REMARK ON 3-D NAVIER-STOKES SYSTEM WITH STRONG DISSIPATION IN ONE DIRECTION

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Abstract. In this paper, we consider 3D anisotropic incompressible Navier-Stokes equations with strong dissipation in the vertical direction. We shall prove that this system has a unique global strong solution and the norm of the vertical component of the velocity field can be controlled by the norm of the corresponding component to the initial data. Similar result can also be obtained for the horizontal components of the vorticity. In particular, we simplify our proofs to the well-posedness result in our previous paper [11, 13].

1. Introduction. The classical incompressible 3-D Navier-Stokes equations in the whole space reads

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p &= 0, \\
\text{div } u &= 0, \\
|u|_{t=0} &= u_0,
\end{align*}
\]

where \( u \) stands for the velocity of the incompressible fluid flow and \( p \) for the scalar pressure function, which guarantees the divergence free condition of the velocity field. \( \nu > 0 \) designates the viscous coefficient of the fluid.

The mathematical study of Navier-Stokes equations dates back to 1934. In the seminal paper [10], Leray proved that for any solenoidal initial velocity in \( L^2 \), (NS) has a global in time weak solution satisfying the following energy inequality

\[
\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2, \quad \forall \ t > 0.
\]

However, the uniqueness and regularity of Leray’s weak solutions is still one of the biggest open problems in the field of mathematical fluid mechanics. we refer [2] for the recent progress concerning the non-unique solution to (NS).

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1Throughout this paper, all the space norms are defined on \( \mathbb{R}^3 \) without specific mention.
With initial data \( u_0 \) in \( L^2 \cap H^1 \), Leray [10] also proved a unique local strong solution to \((NS)\) in the functional space
\[
SS(T) := C([0, T[, L^2 \cap H^1) \cap L^2([0, T[, H^1 \cap H^2)
\] (1.2)
for some positive time \( T \), which can be bounded from below by a constant depending on the \( L^2 \) and \( H^1 \) norms of the initial velocity. Here and in all that follows, we always denote \( H^s \) to be the homogeneous Sobolev space in \( \mathbb{R}^3 \), whose norm is defined by
\[
\| a \|^2_{H^s} := \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{a}(\xi)|^2 \, d\xi.
\]
Moreover, this strong solution can exist globally in time, and there holds
\[
\| u(t) \|^2_{H^s} + \nu \int_0^t \| \nabla u(t') \|^2_{H^s} \, dt' \leq 2\| u_0 \|^2_{H^s}, \quad \forall \, t > 0,
\] (1.3)
provided that the viscous coefficient \( \nu \) is so large that
\[
\nu \gg \| u_0 \|^2_{L^2 \cap H^\frac{1}{2}} \| \nabla u_0 \|^2_{L^2}. \tag{1.4}
\]
We remark that this strong solution is in fact smooth whenever \( t > 0 \) (see [9, 14, 15, 16]).

It is easy to observe that
\[
\| u_0 \|^2_{H^\frac{1}{2}} \leq \| u_0 \|^2_{L^2} \| \nabla u_0 \|^2_{L^2}.
\]
Fujita-Kato [6] constructed local in time unique solution to \((NS)\) with initial data in \( H^\frac{1}{2} \). Furthermore, if \( \| u_0 \|^2_{H^\frac{1}{2}} \ll \nu \), then this solution exists globally in time and satisfy
\[
\| u(t) \|^2_{H^\frac{1}{2}} + \nu \int_0^t \| \nabla u(t') \|^2_{H^\frac{1}{2}} \, dt' \leq \| u_0 \|^2_{H^\frac{1}{2}}, \quad \forall \, t > 0.
\]
This result was extended by Kato [7] for initial data in \( L^3 \), and by Cannone, Meyer and Planchon [3] for initial data in the Besov space \( B_{p,\infty}^{-1+\frac{3}{p}} \) with \( p \in ]3, \infty[ \). The end-point result in this direction was given by Koch and Tataru [8]. They proved that given initial data being sufficiently small in \( \text{BMO}^{-1} \), then \((NS)\) has a unique global solution.

We remark that all these solutions coincide with Leray’s strong solution as long as they exist, and for \( p \in ]3, \infty[ \), the following embedding relations hold:
\[
H^\frac{1}{2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow B_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow \text{BMO}^{-1}(\mathbb{R}^3) \hookrightarrow B_{\infty,\infty}^{-1}(\mathbb{R}^3).
\]
And Bourgain and Pavlović [1] proved the ill-posed of \((NS)\) in \( B_{\infty,\infty}^{-1} \).

We also would like to mention that all the norms to the above spaces, as well as the condition (1.4) are scaling-invariant under the following transformation:
\[
u \lambda (t, x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0, \lambda}(x) = \lambda u_0(\lambda x). \tag{1.5}
\]
It is easy to verify that for any solution \( u \) of \((NS)\) on \([0, T]\), \( u_{\lambda} \) determined by (1.5) is also a solution of \((NS)\) on \([0, T/\lambda^2]\). This scaling-invariant property is very important in the study of Navier-Stokes equations, as was mentioned by Leray in his celebrated work [10].

Our goal here is to study the role of large viscous coefficient in one direction that plays in the global existence of the strong solution for the Navier-Stokes system.

2Throughout this paper, we always use \( a \gg b \) (resp. \( a \ll b \)) to mean that there exists some universal small constant \( \epsilon > 0 \) so that \( a > \epsilon^{-1} b \) (resp. \( a < \epsilon b \)
Let us recall the anisotropic Navier-Stokes equations which describes the evolution of a fluid with “turbulent viscosity” in $\mathbb{R}^3$ (see [4] for instance):

\[
\begin{aligned}
&\frac{\partial u}{\partial t} + u \cdot \nabla u - (\nu_h \Delta u + \nu_\omega \partial_x^2)u + \nabla p = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
&\text{div} \, u = 0, \\
&u|_{t=0} = u_0.
\end{aligned}
\]

\[\text{(ANS)}\]

In particular, when $\nu_\omega = 0$ in (ANS), the authors of [12, 18] improved Fujita and Kato’s result by requiring only two components of the initial velocity being sufficiently small.

**Theorem 1.1** ([12, 18]). Let $u_0 \in B$ with $\text{div} \, u_0 = 0$. Then (ANS) with $\nu_\omega = 0$ has a unique global solution $u \in C([0, \infty[; B)$ with $\nabla u \in L^2([0, \infty[; B)$ provided

\[
\eta \overset{\text{def}}{=} \|u_0^h\|_B \exp(C_1 \nu_h^{-1} \|u_0^3\|_B^2) \ll \nu_h
\]

for some universal positive constant $C_1$. Moreover, for any time $t > 0$, there holds

\[
\|u^1(t)\|_B^2 + \nu_h \int_0^t \|\nabla u^1(t')\|_B^2 \, dt' \leq 2 \exp\left(\frac{C_1}{16}\right) \cdot \eta^2,
\]

and

\[
\|u^3(t)\|_B^2 + \nu_h \int_0^t \|\nabla u^3(t')\|_B^2 \, dt' \leq 2 \|u_0^3\|_B^2 + \nu_h^2,
\]

where $\|a\|_{g^{0.5}} \overset{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta^\ell a\|_{L^2(\mathbb{R}^3)}$ with $\Delta^\ell$ given by (2.4).

We observe that the above result can be re-stated as a well-posedness result with large horizontal viscous coefficient, that is,

**Corollary 1.** Let $u_0 \in B$ with $\text{div} \, u_0 = 0$. Then (ANS) with $\nu_\omega = 0$ has a unique global solution $u \in C([0, \infty[; B)$ with $\nabla u \in L^2([0, \infty[; B)$ provided

\[
\nu_h \gg \|u_0\|_B.
\]

Moreover, for any time $t > 0$, there exists some universal positive constant $C_2$ such that

\[
\|u^1(t)\|_B^2 + \nu_h \int_0^t \|\nabla u^1(t')\|_B^2 \, dt' \leq C_2 \|u_0^1\|_B^2.
\]

We remark that the above result is proved for (ANS) with $\nu_\omega = 0$, it works automatically for the classical Navier-Stokes system. In particular, if the viscous coefficient $\nu$ in (NS) is large enough, then (NS) admits a unique global strong solution $u$. Moreover, the norm to the horizontal components of $u$ is controlled by the norm to the corresponding components of the initial data, that is,

\[
\|u^1(t)\|_B^2 + \nu \int_0^t \|\nabla u^1(t')\|_B^2 \, dt' \leq C_2 \|u_0^1\|_B^2, \quad \forall t > 0.
\]

Now it is natural to ask the following three questions:

Q1: Whether the global well-posedness still holds if we only assume the viscous coefficient in one direction to be sufficiently large?

Q2: Can we bound the norm to one component of the velocity uniformly for all time by the norm of the corresponding component of the initial velocity?

Q3: Can we get a similar type of result for the partial components of the vorticity $\omega \overset{\text{def}}{=} \text{curl} \, u$?
To answer the question Q1, we consider the following anisotropic Navier-Stokes system:

\[
\begin{cases}
    \partial_t u + u \cdot \nabla u - (\Delta_h + \nu_r \partial_h^2)u + \nabla p = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
    \text{div } u = 0, \\
    u|_{t=0} = u_0. 
\end{cases} \tag{ANS_r}
\]

For any given data \(u_0 \in L^2 \cap H^1\), the global well-posedness of (ANS_r) has been proved by us in [11] as well as Paicu and the second author in [13] provided that \(\nu_r\) is large enough. Here we shall first present a alternative simpler proof to this result. Moreover, we shall partially answer the questions Q2 and Q3. The first result of this paper states as follows:

**Theorem 1.2.** Let \(u_0 \in L^2 \cap H^1\) be a solenoidal vector field. Then for any \(K > 0\),

(a) there exists some universal positive constant \(C_3\) such that if

\[ \nu_r \geq 1 + C_3 \max \left\{ \|u_0\|_{L^2}^2 \|\nabla_h u_0\|^2_{L^2}, K^{-\frac{2}{3}} \|u_0\|_{L^2}^2 \|\omega_0^3\|^2_{L^2}, \right. \]

\[ \left. \|u_0\|_{L^2}^2 (\|\nabla u_0^3\|^2_{L^2} + K) \right\} \]  
\tag{1.7}

(ANS_r) has a unique global strong solution \(u \in SS(\infty)\) which satisfies for any \(t > 0\):

\[
\|u^3(t)\|^2_{H^1} + \int_0^t (\|\nabla_h u^3(t')\|^2_{H^1} + \nu_r \|\partial_h u^3(t')\|^2_{H^1}) \, dt' \leq 2 (\|u_0^3\|^2_{H^1} + K). 
\tag{1.8}
\]

(b) there exists some universal positive constant \(C_4\) such that if

\[ \nu_r \geq 1 + C_4 \max \left\{ \|u_0\|_{L^2}^2 \|\nabla_h u_0^3\|^2_{L^2} + K \right\} \frac{7}{5}, \]

\[ \|u_0\|_{L^2}^2 \|\nabla_h u_0^3\|^2_{L^2}, K^{-\frac{2}{3}} \|u_0\|_{L^2}^2 \|\nabla_h u_0^3\|^2_{L^2} \right\} \]  
\tag{1.9}

(ANS_r) have a unique global strong solution \(u \in SS(\infty)\) which satisfies for any \(t > 0\):

\[
\|\nabla_h u^3(t)\|^2_{L^2} + \int_0^t (\|\nabla_h u^3(t')\|^2_{L^2} \|\partial_h\nabla_h u^3(t')\|^2_{L^2}) \, dt' \leq 2 (\|\nabla u_0^3\|^2_{L^2} + K). 
\tag{1.10}
\]

**Remark 1.**

1. The estimate (1.8) can be viewed as a one component version of the estimate (1.3) for strong solution of (NS).
2. Due to the anisotropic viscous coefficients in (ANS_r), it is natural to handle the horizontal derivatives and vertical derivative of the velocity field separately. Notice that the divergence-free condition of \(u\) implies that \(\partial_3 u^3\) equals \(\text{div}_h u^h\), so that \(\partial_3 u^3\) can be regarded as a function of \(u^h\). That’s the reason why we consider the control of \(\nabla_h u^3\) in (1.10).
3. Exactly along the same line to the proof of Theorem 1.2, it is not difficult to show that for any positive constant \(K\) and \(i, j \in \{1, 2, 3\}\), there exists some \(P^K_{i, j}\), which is a polynomial of \(K, \|\partial_i u_0^h\|_{L^2}, \|u_0\|_{L^2}\) and \(\|\nabla u_0^h\|_{L^2}\) such that as long as \(\nu_r \gg P^K_{i, j}\), then for any \(t > 0\), the global strong solution constructed in Theorem 1.2 will in addition satisfy the following estimate:

\[
\|\partial_i u^j(t)\|^2_{L^2} + \int_0^t (\|\nabla_h \partial_i u^j(t')\|^2_{L^2} \|\partial_3 \partial_i u^j(t')\|^2_{L^2}) \, dt' \leq 2 (\|\partial_i u_0^h\|^2_{L^2} + K). 
\tag{1.11}
\]
This means that each component in the matrix $\nabla u$ can be controlled by the corresponding part of the initial data, provided $\nu_\ast$ is sufficiently large.

Before preceding, let us emphasize that the questions Q3 and Q2 are independent, although it follows from the Biot-Savart law that

$$\|\omega\|_{L^p} \leq \|\nabla u\|_{L^p} \leq C \frac{p^2}{p-1} \|\omega\|_{L^p}, \quad \forall 1 < p < \infty,$$

which claims that, roughly speaking, $\omega$ and $\nabla u$ can be controlled by each other. However, it is true only for the whole part of $\omega$, but not for the partial components of $\omega$.

It is easy to derive from $(\text{ANS}_\nu)$ that the vorticity $\omega \defined \text{curl} \, u$ satisfies the following equation

$$\partial_t \omega + u \cdot \nabla \omega - (\Delta_h + \nu_\ast \partial_3^2) \omega = \omega \cdot \nabla u. \quad (1.13)$$

The second result of this paper states as follows:

**Theorem 1.3.** For any solenoidal vector field $u_0 \in L^2 \cap H^1$, there exists some universal positive constant $C_5$ such that if

$$\nu_\ast \geq C_5^2 \|u_0\|_{L^2}^2 \|\nabla u_0\|_{L^2}^2,$$

then $(\text{ANS}_\nu)$ has a unique global strong solution $u \in \mathcal{S}(\infty)$ such that

$$\|\omega(t)\|_{L^2}^2 + \int_0^t \left( \|\nabla \omega(t')\|_{L^2}^2 + \nu_\ast \|\partial_3 \omega(t')\|_{L^2}^2 \right) \, dt' \leq 2 \|\omega_0\|_{L^2}^2, \quad \forall \, t > 0. \quad (1.15)$$

For any positive constant $K$, if in addition

$$\nu_\ast \geq C_6 K^{-\frac{2}{3}} \|u_0\|_{L^2}^2 \|\omega_0\|_{L^2}^2$$

for some universal positive constant $C_6$, then the horizontal vorticity of this global solution $u$ also satisfies

$$\|\omega^h(t)\|_{L^2}^2 + \int_0^t \left( \|\nabla \omega^h(t')\|_{L^2}^2 + \nu_\ast \|\partial_3 \omega^h(t')\|_{L^2}^2 \right) \, dt' \leq 2 \left( \|\omega^h_0\|_{L^2}^2 + K \right), \quad \forall \, t > 0. \quad (1.17)$$

**Remark 2.** It is interesting to compare the conditions for the viscous coefficient in Theorems 1.2 and 1.3 with the classical one (1.4).

Let us end this section with some notations which will be used throughout this context. $C$ stands for some universal constant which may be different in each occurrence. For a Banach space $B$, we shall use the shorthand $L^p_T(B)$ for $\|\cdot\|_{L^p(0,T;B)}$. We use $\mathcal{F}a$ or $\hat{a}$ to denote the Fourier transform of $a$, and $\mathcal{F}^{-1}a$ as its inverse.

2. Ideas of the proof and some technical lemmas. Let us first consider Theorem 1.2. Due to $\text{div} \, u = 0$, we have

$$\text{div}_h u^h \defined \partial_1 u^2 + \partial_2 u^3 = -\partial_3 u^1 \quad \text{and} \quad \text{curl}_h u^h \defined \partial_1 u^2 - \partial_2 u^1 = \omega^3.$$

As a result, we can use Helmholtz decomposition to decompose $u^h$ into

$$u^h = u^h_{\text{curl}} + u^h_{\text{div}}, \quad \text{with} \quad u^h_{\text{curl}} \defined \nabla_h^0 \Delta_h^{-1} \omega^3 \quad \text{and} \quad u^h_{\text{div}} \defined -\nabla_h^0 \Delta_h^{-1} \partial_3 u^3, \quad (2.1)$$

where $\nabla_h \defined (\partial_1, \partial_2)$ and $\nabla_h^0 \defined (-\partial_2, \partial_1)$. 

Hence we can reformulate (ANSv) as
\[
\begin{align*}
\partial_t \omega^3 + u \cdot \nabla \omega^3 - (\Delta_h + \nu_v \partial^3_3)\omega^3 &= \omega^3 \partial_3 u^3 + \partial_3 u^3 \partial_3 u^1 - \partial_1 u^3 \partial_3 u^2, \\
\partial_t u^3 + u \cdot \nabla u^3 - (\Delta_h + \nu_v \partial^3_3)u^3 &= -\partial_3 \Delta^{-1}\left(\sum_{l,m=1}^3 \partial_l u^m \partial_m u^3\right), \\
\omega^3|_{t=0} &= \omega^3_0, \\
u^3|_{t=0} &= \nu^3_0,
\end{align*}
\]
and the $L^2$ estimate of $\nabla_h u^3$ is equivalent to that for $(\omega^3, \partial_3 u^3)$. Indeed, $\|\omega^3\|_{L^2} + \|\partial_3 u^3\|_{L^2} \leq C\|\nabla_h u^3\|_{L^2}$ is obvious to be observed from (2.1). While it is easy to verify that for any $k \in \{1, 2\}$, there holds
\[
\int_{\mathbb{R}^3} \partial_k u^3_{\text{curl}} \cdot \partial_k u^3_{\text{div}} \, dx = 0.
\]
So that we achieve the converse inequality
\[
\|\nabla_h u^3\|_{L^2}^2 = \|\nabla_h u^3_{\text{curl}}\|^2_{L^2} + \|\nabla_h u^3_{\text{div}}\|^2_{L^2} \leq C\|\omega^3\|_{L^2}^2 + \|\partial_3 u^3\|_{L^2}^2.
\]
In general, for any multi-index $\alpha$, we can prove that
\[
\|\partial^\alpha \omega^3\|_{L^2}^2 + \|\partial^\alpha \partial_3 u^3\|_{L^2}^2 \leq C\|\partial^\alpha \nabla_h u^3\|_{L^2}^2 \leq C\|\partial^\alpha \omega^3\|_{L^2}^2 + \|\partial^\alpha \partial_3 u^3\|_{L^2}^2 \quad (2.3)
\]
Before preceding, let us recall some functional spaces that will be used in this paper.

**Definition 2.1.** For any $s, s' \in \mathbb{R}$, the anisotropic Sobolev space $H^{s, s'}$ denotes the space of homogeneous tempered distribution $u$ such that
\[
\|u\|_{H^{s, s'}}^2 \defeq \int_{\mathbb{R}^3} |\xi_h|^{2s}|\xi_3|^{2s'}|\hat{u}(\xi)|^2 \, d\xi < \infty \quad \text{with} \quad \xi_h = (\xi_1, \xi_2).
\]

Next, we shall introduce an anisotropic Besov space $B^{s_1, s_2}_{p, \infty}$, which will be used in the borderline case of the product law introduced in Lemma 2.4 below. Let us first recall some basic facts on anisotropic Littlewood-Paley theory:
\[
\Delta_h^k \hat{u} = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|))\hat{u}, \quad \Delta_3^\alpha \hat{u} = \mathcal{F}^{-1}(\varphi(2^{-|\xi_3|}))\hat{u},
\]
where $\xi_h = (\xi_1, \xi_2)$, $\chi$ and $\varphi$ are smooth functions defined on $[0, \infty)$ such that
\[
\text{Supp } \varphi \subset [3/4, 8/3] \quad \text{and} \quad \forall \tau > 0: \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1,
\]
\[
\text{Supp } \chi \subset [0, 4/3] \quad \text{and} \quad \forall \tau \geq 0: \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1.
\]

**Definition 2.2.** For any $s_1, s_2 \in \mathbb{R}$ and $p \in [1, \infty]$, let us define the anisotropic Besov space $B^{s_1, s_2}_{p, \infty}$ as the space of homogeneous tempered distribution $u$ so that
\[
\|u\|_{B^{s_1, s_2}_{p, \infty}}^2 \defeq \sup_{k, l \in \mathbb{Z}} 2^{ks_1}2^{ls_2}\|\Delta_h^k \Delta_3^l u\|_{L^p} < \infty.
\]

For simplification, we shall use the following norm throughout this paper:
\[
\|u\|_{X(t)}^2 \defeq \|u\|_{L^2}^2 + \|\nabla_h u\|_{L^2}^2 = \|u\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2.
\]

On the other hand, for any smooth enough solution $u$ of (ANSv), by taking $L^2$ inner product of the $u$ equation of (ANSv) with $u$, and using integration by parts and $\text{div } u = 0$, we obtain
\[
\|u(t)\|_{L^2}^2 + 2 \int_0^t \left(\|\nabla_h u(t')\|_{L^2}^2 + \nu_v \|\partial_3 u(t')\|_{L^2}^2\right) \, dt' = \|u_0\|_{L^2}^2, \quad \forall \, t \in [0, T]. \quad (2.6)
\]
Based on (2.6), we shall prove in Section 3 the following A priori estimates, which are the key ingredients to the proof of Theorem 1.2.

**Proposition 1.** Let \((u^3, \omega^3)\) be a smooth enough solution of (2.2) on \([0, T^*]\). Then for any \(t < T^*\), the following estimates hold

\[
\|\omega^3\|_{X(t)}^2 \leq \|\omega_0^3\|_{X(t)}^2 + C \min \left\{ \nu_0^{-\frac{3}{2}}, \|u_0\|_{L^2}^2 \right\} \left( \|\omega^3\|_{X(t)}^2 + \|\nabla u^3\|_{X(t)}^2 \right)
\]

and

\[
\left( \nu_0^{-\frac{3}{2}} + \nu_0^{-1} \right) \|u_0\|_{L^2}^2 \left( \|\omega^3\|_{X(t)}^2 + \|\partial_3 u^3\|_{X(t)}^2 \right)
\]

\[
\|\nabla u^3\|_{X(t)}^2 \leq \|\nabla u_0^3\|_{L^2}^2 + C \|u_0\|_{L^2}^2 \left( \nu_0^{-\frac{3}{2}} \|\omega^3\|_{X(t)}^2 + \|\partial_3 u^3\|_{X(t)}^2 \right) \left( \|\omega^3\|_{X(t)}^2 + \|\partial_3 u^3\|_{X(t)}^2 \right)
\]

In order to prove Theorem 1.2, at least we should show that the estimates (1.8) and (1.10) hold locally in time. This is guaranteed by the following local well-posedness result:

**Proposition 2.** For any solenoidal vector field \(u_0 \in L^2 \cap H^1\) and any positive constant \(K\), there exists some positive time \(T_0^k\) such that \((ANs)\) admits a unique strong solution \(u \in SS(T_0^k)\), satisfying

\[
\|u^i\|_{X(T_0^k)}^2 \leq \|u_0^i\|_{L^2}^2 + K\quad \text{and} \quad \|\partial_i u^3\|_{X(T_0^k)}^2 \leq \|\partial_i u_0^3\|_{L^2}^2 + K, \quad \forall \ i = \{1, 2, 3\}.
\]

**Remark 3.** The above local well-posedness result is not trivial at all. In fact, the classical local well-posedness theory for Navier-Stokes equations only tells you that \((ANs)\) admits a unique strong solution in \(SS(T_1)\) for some positive time \(T_1\), satisfying

\[
\|\nabla u\|_{X(T_1)}^2 \leq 2\|\nabla u_0\|_{L^2}^2.
\]

With the above proposition, we shall use a continuity argument to complete the proof of Theorem 1.2 in Section 3. Similarly, we shall present the proof of Theorem 1.3 in Section 4.

Let us end this section with two lemmas which will be frequently used in what follows. The first one is the Troisi inequality in [17], which can be seen as an anisotropic Sobolev embedding theorem. It states as follows:

**Lemma 2.3.** Let \(1 \leq q_i < \infty (i = 1, \cdots, n)\) with \(\sum_{i=1}^{n} q_i^{-1} > 1\), and \(s = \sum_{i=1}^{n} q_i^{-1} - 1\). Then for any \(a \in C_0^\infty (\mathbb{R}^n)\), there holds

\[
\|a\|_{L^s(\mathbb{R}^n)} \leq C \prod_{i=1}^{n} \|\partial_i a\|_{L^{q_i}(\mathbb{R}^n)}^{\frac{1}{q_i}}.
\]

In particular, for the case \(n = 3\), we have

\[
\|a\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla a\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\partial_3 a\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}},
\]
and
\[ \|a\|_{L^3(\mathbb{R}^3)} \leq \|a\|_{L^4(\mathbb{R}^3)}^{\frac{2}{3}} \|a\|_{L^6(\mathbb{R}^3)}^{\frac{1}{3}} \leq C \|a\|_{L^4(\mathbb{R}^3)}^{\frac{2}{3}} \|\nabla_h a\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}} \|\partial_3 a\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}}. \]

The second one is the product law in anisotropic Sobolev or Besov spaces:

**Lemma 2.4** (A special case of Lemma 4.5 in [5]). For any \( s_1, s_2 < 1 \) with \( s_1 + s_2 > 0 \), and for any \( r_1, r_2 < \frac{1}{2} \) with \( r_1 + r_2 > 0 \), we have
\[ \|ab\|_{H^{r_1+r_2-\frac{1}{2}}|_{H^{s_1, r_1}} + r_2 - \frac{1}{2}} \leq C \|a\|_{H^{s_1, r_1}} \|b\|_{H^{s_2, r_2}}; \]
and for the borderline case when \( s_1 + s_2 = 0 \) with \( 0 \leq s_1 < 1 \), we have
\[ \|ab\|_{H^{r_1+r_2-\frac{1}{2}}|_{H^{s_1, r_1}} + r_2 - \frac{1}{2}} \leq C \|a\|_{H^{s_1, r_1}} \|b\|_{H^{r_2-\frac{1}{2}}}; \]
and for the borderline case when \( r_2 = \frac{1}{2} \), \( -\frac{1}{2} < r_1 < \frac{1}{2} \), we have
\[ \|ab\|_{H^{r_1+r_2-\frac{1}{2}}|_{H^{s_1, r_1}} + r_2 - \frac{1}{2}} \leq C \|a\|_{H^{s_1, r_1}} \|b\|_{H^{r_2-\frac{1}{2}}} \|b\|_{H^{r_2-\frac{1}{2}}}. \]

3. **The proof of Theorem 1.2.** This section is devoted to the proof of Theorem 1.2. We shall first prove Proposition 1 in the first subsection, then the local well-posedness result claimed in Proposition 2 will be presented in the second subsection. With these two propositions at hand, we shall use a continuity argument and the well-known Serrin’s blow-up criteria to show that, under the condition (1.7) (resp. (1.9)), \((AN_{S_2})\) will admit a unique global strong solution in \(SS(\infty)\). Moreover, there holds the uniform in time estimate (1.8) (resp. (1.10)).

3.1. **The proof of Proposition 1.** In this subsection, we shall present the proof of Proposition 1 for smooth enough solution of \(AN_{S_2}\). More precisely, we shall split the proof into the following four lemmas:

**Lemma 3.1.** Under the assumptions of Proposition 1, (2.7) holds for \( t < T^* \).

**Proof.** We first get, by taking \(L^2\) inner product of the \(\omega^3\) equation in (2.2) with \(\omega^3\), that
\[ \frac{1}{2} \frac{d}{dt} \|\omega^3\|_{L^2}^2 + \|\nabla_h \omega^3\|_{L^2}^2 + \nu \|\partial_3 \omega^3\|_{L^2}^2 = \int_{\mathbb{R}^3} \partial_3 u^3 |\omega^3|^2 \, dx - \int_{\mathbb{R}^3} (\partial_3 u^3 \cdot \nabla_h u^3) \omega^3 \, dx. \]

By using integration by parts, we then deduce from Lemma 2.4 that
\[ \int_{\mathbb{R}^3} \partial_3 u^3 |\omega^3|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^3} u^3 \cdot \partial_3 \omega^3 \cdot \omega^3 \, dx \leq C \|\partial_3 \omega^3\|_{L^2} \|\omega^3\|_{L^2} \|u^3\|_{L^6} \]
\[ \leq C \|\omega^3\|_{L^2} \|\nabla_h \omega^3\|_{L^2} \|\partial_3 \omega^3\|_{L^2} \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \overset{\text{def}}{=} I_1, \]
and
\[ \int_{\mathbb{R}^3} (\partial_3 u^3 \cdot \nabla_h \omega^3) \omega^3 \, dx \]
\[ = \int_{\mathbb{R}^3} (u^3 \cdot \nabla_h \partial_3 u^3 + u^3 \cdot \nabla_h \omega^3) \omega^3 \, dx \]
\[ \leq C \left( \|\nabla_h \partial_3 u^3\|_{H^{\frac{1}{2}, 0}} \|\omega^3\|_{H^{\frac{1}{2}, \frac{1}{2}}} + \|\partial_3 \omega^3\|_{L^2} \|\nabla_h u^3\|_{H^{\frac{1}{2}, \frac{1}{2}}} \right) \|u^3\|_{H^{\frac{1}{2}, \frac{1}{2}}} \]
\[ \leq C \left( \|\partial_3 u^3\|_{L^2} \|\nabla_h \partial_3 u^3\|_{L^2} \|\omega^3\|_{L^2} \|\nabla_h \omega^3\|_{L^2} \|\partial_3 \omega^3\|_{L^2} \]
\[ + \|\partial_3 \omega^3\|_{L^2} \|\nabla_h u^3\|_{L^2} \|\nabla_h \partial_3 u^3\|_{L^2} \|\nabla_h \partial_3 u^3\|_{L^2} \right) \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2}. \]
Here and in all that follows, we shall frequently use the following fact that for any $s_1, s_2 \in [0, 1]$ with $s_1 + s_2 \leq 1$,
\[
\|a\|_{H^{s_1-s_2}} \leq \|a\|_{L^2}^{1-s_1} \|\nabla_h a\|_{L^2}^{s_2} \|\partial_3 a\|_{L^2}^{s_2}.
\] (3.4)
By substituting (2.3) with $a = 0$ into (3.3), we obtain
\[
|\int_{\mathbb{R}^3} (\partial_3 u^h \cdot \nabla_h u^3) \omega^3 \, dx| \leq C \left( \|\partial_3 u^3\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u^3\|_{L^2}^{\frac{1}{2}} \|\omega^3\|_{L^2}^{\frac{1}{2}} \|\nabla_h \omega^3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \omega^3\|_{L^2}^{\frac{1}{2}} \right)
\]
\[
+ \|\partial_3 \omega^3\|_{L^2} \|\nabla_h u^3\|_{L^2}^{\frac{1}{2}} \|\nabla^2_h u^3\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u^3\|_{L^2}^{\frac{1}{2}} \right)
\]
\[
\leq \|u_0\|_{L^2} \|\nabla_h u^3\|_{L^2} \left( \|\omega^3\|_{X(t)}^{\frac{1}{2}} + \|\partial_3 u^3\|_{X(t)}^{\frac{1}{2}} \right) \min \left( \|\partial_3 u^3\|_{X(t)}^{\frac{1}{2}} , \nu^{-\frac{3}{2}} \|\nabla_h u^3\|_{X(t)}^{\frac{1}{2}} \right),
\]
from which and (3.5), we conclude the proof of (2.7).

**Lemma 3.2.** Under the assumptions of Proposition 1, (2.8) holds for $t < T^*$. 

**Proof.** We first get, by taking $L^2$ inner product of the $u^i$ equation of (2.2) with $-\partial_3^2 u^3$, that
\[
\frac{1}{2} \frac{d}{dt} \|\partial_3 u^3(t)\|_{L^2}^2 + \|\nabla_h \partial_3 u^3(t)\|_{L^2}^2 + \nu \|\partial_3^2 u^3(t)\|_{L^2}^2 = -P_1 - P_2,
\] (3.6)
where
\[
P_1 \triangleq \left( (\text{Id} + \partial_3^2 \Delta^{-1}) |\partial_3 u^3|^2 + \partial_3^2 \Delta^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell u^m \cdot \partial_m u^\ell \right) \right)_{L^2},
\]
\[
P_2 \triangleq \left( (\text{Id} + 2\partial_3^2 \Delta^{-1}) (\partial_3 u^h \cdot \nabla_h u^3) \right)_{L^2}.
\] (3.7)
We henceforth handle term by term in the righthand side of (3.6).

- The estimate of $P_1$: Since $\partial_3^2 \Delta^{-1}$ is a bounded Fourier multiplier, we infer
\[
|P_1| \leq \left( \left\| \left( |\partial_3 u^3|^2 + \partial_3^2 \Delta^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell u^m \cdot \partial_m u^\ell + 2\partial_\ell u^m \cdot \partial_m u^\ell \right) \right) \right\|_{H^{-\frac{1}{2}}}
\]
\[
\left\| \partial_3^2 \Delta^{-1} \left( \sum_{\ell,m=1}^2 \partial_\ell u_{\text{curl}}^m \cdot \partial_m u_{\text{curl}}^\ell \right) \right\|_{(B_{2,\infty}^{-\frac{1}{2},\infty})_h(L^2)} \| \partial_3 u^3 \|_{(B_{2,1}^\frac{1}{2})_h(L^2)} \cdot (3.8)
\]

It follows from the law of product, Lemma 2.4, and (2.1) that
\[
\left\| (\partial_3 u^3)^2 + \partial_3^2 \Delta^{-1} \left( \sum_{\ell,m=1}^2 \partial_\ell u_{\text{div}}^m \cdot \partial_m u_{\text{div}}^\ell \right) \right\|_{H^{-\frac{1}{2},0}}
\leq C \| \partial_3 u^3 \|^2_{H^{\frac{3}{4},\frac{1}{4}}} + \| \nabla_h u_{\text{div}}^3 \|^2_{H^{\frac{3}{4},\frac{1}{4}}} \leq C \| \partial_3 u^3 \|_{L^2} \| \nabla_h \partial_3 u^3 \|^\frac{1}{2}_{L^2} \| \partial_3^2 u^3 \|^\frac{1}{2}_{L^2},
\]

\[
\left\| \partial_3^2 \Delta^{-1} \left( \sum_{\ell,m=1}^2 \partial_\ell u_{\text{div}}^m \cdot \partial_m u_{\text{curl}}^\ell \right) \right\|_{H^{-\frac{1}{2},0}}
\leq C \| \sum_{\ell,m=1}^2 \partial_\ell u_{\text{div}}^m \cdot \partial_m u_{\text{curl}}^\ell \|_{H^{\frac{3}{4},\frac{1}{4}}} \leq C \| \nabla_h u_{\text{div}}^3 \|_{H^{\frac{3}{4},\frac{1}{4}}} \| \nabla_h u_{\text{curl}}^3 \|^\frac{1}{2}_{L^2} \| \partial_3 \nabla_h u_{\text{curl}}^3 \|^\frac{1}{2}_{L^2},
\]

\[
\leq C (\| \partial_3 u^3 \|_{L^2} \| \nabla_h \partial_3 u^3 \|^\frac{1}{2}_{L^2} \| \partial_3^2 u^3 \|^\frac{1}{2}_{L^2} (\| \omega^3 \|_{L^2} \| \partial_3 \omega^3 \|_{L^2})^\frac{1}{2},
\]

and
\[
\left\| \partial_3^2 \Delta^{-1} \left( \sum_{\ell,m=1}^2 \partial_\ell u_{\text{curl}}^m \cdot \partial_m u_{\text{curl}}^\ell \right) \right\|_{(B_{2,\infty}^{-\frac{1}{2},\infty})_h(L^2)}
\leq \left\| \partial_3^2 \Delta^{-1} \left( \sum_{\ell,m=1}^2 \partial_\ell u_{\text{curl}}^m \cdot \partial_m u_{\text{curl}}^\ell \right) \right\|_{(B_{2,\infty}^{-\frac{1}{2},\infty})_h(L^2)}
\leq C \| \nabla_h u_{\text{curl}}^3 \|_{H^{\frac{3}{4},\frac{1}{4}}} (\| \nabla_h u_{\text{curl}}^3 \|_{H^{\frac{3}{4},\frac{1}{4}}} \| \partial_3 \nabla_h u_{\text{curl}}^3 \|_{L^2})^\frac{1}{2}
\leq C (\| \omega^3 \|_{L^2} \| \partial_3 \omega^3 \|_{L^2})^\frac{1}{2},
\]

Substituting the above two inequalities into (3.8) gives rise to
\[
|P_1| \leq C (\| \partial_3 u^3 \|_{L^2} \| \nabla_h \partial_3 u^3 \|^\frac{1}{2}_{L^2} + \| \omega^3 \|_{L^2} \| \partial_3 \omega^3 \|_{L^2}) (\| \partial_3 u^3 \|^\frac{1}{2}_{L^2} \| \nabla_h \partial_3 u^3 \|^\frac{1}{2}_{L^2}.\]

Thanks to (2.5) and (2.6), by integrating the above inequality over [0, t], we find
\[
\int_0^t |P_1(t')| \, dt' \leq C \left( \| \partial_3 u^3 \|_{L^\infty(L^2)} \| \nabla_h \partial_3 u^3 \|^\frac{1}{2}_{L^2} \right) (\| \partial_3 u^3 \|^\frac{1}{2}_{L^2} \| \nabla_h \partial_3 u^3 \|^\frac{1}{2}_{L^2})
\leq C (\nu^-_1 \| \partial_3 u^3 \|^2_{X(t)} + \nu^-_1 \| \omega^3 \|^2_{X(t)}) \nu^-_2 \| u_0 \|^2_{L^2} \| \nabla_h u^3 \|^\frac{1}{2}_{X(t)}, (3.9)
\]

- The estimate of $P_2$: By using integration by parts, we write
  \[
P_2 = P_{2,1} + P_{2,2}
\]
  with
  \[
P_{2,1} \overset{\text{def}}{=} -(\text{Id} + 2\theta_2^2 \Delta^{-1})(\partial_3 u^h \cdot u^3) \| \nabla_h \partial_3 u^3 \|_{L^2}
\]
  and
  \[
P_{2,2} \overset{\text{def}}{=} -(\text{Id} + 2\theta_2^2 \Delta^{-1})(\partial_3 \nabla u^h \cdot u^3) \| \partial_3 u^3 \|_{L^2}.
\]

Applying the law of product Lemma 2.4 yields
\[ |P_{2,1}| \leq C\|\partial_t u^h \cdot u^3\|_{H^{-\frac{1}{4}} \to H^\frac{1}{4}} \|\nabla_h \partial_t u^3\|_{H^{-\frac{1}{4}} \to H^\frac{1}{4}} \]
\[ \leq C\|\partial_t u^h\|_{H^\frac{1}{4} \to H_{-\frac{1}{4}}} \|u^3\|_{H^\frac{1}{4} \to H^{-\frac{1}{4}}} \|\partial_3^2 u^3\|_{L^2} \|\nabla_h \partial_t u^3\|_{L^2} \]  \tag{3.10}
\[ \leq C\|\partial_t u^h\|_{L^2}^\frac{1}{2} \|\partial_t \nabla_h u^h\|_{L^2}^\frac{1}{2} \|u^3\|_{L^2} \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \|\partial_3^2 u^3\|_{L^2} \|\nabla_h \partial_t u^3\|_{L^2}^\frac{1}{2} \]  \tag{3.11}
Whereas we observe from (2.1) that
\[ \|\partial_t \nabla_h u^h\|_{L^2} \leq C\|\partial_3 u^3\|_{L^2} + \|\partial_3^2 u^3\|_{L^2}. \]
By inserting the above inequality into (3.10), we obtain
\[ |P_{2,1}| \leq C\|\partial_t u^h\|_{L^2}^\frac{1}{2} \left( \|\partial_3 u^3\|_{L^2} + \|\partial_3^2 u^3\|_{L^2} \right) \]
\[ \times \|u^3\|_{L^2} \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \|\partial_3^2 u^3\|_{L^2} \|\nabla_h \partial_t u^3\|_{L^2}^\frac{1}{2}. \]  \tag{3.11}
Along the same line, we have
\[ |P_{2,2}| \leq C\|\partial_3 \text{div}_h u^h \cdot u^3\|_{H^{-\frac{1}{4}} \to H^\frac{1}{4}} \|\partial_3 u^3\|_{H^{-\frac{1}{4}} \to H^\frac{1}{4}} \]
\[ \leq C\|\partial_3 \text{div}_h u^h\|_{H^{-\frac{1}{4}} \to H_{-\frac{1}{4}}} \|u^3\|_{H^\frac{1}{4} \to H^{-\frac{1}{4}}} \|\partial_3^2 u^3\|_{L^2} \|\nabla_h \partial_3 u^3\|_{L^2}^\frac{1}{2} \]
\[ \leq C\|\partial_3 u^h\|_{L^2}^\frac{1}{2} \left( \|\partial_3 u^3\|_{L^2} + \|\partial_3^2 u^3\|_{L^2} \right) \]
\[ \times \|u^3\|_{L^2} \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \|\partial_3^2 u^3\|_{L^2} \|\nabla_h \partial_3 u^3\|_{L^2}^\frac{1}{2}, \]
which together with (3.11) ensures that
\[ |P_2| \leq C\|\partial_3 u^h\|_{L^2}^\frac{1}{2} \left( \|\partial_3 u^3\|_{L^2} + \|\partial_3^2 u^3\|_{L^2} \right) \]
\[ \times \|u^3\|_{L^2} \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \|\partial_3^2 u^3\|_{L^2} \|\nabla_h \partial_3 u^3\|_{L^2}^\frac{1}{2}. \]  \tag{3.13}
Thanks to (2.5) and (2.6), by integrating the above inequality over \([0, t]\), we achieve
\[ \int_0^t |P_2(t')| dt' \leq C\|\partial_3 u^h\|_{L^2(\mathbb{L}^2)} \|\partial_3 u^3\|_{L^2(\mathbb{L}^2)} \|\partial_3^2 u^3\|_{L^2(\mathbb{L}^2)} \|u^3\|_{L^\infty(\mathbb{L}^2)} \]  \tag{3.12}
\[ \times \|\nabla_h u^3\|_{L^2(\mathbb{L}^2)} \|\partial_3 u^3\|_{L^2(\mathbb{L}^2)} \|\partial_3^2 u^3\|_{L^2(\mathbb{L}^2)} \|\nabla_h \partial_3 u^3\|_{L^2(\mathbb{L}^2)} \]
\[ \leq C\nu^\frac{-1}{2} \|u^3\|_{L^2(\mathbb{L}^2)} \left( \|\partial_3 u^3\|_{X(t)} \|\omega^3\|_{X(t)} \right) \|\nabla_h u^3\|_{L^2(\mathbb{L}^2)} \|\nabla_h u^3\|_{L^2(\mathbb{L}^2)} \]
\[ \times \|\nabla_h u^3\|_{L^2(\mathbb{L}^2)} \|\partial_3 u^3\|_{L^2(\mathbb{L}^2)} \|\partial_3^2 u^3\|_{L^2(\mathbb{L}^2)} \|\nabla_h \partial_3 u^3\|_{L^2(\mathbb{L}^2)} \]
\[ \leq C\nu^\frac{-1}{2} \|u^3\|_{L^2(\mathbb{L}^2)} \left( \|\partial_3 u^3\|_{X(t)} + \|\omega^3\|_{X(t)} \right) \|\nabla_h u^3\|_{L^2(\mathbb{L}^2)} \|\nabla_h u^3\|_{L^2(\mathbb{L}^2)} \]
\[ \times \|\nabla_h u^3\|_{L^2(\mathbb{L}^2)} \|\partial_3 u^3\|_{L^2(\mathbb{L}^2)} \|\partial_3^2 u^3\|_{L^2(\mathbb{L}^2)} \|\nabla_h \partial_3 u^3\|_{L^2(\mathbb{L}^2)} \]
By integrating (3.6) over \([0, t]\) for \(t < T^*\), and then inserting (3.9) and (3.12) into the resulting inequality, we conclude the proof of (2.8).
\[ \square \]
Lemma 3.3. Under the assumptions of Proposition 1, (2.9) holds for \(t < T^*\).

Proof. For \(k \in \{1, 2\}\), we first get, by taking \(L^2\) inner product of the \(u^3\) equation of (2.2) with \(\partial_3^2 u^3\), that
\[ \frac{1}{2} \frac{d}{dt} \|\partial_k u^3(t)\|_{L^2} + \|\nabla_h \partial_k u^3(t)\|_{L^2} + \nu \|\partial_3 \partial_k u^3(t)\|_{L^2} = - \sum_{j=1}^3 Q_j, \]  \tag{3.13}
where
\[ Q_1 \overset{\text{def}}{=} \left( \partial_3 \partial_k \Delta^{-1}\left( |\partial_3 u^3|^2 + \sum_{\ell, m=1}^2 \partial_{\ell} u^m \cdot \partial_m u^\ell \right) \right) \|\partial_k u^3\|_{L^2}, \]
\[ Q_2 \overset{\text{def}}{=} \left( 2\partial_3 \partial_k \partial_3 \left( \partial_3 u^h \cdot \nabla_h u^h \right) \right) \|\partial_k u^3\|_{L^2}, \]
\[ Q_3 \overset{\text{def}}{=} \left( \partial_k u \cdot \nabla u^3 \right) \|\partial_k u^3\|_{L^2}. \]  \tag{3.14}
In what follows, we shall handle term by term in the righthand side of (3.13).

The estimate of $Q_1$: Notice that $\partial_3\partial_k\Delta^{-1}$ is a bounded Fourier multiplier, we infer

$$|Q_1| \leq \left\| (\partial_3 u^3)^2 + \sum_{\ell, m=1}^2 \partial_\ell u^m_{\text{div}} \cdot \partial_m u^\ell_{\text{curl}} \right\|_{L^2} \|\partial_k u^3\|_{L^2}$$

$$+ 2 \left\| \partial_3 \partial_k \Delta^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell u^m_{\text{div}} \cdot \partial_m u^\ell_{\text{curl}} \right) \right\|_{H^{0; -\frac{1}{2}}} \|\partial_k u^3\|_{H^{0; \frac{1}{2}}}$$

$$+ \left\| \partial_3 \partial_k \Delta^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell u^m_{\text{curl}} \cdot \partial_m u^\ell_{\text{curl}} \right) \right\|_{L^2} \|\partial_k u^3\|_{L^2}. \quad (3.15)$$

It follows from Lemma 2.3 and (2.1) that

$$\left\| (\partial_3 u^3)^2 + \sum_{\ell, m=1}^2 \partial_\ell u^m_{\text{div}} \cdot \partial_m u^\ell_{\text{curl}} \right\|_{L^2} \leq C\|\partial_3 u^3\|_{L^2}^2 \leq C\|\partial_3 u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2} \|\partial_3^2 u^3\|_{L^2},$$

and

$$\left\| \partial_3 \partial_k \Delta^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell u^m_{\text{div}} \cdot \partial_m u^\ell_{\text{curl}} \right) \right\|_{H^{0; -\frac{1}{2}}} \leq C\left\| \sum_{\ell, m=1}^2 \partial_\ell u^m_{\text{div}} \cdot \partial_m u^\ell_{\text{curl}} \right\|_{H^{0; -\frac{1}{2}} \circ H^{0; 1}} \leq C\|\nabla_h u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2} \|\partial_3^2 u^3\|_{L^2}.$$
\[
\leq C \left( \nu^{-1} \|u_0\|_L^2 \|\nabla_h u^3\|_{X(t)} \|\partial_3 u^3\|_{X(t)}^{\frac{1}{2}} + \nu^{-\frac{3}{2}} \|u_0\|_L^2 \|\omega^3\|_{X(t)}^{\frac{1}{2}} \right) \|\nabla_h u^3\|_{X(t)}
+ \nu^{-1} \|u_0\|_L^2 \|\nabla_h u^3\|_{X(t)}^{\frac{3}{2}} \|\omega^3\|_{X(t)}. \tag{3.16}
\]

**The estimate of \( Q_2 \):** By using integration by parts, we write
\[
Q_2 = -\left( 2 \partial_3 \partial_3 \Delta^{-1} (\partial_3 u^h \cdot u^3) \right)_L^2 - \left( 2 \partial_3 \partial_3 \Delta^{-1} (\partial_3 \text{div}_h u^h \cdot u^3) \right)_L^2
\]
\[
= -\left( 2 \partial_3^2 \Delta^{-1} (\partial_3 u^h \cdot u^3) \right)_L^2 + \left( 2 \partial_3 \Delta^{-1} (\partial_3 \text{div}_h u^h \cdot u^3) \right)_L^2 \equiv Q_{2,1} + Q_{2,2}.
\]

For \( Q_{2,1} \), we deduce from Lemma 2.4 that
\[
|Q_{2,1}| \leq C \|\partial_3^2 \Delta^{-1} (\partial_3 u^h \cdot u^3)\|_L^2 \|\partial_3 \nabla_h u^3\|_L^2 \leq C \|\partial_3 u^h \cdot u^3\|_{H^\frac{1}{2} \cap \frac{1}{2}} \|\partial_3 \nabla_h u^3\|_L^2
\]
\[
\leq C \|\partial_3 u^h\|_{H^\frac{1}{2} \cap \frac{1}{2}} \|u^3\|_{H^\frac{1}{2} \cap \frac{1}{2}} \|\partial_3 \nabla_h u^3\|_L^2
\]
\[
\leq C \|\partial_3 u^h\|_{L^\frac{1}{2}} \|\partial_3 \nabla_h u^3\|_{L^2} \|u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_L^2. \tag{3.17}
\]

Yet observing from (2.1) that
\[
\|\partial_3 \nabla_h u^h\|_{L^2} \leq C \|\partial_3 \omega^3\|_{L^2} + \|\partial_3^2 u^3\|_{L^2}.
\]

As a result, it comes out
\[
|Q_{2,1}| \leq C \|\partial_3 u^h\|_{L^\frac{1}{2}} \left( \|\partial_3 \omega^3\|_{L^2} + \|\partial_3^2 u^3\|_{L^2} \right) \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \|\nabla_h \partial_3 u^3\|_{L^2}. \tag{3.18}
\]

Similarly, we estimate for \( Q_{2,2} \) as follows
\[
|Q_{2,2}| \leq C \|\partial_3 \Delta^{-1} (\partial_3 \text{div}_h u^h \cdot u^3)\|_L^2 \|\partial_3 \nabla_h u^3\|_L^2
\]
\[
\leq C \|\partial_3 \Delta^{-1} (\partial_3 \text{div}_h u^h \cdot u^3)\|_{H^\frac{1}{2} \cap \frac{1}{2}} \|\partial_3 \nabla_h u^3\|_L^2
\]
\[
\leq C \|\partial_3 u^h\|_{L^\frac{1}{2}} \left( \|\partial_3 \omega^3\|_{L^2} + \|\partial_3^2 u^3\|_{L^2} \right) \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_L^2,
\]

which together with (3.18) ensures that
\[
|Q_2| \leq C \|\partial_3 u^h\|_{L^\frac{1}{2}} \left( \|\partial_3 \omega^3\|_{L^2} + \|\partial_3^2 u^3\|_{L^2} \right) \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_L^2.
\]

By integrating the above inequality over \([0, t]\) for \( t < T^* \) and using (2.6), we get
\[
\int_0^t |Q_2(t')| \, dt' \leq C \|\partial_3 u^h\|_{L^\frac{1}{2}} \left( \|\partial_3 \omega^3\|_{L^2} \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \right) \|\nabla_h u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2} \tag{3.19}
\]
\[
\leq C \nu^{-1} \|u_0\|_L^2 \left( \|\omega^3\|_{X(t)} \|\nabla_h u^3\|_{X(t)} \|\partial_3 u^3\|_{X(t)} \right) \|\nabla_h u^3\|_{X(t)} \|\partial_3 \nabla_h u^3\|_{X(t)}.
\]

**The estimate of \( Q_3 \):** We get, by applying Lemma 2.3, that
\[
|Q_3| = \left| \left( \partial_3 u^h \cdot \nabla_h u^3 + \partial_3 u^3 \cdot \partial_3 u^3 \right)_L^2 \right|
\]
\[
\leq C \left( \|\partial_3 u^h\|_{L^2} \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \right) \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \leq C \left( \|\partial_3 u^3\|_{L^2} \|\omega^3\|_{L^2} \right) \|\nabla_h u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2}.
\]

Integrating the above inequality in time and using the energy equality (2.6) yields
\[
\int_0^t |Q_3(t')| \, dt' \leq C \left( \|\partial_3 u^3\|_{L^2} \|\omega^3\|_{L^2} \right) \left( \|\partial_3 u^3\|_{L^2} \|\omega^3\|_{L^2} \right) \|\nabla_h u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2} \tag{3.20}
\]
\[
\times \|\nabla_h u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2}.
\]
We henceforth handle term by term above.

Now integrating the estimate (3.13) in time, and inserting the obtained estimates (3.16), (3.19) and (3.20), we finally achieve the desired estimate (2.9).

Lemma 3.4. Under the assumptions of Proposition 1, (2.10) holds for $t < T^\ast$.

Proof. We first get, by taking $L^2$ inner product of the $u^3$ equation of (2.2) with $-\Delta u^3$, that

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u^3(t) \|_{L^2}^2 + \| \nabla_h \nabla u^3(t) \|_{L^2}^2 + \nu_c \| \partial_3 \nabla u^3(t) \|_{L^2}^2 = -R_1 - R_2 - R_3,
\]

where

\[
R_1 \overset{\text{def}}{=} \left( \partial_3 \nabla^{-1} \left( |\partial_3 u^3|^2 + 2 \sum_{\ell, m=1}^2 \partial_\ell u^m_m \cdot \partial_m u^\ell \right) \right) \| \nabla u^3 \|_{L^2},
\]

\[
R_2 \overset{\text{def}}{=} \left( 2 \partial_3 \nabla^{-1} (\partial_3 u^b \cdot \nabla_h u^3) \right) \| \nabla u^3 \|_{L^2}
\]

and $R_3 \overset{\text{def}}{=} (\nabla u \cdot \nabla u^3) \| \nabla u^3 \|_{L^2}$.

We henceforth handle term by term above.

• The estimate of $R_1$: Notice that $\partial_3 \nabla^{-1}$ is a bounded Fourier multiplier, we infer

\[
|R_1| \leq \left\| (\partial_3 u^3)^2 + 2 \sum_{\ell, m=1}^2 \partial_\ell u^m_m \cdot \partial_m u^\ell \right\|_{H^{-\frac{1}{2}}, 0} \| \nabla u^3 \|_{H^{\frac{1}{2}, 0}}
\]

\[
+ 2 \left\| \partial_3 \nabla^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell u^m_m \cdot \partial_m u^\ell \right) \right\|_{H^{0, -\frac{1}{2}}} \| \nabla u^3 \|_{H^{0, \frac{1}{2}}}
\]

\[
+ \left\| \partial_3 \nabla^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell u^m_m \cdot \partial_m u^\ell \right) \right\|_{H^{-\frac{1}{2}, 0}} \| \nabla u^3 \|_{H^{\frac{1}{2}, 0}}.
\]

It follows from Lemma 2.4 and (2.1) that

\[
\left\| (\partial_3 u^3)^2 + 2 \sum_{\ell, m=1}^2 \partial_\ell u^m_m \cdot \partial_m u^\ell \right\|_{H^{-\frac{1}{2}, 0}} \leq C \| \partial_3 u^3 \|_{H^{\frac{1}{2}, \frac{1}{4}}} + \| \nabla h u^b_{\text{div}} \|_{H^{\frac{1}{2}, \frac{1}{4}}}
\]

\[
\leq C \| \partial_3 u^3 \|_{L^2} \left\| \partial_3 \nabla u^3 \right\|_{L^2} \left\| \partial_3^2 u^3 \right\|_{L^2} \left\| \nabla h \omega \right\|_{L^2}
\]

and

\[
\left\| \partial_3 \nabla^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell u^m_m \cdot \partial_m u^\ell \right) \right\|_{H^{0, -\frac{1}{2}}}
\]

\[
= \left\| \sum_{\ell, m=1}^2 \partial_\ell u^m_m \cdot \partial_m u^\ell \right\|_{H^{-\frac{1}{2}, -\frac{1}{4}}} \leq C \| \nabla_h u^b_{\text{div}} \|_{H^{\frac{1}{2}, \frac{1}{4}}} \| \nabla_h u^b_{\text{curl}} \|_{H^{\frac{1}{2}, 0}}
\]

\[
\leq C \| \partial_3 u^3 \|_{L^2} \left\| \partial_3 \nabla u^3 \right\|_{L^2} \left\| \partial_3^2 u^3 \right\|_{L^2} \left\| \nabla h \omega \right\|_{L^2}.
\]

Yet observing that $\text{div}_h u^b_{\text{curl}} = 0$, we thus write

\[
\left\| \partial_3 \nabla^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell u^m_m \cdot \partial_m u^\ell \right) \right\|_{H^{-\frac{1}{2}, 0}}
\]

\[
= 2 \| \nabla \text{div}_h \left( \partial_3 u^b_{\text{curl}} \otimes u^b_{\text{curl}} \right) \|_{H^{-\frac{1}{2}, 0}} \overset{\text{def}}{=} \tilde{R}_1.
\]
Then it follows from the law of product \((2.13)\) and a similar derivation of \((3.4)\) that

\[
\bar{R}_1 \leq C \|\partial_3 u^h_{\text{curl}} \otimes u^h_{\text{curl}}\|_{H^{\frac{1}{2},0}}^2 \\
\leq C \|\partial_3 u^h_{\text{curl}}\|_{H^{\frac{1}{2},0}} \|u^h_{\text{curl}}\|_{H^{\frac{1}{2},0}} \frac{1}{2} \|\partial_3 u^h_{\text{curl}}\|_{H^{\frac{1}{2},0}}^2 \\
\leq C \|\partial_3 u^h_{\text{curl}}\|_{L^2}^2 \|\partial_3 \nabla h u^h_{\text{curl}}\|_{L^2}^2 \left(\|\nabla h u^h_{\text{curl}}\|_{L^2} \|\partial_3 u^h_{\text{curl}}\|_{L^2} \right)^{\frac{1}{2}} \\
\times \left(\|\nabla h u^h_{\text{curl}}\|_{L^2}^2 \|\partial_3 u^h_{\text{curl}}\|_{L^2} \|\partial_3 \nabla h u^h_{\text{curl}}\|_{L^2} \right)^{\frac{1}{2}} \\
\leq C \|\partial_3 u^h\|_{L^2}^2 \|\omega^3\|_{L^2}^2 \|\partial_3 \omega^3\|_{L^2}.
\]

Substituting the above inequalities into \((3.23)\) gives rise to

\[
|R_1| \leq C \|\partial_3 u^h\|_{L^2}^2 \|\nabla h \partial_3 u^3\|_{L^2}^2 \|\partial_3^2 u^3\|_{L^2}^2 \|\nabla \nabla u^3\|_{L^2} \\
+ C \|\partial_3 u^h\|_{L^2}^2 \|\omega^3\|_{L^2}^2 \|\partial_3 \omega^3\|_{L^2} \|\nabla u^3\|_{L^2} \|\nabla \nabla u^3\|_{L^2} \\
+ C \|\partial_3 u^3\|_{L^2}^2 \|\nabla h \partial_3 u^3\|_{L^2}^2 \|\partial_3^2 u^3\|_{L^2} \|\omega^3\|_{L^2} \|\nabla h \omega^3\|_{L^2} \|\nabla \nabla u^3\|_{L^2} \|\partial_3 \nabla u^3\|_{L^2}.
\]

By integrating the above inequality over \([0,t]\) for \(t < T^*\), we achieve

\[
\int_0^t |R_1(t')| \, dt' \\
\leq C \left(\|\partial_3 u^h\|_{L^2(L^2)}^2 \|\partial_3 u^h\|_{L^2(L^2)} \|\nabla h \partial_3 u^3\|_{L^2(L^2)}^2 \|\partial_3^2 u^3\|_{L^2(L^2)}^2 \\
+ \|\partial_3 u^h\|_{L^2(L^2)} \|\omega^3\|_{L^2(L^2)} \|\partial_3 \omega^3\|_{L^2(L^2)} \|\nabla u^3\|_{L^2(L^2)} \|\nabla \nabla u^3\|_{L^2(L^2)} \\
+ \|\partial_3 u^3\|_{L^2(L^2)} \|\nabla h \partial_3 u^3\|_{L^2(L^2)} \|\partial_3^2 u^3\|_{L^2(L^2)} \|\omega^3\|_{L^2(L^2)} \|\nabla h \omega^3\|_{L^2(L^2)} \|\nabla \nabla u^3\|_{L^2(L^2)} \|\partial_3 \nabla u^3\|_{L^2(L^2)} \right) \\
\leq C \nu \|u^3\|_{X(t)} \|\nabla u^3\|_{X(t)}^2 + \|\omega^3\|_{X(t)} \|\nabla u^3\|_{X(t)}, \tag{3.24}
\]

- The estimate of \(R_2\): By using integration by parts, we write

\[
R_2 = R_{2,1} + R_{2,2}
\]

with

\[
R_{2,1} \overset{\text{def}}{=} - \left(2\nabla^2 \Delta^{-1}(\partial_3 u^h \cdot u^3) \right) \nabla \partial_3 \nabla u^3 \right)_{L^2}
\]

and

\[
R_{2,2} \overset{\text{def}}{=} \left(2\nabla \Delta^{-1}(\partial_3 \nabla h u^h \cdot u^3) \right) \nabla \partial_3 \nabla u^3 \right)_{L^2}.
\]

Exactly along the same line to the derivation of \((3.17)\) to \((3.18)\), we get

\[
|R_{2,1}| \leq C \|\partial_3 u^h\|_{L^2} \left(\|\partial_3 \omega^3\|_{L^2} \|\partial_3^2 u^3\|_{L^2} \|\nabla h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \right). \tag{3.25}
\]

And it follows from the law of product Lemma 2.4 that

\[
|R_{2,2}| \leq C \|\partial_3 \nabla h u^h \cdot u^3\|_{H^{-\frac{1}{2},-\frac{1}{2}}} \|\partial_3 \nabla h u^3\|_{L^2} \\
\leq C \|\partial_3 \div h u^h\|_{H^{-\frac{1}{2},0}} \|u^3\|_{H^{\frac{1}{2},0}} \|\partial_3 \nabla h u^3\|_{L^2} \\
\leq C \|\partial_3 u^h\|_{L^2} \left(\|\partial_3 \omega^3\|_{L^2} \|\partial_3^2 u^3\|_{L^2} \|\nabla h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \right),
\]

which together with \((3.25)\) ensures that

\[
|R_2| \leq C \|\partial_3 u^h\|_{L^2} \left(\|\partial_3 \omega^3\|_{L^2} \|\partial_3^2 u^3\|_{L^2} \right) \|\nabla h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \|\partial_3 \nabla h u^3\|_{L^2}.
\]
Then repeating the calculations in (3.19) yields

\[ \int_0^t |R_2(t')| \, dt' \leq C\nu^{-1} \|u_0\|_{L^2}^\frac{3}{2} \left( \|\omega^3\|_{X(t)}^\frac{3}{2} + \|\partial_3 u^3\|_{X(t)}^\frac{3}{2} \right) \|\nabla u^3\|_{X(t)}^2, \]  

(3.26)

The estimate of \( R_3 \): By distinguishing the horizontal and vertical derivatives, we write

\[ R_3 = R_{3,1} + R_{3,2} \]

with

\[ R_{3,1} \overset{\text{def}}{=} (\nabla_h u^3 \cdot \nabla_h u^3 + \nabla_h u^3 \cdot \partial_3 u^3 \mid \nabla_h u^3)_{L^2} \]

and

\[ R_{3,2} \overset{\text{def}}{=} (\partial_3 u^3 \cdot \nabla_h u^3 + |\partial_3 u^3|^2 \mid \partial_3 u^3)_{L^2}. \]

By applying Lemma 2.3, we obtain

\[ |R_{3,1}| \leq \|\nabla_h u^3\|_{L^2} \|\nabla_h u^3\|_{L^2} \|\nabla_h u^3\|_{L^6} + \|\nabla_h u^3\|_{L^2} \|\partial_3 u^3\|_{L^2} \|\nabla_h u^3\|_{L^6} \leq C(\|\partial_3 u^3\|_{L^2} + \|\omega^3\|_{L^2}) \|\nabla_h u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2}, \]

and by applying Lemma 2.4, we get

\[ |R_{3,2}| \leq \|\partial_3 u^3 \cdot \partial_3 u^3\|_{H^{-\frac{1}{4}, \frac{1}{4}}} + \|\partial_3 u^3 \cdot \nabla_h u^3\|_{H^{-\frac{1}{2}, \frac{1}{2}}} \|\partial_3 u^3\|_{H^{-\frac{1}{4}, \frac{1}{4}}} \leq C(\|\partial_3 u^3\|_{L^2} \|\partial_3 \nabla_h u^3\|_{L^2} \|\nabla^3 u^3\|_{L^2} + \|\partial_3 u^3\|_{L^2} \|\nabla^3 u^3\|_{L^2} \|\nabla^3 u^3\|_{L^2}). \]

By summing up the above two inequalities and then integrating the resulting inequality over \([0, t]\) for \( t < T^* \), and using the energy equality (2.6), we arrive at

\[ \int_0^t |R_3(t')| \, dt' \leq C\nu^{-\frac{3}{4}} \|\omega_0\|_{X(t)}^\frac{3}{2} \|\nabla u^3\|_{X(t)}^\frac{3}{2} + \nu^{-1} \|\omega_0\|_{X(t)}^\frac{3}{2} \|\nabla u^3\|_{X(t)}^\frac{3}{2}. \]  

(3.27)

Integrating (3.21) over \([0, t]\) for \( t < T^* \) and then inserting the estimates (3.24), (3.26) and (3.27) into the resulting inequality, concludes the proof of (2.10).

By summarizing Lemmas 3.1 to 3.4, we conclude the proof of Proposition 1.

3.2. The proof of Proposition 2. For any \( u_0 \in L^2 \cap H^1 \), we get, by applying the classical local well-posedness theory for Navier-Stokes equations (see [4] for instance), that (\(\text{ANS}_c\)) admits a unique strong solution \( u \) in \( \mathcal{S}(T_1) \) for some positive time \( T_1 \), which satisfies

\[ \|u\|_{L^p_t(W^{2,1} \cap H^1)}^2 + \|\nabla_h u\|_{L^p_t(W^{2,1} \cap H^1)}^2 + \nu\|\partial_3 u\|_{L^p_t(W^{2,1} \cap H^1)}^2 \leq 2\|u_0\|_{H^1}^2, \]  

(3.28)

\[ \|\omega\|_{L^p_t(L^2)}^2 + \|\nabla_h \omega\|_{L^p_t(L^2)}^2 + \nu\|\partial_3 \omega\|_{L^p_t(L^2)}^2 \leq 2\|u_0\|_{H^1}^2. \]  

(3.29)
For each $i \in \{1, 2, 3\}$, we write the equations for $\omega^i$ and $\partial_i u^3$ as follows
\[
\partial_t \omega^i + u \cdot \nabla \omega^i - (\Delta h + \nu_3 \partial^2 r_3) \omega^i = \omega \cdot \nabla u^i, \tag{3.30}
\]
and
\[
\partial_t \partial_i u^3 + u \cdot \nabla \partial_i u^3 - (\Delta h + \nu_3 \partial^2 r_3) \partial_i u^3 = -\partial_i u \cdot \nabla u^3 - \partial_i \partial_3 \Delta^{-1} \left( \sum_{\ell, m=1}^3 \partial_{\ell} u^m \partial_m u^i \right). \tag{3.31}
\]

By taking $L^2$ inner product of (3.30) with $\omega$, we find
\[
\frac{1}{2} \frac{d}{dt} \| \omega^i \|^2_{L^2} + \| \nabla_h \omega^i \|^2_{L^2} + \nu \| \partial_3 \omega^i \|^2_{L^2} = \int_{\mathbb{R}^3} (\omega \cdot \nabla u^i) \omega^i \, dx \leq \| \omega \|_{L^2} \| \nabla u \|_{L^6} \leq C \| \omega \|^\frac{3}{2}_{L^2} \| \nabla \omega \|^\frac{1}{2}_{L^2} \| \nabla u \|_{L^6} \leq C. \tag{3.29}
\]

Integrating the above inequality on $[0, t]$ with $t < T_1$ and using the estimates (3.28), (3.29) yields
\[
\| \omega^i \|^2_{X(t)} \leq \| \omega^i_0 \|^2_{L^2} + C \| \omega \|^\frac{3}{2}_{L^2(\mathbb{R}^3)} \| \nabla \omega \|^\frac{1}{2}_{L^2(\mathbb{R}^3)} \| \nabla u \|_{L^6(\mathbb{R}^3)} \cdot t^\frac{1}{4} \leq \| \omega^i_0 \|^2_{L^2} + C \| u_0 \|^3_{H^1} \cdot t^\frac{1}{4}. \tag{3.32}
\]

Similarly, by taking $L^2$ inner product of (3.31) with $\partial_i u^3$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \partial_i u^3 \|^2_{L^2} + \| \nabla_h \partial_i u^3 \|^2_{L^2} + \nu \| \partial_3 \partial_i u^3 \|^2_{L^2} = -\int_{\mathbb{R}^3} (\partial_i u \cdot \nabla u^3) \partial_i u^3 + \partial_i \partial_3 \Delta^{-1} \left( \sum_{\ell, m=1}^3 \partial_{\ell} u^m \partial_m u^i \right) \partial_i u^3 \, dx \leq C \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla u \|_{L^6} \leq C \| u \|^\frac{7}{2}_{H^1} \| \nabla u \|^\frac{3}{2}_{H^1}. \tag{3.30}
\]

Integrating the above inequality on $[0, t]$ for any $t < T_1$ and using the estimates (3.28), (3.29) gives
\[
\| \partial_i u^3 \|^2_{X(t)} \leq \| \partial_i u^3_0 \|^2_{L^2} + C \| u \|^\frac{7}{2}_{L^\infty(\mathbb{R}^3)} \| \nabla u \|^\frac{3}{2}_{L^2(\mathbb{R}^3)} \cdot t^\frac{1}{4} \leq \| \partial_i u^3_0 \|^2_{L^2} + C \| u_0 \|^3_{H^1} \cdot t^\frac{1}{4}. \tag{3.33}
\]

Thanks to (3.32) and (3.33), we take
\[
T^K_0 \overset{\text{def}}{=} \min \left\{ T_1, \frac{K}{C \| u_0 \|^3_{H^1}} \right\}.
\]

Then $T^K_0$ thus obtained meets all the requirements of Proposition 2. This completes the proof of Proposition 2.

3.3. The proof of Theorem 1.2. With the $a$ priori estimates in Proposition 1 and the local well-posedness result in Proposition 2, we are now in a position to complete the proof of Theorem 1.2. As the interesting case only happens when $\nu_3$ is large, we henceforth always assume $\nu_3 \geq 1$ for notational simplification.

Let us first prove that under the assumption (1.9), $(ANS_3)$ has a unique global solution satisfying (1.10). The proof of the corresponding result under the assumption (1.7) is almost the same, which we shall sketch at the end of this section.

Indeed given any $u_0 \in L^2 \cap H^1$, we deduce from the classical local well-posedness theory of Navier-Stokes system (4) that, $(ANS_3)$ has a unique strong solution $u \in \mathcal{S}(T^*)$, where $T^*$ is the maximal time of existence. Moreover, noting that when $\omega^3_0 = \partial_3 u^3_0 = 0$, we must have $u_0 = 0$, then all conclusions become trivial.
Thus we can use Proposition 2 with positive constant \( \|\omega_0^3\|_{X(t)}^2 + \|\partial_3 u_0^3\|_{X(t)}^2 \) and \( K \) respectively to guarantee that for this strong solution, there exists some positive time \( T^K_0 \leq T^* \), such that (1.10) and the following estimate
\[
\|\omega^3\|_{X(t)}^2 + \|\partial_3 u^3\|_{X(t)}^2 \leq 2(\|\omega_0^3\|_{L^2}^2 + \|\partial_3 u_0^3\|_{L^2}^2) 
\] (3.34)
hold for any \( t \in [0, T^K_0] \).

Next, we shall prove that under the assumption (1.9), the estimates (1.10) and (3.34) in fact hold for any \( t \in [0, T^*] \). In order to do so, we shall use a continuity argument. Let us define
\[
T^K_\ast \overset{\text{def}}{=} \sup \{ T \in [0, T^*] : \text{so that (1.10) and (3.34) hold for any } t \in [0, T] \}. 
\] (3.35)

Then \( T^K_\ast > T^K_0 > 0 \), and we are going to prove that \( T^K_\ast = T^* \). Otherwise, if \( T^K_\ast < T^* \), then for any \( t \in [0, T^K_\ast] \), by summing up (2.7) and (2.8), we get
\[
\|\omega^3\|_{X(t)}^2 + \|\partial_3 u^3\|_{X(t)}^2 \leq \|\omega_0^3\|_{L^2}^2 + \|\partial_3 u_0^3\|_{L^2}^2 + C_{\nu_\ast}^3 \|\partial_3 u_0^3\|_{L^2}^2 \|\nabla h u_0^3\|_{L^2} \leq C_{\nu_\ast}^3 \|\partial_3 u_0^3\|_{L^2}^2 \|\nabla h u_0^3\|_{L^2} + K \] (3.36)
Substituting the estimates (1.10) and (3.34) into the above inequality yields
\[
\|\omega^3\|_{X(t)}^2 + \|\partial_3 u^3\|_{X(t)}^2 \leq (1 + C_{\nu_\ast}^3 \|\nabla h u_0^3\|_{L^2} + \sqrt{K}) \bigg(\|\omega_0^3\|_{L^2}^2 + \|\partial_3 u_0^3\|_{L^2}^2\bigg). 
\]
Thus if \( \nu_\ast \) is so large that
\[
C_{\nu_\ast}^3 \|\omega_0^3\|_{L^2}^2 \leq \frac{1}{2}, 
\] (3.37)
then we deduce
\[
\|\omega^3\|_{X(t)}^2 + \|\partial_3 u^3\|_{X(t)}^2 \leq \frac{3}{2} \bigg(\|\omega_0^3\|_{L^2}^2 + \|\partial_3 u_0^3\|_{L^2}^2\bigg), \quad \forall t < T^K_\ast. 
\] (3.38)

On the other hand, we deduce from (2.9), estimates (1.10) and (3.34) that
\[
\|\nabla h u^3\|_{X(t)}^2 \leq \|\nabla h u_0^3\|_{L^2}^2 + C_{\nu_\ast}^3 \|\omega_0^3\|_{L^2}^2 \|\nabla h u_0^3\|_{L^2} + \|\partial_3 u_0^3\|_{L^2}^2 \|\nabla h u_0^3\|_{L^2} + K \) + C_{\nu_\ast}^3 \|\omega_0^3\|_{L^2}^2 \|\nabla h u_0^3\|_{L^2} + \|\partial_3 u_0^3\|_{L^2}^2 \|\nabla h u_0^3\|_{L^2} + \sqrt{K}. 
\] (3.39)
Whereas applying Young’s inequality gives
\[
C_{\nu_\ast}^2 \|\omega_0^3\|_{L^2}^2 \|\nabla h u_0^3\|_{L^2} \leq C_{\nu_\ast}^2 \|\nabla h u_0^3\|_{L^2} + \sqrt{K} \) \leq \frac{1}{4} \bigg(\|\omega_0^3\|_{L^2}^2 + \|\partial_3 u_0^3\|_{L^2}^2 + K\bigg). \] (3.40)
So that if \( \nu_\ast \) is so large that
\[
C_{\nu_\ast}^2 \|\partial_3 u_0^3\|_{L^2}^2 \leq \frac{1}{4}, 
\] (3.41)
we infer from (3.39) that
\[
\|\nabla h u^3\|_{X(t)}^2 \leq \frac{3}{2} \bigg(\|\nabla h u_0^3\|_{L^2}^2 + K\bigg), \quad \forall t < T^K_\ast. \] (3.42)
Clearly the requirements (3.37) and (3.41) are satisfied provided the constant \( C_4 \) in condition (1.9) is chosen to be sufficiently large. Then the estimates (3.38) and (3.42) contract with the definition of \( T^K_\ast \) given by (3.35). This in turn shows that \( T^K_\ast = T^* \), i.e. (1.10) and (3.34) holds uniformly for any \( t \in [0, T^*] \).
It remains to show that $T^* = \infty$. Otherwise, if $T^* < \infty$, we deduce from the estimates (1.10) and (3.44), together with (2.3) that
\[
\|\nabla_h u^h\|_{L^2(\Omega^\infty)}^2 \leq C\|\nabla_h u^h\|_{L^2}^2 \quad \text{and} \quad \|\nabla_h u^3\|_{X(T^*)}^2 \leq 2\left(\|\nabla_h u_0^3\|_{L^2}^2 + K\right). \tag{3.43}
\]
We are thus left to handle the estimate of $\partial_3 u^h$. Indeed by taking $L^2$ inner product of the $u^h$ equation of (ANS) with $-\partial_3^2 u^h$, we get
\[
\frac{1}{2} \frac{d}{dt} \|\partial_3 u^h\|_{L^2}^2 + \|\nabla_h \partial_3 u^h\|_{L^2}^2 + \nu_c \|\partial_3^2 u^h\|_{L^2}^2
\]
\[=- \int_{\mathbb{R}^3} (\partial_3 u \cdot \nabla u^h + \partial_3 \nabla p) \cdot \partial_3 u^h \, dx. \tag{3.44}
\]
It follows from Lemma 2.3 that
\[
\left| \int_{\mathbb{R}^3} (\partial_3 u \cdot \nabla u^h) \cdot \partial_3 u^h \, dx \right| \leq \left( \|\partial_3 u^h\|_{L^2} \|\nabla_h u^h\|_{L^3} + \|\partial_3 u^3\|_{L^3} \|\partial_3 u^h\|_{L^2} \right) \|\partial_3 u^h\|_{L^2}
\]
\[\leq C \|\partial_3 u^h\|_{L^2} \|\nabla_h u^h\|_{L^2}^\frac{1}{2} \|\nabla\nabla_h u^h\|_{L^2} \|\nabla\partial_3 u^h\|_{L^2}. \tag{3.45}
\]
Similarly, we get
\[
\left| \int_{\mathbb{R}^3} (\partial_3^2 \nabla u \cdot \nabla u^h) \cdot \partial_3 u^h \, dx \right| \leq \left\| \partial_3 \Delta_h \nabla u^h \right\|_{L^2} \|\partial_3 u^h\|_{L^2}
\]
\[\leq C \left( \|\partial_3 u^h\|_{L^2} \|\nabla_u u^3\|_{L^3} + \|\nabla_h u^h\|_{L^2} \|\nabla\nabla_h u^h\|_{L^2} \right) \|\partial_3 u^h\|_{L^2}
\]
\[\leq \frac{1}{4} \|\nabla\partial_3 u^h\|_{L^2}^2 + C \left( \|\partial_3 u^h\|_{L^2}^2 \|\nabla_u u^3\|_{L^2} \|\nabla\nabla_h u^h\|_{L^2} + \|\nabla_h u^h\|_{L^2}^3 \|\nabla\nabla_h u^h\|_{L^2} \right). \tag{3.46}
\]
Substituting the above estimate and (3.45) into (3.44) leads to
\[
\frac{d}{dt} \|\partial_3 u^h\|_{L^2}^2 + \|\nabla\partial_3 u^h\|_{L^2}^2
\]
\[\leq C \left( \|\partial_3 u^h\|_{L^2}^2 \|\nabla_h u\|_{L^2} \|\nabla\nabla_h u\|_{L^2} + \|\nabla_h u^h\|_{L^2}^3 \|\nabla\nabla_h u^h\|_{L^2} \right). \tag{3.47}
\]
By applying Gronwall’s inequality and using (3.43) and the energy equality, we find
\[
\|\partial_3 u^h\|_{L^2}^2 + \|\nabla\partial_3 u^h\|_{L^2}^2
\]
\[\leq \left( \|\partial_3 u^h\|_{L^2} + C \|\nabla_h u^h\|_{L^2} \|\nabla\nabla_h u^h\|_{L^2} \right) \exp \left( C \|\nabla_h u\|_{L^2} \|\nabla\nabla_h u\|_{L^2} \right)
\]
\[\leq \left( \|\partial_3 u^h\|_{L^2} + C \|\nabla_h u^h\|_{L^2} \|u_0\|_{L^2} \right) \exp \left( C \|u_0\|_{L^2} \left( \|\nabla_h u_0\|_{L^2} + \sqrt{K} \right) \right). \tag{3.48}
\]
In particular, the estimates (3.43) and (3.46) ensure that
\[
\|u\|_{L^2(\Omega^\infty \cap H^2)} < \infty \quad \text{with} \quad T^* < \infty,
\]
which contradicts with the well-known Serrin’s regularity criteria. Hence $T^* = \infty$, and the estimate (1.10) holds for any $t \in [0, \infty]$.

Finally let us sketch the proof of (1.8) under the assumption (1.7). The proof is basically the same as that of (1.10), so we just list the different key points. Once
again we shall use a continuity argument, but instead of assuming (1.10) and (3.34) hold locally in time, here we assume that (1.8) and the following estimate
\[ \|ω^3\|_{L^2(t)}^2 \leq 2 (\|ω_0^3\|_{L^2}^2 + \|∇u_0^3\|_{L^2}^2) \] (3.47)
hold on \([0, T^*_K]\). Then for \(t \leq T^*_K\), we deduce respectively from (2.7) and (2.10) that
\[ \|ω^3\|_{L^2(t)}^2 \leq \|ω_0^3\|_{L^2}^2 + Cν^{-3/2} \|u_0\|_{L^2}^2 \| ω_0^3 \|_{L^2}^2 \| ∇u_0^3\|_{L^2}^2 (\|∇u_0^3\|_{L^2}^2 + K)^{1/2} \] (3.48)
and
\[ \|∇u^3\|_{L^2(t)}^2 \leq \|∇u_0^3\|_{L^2}^2 Cν^{-3/2} \|u_0\|_{L^2}^2 \| ω_0^3 \|_{L^2}^2 \| ∇u_0^3\|_{L^2}^2 \| ∇u_0^3\|_{L^2}^2 + K \] (3.49)
Then it is easy to verify that, whenever
\[ Cν^{-3/2} \|u_0\|_{L^2}^2 \| ω_0^3 \|_{L^2}^2 \| ∇u_0^3\|_{L^2}^2 \leq \frac{1}{2} \] \[ Cν^{-3/2} \|u_0\|_{L^2}^2 \| ω_0^3 \|_{L^2}^2 \| ∇u_0^3\|_{L^2}^2 \leq \frac{1}{4} \]
hold, which can be guaranteed by choosing \(C_3\) in (1.7) to be sufficiently large, then the estimates (3.48) and (3.49) ensures that
\[ \|ω^3\|_{L^2(t)}^2 \leq \frac{3}{2} (\|ω_0^3\|_{L^2}^2 + \|∇u_0^3\|_{L^2}^2) \] and \[ \|∇u^3\|_{L^2(t)}^2 \leq \frac{3}{2} (\|∇u_0^3\|_{L^2}^2 + K) \]
Thanks to the above estimates and a continuity argument, we can prove that the estimate (1.8) holds globally in time. We thus complete the proof of Theorem 1.2.

4. The proof of Theorem 1.3.

4.1. The proof of the estimate (1.15). For any solenoidal vector field \(u_0 \in L^2 \cap H^1\), it is classical that \((ANS)_ν\) has a unique strong solution \(u \in SS(T)\) for any \(T < T^*\). Then the vorticity \(ω = \text{curl } u\) verifies (1.13).

By taking \(L^2\) inner product of (1.13) with \(ω\), we get
\[ \frac{1}{2} \frac{d}{dt} \|ω(t)\|_{L^2}^2 + \|∇hω(t)\|_{L^2}^2 + ν_κ \|∂₃ω(t)\|_{L^2}^2 = \int_{R^3} (ω \cdot ∇u) \cdot ω(t) \, dx. \] (4.1)
It follows from Lemma 2.3 and (1.12) that
\[ \left| \int_{R^3} (ω \cdot ∇u) \cdot ω \, dx \right| \leq \|∇u\|_{L^6} \|ω\|_{L^3} \|ω\|_{L^6} \]
\[ \leq C [\|ω\|_{L^2} \|∇u\|_{L^2} \|ω\|_{L^2} \|∇hω\|_{L^2} \|∂₃ω\|_{L^2} \|∇hω\|_{L^2} \|∂₃ω\|_{L^2} \]
\[ \leq \frac{κ_ν}{2} \|∂₃ω\|_{L^2}^2 + \frac{1}{2} \|∇hω\|_{L^2}^2 + Cν^{-1} \|ω\|_{L^2} \|∇u\|_{L^2}^2. \]
Substituting the above inequality into (4.1), we obtain
\[ \frac{d}{dt} \|ω(t)\|_{L^2}^2 + \|∇hω(t)\|_{L^2}^2 + ν_κ \|∂₃ω(t)\|_{L^2}^2 \leq Cν^{-1} \|ω(t)\|_{L^2} \|∇u(t)\|_{L^2}^2. \]
By integrating the above inequality over \([0, t]\) for \(T < T^*\) and using the energy equality (2.6), we obtain
\[ \|ω\|_{L^2(t)}^2 \leq \|ω_0\|_{L^2}^2 + Cν^{-1} \|ω\|_{L^2(\infty)} \int_{0}^{t} \|∇u(t')\|_{L^2} \, dt' \]
\[ \leq \|ω_0\|_{L^2}^2 + Cν^{-1} \|u_0\|_{L^2} \|ω\|_{L^2(\infty)}^2. \] (4.2)
Let us denote
\[ T\omega^* = \sup \{ T \in [0, T^*] : \text{so that (1.15) holds for any } t \in [0, T] \}. \tag{4.3} \]
We are going to prove that \( T\omega^* = T^* = \infty \) under the assumption (1.14). Otherwise, for any time \( t < T\omega^* \), we deduce from (4.2) that
\[ \| \omega \|_{L^2}^2(t) \leq \| \omega_0 \|_{L^2}^2 + C\nu^{-1}\| u_0 \|_{L^2}^2\| \omega_0 \|_{L^2}^2. \tag{4.4} \]
In particular if \( \nu \) is so large that
\[ C\nu^{-1}\| u_0 \|_{L^2}^2\| \omega_0 \|_{L^2}^2 \leq \frac{1}{2}, \]
which can be reached provided \( C_3 \) in (1.14) is sufficiently large, we deduce from (4.4) that
\[ \| \omega \|_{L^2}^2(t) \leq \frac{3}{2}\| \omega_0 \|_{L^2}^2, \quad \forall t < T\omega^*. \]
This contracts with the definition of \( T\omega^* \) given by (4.3) unless \( T\omega^* = T^* = \infty \).

4.2. The proof of the estimate (1.17). We first get, by taking \( L^2 \) inner product of (1.13) with \( \omega^h \), that
\[ \frac{1}{2} \frac{d}{dt} \| \omega^h(t) \|_{L^2}^2 + \| \nabla_h \omega^h(t) \|_{L^2}^2 + \nu \| \partial_3 \omega^h(t) \|_{L^2}^2 = A_1 + A_2, \tag{4.5} \]
where
\[ A_1 = \int_{\mathbb{R}^3} (\omega^h \cdot \nabla_h u^h) \cdot \omega^h(t) \, dx \quad \text{and} \quad A_2 = \int_{\mathbb{R}^3} (\omega^3 \partial_3 u^h) \cdot \omega^h(t) \, dx. \]
By using integration by parts, we write
\[ A_1 = -\int_{\mathbb{R}^3} (\text{div}_h \omega^h (u^h \cdot \omega^h) + (\omega^h \cdot \nabla_h \omega^h) \cdot u^h) \, dx. \]
Applying Lemma 2.3 and Young’s inequality leads to
\[ |A_1| \leq \| \nabla_h \omega^h \|_{L^2} \| \omega^h \|_{L^3} \| u^h \|_{L^6} \]
\[ \leq C \| \omega^h \|_{L^2}^2 \| \nabla_h \omega^h \|_{L^2} \| \partial_3 \omega^h \|_{L^2} \| \nabla_h u^h \|_{L^2}^2 \| \partial_3 u^h \|_{L^2}^2 \]
\[ \leq \frac{\nu}{6} \| \partial_3 u^h \|_{L^2}^2 + \frac{1}{6} \| \nabla_h \omega^h \|_{L^2}^2 + C\nu^{-\frac{2}{3}} \| \omega^h \|_{L^2}^2 \| \nabla_h u^h \|_{L^2}^2 \| \partial_3 u^h \|_{L^2}^2. \tag{4.6} \]
To handle the estimate of \( A_2 \), we first use integration by parts to write
\[ A_2 = A_{2,1} + A_{2,2} \quad \text{with} \quad A_{2,1} = -\int_{\mathbb{R}^3} \partial_3 \omega^3 (u^h \cdot \omega^h)(t) \, dx \]
and
\[ A_{2,2} = -\int_{\mathbb{R}^3} \omega^3 (u^h \cdot \partial_3 \omega^h)(t) \, dx. \]
Due to \( \text{div} \omega = 0 \), we observe that \( A_{2,2} \) shares the same estimate as \( A_1 \) in (4.6):
\[ |A_{2,1}| \leq \frac{\nu}{6} \| \partial_3 u^h \|_{L^2}^2 + \frac{1}{6} \| \nabla_h \omega^h \|_{L^2}^2 + C\nu^{-\frac{2}{3}} \| \omega^h \|_{L^2}^2 \| \nabla_h u^h \|_{L^2}^2 \| \partial_3 u^h \|_{L^2}^2. \tag{4.7} \]
Along the same line, we have
\[ |A_{2,2}| \leq \| \partial_3 \omega^h \|_{L^2} \| \omega^3 \|_{L^3} \| u^h \|_{L^6} \]
\[ \leq C \| \partial_3 \omega^h \|_{L^2} \| \omega^3 \|_{L^3} \| \partial_3 \omega^h \|_{L^2} \| \nabla_h \omega^3 \|_{L^2} \| \partial_3 \omega^3 \|_{L^2} \| \partial_3 u^h \|_{L^2}^2 \]
\[ \leq \frac{\nu}{6} \| \partial_3 \omega^h \|_{L^2}^2 + C\nu^{-1} \| \partial_3 \omega^3 \|_{L^2} \| \nabla_h \omega^3 \|_{L^2} \| \partial_3 \omega^3 \|_{L^2} \| \partial_3 u^h \|_{L^2}^2. \tag{4.8} \]
By substituting the obtained estimates (4.6)-(4.8) into (4.5), we obtain
\[
\frac{d}{dt} \|\omega^h\|^2_{L^2} + \|\nabla_h \omega^h\|^2_{L^2} + \nu_\omega \|\partial_3 \omega^h\|^2_{L^2} \\
\leq C \nu_\omega^{-1} \|\omega^h\|^2_{L^2} \|\nabla_h u^h\|^2_{L^2} \|\partial_3 u^h\|^2_{L^2} \\
+ C \nu_\omega^{-1} \|\omega^3\|_{L^2} \|\nabla_h \omega^3\|^2_{L^2} \|\partial_3 \omega^3\|^2_{L^2} \|\nabla_h u^h\|^2_{L^2} \|\partial_3 u^h\|^2_{L^2}. \tag{4.9}
\]
Integrating the above estimate over \([0, t]\) and using (1.12), we infer
\[
\|\omega^h(t)\|^2_{L^2} + \int_0^t \left( \|\nabla_h \omega^h(t')\|^2_{L^2} + \nu_\omega \|\partial_3 \omega^h(t')\|^2_{L^2} \right) dt' \\
\leq \|\omega^h_0\|^2_{L^2} + C \nu_\omega^{-1} \|u_0\|_{L^2} \|\omega_0\|_{L^2} \|\omega^h\|^2_{L^2} \|\nabla_h \omega^3\|^2_{L^2} \|\partial_3 \omega^3\|^2_{L^2} \|\nabla_h u^h\|^2_{L^2} \|\partial_3 u^h\|^2_{L^2} + C \nu_\omega^{-2} \|u_0\|_{L^2} \|\omega_0\|^3_{L^2}. \tag{4.10}
\]
Let us denote
\[
T_{w_h}^* \overset{def}{=} \sup \{ T > 0 : \text{so that (1.17) holds for any } t \in [0, T] \}. \tag{4.11}
\]
(4.10) ensures that \(T_{w_h}^* > 0\). Then for any time \(t < T_{w_h}^*\), we deduce from (4.10) that
\[
\|\omega^h(t)\|^2_{L^2} + \int_0^t \left( \|\nabla_h \omega^h(t')\|^2_{L^2} + \nu_\omega \|\partial_3 \omega^h(t')\|^2_{L^2} \right) dt' \\
\leq \|\omega^h_0\|^2_{L^2} + C \nu_\omega^{-1} \|u_0\|_{L^2} \|\omega_0\|_{L^2} \|\omega^h\|^2_{L^2} \|\nabla_h \omega^3\|^2_{L^2} \|\partial_3 \omega^3\|^2_{L^2} \|\nabla_h u^h\|^2_{L^2} \|\partial_3 u^h\|^2_{L^2} + K \tag{4.12}
\]
So if \(\nu_\omega\) is so large that
\[
C \nu_\omega^{-1} \|u_0\|_{L^2} \|\omega_0\|^2_{L^2} \leq \frac{1}{4} \quad \text{and} \quad C \nu_\omega^{-3} \|u_0\|_{L^2} \|\omega_0\|^3_{L^2} \leq \frac{K}{4}, \tag{4.13}
\]
(4.12) implies
\[
\|\omega^h(t)\|^2_{L^2} + \int_0^t \left( \|\nabla_h \omega^h(t')\|^2_{L^2} + \nu_\omega \|\partial_3 \omega^h(t')\|^2_{L^2} \right) dt' \leq \frac{3}{2} \left( \|\omega^h_0\|^2_{L^2} + K \right), \quad \forall t < T_{w_h}^*.
\]
This contracts with the definition of \(T_{w_h}^*\) given by (4.11) unless \(T_{w_h}^* = \infty\). On the other hand, it is easy to observe that the conditions in (4.13) can be satisfied provided the assumptions (1.14) and (1.16) hold. We thus complete the proof of Theorem 1.3.

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