HALL ALGEBRAS OF EXTRIANGULATED CATEGORIES

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Abstract. Recently, Nakaoka and Palu introduced a notion of extriangulated categories. This is a unification of exact categories and triangulated categories. In this paper, we generalize the definitions of Hall algebras of exact categories and triangulated categories to extriangulated categories.

1. Introduction

Hall algebras first appeared in works of Steinitz [8] and Hall [4] on commutative finite $p$-groups. Later, they reappeared in the work of Ringel [7] on quantum groups. Ringel introduced the notion of the Hall algebra of an abelian category with finite Hom- and Ext$^1$-spaces. By definition, it is a vector space with the basis parameterized by the isomorphism classes of objects in the category, and the structure constants of the multiplication count in a natural way the first extensions with a fixed middle term. The associativity formula of the Hall algebra of abelian categories is easily obtained by counting the filtration of an object. According to [7] and [2], the Hall algebra of a finite dimensional hereditary algebra over a finite filed provides a realization of the half part of the corresponding quantum group.

Hubery [5] proved that the definition of the Hall algebra of abelian categories also applies to exact categories. He proved the associativity formulas of Hall algebras via the push-out and pull-back properties of exact categories. In order to give a realization of the entire quantum group via Hall algebra approach, one tried to define the Hall algebra of triangulated categories. Toën [9] gave a construction of what he called derived Hall algebras for DG-enhanced triangulated categories satisfying certain finiteness conditions. Later, Xiao and Xu [10] showed that Toën’s definition also applies to any triangulated categories satisfying certain finiteness conditions. They proved the associativity formulas of the derived Hall algebra by using only the properties and axioms of triangulated categories, such as the long exact sequence theorem and the octahedron axiom.

Recently, Nakaoka and Palu [6] has introduced a notion of extriangulated categories. This is a unification of exact categories and triangulated categories. The positive and negative higher extensions in extriangulated categories have been defined in [3]. A natural
question has been asked by many people (see for example [1, Section 7]): whether one can define the Hall algebra of extriangulated categories?

In this paper, we give a positive answer to the question above. Basing on the methods in [10], we prove that Toën’s formulas still hold in an extriangulated category. But this is not a trivial generalization. Compared with triangulated categories, the extriangulated categories have no shift functor [1], i.e., for an object $Z$, the object $Z[1]$ does not exist. So, for an object $L$, we do not have the morphism $L \to Z[1]$ or its decomposition (cf. [10, Lemma 2.2]). In order to overcome this trouble, for an $E$-triangle

$$X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{\delta},$$

we consider the decompositions of both $f$ and $g$. Meanwhile, instead of the group action of $Aut(X, Y)$, we consider both the group action of $Aut(X, L)$ and that of $Aut(L, Y)$. Then by combining the two equations respectively provided by the decompositions of $f$ and $g$, we obtain Toën’s formulas in an extriangulated category.

The paper is organized as follows: we summarize some basic definitions and properties of an extriangulated category in Section 2. In Section 3, we prove Toën’s formulas for an extriangulated category under some finiteness conditions. Section 4 is devoted to defining the Hall algebra of extriangulated categories using Toën’s formulas as structure constants.

Throughout this paper, let $k$ be a finite field with $q$ elements, and $\mathcal{C}$ be an essentially small additive $k$-linear category. For an object $X \in \mathcal{C}$, we denote by $End_X$ and $Aut_X$ the endomorphism ring and the automorphism group of $X$; denote by $1 = 1_X$ and $[X]$ the identity morphism and the isomorphism class of $X$, respectively; For a finite set $S$, we denote by $|S|$ its cardinality.

2. Preliminaries

Let us recall some notions and properties concerning extriangulated categories from [6].

Let $\mathcal{C}$ be an additive category and let $E: \mathcal{C}^{op} \times \mathcal{C} \to Ab$ be a biadditive functor. For any pair of objects $A, C \in \mathcal{C}$, an element $\delta \in E(C, A)$ is called an $E$-extension. The zero element $0 \in E(C, A)$ is called the split $E$-extension. For any morphism $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C', C)$, we have $E(C, a)(\delta) \in E(C, A')$ and $E(c, A)(\delta) \in E(C', A)$. We simply denote them by $a_*\delta$ and $c^*\delta$, respectively. A morphism $(a, c): \delta \to \delta'$ of $E$-extensions is a pair of morphisms $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C, C')$ satisfying the equality $a_*\delta = c^*\delta'$. By Yoneda’s lemma, any $E$-extension $\delta \in E(C, A)$ induces natural transformations

$$\delta_2: \mathcal{C}(-, C) \to E(-, A)$$

and

$$\delta^2: \mathcal{C}(A, -) \to E(C, -).$$

For any $X \in \mathcal{C}$, these $(\delta_2)_X$ and $(\delta^2)_X$ are defined by

$$(\delta_2)_X: \mathcal{C}(X, C) \to E(X, A), f \mapsto f^*\delta$$

and

$$(\delta^2)_X: \mathcal{C}(A, X) \to E(C, X), g \mapsto g_*\delta.$$
Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathcal{C}$ are said to be equivalent if there exists an isomorphism $b \in \mathcal{C}(B, B')$ such that the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow & & \downarrow b \\
A & \xrightarrow{x'} & B'
\end{array}
\]

\[
\begin{array}{ccc}
& & C \\
\downarrow & & \downarrow \\
& & C
\end{array}
\]

is commutative. We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{1} B \xrightarrow{0} C]$. In addition, for any $A, C \in \mathcal{C}$, we denote as $0 = [A \xrightarrow{(1)} B \xrightarrow{(0, 1)} C]$. For any two classes $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, we denote as

\[
[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].
\]

**Definition 2.1.** Let $s$ be a correspondence which associates an equivalence class $s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $s$ is called a realization of $\mathbb{E}$ if for any morphism $(a, c) : \delta \rightarrow \delta'$ with $s(\delta) = [\Delta_1]$ and $s(\delta') = [\Delta_2]$, there is a commutative diagram as follows:

\[
\begin{array}{ccc}
\Delta_1 & A & B & C \\
& & & \downarrow b \\
\Delta_2 & A & B & C
\end{array}
\]

A realization $s$ of $\mathbb{E}$ is said to be additive if it satisfies the following conditions:

(a) For any $A, C \in \mathcal{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies $s(0) = 0$.

(b) $s(\delta \oplus \delta') = s(\delta) \oplus s(\delta')$ for any pair of $\mathbb{E}$-extensions $\delta$ and $\delta'$.

Let $s$ be an additive realization of $\mathbb{E}$. If $s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, then the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a conflation, $x$ is called an inflation and $y$ is called a deflation. In this case, we say that $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ is an $\mathbb{E}$-triangle. We will write $A = \text{cocone}(y)$ and $C = \text{cone}(x)$ if necessary. We say an $\mathbb{E}$-triangle is splitting if it realizes 0.

**Definition 2.2.** ([6, Definition 2.12]) We call the triplet $(\mathcal{C}, \mathbb{E}, s)$ an extriangulated category if it satisfies the following conditions:

(ET1) $\mathbb{E} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$ is a biadditive functor.

(ET2) $s$ is an additive realization of $\mathbb{E}$.

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, realized as $s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $s(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. For any commutative square
in $\mathcal{C}$

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \xrightarrow{y} C \\
\downarrow{a} & & \downarrow{b} \\
A' & \xrightarrow{x'} & B' \xrightarrow{y'} C'
\end{array}
\]

there exists a morphism $(a, c): \delta \to \delta'$ which is realized by $(a, b, c)$.

(ET3)$^{op}$ Dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(D, A)$ and $\delta' \in \mathbb{E}(F, B)$ be $\mathbb{E}$-extensions realized by $A \xrightarrow{f} B \xrightarrow{f'} D$ and $B \xrightarrow{g} C \xrightarrow{g'} F$, respectively. Then there exist an object $E \in \mathcal{C}$, a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \xrightarrow{f'} D \\
\downarrow{g} & & \downarrow{d} \\
A & \xrightarrow{h} C \xrightarrow{h'} E \\
\downarrow{g'} & & \downarrow{e} \\
F & \xrightarrow{e} & F
\end{array}
\]

in $\mathcal{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities:

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $\mathbb{E}(F, f')(\delta')$,

(ii) $\mathbb{E}(d, A)(\delta'') = \delta$,

(iii) $\mathbb{E}(E, f)(\delta'') = \mathbb{E}(e, B)(\delta')$.

(ET4)$^{op}$ Dual of (ET4).

According to [3], a morphism $f \in \text{Hom}(X, Y)$ is said to be projective if $\mathbb{E}(f, -) = f^*$ is zero. Dually, a morphism $f \in \text{Hom}(X, Y)$ is injective if $\mathbb{E}(-, f) = f_*$ is zero. A projective morphism in $\mathcal{C}$ which is also a deflation is called a projective deflation. We say that $\mathcal{C}$ has enough projective morphisms if any $C \in \mathcal{C}$ admits a projective deflation $G \xrightarrow{g} C$. Dually, we define that $\mathcal{C}$ has enough injective morphisms.

In what follows, let $\mathcal{C}$ be an extriangulated category such that there are enough projective and injective morphisms in $\mathcal{C}$. Then the higher positive and negative extensions $\mathbb{E}^n$ in $\mathcal{C}$ have been defined in [3], and we have the following

**Proposition 2.3.** ([3]) For any $\mathbb{E}$-triangle $A \longrightarrow B \longrightarrow C \longrightarrow$, the following sequences of natural transformations are exact.

\[
\cdots \to \mathbb{E}^{-1}(A, -) \to \text{Hom}(C, -) \to \text{Hom}(B, -) \to \text{Hom}(A, -) \to \mathbb{E}^1(C, -) \to \cdots,
\]

\[
\cdots \to \mathbb{E}^{-1}(-, C) \to \text{Hom}(-, A) \to \text{Hom}(-, B) \to \text{Hom}(-, C) \to \mathbb{E}^1(-, A) \to \cdots.
\]
3. Toën’s formulas in extriangulated categories

In what follows, we always assume that \( \mathcal{C} \) is an extriangulated category which satisfies the following conditions:

1. \( \mathcal{C} \) has enough projective and injective morphisms.
2. \( \mathcal{C} \) is Krull–Schmidt.
3. \( \dim_k \mathbb{E}^i(X, Y) < \infty \) for any \( X, Y \in \mathcal{C} \) and \( i \in \mathbb{Z} \). Here \( \mathbb{E}^0(X, Y) := \text{Hom}(X, Y) \).
4. \( \mathcal{C} \) is left locally homologically finite, i.e., \( \sum_{i \leq 0} \dim_k \mathbb{E}^i(X, Y) < \infty \).

Given \( X, Y, L \in \mathcal{C} \), let \( W^L_{XY} := \{ (f, g, \delta) \mid X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{-\delta} \text{ is an } \mathbb{E}-\text{triangle} \} \).

For any \( (f, g, \delta) \in W^L_{XY} \), consider the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & L \\
\downarrow{x} & & \downarrow{l} \\
X & \xrightarrow{\gamma} & L \\
\end{array}
\]

\[
\begin{array}{ccc}
L & \xrightarrow{g} & Y \\
\downarrow{\delta} & & \downarrow{y} \\
Y & \xrightarrow{-\delta} & Y \\
\end{array}
\]

with \( (x, l, y) \in \text{Aut } X \times \text{Aut } L \times \text{Aut } Y \). By [6, Proposition 3.7], \( X \xrightarrow{\gamma} L \xrightarrow{y^{-1}x} Y \) is an \( \mathbb{E} \)-triangle. Hence, for any \( (x, l, y) \in \text{Aut } X \times \text{Aut } L \times \text{Aut } Y \), we have a map

\[
\phi_{xly} : W^L_{XY} \longrightarrow W^L_{XY}; \ (f, g, \delta) \mapsto (lfx^{-1}, ygl^{-1}, y^{-1}x \delta).
\]

In what follows, we need to consider several group actions on the set \( W^L_{XY} \). Now, let us give these group actions and the corresponding orbit sets:

The action of \( \text{Aut } X \) on \( W^L_{XY} \) has the orbit set

\[
V^L_{XY} = \{ (f, g, \delta)_X \mid (f, g, \delta) \in W^L_{XY} \},
\]

where

\[
(f, g, \delta)_X = \{ \phi_{x11}((f, g, \delta)) \mid x \in \text{Aut } X \}.
\]

Similarly, the actions of \( \text{Aut } L \) and \( \text{Aut } Y \) on \( W^L_{XY} \) have the orbit sets \( V^L_{XY} \) and \( V^L_{XY} \), respectively.

The action of \( \text{Aut } L \times \text{Aut } Y \) on \( W^L_{XY} \) has the orbit set

\[
V^L_{XY} = \{ (f, g, \delta)^{\wedge} \mid (f, g, \delta) \in W^L_{XY} \},
\]

where

\[
(f, g, \delta)^{\wedge} = \{ \phi_{1ly}((f, g, \delta)) \mid (l, y) \in \text{Aut } L \times \text{Aut } Y \}.
\]

The action of \( \text{Aut } X \times \text{Aut } L \) on \( W^L_{XY} \) has the orbit set

\[
V^L_{XY} = \{ (f, g, \delta)^{\vee} \mid (f, g, \delta) \in W^L_{XY} \},
\]

where

\[
(f, g, \delta)^{\vee} = \{ \phi_{x11}((f, g, \delta)) \mid (x, l) \in \text{Aut } X \times \text{Aut } L \}.
\]
Given $X, Y \in \mathcal{C}$, we set

$$\{X, Y\} := \prod_{i > 0} \mid \mathbb{E}^{-i}(X, Y) \mid^{(-1)^i}.$$ 

Since $\mathcal{C}$ is left locally homologically finite, we have that $\{X, Y\} < \infty$.

In what follows, for the simplicity of notation, we write $(-, -)$ as $\text{Hom}(-, -)$ in $\mathcal{C}$.

**Lemma 3.1.** Let $(f, g, \delta) \in W_{XY}^L$.

1. We have that

   $$|\text{Im} (g, L)| = |(Y, L)| \frac{\{X, L\} \{Y, L\}}{\{L, L\}}$$ 

   and

   $$|\text{Im} (Y, g)| = |(Y, L)| \frac{\{Y, L\}}{|(Y, X)| \{Y, Y\} \{Y, X\}}.$$

2. We have that

   $$|\text{Im} (L, f)| = |(L, X)| \frac{\{L, Y\} \{L, X\}}{\{L, L\}}$$ 

   and

   $$|\text{Im} (f, X)| = |(L, X)| \frac{\{L, X\}}{|(Y, L)| \{Y, X\} \{Y, L\} \{Y, L\}}.$$ 

**Proof.** Applying the functor $\text{Hom}(-, L)$ to the $\mathbb{E}$-triangle $(f, g, \delta)$, we get the exact sequence

$$\cdots \rightarrow \mathbb{E}^{-1}(L, L) \rightarrow \mathbb{E}^{-1}(X, L) \xrightarrow{\delta} \text{Hom} (Y, L) \xrightarrow{(g, L)} \text{Hom} (L, L) \rightarrow \text{Hom} (X, L).$$

Observe that

$$|\text{Im} \delta| = \prod_{i > 0} \frac{\mid \mathbb{E}^{-i}(L, L) \mid^{(-1)^i}}{\mid \mathbb{E}^{-i}(X, L) \mid^{(-1)^i} \mid \mathbb{E}^{-i}(Y, L) \mid^{(-1)^i}} = \frac{\{L, L\}}{\{X, L\} \{Y, L\}}.$$

Hence,

$$|\text{Im} (g, L)| = \frac{|(Y, L)|}{|\text{Im} \delta|} = \frac{|(Y, L)|}{|(Y, L)| \frac{\{X, L\} \{Y, L\}}{\{L, L\}}}.$$

Similarly, applying the functor $\text{Hom}(Y, -)$ to the $\mathbb{E}$-triangle $(f, g, \delta)$, we get the exact sequence

$$\cdots \rightarrow \mathbb{E}^{-1}(Y, L) \rightarrow \mathbb{E}^{-1}(Y, Y) \xrightarrow{\delta} \text{Hom} (Y, X) \rightarrow \text{Hom} (Y, L) \xrightarrow{(Y, g)} \text{Hom} (Y, Y).$$

We obtain that

$$|\text{Im} (Y, g)| = \frac{|(Y, L)|}{|(Y, X)| \mid \text{Im} \delta|} = \frac{|(Y, L)|}{|(Y, X)| \frac{\{Y, L\}}{\{Y, Y\} \{Y, X\}}}.$$

The proof of (2) is similar. \qed
For \((f, g, \delta) \in W_{XY}^L\), we set

\[ G_L(f, g, \delta)_Y = \{ l \in \text{Aut } L \mid (f, g, \delta)_Y = (lf, gl^{-1}, \delta)_Y \}. \]

That is, \(G_L(f, g, \delta)_Y\) is the stabilizer of \((f, g, \delta)_Y \in V_{XY}^L\) under the action of \(\text{Aut } L\). Dually, we set

\[ G_Y(f, g, \delta)_L = \{ y \in \text{Aut } Y \mid (f, g, \delta)_L = (f, yg, y^{-1} \delta)_L \}. \]

Denote by \(\text{ind } (\mathcal{C})\) the set of isomorphism classes of indecomposable objects in \(\mathcal{C}\). For any morphism \(g : L \to Y\), by [10, Lemma 2.2], there exists the decomposition

\[ L = L_1 \oplus L_2, \quad Y = Y_1 \oplus Y_2 \]

and \((l, y) \in \text{Aut } L \times \text{Aut } Y\) such that

\[ ygl = \begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix} : L_1 \oplus L_2 \to Y_1 \oplus Y_2 \]

with \(n_{11}\) being an isomorphism and

\[ n_{22} \in \text{radHom } (L_2, Y_2) := \{ r \in \text{Hom } (L_2, Y_2) \mid r_1 r_2 \text{ is not an isomorphism for any } r_1 : Y_2 \to A \text{ and } r_2 : A \to L_2 \text{ with } A \in \text{ind } (\mathcal{C}) \}. \]

We may denote \(L_1\) and \(Y_1\) by \(L_g\) and \(gY\), respectively. For \(f : X \to Y\), we set

\[ \text{Hom } (Y, Z) f := \{ gf \mid g \in \text{Hom } (Y, Z) \} \]

and

\[ f \text{Hom } (Z, X) := \{ fh \mid h \in \text{Hom } (Z, X) \}. \]

**Lemma 3.2.** Let \((f, g, \delta) \in W_{XY}^L\).

1. \(1 - G_L(f, g, \delta)_Y = \{ l \in \text{End } L \mid l \in \text{Im } (g, L) \text{ and } 1 - l \in \text{Aut } L \}. \)
2. If \((f, g, \delta) = (f_1, g_1, \delta_1) \in V_{XY}^L\). Then

\[ |G_L(f, g, \delta)_Y| = |G_L(f_1, g_1, \delta_1)_Y|. \]

3. We have that

\[ |G_L(f, g, \delta)_Y| = \frac{|\text{Im } (g, L)| |\text{Aut } L_g|}{|\text{End } L_g|}. \]

4. We have that

\[ |G_Y(f, g, \delta)_L| = \frac{|\text{Im } (Y, g)| |\text{Aut } gY|}{|\text{End } gY|}. \]

**Proof.** (1) It is straightforward that \(l \in G_L(f, g, \delta)_Y \iff l \in \text{Aut } L\) and \(f = lf \iff l \in \text{Aut } L\) and \(1 - l = tg\) for some \(t : Y \to L \iff l \in \text{Aut } L\) and \(1 - l \in \text{Im } (g, L)\).

(2) By hypothesis, there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & L \\
\downarrow & & \downarrow \\
X & \xrightarrow{g_1} & Y \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{f} & L \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & Y \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & L \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\delta_1} & Y \\
\end{array}
\]
with \((l, y) \in \text{Aut } L \times \text{Aut } Y\). Consider the map
\[
\alpha : G_L(f, g, \delta)_Y \to G_L(f_1, g_1, \delta_1)_Y; \ l' \mapsto ll' l^{-1}.
\]

By (1), it is easy to see that \(\alpha\) is a bijection.

(3) By [10, Lemma 2.2], there exists the decomposition \(L = L_g \oplus L_2, Y = Y_1 \oplus Y_2\) and \((l, y) \in \text{Aut } L \times \text{Aut } Y\) such that
\[
ygl = \begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix} : L_g \oplus L_2 \to Y_1 \oplus Y_2
\]
with \(n_{11}\) being an isomorphism and \(n_{22} \in \text{radHom}(L_2, Y_2)\). Set \(f' = l^{-1} f, g' = ygl\) and \(\delta' = y^{-1} \delta\). Then \((f, g, \delta)^\wedge = (f', g', \delta')^\wedge\) in \(V_{XY}^L\).

If \(l \in \text{Im}(g', L)\), i.e., there exists a morphism
\[
t = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} : Y_1 \oplus Y_2 \to L_g \oplus L_2
\]
such that
\[
1 - l = 1 - tg' = \begin{pmatrix} 1 - t_{11}n_{11} & -t_{12}n_{22} \\ -t_{21}n_{11} & 1 - t_{22}n_{22} \end{pmatrix}.
\]

(3.1) Since \(n_{22} \in \text{radHom}(L_2, Y_2)\), we have that \(1 - t_{22}n_{22} \in \text{Aut } L_2\). By applying some elementary transformations on the matrix in (3.1), we obtain that \(1 - l \in \text{Aut } L\) if and only if \(1 - t_{11}n_{11} \in \text{Aut } L_g\). By (1) and (2), we have that
\[
|G_L(f, g, \delta)_Y| = |G_L(f', g', \delta')_Y|
\]
\[
= |1 - G_L(f', g', \delta')_Y|
\]
\[
= |\{ l \in \text{End } L \mid l \in \text{Im}(g', L) \text{ and } 1 - l \in \text{Aut } L\}|
\]
\[
= |\text{Aut } L_g| |\text{Hom}(Y_1, L_2)n_{11}| |\text{Hom}(Y_2, L_g)n_{22}| |\text{Hom}(Y_2, L_2)n_{22}|
\]
\[
= \frac{|\text{Im}(g', L)| |\text{Aut } L_g|}{|\text{End } L_g|}
\]
\[
= \frac{|\text{Im}(g, L)| |\text{Aut } L_g|}{|\text{End } L_g|}.
\]

(4) Note that \(y \in G_Y(f, g, \delta)L \Leftrightarrow y \in \text{Aut } Y\) and \(\delta = y^{-1} \delta\Leftrightarrow y \in \text{Aut } Y\) and \(1 - y^{-1} = gt\) for some \(t : Y \to L \Leftrightarrow y \in \text{Aut } Y\) and \(1 - y^{-1} \in \text{Im}(Y, g)\). Then it is proved by the analogous arguments as those for proving (3).

\[\square\]

**Lemma 3.3.** Given \(X, Y, L \in \mathcal{C}\), consider the surjection
\[
\sigma_1 : V_{XY}^L \to V_{XY}^L; \ (f, g, \delta)_Y \mapsto (f, g, \delta)^\wedge.
\]

(1) We have that
\[
\sigma_1^{-1}((f, g, \delta)^\wedge) = \{(lf, gl^{-1}, \delta)_Y \mid l \in \text{Aut } L\}.
\]
(2) We have that

\[ |V_{XY}^L| = \sum_{(f,g,\delta) \in V_{XY}^L} \frac{|\text{Aut } L| |\text{End } L_g|}{|\text{Im } (g, L)| |\text{Aut } L_g|}. \]

(3) We have that

\[ |V_{XY}^F| = \sum_{(f,g,\delta) \in V_{XY}^F} \frac{|\text{Aut } Y| |\text{End } g Y|}{|\text{Im } (Y, g)| |\text{Aut } g Y|}. \]

Proof. (1) Assume that \((f_1, g_1, \delta_1)_Y \in \sigma_1^{-1}((f, g, \delta)^\vee)\), then \((f_1, g_1, \delta_1)^\vee = (f, g, \delta)^\vee\). Hence, we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & L \xrightarrow{g} Y - \delta \\
\downarrow f_1 & & \downarrow y \\
X & \xrightarrow{f_1} & L \xrightarrow{g_1} Y - \delta_1
\end{array}
\]

with \((l, y) \in \text{Aut } L \times \text{Aut } Y\). It follows that \((f_1, g_1, \delta_1) = (lf, ygl^{-1}, y^{-1} \delta)\) and \((f_1, g_1, \delta_1)_Y = (lf, gl^{-1}, \delta)_Y\). Conversely, it is clear that \((lf, gl^{-1}, \delta)^\vee = (f, g, \delta)^\vee\) for any \(l \in \text{Aut } L\).

(2) Since \(\sigma_1\) is surjective, by Lemma 3.2(3) and (1), we have that

\[
|V_{XY}^L| = \sum_{(f,g,\delta) \in V_{XY}^L} |\sigma_1^{-1}((f, g, \delta)^\vee)|
\]

\[
= \sum_{(f,g,\delta) \in V_{XY}^L} \frac{|\text{Aut } L|}{|G_L(f, g, \delta)_Y|}
\]

\[
= \sum_{(f,g,\delta) \in V_{XY}^L} \frac{|\text{Aut } L| |\text{End } L_g|}{|\text{Im } (g, L)| |\text{Aut } L_g|};
\]

(3) It is proved by the analogous arguments as those for proving (2). \(\square\)

Similarly, consider the surjections

\[
\sigma_2 : V_{XY}^L \rightarrow V_{XY}^F; \ (f, g, \delta)_X \mapsto (f, g, \delta)^\vee,
\]

and

\[
\sigma_3 : V_{XY}^L \rightarrow V_{XY}^F; \ (f, g, \delta)_L \mapsto (f, g, \delta)^\vee.
\]

Then we have the following

Lemma 3.4. Given \(X, Y, L \in \mathcal{C}\), we have that

\[
|V_{XY}^L| = \sum_{(f,g,\delta) \in V_{XY}^L} \frac{|\text{Aut } L| |\text{End } f L|}{|\text{Im } (L, f)| |\text{Aut } f L|}
\]

and

\[
|V_{XY}^F| = \sum_{(f,g,\delta) \in V_{XY}^F} \frac{|\text{Aut } X| |\text{End } X_f|}{|\text{Im } (X, f)| |\text{Aut } X_f|}.
\]
We denote by \((X, L_Y)\) the set consisting of inflations \(f : X \to L\) such that \(\text{cone}(f) \cong Y\). Dually, we define \(X(L, Y)\). We denote by \(E(Y, X)_L\) the set consisting of extensions \(\delta \in E(Y, X)\) such that \(s(\delta) = [X \xrightarrow{f} L \xrightarrow{g} Y]\).

**Lemma 3.5.** Given \(X, Y, L \in \mathcal{C}\), the maps

\[
\varphi_1 : V^L_{XY} \to (X, L)_Y; \quad (f, g, \delta)_Y \mapsto f;
\]

\[
\varphi_2 : V^L_{XY} \to E(Y, X)_L; \quad (f, g, \delta)_Y \mapsto \delta;
\]

and

\[
\varphi_3 : V^L_{XY} \to X(L, Y); \quad (f, g, \delta)_Y \mapsto g
\]

are bijections.

**Proof.** Recall that the orbit

\[
(f, g, \delta)_Y = \{(f, yg, y^{-1}\delta) \mid y \in \text{Aut} Y\}.
\]

Thus \(\varphi\) is a well-defined surjection. Now, assume that \(\varphi_1((f, g, \delta)_Y) = \varphi_1((f_1, g_1, \delta_1)_Y)\). Then, by (ET3) and [6 Corollary 3.6], we have the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & L \\
| & | & | \\
X & \xrightarrow{f_1} & L \\
\end{array}
\begin{array}{ccc}
\xrightarrow{g} & \xrightarrow{\delta} & Y \\
\xrightarrow{y} & | & | \\
\xrightarrow{g_1} & \xrightarrow{\delta_1} & Y
\end{array}
\]

with \(y \in \text{Aut} Y\). It follows that \((f, g, \delta)_Y = (f_1, g_1, \delta_1)_Y\). Similarly, we can prove that \(\varphi_2\) and \(\varphi_3\) are bijections. \(\square\)

Now, we have the following key formula.

**Proposition 3.6.** Given \(X, Y, L \in \mathcal{C}\), we have that

\[
\sum_{(f,g,\delta) \in V^L_{XY}} \frac{|\text{End}_g Y|}{|\text{Aut}_g Y|} \frac{1}{|(Y, L)|{(Y, L)}} = \frac{|(X, L)_Y|}{|\text{Aut} L|} \frac{|X, L|}{|(L, L)|} = \frac{|E(Y, X)_L|}{|\text{Aut} Y|} \frac{1}{|(Y, Y)|{(Y, Y)}}.
\]

**Proof.** By Lemma 3.3 and Lemma 3.1, we obtain that

\[
|\text{Im}((g, L))| \frac{|V^L_{XY}|}{|\text{Aut} L|} = \sum_{(f,g,\delta) \in V^L_{XY}} \frac{|\text{End}_{Lg}|}{|\text{Aut}_{Lg}|} = \sum_{(f,g,\delta) \in V^L_{XY}} \frac{|\text{End}_{gY}|}{|\text{Aut}_{gY}|} = |\text{Im}(Y, g)| \frac{|V^L_{XY}|}{|\text{Aut} Y|}.
\]
Using Lemma \ref{lemma:3.1} we have that
\[
\frac{|V_{X,Y}^L|}{|\text{Aut } L|} \frac{1}{|\text{Aut } Y|} \frac{1}{|\{Y,Y\}|(Y,Y)|\{Y,X\}|(Y,L)|\{Y,L\}|}.
\]
By Lemma \ref{lemma:3.5} we complete the proof.

Dually, we also have the following formula.

**Proposition 3.7.** Given $X, Y, L \in \mathcal{C}$, we have that
\[
\sum_{(f,g,\delta) \in V_{X,Y}^L} \frac{|\text{End } X_f|}{|\text{Aut } X|} \frac{1}{|\text{Aut } L|} \frac{1}{|\text{Aut } Y|} \frac{1}{|\{Y,Y\}|(Y,Y)|\{Y,X\}|(Y,L)|\{Y,L\}|}.
\]

**Proof.** Similarly, using Lemma \ref{lemma:3.1} Lemma \ref{lemma:3.4} and Lemma \ref{lemma:3.5}, we complete the proof.

Let us state the main result in this section in the following, which is a generalization of the case for triangulated categories in \cite{10}.

**Theorem 3.8.** Given $X, Y, L \in \mathcal{C}$, we have that
\[
\frac{|((X,Y)_L)|\{L,Y\}|\{L,L\}|}{|\text{Aut } L|} = \frac{|(X,Y)|\{X,L\}|\{X,L\}|}{|\text{Aut } X|}.
\]

**Proof.** By Proposition \ref{prop:3.6} we have that
\[
|(X,Y)_L|\{X,Y\}|\{L,L\}| = \frac{|E(Y,X)_L|}{|\text{Aut } Y|} \frac{1}{\{Y,Y\}|(Y,Y)|\{Y,X\}|} \frac{1}{|\text{Aut } L|}.
\]
By Proposition \ref{prop:3.7} we have that
\[
|X(Y,X)|\{X,L\}|\{L,Y\}| = \frac{|E(Y,X)_L|}{|\text{Aut } X|} \frac{1}{\{X,X\}|(Y,X)|\{Y,X\}|} \frac{1}{|\text{Aut } L|}.
\]
Thus, we conclude that
\[
\frac{|((X,Y)_L)|\{X,Y\}|\{L,L\}|}{|\text{Aut } L|} = \frac{|(X,Y)|\{X,L\}|\{X,L\}|}{|\text{Aut } X|}.
\]
This finishes the proof.

**Corollary 3.9.** (Toën’s formula) Let $\mathcal{C}$ be a triangulated category. For any $X, Y, L \in \mathcal{C}$, we have that
\[
\frac{|(L,Y)_{X[1]}|\{L,Y\}|\{L,L\}|}{|\text{Aut } Y|} = \frac{|(X,Y)_L|\{X,L\}|\{X,L\}|}{|\text{Aut } X|}.
\]

**Proof.** In this case, just noting that $|X(Y,L)| = |L(Y,Y)|$, we complete the proof.

Let $\mathcal{C}$ be an exact category. For $X, Y \in \mathcal{C}$, by \cite{3} Proposition 5.4], we know that $E^{-i}(X,Y) = 0$ for any $i < 0$. Thus,
\[
\{X,Y\} = \prod_{i>0} |E^{-i}(X,Y)|^{(-1)^i} = 1.
\]
Corollary 3.10. Let \( \mathcal{C} \) be an exact category. For any \( X, Y, L \in \mathcal{C} \), we have that
\[
\frac{|W_{X}^{L}|}{|\text{Aut } X||\text{Aut } Y|} = \frac{|x(L, Y)|}{|\text{Aut } Y|} = \frac{|(X, L)_{Y}|}{|\text{Aut } X|}.
\]

Proof. Since each deflation in \( \mathcal{C} \) is an epimorphism, the action of \( \text{Aut } Y \) on \( W_{X}^{L} \) is free. Thus,
\[
|(X, L)_{Y}| = |V_{X}^{L} Y| = \frac{|W_{X}^{L}|}{|\text{Aut } Y|}.
\]
Similarly, \( |x(L, Y)| = \frac{|W_{X}^{L}|}{|\text{Aut } X|} \). By Theorem 3.8, we finish the proof. \( \square \)

Let \( X, Y, L \in \mathcal{C} \) and \( \mathcal{X} \) be a subset of \( W_{X}^{L} \). Then the actions of \( \text{Aut } X \) and \( \text{Aut } Y \) on \( W_{X}^{L} \) naturally induces the actions on \( \mathcal{X} \). We denote the orbit sets of \( \mathcal{X} \) under the actions of \( \text{Aut } X \) and \( \text{Aut } Y \) by \( \mathcal{X}_{X} \) and \( \mathcal{X}_{Y} \), respectively. The following observation is useful for the next section.

Lemma 3.11. Let \( X, Y, L \in \mathcal{C} \) and \( \mathcal{X} \) be a subset of \( W_{X}^{L} \). Then we have that
\[
\frac{|\mathcal{X}_{X}|}{|\text{Aut } Y|} \{ L, Y \} = \frac{|\mathcal{X}_{Y}|}{|\text{Aut } X|} \{ X, L \}.
\]

Proof. It is proved by the analogous arguments as those for proving Theorem 3.8. \( \square \)

4. Hall algebras of extriangulated categories

In this section, we define the Hall algebra of extriangulated categories using the formula in Theorem 3.8.

Definition 4.1. The Hall algebra \( \mathcal{H}(\mathcal{C}) \) of the extriangulated category \( \mathcal{C} \) is a \( \mathbb{Q} \)-space with the basis \( \{ u_{[X]} \mid X \in \mathcal{C} \} \) and the multiplication defined by
\[
u_{[X]} \circ u_{[Y]} = \sum_{[L]} F_{XY}^{L} u_{[L]},
\]
where
\[
F_{XY}^{L} := \frac{|x(L, Y)| \{ L, Y \}}{|\text{Aut } Y| \{ Y, Y \}} = \frac{|(X, L)_{Y}| \{ X, L \}}{|\text{Aut } X| \{ X, X \}}.
\]

Remark 4.2. If \( \mathcal{C} \) is a triangulated category, Definition 4.1 coincides with that in [10]. If \( \mathcal{C} \) is an exact category, by Corollary 3.10 Definition 4.1 coincides with that in [5].

Let us state the main result in this paper as the following

Theorem 4.3. The Hall algebra \( \mathcal{H}(\mathcal{C}) \) of the extriangulated category \( \mathcal{C} \) is an associative algebra.
Before proving Theorem 4.3, we need some preparations.

We set
\[
L'_Z(X \oplus M, L)_Y = \{(f, g) : X \oplus M \to L \mid \text{cone}(f) \cong Y, \text{cocone}(g) \cong Z \text{ and cocone}((f, g)) \cong L'\}
\]
and
\[
z(L', X \oplus M)_Y^L = \{(f, g)^T : L' \to X \oplus M \mid \text{cocone}(f) \cong Z, \text{cone}(g) \cong Y, \text{and cone}((f, g)^T) \cong L\}.
\]

**Lemma 4.4.** (1) The map
\[
\tau_1 : (X, L)_Y \times z(M, L) \to \bigcup_{[L']} L'_Z(X \oplus M, L)_Y; \quad f \times g \mapsto (f, g)
\]
is a bijection.

(2) The map
\[
\tau_2 : z(L', X) \times (L', M)_Y \to \bigcup_{[L]} z(L', X \oplus M)_Y^L; \quad f \times g \mapsto (f, g)^T
\]
is a bijection.

**Proof.** (1) By (ET4)
\[
\begin{array}{c}
Z \\ \downarrow \\
L' \downarrow f \\
X \\
\text{and} \\
Z \\ \downarrow \\
M \downarrow g \\
L'.
\end{array}
\]

By the dual of [6, Corollary 3.16], we obtain that \((f, g) \in L'_Z(X \oplus M, L)_Y\). By definition, it is easy to see that \(\tau_1\) is a bijection. The proof of (2) is similar. \(\square\)

**Lemma 4.5.** Let
\[
L' \overset{(f, g)^T}{\longrightarrow} X \oplus M \overset{(f', g')}{\longrightarrow} L \longrightarrow
\]
be an \(\mathbb{E}\)-triangle. Then \((f, g) \in z(L', X) \times (L', M)_Y\) if and only if \((f', g') \in (X, L)_Y \times z(M, L)\).

**Proof.** Assume that \((f, g) \in z(L', X) \times (L', M)_Y\). By (ET4), there exists a commutative diagram
\[
\begin{array}{c}
Z \\ \downarrow \\
L' \downarrow f \\
X \\
\text{and} \\
Z \\ \downarrow \\
M \downarrow g \\
L'.
\end{array}
\]

By the dual of [6, Corollary 3.16], we know that
\[
L' \overset{(f, g)^T}{\longrightarrow} X \oplus M \overset{(f', g')}{\longrightarrow} L' \longrightarrow
\]
We have that Proposition 4.7.

The maps $\varphi_1 : V(L', X \oplus M, L)_Z \rightarrow z(L', X \oplus M)_Y^L$; $((f,g)^T, (f',g') \in (L', X \oplus M)_Y^L$ and

$$\varphi_2 : V(L', X \oplus M, L)_Z \rightarrow L'_Z(X \oplus M)_Y ; \ (f,g)^T, (f',g') \in (L', X \oplus M)_Y.$$ are bijections.

Proof. The proof is similar to Lemma 3.5.

Proposition 4.7. We have that

$$\frac{|L'(X \oplus M, L)_Y|}{|Aut L|} \frac{|X \oplus M, L|}{\{L, L\}} = \frac{|z(L', X \oplus M)_Y|}{|Aut L'|} \frac{|L', X \oplus M|}{\{L', L'\}}.$$ 

Proof. It is proved by Lemma 3.1 and Lemma 4.6.

Now we are in the position to prove Theorem 3.6.

Proof of Theorem 4.3. We need to prove that

$$u_{[Z]} \circ (u_{[X]} \circ u_{[Y]}) = (u_{[Z]} \circ u_{[X]}) \circ u_{[Y]}$$

for any $X, Y, Z \in \mathcal{C}$. By definition, it is equivalent to proving that

$$\sum_{[L]} F_{XY}^L F_{ZL}^M = \sum_{[L']} F_{XZ}^{L'} F_{LY}^M$$
for $M \in \mathcal{C}$. By Lemma 4.4,

$$
\sum_{[L]} F_{XY}^L F_{ZL}^M = \sum_{[L]} \frac{|(X, L)_Y X, L| |(M, L)_Y M, L|}{|Aut X| X, X |Aut L| L, L} \\
= \frac{1}{|Aut X| X, X} \sum_{[L]} \frac{|(X, L)_Y Z(M, L) X, L|}{|Aut L| L, L} \\
= \frac{1}{|Aut X| X, X} \sum_{[L]} \sum_{[L']} \frac{|(X, L)_Y Z(M, L) X, L|}{|Aut L| L, L}.
$$

Similarly,

$$
\sum_{[L']} F_{ZX}^{L'} F_{LY}^M = \sum_{[L']} \frac{|z(L', X)| L', X |z(M, L)_Y L', L|}{|Aut X| X, X |Aut L'| L', L'} \\
= \frac{1}{|Aut X| X, X} \sum_{[L']} \frac{|z(L', X)| ((L', M)_Y L', M) X, L|}{|Aut L'| L', L'} \\
= \frac{1}{|Aut X| X, X} \sum_{[L']} \sum_{[L]} \frac{|z(L', X)| ((L', M)_Y L', M) X, L|}{|Aut L'| L', L'}.
$$

By Proposition 4.7 we finish the proof. \hfill \Box

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