Resolvability and metric dimension: Graph with order atmost 5

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Abstract
For an ordered subset \( W = \{w_1, w_2, ..., w_k\} \) of \( V(G) \) and a vertex \( v \in V \), the metric representation of \( v \) with respect to \( W \) is a \( k \)-vector, which is defined as \( r(v/W) = \{d(v, w_1), d(v, w_2), ..., d(v, w_k)\} \). The set \( W \) is called a resolving set for \( G \) if \( r(u/W) \neq r(v/W) \) implies that \( u = v \) for all \( u, v \in V(G) \). In this paper, the metric dimension of any simple connected graph with order atmost 5 is determined.

Keywords
Metric dimension, Metric basis, Resolving set.

AMS Subject Classification
Primary: 05C12, Secondary: 05E30.

1. Introduction
The concept of resolving set of a graph and its related properties, such as metric dimension, play a significant role in graph labeling using distance. The concepts of resolving set and minimum resolving set were first introduced in 1975 by P.J.Slater [4] by the names locating set and location number, and in 1976 F.Harary and R.A.Melter [2] introduced these concepts independently but used the term metric dimension rather than location number. The concept of (minimum) resolving set has proved to be useful and/or related to a variety of fields. The minimum metric dimension has applications in the field of robotics. A robot is a mechanical device which is made to move in space with obstructions around. It has neither the concept of direction nor that of visibility. But it is assumed that it can sense the distances to a set of landmarks. Evidently, if the robot knows its distances to a sufficiently large set of landmarks its position in space is uniquely determined. Another applications of resolving sets arise in various areas including coin weighing problem, drug discovery, network discovery and verification, connected joins in graphs and strategies for the mastermind game. The motivation behind this work is due to the large range of application of resolving sets in various fields.

2. Preliminaries
All the graphs considered in this paper are undirected, simple, finite and connected. The order and size of \( G \) are denoted by \( n \) and \( m \) respectively. We use standard terminology, the terms not defined here may found in [2] and [5].

Definition 2.1. Let \( G = (V,E) \) be a connected, undirected graph and \( v_1, v_2, v_3 \in V \). A vertex \( v \) is said to resolve the vertices \( v_1 \) and \( v_3 \) if the distance of \( v_1 \) from \( v_2 \) is different from distance of \( v_3 \) from \( v_2 \).

Definition 2.2. For an ordered subset \( W = \{w_1, w_2, ..., w_k\} \) of \( V(G) \) and for any vertex \( v \in V \), the (metric)representation of \( v \) with respect to \( W \) is the \( k \)-vector which is denoted and defined as \( r(v/W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k)) \). The set \( W \) is called a resolving set for \( G \) if \( r(v_1/W) \neq r(v_2/W) \) implies that \( v_1 = v_2 \) for all \( v_1, v_2 \in V(G) \).

Definition 2.3. A resolving set of minimum cardinality for a graph \( G \) is called a minimum resolving set. A minimum resolving set is usually called a basis for \( G \). The minimum cardinality of a resolving set of \( G \) is called the metric dimension of \( G \) and is denoted by \( \text{dim}(G) \).
Theorem 2.4. [3] A connected graph $G$ of order $n > 2$ has dimension $n - 1$ if and only if $G = K_n$.

Theorem 2.5. [1] For $Q_n, \dim(Q_n)$ is $n$, if $n \leq 4; n - 1$, if $5 \leq n \leq 6; 6$, if $7 \leq n \leq 8; 7$, if $9 \leq n \leq 10$.

Theorem 2.6. [3] A connected graph $G$ of order $n$ has dimension $1$ if and only if $G = P_n$.

Theorem 2.7. [3] The metric dimension of $K_{1,n}$ is $\dim(K_{1,n}) = n - 1$.

Theorem 2.8. [7] For given positive integers $m,n$ the metric dimension of

$$
\dim(P_n + P_m) = \begin{cases} 
1 & \text{if } m = n = 1 \\
2 & \text{if } 2 \leq m \leq 3 \\
\left\lceil \frac{m}{2} \right\rceil + n - 1 & \text{if } n \geq 1, m \geq 4
\end{cases}
$$

Theorem 2.9. [3] The metric dimension of $C_n$ is $\dim(C_n) = 2$.

### 3. Main Results

In this section we consider a simple connected graph $G$ with finite number of vertices and obtained following results.

**Lemma 3.1.** If $G$ be a graph with,

(a) order $1$ or $2$, then $\dim(G) = 1$.

(b) order $3$, then

$$
\dim(G) = \begin{cases} 
1 & \text{if } G = P_3 \\
2 & \text{if } G = K_3
\end{cases}
$$

**Proof:** Let $G$ be a graph with single vertex and upto isomorphism there is only one connected graph of order one, which is isomorphic to $K_1$ or $P_1$. Therefore $\dim(G) = 1$. For the case when $G$ is a connected graph with $2$ vertices then $G \cong P_2$. Therefore $\dim(G) = 1$ by Theorem 2.6.

Let $G$ be a connected graph with $3$ vertices, then the possible graphs are $P_3, C_3, K_3$ or $P_1 + P_2$. By Theorem 2.6, $\dim(P_3) = 1$. Also $C_3 \cong K_3 \cong P_1 + P_2$. Therefore $\dim(C_3) = \dim(K_3) = \dim(P_1 + P_2) = 2$.

**Theorem 3.2.** For a connected graph $G$ with order $4$,

$$
\dim(G) = \begin{cases} 
1 & \text{if } G = P_4 \\
3 & \text{if } G = K_4 \\
2 & \text{if } G \text{ is non-isomorphic to } P_4, K_4
\end{cases}
$$

**Proof:** Given that $G$ be a connected graph with $4$ vertices. Then the possible degree sequence of vertices are $(1,2,2,1), (1,3,2,2), (2,2,2,2), (2,3,2,3), (3,3,3,3)$ and $(3,1,1,1)$ upto isomorphism. For a graph with $4$ vertices degree sequence is independent of its order.

A graph with degree sequence of vertices $(1,2,2,1)$ is isomorphic to the graph $P_4$ and $\dim(P_4) = 1$. $K_4$ be the only graph with degree sequence of vertices $(3,3,3,3)$. Therefore $\dim(K_4) = 3$, by Theorem 2.4. The graph corresponding to the degree sequence $(3,1,1,1)$ is $K_{1,3}$. Therefore $\dim(K_{1,3}) = 2$ by Theorem 2.7. The graph corresponding to the degree sequence $(2,2,2,2)$ is $C_4$. Therefore $\dim(C_4) = 2$, by Theorem 2.9. The graph corresponding to the degree sequence $(2,3,2,3)$ is $K_4 - \{e\}$. Therefore $\dim(K_4 - \{e\}) = 2$.

A graph with degree sequence $(1,3,2,2)$ is as shown in the Figure 1.

**Figure 1**

Graph $G$ with $\dim(G) = 2$

From Figure 1, let $W = \{u_2, u_3\}$, every vertices have distinct metric representation with respect to $W$ and hence $W$ is a resolving set of $G$ and which is minimal. So $\dim(G) = 2$. Therefore for any connected graph with $4$ vertices, metric dimension is determined. hence the result.

**Theorem 3.3.** For a connected graph with $5$ vertices,

$$
\dim(G) = \begin{cases} 
1 & \text{if } G = P_5 \\
3 & \text{if } G = K_5 - \{e\} \text{ or } K_{1,4} \\
4 & \text{if } G = K_5 \\
2 & \text{if } G \text{ is non-isomorphic to } K_5, P_5
\end{cases}
$$

**Proof:** Given that $G$ be a connected graph with $5$ vertices. $G = P_5$ is the only graph with vertex degree sequence $(1,2,2,2,1)$ and from Theorem 2.6, $\dim(G) = 1$. A graph with vertex degree sequence $(4,3,4,3,4)$ is $K_5 - \{e\}$ as shown in the Figure 2.

**Figure 2**

$\dim(K_5 - \{e\}) = 3$.

The set $W = \{u_1, u_2, u_3\}$ form a basis for $K_5 - \{e\}$, since with respect to $W$, every vertices of $G$ have distinct metric representation and $W$ is minimal resolving set. Therefore
If $G$ is the graph with degree sequence $(3,3,3,4,1)$, then $G$ is as shown in the Figure 3. The set $W = \{u_1, u_3, u_4\}$ form a basis for $G$, since every vertex of $G$ have distinct metric representation with respect to $W$ and $W$ is minimal resolving set. So $\dim(G) = 3$.

If $G = K_5$ is the only graph with degree sequence $(4,4,4,4,4)$. Therefore from Theorem 2.4, $\dim(G) = \dim(K_5) = 4$. Also $G = K_{1,4}$ is the only graph with degree sequence $(4,1,1,1,1)$. Therefore from Theorem 2.7, $\dim(G) = \dim(K_{1,4}) = 3$. The remaining possible degree sequences of vertices are $(2,2,2,2,2), (2,3,3,3,1), (2,2,4,2,2),(2,3,3,2,2),(2,3,3,2,2),(4,2,2,2,1,1), (4,2,3,3,1),(4,2,3,3,2), (3,3,3,2,3),(3,3,4,2,4) and (4,3,3,3,3).

A graph with degree sequence $(2,2,2,2,2)$ is $C_5$, therefore by Theorem 2.9, $\dim(C_5) = 2$. Graphs of the remaining degree sequences of vertices are as shown in Figure 4.

In Fig 4(a), Fig 4(b) and Fig 4(d), $W = \{u_1, u_2\}$ is a resolving set and which is minimal, therefore $\dim(G) = 2$. In Fig 4(c), Fig 4(h) and Fig 4(j), $W = \{u_1, u_3\}$ is a resolving set and which is minimal, therefore in this case $\dim(G) = 2$. In Fig 4(e), Fig 4(f) and Fig 4(g) $W = \{u_3, u_4\}$ is a resolving set and which is minimal, therefore in this case $\dim(G) = 2$. Thus, the metric dimension of any connected graph with 5 vertices is determined.

4. Conclusion

In this paper, the metric dimension of any simple connected graph with order atmost 5 is determined.
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