On the Complexity of Decentralized Smooth Nonconvex Finite-Sum Optimization*

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Abstract

We study the decentralized optimization problem \( \min_{x \in \mathbb{R}^d} f(x) \equiv \frac{1}{m} \sum_{i=1}^{m} f_i(x) \), where the local function on the \( i \)-th agent has the form of \( f_i(x) \equiv \frac{1}{n} \sum_{j=1}^{n} f_{i,j}(x) \) and every individual \( f_{i,j} \) is smooth but possibly nonconvex. We propose a stochastic algorithm called DEcentralized probAbilistic Recursive gradiEnt deScenT (DEAREST+) method, which achieves an \( \epsilon \)-stationary point at each agent with the communication rounds of \( \tilde{O}(L \epsilon^{-2}/\sqrt{\gamma}) \), the computation rounds of \( \tilde{O}(n + (L + \min(nL, \sqrt{n/m} L/\gamma)\epsilon^{-2}) \), and the local incremental first-oracle calls of \( \tilde{O}(m n + \min(mnL, \sqrt{mnL})\epsilon^{-2}) \), where \( L \) is the smoothness parameter of the objective function, \( \bar{L} \) is the mean-squared smoothness parameter of all individual functions, and \( \gamma \) is the spectral gap of the mixing matrix associated with the network. We then establish the lower bounds to show that the proposed method is near-optimal. Notice that the smoothness parameters \( L \) and \( \bar{L} \) used in our algorithm design and analysis are global, leading to sharper complexity bounds than existing results that depend on the local smoothness. We further extend DEAREST+ to solve the decentralized finite-sum optimization problem under the Polyak–Lojasiewicz condition, also achieving the near-optimal complexity bounds.

Keywords: decentralized optimization, nonconvex optimization, smoothness parameter, variance reduction, Polyak–Lojasiewicz condition

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1 Introduction

We study the decentralized optimization problem

$$\min_{x \in \mathbb{R}^d} f(x) \triangleq \frac{1}{m} \sum_{i=1}^{m} f_i(x),$$

over a connected network with $m$ agents, where $f_i : \mathbb{R}^d \to \mathbb{R}$ is the local function on the $i$-th agent that has the finite-sum structure with $n$ individual functions as follows

$$f_i(x) \triangleq \frac{1}{n} \sum_{j=1}^{n} f_{i,j}(x).$$

We suppose each individual function $f_{i,j} : \mathbb{R}^d \to \mathbb{R}$ is smooth but possibly nonconvex. This formulation includes a lot of applications in statistics [6, 91], signal processing [23, 56, 74, 75], and machine learning [4, 14, 29, 77, 82]. In decentralized scenario, all of agents target to collaboratively solve problem (1) and each of them can only communicate with its neighbors. We focus on the complexity for achieving the approximate stationary point of the global objective at every agent.

For decentralized optimization, the limitation on the communication protocol leads to the local agents cannot access the exact global information at each round, which leads to the requirement of communication rounds to reduce the consensus error. The gradient tracking [55, 63, 65, 70] is a useful technique to approximate the average of local gradients and make the local first-order estimator accurate. Directly extending (stochastic) gradient descent methods to decentralized setting [20, 21, 43, 48, 54, 72, 76, 84] cannot take the advantage of the popular finite-sum structure in local functions. It is well-known that the algorithms with stochastic recursive gradient estimator [16, 24, 42, 59, 60] can achieve the optimal incremental first-order oracle (IFO) complexity for finding the approximate stationary point of the finite-sum nonconvex function under the mean-squared smooth assumption. Sun et al. [73] first combined stochastic recursive gradient estimator with gradient tracking to solve decentralized nonconvex finite-sum optimization problem by proposing Decentralized Gradient Estimation and Tracking (D-GET). Later, Xin et al. [81] and Zhan et al. [89] proposed GT-SARAH and efficient decentralized stochastic gradient descent (EDSGD) respectively to improve the complexity in terms of the dependency on the numbers of agents and individual functions. Li et al. [39] proposed DEcentralized STochastic REcurSive gradient methodS (DESTRESS), which introduces Chebyshev acceleration [7] to achieve the tighter dependency on the spectral gap of the mixing matrix associated with the network. Metelev et al. [51] further considered the nonconvex problem over the time-varying network. Additionally, Lu and De Sa [48], Yuan et al. [85] studied the tightness of decentralized nonconvex optimization in the online setting.

It is worth noting that existing works [39, 48, 51, 73, 81, 85] for decentralized nonconvex optimization only consider the local smoothness parameters, which may be arbitrary larger than the global ones. Furthermore, their analysis for the computation complexity focuses on one of the overall local incremental first-order oracle (LIFO)
calls and the number of computation rounds. Noticing that in each computation round, a distributed algorithm can make partial agents access their LIFO and allow other agents skip their local computation steps [49, 52]. Therefore, the LIFO calls and the computation rounds should be addressed separately. Recently, Liu et al. [46], Ye et al. [83] studied the distributed optimization by considering the global smoothness dependency and the partial participated protocol, while their results only address the convex problem.

In this paper, we refine the setting of decentralized smooth nonconvex finite-sum optimization (1) by distinguishing the different smoothness parameters and considering the partial participation protocol. We proposed a novel stochastic algorithm called DEcentralized probAbilistic Recursive gradiEnt deScenT (DEAREST$^+$), achieving the $\epsilon$-stationary point at every agent with the communication rounds $\tilde{O}(L\epsilon^{-2}/\sqrt{\gamma})$, the computation rounds of $\tilde{O}(n + (L + \min\{nL, \sqrt{n/\bar{L}}\})\epsilon^{-2})$, and the LIFO calls of $\mathcal{O}(mn + \min\{mnL, \sqrt{mn\bar{L}}\})\epsilon^{-2})$, where $L$ is the smoothness parameter of the global objective function $f$, $\bar{L}$ is the mean-squared smoothness parameter of all individual functions $\{f_{i,j}\}_{i,j=1}^n$, and $\gamma$ is the spectral gap of the mixing matrix associated with the network. We then establish lower complexity bounds with respect to $\epsilon$, $m$, $n$, $L$, $\bar{L}$, and $\gamma$ to show the near-optimality of our method. Notice that the smoothness parameters $L$ and $\bar{L}$ in our results are global, leading to sharper complexity bounds than existing ones that depend on the local smoothness. For single-machine scenario (i.e., $m = 1$), our theory indicates incremental first-order (IFO) complexity of $\mathcal{O}(n + \min\{nL, \sqrt{nL}\})\epsilon^{-2})$, which is a trade-off between the complexity of $\mathcal{O}(nL\epsilon^{-2})$ from vanilla gradient descent [17, 18, 58] and the complexity of $\mathcal{O}(n + \sqrt{nL}\epsilon^{-2})$ from stochastic variance-reduced methods [24, 42, 61, 78, 92, 93]. We further apply DEAREST$^+$ to solve the decentralized finite-sum problem under the Polyak–Lojasiewicz (PL) condition [47, 62, 87], which achieves the $\epsilon$-suboptimal solution at every agent with the communication rounds of $\tilde{O}(\kappa \ln(1/\epsilon))$, the computation rounds $\tilde{O}((n + \kappa + \min\{n\kappa, \sqrt{n/m\bar{\kappa}}\}) \ln(1/\epsilon))$, and the LIFO calls of $\mathcal{O}((mn + \min\{mnL, \sqrt{mn\bar{L}}\}) \ln(1/\epsilon))$, where $\kappa \triangleq L/\mu$ is the global condition number, $\bar{\kappa} \triangleq \bar{L}/\mu$ is the mean-squared condition number, and $\mu$ is the PL parameter. We also provide lower bounds to show above upper complexity bounds under the PL condition are near-optimal.

2 Preliminaries

We formally introduce the notations and problem settings used in this paper.

2.1 Notations

We use bold lower-case letters for vectors and bold upper-case letters for matrices. The notations $\| \cdot \|$ and $\| \cdot \|_2$ are used to denote the Frobenius norm and the spectral norm of the matrix, respectively, as well as the Euclidean norm of the vector. We let $\mathbf{1} = [1, \cdots, 1] \top \in \mathbb{R}^m$ and denote $\mathbf{I} \in \mathbb{R}^{m \times m}$ as the identity matrix. We define
aggregated variables for all agents as

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^{m \times d},$$

(3)

where each $x_i \in \mathbb{R}^{1 \times d}$ are the local variable on the $i$-th agent. We use the lower case with the bar to represent the mean vector, such that

$$\bar{x} = \frac{1}{m} \sum_{i=1}^{m} x_i.$$

We also introduce the matrix for aggregated gradients of local functions $f_1, \ldots, f_m$ as

$$\nabla F(X) = \begin{bmatrix} \nabla f_1(x_1) \\ \vdots \\ \nabla f_m(x_m) \end{bmatrix} \in \mathbb{R}^{m \times d}.$$  

(4)

For ease of presentation, we let the input of a function can be also organized as a row vector, such as $f(\bar{x}), f_i(x_i)$ and $\nabla f_i(x_i)$ for some $i \in [m].$

2.2 Problem Settings

We suppose the formulations (1)–(2) satisfy the following assumptions.

**Assumption 1** (lower bounded). We suppose the objective function $f : \mathbb{R}^d \to \mathbb{R}$ is lower bounded, i.e., we have

$$f^* \triangleq \inf_{x \in \mathbb{R}^d} f(x) > -\infty.$$  

(5)

**Assumption 2** (global smooth). We suppose the differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ is $L$-smooth for some $L > 0$, i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|$$

(6)

for all $x, y \in \mathbb{R}^d$.

**Assumption 3** (global mean-squared smooth). We suppose the individual functions $\{f_{i,j}\}_{i,j=1}^{m,n}$ are $\bar{L}$-mean-squared smooth for some $\bar{L} > 0$, i.e., we have

$$\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \|\nabla f_{i,j}(x) - \nabla f_{i,j}(y)\|^2 \leq \bar{L}^2 \|x - y\|^2$$

(7)

for all $x, y \in \mathbb{R}^d$.

We present the relationship between the smoothness parameters $L$ and $\bar{L}$ as follows.
Proposition 1. The smoothness conditions in Assumptions 2 and 3 have the following relationships:

(a) If the individual functions \( \{f_{i,j}\}_{i,j=1}^{m,n} \) are \( \bar{L} \)-mean-squared smooth, then each of \( \{f_{i,j}\}_{i,j=1}^{m,n} \) and \( \{f_i\}_{i=1}^{m} \) is \( \sqrt{mn\bar{L}} \)-smooth, i.e., we have

\[
\|\nabla f_{i,j}(x) - \nabla f_{i,j}(y)\| \leq \sqrt{mn\bar{L}} \|x - y\|
\]

and

\[
\|\nabla f_i(x) - \nabla f_i(y)\| \leq \sqrt{mn\bar{L}} \|x - y\|
\]

for all \( x, y \in \mathbb{R}^d \), \( i \in [m] \), and \( j \in [n] \).

(b) If the individual functions \( \{f_{i,j}\}_{i,j=1}^{m,n} \) are \( \bar{L} \)-mean-squared smooth, then the objective function \( f \) is \( \bar{L} \)-smooth, i.e., we have

\[
\|\nabla f(x) - \nabla f(y)\| \leq \bar{L} \|x - y\|
\]

for all \( x, y \in \mathbb{R}^d \).

(c) For any \( L > 0 \) and \( \bar{L} > 0 \) such that \( \bar{L} \geq L \), there exist functions \( \{f_{i,j}\}_{i,j=1}^{m,n} \) which satisfy Assumption 2 and 3 with the tight smoothness parameters \( L \) and \( \bar{L} \), respectively.

Remark 1. The statements (a) and (b) of Proposition 1 imply the upper bounds with respect to the tight global smoothness parameter \( \bar{L} \) in Assumption 2 is potential sharper than the upper bounds with respect to the global mean-squared smoothness parameter \( \bar{L} \) in Assumption 3 (global mean-squared smooth). The statement (c) of Proposition 1 means the ratio between the tight parameters \( \bar{L} \) and \( L \) can be arbitrary large, which means considering the difference between \( \bar{L} \) and \( L \) is very necessary in finite-sum nonconvex optimization.

We also present other smoothness assumptions used in related work [39, 51, 73, 81, 89] for comparison.

Assumption 4 (local smooth). We suppose each local function \( f_i : \mathbb{R}^d \to \mathbb{R} \) is \( L_\ell \)-smooth for some \( L_\ell > 0 \), i.e., we have

\[
\|\nabla f_i(x) - \nabla f_i(y)\| \leq L_\ell \|x - y\|
\]

for all \( i \in [m] \) and \( x, y \in \mathbb{R}^d \).

Assumption 5 (local mean-squared smooth). We suppose the individual functions \( \{f_{i,j}\}_{i,j=1}^{n} \) on each agent are \( \bar{L}_\ell \)-mean-squared smooth for some \( \bar{L}_\ell > 0 \), i.e., we have

\[
\frac{1}{n} \sum_{j=1}^{n} \|\nabla f_{i,j}(x) - \nabla f_{i,j}(y)\|^2 \leq \bar{L}_\ell^2 \|x - y\|^2
\]

for all \( i \in [m] \) and \( x, y \in \mathbb{R}^d \).
**Assumption 6** (individual smooth). We suppose each individual function $f_{i,j}$ is $L_{\text{max}}$-smooth for some $L_{\text{max}} > 0$, i.e., we have

$$\|\nabla f_{i,j}(x) - \nabla f_{i,j}(y)\| \leq L_{\text{max}} \|x - y\|$$

for all $i \in [m]$, $j \in [n]$ and $x, y \in \mathbb{R}^d$.

We present the relationship between the assumptions in related work (Assumptions 4 and 5) and ours (Assumptions 2 and 3) in the following proposition.

**Proposition 2.** The smoothness conditions in Assumptions 2–5 holds that:

(a) If each local function $f_i$ is $L_{\ell}$-smooth, then the objective function $f$ is $L_{\ell}$-smooth.

(b) If the individual functions $\{f_{i,j}\}_{i,j=1}^{m,n}$ on each agent are $\bar{L}_{\ell}$-mean-squared smooth, then all of the individual functions $\{f_{i,j}\}_{i,j=1}^{m,n}$ are $\bar{L}_{\ell}$-mean-squared smooth.

(c) For any $L$ and $L_{\ell}$ such that $L_{\ell} \geq L > 0$, there exist functions $\{f_i\}_{i=1}^m$ which satisfy Assumption 2 and 4 with the tight smoothness parameters $L$ and $L_{\ell}$, respectively.

(d) For any $\bar{L}$ and $\bar{L}_{\ell}$ such that $\bar{L}_{\ell} \geq \bar{L} > 0$, there exist functions $\{f_{i,j}\}_{i=1,j=1}^{m,n}$ which satisfy Assumption 3 and 5 with the tight smoothness parameters $\bar{L}$ and $\bar{L}_{\ell}$, respectively.

**Remark 2.** We consider the tight smoothness parameters $L$, $L_{\ell}$, $\bar{L}$, $\bar{L}_{\ell}$, and $L_{\text{max}}$ that satisfy Assumptions 2–6. Then the statements (a) and (c) of Proposition 2 imply $L_{\ell} \geq L$ and $L_{\ell} \geq \bar{L}$, and the statements (c) and (d) of Proposition 2 imply the ratio $L_{\ell}/L$ and $\bar{L}_{\ell}/\bar{L}$ can be arbitrary large. Following the proof of Proposition 2 (Appendix A.2), we can also show that $L_{\text{max}}$ is no smaller than $L$, $L_{\ell}$, $\bar{L}$, and $\bar{L}_{\ell}$, and the ratio $L_{\text{max}}/L$, $L_{\text{max}}/\bar{L}$, and $L_{\text{max}}/\bar{L}_{\ell}$ can be arbitrary large.

**Remark 3.** It is worth noting that ratio between local (individual) and global smoothness parameters can be bounded if all individual functions are convex, which is different from our nonconvex setting. Please see Appendix B for detailed discussion.

Besides the general nonconvex problem, we also study the objective under the additional PL condition [47, 62] as follows.

**Assumption 7.** We suppose the objective function $f : \mathbb{R}^d \to \mathbb{R}$ satisfies the PL condition with the parameter $\mu > 0$, i.e., we have

$$f(x) - \inf_{y \in \mathbb{R}^d} f(y) \leq \frac{1}{2\mu} \|\nabla f(x)\|^2$$

for all $x \in \mathbb{R}^d$.

Under the smoothness and PL conditions, we introduce different types of condition numbers as follows

$$\kappa \triangleq \frac{L}{\mu}, \quad \bar{\kappa} \triangleq \frac{\bar{L}}{\mu}, \quad \kappa_{\ell} \triangleq \frac{L_{\ell}}{\mu}, \quad \bar{\kappa}_{\ell} \triangleq \frac{\bar{L}_{\ell}}{\mu}, \quad \text{and} \quad \kappa_{\text{max}} \triangleq \frac{L_{\text{max}}}{\mu}. \quad (13)$$

Following the discussion in Remark 2, the tight condition numbers hold

$$\kappa_{\text{max}} \geq \kappa_{\ell} \geq \kappa \quad \text{and} \quad \kappa_{\text{max}} \geq \bar{\kappa}_{\ell} \geq \bar{\kappa} \geq \kappa.$$
We characterize the behavior of one communication round on the connected network with \( m \) agents by multiplying the mixing matrix \( W \in \mathbb{R}^{m \times m} \) on the aggregated variable. We impose the following standard assumption for matrix \( W \).

**Assumption 8** ([69]). We suppose the mixing matrix \( W = [w_{ij}] \in \mathbb{R}^{m \times m} \) is symmetric and satisfies \( w_{ij} \neq 0 \) if the \( i \)-th agent and the \( j \)-th agent are connected or \( i = j \). Furthermore, we suppose it holds that \( W1 = W^T1 = 1 \) and \( 0 \leq W \leq 1 \).

Under Assumption 8, the largest eigenvalue of \( W \in \mathbb{R}^{m \times m} \) is 1 and we define the spectral gap of \( W \) as

\[
\gamma = 1 - \lambda_2(W) \in (0, 1],
\]

where \( \lambda_2(W) \) is the second-largest eigenvalue of \( W \).

We consider the black-box optimization procedure based on the local incremental first-order oracle and decentralized communication protocol as follows.

**Definition 1** ([29, 46]). A local incremental first-order oracle (LIFO) algorithm over a network of \( m \) agents satisfies the following constraints:

- **Local memory**: Each agent \( i \) can store past values in a local memory \( M_{1,s} \) at time \( s > 0 \). These values can be accessed and used at time \( s \) by running the algorithm on agent \( i \). Additionally, for all \( i \in [m] \), we have

\[
M_{s}^* = M_{\text{comp},i} \cup M_{\text{comm},i},
\]

where \( M_{\text{comp},i} \) and \( M_{\text{comm},i} \) are the values come from the computation and communication respectively.

- **Local computation**: Each agent \( i \) can access its local first-order oracle \( \{f_{i,j}(x), \nabla f_{i,j}(x)\} \) for given \( x \in M_{s}^* \) and index \( j^* \in [n] \) at time \( s \). That is, for all \( i \in [m] \), we have

\[
M_{\text{comp},i}^* = \text{span}(\{x, \nabla f_{i}(x) : x \in M_{s}^{* - 1}\}),
\]

where \( \text{span}(\cdot) \) is the linear span.

- **Local communication**: Each agent \( i \) can share its value to all or part of its neighbors at time \( s \). That is, for all \( i \in [m] \), we have

\[
M_{\text{comm},i}^* = \text{span}\left(\bigcup_{j \in \text{nbr}_i} M_{s}^{j - \tau}\right),
\]

where \( \text{nbr}_i \) is the set consists of the indices for the neighbors of agent \( i \) and \( \tau < s \).

- **Output value**: Each agent \( i \) can specify one vector in its memory as local output of the algorithm at time \( s \). That is, for all \( i \in [m] \), we have \( x_{s}^* \in M_{s}^* \).

For the general nonconvex setting, we desire to achieve an \( \epsilon \)-stationary point for every agent in expectation, i.e., output the local variables \( x_1, \ldots, x_m \in \mathbb{R}^d \) such that \( \mathbb{E}\|\nabla f(x_i)\| \leq \epsilon \) for all \( i \in [m] \), where \( x_i \) comes from the memory of the \( i \)-th agent.

For the PL condition, we desire to achieve an \( \epsilon \)-suboptimal solution for every agent in expectation, i.e., \( \mathbb{E}[f(x_i) - f^*] \leq \epsilon \) for all \( i \in [m] \).
Algorithm 1 DEAREST$^+$

1: **Input:** initial parameter $\bar{x}^0 \in \mathbb{R}^d$, stepsize $\eta > 0$, probability $p \in (0, 1]$, mini-batch size $b$, numbers of communication rounds $\hat{K}$ and $K$.
2: $X^0 = 1 \bar{x}^0$, $G^0 = \nabla F(x^0)$
3: $S^0 = \text{AccGossip}(G^0, \hat{K})$
4: for $t = 0, \ldots, T - 1$ do
5: $\zeta^t \sim \text{Bernoulli}(p)$
6: $X^t+1 = \text{AccGossip}(X^t - \eta S^t, W, K)$
7: $[\xi^t_1, \ldots, \xi^t_m] \sim \text{Multinomial}(b, q^t)$ with $q^t = 1/\left(mn\right)$
8: parallel for $i = 1, \ldots, m$ do
9: $g^t_{i+1} = \begin{cases} \nabla f_i(x^t_{i+1}), & \text{if } \zeta^t = 1, \\ g^t_i + \frac{1}{n} \sum_{j=1}^n \xi^t_{i,j} (\nabla f_i(x^t_{i+1}) - \nabla f_i(x^t_i)) & \text{otherwise}. \end{cases}$
10: end parallel for
11: $S^t+1 = \text{AccGossip}(S^t + G^t+1 - G^t, K)$
12: end for
13: **Output:** $\begin{cases} x^\text{out}_i \sim \text{Uniform}\left(\{x^0_i, x^1_i, \ldots, x^{T-1}_i\}\right), & \text{general nonconvex,} \\ x^\text{out}_i = x^T_i, & \text{PL condition.} \end{cases}$

Algorithm 2 AccGossip($Y^0, W, K$)

1: **Initialize:** $Y^{-1} = Y^0$, $\eta_y = (1 - \sqrt{1 - \lambda^2(W)})/(1 + \sqrt{1 - \lambda^2(W)})$.
2: for $k = 0, 1, \ldots, K$ do
3: $Y^{k+1} = (1 + \eta_y) W Y^k - \eta_y Y^{k-1}$
4: end for
5: **Output:** $Y^K$.

3 The Algorithm and Main Results

We propose DEcentralized probAbilistic Recursive gradiEnt deScenT (DEAREST$^+$) in Algorithm 1, which is based on variance reduction [24, 42, 61, 78, 92, 93], gradient tracking [38, 55, 63, 65], and the multi-consensus step by Chebyshev acceleration (Algorithm 2) [7, 45]. The random variables $\zeta^t$ and $[\xi^t_1, \ldots, \xi^t_m] \sim \text{Multinomial}(b, q^t)$ in our algorithm are shared by all agents, which can be implemented by using the random number generators with the same seed [40, 46, 68].

The main difference between DEAREST$^+$ and existing decentralized stochastic non-convex optimization methods [39, 48, 51, 73, 81, 85] is that we use the multinomial distribution to mimic the mini-batch sampling on single machine. Specifically, the local
Gradient estimators in line 9 of Algorithm 1 indicate

\[ \frac{1}{m} \sum_{i=1}^{m} g_i^{t+1} = \frac{1}{m} \sum_{i=1}^{m} g_i^t + \frac{1}{b} \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{i,j}^t \left( \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t) \right) \]  

(14)

in the case of \( \zeta^t = 0 \). The multinomial distribution of \( (\xi_{1,1}^t, \ldots, \xi_{m,n}^t) \) means the right-hand side of equation (14) has the identical distribution to

\[ \frac{1}{m} \sum_{i=1}^{m} g_i^t + \frac{1}{b} \sum_{j=1}^{b} \left( \nabla f_{i,j}^{t+1}(x_i^{t+1}) - \nabla f_{i,j}^t(x_i^t) \right), \]  

(15)

where each \( (i,j) \) is uniformly and independently distributed on the index set

\[ \{(i,j) : i \in [m], j \in [n]\}. \]  

(16)

Furthermore, the steps of gradient tracking (line 11 of Algorithm 1) and multi-consensus (Algorithm 2) encourage \( g_i^{t+1}, g_i^t, x_i^{t+1}, \) and \( x_i^t \) to approach \( \bar{g}_{t+1}, \bar{g}_t, \bar{x}_{t+1}, \) and \( \bar{x}_t \) respectively, where \( \bar{g}_t = \frac{1}{m} \sum_{i=1}^{m} g_i^t, \bar{g}_{t+1} = \frac{1}{m} \sum_{i=1}^{m} g_i^{t+1}, \bar{x}_t = \frac{1}{m} \sum_{i=1}^{m} x_i^t \) and \( \bar{x}_{t+1} = \frac{1}{m} \sum_{i=1}^{m} x_i^{t+1} \). Therefore, equation (14) can be regarded as an approximation of

\[ g_i^{t+1} = g_i^t + \frac{1}{b} \sum_{j=1}^{b} \left( \nabla f_{i,j}^{t+1}(x_i^{t+1}) - \nabla f_{i,j}^t(x_i^t) \right), \]  

(17)

which follows the form of the well-known stochastic gradient recursive estimator for the finite-sum problem with \( mn \) individual functions on single machine \([24, 42, 61, 78, 92, 93]\). Additionally, our sampling strategy does not fix the mini-batch size on each agent, which allows the algorithm obtain the LIFO complexity bounds with respect to the global mean-squared smoothness parameter \( \bar{L} \) defined in Assumption 5.

Compared with existing stochastic recursive gradient methods \([24, 39, 42, 51, 61, 73, 78, 81, 89, 92, 93]\), DEAREST \( + \) iterates with the larger mini-batch size and the higher probability to access the exact gradient by considering the difference between the global smoothness parameter \( L \) and the mean-squared smoothness parameter \( \bar{L} \). That is, we take

\[ b = \left\lceil \sqrt{mnL} \right\rceil \quad \text{and} \quad p = \min \left\{ \frac{\bar{L}}{\sqrt{mnL}}, 1 \right\}. \]  

(18)

for Algorithm 1. As a comparison, the stochastic Probabilistic Gradient Estimator (PAGE) method \([42]\) set \( b = \Theta(\sqrt{mn}) \) and \( p = \Theta(1/\sqrt{mn}) \). Intuitively, the larger batch-size and the higher probability for exact gradient computation are potential to reduce the communication rounds of the stochastic distributed algorithms but possibly increase the computational cost. Fortunately, our analysis show that the setting of equation (18) achieves the near-optimal complexity bounds for communication and computation with respect to both \( L \) and \( \bar{L} \).
We present the communication rounds and the computation rounds for finding $\epsilon$-stationary points in the general nonconvex setting, where $\Delta = f(\bar{x}^0) - f^*$ and $\bar{x}^0 \in \mathbb{R}^d$ is the initial point of the algorithm. The design of GT-PAGE focuses on the time-varying network [51]. For the static network in our setting, the communication rounds of GT-PAGE can be improved to $O(L\ell\Delta/\sqrt{\gamma\epsilon^2})$ by introducing the steps of Chebyshev acceleration [7].

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Methods & #Communication & #Computation & Reference \\
\hline
D-GET & $O\left(\frac{\sqrt{\gamma}L_{\max}\Delta}{\gamma\epsilon^2}\right)$ & $O\left(n + \frac{\sqrt{\gamma}L_{\max}\Delta}{\gamma^2\epsilon^2}\right)$ & Sun et al. [73] \\
GT-SARAH & $O\left(\frac{L\ell\Delta}{\gamma\epsilon^2}\right)$ & $O\left(n + \frac{m^{1/3}}{n^{1/3}\gamma} + \sqrt{\frac{m}{n}}L\ell\Delta}{\epsilon^2}\right)$ & Xin et al. [81] \\
EDSGD & $O\left(\frac{L_{\max}\Delta}{\gamma\epsilon^2}\right)$ & $O\left(n + \frac{\sqrt{m}L_{\max}\Delta}{\epsilon^2}\right)$ & Zhan et al. [89] \\
DESTRESS & $O\left(\frac{\sqrt{mn} + L_{\max}\Delta}{\sqrt{m}\epsilon^2}\right)$ & $O\left(n + \frac{\sqrt{mn}L_{\max}\Delta}{\epsilon^2}\right)$ & Li et al. [39] \\
†GT-PAGE & $O\left(\frac{L\ell\Delta}{\gamma\epsilon^2}\right)$ & $O\left(n + \frac{\sqrt{L_{\ell}\Delta}}{\epsilon^2}\right)$ & Metelev et al. [51] \\
DEAREST$^+$ & $O\left(\frac{L\Delta}{\sqrt{\gamma}\epsilon^2}\right)$ & $O\left(n + \frac{(L + \min(nL, \sqrt{n/m}\Delta))\Delta}{\epsilon^2}\right)$ & Corollary 1 \\
\hline
Lower Bound & $\Omega\left(\frac{L\Delta}{\sqrt{\gamma}\epsilon^2}\right)$ & $\Omega\left(n + \frac{(L + \min(nL, \sqrt{n/m}\Delta))\Delta}{\epsilon^2}\right)$ & Theorems 2 and 4 \\
\hline
\end{tabular}
\end{table}

We formally describe our main theoretical result in Theorem 1, and the detailed proofs for results in this section are provided in Section 4.

**Theorem 1.** Under Assumptions 1–3, and 8 with $L \leq \bar{L}$, we run Algorithm 1 with

$$
\eta = \frac{1}{8L}, \quad b = \left\lceil \frac{L\sqrt{mn}}{L} \right\rceil, \quad p = \min\left\{ \frac{L}{\sqrt{mn}L}, 1 \right\}, \quad T = \left\lceil \frac{33L\Delta}{\epsilon^2} \right\rceil, \\
K = O\left(\frac{\ln(mn\bar{L}/L)}{\sqrt{\gamma}}\right) \quad \text{and} \quad \hat{K} = O\left(\frac{\ln(mn\bar{L}/(L\epsilon))}{\sqrt{\gamma}}\right),
$$

where $\Delta = f(\bar{x}^0) - f^*$. Then the output satisfies $E\|\nabla f(x_{out}^i)\| \leq \epsilon$ for all $i \in [m]$.

Based on Theorem 1, we can directly obtain the the communication rounds and the LIFO complexity to achieve the desired approximate stationary points. However, the setting of $[\xi_{1,1}^i, \ldots, \xi_{m,n}^i]^{\top} \sim \text{Multinomial}(b, q1)$ means the agents may access different numbers of LIFO in each iteration of DEAREST$^+$. Therefore, the computation rounds and the LIFO complexity should be analyzed separately. We can bound the expectation of the computation rounds by using the concentration inequality [46, 53]. Finally, we achieve three kinds of upper complexity bounds for DEAREST$^+$ as follows.
Table 2 We present the number of LIFO calls for finding $\epsilon$-stationary points in the general nonconvex setting, where $\Delta = f(\bar{x}_0) - f^*$ and $\bar{x}_0 \in \mathbb{R}^d$ is the initial point of the algorithm.

| Methods     | #LIFO                                      | Reference               |
|-------------|--------------------------------------------|-------------------------|
| D-GET       | $\mathcal{O}\left( mn + \frac{mL_{\text{max}}\Delta}{\gamma^3\epsilon^2} \right)$ | Sun et al. [73]         |
| GT-SARAH    | $\mathcal{O}\left( mn + \left( \frac{n}{\gamma^3} + \frac{m^{1/3}n^{2/3}}{\gamma} + \frac{\sqrt{mn}}{\epsilon^2} \right) L_{\ell}\Delta \right)$ | Xin et al. [81]         |
| EDSGD       | $\mathcal{O}\left( mn + \frac{\sqrt{mn}L_{\text{max}}\Delta}{\epsilon^2} \right)$ | Zhan et al. [89]        |
| DESTRESS    | $\mathcal{O}\left( mn + \frac{\sqrt{mn}L_{\text{max}}\Delta}{\epsilon^2} \right)$ | Li et al. [39]          |
| GT-PAGE     | $\mathcal{O}\left( mn + \frac{m\sqrt{n}L_{\ell}\Delta}{\epsilon^2} \right)$ | Metelev et al. [51]     |
| DEAREST$^+$ | $\mathcal{O}\left( mn + \frac{\min\{mnL, \sqrt{mnL}\} \Delta}{\epsilon^2} \right)$ | Corollary 1              |

Lower Bound $\Omega\left( mn + \frac{\min\{mnL, \sqrt{mnL}\} \Delta}{\epsilon^2} \right)$ Theorem 3

**Corollary 1.** Under the assumptions and settings of Theorem 1, Algorithm 1 can find an expected $\epsilon$-stationary point at every agent with the communication rounds of

$$\mathcal{O}\left( \frac{L\Delta}{\sqrt{\gamma}\epsilon^2} \right),$$

the expected LIFO complexity of

$$\mathcal{O}\left( mn + \frac{\min\{mnL, \sqrt{mnL}\} \Delta}{\epsilon^2} \right),$$

and the expected computation rounds of

$$\mathcal{O}\left( n + \frac{(L + \min\{nL, \sqrt{n/mL}\}) \Delta}{\epsilon^2} \right).$$

We compare our results with related work in Table 1 and 2, where we use the notation $\mathcal{O}(\cdot)$ to hide the logarithmic term with respect to $L$, $L$, $m$, and $n$. Proposition 2 indicates all of our complexity bounds are sharper than the ones in existing work.

**Remark 4.** In the case of $\bar{L} / L < \sqrt{\frac{m}{n}}$, the expected mini-batch size $b = \lfloor L\sqrt{mn}/L \rfloor$ is smaller than the number of agents $m$, which leads to several agents may not perform the LIFO calls in a iteration. The relation between $\bar{L}$ and $L$ in this case also implies the term $L$ before $\min\{nL, \sqrt{n/mL}\}$ in equation (20) cannot be omitted. Hence, the upper bounds on the computation rounds shown in equation (20) is not always proportion to the LIFO complexity shown in equation (19).
Remark 5. The very recent proposed GT-PAGE [51] considers the time-varying network and takes the local mini-batch size $b_\ell = \Theta(\sqrt{n\bar{L}_\ell/L_\ell})$ on every agent (corresponds to mini-batch size $\Theta(m\sqrt{n\bar{L}_\ell}/L_\ell)$ in total), which ignores the naturally existing global smoothness parameters in the problem. In contrast, the mini-batch size $b = \lceil L\sqrt{n\bar{m}/L} \rceil$ used in DEAREST$^+$ is with respect to all agents, which fully leverages the global properties of the objective function to achieve the sharper dependency on smoothness parameters and the number of agents (see Tables 1 and 2).

Remark 6. In the case of $m = 1$ (single machine scenario), Corollary 1 indicates the IFO complexity of $O(n + \min\{nL, \sqrt{nL}\} \Delta \epsilon^{-2})$. Recall that the IFO complexity of vanilla gradient descent [58] and stochastic recursive gradient methods [16, 24, 42, 59, 60] are $O(nL\Delta \epsilon^{-2})$ and $O(n + \sqrt{nL} \Delta \epsilon^{-2})$, respectively. Hence, vanilla gradient descent has the sharper IFO complexity bound than stochastic recursive gradient methods in the case of $L > \sqrt{nL}$. Furthermore, Proposition 1(c) implies the complexity of stochastic recursive gradient methods can be arbitrary more expensive than vanilla gradient descent. In other words, the optimality of the stochastic recursive gradient methods [24, 92] no longer holds if we distinguish between the global smoothness parameter $L$ and the mean-square smoothness parameter $\bar{L}$.

4 The Complexity Analysis for DEAREST$^+$

We start the complexity analysis of DEAREST$^+$ (Algorithm 1) by introduce the following quantities:

- the global gradient estimation error
  \[ U_t \triangleq \left\| \frac{1}{m} \sum_{i=1}^{m} (g_i^t - \nabla f_i(x_i^t)) \right\|^2, \]  
  \[ (21) \]

- the local gradient estimation error
  \[ V_t \triangleq \frac{1}{m} \left\| G_t - \nabla F(X_t) \right\|^2 = \frac{1}{m} \sum_{i=1}^{m} \left\| g_i^t - \nabla f(x_i^t) \right\|, \]  
  \[ (22) \]

- the consensus error
  \[ C_t \triangleq \left\| X_t - \frac{1}{\bar{n}} \bar{x}_t \right\|^2 + \eta^2 \left\| S_t - \bar{s}_t \right\|^2. \]  
  \[ (23) \]

We then define the Lyapunov function
\[ \Phi_t \triangleq f(\bar{x}_t) - f^* + \frac{2\eta}{p} U_t + \frac{\eta}{m^3n^3bp} V_t + \frac{132m^2n^2\bar{L}\eta}{p} C_t. \]  
\[ (24) \]

The remains of this section is organized as follows. In Section 4.1, we provide some basic lemmas. In Section 4.2, we establish the recursion for the Lyapunov function. In Sections 4.3 and 4.4, we formally prove Theorem 1 and Corollary 1, respectively.
4.1 Some Basic Lemmas

We provide some basic lemma for the later analysis of Algorithm 1.

**Lemma 1 ([83, Lemma 3]).** For any $Z \in \mathbb{R}^{m \times d}$, we have
\[
\|Z - 1\bar{z}\| \leq \|Z\| \quad \text{where} \quad \bar{z} = \frac{1}{m} 1^T Z. \tag{25}
\]

**Lemma 2.** For Algorithm 1, we have
\[
\bar{s}^t = \bar{g}^t. \tag{26}
\]

**Proof.** We use induction to prove this lemma by following the analysis in decentralized convex optimization [83, Lemma 2]. The steps in lines 2 and 3 of Algorithm 1 and Proposition 3 means $\bar{s}^0 = \bar{g}^0$. Suppose that we have $\bar{s}^t = \bar{g}^t$, then it holds
\[
\bar{s}^{t+1} = \frac{1}{m} 1^T \text{AccGossip}(S^t + G^{t+1} - G^t, K) \tag{27}
\]
\[
= \bar{s}^t + \bar{g}^{t+1} - \bar{g}^t = \bar{g}^{t+1},
\]
where the first equality is based on line 11 of Algorithm 1; the second equality is based on Proposition 3; the third equality is based on inductive hypothesis $\bar{s}^t = \bar{g}^t$. \hfill \square

The multi-consensus steps of Algorithm 2 holds the following proposition.

**Proposition 3 ([83, Proposition 1]).** Under Assumption 8, Algorithm 2 holds that
\[
\frac{1}{m} 1^T Y^{(K)} = \bar{y}^0 \tag{27}
\]
and
\[
\|Y^{(K)} - 1\bar{y}^0\| \leq c_1 (1 - c_2 \sqrt{\gamma})^K \|Y^0 - 1\bar{y}^0\|, \tag{28}
\]
where $\bar{y}^0 = \frac{1}{m} 1^T Y^0$, $c_1 = \sqrt{\gamma}$, and $c_2 = 1 - 1/\sqrt{2}$.

We describe the decease of function value as follows.

**Lemma 3 (Lemma 2 of Li et al. [42]).** Under Assumption 2, the update
\[
x^+ = x - \eta s
\]
for all $x, s \in \mathbb{R}^d$ and $\eta > 0$ holds that
\[
f(x^+) \leq f(x) - \frac{\eta}{2} \|\nabla f(x)\|^2 - \left(1 - \frac{L}{2\eta}\right)\|x^+ - x\|^2 + \frac{\eta}{2} \|s - \nabla f(x)\|^2. \tag{29}
\]

\(^{1}\)Li et al. [42] assume the individual functions are mean-squared smooth. In fact, all steps in the proof of this lemma still hold even if we only impose the global smoothness assumption (Assumption 2).
Lemma 4. Algorithm 1 holds that
\[
    f(\bar{x}^{t+1}) \leq f(\bar{x}^t) - \frac{\eta}{2} \|\nabla f(\bar{x}^t)\|^2 + \eta U^t + \eta n\bar{L}^2 \eta C^t - \left(\frac{1}{2\eta} - \frac{L}{2}\right) \|\bar{x}^{t+1} - \bar{x}^t\|^2. \quad (30)
\]

Proof. According to Proposition 3 and line 11 of Algorithm 1, we have
\[
    \bar{x}^{t+1} = \frac{1}{m} \sum_{i=1}^{m} g_i^t - \frac{\eta}{2} \|\nabla f(\bar{x}^t)\|^2.
\]

(27)

(26)

(25)

where the last step is based on Lemma 2.

According to Lemma 3 with \(x = \bar{x}^t, x^+ = \bar{x}^{t+1}, \) and \(s = \bar{s}^t, \) we have
\[
    f(\bar{x}^{t+1}) \leq f(\bar{x}^t) - \frac{\eta}{2} \|\nabla f(\bar{x}^t)\|^2 - \left(\frac{1}{2\eta} - \frac{L}{2}\right) \|\bar{x}^{t+1} - \bar{x}^t\|^2 + \frac{\eta}{2} \|\bar{s}^t - \nabla f(\bar{x}^t)\|^2. \quad (32)
\]

We bound the last term of equation (32) as follows
\[
    \|\bar{s}^t - \nabla f(\bar{x}^t)\|^2 \leq \frac{1}{m} \sum_{i=1}^{m} (g_i^t - \nabla f_i(\bar{x}^t))^2 \leq 2 \frac{1}{m} \sum_{i=1}^{m} (g_i^t - \nabla f_i(\bar{x}^t))^2 + 2 \frac{1}{m} \sum_{i=1}^{m} (\nabla f_i(\bar{x}^t) - \nabla f_i(\bar{x}^t))^2
\]

(33)

\[
    \leq 2 \frac{1}{m} \sum_{i=1}^{m} (g_i^t - \nabla f_i(\bar{x}^t))^2 + 2 \frac{2}{m} \sum_{i=1}^{m} \|\nabla f_i(\bar{x}^t) - \nabla f_i(\bar{x}^t)\|^2.
\]

\[
    = 2U^t + 2n\bar{L}^2 C^t,
\]

where the equality is based on Lemma 2; the first inequality uses Young’s inequality; the second inequality uses the fact \(\frac{1}{m} \sum_{i=1}^{m} a_i\|^2 \leq \frac{1}{m} \sum_{i=1}^{m} \|a_i\|^2\) for all \(a_1, \ldots, a_m \in \mathbb{R}^d;\)
the third inequality is based on Proposition 1(a) which implies each \( f_i \) is \( \sqrt{mnL} \)-smooth; the last step is based on the definitions of \( U^t \) and \( C^t \) as equations (21) and (23).

We finish the proof by combining the results of (32) and (33).

\[ \square \]

### 4.2 The Recursion for the Lyapunov Function

Following the notations of Algorithm 1 and Proposition 3, we define

\[
\rho \triangleq c_1 (1 - c_2 \sqrt{\gamma}) K < 1 \quad \text{and} \quad \hat{\rho} \triangleq c_1 (1 - c_2 \sqrt{\gamma}) \hat{K} < 1 \tag{34}
\]

to characterize the convergence of Algorithm 2, where the inequalities \( \rho < 1 \) and \( \hat{\rho} < 1 \) can be guaranteed by the settings of \( K \) and \( \hat{K} \) in Theorem 1 and we present their detailed expressions in the proof of Corollary 1 (Section 4.3).

We then provide recursions for quantities \( C^t \), \( U^t \), and \( V^t \) in following lemmas.

**Lemma 5.** Under the setting of Theorem 1, we have

\[ E[C^{t+1}] \leq 2\rho^2 (27m^3n^3 \bar{L}^2 \eta^2 + 2) E[C^t] + 4\rho^2 pmn \eta^2 E[V^t] + 18\rho^2 m^4n^3 \bar{L}^2 \eta^2 E \| \bar{x}^{t+1} - \bar{x}^t \|^2. \tag{36}\]

**Proof.** The update of \( x^{t+1} \) (line 6 of Algorithm 1) means

\[
\| x^{t+1} - 1 \bar{x}^{t+1} \| = \| \text{AccGossip}(X^t - \eta S^t, K^t) - \frac{1}{m} 11^\top \text{AccGossip}(X^t - \eta S^t, K^t) \| 
\leq \rho \| X^t - \eta S^t \| - \frac{1}{m} 11^\top (X^t - \eta S^t) \| 
\leq \rho \| X^t - \eta S^t - 1(\bar{x}^t - \eta \bar{s}^t) \| 
\leq \rho (\| X^t - 1\bar{x}^t \| + \eta \| S^t - 1\bar{s}^t \| )\tag{35}
\]

where the first inequality is based on Proposition 3 and the definition of \( \rho \); the last step is based on triangle inequality.

Consequently, we applying Young’s inequality and equation (35) to obtain

\[
\| x^{t+1} - 1 \bar{x}^{t+1} \|^2 \leq 2\rho^2 \| X^t - 1\bar{x}^t \|^2 + 2\rho^2 \eta^2 \| S^t - 1\bar{s}^t \|^2 \leq 2\rho^2 C^t. \tag{36}
\]
The update of $g^{t+1}$ (line 9 of Algorithm 1) means
\[ E \|g_i^{t+1} - g_i^t\|^2 = p E \|\nabla f_i(x_i^{t+1}) - g_i^t\|^2 \]
\[ + (1 - p) E \left\| \frac{1}{n} \sum_{j=1}^{n} \frac{\xi_{t,j}}{bq} (\nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t)) \right\|^2. \]  
(37)

For the first term on the right-hand side of equation (37), we have
\[ E \|\nabla f_i(x_i^{t+1}) - g_i^t\|^2 \]
\[ \leq 2E \|\nabla f_i(x_i^{t+1}) - \nabla f_i(x_i^t)\|^2 + 2E \|\nabla f_i(x_i^t) - g_i^t\|^2 \]  
(38)
\[ \leq 2mnL^2E \|x_i^{t+1} - x_i^t\|^2 + E \|\nabla f_i(x_i^t) - g_i^t\|^2, \]
where the first step is based on Young’s inequality; the second step is based on Proposition 1(a).

For the second term on the right-hand side of equation (37), we have
\[ E \left\| \frac{1}{n} \sum_{j=1}^{n} \frac{\xi_{t,j}}{bq} (\nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t)) \right\|^2 \]
\[ \leq \frac{1}{n} \sum_{j=1}^{n} E \left\| \frac{\xi_{t,j}}{bq} (\nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t)) \right\|^2 \]  
(39)
\[ \leq \frac{1}{n} \sum_{j=1}^{n} \frac{1}{q^2} E \|\nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t)\|^2 \]
\[ \leq \frac{1}{n} \sum_{j=1}^{n} \frac{mnL^2}{q^2} E \|x_i^{t+1} - x_i^t\|^2 \]  
(38)
\[ = \frac{m^3n^3L^2}{q^2} E \|x_i^{t+1} - x_i^t\|^2, \]
where the first inequality is based on the fact $\|\frac{1}{n} \sum_{j=1}^{n} a_i\|^2 \leq \frac{1}{n} \sum_{i=1}^{n} a_i^2$ for all $a_1, \ldots, a_n$; the second inequality is based on fact $\xi_{t,j} \leq b$; the last inequality is based on Proposition 1(a) and the setting $q = 1$.

Combining equations (37), (38), and (39), we achieve
\[ E \|g_i^{t+1} - g_i^t\|^2 \]
\[ \leq p(2mnL^2E \|x_i^{t+1} - x_i^t\|^2 + 2E \|\nabla f_i(x_i^t) - g_i^t\|^2) \]
\[ + (1 - p)m^3n^3L^2E \|x_i^{t+1} - x_i^t\|^2 \]  
(40)
\[ \leq 2pE \|\nabla f_i(x_i^t) - g_i^t\|^2 + 3m^3n^3L^2E \|x_i^{t+1} - x_i^t\|^2. \]
Summing equation (40) over $i = 1, \ldots, m$, we obtain
\[
\mathbb{E} \left\| G^{t+1} - G^t \right\|^2 \\
= \sum_{i=1}^{m} \mathbb{E} \left\| g_{i}^{t+1} - g_{i}^t \right\|^2 \\
\overset{(41)}{\leq} 2p \sum_{i=1}^{m} \mathbb{E} \left\| \nabla f_i(x_i^t) - g_{i}^t \right\|^2 + 3m^3n^3\hat{L}^2 \sum_{i=1}^{m} \mathbb{E} \left\| x_{i}^{t+1} - x_i^t \right\|^2 \\
\overset{(25)}{\leq} 2pm\mathbb{E}[V^t] + 3m^3n^3\hat{L}^2 \mathbb{E} \left\| x^{t+1} - x^t \right\|^2 \\
\overset{(34)}{\leq} 2pm\mathbb{E}[V^t] + 9m^3n^3\hat{L}^2 \mathbb{E} \left[ \left\| x^{t+1} - 1\bar{x}^{t+1} \right\|^2 + \left\| 1\bar{x}^{t+1} - 1\bar{x}^t \right\|^2 + \left\| X^t - 1\bar{x}^t \right\|^2 \right] \\
\overset{(35)}{\leq} 2pm\mathbb{E}[V^t] + 9m^3n^3\hat{L}^2 \mathbb{E} \left[ \left\| x^{t+1} - 1\bar{x}^t \right\|^2 + \left\| X^t - 1\bar{x}^t \right\|^2 \right] \\
\overset{(41)}{\leq} 2pm\mathbb{E}[V^t] + 27m^3n^3\hat{L}^2 \left\| X^t - 1\bar{x}^t \right\|^2 + 9m^3n^3\hat{L}^2 \eta^2 \left\| S^t - 1\bar{s}^t \right\|^2 + 9m^4n^3\hat{L}^2 \mathbb{E} \left\| \bar{x}^{t+1} - \bar{x}^t \right\|^2 ,
\]
where the second inequality is based on Young’s inequality; the third inequality uses the result of (35); the last inequality is based on the fact $\rho < 1$.

We also have
\[
\left\| S^{t+1} - 1\bar{s}^{t+1} \right\|^2 \\
= \left\| \text{AccGossip}(S^t + G^{t+1} - G^t, K) - \frac{1}{m} 11^T \text{AccGossip}(S^t + G^{t+1} - G^t, K) \right\|^2 \\
\overset{(28),(34)}{\leq} \rho^2 \left\| S^t + G^{t+1} - G^t - \frac{1}{m} 11^T (S^t + G^{t+1} - G^t) \right\|^2 \\
= 2\rho^2 \left\| S^t - 1\bar{s}^t \right\|^2 + 2\rho^2 \left\| G^{t+1} - G^t - \frac{1}{m} 11^T (G^{t+1} - G^t) \right\|^2 \\
\overset{(25)}{\leq} 2\rho^2 \left\| S^t - 1\bar{s}^t \right\|^2 + 2\rho^2 \left\| G^{t+1} - G^t \right\|^2 ,
\]
where the first inequality is based on Proposition 3 and the definition of $\rho$; the second inequality is based on Young’s inequality and the last step is based on Lemma 1.

Combining equations (41) and (42), we have
\[
\mathbb{E} \left\| S^{t+1} - 1\bar{s}^{t+1} \right\|^2 \\
\overset{(41),(42)}{\leq} 2\rho^2 (9m^3n^3\hat{L}^2 \eta^2 + 1) \mathbb{E} \left\| S^t - 1\bar{s}^t \right\|^2 + 4\rho^2 pm\mathbb{E}[V^t] \\
+ 54\rho^2 m^3n^3\hat{L}^2 \mathbb{E} \left\| X^t - 1\bar{x}^t \right\|^2 + 18\rho^2 m^4n^3\hat{L}^2 \mathbb{E} \left\| \bar{x}^{t+1} - \bar{x}^t \right\|^2 .
\]

(43)
Consequently, results of equations (36) and (43) indicate

\[ E[C^{t+1}] \]

\[ \leq 2\rho^2 E\|X^t - 1\bar{x}^t\|^2 + 2\rho^2 m^2 E\|S^t - 1\bar{s}^t\|^2 + 2\rho^2 (9m^3n^3\bar{L}^2 \eta^2 + 1)\|E\|S^t - 1\bar{s}^t\|^2 + 4\rho^2 mnq^2 E[V^t] = 2\rho^2 (27m^3n^3\bar{L}^2 \eta^2 + 1)E\|X^t - 1\bar{x}^t\|^2 + 2\rho^2 (9m^3n^3\bar{L}^2 \eta^2 + 2)\|E\|S^t - 1\bar{s}^t\|^2 + 4\rho^2 mnq^2 E[V^t] + 18\rho^2 m^4 n^3 \bar{L}^2 \eta^2 E\|\bar{x}^t + 1 - \bar{x}^t\|^2 \]

which finishes the proof.

**Lemma 6.** Under the setting of Theorem 1, we have

\[ E[U^{t+1}] \leq (1 - p)E[U^t] + 9(1 - p)m^2 n^3 \bar{L}^2 E[C^t] + \frac{3(1 - p)\bar{L}^2}{b}E \|\bar{x}^t + 1 - \bar{x}^t\|^2. \]

**Proof.** The setting \([\xi_{1,1}, \ldots, \xi_{m,n}]^\top \sim \text{Multinomial}(b, q1)\) with \(q = 1/(mn)\) implies that we have

\[ \xi_{i,j} \sim \text{Binomial}(b, q), \]

for all \(i \in [m]\) and \(j \in [n]\). This leads to

\[ E_{\xi_{i,j}} \left[ \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j}}{bq} \left( \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t) \right) \right] = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{E[\xi_{i,j}]}{bq} \left( \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t) \right) \]

\[ = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t) \right) \]

\[ = \frac{1}{m} \sum_{i=1}^{m} \left( \nabla f_i(x_i^{t+1}) - \nabla f_i(x_i^t) \right), \]

where we use the fact \(E[\xi_{i,j}] = bq\) for all \(i \in [m]\) and \(j \in [n]\).
Then the update of $g^{t+1}_i$ (line 9 of Algorithm 1) means

$$E[U^{t+1}]$$

$$= pE \left[ \frac{1}{m} \sum_{i=1}^{m} (\nabla f_i(x_i^{t+1}) - \nabla f_i(x_i^t)) \right]^2$$

$$+ (1 - p)E \left[ \frac{1}{m} \sum_{i=1}^{m} \left( g_i^t + \frac{1}{n} \sum_{j=1}^{n} \xi_{i,j}^t \left( \nabla f_i(x_i^{t+1}) - \nabla f_i(x_i^t) \right) \right)^2 \right]$$

$$\overset{(44)}{=} (1 - p)E \left[ \frac{1}{m} \sum_{i=1}^{m} (g_i^t - \nabla f_i(x_i^t))^2 \right]$$

$$+ (1 - p)E \left[ \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j}^t}{b_q} \left( \nabla f_i(x_i^{t+1}) - \nabla f_i(x_i^t) \right) \right]^2$$

$$\overset{(21)}{=} (1 - p)E[U^t]$$

$$+ (1 - p)E \left[ \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j}^t}{b_q} \left( \nabla f_i(x_i^{t+1}) - \nabla f_i(x_i^t) \right) - \frac{1}{m} \sum_{i=1}^{m} (\nabla f_i(x_i^{t+1}) - \nabla f_i(x_i^t)) \right]^2,$$

where the second step is due to the property of Martingale [24, Proposition 1] and equation (44); the last step is based on the definition of $U^t$.

We organize the terms in the second norm term in equation (45) as follows

$$= \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j}^t}{b_q} \left( \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t) \right)$$

$$- \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t) \right)$$

$$= \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j}^t}{b_q} \left( \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t) \right)$$

$$- \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t)$$

$$- \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t) \right)$$

$$- \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(x_i^t).$$
\[
\begin{align*}
&= \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j} t}{bq} - 1 \right) (\nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(\bar{x}_{i}^{t+1})) \\
&- \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j} t}{bq} - 1 \right) (\nabla f_{i,j}(x_{i}^{t}) - \nabla f_{i,j}(\bar{x}_{i}^{t})) \\
&+ \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{i,j} t (\nabla f_{i,j}(\bar{x}_{i}^{t+1}) - \nabla f_{i,j}(\bar{x}_{i}^{t})) \\
&- \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (\nabla f_{i,j}(\bar{x}_{i}^{t+1}) - \nabla f_{i,j}(\bar{x}_{i}^{t})).
\end{align*}
\]

Taking the square of norm on above equation, we achieve

\[
\begin{align*}
&\leq 3 \left\| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{i,j} t (\nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(\bar{x}_{i}^{t+1})) \right\|^2 \\
&+ 3 \left\| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{i,j} t (\nabla f_{i,j}(x_{i}^{t}) - \nabla f_{i,j}(\bar{x}_{i}^{t})) \right\|^2 \\
&+ 3 \left\| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{i,j} t (\nabla f_{i,j}(\bar{x}_{i}^{t+1}) - \nabla f_{i,j}(\bar{x}_{i}^{t})) \right\|^2 \\
&\leq 3 \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{\xi_{i,j} t}{bq} - 1 \right)^2 \left\| \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(\bar{x}_{i}^{t+1}) \right\|^2 \\
&+ 3 \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{\xi_{i,j} t}{bq} - 1 \right)^2 \left\| \nabla f_{i,j}(x_{i}^{t}) - \nabla f_{i,j}(\bar{x}_{i}^{t}) \right\|^2 \\
&+ 3 \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{\xi_{i,j} t}{bq} - 1 \right)^2 \left\| \nabla f_{i,j}(\bar{x}_{i}^{t+1}) - \nabla f_{i,j}(\bar{x}_{i}^{t}) \right\|^2 \\
&\overset{(46)}{=} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{i,j} t (\nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(\bar{x}_{i}^{t})) - \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (\nabla f_{i,j}(\bar{x}_{i}^{t+1}) - \nabla f_{i,j}(\bar{x}_{i}^{t})).
\end{align*}
\]

where the first step is based on Young’s inequality; the last step is based on the fact

\[
\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \leq \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \|a_{i,j}\|^2 \quad \text{for all} \quad a_{1,1}, \ldots, a_{m,n} \in \mathbb{R}^d.
\]
We bound the terms in the last step in equation (46) as follows:

- For the first term, we have

\[
\frac{3}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{\xi_{i,j}^t}{bq} - 1 \right)^2 \left\| \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(\bar{x}^{t+1}) \right\|^2 \leq \frac{3}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{q^2} \left\| \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(\bar{x}^{t+1}) \right\|^2 \leq \frac{3}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{\xi_{i,j}^t}{bq} - 1 \right)^2 \left\| \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(\bar{x}^{t+1}) \right\|^2 \leq \frac{3nL^2}{q^2} \| x^{t+1} - \bar{x}^{t+1} \|^2,
\]

where the first inequality is based on the fact \( \xi_{i,j}^t \in \{0, 1, \ldots, b\} \) that leads to

\[
\left| \frac{\xi_{i,j}^t}{bq} - 1 \right| \leq \max \left\{ \left| \frac{b}{bq} - 1 \right|, 1 \right\} \leq \left| \frac{1}{q} - 1 \right| + 1 = \frac{1}{q}; \quad (48)
\]

the second inequality is based on Proposition 1(a).

- For the second term, we follow the derivation of equation (47) to achieve

\[
\frac{3nL^2}{q^2} \| x^{t+1} - \bar{x}^{t+1} \|^2. \quad (49)
\]

- For the third term, the setting \( [\xi_{1,1}^t, \ldots, \xi_{m,n}^t]^\top \sim \text{Multinomial}(b, q) \) with parameter \( q = 1/(mn) \) implies we have

\[
\xi_{i,j}^t \sim \text{Binomial}(b, q),
\]

for all \( i \in [m] \) and \( j \in [n] \). This leads to

\[
\mathbb{E}[\xi_{i,j}^t] \left[ \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j}^t}{bq} (\nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(\bar{x}^{t+1})) \right] = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{E}[\xi_{i,j}^t] (\nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(\bar{x}^{t+1})) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \nabla f_{i,j}(x_i^{t+1}) - \nabla f_{i,j}(\bar{x}^{t+1}) \right),
\]

where we use the fact \( \mathbb{E}[\xi_{i,j}^t] = bq \) for all \( i \in [m] \) and \( j \in [n] \).
The setting $[\xi_{1,1}, \ldots, \xi_{m,n}]^{\top} \sim \text{Multinomial}(b, q_1)$ with parameter $q = 1/(mn)$ also implies the random vector

$$\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j}^t}{b q_1} (\nabla f_{i,j}(\bar{x}^{t+1}) - \nabla f_{i,j}(\bar{x}^{t}))$$

has the identical distribution to the random vector

$$\frac{1}{b} \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{i,j}^t (\nabla f_{i,j}(\bar{x}^{t+1}) - \nabla f_{i,j}(\bar{x}^{t}))$$

where each pair $(i^t_k, j^t_k)$ is independently and uniformly sampled from the set

$$\{(i, j) : i \in [m], j \in [n]\}.$$

In the view of equations (50), (51), and (52), we bound the variance of equation (51) as follows

$$E_{\xi_{i,j}} \left\| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j}^t}{b q_1} (\nabla f_{i,j}(\bar{x}^{t+1}) - \nabla f_{i,j}(\bar{x}^{t})) - \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (\nabla f_{i,j}(\bar{x}^{t+1}) - \nabla f_{i,j}(\bar{x}^{t})) \right\|^2$$

$$= E_{i_k, j_k} \left\| \frac{1}{b} \sum_{k=1}^{b} (\nabla f_{i_k,j_k}(\bar{x}^{t+1}) - \nabla f_{i_k,j_k}(\bar{x}^{t+1})) - \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (\nabla f_{i,j}(\bar{x}^{t+1}) - \nabla f_{i,j}(\bar{x}^{t})) \right\|^2$$

$$\leq \frac{1}{b} E_{i_k, j_k} \left\| \nabla f_{i_k,j_k}(\bar{x}^{t+1}) - \nabla f_{i_k,j_k}(\bar{x}^{t}) - \mathbb{E}[\nabla f_{i_k,j_k}(\bar{x}^{t+1}) - \nabla f_{i_k,j_k}(\bar{x}^{t})] \right\|^2$$

$$\leq \frac{1}{b} E_{i_k, j_k} \left\| \nabla f_{i_k,j_k}(\bar{x}^{t+1}) - \nabla f_{i_k,j_k}(\bar{x}^{t}) \right\|^2$$

$$(53)$$

where the first equality is based on the distributions of equation (51) and (52) are identical; the second equality is based on the definition of $(i_k, j_k)$; the first inequality is based on equations (50) and (51); the second inequality is based on Assumption 3.
Combining the equations (46), (47), (49), and (53), we have

\[
\mathbb{E}\left\| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\xi_{i,j}}{b q} \left( \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t}) \right) - \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \nabla f_{i,j}(x_{i+1}^{t}) - \nabla f_{i,j}(x_{i}^{t}) \right) \right\|^2
\]

\[
\leq \frac{3n\bar{L}^2}{q^2} \left( \| X^{t+1} - x^{t+1} \|^2 + 3n\bar{L}^2 \| X^t - x^t \|^2 + \frac{3L^2}{b} \| x^{t+1} - x^t \|^2 \right)
\]

\[
\leq \left[ \frac{6\rho^2 n\bar{L}^2}{q^2} \left( \| X^t - x^t \|^2 + \eta^2 \| S^t - 1s^t \|^2 \right) + \frac{3n\bar{L}^2}{q^2} \| X^t - x^t \|^2 + \frac{3L^2}{b} \| x^{t+1} - x^t \|^2 \right]
\]

\[
\leq \left[ \frac{9n\bar{L}^2}{q^2} C^t + \frac{3\bar{L}^2}{b} \| x^{t+1} - x^t \|^2 \right],
\]

(54)

where the second inequality is based on equation (36); the third inequality is based on the Young’s inequality; the last inequality is based on the definition of $C^t$ and the fact $p \leq 1$.

Combining equations (45) and (54), we have

\[
\mathbb{E}[U^{t+1}] \leq (1 - p)\mathbb{E}[U^t] + \frac{9(1-p)n\bar{L}^2}{q^2} \mathbb{E}[C^t] + \frac{3(1-p)\bar{L}^2}{b} \mathbb{E}\| x^{t+1} - x^t \|^2
\]

\[
\leq (1 - p)\mathbb{E}[U^t] + 9(1-p)n^2m^3\bar{L}^2\mathbb{E}[C^t] + \frac{3(1-p)\bar{L}^2}{b} \mathbb{E}\| x^{t+1} - x^t \|^2,
\]

where the last step is based on the setting $q = 1/(mn)$. Therefore, we finish the proof.

**Lemma 7.** Under the setting of Theorem 1, we have

\[
\mathbb{E}[V^{t+1}] \leq (1 - p)\mathbb{E}[V^t] + 36(1-p)n^2m^3\bar{L}^2\mathbb{E}[C^t] + 9(1-p)n^3m^3\bar{L}^2\mathbb{E}\| x^{t+1} - x^t \|^2,
\]

Proof. The setting of $[\xi_{1}^{t}, \ldots, \xi_{m,n}^{t}] \sim \text{Multinomial}(b, q1)$ with $q = 1/(mn)$ implies

\[
\xi_{i,j}^t \sim \text{Binomial}(b, q),
\]

for all $i \in [m]$ and $j \in [n]$. This leads to

\[
\mathbb{E}[\xi_{i,j}^t] \left[ \frac{1}{n} \sum_{j=1}^{n} \frac{\xi_{i,j}^t}{bq} \left( \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t}) \right) \right]
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[\xi_{i,j}^t] \left( \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t}) \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t}) \right) = \nabla f_{i}(x_{i}^{t+1}) - \nabla f_{i}(x_{i}^{t}).
\]

(55)
The update of $g_{i}^{t+1}$ (line 9 of Algorithm 1) means

$$
\mathbb{E} \left\| g_{i}^{t+1} - \nabla f_{i}(x_{i}^{t+1}) \right\|^2
= p \mathbb{E} \left\| \nabla f_{i}(x_{i}^{t+1}) - \nabla f_{i}(x_{i}^{t}) \right\|^2
+ (1 - p) \mathbb{E} \left\| g_{i}^{t} + \frac{1}{n} \sum_{j=1}^{n} \xi_{i,j} \left( \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t}) \right) - \nabla f_{i}(x_{i}^{t+1}) \right\|^2
\overset{(55)}{=} (1 - p) \mathbb{E} \left\| g_{i}^{t} - \nabla f_{i}(x_{i}^{t}) \right\|^2
+ (1 - p) \mathbb{E} \left\| \frac{1}{n} \sum_{j=1}^{n} \xi_{i,j} \left( \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t}) \right) - \nabla f_{i}(x_{i}^{t+1}) \right\|^2,
$$

(56)

where the second equality uses the property of Martingale [24, Proposition 1] and equation (55).

Similar to the derivation of equation (46) in the proof of Lemma 6, we can bound the last term in equation (56) as

$$
\mathbb{E} \left\| \frac{1}{n} \sum_{j=1}^{n} \xi_{i,j} \left( \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t}) \right) - \nabla f_{i}(x_{i}^{t}) \right\|^2
\leq \frac{3}{n} \sum_{j=1}^{n} \left( \frac{\xi_{i,j}}{bq} - 1 \right)^2 \left\| \nabla f_{i,j}(x_{i}^{t}) - \nabla f_{i,j}(x_{i}^{t+1}) \right\|^2
+ \frac{3}{n} \sum_{j=1}^{n} \left( \frac{\xi_{i,j}}{bq} - 1 \right)^2 \left\| \nabla f_{i,j}(x_{i}^{t}) - \nabla f_{i,j}(x_{i}^{t}) \right\|^2
+ \frac{3}{n} \sum_{j=1}^{n} \xi_{i,j} \left( \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t}) \right) - \nabla f_{i}(x_{i}^{t}) \right\|^2
\overset{(57)}{=}
$$

Following the derivation of equations (47)-(49) in the proof of Lemma 6, we can bound the first two terms on the right-hand side of equation (57) as

$$
\frac{3}{n} \sum_{j=1}^{n} \left( \frac{\xi_{i,j}}{bq} - 1 \right)^2 \left\| \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t+1}) \right\|^2
\leq \frac{3}{n} \sum_{j=1}^{n} \frac{1}{q^2} \left\| \nabla f_{i,j}(x_{i}^{t+1}) - \nabla f_{i,j}(x_{i}^{t+1}) \right\|^2
\overset{(58)}{=}
$$
and
\[
\frac{3}{n} \sum_{j=1}^{n} \left( \frac{\xi_{i,j}}{bq} - 1 \right)^2 \| \nabla f_{i,j}(\bar{x}_1^t) - \nabla f_{i,j}(\bar{x}_t) \|^2 \leq \frac{3mnL^2}{q^2} \| \bar{x}_t^t - \bar{x}_t^t \|^2.
\] (59)

For the third term on the right-hand side of equation (57), we have
\[
3 \left\| \frac{1}{n} \sum_{j=1}^{n} \xi_{i,j} \left( \nabla f_{i,j}(\bar{x}_1^{t+1}) - \nabla f_{i,j}(\bar{x}_t^t) \right) - \frac{1}{n} \sum_{j=1}^{n} \left( \nabla f_{i,j}(\bar{x}_1^{t+1}) - \nabla f_{i,j}(\bar{x}_t^t) \right) \right\|^2
\]
\[
= \left\| \frac{1}{n} \sum_{j=1}^{n} \left( \frac{\xi_{i,j}}{bq} - 1 \right) \left( \nabla f_{i,j}(\bar{x}_1^{t+1}) - \nabla f_{i,j}(\bar{x}_t^t) \right) \right\|^2
\]
\[
\leq 3 \sum_{j=1}^{n} \left( \frac{\xi_{i,j}}{bq} - 1 \right)^2 \| \nabla f_{i,j}(\bar{x}_1^{t+1}) - \nabla f_{i,j}(\bar{x}_t^t) \|^2
\] (60)
\[
\leq \frac{3}{n} \sum_{j=1}^{n} \frac{1}{q^2} \| \nabla f_{i,j}(\bar{x}_1^{t+1}) - \nabla f_{i,j}(\bar{x}_t^t) \|^2
\]
\[
\leq \frac{3}{n} \sum_{j=1}^{n} \frac{mnL^2}{q^2} \| x_1^{t+1} - x_t^t \|^2 = \frac{3mnL^2}{q^2} \| x_1^{t+1} - x_t^t \|^2
\]
\[
\leq \frac{9mnL^2}{q^2} \left( \| x_1^{t+1} - \bar{x}_t^t \|^2 + \| x_1^{t+1} - \bar{x}_t^t \|^2 + \| \bar{x}_t^t - x_t^t \|^2 \right),
\]
where the first inequality is based on the fact \( \| \frac{1}{n} \sum_{j=1}^{n} a_j \|^2 \leq \frac{1}{n} \sum_{j=1}^{n} \| a_j \|^2 \) for all \( a_1, \ldots, a_n \in \mathbb{R}^d \); the second inequality is based on equation (48); the third inequality is based on Proposition 1(a); the last inequality is based on Young’s inequality.

Combining equations (57), (58), (59), and (60), we achieve
\[
\left\| \frac{1}{n} \sum_{j=1}^{n} \xi_{i,j} \left( \nabla f_{i,j}(\bar{x}_1^{t+1}) - \nabla f_{i,j}(\bar{x}_t^t) \right) - \frac{1}{n} \sum_{j=1}^{n} \left( \nabla f_{i,j}(\bar{x}_1^{t+1}) - \nabla f_{i,j}(\bar{x}_t^t) \right) \right\|^2
\]
\[
\leq \frac{3mnL^2}{q^2} \left( \| x_1^{t+1} - \bar{x}_t^t \|^2 + \| x_t^t - \bar{x}_t^t \|^2
\]
\[
+ 3 \| x_1^{t+1} - \bar{x}_t^t \|^2 + 3 \| \bar{x}_t^t - \bar{x}_t^t \|^2 + 3 \| \bar{x}_t^t - x_t^t \|^2 \right)
\]
\[
= \frac{3mnL^2}{q^2} \left( \| x_1^{t+1} - \bar{x}_t^t \|^2 + 4 \| x_t^t - \bar{x}_t^t \|^2 + 3 \| \bar{x}_t^t - x_t^t \|^2 \right).
\] (61)
Combining equations (56) and (61), we have
\[
\begin{align*}
   & \mathbb{E} \left[ \|g_i^{t+1} - \nabla f_i(x_i^{t+1})\|^2 \right] \\
\leq & \underbrace{(1 - p) \mathbb{E} \left[ \|g_i^t - \nabla f_i(x_i^t)\|^2 \right]}_{\text{First case, and the algorithm always iterates with the exact local gradient in second case}} \\
& + \frac{3(1 - p) \eta n L^2}{q^2} \mathbb{E} \left[ 4 \|x_i^{t+1} - \bar{x}^{t+1}\|^2 + 4 \|x_i^t - \bar{x}^t\|^2 + 3 \|\bar{x}^{t+1} - \bar{x}^t\|^2 \right] \\
\end{align*}
\]
(62)

Taking the average on equation (62) over \( i = 1, \ldots, m \), we obtain
\[
\begin{align*}
   & \mathbb{E}[V^{t+1}] \\
= & \frac{1}{m} \sum_{i=1}^{m} \mathbb{E} \left[ \|g_i^{t+1} - \nabla f_i(x_i^{t+1})\|^2 \right] \\
\leq & \frac{1}{m} \sum_{i=1}^{m} \mathbb{E} \left[ \|g_i^t - \nabla f_i(x_i^t)\|^2 \right] \\
& + \frac{3(1 - p) \eta n L^2}{q^2} \sum_{i=1}^{m} \mathbb{E} \left[ 4 \|x_i^{t+1} - \bar{x}^{t+1}\|^2 + 4 \|x_i^t - \bar{x}^t\|^2 + 3 \|\bar{x}^{t+1} - \bar{x}^t\|^2 \right] \\
\leq & (1 - p) \mathbb{E}[V^t] + \frac{3(1 - p) \eta n L^2}{q^2} \mathbb{E} \left[ 4 \|x^{t+1} - 1x^{t+1}\|^2 + 4 \|x^t - 1x^t\|^2 + 3m \mathbb{E} \|\bar{x}^{t+1} - \bar{x}^t\|^2 \right] \\
= & (1 - p) \mathbb{E}[V^t] + 3(1 - p)m^2 n^3 L^2 \mathbb{E} [8C^t + 4C^t + 3m \mathbb{E} \|\bar{x}^{t+1} - \bar{x}^t\|^2] \\
\leq & (1 - p) \mathbb{E}[V^t] + 36(1 - p)m^2 n^3 \eta L^2 \mathbb{E}[C^t] + 9(1 - p)m^3 n^3 \mathbb{E} \|\bar{x}^{t+1} - \bar{x}^t\|^2,
\end{align*}
\]
where the last inequality is based on equation (36) and the definition of \( C^t \). Therefore, we finish the proof.

4.3 The Proofs of Theorem 1

We prove Theorem 1 by considering the cases of \( \bar{L} < \sqrt{mn}L \) and \( \bar{L} \geq \sqrt{mn}L \), respectively. Note that DEAREST\textsuperscript{+} (Algorithm 1) only apply variance reduction in the first case, and the algorithm always iterates with the exact local gradient in second ones.

**Proof. Part I:** We first consider the case of \( \bar{L} < \sqrt{mn}L \). We scale the results of Lemmas 5–7 as
\[
\begin{align*}
   & \frac{2\eta}{p} \mathbb{E}[V^{t+1}] \leq \frac{2(1 - p)\eta}{p} \mathbb{E}[V^t] + \frac{18(1 - p)m^2 n^3 \eta L^2}{p} \mathbb{E}[C^t] + \frac{6(1 - p)\eta L^2}{bp} \mathbb{E} \|\bar{x}^{t+1} - \bar{x}^t\|^2 \\
   & \frac{\eta}{m^3 n^3 bp} \mathbb{E}[V^{t+1}] \leq \frac{(1 - p)\eta}{m^3 n^3 bp} \mathbb{E}[V^t] + \frac{36(1 - p)\eta L^2}{mbp} \mathbb{E}[C^t] + \frac{9(1 - p)\eta L^2}{bp} \mathbb{E} \|\bar{x}^{t+1} - \bar{x}^t\|^2
\end{align*}
\]
\[
\frac{132m^2 n^3 \hat{L} \eta}{p} \mathbb{E}[C^{t+1}] \leq \frac{264\rho^2 (27m^3 n^3 \hat{L}^2 \eta^2 + 2)m^2 n^3 \hat{L} \eta}{p} \mathbb{E}[C^t] + \frac{2376\rho^2 m^6 n^6 \hat{L} \eta^3}{p} \mathbb{E} \left\| \bar{x}^{t+1} - \bar{x}^t \right\|^2.
\]

Combining with Lemma 4, we have

\[
\mathbb{E} \Phi^{t+1}
\]

\[
\overset{(24)}{=} \mathbb{E} \left[ f(\bar{x}^{t+1}) - f^* + \frac{2n}{p} U^{t+1} + \frac{\eta}{m^4 n^2 b \rho} V^{t+1} + \frac{132m^2 n^3 \hat{L} \eta}{p} C^{t+1} \right]
\]

\[
\leq \mathbb{E} \left[ f(\bar{x}^t) - f^* + \frac{\eta}{2} \left\| \nabla f(\bar{x}^t) \right\|^2 \right] + \frac{\eta}{m^4 n^2 b \rho} \mathbb{E} \left[ U^t \right] + \frac{18(1-p)m^2 n^3 \hat{L} \eta}{p} \mathbb{E} [C^t] + \frac{6(1-p)\hat{L} \eta}{b \rho} \mathbb{E} \left\| \bar{x}^{t+1} - \bar{x}^t \right\|^2
\]

\[
+ \frac{2(1-p)\eta}{p} \mathbb{E} [V^t] + \frac{36(1-p)\hat{L} \eta}{p} \mathbb{E} [C^t] + \frac{9(1-p)\hat{L} \eta}{b \rho} \mathbb{E} \left\| \bar{x}^{t+1} - \bar{x}^t \right\|^2
\]

\[
+ \frac{264\rho^2 (27m^3 n^3 \hat{L}^2 \eta^2 + 2)m^2 n^3 \hat{L} \eta}{p} \mathbb{E} [C^t] + \frac{528\rho^2 m^3 n^3 \hat{L} \eta^3}{p} \mathbb{E} [V^t]
\]

\[
+ \frac{2376\rho^2 m^6 n^6 \hat{L} \eta^3}{p} \mathbb{E} \left\| \bar{x}^{t+1} - \bar{x}^t \right\|^2,
\]

where the first inequality is based on Lemma 4–7 and the second inequality is based on parameter settings shown in the statement of Theorem 1 and

\[
K = \left[ \frac{2 + \sqrt{2}}{2 \sqrt{r}} \ln \left( 14 \max \left\{ \frac{33m^{6.5} n^{6.5} \hat{L}^3}{4 \hat{L}^3}, \frac{33(27m^3 n^3 \hat{L}^2 + 128 \hat{L}^2)}{608 \hat{L}^2} \right\} \right) \right].
\]
Summing over equation (63) with $t = 0, \ldots, T - 1$, we have

$$E[\Phi^t] \leq E \left[ \Phi^0 - \frac{\eta}{2} \sum_{t=0}^{T-1} \|\nabla f(\bar{x}^t)\|^2 - \frac{m^2 n^3 L^2}{p} C^t \right].$$

This implies

$$E \left[ \frac{1}{T} \sum_{t=0}^{T-1} \|\nabla f(\bar{x}^t)\|^2 \right] \leq E \left[ \frac{2(\Phi^0 - \Phi^t)}{\eta T} - \frac{2m^2 n^3 L^2}{p T} \sum_{t=0}^{T-1} C^t \right].$$

(64)

For the first term on the right-hand side of equation (64), we have

$$\Phi^0 - \Phi^t \leq f(\bar{x}^0) - f^* + \frac{2\eta}{p} U_0 + \frac{\eta}{m^3 n^3 b p} V_0 + \frac{132m^2 n^3 \bar{L}^2 \eta}{p} C_0 \tag{24}$$

$$- \left( f(\bar{x}^1) - f^* + \frac{2\eta}{p} U^t + \frac{\eta}{m^3 n^3 b p} V^t + \frac{132m^2 n^3 \bar{L}^2 \eta}{p} C^t \right) \leq f(\bar{x}^0) + \frac{2\eta}{p} U_0 + \frac{\eta}{m^3 n^3 b p} V_0 + \frac{132m^2 n^3 \bar{L}^2 \eta}{p} C_0 - f^* \tag{5}$$

$$= f(\bar{x}^0) - f^* + \frac{132m^2 n^3 \bar{L}^2 \eta}{p} C_0 \tag{65}$$

$$\leq f(\bar{x}^0) - f^* + \frac{132m^2 n^3 \bar{L}^2 \eta^3}{p} \|S^0 - 1 s^0\|^2, \tag{23}$$

where the first inequality is based on the fact $U_t, V_t, C_t \geq 0$ and Assumption 1; the second equality is based on the fact $U^0 = V^0 = 0$; the second inequality is based on the definition of $C^t$.

For the second term on the right-hand side of equation (64), we have

$$\frac{2m^2 n^3 \bar{L}^2}{p T} \sum_{t=0}^{T-1} C^t \leq \frac{2m^2 n^3 \bar{L}^2}{p T} \sum_{t=0}^{T-1} \|X^t - 1 \bar{x}^t\|^2 \geq \frac{2L^2}{T} \sum_{t=0}^{T-1} \|X^t - 1 \bar{x}^t\|^2, \tag{66}$$

where the first inequality is based on the definition of $C^t$; the second inequality is based on the setting $p \leq 1$ and the assumption $\bar{L} \geq L$.

Combining equations (64), (65), and (66), we have

$$E \left[ \frac{1}{T} \sum_{t=0}^{T-1} \|\nabla f(\bar{x}^t)\|^2 \right] \leq \frac{2}{\eta T} \left( f(\bar{x}^0) - f^* + \frac{132m^2 n^3 \bar{L}^2 \eta^3}{p} \|S^0 - 1 s^0\|^2 \right) - \frac{2L^2}{T} \sum_{t=0}^{T-1} \|X^t - 1 \bar{x}^t\|^2. \tag{67}$$

(23)
Therefore, the output $x_{\text{out}}^i$ satisfies
\[ E \| \nabla f(x_{\text{out}}^i) \|^2 \]
\[ = \frac{1}{T} \sum_{t=0}^{T-1} E \| \nabla f(x_t^i) \|^2 \]
\[ \leq \frac{2}{T} \sum_{t=0}^{T-1} E \left[ \| \nabla f(\tilde{x}^t) \|^2 + \| \nabla f(x_t^i) - \nabla f(\tilde{x}^t) \|^2 \right] \]
\[ \leq \frac{2}{T} \sum_{t=0}^{T-1} E \| \nabla f(\tilde{x}^t) \|^2 + \frac{2L^2}{T} \sum_{t=0}^{T-1} E \| x_t^i - \bar{x}^t \|^2 \]
\[ \leq \frac{2}{T} \sum_{t=0}^{T-1} E \| \nabla f(\tilde{x}^t) \|^2 + \frac{2L^2}{T} \sum_{t=0}^{T-1} E \| X^t - 1\bar{x}^t \|^2 \]
\[ \leq \frac{4}{\eta T} \left( f(\tilde{x}^0) - f^* + 132m^2n^3\bar{L}^2\eta^2 \| S^0 - 1S^0 \|^2 \right) + \frac{4L^2}{T} \sum_{t=0}^{T-1} \| X^t - 1\bar{x}^t \|^2 \]
\[ + \frac{2L^2}{T} \sum_{t=0}^{T-1} E \| X^t - 1\bar{x}^t \|^2 \]
\[ \leq \frac{32}{33} \epsilon^2 + \frac{1}{33} \epsilon^2 = \epsilon^2, \]  
for all $i \in [m]$ where the first inequality is based on Young’s inequality; the second inequality is based on Assumption 2; the third inequality is based on equation (67); the last inequality is based on the parameter settings shown in the statement of Theorem 1 and taking
\[ \hat{K} = \left[ \frac{2 + \sqrt{2}}{2\sqrt{\eta}} \ln \left( 1 + \frac{1089m^2n^3\bar{L}^2}{4L(33)(f(\tilde{x}^0) - f^*) + \epsilon^2} \sum_{i=1}^{m} \| \nabla f_i(\tilde{x}^0) - \nabla f(\tilde{x}^0) \|^2 \right) \right]. \]

Finally, we apply Jensen’s inequality to obtain
\[ E \| \nabla f(x_{\text{out}}^i) \| \leq \sqrt{E \| \nabla f(x_{\text{out}}^i) \|^2} \leq \epsilon. \]  
\[ (69) \]

**Part II:** We then consider the case of $\bar{L} \geq \sqrt{mnL}$. Note that the setting $p = 1$ leads to Algorithm 1 always holds
\[ g_t^i = \nabla f_i(x_t^i), \]
which implies $U^t = V^t = 0$ for all $t$ and the Lyapunov can be simplified to
\[ \Phi^t \triangleq f(\bar{x}^t) - f^* + 132m^2n^3\bar{L}^2\eta C^t. \]  
\[ (70) \]
Similar to the derivation of equation (63), we have

\[ E[\Phi^{t+1}] \]

\[ \leq E \left[ f(\tilde{x}^{t+1}) - f^* + 132m^2n^3L^2\eta C^{t+1} \right] \]

\[ \leq E \left[ f(\tilde{x}^t) - f^* - \frac{\eta}{2} \| \nabla f(\tilde{x}^t) \|^2 + nL^2\eta C^t - \left( \frac{L}{2} - \frac{L}{2} \right) \| \tilde{x}^{t+1} - \tilde{x}^t \|^2 \right. \]

\[ + 264\rho^2(27m^3n^3L^2\eta^2 + 2)m^2n^3L^2\eta E[\mathcal{C}^t] + 2376\rho^2m^6n^6L^4\eta^3E \| \tilde{x}^{t+1} - \tilde{x}^t \|^2 \]

\[ = E \left[ f(\tilde{x}^t) - f^* - \frac{\eta}{2} \| \nabla f(\tilde{x}^t) \|^2 \right. \]

\[ + (nL^2\eta + 264\rho^2(27m^3n^3L^2\eta^2 + 2)m^2n^3L^2\eta) E[\mathcal{C}^t] \]

\[ - \left( \frac{1}{2\eta} - \frac{L}{2} - 2376\rho^2m^6n^6L^4\eta^3 \right) E \| \tilde{x}^{t+1} - \tilde{x}^t \|^2 \]

\[ \leq E \left[ f(\tilde{x}^t) - f^* - \frac{\eta}{2} \| \nabla f(\tilde{x}^t) \|^2 + (132m^2n^3L^2\eta - m^2n^3L^2\eta) \mathcal{C}^t \right] \]

\[ = E \left[ \Phi^t - \frac{\eta}{2} \| \nabla f(\tilde{x}^t) \|^2 - m^2n^3L^2\eta \mathcal{C}^t \right], \] (71)

where the last inequality is based on the parameter settings shown in the statement of Theorem 1 and taking

\[ K = \left\lceil \frac{2 + \sqrt{2}}{2\sqrt{\gamma}} \ln \left( 14 \max \left\{ \frac{33(27m^3n^3L^2 + 128L^2)}{1040L^2}, \frac{297m^6.5n^6.5L^3}{112L^3} \right\} \right) \right\rceil. \]

We then follow the derivation of equations (64)–(69) to achieve

\[ E \| \nabla f(\tilde{x}_{\text{out}}^i) \| \leq \epsilon, \]

for all \( i \in [m] \).

**Remark 7.** It is worth noting that Theorem 1 guarantees that each agent can obtain an \( \epsilon \)-stationary point \( x_{\text{out}}^i \) in expectation. We achieve this result by retaining the negative term \( -m^2n^3L^2\eta p^{-1}\mathcal{C}^t \) in the last line of equation (71), which is helpful to cancel the terms related to consensus error in the output \( \{x_{\text{out}}^i\}_{i=1}^m \). In contrast, the previous work only guarantee one of the agents can obtain an \( \epsilon \)-stationary point [39, 81], or the average of vectors from all agents is an \( \epsilon \)-stationary point [48, 51, 73, 85, 89].

### 4.4 The Proofs of Corollary 1

We prove Corollary 1 by considering the communication rounds, the local incremental first-oracle calls, and the computation rounds, respectively.
4.4.1 The Upper Bound on Communication Rounds

We upper bound the communication rounds as follows

\[ \hat{K} + KT = O \left( \frac{\ln(mnL/(Le))}{\sqrt{\gamma}} + \frac{\ln(mnL/L)}{\sqrt{\gamma}} \cdot \frac{L}{e^2} \right) = \tilde{O} \left( \frac{L}{\sqrt{\gamma}e^2} \right), \]

where we use the settings of \( \hat{K}, K, \) and \( T \) in Theorem 1.

4.4.2 The Upper Bound on LIFO Calls

In the case of \( \bar{L} \leq \sqrt{mnL} \), we upper bound the number of LIFO calls as follows

\[
\begin{align*}
& mn + (pm + (1-p)b)T \\
\leq & mn + (pmn + b)T \\
= & \mathcal{O} \left( mn + \left( \frac{L}{\sqrt{mnL}} \cdot mn + \left\lfloor \frac{L\sqrt{mnL}}{L} \right\rfloor \right) \frac{L}{e^2} \right) \\
= & \mathcal{O} \left( mn + \frac{\sqrt{mnL}}{e^2} \right),
\end{align*}
\]

where we use the settings of \( p, n, \) and \( T \) in Theorem 1. In the case of \( \bar{L} > \sqrt{mnL} \), the number of LIFO calls can be upper bounded by \( mnT = \mathcal{O}(mn + mnL \epsilon^{-2}) \).

Combining above two cases, the overall LIFO complexity of Algorithm 1 under the settings of Theorem 1 is no more than \( \mathcal{O}(mn + \min\{mnL, \sqrt{mnL}\} \epsilon^{-2}) \).

4.4.3 The Upper Bound on Computation Rounds

In the case of \( \bar{L} > \sqrt{mnL} \), the agents always perform the full gradient at each iteration, which means the number of computation round is no more than

\[ n + Tn = \mathcal{O} \left( n + \frac{nL}{e^2} \right). \tag{72} \]

The remains in this subsection considers the case of \( \bar{L} \leq \sqrt{mnL} \). The main issue is the case of \( \zeta_t = 0 \), leading to

\[ g_{t+1} = g_t + \frac{1}{n} \sum_{j=1}^{n} \xi_{t,j} \left( \nabla f_{i,j}(x_{t+1}^j) - \nabla f_{i,j}(x_t^j) \right). \tag{73} \]

The setting

\[ [\xi_{1,1}^t, \ldots, \xi_{m,n}^t] \sim \text{Multinomial} (b, q1) \]
with $q = 1/(mn)$ means the update (73) can be regarded as the procedure of sampling $b$ index pairs $(i^t_1, j^t_1), \ldots, (i^t_b, j^t_b)$ from the set

$$\{(i, j) : i \in [m], j \in [n]\}$$

independently and uniformly with replacement and performing the computation

$$g^t_{i+1} = g^t_i + \sum_{k=1}^{b} \frac{\tau^t_i(k)}{bq} (\nabla f_{i^t_k, j^t_k}(\mathbf{x}^t_{i^t_k}) - \nabla f_{i^t_k, j^t_k}(\mathbf{x}^t_{i^t_k}))$$
on each node, where we define

$$\tau^t_i(k) \triangleq 1(i_k = i)$$

for $i \in [m]$ and $k \in [b]$ to present whether the $k$-th index pair $(i^t_k, j^t_k)$ in iteration $t$ corresponds to the component function on node $i$.

Since there are $mn$ individual functions in total and each node has $n$ individual functions, the uniform distribution of $(i^t_k, j^t_k)$ and their independence imply

$$\tau^t_i(k) \sim \text{Bernoulli}(nq)$$

and $\tau^t_1(1), \ldots, \tau^t_b(b)$ are mutually independent. Therefore, the procedure of Algorithm 1 means the number of LIFO calls on node $i$ at the $t$-th iteration is no more than

$$2 \sum_{k=1}^{b} \tau^t_i(k).$$

We denote

$$Y^t_i \triangleq \sum_{k=1}^{b} \tau^t_i(k),$$

which holds that $Y^t_i \sim \text{Binomial}(b, nq)$ because of the distribution (75).

To analyze the expected computation rounds in the $t$-th iteration, we only need to upper bound the quantity

$$\mathbb{E} \left[ 2 \max_{i \in [m]} Y^t_i \right].$$

In a recent work, Liu et al. [46] analyzed the computation rounds of Katyusha-type methods [3, 34, 64, 71] in decentralized convex optimization by establishing the upper bound of the quantity in the form of (77) with the mutually independent binomial variables $Y^t_1, \ldots, Y^t_m$. Although the random variables $Y^t_1, \ldots, Y^t_m$ are not mutually independent in our setting, we can also follow the analysis of Liu et al. [46] to bound the quantity (77). For the completeness, we present the details as follows.
Lemma 8 (Theorem 4.1 of Motwani and Raghavan [53]). Suppose that the random variables $X_1, ..., X_b$ are independent and each $X_k$ is distributed to Bernoulli($p_k$) for some $p_k \in [0, 1]$. We let $Z = \sum_{k=1}^{b} X_k$ and $\nu = \mathbb{E}[Z]$. Then for any $\delta > 0$, it holds

$$
P(Z \geq (1 + \delta) \nu) \leq \left( \frac{\exp(\delta)}{(1 + \delta)^{1+\delta}} \right)^{\nu}.
$$

(78)

Based on Lemma 8, we further achieve the following upper bound for the sum of Bernoulli variables with high probability.

**Lemma 9.** Under the settings and notations of Lemma 8, For any constant $a \geq \exp(2)$, we have

$$
P \left( Z \geq 2e \max \{ \mathbb{E}[Z], (\ln a)^2 \} \right) \leq \frac{1}{a^2}.
$$

(79)

**Proof.** We denote $\nu = \mathbb{E}[Z] = \sum_{k=1}^{b} p_k$. We consider the upper bounds for different cases of $\nu$ as follows:

(a) In the case of $\nu \geq \ln a$, we apply Lemma 8 with $\delta = 2e - 1$ to achieve

$$
P(Z \geq 2e\nu) \leq \left( \frac{\exp(2e - 1)}{(2e)^{2e}} \right)^{\nu} = \left( \frac{1}{2^{2e}} \right)^{\nu} \leq \frac{1}{a^2},
$$

where the last step can be verified by taking logarithm on both sides and the fact $2 \leq \ln (2^{2e}) = 1 + 2e \ln 2$.

(b) If $1 \leq \nu \leq \ln a$, we have

$$
P \left( Z \geq 2e(\ln a)^2 \right) \leq P(Z \geq (2e \ln a)\nu).
$$

According to Lemma 8 with $\delta = 2e \ln a - 1$, we have

$$
P \left( Z \geq (2e \ln a)\nu \right) \leq \left( \frac{\exp(2e \ln a - 1)}{(2e \ln a)^{2e \ln a}} \right)^{\nu} = \left( \frac{1}{(2 \ln a)^{2e \ln a} \exp(2e \ln a)} \right)^{\nu} \leq \frac{1}{a^2},
$$

where the second inequality is because $a \geq e^2$ and the last inequality holds as $\nu \geq 1$. 

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(c) If $\nu < 1$, we let $X'_1 \sim \text{Bernoulli}(p'_1)$ with $p'_1 = 1 - \nu + \mathbb{E}[X_1]$ which is independent with $X_1, \ldots, X_b$, and denote $Z' = \sum_{j=2}^{b} X_j + X'_1$. It is clear that for any $c \geq 0$, it holds
\[
\mathbb{P}(Z \geq c) \leq \mathbb{P}(Z' \geq c). \tag{80}
\]
We also can verify that
\[
\mathbb{E}[Z'] = \mathbb{E} \left[ \sum_{j=2}^{b} X_j + X'_1 \right] = \sum_{j=2}^{b} p_j + 1 - \nu + p_1 = 1. \tag{81}
\]
Therefore, applying Lemma 8 on random variable $Z'$ with $\delta = 2e \ln a - 1$ leads to that
\[
\mathbb{P} (Z \geq (2e \ln a)) = \mathbb{P} (Z \geq (2e \ln a) \cdot \mathbb{E} [Z']) \leq \mathbb{P} (Z' \geq (2e \ln a) \cdot \mathbb{E} [Z']) \leq \left( \frac{\exp(2e \ln a - 1)}{(2e \ln a)^{2e \ln a}} \right)^{\mathbb{E}[Z']}
\]
\[
= \frac{\exp(2e \ln a - 1)}{(2e \ln a)^{2e \ln a}} \exp(2e \ln a) = \frac{1}{(2e \ln a)^{2e \ln a}} \leq \frac{1}{(2e \ln a)^{2e \ln a} e} \leq \frac{1}{a^2},
\]
where the second last inequality is because $a \geq e^2$. Combining all three cases, we obtain
\[
\mathbb{P} (Z \geq 2e \max \{\mathbb{E}[Z], (\ln a)^2\}) = \mathbb{P} (Z \geq \max \{2e \nu, 2e(\ln a)^2\}) \leq \frac{1}{a^2}
\]
for all $a \geq e^2$, which finishes the proof. \hfill \Box

Recall that it always holds $Y^t_i \leq b$ for all $i \in [m]$, then we can apply Lemma 9 to upper bound the expectation of $\max_{i \in [m]} Y^t_i$.

**Lemma 10.** Following the definitions in equations (74) and (76), we have
\[
\mathbb{E} \left[ \max_{i \in [m]} Y^t_i \right] \leq 4e \max \{bnq, (2 + \ln mb)^2\} + 2. \tag{82}
\]
Proof. Notice that the definition of $Y_i^t$ in equation (76) means it is a summation of Bernoulli random variables. Thus, we can apply Lemma 9 with $a = mb \exp(2)$ and $Z = Y_i^t$ for all $i \in [m]$ to achieve

$$
\mathbb{P}(Y_i^t \geq 2e \max \{b, (2 + \ln mb)^2\}) = \mathbb{P}(Y_i^t \geq 2e \max \{b, (2 + \ln mb)^2\}) \leq \frac{1}{mb^2 \exp(4)},
$$

(83)

where the first step is based on the fact $\mathbb{E}[Y_i^t] = bq$.

Then we apply Boole’s inequality to achieve

$$
\mathbb{P}\left(\max_{i \in [m]} Y_i^t \geq 2e \max \{b, (2 + \ln mb)^2\}\right)
= \mathbb{P}\left(\exists i \in [m] \text{ such that } Y_i^t \geq 2e \max \{b, (2 + \ln mb)^2\}\right)
\leq \sum_{i=1}^{m} \mathbb{P}(Y_i^t \geq 2e \max \{b, (2 + \ln mb)^2\})
\leq \sum_{i=1}^{m} \frac{1}{mb^2 \exp(4)} = \frac{1}{mb^2 \exp(4)}.
$$

(84)

Since we have $Y_i^t \leq b$, the conditional expectation formula implies

$$
\mathbb{E}\left[\max_{i \in [m]} Y_i^t\right]
= \mathbb{P}\left(\max_{i \in [m]} Y_i^t < 2e \max \{b, (2 + \ln mb)^2\}\right)
\cdot \mathbb{E}\left[\max_{i \in [m]} Y_i^t \middle| \max_{i \in [m]} Y_i^t < 2e \max \{b, (2 + \ln mb)^2\}\right]
+ \mathbb{P}\left(\max_{i \in [m]} Y_i^t \geq 2e \max \{b, (2 + \ln mb)^2\}\right)
\cdot \mathbb{E}\left[\max_{i \in [m]} Y_i^t \middle| \max_{i \in [m]} Y_i^t \geq 2e \max \{b, (2 + \ln mb)^2\}\right]
\leq \left(1 - \frac{1}{mb^2 \exp(4)}\right) \cdot 2e \max \{b, (2 + \ln mb)^2\} + \frac{1}{mb^2 \exp(4)} \cdot b
\leq 2e \max \{b, (2 + \ln mb)^2\} + 1
$$

Hence, we have

$$
2\mathbb{E}\left[\max_{i \in [m]} Y_i^t\right] \leq 4e \max \{b, (2 + \ln mb)^2\} + 2 = O\left(bq + (\ln mb)^2\right),
$$

which concludes the proof. \qed

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In the case of $\zeta_t = 0$, Lemma 10 means the expected number of computation rounds is no more than

$$2E \left[ \max_{i \in [m]} Y^t_i \right] \leq 4e \max \{bnq, (2 + \ln mb)^2\} + 2 = O \left( bnq + (\ln mb)^2 \right)$$

In the case of $\zeta_t = 1$, it is obviously that the number of computation rounds is $n$. Therefore, the overall expected number of computation rounds for Algorithm 1 in the case of $\bar{L} \leq \sqrt{mnL}$ is no more than

$$n + \left( (1 - p)E \left[ \sum_{t=1}^{T} \max_{i \in [m]} Y^t_i \right] + pm \right) T$$

$$\leq O \left( n + (pm + (1 - p)(bnq + (\ln mb)^2))T \right)$$

$$\leq O \left( n + (pm + bnq + (\ln mb)^2)T \right)$$

$$\leq O \left( n + \left( \frac{nL}{\sqrt{mnL}} + \frac{\bar{L}\sqrt{mn}}{mnL} \right) \left( \ln \frac{m\bar{L}}{L} \right)^2 L^{-2} \right)$$

$$= \tilde{O} \left( n + \left( \sqrt{\frac{n}{m}} \bar{L} + L \right) e^{-2} \right),$$

where we use the parameter settings of $b$, $p$, and $T$ in Theorem 1. Combining the upper bound (72) in the case of $\bar{L} \leq \sqrt{mnL}$, the expected number of computation rounds of our method can be upper bounded by

$$\min \left\{ O \left( n + nLe^{-2} \right), \tilde{O} \left( n + \left( \sqrt{\frac{n}{m}} \bar{L} + L \right) e^{-2} \right) \right\}$$

$$= \tilde{O} \left( n + \left( L + \min \left\{ nL, \sqrt{\frac{n}{m}} \bar{L} \right\} \right) e^{-2} \right).$$

Hence, we finish the proof of Corollary 1.

5 The Lower Bounds for General Nonconvex Case

This section provide the lower bounds for the communication rounds, the LIFO calls, and the computation rounds for finding the approximate first-order stationary point in decentralized smooth nonconvex finite-sum optimization problem by the algorithm class shown in Definition 1. Without the loss of generality, we always assume the algorithm iterates with initial point $\bar{x}^0 = \mathbf{0}$ in this section. Otherwise, we can take functions $\{f_{i,j}(x - \bar{x}^0)\}_{i,j=1}^{m,n}$ into considerations. Compared with most of existing lower bounds only consider one of smoothness parameters $[1, 8, 17, 18, 29, 48, 80, 85]$, we simultaneously address the global smoothness parameter $L$ and the mean-squared smoothness parameter $\bar{L}$ in each of our lower bounds.
For our later analysis, we introduce the nonconvex function \( f^{nc}_T : \mathbb{R}^T \rightarrow \mathbb{R} \) provided by Carmon et al. [17] as follows

\[
f^{nc}_T(x) := -\Psi(1)\Phi(x_1) + \sum_{k=2}^{T} (\Psi(-x_{k-1})\Phi(-x_k) - \Psi(x_{k-1})\Phi(x_k)),
\]

(85)

where \( x = [x_1, \ldots, x_T]^T \) and the component functions are defined as

\[
\Psi(x) \triangleq \begin{cases} 
0, & x \leq 1/2, \\
\exp \left(1 - \frac{1}{(2x - 1)^2}\right), & x > 1/2,
\end{cases}
\]

and \( \Phi(x) := \sqrt{e} \int_{-\infty}^{x} \exp \left(-\frac{1}{2} t^2\right) dt \).

We denote

\[
\text{prog}(x) \triangleq \max\{i \geq 0 : |x_i| > 0\},
\]

where \( x = [x_1, \ldots, x_T]^T \) and we let \( x_0 = 1 \). The functions \( \Phi(\cdot) \) and \( \Psi(\cdot) \) defined equation (86) have the following properties.

**Lemma 11** ([8, Observation 2]). The function value and (second-order) derivatives of \( \Phi \) and \( \Psi \) defined in equation (86) satisfy

\[
0 \leq \Psi \leq e, \quad 0 \leq \Psi' \leq \sqrt{54/e}, \quad |\Psi''| \leq 32.5,
\]

\[
0 \leq \Phi \leq \sqrt{2\pi e}, \quad 0 \leq \Phi' \leq \sqrt{e}, \quad \text{and} \quad |\Phi''| \leq 1.
\]

We can decompose the function \( f^{nc}_T \) defined in equation (85) as

\[
f^{nc}_T(x) = f^{nc,1}_T(x) + f^{nc,2}_T(x),
\]

(87)

where

\[
f^{nc,1}_T(x) = -\Psi(1)\Phi(x_1) + \sum_{k \text{ is odd; } 2 \leq k \leq T} (\Psi(-x_{k-1})\Phi(-x_k) - \Psi(x_{k-1})\Phi(x_k)),
\]

and

\[
f^{nc,2}_T(x) = \sum_{k \text{ is even; } 2 \leq k \leq T} (\Psi(-x_{k-1})\Phi(-x_k) - \Psi(x_{k-1})\Phi(x_k)).
\]

(88)

Lemma 11 implies the following properties for functions \( f^{nc,1}_T, f^{nc,2}_T \), and \( f^{nc}_T \).

**Lemma 12.** The functions \( f^{nc,1}_T \) and \( f^{nc,2}_T \) defined in equation (88) is \( l_0 \)-smooth with \( l_0 = 152 \).
Proof. For all $x \in \mathbb{R}^T$, we have
\[
\|\nabla^2 f^{nc,1}_{T}(x)\|_2 \leq \sqrt{\|\nabla^2 f^{nc,1}_{T}(x)\|_1 \|\nabla^2 f^{nc,1}_{T}(x)\|_\infty} = \|\nabla^2 f^{nc,1}_{T}(x)\|_\infty
\]
\[
= \max_{i \in [T]} \sum_{j=1}^{T} \|\nabla^2 f^{nc,1}_{T,J}(x)\| = \max_{i \in [T]} \sum_{j=1}^{T} |\nabla^2 f^{nc,1}_{T,J}(x)|
\]
\[
\leq \max_{i \in [T]} \|\nabla^2 f^{nc,1}_{T,J}(x)\| + \max_{i \in [T-1]} |\nabla^2 f^{nc,1}_{T,J}(x)| + \max_{i \in \{2, \ldots, T\}} |\nabla^2 f^{nc,1}_{T,J}(x)|
\]
\[
\leq \sup_{z \in \mathbb{R}} |\Phi''(z)| \sup_{z \in \mathbb{R}} |\Psi(z)| + \sup_{z \in \mathbb{R}} |\Phi'(z)| \sup_{z \in \mathbb{R}} |\Psi(z)| + 2 \sup_{z \in \mathbb{R}} |\Phi'(z)| \sup_{z \in \mathbb{R}} |\Psi'(z)|
\]
\[
\leq 152,
\]
where the first inequality is based on H"older’s inequality; the first equality is due to the Hessian being symmetric; the second equality is based on the definition of the infinity norm; the third equality is due to the definition of $f^{nc,1}_T$ in equation (88); the second inequality is based on the chain rule; the last step is based on Lemma 11. Therefore, the function $f^{nc,1}_T$ is 152-smooth. Similarly, we can prove the function $f^{nc,2}_T$ is also 152-smooth, which finishes the proof. \qed

Lemma 13 ([8, Lemma 2]). The function $f^{nc}_T$ defined in equation (85) satisfies:
(a) The function value of $f^{nc}_T$ holds $f^{nc}_T(0) - \inf_{y \in \mathbb{R}^T} f^{nc}_T(y) \leq 12T$.

(b) The gradient of $f^{nc}_T$ is $l_0$-Lipschitz continuous, where $l_0 = 152$.

(c) For all $x = [x_1, \ldots, x_T] \in \mathbb{R}^T$, we have $\text{prog}(\nabla f^{nc}_T(x)) \leq \text{prog}(x) + 1$.

(d) For all $x = [x_1, \ldots, x_T] \in \mathbb{R}^T$ such that $\text{supp}(x) \subseteq \{1, 2, \ldots, T - 1\}$, we have $\|\nabla f^{nc}_T(x)\| > 1$.

We also provide the following lemmas for our later analysis, extending the results of Lemmas 5.1 and 5.2 from Zhou and Gu [92].

Lemma 14. Suppose that the function $g : \mathbb{R}^m \rightarrow \mathbb{R}$ is $\tilde{L}$-smooth and has lower bound $g^* = \inf_{y \in \mathbb{R}^m} {g(y)}$, then the function $\tilde{g}(x) = \alpha g(\beta x)$ is $\alpha \beta \tilde{L}$-smooth and satisfies
\[
\|\nabla \tilde{g}(x)\| = \alpha \beta \|\nabla g(\beta x)\| \quad \text{and} \quad \tilde{g}(0) - \tilde{g}^* = \alpha (g(0) - g^*)
\]
for all $\alpha, \beta > 0$, where $\tilde{g}^* = \inf_{y \in \mathbb{R}^m} \tilde{g}(y)$. If we further suppose the function $g$ is $\tilde{\mu}$-$PL$, then the function $\tilde{g}$ is $\alpha \beta \tilde{\mu}$-$PL$.

Proof. For all $x, y \in \mathbb{R}^m$, the smoothness of $g : \mathbb{R}^m \rightarrow \mathbb{R}$ implies
\[
\|\nabla \tilde{g}(x) - \nabla \tilde{g}(y)\| = \alpha \beta \|\nabla g(\beta x) - \nabla g(\beta y)\| \leq \alpha \beta \tilde{L} \|\beta x - \beta y\| \leq \alpha \beta^2 \tilde{L} \|x - y\|
\]
which means $\tilde{g}$ is $\alpha \beta^2 \tilde{L}$-smooth.

We can verify that $\tilde{g}(0) = \alpha g(0)$ and $\tilde{g}^* = \alpha g^*$, which means
\[
\tilde{g}(0) - \tilde{g}^* = \alpha (g(0) - g^*).
\]

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For any $x \in \mathbb{R}^{mn}$, the PL condition of $g : \mathbb{R}^m \to \mathbb{R}$ implies
\[
\|\nabla \hat{g}(x)\|^2 = \alpha^2 \beta^2 \|\nabla g(\beta x)\|^2 \geq 2\alpha^2 \beta^2 \hat{\mu}(g(\beta x) - g^*) = 2\alpha^2 \beta^2 \hat{\mu}(x - \hat{g}^*),
\]
which means $\hat{g}$ is $\alpha^2 \beta^2 \hat{\mu}$-PL. Hence, we finish the proof.

Lemma 15. Given a function $g : \mathbb{R}^T \to \mathbb{R}$ that is $\hat{L}$-smooth, we define $f_i(x) = g(U^{(i)}x)$ for $x \in \mathbb{R}^{mT}$ and $i \in [m]$, where $U^{(i)} = [e_{(i-1)T+1}, \ldots, e_{iT}]^T \in \mathbb{R}^{T \times mT}$ and we denote $e_k = [0, \ldots, 1, \ldots, 0]^T \in \mathbb{R}^{mT}$ as the k-th standard basis vector in Euclidean space $\mathbb{R}^{mT}$, then the functions $\{f_i : \mathbb{R}^{mT} \to \mathbb{R}\}_{i=1}^m$ are $\hat{L}/\sqrt{m}$-mean-squared smooth, and the function $f = \frac{1}{m} \sum_{i=1}^m f_i$ is $\hat{L}/m$-smooth and satisfies
\[
f(0) - \inf_{y \in \mathbb{R}} f(y) = g(0) - \inf_{y \in \mathbb{R}} g(y).
\]
If we further suppose the function $g$ is $\hat{\mu}$-PL, then the function $f$ is $\hat{\mu}/n$-PL.

Proof. For any $x, y \in \mathbb{R}^{mn}$, the smoothness of $g : \mathbb{R}^m \to \mathbb{R}$ implies
\[
\|\nabla f_i(x) - \nabla f_i(y)\| = \|\nabla g(U^{(i)}x) - \nabla g(U^{(i)}y)\|
\leq \hat{L} \|U^{(i)}(x-y)\| \leq \hat{L} \|x-y\|,
\]
and
\[
\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(x) - \nabla f_i(y)\|^2 = \frac{1}{n} \sum_{i=1}^n \|\nabla g(U^{(i)}x) - \nabla g(U^{(i)}y)\|^2
\leq \frac{\hat{L}^2}{n} \|x-y\|^2.
\]
This implies each $f_i$ is $\hat{L}$-smooth and $\{f_i\}_{i=1}^n$ are $\hat{L}/\sqrt{n}$-mean-squared smooth.
Consider the facts $f(0) = g(0)$ and
\[
\inf_{x \in \mathbb{R}^{mn}} g(U^{(i)}x) = \inf_{x \in \mathbb{R}^{mn}} g(U^{(i)}x) = n \inf_{y \in \mathbb{R}^m} g(y),
\]
then we have $f(0) - \inf_{y \in \mathbb{R}} f(y) = g(0) - \inf_{y \in \mathbb{R}} g(y)$.

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For all \( x \in \mathbb{R}^m \), the PL condition of \( g: \mathbb{R}^m \rightarrow \mathbb{R} \) implies

\[
\|\nabla f(x)\|^2 = \frac{1}{n^2} \left\| \sum_{i=1}^{n} \nabla f_i(x) \right\|^2 \\
= \frac{1}{n^2} \left\| \sum_{i=1}^{n} (U^{(i)})^\top \nabla g(U^{(i)}x) \right\|^2 \\
= \frac{1}{n^2} \sum_{i=1}^{n} \|\nabla g(U^{(i)}x)\|^2 \geq \frac{2\hat{\mu}}{n^2} \sum_{i=1}^{n} \left( g(U^{(i)}x) - \inf_{x \in \mathbb{R}^m} g(U^{(i)}x) \right) \\
= \frac{2\hat{\mu}}{n} \left( f(x) - \inf_{y \in \mathbb{R}^m} f(y) \right),
\]

which means \( f \) is \( \hat{\mu}/n \)-PL. Hence, we finish the proof.

5.1 The Lower Bound on Communication Rounds

The communication complexity of the decentralized optimization depends on the topology of the network. We let \( \mathcal{G} = \{V, \mathcal{E}\} \) be the graph associated with the network in the problem, where the node set \( V = \{1, \ldots, m\} \) corresponds to the \( m \) agents and the edge set \( \mathcal{E} = \{(i, j) : \text{agents } i \text{ and } j \text{ are connected}\} \) describes the connectivity of the agents. We use use the notation \( \text{dia}(\mathcal{G}) \) to present the diameter of graph \( \mathcal{G} \).

We introduce the ring-lattice graph and its properties as follows.

**Definition 2** (Ring-Lattice Graph [79]). Given any positive integers \( m, k \) such that \( k \) is even and \( 2 \leq k < m - 1 \), the \( k \)-regular ring-lattice graph over nodes \( \{1, \ldots, m\} \), denoted by \( \mathcal{R}_{m,k} \), is an undirected graph in which each node \( i \in [m] \) is connected with the set of nodes \( \{i + \ell \mod m : \ell \in \mathbb{Z}, 1 \leq |\ell| \leq k/2\} \) where \((i + \ell) \mod m\) is defined as

\[
(i + \ell) \mod m = \begin{cases} 
  i + \ell & \text{if } 1 \leq i + \ell \leq m; \\
  i + \ell + m & \text{if } i + \ell < 1; \\
  i + \ell - m & \text{if } i + \ell > m.
\end{cases}
\]

Additionally, we use notation \( L_{m,k} \in \mathbb{R}^{m \times m} \) to present the Laplacian matrix of the ring-lattice graph \( \mathcal{R}_{m,k} \) and \( \lambda_{m-1}(L_{m,k}) \) to present its second smallest eigenvalue.

**Lemma 16** ([85, Lemma 1]). For any two nodes \( i, j \in [m] \) in the ring-lattice graph \( \mathcal{R}_{m,k} \), the distance between nodes \( i \) and \( j \) is

\[
\left\lfloor \frac{2 \min\{j - i, i + m - j\}}{k} \right\rfloor.
\]

In particular, the diameter of \( \mathcal{R}_{m,k} \) is \( \lfloor 2|m/2|/k \rfloor = \Theta(m/k) \).
Recently, Yuan et al. [85] showed that for all fixed \( m \geq 2 \) and \( \gamma \in [1 - \cos(\pi/m), 1] \), we can always establish a mixing matrix associated with some ring-lattice graph that has the spectral gap \( \gamma \).

**Lemma 17 ([85, Theorem 1]).** For any \( m \geq 2 \) and \( \gamma \in [1 - \cos(\pi/m), 1] \), there exists a ring-lattice graph \( G = \{V, E\} \) with an associated mixing matrix \( W \in \mathbb{R}^{m \times m} \) such that \( |V| = m \), \( 1 - \lambda_2(W) = \gamma \), and \( \text{dia}(G) = \Omega(1/\sqrt{\gamma}) \).

Based on above results, we can construct the graph whose nodes contains two disjoint subsets with the specific distance.

**Lemma 18.** For any \( m \geq 2 \) and \( \gamma \in [1 - \cos(\pi/m), 1] \), there exists a ring-lattice graph \( G = \{V, E\} \) with associated mixing matrix \( W \in \mathbb{R}^{m \times m} \) and \( V_1, V_2 \subseteq V \) such that \( |V| = m \), \( |V_1| \geq m/3 \), \( |V_2| \geq m/3 \), \( 1 - \lambda_2(W) = \gamma \), and the distance between \( V_1 \) and \( V_2 \) satisfies \( \text{dist}(V_1, V_2) = \Omega(1/\sqrt{\gamma}) \). Specifically, such graph \( G \) and matrix \( W \) can be constructed by

\[
G = \begin{cases} \mathcal{R}_{m, m-1}, & \gamma \in \max\{1 - \cos(\pi/m), 1 - \cos(\pi/9)\}, 1], \\ \mathcal{R}_{m, k}, & \text{otherwise} \end{cases}
\]

and

\[
W = \begin{cases} \frac{\gamma}{m} 11^T + (1 - \gamma)I, & \gamma \in \max\{1 - \cos(\pi/m), 1 - \cos(\pi/9)\}, 1], \\ I - \frac{\gamma}{\lambda_{m-1}(L_{m,k})} L_{m,k}, & \text{otherwise}, \end{cases}
\]

where

\[
k = 2 \left\lfloor m \sqrt{\frac{3\gamma}{\pi^2(1 - \pi^2/12)}} \right\rfloor.
\]

**Proof.** Based on Lemma 17, there exists a ring-lattice graph \( G = \{V, E\} \) associated with the mixing matrix \( W \in \mathbb{R}^{m \times m} \) such that \( |V| = m \), \( 1 - \lambda_2(W) = \gamma \), and \( \text{dia}(G) = \Omega(1/\sqrt{\gamma}) \). Let \( V_1 = \{1, \ldots, \lfloor m/3 \rfloor\} \) and \( V_2 = \{\lceil m/2 \rceil + 1, \ldots, \lfloor m/2 \rfloor + \lfloor m/3 \rfloor\} \), which means \( |V_1| \geq m/3 \) and \( |V_2| \geq m/3 \). Based on Lemma 16, the distance between node sets \( V_1 \) and \( V_2 \) satisfies

\[
\text{dist}(V_1, V_2) = \left[ \frac{2 \min_{i \in V_1, j \in V_2} \{j - i, i + m - j\}}{k} \right] \geq \left[ \frac{2 \min\{\lfloor m/2 \rfloor + 1 - \lfloor m/3 \rfloor, 1 + m - \lfloor m/2 \rfloor - \lfloor m/3 \rfloor\}}{k} \right] \geq \left[ \frac{2 \min\{m/2 - 1/2 + 1 - m/3 - 2/3, 1 + m - m/2 - m/3 - 2/3\}}{k} \right] = \left[ \frac{m - 1}{3k} \right] = \Omega \left( \frac{m}{k} \right) = \Omega(\text{dia}(G)) = \Omega \left( \frac{1}{\sqrt{\gamma}} \right).
\]
Applying Lemma 18, we provide the lower bound on communication rounds for our smooth finite-sum decentralized optimization problem.

**Theorem 2.** For any $L > 0, \bar{L} > 0, m \geq 2, n > 0, \Delta > 0, \gamma \in [1 - \cos(\pi/m), 1]$ and $\epsilon > 0$ with $\bar{L} \geq L$ and $\epsilon = O(\sqrt{\Delta L})$, there exist matrix $W \in \mathbb{R}^{m \times m}$ with spectral gap $\gamma$ and $L$-mean-squared smooth functions $\{f_{i,j} : \mathbb{R}^d \to \mathbb{R}\}_{i,j=1}^{m,n}$ with $d = O(\Delta L^{-2})$ such that their average $f = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}$ is $L$-smooth with $f(0) - f^* \leq \Delta$. In order to find an $\epsilon$-stationary point of the function $f$, any LIFO algorithm needs at least $\Omega(\Delta L^{-2}/\sqrt{\gamma})$ communication rounds.

**Proof.** According to Lemma 18, there exists a ring-lattice graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with the associated mixing matrix $W \in \mathbb{R}^{m \times m}$ and $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$ such that $|\mathcal{V}| = m, 1 - \lambda_2(W) = \gamma, |\mathcal{V}_1| \geq m/3, |\mathcal{V}_2| \geq m/3$, and the distance between the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ satisfies $\text{dist}(\mathcal{V}_1, \mathcal{V}_2) = \Omega(1/\sqrt{\gamma})$.

We consider the nonconvex function $f_T^{nc}$ defined in equation (85) with

$$T = \left\lceil \frac{\Delta L}{36d_0 \epsilon^2} \right\rceil,$$

where $\epsilon \leq \sqrt{\frac{\Delta L}{36d_0}}$, and $l_0 = 152$. (90)

According to Lemma 14 with

$$g(x) = f_T^{nc}(x), \quad \alpha = \frac{3d_0 \epsilon^2}{L}, \quad \text{and} \quad \beta = \frac{L}{3d_0 \epsilon},$$

we can conclude the function $\hat{g}(x) = \alpha f_T^{nc}(\beta x)$ is $\alpha \beta^2 l_0$-smooth and satisfies

$$\|\nabla \hat{g}(x)\| = \alpha \beta \|\nabla f_T^{nc}(\beta x)\|$$

and

$$\hat{g}(0) - \inf_{y \in \mathbb{R}^d} \hat{g}(y) = \alpha \left(f_T^{nc}(0) - \inf_{y \in \mathbb{R}^d} f_T^{nc}(y)\right) \leq 12\alpha T,$$  

where the inequality is based on Lemma 13(a).

We then construct the hard instance based on the decomposition $f_T^{nc} = f_T^{nc,1} + f_T^{nc,2}$ shown in equations (87) and (88). Specifically, we take $f_{i,j}$ as

$$f_{i,j}(x) = \begin{cases} \frac{\alpha m}{|\mathcal{V}_1|} f_T^{nc,1}(\beta x), & i \in \mathcal{V}_1, \\ \frac{\alpha m}{|\mathcal{V}_2|} f_T^{nc,2}(\beta x), & i \in \mathcal{V}_2, \\ 0, & \text{otherwise}, \end{cases}$$  

where $f_T^{nc,1}$ and $f_T^{nc,2}$ are defined in equation (88). According to Lemma 12, Lemma 14, and equation (94), each individual function $f_{i,j}$ is $\alpha \beta^2 m l_0 / \min(|\mathcal{V}_1|, |\mathcal{V}_2|)$-smooth. Therefore, functions $\{f_{i,j}\}_{i,j=1}^{m,n}$ are $\alpha \beta^2 m l_0 / \min(|\mathcal{V}_1|, |\mathcal{V}_2|)$-mean-squared smooth.
Additionally, the definition of $f_{i,j}$ in equation (94) means

$$f(x) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}(x) = \hat{g}(x), \quad (95)$$

which is $\alpha \beta^2 l_0$-smooth.

The parameter settings in equations (90) and (91) imply

$$\frac{\alpha \beta^2 l_0}{\min\{|V_1|, |V_2|\}} = \frac{mL}{3 \min\{|V_1|, |V_2|\}} \leq L \leq \bar{L}, \quad \text{and} \quad 12 \alpha T \leq \Delta.$$  

Therefore, the function $f = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}$ is $L$-smooth and satisfies $f(0) - f^* \leq \Delta$
and the function set $\{f_{i,j}\}$ is $L$-mean-squared smooth, which satisfies our requirements.

Now we show that any LIFO algorithm require at least

$$(T - 1)\text{dist}(V_1, V_2) = \Omega \left( \frac{L \Delta}{\sqrt{\gamma \epsilon^2}} \right), \quad (96)$$

communication rounds to achieve an $\epsilon$-stationary point of $f(\cdot)$.

According to Corollary 13, equations (91), (92) and (95), we have

$$\|\nabla f(x)\| > \alpha \beta = \epsilon.$$  

for all $x = [x_1, \ldots, x_T]^\top \in \mathbb{R}^d$ such that $x_T = 0$.

We let

$$\text{nnz}(s,i) \triangleq \max \left\{ t \in \{0, 1, \ldots, T\} : \right. \left. \exists y = [y_1, \ldots, y_T]^\top \in M^*_s \text{ such that } y_t \neq 0 \right\}, \quad (97)$$

where we denote $y_0 = 1$.

Consider the definitions of $f_{i,j}$, $f_{nc,1}$ and $f_{nc,2}$ in equations (88) and (94). Notice that the fact $\Psi(0) = \Psi'(0) = 0$ implies the partial derivatives of

$$\Psi(-x_{k-1})\Phi(-x_k) - \Psi(x_{k-1})\Phi(-x_k)$$

with respect to $x_{k-1}$ or $x_k$ are zero when $x_{k-1} = x_k = 0$. This means the computation of any LIFO algorithm holds:

1. If $i \in V_1$ and nnz$(s,i)$ is even, one step of local computation can increase at most one dimension for memory of node $i$.
2. If $i \in V_2$ and nnz$(s,i)$ is odd, one step of local computation can increase at most one dimension for memory of node $i$.
3. Otherwise, one step of local computation cannot increase the dimension for memory of node $i$.
In summary, we have
\[
\text{nnz}(s + 1, i) \leq \begin{cases} 
\text{nnz}(s, i) + 1, & \text{if } i \in V_1, \text{nnz}(s, i) \equiv 0 \pmod{2}, \\
\text{nnz}(s, i) + 1, & \text{if } i \in V_2, \text{nnz}(s, i) \equiv 1 \pmod{2}, \\
\text{nnz}(s, i), & \text{otherwise.}
\end{cases}
\] (98)

We consider the cost to reach the second coordinate from the initial status that \( M_0^i = \{0\} \) for all \( i \in [m] \). According to equation (134), we need to let a node in \( V_2 \) reach the first coordinate, which requires at least one local computation step on some node in \( V_1 \) first. Then, according to definitions of LIFO algorithm (Definition 1), one must perform at least \( \text{dist}(V_1, V_2) \) local communication rounds for a node in \( V_2 \) to receive the information of the first coordinate from some node in \( V_1 \). After above rounds, we can perform at least 1 computation on nodes in \( V_2 \) to reach the second coordinate. In summary, to reach the second coordinate requires at least \( \text{dist}(V_1, V_2) \) local communication rounds. Similarly, to reach the \( k \)-th coordinate, any LIFO algorithm must perform at least \( (k - 1)\text{dist}(V_1, V_2) \) local communication rounds. Therefore, to achieve the \( \epsilon \)-stationary point, we require \( x_T \neq 0 \) that needs at least
\[
(T - 1)\text{dist}(V_1, V_2) = \Omega \left( \frac{L\Delta}{\sqrt{\gamma\epsilon^2}} \right)
\]
communications rounds.

Remark 8. The results of Corollary 1 and Theorem 2 show that the complexity on communication rounds of DEAREST is near-optimal for all \( \gamma \in [1 - \cos(\pi/m), 1] \) and \( \bar{L} \geq L \). As pointed by Yuan et al. [85], such result is more general than the lower bounds established by the linear graph [29, 48, 68, 69] that only work in the case of \( \gamma = 1 - \cos(\pi/m) \), since \( \cos(\pi/m) \) approaches to 1 for large \( m \). More importantly, our results indicate the communication complexity in distributed nonconvex optimization essentially depends on the global smoothness parameter \( L \), rather than the mean-squared or the local smoothness parameters which used in existing work [39, 48, 51, 73, 81, 85, 89].

5.2 The Lower Bound on LIFO calls

It is natural that the lower bound on the LIFO calls for decentralized finite-sum optimization problem (1) with local functions (2) can be established by considering the IFO complexity of the finite-sum optimization problem on single machine with \( N = mn \) individual functions. Unlike existing lower bounds on IFO complexity \( L \) [1, 24, 35, 42, 80, 92] only consider the mean-squared smoothness parameter, our constructions address both \( L \) and \( \bar{L} \), which is formally presented as follows.

Theorem 3. For all \( \bar{L} \geq 0, L > 0, N > 0, \Delta > 0 \) and \( \epsilon > 0 \) with \( \bar{L} \geq L \) and \( \epsilon = O((\min\{L, \bar{L}/\sqrt{N}\})\Delta^{1/2}) \), there exist \( L \)-mean-squared smooth functions \( \{\tilde{f}_i : \mathbb{R}^d \to \mathbb{R}\}_{i=1}^N \) such that the function \( \tilde{f} = \frac{1}{N} \sum_{i=1}^N \tilde{f}_i \) is \( L \)-smooth and satisfies that \( \tilde{f}(0) - \inf_{y \in \mathbb{R}^d} \tilde{f}(y) \leq \Delta \). In order to find an \( \epsilon \)-stationary point of \( \tilde{f} \), any IFO algorithm (defined in Appendix C) needs at least \( \Omega(N + \min\{NL, \sqrt{NL}\Delta\epsilon^{-2}\}) \) IFO calls.
Proof. We consider the nonconvex function $f_{nc}^T$ defined in equation (85) with

$$T = \left\lceil \frac{\min\{L, \bar{L}/\sqrt{N}\}\Delta}{24l_0^2} \right\rceil,$$

where $\epsilon \leq \sqrt{\frac{\min\{L, \bar{L}/\sqrt{N}\}\Delta}{24l_0}}$ and $l_0 = 152$. (99)

According to Lemma 14 with

$$g(x) = f_{nc}^T(x), \quad \alpha = \frac{2l_0\epsilon^2}{\min\{L, L/\sqrt{N}\}}, \quad \text{and} \quad \beta = \frac{\min\{L\sqrt{N}, \bar{L}\}}{\sqrt{2l_0\epsilon}},$$

we can conclude the function $\hat{g}(x) = \alpha f_{nc}^T(\beta x)$ is $\alpha\beta l_0$-smooth and holds

$$\|\nabla \hat{g}(x)\| = \alpha\beta \|\nabla f_{nc}^T(\beta x)\|$$

and

$$\hat{g}(0) - \inf_{y \in \mathbb{R}^T} \hat{g}(y) = \alpha \left( f_{nc}^T(0) - \inf_{y \in \mathbb{R}^T} f_{nc}^T(y) \right) \leq 12\alpha T,$$

where the inequality is based on Lemma 13(a).

We construct the hard instance according to Lemma 15 with $g(x) = \hat{g}(x)$, which results the functions

$$\tilde{f}_i(x) = \alpha f_{nc}^T(\beta U(i)x) \quad \text{and} \quad \tilde{f}(x) = \frac{1}{N} \sum_{i=1}^N \tilde{f}_i(x)$$

such that $\{\tilde{f}_i\}_{i=1}^N$ is $\alpha\beta^2 l_0/\sqrt{N}$-mean-squared smooth and $\tilde{f}$ is $\alpha\beta^2 l_0/N$-smooth with

$$\tilde{f}(x^0) - \tilde{f}^* = \hat{g}(0) - \inf_{y \in \mathbb{R}^T} \hat{g}(y) \leq 12\alpha T,$$

where $\tilde{f}^* = \inf_{y \in \mathbb{R}^T} \tilde{f}(y)$.

The settings of $\alpha$, $\beta$ and $T$ in equations (99) and (100) imply

$$\frac{\alpha\beta^2 l_0}{\sqrt{N}} \leq L, \quad \frac{\alpha\beta^2 l_0}{N} \leq L, \quad \text{and} \quad 12\alpha T \leq \Delta.$$

Therefore, the function set $\{\tilde{f}_i\}_{i=1}^N$ is $L$-average smooth and the function $\tilde{f}$ is $L$-smooth with $\tilde{f}(x^0) - \tilde{f}^* \leq \Delta$, which satisfies our requirements.

Now we show that any IFO algorithm require at least $NT/2$ IFO calls to achieve an $\epsilon$-stationary point $\tilde{x}$ of the function $f$. According to equation (101), Lemma 13(d) and Lemma 14, for all $x \in \mathbb{R}^{NT}$ with $\text{supp}(U(i)x) \subseteq \{1, 2, \ldots, T-1\}$, we have

$$\|\nabla f_j(x)\| = \|\alpha\beta(U(j))^T \nabla f_{nc}^T(\beta U(j)x)\| = \alpha\beta \|\nabla f_{nc}^T(\beta U(j)x)\| > \alpha\beta.$$ (104)
We consider the vector \( x \in \mathbb{R}^{NT} \) which is achieved by an IFO algorithm with at most \( \lfloor NT/2 \rfloor \) IFO calls. The zero-chain property shown in Lemma 13(c) implies such vector \( x \) has at most \( \lfloor NT/2 \rfloor \) non-zero entries. We partition \( x \in \mathbb{R}^{NT} \) into \( N \) vectors \( y^{(1)}, \ldots, y^{(N)} \in \mathbb{R}^{T} \) such that \( y^{(j)} = U^{(j)}x \in \mathbb{R}^{T} \), then there at least \( N/2 \) vectors in \( \{ y^{(j)} \}_{j=1}^{N} \) such that each of them has at least one zero entry. The zero-chain property (Lemma 13(c)) means there exists index set \( I \subseteq [N] \) with \( |I| \geq N/2 \) such that each \( j \in I \) satisfies \( \text{supp}(y^{(j)}) \subseteq \{ 1, 2, \ldots, T-1 \} \). Therefore, we have

\[
\| \nabla f(x) \| = \frac{1}{N} \sqrt{ \sum_{i=1}^{N} \| \nabla f_i(x) \|^2 }
\]

\[
\geq \frac{1}{N} \sqrt{ \sum_{j \in I} \| \nabla f_j(x) \|^2 } \quad \text{(104)} > \frac{1}{N} \sqrt{ \frac{N}{2} } \alpha \beta = \epsilon.
\]

Hence, any IFO algorithm requires at least

\[
\left\lfloor \frac{NT}{2} \right\rfloor + 1 = \Omega \left( \min \{ NL, \sqrt{NL} \} \right)
\]

IFO calls to achieve an \( \epsilon \)-stationary point of function \( \tilde{f}(\cdot) \) defined in equation (103).

Then, we consider the lower bound dominated by \( N \) and construct the another hard instance. We assume that \( \epsilon \leq \sqrt{L\Delta}/2 \). We follow the functions provided by Li et al. [42], i.e., define \( \tilde{f}_i : \mathbb{R}^d \to \mathbb{R} \) and \( \tilde{f} : \mathbb{R}^d \to \mathbb{R} \) as

\[
\tilde{f}_i(x) = c(v_i, x) + \frac{L}{2} \| x \|^2 \quad \text{and} \quad \tilde{f}(x) = \frac{1}{N} \sum_{i=1}^{N} \tilde{f}_i(x).
\]

(105)

for all \( i \in [N] \), where \( v_i = e_i, c = 2c\sqrt{N} \) and \( d = N \). We can verify that for all \( x, y \in \mathbb{R}^d \), it holds

\[
\| \nabla \tilde{f}(x) - \nabla \tilde{f}(y) \| = \left\| \left( \frac{c}{N} \sum_{i=1}^{N} v_i + Lx \right) - \left( \frac{c}{N} \sum_{i=1}^{N} v_i + Ly \right) \right\| = L\| x - y \|
\]

and

\[
\frac{1}{N} \sum_{i=1}^{N} \| \nabla \tilde{f}_i(x) - \tilde{f}_i(y) \|^2
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \| (cv_i + Lx) - (cv_i + Ly) \|^2
\]

\[
= L^2 \| x - y \|^2 \leq \bar{L}^2 \| x - y \|^2.
\]
Therefore, the function $\tilde{f}$ is $L$-smooth and the functions \( \{\tilde{f}_i\}_{i=1}^N \) are $\bar{L}$-average smooth. Since we suppose the IFO algorithm start with the initial point $x^0 = 0$, it holds

$$\tilde{f}(x^0) - \inf_{y \in \mathbb{R}^d} \tilde{f}(y) = 0 - \left( \frac{c}{N} \sum_{i=1}^N (v_i, x^*) + \frac{L}{2} ||x^*||^2 \right) = \frac{c^2 d}{2LN^2} \leq \Delta,$$

where $x^* = -\frac{c}{L} \sum_{i=1}^N v_i$ is the minimizer of $\tilde{f}$. Following Theorem 2 of Li et al. [42], we can show that any IFO algorithm requires at least IFO calls of $\Omega(N)$ to achieve an $\epsilon$-stationary point of the function $\tilde{f}$ defined in equation (105).

Combining results of above two hard instances, we achieve the lower bound on the IFO complexity of

$$\Omega \left( N + \min\{NL, \sqrt{NL\Delta}\} \right).$$

\[\square\]

Remark 9. Applying Theorem 3 with $N = mn$ means the LIFO complexity shown in Corollary 1 is optimal.

5.3 The Lower Bound on Computation Rounds

As pointed in Remark 4, the communication rounds are not always proportion to the LIFO calls. Therefore, we detailed analyze this lower bound as follows.

**Theorem 4.** For any $\bar{L} > 0$, $L > 0$, $m > 0$, $n > 0$, $\Delta > 0$, and $\epsilon > 0$ with $\bar{L} \geq L$ and $\epsilon = \mathcal{O}\left((\min\{L, \bar{L}\sqrt{N}\} \Delta)^{1/2}\right)$, there exist $\bar{L}$-mean-squared smooth functions $\{f_{i,j} : \mathbb{R}^d \rightarrow \mathbb{R}\}$ such that the function $f = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f_{i,j}$ is $L$-smooth with $f(0) - f^* \leq \Delta$. In order to find an $\epsilon$-stationary point of $f$, any LIFO algorithm needs at least $\Omega(n + (L + \min\{nL, \sqrt{n/m}\bar{L}\})\epsilon^{-2})$ computation rounds.

**Proof.** We first show the lower bound in the view of linear speed-up. According to Theorem 3 with $N = mn$ and $\epsilon = \mathcal{O}\left((\min\{L, \bar{L}\sqrt{N}\} \Delta)^{1/2}\right)$, there exist $\bar{L}$-mean-squared smooth functions $\{f_{i,j}\}_{i=1,j=1}^{m,n}$ such that the function $f = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f_{i,j}$ is $L$-smooth and satisfies $f(x^0) - f^* \leq \Delta$. In order to find an $\epsilon$-stationary point of problem $f$, any IFO algorithm needs at least

$$\Omega \left( mn + \min\{mnL, \sqrt{mn\bar{L}}\} \right),$$

IFO calls. For the distributed setting with $m$ agents, any LIFO algorithm can perform at most $m$ LIFO calls in per computation rounds. Therefore, the IFO lower bound in equation (106) directly implies the lower bound on the computation rounds of

$$\Omega \left( \frac{mn}{m} + \frac{\min\{mnL, \sqrt{mn\bar{L}}\}}{me^2} \right) = \Omega \left( n + \frac{\min\{nL, \sqrt{n/m\bar{L}}\}}{e^2} \right),$$

(107)
which nearly matches the upper bound (20) in Corollary 1 when \( \bar{L}/L < \sqrt{m/n} \).

The remain of the proof only needs to provide a hard instance with the lower bound of \( \Omega(\bar{L} \epsilon^2 - \Delta) \) when \( \bar{L}/L < \sqrt{m/n} \). We consider the nonconvex function \( f_T^{\text{nc}} \) defined in equation (85) with

\[
T = \left\lfloor \frac{L \Delta}{12l_0 \epsilon^2} \right\rfloor, \quad \text{where } \epsilon \leq \sqrt{\frac{L \Delta}{12l_0}} \text{ and } l_0 = 152. \tag{108}
\]

According to Lemma 14 with

\[
g(x) = f_T^{\text{nc}}(x), \quad \alpha = \frac{l_0 \epsilon^2}{L} \quad \text{and} \quad \beta = \frac{L}{l_0 \epsilon}, \tag{109}
\]

we can conclude the function \( \hat{g}(x) = \alpha f_T^{\text{nc}}(\beta x) \) is \( \alpha \beta^2 l_0 \)-smooth and satisfies

\[
\|\nabla \hat{g}(x)\| = \alpha \beta \|\nabla f_T^{\text{nc}}(\beta x)\| \tag{110}
\]

and

\[
\hat{g}(0) - \inf_{y \in \mathbb{R}^T} \hat{g}(y) = \alpha \left( f_T^{\text{nc}}(0) - \inf_{y \in \mathbb{R}^T} f_T^{\text{nc}}(y) \right) \leq 12 \alpha T, \tag{111}
\]

where the inequality is based on Lemma 13(a).

We construct the hard instance as

\[
f_{i,j}(x) = \hat{g}(x) \quad \text{and} \quad f(x) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}(x) \tag{112}
\]

such that \( \{f_{i,j}\}_{i=1,j=1}^{m,n} \) is \( \alpha \beta^2 l_0 \)-mean-squared smooth and \( f \) is \( \alpha \beta^2 l_0 \)-smooth with

\[
f(x^0) - f^* = \hat{g}(0) - \inf_{y \in \mathbb{R}^T} \hat{g}(y) \leq 12 \alpha T. \tag{111}
\]

The setting of \( \alpha, \beta \) and \( T \) implies

\[
\alpha \beta^2 l_0 = \frac{L}{12l_0} \quad \text{and} \quad 12 \alpha T \leq \Delta. \tag{109}
\]

Therefore, the function set \( \{f_{i,j}\}_{i=1,j=1}^{m,n} \) is \( L \)-average smooth and the function \( f \) is \( L \)-smooth with \( f(x^0) - f^* \leq \Delta \), which satisfies our requirements.

Now we show that any LIFO algorithm require at least \( T - 1 \) computation rounds to achieve an \( \epsilon \)-stationary point of the function \( f(\cdot) \) at each node. According to equations (110)-(112), Lemmas 13(d) and 14, for all vector \( x_i \in \mathbb{R}^T \) that satisfies \( \text{supp}(x_i) \subseteq \{1, 2, \cdots, T-1\} \), we have

\[
\|\nabla f(x_i)\| = \|\alpha \beta \nabla f_T^{\text{nc}}(\beta x_i)\| > \alpha \beta. \tag{113}
\]
We consider the vectors $x_1, \ldots, x_m \in \mathbb{R}^T$ which are achieved by an LIFO algorithm with at most $T - 1$ computation rounds. Lemma 13(c) implies such $x_i$ has at most $T - 1$ non-zero entries, that is $\text{supp}(x_i) \subseteq \{1, 2, \cdots, T - 1\}$ for all $i \in [m]$. Therefore, we have

$$\|\nabla f(x_i)\| > \alpha \beta = \epsilon.$$ 

Hence, any LIFO algorithm requires at least

$$T = \Omega\left(\frac{L \Delta}{\epsilon^2}\right)$$

(114)

computation rounds to achieve an $\epsilon$-stationary point of function $f$. Combing results of (107) and (114), we finish the proof.

**Remark 10.** In a very recent work, Metelev et al. [51] claimed the lower bounds on the communication rounds and the computation rounds with respect to local smoothness parameters $L_i$ and $L_\ell$ (see Assumptions 4 and 5) respectively. However, their analysis essentially only obtains the lower bound on the computation rounds of $\Omega(n L_i \Delta \epsilon^{-2})$ [51, at the end of page 45], and the desired lower bound of $\Omega(n + \sqrt{n} L_\ell \epsilon^{-2})$ shown in their statement [51, Corollary 4.6] implicitly requires the additional assumption of $L_i/L_\ell \leq O(\sqrt{n})$. Recall that the discussion in Remark 1 shows the ratio $L_i/L_\ell$ (consider $L/L$ when $m = 1$) may be larger than $O(\sqrt{n})$, which is not included in the analysis of Metelev et al. [51]. In contrast, all of our theorems for lower bounds are valid for any $L > 0$ and $\bar{L} > 0$ such that $\bar{L} \geq L$, which is general since it always holds for the tight $L$ and $\bar{L}$ (see Remark 1).

6 The Results under the PL Condition

The PL condition (Assumption 7) suggests the function value gap is dominated by the square of gradient norm, which avoids the hardness of finding global solution in general smooth nonconvex optimization [57] and leads to the gradient descent method linearly converging to the global minimum without the convexity [33, 47, 62]. It covers a lot of popular applications, such as deep neural networks [5, 44, 88], reinforcement learning [2, 27, 50, 86], optimal control [15, 25] and matrix recovery [11, 28, 41].

For the single machine scenario, Lei et al. [36], Reddi et al. [66] considered the finite-sum optimization under the PL condition by proposing SVRG-type methods [32], which achieves the $\epsilon$-suboptimal solution with at most $O((N + N^{2/3} \bar{\kappa}) \ln(1/\epsilon))$ IFO calls, where $N$ is the number of individual functions and $\bar{\kappa}$ is defined in equation (13). Later, Li et al. [42], Wang et al. [78], Zhou et al. [94] improved the IFO upper bound to $O((N + \sqrt{N\bar{\kappa}}) \ln(1/\epsilon))$ by stochastic recursive gradient estimator. Additionally, Yue et al. [87] established the tight lower bound on the exact first-order oracle complexity of $\Omega(\kappa \ln(1/\epsilon))$ for minimizing the PL function, where $\kappa$ is defined in equation (13).

In this section, we show the proposed DEAREST$^+$ (Algorithm 1) also achieves the near optimal complexity bounds for the decentralized smooth finite-sum problem under the PL condition. We present the complexity analysis for DEAREST$^+$ under the PL condition in Section 6.1 and provide the corresponding lower bounds in Section 6.2,
Table 3 We present the communication rounds and the computation rounds for finding $\epsilon$-suboptimal solutions under the PL condition, where $\Delta = f(\bar{x}_0) - f^*$ and $\bar{x}_0 \in \mathbb{R}^d$ is the initial point of the algorithm.

| Methods     | #Communication          | #Computation          | Reference          |
|-------------|-------------------------|-----------------------|--------------------|
| DGD-GT      | $\mathcal{O}\left(\kappa \ln(\Delta/\epsilon) \over \sqrt{\gamma}\right)$ | $\mathcal{O}\left(n\kappa \ln \left(\frac{\Delta}{\epsilon}\right)\right)$ | Bai et al. [10]    |
| DRONE       | $\mathcal{O}\left(\bar{k} \ln(\Delta/\epsilon) \over \sqrt{\gamma}\right)$ | $\mathcal{O}\left(n\bar{k} \ln \left(\frac{\Delta}{\epsilon}\right)\right)$ | Bai et al. [10]    |
| DEAREST$^+$ | $\mathcal{O}\left(\kappa \ln(\Delta/\epsilon) \over \sqrt{\gamma}\right)$ | $\mathcal{O}\left((n + \kappa + \min\left\{n\kappa, \sqrt{\frac{n}{m}} \bar{k}\right\}) \ln \left(\frac{\Delta}{\epsilon}\right)\right)$ | Corollary 2        |

Lower Bounds $\Omega\left(\kappa \ln(\Delta/\epsilon) \over \sqrt{\gamma}\right)$ $\Omega\left(n + \kappa + \min\left\{n\kappa, \sqrt{\frac{n}{m}} \bar{k}\right\} \ln \left(\frac{\Delta}{\epsilon}\right)\right)$ This work

Table 4 We present the number of LIFO calls for finding $\epsilon$-suboptimal solutions under the PL condition, where $\Delta = f(\bar{x}_0) - f^*$ and $\bar{x}_0 \in \mathbb{R}^d$ is the initial point of the algorithm.

| Methods     | #LIFO                        | Reference          |
|-------------|------------------------------|--------------------|
| DGD-GT      | $\mathcal{O}\left(mn\kappa \ln \left(\frac{\Delta}{\epsilon}\right)\right)$ | Bai et al. [10]    |
| DRONE       | $\mathcal{O}\left((mn + \sqrt{mn}\kappa) \ln \left(\frac{\Delta}{\epsilon}\right)\right)$ | Bai et al. [10]    |
| DEAREST$^+$ | $\mathcal{O}\left((mn + \min\left\{mn\kappa, \sqrt{mn}\bar{k}\right\}) \ln \left(\frac{\Delta}{\epsilon}\right)\right)$ | Corollary 2        |

Lower Bounds $\Omega\left(mn + \min\left\{mn\kappa, \sqrt{mn}\bar{k}\right\} \ln \left(\frac{\Delta}{\epsilon}\right)\right)$ This work

which extends and improves the results in our conference paper [10] in the following aspects:

(a) This manuscript considers the finite-sum structure (2) in local functions, while our conference paper [10] only considers the oracle of accessing the exact local gradient which can be regarded as the special case of $n = 1$.

(b) This manuscript distinguishes the difference between $L$ and $\bar{L}$ in the algorithm design and analysis, while our conference paper [10] only addresses the mean-squared smoothness parameter $\bar{L}$.

(c) This manuscript proves the algorithm can find $x_T^i$ such that $\mathbb{E}[f(x_T^i) - f^*] \leq \epsilon$ for every agent, while our conference paper [10] only shows $\mathbb{E}[f(x_{\text{out}}) - f^*] \leq \epsilon$, where $x_{\text{out}}$ is uniformly sampled from vectors $\{x_T^i\}_{i=1}^m$ on all $m$ agents.

We summarize the theoretical results in this section and our conference paper in Tables 3 and 4. To the best of our knowledge, no previous work specifically studies the complexity of decentralized optimization problem under the PL condition.
6.1 The Complexity Analysis under the PL Condition

The complexity analysis for DEAREST+ (Algorithm 1) under the PL condition also distinguishes the between $L$ and $L$ ($\bar{k}$ and $\bar{k}$). We formally present the results as follows.

**Theorem 5.** Under Assumptions 1–3 and 7–8 with $\mu < L \leq \bar{L}$, we run Algorithm 1 with

$$
\eta = \frac{1}{8L}, \quad b = \left[ L \min \left( \bar{k}, \sqrt{\frac{\ln m}{L}} \right) \right], \quad p = \min \left\{ L \max \left\{ \frac{1}{\bar{k}}, \frac{1}{\sqrt{mn}} \right\}, 1 \right\},
$$

$$
T = \left\lceil 16k \ln \left( \frac{2c}{\epsilon} \right) \right\rceil, \quad K = \mathcal{O}\left( \frac{\ln(mnk\sqrt{L})}{\sqrt{\gamma}} \right), \quad \text{and} \quad \bar{K} = \mathcal{O}\left( \frac{\ln(mnk\bar{L}/(Lc))}{\sqrt{\gamma}} \right),
$$

where $\Delta = f(\bar{\theta}) - f^\ast$. Then the output satisfies $\mathbb{E}[f(x_T^\ast) - f^\ast] \leq \epsilon$ for all $i \in [m]$.

**Proof.** Part I: We first consider the case of $L < \sqrt{mn}L$. Following the derivation of equation (63) in the analysis of general case and the PL condition (12), we have

$$
\mathbb{E}[\Phi^{i+1}] \leq \mathbb{E}\left[ \left( 1 - \frac{\mu}{2} \right) f(x_t^\ast) - f^\ast - \frac{\eta}{2} \| \nabla f(x_t^\ast) \|^2 \right] + \left( \eta + \frac{2(1-p)\eta}{p} \right) \mathbb{E}[U^t]
$$

$$
+ \left( \frac{1}{m^3n^4bp} + 264p^227m^3nL^2\gamma \right) \mathbb{E}[V^t]
$$

$$
+ \left( \frac{1}{b} - \frac{L}{2} \right) \left( \frac{1}{n} \right) \mathbb{E}\left[ \| x_t^{t+1} - x^t \| \right]^2
$$

$$
\leq \mathbb{E}\left[ \left( 1 - \frac{\eta}{2} \right) f(x_t^\ast) - f^\ast - \frac{\eta}{2} \| \nabla f(x_t^\ast) \|^2 \right] + \left( \eta + \frac{2(1-p)\eta}{p} \right) \mathbb{E}[U^t]
$$

$$
+ \left( \frac{1}{m^3n^4bp} + 264p^227m^3nL^2\gamma \right) \mathbb{E}[V^t]
$$

$$
+ \left( \frac{1}{b} - \frac{L}{2} \right) \left( \frac{1}{n} \right) \mathbb{E}\left[ \| x_t^{t+1} - x^t \| \right]^2
$$

$$
\leq \mathbb{E}\left[ \left( 1 - \frac{\eta}{2} \right) f(x_t^\ast) - f^\ast - \frac{\eta}{2} \| \nabla f(x_t^\ast) \|^2 \right] - \frac{L^2\eta}{4} \mathbb{E}\left[ \| x_t^{t+1} - x^t \| \right]^2
$$

$$
\leq \mathbb{E}\left[ \left( 1 - \frac{\eta}{2} \right)^2 \Phi^t \right],
$$

(115)

where the second inequality uses the fact $p \geq 8\mu$ from the settings of $\eta = 1/(8L)$ and $p = (L/L) \max \{ \mu/L, 1/\sqrt{mn} \}$; the last step is based on the parameter settings in the statement of Theorem 5 and taking

$$
K = \left\lceil 2 + \frac{\sqrt{2}}{2\sqrt{\gamma}} \ln \left( 14 \max \left\{ \frac{44m^6n^6(\bar{L}^2 + L^2)}{5L^3\mu}, \frac{33(27m^3n^3\bar{L}^2 + 12L^2)}{2(308L + 183\mu)}, \frac{297m^6n^6\bar{L}^3}{51L^3} \right\} \right) \right\rceil.
$$

51
We split the function value gap at the $i$-th node as

$$
\mathbb{E}[f(x_i^T) - f^*] = \mathbb{E}[f(x_i^T) - f^*] + \mathbb{E}[f(x_i^T) - f(x_i^T)].
$$

(116)

For the first term on the right-hand side of equation (116), we have

$$
\mathbb{E}[f(x_i^T) - f^*] 
\leq \mathbb{E}[\Phi^T - 132L^2\eta C^T]
$$

(115)

\begin{align}
&\leq \mathbb{E}\left[\left(1 - \frac{\mu\eta}{2}\right)\Phi^T - \frac{\eta}{4}\|\nabla f(x_i^T)\|^2 - \frac{L^2\eta}{4}\|x_i^T - x_i^T\|^2 - 132L^2\eta C^T\right].
\end{align}

(117)

For the second term on the right-hand side of equation (116), we have

$$
\mathbb{E}[f(x_i^T) - f(x_i^T)]
\leq \mathbb{E}\left[\langle \nabla f(x_i^T), x_i^T - x_i^T \rangle + \frac{L}{2}\|x_i^T - x_i^T\|^2\right]
$$

$$
\leq \mathbb{E}\left[\frac{\eta}{8}\|\nabla f(x_i^T)\|^2 + \frac{2}{\eta}\|x_i^T - x_i^T\|^2 + \frac{L}{2}\|x_i^T - x_i^T\|^2\right]
$$

(118)

$$
\leq \mathbb{E}\left[\frac{\eta}{4}\|\nabla f(x_i^T)\|^2 + \frac{\eta}{4}\|\nabla f(x_i^T) - \nabla f(x_i^T)\|^2 + \left(\frac{2}{\eta} + \frac{L}{2}\right)C_T\right]
$$

where the first and the last inequalities are based on Assumption 2; the second inequality is based on the Cauchy–Schwarz inequality; the third inequality is based on the Young’s inequality and the definition of $C_T$ as equation (23).

Combining equations (116), (117) and (118), we have

$$
\mathbb{E}[f(x_i^T) - f^*]
\leq \mathbb{E}\left[\left(1 - \frac{\mu\eta}{2}\right)\Phi^T - \left(132L^2\eta - \frac{2}{\eta} - \frac{L}{2}\right)C_T\right]
$$

where the last step is based on the setting of $\eta = 1/(8L)$, the assumption $\bar{L} \geq L$, and equation (115).
We also have

\[
\Phi_0 = f(\bar{x}^0) - f^* + \frac{2\eta}{p}U_0 + \eta \frac{\eta}{m^3n^3b^p}V_0 + \frac{132m^2n^3{\bar{L}}^2\eta}{p}C_0
\]

\[
\leq f(\bar{x}^0) - f^* + \frac{132m^2n^3{\bar{L}}^2\eta}{p} \|S^0 - 1s^0\|^2
\]

\[
\leq 2(f(\bar{x}^0) - f^*) + \epsilon,
\]

where the last step is based on the setting

\[
\hat{K} = \left[\frac{2 + \sqrt{2}}{2\sqrt{\gamma}} \ln \left(1 + \frac{231m^2n^3{\bar{L}}^2}{64L^3}\max\{\bar{k}, \sqrt{mn}\} \sum_{i=1}^{m} \left\|\nabla f_i(\bar{x}^0) - \nabla f(\bar{x}^0)\right\|^2\right)\right].
\]

Therefore, we can achieve

\[
\mathbb{E}[f(x_1^T) - f^*] \leq \epsilon
\]

for all \(i \in [m]\) by taking

\[
T = \left[\frac{16L}{\mu} \ln \left(\frac{2(f(\bar{x}^0) - f^*)}{\epsilon}\right)\right].
\]

**Part II:** We then consider the case of \(\bar{L} \geq \sqrt{mnL}\). Note that the setting \(p = 1\) leads to Algorithm 1 always holds

\[
g_i^t = \nabla f_i(x_i^t),
\]

which implies \(U^t = V^t = 0\) for all \(t\) and the Lyapunov function can be simplified to

\[
\Phi^t = f(\bar{x}^t) - f^* + 132m^2n^3{\bar{L}}^2\eta C^t.
\]

Following the derivation of equations (63) and (115), we have

\[
\mathbb{E}[\Phi^{t+1}]
\]

\[
= \mathbb{E}\left[f(\bar{x}^{t+1}) - f^* + 132m^2n^3{\bar{L}}^2\eta C^{t+1}\right]
\]

\[
\leq \mathbb{E}[(1 - \eta\mu)(f(\bar{x}^t) - f^*)] + (n\bar{L}^2\eta + 264\rho^2(27m^3n^3{\bar{L}}^2\eta^2 + 2)m^2n^3{\bar{L}}^2\eta)\mathbb{E}[C^t]
\]

\[
- \left(\frac{1}{2\eta} - \frac{L}{2} - 2376\rho^2m^6n^6\tilde{L}_t^4\eta^3\right) \mathbb{E}\left[\|x^{t+1} - x^t\|^2\right]
\]

\[
\leq \mathbb{E}\left[(1 - \eta\mu)\Phi^t - (1 - \eta\mu)m^2n^3{\bar{L}}^2\eta C^t\right].
\]
We then follow the derivation of equations (115)-(120) to achieve

$$\mathbb{E}[f(x^T_i) - f^*] \leq \epsilon.$$  \hspace{1cm} (122)

for all $i \in [m]$.

Similar to the analysis of Corollary 2, we can obtain the upper complexity bounds for DEAREST$^+$ (Algorithm 1) under the PL condition as follows.

**Corollary 2.** Under the assumptions and settings of Theorem 5, Algorithm 1 can find an $\epsilon$-suboptimal solution at every agent with the communication rounds of

$$\tilde{O}\left(\frac{\kappa}{\sqrt{\gamma}} \ln \left(\frac{\Delta}{\epsilon}\right)\right),$$

expected LIFO complexity of

$$\mathcal{O}\left((mn + \min\{mn\kappa, \sqrt{mn\bar{\kappa}}\}) \ln \left(\frac{\Delta}{\epsilon}\right)\right),$$

and the expected computation rounds of

$$\tilde{O}\left((n + \kappa + \min\{nk, \sqrt{\frac{n}{m}}\}) \ln \left(\frac{\Delta}{\epsilon}\right)\right).$$

**Proof.** The communication rounds can be upper bounded by

$$\hat{K} + KT = \mathcal{O}\left(\frac{\ln(mn\bar{L}/(Le))}{\sqrt{\gamma}} + \frac{\ln(mnL)}{\sqrt{\gamma}} \cdot \kappa \ln \left(\frac{1}{\epsilon}\right)\right) = \tilde{O}\left(\frac{\kappa \ln(1/\epsilon)}{\sqrt{\gamma}}\right),$$

where we use the settings of $T$, $K$, and $\hat{K}$ in Theorem 5.

We then consider the LIFO complexity. In the case of $\bar{L} > \sqrt{mnL}$, the number of LIFO calls can be upper bounded by

$$mnT = \mathcal{O}\left(mn\kappa \ln \left(\frac{1}{\epsilon}\right)\right),$$

where we use the setting of $T$ in Theorem 5. In the case of $\bar{L} \leq \sqrt{mnL}$, the number of LIFO calls can be upper bounded by

$$mn + (pm + (1 - p)b)T \leq mn + (pmn + b)T$$

$$= \mathcal{O}\left(mn + \left(\frac{\bar{L}}{L} \max\left\{\frac{1}{\bar{\kappa}}, \frac{1}{\sqrt{mn}}\right\} \cdot mn + \left(\frac{L}{\bar{L}} \min\{\bar{\kappa}, \sqrt{mn}\}\right) \cdot \kappa \ln \left(\frac{1}{\epsilon}\right)\right)$$

$$= \mathcal{O}\left(mn + \sqrt{mn\bar{\kappa}} \ln \left(\frac{1}{\epsilon}\right)\right).$$
where we use the settings of \( b, p, \) and \( T \) in Theorem 5. Therefore, the overall LIFO complexity is no more than
\[
\mathcal{O}\left((mn + \min\{mn\kappa, \sqrt{mn}\kappa}\} \ln \left(\frac{1}{\epsilon}\right)\right),
\]

We finally consider the computation rounds. In the case of \( \bar{L} > \sqrt{mnL} \), the number of computation round is no more than
\[
n + nT = \mathcal{O}\left(n + \frac{\bar{L}}{\mu} \ln \left(\frac{1}{\epsilon}\right) \cdot n\right) \leq \mathcal{O}\left(n\kappa \ln \left(\frac{1}{\epsilon}\right)\right).
\] (123)

In the case of \( \bar{L} \leq \sqrt{mnL} \), we following the analysis in Section 4.4.3 which implies the overall expected number of computation rounds is no more than
\[
n + \left((1 - p)E\left[\sum_{t=1}^{T} \max_{i \in [m]} Y^t_i\right]+pn\right)T
\leq \mathcal{O}\left(n + (pn + (1 - p)(bnq + (ln mb)^2))T\right)
\leq \mathcal{O}\left(n + (pn + bnq + (ln mb)^2)T\right)
\leq \mathcal{O}\left(n + \left(\frac{\bar{L}}{L} \max\{\frac{1}{\kappa}, \frac{1}{\sqrt{mn}}\} \cdot n + \frac{\bar{L}}{L} \min\{\bar{\kappa}, \sqrt{mn}\} \cdot \frac{n}{mn} + \left(\min\{\bar{\kappa}, \sqrt{mn \ln m\bar{L}} / L\}\right)^2\right)T\right)
\leq \mathcal{O}\left((n + \kappa + \sqrt{\frac{n}{m} \kappa}) \ln \left(\frac{1}{\epsilon}\right)\right),
\]
where we use \( q = 1/(mn) \) and the settings of \( b, p, \) and \( T \) in Theorem 5. Combining the upper bound (123) in the case of \( \bar{L} \leq \sqrt{mnL} \), the expected number of computation rounds of our method can be upper bounded by
\[
\min\left\{\mathcal{O}\left(n\kappa \ln \left(\frac{1}{\epsilon}\right)\right), \mathcal{O}\left(n + \kappa + \sqrt{\frac{n}{m} \kappa} \ln \left(\frac{1}{\epsilon}\right)\right)\right\}
= \mathcal{O}\left(n + \kappa + \min\left\{\kappa, \sqrt{\frac{n}{m} \bar{\kappa}}\right\} \ln \left(\frac{1}{\epsilon}\right)\right).
\]

\(\square\)

### 6.2 The Lower Bounds under the PL Condition

This section provides the lower bounds for decentralized finite-sum optimization under the PL condition. Compared with the construction of Yue et al. [87], we additionally consider the communication in the network, the finite-sum structure in objective, and the difference between global and mean-squared smoothness. We present the results of communication
We introduce the functions \( \psi_\theta : \mathbb{R} \rightarrow \mathbb{R}, q_{T,t} : \mathbb{R}^{Tt} \rightarrow \mathbb{R} \) and \( g_{T,t} : \mathbb{R}^{Tt} \rightarrow \mathbb{R} \) provided by Yue et al. [87], that is

\[
\psi_\theta(x) = \begin{cases} 
\frac{1}{2}x^2, & x \leq \frac{31}{32}\theta, \\
\frac{1}{2}x^2 - 16(x - \frac{31}{32}\theta)^2, & \frac{31}{32}\theta < x \leq \theta, \\
\frac{1}{2}x^2 - \frac{1}{32}\theta^2, & x < \frac{31}{32}\theta, \\
\frac{1}{2}x^2 + 16(x - \frac{31}{32}\theta)^2, & \theta < x \leq \frac{33}{32}\theta, \\
\frac{1}{2}x^2 - \frac{33}{32}\theta^2, & x > \frac{33}{32}\theta,
\end{cases}
\]

(124)

\[
q_{T,t}(x) = \frac{1}{2} \sum_{i=0}^{Tt-1} \left( \frac{7}{8}x_{iT} - x_{iT+1} \right)^2 + \sum_{j=1}^{T-1} \left( x_{iT+j+1} - x_{iT+j} \right)^2,
\]

and \( g_{T,t}(x) = q_{T,t}(b - x) + \sum_{i=1}^{Tt} b_i - x_i, \)

where we define \( x_0 = 0 \) and \( b \in \mathbb{R}^{Tt} \) with \( b_{kT+\tau} = (7/8)^k \) for \( k \in \{0\} \cup [T-1] \) and \( \tau \in [T] \). We can verify that

\[
g_{T,t}^* = \inf_{y \in \mathbb{R}^{Tt}} g_{T,t}(y) = 0.
\]

The following lemma shows the function \( g_{T,t} \) holds the zero-chain property [16, 58] and describes its smoothness, PL parameter and optimal function value gap, which results the tight lower bound of full-batch first-order methods [87].

**Lemma 19** ([87, Section 4]). The function \( g_{T,t} : \mathbb{R}^{Tt} \rightarrow \mathbb{R} \) holds that:

(a) For any \( x \in \mathbb{R}^{Tt} \), it holds \( \text{prog}(\nabla g_{T,t}(x)) \leq \text{prog}(x) + 1 \).

(b) The function \( g_{T,t} \) is 3T-smooth.

(c) The function \( g_{T,t} \) is \( 1/(aT) \)-PL with \( a = 19708 \).

(d) The function \( g_{T,t} \) satisfies that \( g_{T,t}(0) - g_{T,t}^* \leq 3T \).

(e) For any \( \delta < 0.01 \), \( t = 2\lceil \log_{8/7} 2/(3\delta) \rceil \), and \( x \in \mathbb{R}^{Tt} \) satisfying that \( \text{supp}(x) \subseteq \{1, 2, \ldots, Tt/2\} \), it holds that \( g_{T,t}(x) - g_{T,t}^* > 3T\delta \).

We can decompose the function \( g_{T,t} : \mathbb{R}^{Tt} \rightarrow \mathbb{R} \) defined in equation (124) by introducing the functions \( q_1 : \mathbb{R}^{Tt} \rightarrow \mathbb{R}, q_2 : \mathbb{R}^{Tt} \rightarrow \mathbb{R} \) and \( r : \mathbb{R}^{Tt} \rightarrow \mathbb{R} \) as

\[
q_1(x) = \frac{1}{2} \sum_{i=1}^{Tt/2} (x_{2i-1} - x_{2i})^2,
\]

(125)

\[
q_2(x) = \frac{1}{2} \sum_{i=0}^{Tt-1} \left( \frac{7}{8}x_{iT} - x_{iT+1} \right)^2 + \sum_{j=1}^{(i+1)T/2 - 1} (x_{2j} - x_{2j+1})^2,
\]

(126)

\[
r(x) = \sum_{i=1}^{Tt} b_i - x_i.
\]

(127)
where we suppose $T$ is even and let $x_0 = 0$. Then we can verify that the function $g_{T,t}(\cdot)$ can be written as

$$g_{T,t}(x) = q_1(b - x) + q_2(b - x) + r(x).$$

**Lemma 20.** The functions $q_1$ and $q_2$ are 2-smooth and the function $r$ is 33-smooth.

**Proof.** The smoothness of $r$ can be achieved from the proof of Lemma 4 provided by Yue et al. [87]. Since the functions $q_1$ and $q_2$ are quadratic and hold $q_1(x) \leq x^\top x$ and $q_2(x) \leq x^\top x$ for all $x \in \mathbb{R}^T$, they are 2-smooth. $\square$

We now provide the lower bounds on the communication rounds under the PL condition in the following theorem.

**Theorem 6.** For any $L > 0$, $\bar{L} > 0$, $\mu > 0$, $m \geq 2$, $n > 0$, $\Delta > 0$, $\gamma \in [1 - \cos(\pi/m), 1]$, and $\epsilon > 0$ with $\bar{L} \geq L \geq 78\mu\epsilon$, $\epsilon < 0.01\Delta$, and $a = 19708$, there exist matrix $W \in \mathbb{R}^{m \times m}$ with $1 - \lambda_2(W) = \gamma$ and $L$-mean-squared smooth functions $\{f_{i,j} : \mathbb{R}^d \to \mathbb{R}\}$ such that $f_{i,j}$ is $L$-smooth and $\mu$-PL with $f_0(y) - \inf_{x \in \mathbb{R}^d} f(x) \leq \Delta$. In order to find an $\epsilon$-suboptimal solution of problem $\min_{x \in \mathbb{R}^d} f(x)$, any LIFO algorithm needs at least $\Omega(\kappa/\sqrt{\gamma \ln(1/\epsilon)})$ communication rounds.

**Proof.** According to Lemma 18, there exists a ring-lattice graph $G = \{V, E\}$ with the associated mixing matrix $W \in \mathbb{R}^{m \times m}$ and $V_1, V_2 \subseteq V$ such that $|V| = m$, $1 - \lambda_2(W) = \gamma$, $|V_1| \geq m/3$, $|V_2| \geq m/3$ and the distance between $V_1$ and $V_2$ satisfies $\text{dist}(V_1, V_2) = \Omega(1/\sqrt{\gamma})$.

We consider the function $g_{T,t}$ defined in equation (124) with

$$T = 2 \left\lfloor \frac{\kappa}{18a} \right\rfloor \quad \text{and} \quad t = 2 \left\lfloor \log_2 \frac{2\Delta}{3\epsilon} \right\rfloor,$$

where $L \geq 78\mu\epsilon$ and $\epsilon < 0.01\Delta$. (128)

According to Lemma 14 with

$$g(x) = g_{T,t}(x), \quad \alpha = \frac{\Delta}{3T} \quad \text{and} \quad \beta = \sqrt{\frac{LT}{13\Delta}},$$

we can conclude the function $\hat{g}(x) = \alpha g_{T,t}(\beta x)$ is $37a\beta^2$-smooth, and it holds

$$\hat{g}(0) - \inf_{y \in \mathbb{R}^d} \hat{g}(y) = \alpha \left( g_{T,t}(0) - \inf_{y \in \mathbb{R}^d} g_{T,t}(y) \right) \leq 3\epsilon T,$$

where the inequality is based on Lemma 19(d).

We then construct the hard instance based on the decomposition

$$g_{T,t}(x) = q_1(b - x) + q_2(b - x) + r(x),$$

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where \( q_1, q_2 \) and \( r \) are defined in equations (125), (126), and (127). Specifically, we take \( f_{i,j} \) as

\[
f_{i,j}(x) = \begin{cases} 
\alpha r(\beta x) + \frac{\alpha m}{|V_1|} q_1(b - \beta x), & i \in V_1, \\
\alpha r(\beta x) + \frac{\alpha m}{|V_2|} q_2(b - \beta x), & i \in V_2, \\
\alpha r(\beta x), & \text{otherwise}
\end{cases}
\] (131)

According to Lemma 14, Lemma 20 and equation (131), each component function \( f_{i,j} \) is \( \alpha \beta^2 \left( 33 + 2m/\min\{|V_1|, |V_2|\} \right) \)-smooth. Therefore, the function set \( \{ f_{i,j}\}_{i,j=1}^{m,n} \) is \( \alpha \beta^2 \left( 33 + 2m/\min\{|V_1|, |V_2|\} \right) \)-mean-squared smooth.

The definition of \( f_{i,j} \) in equation (131) also means

\[
f(x) = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f_{i,j}(x) = \hat{g}(x).
\]

The settings of \( \alpha, \beta, T, t, |V_1| \) and \( |V_2| \) in equations (128) and (130) imply

\[
37\alpha \beta^2 = \frac{37L}{39} \leq L, \quad \frac{\alpha \beta^2}{aT} \geq \mu, \quad 3\alpha T \leq \Delta,
\]

and

\[
\alpha \beta^2 \left( 33 + \frac{2m}{\min\{|V_1|, |V_2|\}} \right) \leq L \leq \bar{L}.
\]

Therefore, the global objective \( f = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f_{i,j} \) is \( L \)-smooth, \( \mu \)-PL and satisfies \( f(0) - f^* \leq \Delta \) and the function sets \( \{ f_{i,j}\}_{i,j=1}^{m,n} \) are \( \bar{L} \)-mean-squared smooth, which satisfies our requirements.

Now we show that any LIFO algorithm require at least

\[
\frac{Tt}{2} \text{dist}(V_1, V_2) = \Omega \left( \frac{\kappa}{\sqrt{\gamma}} \ln \left( \frac{1}{\epsilon} \right) \right)
\] (132)

communication rounds to to achieve an \( \epsilon \)-suboptimal solution \( \hat{x} \) of \( f(\cdot) \).

We take \( \delta = \epsilon/\Delta < 0.01 \). According to Lemma 19(e), we have

\[
\|\nabla f(x)\| > 3\alpha T \delta = \epsilon.
\]

for all \( x = [x_1, \ldots, x_{Tt}]^\top \in \mathbb{R}^d \) such that \( \text{prog}(x) \leq Tt/2 \). We let

\[
\widehat{\min}(s, i) \triangleq \max \{ i \in \{0, 1, \ldots, Tt\} : \text{there exists } y = [y_1, \ldots, y_{Tt}]^\top \in \mathcal{M}_s^+ \text{ such that } y_i \neq 0 \},
\] (133)

where we denote \( y_0 = 1 \).
Consider the definitions of $f_{i,j}$, $q_1$, $q_2$ and $r$ in equations (125), (126), (127) and (131). Notice that the fact $\psi_i'(b_i)(b_i) = 0$ implies the partial derivative of $r(x)$ with respect to $x_k$ is zero when $x_k = 0$. This means the computation of any LIFO algorithm holds:

1. If $i \in V_1$ and $\text{nnz}(s,i)$ is odd, one step of local computation can increase at most one dimension for memory of node $i$.
2. If $i \in V_2$ and $\text{nnz}(s,i)$ is even, one step of local computation can increase at most one dimension for memory of node $i$.
3. Otherwise, one step of local computation cannot increase the dimension for memory of node $i$.

In summary, we have

$$\text{nnz}(s+1,i) \leq \begin{cases} \text{nnz}(s,i) + 1, & \text{if } i \in V_1, \text{nnz}(s,i) \equiv 1 \pmod{2}, \\ \text{nnz}(s,i) + 1, & \text{if } i \in V_2, \text{nnz}(s,i) \equiv 0 \pmod{2}, \\ \text{nnz}(s,i), & \text{otherwise.} \end{cases}$$

(134)

We consider the cost to reach the second coordinate from the initial status that $M_0^i = \{0\}$ for all $i \in [m]$. According to equation (134), we need to let a node in $V_1$ reach the first coordinate, which requires at least one local computation step on some node in $V_2$ first. According to the definition of LIFO algorithm (Definition 1), one must perform at least $\text{dist}(V_1,V_2)$ communication rounds for a node in $V_1$ to receive the information of the first coordinate from some node in $V_2$. After above rounds, we can perform at least 1 computation on nodes in $V_1$ to reach the second coordinate. In summary, to reach the second coordinate requires at least $\text{dist}(V_1,V_2)$ communication rounds.

Similarly, to reach the $k$-th coordinate, any LIFO algorithm must perform at least $(k-1)\text{dist}(V_1,V_2)$ communication rounds. Thus, to attain the condition

$$\text{prog}(x) > \frac{Tt}{2},$$

one needs at least

$$\frac{Tt}{2} \text{dist}(V_1,V_2) = \Omega \left( \frac{\kappa}{\sqrt{\gamma}} \ln \left( \frac{1}{\epsilon} \right) \right)$$

communications rounds.

We then provide the lower bound on the IFO complexity for the finite-sum optimization problem with $N$ individual functions under the PL condition, which directly implies the corresponding lower bound on the LIFO complexity in decentralized setting by taking $N = mn$.

**Theorem 7.** For any $\bar{L} > 0$, $L > 0$, $\mu > 0$, $N > 0$, $\Delta > 0$, and $\epsilon > 0$ with $\epsilon < 0.005\Delta$ and $\bar{L} \geq L > \mu$, there exists $L$-mean-squared smooth functions $\{f_i : \mathbb{R}^d \to \mathbb{R}\}_{i=1}^N$ such that the function $\bar{f} = \frac{1}{N} \sum_{i=1}^N f_i$ is $L$-smooth, $\mu$-PL and holds $\bar{f}(x^0) - \inf_{y \in \mathbb{R}^d} \bar{f}(y) \leq \Delta$. In order to find an $\epsilon$-suboptimal solution of problem $\min_{x \in \mathbb{R}^d} \bar{f}(x)$, any IFO algorithm (defined in Appendix C) needs at least $\Omega (N + \min\{N\kappa, \sqrt{N\kappa} \ln(1/\epsilon)\})$ IFO calls.
Proof. We first consider the case of \( \min\{\bar{\kappa}, \kappa \sqrt{N}\} \geq 37a\sqrt{N} \). We apply the function \( g_{T,t} \) defined in equation (124) with

\[
T = \left\lfloor \min\{\bar{\kappa}, \kappa \sqrt{N}\} \right\rfloor / 37a\sqrt{N} \quad \text{and} \quad t = 2 \left\lfloor \log_2 \frac{\Delta}{3\epsilon} \right\rfloor, \quad \text{where} \quad \epsilon < 0.005\Delta.
\]

(135)

According to Lemma 14 with \( g(x) = g_{T,t} \), \( \alpha = \Delta / 3T \) and \( \beta = \sqrt{3 \min\{\sqrt{NL}, NL\} / 37\Delta} \), we can conclude the function \( \hat{g}(x) = \alpha g_{T,t}(\beta x) \) is \( 37\alpha\beta^2 \)-smooth, and it holds

\[
\hat{g}(0) - \inf_{y \in \mathbb{R}^T} \hat{g}(y) = \alpha \left( g_{T,t}(0) - \inf_{y \in \mathbb{R}^T} g_{T,t}(y) \right) \leq 3\alpha T,
\]

(137)

where the inequality is based on Lemma 19(d).

We construct the hard instance according to Lemma 15 with \( g(x) = \hat{g}(x) \), which results the functions

\[
\tilde{f}_i(x) = \alpha g_{T,t}(\beta U^i x) \quad \text{and} \quad \tilde{f}(x) = \frac{1}{N} \sum_{i=1}^{N} \tilde{f}_i(x)
\]

(138)

such that the functions \( \{\tilde{f}_i\}_{i=1}^{N} \) are \( \bar{L} \)-mean-squared smooth and the function \( \tilde{f} \) is \( 37\alpha\beta^2/N \)-smooth and \( \alpha\beta^2/(aNT) \)-PL with

\[
\tilde{f}(x^0) - \tilde{f}^* = \hat{g}(0) - \inf_{y \in \mathbb{R}^T} \hat{g}(y) \leq 3\alpha T,
\]

where \( \hat{f}^* = \inf_{y \in \mathbb{R}^T} \hat{f}(y) \).

The settings of \( \alpha, \beta, T \) and \( t \) in equations (135) and (136) imply

\[
\frac{37\alpha\beta^2}{\sqrt{N}} \leq \bar{L}, \quad \frac{37\alpha\beta^2}{N} \leq L, \quad \frac{\alpha\beta^2}{anT} \geq \mu \quad \text{and} \quad 3\alpha T \leq \Delta.
\]

(136)

(137)

Therefore, the function set \( \{\tilde{f}_i\}_{i=1}^{N} \) is \( \bar{L} \)-average smooth and the function \( f \) is \( L \)-smooth and \( \mu \)-PL with \( \tilde{f}(x^0) - \tilde{f}^* \leq \Delta \), which satisfies our requirements.

Now we show that any IFO algorithm require at least \( \lceil NT/4 \rceil + 1 \) IFO calls to achieve an \( \epsilon \)-suboptimal solution \( \hat{x} \) of the function \( f \). We take \( \delta = 2\epsilon / \Delta \). According to Lemmas 14 and 19(e), for all \( x \in \mathbb{R}^{NT} \) with \( \text{supp}(U^{(i)}x) \subseteq \{1, 2, \cdots, Tt/2\} \), we have

\[
f_i(x) - f_i^* = \alpha g_{T,t}(\beta U^{(i)}x) - \alpha g_{T,t}^* > 3\alpha T\delta = 2\epsilon.
\]

(139)
We consider the vector $x \in \mathbb{R}^{NTt}$ which is achieved by an IFO algorithm with at most $\lfloor NTt/4 \rfloor$ IFO calls. Lemma 19(a) implies such vector $x$ has at most $\lfloor NTt/4 \rfloor$ non-zero entries. We partition $x \in \mathbb{R}^{NTt}$ into $N$ vectors $y^{(1)}, \ldots, y^{(N)} \in \mathbb{R}^{Tt}$ such that $y^{(j)} = U^{(j)}x \in \mathbb{R}^{Tt}$, then there are at least $\lceil N/2 \rceil$ vectors in $\{y^{(j)}\}_{j=1}^{N}$ such that each of them has at least $Tt/2$ zero entries. Lemma 19(a) means there exists index set $I \subseteq [N]$ with $|I| \geq \lceil N/2 \rceil$ such that each $j \in I$ satisfies $\text{supp}(y^{(j)}) \subseteq \{1, 2, \ldots, Tt/2\}$. Therefore, we have

$$f(x) - f^* = \frac{1}{N} \sum_{i=1}^{N} f_i(x) - \alpha g_{T,t}^{*}$$

$$\geq \frac{1}{N} \sum_{i \in I} (f_i(x) - \alpha g_{T,t}^{*})$$

$$\geq \frac{1}{N} \cdot \lceil \frac{N}{2} \rceil \cdot 2\epsilon \geq \epsilon.$$ 

Hence, any IFO algorithm requires at least

$$\left\lfloor \frac{NTt}{4} \right\rfloor + 1 = \Omega \left( \min \{\kappa N, \bar{\kappa} \sqrt{N} \} \ln \left( \frac{1}{\epsilon} \right) \right)$$

IFO calls to achieve an $\epsilon$-suboptimal solution of function $\tilde{f}(\cdot)$ defined in equation (138).

We then consider the case $\min \{\bar{\kappa}, \kappa \sqrt{N} \} < 37a \sqrt{N}$ and construct the another hard instance. We assume that $\epsilon < \Delta/2$. We follow the functions provided by Li et al. [42], i.e., define $\tilde{f}_i : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\tilde{f} : \mathbb{R}^d \rightarrow \mathbb{R}$ as

$$\tilde{f}_i(x) = c\langle v_i, x \rangle + \frac{L}{2} \|x\|^2 \quad \text{and} \quad \tilde{f}(x) = \frac{1}{N} \sum_{i=1}^{N} \tilde{f}_i(x).$$

(140)

for all $i \in [N]$, where $c = \sqrt{L \Delta}$, $d = 2N^2$ and

$$v_i = \left[ I\left( \left\lfloor \frac{1}{2N} \right\rfloor = i \right), I\left( \left\lfloor \frac{2}{2N} \right\rfloor = i \right), \ldots, I\left( \left\lfloor \frac{2N}{2N} \right\rfloor = i \right) \right]^\top \in \mathbb{R}^d.$$ 

We can verify that $\nabla^2 f(x) = L \mathbf{I} \succeq \mu \mathbf{I}$ for any $x \in \mathbb{R}^d$, which means the function $\tilde{f}$ is $L$-smooth and $\mu$-strongly convex, also $\mu$-PL. We also have

$$f^* = \frac{1}{N} \sum_{i=1}^{N} \left( c\langle u_i, x^* \rangle + \frac{L}{2} \|x^*\|^2 \right)$$

$$= \frac{c}{N} \sum_{i=1}^{N} \langle u_i, x^* \rangle + \frac{L}{2} \|x^*\|^2$$

$$= - \frac{c^2}{2L N^2} \left\| \sum_{i=1}^{N} u_i \right\|^2 = - \frac{c^2}{L},$$

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where $x^* = -(c/NL)\mathbf{1}$ is the minimizer of $f$. Then the optimal function value gap holds

$$f(x^0) - f^* = 0 - f^* = \frac{c^2}{L} = \Delta.$$ 

We consider any IFO algorithm with initial point $0$ the $t$-th IFO calls, which holds

$$x^t \in \text{span} \{\nabla f_{i_0}(x^0), \ldots, \nabla f_{i_{t-1}}(x^{t-1})\} = \text{span} \{u_{i_0}, \ldots, u_{i_{t-1}}\},$$

where $i_\tau \in [N]$ is the index of individual which is accessed at the $\tau$-th IFO calls. Since each $u_{i_\tau}$ has $2N$ nonzero entries, any vector $x \in \mathbb{R}^d$ achieved by at most $N/2$ IFO calls has at least $d - \frac{N}{2} \cdot 2N = 2N^2 - N^2 = N^2$ zero entries. Let $I_0 = \{j \in [2n^2] : x_j = 0\}$, then we have $|I| \geq N^2$. Based on the construction of $f_i$ and $u_i$, we have

$$f(x) - f^* = \frac{1}{N} \sum_{i=1}^{N} \left( c(u_i, x) + \frac{L}{2} \|x\|_2 \right) - \left( -\frac{c^2}{L} \right)$$

$$= \frac{2N^2}{N} \left( \frac{c}{N} x_j^2 + \frac{L}{2} x_j^2 + \frac{c^2}{2LN^2} \right)$$

$$= \frac{2N^2}{N} \left( \frac{c}{N} x_j^2 + \frac{L}{2} x_j^2 + \frac{c^2}{2LN^2} \right) + \sum_{j \in I_0} \left( \frac{c}{N} x_j^2 + \frac{L}{2} x_j^2 + \frac{c^2}{2LN^2} \right)$$

$$\geq N^2 \cdot \frac{c^2}{2LN^2} + \sum_{j \notin I_0} \left( x_j + \frac{c}{N} \right)^2$$

$$\geq \frac{\Delta}{2} > \epsilon,$$

Hence, achieving an $\epsilon$-suboptimal solution requires at least $N/2 + 1 = \Omega(N)$ IFO calls.

Combining the results in above two hard instances, we achieve the lower bound on IFO complexity of

$$\Omega \left( N + \min \{N\kappa, \sqrt{N\kappa} \} \ln \left( \frac{1}{\epsilon} \right) \right).$$

Finally, we show the lower bound on the communication rounds under the PL condition.
Theorem 8. For any $\bar{L} > 0$, $L > 0$, $m > 0$, $n > 0$, $\Delta > 0$, and $\epsilon > 0$ with $\epsilon < 0.005\Delta$, $\bar{L} \geq L \geq 37a\mu$, and $a = 19708$, there exists $\bar{L}$-mean-squared smooth functions $\{f_{i,j} : \mathbb{R}^d \to \mathbb{R}\}$ such that the function $f = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}$ is $L$-smooth and $\mu$-PL with $f(x^0) - \inf_{y \in \mathbb{R}^d} f(y) \leq \Delta$. In order to find an $\epsilon$-suboptimal solution of problem $\min_{x \in \mathbb{R}^d} f(x)$, any LIFO algorithm needs at least the computation rounds of $\Omega(n + (\kappa + \min\{\kappa n, \sqrt{n/m}\}) \ln(1/\epsilon))$.

Proof. We first show the lower bound in the view of linear speed-up. According to Theorem 7 with $N = mn$ and $\epsilon < 0.005\Delta$, there exist $\bar{L}$-mean-squared smooth function set $\{f_{i,j}\}_{i=1,j=1}^{m,n}$ such that the function $f(\cdot) = \frac{1}{N} \sum_{i=1}^{N} f_{i}(\cdot)$ is $L$-smooth, $\mu$-PL and satisfies $f(x^0) - f^* \leq \Delta$. In order to find an $\epsilon$-suboptimal solution of problem $\min_{x \in \mathbb{R}^d} f(x)$, any IFO algorithm needs at least

$$\Omega \left( mn + \min\{mn\kappa, \sqrt{mn\bar{\kappa}}\} \ln \left( \frac{1}{\epsilon} \right) \right) \quad (141)$$

IFO calls. For the distributed setting with $m$ nodes, any DIFO algorithm can perform at most $m$ LIFO calls in per computation round. Therefore, we achieve the lower complexity bound of on the computation rounds of

$$\Omega \left( \frac{mn}{m} + \min\left\{ \frac{mn\kappa}{m}, \sqrt{mn\bar{\kappa}} \right\} \ln \left( \frac{1}{\epsilon} \right) \right) = \Omega \left( n + \min\left\{ n\kappa, \sqrt{n\bar{\kappa}} \right\} \ln \left( \frac{1}{\epsilon} \right) \right). \quad (142)$$

Now show the lower bound $\Omega(\kappa \ln(1/\epsilon))$ on the computation rounds. It is worth noting that it is necessary since the case of $\bar{L}/L < \sqrt{m/n}$ leads to $\kappa > \min\{\kappa n, \bar{\kappa} \sqrt{n/m}\}$.

We consider the PL function $g_{T,t}$ defined in equation (124) with

$$T = \left\lfloor \frac{\kappa}{37a} \right\rfloor, \quad t = 2 \left\lfloor \log_{\frac{37a}{2\Delta}} \frac{2\Delta}{3\epsilon} \right\rfloor,$$  

where $\epsilon < 0.01\Delta$ and $a = 19708$. \quad (143)

According to Lemma 14 with

$$g(x) = g_{T,t}(x), \quad \alpha = \frac{\Delta}{3T} \quad \text{and} \quad \beta = \sqrt{\frac{3LT}{37\Delta}},$$  

we can conclude the function $\hat{g}(x) = \alpha g_{T,t}(\beta x)$ is $37a\beta^2$-smooth and $\alpha \beta^2/(\alpha T)$-PL, $\hat{g}(x) - \inf_{y \in \mathbb{R}^d} \hat{g}(y) = \alpha \left( g_{T,t}(x) - \inf_{y \in \mathbb{R}^d} g_{T,t}(y) \right)$, and it holds

$$\hat{g}(0) - \inf_{y \in \mathbb{R}^d} \hat{g}(y) = \alpha \left( g_{T,t}(0) - \inf_{y \in \mathbb{R}^d} g_{T,t}(y) \right) \leq 3\alpha T, \quad (145)$$

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where the inequality is based on Lemma 19(d). We construct the hard instance as

\[ f_{i,j}(x) = \hat{g}(x) \quad \text{and} \quad f(x) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}(x) \]  

such that \( \{f_{i,j}\}_{i=1,j=1}^{m,n} \) is \( 37\alpha \beta^2 \)-mean-squared smooth and \( f \) is \( 37\alpha \beta^2 \)-smooth with

\[ f(x^0) - f^* = \hat{g}(0) - \inf_{y \in R^T} \hat{g}(y) \leq 3\alpha T. \]  

The settings of \( \alpha, \beta, T \) and \( t \) in equations (143) and (144) imply

\[ 37\alpha \beta^2 = L \leq \bar{L}, \quad \frac{\alpha \beta^2}{aT} \geq \mu \quad \text{and} \quad 3\alpha T \leq \Delta. \]  

Therefore, the global objective \( f = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j} \) is \( L \)-smooth, \( \mu \)-PL and satisfies \( f(0) - f^* \leq \Delta \), and the function set \( \{f_{i,j}\} \) is \( \bar{L} \)-mean-squared smooth, which satisfies our requirements.

We then show that any DIFO algorithm require at least \( Tt/2 + 1 \) computation rounds to achieve an \( \epsilon \)-suboptimal solution of the problem \( \min_{x \in R^d} f(x) \). We take \( \delta = \epsilon / \Delta \). According to Lemma 14 and 19(e), for all \( x_i \in R^{Tt} \) with \( \text{supp}(x_i) \subseteq \{1, 2, \cdots, Tt/2\} \), we have

\[ f(x_i) - f^* = \alpha g_{T,t}^*(\beta x_i) - \alpha g_{T,t}^* > 3\alpha T\delta = \epsilon. \]

We consider the vectors \( x_1, \ldots, x_m \in R^{Tt} \) which are achieved by a DIFO algorithm with at most \( Tt/2 \) computation rounds. Lemma 19(a) implies such vector \( x_i \) has at most \( Tt/2 \) non-zero entries, that is \( \text{supp}(x_i) \subseteq \{1, 2, \cdots, Tt/2\} \) for all \( i \in [m] \). Therefore, we have

\[ f(x_i) - f^* > \epsilon. \]

Hence, any LIFO algorithm requires at least

\[ \frac{Tt}{2} + 1 = \Omega \left( \kappa \ln \left( \frac{1}{\epsilon} \right) \right) \]

computation rounds to achieve an \( \epsilon \)-suboptimal solution of function \( f(\cdot) \) defined in equation (146) at local nodes.

Combining above two hard instance, we obtain the lower bound on the communication rounds of

\[ \Omega \left( n + \left( \kappa + \min \left\{ n \kappa, \sqrt{\frac{n}{\kappa}} \right\} \ln \left( \frac{1}{\epsilon} \right) \right) \right). \]

\[ \square \]
7 Numerical Experiments

In this section, we validate our theory by conducting the numerical experiments on the following problems:

(a) Hard Instances: We consider the hard instance for the general nonconvex case defined in equation (103) with $T = 10^2$ and $\beta = 10^3$ and the hard instance for the PL case defined in equation (129) with $T = 100$, $t = 5$, and $\beta = 10^3$.

(b) Linear Regression: We consider the (regularized) linear regression, which corresponds to the individual function

$$f_{i,j}(x) = \frac{1}{2}(a_{i,j}^\top x - b_{i,j})^2 + \lambda r_\alpha(x),$$

where

$$r_\alpha(x) = \sum_{k=1}^{d} \frac{\alpha x_k^2}{1 + \alpha x_k^2}$$

is the nonconvex regularizer [6] with $\alpha = 10$, $a_{i,j} \in \mathbb{R}^d$ is the feature of the $j$-th sample on the agent $i$, $b_{i,j} \in \mathbb{R}$ is the corresponding label, and $\lambda \geq 0$ is hyperparameter. We set $\lambda = 10^{-8}$ for the general nonconvex case and $\lambda = 0$ for the PL case.

(c) Logistic Regression: We consider the (regularized) logistic regression, which corresponds to the individual function

$$f_{i,j}(x) = \ln(1 + \exp(-b_{i,j}a_{i,j}^\top x)) + \lambda r_\alpha(x),$$

where $r_\alpha$ is defined in equation (148) with $\alpha = 10$, $a_{i,j} \in \mathbb{R}^d$ is the feature of the $j$-th sample on the agent $i$, $b_{i,j} \in \{-1, 1\}$ is the corresponding label, and $\lambda \geq 0$ is hyperparameter. We also set $\lambda = 10^{-8}$ for the general nonconvex case and $\lambda = 0$ for the PL case.

We include the experiments on problems of linear regression and logistic regression on datasets “Wikipedia Math Essential” ($mn = 1,056$, $d = 730$) [67] and “RCV1.binary” [19, 37] ($N = 20,096$, $d = 47,236$), respectively. We perform all experiments on the ring graph with 16 agents (i.e., $m = 16$).

For the general nonconvex case, we compare DEAREST$^+$ (Algorithm 1) with the classical Decentralized Stochastic Gradient Descent (DSGD) [43, 54] and the state-of-the-art variance reduced method DESTRESS [39]. For the PL case, we compare DEAREST$^+$ (Algorithm 1) with DSGD [43, 54] and DRONE [10]. We tune the stepsize $\eta$ from $\{10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$ for all methods and the mini-batch size $b$ from $\{2^4, 2^5, 2^6, 2^7\}$ for DSGD, DESTRESS, and DEAREST$^+$. For DEAREST$^+$, we simply set the probability for full gradient computation as $p = b/(mn)$ according to the theoretical setting in Theorem 1. For DRONE, we set the mini-batch size of agents be 4 and tune probability for the full participation from $\{0.1, 0.3, 0.5, 0.9\}$. 

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We present the experimental results of the communication rounds, the LIFO calls, and the computation rounds against the gradient norm for the general nonconvex problems in Figures 1–3. We can observe that the proposed DEAREST$^+$ outperforms the baselines in terms of all three metrics. We also present the experimental results of the communication rounds, the LIFO calls, and the computation rounds against the objective function value for the PL problems in Figures 4–6. We can observe that the proposed DEAREST$^+$ outperforms the baselines in terms of the LIFO calls and the computation rounds. Additionally, DRONE has the comparable communication rounds to DEAREST$^+$. This is because of the iteration scheme of DRONE can be regarded as the special case of DEAREST$^+$ with $n = 1$. Although the analysis of DRONE in our conference paper [10] only considers the mean-squared smoothness, this method with the appropriate parameter setting (Theorem 5 with $n = 1$) can achieve the communication complexity depends on the global smoothness dependency.
8 Conclusion

In this paper, we study the complexity of decentralized smooth nonconvex finite-sum optimization by considering both the smoothness of the global objective and the mean-squared smoothness of all individual functions. We propose DEAREST$^+$ method, which simultaneously achieves the (near) optimal complexity in terms of the communication rounds, the LIFO calls, and the computation rounds. We also extend our results to PL condition and show the (near) optimality in such case. In future work, it is possible to apply our ideas to establish the sharper complexity bounds for the decentralized optimization problem over time-varying network [51] and consider the problem in online setting [8, 48, 85]. We are also interested in extending our results to address the functions that satisfy the Kurdyka–Łojasiewicz inequality [9, 12, 13, 26, 31, 95].
Appendix A  The Proofs in Section 2.2

This section provides the proofs for the relationships among the smoothness parameters shown in Section 2.2.

### A.1 The Proof of Proposition 1

**Proof.**

**Part (a):** For any $i \in [m]$ and $j \in [n]$, we have

$$
\| \nabla f_{i,j}(x) - \nabla f_{i,j}(y) \|_2 \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \| \nabla f_{i,j}(x) - \nabla f_{i,j}(y) \|_2 \leq mn\bar{L}^2 \| x - y \|_2 \quad (A1)
$$

for all $x, y \in \mathbb{R}^d$, where the last step is based on \( \{ f_{i,j} \} \) is \( \bar{L} \)-mean-squared-smooth. Therefore, each individual function $f_{i,j}$ is \( \sqrt{mn}\bar{L} \)-smooth and we have proved equation (8). Consequently, we have

$$
\| \nabla f_i(x) - \nabla f_i(y) \|_2 = \left\| \frac{1}{n} \sum_{j=1}^{n} (\nabla f_{i,j}(x) - \nabla f_{i,j}(y)) \right\|_2 
\leq \frac{1}{n} \sum_{j=1}^{n} \| \nabla f_{i,j}(x) - \nabla f_{i,j}(y) \|_2 \quad (8)
\leq mn\bar{L}^2 \| x - y \|_2
$$

for all $x, y \in \mathbb{R}^d$, where the first inequality is based on the fact $\left\| \frac{1}{n} \sum_{i=1}^{n} a_i \right\|_2 \leq \frac{1}{n} \sum_{i=1}^{n} \| a_i \|_2$ for all $a_1, \ldots, a_n \in \mathbb{R}^d$; the second inequality is based on equation (A1).

**Part (b):** The mean-squared-smoothness condition means

$$
\| \nabla f(x) - \nabla f(y) \|_2 = \left\| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \nabla f_{i,j}(x) - \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \nabla f_{i,j}(y) \right\|_2 
\leq \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \| \nabla f_{i,j}(x) - \nabla f_{i,j}(y) \|_2 \quad (7)
\leq \bar{L}^2 \| x - y \|_2
$$

for all $x, y \in \mathbb{R}^d$, where the first inequality is based on the fact

$$
\left\| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \right\|_2 \leq \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \| a_{i,j} \|_2
$$

for all $a_{1,1}, \ldots, a_{m,n} \in \mathbb{R}^d$; the last step is based on \( \{ f_{i,j} \} \) is \( \bar{L} \)-mean-squared-smooth. Therefore, the objective $f$ is $\bar{L}$-smooth.
Part (c): If the number of components $mn$ is even, we let
\begin{align*}
  f_{i,j}(x) &= \begin{cases} 
    (1 + c_{\text{even}})g(x), & \text{if } i + j \text{ is even,} \\
    (1 - c_{\text{even}})g(x), & \text{if } i + j \text{ is odd,}
  \end{cases} \\
  \text{(A2)}
\end{align*}

where $g(x) = \frac{L}{2} \|x\|^2$ and
\begin{align*}
  c_{\text{even}} &= \sqrt{\frac{L^2}{L^2} - 1}. \\
  \text{(A3)}
\end{align*}

Then we have
\begin{align*}
  f(x) &= \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}(x) \\
  \tag{A2} &= \frac{1}{mn} \left( \frac{mn(1 + c_{\text{even}})}{2} + \frac{mn(1 - c_{\text{even}})}{2} \right) g(x) \\
  \tag{A3} &= g(x) = \frac{L}{2} \|x\|^2
\end{align*}

that means
\[
  \|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|,
\]

and
\begin{align*}
  \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \|\nabla f_{i,j}(x) - \nabla f_{i,j}(y)\|^2 \\
  \tag{A2} &= \frac{1}{mn} \left( \frac{mn(1 + c_{\text{even}})^2}{2} + \frac{mn(1 - c_{\text{even}})^2}{2} \|\nabla g(x) - \nabla g(y)\|^2 \right) \\
  \tag{A3} &= \left( \frac{(1 + c_{\text{even}})^2}{2} + \frac{(1 - c_{\text{even}})^2}{2} \right) \|\nabla g(x) - \nabla g(y)\|^2 \\
  &= (1 + c_{\text{even}}^2) L^2 \|x - y\|^2 = \bar{L}^2 \|x - y\|^2
\end{align*}

for all $x, y \in \mathbb{R}^d$. Therefore, the functions $\{f_{i,j}\}$ satisfy Assumption 2 and 3 with the tight $L$ and $\bar{L}$, respectively.

If the number of components $mn$ is odd, we let
\begin{align*}
  f_{i,j}(x) &= \begin{cases} 
    g(x), & \text{if } (i,j) = (1,1), \\
    (1 + c_{\text{odd}})g(x), & \text{if } (i,j) \neq (1,1) \text{ and } i + j \text{ is even,} \\
    (1 - c_{\text{odd}})g(x), & \text{if } (i,j) \neq (1,1) \text{ and } i + j \text{ is odd,}
  \end{cases} \\
  \text{(A4)}
\end{align*}

for all $x, y \in \mathbb{R}^d$. Therefore, the functions $\{f_{i,j}\}$ satisfy Assumption 2 and 3 with the tight $L$ and $\bar{L}$, respectively.

If the number of components $mn$ is odd, we let
\begin{align*}
  f_{i,j}(x) &= \begin{cases} 
    g(x), & \text{if } (i,j) = (1,1), \\
    (1 + c_{\text{odd}})g(x), & \text{if } (i,j) \neq (1,1) \text{ and } i + j \text{ is even,} \\
    (1 - c_{\text{odd}})g(x), & \text{if } (i,j) \neq (1,1) \text{ and } i + j \text{ is odd,}
  \end{cases} \\
  \text{(A4)}
\end{align*}

for all $x, y \in \mathbb{R}^d$. Therefore, the functions $\{f_{i,j}\}$ satisfy Assumption 2 and 3 with the tight $L$ and $\bar{L}$, respectively.
where \( g : \mathbb{R}^d \to \mathbb{R} \) is some \( L \)-smooth function and
\[
c_{\text{odd}} = \sqrt{\frac{mnL^2/L^2 - 1}{mn - 1}} - 1. \tag{A5}
\]

Then we have
\[
f(x) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}(x)
\]
\[
= \frac{1}{mn} \left( g(x) + \frac{(mn-1)(1+c_{\text{odd}})}{2} g(x) + \frac{(mn-1)(1-c_{\text{odd}})}{2} g(x) \right)
\]
\[
= g(x) = \frac{L}{2} \|x\|^2
\]

and
\[
\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \|\nabla f_{i,j}(x) - \nabla f_{i,j}(y)\|^2
\]
\[
= \frac{1}{mn} \left( 1 + \frac{(mn-1)(1+c_{\text{odd}})^2}{2} + \frac{(mn-1)(1-c_{\text{odd}})^2}{2} \right) \|\nabla g(x) - \nabla g(y)\|^2 \tag{A4}
\]
\[
= \frac{1}{mn} \left( 1 + \frac{mn-1}{2} \left( 2 + \frac{2(mnL^2/L^2 - 1)}{mn-1} - 2 \right) \right) \|\nabla g(x) - \nabla g(y)\|^2 \tag{A5}
\]
\[
= \frac{1}{mn} \left( 1 + \frac{mn-1}{2} \cdot \frac{2(mnL^2/L^2 - 1)}{mn-1} \right) \|\nabla g(x) - \nabla g(y)\|^2
\]
\[
= \frac{1}{mn} \cdot \frac{mn\bar{L}^2}{L^2} \cdot L^2 \|x - y\|^2 \leq \bar{L}^2 \|x - y\|^2
\]

for all \( x, y \in \mathbb{R}^d \). Therefore, the functions \( \{f_{i,j}\} \) satisfy Assumption 2 and 3 with the tight \( L \) and \( \bar{L} \), respectively. \( \square \)

**A.2 The Proof of Proposition 2**

**Proof. Part (a):** The local smoothness condition means
\[
\|\nabla f(x) - \nabla f(y)\|^2
\]
\[
= \left\| \frac{1}{m} \sum_{i=1}^{m} \nabla f_i(x) - \frac{1}{m} \sum_{i=1}^{m} \nabla f_i(y) \right\|^2
\]
\[
\leq \frac{1}{m} \sum_{i=1}^{m} \|\nabla f_i(x) - \nabla f_i(y)\|^2 \leq L_t^2 \|x - y\|^2, \tag{10}
\]

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where the first inequality is based on the fact \( \frac{1}{m} \sum_{i=1}^{m} a_i \| a_i \|^2 \leq \frac{1}{m} \sum_{i=1}^{m} \| a_i \|^2 \) for all \( a_1, \ldots, a_m \in \mathbb{R}^d \); the last step is based on each \( f_i \) is \( L_\ell \)-smooth (Assumption 4). Therefore, the objective \( f \) is \( L_\ell \)-smooth.

**Part (b):** The local mean-squared smoothness condition means

\[
\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \| \nabla f_{i,j}(x) - \nabla f_{i,j}(y) \|^2 \leq \frac{1}{m} \sum_{i=1}^{m} \bar{L}_\ell^2 \| x - y \|^2 = \bar{L}_\ell^2 \| x - y \|^2 ,
\]

where the first inequality is based on the functions \( \{ f_{i,j} \}_{j=1}^{n} \) are \( \bar{L}_\ell \)-mean-squared smooth for all \( i \in [m] \) (Assumption 5). Therefore, the individual functions \( \{ f_{i,j} \}_{i,j=1}^{m,n} \) are \( \bar{L}_\ell \)-mean-squared smooth.

**Part (c):** We let

\[
f_i(x) = \begin{cases} 
\frac{L_\ell}{2} \| x \|^2 , & \text{if } i = 1, \\
\frac{mL - L_\ell}{2(m-1)} \| x \|^2 , & \text{if } i \geq 2.
\end{cases}
\]

Then we have

\[
f(x) = \frac{1}{m} \sum_{i=1}^{m} f_i(x) = \frac{1}{m} \left( \frac{L_\ell}{2} + \frac{(m-1)(mL - L_\ell)}{2(m-1)} \right) \| x \|^2 = \frac{L_\ell}{2} \| x \|^2 .
\]

Additionally, the function \( f_1 \) has the largest (tight) smoothness parameter \( L_\ell \) among the functions \( \{ f_i \}_{i=1}^{m} \), since it holds

\[ L_\ell \geq \frac{mL - L_\ell}{m-1}. \]

Therefore, the functions \( \{ f_i \}_{i=1}^{m} \) satisfy Assumption 2 and 4 with the tight \( L \) and \( L_\ell \), respectively.

If the number of agents \( m \) is odd, we let

\[
f_i(x) = \begin{cases} 
(1 + c_{\text{even}})g(x) , & \text{if } i + j \text{ is even}, \\
(1 - c_{\text{even}})g(x) , & \text{if } i + j \text{ is odd},
\end{cases}
\]

where \( g(x) = \frac{L_\ell}{2} \| x \|^2 \) and

\[
c_{\text{even}} = \sqrt{\frac{L_\ell^2}{L^2} - 1} .
\]
Part (d): We let
\[ f_{i,j}(x) = \begin{cases} \frac{\bar{L}}{2} \|x\|^2, & \text{if } i = 1, \\ \frac{m \bar{L} - \bar{L}}{2(m-1)} \|x\|^2, & \text{if } i \geq 2 \end{cases} \]
for all \( j \in [n] \). We then have
\[
\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \| \nabla f_{i,j}(x) - \nabla f_{i,j}(y) \|^2
= \frac{1}{m} \left( \bar{L} + \frac{(m-1)(m \bar{L} - \bar{L})}{m-1} \right) \| x - y \|^2 = \bar{L} \| x - y \|^2.
\]
Additionally, we have
\[
\frac{1}{n} \sum_{j=1}^{n} \| \nabla f_{i,j}(x) - \nabla f_{i,j}(y) \|^2
= \begin{cases} \frac{\bar{L}}{2} \| x - y \|^2, & \text{if } i = 1, \\ \frac{m \bar{L} - \bar{L}}{m-1} \| x - y \|^2, & \text{if } i \geq 2, \end{cases}
\]
which means the function set \( \{f_{i,j}\}_{j=1}^{n} \) has the largest (tight) smoothness parameter \( \bar{L} \) among the function sets \( \{f_{1,j}\}_{j=1}^{n}, \ldots, \{f_{m,j}\}_{j=1}^{n} \), since it holds
\[
\bar{L} \geq \frac{m \bar{L} - \bar{L}}{m-1}.
\]
Therefore, the functions \( \{f_{i,j}\}_{i,j=1}^{m,n} \) satisfy Assumption 3 and 5 with the tight \( L \) and \( L_{\ell} \), respectively.

Appendix B The Finite-Sum Optimization with Convex Individual Functions

Recall that Remark 6 says that the IFO complexity of vanilla gradient descent may be sharper than the stochastic recursive gradient method since the ratio \( \bar{L}/L \) can be arbitrary large. However, the similar result does not hold in the convex case. We consider the finite-sum optimization problem
\[
\min_{x \in \mathbb{R}^d} f^c(x) = \frac{1}{n} \sum_{i=1}^{n} f^c_i(x),
\]
where \( f^c : \mathbb{R}^d \to \mathbb{R} \) is \( L \)-smooth and \( \mu \)-strongly convex, and each \( f^c_i : \mathbb{R}^d \to \mathbb{R} \) is \( L_{\max} \)-smooth and convex. It is well-known that accelerated gradient descent (AGD) [58] can find an \( \epsilon \)-suboptimal solution of problem (B8) with \( O(\sqrt{L/\mu \ln(1/\epsilon)}) \) exact gradient
calls, corresponding to the IFO complexity of $O(n\sqrt{L/\mu} \ln(1/\epsilon))$. On the other hand, the stochastic variance reduced gradient methods (e.g., Katyusha [3]) can achieve the IFO complexity of $O((n + \sqrt{nL_{\max}/\mu}) \ln(1/\epsilon))$, which is always better or equal to the result of AGD since the ratio $L_{\max}/L$ (for the tight $L_{\max}$ and $L$) is no larger than $n$ in the convex setting.

**Proposition 4.** Suppose the differentiable functions $f_1^c, \ldots, f_n^c : \mathbb{R}^d \to \mathbb{R}$ are convex, and the function $f^c = \frac{1}{n} \sum_{i=1}^n f_i^c$ is $L$-smooth, then each $f_i^c$ is $nL$-smooth.

**Proof.** For all $x, y \in \mathbb{R}^d$, the convexity of the individual function means

$$f_j^c(y) - f_j^c(x) - \langle \nabla f_j^c(x), y - x \rangle \geq 0 \quad (B9)$$

for all $j \in [n]$. Combining the smoothness of $f^c = \frac{1}{n} \sum_{i=1}^n f_i^c$, we have

$$0 \leq f^c(x) - f^c(y) - \langle \nabla f^c(x), y - x \rangle \leq \frac{L}{2} \|x - y\|^2. \quad (B10)$$

Then we have

$$f_i^c(y) - f_i^c(x) - \langle \nabla f_i^c(x), y - x \rangle \leq \frac{nL}{2} \|x - y\|^2,$$

for all $i \in [n]$ and $x, y \in \mathbb{R}^d$, which finishes the proof. \qed

**Appendix C  The IFO Algorithm**

We consider solving the finite-sum optimization problem

$$\min_{x \in \mathbb{R}^d} f(x) \triangleq \frac{1}{N} \sum_{i=1}^N f_i(x)$$

where each $f_i : \mathbb{R}^d \to \mathbb{R}$ is differentiable. We formally present the definition of the incremental first-order oracle (IFO) algorithm as follows [3, 22, 24, 30, 32, 42, 69, 90, 92].

**Definition 3.** An IFO algorithm for given initial point $x^0$ is defined as a measurable mapping from functions $\{f_i\}_{i=1}^N$ to an infinite sequence of point and index pairs $\{(x^t, i_t)\}_{t=0}^{\infty}$ with random variable $i_t \in [N]$, which satisfies

$$x^t \in \text{span}(\{x^0, \ldots, x^{t-1}, \nabla f_{i_0}(x^0), \ldots, \nabla f_{i_{t-1}}(x^{t-1})\}),$$

where $\text{span}(\cdot)$ denotes the linear span and $i_t$ denotes the index of individual function chosen at the $t$-th step.
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