Introduction to Coherent States and
Quantum Information Theory

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Abstract

The purpose of this paper is to introduce several basic theorems of coherent states
and generalized coherent states based on Lie algebras su(2) and su(1,1), and to give
some applications of them to quantum information theory for graduate students
or non–experts who are interested in both Geometry and Quantum Information
Theory.

In the first half we make a general review of coherent states and generalized
coherent states based on Lie algebras su(2) and su(1,1) from the geometric point of
view and, in particular, prove that each resolution of unity can be obtained by the
curvature form of some bundle on the parameter space.

We also make a short review of Holonomic Quantum Computation (Computer)
and show a geometric construction of the well–known Bell states by making use of
generalized coherent states.

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In the latter half we apply a method of generalized coherent states to some important topics in Quantum Information Theory, in particular, swap of coherent states and cloning of coherent ones.

We construct the swap operator of coherent states by making use of a generalized coherent operator based on su(2) and show an “imperfect cloning” of coherent states, and moreover present some related problems.

We also present a problem on a possibility of calculation or approximation of coherent state path integrals on Holonomic Quantum Computer.

In conclusion we state our dream, namely, a construction of Geometric Quantum Information Theory.
1 Introduction

This paper is the pair to the preceding one [14] and the aim is to introduce geometric aspects of coherent states and generalized coherent ones based on Lie algebras $su(1,1)$ and $su(2)$ and to apply them to quantum information theory for graduate students or non–experts (in this field) who are interested in both Geometry and Quantum Information Theory.

Coherent states or generalized coherent states play a crucial role in quantum physics, in particular, quantum optics, see [1] and its references or [2]. They also play an important one in mathematical physics, see [3]. For example, they are very useful in performing stationary phase approximations to path integral, see [4], [5] and [6].

In the theory of coherent states or generalized coherent ones the resolution of unity is just a key concept, see [1]. Is it possible to understand this fact from the geometric point of view? For a set of coherent or generalized coherent states we can define a projector from the manifold consisting of parameters of them to infinite–dimensional Grassmann manifold (called classifying spaces in K–Theory). Making use of this projector we can calculate several geometric quantities such as Chern characters, see for example [9]. In particular, we prove that each resolution of unity can be obtained by the curvature form of some bundle on the parameter space.

Let us turn to Quantum Information Theory (QIT). The main subjects in QIT are

(i) Quantum Computation

(ii) Quantum Cryptgraphy

(iii) Quantum Teleportation

As for general introduction to QIT see [11], [12] and [13], [14]. The aim of this paper is to apply geometric methods to QIT, or more directly

A Geometric Construction of Quantum Information Theory.
We are developing the theory of geometric quantum computation called Holonomic Quantum Computation, see [20], [21], [22] and [15]–[19], and we are also studying geometric construction of the Bell states or the generalized Bell ones, see [40], [41]. We are interested in geometric method of Homodyne Tomography [24], [25] or geometric one of Quantum Cryptography [26], [27].

On the other hand, the method of path integral plays a very important role in Quantum Mechanics or Quantum Field Theory. However it is not easy to calculate complicated path integrals with classical computers. We are interested in it from the quantum information theory’s point of view. That is, can we calculate or approximate some path integral in polynomial times with Quantum Computers (Holonomic Quantum Computer especially)? Unfortunately we cannot answer this question, however we believe this problem becomes crucial for Quantum Computers.

By the way it seems to the author that our calculations suggest some profound relation to recent non–commutative differential geometry or non–commutative field theory, see [28] or [29]. This is very interesting, but beyond the scope of this paper. We show the relation diagramatically

\[
\begin{align*}
\text{Classical Information Theory} & \iff \text{Classical Geometry} \\
\downarrow & \quad 21 \text{ Century} \quad \downarrow \\
\text{Quantum Information Theory} & \iff \text{Quantum Geometry}
\end{align*}
\]

We expect that some readers would develop this subject.

In the latter half of this paper we treat special topics in Quantum Information Theory, namely, swap of coherent states and cloning of coherent states. It is not difficult to construct a universal swap operator (see Appendix), however for coherent states we can construct a special and better one by making use of a generalized coherent operator based on \(su(2)\). On the other hand, to construct a cloning operator is of course not easy by the no cloning theorem [43]. However for coherent states we can make an approximate cloning
(“imperfect cloning” in our terminology) by making use of the same coherent operator
based on $su(2)$. This and some method in [37] may develop a better approximate cloning
method. We also present some related problems on these topics.

We have so many problems to be solved in the near future. The author expects strongly
that young mathematical physicists or information theorists will take part in this fruitful
field.

The contents of this paper are as follows:

1 Introduction

2 Coherent States

3 Generalized Coherent States Based on $su(1,1)$
   3.1 General Theory
   3.2 Some Formulas
   3.3 A Supplement
   3.4 Barut–Girardello Coherent States

4 Generalized Coherent States Based on $su(2)$
   4.1 General Theory
   4.2 Some Formulas
   4.3 A Supplement

5 Schwinger’s Boson Method

6 Universal Bundles and Chern Characters

7 Calculations of Curvature Forms
   7.1 Coherent States
   7.2 Generalized Coherent States Based on $su(1,1)$
7.3 Generalized Coherent States Based on $su(2)$

8 Holonomic Quantum Computation

  8.1 One–Qubit Case
  4.2 Two–Qubit Case

9 Geometric Construction of Bell States

  9.1 Review on General Theory
  9.2 Review on Projective Spaces
  9.3 Bell States Revisited

10 Topics in Quantum Information Theory

  10.1 Some Useful Formulas
  10.2 Swap of Coherent States
  10.3 Imperfect Cloning of Coherent States
  10.4 Swap of Squeezed–like States ?
  10.5 A Comment

11 Path Integral on a Quantum Computer

12 Discussion and Dream

Appendix

  A Universal Swap Operator
  B Calculation of (250)
  C Representation from $SU(2)$ to $SO(3)$
2 Coherent States

We make a review of some basic properties of displacement (coherent) operators within our necessity. For the proofs see [3] or [1].

Let \( a(a^\dagger) \) be the annihilation (creation) operator of the harmonic oscillator. If we set \( N \equiv a^\dagger a \) (number operator), then

\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a^\dagger, a] = -1. \tag{1}
\]

Let \( \mathcal{H} \) be a Fock space generated by \( a \) and \( a^\dagger \), and \( \{|n\rangle| n \in \mathbb{N} \cup \{0\}\} \) be its basis. The actions of \( a \) and \( a^\dagger \) on \( \mathcal{H} \) are given by

\[
a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle, \quad N|n\rangle = n|n\rangle \tag{2}
\]

where \( |0\rangle \) is a normalized vacuum \( (a|0\rangle = 0 \) and \( \langle 0|0\rangle = 1 \). From (2) state \( |n\rangle \) for \( n \geq 1 \) are given by

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{3}
\]

These states satisfy the orthogonality and completeness conditions

\[
\langle m|n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1 \tag{4}
\]

Let us state coherent states. For the normalized state \( |z\rangle \in \mathcal{H} \) for \( z \in \mathbb{C} \) the following three conditions are equivalent:

(i) \( a|z\rangle = z|z\rangle \) and \( \langle z|z\rangle = 1 \) \hspace{1cm} \tag{5}

(ii) \( |z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle = e^{-|z|^2/2} e^{za^\dagger}|0\rangle \) \hspace{1cm} \tag{6}

(iii) \( |z\rangle = e^{za^\dagger -\bar{z}a}|0\rangle \). \hspace{1cm} \tag{7}

In the process from (6) to (7) we use the famous elementary Baker-Campbell-Hausdorff formula

\[
e^{A+B} = e^{-\frac{1}{2}[A,B]}e^A e^B \tag{8}
\]

whenever \([A, [A, B]] = [B, [A, B]] = 0\), see [1] or [3]. This is the key formula.
**Definition**  The state $|z\rangle$ that satisfies one of (i) or (ii) or (iii) above is called the coherent state.

The important feature of coherent states is the following resolution (partition) of unity.

\[
\int_C \frac{[d^2 z]}{\pi}|z\rangle \langle z| = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1,
\]

where we have put $[d^2 z] = d(\text{Re} z) d(\text{Im} z)$ for simplicity. We note that

\[
|\langle z|w\rangle| = e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 + \bar{z}w} \implies |\langle z|w\rangle| = e^{-\frac{1}{2}|z-w|^2},
\]

so $|\langle z|w\rangle| < 1$ if $z \neq w$ and $|\langle z|w\rangle| \ll 1$ if $z$ and $w$ are separated enough. We will use this fact in the following.

Since the operator

\[
D(z) = e^{za^\dagger - \bar{z}a}
\]

for $z \in \mathbb{C}$ is unitary, we call this a displacement (coherent) operator. For these operators the following properties are crucial. For $z, w \in \mathbb{C}$

\[
D(z)D(w) = e^{\bar{z}w - \bar{w}z} D(w)D(z),
\]

\[
D(z + w) = e^{-\frac{1}{2}(z\bar{w} - \bar{z}w)} D(z)D(w).
\]

Here we list some basic properties of this operator.

(a) **Matrix Elements**  The matrix elements of $D(z)$ are

(i) $n \leq m$  \quad $\langle n|D(z)|m\rangle = e^{-\frac{1}{2}|z|^2} \sqrt{\frac{n!}{m!}} (-\bar{z})^{m-n} L_n^{(m-n)}(|z|^2),$

(ii) $n \geq m$  \quad $\langle n|D(z)|m\rangle = e^{-\frac{1}{2}|z|^2} \sqrt{\frac{m!}{n!}} z^{n-m} L_m^{(n-m)}(|z|^2),$

where $L_n^{(\alpha)}$ is the associated Laguerre’s polynomial defined by

\[
L_n^{(\alpha)}(x) = \sum_{j=0}^{n} (-1)^j \binom{n+\alpha}{n-j} \frac{x^j}{j!}.
\]

(b) **Trace Formula**  We have

\[
\text{Tr} U(z) = \pi \delta^2(z) \equiv \pi \delta(x)\delta(y) \quad \text{if } z = x + iy.
\]

7
(c) Glauber Formula  Let $A$ be any observable. Then we have

$$A = \int_C \frac{[d^2z]}{\pi} \text{Tr}[AD^\dagger(z)]D(z)$$  \hspace{1cm} (18)

This formula plays an important role in the field of homodyne tomography, \cite{24} and \cite{25}.

(d) Projection on Coherent State  The projection on coherent state $|z\rangle$ is given by $|z\rangle\langle z|$. But this projection has an interesting expression :

$$|z\rangle\langle z| =: e^{-(a-z)\dagger(a-z)} : (19)$$

where the notation $:\,$ means normal ordering.

This formula has been used in the field of quantum cryptography, \cite{26} and \cite{27}. We note that

$$|z\rangle\langle w| \neq : e^{-(a-z)\dagger(a-w)} :$$

for $z, \ w \in \mathbb{C}$ with $z \neq w$.

3 Generalized Coherent States Based on $su(1,1)$

In this section we introduce some basic properties of generalized coherent operators based on $su(1,1)$, see \cite{28} or \cite{29}.

3.1 General Theory

We consider a spin $K \ (>0)$ representation of $su(1,1) \subset sl(2, \mathbb{C})$ and set its generators

$${\{K_+, K_-, K_3\} \ ((K_+)\dagger = K_-)},$$

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \hspace{1cm} (20)$$

We note that this (unitary) representation is necessarily infinite dimensional. The Fock space on which $\{K_+, K_-, K_3\}$ act is $\mathcal{H}_K \equiv \{|K, n\rangle|n \in \mathbb{N} \cup \{0\}\}$ and whose actions are

$$K_+|K, n\rangle = \sqrt{(n+1)(2K+n)}|K, n+1\rangle,$$
\[ K_- |K, n\rangle = \sqrt{n(2K + n - 1)} |K, n - 1\rangle, \]
\[ K_3 |K, n\rangle = (K + n) |K, n\rangle, \]
(21)

where \(|K, 0\rangle\) is a normalized vacuum \((K_- |K, 0\rangle = 0\) and \(\langle K, 0 |K, 0\rangle = 1\). We have written \(|K, 0\rangle\) instead of \(|0\rangle\) to emphasize the spin \(K\) representation, see [4]. We also denote by \(1_K\) the unit operator on \(\mathcal{H}_K\). From (21), states \(|K, n\rangle\) are given by

\[ |K, n\rangle = \frac{(K_+)^n}{\sqrt{n!(2K)_n}} |K, 0\rangle, \]
(22)

where \((a)_n\) is the Pochammer’s notation \((a)_n \equiv a(a + 1) \cdots (a + n - 1)\). These states satisfy the orthogonality and completeness conditions

\[ \langle K, m |K, n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |K, n\rangle \langle K, n| = 1_K. \]
(23)

Now let us consider a generalized version of coherent states:

**Definition** We call a state

\[ |w\rangle \equiv e^{wK_+ - \bar{w}K_-} |K, 0\rangle \quad \text{for} \quad w \in \mathbb{C}. \]
(24)

the generalized coherent state based on \(su(1, 1)\), [2].

We note that this is the extension of (7) not (5), see [3]. For this the following disentangling formula is well-known:

\[ e^{wK_+ - \bar{w}K_-} = e^{\zeta K_+ + \bar{\zeta} K_-} \quad \text{or} \]
\[ = e^{-\bar{\zeta} K_-} e^{-\log(1-|\zeta|^2)K_3} e^{\zeta K_+}. \]
(25)

where

\[ \zeta = \zeta(w) \equiv \frac{wtanh(|w|)}{|w|} \quad (\implies |\zeta| < 1). \]
(26)

This is the key formula for generalized coherent operators. Therefore from (21)

\[ |w\rangle = (1 - |\zeta|^2)^K e^{\zeta K_+} |K, 0\rangle \equiv |\zeta\rangle. \]
(27)

This corresponds to the right hand side of (6). Moreover since

\[ e^{\zeta K_+} |K, 0\rangle = \sum_{n=0}^{\infty} \frac{(2K)_n}{n!} \zeta^n |K, 0\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{(2K)_n}{n!}} \frac{\zeta^n K_+^n}{\sqrt{(2K)_n n!}} |K, 0\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{(2K)_n}{n!}} \zeta^n |K, n\rangle \]
(27)
we have
\[ |w\rangle = (1 - |\zeta|^2)^K \sum_{n=0}^{\infty} \sqrt{\frac{(2K)^n}{n!}} \zeta^n |K, n\rangle \equiv |\zeta\rangle. \quad (28) \]

This corresponds to the left hand side of (3). Then the resolution of unity corresponding to (9) is
\[ \int \frac{2K - 1}{\pi} \frac{\tanh(|w|)[d^2w]}{(1 - \tanh^2(|w|))|w|} |w\rangle\langle w| = \int \frac{2K - 1}{\pi} \frac{\sinh(2|w|)[d^2w]}{2|w|} |w\rangle\langle w| \]
\[ = \int_D \frac{2K - 1}{\pi} \frac{[d^2\zeta]}{(1 - |\zeta|^2)^2} |\zeta\rangle \langle \zeta| = \sum_{n=0}^{\infty} |K, n\rangle\langle K, n| = 1_K, \quad (29) \]
where \( C \to D : w \mapsto \zeta = \zeta(w) \) and \( D \) is the Poincare disk in \( C \), see (30).

Here let us construct an example of spin \( K \)-representations.

If we set
\[ K_+ \equiv \frac{1}{2} (a^\dagger)^2, \quad K_- \equiv \frac{1}{2} a^2, \quad K_3 \equiv \frac{1}{2} (a^\dagger a + \frac{1}{2}) \quad (30) \]
then we have

\[ [K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \quad (31) \]

That is, the set \( \{K_+, K_-, K_3\} \) gives a unitary representation of \( su(1, 1) \) with spin \( K = 1/4 \) and \( 3/4 \). Now we also call an operator
\[ S(w) = e^{\frac{1}{2} \{w(a^\dagger)^2 - \bar{w}a^2\}} \text{ for } w \in C \quad (32) \]
the squeezed operator.

### 3.2 Some Formulas

We make some preliminaries for the following section. For that we list some useful formulas on generalized coherent states based on \( su(1, 1) \). Since the proofs are not so difficult, we leave them to the readers.

**Formulas** For \( w_1, w_2 \) we have
\[ (1) \quad \langle w_1 | w_2 \rangle = \left\{ \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{1 - \zeta_1 \zeta_2} \right\}^K, \quad (33) \]
(ii) \[ \langle w_1|K_+|w_2 \rangle = \langle w_1|w_2 \rangle \frac{2K \bar{\zeta}_1}{1 - \zeta_1 \zeta_2}, \]  
(34)

(iii) \[ \langle w_1|K_-|w_2 \rangle = \langle w_1|w_2 \rangle \frac{2K \zeta_2}{1 - \zeta_1 \zeta_2}, \]  
(35)

(iv) \[ \langle w_1|K_-K_+|w_2 \rangle = \langle w_1|w_2 \rangle \frac{2K + 4K^2 \bar{\zeta}_1 \zeta_2}{(1 - \zeta_1 \zeta_2)^2}. \]  
(36)

where

\[ \zeta_j = \frac{w_j \tanh(|w_j|)}{|w_j|} \quad \text{for} \quad j = 1, 2. \]  
(37)

When \( w_1 = w_2 \equiv w \), then \( \langle w|w \rangle = 1 \), so we have

\[ \langle w|K_+|w \rangle = \frac{2K \bar{\zeta}}{1 - |\zeta|^2}, \quad \langle w|K_-|w \rangle \frac{2K \zeta}{1 - |\zeta|^2}, \]  
(38)

\[ \langle w|K_-K_+|w \rangle = \frac{2K + 4K^2 |\zeta|^2}{(1 - |\zeta|^2)^2}. \]  
(39)

Here let us make a comment. From (33)

\[ |\langle w_1|w_2 \rangle|^2 = \left\{ \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{|1 - \bar{\zeta}_1 \zeta_2|^2} \right\}^{2K}, \]  

so we want to know the property of

\[ \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{|1 - \bar{\zeta}_1 \zeta_2|^2}. \]

It is easy to see that

\[ 1 - \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{|1 - \bar{\zeta}_1 \zeta_2|^2} = \frac{|\zeta_1 - \zeta_2|^2}{|1 - \bar{\zeta}_1 \zeta_2|^2} \geq 0 \]  
(40)

and (40) = 0 if and only if (iff) \( \zeta_1 = \zeta_2 \). Therefore

\[ |\langle w_1|w_2 \rangle|^2 = \left\{ \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{|1 - \bar{\zeta}_1 \zeta_2|^2} \right\}^{2K} \leq 1 \]  
(41)

because \( 2K > 1 \) (\( 2K - 1 > 0 \) from (29)). Of course

\[ |\langle w_1|w_2 \rangle| = 1 \quad \text{iff} \quad \zeta_1 = \zeta_2 \quad \text{iff} \quad w_1 = w_2. \]  
(42)

by (37).
3.3 A Supplement

Before ending this section let us make a brief comment on generalized coherent states (24). The coherent states $|z\rangle$ has been defined by (5): $a|z\rangle = z|z\rangle$. Why do we define the generalized coherent states $|w\rangle$ as $K_-|w\rangle = w|w\rangle$ because $K_-$ is an annihilation operator corresponding to $a$? First let us try to calculate $K_-|w\rangle$ making use of (27).

$$K_-|w\rangle = (1 - |\zeta|^2)K_- e^{\zeta K_+}|K,0\rangle + (1 - |\zeta|^2)K e^{\zeta K_+} e^{-\zeta K_+}K_- e^{\zeta K_+}|K,0\rangle.$$  

Here it is easy to see

$$e^{-\zeta K_+} K_- e^{\zeta K_+} = \sum_{n=0}^{\infty} \frac{1}{n!} [\zeta K_+, [-\zeta K_+, [\cdot, \cdot, [-\zeta K_+, K_-] \cdot \cdot \cdot]]] = K_- + 2\zeta K_3 + \zeta^2 K_+,$$

from the relations (20), so that

$$K_-|w\rangle = (1 - |\zeta|^2)K_- e^{\zeta K_+}|K,0\rangle = (1 - |\zeta|^2)K e^{\zeta K_+} e^{-\zeta K_+}K_- e^{\zeta K_+}|K,0\rangle.$$  

This equation is completely different from (5).

3.4 Barut–Girardello Coherent States

Now let us make a brief comment on Barut–Girardello coherent states, [33]. The states $||w\rangle\rangle (w \in \mathbb{C})$ defined by

$$K_-||w\rangle\rangle = w||w\rangle\rangle$$  

(46)
are called the Barut–Girardello coherent states. This definition is a natural generalization of (3) because $K_-$ is an annihilation operator. In the preceding section we denoted by a capital letter $K$ a level of the representation of $su(1,1)$. But to avoid some confusion in this subsection we use a small letter $k$ instead of $K$.

The solution is easy to find and given by

$$||w|| = \sum_{n=0}^{\infty} \frac{w^n}{\sqrt{n!(2k)_n}} |k,n\rangle \quad (47)$$

top the normalization factor. Compare this with (28). Let us determine the inner product.

$$\langle\langle w||w'\rangle\rangle = \sum_{n=0}^{\infty} \frac{(\bar{w}w')^n}{n!(2k)_n} = \sum_{n=0}^{\infty} \frac{(\sqrt{\bar{w}w'})^{2n}}{n!(2k)_n}$$

Noting that

$$(2k)_n = \frac{\Gamma(2k+n)}{\Gamma(2k)} \implies \frac{1}{(2k)_n} = \frac{\Gamma(2k)}{\Gamma(2k+n)}$$

we have

$$\langle\langle w||w'\rangle\rangle = \Gamma(2k)(\sqrt{\bar{w}w'})^{-2k+1}I_{2k-1}(2\sqrt{\bar{w}w'})$$

where $I_{\nu}(z)$ is the modified Bessel function of the first kind:

$$I_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)}.$$  

Therefore

$$||w|| = \langle\langle w||w'\rangle\rangle^{1/2} = \left\{\Gamma(2k)|w|^{-2k+1}I_{2k-1}(2|w|)\right\}^{1/2}. \quad (48)$$

This gives the normalization factor of (17). Therefore the normalized solution of (16) corresponding to (3) is given by

$$||w|| = \left\{\Gamma(2k)|w|^{-2k+1}I_{2k-1}(2|w|)\right\}^{-1/2} \sum_{n=0}^{\infty} \frac{w^n}{\sqrt{n!(2k)_n}} |k,n\rangle. \quad (49)$$

Next we show the resolution of unity.

$$\int_{C} d\mu(\bar{w},w)||w||\langle\langle w|| = 1_k, \quad (50)$$

$$d\mu(\bar{w},w) = \frac{2K_{2k-1}(2|w|)}{\pi\Gamma(2k)} [d^2w]. \quad (51)$$
where $K_\nu(z)$ is the modified Bessel function whose integral representation is given by

$$K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(\nu + 1/2)} \left(\frac{z}{2}\right)^\nu \int_1^\infty dy e^{-zy}(y^2 - 1)^{\nu-1/2}, \quad \nu > -\frac{1}{2}.$$

The proof of (50) is not so easy, so we give the proof.

$$\int_C d\mu(\bar{w}, w)||w\rangle \langle w|| = \sum_{n=0}^\infty \sum_{m=0}^\infty \int_C d\mu(\bar{w}, w) \frac{\bar{w}^n w^m}{n!(2K)_n m!(2K)_m} |k, n\rangle \langle k, m|$$ (52)

where we have integrated on $\theta$ making use of $w = re^{i\theta}$. Here

$$\int_0^\infty d\mu(r)r^{2n} = \frac{4}{\Gamma(2k)} \int_0^\infty dr r^{2k+2n}K_{2k-1}(2r)$$

$$= \frac{4}{\Gamma(2k)} \frac{1}{4} \Gamma(\frac{2k+2n+1+2k-1}{2})\Gamma(\frac{2k+2n+1-2k+1}{2})$$

$$= \frac{\Gamma(2k+n)}{\Gamma(2k)} \Gamma(n+1) = (2k)_n n!,$$

where we have used the famous formula

$$\int_0^\infty dx x^{\mu-1}K_\nu(ax) = \frac{1}{4} \left(\frac{2}{a}\right) \Gamma(\frac{\mu + \nu}{2})\Gamma(\frac{\mu - \nu}{2}) \quad a > 0, \quad \text{Re}\mu > |\text{Re}\nu|.$$

For the proof see [34]; Appendix B. Therefore we have

$$\int_C d\mu(\bar{w}, w)||w\rangle \langle w|| = \sum_{n=0}^\infty \frac{1}{n!(2K)_n} |k, n\rangle \langle k, n| = \sum_{n=0}^\infty |k, n\rangle \langle k, n| = 1_k.$$

Their states have several interesting structures, but we don’t consider them in this paper.

See [34], [35] and [36] as for further developments and applications.

4 Generalized Coherent States Based on su(2)

In this section we introduce some basic properties of generalized coherent operators based on $su(2)$, see [4] or [3].
4.1 General Theory

We consider a spin \( J \) \((> 0)\) representation of \( su(2) \subset sl(2, \mathbb{C}) \) and set its generators \( \{J_+, J_-, J_3\} \) \((J_+)^\dagger = J_-\),

\[
[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3. \tag{53}
\]

We note that this (unitary) representation is necessarily finite dimensional. The Fock space on which \( \{J_+, J_-, J_3\} \) act is \( \mathcal{H}_J \equiv \{|J, m\rangle | 0 \leq m \leq 2J\} \) and whose actions are

\[
J_+|J, m\rangle = \sqrt{(m+1)(2J-m)}|J, m+1\rangle, \\
J_-|J, m\rangle = \sqrt{m(2J-m+1)}|J, m-1\rangle, \\
J_3|J, m\rangle = (-J+m)|J, m\rangle, \tag{54}
\]

where \( |J, 0\rangle \) is a normalized vacuum \((J_-|J, 0\rangle = 0 \text{ and } \langle J, 0|J, 0\rangle = 1)\). We have written \( |J, 0\rangle \) instead of \( |0\rangle \) to emphasize the spin \( J \) representation, see [4]. We also denote by \( 1_J \) the unit operator on \( \mathcal{H}_J \). From (21), states \( |J, m\rangle \) are given by

\[
|J, m\rangle = \frac{(J_+)^m}{\sqrt{m!2J_P_m}}|J, 0\rangle, \tag{55}
\]

where \( 2J_P_m = (2J)(2J-1)\cdots(2J-m+1) \). These states satisfy the orthogonality and completeness conditions

\[
\langle J, m|J, n\rangle = \delta_{mn}, \quad \sum_{m=0}^{2J}|J, m\rangle\langle J, m| = 1_J. \tag{56}
\]

Now let us consider a generalized version of coherent states:

**Definition** We call a state

\[
|v\rangle \equiv e^{vJ_+ - \bar{v}J_-}|J, 0\rangle \quad \text{for} \quad v \in \mathbb{C}. \tag{57}
\]

the generalized coherent state based on \( su(2) \), [32].

We note that this is the extension of (7) not (5), see [3]. For this the following disentangling formula is well-known:

\[
e^{vJ_+ - \bar{v}J_-} = e^{\eta J_+}e^{\log(1+|\eta|^2)J_3}e^{-\bar{\eta}J_-} \quad \text{or} \quad e^{-\bar{\eta}J_-}e^{-\log(1+|\eta|^2)J_3}e^{\eta J_+}. \tag{58}
\]
where
\[ \eta = \eta(v) \equiv \frac{v \tan(|v|)}{|v|}. \]  
(59)

This is the key formula for generalized coherent operators. Therefore from (54)
\[ |v\rangle = \frac{1}{(1 + |\eta|^2)^{J}} e^{\eta J_{+}} |J, 0\rangle \equiv |\eta\rangle. \]  
(60)

This corresponds to the right hand side of (6). Moreover since
\[ e^{\eta J_{+}} |J, 0\rangle = \sum_{m=0}^{\infty} \frac{\eta^{m}}{m!} J_{+}^{m} |J, 0\rangle = \sum_{m=0}^{2J} \sqrt{2J_{m}} \frac{\eta^{m} J_{m}}{m!} |J, m\rangle \]
\[ = \sum_{m=0}^{2J} \sqrt{2J_{m}} \eta^{m} |J, m\rangle \]
we have
\[ |v\rangle = \frac{1}{(1 + |\eta|^2)^{J}} \sum_{m=0}^{2J} \sqrt{2J_{m}} \eta^{m} |J, m\rangle \equiv |\eta\rangle. \]  
(62)

This corresponds to the left hand side of (6). Then the resolution of unity corresponding to (9) is
\[ \int_{C} \frac{2J + 1}{\pi} \frac{\tan(|v|)[d^2v]}{(1 + \tan^2(|v|) |v|)} |v\rangle \langle v| = \int_{C} \frac{2J + 1}{\pi} \frac{\sin(2|v|)[d^2v]}{2|v|} |v\rangle \langle v| = \int_{C} \frac{2J + 1}{\pi} \frac{[d^2\eta]}{(1 + |\eta|^2)^2} |\eta\rangle \langle \eta| = \sum_{m=0}^{2J} |J, m\rangle \langle J, m| = 1_{J}, \]  
(63)

where \( C \rightarrow C \subset \mathbb{C}P^1 : v \mapsto \eta = \eta(v) \), see [30].

4.2 Some Formulas

We make some preliminaries for the following section. For that we list some useful formulas on generalized coherent states based on \( su(2) \). Since the proofs are not so difficult, we leave them to the readers.

Formulas For \( v_{1}, v_{2} \) we have
\[ \langle v_{1}|v_{2}\rangle = \left\{ \frac{(1 + \bar{\eta}_{1}\eta_{2})^{2}}{(1 + |\eta_{1}|^2)(1 + |\eta_{2}|^2)} \right\}^{J}, \]  
(64)
\[ \langle v_{1}|J_{+}|v_{2}\rangle = \langle v_{1}|v_{2}\rangle \frac{2J\bar{\eta}_{1}}{1 + \bar{\eta}_{1}\eta_{2}}, \]  
(65)
(iii) \[ \langle v_1|J_-|v_2 \rangle = \langle v_1|v_2 \rangle \frac{2J\eta_2}{1 + \bar{\eta}_1\eta_2}, \] 

(66)

(iv) \[ \langle v_1|J_-J_+|v_2 \rangle = \langle v_1|v_2 \rangle \frac{2J + 4J^2\bar{\eta}_1\eta_2}{(1 + \bar{\eta}_1\eta_2)^2}. \] 

(67)

where

\[ \eta_j = \frac{v_j\tan(|v_j|)}{|v_j|} \text{ for } j = 1, 2. \] 

(68)

When \( v_1 = v_2 \equiv v \), then \( \langle v|v \rangle = 1 \), so we have

\[ \langle v|J_+|v \rangle = \frac{2J\bar{\eta}}{1 + |\eta|^2}, \quad \langle v|J_-|v \rangle = \frac{2J\eta}{1 + |\eta|^2}, \] 

(69)

\[ \langle v|J_-J_+|v \rangle = \frac{2J + 4J^2|\eta|^2}{(1 + |\eta|^2)^2}. \] 

(70)

Here let us make a comment. From (64)

\[ |\langle v_1|v_2 \rangle|^2 = \left\{ \frac{|1 + \bar{\eta}_1\eta_2|^2}{(1 + |\eta_1|^2)(1 + |\eta_2|^2)} \right\}^{2J}, \] 

so we want to know the property of

\[ \frac{|1 + \bar{\eta}_1\eta_2|^2}{(1 + |\eta_1|^2)(1 + |\eta_2|^2)}. \]

It is easy to see that

\[ 1 - \frac{|1 + \bar{\eta}_1\eta_2|^2}{(1 + |\eta_1|^2)(1 + |\eta_2|^2)} = \frac{|\eta_1 - \eta_2|^2}{(1 + |\eta_1|^2)(1 + |\eta_2|^2)} \geq 0 \] 

(71)

and (71) = 0 if and only if (iff) \( \eta_1 = \eta_2 \). Therefore

\[ |\langle v_1|v_2 \rangle|^2 = \left\{ \frac{|1 + \bar{\eta}_1\eta_2|^2}{(1 + |\eta_1|^2)(1 + |\eta_2|^2)} \right\}^{2J} \leq 1 \] 

(72)

because \( 2J > 1 \) (from (63)). Of course

\[ |\langle v_1|v_2 \rangle| = 1 \text{ iff } \eta_1 = \eta_2 \text{ iff } v_1 = v_2. \] 

(73)

by (68).
4.3 A Supplement

Before ending this section let us make a brief comment on generalized coherent states \((\text{(57)})\). The coherent states \(|z\rangle\) has been defined by \((\text{(5)})\) : \(a|z\rangle = z|z\rangle\). Why do we define the generalized coherent states \(|w\rangle\) as \(J_-|v\rangle = v|v\rangle\) because \(J_-\) is an annihilation operator corresponding to \(a\)? First let us try to calculate \(J_-|v\rangle\) making use of \((\text{60})\).

\[
J_-|v\rangle = (1 + |\eta|^2)^{-J}e^{\eta J_+}|J,0\rangle = (1 + |\eta|^2)^{-J}e^{\eta J_+}e^{-\eta J_+}J_-e^{\eta J_+}|J,0\rangle.
\]

Here it is easy to see

\[
e^{-\eta J_+}J_-e^{\eta J_+} = \sum_{m=0}^{\infty} \frac{1}{m!}[-\eta J_+,[[-\eta J_+,[\cdots,[-\eta J_+,J_-]\cdots]]]
\]

from the relations \((\text{53})\), so that

\[
J_-|v\rangle = (1 + |\eta|^2)^{-J}e^{\eta J_+}(J_- - 2\eta J_3 - \eta^2 J_+)|J,0\rangle
\]

because \(J_-|J,0\rangle = 0\) and \(J_3|J,0\rangle = -J|J,0\rangle\). Namely we have the equation

\[
(J_- + \eta^2 J_+)|v\rangle = 2J\eta|v\rangle, \quad \text{where} \quad \eta = \frac{\nu\tan(|\nu|)}{|\nu|}.
\]

or more symmetrically

\[
(\eta^{-1}J_- + \eta J_+)|v\rangle = 2J|v\rangle, \quad \text{where} \quad \eta = \frac{\nu\tan(|\nu|)}{|\nu|}.
\]

This equation is completely different from \((\text{5})\).

5 Schwinger’s Boson Method

Here let us construct the spin \(J\) and \(K\)–representations making use of Schwinger’s boson method.
We consider the system of two-harmonic oscillators. If we set

\[ a_1 = a \otimes 1, \ a_1^{\dagger} = a^{\dagger} \otimes 1; \ a_2 = 1 \otimes a, \ a_2^{\dagger} = 1 \otimes a^{\dagger}, \] (77)

then it is easy to see

\[ [a_i, a_j] = [a_i^{\dagger}, a_j^{\dagger}] = 0, \ [a_i, a_j^{\dagger}] = \delta_{ij}, \ i, j = 1, 2. \] (78)

We also denote by \( N_i = a_i^{\dagger} a_i \) number operators.

Now we can construct representation of Lie algebras \( su(2) \) and \( su(1,1) \) by making use of Schwinger’s boson method, see [4], [5]. Namely if we set

\[ su(2) : \ J_+ = a_1^{\dagger} a_2, \ J_- = a_2^{\dagger} a_1, \ J_3 = \frac{1}{2} (a_1^{\dagger} a_1 - a_2^{\dagger} a_2), \] (79)
\[ su(1,1) : \ K_+ = a_1^{\dagger} a_2^{\dagger}, \ K_- = a_2 a_1, \ K_3 = \frac{1}{2} (a_1^{\dagger} a_1 + a_2^{\dagger} a_2 + 1), \] (80)

then we have

\[ su(2) : \ [J_3, J_+] = J_+, \ [J_3, J_-] = -J_-, \ [J_+, J_-] = 2J_3, \] (81)
\[ su(1,1) : \ [K_3, K_+] = K_+, \ [K_3, K_-] = -K_-, \ [K_+, K_-] = -2K_3. \] (82)

In the following we define (unitary) generalized coherent operators based on Lie algebras \( su(2) \) and \( su(1,1) \).

**Definition** We set

\[ su(2) : \ U_J(v) = e^{vJ_+ - \bar{v}J_-} \text{ for } v \in \mathbb{C}, \] (83)
\[ su(1,1) : \ U_K(w) = e^{wK_+ - \bar{w}K_-} \text{ for } w \in \mathbb{C}. \] (84)

For the latter convenience let us list well-known disentangling formulas once more. We have

\[ su(2) : \ U_J(v) = e^{vJ_+} e^{\log(1+|v|^2)J_3} e^{-\bar{v}J_-}, \text{ where } \eta = \frac{v \tan(|v|)}{|v|}, \] (85)
\[ su(1,1) : \ U_K(w) = e^{\zeta K_+} e^{\log(1-|\zeta|^2)K_3} e^{-\bar{\zeta}K_-}, \text{ where } \zeta = \frac{w \tanh(|w|)}{|w|}. \] (86)

As for a generalization of these formulas see [32].
Now let us make some mathematical preliminaries for the latter sections. We have easily

$$U_J(t)a_1U_J(t)^{-1} = \cos(|t|)a_1 - \frac{tsin(|t|)}{|t|}a_2,$$  \hspace{1cm} (87)

$$U_J(t)a_2U_J(t)^{-1} = \cos(|t|)a_1 + \frac{\bar{tsin}(|t|)}{|t|}a_2,$$  \hspace{1cm} (88)

so the map \((a_1, a_2) \longrightarrow (U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1})\) is

$$(U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1}) = (a_1, a_2) \begin{pmatrix} \cos(|t|) & \bar{tsin}(|t|) \\ -\frac{tsin(|t|)}{|t|} & \cos(|t|) \end{pmatrix}.$$

We note that

$$\begin{pmatrix} \cos(|t|) & \bar{tsin}(|t|) \\ -\frac{tsin(|t|)}{|t|} & \cos(|t|) \end{pmatrix} \in SU(2).$$

On the other hand we have easily

$$U_K(t)a_1U_K(t)^{-1} = \cosh(|t|)a_1 - \frac{tsinh(|t|)}{|t|}a_\dagger_2,$$  \hspace{1cm} (89)

$$U_K(t)a_\dagger_2U_K(t)^{-1} = \cosh(|t|)a_\dagger_2 - \frac{\bar{tsinh}(|t|)}{|t|}a_1,$$  \hspace{1cm} (90)

so the map \((a_1, a_\dagger_2) \longrightarrow (U_K(t)a_1U_K(t)^{-1}, U_K(t)a_\dagger_2U_K(t)^{-1})\) is

$$(U_K(t)a_1U_K(t)^{-1}, U_K(t)a_\dagger_2U_K(t)^{-1}) = (a_1, a_\dagger_2) \begin{pmatrix} \cosh(|t|) & -\bar{tsinh}(|t|) \\ -\frac{tsinh(|t|)}{|t|} & \cosh(|t|) \end{pmatrix}.$$

We note that

$$\begin{pmatrix} \cosh(|t|) & -\bar{tsinh}(|t|) \\ -\frac{tsinh(|t|)}{|t|} & \cosh(|t|) \end{pmatrix} \in SU(1, 1).$$

Before ending this section let us ask a question.

What is a relation between \((84)\) and \((32)\) of generalized coherent operators based on \(su(1,1)\)?

The answer is given by :

**Formula** We have

$$U_J(-\frac{\pi}{4})S_1(w)S_2(-w)U_J(-\frac{\pi}{4})^{-1} = U_K(w),$$  \hspace{1cm} (91)
where \( S_j(w) = \{32\} \) with \( a_j \) instead of \( a \), see \[25\].

Namely, \( U_K(w) \) is given by “rotating” the product \( S_1(w)S_2(-w) \) by \( U_J(-\frac{\pi}{4}) \).

**Proof** It is easy to see

\[
U_J(t)S_1(w)S_2(-w)U_J(t)^{-1} = U_J(t)e^{\frac{\pi}{2}(a_1^2-\alpha^2)-(a_2^2-\beta^2)}U_J(t)^{-1} = e^{X} \quad (92)
\]

where

\[
X = \frac{w}{2}\left\{ (U_J(t)a_1^\dagger U_J(t)^{-1})^2 - (U_J(t)a_2^\dagger U_J(t)^{-1})^2 \right\} - \bar{w}\left\{ (U_J(t)a_1 U_J(t)^{-1})^2 - (U_J(t)a_2 U_J(t)^{-1})^2 \right\}. \quad (93)
\]

From (87) and (88) we have

\[
X = \frac{w}{2}\left\{ \left( \cos^2(|t|) - \frac{t^2 \sin^2(|t|)}{|t|^2} \right) (a_1^\dagger)^2 - \left( \cos^2(|t|) - \frac{t^2 \sin^2(|t|)}{|t|^2} \right) (a_2^\dagger)^2 - \frac{(t + \bar{t}) \sin(2|t|)}{|t|} a_1^\dagger a_2^\dagger \right\} - \bar{w}\left\{ \left( \cos^2(|t|) - \frac{t^2 \sin^2(|t|)}{|t|^2} \right) a_1^2 - \left( \cos^2(|t|) - \frac{t^2 \sin^2(|t|)}{|t|^2} \right) a_2^2 - \frac{(t + \bar{t}) \sin(2|t|)}{|t|} a_1 a_2 \right\}. \quad (94)
\]

Here we set \( t = -\frac{\pi}{4} \), then

\[
X = \frac{w}{2}(2a_1^\dagger a_2^\dagger) - \bar{w}(2a_1 a_2) = wa_1^\dagger a_2^\dagger - \bar{w} a_1 a_2 \implies e^{X} = U_K(w).
\]

Namely, we obtain the formula.

Next let us prove the following

**Formula**

\[
U_J(t)S_1(\alpha)S_2(\beta)U_J(t)^{-1} = U_J(t)e^{\frac{\alpha}{2}(a_1^2-\alpha^2)+\frac{\beta}{2}(a_2^2-\beta^2)}U_J(t)^{-1} = e^{X} \quad (95)
\]

where

\[
X = \frac{\alpha}{2}(U_J(t)a_1^\dagger U_J(t)^{-1})^2 - \frac{\bar{\alpha}}{2}(U_J(t)a_1 U_J(t)^{-1})^2 + \frac{\beta}{2}(U_J(t)a_2^\dagger U_J(t)^{-1})^2 - \frac{\bar{\beta}}{2}(U_J(t)a_2 U_J(t)^{-1})^2.
\]

21
From (87) and (88) we have
\[
X = \frac{1}{2} \left\{ \cos^2(|t|) \alpha + \frac{t^2 \sin^2(|t|)}{|t|^2} \beta \right\} (a_1^\dagger)^2 - \frac{1}{2} \left\{ \cos^2(|t|) \bar{\alpha} + \frac{\bar{t}^2 \sin^2(|t|)}{|t|^2} \bar{\beta} \right\} a_1^2
\]
\[
+ \frac{1}{2} \left\{ \cos^2(|t|) \beta + \frac{t^2 \sin^2(|t|)}{|t|^2} \alpha \right\} (a_2^\dagger)^2 - \frac{1}{2} \left\{ \cos^2(|t|) \bar{\beta} + \frac{\bar{t}^2 \sin^2(|t|)}{|t|^2} \bar{\alpha} \right\} a_2^2
\]
\[
+ (\beta t - \alpha \bar{t}) \frac{\sin(2|t|)}{2|t|} a_1^\dagger a_2^\dagger - (\bar{\beta} \bar{t} - \bar{\alpha} t) \frac{\sin(2|t|)}{2|t|} a_1 a_2.
\] (96)

If we set
\[
\beta t - \alpha \bar{t} = 0 \iff \beta t = \alpha \bar{t},
\] (97)
then it is easy to check
\[
\cos^2(|t|) \alpha + \frac{t^2 \sin^2(|t|)}{|t|^2} \beta = \alpha, \quad \cos^2(|t|) \beta + \frac{\bar{t}^2 \sin^2(|t|)}{|t|^2} \alpha = \beta,
\]
so, in this case,
\[
X = \frac{1}{2} \alpha (a_1^\dagger)^2 - \frac{1}{2} \bar{\alpha} a_1^2 + \frac{1}{2} \beta (a_2^\dagger)^2 - \frac{1}{2} \bar{\beta} a_2^2.
\]

Therefore
\[
U_J(t) S_1(\alpha) S_2(\beta) U_J(t)^{-1} = S_1(\alpha) S_2(\beta).
\] (98)

That is, \(S_1(\alpha) S_2(\beta)\) commutes with \(U_J(t)\) under the condition (97). We use this formula in the following.

### 6 Universal Bundles and Chern Characters

In this section we introduce some basic properties of pull–backed ones of universal bundles over the infinite–dimensional Grassmann manifolds and Chern characters, see [9].

Let \(\mathcal{H}\) be a separable Hilbert space over \(\mathbb{C}\). For \(m \in \mathbb{N}\), we set
\[
St_m(\mathcal{H}) \equiv \left\{ V = (v_1, \cdots, v_m) \in \mathcal{H} \times \cdots \times \mathcal{H} \mid V^\dagger V \in GL(m; \mathbb{C}) \right\}.
\] (99)
This is called a (universal) Stiefel manifold. Note that the unitary group \(U(m)\) acts on \(St_m(\mathcal{H})\) from the right:
\[
St_m(\mathcal{H}) \times U(m) \longrightarrow St_m(\mathcal{H}) : (V, a) \longmapsto V a.
\] (100)
Next we define a (universal) Grassmann manifold

\[ \text{Gr}_m(\mathcal{H}) \equiv \left\{ X \in M(\mathcal{H}) \mid X^2 = X, X^\dagger = X \text{ and } \text{tr}X = m \right\} , \quad (101) \]

where \( M(\mathcal{H}) \) denotes a space of all bounded linear operators on \( \mathcal{H} \). Then we have a projection

\[ \pi : St_m(\mathcal{H}) \longrightarrow \text{Gr}_m(\mathcal{H}) , \quad \pi(V) \equiv V(V^\dagger V)^{-1}V^\dagger , \]

compatible with the action \( (100) \) \( \pi(V a) = V a^{-1}(V^\dagger V)^{-1}a(V a)^\dagger = \pi(V) \).

Now the set

\[ \{U(m), St_m(\mathcal{H}), \pi, \text{Gr}_m(\mathcal{H})\} , \quad (103) \]

is called a (universal) principal \( U(m) \) bundle, see \[9\] and \[14\]. We set

\[ E_m(\mathcal{H}) \equiv \{(X, v) \in \text{Gr}_m(\mathcal{H}) \times \mathcal{H} \mid Xv = v\} . \quad (104) \]

Then we have also a projection

\[ \pi : E_m(\mathcal{H}) \longrightarrow \text{Gr}_m(\mathcal{H}) , \quad \pi((X, v)) \equiv X . \quad (105) \]

The set

\[ \{C^m, E_m(\mathcal{H}), \pi, \text{Gr}_m(\mathcal{H})\} , \quad (106) \]

is called a (universal) \( m \)-th vector bundle. This vector bundle is one associated with the principal \( U(m) \) bundle \( (103) \).

Next let \( \mathcal{M} \) be a finite or infinite dimensional differentiable manifold and the map

\[ P : \mathcal{M} \longrightarrow \text{Gr}_m(\mathcal{H}) \]

be given (called a projector). Using this \( P \) we can make the bundles \( (103) \) and \( (106) \) pullback over \( \mathcal{M} : \)

\[ \{U(m), St_m(\mathcal{H}), \pi, \text{Gr}_m(\mathcal{H})\} \equiv P^* \{U(m), St_m(\mathcal{H}), \pi, \text{Gr}_m(\mathcal{H})\} , \quad (108) \]

\[ \{C^m, E_m(\mathcal{H}), \pi, \text{Gr}_m(\mathcal{H})\} \equiv P^* \{C^m, E_m(\mathcal{H}), \pi, \text{Gr}_m(\mathcal{H})\} , \quad (109) \]
see [3]. (109) is of course a vector bundle associated with (108).

For this bundle the (global) curvature (2–) form \( \Omega \) is given by

\[
\Omega = P dP \wedge dP
\]  

(110)

making use of (107), where \( d \) is the usual differential form on \( \Omega \). For the bundles Chern characters play an essential role in several geometric properties. In this case Chern characters are given by

\[ \Omega, \Omega^2, \cdots, \Omega^{m/2}; \quad \Omega^2 = \Omega \wedge \Omega, \text{ etc.} \]  

(111)

where we have assumed that \( m = \dim \mathcal{M} \) is even. In this paper we don’t take the trace of (111), so it may be better to call them densities for Chern characters.

To calculate these quantities in infinite–dimensional cases is not so easy. In the next section let us calculate these ones in the special cases.

Let us now define our projectors for the latter aim. In the following, for \( \mathcal{H} \) we treat \( \mathcal{H} = \mathcal{H} \) in section 2, \( \mathcal{H} = \mathcal{H}_K \) in section 3 and \( \mathcal{H} = \mathcal{H}_J \) in section 4 at the same time. For \( u_1, u_2, \cdots, u_m \in \mathbb{C} \) we consider a set of coherent or generalized coherent states \( \{ |u_1\rangle, |u_2\rangle, \cdots, |u_m\rangle \} \) and set

\[
V_m(\mathbf{u}) = (|u_1\rangle, |u_2\rangle, \cdots, |u_m\rangle) \equiv V_m
\]  

(112)

where \( \mathbf{u} = (u_1, u_2, \cdots, u_m) \). Since \( V_m^\dagger V_m = \langle u_k | u_j \rangle \in M(m, \mathbb{C}) \), we define

\[
\mathcal{D}_m \equiv \{ \mathbf{u} \in \mathbb{C}^m \mid \det(V_m^\dagger V_m) \neq 0 \}.
\]  

(113)
We note that $D_m$ is an open set in $\mathbb{C}^m$. For example, for $m = 1$ and $m = 2$

$$V_1^\dagger V_1 = 1,$$

$$\det(V_2^\dagger V_2) = \begin{vmatrix} 1 & a \\ \bar{a} & 1 \end{vmatrix} = 1 - |a|^2 \geq 0,$$

where $a = \langle u_1 | u_2 \rangle$. So from (10), (42) and (73) we have

$$D_1 = \{ u \in \mathbb{C} \mid \text{no conditions} \} = \mathbb{C}, \quad (114)$$

$$D_2 = \{ (u_1, u_2) \in \mathbb{C}^2 \mid u_1 \neq u_2 \}. \quad (115)$$

For $D_m$ ($m \geq 3$) it is not easy for us to give a simple condition like (115).

**Problem** For the case $m = 3$ make the condition (113) clear like (115).

At any rate $V_m \in \text{St}_m(\mathcal{H})$ for $u \in D_m$. Now let us define our projector $P$ as follows:

$$P : D_m \longrightarrow \text{Gr}_m(\mathcal{H}), \quad P(u) = V_m(V_m^\dagger V_m)^{-1}V_m^\dagger. \quad (116)$$

In the following we set $V = V_m$ for simplicity. Let us calculate (110). Since

$$dP = V(V^\dagger V)^{-1}dV\{1 - V(V^\dagger V)^{-1}V^\dagger\} + \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV(V^\dagger V)^{-1}V^\dagger \quad (117)$$

where $d = \sum_{j=1}^m \left( du_j \frac{\partial}{\partial u_j} + d\bar{u}_j \frac{\partial}{\partial \bar{u}_j} \right)$, we have

$$PdP = V(V^\dagger V)^{-1}dV\{1 - V(V^\dagger V)^{-1}V^\dagger\}$$

after some calculation. Therefore we obtain

$$PdP \wedge dP = V(V^\dagger V)^{-1}[dV\{1 - V(V^\dagger V)^{-1}V^\dagger\}dV](V^\dagger V)^{-1}V^\dagger. \quad (118)$$

Our main calculation is $dV\{1 - V(V^\dagger V)^{-1}V^\dagger\}dV$, which is rewritten as

$$dV^\dagger\{1 - V(V^\dagger V)^{-1}V^\dagger\}dV = [\{1 - V(V^\dagger V)^{-1}V^\dagger\}dV]^\dagger [\{1 - V(V^\dagger V)^{-1}V^\dagger\}dV] \quad (119)$$

since $Q \equiv 1 - V(V^\dagger V)^{-1}V^\dagger$ is also a projector ($Q^2 = Q$ and $Q^\dagger = Q$). Therefore the first step for us is to calculate the term

$$\{1 - V(V^\dagger V)^{-1}V^\dagger\}dV. \quad (120)$$
Let us summarize our process of calculations:

1–st step \( \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV \cdots (120) \),

2–nd step \( dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV \cdots (119) \),

3–rd step \( V(V^\dagger V)^{-1}[dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV](V^\dagger V)^{-1}V^\dagger \cdots (118) \).

7 Calculations of Curvature Forms

In this section we only calculate the curvature forms \((m = 1)\). The calculations even for the case \(m = 2\) are complicated enough, see [7] and [8]. For \(m \geq 3\) calculations may become miserable.

7.1 Coherent States

In this case \( \langle z|z \rangle = 1 \), so our projector is very simple to be

\[ P(z) = |z\rangle\langle z| . \] (121)

In this case the calculation of curvature is relatively simple. From (118) we have

\[ PdP \wedge dP = |z\rangle\{d\langle z|(1 - |z\rangle\langle z|)d|z\rangle\}\langle z| = |z\rangle\langle z|\{d\langle z|(1 - |z\rangle\langle z|)d|z\rangle\}. \] (122)

Since \( |z\rangle = \exp(-\frac{1}{2}|z|^2)\exp(za^\dagger)|0\rangle \) by (1),

\[ d|z\rangle = \left\{a^\dagger - \frac{z}{2}\right\}d\bar{z} - \frac{z}{2}d\bar{z} \right\}|z\rangle = \left\{a^\dagger d\bar{z} - \frac{1}{2}(\bar{z}d\bar{z} + zd\bar{z}) \right\}|z\rangle = \left\{a^\dagger d\bar{z} - \frac{1}{2}d(|z|^2) \right\}|z\rangle , \]

so that

\[ (1 - |z\rangle\langle z|)d|z\rangle = (1 - |z\rangle\langle z|)a^\dagger|z\rangle d\bar{z} = (a^\dagger - \langle z|a^\dagger|z\rangle)|z\rangle d\bar{z} = (a - z)^\dagger d\bar{z}|z\rangle \]

because \( (1 - |z\rangle\langle z|)|z\rangle = 0 \). Similarly \( d\langle z|(1 - |z\rangle\langle z|) = \langle z|(a - z)d\bar{z} \).

Let us summarize:

\[ (1 - |z\rangle\langle z|)d|z\rangle = (a - z)^\dagger d\bar{z}|z\rangle , \quad d\langle z|(1 - |z\rangle\langle z|) = \langle z|(a - z)d\bar{z} . \] (123)
Now we are in a position to determine the curvature form (122).

\[
d\langle z| (1 - |z\rangle|z)\rangle d|z\rangle = \langle z| (a - z)(a - \bar{z})^\dagger |z\rangle d\bar{z} \wedge dz = d\bar{z} \wedge dz
\]

after some algebra. Therefore

\[
\Omega = P dP \wedge dP = |z\rangle \langle z| d\bar{z} \wedge dz.
\] (124)

From this result we know

\[
\frac{\Omega}{2\pi i} = |z\rangle \langle z| \frac{dx \wedge dy}{\pi}
\]

when \(z = x + iy\). This just gives the resolution of unity in (11).

### 7.2 Generalized Coherent States Based on \(su(1, 1)\)

In this case \(\langle w|w\rangle = 1\), so our projector is very simple to be

\[
P(w) = |w\rangle \langle w|.
\] (125)

In this case the calculation of curvature is relatively simple. From (118) we have

\[
P dP \wedge dP = |w\rangle \{d\langle w| (1_K - |w\rangle \langle w|) d|w\rangle\} \langle w| = |w\rangle \langle w| \{d\langle w| (1_K - |w\rangle \langle w|) d|w\rangle\},
\] (126)

where \(d = dw \frac{\partial}{\partial w} + d\bar{w} \frac{\partial}{\partial \bar{w}}\). Since \(|w\rangle = (1 - |\zeta|^2)^K \exp(\zeta K_+)|K, 0\rangle\) by (27),

\[
d|w\rangle = \left\{d\zeta K_+ + K \log(1 - |\zeta|^2)\right\} |w\rangle
\] (127)

by some calculation, so that

\[
(1_K - |w\rangle \langle w|) d|w\rangle = (1_K - |w\rangle \langle w|) K_+ |w\rangle d\zeta = (K_+ - \langle w|K_+|w\rangle) |w\rangle d\zeta = \left(K_+ - \frac{2K\zeta}{1 - |\zeta|^2}\right) d\zeta |w\rangle
\] (128)

because \((1_K - |w\rangle \langle w|) |w\rangle = 0\). Similarly we have

\[
d\langle w| (1_K - |w\rangle \langle w|) = \langle w| \left(K_- - \frac{2K\zeta}{1 - |\zeta|^2}\right) d\bar{\zeta}
\] (129)
Now we are in a position to determine the curvature form (126).

\[
\frac{d}{d\langle w|} \left( 1 - \langle w|w \rangle \right) \langle w|w \rangle \left( K_+ - \frac{2K\zeta}{1 - |\zeta|^2} \right) \langle w|d\zeta \wedge d\zeta \\
= \left\{ \langle w|K_- K_+ |w \rangle - \frac{2K\zeta}{1 - |\zeta|^2} \langle w|K_+ |w \rangle - \frac{2K\zeta}{1 - |\zeta|^2} \langle w|K_- |w \rangle + \frac{4K^2|\zeta|^2}{(1 - |\zeta|^2)^2} \right\} d\zeta \wedge d\zeta
\]

(130)
after some algebra with (38) and (39). Therefore

\[
\Omega = \frac{dP \wedge dP}{\langle w|w \rangle} = \frac{2Kd\zeta \wedge d\zeta}{(1 - |\zeta|^2)^2}.
\]

(131)

From this result we know

\[
\frac{\Omega}{2\pi i} = \frac{2Kd\zeta_1 \wedge d\zeta_2}{\pi (1 - |\zeta|^2)^2} \langle w|w \rangle = \frac{2Kd\zeta_1 \wedge d\zeta_2}{\pi (1 - |\zeta|^2)^2} |\zeta\rangle \langle \zeta|
\]

by (27) when \( \zeta = \zeta_1 + \sqrt{-1}\zeta_2 \). If we define a constant

\[
C_K = \frac{2K - 1}{2K},
\]

(132)
then we have

\[
C_K \frac{\Omega}{2\pi i} = \frac{2K - 1}{\pi} \frac{d\zeta_1 \wedge d\zeta_2}{(1 - |\zeta|^2)^2} |\zeta\rangle \langle \zeta|
\]

(133)
This gives the resolution of unity in (29). But the situation is a bit different from [7] in which the constant corresponding to \( C_K \) is just one.

**Problem** What is a (deep) meaning of \( C_K \)?

### 7.3 Generalized Coherent States Based on \( su(2) \)

In this case \( \langle v|v \rangle = 1 \), so our projector is very simple to be

\[
P(v) = |v\rangle \langle v|.
\]

(134)
In this case the calculation of curvature is relatively simple. From (118) we have

\[
PdP \wedge dP = |v\rangle \{ d\langle v|(1J - |v\rangle \langle v|)d|v\rangle \} \langle v| = |v\rangle \langle v| \{ d\langle v|(1J - |v\rangle \langle v|)d|v\rangle \},
\]

(135)
where $d = dv \frac{\partial}{\partial v} + dv \frac{\partial}{\partial \bar{v}}$. Since $|v\rangle = (1 + |\eta|^2)^{-1} \exp(\eta J_+) |J,0\rangle$ by (60),

$$d|v\rangle = \left\{ d\eta J_+ - J d\log(1 + |\eta|^2) \right\} |v\rangle$$

(136)

by some calculation, so that

$$(1 - |v\rangle\langle v|)d|v\rangle = (1 - |v\rangle\langle v|)J_+ |v\rangle d\eta = (J_+ - \langle v|J_+ |v\rangle) |v\rangle d\eta$$

(137)

because $(1 - |v\rangle\langle v|) |v\rangle = 0$. Similarly we have

$$d\langle v| (1 - |v\rangle\langle v|) = \langle v| \left( J_- - \frac{2J\bar{\eta}}{1 + |\eta|^2} \right) d\bar{\eta}$$

(138)

Now we are in a position to determine the curvature form (135).

$$d\langle v| (1 - |v\rangle\langle v|)d|v\rangle$$

$$= \langle v| \left( J_- - \frac{2J\eta}{1 + |\eta|^2} \right) \left( J_+ - \frac{2J\bar{\eta}}{1 + |\eta|^2} \right) |v\rangle d\bar{\eta} \wedge d\eta$$

$$= \left\{ \langle v|J_- J_+|v\rangle - \frac{2J\bar{\eta}}{1 + |\eta|^2} \langle v|J_+|v\rangle \right\} d\bar{\eta} \wedge d\eta$$

$$= \frac{2J}{(1 + |\eta|^2)^2} \bar{\eta} \wedge d\eta$$

(139)

after some algebra with (69) and (70). Therefore

$$\Omega = P \wedge dP = |v\rangle \langle v| \left( \frac{2Jd\bar{\eta} \wedge d\eta}{1 + |\eta|^2} \right).$$

(140)

From this result we know

$$\frac{\Omega}{2\pi i} = \frac{2J}{\pi} \frac{d\eta_1 \wedge d\eta_2}{(1 + |\eta|^2)^2} |v\rangle \langle v| = \frac{2J}{\pi} \frac{d\eta_1 \wedge d\eta_2}{(1 + |\eta|^2)^2} |\eta\rangle \langle \eta|$$

by (60) when $\eta = \eta_1 + \sqrt{-1} \eta_2$. If we define a constant

$$C_J = \frac{2J + 1}{2J},$$

(141)

then we have

$$C_J \frac{\Omega}{2\pi i} = \frac{2J + 1}{\pi} \frac{d\eta_1 \wedge d\eta_2}{(1 + |\eta|^2)^2} |\eta\rangle \langle \eta|.$$
**Problem** What is a (deep) meaning of $C_J$?

### 8 Holonomic Quantum Computation

In this section we introduce the concept of Holonomic Quantum Computation, see [19].

Let $\mathcal{M}$ be a parameter space and we denote by $\lambda$ its element. Let $\lambda_0$ be a fixed reference point of $\mathcal{M}$. Let $H_{\lambda}$ be a family of Hamiltonians parameterized by $\mathcal{M}$ which act on a Fock space $\mathcal{H}$. We set $H_0 = H_{\lambda_0}$ for simplicity and assume that this has a $m$-fold degenerate vacuum:

$$H_0 v_j = 0, \quad j = 1 \sim m. \tag{143}$$

These $v_j$’s form a $m$-dimensional vector space. We may assume that $\langle v_i | v_j \rangle = \delta_{ij}$. Then $(v_1, \cdots, v_m) \in St_m(\mathcal{H})$ and

$$F_0 \equiv \left\{ \sum_{j=1}^m x_j v_j | x_j \in \mathbb{C} \right\} \cong \mathbb{C}^m. \tag{143}$$

Namely, $F_0$ is a vector space associated with o.n.basis $(v_1, \cdots, v_m)$.

Next we assume for simplicity that a family of unitary operators parameterized by $\mathcal{M}$

$$W : \mathcal{M} \rightarrow U(\mathcal{H}), \quad W(\lambda_0) = \text{id}. \tag{144}$$

is given and $H_\lambda$ above is given by the following isospectral family

$$H_\lambda \equiv W(\lambda) H_0 W(\lambda)^{-1}. \tag{145}$$

In this case there is no level crossing of eigenvalues. Making use of $W(\lambda)$ we can define a projector

$$P : \mathcal{M} \rightarrow Gr_m(\mathcal{H}), \quad P(\lambda) \equiv W(\lambda) \left( \sum_{j=1}^m v_j v_j^\dagger \right) W(\lambda)^{-1} \tag{146}$$

and have the pullback bundles over $\mathcal{M}$ from (108) and (109)

$$\left\{ U(m), \tilde{\mathcal{S}}t, \pi_{\tilde{\mathcal{S}}t}, \mathcal{M} \right\}, \quad \left\{ \mathbb{C}^m, \tilde{E}, \pi_{\tilde{E}}, \mathcal{M} \right\}. \tag{147}$$
For the latter we set

\[ |\text{vac}⟩ = (v_1, \cdots, v_m). \]  
(148)

In this case a canonical connection form \( \mathcal{A} \) of \( \{U(m), \widetilde{St}, \pi_{\widetilde{St}}, \mathcal{M}\} \) is given by

\[ \mathcal{A} = ⟨|\text{vac}\rangle W(λ)^{-1}dW(λ)|\text{vac}⟩, \]  
(149)

where \( d \) is a usual differential form on \( \mathcal{M} \), and its curvature form by

\[ \mathcal{F} ≡ d\mathcal{A} + \mathcal{A} ∧ \mathcal{A}, \]  
(150)

see [10] and [9].

Let \( γ \) be a loop in \( \mathcal{M} \) at \( λ_0 \),

\[ γ : [0, 1] \rightarrow \mathcal{M}, \quad γ(0) = γ(1). \]

For this \( γ \) a holonomy operator \( Γ_\mathcal{A} \) is defined as the path–ordered integral of \( \mathcal{A} \) along \( γ \):

\[ Γ_\mathcal{A}(γ) = \mathcal{P}\exp \left\{ \oint_γ \mathcal{A} \right\} \in U(m), \]  
(151)

where \( \mathcal{P} \) means path-ordered. See [4].

This acts on the fiber \( F_0 \) at \( λ_0 \) of the vector bundle \( \{C^m, E, \pi_E, \mathcal{M}\} \) as follows : \( x \rightarrow Γ_\mathcal{A}(γ)x \). The holonomy group \( Hol(\mathcal{A}) \) is in general subgroup of \( U(m) \). In the case of \( Hol(\mathcal{A}) = U(m) \), \( \mathcal{A} \) is called irreducible, see [9].

In the Holonomic Quantum Computation we take

\[ \text{Encoding of Information} \implies x \in F_0, \]

\[ \text{Processing of Information} \implies Γ_\mathcal{A}(γ) : x \rightarrow Γ_\mathcal{A}(γ)x. \]  
(152)
Quantum Computational Bundle

8.1 One–Qubit Case

Let $H_0$ be a Hamiltonian with nonlinear interaction produced by a Kerr medium, that is

$$H_0 = \hbar X N (N - 1),$$

where $X$ is a certain constant, see [2] and [22]. The eigenvectors of $H_0$ corresponding to 0 is $\{|0\rangle, |1\rangle\}$, so its eigenspace is $\text{Vect} \{ |0\rangle, |1\rangle \} \cong \mathbb{C}^2$. The vector space $\text{Vect} \{ |0\rangle, |1\rangle \}$ is called 1-qubit (quantum bit) space and we set

$$F_0 = \text{Vect} \{ |0\rangle, |1\rangle \} \quad \text{and} \quad |\text{vac}\rangle = (|0\rangle, |1\rangle).$$
Now we consider the following isospectral family of $H_0$:

\[ H_{(\alpha, \beta)} = W(\alpha, \beta)H_0W(\alpha, \beta)^{-1}, \]  
\[ W(\alpha, \beta) = D(\alpha)S(\beta). \]  

(154) \hspace{1cm} (155)

In this case

\[ \mathcal{M} = \{ (\alpha, \beta) \in \mathbb{C}^2 \} \]  

(156)

and we want to calculate

\[ \mathcal{A} = \langle \text{vac}|W^{-1}dW|\text{vac}\rangle \]  

(157)

where

\[ d = d\alpha \frac{\partial}{\partial \alpha} + d\bar{\alpha} \frac{\partial}{\partial \bar{\alpha}} + d\beta \frac{\partial}{\partial \beta} + d\bar{\beta} \frac{\partial}{\partial \beta}. \]  

(158)

Since $\mathcal{A}$ is anti-hermitian ($\mathcal{A}^\dagger = -\mathcal{A}$), we can write

\[ \mathcal{A} = A_\alpha d\alpha + A_\beta d\beta - A_\alpha^\dagger d\bar{\alpha} - A_\beta^\dagger d\bar{\beta} \]  

(159)

where

\[ A_\alpha = \langle \text{vac}|W^{-1}\frac{\partial W}{\partial \alpha}|\text{vac}\rangle \hspace{1cm} A_\beta = \langle \text{vac}|W^{-1}\frac{\partial W}{\partial \beta}|\text{vac}\rangle. \]

The calculation of $A_\alpha$ and $A_\beta$ is as follows (13):

\[ A_\alpha = \frac{\bar{\alpha}}{2}L + \text{cosh}(|\beta|)F + \frac{\bar{\beta}\text{sinh}(|\beta|)}{|\beta|}E, \]  

(160)

\[ A_\beta = \frac{\bar{\beta}(-1 + \text{cosh}(2|\beta|))}{4|\beta|^2} \left( K + \frac{1}{2}L \right), \]  

(161)

where

\[ E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hspace{1cm} F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hspace{1cm} K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hspace{1cm} L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Then making use of these ones we can show that the holonomy group generated by (157) is irreducible in $U(2)$, namely just $U(2)$, see [21] and [13]. This is very crucial fact to Holonomic Quantum Computation.
8.2 Two–Qubit Case

We consider the system of two particles, so the Hamiltonian that we treat in the following is

$$H_0 = \hbar X_{N_1}(N_1 - 1) + \hbar X_{N_2}(N_2 - 1). \quad (162)$$

The eigenspace of 0 of this Hamiltonian becomes therefore

$$F_0 = \text{Vect}\{\ket{0}, \ket{1}\} \otimes \text{Vect}\{\ket{0}, \ket{1}\} \cong \mathbb{C}^4. \quad (163)$$

We set \(|\text{vac}\rangle = (\ket{0, 0}, \ket{0, 1}, \ket{1, 0}, \ket{1, 1})\).

Next we consider the following isospectral family of \(H_0\):

$$H_{(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2)} = W(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2)H_0W(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2)^{-1}, \quad (164)$$

$$W(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2) = W_1(\alpha_1, \beta_1)O_{12}(\lambda, \mu)W_2(\alpha_2, \beta_2). \quad (165)$$

where

$$O_{12}(\lambda, \mu) = U_J(\lambda)U_K(\mu), \quad W_j(\alpha_j, \beta_j) = D_j(\alpha_j)S_j(\beta_j) \quad \text{for} \quad j = 1, 2. \quad (166)$$

In this case

$$\mathcal{M} = \{ (\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2) \in \mathbb{C}^6 \} \quad (167)$$

and we have only to calculate the following

$$\mathcal{A} = \langle \text{vac}|W^{-1}dW|\text{vac}\rangle, \quad (168)$$

where

$$d = d\alpha_1 \frac{\partial}{\partial \alpha_1} + d\beta_1 \frac{\partial}{\partial \beta_1} + d\lambda \frac{\partial}{\partial \lambda} + d\mu \frac{\partial}{\partial \mu} + d\bar{\alpha}_1 \frac{\partial}{\partial \bar{\alpha}_1} + d\bar{\beta}_1 \frac{\partial}{\partial \bar{\beta}_1} + d\bar{\lambda} \frac{\partial}{\partial \bar{\lambda}} + d\bar{\mu} \frac{\partial}{\partial \bar{\mu}} + d\bar{\alpha}_2 \frac{\partial}{\partial \bar{\alpha}_2} + d\bar{\beta}_2 \frac{\partial}{\partial \bar{\beta}_2}. \quad (169)$$
The calculation of (168) is not easy, but we can determine it, see [15], [16] and [18] for the details. But we cannot determine its curvature form which is necessary to look for the holonomy group (Ambrose–Singer theorem) due to too complication. Then the essential point is

**Problem**  Is the connection form (168) irreducible in $U(4)$?

Our analysis in [18] shows that the holonomy group generated by $\mathcal{A}$ may be $SU(4)$ not $U(4)$. To obtain $U(4)$ a sophisticated trick \cdots higher dimensional holonomies [44] \cdots may be necessary. See also [23].

## 9 Geometric Construction of Bell States

In this section we introduce the geometric construction of Bell states by making use of coherent states based on $su(2)$, [40]. One of purpose of Quantum Information Theory is to clarify a role of entanglement of states, so that we would like to look for geometric meaning of entanglement.

The famous Bell states ([15], [16]) given by

\[
\frac{1}{\sqrt{2}}(\ket{0} \otimes \ket{0} + \ket{1} \otimes \ket{1}),
\]

\[
\frac{1}{\sqrt{2}}(\ket{0} \otimes \ket{0} - \ket{1} \otimes \ket{1}),
\]

\[
\frac{1}{\sqrt{2}}(\ket{0} \otimes \ket{1} + \ket{1} \otimes \ket{0}),
\]

\[
\frac{1}{\sqrt{2}}(\ket{0} \otimes \ket{1} - \ket{1} \otimes \ket{0})
\]

are typical examples of entanglement. It is very interesting that these play an essential role in Quantum Teleportation, see [11]. We would like to reconstruct these states in a geometric manner.
9.1 Review on General Theory

Let us make a review of [42] and rewrite the result with our method. Let $G$ be a compact linear Lie group (for example $G = U(n)$) and consider a coherent state representation of $G$ whose parameter space is a compact complex manifold $S = G/H$, where $H$ is a subgroup of $G$. For example $G = U(n)$ and $H = U(k) \times U(n-k)$, then $S = U(n)/U(k) \times U(n-k) \cong G_k(C^n)$, which is called a complex Grassmann manifold, [3], [6]. Let $Z$ be a local coordinate on $S$ and $|Z\rangle$ a generalized coherent state in some representation space $V (\cong C^K$ for some high $K \in \mathbb{N})$. Then we have, by definition, the measure $d\mu(Z, Z^\dagger)$ that satisfies the resolution of unity

$$\int_S d\mu(Z, Z^\dagger)|Z\rangle\langle Z| = 1_V \quad \text{and} \quad \int_S d\mu(Z, Z^\dagger) = \dim V. \quad (174)$$

Next we define an anti-automorphism $\♭ : S \rightarrow S$. We call $Z \rightarrow Z^\♭$ an anti-automorphism if and only if

(i) $Z \rightarrow Z^\♭$ induces an automorphism of $S$, \quad (175)

(ii) $\♭$ is an anti-map, namely $\langle Z^\♭|W\rangle = \langle W|Z\rangle$. \quad (176)

Now let us redefine the generalized Bell state in [42] as follows:

**Definition** The generalized Bell state is defined as

$$||B\rangle = \frac{1}{\sqrt{\dim V}} \int_S d\mu(Z, Z^\dagger)|Z\rangle \otimes |Z^\♭\rangle. \quad (177)$$

Then we have

$$\langle\langle B|B\rangle = \frac{1}{\dim V} \int_S \int_S d\mu(Z, Z^\dagger)d\mu(W, W^\dagger)(\langle Z| \otimes \langle Z^\♭\rangle)(|W\rangle \otimes |W^\♭\rangle)$$

$$= \frac{1}{\dim V} \int_S \int_S d\mu(Z, Z^\dagger)d\mu(W, W^\dagger)(\langle Z|W\rangle \langle Z^\♭|W^\♭\rangle)$$

$$= \frac{1}{\dim V} \int_S \int_S d\mu(Z, Z^\dagger)d\mu(W, W^\dagger)(\langle Z|W\rangle \langle W|Z\rangle)$$

$$= \frac{1}{\dim V} \int_S d\mu(Z, Z^\dagger) = 1,$$

where we have used (174) and (176).
9.2 Review on Projective Space

We make a review of complex projective spaces, [9], [5] and [18]. For \( N \in \mathbb{N} \) the complex projective space \( \mathbb{C}P^N \) is defined as follows: For \( \zeta, \mu \in \mathbb{C}^{N+1} - \{0\} \) \( \zeta \) is equivalent to \( \mu \) \((\zeta \sim \mu)\) if and only if \( \zeta = \lambda \mu \) for \( \lambda \in \mathbb{C} - \{0\} \). We show the equivalent relation class as \([\zeta]\) and set \( \mathbb{C}P^N \equiv \mathbb{C}^{N+1} - \{0\} / \sim \). When \( \zeta = (\zeta_0, \zeta_1, \ldots, \zeta_N) \) we write usually as \([\zeta] = [\zeta_0 : \zeta_1 : \cdots : \zeta_N] \). Then it is well-known that \( \mathbb{C}P^N \) has \( N + 1 \) local charts, namely

\[
\mathbb{C}P^N = \bigcup_{j=0}^{N} U_j, \quad U_j = \{[\zeta_0 : \cdots : \zeta_j : \cdots : \zeta_N] | \zeta_j \neq 0\}. \tag{178}
\]

Since \((\zeta_0, \cdots, \zeta_j, \cdots, \zeta_N) = \zeta_j \left(\frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, 1, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_N}{\zeta_j}\right)\), we have the local coordinate on \( U_j \)

\[
\left(\frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_N}{\zeta_j}\right). \tag{179}
\]

However the above definition of \( \mathbb{C}P^N \) is not tractable, so we use the well-known expression by projections

\[
\mathbb{C}P^N \cong G_1(\mathbb{C}^{N+1}) = \{P \in M(N+1; \mathbb{C}) | P^2 = P, P^\dagger = P \text{ and } \text{tr} P = 1\} \tag{180}
\]

and the correspondence

\[
[\zeta_0 : \zeta_1 : \cdots : \zeta_N] \Leftrightarrow \frac{1}{|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_N|^2} \begin{pmatrix}
|\zeta_0|^2 & \zeta_0 \bar{\zeta}_1 & \cdots & \zeta_0 \bar{\zeta}_N \\
\zeta_1 \bar{\zeta}_0 & |\zeta_1|^2 & \cdots & \zeta_1 \bar{\zeta}_N \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_N \bar{\zeta}_0 & \zeta_N \bar{\zeta}_1 & \cdots & |\zeta_N|^2
\end{pmatrix} \equiv P. \tag{181}
\]

If we set

\[
|\zeta| = \frac{1}{\sqrt{\sum_{j=0}^{N} |\zeta_j|^2}} \begin{pmatrix}
\zeta_0 \\
\zeta_1 \\
\vdots \\
\zeta_N
\end{pmatrix}, \tag{182}
\]

37
then we can write the right hand side of (181) as

\[ P = |\zeta\rangle\langle\zeta| \quad \text{and} \quad \langle\zeta|\zeta\rangle = 1. \] (183)

For example on \( U_1 \)

\[ (z_1, z_2, \cdots, z_N) = \left( \frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}, \cdots, \frac{\zeta_N}{\zeta_0} \right), \]

we have

\[
P(z_1, \cdots, z_N) = \frac{1}{1 + \sum_{j=1}^{N} |z_j|^2} \begin{pmatrix}
1 & \bar{z}_1 & \cdots & \bar{z}_N \\
z_1 & |z_1|^2 & \cdots & z_1 \bar{z}_N \\
\vdots & \vdots & \ddots & \vdots \\
z_N & z_N \bar{z}_1 & \cdots & |z_N|^2
\end{pmatrix}
\]

\[ = |(z_1, z_2, \cdots, z_N)\rangle\langle(z_1, z_2, \cdots, z_N)|, \] (184)

where

\[
|z_1, z_2, \cdots, z_N\rangle = \frac{1}{\sqrt{1 + \sum_{j=1}^{N} |z_j|^2}} \begin{pmatrix}
1 \\
z_1 \\
\vdots \\
z_N
\end{pmatrix}
\] (185)

Let us give a more detail description for the cases \( N = 1 \) and \( 2 \).

(a) \( N = 1 \) :

\[
P(z) = \frac{1}{1 + |z|^2} \begin{pmatrix}
1 & \bar{z} \\
z & |z|^2
\end{pmatrix} = |z\rangle\langle z|,
\]

where \(|z\rangle = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix}\), \( z = \frac{\zeta_1}{\zeta_0} \), on \( U_1 \), (186)

\[
P(w) = \frac{1}{|w|^2 + 1} \begin{pmatrix}
|w|^2 & w \\
w \bar{w} & 1
\end{pmatrix} = |w\rangle\langle w|,
\]

where \(|w\rangle = \frac{1}{\sqrt{|w|^2 + 1}} \begin{pmatrix} w \\ 1 \end{pmatrix}\), \( w = \frac{\zeta_0}{\zeta_1} \), on \( U_2 \). (187)
(b) $N = 2$:

$$P(z_1, z_2) = \frac{1}{1 + |z_1|^2 + |z_2|^2} \begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & |z_1|^2 & z_1 \bar{z}_2 \\ z_2 & \bar{z}_1 & |z_2|^2 \end{pmatrix} = |(z_1, z_2)\rangle\langle(z_1, z_2)|,$$

where $|(z_1, z_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} \begin{pmatrix} 1 \\ z_1 \end{pmatrix}$, $(z_1, z_2) = \left( \frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0} \right)$ on $U_1$.

$$P(w_1, w_2) = \frac{1}{|w_1|^2 + 1 + |w_2|^2} \begin{pmatrix} |w_1|^2 & w_1 & w_1 \bar{w}_2 \\ w_2 \bar{w}_1 & 1 & \bar{w}_2 \\ w_2 & w_2 |w_2|^2 \end{pmatrix} = |(w_1, w_2)\rangle\langle(w_1, w_2)|,$$

where $|(w_1, w_2)\rangle = \frac{1}{\sqrt{|w_1|^2 + 1 + |w_2|^2}} \begin{pmatrix} w_1 \\ 1 \\ w_2 \end{pmatrix}$, $(w_1, w_2) = \left( \frac{\zeta_0}{\zeta_1}, \frac{\zeta_2}{\zeta_1} \right)$ on $U_2$.

$$P(v_1, v_2) = \frac{1}{|v_1|^2 + |v_2|^2 + 1} \begin{pmatrix} |v_1|^2 & v_1 \bar{v}_2 & v_1 \\ v_2 \bar{v}_1 & |v_2|^2 & v_2 \\ \bar{v}_1 & \bar{v}_2 & 1 \end{pmatrix} = |(v_1, v_2)\rangle\langle(v_1, v_2)|,$$

where $|(v_1, v_2)\rangle = \frac{1}{\sqrt{|v_1|^2 + |v_2|^2 + 1}} \begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix}$, $(v_1, v_2) = \left( \frac{\zeta_0}{\zeta_2}, \frac{\zeta_1}{\zeta_2} \right)$ on $U_3$.

### 9.3 Bell States Revisited

In this subsection we show that (177) coincides with the Bell states (170)–(173) by choosing anti-automorphism $\flat$ suitably.

We treat first of all the case of spin $\frac{1}{2}$. From here we identify

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
so we have
\[ |\eta\rangle = \frac{1}{\sqrt{1 + |\eta|^2}} (|0\rangle + \eta|1\rangle) = \frac{1}{\sqrt{1 + |\eta|^2}} \left( \begin{array}{c} 1 \\ \eta \end{array} \right). \] (191)

In this case we consider the following four anti-automorphisms (173) and (176):

\begin{align*}
(1) & \quad \eta^b = \bar{\eta} \\
(2) & \quad \eta^b = -\bar{\eta} \\
(3) & \quad \eta^b = \frac{1}{\bar{\eta}} \\
(4) & \quad \eta^b = -\frac{1}{\bar{\eta}}.
\end{align*}

(192)

Now by making use of these we define

**Definition**

\begin{align*}
(1) & \quad |B\rangle = \frac{1}{\sqrt{2}} \int_C d\mu(\eta, \bar{\eta}) |\eta\rangle \otimes |\bar{\eta}\rangle, \\
(2) & \quad |B\rangle = \frac{1}{\sqrt{2}} \int_C d\mu(\eta, \bar{\eta}) |\eta\rangle \otimes |-\bar{\eta}\rangle, \\
(3) & \quad |B\rangle = \frac{1}{\sqrt{2}} \int_C d\mu(\eta, \bar{\eta}) |\eta\rangle \otimes |1/\bar{\eta}\rangle, \\
(4) & \quad |B\rangle = \frac{1}{\sqrt{2}} \int_C d\mu(\eta, \bar{\eta}) |\eta\rangle \otimes |-1/\bar{\eta}\rangle,
\end{align*}

(193) – (196)

where we have put for simplicity
\[ d\mu(\eta, \bar{\eta}) = \frac{2}{\pi} \frac{[d^2\eta]}{(1 + |\eta|^2)^2}. \]

It is easy to see from (180) and (187)

\begin{align*}
(1) & \quad |\eta^b\rangle = |\bar{\eta}\rangle = \frac{1}{\sqrt{1 + |\eta|^2}} (|0\rangle + \eta|1\rangle), \\
(2) & \quad |\eta^b\rangle = |-\bar{\eta}\rangle = \frac{1}{\sqrt{1 + |\eta|^2}} (|0\rangle - \bar{\eta}|1\rangle), \\
(3) & \quad |\eta^b\rangle = |1/\bar{\eta}\rangle = \frac{1}{\sqrt{1 + |\eta|^2}} (\bar{\eta}|0\rangle + |1\rangle), \\
(4) & \quad |\eta^b\rangle = |-1/\bar{\eta}\rangle = \frac{1}{\sqrt{1 + |\eta|^2}} (-\bar{\eta}|0\rangle + |1\rangle),
\end{align*}

(197) – (200)

Then making use of elementary facts

\begin{align*}
\frac{2}{\pi} \int_C \frac{[d^2\eta]}{(1 + |\eta|^2)^2} \frac{1}{1 + |\eta|^2} &= \frac{2}{\pi} \int_C \frac{[d^2\eta]}{(1 + |\eta|^2)^2} \frac{|\eta|^2}{1 + |\eta|^2} = 1, \\
\frac{2}{\pi} \int_C \frac{[d^2\eta]}{(1 + |\eta|^2)^2} \eta = \frac{2}{\pi} \int_C \frac{[d^2\eta]}{(1 + |\eta|^2)^2} \bar{\eta} = 0,
\end{align*}

(198) – (200)

40
we obtain easily

\[(1) \quad |\langle B \rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle),\]

\[(2) \quad |\langle B \rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle),\]

\[(3) \quad |\langle B \rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle),\]

\[(4) \quad |\langle B \rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).\]

We just recovered the Bell states (170)–(173)!! We can say that four $|\langle B \rangle\rangle$ in Definition are overcomplete expression (making use of generalized coherent states) of the Bell states. This is an important point of view.

Since we consider the case of higher spin $J$, we write $|\eta\rangle$ as

$$|\eta\rangle_J = \frac{1}{\left(1 + |\eta|^2\right)^{J}} \sum_{k=0}^{2J} \sqrt{2 J C_k} \eta^k |k\rangle.$$  

(205)

to emphasize the dependence of spin $J$. Here we have set $|k\rangle = |J, k\rangle$ for simplicity. From the above result it is very natural to define Bell states with spin $J$ as follows because the parameter space is the same $CP^1$:

**Definition**

\[(1) \quad |\langle B \rangle\rangle = \frac{1}{\sqrt{2J + 1}} \int_{C} d\mu(\eta, \bar{\eta}) |\eta\rangle_J \otimes |\bar{\eta}\rangle_J,\]

\[(2) \quad |\langle B \rangle\rangle = \frac{1}{\sqrt{2J + 1}} \int_{C} d\mu(\eta, \bar{\eta}) |\eta\rangle_J \otimes |\bar{\eta}\rangle_J,\]

\[(3) \quad |\langle B \rangle\rangle = \frac{1}{\sqrt{2J + 1}} \int_{C} d\mu(\eta, \bar{\eta}) |\eta\rangle_J \otimes |1/\bar{\eta}\rangle_J,\]

\[(4) \quad |\langle B \rangle\rangle = \frac{1}{\sqrt{2J + 1}} \int_{C} d\mu(\eta, \bar{\eta}) |\eta\rangle_J \otimes |1/\bar{\eta}\rangle_J,\]

(206)

(207)

(208)

(209)

where

$$d\mu(\eta, \bar{\eta}) = \frac{2J + 1}{\pi} \frac{[d^2\eta]}{(1 + |\eta|^2)^2}.$$  

Let us calculate $|\bar{\eta}\rangle_J$, $|\bar{\eta}\rangle_J$, $|1/\bar{\eta}\rangle_J$ and $|1/\bar{\eta}\rangle_J$. It is easy to see

\[(1) \quad |\bar{\eta}\rangle_J = \frac{1}{\left(1 + |\eta|^2\right)^{J}} \sum_{k=0}^{2J} \sqrt{2 J C_k} \eta^k |k\rangle.$$  

(210)

From this lemma and the elementary facts

\[
\frac{2J + 1}{\pi} \int_{C} \frac{[d^2 \eta]}{(1 + |\eta|^2)^2} \frac{|\eta|^{2k}}{(1 + |\eta|^2)^{2J}} = \frac{1}{2J C_k} \quad \text{for} \quad 0 \leq k \leq 2J,
\]

we can give explicit forms to the Bell states with spin \( J \):

1. \(|\langle B \rangle\rangle = \frac{1}{\sqrt{2J + 1}} \sum_{k=0}^{2J} |k\rangle \otimes |k\rangle,
2. \(|\langle B \rangle\rangle = \frac{1}{\sqrt{2J + 1}} \sum_{k=0}^{2J} (-1)^k |k\rangle \otimes |k\rangle,
3. \(|\langle B \rangle\rangle = \frac{1}{\sqrt{2J + 1}} \sum_{k=0}^{2J} |k\rangle \otimes |2J - k\rangle,
4. \(|\langle B \rangle\rangle = \frac{1}{\sqrt{2J + 1}} \sum_{k=0}^{2J} (-1)^k |k\rangle \otimes |2J - k\rangle.

We obtained the Bell states with spin \( J \) which are a natural extension of usual ones (\( J = 1/2 \)).

A comment is in order. For the case \( J = 1 \):

1. \( \frac{1}{\sqrt{3}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle),
2. \( \frac{1}{\sqrt{3}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle),
3. \( \frac{1}{\sqrt{3}}(|0\rangle \otimes |2\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |0\rangle),
4. \( \frac{1}{\sqrt{3}}(|0\rangle \otimes |2\rangle - |1\rangle \otimes |1\rangle + |2\rangle \otimes |0\rangle).

It is easy to see that they are not linearly independent, so that only this case is very special (peculiar).

**Comment** We cannot give a geometric construction of Bell states by making use of generalized coherent states based on \( su(1, 1) \) (Lie algebra of non–compact Lie group).
Because the parameter space in this case is a Poincare disk \( D = \{ \zeta \in \mathbb{C} \mid |\zeta| < 1 \} \) and the measure on it is given by
\[
d\mu(\zeta, \zeta) = \frac{2K - 1}{\pi} \frac{[d^2 \zeta]}{(1 - |\zeta|^2)^2},
\]
see (28). Therefore we have
\[
\int_D d\mu(\zeta, \zeta) = (2K - 1) \int_0^1 dr \frac{1}{(1 - r)^2} = \infty !
\]
Compare this with (174). This is a reason why we cannot determine a normalization.

### 10 Topics in Quantum Information Theory

In this section we don’t introduce a general theory of quantum information theory (see for example [11]), but focus our attention to special topics of it, that is,

- swap of coherent states
- cloning of coherent states

Because this is just a good one as examples of applications of coherent and generalized coherent states and our method developed in the following may open a new possibility.

First let us define a swap operator:

\[
S : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}, \quad S(a \otimes b) = b \otimes a \quad \text{for any } a, b \in \mathcal{H}
\]

(218)

where \( \mathcal{H} \) is the Fock space in Section 2.

It is not difficult to construct this operator in a universal manner, see Appendix A. But for coherent states we can construct a better one by making use of generalized coherent operators in the preceding section.

Next let us introduce no cloning theorem, [43]. For that we define a cloning (copying) operator \( C \) which is unitary

\[
C : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}, \quad C(h \otimes |0\rangle) = h \otimes h \quad \text{for any } h \in \mathcal{H} .
\]

(219)
It is very known that there is no cloning theorem

**“No Cloning Theorem”** We have no \( C \) above.

The proof is very easy (almost trivial). Because \( 2h = h + h \in \mathcal{H} \) and \( C \) is a linear operator, so

\[
C(2h \otimes |0\rangle) = 2C(h \otimes |0\rangle).
\]  

(220)

The LHS of (220) is

\[
C(2h \otimes |0\rangle) = 2h \otimes 2h = 4(h \otimes h),
\]

while the RHS of (220)

\[
2C(h \otimes |0\rangle) = 2(h \otimes h).
\]

This is a contradiction. This is called no cloning theorem.

Let us return to the case of coherent states. For coherent states \(|\alpha\rangle\) and \(|\beta\rangle\) the superposition \( |\alpha\rangle + |\beta\rangle \) is no longer a coherent state, so that coherent states may not suffer from the theorem above.

**Problem** Is it possible to clone coherent states?

At this stage it is not easy, so we will make do with approximating it (imperfect cloning in our terminology) instead of making a perfect cloning.

We write notations once more.

\[
\begin{align*}
\text{Coherent States} & \quad |\alpha\rangle = D(\alpha)|0\rangle \quad \text{for} \quad \alpha \in \mathbb{C} \\
\text{Squeezed-like States} & \quad |\beta\rangle = S(\beta)|0\rangle \quad \text{for} \quad \beta \in \mathbb{C}
\end{align*}
\]

### 10.1 Some Useful Formulas

We list and prove some useful formulas in the following. Now we prepare some parameters \( \alpha, \epsilon, \kappa \) in which \( \epsilon, \kappa \) are free ones, while \( \alpha \) is unknown one in the cloning case. Let us
unify the notations as follows.

\[ \alpha : \text{(unknown)} \quad \alpha = |\alpha|e^{i\chi}, \quad (221) \]
\[ \epsilon : \text{known} \quad \epsilon = |\epsilon|e^{i\phi}, \quad (222) \]
\[ \kappa : \text{known} \quad \kappa = |\kappa|e^{i\delta}, \quad (223) \]

Let us start.

(i) First let us calculate
\[ S(\epsilon)D(a)S(\epsilon)^{-1}. \quad (224) \]
For that we show
\[ S(\epsilon)aS(\epsilon)^{-1} = cosh(|\epsilon|)a - e^{i\phi}sinh(|\epsilon|)a^\dagger. \quad (225) \]
Proof is as follows. For \( X = \left(1/2\right)\{\epsilon(a^\dagger)^2 - \bar{\epsilon}a^2\} \) we have easily \([X, a] = -\epsilon a^\dagger\) and \([X, a^\dagger] = -\bar{\epsilon}a\), so
\[
S(\epsilon)aS(\epsilon)^{-1} = e^{X}ae^{-X} = a + [X, a] + \frac{1}{2!}[X, [X, a]] + \frac{1}{3!}[X, [X, [X, a]]] + \cdots \\
= a - \epsilon a^\dagger + \frac{|\epsilon|^2}{2!}a - \frac{\epsilon|\epsilon|^2}{3!}a^\dagger + \cdots \\
= \left\{1 + \frac{|\epsilon|^2}{2!} + \cdots \right\}a - \frac{\epsilon}{|\epsilon|}\left\{|\epsilon| + \frac{|\epsilon|^3}{3!} + \cdots \right\}a^\dagger \\
= cosh(|\epsilon|)a - \frac{\epsilon sinh(|\epsilon|)}{|\epsilon|}a^\dagger = cosh(|\epsilon|)a - e^{i\phi}sinh(|\epsilon|)a^\dagger.
\]
From this it is easy to check
\[
S(\epsilon)D(\alpha)S(\epsilon)^{-1} = D\left(\alpha S(\epsilon)a^\dagger S(\epsilon)^{-1} - \bar{\alpha}S(\epsilon)aS(\epsilon)^{-1}\right) \\
= D\left(cosh(|\epsilon|)\alpha + e^{i\phi}sinh(|\epsilon|)\bar{\alpha}\right). \quad (226)
\]
Therefore
\[
S(\epsilon)D(\alpha)S(\epsilon)^{-1} = \left\{
\begin{array}{ll}
D(e^{i\epsilon|\alpha}) & \text{if } \phi = 2\chi \\
D(e^{-i\epsilon|\alpha}) & \text{if } \phi = 2\chi + \pi
\end{array}
\right. \quad (227)
\]
By making use of this formula we can change a scale of \( \alpha \).

(ii) Next let us calculate
\[ S(\epsilon)S(\alpha)S(\epsilon)^{-1}. \quad (228) \]
From the definition

\[ S(\epsilon)S(\alpha)S(\epsilon)^{-1} = S(\epsilon)\exp \left\{ \frac{1}{2} (\alpha(a^\dagger)^2 - \bar{\alpha}a^2) \right\} S(\epsilon)^{-1} = e^{Y/2} \]

where

\[ Y = \alpha \left( S(\epsilon)a^\dagger S(\epsilon)^{-1} \right)^2 - \bar{\alpha} \left( S(\epsilon)aS(\epsilon)^{-1} \right)^2. \]

From (225) and after some calculations we have

\[ Y = \left\{ \cosh^2(|\epsilon|)\alpha - e^{2i\phi} \sinh^2(|\epsilon|)\bar{\alpha} \right\} (a^\dagger)^2 \]
\[ + \left( -e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha} \right) \sinh(2|\epsilon|)(a^\dagger a + aa^\dagger) \]
\[ = \left\{ \cosh^2(|\epsilon|)\alpha - e^{2i\phi} \sinh^2(|\epsilon|)\bar{\alpha} \right\} (a^\dagger)^2 \]
\[ + \left( -e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha} \right) \sinh(2|\epsilon|)(a^\dagger a + \frac{1}{2}) \quad (\Leftarrow [a, a^\dagger] = 1), \]

or

\[ \frac{1}{2} Y = \left\{ \cosh^2(|\epsilon|)\alpha - e^{2i\phi} \sinh^2(|\epsilon|)\bar{\alpha} \right\} K_+ - \left\{ \cosh^2(|\epsilon|)\bar{\alpha} - e^{-2i\phi} \sinh^2(|\epsilon|)\alpha \right\} K_- \]
\[ + \left( -e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha} \right) \sinh(2|\epsilon|)K_3 \]  \hfill (229)

with \( \{K_+, K_-, K_3\} \) in (30). This is our formula.

Now

\[ -e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha} = |\alpha|(-e^{-i(\phi-\chi)} + e^{i(\phi-\chi)}) = 2i|\alpha|\sin(\phi - \chi), \]

so if we choose \( \phi = \chi \), then \( e^{2i\phi}\bar{\alpha} = e^{2\chi}e^{-i\chi}|\alpha| = \alpha \) and

\[ \cosh^2(|\epsilon|)\alpha - e^{2i\phi} \sinh^2(|\epsilon|)\bar{\alpha} = \left( \cosh^2(|\epsilon|) - \sinh^2(|\epsilon|) \right) \alpha = \alpha \]

, and finally

\[ Y = \alpha(a^\dagger)^2 - \bar{\alpha}a^2. \]

That is,

\[ S(\epsilon)S(\alpha)S(\epsilon)^{-1} = S(\alpha) \iff S(\epsilon)S(\alpha) = S(\alpha)S(\epsilon). \]

The operators \( S(\epsilon) \) and \( S(\alpha) \) commute if the phases of \( \epsilon \) and \( \alpha \) coincide.
(iii) Third formula is: For \( V(t) = e^{itN} \) where \( N = a^\dagger a \) (a number operator)

\[
V(t)D(\alpha)V(t)^{-1} = D(e^{it}\alpha).
\]

(230)

The proof is as follows.

\[
V(t)D(\alpha)V(t)^{-1} = \exp\left(\alpha V(t)a^\dagger V(t)^{-1} - \bar{\alpha} V(t)aV(t)^{-1}\right).
\]

It is easy to see

\[
V(t)aV(t)^{-1} = e^{itN}ae^{-itN} = a + [itN, a] + \frac{1}{2!}[itN, [itN, a]] + \cdots
\]

\[
= a + (-it)a + \frac{(-it)^2}{2!}a + \cdots
\]

\[
= e^{-it}a.
\]

Therefore we obtain

\[
V(t)D(\alpha)V(t)^{-1} = \exp\left(\alpha e^{it}a^\dagger - \bar{\alpha} e^{-it}a^\dagger\right) = D(e^{it}\alpha).
\]

This formula is often used as follows.

\[
|\alpha\rangle \rightarrow V(t)|\alpha\rangle = V(t)D(\alpha)V(t)^{-1}V(t)|0\rangle = D(e^{it}\alpha)|0\rangle = |e^{it}\alpha\rangle,
\]

(231)

where we have used

\[
V(t)|0\rangle = |0\rangle
\]

becase \( N|0\rangle = 0 \). That is, we can add a phase to \( \alpha \) by making use of this formula.

10.2 Swap of Coherent States

The purpose of this section is to construct a swap operator satifying

\[
|\alpha_1\rangle \otimes |\alpha_2\rangle \rightarrow |\alpha_2\rangle \otimes |\alpha_1\rangle.
\]

(232)

Let us remember \( U_J(\kappa) \) once more

\[
U_J(\kappa) = e^{\kappa a_1^\dagger a_2 - \bar{\kappa} a_1 a_2^\dagger} \quad \text{for} \quad \kappa \in \mathbb{C}.
\]
We note an important property of this operator:

\[ U_J(\kappa)|0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle. \]  \hspace{1cm} (233)

The construction is as follows.

\[
U_J(\kappa)|\alpha_1\rangle \otimes |\alpha_2\rangle = U_J(\kappa)D(\alpha_1) \otimes D(\alpha_2)|0\rangle \otimes |0\rangle = U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)|0\rangle \otimes |0\rangle \\
= U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle \\
= U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle \quad \text{by (233),} \tag{234}
\]

and

\[
U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1} = U_J(\kappa)\exp \left\{ \alpha_1 a_1^{\dagger} - \bar{\alpha}_1 a_1 + \alpha_2 a_2^{\dagger} - \bar{\alpha}_2 a_2 \right\} U_J(\kappa)^{-1} \\
= \exp \left\{ \alpha_1 (U_J(\kappa)a_1 U_J(\kappa)^{-1})^{\dagger} - \bar{\alpha}_1 U_J(\kappa)a_1 U_J(\kappa)^{-1} \\
+ \alpha_2 (U_J(\kappa)a_2 U_J(\kappa)^{-1})^{\dagger} - \bar{\alpha}_2 U_J(\kappa)a_2 U_J(\kappa)^{-1} \right\} \\
\equiv \exp(X). \tag{235}
\]

From (237) and (238) we have

\[
X = \left\{ \cos(\kappa)|\alpha_1\rangle + \frac{\sin(\kappa)}{|\kappa|}|\alpha_2\rangle \right\} a_1^{\dagger} - \left\{ \cos(\kappa)\bar{\alpha}_1 \right\} a_1 \\
+ \left\{ \cos(\kappa)|\alpha_2\rangle - \frac{\sin(\kappa)}{|\kappa|}\bar{\alpha}_2 \right\} a_2^{\dagger} - \left\{ \cos(\kappa)\bar{\alpha}_2 \right\} a_2,
\]

so

\[
\exp(X) = D_1 \left( \cos(|\kappa|)\alpha_1 + \frac{\sin(|\kappa|)}{|\kappa|}\alpha_2 \right) D_2 \left( \cos(|\kappa|)\alpha_2 - \frac{\sin(|\kappa|)}{|\kappa|}\alpha_1 \right) \\
= D \left( \cos(|\kappa|)\alpha_1 + \frac{\sin(|\kappa|)}{|\kappa|}\alpha_2 \right) \otimes D \left( \cos(|\kappa|)\alpha_2 - \frac{\sin(|\kappa|)}{|\kappa|}\alpha_1 \right).
\]

Therefore we have from (235)

\[
|\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |\cos(|\kappa|)\alpha_1 + \frac{\sin(|\kappa|)}{|\kappa|}\alpha_2 \rangle \otimes |\cos(|\kappa|)\alpha_2 - \frac{\sin(|\kappa|)}{|\kappa|}\alpha_1 \rangle.
\]

If we write \( \kappa \) as \( |\kappa|e^{i\delta} \), then the above formula reduces to

\[
|\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |\cos(|\kappa|)\alpha_1 + e^{i\delta}\sin(|\kappa|)\alpha_2 \rangle \otimes |\cos(|\kappa|)\alpha_2 - e^{-i\delta}\sin(|\kappa|)\alpha_1 \rangle.
\]
Here if we choose $\sin(|\kappa|) = 1$, then

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \rightarrow |e^{i\delta}\alpha_2\rangle \otimes |-e^{-i\delta}\alpha_1\rangle = |e^{i\delta}\alpha_2\rangle \otimes |e^{-i(\delta+\pi)}\alpha_1\rangle.$$ 

Now by operating the operator $V = e^{-i\delta N} \otimes e^{i(\delta+\pi) N}$ where $N = a^\dagger a$ from the left (see (231)) we obtain the swap

$$|\alpha_1\rangle \otimes |\alpha_2\rangle \rightarrow |\alpha_2\rangle \otimes |\alpha_1\rangle.$$ 

A comment is in order. In the formula we set $\alpha_1 = \alpha$ and $\alpha_2 = 0$, then the formula reduces to

$$U_J(\kappa) D_1(\alpha) U_J(\kappa)^{-1} = D_1(\cos(|\kappa|)\alpha) D_2(-e^{-i\delta} \sin(|\kappa|)\alpha).$$  \hspace{1cm} (236)

### 10.3 Imperfect Cloning of Coherent States

We cannot clone coherent states in a perfect manner likely

$$|\alpha\rangle \otimes |0\rangle \rightarrow |\alpha\rangle \otimes |\alpha\rangle \quad \text{for} \, \alpha \in \mathbb{C}. \hspace{1cm} (237)$$

Then our question is: is it possible to approximate? We show that we can at least make an “imperfect cloning” in our terminology against the statement of [37].

Let us start. The method is almost same with one in the preceding subsection, but we repeat it once more. Operating the operator $U_J(\kappa)$ on $|\alpha\rangle \otimes |0\rangle$

$$U_J(\kappa)|\alpha\rangle \otimes |0\rangle = U_J(\kappa) \{ D(\alpha) \otimes 1 \} |0\rangle \otimes |0\rangle = U_J(\kappa) D_1(\alpha)|0\rangle \otimes |0\rangle \quad \text{by} \, (233)$$

$$=U_J(\kappa) D_1(\alpha) U_J(\kappa)^{-1} U_J(\kappa)|0\rangle \otimes |0\rangle = U_J(\kappa) D_1(\alpha) U_J(\kappa)^{-1} |0\rangle \otimes |0\rangle \quad \text{by} \, (233)$$

$$=D_1(\cos(|\kappa|)\alpha) D_2(-e^{-i\delta} \sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle \quad \text{by} \, (236)$$

$$=D_1(\cos(|\kappa|)\alpha) D_2(e^{-i(\delta+\pi)} \sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle$$

$$=\{ D(\cos(|\kappa|)\alpha) \otimes D(e^{-i(\delta+\pi)} \sin(|\kappa|)\alpha) \} |0\rangle \otimes |0\rangle.$$

Operating the operator $1 \otimes e^{i(\delta+\pi) N}$ on the last equation

$$D(\cos(|\kappa|)\alpha) \otimes e^{i(\delta+\pi) N} D(e^{-i(\delta+\pi)} \sin(|\kappa|)\alpha)|0\rangle \otimes |0\rangle$$
Namely we have constructed

\[ |\alpha\rangle \otimes |0\rangle \longrightarrow |\cos(|\kappa|)\alpha\rangle \otimes |\sin(|\kappa|)\alpha\rangle. \]  

(238)

This is an “imperfect cloning” what we have called.

**A comment is in order.** The authors in [37] state that the “perfect cloning” (in their terminology) for coherent states is possible. But it is not correct as shown below.

Before starting let us prepare a notation for simplicity (220):

\[ S(\epsilon)D(\alpha)S(\epsilon)^{-1} = D(\tilde{\alpha}), \quad \tilde{\alpha} \equiv \cosh(|\epsilon|)\alpha + e^{i\phi}\sinh(|\epsilon|)\bar{\alpha}. \]

Operating the operator \( S(\epsilon) \otimes S(e^{-2i\delta}) \) from the left

\[
S(\epsilon) \otimes S(e^{-2i\delta})|\alpha\rangle \otimes |0\rangle = \left\{ S(\epsilon) \otimes S(e^{-2i\delta}) \right\} \left\{ D(\alpha) \otimes 1 \right\} |0\rangle \otimes |0\rangle \\
= S(\epsilon)D(\alpha) \otimes S(e^{-2i\delta})|0\rangle \otimes |0\rangle \\
= S(\epsilon)D(\alpha)S(\epsilon)^{-1}S(\epsilon) \otimes S(e^{-2i\delta})|0\rangle \otimes |0\rangle \\
= D(\tilde{\alpha})S(\epsilon) \otimes S(e^{-2i\delta})|0\rangle \otimes |0\rangle \\
= \left\{ D(\tilde{\alpha}) \otimes 1 \right\} \left\{ S(\epsilon) \otimes S(e^{-2i\delta}) \right\} |0\rangle \otimes |0\rangle \\
= D_1(\tilde{\alpha}) \left\{ S(\epsilon) \otimes S(e^{-2i\delta}) \right\} |0\rangle \otimes |0\rangle.
\]

Operating the operator \( U_J(\kappa) \) (remember that \( \kappa = |\kappa|e^{i\delta} \)) from the left

\[
U_J(\kappa)D_1(\tilde{\alpha}) \left\{ S(\epsilon) \otimes S(e^{-2i\delta}) \right\} |0\rangle \otimes |0\rangle \\
= U_J(\kappa)D_1(\tilde{\alpha}) \left\{ S(\epsilon) \otimes S(e^{-2i\delta}) \right\} U_J(\kappa)^{-1}U_J(\kappa)|0\rangle \otimes |0\rangle
\]
where we have chosen in the last step

\[ \operatorname{Problem} \]

modify that more subtly by making use of (227).

Therefore one cannot follow their method from this stage.

But as stated above their method is simple and very interesting, so it may be possible to modify that more subtly by making use of (227).

\textbf{Problem} Is it possible to make a “perfect cloning” in the sense of [37]?
10.4 Swap of Squeezed–like States?

We would like to construct an operator like

\[ |\beta_1\rangle \otimes |\beta_2\rangle \longrightarrow |\beta_2\rangle \otimes |\beta_1\rangle. \] (243)

In this case we cannot use an operator \( U_J(\kappa) \). Let us explain the reason.

Similar to (234)

\[ U_J(\kappa)|\beta_1\rangle \otimes |\beta_2\rangle = U_J(\kappa)S(\beta_1) \otimes S(\beta_2)|0\rangle \otimes |0\rangle \]
\[ = U_J(\kappa)S_1(\beta_1)S_2(\beta_2)|0\rangle \otimes |0\rangle \]
\[ = U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle. \] (244)

On the other hand by (95)

\[ U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1} = e^X, \]

where

\[ X = \frac{1}{2} \left\{ \cos^2(|\kappa|) \beta_1 + \frac{\kappa^2 \sin^2(|\kappa|)}{|\kappa|^2} \beta_2 \right\} (\dagger a_1)^2 - \frac{1}{2} \left\{ \cos^2(|\kappa|) \beta_1 + \frac{\bar{\kappa}^2 \sin^2(|\kappa|)}{|\kappa|^2} \bar{\beta}_2 \right\} a_1^2 \]
\[ + \frac{1}{2} \left\{ \cos^2(|\kappa|) \beta_2 + \frac{\bar{\kappa}^2 \sin^2(|\kappa|)}{|\kappa|^2} \bar{\beta}_1 \right\} (\dagger a_2)^2 - \frac{1}{2} \left\{ \cos^2(|\kappa|) \beta_2 + \frac{\kappa^2 \sin^2(|\kappa|)}{|\kappa|^2} \bar{\beta}_1 \right\} a_2^2 \]
\[ + (\beta_2\bar{\kappa} - \bar{\beta}_1\kappa)^2 \frac{\sin(2|\kappa|)}{2|\kappa|} \bar{a}_1 a_1 a_2^\dagger - (\bar{\beta}_2 \bar{\kappa} - \bar{\beta}_1 \kappa) \frac{\sin(2|\kappa|)}{2|\kappa|} \bar{a}_1 a_1 a_2^\dagger. \]

Here an extra term containing \( \dagger a_1 a_2^\dagger \) appeared. To remove this we must set \( \beta_2 \kappa - \bar{\beta}_1 \bar{\kappa} = 0 \), but in this case we meet

\[ U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1} = S_1(\beta_1)S_2(\beta_2) \]

by (98). That is, there is no change.

We could not construct an operator likely in the subsection 8.1 in spite of very our efforts, so we present

**Problem** Is it possible to find an operator such as \( U_J(\kappa) \) in the preceding subsection for performing the swap?
10.5 A Comment

We have used in the process of proofs both a displacement operator $D(\alpha)$ and a squeezed one $S(\epsilon)$

$$D(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a), \quad S(\epsilon) = \exp\left(\frac{1}{2}(\epsilon(a^\dagger)^2 - \bar{\epsilon} a^2)\right)$$

as a product operator

$$S(\epsilon)D(\alpha).$$  \hspace{1cm} (245)

We note that this product operator with the parameter space $\{(\alpha, \epsilon) \in \mathbb{C}^2\}$ plays a crucial role in our Holonomic Quantum Computation (Computer), see section 8.1.

Similarly, the product operator

$$U_K(w)U_J(v)$$  \hspace{1cm} (246)

with the parameter space $\{(v, w) \in \mathbb{C}^2\}$ also plays a crucial role in it, see section 8.2.

We believe that Holonomic Quantum Computation and our geometric method (involving swap or imperfect cloning) in Quantum Information Theory are well-matched.

11 Path Integral on A Quantum Computer

In this section we present a very important problem (at least to the author) about the possibility of calculation of path integral on a Quantum Computer.

The path integral method plays an essential role in Quantum Mechanics or Quantum Field Theory. But it is, in general, not easy to calculate it except for Gaussian cases. Some specialists must, in a perturbation theory, calculate many Feynman’s graphs by making use of a classical computer(s). This is a hard and painful task.

Now let us present our general problem.

**Problem** Is it possible to calculate a path integral in polynomial times by making use of a quantum computer?
For this subject refer \[38\] and its references. But our method or interest is a bit different from \[38\]. To match our method with path integrals we should use coherent state path integral method, see \[4\], \[5\], \[6\].

To calculate a physical quantity such as a trace formula of the Hamiltonian we, for example, give a coherent state path integral expression to it. We want to calculate it, but it is usually not easy to do so. Therefore we have to make do with some approximation. Then our next problem is

**Problem**  Is it possible to give it in polynomial times with Holonomic Quantum Computer?

For the readers who are not familiar with coherent state path integral method let us show a simple, but very instructive example, \[39\].

Let us consider the Hamiltonian of harmonic oscillator

\[
H = \omega N = \omega a^\dagger a,
\]

where we have omitted the constant term for simplicity. The eigenvalues of \(H\) are well-known to be \(\{n\omega \mid n = 0, 1, \cdots\}\) and its trace formula is given as

\[
\text{tr} \ e^{-iTH} = \sum_{n=0}^{\infty} e^{-in\omega T} = \frac{1}{1 - e^{-i\omega T}} \quad \text{(Abel sum)}.
\]

Let us give a coherent state path integral expression to this trace formula. Making use of the resolution of unity \[1\] we obtain

\[
\text{tr} \ e^{-iTH} = \text{tr} \ 1 e^{-iTH} = \text{tr} \int_C \frac{[d^2 z]}{\pi} \langle z | e^{-iTH} | z \rangle = \int_C \frac{[d^2 z]}{\pi} \langle z | e^{-iTH} | z \rangle.
\]

This is just an analytical expression of the trace formula. It is easy to calculate this directly, but we give this a path integral expression. Noting

\[
e^X = \lim_{N \to \infty} \left(1 + \frac{X}{N}\right)^N,
\]

we have

\[
\text{RHS of (249)} = \lim_{N \to \infty} \int_C \frac{[d^2 z]}{\pi} \langle z | (1 - i\Delta t H)^N | z \rangle,
\]

\[
(250)
\]
where we have set $\Delta t = T/N$.

By inserting the resolution of unity (11) at each step likely

$$
(1 - i\Delta tH)^N = (1 - i\Delta tH) 1 (1 - i\Delta tH) 1 \cdots (1 - i\Delta tH) 1 (1 - i\Delta tH)
$$

1 = \int_C \frac{[d^2z_j]}{\pi} |z_j]\langle z_j|$ for any $1 \leq j \leq N - 1$,

we have

$$
\text{RHS of (249)} = \lim_{N \to \infty} \int_{PBC} \prod_{j=1}^N \frac{[d^2z_j]}{\pi} \prod_{j=1}^N \langle z_j|1 - i\Delta tH|z_{j-1}\rangle, \tag{251}
$$

where PBC (periodic boundary condition) means $z_N = z_0 = z$. We note that the choice of $\{z_1, z_2, \cdots, z_{N-1}\}$ is random.

Let us calculate the term $\langle z_j|1 - i\Delta tH|z_{j-1}\rangle$:

$$
\langle z_j|1 - i\Delta tH|z_{j-1}\rangle = \langle z_j|z_{j-1}\rangle - i\Delta t\langle z_j|H|z_{j-1}\rangle
$$

$$
= \langle z_j|z_{j-1}\rangle \left\{ 1 - i\Delta t \frac{\langle z_j|H|z_{j-1}\rangle}{\langle z_j|z_{j-1}\rangle} \right\}
$$

$$
= \langle z_j|z_{j-1}\rangle \exp \left\{ -i\Delta t \frac{\langle z_j|H|z_{j-1}\rangle}{\langle z_j|z_{j-1}\rangle} \right\} \text{ up to } O((\Delta t)^2).
$$

On the other hand from (12) and (13)

$$
\langle z_j|z_{j-1}\rangle = \exp \left( -\frac{1}{2}|z_j|^2 - \frac{1}{2}|z_{j-1}|^2 + \bar{z}_jz_{j-1} \right),
$$

$$
\langle z_j|H|z_{j-1}\rangle = \omega \langle z_j|a^\dagger a|z_{j-1}\rangle = \omega \bar{z}_jz_{j-1}\langle z_j|z_{j-1}\rangle, \quad \frac{\langle z_j|H|z_{j-1}\rangle}{\langle z_j|z_{j-1}\rangle} = \omega \bar{z}_jz_{j-1},
$$

so we have

$$
\langle z_j|1 - i\Delta tH|z_{j-1}\rangle = \exp \left\{ -\frac{1}{2}|z_j|^2 - \frac{1}{2}|z_{j-1}|^2 + \bar{z}_jz_{j-1} - i\omega \Delta t \bar{z}_jz_{j-1} \right\}
$$

after some algebra. Here taking the periodic boundary condition $z_N = z_0$ it is easy to see

$$
\prod_{j=1}^N \langle z_j|1 - i\Delta tH|z_{j-1}\rangle = \exp \left\{ - \sum_{j=1}^N \{ \bar{z}_j(z_j - z_{j-1}) + i\omega \Delta t \bar{z}_jz_{j-1} \} \right\}.
$$

Therefore from this we reach

$$
\text{RHS of (249)} = \lim_{N \to \infty} \int_{PBC} \prod_{j=1}^N \frac{[d^2z_j]}{\pi} \exp \left\{ - \sum_{j=1}^N \{ \bar{z}_j(z_j - z_{j-1}) + i\omega \Delta t \bar{z}_jz_{j-1} \} \right\}. \tag{252}
$$
This is just the coherent state path integral expression of trace formula of the harmonic oscillator. As for calculation of (252) see Appendix B.

12 Discussion and Dream

Here we state our dream once more. The main subjects of Quantum Information Theory are

- Quantum Computer (Computation)
- Quantum Cryptgraphy
- Quantum Teleportation

The purpose is to understand Quantum Information Theory from the geometric point of view or, more clearly, to construct

**Geometric Quantum Information Theory.**

For example,

- **Geometric** Quantum Computer (Computation)
- **Geometric** Quantum Cryptgraphy
- **Geometric** Quantum Teleportation

Geometric understanding of several concepts is very important because we can view them from global point of view. We believe that we have taken a first step towards this dream.

*Acknowledgment.* The author wishes to thank Kunio Funahashi for useful comments.
Appendix

Appendix A  Universal Swap Operator

Let us construct the swap operator in a universal manner

\[ U : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}, \quad U(a \otimes b) = b \otimes a \quad \text{for any } a, b \in \mathcal{H} \]

where \( \mathcal{H} \) is an infinite-dimensional Hilbert space. Before constructing it we show in the finite-dimensional case.

For \( a, b \in \mathbb{C}^2 \) then

\[
\begin{pmatrix}
a \otimes b \\
a_1 b \\
a_2 b
\end{pmatrix} =
\begin{pmatrix}
a_1 b_1 \\
a_1 b_2 \\
a_2 b_1 \\
a_2 b_2
\end{pmatrix},
\begin{pmatrix}
b \otimes a \\
b_1 a_1 \\
b_1 a_2 \\
b_2 a_1 \\
b_2 a_2
\end{pmatrix} =
\begin{pmatrix}
a_1 b_1 \\
a_2 b_1 \\
a_1 b_2 \\
a_2 b_2
\end{pmatrix},
\]

so it is easy to see

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_1 b_1 \\
a_1 b_2 \\
a_2 b_1 \\
a_2 b_2
\end{pmatrix} =
\begin{pmatrix}
a_1 b_1 \\
a_2 b_1 \\
a_1 b_2 \\
a_2 b_2
\end{pmatrix}.
\]

That is, the swap operator is

\[
U =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\tag{253}
\]

In this case we can happen to write \( U \) as

\[
U = \frac{1}{2} \left( 1 \otimes 1 + \sum_{j=1}^{3} \sigma_j \otimes \sigma_j \right),
\tag{254}
\]

where \( \{\sigma_1, \sigma_1, \sigma_1\} \) are Pauli matrices. But unfortunately we cannot extend this formula further.
It is not easy for us to conjecture its general form from this swap operator. Let us try for
\( n = 3 \). The result is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_1b_1 \\
a_1b_2 \\
a_1b_3 \\
a_2b_1 \\
a_2b_2 \\
a_2b_3 \\
a_3b_1 \\
a_3b_2 \\
a_3b_3
\end{pmatrix}
= \begin{pmatrix}
a_1b_1 \\
a_2b_1 \\
a_3b_1 \\
a_1b_2 \\
a_2b_2 \\
a_3b_2 \\
a_1b_3 \\
a_2b_3 \\
a_3b_3
\end{pmatrix}.
\]

Here we rewrite the swap operator above as follows.

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{255}
\]

Now, from the above form we can conjecture the general form of the swap operator.

We note that
\[
(1 \otimes 1)_{ij,kl} = \delta_{ik}\delta_{jl}, \tag{256}
\]
so after some trials we conclude

\[
U : \mathbb{C}^n \otimes \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathbb{C}^n
\]
as
\[ U = (U_{ij,kl}) ; \quad U_{ij,kl} = \delta_i^l \delta_j^k, \quad (257) \]

where \( ij = 11, 12, \cdots, 1n, 21, 22, \cdots, 2n, \cdots, n1, n2, \cdots, nn. \)

The proof is simple and as follows.

\[ (a \otimes b)_{ij} = a_i b_j \longrightarrow \{ U(a \otimes b) \}_{ij} = \sum_{kl=1}^{nn} U_{ij,kl} a_k b_l = \sum_{kl=1}^{nn} \delta_i^l \delta_j^k a_k b_l = \sum_{l=1}^{n} \delta_i^l b_l \sum_{k=1}^{n} \delta_j^k a_k = b_i a_j = (b \otimes a)_{ij}. \]

At this stage there is no problem to take a limit \( n \to \infty. \)

Let \( \mathcal{H} \) be a Hilbert space with a basis \( \{ e_n \} \) \( (n \geq 1) \). Then the universal swap operator is given by

\[ U = (U_{ij,kl}) ; \quad U_{ij,kl} = \delta_i^l \delta_j^k, \quad (258) \]

where \( ij = 11, 12, \cdots, \cdots. \)

**Appendix B  Calculation of (252)**

Let us calculate (252) explicitly. Noting \( z_N = z_0 \) and rewriting

\[
\sum_{j=1}^{N} \{ \bar{z}_j (z_j - z_{j-1}) + i \omega \Delta t \bar{z}_j z_{j-1} \}
\]

\[
= \sum_{j=1}^{N} \{ \bar{z}_j z_j - \bar{z}_j (1 - i \omega \Delta t) z_{j-1} \} = (\bar{z}_N, \bar{z}_{N-1}, \cdots, \bar{z}_1) A \begin{pmatrix} z_N \\ z_{N-1} \\ \vdots \\ z_1 \end{pmatrix} \equiv Z^\dagger A Z,
\]
where

\[
A = \begin{pmatrix}
1 & -(1 - i\omega \Delta t) & 0 & \cdots & 0 & 0 \\
0 & 1 & -(1 - i\omega \Delta t) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -(1 - i\omega \Delta t) \\
-(1 - i\omega \Delta t) & 0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

we have

\[
\int_{P_{BC}} \prod_{j=1}^{N} \frac{[d^2z_j]}{\pi} \exp \left\{ -\sum_{j=1}^{N} \{ \bar{z}_j (z_j - z_{j-1}) + i\omega \Delta t \bar{z}_j z_{j-1} \} \right\}
\]

\[
= \int \prod_{j=1}^{N} \frac{[d^2z_j]}{\pi} \exp \left( -\mathbf{Z}^\dagger \mathbf{A} \mathbf{Z} \right) = \frac{1}{\det \mathbf{A}},
\]

where we have used that \( A \) is a normal matrix (\( X^\dagger X = XX^\dagger \)). Since it is easy to see

\[
\det \mathbf{A} = 1 - (1 - i\omega \Delta t)^N,
\]

we obtain

\[
(252) = \lim_{N \to \infty} \frac{1}{1 - (1 - i\omega \Delta t)^N} = \lim_{N \to \infty} \frac{1}{1 - (1 - \frac{i\omega T}{N})^N} = \frac{1}{1 - e^{-i\omega T}}. \quad (259)
\]

This is just (248).

**Appendix C  Representation from \( SU(2) \) to \( SO(3) \)**

We in this appendix give a useful expression to the well–known representation from \( SU(2) \) to \( SO(3) \). This result is no direct relation to the text of this paper, but may become useful in the near future. Now let us define

\[
\rho : SU(2) \longrightarrow SO(3).
\]

First of all we note a simple fact :

\[
g = \begin{pmatrix}
a + ib & c + id \\
-c + id & a - ib \\
\end{pmatrix}, \quad a, b, c, d \in \mathbb{R}
\]
where
\[ g \in SU(2) \iff a^2 + b^2 + c^2 + d^2 = 1. \]

Let us set \( \{ \sigma_1, \sigma_2, \sigma_3 \} \) Pauli matrices
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
and set
\[ \tau_j = \frac{1}{2} \sigma_j \quad \text{for} \quad j = 1, 2, 3. \]

The representation \( \rho \) is given as follows: it is easy to see
\[
\begin{align*}
g^{-1} \tau_1 g &= (a^2 - b^2 - c^2 + d^2) \tau_1 + 2(ab + cd) \tau_2 - 2(ac - bd) \tau_3, \\
g^{-1} \tau_2 g &= -2(ab - cd) \tau_1 + (a^2 - b^2 + c^2 - d^2) \tau_2 + 2(ad + bc) \tau_3, \\
g^{-1} \tau_3 g &= 2(ac + bd) \tau_1 - 2(ad - bc) \tau_2 + (a^2 + b^2 - c^2 - d^2) \tau_3,
\end{align*}
\]
so we have
\[
\left( g^{-1} \tau_1 g, g^{-1} \tau_2 g, g^{-1} \tau_3 g \right) = (\tau_1, \tau_2, \tau_3) \rho(g)
\]
where
\[
G \equiv \rho(g) = \begin{pmatrix}
a^2 - b^2 - c^2 + d^2 & -2(ab - cd) & 2(ac + bd) \\
2(ab + cd) & a^2 - b^2 + c^2 - d^2 & -2(ad - bc) \\
-2(ac - bd) & 2(ad + bc) & a^2 + b^2 - c^2 - d^2
\end{pmatrix}. \quad (260)
\]

Here let us transform the above \( G \). Noting that
\[
\begin{align*}
a^2 - b^2 - c^2 + d^2 &= 1 - 2(b^2 + c^2), \\
a^2 - b^2 + c^2 - d^2 &= 1 - 2(b^2 + d^2), \\
a^2 + b^2 - c^2 - d^2 &= 1 - 2(c^2 + d^2),
\end{align*}
\]
from \( a^2 + b^2 + c^2 + d^2 = 1 \), we have
\[
G = \begin{pmatrix}
1 - 2(b^2 + c^2) & -2(ab - cd) & 2(ac + bd) \\
2(ab + cd) & 1 - 2(b^2 + d^2) & -2(ad - bc) \\
-2(ac - bd) & 2(ad + bc) & 1 - 2(c^2 + d^2)
\end{pmatrix}
\]

61
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} + \begin{pmatrix}
-2(b^2 + c^2) & -2(ab - cd) & 2(ac + bd) \\
2(ab + cd) & -2(b^2 + d^2) & -2(ad - bc) \\
-2(ac - bd) & 2(ad + bc) & -2(c^2 + d^2) \\
\end{pmatrix} \\
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} + \begin{pmatrix}
0 & -2ab & 2ac \\
2ab & 0 & -2ad \\
-2ac & 2ad & 0 \\
\end{pmatrix} + \begin{pmatrix}
-2(b^2 + c^2) & 2cd & 2bd \\
2cd & -2(b^2 + d^2) & 2bc \\
2bd & 2bc & -2(c^2 + d^2) \\
\end{pmatrix} \\
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} + 2a \begin{pmatrix}
0 & -b & c \\
b & 0 & -d \\
-c & d & 0 \\
\end{pmatrix} + 2 \begin{pmatrix}
-(b^2 + c^2) & cd & bd \\
-cd & -(b^2 + d^2) & bc \\
bd & bc & -(c^2 + d^2) \\
\end{pmatrix}
\]

If we define \( M \) as
\[
M = \begin{pmatrix}
0 & -b & c \\
b & 0 & -d \\
-c & d & 0 \\
\end{pmatrix}
\] (261)

then easily
\[
M^2 = \begin{pmatrix}
-(b^2 + c^2) & cd & bd \\
cd & -(b^2 + d^2) & bc \\
bd & bc & -(c^2 + d^2) \\
\end{pmatrix},
\]
so we finally obtain
\[
G = 1 + 2aM + 2M^2. \quad (262)
\]

This equation is very simple and interesting.

The author could not find standard books (not papers) in representation theory which write this equation.
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