Non-commutative/non-associative IIA (IIB) geometries from $Q$- and $R$-branes and their intersections

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Abstract

In this paper we discuss the construction of non-geometric $Q$- and $R$-branes as sources of non-geometric $Q$- and $R$-fluxes in string compactifications. The non-geometric $Q$-branes, being obtained via T-duality from the NS 5-brane or respectively from the KK-monopole, are still local solutions of the standard NS action, where however the background fields $G$ and $B$ possess non-geometric global monodromy properties. We show that using double field theory and redefined background fields $\tilde{G}$ and $\beta$ as well as their corresponding effective action, the $Q$-branes are locally and globally well behaved solutions. Furthermore the $R$-brane solution can be at least formally constructed using dual coordinates. We derive the associated non-geometric $Q$- and $R$-fluxes and discuss that closed strings moving in the space transversal to the world-volumes of the non-geometric branes see a non-commutative or a non-associative geometry.

In the second part of the paper we construct intersecting $Q$- and $R$-brane configurations as completely supersymmetric solutions of type IIA/B supergravity with certain $SU(3) \times SU(3)$ group structures. In the near horizon limit the intersecting brane configurations lead to type II backgrounds of the form $AdS_4 \times M_6$, where the six-dimensional compact space $M_6$ is a torus fibration with various non-geometric $Q$- and $R$-fluxes in the compact directions. It exhibits an interesting non-commutative and non-associative geometric structure. Furthermore we also determine some of the effective four-dimensional superpotentials originating from the non-geometric fluxes.
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A $Q$-branes as solution of the NS action
1 Introduction

Non-geometric string backgrounds are interesting because of several reasons. Already early examples for classes of non-geometric string constructions, such as covariant lattices [1], fermionic string constructions [2,3] or asymmetric orbifolds [4], have shown that non-geometric string backgrounds are abundant and provide generic points in the string landscape. More recently it became clear that generalized complex geometries, which include T-folds (spaces that are globally not well-defined) [5–8] and other non-geometric string backgrounds, have very interesting mathematical properties. They generalize Calabi-Yau manifolds to spaces with several kind of geometric and non-geometric fluxes and generalized $SU(3) \times SU(3)$ group structures. Finally, non-geometric string backgrounds naturally arise in the context of doubled field theory [9] and T-duality, as they can be constructed by applying chains of T-duality transformations to geometric flux backgrounds.

Another reason for the importance of non-geometric string background is the observation that non-geometric fluxes are part of the effective 4D (super)-potential. Even without reference to an underlying non-geometric compactification from 10 dimensions, non-geometric fluxes already arise from the requirement of T-duality invariance of the effective scalar potential. In particular there is an interesting relation to the gauge algebra of gauged supergravity theories and the (non)-geometric fluxes in their effective potential. In many cases, the presence of these fluxes is important for the moduli stabilization process, and one expects to obtain phenomenologically interesting string ground states from supergravity potentials with non-geometric fluxes. Furthermore it was recently shown [10–12] that 4D non-geometric fluxes can be expressed by a new ten-dimensional effective action, where the standard $H$-flux term is replaced by terms, which encapsulate the non-geometric fluxes. This new ten-dimensional effective action is indeed well defined for non-geometric T-fold spaces. It can be obtained by a certain field redefinition which is closely motivated by T-duality and double field theory.

Finally, it was discovered that closed string coordinates in a space that is "deformed" by non-geometric fluxes, become non-commutative and also non-associative [15–19]. The non-commutativity of closed strings in non-geometric $Q$-flux backgrounds is a non-local effect, where the closed string commutator is proportional to the non-geometric flux times a winding number (dual momentum) [16] [18]:

$$[X^i_Q(\tau, \sigma), X^j_Q(\tau, \sigma)] \simeq Q^{ij}_k \tilde{p}^k,$$  \hspace{1cm} (1.1)

with $X^i_Q(\tau, \sigma)$, $X^k_Q(\tau, \sigma)$ being the closed string coordinates in the $i$, $j$-directions and $\tilde{p}^k$ the dual momentum in the $k$-th direction. The reason for the observed non-commutativity is that the closed string acquires mixed boundary (monodromy) conditions (which are reminiscent of mixed D-N boundary conditions for open strings in the presence of $F$-flux) in the presence of non-geometric fluxes of the following form:

$$X^i_Q(\tau, \sigma + 2\pi) = X^i_Q(\tau, \sigma) + Q^{ij}_k \tilde{p}^k \bar{X}^j_Q(\tau, \sigma),$$ \hspace{1cm} (1.2)

where $\bar{X}^j_Q(\tau, \sigma)$ denotes the dual string coordinate in the $j$-th direction. More generally this non-commutativity is measured by a Wilson line operator of the $Q$-flux around holonomy.

\(^1\text{An alternative field redefinition was discussed in [15][16].}\)
circles of the non-geometric backgrounds [12]:

$$[X^i, X^j] \sim \int_{C_k} Q^j_k(X) \, dX^k.$$  

(1.3)

For so-called $R$-fluxes, which do not possess a description in terms of a local background, the closed string commutator is proportional to the momentum of the string:

$$[X^i_R(\tau, \sigma), X^j_R(\tau, \sigma)] \simeq R^{ijk} p_k.$$  

(1.4)

Here $X^i_R(\tau, \sigma)$ denotes the closed string coordinate in the $R$-flux background, and the $p_k$ are the ordinary momenta. This then also leads to the closed string non-associativity in terms of a non-vanishing 3-bracket in the presence of $R$-fluxes:

$$[X^i_R(\tau, \sigma), X^j_R(\tau, \sigma), X^k_R(\tau, \sigma)] \simeq R^{ijk}.$$  

(1.5)

These closed string commutation and 3-bracket relations also indicate a very interesting and new phase space structure, which can be also derived and quantized using membrane sigma models [20].

**Intersecting $Q$- and $R$-branes:**

As it is well known, NS 5-branes are supersymmetric solutions of the standard NS effective action of type IIA/B supergravity. They act as microscopic brane sources for the $H$-fluxes. Their T-dual configurations are the Kaluza-Klein monopoles, which are the sources for the geometric $f$-fluxes. Hence it is natural to ask, are there also microscopic sources for the non-geometric $Q$- and $R$-fluxes? As we will show, these branes, which we will call $Q$- and $R$-branes can be constructed by T-duality. More concretely, the $Q$-branes follow from one T-duality transformation acting in the direction transversal to the Kaluza-Klein monopole configuration. Their corresponding harmonic functions depend logarithmically on the two transverse directions (similar to $D7$-branes). As we will discuss, the $Q$-brane is the source for a non-geometric $Q$-flux. Hence it is also the source for closed string non-commutativity, as the space along the two "nut-directions" of the $Q$-brane becomes non-commutative for the closed string coordinates. In addition we will also discuss that the $Q$-branes, being non-geometric solutions of the standard NS effective action, are at the same time also solutions of the new effective action [10] [12] for non-geometric string backgrounds. In fact, in terms of the redefined background fields, where the metric $g_{ij}$ is replaced by a dual metric $\tilde{g}_{ij}$, and the $B_{ij}$-field gets replaced by a bi-vector $\beta^{ij}$, the $Q$-brane solution looks like an ordinary brane with a metric, which is well-defined under coordinate transformations.

The $R$-branes are still more speculative. T-duality strongly suggests that these 8-dimensional object should exist, and it is conceivable that they can be constructed in more concrete terms using the doubled field theory formalism. In fact, the $R$-brane metric also depends on the dual coordinates and hence cannot be given as a local function of original coordinates. In any case, being the microscopic sources for the $R$-fluxes, the coordinates along a three-dimensional subspace of their world-volume are argued to be non-associative. So after three

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\(^{2}\)The $Q$-brane solutions have been constructed before, called higher KK respectively defect branes [21] [22] or exotic branes [23] [24].
T-dualities, we are led to the following T-duality chain of brane solutions:

\[
\text{NS 5 - brane} \xrightarrow{T_1} \text{KK monopole} \xrightarrow{T_2} \text{Q - brane} \xrightarrow{T_3} \text{R - brane}
\]  

(1.6)

The construction of \( Q \)- and \( R \)-branes is very useful and closely related to the problem of obtaining 6-dimensional non-commutative and non-associative spaces, which provide consistent supersymmetric compactifications for the type IIA/IIB superstring. So far, closed string non-commutativity and non-associativity was discussed for two classes of non-geometric string backgrounds. The first example is a 3-dimensional T-fold, being a torus fibration with elliptic \( \mathbb{Z}_4 \) monodromy supplemented by non-constant fluxes [16]. In [18] this non-commutative background was further extended to a full CFT construction, which describes a 6-dimensional, freely acting asymmetric orbifold.

Another class of 3-dimensional non-commutative and non-associative string backgrounds is given by the well-known chain of three T-duality transformations:

\[
H_{ijk} \xrightarrow{T_i} f_{jk} \xrightarrow{T_j} Q_{k}^{ij} \xrightarrow{T_k} R^{ijk}.
\]  

(1.7)

Starting from a flat 3-torus with constant \( H \)-flux one successively gets a 3-dimensional twisted torus with geometric flux, a 3-dimensional T-fold with constant \( Q \)-flux and finally a space with constant \( R \)-flux. The corresponding flux sources are given by NS 5-branes, KK monopoles and by \( Q \)- and \( R \)-branes. However as they stand, these 3-dimensional spaces are not consistent, supersymmetric solutions of type IIA/IIB superstring theory. Simple products of two such spaces also do not lead to consistent 6-dimensional backgrounds. In order to generalize this chain of 3-dimensional spaces to consistent, 6-dimensional, supersymmetric solutions of type IIA/IIB supergravity, we will utilize intersecting NS 5-branes, intersecting KK monopoles, intersecting \( Q \)-branes and intersecting \( R \)-branes. In particular we will argue that intersecting \( Q \)- and \( R \)-branes make physically perfect sense and lead to supersymmetric ground states. We will discuss various intersecting \( Q \)-brane configurations and their corresponding non-vanishing closed string commutators. We will also discuss several intersecting \( R \)-brane configurations and their related 3-brackets.

In the near horizon limit of all these intersecting brane configurations the 10-dimensional supersymmetric geometries will be always of the form\(^3\):

\[
M_{10} = \text{AdS}_4 \times M_6^{H,f,Q,R}.
\]  

(1.8)

As we will discuss, the allowed internal 6-dimensional spaces \( M_6^{H,f,Q,R} \) are equipped with \( H, f, Q, R \)-fluxes and can be derived from chains of consecutive T-dualities as follows:

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\(^3\)The geometric spaces \( M_6^{H,f} \) were already constructed in [20, 21].
These six-dimensional spaces must possess specific $SU(3) \times SU(3)$ group structures in order to satisfy the type IIA/B supersymmetry conditions. Specifically, $M_H^6$ is a flat 3-torus with $H$-fluxes in four different directions, and $M_f^6$ a geometric space with four different geometric fluxes. This geometric space is just the Nilmanifold $N_{4,7}$, which is known to provide a nice example for supersymmetric non-Calabi-Yau compactification with $SU(3)$ group structure. $M_Q^6$ and $M_R^6$ will be new non-geometrical spaces with particular $SU(3) \times SU(3)$ group structures, which arise from the intersection of four $Q$-branes or, respectively, with four $R$-branes. In addition there are also several other allowed spaces with mixed geometrical and non-geometrical fluxes, such as backgrounds with $H$-, $f$- and $Q$-fluxes. In all the cases that involve intersections of $Q$- and/or $R$-branes one obtains an interesting pattern of different commutators and/or 3-brackets.

The paper is organized as follows. In the next section we want to recall the construction of the NS 5-branes and KK monopoles as solutions of the standard NS effective action. Then we will move on via T-duality to the construction of the non-geometric $Q$- and $R$-branes, where we will show that these branes are good solutions of the new effective NS action for non-geometric backgrounds. This also allows a simple derivation of the corresponding $Q$- and $R$-fluxes caused by these brane solutions. In section three we continue to the configurations of four intersecting branes. Taking the near horizon limit and performing a suitable rescaling of the coordinates, $AdS_4 \times M_6^{H,f,Q,R}$ geometries are derived. As we will see, the study of the intersecting $Q$- and $R$-branes provides a simple and elegant way to derive all non-geometric flux backgrounds and also the commutation relations of the internal coordinates. We will also provide a brief discussion about the form of the supersymmetry conditions for the intersecting non-geometric branes. In fact, using the redefined background fields $\tilde{G}$, $\beta$, $Q$ and $R$, the supersymmetry conditions can be written in a very short form, in analogy to the supersymmetry conditions for spaces with non-vanishing $H$-field background. In addition will also briefly discuss the effective four-dimensional superpotentials, which follow from the compactification on the considered geometric as well as non-geometric spaces. Specifically, these compactifications will lead to effective IIA/IIB flux superpotentials $[7, 28–31]$, which depend on the dilaton $S$, the Kähler moduli $T_i$ and the complex structure moduli $U_m$. We will derive the moduli dependence of the geometric as well as non-geometric IIA flux superpotentials.
2 Geometric and non-geometric NS brane solutions

2.1 Geometric brane solutions

In the following, we will first briefly recall the NS 5-brane solution and the T-dual Kaluza-Klein monopole.

2.1.1 The NS 5-brane

Let us start with the standard effective action of type IIA/B superstrings, where we include only the NS background fields, namely the metric $G$, the antisymmetric tensor field $B$, its associated 3-form field $H = dB$ and the dilaton $\phi$:

$$S = \int d^{10}x \sqrt{|G|} \left( R + 4(\partial \phi)^2 - \frac{1}{12} H_{ij k} H^{ij k} \right).$$

As it is very well known, the NS 5-brane is a solution of the field equations of the NS effective action (2.1). It acts as the source for the 3-form $H$-field flux. In the string frame, the NS 5-brane is described by the following metric, antisymmetric tensor field and dilaton configuration:

$$ds_{NS5}^2 = \sum_{i=0}^{2} (dx^i)^2 + h(r) \sum_{k=1}^{4} (dx^k)^2 \quad (i = 0, \ldots, 5 \text{ and } k = 1, \ldots, 4)$$

$$e^\phi = \sqrt{h(r)}$$

$$H_{mnp} = \epsilon_{mnpq} \partial_q h(r)$$

with the harmonic function $h$ given as $h(r) = 1 + \frac{H}{r^2}$ ($r^2 = \sum x_i^2$).

In order to make contact with the $H$-flux backgrounds, which will be discussed in section three, we assume that the NS 5-brane is wrapping three internal, compact directions, e.g. $y^1, y^5, y^6$, and it forms a domain wall in the four-dimensional uncompactified space-time, where we denote the four uncompactified coordinates by $x^\mu$ ($\mu = 0, \ldots, 3$) and the six compact ones by $y^i$ ($i = 1, \ldots, 6$). The metric in the six internal directions then takes the form

$$ds_{NS5}^2 = h(r) \sum_{i=1,2,3} (dy^i)^2 + \sum_{i=4,5,6} (dy^i)^2.$$  

The corresponding 5-brane geometry is depicted in the following table:

| $x^0$ | $x^1$ | $x^2$ | $x^3$ | $y^1$ | $y^2$ | $y^3$ | $y^4$ | $y^5$ | $y^6$ |
|-------|-------|-------|-------|------|------|------|------|------|------|
| NS5   | $\otimes$ | $\otimes$ | $\otimes$ | | | | | | |

The four-dimensional domain wall structure will be always valid in all brane configurations to be discussed in the following. In section three we will consider the case that the $H$-field has only legs in the transversal compact space, i.e. $H_{y^1, y^2, y^3} = H$. Furthermore we will consider the intersecting of four different branes, such that there remains only one common transversal direction, denoted by $x^3$. This will be achieved by assuming that the harmonic function $h(r)$
linearly depends only on the radial direction of the four-dimensional domain wall, associated with the coordinate $x^3$. Thus in this case

$$h(x^3) = H x^3 \quad (2.4)$$

and after rescaling of the coordinates, the internal six-dimensional part of the metric eq.(2.3) will become the flat metric of $T^6$.

### 2.1.2 The Kaluza-Klein monopole

Now we take the background eq.(2.2) of the NS 5-brane and perform a T-duality transformation along the transversal compact direction $y^1 \equiv y$. As it is well-known, the T-dual configuration is given by the Kaluza-Klein monopole. It is a purely geometrical configuration without $H$-field and dilaton $\phi$, whose metric can be brought into the following form:

$$ds_{KK}^2 = \sum_{\mu=0,1,2} (dx^\mu)^2 + \sum_{i=4,5,6} (dy^i)^2 + \frac{1}{h(r)} \left( dy^3 + \sum_{i=2,3} A_i dy^i \right)^2 + h(r) \left( (dx^3)^2 + \sum_{i=2,3} (dy^i)^2 \right) \quad (2.5)$$

Here $A_i dy^i$ is a one-form gauge field that corresponds to the off-diagonal metric component of the KK monopole. The direction $y$ is now an isometry of the solution, as the harmonic function $h(r) = 1 + \frac{f}{r^2}$, $r^2 = (x^3)^2 + (y^2)^2 + (y^3)^2$ does not anymore depend on $y$. However this solution does not correspond to a real six-dimensional brane, but the $y$-direction is referred to be the nut direction of the KK monopole. The corresponding nut charge is given by the parameter $f$ of the harmonic function. T-duality with the NS 5-brane implies the connection $A_i = B_{y,y^i}$ between the one-form gauge field of the KK monopole and the Kalb-Ramond field of the NS 5-brane. There is always a gauge of the Kalb-Ramond field in which $B_{y,y^3} = A_3$ vanishes. The remaining component $A_2$ is connected to the harmonic function $h$ by

$$A_2 = \int dy^3 \partial_{x^3} h. \quad (2.6)$$

The KK monopole configuration is shown in the following table, with the dot denoting the nut direction:

|       | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $y^1$ | $y^2$ | $y^3$ | $y^4$ | $y^5$ | $y^6$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| KK    | ⊗     | ⊗     | ⊗     | •     | ⊗     | ⊗     | ⊗     |       | ⊗     | ⊗     |

Setting the harmonic function $h$ according to eq.(2.4) and identifying $f \equiv H r$ the metric in the internal compact directions will take the form:

$$ds_{KK int}^2 = \frac{1}{f x^3} \left( dy + f y^3 dy^2 \right)^2 + f x^3 \sum_{i=2,\ldots,6} (dy^i)^2. \quad (2.7)$$

It is not difficult to see that for a constant $x^3$ this metric is identical to the metric of the simplest 6-dimensional Nilmanifold, namely $N_{5,2}$. The corresponding metric flux can be immediately read off and is given by the following non-vanishing flux component:

$$f_{y^3}^y = f. \quad (2.8)$$

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4 The T-duality between the NS 5-brane and the KK monopole was discussed using the double geometry formalism in [27].
2.2 Non-geometric brane solutions

2.2.1 Effective actions for non-geometric backgrounds

Non-geometric backgrounds can be nicely described via the framework of doubled field theory (DFT). DFT was introduced in [9]. In this theory, T-duality is turned into a manifest symmetry by doubling the coordinates at the level of the effective space-time action for string theory. T-duality relates momentum and winding modes of a closed string moving on a torus $T^D$ via the T-duality group $O(D, D)$. When the coordinates are doubled, this duality symmetry can be made manifest. Thus, in DFT every conventional coordinate $x^i$, associated to momentum modes, is complemented by a dual coordinate $\tilde{x}^i$, associated to winding modes. The coordinates combine into a fundamental $O(D, D)$ vector $X^M = (\tilde{x}^i, x^i)$. As explained in [10–12], we now like to consider the following field redefinition of the metric $G$ and the $B$-field:

$$\tilde{G}^{-1} + \beta^{-1} = \tilde{E}^{-1} = G + B,$$

(2.9)

where we have introduced

$$\tilde{E}^{ij} = \tilde{G}^{ij} + \beta^{ij}.$$  

(2.10)

Here $\beta^{ij}$ is a bi-vector. We also redefine $\phi$:

$$\sqrt{|G|} e^{-2\phi} = e^{-2\tilde{\phi}} = \sqrt{|\tilde{G}|} e^{-2\tilde{\phi}}.$$  

(2.11)

The redefinitions (2.9) has the form of a T-duality transformation over all $D$ coordinates along the tours $T^D$. Now, the $Q$-flux is defined as

$$Q^{jk} = \partial_j \beta^{jk}. $$  

(2.12)

For non-geometric situations, where the metric $B$ and the $G$-field are only locally but not globally defined, the $Q$-flux is nevertheless a globally well-defined object. However note that, being a partial derivative of a bi-vector, $Q$ is in general not a tensor. But, as shown in [11, 12], the proper geometrical interpretation of $Q$ is playing the role of a connection, which allows us to construct a derivative for the dual $\tilde{x}$ coordinates that is covariant with respect to the $x$ diffeomorphisms. In case $\beta$ is satisfying the simplifying condition

$$\beta^{ij} \partial_j = 0 \, (2.13)$$

when acting on arbitrary fields, the $Q$-flux actually behaves like a tensor.

In [10-12], we have proposed an effective action for (the NSNS sector of) non-geometric backgrounds, given in terms of the metric $\tilde{G}^{ij}$, the bivector $\beta^{ij}$ and the dilaton $\tilde{\phi}$. In case $\beta$ is satisfying the condition (eq. (2.13)), the effective action for $\tilde{G}^{ij}, \beta^{ij}, \tilde{\phi}$ takes the form [10]

$$S = \int d^{10}x \sqrt{|\tilde{G}|} e^{-2\tilde{\phi}} \left[ \tilde{R} + 4(\partial \tilde{\phi})^2 - \frac{1}{4} Q^2 \right],$$

(2.14)

where $(\partial \tilde{\phi})^2$ and $Q^2$ are simply the squares contracted with $\tilde{G}$. Let us emphasize that this action has the same form as the standard NS action (2.1). As we will see in the following, although the $Q$-branes are locally still solutions of the standard action (2.1), the action (2.14) is much better suited to describe the $Q$-brane solutions than the standard NS one.

The non-geometric $R$-flux proposed in [11,12] (see also [32,33]) has the general form

$$R^{ijk} = 3 \tilde{D}^{[i} \beta^{jk]} = 3(\tilde{\partial}^{[i} \beta^{jk]} + \beta^{[i} \partial_{[i} \beta^{jk]}),$$

(2.15)
where $\tilde{\partial}$ denotes the derivative with respect to the dual coordinate. If the simplifying condition eq.\((2.13)\) is satisfied, the second term does not contribute, while the first gives

$$R^{ijk} = 3 \tilde{\partial}^{[i} \beta^{jkl]}.$$  \hspace{1cm} (2.16)

Since the $R$-flux also contains $\tilde{\partial}^i$-derivatives, one has to use the full DFT effective action that in general contains coordinates as well as dual coordinates:

$$S_{\text{DFT}} = \int d^{10} x d^{10} \tilde{x} \sqrt{|\tilde{G}|} e^{-2\phi} \left[ \tilde{R} + 4(\tilde{\partial} \phi)^2 - \frac{1}{4} R^2 + \ldots \right]$$  \hspace{1cm} (2.17)

As we will see in the following, the $R$-brane solution will indeed depend on the dual coordinate $\tilde{x}_i$ in one of the directions, but not at the same time on coordinate $x^i$ in the same direction of the compact space.

Now we will move on to solutions of this new effective actions (\(2.14)\) and (\(2.17)\): First we consider $Q$-branes, which are globally well defined solutions of (\(2.14)\). Afterwards we describe $R$-branes which are closely connected to (\(2.17)\).

### 2.2.2 The non-commutative $Q$-brane configuration

T-dualizing along a direction perpendicular to a KK-monopole will result in a non-geometric background. Specifically, starting from a single KK-monopole shown in the previous subsection, we assume that the metric eq.\((2.15)\) does not depend on the coordinate $y^2$ (hence the KK monopole gets smeared in this direction). Now we can perform a T-duality transformation along the direction $y^2 \equiv y'$. Using the Buscher rules [34, 35], this operation leads to the following metric:

$$ds_Q^2 = \sum_{\mu=0,1,2} (dx^\mu)^2 + \sum_{i=4,5,6} (dy^i)^2 + \frac{h(r)}{h(r)^2 + A_2^2} (dy^2 + dy'^2) + h(r) \left( (dx^3)^2 + (dy^3)^2 \right).$$  \hspace{1cm} (2.18)

In addition there are also a non-vanishing $B$-field and a dilaton of the following form:

$$B_{y,y'} = \frac{A_2}{h(r)^2 + A_2^2}, \quad e^\phi = \sqrt{\frac{h(r)}{h(r)^2 + A_2^2}}.$$  \hspace{1cm} (2.19)

The metric (\(2.18)\) has the form of a 7-brane, but now there are two nut directions $y$ and $y'$ in the metric. The harmonic function $h$ only depends on two transversal coordinates $x^3$ and $y^3$, and therefore we now get the logarithmic dependence

$$h(r) = \ln r, \quad r^2 = (x^3)^2 + (y^3)^2.$$  \hspace{1cm} (2.20)

on the transversal coordinates. The logarithmic divergence of $h$ implies that this co-dimension two brane is ill-defined as a single brane. It does not lead to a finite energy solution. However when we will consider intersecting branes in the next section, we will obtain configurations that make physically good sense. In addition, since shifting the periodic coordinate $y^3$ by $2\pi$ does not correspond to a standard diffeomorphism of the background, but acts as a T-duality transformation, this "7-brane" configuration is non-geometric. Its form is depicted in the following table, where the dots denote the two nut directions:

\footnote{This background was already considered in [21, 24].}
For the harmonic function \( h \) in eq. (2.4) and \( Q \equiv f \equiv H \), the two functions \( h \) and \( A_2 \) become
\[
h = Qx^3, \quad A_2 = Qy^3, \tag{2.21}
\]
and the metric in the internal six directions can be written as
\[
ds_{Q\text{int}}^2 = \sum_{i=4,5,6} (dy^i)^2 + Qx^3 dy^3 + \frac{Qx^3}{(Qx^3)^2 + (Qy^3)^2} (dy^2 + dy'^2). \tag{2.22}
\]
Assuming a constant \( x^3 \), this is nothing else than the metric of the non-geometric T-fold, which we like to call \( N_{Q}^2 \), since it is the T-dual to the Nilmanifold \( N_{5,2} \). Thus this brane acts as the source of the non-geometric flux \( Q \). We will therefore call it a \( Q \)-brane. Its metric (2.22) is equipped with an additional \( B \)-field, given now as:
\[
B_{y,y'} = -\frac{Qy^3}{(Qx^3)^2 + (Qy^3)^2}. \tag{2.23}
\]

The \( Q \)-brane background, which is specified by eq. (2.18) together with eq. (2.19), is locally a solution of the standard NS action eq. (2.1) (see appendix A). It was also shown in [24] that this configuration preserves half of the type IIA/B supersymmetries. However it is much simpler to discuss this solution using the redefined background parameters \( \tilde{G} \) and \( \beta \).

Specifically, using the field redefinition eq. (2.10), we obtain for \( \tilde{G} \) and \( \beta \):
\[
d_{Q}^2 = \sum_{\mu=0,1,2} (dx^\mu)^2 + \frac{1}{h(r)} \left( dy^2 + dy'^2 \right) + h(r) \left( (dx^3)^2 + (dy^3)^2 \right) + \sum_{i=4,5,6} (dy^i)^2, \\
\beta_{y,y'}^{Q} = -A_2, \\
e^{\tilde{\phi}} = \frac{1}{\sqrt{h(r)}}. \tag{2.24}
\]
Instead of an \( H \)-field, the redefined background possesses a non-vanishing \( Q \)-flux, which can be easily computed using eq. (2.12):
\[
Q_{y,y'}^3 = \partial_{y^3} \beta_{y,y'}^{Q} = -Q, \tag{2.25}
\]
where the bi-vector \( \beta_{y,y'}^{Q} \) is satisfying the simplifying constraint (2.13). In appendix A we show that this background is indeed a solution of the redefined effective action eq. (2.14). Furthermore the redefined background eq. (2.24) now behaves well-defined with respect to shifts of the periodic coordinate \( y^3 \) by \( 2\pi \).

Let us now discuss the non-commutative closed string geometry of \( Q \)-brane solution. Since it carries the \( Q \)-flux \( Q_{y,y'}^3 \), the directions \( y \) and \( y' \) possess non-trivial monodromy properties, when going around the circle in the \( y^3 \) direction. This leads to the following closed string boundary conditions, which mix the coordinates with the dual coordinates in the \( y \) and \( y' \) directions of the closed string [19]:
\[
Y(\tau, \sigma + 2\pi) = Y(\tau, \sigma) + Q_{y,y'}^3 \tilde{p}^3 \tilde{Y}'(\tau, \sigma), \\
Y'(\tau, \sigma + 2\pi) = Y'(\tau, \sigma) - Q_{y,y'}^3 \tilde{p}^3 \tilde{Y}(\tau, \sigma). \tag{2.26}
\]
It follows that a closed string in the field of the $Q$-brane sees a non-commutative geometry in the $y, y'$-directions:

$$[Y(\tau, \sigma), Y'(\tau, \sigma)] \simeq Q \widetilde{p}^3. \quad (2.27)$$

Here $\widetilde{p}_3$ is the dual momentum in the $y^3$-direction.

### 2.2.3 The non-associative $R$-brane configuration

Let us now come to the final step in the T-duality chain eq.(1.9). Starting from the $Q$-brane, we will assume that the function $h(r)$ does not depend anymore on the coordinate $y^3$. Hence $h(r)$ will be a linear function in the remaining transversal coordinate $x^3$. Then the T-duality in $y^3 = y''$ leads to the following 8-dimensional $R$-brane configuration:

\[
\begin{array}{cccccccc}
\text{R} & x^0 & x^1 & x^2 & x^3 & y & y' & y'' & y^4 & y^5 & y^6
\end{array}
\]

This brane configuration apparently possess three nut directions. However, since we are now doing a T-duality along a non-isometry direction, namely $y''$, the $R$-brane does not possess a local metric in the original coordinates. But, as discussed in [12], double field theory [9] has a proposal on how to T-dualise along a direction which is not an isometry: we just need here to formally replace the coordinate $y''$ by its dual coordinate $\widetilde{y}''$, in analogy to the replacement of the momentum by its dual quantity, namely the winding. Performing this replacement, we get using the redefined background fields $\widetilde{G}$ and $\beta$ the following expressions:

\[
\begin{align*}
\delta_{R}^3 & = \sum_{\mu=0,1,2} (dx^\mu)^2 + \frac{1}{h(r)} (dy^2 + dy'^2) + h(r) \left( (dx^3)^2 + dy''^2 \right) + \sum_{i=4,5,6} (dy^i)^2, \\
\beta_{y'y''} & = -R\widetilde{y}'', \\
e^{-\phi} & = \frac{1}{\sqrt{h(r)}},
\end{align*}
\]

where after T-duality we have denoted the parameter of the solution by $R$, with $R \equiv Q \equiv f \equiv H$. Using eq.(2.10), the corresponding $R$-flux is given as

$$R^{y'y''} = \partial_{y''} \beta_{R}^{y'y''} = -R. \quad (2.29)$$

A closed string in the field of the $R$-brane sees a non-commutative and non-associative geometry in the $y, y', y''$-directions. Specifically we obtain the following non-vanishing commutators and 3-brackets with $p, p', p''$ being the momenta in the $y, y', y''$-directions:

\[
\begin{align*}
[Y(\tau, \sigma), Y'(\tau, \sigma)] & \simeq R \ p'' \\
[Y''(\tau, \sigma), Y(\tau, \sigma)] & \simeq R \ p' \\
[Y'(\tau, \sigma), Y''(\tau, \sigma)] & \simeq R \ p \\
[Y(\tau, \sigma), Y'(\tau, \sigma), Y''(\tau, \sigma)] & \simeq R.
\end{align*}
\]

(2.30)
3 Type IIA/B \( AdS_4 \times M_6 \) backgrounds from intersecting NS 5-branes, KK monopoles, \( Q \)- and \( R \)-branes

3.1 Geometric intersecting branes

In the absence of sources and higher-order derivative corrections, supersymmetric backgrounds of type II supergravity of the form \( \mathbb{R}^{1,3} \times M_6 \) require the internal manifold \( M_6 \) to be Calabi-Yau. Turning on the background fluxes while preserving maximal symmetry in the four non-compact dimensions forces the background to be of the form of a warped product \( AdS_4 \times w.M_6 \), where \( M_6 \) is no longer a Calabi-Yau. The departure from the Calabi-Yau condition in the presence of fluxes can be described by \( SU(3) \) group structures and associated torsion classes \([-36, 37]\) or more generally by reformulating the supersymmetry conditions in the framework of generalized geometry \([-38, 39]\). This leads to the statement that \( M_6 \) must possess a pair of compatible pure spinors obeying certain differential conditions \([-40]\).

Let us very briefly recall the supersymmetry conditions on the background \((G, B, \phi)\) for type IIA/IIB compactifications. We perform a four-plus-six space-time split, according to which the ten-dimensional metric takes the warped-product form:

\[
ds^2 = e^{2A(y)} ds_4^2 + g_{ij} dy^i dy^j,
\]

where \(\exp(2A(y))\) is the warp factor, \(ds_4^2\) is the line element of \(AdS_4\) and \(g_{ij}\) is the internal-manifold metric. (We denote the four coordinates of \(AdS_4\) by \(x^\mu (\mu = 0, \ldots, 3)\), whereas the six compact coordinates are denoted by \(y^i (i = 1, \ldots, 6)\).) Within the framework of generalized complex geometry, supersymmetry requires that the structure group of the generalized tangent bundle to be \(SU(3) \times SU(3)\). The supersymmetry generators \(\eta^{(1)}\) and \(\eta^{(2)}\) are then collected into two spinor bilinears, which can be associated with two polyforms of definite degree

\[
\Psi_+ = \frac{8}{|a||b|} \eta_+^{(1)} \otimes \eta_+^{(2)*}, \quad \Psi_- = \frac{8}{|a||b|} \eta_-^{(1)} \otimes \eta_-^{(2)*}.
\]

In order to obtain similar equations in IIA and IIB one redefines

\[
\Psi_1 = \Psi_+, \quad \Psi_2 = \Psi_-,
\]

with upper/lower sign for IIA/IIB. With these definitions the supersymmetry conditions take the following schematic form in both IIB and IIA, where we neglected all possible Ramond fluxes as well as a warp factor and the dilaton

\[
d_H \Psi_1 = 0, \\
d_H \Psi_2 = 0,
\]

where

\[
d_H = d + H \wedge .
\]

\(AdS_4\) vacua in Type IIA:

Necessary and sufficient conditions for supersymmetric vacua of the form \(AdS_4 \times M_6\) have been established in the case of (massive) IIA backgrounds with constant dilaton and warp factor in \([41]\). First we discuss IIA solutions with strict \(SU(3)\) group structure for the
cases of geometric spaces with $H$-fluxes and/or geometric fluxes, where we mainly follow the discussion in \cite{26,41}. Neglecting the Ramond forms, the solutions are given by

$$H(y) = He^{\phi}\Re\Omega. \quad (3.6)$$

Furthermore, $(J, \Omega)$ is the SU(3) group structure of the internal six-manifold, i.e. $J$ is a real two-form, and $\Omega$ is a decomposable complex three form such that:

$$\Omega \wedge J = 0,$$

$$\Omega \wedge \Omega^* = \frac{4i}{3}J^3 \neq 0. \quad (3.7)$$

The SU(3) group structure is related to the two polyforms by making the following identification:

$$\Psi_- = -\Omega, \quad \Psi_+ = e^{iJ}, \quad (3.8)$$

Supersymmetry is then requiring that the only non-zero torsion classes of the internal manifold are $W^-_1, W^-_2$ which are defined such that:

$$dJ = -\frac{3}{2}iW^-_1\Re\Omega,$$

$$d\Omega = W^-_1 J \wedge J + W^-_2 \wedge J. \quad (3.9)$$

This requirement largely restricts the number of known solutions. By taking the internal six-dimensional space to be a 6-dimensional Nilmanifold $M_6$, it suffices to look for all possible Nilmanifolds, whose only non-zero torsion classes are $W^-_{1,2}$. As we will discuss in the following, there are exactly two possibilities in type IIA, namely the six-torus and the Nilmanifold $N_{4,7}$ of Table 4 of \cite{42} (also known as the Iwasawa manifold).

**AdS$_4$ vacua in Type IIB:**

For IIB on the other hand, where the above definitions of $\Psi_1$ and $\Psi_2$ are switched, we are interested in the case for which

$$\eta_+^{(2)} = V\gamma_i\eta_-^{(1)}. \quad (3.10)$$

This condition will correspond to spaces with static SU(2) group structure, and we define the following SU(2)-structure quantities

$$\omega_2 = J + 2iV \wedge V^*, \quad \Omega_2 = iV^*\Omega \quad (3.11)$$

where $J$ and $\Omega$ form the SU(3)-structure associated to $\eta_+ = |a|^{-1}\eta_+^{(1)}$. We then find for the pure spinors

$$\Psi_+ = -e^{2V \wedge V^*}\Omega_2,$$

$$\Psi_- = -2V \wedge e^{i\omega_2}. \quad (3.12)$$

Plugging this ansatz in (3.4), one finds equations for the SU(2)-structure quantities $V$, $\omega_2$ and $\Omega_2$.

We will now first consider six-dimensional geometric background, which arise from the intersection of NS 5-branes and KK monopoles.

---

6The parameter $H$ is proportional to the mass parameter $m$ of massive IIA supergravity $H = \frac{2}{5}m$. Furthermore $m$ is related to $W^{-1}$, the radius of AdS$_4$, in the following way: $W = -\frac{4}{5}e^{m}$. In general, this equation also contains the Freund-Rubin parameter.

7Consistent supersymmetric IIA solutions will also require the existence of Ramond 2-form and 4-form fluxes.
3.1.1 Type IIA: six-torus with $H$-flux

Here the six-dimensional space $M_6$ is just the flat torus $T^6$. Let us define a left-invariant basis \{e^i\} such that:

$$de^i = 0, \quad i = 1, \ldots, 6.$$  \hfill (3.13)

On the torus we can just choose $e^i = dy^i$, where $y^i$ are the internal coordinates. The SU(3)-structure is given by

$$J = e^{12} + e^{34} + e^{56},$$

$$\Omega = (ie^1 + e^2) \wedge (ie^3 + e^4) \wedge (ie^5 + e^6),$$  \hfill (3.14)

Hence all torsion classes trivially vanish in this case. In addition, there are four non-vanishing $H$-field components:

$$H_{y^2,y^4,y^6} = H_{y^2,y^5,y^3} = H_{y^1,y^6,y^3} = H_{y^1,y^5,y^4} = H.$$  \hfill (3.15)

The corresponding $B$-field components are

$$B_{y^2,y^4} = -B_{y^1,y^3} = Hy^6, \quad B_{y^1,y^4} = B_{y^2,y^3} = -Hy^5.$$  \hfill (3.16)

As shown in [25][26], this background can be obtained from the intersection of four different NS 5-branes, as shown in the following table.\footnote{In addition, the complete brane configuration also contains three intersecting D4-branes and one D8-brane, which are the sources of the omitted Ramond 4-form fluxes and the IIA mass parameter $m$, as well as four (smeared) orientifold O6-planes.}

|     | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $y^1$ | $y^2$ | $y^3$ | $y^4$ | $y^5$ | $y^6$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| NS5 | ☒     | ☒     | ☒     | ☒     | ☒     | ☒     |       |       |       |       |
| NS5’| ☒     | ☒     | ☒     |       |       |       | ☒     | ☒     | ☒     |       |
| NS5”| ☒     | ☒     | ☒     |       | ☒     |       | ☒     |       |       | ☒     |
| NS5”’| ☒ | ☒ | ☒ | ☒ | ☒ | ☒ | ☒ | ☒ | ☒ | ☒ |

The full intersecting brane solution patches two asymptotic regions: a near-horizon region of $AdS_4 \times T^6$ geometry and a flat region at infinity. Note that the harmonic function of the intersecting branes only depend on the non-compact (radial) coordinate of the four-dimensional $AdS_4$ space, denoted by $x^3$. As shown in [25], in the near-horizon limit $x^3 \to 0$ of the solution, the brane system above is replaced by fluxes. After rescaling of the coordinates, the near-horizon limit of the solution precisely corresponds to the flat six-torus $T^6$ with $H$-field, as shown eq.(3.15).

Furthermore, in type IIA (orientifold) compactifications, the effective, 4-dimensional superpotential has the following general structure:

$$H - \text{flux} : \quad W_{H}^{IIA} = a_0 S + b_m U_m.$$  \hfill (3.17)

Here we obtain from the NS sector:

$$W_{H}^{IIA} = H_{246}S + H_{235}U_1 + H_{145}U_2 + H_{136}U_3.$$  \hfill (3.18)
Note that the superpotential with all NS and Ramond fluxes, which follow from the complete intersection of NS 5-branes branes plus the additional D-branes, stabilizes all moduli in an supersymmetric $AdS_4$ minimum. In this context it is interesting to see that the values of the stabilized moduli fields in the effective field theory are identical to the asymptotic values of these fields in the near horizon limit of the intersecting brane geometries \[25\].

3.1.2 Type IIA: Iwasawa manifold

Let us now perform two T-duality transformations along the $y^1$ and $y^2$ directions of the internal manifolds. These will transform all four $H$-field components into the following geometric fluxes:

$$ f_{46}^2 = -f_{36}^1 = -f_{15}^1 = -f_{35}^2 = f $$

where by T-duality $f \equiv H$. Then the 6-dimensional metric takes the following form

$$ ds^2 = \sum_i \left(dy^i + f_{jk} y^j dy^k\right)^2 = \left(dy^1 - fy^6 dy^3 - fy^5 dy^4\right)^2 + \left(dy^2 + fy^6 dy^4 - fy^5 dy^3\right)^2 + \sum_{i=3,4,5,6} (dy^i)^2. $$

The corresponding 6-dimensional space is just the Nilmanifold $N_{4,7}$, known as Iwasawa manifold, as can be seen as follows. The left-invariant basis is now defined by:

$$ de^a = 0, \quad a = 3, \ldots, 6, $$

$$ de^1 = f(e^{45} - e^{36}) , $$

$$ de^2 = f(e^{46} + e^{35}) , $$

and is usually denoted by $(45 - 36, 46 + 35, 0, 0, 0, 0)$. Up to basis transformations there is a unique SU(3)-structure satisfying the supersymmetry conditions:

$$ J = e^{12} + e^{34} + e^{65}, $$

$$ \Omega = (e^1 + i e^2) \wedge (e^3 - i e^4) \wedge (e^5 + i e^6). $$

In the left-invariant basis, the metric is simply given by $g = \text{diag}(1,1,1,1,1,1)$, and the torsion classes can be read off from $dJ, d\Omega$ as follows:

$$ W_1^- = \frac{-2if}{3}, $$

$$ W_2^- = \frac{-4if}{3} \left(2e^{12} + e^{34} + e^{56}\right), $$

while all other torsion classes vanish. We can verify that $dW_2^-$ is proportional to $\Re\Omega$:

$$ dW_2^- = -\frac{8if^2}{3} \Re\Omega. $$

This geometric space corresponds to the near horizon geometry of the intersection of four different KK monopoles as shown in the following table (We do not display the non-compact directions anymore, but let us keep in mind that they form domain walls).
The IIA effective superpotential of geometric fluxes has the following general structure:

\[ f - \text{flux} : \quad W_{f}^{IA} = a_i S T_i + b_{im} T_i U_m . \]  

Using precisely the fluxes in eq. (3.19) we obtain

\[ W_{f}^{IA} = f_2^{46} S T_1 + f_2^{35} T_1 U_1 + f_1^{45} T_1 U_2 + f_1^{36} T_1 U_3 . \]  

### 3.1.3 Type IIB: the Nilmanifold \( N_{5,1} \)

Starting from the flat 6-torus with \( H \)-flux we can also perform only one T-duality along the \( y^1 \) direction (or from the Iwasawa manifold via T-duality along \( y^2 \)). This leads us to a IIB background with the following two geometric fluxes:

\[ -f_1^{36} = -f_1^{45} = f . \]  

In addition two \( H \)-fluxes survive the T-duality transformation:

\[ H_{246} = -H_{235} = H . \]  

Now the metric is given by

\[ ds^2 = \left( dy^1 - f y^6 dy^3 - f y^5 dy^4 \right)^2 + \sum_{i=2,3,4,5,6} (dy^i)^2 , \]

which corresponds to the Nilmanifold \( N_{5,1} \), defined left-invariant basis

\[ de^a = 0, \quad a = 2, 3, \ldots, 6 , \]

\[ de^1 = f(e^{45} - e^{36}) . \]

For the \( SU(2) \) group structure quantities one obtains

\[ e^{i\theta} V = \frac{1}{2} \left( e^1 + i e^2 \right) , \]

\[ \omega_2 = -e^{36} + e^{45} , \]

\[ \Omega_2 = -i e^{i\theta} (i e^6 + e^3) \land (i e^4 + e^5) . \]
The chain of T-dualities for the three IIA/IIB geometrical brane intersections and their near horizon geometries obtained so far is summarized as follows:

$$T^6 + 4H_{IIA} \xrightarrow{T_1} N_5, 1 + 2H_{IIB} \xrightarrow{T_2} N_4, 7_{IIA}$$  \hspace{1cm} (3.30)

### 3.2 Non-geometric spaces: Intersecting $Q$-branes and $R$-branes

Let us start this section by briefly discussing the supersymmetry conditions on 6D spaces with non-geometric fluxes. This can be done most elegantly using the redefined background fields $\tilde{G}, \beta, \tilde{\phi}$. Since, under the simplifying condition (2.13), the old action (2.1) and the new action (2.14) have the same form, supersymmetry still requires the structure group of the generalized tangent bundle to be $SU(3) \times SU(3)$. However now the supersymmetry conditions are evaluated with respect to the space determined by $\tilde{G}, \beta, \tilde{\phi}$. Hence we can again define two polyforms

$$\tilde{\Psi}^+_1 = \frac{8}{|a||b|} \eta_+^{(1)} \otimes \eta_+^{(2)} \dagger, \quad \tilde{\Psi}^- = \frac{8}{|a||b|} \eta_-^{(1)} \otimes \eta_-^{(2)} \dagger,$$  \hspace{1cm} (3.31)

where $\eta^{(1,2)}$ are spinors on the redefined background space. The supersymmetry conditions then take the following schematic form for the case of $Q$-flux under the simplifying condition (2.13)\(^{10}\)

$$d_Q \tilde{\Psi}_1 = 0,$$
 $$d_Q \tilde{\Psi}_2 = 0,$$  \hspace{1cm} (3.32)

where now

$$d_Q = d + Q \cdot .$$  \hspace{1cm} (3.33)

Here the $\cdot$ defines the proper contraction of the $Q$-flux on vectors and forms (see also [43] for a discussion about supersymmetry, T-duality and non-geometric backgrounds).

For explicit examples the generalized $SU(3) \times SU(3)$ structure group will be reduced to a space with $SU(3)$ or with $SU(2)$ group structure. As we will now discuss, the IIA non-geometric background with four $Q$-fluxes will be a space with flat metric $\tilde{G}_{ij}$ (a six-dimensional torus) after the field redefinition. On the other hand, the IIB space with two geometric fluxes

\(^9\)Since the field redefinition eq.(2.9), which involves an exchange of 2-form $B_{ij}$ with the bi-vector $\beta^{ij}$, basically exchanges the tangent bundle with the co-tangent bundle, the two $SU(3)$ group factors also get exchanged.

\(^{10}\)The general case and also the case of non-vanishing $R$-flux is more involved and will be treated by D. Andriot and A. Betz.
and two \( Q \)-fluxes will possess a \( SU(2) \) group structure just like its T-dual IIB counterpart, namely the Nilmanifold \( N_{5,1} \) equipped with two \( H \)-fluxes.

At the end of this subsection let us also discuss the non-trivial monodromy properties and the associated non-commutative geometry structure of the intersecting brane configurations. All considered spaces \( M_6 \) are fibrations of a four-dimensional torus \( T^4 \) with coordinates \( y^i (i = 1, \ldots, 4) \) over a two-dimensional torus with coordinates denoted by \( y^m (m = 5, 6) \). The monodromy properties of \( M_6 \) are specified by the fluxes \( H, f, Q \), when going around two circles in the directions \( y^5 \) and \( y^6 \). From these we can then derive the following closed string boundary (monodromy) conditions:

\[
Y^m(\tau, \sigma + 2\pi) = Y^m(\tau, \sigma) + 2\pi \tilde{p}^m,
Y^i(\tau, \sigma + 2\pi) = Y^i(\tau, \sigma) + f^i_{jm} \tilde{p}^m Y^j(\tau, \sigma) + Q^i_{jm} \tilde{p}^m \tilde{Y}^j(\tau, \sigma),
\tilde{Y}_i(\tau, \sigma + 2\pi) = \tilde{Y}_i(\tau, \sigma) - f^i_{jm} \tilde{p}^m \tilde{Y}^j(\tau, \sigma) + H^{ijm} \tilde{p}^m Y^j(\tau, \sigma). \tag{3.34}
\]

Here the \( \tilde{p}^m \) are the winding numbers of the closed string in the \( y^m \)-directions. Following [19], we can derive from these closed string boundary conditions the following non-vanishing commutators among the closed string coordinates in the directions \( y^i (i = 1, \ldots, 4) \):

\[
[Y^i(\tau, \sigma), Y^j(\tau, \sigma)] \simeq Q^i_{jm} \tilde{p}^m. \tag{3.35}
\]

### 3.2.1 Type IIA: four \( Q \)-fluxes

The metric and the \( H \)-field of four intersection Q-Branes

| \( Q \) | \( y^1 \) | \( y^2 \) | \( y^3 \) | \( y^4 \) | \( y^5 \) | \( y^6 \) |
|---|---|---|---|---|---|---|
| \( Q \) | \( \times \) | \( \cdot \) | \( \times \) | \( \cdot \) | \( \times \) | \( \cdot \) |
| \( Q' \) | \( \times \) | \( \cdot \) | \( \times \) | \( \cdot \) | \( \times \) |
| \( Q'' \) | \( \cdot \) | \( \times \) | \( \cdot \) | \( \times \) | \( \times \) |
| \( Q''' \) | \( \cdot \) | \( \times \) | \( \cdot \) | \( \times \) | \( \cdot \) |

are derived form the D4/D8/NS5 solution presented in [25]. This solution is parameterized by eight different functions: \( H^{D8}_i \) for the D8-brane, \( H^{D4}_i (i = 1, 2, 3) \) for the three D4-branes and \( H^{NS5}_i (i = 1, 2, 3, 4) \) for the four NS 5-branes. To adapt this solution to the one introduced in section 3.1.1 we use

\[
H_{246} = \partial_{x^3} H_1^{NS5} (H^{D8})^{-1} \quad H_{235} = -\partial_{x^3} H_2^{NS5} (H^{D8})^{-1}
H_{136} = -\partial_{x^3} H_3^{NS5} (H^{D8})^{-1} \quad H_{145} = -\partial_{x^3} H_4^{NS5} (H^{D8})^{-1}
\]

which differs by the sign of \( H_{246} \) from [25]. T-dualizing in the \( y^1, y^2, y^3 \) and \( y^4 \) directions and demanding

\[
H_1^{NS5} H_2^{NS5} \partial_{x^3} H_3^{NS5} \partial_{x^3} H_4^{NS5} = H_3^{NS5} H_4^{NS5} \partial_{x^3} H_1^{NS5} \partial_{x^3} H_2^{NS5}
\]

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lead to the internal metric
\[
\begin{align*}
\tilde{d}s_Q^2 & = \left( H_{D8}^3 \prod_{i=1}^{3} H_i^{D4} \right)^{1/2} \left[ \frac{H_{NS5}^4 H_{NS5}^4}{H_{D4}^2} (dy^1)^2 + \frac{H_{NS5}^4 H_{NS5}^4}{H_{D4}^2} (dy^2)^2 + \frac{H_{NS5}^4 H_{NS5}^4}{H_{D4}^2} (dy^3)^2 + \\
& \quad \frac{H_{NS5}^4 H_{NS5}^4}{H_{D4}^2} (dy^4)^2 + \frac{H_{NS5}^4 H_{NS5}^4}{H_{D4}^2} (dy^5)^2 + \frac{H_{NS5}^4 H_{NS5}^4}{H_{D4}^2} (dy^6)^2 \right] \\
\end{align*}
\]
with
\[
x = \left( \prod_{i=1}^{4} H_i^{NS5} \right) + \frac{1}{H_{D8}} \left( \partial_{x^3} H_2^{NS5} \partial_{x^3} H_4^{NS5} (y^5)^2 + \partial_{x^3} H_1^{NS5} \partial_{x^3} H_3^{NS5} (y^6)^2 \right)
\]
and the B-field
\[
\begin{align*}
B_{31} & = \frac{\partial_{x^3} H_1^{NS5} y^6}{x} & B_{32} & = \frac{\partial_{x^3} H_2^{NS5} y^5}{x} \\
B_{42} & = \frac{\partial_{x^3} H_3^{NS5} y^6}{x} & B_{41} & = -\frac{\partial_{x^3} H_4^{NS5} y^5}{x}
\end{align*}
\]
of four intersection Q-branes. As already stressed, it is often more convenient to use the redefined metric \( \tilde{G} \) and the bi-vector \( \beta \) for Q-brane configurations. After applying the corresponding transformation in eq. (2.10), one gets
\[
\begin{align*}
\tilde{d}s_Q^2 & = \left( H_{D8}^3 \prod_{i=1}^{3} H_i^{D4} \right)^{1/2} \left[ \frac{H_{D4}^2 H_3^{D4}}{H_{D8}^2} \left( H_3^{NS5} H_4^{NS5} (dy^1)^2 + H_1^{NS5} H_2^{NS5} (dy^2)^2 \right) + \\
& \quad \frac{H_{D4}^2 H_3^{D4}}{H_{D8}^2} \left( H_2^{NS5} H_3^{NS5} (dy^3)^2 + H_1^{NS5} H_4^{NS5} (dy^4)^2 \right) + \\
& \quad \frac{H_{D4}^2 H_3^{D4}}{H_{D8}^2} \left( \frac{1}{H_2^{NS5} H_4^{NS5}} (dy^5)^2 + \frac{1}{H_1^{NS5} H_3^{NS5}} (dy^6)^2 \right) \right] \\
\end{align*}
\]
and
\[
\begin{align*}
\beta_{31} & = -\frac{\partial_{x^3} H_1^{NS5} y^6}{H_{D8}^2} & \beta_{32} & = -\frac{\partial_{x^3} H_2^{NS5} y^5}{H_{D8}^2} \\
\beta_{42} & = -\frac{\partial_{x^3} H_3^{NS5} y^6}{H_{D8}^2} & \beta_{41} & = \frac{\partial_{x^3} H_4^{NS5} y^5}{H_{D8}^2},
\end{align*}
\]
which have the form of the D4/D8/NS5 solution we started with. To obtain AdS\(_4\) on the uncompactified space, one takes the near horizon limit \( x^3 \to 0 \) which is equivalent to
\[
H_{D8} \to \frac{2}{3} (x^3)^{-5/3}, \quad H_{i}^{D4} \to \frac{2}{3 Q^4} \quad \text{and} \quad H_{i}^{NS5} \to Q (x^3)^{-2/3}.
\]
In this limit
\[
\tilde{d}s_Q = \sum_{i=1}^{6} (dy^i)^2
\]
is the metric of a flat, 6-dimensional torus with
\[
\beta_{31} = -\beta_{41} = Q y^6 \quad \text{and} \quad -\beta_{32} = -\beta_{41} = Q y^5.
\]
Having shown that the near horizon limit of four intersecting $Q$-branes leads to the metric of flat six-dimensional space, let us now start from the Iwasa manifold, the near horizon limit of four KK-monopoles, and perform two T-duality transformations along the $y^3$ and $y^4$ directions of the internal manifolds. The 6-dimensional metric takes the form

$$ds_Q^2 = \frac{1}{x} \sum_{i=1}^{4} (dy^i)^2 + \sum_{j=5,6} (dy^j)^2$$  \hspace{1cm} (3.36)$$

where $x = 1 + Q^2 \left( (y^5)^2 + (y^6)^2 \right)$. The corresponding $B$-field components are

$$-B_{24} = B_{13} = \frac{Qy^6}{x}, \quad B_{14} = B_{23} = \frac{Qy^5}{x}. \hspace{1cm} (3.37)$$

Since we are dealing with a non-geometric background, it is more convenient to re-express the metric $G$ by its redefined metric $\tilde{G}$ and $B$-field in terms of the bi-vector $\beta$. Using eq.(2.9), we obtain

$$\tilde{G}_{ij} = \delta_{ij}, \quad \beta^{24} = -\beta^{13} = Qy^6, \quad -\beta^{14} = -\beta^{23} = Qy^5. \hspace{1cm} (3.38)$$

Since $\beta^{ij}$ is linear in $y^5$ reps. in $y^6$, it is easy to compute the corresponding $Q$-fluxes. These will transform all four $f$-fluxes into the non-geometric fluxes

$$Q^{24}_6 = -Q^{13}_5 = -Q^{14}_5 = -Q^{23}_5 = Q \hspace{1cm} (3.39)$$

where by T-duality $Q \equiv f \equiv H$. Hence all four $f$-fluxes are transformed into the non-geometric fluxes.

With these four $Q$-fluxes, the corresponding closed string background is non-commutative in the following directions:

$$[Y^2(\tau, \sigma), Y^3(\tau, \sigma)] \simeq Q^{23}_5 \tilde{p}^5, \quad [Y^1(\tau, \sigma), Y^4(\tau, \sigma)] \simeq Q^{14}_5 \tilde{p}^5, \quad [Y^1(\tau, \sigma), Y^3(\tau, \sigma)] \simeq Q^{13}_6 \tilde{p}^6, \quad [Y^2(\tau, \sigma), Y^4(\tau, \sigma)] \simeq Q^{24}_6 \tilde{p}^6. \hspace{1cm} (3.40)$$

The $Y^i(\tau, \sigma)$ are the internal closed string coordinates, and the $\tilde{p}^5$ and $\tilde{p}^6$ are the dual (winding) momenta in the 5.6-directions of internal space.

Concerning the 4D low energy action, as already discussed in [7], the IIA superpotential with non-geometric $Q$-flux has the following general structure:

$$Q - \text{flux} : \quad W_{Q}^{IIA} = a_{ij} ST_i T_j + c_{ijm} T_i T_j U_m. \hspace{1cm} (3.41)$$

With our choice of four $Q$-fluxes we then obtain

$$W_{Q}^{IIA} = Q^{24}_6 ST_1 T_2 + Q^{23}_5 T_1 T_2 U_1 + Q^{14}_5 T_1 T_2 U_2 + Q^{13}_6 T_1 T_2 U_3. \hspace{1cm} (3.42)$$

### 3.2.2 Type IIA: one $H$-flux, two $f$-fluxes, one $Q$-flux

We can consider also intersections among $Q$-branes and other (geometrical) brane configurations. Applying one T-duality along $y^3$ on the intersection configuration of two KK monopoles and two NS 5-branes, we obtain the intersection of one NS 5-brane, two KK monopoles and one $Q$-brane:
This is equivalent to one T-duality transformations along $y^3$ when starting from the IIB Nilmanifold $N_{5,1}$. The resulting IIA background has the two geometric fluxes

$$f_{25}^3 = -f_{15}^1 = f.$$  \hfill (3.43)

In addition the $H$-flux

$$H_{246} = H$$  \hfill (3.44)

survives the T-duality transformation. Finally, the new flux, which we gain after the T-duality, is non-geometric $Q$-flux

$$Q_{6}^{13} = -Q.$$  \hfill (3.45)

The 6-dimensional metric takes the following form

$$ds_{H,f,Q}^2 = \frac{1}{x} \left[ (dy^3 + fy^5dy^2)^2 + (dy^1 - fy^5dy^4)^2 \right] + \sum_{i=2,4,5,6}^{} (dy^i)^2$$  \hfill (3.46)

where $x = 1 + Q^2(y^6)^2$. The non-vanishing components of the $B$-field are

$$B_{12} = B_{34} = \frac{fx^5 Q y^6}{x}, \quad B_{13} = \frac{Q y^6}{x} \quad \text{and} \quad B_{24} = \left( H + \frac{(fx^5)^2 Q y^6}{x} \right) y^6.$$  \hfill (3.47)

The corresponding closed string background is non-commutative in the following directions:

$$[Y^1(\tau, \sigma), Y^3(\tau, \sigma)] \simeq Q_{6}^{13} \tilde{p}^6.$$  \hfill (3.48)

### 3.2.3 Type IIB: two $f$-fluxes, two $Q$-fluxes

We can also consider intersections of two KK monopoles and two $Q$-branes. They arise from the intersection of four KK monopoles after T-duality in $y^3$:

|        | $y^1$ | $y^2$ | $y^3$ | $y^4$ | $y^5$ | $y^6$ |
|--------|-------|-------|-------|-------|-------|-------|
| NS5   | $\otimes$ | $\otimes$ | $\otimes$ |       |       |       |
| KK'   | $\otimes$ |       | $\otimes$ | $\bullet$ |       | $\otimes$ |
| Q''   |       | $\otimes$ | $\bullet$ | $\otimes$ |       | $\otimes$ |
| KK''  | $\bullet$ | $\otimes$ | $\otimes$ | $\otimes$ |       | $\otimes$ |

The near horizon geometry of this brane intersection can be obtained by direct computation, which are similar to the ones shown in section 3.2.1. Alternatively, starting from the
Iwasawa manifold, let us perform one T-duality transformation along the $y^3$ direction. This transformation will keep the following two $f$-fluxes untouched:

$$f_{15}^2 = -f_{15}^1 = f.$$  \hspace{1cm} (3.49)

In addition the remaining two $f$-fluxes are transformed into the two non-geometric fluxes

$$Q_{6}^{13} = Q_{5}^{23} = Q.$$  \hspace{1cm} (3.50)

The 6-dimensional metric takes the following form

$$d\tilde{s}^2_{f,Q} = (dy^4 - f y^5 dy^4)^2 + (dy^2 + f y^6 dy^4)^2 + \frac{\left(\frac{dy^3}{x} - \left(Q y^6 dy^1 + Q y^5 dy^2\right)\right)^2}{x} + \sum_{i=4,5,6} (dy^i)^2$$  \hspace{1cm} (3.51)

where $x = 1 + Q^2 \left((y^5)^2 + (y^6)^2\right)$. The non-vanishing components of the $B$-field are

$$B_{13} = \frac{Q y^6}{x}, \quad B_{23} = \frac{Q y^5}{x}.$$  \hspace{1cm} (3.52)

Let us again compute the background obtained after the field redefinition eq.(2.9). For the redefined metric we obtain

$$d\tilde{s}^2_{f,Q} = (dy^4 + f y^5 dy^4 - f y^6 dy^4)^2 + \sum_{i=1,2,3,5,6} (dy^i)^2$$  \hspace{1cm} (3.53)

and the bi-vector $\beta$ becomes in this case

$$\beta^{13} = -Q y^6, \quad \beta^{23} = -Q y^5.$$  \hspace{1cm} (3.54)

This is in good agreement with the $Q$-fluxes in eq.(3.50). The diffeomorphism

$$y^1 = y^4, \quad y^2 = y^3, \quad y^3 = -y^1, \quad y^4 = y^2, \quad y^5 = y^6 \quad \text{and} \quad y^6 = y^5$$  \hspace{1cm} (3.55)

combined with $H \equiv Q$ transforms $\tilde{s}_{f,Q}$ and $\beta$ into the metric and the $H$-flux of the Nilmanifold $N_{5,1}$ presented in section 3.1.3 Therefore, like $N_{5,1}$, the background described in this section exhibits a $SU(2)$ group structure.

This space is non-commutative in the following directions:

$$[Y^2(\tau,\sigma), Y^3(\tau,\sigma)] \simeq Q_{5}^{23} \hat{p}^5,$$

$$[Y^1(\tau,\sigma), Y^3(\tau,\sigma)] \simeq Q_{6}^{13} \hat{p}^6.$$  \hspace{1cm} (3.56)

### 3.2.4 Type IIA: four $R$-fluxes

Now we are finally coming to 6D spaces with non-vanishing $R$-fluxes. Starting from the flat torus with four $H$-fluxes, let us perform a T-duality in all six internal directions. This results in the following intersection pattern of four $R$-branes:
Using double geometry, the metric and the $B$-field of this configuration is obtained from the background of four intersecting $Q$-branes by replacing $y^5$ and $y^6$ by its dual coordinates $\tilde{y}^5$ and $\tilde{y}^6$. Therefore we simply obtain from eqs. (3.36), (3.37) and (3.38):

$$ds_R^2 = \frac{1}{\tilde{x}} \sum_{i=1}^{4} (dy^i)^2 + \sum_{j=5,6} (d\tilde{y}^j)^2$$

(3.57)

where $\tilde{x} = 1 + R^2 \left( (\tilde{y}^5)^2 + (\tilde{y}^6)^2 \right)$. The corresponding $B$-field components are

$$-B_{24} = B_{13} = \frac{R\tilde{y}^6}{\tilde{x}}, \quad B_{14} = B_{23} = \frac{R\tilde{y}^5}{\tilde{x}}.$$  

(3.58)

and

$$\beta^{24} = -\beta^{13} = R\tilde{y}^6, \quad -\beta^{14} = -\beta^{23} = R\tilde{y}^5,$$

(3.59)

where by T-duality $R \equiv Q \equiv f \equiv H$. The associated $R$-fluxes are

$$R^{246} = -R^{136} = -R^{145} = -R^{235} = R.$$  

(3.60)

One can easily work out the corresponding commutators and three-brackets. The phase space structure of the four $R$-brane intersection is quite interesting; e.g. for the 3-brackets among the closed string coordinates $Y^i(\tau, \sigma)$ one obtains

$$[Y^2(\tau, \sigma), Y^4(\tau, \sigma), Y^6(\tau, \sigma)] \simeq R^{246},$$
$$[Y^2(\tau, \sigma), Y^3(\tau, \sigma), Y^5(\tau, \sigma)] \simeq R^{235},$$
$$[Y^1(\tau, \sigma), Y^3(\tau, \sigma), Y^6(\tau, \sigma)] \simeq R^{136},$$
$$[Y^1(\tau, \sigma), Y^4(\tau, \sigma), Y^5(\tau, \sigma)] \simeq R^{145}.$$  

(3.61)

The associated commutators also can be written down without big effort.

Finally, the IIA superpotential with non-geometric $R$-flux has the following general structure:

$$R - \text{flux:} \quad W_{IIA}^R = a_{ijk}ST_iT_jT_k + c_{ijkm}T_iT_jT_kU_m.$$  

(3.62)

Here we obtain:

$$W_{IIA}^R = R^{246}ST_1T_2T_3 + R^{235}T_1T_2T_3U_1 + R^{145}T_1T_2T_3U_2 + R^{136}T_1T_2T_3U_3.$$  

(3.63)
3.2.5 Type IIB: two $Q$-fluxes, two $R$-fluxes

For completeness let us also perform a T-duality in the internal directions $y^1, y^2, y^3, y^4, y^5$, starting from the flat torus with four $H$-fluxes. This results in the following brane intersection pattern:

\[
\begin{array}{cccccc}
 y^1 & y^2 & y^3 & y^4 & y^5 & y^6 \\
 Q & \otimes & \bullet & \otimes & \bullet & \otimes \\
 R' & \otimes & \bullet & \bullet & \otimes & \bullet \\
 Q'' & \bullet & \otimes & \bullet & \otimes & \otimes \\
 R''' & \bullet & \otimes & \otimes & \bullet & \bullet \\
\end{array}
\]

The corresponding non-geometric fluxes are

\[Q_6^{24} = -Q_6^{13} = Q, \quad -R_6^{145} = -R_6^{235} = R.\] (3.64)

This space is non-commutative/non-associative in the following directions:

\[
\begin{align*}
[Y^1(\tau, \sigma), Y^3(\tau, \sigma)] & \simeq Q_6^{13} \tilde{p}^6 \\
[Y^2(\tau, \sigma), Y^4(\tau, \sigma)] & \simeq Q_6^{24} \tilde{p}^6 \\
[Y^2(\tau, \sigma), Y^3(\tau, \sigma), Y^6(\tau, \sigma)] & \simeq R_6^{235}. \quad (3.65)
\end{align*}
\]

4 Conclusions and summary

In this paper we have discussed the construction of non-geometric $Q$- and $R$-branes as microscopic sources of non-geometric $Q$- and $R$-fluxes. Having constructed these non-geometric branes via T-duality from the NS 5-brane and the purely geometric KK-monopole background, we showed that doubled field theory and the use of the redefined background fields $\tilde{G}$ and $\beta$ leads to a very simple and elegant construction of these brane solutions. This also allows for straightforward derivation of the associated $Q$- and $R$-background fields as certain derivatives of the bi-vector $\beta$ background. Since the non-geometric branes carry an elementary unit of $Q$- and $R$-charge, it follows from the nontrivial monodromy properties around these branes, that the transversal closed string geometry is non-commutative or, respectively, non-associative.

In the second part of the paper we constructed supersymmetric type IIA and type IIB intersecting brane configurations, consisting of NS 5-branes, KK-monopoles, $Q$- and $R$-branes. They lead to consistent six-dimensional compact space with various combinations of non-geometric fluxes and an interesting non-commutative or non-associative geometric structure. We can summarize the web of T-dualities that led to all considered geometrical and non-geometrical brane intersection spaces as follows:
In the corresponding effective four-dimensional theories, all moduli of the associated 6D compact spaces are fixed by the combinations of geometric and/or non-geometric fluxes, when also including Ramond fluxes from additional intersecting D-branes. The corresponding potential has in its minimum a negative energy, confirming the fact that in the near horizon limit of the interesting branes the four-dimensional space is given by $AdS_4$.

**Acknowledgments**

This work is supported by the Munich Excellence Cluster for Fundamental Physics "Origin and Structure of the Universe". We like to thank D. Andriot, A. Betz, O. Hohm, M. Larfors, D. Tsimpis and T. Weigand for very useful discussions and comments on the manuscript. D.L. likes to thank the theory group of CERN and also the MIT Center for Theoretical Physics for hospitality, where part of this work was performed.
\section{Q-branes as solution of the NS action}

Setting the variation $\delta S$ of the standard NS action in eq. (2.1) to zero, leads to the field equations

\begin{align}
\mathcal{R} + 4(\nabla^2 \phi - (\partial \phi)^2) &= \frac{1}{12} H_{ijk} H^{ijk} \quad \text{(A.1)} \\
R_{ij} + 2 \nabla_i \partial_j \phi &= \frac{1}{4} H_{ijm} H^{mn} \quad \text{(A.2)} \\
2 \partial^i \phi H_{ijk} &= \nabla^i H_{ijk} \quad \text{(A.3)}
\end{align}

for the metric $G_{ij}$, the asymmetric Kalb-Ramond field $B_{ij}$ with the $H$-flux $H_{ijk} = \nabla_{[i} B_{jk]}$ and the dilaton $\phi$. When evaluating the various terms of these field equations, it is sufficient to consider only the orthogonal coordinates $x^3$, $y^1$, $y^2$ and $y^3$ of the Q-brane. The remaining, transversal coordinates are trivial and do not give a contribution. Using eqs. (2.18) and (2.19), we start with

\[
G = \begin{pmatrix}
    h & 0 & 0 & 0 \\
    0 & \frac{h}{h^2 + A_2^2} & 0 & 0 \\
    0 & 0 & \frac{h}{h^2 + A_2^2} & 0 \\
    0 & 0 & 0 & \frac{h}{h^2 + A_2^2}
\end{pmatrix}, \quad B = \begin{pmatrix}
    0 & 0 & A & 0 \\
    0 & 0 & - \frac{A}{h^2 + A_2^2} & 0 \\
    0 & 0 & 0 & 0 \\
\end{pmatrix} \quad \text{and} \quad \phi = \log \sqrt{\frac{h}{h^2 + A_2^2}}
\]

where $h(x^3, y^3)$ is an arbitrary harmonic function and $A_2(x^3, y^3)$ is linked to $h$ by eq. (2.6). This two functions $h$ and $A_2$ parameterize the Q-brane completely. They fulfill the Cauchy-Riemann equations

\[ h^{(1,0)} = A_2^{(0,1)} \quad \text{and} \quad h^{(0,1)} = -A_2^{(1,0)}, \]

where we used the convenient abbreviation $h^{(n,m)} = \partial^n x^m h$. This property of $h$ and $A_2$ allows to express the Q-brane as an hermitian manifold which is parameterized by a single holomorphic function

\[ f(z^2) = h + iA_2 \quad \text{with} \quad z^2 = x^3 + iy^3. \]

The metric on this hermitian manifold

\[ ds_h^2 = \frac{1}{f(z^2)^2} dz^1 \otimes dz^1 + f(z^2) dz_2 \otimes \overline{dz_2} \quad \text{(A.4)} \]

reproduces for $z_2 = y^1 + iy^2$ to the metric $ds^2 = 1/2(ds_h^2 + \overline{ds_h^2})$ corresponding to $G_{ij}$ and the 2-form flux $B = i/2(ds_h^2 - \overline{ds_h^2}) = B_{ij} dy^i \wedge dy^j$.

To simply the following calculation, we use the substitution rules

\[ A^{(0,1)} \rightarrow h^{(1,0)}, \quad A^{(1,0)} \rightarrow -h^{(0,1)}, \quad A^{(2,0)} \rightarrow -h^{(1,1)} \quad \text{and} \quad A^{(0,2)} \rightarrow h^{(1,1)} \]

in order to eliminate all $A_2$-derivatives from the field equations. Additionally the substitution $h^{(0,2)} \rightarrow -h^{(2,0)}$ takes into account that $h$ is a harmonic function. Using the Mathematica tensor package xAct, we computed all terms of the field equations. The terms

\[
\mathcal{R} = 3 \frac{h^{(1,0)}^2 + h^{(0,1)}^2}{2h^3}, \quad \nabla^2 \phi = 0, \quad (\partial \phi)^2 = \frac{h^{(1,0)}^2 + h^{(1,0)}^2}{2h^3}, \\
H_{ijk} H^{ijk} = 6 \frac{h^{(1,0)}^2 + h^{(0,1)}^2}{h^3}, \quad (\partial^i \phi) H_{ijk} = 0 \quad \text{and} \quad \nabla^i H_{ijk} = 0
\]
of eqs. (A.1) and (A.3) can be reduced to a simple form which only depends on $h$ and its first derivative. This is not the case for terms of eq. (A.2) which are much more complex and therefore not presented here. Nevertheless they fulfill, like the terms given above, the field equation derived from the standard NS action.

Now we like to discuss the solutions of the redefined NS action in eq. (2.4). The principle of stationary action $\delta S = 0$ leads for the redefined NS action to the following redefined field equations [10]

$$\tilde{\mathcal{R}} + 4(\nabla^2 \tilde{\phi} - (\partial \tilde{\phi})^2) = \frac{1}{4} Q^i j_k Q^{j_k}$$  \hspace{1cm} (A.5)

$$\tilde{R}_{ij} + 2\nabla_i \partial_j \tilde{\phi} = \frac{1}{4} \left( Q^{mn} Q_{jmn} - 2Q_{mi} Q^{mn} \right)$$ \hspace{1cm} (A.6)

$$2(\partial^i \tilde{\phi}) Q^{jk} = \nabla^i Q^{jk}$$ \hspace{1cm} (A.7)

where $Q$ denotes the $Q$-flux $Q^{jk} = \partial_i \beta^{jk}$. Like for the field equations of the standard NS action, we only consider the orthogonal coordinates of the $Q$-brane whose configuration

$$\tilde{G} = \left( \begin{array}{ccc} h & 0 & 0 \\ 0 & \frac{1}{h} & 0 \\ 0 & \frac{1}{h} & 0 \\ 0 & 0 & h \end{array} \right), \quad \beta = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -A \\ 0 & A & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \text{and} \quad \tilde{\phi} = \log \sqrt{\frac{1}{h}}$$

is given by eq. (2.24). Again the above introduced substitutions are applied during the calculations of the individual terms of the redefined field equations, leading to

$$\tilde{\mathcal{R}} = -\frac{5}{2h^3} \frac{h(1,0)^2 + h^2}{2h^2}, \quad \nabla^2 \tilde{\phi} = \frac{h(1,0)^2 + h(0,1)^2}{h^3}, \quad (\partial \tilde{\phi})^2 = \frac{h(1,0)^2 + h(0,1)^2}{4h^3},$$

$$Q^{jk} Q_{jk} = 2 \frac{h(1,0)^2 + h^2}{h^3}, \quad (\partial \tilde{\phi})^2 = 0 \quad \text{and} \quad \nabla^\mu Q^{jk} = 0$$

for eqs. (A.5) and (A.7). Additionally on get the following terms

$$\tilde{R}_{ij} = \left( \begin{array}{cccc} \frac{2h(0,1)^2 - 3h(1,0)^2 + 2hh(2,0)}{2h^2} & 0 & 0 & 2hh(1,1) - 5h(0,1)h(1,0) \\
0 & \frac{h(0,1)^2 + h(1,0)^2}{h^2} & 0 & 2h^2 \\
0 & 0 & -\frac{h(0,1)^2 + h(1,0)^2}{h^2} & 0 \\
\frac{2hh(1,1) - 5h(0,1)h(1,0)}{2h^2} & 0 & 0 & -\frac{3h(0,1)^2 - 2h(1,0)^2 + 2hh(2,0)}{2h^2} \end{array} \right)$$

$$\nabla_i \partial_j \tilde{\phi} = \left( \begin{array}{cccc} \frac{h(0,1)^2 - 3h(1,0)^2 + 2hh(2,0)}{4h^3} & 0 & 0 & 2h^2 \\
0 & \frac{h(0,1)^2 + h(1,0)^2}{4h^3} & 0 & 2h^2 \\
0 & 0 & \frac{h(0,1)^2 + h(1,0)^2}{4h^3} & 0 \\
\frac{2h(0,1)h(1,0) - hh(1,1)}{2h^2} & 0 & 0 & 2h^2 \end{array} \right)$$

$$Q^{mn} Q_{jmn} - 2Q_{mi} Q^{mn} = \left( \begin{array}{cccc} \frac{2h(0,1)^2}{h^2} & 0 & 0 & -2h(0,1)h(1,0) \\
0 & \frac{2h(0,1)^2 + h(1,0)^2}{h^2} & 0 & 0 \\
0 & 0 & \frac{h(0,1)^2 + h(1,0)^2}{h^2} & 0 \\
\frac{2h(0,1)h(1,0) - hh(1,1)}{2h^2} & 0 & 0 & \frac{2h(0,1)^2}{h^2} \end{array} \right)$$

for eq. (A.6). As expected, these results prove that the redefined $Q$-brane satisfies the redefined field equations.
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