Inclusive S- and P-wave charmonium productions in the bottomonium ground state $\eta_b$ decay are calculated at the leading order in the strong coupling constant $\alpha_s$ and quarkonium internal relative velocity $v$ in the framework of the NRQCD factorization approach. We find the contribution of $\eta_b \rightarrow \chi_{cJ} + g$ followed by $\chi_{cJ} \rightarrow J/\psi + \gamma$ is also very important to the inclusive $J/\psi$ production in the $\eta_b$ decays, which maybe helpful to the investigation of the color-octet mechanism in the inclusive $J/\psi$ production in the $\eta_b$ decays in the forthcoming LHCb and SuperB. As a complementary work, we also study the inclusive production of $\eta_c$, and $\chi_{cJ}$ in the $\eta_b$ decays, which may help us understand the $X(3940)$ and $X(3872)$ states.

I. INTRODUCTION

The existence of the spin-singlet state $\eta_b$, which is the ground state of $b\bar{b}$ system, is a solid prediction of the non-relativistic quark model. Since the discovery of its spin-tritlet partner $Y$, people have make great efforts to search for it in various experimental environments, such as in $e^+e^-$ collisions at CLEO [1], in $\gamma\gamma$ collisions at LEP II [2] and in $pp$ collisions at Tevatron [3]. Unfortunately, no evident signal was seen in these attempts. Recently, a significant progress has been achieved by Babar collaboration. After analysing about 10$^8$ data, they observed $\eta_b$ in the photon spectrum of $\Upsilon(3S) \rightarrow \gamma \eta_b$ with a signal of $10\sigma$ significance. They found the hyperfine $\Upsilon(1S) - \eta_b$ mass splitting is 71.4$^{+2.3}_{-3.0}$(stat) $\pm$ 2.7(syst) MeV. Soon after, it was also seen in $\Upsilon(2S) \rightarrow \gamma \eta_b$ by another group in Babar, and the mass splitting is determined to be 67.4$^{+4.8}_{-4.6}$(stat) $\pm$ 2.0(syst) MeV.

On the theoretical side, considerable works have been done to study its properties. The mass of $\eta_b$ has been predicted by potential model [6], effective theory [7] and Lattice QCD [8]. And the recent determinations of $\Upsilon(1S) - \eta_b$ mass splitting in the range of 40 - 60 MeV [4-6, 10-12] are consistent with the Babar’s results. Aside from its mass, the production and decay properties of $\eta_b$ have also been considered. The number of $\eta_b$ produced in $e^+e^- \rightarrow \gamma + \eta_b$ at B-factories [13] is found to exceed that produced at LEP II by about an order of magnitude. In Ref. [14], the authors calculated the production rates of $\eta_b$ at Tevatron Run II and suggested to detect it through the decay of $\eta_b \rightarrow J/\psi J/\psi$, while in Ref. [15], it was thought that the double $J/\psi$ channel might be overestimated and it was suggested the $\eta_b \rightarrow D^*D^{(*)}$ channel to be the most promising channels. An explicit calculation of $\eta_b \rightarrow J/\psi J/\psi$ at NLO in $v^2$ [16] and NLO in $\alpha_s$ shown that this branching fraction is of $10^{-8}$ order which is about four orders of magnitude smaller than that given in Ref. [14]. And the author in Ref. [18] argued the effect of final state interactions in $\eta_b \rightarrow DD^* \rightarrow J/\psi J/\psi$ was also important. Some other exclusive decay modes such as $\eta_b \rightarrow \gamma J/\psi$ [19, 20] and $\eta_b$ decays into double charmonium [21] and inclusive decays, e.g. $\eta_b \rightarrow cc\bar{c}c$, [17] and $\eta_b \rightarrow J/\psi + X$ have also been taken into account.

However, comparing to the $c\bar{c} S_0$ state $\eta_c$, our knowledge about $\eta_b$ is quite limited, and doing some further works is necessary. In this paper, we will systemically study the inclusive decays of $\eta_b$ into S- and P-wave charmonium states. The motivations of this work are fourfold. First, in these processes, the typical energy scale $m_b$ in the initial state and $m_c$ in the final state are both much larger than the QCD scale $\Lambda_{QCD}$, so we can calculate the decay widths perturbatively and the non-perturbative effect plays a minor role, which will reduce the theoretical uncertainties. Second, the branching fraction of the inclusive decay process is much larger than that of the exclusive process, which makes the test of theoretical prediction for the inclusive process be more feasible. Third, in Ref. [22], Hao et al. have calculated the branching ratio of $\eta_b \rightarrow J/\psi + X$ and found the contribution of the color-octet process $\eta_b \rightarrow c\bar{c}cS(1388)^0 + g$ is larger than the one of the color singlet process by about an order. Since the color-octet process also contributes to P-wave states $\chi_{cJ}$ production, in which the $\chi_{c1}$ and $\chi_{c2}$ has about 36% and 20% branching ratio to $J/\psi$ respectively, so we expect that the contribution of $\eta_b \rightarrow \chi_{cJ} + X$ process followed by $\chi_{cJ} \rightarrow J/\psi + \gamma$ might also be important for inclusive $J/\psi$ production in $\eta_b$ decay. Fourth, in recent years, many charmonium or charmonium-like states have been found at B-factory (see Ref. [23, 24, 25] for a review). In
the further coming LHCb and Super-B, when accumulating enough data, it might be possible to observe the interesting decays of $\eta_b$ to $X(3940)$ or $X(3872)$ etc..

The $J/\psi$ inclusive production has already been studied\cite{22}, and the $J/\psi(\eta_c, \chi_c)$ production in association with $c\bar{c}$ pair has been discussed in our previous work\cite{20}. As important supplements, here we are going to consider the contribution of $\eta_b \to \eta_c(\chi_c J) + gg$ process in the non-relativistic limit at leading order in $\alpha_s$.

II. NRQCD FACTORIZATION FORMULISM

Because of the non-relativistic nature of $b\bar{b}$ and $c\bar{c}$ systems, we adopt the non-relativistic QCD (NRQCD) effective theory\cite{27} to calculate the inclusive decay widths of $\eta_b$ to charmonium states. In NRQCD, the inclusive decay and production of heavy quarkonium are factorized into the production of short distance coefficient and the corresponding long distance matrix element. The short distance coefficient can be calculated perturbatively through the expansion of the QCD coupling constant $\alpha_s$. The non-perturbative matrix element, which describes the possibility of the $Q\bar{Q}$ pair transforming into the bound state, is weighted by the relative velocity $v_Q$ of the heavy quarks in the heavy meson rest frame.

In the framework of NRQCD, at leading order in $v_b$ and $v_c$, for the $S$-wave heavy quarkonium production and decay, only the the $Q\bar{Q}$ pair in color-singlet contributes. For $P$-wave $\chi_c J$ production, the color singlet $P$-wave matrix elements and color-octet $S$-wave matrix element are both in the same order of $v_c$. Then the factorization formulas for the processes under consideration in this work are given by:

\begin{equation}
\Gamma(\eta_b \to \eta_c + gg) = \hat{\Gamma}(b\bar{b}(1S_0^1) \to c\bar{c}(1S_0^1) + X) \langle \eta_b | O_0(1S_0^1) | \eta_b \rangle \langle O_0^c(1S_0^1) \rangle,
\end{equation}

\begin{equation}
\Gamma(\eta_b \to \chi_c J + X) = \hat{\Gamma}_1(b\bar{b}(1S_0^1) \to c\bar{c}(3P_J^1) + X) \langle \eta_b | O_b(1S_0^1) | \eta_b \rangle \langle O_1^c(3P_J^1) \rangle + \hat{\Gamma}_8(b\bar{b}(1S_0^1) \to c\bar{c}(3S_1^8) + X) \langle \eta_b | O_b(1S_0^1) | \eta_b \rangle \langle O_8^c(3S_1^8) \rangle,
\end{equation}

where the $\hat{\Gamma}$s are the short-distance factors and $\langle \eta_b | O_0(1S_0^1) | \eta_b \rangle$, $\langle O_0^c(1S_0^1) \rangle$, $\langle O_1^c(3P_J^1) \rangle$ and $\langle O_8^c(3S_1^8) \rangle$ are the long-distance matrix elements. During our calculation of the short distance coefficients associating with the $P$-wave color-singlet matrix elements, there will appear infrared divergence. This divergence will be absorbed into the color octet matrix element $\langle O_8^c(3S_1^8) \rangle$.

III. $\eta_b \to \eta_c + gg$

We first consider the $S$-wave $\eta_c$ production from $\eta_b$ decay. At leading order in $\alpha_s$, there are eight Feynman diagrams for $(b\bar{b}(1S_0^1) \to c\bar{c}(1S_0^1) + gg$. The typical one is shown in Fig.1a. The general form of the short distance coefficient can be expressed as:

\begin{equation}
\hat{\Gamma}(b\bar{b}(1S_0^1) \to c\bar{c}(1S_0^1) + gg) = \frac{\alpha_s}{m_b^2} f(r),
\end{equation}

where $r = m_c/m_b$ is a dimensionless parameter. Since there is no infrared divergence, we calculate $f(r)$ directly using the standard covariant projection technique\cite{28}. Given $m_b = 4.65$GeV, $m_c = 1.5$GeV, we get $f(r) = 23.1$. In NRQCD, up to $\alpha_s$ order, the relations between the color singlet matrix elements and the non-relativistic wave functions are:\cite{27}

\begin{equation}
\langle \eta_b | O_0(1S_0^1) | \eta_b \rangle = \frac{1}{4\pi} |R_{1S}^b(0)|^2 (1 + O(v_b^4)),
\end{equation}

\begin{equation}
\langle O_0^c(1S_0^1) \rangle = \frac{1}{4\pi} |R_{1S}^c(0)|^2 (1 + O(v_c^4)).
\end{equation}

In order to compare with our previous work before, we choose the same numerical values with $m_b = 4.65$GeV, $m_c = 1.5$GeV, $\alpha_s = 0.22$, $|R_{1S}^c(0)|^2 = 0.81$GeV$^2$, and $|R_{1S}^b(0)|^2 = 6.477$GeV$^2$\cite{29}. Then we get

\begin{equation}
\Gamma(\eta_b \to \eta_c + gg) = 0.83 \text{ kev}.
\end{equation}

\footnote{For the color-singlet four-fermion operators, there is a additional $\frac{1}{\Lambda_{QCD}}$ factor compared to these in Ref.\cite{27}.}
In our previous work, we got \( \Gamma(\eta_c \rightarrow g g) \approx 9.67 \text{MeV} \). (5)

In our previous work, we got \( \Gamma(\eta_c \rightarrow g g) \approx 0.27 \text{keV} \). So the branching ratio of inclusive decay of \( \eta_c \) into \( \eta_b \) is

\[
\text{Br}(\eta_b \rightarrow \eta_c + X) = 1.1 \times 10^{-4},
\]

in which the contribution of \( g g \) process is about 3 times larger than that of the \( c \bar{c} \) process. The re-scaled energy distribution curve \( d\Gamma/dx_{13} \) for \( \eta_b \rightarrow \eta_c + X \) is shown in Fig.[2], where \( x_1 \) is the ratio of \( \eta_c \) energy \( E_{\eta_c} \) to \( m_b \).

Recently the \( X(3940) \) state was observed by the Belle Collaboration in the recoiling spectrum of \( J/\psi \) in \( e^+e^- \) annihilation \[30\]. It is most likely to be a \( \eta_c(3S) \) state \[31\]. In the non-relativistic limit, the only difference between \( \eta_c \) and \( \eta_c(3S) \) is the value of wave function. If \( X(3940) \) is the \( \eta_c(3S) \) state, we predict the branching ratio of \( X(3940) \) production in \( \eta_b \) decay to be

\[
\text{Br}(\eta_b \rightarrow X(3940) + X) \approx 0.62 \times 10^{-4}.
\]

To obtain the prediction, we have chose \( |R_{3S}^0(0)|^2 = 0.455\text{GeV}^3 \) to take the place of \( |R_{1S}^c(0)|^2 = 0.81\text{GeV}^3 \).

IV. \( \eta_b \rightarrow \chi_{cJ} + g g \)

As mentioned above, the color singlet short distance coefficients are infrared divergent in full QCD calculation. We will adopt the dimensional regularization scheme to regularize the divergence. To absorb the divergence into the color-octet matrix elements \( \langle O_{c\bar{c}}^{c\bar{c}}(3S^1_c) \rangle \), we are necessary to calculate the color-octet short distance coefficient in \( D = 4 - 2\epsilon \) dimensions. The \( b\bar{b}(1S^1) \rightarrow c\bar{c}(3S^1_c) + g \) process includes two Feynman diagram, one of which is shown in Fig.[1b]. Using the \( D \) dimension spin projector expression \[32\], at leading order in \( \alpha_s \), the short distance factor is given by

\[
\hat{\Gamma}(b\bar{b}(1S^1_0) \rightarrow c\bar{c}(3S^1_0) + g) = \frac{4\pi\alpha_s^3}{24m_b^2r^3}\frac{\Phi_2}{(D-2)(D-3)},
\]

where \( \Phi_2 = (\frac{m_b^2}{m_b^2})^{\Gamma(1-x)(1-\epsilon)} \) is the 2-body phase space in \( D \)-dimensions.

The calculation of the color-singlet coefficient in full QCD is a little more complicate. The Feynman diagrams for \( b\bar{b}(1S^1_0)(P) \rightarrow c\bar{c}(3S^1_0)(p_1) + g(p_2)g(p_3) \) is the same as those for \( \eta_c \) production process. Such \( 1 \rightarrow 3 \) process can be described by the following invariants:

\[
x_1 = \frac{2P \cdot p_1}{M^2}, \sum x_i = 2,
\]

where \( M = 2m_b \). In \( D = 4 - 2\epsilon \) dimensions, the three-body phase space is given by

\[
d\Phi_2 = KK = \frac{4\pi^{3/2}}{3\epsilon^2} M^3, \quad b \bar{b} = (2 - x_1)/2 \quad \text{and} \quad K = \frac{32\pi^{1/2}}{\Gamma(2-2\epsilon)}.
\]

In calculating the amplitude, we put the four diagrams with the gluon carrying a momentum of \( p_2 \) emitted from charm quark line together and label their total amplitude with \( M_2 \) and the total amplitude of the left four diagrams with the gluon carrying a momentum of \( p_3 \) emitted from charm quark line is represented by \( M_3 \). The total amplitude \( M = M_2 + M_3 \) and \( |M| = |M_2|^2 + |M_3|^2 + 2\text{Re}(M_2M_3) \).

As being illustrated in Ref.[27], for P-wave case when \( p_1(i = 2, 3) \) goes to zero, there will be singularities in \( M_1 \). However, because of the four-momentum conservation, \( p_2 \) and \( p_3 \) can not be soft simultaneously in the phase space. Therefore, the integration of the interference term \( 2\text{Re}(M_2M_3) \) is finite. We could perform it in 4-dimensions directly. For the symmetry of the two gluons, the result of phase space integration for \( |M_2|^2 \) and \( |M_3|^2 \) are equal to each other. We only need to calculate one of them. The total \( \hat{\Gamma} \) then could be written as

\[
\hat{\Gamma}_1 = 2\hat{\Gamma}_M + \hat{\Gamma}_{\text{int}}.
\]

where \( \Gamma_M \) and \( \Gamma_{\text{int}} \) are the contribution related to \( |M_2|^2 \) and \( 2\text{Re}(M_2M_3) \) respectively.

We now present how we calculate \( \hat{\Gamma}_M \) in detail. The denominator of charm-quark propagator in Fig.[1a] is

\[
(p_2 - p_3)^2 - m_c^2 = -2p_2 \cdot p_3 \mid_{q_c=0} \propto (1 + r^2 - x_1 - x_2),
\]

where \( p_c = \frac{p_2 - p_3}{2} - q_c \) is the momentum of anti-charm quark and \( q_c \) is the relative momentum of \( c \) and \( \bar{c} \).
When \( \overline{c} \overline{e} \) in P-wave configuration, we need to know the first derivative of the amplitude with respect to \( q_e \). Then in the non-relativistic limit, there will be three kinds of the divergences in \( |M_2|^2 \), which are proportional to \( \frac{x_3^2}{(1+r^2-x_1-x_2)^2} \) or \( \frac{x_3^2}{(1+r^2-x_1-x_2)} \). These terms will be divergent at point \( (x_1, x_2) = (1 + r^2 , 0) \) which are not easily to be integrated out. We introduce two new variables \( (x_1', x_2') \), defined by

\[
x_1' = x_1, x_2' = 1 - \frac{1 + r^2 - x_1}{x_2}.
\]

(13)

In the variables \( x_1' \) and \( x_2' \), the phase space is re-expressed as:

\[
d\Phi_{(3)} = \frac{\pi^{2r} m_b^{2-4\epsilon}}{32\pi^3 \Gamma(2 - 2\epsilon)} \int_{2r}^{1+r^2} dx_1' \int_{1-(bb+aa)}^{1+(bb-aa)} dx_2' (1 - x_2')^2 (1 + r^2 - x_1)^{1-2\epsilon}((aa + bb - x_2)((1 - x_2' - \frac{1}{bb + aa})^{-\epsilon}),
\]

(14)

where \( aa = \sqrt{x_1'-1}, bb = -(x_1'-1) \) and \( x_2 = 1 + x_2' - x_1' \).

And the three divergence structures are changed to be the form of \( \frac{-1}{x_2' (1 + r^2 - x_1'} \), \( \frac{-1}{x_2' (1 + r^2 - x_2') \) and \( \frac{-1}{x_2' (1 + r^2 - x_2') \) respectively, which are all proportional to \( \frac{1}{(1+r^2-x_1')^2} \).

Then \( |M_2|^2 \) could be expanded as

\[
|M_2|^2 = \frac{f_1(1 + r^2, x_1', x_2', \epsilon)}{(1 + r^2 - x_1')^2} + f_2(x_1', x_2', \epsilon).
\]

(15)

Accordingly,

\[
\hat{\Gamma}_{M_2} = \hat{\Gamma}_{M_2}^{\text{div}} + \hat{\Gamma}_{M_2}^{\text{fin}},
\]

(16)

Where \( \hat{\Gamma}_{M_2}^{\text{fin}} \) is finite and can be calculate in \( D = 4 \) dimensions. The phase space integration of the first term in Eq. (15) is expressed as

\[
\int d\Phi_{(3)} \frac{f_1(1 + r^2, x_1', \epsilon)}{(1 + r^2 - x_1')^2} = KK \int_{2r}^{1+r^2} dx_1' g(x_1', \epsilon) \frac{(1 - x_1')^{1+2\epsilon}}{(1 + r^2 - x_1')^{1+2\epsilon}},
\]

(17)

where

\[
g(x_1', \epsilon) = \int_{1-(bb+aa)}^{1+(bb-aa)} \frac{f_1(1 + r^2, x_2', \epsilon)}{(1 - x_1')^2} ((aa + bb - x_2)((1 - x_2' - \frac{1}{bb + aa})^{-\epsilon}) dx_2'.
\]

(18)

Furthermore, the integrals in Eq. (17) can be written into the sum of two terms defined by:

\[
\int_{2r}^{1+r^2} dx_1' g(x_1', \epsilon) = \int_{2r}^{1+r^2} dx_1' g(1 + r^2, \epsilon) \frac{(1 + r^2 - x_1')^{1+2\epsilon}}{(1 + r^2 - x_1')^{1+2\epsilon}},
\]

(19)

The first term on the right side includes \( \delta \) pole, and the second term is finite. Therefore we only need to keep the \( O(\epsilon) \) contribution when calculating \( g(1 + r^2, \epsilon) \) and the second term can evaluated directly by setting \( \epsilon = 0 \).

Putting Eq.(11) and (16) into together, we get

\[
\hat{\Gamma}_1 = 2(\hat{\Gamma}_{M_2}^{\text{div}} + \hat{\Gamma}_{M_2}^{\text{fin}}) + \hat{\Gamma}_{\text{Int}}.
\]

(20)

\( \hat{\Gamma}_{M_2}^{\text{div}} \) is calculated analytically, and \( \hat{\Gamma}_{M_2}^{\text{fin}} \) and \( \hat{\Gamma}_{\text{Int}} \) are calculated numerically. For \( J = 0, 1, 2 \), the expressions for \( \hat{\Gamma}_{M_2}^{\text{div}} \) are

\[
\hat{\Gamma}_{M_2}^{\text{div}} = \frac{128 (-1 + r^2) C_A C_F (\alpha_s \pi \mu^{2\epsilon})^4 K K}{81 m_b^2 r^5 \epsilon} + \frac{64 C_A C_F \pi^4 \alpha_s^4 K K (4 - 4 r^2 + 24 (1 - r^2 (3 - 3 r^2 + r^4)) \log(1 - r^2) + 12 (1 + r^6) \log(r) + 12 (1 - 3 r^2 + 3 r^4 - r^6) \log(r)}{243 m_b^2 r^5 (-1 + r^2)^3} (J = 0)
\]

(21a)

\[
\hat{\Gamma}_{M_2}^{\text{div}} = \frac{128 (-1 + r^2) C_A C_F (\alpha_s \pi \mu^{2\epsilon})^4 K K}{81 m_b^2 r^5 \epsilon} + \frac{128 C_A C_F \pi^4 \alpha_s^4 K K}{243 m_b^2 r^3 (-1 + r^2)^3} (2 - 9 r^2 + 9 r^4 - 2 r^6 + 3 (2 - 3 r^2 - 3 r^4 + 2 r^6) \log(r) + 12 (1 - 3 r^2 + 3 r^4 - r^6) \log(r)} (J = 1)
\]

(21b)

\[
\hat{\Gamma}_{M_2}^{\text{div}} = \frac{128 (-1 + r^2) C_A C_F (\alpha_s \pi \mu^{2\epsilon})^4 K K}{81 m_b^2 r^5 \epsilon} + \frac{128 C_A C_F \pi^4 \alpha_s^4 K K}{1215 m_b^2 r^5 (-1 + r^2)^3} (10 - 27 r^2 + 27 r^4 - 10 r^6 + 3 (10 - 9 r^2 - 9 r^4 + 10 r^6) \log(r) - 60 (1 + r^2)^3 \log(1 - r^2)) (J = 2).
\]

(21c)

The \( C_A = 3 \) and \( C_F = 4/3 \) in above equations are the color factors. It can be seen that for different \( J \) the diver-
gence part of $\tilde{\Gamma}_{\mu\mu}$ are the same, which will be absorbed into the color-octet matrix element. And $2\Gamma_{M_2}^{\text{fin}} + \Gamma_{\text{Int}}$ are

$$2\Gamma_{M_2}^{\text{fin}} + \Gamma_{\text{Int}} = \frac{\alpha_s}{m_b^2} A_J(r) \text{ (for } J = 0, 1, 2). \tag{22}$$

When $r = 1.5/4.65$, we obtain $A_0(r) \simeq -9.71 \times 10^2$, $A_1(r) \simeq -2.66 \times 10^2$ and $A_2(r) \simeq -6.06 \times 10^2$.

To cancel the infrared divergence of $\Gamma_{M_2}^{\text{div}}$, we also need to take into account the renormalization of $\langle O^{c\bar{f}c}_{\alpha}(3 S_1^0) \rangle^\Lambda$. In $\overline{MS}$ scheme, it is given by $\overline{24, 32}$

$$\langle O^{c\bar{f}c}_{\alpha}(3 S_1^0) \rangle^\Lambda = \langle O^{c\bar{f}c}_{\alpha}(3 S_1^0) \rangle^{\text{(Born)}} - \frac{4\alpha_s C_F}{3\pi m_c^2} \left( 1 + \log 4\pi - \gamma_E \right) \left\{ \left( \frac{\mu}{\Lambda} \right)^2 \sum_{J=0}^2 \langle O^{c\bar{f}c}_{\alpha}(3 P_J^0) \rangle \right\}. \tag{23}$$

Combining the results in Eq.(8,21,22,23), we finally obtain the infrared safe expressions for inclusive decay of $\eta_b$ into $\chi_{cJ}(J = 0, 1, 2)$ states

$$\Gamma(\eta_b \to \chi_{cJ} + X) = \Gamma_0^J + \Gamma_1^J, \tag{24}$$

where $\Gamma_0^J$ is

$$\frac{2\pi^2 \alpha_s^3(1 - r^2)}{9m_b^3 r^3} \langle \eta_b | \cal{O}_b^{(1 S_0^0)} | \eta_b \rangle \langle \cal{O}^{c\bar{f}c}_{\alpha}(3 S_1^0) \rangle, \tag{25}$$

and $\Gamma_1^J$ are

$$\begin{align*}
\Gamma_1^0 &= \frac{8\pi \alpha_s^4 \langle \eta_b | \cal{O}_b^{(1 S_0^0)} | \eta_b \rangle \langle \cal{O}^{c\bar{f}c}_{\alpha}(3 P_0^0) \rangle}{243 m_b^6 r^3 (1 - r^2)^2} \left( 12(r^6 + 1) \log(r) + 24(1 - 3r^2 + 3r^4 - r^6) \log(1 - r^2) + 2(1 - r^2)(6 \log 2 - 5)r^4 - 4(3 \log 2 - 4)r^2 + 6 \log 2 - 5 + 6(1 - r^2)^2 \log\left( \frac{m_b}{\mu} \right) \right) + \frac{243 r^5 (1 - r^2)^2 A_0(r)}{8\pi}, \tag{26a} \\
\Gamma_1^1 &= \frac{16\pi \alpha_s^4 \langle \eta_b | \cal{O}_b^{(1 S_0^0)} | \eta_b \rangle \langle \cal{O}^{c\bar{f}c}_{\alpha}(3 P_1^0) \rangle}{243 m_b^6 r^3 (1 - r^2)^2} \left( 3(2r^6 - 3r^4 - 3r^2 + 2) \log(r) + 12(1 - r^2)^3 \log(1 - r^2) + (1 - r^2)(6 \log 2 - 5)r^4 + (7 - 12 \log 2)r^2 + 6 \log 2 - 5 + 6(1 - r^2)^2 \log\left( \frac{m_b}{\mu} \right) \right) + \frac{243 r^5 (1 - r^2)^2 A_1(r)}{16\pi}, \tag{26b} \\
\Gamma_1^2 &= \frac{16\pi \alpha_s^4 \langle \eta_b | \cal{O}_b^{(1 S_0^0)} | \eta_b \rangle \langle \cal{O}^{c\bar{f}c}_{\alpha}(3 P_2^0) \rangle}{1215 m_b^6 r^3 (1 - r^2)^2} \left( 3(10r^6 - 9r^4 - 9r^2 + 10) \log(r) + 60(1 - r^2)^3 \log(1 - r^2) + (1 - r^2)(56 \log 2 - 5)r^4 + (53 - 60 \log 2)r^2 + 5(6 \log 2 - 5) + 30(1 - r^2)^2 \log\left( \frac{m_b}{\mu} \right) \right) + \frac{1215 r^5 (1 - r^2)^2 A_2(r)}{16\pi}. \tag{26c}
\end{align*}$$

It can be seen that the contribution of $P$-wave color-singlet is dependent on the factorization scale $\mu_A$. When combining it with the color-octet $S$-wave contribution, in which the matrix element also depends on $\mu_A$, the $\mu_A$-dependence will be canceled.

To give numerical predictions, we also need to know the values of the long-distance matrix elements. The color octet matrix elements can be studied in lattice simulations, fitted to experimental data phenomenologically or determined through some other non-perturbative ways. Here we determined their numerical values with the help of operator evolution equations. In the decay process, the solution of the operator evolution equations are $\overline{24}$

$$\langle \chi_{cJ} | \cal{O}_b^{(3 S_1^0; \mu_b)} | \chi_{cJ} \rangle = \langle \chi_{cJ} | \cal{O}_b^{(3 S_1^0; \mu_b)} | \chi_{cJ} \rangle + \frac{8 C_F}{3\beta_0 \mu_c^2} \ln \frac{\alpha_s(\mu_A)}{\alpha_s(\mu_b)} \langle \chi_{cJ} | \cal{O}_1^{(3 P_J)} | \chi_{cJ} \rangle, \tag{27}$$

where $\beta_0 = \frac{11\chi_c - 2N_c}{6}$. We then naively relate the matrix element of production operator $\cal{O}_n^H$ and that of the decay operator $\cal{O}_n$ using

$$\langle \cal{O}_n^H \rangle \approx 2J + 1 \langle \cal{O}_n | H \rangle. \tag{28}$$

When $\mu_A \gg \mu_{\mu_b}$, the evolution term will be dominated, and the contribution of the initial matrix elements can be neglected. Since the operator evolution hold only down to scale $m_{\mu_b}$, we set the lower bound $\mu_{\mu_b} = m_{c\bar{c}}$ and choose $v^2 = 0.3$. And we set $\mu_A = 2m_c$ since the divergence comes from the soft gluons linked with the $c\bar{c}$ pair. The $P$-wave color singlet matrix elements can be estimated through their relations with the first derivative of the non-relativistic wave function at origin which, in
non-relativistic limit, is given by
\[
\langle O_c^{XJ} (\ell^{P[1]}_J) \rangle \approx \frac{3(2J+1)}{4\pi} |R'_c(0)|^2.
\] (29)

Setting \( N_f = 3 \), \( \Lambda_{QCD} = 390\text{MeV} \) and \( |R'_c(0)|^2 = 0.075\text{GeV}^5 \), we obtain
\[
\Gamma(\eta_b \rightarrow \chi_{cJ} + gg) = (0.17, 1.55, 1.76)\text{keV} \quad \text{(for} J = 0, 1, 2) .
\] (30)
The \( \eta_b \rightarrow \chi_{cJ} + c\bar{c} \) processes have been considered in our previous work, in which both the color-singlet and color-octet contributions have been included but with different values of the color-octet matrix elements \cite{32}. If we use the color-octet matrix elements determined in this work, the results now become:
\[
\Gamma(\eta_b \rightarrow \chi_{cJ} + c\bar{c}) =
\begin{cases}
(4.54, 4.21, 4.28) 	imes 10^{-2}\text{keV} & \text{(for} J = 0, 1, 2) .
\end{cases}
\] (31)

which are about an order of magnitude less than the widths of \( \eta_b \rightarrow \chi_{cJ} + gg \) processes respectively. Including the contribution of the associate processes, we then predict that the branching ratios for \( \eta_b \) inclusive decay into \( \chi_{cJ} \) are
\[
\text{Br}(\eta_b \rightarrow \chi_{cJ} + X) =
\begin{cases}
(0.22, 1.65, 1.87) 	imes 10^{-4} & \text{(for} J = 0, 1, 2) .
\end{cases}
\] (32)

The \( X(3872) \) state was discovered in \( pp \) collisions at Tevatron \cite{32} and \( B \) decay at Belle \cite{34}. Until now, people have not found an convincing explanation about it yet. The authors in \cite{32} suggest it is a \( \chi_{c1}(2P) \) state. If it is a \( \chi_{c1}(2P) \) state, we roughly predict
\[
\text{Br}(\eta_b \rightarrow X(3872) + X) = 2.25 	imes 10^{-4} ,
\] (33)
where we have chose \( |R'_c(0)|^2 = 0.102\text{GeV}^5 \) and assumed the ratio between color-singlet and color-octet matrix elements does not change for \( 2P \) state.

\[
\frac{\langle O_c^{XJ} (\ell^{S[8]}_J) \rangle}{\langle O_c^{XJ} (\ell^{P[1]}_J) \rangle} = \frac{\langle O_c^{X(3872)} (\ell^{S[8]}_J) \rangle}{\langle O_c^{X(3872)} (\ell^{P[1]}_J) \rangle} .
\] (34)

In \cite{22}, the authors have studied the \( \eta_b \rightarrow J/\psi + X \) process with \( \Gamma(\eta_b \rightarrow J/\psi + X) = 2.29\text{keV} \). They found the contribution of color-octet process \( \eta_b \rightarrow J/\psi + color-octet + X \) is more than one order of magnitude larger than that of the color-singlet contribution. Since \( \chi_{c1} \) and \( \chi_{c2} \) could also decay to \( J/\psi + \gamma \) with \( \text{Br}(\chi_{c1} \rightarrow J/\psi + \gamma) = 36\% \) and \( \text{Br}(\chi_{c2} \rightarrow J/\psi + \gamma) = 20\% \). The branching ratio of \( \chi_{c0} \rightarrow J/\psi + \gamma \) is so small that the contribution of this process can be neglected. Then re-scaling our result by the values of parameters in Ref. \cite{22}, we find the \( \chi_{cJ} \) feed-down contribution to the decay of \( \eta_b \) into \( J/\psi \) is:
\[
\Gamma(\eta_b \rightarrow (J/\psi + \gamma)_{\chi_{cJ}} + X) = 0.71\text{keV} ,
\] (35)
which is about three times larger than that of color-singlet process. Therefore in the future experiment, when measuring the \( J/\psi \) production in \( \eta_b \) decay, the contribution of \( \eta_b \) decays into \( \chi_{cJ} \) followed by \( \chi_{cJ} \rightarrow J/\psi + \gamma \) are also important.

V. SUMMARY

In this work, we have studied the inclusive production of charmonium state \( \eta_b, \chi_{cJ} \) in the decay of ground bottomonium state \( \eta_b \) within the framework of NRQCD factorization formula. We find for the \( P \)-wave states \( \chi_{cJ} \) case, the color-singlet processes \( bb(1S_0) \rightarrow c\bar{c}(3P[1]) + gg \) include infrared divergence. We show that such divergence can be absorbed into the \( S \)-wave color-octet matrix element. To give numerical predictions, we use the potential model results to determine the color-singlet matrix elements and estimate the color-singlet matrix elements with the help of operator evolution equations naively. We find that the branching ratios of \( \eta_b \) decays into \( \eta_c \) or \( \chi_{cJ} \) plus anything are all of \( 10^{-4} \) order. Furthermore, we also give the branching ratios of \( \eta_b \rightarrow X(3940) + X \) and \( \eta_b \rightarrow X(3872) + X \), if the \( X(3940) \) and \( X(3872) \) are the excited \( \eta_c(3S) \) and \( \chi_{c1}(2P) \) states respectively. In Ref. \cite{22}, the authors investigated the color-octet mechanism for \( J/\psi \) production in \( \eta_b \) decay, our results show that the \( J/\psi \) production from \( \chi_{cJ} \) feed-down is also important, since it is about three times larger than the direct \( J/\psi \) production via color-singlet channel. These theoretical predictions may not be observed in experiment for the time being, but will be very helpful to study the \( \eta_b \)'s properties in the future experiment such as Super-B.

Acknowledgement

We would like to thank Yu Jia for helpful discussions. The author Zhi-Guo He also thanks to the organization of the ”Effective Field Theories in Particle and Nuclear Physics” by KITPC Beijing.

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