Perspective

2D $F(R)$ gravity and AdS$_2$/CFT$_1$ correspondence

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Abstract – We studied the canonical structure of 2D $F(R)$ gravity. Its equivalence with Jackiw-Teitelboim gravity is demonstrated when no matter is present. Then, due to AdS$_2$/CFT$_1$ correspondence, the singular $D \rightarrow 2$ limit of $F(R)$ gravity is also studied. It is shown that in such a limit AdS$_2$/CFT$_1$ correspondence is not realized.

Introduction. – For the resolution of the problem of the early- and late-time acceleration of the universe, several modified gravities have been developed (for a general review see [1,2]). Among these models of modified gravity theories, the most realistic ones are different versions of $F(R)$ gravity, which may even describe the whole universe’s evolution from inflation to the dark energy epoch [3].

From another point, recently the wonderful equivalence between 2D Jackiw-Teitelboim (JT) gravity [4,5] and the so-called Sachdev-Ye-Kitaev (SYK) models have been discovered in refs. [6,7]. In the present letter, we study the canonical structure of 2D $F(R)$ gravity and its relation to JT gravity. It is shown that due to its equivalence with the JT gravity, the theory enjoys AdS$_2$/CFT$_1$ correspondence. Its singular $D \rightarrow 2$ limit is also discussed. It is demonstrated that gravity theory in such a limit does not have AdS$_2$ solutions. As a result, in this case, there is no AdS$_2$/CFT$_1$ correspondence, and the relation with the SYK models is not realized.

Let us now show that $F(R)$ gravity is a unitary one and is equivalent to other 2D theories like the dilaton gravity and the JT gravity when no matter is included.

The action of $F(R)$ gravity in $D$-dimensions,

$$S = \frac{1}{2\kappa^2} \int dx^D \sqrt{-g} F(R), \quad (1)$$

can be rewritten as

$$S = \frac{1}{2\kappa^2} \int dx^D \sqrt{-g} (\phi R - V(\phi)). \quad (2)$$

In fact, by the variation of the action (2) with respect to $\phi$, we find $R = V'(\phi)$, which can be solved with respect to $\phi$, $\phi = \phi(R)$. By substituting the expression into the action (2), we obtain the action (1) with $F(R) = \phi(R) R - V(\phi(R))$. The model (2) is a kind of dilaton gravity, whose relations with the Sachdev-Ye-Kitaev (SYK) model have been well studied [8,9] in two dimensions.

By the variation of the action (2) with respect to the metric, we obtain

$$0 = \phi \left( \frac{1}{2} g_{\mu \nu} R - R_{\mu \nu} \right) + \nabla_{\nu} \nabla_{\nu} \phi - g_{\mu \nu} \Box \phi - \frac{1}{2} g_{\mu \nu} V(\phi). \quad (3)$$

If we consider the scale transformation $g_{\mu \nu} \rightarrow e^\sigma g_{\mu \nu}$, the action (2) is transformed as

$$S = \frac{1}{2\kappa^2} \int dx^D \sqrt{-g} e^{\frac{D}{2} \sigma} \left\{ \phi \left( R - (D - 2) \Box \sigma \right) - (D - 3)(D - 2) \frac{\partial^\mu \sigma \partial_\mu \sigma}{4} e^{-\sigma} - V(\phi) \right\}. \quad (4)$$

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Except for the two-dimensional case, $D \neq 2$, by choosing $\sigma$ so that $e^{2/3} \rho = 1$, we obtain the well-known scalar-tensor theory,

$$S = \frac{1}{2\kappa^2} \int d^D \sqrt{-g} \left\{ R - \frac{(D-3)(D-2)}{4} \partial^\nu \sigma \partial_{\nu} \sigma - e^{2\sigma} V \left( \phi = e^{\frac{2\sigma}{3}} \right) \right\}. \quad (5)$$

Therefore, general scalar-tensor theory is equivalent to $F(R)$ gravity.

The scalar-tensor theory (5) shows that the model does not include any ghosts. On the other hand, in two dimensions, $D = 2$, the action (4) has the following form:

$$S = \frac{1}{2\kappa^2} \int d^2 \sqrt{-g} \left\{ \phi R - e^{\sigma} V(\phi) \right\}, \quad (6)$$

and we cannot rewrite the action in the scalar-tensor form as in (5). One can choose, however, $\sigma$ as $e^{\sigma} V(\phi) = \Lambda$ with a constant $\Lambda$, which can be identified with a cosmological constant. For the choice $e^{\sigma} V(\phi) = \Lambda$, the action (6) reduces,

$$S = \frac{1}{2\kappa^2} \int d^2 \sqrt{-g} \left\{ \phi R - \Lambda \right\}. \quad (7)$$

Instead of $e^{\sigma} V(\phi) = \Lambda$, we may choose another form of $\sigma$, as $e^{\sigma} V(\phi) = \phi \Lambda$ with a constant $\Lambda$, again. Under the choice $e^{\sigma} V(\phi) = \phi \Lambda$, the action (6) reduces,

$$S = \frac{1}{2\kappa^2} \int d^2 \sqrt{-g} \phi \left\{ R - \Lambda \right\}, \quad (8)$$

which is nothing but the action of the JT gravity [4,5]. Therefore, $F(R)$ gravity without matter in two dimensions is equivalent to the Jackiw-Teitelboim gravity. More specific choice could be given by $\sqrt{-g} e^{\sigma} V(\phi) = L^2$ with a constant $L$.

We have shown the equivalence between $F(R)$ gravity and other gravity theories in two dimensions on the classical level. If we consider the quantum theory by using the path integral formalism, we need to define the integration measures for the fields.

**Canonical Structure of 2D $F(R)$ gravity.** In two dimensions, one can choose the conformal gauge condition, $g_{\mu \nu} = e^{2\gamma} \eta_{\mu \nu}$, where $\eta_{\mu \nu}$ is the metric in the flat two-dimensional space-time ($\eta_{\mu \nu} = \delta_{\mu \nu}$). Then (0,0), (1,1), and (0,1) components of eq. (3) are

$$0 = \phi_{,11} - \gamma_{,0} \phi_{,0} - \gamma_{,1} \phi_{,1} + \frac{e^{2\gamma}}{2} V(\phi),$$
$$0 = \phi_{,00} - \gamma_{,0} \phi_{,0} - \gamma_{,1} \phi_{,1} - \frac{e^{2\gamma}}{2} V(\phi),$$
$$0 = \phi_{,01} - \gamma_{,0} \phi_{,1} - \gamma_{,1} \phi_{,0},$$

and the equation $R = V'(\phi)$ for the action (2) has the following form:

$$0 = 2 \left( -\gamma_{,00} + \gamma_{,11} \right) + e^{2\gamma} V'(\phi). \quad (10)$$

Under the conformal gauge condition $g_{\mu \nu} = e^{2\gamma} \eta_{\mu \nu}$, one may write the action (2) which is completely equivalent to the action of the $F(R)$ gravity,

$$S = \frac{1}{2\kappa^2} \int d^2 \sqrt{-\eta} \left\{ -2 \phi \eta_{\mu \nu} \gamma_{,\mu \nu} - e^{2\gamma} V(\phi) \right\} = \frac{1}{2\kappa^2} \int d^2 \left\{ 2 \eta^{\rho \sigma} \phi \gamma_{,\rho \sigma} - e^{2\gamma} V(\phi) \right\}. \quad (11)$$

The variation of the action with respect to $\gamma$ gives a combination of the first two equations in (9), that is, the trace part, $0 = \phi_{,00} - \phi_{,11} - e^{2\gamma} V(\phi)$. The combination of the first two equations in (9) independent of the above trace equation is given by $0 = \phi_{,00} + \phi_{,11} - 2 \gamma_{,0} \phi_{,0} - 2 \gamma_{,1} \phi_{,1}$. The combination of the kinetic term $\phi \eta^{\rho \sigma} \gamma_{,\rho \sigma}$ in the action (11) could indicate that there appears a ghost, which may be special in two dimensions.

To investigate if there could be a ghost or not, we use the Hamiltonian formalism. The action (11) shows that the conjugate momenta $\pi_{\phi}$ and $\pi_{\gamma}$ which are conjugate to $\phi$ and $\gamma$ are given by $\pi_{\phi} = -\frac{e^{-\gamma}}{2} \phi_{,0}$ and $\pi_{\gamma} = -\frac{e^{-\gamma}}{2} \phi_{,1}$, respectively, and the Hamiltonian is

$$H = \int d\gamma \left\{ -\kappa^2 \phi \pi_{\gamma} - \frac{1}{\kappa^2} \left( \phi_{,11} - \frac{1}{2} e^{2\gamma} V(\phi) \right) \right\}. \quad (12)$$

The Hamiltonian density is the first equation in (9) times $\frac{1}{e^{2\gamma}}$ and the total derivative term

$$H \equiv -\kappa^2 \pi_{\phi} \pi_{\gamma} - \frac{1}{\kappa^2} \left( \phi_{,11} - \frac{1}{2} e^{2\gamma} V(\phi) \right) = \frac{1}{\kappa^2} \left( -\gamma_{,0} \phi_{,0} - \gamma_{,1} \phi_{,1} + \frac{1}{2} e^{2\gamma} V(\phi) \right) + \partial_1 \left( \frac{\phi_{,1}}{\kappa^2} \right).$$

Therefore the Hamiltonian constraint is not given by $H = 0$ but by the first equation (9).

The first and last equations in (9) can be regarded with the constraint equations

$$0 = C_{(1)} \equiv \phi_{,11} - \kappa^4 \pi_{\phi} \pi_{\gamma} - \gamma_{,1} \phi_{,1} + \frac{e^{2\gamma}}{2} V(\phi),$$
$$0 = C_{(2)} \equiv \pi_{\gamma} \phi_{,1} + \gamma_{,1} \phi_{,1}.$$  

In order to find the secondary constraints, we define the Poisson bracket for two physical quantities $F(x_{(1)})$ and $G(x_{(2)})$ as follows:

$$[F(x_{(1)}) , G(x_{(2)}) ] \equiv$$

$$\int d\gamma \left\{ \frac{\partial F(x_{(1)})}{\partial \phi(x)} \frac{\partial G(x_{(2)})}{\partial \phi(x)} - \frac{\partial F(x_{(1)})}{\partial \phi(x)} \frac{\partial G(x_{(2)})}{\partial \phi(x)} \right\}$$

$$+ \frac{\partial F(x_{(1)})}{\partial \gamma(x)} \frac{\partial G(x_{(2)})}{\partial \gamma(x)} - \frac{\partial F(x_{(1)})}{\partial \gamma(x)} \frac{\partial G(x_{(2)})}{\partial \gamma(x)} \right\}. \quad (15)$$

By using the Poisson bracket, we find

$$[C_{(1)}(x_{(1)}) , H] = \kappa^2 C_{(2,1)}(x_{(1)}) ,$$
$$[C_{(2)}(x_{(1)}) , H] = \frac{1}{\kappa^2} C_{(1,1)}(x_{(1)}) .$$

(16)
Therefore if \( C_{(1)} = C_{(2)} = 0 \), we obtain \( [C_{(1)} (x_{(1)}), H] = [C_{(2)} (x_{(1)}), H] = 0 \) and there does not appear any secondary constraint. Hence we have four canonical variables \( \phi, \pi_\phi, \gamma, \) and \( \pi_\gamma \), and two constraints, we have two independent canonical variables. This shows that although the action (11) seems to generate a ghost, the constraints might delete the ghost.

**2D \( F(R) \) gravity and AdS\(_2\)/CFT\(_1\) correspondence.** – We now discuss the correspondence between 2D anti-de Sitter gravity and the conformal field theory in one dimension, that is, the AdS\(_2\)/CFT\(_1\) correspondence.

The conformal field theory in one dimension is nothing but quantum mechanics with conformal or scale invariance. A candidate of the conformal quantum mechanics is given by the SYK model \([6,7]\), which is given by 2\( N \) Majorana fermions \( \psi^a, a = 1, 2, \cdots, 2N \) and the Hamiltonian is given by

\[
H = \sum_{a,b,c,d=1}^{2N} J_{abcd} 4! \psi^a \psi^b \psi^c \psi^d.
\]

(17)

Here \( J_{abcd} \) is a random source whose average vanishes, \( J_{abcd} = 0 \), and the average of the square of \( J_{abcd} \) is given by \( \sum_{a,b,c,d=1}^{2N} J_{abcd}^2 = \frac{3J^2}{(2N)!} \) with a constant \( J \). The AdS\(_2\)/CFT\(_1\) correspondence of this model has been well studied \([8,9]\).

In the model, there appears a conformal symmetry at low energy. Interestingly, this model saturates the chaos bound \([10,11]\). The conformal symmetry in \( d \) dimensions is identical with the isometry of the AdS space-time in \( D = d + 1 \) dimensions. In this section, we check those 2D gravity models have AdS space-time as a solution.

In the case of \( F(R) \) gravity (1), the variation over the metric gives

\[
0 = \frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F(R) + \nabla_\mu \nabla_\nu F(R) - g_{\mu\nu} \nabla^2 F(R).
\]

(18)

Here \( F(R) \equiv \frac{dF(R)}{dR} \). Because AdS space-time is the Einstein manifold, where the Ricci tensor \( R_{\mu\nu} \) is covariantly constant, \( \nabla_\mu R_{\mu\nu} = 0 \), we may assume \( R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} \). As in two dimensions, here \( R_{\mu\nu} \) is a constant corresponding to the scalar curvature. Then eq. (18) reduces to the form

\[
0 = F(R_0) - R_0 F(R_0),
\]

which is nothing but the condition \( 0 = F(R_0) - R_0 F(R_0) \) as expected.

As AdS\(_2\)/CFT\(_1\) correspondence between the gravity model (2) and the SYK model (17) has been established in \([8]\), the equivalence between the model in (2) and the SYK model (17) has been proved that the \( F(R) \) gravity model (1) in two dimensions is dual to the SYK model (17).

**Singular \( D \to 2 \) limit in \( F(R) \) gravity.** – In two dimensions, the Hilbert-Einstein action is a total derivative and the action does not contribute to any dynamics. There was, however, a proposal that Einstein’s theory in the limit of \( D \to 2 \) with a singular rescaling of the gravitational coupling constant as \( 1/(D - 2) \) \([12]\) may lead to non-trivial gravitational dynamics. In this section, we consider the \( D \to 2 \) limit in the scalar-tensor theory (5).

By the variation of the action (5) with respect to the metric, we obtain

\[
0 = \frac{1}{2\kappa^2} \left[ \frac{1}{2} g_{\mu\nu} \left( R - \frac{(D - 3)(D - 2)}{4} \partial^\rho \sigma \partial_\rho \sigma \right) - e^{\frac{2}{\sigma}} V \left( \phi = e^{-\frac{D-2}{2}\sigma} \right) \right] - R_{\mu\nu} + \frac{(D - 3)(D - 2)}{4} \partial_\mu \sigma \partial_\nu \sigma,
\]

(21)

and by the variation with respect to \( \sigma \), we find

\[
0 = \frac{(D - 3)(D - 2)}{2} \nabla^2 \sigma - \frac{D - 2}{2} e^{\frac{2}{\sigma}} V \left( \phi = e^{-\frac{D-2}{2}\sigma} \right).
\]

(22)

By multiplying (21) with \( g^{\mu\nu} \), we obtain

\[
0 = \frac{1}{2\kappa^2} \left[ \frac{D - 2}{2} \left( R - \frac{(D - 3)(D - 2)}{4} \partial^\rho \sigma \partial_\rho \sigma \right) - D - 2 \right] e^{\frac{2}{\sigma}} V \left( \phi = e^{-\frac{D-2}{2}\sigma} \right).
\]

(23)

Therefore in the limit of \( D \to 2 \), one gets \( 0 = e^\sigma V (\phi = 1) \), which gives \( V (\phi = 1) = 0 \), which is not satisfied for general \( V (\phi) \).

There are several limits to obtaining non-trivial solutions. An example is given by the limit \( D \to 2 \) after redefining \( \kappa^2 = \tilde{\kappa}^2 (D - 2) \) and \( \sigma = \tilde{\sigma} + \frac{2}{\tilde{\kappa}} \ln (D - 2) \) and keeping \( \tilde{\kappa} \) and \( \tilde{\sigma} \) finite. In the limit, eqs. (23) and (22)
have the following non-trivial forms:
\[ 0 = \frac{1}{2\kappa^2} \left( \frac{1}{2} R - e^{\phi} V (\phi = 1) \right), \]
\[ 0 = -\frac{1}{2} \nabla^2 \sigma - e^{\phi} V (\phi = 1), \]
which could give non-trivial solutions.

We now consider the solution of the equations in eq. (24). In the conformal gauge \( g_{\mu\nu} = e^{2\gamma} \eta_{\mu\nu}, \) we find that the equations in (24) have the following non-trivial forms:
\[ 0 = \gamma_{\mu0} - \gamma_{11} - e^{\phi + 2\gamma} V_0, \]
\[ 0 = \frac{1}{2} (\sigma_{\mu0} - \sigma_{11}) - e^{\phi + 2\gamma} V_0. \]
Here \( V_0 \equiv V (\phi = 1). \) By combining the above equations in (25), we obtain
\[ 0 = \gamma_{\mu0} - \gamma_{11} - \frac{1}{2} (\sigma_{\mu0} - \sigma_{11}), \]
\[ 0 \equiv \frac{1}{2} (\rho_{\mu0} - \rho_{11}) - 2V_0 \rho e^\phi. \]

Here \( \rho \equiv \sigma + 2\gamma. \) We should note that the second equation of (26) is nothing but the Liouville equation, whose solutions are well known. As an example, we review the static case, where we find \( \frac{d}{dt} (\frac{1}{2} \rho_{11} - 2V_0 e^\phi) \) and therefore \( \frac{d}{dt} \rho_{11} + 2V_0 e^\phi = C \) (\( C \) is a constant), that is,
\[ x = -\frac{\sqrt{2\kappa_0}}{C} (\theta - \theta_0) \left( e^{-\frac{\gamma}{\kappa_0}} - \sqrt{\frac{2\kappa_0}{C}} \cosh \theta \right). \] Here \( \theta_0 \) is a constant of the integration. Therefore we find 
\( e^{-\rho} = \frac{2\kappa_0}{C} \cosh^2 (\frac{2C}{\sqrt{2\kappa_0}} (x - x_0)). \) Here \( x_0 \equiv \frac{4\kappa_0}{\sqrt{2\kappa_0}}. \) The general solution of the first equation in (26) when the solution does not depend on \( t \) is
\[ \tilde{\sigma} = 2\gamma - 4\gamma_0 - 4\gamma_1 x, \]
where \( \gamma_0 \) and \( \gamma_1 \) are constants and the factor \(-4\) is just for convenience. Therefore we find \( \rho = 4(\gamma - \gamma_0 - \gamma_1 x), \) which tells
\[ e^{2\gamma} = \frac{C e^{\gamma + \gamma_1 x}}{2V_0 \cosh (\frac{2C}{\sqrt{2\kappa_0}} (x - x_0))}. \] The conformal gauge condition \( g_{\mu\nu} = e^{2\gamma} \eta_{\mu\nu} \) shows that the metric is given by
\[ ds^2 = \frac{C e^{\gamma + \gamma_1 x}}{2V_0 \cosh (\frac{2C}{\sqrt{2\kappa_0}} (x - x_0))} (\, dt^2 + \, dx^2). \]

Even if we consider the adjustments of the parameters \( \gamma_0, \gamma_1, V_0, C, \) and \( x_0, \) or redefinition of the coordinates \( t \) and \( x, \) we cannot obtain the AdS2 space-time. In the AdS2 space-time, the scalar curvature \( R \) is constant. In order for \( R \) to be constant, the equation \( 0 = e^{\phi} V (\phi = 1) \) shows that \( \sigma \) must be constant but the first equation in (24) does not allow \( \tilde{\sigma} \) to be constant. Therefore the AdS2 space-time cannot be a solution of the \( D \to 2 \) model even asymptotically.

To be sure, we may consider the modification of the \( D \to 2 \) limit. For example, if we redefine \( \tilde{\sigma} = \epsilon \eta - \ln \epsilon \) and \( V (\phi = 1) = C e^\eta, \) with constants \( \epsilon \) and \( U_0, \) in the limit of \( \epsilon \to 0, \) the first equation in eq. (24) reduces to the form
\[ 0 = \frac{1}{2\kappa^2} \left[ \frac{1}{2} R - U_0 \right]. \] This equation seems to tell that the scalar curvature \( R \) is a constant \( R = 2U_0 \) and therefore the AdS space-time could be a solution. In the limit of \( \epsilon \to 0, \) however, the second equation in (24) gives \( 0 = U_0 \) and therefore the scalar curvature \( R \) vanishes, \( R \to 0. \)

This demonstrates that AdS space-time could not be a solution of the equations in (24) in any limit even asymptotically. Therefore this model could not be related to the SYK model and the perspectives of the AdS2/CFT1 correspondence are not clear in the singular \( D \to 2 \) gravity model here.

In summary, we have shown that \( F(R) \) gravity in two dimensions is equivalent to several gravity theories like JT gravity if we neglect the matter. The equivalence between JT gravity and the SYK models, which have conformal symmetry in one dimension, has been well discussed from the viewpoint of the AdS2/CFT1 correspondence. Therefore \( F(R) \) gravity in two dimensions is also equivalent to the SYK models.

We also discussed the singular \( D \to 2 \) limit in the \( F(R) \) gravity and we have shown that such a limit does not have AdS2 solutions and therefore there is no AdS2/CFT correspondence in this case. However, the account of matter or quantum effects may qualitatively change this picture.

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