Large deviations for slow-fast stochastic partial differential equations

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Abstract
A large deviation principle is derived for stochastic partial differential equations with slow-fast components. The result shows that the rate function is exactly that of the averaged equation plus the fluctuating deviation which is a stochastic partial differential equation with small Gaussian perturbation. This also confirms the effectiveness of the approximation of the averaged equation plus the fluctuating deviation to the slow-fast stochastic partial differential equations.

1 Introduction
Uncertainties (noise) is widely recognized in modeling, analyzing, simulating and predicting complex phenomena [2, 18, 25, 35] e.g.]. Noise causes rare events in nonlinear stochastic system describing the metastability of the system [19, 22, 20]. The classic example is a tunnelling event between two stable points of a macroscopic system. This rare event eventually occurs after a long time scale by an addition of a small external noise to the macroscopic system, but the probability of such rare event converges to zero as the strength of noise tends to zero. We need to understand the rate of such convergence. The theory of large deviations provide a powerful tool to give an

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estimate to the rate of such convergence, shown to be exponential for finite dimensional stochastic systems \[36, 20, 34, \text{e.g.}\]. To study the metastability of a macroscopic system with small noise we must build the large deviation principle (LDP).

Stochastic partial differential equations (SPDEs) are appropriate mathematical models for many multiscale systems with uncertain influences \[38\]. The LDP for SPDEs has been studied by many people \[7, 11, 13, 14, 16, 21, 26, 27, 32, \text{e.g.}\] under a different framework to that used here. However, there are very few results on the LDP for nonlinear stochastic system with two widely separated timescales, as often appears in a complex system. Freidlin and Wentzell \[20\] first studied the LDP for finite dimensional slow-fast stochastic system with partial coupling. They used a bounded assumption on the nonlinearity. Then Veretennikov \[37\] built a LDP for the full coupled case with bounded assumptions on nonlinearity. Later still, under a bounded assumption, Ioffe studied the LDP for stochastic reaction-diffusion equation with rapidly oscillating random noise in the special case when there is no coupling between the slow component and fast component \[23, 24\]. However, there appear to be no other previous LDPs for slow-fast coupled stochastic partial differential equations.

This article establishes the Freidlin–Wentzell LDP for a class of SPDEs with stochastic fast component and deterministic slow component. Let \(D\) be an open bounded interval and \(L^2(D)\) be the Lebesgue space of square integrable real valued functions on \(D\). Consider the following pair of stochastically forced, coupled, reaction-diffusion SPDEs for any \(\epsilon > 0\)

\[
\partial_t u^\epsilon = \partial_{xx} u^\epsilon + f(u^\epsilon, v^\epsilon), \quad u^\epsilon(0) = u_0 \in L^2(D) \tag{1}
\]

\[
\partial_t v^\epsilon = \frac{1}{\epsilon} \left[ \partial_{xx} v^\epsilon + g(u^\epsilon, v^\epsilon) \right] + \frac{\sigma}{\sqrt{\epsilon}} \partial_t W(t), \quad v^\epsilon(0) = v_0 \in L^2(D) \tag{2}
\]

with zero Dirichlet boundary condition on \(\partial D\). Here \(W(t)\) is an \(L^2(D)\) valued Wiener process defined on a complete probability space \((\Omega, \mathcal{F}, P)\) as detailed in the next section. In physical applications this supposition is that the noise directly drives microscopic modes \(v^\epsilon\)—the noise only emerges in the macroscopic modes \(u^\epsilon\) through nonlinear coupling.

If parameter \(\epsilon > 0\) is small so that \(v^\epsilon\) fluctuates rapidly, then an effective approximated system is desirable. Under some appropriate assumptions one averages the slow part \(u^\epsilon\) over the fast part \(v^\epsilon\) which yields the following so-called averaged equation describing the dynamics of the system on a slow time scale

\[
\partial_t u = \partial_{xx} u + \bar{f}(u), \quad u(0) = u_0, \quad u|_{\partial D} = 0. \tag{3}
\]

Here \(\bar{f}(u)\) is the average of \(f(u, v)\) over the distribution of the fast part \(v\).
Cerrai et al. [8, 9, e.g.] recently developed more on the averaging principle of SPDEs. Wang and Roberts [39] very recently gave a further approximation result via a martingale discussion which shows that the deviation \( u'(t) - u(t) \) is approximated by \( \sqrt{\epsilon} z(t) \), for some Guassian process \( z(t) \), in the case that fast component is coupled with slow component and without any Lipschitz assumption on the slow component. Cerria [10] obtained the same result for the special case where the nonlinearity is Lipschitz and there is no coupling of the slow component to the fast component. Then the deviation estimate shows that \( u' \) approximates \( u \) with a small Gaussian perturbation and this suggests a LDP for \( \{u'\}_\epsilon \). By studying the LDP for some auxiliary systems, we prove the LDP for \( \{u'\}_\epsilon \), Theorem 9. Moreover, the rate function for the LDP of \( \{u'\}_\epsilon \) in the main result is exactly that of \( \{\tilde{u}'\}_\epsilon \) solving (10)–(11), which is the averaged equation plus deviation up to errors of \( O(\epsilon) \), Section 2. Our results further confirms the effectiveness of the averaged equation plus deviation to approximate slow-fast SPDEs (1)–(2).

Recently, a weak convergence approach, which avoids giving some technical exponential tightness estimates, was applied to obtain LDP for SPDEs [17, 41, e.g.]. But this approach does not work here because a drift transformation leads the fast system to become a non-autonomous system for which one cannot average the slow part over the fast part. For this here we still give some exponential tight estimates, Section 3 and then by some contraction principles and an approximation we obtain LDP for some auxiliary systems, Section 4. An approximation shows that the slow-fast stochastic system (1)–(2) is comparable with the auxiliary systems near some functions, Section 5 and we derive the LDP for \( \{u'\}_\epsilon \).

Section 6 presents an example slow-fast reaction-diffusion SPDE to illustrate the LDP theory. Section 7 then explores the parameter regime near a stochastic pitchfork bifurcation in this example. Constructing the stochastic ‘superslow’ manifold and the evolution thereon confirms that there is indeed a close correspondence between the original example system and the LDP averaged system.

2 Preliminaries

Let \( H = L^2(D) \) with \( L^2 \)-norm denoted by \( \| \cdot \|_0 \) and inner product by \( \langle \cdot, \cdot \rangle \). Define the linear operator \( A = \partial_{xx} \) with zero Dirichlet boundary condition on \( D \). Then operator \( A \) is the generator of a compact analytic semigroup \( e^{At} \), \( t \geq 0 \), on \( H \). Moreover, denote by \( \{e_i\}_{i=1}^\infty \), which forms a complete standard orthogonal basis in \( H \), a family of eigenfunctions of \( A \) and \( -Ae_i = \lambda_i e_i \), \( \lambda_i > 0 \), \( i = 1, 2, \ldots \). For any \( \alpha > 0 \) and \( u \in H \) define \( \|u\|_\alpha = \|A^{\alpha/2}u\|_0 \). Then let
be the space of the closure of $C_0^\infty(D)$, the space of smooth functions with compact support on $D$, under the norm $\| \cdot \|_\alpha$. Furthermore, let $H^{-\alpha}$ denote the dual space of $H_0^\alpha$. Also we are given $H$ valued Wiener processes $W(t)$, $t \geq 0$, defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ [27]. Denote by $\mathbb{E}$ the expectation operator with respect to $\mathbb{P}$. We consider the SPDEs of the form (1)–(2) with separated time scale and with $\sigma \neq 0$ is an arbitrary real number parametrising the strength of the noise. We adopt the following four hypotheses.

**H_1** $f(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in both $x$ and $y$ with Lipschitz constant $L_f$ and for all $x, y \in \mathbb{R}$

$$|f(x, y)|^2 \leq ax^2 + by^2 + c, \quad f(x, y)x \leq ax^2 + bxy + c,$$

for some positive constants $a, b$ and $c$.

**H_2** $g(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in both variables with Lipschitz constant $L_g$. For any $x, y \in \mathbb{R}$

$$g(x, y)y \leq -dy^2 + exy$$

for some positive constants $d$ and $e$.

**H_3** $L_g < \lambda_1$.

**H_4** $W$ is a Q-Wiener processes which has the following series expansion

$$W(t, x) = \sum_{i=1}^{\infty} \sqrt{q_i} e_i(x) \beta_i(t)$$

and

$$Qe_i = q_i e_i, \quad i = 1, 2, \ldots$$

Moreover, $\text{tr} Q < \infty$.

By the above assumptions we have the following results on the averaging approximation to the slow-fast SPDE (1)–(2) [39].

**Theorem 1.** Assume $H_2$. For any fixed slow part $u \in H$, the fast system (2) has a unique stationary solution, $\eta^{\epsilon,u}(t)$, with distribution $\mu^u$ independent of $\epsilon$. Moreover, the stationary measure $\mu^u$ is exponentially mixing. Also, $\eta^{\epsilon,u}$ is differentiable with respect to $u$ with Fréchet derivative

$$D_u \eta^{\epsilon,u} \leq D_v$$

for some positive constant $D_v$ which is independent of $\epsilon$ and $u$. 

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Given the stationary measure $\mu^u$ for the fast part, we define the following deterministic averaged equation

$$\begin{align*}
du &= [Au + \bar{f}(u)] \, dt, \\
u(0) &= u_0,
\end{align*}$$

(5)

where the average

$$\bar{f}(u) = \int_H f(u, v) \mu^u(dv).$$

(7)

Denote by $\rho_0T$ the metric on space $C(0, T; H)$ with

$$\rho_0T(u, v) = \max_{0 \leq t \leq T} \|u(t) - v(t)\|_0$$

for all $u, v \in C(0, T; H)$,

then we have the following theorem.

**Theorem 2.** Assume $H_1 - H_4$. Given some $T > 0$, for any $u_0 \in H$, solutions $u^\epsilon(t, u_0)$ of (1) converges in probability to $u$ in $C(0, T; H)$ which solves (5)–(6). Moreover, the convergence rate is $1/2$; that is, for any $\kappa > 0$

$$\Pr \left\{ \rho_0T(u^\epsilon, u) \leq C_T^\kappa \sqrt{\epsilon} \right\} > 1 - \kappa$$

for some positive constant $C_T^\kappa > 0$.

Now define the deviation between solutions $u^\epsilon$ and averaged solution $u$,

$$z^\epsilon(t) = \frac{1}{\sqrt{\epsilon}} (u^\epsilon - u).$$

(8)

For $\epsilon = 1$ we write $\eta^{1,u} = \eta^u$. Then we have

**Theorem 3.** The deviation $z^\epsilon$ converges in distribution to a stochastic process $z$ in the space $C(0, T; H)$ which solves the SPDE

$$\dot{z} = Az + \bar{f}'_u(u) z + \sqrt{B(u)} \dot{W}$$

(9)

where $B(u) : H \to H$ is Hilbert–Schmidt with

$$B(u) = 2 \int_0^\infty \mathbb{E} \left[ (f(u, \eta^u(t)) - \bar{f}(u)) \otimes (f(u, \eta^u(0)) - \bar{f}(u)) \right] \, dt$$

$$\bar{f}'_u(u) = \int_H f'_u(u, v) \mu^u(dv)$$

and $\dot{W}(t)$ is an $H$-valued cylindrical Wiener process with covariance operator $\text{Id}_H$. 

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Remark 4. Assume \( \{e_i\}_{i=1}^{\infty} \) be a standard eigenbasis of \( H \), then \( B(u) \) has the following series form

\[
B(u) = 2 \sum_{i,j=1}^{\infty} \int_0^{\infty} \mathbb{E} \left[ \langle f(u, \eta^n(t)) - \bar{f}(u), e_i \rangle \langle f(u, \eta^n(0)) - \bar{f}(u), e_j \rangle \right] dt e_i \otimes e_j
\]

where \( e_i \otimes e_j \) is the tensor product of \( e_i \) and \( e_j \).

Then formally we write the following averaged equation plus deviation up to errors of \( O(\epsilon) \) as

\[
d\tilde{u}^\epsilon = \left[ A\tilde{u}^\epsilon + \tilde{f}(\tilde{u}^\epsilon) \right] dt + \sqrt{\epsilon} \sqrt{B(\tilde{u}^\epsilon)} dW(t), \tag{10}
\]

\[
\tilde{u}^\epsilon(0) = u_0. \tag{11}
\]

A family of random process \( \{u^\epsilon\}_\epsilon \) in space \( C(0,T; H) \) is said to satisfy the LDP with rate function \( I \) if

1. (lower bound) for any \( \varphi \in C(0,T; H) \) and any \( \delta, \gamma > 0 \), there is an \( \epsilon_0 > 0 \) such that for \( 0 < \epsilon < \epsilon_0 \)

\[
\mathbb{P}\{\rho_{0T}(u^\epsilon, \varphi) \leq \delta\} \geq \exp \left\{ -\frac{I(\varphi) + \gamma}{\epsilon} \right\},
\]

2. (upper bound) for any \( r > 0 \) and any \( \delta, \gamma > 0 \), there is a \( \epsilon_0 > 0 \) such that for any \( 0 < \epsilon < \epsilon_0 \)

\[
\mathbb{P}\{\rho_{0T}(u^\epsilon, K_T(r)) \geq \delta\} \leq \exp \left\{ -\frac{r - \gamma}{\epsilon} \right\}.
\]

Here the level set \( K_T(r) := \{ \varphi \in C(0,T; H) : I(\varphi) \leq r \} \). If \( K_T(r) \) is compact, then rate function \( I \) is called a good one.

There are many known results on the LDP for those SPDEs in the form of (10)–(11) as \( \epsilon \to 0 \), as mentioned in Section 1. Thus one may expect to derive LDP for \( \{u^\epsilon\}_\epsilon \) from (10)–(11). However, it is difficult to obtain an exponential approximation in probability between \( u^\epsilon \) and \( \tilde{u}^\epsilon \) as \( \epsilon \to 0 \), which is needed to pass the LDP of \( \{\tilde{u}^\epsilon\}_\epsilon \) to \( \{u^\epsilon\}_\epsilon \). But here our result shows that the rate function for \( \{u^\epsilon\}_\epsilon \) as \( \epsilon \to 0 \) is exactly that of \( \{\tilde{u}^\epsilon\}_\epsilon \). This also shows system (10)–(11) is indeed an effective approximate description of the macroscopic behaviour of the slow-fast system (1)–(2), even when one considers the exit problem caused by small noise perturbation for \( u^\epsilon \) in the system of slow-fast SPDE (1)–(2).

For our purposes we study the LDP for a series of auxiliary slow-fast stochastic systems by a contraction principle and an approximation argument. Then a controlled approximation derives the LDP for \( \{u^\epsilon\}_\epsilon \). We first give some contraction principles [15] which are used in our approach.
Lemma 5. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff topological spaces and $\Phi : \mathcal{X} \to \mathcal{Y}$ a continuous function. Consider a good rate function $I : \mathcal{X} \to [0, \infty]$. Then

1. for each $y \in \mathcal{Y}$,
   \[ \tilde{I}(y) = \inf_{x \in \mathcal{X}} \{ I(x) : y = \Phi(x) \} \]
   is a good rate function on $\mathcal{Y}$, where the infimum over an empty set is taken as $\infty$;

2. if $I$ controls the LDP associated with a family of probability measures $\{\mu_\epsilon\}_\epsilon$ on $\mathcal{X}$, then $\tilde{I}$ controls the LDP associated with the family of probability measures $\{\tilde{\mu}_\epsilon \circ \Phi^{-1}\}_\epsilon$ on $\mathcal{Y}$.

To introduce a generalized contraction principle we give the following definitions.

Definition 6. Let $(\mathcal{Y}, d)$ be a metric space. Then the probability measures $\{\mu_\epsilon\}$ and $\{\tilde{\mu}_\epsilon\}_\epsilon$ on $\mathcal{Y}$ are called exponentially equivalent if there exists a probability space $\{\Omega, \mathcal{B}_\epsilon, \mathbb{P}_\epsilon\}$ and two families of $\mathcal{Y}$-valued random variables $\{Z_\epsilon\}_\epsilon$ and $\{\tilde{Z}_\epsilon\}_\epsilon$ with joint laws $\{\mathbb{P}_\epsilon\}_\epsilon$ and marginals $\{\mu_\epsilon\}$ and $\{\tilde{\mu}_\epsilon\}_\epsilon$, respectively, such that for each $\delta > 0$, the set $\{\omega : (\tilde{Z}_\epsilon, Z_\epsilon) \in \Gamma_\delta\}$ is $\mathcal{B}_\epsilon$ measurable, and
   \[ \limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}_\epsilon(\Gamma_\delta) = -\infty \]
   where $\Gamma_\delta = \{((\tilde{y}, y) : d(\tilde{y}, y) > \delta\} \subset \mathcal{Y} \times \mathcal{Y}$.

As far as the LDP is concerned, exponentially equivalent measures are indistinguishable.

Lemma 7. If an LDP with good rate function $I(\cdot)$ holds for the probability measures $\{\mu_\epsilon\}$, which are exponentially equivalent to $\{\tilde{\mu}_\epsilon\}_\epsilon$, then the same LDP holds for $\{\tilde{\mu}_\epsilon\}_\epsilon$.

Furthermore, we need the following assumption

$H_5$ There is a positive constant $c_0$ and $c_1$ such that
   \[ \langle B(\varphi) h, h \rangle \geq c_0 \| h \|_0^2 \quad \text{and} \quad \langle DB(\varphi) h, h \rangle < c_1 \| h \|_0^2, \quad \text{for all} \ \varphi, h \in H \]
   where $DB$ is the Fréchet derivative of $B$.

Remark 8. Under the above assumption, $\sqrt{B(\varphi)}$ is Lipschitz continuous in $\varphi$. The following is one simple example of $f$ such that $B(\varphi)$ satisfies assumption $H_5$,
   \[ f(u, v) = f_1(u) + f_2(v) \]
with \( f_1 \) and \( f_2 \) both Lipschitz continuous. If the stationary Gaussian process \( \eta^u = \eta \) is independent of \( u \), by the expression of \( B(u) \) in Theorem 3
\[
B(u) = 2 \int_0^\infty \mathbb{E} \left[ (f_2(\eta(t)) - \mathbb{E}f_2(\eta)) \otimes (f_2(\eta(0)) - \mathbb{E}f_2(\eta)) \right] dt.
\]
Then \( B(u) \), which is independent of \( u \), satisfies assumption \( H_5 \). Section 6 details one example where \( f_2(v) \) is linear in the fast \( v \).

Now for the slow-fast stochastic system (1) we define the following skeleton equation
\[
\dot{\varphi} = A\varphi + \bar{f}(\varphi) + \sqrt{B(\varphi)}h, \quad \varphi(0) = u_0. \tag{12}
\]
And define the rate function
\[
I_u(\varphi) = \inf_{h \in L^2(0,T;H)} \left\{ \frac{1}{2} \int_0^T \|h(s)\|_H^2 \, ds : \varphi = \varphi^h \right\} \tag{13}
\]
where \( \varphi^h \) solves (12) and with \( \inf \emptyset = +\infty \). Then we give our main result.

**Theorem 9.** Assume \( H_1-H_5 \). For any \( T > 0 \), \( \{\tilde{u}^\epsilon\}_\epsilon \) satisfies the LDP with good rate function \( I_u \) on space \( C(0,T;H) \).

**Remark 10.** By the large deviations principle for stochastic evolutionary equations [27], under assumptions \( H_1-H_5 \), \( \{\tilde{u}^\epsilon\}_\epsilon \) satisfies the LDP with rate function \( I_u(\varphi) \) in space \( C(0,T;H) \) for any \( T > 0 \). So the averaged equation plus deviation (10)–(11) predicts the metastability of (1)–(2).

### 3 Exponential tightness

We prove some exponential tightness results which are crucial for the LDP estimates.

For any \( \psi \in C(0,T;H) \) consider the following system
\[
d\tilde{u}^{\epsilon,\psi} = \left[ A\tilde{u}^{\epsilon,\psi} + f(\psi, \tilde{v}^{\epsilon,\psi}) \right] \, dt, \quad \tilde{u}^{\epsilon,\psi}(0) = u_0, \tag{14}
\]
\[
d\tilde{v}^{\epsilon,\psi} = \frac{1}{\epsilon} \left[ A\tilde{v}^{\epsilon,\psi} + g(\psi, \tilde{v}^{\epsilon,\psi}) \right] \, dt + \frac{1}{\sqrt{\epsilon}} dW(t), \quad \tilde{v}^{\epsilon,\psi}(0) = v_0. \tag{15}
\]

Then we have the following exponential tightness result.

**Lemma 11.** Fix \( \psi \in C(0,T;H) \). For any \( T > 0 \) and \( \epsilon_0 > 0 \), \( \{\tilde{u}^{\epsilon,\psi}\}_{0 \leq \epsilon \leq \epsilon_0} \) is exponentially tight in space \( C(0,T;H) \).
Proof. This result follows via a uniform estimate to $\tilde{u}^{x,\psi}$ in space $C(0, T; H^1_0) \cap C^\alpha(0, T; H)$ for some $\alpha > 0$. By 

$$
\tilde{u}^{x,\psi}(t) = u_0 + \int_0^t A\tilde{u}^{x,\psi}(s) \, ds + \int_0^t f(\psi(s), \tilde{v}^{x,\psi}(s)) \, ds.
$$

Then by the increasing property of $f$

$$
\|\tilde{u}^{x,\psi}(t) - \tilde{u}^{x,\psi}(\tau)\|^2 \\
\leq 2 \left[ \int_\tau^t \| A\tilde{u}^{x,\psi}(s) \|_0 \, ds \right]^2 + 2 \left[ \int_\tau^t \| f(\psi(s), \tilde{v}^{x,\psi}(s)) \|_0 \, ds \right]^2 \\
\leq 2| t - \tau | \int_0^T \| A\tilde{u}^{x,\psi}(s) \|_0^2 \, ds + 2| t - \tau | \int_\tau^t \| f(\psi(s), \tilde{v}^{x,\psi}(s)) \|_0^2 \, ds \\
\leq 2| t - \tau | \int_0^T \| A\tilde{u}^{x,\psi}(s) \|_0^2 \, ds + 2| t - \tau | \int_0^T \left[ a\| \psi(s) \|_0^2 + \| \tilde{v}^{x,\psi}(s) \|_0^2 + c \right] \, ds,
$$

and

$$
\frac{1}{2} \frac{d}{dt} \| \tilde{u}^{x,\psi} \|_1^2 = - \| A\tilde{u}^{x,\psi} \|_0^2 - \langle f(\psi, \tilde{v}^{x,\psi}), A\tilde{u}^{x,\psi} \rangle \\
\leq - \frac{1}{2} \| A\tilde{u}^{x,\psi} \|_0^2 + \frac{1}{2} \| f(\psi, \tilde{v}^{x,\psi}) \|_0^2 \\
\leq - \frac{1}{2} \| A\tilde{u}^{x,\psi} \|_0^2 + \frac{1}{2} \left[ a\| \psi \|_0^2 + \| \tilde{v}^{x,\psi} \|_0^2 + c \right].
$$

That is,

$$
\int_0^T \| A\tilde{u}^{x,\psi}(s) \|_0^2 \, ds + \sup_{0 \leq s \leq T} \| \tilde{u}^{x,\psi}(s) \|_1^2 \\
\leq \| u_0 \|_1^2 + a \int_0^T \| \psi(s) \|_0^2 \, ds + \int_0^T \| \tilde{v}^{x,\psi}(s) \|_0^2 \, ds + cT.
$$

Then to give a uniform estimate in space $C(0, T; H^1_0) \cap C^\alpha(0, T; H)$ for some $\alpha > 0$, we just need to estimate $\int_0^T \| \tilde{v}^{x,\psi}(s) \|_0^2 \, ds$.

Next we estimate $\int_0^T \| \tilde{v}^{x,\psi} \|_0^2 \, ds$. The Lipschitz property of $g$ and applying the Itô formula to $\| \tilde{v}^{x,\psi} \|_0^2$ yield for some positive constants $c'$ and $c''$

$$
\frac{1}{2} \frac{d}{dt} \| \tilde{v}^{x,\psi} \|_0^2 = - \frac{1}{\epsilon} \| \tilde{v}^{x,\psi} \|_1^2 + \frac{1}{\epsilon} \langle g(\psi, \tilde{v}^{x,\psi}), \tilde{v}^{x,\psi} \rangle + \frac{1}{\sqrt{\epsilon}} \langle \tilde{v}^{x,\psi}, \tilde{W} \rangle + \frac{1}{2\epsilon} \text{tr} \, Q \\
\leq - \frac{c'}{\epsilon} \| \tilde{v}^{x,\psi} \|_0^2 + \frac{c''}{\epsilon} \| \psi \|_0^2 + \frac{1}{2\epsilon} \text{tr} \, Q + \frac{1}{\sqrt{\epsilon}} \langle \tilde{v}^{x,\psi}, \tilde{W} \rangle.
$$

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Then
\[
\epsilon \| \tilde{v}^{\epsilon, \psi}(t) \|_1^2 + 2c' \int_0^t \| \tilde{v}^{\epsilon, \varphi}(s) \|_1^2 ds - \epsilon \| v_0 \|_0^2 \\
\leq 2c'' \int_0^t \| \psi(s) \|_0^2 ds + \text{tr } Q t + 2\sqrt{\epsilon} \int_0^t \langle \tilde{v}^{\epsilon, \psi}(s), dW(s) \rangle.
\]

Now define \( M^\epsilon_t = \int_0^t \langle \tilde{v}^{\epsilon, \psi}(s), dW(s) \rangle \) and \( \lambda_0 = c' \lambda_1 / 2 q_{\text{max}} \) with \( q_{\text{max}} = \max_i q_i \). And denote by \( \langle M \rangle^\epsilon_t \) the covariance of \( M^\epsilon_t \). Then we have
\[
c' \int_0^t \| \tilde{v}^{\epsilon, \psi}(s) \|_1^2 ds \leq \| v_0 \|_0^2 + 2c'' \int_0^t \| \psi(s) \|_0^2 ds + \text{tr } Q t + \sqrt{\epsilon} M^\epsilon_t - \frac{\lambda_0}{2} \langle M \rangle^\epsilon_t \\
+ \frac{\lambda_0}{2} \langle M \rangle^\epsilon_t - c' \int_0^t \| \tilde{v}^{\epsilon, \psi}(s) \|_1^2 ds.
\]

And by the exponential martingale inequality we have
\[
P \left\{ \sqrt{\epsilon} M^\epsilon_t - \frac{\lambda_0}{2} \langle M \rangle^\epsilon_t > \delta \right\} = P \left\{ \frac{\lambda_0}{\sqrt{\epsilon}} M^\epsilon_t - \frac{\lambda_0^2}{2 \epsilon} \langle M \rangle^\epsilon_t > \frac{\delta \lambda_0}{\epsilon} \right\} \leq e^{-\delta \lambda_0 / \epsilon}
\]
which yields the exponential tightness of \( \{ \tilde{u}^{\epsilon, \psi} \}^\epsilon \).

\[ \square \]

**Lemma 12.** For any \( T > 0 \), \( \{ u^\epsilon \}^\epsilon \) is exponentially tight in space \( C(0, T; H) \).

**Proof.** Follow the proof of Lemma [11], we just need to estimate \( \int_0^T \| u^\epsilon(s) \|_0^2 ds + \int_0^T \| v^\epsilon(s) \|_1^2 ds \). By our assumption we estimate
\[
\int_0^T \| u^\epsilon(s) \|_0^2 ds + \int_0^T \| v^\epsilon(s) \|_1^2 ds.
\]

For this, applying the Itô formula to \( \| v^\epsilon \|_0^2 \) and by the assumption on \( g \) we have
\[
\frac{1}{2} \frac{d}{dt} \| v^\epsilon \|_0^2 \leq -\frac{1}{\epsilon} \| v^\epsilon \|_1^2 - \frac{d}{dt} \| v^\epsilon \|_0^2 + \frac{e}{\epsilon} \langle u^\epsilon, v^\epsilon \rangle + \frac{1}{2 \epsilon} \text{tr } Q + \frac{1}{\sqrt{\epsilon}} \langle v^\epsilon, \dot{W} \rangle.
\] \tag{16}

Moreover,
\[
\frac{1}{2} \frac{d}{dt} \| u^\epsilon \|_0^2 \leq -\| u^\epsilon \|_1^2 - a \| u^\epsilon \|_0^2 + b \langle u^\epsilon, v^\epsilon \rangle + c.
\] \tag{17}
Then $\epsilon(16) + (17)$ yields
\[
2 \int_0^T \|u^\epsilon(s)\|_1^2 \, ds + 2 \int_0^T \|v^\epsilon(s)\|_1^2 \, ds \\
\leq \|u_0\|_0^2 + \|v_0\|_0^2 + C_T + tr\, Q_T + \sqrt{\epsilon} \int_0^t \langle v^\epsilon(s), dW(s) \rangle.
\]

Then a similar argument as for Lemma [11] yields the exponential tightness. The proof is complete. \qed

4 LDP for some auxiliary systems

Next we study the LDP for some auxiliary systems.

First, let $\psi \in H$, which is independent of time, and consider the following system
\[
\begin{align*}
    du^{\epsilon,\psi} &= \left[ Au^{\epsilon,\psi} + f(\psi, v^{\epsilon,\psi}) \right] \, dt, \quad u^{\epsilon,\psi}(0) = u_0, \quad (18) \\
    dv^{\epsilon,\psi} &= \frac{1}{\epsilon} \left[ Av^{\epsilon,\psi} + g(\psi, v^{\epsilon,\psi}) \right] \, dt + \frac{1}{\sqrt{\epsilon}} dW(t), \quad v^{\epsilon,\psi}(0) = v_0. \quad (19)
\end{align*}
\]

We study the LDP for $\{u^{\epsilon,\psi}\}_\epsilon$. For this define
\[
\xi^{\epsilon,\psi}(t) = \int_0^t \left[ f(\psi, v^{\epsilon,\psi}(s)) - \bar{f}(\psi) \right] \, ds.
\]

Then
\[
\dot{u}^{\epsilon,\psi} = Au^{\epsilon,\psi} + \bar{f}(\psi) + \dot{\xi}^{\epsilon,\psi}, \quad u^{\epsilon,\psi}(0) = u_0. \quad (20)
\]

Now in order to obtain the LDP for $\{u^{\epsilon,\psi}\}_\epsilon$ we study the LDP for $\{\bar{u}^{\epsilon,\psi}\}_\epsilon$, which solves
\[
\dot{\bar{u}}^{\epsilon,\psi} = A\bar{u}^{\epsilon,\psi} + \bar{f}(\psi) + \dot{\bar{\xi}}^{\epsilon,\psi}, \quad \bar{u}^{\epsilon,\psi}(0) = u_0 \quad (21)
\]
with
\[
\bar{\xi}^{\epsilon,\psi}(t) = \int_0^t \left[ f(\psi, \eta^{\epsilon,\psi}(s)) - \bar{f}(\psi) \right] \, ds
\]
where $\eta^{\epsilon,\psi}$ is defined in Theorem [1]. Now define the skeleton equation to (21) with $\psi \in H$
\[
\dot{\varphi}^{\psi,h} = A\varphi^{\psi,h} + \bar{f}(\psi) + \sqrt{B(\psi)}h, \quad \varphi^{\psi,h}(0) = u_0 \quad (22)
\]
for \( h \in L^2(0, T; H) \). Equation (22) is a linear equation, so for any \( \psi \in H \) and \( h \in L^2(0, T; H) \) there is a unique \( \varphi^{\psi, h} \in C(0, T; H) \) solving (22). Then define the functional

\[
I^\psi_u(\varphi) = \inf_{h \in L^2(0, T; H)} \left\{ \frac{1}{2} \int_0^T \|h(s)\|^2_0 ds : \varphi = \varphi^{\psi, h} \right\}.
\]

We prove the following LDP result for \( \{\bar{u}^{\psi, \epsilon}\}_\epsilon \).

**Theorem 13.** Fix \( \psi \in H \). For any \( T > 0 \), \( \{\bar{u}^{\psi, \epsilon}\}_\epsilon \) satisfies LDP in space \( C(0, T; H) \) with a good rate function \( I^\psi_u \). Moreover \( I^\psi_u \) is lower semicontinuous in \( \psi \in H \).

**Proof.** First by the same discussion in the proof to Lemma 11, for any \( T > 0 \), \( \{\bar{u}^{\psi, \epsilon}\}_\epsilon \) is exponential tight in space \( C(0, T; H) \). Then there exists \( \{K_R\}_R \) which is a nondecreasing family of compact sets such that

\[
P\{\bar{u}^{\psi, \epsilon} \in K_R\} \geq 1 - e^{-R/\epsilon}.
\] (23)

Now for any fixed \( N \geq 1 \), let \( H_N := \text{span}\{e_i : i = 1, \ldots, N\} \) and \( P_N : H \to H_N \) be the orthogonal projection. Then \( \xi_N^{\psi} := P_N \xi^{\psi, \epsilon} \in C(0, T; H_N) \) satisfies LDP with good rate function [20]

\[
I_N^\psi(\varphi_N) = \inf_{h_N} \left\{ \frac{1}{2} \int_0^T \|h_N(t)\|^2_{H_N} dt : \sqrt{B_N(\psi)}h_N = \varphi_N \right\}
\]

where \( \varphi_N \in C(0, T; H_N) \) and

\[
B_N(\psi) = 2 \int_0^\infty \mathbb{E} \left[ \left( (P_N f(\psi, \eta^{\psi}(t)) - P_N \bar{f}(\psi)) \otimes (P_N f(\psi, \eta^{\psi}(0)) - P_N \bar{f}(\psi)) \right) \right] dt.
\]

Introduce process \( \bar{u}_N^{\epsilon, \psi} \) solving

\[
\dot{\bar{u}}_N^{\epsilon, \psi} = A\bar{u}_N^{\epsilon, \psi} + f_N(\psi) + \xi_N^{\epsilon, \psi}, \quad \bar{u}_N^{\epsilon, \psi}(0) = P_N u_0
\]

with \( \bar{f}_N = P_N \bar{f} \). By the continuity of the map \( \xi_N^{\epsilon, \psi} \mapsto \bar{u}_N^{\epsilon, \psi} \) in space \( C(0, T; H) \) and the contraction principle Lemma [5], \( \{\bar{u}_N^{\epsilon, \psi}\}_\epsilon \) satisfies LDP with a good rate function

\[
I^\psi_{u, N}(\varphi) = \inf_{h \in L^2(0, T; H)} \left\{ \frac{1}{2} \int_0^T \|h(s)\|^2_0 ds : \varphi_N = \varphi_N^{\psi, h} \right\}
\]

where \( \varphi_N^{\psi, h} \in C(0, T; H) \) solves the following equation

\[
\dot{\varphi}_N^{\psi, h} = A\varphi_N^{\psi, h} + f_N(\psi) + \sqrt{B_N(\psi)}h, \quad \varphi_N^{\psi, h}(0) = P_N u_0.
\]
Moreover $\varphi_N^{\psi,h} \to \varphi^{\psi,h}$ as $N \to \infty$ in space $C(0,T;H)$. Then we have

\[ I_{\psi}^{\varphi} = \lim_{N \to \infty} I_{\psi,N}^{\varphi} = \sup_{N} I_{\psi,N}^{\varphi}, \quad \varphi \in C(0,T;H) \]  

which is a good rate function.

Now for any $\gamma, \delta > 0$, $\varphi \in C(0,T;H)$ and $R > 0$, for $\bar{u}^{\epsilon,\psi} \in K_R$, there is $N_R(\delta, \varphi) > 0$ such that

\[ \rho_0 T(\bar{u}^{\epsilon,\psi}, \bar{u}^{\epsilon,\psi}_N) + \rho_0 T(\bar{u}^{\epsilon,\psi}_N, P_N \varphi) < \delta / 2, \quad N > N_R(\delta, \varphi). \]

Then there is $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$

\[ \mathbb{P} \left\{ \rho_0 T(\bar{u}^{\epsilon,\psi}, \varphi) < \delta \right\} \]

\[ \geq \mathbb{P} \left\{ \rho_0 T(\bar{u}^{\epsilon,\psi}_N, P_N \varphi) < \delta / 2, \quad N > N_R(\delta, \varphi) \mid \bar{u}^{\epsilon,\psi} \in K_R \right\} \]

\[ \geq \mathbb{P} \left\{ \rho_0 T(\bar{u}^{\epsilon,\psi}_N, P_N \varphi) < \delta / 2 \right\} \mathbb{P} \left\{ \bar{u}^{\epsilon,\psi} \in K_R \mid \rho_0 T(\bar{u}^{\epsilon,\psi}_N, P_N \varphi) < \delta / 2 \right\} \]

\[ \geq \exp \left\{ - \frac{I_{\psi}^{\varphi}(P_N \varphi) + \gamma}{\epsilon} \right\} \mathbb{P} \left\{ \bar{u}^{\epsilon,\psi} \in K_R \mid \rho_0 T(\bar{u}^{\epsilon,\psi}_N, P_N \varphi) < \delta / 2 \right\}. \]

Passing to the limit $R \to \infty$ and noticing that $\lim_{R \to \infty} \mathbb{P} \{ \bar{u}^{\epsilon,\psi} \in K_R \} = 1$, the above estimate yields the lower bound estimate.

Now for any $r > 0$ and $\gamma > 0$, there is an $\epsilon_1 > 0$ and $R > 0$ such that for any $0 < \epsilon < \epsilon_1$

\[ \mathbb{P} \{ \bar{u}^{\epsilon,\psi} \in K^*_R \} \leq \exp \left\{ - \frac{r - \gamma}{\epsilon} \right\}. \]

Let $K_T(r) = \{ \varphi \in C(0,T;H) : I_{\psi}^{\varphi}(\varphi) \leq r \}$ and $K_{T,N}(r) = \{ \varphi_N \in C(0,T;H_N) : I_{\psi,N}^{\varphi}(\varphi_N) \leq r \}$. Then for $\bar{u}^{\epsilon,\psi} \in K_R$, there is $N(R,r) < 0$ such that for $N > N(R,r)$

\[ \mathbb{P} \left\{ \rho_0 T(\bar{u}^{\epsilon,\psi}, K_T(r)) > \delta \mid \bar{u}^{\epsilon,\psi} \in K_R \right\} \]

\[ \leq \mathbb{P} \left\{ \rho_0 T(\bar{u}^{\epsilon,\psi}_N, K_{T,N}(r)) > \delta / 2 \mid \bar{u}^{\epsilon,\psi} \in K_R \right\} \]

\[ \leq \exp \left\{ - \frac{r - \gamma}{\epsilon} \right\} \]  

for $\epsilon$ is small enough. Then (26)–(27) yield the upper bound estimate.

Next we prove the second result. This is followed by proving the set

\[ \Psi(r) := \{ \psi \in H : I_{\psi}^{\varphi}(\varphi) \leq r \} \]
is closed for any \( r > 0 \) and \( \varphi \in C(0, T; H) \). For this proof, let \( \{\psi_n\}_n \subset \Psi(r) \) and \( \psi_n \to \psi \) in space \( H \). Moreover, there is a sequence \( h_n \in L^2(0, T; H) \) such that \( \varphi_{\psi_n, h_n} = \varphi \) with
\[
\dot{\varphi}_{\psi_n, h_n} = A\varphi_{\psi_n, h_n} + \bar{f}(\psi_n) + \sqrt{B(\psi_n)}h_n, \quad \varphi_{\psi_n, h_n}(0) = u_0
\]
and
\[
\frac{1}{2} \int_0^T \|h_n(s)\|^2 \, ds < r + \frac{1}{n}.
\]
Then there is a subsequence, which we still denote by \( h_n \), weakly convergent to some \( h \in L^2(0, T; H) \). We show that \( \varphi_{\psi, h} = \varphi \) with
\[
\dot{\varphi}_{\psi, h} = A\varphi_{\psi, h} + \bar{f}(\psi) + \sqrt{B(\psi)}h, \quad \varphi_{\psi, h}(0) = u_0
\]
and
\[
\frac{1}{2} \int_0^T \|h(s)\|^2 \, ds \leq r.
\]
By the assumptions \( H_1 \) and \( H_5 \), we have
\[
\bar{f}(\psi_n) \to \bar{f}(\psi) \quad \text{in} \quad H
\]
and
\[
\sqrt{B(\psi_n)} \to \sqrt{B(\psi)} \quad \text{in} \quad \mathcal{L}(H).
\]
Then
\[
\sqrt{B(\psi_n)}h_n \to \sqrt{B(\psi)}h \quad \text{weakly in} \quad L^2(0, T; H).
\]
The proof is complete.

By the above result we have the following corollary.

**Corollary 14.** For fixed \( \psi \in H \), \( \{u^{\epsilon, \psi}\}_\epsilon \) satisfies LDP with good rate function \( I^\psi_u \) in space \( C(0, T; H) \) for any \( T > 0 \).

**Proof.** We show that \( u^{\epsilon, \psi} \) is exponentially equivalent in probability to \( \bar{u}^{\epsilon, \psi} \).

Let \( U^{\epsilon, \psi} = u^{\epsilon, \psi} - \bar{u}^{\epsilon, \psi} \), then
\[
U^{\epsilon, \psi} = AU^{\epsilon, \psi} + \xi^{\epsilon, \psi} - \bar{\xi}^{\epsilon, \psi}, \quad U^{\epsilon, \psi}(0) = 0. \]

Notice that
\[
\xi^{\epsilon, \psi}(t) - \bar{\xi}^{\epsilon, \psi}(t) = \int_0^t \left[ f(\psi, v^{\epsilon, \psi}(s)) - f(\psi, \eta^{\epsilon, \psi}(s)) \right] \, ds.
\]
Then for any $T > 0$

$$\sup_{0 \leq t \leq T} \|u^{\epsilon,\psi}(t) - \bar{u}^{\epsilon,\psi}(t)\|^2_0 \leq C_{\lambda_1} \sup_{0 \leq t \leq T} \int_0^t \|f(\psi, v^{\epsilon,\psi}(s)) - f(\psi, \eta^{\epsilon,\psi}(s))\|_0 ds$$

$$\leq LfC_{\lambda_1} \int_0^T \|v^{\epsilon,\psi}(s) - \eta^{\epsilon,\psi}(s)\|_0 ds \quad (28)$$

for some positive constant $C_{\lambda_1}$ which depends on $\lambda_1$. Let $\zeta^{\epsilon,\psi} = v^{\epsilon,\psi} - \eta^{\epsilon,\psi}$,

$$\dot{\zeta}^{\epsilon,\psi} = \frac{1}{\epsilon} A\zeta^{\epsilon,\psi} + \frac{1}{\epsilon} [g(\psi, v^{\epsilon,\psi}) - g(\psi, \eta^{\epsilon,\psi})], \quad \zeta^{\epsilon,\psi}(0) = v_0 - \eta^{\epsilon,\psi}(0).$$

Then

$$\frac{1}{2} \frac{d}{dt}\|\zeta^{\epsilon,\psi}(t)\|^2_0 \leq -\lambda_1 - \frac{Lg}{\epsilon}\|\zeta^{\epsilon,\psi}(t)\|^2_0.$$

By assumption $H_3$ we have

$$\|\zeta^{\epsilon,\psi}(t)\|^2_0 \leq e^{-2c/\epsilon}\|v_0 - \eta^{\epsilon,\psi}(0)\|^2_0$$

for some positive constant $c$ which is independent of $\epsilon$. Then by (28) we have for any $T > 0$

$$\sup_{0 \leq t \leq T} \|u^{\epsilon,\psi}(t) - \bar{u}^{\epsilon,\psi}(t)\|_0 \leq LfC_{\lambda_1} \int_0^T e^{-sc/\epsilon}\|v_0 - \eta^{\epsilon,\psi}(0)\|_0 ds$$

$$\leq \epsilon C\|v_0 - \eta^{\epsilon,\psi}\|_0$$

for some positive constant $C$ which is independent of $\epsilon$.

Then we have for any $\delta > 0$,

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} \|u^{\epsilon,\psi}(t) - \bar{u}^{\epsilon,\psi}(t)\|_0 > \delta\right\} \leq \mathbb{P}\{\epsilon C\|v_0 - \eta^{\epsilon,\psi}(0)\|_0 > \delta\}.$$

And by the Guassian property of $\eta^{\epsilon,\psi}(0)$, we have for any $\delta > 0$

$$\lim_{\epsilon \to 0} \epsilon \ln \mathbb{P}\left\{\sup_{0 \leq t \leq T} \|u^{\epsilon,\psi}(t) - \bar{u}^{\epsilon,\psi}(t)\|_0 > \delta\right\} = -\infty.$$

Then the generalized contraction principle Lemma 7 completes the proof.

Now we consider the special case that $\psi$ is a step function which we denote by $\psi^n$. Let $(\tilde{u}^{\epsilon,\psi^n}, \tilde{v}^{\epsilon,\psi^n})$ solve (14)–(15) with $\psi$ replaced by $\psi^n$, we show that $\{\tilde{u}^{\epsilon,\psi^n}\}_\epsilon$ satisfies the LDP with a good rate function. Typically we choose

$$\psi^n(t) = \sum_{i=0}^{n-1} \psi^n_i \chi_{[t_i, t_{i+1}]} \quad (29)$$
with \( \psi^n_i \in H \) and \( t_i = iT/n \), \( i = 0, 1, \ldots, n - 1 \). Let \( \tilde{u}^{\epsilon, \psi^n}_i \) and \( \tilde{v}^{\epsilon, \psi^n}_i \) be the restriction of \( u^{\epsilon, \psi^n} \) and \( v^{\epsilon, \psi^n} \) on the time interval \([t_i, t_{i+1}]\), respectively, then we have

\[ \dot{\tilde{u}}^{\epsilon, \psi^n}_i = A\tilde{u}^{\epsilon, \psi^n}_i + f(\psi^n_i, \tilde{v}^{\epsilon, \psi^n}_i) \]

which satisfies the LDP in space \( C([t_i, t_{i+1}], H) \) with rate function \( I_{u}^{\psi^n}_i \) by Corollary 14. Then we have the following result

**Theorem 15.** For any \( n \) and step function \( \psi^n \in C(0, T; H) \) in the form (29), \( \{\tilde{u}^{\epsilon, \psi^n}\} \) satisfies LDP with good rate function

\[ I_{u}^{\psi^n}(\varphi) = \sum_{i=0}^{n-1} I_{u}^{\psi^n}_i = \inf_{h \in L^2(0, T; H)} \left\{ \frac{1}{2} \int_0^T \|h(s)\|^2_0 ds : \varphi = \varphi^{\psi^n, h} \right\} \]

where \( \varphi^{\psi^n, h} \) solves (22) with \( \psi \) replaced by step function \( \psi^n \).

**Proof.** For any \( \varphi \in C(0, T; H) \) with \( I_{u}^{\psi^n}(\varphi) < \infty \), let \( \varphi_i(t) = \varphi(t)\chi_{[t_i, t_{i+1}]} \in C([t_i, t_{i+1}]; H) \), then for any \( \delta, \gamma > 0 \), there is an \( \epsilon_0 > 0 \) such that for any \( 0 < \epsilon < \epsilon_0 \),

\[ \mathbb{P} \left\{ \max_{t_i \leq t \leq t_{i+1}} \|\tilde{u}^{\epsilon, \psi^n}(t) - \varphi_i(t)\|_0 \leq \delta \right\} \geq \exp \left[ -\frac{I_{u}^{\psi^n}(\varphi_i) + \frac{\gamma}{n}}{\epsilon} \right]. \]

Moreover, by the assumption on \( f \), for \( n \) large enough for any \( \delta > 0 \)

\[ \left\{ \max_{t_i \leq t \leq t_{i+1}} \|\tilde{u}^{\epsilon, \psi^n}(t) - \psi_i(t)\|_0 \leq \delta \right\} \supset \left\{ \|\tilde{u}^{\epsilon, \psi^n}(t_i) - \psi_i(t_i)\|_0 \leq \delta'(\delta) \right\} := A_i \]

for some \( \delta'(\delta) \) small enough. Now for fixed time \( t_i \), \( \tilde{u}^{\epsilon, \psi^n}(t_i) \) is an \( H \)-valued
random variable. Then for small enough $\delta = \delta'(\delta)$
\[
\mathbb{P} \left\{ \rho_T(\tilde{u}^{\varphi, n}, \varphi) \leq \delta \right\} = \mathbb{P} \left\{ \max_{1 \leq i \leq n} \| \tilde{u}^{\varphi, n}(t_i) - \varphi(t_i) \|_0 \leq \delta \right\} \\
\geq \mathbb{P} \left\{ \max_{1 \leq i \leq n} \| \tilde{u}^{\varphi, n}(t_i) - \varphi(t_i) \|_0 \leq \delta' \right\} = \mathbb{P} \{ A_1 A_2 \ldots A_n \} \\
= \mathbb{P} \{ A_1 \} \mathbb{P} \{ A_2 \mid A_1 \} \mathbb{P} \{ A_3 \mid A_1 A_2 \} \ldots \mathbb{P} \{ A_n \mid A_1 A_2 \ldots A_{n-1} \} \\
\geq \mathbb{P} \left\{ \max_{t_0 \leq t \leq t_1} \| \tilde{u}^{\varphi, 0}(t) - \varphi_0(t) \|_0 \leq \delta' \right\} \\
\times \mathbb{P} \left\{ \max_{t_1 \leq t \leq t_2} \| \tilde{u}^{\varphi, 1}(t) - \varphi_1(t) \|_0 \leq \delta' \mid A_1 \right\} \\
\times \cdots \times \mathbb{P} \left\{ \max_{t_{n-1} \leq t \leq t_n} \| \tilde{u}^{\varphi, n-1}(t) - \varphi_{n-1}(t) \|_0 \leq \delta' \mid A_1 A_2 \ldots A_{n-1} \right\} \\
\geq \prod_{i=1}^{n} \exp \left[ -\frac{I^{\psi,(\varphi)}_{u} + \gamma}{\epsilon} \right] = \exp \left[ -\frac{I^{\psi,(\varphi)}_{u} + \gamma}{\epsilon} \right].
\]

Now for any $r > 0$ and $\delta > 0$,
\[
\mathbb{P} \left\{ \rho_T(\tilde{u}^{\varphi, n}, K_T(r)) > \delta \right\} \leq \mathbb{P} \left\{ \rho_T(\tilde{u}^{\varphi, n}, K_{[t_i, t_{i+1}]}(r)) > \delta \right\} \text{ for some } 0 \leq i \leq n-1 \\
\leq \exp \left[ -\frac{r - \gamma}{\epsilon} \right].
\]

The proof is complete.

The lower semicontinuous property of $I^{\psi}_{u}$ for any $\psi \in C(0, T; H)$ is needed in our approach. We have

**Lemma 16.** For any $T > 0$, $I^{\psi}_{u}(\varphi)$ is lower semicontinuous in both $\varphi$ and $\psi \in C(0, T; H)$.

**Proof.** The lower semicontinuous property in $\psi$ is followed by a similar discussion in the proof of second part of Theorem 13 by the fact that both $\tilde{f}(\psi)$ and $\sqrt{B(\psi)}$ are continuous in $\psi$ in space $C(0, T; H)$. And the lower semicontinuous property in $\varphi$ is followed by the same discussion for the usual evolutionary equation [11].

For every $E \subset C(0, T; H)$, denote by Int($E$) the interior of $E$ and Cl($E$) the closure of $E$. Then by the above result and the lower semicontinuity of $I^{\psi}_{u}(\varphi)$ in $\varphi$ we then have, for any step function $\psi \in C(0, T; H)$ in the form (29).

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Corollary 17. For any $E \subset C(0, T; H)$,
\[
-\inf_{\varphi \in \text{Int}(E)} \tilde{I}_u^\psi(\varphi) \leq \liminf_{\epsilon \to 0} \epsilon \ln \mathbb{P}\{u^{\epsilon, \psi} \in E\} \leq \limsup_{\epsilon \to 0} \epsilon \ln \mathbb{P}\{u^{\epsilon, \psi} \in E\} \leq -\inf_{\varphi \in \text{Cl}(E)} \tilde{I}_u^\psi(\varphi). \]

Now for our purpose we need the following result

Theorem 18. For any $h \in L^2(0, T; H)$, the skeleton equation (12) has a unique solution $\varphi \in C(0, T; H)$.

Proof. By the assumptions on $f$ both $\bar{f}(\varphi)$ and $\sqrt{\mathcal{B}(\varphi)}$ are Lipschitz continuous. Then the result follows by a standard discussion on deterministic pde [11].

Then we define the rate function $I_u(\varphi) = I_u^\psi(\varphi)$ for any $\varphi \in C(0, T; H)$. Furthermore, for any $\psi \in C(0, T; H)$, the following relation between rate functions $I_u^\psi$ and $I_u$ is needed to derive the ldp for $\{u^\epsilon\}_\epsilon$.

Lemma 19. Let $\psi^n$ be a family of step functions uniformly converging to $\varphi \in C(0, T; H)$ as $n \to \infty$. Then there is a family of functions $\varphi^n \in C(0, T; H)$ converging to $\varphi$, such that
\[
\limsup_{n \to \infty} I_u^{\psi^n}(\varphi^n) \leq I_u(\varphi). \]

Proof. Suppose $I_u(\varphi) < r < \infty$. Otherwise the result is clear.

By the definition of $I_u(\varphi)$, there exists a function $h$ and a sequence $h^n \in L^2(0, T; H)$ such that
\[
\varphi(t) = S(t)u_0 + \int_0^t S(t-s)\bar{f}(\varphi(s)) \, ds + \int_0^t \sqrt{B(\varphi(s))}h(s) \, ds \tag{30}
\]
and $h^n \to h$ in $L^2(0, T; H)$ with
\[
\frac{1}{2} \int_0^T \|h^n(s)\|^2 \, ds \leq I_u(\varphi) + \frac{1}{n}.
\]
Then define
\[
\varphi^n(t) = S(t)u_0 + \int_0^t S(t-s)\bar{f}(\psi^n(s)) \, ds + \int_0^t \sqrt{B(\psi^n(s))}h^n(s) \, ds.
\]
By the Lipschitz property of $f$, the definition of $\sqrt{\mathcal{B}(\psi)}$ and that $\psi^n \to \varphi$ in space $C(0, T; H)$ we have $\varphi^n \to \varphi$ in $C(0, T; H)$ as $n \to \infty$

and
\[
\limsup_{n \to \infty} I_u^{\psi^n}(\varphi^n) \leq I_u(\varphi). \]

This completes the proof. \qed

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5 LDP for slow-fast stochastic partial differential equations

Now we show that \( \{u^\varepsilon\}_\varepsilon \) satisfies the LDP with a good rate function \( I_u \). To do this we prove a special relationship between \( u^\varepsilon \) and \( \tilde{u}^\varepsilon,\psi \) in space \( C(0,T;H) \). The relationship shows that \( \tilde{u}^\varepsilon,\psi \) is comparable with \( u^\varepsilon \) near \( \psi \) and implies the LDP. Here we do not restrict \( \psi \in C(0,T;H) \) to be a step function.

For this let \( U^\varepsilon,\psi = u^\varepsilon - \tilde{u}^\varepsilon,\psi \), then

\[
\dot{U}^\varepsilon,\psi = AU^\varepsilon,\psi + f(u^\varepsilon, v^\varepsilon) - f(\psi, \tilde{v}^\varepsilon,\psi), \quad U^\varepsilon,\psi(0) = 0. \tag{31}
\]

In the mild sense

\[
U^\varepsilon,\psi(t) = \int_0^t S(t-s) \left[ f(u^\varepsilon(s), v^\varepsilon(s)) - f(\psi(s), \tilde{v}^\varepsilon,\psi(s)) \right] ds
= \int_0^t S(t-s) \left[ f(u^\varepsilon(s), v^\varepsilon(s)) - f(\psi(s), v^\varepsilon(s)) \right] ds
+ \int_0^t S(t-s) \left[ f(\psi(s), v^\varepsilon(s)) - f(\psi(s), \tilde{v}^\varepsilon,\psi(s)) \right] ds.
\]

By the assumptions on \( f \) and the analysis on fast motion \( \tilde{v}^\varepsilon,\psi \)

\[
\|f(u^\varepsilon, v^\varepsilon) - f(\psi, v^\varepsilon)\|_0 \leq L_f \|u^\varepsilon - \psi\|_0,
\]

\[
\|f(\psi, v^\varepsilon) - f(\psi, \tilde{v}^\varepsilon,\psi)\|_0 \leq L_f \|v^\varepsilon - \tilde{v}^\varepsilon,\psi\|_0 \leq L_f D_v \|u^\varepsilon - \psi\|_0.
\]

Then there is a positive constant \( C_T \) such that

\[
\|u^\varepsilon - \tilde{u}^\varepsilon,\psi\|_{C(0,T;H)} \leq C_T \|u^\varepsilon - \psi\|_{C(0,T;H)}. \tag{32}
\]

Rewrite (31) as

\[
\dot{U}^\varepsilon,\psi = AU^\varepsilon,\psi + f(\tilde{u}^\varepsilon,\psi, \tilde{v}^\varepsilon,\psi) - f(\psi, \tilde{v}^\varepsilon,\psi) + f(u^\varepsilon, v^\varepsilon) - f(\tilde{u}^\varepsilon,\psi, v^\varepsilon)
+ f(\tilde{u}^\varepsilon,\psi, v^\varepsilon) - f(\tilde{u}^\varepsilon,\psi, \tilde{v}^\varepsilon,\psi)
\]

Then by the Lipschitz property of \( f \) we have one positive constant \( L \) depending on \( L_f \) and \( D_v \) such that

\[
\frac{1}{2} \frac{d}{dt} \|U^\varepsilon,\psi\|_0^2 \leq L \|U^\varepsilon,\psi\|_0^2 + L_f (1 + D_v) \|\tilde{u}^\varepsilon,\psi - \psi\|_0^2
\]

which yields that

\[
\|\tilde{u}^\varepsilon,\psi - u^\varepsilon\|_{C(0,T;H)} \leq C_T \|\tilde{u}^\varepsilon,\psi - \psi\|_{C(0,T;H)} \tag{33}
\]

for some positive constant \( C_T \). Now by relations (32) and (33) we prove the main result.
Proof of Theorem 9. We follow Freidlin and Wentzell’s [20] approach to obtain LDP for slow-fast random ordinary differential equations. For any $\varphi \in C(0, T; H)$ with $I_u(\varphi) < \infty$ and any $\gamma, \delta > 0$, by Lemma 19 we can choose a step function $\psi^n$ and a function $\varphi^n$ such that

$$\rho_0 T(\varphi^n, \varphi) < \frac{1}{n}, \quad \max_{0 \leq t \leq T} \|\psi^n(t) - \varphi(t)\|_0 < \frac{1}{n} \quad \text{and} \quad I_{u_\psi}^{\psi^n} (\varphi^n) < I_u(\varphi) + \frac{1}{n}.$$ 

Now by (33)

$$\rho_0 T(u', \varphi) \leq \rho_0 T(\tilde{u}'_{\psi^n}, \varphi^n) + \rho_0 T(u', \tilde{u}'_{\psi^n}) + \rho_0 T(\varphi^n, \varphi)$$

$$\leq \rho_0 T(\tilde{u}'_{\psi^n}; \varphi^n) + C_T \rho_0 T(\tilde{u}'_{\psi^n}, \psi^n) + \rho_0 T(\varphi^n, \varphi)$$

$$\leq \rho_0 T(\tilde{u}'_{\psi^n}, \varphi^n) + C_T \rho_0 T(\tilde{u}'_{\psi^n}, \varphi^n)$$

$$+ C_T \max_{0 \leq t \leq T} \|\psi^n(t) - \varphi^n(t)\|_0 + \rho_0 T(\varphi^n, \varphi)$$

$$\leq (1 + C_T) \rho_0 T(\tilde{u}'_{\psi^n}, \varphi^n) + (2C_T + 1) \frac{1}{n}. $$

Then for any $\delta > 0$ and $\gamma > 0$, by Theorem 15 there is an $\epsilon_1 > 0$ such that for any $0 < \epsilon < \epsilon_1$ we have the following lower bound estimate by choosing $n$ large enough

$$\mathbb{P}\{ \rho_0 T(u', \varphi) \leq \delta \}$$

$$\geq \mathbb{P}\{ (1 + C_T) \rho_0 T(\tilde{u}'_{\psi^n}, \varphi^n) \leq \delta/2 \}$$

$$\geq \exp \left\{ - \frac{I_{u_\psi}^{\psi^n}(\varphi^n) + \gamma}{\epsilon} \right\} \geq \exp \left\{ - \frac{I_u(\varphi) + 2\gamma}{\epsilon} \right\} .$$

Now we prove the upper bound estimate. First, by the exponential tightness of $\{u'\}_\epsilon$ in space $C(0, T; H)$, for any $r > 0$ there is a compact set $K_r$ such that

$$\limsup_{\epsilon \to 0} \epsilon \ln \mathbb{P}\{ u' \in K_r^\epsilon \} \leq -r .$$

Then for any $\gamma > 0$ there is an $\epsilon_2 > 0$ such that for any $0 < \epsilon < \epsilon_2$

$$\mathbb{P}\{ u' \in K_r^\epsilon \} \leq \exp \left\{ - \frac{r - \gamma}{\epsilon} \right\} .$$

As $K_r$ is compact, choose a finite $\delta'$-net in $K_r$ with $\delta' < \delta$ and let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be the elements of this net, not belonging to $K_T(r)$. Then

$$\mathbb{P}\{ \rho_0 T(u', K_T(r)) > \delta \} \leq \sum_{i=1}^n \mathbb{P}\{ \rho_0 T(u', \varphi_i) < \delta' \} + \mathbb{P}\{ u' \in K_r^\epsilon \} .$$

20
Now we choose step functions $\psi_1, \psi_2, \ldots, \psi_n$ such that

$$\rho_{0T}(\psi_i, \varphi_i) < \delta', \quad i = 1, 2, \ldots, n.$$ 

Then by the inequality (33) for $i = 1, 2, \ldots, n$

$$\mathbb{P} \{ \rho_{0T}(u^\epsilon, \varphi_i) < \delta' \} \leq \mathbb{P} \{ \rho_{0T}(\tilde{u}^{\epsilon, \psi_i}, \varphi_i) < 2(C_T + 1)\delta' \}.$$

And by Corollary [17] for $i = 1, 2, \ldots, n$ we have

$$\mathbb{P} \{ \rho_{0T}(\tilde{u}^{\epsilon, \psi_i}, \varphi_i) < 2(C_T + 1)\delta' \} \leq \exp \left\{ -\frac{1}{\epsilon} \inf \{ I_{\psi_i}(\varphi): \rho_{0T}(\varphi, \varphi_i) < 2(C_T + 1)\delta' \} - \gamma \right\}. $$

By the semicontinuity of the functional $I_{\psi_i}^u$ in $\psi$, Lemma [16], that for any $\gamma > 0$, there is $\delta'$ such that

$$I_{\psi_i}^u(\varphi) > r - \gamma/2 \quad \text{for} \quad \rho_{0T}(\varphi, \varphi_i) < 2(C_T + 1)\delta' \quad \text{and} \quad I_{\psi_i}^u(\varphi_i) > r.$$ 

Then by the choice of $\varphi_i \notin K_T(r)$ we have

$$\mathbb{P} \{ \rho_{0T}(\tilde{u}^{\epsilon, \psi_i}, \varphi_i) < 2(C_T + 1)\delta' \} \leq \exp \left\{ -\frac{r - \gamma}{\epsilon} \right\}$$

for $\epsilon$ small enough. The proof is complete. 

\[ \square \]

6 An example of slow-fast stochastic reaction-diffusion equation

Next we consider the following slow-fast SPDE on the domain $(0, L)$ with zero Dirichlet boundary condition

\begin{align*}
\partial_t u^\epsilon &= \partial_{xx} u^\epsilon + \lambda \sin u^\epsilon - v^\epsilon, \quad u^\epsilon(0) = u_0, \quad (34) \\
\epsilon \partial_t v^\epsilon &= \partial_{xx} v^\epsilon - v^\epsilon + u^\epsilon + \sqrt{\epsilon} \sigma \partial_t W(t), \quad v^\epsilon(0) = v_0 \quad (35)
\end{align*}

where $W$ is $L^2(0, L)$-valued $Q$-Wiener process and $\lambda, \sigma > 0$ are constants. As usual, the small parameter $\epsilon$ measures the separation of time scales between the fast modes $v$ and the slow modes $u$. For small $\epsilon$ our LDP theory gives the SPDE (37) as an appropriate (weak) model for the stochastic dynamics of the slow modes $u$.

Apply the LDP theory to this system. Note that the nonlinear reaction/interaction function $f(u, v) = \lambda \sin u - v$ is Lipschitz continuous. Denote the
operator $A = \partial_{xx}$ with zero boundary condition on $(0, L)$. Now for fixed $u$ the fast system (35) has a unique stationary solution $\eta^{u}$ with distribution

$$
\mu_u = \mathcal{N}\left((I - A)^{-1}u, \sigma^2 \frac{(I - A)^{-1}Q}{2}\right).
$$

Then

$$
\tilde{f}(u) = \lambda \sin u - (I - A)^{-1}u.
$$

Let $\eta^{u}$ be the stationary solution of

$$
\partial_t v = \partial_{xx} v - v + u + \sigma \partial_t W(t)
$$

for any fixed $u \in L^2(0, L)$. Then $\eta^{u}$ distributes as $\mu_u$ and

$$
B(u) = 2E \int_0^\infty [\eta^{u}(t) - (I - A)^{-1}u] \otimes [\eta^{u}(0) - (I - A)^{-1}u] dt.
$$

Noticing that

$$
E\eta^{u}(t) \otimes \eta^{u}(0) = \sigma^2 \exp\left\{- (I - A)t\right\} \left[\frac{(I - A)^{-1}Q}{2}\right] + (I - A)^{-2}u \otimes u
$$

we then have

$$
\sqrt{B(u)} = (I - A)^{-1}\sigma \sqrt{Q}
$$

which is independent of $u$ and satisfies assumption $\mathbf{H}_5$. By Theorem 9, $\{u^\epsilon\}$ satisfies LDP on space $C(0, T; H)$ with good rate function

$$
I_u(\varphi) = \inf_{h \in L^2(0, T; H)} \left\{ \frac{1}{2} \int_0^T \|h(s)\|^2_0 ds : \varphi = \varphi^h \right\}
$$

with $\varphi^h$ solving

$$
\dot{\varphi} = A\varphi^h + \lambda \sin \varphi - (I - A)^{-1}\varphi + (I - A)^{-1}\sigma \sqrt{Q}h, \quad \varphi(0) = u_0.
$$

Furthermore, the rate function is

$$
I_u(\varphi) = \frac{1}{2} \int_0^T \left\| \frac{I - A}{\sqrt{Q}} \left[ \dot{\varphi}(s) - A\varphi(s) - \lambda \sin \varphi + (I - A)^{-1}\varphi \right] \right\|^2_0 ds \quad (36)
$$

for $\varphi$ is absolute continuous. Otherwise $I_u(\varphi) = \infty$.

Now we write out the averaged equation plus the deviation for (34)–(35) as

$$
d\bar{u}^\epsilon = [A\bar{u}^\epsilon + \lambda \sin \bar{u}^\epsilon - (I - A)^{-1}\bar{u}^\epsilon] dt + \sqrt{\epsilon} \sigma (I - A)^{-1} \sqrt{Q}dW(t) \quad (37)
$$
Then by the LDP for stochastic evolutionary equation \[27\], \( \{\tilde{u}^\epsilon\} \) satisfies LDP with rate function \( I_u(\varphi) \) defined in \(36\). This shows that the averaged equation plus deviation \(37\) does describe the metastability of \( \{u^\epsilon\} \), solving \(34\) for small \( \epsilon \). Moreover, for large enough parameter \( \lambda \), the SPDE model \(37\) has two stable states near zero for \( \epsilon = 0 \). When \( \epsilon \neq 0 \), noise causes orbits near one stable state to the position near the other one which shows the metastability of the system \(37\). So the slow-fast stochastic \(34\)–\(35\) also has such metastability described by system \(37\). The description of such tunnelling of the orbit needs detail analysis by the LDP which is left for future work.

7 Stochastic centre manifold models confirm the LDP example

This section uses the example slow-fast SPDE \(34\)–\(35\) to verify the LDP theory in a significant parameter regime. Without loss of generality set the non-dimensional domain length \( L = \pi \). In the absence of noise the SPDE \(34\)–\(35\) undergoes a deterministic pitchfork bifurcation from the trivial field \( u = v = 0 \), as parameter \( \lambda \) crosses the critical value 3/2, to two nontrivial fixed points \( u \approx \sqrt{\lambda - 3/2} \sin x \). Consequently, with noise, a stochastic pitchfork bifurcation takes place in the vicinity of parameter \( \lambda \approx 3/2 \) \[3 \[35\] \[4\] e.g.\]. As shown schematically in Figure 1 here we establish that the stochastic bifurcation dynamics of the original slow-fast SPDE \(34\)–\(35\) and that of the LDP averaged SPDE \(37\) are identical to the expected \( \mathcal{O}(\epsilon) \) error.

As indicated in Figure 1 we make a wide ranging comparison of the dynamics by constructing and comparing the stochastic centre manifolds, and the evolution thereon, of the two SPDE systems near this stochastic bifurca-
tion. We explore dynamics in the vicinity of the stochastic bifurcation by setting the parameter $\lambda = \frac{3}{2} + \lambda'$ for any small enough bifurcation parameter $\lambda'$. For small $\lambda'$ and small $\epsilon$ there are three time scales in the example fast-slow SPDE (34)–(35): the $v$ modes quasi-equilibrate on the fast time scale of $O(\epsilon)$; almost all of the $u$ modes evolve on the slow time scale of $O(1)$; but the $\sin x$ mode in $u$ evolves on the superslow, long time scale of $O(1/\lambda')$. The LDP averaged SPDE has just the latter two time scales in this parameter regime. The interactions among three time scales is a major complicating factor in constructing the stochastic centre manifold and the evolution thereon. Because of the three time scales, and to be consistent with the terminology of earlier sections, we henceforth refer to the stochastic superslow manifold as it is the evolution on the superslow, long time scales of $O(1/\lambda')$ that we encompass and compare in this section.

The stochastic superslow manifold (SSM) is based from the linear dynamics exactly at critical and, for simplicity, based from no noise [29, 1, e.g.]. Exactly at critical, and with no noise, $\lambda' = \sigma = 0$, both the SPDEs (34)–(35) and SPDE (37), have centre subspaces about the origin: $u = a \sin x$, $v = \frac{1}{2} a \sin x$ and $\bar{u} = \bar{a} \sin x$, respectively. Stochastic centre manifold theory [5, 1, e.g.] then asserts that in a domain of small but finite amplitudes $a$ and $\bar{a}$, and small but finite noise $\sigma$ and small but finite parameter $\lambda'$, there exists an emergent SSM: solutions are attracted to the SSM roughly as $\exp(-\frac{27}{10} t)$, and then evolve on the superslow long time scale. We compare the slow-fast SPDE (34)–(35) with the LDP averaged SPDE (37) via construction of their SSM models.

Constructing SSMs and the evolution thereon has many technical challenges reported in detail elsewhere [12, 29, 31, 40, e.g.]. The technicalities are even more challenging here due to the three time scales in the slow-fast SPDE (34)–(35) when, as we assume, the parameter $\epsilon$ is small. The construction procedure used herein is detailed in a separate technical report [30] that all can check, reproduce and perhaps modify to other problems in the same class. For our purposes we appeal to a little more of the theory of stochastic centre manifolds: Arnold [1], building on the work of Boxler [6], assures us that if the SPDEs are satisfied to some order of residual in the small parameters, then the SSMs and the evolution thereon are constructed to the same order of error. Thus one may confirm the veracity of the following SSMs by substituting the expressions into the SPDEs and verifying that the residuals are as asymptotically small as required.

To reduce complicating detail but retain significant information in the example, we truncate the noise to its first three spatial modes: $W = \varphi_1 \sin x + \varphi_2 \sin 2x + \varphi_3 \sin 3x$ where $\varphi_i$ denote formal derivatives of independent Wiener processes. Including more noise modes appears to just greatly increase detail,
without adding any significant change to the nature of the interactions seen among these three modes.

**Slow-fast SPDE** \([34]-[35]\) In six iterations, computer algebra \([30]\) constructs the stochastic superslow manifold model. In terms of the superslow evolving amplitude \(a(t)\), where \(u \approx a \sin x\) and \(v \approx \frac{1}{2} a \sin x\), the stochastic bifurcation SDE for the amplitude is

\[
\dot{a} = \lambda' (1 + \frac{1}{4} \epsilon) a - \left( \frac{3}{16} + \frac{1}{8} \lambda' + \frac{3}{64} \epsilon \right) a^3 + \frac{91}{9728} a^5 \\
- \sqrt{\epsilon} \sigma \left[ \left( \frac{1}{2} + \frac{1}{8} \epsilon \right) \varphi_1 + \frac{3}{1216} a^2 \varphi_3 \right] \\
+ \epsilon \sigma^2 a \left[ -\frac{1}{180} \varphi_2 e^{-\frac{27}{10} \epsilon t} \star \varphi_2 + \frac{3}{1216} \varphi_1 e^{-\frac{35}{3} \epsilon t} \star \varphi_3 - \frac{3}{6080} \varphi_3 e^{-\frac{35}{3} \epsilon t} \star \varphi_3 \right] \\
+ \mathcal{O}(a^6 + \lambda^2 + \epsilon^3 + \sigma^6) \tag{38}
\]

In this and other expressions, convolutions \(e^{-\alpha t} \star \varphi = \int_{0}^{\infty} e^{-\alpha s} \varphi(t-s) \, ds\).

The corresponding SSM has slow field \(u = a \sin x + \frac{5}{608} a^3 \sin 3x + \frac{1}{2} \sqrt{\epsilon} \sigma \sin x \, e^{-\frac{2}{\epsilon} t} \star \varphi_1 \)

\[
- \sqrt{\epsilon} \sigma \left[ \frac{1}{5} \sin 2x \left( e^{-\frac{27}{10} \epsilon t} \star -e^{-\frac{5}{2} \epsilon t} \star \right) \varphi_2 + \frac{1}{10} \sin 3x \left( e^{-\frac{35}{3} \epsilon t} \star -e^{-\frac{10}{3} \epsilon t} \star \right) \varphi_3 \right] \\
+ \mathcal{O}(a^4 + \lambda^2 + \epsilon^2 + \sigma^4), \tag{39}
\]

and the fast field

\[
v = \frac{1}{2} a \sin x + \frac{1}{1216} a^3 \sin 3x \\
+ \frac{\sigma}{\sqrt{\epsilon}} \sin x \left[ \left( 1 + \frac{1}{2} \epsilon \right) e^{-\frac{2}{\epsilon} t} \star + \frac{1}{2} e^{-\frac{2}{\epsilon} t} \star e^{-\frac{2}{\epsilon} t} \star \right] \varphi_1 \\
+ \frac{\sigma}{\sqrt{\epsilon}} \sin 2x \left[ \left( 1 + \frac{1}{25} \epsilon \right) e^{-\frac{5}{2} \epsilon t} \star - \frac{1}{25} e^{-\frac{27}{10} \epsilon t} \star + \frac{1}{5} e^{-\frac{5}{2} \epsilon t} \star e^{-\frac{2}{\epsilon} t} \star \right] \varphi_2 \\
+ \frac{\sigma}{\sqrt{\epsilon}} \sin 3x \left[ \left( 1 + \frac{1}{100} \epsilon \right) e^{-\frac{10}{3} \epsilon t} \star - \frac{1}{100} e^{-\frac{35}{3} \epsilon t} \star e^{-\frac{10}{3} \epsilon t} \star \right] \varphi_3 \\
+ \mathcal{O}(a^4 + \lambda^2 + \epsilon^2 + \sigma^4). \tag{40}
\]

This fast field \(v\) has large \(\mathcal{O}(1)\) fluctuations, through terms like \(\frac{1}{\sqrt{\epsilon}} e^{-\frac{2}{\epsilon} t} \ast\), because convolution over the fast time scale, \(e^{-\beta t/\epsilon} \ast\), is \(\mathcal{O}(\sqrt{\epsilon})\). However, note that repeated such convolution, \(e^{-\beta t/\epsilon} \ast e^{-\beta t/\epsilon} \ast\), is \(\mathcal{O}(\epsilon^{3/2})\) \([28]\) equation (27)].
LDP averaged SPDE (3.7) In just four iterations, computer algebra \[30\] constructs the stochastic superslow evolution to be

\[
\dot{\bar{a}} = \lambda' \bar{a} - \left( \frac{a}{16} + \frac{1}{8} \lambda' \right) \bar{a}^3 + \frac{91}{9728} \bar{a}^5
- \sqrt{\epsilon} \sigma \left[ \frac{1}{2} \varphi_1 + \frac{3}{1216} \bar{a}^2 \varphi_3 \right]
+ \epsilon \sigma^2 \bar{a} \left[ -\frac{1}{130} \varphi_2 e^{-\frac{27}{10} t} \ast \varphi_2 + \frac{3}{1216} \varphi_1 e^{-\frac{38}{5} t} \ast \varphi_3 - \frac{3}{6080} \varphi_3 e^{-\frac{38}{5} t} \ast \varphi_3 \right]
+ \mathcal{O} \left( \bar{a}^6 + \lambda'^2 + \epsilon^2 + \sigma^6 \right).
\]

The corresponding SSM is

\[
\ddot{u} = a \sin x + \frac{5}{608} \bar{a}^3 \sin 3x
- \sqrt{\epsilon} \sigma \left[ \frac{1}{5} \sin 2x e^{-\frac{27}{10} t} \ast \varphi_2 + \frac{1}{10} \sin 3x e^{-\frac{38}{5} t} \ast \varphi_3 \right]
+ \mathcal{O} \left( \bar{a}^4 + \lambda'^2 + \epsilon^2 + \sigma^4 \right).
\]

**Compare the two superslow models** First compare the slow field \(u\) for the slow-fast SPDE, (3.8), with the slow field for the LDP averaged SPDE, (4.2). The differences are the \(\mathcal{O}(\epsilon)\) terms in the fast time convolutions \(\sqrt{\epsilon} e^{-\beta t/\epsilon} \ast \cdot\). Since \(\ddot{u}\) is \(u\) averaged over fast fluctuations, these differences are acceptable, and also ensure that the two amplitudes correspond: \(\ddot{a} = a + \mathcal{O}(\epsilon)\).

Second, compare the evolution of the amplitudes, (3.8) and (4.1). The only differences are in terms \(\mathcal{O}(\epsilon)\), as indicated schematically on the right-hand side of Figure [1]. Thus the dynamics of the two superslow SDE are within the claimed accuracy of the LDP. We conclude that this section verifies that in a parameter regime at least near the stochastic bifurcation, the LDP averaging approximation is correct. Although the LDP averaging only assures us that the slow model is correct in a weak sense, the strong identity between the convolutions appearing in the two SSMs, (3.8)–(3.9) and (4.1)–(4.2), suggests the correspondence between the LDP averaged system and the original system is generally stronger.

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