On the Uniqueness of Black Hole Attractors

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**Abstract**

We examine the attractor mechanism for extremal black holes in the context of five dimensional $N = 2$ supergravity and show that attractor points are unique in the extended vector multiplet moduli space. Implications for black hole entropy are discussed.
1. Introduction

BPS black holes in four and five dimensional $N = 2$ supergravity have been much studied using the attractor mechanism. To construct a black hole solution, one specifies the charges and asymptotic values of the moduli. The moduli then evolve as a function of the radius until they reach a minimum of the central charge at the horizon. This minimum value determines the entropy of the black hole. Consequently, it is important to know if different values of the central charge can be attained at different local minima, or perhaps even more basically if multiple local minima are allowed. If uniqueness fails, one would be led to believe that the degeneracy of BPS states does not solely depend on the charges.

As it turns out, in the five dimensional case we can show that at most one critical point of the central charge can occur; that is the subject of this paper. The structure of the argument is simple. Remarkably, the extended Kähler cone turns out to be convex. Therefore we can take a straight line between two supposed minima and analyse the (correctly normalised) central charge along this line. The central charge cannot have two minima when restricted to this line, yielding a contradiction.

We would like to emphasize that although Calabi-Yau spaces will be in the back of our minds for most of this paper, the geometric statements have clear analogues in five dimensional supergravity, so that our arguments are independent of the presence of a Calabi-Yau. We will point out some of the parallel interpretations where they occur.

In section two we provide two arguments for uniqueness in a single Kähler cone. In section three we tackle the extended Kähler cone, and in section four we discuss some of the implications. The reader may wish to start with section four before moving on to the arguments of sections two and three.

2. Single Kähler cone

2.1. Review of the attractor mechanism

Let us recall the basic setting for the five-dimensional attractor problem. We consider M-theory compactified on a Calabi-Yau threefold. As the low-energy effective theory we obtain five dimensional $N = 2$ supergravity with $h^{1,1} - 1$ vector multiplets, $h^{2,1} + 1$ hypermultiplets and the gravity multiplet \([1]\). The vector multiplets each contain one real scalar, so the vector moduli space is $h^{1,1} - 1$ dimensional. From the point of view of Calabi-Yau compactification these scalars have a simple geometrical interpretation. Let us
denote the Calabi-Yau three-fold by \( X \) and expand two-cycles \( Q \) of \( X \) as \( Q = Q_i e^i \), where \( e^i \) is a basis for \( H_2(X, \mathbb{Z}) \). The dual basis for \( H^2(X, \mathbb{Z}) \) will be written with lower indices, \( e_j \), so that \( \langle e_j e^i \rangle = \delta^i_j \). The Calabi-Yau Kähler class \( k \) can be expanded as \( k = k^i e_i \) which gives an \( h^{1,1} \) dimensional space of parameters. One parameter corresponding to the total volume of the Calabi-Yau is part of the universal hypermultiplet and the rest of the parameters corresponding to the sizes of cycles in the Calabi-Yau describe precisely the moduli space of vector multiplets.

The dynamics of vector multiplets in \( N = 2 \) supergravity is completely governed by the prepotential \( F(k) \), which is a homogeneous cubic polynomial in vector moduli co-ordinates \( k^i \). It is a special property of five dimensional supergravity that there are no nonperturbative quantum corrections to this prepotential \[2\]. Geometrically the prepotential is simply the volume of \( X \) in terms of \( k \)

\[ F(k) \equiv \frac{1}{6} \int_X k \wedge k \wedge k = \frac{1}{6} k^i k^j k^l d_{ijl} \tag{2.1} \]

where \( d_{ijl} \) denote the triple intersection numbers of \( X \)

\[ d_{ijl} = \int_X e_i \wedge e_j \wedge e_l. \]

In order to abbreviate the formulae, let us introduce the following notation:

\[ a \cdot b \equiv a_m b^m \]
\[ a \cdot b \cdot c \equiv a^i b^j c^l d_{ijl} \]
\[ k^3 \equiv k \cdot k \cdot k \]

To decouple the universal hypermultiplet coordinate we need to impose the constraint \( F(k) = 1 \) which gives us the vector moduli space as a hypersurface inside the Kähler cone. Alternatively, we will sometimes think of the moduli space as a real projectivisation of the Kähler cone. In this case we have to consider functions invariant under overall rescaling of \( k \)'s.

The prepotential defines the metric on moduli space as well as the gauge coupling matrix for the five-dimensional gauge fields. Including the graviphoton, there are exactly \( h^{1,1} U(1) \) gauge fields in the theory and their moduli-dependent gauge coupling matrix is given by \[3\]

\[ G_{ij} = -\frac{1}{2} \frac{\partial^2}{\partial k^i \partial k^j} \log F(k) = -\frac{1}{2} \frac{\partial^2}{\partial k^i \partial k^j} \log k^3 \tag{2.2} \]
The moduli space metric $g_{ij}$ is just the restriction of $G_{ij}$ to the hypersurface $F(k) = 1$. In the supergravity Lagrangian $G_{ij}$ and $g_{ij}$ multiply kinetic terms for the gauge fields and the moduli fields respectively. It is then very important that both metrics should be positive-definite inside the physical moduli space. The tangent space to the $F(k) = 1$ hypersurface is given by vectors $\Delta k$ such that $k \cdot k \cdot \Delta k = 0$. Then positivity of $g_{ij}$ requires

$$\Delta k^i g_{ij} \Delta k^j \equiv \Delta k^i G_{ij} \Delta k^j = -3 \kappa \Delta k \cdot \Delta k > 0. \quad (2.3)$$

Next we consider BPS states with given electric charges. The vector of electric charges with respect to the $h^{1,1} U(1)$ gauge fields can be thought of as an element $Q$ of $H_2(X, \mathbb{Z})$. In M-theory language these BPS states are M2-branes wrapped on a holomorphic cycle in the class $Q$. For large charges we can represent them in supergravity by certain extremal black hole solutions [6]. The structure of these solutions is as follows. As one moves radially towards the black hole the vector multiplet moduli fields $k^i$ vary. They follow the gradient flow of the function $Z \equiv Q \cdot k$:

$$\partial_\tau U = \frac{1}{6} e^{-2U} Z$$
$$\partial_\tau k^i = -\frac{1}{2} e^{-2U} G^{ij} D_j Z$$

Here $U \equiv U(r)$ is the function determining the five dimensional metric

$$ds^2 = -e^{-4U} dt^2 + e^{2U} (dr^2 + r^2 d\Omega^2),$$

$\tau = 1/r^2$ and the covariant derivative is

$$D_j = \partial_j - \frac{1}{6} k^i k^l d_{ijl}.$$

Geometrically $Z$ is just the volume of the holomorphic cycle $Q$ in the Calabi-Yau with Kähler class $k$. We will refer to $Z(k)$ as the central charge because when evaluated at infinity, $Z$ is indeed the electric central charge of the $N = 2$ algebra. As we approach the horizon of the black hole at $r = 0$ or $\tau = \infty$, the central charge rolls into a local minimum and the moduli stabilise there, let us call that point $k_0$. The area of the horizon and thus the entropy of the black hole are determined only by the minimal value of the central charge [4,5] :

$$S = \frac{\pi^2}{12} Z_0^{3/2} = \frac{\pi^2}{12} \left( \int_Q k_0 \right)^{3/2} = \frac{\pi^2}{12} (Q \cdot k_0)^{3/2}. \quad (2.4)$$
Those points in the moduli space where the central charge attains a local minimum for a fixed electric charge $Q$ are called attractor points.

The microscopic count of the number of BPS states with given charge has been performed for the special case of compactifications of M-theory on elliptic Calabi-Yau threefolds \cite{7}. There the attractor point for any charge vector $Q$ was found explicitly and the resulting entropy prediction (2.4) agreed with the microscopic count for large charges.

It was pointed out in \cite{8} that in the case of general Calabi-Yau compactifications an attractor point is not necessarily unique, and in principle for making an entropy prediction one needs to specify not only the charges of the black hole, but also an attractor basin, that is one needs to specify a region in moduli space in which all the points flow to a given attractor point along a path from infinity to the horizon. In the remainder of this article we show that if a minimum of the central charge exists, the attractor basin for this minimum covers the entire moduli space. There cannot be a second local minimum and so the specification of the charges of the black hole is sufficient for determining the attractor point.

2.2. Geometric argument

To find an attractor point explicitly, one needs to extremise the central charge subject to the constraint $k \cdot k \cdot k = 1$, which leads directly to the five dimensional attractor equation \cite{9}

$$Q_i = (Q \cdot k) k^j k^l d_{ijl}$$

In differential form notation, it reads

$$[Q] = \left( \int_Q k \right) [k \wedge k]. \tag{2.5}$$

Here $[Q]$ is a four-form which is Poincaré dual to the two-cycle $Q$. For convenience, we will leave out the square brackets in what follows.

Let us recall some standard facts about the Lefschetz decomposition (see for instance \cite{9}). On any Kähler manifold the Kähler class is a harmonic form of type $(1,1)$. It can therefore be used to define an action on the cohomology. We define the raising operator to be the map from $H^{p,q}(X, \mathbb{C})$ to $H^{p+1,q+1}(X, \mathbb{C})$ obtained by wedging with $k$

$$L_k \alpha = k \wedge \alpha$$
and similarly the lowering operator to be the map from $H^{p,q}(X, \mathbb{C})$ to $H^{p-1,q-1}(X, \mathbb{C})$ obtained by contracting with $k$

$$\Lambda_k \alpha = \iota_k \alpha.$$  

The commutator sends forms of type $(p,q)$ to themselves up to an overall factor:

$$[L, \Lambda] = (p + q - n)I$$  

(2.6)

where $n$ is the complex dimension of $X$. Thus $L$, $\Lambda$ and $(p+q-n)I$ form an $sl(2, \mathbb{R})$ algebra and the cohomology of $X$ decomposes as a direct sum of irreducible representations. When $X$ is a Calabi-Yau threefold, the decomposition is

$$H^*(X, \mathbb{C}) = 1(3/2) \oplus (h^{1,1} - 1)(1/2) \oplus (2h^{2,1} + 2)(0).$$

The spin $3/2$ represenatation corresponds to $\{1, k, k^2, k^3\}$. There can be no spin $0$ represenations in $H^{1,1}(X, \mathbb{C})$ because if $\alpha$ is of type (1,1) and $L_k \alpha = 0$ then by equation (2.6) we deduce that $\alpha$ is zero. In particular, the raising operator $L_k$ maps classes of type (1,1) isomorphically onto classes of type (2,2). We will use this fact in the following argument.

To prove that a single Kähler cone supports at most one attractor point, assume to the contrary that there are two such points, $k_0$ and $k_1$, satisfying (2.3). Then we may rescale $k_0$ or $k_1$ by a positive factor such that

$$k_0 \wedge k_0 = \pm k_1 \wedge k_1.$$  

(2.7)

First we fix the sign in the above equation. Since $\frac{1}{2}(k^0 + k^1)$ is inside the Kähler cone, $k^0 + k^1$ is an admissible Kähler class and $\int_X (k^0 + k^1)^\wedge 3$ is positive. Assuming the sign in (2.7) is minus, one may expand $(k_0 + k_1)^\wedge 3$ and deduce that

$$\int (k_0 + k_1)^\wedge 3 = -2 \int (k_0)^\wedge 3 - 2 \int (k_1)^\wedge 3 < 0,$$

which is impossible, so the sign is a plus. Therefore we have

$$(k_0 + k_1) \wedge (k_0 - k_1) = 0.$$  

As discussed above, $L_{k_0+k_1}$ cannot annihilate any classes of type (1,1) because $k^0 + k^1$ is an allowed Kähler class. We conclude that $k_0 - k_1$ must vanish.

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1 We thank C. Vafa for this argument.
2.3. Physical argument

One doesn’t really need the attractor equation to prove that in a single cone multiple critical points cannot occur. Another argument makes use of the simple properties of the prepotential \((2.1)\) and gives further insight into the behaviour of the central charge function.

Let us examine the behaviour of the central charge along straight lines in the Kähler cone. First take any two points \(k_0\) and \(k_1\) in the Kähler cone. By convexity of the cone we can take a straight line from \(k_0\) to \(k_1\),

\[
k(t) = k_0 + t \Delta k, \quad \Delta k = k_1 - k_0. \tag{2.8}
\]

For \(t\) between 0 and 1 and a little bit beyond those values \(k(t)\) certainly lies in the Kähler cone, but it no longer satisfies \(k(t)^3 = 1\). To cure this we think of the moduli space as a real projectivisation of the Kähler cone and define the central charge everywhere in the cone by normalising \(k\):

\[
Z(k) = \frac{\int_X k}{(\int_X k \wedge k \wedge k)^{1/3}} = \frac{Q \cdot k}{(k^3)^{1/3}} \tag{2.9}
\]

Then the central charge along the straight line (2.8) is just

\[
Z(t) = \frac{Q \cdot k(t)}{(k(t)^3)^{1/3}}. \tag{2.10}
\]

Let us also assume that the central charge is positive at \(k_0\), i.e. \(Z(0) > 0\). Otherwise we would consider the same problem with the opposite charge.

Differentiating \(Z(t)\), one finds for the first derivative

\[
Z'(t) = (k^3)^{-4/3} \left((Q \cdot \Delta k)(k^3) - (Q \cdot k)(\Delta k \cdot k \cdot k)\right). \tag{2.11}
\]

Now suppose \(Z(t)\) has a critical point \(t_c\) where \(Z'(t_c) = 0\). Then the second derivative at \(t_c\) can be expressed as

\[
Z''(t_c) = 2(k^3)^{-4/3}(Q \cdot k) \left( \frac{(\Delta k \cdot k \cdot k)^2}{k^3} - \Delta k \cdot \Delta k \cdot k \right)
= 2(k^3)^{-4/3}(Q \cdot k)(B_{ij} \Delta k^i \Delta k^j).
\]

The bilinear form \(B_{ij}\) has the following properties: exactly one of its eigenvalues is zero (namely in the \(k\)-direction) and the other eigenvalues are positive. In the language of
Calabi-Yau geometry this holds because \( \int_X k \wedge \Delta k \wedge \Delta k < 0 \) for any \( \Delta k \) that satisfies \( \int_X k \wedge k \wedge \Delta k = 0 \). Now if \( \Delta k \) would be proportional to \( k(t) = k_0 + t\Delta k \) for some \( t \), we would find that \( k_0 \) and \( k_1 \) are in fact equal. Thus \( \Delta k \) necessarily has a piece that is orthogonal to \( k(t) \) and so

\[
B_{ij} \Delta k^i \Delta k^j > 0 \text{ strictly.}
\]

In the language of supergravity the above statement follows from the fact that the form \( B \) is proportional to the metric when restricted to directions tangent to the moduli space, for which \( k \cdot \Delta k = 0 \), see (2.3). The zero eigenvalue in the \( k \)-direction is simply the scale invariance of \( Z(k) \).

The above inequality can be expressed in the following words: for positive central charge any critical point along any straight line is in fact a local minimum! For negative \( Z \) every critical point along a straight line is a local maximum. It is well known that a critical point of the central charge is a minimum when considered as a function of all moduli. Here we have a much stronger statement. We see that on a one-dimensional subspace (a projection of a straight line) any critical point is in fact a local minimum.

We can use the above observation as follows. The central charge \( Z \) has a local minimum at the attractor point \( k_0 \) by definition. Therefore it must grow continuously on any straight line emanating from \( k_0 \) and can never achieve a second local minimum. Moreover, we see that the central charge has a global minimum at \( k_0 \) in the entire Kähler cone.

Let us remark on another consequence of our observation. Consider level sets of the central charge function, i.e sets where \( Z < a \) for some constant \( a \). Note that all such sets are necessarily convex. For otherwise, if we could connect two points inside a level set by a line segment venturing outside it, there would be a maximum of the central charge on that line segment, which contradicts the above observation.

3. Extended Kähler cone

3.1. Review

The single Kähler cone we have just discussed is only a part of the full vector moduli space. Some of the boundaries of the Kähler cone correspond to actual boundaries of the moduli space. At other boundaries the Calabi-Yau undergoes a flop transition, that is a

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\(^{2}\) See for instance [9], page 123. Note the misprint there; it should say \( 2k = p + q \).

\(^{3}\) See [10] for discussion.
curve collapses to zero size, but one may continue through the wall and arrive in a different geometric phase. There one has another Calabi-Yau which is birationally equivalent to the original one. They share the same Hodge numbers but have different triple intersection numbers. In terms of the original Kähler moduli, the collapsed curve has a finite but negative area on the other side of the wall. In five dimensions this has the interpretation of a phase transition where a BPS hypermultiplet goes from positive to negative mass.

The union of the Kähler cones of all Calabi-Yaus related to each other through a sequence of flop transitions is called the extended Kähler cone. Geometrically one cannot go beyond the boundaries of this extended cone. It has also been argued that at the boundaries that are at finite distance the physical vector moduli space ends [10].

After we cross the wall into an adjacent cone we may take linear combinations of Kähler parameters $k^i$ in order to get an acceptable set of moduli that yield positive areas for two- and four-cycles. But we will find it more convenient to stick to the original $k^i$, even though they can sometimes yield negative areas outside the original cone. By induction, we may still use the $k^i$ if we pass through a second flop transition into a third cone, and so on.

The Calabi-Yau in the adjacent cone has different intersection numbers, which means that we have to adjust the prepotential for the new cone. It is well known how the prepotential changes when one passes through a wall: if we denote by $m$ the area of the collapsing curve that is negative on the other side of the wall, then [11]

$$k \cdot k \cdot k \rightarrow k \cdot k \cdot k - (\#P^1) m^3.$$  \hspace{1cm} (3.1)

Here $\#P^1$ stands for the number of $P^1$'s shrinking to zero size at the wall. Intuitively, the growing curve should contribute a positive number to the volume of the Calabi-Yau for $m < 0$, hence the minus sign in (3.1). This sign is crucial for proving uniqueness of attractor points in the extended Kähler cone.

Physical quantities experience only a mild change at the flop transition [3]. In particular from (3.1) we see that the prepotential is twice continuously differentiable. The central charge $Z$ is also twice continuously differentiable. The metric, which involves second derivatives of the prepotential, is only continuous.
3.2. Convexity of the extended cone

The extended Kähler cone has an alternative description in terms another cone, as we will explain below. The advantage of this description comes from the fact that this other cone is manifestly convex, hence so is the extended Kähler cone. We will only give a brief sketch of the argument here and simply use the result in the remainder of the paper. For a detailed proof, one may consult the mathematics literature [12] (see also [11]).

There is a one to one correspondence between real cohomology classes of type (1,1) and line bundles. Namely, given such a class $\omega$, one may find a line bundle $L[\omega]$ such that its first Chern class is $\omega$, and conversely. In order to apply some standard constructions in algebraic geometry, we will assume that $\omega$ is a rational class, that is it is a class in the intersection of $H^{1,1}(X, \mathbb{C})$ and $H^2(X, \mathbb{Q})$.

With appropriate restrictions, some high tensor power of $L[\omega]$ will have sufficiently many holomorphic sections to define a ‘good’ map to some projective space, as follows: choose a basis of holomorphic sections $s_0, s_1, \ldots, s_n$. Then we get a map $f[\omega]$ from $X$ to $\mathbb{P}^n$ by sending a point $p$ in $X$ to the equivalence class $[s_0(p), s_1(p), \ldots, s_n(p)]$. Let us call the image $Y$. Some points $p$ may be a common zero for all the $s_i$’s. The collection of such points is called the base locus of $L[\omega]$. Under $f[\omega]$ the base locus is mapped to the origin in $\mathbb{C}^{n+1}$, so when one projectivises $\mathbb{C}^{n+1}$ cycles in the base locus may get contracted and points may get smeared out. Away from the base locus the map $f[\omega]$ is an isomorphism.

In order to insure that the image will be a Calabi-Yau that is related to $X$ by flop transitions at most, we require that $[\omega]$ is movable, which means that the base locus is of complex codimension at least two in $X$. This condition means that the map $f[\omega]$ is an isomorphism in codimension one, i.e. the most that can happen is that some two-cycles contract or some points expand to two-cycles. In particular, the canonical class of $Y$ must be trivial, so $Y$ is a Calabi-Yau. Finally, by construction the holomorphic sections of (some multiple of) $L[\omega]$ get transformed into hyperplane sections, which are the sections of the line bundle corresponding to the Kähler class on $Y$. So the pull-back of the Kähler class on $Y$ is precisely (some multiple of) $[\omega]$. It is well known that when $[\omega]$ is taken to be in the original Kähler cone of $X$, $f[\omega]$ will give a smooth embedding of $X$ in projective space.

Hopefully we have made it plausible that for any rational class $[\omega]$ of type (1,1) on $X$, provided it is movable, we can find a Calabi-Yau $Y$ which has Kähler class (a multiple of) $[\omega]$ and is related to $X$ by flop transitions. Conversely, given a Calabi-Yau $Y$ with rational

\footnote{We would like to thank D. Morrison for pointing this out to us.}
Kähler class $[\omega]$ and related to $X$ by flops, it can been shown that the transform of $[\omega]$ on $X$ is movable. Therefore the extended rational Kähler cone is precisely the rational cone generated by movable classes of type (1,1). We want to discuss why the property of movability is preserved under positive linear combinations.

To see this, take two classes $[\omega_1]$ and $[\omega_2]$ and consider the sum $[\omega] = m[\omega_1] + n[\omega_2]$ where $m$ and $n$ are positive rational numbers. Since we may multiply $[\omega]$ by any integer, we may assume that $m$ and $n$ are themselves integer and moreover large enough for the following argument to apply. The corresponding line bundle is

$$L_{[\omega]} = L_{[\omega_1]}^\otimes m \otimes L_{[\omega_2]}^\otimes n.$$  

Take a basis of holomorphic sections $s_i$, $i = 0 \ldots n_1$, for $L_{[\omega_1]}^\otimes m$. This defines a map to $\mathbb{P}^{n_1}$. Similarly choose basis $t_j$, $j = 0 \ldots n_2$, for $L_{[\omega_2]}^\otimes n$. Then $L_{[\omega]}$ defines a map to $\mathbb{P}^{n_1 n_2 + n_1 + n_2}$ by means of the sections $s_i \otimes t_j$. Thus the base locus of $L_{[\omega]}$ is the union of the base loci of $L_{[\omega_1]}^\otimes m$ and $L_{[\omega_2]}^\otimes n$, and in particular is of codimension at least two.

So we conclude that the cone generated by movable classes is convex, and therefore that the extended Kähler cone is convex.

3.3. Uniqueness in the extended cone

Armed with the knowledge that the extended Kähler cone is convex, we may try to employ the argument that was used successfully in section two. We start with the local minimum of the central charge at the attractor point $k_0$. Then on straight lines emanating from $k_0$ the critical points of $Z$ are all minima as long as they are inside some Kähler cone. However, a critical point may lie on the boundary between two cones in which case our argument that it must be a minimum doesn’t apply. Recall that we needed the form $B_{ij}$ to be positive in the direction of the line. This followed from the physical requirement of the positivity of the metric inside the moduli space, so that $B_{ij}$ is positive-definite for all directions tangent to the moduli space. But continuity of the metric alone does not prevent it from acquiring a zero eigenvalue on the flopping wall and so in principle $B_{ij}$ may become degenerate there. We are not aware of a proof of nondegeneracy of the metric (or a counterexample). Thus it may be possible for a critical point of the central charge along a straight line to have vanishing second derivative when it lies on the wall between two Kähler cones. Depending on the details of the behaviour of $Z'$ such a critical point may be a local minimum, maximum or an inflection point.
By the same token, it may even be that an attractor point \( k_0 \) is not a local minimum of the central charge on the extended moduli space, but rather a saddle point. We therefore switch to a more direct argument. We will first consider the case when \( k_0 \) itself is not on the wall and consequently it is a local minimum.

Let us again examine the central charge function along a straight line (2.8) between an attractor point \( k_0 \) and some other point \( k_1 \) inside the extended moduli space. If \( k_1 \) were another attractor point, then it would also be a critical point of \( Z(t) \) at \( t = 1 \), which is why we will be looking at the critical points of \( Z(t) \). As we know, the only source of trouble are the points where our line crosses walls of the Kähler cone. Let us first consider the case where only a single wall is crossed between \( t = 0 \) and \( t = 1 \). Suppose that the intersection is at \( t = t_f \). Rather than looking at the central charge \( Z(t) \) itself, we will examine its derivative. The derivative of \( Z \) was

\[
Z'(t) = (k^3)^{-4/3} \left( (Q \cdot \Delta k)(k^3) - (Q \cdot k)(\Delta k \cdot k \cdot k) \right).
\]  

The cubic terms in the second term of \( Z' \) cancel, so we put

\[
Z'(t) = (k^3)^{-4/3} R(t)
\]

where \( R(t) \) is a polynomial of degree two. As \((k^3)^{-4/3}\) is nonnegative, we will focus on \( R(t) \). By assumption \( t = 0 \) is a local minimum for \( Z \), i.e. it is a root for \( R(t) \) where \( R' > 0 \).

Recall that at the attractor point we start with a positive value of the central charge. It is important that for positive \( Z \) the only physical root of \( R(t) \) is the one where the first derivative is positive or possibly zero if such a root is on the flopping wall. The other root of \( R(t) \) is not physical, i.e. it lies outside the original Kähler cone. Thus \( Z(t) \) has only one critical point in a cone, which is another proof of uniqueness for a single cone. Now we will see what happens in the adjacent cone.

First we need to know the point where \( Z(t) \sim Q \cdot k(t) \) vanishes, call it \( t_0 \). With this definition we may write

\[
Q \cdot k(t) = Q \cdot k_0 + t Q \cdot \Delta k = Q \cdot k_0 (1 - t/t_0)
\]

When we cross the flopping wall at \( t = t_f \) the prepotential changes as (3.1)

\[
k^3 \rightarrow k^3 + c(t - t_f)^3
\]
where $c > 0$. So we must also modify $Z'(t)$ for $t > t_f$:

$$Z'(t)|_{t > t_f} = (k^3 + c(t - t_f)^3)^{-4/3} \left( R(t) + c(Q \cdot \Delta k - Q \cdot k_0)(t - t_f)^2 \right)$$

$$= (k^3 + c(t - t_f)^3)^{-4/3} \left( R(t) + c \frac{Q \cdot k_0}{t_0} \left( \frac{t_f}{t_0} - 1 \right) (t - t_f)^2 \right)$$

$$= (k^3 + c(t - t_f)^3)^{-4/3} P(t).$$

We are interested in the physical roots of $P(t)$ for $t \geq t_f$. As long as $Z(t)$ is positive those are the roots where $P' \geq 0$. $Z(t)$ is positive for all $t > 0$ if $t_0 < 0$ while if $t_0 > 0$ it is only positive for $t < t_0$. In both cases the constant $A \equiv c Q \cdot k_0 \left( \frac{t_f}{t_0} - 1 \right)$ is negative. First we show that for $A < 0$ there are no physical roots of $P(t)$ for $t \geq t_f$. For that we simply find the root $t_r$ where $P' \geq 0$ and show that $t_r < t_f$. It will also imply that $Z(t)$ may not start decreasing and therefore cannot become negative inside the extended moduli space.

We write $R(t) = at^2 + bt$ where $b > 0$ since $t = 0$ is a minimum of $Z$. Then we need to find roots of the quadratic equation

$$P(t) = at^2 + bt + A(t - t_f)^2 = 0$$

The root where the derivative $P'(t)$ is nonnegative, if it exists, is always given by

$$t_r = \frac{2At_f - b + \sqrt{b^2 + 4t_f(-A)(b + at_f)}}{2(a + A)}$$

Then there are four cases to consider depending on the value of the coefficient $a$.

1. $a \leq -b/t_f < 0$, note that $a + A < 0$ and $b + at_f < 0$, see Fig. 1. In this case there may be no roots, but if there are, we have

$$t_r < \frac{2At_f - b}{2(a + A)} \leq \frac{2At_f + at_f}{2(a + A)} < t_f.$$

This case is not physical, because the second root of $R(t)$ is inside the original Kähler cone which gives unphysical maximum of the central charge. We include this case for future reference.
2. \(-b/t_f < a < -A\), still \(a + A < 0\) but \(b + at_f > 0\). Now there are always roots and we have

\[
t_r < \frac{2At_f - b + \sqrt{b^2 + 4t_f a(b + at_f)}}{2(a + A)} \\
= \frac{2At_f - b + |b + 2at_f|}{2(a + A)} \\
\leq \frac{2At_f - b + b + 2at_f}{2(a + A)} \\
= t_f.
\]

(3.4)

3. \(a = -A > 0\). In this case \(P(t)\) is linear. It has only one root, which satisfies

\[
t_r = \frac{a t_f^2}{b + 2at_f} < t_f.
\]

4. \(-A < a\), see Fig. 2. As in item 2 we replace \(-A\) by \(a\) inside the square root. The resulting equations are exactly the same as in (3.4).
What we have found is that $P(t)$ doesn’t have physical roots in the new Kähler cone. This means that $Z(t)$ doesn’t have critical points there and therefore continues to grow as we go away from the attractor. It follows that we need not consider the case when the central charge is negative.

Crossing several walls can now be handled by induction. The polynomial $P(t)$ after the wall plays the role of $R(t)$ for the next crossing. We have shown that $P(t)$ has a root where $P’ > 0$, but it lies to the left of the new wall, therefore we are in the same situation as we started.

Let us comment on the (im)possibility of the critical point of $Z(t)$ which lies on the flopping wall and is a local maximum. It would correspond to the situation where $R$ and $R'$ both vanish (recall that $R'$ cannot be negative at the critical point by continuity and positivity of the metric away from the flopping wall). Such a point can be described by a situation in item 4 above with $b = 0$, $t_f = 0$ and both roots of $R(t)$ at 0. As $R(t)$ is positive to the left of the wall and $P(t)$ is negative to the right of it, this critical point is a local maximum. However, $R(t)$ cannot have a double zero at $t_f$ because we have shown that it always has one root strictly to the left of the wall. Therefore such critical points do not arise.

Finally, we are left to consider the case when $k_0$ itself lies on the flopping wall. Then $P(t)$ may be either positive or negative to the right of the wall. In the former case $Z(t)$ starts growing for $t > 0$ and the analysis we have made earlier carries over with no changes. In the latter case $k_0$ may be a local maximum in some directions precisely as described in the previous paragraph. The difference is that we now begin in this situation and cannot argue that $R(t)$ does not have a double root on the wall.

In such a case the central charge decreases from $t = 0$ and it may eventually become negative. While it is still positive the first wall crossing is described essentially by the situation in item 1 above, with $b = 0$. It is then clear that $P(t)$ will have no roots after all subsequent wall crossings while $Z(t) > 0$. Moreover, in every cone $P(t)$ will be a downward-pointing parabola with the apex to the left of the left wall.

After the central charge becomes negative the discussion changes in two ways. First, when crossing the wall the constant $A$ in the change from $R(t)$ to $P(t)$ becomes positive:

$$R(t) \rightarrow P(t) = R(t) + A(t - t_f)^2, \quad A > 0.$$ 

And second, the physical critical points are now the roots of $R(t)$ where the first derivative is negative or possibly zero if the root is on the flopping wall.
Now, \( Z(t) \) is decreasing when it becomes negative, therefore the first critical point after that may only be on the wall of some cone such that \( R(t) \) has a double zero. If we assume that this critical point exists, right before it \( R(t) \) would be a downward pointing parabola with a double zero on the wall. We can move backwards from it and reconstruct the polynomials \( R(t) \) and \( P(t) \) in all the preceding cones. Taking into account that the constant \( A \) is now positive but has to be subtracted, we see that while \( Z \) is negative \( R(t) \) in every cone is a downward pointing parabola with no roots and apex to the right of the right wall. In the cone where \( Z \) crosses zero we obtain a contradiction with the previous analysis where we have found that the apex of the parabola should be to the left of the left wall. Therefore, there cannot be multiple attractor points even when they lie on the walls between Kähler cones.

4. Discussion

In this paper we have demonstrated that a critical point of the central charge \( Z \) is unique if it exists. Moreover, if \( Z \) has a minimum (maximum) at the critical point then it will grow (decrease) along straight lines emanating from the critical point. In this section we will discuss two implications of our result.

If one restricts the moduli to lie inside a single Kähler cone then uniqueness is not surprising. The reason for this derives from the microscopic interpretation of entropy: it should be possible to reproduce the entropy of a BPS black hole by a microscopic count of degenerate BPS states. For the Calabi-Yau black holes considered in this paper, we would have to count the degeneracy\(^5\) of holomorphic curves within the class specified by the charge vector. As we have seen in section two, the macroscopic entropy predicted by the attractor mechanism is \( S = \frac{\pi^2}{12} Z_{0}^{3/2} \). Thus one expects that the degeneracy of BPS states for large charges, when supergravity should give a good description, asymptotically approaches \( e^{\frac{\pi}{24} Z_{0}^{3/2}} \). In [7] the count was done for the special case of elliptic threefolds. The attractor equation in that case could be solved explicitly and the attractor point was therefore unique (at least in a single Kähler cone). But even for a general Calabi-Yau one should not have expected multiple attractor points to occur in a single Kähler cone. Supergravity is well-behaved when the moduli vary only over a single cone and so the existence of two black hole solutions with different entropy should have its origin in the

\(^5\) See [7] for a discussion of the correct quantity to consider.
possibility of counting different BPS state degeneracies. But the number of holomorphic curves does not change inside a Kähler cone, so neither should the number of BPS states. So at least inside a single Kähler cone, it is clear that the entropy should be completely fixed by specification of the charges of the black hole.

In the extended moduli space however this is not so clear: the number of curves does change as one crosses a flopping wall. One could therefore interpret the walls of a Kähler cone as a hypersurface of marginal stability, analogous to the curve of marginal stability in Seiberg-Witten theory. The puzzle is this: suppose one starts with asymptotic moduli at some point very far away from the attractor point. Then somehow the attractor point seems to be aware of the degeneracy of BPS states for the Calabi-Yau associated with the asymptotic moduli. But when we choose the asymptotic moduli in a cone that is different from the cone where the attractor point lies, the degeneracies in the two cones will in general not be the same, so there is no a priori reason for the absence of multiple critical points. In the light of our result, one possibility is that the number of curves changes only very mildly across a transition, mild enough so that the asymptotic degeneracy in the limit of large charges is not affected. It would be interesting to check this mathematically.

The existence of multiple attractor points was also thought to be desirable for the construction of domain walls in five dimensional $N = 2$ gauged supergravity, along the lines of [13]. In that setup the goal is not to minimise the central charge, but to find extrema of the scalar potential of gauged supergravity, which is

$$V = -6(W^2 - \frac{3}{4} g^{ij} \partial_i W \partial_j W).$$

In the above, $W = Q_i k^i$ where $k^i$ are the usual Kähler moduli and $Q_i$ are the gravitino and gaugino charges under the $U(1)$ that is being gauged. Even though the interpretation is different, $W$ is numerically the same as what we have called $Z$ before and the supersymmetric critical points of $V$ are also critical points of $W$ [7]. At a critical point $W_0$ of $W$ the five dimensional supergravity solution is anti-De Sitter space with cosmological constant equal to $-6W_0^2$. To construct a domain wall, one would like to have two critical points $k_0$ and $k_1$ of $W$. Then one could write down a supergravity solution that interpolates between two different anti-De Sitter vacua, with the asymptotic values of the moduli being $k_0$ on one side of the wall and $k_1$ on the other. It was hoped that this might lead to a supergravity realisation of the Randall-Sundrum scenario [14]. Unfortunately as we have seen, this construction does not appear to be possible, at least in its simplest form, because of the absence of multiple (supersymmetric) critical points.
Finally, the attractor mechanism in four dimensions is somewhat similar to the five dimensional mechanism considered in this paper. It would be interesting if our methods could be used to shed some light on this important problem as well.

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