Conservation laws for colliding branes with induced gravity

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Abstract

We derive conservation laws for collisions of self-gravitating n-branes (or n-dimensional shells) in an (n + 2) dimensional spacetime including induced gravity on the brane. Previous work has shown how geometrical identities in general relativity enforce conservation of energy-momentum at collisions. The inclusion of induced gravity terms introduces a gravitational self-energy on the brane which permits energy-momentum conservation of matter fields on the brane to be broken, so long as the total energy-momentum, including induced gravity terms, is conserved. We give simple examples with two branes (one ingoing and one outgoing) and three branes.

1 Introduction

Understanding the hierarchy problem is one the major theoretical challenges in modern physics [1, 2, 3]. It can be formulated in different ways: why the corrections to the Higgs boson mass are so big by comparison to its bare mass? why there exist very different mass scales in nature? why gravity is so weak by comparison to standard model forces? Indeed the electroweak scale of standard model fields is hundreds of GeV while the Planck scale -where quantum gravity is expected to be dominate- is approximately $10^{19}$ GeV.

To solve this problem, following the original idea of Kaluza [4] and Klein [5], many models allowing for extra dimensions have been proposed. Some of these allow large extra dimensions [6, 7] by requiring matter fields to be restricted to lower-dimensional branes while allowing gravity to propagate in a higher-dimensional bulk. Other studies have attempted to solve the hierarchy problem using warped extra dimensions models [8, 9]. This motivates the study of branes as cosmological objects [10, 11, 12] (for reviews see Ref. [13, 14, 15]). In particular, it has been proposed [16, 17] that our universe could be the result of the collision of branes, in the so-called ekpyrotic / pyrotechnic scenario. Thus it is natural to investigate collision of self-gravitating branes (or shells) embedded in higher dimensions [18, 19, 20, 21, 22, 23, 24, 25]. It has previously been shown [21] that for self-gravitating branes in general relativity a simple geometrical identity enforces conservation of energy and momentum at collisions.

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In Ref. \[12\], it was argued that it is possible to generate 4-dimensional Newtonian gravity from a static 3-brane embedded in 5 dimensional Minkowski spacetime by including the effect of induced gravity on the brane, as might be generated by quantum corrections to classical relativity, allowing an infinite size flat extra dimension.

In this paper we will investigate the effect of induced gravity on branes upon energy-momentum conservation rules for branes collisions. In the first section, we will first recall the derivation of conservation laws for colliding branes in classical general relativity \[21\]. After deriving new conservation rules, some examples of violation of the principle of matter conservation will be given.

## 2 Classical conservation laws

We consider \(n\)-dimensional branes living in a \(n + 2\) dimensional spacetime. Let us assume that we have \(N\) branes which separate the bulk into \(N\) different space-time regions. The dynamics is given by the effective action

\[
S = \sum_{i=1}^{N} \left( S_{EH}^{(i)} + S_{brane}^{(i)} + S_{matter}^{(i)} \right),
\]

where in each bulk space-time region we have the Einstein-Hilbert action

\[
S_{EH}^{(i)} = -\frac{1}{2\kappa^2} \int_{B_i} dx^{n+2} \sqrt{-g} \mathcal{R}_{(n+2)} - 2\Lambda
\]

and on each brane we have

\[
S_{brane}^{(i)} = -\frac{1}{2\kappa^2} \int_{b_i} dx^{n+1} \sqrt{-h} K
\]

and

\[
S_{matter}^{(i)} = \int_{b_i} dx^{n+1} \sqrt{-h} L_{matter}.
\]

where \(\mathcal{R}_{(n+2)}\) is the Ricci scalar in \(n + 2\) dimensions, \(\Lambda\) the cosmological constant, \(\kappa^2\) the coupling between matter and gravity, \(L_{matter}\) the matter Lagrangian, e.g. the standard model, and \(K\) the trace of the intrinsic curvature associated with the Gibbons-Hawking boundary term \[26\] at the brane. Note that we do not explicitly include any brane tension in the above and consider it to be part of the matter Lagrangian on the brane.

The collision of \(N_{in}\) ingoing branes giving rise to \(N_{out}\) outgoing branes is shown in Figure 1. The bulk regions are empty and can be described by the Schwarzschild-(anti)-de Sitter metric

\[
ds^2 = -f(R)dt^2 + \frac{dR^2}{f(R)} + R^2 d\Omega_n^2,
\]

where the “orthogonal” metric \(d\Omega_n^2 = \gamma_{ij} dx^i dx^j\) is the line-element for a maximally symmetric \(n\)-space and does not depend on either \(T\) or \(R\). The function \(f\) is

\[
f(R) = k - \frac{M}{R^n} + \left( \frac{R}{\ell} \right)^2,
\]

with \(k\) the curvature of the space, \(M\) the Schwarzschild mass and \(\ell\) the (anti)-de Sitter curvature length. Thus a brane at the boundary of this region (i.e. the bulk) is described by a two-dimensional
trajectory \((T(\tau), R(\tau))\), where \(\tau\) is the proper time on the brane. It is then possible to define the two-dimensional velocity vector \(u^a = (\dot{T}, \dot{R})\), with dots denoting derivative by respect to the proper time \(\tau\). One can introduce a basis of normalized vectors: \(e_T = f^{-1/2} \frac{\partial}{\partial T}\) and \(e_R = f^{1/2} \frac{\partial}{\partial R}\). We further define a local Lorentz factor \(\gamma = -e_T \cdot u\) and a relative velocity \(\beta\) given by: \(\gamma \beta = -e_R \cdot u\). To characterize the motion of a brane \(B_{2k-1}\) with respect to the static region or bulk \(R_{2k}\), we adopted the following definitions for quantities associated with a Lorentz angle:

\[
\gamma_{2k-1|2k} = \cosh(\alpha_{2k-1|2k}) = \sqrt{1 + \frac{\dot{R}_{2k-1}^2}{f_{2k}}},
\]

(7)

and

\[
\gamma_{2k-1|2k} \beta_{2k-1|2k} = \sinh(\alpha_{2k-1|2k}) = \epsilon_{2k} \frac{\dot{R}_{2k-1}}{\sqrt{f_{2k}}},
\]

(8)

where \(\dot{R}\) denotes the derivative of \(R\) with respect to \(\tau\) and \(f\) is the function describing the metric. Note that \(\epsilon\) enables us to fix the convention to draw this situation. If \(R\) decreases from “left” to “right” then \(\epsilon = +1\) and \(\epsilon = -1\) otherwise. Moreover by analogy with special relativity, \(\gamma\) and \(\beta\) defined above satisfy \(\gamma = 1/\sqrt{1 - \beta^2}\). The junction condition \([27, 28]\) which represents the jump of extrinsic curvature between two spacetime regions is

\[
[K_{AB}] = -\kappa^2 \left( S_{AB} - \frac{S}{n} g_{AB} \right),
\]

(9)

where \(S_{AB}\) is the energy-momentum tensor derived from the matter Lagrangian on the brane. For the orthogonal part, the extrinsic curvature components are \(K_{ij} = (\epsilon_{2k}/R) \sqrt{f_{2k} + \dot{R}_{2k-1}^2} g_{ij}\), The junction condition becomes

\[
\epsilon_{2k} \sqrt{f_{2k} + \dot{R}_{2k-1}^2} - \epsilon_{2k-2} \sqrt{f_{2k-2} + \dot{R}_{2k-1}^2} = \kappa^2 \rho_{2k-1} R,
\]

(10)

with \(\rho_{2k-1}\) the comoving energy density associated with the brane \(B_{2k-1}\). Note the absence of index on \(R\); this is due to the fact that \(R\) (the radial position) is the same for any brane meeting at the

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Figure 1: Collision of \(N_{in}\) ingoing branes giving rise to \(N_{out} = N - N_{in}\) outgoing branes. This is a system of \(N\) branes.
same point at the same time. Following Ref. [21] we define the rescaled brane density as

$$\tilde{\rho}_{2k-1} \equiv \pm \frac{\kappa^2}{n} \rho_{2k-1} R,$$

with the plus sign for ingoing branes and the minus sign for outgoing branes.

Geometrical consistency [21] leads to simple conservation rules for the branes. The energy conservation law is

$$\sum_{k=1}^{N} \tilde{\rho}_{2k-1} \gamma_{|2k-1} = 0,$$

while the momentum conservation law is

$$\sum_{k=1}^{N} \tilde{\rho}_{2k-1} \gamma_{|2k-1} \beta_{|2k-1} = 0,$$

for any value of the index $j$.

3 Modified conservation laws

3.1 Conservation laws

We now consider the effect of induced gravity on the brane, modifying the “classical” description. By induced gravity we mean an Einstein-Hilbert action in $n + 1$ dimensions on each $n$-brane. The assumed action becomes

$$S = \sum_{i=1}^{N} \left( S_{EH(n+2)}^{(i)} + S_{EH(n+1)}^{(i)} + S_{brane}^{(i)} + S_{matter}^{(i)} \right),$$

where on each brane

$$S_{EH(n+1)}^{(i)} = - \frac{1}{2\mu^2} \int_{B_i} dx^{n+1} \sqrt{-h} R^{(n+1)};$$

with $R^{(n+1)}$ the Ricci scalar in $n + 1$ dimensions and $\mu^2$ the coupling between matter and gravity. The junction condition is then:

$$[K_{AB}] = -\kappa^2 \left( (S_{AB} - S_{n g_{AB}}) + \frac{1}{\mu^2} (\tilde{U}_{AB} - \tilde{U} \frac{n}{n g_{AB}}) \right)$$

with $\tilde{U}_{AB}$ the Einstein tensor on the brane $(n + 1$ dimensional spacetime) defined as

$$\tilde{U}_{00} = + \frac{n(n-1)}{2} \frac{\delta(\chi)}{\mu^2} \left( \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} \right)$$

and

$$\tilde{U}_{ij} = -(n-1) \frac{\delta(\chi)}{\mu^2} \gamma_{ij} \left( R^2 \left( \frac{n-2}{2} \frac{\dot{R}^2}{R^2} + \frac{\dot{R}}{R} \right) + k \frac{(n-2)}{2} \right).$$

Finally by contracting the indices, the junction condition becomes

$$\epsilon_+ \sqrt{f_+ + \dot{R}^2} - \epsilon_- \sqrt{f_- + \dot{R}^2} = R_{n}^{\mu_2} \rho_{2k-1},$$
with the effective energy density defined as

\[
\rho'_{2k-1} = \rho_{2k-1}^{\text{matter}} - \frac{(n-1) n}{2 \mu^2} \left( \frac{\dot{R}^2_{2k-1}}{R^2} + \frac{k}{R^2} \right). 
\]

Here \( \rho_{2k-1}^{\text{matter}} \) denotes the \( \rho_{2k-1} \) of the previous section i.e. the brane energy density. Recovering the classical case is thus equivalent to set \( \rho_{2k-1}^{\text{gravity}} = 0 \). Note that this result is in accordance with Ref. [29] where the junction condition has been derived for the case \( n = 3 \). According to this definition it is possible to derive modified conservation laws for energy and momentum for colliding branes by following the same procedure than in Ref. [21]. The modified energy and momentum conservation laws are then:

\[
\sum_{k=1}^{N} \rho'_{2k-1} \gamma_{2k-1j} = 0, 
\]

and

\[
\sum_{k=1}^{N} \rho'_{2k-1} \gamma_{2k-1j} \beta_{2k-1j} = 0. 
\]

for any value of the index \( j \).

### 3.2 Nil-brane

We introduce here the concept of nil-brane which will be used abundantly in the following. By saying nil-brane or vacuum we mean, \( \rho' = 0 \) and \( \rho_{2k-1}^{\text{matter}} = 0 \) which implies \( \rho_{2k-1}^{\text{gravity}} = 0 \). According to the definition of \( \rho' \) in equation (20), it appears (for \( n \neq 0, 2 \)) that it gives constraints on the geometry of the bulk. It implies that for a nil-brane the curvature of the space\( ^2 \) is

\[
k = -\dot{R}^2 \leq 0.
\]

This shows that when nil-branes are considered the spatial curvature of the \( n \)-dimensional space will always be negative or zero (in which case \( \dot{R} = 0 \)).

### 3.3 Diagrammatic description

In order to simplify the understanding of the different cases that will be discussed in the following, we have adopted a diagrammatic description of the collisions. As shown in Figure \( 2 \) standard branes will be described by solid lines. By standard branes, it is understood \( \rho' \neq 0 \). Dotted lines will represent nil-branes while solid-wavy lines will represent branes with \( \rho' = 0 \) but with a non zero matter density (i.e. \( \rho_{2k-1}^{\text{matter}} \neq 0 \)).

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1 We here dropped the normalisation factor to simplify notation. For our purpose, this would not play any role because this normalisation factor would be identical for every branes and thus would always simplify.

2 We require that the curvature of the \( n \)-space must be the same for two regions separated by a brane.
4 Two branes

We will now focus on particular cases of branes collisions. The first case studied here is the simplest non trivial one, i.e. a two branes system with one ingoing brane and one outgoing brane. First, the classical case will be studied. Then the addition of induced gravity will be considered. It will exhibit new effects, made possible by the use of modified rules.

4.1 General equations

The ingoing brane will be denoted by the subscript “a” while the outgoing brane will be denoted by the subscript “b”. Form the conservation laws (21) and (22), one gets

$$\rho'_b = \rho'_a \gamma_{a|b}$$

(24)

and

$$\rho'_b \gamma_{b|c} \beta_{b|c} = \rho'_a \gamma_{a|c} \beta_{a|c}.$$  

(25)

To satisfy these equations in general, regardless of the values (non zero) of $\rho'_a$ and $\rho'_b$ the following equation has to be satisfied:

$$\left( \gamma_{a|b} \gamma_{b|c} \beta_{b|c} - \gamma_{a|c} \beta_{a|c} \right) = 0.$$  

(26)

It gives a non trivial condition on the expansion rate of the different branes (the expansion rate being defined as: $H = \dot{R}^2 / R^2$). A particular solution to fulfill this set of equations is to pick $\rho'_b = 0$. This leads to $\rho'_a = 0$, $\gamma_{a|b} \neq 0$ by definition. We will focus later on this solution.

4.2 Standard behaviours

4.2.1 Classical case

For the classical case, the definition of the effective energy density is

$$\rho' = \rho^{\text{matter}},$$  

(27)

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As the careful reader might have noticed, we have here chosen “angles” between two branes and not only between a bulk and a brane. This is possible by using the fact that $\alpha_{a|b} = \alpha_{a|I} + \alpha_{I|b}$ with $I$ the region between the branes “a” and “b”.

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Figure 2: Solid lines describe standard branes, dotted lines describe nil-branes and solid-wavy describe branes with $\rho' = 0$ but with a non zero matter density.
i.e. the usual matter density. The constraint on expansion rate is the same (see equation \((26)\)). It has to be stressed that there can be an augmentation of matter in the classical case in the sense that \(\rho_a^{\text{matter}}\) can be different from \(\rho_b^{\text{matter}}\). This difference is “compensated” by the motion of the brane. From equations \((24)\) and \((25)\), the strict conservation of matter density, \(\rho_a^{\text{matter}} = \rho_b^{\text{matter}}\), leads to the fact that the trajectories of the branes “a” and “b” are similar. It means that \(\dot{R}_a = \dot{R}_b\) (by definition of \(\gamma\), see equation \((7)\)). The two branes are indistinguishable. Thus there is an augmentation of matter density when trajectories evolve and strict conservation if not.

What is not possible in the classical case is the creation of matter from nothing i.e. the vacuum. Indeed, if one considers the case \(\rho_a^{\text{matter}} = 0\), it necessarily implies \(\rho_b^{\text{matter}} = 0\). It means that no matter can be created instantly. It thus leads to the diagrams showed in Figure 3.

### 4.2.2 Induced gravity case

In the previous section, it has been shown that the matter density could evolve. This is still true when induced gravity is considered. The strict conservation of matter density \((\rho_a^{\text{matter}} = \rho_b^{\text{matter}})\) leads to a non trivial relation between the “gravitational” density and the trajectory namely:

\[
\rho_b^{\text{gravity}}\gamma_{bc}\beta_{bc}(1 - \gamma_{bc}) - \rho_a^{\text{gravity}}\gamma_{ac}\beta_{ac}(1 - \gamma_{ac}) = \\
\rho_a^{\text{gravity}}(\gamma_{bc}\beta_{bc} - \gamma_{ac}\beta_{ac}) - \rho_b^{\text{gravity}}\gamma_{ab}(\gamma_{bc}\beta_{bc} - \gamma_{ac}\beta_{ac}).
\]

(28)

If one now considers the case where \(\rho_a' = 0\) implying \(\rho_b' = 0\), the conservation laws become

\[
\rho_a^{\text{matter}} - \frac{(n-1)}{2} \frac{n}{\mu^2} \left( \frac{\dot{R}_a^2}{R_a^2} + \frac{k}{R_a^2} \right) = \rho_b^{\text{matter}} - \frac{(n-1)}{2} \frac{n}{\mu^2} \left( \frac{\dot{R}_b^2}{R_b^2} + \frac{k}{R_b^2} \right) = 0.
\]

(29)

Demanding a strict preservation of matter density leads to:

\[
\rho^{\text{gravity}} = \frac{(n-1)}{2} \frac{n}{\mu^2} \left( \frac{\dot{R}_a^2}{R_a^2} + \frac{k}{R_a^2} \right) = \frac{(n-1)}{2} \frac{n}{\mu^2} \left( \frac{\dot{R}_b^2}{R_b^2} + \frac{k}{R_b^2} \right).
\]

(30)

Figure 3: On the left hand side, a brane with no change of trajectory and a strict conservation of matter (branes “a” and “b” are the same). On the right hand side, augmentation of matter density through evolution of trajectory.
This implies (if \( \mu \) is assumed to be constant) that \( \dot{R}_a = \dot{R}_b \) meaning that both branes have the same velocity \( i.e. \) they are identical.

Nonetheless one can imagine that the gravitational constant \( \mu \) can vary at the collision point. Thus we could imagine having a change of trajectory \( i.e. \) \( \dot{R}_a \neq \dot{R}_b \) at the collision point if it is compensated by the variation of \( \mu \).

For now we retain that the introduction of induced gravity still allows strict conservation of matter density as well as its augmentation. The conditions for the evolution of matter density or its strict conservation are more complicated in this case. Nonetheless, it still gives the possibility to recover “standard” behaviours from the classical case (these cases are described by Figure 3).

4.3 New mechanisms for evolution of matter density

As pointed out previously, one solution is to pick \( \rho'_a = \rho'_b = 0 \). In this section, the effective density \( \rho' \) will always be assumed to be null. For the sake of clarity we recall that the setting studied is encompassed in the following equations (see equation (20)):

\[
\rho'_a = \rho'_b = 0.
\]

The previous section described cases of matter evolution whether or not induced gravity was considered. The cases where the introduction of induced gravity gives the opportunity to create matter instantly will now be studied.

According to the energy/momentum conservation laws, we identified two new mechanisms for augmentation and instant creation of matter. The first case is the case where the branes “a” and “b” are identical. By identical we mean that \( \dot{R}_a = \dot{R}_b \) \( i.e. \) there is no change in the trajectory after the collision point. We will refer to this possibility as the “creation of matter through modification of gravity”. The second case is the one where \( \dot{R}_a \neq \dot{R}_b \) meaning there is a change in the trajectory at the collision point. This case we will denoted by “creation of matter through change of trajectory”.

4.3.1 Augmentation of matter through modification of gravity

Concerning the case of identical branes \( \dot{R}_a = \dot{R}_b \), a way of creating matter is to assume a possible modification of the induced gravity on the brane through its trajectory. In fact, it is possible to vary the gravitational constant \( \mu \) in equation (15) at the “collision point”.

Indeed by looking at equation (31) if \( \mu \) decreases then \( \rho^{\text{gravity}} \) would have to increase. Hence, the matter density would increase to conserve -at the collision point- the null effective energy density. Thus a decrease of the gravitational constant could lead to a creation of matter on the brane. Note that with this mechanism it is not possible to start from a nil-brane \( i.e. \) a vacuum state \( (\rho' = 0) \) and then giving rise to a brane with non zero matter density. Finally the junction condition (see equation (19)) still holds but does not give further information.

This mechanism has similar effects to the classical evolution of matter density but it is different in nature. Indeed, here the augmentation of matter is not constrained by the trajectories of the branes but by the value of the gravitational constant. It is possible to have \( \rho'^{\text{matter}} = \rho^{\text{gravity}} \) for identical trajectories \( \dot{R}_a = \dot{R}_b \). This was impossible in the classical case (see equation (24)). Finally, this mechanism allows an augmentation of matter density but by no way an instant creation of matter from the vacuum.
4.3.2 Possible instant creation of matter through change of trajectory

As demonstrated in the preceding section, a modification of gravity allows an increase of the matter density already existing on the brane. But for a two branes system, this mechanism cannot create branes instantly from a vacuum states i.e. a nil-brane in our verbiage. This subsection will focus on the creation of a single brane from a nil-brane. It is focused on the possibility to have a change in the trajectory at the collision point i.e. $\dot{R}_a \neq \dot{R}_b$. In the case of a null effective energy density the junction condition (see equation (19)) becomes

$$\epsilon_+ \sqrt{f_+ + \dot{R}^2} - \epsilon_- \sqrt{f_- + \dot{R}^2} = 0. \quad (32)$$

According to this equation two sub-cases appear: the so-called symmetric case ($\epsilon_+ \epsilon_+ = 1$) and the orbifold case ($\epsilon_+ \epsilon_+ = -1$ and $f_+ = f_+^4$) We will see that both cases do not necessarily lead to instant creation of matter. An important point in this section is that the incoming brane "$a$" will always be a nil-brane, meaning $\rho_a^\prime = \rho_{a \text{ matter}} = 0$. The conservation equations becomes (see equation (29)):

$$\frac{(n-1)n}{2 \mu^2} \left( \frac{\dot{R}_a^2}{R_a^2} + \frac{k}{R_a^2} \right) \rho_b^\text{matter} - \frac{(n-1)n}{2 \mu^2} \left( \frac{\dot{R}_b^2}{R_b^2} + \frac{k}{R_b^2} \right) = 0. \quad (33)$$

Here, no change of gravity is assumed that is to say, the gravitational constant $\mu$ is assumed to be constant at the collision point.

- **Symmetric case.**

  By symmetric case we mean the setting where $\epsilon_- \epsilon_+ = +1$. By looking at equation (32), the equation describing the “boundary condition” of the two regions surrounding a given brane, one obtains

$$f_+(R_{\text{brane}}) = f_-(R_{\text{brane}}) \forall R_{\text{brane}}, \quad (34)$$

with $R_{\text{brane}}$ the trajectory of the brane. This being true for any point of the trajectory then it gives: $f'_+(R_{\text{brane}}) = f'_-(R_{\text{brane}})$ with $f'$ the derivative of the function $f : R \rightarrow f(R)$. Thus if one performs a Taylor expansion of $f_+/-(R)$, for $R$ sufficiently close to the brane trajectory $R_{\text{brane}}$, it gives:

$$f_+/-(R) = f_+/-(R_{\text{brane}}) + f'_+/-(R_{\text{brane}}) (R - R_{\text{brane}}). \quad (35)$$

Thus, it can be inferred that:

$$f_+(R) = f_-(R) \forall R. \quad (36)$$

It means that the spacetime is identical on both sides of the brane. Moreover the conservation rules (see equation (33)) gives

$$\rho_b^\text{matter} = \frac{n(n-1)}{2 \mu^2 R^2} \left( \dot{R}_b^2 - \dot{R}_a^2 \right) \neq 0. \quad (37)$$

Hence it appears that the creation of matter on the outgoing brane is due to the change of velocity of the brane. Recall that for the incoming brane, it was assumed $\rho_a^\text{matter} = 0 \ i.e. \ there \ was \ no \ matter \ on \ it$. Thanks to this new conservation laws, it is now possible to generate a non zero matter density on the outgoing brane through a change of trajectory at the “collision point”. The outgoing brane is no more a nil-brane (see Figure 4).

\[\text{Note that equation (32) necessarily lead to identical spacetime on both side of the brane. Indeed for nil-branes, it is understandable that “both sides” are identical because a nil-brane is the vacuum.}\]
• Orbifold case.

Note that -strictly speaking- this case should not be in this section devoted to the creation of matter density. Indeed, this setting will not lead to creation of matter when one considers only two branes. The orbifold case stands in this context for \( \epsilon_- \epsilon_+ = -1 \) and \( f_+ = f_- \) (spacetime are identical on both side). Here, equation \( (32) \) gives

\[
f(R) + \dot{R}^2 = 0.
\]

According to equations \( (6) \) and \( (15) \), it gives a “new” definition of \( \rho^{\text{gravity}} \):

\[
\rho^{\text{gravity}} = \frac{(n-1)}{2} \frac{n}{\mu^2} \frac{1}{R^2} \left( \frac{M}{R^{n-1}} \pm \left( \frac{R}{l} \right)^2 \right).
\]

Knowing that \( \rho_a^{\text{gravity}} = 0 \) (nil-brane for incoming brane), it necessarily leads to \( \rho_b^{\text{gravity}} = 0 \) because \( \rho^{\text{gravity}} \) will not change at the collision point. Indeed \( \rho^{\text{gravity}} \) depends only on \( R \). \( R \) being the position of the collision, there are no possibilities for a modification of the “gravitational” density. Note that the variation of \( \mu \) would still lead to \( \rho_b^{\text{matter}} = 0 \). Thus in the case of \( \mathbb{Z}_2 \) symmetric two branes system, it is impossible to create instantly single brane (and thus matter) from the vacuum (i.e. nil-brane). The \( \mathbb{Z}_2 \) symmetry then seems to prevent instant creation of matter from vacuum. More than that, it ensures a strict conservation of matter density and the trajectory will not evolve (see Figure 5).

5 Three branes

This section is devoted to the study of three branes system. Here the ingoing brane will be denoted by the subscript “\( c \)” while the two outgoing branes will be respectively denoted by the subscript “\( a \)” and “\( b \)”. This situation is described in Figure 6. We will follow the same procedure as before by first reviewing the case where there is no violation of matter conservation and no brane creation in both classical and modified case.

![Figure 4: A nil-brane “\( a \)” giving rise to a brane “\( b \)” where \( \rho'_b = 0 \) and \( \rho_{\text{matter}}^b \neq 0 \) i.e. a case of instant creation of matter.](image-url)
5.1 General equations

In a frame in which the "c" brane is stationary, the momentum and energy conservation laws (see equations (21) and (22)) give:

\[ 0 = \rho'_a \gamma_a |c| \beta_a |c| + \rho'_b \gamma_b |c| \beta_b |c|, \]  (40)

and

\[ \rho'_c = \rho'_a \gamma_a |c| + \rho'_b \gamma_b |c|. \]  (41)

These equations thus give a non trivial condition between the effective energy density and the expansion rate of the three branes.

By considering the case where the effective energy density of the incoming brane is null \( \rho'_c = 0 \), one obtains

\[ 0 = \rho'_a \gamma_a |c| \beta_a |c| + \rho'_b \gamma_b |c| \beta_b |c|, \]  (42)

Figure 5: A \( \mathbb{Z}_2 \) symmetric brane will always respect a strict conservation of matter density and its trajectory will never evolve.

Figure 6: One ingoing brane yielding two outgoing brane in the classical case.
and
\[ \rho_a' \gamma_{a|c} + \rho_b' \gamma_{b|c} = 0. \]  
(43)

Given that \( \gamma \neq 0 \), it implies
\[ \rho_b' (\beta_{a|c} - \beta_{b|c}) = 0. \]  
(44)

Finally, either brane \( "a" \) and \( "b" \) have the same trajectory (i.e., \( \beta_{a|c} = \beta_{b|c} \) and hence \( \dot{R}_a = \dot{R}_b \)) and \( \rho_a' = -\rho_b' \) or \( \rho_a' = \rho_b' = 0 \). Having the same trajectory means that they merge. Thus this merged brane have an effective energy density \( \rho_{ab}' = \rho_a' + \rho_b' = 0 \). It corresponds to the system of a 2 branes previously studied.

Thus for a 3 branes system, if the incoming brane has a null effective energy density, then the outgoing branes have both a null effective energy density or merge into a brane of null effective energy density.

- Orbifold case.

We present here the conservation laws in the orbifold case for a three branes system. One incoming \( \mathbb{Z}_2 \) brane \( "c" \) and one normal incoming brane \( "b" \) are considered. The outgoing brane \( "a" \) is also considered \( \mathbb{Z}_2 \) symmetric. The equations are:
\[ 0 = \rho_b' \gamma_{b|c} \beta_{b|c} \]  
(45)

and
\[ \rho_c' = \rho_a' + 2 \rho_b' \gamma_{b|c}. \]  
(46)

This case is displayed in Figure 7.

5.2 Classical behaviours

In this part, it will be shown that modified conservation laws can also mimic the behaviours of three branes systems with classical conservation laws.

\[ ^5 \text{Just as a remind, due to the } \mathbb{Z}_2 \text{ symmetry of branes } "a" \text{ and } "c", \alpha_{a|c} = 0. \]

Figure 7: A \( \mathbb{Z}_2 \) symmetric brane \( "c" \) collides a “normal” brane \( "b" \) to give rise to a \( \mathbb{Z}_2 \) symmetric brane \( "a" \).
5.2.1 Classical case

As explained previously, in the classical case the contribution from the induced gravity is null. Thus equations (40) and (41) becomes:

\[ 0 = \rho'^{\text{matter}}_a \gamma_{a[c]} \beta_{a[c]} + \rho'^{\text{matter}}_b \gamma_{b[c]} \beta_{b[c]}, \] (47)

and

\[ \rho'^{\text{matter}}_c = \rho'^{\text{matter}}_a \gamma_{a[c]} + \rho'^{\text{matter}}_b \gamma_{b[c]}. \] (48)

Of course, as for the two branes systems, there can be augmentation of matter i.e. \( \rho_c < \rho_a + \rho_b \). This augmentation being “compensated” by the dynamics of the branes. If one now demands a strict conservation of matter density i.e. \( \rho_c = \rho_a + \rho_b \), it leads to

\[ \gamma_{a[c]} \beta_{a[c]} (1 - \gamma_{a[c]}) = \gamma_{b[c]} \beta_{b[c]} (1 - \gamma_{b[c]}). \] (49)

This means that the trajectories of “a” and “b” are similar i.e. there is just one outgoing brane. This is thus the two branes case and the conclusion still hold: if there is strict conservation of matter density, there is no change of trajectory at the “collision point”.

Finally one considers that the ingoing branes has \( \rho'_c = 0 \) -thanks to our previous calculation- it implies \( \rho'^{\text{matter}}_a = \rho'^{\text{matter}}_b = 0 \). Thus -as usual in the classical case- instant creation of matter is impossible. The conclusion for three branes systems are thus similar to the one for two branes systems.

5.2.2 Induced gravity case

By considering induced gravity, equations (40) and (41) thus becomes

\[ 0 = \left( \rho'^{\text{matter}}_a - \frac{(n-1)}{2} \frac{n \mu^2}{2 \mathcal{R}^2} \right) \gamma_{a[c]} \beta_{a[c]} + \left( \rho'^{\text{matter}}_b - \frac{(n-1)}{2} \frac{n \mu^2}{2 \mathcal{R}^2} \right) \gamma_{b[c]} \beta_{b[c]}, \] (50)

and

\[ \left( \rho'^{\text{matter}}_c - \frac{(n-1)}{2} \frac{n \mu^2}{2 \mathcal{R}^2} + \frac{N}{2 \mathcal{R}^2} \right) \gamma_{a[c]} \beta_{a[c]} + \left( \rho'^{\text{matter}}_c - \frac{(n-1)}{2} \frac{n \mu^2}{2 \mathcal{R}^2} + \frac{N}{2 \mathcal{R}^2} \right) \gamma_{b[c]} \beta_{b[c]}, \] (51)

We will not perform an extensive study of this case. Indeed, no new effects or mechanism concerning the strict conservation of matter density (\( \rho_c = \rho_a + \rho_b \)) have been observed. This statement holds when \( \rho'_c \neq 0 \) and \( \rho'_c = 0 \).

5.3 New behaviours

We will proceed as for the two branes system cases. We will evoke separately the two mechanisms.
5.3.1 Augmentation of matter through change of gravity

In the section devoted to the two branes system, it was pointed out that matter density could be augmented by a modification of the induced gravitational coupling, \( \mu^2 \), on the brane. The mechanism still holds in the context of three branes system. In this section it was demonstrated that this mechanism could make possible an unconstrained augmentation of the matter density. But it could not lead to the creation of a single brane from a nil-brane.

Here the incoming brane “c” with \( \rho_c' = 0 \) is not a nil-brane i.e. \( \rho_c^{\text{matter}} \neq 0 \). By looking at equations (50) and (51), one can see that this mechanism of matter augmentation thanks to the modification of the gravitational constant still works. One could imagine having \( \rho_c^{\text{matter}} < \rho_a^{\text{matter}} + \rho_b^{\text{matter}} \) (as in the classical case). Indeed, an increase of \( \mu \) could lead to an increase of both \( \rho_a^{\text{matter}} \) and \( \rho_b^{\text{matter}} \). Until now, it was assumed that the incoming brane was not a nil-brane.

If one considers that the incoming brane is a nil-brane, it leads to

\[
\rho'_a = \rho_a^{\text{matter}} - \frac{(n-1)}{2} \frac{n}{\mu^2} \frac{1}{R^2} \left( \dot{R}_a^2 - \dot{R}_c^2 \right) = 0,
\] (52)

and

\[
\rho'_b = \rho_b^{\text{matter}} - \frac{(n-1)}{2} \frac{n}{\mu^2} \frac{1}{R^2} \left( \dot{R}_b^2 - \dot{R}_c^2 \right) = 0.
\] (53)

Thus if \( \mu \) is evolving, an instant creation of matter is possible. This makes now possible the creation a single branes thanks to this mechanism. This is only true under the assumption of change of trajectory. The trajectories of the outgoing branes should be different from the trajectory of the ongoing brane otherwise \( \rho^{\text{matter}} = 0 \) for any branes involved.

Hence, the mechanism of change of gravity allows augmentation of the matter density as in the classical case. It also creates single branes with matter density for three branes systems if there is a change of trajectory.

5.3.2 Instant creation of matter through change of trajectory

As previously for the “change of trajectory” mechanism, the incoming brane “c” will be a nil-brane and “a” and “b” the outgoing branes. In this section it will again be assumed that \( \rho_c' = \rho_c^{\text{matter}} = 0 = \rho_c^{\text{gravity}} \).

- Symmetric case.

As previously explained, assuming \( \rho_c' = 0 \) implies that \( \rho'_a = \rho'_b = 0 \). The solutions are thus identical to the one of equations (52) and (53) with \( \rho_c^{\text{matter}} = 0 \). In the former section, we just mentioned the role of \( \mu \) for the creation of matter. It is clear that if the trajectory of the ingoing brane is different from the trajectory of one of the outgoing branes, the matter density of one of the outgoing brane will be non zero. This situation is depicted on Figure 8. It means that if the outgoing branes do not have the same trajectory, there will necessary be instant creation of matter. This constitutes a stronger statement than before. If nil-branes exists and can give rise to two distinctive branes then there will necessarily be instant creation of matter. Note again that if the trajectories of the two outgoing branes are similar, the situation is described by the two branes system. Same trajectory being here interpreted as a merging of the two branes.

- Orbifold case.

In this setting, the incoming \( \mathbb{Z}_2 \) brane is denoted by “c” and the normal incoming brane is denoted by “b”. The outgoing brane “a” is also considered as \( \mathbb{Z}_2 \) symmetric (see Figure 9). The
Figure 8: A nil-brane “c” giving rise to two zero effective density branes “a” and “b”.

equations for the conservation laws are (45) and (46). They give $\rho'_c = \rho'_a$ i.e. the effective matter density for the two $\mathbb{Z}_2$ branes are identical. It has been shown previously (see equation (6)) that

Figure 9: A $\mathbb{Z}_2$ symmetric nil-brane (c) collides with a brane (b) with null effective energy density but with non zero matter density to give rise to a $\mathbb{Z}_2$ symmetric nil-brane (a).

the “gravitational” density $\rho^{\text{gravity}}$ of a $\mathbb{Z}_2$ brane is entirely determined by the parameters at the collision point. In particular, when collisions of only $\mathbb{Z}_2$ branes are considered, $\rho^{\text{gravity}}$ has to be identical for every $\mathbb{Z}_2$ brane at the collision point. Thus when collisions including $\mathbb{Z}_2$ branes are considered, all the $\mathbb{Z}_2$ branes have necessarily the same matter density. In the case studied here with one incoming and one outgoing $\mathbb{Z}_2$ brane, there will necessarily be conservation of matter between those two branes.\(^6\)

Equations (45) and (46) implies: $\rho'_b = \rho'_b^{\text{matter}} - \rho'_b^{\text{gravity}} = 0$. Nothing constraints the value of the “gravitational” density of the normal brane. In particular, $\rho'_b^{\text{matter}}$ can be anything. This

\(^{6}\)We did not study a system of more that three $\mathbb{Z}_2$ branes because one dimension of spacetime would disappear at the collision.
constitutes an instant decay of matter disappearing into the vacuum\[7\]. Note that if another “normal” brane is added, the conclusion would still hold i.e. no instant creation of matter between the two \(Z_2\) (a strict conservation) but possibly for the two (or more) non-\(Z_2\) branes.

6 Conclusion

We have presented here a detailed study of conservation laws for colliding \(n\)-branes (or \(n\)-dimensional shells) in arbitrary dimension. After reviewing the classical rules, we have derived new conservation rules when considering self-gravitating \(n\)-branes by including induced gravity on the brane. We have exhibited various simple examples with two or three branes for both the classical case and the induced gravity case. These examples showed that the inclusion of induced gravity leads to drastically different behaviours. In particular these examples shed light on a possible breakdown of energy-momentum conservation. This seems to violate the principle of matter conservation.

Indeed we found two mechanisms that lead to undesirable behaviours. The first allows for an unconstrained augmentation of matter density through its trajectory. The second, allows instant creation of single branes from the vacuum. We noticed that the \(Z_2\) symmetry seems to play a special role in this context. Indeed, when considering two branes systems, it automatically ensures the strict conservation of matter density. Nonetheless, the existence of nil-branes in this context leads -in general- to an unacceptable violation of matter conservation. This could in particular lead to unconstrained instant creation of \(n\)-branes form the vacuum. This is a common behaviour in quantum theory but we have made here a classical calculation.

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\[7\]Note that “time” can be reversed and the normal incoming brane can become an outgoing brane. The instant decay would become an instant creation of matter.
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