Total Current Fluctuations in ASEP

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Abstract
A limit theorem for the total current in the asymmetric simple exclusion process (ASEP) with step initial condition is proved. This extends the result of Johansson on TASEP to ASEP.

1 Introduction

The asymmetric simple exclusion process (ASEP) is a continuous time Markov process of interacting particles on the integer lattice $\mathbb{Z}$ subject to two rules: (1) A particle at $x$ waits an exponential time with parameter one (independently of all other particles) and then it chooses $y$ with probability $p(x, y)$; (2) If $y$ is vacant at that time it moves to $y$, while if $y$ is occupied it remains at $x$ and restarts its clock. The adjective “simple” refers to the fact that the allowed jumps are one step to the right, $p(x, x+1) = p$, or one step to the left $p(x, x-1) = 1 - p = q$. The asymmetric condition is $p \neq q$ so there is a net drift of particles. The special cases $p = 1$ (particles hop only to the right) or $q = 1$ (particles hop only to the left) are called the T(totally)ASEP. The dynamics are uniquely determined once we specify the initial state, which may be either deterministic or random. A rigorous construction of this infinite particle process can be found in Liggett [7].

Since its introduction by Spitzer [12], the ASEP has remained a popular model among probabilists and physicists because it is one of the simplest nontrivial processes modeling nonequilibrium phenomena. (For recent reviews see [4, 8, 10, 13].) If initially the particles are located at $\mathbb{Z}^+ = \{1, 2, \ldots\}$, called the step initial condition, and if $p < q$, then there will be on average a net flow of particles, or current, to the left. More precisely, we introduce the total current $I$ at position $x \leq 0$ at time $t$:

$$I(x, t) := \# \text{ of particles } \leq x \text{ at time } t.$$
With step initial condition, it has been known for some time (see, e.g., Theorem 5.12 in [7]) that if we set \( \gamma := q - p > 0 \) and \( 0 \leq c \leq \gamma \), then the current \( I \) satisfies the strong law
\[
\lim_{t \to \infty} \frac{I([-ct], t)}{t} = \frac{1}{4\gamma} (\gamma - c)^2 \quad \text{a.s.}
\]

The natural next step is to examine the current fluctuations
\[ I(x, t) - \frac{1}{4\gamma} (\gamma - c)^2 t \quad \text{(1)} \]
for large \( x \) and \( t \). Physicists conjectured [6], and Johansson proved for TASEP [5], that to obtain a nontrivial limiting distribution the correct normalization of (1) is cube root in \( t \). For TASEP Johansson not only proved that the fluctuations are of order \( t^{1/3} \) but found the limiting distribution function. Precisely, for \( 0 \leq \nu < 1 \) we have
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{I([-\nu t], t) - a_1 t}{a_2 t^{1/3}} \leq s \right) = 1 - F_2(-s), \quad \text{(2)}
\]
where
\[
a_1 = \frac{1}{4} (1 - \nu)^2, \quad a_2 = 2^{-4/3} (1 - \nu^2)^{2/3}, \quad \text{(3)}
\]
and \( F_2 \) is the limiting distribution of the largest eigenvalue in the Gaussian Unitary Ensemble [14].

The proof of this relied on the fact that TASEP is a determinantal process [5, 11, 13]. However, universality arguments suggest that (2) should extend to ASEP with step initial condition even though ASEP is not a determinantal process. When the initial state is the Bernoulli product measure, it has been recently proved, using general probabilistic arguments, that the correct normalization remains \( t^{1/3} \) for a large class of stochastic models including ASEP [1, 2, 3, 9].

In this paper we show that (2) does extend to ASEP.

**Theorem.** For ASEP with step initial condition we have, for \( 0 \leq \nu < 1 \),
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{I([-\nu t], t/\gamma) - a_1 t}{a_2 t^{1/3}} \leq s \right) = 1 - F_2(-s),
\]
where \( \gamma = q - p \) and \( a_1 \) and \( a_2 \) are given by (3).

This theorem is a corollary, as we show below, of earlier work by the authors [15].

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1. The value of \( a_2 \) given in (3) corrects a misprint in Corollary 1.7 of [5].
2. With step initial condition and \( x > 0 \) the total current equals the number of particles to the left of \( x \) at time \( t \) minus \( x \). In what follows we shall require only that \( |\nu| < 1 \). Therefore the statement of the Theorem holds for all such \( \nu \) if when \( \nu < 0 \) the value of \( a_1 \) is decreased by \( |\nu| \).
2 Proof of the Theorem

We denote by \( x_m(t) \) the position of the \( m \)-th left-most particle (thus \( x_m(0) = m \in \mathbb{Z}^+ \)). We are interested in the probability of the event

\[
\{ I(x, t) = m \} = \{ x_m(t) \leq x, x_{m+1}(t) > x \}. \tag{4}
\]

The sample space consists of the four disjoint events \( \{ x_m(t) \leq x, x_{m+1}(t) \leq x \} \), \( \{ x_m(t) > x, x_{m+1}(t) > x \} \), \( \{ x_m(t) \leq x, x_{m+1}(t) \leq x \} \) and because of the exclusion property we have

\[
\{ x_m(t) \leq x, x_{m+1}(t) \leq x \} = \{ x_{m+1}(t) \leq x \},
\]

\[
\{ x_m(t) > x, x_{m+1}(t) > x \} = \{ x_m(t) > x \},
\]

\[
\{ x_m(t) > x, x_{m+1}(t) \leq x \} = \emptyset.
\]

These observations and (4) give (the intuitively obvious)

\[
\mathbb{P}(I(x, t) = m) = \mathbb{P}(x_m(t) \leq x) - \mathbb{P}(x_{m+1}(t) \leq x).
\]

Since \( \mathbb{P}(I(x, t) = 0) = \mathbb{P}(x_1(t) > x) \), we have

\[
\mathbb{P}(I(x, t) \leq m) = 1 - \mathbb{P}(x_{m+1}(t) \leq x).
\]

Thus, since \( x \) and \( x_{m+1}(t) \) are integers, the statement of the Theorem is equivalent to the statement that

\[
\lim_{t \to \infty} \mathbb{P}(x_{m+1}(t/\gamma) \leq -vt) = F_2(s),
\]

when \( m = [a_1 t - a_2 s t^{1/3}] \). In fact, we shall show that

\[
\lim_{t \to \infty} \mathbb{P}(x_m(t/\gamma) \leq -vt) = F_2(s), \tag{5}
\]

when

\[
m = a_1 t - a_2 s t^{1/3} + o(t^{1/3}). \tag{6}
\]

Let

\[
\sigma = \frac{m}{t}, \quad c_1 = -1 + 2\sqrt{\sigma}, \quad c_2 = \sigma^{-1/6}(1 - \sqrt{\sigma})^{2/3}.
\]

It was proved in [15] that when \( 0 \leq p < q \),

\[
\lim_{t \to \infty} \mathbb{P}(x_m(t/\gamma) \leq c_1 t + s c_2 t^{1/3}) = F_2(s) \tag{7}
\]

uniformly for \( \sigma \) in a compact subset of \( (0, 1) \).

To obtain (5) from this we determine \( \sigma \) so that

\[
-vt = c_1 t + s c_2 t^{1/3}.
\]
Thus,
\[ v = 1 - 2\sqrt{\sigma} - s \sigma^{-1/6} (1 - \sqrt{\sigma})^{2/3} t^{-2/3}. \]

Solving, we get
\[
\left( \frac{1 - v}{2} \right)^2 = \sigma + s \sigma^{1/3} (1 - \sqrt{\sigma})^{2/3} t^{-2/3} + O \left( t^{-4/3} \right),
\]
from which we deduce
\[
\sigma = \left( \frac{1 - v}{2} \right)^2 - s \left( \frac{1 - v}{2} \right)^{2/3} \left( \frac{1 + v}{2} \right)^{2/3} t^{-2/3} + O \left( t^{-4/3} \right)
\]
\[
= \left( \frac{1 - v}{2} \right)^2 - s 2^{-4/3} (1 - v^2)^{2/3} t^{-2/3} + O \left( t^{-4/3} \right). \]

By the uniformity of (\ref{eq:uniformity}) in \( \sigma \) we get the same asymptotics if we replace the \( \sigma \) we just computed by any \( \sigma \) satisfying
\[
\sigma = \left( \frac{1 - v}{2} \right)^2 - s 2^{-4/3} (1 - v^2)^{2/3} t^{-2/3} + o(t^{-2/3}).
\]

Since this is exactly the statement that \( m = \sigma t \) satisfies \cite{3}, we see that the Theorem is established.

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\footnote{ Since the condition on \( \sigma \) is \( 0 < \sigma < 1 \), the corresponding condition on \( v \) is \( |v| < 1 \), as was stated in footnote 2.}
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