Research Article
A New Notion of Fuzzy Local Function and Some Applications

I. M. Taha\textsuperscript{1,2} and S. E. Abbas\textsuperscript{2}

\textsuperscript{1}Department Basic Sciences, Higher Institute of Engineering and Technology, Menoufia, Egypt
\textsuperscript{2}Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt

Correspondence should be addressed to I. M. Taha; imtaha2010@yahoo.com

Received 11 January 2022; Revised 18 April 2022; Accepted 3 May 2022; Published 13 June 2022

Academic Editor: Ferdinando Di Martino

Copyright © 2022 I. M. Taha and S. E. Abbas. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, a new notion of fuzzy local function called $r$-fuzzy local function is introduced, and some properties are given in a fuzzy ideal topological space $(X, \tau, \ell)$ in Šostak sense. After that, the concepts of fuzzy upper (resp., lower) almost $\ell$-continuous, weakly $\ell$-continuous, and almost weakly $\ell$-continuous multifunctions are introduced. Some properties and characterizations of these multifunctions along with their mutual relationships are discussed with the help of examples.

1. Introduction
Zadeh [1] introduced the basic idea of a fuzzy set as an extension of classical set theory. The basic notions of fuzzy sets have been improved and applied in different directions. Along this direction, we can refer [2–4]. A fuzzy multifunction (multivalued mapping) is a fuzzy set valued function [5–8]. The difference between fuzzy multifunctions and fuzzy functions has to do with the definition of an inverse image. For a fuzzy function, there is one inverse but for a fuzzy multifunction there are two types of inverses. By these two definitions of inverse, we can define the continuity of fuzzy multifunction. Fuzzy multifunctions have many applications, for instance, decision theory and artificial intelligence. Taha [9–11] introduced and studied the concepts of $r$-fuzzy $\ell$-open, $r$-generalized fuzzy $\ell$-open, and $r$-fuzzy $\delta \ell$-open sets in a fuzzy ideal topological space $(X, \tau, \ell)$ in Šostak sense [12]. Moreover, Taha [10, 11, 13] introduced and studied the concepts of fuzzy upper and lower $\alpha \ell$-continuous (resp., $\beta \ell$-continuous, $\delta \ell$-continuous, and generalized $\ell$-continuous) multifunctions via fuzzy ideals [14].

We lay out the remainder of this article as follows. Section 2 contains some basic definitions and results that help in understanding the obtained results. In Section 3, we introduce and study the notion of $r$-fuzzy local function by using fuzzy ideal topological space and the definition of fuzzy difference between two fuzzy sets. Additionally, from the definition of $r$-fuzzy local function, we introduce a stronger form of fuzzy upper (resp., lower) precontinuous multifunctions [15], namely, fuzzy upper (resp., lower) $\ell$-continuous multifunctions. In Sections 4, 5 and 6 we introduce the concepts of fuzzy upper (resp., lower) almost $\ell$-continuous, weakly $\ell$-continuous, and almost weakly $\ell$-continuous multifunctions. Several properties of these new multifunctions are established. Finally, Section 7 gives some conclusions and suggests some future works.

2. Preliminaries
In this section, we present the basic definitions which we need in the next sections. Throughout this paper, $X$ refers to an initial universe. The family of all fuzzy sets in $X$ is denoted by $I^X$ and for $\lambda \in I^X$, $\lambda^c(x) = 1 - \lambda(x)$ for all $x \in X$ (where $I = [0, 1]$ and $(I') = (0, 1]$). For $t \in I$, $t^c(x) = t$ for all $x \in X$. All other notations are standard notations of fuzzy set theory. Let us define the fuzzy difference between two fuzzy sets $\lambda, \mu \in I^X$ as follows:

$$\lambda \nabla \mu = \begin{cases} 
0, & \text{if } \lambda \leq \mu, \\
\lambda \land \mu^c, & \text{otherwise}.
\end{cases} \quad (1)$$

Hindawi
Advances in Fuzzy Systems
Volume 2022, Article ID 8954163, 10 pages
https://doi.org/10.1155/2022/8954163
Recall that a fuzzy ideal \( \ell \) on \( X \) [14] is a map \( \ell : I^X \rightarrow I \) that satisfies the following conditions: (i) \( \forall \lambda, \mu \in I^X \) and \( \lambda \leq \mu \Rightarrow \ell(\mu) \leq \ell(\lambda) \); (ii) \( \forall \lambda, \mu \in I^X \Rightarrow \ell(\lambda \lor \mu) \geq \ell(\lambda) \lor \ell(\mu) \). Also, \( \ell \) is called proper if \( \ell(1) = 0 \) and there exists \( \mu \in I^X \) such that \( \ell(\mu) > 0 \). The simplest fuzzy ideals on \( X, \ell_0 \), and \( \ell \) are defined as follows:

\[
\ell_0(\lambda) = \begin{cases} 
1, & \text{if } \lambda = \emptyset \\
0, & \text{otherwise} 
\end{cases}
\]

and \( \ell_1(\lambda) = 1 \lor \lambda \in I^X \). If \( \ell_1 \) and \( \ell_2 \) are fuzzy ideals on \( X \), we say that \( \ell_1 \) is finer than \( \ell_2 \) (\( \ell_2 \) is coarser than \( \ell_1 \)), denoted by \( \ell_1 \preceq \ell_2 \), iff \( \ell_2(\lambda) \leq \ell_1(\lambda) \forall \lambda \in I^X \). Let \( r \) be a fuzzy topology on \( X \) in Šostak sense, the triple \( (X, r, \ell) \) is called fuzzy ideal topological space.

A mapping \( F : X \rightarrow Y \) is called a fuzzy multifunction ([15, 16]) iff \( F(x) \in I^Y \) for each \( x \in X \). The degree of membership of \( y \) in \( F(x) \) is denoted by \( F(x)(y) = G_F(x, y) \) for any \( (x, y) \in X \times Y \). Also, \( F \) is normalized iff for each \( x \in X \), there exists \( y_0 \in Y \) such that \( G_F(x, y_0) = 1 \) and \( F \) is Crisp iff \( G_F(x, y) = 1 \) for each \( x \in X \) and \( y \in Y \). The upper inverse \( F^+(\mu) \), the lower inverse \( F^-(\mu) \) of \( \mu \in I^Y \), and the image \( F(\lambda) \) of \( \lambda \in I^X \) are defined as follows: \( F^+(\mu)(x) = \land \{ G_F(x, y) \mid y \in Y, G_F(x, y) \geq \mu \} \), \( F^-(\mu)(x) = \lor \{ G_F(x, y) \mid y \in Y, G_F(x, y) \leq \mu \} \), and \( F(\lambda)(y) = \lor \{ G_F(x, y) \mid x \in X, \lambda(x) \geq \mu(y) \} \). All definitions and properties of image, lower, and upper are found in [15].

3. Fuzzy Ideal and \( r \)-Fuzzy Local Function

**Definition 1.** Let \( (X, r, \ell) \) be a fuzzy ideal topological space, \( \lambda \in I^X \) and \( r \in I^\ell \). Then, the \( r \)-fuzzy local function \( \lambda^r \) of \( \lambda \) is defined as follows:

\[
\lambda^r = \land \{ \mu \in I^X : \ell(\lambda \land \mu) \geq r, \ell(\mu) \geq r \}.
\]

**Remark 1**

(1) If we take \( \ell = \ell_0 \) for each \( \lambda \in I^X \), we have

\[
\lambda^r = \land \{ \mu \in I^X : \lambda \leq \mu, \ell(\mu) \geq r \} = C_r(\lambda, r).
\]

(2) If we take \( \ell = \ell_1 \) (resp., \( \ell(\lambda) \geq r \)) for each \( \lambda \in I^X \), we have \( \lambda^r = \emptyset \).

We will occasionally write \( \lambda^r \) or \( \lambda^r(\ell) \) for \( \lambda^r(\ell, r) \).

**Theorem 1.** Let \( (X, r, \ell) \) be a fuzzy ideal topological space and \( \ell_1 \) and \( \ell_2 \) be two fuzzy ideals on \( X \). Then, for any fuzzy sets \( \lambda, \nu \in I^X \),

1. \( 0^r_\nu = 0 \)
2. \( 0^r = 0 \)
3. \( \lambda \leq \mu \Rightarrow \lambda^r \leq \nu^r \)
4. \( \lambda \leq \mu \Rightarrow \lambda^r \leq \nu^r \)
5. \( (\lambda^r)_\nu \leq \lambda^r \) and \( (\lambda^r)^r \neq (\lambda^r)^r \)
6. \( (\lambda \lor \mu)^r = \lambda^r \lor \mu^r \) and \( (\lambda \lor \mu)^r \leq \lambda^r \lor \mu^r \)
7. \( \ell(\lambda \lor \mu) \geq \ell(\lambda) \lor \ell(\mu) \)
8. \( \tau(\tau(\ell)) \leq \tau(\ell) \leq \tau(\ell) \)

**Proof**

(1) From Definition 1, we have \( 0^r_\nu = 0 \).

(2) Suppose that \( \lambda^r \leq \nu^r \) and \( \lambda \leq \nu \). By the definition of \( \nu^r \), there exists \( \mu \in I^X \) with \( \nu^r \leq \mu \), \( \ell(\lambda \land \mu) \geq r \), and \( \ell(\mu) \geq r \) such that \( \lambda^r \leq \mu^r \).

Hence, \( \lambda \leq \nu \) implies \( \lambda^r \leq \nu^r \). Hence, \( \lambda^r \leq \nu^r \).

(3) Suppose that \( \lambda^r \leq \nu^r \). By the definition of \( \lambda^r \), there exists \( \mu \in I^X \) with \( \lambda^r \leq \mu^r \).

Hence, \( \lambda^r \leq \nu^r \).

(4) From Definition 2, we have \( \lambda^r \leq \nu^r \).

(5) By (4), we have \( \lambda^r \leq \nu^r \).

(6) Suppose \( \lambda^r \leq \nu^r \). By the definition of \( \nu^r \), there exists \( \mu \in I^X \) with \( \nu^r \leq \mu^r \).

Hence, \( \lambda^r \leq \nu^r \).

**Example 1.** Define \( r, \ell : I^X \rightarrow I \) as follows:

\[
\ell(\emptyset) = 0, \quad \ell(1) = 1, \quad \ell(\lambda \lor \mu) = \ell(\lambda) \lor \ell(\mu) \leq \ell(\lambda) \lor \ell(\mu) \leq r \frac{\ell(\lambda) \lor \ell(\mu) \geq r}.\]
Lemma 1. Let $(X, r, \ell)$ be a fuzzy ideal topological space, and 
\[ \{\lambda_j : j \in J\} \subseteq I^X. \] Then, \( \lor_{j \in J} (\lambda_j)^* \leq (\lor_{j \in J} (\lambda_j)^* \ (\text{resp.}, \ (\land_{j \in J} (\lambda_j)^* \leq (\land_{j \in J} (\lambda_j)^* \}. \]

Proof. Since \( \lambda_j \leq \lambda \), for each \( j \in J \), by Theorem 1 (2), we have 
\( (\lambda_j)^* \leq (\lor_{j \in J} (\lambda_j)^* \). This implies \( \lor_{j \in J} (\lambda_j)^* \leq (\lor_{j \in J} (\lambda_j)^* \). Other case is similarly proved. \( \square \)

Definition 2. Let \((X, r, \ell)\) be a fuzzy ideal topological space. Then, for each \( \lambda \in I^X \) and \( r \in \Gamma \), we define an operator 
\( \text{Cl}^*: I^X \times \Gamma \rightarrow I^X \) as follows:
\[ \text{Cl}^*(\lambda, r) = \lambda \land \lambda^*_r. \] (6)

The fuzzy topology generated by \( \text{Cl}^* \) is denoted by \( \tau^* \), i.e., \( \tau^*(\lambda) = \lor_{r \in \Gamma}(\lambda \land \lambda_r^*) \) and \( \tau \leq \tau^* \). Now, if \( \ell = \ell_0 \), then \( \text{Cl}^*(\lambda, r) = \lambda \land \lambda^*_r = \lambda \land C_\ell(\lambda, r) = C_\ell(\lambda, r) \) for each \( \lambda \in I^X \). So, \( \tau^* = \tau \). Again if \( \ell = \ell_1 \) (resp., \( \ell(\lambda) \geq \lambda \)), then 
\( \text{Cl}^*(\lambda, r) = \lambda \) for each \( \lambda \in I^X \).

Theorem 2. Let \((X, r, \ell)\) be a fuzzy ideal topological space. Then, for any fuzzy sets \( \lambda, \nu \in I^X \) and \( r, t, \eta \in \Gamma \), the operator 
\( \text{Cl}^*: I^X \times \Gamma \rightarrow I^X \) satisfies the following properties:

(1) \( \text{Cl}^*(0, r) = 0 \)
(2) \( \lambda \leq \text{Cl}^*(\lambda, r) \leq C_\ell(\lambda, r) \)
(3) If \( \lambda \leq \nu \), then \( \text{Cl}^*(\lambda, r) \leq \text{Cl}^*(\nu, r) \)
(4) \( \text{Cl}^*(\lambda \lor \nu, r) = \text{Cl}^*(\lambda, r) \lor \text{Cl}^*(\nu, r) \)
(5) \( \text{Cl}^*(\lambda \land \nu, r) \leq \text{Cl}^*(\lambda, r) \land \text{Cl}^*(\nu, r) \)
(6) \( \text{Cl}^*(\text{Cl}^*(\lambda, r), r) = \text{Cl}^*(\lambda, r) \)

Proof

(1) Hence, \( \text{Cl}^*(0, r) = 0 \lor 0^*_r \) and \( 0^*_r = 0 \) implies 
\( \text{Cl}^*(0, r) = 0 \).

(2) Hence, \( \text{Cl}^*(\lambda, r) = \lambda \land \lambda^*_r \) implies \( \lambda \leq \text{Cl}^*(\lambda, r) \). Since \( \lambda \leq C_\ell(\lambda, r) \) and from Theorem 1 (4), we have 
\( \lambda^*_r \leq \lambda \land \lambda^*_r \) implying 
\( \text{Cl}^*(\lambda, r) \leq C_\ell(\lambda, r) \). Thus, 
\( \lambda \leq \text{Cl}^*(\lambda, r) \leq C_\ell(\lambda, r) \).

(3) From \( \lambda \leq \nu \) and Theorem 1 (2), we have 
\( \lambda \land \lambda^*_r \leq \nu \land \nu^*_r \), i.e., 
\( \text{Cl}^*(\lambda, r) \leq \text{Cl}^*(\nu, r) \).

(4) By the definition of \( \text{Cl}^* \) and from Theorem 1 (6), we have 
\( \text{Cl}^*(\lambda \lor \nu, r) = (\lambda \lor \nu) \land (\lambda^*_r \lor \nu^*_r) = (\lambda \lor \nu) \land (\lambda^*_r \lor \nu^*_r) = \text{Cl}^*(\lambda, r) \lor \text{Cl}^*(\nu, r) \).

(5) Hence, \( \lambda \land \nu \leq \lambda \land \nu \) implies 
\( \text{Cl}^*(\lambda \land \nu, r) \leq \text{Cl}^*(\lambda, r) \) and 
\( \text{Cl}^*(\lambda \lor \nu, r) \leq \text{Cl}^*(\nu, r) \).

(6) From (2) and (3), we have 
\( \text{Cl}^*(\lambda, r) \leq \text{Cl}^*(\text{Cl}^*(\lambda, r), r) \).

(7) Hence, \( \lambda \land \nu \leq \lambda \land \nu \) implies 
\( \text{Cl}^*(\lambda \land \nu, r) \leq \text{Cl}^*(\lambda, r) \).

The following theorem is similarly proved, as in Theorem 2. \( \square \)

Theorem 3. Let \((X, r, \ell)\) be a fuzzy ideal topological space. Then, for each \( \lambda \in I^X \) and \( r \in \Gamma \), we define an operator 
\( \text{int}^* : I^X \times \Gamma \rightarrow I^X \) as follows:
\[ \text{int}^*(\lambda, r) = \lambda \land \lambda^*_r. \] (7)

For each \( \lambda, \nu \in I^X \), the operator \( \text{int}^* \) satisfies the following properties:

(1) \( \text{int}^*(1, r) = 1 \)
(2) \( I_\epsilon(\lambda, r) \leq \text{int}^*(\lambda, r) \leq \lambda \)
(3) If \( \lambda \leq \nu \), then \( \text{int}^*(\lambda, r) \leq \text{int}^*(\nu, r) \)
(4) \( \text{int}^*(\text{int}^*(\lambda, r), r) = \text{int}^*(\lambda, r) \)
(5) \( \text{int}^*(\lambda \land \nu, r) = \text{int}^*(\lambda, r) \land \text{int}^*(\nu, r) \)
(6) \( \text{int}^*(\lambda, r) = I_\ell(\lambda, r, \ell) \)
(7) \( \text{int}^*(\lambda, r) = (\text{Cl}^*(\lambda, r))^c \)

Definition 3. A fuzzy multifunction \( F : (X, r, \ell) \rightarrow (Y, \eta) \) is called

(1) Fuzzy upper (resp., lower) \( \ell \)-continuous at a fuzzy point \( x \in \text{dom}(F) \) iff \( x_\mu \in F^\mu(\mu) \) (resp., \( x_\mu \in F^\mu(\mu) \)) for each \( \mu \in I^r, \eta(\mu) \geq r \), there exists \( \lambda \in I^X, \tau(\lambda) \geq r \), and \( x_\lambda \in \lambda \) such that \( \lambda \land \text{dom}(F) \subseteq F^\mu(\mu)^c \) (resp., \( \lambda \leq F^\mu(\mu)^c \)).

(2) Fuzzy upper \( \ell \)-continuous (resp., fuzzy lower \( \ell \)-continuous) iff it is fuzzy upper \( \ell \)-continuous (resp., fuzzy lower \( \ell \)-continuous) at every \( x_\mu \in \text{dom}(F) \).

Remark 2. If \( F \) is normalized, then \( F \) is fuzzy upper \( \ell \)-continuous at a fuzzy point \( x_\mu \in \text{dom}(F) \) iff \( x_\mu \in F^\mu(\mu) \) for each \( \mu \in I^r, \eta(\mu) \geq r \) there exists \( \lambda \in I^X, \tau(\lambda) \geq r \) and \( x_\lambda \in \lambda \) such that \( \lambda \leq F^\mu(\mu)^c \).
Theorem 4. Let $F: (X, \tau, \ell) \rightarrow (Y, \eta)$ be a fuzzy multifunction (resp., normalized fuzzy multifunction). Then, $F$ is fuzzy lower (resp., upper) $\ell$-continuous if $F^l(\mu) \subseteq I_\tau([F^l(\mu)]^*, r)$ (resp., $F^u(\mu) \subseteq I_\tau([F^u(\mu)]^*, r)$) for each $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $r \in I_\tau$.

Proof. ($\Rightarrow$) Let $x_t \in \text{dom}(F), \mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_t \in \lambda$. Since there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \in [F^l(\mu)]^*$, it follows that $x_t \in \lambda \in [F^l(\mu)]^*$. Hence $F^l(\mu) \subseteq I_\tau([F^l(\mu)]^*, r)$ and hence $F^l(\mu) \subseteq I_\tau([F^l(\mu)]^*, r)$.

($\Leftarrow$) Let $x_t \in \text{dom}(F), \mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_t \in \lambda$. Then, $F^l(\mu) \subseteq I_\tau([F^l(\mu)]^*, r)$ and hence $x_t \in \lambda \in [F^l(\mu)]^*$. Thus, $F$ is fuzzy lower $\ell$-continuous. Other case is similarly proved. \hfill \Box

Remark 3

(1) Every fuzzy lower (resp., upper) $\ell$-continuous multifunction is fuzzy lower (resp., upper) pre-

(2) If we take $\ell = \ell_0$, we have $F$ as fuzzy lower (resp., upper) $\ell$-continuous if it is fuzzy lower (resp., upper) pre-

(3) Fuzzy upper (resp., lower) $\ell$-continuity and fuzzy upper (resp., lower) semicontinuity [15] are independent notions, as shown by example 2.

Example 2. Let $X = \{x_1, x_2\}, Y = \{y_1, y_2, y_3\}$, and $F: X \rightarrow Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.1, G_F(x_1, y_2) = 1.0, G_F(x_1, y_3) = 0.3, G_F(x_2, y_1) = 0.5, G_F(x_2, y_2) = 0.1$, and $G_F(x_2, y_3) = 1.0$. Define $\tau_1, \tau_2, \ell^1, \ell^2: I^X \rightarrow I$ and $H: I^Y \rightarrow I$ as follows:

\[
\tau_1(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{1}{2}, & \text{if } \lambda = 0.2, \\
\frac{3}{4}, & \text{if } \lambda = 0.3, \\
0, & \text{otherwise}, 
\end{cases} \\
\tau_2(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{1}{2}, & \text{if } \lambda = 0.3, \\
0, & \text{otherwise}, 
\end{cases} \\
\ell^1(\nu) = \begin{cases} 
1, & \text{if } \nu = 0, \\
\frac{2}{3}, & \text{if } 0 < \nu < 0.3, \\
0, & \text{otherwise}, 
\end{cases} \\
\ell^2(\nu) = \begin{cases} 
1, & \text{if } \nu = 0, \frac{1}{2} \text{ if } 0 < \nu < 0.2, \\
0, & \text{otherwise}, 
\end{cases}
\]

Then,

(1) $F: (X, \tau, \ell^1) \rightarrow (Y, \eta)$ is fuzzy upper (resp., lower) semicontinuous, but it is not fuzzy upper (resp., lower) $\ell$-continuous because $F^u(0.2) = 0.2$ (resp., $F^l(0.2) = 0.2$) and $0.2 \leq I_\tau(0.2/12, 1/2) = 0$

(2) $F: (X, \tau, \ell^2) \rightarrow (Y, \eta)$ is fuzzy upper (resp., lower) $\ell$-continuous, but it is not fuzzy upper (resp., lower) semicontinuous because $F^u(0.2) = 0.2$ (resp., $F^l(0.2) = 0.2$) and $0.2 \leq I_\tau(0.2/12, 1/2) = 0$.3 but $0.2 \notin I_\tau(0.2/12, 1/2) = 0$

Corollary 1. Let $F: (X, \tau, \ell) \rightarrow (Y, \eta)$ and $H: (Y, \eta) \rightarrow (Z, \mu)$ be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, $H \circ F$ is fuzzy lower (resp., upper) $\ell$-continuous if $F$ is fuzzy lower (resp., upper) $\ell$-continuous and $H$ is fuzzy lower (resp., upper) semicontinuous.

4. Fuzzy Upper and Lower Almost $\ell$-Continuity

Definition 4. A fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$ is called

(1) Fuzzy upper almost $\ell$-continuous at a fuzzy point $x_t \in \text{dom}(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \in \text{dom}(F) \subseteq F^l(\mu) \subseteq I^y(\mu, r)$.

(2) Fuzzy lower almost $\ell$-continuous at a fuzzy point $x_t \in \text{dom}(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \in F^u(\mu) \subseteq I^y(\mu, r)$.

(3) Fuzzy upper almost $\ell$-continuous (resp., fuzzy lower almost $\ell$-continuous) iff it is fuzzy upper almost $\ell$-continuous (resp., fuzzy lower almost $\ell$-continuous) at every $x_t \in \text{dom}(F)$.

Remark 4

(1) If $F$ is normalized, $F$ is fuzzy upper almost $\ell$-continuous at a fuzzy point $x_t \in \text{dom}(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \in F^l(\mu) \subseteq I^y(\mu, r)$.r.
(2) Fuzzy upper (lower) semicontinuity ⇒ fuzzy upper (lower) almost \(\ell\)-continuity ⇒ fuzzy upper (lower) almost continuity [17].

(3) Fuzzy upper (lower) almost \(\ell\)-continuity ⇔ fuzzy upper (lower) almost continuity.

**Theorem 5.** For a fuzzy multifunction \(F: (X, r) \rightarrow (Y, \eta, \ell)\), \(\mu \in I^Y\), and \(r \in I_\omega\), the following statements are equivalent:

1. \(F\) is fuzzy lower almost \(\ell\)-continuous.
2. \(F^2(\mu) \leq I_{\tau}(F^1(I_\eta(Cl^*_\mu(r), r)), r), \) if \(\eta(\mu) \geq r\)
3. \(C_\ell(F^1(Cl^*_\mu(r), r), r) \leq F^2(\mu), \) if \(\eta(\mu) \geq r\)

**Proof**

1. (1) \(\Rightarrow\) (2) Let \(x_t \in \text{dom}(F), \mu \in I^Y\) with \(\eta(\mu) \geq r\) and \(x_t \in F^1(\mu)\). Then, there exists \(\lambda \in I^X\) with \(\tau(\lambda) \geq r\) and \(x_t \in \lambda\) such that \(\lambda \leq F^1(I_\eta(Cl^*_\mu(r), r)),\) i.e., \(\lambda \leq F^1(F^1(I_\eta(Cl^*_\mu(r), r), r))\).

2. (2) \(\Rightarrow\) (3) Let \(\mu \in I^Y\) with \(\eta(\mu) \geq r\). Then, by (2), \(F^2(\mu)^\ell = F^3(\mu^\ell) \leq I_{\tau}(F^1(I_\eta(Cl^*_\mu(r), r)), r) = (C_\ell(F^1(Cl^*_\mu(r), r), r))\). Thus, \(C_\ell(F^1(Cl^*_\mu(r), r), r) \leq F^2(\mu)\).

3. (3) \(\Rightarrow\) (1) Let \(x_t \in \text{dom}(F), \mu \in I^Y\) with \(\eta(\mu) \geq r\) and \(x_t \in F^1(\mu)\). Then, by (3), \(I_{\tau}(F^1(I_\eta(Cl^*_\mu(r), r), r)) \leq F^2(\mu)^\ell = (F^2(\mu))\), i.e., \(F^2(\mu) \leq I_{\tau}(F^1(I_\eta(Cl^*_\mu(r), r), r))\). Hence, \(F\) is fuzzy lower almost \(\ell\)-continuous.

The following theorem is similarly proved as in Theorem 5.

**Theorem 6.** For a normalized fuzzy multifunction \(F: (X, r) \rightarrow (Y, \eta, \ell), \mu \in I^Y\) and \(r \in I_\omega\), the following statements are equivalent:

1. \(F\) is fuzzy upper almost \(\ell\)-continuous.
2. \(F^1(\mu) \leq I_{\tau}(F^0(I_\eta(Cl^*_\mu(r), r), r), r), \) if \(\eta(\mu) \geq r\)
3. \(C_\ell(F^0(Cl^*_\mu(r), r), r) \leq F^1(\mu), \) if \(\eta(\mu) \geq r\)

**Example 3.** Define a fuzzy multifunction \(F: X \rightarrow Y\) as in example 2, \(\tau_1, \tau_2: I^X \rightarrow I\) and \(\eta_1, \eta_2, \ell^1, \ell^2: I^Y \rightarrow I\) as follows:

\[
\tau_1(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{3}{4}, & \text{if } \lambda = 0.3, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\tau_2(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{1}{2}, & \text{if } \lambda = 0.5, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\eta_1(\mu) = \begin{cases} 
1, & \text{if } \mu \in [0, 1], \\
\frac{1}{2}, & \text{if } \mu = 0.2, \\
2, & \text{if } \mu = 0.3, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\eta_2(\mu) = \begin{cases} 
1, & \text{if } \mu \in [0, 1], \\
\frac{1}{2}, & \text{if } \mu = 0.5, \\
3, & \text{if } \mu = 0.4, \\
0, & \text{otherwise,}
\end{cases}
\]

Then,

1. \(F: (X, \tau_1) \rightarrow (Y, \eta_1, \ell^1)\) is fuzzy upper (resp., lower) almost \(\ell\)-continuous, but it is not fuzzy upper (resp., lower) semicontinuous because \(0.2 \leq F^0(0.2, 1/2) \leq I_{\tau_1}(F^0(I_\eta(Cl^*_\mu(r), 1/2), 1/2), 1/2) = 0.3\) and \(0.3 \leq F^0(0.3, 1/2) \leq I_{\tau_1}(F^0(I_\eta(Cl^*_\mu(r), 1/2)), 1/2) = 0.3\), but \(0.2 \notin F^0(I_\eta(Cl^*_\mu(r), 1/2), 1/2) = 0.3\).

2. \(F: (X, \tau_2) \rightarrow (Y, \eta_2, \ell^2)\) is fuzzy upper (resp., lower) almost continuous, but it is not fuzzy upper (resp., lower) almost \(\ell\)-continuous because \(0.4 \leq F^0(0.4, 1/2) \leq I_{\tau_2}(F^0(I_\eta(Cl^*_\mu(r), 0.4, 1/2), 1/2), 1/2) = 0.5\) and \(0.3 \leq F^0(0.3, 1/2) \leq I_{\tau_2}(F^0(I_\eta(Cl^*_\mu(r), 0.5, 1/2)), 1/2) = 0.5\), but \(0.4 \
otin F^0(I_\eta(Cl^*_\mu(r), 0.4, 1/2), 1/2) = 0.5\).
Theorem 7. For a fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$, $\mu \in \mathcal{I}^Y$, and $r \in \mathcal{I}_\omega$, the following statements are equivalent:

1. $F$ is fuzzy lower almost $\ell$-continuous.
2. $\tau(F^\ell(\mu)) \geq r$ if $\mu = I_{\eta}(CL^\ell(\mu, r), r)$.
3. $\tau(F^\ell(I_{\eta}(CL^\ell(\mu, r), r))) \geq r$ if $\eta(\mu) \geq r$.

Proof

$(\Rightarrow)$ Let $\mu \in \mathcal{I}^Y$ with $\mu = C_\eta(\text{int}^* \mu, r)$, then $\eta(\mu) \geq r$. By Theorem 5

$(\Leftarrow)$ $F^\ell(\mu) \leq I_{\eta}(F^\ell(I_{\eta}(CL^\ell(\mu, r), r)), r) = I_{\eta}(F^\ell(\mu), r)$.

Thus, $\tau(F^\ell(\mu)) \geq r$.

$(\Leftrightarrow)$ $(3)$ is obvious.

$(\Rightarrow)$ Let $x_\ell \in \text{dom}(F)$, $\mu \in \mathcal{I}^Y$ with $\mu(\eta) \geq r$ and $x_\ell \in F^\ell(\mu)$. Then, by $(3)$ and $\mu \leq I_{\eta}(CL^\ell(\mu, r), r)$, $\tau(F^\ell(I_{\eta}(CL^\ell(\mu, r), r))) \geq r$ and $x_\ell \in F^\ell(\mu) \leq I_{\eta}(F^\ell(I_{\eta}(CL^\ell(\mu, r), r)), r)$. Thus, $F$ is fuzzy lower almost $\ell$-continuous.

The following theorems are similarly proved as in Theorem 7. \qed

Theorem 8. For a fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$, $\mu \in \mathcal{I}^Y$, and $r \in \mathcal{I}_\omega$, the following statements are equivalent:

1. $F$ is fuzzy lower almost $\ell$-continuous.
2. $\tau(F^\ell(\mu)) \geq r$ if $\mu = C_\eta(\text{int}^* \mu, r, r)$.
3. $\tau((F^\ell(I_{\eta}(\text{int}^* \mu, r, r)))) \geq r$ if $\eta(\mu) \geq r$.

Theorem 9. For a normalized fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$, $\mu \in \mathcal{I}^Y$ and $r \in \mathcal{I}_\omega$, the following statements are equivalent:

1. $F$ is fuzzy upper almost $\ell$-continuous.
2. $\tau(F^\ell(\mu)) \geq r$ if $\mu = I_{\eta}(CL^\ell(\mu, r), r)$.
3. $\tau(F^\ell(I_{\eta}(CL^\ell(\mu, r), r))) \geq r$ if $\eta(\mu) \geq r$.

Theorem 10. For a normalized fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$, $\mu \in \mathcal{I}^Y$, and $r \in \mathcal{I}_\omega$, the following statements are equivalent:

1. $F$ is fuzzy upper almost $\ell$-continuous.
2. $\tau(F^\ell(\mu)) \geq r$ if $\mu = I_{\eta}(\text{int}^* \mu, r, r)$.
3. $\tau((F^\ell(I_{\eta}(\text{int}^* \mu, r, r)))) \geq r$ if $\eta(\mu) \geq r$.

Theorem 11. Let $F: (X, \tau) \rightarrow (Y, \eta, \ell)$ be a fuzzy multifunction. Then, $F$ is fuzzy lower almost $\ell$-continuous iff $C_\eta(F^\ell(\mu), r) \leq F^\ell(C_\eta(\mu, r))$ for any $\mu \in \mathcal{I}^Y$ with $\mu \leq C_\eta(\text{int}^* \mu, r)$, and $r \in \mathcal{I}^\ell$.

Proof. $(\Rightarrow)$ Let $\mu$ be a fuzzy lower almost $\ell$-continuous. Then, for any $\mu \in \mathcal{I}^Y$ with $\mu \leq C_\eta(\text{int}^* \mu, r)$, $(\nu) \geq r$ (say), where $\nu = C_\eta(\text{int}^* (\nu, r), r)$. By Theorem 8, $\tau((F^\ell(\nu))) \geq r$ and thus $C_\eta(F^\ell(\nu), r) \leq C_\eta(F^\ell(\nu), r) = F^\ell(C_\eta(\text{int}^* (\nu, r), r)) \leq F^\ell(C_\eta(\mu, r))$.

$(\Leftarrow)$ Let $\mu \in \mathcal{I}^Y$ with $\mu = C_\eta(\text{int}^* \mu, r)$, then $\mu \leq C_\eta(\text{int}^* \mu, r)$ and $C_\eta(F^\ell(\mu), r) \leq F^\ell(C_\eta(\mu, r)) = F^\ell(\mu)$. Therefore, we obtain $\tau(F^\ell(\mu)) \geq r$. Thus, by Theorem 8, $F$ is fuzzy lower almost $\ell$-continuous.

The following theorem is similarly proved as in Theorem 11. \qed

Theorem 12. Let $F: (X, \tau) \rightarrow (Y, \eta, \ell)$ be a normalized fuzzy multifunction. Then, $F$ is fuzzy upper almost $\ell$-continuous iff $C_\eta(F^\ell(\mu), r) \leq F^\ell(C_\eta(\mu, r))$ for any $\mu \in \mathcal{I}^Y$ with $\mu \leq C_\eta(\text{int}^* \mu, r)$, and $r \in \mathcal{I}^\ell$.

Theorem 13. Let $\{F_i: (X, \tau) \rightarrow (Y, \eta, \ell)\}_{i \in I}$ be a family of fuzzy lower almost $\ell$-continuous. Then, $\cup_{i \in I} F_i$ is fuzzy lower almost $\ell$-continuous.

Proof. Let $\mu \in \mathcal{I}^Y$. Then, $(\cup_{i \in I} F_i)(\mu) = \vee(F_i(\mu))$. Since $(\cup_{i \in I} F_i)(\mu) = \vee(F_i(\mu)) \geq r$ for any $\mu = I_{\eta}(CL^\ell(\mu, r), r)$ and $r \in \mathcal{I}^{\ell}$. Then, $\forall r \in \mathcal{I}^\ell$ $\mu(\tau(F_i(\mu))) \geq \tau(F_i(\mu)) \geq r$. Hence, $\cup_{i \in I} F_i$ is fuzzy lower almost $\ell$-continuous. \qed

Theorem 14. Let $F_1$ and $F_2: (X, \tau) \rightarrow (Y, \eta, \ell)$ be two normalized fuzzy upper almost $\ell$-continuous. Then, $F_1 \cup F_2$ is fuzzy upper almost $\ell$-continuous.

Proof. Let $\mu \in \mathcal{I}^Y$. Then, $(F_1 \cup F_2)(\mu) = F_1(\mu) \lor F_2(\mu)$. Since $F_1$ and $F_2$ are two normalized fuzzy upper almost $\ell$-continuous, $\tau(F_1(\mu)) \geq r$ for any $\mu = I_{\eta}(CL^\ell(\mu, r), r)$ and $r \in \mathcal{I}^{\ell}$. Then, $\tau((F_1 \cup F_2)(\mu)) = \tau(F_1(\mu) \lor F_2(\mu)) \geq r$. Hence, $F_1 \cup F_2$ is fuzzy upper almost $\ell$-continuous. \qed

Corollary 2. Let $F: (X, \tau) \rightarrow (Y, \eta)$ and $H: (Y, \eta) \rightarrow (Z, \gamma)$ be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, $H^\ell$ is fuzzy lower (resp., upper) almost $\ell$-continuous if $F$ is fuzzy lower (resp., upper) almost $\ell$-continuous and $H$ is fuzzy lower (resp., upper) semicontinuous.

5. Fuzzy Upper and Lower Weakly $\ell$-Continuity

Definition 5. A fuzzy multifunction $F: (X, \tau) \rightarrow (Y, \eta, \ell)$ is called

1. Fuzzy upper weakly $\ell$-continuous at a fuzzy point $x_\ell \in \text{dom}(F)$ iff $x_\ell \in F^\ell(\mu)$ for each $\mu \in \mathcal{I}^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in \mathcal{I}^Y$ with $\tau(\lambda) \geq r$ and $x_\ell \in \lambda$ such that $\lambda \in \text{dom}(F) \subseteq F^\ell(\text{int}^* \mu, r)$.

2. Fuzzy lower weakly $\ell$-continuous at a fuzzy point $x_\ell \in \text{dom}(F)$ iff $x_\ell \in F^\ell(\mu)$ for each $\mu \in \mathcal{I}^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in \mathcal{I}^Y$ with $\tau(\lambda) \geq r$ and $x_\ell \in \lambda$ such that $\lambda \subseteq F^\ell(\text{int}^* \mu, r)$.

3. Fuzzy upper weakly $\ell$-continuous (resp., fuzzy lower weakly $\ell$-continuous) iff it is fuzzy upper weakly $\ell$-continuous (resp., fuzzy lower weakly $\ell$-continuous) at every $x_\ell \in \text{dom}(F)$. 

Remark 5

(1) If $F$ is normalized, $F$ is fuzzy upper weakly $\ell$-continuous at a fuzzy point $x_i \in \text{dom}(F)$ iff $x_i \in F^u(\mu)$ for each $\mu \in I^X$ and $\eta(\mu) \geq r$, there exists $\lambda \in I^Y$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq F^u(C^r(\mu, r))$.

(2) Fuzzy upper (lower) almost $\ell$-continuity $\Rightarrow$ fuzzy upper (lower) weakly $\ell$-continuity $\Rightarrow$ fuzzy upper (lower) weakly continuity [17].

(3) Fuzzy upper (lower) weakly $\ell_r$-continuity $\Rightarrow$ fuzzy upper (lower) weakly continuity.

Theorem 15. A fuzzy multifunction $F : (X, \tau) -\rightarrow (Y, \eta, \ell)$ is fuzzy lower weakly $\ell$-continuous iff $F^u(\mu) \subseteq I_\ell(F^u(C^r(\mu, r), r))$, for each $\mu \in I^X$, $\eta(\mu) \geq r$, and $r \in I_Y$.

Proof. (\Leftarrow) Let $x_i \in \text{dom}(F)$, $\mu \in I^X$ with $\eta(\mu) \geq r$ and $x_i \in F^u(\mu)$. Then, there exists $\lambda \in I^Y$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq F^u(C^r(\mu, r))$. Thus, $x_i \in \lambda \leq I_\ell(F^u(C^r(\mu, r), r))$ and hence $F^u(\mu) \subseteq I_\ell(F^u(C^r(\mu, r), r))$. Hence, $F$ is fuzzy lower weakly $\ell$-continuous.

The following theorem is similarly proved as in Theorem 15.

\[
\begin{aligned}
\ell^1(v) &= \begin{cases} 
1, & \text{if } v = 0, \\
\frac{1}{2}, & \text{if } 0 < v \leq 0.4, \\
0, & \text{otherwise},
\end{cases} \\
\ell^2(v) &= \begin{cases} 
1, & \text{if } v = 0, \\
\frac{1}{2}, & \text{if } 0 < v \leq 0.2, \\
0, & \text{otherwise}.
\end{cases}
\end{aligned}
\]

Let $x_i \in \text{dom}(F)$, $\mu \in I^X$ with $\eta(\mu) \geq r$ and $x_i \in F^u(\mu)$. Then, $x_i \in F^u(\mu) \subseteq I_\ell(F^u(C^r(\mu, r), r))$. Thus $x_i \in I_\ell(F^u(C^r(\mu, r), r)) \subseteq F^u(C^r(\mu, r), r)$. Hence, $F$ is fuzzy lower weakly $\ell$-continuous.

Theorem 16. A normalized fuzzy multifunction $F : (X, \tau) -\rightarrow (Y, \eta, \ell)$ is fuzzy upper weakly $\ell$-continuous iff $F^u(\mu) \subseteq I_\ell(F^u(C^r(\mu, r), r))$, for each $\mu \in I^X$ with $\eta(\mu) \geq r$ and $r \in I_Y$.

Example 4. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X -\rightarrow Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.8, G_F(x_1, y_2) = 0.3, G_F(x_1, y_3) = 0.3, G_F(x_2, y_1) = 0.1, G_F(x_2, y_3) = 1.0, G_F(x_3, y_1) = 0.1, G_F(x_3, y_2) = 0.2, G_F(x_3, y_3) = 1.0$. Define $\mu_1 \in I^X$ and $\mu_2 \in I^X$ as follows: $\mu_1 = [x_1/0.4, x_2/0.1, x_3/0.2]$ and $\mu_2 = [y_1/0.4, y_2/0.1, y_3/0.2]$. Define $\tau : I^X \rightarrow I$ and $\eta, \ell^1, \ell^2 : I^Y \rightarrow I$ as follows:

\[
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1], \\
\frac{1}{2}, & \text{if } \lambda = 0.5, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\eta(\mu) = \begin{cases} 
3, & \text{if } \mu = \mu_2, \\
4, & \text{if } \mu = \mu_1, \\
0, & \text{otherwise},
\end{cases}
\]

Then,

(1) $F : (X, \tau) -\rightarrow (Y, \eta, \ell^1)$ is fuzzy upper (resp., lower) weakly $\ell$-continuous, but it is not fuzzy upper (resp., lower) weakly $\ell$-continuous because $\mu_1 \neq F^u(\mu_2)$ $\subseteq I_\ell(F^u(C^r(\mu_2, 1/2), 1/2))$, $1/2 = 0.5$. but $\mu_1 \neq F^u(\mu_2) \subseteq I_\ell(F^u(C^r(\mu_2, 1/2), 1/2))$, $1/2 = 0.5$. but $\mu_1 \neq F^u(\mu_2) \subseteq I_\ell(F^u(C^r(\mu_2, 1/2), 1/2))$, $1/2 = 0.5$. but $\mu_1 \neq F^u(\mu_2) \subseteq I_\ell(F^u(C^r(\mu_2, 1/2), 1/2))$, $1/2 = 0.5$. but $\mu_1 \neq F^u(\mu_2) \subseteq I_\ell(F^u(C^r(\mu_2, 1/2), 1/2))$, $1/2 = 0.5$.

(2) $F : (X, \tau) -\rightarrow (Y, \eta, \ell^2)$ is fuzzy upper (resp., lower) weakly $\ell$-continuous, but it is not fuzzy upper (resp., lower) almost $\ell$-continuous because $\mu_1 = F^u(\mu_2)$ $\subseteq I_\ell(F^u(C^r(\mu_2, 1/2), 1/2))$, $1/2 = 0.5$. but $\mu_1 = F^u(\mu_2) \subseteq I_\ell(F^u(C^r(\mu_2, 1/2), 1/2))$, $1/2 = 0.5$.

Theorem 17. A fuzzy multifunction $F : (X, \tau) -\rightarrow (Y, \eta, \ell)$ is fuzzy lower weakly $\ell$-continuous iff $C_r(F^u(\text{int}^*(\mu, r)), r) \subseteq F^u(\mu)$ for each $\mu \in I^X$ with $\eta(\mu) \geq r$.

Proof. (\Leftarrow) Let $\mu \in I^X$ with $\eta(\mu) \geq r$. Then, by Theorem 15, $F^u(\mu) \subseteq I_\ell(F^u(C^r(\mu, r), r)) = C_r(F^u(\text{int}^*(\mu, r)), r)$. Thus, $C_r(F^u(\text{int}^*(\mu, r)), r) \subseteq F^u(\mu)$.

(\Rightarrow) Let $x_i \in \text{dom}(F)$, $\mu \in I^X$ with $\eta(\mu) \geq r$ and $x_i \in F^u(\mu)$. Then, $I_\ell(F^u(C^r(\mu, r), r)) = C_r(F^u(\text{int}^*(\mu, r)), r)$.

Theorem 18. A fuzzy multifunction $F : (X, \tau) -\rightarrow (Y, \eta, \ell)$ is fuzzy upper weakly $\ell$-continuous multifunction and $F(\lambda) \subseteq I_\ell(\text{Cl}^*(\mu, r), r)$ for each $\lambda \in I^X$, then $F$ is fuzzy upper almost $\ell$-continuous.

Proof. Let $x_i \in \text{dom}(F)$, $\mu \in I^X$, $\eta(\mu) \geq r$, and $x_i \in F^u(\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq F^u(\text{Cl}^*(\mu, r), r)$, $F(\lambda) \subseteq F^u(\text{Cl}^*(\mu, r), r)$. Since $F^u(\mu) \subseteq I_\ell(\text{Cl}^*(\mu, r), r)$, $\lambda \leq F^u(\mu)$ and $F^u(\mu) \subseteq F^u(\text{Cl}^*(\mu, r), r)$. Hence, $F^u(\mu) \subseteq F^u(\text{Cl}^*(\mu, r), r)$. Then, $F$ is fuzzy upper almost $\ell$-continuous.
Corollary 3. Let $F : (X, r) \rightarrow (Y, \eta, \ell)$ and $H : (Y, \eta) \rightarrow (Z, \gamma)$ be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, $H \circ F$ is fuzzy lower (resp., upper) weakly $\ell$-continuous if $F$ is fuzzy lower (resp., upper) weakly $\ell$-continuous and $H$ is fuzzy lower (resp., upper) semicontinuous.

Theorem 20. Let $F : (X, r) \rightarrow (Y, \eta, \ell)$ be a fuzzy lower weakly $\ell$-continuous multifunction. Then, $F^\omega(\mu) \leq I_x (F^\lambda (C^\mu (\mu, r)), r)$ for any $\mu \in I^Y$ with $\mu \leq I_\eta (C^\mu (\mu, r), r)$ and $r \in I^r$.

Proof. Let $F$ be a fuzzy lower weakly $\ell$-continuous and $\mu \leq I_x (C^\mu (\mu, r), r)$. Then, if $x_i \in F^\lambda (\mu) \leq F^\omega (I_\eta (C^\mu (\mu, r), r))$, there exists $\lambda \in I^X$ with $\tau (\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq F^\omega (I_x (C^\mu (\mu, r), r)) \leq F^\omega (C^\mu (\mu, r))$. Thus, $\lambda \leq I_x (F^\omega (C^\mu (\mu, r)), r)$ and $F^\lambda (\mu) \leq I_x (F^\omega (C^\mu (\mu, r)), r)$.

The following theorem is similarly proved as in Theorem 20.

Theorem 21. Let $F : (X, r) \rightarrow (Y, \eta, \ell)$ be a normalized fuzzy upper weakly $\ell$-continuous multifunction. Then, $F^\upsilon(\mu) \leq I_x (F^\upsilon (C^\upsilon (\mu, r)), r)$ for any $\mu \in I^Y$ with $\mu \leq I_\eta (C^\mu (\mu, r), r)$ and $r \in I^r$.

6. Fuzzy Upper and Lower Almost Weakly $\ell$-Continuity

Definition 6. A fuzzy multifunction $F : (X, r) \rightarrow (Y, \eta, \ell)$ is called

(1) Fuzzy upper almost weakly $\ell$-continuous at a fuzzy point $x_i \in \text{dom} (F)$ iff $x_i \in F^\lambda (\mu)$ for each $\mu \in I^Y$ and $\eta (\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau (\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \text{dom} (F) \leq C_\mu (F^\lambda (C^\mu (\mu, r), r)).$

(2) Fuzzy lower almost weakly $\ell$-continuous at a fuzzy point $x_i \in \text{dom} (F)$ iff $x_i \in F^\lambda (\mu)$ for each $\mu \in I^Y$ and $\eta (\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau (\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq C_\mu (F^\lambda (C^\mu (\mu, r), r))$.

(3) Fuzzy upper almost weakly $\ell$-continuous (resp., fuzzy lower almost weakly $\ell$-continuous) iff it is fuzzy upper almost weakly $\ell$-continuous (resp., fuzzy lower almost weakly $\ell$-continuous) at every $x_i \in \text{dom} (F)$.

Remark 6

(1) If $F$ is normalized, then $F$ is fuzzy upper almost weakly $\ell$-continuous at a fuzzy point $x_i \in \text{dom} (F)$ iff $x_i \in F^\lambda (\mu)$ for each $\mu \in I^Y$ and $\eta (\mu) \geq r$, there exists $\lambda \in I^X$ with $\tau (\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq C_\mu (F^\lambda (C^\mu (\mu, r), r))$.

(2) Fuzzy upper (lower) weakly $\ell$-continuity $\Rightarrow$ fuzzy upper (lower) almost weakly $\ell$-continuity $\Rightarrow$ fuzzy upper (lower) almost weakly continuity [17]

(3) Fuzzy upper (lower) almost weakly $\ell_0$-continuity $\Leftrightarrow$ fuzzy upper (lower) almost weakly continuity

Theorem 22. For a fuzzy multifunction $F : (X, r) \rightarrow (Y, \eta, \ell)$, $\mu \in I^Y$, and $r \in I_\omega$, the following statements are equivalent:

1. $F$ is fuzzy lower almost weakly $\ell$-continuous
2. $F^\lambda (\mu) \leq I_x (C^\lambda (F^\mu (C^\rho (\mu, r)), r), r)$ if $\eta (\mu) \geq r$
3. $C_\mu (I_x (F^\upsilon (C^\upsilon (\mu, r)), r), r) \leq F^\upsilon (\mu)$ if $\eta (\upsilon) \geq r$

Proof

(1) $\Rightarrow$ (2) Let $x_i \in \text{dom} (F)$, $\mu \in I^Y$ with $\eta (\mu) \geq r$ and $x_i \in F^\lambda (\mu)$. Then, there exists $\lambda \in I^X$ with $\tau (\lambda) \geq r$ and $x_i \in \lambda$ such that $\lambda \leq C_\mu (F^\lambda (C^\mu (\mu, r), r))$. Thus, $x_i \in I_x (C^\lambda (F^\mu (C^\rho (\mu, r)), r), r)$ and $F^\lambda (\mu) \leq I_x (C^\lambda (F^\mu (C^\rho (\mu, r)), r), r)$.

(2) $\Rightarrow$ (3) Let $\mu \in I^Y$ with $\eta (\upsilon) \geq r$. Then, by (2), $F^\upsilon (\mu) = F^\upsilon (\mu^\upsilon) \leq I_x (C^\upsilon (F^\upsilon (\mu^\upsilon), r), r) = C^\upsilon (I_x (F^\upsilon (C^\upsilon (\mu^\upsilon), r), r), r) = C^\upsilon (I_x (F^\upsilon (C^\upsilon (\mu^\upsilon), r), r), r)$. Thus, $C_\mu (I_x (F^\upsilon (C^\upsilon (\mu^\upsilon), r), r), r) \leq F^\upsilon (\mu)$.

(3) $\Rightarrow$ (1) Let $x_i \in \text{dom} (F)$, $\mu \in I^Y$ with $\eta (\mu) \geq r$ and $x_i \in F^\lambda (\mu)$. Then, by (3), $I_x (C^\lambda (F^\mu (C^\rho (\mu, r)), r), r) \leq F^\upsilon (\mu)$. Therefore, $x_i \in I_x (C^\lambda (F^\mu (C^\rho (\mu, r)), r), r) \leq C^\upsilon (F^\upsilon (C^\upsilon (\mu^\upsilon), r), r)$. Thus, $F$ is fuzzy lower almost weakly $\ell$-continuous.

The following theorem is similarly proved as in Theorem 22.

Theorem 23. For a normalized fuzzy multifunction $F : (X, r) \rightarrow (Y, \eta, \ell)$, $\mu \in I^Y$ and $r \in I_\omega$, the following statements are equivalent:

1. $F$ is fuzzy upper almost weakly $\ell$-continuous
2. $F^\mu (\mu) \leq I_x (C^\mu (F^{\mu\upsilon} (C^{\upsilon\mu} (\mu, r)), r), r)$ if $\eta (\mu) \geq r$
3. $C_\mu (I_x (F^{\mu\upsilon} (C^{\upsilon\mu} (\mu, r)), r), r) \leq F^\mu (\mu)$ if $\eta (\upsilon) \geq r$

Example 5. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, and $F : X \rightarrow Y$ be a fuzzy multifunction defined by $G_x (x_1, y_1) = 0.8, G_x (x_1, y_2) = 0.3, G_x (x_1, y_3) = 0.3, G_y (x_2, y_1) = 0.1, G_y (x_2, y_2) = 1.0, G_y (x_2, y_3) = 0.1, G_y (x_3, y_1) = 0.2, G_y (x_3, y_2) = 0.2, G_y (x_3, y_3) = 1.0$. Define $\mu_1 \in I^X$ and $\mu_2 \in I^Y$ as follows: $\mu_1 = [0.1, 0.2, 0.1, 0.2]$ and $\mu_2 = [0.5, 0.4, 0.1, 0.2]$. Define $\tau_1, \tau_2 : I^X \rightarrow I$ and $\eta, \ell : I^Y \rightarrow I$ as follows:

\[
\tau_1 (\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1] \\
\frac{1}{2}, & \text{if } \lambda = 0.5 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\tau_2 (\lambda) = \begin{cases} 
1, & \text{if } \lambda \in [0, 1] \\
\frac{1}{2}, & \text{if } \lambda = 0.3 \\
0, & \text{otherwise}
\end{cases}
\]
Then,

(1) \( F: (X, r_2) \rightarrow (Y, \eta, \ell) \) is fuzzy upper (resp., lower) almost weakly \( \ell \)-continuous, but it is not fuzzy upper (resp., lower) almost weakly \( \ell \)-continuous

(2) \( F: (X, r_1) \rightarrow (Y, \eta, \ell) \) is fuzzy upper (resp., lower) almost weakly \( \ell \)-continuous, but it is not fuzzy upper (resp., lower) weakly \( \ell \)-continuous

**Theorem 24.** Let \( F: (X, r) \rightarrow (Y, \eta, \ell) \) be a normalized fuzzy multifunction and \( F \) be a fuzzy upper almost weakly \( \ell \)-continuous and fuzzy lower almost \( \ell \)-continuous. Then, \( F \) is fuzzy upper weakly \( \ell \)-continuous.

**Proof.** Let \( \mu \in I^Y \) with \( \eta(\mu) \geq r \) and \( F \) be a fuzzy upper almost weakly \( \ell \)-continuous. Then, by Theorem 23 (2), \( F^u(\mu) \subseteq I_\ell(C_\eta(F^u(\mu), r), r) \). Since \( C_\eta(\mu, r) = C_\eta(\int^* C_\eta(\mu, r), r) \), it follows from Theorem 8 (2) that \( r_1((F^u(\mu), r)) \geq r \) and \( F^u(\mu) \subseteq I_\ell(C_\eta(F^u(\mu), r), r) \). Thus, by Theorem 16, \( F \) is fuzzy upper weakly \( \ell \)-continuous.

24. The following theorem is similarly proved as in Theorem

**Theorem 25.** Let \( F: (X, r) \rightarrow (Y, \eta, \ell) \) be a normalized fuzzy multifunction and \( F \) be a fuzzy lower almost weakly \( \ell \)-continuous and fuzzy upper almost \( \ell \)-continuous. Then, \( F \) is fuzzy lower weakly \( \ell \)-continuous.

**Corollary 4.** Let \( F: (X, r) \rightarrow (Y, \eta, \ell) \) and \( H: (Y, \eta) \rightarrow (Z, \gamma) \) be two fuzzy multifunctions (resp., normalized fuzzy multifunctions). Then, \( H \circ F \) is fuzzy lower (resp., upper) almost weakly \( \ell \)-continuous if \( F \) is fuzzy lower (resp., upper) almost weakly \( \ell \)-continuous and \( H \) is fuzzy lower (resp., upper) semicontinuous.

### 7. Conclusion and Future Work

In the present paper, based on fuzzy operators \( \alpha, \beta, id_X: I^X \times I^\ell \rightarrow I^X \), and \( \delta, \theta, \partial, id_Y: I^Y \times I^\ell \rightarrow I^Y \), we can give a generalized form of fuzzy (resp., normalized fuzzy) multifunction as \( F: (X, r, \ell) \rightarrow (Y, \eta, \ell) \) is fuzzy lower (resp., upper) \( (\alpha, \beta, \theta, \partial, \ell^1, \ell^2) \)-continuous multifunction if for every \( \mu \in I^Y \) with \( \eta(\mu) \geq r \) and \( r \in I_\ell \), \( \ell^1(\alpha(F^l(\delta(\mu, r)), r)) \geq \eta(\mu) \) (resp., \( \ell^1(\alpha(F^u(\delta(\mu, r)), r)) \geq \eta(\mu) \)). It is clear that

1. Fuzzy lower (resp., upper) \((id_X, I_\ell, I_0(\ell^1)), id_Y, \ell^1, \ell^2)\)-continuous is fuzzy lower (resp., upper) almost \( \ell \)-continuous multifunction
2. Fuzzy lower (resp., upper) \((id_X, I_\ell, CP, id_Y, \ell^1, \ell^2)\)-continuous is fuzzy lower (resp., upper) weakly \( \ell \)-continuous multifunction
3. Fuzzy lower (resp., upper) \((id_X, I_\ell, CL, id_Y, \ell^1, \ell^2)\)-continuous is fuzzy lower (resp., upper) almost weakly \( \ell \)-continuous multifunction

In the upcoming work, we will define some new separation axioms on fuzzy ideal topological spaces. Also, we shall discuss the concepts given here in the frames of fuzzy soft \( r \)-minimal spaces [3]. We hope that this work will contribute fuzzy ideal topological structure studies.

**Data Availability**

No data were used to support the findings of this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**

[1] L. A. Zadeh, “Fuzzy sets,” Information and Control, vol. 8, no. 3, pp. 338–353, 1965.
[2] G. J. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic,” Theory and Applications, Prentice-Hall, New Jersey, 1995.
[3] I. M. Taha, “Compactness on fuzzy soft \( r \)-minimal spaces,” International Journal of Fuzzy Logic and Intelligent Systems, vol. 21, no. 3, pp. 251–258, 2021.
[4] I. M. Taha, “On upper and lower generalized semi-continuous fuzzy multifunctions,” The Journal of Mathematics and Computer Science, vol. 25, no. 03, pp. 251–258, 2021.
[5] J. Albrycht and M. Matłoka, “On fuzzy multi-valued functions Part 1: introduction and general properties,” Fuzzy Sets and Systems, vol. 12, no. 1, pp. 61–69, 1984.
[6] N. S. Papageorgiou, “Fuzzy topology and fuzzy multifunctions,” Journal of Mathematical Analysis and Applications, vol. 109, no. 2, pp. 397–425, 1985.
[7] E. Tsiporkova-Hristoskova, B. De Baets, and E. Kerre, “A fuzzy inclusion based approach to upper inverse images under fuzzy multivalued mappings,” Fuzzy Sets and Systems, vol. 85, no. 1, pp. 93–108, 1997.
[8] E. Tsiporkova, B. De Baets, and E. Kerre, “Continuity of fuzzy multivalued mappings,” Fuzzy Sets and Systems, vol. 94, no. 3, pp. 335–348, 1998.
[9] I. M. Taha, “On \( r \)-fuzzy \( \ell \)-open sets and continuity of fuzzy multifunctions via fuzzy ideals,” Journal of Mathematical and Computational Science, vol. 10, no. 6, pp. 2613–2633, 2020.
[10] I. M. Taha, “On \( r \)-generalized fuzzy \( \ell \)-closed sets: properties and applications,” Journal of Mathematics, vol. 2021, Article ID 4483481, 8 pages, 2021.
[11] I. M. Taha, “\( \delta \)-fuzzy \( \ell \)-continuous and fuzzy upper (lower) \( \delta \)-\( \ell \)-continuity via fuzzy idealization,” The Journal of Mathematics and Computer Science, vol. 25, no. 1, pp. 1–9, 2022.
[12] A. P. Šostak, “On a fuzzy topological structure,” in Proceedings of the 13th winter School on Abstract Analysis, Section of Topology, pp. 89–103, Circolo Matematico di Palermo, Palermo, 1985.

[13] I. M. Taha, “On fuzzy upper and lower α-χ-continuity and their decomposition,” Journal of Mathematical and Computational Science, vol. 11, no. 1, pp. 427–441, 2021.

[14] A. A. Ramadan, Y. C. Kim, and M. K. Ei-Gayyar, “Smooth ideals,” Journal of Korean Institute of Intelligent Systems, vol. 12, no. 1, pp. 90–95, 2002.

[15] S. E. Abbas, M. A. Hebeshi, and I. M. Taha, “On fuzzy upper and lower semi-continuous multifunctions,” Journal of Fuzzy Mathematics, vol. 22, no. 4, pp. 951–962, 2014.

[16] I. Beg, “Continuity of fuzzy multifunctions,” Journal of Applied Mathematics and Stochastic Analysis, vol. 12, no. 1, pp. 17–22, 1999.

[17] S. E. Abbas, M. A. Hebeshi, and I. M. Taha, “On upper and lower almost weakly continuous fuzzy multifunctions,” Iranian Journal of Fuzzy Systems, vol. 12, no. 1, pp. 101–114, 2015.