Continuous and discontinuous transitions in generalized $p$-spin glass models

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Abstract

We investigate the generalized $p$-spin models that contain arbitrary diagonal operators $\hat{U}$ with no reflection symmetry. We derive general equations that give an opportunity to uncover the behavior of the system near the glass transition at different (continuous) $p$. The quadrupole glass with $J = 1$ is considered as an illustrative example. It is shown that the crossover from continuous to discontinuous glass transition to the one-step replica breaking solution (1RSB) takes place at $p = 3.3$ for this model. For $p < 2 + \Delta p$, where $\Delta p = 0.5$ is a finite value, the stable 1RSB solution disappears. This behavior is very different from that of the $p$-spin Ising glass model.

Keywords: glass transition, spin glasses, frustration, replica symmetry breaking

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(Some figures may appear in colour only in the online journal)

1. Introduction

The theory of spin glasses has been introduced as an attempt to describe unordered equilibrium freezing of spins in actual dilute magnetic systems with disorder and frustration. This problem was soon partially solved at the mean-field level [1, 2]: using the replica trick to average over disorder, the replica symmetric (RS) solution was obtained. However, it was soon shown that an adequate description of the low-temperature phase requires a breaking of the replica symmetry.
In the paper of Sherrington and Kirkpatrick [2] (SK) it was already shown that the RS ansatz is not the correct one. It leads to a negative zero temperature entropy. In a subsequent paper of de Almeida and Thouless of 1978 [3] it was shown that the RS ansatz gives an unstable solution in the low-temperature phase, hence calling for replica symmetry breaking (RSB). Different approaches to RSB were considered (see, e.g. [4–6]). Two-group breaking was proposed by Bray and Moore [5]. Parisi introduced the method of RSB step by step with the limit—full RSB (FRSB) when the glass order parameter becomes a continuous non-decreasing function \( q(x) \) of a parameter \( 0 \leq x \leq 1 \). It provides the hierarchical distribution of pure states overlaps probability \( P(q) \) through \( P(q) = d\xi/dq \) [6, 7].

Now it is largely believed that a similar approach occurs in general glass models. However, the details of the particular RSB scheme in different spin-like glasses and the dependence on the properties of models are far from being understood and remain the focus of intense investigations [8–18].

The \( p \)-spin spin-glass model of randomly interacting \( p \)-Ising spins was introduced as a natural generalization of the SK model [19–21]. It was shown that in the limit \( p \to \infty \) the first step of RSB (1RSB) gives a solution which remains stable down to zero temperature. In [22] a detailed investigation of the model was performed for \( p > 2 \). It was shown in particular for \( p = 2 + \epsilon \) with small \( \epsilon \) that the 1RSB solution appears at a certain \( T_c \) with a jump in the order parameter and remains stable at a certain interval near \( T_c \). Both of these values, the jump at \( T_c \) and the range of 1RSB solution stability, tend to zero as \( p \to 2 \).

Similar behavior as in the \( p \to \infty \) model of interacting hard Ising spins was observed in the spherical \( p \)-spin models with \( p > 2 \) [23]. These models, as well as soft Potts models, were supposed to be the prototypes of structural glasses [24–26]. It is worthwhile noting that in Potts’ glass the jump of the order parameter at the transition to the 1RSB state exists beginning at \( p = 5 \) [27] while the 1RSB solution remains stable in a pronounced interval already for \( p = 3 \) [28]. As for the ‘spherical’ Potts model with \( p = 3 \), its ground state is RS [29] as in the spherical SK model [30].

It should be noted that for a long time the discontinuity of the order parameter at the transition and the stability of the 1RSB solution were associated with the absence of time-reversal (reflection) symmetry. This problem is still relevant today [31, 32]. The usual paradigm is that the absence of reflection symmetry should be incorporated into the structure of the Hamiltonian. However, it can sometimes be caused by the characteristics of the interacting operators themselves. This was so in our recent investigations (see for example [33, 34]). We can consider a generalization of the spin model of Ising spins where arbitrary diagonal operators \( \hat{U} \) are used instead of Ising spins. In this way, a number of real physical systems can be described (see for example the reviews [33, 35]). The operators have a different physical origin depending on the problem under study. For example, Ising spin should be replaced with the molecule multipole moment if freezing of the orientational order is the target of the investigation [33, 35–37].

Recently, the generalized \( p \)-spin model with \( p = 3 \) has been considered with a type of quadrupole operator instead of Ising spins [37]. This model describes, in agreement with experiments [38], the high-pressure orientational glass phase in solid molecular ortho-\( D_2 \) and para-\( H_2 \), where the interactions of more than two particles play an important role.

1.1. Our results

In this paper we investigate the generalized \( p \)-spin models that contain arbitrary diagonal operators \( \hat{U} \) instead of Ising spins and focus on the case where \( \hat{U} \) has no reflection symmetry; that is, \( \text{Tr} \hat{U}^{(2k+1)} \neq 0 \) for all integer \( k > 0 \). Now the glass freezing scenario is completely
Figure 1. The sketch of the ‘phase diagram’ of the generalized \( p \)-operator model, (a) without reflection symmetry (our main result) and (b) with reflection symmetry (this is, for example, a \( p \)-spin glass model of randomly interacting \( p \)-Ising spins). Here \( k \) is an integer while \( \Delta \) and \( p_c \) are model-dependent constants that we find analytically and numerically for a \( p \)-quadrupole model.

different from the Ising \( p \)-spin case. There is no temperature interval, in particular, where the RS solution coincides with the ‘para-state’ of the pure system.

We obtain general equations for order parameters near the glass transition for arbitrary real \( p > 2 \) using expansions of the effective free energy and the bifurcation theory [39]. That allowed us to uncover the behavior of the generalized \( p \)-spin models depending on the continuous parameter \( p \) and the symmetry properties of the operator \( \hat{U} \). This is one of the main results of our paper and we represent it qualitatively in figure 1(a). We have found analytically and numerically the critical value of \( p \) where a crossover from continuous to discontinuous transition to the 1RSB solution takes place. In the case \( \text{Tr} \hat{U}_{(2k+1)} \neq 0 \), the problem has the property of continuity in the parameter \( p \). The results we have found differ from the corresponding results for the case \( \text{Tr} \hat{U}_{(2k+1)} = 0 \) [34] when point \( p = 2 \) is a distinguished point, as shown in figure 1(b).

As an illustrative example we consider the quadrupole glass with \( J = 1 \). We investigate the temperature dependence of the order parameters, as shown in figures 2 and 3, and scan the range of 1RSB stability depending on \( p \) for this model, as shown in figure 4. A stable 1RSB solution appears for \( p > 2.5 \). We show that the crossover from continuous to discontinuous transition occurs at \( p = 3.3 \), as shown in figure 2.

1.2. Structure of the paper

In section 2.1 we give the standard starting equations for the generalized \( p \)-spin model and discuss them in section 2.2. In section 2.3 we investigate the stability and examine the continuity depending on the parameter \( p \). We provide general expansions of the effective free energy near the transition (as was done, for example, in [33, 34]) and finally arrive at equations (25) and (26). These expressions lead us to the ‘phase diagram’ of the generalized \( p \)-operator model shown in figure 1 that schematically shows the main results of our paper. In section 2.4 we consider as an illustrative example the \( p \)-quadrupole glass. In section 3 we provide our conclusions.

2. Generalized \( p \)-spin model

2.1. Main equations

The Hamiltonian of the \( p \)-operator model in general looks like:

\[
H = - \sum_{i_1 \leq i_2 \leq \ldots \leq i_p} J_{i_1 \ldots i_p} \hat{U}_{i_1} \hat{U}_{i_2} \ldots \hat{U}_{i_p},
\]

(1)
Figure 2. Graph (a) and inset in (b) show temperature dependence of the glass 1RSB order parameter $v_{1RSB}$ while (b) corresponds to the 1RSB order parameter $m_{1RSB}$ for the $p$-quadrupole model. In the case in hand, $\hat{U} = 3J_z^2 - 2$ is the quadrupole moment of the molecule with the angular momentum operator $J_z$. Here we focus on $J = 1$, so $J_z = \{0, \pm 1\}$. When $p = 3.3$, the transition from the continuous to discontinuous scenario takes place, $m_{1RSB} = 1$ at the branch point $T_0$ and $v_{1RSB} \sim \sqrt{T_0 - T}$ near $T_0$.

where $\hat{U}$ is now an arbitrary diagonal operator with $\text{Tr} \hat{U} = 0$, $N$ is the number of sites on the lattice, $i = 1, 2, \ldots, N$, and $p$ gives the number of interacting particles. The coupling strengths are independent random variables with Gaussian distribution

$$P(J_{i_1 \ldots i_p}) = \frac{\sqrt{N^{p-1}}}{\sqrt{p! \pi J}} \exp \left\{ -\frac{(J_{i_1 \ldots i_p})^2 N^{p-1}}{p!J^2} \right\}. \quad (2)$$

Using the replica trick in the standard way, as shown in [7] for example, we write in general the free energy averaged over disorder [34, 37]:

$$\langle F \rangle / NkT = \lim_{n \to 0} \frac{1}{n} \max \left\{ (p - 1) \frac{\ell^2}{4} \sum_a (n^a)^p + (p - 1) \frac{\ell^2}{2} \sum_{a > b} (q^{a,b})^p - \ln \text{Tr}_{(\ell^n)} \exp \hat{\theta} \right\}. \quad (3)$$
The saddle point conditions give the glass order parameter

\[ q_{\alpha \beta} = \frac{\text{Tr}[\tilde{U}_\alpha \tilde{U}_\beta \exp(\tilde{\theta})]}{\text{Tr}[\exp(\tilde{\theta})]}, \]  

where

\[ \tilde{\theta} = pt^2 \sum_{\alpha > \beta} (q_{\alpha \beta})^{(p-1)} \tilde{U}_\alpha \tilde{U}_\beta + \frac{p^2}{4} \sum_{\alpha} (w_\alpha)^{(p-1)} (\tilde{U}_\alpha)^2. \]  

Here, \( t = J/kT \).

The saddle point conditions give the glass order parameter

\[ q_{\alpha \beta} = \frac{\text{Tr}[\tilde{U}_\alpha \tilde{U}_\beta \exp(\tilde{\theta})]}{\text{Tr}[\exp(\tilde{\theta})]} \]  

Figure 3. Temperature dependence of the order parameters, \( \lambda_{\text{RS}} \) and \( \lambda_{\text{1RSB repl}} \), of the \( p \)-quadrupole model for the boundary value \( p = 3.3 \) where the crossover from continuous to discontinuous 1RSB transition takes place. We should emphasize that 1RSB order parameter \( \nu_{\text{1RSB}} \) approaches zero nonlinearly.

Figure 4. Temperature dependence of \( \lambda_{\text{RS repl}} \) and \( \lambda_{\text{1RSB repl}} \) for \( p \)-quadrupole glass. For \( p > 2.5 \) there is a domain where \( \lambda_{\text{1RSB repl}} > 0 \) and so a stable 1RSB solution appears. At \( p = 3.3 \) the transition from continuous to discontinuous (jumpwise) scenario of the order parameters evolution with temperature takes place.
and the auxiliary order parameter
\[ w^\mu = \text{Tr}[(\hat{U}^\mu)^2 \exp(\hat{\theta})]/\text{Tr}[\exp(\hat{\theta})]. \]  
(6)

We also introduce by analogy the regular order parameter
\[ x^\mu = \text{Tr}[(\hat{U}^\mu)^2 \exp(\hat{\theta})]/\text{Tr}[\exp(\hat{\theta})]. \]  
(7)

Using the standard procedure [7] we perform the first stage of the RSB \((n \text{ replicas are divided into } n/m_{\text{RSB}} \text{ groups with } m_{\text{RSB}} \text{ replicas in each})\) and obtain the expression for the free energy. Order parameters are denoted by \(q^\mu = r_{\text{RSB}}\) if \(\alpha\) and \(\beta\) are from different groups and \(q^\mu = r_{\text{RSB}} + v_{\text{RSB}}\) if \(\alpha\) and \(\beta\) belong to the same group. So
\[
F_{\text{RSB}} = -NkT \left\{ m_{\text{RSB}}(p-1) \frac{r_{\text{RSB}}^2}{4} + (1 - m_{\text{RSB}})(p-1)\frac{(r_{\text{RSB}} + v_{\text{RSB}})^2}{4} \right. \\
+ \left. t^2(p-1) \frac{w_{\text{RSB}}}{4} + \frac{1}{m_{\text{RSB}}} \int d\xi G \ln \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \right\}. 
\]  
(8)

Here
\[
\hat{\theta}_{\text{RSB}} = zT \int \frac{p_{\text{RSB}}(p-1)}{2} \hat{U} + sT + \int \frac{p}{2} \left[ (r_{\text{RSB}} + v_{\text{RSB}})(p-1) - r_{\text{RSB}}(p-1) \right] \hat{U} \\
+ \int \frac{p}{4} \left[ (r_{\text{RSB}} + v_{\text{RSB}})(p-1) \right] \hat{U}^2, 
\]  
(9)
and \(\int d\xi G = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} \exp(-\xi^2/2)\). The extremum conditions for \(F_{\text{RSB}}\) yield equations for the glass order parameters \(r_{\text{RSB}}\) and \(v_{\text{RSB}}\), the additional order parameter \(w_{\text{RSB}}\), the regular order parameter \(x_{\text{RSB}}\) and the parameter \(m_{\text{RSB}}:\)

\[
r_{\text{RSB}} = \int d\xi G \left\{ \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \right\} \left\{ \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \right\}, 
\]  
(10)

\[
v_{\text{RSB}} = \int d\xi G \frac{\int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \}}{\int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \}} \left\{ \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \right\} \left\{ \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \right\}, 
\]  
(11)

\[
w_{\text{RSB}} = \int d\xi G \frac{\int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \}}{\int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \}} \left\{ \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \right\} \left\{ \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \right\}, 
\]  
(12)

\[
m_{\text{RSB}} \frac{t^2}{4}(p-1)[(r_{\text{RSB}} + v_{\text{RSB}})^2 - (r_{\text{RSB}})^2] = -\frac{1}{m_{\text{RSB}}} \int d\xi G \ln \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \left\{ \int d\eta G \{ \text{Tr} \exp(\hat{\theta}_{\text{RSB}}) \} \right\}. 
\]  
(13)

The expressions for the RS approximation can be found technically from equations (10)–(12), setting \(v_{\text{RSB}} = 0\). Then for the glass order parameter \(q_{\text{RS}}\) we have:

\[
q_{\text{RS}} = \int d\xi G \left\{ \frac{\text{Tr} \hat{U} \exp(\hat{\theta}_{\text{RS}})}{\text{Tr} \exp(\hat{\theta}_{\text{RS}})} \right\}^{\frac{1}{2}}, 
\]  
(14)

where
\[
\hat{\theta}_{\text{RS}} = zT \int \frac{p_{\text{RS}}(p-1)}{2} \hat{U} + t^2 \int \frac{p_{\text{RS}}(p-1) - q_{\text{RS}}(p-1)}{4} \hat{U}^2. 
\]  
(15)
2.2. Discussion

We should emphasize that the characteristic properties of the systems in hand develop themselves already in the RS approximation. If \( \hat{U} \) has no reflection symmetry then the nonlinear integral equation for the RS glass order parameter (14)–(15) simply has no trivial solutions at any temperature, because the integrand is nonsymmetric due to the cubic terms in the free energy expansion [33, 35]. Both physical RS order parameters, glass (5) and regular (7), increase smoothly as the temperature decreases from the high-temperature nonzero values (see also section 2.4 below). There is no temperature interval where the RS solution coincides with the ‘para-state’ of the pure system. At the point where the RS solution becomes unstable (the bifurcation point \( T_0 \) of equation (11) for the glass order parameter), the 1RSB branch appears. The 1RSB glass parameters \( v_{\text{1RSB}} \) and \( r_{\text{1RSB}} \) appear continuously at the bifurcation point \( T_0 \).

We find the 1RSB order parameter \( m_{\text{1RSB}} \) analytically at the branching point \( T_0 \) and express the result through RS order parameters. If \( m_{\text{1RSB}} \leq 1 \) at the branching point then the 1RSB solution has physical meaning (if stable) near \( T_0 \), as shown in figure 2. The condition \( m_{\text{1RSB}} > 1 \) at \( T_0 \) implies the unphysical region and so then \( T_0 \) is not the actual point of the transition. There, 1RSB glass parameters show recurrent behavior with many-valued dependence on temperature. As a result we have a jump of physical glass order parameters at \( T_{\text{1RSB}} \) where \( m_{\text{1RSB}} = 1 \), as shown in figure 2. In a sense, the 1RSB solution behaves as in the ordinary \( p \)-spin model here, but in the external field [40]. This is due to the fact that the sub-algebra of commuting diagonal matrices to which \( \hat{U} \) belongs must have a dimension greater than 2, so that \( \hat{U}^2 \) can contain \( \hat{U} \) (or some other operator of the sub-algebra).

In contrast, reflection symmetry of the operators \( \hat{U} \) should lead to the vanishing of a number of terms in the free energy, so that the RS solution for the order parameters is zero at high temperature. As a result, the behavior of the 1RSB solution for the order parameters is like in the ordinary \( p \)-spin model of Ising spins. In this case for \( p = 2 \) we have FRSB appearing only at the transition point [41] and ultrametricity of the space of states [42].

2.3. Stability of the 1RSB solution

2.3.1. Replicon mode. The stability of RS and nRSB solutions can be tested by the investigation of the Gaussian fluctuation contribution to the free energy near this solution, as described in [7]. The saddle point solution is stable if all the eigen modes of the fluctuation propagator are positive. The most important mode is the so-called replicon mode [3, 35] since its sign is usually very sensitive to the RSB and to temperature. It is just \( \lambda_{\text{1RSB repl}} \) that enters the free energy with \( v_{\text{1RSB repl}} \), where \( v_{\text{1RSB repl}} \) defines the novel intragroup difference of the order parameters [34]. For example, the RS solution is stable unless the corresponding replicon mode energy \( \lambda_{\text{RS repl}} > 0 \). The RS solution can break at the temperature \( T_0 \) determined by the equation \( \lambda_{\text{RS repl}} = 0 \), where

\[
\lambda_{\text{(RS repl)}} = 1 - \frac{2}{(p-1)q_{\text{es}}^{p-2}} \int dG \left[ \frac{\text{Tr}(\hat{U}^2 e^{\hat{G}})}{\text{Tr} e^{\hat{G}}} - \left( \frac{\text{Tr} \hat{U} e^{\hat{G}}}{\text{Tr} e^{\hat{G}}} \right)^2 \right]^2. 
\]

The equation \( \lambda_{\text{(RS repl)}} = 0 \) can be obtained as the branching condition for (11); i.e., as the condition that a small solution with 1RSB can appear.

From equation (16), it follows, in particular, that the point \( p = 2 \) is a singular point in the case \( \text{Tr} \hat{U}^{(2k+1)} = 0 \) (since \( q_{\text{es}} = 0 \), as shown in figure 1. Let us recall that we are investigating now the case of the absence of reflection symmetry, so that the case \( q_{\text{es}} = 0 \) is excluded.

In our case \( \text{Tr} \hat{U}^{(2k+1)} \neq 0 \) for \( k > 0 \) and the high-temperature expansion of the equation for the order parameter \( q_{\text{es}} \) does not give the trivial solution. Using the condition \( q_{\text{es}} \neq 0 \), we
can find that $T_0 \neq 0$ in contrast with the reflection symmetry case. Therefore solutions with unbroken symmetry may appear continuously not only for $p = 2$.

It is important that $q_{\text{rs}} = 0$ is excluded when we investigate the case without the reflection symmetry of $\hat{U}$. In this case, the problem has the property of continuity over the parameter $p$.

If we break the RS once more then we obtain the corresponding expressions for the free energy and the order parameters. The bifurcation condition $\lambda_{\text{rsb}} = 0$ that determines the temperature $T = T_2$ follows from the condition that a nontrivial small solution for the 2RSB intragroup glass order parameter appears as $v_{\text{rsb}} \to 0$. So,

$$
\lambda_{\text{rsb}} = \frac{1 - r^2 p(p - 1)(r_{\text{rsb}} + v_{\text{rsb}})^{(p-2)}}{2} \int d^G x \left[ \text{Tr} \exp(\hat{\theta}_{\text{rsb}}) \right] \frac{\text{Tr}[\hat{U} \exp(\hat{\theta}_{\text{rsb}})]}{\text{Tr}[\exp(\theta_{\text{rsb}})]} \left( \frac{\text{Tr}[\hat{U} \exp(\hat{\theta}_{\text{rsb}})]}{\text{Tr}[\exp(\theta_{\text{rsb}})]} \right)^2. \tag{17}
$$

Note that equation (17) depends only on the 1RSB solution. The equation $\lambda_{\text{rsb}} = 0$ always has the solution for $v_{\text{rsb}} = 0$, which determines the point $T_0$ and coincides with the solution of equation (16) $\lambda_{\text{rsb}} = 0$, as shown in figure 4.

In addition to the point $T_0$, one more bifurcation point ($T_2$) defined by $\lambda_{\text{rsb}} = 0$ may exist as $v_{\text{rsb}} \neq 0$, and the 2RSB solution can appear at this point. A transition to the FRSB state or to a stable 2RSB state may take place at the point $T_2$.

### 2.3.2. Free energy expansion.

We expand the free energy (3)–(4) up to the third order to get a general 1RSB solution near the bifurcation point $T_0$ where it slightly deviates from the RS solution. We assume that the deviations $\delta p^{\rho}$ from $q_{\text{rs}}$ and $\rho$ from $w_{\text{rs}}$ are small. We use the notation $\Delta F$ for the difference between the free energy $F_{\text{rsb}}$ and its RS value $F_{\text{rs}}^{\text{rsb}}$. So,

$$
\frac{\Delta F}{NkT} = \frac{t^2 p(p - 1)}{4} q_{\text{rs}}^{(p-2)} \lambda_{\text{rsb}} \left( v_{\text{rsb}} - r - (m_{\text{rsb}} - 1)v_{\text{rsb}}^2 \right) - \frac{t^2}{2} L [r - (m_{\text{rsb}} - 1)v_{\text{rsb}}]^2 - t^4 \left\{ C [r - (m_{\text{rsb}} - 1)v_{\text{rsb}}]^3 + B [r - (4m_{\text{rsb}} - 1)v_{\text{rsb}}] \right\} + B_4 v_{\text{rsb}}^4 (m_{\text{rsb}} - 1)(2m_{\text{rsb}} - 1) + \Psi(\rho) + \cdots. \tag{18}
$$

where $t = t_0 + \Delta t$, $r_{\text{rsb}} = q_{\text{rs}} + r$, $w_{\text{rsb}} = w_{\text{rs}} + \rho$ and the expressions for $L, C, D, B_3, B_4$ and $\Psi$ are combinations of operators averaged over the RS solution (the exact expressions can be found here in the appendix and in the appendix of [34]).

Then one can see that $L|_{k=0} \neq 0$ for $k \geq 1$ or $r^{(2k+1)} \neq 0$ for $k \neq 0$. Using this fact we obtain from the extremum conditions for the free energy (19) that the branching can take place only if the solution is of the following form:

$$
r - (m_{\text{rsb}} - 1)v_{\text{rsb}} = 0 + o(\Delta t)^2. \tag{19}
$$

This condition states that there is no linear term for the glass order parameters (see [39]). There is no other linear term because $\lambda_{\text{rsb}}|_{v_{\text{rsb}}=0}$ at the bifurcation point. From the extremum condition we get $\rho \sim [r - (m_{\text{rsb}} - 1)v_{\text{rsb}}]$ and $\Psi(\rho) = 0 + o(\Delta t)^2$; see [33, 34]. Finally, we have:

$$
(2m_{\text{rsb}} - 1)Z \Delta T = t_0^6 \left( (2m_{\text{rsb}} - 1)(-B_4 + m_{\text{rsb}})(-B_3 + 2B_4) \right) + m_{\text{rsb}}(m_{\text{rsb}} - 1)(-B_3 + 2B_4) v_{\text{rsb}}, \tag{21}
$$

$$
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$$

where $t = t_0 + \Delta t$, $r_{\text{rsb}} = q_{\text{rs}} + r$, $w_{\text{rsb}} = w_{\text{rs}} + \rho$ and the expressions for $L, C, D, B_3, B_4$ and $\Psi$ are combinations of operators averaged over the RS solution (the exact expressions can be found here in the appendix and in the appendix of [34]).

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where
\[ Z = \frac{t^2}{4} \frac{p(p-1)}{2} \frac{d[q_{ks}^{(p-2)} \lambda_{\text{repl}}]}{dT}. \] (22)

Here \( B_3, B_4 \) and \( Z \) are taken at \( T = T_0 \). Then we find from (20) and (21) at the branch point \( T_0 \) where the 1RSB solution appears (the cases \( m_{\text{repl}} = 0 \) and \( m_{\text{repl}} = 1 \) should be investigated separately):
\[ m_{\text{repl}} = B_4/B_3, \] (23)
and
\[ r_{\text{repl}} = q_{ks} + (m_{\text{repl}} - 1)v_{\text{repl}}, \quad v_{\text{repl}} = K\Delta t \] (24)
in the neighborhood of \( T_0 \). The coefficient of proportionality \( K \) and \( m_{\text{repl}} \) depends only on the RS solution at \( T_0 \). The exact expression for \( K \) is rather lengthy and can be found in [34]. The 1RSB solution branches smoothly from the RS solution.

From equations (20)–(21) and (23), we obtain
\[ 2Z\Delta T = 3\epsilon_0^2 m_{\text{repl}}(m_{\text{repl}} - 1)B_3 v_{\text{repl}}, \] (25)
where
\[ B_3 = \left[ \frac{p(p-1)}{2} q_{ks}^{(p-2)} \right]^3 \frac{1}{6} \int dG \left\{ \frac{\text{Tr}(\hat{U} \hat{\epsilon} \hat{\epsilon})}{\text{Tr} \hat{\epsilon} \hat{\epsilon}} - \left[ \frac{\text{Tr} \hat{U} \hat{\epsilon} \hat{\epsilon}}{\text{Tr} \hat{\epsilon} \hat{\epsilon}} \right]^2 \right\}^3 \geq 0. \] (26)

The last inequality follows from the Cauchy–Schwarz inequality \((\sum_n A_n^2)(\sum_n B_n^2) \geq (\sum_n A_n B_n)^2\).

2.3.3. Discussion. The qualitative analysis of these solutions near \( T_0 \) was performed using this last set of equations (22)–(25).

From equation (25) we see that the value of the parameter \( m_{\text{repl}} \) at \( T = T_0 \) determines the behavior of the glass order parameter \( v_{\text{repl}} \). For values of \( 0 < m_{\text{repl}} < 1 \) and \( m_{\text{repl}} > 1 \) there is a linear dependence on \( \Delta T \). But the sign of the coefficient of proportionality depends on \( m_{\text{repl}} \). Really, \( \lambda_{\text{repl}} \) is an increasing function of temperature, equal to zero at \( T_0 \), and it changes its sign at this point because the sign of \( \lambda_{\text{repl}} \) determines the stability of the RS solution. The order parameter \( q_{\text{repl}} \) is a positive and slowly varying function of temperature in the vicinity of \( T = T_0 \). So, we have \( Z > 0 \) near \( T_0 \); see equation (22).

In the first case, when \( 0 < m_{\text{repl}} < 1 \), the solution goes to the left side of the point \( T = T_0 \) and so the 1RSB solution appears continuously. The resulting solution can be stable only when \( \lambda_{\text{repl}} > 0 \). In the second case, when \( m_{\text{repl}} > 1 \), a positive solution occurs at \( T \geq T_0 \). The resulting solution is a nonphysical solution near \( T = T_0 \) since \( m_{\text{repl}} > 1 \) has no physical meaning; see figure 2 for an illustration. The 1RSB solution can only occur abruptly when \( m_{\text{repl}}(T) \) yields the value of \( m_{\text{repl}}(T) = 1 \).

If we obtain \( m_{\text{repl}} = 1 \) at \( T = T_0 \) at a certain value of \( p \), thus we find the point where a crossover from continuous to discontinuous transition to the 1RSB solution takes place. In this case we see from the equation (25) that it is necessary to consider fourth-order terms in the expansion of the free energy (19). One can show that in this case \( v_1^2 \sim \Delta T \); see figure 3 for an illustration.

2.4. Quadrupole glass with \( J = 1 \)

2.4.1. Results. The quadrupole glass with \( J = 1 \) and \( J_z = \{0, \pm 1\} \) is the simplest example of the system without reflection symmetry. In this case \( \hat{U} = 3J_z^2 - 2 \) is the quadrupolar moment
of the molecule. It is worth emphasizing that the smooth increase of the glass and regular order parameters from high to low temperature obtained in the frame of the simplest version of such a glass [43] was confirmed by the experiment in [44]. Now we present a calculation based on equations (10)–(14) and (22)–(25) for the model with varying parameters.

For $p = 3$, the 1RSB solution is stable near $T_0$, it branches continuously at the bifurcation point $T_0 = T_{\text{1RSB}}$ and changes smoothly on cooling below $T_0$, as shown in figure 2. We find that $m_{\text{1RSB}} = 0.88 < 1$ at the branch point. It follows from equation (25) that $v_{\text{1RSB}} \sim -(T - T_0)$ near the branch point.

For $p = 4, p = 5$, etc, the solution appears smoothly but in these cases, the condition $\lambda_{1\text{RSB, repl}} = 0$ does not determine the small physical solution in the neighborhood of the branch point. Namely, $m_{\text{1RSB}} > 1$ at the branch point and the nonphysical branch of the free energy appears at $T = T_0$. So, as seen in equation (25), the order parameter is linear again: $v_{\text{1RSB}} \sim (T - T_0)$. In fact, the transition from the RS to the 1RSB solution is discontinuous at the point $T_{\text{1RSB}} > T_0$, determined by the condition $m_{\text{1RSB}} = 1$. At this point, $F_{\text{RS}} = F_{\text{1RSB}}$. The RS solution is stable above $T_{\text{1RSB}}$ while $m_{\text{1RSB}} < 1$ for $T < T_{\text{1RSB}}$ and the corresponding physical 1RSB solution corresponds to larger (preferable) free energy than the RS solution.

At $p = 3.3$ the crossover from the continuous to the discontinuous scenario takes place. In this case, $m_{\text{1RSB}} = 1$ at the branch point. Equation (20) becomes an identity. The right-hand sides of equations (21) and (25) become zero. It is therefore necessary to consider the terms of the fourth order of glass order parameters in the expansion of the free energy (19) near $T_0$.

Hence now we obtain the nonlinear behavior of the order parameter near the transition point: $v_{\text{1RSB}} \sim \sqrt{T_0 - T}$, as shown in figures 2 and 3. It is important that the positivity of $\lambda_{1\text{RSB, repl}}$ is a necessary condition for the solution in these models to be stable with respect to subsequent RSB [34].

The $p$-quadrupole glass model for $p < 2 + \Delta p$, where $\Delta p = 0.5$ is a finite value, behaves in exactly the same way as it does in the case of pair interaction. For $p < 2.5$, the stable 1RSB solution disappears, as shown in figures 2–4. We get $\lambda_{1\text{RSB, repl}} \leq 0$ for $T < T_0$. This behavior is expected due to the continuity reasons.

In the case of $\text{Tr} U^{(2k+1)} \neq 0$, the problem has the property of continuity over the parameter $p$. These results differ from the corresponding results for the case $\text{Tr} U^{(2k+1)} = 0$, when the point $p = 2$ is a singular point, as shown in equation (16), since $q_{\text{fs}} = 0$.

### 2.4.2. Discussion.

Let us note that qualitatively similar behavior in certain aspects shows the Potts model for 3, 4 and 5 states [27]. The form of the series for $\Delta F = F_{\text{fs}} - F_{\text{RS}}$ (19) over the small deviations $kq^{q+\epsilon}$ from $q_{\text{fs}}$ is one and the same for different models. In the case of Potts’ spin glass model the reflection symmetry is absent. However, $L = 0$, because it is the zero RS solution that bifurcates. As $L_{1\text{RSB, repl}} = 0$, then the condition $r = (m - 1)\psi_{\text{1RSB}} = 0$ is not fulfilled ($r_{\text{1RSB}} = r = 0$ [45]). But equations similar to (20), (21) and (23) do exist. Then the crossover, at $p = 4$, from continuous to jumpwise behavior with the increase of the number of states can be traced analytically. There also exists a domain of stability where the 1RSB solution remains stable under further RSBs [28].

We also remind that it was shown for the $p$-Ising spin-glass model [19–22] that the 1RSB solution appears at a certain $T_c$ with a jump in the order parameter for $p = 2 + \epsilon$ with small $\epsilon$, and remains stable at a certain interval near $T_c$. The jump at $T_c$ and the range of stability of the 1RSB solution tend to zero as $p \to 2$.

5. We have $Z > 0$ near $T_0$.
6. For $p = 4$ we have $m_{\text{1RSB}} = 1.17$, for $p = 5$ we have $m_{\text{1RSB}} = 1.3$.
7. We remind that $L$ is defined in equations (19) and (A.2).
8. We remind that $r_{\text{1RSB}} = q_{\text{fs}} + r$. 

10
3. Conclusions

In this paper we investigate the generalized $p$-spin models that contain arbitrary diagonal operators $\hat{U}$ instead of Ising spins. We focus our attention mainly on the case when $\hat{U}$ does not have reflection symmetry (such systems as a whole have no time-reversal symmetry).

We derive general equations that allow analytical investigation of the qualitative behavior of the system near the glass transition at different (continuous) $p$. The main results are schematically shown in figure 1(a).

For the quadrupole glass with $J = 1$ the detailed quantitative analysis is performed. At $p = 3.3$ it is shown that $m_{RSB} = 1$ at the branch point, and the crossover from continuous to discontinuous transition takes place. For $p < 2 + \Delta p$ (where $\Delta p = 0.5$ is a finite value) we get $\lambda_{(RSB)} \leq 0$ for $T < T_0$ (stable 1RSB solution disappears). This behavior differs from the corresponding behavior for the conventional $p$-spin Ising glass model.

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Appendix. $W$, $L$ and $B_4$

\[ W = \left[ \frac{p(p-1)}{2} q_{ks}^{(p-2)} \right] \left\{ \langle \hat{U}_1^2 \hat{U}_2^3 \rangle - 2\langle \hat{U}_1^2 \hat{U}_2 \hat{U}_3 \rangle + \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \rangle \right\} \]

\[ = \int d\vec{\theta} \left\{ \frac{\text{Tr}(\hat{U}^2 e^{i\hat{U}_0})}{\text{Tr} e^{i\hat{U}_0}} - \left[ \frac{\text{Tr} \hat{U} e^{i\hat{U}_0}}{\text{Tr} e^{i\hat{U}_0}} \right]^2 \right\}^2. \]  \hspace{1cm} (A.1)

Notation used below are built according to the same rules used in equation (A.1):

\[ L = \left[ \frac{p(p-1)}{2} q_{ks}^{(p-2)} \right] \left\{ \langle \hat{U}_1^2 \hat{U}_2 \hat{U}_3 \rangle - \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \rangle \right\} \geq 0 \]  \hspace{1cm} (A.2)

similar to (26):

\[ t^6 B_4 = \left[ \frac{p(p-1)}{2} q_{ks}^{(p-2)} \right] \left\{ \frac{1}{3} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \rangle - \langle \hat{U}_1^2 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \rangle \right\} \]

\[ + \frac{1}{3} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \rangle + \frac{3}{4} \langle \hat{U}_1^2 \hat{U}_2 \hat{U}_3 \hat{U}_4 \rangle - \frac{1}{2} \langle \hat{U}_1^3 \hat{U}_2 \hat{U}_3 \rangle + \frac{1}{12} \langle \hat{U}_1^4 \hat{U}_2 \rangle \]

\[ - t^2 p(p-1)(p-2)q_{ks}^{(p-3)} \left[ 1 - t^2 \frac{3}{2} W \right]. \]

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