A Mathematical Model for Whorl Fingerprint

Abstract

In this paper, different classes of the whorl fingerprint are discussed. A general dynamical system with a parameter \( \theta \) is created using differential equations to simulate these classes by varying the value of \( \theta \). The global dynamics is studied, and the existence and stability of equilibria are analyzed. The Maple is used to visualize fingerprint’s orientation image as a smooth deformation of the phase portrait of a planar dynamical system.

Keywords Whorl fingerprint · Concentric whorl · Spiral whorl · Composite whorl with "S" core · Simulations of the whorl fingerprint using general dynamical system.

1 Introduction

Fingerprints are a set of raised lines that form unique patterns on the pads of the fingers and thumbs. Everyone leaves parts or entire fingerprints on many things through our daily activities by touching cups, doors, books, etc., so studying fingerprints is important in security especially no two people have been found to have the same fingerprints. One of the early studies about fingerprints appeared in 1892 by Sir Francis Galton in his book, finger prints [1]. There are three general types of fingerprints; loop, whorl, and arch. The whorl type occurs in about 25–35% of all fingerprints, see [3, 2, 1].

The whorl patterns display in the way that the ridges in the center tend to show a circular orientation with a core to whorl and two deltas in the right and left sides. The focus of this paper is on three basic categories of whorl fingerprint which are concentric whorl, spiral whorl, and composite whorl with "S" core:

- Concentric whorl, this pattern represents the most basic form of a whorl in which the core is circular or elliptical in the center of the fingerprint, see picture (a) in figure 1.
- Spiral whorl, the ridges flow are in winding way in the center making a spiral core, see picture (b) in figure 1.
- Composite whorl with "S" core, this pattern twists its ridges in the way that forms a core in "S" shape in the center, see picture (c) in figure 1.

Few studies talked about modeling of the whorl fingerprint specially using the phase portraits of a system of differential equations. The idea of phase portraits in texture modelling can be seen in [4] where a characterizing oriented patterns was proposed using the qualitative differential equation theory to analyze real texture images, but no fingerprint image seems to have been considered. In the thesis by Ford [5], the complex flows were divided into simpler components which are modeled by linear phase portraits and then combined to obtain a model for the entire flow field, this idea was applied to fingerprints in [6].

Fingerprint can be captured as graphical ridge and valley patterns, so the global representation of the above categories of fingerprint’s flow-like patterns as a smooth deformation of the phase portrait of a system of differential equations is considered in this paper. Using differential equations in the formalization of the general dynamical system that
Figure 1: (a) Concentric whorl, (b) spiral whorl and (c) composite whorl

describes concentric, spiral, and composite whorl fingerprint requires understanding the behaviour of the ridges and interpreting the deltas and cores that appear in these classes, and how the singular points that represent the core to whorl and deltas in the patterns of phase portraits look like in the considered model.

The system of differential equations basically has two types of singular points; first type is nondegenerate singular point (The Jacobian matrix of the system has no zero eigenvalues), the second type is degenerate singular point (The Jacobian matrix has at least one zero eigenvalue). For the core to whorl can be represented as center or spiral singular point in the system of differential equations, but the delta can be interpreted as a singular point with three hyperbolic sectors, and there is no such singular point exist. So the closer singular point for the delta is the cusp with cutting appropriate edge of it. To know more information about the cusp, we present the following theorem, including determining some types of degenerate equilibrium points of the planar system that are found in [7]. Assume that the origin is an isolated critical point of the planar system

\[
\begin{align*}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y).
\end{align*}
\]  

where \(P(x, y)\) and \(Q(x, y)\) are analytic in some neighborhood of the origin, and consider the case when the matrix \(A = Df(0)\) has two zero eigenvalues i.e., \(\det(A) = 0, \text{tr}(A) = 0\), but \(A \neq 0\), in this case the system (1) can be put in the normal form

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= a_k x^k[1 + h(x)] + b_n x^n y[1 + g(x)] + y^2 R(x, y). 
\end{align*}
\]  

where \(h(x)\), \(g(x)\) and \(R(x, y)\) are analytic in a neighborhood of the origin, \(h(0) = g(0) = 0, k \geq 2, a_k \neq 0\) and \(n \geq 1\).

**Theorem 1.1** (Theorem 2 and theorem 3, page 151 [7]).

1. Let \(k = 2m + 1\) with \(m \geq 1\) in (2) and let \(\lambda = b_n^2 + 4(m + 1)a_k\). Then if \(\lambda > 0\), the origin is a (topological) saddle. If \(\lambda < 0\), the origin is
   - a focus or a center if \(b_n = 0\) and also if \(b_n \neq 0\) and \(n > m\) or if \(n = m\) and \(\lambda < 0\),
   - a node if \(b_n \neq 0\) and \(n \geq m\) and also if \(b_n \neq 0\) and \(n < m\) and \(\lambda \geq 0\),
   - a critical point with an elliptic domain if \(b_n \neq 0\), \(n\) is an odd number and \(n < m\) and also if \(b_n \neq 0\), \(n\) is an odd number and \(n = m\) and \(\lambda \geq 0\).

2. Let \(k = 2m\) with \(m \geq 1\) in (2). Then the origin is
   - a cusp if \(b_n = 0\) and also if \(b_n \neq 0\) and \(n \geq m\),
   - a saddle-node if \(b_n \neq 0\) and \(n < m\).

It is clear now from the above theorem that if the Jacobian matrix \(Df(x_0)\) has two zero eigenvalues, then the critical point \(x_0\) is either a focus, a center, a node, a (topological) saddle, a saddle-node, a cusp, or a critical point with an elliptic domain.
In [8], Zinoun used the Taylor polynomials and the normal forms then applied the theorem (1.1) to formulate some systems of the classes of the whorl fingerprint such as the concentric whorl fingerprint

\[ \dot{x} = y, \]
\[ \dot{y} = -x(x^2 - 1)^2. \]  

(3)

and the spiral whorl fingerprint

\[ \dot{x} = y, \]
\[ \dot{y} = (y - x/2)(x^2 - 1)^2. \]  

(4)

In this research, the spiral whorl fingerprint is developed, a new model for composite whorl with S core is suggested and all above categories of the whorl fingerprint are generalized in a dynamical system with a parameter \( \theta \) as follows

\[ \ddot{x} - (\theta \dot{x} - x)(x^2 - 1)^2 = 0, \quad \theta \in \mathbb{R}. \]  

(5)

Equation (5) can be written as a first order system

\[ \dot{x} = y, \]
\[ \dot{y} = -x(x^2 - 1)^2 + \theta y(x^2 - 1)^2, \quad \theta \in \mathbb{R}. \]  

(6)

This paper is organized as follows: In the next section, we study the stability of the equilibria of the system (6). In section 3, interesting simulations are shown for the three categories of the whorl fingerprint. A brief results are summarized in section 4.

2 Steady States and Their Stability

To study the stability of the system (6), we find the equilibria first, which are the solutions of the following equations:

\[ 0 = y, \]  

(7)

\[ 0 = -x(x^2 - 1)^2 + \theta y(x^2 - 1)^2. \]  

(8)

and are given by \( E_0 = (0, 0), E_1(1, 0) \), and \( E_2 = (-1, 0) \) which are called equilibrium points or singular points. The Jacobian matrix of (6) takes the form

\[ J = \begin{bmatrix} 0 & 1 \\ -x(x^2 - 1)^2 + 4(\theta y - x)(x^2 - 1)x & \theta (x^2 - 1)^2 \end{bmatrix}. \]  

(9)

The Jacobian matrix evaluated at the equilibrium point \( E_0 = (0, 0) \) is

\[ J(E_0) = \begin{bmatrix} 0 & 1 \\ -1 & \theta \end{bmatrix}. \]  

(10)

We summarize the stability of \( E_0 \) in the following theorem.

**Theorem 2.1.** The equilibrium point \( E_0 = (0, 0) \) is stable if \( \theta < 0 \), unstable if \( \theta > 0 \), and center if \( \theta = 0 \).

**Proof.** From the Jacobian matrix (10), the eigenvalues of \( J(E_0) \) are

\[ \lambda_{1,2} = \frac{\theta \pm \sqrt{\theta^2 - 4}}{2}. \]  

(11)

Notice that the eigenvalues of the equilibrium point \( E_0 = (0, 0) \) depend only on the parameter \( \theta \). When \( \theta < 0 \), the real parts of the eigenvalues are negative, so \( E_0 \) is stable. Similarly, when \( \theta > 0 \), the real parts of the eigenvalues are positive, and \( E_0 \) is unstable. Finally, when \( \theta = 0 \), we get \( \lambda_{1,2} = \pm i \), and so \( E_0 \) is center.

The equilibrium points \( E_1 = (-1, 0) \) and \( E_2 = (1, 0) \) have the same Jacobian matrix

\[ J(E_1) = J(E_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \]  

(12)

We summarize the stability of \( E_1 = (-1, 0) \) and \( E_2 = (1, 0) \) in the following theorem

**Theorem 2.2.** The equilibrium points \( E_1 = (-1, 0) \) and \( E_2 = (1, 0) \) are cusps.
Proof. The eigenvalues in this case are \( \lambda_{1,2} = 0 \) which means degenerate equilibrium points, i.e., \( \det(J) = 0, \text{tr}(J) = 0 \), but \( J \neq 0 \). In this case it is shown in the introduction that the system can be put in the normal form to which functions can be reduced in a neighbourhood of the degenerate critical points \( E_1 = (-1, 0) \) and \( E_2 = (1, 0) \) as the following

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= \pm \alpha (x \pm 1)^2 + o((x \pm 1)^3).
\end{align*}
\]  

Apply theorem (1.1), \( k = 2, b_n = 0, h(x) = 0 \) and \( \alpha > 0 \) which gives that both \( E_1 = (-1, 0) \) and \( E_2 = (1, 0) \) are cusps.

3 Simulations and Numerical Results of the Whorl Fingerprint

In this section, we display the phase portrait of the system (6) using particular values of the parameter \( \theta \) and match them with the images in the above categories of the whorl fingerprint. We get similar shape to the concentric whorl at the value \( \theta = 0 \), around and closed to \( \theta = 0 \), we get shapes look like to the spiral whorl and when we increase the value of the parameter \( \theta \) to be around one, we get phase portrait closed to the composite whorl with "S" core. For more details let us go over these cases using Maple software.

3.1 Concentric whorl class

The basic features of the concentric whorl are the existence of the circular or elliptical ridges in the middle which are represented by the center in the phase portrait, the two deltas which are represented by the two cusps in the phase portrait in the right and left sides, and we can draw two connections between the deltas through the ridges which are represented by separatrices between the cusps from above and below half planes in the phase portrait. Let us put \( \theta = 0 \) in the system (6), we get the following system

Example 3.1.

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x(x^2 - 1)^2.
\end{align*}
\]  

Figure 2 shows the phase portrait of the example 3.1 using Maple and the image that represents the concentric whorl. It easy to compare and matching center with circular ridges in the middle, the cusps with the deltas, and the separatrices with the connections in both pictures.

![Figure 2: Simulation of the concentric whorl fingerprint: (a) phase portrait of example 3.1 and (b) image of the concentric whorl fingerprint.](image_url)
3.2 Spiral whorl class

In this type, we go over two kinds of the spiral whorl:

- UR-LL spiral whorl in which there exist an upper right connection (UR connection) between the spiral core and the right delta and a lower left connection (LL connection) between the spiral core and the left delta, see picture (b) in figure 3.

- LR-UL spiral whorl in which there exist a lower right connection (LR connection) between the spiral core and the right delta and an upper left connection (UL connection) between the spiral core and the left delta, see picture (b) in figure 4.

To get the first kind of spiral, substitute $\theta = 0.2$ in the system (6), we get the following system

**Example 3.2.**

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x(x^2 - 1)^2 + 0.2y(x^2 - 1)^2.
\end{align*}
\]

The phase portrait of the example 3.2 is shown in figure 3. We can see the similarity between the phase portrait and the image that represents UR-LL spiral whorl by comparing the focus with the spiral core, the cusps with deltas, the upper right orbit (UR orbit) between the focus and the right cusp with the upper right connection (UR connection), and the lower left orbit (LL orbit) between the focus and the left cusp with the lower left connection (LL connection).

![Figure 3: Simulation of the UR-LL spiral whorl fingerprint: (a) phase portrait of example 3.2 and (b) image of the UR-LL spiral whorl fingerprint.](image)

To get the kind LR-UL spiral whorl, we should think how to switch the connections in the figure 3 up side down, this can be happened if we use negative value of $\theta$, we use $\theta = -0.2$ in the system (6) which is explained in example 3.3.

**Example 3.3.**

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x(x^2 - 1)^2 - 0.2y(x^2 - 1)^2.
\end{align*}
\]

The phase portrait of example 3.3 is illustrated in figure 4 which achieves our target.
3.3 Composite whorl with S core class

The composite whorl with "S" core is characterized with a center looks like an "S", and two cusps in the two sides. If $\theta$ in the system (6) grows up around one, the flow is twisted enough to create "S" in the middle with keeping the cusps in both sides, for example consider system (6) with $\theta = 0.9$ we get the following system

**Example 3.4.**

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x(x^2 - 1)^2 + 0.9y(x^2 - 1)^2.
\end{align*}
\]  

(17)

Notice the flow in the phase portrait of the example 3.4 and the friction ridges of the composite whorl with "S" are almost similar.
4 Conclusion

The dynamical system (6) with the parameter $\theta$ is a good source to generate different categories of the whorl fingerprint. We have noticed that the shape of the flow of this dynamical system at a particular value of the parameter $\theta$ and the shape of the ridges in the corresponding image of a category of the whorl fingerprint are almost identical to each other. The flexibility of the system (6) enable us to simulate a class of whorl depending on how much this class is twisted in either a clockwise direction or counterclockwise direction.

References

[1] F. Galton. Finger prints. In MacMillan and co., London 1892.
[2] Henry ER. Classification and uses of finger prints. In London: George Rutledge & Sons Ltd. 1990.
[3] Arent de Jongh Ph.D. Anko R. Lubach Sheryl L. Lie Kwie M.A. Ivo Alberink Ph.D. Measuring the Rarity of Fingerprint Patterns in the Dutch Population Using an Extended Classification Set. In Journal of Forensic Sciences 2019.
[4] Rao A. R. . The analysis of oriented textures through phase portraits. In: A Taxonomy for Texture Description and Identification. In Springer Series in Perception Engineering, Springer, New York, NY. 1990.
[5] Ford, Ralph. Image models for flow field analysis, representation, and compression: A dynamical systems approach. In Springer Series in Perception Engineering, Springer, New York, NY. 1990.
[6] J. Li and W.-Y. Yau. Constrained nonlinear phase portrait model of fingerprint orientation and its application to fingerprint classification. In Advanced Topics in Biometrics. pages 177–226, 2003.
[7] Perko, Lawrence. Can a Fingerprint be Modelled by a Differential Equation ? In arXiv 2018.
[8] Fouad Zinoun. Differential equations and dynamical systems. In Springer-Verlag, New York 2001.
[9] C. O. Folorunso, O. S. Asaolu, and O. P. Popoola. A Review of Voice-Base Person Identification: State-of-the-Art. In Covenant J. Eng. Technol. 2019.
[10] C. Lin and A. Kumar. A CNN-based framework for comparison of contactless to contact-based fingerprints. In IEEE Trans. Inf. Forensics Secur. pages 662–676, 2018.