PERMUTATION GROUPS CONTAINING INFINITE LINEAR GROUPS AND REDUCTS OF INFINITE DIMENSIONAL LINEAR SPACES OVER THE TWO ELEMENT FIELD

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Abstract. Let $\mathbb{F}_2^\omega$ denote the countably infinite dimensional vector space over the two element field and $\text{GL}(\omega, 2)$ its automorphism group. Moreover, let $\text{Sym}(\mathbb{F}_2^\omega)$ denote the symmetric group acting on the elements of $\mathbb{F}_2^\omega$. It is shown that there are exactly four closed subgroups, $G$, such that $\text{GL}(\omega, 2) \leq G \leq \text{Sym}(\mathbb{F}_2^\omega)$. As $\mathbb{F}_2^\omega$ is an $\omega$-categorical (and homogeneous) structure, these groups correspond to the first order definable reducts of $\mathbb{F}_2^\omega$. These reducts are also analyzed. In the last section the closed groups containing the infinite symplectic group $\text{Sp}(\omega, 2)$ are classified.

1. Introduction

Let $\mathbb{F}_2^\omega$ denote the countably infinite dimensional vector space over the two element field and $\text{GL}(\omega, 2)$ its automorphism group. Moreover, let $\text{Sym}(\mathbb{F}_2^\omega)$ denote the symmetric group acting on the elements of $\mathbb{F}_2^\omega$. In this paper the closed subgroups of $\text{Sym}(\mathbb{F}_2^\omega)$ containing $\text{GL}(\omega, 2)$ are investigated. In the last section this results are extended to closed groups containing the infinite symplectic group $\text{Sp}(\omega, 2)$. Closure of subgroups means being closed in the topology of pointwise convergence. For an infinite set $\Omega$ a subgroup $H \leq \text{Sym}(\Omega)$ is closed if the following condition holds: for every $\pi \in \text{Sym}(\Omega)$ and every finite subset $S \subset \Omega$ if there is some $\sigma \in H$ such that $\sigma|_S = \pi|_S$ then $\pi \in H$.

For finite dimensional projective spaces it is almost independently shown in [1], [10] and [14] that if $\text{PSL}(n, q)$, the special linear group is contained in a subgroup $G$ of the symmetric group acting on the points of the projective space, then $G$ is either the alternating or the full symmetric group, or $G$ is contained in the twisted projective linear group. If we investigate the infinite version of this theorem, we have to consider the following: The alternating group has an infinite counterpart - the group of finite support even permutations but it is not closed. As $A_n \geq S_{n-2}$ for every finite integer $n$, the even permutations on a set of size $\omega$ generate all permutations on every finite subset: for every finite $S \subset \omega$ and $\pi \in \text{Sym}(S)$ there is some $\mu \in \text{Sym}(\omega)$ such that $\mu|_S = \pi$. Hence the closure of the subgroup generated by the even permutations in $\text{Sym}(\omega)$ is $\text{Sym}(\omega)$ itself. Similarly, "$\text{PSL}(\omega, q)$" does not make sense, also, the closure of the subgroup generated by the permutations in $\text{Sym}(P(\omega, q))$ generated by the matrices of determinant 1 is $\text{PGL}(\omega, q)$.

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In this paper we consider the countably infinite dimensional vector space, \( \mathbb{F}_2^\omega \), and prove that if the automorphism group of the countably infinite dimensional vector space, \( \text{GL}(\omega, 2) \) is contained in a closed (nontrivial) subgroup \( G \) of the symmetric group \( \text{Sym}(\mathbb{F}_2^\omega) \), then \( G \) is either the affine group or \( \text{Sym}_0(\mathbb{F}_2^\omega) \), the stabilizer of 0. As for the two element field the linear and projective linear groups are essentially the same, as a corollary, we obtain that the projective linear group is a maximal closed group of \( \text{Sym}(\mathbb{F}_2^\omega) \). We also obtain that the closed groups containing the infinite symplectic group \( \text{Sp}(\omega, 2) \) are exactly the groups containing \( \text{GL}(\omega, 2) \), the group \( \text{Sp}(\omega, 2) \) itself and the group generated by \( \text{Sp}(\omega, 2) \) and the group of translations.

One would expect similar characterisation for all finite fields of prime size. We show that contrary to the expectations there are several more such closed subgroups for primes greater than 2. We exhibit \( d(p-1) \) many closed supergroups of \( \text{GL}(\omega, p) \) for every prime \( p > 2 \), where \( d(n) \) denotes the number of the divisors of \( n \). This shows that for fields of larger size the problem is far more complicated.

The vector space \( \mathbb{F}_2^\omega \) is homogeneous in the sense that every (partial) automorphism between two finite substructures can be extended to an automorphism of \( \mathbb{F}_2^\omega \). For more details on the model theoretical background consult \([8]\). Closed supergroups of automorphism groups of homogeneous structures have special importance. Closed supergroups of the automorphism group are in a Galois-connection with the first order definable reducts of the structure. A relational structure is first order definable from an other one if it has the same underlying set and can be defined by first order sentences. First order interdefinability is an equivalence relation, and the classes dually correspond to the closed subgroups of the full symmetric group containing the automorphism group of the structure.

For \( \omega \)-categorical structures this is a bijection between the equivalence classes of reducts and the closed groups containing the automorphism group of the structure. Since the structures investigated in this paper are \( \omega \)-categorical the description of the reducts up to first order interdefinability and the description of the closed groups containing the automorphism group of a given structure is equivalent.

The \( \omega \)-categoricity can be checked using the following theorem \([8]\):

**Theorem 1.1** (Engeler, Ryll-Nardzewski, Svenonius). A countable structure is \( \omega \)-categorical if and only if its automorphism group is oligomorphic i. e. for every \( n \) the automorphism group acting on the ordered \( n \)-tuples of the elements of the structure has finitely many orbit.

Until recently only sporadic examples of first order definable reducts were known. In \([6]\) the reducts of the dense linear order are determined. For the random graph and random hypergraphs Thomas has determined their reducts \([16], [17]\). Both the random graph and the dense linear order have 5 reducts. In \([9]\) it is shown that the “pointed” linear order has 116 reducts. Thomas has conjectured that any homogeneous structure on a finite relational language has finitely many reducts. Later in \([4]\) and \([5]\) a general technique was introduced to investigate first order definable reducts of homogeneous structures on a finite language. Then several structures were analyzed from this aspect: the pointed Henson-graphs \([15]\), the random poset \([12], [13]\), equality \([2]\) and the random graph revisited \([3]\).

It is argued in \([11]\) that the homogeneous vector spaces and the affine spaces cannot be defined by any finite relational language. Hence our work is a first attempt to classify first order definable reducts of homogeneous structures on an infinite relational language.
We aimed to have our proofs as elementary as possible. All finitary versions of our theorem and recent results on homogeneous structures use rather difficult techniques. Although it is tempting to refer to those results, we kept this paper self-contained in this sense.

2. SUPERGROUPS OF $GL_2^2$

Let us introduce some notation, first. Let the group $G$ act on the set $\Omega$. For an element $s \in \Omega$ let $G_s$ denote the stabilizer of $s$ in $G$ and for $s_1, s_2, \ldots, s_n$ let $G(s_1, s_2, \ldots, s_n)$ denote the elementwise stabilizer of the elements. Let $Aff(\omega, 2)$ denote the affine space obtained from $F_2^2$. Let $AGL$ denote the automorphism group of $Aff(\omega, 2)$. The group $AGL$ is generated by $GL(\omega, 2)$ and the translations. Let $a \in F_2^2$. The translation by $a$, denoted by $t_a$, is defined by $v + a$. Note that $GL(\omega, 2)$ along with any translation generates $AGL$. We shall use the elementary facts about linear algebra (as the extendability of a map from a basis to a linear transformation) without any reference.

We summarize the necessary information about $AGL$.

**Lemma 2.1.** The group $AGL$ contains exactly those $f$ permutations of $\text{Sym}(F_2^2)$ that preserve the ternary addition: for all $a, b, c \in F_2^2$ the equality $(a + b + c)^f = a^f + b^f + c^f$ holds, or alternatively if $a + b + c + d = 0$ we have $a^f + b^f + c^f + d^f = 0$. In particular $AGL$ is closed.

**Proof.** Let $f$ be such that for all $a, b, c \in F_2^2$ the equality $(a + b + c)^f = a^f + b^f + c^f$ hold. Let the map be defined by $x^g = x^f + 0^f$. Then $g \in GL(\omega, 2)$, because $0^g = 0^f + 0^f = 0$ and $(a + b)^g = (a + b + 0)^f + 0^f = a^f + b^f + 0^f + 0^f = a^g + b^g$.

Since $x^f = x^g + 0^f$ the permutation $f$ is contained in $AGL$, it is the composition of the vector space automorphism $g$ and the translation $t_0^f$.

The other direction is obvious as both the elements of $GL(\omega, 2)$ and the translations preserve the ternary addition. \hfill $\square$

The following lemma will be applied several times.

**Lemma 2.2.** Let us assume that $G \leq \text{Sym}(F_2^2)_0$ and $G$ acts $n$-transitively on $F_2^2 \setminus \{0\}$.

Moreover, let us assume that for every $(x_1, x_2 \ldots x_k) \in F_2^k \setminus \{0\}$ finite tuple of elements and every $y \in F_2^k \setminus \langle x_1, x_2 \ldots x_k \rangle$ the element $y$ has an infinite orbit in the pointwise stabilizer of the tuple $(x_1, x_2 \ldots x_k)$ in $G$.

Let $a_1, a_2 \ldots a_n, a_{n+1} \in F_2^k \setminus \{0\}$ be distinct elements such that $a_{n+1} \notin \langle a_1, a_2 \ldots a_n \rangle$. Then there is an $h \in G$ such that $a_1^h, a_2^h, \ldots, a_n^h, a_{n+1}^h$ are linearly independent.

**Proof.** Since $G$ acts $n$-transitively on $F_2^k \setminus \{0\}$ we can choose an element $g \in G$ such that $a_1^g, a_2^g, \ldots, a_n^g$ are linearly independent. Let $W = \langle a_1^g, a_2^g, \ldots, a_n^g \rangle$. Now, $W^{g^{-1}}$ is a finite set, and the orbit of $a_{n+1}$ in the stabilizer of $\{a_1, a_2 \ldots a_n\}$ is infinite, hence we can choose an $h \in G(a_1, a_2 \ldots a_n)$ such that $a_{n+1}^h \notin W^{g^{-1}}$. Then $a_i^h = a_i^g$ for $1 \leq i \leq n$ and they are independent, moreover $a_{n+1}^h \notin W$, hence $a_1^h, a_2^h, \ldots, a_n^h, a_{n+1}^h$ are linearly independent. Thus $hg$ satisfies the condition of the Lemma. \hfill $\square$

**Theorem 2.3.** Let us assume that $GL(\omega, 2) \leq G \leq \text{Sym}(F_2^2)_0$ and $G$ is closed. Then $G = \text{Sym}(F_2^2)_0$
Proof. It is enough to prove that $G$ is $n$-transitive for every finite $n$.

At first we show 3-transitivity. The linear group $\text{GL}(\omega, 2)$ acts transitively on the 3-element independent sets, hence it is enough to show that any three vectors $a, b, c \in \mathbb{F}_2$ can be mapped to and independent set. The vectors $a, b, c$ are dependent exactly if $a + b = c$. The condition $\text{GL}(\omega, 2) \leq G$ implies that there are $a', b', c' \in \mathbb{F}_2$ and $g \in G$ such that $a'+b' = c'$ and $a'^g+b'^g \neq c'^g$. Now, consider a map $h \in \text{GL}(\omega, 2)$ mapping $a, b, c$ to $a', b', c'$, respectively. The map $hg$ maps $a, b, c$ to an independent set.

Now, we prove $n$-transitivity by induction. We show that every set of $n+1$ vectors can be mapped to an independent set. Let $a_1, a_2, \ldots, a_n, a_{n+1} \in \mathbb{F}_2^n \setminus \{0\}$ be dependent distinct elements. By the $n$-transitivity we may assume that $a_n = a_1 + a_2 + \ldots + a_{n-1}$ and there is an $h \in G$ such that $\{a_i^h| i = 1, 2, \ldots, n\}$ is a linearly independent set. If $a_{n+1} \notin \langle a_1, a_2, \ldots, a_n \rangle$, then by Lemma 2.2 we are done. If $a_{n+1} \in \langle a_1, a_2, \ldots, a_n \rangle$, then $a_{n+1} = \sum_{1}^{n-1} \varepsilon_i a_i$ where at least two, but not all $\varepsilon_i$ are equal to 1. Indeed, assume that there is a unique $i$ such that $\varepsilon_i = 1$, then $a_{n+1} = a_i$ would hold, and if all of them were equal to 1, then $a_{n+1} = \sum_{1}^{n-1} a_i = a_n$ would contradicting that the vectors are distinct. Let $\varepsilon_j = 1$ and $\varepsilon_k = 0$ for some $j, k < n$. Then there is a linear map $g$ flipping $a_j$ and $a_k$ and fixing every $a_i$, where $i < n$ and $i \neq j, k$. Now, $\{a_i^h| i = 1, 2, \ldots, n\} = \{a_i^{gh}| i = 1, 2, \ldots, n\}$ is an independent set and $a_{n+1}^h \neq a_{n+1}^{gh}$. If any of the latter two elements is not in $\langle a_1^h, a_2^h, \ldots, a_n^h \rangle$ then we are done by Lemma 2.2. Otherwise we may assume that $a_{n+1}^h = \sum_{1}^{n} \xi_i a_i^h$, where there is an $l$ such that $\xi_l = 0$. Now, $a_l^h \notin \langle a_i^h| 1 \leq i \leq n+1, i \neq l \rangle$ and we are done again, by Lemma 2.2.

Now, we consider the case when 0 is not fixed by $G$.

Lemma 2.4. Let us assume that $G \leq \text{Sym}(\mathbb{F}_2^n)$, and $G_0 = G \cap (\text{Sym}(\mathbb{F}_2^n))_0 \leq \text{GL}(\omega, 2)$.

Moreover, let us assume that for every $(x_1, x_2, x_3) \in \mathbb{F}_2^n \setminus \{0\}$ finite tuple of elements and every $y \in \mathbb{F}_2^n \setminus \langle x_1, x_2, x_3 \rangle$ the element $y$ has an infinite orbit in the pointwise stabilizer of the tuple $(x_1, x_2, x_3)$ in $G$. Then $G \leq \text{AGL}$.

Proof. Recall that the affine group AGL is the set of elements of $\text{Sym}(\mathbb{F}_2^n)$ preserving the ternary addition, or alternatively, preserving the 4-tuples $(a, b, c, d)$ satisfying $a + b + c + d = 0$.

The pointwise stabilizer of an injective 3-tuple $(0, a, b)$ fixes the element $a + b$ because this stabilizer fixes the 0 so it is in GL. By the assumption of the lemma this stabilizer does not fix any other elements in $\mathbb{F}_2^n \setminus \{0, a, b\}$. Because of the transitivity this implies that the pointwise stabilizer of any injective 3-tuple $(a, b, c)$ must have exactly one fixed point in $\mathbb{F}_2^n \setminus \{a, b, c\}$. Let us denote this fix point by $F(a, b, c)$. By the assumption of the lemma we know that $F(a, b, c)$ must be contained in the subspace $\langle a, b, c \rangle$. In particular it implies that if $c = a + b$ and $a, b, c \neq 0$, then $F(a, b, c) = 0$.

Now, let $a, b, c \in \mathbb{F}_2^n$ be linearly independent elements. Then $F(a, b, c) \in \langle a, b, c \rangle$. We claim that $F(a, b, c) = a + b + c$. Suppose not. Then $F(a, b, c) \in \{0, a + b, a + c, b + c\}$. By symmetry we can assume that $F(a, b, c)$ is either 0 or $a + b$. In any case $c \notin \langle a, b, F(a, b, c) \rangle$, hence $c \neq F(a, b, F(a, b, c))$. The group $G$ is transitive,
thus there exists a permutation \( g \in G \) such that \( a^g = 0 \). By the definition of the function \( F \) we know that \( F(x^g, y^g, z^g) = (F(x, y, z))^g \) holds for any injective 3-tuples \((x, y, z)\). Therefore

\[
F(a, b, F(a, b, c)) = (F(a^g, b^g, F(a^g, b^g, c^g))^g^{-1} =
= (F(0, b^g, F(0, b^g, c^g))^g^{-1} = (F(0, b^g, b^g + c^g)^g^{-1} = (c^g)^g^{-1} = c.
\]

This is a contradiction, hence \( F(a, b, c) \) must be \( a + b + c \). We obtained that the group \( G \) preserves the ternary addition \( x + y + z \), therefore \( G \leq AGL \). \( \square \)

**Theorem 2.5.** Let us assume that \( GL(\omega, 2) \subsetneq G \leq Sym(\mathbb{F}_2^\omega) \), and \( G \) is closed not fixing 0. Then \( G = AGL \) or \( G = Sym(\mathbb{F}_2^\omega) \)

**Proof.** The stabilizer of the 0 in \( G \) is GL or Sym\(_2\). If it is Sym\(_2\), then \( G \) must be Sym because \( G \) does not fix the 0. Assume that the stabilizer of the 0 in \( G \) is GL. Then the assumptions in Lemma 2.4 hold for \( G \). So \( G \) is a subgroup of AGL containing GL. Using that AGL is a semidirect product of \( T \) and GL, and AGL is generated by GL and any translation (except the identity), and \( G \) cannot be GL because \( G \) does not fix the 0. We obtained that in this case \( G \) must be the group AGL. \( \square \)

### 3. Orbits and reducts

In Section 2 we have found the closed supergroups of \( GL(\omega, 2) \) in \( Sym(\mathbb{F}_2^\omega) \). Each of them corresponds to a first order definable reduct of the vector space \( \mathbb{F}_2^\omega \).

**Corollary 3.1.** The vector space \( \mathbb{F}_2^\omega \) has 4 first order definable reducts:

1. \( \mathbb{F}_2^\omega \), corresponding to the group \( GL(\omega, 2) \),
2. The affine space corresponding to AGL,
3. The structure, with one unary relation, \( \{0\} \) corresponding to \( Sym_2(\mathbb{F}_2) \),
4. The trivial (no relation) structure, corresponding to \( Sym(\mathbb{F}_2) \).

Each reduct is homogeneous by looking at its automorphism group. On the other hand, as we mentioned earlier, neither \( \mathbb{F}_2^\omega \), nor the affine space are homogeneous on a finite language. As AGL is 3 transitive, the first interesting problem is to describe the 4-types in the affine space.

**Lemma 3.2.** The affine group AGL has the following orbits on the 4-tuples of the affine space:

1. \((a, a, a, a)\) for any \( a \in \mathbb{F}_2^\omega \)
2. \((a, a, a, b)\) for any \( a, b \in \mathbb{F}_2^\omega \), where \( a \neq b \) and all its cyclic permutations.
3. \((a, a, b, b)\) for any \( a, b \in \mathbb{F}_2^\omega \), where \( a \neq b \)
4. \((a, b, a, b)\) for any \( a, b \in \mathbb{F}_2^\omega \), where \( a \neq b \)
5. \((a, b, b, a)\) for any \( a, b \in \mathbb{F}_2^\omega \), where \( a \neq b \)
6. \((a, a, b, c)\) for any \( a, b, c \in \mathbb{F}_2^\omega \), where \( \{|a, b, c|\} = 3 \), and all six permutations of this tuple.
7. \((a, b, c, d)\) for any \( a, b, c, d \in \mathbb{F}_2^\omega \), where \( \{|a, b, c, d|\} = 4 \), and \( a+b+c+d \neq 0 \).
8. \((a, b, c, d)\) for any \( a, b, c, d \in \mathbb{F}_2^\omega \), where \( \{|a, b, c, d|\} = 4 \), and \( a+b+c+d = 0 \).

**Proof.** The first six items follow from the 3-transitivity of AGL. For items 7 and 8 let \( a, b, c, d \) be distinct elements of the affine space. Again, by the 3-transitivity of AGL we may assume that \( a, b, c \) are linearly independent. If \( d \notin \langle a, b, c \rangle \), then
Definition 4.1. Let \( P \). We can add the binary relations bilinear form, then it is no longer a first order structure. To get around this problem this definition is that if we enrich the structure of a vector space with a symplectic \( \cdot \) vector space automorphisms which preserve the symplectic bilinear product \( F \). For \( F \) vector spaces over \( \mathbb{F}_q \) dimensional counterpart of the finite symplectic groups.

Equivalently: find the \( \omega \), \( \omega \), \( \omega \), \( \omega \) the group \( GL(\omega, q) \) as the set of permutations which preserve the relation \( \sim_H \). This group acts on \( V \) as the symmetric group, so it is not the group \( GL(\omega, q) \), and preserves a nontrivial relation, so it can not be \( Sym(\mathbb{F}_q^\omega) \) either.

Hence, finding the closed supergroups of \( GL(\omega, q) \) for \( q > 2 \) will require different techniques.

Problem 3.4. Find the closed groups \( G \) satisfying \( GL(\omega, q) \leq G \leq Sym(\mathbb{F}_q^\omega) \). Equivalently: find the first order definable reducts of the vector space \( \mathbb{F}_q^\omega \).

4. Vector space endowed with a symplectic bilinear product

In this section we will investigate the closed groups containing the infinite dimensional counterpart of the finite symplectic groups.

Fraïssé’s theorem states that every homogeneous structure can be obtained as the Fraïssé-limit of its age (the class of its finitely generated substructures) \([7]\). We would like to define the structure \( \mathbb{F}_2^\omega (\tilde{\cdot}, \cdot) \) as the Fraïssé-limit of finite dimensional vector spaces over \( \mathbb{F}_2 \) endowed with a symplectic bilinear form \( \cdot \) . The problem with this definition is that if we enrich the structure of a vector space with a symplectic bilinear form, then it is no longer a first order structure. To get around this problem we can add the binary relations \( P_i(x, y) \) to the vector spaces \( \mathbb{F}_2^\omega \) which will express \( x \cdot y = i \) (\( i = 0, 1 \)). In this case the automorphisms of this structure are exactly those vector space automorphisms which preserve the symplectic bilinear product \( \cdot \) .

Definition 4.1. Let \( \mathcal{F} \) denote the class of finite dimensional vector spaces over \( \mathbb{F}_2 \) with binary relations \( P_0, P_1 \) for which the following axioms hold:

1. \( \forall x, y (P_0(x, y) \leftrightarrow \neg P_1(x, y)) \)
2. For all \( i, j \in \mathbb{F}_2 \) the formula \( \forall x, y, z (P_i(x, z) \land P_j(y, z) \rightarrow P_{i+j}(x + y, z)) \).
3. For all \( i, j \in \mathbb{F}_2 \) the formula \( \forall x, y, z (P_i(x, y) \land P_j(x, z) \rightarrow P_{i+j}(x, y + z)) \).
4. \( \forall x (P_0(x, x)) \).

Note that non-degenerateness is not required. In \([7]\) a Fraïssé-class is defined as a class of finitely generated structures satisfying the three properties described below. For every such class Fraïssé’s theorem guarantees the existence of a countable

\((a, b, c, d) \) belongs to case \([4]\). If \( d \in \langle a, b, c \rangle \), then as \( a, b, c \) are distinct either \( d \) is the sum of two or the sum of three elements. In the first case we may assume that \( a + b = d \). Then \( \langle a, b \rangle = \{a, b, d, 0\} \) and so \( c \notin \langle a, b, d \rangle \). Then, by Lemma 2.2 ( \( a, b, c, d \) ) belongs to case \([7]\). In the second case \( a + b + c = d \), or equivalently \( a + b + c + d = 0 \), hence we are in case \([8]\). \( \square \)

And now we present an example of a reduct of \( \mathbb{F}_q^\omega \) for \( q > 2 \).

Example 3.3. Let \( H < \mathbb{F}_q^\omega \), a subgroup of the multiplicative group of \( \mathbb{F}_q \) and let \( |H| = k \). Note that \( H \) is cyclic and \( k|p-1 \). As \( H \) is a subgroup of the multiplicative group, every \( h \in H \) acts by multiplication on \( \mathbb{F}_q^\omega \). For convenience we shall write \( h \cdot v \) instead of \( v^h \) to distinguish the action of \( H \) on \( \mathbb{F}_q^\omega \). Define the relation \( \sim_H \) on \( \mathbb{F}_q^\omega \) \( \setminus \{0\} \) in the following way: for \( a, b \in \mathbb{F}_q^\omega \setminus \{0\} \) let \( a \sim_H b \) if there is an \( h \in H \) such that \( h \cdot a = b \). The relation \( \sim_H \) is an equivalence relation and every \( \sim_H \) class is contained in a 1-dimensional subspace of \( \mathbb{F}_q^\omega \). Let \( \mathcal{V} \) denote the set of equivalence classes of \( \sim_H \).

Now, we define the subgroup \( G < Sym(\mathbb{F}_q^\omega) \) as the set of permutations which preserve the relation \( \sim_H \). This group acts on \( \mathcal{V} \) as the symmetric group, so it is not the group \( GL(\omega, q) \), and preserves a nontrivial relation, so it can not be \( Sym(\mathbb{F}_q^\omega) \) either.

Hence, finding the closed supergroups of \( GL(\omega, q) \) for \( q > 2 \) will require different techniques.
homogeneous structure such that its class of finitely generated substructures is precisely the given class. The three required properties:

- (HP) Hereditary property: if a structure $S$ is in $\mathcal{F}$ then every finitely generated substructure of $S$ is in $\mathcal{F}$.
- (JEP) Joint embedding property: if $S_1$ and $S_2$ are two structures from $\mathcal{F}$ then there is a structure $S_3$ in $\mathcal{F}$ such that both $S_1$ and $S_2$ can be embedded into it.
- (AP) Amalgamation property: if $S_1, S_2$ and $S_3$ are three structures from $\mathcal{F}$ and $\phi_1$ and $\phi_2$ are embeddings of $S_3$ into $S_1$ and $S_2$ respectively then there exist a structure $S_4$ in $\mathcal{F}$ and embeddings $\psi_1 : S_1 \to S_4$ and $\psi_2 : S_2 \to S_4$ such that the embeddings $\phi_1 \circ \psi_1$ and $\phi_2 \circ \psi_2$ are the same ($\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$ embeds $S_3$ into $S_4$).

These three properties are satisfied by $\mathcal{F}$.

Now the definition of $\mathbb{F}_2^2(+, \cdot)$ is as follows.

**Definition 4.2.** Let $\mathcal{V} = \mathbb{F}_2^2(+, \cdot)$ be the Fraïssé-limit of the class $\mathcal{F}$ defined in Definition [4.1]. We will define the function $\cdot : \mathcal{V} \times \mathcal{V} \to \mathbb{F}_2$ as follows:

$$x \cdot y = i \text{ iff } P_i(x, y) \text{ holds.}$$

Because of the axioms in Definition [4.1], the function $\cdot$ is well-defined and is a non-degenerate symmetric bilinear form on $\mathbb{F}_2^2$. Moreover, the automorphism group of $\mathbb{F}_2^2(+, \cdot)$ as a first order structure is the group of those vector space automorphisms which preserve the symmetric bilinear product $\cdot$.

**Proposition 4.3.** The structure $\mathcal{V} = \mathbb{F}_2^2(+, \cdot)$ is homogeneous, and $\omega$-categorical.

**Proof.** The homogeneity is guaranteed by Fraïssé’s theorem. The $\omega$-categoricity is equivalent to the oligomorphy of the automorphism group [1.1]. The automorphism group will be oligomorphic because for every $n$ there are finitely many (possibly 0) isomorphism types of finitely generated substructures, and by homogeneity two isomorphic substructures lie in the same orbit of the automorphism group. \(\square\)

We will denote the automorphism group of the structure $\mathcal{V} = \mathbb{F}_2^2(+, \cdot)$ by $\text{Aut}(\mathbb{F}_2^2(+, \cdot)) = \text{Sp}$.

**Proposition 4.4.** If $a_1, a_2, \ldots, a_n \in \mathcal{V}$ are linearly independent, then for all $i_1, i_2, \ldots, i_n \in \mathbb{F}_2$ there exists an element $w \notin \langle a_1, \ldots, a_n \rangle$ in $\mathcal{V}$ such that $a_j \cdot w = i_j$ for all $j = 1, 2, \ldots, n$.

**Proof.** This is a special case of the extension property of homogeneous structures. Let $b_1, b_2, \ldots, b_{n+1}$ be a base of the vector space $\mathbb{F}_2^{n+1}$. We will define $a \cdot$ bilinear form on $\mathbb{F}_2^{n+1}$ to do so is enough to define the values $b_j \cdot b_k$ for all possible pairs of base elements. Let $b_j \cdot b_k$ be

- 0 if $j = k$
- 0 if $j \neq k$, $j \leq n$, $k \leq n$ and $P_0(a_j, a_k)$ holds in $\mathcal{V}$
- 1 if $j \neq k$, $j \leq n$, $k \leq n$ and $P_1(a_j, a_k)$ holds in $\mathcal{V}$
- $x$ if $j \leq n$, $k = n + 1$ and $i_j = x$
- $x$ if $k \leq n$, $j = n + 1$ and $i_k = x$

We define the relations $P_0$ and $P_1$ on $\mathbb{F}_2^{n+1}$ as usual, and this yields a structure from the class $\mathcal{F}$ [1.1]. This structure will be denoted by $F_1$. The structure $\mathcal{V}$ has a substructure isomorphic to $F_1$: this substructure will be denoted by $F_2$. Denote the
substructure of \( V \) generated by the elements \( a_1, a_2, \ldots, a_n \in V \) by \( F_3 \). Denote the substructure of \( F_2 \) (and thus of \( V \)) generated by the elements \( \phi(b_1), \phi(b_2), \ldots, \phi(b_n) \) where \( \phi \) is an isomorphism from \( F_1 \) to \( F_2 \) by \( F_4 \). Then \( F_2 \) and \( F_4 \) are isomorphic so there is a \( \psi \) automorphism of \( V \) extending the isomorphism between them. This \( \psi \) maps \( F_2 \) onto a substructure of \( V \) containing \( F_3 \): the element \( \psi(\phi(b_{n+1})) \) will be a good choice for \( w \).

**Proposition 4.5.** \( Sp = \text{Aut}(P_0) \).

*Proof.* The inclusion "\( \subset \)" is obvious. For the other containment, we have to show that the binary function \( + \) and the relation \( P_1 \) are first order definable from the relation \( P_0 \). The latter is obvious since \( P_1(x,y) \iff \neg P_0(x,y) \). We know that \( \cdot \) is non-degenerate, hence \( x = 0 \) holds if and only if \( \forall y(P_0(x,y)) \) holds. This implies that 0 is definable from \( P_0 \). Now, we claim that for all \( x, y, z \in V \setminus \{0\} \) the equality \( x + y = z \) holds if and only if

\[
z \neq x \land z \neq y \land \forall w((P_0(w,x) \land P_0(w,y)) \rightarrow P_0(w,z))
\]

holds.

At first, assume \( x, y, z \in V \setminus 0 \) and \( x + y = z \). Then \( z \) is not equal to \( x \) or \( y \) and if \( w \cdot x = w \cdot y = 0 \), then \( w \cdot z = w \cdot x + w \cdot y = 0 \). Now assume that \( x, y, z \in V \setminus 0 \) and \( z \) is not equal to \( x \) or \( y \) or \( x + y \). Then \( z \not\in \langle x, y \rangle \), thus by Lemma 4.4 it follows that there exists a \( w \in V \) such that \( w \) is orthogonal to \( x, y \) but it is not orthogonal to \( z \). Therefore \( \forall w((P_0(w,x) \land P_0(w,y)) \rightarrow P_0(w,z)) \) does not hold.

Now, the relation \( x + y = z \) can be defined in general as follows:

\[
x + y = z \iff (x = y \land z = 0) \lor (y = z \land x = 0) \lor (z = x \land y = 0) \lor \\
\land (x \neq 0 \land y \neq 0 \land z \neq 0) \land z \neq x \land z \neq y \land \forall w((P_0(w,x) \land P_0(w,y)) \rightarrow P_0(w,z)).
\]

As we have seen, the relation \( x = 0 \) is definable from \( P_0 \), thus this gives us a first order definition of \( + \) from the relation \( P_0 \). \( \square \)

First we will deal with the groups fixing the 0.

Let \( B \) denote a basis of the vector space \( V \). Then the relation \( P_0 \) defines a graph \( G \) on the domain \( B \): two elements of \( B \) will be connected with an edge if and only if \( P_1(x,y) \) (that is \( P_0(x,y) \) does not hold). It is easy to see that in this case \( G \) is an undirected graph without loops.

**Proposition 4.6.** We can choose the basis \( B \) in such a way that the graph \( G \) defined as above will be isomorphic to the random graph.

*Proof.* We will construct a basis with the given property using the back-and-forth method. Let \( R \) denote the random graph and enumerate all of its vertices as \( a_1, a_2, a_3, \ldots \). Enumerate the elements of \( V \) as \( b_1, b_2, b_3, \ldots \). We will denote the graph obtained from a subset \( S \) of \( V \) by connecting elements \( a, b \in S \) with an edge if and only if \( P_1(a,b) \) holds by \( G(S) \).

We will construct an \( R_i \) sequence of finite subgraphs of \( R \) and an \( S_j \) sequence of finite linearly independent subsets of \( V \) such that

\[
R_1 \subset R_2 \subset R_3 \quad \text{and} \quad R = \cup R_i
\]

\[
S_1 \subset S_2 \subset S_3 \quad \text{and} \quad S = \cup S_j \quad \text{is a basis in} \ V
\]

\[
R_i \cong G(S_i) \quad \text{for every} \ i
\]
Let $R_1$ be an arbitrary vertex of $R$ and $S_1$ be an arbitrary nonzero element of $V$. We will use recursion:

- If $i$ is even and $R_j$ and $S_j$ are already defined for all $j < i$. We choose $R_i$ as the subgraph of $R$ determined by the vertices of $R_{i-1}$ and that vertex of $R \setminus R_{i-1}$ which has the least index in the series $a_1, a_2, a_3, \ldots$. Then we can choose an element $b_i$ from $V \setminus S_{i-1}$ such that $R_i \cong G(S_{i-1} \cup \{b_i\})$ by Proposition 4.4. Let $S_i = S_{i-1} \cup \{b_i\}$.
- If $i$ is odd and $R_j$ and $S_j$ are already defined for all $j < i$. We choose $S_i$ as $S_{i-1} \cup b_k$ where $b_k$ is that element of $V$ which has the least index in the sequence $b_1, b_2, b_3, \ldots$ amongst the elements linearly independent from $S_{i-1}$.

Then we can choose an $R_i \supset R_{i-1}$ subgraph of $R$ such that $R_i \cong G(S_i)$ because of the well-known extension property of the random graph.

The subset $S$ of $V$ will be a basis satisfying the requirements of the Lemma. 

Now, let us fix a basis $B$ with the above property, and let us fix the corresponding graph $G$ as well. If $G$ is an arbitrary permutation group acting on the domain set of $V$ then let $G^B$ denote closure of the action of the group $G_B$ on $B$ in the symmetric group $\text{Sym}(V)$. Here $G_B$ denotes the setwise stabilizer of $B$ in $G$. Note that using this notation $Sp^B = \text{Aut}(G)$.

In the following few lemmas we will show that if an element $g$ violates a certain type of relation on linearly independent elements of $V$, then it can be realized in the group $\langle Sp, g \rangle^B$.

**Lemma 4.7.** Let $S$ be a finite subset of $B$. Suppose we have a function $g \in \text{Sym}(V)$ such that $a^g$ is in $B$ for all $a \in S$. Let $b \in B$ be an arbitrary element. Then there exists an element $g' \in \langle g, Sp \rangle$ such that $g'$ agrees with $g$ on $S$ and $b^g$ is in $B$.

**Proof.** If $b \notin \langle S \rangle$, then the statement of the lemma is trivial. Suppose that it is not the case. Then $b \notin \langle S \rangle$, hence by the homogeneity of $V$ the orbit of $b$ in the pointwise stabilizer of $S$ in $Sp$ is infinite. In particular there exists an element $x$ in this orbit such that $x^g \notin \langle S^g \rangle$ so the elements of $S^g \cup \{x^g\}$ are linearly independent. By the universal property of the graph $G$ there exists an $y \in B$ such that $P_0(a^g, x^g) \Rightarrow P_0(a^g, y)$ for all $a \in S$. Then since both $S^g \cup \{x^g\}$ and $S^g \cup \{y^g\}$ are linearly independent sets, it follows that there exists an automorphism $\gamma$ of $V$ such that $a^\gamma = a$ for all $a \in S$ and $x^\gamma = y$. By the definition of $x$ we know that there exists an automorphism $\delta$ of $V$ such that $a^\delta = a$ for all $a \in S$ and $b^\delta = x$. Then the permutation $g' = d\gamma \delta \in \langle g, Sp \rangle$ will satisfy our requirements. 

**Corollary 4.8.** Let $S \subseteq T_1, T_2$ be finite subsets of $B$. Suppose we have a function $g \in \text{Sym}(V)$ such that $a^g$ and $a^{g^{-1}}$ are in $B$ for all $a \in S$. Then there exists an element $g' \in \langle g, Sp \rangle$ such that

- $g'$ and $g$ agree on $S$,
- $g^{-1}$ and $g^{-1}$ agree on $S$,
- For all $a \in T_1 : a^g \in B$,
- For all $a \in T_2 : a^{g^{-1}} \in B$.

**Proof.** Let $n := |T_1 \setminus S| + |T_2 \setminus S|$. In the proof we will use induction on the value of $n$. If $n = 0$, then the statement is trivial. Now, assume $n > 0$. By switching to the inverse function if it is necessary, we can assume that $T_1 \setminus S \neq \emptyset$. Then let $b \in T_1 \setminus S$ an arbitrary element. By the induction hypotheses we know that there exists an element $g'' \in \langle g, Sp \rangle$ such that
• $g''$ and $g$ agrees on $S$,
• $g''^{-1}$ and $g^{-1}$ agrees on $S$,
• For all $a \in T_1 \setminus \{b\} : a^{g''} \in B$,
• For all $a \in T_2 : a^{g''^{-1}} \in B$.

Now, by applying Lemma 4.7 to the set $S' = T_1 \setminus \{b\} \cup T_2^{g''^{-1}}$ and the element $b$ we obtain a permutation $g' \in \langle g', \text{Sp} \rangle \subset \langle g, \text{Sp} \rangle$ such that $g'$ agrees with $g''$ on $S'$ and $b' \in B$. It is easy to check that this $g'$ satisfies the conditions of the corollary. \hfill $\square$

**Lemma 4.9.** Let $\Phi(x_1, x_2, \ldots, x_n)$ be an arbitrary quantifier-free formula in the language $L = \{P_0\}$, and assume that for an element $g \in \text{Sym}(V)$ there exist $a_1, a_2, \ldots, a_n \in V$ such that

- $a_1, a_2, \ldots, a_n$ are linearly independent,
- $a_1^g, a_2^g, \ldots, a_n^g$ are linearly independent,
- $\Phi(a_1, a_2, \ldots, a_n)$ is true,
- $\Phi(a_1^g, a_2^g, \ldots, a_n^g)$ is false.

Then there exists an $h \in \langle \text{Sp}, g' \rangle^B$ and $b_1, b_2, \ldots, b_n \in B$ such that $\Phi(b_1, b_2, \ldots, b_n)$ is true, but $\Phi(b_1^h, b_2^h, \ldots, b_n^h)$ is false.

**Proof.** Let $G$ denote the following graph: the vertices of $G$ are $a_1, a_2, \ldots, a_n$ and two vertices $a_i$ and $a_j$ are connected if and only if $P_0(a_i, a_j)$ is false. Then the graph $G$ is a finite undirected graph without loops, therefore it can be embedded into $\mathcal{G}$. Let $\psi$ be an arbitrary embedding of $G$ into $\mathcal{G}$. Then $\Phi(a_1^\psi, a_2^\psi, \ldots, a_n^\psi)$ is true because $\Phi$ is a quantifier-free formula. Since both $\{a_1, a_2, \ldots, a_n\}$ and $\{a_1^\psi, a_2^\psi, \ldots, a_n^\psi\}$ are linearly independent sets it follows from the homogeneity that $\psi$ extends to an automorphism of $\mathcal{V}$. We will denote this automorphism by $\psi$ as well. Let $G'$ be the graph on the vertices $a_1^\psi, a_2^\psi, \ldots, a_n^\psi$ defined similarly, and let $\psi'$ be an embedding of $G'$ into $\mathcal{G}$. Then $\Phi(a_1^{\psi'}, a_2^{\psi'}, \ldots, a_n^{\psi'})$ is false and $\psi'$ extends to an automorphism of $\mathcal{V}$ as well.

Let $b_1, b_2, b_3, \ldots$ be an enumeration of $B$ such that $b_i = a_i^\psi$ for $i = 1, 2, \ldots, n$. Then by using Corollary 4.8 we can define a sequence of functions $h_n, h_{n+1}, h_{n+2}, \ldots \in \langle \text{Sp}, g' \rangle^B$ by recursion such that

- $h_n$ and $\psi^{-1}g\psi'$ agrees on $\{b_1, b_2, \ldots, b_n\}$,
- $h_n^{-1}$ and $\psi'^{-1}g^{-1}\psi$ agrees on $\{b_1, b_2, \ldots, b_n\}$,
- $h_{k+1}$ and $h_k$ agrees on $\{b_1, b_2, \ldots, b_k\}$,
- $h_{k+1}^{-1}$ and $h_k^{-1}$ agrees on $\{b_1, b_2, \ldots, b_k\}$,
- For all $i \leq k : b_i^{h_k} \in B$,
- For all $i \leq k : b_i^{h_k^{-1}} \in B$.

The sequences $h_k$ and $h_k^{-1}$ restricted to $B$ are convergent, therefore there exists an $h \in \langle \text{Sp}, g' \rangle^B$ such that $h|_{\{b_1, b_2, \ldots, b_k\}} = h_k|_{\{b_1, b_2, \ldots, b_k\}}$ for all $k \geq n$. In particular $h|_{\{b_1, b_2, \ldots, b_k\}} = h_k|_{\{b_1, b_2, \ldots, b_k\}} = \psi^{-1}g\psi'|_{\{b_1, b_2, \ldots, b_k\}}$. So the formula $\Phi(b_1^h, b_2^h, \ldots, b_k^h)$ is false which finishes the proof of Lemma 4.9. \hfill $\square$

**Definition 4.10.** Let $\diamond(a, b, c, d)$ denote the following 4-ary relation:

$$\diamond(a, b, c, d) \iff a \cdot b + b \cdot c + c \cdot d + d \cdot a = 1$$

And $a, b, c$ and $d$ are pairwise disjoint nonzero elements.
Lemma 4.11. Let $\text{Sp} \leq G \leq (\text{Sym}(V))_0$ be a closed group. If $G$ does not preserve $\circ$ then $G^B = \text{Sym}(B)$. Moreover, in this case for every $n$ the group $G$ acts transitively on the $n$-element linearly independent sets.

Proof. If $G$ does not preserve $\circ$ then there is a permutation $g \in G$ and pairwise disjoint elements $a, b, c, d \in V \setminus \{0\}$ such that $\circ(a, b, c, d)$ holds but $\circ(a^g, b^g, c^g, d^g)$ does not hold. We can assume that $a, b, c$ and $d$ are linearly independent: there is an element $s \in V \setminus \{0\}$ such that $s \notin \langle a, b, c, d \rangle$ and $s^g \notin \langle a^g, b^g, c^g, d^g \rangle$. Observe that exactly zero or two holds from the formulas $\circ(a, b, c, d)$, $\circ(a, b, s, d)$ and $\circ(s, b, c, d)$, and similarly exactly zero or two holds from the formulas $\circ(a^g, b^g, c^g, d^g)$, $\circ(a^g, b^g, s^g, d^g)$ and $\circ(s^g, b^g, c^g, d^g)$. Using that exactly one formula from $\circ(a, b, c, d)$ and $\circ(a^g, b^g, c^g, d^g)$ holds we can conclude that $g$ does not preserve the relation $\circ$ on $(a, b, s, d)$ or on $(s, b, c, d)$. Similarly we can replace $b$ or $d$ by an element $r$ such that $r$ and $r^g$ are linearly independent from the previous vectors. The elements $s, r$ and the remaining ones $x \in \{a, c\}$ and $y \in \{b, d\}$ will be linearly independent and their images under $g$ will also be independent.

The relation $\circ$ was defined by a quantifier-free formula 4.10 so we can apply Lemma 4.9 which yields that the group $\text{Aut}(G) \leq G^B \leq \text{Sym}(G)$ does not preserve the relation $\circ$. Using the classification obtained by Thomas in [16] all closed groups containing $\text{Aut}(G)$ preserve $\circ$ except the group $\text{Sym}(B)$ so $G^B = \text{Sym}(B)$.

We can prove that for every $n$ the group $G$ acts transitively on the $n$-element linearly independent sets by showing that any finite linearly independent sets can be mapped into $B$ by an automorphism of $V$. Now let $S$ be a finite set of linearly independent elements in $V$. Let us consider the graph $G$ the vertices of which are the elements of $S$ and two vertices $a, b$ are connected if and only if $P_0(a, b)$ is false. Then by the universality of $\mathcal{G}$ the graph $G$ embeds into $\mathcal{G}$. Let $\psi$ be an arbitrary embedding of $G$ into $\mathcal{G}$. Then by the homogeneity of $\psi$ can be extended to an automorphism of $V$. This automorphism maps $S$ into $B$. \)

Lemma 4.12. If a permutation $g \in (\text{Sym}(V))_0$ preserves the relation $\circ$ then it is linear.

Proof. First we define the 5-ary relation $\circ(a, b, c, d, e)$:

$$\circ(a, b, c, d, e) \iff \text{the number of unordered pairs } \{x, y\} \text{ such that } x \neq y, x, y \in \{a, b, c, d, e\}, x \cdot y = 1 \text{ is odd}$$

and $a, b, c, d$ and $e$ are pairwise disjoint nonzero elements.

The relation $\circ$ is first order definable from the relation $\circ$: $\circ(a, b, c, d, e)$ holds if and only if the number of true formulas amongst $\circ(a, b, c, d), \circ(a, b, e), \circ(a, b, d, e)$ and $\circ(b, c, d, e)$ is odd. If $a, b$ and $c$ are linearly independent elements then the truth value of the formula $\circ(a, b, c, a + b + c, x)$ will be the same regardless of the choice of $x$. This is true because exactly zero or two of $a \cdot x, b \cdot x, c \cdot x$ and $(a + b + c) \cdot x$ can be $1$. On the other hand, if $a, b, c$ and $d$ are four linearly independent element then by Proposition 4.3 we can choose $x$ and $y$ such that $\circ(a, b, c, d, x)$ holds and $\circ(a, b, c, d, y)$ does not hold.

Let $h \in (\text{Sym}(V))_0$ be a non-linear permutation. We will show that $h$ does not preserve the relation $\circ$. It suffices to show that $h$ does not preserve $\circ$ because $\circ$ is first order definable from $\circ$. We can assume there are elements $a, b, a + b \in V \setminus \{0\}$ such that $a^h + b^h \neq (a + b)^h$ (if this is not the case then we can change
to work with $h^{-1}$ instead of $h$ because a permutation and its inverse both preserve or do not preserve a given relation). There exist $x$ and $y$ such that $\bigcirc(a, b, a + b, (a^h + b^h + (a + b)^h)h^{-1}, x)$ holds and $\bigcirc(a, b, a + b, (a^h + b^h + (a + b)^h)h^{-1}, y)$ does not hold. The truth value of $\bigcirc(a^h, b^h, (a + b)^h, a^h + b^h + (a + b)^h, x^h)$ and $\bigcirc(a^h, b^h, (a + b)^h, a^h + b^h + (a + b)^h, y^h)$ must be the same. So $h$ can not preserve $\preceq$.

So every permutation preserving $\preceq$ must be linear. □

**Theorem 4.13.** If $\text{Sp} \leq G \leq \text{Sym}(\mathcal{V})_0$ is a closed group then either $G = \text{Sp}$, $G = \text{GL}$ or $G = \text{Sym}(\mathcal{V})_0$.

**Proof.** First suppose $\text{Sp} \leq G \leq \text{GL}$. Then by Proposition 4.5 there is an element $g \in G$ which does not preserve the relation $P_0$. Now, let $a, b \in \mathcal{V} \setminus \{0\}$ elements such that $a \cdot b \not= a^g \cdot b^g$. We can assume that $a \cdot b = 0$ and $a^g \cdot b^g = 1$. Then let us choose an element $c \in \mathcal{V} \setminus \{a, b\}$ such that $c \cdot a = c \cdot b = 0$. Then $a \cdot c = c \cdot b = b \cdot (a + c) = (a + c) \cdot a = 0$ so $\bigcirc(a, c, b, a + c)$ does not hold. Furthermore $a^g \cdot c^g + c^g \cdot b^g + b^g \cdot (a + c)^g + (a + c)^g \cdot a^g = a^g \cdot b^g = 1$ therefore $g$ does not preserve the relation $\preceq$. Then by Lemma 4.11 we know that for all $n$ the group $G$ acts transitively on $n$-element linearly independent sets. Now, let $S$ and $T$ finite dimensional subspaces of $\mathcal{V}$ of the same dimension, and let $\psi$ be a $S \rightarrow T$ linear isomorphism. We need to show that there exists a linear automorphism $h \in G$ which extends $\psi$ because this extension property characterizes $\mathbb{F}_2^2$. Let $b_1, b_2, \ldots, b_n$ be a basis of $S$. Then we know that there exists an element $h \in G$ such that $b_i^h = b_i^\psi$ for all $i = 1, 2, \ldots, n$ since $G$ acts transitively on $n$-element linearly independent sets.

The transformation $h$ is linear as well, hence $h|_S = \psi$. So in this case GL $\leq G$.

Now suppose $G \leq \text{GL}$. This means that $G$ does not preserve the relation $\preceq$ by Lemma 4.12 using Lemma 4.2 we get that for all $n$ the group $G$ acts transitively on $n$-element linearly independent sets. We will prove that $G$ is $n$-transitive on $\mathcal{V} \setminus \{0\}$ by induction modifying the proof of Theorem 2.3.

The group $G$ is 2-transitive because $G$ acts transitively on two-element linearly independent sets.

We will show the 3-transitivity. The group $G$ acts transitively on the 3-element independent sets, hence it is enough to show that any three vectors $a, b, c \in \mathcal{V} \setminus \{0\}$ can be mapped to an independent set. Let $a, b, c \in \mathcal{V} \setminus \{0\}$ and assume they cannot be mapped into an independent system. Then all 3-tuples $(x, y, z)$ such that $x + y = z$ are on the same orbit because the group $\text{Sp}$ has two orbits on such tuples: on the first orbit $P_0$ holds for all pairs of elements of the tuple, and on the second orbit $P_1$ holds for all pairs of elements of the tuple. Since $G$ is 2-transitive $(a, b, c)$ can be mapped to $(x, y, z)$ where $a \cdot b \not= x \cdot y$, using the assumption that $(a, b, c)$ cannot be mapped to an independent system $x + y = z$ so there is only one orbit on the linearly dependent 3-tuples in $G$. The condition $G \leq \text{GL}$ implies that there are $a', b', c' \in \mathcal{V} \setminus \{0\}$ and $g \in G$ such that $a' + b' = c'$ and $a'^g + b'^g \not= c'^g$. Now, consider a map $h \in G$ mapping $a, b, c$ to $a', b', c'$, respectively. This $h$ map exists because all 3-tuples such that $a + b = c$ must lie on the same orbit. The map $h\gamma$ maps $a, b, c$ to an independent set. This contradicts our assumption that $(a, b, c)$ cannot be mapped to an independent set.

Now, we prove $n$-transitivity by induction. We show that every set of $n + 1$ vectors can be mapped to an independent set. Let $a_1, a_2, \ldots, a_n, a_{n+1} \in \mathcal{V} \setminus \{0\}$ be dependent distinct elements. By the $n$-transitivity we may assume that $a_n = a_1 + a_2 + \ldots + a_{n-1}$ and $a_1 \cdot a_j = 0$ for all $1 \leq i < j \leq n$. We can also assume that there is an $h \in G$
such that \( \{ a^i_b | i = 1, 2, \ldots, n \} \) is a linearly independent set such that \( a^i_b \cdot a^j_b = 0 \) for all \( 1 \leq i < j \leq n \). If \( a_{n+1} \notin \langle a_1, a_2 \ldots a_n \rangle \), then by Lemma 2.2 we are done. If \( a_{n+1} \in \langle a_1, a_2 \ldots a_n \rangle \), then \( a_{n+1} = \sum_{1}^{n-1} \varepsilon_i a_i \) where at least two, but not all \( \varepsilon_i \) are equal to 1. Indeed, assume that there is a unique \( i \) such that \( \varepsilon_i = 1 \), then \( a_{n+1} = a_i \) would hold, and if all of them were equal to 1, then \( a_{n+1} = \sum_{1}^{n-1} a_i = a_n \) would hold contradicting that the vectors are distinct. Let \( \varepsilon_j = 1 \) and \( \varepsilon_k = 0 \) for some \( j, k < n \). Then there is a map \( g \) in the group \( Sp \) flipping \( a_j \) and \( a_k \) and fixing every \( a_i \), where \( i < n \) and \( i \neq j, k \). Here we used that \( a_i \cdot a_j = 0 \) for all \( 1 \leq i < j \leq n \) which also implies \( a_i \cdot a_{n+1} = 0 \) for all \( 1 \leq i \leq n \) because \( \cdot \) is bilinear. Now, \( \{ a^i_b | i = 1, 2, \ldots, n \} = \{ a^{ab}_i | i = 1, 2, \ldots, n \} \) is an independent set and \( a^{ab}_{n+1} \neq a^{ab}_{n+1} \). If any of the latter two elements is not in \( \langle a^1_b, a^2_b, \ldots a^n_b \rangle \) then we are done by Lemma 2.2. Otherwise we may assume that \( a^{ab}_{n+1} = \sum_{1}^{n} \xi_i a^i_b \), where there is an \( l \) such that \( \xi_l = 0 \). Now, \( a^l_b \notin \langle a^{ab}_i | 1 \leq i \leq n+1, i \neq l \rangle \) and we are done again, by Lemma 2.2. \( \Box \)

We have finished the classification of the closed groups containing \( Sp \) which preserve the 0. We will continue with the classification of groups not fixing the 0. Using Theorem 4.13 and Theorem 2.5 we can restrict our attention to the groups where the stabilizer of the 0 is exactly the group \( Sp \). There is only one closed group which contains \( Sp \) in addition to those already mentioned and it can be obtained as the automorphism group of the relation defined below 4.14.

**Lemma 4.14.** Let \( \nabla(a, b, c) \) denote the following ternary relation:

\[
\nabla(a, b, c) \iff a \cdot b + b \cdot c + c \cdot a = 1
\]

\( a, b \) and \( c \) are pairwise disjoint elements.

The automorphism group of this relation can be obtained as a semidirect product:

\[
\text{Aut}(\nabla) = T \rtimes Sp
\]

Moreover, \( \text{Aut}(\nabla) \) is a minimal supergroup: there is no \( Sp \subsetneq G \subsetneq \text{Aut}(\nabla) \) closed group.

**Proof.** The group \( \text{Aut}(\nabla) \) will be denoted by \( \Delta \). The following calculation shows that every translation preserves \( \nabla \): \( t_x(a) \cdot t_x(b) + t_x(b) \cdot t_x(c) + t_x(c) \cdot t_x(a) = (a + x) \cdot (b + x) + (b + x) \cdot (c + x) + (c + x) \cdot (a + x) = a \cdot b + b \cdot c + c \cdot a \).

\( \Delta \leq AGL \) because \( Sp(\omega, 2) \subsetneq \Delta \leq \text{Sym}(F_2^\omega) \) and \( \Delta_0 = \Delta \cap (\text{Sym}(\nabla))_0 = Sp \) so we can use Lemma 2.4.

Using that \( AGL = T \rtimes GL \) and \( Sp \leq GL \) and \( T \leq \Delta \leq AGL \) we can conclude that \( \text{Aut}(\nabla) = T \rtimes Sp \). Since \( Sp \) acts transitively on \( V \setminus \{ 0 \} \) every non-identical translation can be conjugated to any other non-identical translation by an element of \( Sp \). If \( a, b \in V \setminus \{ 0 \} \) and \( \phi \in Sp \) such that \( \phi(a) = b \) then \( \phi t_b \phi^{-1}(x) = \phi(\phi^{-1}(x) + b) = x + \phi^{-1}(b) = t_a(x) \). So there is no \( Sp \subsetneq G \subsetneq \Delta \) closed group. \( \Box \)
Let us assume that Lemma 4.15.

Proof. Easy consequence of Lemma 2.4. \hfill \Box

Lemma 4.16. Let us assume that \( Sp(\omega, 2) \preceq G \preceq \Sym(\mathbb{F}_2^d) \), and \( G_0 = G \cap (\Sym(V))_0 = Sp \). Then \( G \preceq AGL \).

Proof. Let \( (x, y, z) \) an arbitrary injective 3-tuple of \( V \). Since \( Sp(\omega, 2) \) is transitive on \( V \backslash 0 \), and \( G \) does fix 0, it follows that \( G \) is transitive. This means that \( (x^g, y^g, z^g) = (0, a, b) \) for some \( a, b \in V \backslash 0 \). The group \( Sp \) has exactly two orbits on injective 3-tuples of the form \((0, u, v)\). These orbit are distinguished by the product \( u \cdot v \). This implies that \( G \) has at most 2 orbits on injective 3-tuples. We also need to show that \( G \) cannot be 3-transitive. Suppose for contradiction that \( G \) is 3-transitive. Then there exist distinct elements \( a, b \in V \backslash 0 \) and a permutation \( g \in G \) such that \( 0^g = 0 \) and \( a \cdot b \neq a^g \cdot b^g \), but this contradicts the fact that \( G_0 = G \cap (\Sym(V))_0 = Sp \).

For the second statement of the Lemma it is enough to show that for any \( a, b \in V \backslash 0 \) distinct elements the set \( \{0, a, b\} \) can be mapped into a linearly independent set. Now, let \( a \neq b \) be arbitrary elements in \( V \). Then there exist elements \( a_1, a_2, a_3, a_4 \in V \setminus 0 \) such that \( a_i \cdot a_j = a \cdot b \) for all \( 1 \leq i < j \leq 4 \). Let \( g \in G \) a permutation which does not fix 0. Then \( \dim(0^g, a_1^g, a_2^g, a_3^g) \geq 3 \), thus the elements \( 0^g, a_1^g, a_2^g, a_3^g \) are linearly independent for some \( 1 \leq i < j \leq 4 \). Since \( a_i \cdot a_j = a \cdot b \), there exists \( h \in Sp \subset G \) such that \( a^h = a_i, b^h = a_j \). Then the permutation \( hg \) maps the set \( \{0, a, b\} \) into a linearly independent set. \hfill \Box

Lemma 4.17. Let us assume that \( Sp(\omega, 2) \preceq G \preceq Sym(\mathbb{F}_2^d) \), and \( G_0 = G \cap (\Sym(V))_0 = Sp \). Let \( (a, b, c, d) \) be an injective 4-tuple of \( V \) such that \( a+b+c+d = 0 \). Then for any 3-element subset \( \{s, t, u\} \subset \{a, b, c, d\} \) the tuples \( (s, t, u) \) and \( (a, b, c) \) are on the same orbit of \( G \).

Proof. By Lemma 4.15 and 4.16 we get that the set of injective 4-tuples \( (a, b, c, d) \) forming two dimensional affine subspace of \( V \) is a union of two 4-orbits of \( G \). This means that we only need to show the statement of the lemma for one single 4-tuple. Let \( a, b, c \) be linearly independent elements of \( V \) such that \( a \cdot b = a \cdot c = b \cdot c = 0 \). Then the statement of the lemma is obvious. \hfill \Box

Lemma 4.18. Let us assume that \( Sp \preceq G \preceq \Sym(\mathbb{F}_2^d) \), and \( G_0 = G \cap (\Sym(V))_0 = Sp \). Then \( G = \Delta \).
Proof. Let $G$ be a group satisfying the conditions of the lemma. Assume $G \not\subseteq \Delta$. Then by Lemma 4.14 $G \not\subseteq \Delta$, so there is a permutation $g \in G$ which does not preserve the relation $\nvdash$. We claim that this can be realized by linearly independent element, i.e. there are elements $a, b$ and $c$ such that \{a, b, c\} and $\{a^g, b^g, c^g\}$ are linearly independent sets, and $a \cdot b + b \cdot c + c \cdot a \neq a^g \cdot b^g + b^g \cdot c^g + c^g \cdot a^g$.

By the definition of $g$ we know that there are elements $a, b, c \in V$ such that $a \cdot b + b \cdot c + c \cdot a \neq a^g \cdot b^g + b^g \cdot c^g + c^g \cdot a^g$. By Lemma 4.17 we know that for all $u, v \in V \setminus 0$ the tuples $(0, u, v)$ and $(u, v, u + v)$ lie on the same orbit. Note that

$$\nabla(0, u, v) \iff \nabla(u, v, u + v) \iff P_1(u, v)$$

for any elements $u, v \in V \setminus 0$. Using these observations we can assume that $a, b, c, a^g, b^g, c^g \neq 0$. Now, pick an element $d$ such that both $d$ and $d^g$ are linearly independent from $\{a, b, c, a^g, b^g, c^g\}$. It is easy to check that for any injective 4-tuple $(x, y, z, v)$ an even number of the formulas $\nabla(x, y, z), \nabla(y, z, v), \nabla(z, v, x)$ and $\nabla(v, x, y)$ hold. This implies that at least one of the tuples $(d, a, b), (d, b, c)$ and $(d, c, a)$ satisfies that $g$ does not preserve the relation $\nabla$ on it (exactly one of the tuple and its image under $g$ is in $\nabla$) and these tuples contain linearly independent elements. This proves our claim.

Now, we are ready to prove the statement of the lemma. For $i = 0, 1, 2, 3$ let $T_i$ denote the set of those linearly independent tuples $(a, b, c)$ tuples where exactly $i$ of the relations $P_1(a, b), P_1(b, c)$ and $P_1(c, a)$ hold. Then by Lemma 4.17 it follows that each set $T_i$ is contained in some 3-orbit of $G$. We would like to determine that which of these sets can be contained in the same orbit. At first, we show that $T_1$ and $T_3$ are on the same orbit. For this let us choose linearly independent elements $a, b, c$ in $V$ such that $a \cdot b = a \cdot c = b \cdot c = 1$. Then by Lemma 4.17 the tuples $(a, b, c)$ and $(a, b, a + b + c)$ lie on the same orbit, and it is easy to see that $(a, b, c) \in T_3$ and $(a, b, a + b + c) \in T_1$.

We have seen that there are elements $a, b$ and $c$ such that $\{a, b, c\}$ and $\{a^g, b^g, c^g\}$ are linearly independent sets, and $a \cdot b + b \cdot c + c \cdot a \neq a^g \cdot b^g + b^g \cdot c^g + c^g \cdot a^g$. This implies that either $T_0$ or $T_2$ is contained in the same orbit as $T_3$. We will deduce a contradiction in both cases. By Lemma 4.16 we know that it is not possible that all $T_i$s are contained in the same orbit.

Case 1. $T_0$ and $T_3$ are contained in the same 3-orbit of $G$. Then we know that there exists a 3-tuple $(a, b, c) \in T_3$ and a permutation $g \in G$ such that $(a^g, b^g, c^g) \in T_0$.

Now, pick an element $d \in V$ such that $d \notin \langle a, b, c \rangle$, $d^g \notin \langle a^g, b^g, c^g \rangle$, and $d \cdot a = 0, d \cdot b = 0, d \cdot c = 1$. If $d^g \cdot a^g = d^g \cdot b^g = 1$, then $(a, b, d) \in T_3$ and $(a^g, b^g, d^g) \in T_2$, which is impossible. Hence $d^g \cdot a^g \neq 0$ or $d^g \cdot b^g$. By symmetry we can assume that the latter holds. Then $(b, c, d) \in T_2$ and $(b^g, c^g, d^g) \in T_3 \cup T_5$. This is also impossible because we already know that $T_0, T_1$ and $T_3$ are contained in the same orbit.

Case 2. $T_2$ and $T_3$ are contained in the same 3-orbit of $G$. Let $\{a, b\}$ and $\{c, d\}$ be two linearly independent sets such that $a \cdot b = 0$ and $c \cdot d = 1$. The group $G$ is transitive and the $G_0$ is transitive on $V$, therefore $G$ is 2-transitive as well. In particular, there exists a permutation $g \in G$ such that $a^g = c$ and $b^g = d$. Now, pick an element $d \in V$ such that $d \notin \langle a, b \rangle$, $d^g \notin \langle c, d \rangle$ and $d \cdot a = -b = 0$. Then $(a, b, d) \in T_0$ and $(a^g, b^g, d^g)$. This is again a contradiction since we already know that $T_1, T_2$ and $T_3$ are contained in the same orbit.

We arrived to a contradiction in all cases, which proves the statement of the lemma.
Theorem 4.19. The closed supergroups containing $\text{Sp}$ are exactly the following groups:

(1) The group $\text{Sp}$
(2) The group $\Delta$
(3) The group $\text{GL}$
(4) The group $\text{AGL}$
(5) The group $\text{Sym}_0$
(6) The group $\text{Sym}$

Proof. The groups fixing the 0 were described in Theorem 4.13: this groups are exactly $\text{Sp}$, $\text{GL}$ and $\text{Sym}_0$. The groups not fixing the 0 can be classified according to the stabilizer of the 0 in them. The groups where this stabilizer contains $\text{GL}$ were described in in Theorem 2.5: this groups are exactly $\text{AGL}$ and $\text{Sym}$. The only group not fixing the 0 where the stabilizer of the 0 is $\Delta$: this was proved in Lemma 4.18.

□

References

[1] Bhattacharyya, P. (1981). On groups containing the projective special linear group. Arch. Math. 37:295-299.
[2] Bodirsky, M., Chen, H., Pinsker, M. (2010). The reducts of equality up to primitive positive interdefinability. Journal of Symbolic Logic. 75(4):1249-1292.
[3] Bodirsky, M., Pinsker, M. (2010). Minimal functions on the random graph. Israel Journal of Mathematics. to appear
[4] Bodirsky, M., Pinsker, M., Tsankov, T. Decidability of definability. Journal of Symbolic Logic. 78:1036-1054.
[5] Bodirsky, M., Pinsker, M. (2010). Minimal functions on the random graph. Journal of Symbolic Logic. 78:1036-1054.
[6] Cameron, P. J. (1976). Transitivity of permutation groups on unordered sets. Mathematische Zeitschrift. 148:127-139.
[7] Fraïssé, R. (1953). Sur certaines relations qui généralisent l’ordre des nombres rationnels. Comptes Rendus d’Académie des Sciences de Paris 237. 540-542.
[8] Hodges, W. (1993). Model theory. Cambridge University Press, Cambridge.
[9] Junker, M., Ziegler, M. (2008). The 116 reducts of $(\mathbb{Q}; <; a)$. Journal of Symbolic Logic. 74(3):861-884.
[10] Kantor, W. M., McDonough, T. P., (1974). On the maximality of $\text{PSL}(d + 1, q), d \geq 2$. J. London Math. Soc. (2) 8:426.
[11] Macpherson, D. (2011). A survey of homogeneous structures. Discrete Mathematics. 311(15):1599-1634.
[12] Pach, P. P., Pinsker, M., Pongrác, A., Szabó, Cs. (2013). A new transformation of partially ordered sets. J. Comb. Theory A. 120(7):1450-1462.
[13] Pach, P. P., Pinsker, M., Pluhár, G., Pongrác, A., Szabó, Cs. (2014). Reducts of the random partial order. Advances in Mathematics to appear
[14] Pogorelov, P. A. (1974). Maximal subgroups of symmetric groups that are defined on projective spaces over finite fields. (Russian) Mat. Zametki. 16:91-100.
[15] Pongrác, A. (2013). Reducts of the Henson graphs with a constant. Annals of Pure and Applied Logic. to appear
[16] Thomas, S. (1991). Reducts of the random graph. Journal of Symbolic Logic. 56(1):176-181.
[17] Thomas, S. (1996). Reducts of random hypergraphs. Annals of Pure and Applied Logic. 80(2):165-193.
