ASYMPTOTIC BEHAVIOR OF A NONLOCAL PARABOLIC PROBLEM
IN OHMIC HEATING PROCESS

QILIN LIU, FEI LIANG, AND YUXIANG LI

Abstract. In this paper, we consider the asymptotic behavior of the nonlocal parabolic problem

$$u_t = \Delta u + \frac{\lambda f(u)}{\left( \int_\Omega f(u) dx \right)^p}, \quad x \in \Omega, \ t > 0,$$

with homogeneous Dirichlet boundary condition, where \( \lambda > 0, \ p > 0, \ f \) is nonincreasing. It is found that: (a) For \( 0 < p \leq 1 \), \( u(x, t) \) is globally bounded and the unique stationary solution is globally asymptotically stable for any \( \lambda > 0 \); (b) For \( 1 < p < 2 \), \( u(x, t) \) is globally bounded for any \( \lambda > 0 \); (c) For \( p = 2 \), if \( 0 < \lambda < 2|\partial\Omega|^2 \), then \( u(x, t) \) is globally bounded, if \( \lambda = 2|\partial\Omega|^2 \), there is no stationary solution and \( u(x, t) \) is a global solution and \( u(x, t) \to \infty \) as \( t \to \infty \) for all \( x \in \Omega \); (d) For \( p > 2 \), there exists a \( \lambda^* > 0 \) such that for \( \lambda > \lambda^* \), or for \( 0 < \lambda \leq \lambda^* \) and \( u_0(x) \) sufficiently large, \( u(x, t) \) blows up in finite time for all \( x \in \Omega \); (e) For \( p > 2 \), there exists a \( \lambda^* > 0 \) such that for \( \lambda > \lambda^* \), or for \( 0 < \lambda \leq \lambda^* \) and \( u_0(x) \) sufficiently large, \( u(x, t) \) blows up in finite time. Moreover, some formal asymptotic estimates for the behavior of \( u(x, t) \) as it blows up are obtained for \( p \geq 2 \).

1. Introduction

In this paper, we study the asymptotic behavior of the following nonlocal parabolic problem

$$\begin{cases}
    u_t = \Delta u + \frac{\lambda f(u)}{\left( \int_\Omega f(u) dx \right)^p}, & x \in \Omega, \ t > 0, \\
    u(x, t) = 0, & x \in \partial\Omega, \ t > 0, \\
    u(x, 0) = u_0(x), & x \in \Omega,
\end{cases} \quad (1.1)$$

where \( \lambda > 0 \) and \( u(x, t) = u(x, t, \lambda) \) stands for the dimensionless temperature of a conductor when an electric current flows through it \([3, 7, 10]\). \( \Omega \) is a bounded domain of \( R^n \) with \( C^2 \) boundary. The nonlinear function \( f(s) \) satisfies the condition

$$f(s) > 0, \ f'(s) < 0, \ s \geq 0, \ \int_0^\infty f(s) ds < \infty,$$ \quad (1.2)

and represents, depending on the problem, either the electrical conductivity or the electrical resistance of the conductor. Condition \((1.2)\) permits us to use comparison methods, see \([7, 8, 9, 10]\). Also for simplicity, we assume \( u_0(x) \) is continuous with \( u_0(x) = 0, \ x \in \partial\Omega \) and \( u_0(x) \geq 0, \ x \in \Omega \). Without loss of generality, we may assume that \( \int_0^\infty f(s) ds = 1 \).

A similar study had been undertaken in \([7, 8, 10, 5, 6]\) for the nonlocal reaction-diffusion problem \((1.1)\) for \( p = 2 \). Lacey \([7, 8]\) and Tzanetis \([10]\) proved the occurrence of blow-up.
for the one-dimensional problem and for the two-dimensional radially symmetric problem, respectively. First they estimated the supremum $\lambda^*$ of the spectrum of the related steady-state problem and then they proved the blow-up, for $\lambda > \lambda^*$, by constructing some blowing-up lower solutions. Using some ideas of Bebernes and Lacey [1], Kavallaris and Tzanetis [6] generalized the blow-up results for $\lambda > \lambda^*$ and dimensions $n \geq 2$ if $u_0$ is sufficiently large and $f(s)$ satisfies
\[
\int_0^\infty [sf(s) - s^2f'(s)]ds < \infty. \tag{1.3}
\]
Kavallaris and Lacey [5] showed that the solution $u^*(x, t) = u(x, t, \lambda^*)$ is global in time and diverges in the sense $\|u^*(\cdot, t)\|_\infty \to \infty$ as $t \to \infty$ when $n = 1, \Omega = (-1, 1)$ and $f(s)$ satisfies (1.2) or $n = 2, \Omega = \{(x, y) \in R^2 : x^2 + y^2 < 1\}$ and $f(s) = e^{-s}$. Moreover, it is proved that this divergence is global, i.e. $u^*(x, t) \to \infty$ as $t \to \infty$ for all $x \in \Omega$.

Throughout this paper, we always assume that the domain $\Omega$ satisfies the following condition:

(H) $\Omega \subset R^n$ is a convex smooth bounded domain; for any point $y_0 \in \partial\Omega$, there exists a hyperplane $S_{y_0}$ such that $S_{y_0}$ is tangent to $\Omega$ at $y_0$ ($\{y_0\} = S_{y_0} \cap \partial\Omega$).

The main purpose of this paper is to generalize and improve the results for dimensions $n \geq 2$ and $p > 0$ obtained in [7, 8, 10, 5, 6]. Our main results read as follows.

- If $0 < p \leq 1$, then $u(x, t)$ is globally bounded and there exists a unique stationary solution which is globally asymptotically stable for any $\lambda > 0$.
- If $1 < p < 2$, then $u(x, t)$ is globally bounded for any $\lambda > 0$.
- Assume $p = 2$, let $\lambda^* = 2|\partial\Omega|^2$. If $0 < \lambda < \lambda^*$, $u(x, t)$ is globally bounded. If $\lambda = \lambda^*$, there is no stationary solution and $u^*(x, t)$ is a global-in-time solution and $u^*(x, t) \to \infty$ as $t \to \infty$ for all $x \in \Omega$. If $\lambda > \lambda^*$, there is no stationary solution and $u(x, t)$ blows up globally in finite time $T$ but the condition (1.3) and $u_0$ sufficiently large are not required.
- If $p > 2$, then there exists a critical value $\lambda^*$ such that for $\lambda > \lambda^*$ or for any $0 < \lambda \leq \lambda^*$ and $u_0(x)$ sufficiently large, $u(x, t)$ blows up globally in finite time $T$.
- We also obtain some formal asymptotic estimates for the local behavior of $u(x, t)$ as it blows up for $p \geq 2$.

This paper is organized as follows. In Section 2 we consider the steady-state problem corresponding to (1.1). In Section 3, we investigate the behavior of some critical solutions of the equation (1.1) for $p = 2$. Section 4 is devoted to some formal asymptotic estimates for the local behavior of $u(x, t)$ as it blows up in finite time for $p \geq 2$.

2. Steady-state problem

The steady states of the problem (1.1) play an important role in the description of the asymptotic behavior of the solutions of (1.1) and the construction of the lower and upper solutions, so we first consider the stationary problem of (1.1). The stationary problem corresponding to (1.1) is
\[
\Delta w + \frac{\lambda f(w)}{\int_{\Omega} f(w)dx} = 0, \quad x \in \Omega; \quad w = 0, \quad x \in \partial\Omega. \tag{2.1}
\]

In order to study the nonlocal problem (2.1), let us first consider the following local problem:
\[
\Delta w + \mu f(w) = 0, \quad x \in \Omega; \quad w = 0, \quad x \in \partial\Omega, \tag{2.2}
\]
where $\mu \geq 0$ and $f(s)$ satisfies (1.2). It is well-known that the basic theory of monotone schemes can be carried out for the problem (2.2). Therefore, there exists a solution in $H^1_0(\Omega)$. Moreover, the straightforward argument, based on the coercivity of $-\Delta$ with Dirichlet boundary condition, implies that (2.2) has a unique positive solution $w^\Omega_\mu$ in $H^1_0(\Omega)$. The above arguments are classical and known in the literature [4].

In order to establish a relationship between the local problem (2.2) and the nonlocal problem (2.1), we define a real function $\lambda(\mu)$ by
\[
\lambda(\mu) = \mu \left( \int_{\Omega} f(w^\Omega_\mu) dx \right)^p,
\]
for any $\mu \geq 0$. This function is well defined due to the positive character of $w^\Omega_\mu$. From the analyticity of the solutions $w^\Omega_\mu$ on $\mu$, we deduce that the function $\lambda(\mu)$ is analytical on $\mu$.

It is easy to see the relation between the solutions of problem (2.2) and the problem (2.1).

**Theorem 2.1.** If $w$ is a solution of problem (2.1) for $\lambda = \lambda_0$, then $w$ is a solution of problem (2.2) for $\mu = \lambda_0 \left( \int_{\Omega} f(w^\Omega_\mu) dx \right)^p$. Conversely, if $w$ is a solution of problem (2.2) for $\mu = \mu_0$, then $w$ is a solution of problem (2.1) for $\lambda = \lambda(\mu_0)$.

Theorem 2.1 allows us to study problem (2.1) by analyzing the behavior of the function $\lambda(\mu)$. This is the key idea to solve problem (2.1). Now we give some qualitative properties of the profile of the bifurcation diagram of the local problem (2.2).

**Lemma 2.2.** Let $w^\Omega_\mu$ be the solution of (2.2), then
\begin{enumerate}
  \item $\partial w^\Omega_\mu / \partial \mu > 0$ for $x \in \Omega$.
  \item $\lim_{\mu \to \infty} w^\Omega_\mu(x) / \Phi^1_\Omega(x) \to \infty$, uniformly in $\Omega$, where $\Phi^1_\Omega(x)$ is the first normalized eigenfunction of $-\Delta$ in $H^1_0(\Omega)$.
\end{enumerate}

The proof follows the same line as in [2], so we omit it.

Now we are going to prove that the solution of (2.1) is unique for any $0 < p \leq 1$.

**Theorem 2.3.** For any $0 < p \leq 1$, there exists a unique solution of the problem (2.1) for any $\lambda \geq 0$.

**Proof.** Let us prove that $\lambda(\mu)$ is strictly increasing. Integrating the equation (2.2) over $\Omega$, we have
\[
\int_{\partial\Omega} \frac{\partial w}{\partial \nu} ds + \lambda^\frac{1}{p} \mu \frac{1}{p-1} = 0,
\]
where $\partial / \partial \nu$ is the outward normal derivative, which implies
\[
\lambda(\mu) = \mu^{1-p} \left( \int_{\partial\Omega} \frac{\partial w}{\partial \nu} ds \right)^p.
\]
By $0 < p \leq 1$, $w_\mu = 0$ on $\partial\Omega$ and Lemma 2.2, we get
\[
\lambda'(\mu) > 0 \text{ for } \mu > 0 \text{ and } \lim_{\mu \to \infty} \lambda(\mu) = \infty.
\]
The proof is completed.

The following results give us a way to construct sub-solution of $w^\Omega_\mu$ in order to estimate from above the function $\lambda(\mu)$.

**Lemma 2.4.** Let $\Omega' \subset \Omega$. Then $w^\Omega_{\mu'} \leq w^\Omega_\mu$ on $\Omega'$ for any $\mu > 0$. 

We omit the proof.

We need a lemma concerning the solution to the problem on a ball
\[ \Delta w + \mu f(w) = 0, \quad x \in B; \quad w = 0, \quad x \in \partial B. \] (2.5)

**Lemma 2.5.** (See [10, Lemma 5.1]) Let \( f(s) \) satisfy (1.2), \( \int_0^\infty f(s)ds = 1 \) and \( w^B_\mu \) is a solution of (2.5), then we have
\[ -\frac{1}{\sqrt{\mu}} \frac{dw^B_\mu(R)}{dr} < \sqrt{2}, \quad \lim_{\mu \to \infty} -\frac{1}{\sqrt{\mu}} \frac{dw^B_\mu(R)}{dr} = \sqrt{2}, \] (2.6)
where \( B = \{ x \in R^n : |x - x_0| < R \}, \) \( r = |x - x_0| \).

**Theorem 2.6.** Let \( f(s) \) satisfy (1.2), \( \int_0^\infty f(s)ds = 1, \) and \( \Omega \) is a bounded domain satisfying (H). Then the following assertions hold.

1. For \( 1 < p < 2, \) there exists at least one solution of the problem (2.7) for any value \( \lambda > 0 \).
2. For \( p = 2, \) let \( \lambda^* = 2|\partial \Omega|^2, \) then there exists at least one solution of the problem (2.1) for \( 0 < \lambda < \lambda^* \) and no solution for \( \lambda \geq \lambda^* \). Moreover, \( \lambda(\mu) < 2|\partial \Omega|^2 \) for \( \mu > 0 \) and \( \lim_{\mu \to \infty} \lambda(\mu) = 2|\partial \Omega|^2 \).
3. For \( p > 2, \) there exists a critical value \( \lambda^* > 0 \) such that there exist at least two solutions of the problem (2.7) for \( 0 < \lambda < \lambda^* \), at least one solution for \( \lambda = \lambda^* \) and no solution for \( \lambda > \lambda^* \). Moreover, \( \lim_{\mu \to \infty} \lambda(\mu) = 0 \).

**Proof.** Let \( y_0 \in \partial \Omega \). Without loss of generality we assume that \( y_0 = 0 \) and the hyperplane \( \{ x \in R^n : x_1 = 0 \} \) is tangent to \( \Omega \) at \( y_0 \). By (H), there exist two balls \( \Omega_1, \Omega_2 (\Omega_1 \subset \Omega \subset \Omega_2) \) which are tangent to \( \Omega \) at \( y_0 \), where \( \Omega_i = \{ x \in R^n : |x - y_i| < R_i, y_i = (L_i, 0') \} \). Lemma [2.4] implies that \( w_{\mu}^{\Omega_1} \geq w_{\mu}^{\Omega_2} \) on \( \Omega_1 \) and \( w_{\mu}^{\Omega_2} \geq w_{\mu}^{\Omega} \) on \( \Omega \). Applying Lemma 2.5, we conclude that
\[ \sqrt{2} > -\frac{1}{\sqrt{\mu}} \frac{dw_{\mu}^{\Omega_2}(0)}{dx_1} \geq -\frac{1}{\sqrt{\mu}} \frac{dw_{\mu}^{\Omega}(0)}{dx_1} = -\frac{1}{\sqrt{\mu}} \frac{dw_{\mu}^{\Omega_1}(0)}{dx_1}, \quad \mu > 0 \]
and
\[ \sqrt{2} = -\lim_{\mu \to \infty} \frac{1}{\sqrt{\mu}} \frac{dw_{\mu}^{\Omega_2}(0)}{dx_1} \geq -\lim_{\mu \to \infty} \frac{1}{\sqrt{\mu}} \frac{dw_{\mu}^{\Omega}(0)}{dx_1} \geq -\lim_{\mu \to \infty} \frac{1}{\sqrt{\mu}} \frac{dw_{\mu}^{\Omega_1}(0)}{dx_1} = \sqrt{2}, \]
which imply
\[ -\frac{1}{\sqrt{\mu}} \frac{dw_{\mu}^{\Omega}(0)}{dx_1} < \sqrt{2} \quad \text{and} \quad -\lim_{\mu \to \infty} \frac{1}{\sqrt{\mu}} \frac{dw_{\mu}^{\Omega}(0)}{dx_1} = \sqrt{2}. \]
Since \( y_0 \) is arbitrary, it follows that
\[ -\frac{1}{\sqrt{\mu}} \int_{\partial \Omega} \frac{\partial w_{\mu}^{\Omega}}{\partial \nu} ds < \sqrt{2}|\partial \Omega| \quad \text{for} \quad \mu > 0 \quad \text{and} \quad -\lim_{\mu \to \infty} \frac{1}{\sqrt{\mu}} \int_{\partial \Omega} \frac{\partial w_{\mu}^{\Omega}}{\partial \nu} ds = \sqrt{2}|\partial \Omega|. \]

By (2.4), we obtain
(i) If \( 0 < p < 2, \) then \( \lim_{\mu \to \infty} \lambda(\mu) = \infty. \)
(ii) If \( p = 2, \) then \( \lambda(\mu) < 2|\partial \Omega|^2 \) for \( \mu > 0 \) and \( \lim_{\mu \to \infty} \lambda(\mu) = 2|\partial \Omega|^2. \)
(iii) If \( p > 2, \) then \( \lim_{\mu \to \infty} \lambda(\mu) = 0. \)

The proof is completed. \( \square \)

Let \( \mu \) in (2.2) be a function of \( t. \) Now we give some conditions of \( \mu(t) \) in order for \( w(x; \mu(t)) \) to be a lower or an upper solution of (1.1). We first give a lemma.
Lemma 2.7. \( w(x; \mu) \) is the solution of (2.3), then \( w\mu > 0 \) in \( \Omega \) and \( w\mu \) is bounded.

Proof. \( w\mu \) satisfies

\[
\begin{cases}
\Delta w\mu + \mu f'(w)w\mu + f(w) = 0, & x \in \Omega, \\
w\mu(x) = 0, & x \in \partial \Omega.
\end{cases}
\]

Since \( f'(s) < 0 \), the coefficient of \( w\mu \) in this equation is negative. By the maximum principle we obtain \( w\mu > 0 \). Also \( w\mu \) is finite, indeed for a fixed \( \mu \), any sufficiently large constant is an upper solution, \( 0 \leq w\mu \leq C \).

Using Lemma 2.7,

\[
\inf_{x \in \Omega} \frac{f(w)}{w\mu} > 0,
\]

since \( f(w) \) is bounded and away from zero. Denote \( v(x, t) = w(x; \mu(t)) \), then

\[
v_t - \Delta v - \frac{\lambda f(v)}{\left( \int_{\Omega} f(v)dx \right)^{p}} = w\mu'(t) - \frac{(\lambda - \lambda(\mu)) f(w)}{\left( \int_{\Omega} f(w)dx \right)^{p}}.
\]

Let \( \mu(t) \) be the solution of

\[
\mu'(t) = \frac{\lambda - \lambda(\mu)}{\left( \int_{\Omega} f(w)dx \right)^{p}} \inf_{x \in \Omega} \frac{f(w)}{w\mu}, \quad \mu(0) = \mu_0. \tag{2.7}
\]

If there exists \( \mu_0 \) such that

\[
\lambda \leq \lambda(\mu_0) \quad \text{and} \quad w(x; \mu_0) \geq u_0(x),
\]

then \( v(x, t) \) is decreasing and satisfies

\[
v_t - \Delta v - \frac{\lambda f(v)}{\left( \int_{\Omega} f(v)dx \right)^{p}} \geq 0.
\]

So \( v(x, t) \) is a decreasing upper solution of (1.1).

If there exists \( \mu_0 \) such that

\[
\lambda \geq \lambda(\mu_0) \quad \text{and} \quad w(x; \mu_0) \leq u_0(x),
\]

then \( v(x, t) \) is increasing and satisfies

\[
v_t - \Delta v - \frac{\lambda f(v)}{\left( \int_{\Omega} f(v)dx \right)^{p}} \leq 0.
\]

So \( v(x, t) \) is an increasing lower solution of (1.1).

The above preparations in hand, we can discuss the behavior of the solution of (1.1).

Theorem 2.8. Assume \( 0 < p \leq 1 \), then the solution of (1.1) is globally bounded and the unique steady state is globally asymptotically stable for any \( \lambda > 0 \).

Proof. From Theorem 2.3, for fixed \( \lambda \), there is a unique steady state \( w(x; \mu_1) \) of (2.1) with \( \lambda = \lambda(\mu_1) \). Take \( \overline{\mu}(t) \) satisfying (2.7) with \( \mu(0) = \overline{\mu}_0 \). For any initial data \( u_0(x) > 0 \), we can select \( \overline{\mu}_0 \) to satisfy \( w(x; \overline{\mu}_0) \geq u_0(x) \). This can clearly be done if we require that \( u_0(x) \) and \( u_0'(x) \) are bounded(see[8]). We also choose \( \overline{\mu}_0 > \mu_1 \). Since \( \lambda(\mu) > 0 \), we have

\[
\lambda \leq \lambda(\overline{\mu}_0) \quad \text{and} \quad w(x; \overline{\mu}_0) \geq u_0(x),
\]
thus \( \overline{u}(t) \) is decreasing and \( \overline{v}(t) \to \mu_1 \) as \( t \to \infty \). So \( \overline{v}(x, t) = w(x; \overline{u}(t)) \) is a decreasing upper solution of the problem and

\[
\overline{v}(x, t) \to w(x; \mu_1), \quad \text{as} \quad t \to +\infty.
\]

On the other hand, take \( \underline{\mu}(t) \) satisfying \( (2.7) \) with \( \mu(0) = \underline{\mu}_0 \). Since

\[
\lambda(\mu) \to 0, \quad \text{as} \quad \mu \to 0,
\]

we can select \( \underline{\mu}_0 \), sufficiently small such that

\[
\lambda \geq \lambda(\underline{\mu}_0) \quad \text{and} \quad w(x; \underline{\mu}_0) \leq u_0(x),
\]

thus \( \underline{\mu}(t) \) is increasing and \( \underline{\mu}(t) \to \mu_1 \) as \( t \to \infty \). So \( \underline{\mu}(x, t) = w(x; \underline{\mu}(t)) \) is an increasing lower solution of the problem and

\[
\underline{\mu}(x, t) \to w(x; \mu_1), \quad \text{as} \quad t \to +\infty.
\]

Since \( \underline{\mu} \leq u(x, t) \leq \overline{v} \) and both \( v \) and \( \overline{v} \) tend to \( w(x; \mu_1) \) as \( t \to \infty \), we see that \( u(x, t) \) exists globally and \( u(x, t) \to w(x; \mu_1) \) as \( t \to \infty \). The above procedure holds for any initial data \( u_0(x) \), from which it follows that the solution \( w(x; \mu_1) \) is globally asymptotically stable. The proof is completed. \( \square \)

**Theorem 2.9.** If \( 1 < p < 2 \) and \( \int_0^\infty f(s) ds = 1 \), then \( u(x, t) \) is globally bounded for any \( \lambda > 0 \).

**Proof.** For the global boundedness of \( u(x, t) \), it suffices to construct an upper solution which is globally bounded. Select \( \mu_0 \) so large that

\[
\lambda \leq \lambda(\mu_0) \quad \text{and} \quad w(x; \mu_0) \geq u_0(x),
\]

then \( \mu(t) \), the solution of \( (2.7) \), is decreasing and therefore \( w(x; \mu(t)) \) is a globally bounded upper solution. \( \square \)

For \( p = 2 \), we have the similar result.

**Theorem 2.10.** If \( p = 2 \), \( \int_0^\infty f(s) ds = 1 \) and \( 0 < \lambda < 2|\partial \Omega|^2 \), then \( u(x, t) \) is globally bounded for any initial data.

3. **Behavior of solutions of problem \( (1.1) \) for \( p = 2 \)**

In this Section, we study the behavior of solutions of the following nonlocal parabolic problem:

\[
\begin{cases}
  u_t = \Delta u + \frac{2|\partial \Omega|^2 f(u)}{\left( \int_\Omega f(u) dx \right)^2}, & x \in \Omega, \ t > 0, \\
  u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]

(3.1)

where \( f \) satisfies \( (1.2) \) and \( \int_0^\infty f(s) ds = 1 \). By Theorem 2.6, it follows that \( \lambda(\mu) < 2|\partial \Omega|^2 \) for all \( \mu > 0 \), then we can find an increasing lower solution \( v = w(x; \mu(t)) \) with \( \mu(t) \to \infty \) as \( t \to T \leq \infty \). Thus \( u(x, t) \) is unbounded. Moreover, \( u(x, t) \) is globally unbounded. Indeed, if \( T = \infty \), from Lemma 2.2, \( u(x, t) \) is globally unbounded; if \( T < \infty \), \( u(x, t) \) is globally blow-up (see the proof of Theorem 4.1 for details).

Now we will prove that \( \|u(\cdot, t)\|_\infty \to \infty \) as \( t \to \infty \), i.e. \( T = \infty \). It is sufficient to construct an upper solution \( V(x, t) \) to problem \( (3.1) \) which is global in time and unbounded. Without
loss of generality, we assume that the hyperplane \( \{ x : x_1 = 1 \} \) is tangent to \( \Omega \) at \((1,0')\), and \( \Omega \) lies in the half-space \( \{ x : x_1 < 1 \} \). Let \( d(x) = \text{dist}(x, \partial \Omega) \). Set

\[
\begin{aligned}
V(x, t) &= w(y(x, t); \mu(t)), \quad 0 \leq d(x) \leq \varepsilon(t), \quad x \in \Omega, \ t > 0 \\
V(x, t) &= M(t) = \max_{0 \leq d(x) \leq \varepsilon(t)} w(y(x, t); \mu(t)), \quad d(x) \geq \varepsilon(t), \quad x \in \Omega, \ t > 0,
\end{aligned}
\tag{3.2}
\]

where \( 0 \leq y(x, t) = d(x)/\varepsilon(t) \leq 1, \varepsilon(t) > 0 \) is a function to be chosen later and \( w(y(x, t); \mu(t)) \) satisfies

\[
w_{yy} + \mu(t)f(w) = 0, \quad 0 < y < 1, \ t > 0; \quad w(0; \mu(t)) = w'(1; \mu(t)) = 0,
\tag{3.3}
\]
or equivalently

\[
w_{rr} + \frac{\mu(t)}{\varepsilon^2(t)}f(w) = 0, \quad r = d(x), \quad 0 \leq r \leq \varepsilon(t), \ t > 0; \quad w(0) = \frac{dw}{dr}|_{r=\varepsilon(t)} = 0,
\tag{3.4}
\]

and

\[
\begin{aligned}
\Delta w - \frac{\Delta d}{\varepsilon} \frac{dw}{dy} + \frac{\mu}{\varepsilon^2} f(w) &= 0, \quad 0 \leq d(x) \leq \varepsilon(t), \ t > 0; \\
w(y(x, t); \mu(t)) &= 0, \quad x \in \partial \Omega, \ t > 0, \quad \frac{dw}{dr}|_{r=\varepsilon(t)} = 0,
\end{aligned}
\tag{3.5}
\]

which implies

\[
\begin{aligned}
\frac{d^2 w(y(x_1, 0'); \mu(t))}{dx_1^2} + \frac{\mu}{\varepsilon^2} f(w(y(x_1, 0'); \mu(t))) &= 0, \quad \delta(t) < x_1 < 1, \ t > 0; \\
w(y(1, 0'); \mu(t)) &= 0, \quad \frac{dw(y(\delta(t), 0'); \mu(t))}{dx_1} = 0,
\end{aligned}
\tag{3.6}
\]

where \( \varepsilon(t) = 1 - \delta(t) \).

From the definition of \( w \), it is obvious that \( w, w_r \) are continuous at \( r = \varepsilon(t) \). We can choose \( \mu(0)(\text{or equivalently } M(0)) \) sufficiently large so that \( V(x, 0) \geq u_0(x) \)(such a choice is possible since \( w \to \infty \) as \( \mu \to \infty \) and provided that \( u_0(x), u'_0(x) \) are bounded).

For any \( \varepsilon > 0 \), set \( \Omega_\varepsilon = \{ x \in \Omega : 0 < d(x) < \varepsilon(t) \} \). To prove that \( V(x, t) \) is an upper solution, we need some preliminary results.

Problem (3.4) and (3.6) imply that

\[
w_r(0) = \frac{\sqrt{2\mu}}{\varepsilon} \sqrt{\int_0^M f(s)ds},
\tag{3.7}
\]

and

\[
\int_{\delta(t)}^1 f(w(y(x_1, 0'); \mu(t))) dx_1 = -\frac{\varepsilon^2 dw(y(1, 0'); \mu(t))}{\mu} dx_1.
\tag{3.8}
\]

From (3.4), we get

\[
\frac{w_r}{\sqrt{F(w) - F(M)}} = \frac{\sqrt{2\mu}}{\varepsilon},
\tag{3.9}
\]

where \( F(s) = \int_s^\infty f(\sigma)d\sigma > 0 \). Relation (3.9) gives

\[
\sqrt{\mu(M)} = \frac{\sqrt{2}}{2} \int_0^M \frac{ds}{\sqrt{F(s) - F(M)}}.
\tag{3.10}
\]
For $s \leq M$, we have $F(s) - F(M) = f(\theta)(M - s)$, $\theta \in [s, M]$ and due to $f'(s) < 0$ for $s \geq 0$, we get

$$(M - s)f(M) \leq F(s) - F(M) \leq (M - s)f(s). \quad (3.11)$$

Then

$$\sqrt{\mu(M)} \leq \frac{\sqrt{2}}{2} \int_0^M (M - s)^{-\frac{1}{2}} f^{-\frac{1}{2}}(M) ds \leq \sqrt{\frac{2M}{f(M)}},$$

and hence

$$\mu(M)f(M) \leq 2M \quad \text{for } M > 0. \quad (3.12)$$

However,

$$Mf(M) \leq 2 \int_{M/2}^M f(s) ds \leq 2 \int_{M/2}^\infty f(s) ds \quad \text{and} \quad \int_{M/2}^\infty f(s) ds \to 0 \quad \text{as } M \to \infty,$$

so $Mf(M) \to 0$ as $M \to \infty$ and due to (3.12) we finally get

$$\sqrt{\mu(M)f(M)} \to 0 \quad \text{as } M \to \infty. \quad (3.13)$$

Next we claim that $\lim_{M \to \infty} \sqrt{2\mu/M} = \infty$. Indeed, by (1.2) and (3.10), we obtain

$$\frac{\sqrt{2\mu}}{M} \geq \frac{\int_0^M (M - s)^{-\frac{1}{2}} f^{-\frac{1}{2}}(s) ds}{M} = \int_0^1 \frac{s^{\frac{1}{2}}(1 - s)^{-\frac{1}{2}}}{(Msf(M))^\frac{1}{2}} ds.$$

Taking into account $sf(s) \to 0$ as $s \to \infty$, we deduce that $\lim_{M \to \infty} \sqrt{2\mu/M} = \infty$, i.e.

$$\lim_{M \to \infty} M/\sqrt{2\mu} = 0. \quad (3.14)$$

As is indicated in [1], $d(x)$ is smooth and more precisely $|\Delta d| \leq K$, for some $K$, in a neighborhood of the boundary if $\partial \Omega$ is smooth. In particular, such a neighborhood $\Omega_\varepsilon$ consists of all $x \in \Omega$ such that $d(x, \partial \Omega) \leq \varepsilon(t)$ where $\varepsilon(t)$ is chosen small enough.

Integrating (3.5) over $\Omega_\varepsilon$ we obtain

$$\int_{\Omega_\varepsilon} f(w) dx = -\frac{\varepsilon^2}{\mu} \int_{\partial \Omega} \frac{\partial w}{\partial \nu} ds + \frac{\varepsilon}{\mu} \int_{\Omega_\varepsilon} \Delta d \frac{dw}{dy} dx \quad \varepsilon \geq 0,$$

$$= \varepsilon |\partial \Omega| \frac{w_r(0)}{\mu} + \frac{\varepsilon}{\mu} \int_{\Omega_\varepsilon} \Delta d \frac{dw}{dy} dx,$$

$$\geq \varepsilon |\partial \Omega| \sqrt{\frac{2\mu}{\varepsilon}} \int_0^M f(s) ds + \frac{\varepsilon}{\mu} \int_{\Omega_\varepsilon} \Delta d \frac{dw}{dy} dx \quad (Using \ \frac{dw}{dy} \geq 0),$$

$$\geq \varepsilon |\partial \Omega| \sqrt{\frac{2\mu}{\varepsilon}} \int_0^M f(s) ds + \frac{\varepsilon^2 |\partial \Omega| K}{\mu} \int_{x_1}^1 \frac{dw((x_1, 0'); \mu(t))}{dx_1} dx_1,$$

$$= \varepsilon |\partial \Omega| \sqrt{\frac{2\mu}{\varepsilon}} \int_0^M f(s) ds - \varepsilon^2 |\partial \Omega| K \frac{M}{\mu}. $$
which implies
\[
\int_{\Omega} f(V) dx = \int_{\Omega \setminus \Omega_{\varepsilon}} f(M) dx + \int_{\Omega_{\varepsilon}} f(w) dx \\
\geq |\Omega \setminus \Omega_{\varepsilon}| f(M) + \varepsilon |\partial \Omega| \sqrt{\frac{2}{\mu}} \sqrt{\int_{0}^{M} f(s) ds} - \varepsilon^{2} |\partial \Omega| K \frac{M}{\mu}.
\]

Our construction of upper solution \( V \) depends strongly on the behavior of the function
\[
g(s) = \frac{f(s) \sqrt{\mu(s)}}{F(s)} > 0.
\]

Since (3.13) holds and \( F(M) \to 0 \) as \( M \to \infty \), we distinguish two cases for the behavior of \( g(M) \). More precisely the following holds:

**Theorem 3.1.** Let \( f(s) \) satisfy (L2), \( \int_{0}^{\infty} f(s) ds = 1 \), \( \lim \inf_{s \to \infty} g(s) > C > 0 \) and \( \lim \inf_{s \to \infty} \mu(s)f(s) = C_{0} > 0 \) (e.g. \( f(s) = e^{-s} \)) or \( \lim \inf_{s \to \infty} \mu(s)f(s)/s = C_{1} > 0 \) (\( C_{1} \leq 2 \), e.g. \( f(s) = b(1+s)^{-1-b}, b > 0 \)). \( \Omega \) is a bounded domain satisfying (H). Then the function \( V(x,t) \) is an upper solution to problem (2.1) and exists for all \( t > 0 \).

In order to prove Theorem 3.1, we first derive a number of preliminary facts on \( d(x) \).

**Lemma 3.2.** Assume \( x_{0} = (x_{10}, x_{20}, \ldots, x_{n0}) \), \( \Omega_{i} = \{x \in \mathbb{R}^{n} : |x - x_{0}| < R_{i}\}, i = 1, 2 \) and \( R_{1} > R_{2} \). Let \( d(x) = \text{dist}(x, \partial \Omega_{1}), x \in \Omega_{1} \setminus \Omega_{2} \). Then \( \Delta d(x) = (1 - n)/(|x - x_{0}|) \).

**Lemma 3.3.** \( \Omega \) is a bounded domain satisfying (H). Then there exists \( \varepsilon > 0 \) such that \( \Delta d \leq 0 \) for \( x \in \Omega_{\varepsilon} \).

**Proof.** Here we only consider the case of \( n = 2 \). As for \( n = 1 \) or \( n \geq 3 \), the proof is completely similar. Divide \( \partial \Omega \) into \( m \) parts and taking \( m \) large enough such that the largest arc is sufficiently small. Let \( A_{1}, A_{2}, \ldots, A_{m} \) be the division points. For any arc \( \overline{A_{i}A_{i+1}}(1 \leq i \leq m - 1) \), choosing \( C \in \overline{A_{i}A_{i+1}} \) such that \( |\overline{A_{i}C}| = |\overline{C_{i}A_{i+1}}| \). By the definition of \( \Omega \), there exists a circle \( \Omega_{1} = \{x \in \mathbb{R}^{2} : |x - x_{0}| < R_{1}\} \) such that \( \Omega_{1}(\Omega \subset \Omega_{1}) \) is tangent to \( \Omega \) at the point \( C \). Taking \( A_{i}', A_{i+1}' \in \partial \Omega_{1} \) such that the segments \( A_{i}'x_{0}, A_{i+1}'x_{0} \) intersect \( \partial \Omega \) at \( A_{i}, A_{i+1} \), respectively. Since \( \overline{A_{i}A_{i+1}} \) is sufficiently small, we have \( \overline{A_{i}A_{i+1}} \sim \overline{A_{i}'A_{i+1}'} \). From Lemma 3.2, there exists a constant \( \varepsilon_{A_{i}A_{i+1}} > 0 \) such that
\[
\Delta d(x) \leq \frac{-1}{2|x - x_{0}|} < 0, \quad x \in \left\{x \in \Omega : d(x, \overline{A_{i}A_{i+1}}) < \varepsilon_{A_{i}A_{i+1}}\right\}.
\]
Set \( \varepsilon = \min\{\varepsilon_{A_{i}A_{i+1}}, \varepsilon_{A_{i}A_{i+1}}, i = 1, 2, \ldots, m - 1\} \). Then
\[
\Delta d(x) \leq 0, \quad x \in \Omega_{\varepsilon} = \{x \in \Omega : d(x, \partial \Omega) < \varepsilon\}.
\]
The proof is completed. \( \square \)

Now we give the proof of Theorem 3.1.
Proof. Case 1: We assume $f(s)$ to be that $\liminf_{s \to \infty} g(s) > C > 0$ and $\lim_{s \to \infty} \mu(s)f(s) = C_0 > 0$. Then taking into account the relation (3.15), for $d(x) \geq \varepsilon(t)$, we get

$$F(V) = V_t - \Delta V - \frac{2|\partial \Omega|^2 f(V)}{(\int_\Omega f(V)dx)^2} \geq \tilde{M}(t) - \frac{2|\partial \Omega|^2 f(M)}{(\int \Omega \setminus \Omega_c |f(M)| + \varepsilon|\partial \Omega| \sqrt{\frac{2}{\pi}} \int_0^M f(s)ds - \varepsilon^2|\partial \Omega| K \frac{M}{\sqrt{\mu}})^2} \geq \tilde{M}(t) - \frac{\mu(M)f(M)}{\varepsilon^2(\frac{\sqrt{|\mu f(M)|}}{2\sqrt{2}\varepsilon|\partial \Omega|} + \int_0^M f(s)ds - \frac{K\varepsilon M}{\sqrt{2\mu}})^2} \geq \tilde{M}(t) - \frac{\mu(M)f(M)}{\varepsilon^2(\log |\mu f(M)| / 2\varepsilon|\partial \Omega| + \int_0^M f(s)ds - \frac{K\varepsilon M}{\sqrt{2\mu}})^2} \text{ for } \varepsilon(t) \ll 1.$$  

Choosing $K_1 = (C_0|\Omega|)/(8K|\partial \Omega|)$ and $\varepsilon(t) = (K_1/M)^{1/2}$, we have $0 < \varepsilon(M) \ll 1$ for $M \gg 1$. Moreover, from (3.13) and $\lim_{M \to \infty} \mu(M)f(M) = C_0$, we obtain

$$\frac{|\Omega|\sqrt{\mu f(M)}}{2\sqrt{2\varepsilon|\partial \Omega|}} + \int_0^M f(s)ds - \frac{K\varepsilon M}{\sqrt{2\mu}} \geq \frac{|\Omega|\sqrt{\mu f(M)}}{2\sqrt{2\varepsilon|\partial \Omega|}} + \int_0^M f(s)ds - \frac{\sqrt{2K_1}K\sqrt{\mu f(M)}}{C_0\varepsilon} = \frac{|\Omega|\sqrt{\mu f(M)}}{4\sqrt{2\varepsilon|\partial \Omega|}} + \int_0^M f(s)ds \text{ for } M \gg 1.$$  

Since

$$\frac{|\Omega|\sqrt{\mu f(M)}}{4\sqrt{2\varepsilon|\partial \Omega|} F(M)} = \frac{|\Omega|\sqrt{\mu f(M)}}{4\sqrt{2\varepsilon|\partial \Omega|}(1 - \int_0^M f(s)ds)} \geq \frac{|\Omega|C}{4\sqrt{2\varepsilon|\partial \Omega|}} > 1 \text{ for } M \gg 1,$$

which implies

$$\frac{|\Omega|\sqrt{\mu f(M)}}{4\sqrt{2\varepsilon|\partial \Omega|}} + \int_0^M f(s)ds > 1 \text{ for } M \gg 1. \quad (3.16)$$

Taking $M(t)$ to satisfy

$$\tilde{M}(t) = \frac{\mu(M)f(M)}{\varepsilon^2(M)} \text{, } t > 0,$$

we obtain

$$F(V) > \tilde{M}(t) - \frac{\mu(M)f(M)}{\varepsilon^2(M)} = 0 \text{ for } d(x) \geq \varepsilon(t)(x \in \Omega) \text{ and } M \gg 1.$$  

By integrating (3.17), we have

$$\int_{M(0)}^{M(t)} \frac{\varepsilon^2(s)}{\mu(s)f(s)} ds = t,$$

and taking into account $\lim_{s \to \infty} \mu(s)f(s) = C_0$, we obtain

$$\frac{K_1}{1 + C_0} \int_{M(0)}^{M(t)} \frac{1}{s} ds < t \text{ for } M(0) \gg 1.$$

The last inequality implies that if $M(t) \to \infty$ then $t \to \infty.$
Also for $0 < d(x) \leq \varepsilon(t) (x \in \Omega)$, we have

\[
\mathcal{F}(V) = w_\mu(y(x,t); \mu(t)) \dot{\mu}(t) - \frac{dw(y(x,t); \mu(t))}{dy} \dot{y}(t) - \Delta w - \frac{2|\partial \Omega|^2 f(w)}{\int_{\Omega} f(V)dx^2} = w_\mu(y(x,t); \mu(t)) \dot{\mu}(t) - \frac{dw(y(x,t); \mu(t)) d(x)}{\varepsilon^2 \varepsilon(t)} - \frac{\Delta d}{\varepsilon} + \frac{\mu f(w)}{\varepsilon^2} - \frac{2|\partial \Omega|^2 f(w)}{\int_{\Omega} f(V)dx^2}.
\]

Since $w_\mu > 0$, $\dot{\mu}(t) > 0$, $\dot{\varepsilon}(t) < 0$, $dw/dy \geq 0$ and $\Delta d(t) \leq 0$ for $M \gg 1$, we have

\[
\mathcal{F}(V) \geq \frac{\mu f(w)}{\varepsilon^2} - \frac{2|\partial \Omega|^2 f(w)}{\int_{\Omega} f(V)dx^2} \geq \frac{\mu f(w)}{\varepsilon^2} \left(1 - \frac{1}{\frac{|\Omega|}{4\varepsilon^2 |\partial \Omega|} + \int_0^M f(s)ds^2}\right) > 0 \text{ for } M \gg 1.
\]

Case 2: Now let $f$ be such $\lim \inf_{s \to \infty} \mu(s)f(s)/s = C_1 > 0 (C_1 \leq 2)$ and $\lim \inf_{s \to \infty} g(s) > C > 0$. For this case it is enough to consider $\varepsilon(t)$ to be constant such that $\Delta d \leq 0$ for $x \in \Omega_\varepsilon$. Moreover, we choose $\varepsilon$ to satisfy

\[
\frac{|\Omega \setminus \Omega_\varepsilon|}{\sqrt{2\varepsilon |\partial \Omega|}} - \frac{\sqrt{2K\varepsilon}}{C_1} > 1.
\]

For $d(x) \geq \varepsilon(x \in \Omega)$, we have

\[
\mathcal{F}(V) \geq \dot{M}(t) - \frac{2|\partial \Omega|^2 f(M)}{(\varepsilon^2(|\Omega \setminus \Omega_\varepsilon| f(M) + \varepsilon |\partial \Omega| \sqrt{\frac{2}{\varepsilon^2}} \int_0^M f(s)ds - \varepsilon^2 |\partial \Omega| K M f(s)ds)}
\]

\[
\geq \dot{M}(t) - \frac{\mu(M) f(M)}{\varepsilon^2 (\sqrt{\frac{|\Omega \setminus \Omega_\varepsilon| f(M)}{2\varepsilon |\partial \Omega|}} + \int_0^M f(s)ds - K \frac{\varepsilon f(M)}{\sqrt{\pi}})^2}
\]

\[
\geq \dot{M}(t) - \frac{\mu(M) f(M)}{\varepsilon^2 (\sqrt{\frac{f(M) \mu(M)}{C}} + \int_0^M f(s)ds)^2} \text{ for } M \gg 1.
\]

Since

\[
\frac{f(M) \sqrt{\mu(M)}}{CF(M)} = \frac{f(M) \sqrt{\mu(M)}}{C(1 - \int_0^M f(s)ds)} > \frac{1}{C} C = 1,
\]

which implies

\[
\frac{f(M) \sqrt{\mu(M)}}{C} + \int_0^M f(s)ds > 1.
\]

Hence $\mathcal{F}(V) > 0$ for $d(x) \geq \varepsilon$ and $M \gg 1$, provided that $M(t)$ satisfies

\[
\dot{M}(t) = \frac{\mu(M) f(M)}{\varepsilon^2}, \quad t > 0. \quad \text{(3.18)}
\]

By integrating (3.18), we have

\[
\int_{M(0)}^{M(t)} \frac{\varepsilon^2}{\mu(s)f(s)} ds = t,
\]
and taking into account (3.12) we obtain
\[
\frac{\varepsilon^2}{2} \int_{M(0)}^{M(t)} \frac{1}{s} ds \leq t,
\]
which implies that if \(M(t) \to \infty\) then \(t \to \infty\).

For \(0 \leq d(x) \leq \varepsilon\), we have
\[
\mathcal{F}(V) = w_\mu(y(x,t); \mu(t)) \mu(t) \geq \frac{\mu(M)f(w)}{\varepsilon^2} - \frac{f(w)\mu(M)}{\varepsilon^2(\int_M f(V)dx)^2} > 0 \quad \text{for} \quad M \gg 1.
\]

Therefore, we finally get that, in each case, \(V(x,t)\) is an upper solution to problem (3.1) for all \(t > 0\). The proof is completed.

Thus we formulate this main result of this Section in the following theorem.

**Theorem 3.4.** If \(f(s)\) satisfies the hypotheses of Theorem 3.1, and \(\Omega\) satisfies (H), then \(u(x,t)\) is a global-in-time solution to problem (1.1) and \(u(x,t) \to \infty\) as \(t \to \infty\), i.e. \(u(x,t)\) diverges globally in \(\Omega\).

### 4. Asymptotic behavior of the blow-up solutions

In this Section, we deal with the blow-up solutions of problem (1.1).

**Theorem 4.1.** Let \(f(s)\) satisfy (1.2), \(\int_0^\infty f(s)ds = 1\), \(p = 2\) and \(\Omega\) satisfy (H). If \(\lambda > \lambda^* = 2|\partial \Omega|^2\), the solution of the problem (1.1) blows up globally in finite time \(T\).

**Proof.** By Theorem 2.6 in the case of \(\lambda > \lambda^* = 2|\partial \Omega|^2\) and \(\int_0^\infty f(s)ds = 1\), there is no steady solution to (2.2). Since \(\lambda(\mu) < \lambda\) for any \(\mu > 0\), we can find an increasing lower solution \(v(x,t) = w(x,\mu(t))\) with \(\mu\) and \(v \to \infty\) as \(t \to \infty\). Thus \(u(x,t)\) is globally unbounded. We shall show that \(T < \infty\). Therefore, we look for a lower solution \(V(x,t)\) which blows up at a finite time (\(V(x,t)\) satisfy (3.2)–(3.6)). From (3.6) and (3.8), we have
\[
\int_\Omega f(V)dx = \int_{b_1(t)} f(M)dx + \int_{\delta(t)} f(w)dx \leq |\Omega|f(M) + |\partial \Omega| \int_{\delta(t)} f(w(x_1,0'); \mu(t))dx_1
\]
\[
\leq |\Omega|f(M) + |\partial \Omega| \varepsilon \sqrt{\frac{2}{\mu}} + \sqrt{2|\partial \Omega|f(M)(\frac{|\Omega|}{\sqrt{2|\partial \Omega|}} + \alpha)},
\]
on choosing \(\alpha = \varepsilon/(\sqrt{\mu}(f(M)))\), where \(\alpha\) is a suitable chosen constant; in particular choose \(\alpha > |\Omega|/(\sqrt{\lambda} - \sqrt{2|\partial \Omega|})\) for \(\lambda > \lambda^* = 2|\partial \Omega|^2\). Such an \(\alpha\) gives
\[
3\Lambda = \frac{(|\Omega| + \sqrt{2\alpha|\partial \Omega|})^2 - 1}{\alpha^2} > 0.
\]

From (3.13), we also note that with such a fixed \(\alpha\), \(\varepsilon \to 0\) as \(M \to \infty\). Integrating (3.9) on \((0,r)\), we get
\[
\int_0^r \frac{ds}{\sqrt{F(s) - F(M)}} = \frac{\sqrt{2\mu r}}{\varepsilon} = \frac{\sqrt{2}}{\alpha f(M)}.
\]
For $x \in \Omega \setminus \Omega_{\varepsilon}$, 
\[
\mathcal{F}(V) = \dot{M} - \frac{\lambda f(M)}{\int_\Omega f(V) \, dx} \leq \dot{M} - \frac{\lambda}{2|\partial\Omega|^2 f(M) \left( \frac{|\Omega|}{\sqrt{2|\partial\Omega|}} + \alpha \right)^2} \leq \dot{M} - \frac{\Lambda}{f(M)} \leq 0,
\]
on choosing $\dot{M} \leq \Lambda/f(M)$.

For $x \in \Omega_{\varepsilon}$, we first differentiate (4.1) with respect to $t$ and get
\[
w_t = -f'(M) \frac{\dot{M}(t)[F(w) - F(M)]}{f(M)} \int_0^w \frac{ds}{\sqrt{F(s) - F(M)}} + \frac{1}{2} f(M) \dot{M}(t)[F(w) - F(M)] \int_0^w [F(s) - F(M)]^{-\frac{3}{2}} ds := A + B.
\]
For $A$, from (3.11) we have
\[
A = -\frac{f'(M)}{f(M)} \dot{M}(t)[F(w) - F(M)] \int_0^w \frac{ds}{\sqrt{F(s) - F(M)}} \leq -\frac{2f'(M)}{f^2(M)} M \dot{M}(t) f^\frac{1}{2}(w) \leq \frac{\Lambda f(w)}{f^2(M)},
\]
provided that
\[
\dot{M}(t) \leq -\frac{\Lambda}{2M f'(M)}
\]
and taking into account that $f'(s) \leq 0$ so that $f(w)/f(M) \geq 1$ for $w \leq M$. For $B$ we have
\[
B = \frac{1}{2} f'(M) M \dot{M}(t)[F(w) - F(M)] \int_0^w [F(s) - F(M)]^{-\frac{3}{2}} ds \leq \frac{f^\frac{1}{2}(w)}{f^\frac{1}{2}(M)} \dot{M}(t) \leq \frac{\Lambda f(w)}{f^2(M)},
\]
provided that
\[
\dot{M}(t) \leq \frac{\Lambda}{f(M)}.
\]
Also, using (3.5) and (3.9), we have the estimate
\[
-\Delta w = -w_r \Delta d + \frac{\mu}{\varepsilon^2} f(w) \leq Kw_r + \frac{\mu}{\varepsilon^2} f(w) \quad \text{(Using $|\Delta d| \leq K$)}
\]
\[
= \frac{K \sqrt{2\mu}}{\varepsilon}[F(w) - F(M)]^\frac{1}{2} + \frac{f(w)}{\alpha^2 f^2(M)} \leq \frac{\sqrt{2}K (M f(M))^\frac{1}{2} f(w)}{f^2(M)} + \frac{f(w)}{\alpha^2 f^2(M)} \leq \frac{\Lambda f(w)}{f^2(M)} + \frac{f(w)}{\alpha^2 f^2(M)}, \quad \text{for } M \gg 1,
\]
since $M f(M) \to 0$ as $M \to \infty$. Thus for $x \in \Omega_{\varepsilon}$ if
\[
0 \leq \dot{M}(t) = \min \left\{ \frac{\Lambda}{f(M)}, -\frac{\Lambda}{2M f'(M)} \right\}
\]
also for $x \in \Omega_{\varepsilon}$.
and using the previous estimate we obtain
\[
\mathcal{F}(V) = w_t - \Delta w - \frac{\lambda f(w)}{(\int_{\Omega} f(V)dx)^2}
= A + B - w_t \Delta d + \frac{\mu}{\varepsilon^2} f(w) - \frac{\lambda f(w)}{(\int_{\Omega} f(V)dx)^2}
\leq 3A f(w) + \frac{f(w)}{\alpha^2 f^2(M)} - \frac{\lambda f(w)}{2 |\partial \Omega|^2 f^2(M)(\frac{|\Omega|}{\sqrt{2|\partial \Omega|}} + \alpha)^2} = 0.
\]
Also \(V(x, t) = u(x, t) = 0\) on the boundary \(\partial \Omega\) and taking \(V(x, 0) \leq u_0(x)\), the function \(V(x, t)\) is a lower solution to the problem \((1.1)\). Hence \(u(x, t) \geq V(x, t)\) for \(M\) is large enough (after some time at which \(u(x, t)\) is sufficiently large if \(T = \infty\)).

Now we show that \(u(x, t)\) blows up in finite time. Indeed, from (4.2) we have
\[
\Lambda \frac{dt}{dM} = \max\{f(M), -2Mf'(M)\} \leq f(M) - 2Mf'(M) \quad (f'(s) \leq 0)
\]
or
\[
\Lambda t \leq \int_0^M [f(s) - 2sf'(s)] ds < \infty,
\]
since \(Mf(M) \to 0\) as \(M \to \infty\) and \(\int_0^\infty f(s) ds = 1\). Hence \(V(x, t)\) blows up at \(t^* < \infty\) and \(u(x, t)\) must blow up at \(T \leq t^* < \infty\).

As for the blow-up is global from the fact
\[
\int_{\Omega} f(u)dx \to 0 \quad \text{as} \quad t \to T.
\]
Indeed,
\[
\dot{M} \leq \frac{\lambda f(M)}{(\int_{\Omega} f(u)dx)^2} = h(t),
\]
giving
\[
M(t) - M(0) \leq \int_0^t h(s) ds \to \infty \quad \text{as} \quad t \to T.
\]
This implies \(\int_{\Omega} f(u)dx \to 0\) as \(t \to T\) since \(f(s)\) is bounded. Thus, for \(\lambda > \lambda^* = 2|\partial \Omega|^2\), \(u(x, t)\) blows up globally. The proof is completed.

**Theorem 4.2.** Let \(f(s)\) satisfy (1.2), \(\int_0^\infty f(s) ds = 1\), \(p > 2\) and \(\Omega\) satisfy (H). Then there exists a critical value \(\lambda^*\) such that for \(\lambda > \lambda^*\) or for any \(0 < \lambda \leq \lambda^*\) but with initial data sufficiently large, the solution of the problem \((1.1)\) blows up globally in finite time \(T\).

**Proof.** Using Theorem 2.6 we know that for \(\lambda > \lambda^*\) or for any \(0 < \lambda \leq \lambda^*\) but with initial data \(u_0\) more than the greater steady state \(u(x, t)\) is globally unbounded (see(8)). In order to prove \(u(x, t)\) blows up in finite time \(T < \infty\), we also look for a lower solution \(V(x, t)\) to satisfy (3.2)–(3.6). Then
\[
\int_{\Omega} f(V)dx = \int_{\Omega \setminus \Omega_\epsilon} f(M)dx + \int_{\Omega_\epsilon} f(w)dx \leq |\Omega| f(M) + |\partial \Omega| \int_{\delta(t)} f(w(x_1, 0); \mu(t)) dx_1
\leq |\Omega| f(M) + |\partial \Omega| \varepsilon \sqrt{\frac{2}{\mu}} = \sqrt{2} |\partial \Omega| f(M) (\frac{|\Omega|}{\sqrt{2|\partial \Omega|}} + 1),
\]
on choosing \(\varepsilon = \sqrt{\mu f(M)}\). From (3.13), we also note that \(\varepsilon \to 0\) as \(M \to \infty\).
For \( x \in \Omega \setminus \Omega_\epsilon \),
\[
\mathcal{F}(V) = \dot{M} - \frac{\lambda f(M)}{(\int \Omega f(V) dx)^p} \leq \dot{M} - \frac{\lambda}{(\sqrt{2} |\partial \Omega|)^p f^{p-1}(M)(\frac{|\Omega|}{\sqrt{2} |\partial \Omega|} + 1)^p} 
\]
\[
\leq \dot{M} - \frac{1}{f(M)} \leq 0 \quad \text{for } M \gg 1,
\]
on choosing \( \dot{M} \leq 1/f(M) \) and taking into account \( p > 2 \) and \( f(M) \to 0 \) as \( M \to \infty \).

For \( x \in \Omega_\epsilon \), similar to the proof of Theorem 4.1, we have \( w_t = A + B \). For \( A \), from (3.11) we have
\[
A = -\frac{f'(M)}{f(M)} \dot{M}(t)[F(w) - F(M)]^\frac{1}{2} \int_0^w \frac{ds}{\sqrt{F(s) - F(M)}} 
\]
\[
\leq -\frac{2f'(M)}{f^2(M)} M \dot{M}(t) f^\frac{1}{2}(w) \leq \frac{f(w)}{f^2(M)},
\]
provided that
\[
\dot{M}(t) \leq -\frac{1}{2Mf'(M)}.
\]
For \( B \) we have
\[
B = \frac{1}{2} f(M) \dot{M}(t)[F(w) - F(M)]^\frac{1}{2} \int_0^w [F(s) - F(M)]^{-\frac{1}{2}} ds 
\]
\[
\leq \frac{f^\frac{1}{2}(w)}{f^\frac{1}{2}(M)} \dot{M}(t) \leq \frac{f(w)}{f^2(M)},
\]
provided that
\[
\dot{M}(t) \leq \frac{1}{f(M)}.
\]
Also, using (3.5) and (3.9), we have the estimate
\[
-\Delta w = -w_r \Delta d + \frac{\mu}{\epsilon^2} f(w) \leq K w_r + \frac{\mu}{\epsilon^2} f(w) \quad \text{(Using } |\Delta d| \leq K) 
\]
\[
= \frac{K \sqrt{2} \mu}{\epsilon} [F(w) - F(M)]^\frac{1}{2} + \frac{f(w)}{f^2(M)} 
\]
\[
\leq \sqrt{2} K \left( \frac{M f(M)}{f^2(M)} \right)^\frac{1}{2} f(w) + \frac{f(w)}{f^2(M)} 
\]
\[
\leq \frac{2f(w)}{f^2(M)}, \quad \text{for } M \gg 1,
\]
since \( M f(M) \to 0 \) as \( M \to \infty \). Thus for \( x \in \Omega_\epsilon \) if
\[
0 \leq \dot{M}(t) = \min\left\{ \frac{1}{f(M)}, -\frac{1}{2Mf'(M)} \right\}
\]
and using the previous estimate we obtain
\[ F(V) = w_t - \Delta w - \frac{\lambda f(w)}{\int_{\Omega} f(V) dx}^p \]
\[ = A + B - w_t \Delta d + \frac{\mu}{2} f(w) - \frac{\lambda f(w)}{\int_{\Omega} f(V) dx}^p \]
\[ \leq \frac{4f(w)}{f^2(M)} - \frac{\lambda f(w)}{2\delta^p(M)f^p(M)(\frac{\delta}{\sqrt{2k|\Omega|}} + 1)^p} \leq 0 \quad \text{for } M \gg 1, \]
since \( p > 2 \) and \( f(M) \to 0 \) as \( M \to \infty \).

Also \( V(x, t) = u(x, t) = 0 \) on the boundary \( \partial \Omega \) and taking \( V(x, 0) \leq u_0(x) \), the function \( V(x, t) \) is a lower solution to the problem \( \text{(1.1)} \). Hence \( u(x, t) \geq V(x, t) \) for \( M \) is large enough (after some time at which \( u \) is sufficiently large if \( T = \infty \)).

The rest proof is same as the Theorem 4.1, so we omit it here. \( \square \)

Now we will consider the Dirichlet problem, which we rewrite \( \text{(1.1)} \) as
\[
\begin{align*}
\begin{cases}
u_t = \Delta u + g(t) f(u), & x \in \Omega, \ t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\end{align*}
\]
where \( g(t) = \lambda/(\int_{\Omega} f(u) dx)^p \), \( \Omega \) is defined as Theorem 2.6.

We seek a formal asymptotic approximation for \( u(x, t) \) near the blow-up time \( T \), still taking \( f \) to be decreasing and to satisfy \( \int_0^\infty f(s) ds = 1 \). Set \( M(t) = \max_{x \in \Omega} u(x, t) \).

As in [6], we obtain that \( \lim_{t \to T} f(t) = \infty \) and \( u(x, t) \sim M \) except in some boundary layers near \( \partial \Omega \). In the main core(outer) region we neglect \( \Delta u \), so
\[ \frac{dM}{dt} \sim g(t) f(M) \]
and significant contributions to the integral \( \int_{\Omega} f(u) dx \) can come from the largest(core) region which has volume \( \sim |\Omega|/(\text{contribution} \sim |\Omega| f(M)) \) and from the boundary layers where \( f \) is large, \( f(u) \) is \( O(1) \) where \( u(x, t) \) is \( O(1) \). If the boundary layers have volume \( O(\delta) \), for some small \( \delta \), then to obtain a balance involving \( \Delta u \), either \( \delta^{-2} = O(g) \) or \( \delta^{-2} = O((T - t)^{-1}) \), whichever is the larger, see [8].

Supposing that \( g(t) \ll (T - t)^{-1} \) for \( t \to T \) the contribution to the the integral from the boundary layer is \( O(\delta) = O((T - t)^{\frac{1}{2}}) \), whereas
\[ \int_{\Omega} f(u) dx = O(g(t)^{-\frac{1}{2}}) \gg (T - t)^{\frac{1}{2}} \geq (T - t)^{\frac{1}{2}} \quad \text{as} \quad t \to T. \]
This suggests that the core dominates and
\[ \int_{\Omega} f(u) dx \sim |\Omega| f(M). \]
Then
\[ g(t) \sim \frac{\lambda}{|\Omega|^p f^p(M)}, \quad f(M) \sim \frac{1}{|\Omega|} \left( \frac{\lambda}{g} \right)^{\frac{1}{p}}, \]
and
\[ \frac{dM}{dt} \sim g(t) f(M) \sim \frac{1}{|\Omega|} \lambda^\frac{1}{p} g^{\frac{p-1}{p}} \ll (T - t)^{-\frac{p}{p}} \quad \text{for} \quad t \to T. \]
This would indicate that \( M \) is actually bounded as \( t \to T \). Contradicting the occurrence of blow-up.

Next we suppose that \( g(t) = O((T - t)^{-1}) \) for \( t \to T \). Since
\[
|\Omega| f(M) \lesssim \int_{\Omega} f(u) dx = (\frac{\lambda}{g})^{\frac{1}{p}},
\]
we must have \( f(M) \leq O((T - t)^{\frac{1}{p}}) \). Again,
\[
\frac{dM}{dt} \sim g(t) f(M) \leq O((T - t)^{\frac{1}{p}}),
\]
which contradicts the assumption of blow-up. There remains only one possibility:
\[
g(t) \gg (T - t)^{-1} \quad \text{for} \quad t \to T.
\]
The boundary layer has volume \( O(g(t)^{-\frac{1}{2}}) \ll (T - t)^{\frac{1}{2}} \), where \( u(x, t) \) is \( O(1) \) and \( u_t \) is negligible compare to \( \Delta u \). There has to be a balance between \( \Delta u \) and \( g(t)f(u) \), that is,
\[
-\Delta u \sim g(t)f(u).
\]
Without loss of generality, we assume that the hyperplane \( \{ x \in \mathbb{R}^n : x_1 = 1 \} \) is tangent to \( \Omega \) at \( y_0(y_0 = (1, 0')) \), and \( \Omega \) lies in the half-space \( \{ x : x_1 < 1 \} \). Writing \( x_1 = 1 - y/\sqrt{g}(y/\sqrt{g} \ll 1) \) gives
\[
\begin{align*}
-u_{yy}((y, 0')) &\sim f(u((y, 0'))), \quad y > 0, \\
u((y, 0')) &= 0, \quad y = 0, \\
u((y, 0')) &\gg 1 \gg u_y((y, 0')), \quad y \gg 1.
\end{align*}
\]
Multiplying both sides of (4.3) by \( u_y((y, 0')) \) and integrating, we get
\[
u_y^2((y, 0')) \sim 2F(u((y, 0'))),
\]
where \( F(u((y, 0'))) = \int_{u((y, 0'))}^{\infty} f(s) ds \). Integrating again gives \( u((y, 0')) \sim U(y) \), where
\[
\sqrt{2y} = \int_0^{U(y)} F^{-\frac{1}{2}}(s) ds.
\]
Since \( y_0 \) is arbitrary, it follows from (4.4) that the boundary layers contribute to a total amount
\[
\int_{d(x, \partial \Omega) \leq y/\sqrt{g}} f(u) dx \sim |\partial \Omega| \int_{x_1}^{1} f(u((x_1, 0'))) dx_1 \sim \frac{|\partial \Omega|}{\sqrt{g}} \int_0^{\infty} f(U(y)) dy,
\]
this is automatically of the correct size \( g(t) = \lambda/\int_{\Omega} f(u) dx \). It should also be observed that
\[
\int_0^{\infty} f(U(y)) dy = U'(0).
\]
Looking at the following steady problem
\[
w'' + \mu f(w) = 0, \quad -1 < x < 1; \quad w(\pm 1) = 0.
\]
Set \( M(\mu) = \max_{-1 < x < 1} w(x) = w(0) \) and \( x = 1 - y/\sqrt{\mu} \), then
\[
\frac{d^2 w}{dy^2} + f(w) = 0, \quad w(0) = 0, \quad \frac{dw}{dy}|_{y = \sqrt{\mu}} = 0, \quad w(\sqrt{\mu}) = M.
\]
From Lemma 2.5 (in case of $n = 1$), we have
\[
\lim_{\mu \to \infty} -\frac{1}{\sqrt{\mu}} \frac{dw(1)}{dx} = \sqrt{2}.
\]
which implies
\[
\lim_{\mu \to \infty} \frac{dw(0)}{dy} = \sqrt{2},
\]
and it appears that the problem in limit of large $\mu$ is the same as the asymptotic problem (4.3). Thus,
\[
\int_{0}^{\infty} f\left(\Omega\right) dy = U'\left(0\right) = \lim_{\mu \to \infty} \frac{dw(0)}{dy} = \sqrt{2}.
\]
We deduce that the contribution to $\int_{\Omega} f(u) dx$ from the boundary layers $\sim \sqrt{2}|\partial\Omega|/\sqrt{g}$.

Now
\[
\int_{\Omega} f(u) dx \sim |\Omega| f(M) + \sqrt{2}|\partial\Omega|/\sqrt{g}
\]
and
\[
g \sim \frac{\lambda}{(|\Omega| f(M) + \sqrt{2}|\partial\Omega|/\sqrt{g})^p}
\]
for $t \to T(g, M \to \infty)$.
We see that
\[
\lambda^{\frac{1}{p}} \sim g^{\frac{1}{p}} (|\Omega| f(M) + \sqrt{2}|\partial\Omega|/\sqrt{g}) = |\Omega| f(M) g^{\frac{1}{p}} + \sqrt{2}|\partial\Omega| g^{\frac{2-p}{2p}},
\]
i.e.,
(i) If $p = 2$, then $f(M) \sim \frac{\sqrt{\lambda} - \sqrt{2}|\partial\Omega|}{|\Omega|\sqrt{g}}$.
(ii) If $p > 2$, then $f(M) \sim \frac{1}{|\Omega|} (\frac{\lambda}{g})^{\frac{1}{p}}$.
Therefore, in the core region $u(x, t) \sim M$ which satisfies
\[
\frac{dM}{dt} \sim g(t) f(M) \sim \frac{\Lambda_1^2}{f(M)} \quad \text{if} \quad p = 2,
\]
where $\Lambda_1 = (\sqrt{\lambda} - \sqrt{2}|\partial\Omega|)/|\Omega|$, and
\[
\frac{dM}{dt} \sim g(t) f(M) \sim \frac{\Lambda_2}{f^{p-1}(M)} \quad \text{if} \quad p > 2,
\]
where $\Lambda_2 = \lambda/|\Omega|^p$.

Remark 4.1. By (4.5) and (4.6), we obtain that the significant contributions to integral $\int_{\Omega} f(u) dx$ come from the largest core region and the boundary layers where $f$ is large if $p = 2$, but the core dominates for $p > 2$.

Let us consider two examples.

Example 1. Suppose $f(s)$ is decreasing, $\int_{0}^{\infty} f(s) ds = 1$, $f(s) \sim B/s^{1+b}$ as $s \to \infty$ for some positive constants $b$ and $B$.

For $p = 2$,
\[
\frac{dM}{dt} \sim \frac{\Lambda_1^2}{f(M)},
\]
which implies

\[ M \sim \left( \frac{b \Lambda_1^2}{B} \right)^{-\frac{1}{t}} (T - t)^{-\frac{1}{2}}. \]

For \( p > 2 \),

\[ \frac{dM}{dt} \sim \frac{\Lambda_2}{f^{p-1}(M)}, \]

which follows that

\[ M \sim \left( \frac{(1 + b)(p - 1) - 1}{B^{p-1}} \right) \Lambda_2 \left( T - t \right)^{\frac{1}{1-(1+b)(p-1)}} \frac{1}{1-(1+b)(p-1)}. \]

**Example 2.** \( f(s) = e^{-s} \).

For \( p = 2 \),

\[ \frac{dM}{dt} \sim \frac{\Lambda_1^2}{e^{-M}}, \]

which implies

\[ M \sim - \ln(T - t) - 2 \ln \Lambda_1. \]

For \( p > 2 \),

\[ \frac{dM}{dt} \sim \frac{\Lambda_2}{e^{(1-p)M}}, \]

that is,

\[ M \sim \frac{1}{1-p} \ln \left( (p - 1) \Lambda_2 \right) + \frac{1}{1-p} \ln(T - t). \]

**References**

[1] J.W. Bebernes and A.A. Lacey, *Global existence and finite-time blow-up for a class of nonlocal parabolic problems*, Adv. Differential Equations 2 (1997), 927-953.

[2] J.A. Carrillo, *On a nonlocal elliptic equation with decreasing nonlinearity arising in plasma physics and heat conduction*, Nonlinear Anal. TMA 32 (1998), 97-115.

[3] A.C. Fowler, I. Frigaard and S.D. Howison, *Temperature surges in current-limiting circuit devices*, SIAM J. Appl. Math 52 (1992), 998-1011.

[4] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag Berlin, 1977.

[5] N.I. Kavallaris, A.A. Lacey and D.E. Tzanetis, *Global existence and divergence of critical solutions of a nonlocal parabolic problem in Ohmic heating process*, Nonlinear Anal. TMA 58 (2004), 787-812.

[6] N.I. Kavallaris and D.E. Tzanetis, *On the blow-up of the nonlocal thermistor problem*, Proc. Edinb. Math. Soc. 50 (2007), 389-409.

[7] A.A. Lacey, *Thermal runaway in a nonlocal problem modelling Ohmic heating. Part I: Model derivation and some special cases*, European J. Appl. Math 6 (1995), 127-144.

[8] A.A. Lacey, *Thermal runaway in a nonlocal problem modelling Ohmic heating. Part II: General proof of blow-up and asymptotics of runaway*, European J. Appl. Math 6 (1995), 201-224.

[9] D.H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. 21 (1972), 979-1000.

[10] D.E. Tzanetis, *Blow-up of radially symmetric solutions of nonlocal problem modelling Ohmic heating*, Electron. J. Diff. Eqns 11 (2002), 1-26.
DEPARTMENT OF MATHEMATICS, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI 200240, PR CHINA
E-mail address: liuqlseu@yahoo.com.cn

DEPARTMENT OF MATHEMATICS, SOUTHEAST UNIVERSITY, NANJING, 210096, JIANGSU, PR CHINA
E-mail address: liangfei1980@yahoo.com.cn

DEPARTMENT OF MATHEMATICS, SOUTHEAST UNIVERSITY, NANJING, 210096, JIANGSU, PR CHINA
E-mail address: lieyx@seu.edu.cn