Commutative rings with one-absorbing factorization

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\textbf{ABSTRACT}

Let $R$ be a commutative ring with nonzero identity. Yassine et al. defined the concept of 1-absorbing prime ideals as follows: a proper ideal $I$ of $R$ is said to be a 1-absorbing prime ideal if whenever $xyz \in I$ for some nonunit elements $x, y, z \in R$, then either $xy \in I$ or $z \in I$. We use the concept of 1-absorbing prime ideals to study those commutative rings in which every proper ideal is a product of 1-absorbing prime ideals (we call them OAF-rings). Any OAF-ring has dimension at most one and local OAF-domains $(D, M)$ are atomic such that $M^2$ is universal.

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1. Introduction

Throughout this article, all rings are commutative with nonzero identity and all modules are unital. Let $\mathbb{N}$ denote the set of positive integers. For $m \in \mathbb{N}$, let $[1, m] = \{n \in \mathbb{N} | 1 \leq n \leq m\}$. Let $R$ be a ring. An ideal $I$ of $R$ is said to be proper if $I \neq R$. The radical of $I$ is denoted by $\sqrt{I} = \{x \in R | x^n \in I$ for some $n \in \mathbb{N}\}$. We denote by Min$(I)$ the set of minimal prime ideals over the ideal $I$. The concept of prime ideals plays an important role in ideal theory and there are many ways to generalize it.

In [9], Badawi introduced and studied the concept of 2-absorbing ideals which is a generalization of prime ideals. An ideal $I$ of $R$ is a 2-absorbing ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In this case, $\sqrt{I} = P$ is a prime ideal with $P^2 \subseteq I$ or $\sqrt{I} = P_1 \cap P_2$ where $P_1, P_2$ are incomparable prime ideals with $P_1P_2 \subseteq I$, cf. [9, Theorem 2.4]. In [8], Anderson and Badawi introduced the concept of $n$-absorbing ideals as a generalization of prime ideals where $n$ is a positive integer. An ideal $I$ of $R$ is called an $n$-absorbing ideal of $R$, if whenever $a_1, a_2, \ldots, a_{n+1} \in R$ and $\prod_{i=1}^{n+1} a_i \in I$, then there are $n$ of the $a_i$’s whose product is in $I$. In this case, due to Choi and Walker [13, Theorem 1], $(\sqrt{I})^n \subseteq I$.

In [23], Mukhtar et al. studied the commutative rings whose ideals have a TA-factorization. A proper ideal is called a TA-ideal if it is a 2-absorbing ideal. By a TA-factorization of a proper ideal $I$, we mean an expression of $I$ as a product $\prod_{i=1}^{n} I_i$ of TA-ideals. Mukhtar et al. prove that...
any TAF-ring has dimension at most one and the local TAF-domains are atomic pseudo-valuations domains. Recently in [1], Ahmed et al. studied commutative rings whose proper ideals have an \(n\)-absorbing factorization. Let \(I\) be a proper ideal of \(R\). By an \(n\)-absorbing factorization of \(I\) we mean an expression of \(I\) as a product \(\prod_{i=1}^{n} I_i\) of proper \(n\)-absorbing ideals of \(R\). Ahmed et al. called \(AF\text{-dim}(R)\) (absorbing factorization dimension) the minimum positive integer \(n\) such that every ideal of \(R\) has an \(n\)-absorbing factorization. If no such \(n\) exists, set \(AF\text{-dim}(R) = \infty\). An FAF-ring (finite absorbing factorization ring) is a ring such that \(AF\text{-dim}(R) < \infty\). Recall that a general ZPI-ring is a ring whose proper ideals can be written as a product of prime ideals. Therefore, \(AF\text{-dim}(R)\) measures, in some sense, how far \(R\) is from being a general ZPI-ring, cf. [1, Proposition 3]. By dim\((R)\), we denote the Krull dimension of \(R\).

In [25], Yassine et al. introduced the concept of a 1-absorbing prime ideal which is a generalization of a prime ideal. A proper ideal \(I\) of \(R\) is a 1-absorbing prime ideal (our abbreviation OA-ideal) if whenever we take nonunit elements \(a, b, c \in R\) with \(abc \in I\), then \(ab \in I\) or \(c \in I\). In this case, \(\sqrt{I} = P\) is a prime ideal, cf. [25, Theorem 2.3]. And if \(R\) is a ring in which exists an OA-ideal that is not prime, then \(R\) is a local ring, that is a ring with one maximal ideal.

Let \(I\) be a proper ideal of \(R\). By an OA-factorization of \(I\), we mean an expression of \(I\) as a product \(\prod_{i=1}^{n} I_i\) of OA-ideals. The aim of this note is to study the commutative rings whose proper ideals (resp., proper principal ideals, resp., proper 2-generated ideals) have an OA-factorization.

We call \(R\) a 1-absorbing prime factorization ring (OAF-ring) if every proper ideal has an OA-factorization. An OAF-domain is a domain which is an OAF-ring. Our article consists of five sections (including the introduction).

In the next section, we characterize OA-ideals (Lemma 2.1) and we prove that if \(I\) is an OA-ideal, then \(I\) is a primary ideal. We also show that the OAF-ring property is stable under factor ring (resp., fraction ring) formation (Propositions 2.2 and 2.3). Furthermore, we investigate OAF-rings with respect to direct products (Corollary 2.5) and polynomial ring extensions (Corollary 2.6). We prove that the general ZPI-rings are exactly the arithmetical OAF-rings (Theorem 2.8).

The third section consists of a collection of preparational results which will be of major importance in the fourth section. For instance, we show that the Krull dimension of an OAF-ring is at most one (Theorem 3.5).

The fourth section contains the main results of our article. Among other results, we provide characterizations of OAF-rings (Theorem 4.2), rings whose proper principal ideals have an OA-factorization (Corollary 4.3) and rings whose proper (principal) ideals are OA-ideals (Proposition 4.5).

In the last section, we study the transfer of the various OA-factorization properties to the trivial ring extension.

### 2. Characterization of OA-ideals and simple facts

We start with a characterization of OA-ideals. Recall that a ring \(R\) is a Q-ring (cf. [3]) if every proper ideal of \(R\) is a product of primary ideals.

**Lemma 2.1.** Let \(R\) be a ring with Jacobson radical \(M\) and \(I\) be an ideal of \(R\).

1. If \(R\) is not local, then \(I\) is an OA-ideal if and only if \(I\) is a prime ideal.
2. If \(R\) is local, then \(I\) is an OA-ideal if and only if \(I\) is a prime ideal or \(M^2 \subseteq I \subseteq M\).
3. Every OA-ideal is a primary TA-ideal. In particular, every OAF-ring is both a Q-ring and a TAF-ring.

**Proof.** (1) This follows from [25, Theorem 2.4].

(2) Let \(R\) be local. Then, \(M\) is the maximal ideal of \(R\).
Let $I$ be an OA-ideal such that $I$ is not a prime ideal. Since $I$ is proper, we infer that $I \subseteq M$. Since $I$ is not prime, there are $a, b \in M \setminus I$ such that $ab \in I$. To prove that $M^2 \subseteq I$, it suffices to show that $xy \in I$ for all $x, y \in M$. Let $x, y \in M$. Then, $xya \in I$. Since $xy, a, b \in M, b \not\in I$ and $I$ is an OA-ideal, it follows that $xya \in I$. Again, since $x, y, a \in M, a \not\in I$ and $I$ is an OA-ideal, we have that $xy \in I$.

Clearly, $I$ is a prime ideal, then $I$ is an OA-ideal. Now let $M^2 \subseteq I \subseteq M$. Then, $I$ is proper. Let $a, b, c \in M$ be such that $abc \in I$. Then, $ab \in M^2 \subseteq I$. Therefore, $I$ is an OA-ideal.

Let $I$ be an OA-ideal. It is an immediate consequence of (1) and (2) that $I$ is a primary ideal. Now let $a, b, c \in R$ be such that $abc \in I$. We have to show that $ab \in I$ or $ac \in I$ or $bc \in I$.

First let $a$ or $b$ or $c$ be a unit of $R$. Without restriction let $a$ be unit of $R$. Since $abc \in I$, we infer that $bc \in I$.

Now let $a$, $b$, and $c$ be nonunits. Then, $ab \in I$ or $c \in I$. If $c \in I$, then $ac \in I$. The in particular statement is clear.

**Proposition 2.2.** Let $R$ be an OAF-ring and $I$ be a proper ideal of $R$. Then, $R/I$ is an OAF-ring.

**Proof.** Let $J$ be a proper ideal of $R$ which contains $I$. Let $J = \prod_{i=1}^{m} J_i$ be an OA-factorization. Then, $J/I = \prod_{i=1}^{m} (J_i/I)$. It suffices to show that $J_i/I$ is an OA-ideal for each $i \in [1, m]$. Let $i \in [1, m]$ and let $a, b, c \in R$ be such that $a, b, c$ are three nonunit elements of $R/I$ and $\bar{abc} \in J_i/I$. Clearly, $a, b, c$ are nonunit elements of $R$ and $abc \in J_i$. Since $J_i$ is an OA-ideal of $R$, we get that $ab \in J_i$ or $c \in J_i$ which implies that $\bar{ab} \in J_i/I$ or $\bar{c} \in J_i/I$. Therefore, $R/I$ is an OAF-ring.

**Proposition 2.3.** Let $S$ be a multiplicatively closed subset of $R \setminus 0$. If $R$ is an OAF-ring, then $S^{-1}R$ is an OAF-ring. In particular, $R_M$ is an OAF-ring for every maximal ideal $M$ of $R$.

**Proof.** Let $J$ be a proper ideal of $S^{-1}R$. Then, $J = S^{-1}I$ for some proper ideal $I$ of $R$ with $I \cap S = \emptyset$. Let $I = \prod_{i=1}^{m} I_i$ be an OA-factorization. Then, $J = \prod_{i=1}^{m} (S^{-1}I_i)$ where each $S^{-1}I_i$ which is proper is an OA-ideal by [25, Theorem 2.18]. Thus, $S^{-1}R$ is an OAF-ring. The in particular statement is clear.

Let $R$ be a ring. Then, $R$ is said to be a $\pi$-ring if every proper principal ideal of $R$ is a product of prime ideals. We say that $R$ is a unique factorization ring (in the sense of Fletcher, cf. [4]) if every proper principal ideal of $R$ is a product of principal prime ideals. A unique factorization domain is an integral domain which is a unique factorization ring.

**Remark 2.4.** Let $R$ be a non local ring.

1. $R$ is a general ZPI-ring if and only if $R$ is an OAF-ring.
2. $R$ is a $\pi$-ring if and only if each proper principal ideal of $R$ has an OA-factorization.
3. $R$ is a unique factorization ring if and only if each proper principal ideal of $R$ is a product of principal OA-ideals.

**Proof.** This is an immediate consequence of Lemma 2.1(1).

In the light of the above remark, we give the next result.

**Corollary 2.5.** Let $R_1$ and $R_2$ be two rings and $R = R_1 \times R_2$ be their direct product. The following statements are equivalent.

1. $R$ is an OAF-ring.
2. $R$ is a general ZPI-ring.
3. $R_1$ and $R_2$ are general ZPI-rings.
Proof. This follows from Remark 2.4(1) and [21, Exercise 6(g), page 223].

Let \( R \) be a ring. Then, \( R \) is called a von Neumann regular ring if for each \( x \in R \) there is some \( y \in R \) with \( x = x^2 y \). The ring \( R \) is von Neumann regular if and only if \( R \) is a zero-dimensional reduced ring (see [19, Theorem 3.1, page 10]).

**Corollary 2.6.** Let \( R \) be a ring. The following statements are equivalent.

1. \( R[X] \) is an OAF-ring.
2. \( R \) is a Noetherian von Neumann regular ring.
3. \( R \) is a finite direct product of fields.

**Proof.** Observe that the polynomial ring \( R[X] \) is never local, since \( X \) and \( 1 - X \) are nonunit elements of \( R[X] \), but their sum is a unit. Consequently, \( R[X] \) is an OAF-ring if and only if \( R[X] \) is a general ZPI-ring by Remark 2.4(1). The rest is now an easy consequence of [2, Theorem 6 and Corollary 6.1], [21, Exercise 10, page 225] and Hilbert’s basis theorem. \( \square \)

Let \( R \) be a ring and \( I \) be an ideal of \( R \). Then, \( I \) is called divided if \( I \) is comparable to every ideal of \( R \) (or equivalently, \( I \) is comparable to every principal ideal of \( R \)).

**Lemma 2.7.** Let \( R \) be a local ring with maximal ideal \( M \) such that \( M^2 \) is divided. The following statements are equivalent.

1. Each two principal OA-ideals which contain \( M^2 \) are comparable.
2. For each OA-ideal \( I \) of \( R \), we have that \( I \) is a prime ideal or \( I = M^2 \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( I \) be an OA-ideal of \( R \) such that \( I \) is not a prime ideal of \( R \). Then, \( M^2 \subseteq I \subseteq M \) by Lemma 2.1(2). Assume that \( M^2 \subseteq I \). Let \( x \in I \setminus M^2 \) and let \( y \in M \setminus I \). Then, \( x, y \notin M^2 \), and thus, \( M^2 \subseteq xR, yR \) (since \( M^2 \) is divided). It follows that \( xR \) and \( yR \) are (principal) OA-ideals of \( R \) by Lemma 2.1(2). Since \( y \notin xR \) and \( xR \) and \( yR \) are comparable, we infer that \( xR \subseteq yR \). Consequently, there is some \( z \in M \) such that \( x = yz \), and hence \( x \in M^2 \), a contradiction. Therefore, \( I = M^2 \).

(2) \( \Rightarrow \) (1): This is obvious. \( \square \)

Let \( R \) be a ring. An ideal \( I \) of \( R \) is called 2-generated if \( I = xR + yR \) for some (not necessarily distinct) \( x, y \in R \). Note that every principal ideal of \( R \) is 2-generated. We say that \( R \) is a chained ring if each two ideals of \( R \) are comparable under inclusion. Moreover, \( R \) is said to be an arithmetical ring if \( R_M \) is a chained ring for each maximal ideal \( M \) of \( R \).

**Theorem 2.8.** Let \( R \) be a ring. The following statements are equivalent.

1. \( R \) is a general ZPI-ring
2. \( R \) is an arithmetical OAF-ring.
3. \( R \) is an arithmetical ring and each proper principal ideal of \( R \) has an OA-factorization.

**Proof.** First, we show that if \( R \) is an arithmetical \( \pi \)-ring, then \( R \) is a general ZPI-ring. Let \( R \) be an arithmetical \( \pi \)-ring and let \( M \) be a maximal ideal of \( R \). It is straightforward to show that \( R_M \) is a \( \pi \)-ring. Moreover, \( R_M \) is a chained ring, and hence every 2-generated ideal of \( R_M \) is principal. Therefore, every proper 2-generated ideal of \( R_M \) is a product of prime ideals of \( R_M \). Consequently, \( R_M \) is a general ZPI-ring by [22, Theorem 3.2]. This implies that \( \dim(R_M) \leq 1 \) by [21, page 205]. We infer that \( \dim(R) \leq 1 \), and thus \( R \) is a general ZPI-ring by [16, Theorems 39.2, 46.7, and 46.11].

(1) \( \Rightarrow \) (2) \( \Rightarrow \) (3): This is obvious.
ideal of $R$ is finite and each prime ideal of $R$ is principal.

**Proof.** Let $R$ be a local ring such that each proper principal ideal of $R$ has an OA-factorization. Then, each nonmaximal minimal prime ideal of $R$ is principal. Let $R$ be a local ring with maximal ideal $M$ such that $R$ is an integral domain.

**Lemma 3.2.** Let $R$ be a local ring such that each proper principal ideal of $R$ has an OA-factorization. Then, each nonmaximal minimal prime ideal of $R$ is principal.

**Proof.** Let $I$ be a nonmaximal minimal prime ideal of $R$. By [2, Theorem 1], it is sufficient to show that $I$ is a multiplication ideal.

Let $x \in P$ and let $xR = \prod_{i=1}^{n} I_i$ be an OA-factorization. There is some $j \in [1, n]$ such that $I_j \subseteq P$. By Lemma 2.1(2), we have that $P = I_j$, and hence $xR = PJ$ for some ideal $J$ of $R$. We infer that $xR = P(xR : P)$. Let $I$ be an ideal of $R$ such that $I \subseteq P$. Then, $I = \sum_{y \in I} yR = \sum_{y \in I} P(yR : P) = P \sum_{y \in I} (yR : P)$, and thus, $P$ is a multiplication ideal.

The next result is a generalization of [16, Theorem 46.8] and its proof is based on the proof of the same result.

**Proposition 3.3.** Let $R$ be a local ring with maximal ideal $M$ such that $\dim(R) \geq 1$ and every proper principal ideal of $R$ has an OA-factorization. Then, $R$ is an integral domain and if $\dim(R) \geq 2$, then $R$ is a unique factorization domain.

**Proof.** Let $N$ be the nilradical of $R$. It follows from Proposition 3.1 and Lemma 3.2 that $\min(I)$ is finite and each $P \in \min(0)$ is principal.

Claim: Every proper principal ideal of $R/N$ has an OA-factorization. Let $I$ be a proper principal ideal of $R/N$. Then, $I = (xR + N)/N$ for some $x \in M$. Let $xR = \prod_{i=1}^{n} I_i$ be an OA-factorization. We infer that $I = (xR)/N = (\prod_{i=1}^{n} I_i)/N = \prod_{i=1}^{n} (I_i/N)$. It suffices to show that $I_i/N$ is an OA-Ideal of $R/N$ for each $i \in [1, n]$. If $I_i$ is a prime ideal of $R$, then $N \subseteq I_i$, and hence, $I_i/N$ is a prime ideal of $R/N$. Now let $I_i$ be not a prime ideal of $R$. By Lemma 2.1(2), we have that $M^2 \subseteq I_i \subseteq M$. Note that $R/N$ is local with maximal ideal $M/N$. Since $(M/N)^2 = M^2/N \subseteq I_i/N \subseteq M/N$, it follows by Lemma 2.1(2) that $I_i/N$ is an OA-ideal of $R/N$. This proves the claim.

Case 1: $R$ is one-dimensional. We prove that $R$ is an integral domain. If every OA-ideal of $R$ is a prime ideal, then $R$ is a $\pi$-ring, and hence, $R$ is an integral domain by [16, Theorem 46.8]. Now let not every OA-ideal of $R$ be a prime ideal. It follows from Lemma 2.1(2) that $M$ is not idempotent. Set $L = M^2 \cup \cup_{Q \in \min(0)} Q$. Next we prove that $M^2 \subseteq xR$ for each $x \in R \setminus L$. Let $x \in R \setminus L$. 

(3) $\Rightarrow$ (1): It is sufficient to show that $R$ is a $\pi$-ring. If $R$ is not local, then $R$ is a $\pi$-ring by Remark 2.4(2). Therefore, we can assume that $R$ is local with maximal ideal $M$. Since $R$ is local, we have that $R$ is a chained ring. Therefore, $M^2$ is divided and each two OA-ideals of $R$ are comparable. We infer by Lemma 2.7 that each OA-ideal of $R$ is a product of prime ideals. Now it clearly follows that $R$ is a $\pi$-ring. 

\[ \square \]
Without restriction let $x$ be a nonunit. Note that $xR$ cannot be a product of more than one OA-ideal, and hence, $xR$ is an OA-ideal. By Lemma 2.1(2), we have that $M^2 \subseteq xR$.

Now we show that $P \subseteq M^2$ for each $P \in \text{Min}(0)$. Let $P \in \text{Min}(0)$. Assume that $P \not\subseteq M^2$. Let $w \in R \setminus P$. Then, $P + wR \not\subseteq L$ by the prime avoidance lemma, and thus there is some $v \in (P + wR) \setminus L$. It follows that $M^2 \not\subseteq vR \subseteq P + wR$. Since $P$ is a nonmaximal prime ideal, we have that $R/P$ has no simple $R/P$-submodules, and hence $\cap_{y \in R \setminus P} (P + yR) = P$. (Note that if $\cap_{y \in R \setminus P} (P + yR) \neq P$, then $\cap_{y \in R \setminus P} (P + yR)/P$ is a simple $R/P$-submodule of $R/P$.) This implies that $M^2 \subseteq \cap_{y \in R \setminus P} (P + yR) = P$, and thus $P = M$, a contradiction.

Let $Q \in \text{Min}(0)$. By the prime avoidance lemma, there is some $z \in M \setminus L$. We infer that $Q \subseteq M^2 \subseteq zR$. Consequently, $Q = zQ$. Since $Q$ is principal, it follows that $Q = 0$ (e.g. by Nakayama’s lemma), and hence $R$ is an integral domain.

Case 2: $\dim(R) \geq 2$ and $R$ is reduced. We show that $R$ is a unique factorization domain. There is some nonmaximal nonminimal prime ideal $Q$ of $R$. By the prime avoidance lemma, there is some $x \in Q \setminus \cup_{P \in \text{Min}(0)} P$. Since $R$ is reduced, we have that $\cap_{L \in \text{Min}(0)} L = 0$. If $y \in R$ is nonzero with $xy = 0$, then $y \not\subseteq L$ and $xy \in L$ for some $L \in \text{Min}(0)$, and hence $x \in L$, a contradiction. We infer that $x$ is a regular element of $R$. Let $xR = \prod_{j=1}^{n} I_j$ be an OA-factorization. Then, $I_j \subseteq Q$ for some $j \in [1, n]$. Since $x$ is regular, $I_j$ is invertible, and hence, $I_j$ is a regular principal ideal (because invertible ideals of a local ring are regular principal ideals). Since $I_j \subseteq Q$ and $Q \not= M$, we have that $I_j$ is a prime ideal by Lemma 2.1(2). Consequently, $P \subseteq I_j$ for some $P \in \text{Min}(0)$. Since $I_j$ is regular, we infer that $P \subseteq I_j$, and hence $P = PI_j$ (since $I_j$ is principal). It follows (e.g. from Nakayama’s lemma) that $P = 0$ (since $P$ is principal). We obtain that $R$ is an integral domain.

To show that $R$ is a unique factorization domain, it suffices to show by [4, Theorem 2.6] that every nonzero prime ideal of $R$ contains a nonzero principal prime ideal. Since $\dim(R) \geq 2$ and $R$ is local, we only need to show that every nonzero nonmaximal prime ideal of $R$ contains a nonzero principal prime ideal. Let $L$ be a nonzero nonmaximal prime ideal of $R$ and let $z \in L$ be nonzero. Let $zR = \prod_{k=1}^{m} J_k$ be an OA-factorization. Then, $J_\ell \subseteq L$ for some $\ell \in [1, m]$. Since $R$ is an integral domain, $zR$ is invertible, and hence, $J_\ell$ is invertible. Therefore, $J_\ell$ is nonzero and principal (since $R$ is local). Since $L \not= M$, it follows from Lemma 2.1(2) that $J_\ell$ is a prime ideal.

Case 3: $\dim(R) \geq 2$. We have to show that $R$ is a unique factorization domain. Note that $R/N$ is a reduced local ring with maximal ideal $M/N$ and $\dim(R/N) \geq 2$. Moreover, each proper principal ideal of $R/N$ has an OA-factorization by the claim. It follows by Case 2 that $R/N$ is a unique factorization domain, and thus $N$ is the unique minimal prime ideal of $R$. Since $R/N$ is a unique factorization domain and $\dim(R/N) \geq 2$, $R/N$ possesses a nonzero nonmaximal principal prime ideal. We infer that there is some nonminimal nonmaximal prime ideal $Q$ of $R$ such that $Q/N$ is a principal ideal of $R/N$. Consequently, there is some $q \in Q$ such that $Q = qR + N$. Let $qR = \prod_{j=1}^{n} I_j$ be an OA-factorization. Then, $I_j \subseteq Q$ for some $j \in [1, n]$. Since $Q \not= M$, we infer by Lemma 2.1(2) that $I_j$ is a prime ideal of $R$. Therefore, $Q = qR + N \subseteq I_j \subseteq Q$, and hence $I_j = Q$.

Assume that $Q \not= qR$. Then, $qR = qJ$ for some proper ideal $J$ of $R$. It follows that $q \in qR = (qR + N)J \subseteq qJ + N$, and thus $q(1 - a) \in N$ for some $a \in J$. Since $a$ is a nonunit of $R$, we obtain that $q \in N$. This implies that $Q = qR + N = N$, a contradiction. We infer that $Q = qR$. Since $N \subset Q$ and $N$ is a prime ideal of $R$, we have that $N = NQ$. Consequently, $N = 0$ (e.g. by Nakayama’s lemma, since $N$ is principal), and thus $R \cong R/N$ is a unique factorization domain. □

**Proposition 3.4.** Let $R$ be a local ring with maximal ideal $M$ such that each proper 2-generated ideal of $R$ has an OA-factorization. Then, $\dim(R) \leq 2$ and each nonmaximal prime ideal of $R$ is principal.

**Proof.** First we show that $\dim(R_P) \leq 1$ for each nonmaximal prime ideal $P$ of $R$. Let $P$ be a nonmaximal prime ideal and let $I$ be a proper 2-generated ideal of $R_P$. Observe that $I = I_P$ for some...
2-generated ideal $J$ of $R$ with $J \subseteq P$. Let $J = \prod_{i=1}^n J_i$ be an OA-factorization. Then, $I = I_P = \prod_{i=1}^n (J_i)_P = \prod_{i=1, i \in P} (J_i)_P$. If $i \in [1, n]$ is such that $J_i \subseteq P$, then $J_i$ is a prime ideal of $R$ by Lemma 2.1(2), and thus $(J_i)_P$ is a prime ideal of $R_P$. We infer that $I$ is a product of prime ideals of $R_P$. It follows from [22, Theorem 3.2], that $R_P$ is a general ZPI-ring. It is an easy consequence of [21, page 205] that $\dim(R_P) \leq 1$.

This implies that $\dim(R) \leq 2$. It remains to show that every nonmaximal prime ideal of $R$ is principal. Without restriction let $\dim(R) \geq 1$. It follows from Proposition 3.3 that $R$ is either a one-dimensional domain or a two-dimensional unique factorization domain. In any case, we have that each nonmaximal prime ideal of $R$ is principal.

In the next result, we will prove a generalization of the fact that every OAF-ring has Krull dimension at most one.

**Theorem 3.5.** Let $R$ be a ring such that every proper 2-generated ideal of $R$ has an OA-factorization. Then, $\dim(R) \leq 1$.

*Proof.* If every OA-ideal of $R$ is a prime ideal, then $R$ is a general ZPI-ring by [22, Theorem 3.2], and hence, $\dim(R) \leq 1$ by [21, page 205]. Now let not every OA-ideal of $R$ be a prime ideal. We infer by Lemma 2.1 that $R$ is local and the maximal ideal of $R$ is not idempotent. Let $M$ be the maximal ideal of $R$. It suffices to show that if $Q$ is a nonmaximal prime ideal of $R$, then $Q = 0$. Let $Q$ be a nonmaximal prime ideal of $R$.

Assume that $Q \not\subseteq M^2$. Since $\dim(R) \leq 2$ by Proposition 3.4, there is some prime ideal $P$ of $R$ such that $Q \subseteq P$ and $\dim(R/P) = 1$. Next we show that $M^2 \subseteq P + yR$ for each $y \in R \setminus P$. Let $y \in R \setminus P$ and set $J = P + yR$. Without restriction let $J \subseteq M$. Note that $J$ is 2-generated by Proposition 3.4. Since $J \not\subseteq M^2$, $J$ cannot be a product of more than one OA-ideal, and thus, $J$ is an OA-ideal of $R$. Since $P \subseteq J \subseteq M$, we have that $J$ is not a prime ideal of $R$, and thus $M^2 \subseteq J$ by Lemma 2.1(2). Moreover, $R/P$ is an integral domain that is not a field. Consequently, $R/P$ does not have any simple $R/P$-submodules, which implies that $P = \cap_{x \in R \setminus P} (P + xR)$. (Observe that if $\cap_{x \in R \setminus P} (P + xR) \neq P$, then $\cap_{x \in R \setminus P} (P + xR)/P$ is a simple $R/P$-submodule of $R/P$.) Therefore, $M^2 \subseteq \cap_{x \in R \setminus P} (P + xR) = P$, and hence $P = M$, a contradiction. We infer that $Q \subseteq M^2$.

There is some $z \in M \setminus M^2$ (since $M$ is not idempotent). Since $zR$ is a product of OA-ideals, we have that $zR$ is an OA-ideal of $R$. As shown before, $L \subseteq M^2$ for each nonmaximal prime ideal $L$ of $R$, and thus $zR$ is not a nonmaximal prime ideal. Consequently, $Q \subseteq M^2 \subseteq zR$ by Lemma 2.1(2), and hence $Q = zQ$. Since $Q$ is principal by Proposition 3.4, it follows (e.g. by Nakayama’s lemma) that $Q = 0$. 

**Lemma 3.6.** Let $D$ be a local domain with maximal ideal $M$. Then, each proper principal ideal of $D$ has an OA-factorization if and only if $D$ is atomic and each irreducible element generates an OA-ideal. If these equivalent conditions are satisfied, then $\cap_{n \in \mathbb{N}} P^n = 0$ for each height-one prime ideal $P$ of $D$.

*Proof.* ($\Rightarrow$) Let each proper principal ideal of $D$ have an OA-factorization. If $D$ is a unique factorization domain, then $D$ is atomic and each irreducible element generates a prime ideal. Now let $D$ be not a unique factorization domain. Then, $\dim(D) = 1$ by Proposition 3.3.

Assume that $M^2$ is principal. Then, $M$ is invertible, and hence $M$ is principal (since $D$ is local). Note that $D$ is a DVR (since $\dim(D) = 1$), and hence, $D$ is a unique factorization domain, a contradiction.

We infer that $M^2$ is not principal. We show that $D$ is atomic. Let $y \in D$ be a nonzero nonunit. Then, $yD = \prod_{i=1}^n I_i$ for some principal OA-ideals $I_i$. There are nonzero nonunits $x_i \in D$ such that $y = \prod_{i=1}^n x_i$ and $I_j = x_jD$ for each $j \in [1, n]$. Let $i \in [1, n]$. If $I_i$ is a prime ideal, then $x_i$ is a prime
element, and thus \( x_i \) is irreducible. Now let \( I_i \) not be a prime ideal. It follows from Lemma 2.1(2) that \( M^2 \subseteq I_i \). Since \( M^2 \) is not principal, we have that \( x_i \notin M^2 \). Therefore, \( x_i \) is irreducible.

Finally, let \( z \in D \) be irreducible. Then, \( zD = \prod_{i=1}^{n} J_i \) for some principal OA-ideals \( J_i \). Since \( zD \) is maximal among the proper principal ideals of \( D \), we obtain that \( zD = J_j \) for some \( j \in [1, n] \).

(\( \Leftarrow \)) Let \( D \) be atomic such that each irreducible element generates an OA-ideal. Let \( I \) be a proper principal ideal of \( D \). Without restriction let \( I \) be nonzero. Then, \( I = xD \) for some nonzero nonunit \( x \in D \). Observe that \( x = \prod_{i=1}^{n} x_i \) for some irreducible elements \( x_i \in D \). It follows that \( \prod_{i=1}^{n} x_i D \) is an OA-factorization of \( I \).

Now let the equivalent conditions be satisfied and let \( P \) be a height-one prime ideal of \( D \). First let \( P \neq M \). Then, \( D \) is a unique factorization domain by Proposition 3.3, and hence, \( P \) is principal. Therefore, \( \cap_{n \in \mathbb{N}} P^n \) is a prime ideal of \( D \) by \([5, \text{Theorem 2.2(1)}]\). Since \( \cap_{n \in \mathbb{N}} P^n \subset P \), we infer that \( \cap_{n \in \mathbb{N}} P^n = 0 \).

Now let \( P = M \). Assume that \( \cap_{n \in \mathbb{N}} M^n \neq 0 \) and let \( x \in \cap_{n \in \mathbb{N}} M^n \) be nonzero. Then, \( xD \) is a product of \( m \) OA-ideals of \( D \) for some positive integer \( m \). We infer by Lemma 2.1(2) that \( M^{2m} \subseteq xD \), and hence, \( M^{2m} \subseteq xD \subseteq M^{4m} \subseteq M^{2m} \). This implies that \( xD = M^{2m} = M^{4m} = x^2 D \), and thus, \( x \) is a unit of \( D \), a contradiction. Therefore, \( \cap_{n \in \mathbb{N}} M^n = 0 \).

**Lemma 3.7.** Let \( R \) be a local ring with maximal ideal \( M \) such that \( M^2 \) is divided and such that either \( M \) is nilpotent or \( R \) is an integral domain with \( \cap_{n \in \mathbb{N}} M^n = 0 \). Then, \( R \) is an OA-ring and every proper principal ideal of \( R \) is a product of principal OA-ideals.

**Proof.** If \( M \) is idempotent, then \( M = 0 \), and hence, \( R \) is a field and both statements are clearly satisfied. Now let \( M \) be not idempotent. There is some \( x \in M \setminus M^2 \). In what follows, we freely use the fact that if \( N \) is an ideal of \( R \) and \( z \in R \) such that \( N \subseteq zR \), then \( N = z(N : zR) \), and hence, \( N = zI \) for some ideal \( I \) of \( R \).

Next we prove that \( M^2 = xM \) and \( xR \) is an OA-ideal of \( R \). Since \( x \notin M^2 \) and \( M^2 \) is divided, we have that \( M^2 \subseteq xR \subseteq M \). Therefore, \( xR \) is an OA-ideal by Lemma 2.1(2). Since \( M^2 \subset xR \), there is some proper ideal \( J \) of \( R \) with \( M^2 = xJ \), and thus \( M^2 \subseteq xM \). Obviously, \( xM \subseteq M^2 \), and hence \( M^2 = xM \).

Now we show that \( R \) is an OA-ring. Let \( I \) be a proper ideal of \( R \). First let \( I = 0 \). If \( M \) is nilpotent, then \( I \) is obviously a product of OA-ideals. If \( R \) is an integral domain, then \( I \) is an OA-ideal. Now let \( I \) be nonzero. In any case, there is a largest positive integer \( n \) such that \( I \subseteq M^n \). Observe that \( I \subseteq M^n = x^{n-1}M \subseteq x^{n-1}R \). Consequently, \( I = x^{n-1}I = (xR)^{n-1}L \) for some proper ideal \( L \) of \( R \). Assume that \( L \subseteq M^2 \). Note that \( L \subseteq M^2 = xM \subseteq xR \). This implies that \( L = xA \) for some proper ideal \( A \) of \( R \), and hence \( I = x^nA \subseteq x^nM = M^{n+1} \), a contradiction. We infer that \( M^2 \subseteq L \) (since \( M^2 \) is divided). It follows from Lemma 2.1(2) that \( L \) is an OA-ideal. In any case, \( I \) is a product of OA-ideals.

Finally, we prove that every proper principal ideal of \( R \) is a product of principal OA-ideals. Let \( y \in M \). First let \( y = 0 \). If \( M \) is nilpotent, then \( x^k = 0 \) for some \( k \in \mathbb{N} \), and thus \( yR = (xR)^k \) is a product of principal OA-ideals. If \( R \) is an integral domain, then \( yR \) is a principal OA-ideal. Now let \( y \) be nonzero. There is some greatest \( \ell \in \mathbb{N} \) such that \( y \in M^\ell \). Therefore, \( y = x^{\ell-1}z \) for some \( z \in M \). If \( z \in M^2 \), then \( z = xv \) for some \( v \in M \), and hence \( y = x^\ell v \in M^{\ell+1} \), a contradiction. We infer that \( z \notin M^2 \), and thus \( M^2 \subseteq zR \subseteq M \). It follows from Lemma 2.1(2) that \( zR \) is an OA-ideal of \( R \). Consequently, \( yR = (xR)^{\ell-1}(zR) \) is a product of principal OA-ideals.

**4. Characterization of OA\-rings and related concepts**

First, we recall several definitions and discuss the factorization theoretical properties of local one-dimensional OA-domains. Let \( D \) be an integral domain with quotient field \( K \). Then, \( \bar{D} = \{ x \in \)
K] there is some nonzero \( c \in D \) such that \( cx^n \in D \) for all \( n \in \mathbb{N} \) is called the complete integral closure of \( D \). Let \( (D : \hat{D}) = \{ x \in D \mid x\hat{D} \subseteq D \} \) be the conductor of \( D \) in \( \hat{D} \). The domain \( D \) is called completely integrally closed if \( D = \hat{D} \) and \( D \) is said to be seminormal if for all \( x \in K \) such that \( x^2, x^3 \in D \), it follows that \( x \in D \). Note that every completely integrally closed domain is seminormal. We say that \( D \) is a finitely primary domain of rank one if \( D \) is a local one-dimensional domain such that \( \hat{D} \) is a DVR and \( (D : \hat{D}) \neq 0 \). For each subset, \( X \subseteq K \) let \( X^{-1} = \{ x \in K \mid xX \subseteq D \} \) and \( X_v = (X^{-1})^{-1} \). An ideal \( I \) of \( D \) is called divisorial if \( I_v = I \). Moreover, \( D \) is called a Mori domain if \( D \) satisfies the ascending chain condition on divisorial ideals. It is well known that every unique factorization domain and every Noetherian domain is a Mori domain (see [14, Corollary 2.3.13] and [11, page 57]). We say that \( D \) is half-factorial if \( D \) is atomic and each two factorizations of each nonzero element of \( D \) into irreducible elements are of the same length. Finally, \( D \) is called a C-domain if the monoid of nonzero elements of \( D \) (i.e. \( D \setminus \{0\} \)) is a C-monoid. For the precise definition of C-monoids, we refer to [14, Definition 2.9.5].

Let \( D \) be a local domain with quotient field \( K \) and maximal ideal \( M \). Set \( (M : M) = \{ x \in K \mid xM \subseteq M \} \). Then, \( (M : M) \) is called the ring of multipliers of \( M \). Moreover, \( M^2 \) is said to be universal if \( M^2 \subseteq uD \) for each irreducible element \( u \in D \).

**Theorem 4.1.** Let \( D \) be a local domain with maximal ideal \( M \) such that \( D \) is not a field. The following statements are equivalent.

1. \( D \) is an OAF-domain.
2. \( D \) is a TAF-domain.
3. \( D \) is one-dimensional and every proper principal ideal has an OA-factorization.
4. \( D \) is one-dimensional and atomic and every irreducible element generates an OA-ideal.
5. \( D \) is atomic such that \( M^2 \) is universal.
6. \( (M : M) \) is a DVR with maximal ideal \( M \).
7. \( D \) is a seminormal finitely primary domain of rank one.

If these equivalent conditions are satisfied, then \( D \) is a half-factorial C-domain and a Mori domain.

**Proof.** (1) \( \Rightarrow \) (2): This follows from Lemma 2.1(3).

(1) \( \Rightarrow \) (3): By Theorem 3.5, \( D \) is one-dimensional. The rest of assertion (3) is clear.

(2) \( \iff \) (5) \( \iff \) (6): This follows from [23, Theorem 4.3].

(3) \( \iff \) (4): This is an immediate consequence of Lemma 3.6.

(4) \( \Rightarrow \) (5): Let \( y \in D \) be an irreducible element. Since \( yD \) is an OA-ideal and \( \sqrt{yD} = M \), we deduce from Lemma 2.1(2) that \( M^2 \subseteq yD \). Hence, \( M^2 \) is universal.

(5) + (6) \( \Rightarrow \) (1): It follows from [6, Theorem 5.1] that \( M^2 \) is comparable to every principal ideal of \( D \), and thus \( M^2 \) is divided. Since \( (M : M) \) is a DVR with maximal ideal \( M \), we have that \( \bigcap_{n \in \mathbb{N}} M^n = \{0\} \). Consequently, \( D \) is an OAF-domain by Lemma 3.7.

(5) + (6) \( \Rightarrow \) (7): First we show that \( D \) is finitely primary of rank one. Let \( P \) be a nonzero prime ideal of \( D \). Then, \( P \) contains an irreducible element \( y \in D \), and hence, \( M^2 \subseteq yD \subseteq P \). Therefore, \( P = M \), and thus, \( D \) is one-dimensional. It remains to show that \( \hat{D} \) is a DVR and \( (D : \hat{D}) \neq 0 \). Since \( (M : M) \) is a DVR, we have that \( (M : M) \) is completely integrally closed. Observe that \( D \subseteq (M : M) \subseteq \hat{D} \), and hence \( \hat{D} \subseteq (M : M) = (M : M) \). Therefore, \( \hat{D} = (M : M) \) is a DVR. Since \( M \hat{D} = M(M : M) \subseteq M \subseteq D \) and \( M \neq 0 \), we infer that \( (D : \hat{D}) \neq 0 \).

Next we show that \( D \) is seminormal. Let \( V \) be the group of units of \( \hat{D} \). Let \( K \) be the field of quotients of \( D \) and let \( x \in K \) be such that \( x^2, x^3 \in D \). Then, \( x^2, x^3 \in \hat{D} \). Since \( \hat{D} \) is a DVR, \( \hat{D} \) is seminormal, and thus \( x \in \hat{D} \). In particular, \( x \in M \) or \( x \in V \). If \( x \in M \), then \( x \in D \). Now let \( x \in V \). Note that \( V \cap D \) is the group of units of \( D \) (by [24, Corollary 1.4] and [12, Proposition 2.1]), and thus \( x^2 \) and \( x^3 \) are units of \( D \). Therefore, \( x = x^{-2}x^3 \) is a unit of \( D \), and hence \( x \in D \).
(7) ⇒ (6): By [15, Lemma 3.3.3], we have that $M$ is the maximal ideal of $\hat{D}$. If $x \in \hat{D}$, then $xM \subseteq M$ (since $M$ is an ideal of $\hat{D}$). It is straightforward to show that $(M : M) \subseteq \hat{D}$. We infer that $(M : M) = \hat{D}$ is a DVR.

Now let the equivalent statements of Theorem 4.1 be satisfied. It remains to show that $D$ is a half-factorial C-domain and a Mori domain. It follows from [6, Theorem 6.2] that $D$ is a half-factorial domain. Obviously, $V$ is a subgroup of finite index of $V$ and $VM \subseteq \hat{D}M = (M : M)M \subseteq M$. It follows from [18, Corollary 2.8] and [14, Corollary 2.9.8] that $D$ is a C-domain. Moreover, $D$ is a Mori domain by [18, Proposition 2.5.1].

We want to point out that a local one-dimensional OAF-domain need not be Noetherian. Let $K \subseteq L$ be a field extension such that $[L : K] = \infty$ and let $D = K + XL[[X]]$. Then, $D$ is a local one-dimensional domain with maximal ideal $M = XL[[X]]$ and $(M : M) = L[[X]]$ is a DVR with maximal ideal $M$. Consequently, $D$ is an OAF-domain by Theorem 4.1. Since $[L : K] = \infty$, it follows that $D$ is not Noetherian.

An integral domain $D$ is called a Cohen-Kaplansky domain if $D$ is atomic and $D$ has only finitely many irreducible elements up to associates. It follows from [6, Example 6.7] that there exists a local half-factorial Cohen-Kaplansky domain with maximal ideal $M$ for which $M^2$ is not universal. We infer by Theorem 4.1 that the aforementioned domain is not an OAF-domain.

**Theorem 4.2.** Let $R$ be a ring with Jacobson radical $M$. The following statements are equivalent.

1. $R$ is an OAF-ring.
2. Each proper 2-generated ideal of $R$ has an OA-factorization.
3. $\dim(R) \leq 1$ and each proper principal ideal has an OA-factorization.
4. $R$ satisfies one of the following conditions.
   - (A) $R$ is a general ZPI-ring.
   - (B) $R$ is a local domain, $M^2$ is divided and $\bigcap_{n \in \mathbb{N}} M^n = 0$.
   - (C) $R$ is local, $M^2$ is divided and $M$ is nilpotent.

**Proof.**

(1) ⇒ (2): This is obvious.

(2) ⇒ (3): This is an immediate consequence of Theorem 3.5.

(3) ⇒ (4): First let each OA-ideal of $R$ be a prime ideal. Then, $R$ is a $\pi$-ring. By [16, Theorems 39.2, 46.7, and 46.11], $R$ is a general ZPI-ring. Now let there be an OA-ideal of $R$ which is not a prime ideal. It follows from Lemma 2.1 that $R$ is local with maximal ideal $M$ and $M$ is not nilpotent. Note that if $x \in M \setminus M^2$, then $xR$ cannot be a product of more than one OA-ideal, and hence $xR$ is an OA-ideal.

Case 1: $R$ is zero-dimensional. Let $x \in M \setminus M^2$. Then, $xR$ is an OA-ideal. We infer by Lemma 2.1(2) that $M^2 \subseteq xR$. Consequently, $M^2$ is divided. It follows from Lemma 2.1 that $M^2 \subseteq I$ for each OA-ideal $I$ of $R$. Since $0$ is a product of OA-ideals, we have that $0$ contains a power of $M$. This implies that $M$ is nilpotent.

Case 2: $R$ is one-dimensional. It follows from Proposition 3.3 that $R$ is an integral domain, and hence $\bigcap_{n \in \mathbb{N}} M^n = 0$ by Lemma 3.6. It remains to show that $M^2$ is divided. Let $x \in R \setminus M^2$. Without restriction let $x$ be a nonunit. Then, $xR$ is an OA-ideal. By Lemma 2.1(2), we have that $M^2 \subseteq xR$.

(4) ⇒ (1): Clearly, every general ZPI-ring is an OAF-ring. The rest follows from Lemma 3.7. □

**Corollary 4.3.** Let $R$ be a ring with Jacobson radical $M$. The following statements are equivalent.

1. Each proper principal ideal of $R$ has an OA-factorization.
2. $R$ is a $\pi$-ring or an OAF-ring.
3. $R$ satisfies one of the following conditions.
(A) $R$ is a $\pi$-ring.
(B) $R$ is a local domain, $M^2$ is divided and $\cap_{n\in\mathbb{N}} M^n = 0$.
(C) $R$ is local, $M^2$ is divided and $M$ is nilpotent.

**Proof.** (1) $\implies$ (2): If $R$ is not local, then $R$ is a $\pi$-ring by Remark 2.4(2). Now let $R$ be local. If $\dim(R) \geq 2$, then $R$ is a unique factorization domain by Proposition 3.3, and hence, $R$ is a $\pi$-ring. If $\dim(R) \leq 1$, then $R$ is an OAF-ring by Theorem 4.2.

(2) $\implies$ (1): This is obvious.

(2) $\iff$ (3): This is an immediate consequence of Theorem 4.2 and the fact that every general ZPI-ring is a $\pi$-ring. \hfill \Box

**Corollary 4.4.** Let $R$ be a ring with Jacobson radical $M$. The following statements are equivalent.

1. Each proper principal ideal of $R$ is a product of principal OA-ideals.
2. $R$ satisfies one of the following conditions.
   - $R$ is a unique factorization ring.
   - $R$ is a local domain, $M^2$ is divided and $\cap_{n\in\mathbb{N}} M^n = 0$.
   - $R$ is local, $M^2$ is divided and $M$ is nilpotent.

**Proof.** (1) $\implies$ (2): If $R$ is not local, then $R$ is a unique factorization ring by Remark 2.4(3). If $R$ is local, then the statement follows from Corollary 4.3 and the fact that every local $\pi$-ring is a unique factorization ring ([4, Corollary 2.2]).

(2) $\implies$ (1): Obviously, if $R$ is a unique factorization ring, then each proper principal ideal of $R$ is a product of principal OA-ideals. The rest is an immediate consequence of Lemma 3.7. \hfill \Box

In Lemma 2.1, we saw that if $R$ is a local ring with maximal ideal $M$ and $I$ is an ideal of $R$ such that $M^2 \subseteq I$, then $I$ is an OA-ideal of $R$. Now we will give a characterization of the rings for which every proper (principal) ideal is an OA-ideal.

**Proposition 4.5.** Let $R$ be a ring with Jacobson radical $M$. The following statements are equivalent.

1. Every proper ideal of $R$ is an OA-ideal.
2. Every proper principal ideal of $R$ is an OA-ideal.
3. $R$ is local and $M^2 = 0$.

**Proof.** (1) $\implies$ (2): This is obvious.

(2) $\implies$ (3): Assume that $R$ is not local. Then, every proper principal ideal of $R$ is a prime ideal by Lemma 2.1(1). Consequently, $R$ is an integral domain. If $x \in R$ is a nonunit, then $x^2R$ is a prime ideal, and hence $x^2R = xR$ and $x = 0$. Therefore, $R$ is a field, a contradiction. This implies that $R$ is local with maximal ideal $M$. We infer by Lemma 2.1(2) that $0$ is a prime ideal or $M^2 = 0$.

Assume that $M^2 \neq 0$. Then, $R$ is an integral domain and there is some nonzero $x \in M^2$. It follows from Lemma 2.1(2) that $x^2R$ is a prime ideal or $M^2 \subseteq x^2R$. If $x^2R$ is a prime ideal, then $x^2R = xR$. If $M^2 \subseteq x^2R$, then $M^2 \subseteq x^2R \subseteq xR \subseteq M^2$, and thus $x^2R = xR$. In any case, we have that $x^2R = xR$, and hence, $x$ is a unit (since $x$ is regular), a contradiction.

(3) $\implies$ (1): This is an immediate consequence of Lemma 2.1(2). \hfill \Box

**5. OA-factorization properties and trival ring extensions**

Let $A$ be a ring and $E$ be an $A$-module. Then, $A \ltimes E$, the trivial (ring) extension of $A$ by $E$, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is
defined by \((a,e)(b,f) = (ab, af + be)\) for all \(a, b \in A\) and all \(e, f \in E\). (This construction is also known by other terminology and other notation, such as the idealization \(A(+)E\).) The basic properties of trivial ring extensions are summarized in the textbooks [17, 19]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [7, 10, 20]). We say that \(E\) is divisible if \(E = aE\) for each regular element \(a \in A\).

We start with the following lemma.

**Lemma 5.1.** Let \(A\) be a ring, \(I\) be an ideal of \(A\) and \(E\) be an \(A\)-module. Let \(R = A \times E\) be the trivial ring extension of \(A\) by \(E\).

1. \(I \propto E\) is an OA-ideal of \(R\) if and only if \(I\) is an OA-ideal of \(A\).
2. Assume that \(A\) contains a nonunit regular element and \(E\) is a divisible \(A\)-module. Then, the OA-ideals of \(R\) have the form \(L \propto E\) where \(L\) is an OA-ideal of \(A\).

**Proof.** (1) This follows immediately from [25, Theorem 2.20].

(2) Let \(J\) be an OA-ideal of \(R\). Our aim is to show that \(0 \propto E \subseteq J\). Let \(e \in E\) and let \(a \in A\) be a nonunit regular element. Then, \(e = af\) for some \(f \in E\) and thus \((a,0)(0,f) = (0,e) \subseteq J\).

Since \(J\) is an OA-ideal, we conclude that \((a,0)(0,f) = (0,e) \subseteq J\) or \((0,e) \subseteq J\) which implies that \(0 \propto E \subseteq J\). Therefore, \(J = L \propto E\) with \(L = \{b \in A | (b,g) \subseteq J\} \text{ for some } g \in E\) and \(L\) is an ideal of \(A\) by [7, Theorems 3.1 and 3.3(1)]. Now the result follows from (1).

**Corollary 5.2.** Let \(A\) be an integral domain that is not a field, \(E\) be a divisible \(A\)-module and \(R = A \times E\). Then, the OA-ideals of \(R\) have the form \(I \propto E\) where \(I\) is an OA-ideal of \(A\).

Next, we study the transfer of the OAF-ring property to the trivial ring extension.

**Theorem 5.3.** Let \(A\) be a ring with Jacobson radical \(M\), \(E\) be an \(A\)-module and \(R = A \times E\).

(1) \(R\) is an OAF-ring if and only if one of the following conditions is satisfied.

- \((A)\) \(A\) is a general ZPI-ring, \(E\) is cyclic and the annihilator of \(E\) is a product of idempotent maximal ideals of \(A\).
- \((B)\) \(A\) is local, \(M^2\) is divided, \(E = 0\) and either \(M\) is nilpotent or \(A\) is a domain with \(\cap_{n \in \mathbb{N}} M^n = 0\).
- \((C)\) \(A\) is local, \(M^2 = 0\), \(ME = aE\) for each nonzero \(a \in M\) and \(ME = Mx\) for each \(x \in E \setminus ME\).

In particular, if \(R\) is an OAF-ring, then \(A\) is an OAF-ring.

(2) Every proper ideal of \(R\) is an OA-ideal if and only if \(A\) is local, \(M^2 = 0\) and \(ME = 0\).

**Proof.** (1) \((\Rightarrow)\) First let \(R\) be an OAF-ring. By Theorem 4.2, it follows that (a) \(R\) is a general ZPI-ring or (b) \(R\) is local with maximal ideal \(N\), \(N^2\) is divided and \((N\) is nilpotent or \(R\) is a domain such that \(\cap_{n \in \mathbb{N}} N^n = 0\)). If \(R\) is a general ZPI-ring, then condition \((A)\) is satisfied by [7, Theorem 4.10].

From now on let \(R\) be local with maximal ideal \(N\) such that \(N^2\) is divided. Observe that \(A\) is local with maximal ideal \(M = M \propto E\) by [7, Theorem 3.2(1)]. If \(R\) is a domain such that \(\cap_{n \in \mathbb{N}} N^n = 0\), then \(E = 0\) (for if \(z \in E\) is nonzero, then \((0,z)\) is a nonzero zero-divisor of \(R\)), and hence \(A \cong R\) is a domain, \(M^2\) is divided and \(\cap_{n \in \mathbb{N}} M^n = 0\).

Now let \(N\) be nilpotent. If \(E = 0\), then \(A \cong R\) and thus \(M^2\) is divided and \(M\) is nilpotent. From now on let \(E\) be nonzero. There is some \(k \in \mathbb{N}\) such that \(N^k = 0\). Note that \(N^2 = M^2 \propto ME\) and \(N^k = M^k \propto M^{k-1}E\), and thus \(M^k = 0\). Since \(N^2\) is divided, we have that \(0 \propto E \subseteq N^2\) or \(N^2 \subseteq 0 \propto E\). If \(0 \propto E \subseteq N^2\), then \(E = ME\), and hence \(E = M^kE = 0\), a contradiction. Therefore, \(N^2 \subseteq 0 \propto E\), which implies that \(M^2 = 0\).
Let \( a \in M \) be nonzero. Then, \( (a,0) \notin N^2 \), and hence \( N^2 \subseteq (a,0)R = aA \propto aE \). Consequently, \( ME \subseteq aE \), and thus \( ME = aE \). Finally, let \( x \in E \setminus ME \). Then, \( (0,x) \notin N^2 \). We infer that \( N^2 \subseteq (0,x)R = 0 \propto Ax \). This implies that \( ME \subseteq Ax \). If \( ME \nsubseteq Mx \), then \( bx \in ME \) for some unit \( b \in A \), and hence \( x \in ME \), a contradiction. It follows that \( ME \subseteq Mx \), which clearly implies that \( ME = Mx \).

(\( \Leftarrow \)) Next we prove the converse. If condition (A) is satisfied, then \( R \) is a general ZPI-ring by [7, Theorem 4.10], and thus \( R \) is an OAF-ring. If condition (B) is satisfied, then \( A \) is an OAF-ring by Theorem 4.2, and hence \( R \cong A \) is an OAF-ring. Now let condition (C) be satisfied. Set \( N = M \propto E \). Then, \( R \) is local with maximal ideal \( N \) by [7, Theorem 3.2(1)]. By Theorem 4.2, it suffices to show that \( N \) is nilpotent and \( N^2 \) is divided. Since \( M^2 = 0 \), we obtain that \( N^3 = M^3 \propto M^2E = 0 \), and thus \( N \) is nilpotent. It remains to show that \( N^2 \subseteq (a,x)R \) for each \( (a,x) \in R \setminus N^2 \).

Let \( a \in A \) and \( x \in E \) be such that \( (a,x) \notin N^2 \). Since \( N^2 = 0 \propto ME \), we have to show that \( 0 \propto ME \subseteq (a,x)R \). If \( a \) is a unit of \( A \), then \( (a,x) \) is a unit of \( R \) by [7, Theorem 3.7] and the statement is clearly true. Let \( z \in 0 \propto ME \). Then, \( z = (0,y) \) for some \( y \in ME \).

Case 1: \( a \) is a nonzero nonunit. Since \( ME = aE \), there is some \( v \in E \) such that \( y = av \). Observe that \( z = (0,av) = (a,x),(0,v) \in (a,x)R \).

Case 2: \( a = 0 \). Then, \( x \in E \setminus ME \) (since \( (a,x) \notin N^2 \)). Since \( ME = Mx \), there is some \( b \in M \) such that \( y = bx \). It follows that \( z = (0,bx) = (a,x)(b,0) \in (a,x)R \).

The in particular statement now follows from Theorem 4.2.

(2) First let every proper ideal of \( R \) be an OA-ideal. By Proposition 4.5, we have that \( R \) is local with maximal ideal \( N \) and \( N^2 = 0 \). It follows that \( A \) is local with maximal ideal \( M \) and \( N = M \propto E \) by [7, Theorem 3.2(1)]. Moreover, \( 0 = N^2 = M^2 \propto ME \), and hence \( M^2 = 0 \) and \( ME = 0 \).

Conversely, let \( A \) be local, \( M^2 = 0 \) and \( ME = 0 \). Set \( N = M \propto E \). Then, \( R \) is local with maximal ideal \( N \) by [7, Theorem 3.2(1)] and \( N^2 = M^2 \propto ME = 0 \). We infer by Proposition 4.5 that each proper ideal of \( R \) is an OA-ideal.

\[ \square \]

\textbf{Corollary 5.4.} Let \( A \) be an integral domain, \( E \) be a nonzero \( A \)-module and \( R = A \propto E \). The following statements are equivalent.

\begin{enumerate}
\item \( R \) is an OAF-ring.
\item \( A \) is a field.
\item Every proper ideal of \( R \) is an OA-ideal.
\end{enumerate}

\textbf{Proof.} (1) \( \Rightarrow \) (2): It follows from Theorem 5.3(1) that \( A \) is a general ZPI-ring and the annihilator of \( E \) is a product of idempotent maximal ideals of \( A \) or that \( A \) is local with maximal ideal \( M \) such that \( M^2 = 0 \).

First, let \( A \) be a general ZPI-ring such that the annihilator of \( E \) is a product of idempotent maximal ideals of \( A \). Note that \( A \) is a Dedekind domain, and thus, the only proper idempotent ideal of \( A \) is the zero ideal. Since \( E \) is nonzero, the annihilator of \( E \) is a proper ideal of \( A \), and hence \( A \) must possess an idempotent maximal ideal. We infer that the zero ideal is a maximal ideal of \( A \), and thus \( A \) is a field.

Now let \( A \) be local with maximal ideal \( M \) such that \( M^2 = 0 \). Since \( A \) is an integral domain, it follows that \( M = 0 \), and hence \( A \) is a field.

(2) \( \Rightarrow \) (3): Set \( M = 0 \). Then, \( A \) is local with maximal ideal \( M \), \( M^2 = 0 \), and \( ME = 0 \). Now the statement follows from Theorem 5.3(2).

(3) \( \Rightarrow \) (1): This is obvious. \[ \square \]

\textbf{Remark 5.5.} In general, if \( A \) is an OAF-ring and \( E \) is an \( A \)-module, then \( A \propto E \) need not be an OAF-ring. Indeed, let \( A \) be an OAF-domain that is not a field and let \( E \) be a nonzero \( A \)-module. By Corollary 5.4, \( A \propto E \) is not an OAF-ring.
Corollary 5.6. Let $A$ be a local ring with maximal ideal $M$ and $E$ be a nonzero $A$-module such that $ME = 0$. Set $R = A \sim E$. The following statements are equivalent.

1. $R$ is an OAF-ring.
2. $M^2 = 0$.
3. Every proper ideal of $R$ is an OA-ideal.

Proof. (1) $\Rightarrow$ (2): Assume that $M^2 \neq 0$. By Theorem 5.3(1), $A$ is a local general ZPI-ring and $M$ is idempotent (since the annihilator of $E$ is a nonempty product of idempotent maximal ideals of $A$ and $M$ is the only maximal ideal of $A$). We infer by [21, Corollary 9.11] that $A$ is a Dedekind domain or each proper ideal of $A$ is a power of $M$ (because local rings are indecomposable). If $A$ is a Dedekind domain, then clearly $M^2 = M = 0$ (since $M$ is idempotent and a Dedekind domain has no nonzero proper idempotent ideals). Moreover, if every proper ideal of $A$ is a power of $M$, then again $M^2 = M = 0$ (since $M$ is idempotent). In any case, we obtain that $M^2 = 0$, a contradiction.

(1) $\iff$ (2) $\iff$ (3): This follows from Theorem 5.3. □

Example 5.7. Let $A$ be a local principal ideal ring with maximal ideal $M$ such that $A$ is not a field and $M^2 = 0$ (e.g. $A = \mathbb{Z}/4\mathbb{Z}$). Set $R = A \sim A$. Then, $R$ is an OAF-ring, and yet not every proper ideal of $R$ is an OA-ideal.

Proof. Since $M \neq 0$, it follows from Theorem 5.3(2) that not every proper ideal of $R$ is an OA-ideal. By Theorem 5.3(1), it remains to show that $M = aA$ for each nonzero $a \in M$ and $M = Mx$ for each $x \in A \backslash M$. Note that $M = zA$ for some $z \in M$. If $a \in M$ is nonzero, then $a = zb$ for some $b \in A$. Clearly, $b \notin M$, and thus $b$ is a unit of $A$, which clearly implies that $M = zA = aA$. Finally, if $x \in A \backslash M$, then $x$ is a unit of $A$, and thus $M = Mx$. □

Remark 5.8. Let $A$ be a ring with Jacobson radical $M$, $E$ be an $A$-module and $R = A \sim E$. Then, each proper principal ideal of $R$ has an OA-factorization if and only if one of the following conditions is satisfied.

1. $A$ is a $\pi$-ring, $E$ is cyclic and the annihilator of $E$ is a (possibly empty) product of idempotent maximal ideals of $A$.
2. $A$ is local, $M^2$ is divided, $E = 0$ and either $M$ is nilpotent or $A$ is a domain with $\cap_{n \in \mathbb{N}} M^n = 0$.
3. $A$ is local, $M^2 = 0$, $ME = aE$ for each nonzero $a \in M$ and $ME = Mx$ for each $x \in E \backslash ME$.

Proof. This can be proved along similar lines as Theorem 5.3(1) by using Corollary 4.3 and [7, Theorems 3.2(1) and 4.10]. □

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