How Perfect a Gluon Plasma Can Be in Perturbative QCD?

Jiunn-Wei Chen,1 Jian Deng,2 Hui Dong,2 and Qun Wang3

1Department of Physics and Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan
2School of Physics, Shandong University, Shandong 250100, People’s Republic of China
3Interdisciplinary Center for Theoretical Study and Department of Modern Physics, University of Science and Technology of China, Anhui 230026, People’s Republic of China

The shear viscosity to entropy density ratio, $\eta/s$, characterizes how perfect a fluid is. We calculate the leading order $\eta/s$ of a gluon plasma in perturbation using the kinetic theory. The leading order contribution only involves the elastic $gg \leftrightarrow gg$ (22) process and the inelastic $gg \leftrightarrow ggg$ (23) process. The Hard-Thermal-Loop (HTL) treatment is used for the 22 matrix element, while the exact matrix element in vacuum is supplemented by the gluon Debye mass insertion for the 23 process. Also, the asymptotic mass is used for the external gluons in the kinetic theory. The errors from not implementing HTL and the Landau-Pomeranchuk-Migdal effect in the 23 process, and from the uncalculated higher order corrections, are estimated. Our result for $\eta/s$ lies between that of Arnold, Moore and Yaffe (AMY) and Xu and Greiner (XG). Our result shows that although the finite angle contributions are important at intermediate $\alpha_s$ ($\alpha_s \sim 0.01-0.1$), the 22 process is still more important than 23 when $\alpha_s \lesssim 0.1$. This is in qualitative agreement with AMY’s result. We find no indication that the proposed perfect fluid limit $\eta/s \simeq 1/(4\pi)$ can be achieved by perturbative QCD alone.

I. INTRODUCTION

A perfect fluid is a system with zero shear and bulk viscosities, $\eta$ and $\zeta$, and no dissipation. These conditions can be satisfied for a superfluid at zero temperature where only the super fluid component exists, but a sharper description is with the dimensionless ratios $\eta/s$ and $\zeta/s$, where $s$, the entropy density, vanishes for the superfluid component as well. While $\zeta/s$ can still be zero for scaling invariant systems, the situation for $\eta/s$ is more subtle.

In general, stronger interactions implies a smaller $\eta$. Thus, a perfect fluid with the smallest $\eta/s$ is likely to be strongly interacting which requires non-perturbative tools to compute it. The anti-de-Sitter space/conformal field theory correspondence (AdS/CFT) [1–3] allows the $\eta/s$ of strongly interacting CFT’s to be computed in weakly interaction gravitational theories. A universal number $\eta/s = 1/(4\pi)$ is found for every CFT with a gravity dual in the large $N$, with $N$ the size of the gauge group, and infinite t’Hooft coupling limit $\lambda \to \infty$. With this result, together with the connection to the uncertainty principle through the relation $\eta/s \sim \Delta E \Delta t$, with $\Delta E$ and $\Delta t$ the mean energy and life time of quasiparticles, Kovtun, Son, and Starinets (KSS) [5] conjectured that the strongly interacting CFT value $1/(4\pi)$ is the minimum bound for $\eta/s$ for all physical systems.

Theoretically, there are several attempts to evade this bound. It is found that $\eta/s$ can be as small as possible (but still non-negative) in a carefully engineered meson system [6, 8], although the system is meta-stable. Also, in strongly interacting CFTs, $1/N$ corrections can be negative [9, 10] and can modify the $\eta/s$ bound slightly [11, 12].

Experimentally, there are intensive interests to find the most perfect fluid (see [12, 14] for recent reviews). The smallest $\eta/s$ known so far is realized in a system of hot and dense matter thought to be quark gluon plasma just above the phase transition temperature produced at RHIC [15–17] with $\eta/s = 0.1 \pm 0.1$(theory) $\pm 0.08$(experiment) [18]. A robust upper limit $\eta/s < 5 \times 1/(4\pi)$ was extracted by another group [19] and a lattice computation of gluon plasma yields $\eta/s = 0.134(33)$ [20]. Progress has been made in cold unitary fermi gases as well. An analysis of the damping of collective oscillations gives $\eta/s \gtrsim 0.5$ [21, 22]. Even smaller values of $\eta/s$ are indicated by recent data on the expansion of rotating clouds [23, 24] but more careful analyses are needed [25, 26].

Even if the $1/(4\pi)$ bound for $\eta/s$ turns out to be invalid, it is still interesting to use it as a benchmark value for the perfection of fluids. It was found that based on the perturbative QCD (PQCD) analysis of Arnold, Moore and Yaffe (AMY) [27, 28], the measured $\eta/s$ at RHIC cannot be explained by PQCD (for a recent review, see, e.g., [29]). This strongly interacting QGP picture is very different from the conventional picture of weakly interacting QGP and is considered as one of the most surprising discoveries at the RHIC.

However, a recent perturbative QCD calculation of $\eta/s$ of a gluon plasma by Xu and Greiner (XG) [30] shows that the dominant contribution comes from the inelastic $gg \leftrightarrow ggg$ (23) process instead of the elastic $gg \to gg$ (22) process. In particular, the 23 process is 7 times more important than 22. Thus, $\eta/s \simeq 1/4\pi$ can be achieved when the strong coupling constant $\alpha_s \simeq 0.6$. In Ref. [31], XG and their collaborators improve their calculation using the Kubo relation and give smaller contribution from the 23 process: about 5 times (2–9 times) the 22 process. Thus, the conventional weakly interacting QGP could still be valid. This is in sharp contrast to AMY’s result where the 23 process only gives $\sim 10\%$ correction to the 22 process.
Both XG and AMY use kinetic theory for their calculations. The main differences are (i) XG uses a parton cascade model \[32\] to solve the Boltzmann equation and, for technical reasons, gluons are treated as a classical gas instead of a bosonic gas. On the other hand, AMY solves the Boltzmann equation for a bosonic gas. (ii) AMY approximates the $N g \rightarrow (N + 1) g$ processes, $N = 2, 3, 4 \ldots$, by the $g \leftrightarrow gg$ splitting in the collinear limit where the two gluon splitting angle is higher order. XG uses the soft gluon bremsstrahlung limit where one of the gluon momenta in the final state of $gg \rightarrow gg$ is soft but it can have a large splitting angle with its mother gluon.

In an earlier attempt to resolve the discrepancy between XG’s and AMY’s results \[33\], a Boltzmann equation computation of $\eta$ is carried out without taking the classical gluon approximation (like AMY’s approach) but the soft gluon bremsstrahlung limit is applied to the 23 matrix element (like XG’s approach, modulo a factor 2 in the 23 matrix element squared; see \[33\] for details). It was found that the classical gas approximation does not cause a significant error in $\eta/s$ (although the individual errors on $\eta$ and $s$ are larger). However, the result is sensitive to whether the soft gluon bremsstrahlung limit is imposed on the phase space or not. If this limit is imposed, the result is closer to AMY’s; if not, the result is closer to XG’s. This raises the concern whether this approximation is good for computing $\eta$.

The goal of this paper is to settle this issue by removing both the soft gluon bremsstrahlung approximation and the collinear approximation to the 23 process. The leading order [$O(\alpha_s^2)$] contribution to $\eta$ only involves the 22 and 23 processes \[28\] (the power counting for 22, 23 and other processes are reproduced in \[33\]). In this paper, the Hard-Thermal-Loop (HTL) treatment is used for the 22 matrix element, while the exact matrix element in vacuum is supplemented by the gluon Debye mass insertion for the 23 process. Also, the Debye mass is used for the external gluon mass in the kinetic theory as well. The errors from not implementing HTL and the Landau-Pomeranchuk-Migdal effect in the 23 process, and from the uncalculated higher order corrections, are also estimated.

**II. KINETIC THEORY BEYOND THE SOFT OR COLLINAR GLUON APPROXIMATIONS**

Using the Kubo formula, $\eta$ can be calculated through the linearized response function of a thermal equilibrium state

$$\eta = \frac{1}{\pi} \int_{-\infty}^{0} dt' \int_{-\infty}^{t'} dt \int d^3x \langle [T^{ij}(0), T^{ij}(x, t)] \rangle,$$

where $T^{ij}$ is the spatial part of the off-diagonal energy momentum tensor. In a leading order (LO) expansion of the coupling constant, there are an infinite number of diagrams \[34, 35\]. However, it is proven that the summation of the LO diagrams in a weakly coupled $\phi^4$ theory \[34, 35\] or in hot QED \[39\] is equivalent to solving the linearized Boltzmann equation with temperature-dependent particle masses and scattering amplitudes. The conclusion is expected to hold in weakly coupled systems and can as well be used to compute the LO transport coefficients in QCD-like theories \[27, 28\], hadronic gases \[40–45\] and weakly coupled scalar field theories \[34, 37–43\].

The Boltzmann equation of a hot gluon plasma describes the evolution of the color and spin averaged gluon distribution function $f_p(x)$ which is a function of space-time $x = (t, x)$ and momentum $p = (E_p, \mathbf{p})$. The infinitesimal deviation of $f_p(x)$ from its equilibrium value $f_p^0 = (e^{p/T} - 1)^{-1}$ is denoted as

$$f_p = f_p^0 [1 - \chi_p (1 + f_p^0)],$$

where $\chi_p \equiv \chi(x, p)$ can be parametrized as

$$\chi_p = \frac{A(p)}{T} \nabla \cdot \mathbf{v} + \frac{B_{ij}(p)}{T} \frac{1}{2} \left( \frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right),$$

at the leading order of the derivative expansion of the fluid velocity $v(x) = (v^0, \mathbf{v})$. $T = T(x)$ is the local temperature, $\hat{p}$ is the unit vector in the $p$ direction. $B_{ij}(p) = B(p)(\hat{p} \hat{p} \delta_{ij} - \frac{1}{3} \delta_{ij})$. $A(p)$ and $B(p)$ are functions of $p$ which will be fixed by the Boltzmann equation corresponding to the bulk and shear viscosities, respectively. In this work we will just focus on the shear viscosity calculation.

The Boltzmann equation \[48, 53\] for the gluon plasma reads

$$\frac{p^\mu}{E_p} \partial_\mu f_p = \frac{1}{N_g} \sum_{(n,l)} \frac{1}{N(n,l)} \int_{(l-1)(n-1)} d\Gamma_{l-1l+1} \delta_{l1} \prod_{r=1}^{n-1} f_r \prod_{s=l+1}^{n-1-f_s} (1 + f_s) - f_p \prod_{r=1}^{l} (1 + f_r) \prod_{s=l+1}^{n-1} f_s,$$

(3)
where the collision rates are given by

\[
d\Gamma_{1\rightarrow(l+1)\ldots(n-1)p} = \prod_{j=1}^{n-1} \frac{d^3p_j}{(2\pi)^32E_j} \frac{1}{2E_p} |M_{1\rightarrow(l+1)\ldots(n-1)p}|^2 (2\pi)^4 \delta^4(\sum_{r=1}^l p_r - \sum_{s=l+1}^{n-1} p_s - p). \tag{5}
\]

\(N_g = 16\) is the color and spin degeneracy of a gluon. The \(i\)-th gluon is labeled as \(i\) while the \(n\)-th gluon is labeled as \(p\). For a process with \(i\) initial and \((n-1)\) final gluons, the symmetry factor \(N(n,l) = l!((n-l-1)!)\). For example, processes \(12 \rightarrow 3p\), \(12 \rightarrow 3p\), \(123 \rightarrow 4p\) yield \((n,l) = (4,2), (5,2), (5,3)\) and \(N(n,l) = 2, 4, 6\), respectively.

In vacuum, the matrix element squared for the 22 process is

\[
|M_{12\rightarrow34}|^2 = \frac{9}{2} (4\pi)^2 N_g^2 \alpha_s^2 \left(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2}\right) ,
\tag{6}
\]

where \(\alpha_s = g^2/(4\pi)\) is the strong coupling constant, and \((s,t,u)\) are the Mandelstam variables \(s = (p_1 + p_2)^2\), \(t = (p_1 - p_3)^2\) and \(u = (p_1 - p_4)^2\).

For the 23 process \([54, 55]\), under the convention \(\sum_{i=1}^5 p_i = 0\), we have

\[
|M_{12345\rightarrow6}|^2 = |M_{0\rightarrow12345}|^2 = 54\pi^3 N_g^2 u^3 \left[(12)^4 + (13)^4 + (14)^4 + (15)^4 + (23)^4 \right.
+ (24)^4 + (25)^4 + (34)^4 + (35)^4 + (45)^4\]
\[× \sum_{\text{perm}(1,2,3,4)} \frac{1}{(12)(23)(34)(45)(51)}, \tag{7}\]

where \((ij) \equiv p_i\cdot p_j\) and the sum is over all permutations of \(\{1, 2, 3, 4\}\). To convert to the convention \(p_1 + p_2 = p_3 + p_4 + p_5\), we just perform the replacement:

\[
|M_{12\rightarrow345}|^2 = |M_{0\rightarrow12345}|^2 \bigg|_{p_1\rightarrow-p_1, p_2\rightarrow-p_2} ,
|M_{345\rightarrow12}|^2 = |M_{12345\rightarrow6}|^2 \bigg|_{p_1\rightarrow-p_1, p_2\rightarrow-p_2} . \tag{8}
\]

In the medium, the gluon thermal mass effect serves as the infrared (IR) cut-off to regularize IR sensitive observables. The most singular part of Eq.\((6)\) comes from the collinear region (i.e. either \(t \approx 0\) or \(u \approx 0\)) which can be regularized by the Hard-Thermal-Loop (HTL) corrections to the gluon propagators \([56, 57]\) and yields \([58]\),

\[
|M_{12\rightarrow34}|^2 \approx \frac{1}{4} (12\pi\alpha_s)^2 N_g^2 (4E_1E_2)^2 \left| \frac{1}{q^2 + \Pi_L} - \frac{(1 - \pi^2) \cos \phi}{q^2 (1 - \pi^2) + \Pi_T} \right|^2 , \tag{9}\]

where \(q = p_1 - p_3 = (q_0, \mathbf{q})\), \(\pi = q_0/|\mathbf{q}|\) and \(\phi\) is the angle between \(\mathbf{p}_1 \times \mathbf{q}\) and \(\mathbf{p}_2 \times \mathbf{q}\). The HTL self-energies \(\Pi_L\) (longitudinal) and \(\Pi_T\) (transverse) are given by

\[
\Pi_L = m_D^2 \left[1 - \frac{\pi}{2} \ln \frac{1 + \pi}{1 - \pi} + i \pi \right] ,
\Pi_T = m_D^2 \left[\frac{\pi^2}{2} + \frac{\pi}{4} (1 - \pi^2) \ln \frac{1 + \pi}{1 - \pi} - i \frac{\pi}{4} (1 - \pi^2)\right]. \tag{10}\]

The external gluon mass \(m_\infty\) (i.e. the asymptotic mass) is the mass for an on-shell transverse gluon, and \(m_\infty^2 = \Pi_T (|\pi| = 1) = m_D^2/2\) both in the HTL approximation and in the full one-loop result.

Previous investigations of the thermodynamics within resummed perturbation theory showed that the most important plasma effects are the thermal masses \(\sim gT\) acquired by the hard thermal particles \([59, 61]\). So a simpler (though less accurate) treatment for the regulator is to insert the Debye mass \(m_D = (4\pi\alpha_s)\pi/2T\) to the gluon propagator such that in the center-of-mass (CM) frame,

\[
|M_{12\rightarrow34}|_{CM} \approx (12\pi\alpha_s)^2 N_g^2 \frac{s^2}{(q_T^2 + m_D^2)} , \tag{11}\]

where \(s = M_{12}^2 - M_{34}^2\).
we can rewrite one momentum configuration in the CM frame in terms of variable $p$. Taking the large $s$ order in $B$ formula \[62\] after taking $m$ where the permutation is over all final state gluon configurations. We see that Eq. (16) reduces to the Gunion-Bertsch denominator of Eq. (7), should be modified to

$$\sum$$

if the $(ij)$ combination is set by $T$ and is $O(T^8)$. So we can still apply the substitution of Eq.(12), even if the $(ij)$ factors might not have the inverse propagator form. The error is $\sim m_D^2 (ij)^3 = O(\alpha_s T^8)$, which is higher order in $\alpha_s$.

It is instructive to show that Eqs. (7,8) and (12) give the correct soft bremsstrahlung limit. Using the light-cone variable

$$p = (p^+, p^-, p_\perp)$$

we can rewrite one momentum configuration in the CM frame in terms of $p, p', q$ and $k$: $p_1 = p, p_2 = p', p_3 = p + q - k, p_4 = p' - q$ and $p_5 = k, with

$$p = (\sqrt{s}, m^2_g/\sqrt{s}, 0, 0),$$

$$p' = (m^2_g/\sqrt{s}, \sqrt{s}, 0, 0),$$

$$k = (y\sqrt{s}, (k_\perp^2 + m^2_g)/y\sqrt{s}, k_\perp, 0),$$

$$q = (q^+, q^-, q_\perp).$$

The on-shell condition $p^2 = p'^2 = m^2_g$ yields

$$q^+ \approx -q_\perp^2/\sqrt{s},$$

$$q^- \approx \frac{k_\perp^2 + yq_\perp^2 - 2y k_\perp \cdot q_\perp + (1 - y + y^2)m^2_g}{y(1 - y)\sqrt{s}}.$$  

Taking the large $s$ limit, then the $y \to 0$, we obtain

$$|M_{12\to345}|^2_{CM} = \sum_{\text{perm}(3,4,5)} \frac{3456 \pi^3 \alpha_s^3}{3 \sqrt{s}} \frac{k^2 + m^2_g}{(k_\perp^2 + m^2_g)(q_\perp^2 + m^2_D)} \frac{[k_\perp - q_\perp]^2 + m^2_D}{s^2},$$

where the permutation is over all final state gluon configurations. We see that Eq. (16) reduces to the Gunion-Bertsch formula \[62\] after taking $m_D, m_g \to 0$. A similar derivation of can be found in Ref. \[63, 64\].

III. BEYOND VARIATION – SOLVING FOR $\eta$ SYSTEMATICALLY

Following the standard procedure, the shear viscosity is related to $B(p)$ by,

$$\eta = \frac{N_g}{10T} \int \frac{d^3p}{(2\pi)^3 E_p} f^{eq}_p (1 + f^{eq}_p) \left( \frac{p^i p^j - \frac{1}{3} \delta_{ij} p^2}{2N_g} \right) B_{ij}(p).$$

$B_{ij}(p)$ satisfies the constraint derived from the linearized Boltzmann equation,

$$p^i p^j - \frac{1}{3} \delta_{ij} p^2 = \frac{E_p}{2N_g} \int_{123} d\Gamma_{12\to3p}^{eq} f^{eq}_{12} (1 + f^{eq}_{12}) (f^{eq}_{12})^{-1}.$$
\[ \times [B_{ij}(p) + B_{ij}(p_3) - B_{ij}(p_1) - B_{ij}(p_2)] + \frac{E_p}{4Ng} \int_{1234} d\Gamma (1 + f_p)(1 + f_p^q) f_i^q f_j^q (1 + f_p^q)^{-1} \times [B_{ij}(p) + B_{ij}(k_4) + B_{ij}(k_3) - B_{ij}(k_2) - B_{ij}(k_1)] + \frac{E_p}{6Ng} \int_{1234} d\Gamma (1 + f_p)(1 + f_p^q) f_i^q f_j^q (1 + f_p^q)^{-1} \times [B_{ij}(p) + B_{ij}(k_4) - B_{ij}(k_3) - B_{ij}(k_2) - B_{ij}(k_1)]. \] (18)

However, solving \( B_{ij} \) using this equation is technically challenging. It is easier to perform a projection (or convolution) to the above equation, then solve for the less restricted \( B_{ij} \). We will discuss the procedure below.

By plugging Eq. (18) into Eq. (17), we obtain \( \eta \) in a bilinear form of \( B_{ij} \),

\[ \eta = \frac{1}{80T} \int \prod_{i=1}^{5} \frac{d^3k_i}{(2\pi)^32E_i} |M_{1234}|^2 (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - k_4) \times (1 + f_1^q) (1 + f_2^q) f_3^q f_4^q \times [B_{ij}(k_4) + B_{ij}(k_3) - B_{ij}(k_2) - B_{ij}(k_1)] \]
\[ + \frac{1}{120T} \int \prod_{i=1}^{5} \frac{d^3k_i}{(2\pi)^32E_i} |M_{12345}|^2 (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - k_4 - k_5) \times (1 + f_1^q) (1 + f_2^q) f_3^q f_4^q f_5^q \times [B_{ij}(k_5) + B_{ij}(k_4) + B_{ij}(k_3) - B_{ij}(k_2) - B_{ij}(k_1)]. \] (19)

Then one can solve for \( B_{ij} \) which equates the right hand sides of Eqs. (17) and (19). This is nothing but a projection of Eq. (18). The resulting solution is not unique because only the projected equation but not the equation itself is satisfied. However, it is proven \([28, 66]\) that the true solution of \( B_{ij} \), i.e., the solution satisfying Eq. (18), would give the maximum value of \( \eta \). Thus, solving for \( \eta \) becomes a variational problem.

Recently, an algorithm is developed to find the true solution of \( B_{ij} \) systematically \([60]\). Thus, this approach is no more variational but systematic. Here we outline the procedure. First, expanding \( B(p) \) using a specific set of orthogonal polynomials \([42, 43]\):

\[ B(p) = \frac{E_p}{2E_p} \int (2\pi)^5 f_p^q (1 + f_p^q)|p|^2(E_p/T)^y B^{(r)}(E_p/T)B^{(s)}(E_p/T) = T^4 \delta_{rs}. \] (21)

Then Eq. (19) can be written in a compact form,

\[ \eta = \langle B | F | B \rangle = \sum_{r,s=0}^{\text{max}} b_r b_s \langle B^{(r)} | F | B^{(s)} \rangle, \] (22)

while Eq. (17) gives

\[ \eta = \sum_{r=0}^{\text{max}} b_r \frac{N_g}{15T} \int \frac{d^3p}{(2\pi)^3 E_p} f_p^q (1 + f_p^q)|p|^2(E_p/T)^y B^{(r)}(E_p/T) \]
\[ = \sum_{r=0}^{\text{max}} b_r L^{(r)} \delta_{r0} = b_0 L^{(0)}, \] (23)

with

\[ L^{(0)} = \frac{N_g}{15T} B^{(0)} \int \frac{d^3p}{(2\pi)^3 E_p} f_p^q (1 + f_p^q)|p|^2(E_p/T)^y. \] (24)
From Eqs. (22) and (23), we can find \( b_0 \) by solving the equation

\[
L^{(r)} \delta_{r0} = \sum_{s=0}^{r_{\text{max}}} b_s \left\langle B^{(r)} | F | B^{(s)} \right\rangle,
\]

and then determine \( \eta \) from Eq. (23). [Note that there could be more than one solution satisfying Eqs. (22) and (23), but they all give the same \( \eta \).]

In Ref. [66], it is proven that this procedure gives a monotonically increasing value of \( \eta \) with increasing \( r_{\text{max}} \). Thus, one can systematically approach the true value of \( \eta \) by adding more terms in the expansion of Eq. (20). We find good convergence in this algorithm. From \( r_{\text{max}} = 1 \) to 2, \( \eta \) changes by less than 2% for \( \alpha_s \leq 0.3 \). Better convergence is found for smaller \( \alpha_s \).

**IV. NUMERICAL RESULTS**

**A. Leading-Log result**

The leading order \([O(\alpha_s^{-2})]\) contribution to \( \eta \) only involves the 22 and 23 processes [28]. The 22 collision rate is larger than 23 by a \((\ln \alpha_s)\) factor. In the leading-log (LL) approximation, one just needs to focus on the small \( q_T \) contribution from the 22 process. Furthermore, it was shown in [67, 68] that using the HTL regulator (9) gives the same LL viscosity to that using the \( m_D \) regulator [41]. Thus, after performing the small \( q_T \) expansion to Eq. (19), we obtain

\[
\eta_{\text{LL}} \simeq 27.1 \frac{T^3}{g^4 \ln(1/g)},
\]

which coincides with that of [27] to significant digits shown above. Using the entropy density for non-interacting gluons, \( s = N_g \frac{2\pi^2}{45} T^3 \) (for \( m_g = 0 \)), we obtain

\[
\frac{\eta_{\text{LL}}}{s} \simeq \frac{3.9}{g^4 \ln(1/g)}.
\]

This will be used to check our numerical result later.

**B. \( \eta_{22} \) - Shear viscosity with the 22 process only**

To study the effect of the HTL regulator, \( \eta_{22} \) (i.e. \( \eta \) with the 22 process only) with the HTL and \( m_D \) for the internal gluon masses, respectively, are shown in Fig. 1. The LL result \( \eta_{\text{LL}} \) and AMY’s \( \eta_{22} \) [denoted as \( \eta_{22(\text{AMY})} \)] [27, 28] are also shown. The external gluon mass \( m_g \), used in kinematics and in \( f_p^{eq} \) such that \( E_p = \sqrt{\mathbf{p}^2 + m_g^2} \), is a higher order effect in \( \eta_{22} \). Changing \( m_g \) from 0 to \( m_D = \sqrt{2} m_\infty \) yields an \( O(m_g^2/T^2) = O(\alpha_s) \) variation to \( \eta_{22} \). This is confirmed numerically in the left panel of Fig. 1. It is a good check to our numerical calculation that \( \eta_{22(\text{HTL})} \) and \( \eta_{22(\text{MD})} \) both converge to \( \eta_{\text{LL}} \) in small \( \alpha_s \), and \( \eta_{22(\text{HTL})} \) agrees well with \( \eta_{22(\text{AMY})} \) when \( m_g = 0 \) is used to conform with the AMY result.

About the HTL effect, \( \eta_{22(\text{HTL})}/\eta_{22(\text{MD})} \) is quite close to unity at \( \alpha_s = 10^{-6} \), see the right panel of Fig. 1. This ratio gets smaller at larger \( \alpha_s \) and reaches 0.65 at \( \alpha_s = 0.1 \) with little \( m_g \) dependence (each \( \eta_{22(\text{HTL})}/\eta_{22(\text{MD})} \) is evaluated with the same \( m_g \)). This means the error is \( \sim 30\% \) in the shear viscosity at \( \alpha_s = 0.1 \) if we use \( m_D \) as the regulator for the gluon propagator instead of the HTL propagator.

**C. \( \eta_{22+23} \) - Shear viscosity with the 22 and 23 processes**

In our full calculation, we use the HTL propagator for the 22 process. However, for technical reasons, we use the internal gluon mass \( m_D \) for the 23 process. More specifically, we use matrix elements of Eqs. (14, 15, 17), \( E_p = \sqrt{\mathbf{p}^2 + m_g^2} \) in kinematics and \( f_p^{eq} \) for external gluons, and \( m_g = m_\infty \). If the external gluons are massless but the internal gluons are massive, then the \( 1/[(p_1 + p_2)^2 - m_D^2] \) factor could diverge. Using \( m_g = m_\infty \), each term in Eq. (7) is non-negative.
FIG. 1: $\eta_{22}$ over the entropy density (left panel) and $\eta_{22}$ over $\eta_{22}(MD)$ (right panel) in various treatments. ‘LL’ is the leading log result of Eq. (20). ‘HTL’ is the result using the full HTL matrix element of Eq. (9). ‘MD’ is the result using $m_D$ as the regulator as in Eq. (11). ‘AMY’ is AMY’s result.

FIG. 2: Left panel: $\eta_{22}/s$ and $\eta_{22+23}/s$ for various cases. Right panel: the ratio of our result to AMY’s. ‘22(HTL)+23’ denotes $\eta_{22+23}$ where $m_g = m_\infty$, the full HTL matrix element (9) is used for the 22 process, and the matrix elements of Eqs. (7-9,12) are used for the 23 process. ‘22(AMY)+23(AMY)’ denotes AMY’s result for $\eta_{22+23}$ (we have used the same $s$ in $\eta_{22+23}/s$ as ours with $m_g = m_\infty$). The range of the ‘recommended value’ of $\eta_{22+23}$ is bounded by $\eta_-$ (lower bound) and $\eta_+$ (upper bound).

In AMY and XG, external gluon masses were not included ($m_g = 0$). This divergence was avoided by keeping only the most singular matrix elements in the small $k_\perp$, $q_\perp$ limit, and taking the collinear approximation (AMY) or regulating the gluon bremsstrahlung infrared divergence by the Landau-Pomeranchuk-Migdal (LPM) effect (XG) which will be discussed in Section 11

We show $\eta_{22}/s$ and $\eta_{22+23}/s$ in the left panel of Fig. 2, where the HTL propagator is used for 22 and the external gluon mass $m_g = m_\infty$. We also show AMY’s result for $\eta_{22+23}/s$ for comparison. In the left panel, we see that our result for $\eta_{22+23}/s$ deviates from XG’s significantly, e.g. at $\alpha_s = 0.1/0.01/0.001$ our result gives 2.3/103/7020 but XG give about 0.45/20/1200 (read off from Ref. 30) respectively, i.e. our result is about 5 to 6 times as large as XG’s. In Ref. 31 they improve their calculation by using the Kubo relation and give larger values 0.795/60 at $\alpha_s = 0.1/0.01$, so our result is about 2 to 3 times theirs. In the right panel of Fig. 2, we see that the ratio of our result to AMY’s approaches unity at $\alpha_s \lesssim 10^{-4}$ and $\sim 0.8$ at $\alpha_s \approx 0.004$. The deviation in moderate $\alpha_s$ is partly due to the finite angle, non-collinear 3-body configurations in the 23 process described by the full matrix element 7 and partly due to the gluon mass. We have also included a theoretical error band for $\eta_{22+23}$ which will be discussed in Section 11

The effect of the 23 process can be seen more clearly in the ratio $\eta_{22+23}/\eta_{22}$ shown in Fig. 3 We have plotted $\eta_{22}(HTL)+23/\eta_{22}(HTL)$ together with AMY’s and XG’s result for comparison. Our result shows that the 22 process dominates at small $\alpha_s$. When $\alpha_s$ increases, $\eta_{22+23}/\eta_{22}$ decreases and the central value reaches the minimum of $\sim 0.6$ (which means the 23 collision rate is $\sim 60\%$ of the 22 one) at $\alpha_s \approx 0.1$ and then increases again for $\alpha_s > 0.1$ Thus, our result shows: (1) the 22 process is more important than 23 when $\alpha_s \lesssim 0.1$; (2) the finite angle contributions are important at intermediate $\alpha_s$ ($\alpha_s \sim 0.01-0.1$). We see that AMY’s result which employs the collinear approximation for the $1 \leftrightarrow 2$ process (corresponding to our 23 process), gives $\eta_{22+23}/\eta_{22}$ close to unity. This implies their 23 collisions
is just a small perturbation to the 22 collisions. XG’s result in Ref. [30] which employs the soft gluon bremsstrahlung approximation, however, gives $\eta_{22+23}/\eta_{22} \simeq [0.11, 0.16]$ around 1/8. This implies their 23 collision rate is about 7 times the 22 one. Their improved result in Ref. [31] gives $\eta_{22+23}/\eta_{22} \simeq [0.1, 0.3]$ corresponding to the 23 collision rate that is about 2 to 9 times (on average 5 times) the 22 one. Our result takes neither of the approximations, lies between AMY’s and XG’s result, but it agrees with AMY’s result qualitatively but not with XG’s.

D. Error estimation

Our $\eta_{22(HTL)+23}/s$ is tabulated in Table I. The error assignment is based on the following error analyses for $\eta_{22+23}$:

(a) HTL corrections for the 23 process. From our $\eta_{22}$ error analysis, we assign a $\sim 30\%$ error at $\alpha_s = 0.1$ to the 23 contribution for not implementing the HTL approach to the 23 collisions. The error will be smaller at smaller $\alpha_s$ if the scaling for $\eta_{22}$ holds also for the error. Since the HTL effect tends to reduce the magnetic screening effect which lowers the IR cut-off and enhances the 23 collision rate, the HTL correction tends to reduce $\eta_{22+23}$.

(b) LPM effect. We will try to estimate the error from neglecting the LPM effect. An intuitive explanation of this effect was given in Ref. [69]. for the soft bremsstrahlung gluon with transverse momentum $k_T$, the mother gluon has a transverse momentum uncertainty $\sim k_T$ and a size uncertainty $\sim 1/k_T$. It takes the bremsstrahlung gluon the formation time $t \sim 1/(k_T v_T)$ to fly far enough from the mother gluon to be resolved as a radiation. But if the formation time is longer than the mean free path $l_{mfp} \approx O(\alpha_s^{-1})$, then the radiation is incomplete and it would be resolved as $gg \rightarrow gg$ instead of $gg \rightarrow ggg$. Thus, the resolution scale is set by $t \leq l_{mfp}$. This yields an IR cut-off $k_T^2 \geq E_k/l_{mfp} \approx O(\alpha_s)$ on the phase space [70]. Thus, the LPM effect reduces the 23 collision rate and will increase $\eta_{22+23}$. Our previous calculation using the Gunion-Bertsch formula shows that implementing the $m_g$ regulator gives a very close result to the LPM effect [33]. Thus, we will estimate the size of the LPM effect by increasing the external gluon mass $m_g$ from $m_\infty$ to $m_D$.

(c) Higher order effect. The higher order effect is parametrically suppressed by $O(\alpha_s)$, but the size is unknown. Computing this effect requires a treatment beyond the Boltzmann equation [33] and the inclusion of the 33 and 24 processes. We just estimate the effect to be $\alpha_s$ times the leading order which is $\sim 10\%$ at $\alpha_s = 0.1$.

Combining the above analyses, we consider errors from (a) to (c). To compute a recommended range for $\eta/s$, we will work with the $R_{22}$ and $R_{23}$ collision rates defined as

$$R_{22}^{-1} \equiv \eta_{22(HTL)},$$

$$R_{22+23}^{-1} \equiv \eta_{22(HTL)+23}.$$  \hspace{1cm} (28)

Using HTL instead of $m_D$ in for the gluon propagator enhances the 22 rate by a factor of

$$\delta \equiv \frac{R_{22(HTL)}}{R_{22(MD)}} \equiv \frac{\eta_{22(MD)}}{\eta_{22(HTL)}}.$$  \hspace{1cm} (29)
We will assume that the same enhancement factor appears in 23 rate as well, such that
\[
\delta \simeq \frac{R_{23(HTL)}}{R_{23(MD)}}.
\]
(30)

On the other hand, the LPM effect is estimated to suppress the 23 rate by a factor of
\[
\gamma = \frac{R_{23} (m_g = m_D)}{R_{23} (m_g = m_{\infty})}.
\]
(31)

Combining the estimated HTL and LPM corrections to the 23 rate, the 22+23 rate is likely to lie in the range \([R_{22} + R_{23}, R_{22} + \gamma \delta R_{23}],\) while the higher order effect gives \(\pm \alpha_s (R_{22} + R_{23})\) corrections to the rate. Without further information, the errors are assumed to be Gaussian and uncorrelated, the total rate is
\[
\left( R_{22} + \frac{\gamma \delta + 1}{2} R_{23} \right) \pm \left( \frac{\gamma \delta - 1}{2} R_{23} \right) \pm \alpha_s (R_{22} + R_{23}),
\]
(32)

and the recommended upper \((\eta_+\)) and lower \((\eta_-\)) range for \(\eta_{22+23}\) are
\[
\eta_{\pm} = \left( \frac{1}{R_{22} + \frac{\gamma \delta + 1}{2} R_{23}} \mp \sqrt{\left( \frac{\gamma \delta - 1}{2} R_{23} \right)^2 + \alpha_s^2 (R_{22} + R_{23})^2} \right).
\]
(33)

The \(\eta_\pm\) values are shown in the right panel of Fig. 2 and in Fig. 3.

Our final \(\eta/s\) result is as presented in Table I. At \(\alpha_s = 0.1,\) \(\eta/s \simeq [2.08, 2.58],\) which is closer to 2.85 of AMY than to XG’s 0.5 in Ref. [30] and 0.795 \pm 0.025 in Ref. [31]. At \(\alpha_s = 0.3,\) we have \(\eta/s \simeq [0.43, 0.759],\) which is compatible to 0.6 of AMY but not to XG’s 0.13 in Ref. [30] and 0.166 \pm 0.025 in Ref. [31].

TABLE I: \(\eta/s\) values for our \(\eta_{22(HTL)+23}/s\) result and the range of our ‘recommended values’ bounded by \(\eta_-/s\) and \(\eta_+/s\).

| \(\alpha_s\) | \(\eta_{22(HTL)+23}\) | \(\eta_-\) | \(\eta_+\) | \(\alpha_s\) | \(\eta_{22(HTL)+23}\) | \(\eta_-\) | \(\eta_+\) |
|-------------|-----------------|--------|--------|-------------|-----------------|--------|--------|
| 1.00E-6     | 3.75E+9         | 3.75E+9| 3.75E+9| 0.100      | 2.30            | 2.08   | 2.58   |
| 1.58E-6     | 1.54E+9         | 1.54E+9| 1.54E+9| 0.125      | 1.67            | 1.48   | 1.91   |
| 2.51E-6     | 6.32E+8         | 6.32E+8| 6.34E+8| 0.150      | 1.30            | 1.13   | 1.52   |
| 3.98E-6     | 2.60E+8         | 2.60E+8| 2.61E+8| 0.175      | 1.06            | 0.909  | 1.27   |
| 6.31E-6     | 1.07E+8         | 1.07E+8| 1.08E+8| 0.200      | 0.888           | 0.752  | 1.09   |
| 1.00E-5     | 4.42E+7         | 4.42E+7| 4.44E+7| 0.225      | 0.765           | 0.638  | 0.971  |
| 1.58E-5     | 1.83E+7         | 1.82E+7| 1.84E+7| 0.250      | 0.672           | 0.551  | 0.880  |
| 2.51E-5     | 7.57E+6         | 7.52E+6| 7.61E+6| 0.275      | 0.600           | 0.484  | 0.811  |
| 3.98E-5     | 3.14E+6         | 3.12E+6| 3.16E+6| 0.300      | 0.542           | 0.430  | 0.759  |
| 6.31E-5     | 1.30E+6         | 1.29E+6| 1.31E+6| 0.325      | 0.495           | 0.386  | 0.720  |
| 1.00E-4     | 5.43E+5         | 5.38E+5| 5.47E+5| 0.350      | 0.456           | 0.349  | 0.691  |
| 1.58E-4     | 2.26E+5         | 2.24E+5| 2.28E+5| 0.375      | 0.423           | 0.318  | 0.669  |
| 2.51E-4     | 9.47E+4         | 9.38E+4| 9.55E+4| 0.400      | 0.395           | 0.291  | 0.654  |
| 3.98E-4     | 3.97E+4         | 3.93E+4| 4.01E+4| 0.425      | 0.371           | 0.268  | 0.644  |
| 6.31E-4     | 1.67E+4         | 1.65E+4| 1.68E+4| 0.450      | 0.350           | 0.248  | 0.639  |
| 1.00E-3     | 7.02E+3         | 7.01E+3| 7.11E+3| 0.475      | 0.332           | 0.230  | 0.638  |
| 1.58E-3     | 2.97E+3         | 2.93E+3| 3.01E+3| 0.500      | 0.316           | 0.214  | 0.642  |
| 2.51E-3     | 1.27E+3         | 1.25E+3| 1.29E+3| 0.525      | 0.302           | 0.200  | 0.649  |
| 3.98E-3     | 542.            | 532.   | 553.   | 0.550      | 0.289           | 0.187  | 0.659  |
| 6.31E-3     | 234.            | 229.   | 240.   | 0.575      | 0.278           | 0.175  | 0.673  |
| 1.00E-2     | 103.            | 99.9   | 106.   | 0.600      | 0.267           | 0.164  | 0.691  |
| 1.58E-2     | 45.6            | 44.1   | 47.3   | 0.625      | 0.258           | 0.153  | 0.711  |
| 2.51E-2     | 20.7            | 19.8   | 21.7   | 0.650      | 0.250           | 0.144  | 0.736  |
| 3.98E-2     | 9.63            | 9.08   | 10.2   | 0.675      | 0.242           | 0.135  | 0.764  |
| 6.31E-2     | 4.62            | 4.28   | 5.01   | 0.700      | 0.235           | 0.127  | 0.795  |
We have only carried out the leading order \([O(\alpha_s^{-2})]\) \(\eta/s\) in the \(\alpha_s\) expansion. Without computing the higher order contribution, it is hard to tell at what value of \(\alpha_s\) the perturbation starts to break down. In the above section, we have naively assumed the higher order contribution to be the leading order times \(\alpha_s\), i.e. the expansion breaks down at \(\alpha_s \simeq 1\). However, explicit computations of thermal dynamical quantities and transport coefficients showed that the break down might happen at smaller \(\alpha_s\) \([59-61]\) (the screening mass computation breaks down at \(\alpha_s \simeq 0.1\) \([60]\) and the heavy quark diffusion constant computation breaks down at \(\alpha_s \simeq 0.01\) \([61]\)).

Looking more closely to our leading order \(\eta/s\) shown in the left panel of Fig. 2, the \(\alpha_s\) dependence of \(\eta/s\) changes quantitatively at \(\alpha_s \simeq 0.1\). This could be a sign that higher order \(\alpha_s\) dependence has become as important as the leading order one. Thus, the higher order corrections could be bigger than our previous estimation, and our result might be only reliable when \(\alpha_s \lesssim 0.1\).

Having said this, it is interesting that our \(\eta/s\) bends slightly upward at \(\alpha_s \gtrsim 0.1\) as if \(\eta/s\) is trying to avoid going below the conjectured \(1/4\pi\) bound. Several models have proposed to described the microscopic picture in the non-perturbative region \([71-74]\). And a similar result to ours is obtained in a recent calculation \([75]\) based on one kind of simplification of the 23 matrix element \([63]\).

Our result implies that the proposed perfect fluid limit \(\eta/s \simeq 1/(4\pi)\) cannot be achieved by perturbative QCD alone.

V. CONCLUSIONS

We have calculated the leading order \([O(\alpha_s^{-2})]\) \(\eta/s\) of a gluon plasma in perturbation using the kinetic theory. The leading order contribution only involves the 22 and 23 processes. The leading order contribution only involves the 22 and 23 processes. The Hard-Thermal-Loop (HTL) propagator has been used for the 22 matrix element, while the exact matrix element in vacuum is supplemented by the Debye mass \(m_D\) for gluon propagators for the 23 process. Also, the asymptotic mass \(m_\infty = m_D/\sqrt{2}\) is used for the external gluon mass in the kinetic theory as well. The errors from not implementing HTL and the Landau-Pomeranchuk-Migdal effect in the 23 process, and from the uncalculated higher order \([O(\alpha_s^{-1})]\) corrections, have been estimated.

Our result for \(\eta/s\) lies between that of Arnold, Moore and Yaffe (AMY) and Xu and Greiner (XG). Our result shows that although the finite angle contributions are important at intermediate \(\alpha_s\) (\(\alpha_s \sim 0.01-0.1\)), the 22 process is still more important than 23 when \(\alpha_s \lesssim 0.1\). This is in qualitative agreement with AMY’s result. We find no indication that the proposed perfect fluid limit \(\eta/s \simeq 1/(4\pi)\) can be achieved by perturbative QCD alone.

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