ON THE MINIMAL NUMBER OF CRITICAL POINTS OF FUNCTIONS ON $h$-COBORDISMS

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Abstract. Let $(W, M_0, M_1)$ be a non-trivial $h$-cobordism (i.e., the Whitehead torsion of $(W, M_0)$ is non-zero) with $\dim W > 5$. We prove that every smooth function $f : W \to [0, 1]$, $f(M_0) = 0$, $f(M_1) = 1$ has at least 2 critical points. This estimate is sharp: $W$ possesses a function as above with precisely two critical points.

Introduction

Let $(W, M_0, M_1)$ be an $h$-cobordism, [3]. Here $W$ is always assumed to be smooth, connected and compact and $M_i, i = 0, 1$ is always assumed to be closed. Recall that an $h$-cobordism $(W, M_0, M_1)$ is called trivial if there is a diffeomorphism $(W, M_0, M_1) \cong (M \times [0, 1], M_0, M_0)$. We say that a function (not necessarily Morse) $f : W \to [0, 1]$ is regular if $f^{-1}(M_0) = 0$, $f^{-1}(M_1) = 1$ and both values 0 and 1 are regular values of $f$. It is well known that an $h$-cobordism $(W, M_0, M_1)$ is trivial if and only if $W$ possesses a regular function without critical points. In this note we prove the following theorem.

Theorem Let $(W, M_0, M_1)$ be a non-trivial $h$-cobordism with $\dim W \geq 6$. Then every regular function on $W$ has at least two critical points. Moreover, this estimate is sharp: $W$ possesses a regular function with precisely two critical points.

We denote by $I$ the closed interval $[0, 1]$.

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1. Preliminaries

Let $f : W \to I$ be a regular Morse function on an $h$-cobordism $(W, M_0, M_1)$. Choose a Riemannian metric on $W$ and consider integral trajectories for the vector field $-\text{grad} \, f$, the so-called anti-gradient trajectories. We say that an anti-gradient trajectory $a = a(t)$ is a special trajectory from $p$ to $q$ if $\lim_{t \to -\infty} a(t) = p$ and $\lim_{t \to +\infty} a(t) = q$ where $p$ and $q$ are critical points of $f$ such that the index of $p$ is one more than the index of $q$. We can and shall assume that the number of special trajectories is finite (this is true for generic function and metric).

For every critical point of $f$ we fix orientations of unstable disks (left-hand disks in terminology of [3]). Then every unstable sphere (in a certain level) gets an orientation. Moreover, every stable sphere gets a coorientation, i.e., an orientation of its normal bundle in the corresponding level set. Now, for every special trajectory $a$ from $p$ to $q$ we define the number $\varepsilon(a) = \pm 1$ as follows. Take $c \in f(q), f(p)$. Then our trajectory $a$ meets the level $f^{-1}(c)$ in a certain point $x$, which is a point of transversal intersection of the corresponding stable and unstable spheres. We define $\varepsilon(a)$ to be the intersection index at $x$.

2. Whitehead torsion

Given a ring $R$, we define a based $R$-module to be a free finite generated left $R$-module $M$ with a fixed $R$-free basis.

Recall the definition of the Whitehead torsion of an $h$-cobordism $(W, M_0, M_1)$. Given a group $\pi$, let $A = A(\pi)$ denote the set of long exact sequences

$$
\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to 0
$$

such that each $C_i$ is a based $\mathbb{Z}[\pi]$-module and all but finite number of modules $C_i$ are zero modules. Furthermore, each $\partial_i$ is a $\mathbb{Z}[\pi]$-module homomorphism. Let us call the exact sequence of based $\mathbb{Z}[\pi]$-modules trivial if it has only two non-zero terms and the corresponding isomorphism is given by the identity matrix. The term-wise direct sum operation converts $A$ into an abelian semigroup. Let $R$ be the equivalence relation on $A$ generated by the following operations:

- interchanging of the elements;
- replacement of a basis element by the sum of this element with the multiple of another basis element;
- addition of the trivial exact sequence;
- multiplication of any basis element by the element $\pm g, g \in \pi$. 

The above mentioned operation in $A$ induces a group structure in $A/R$. This groups is called the Whitehead group of $\pi$ and is denoted by $\text{Wh}(\pi)$, \[4\]. It turns out to be that $\text{Wh}(\pi)$ is a functor of $\pi$. In particular, every homomorphism $\varphi : \pi \to G$ induces a homomorphism $\text{Wh}(\varphi) : \text{Wh}(\pi) \to \text{Wh}(G)$. Namely, the homomorphism $\varphi$ yields the homomorphism $Z[\varphi] : Z[\pi] \to Z[G]$ of group rings, and for every based $Z[\pi]$-module $C$ we can form the based $Z[G]$-module $C \otimes_{Z[\varphi]} Z[G]$. The sequence $\{C_n \otimes_{Z[\varphi]} Z[G]\}$ turns out to be exact because all the $C_n$‘s are free, etc.

For every $h$-cobordism $(W, M_0, M_1)$ with $\pi_1(W) = \pi$ the Whitehead torsion $\tau(W, M_0, M_1) \in \text{Wh}(\pi)$ is defined as follows. Consider a regular Morse function $f : W \to I$, Riemannian metric, etc. as in $\S 4$. Fix a point $x_0 \in W$ and, for every critical point $p$ of $f$, choose a path $u(p)$ from $x_0$ to $p$. Given a special trajectory $a$ from $p$ to $q$, we define a path $v = v_a : I \to W$ as follows. Let $\lambda(t) : \mathbb{R} \to [0, 1]$ be a function such that $\lim_{t \to 0} \lambda(t) = -\infty$, $\lim_{t \to 1} \lambda(t) = +\infty$. We set $v(0) = p$, $v(1) = q$, $v(t) = a(\lambda(t))$. Now, consider the loop $u(p) \circ v(\lambda(q))^{-1}$ (where $\circ$ denotes the composition of paths) and define $g(a) \in \pi = \pi_1(W)$ as the based homotopy class of the loop.

Let $p_1, \ldots, p_k$ be all the critical points of the index $n$. Define $C_n$ to be the free $Z[\pi]$-module generated by symbols $[p_1], \ldots, [p_k]$. In other words, $C_n$ consists of formal linear combinations

$$\sum_{i=1}^{k} \alpha_i[p_i], \quad \alpha_i \in Z[\pi].$$

We define the differential $\partial_n : C_n \to C_{n-1}$ to be a $Z[\pi]$-module homomorphism such that

$$\partial_n[p] = \sum_{q} \sum_{a \in T(p, q)} \varepsilon(a)g(a)[q]$$

where $q$ runs over all critical points of the index $n - 1$ and $T(p, q)$ is the set of special trajectories from $p$ to $q$.

It follows from the Morse theory that $H_*(\{C_n, \partial_n\}) = H_*(\widetilde{W}, \widetilde{M}_0)$ where $(\widetilde{W}, \widetilde{M}_0)$ is the universal covering of the pair $(W, M_0)$. Since $M_0$ is a deformation retract of $W$, we conclude that $\widetilde{M}_0$ is a deformation retract of $\widetilde{W}$, and therefore the complex $\{C_n, \partial_n\}$ is acyclic, i.e. the sequence

$$\cdots \longrightarrow C_n \overset{\partial_n}{\longrightarrow} C_{n-1} \longrightarrow \cdots \longrightarrow 0$$
is exact. Thus, the above sequence determines a certain element \( \tau = \tau(W, M_0) \in \text{Wh}(\pi) \), the so-called Whitehead torsion of the \( h \)-cobordism \((W, M_0, M_1)\).

According to well-known Barden–Mazur–Stallings Theorem, [1, 2, 5], an \( h \)-cobordism \((W, M_0, M_1)\) with \( \dim W \geq 6 \) is trivial if and only if \( \tau(W, M_0) = 0 \).

2.1. Lemma. Suppose that an \( h \)-cobordism \((W, M_0, M_1)\) possesses a regular Morse function \( f \) such that all the critical points and special trajectories of \( f \) are contained in a simply connected domain \( U \) of \( W \). Then \( \tau(W, M_0) = 0 \).

Proof. Since \( \tau(W, M_0) \) does not depend on the choice of the based point \( x_0 \) and the paths \( u(p) \), we can assume that \( x_0 \in U \) and every path \( u(p) \) belongs to \( U \). Then, for every special trajectory \( a \), \( ga \) is the neutral element of \( \pi = \pi_1(W) \). Thus,

\[
\tau(W, M_0) \in \text{Im}\{\text{Wh}(j) : \text{Wh}\{e\} \to \text{Wh}(\pi)\}
\]

where \( j : \{e\} \to \pi \) is the inclusion of the trivial subgroup. But it follows from the elementary linear algebra that \( \text{Wh}\{e\} = 0 \), see e.g. [4]. Thus, \( \tau(W, M_0) = 0 \).

3. Proof of the theorem

Let \( f : M \to \mathbb{R} \) be a smooth function (not necessarily Morse) on a Riemannian manifold \( M \). Let \( U \) be an open ball in \( M \) and suppose that \( U \) contains precisely one critical point \( o \).

3.1. Lemma. There exists a regular function \( g \) which is \( C^\infty \)-closed to \( f \) in the Whitney topology and such that every special \( g \)-trajectory is contained in \( U \) whenever its ends are contained in \( U \).

Proof. Let \( D(r) = \{m \in M \mid d(m, o) < r \} \) where \( d \) is the distance function on \( M \). We can and shall assume that the injectivity radius at \( p \) is at least one and that \( U = D(1) \). Then there constants positive constants \( C \) and \( E \) such that, for every function \( g \) which is \( C^\infty \)-closed to \( f \), the following estimates holds in \( D(1) \setminus D(1/2) \):

\[
|\text{grad } g| \geq E, \quad |L_{\text{grad } f}d(m, o)| \leq C.
\]

Choose a function \( g \) closed to \( f \) let \( p \) and \( q \) be two critical points of \( g \) which belongs to \( U \). Suppose that there is a special trajectory \( a(t) \) from \( p \) to \( q \) which meets the boundary of \( D(3/4) \). We claim that in this case

\[
g(p) - g(q) \geq \frac{E^2}{4C}.
\]
Indeed, since \( L_{\text{grad} f} \, d(m, o) \leq C \), we conclude that
\[
a \left[ t - \frac{1}{4C}, t + \frac{1}{4C} \right]
\]
does not meet \( D(1/2) \) whenever \( a(t) \notin D(3/4) \). So, if \( a(t_0) \notin D(3/4) \) then
\[
g(p) - g(q) \geq \int_{t_0 - \frac{1}{4C}}^{t_0 + \frac{1}{4C}} |\text{grad} \, g|^2 dt \geq \frac{E^2}{4C}.
\]

Now we can finish the proof as follows. Since \( f \) has only one critical point, there exists \( g \) closed to \( f \) and such that \( g(p) - g(q) \) is small enough for every critical point \( p \) and \( q \) of \( g \). This is a contradiction.

3.2. Corollary. If an \( h \)-cobordism \( (W, M_0, M_1) \) possesses a regular function \( f \) with one critical point \( p \), then \( \tau(W, M_0) = 0 \). In particular, if \( \dim W \geq 6 \) then the \( h \)-cobordism is trivial.

Proof. Because of Lemma 3.1, we can perturb the function \( f \) in a small neighborhood of the critical point and get a function \( f_1 \) such that all its critical points and special trajectories belong to a disk neighborhood of \( p \). Now the result follows from Lemma 2.1.

3.3. Proposition. Every \( h \)-cobordism \( (W, M_0, M_1) \), \( \dim W \geq 6 \) possesses a regular function with at most 2 critical points.

Proof. Consider a regular Morse function \( f : W \to I \). Asserting as in [1, Lemme 1] and [3, §4], we can modify \( f \) and to get a regular Morse function which has at most two critical levels \( a, b \), \( a < b \) and index of each of critical points is equal to 2 or 3. Because of this, every critical level is path connected. Now, following [1, Th. 2.7 and Prop.2.9], we can contract the critical points in each of levels and get a regular function with at most 2 critical points.

Clearly, Corollary 3.2 and Proposition 3.3 together imply the Theorem.

3.4. Remarks. 1. Asserting as in 3.2, one can show that, for every regular function \( f \) on a non-trivial \( h \)-cobordism, the number of critical levels of \( f \) is at least 2 provided that all the critical points of \( f \) are isolated.

2. Every \( h \)-cobordism \( (W, M_0, M_1) \) possesses a regular function with 1 critical level. Namely, choose collars of the boundary components and define \( f \) to be constant on complements of collars and depending
on “vertical” coordinate only for collars. In greater detail, consider a smooth function

\[ \varphi : I \to I, \quad \varphi(t) = \begin{cases} t & \text{if } 0 \leq t \leq \varepsilon/4 \text{ or } 1 - \varepsilon/4 \leq t \leq 1, \\ 1/2 & \text{if } \varepsilon/2 \leq t \leq 1 - \varepsilon/2 \end{cases} \]

for \( \varepsilon > 0 \) small enough. Choose collars \( M_0 \times [0, \varepsilon] \) and \( M_1 \times [1 - \varepsilon, 1] \) and define \( f : W \to I \) by setting

\[ f(x) = \begin{cases} \varphi(t) & \text{if } x = (m, t) \in M_0 \times [0, \varepsilon], \\ \varphi(t) & \text{if } x = (m, t) \in M_1 \times [1, 1 - \varepsilon], \\ 1/2 & \text{else.} \end{cases} \]

3. Every trivial \( h \)-cobordism \((M \times I, M_0, M_1)\) possesses a regular function with 1 critical point. Indeed, consider a function \( \varphi : M \to I \) such that \( \varphi^{-1}(1) \) is a point \( m_0 \) (and therefore \( m_0 \) is a critical point of \( \varphi \)) and define

\[ f : M \times I \to I, \quad f(m, t) = (t - 1/2)(1 - \varphi(m)) + \varphi(m)(t - 1/2)^3. \]

It is easy to see that \( f \) has just one critical point \((m_0, 1/2)\).

4. Notice that, for every \( h \)-cobordism \((W, M_0, M_1)\), the relative Lusternik–Schnirelmann category \( \text{cat}(W, M_0) = 0 \), while every regular function on any non-trivial \( h \)-cobordism \((W, M_0, M_1)\) has at least two critical points.

5. It is easy to see that, because of the collar theorem, the regularity condition for \( f \) in the Theorem can be weaken as follows: \( f(M_0) = 0 \) and \( f(M_1) = 1 \).

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