No Inner-Horizon Theorem for Black Holes with Charged Scalar Hair

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We establish a no inner-horizon theorem for black holes with charged scalar hair. Considering a general gravitational theory with a charged scalar field, we prove that there exists no inner Cauchy horizon for both spherical and planar black holes with non-trivial scalar hair. The hairy black holes approach a spacelike singularity at late interior time. This result is independent of the form of scalar potentials as well as the UV completion of spacetime. For a large class of solutions, we show that the geometry near the singularity resembles the Kasner type solution. When the potentials become important, the behaviors that are quite distinct from the Kasner form are observed. For the hyperbolic horizon case, we show that it only has at most one inner horizon. All these features are also valid for the Einstein gravity coupled with a neutral scalar.

As one of the most fundamental objects among all gravitational objects, black holes play a central role in understanding the nature of gravity. The development of black hole physics has uncovered a deep and intrinsic relationship between gravitation, thermodynamics and quantum theory and has provided most of our present physical understandings of the quantum phenomena in strong gravity regime. In recent years there has been dramatic progress in understanding black hole physics both from theoretical and experimental approaches. In particular, thanks to the innovation and progress of observation techniques, we are able to directly detect the gravitational waves from a binary black hole coalescence [1] and to take a photo of the shadow of a black hole [2, 3], opening a new window in the study of gravity, astrophysics and cosmology.

While the exterior physics of black holes has been extensively investigated in the literature, in particular, the establish of black hole thermodynamics and uniqueness theorems (see e.g. Refs. [4] and [5], for reviews), the interior structure of black holes behind the event horizon has not been well understood. Nevertheless, exploring the internal structure of black holes and spacetime singularities inside are intriguing and fundamental topics in general relativity and can provide a better understanding of black hole physics, gravitation and quantum physics. For example, the existence of the inner (Cauchy) horizon violates the predictability in general relativity and motivates the strong cosmic censorship (SCC) conjecture (see e.g. Refs. [6, 7]). Recent progress suggested that the black hole information paradox could be solved by including “island” that lies in the interior of black hole [8, 9].

The most studied theory in general relativity involves a Maxwell field, for which Schwarzschild, Reissner-Nordström (RN) and Kerr-Newmann black holes are three well-known solutions. The neutral Schwarzschild black hole has an event horizon and a spacelike singularity inside, while the later two have one additional Cauchy horizon that appears to violate SCC and a timelike singularity at the center, due to the presence of non-trivial electric charge and angular momentum. On the other hand, scalar fields should be one of the simplest types of “matter” considered in the literature and plays important role in particle physics, cosmology and gravitational physics. Due to their simplicity, it is quite natural to consider scalar fields when testing some no-go ideas as a first step. Recently, it has been argued that SCC can be violated by turning on linear scalar field perturbations of RN black holes in de Sitter (dS) space [10].

One anticipates that the black hole interior would be dramatically affected in presence of scalar hair. Indeed, it has been recently shown that [11] there is no inner Cauchy horizon for some kind of charged black holes with a neutral scalar. However, the result relies on a strong requirement for the scalar potential (the scalar mass-square should be negative) and breaks down for charged scalar case (the absence of inner horizon for planar black holes with charged scalar was discussed in Ref. [12] more recently, see Note added also). In this Letter we will establish a stronger no inner-horizon result for both the Einstein-scalar and the Einstein-Maxwell-charged scalar theories. Our results are quite generic and are independent of the form of scalar potentials as well as the UV completion of spacetime. We will also discuss the asymptotical solutions near the singularity. In addition to the Kasner form of solutions for which the kinetic terms dominate the dynamics, we will show numerical evidence for the existence of novel oscillating behaviors all the way down to the singularity when the potential terms become important to the geometry.

The model.—We consider a $(d + 2)$-dimensional theory with gravity coupled with a Maxwell field $A_\mu$ and a charged scalar field $\Psi$:

\begin{equation}
S = \frac{1}{2\kappa_N^2} \int d^{d+2}x \sqrt{-g} \left[ R + \mathcal{L}_M \right],
\end{equation}

\begin{equation}
\mathcal{L}_M = -\frac{Z(|\Psi|^2)}{4} F_{\mu\nu} F^{\mu\nu} - (D_\mu \Psi)^* D^\mu \Psi - V(|\Psi|^2),
\end{equation}
where \( F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} \) and \( D_{\mu} = \nabla_{\mu} - iq A_{\mu} \) with \( q \) the charge of the scalar field. \( Z \) and \( V \) are arbitrary smooth functions of \( |\Psi|^2 \). One only demands \( Z \) to be positive to ensure positivity of the kinetic term for \( A_{\mu} \), and take \( Z(0) = 1 \) without loss of generality. The Einstein-scalar theory is obtained by turning off \( A_{\mu} \) and most of our discussion below will apply to black holes in the Einstein-scalar case [13]. Note that the spacetime can be asymptotically flat, anti-de Sitter (AdS), dS or other geometries, depending on the choice of the scalar potential \( V \) as well as the coupling \( Z \).

We are interested in static charged black holes that are homogeneous and isotropic, so the ansatz for metric and matter fields can be written as

\[
ds^2 = \frac{1}{z^2} \left[ -f(z)e^{-\chi(z)}dt^2 + \frac{dz^2}{f(z)} + d\Sigma_{d,k}^2 \right],
\]

where the \( d \)-dimensional line element \( d\Sigma_{d,k}^2 \) is

\[
d\Sigma_{d,k}^2 = \begin{cases} 
  d\theta^2 + \sin^2 \theta d\Omega_{d-1}^2, & k = 1, \\
  \sum_{i=1}^{d} dx_i^2, & k = 0, \\
  d\theta^2 + \sinh^2 \theta d\Omega_{d-1}^2, & k = -1,
\end{cases}
\]

with \( d\Omega_{d-1}^2 \) the line element of \((d-1)\)-dimensional unit sphere. We assume that the black hole boundary is at \( z = 0 \) and the singularity at \( z \to \infty \), but the precise location of the boundary and singularity is not important in our discussion. For black hole spacetimes obeying the dominant energy condition only the spherical horizon case is allowed [21]. The well known case that breaks the dominant energy condition only the spherical horizon is not important in our discussion. For black hole spacetimes obeying the dominant energy condition only the spherical horizon case is allowed [21]. The well known case that breaks the dominant energy condition is to introduce a negative cosmological constant, for which the topology of a black hole horizon can be flat or hyperbolic. In particular, the planar case has been widely investigated in the application of holographic duality to strongly coupled systems [22–25].

The equations of motion are given by

\[
z^{d+2}e^{\chi/2}(e^{-\chi/2}z^{-d}f\psi')' = \left[ V_{\text{eff}}(\psi^2) - \frac{q^2 z^2 e^\chi A_t^2}{f} \right] \psi,
\]

\[
z^d [Z(\psi^2)e^{\chi/2}z^{-d-2}A_t'] = 2\frac{V_{\text{eff}}(\psi^2) - A_t}{f},
\]

\[
\frac{d}{dz} \chi = z\psi^2 + \frac{ze^\chi q^2 \psi^2 A_t^2}{f^2},
\]

\[
\frac{d}{dz} f' - \frac{z}{f} \psi^2 - \frac{d(d+1)}{2z} = \frac{V_{\text{eff}}(\psi^2)}{2zf} - \frac{k(d(d-1))z}{2f}
\]

\[
+ \frac{ze^\chi q^2 \psi A_t^2}{2f^2} + \frac{Z(\psi^2)z^3e^\chi A_t^2}{2f},
\]

where a prime denotes the derivative with respect to \( z \) and the effective potential \( V_{\text{eff}}(x) = V(x) - \frac{1}{2} Z(x) z^4 e^\chi A_t^2 \) with \( \dot{V}_{\text{eff}}(x) = dV_{\text{eff}}(x)/dx \). Without loss of generality, we have chosen \( \psi(z) \) to be real. In general, the above coupled equations do not have analytical solutions, so one has to solve the system numerically.

**Proof of no inner-horizon.**— Suppose that there were two horizons, including the event horizon at \( z_H \) and the inner horizon at \( z_I \), for which \( f(z_H) = f(z_I) = 0 \) with \( z_H < z_I \). In the present study we consider black holes with finite temperature. The structure of the black hole is shown schematically in Fig. 1. The blackening function \( f(z) \) turns from positive to negative towards the interior near \( z_H \), while it from negative to positive near the inner horizon \( z_I \) (red curve) or \( z_I \) is a local maximum (green curve). Therefore, one has

\[
f'(z_H) < 0, \quad f'(z_I) \geq 0.
\]

Furthermore, to have a regular horizon, the metric and matter fields should be sufficiently smooth near the horizon. For the black hole solution with non-trivial charged scalar \( \psi \), the equations of motion imply the condition:

\[
A_t(z_H) = A_t(z_I) = 0,
\]

with both \( \psi \) and \( \chi \) finite at two horizons.

Before going to prove the no inner-horizon theorem, we briefly show why the discussion of Ref. [11] does not work for the charged scalar case. Following Ref. [11], we obtain from Eq. (4) that

\[
0 = \int_{z_H}^{z_I} \frac{f e^{x/2} \psi \psi'}{z^d} \left[ V_{\text{eff}}(\psi^2) + 2zf \psi^2 - \frac{q^2 z^2 e^\chi A_t^2 \psi^2}{f} \right].
\]

For the neutral case with \( q = 0 \), the integrand in the second line is non-positive over the range \((z_H, z_I)\) provided \( V_{\text{eff}} < 0 \) [26]. Therefore, the only way for two horizons is for \( \psi = 0 \). However, for the charged scalar case with non-zero \( q \), there is an additional contribution which is positive. So we cannot rule out the existence of inner horizon even when \( V_{\text{eff}} < 0 \). Nevertheless, we will show that there is no inner horizon for a charged black hole.
with the curvature of its horizon be non-negative. The hyperbolic case is a bit complicated, but we are able to show that it has at most one inner horizon with non-vanishing surface gravity.

Our key observation for the background (2) is the existence of the conserved quantity:

$$Q(z) = z^{2-d}e^{\chi/2} [z^{-2}(f e^{-\chi})' - Z A_I A'_I] + 2k(d-1) \int_{z_l}^{z} y^{-d}e^{-\chi(y)/2}dy.$$  \hspace{1cm} (11)

Making use of Eqs. (4)-(7), one can verify that $Q$ is radially independent, i.e. $Q'(z) = 0$. For the planar case with $k = 0$, $Q$ was constructed in the literature (see e.g. Refs. [27, 28]) and has been recently used to demonstrate the absence of inner horizon in the minimal model of holographic superconductor [12]. The form of $Q$ with $k = 0$ is due to a particular scaling symmetry that is only valid for the planar topology [27]. Intriguingly, we manage to construct a radially conserved $Q$ in Eq. (11) for non-planar cases even the scaling symmetry breaks down.

Evaluating $Q$ both at the event and inner horizons, we obtain

$$Q(z_j) = \frac{f'(z_j)}{z_j^2}e^{-\chi(z_j)/2} + 2k(d-1) \int_{z_l}^{z_j} y^{-d}e^{-\chi(y)/2}dy,$$

where the subscript $j = (H, I)$ and we have used Eq. (9). Since $Q(z_H) = Q(z_I)$, we have

$$Q'(z_j) = \frac{f''(z_j)}{z_j^2}e^{-\chi(z_j)/2} - \frac{f'(z_j)}{z_j^2}e^{-\chi(z_I)/2} = 2k(d-1) \int_{z_H}^{z_I} y^{-d}e^{-\chi(y)/2}dy.$$

It is obvious that the left hand side is negative because of Eq. (8). For black holes with spherical ($k = 1$) and planar ($k = 0$) topologies, the right hand side of Eq. (13) is non-negative. Therefore, smooth inner Cauchy horizon is never able to form for spherical and planar black holes with charged scalar hair.

For the hyperbolic case with $k = -1$, since both sides of Eq. (13) share the same sign, it is possible to develop an inner horizon, provided Eqs. (10) and (13) are satisfied. A concrete example for the hyperbolic black hole with an inner horizon is presented in Supplemental Material [29]. Nevertheless, we can show that for the hyperbolic case, there only exists at most one inner horizon with nonzero surface gravity. The proof contains two steps. We first prove the horizon $z_I$ must be a single root. Otherwise, we must have $f''(z_I) \leq 0$. Computing $Q'(z_I)$, we find

$$Q'(z_I) = \frac{f''(z_I)}{z_I^2}e^{-\chi(z_I)/2} - Z z_I^{-2} e^{\chi(z_I)/2} A'_I(z_I)^2 - 2(d-1) z_I^{-d} e^{-\chi(z_I)/2} < 0.$$  \hspace{1cm} (14)

This is contradictory to the fact that $Q'(z) = 0$. Thus, we find $z_I$ should be a single root with $f'(z_I) > 0$. Secondly, suppose there is a second inner horizon appearing at $z = z_{II} > z_I$. It is then obvious that $f'(z_{II}) \leq 0$ with $f(z_{II}) = 0$. Using similar discussion, we obtain

$$\frac{f'(z_I)}{z_I^2}e^{-\chi(z_I)/2} - \frac{f'(z_{II})}{z_{II}^2}e^{-\chi(z_{II})/2} = -2(d-1) \int_{z_I}^{z_{II}} y^{-d}e^{-\chi(y)/2}dy,$$

for $k = -1$. While the right hand side is negative, the left hand side is positive, since $f'(z_I) > 0$ and $f'(z_{II}) \leq 0$. Thus, the second inner horizon does not exist.

**Geometry near singularity.**—After knowing the inner structure behind the event horizon, we are now interested in the dynamics near the singularity. Since there involves the dynamics in nonlinear regimes, the behavior should be in general sensitive to the details of the model. More recently, a simple model in AdS spacetime with the planar topology was discussed in Ref. [12] where various dynamics was observed and was found to be very sensitive to the temperature and model parameters. Instead of dealing with specific models, we aim to provide some generic features of the geometry near the singularity. Currently, we assume that the singularity only appears at $z \to \infty$.

To characterize the charge degrees of freedom behind the surface generating a nonzero electric flux in the deep interior, we introduce [30]

$$Q(z) = \frac{1}{2\kappa_N^2} \int_{\Sigma} Z \cdot F = -\frac{1}{2\kappa_N^2} \int_{\Sigma} Z^2 - d e^{\chi/2} A'_I,$$

where $\omega(d)$ is the volume of the section with $t$ and $z$ fixed and we have used the ansatz of Eq. (2). For the hyperbolic case it is possible to have an inner horizon (see Fig. 1). We find that this timelike singularity of a charged hyperbolic black hole always carries charge, i.e. $Q(z \to \infty) \neq 0$. More precisely, one can prove that behind the inner horizon $z_I$, $Q(z^2)$ is monotonically increasing towards the singularity (see Supplemental Material [29]).

For other cases we have shown that there exists no inner horizon and the spacetime ends at a spacelike singularity. The geometry near the singularity depends on the details of a model one considers. We shall specify $Z = 1$ to simplify the discussion. For the model in which the kinetic term of scalar dominates the dynamics, we can neglect the potential $V(\psi^2)$. Then, solving for Eqs. (4)-(7) yields [29]

$$\psi = \sqrt{\omega} \ln z + \cdots, \quad A'_I = E_z z^{d-2-\alpha^2} + \cdots, \quad e^{\chi} = \chi z^{2\alpha^2} + \cdots, \quad f = -f_z z^{1+d+\alpha^2} + \cdots,$$

as $z \to \infty$, with $(\alpha, E_z, \chi_z, f_z)$ constants. One can check that $Q(z \to \infty)$ approaches a constant, so the space-
like singularity in the present case carries a finite charge. Changing the $z$ coordinate to the proper time $\tau$ via $\tau \sim z^{-(1+d+\alpha^2)/2}$, we then obtain

$$ds^2 = -d\tau^2 + c_1 \tau^{2p_1} dt^2 + c_2 \tau^{2p_2} \cdot d\Sigma_{d,k}^2,$$

$$\psi(z) = -p_\psi \ln \tau,$$  \hspace{1cm} (18)

where

$$p_\psi = \frac{1 - d + \alpha^2}{1 + d + \alpha^2}, \quad p_s = \frac{2}{1 + d + \alpha^2}, \quad p_\psi = \frac{2\sqrt{d} \alpha}{1 + d + \alpha^2}.$$  \hspace{1cm} (19)

One immediately finds that

$$p_\psi + dp_s = 1, \quad p_\psi^2 + dp_s^2 + p_\psi^2 = 1,$$

and therefore the geometry has the Kasner form $[31, 32]$. The case for $d = 2$ was recently discussed in Ref. [12].

As we have emphasized, for above discussion to be consistent, one should require the kinetic term of scalar to be dominant. In particular, one should at least has the following constraint:

$$\lim_{z \to \infty} \frac{|V(z^2)|}{z^{d+1+\alpha^2}} \ll 1,$$  \hspace{1cm} (21)

which allows the scalar potential $V$ to be arbitrary algebraic functions, including polynomial functions. For example, the model in Ref. [12] considered $V = m^2 \psi^2$ and the singularity was found to be Kasner form (18). However, if the scalar potential becomes important to the singularity geometry, the Kasner form will not solve the system and the asymptotic behavior around the singularity would depend on the detailed form of potentials.

We consider four different scalar potentials and present the numerical results in Fig. 2 [33]. To present our numerical data, we introduce $R_1 = z \psi', R_2 = \ln \left( \frac{z^2}{z^2 - h} \right), R_3 = 4z^{2-d} e^{\chi/2} A'_i$, with $h = e^{-\chi/2} f / z^{1+d}$. For the Kasner solution, one has $\lim_{z \to \infty} R_i(z) = \text{const.}$ with $i = 1, 2, 3$. When the kinetic term dominates the dynamics, the numerical behavior satisfies the universal asymptotic form (17) [Figs. 2(a) and 2(b)]. In contrast, when the condition is not satisfied, in particular, Eq. (21) is violated, numerical results exhibit behaviors that are quite distinct from the Kasner form. For potentials with exponential forms, we observe behaviors with strong oscillations all the way down to the singularity [Figs. 2(c) and 2(d)]. Numerical details and more examples are provided in Supplemental Material [29].

Discussion.—We have shown a no inner-horizon theorem for black holes with charged scalar hair and discussed the possible asymptotic geometries near the singularity. The key step is to have constructed the radially conserved quantity $\mathcal{Q}$ of Eq. (11). Note that to prove the theorem we do not make direct use of the scalar potential $V$, thus our result applies to quite generic space-times. Since Eqs. (13) and (15) do not depend on $A_i$, our no inner-horizon results also apply to the Einstein-scalar and Einstein-Maxwell-charged scalar theories. Our theorem has some direct significance for the SCC by showing that the Cauchy horizon in large classes of theories will be definitely removed by a non-trivial SCC. We then discussed the geometry near the singularity. The Kasner form of solutions has been obtained when the potential terms can be neglected. However, we have found strong numerical evidence for the existence of novel oscillating behavior all the way down to the singularity when the potential terms become important to the geometry.

Although one could engineer complicated hairy black holes by adjusting the scalar potential, our results suggest that the inner structure behind hairy black holes is pretty simple. There is no way to construct spherical and planar black holes with a Cauchy horizon for both Einstein-scalar and Einstein-Maxwell-charged scalar theories. Our theorem has some direct significance for the SCC by showing that the Cauchy horizon in large classes of theories will be definitely removed by a non-trivial scalar hairy. For example, it has been recently argued that SCC can be violated under the linear charged scalar perturbation of RN black holes in dS space [34–36]. Nevertheless, this linear perturbation result breaks down by considering non-linear effect of the charged scalar field.
Notice that as the energy accumulates near the inner horizon, non-linear effects from the scalar field cannot be neglected. The backreaction of the charged scalar field could sufficiently modify the background geometry and the resulted hairy black hole, if exists, has no Cauchy horizon, as shown in this Letter [37].

In the present study, we have limited ourselves to black holes with maximally symmetric horizon, it would be interesting to consider more general cases with inhomogeneous spacetimes and also with additional forms of matter. We have shown that some hyperbolic black holes can have an inner horizon with timelike singularity. It is interesting to further understand the interior of hyperbolic case. The dynamics near singularity seems to allow different behaviors from the Kasner form (18). It is desirable to understand this feature in the future.

Note added.—While this work was being completed, the work [12] appeared in arXiv, which discusses the interior dynamics for planar black holes by considering the minima holographic superconductor. Similar construction was used to prove the no Cauchy horizon feature for minima holographic superconductor. This Letter [37] appeared in arXiv, which discusses the in-

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The anstatz for the background is given by

$$V. Cardoso, J. L. Costa, K. Destounis, P. Hintz and E. Kasner, “Geometrical theorems on Einstein’s cosmological equations,” Am. J. Math. 43, 217-221 (1921).

Eqs. (4)-(7) have been solved numerically outside black holes ($z < z_H$) in the literature. To continue behind the event horizon it is simple to switch to ingoing coordinate, for which the equations of motion do not change.

V. Cardoso, J. L. Costa, K. Destounis, P. Hintz and A. Jansen, “Strong cosmic censorship in charged black-hole spacetimes: still subtle,” Phys. Rev. D 98, no.12, 124025 (2018) [arXiv:1808.03631 [gr-qc]].

Example for Hyperbolic Black Holes with Inner Horizon

For the hyperbolic black holes, our method adopted in the main text can not be appropriate to prove the black hole no-inner horizon or that the inner horizon of black hole with charged scalar hairs may exist. In this section, we show that for the hyperbolic case ($k = -1$) it is indeed possible to have an inner horizon.
We now present a concrete numerical example for the hyperbolic black hole with an inner Cauchy horizon. The four dimensional model \((d = 2)\) is given by
\[
V(\psi^2) = -6 + m^2 \psi^2, \quad Z = 1, \tag{29}
\]
with \(m^2 = -0.18388\) and \(q = 1.5\). At the event horizon \(z_H = 1.193936\), we choose the following initial conditions:
\[
\psi(z_H) \approx 1.10683410, \quad \psi'(z_H) \approx 0.115816263, \quad \tilde{Q}(z_H) \approx 0.650999915, \quad \chi(z_H) = h(z_H) = A_t(z_H) = 0. \tag{30}
\]
In order to have much higher numerical stability and to improve the error control, we further recombine the functions into the following form:
\[
h := f e^{-\chi/2} z^{1-d}, \quad \tilde{Q} := -\frac{2 \kappa^2 N}{\omega(d)} Q = z^{2-d} e^{\chi/2} A'_t. \tag{31}
\]
By using these new variables, we can rewrite Eqs. (25)-(28) into the following form and then numerically solve the variables \(\{\chi, h, \psi, \tilde{Q}, A_t\}\).
\[
\chi' = \frac{2}{d} \left[ \frac{\psi^2 A_t^2 q^2}{h^2 z^{2d+1}} + z \psi'^2 \right], \tag{32}
\]
\[
h' = -\frac{(d-1)k}{z^d} e^{-\chi/2} + \frac{e^{-\chi/2}}{d} \left( \frac{\tilde{Q}_z^2 z^{d-2}}{2} + \frac{V(\psi^2)}{z^{d+2}} \right), \tag{33}
\]
\[
\tilde{Q}' = \frac{2 \psi^2 A_t q^2}{z^{2d+1} h}, \quad A'_t = \tilde{Q} e^{-\chi/2} z^{d-2}, \tag{34}
\]
\[
\psi'' = -\left( \frac{h'}{h} + \frac{1}{z} \right) \psi' + \left( \frac{V(\psi^2)e^{-\chi/2}}{z^{d+3} h} - \frac{A_t^2 q^2}{h^2 z^{2d+2}} \right) \psi, \tag{35}
\]
where we have set \(Z = 1\).

FIG. 3. Numerical solution for hyperbolic case \((k = -1)\) with the boundary conditions (30). The hairy black hole has the event horizon at \(z_H = 1.193936\). There is a Cauchy horizon at \(z_I \approx 4.15699837\) and all functions are smooth at two horizons. From the right panel, we see that \(f(0) = 1, \psi(0) = 0, \chi(0)\) and \(A_t(0)\) are both finite, which implies that this solution is indeed an asymptotically AdS black hole. We have considered the four dimensional model with \(V(\psi^2) = -6 - 0.18388 \psi^2, Z = 1\) and \(q = 1.5\).

We then integrate Eqs. (32)-(35) with \(k = -1\) numerically from \(z = z_H + \varepsilon_1\) to \(z \to \infty\) (here we set \(\varepsilon_1 = 10^{-9}\)). We use solver \texttt{ode45} of MATLAB with accuracy control \texttt{"odeset('RelTol',1e-13,'AbsTol',1e-13')}\). As Eqs. (32)-(35) will meet coordinate singularity at \(z = z_I\), numerical solver will fail at this singularity and cannot directly pass
through this point. To overcome this issue, we slightly modify Eqs. (32), (34) and (35) into the following form:

\[
\chi' = \frac{2}{d} \left[ \psi^2 A_l q^2 z + \left( h + i\varepsilon \right)^2 z^{d+1} \right], \quad \bar{Q}' = \frac{2\psi^2 A_l q^2}{z^{2d+1}(h + i\varepsilon)},
\]

\[
\psi'' = -\left( \frac{h'}{h + i\varepsilon} + \frac{1}{z} \right) \psi' + \left( \frac{\check{V}(\psi^2)e^{-\chi/2}}{z^{d+2}(h + i\varepsilon)} - \frac{A_l^2 q^2}{(h + i\varepsilon)^2 z^{2d+2}} \right) \psi.
\]

This leads to the fact that the solutions all have imaginary parts with order \( O(\varepsilon) \). Then we carefully tune \( \varepsilon \) from \( 10^{-3} \) to \( 10^{-9} \) to verify the convergency of real parts for variables \( \{h(z), \psi(z), A_l(z), \chi(z), \bar{Q}(z), \psi'(z)\} \). For example, when \( \varepsilon = 10^{-5}, 10^{-7}, 10^{-9} \), we have

\[
\psi(6) \approx 1.13489227 + 3 \times 10^{-6}i, \quad 1.13540830 + 6 \times 10^{-7}i, \quad 1.13541591 + 1.0 \times 10^{-8}i,
\]

respectively. Thus, the real parts will convtergent and we can ignore all imaginary parts when \( \varepsilon \) is small enough.

The numerical results are shown in Fig. 3. It is clear that the inner horizon appears at \( z \approx \varepsilon \). Thus, the real parts will converge and we can ignore all imaginary parts when \( \varepsilon \) is small enough.

The properties of Taylor expansion:

\[
\psi = \psi_n \delta^n + \psi_{n+1} \delta^{n+1} + \cdots, \quad n \geq 1,
\]

with \( \psi_n \neq 0 \) and \( \delta = z - z_i \). Similarly, \( f \) has the expansion

\[
f = f_l \delta^l + f_{l+1} \delta^{l+1} + \cdots, \quad l \geq 1.
\]

With \( f_l \neq 0 \). Since we have assumed \( A_l(z_i) \neq 0 \), smoothness of Eq. (27) implies \( n \geq l \). Up to the leading order, Eq. (25) becomes

\[
\psi_n n(n-1)\delta^{n-2} = -n l \psi_n \delta^{n-2} - \frac{e^{\chi(z_i)} A_l^2(z_i) q^2 \delta^{n-2}}{f_l^2},
\]

where we have used \( Z(0) = 1 \). Then we find that

\[
- e^{\chi(z_i)} A_l^2(z_i) q^2 = f_l^2 n(n-1+l)\delta^{2l-2}.
\]

This is impossible because the left side is negative, while the right side is not negative. Therefore, we arrive at the result of Lemma 1. \( \square \)

**Lemma 2:** In an interval \( (z_1, z_2) \), if \( f(z) \geq 0 \), then \( A_l(z)^2 \) has no local maximum.

**Properties of Gauge Sector**

The properties of gauge potential \( A_l \) play a crucial role in understanding the structure of interior geometry behind the event horizon. In this section we will discuss some general features for the gauge sector. In particular, we will present a few lemmas of the gauge sector, which are important to build our no inner-horizon theorem in the main text.

**Lemma 1:** For hairy charged black hole (i.e. \( \psi \) and \( A_l \) are not zero somewhere), \( A_l(z_i) \) must be zero at any horizon \( z = z_i \) where \( f(z_i) \) vanishes.

**Proof of Lemma 1:** This proof uses the smoothness of the spacetime geometry away from the singularity. When \( f(z_i) = 0 \), to insure the smoothness around \( z = z_i \) for other functions, we have the following two choices:

(a) \( A_l(z_i) = 0 \), or (b) \( A_l(z_i) \neq 0 \), \( \psi(z_i) = 0 \).

What we need is to exclude the second case. We first assume that case (b) is true. As \( \psi(z) \) is smooth and nonzero somewhere, we have the following Taylor expansion:

\[
\psi = \psi_n \delta^n + \psi_{n+1} \delta^{n+1} + \cdots, \quad n \geq 1,
\]

with \( \psi_n \neq 0 \) and \( \delta = z - z_i \). Similarly, \( f \) has the expansion

\[
f = f_l \delta^l + f_{l+1} \delta^{l+1} + \cdots, \quad l \geq 1.
\]

With \( f_l \neq 0 \). Since we have assumed \( A_l(z_i) \neq 0 \), smoothness of Eq. (27) implies \( n \geq l \). Up to the leading order, Eq. (25) becomes

\[
\psi_n n(n-1)\delta^{n-2} = -n l \psi_n \delta^{n-2} - \frac{e^{\chi(z_i)} A_l^2(z_i) q^2 \delta^{n-2}}{f_l^2},
\]

where we have used \( Z(0) = 1 \). Then we find that

\[
- e^{\chi(z_i)} A_l^2(z_i) q^2 = f_l^2 n(n-1+l)\delta^{2l-2}.
\]

This is impossible because the left side is negative, while the right side is not negative. Therefore, we arrive at the result of Lemma 1. \( \square \)
Proof of Lemma 2: We only need to consider the case for which $A_t$ is not a constant in the interval $(z_1, z_2)$. We write Eq. (26) into the following form

$$\left( A_t^2 \right)^{''} - \left( \frac{d - 2}{z} - \frac{1}{2} \chi' + \frac{d \ln Z(\psi^2)}{dz} \right) \left( A_t^2 \right)'^{'} = \frac{4\psi^2 A_t^2 q^2}{z^2 f Z(\psi^2)} + 2A_t^2.$$  \hspace{1cm} (42)

Then in the interval $(z_1, z_2)$ with $f > 0$, we see that

$$\left( A_t^2 \right)^{''} - \left( \frac{d - 2}{z} - \frac{1}{2} \chi' + \frac{d \ln Z(\psi^2)}{dz} \right) \left( A_t^2 \right)'^{'} \geq 0. \hspace{1cm} (43)$$

If there is a local maximum which locates at $z_m \in (z_1, z_2)$, we have the following Taylor expansion

$$A_t^2 = A_0 - A_s(z - z_m)^2 + \cdots,$$  \hspace{1cm} (44)

with $A_s > 0$ and the integer $s \geq 1$. Taking it into Eq. (43), we find

$$- 2s(2s - 1)A_s(z - z_m)^{2s-2} + O((z - z_m)^{2s-1}) \geq 0.$$  \hspace{1cm} (45)

As the left side of Eq. (45) is negative, Eq. (45) cannot be satisfied. Thus, there is no local maximum in the interval $(z_1, z_2)$. In addition, if $A_t(z)$ is continuous at two endpoints, then the maximum of $A_t(z)^2$ in the interval $[z_1, z_2]$ is given by $\max\{A_t(z_1)^2, A_t(z_2)^2\}$. \hfill $\Box$

The charge degrees of freedom behind the surface generate a non-zero electric flux and can be characterized by

$$Q(z) = \frac{1}{2\kappa_N} \int_{\Sigma} Z(\mid \Psi \mid^2) \ast F = -\frac{\omega(d)}{2\kappa_N^2} Z(\psi^2) z^{2-d} e^{\chi/2} A_t',$$  \hspace{1cm} (46)

where $\omega(d)$ is the volume of the section with $t$ and $z$ fixed and we have used the ansatz (24). We have the following Lemma for $Q(z)$.

Lemma 3: In an interval $(z_1, z_2)$ with $f(z) \geq 0$ and $A_t(z_1) = 0$, $Q(z)^2$ is monotonic increasing in the interval $(z_1, z_2)$.

Proof of Lemma 3: We first rewrite Eq. (26) as

$$(z^{2-d} e^{\chi/2} Z A_t')' = \frac{2e^{\chi/2} \psi^2 q^2}{z^d f} A_t.$$  \hspace{1cm} (47)

Making use of Eq. (26), we then have

$$Q' = \frac{\omega(d)}{2\kappa_N^2} \frac{2e^{\chi/2} \psi^2 q^2}{z^d f} A_t \Rightarrow QQ' = -Q \frac{\omega(d)}{2\kappa_N^2} \frac{2e^{\chi/2} \psi^2 q^2}{z^d f} A_t = \frac{\omega(d)}{2\kappa_N^2} \frac{Z z^{2} e^{\chi/2} \psi^2 q^2}{z^d f} A_t A_t'.$$  \hspace{1cm} (48)

Thus, we obtain

$$(Q^2)' = \frac{\omega(d)}{2\kappa_N^2} \frac{Z z^{2} e^{\chi/2} \psi^2 q^2}{z^d f} (A_t^2)' \hspace{1cm} (49)$$

From Lemma 2, in the interval $(z_1, z_2)$ with $f(z) \geq 0$, $A_t^2$ has no local maximum. Since $A_t(z_1) = 0$, one obtains $(A_t^2)' \geq 0$ (see Fig. 4 for a schematic explanation). Then, we immediately have

$$(Q^2)' \geq 0, \hspace{1cm} (50)$$

in an interval $(z_1, z_2)$ with $f(z) \geq 0$ and $A_t(z_1) = 0$. Note that when $f = 0$, Eq. (48) seems to be singular. Nevertheless, the smoothness of equations of motion (25)-(28) insures that this is a removable singularity. Thus the desired result follows.

One interesting feature is that the timelike singularity must be charged. The hairless solution is described by the Reissner-Nordström black hole, for which there is no any medium to conduct the charge, thus the charge has to stay...
As we have shown in the main text that for a large class of hairy black holes there exists no inner horizon and the spacetime ends at a spacelike singularity. In this section, we present the analytical analyzes as well as numerical details for constructing the geometry near the spacelike singularity. To simplify our discussion, we shall specify $Z = 1$ so that the theory becomes a standard Einstein-Maxwell-scalar theory.

We begin our discussion with the case that the potential of scalar field can be neglected near the singularity. We will show that in this case the spacelike singularity always has the asymptotic Kasner geometry. We then discuss what will happen if this condition is broken.

### Analytical analyze

Near the singularity, the functions will be divergent, which challenges the numerical precision. So we first give some analytic discussion about the geometry near the singularity.

When the contribution from the scalar effective potential can be neglected, we obtain the following equations from Eqs. (32)-(35) near the singularity as $z \to \infty$:

\begin{align}
\chi' &= 2 \frac{d}{2d+1} \left[ \psi^2 A_t^2 q^2 \frac{z^d}{h z^{2d+1}} + z \psi' \right], \\
h' &= \frac{2z^d}{2d+1} e^{-\chi/2}, \\
\bar{Q}' &= \frac{2 \psi^2 A_t q^2}{z^{2d+1} h}, \quad A_t' = \bar{Q} e^{-\chi/2} z^{d-2}, \\
\psi'' &= - \left( \frac{h'}{h} + \frac{1}{z} \right) \psi' - \frac{A_t^2 q^2}{h^2 z^{2d+2}} \psi.
\end{align}

It is obvious from Eq. (53) that $h(z)$ is a monotonic increasing function when $z$ is large enough. Since we have assumed that the singularity is spacelike, we have $h < 0$ as well as

\begin{equation}
\lim_{z \to \infty} h(z) = -h_0 < 0.
\end{equation}
Then, we find $O(h') < O(1/z)$. Therefore, we obtain from Eq. (54) that
\[ A_i^2 = 2dh'e^{-\chi/2}z^{d-2} < O(z^{d-3}), \] (57)
and thus $|A_i| < |O(z^{(d-1)/2})|$. Here we have used the fact that $e^{-\chi/2} \leq O(z^0)$ since $\chi$ is a monotonic increasing function and is finite at the event horizon. This leads to that the coefficient of the last term of Eq. (55) satisfies
\[ \frac{A_i^2q^2}{h^2z^{2d+2}} < O(1/z^{d+3}) < O(1/z^2). \] (58)
Therefore, the last term of Eq. (55) can be neglected. The solutions of $\psi$ in large $z$ limit then reads
\[ \psi(z) \sim \sqrt{d}\alpha \ln z, \] (59)
with $\alpha$ a constant. Taking it into Eq. (52), we can find $\chi' = 2\alpha^2/z$. Finally, we obtain a simple result for the geometry near the singularity.
\[ \psi = \sqrt{d}\alpha \ln z + \cdots, \quad A'_i = E_s z^{d-2-\alpha^2} + \cdots, \]
\[ \chi = 2\alpha^2 \ln z + \cdots, \quad f = -f_s z^{3+d+\alpha^2} + \cdots, \] (60)
where $E_s$ and $f_s$ are constants.

Changing the $z$ coordinate to the proper time $\tau$ via $\tau \sim z^{-(1+d+\alpha^2)/2}$, we obtain
\[ ds^2 = -d\tau^2 + c_\tau \tau^{2p_s}dt^2 + c_{\alpha}\tau^{2p_{\alpha}}d\Sigma_{d,k}^2, \quad \psi(z) = -p_{\psi}\ln \tau, \] (61)
with
\[ p_t = \frac{1 - d + \alpha^2}{1 + d + \alpha^2}, \quad p_s = \frac{2}{1 + d + \alpha^2}, \quad p_{\psi} = \frac{2\sqrt{d}\alpha}{1 + d + \alpha^2}. \] (62)
One can immediately check that
\[ p_t + dp_s = 1, \quad p_t^2 + dp_s^2 + p_{\psi}^2 = 1, \] (63)
and therefore the geometry around the spacelike singularity has the Kasner form. We now arrive at the conclusion: when the scalar potential can be neglected in the spacelike singularity, the asymptotic solutions are of Kasner type in Eq. (61).

Note that in above discussion we have assumed that the scalar kinetic term should dominate the dynamics near the singularity. One should at least has the following constraint:
\[ \lim_{z \to \infty} \frac{|V(\psi^2)|}{z^{d+1+\alpha^2}} \ll 1. \] (64)
In particular, it allows the scalar potential $V$ to be arbitrary algebraic functions, including polynomial functions. However, if one chooses a potential that diverges exponentially or even worse, the condition (64) will be broken and we can not obtain (60). For example, we take
\[ V(\psi^2) = P(\psi^2) + \sinh(\gamma \psi^2), \] (65)
with $P(x)$ a polynomial. When $\gamma = 0$, we expect to obtain (60) no matter how high order of polynomial one considers. In contrast, once we choose $\gamma > 0$, the asymptotic solution will be different from (60). This can be understood as follows. Suppose we are in the Kasner regime, where we have $\psi \sim \sqrt{d}\alpha \ln z$ at large $z$. After including the second term of Eq. (65), one has
\[ \frac{|V(\psi^2)|}{z^{d+1+\alpha^2}} \sim \frac{e^{d\gamma\alpha^2(\ln z)^2}}{z^{d+1+\alpha^2}} > \frac{e^{d\gamma\alpha^2(\kappa \ln z)}}{z^{d+1+\alpha^2}} = \frac{z^{\kappa\gamma\alpha^2}}{z^{d+1+\alpha^2}}. \] (66)
Here $\kappa$ is a constant for which we only demand $\kappa < \ln z$. Therefore, $\kappa$ can be sufficiently large as we approach the singularity. In particular, when $\kappa > \frac{d+1+\alpha^2}{\gamma\alpha^2}$, the numerator of Eq. (66) is larger than the denominator and therefore
FIG. 5. Numerical results for different polynomial scalar potentials. \( \{R_1, R_2, R_3\} \) all approach to be constant values as \( z/z_H \to \infty \), confirming the Kasner form geometry near the singularity. We have fixed \( Z = 1 \).

The constraint (64) is broken. As a consequence, no matter how small the value of \( \gamma \) is, the Kasner solution is expected to be modified when \( z > z_c \), where the critical point is given by

\[
  z_c = c_0 e^{\frac{\gamma d + \alpha_2}{\gamma d + \alpha_2}},
\]

where \( c_0 \) is a coefficient that depends on a model one considers. To be self-consistent, we need \( z_c \gg z_H \) such that the solutions are approximately in the Kasner regime when \( \gamma = 0 \).

**Numerical check**

In this section we examine the geometry inside hairy charged black holes by numerically solving Eqs. (32)-(35). We are particularly interested in the behavior near the singularity. We also would like to check our analytic results obtained in the last part.

Note that the Kasner type solution means \( \{h, Q, A_t, z\chi', z\psi'\} \) approach to constant. To present our numerical data, we further introduce

\[
  R_1 = z\psi', \quad R_2 = \ln \left( \frac{z^2_H - h}{z^2} \right), \quad R_3 = 4\hat{Q},
\]

for which the Kasner solution yields

\[
  \lim_{z \to \infty} R_i(z) = \text{const.}, \quad i = 1, 2, 3.
\]

We first consider the class of potentials

\[
  V(\psi^2) = -6 + \psi^{2n} + \sinh(\gamma \psi^2),
\]

and numerically solve Eqs. (32)-(35) by using the Runge-Kutta method. We consider the planar black holes with \( k = 0 \) and specify \( d = 2, z_h = 1, q = 1 \). We choose \( A'_1(z_h) = 1, \chi(r_h) = 0 \) and \( \psi(r_h) = 1/2 \) in all cases. As we only care about the inner geometry, we do not consider the UV completion for (70). We first present the numerical result for the polynomial case, for which we set \( \gamma = 0 \) and vary \( n \). According to the discussion above, we anticipate Kasner type solutions (60) as \( z \to \infty \). In Fig. 5 we show our numerical results for \( V = -6 + \psi^2 \) and \( V = -6 + \psi^{20} \). As expected, the functions \( \{R_1, R_2, R_3\} \) all approach to constant when \( z/z_H \to \infty \), confirming the Kasner form near the singularity.

As the value of \( \gamma \) is increased, the contribution from \( V \) to the geometry near the singularity becomes more and more important. The numerical results are presented in Fig. 6. For non-polynomial potentials which violate (64), the asymptotic solutions are different from the Kasner behavior (60). We observe some strong oscillating behavior all the way down to the singularity. The oscillating behavior can be found more clearly from the behavior of \( \psi \), which is shown in Fig. 7. We see that the scalar field does no longer satisfy the asymptotic behavior shown in Eq. (60).
FIG. 6. Numerical results for non-polynomial potentials. The behaviors near the singularity are different from the Kasner form. The oscillating behavior appears when $z$ is large. The observation of intensity oscillation becomes more and more manifest as $\gamma$ is increased.

Interestingly, our numerical result implies that the asymptotic solution of $\psi$ approximately obeys
\begin{equation}
\psi \sim T_0(z) T_1[\ln(z/z_H)],
\end{equation}
where $T_0(x)$ is a slow-varied function and $T_1[\ln(z/z_H)]$ a periodic function of $\ln(z/z_H)$ (see the left panel of Fig. 7). We have carefully checked the convergence of our numerical results.

One may worry that for the choice of potentials in Eq. (70), the asymptotic geometry near the black hole boundary is different for different $\gamma$. To avoid this issue, we consider the second class of potentials
\begin{equation}
V(\psi^2) = -6 + (1 - \gamma)\psi^2 + \sinh(\gamma \psi^2),
\end{equation}
for which the black holes are asymptotic AdS as $z \to 0$ and near the AdS boundary $V$ behaves as $V = -6 + \psi^2 + \cdots$. We obtain very similar oscillating behavior as the first class of potentials, see Fig. 8.

For a potential of Eq. (65), as we have discussed around Eq. (66), once $\gamma$ is turned on, no matter how small it is, the Kasner form (60) would be broken. Numerically, it is not easy to verify this feature for small $\gamma$. This is because the exponential term will play role only when the scalar field is large enough, which means one has to solve the equations to sufficiently large $z$. Nevertheless, one anticipates from numerics that there exists a critical value $z_c$, beyond which the the Kasner behavior would be modified. To check this point, we consider the class of potentials in Eq. (72) and numerically study how $z_c$ depends on the parameter $\gamma$. We indeed observe expected features from our numerics shown in Fig. 9. In particular, from the left panel of Fig. 9, one finds that there is a critical value of $z$ at which the Kasner
solution ceases to be valid for a given small value of \( \gamma \). We then numerically find the relationship between \( z_c \) and \( \gamma \), which is shown in the right panel of Fig. 9. We obtain that for small \( \gamma \), there is a scaling behavior

\[
\gamma \ln(z_c/z_H) = \text{const.},
\]

suggesting that the oscillating behavior will appear for large enough \( z \), no matter how small \( \gamma \) is. Interestingly, the scaling relation (73) is exactly we we have obtained in Eq. (67). We also compare the numerical results with our theoretical prediction (67) quantitatively. We obtain \( \alpha \) by using the relation \( \psi = \sqrt{4\alpha} \ln z \) when \( z \approx z_c \) (the linear region in the left panel of Fig. 9). After fitting the coefficient \( z_0 \) in Eq. (67), we find that the numerical results match with the theoretical prediction (67) quite well (see the solid line in the right panel of Fig. 9).