Induced representations arising from a character with finite orbit in a semidirect product

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Abstract. Making use of a unified approach to certain classes of induced representations, we establish here a number of detailed spectral theoretic decomposition results. They apply to specific problems from noncommutative harmonic analysis, ergodic theory, and dynamical systems. Our analysis is in the setting of semidirect products, discrete subgroups, and solenoids. Our applications include analysis and ergodic theory of Bratteli diagrams and their compact duals; of wavelet sets, and wavelet representations.

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1. Introduction

The purpose of the present paper is to demonstrate how a certain induction (from representation theory) may be applied to a number of problems in dynamics, in spectral theory and harmonic analysis; yielding answers in the form of explicit invariants, and equivalences. In more detail, we show that a certain family of induced representations forms a unifying framework. The setting is unitary representations, infinite-dimensional semidirect products and crossed products. While the details in our paper involve a variant of...
Mackey’s theory, our approach is extended, and it is constructive and algorithmic. Our starting point is a family of imprimitivity systems. By this we mean a pair: a unitary representation, and a positive operator-valued mapping, subject to a covariance formula for the two (called imprimitivity). One of its uses is a characterization of those unitary representations of some locally compact group $G$ “which arise” as an induction from a subgroup; i.e., as induced from of a representation of a specific subgroup of $G$. (Here, the notation “which arise” means “up to unitary equivalence.”)

Below we give a summary of notation and terminologies, and we introduce a family of induced representations, see, e.g., [Jor88, Mac88, Ørs79].

The framework below is general; the context of locally compact (nonabelian) groups. But we shall state the preliminary results in the context of unimodular groups, although it is easy to modify the formulas and the results given to nonunimodular groups. One only needs to incorporate suitable factors on the respective modular functions, that of the ambient group and that of the subgroup. Readers are referred to [Ørs79] for additional details on this point. Another reason for our somewhat restricted setting is that our main applications below will be to the case of induction from suitable abelian subgroups. General terminology: In the context of locally compact abelian groups, say $B$, we shall refer to Pontryagin duality (see, e.g., [Rud62]); and so in particular, when the abelian group $B$ is given, by “the dual” we mean the group of all continuous characters on $B$, i.e., the one-dimensional unitary representations of $B$. We shall further make use of Pontryagin’s theorem to the effect that the double-dual of $B$ is naturally isomorphic to $B$ itself.

Our results are motivated in part by a number of noncommutative harmonic analysis issues involved in the analysis of wavelet representations, and wavelet sets; see especially [LiPT01]; and also [LaSST06, CM11, CM13, CMO14, MO14].

**Definition 1.1.** Let $G$ be a locally compact group, $B \subset G$ a closed subgroup; both assumed unimodular. Given $V \in \text{Rep}(B, \mathcal{H}_V)$, a representation of $B$ in the Hilbert space $\mathcal{H}_V$, let

$$U = \text{Ind}^G_B (V)$$

be the **induced representation**. $U$ acts on the Hilbert space $\mathcal{H}_U$ as follows: $\mathcal{H}_U$ consists of all measurable functions $f : G \to \mathcal{H}_V$ s.t.

$$f(bg) = V_b f(g), \forall b \in B, \forall g \in G,$$

$$\|f\|_{\mathcal{H}_U}^2 = \int_{B \setminus G} \|f(g)\|^2 d\mu_{B \setminus G}(g) < \infty,$$

w.r.t. the measure on the homogeneous space $B \setminus G$, i.e., the Hilbert space carrying the induced representation. Set

$$(U_g f)(x) := f(xg), \quad x, g \in G.$$
Remark 1.2. If the respective groups $B$ and $G$ are nonunimodular, we select respective right-invariant Haar measures $db$, $dg$; and corresponding modular functions $\Delta_B$ and $\Delta_G$. In this case, the modification to the above is that Equation (1.2) will instead be

\[
(1.5) \quad f(bg) = \left( \frac{\Delta_B(b)}{\Delta_G(b)} \right)^{1/2} V_b f(g); \quad b \in B, \ g \in G,
\]

thus a modification in the definition of $\mathcal{H}(\text{Ind}_B^G(V))$.

We recall the following theorems of Mackey [Mac88, Ørs79].

Theorem 1.3 (Mackey). $U = \text{Ind}_B^G(V)$ is a unitary representation.

Theorem 1.4 (Imprimitivity Theorem). A representation $U \in \text{Rep}(G, \mathcal{H})$ is induced iff $\exists$ a positive operator-valued mapping

\[
(1.6) \quad \pi : C_c(B \setminus G) \to \mathcal{B}(\mathcal{H}) \ s.t.
\]

\[
(1.7) \quad U_g \pi(\varphi) U_g^* = \pi(R_g \varphi)
\]

$\forall g \in G, \ \forall \varphi \in C_c(B \setminus G);$ where $\pi$ is nondegenerate, and $R_g$ in (1.7) denotes the right regular action.

Proof. See [Ørs79, Jor88]. □

Theorem 1.5. Given $U \in \text{Rep}(G, \mathcal{H})$, the following are equivalent:

1. $\exists \pi$ s.t. (1.7) holds.
2. $\exists V \in \text{Rep}(B, \mathcal{H}_V) \ s.t. \ U \cong \text{Ind}_B^G(V)$.
3. $L_G(U, \pi) \cong L_B(V)$.
4. If $V_i \in \text{Rep}(B, \mathcal{H}_{V_i})$, $i = 1, 2$ and $U_i = \text{Ind}_B^G(V_i)$, $i = 1, 2$, then

\[
L_G((U_1, \pi_1); (U_2, \pi_2)) \cong L_B(V_1, V_2);
\]

i.e., all intertwining operators $V_1 \to V_2$ lift to the pair $(U_i, \pi_i)$, $i = 1, 2$.
(Here, “$\cong$” denotes unitary equivalence.) Specifically,

\[
(1.8) \quad L_G((U_1, \pi_1); (U_2, \pi_2))
\]

\[
= \left\{ W : \mathcal{H}_{U_1} \to \mathcal{H}_{U_2} \mid WU_1(g) = U_2(g) W, \forall g \in G, \ \text{and} \right. \]

\[
U_{\pi_1}(\varphi) = \pi_2(\varphi) W
\]

and

\[
(1.9) \quad L_B(V_1, V_2) = \left\{ w : \mathcal{H}_{V_1} \to \mathcal{H}_{V_2} \mid wV_1(b) = V_2(b) w, \forall b \in B \right\}.
\]

Proof. See [Ørs79, Jor88, Mac88]. □

We also recall the following result:
Lemma 1.6. Let $B$ be a lattice in $\mathbb{R}^d$, let $\alpha$ be an action of $\mathbb{Z}$ on $B$ by automorphisms, and let $(K = \hat{B}, \hat{\alpha})$ represent the dual action of $\mathbb{Z}$ on the compact abelian group $K$. Fix a $d \times d$ matrix $A$ preserving the lattice $B$ with $\text{spec}(A) \subset \{ \lambda : |\lambda| > 1 \}$.

If $\alpha = \alpha_A \in \text{Aut}(B)$, then $\hat{\alpha} \in \text{Aut}(K)$ is ergodic, i.e., if $\nu$ is the normalized Haar measure on $K$ and if $E \subset K$ is measurable s.t. $\hat{\alpha}E = E$, then $\nu(E)(1 - \nu(E)) = 0$.

Proof. By Halmos–Rohlin’s theorem (see [BreJ91]), we must show that if $b \in B$ and $A^nb = b$ for some $n \in \mathbb{N}$, then $b = 0$. This then follows from the assumption on the spectrum of $A$. □

2. The representation $\text{Ind}_G^B(\chi)$

We now specify our notation and from this point on will restrict our study to the following setting:

(i) $B$: a discrete abelian group (written additively);
(ii) $\alpha \in \text{Aut}(B)$;
(iii) $G_\alpha := B \rtimes_\alpha \mathbb{Z}$;
(iv) $K := \hat{B}$, the compact dual group;
(v) $\hat{\alpha} \in \text{Aut}(\hat{B})$, dual action;
(vi) $L_\alpha := K \rtimes_{\hat{\alpha}} \mathbb{Z}$, the $C^*$-algebra crossed product [BreJ91, p.299]; also written as $C^*(K) \rtimes_{\hat{\alpha}} \mathbb{Z}$.

More generally, let $K$ be a compact Hausdorff space, and $\beta : K \to K$ a homeomorphism; then we study the $C^*$-crossed product $C^*(K) \rtimes_{\beta} \mathbb{Z}$ (see [BreJ91]).

We define the induced representation

$$U^\chi := \text{Ind}_B^G(\chi)$$

where $\chi \in K$, i.e., a character on $B$, and

$$G := B \rtimes_\alpha \mathbb{Z}$$

as a semi-direct product. Note that $U^\chi$ in (2.1) is induced from a one-dimensional representation.

Below, $\{\delta_k\}_{k \in \mathbb{Z}}$ denotes the canonical basis in $l^2(\mathbb{Z})$, i.e.,

$$\delta_k = (\ldots, 0, 0, 1, 0, 0, \ldots)$$

with “1” at the $k^{th}$ place.

Lemma 2.1. The representation $\text{Ind}_G^B(\chi)$ is unitarily equivalent to

$$U^\chi : G \to \mathcal{B}(l^2(\mathbb{Z})), \text{ where}$$

$$U^\chi_{(j,b)}\delta_k = \chi(\alpha_{k-j}(b))\delta_{k-j}, \quad (j, b) \in G.$$
Proof. First note that both groups $B$ and $G = B \times_{\alpha} \mathbb{Z}$ are discrete and unimodular, and the respective Haar measures are the counting measure. Since $B \backslash G \cong \mathbb{Z}$, so $\mathcal{H}^\chi = L^2(B \backslash G) \cong l^2(\mathbb{Z})$. Recall the multiplication rule in $G$:

\[(k, c) \, (j, b) = (k + j, \alpha_k(b) + c)\]

where $j, k \in \mathbb{Z}$, and $b, c \in B$.

The representation $\text{Ind}_B^G(\chi)$ acts on functions $f : G \to \mathbb{C}$ s.t.

\[f((j, b)) = \chi(b) \, f(j, 0)\]

and

\[\|f\|^2_{\mathcal{H}^\chi} = \sum_{j \in \mathbb{Z}} |f(j, 0)|^2.\]

Moreover, the mapping

\[l^2 \ni \xi \mapsto W\xi \in \mathcal{H}^\lambda\]

given by

\[(W\xi)(j, b) = \chi(b) \, \xi_j\]

is a unitary intertwining operator, i.e.,

\[\text{Ind}_B^G(\chi) \, W\xi = WU^\chi \xi\]

where

\[\left(U^\chi_{(j, b)} \xi\right)_k = \chi(\alpha_k(b)) \, \xi_{k+j}\]

\[\forall j, k \in \mathbb{Z}, \forall b \in B, \forall \xi \in l^2(\mathbb{Z}).\] Note that (2.11) is equivalent to (2.4).

To verify (2.10), we have

\[\left(\text{Ind}_B^G(\chi) \, W\xi\right)(k, c) = W\xi((k, c)(j, b)) = W\xi(k + j, \alpha_k(b) + c) = \chi(\alpha_k(b) + c) \, W\xi(k + j, 0) = \chi(\alpha_k(b) + c) \, \xi_{k+j}\]

and

\[\left(WU^\chi_{(j, b)} \xi\right)(k, c) = \chi(c) \, \left(U^\chi_{(j, b)} \xi\right)_k = \chi(c) \, \chi(\alpha_k(b)) \, \xi_{k+j} = \chi(\alpha_k(b) + c) \, \xi_{k+j}.\]

In the last step we use that $\chi(\alpha_k(b) + c) = \chi(\alpha_k(b)) \, \chi(c)$, which is just the representation property of $\chi : B \to \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$,

\[\chi(b_1 + b_2) = \chi(b_1) \, \chi(b_2), \quad \forall b_1, b_2 \in B.\]
Remark 2.2. In summary, the representation $\text{Ind}_B^G(\chi)$ has three equivalent forms:

1. On $l^2(\mathbb{Z})$, 
   \[(\text{Ind}_B^G(\chi))_k = \chi(\alpha_k(b)) \xi_{k+j}, \quad (\xi_k) \in l^2(\mathbb{Z}). \]

2. In the ONB $\{\delta_k \mid k \in \mathbb{Z}\}$, 
   \[\text{Ind}_B^G(\chi)(\delta_k) = \chi(\alpha_{k-j}(b)) \delta_{k-j}, \quad k \in \mathbb{Z}. \]

3. On $\mathcal{H}^\chi$, consisting of functions $f : G \rightarrow \mathbb{C}$ s.t. 
   \[f(j,b) = \chi(b) f(j,0), \quad \|f\|^2_{\mathcal{H}^\chi} = \sum_{j \in \mathbb{Z}} |f(j,0)|^2 < \infty \]
   where
   \[\left(\text{Ind}_B^G(\chi)(\delta_k) f\right)(k,c) = \chi(\alpha_k(b)+c) f(k+j,0), \quad \text{with } (j,b),(k,c) \in G = B \rtimes \alpha \mathbb{Z}, \text{ i.e., } j,k \in \mathbb{Z}, b,c \in B.\]

3. Irreducibility

Let $G = B \rtimes \alpha \mathbb{Z}$, and $\text{Ind}_B^G(\chi)$ be the induced representation. Our main results are summarized as follows:

1. If $\chi \in K = \hat{B}$ has infinite order, then $\text{Ind}_B^G(\chi) \in \text{Rep}_{\text{irr}}(G; l^2(\mathbb{Z}))$, i.e., $\text{Ind}_B^G(\chi)$ is irreducible. We have
   \[\text{Ind}_B^G(\chi)(\delta_k) = D \chi(b) T_j \]
   where $D \chi(b)$ is diagonal, and $T_j : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$, $(T_j \xi)_k = \xi_{k+j}$, for all $\xi \in l^2(\mathbb{Z})$.

2. Suppose $\chi$ has finite order $p$, i.e., $\exists p$ s.t. $\hat{\alpha}^p \chi = \chi$, but $\hat{\alpha}^k \chi \neq \chi$, \forall $1 \leq k < p$. Set $U^\chi_p = \text{Ind}_B^G(\chi)$, then
   \[U^\chi_p(j,b) = D \chi(b) P^j \]
   where $T_j \delta_k = \delta_{k-j}$ and $P$ = the $p \times p$ permutation matrix. See (3.9). Details below.

Lemma 3.1. Set $(T_j \xi)_k = \xi_{k+j}$, for $k,j \in \mathbb{Z}$, $\xi \in l^2$; equivalently, $T_j \delta_k = \delta_{k-1}$. Then

1. The following identity holds:
   \[T_n U^\chi = U^{\hat{\alpha}^n} \chi T_n, \quad \forall n \in \mathbb{Z}, \forall \chi \in K. \]
(2) For \( \omega = (\omega_i) \in l^\infty (\mathbb{Z}) \), set
\[
(\pi (\omega) \xi)_k = \omega_k \xi_k, \quad \xi \in l^2 (\mathbb{Z}); \text{ or}
\]
\[
(3.4) \quad \pi (\omega) \delta_k = \omega_k \delta_k.
\]
Then
\[
(3.5) \quad U^\chi (j,b) \pi (\omega) (U^\chi (j,b))^* = \pi (T_j \omega)
\]
\( \forall (j,b) \in G, \forall \chi \in K. \)

(3) Restriction to the representation:
\[
(3.6) \quad U^\chi \bigg|_B = \sum_{j \in \mathbb{Z}} (\hat{\alpha}^j \chi).
\]
Recall that \( G := B \rtimes_\alpha \mathbb{Z} \), and so the subgroup \( B \) corresponds to \( j = 0 \) in \( \{ (j,b) \mid j \in \mathbb{Z}, b \in B \} \). Note that \( \hat{\alpha}^j \chi \) is a one-dimensional representation of \( B \).

**Proof.** One checks that
\[
\begin{align*}
\left( T_n U^\chi (j,b) \xi \right)_k &= \left( U^\chi (j,b) \xi \right)_{k+n} = \chi (\alpha^{k+n} (b)) \xi_{k+n+j}, \text{ and} \\
\left( U^{\hat{\alpha}^j \chi} (j,b) T_n \xi \right)_k &= (\hat{\alpha}^j \chi) \chi (\alpha (b)) (T_n \xi)_{k+j} = \chi (\alpha^{k+n} (b)) \xi_{k+n+j}.
\end{align*}
\]
The other assertions are immediate. \( \square \)

**Corollary 3.2.** If \( \hat{\alpha}^p \chi = \chi \), then \( T_p \) commutes with \( U^\chi \). In this case,
\[
(3.7) \quad U^\chi (j,b) \delta_k = \hat{\alpha}^{k-j} \chi (b) \delta_{k-j}
\]
induces an action on \( l^2 (\mathbb{Z}/p\mathbb{Z}) \). If \( m \) is fixed, the same representation is repeated, where the same action occurs in the sub-band
\[
(3.8) \quad l^2 (\{ \delta_{j+mp} \mid 0 \leq j < p \}).
\]

**Proof.** Immediate from Lemma 3.1. \( \square \)

The \( p \)-dimensional representation in each band \( \{ \delta_{j+mp} \mid 0 \leq j < p \} \) may be given in matrix form:

**Lemma 3.3.** Let \( \chi \in K \), and assume the orbit of \( \chi \) under the action of \( \hat{\alpha} \) has finite order \( p \), i.e., \( \hat{\alpha}^p \chi = \chi \) and \( \hat{\alpha}^k \chi \neq \chi \), \( 1 \leq k < p \). For \( b \in B \), set
\[
D^\chi (b) = \begin{pmatrix}
\chi (b) & \chi (\alpha (b)) & \cdots & 0 \\
0 & \chi (\alpha (b)) & \cdots & 0 \\
& & \ddots & \ddots \\
& & & \chi (\alpha^{p-1} (b))
\end{pmatrix} = \text{diag} (\chi (\alpha^i (b)))_{0}^{p-1}, \]
a \( p \times p \) diagonal matrix. If \( \chi = (\chi_1, \ldots, \chi_n) \), set \( D^\chi (b) = \text{diag} (\chi_i (b))_{n \times n} \).
For the permutation matrix \( P \), we shall use its usual \( p \times p \) matrix representation

\[
P = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}_{p \times p}.
\]

(3.9)

We have that:

1. The following identity holds:

\[
PD_{\chi}(b) = D_{\chi}(\alpha(b))P.
\]

(3.10)

2. For all \((j, b) \in G = B \rtimes_{\alpha} \mathbb{Z}\), let

\[
U_{p}^{(\chi)}(j, b) := D_{\chi}(b)P^{j}.
\]

(3.11)

Then \(U_{p}^{\chi} \in \text{Rep}(G, \mathbb{C}^p)\), i.e.,

\[
U_{p}^{\chi}(j, b)U_{p}^{\chi}(j', b') = U_{p}^{\chi}(j + j', \alpha_{j}(b') + b)
\]

for all \((j, b), (j', b') \in G\).

**Proof.** It suffices to verify (3.10), and the rest of the lemma is straightforward. Set \(\chi_{k} := \chi(\alpha^{k}(b)), 0 \leq k < p\), and

\[
D_{\chi}(b) = \text{diag}(\chi_{0}, \chi_{1}, \ldots, \chi_{p-1}).
\]

(3.12)

The assertion in (3.10) follows from a direct calculation. We illustrate this with \(p = 3\): see Example 3.4 below.

**Example 3.4.** For \(p = 3\), (3.10) reads:

\[
PD_{\chi}(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \chi_{0} & 0 & 0 \\ 0 & \chi_{1} & 0 \\ 0 & 0 & \chi_{2} \end{pmatrix} = \begin{pmatrix} 0 & \chi_{1} & 0 \\ 0 & 0 & \chi_{2} \\ \chi_{0} & 0 & 0 \end{pmatrix}.
\]

\[
D_{\chi}(\alpha(b))P = \begin{pmatrix} \chi_{1} & 0 & 0 \\ 0 & \chi_{2} & 0 \\ 0 & 0 & \chi_{0} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \chi_{1} & 0 \\ 0 & 0 & \chi_{2} \\ \chi_{0} & 0 & 0 \end{pmatrix}.
\]

**Lemma 3.5.** Let \(\chi \in K\), assume the orbit of \(\chi\) under the action of \(\hat{\alpha}\) has finite order \(p\). Let \(U_{p}^{\chi}\) be the corresponding representation. Then \(U_{p}^{\chi} \in \text{Rep}_{\text{irr}}(G, l^{2}(\mathbb{Z}/p\mathbb{Z}))\), i.e., \(U_{p}^{\chi}\) is irreducible.

**Proof.** Recall \(U_{p}^{\chi}(j, b) = D_{\chi}(b)P^{j}, \forall (j, b) \in G\); see (3.11).
Let $A : l^2(Z_p) \to l^2(Z_p)$ be in the commutant of $U_{\chi}^\ast$, so $A$ commutes with $P$. It follows that $A$ is a Toeplitz matrix

\[
A = \begin{pmatrix}
A_0 & A_2 & \cdots & A_{p-1} \\
A_{-1} & A_0 & A_2 & \cdots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & A_{-p+1} & \cdots & A_0
\end{pmatrix}
\]

relative to the ONB $\{\delta_0, \delta_1, \ldots, \delta_{p-1}\}$, where $A_{i,j} := \langle \delta_i, A\delta_j \rangle$. Note that

\[
A_{i,j} = \langle \delta_i, A\delta_j \rangle = \langle \delta_i, P\delta_{j+1} \rangle = \langle P^i\delta_i, A\delta_{j+1} \rangle = \langle \delta_{i+1}, A\delta_{j+1} \rangle = A_{i+1,j+1}
\]

where $i + 1, j + 1$ are the additions in $Z_p = Z/pZ$, i.e., $+ \mod p$.

If $A$ also commutes with $D$, then

\[
A_k \left( \chi(b) - \chi(\alpha^k(b)) \right) = 0
\]

for all $k = 1, 2, \ldots, p - 1$, and all $b \in B$. But since $\chi, \widehat{\alpha}\chi, \ldots, \widehat{\alpha}^{p-1}\chi$ are distinct, $A_k = 0, \forall k \neq 0$, and so $A = A_0 I_{p \times p}, A_0 \in \mathbb{C}$.

**Proposition 3.6.** The orbit of $\chi \in K$ under the action of $\widehat{\alpha}$ has finite order $p$ if and only if

\[
\text{Ind}_{B}^G(\chi) = \sum (p\text{-dimensional irreducibles}).
\]

**Proof.** This follows from an application of the general theory; more specifically from an application of Mackey’s imprimitivity theorem, in the form given to it in [Ørs79]; see also Theorem 1.4 above.

**Lemma 3.7.** $\text{Ind}_{B}^G(\chi)$ is irreducible iff the orbit of $\chi$ under the action of $\widehat{\alpha}$ has finite order $p$, i.e., iff the set $\{\widehat{\alpha}^k\chi \mid k \in Z\}$ consists of distinct points.

**Proof.** We showed that if $\chi$ has finite order $p$, then the translation $T_p$ operator commutes with $U_{\chi} := \text{Ind}_{B}^G(\chi)$, where $T_p\delta_k = \delta_{k-p}$, $k \in Z$. Hence assume $\chi$ has infinite order.

Consider the cases $U_{\chi}(j,b)$ of

1. $b = 0$. $U_{\chi}(j,0) = T_j, j \in Z$;
2. $j = 0$. $U_{\chi}(0,b) = D_\chi(b), \delta_k \mapsto \chi(\alpha_k(b))\delta_k$, where $\chi(\alpha_k(b)) = (\widehat{\alpha}^k\chi)(b)$, and $D_\chi = (\widehat{\alpha}^k\chi) = \text{diagonal matrix}$.

Thus if $A \in B(l^2(Z))$ is in the commutant of $U_{\chi}$, then

\[
(A\xi)_k = \sum_{j \in Z} \eta_{k-j}\xi_j, \quad \eta \in l^\infty.
\]
Theorem 3.8. Let $Ind_B^G(\chi)$ be the induced representation in (2.1)–(2.2).

1. $Ind_B^G(\chi)$ is irreducible iff $\chi$ has no finite periods.
2. Suppose the orbit of $\chi$ under the action of $\hat{\alpha}$ has finite order $p$, i.e., $\hat{\alpha}^p\chi = \chi$, and $\hat{\alpha}^k\chi \neq \chi$, for all $1 \leq k < p$. Then the commutant is as follows:

$$M_p := \{Ind_B^G(\chi)\}' \cong \{f(z^p) \mid f \in L^\infty(T)\}$$

where $\cong$ in (3.16) denotes unitary equivalence.

Theorem 3.9. Let $G = B \rtimes_\alpha \mathbb{Z}$, $K = \hat{B}$, $\hat{\alpha} \in \text{Aut}(K)$. Assume the orbit of $\chi$ under the action of $\hat{\alpha}$ has finite order $p$, i.e., $\hat{\alpha}^p\chi = \chi$, $\hat{\alpha}^k\chi \neq \chi$, for all $1 \leq k < p$. Then the representation $Ind_B^G(\chi)$ has abelian commutant

$$M_p = \{Ind_B^G(\chi)\}'$$

and $M_p$ does not contain minimal projections.

Proof. Follows from the fact that $M_p$ is $L^\infty$ (Lebesgue) as a von Neumann algebra, and this implies the conclusion.

Details: Recall that $l^2(\mathbb{Z}) \simeq L^2(T)$, where

$$l^2 \ni \xi \mapsto f_\xi(z) = \sum_{j \in \mathbb{Z}} \xi_j z^j \in L^2(T)$$

and

$$T\xi \mapsto z^{-j} f_\xi(z), \quad z \in \mathbb{T}.$$ 

Note that $Ind_B^G(\chi)$ may be realized on $l^2(\mathbb{Z})$ or equivalently on $L^2(T)$ via (3.17), where

$$\|f_\xi\|^2 = \int_T |f_\xi|^2 = \sum_{k \in \mathbb{Z}} |\xi_k|^2.$$ 

On $l^2(\mathbb{Z})$, we have

$$Ind_B^G(\chi)_{(j,b)} = D_\chi(b) T_j;$$

See (3.1), and Lemma 3.3.

And on $L^2(T)$, we have

$$Ind_B^G(\chi)_{(j,b)} = \hat{D}_\chi(b) \hat{T}_j,$$

where $\hat{D}_\chi(b)$ denotes rotation on $T$, extended to the solenoid; and $\hat{T}_j$ multiplication by $z^{-j}$ acting on $L^2(T)$.

By (3.16), $M_p$ is abelian and has no minimal projections. Note projections in $L^\infty$ are given by $P_E = \text{multiplication by } \chi_E(z^p)$, where $E$ is measurable in $T$. □
4. Super-representations

By “super-representation” we will refer here to a realization of noncommutative relations “inside” certain unitary representations of suitable groups acting in enlargement Hilbert spaces; this is in the sense of “dilation” theory [FK02], but now in a wider context than is traditional. Our present use of “super-representations” is closer to that of [BiDP05, DJ08, DJP09, DLS11].

**Definition 4.1.** Let $G$ be a locally compact group, $B$ a given subgroup of $G$, and let $\mathcal{H}_0$ be a Hilbert space. Let $U_0 : G \to \mathcal{B} (\mathcal{H}_0)$ be a positive definite operator-valued mapping, i.e., for all finite systems $\{c_j\}_{j=1}^n \subset \mathbb{C}, \{g_j\}_{j=1}^n \subset G$, we have
\[
\sum_j \sum_k c_j c_k U_0 (g_j^{-1} g_k) \geq 0
\]
in the usual ordering of Hermitian operators.

If there is a Hilbert space $\mathcal{H}$, an isometry $V : \mathcal{H}_0 \to \mathcal{H}$, and a unitary representation $U : G \to \text{unitary operators on } \mathcal{H}$ such that
\[
(1) \quad U \text{ is induced from a unitary representation of } B,
\]
\[
(2) \quad U_0 (g) = V^* U (g) V, \quad g \in G,
\]
then we say that $U$ is a super-representation.

Let $B$ be discrete, abelian as before, $K = \hat{B}$, $\alpha \in \text{Aut} (B)$, $\hat{a} \in \text{Aut} (K)$, and $G = B \rtimes \alpha \mathbb{Z}$. Set:
\[
\begin{align*}
\text{• Rep} (G) & : \text{unitary representations of } G; \\
\text{• Rep}_{\text{irr}} (G) & : \text{irreducible representations in Rep} (G).
\end{align*}
\]
For $\chi \in K$, let $O (\chi)$ be the orbit of $\chi$, i.e.,
\[
(4.1) \quad O (\chi) = \{ \hat{a}^j (\chi) \mid j \in \mathbb{Z} \}.
\]

Given $U \in \text{Rep} (G)$, let $\text{Class} (U) = \text{the equivalence class of all unitary representations equivalent to } U$; i.e.,
\[
(4.2) \quad \text{Class} (U) = \{ V \in \text{Rep} (G) \mid V \simeq U \}
\]
\[
= \{ V \in \text{Rep} (G) \mid \exists W, \text{unitary s.t. } WV_g = U_g W, \ g \in G \}.
\]

For $U_1, U_2 \in \text{Rep} (G)$, set
\[
(4.3) \quad L (U^{(1)}, U^{(2)}) = \left\{ W : \mathcal{H} (U_1) \to \mathcal{H} (U_2) \mid W \text{ bounded s.t. } \right\}
\]
\[
WU_1^{(1)} = U_2^{(2)} W, \ g \in G \}\}
\]

**Theorem 4.2.** The mapping \{ set of all orbits $O (\chi)$ \} $\to \text{Class} (\text{Rep} (G))$

\[
K \ni \chi \mapsto U^\chi := \text{Ind}_B^G (\chi) \in \text{Rep} (G)
\]
passes to
\[
(4.4) \quad O (\chi) \mapsto \text{Class} (U^\chi).
\]
Proof. (Sketch) By Lemma 3.1 Equation (3.3), if $\chi \in K$, $j \in \mathbb{Z}$, then $T_j U^\chi T_j^* = U^{\tilde{\alpha}^j \chi}$, and so Class $(U^\chi)$ depends only on $O(\chi)$ and not on the chosen point in $O(\chi)$.

Using (3.4)–(3.6), we can show that $U^\chi$ is irreducible, but if $O(\chi)$ is finite then we must pass to the quotient $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ and realize $U^\chi$ on $l^2(\mathbb{Z}_p)$, as a finite-dimensional representation. □

Theorem 4.3. There is a natural isomorphism:

$$L_G(U^\chi_1, U^\chi_2) \cong L_B(\chi_2, U^\chi_1|_B)$$

$$= \sum_{j \in \mathbb{Z}} L_B(\chi_2| \tilde{\alpha}^j(\chi_1)) = \# \{ j \mid \chi_2 = \tilde{\alpha}^j(\chi_1) \}.$$ 

Proof. Follows from Frobenius reciprocity. □

Another application of Frobenius reciprocity:

Theorem 4.4. Let $\chi \in K$, and assume the orbit of $\chi$ under the action of $\tilde{\alpha}$ has finite order $p$, i.e., $\tilde{\alpha}^p \chi = \chi$, and $\tilde{\alpha}^k \chi \neq \chi$ if $1 \leq k < p$. Let $U_p^\chi \in \text{Rep}(G, l^2(\mathbb{Z}_p))$. (Recall that $U_p^\chi$ is irreducible, $\dim U_p^\chi = p$.) Then

$$L_G(U_p^\chi, \text{Ind}_B^G(\chi)) \cong L_B(\chi, U_p^\chi|_B)$$

and

$$(4.5) \quad \dim L_B(\chi, U_p^\chi|_B) = 1.$$ 

Proof. Since $U_p^\chi(j, b) := D(\chi)(b) P^j$ (see (3.11)), where $P$ is the permutation matrix on $\mathbb{Z}_p$, we get

$$U_p^\chi|_B = \bigoplus_{0 \leq k < p} \tilde{\alpha}^k \chi;$$

see (3.6). But notice that all the $p$ characters $\chi, \tilde{\alpha}(\chi), \ldots, \tilde{\alpha}^{p-1}(\chi)$ are distinct, so (4.5) holds since $L_B(\chi, \tilde{\alpha}^k(\chi)) = 0$ if $k \neq 0 \mod p$. □

We have also proved the following:

Theorem 4.5. Let $\chi \in K$, and assume the orbit of $\chi$ under the action of $\tilde{\alpha}$ has finite order $p$. Assume $G$ is compact. Let $U_p^\chi \in \text{Rep}_{\text{irr}}(G, l^2(\mathbb{Z}_p))$. Then $U_p^\chi$ is contained in $\text{Ind}_B^G(\chi)$ precisely once, i.e.,

$$\dim L_G(U_p^\chi, \text{Ind}_B^G(\chi)) = 1.$$ 

But this form of Frobenius reciprocity only holds for certain groups $G$, e.g., when $G$ is compact. Now for $G = B \rtimes \alpha \mathbb{Z}$, the formal Frobenius reciprocity breaks down, and in fact:

Theorem 4.6. $L_G(U_p^\chi, \text{Ind}_B^G(\chi)) = 0$ if $\chi \in K$ is an element of finite order $p$. 

Proof. We sketch the details for \( p = 3 \) to simplify notation. For \( p = 3 \),

\[
P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D_\chi (b) = \begin{pmatrix} \chi (b) & 0 & 0 \\ 0 & \chi (\alpha (b)) & 0 \\ 0 & 0 & \chi (\alpha_2 (b)) \end{pmatrix}
\]

and

\[
U_p^\chi (j, b) = D_\chi (b) P^j
\]

while

\[
\left( \text{Ind}_B^G (\chi) \right)_{(j,b)} \xi_k = \chi (\alpha_k (b)) \xi_{k+j}
\]

\( \forall (j, b) \in G, \forall k \in \mathbb{Z}, \forall \xi \in L^2 (\mathbb{Z}_p) \).

Let \( W \in L_G (U_p^\chi, \text{Ind}_B^G (\chi)) \), and \( u_0, u_1, u_2 \) be the canonical basis in \( \mathcal{H} (U_p^\chi) = \mathbb{C}^3 \), where

\[
U_p^\chi (j, b) u_k = \chi (\alpha_{k+2j} (b)) u_{k+2j} \mod 3
\]

\( \forall (j, b) \in G, k \in \{0, 1, 2\} \simeq \mathbb{Z}/3\mathbb{Z} \).

Set \( W u_k = \xi^{(k)} = (\xi^{(k)}_s)_{s \in \mathbb{Z}} \in L^2 (\mathbb{Z}) \), where \( \|\xi^{(k)}\|^2 = \sum_{s \in \mathbb{Z}} |\xi^{(k)}_s|^2 < \infty \).

It follows that

\[
WU_p^\chi (j, b) u_k = \text{Ind}_B^G (\chi)_{(j,b)} W u_k
\]

\( \forall (j, b) \in G, k \in \{0, 1, 2\} \). Thus

\[
\chi (\alpha_{k+2j} (b)) \xi^{(k+2j)}_s = \chi (\alpha_s (b)) \xi^{(k)}_{s+j}, \quad \forall s, j \in \mathbb{Z}.
\]

Now set \( j = 3t \in 3\mathbb{Z} \), and we get

\[
\chi (\alpha_k (b)) \xi^{(k)}_s = \chi (\alpha_s (b)) \xi^{(k)}_{s+3t}
\]

and

\[
|\xi^{(k)}_s| = |\xi^{(k)}_{s+3t}|, \quad \forall s, t \in \mathbb{Z}.
\]

Since \( \xi^{(k)} \in L^2 (\mathbb{Z}) \), \( \lim_{t \to \infty} \xi^{(k)}_{s+3t} = 0 \). We conclude from (4.7) that \( \xi^{(k)} = 0 \) in \( L^2 (\mathbb{Z}) \).

\[\square\]

Remark 4.7. The decomposition of \( \text{Ind}_B^G (\chi) : G \longrightarrow B \left( L^2 (\mathbb{Z}) \right) \) is still a bit mysterious. Recall this representation commutes with \( T_3 : (\xi_k) \longmapsto (\xi_{k+3}) \); or equivalently via \( f (z) \longmapsto z^3 f (z) \). So it is not irreducible.

The reason for \( L_G (U_p^\chi, \text{Ind}_B^G (\chi)) = 0 \), e.g., in the case of \( p = 3 \), is really that there is no isometric version of the \( 3 \times 3 \) permutation matrix \( P \) in \( T \), where \( T \delta_k = \delta_{k-1} \).

If \( W : \mathbb{C}^3 \longrightarrow L^2 (\mathbb{Z}) \) is bounded, \( WP = TW \), then applying the polar decomposition

\[
W = (W^* W)^{1/2} V
\]

with \( V : \mathbb{C}^3 \to L^2 \text{ isometric, and } VP = TV, \) or

\[
P = V^* TV, \quad P^j = V^* T_j V.
\]
So $T_j$ has the form

$$T_j = \begin{pmatrix} P_j^* & \ast \\ 0 & \ast \end{pmatrix}.$$ 

Pick $u \in \mathbb{C}^3$; then $\langle u, T_j u \rangle = \langle u, P^j u \rangle$. However, $\langle u, T_j u \rangle \to 0$ by Riemann–Lebesgue; while $\langle u, P^j u \rangle \not\to 0$ since $P^3 = I$.

**Definition 4.8.** A group $L$ acts on a set $S$ if there is a mapping

$$L \times S \to S, \quad (\lambda, s) \mapsto \lambda[s], \quad \lambda \lambda'[s] = \lambda[\lambda'[s]]$$

and $\lambda^{-1}$ is the inverse of $S \ni s \mapsto \lambda[s] \in S$, a bijection. Often $S$ will have the structure of a topological space or will be equipped with a $\sigma$-algebra of measurable sets.

**Remark 4.9.** While we have stressed discrete decompositions of induced representations, the traditional literature has stressed direct integral decompositions, see, e.g., [Mac88]. A more recent use of continuous parameters in decompositions is a construction by J. Packer et al. [LiPT01] where “wavelet sets” arise as sets of support for direct integral measures. In more detail, starting with a wavelet representation of a certain discrete wavelet group (an induced representation of a semidirect product), the authors in [LiPT01] establish a direct integral where the resulting measure is a subset of $\mathbb{R}$ called “wavelet set.” These wavelet sets had been studied earlier, but independently of representation theory. In the dyadic case, a wavelet set is a subset of $\mathbb{R}$ which tiles $\mathbb{R}$ itself by a combination of $\mathbb{Z}$-translations, and scaling by powers of 2. In the Packer et al. case, $\mathbb{R} \hookrightarrow K$, and

$$E \text{ wavelet set} \iff \int_E \oplus \mathcal{I}^G_B (\chi_t) \, dt \text{ is the wavelet representation in } L^2(\mathbb{R}).$$

**Theorem 4.10.** The group $L := K \rtimes \hat{\mathbb{Z}}$ acts on the set $\text{Rep}(G)$ by the following assignment:

Arrange it so that all the representations $U^x := \text{Ind}^G_B (\chi)$, $\chi \in K$, acts on the same $l^2$-space. Then $\text{Rep}(G) \sim \text{Rep}(G, l^2)$.

**5. Induction and Bratteli diagrams**

A Bratteli diagram is a group $G$ with vertices $V$, and edges

$$E \subset V \times V \setminus \{\text{diagonal}\}.$$ 

It is assumed that

$$V = \bigcup_{n=0}^{\infty} V_n,$$

as a disjoint union in such a way that the edges $E$ in $G$ can be arranged in a sequence of lines $V_n \to V_{n+1}$, so no edge links pairs of vertices at the same level $V_n$. With a system of inductions and restrictions one then creates these diagrams.
Let $G$ be created as follows: For a given $V_n$, let the vertices in $V_n$ represent irreducible representations of some group $G_n$, and assume $G_n$ is a subgroup in a bigger group $G_{n+1}$. For the vertices in $V_{n+1}$, i.e., in the next level in the Bratteli diagram $G$, we take the irreducible representations occurring in the decomposition of each of the restrictions:

$$\text{Ind}_{G_n}^{G_{n+1}}(L)|_{G_n},$$

so the decomposition of the restriction of the induced representation; see Figure 5.1.

Multiple lines in a Bratteli diagram count the occurrence of irreducible representations with multiplicity; these are called \textit{multiplicity lines}.

Counting multiplicity lines at each level $V_n \rightarrow V_{n+1}$ we get a so called \textit{incidence matrix}. For more details on the use of Bratteli diagrams in representation theory, see [BraJKR00, BraJKR01, BraJKR02, BraJO04, BeKwY14, BeKa14, BeJ15].

**Figure 5.1.** Part of a Bratteli diagram.

While there is a host of examples from harmonic analysis and dynamical systems; see the discussion above, we shall concentrate below on a certain family of examples when the discrete group $B$ in the construction (from Sections 2 and 3) arises as a discrete Bratteli diagram, and, as a result, its Pontryagin dual compact group $K$ is a solenoid; also often called a compact Bratteli diagram.

Example 5.2 below illustrates this in the special case of constant incidence diagrams, but much of this discussion applies more generally.

**Example 5.1.** Let $B = \mathbb{Z} \left\{ \frac{1}{2} \right\}$, dyadic rationals in $\mathbb{R}$. Let $\alpha_2 = \text{multiplication by } 2$, so that $\alpha_2 \in \text{Aut}(B)$, and set $G_2 := B \rtimes_{\alpha_2} \mathbb{Z}$.

Recall the Baumslag–Solitar group $G_2$ with two generators $\{u,t\}$, satisfying $utu^{-1} = t^2$. With the correspondence,

$$u \leftrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
\( \alpha_2 \) acts by conjugation,

\[
\begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
2^{-1} & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}.
\]

In particular, \( T^k = \begin{pmatrix}
1 & k \\
0 & 1
\end{pmatrix}, \ k = 1, 2, 3, \ldots. \)

We have

\[
\text{Ind}_{G^2 \mathbb{Z}[\frac{1}{2}]}^G (\chi) \begin{pmatrix}
2^j & b \\
0 & 1
\end{pmatrix} = D\chi (b) T^j.
\]

Moreover,

\[
\{2^j : j \in \mathbb{Z}\} \simeq \mathbb{Z}[\frac{1}{2}] \setminus G_2 \simeq \mathbb{Z}. \]

Set \( (T\xi)_k := \xi_{k+1}, \ \xi \in l^2(\mathbb{Z}), \) and let

\[
D\chi (b) = \begin{pmatrix}
\ldots & \chi (b) \\
\chi (2b) & \chi (3b)
\end{pmatrix}, \ \forall b \in \mathbb{Z}[\frac{1}{2}].
\]

Note that

\[
TD\chi (b) = D\chi (\alpha_2 (b)) T = D\alpha_2 \chi (b) T, \ \forall b \in \mathbb{Z}[\frac{1}{2}].
\]

**Example 5.2.** Let \( A \) be a \( d \times d \) matrix over \( \mathbb{Z} \), \( \det A \neq 0; \) then

\[
\mathbb{Z}^d \hookrightarrow A^{-1}\mathbb{Z}^d \hookrightarrow A^{-2}\mathbb{Z}^d \hookrightarrow \ldots
\]

and set

\[
B_A := \bigcup_{k \geq 0} A^{-k}\mathbb{Z}^d.
\]

Let \( \alpha_A = \) multiplication by \( A \), so that \( \alpha_A \in \text{Aut} (B_A) \), and \( G_A := B_A \rtimes \alpha_A \mathbb{Z}. \)

The corresponding standard wavelet representation \( U_A \) will now act on the Hilbert space \( L^2 (\mathbb{R}^d) \) with \( d \)-dimensional Lebesgue measure; and \( U_A \) is a unitary representation of the discrete matrix group \( G_A = \{(j, \beta)\}_{j, \beta \in B_A} \), specified by

\[
(j, \beta) (k, \gamma) = (j + k, \beta + A^j \gamma), \quad \text{and}
\]

defined for all \( j, k \in \mathbb{Z} \), and \( \beta, \gamma \in B_A \). Alternatively, the group from (5.1) may be viewed in matrix form as follows:

\[
(j, \beta) \rightarrow \begin{pmatrix}
A^j & \beta \\
0 & 1
\end{pmatrix}, \quad \text{and} \quad (k, \gamma) \rightarrow \begin{pmatrix}
A^k & \gamma \\
0 & 1
\end{pmatrix}.
\]

The wavelet representation \( U_A \) of \( G_A \) acting on \( L^2 (\mathbb{R}^d) \) is now

\[
U_A (A) (x) := (\det A)^{-\frac{d}{2}} f (A^{-j} (x - \beta)),
\]

defined for all \( f \in L^2 (\mathbb{R}^d), \forall j \in \mathbb{Z}, \forall \beta \in B_A, \) and \( x \in \mathbb{R}^d. \)
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