A PROCEDURE FOR CONSTRUCTING PEAK FUNCTIONS

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Abstract. We extend Bishop’s one-fourth three-fourths principle for constructing peak functions belonging to a uniform algebra to a situation where the “approximate barriers” associated with the Bishop construction are not uniformly bounded.

1. Introduction

In this paper, we revisit a procedure for constructing peak functions devised by Bishop. Specifically, let $\Omega$ be a bounded domain in $\mathbb{R}^n$ (or $\mathbb{C}^n$), and let $E$ be a closed subspace of $C(\overline{\Omega})$ : the class of all complex-valued functions that are continuous on $\overline{\Omega}$. Let $x \in \overline{\Omega}$; we say that $f$ (here $f \in E$) peaks at $x$ if $f(x) = 1$ and $|f(y)| < 1 \forall y \in \overline{\Omega} \setminus \{x\}$, and we call $f$ a peak function of class $E$.

We present a procedure for constructing a peak function of class $E$.

The procedure now known as Bishop’s one-fourth three-fourths principle [2] (also see [5, Theorem II/11.1] ) says that given a compact metric space $X$, a uniform algebra $A$ on $X$, and a point $x \in X$, if for each neighbourhood $U$ of $x$, we could find a $f_U \in A$ such that : i) $f_U(x) = 1$, ii) The sup-norms $\sup_{X} |f_U|$ are uniformly bounded, and iii) $|f_U(y)| \leq \alpha \forall y \in X \setminus U$ with a uniform constant $0 < \alpha < 1$, then we could construct a function $F \in A$ that peaked at $x$. Our result first exploits the fact that the result just described can be extended beyond uniform algebras to closed subspaces $E \subset_{\text{closed}} C$. Secondly, it develops a Bishop-type construction in a setting where condition (ii) is replaced by a weaker analogue : $\sup_{X} |f_U| \lesssim \psi(\text{diam}(U)^{-1})$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, and $\psi(x) \nearrow +\infty$ sufficiently gradually as $x \to +\infty$. This sort of of bound is motivated by applications in which the condition (ii) is difficult to verify. We make all of this precise in the following

**Theorem 1.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ (or $\mathbb{C}^n$), and let $x \in \overline{\Omega}$. Let $E$ be a closed subspace of $C(\overline{\Omega})$. Suppose there exist constants

$$0 < \alpha < 1,$$

$$0 < s \leq 1, \ 0 < t < 1, \ \text{and}$$

$$0 < A < 1, \ C > 0$$

such that for each neighbourhood $U$ of $x$ with $r_x(U) < 1$, there exists a function $f_U \in E$ with the properties

$$1) \ f_U(x) = 1;$$

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2) \(|f_U(y)| \leq \alpha \forall y \in \overline{\Omega} \setminus U;\)
3) \(|f_U(y)| \leq C \log \left( \frac{1}{r_x(U)} \right) \forall y \in U;\) and
4) \(\{y \in \overline{\Omega}: |f_U(y)| < 1 + \epsilon\} \supset B(x; A r_x(U) \epsilon), 0 < \epsilon < 1.\)

Then there exists a function \(F \in E\) which peaks at \(x.\)

The expression \(r_x(U)\) in the above theorem is defined as
\[
r_x(U) := \sup_{y \in U} |y - x|,
\]
where \(|\cdot|\) denotes the Euclidean norm, while the set \(B(y; r)\) is defined as \(B(y; r) := \overline{\Omega} \cap B(y; r),\) where \(B(y; r)\) is the open Euclidean ball centered at \(y \in \overline{\Omega}\) and having radius \(r.\)

**Remark 1.2.** The above theorem, suitably restated, is true if \(\overline{\Omega}\) is replaced by a compact metric space \(X\) – we merely state the theorem in a setting that is closest to its applications.

We comment briefly on the motivation behind the precise form of condition (4) in Theorem 1.1. This condition is motivated by certain applications in multivariate complex analysis – such as when \(D\) is a pseudoconvex domain in \(\mathbb{C}^2\) having a sufficiently “nice” boundary, \(x = 0 \in \partial D, \overline{\Omega} = D \cap \overline{V}, E = O(D \cap V) \cap C(\overline{\Omega})\) (where \(V\) is an appropriately chosen, small \(\mathbb{C}^2\)-neighbourhood of 0), and one constructs a function of class \(E\) that peaks at \(x = 0;\) refer to Fornaess & Sibony [4], and Fornaess & McNeal [3]. In both these constructions, the relevant \(f_U\)'s are scalings of a single function \(f\) that satisfies (1) and (2) with \(U = \mathbb{B}^2(0; 1) \cap \overline{D}\) and \(x = 0,\) and \(f\) satisfies a Hölder condition with exponent \(s = 1.\) The scalings \(f_U\) then satisfy \(|f_U(x) - f_U(y)| \lesssim r_x(U)^{-1} |y|,\) which implies the condition (4) with \(s = 1\) (but is not equivalent to (4) above). We ought to clarify here that the Hölder condition just stated, combined with condition (1), forces the relevant \(f_U\)'s to be uniformly bounded. Thus, in the constructions [3] and [4], a variant of Bishop’s original procedure suffices. But a different procedure is needed in order to construct (local) peak functions on far more complicated domains in \(\mathbb{C}^n, n > 2\) (such as the examples in [1] – known as the non-semiregular domains – where uniform boundedness of the relevant \(f_U\)'s, constructed in analogy with [4], fails. However, a weaker “Hölder-type” condition, i.e. condition (4), does hold in some of these domains, whereby Theorem 1.1 can be used. Details of this last application will appear elsewhere.

2. **The Proof of Theorem 1.1**

To prove our theorem, we will need the following two lemmas.

**Lemma 2.1.** Let \(\psi: [0, \infty) \to (\epsilon, \infty)\) be a continuous function (with \(\epsilon > 0\)), and assume that \(\psi(x) \approx x^{1+\delta}\) for large \(x\) (here, \(\delta\) is a small positive number). Define
\[
g(x) := \exp \left\{ -k \int_0^x \psi(s)^{-t} \, ds \right\},
\]
where \(k > 0\) and \(t\) is such that \((1 + \delta)t < 1.\) Then
1) There are constants $A_1, A_2 > 0$ such that

\[ 0 < g(x) \leq A_1 \exp \left\{ -\frac{kx^{1-(1+\delta)t}}{A_2(1-(1+\delta)t)} \right\}. \]

2) $g$ satisfies the equation

\[ g(x) = k \int_x^\infty \frac{g(s)}{\psi(s)^t} \, ds. \]

**Proof.** Part (1) follows easily from the fact that there exist constants $M, A_2, A_3 > 0$ such that

\[ A_1/t \frac{s}{1+\delta} \leq \psi(s) \leq A_1/t \frac{s}{1+\delta} \quad \forall s \geq M. \]

To prove part (2), we use the estimate (2.1), to see that

\[
\begin{align*}
0 < k \int_x^\infty \frac{g(s)}{\psi(s)^t} \, ds \leq & \begin{cases} 
\text{const.} + \frac{kA_1}{A_3} \int_M^\infty s^{-(1+\delta)t} \exp \left\{ -\frac{k}{A_2(t+(1-\delta)t)} s^{1-(1-\delta)t} \right\} \, ds, & \text{if } x < M, \\
\frac{kA_1}{A_3} \int_x^\infty s^{-(1+\delta)t} \exp \left\{ -\frac{k}{A_2(t+(1-\delta)t)} s^{1-(1-\delta)t} \right\} \, ds, & \text{if } x \geq M,
\end{cases}
\end{align*}
\]

whence the integral on the right-hand side of (2.2) is a convergent integral. We make the following change of variable

\[ u(s) = k \int_0^s \psi(\tau)^{-t} \, d\tau. \]

Since $\psi(\tau)^{-t} \geq \tau^{-(1+\delta)t}/A_2 \quad \forall \tau \geq M$, and $(1+\delta)t < 1$, we have

\[ \lim_{s \to \infty} u(s) = \infty. \]

From (2.3) and (2.4), we have

\[ k \int_x^\infty \frac{g(s)}{\psi(s)^t} \, ds = \int_{u(x)}^\infty e^{-u} \, du = e^{-u(x)} = g(x). \]

This completes the proof. \qed

**The proof of Theorem 1.1.** We begin by observing that we may as well assume that $1/2 < t < 1$, and that $x = 0$. We may henceforth assume – by raising the value of $C$ if necessary – that $s - (1-\alpha)/C > 0$. This allows us to choose a $D \in (0, 1)$, $D$ sufficiently small, so that:

\[ \frac{1-\alpha}{2} D^{(1-\alpha)/C} \geq D^s. \]

Recall that $A \in (0, 1)$. Using this fact, we define

\[ \log(\varepsilon_k/D) := (\log A) \sum_{j=1}^k j^{-1+p}, \quad k = 1, 2, \ldots \]

\[ U_k := \begin{cases} B(0; D), & \text{if } k = 1, \\
B(0; A r_x(U_{k-1} \varepsilon_{k-1})), & \text{if } k \geq 2,
\end{cases} \]

where we choose $p$ to satisfy $0 < p < 1$. Notice that $\varepsilon_k \downarrow 0$ as $k \to \infty$. We choose $p$ to be so close to 0 that $(1 + p)t < 1$. This is always possible because $t < 1$. In fact, we shall demand that:
• \((1 + p)t = q\), with \(q \in (0, 1)\); and
• \((1 - q) = p/M\), where \(M\) satisfies \(M \geq C\), and whose precise value will be stated later in this proof.

These imply that

\[
q = \frac{t + Mt}{1 + Mt}, \quad p = \frac{M(1 - t)}{1 + Mt}.
\]

Observe that since we have assumed that \(1/2 < t < 1\), the value of \(p\) will indeed be less than 1.

We now construct a sequence of functions \(\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{E}\) by induction as follows:

Define \(f_1 := f_{U_1}\) and write

\[
W_1 = \{y \in \Omega : |f_1(y)| \geq 1 + \varepsilon_1\}.
\]

Due to the condition (4) of our hypothesis (note that \(\varepsilon_1 < 1\)), \(W_1 \subset (\Omega \setminus U_1)\). If we define \(f_2 := f_{U_2}\), it follows from the hypotheses of our theorem that:

(a) \(f_2(0) = 1\);
(b) \(\{y \in \Omega : |f_2(y)| < 1 + \varepsilon_2\} \supset U_3\);
(c) \(|f_2(y)| \leq C\log [1/r_x(U_2)] \forall y \in U_2\); and
(d) \(|f_2(y)| \leq \alpha \forall y \in (\Omega \setminus U_2)\).

Assume that we have found \(f_2, \ldots, f_m \in \mathcal{E}\) such that, defining

\[
W_m = \{y \in \Omega : \max_{1 \leq j \leq m} |f_j(y)| \geq 1 + \varepsilon_m\},
\]

they satisfy

(a) \(f_j(0) = 1, \ j = 2, \ldots, m\);
(b) \(\{y \in \Omega : |f_j(y)| < 1 + \varepsilon_j\} \supset U_{j+1}, \ j = 2, \ldots, m\);
(c) \(|f_j(y)| \leq C\log [1/r_x(U_j)] \forall y \in U_j\) and \(j = 2, \ldots, m\); and
(d) \(|f_j(y)| \leq \alpha \forall y \in (\Omega \setminus U_j)\) and \(j = 2, \ldots, m\).

Notice that by (b)\(_m\), \(W_m \subset (\Omega \setminus U_{m+1})\). If we define \(f_{m+1} := f_{U_{m+1}}\), then, by our hypotheses, \(\{f_2, \ldots, f_m, f_{m+1}\} \subset \mathcal{E}\) satisfies (a)\(_{m+1}\)–(d)\(_{m+1}\).

It is easy to check that for \(m \geq 2\)

\[
\log \frac{1}{r_x(U_m)} = m \log \left(\frac{1}{D}\right) + \log \left(\frac{1}{A}\right) \left\{ (m - 1) + \sum_{j=1}^{m-1} (m - j)^{-1+p} \right\}
\]

\[(2.7) \quad = m \log \left(\frac{1}{D}\right) + \log \left(\frac{1}{A}\right) \left\{ (m - 1) + m \sum_{j=1}^{m-1} j^{-1+p} - \sum_{j=1}^{m-1} j^p \right\}.
\]
Notice that by estimating the sums obtained above by integrals, we have

\[
\frac{m^p - 1}{p} < \sum_{j=1}^{m-1} j^{-1+p} < \frac{m^p - 1}{p} + (1 - m^{-1+p}),
\]

\[
\frac{m^{p+1} - 1}{p + 1} + (1 - m^p) < \sum_{j=1}^{m-1} j^p < \frac{m^{p+1} - 1}{p + 1}, \quad m \geq 2.
\]

From (2.7) and (2.8), we get

\[
\log \frac{1}{r_x(U_m)} \leq m \log \left( \frac{1}{D} \right) + \log \left( \frac{1}{A} \right) \left\{ 2(m - 1) + \frac{m^1 + p - (p + 1)m + p}{p(p + 1)} \right\} \forall m \geq 2.
\]

Therefore, there exists a constant \( L > 0 \) that is independent of \( p \in (0, 1) \) such that

\[
\log \frac{1}{r_x(U_m)} \leq m \log \left( \frac{1}{D} \right) + \log \left( \frac{1}{A} \right) Lm^{1+p} \frac{1}{p(p + 1)} \forall m \geq 1.
\]

Define

\[
\psi(\tau) := \begin{cases} \tau \log(1/D) + \log(1/A) \frac{Lm^{1+p}}{p(p + 1)} & \text{if } \tau \geq 1, \\ \psi(1), & \text{if } 0 \leq \tau < 1. \end{cases}
\]

Note that \( \psi(m) \geq \log[1/r_x(U_m)] \forall m \in \mathbb{N} \). Finally, define

\[
F(y) := \sigma^{-1} \left[ \sum_{j=1}^{\infty} \sigma_j f_j(y) \right],
\]

where

\[
g(x) := \exp \left\{ -\frac{1 - \alpha}{2M} \int_0^x \psi(s)^{-t} \, ds \right\} \quad \text{and} \quad \sigma_j := \frac{g(j)}{M \psi(j)^t},
\]

\[
\sigma := \sum_{j=1}^{\infty} \sigma_j.
\]

It is easy to check that the last series above is rapidly convergent. To see this, we apply Lemma 2.1 to \( \psi \). By the manner in which \( p \) and \( q \) are defined, we have the estimate

\[
0 < \sigma_j \leq \frac{A_1}{M \psi(j)^t} \exp \left\{ -\frac{1 - \alpha}{2MA_2(1-q)} j^{1-q} \right\},
\]

where \( A_1 \) and \( A_2 \) are the constants given by Lemma 2.1. Thus, by item (3) of our hypothesis,

\[
0 < \sigma_j \sup_{\Omega} |f_j| \leq \frac{CA_1}{M} \exp \left\{ -\frac{1 - \alpha}{2MA_2(1-q)} j^{1-q} \right\}.
\]

Since the right-hand side of the above estimate constitutes a summable series, as \( j \) varies over \( \mathbb{N} \), we conclude that the right-hand side of (2.10) converges uniformly on \( \Omega \). Therefore \( F \in \mathcal{E} \).

We claim that \( F \) defined by (2.10) peaks at \( x \). Before proving this assertion, we choose an appropriately large value for \( M \), which links \( p \) and \( q \) via the relation

\[
(1 - q) = p/M.
\]
We choose $M$ to be so large – i.e. $q < 1$ to be so close to 1 – that:

$$M \geq C, \quad \text{and}$$

$$
\exp \left\{ - \frac{1 + Mt}{1 - t} \frac{1 - \alpha}{2M} \left[ (m + 1)^{p/M} - 2^{p/M} \right] \right\}
\geq \exp \left\{ - \log \frac{1}{A} s(1 + Mt) \frac{1}{M(1 - t)} \left[ (m - 1)^{p/M} - 1 \right] \right\} \quad \forall m \geq 3.
$$

With this choice of $p$ and $q$, we can make the following claims:

**Claim 1.** $(C \psi(m)^t - 1)\sigma_m < (1 - \alpha) \sum_{j \geq m+1} \sigma_j / 2$ for each $m \in \mathbb{N}$.

**Claim 2.** $(1 - \alpha) \sum_{j \geq m+1} \sigma_j / 2 > \varepsilon_{m-1}^* \sum_{j=1}^{m-1} \sigma_j$ for each $m \geq 2$.

We defer the proofs of these claims to the end of this section. Assuming that these claims are true, we can show that $F$ peaks at 0. We first consider $y \neq 0$ and $y \notin \bigcup_{j \in \mathbb{N}} W_j$. Then, $|f_j(y)| \leq 1 \forall j \in \mathbb{N}$. However, $y \in (\Omega \setminus U_{j_0})$ for some $j_0 \in \mathbb{N}$, whence, by condition (d)$_m$ : $|f_{j_0}(y)| \leq \alpha < 1$. Consequently, $|F(y)| < 1$. This leaves us with the case $y \in \bigcup_{j \in \mathbb{N}} W_j$ to analyze. Since $\{W_j\}_{j \in \mathbb{N}}$ is a strictly increasing sequence of closed sets, either $y \in W_1$ or there exists $m \in \mathbb{N}$ such that $y \in W_m$ but $y \notin W_j \forall j \leq m-1$. In the former case, it is clear by construction that $|F(y)| \leq \alpha < 1$. The latter case results in the following estimates

$$
\begin{align*}
|f_j(y)| &< 1 + \varepsilon_{m-1}^*, \quad 1 \leq j < m, \\
|f_m(y)| &\leq C \log^t [1/r_x(U_m)] = C \psi(m)^t, \\
|f_j(y)| &\leq \alpha \quad \forall j \geq m + 1.
\end{align*}
$$

Then

$$
|F(y)| \leq \sigma^{-1} \left\{ (1 + \varepsilon_{m-1}^*) \sum_{j=1}^{m-1} \sigma_j + C \psi(m)^t \sigma_m + \alpha \sum_{j=m+1}^{\infty} \sigma_j \right\}
\leq \sigma^{-1} \left\{ \sum_{j=1}^{m-1} \sigma_j + \frac{1 - \alpha}{2} \sum_{j=m+1}^{\infty} \sigma_j \right\}
+ \left( \sigma_m + \frac{1 - \alpha}{2} \sum_{j=m+1}^{\infty} \sigma_j \right) + \alpha \sum_{j=m+1}^{\infty} \sigma_j = 1. \quad \text{(from Claims 1 & and 2)}
$$

Thus, $|F(y)| < 1 \forall y \neq 0$, whence $F$ peaks at 0.

To complete our proof, we first present
The proof of Claim 1: We compute:

\[
(C\psi(m)^t - 1)\sigma_m = \frac{C}{M} g(m) - \frac{g(m)}{M\psi(m)^t}
\]

\[
< \frac{C(1-\alpha)}{2M} \int_m^\infty \frac{g(s)}{M\psi(s)^t} \, ds - \frac{1-\alpha}{2} \sigma_m \quad \text{(applying Lemma 2.1-2)}
\]

\[
< \frac{1-\alpha}{2} \sum_{j=m}^\infty \frac{g(j)}{M\psi(j)^t} - \frac{1-\alpha}{2} \sigma_m \quad \text{(estimating from above by a series)}
\]

\[
= \frac{1-\alpha}{2} \sum_{j=m+1}^\infty \sigma_j.
\]

This proves Claim 1.

And finally, we present

The proof of Claim 2: Notice that when \( m \geq 2 \),

\[
\frac{1-\alpha}{2} \sum_{j=1}^{m-1} \sigma_j > \frac{1-\alpha}{2} \int_0^{m-1} \frac{g(s)}{M\psi(s)^t} \, ds
\]

\[
= \frac{1-\alpha}{2} \exp \left\{ - \frac{1-\alpha}{2M} \int_0^{m-1} (\psi(s)^t) \, ds \right\}
\]

\[
\geq \frac{1-\alpha}{2} D^{(1-\alpha)/M} \exp \left\{ - \frac{1-\alpha}{2M} \int_2^{m+1} s^{-t(1+p)} \, ds \right\} \quad \text{(since \( s^{1+p} \leq \psi(s) \forall s \geq 2 \)}
\]

\[
\geq \frac{1-\alpha}{2} D^{(1-\alpha)/C}
\]

\[
\times \exp \left\{ - \frac{1-\alpha}{2M} \frac{(m+1)^{1-q} - 2^{1-q}}{1-q} \right\} \quad \text{(since \( C \leq M \) and \( 0 < D < 1 \)}
\]

At this stage, we use the condition (recall that \( m \geq 2 \)) to get

\[
\frac{1-\alpha}{2} D^{(1-\alpha)/C} \exp \left\{ - \frac{1-\alpha}{2M} \frac{(m+1)^{1-q} - 2^{1-q}}{1-q} \right\}
\]

\[
= \frac{1-\alpha}{2} D^{(1-\alpha)/C}
\]

\[
\times \exp \left\{ - \frac{1+Mt}{1-t} \frac{1-\alpha}{2M} \left[ (m+1)^{p/M} - 2^{p/M} \right] \right\}
\]

\[
\geq D^\epsilon \exp \left\{ - \log \frac{1}{A} \frac{1-\alpha}{p} (m-1)^{p-1} \right\} \quad \text{(using the fact \( (1+Mt)/M(1-t) = 1/p \)}
\]

\[
\geq D^\epsilon A^\epsilon \sum_{j \leq (m-1)} j^{-1+p}
\]

\[
= \epsilon_{m-1}^s.
\]
Note that the first inequality makes use of the condition (2.5). Comparing (2.13) with (2.12) we conclude that
\[ \frac{1 - \alpha}{2} \sum_{j=m+1}^{\infty} \sigma_j > \varepsilon_{m-1} \sum_{j=1}^{m-1} \sigma_j \quad \forall m \geq 2, \]
which is precisely Claim 2.

This concludes our proof. □

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