D-brane effective field theory from string field theory

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Abstract

Open string field theory is considered as a tool for deriving the effective action for the massless or tachyonic fields living on D-branes. Some simple calculations are performed in open bosonic string field theory which validate this approach. The level truncation method is used to calculate successive approximations to the quartic terms $\phi^4$, $(A^\mu A_\mu)^2$ and $[A^\mu, A_\nu]^2$ for the zero momentum tachyon and gauge field on one or many bosonic D-branes. We find that the level truncation method converges for these terms within 2-4% when all massive fields up to level 20 are integrated out, although the convergence is slower than exponential. We discuss the possibility of extending this work to determine the structure of the nonabelian Born-Infeld theory describing the gauge field on a system of many parallel bosonic or supersymmetric D-branes. We also describe a brane configuration in which tachyon condensation arises in both the gauge theory and string field theory pictures. This provides a natural connection between recent work of Sen and Zwiebach on tachyon condensation in string field theory and unstable vacua in super Yang-Mills and Born-Infeld field theory.

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1 Introduction

String field theory provides an off-shell formulation of string theory which has the potential to address nonperturbative questions in a systematic fashion. A particularly elegant formulation of covariant string field theory for an open bosonic string was provided by Witten [1]. In this string field theory, the entire classical action is contained in a pair of terms which are quadratic and cubic in the string field. The simplicity of this theory makes it feasible to perform interesting off-shell calculations.

The bosonic open string has a tachyonic instability around the usual string vacuum. Early work by Kostelecky and Samuel [2] and by Kostelecky and Potting [3] indicated that string field theory could be used to describe the condensation of the tachyon, leading to another more stable vacuum. Since, however, the structure of the vacuum and the role of the tachyonic instability were poorly understood at that time, the significance of this result was not generally appreciated. Recently, renewed interest in tachyonic instabilities related to D-branes led Sen to propose that the condensation of a tachyon in the open bosonic string should be understood as the decay of an unstable D-brane in the bosonic string theory. The vacuum energy of the open bosonic string field theory in the “true” vacuum should thus differ from that in the unstable vacuum by precisely the mass energy of the unstable D-brane, and Sen argued that this mass difference should be calculable using the bosonic open string field theory [5]. This conclusion was given very strong support by a recent paper by Sen and Zwiebach [6], in which the authors demonstrated that when the string field theory is truncated to states of level 4, the energy at the minimum of the potential differs from the energy in the unstable vacuum by 99% of the energy of the bosonic D-brane.

The result of Sen and Zwiebach and the earlier work of Kostelecky, Samuel and Potting suggest that string field theory may be a much more effective tool for asking nonperturbative and off-shell questions about string theory than was previously suspected. While string field theory has been studied for quite some time, there are still many fundamental questions about the structure of these theories. Even in the context of the simplest string field theory, that of the open bosonic string, the convergence properties of the theory are poorly understood. The work in this paper provides new evidence that string field theory provides a systematic and completely convergent framework in which to study many features of string theory.

The main goal of this paper is to initiate a systematic study of how effective field theories on D-branes can be derived from open string field theory. We focus on some of the simplest terms in the effective actions for the tachyon and massless vector field in the open bosonic string, namely the quartic terms $\phi^2$ and $(A^\mu A_\mu)^2$ in the abelian theory and $[A_\mu, A_\nu]^2$ in the nonabelian theory. We perform a systematic truncation of string field theory, including all fields up to a fixed level $n$ for various values of $n \leq 20$. We find that the contributions to each of the quartic terms from fields at level $n$ decrease monotonically and appear to give completely convergent series with slower than exponential convergence.

These results increase significantly our confidence in the viability of string field theory.

1 Another approach to describing the stable vacuum was taken in [4]
as a tool for calculating higher order terms in the action for the massless/tachyonic fields on the D-brane. We discuss briefly how these results may be extended to study the structure of the nonabelian Born-Infeld theory, and we suggest that further insight into the process of tachyon condensation may be gleaned from consideration of simple D-brane models in which the tachyon, the unstable vacuum and the stable vacuum can all be described in the language of supersymmetric Yang-Mills theory.

Except for the discussion of tachyon condensation in the last section, this paper primarily focuses on the open bosonic string field theory. It would obviously be of great interest to develop an analogous systematic approach to computations in supersymmetric open [7] and closed [8] string field theory. While Witten’s formulation of open string field theory can be extended to the supersymmetric theory [9, 10], this formalism may be problematic due to the necessity of considering higher order contact interactions [10]. Several alternative formulations of open superstring field theory have been suggested [11, 12]. Recent work of Berkovits [13], in which he finds a similar result to that of Sen and Zwiebach for tachyon condensation in a supersymmetric theory, indicates that his alternative formulation of the supersymmetric open string field theory [12] may present a viable framework for systematic calculations.

In Section 2 we review the basic structure of open bosonic string field theory and describe the calculation of the quartic terms in the tachyon and vector field potential at zero momentum. Some details of the calculations, which were carried out using the symbolic manipulation program Mathematica, are described in the appendices. In section 3 we discuss the possible extension of this work to studying the nonabelian Born-Infeld theory and tachyon condensation in D-brane systems.

2 Open bosonic string field theory

In this section we perform some simple examples of calculations in the open bosonic string field theory to demonstrate the effectiveness of this approach in finding terms in the effective action for massless or tachyonic fields on a D-brane. We begin in subsection 2.1 with a brief review of bosonic open string field theory and the Fock space representation of the Witten vertex, in order to fix conventions and notation. In subsection 2.2 we calculate the terms in the effective action which are quartic in the tachyon by integrating out all non-tachyonic fields. In subsection 2.3 we calculate the terms quartic in the gauge field by integrating out the tachyon and all massive fields. For all these calculations we describe the first few terms analytically and summarize the results of the summation of higher level terms. Some details of the exact calculations, which were performed using the symbolic manipulation program Mathematica are given in the appendices.
2.1 Review of notation and formalism

As formulated by Witten [1], covariant open string field theory is described in terms of a string field $\Phi$ which contains a component field for every state in the first-quantized string Fock space. The string field theory action is

$$S = \frac{1}{2\alpha'} \int \Phi \star Q \Phi + \frac{g}{3!} \int \Phi \star \Phi \star \Phi,$$  

(1)

where $Q$ is the BRST operator and $\star$ is the string field theory star product.

Throughout this paper we use the conventions of Kostelecky and Samuel from [2]; for a more detailed review of open string field theory see [14, 15]. We summarize here briefly the basic formalism we will need, following [16, 17, 18, 2]. The open bosonic string has a system of 26 matter oscillators $\alpha_n^\mu$ and ghost oscillators $b_n, c_n$, which act on the vacuum $|0\rangle$ through

$$\alpha_n^\mu |0\rangle = 0, \quad n \geq 0$$
$$b_n |0\rangle = 0, \quad n \geq 0$$
$$c_n |0\rangle = 0, \quad n > 0$$

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu} \delta_{n+m}$$
$$\{b_n, c_m\} = \delta_{n+m}$$

The vacuum $|0\rangle$ is related to the SL(2,R) invariant vacuum $|\Omega\rangle$ through

$$|0\rangle = c_1 |\Omega\rangle.$$  

The string field $\Phi$ is restricted to a sum over states with ghost number 1 in the classical action. This restricts us to states in which an equal number of $b$ and $c$ raising operators act on the vacuum $|0\rangle$. We denote the Fock space of physical states by $\mathcal{H}$ and its dual space by $\mathcal{H}^*$. We will work in Feynman-Siegel gauge, where we can impose a further restriction to states $|S\rangle$ satisfying

$$b_0 |S\rangle = 0$$

The level of a state is defined to be the sum of the oscillator numbers $n$ used to produce the state from the vacuum $|0\rangle$; thus, the vacuum is the unique state at level 0, $\alpha_{-1}^\mu |0\rangle$ are the allowed states at level 1 (of ghost number 1), etc. These are the states associated with the tachyon and gauge field on the D-brane. In our calculations, we will only be interested in fields which couple through the cubic string field vertex to two tachyons or two gauge fields. Thus, we need only consider massive fields which have both an even level number and an even number of space-time indices. In the Feynman-Siegel gauge, the leading terms of interest in an explicit expansion of the string field $\Phi$ are

$$\Phi = \left( \phi + A_\mu \alpha_{-1}^\mu + \frac{1}{\sqrt{2}} B_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu + \beta b_{-1} c_{-1} + \cdots \right) |0\rangle$$  

(2)
It is possible to write both the quadratic kinetic term and the cubic interaction term from the string field action \( [1] \) in terms of the Fock space representation of the string field. The quadratic term evaluated on a state \( |S\rangle \) is simply

\[
\int |S\rangle \ast Q |S\rangle = \langle S | c_0 \left( \alpha' p^2 + \frac{1}{2} M^2 \right) |S\rangle
\]

where \( \frac{1}{2} M^2 \) is the string mass operator, given by the level of \( S \) minus 1, \( p \) is the momentum of the state \( S \), \( \langle S | \) is the BPZ dual state to \( |S\rangle \) produced by acting with the conformal transformation \( z \to -1/z \), and the dual vacuum satisfies \( \langle 0 | c_0 | 0 \rangle = 1 \).

From (3) we can write the kinetic terms in the string field action

\[
S_2 = \frac{1}{2 \alpha'} \int \Phi \ast Q \Phi
\]

\[
= \frac{1}{2} \int d^{26} x \left[ \frac{1}{\alpha'} \left( -\phi^2 + B_{\mu\nu} B^{\mu\nu} - \beta^2 + \ldots \right) + \partial_\mu \phi \partial^\mu \phi + \partial_\mu A_\nu \partial^\mu A^\nu + \partial_\mu B_{\nu\lambda} \partial^\mu B^{\nu\lambda} - \partial_\mu \beta \partial^\mu \beta + \ldots \right]
\]

(4)

for the terms of interest up to level 2.

The cubic interaction terms in the string field action can be described in terms of Witten’s vertex operator \( V \) through

\[
\int \Phi \ast \Phi \ast \Phi = \langle V | (\Phi \otimes \Phi \otimes \Phi)
\]

where \( V \) is a state in the tensor product space \( \mathcal{H}_s \otimes \mathcal{H}_s \otimes \mathcal{H}_s \). An explicit representation of this state in the string Fock space was found in [16, 17, 18] and can be written as

\[
\langle V \rangle = \delta(p(1) + p(2) + p(3))(\langle 0| c_0^{(1)} \otimes \langle 0| c_0^{(2)} \otimes \langle 0| c_0^{(3)} \rangle) \exp \left( \frac{1}{2} \alpha^{(r)}_n \eta_{nm} \alpha^{(s)}_n + c^{(r)}_n X^{rs}_{nm} b^{(s)}_n \right).
\]

The indices \( r, s \) take values from 1-3 and indicate which Fock space the oscillators act in. The coefficients \( N^{rs}_{nm}, X^{rs}_{nm} \) are given in terms of the 6-string Neumann functions \( \bar{N}^{rs}_{nm} \), \( 1 \leq r, s \leq 6 \) through

\[
N^{rs}_{nm} = \frac{1}{2} (\bar{N}^{rs}_{nm} + \bar{N}^{r(s+3)}_{nm} + \bar{N}^{(r+3)s}_{nm} + \bar{N}^{(r+3)(s+3)}_{nm})
\]

\[
X^{rs}_{nm} = -m(\bar{N}^{rs}_{nm} - \bar{N}^{r(s+3)}_{nm}), \quad s = r, r + 2
\]

\[
X^{rs}_{nm} = m(\bar{N}^{rs}_{nm} - \bar{N}^{r(s+3)}_{nm}), \quad s = r + 1
\]

(7)

For \( s < r \) the values of \( X^{rs}_{nm} \) are fixed by using (3) and the cyclic symmetry of the coefficients under \( r \to (r \text{ mod } 3) + 1 \), \( s \to (s \text{ mod } 3) + 1 \). When all momenta are zero we only need the Neumann functions with \( n, m > 0 \), which are given by

\[
\bar{N}^{rs}_{nm} = \frac{1}{nm} \int_{\pi} dz \int_{\pi} dw \frac{1}{2\pi i} \frac{1}{(z - w)^2} (-1)^{n(r-1) + m(s-1)} (f(z))^{(-1)^r n} (f(w))^{(-1)^s m}
\]

(8)

with

\[
f(z) = \frac{z(z^2 - 3)}{3z^2 - 1}
\]

(9)
Figure 1: Tree diagram contributing to $\phi^4$ term

and

$$z^1, \ldots, z^6 = \sqrt{3}, 1/\sqrt{3}, 0, -1/\sqrt{3}, -\sqrt{3}, \infty.$$  \hspace{1cm} (10)

A table of the coefficients $N_{nm}^{rs}, X_{nm}^{rs}$ with $n, m < 9$ is given in Appendix A.

From (3) the cubic interaction terms in the string field theory can be written down for an arbitrary set of 3 fields. For the fields appearing in (2), the interactions at zero momentum are

$$S_3 = \kappa g \left( \phi^3 - \frac{5}{3^2 \sqrt{2}} B^\mu \phi^2 - \frac{11}{3^2} \beta \phi^2 \right)$$  \hspace{1cm} (11)

$$+ \frac{2^4}{3^2} \phi A_\mu A^\mu - \frac{5 \cdot 2^3 \sqrt{2}}{3^5} B^\mu A_\nu A^\nu + \frac{2^8 \sqrt{2}}{3^5} B^{\mu \nu} A_\mu A_\nu - \frac{11 \cdot 2^4}{3^5} \beta A_\mu A^\mu + \cdots$$

where

$$\kappa = \frac{3^{7/2}}{2^7}$$

### 2.2 Terms quartic in tachyon field

Kostelecky and Samuel proposed in [2] that the answers to many physical questions in string field theory might be very well approximated by truncating the string field action at a finite level. As a first check of this proposal they considered the effective quartic term in the tachyon field $\phi^4$ which arises from integrating out all the massive scalar fields in the theory. For each massive scalar field $\psi$ whose quadratic term and coupling to $\phi^2$ are given by

$$S_\psi = \frac{a}{2} \psi^2 + c \psi \phi^2$$  \hspace{1cm} (12)

there is a term in the effective potential for $\phi$ of the form

$$S_{\phi^4} = -\frac{c^2}{2a} \phi^4.$$  

The (tree-level) diagram for such a term is shown in Figure 1. Thus, for example, the contributions from the level 2 fields $\beta, B^\mu_\nu$ to the $\phi^4$ term in the effective tachyon potential are seen from (4,11) to be

$$\kappa^2 g^2 \alpha' \left( \frac{1}{2} \frac{11}{3^2} \phi^4 - 26 \cdot \frac{1}{2} \cdot \frac{5}{3^2 \sqrt{2}} \phi^4 \right) \phi^4 = -\kappa^2 g^2 \alpha' \left( \frac{34}{27} \right) \phi^4.$$  \hspace{1cm} (13)
The exact quartic term in the tachyon potential was determined in [19], and is given by

\[ \kappa^2 g^2 \alpha' \gamma \phi^4 \]

where \(\gamma \approx -1.75 \pm 0.02\) (14)

The quartic term produced by integrating out the level 2 fields (13) represents 72% of the total (14). In [2], Kostelecky and Samuel calculated the contribution from level 4 fields, and showed that these produce an additional 12% of the total quartic term. As a first test of how well the level truncation method does in reproducing quartic terms in the effective action on a D-brane, and as a test of our algorithm for calculating contributions from fields at arbitrary level, we have used Mathematica to carry out this truncation exactly up to order 20. The results of this calculation are summarized in Table 1. A more detailed description of the contributions from each field up to level 6 is given in Appendix B. We denote by \(\hat{\gamma}^{(n)}\) the contribution to \(\gamma\) from fields at level \(n\), and by \(\gamma^{(n)}\) the cumulative \(\gamma\) found by adding the contribution from all fields at levels \(m \leq n\). After summing the contributions from all 13,112 fields at levels \(\leq 20\), we find that the exact result from [19] is reproduced to within approximately 4%. A graph of the coefficient of the quartic term coming from the successive level truncation approximations is shown in Figure 2. There are several significant aspects of how the successive approximations to this term in the effective action behave as the level number increases. First, the approximations seem to be completely convergent, with no sign of bad behavior at high level as one might expect if the series were only asymptotic but not convergent. Second, the successive approximations seem to have slower than exponential falloff, but do not seem to follow a simple power law. It would be very interesting to have

| \(n\) | \# of fields | \(\hat{\gamma}^{(n)}\) | \(\gamma^{(n)}\) | \(\gamma^{(n)}/\gamma_{\text{exact}}\) |
|---|---|---|---|---|
| 2 | 2 | \(-\frac{2.17}{3^3}\) \(\approx -1.25926\) | -1.259 | 0.72 ± 0.01 |
| 4 | 7 | \(-\frac{1399}{3^5}\) \(\approx -0.21323\) | -1.472 | 0.84 ± 0.01 |
| 6 | 20 | \(-\frac{2^2 \cdot 7.643463}{3^{16} \cdot 5}\) \(\approx -0.08371\) | -1.556 | 0.89 ± 0.01 |
| 8 | 55 | \(-\frac{167 \cdot 1846847}{2 \cdot 3^{30}}\) \(\approx -0.04423\) | -1.600 | 0.92 ± 0.01 |
| 10 | 139 | \(-\frac{5.193 \cdot 241 - 1341187}{2^{33} \cdot 3^{30}}\) \(\approx -0.02727\) | -1.628 | 0.93 ± 0.01 |
| 12 | 331 | \(-\frac{5 \cdot 2012243 - 232794533}{3^{44} \cdot 3^{11}}\) \(\approx -0.01848\) | -1.646 | 0.94 ± 0.01 |
| 14 | 747 | \(-\frac{2^2 \cdot 11 \cdot 1775214529932629}{3^{77} \cdot 3^{13}}\) \(\approx -0.01334\) | -1.660 | 0.95 ± 0.01 |
| 16 | 1618 | \(-\frac{298001292970739836603}{2 \cdot 3^{33} \cdot 5}\) \(\approx -0.01009\) | -1.670 | 0.96 ± 0.01 |
| 18 | 3375 | \(-\frac{2^2 \cdot 7 \cdot 21159416989 - 1463230224529}{3^{32} \cdot 17}\) \(\approx -0.00789\) | -1.677 | 0.96 ± 0.01 |
| 20 | 6818 | \(-\frac{3 \cdot 1061 \cdot 1319 - 164689 - 26468217099651}{3^{29} \cdot 19}\) \(\approx -0.00634\) | -1.684 | 0.96 ± 0.01 |

Table 1: Contributions at each level to coefficient of \(\phi^4\)
a better understanding of the convergence properties of these terms from a study of the asymptotic properties of the states and their couplings.

2.3 Terms quartic in gauge field

Now let us turn to the massless gauge field $A^\mu$ associated with the spin 1 states $\alpha_{-1}^\mu |0\rangle$. Again, we are interested in finding quartic terms in the gauge field. Let us first consider the abelian U(1) theory associated with a single bosonic D-brane. In this theory the only Lorentz invariant term which is quartic in $A^\mu$ is $(A^\mu A_\mu)^2$. Such a term is indeed induced by integrating out the tachyon at tree level. From the vertices in (11) of the form $\phi A^\mu A_\mu$ we see that there is a term in the effective action of the U(1) theory

$$\kappa^2 g^2 \alpha' \left( \frac{1}{2} \frac{24}{32} \right) (A^\mu A_\mu)^2 = \kappa^2 g^2 \alpha' \left( \frac{128}{81} \right) (A^\mu A_\mu)^2$$

(15)

The appearance of this term in the effective action for the world-volume gauge field may seem rather surprising, as this term does not appear as the 0-momentum part of any expression which is invariant under the U(1) gauge symmetry we expect to have in the effective gauge theory found after integrating out the tachyonic and massive string modes. Indeed, the gauge noninvariance of this term suggests that contributions to the quartic term from higher string modes should cancel this term, so that in the true effective U(1) gauge theory there is no $A^4$ term. This argument is not completely conclusive, however, since by choosing Feynman-Siegel gauge we have broken the gauge invariance of the effective U(1) theory. For example, the kinetic term for the vector field in Feynman-Siegel gauge is simply $\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu$, which is the kinetic term for the massless vector field in Lorentz gauge $\partial_\mu A^\mu = 0$. 

Figure 2: Approximations to coefficient of $\phi^4$ term
A related argument for the conclusion that the $A^4$ term should vanish in the abelian theory is that if we compactify the theory in some set of dimensions and then perform a series of T-duality symmetries which change the Neumann boundary conditions on the open string to Dirichlet boundary conditions in those directions, we have an effective theory of a bosonic D-brane of dimension $p < 25$ whose position in the transverse spatial directions is described by a set of coordinates $X^\mu$ which are T-dual to the constant gauge field components $A^\mu$ giving rise to non-trivial holonomies around the compact directions in original picture. The quartic term in the T-dual picture analogous to (15) is of the form $(X^\mu X_\mu)^2$; this term breaks translation invariance, perhaps even a worse transgression than the breaking of gauge invariance by (15).

It may seem strange that we have integrated out the tachyon field and kept the massless vector field. The usual approach to deriving an effective action in a theory with fields at many energy scales is to integrate out the fields with very large masses keeping the light or massless fields. This gives a low-energy effective theory for the light fields in the theory which has a clear physical significance. It is less clear how to interpret the action for the vector field on a bosonic D-brane which arises when we integrate out the tachyon, which has negative mass squared. Because the energy scale of the tachyon is the same as that of the massive string states, however, it seems most reasonable to integrate out the tachyon field along with the massive states when constructing an effective action for the vector field. As we will discuss in the next section, we expect that this effective action should take the form of a bosonic Born-Infeld action.

In any case, the effective action which appears when integrating out the tachyon as well as the massive fields in the theory seems to be quite well behaved, at least at quartic order. A consideration of the effects of the fields of level 2 and higher bears out the conjecture that the term (15) is completely cancelled by the infinite tower of massive string states. At level 2, there are contributions to the quartic gauge field term from the fields $\beta, B_{\mu\nu}$. The diagrams associated with these contributions are depicted in Figure 3. The total contribution from these diagrams to the quartic term in the gauge field is

$$- \kappa^2 g^2 \alpha' \left( \frac{71168}{59049} \right) (A^\mu A_\mu)^2. \quad (16)$$

This term cancels 76% of the spurious term generated by the tachyon. It is instructive to consider how this term is computed, so we include some of the details in this particular case. Let us consider in particular those terms with two distinct pairs of indices on the gauge field, such as $A_i A_i A_j A_j$ with $i \neq j$. For terms of this type, the field $\beta$ and the diagonal components $B_{\mu\mu}$ with $\mu \neq i, j$ contribute in the same fashion as we have found previously through the couplings of the forms $\beta(A^\mu A_\mu)$ and $B_{\nu\nu} (A^\mu A_\mu)$ (diagrams (a) and (b) in Figure 3), giving terms in the effective action

$$\kappa^2 g^2 \alpha' \left( \frac{1}{2} \left( \frac{11 \cdot 2^4}{3^5} \right)^2 - 24 \cdot \frac{1}{2} \left( \frac{5 \cdot 2^7}{3^5} \right)^2 \right) (2A_i^2 A_j^2) = -\kappa^2 g^2 \alpha' \left( \frac{22912}{59049} \right) (2A_i A_i A_j A_j). \quad (17)$$
A^{\mu} \quad \beta \quad A^{\nu} \quad (a) \quad A^{\mu} \quad B^{\lambda}_{\lambda} \quad A^{\nu} \quad (b) \quad A^{\mu} \quad B_{\mu \nu} \quad A^{\mu} \quad (c) \quad A^{\mu} \quad B_{\mu \nu} \quad A^{\nu} \quad (d)

Figure 3: Diagrams contributing to terms quartic in vector field at level 2

The fields $B_{ii}$ and $B_{jj}$ together contribute

$$\kappa^2 g^2 \alpha' \left( -2 \cdot \left( \frac{32 - 5}{3^5} \right) \cdot \left( \frac{27^2}{3^5} \right) \right) \left( A_i^2 A_j^2 \right) = \kappa^2 g^2 \alpha' \left( \frac{17280}{59049} \right) \left( 2A_i A_j A_i A_j \right). \quad (18)$$

In addition to these terms, there are terms arising from the field $B_{ij}$, associated with diagrams (c) and (d) in Figure 3. These additional terms come to

$$\kappa^2 g^2 \alpha' \left( \frac{1}{2} \left( \frac{2 \cdot 2^8 \sqrt{2}}{3^5} \right)^2 \right) \left( A_i A_j \right)^2 = -\kappa^2 g^2 \alpha' \left( \frac{65536}{59049} \right) \left( 2A_i A_j A_i A_j \right). \quad (19)$$

Summing these, we arrive at the total coefficient described by (16). We have performed a similar analysis for the terms of the form $A_4^4$, and find the same result for the overall coefficient, as is necessitated by Lorentz invariance.

We have thus explicitly shown that by including the fields at level 2, we cancel 76% of the spurious $A^4$ term in the U(1) theory induced by the tachyon. We have continued this calculation further using Mathematica, determining the exact contribution to the quartic term arising from all fields up to level 16. Indeed, we find that the contributions at each level serve to decrease the total coefficient of the quartic term, so that the term arising from the tachyon has been 97% cancelled at level 16. Before presenting the results of this calculation, we briefly discuss the modifications which arise in the nonabelian theory.

Let us consider the nonabelian theory which arises when we add Chan-Paton factors to the strings in the original string field theory. If the Chan-Paton factors range from 1 through $N$ then all the fields in the string field expansion become $N \times N$ matrices. The gauge field, in particular, becomes a U(N) gauge field. We expect that the effective action for this gauge field will have a leading term of the form $F^2$, which in the 0-momentum theory reduces to a term of the form

$$\left[ A_\mu, A_\nu \right]^2 \quad (20)$$
Thus, we should hope that by keeping track of the nonabelian structure of the fields in the U(N) theory, we should find that instead of vanishing identically, the quartic term in the effective gauge field action will take the form (20). If string field theory is sufficiently well behaved, we might hope to find that summing the contributions to this term from successive level truncation will give us a series of terms which converge to a finite value for the coefficient of the term (20).

In the nonabelian theory with Chan-Paton indices, the quadratic and cubic terms (4,11) in the string field action are modified by simply taking the trace of each term. It is important to note, however, that this trace must be taken before the order of the fields in the term is modified. Because the Neumann coefficients $N_{rs}^{nm}$ are not necessarily symmetric under a reversal of order in the 3 strings at a string vertex, it is necessary to separately compute the coupling for each ordering of the fields in the vertex. For the terms describing the couplings of the level 2 fields to a pair of gauge fields, however, this subtlety is not important, and the relevant coupling terms are simply

$$-\frac{5 \cdot 2^3 \sqrt{2}}{3^5} \text{Tr} \left( B_{\mu}^\mu A_\nu A_\nu^\nu \right) + \frac{2^7 \sqrt{2}}{3^5} \text{Tr} \left( B^{\mu\nu} A_\mu A_\nu + B^{\mu\nu} A_\mu A_\nu \right) - \frac{11 \cdot 2^4}{3^5} \text{Tr} \left( \beta A_\mu A_\mu \right)$$  \hspace{1cm} (21)

We can now go through the calculation of the terms quartic in $A_\mu$ for the nonabelian theory just as we did above in the abelian case. The term (13) produced by integrating out the tachyon is unchanged except for the addition of an overall trace. Now, however, we find that the term (11) decomposes into a sum of two terms with distinct ordering of the $A$'s

$$\kappa^2 g^2 \alpha' \left[ \gamma_1 \text{Tr} \left( A_\mu A_\mu A_\nu A_\nu \right) + \gamma_2 \text{Tr} \left( A_\mu A_\nu A_\mu A_\nu \right) \right]$$  \hspace{1cm} (22)

where

$$\gamma_1 = -\frac{38400}{59049}, \quad \gamma_2 = -\frac{32768}{59049}$$

In this decomposition, the diagrams (a, b, c) of Figure 3 contribute to $\gamma_1$ while diagram (d) contributes to $\gamma_2$. In general, this will be the situation at every level, with nonzero contributions to both coefficients $\gamma_1$ and $\gamma_2$. We have computed these coefficients from terms at all levels up to 16. These results are summarized in Table 2. A more detailed description of the contribution from each field up to level 6 is given in Appendix B. An important feature of these results is the behavior of the sum and difference of the coefficients $\gamma_1$, $\gamma_2$,

$$\gamma_\pm = \frac{\gamma_1 \pm \gamma_2}{2}.$$  \hspace{1cm} (23)

The only way to form a gauge/translation invariant combination of the two quartic terms in (22) is to have a term of the form (20), which appears when $\gamma_+ = 0$. When this is the case, (22) can be rewritten in the form

$$- \kappa^2 g^2 \alpha' \left( \frac{\gamma_-}{2} \right) \text{Tr} \left( [A_\mu, A_\nu]^2 \right).$$  \hspace{1cm} (24)
\[
\begin{array}{|c|cccc|cccc|}
\hline
n & \hat{\gamma}_1^{(n)} & \hat{\gamma}_2^{(n)} & \gamma_1^{(n)} & \gamma_2^{(n)} & \gamma_+^{(n)} & \gamma_-^{(n)} \\
\hline
\text{table 0} & \frac{2}{3\pi} \approx 1.58020 & 0 \approx 0.00000 & 1.580 & 0.000 & 0.7901 & -0.790 \\
2 & -\frac{2^{9.52}}{3^9} \approx -0.65031 & -\frac{2^{15}}{3^{10}} \approx -0.55493 & 0.930 & -0.555 & 0.1875 & -0.742 \\
4 & -\frac{2^{9.6290}}{3^{15}} \approx -0.11113 & -\frac{2^{17}}{3^{17}} \approx -0.08221 & 0.819 & -0.637 & 0.0908 & -0.728 \\
6 & \approx -0.03878 & \approx -0.02528 & 0.780 & -0.662 & 0.0588 & -0.721 \\
8 & \approx -0.01933 & \approx -0.01150 & 0.761 & -0.674 & 0.0434 & -0.717 \\
10 & \approx -0.01157 & \approx -0.00645 & 0.749 & -0.680 & 0.0344 & -0.715 \\
12 & \approx -0.00771 & \approx -0.00409 & 0.741 & -0.684 & 0.0285 & -0.713 \\
14 & \approx -0.00551 & \approx -0.00282 & 0.736 & -0.687 & 0.0243 & -0.712 \\
16 & \approx -0.00414 & \approx -0.00206 & 0.732 & -0.689 & 0.0212 & -0.711 \\
\hline
\end{array}
\]

Table 2: Contributions at each level to coefficients of terms of order $A^4$

Figure 4: Approximations to coefficients of $A^4$ terms
The successive approximations to $\gamma_+$ at levels $n \leq 16$ are graphed in Figure 4. From Table 2 and Figure 4, we see that indeed as the contribution from all levels is considered,

$$
\begin{align*}
\gamma_+ &\to 0 \\
\gamma_- &\to \approx 0.70
\end{align*}
$$

(25)

At level 16, the $\gamma_+$ term produced by the tachyon has been cancelled to within 3%, and the coefficient of $\gamma_-$ has converged to within approximately 2% of its apparent asymptotic value. Thus, we see that string field theory satisfies the important consistency check that the effective U(N) vector theory has a gauge-invariant quartic term for $A^\mu$ in the effective theory arising when all massive fields are integrated out, even though gauge invariance is apparently broken when only a subset of the fields are integrated out. The results of this calculation seem to have nice convergence properties, with slower than exponential convergence just as the terms discussed previously for the tachyon. It would be nice to have a better theoretical understanding of the relative rates of convergence of these terms in the effective action.

It is interesting to compare our results for the coefficient of the $[A_\mu, A_\nu]^2$ term with the kinetic and cubic terms for the vector field in the string field theory action

$$
\frac{1}{2}(\partial_\mu A_\nu)^2 - ig\kappa \sqrt{\alpha'} \left( \frac{2^3 \sqrt{2}}{3^{3/2}} \right) \partial_\mu A_\nu [A^\mu, A^\nu].
$$

(26)

From these terms, we might expect to find a quartic term of the form (24) with coefficient

$$
\gamma_-^2 = \frac{2^9}{3^7} \approx 0.234
$$

(27)

This result disagrees with (23) by a significant margin, perhaps by a factor of 3 or $2\sqrt{2}$. This discrepancy is somewhat puzzling. One obvious possibility is that there is an error somewhere in the calculation. There are numerous factors of 2 and $\sqrt{2}$ appearing in the calculation of (26), but most of these would affect (27) by an overall factor of 2, so it is difficult to see how the needed factor could arise. This discrepancy does not depend on the details of the calculation at high level; indeed, the result (27) seems to already be ruled out from the explicit calculations of the quartic term at level 0 and level 2 given in detail above, assuming that the results for $\gamma_2$ from successive level truncation are monotonically decreasing as they seem to be. Another possible explanation of the discrepancy is that the gauge of the U(1) theory has been fixed in such a way as to introduce spurious terms either at order $\partial A[A, A]$ or at order $[A, A]^2$. We leave a resolution of this puzzle as an outstanding problem, but in any case it seems likely that this discrepancy can be resolved without requiring dramatic changes in the picture we have presented here.

### 3 Further directions

We conclude with a brief discussion of several further issues. In subsection 3.1 we discuss the calculation of higher-order terms in the gauge field effective action, which for the U(N)
theory may give a form of the nonabelian Born-Infeld theory. In subsection 3.2 we discuss the extension of this approach to the supersymmetric theory and a class of configurations in which a direct connection can be made between descriptions of tachyon condensation in string theory and in D-brane world-volume gauge theories.

### 3.1 Nonabelian Born-Infeld

In supersymmetric string theories, infinite flat D-branes are stable BPS objects whose fluctuations are described in flat space by a Born-Infeld action of the form \[ S \sim -T_p \int d^{p+1} \xi \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})} \] (28)

where \( p \) is the dimension of the D-brane, \( T_p \) is the brane tension and \( F_{\mu\nu} \) is the U(1) field on the world volume of the D-brane. Expanding this action in \( F \) gives

\[
S \sim T_p \int \left( -1 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} \left( F_{\mu\nu} F^{\rho\lambda} F_{\rho\lambda\sigma} F^{\sigma\mu} - \frac{1}{4} (F_{\mu\nu} F^{\mu\nu})^2 \right) + \cdots \right) \quad (29)
\]

A similar action describes the dynamics of a D-brane in the bosonic theory. When there are multiple parallel D-branes of the same type, just as the leading U(1) Yang-Mills theory is extended to a nonabelian U(N) Yang-Mills theory, the Born-Infeld action should be extended to a Nonabelian Born-Infeld (NBI) action. Determining the exact structure of this nonabelian action is a long-standing problem in string theory. Because the field strength components \( F_{\mu\nu} \) do not commute, the ordering of the terms in the nonabelian generalization of the action (28) is not well-defined without further information. In the abelian theory, (28) is corrected by terms of higher order in \( \alpha' \) containing derivatives of \( F \). In the nonabelian theory, it is not possible to separate these correction terms from the ordering ambiguity just mentioned since \( [D_\mu D_\nu] F_{\lambda\sigma} \sim F_{\mu\nu} F_{\lambda\sigma} \). It was proposed by Tseytlin in [22] that the nonabelian action at leading order in \( \alpha' \) should be given by first expanding in \( F \) as in the abelian theory, and then taking a symmetrized trace in which all possible orderings of the components \( F_{\mu\nu} \) are given an equal weighting. This prescription amounts to treating all commutators of \( F \)'s as corrections of higher order in \( \alpha' \). In the supersymmetric case, this proposal agrees with calculations which have been performed in string theory [23, 20] and M(atrix) theory [24, 25] for the \( F^4 \) terms in (29), and for the terms linearly coupling the NBI action to supergravity background fields [26, 27, 28]. For a review of recent work on the nonabelian Born-Infeld action, see [29].

At this point there is no systematic understanding of the \( \alpha' \) corrections to the Born-Infeld action, either in the abelian or nonabelian theories (for some discussion of these terms, see [30, 31, 32]). These terms are not uniquely defined due to the possibility of changing the apparent form of the effective action through field redefinitions. A particularly interesting set of corrections are those which appear for constant but noncommuting field strengths \( F \). Such terms are necessary in order to have agreement between the Born-Infeld and string theory descriptions of certain brane configurations [33, 34].
In the supersymmetric nonabelian Born-Infeld theory, there are no corrections to (29) up to order $F^5$. In the bosonic theory, on the other hand, there are corrections containing commutators of $F$’s at orders $F^3$ [25], $F^4$ [20] and $F^5$ [36]. An interesting application of the level truncation method in string field theory would be to extend the analysis of this paper past the quartic terms in the gauge field and to derive the higher order terms in the effective action of the nonabelian gauge field on a bosonic or supersymmetric D-brane. If the action is indeed to take the form of the nonabelian Born-Infeld action, we expect that the terms at order $A^6$ will all vanish in the supersymmetric theory, while in the bosonic theory we should find a term of the form

$$F^\nu_\lambda \left[F^\lambda_\mu, F^\mu_\nu\right].$$

While only four topological types of diagrams contribute to these terms, this is a somewhat more challenging calculation than the determination of the quartic terms performed in this paper, since the cubic vertices between an arbitrary set of 3 fields must be used in the calculation if we include all diagrams with fields below a fixed level. It would be very interesting to see if the perturbative calculation of [25] can be correctly reproduced from open bosonic string field theory. The next interesting question is whether the terms of order $A^8$ have the characteristic structure $\text{Tr} \left( [A, A]^4 - ([A, A]^2)^2 / 4 \right)$ of the Born-Infeld action (29). According to [20], in the bosonic theory we should find in addition to these terms a correction of the form

$$F^\nu_\lambda F^\lambda_\sigma \left[F^\mu_\nu, F^\sigma_\mu\right].$$

At order $A^{10}$ we again expect a nonzero result containing commutator terms [36]. Assuming that all these terms can be calculated and agree with the predictions of perturbative calculations, the next interesting question comes in at order $A^{12}$, corresponding to order $F^6$ in the nonabelian Born-Infeld action. If it were possible to use level truncation to convincingly demonstrate the structure of these terms, it would represent a new piece of information about the nonabelian Born-Infeld action. While these calculations would be fairly complicated, they should in principle be quite tractable with the aid of automated symbolic manipulation tools, including all fields at least up to level 8 or 10. Whether a calculation at this level will be sufficient to fix the structure of the higher order terms in the action is an interesting open problem for future research. If these terms can indeed be calculated in the bosonic theory it may help us to understand the structure of higher-derivative corrections in this theory. If this approach can be extended to the supersymmetric theory, it might give valuable insight into the structure of the full supersymmetric nonabelian Born-Infeld theory.

One subtlety which may complicate the interpretation of the higher order terms in the zero-momentum vector field action is the issue of gauge invariance. In particular, since the kinetic terms in the vector field action are not gauge invariant, there is no reason why the zero momentum part of the action must preserve gauge invariance. Since, however, as discussed above the gauge invariance of the zero-momentum sector is T-dual to translation invariance, we are fairly confident that all the terms in the action which do not have momentum dependence should have the appropriate dependence on $[A_\mu, A_\nu]$ for a Born-Infeld action. Only
further calculations or a better understanding of the role of the Feynman-Siegel gauge in the effective vector field theory will resolve this question completely.

It is also worth pointing out in this context that it may be simpler to approach the Born-Infeld action by including momentum dependence, so that the $A^{12}$ term of interest, representing a 12-string interaction, can instead be determined by calculating a 6-string interaction with 6 powers of momentum of the form $(\partial A)^6$. While this may indeed be an easier approach, it is still necessary to include a fairly general class of cubic vertices at this order, and the simplification in diagrams may not make up for the added complications of including momentum. It is also more likely that the momentum-dependent terms of the $F^6$ part of the action will be difficult to interpret due to the issues of gauge-dependence mentioned above. It will be quite interesting, however, to see whether either of these approaches can give new information about the nonabelian Born-Infeld action.

### 3.2 Tachyon condensation

A fundamental question about the open bosonic string field theory, which has been the subject of much study since the early days of the subject, is that of understanding the mechanism of tachyon condensation. It is important to understand whether there is a stable vacuum in the theory which can be found by giving the fields expectation values, and if such a vacuum exists it is of interest to understand the dynamics of fluctuations around that true vacuum for the theory. There are many other scenarios in which similar questions arise, including in particular the tachyonic closed bosonic string theory and brane-antibrane systems in supersymmetric string theories. In the case of brane-antibrane systems in type II string theory, we have a fairly good understanding of the nature of the ground state: a D-brane and anti D-brane of the same dimension should annihilate into the ground state of the theory. An interesting aspect of some brane-antibrane systems is that they can be described in field theory as well as in string theory. In these cases, the mechanism of tachyon condensation can be seen explicitly in the field theory. It would be very interesting to make a more explicit connection between the string field theory and field theory descriptions of these configurations; we devote the last part of this paper to sketching part of this connection.

As mentioned in the previous subsection, the world-volume dynamics of a D-brane in type II superstring theory is described by a supersymmetric Born-Infeld type action. In the low-energy limit, this reduces to super Yang-Mills theory in flat space. A parallel brane and antibrane of the same dimension which come close enough together develop a tachyonic instability [37]. It is believed [3] that the brane and antibrane should annihilate in a process which may be understood as the tachyon and other string fields developing nonzero expectation values, leading to a vacuum which is nonperturbative in the original open string field theory. If the brane and antibrane are both free of additional structure, it is believed that the resulting vacuum is the true vacuum of the type II string theory.

By placing fluxes on one or both of the D-branes, additional charges are added to the configuration representing lower-dimensional D-branes [38] which may persist after the initial
brane and antibrane have annihilated. This picture forms the basis for Witten’s suggestion \cite{39, 40} that K-theory is the natural language for describing the topological quantum numbers associated with configurations of multiple D-branes. A parallel brane and antibrane can be interpreted as two branes with opposite orientation, and without the addition of further charges or fluxes cannot be given a simple description in terms of super Yang-Mills theory. After the addition of further brane charges encoded in fluxes on the brane and antibrane, however, it is possible in certain circumstances to describe the brane-antibrane annihilation process completely in the language of super Yang-Mills theory.

A configuration of this type was discussed in \cite{33}. For concreteness we will frame the following discussion in terms of the example discussed there. Consider compactifying type IIA string theory on a 2-torus with sides of length $L_1, L_2$. Let us wrap two D2-branes around the torus with the same orientation. We place +1 unit of magnetic flux on one D-brane and −1 unit of magnetic flux on the other. This corresponds to placing a D0-brane and an anti D0-brane on the two D2-branes. Under a pair of T-dualities, this configuration goes to a D2-brane and an anti D2-brane each with a unit of D0-brane charge. This is an example of the type of system mentioned above with a tachyonic instability leading to a vacuum with residual D-branes. Under a single T-duality, the two D2 ± D0 branes become a pair of intersecting diagonally wrapped D1-branes.

The spectrum of string excitations around this background was found in the intersecting D1-brane language by Berkooz, Douglas and Leigh \cite{41}. When the branes are coincident in the directions transverse to the torus, the spectrum contains a tachyon and an infinite tower of massive states with level spacing proportional to the angle $\theta$ between the D1-branes. The angle $\theta$ is related to the magnetic flux density $F$ on the branes in the D2-brane picture through

$$\tan(\theta/2) = 2\pi\alpha' F.$$

A particularly interesting aspect of this configuration is that the tachyonic instability as well as the mechanism of tachyon condensation can be completely understood within the framework of gauge theory. In our original system with two D2-branes of the same orientation, we can define a U(2) gauge field with boundary conditions

\begin{align*}
A_1(x_1 + L_1, x_2) &= e^{2\pi i(x_2/L_2)\tau_3}A_1(x_1, x_2)e^{-2\pi i(x_2/L_2)\tau_3} \\
A_1(x_1, x_2 + L_2) &= A_1(x_1, x_2) \\
A_2(x_1 + L_1, x_2) &= e^{2\pi i(x_2/L_2)\tau_3}A_2(x_1, x_2)e^{-2\pi i(x_2/L_2)\tau_3} + \left(\frac{2\pi}{L_2}\right)\tau_3 \\
A_2(x_1, x_2 + L_2) &= A_2(x_1, x_2)
\end{align*}

Although these boundary conditions look nontrivial, they are equivalent to trivial boundary conditions through a gauge transformation. We now consider the gauge field background

\begin{align*}
A_1^0 &= 0 \\
A_2^0 &= \frac{2\pi}{L_1L_2}x_1\tau_3.
\end{align*}
This corresponds to placing a unit of flux on the first D2-brane and a unit of anti flux on the second D2-brane. The spectrum of excitations around this background in the Yang-Mills theory was originally studied by Van Baal \[12\]. (Van Baal worked on \(T^4\), but we can reduce to the \(T^2\) case by setting the curvature \(F_{34}\) to vanish.) This spectrum indeed contains a tachyon, as well as an infinite series of massive modes with level spacing proportional to \(F\). In the small \(F\) limit the Yang-Mills and string theory spectra coincide; in \[33\] it is shown that under certain circumstances the symmetrized trace formulation of nonabelian Born-Infeld theory produces combinatorial relations which transform \(F\) into \(\theta \sim \tan^{-1} F\).

The gauge field fluctuations around the background \((30)\) can be written in terms of theta functions on the torus. The tachyonic modes are given explicitly in \[33\], and the theta functions for all the other modes are given by Troost in \[43\]. In this configuration, the entire process by which the tachyonic mode condenses and the entire system of modes acquire expectation values, leading to a nonperturbative vacuum, can be seen in the language of Yang-Mills theory. Both the initial and final configurations have diagonal gauge fields, so that the energies can be computed using abelian Born-Infeld. The gap between initial and final energies in the Born-Infeld theory is precisely the difference in energies between a pair of bound D2 \(\pm\) D0 brane systems and a pair of D2-branes.

We propose that this system may be an excellent model with which learn more about the mechanism of tachyon condensation in string field theory. The infinite tower of theta function modes in the gauge theory picture precisely corresponds to the infinite tower of string modes stretching between the branes in the D1-brane picture. It should in principle be possible to go from the known spectrum of excitations on the string side to an open string field theory. The effective action of the set of string fields which correspond to the Yang-Mills fluctuations should then at quartic order precisely reproduce the Yang-Mills action in terms of the diagonal and theta function fluctuations on the D2-brane world-volume. At higher order, as discussed in the preceding section, we should see the structure of the nonabelian Born-Infeld theory which allows the initial unstable but abelian vacuum to decay into the final stable vacuum. Some progress in understanding tachyon condensation in this system from the string theory side was made by Gava, Narain and Sarmadi \[44\], who showed that when the tachyon condenses to the minimum of the potential arising when quartic terms computed from string theory are included, the energy gap is on the same order as the energy differential between the unstable and stable brane configurations. So far, however, this configuration is not completely understood in the language of string field theory. Indeed, as mentioned above there are technical challenges remaining to finding a consistent calculable formulation of open or closed superstring field theory, but it may be that the parallel presented here with tachyonic configurations in gauge theory will be a helpful clue to resolving some of these difficulties.
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### A Tables of coefficients $N_{nm}^{rs}$, $X_{nm}^{rs}$

These are the coefficients appearing in the Witten vertex for the bosonic string, computed from equations (3, 8).

| $n$ | $m$ | $N_{nm}^{11}$ | $N_{nm}^{12}$ | $N_{nm}^{13}$ |
|-----|-----|---------------|---------------|---------------|
| 1   | 1   | $-5/27$       | $16/27$       | $16/27$       |
| 1   | 2   | $0$           | $32 \sqrt{3}/243$ | $-32 \sqrt{3}/243$ |
| 2   | 1   | $0$           | $-32 \sqrt{3}/243$ | $32 \sqrt{3}/243$ |
| 1   | 3   | $32/729$      | $-16/729$     | $-16/729$     |
| 2   | 2   | $13/486$      | $-64/243$     | $-64/243$     |
| 3   | 1   | $32/729$      | $-16/729$     | $-16/729$     |
| 1   | 4   | $0$           | $-64 \sqrt{3}/2187$ | $64 \sqrt{3}/2187$ |
| 2   | 3   | $0$           | $-160 \sqrt{3}/2187$ | $160 \sqrt{3}/2187$ |
| 3   | 2   | $0$           | $160 \sqrt{3}/2187$ | $-160 \sqrt{3}/2187$ |
| 4   | 1   | $0$           | $64 \sqrt{3}/2187$ | $-64 \sqrt{3}/2187$ |
| 1   | 5   | $-416/19683$  | $208/19683$   | $208/19683$   |
| 2   | 4   | $-256/19683$  | $128/19683$   | $128/19683$   |
| 3   | 3   | $-803/59049$  | $10288/59049$ | $10288/59049$ |
| 4   | 2   | $-256/19683$  | $128/19683$   | $128/19683$   |
| 5   | 1   | $-416/19683$  | $208/19683$   | $208/19683$   |
| 1   | 6   | $0$           | $800 \sqrt{3}/59049$ | $-800 \sqrt{3}/59049$ |
| 2   | 5   | $0$           | $352 \sqrt{3}/19683$ | $-352 \sqrt{3}/19683$ |
| 3   | 4   | $0$           | $3136 \sqrt{3}/59049$ | $-3136 \sqrt{3}/59049$ |
| 4   | 3   | $0$           | $-3136 \sqrt{3}/59049$ | $3136 \sqrt{3}/59049$ |
| 5   | 2   | $0$           | $-352 \sqrt{3}/19683$ | $352 \sqrt{3}/19683$ |
| 6   | 1   | $0$           | $-800 \sqrt{3}/59049$ | $800 \sqrt{3}/59049$ |
| 1   | 7   | $2272/177147$ | $-1136/177147$ | $-1136/177147$ |
| 2   | 6   | $1408/177147$ | $-704/177147$ | $-704/177147$ |
| 3   | 5   | $1504/177147$ | $-752/177147$ | $-752/177147$ |
| 4   | 4   | $5125/708588$ | $-22784/177147$ | $-22784/177147$ |
| 5   | 3   | $1504/177147$ | $-752/177147$ | $-752/177147$ |
| 6   | 2   | $1408/177147$ | $-704/177147$ | $-704/177147$ |
| 7   | 1   | $2272/177147$ | $-1136/177147$ | $-1136/177147$ |
| 1   | 8   | $0$           | $-38272 \sqrt{3}/4782969$ | $38272 \sqrt{3}/4782969$ |
| 2   | 7   | $0$           | $-41312 \sqrt{3}/4782969$ | $41312 \sqrt{3}/4782969$ |
| 3   | 6   | $0$           | $-203296 \sqrt{3}/14348907$ | $203296 \sqrt{3}/14348907$ |
| 4   | 5   | $0$           | $-195136 \sqrt{3}/4782969$ | $195136 \sqrt{3}/4782969$ |
| 5   | 4   | $0$           | $195136 \sqrt{3}/4782969$ | $-195136 \sqrt{3}/4782969$ |
| 6   | 3   | $0$           | $203296 \sqrt{3}/14348907$ | $-203296 \sqrt{3}/14348907$ |
| 7   | 2   | $0$           | $41312 \sqrt{3}/4782969$ | $-41312 \sqrt{3}/4782969$ |
| 8   | 1   | $0$           | $38272 \sqrt{3}/4782969$ | $-38272 \sqrt{3}/4782969$ |
| $n$ | $m$ | $X_{nm}^{11}$ | $X_{nm}^{12}$ | $X_{nm}^{13}$ |
|-----|-----|-------------|-------------|-------------|
| 1   | 1   | 11/27       | 8/27        | 8/27        |
| 1   | 2   | 0           | $40\sqrt{3}/243$ | $-40\sqrt{3}/243$ |
| 2   | 1   | 0           | $-80\sqrt{3}/243$ | $80\sqrt{3}/243$ |
| 1   | 3   | $-80/729$   | $40/729$    | $40/729$    |
| 2   | 2   | $-19/243$   | $-112/243$  | $-112/243$  |
| 3   | 1   | $-80/243$   | $40/243$    | $40/243$    |
| 1   | 4   | 0           | $-104\sqrt{3}/2187$ | $104\sqrt{3}/2187$ |
| 2   | 3   | 0           | $-304\sqrt{3}/2187$ | $304\sqrt{3}/2187$ |
| 3   | 2   | 0           | $152\sqrt{3}/729$ | $-152\sqrt{3}/729$ |
| 4   | 1   | 0           | $416\sqrt{3}/2187$ | $-416\sqrt{3}/2187$ |
| 1   | 5   | $1136/19683$ | $-568/19683$ | $-568/19683$ |
| 2   | 4   | $800/19683$ | $-400/19683$ | $-400/19683$ |
| 3   | 3   | $2099/19683$ | $8792/19683$ | $8792/19683$ |
| 4   | 2   | $1600/19683$ | $-800/19683$ | $-800/19683$ |
| 5   | 1   | $5680/19683$ | $-2840/19683$ | $-2840/19683$ |
| 1   | 6   | 0           | $1528\sqrt{3}/59049$ | $-1528\sqrt{3}/59049$ |
| 2   | 5   | 0           | $688\sqrt{3}/19683$ | $-688\sqrt{3}/19683$ |
| 3   | 4   | 0           | $3320\sqrt{3}/19683$ | $-3320\sqrt{3}/19683$ |
| 4   | 3   | 0           | $-13280\sqrt{3}/59049$ | $13280\sqrt{3}/59049$ |
| 5   | 2   | 0           | $-1720\sqrt{3}/19683$ | $1720\sqrt{3}/19683$ |
| 6   | 1   | 0           | $-3056\sqrt{3}/19683$ | $3056\sqrt{3}/19683$ |
| 1   | 7   | $-6640/177147$ | $3320/177147$ | $3320/177147$ |
| 2   | 6   | $-4640/177147$ | $2320/177147$ | $2320/177147$ |
| 3   | 5   | $-3568/59049$ | $1784/59049$ | $1784/59049$ |
| 4   | 4   | $-8251/177147$ | $-8448/177147$ | $-8448/177147$ |
| 5   | 3   | $-17840/177147$ | $8920/177147$ | $8920/177147$ |
| 6   | 2   | $-4640/59049$ | $2320/59049$ | $2320/59049$ |
| 7   | 1   | $-46480/177147$ | $23240/177147$ | $23240/177147$ |
| 1   | 8   | 0           | $-82040\sqrt{3}/4782969$ | $82040\sqrt{3}/4782969$ |
| 2   | 7   | 0           | $-86288\sqrt{3}/4782969$ | $86288\sqrt{3}/4782969$ |
| 3   | 6   | 0           | $-239720\sqrt{3}/4782969$ | $239720\sqrt{3}/4782969$ |
| 4   | 5   | 0           | $-763424\sqrt{3}/4782969$ | $763424\sqrt{3}/4782969$ |
| 5   | 4   | 0           | $954280\sqrt{3}/4782969$ | $-954280\sqrt{3}/4782969$ |
| 6   | 3   | 0           | $479440\sqrt{3}/4782969$ | $-479440\sqrt{3}/4782969$ |
| 7   | 2   | 0           | $302008\sqrt{3}/4782969$ | $-302008\sqrt{3}/4782969$ |
| 8   | 1   | 0           | $656320\sqrt{3}/4782969$ | $-656320\sqrt{3}/4782969$ |

The coefficients $N_{rs}^{nm}, X_{rs}^{nm}$ have a cyclic symmetry under $r \rightarrow (r \mod 3) + 1$, $s \rightarrow (s \mod 3) + 1$, so we only give the coefficients for $r = 1$.  

20
## B Some details of the calculations up to level 6

This table describes the contributions of each state to the quartic terms in the effective tachyon and gauge field potential. Note that, like $B_{\mu \nu}$, many fields can be contracted in multiple ways with a pair of scalar or vector fields. The contribution calculated includes interactions from all such contractions.

| state       | $\hat{\gamma}$ | $\hat{\gamma}_1$ | $\hat{\gamma}_2$ |
|-------------|-----------------|-------------------|-------------------|
| $|0\rangle$  |                 | $\frac{27}{3^{10}} \approx 1.5802$ | 0                 |
| $b_{-1}c_{-1}|0\rangle$ | $\frac{112}{3^{25}} \approx 0.7469$ | $\frac{27}{3^{10}} \approx 0.2623$ | 0                 |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\mu}|0\rangle$ | $\frac{-5^{212}13}{3^{237}} \approx -2.0062$ | $\frac{-27^{212}421}{3^{199}} \approx -0.9126$ | $\frac{-215^{212}}{3^{199}} \approx -0.5549$ |
| $b_{-1}c_{-3}|0\rangle$ | $\frac{27^{212}}{3^{199}} \approx 0.0542$ | $\frac{215^{212}}{3^{199}} \approx 0.0190$ | 0                 |
| $b_{-2}c_{-2}|0\rangle$ | $\frac{19^{212}}{3^{199}} \approx 0.0092$ | $\frac{215^{212}}{3^{199}} \approx 0.0032$ | 0                 |
| $b_{-3}c_{-1}|0\rangle$ | $\frac{27^{212}}{3^{199}} \approx 0.0542$ | $\frac{215^{212}}{3^{199}} \approx 0.0190$ | 0                 |
| $\alpha_{-1}^{\rho\nu} \alpha_{-1}^{\nu\rho}|0\rangle$ | $\frac{-2^{128}}{3^{216}} \approx -0.2254$ | $\frac{-2^{128}}{3^{216}} \approx -0.0746$ | $\frac{-2^{128}}{3^{216}} \approx -0.0015$ |
| $\alpha_{-2}^{\mu\nu} \alpha_{-2}^{\nu\mu}|0\rangle$ | $\frac{-13^{212}}{3^{237}} \approx -0.0558$ | $\frac{-2^{128}}{3^{216}} \approx -0.0259$ | $\frac{-2^{128}}{3^{216}} \approx -0.0162$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-1}c_{-1}|0\rangle$ | $\frac{5^{212}}{3^{237}} \approx 0.1110$ | $\frac{2^{128}}{3^{216}} \approx 0.0505$ | $\frac{2^{128}}{3^{216}} \approx 0.0307$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} \alpha_{-1}^{\rho\mu}|0\rangle$ | $\frac{-5^{212}}{3^{237}} \approx -0.0172$ | $\frac{-2^{128}}{3^{216}} \approx -0.1024$ | $\frac{-2^{128}}{3^{216}} \approx -0.0952$ |
| $b_{-1}c_{-5}|0\rangle$ | $\approx 0.0150$ | $\approx 0.0053$ | 0                 |
| $b_{-2}c_{-4}|0\rangle$ | $\approx 0.0030$ | $\approx 0.0010$ | 0                 |
| $b_{-3}c_{-3}|0\rangle$ | $\approx 0.0102$ | $\approx 0.0036$ | 0                 |
| $b_{-4}c_{-2}|0\rangle$ | $\approx 0.0030$ | $\approx 0.0010$ | 0                 |
| $b_{-5}c_{-1}|0\rangle$ | $\approx 0.0150$ | $\approx 0.0053$ | 0                 |
| $b_{-2}b_{-1}c_{-2}c_{-1}|0\rangle$ | $\approx -0.0009$ | $\approx -0.0003$ | 0                 |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\mu}|0\rangle$ | $\approx -0.0523$ | $\approx -0.0173$ | $\approx -0.0004$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho}|0\rangle$ | $\approx -0.0317$ | $\approx -0.0105$ | $\approx -0.0019$ |
| $\alpha_{-1}^{\rho\nu} \alpha_{-1}^{\nu\rho}|0\rangle$ | $\approx -0.0241$ | $\approx -0.0084$ | $\approx -0.0000$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-1}c_{-3}|0\rangle$ | $\approx 0.0145$ | $\approx 0.0066$ | $\approx 0.0040$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-2}c_{-2}|0\rangle$ | $\approx 0.0025$ | $\approx 0.0011$ | $\approx 0.0007$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-3}c_{-1}|0\rangle$ | $\approx 0.0145$ | $\approx 0.0066$ | $\approx 0.0040$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-1}c_{-2}|0\rangle$ | 0 | 0 | 0 |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-2}c_{-1}|0\rangle$ | 0 | 0 | 0 |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-1}c_{-1}|0\rangle$ | $\approx 0.0225$ | $\approx 0.0074$ | $\approx 0.0002$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-2}c_{-1}|0\rangle$ | $\approx 0.0056$ | $\approx 0.0026$ | $\approx 0.0016$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-1}c_{-1}|0\rangle$ | $\approx -0.0070$ | $\approx -0.0290$ | $\approx -0.0188$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-1}b_{-1}c_{-1}|0\rangle$ | $\approx -0.0006$ | $\approx -0.0091$ | $\approx -0.0085$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-1}b_{-1}c_{-1}|0\rangle$ | $\approx 0.0017$ | $\approx 0.0102$ | $\approx 0.0095$ |
| $\alpha_{-1}^{\mu\nu} \alpha_{-1}^{\nu\rho} b_{-1}b_{-1}c_{-1}|0\rangle$ | $\approx -0.0003$ | $\approx -0.0149$ | $\approx -0.0157$ |
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