Analytic solutions of the geodesic equation for
Reissner-Nordström (Anti-)de-Sitter black holes surrounded by
different kinds of regular and exotic matter fields

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March 29, 2019

Abstract

The purpose of this study is the derivation of the equation of motion for particles and light in
the spacetime of Reissner-Nordström (Anti-)de-Sitter black holes in the background of different kinds
of regular and exotic matter fields. The complete analytical solutions of the geodesic equations are
given in terms of the elliptic Weierstraß $\wp$-function and the hyperelliptic Kleinian $\sigma$-function. Finally
after analyzing the geodesic motion of test particles and light using parametric diagrams and effective
potentials, we present a list of all possible orbits.

1 Introduction

Recent high precession astronomical observations confirm the fact that the universe is expanding at an
accelerating rate. The observations also predict the existence of some form of energy with a large negative
pressure called dark energy [1, 2, 3, 4, 5]. Dark energy is supposed to be the origin behind the accelerated
expansion of the universe at a global scale. There are two proposed forms of dark energy. The first
and simplest candidate is the cosmological constant [6]. A different possibility are dynamical scalar field
models such as quintessence [7], chameleon [8], K-essence [9], tachyon [10], phantom [11], and dilaton [12].

In an astrophysical scenario, this kind of dark energy should cause gravitational effects (e.g. deflection of
light coming from a distance star) and should therefore be taken into consideration in this context. The
study of the geodesic motion is one way to understand the physical effects of the gravitational field and
the influence of dark matter on the geometry of the spacetime.

Although the majority of gravitational effects can be discussed using approximations and numerical
methods, a robust and systematic study of all effects can only be achieved by using analytical methods. The
motion of test particles provides an experimentally feasible way to study the gravitational fields of black
holes. Predictions about observables (such as light deflection, Shapiro time-delay, perihelion shift) can
be studied and compared with observations. In 1931 Hagihara [13] solved the geodesic equation for the
Schwarzschild spacetime using the Weierstrass elliptic function. This analytical tool was further advanced
by the application of hyperelliptic functions when higher order polynomials ($> 4$) occur in the geodesic
equation. Here functions are based on the solution of the Jacobi inversion problem [14, 15]. There are
many works dedicated to the literature of analytic study in the geodesic motion. It is not possible to
include due to the large volume of existing work but we will mention a few of them we found interesting.
In four dimensions it was applied to the Schwarzschild de-Sitter [16], Kerr de-Sitter spacetime [17]
and in higher dimensions to Schwarzschild, Schwarzschild Anti-de-Sitter, Reissner Nordström, Reissner
Nordström Anti-de-Sitter [18] and Myers-Perry spacetimes [19, 20]. Moreover this analytical method are
used for some spacetime such as $f(R)$ gravity [21], GMGHS black holes [22], BTZ [23]. Static cylindrically symmetric conformal spacetime [24]. Kerr-Newman-(Λ)dS spacetime and the rotating charged black hole spacetime in $f(R)$ gravity [25], Einstein-Maxwell-dilaton-axion black holes [26], $U(1)^2$ dyonic rotating black holes [27].

In this paper we discuss the geodesic motion of test particles and light rays in the Reissner-Nordström (Anti-)de-Sitter spacetime in the presence of different regular and exotic matter fields. We provide our results here in terms of the Weierstraß $\wp$-function and in terms of derivatives of the Kleinian $\sigma$-function.

The outline of this paper is as follows: In section 2 we briefly review the metric of a Reissner-Nordström (Anti-)de-Sitter spacetime surrounded by various matter fields. In section 3 we study possible orbits using parametric diagrams and effective potentials for four surrounding matter fields. In section 4 we present the analytical solution and plot some of the possible orbits in Section 5. Section 6 is dedicated to conclusion.

### 2 The spacetime structure

In 2003 Kiselev [28] solved the Einstein-Maxwell field equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}$$

in the presence of a cosmological constant $\Lambda$. The additivity and linearity conditions on the quintessence dark energy stress tensor imply

$$T_t^t = T_r^r = \rho_q$$

$$T_\theta^\theta = T_\varphi^\varphi = -\frac{1}{2}(3\omega_q + 1)$$

where,

$$\rho_q = -\frac{a}{2}\frac{3\omega_q}{r^{3(\omega_q + 1)}}$$

and $\rho_q$ is the density of quintessence field, $\omega_q$ is the state parameter and $a$ is the normalization parameter related to the density of quintessence field. The metric of charged (Anti-)de-Sitter black hole immersed in quintessence dark energy as explained by Kiselev can be expressed by

$$ds^2 = -Y(r)dt^2 + \frac{dr^2}{Y(r)} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

with

$$Y(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{a}{r^{\omega_q + 1}} - \frac{\Lambda}{3}r^2.$$  

If we set $\Lambda = 0$, $Q = 0$ and put $\omega_q = -\frac{2}{3}$ the spacetime reduces to the Schwarzschild solution surrounded by quintessence whose geodesics were studied in [29]. Null trajectories of charged black holes surrounded by quintessence were carried out in [30], thermodynamic properties and the Joule-Thomson effect was investigated in [31] and [32] respectively. The Shadow of rotating charged black holes with quintessence was considered in [33]. Other appropriate choices of $\omega_q$ allow us to obtain some well known cases such as radiation field for $\omega_q = \frac{1}{3}$, extraordinary quintessence for $\omega_q = -1$ named as effective cosmological constant. In this universe it could be possible that black holes are surrounded by regular $\omega_q = \frac{1}{3}$ as well as exotic matter fields e.g. $\omega_q = [-\frac{1}{3}, -\frac{2}{3}, -1]$. Throughout this paper we restrict our analysis to these four cases, whereas two of them can be formulated as well known special cases. Furthermore we only deal with spacetimes, where all of the black hole parameters are non-zero. A crucial point of the spacetime structure is the existence of various numbers of horizons. For $\omega_q = -1$, we can define an effective cosmological constant $\Lambda' = \Lambda + a$. If this parameter is negative, then the spacetime reduces to the (Anti-)de-Sitter spacetime. Here the possible horizons are the event horizon, the Cauchy horizon and the cosmological horizon. If the set $\omega_q = -\frac{2}{3}$ an additional horizon can appear for a negative cosmological constant generated by the quintessence field. For $\omega = \pm\frac{1}{3}$ also up to three horizons are possible, whereas the + sign corresponds to a Reissner-Nordström (Anti-)de-Sitter black hole with the effective charge $Q^2 - a$. 
3 The Geodesic Equation

In this section we derive the geodesic equation for the Reissner-Nordström (Anti-)de-Sitter black hole in the presence of various matter fields both for particles and light. To begin with we follow the standard Lagrangian procedure. The general form of the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0,$$

where $\Gamma^\mu_{\rho\sigma}$ is the Christoffel symbol of second kind and $\lambda$ is an affine parameter along the geodesic, which corresponds to the proper time for massive particles. The spacetime considered in this paper is given by the metric function in equation (5). The Lagrangian reads

$$\mathcal{L} = \frac{1}{2} \left[ -Y(r) \left( \frac{dt}{d\lambda} \right)^2 + \frac{1}{Y(r)} \left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \frac{d\varphi}{d\lambda} \right)^2 \right],$$

where $\epsilon$ takes the value 1 and 0 for massive and massless particle respectively. Since the spacetime is spherically symmetric, we can restrict the analysis to the equatorial plane only. Furthermore using the Euler-Lagrange equation we obtain the constants of motion.

$$P_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -Y(r) \dot{t} = -E$$

$$P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = r^2 \dot{\varphi} = L,$$

where $E$ is the energy and $L$ is the angular momentum of the particle. Using the above conserved quantities the geodesic equation can be written as an ordinary differential equation which involves $\varphi$

$$\left( \frac{dr}{d\varphi} \right)^2 = \frac{r^4}{L^2} \left[ E^2 - \left( \epsilon + \frac{L^2}{r^2} \right) \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{a}{r^{3\omega_q+1} - \frac{\Lambda M^2}{3}} \right) \right].$$

To obtain the turning points of the geodesic motion, an effective potential can be introduced easily

$$V = \left( \epsilon + \frac{L^2}{r^2} \right) \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{a}{r^{3\omega_q+1} - \frac{\Lambda M^2}{3}} \right).$$

For a simplified analysis of the geodesic equation it is convenient to introduce dimensionless quantities as follows

$$\tilde{r} = \frac{r}{M}, \quad \tilde{L} = \frac{M^2}{L^2}, \quad \tilde{Q} = MQ, \quad \tilde{a} = aM, \quad \tilde{\Lambda} = \frac{\Lambda M^2}{3}.$$

Now the scaled form of Eqn. (3) is given by

$$\left( \frac{d\tilde{r}}{d\tilde{\varphi}} \right)^2 = \epsilon \tilde{\Lambda} \tilde{L} \tilde{r}^6 + \left( (E^2 - \epsilon) \tilde{L} + \tilde{\Lambda} \right) \tilde{r}^4 + 2\epsilon \tilde{L} \tilde{r}^3$$

$$- \left( \epsilon \tilde{L} \tilde{Q}^2 + 1 \right) \tilde{r}^2 + 2\tilde{r} - \tilde{Q}^2 + \frac{\epsilon \tilde{a} \tilde{L}}{\tilde{r}^{3\omega_q-3} + \tilde{a}} = R(\tilde{r}).$$

The above equation implies that $R(\tilde{r}) \geq 0$ is a necessary condition for the existence of geodesic motion and thus the real zeros of $R(\tilde{r})$ are the turning points of the geodesic motion. In the forthcoming chapters we will study the analytic solutions of the geodesic equation obtained by varying the state parameter $\omega_q$.

3.1 Preliminary discussion on the possible orbit types

Eqn. (11) suggests that the shape of an orbit depends on the energy $E$ and the angular momentum $\tilde{L}$ of the test particle or light ray as well as the state parameter $\omega_q$ and the normalization parameter $\tilde{a}$. The physically acceptable regions of the geodesic motion are given by the real values of $\tilde{r}$ for which $R(\tilde{r}) \geq 0$ or equivalently $E^2 > V_{eff}$. In the following we present a list of the possible orbit types. We consider $\tilde{r}_+$ and $\tilde{r}_-$ as the Event and Cauchy horizon respectively such that $\tilde{r}_+ > \tilde{r}_-$. We distinguish between the following orbit types
1. **Escape orbit (EO)** with range $\tilde{r} \in [\tilde{r}_1, \infty)$, where $\tilde{r}_1 > \tilde{r}_+$. 

2. **Bound orbit (BO)** with range $\tilde{r} \in [\tilde{r}_1, \tilde{r}_2]$, where $\tilde{r}_1, \tilde{r}_2 > \tilde{r}_+$. 

3. **Two-world escape orbit (TEO)** with range $\tilde{r} \in [\tilde{r}_1, \infty)$, where $0 < \tilde{r}_1 < \tilde{r}_-$. 

4. **Many-world bound orbit (MBO)** with range $\tilde{r} \in [\tilde{r}_1, \tilde{r}_2]$, where $0 < \tilde{r}_1 \leq \tilde{r}_- \text{ and } \tilde{r}_2 \geq \tilde{r}_+$. 

### 3.2 Parametric Diagrams

The $\tilde{r} - \varphi$ equation for each kind of surrounding matter fields can be obtained from equation (11) through a suitable choice of $\omega_q$. If for a given set of parameters $\tilde{\Lambda}, \tilde{a}, \tilde{Q}, \epsilon, E^2, \tilde{L}$, the polynomial (11) has $n$ positive real zeros, then for varying $E^2$ and $\tilde{L}$ this number can only change if two zeros merge to one. Solving $R(\tilde{r}) = 0$ and $\frac{dR(\tilde{r})}{dE^2} = 0$ for $E^2$ and $\tilde{L}$ for different matter fields we obtain the parametric diagrams shown in Fig. 1 and Fig. 2.

![Parametric $\tilde{L}$-$E^2$ diagrams](image)

(a) $\epsilon = 1, \tilde{a} = 0.8, \tilde{\Lambda} = -0.001$ and $\tilde{Q} = 0.1$

(b) $\epsilon = 0, \tilde{a} = 0.8, \tilde{\Lambda} = -0.001$ and $\tilde{Q} = 0.1$

(c) $\epsilon = 1, \tilde{a} = 0.1, \tilde{\Lambda} = 0.001$ and $\tilde{Q} = 0.1$

(d) $\epsilon = 0, \tilde{a} = 0.1, \tilde{\Lambda} = 0.001$ and $\tilde{Q} = 0.1$

Figure 1: Parametric $\tilde{L}$-$E^2$ diagrams for particles and light for the state parameter $\omega_q = -\frac{4}{5}$. The numbers in brackets denote the number of positive real zeros of the polynomial $R$. 


3.3 Effective potentials

Another possibility to determine the location of the zeros of the polynomial $R(\tilde{r})$ (e.g., the turning points of the geodesic motion) is to use the effective potential $V$ defined in Eqn. 10. In Fig. 3 and 4 we illustrated the effective potential for the two values of $\omega_q$ each for test particles and light. Here the intersections of the different particle energies $E$ with the effective potential correspond to the turning points of the particle motion.
Figure 3: Effective potential $V$ and various energies for different kinds of orbits for the state parameter $\omega_q = -\frac{3}{4}$. The blue line shows the potential and the grey area is forbidden, because here $R(\tilde{r})$ becomes negative. The turning points of the particle are denoted by red dots.

(a) $\epsilon = 1$, $\tilde{a} = 0.03$, $\tilde{\Lambda} = -0.01$, $\tilde{Q} = 0.1$ and $\tilde{L} = 0.04$

(b) $\epsilon = 0$, $\tilde{a} = 0.03$, $\tilde{\Lambda} = -0.01$, $\tilde{Q} = 0.1$ and $\tilde{L} = 0.04$

(c) $\epsilon = 1$, $\tilde{a} = 0.03$, $\tilde{\Lambda} = 0.01$, $\tilde{Q} = 0.1$ and $\tilde{L} = 0.005$

(d) $\epsilon = 0$, $\tilde{a} = 0.03$, $\tilde{\Lambda} = 0.01$, $\tilde{Q} = 0.1$ and $\tilde{L} = 0.04$

(e) $\epsilon = 1$, $\tilde{a} = 0.1$, $\tilde{\Lambda} = -0.0025$, $\tilde{Q} = 0.005$ and $\tilde{L} = 0.2$

(f) $\epsilon = 0$, $\tilde{a} = 0.1$, $\tilde{\Lambda} = -0.0025$, $\tilde{Q} = 0.005$ and $\tilde{L} = 0.08$
Figure 4: Effective potential $V$ and various energies for different kinds of orbits for the state parameter $\omega_q = -\frac{1}{3}$. The blue line shows the potential and the grey area is forbidden, because here $R(f)$ becomes negative. The turning points of the particle are denoted by red dots.

Combining the parametric diagrams and the effective potentials we can present a list of all possible orbits for each value of $\omega_q$ in tables 1 and 2.
| type | zeros | $\epsilon_{\text{sign}(\Lambda)}$ | range of $\tilde{r}$ | orbit |
|------|-------|-----------------|-----------------|-------|
| A    | 1     | $0_{\pm}$       | $\bullet$       | TEO   |
| B    | 2     | $1_{\pm}, 0_{\pm}$ | $\bullet$       | MBO   |
| C    | 3     | $1_{\pm}, 0_{\pm}$ | $\bullet$       | MBO, TEO |
| D    | 4     | $1_{\pm}, 0_{\pm}$ | $\bullet$       | MBO, TEO |
| E    | 1     | $0_{-}$          | $\bullet$       | TEO   |
| F    | 2     | $1_{-}$          | $\bullet$       | MBO   |
| G    | 3     | $0_{-}$          | $\bullet$       | MBO, EO |
| H    | 4     | $0_{-}$          | $\bullet$       | MBO, BO |
| I    | 4     | $1_{-}$          | $\bullet$       | MBO, MBO |

Table 1: Possible types of orbits in the Reissner-Nordström (Anti-)de-Sitter spacetime surrounded by a matter field with $\omega_q = -\frac{3}{2}$.

The thick lines represent the range of $\tilde{r}$, $\tilde{r} = 0$ is represented by a blank circle and the horizons by two vertical lines. Note that the cosmological horizon in types A to D is only present for $\tilde{\Lambda} > 0$. The turning points are indicated by thick dots.

| type | zeros | $\epsilon_{\text{sign}(\Lambda)}$ | range of $\tilde{r}$ | orbit |
|------|-------|-----------------|-----------------|-------|
| A    | 1     | $1_{\pm}, 0_{\pm}$ | $\bullet$       | TEO   |
| B    | 2     | $1_{-}, 0_{-}$    | $\bullet$       | MBO   |
| C    | 3     | $1_{\pm}, 0_{\pm}$ | $\bullet$       | MBO, EO |
| D    | 4     | $1_{-}$          | $\bullet$       | MBO, BO |

Table 2: Possible types of orbits in the Reissner-Nordström (Anti-)de-Sitter spacetime surrounded by a matter field with $\omega_q = -\frac{1}{3}$.

The thick lines represent the range of $\tilde{r}$, $\tilde{r} = 0$ is represented by a blank circle and the horizons by two vertical lines. Note that the cosmological horizon is only present for $\tilde{\Lambda} > 0$. The turning points are indicated by thick dots.

### 4 Solution of the geodesic equation

In this section we present the analytical solution of the geodesic equation (Eqn. 11) for each analyzed value of $\omega_q$. The right hand side of Eqn. 11 is a polynomial of order 6

$$P_6(\tilde{r}) = \sum_{i=1}^{6} a_i \tilde{r}^i,$$

with the coefficients $a_i$ depending on the parameters of the black hole and on the value of $\omega_q$. The order of $P_6(\tilde{r})$ can be reduced by one with the substitution

$$\tilde{r} = \pm \frac{1}{x} + \tilde{r}_6,$$

where $\tilde{r}_6$ is a zero of $P_6(\tilde{r})$. With this substitution Eqn. 11 becomes

$$\frac{\left(x \frac{dx}{d\varphi}\right)^2}{\sum_{i=0}^{5} b_i x^i} = P_5(x),$$

with adjusted coefficients. By separating the variables, we can reformulate Eqn. 14 in terms of a hyper-elliptic integral of the first kind

$$\varphi - \varphi_{\text{in}} = \int_{x_{\text{in}}}^{x} \frac{x' dx'}{\sqrt{P_5(x')}}.$$

Eqn. 15 can be solved in terms of derivatives the Kleinian $\sigma$-function

$$x = -\frac{\sigma_1(\tilde{\varphi}_{\infty})}{\sigma_2(\tilde{\varphi}_{\infty})},$$
where $\sigma_i = \frac{\partial \sigma_i}{\partial \sigma_1}$ and

$$\varphi_{\infty} = \left(-\int_x^\infty \frac{dx}{\sqrt{P_5(x)}}, \varphi - \varphi_{\infty} - \int_{\varphi_{\infty}}^{\varphi} \frac{x dx}{\sqrt{P_5(x)}} \right)^T. \quad (17)$$

By resubstituting $x$ into 13 we obtain the full solution

$$\tilde{\tau}(\varphi) = \pm \frac{\sigma_2(\varphi_{\infty})}{\sigma_1(\varphi_{\infty})} + \tilde{r}_6. \quad (18)$$

If we only look at orbits for light (e.g. $\epsilon = 0$) the solution simplifies tremendously. In that case the right-hand side of Eqn. 11 reduces to a fourth order polynomial. With a similar substitution as before

$$\tilde{\tau} = \pm \frac{1}{x} + \tilde{r}_4, \quad (19)$$

where $\tilde{r}_4$ is a zero of $R(\tilde{\tau})$, Eqn. 11 simplifies to

$$\left(\frac{dx}{d\varphi}\right)^2 = \sum_{i=0}^4 c_i x^i. \quad (20)$$

With a further substitution $x = \frac{1}{b_3} (4z - \frac{b_2}{4})$ this reduces to the standard Weierstraß form

$$\left(\frac{dz}{d\varphi}\right)^2 = 4z^3 - g_2 z - g_3. \quad (21)$$

This equation can be solved in terms of the Weierstraß $\wp$-function

$$z(\varphi) = \wp(\varphi - \varphi'; g_2, g_3). \quad (22)$$

Here $\varphi' = \varphi_{\infty} + \int_{\varphi_{\infty}}^{\varphi} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$ only depends on the initial values $\varphi_{\infty}$ and $z_{\infty}$. A resubstitution leads to the full solution

$$\tilde{\tau} = \pm \frac{c_3}{4\wp(\varphi - \varphi'; g_2, g_3)} + \tilde{r}_4. \quad (23)$$

### 5 Orbits

With the help of Eqn. 18 we can visualize examples of possible orbits in the analyzed spacetime. Fig. 5 shows various kinds of orbits for different values of the spacetime parameters.
(a) $\epsilon = 1$, $\omega_q = -1$, $\alpha = 0.001$, $\tilde{A} = -0.07$, $\tilde{Q} = 0.1$, $\tilde{L} = 0.01$, $E = 3.41$: Many-World Bound Orbit

(b) $\epsilon = 0$, $\omega_q = -\frac{\pi}{2}$, $\alpha = 0.03$, $\tilde{A} = -0.01$, $\tilde{Q} = 0.1$, $\tilde{L} = 0.04$, $E = \sqrt{0.33}$: Escape Orbit

(c) $\epsilon = 1$, $\omega_q = -\frac{4}{3}$, $\alpha = 0.03$, $\tilde{A} = -0.01$, $\tilde{Q} = 0.7$, $\tilde{L} = 0.4$, $E = 1.5$: Many-World Bound Orbit

(d) $\epsilon = 0$, $\omega_q = -\frac{4}{3}$, $\alpha = -0.03$, $\tilde{A} = -0.0001$, $\tilde{Q} = 0.1$, $\tilde{L} = 0.04$, $E = 1.1$: Bound Orbit

(e) $\epsilon = 0$, $\omega_q = -\frac{5}{2}$, $\alpha = 0.1$, $\tilde{A} = -0.0025$, $\tilde{Q} = 0.005$, $\tilde{L} = 0.08$, $E = \sqrt{0.12}$: Two-World Escape Orbit

(f) $\epsilon = 1$, $\omega_q = -\frac{5}{2}$, $\alpha = 0.1$, $\tilde{A} = -0.00025$, $\tilde{Q} = 0.005$, $\tilde{L} = 0.08$, $E = 1.65$: Many-World Bound Orbit

Figure 5: Orbits of test particles and light in the Reissner-Nordström Anti-de-Sitter spacetime surrounded by various regular and exotic matter fields. The blue line corresponds to the geodesic and the dashed black lines denote the black hole horizons.
6 Conclusion

In this paper we studied the spacetime of a Reissner-Nordström (Anti-)de-Sitter black hole in the presence of different kinds of matter fields depending on different choices of the state parameter \( \omega_q \) with the help of geodesics. After a short review of the structure of the spacetime, we derived the equation of motion for null and timelike geodesics for each separate matter fields. The analytical solutions were presented in terms of the elliptic Weierstraß \( \wp \)-function for null geodesics and in terms of derivatives of the Kleinian \( \sigma \)-function for timelike geodesics. Using effective potential techniques and parametric diagrams, a complete set of orbit types were analyzed for massive and massless test particles moving on geodesics. The derived orbits depend on the particles energy, angular momentum, cosmological constant, normalization parameter and on the state parameter. The analytical solutions obtained in section 4 are valid for all four values of \( \omega_q \). The possible orbits for the spacetimes characterized by \( \omega_q = \frac{1}{3} \) and \( \omega_q = -1 \) are extensively studied in [35] with the replacement of \( Q^2 \) by \( Q^2 - a \) and \( \Lambda \) by \( \Lambda + a \). In the view of geodesic motion taking place in the spacetime surrounded by irregular matter fields the possible orbit types are TEO, MBO, BO and EO for \( \omega_q = -\frac{2}{3} \) and for \( \omega_q = -\frac{1}{3} \). In the case of \( \omega_q = -\frac{2}{3} \) it is possible to find MBO and TEO that crosses all four horizons. The results obtained in this paper are used to calculate the exact orbits and their properties. Furthermore, observables like periastron shift of bound orbits or the light deflection of escape orbits can be investigated. It would be interesting to extend the equation of motion and their solutions in the case of electrically and magnetically charged particles around the black hole considered in this paper. Another project for future work could be to study the influence of regular as well as exotic matter fields on the shadow cast by Reissner-Nordström (Anti-)de-Sitter black hole.

7 Acknowledgements

A.K.C and A.R would like to express sincere gratitude to Dr. P. P. Pradhan who first made a strong impression on research related to the analytic study of black hole geodesic motion. K.F. would like to thank Jutta Kunz for interesting discussions and suggestions on the outline of the paper and gratefully acknowledges support by the DFG, within the Research Training Group Models of Gravity.

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