum_{t \in \mathbb{R}_n} \frac{\alpha(t)}{t!} = \frac{(n - 1)!}{2^{n-1}}.$$

\[\square\]

We study the general moment problem $B(t) = (1/t!)^{l+1}$, $l \in \mathbb{N}$, by working directly on a suitable differential equation as follows: the operator $L(\theta + 1)$ takes the form $L(\theta + 1) = 1/(\theta + 1)^l$. Assume that the differential operator $L(\theta + 1)$ is invertible. Then the formal systems becomes

$$L(\theta + 1)^{-1} \frac{d}{dz} Y = \psi(Y). \quad (7)$$

Consider again the second moment problem for the tree factorial, with $B_k \equiv 1/k^2$ and $L(k) \equiv 1/k$. Choose $L$ such that $L(z) = 1/z$; the inverse operator might be equal to $L(\theta + 1)^{-1} = \theta + 1$, and, if this is the case,

$$(\theta + 1) \frac{d}{dz} Y = \psi(Y),$$

with $(\theta + 1)(d/dz) = z(d^2/dz^2) + (d/dz)$, see Example \[10\].

9
More generally, if one considers the moment of order \( l + 1 \in \mathbb{N} \) of the inverse tree factorial, the choice \( L(k) = 1/k^l \) should give \( (\theta + 1)^l \frac{d}{dz} Y(z) = \psi(Y) \). Our result, Theorem 2 below shows that the formalism of inversion is correct in term of power series. This result sheds some light and extends the computations done in [6] for the second moment, and its proof avoids computations in the Butcher’s group.

**Theorem 2** The B-series \( Y(z) \) associated with the moment of order \((l + 1)\) of the inverse tree factorial satisfies the differential equation

\[
(\theta + 1)^l \frac{d}{dz} Y = \psi(Y).
\]

**Proof:** Let

\[
Y(z) = \sum_{t \in \mathcal{R}} \frac{z^{|t|}}{|t|!} \alpha(t) \frac{1}{t!^{l+1}} t! \delta_t.
\]

Then

\[
\psi(Y(z)) = \sum_{k \geq 0} \frac{\psi^{(k)}(0)}{k!} \sum_{(t_1 \cdots t_k) \in \mathcal{R}^k} \frac{z^{\sum_i |t_i|}}{|t_1|! \cdots |t_k|!} \alpha(t_1) \cdots \alpha(t_k) (t_1! \cdots t_k!) \cdot \delta_{t_1} \cdots \delta_{t_k}.
\]

For given \((t_1 \cdots t_k) \in \mathcal{R}^k\), set \( t = B_+(t_1, \cdots, t_k) \). Then \( \sum_i |t_i| = |t| - 1 \), \( \psi^{(k)}(0) \delta_{t_1} \cdots \delta_{t_k} = \delta_t \), and \((t_1! \cdots t_k!) = t! / |t|\). The associated term becomes

\[
\frac{z^{|t|-1} |t|^l}{t!} \cdot \alpha(t_1) \cdots \alpha(t_k) \frac{1}{|t_1|! \cdots |t_k|!} \cdot \delta_{t_1} \cdots \delta_{t_k}.
\]

Proceeding as in the proof of Theorem 1, we get that the contribution associated with \( t \in \mathcal{R} \) is given by

\[
\frac{z^{|t|-1} |t|^l}{t!} \cdot \delta_t \cdot \frac{|t| \cdot \alpha(t)}{|t|!} = \frac{z^{|t|-1} |t|^l}{(|t| - 1)!} \cdot \frac{2^{|t|}}{t!} \cdot \alpha(t) \cdot \delta_t.
\]

On the other hand,

\[
(\theta + 1)^l \frac{d}{dz} Y(z) = (\theta + 1)^l \sum_{t \in \mathcal{R}} \frac{z^{|t|-1}}{|t|!(|t| - 1)!} \cdot \alpha(t) \cdot \frac{1}{t!^l} \cdot \delta_t
\]

\[
= \sum_{t \in \mathcal{R}} \frac{|t|^l z^{|t|-1}}{|t|!(|t| - 1)!} \cdot \alpha(t) \cdot \frac{1}{t!^l} \cdot \delta_t.
\]

\( \square \)

In the next section, we show that traces of certain products of Wigner matrices (see for example [24]) provide natural examples of bare Green functions.
5 Wigner processes

Definition 11 The $N$-dimensional random matrices $\Gamma_N := (\gamma_{i,j})_{1 \leq i,j \leq N}$ are called Wigner matrices of variance $\beta^2$ if the following holds.

- Each $\Gamma_N$ is symmetric, that is, $\gamma_{i,j} = \gamma_{j,i}$.
- For $i \leq j$, the random variables $\gamma_{i,j}$ are independent and centered.
- For $i \neq j$, $E(\gamma_{i,j}^2) = \beta^2$.
- For any $k \geq 2$, $E(|\gamma_{i,j}|^k) \leq c_k$, where $c_k$ is independent of $i \leq j$.

Definition 12 The sequence $\Gamma_N(k) := (\gamma_{i,j}(k))_{1 \leq i,j \leq N}$ of $N$-dimensional random matrices, indexed by $k \geq 1$, is called a Wigner process of variance $\beta^2$ and correlation function $r$, $r(k,k) = 1$, $|r(k,m)| \leq 1$ and $r(k,m) = r(m,k)$ if the following holds.

- Each $\Gamma_N(k)$ is a Wigner matrix of variance $\beta^2$ in the sense of definition 11.
- For $i \leq j$, each process $(\gamma_{i,j}(k))_k$ is independent of the others.
- For $i \neq j$, the process $(\gamma_{i,j}(k))_k$ is $r$-correlated, that is, for any $k \geq m$,
  \[ E(\gamma_{i,j}(k)\gamma_{i,j}(m)) := \beta^2 r(k,m). \] (9)

A Wigner process is stationary when $r$ is such that $r(k,m) = r(|k-m|)$.

Let $D_N$ be a sequence of random diagonal matrices, with independent and identically distributed entries of law $\mu$, having finite moment $\mu_k = \mu(X^k)$, $k \geq 1$, with $\mu_1 = 1$. Let

\[ Q_N^k := N^{-k/2}D_N \prod_{m=1}^{k} (\Gamma_N(m)D_N), \]

and set

\[ B_N^k(r) = N^{-1}E(\text{tr}(Q_N^k)). \]

Involutions, Dyck paths and rooted plane trees

For $k \geq 1$, $[k] := \{1,2,\ldots,k\}$, $I(k)$ is the set of the involutions of $[k]$ with no fixed point, $J(k)$ is the subset of $I(k)$ of the involutions $\sigma$ with
no crossing. This means that the configurations $i < j < \sigma(i) < \sigma(j)$ do not appear in $\sigma \in \mathcal{J}(k)$. Let $i \in \text{cr}(\sigma)$ denote the fact that $i < \sigma(i)$. Let $\mathcal{D}(2k)$ be the set of the Dyck paths of length $2k$, that is, of the sequences $c := (c_n)_{0 \leq n < 2k}$ of nonnegative integers such that $c_0 = c_{2k} = 0$, $c_n - c_{n-1} = \pm 1$, $n \in [2k]$. Thus, exactly $k$ indices $n \in [2k]$ correspond to ascending steps $(c_{n-1}, c_n)$, that is, to steps when $c_n = c_{n-1} + 1$. We denote this by $n \in \text{asc}(c)$. The $k$ others indices correspond to descending steps, that is, to steps when $c_n = c_{n-1} - 1$, and we denote this by $n \in \text{desc}(c)$. We make use of bijections between $\mathcal{D}(2k)$ and $\mathcal{J}(2k)$ [2]. If $c \in \mathcal{D}(2k)$, $\phi(c) := \sigma \in \mathcal{J}(2k)$ is an involution which maps each element of $\text{desc}(c)$ to a smaller element of $\text{asc}(c)$. Thus, $\text{cr}(\sigma) = \text{asc}(c)$. More specifically, if $n \in \text{desc}(c)$, $\sigma(n)$ is the greatest $m \leq n$ such that $(c_{m-1}, c_m) = (c_n, c_{n-1})$. Finally, the set $\mathcal{D}(2k)$ is in bijection with $\mathcal{F}_{k+1}$, the set of rooted plane trees on $k + 1$ nodes, where the bijection is given by the walk on the tree from the right to the left (see for example [25]). Let $\sigma_t$ denote the involution of $\mathcal{J}(2(|t| - 1))$ corresponding to $t \in \mathcal{F}$. Given $t \in \mathcal{F}_{k+1}$, consider the walk on $t$ from the right to the left: every edge $(v, w)$ with $w \in \text{ch}(v)$, is crossed at some instant $s_v \in [2k]$ as $(v \rightarrow w)$ and at a later time $s_w \in [2k]$ as $(w \rightarrow v)$. Clearly, $s_w = s_v + 2(|t_w| - 1) + 1$, where $t_w$ is the subtree of $t$ rooted at node $w$, that is the subgraph of $t$ induced by the nodes $u$ with $u \geq w$. $\sigma_t$ is such that $\sigma_t(s_v) = s_w$ and vice versa.

![Fig 1. Bijections between $\mathcal{F}_{k+1}$, $\mathcal{D}(2k)$ and $\mathcal{J}(2k)$ ($s_v = 2$ and $s_w = 5$).](image)

**Proposition 13** Assume that the covariance $r$ is such that $r(l, m) = r(|l - m|)$. Then, the functional $B^r \in B$ given by the weights

$$B^r_k = \beta^2 r(2k - 1), \forall k \geq 1,$$

is such that

$$\frac{B^r(t)}{B^r_{|t|}} = \prod_{i \in \text{cr}(\sigma_t)} (\beta^2 r(i, \sigma_t(i))).$$

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Proof: Let $t \in \mathcal{F}$. Let $s_v < s_w$ be the instants where the oriented edges $(v \to w)$ and $(w \to v)$, $w \in \text{ch}(v)$, are crossed during the walk on the tree. $r(s_v, \sigma_t(s_v)) = r(s_w - s_v) = r(2|t_w| - 1)$, and thus $\beta^2 r(s_v, \sigma_t(s_v)) = B^r_{t_w}$. Finally, $\prod_{i \in \text{cr}(\sigma_t)} \beta^2 r(i, \sigma_t(i)) = \prod_{w \neq \text{root}} B^r_{|t_w|} = B^r(t)/B^r_0$, as required.

As we have just seen, every Wigner process with covariance $r$ such that $r(l, m) = r(|l - m|)$ produces a bare Green function $B^r \in \mathcal{B}$. The converse is not true, that is, there exists $B \in \mathcal{B}$ such that $B$ is not of the form $B = B^r$ for some covariance function $r$. Set $\mathcal{B}^w = \{B \in \mathcal{B}; \exists \text{ a covariance } r \text{ with } B = B^r\}$.

Let $\psi_\mu$ be the generating function of the weight sequence $\psi_k = \mu_{k+1}$, and let $\omega_\mu(t)$, $t \in \mathcal{F}$ be the associated weight function.

**Theorem 3** Let $(\Gamma_N(k))_{k \geq 1}$ be a stationary Wigner process of covariance function $r$ and variance $\beta^2$, and let $D_N$ be a sequence of random diagonal matrices, independent of the Wigner process, with i.i.d. entries $\lambda_j$ of law $\mu$, with $\mu_1 = \mu(\lambda) = 1$ and finite moments $\mu_k = \mu(\lambda^k)$, $\forall k$. Then

$$B^N_{2k}(r) \longrightarrow B_{2k}(r) = \frac{1}{B^r_{k+1}} \sum_{t \in \mathcal{F}_{k+1}} B^r(t) \omega_\mu(t),$$

and $B^N_{2k+1}(r) \longrightarrow 0$, $N \to \infty$. Assume that the covariance is such that there exists a power series $L^r(z)$ with $B_k^r = L^r(k)/k$, $\forall k$. Then the formal power series

$$Y(z) = \sum_{k \geq 1} z^k B_k^r B_{2(k-1)}(r),$$

solves

$$Y' = L^r(\theta + 1) \psi_\mu(Y).$$

Moreover

$$\sum_{k \geq 1} z^k B_{2(k-1)}(r) = z \psi_\mu(Y). \quad (10)$$

**Example 14**

Let $B(t) = 1/t!$. If a tree $t$ has $n$ nodes and $n - 1$ edges, then the requirement $B_n = 1/n$ is satisfied iff $\beta^2 r(2n - 1) = 1/n$, that is $r$ must be such that $\beta^2 r(k) = 2/(k + 1)$, $k \in 2\mathbb{N} + 1$. By construction, $r(0) = 1$ and therefore $\beta^2 = 2$. $1/(x + 1)$ is positive definite, which implies that $B(t) = 1/t!$ is element of $\mathcal{B}^w$. Next, from Theorem 2 the generating function $Y(z) = \sum_{t \in \mathcal{F}} z^{|t|} B^r(t) \omega_\mu(t)$ is solution of the system $(d/dz)Y(z) = \psi_\mu(Y)$. 

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Assume that $\mu$ is the point mass $\delta_1$, that is each matrix $D_N$ is the identity matrix of size $N$, with $\psi_{\mu}(z) = 1/(1-z)$. The solution of the system is $Y(z) = 1 - \sqrt{1-2z} = 2\bar{Y}(z/2)$, where $\bar{Y}$ is the series given in Example 9. On the other hand, Proposition 13 and Theorem 3 show that $Y(z) = \sum_{k \geq 1} z^kB_k^2B_{2(k-1)}(r)$. Therefore the limiting mean normalized trace $B_{2k}(r)$ of the product of correlated random matrices $\prod_{m=1}^{2k} \Gamma_N(m)$ is such that $B_{2k}(r) = E(Z^{2k})/k!$, where $Z$ denotes a normal N(0,1) random variable.

\[ \square \]

**Example 15**

Consider as in Example 9 the special case where $L(z) = z$. The associated inductive parameter (see Remark 8) is the tree size. The covariance $r$ is constant with $r(k) \equiv 1$, and $B_k^N \equiv 1$. Then the generating function $Y$ is solution of the fixed point equation $Y(z) = z\psi_{\mu}(Y(z))$ (either by Theorem 11 or by (10)). Notice that in this situation, $\Gamma_N(m) \equiv \Gamma_N(1)$, and thus $B_k^N(r)$ describes the mean normalized moment of the spectral measure of the random matrix $D_N(\Gamma_N(1)D_N)^k$. This example can be extended by considering $L(z) = z\rho^z$, for some $0 < \rho \leq 1$. When $D_N$ is the identity matrix, $Y(z)$ is related to the Rogers-Ramanujan continued fraction [19], and corresponds to the generating function associated with path length, see [11, 25].

\[ \square \]

**Proof of Theorem 5** The first part is a generalization of Theorem 1 of [19]. Set $\tilde{\gamma}_{ij}(m) = \gamma_{ij}(m)\lambda_j$, and $\tilde{\Gamma}_N(m) = \Gamma_N(m)D_N$. The mean normalized trace adds the contributions $E(i) = E(\lambda_{i_0}\tilde{\gamma}_{i_0i_1}\cdots\tilde{\gamma}_{i_{k-1}i_k})$, for paths $i = (i_0)_{0 \leq l \leq k}$, with $i_l \in [N]$ and $i_0 = i_k$. The $\tilde{\gamma}_{ij}$ are centered, so that any edge $(i, j)$ appearing once appears at least twice. Given $i$, define $\varepsilon_l = 1$ and $\varepsilon_l = -1$ otherwise, and consider the walk $c = (c_l)$ defined by $c_l = \sum_{j=1}^{l} \varepsilon_j$, with $c_k \leq 0$. The support of $i$ is $s(i) = \{i_l; 0 \leq l \leq k\}$, of size $s = |s(i)|$, with $s \leq 1 + k/2$. The contribution $E(i)$ is independent of the labels $i_l$; they are $N(N-1)\cdots(N-s+1)$ labellings giving the same walk $c$, with the same contribution. Thus, the normalization $N^{-1+(1+k)/2}$ shows that the only walks surviving in the large $N$ limit are those with $s = 1 + k/2$. This shows that $B_k^N(r) \to 0$ when $k$ is odd. Concerning $B_k^N(r)$, $s = 1 + k$ means that every edge occurring in the path occurs exactly twice, in opposite directions. $c$ is a Dyck path of $D(2k)$; let $t \in F$ be the associated rooted plane tree, with involution $\sigma_t$. Using the right to left walk on $t$ and the independence of the random variables, the contribution $E(i)$ of any path leading to $c$ or $t$ is $E(i) = \prod_{m \in cr(\sigma_t)} E(\gamma(m)\gamma(\sigma_t(m)))E(\prod_v \lambda_v^{d(v)+1})$ where $d(v) = |ch(v)|$. 

\[ \square \]
From Proposition 13 one obtains \( E(i) = (B^r(t)/B_{k+1}^r) \prod_v \mu_{d(v)+1} \), with \( B_{2k}(r) = \sum_{t \in F_{k+1}} (B^r(t)/B_{k+1}^r) \prod_v \mu_{d(v)+1} \), as required. (10) is a consequence of the multiplicative form of bare Green functions and of Lemma 1.9, chap. III.1 of [14].

□

These results show that the elements of \( B^w \) appear naturally in the computation of normalized traces of products of large random matrices (see for example [23]). In the next Section we illustrate B-series by considering triangular operators from free probability.

6 On Dykema-Haagerup triangular operator

Let \( B \) be an algebra and \( A \) be a \( B \) bi-module. Let \( \kappa : A \times A \rightarrow B \) be a bilinear map. We follow [22] by defining the product \( a_1 \ast \kappa a_2 = \kappa(a_1, a_2) \), \( a_1, a_2 \in A \), and setting

- \( i \) \((ba_1) \ast \kappa a_2 = b(a_1 \ast \kappa a_2)\),
- \( ii \) \((a_1b) \ast \kappa a_2 = a_1 \ast \kappa (ba_2)\),
- \( iii \) \(a_1 \ast \kappa (a_2b) = (a_1 \ast \kappa a_2)b\).

Let \( \sigma \in J(2n) \) be an involution of \([2n]\) without fixed point and without crossing. Given a word \( a = a_1 \cdots a_{2n} \) in \( A \), \( \sigma \) induces parentheses on \( a \), and the previous rules permit the evaluation of this parenthesized word. This extends to a map \( \kappa_\sigma \) on \( A^{2n} \). Sniady defines such maps to prove a conjecture of Dykema and Haagerup on generalized circular elements. Let \((B \subset A, E)\) be an operator valued probability space, that is \( A \) is a unital \( \ast \)-algebra, \( B \subset A \) an unital \( \ast \)-subalgebra and \( E : A \rightarrow B \) be a conditional expectation (linear, \( E(1) = 1 \), and \( E(b_1ab_2) = b_1E(a)b_2 \), \( \forall b_1, b_2 \in B, a \in A \)).

**Definition 16** \( T \in A \) is a generalized circular element if there is a bilinear map \( \kappa \) satisfying the rules i), ii) and iii) such that

\[
E(b_1T^{s_1}b_2T^{s_2} \cdots b_{2n}T^{s_{2n}}) = \sum_{\sigma \in J(2n)} \kappa_\sigma(b_1T^{s_1}, \cdots , b_{2n}T^{s_{2n}}),
\]

\[
E(b_1T^{s_1}b_2T^{s_2} \cdots b_{2n+1}T^{s_{2n+1}}) = 0,
\]

\( \forall b_1, \cdots , b_{2n+1} \in B \) and \( \forall s_1, \cdots , s_{2n+1} \in \{1, \ast\} \).

The triangular operator \( T \) of Dykema and Haagerup is obtained from \( B = \mathbb{C}[x] \), the \( \ast \)-algebra of complex polynomials of one variable by setting

\[
[k(T, bT^\ast)](x) = \int_x^1 b(s)ds,
\]
\[ [\kappa(T^*, bT)](x) = \int_0^x b(s)ds, \]
\[ [\kappa(T, bT)](x) = [\kappa(T^*, bT^*)](x) = 0. \]

\( T \) is the limit for the convergence of *-moments of large upper triangular random matrices \( T_N \) \((\text{10})\). Define a trace \( \tau \) as (see \((\text{22})\))

\[ \tau(a) = \tau(E(a)), \quad \tau(b) = \int_0^1 b(s)ds. \]

In what follows, we use P-series (where P stands for partitioned differential systems, see \((\text{13})\)). We follow \((\text{6})\), and adapt his notations to P-series. Given some function \( \psi \), and two kernels \( (a^x(u,v))_{u,v \in [0,1]} \) and \( (a^y(u,v))_{u,v \in [0,1]} \), consider the iterated integrals \( \phi^x_u(t) \) and \( \phi^y_u(t) \) which are functionals over \( \mathcal{R} \) defined by \( \phi^x_u(*) = \phi^y_u(*) = 1 \), and, for \( t = B_+(t_1, \cdots, t_k) \),

\[
\phi^x_u(t) = \prod_{i=1}^k \int_0^1 a^x(u,v)\phi^y_v(t_i)dv, \\
\phi^y_u(t) = \prod_{i=1}^k \int_0^1 a^y(u,v)\phi^x_v(t_i)dv.
\]

**Lemma 17** Let \( a^x(u,v) = I_{[0,u]}(v) \) and \( a^y(u,v) = I_{[u,1]}(v) \). Then

\[
\tau((TT^*)^n) = \sum_{t \in \mathcal{F}_{n+1}} \int_0^1 \phi^x_v(t)dv = \sum_{t \in \mathcal{F}_{n+1}} \int_0^1 \phi^y_v(t)dv.
\]

**Proof:** The word \( W = (TT^*) \cdots (TT^*) \) is of the generic form with \( b_1 = \cdots b_{2n} = 1 \) (Definition \((\text{16})\)). Let \( t \in \mathcal{F}_{n+1} \) with associated involution \( \sigma_t \) (see Section \((\text{5})\)). Let \( s_v \) and \( s_w \) be the instants where the walk on \( t \) crosses the oriented edges \( (v \to w) \) and \( (w \to v) \), with \( w \in \text{ch}(v) \). We colour these edges by giving colour ‘1’ to \( (v \to w) \) when the symbol in \( W \) located at position \( s_v \) is \( T \), and give the colour ‘*’ otherwise. Clearly, both edges have different colours, and the elements of the set of edges \{ \( (v \to w); \ w \in \text{ch}(v) \} \) (the children of \( v \) in \( t \)) have the same colour. The result is then a consequence of the definition of the product with the rules i), ii) and iii).

\(\square\)

**Remark 18** Iterated integrals are natural objects to consider in the setting of Butcher’s Theory. For example, in the framework of Theorem \((\text{7})\) the iterated integrals \( \phi_u(t) \) defined by \( \phi_u(t) = \prod_{i=1}^k \int_0^a L(\theta + 1)(\phi_v(t_i))dv, \) when \( t = B_+(t_1, \cdots, t_k) \), are such that \( \phi_1(t) = B(t), \ \forall t \in \mathcal{F}. \)
Proposition 19  The P-series

\[ X_u(s) = X_0 + \sum_{t \in \mathbb{R}} \frac{s^{|t|}}{|t|!} \alpha(t) t! \delta_t \int_0^1 a^x(u, v) \phi^x_v(t) dv, \]

and

\[ Y_u(s) = Y_1 + \sum_{t \in \mathbb{R}} \frac{s^{|t|}}{|t|!} \alpha(t) t! \delta_t \int_0^1 a^y(u, v) \phi^y_v(t) dv, \]

are solutions of the integral system

\[ X_u(s) = X_0 + s \int_0^1 a^x(u, v) \psi(Y_v(s)) dv, \]

\[ Y_u(s) = Y_1 + s \int_0^1 a^y(u, v) \psi(X_v(s)) dv. \]

Proof: This is consequence of Butcher’s general theory (see [4]). To prove it more directly, proceed as in the proof of Theorem [4].

\[ \square \]

Corollary 20  Let \( X_0 = Y_1 = 0 \). Assume that \( a^x(u, v) = I_{[0,u]}(v) \) and \( a^y(u, v) = I_{[u,1]}(v) \). Suppose that \( \psi(z) = 1/(1 - z) \). Then

\[ Y_0(s) = \sum_{t \in \mathbb{R}} \frac{s^{|t|}}{|t|!} \alpha(t) t! \delta_t \int_0^1 \phi^x_v(t) dv = \sum_{t \in \mathcal{F}} s^{|t|} \tau(TT^*)^{|t| - 1}. \]

This result shows that the generating function of the *-moments of the operator \( TT^* \) can be obtained by solving the system given in Proposition 19. We recover in this way a result of [10], Lemmas 8.5 and 8.8.

Lemma 21  In the setting of Corollary 20, the generating function \( Y_0(s) \) solves

\[ G \left( \frac{s}{1 - Y_0(s)} \right) = s, \]  \hspace{1cm} (11)  

where \( G(z) = z \exp(-z) \), that is, \( L(s) = s/(1 - Y_0(s)) \) and \( G \) are inverse with respect to composition. Moreover \( \tau(TT^*)^n = n^n/(n + 1)! \).

Proof: We solve the integral system by looking for solutions of the form \( X_u(s) = 1 - \exp(\lambda u) \) and \( Y_u(s) = 1 - \exp(\lambda'(u - 1)) \), with \((d/du)X_u(s) = s/(1 - Y_u(s))\) and \((d/du)Y_u(s) = -s/(1 - X_u(s))\). We deduce that \( \lambda' = -\lambda \) is solution of the equation \( \lambda + s \exp(-\lambda) = 0 \). The formula for the moments of \( TT^* \) is a consequence of Lagrange’s inversion formula.

\[ \square \]
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