Adjacent-vertex-distinguishing proper total colouring number of $\overline{K_m} \lor P_n$

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Abstract. Coloring is a fundamental problem in scientific computation and engineering design. In recent years, a variety of colorings frequently appeared and solved many problems in production. For example, adjacent-vertex-distinguishing proper total coloring, adjacent-vertex-distinguishing proper edge coloring, smarandachely-adjacent-vertex-distinguishing proper edge coloring. It is an important also difficult problem to discuss the coloring numbers of a given graph class. And we focus on the adjacent-vertex-distinguishing proper total coloring numbers in this paper. We study the adjacent-vertex-distinguishing proper total coloring numbers of joint graphs $\overline{K_m} \lor P_n$.

1 Introduction
Coloring problem in graph theory is one of the most famous NP-complete problems. Four-color-conjecture which is one of the world’s three major mathematical conjectures says that each map can only use four colors to dye, and no two adjacent areas dyed same color. In the spring of 1976, with the help of the computer, the four-color-conjecture was proved. The conjecture finally became a theorem. The significance of graph coloring theory is much more than that. Known to all, coloring problems can solve many problems such as scheduling problem, time tabling, transportation, arrangement, circuit design and storage problems.

In recent years, more and more colouring concepts was put forward by experts of graph theory, such as adjacent-vertex-distinguishing proper edge colouring, adjacent-vertex-distinguishing proper total colouring, smarandachely adjacent-vertex-distinguishing proper edge colouring. Symbol $\Delta$ in the paper always denote the maximum degree of the graphs discussed.

Definition 1[1] A k-proper total colouring for a graph $G$ is a mapping $f$ from $V(G) \cup E(G)$ to $\{1, 2, \ldots, k\}$ such that:

1) $\forall u, v \in V(G)$, if $uv \in E(G)$, then $f(u) \neq f(v)$;
2) $\forall e_i, e_j \in E(G), e_i \neq e_j$, if $e_i, e_j$ have a common end vertex, then $f(e_i) \neq f(e_j)$;
3) $\forall u \in V(G), e \in E(G)$, if $u$ is the end vertex of $e$, then $f(u) \neq f(e)$.

Let $f$ be a k-proper-total-colouring of $G$. Denote $C(u) = \{f(u)\} \cup \{f(\nu)\} \vert \nu \in V(G) \land \nu \in E(G)$ for every vertices $u \in V(G)$, if $\forall u, v \in V(G), uv \in E(G)$, we have $C(u) \neq C(v)$, then $f$ is called a k-adjacent-vertex-distinguishing proper total colouring, short for k-AVDTC.
The number \( \min \{ k \mid \chi(G) \text{ has a } k\text{-adjacent-vertex-distinguishing proper total colouring} \} \) is called the adjacent –vertex-distinguishing proper total colouring number and denoted by \( \chi_{av}(G) \). This concept was put forward by Zhang Zhongfu in [1], and researched by many students and researchers of graph theory. Zhang Zhongfu put forward a conjecture on it such that:

**Conjecture 1**[1] For every connected graph \( G \) with order at least 2, we have \( \chi_{av}(G) \leq \Delta + 3 \).

In the same time, Zhang Zhongfu put forward an open question in [1] such that:

**Question:** If \( H \) is a subgraph of \( G \), when there is \( \chi_{av}(H) \leq \chi_{av}(G) \) ?

Huang Danjun give an upper bound of the adjacent-vertex-distinguishing proper total colouring number when the maximum degree of the graph \( \Delta \geq 3 \) [4].

**Theorem:** If \( \Delta \geq 3 \), then \( \chi_{av}(G) \leq 2\Delta(G) \).

**Question:** For which kind of graph \( G \) the upper bound of the \( \chi_{av}(G) \) can reduce to \( 2\Delta(G) \) – 1.

Papers [5-8] discussed the adjacent-vertex-distinguishing proper total colouring of the complete multiple graphs, generalized halin graphs, outer planar graphs and unicyclic graphs respectively.

Determining the adjacent-vertex-distinguishing proper total colouring number of a given type of graphs is a main problem in the question, and common methods that used in colouring problems are such that: giving the specific methods of the colouring; combination analysis methods; probabilistic methods.

**Lemma 1** If two arbitrary distinct vertices of maximum degree in \( G \) are not adjacent, then \( \chi_{av}(G) \geq \Delta + 1 \); If \( G \) has two distinct vertices of maximum degree which are adjacent, then \( \chi_{av}(G) \geq \Delta + 2 \).

**Proof** For every graph \( G \), colours that are used in the vertices and it’s adjacent edges are more than \( \Delta + 1 \), so we get the result such as \( \chi_{av}(G) \geq \Delta + 1 \). If there are two vertices whose degree are the maximum of the graph who is denoted by \( u \) and \( v \), then the cardinal numbers of the colour sets for \( u \) and \( v \) satisfied that \( |C(u)| = |C(v)| = \Delta + 1 \), but to be a \( k\)–adjacent-vertex-distinguishing proper total colouring, There must be the condition such as \( C(u) \neq C(v) \), so \( |C(u) \cup C(v)| \geq \Delta + 1 + 1 = \Delta + 2 \), this shows the results \( \chi_{av}(G) \geq \Delta + 2 \). Now, lemma 1 is proved.

In the following paper, we will study the adjacent- vertex-distinguishing proper total colouring of the graph \( \overline{K}_m \lor P_n \) which is a joint graph which are jointed by \( \overline{K}_m \) and \( P_n \). \( \overline{K}_m \) denotes graphs with order \( m \) and also have no edge. \( P_n \) denotes the path graphs with order \( n \). That is to say

\[
\overline{K}_m = (V_1, E_1), E_1 = \{u_1, u_2, \cdots, u_m \}, E_2 = \Phi, P_n = (V_2, E_2), V_2 = \{v_1, v_2, \cdots, v_n \}, E_2 = \{(v_{i-1}, v_i) \mid 2 \leq i \leq n\}
\]

So we get the graph \( \overline{K}_m \lor P_n \) such that:

\[
\overline{K}_m \lor P_n = (V, E)
\]

\[
V = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\},
\]

\[
E = \{(v_{i-1}, v_i) \mid 1 \leq i \leq n\} \cup \{(u_j, v_i) \mid 1 \leq j \leq m, 1 \leq i \leq n\}
\]

This paper use the methods of apagoge, construction and direct proving.

### 2 Adjacent-vertex-distinguishing proper total colouring of \( \overline{K}_2 \lor P_n \).

In this part, we consider the situation such as \( m = 2 \). Then for easy calculation, we define the vertex set \( V_i = \{v_0, w_0\} \), then we can get the results such that:
Theorem 1: $\chi_{aw}(\overline{K_2 \lor P_2}) = 5$

Proof: There are two adjacent vertex $v_0$ and $w_0$ whose degree (= 3) are the maximum degree, so concluded by Lemma 1, we get such that:

$\chi_{aw}(\overline{K_2 \lor P_2}) \geq 5$

Then we let $f$ be a mapping from $V(\overline{K_2 \lor P_2}) \cup E(\overline{K_2 \lor P_2})$ to $\{1, 2, 3, 4, 5\}$ such as follows:

$f(v_0) = f(w_0) = f(v_1v_2) = 5$ ;
$f(v_0v_1) = 1$
$f(v_0v_2) = f(v_1) = 2$
$f(w_0v_1) = f(v_2) = 3$
$f(w_0v_2) = 4$

By comparing the colour sets of all vertices for the graph, it is easy can be seen that $f$ is a AVDTC, so

$\chi_{aw}(\overline{K_2 \lor P_2}) = 5$.

Theorem 2: $\chi_{aw}(\overline{K_2 \lor P_3}) = 5$

Proof: we can see that vertex $v_3$ is the only vertex whose degree (= 4) is the maximum degree of the graph. so concluded by Lemma 1, we get the result such as:

$\chi_{aw}(\overline{K_2 \lor P_3}) \geq 5$

Then we let $f$ be a mapping from $V(\overline{K_2 \lor P_3}) \cup E(\overline{K_2 \lor P_3})$ to $\{1, 2, 3, 4, 5\}$ as follows:

$f(v_0) = f(w_0) = f(v_2v_3) = 5$
$f(v_0v_2) = f(w_0v_3) = f(v_1) = 2$
$f(v_0v_1) = f(w_0v_2) = 1$
$f(w_0v_1) = f(v_0v_3) = f(v_2) = 3$
$f(v_1v_2) = f(v_3) = 4$

Listing the colour sets of all vertices for the mapping, we can see that $f$ is a AVDTC, so

$\chi_{aw}(\overline{K_2 \lor P_3}) = 5$.

Theorem 3: $\chi_{aw}(\overline{K_2 \lor P_4}) = 6$

Proof: There are two adjacent vertex $v_2$ and $v_3$ whose degree (= 4) are the maximum degree, so concluded by Lemma 1, we get the result such as:

$\chi_{aw}(\overline{K_2 \lor P_4}) \geq 6$

Then we let $f$ be a mapping from $V(\overline{K_2 \lor P_4}) \cup E(\overline{K_2 \lor P_4})$ to $\{1, 2, 3, 4, 5, 6\}$ as follows:

$f(v_0) = f(w_0) = f(v_3v_4) = 6$
$f(v_0v_1) = f(w_0v_2) = 1$
$f(v_0v_2) = f(w_0v_3) = f(v_1) = 2$
$f(w_0v_4) = f(v_0v_3) = f(v_2) = 3$
$f(v_1v_2) = f(v_0v_4) = f(v_3) = 4$
$f(v_2v_3) = f(w_0v_1) = f(v_4) = 5$
By listing colour sets of all vertices for the mapping $f$, we can see that $f$ is an AVDTC, so

$$\chi_{av}((\overline{K}_2 \lor P_n)) = 6.$$  

**Theorem 4** If $n > 4$, $\chi_{av}((\overline{K}_2 \lor P_n)) = n + 1$

**Proof** From the graph it can be seen that vertex $v_0$ and $w_0$ all have the maximum degree (=$n$), and the other vertex's degree are less than $n$, so concluded by Lemma 1, it can reach the conclusion such as:

$$\chi_{av}((\overline{K}_2 \lor P_n)) \geq n + 1$$

Then we let $f$ be a mapping from $V((\overline{K}_2 \lor P_n) \cup E((\overline{K}_2 \lor P_n))$ to $\{1,2,3,4,5,6\}$ such as:

$$f(v_0) = f(w_0) = n + 1$$

$$f(v_i) = i + 1 \quad (1 \leq i \leq n - 1)$$

$$f(v_0v_i) = i \quad (1 \leq i \leq n)$$

$$f(v_0v_{i+1}) = i - 1 \quad (2 \leq i \leq n - 1)$$

$$f(v_iv_2) = n$$

$$f(w_0v_i) = i + 2 \quad (1 \leq i \leq n - 2)$$

$$f(w_0v_{n-1}) = 1$$

$$f(w_0v_n) = 2$$

By definition of $f$, then we can get

$$C(v_0) = C(w_0) = \{1,2,3,\ldots, n - 1, n, n + 1\}$$

$$C(v_1) = \{2,3,n,n+1\}$$

$$C(v_2) = \{1,2,3,4,n\}$$

$$C(v_3) = \{1,2,3,4,5\}$$

$$C(v_4) = \{2,3,4,5,6\}$$

......

$$C(v_i) = \{i - 2, i - 1, i, i + 1, i + 2\} \quad (3 \leq i \leq n - 2)$$

$$C(v_{n-1}) = \{1, n - 3, n - 2, n - 1, n\}$$

$$C(v_n) = \{1,2,n-2,n\}$$

Obviously, $f$ is an AVDTC, so

$$\chi_{av}((\overline{K}_2 \lor P_n)) = n + 1.$$  

**3 Adjacent-vertex-distinguishing proper total colouring of $\overline{K}_m \lor P_n$ when $m \geq 3$**

In this part, we consider the situation such that when $m \geq 3$. And we get the results that typed below.

First, we consider the simplest situation when $n = 2$. Then $V_1 = \{u_1, u_2, \ldots, u_m\}$, $V_2 = \{v_1, v_2\}$.

We get the following conclusion.

**Theorem 5** If $m \geq 3$, $\chi_{av}((\overline{K}_m \lor P_2)) = m + 3$.

**Proof** The degrees of the graph are shown as below:

$$d(u_i) = d(u_2) = \cdots = d(u_m) = 2, \quad d(v_1) = d(v_2) = m + 1.$$
Because of $m \geq 3$, the maximum degree of graph $\overline{K}_m \lor P_2$ is $m+1$, and there are two vertices who have maximum degree are adjacent.

So concluded by Lemma 1, $\chi_{at}(\overline{K}_m \lor P_2) \geq m+3$

Then we let $f$ be a mapping from $V(\overline{K}_m \lor P_2) \cup E(\overline{K}_m \lor P_2)$ to $\{1, 2, 3, \ldots, m, m+1, m+2\}$ such as:

\[
\begin{align*}
    f(u_1) &= f(u_2) = \cdots = f(u_m) = m+1, \\
    f(v_1) &= 1, f(v_2u_1) = 2 \\
    f(v_2u_2) &= 2, f(v_3u_2) = 3 \\
    f(v_2u_3) &= 3, f(v_2u_4) = 4 \\
    \ldots \\
    f(v_{m-1}u_{m-1}) &= m-1, f(v_{m-1}u_m) = m \\
    f(v_m) &= m+2, f(v_2) = m+3, \\
    f(v_{m+1}) &= m+1.
\end{align*}
\]

We can see that $f$ is a $m+3$-proper-total-colouring of the graph $\overline{K}_m \lor P_2$, and we list the colour sets of all vertices such that:

$C(v_i) = \{1, 2, 3, \ldots, m-1, m+1, m+2\}$,
$C(v_i) = \{1, 2, 3, \ldots, m-1, m+1, m+2\}$,
$C(u_i) = \{1, 2, m+1\}, C(u_2) = \{2, 3, m+1\}$,
$C(u_i) = \{3, 4, m+1\}, C(u_4) = \{4, 5, m+1\}$,
$C(u_{m-1}) = \{m-1, m+1\}, C(u_m) = \{1, m+1\}$.

Then all vertices that are adjacent have different colour sets, so $f$ is an AVDTC of $\overline{K}_m \lor P_2$, so the conclusion is proved.

Next we discuss the adjacent-vertex-distinguishing proper total colouring of graphs $\overline{K}_m \lor P_n$ when $m = n \geq 3$ and get results such as:

Theorem 6 If $m = n = 3$, $\chi_{at}(\overline{K}_m \lor P_n) = 6$.

In fact, if $m = n = 3$, as we known, graph $\overline{K}_m \lor P_n$ is equal to graph $\overline{K}_3 \lor P_3$. The maximum degree of the graph is 5. So, deduced from lemma1 we get that $\chi_{at}(\overline{K}_3 \lor P_3) \geq 6$. Now we define a mapping $f$ from $V(\overline{K}_3 \lor P_3) \cup E(\overline{K}_3 \lor P_3)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows:

\[
\begin{align*}
    f(u_1) &= f(u_2) = 1, f(u_3) = 5, \\
    f(v_1) &= 4, f(v_2) = 6, f(v_3) = 2, \\
    f(u_1v_1) &= 3, f(u_1v_2) = 4, f(u_1v_3) = 5, \\
    f(u_2v_1) &= 2, f(u_2v_2) = 3, f(u_2v_3) = 6, \\
    f(u_3v_1) &= 1, f(u_3v_2) = 2, f(u_3v_3) = 3, \\
    f(v_3v_1) &= 5, f(v_3v_2) = 1, \\
\end{align*}
\]

According to the comparison to the colour sets of all vertices, we can deduce that $f$ is an AVDTC of graph $\overline{K}_3 \lor P_3$, so $\chi_{at}(\overline{K}_3 \lor P_3) = 6$. 

5
Theorem 7 If \( m \geq 4 \), \( \chi'_{av}(\overline{K}_m \vee P_m) = m + 4 \).

Proof Because of \( m = n \geq 4 \), then the degree of all vertices are such that:
\[
d(u_1) = d(u_2) = \cdots = d(u_{m}) = m,
\]
\[
d(v_1) = d(v_2) = m + 1,
\]
\[
d(v_3) = d(v_{m-1}) = m + 2,
\]
The maximum degree of the graph is \( m + 2 \), and there are maximum degree vertices are adjacent, so deduce from lemma 1, we can get \( \chi'_{av}(\overline{K}_m \vee P_n) \geq m + 4 \).

Now we define a mapping \( f \) from \( V(\overline{K}_m \vee P_n) \cup E(\overline{K}_m \vee P_n) \) to \( \{1, 2, \cdots, m + 2, m + 3, m + 4\} \) as follows:
\[
f(u_1) = f(u_2) = \cdots = f(u_m) = m + 1,
\]
\[
f(u_1, v_1) = 1, f(u_1, v_2) = 2, f(u_1, v_3) = 3, \cdots, f(u_1, v_{m-1}) = m - 1, f(u_1, v_m) = m,
\]
\[
f(u_2, v_1) = 2, f(u_2, v_2) = 3, f(u_2, v_3) = 4, \cdots, f(u_2, v_{m-1}) = m, f(u_2, v_m) = m + 1,
\]
\[
f(u_3, v_1) = 3, f(u_3, v_2) = 4, f(u_3, v_3) = 5, \cdots, f(u_3, v_{m-1}) = m, f(u_3, v_m) = m + 2,
\]
\[
\cdots
\]
\[
f(u_m, v_1) = m, f(u_m, v_2) = 1, f(u_m, v_3) = 2, \cdots, f(u_m, v_{m-1}) = m - 3, f(u_m, v_m) = m - 2,
\]
\[
f(v_{2k-1}, v_{2k}) = m + 1, f(v_{2k}, v_{2k+1}) = m + 4,
\]
\[
f(v_{2k+1}) = m + 2, f(v_{2k}) = m + 3.
\]
We calculate the colour set of all vertices and the results are as below:
\[
C(u_1) = \{1, 2, 3, \cdots, m, m + 1\}, 1 \leq i \leq m,
\]
\[
C(v_1) = C(u_1) \cup \{m + 2\}
\]
\[
C(v_k) = C(u_k) \cup \{m + 3, m + 4\}, 1 \leq k \leq \frac{m}{2} - 1,
\]
\[
C(v_{2k}) = C(u_k) \cup \{m + 2, m + 4\}, 1 \leq k \leq \frac{m - 1}{2} - 1,
\]
\[
C(v_m) = C(u_k) \cup \{m + 3\} \text{ when } m \text{ is and even number; or}
\]
\[
C(v_m) = C(u_k) \cup \{m + 2, m + 4\} - \{m + 1\} \text{ if } m \text{ is odd.}
\]

According to the comparison to the colour sets of all vertices, we can deduce that \( f \) is an AVDTC of graph \( \overline{K}_m \vee P_n \), so \( \chi'_{av}(\overline{K}_m \vee P_n) = m + 4 \).

4 Conclusion
In this paper, we get the adjacent –vertex-distinguishing proper total colouring numbers of the graph \( \overline{K}_m \vee P_n \) when \( m = 2 \) and \( m = n \geq 3 \). Next we can research on graphs \( \overline{K}_m \vee P_n \) when \( m > n \geq 3 \) or other situations.

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