Decoupling Transformations in Path Integral Bosonization

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Abstract

We construct transformations that decouple fermionic fields in interaction with a gauge field, in the path integral representation of the generating functional. Those transformations express the original fermionic fields in terms of non-interacting ones, through non-local functionals depending on the gauge field. This procedure, holding true in any number of spacetime dimensions both in the Abelian and non-Abelian cases, is then applied to the path integral bosonization of the Thirring model in 3 dimensions. Knowledge of the decoupling transformations allows us, contrarily to previous bosonizations, to obtain the bosonization with an explicit expression of the fermion fields in terms of bosonic ones and free fermionic fields. We also explain the relation between our technique, in the two dimensional case, and the usual decoupling in 2 dimensions.

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1 Introduction

Any physical system can be studied by using different sets of variables which yield, of course, mathematically different descriptions. Different choices of variables must however be physically equivalent, in the sense that they should describe one and the same system. An extreme manifestation of this property appears in models which can be defined in terms of either fermionic or bosonic variables. The equivalence between these two formulations is made explicit by the so called ‘bosonization rules’, that map fermionic into bosonic variables.

Bosonization turns out to be a very useful tool indeed in order to understand and in some cases even to solve some non-trivial interacting Quantum Field Theory models in two spacetime dimensions. It is an interesting fact that there is no theoretical obstacle to the extension of this procedure to higher dimensions. Indeed, there has recently been some progress in the application, although in an approximated form, of a path integral bosonization procedure to theories in more than two spacetime dimensions [1]-[11], dealing with both the Abelian and the non-Abelian cases. An important difference between these works and two dimensional bosonization [12]-[18] is that, in the latter case, the fermionic determinant can be evaluated by performing a decoupling transformation of the fermions [19]-[20]. The redefined fermionic fields (in the massless case) are free, and the effects of the interaction manifest themselves through the existence of an anomalous Jacobian which depends on the gauge field. We remark that no such a thing had yet been suggested for the higher dimensional case; the present work deals with such an extension. In this paper we shall complement the usual path integral bosonization technique by showing how to redefine the fermionic fields in the path integral, in order to decouple them from the gauge field. This step, we believe, fills a gap in this approach to bosonization in higher dimensions, rendering the whole process entirely analogous to its well known two dimensional counterpart. The main point of our work is that it gives, contrarily to previous bosonizations in 3 dimensions, the bosonization with an explicit expression for the fermionic fields in terms of bosonic ones and free fermionic fields. This procedure is moreover generalizable to higher dimensions.

Let us briefly review, for the Abelian case in 3 dimensions, the usual approach to bosonization, which starts from the definition of the generating
where $W(J)$ is the generating functional of Euclidean connected correlation functions of fermionic currents. The bosonized action $S_{bos}(A)$ is then given by a generalised functional Fourier transform of $Z(b) = \exp[-W(b)]$:

$$\exp[-S_{bos}(A)] = \int \mathcal{D}b_\mu \exp \left[ -W(b) + i \int d^3x \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu A_\lambda \right],$$

and the bosonization rule for the fermionic bilinear $\bar{\Psi} \gamma_\mu \Psi$ is

$$\bar{\Psi}(x) \gamma_\mu \Psi(x) \rightarrow i \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda(x).$$

The fact that the functional $W(J)$ proceeds from fermionic matter field seems to be here a matter of no relevance, the crucial property invoked being gauge invariance of the fermionic determinant instead. This is quite different to what happens in two dimensional path integral bosonization, where the fermionic fields are redefined in terms of new, decoupled fermions, through transformation functions that depend on the bosonic gauge fields. In this paper, we give explicit transformation functions that achieve the same goal as in the two dimensional case.

A difficulty renders the bosonization procedure in more than two space-time dimensions non-exact. It is our inability to compute exactly a fermionic determinant in those cases. This allows us to determine only approximated bosonized actions, which usually result from some perturbative or low-momentum expansion. We shall see that the same can be said about the decoupling transformations, one can consistently decouple the fermions up to some order in the relevant coupling constant of the model being considered.

The structure of this paper is as follows: In section 2 we explain the mechanism of decoupling on a general footing, discussing both Abelian and non-Abelian cases, in the massless and massive situations. The application of the general results to the path-integral bosonization of the three dimensional Thirring model is studied in section 3, and in section 4 we explain the differences and similarities between our approach, when restricted to the two dimensional case, and the one which makes use of the anomalous (Fujikawa) Jacobian method.
2 Decoupling the fermions by a change of variables

2.1 Massless case

The generating functional $Z(A)$ for massless fermionic fields in the presence of an external gauge field $A$ in a $D$-dimensional Minkowski spacetime is defined by

$$Z(A) = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \ e^{iS_F(\bar{\Psi}, \Psi; A)}$$

where

$$S_F(\bar{\Psi}, \Psi; A) = \int d^Dx \ \bar{\Psi}(x) i \ \bar{D} \ \Psi(x)$$

$$\bar{D} = \gamma_\mu D_\mu \quad D_\mu = \partial_\mu + e A_\mu$$

and

$$A = -A^\dagger \equiv \left\{ \begin{array}{ll}
i A_\mu & \text{in the Abelian case} \\
A_\mu^a \tau_a & \text{in the non Abelian case} \\end{array} \right.$$  

where $\tau_a$ are (anti-hermitian) generators of the Lie algebra of the non-Abelian gauge group, and both $A_\mu$ and $A_\mu^a$ are real. Dirac’s $\gamma$-matrices are assumed to satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \quad \gamma_0^\dagger = \gamma_0 \quad \gamma_j^\dagger = -\gamma_j.$$  

We shall show that, by a change of variables, one can decouple the fermions from the external field $A$. Then we redefine the fermionic fields $\Psi(x), \bar{\Psi}(x)$ in terms of new ones $\chi(x), \bar{\chi}(x)$, as follows

$$\Psi(x) = F(e \bar{\vartheta}^{-1} A) \chi(x)$$

$$\bar{\Psi}(x) = \bar{\chi}(x) F(e \vartheta^{-1} A)$$

where

$$F(e \vartheta^{-1} A) = \gamma_0 \left[ F(e \vartheta^{-1} A) \right]^\dagger \gamma_0$$

and $F$ is a function which, we shall assume, is defined by a power series expansion in $e$

$$F(e \vartheta^{-1} A) = 1 + \sum_{n=1}^{\infty} \alpha_n e^n (\vartheta^{-1} A)^n.$$  

\footnote{We shall later on explain the analogous procedure for the Euclidean spacetime case.}
with real coefficients $\alpha_n$. The fact that we shall understand the transformation (8) as given by the series (10) is the reason why we call the whole technique 'perturbative'.

It is important to note that the operator $-\partial^{-1}$ introduced in (8) must be understood as acting on both $A$ and the fermionic field $\chi$, not on $A$ alone. Namely,

$$\mathcal{F}(e\vartheta^{-1} A) \neq \mathcal{F}(e(\vartheta^{-1} A)), \quad (11)$$

where the outer parenthesis denotes functional dependence, while the inner one delimits the action of the operator $\vartheta^{-1}$. This fact and the reality of the $\alpha_n$’s implies that the transformation formula for $\bar{\Psi}(x)$ may also be written more explicitly as

$$\bar{\Psi}(x) = \bar{\chi}(x) \mathcal{F}(e A \vartheta^{-1}) \quad (12)$$

and this property turns out to be crucial in the decoupling. We then see that the new fields will have a free action as long as the equation

$$\int d^D x \bar{\chi} \mathcal{F}(e A \vartheta^{-1}) i \vartheta \mathcal{F}(e \vartheta^{-1} A) \chi(x) = \int d^D x \bar{\chi} i \vartheta \chi(x) \quad (13)$$

may be satisfied. This implies

$$\mathcal{F}(e A \vartheta^{-1}) \vartheta \mathcal{F}(e \vartheta^{-1} A) = \vartheta \quad (14)$$

which, by using the power series expansion (12), yields a set of relations for the unknown coefficients $\alpha_n$. They may be written as

$$0 = e(1 + 2\alpha_1) A + \sum_{n=1}^{\infty} 2(\alpha_n + \alpha_{n+1})e^{n+1}T^{(n)} + \sum_{n=1}^{\infty} (\sum_{m=1}^{n} \alpha_{n+1-m} \alpha_m)e^{n+1}T^{(n)} + \sum_{n=2}^{\infty} (\sum_{m=1}^{n-1} \alpha_{n-m} \alpha_m)e^{n+1}T^{(n)} \quad (15)$$

where

$$T^{(n)} = A(\vartheta^{-1} A)^n = (A \vartheta^{-1})^n A = A \vartheta^{-1} A \cdots \vartheta^{-1} A \quad (16)$$

namely, the operator $T^{(n)}$ consists of a product of alternating factors $\vartheta^{-1}$ and $A$, starting and ending with an $A$, and containing $n$ factors $\vartheta^{-1}$ altogether. The crucial property that makes the decoupling possible is that any redefinition like (8), (even with arbitrary coefficients $\alpha_n$), modifies the action by adding a series of terms, which are always proportional to one of the $T^{(n)}$’s.
Equation (15) implies the recurrence relations

\[ \alpha_{n+1} = -\frac{3}{4} \alpha_n - \frac{1}{2} \sum_{k=1}^{n-1} (\alpha_{k+1} + \alpha_k) \alpha_{n-k} \quad \forall n > 1 \]  

(17)

plus the initial conditions

\[ \alpha_1 = -\frac{1}{2}, \quad \alpha_2 = \frac{3}{8}. \]  

(18)

The solution to the recurrence relations (17), with conditions (18), is

\[ \alpha_n = \binom{-\frac{1}{2}}{n} \]  

(19)

where

\[ \binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{1}{2}-1) \cdots (-\frac{1}{2}-n+1)}{n!}. \]  

(20)

Thus we conclude that \( \mathcal{F}(x) \) can be thought of as the power series expansion of the function \( (1 + x)^{-\frac{1}{2}} \) around \( x = 0 \), with a radius of convergence equal to 1,

\[ \mathcal{F}(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n = (1 + x)^{-\frac{1}{2}}, \quad |x| < 1. \]  

(21)

Condition \( |x| < 1 \) is equivalent to saying that we are in the perturbative regime. Whenever we write, henceforward, \( \mathcal{F}(x) = (1 + x)^{-\frac{1}{2}} \), we shall have in mind its power series expansion as given by (21).

When performing the transformations

\[ \Psi(x) = (1 + e^{-1} A)^{-\frac{1}{2}} \chi(x) \]
\[ \bar{\Psi}(x) = \bar{\chi}(x)(1 + e A^{-1})^{-\frac{1}{2}}, \]  

(22)

we can then affirm that the new fermionic action shall be free, but, on the other hand, there will appear a Jacobian \( J(A) \), due to the change of (Grassmann) variables,

\[ \mathcal{Z}(A) = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{iS_F(\Psi, \bar{\Psi}; A)} \]

\[ = \int \mathcal{D}\bar{\chi} \mathcal{D}\chi J(A) e^{iS_F(\bar{\chi}, \chi; 0)} \]  

(23)
where,

\[ S_F(\bar{\chi}, \chi; 0) = \int d^D x \bar{\chi}(x) i \not{D} \chi(x) \]  

(24)

and

\[ J(A) = J_{\chi}(A) J_{\bar{\chi}}(A), \]

\[ J_{\chi}(A) = \left\{ \det\left( (1 + e \not{A}^{-1} \not{\Lambda})^{-\frac{1}{2}} \right) \right\}^{-1} \]

\[ J_{\bar{\chi}}(A) = \left\{ \det\left( (1 + e \not{\Lambda}^{-1} \not{A})^{-\frac{1}{2}} \right) \right\}^{-1}. \]  

(25)

We remark that these Jacobians are different from one, but nevertheless not anomalous, in the sense that, as the field transformations are not unitary, they shouldn’t be equal to one.

From the two self-evident relations

\[ \det{\not{D}} = \det{\not{\Lambda}} \det(1 + e \not{A}^{-1} \not{\Lambda}) \]

\[ \det{\not{D}} = \det(1 + e \not{\Lambda}^{-1} \not{A}) \det{\not{\Lambda}} \]

we deduce

\[ J(A) = \frac{\det{\not{D}}}{\det{\not{\Lambda}}}. \]  

(27)

Recalling (24), we see that, as it should be, multiplying the Jacobian by the free determinant we reconstruct the determinant of \( \not{D} \).

The use of transformations (22) would, in principle, allow us to decouple the fermions completely from the external field \( A \). This, however, is not possible to use in the applications, since it would require the evaluation of the Jacobian (27) exactly, something not possible in more than 1 + 1 dimensions.

We may, however, still use a finite order approximation to (22), in order to achieve partial decoupling. For example, we can perform a transformation that corresponds to the truncation to second order in \( e \) of (22). This will lead to an action for the new fermions containing interaction terms of order three or higher; namely,

\[ \Psi(x) = \left( 1 - \frac{1}{2} e \not{A} \not{A}^{-1} \not{\Lambda} + \frac{3}{8} e^2 \not{A} \not{\Lambda}^{-1} \not{A} \right) \chi(x) \]

\[ \bar{\Psi}(x) = \bar{\chi}(x) \left( 1 - \frac{1}{2} e \not{\Lambda} \not{\Lambda}^{-1} \not{A} + \frac{3}{8} e^2 \not{\Lambda} \not{A}^{-1} \not{\Lambda} \right) \]

(28)

and

\[ S_F \to \int d^D x \bar{\chi}(i \not{D} \chi + O(e^3)) \chi. \]  

(29)
The use of the decoupling transformations in the Euclidean spacetime case could of course be implemented by first rotating to Minkowski spacetime, then performing the transformations (22), and at the end coming back to Euclidean. The new fermions will of course remain decoupled after the rotation. If one wanted to implement the transformations directly in Euclidean spacetime, one should face the problem that the Dirac matrices have different hermiticity properties than in Minkowski spacetime, and that spoils the transformation rule for the adjoint spinor, since under the transformation

$$\Psi(x) = \mathcal{F}(e\nabla^{-1}A)\chi(x)$$  \hspace{1cm} (30)

it is still true that

$$\bar{\Psi}(x) = \bar{\chi}(x)\mathcal{F}(e\nabla^{-1}A)$$  \hspace{1cm} (31)

but

$$\gamma_0 \left[ \mathcal{F}(e\nabla^{-1}A) \right]^\dagger \gamma_0 = \mathcal{F}(e\tilde{\nabla}\tilde{A}^{-1}) \neq \mathcal{F}(e\tilde{A}\tilde{\nabla}^{-1})$$  \hspace{1cm} (32)

where, for any Euclidean vector $v_\mu$, we denote $\tilde{v}_\mu = (-v_1, -v_2, v_3)$.

Although not necessary in principle, one can, however, still give an explicit decoupling transformation in the Euclidean case. To that end, we employ the trick used in [21], where it was introduced in a slightly different context. That trick amounts to make in the Euclidean path integral representing the fermionic determinant

$$Z(A) = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{-\int d^Dx \bar{\Psi}(x)\mathcal{D}\Psi(x)} = \det \mathcal{D},$$  \hspace{1cm} (33)

the change $\bar{\Psi}(x) \rightarrow \bar{\Psi}(x)\gamma_3$, which of course does not change the value of the determinant. Then one see that the fermionic action is

$$S_F = \int d^Dx \bar{\Psi}(x)\gamma_3 \mathcal{D}\Psi(x)$$  \hspace{1cm} (34)

and, under the decoupling transformations

$$\Psi(x) = (1 + e\nabla^{-1}A)^{-\frac{1}{2}}\chi(x)$$

$$\bar{\Psi}(x) = \bar{\chi}(x)\gamma_3(1 + e\tilde{A}\tilde{\nabla}^{-1})^{-\frac{1}{2}}\gamma_3$$  \hspace{1cm} (35)

\footnote{One could of course have just said that one wanted to evaluate the determinant of $\gamma_3 \mathcal{D}$.}
the new action is
\[ S_F \to \int d^D x \overline{\chi}(x) \gamma_3 \not\partial \chi(x) \] (36)
as can be easily checked. Note that the presence of \( \gamma_3 \) in the new action is
as irrelevant as in the old one. To avoid writing the \( \gamma_3 \), when we use the
Euclidean formalism in section 3, we work with \( \Psi^\dagger \) instead of \( \overline{\Psi} \),
\[ S_F = \int d^D x \Psi^\dagger(x) \not D \Psi(x) \] (37)
and give the transformations in terms of those fields. The procedure is, of
course, equivalent.

2.2 Massive case

The massive case admits, within our decoupling approach, two different treat-
ments. Beginning from the massive fermionic action
\[ S_F(\overline{\Psi}, \Psi; A) = \int d^D x \overline{\Psi}(x) (i \not D - M) \Psi(x) , \] (38)
one possibility is to perform the analogous of the redefinitions (22) but in-
cluding the mass term into the inverse of the free Dirac operator, i.e.,
\[ \Psi(x) = [1 + e(\not D + iM)^{-1} \not A]^{-\frac{1}{2}} \chi(x) \]
\[ \overline{\Psi}(x) = \overline{\chi}(x) [1 + e \not A(\not D + iM)^{-1}]^{-\frac{1}{2}} . \] (39)
This way of transforming the fermionic fields leads, after some algebra that
is essentially equal to the one of the massless case, to the decoupled fermionic
fields \( \overline{\chi} \) and \( \chi \), with a massive free action
\[ S_F(\overline{\chi}, \chi) = \int d^D x \overline{\chi}(x) (i \not D - M) \chi(x) , \] (40)
and to the Jacobian
\[ J(A) = \frac{\det(\not D + iM)}{\det(\not \partial + iM) } ; \] (41)
this way of decoupling, applied in its second order approximated version,
is the one we use in the next section to study the Thirring model in 2 + 1
dimensions.
There is, however, another way of transforming the fermions, which is in fact more similar to the one used in two dimensions. It consists in using the transformations (22), corresponding to the massless case, for the massive action. The new fermions are no longer decoupled, but all the interactions are proportional to the mass, and the new action is a suitable starting point for performing an expansion in powers of the mass, as it happens to be the case in two dimensions

\[
S_F(\bar{\chi}, \chi; A) = \int d^D x \bar{\chi}(x) i \not\partial \chi(x)
- M \int d^D x \bar{\chi}(x) \left\{ [1 + e(\not\partial + iM)^{-1} A][1 + e A(\not\partial + iM)^{-1}]} \right\}^{-\frac{1}{2}} \chi(x) .
\] (42)

In Euclidean spacetime, as for the massless case, the continuation to Minkowski prior to decoupling is a possible path. However, one can also work directly in Euclidean spacetime as explained for the massless case. One then uses the $\gamma_3$ trick. If the first version of decoupling (39) is used, due to the presence of the mass term, we must also make a "smooth continuation" (in the sense of [22]) to imaginary masses before decoupling, and then rotate back to real values. In perturbation theory, there is no obstruction to that. Then the transformations in terms of $\Psi$ and $\Psi^\dagger$ read

\[
\Psi(x) = [1 + e(\not\partial + M)^{-1} A]^{-\frac{1}{2}} \chi(x)
\Psi^\dagger(x) = \chi^\dagger(x) [1 + e A(\not\partial + M)^{-1}]^{-\frac{1}{2}}.
\] (43)

where $M$ is regarded as imaginary.

We end up this section by making a consistency check, valid for both the massive and the massless cases. Namely, that the expectation value of the fermionic current $\bar{\Psi}\gamma_\mu \Psi$, when the fermions are written in terms of the $\chi$ and $\bar{\chi}$, yields the proper result. For the sake of concreteness, we deal the case of the massive action (38). The expectation value of that current, when evaluated in terms of the decoupled fermions is

\[
\langle \bar{\Psi}(x)\gamma_\mu \Psi(x) \rangle =
\langle \bar{\chi}(x)[1 + e A(\not\partial + iM)^{-1}]^{-\frac{1}{2}} \gamma_\mu[1 + e(\not\partial + iM)^{-1} A]^{-\frac{1}{2}} \chi(x) \rangle , \quad (44)
\]
or

\[
\langle \bar{\Psi}(x)\gamma_\mu \Psi(x) \rangle =
\]
\[-\text{Tr} \int d^Dy d^Dz \left\{ \langle y| [1 + e \mathcal{A}(\mathcal{D} + iM)^{-1}]^{-\frac{1}{2}} |x\rangle \gamma_\mu \right. \\
\left. \langle x| [1 + e(\mathcal{D} + iM)^{-1} \mathcal{A}]^{-\frac{1}{2}} |z\rangle \langle \chi(z)\bar{\chi}(y)\rangle \right\} \] (45)

where the usual notation $\langle x| \cdots |y\rangle$ denotes kernels of functional operators in coordinate space.

Since the new fermions are free,

\[\langle \chi(z)\bar{\chi}(y)\rangle = \langle z|(\mathcal{D} + iM)^{-1}|y\rangle .\] (46)

Using this, and after some straightforward algebra, we obtain

\[\langle \bar{\Psi}(x)\gamma_\mu \Psi(x)\rangle = -\text{Tr} \left[ \gamma_\mu \langle x|[(\mathcal{D} + ie - A + iM)^{-1}]|x\rangle \right] \] (47)

which is the proper expression for the expectation value of the current in an external field. Also, (47) can of course be written in terms of the fermionic determinant:

\[\langle \bar{\Psi}(x)\gamma_\mu \Psi(x)\rangle = \frac{\delta}{e\delta A_\mu(x)} \ln \det(i\mathcal{D} - M) .\] (48)
3 The Thirring model in $2 + 1$ dimensions

We shall here follow the same strategy as in [18] for the $1 + 1$ dimensional Thirring model, but with the necessary changes due to the fact that we are now in $2 + 1$ dimensions, and the decoupling is, in consequence, non-exact.

We define the generating functional of current correlation functions for the Thirring model in $2 + 1$ Euclidean dimensions by

$$Z(J) = \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi \exp\left\{ - \int d^3x [\Psi^\dagger (\partial + J + M)\Psi + \frac{1}{2} g (\Psi^\dagger \gamma_\mu \Psi)^2] \right\} \tag{49}$$

where $g$ is the coupling constant and $J_\mu$ is a source introduced in order to generate fermionic current correlation functions. In order to render the fermionic action quadratic, we introduce an auxiliary vector field $A_\mu$, such that the generating functional now reads

$$Z(J) = \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi \mathcal{D}A_\mu \exp\left\{ - \int d^3x [\Psi^\dagger (\partial + \sqrt{i} \gamma_\mu A + J + M)\Psi + \frac{1}{2} (A_\mu)^2] \right\} \tag{50}$$

Making now a shift in the vector field $A_\mu \to A'_\mu = i \sqrt{g} A_\mu + J_\mu$,

$$Z(J) = \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi \mathcal{D}A'_\mu \exp\left\{ - \int d^3x [\Psi^\dagger (\partial + A' + M)\Psi - \frac{1}{2g} (A' - J)^2] \right\} \tag{51}$$

Now we proceed to make a decomposition of the field $A'$ into the curl of a vector field $\Phi_\mu$, plus the gradient of a scalar field $\varphi$,

$$A'_\mu = i(\epsilon_{\mu\nu\lambda} \partial_\nu \Phi_\lambda + \partial_\mu \varphi) \tag{52}$$

We note that we can shift the vector field $\Phi_\mu$ by a gradient without affecting the configuration of $A'_\mu$. This freedom is due to the fact that $A'_\mu$ has three components, while on the right hand side of (52) there appear to be four (three from $\Phi_\mu$ plus one from $\varphi$). This apparent contradiction is solved by the above mentioned ‘gauge invariance’ under transformations of $\Phi_\mu$, which allows us to impose a gauge fixing condition on $\Phi_\mu$, in order to leave only two components. We choose the Lorentz condition

$$\partial \cdot \Phi = 0 \tag{53}$$
which shall, of course, be included in the generating functional. We then get an expression for $Z(J)$ as an integral over the fermions $\Psi^\dagger, \Psi$ and the bosonic fields $\Phi_\mu, \varphi$

$$Z(J) = \int D\Psi^\dagger D\Psi D\Phi_\mu D\varphi \delta(\partial \cdot \Phi) \exp \left\{ -\int d^3x [\Psi^\dagger (\vartheta + i\gamma_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \Phi_\lambda + i \partial_\varphi + M)\Psi - \frac{1}{2g} (i\epsilon_{\mu\nu\lambda} \partial_\nu \Phi_\lambda + i \partial_\mu \varphi - J_\mu)^2] \right\}.$$ (54)

By a redefinition of the fermions,

$$\Psi(x) = e^{-i\varphi(x)} \Psi'(x)$$
$$\Psi^\dagger(x) = (\Psi'^\dagger(x)) e^{i\varphi(x)}$$ (55)

we entirely decouple them from the scalar field $\varphi$:

$$Z(J) = \int D\Psi'^\dagger D\Psi' D\Phi_\mu D\varphi \delta(\partial \cdot \Phi) \exp \left\{ -\int d^3x [\Psi'^\dagger (\vartheta + i\gamma_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \Phi_\lambda + M)\Psi' + \frac{1}{2g} (i\epsilon_{\mu\nu\lambda} \partial_\nu \Phi_\lambda + i \partial_\mu \varphi - J_\mu)^2 \right\}.$$ (56)

At this point we make use of the results of the previous section, to perform a new redefinition of the fermions, corresponding to (43), this time to decouple them from $\Phi_\mu$

$$\Psi'(x) = [1 + (\partial + M)^{-1}(\partial \Phi)]^{-\frac{1}{2}} \chi(x)$$
$$\Psi'^\dagger(x) = \chi'^\dagger(x)[1 + (\partial \Phi)(\partial + M)^{-1}]^{-\frac{1}{2}},$$ (57)

where we have used

$$i\gamma_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \Phi_\lambda = (\partial \Phi)$$ (58)

which holds as a consequence of the Dirac algebra in 3 dimensions when the Lorentz condition for $\Phi$ is used. Transformation (57) introduces a $\Phi$ dependent Jacobian,

$$Z(J) = \int D\chi^\dagger D\chi D\Phi_\mu D\varphi J(\Phi) \delta(\partial \cdot \Phi)$$

13
\[
\exp \left\{ - \int d^3x [\chi^\dagger (\partial + M) \chi + \frac{1}{2g} \Phi_\mu (-\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu) \Phi_\nu \right.
\]
\[
+ \frac{1}{2g} (\partial_\mu \varphi)^2 - \frac{1}{2g} J^2 + i J_\mu (\partial_\mu \varphi + \epsilon_{\mu\nu\lambda} \partial_\nu \Phi_\lambda)] \right\} \]
\]
(59)

where
\[
J(\Phi) = \text{det}[1 + (\partial + M)^{-1}(\partial \Phi)] .
\]
(60)

As already explained, it is not possible to evaluate the Jacobian \(J(\Phi)\) exactly. We shall instead use the quadratic approximation of \([9]\), which yields for this object
\[
J(\Phi) = \exp[-W(\Phi)]
\]
(61)

with
\[
W(\Phi) = \int d^3x \Phi_\mu [C^+ P_+ + C^- P_-]_{\mu\nu} \Phi_\nu
\]
(62)

where the \(C_\pm\) are scalar operator functions \([9]\) and the \(P_\pm\) are two of the three orthogonal projectors used in the calculation:
\[
[P_\pm]_{\mu\nu} = \frac{1}{2} (\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \pm \frac{i}{\sqrt{-\partial^2}} \epsilon_{\mu\nu\lambda} \partial_\lambda) \quad Q_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\partial^2} .
\]
(63)

We shall also need the transverse projector
\[
P = P_+ + P_- = \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} .
\]
(64)

Integrating out the field \(\varphi\),
\[
\mathcal{Z}(J) = \int \mathcal{D}\chi^\dagger \mathcal{D}\chi \mathcal{D}\Phi_\mu \delta(\partial \cdot \Phi) \exp \left\{ - \int d^3x [\chi^\dagger (\partial + M) \chi
\]
\[
+ W(\Phi) - \frac{1}{2g} \Phi_\mu \partial^2 P_{\mu\nu} \Phi_\nu - \frac{1}{2g} J_\mu \partial_\mu \Phi_\nu + \frac{i}{g} J_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \Phi_\lambda \right\} .
\]
(65)

The term quadratic in the source \(J_\mu\) can be, as in 1 + 1 dimensions, linearized by the introduction of an auxiliary field. In 2 + 1 dimensions it has to be a vector field \(\xi_\mu\), and the new expression for \(\mathcal{Z}\) becomes
\[
\mathcal{Z}(J) = \int \mathcal{D}\chi^\dagger \mathcal{D}\chi \mathcal{D}\Phi_\mu \mathcal{D}\xi_\mu \delta(\partial \cdot \Phi)
\]

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\[
\exp \left\{ - \int d^3x [\chi^\dagger (\partial + M) \chi + W(\Phi) - \frac{1}{2g} \Phi_\mu \partial^2 P_{\mu\nu} \Phi_\nu + \frac{\lambda}{2} (\partial \cdot \xi)^2 \\
+ \frac{\alpha}{2} \xi_\mu \partial^2 P_{\mu\nu} \xi_\nu + i J_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \left( \frac{1}{g} \Phi_\lambda - \beta \xi_\lambda \right) ] \right\} \tag{66}
\]

where the constants \( \alpha \) and \( \beta \) satisfy the relation:

\[
\frac{\beta^2}{\alpha} = \frac{1}{g} \tag{67}
\]

and \( \lambda \neq 0 \) is arbitrary.

We then introduce two vector fields \( \theta_\mu \) and \( \theta_\mu' \), defined in terms of the auxiliary fields \( \Phi_\mu \) and \( \xi \) such that: the transverse part of \( \theta_\mu \) is uniquely determined by the requirement that it should be the one that couples to the external current, and for the longitudinal part we choose it to be equal to the one of the field \( \xi \), i.e.,

\[
\theta_\mu = \frac{1}{g} P_{\mu\nu} \Phi_\nu - \beta \xi_\mu, \tag{68}
\]

and \( \theta_\mu' \) is instead defined in terms of \( \Phi_\mu \) and \( \xi \) in such a way that the new action for the fields \( \theta_\mu \) and \( \theta_\mu' \) contains no mixing term. There are many consistent ways of achieve this, however giving the same bosonic action for the relevant field \( \theta \), but the simplest we found is

\[
\theta_\mu' = \Phi_\mu - \left[ \frac{\partial^2}{C_+} P_+ + \frac{\partial^2}{C_-} P_- \right]_{\mu\nu} \theta_\nu \tag{69}
\]

with \( \theta_\mu \), of course, as given in (68). Note that, by virtue of (69), the longitudinal part of \( \theta_\mu' \) is equal to the one of \( \Phi \),

\[
\partial \cdot \Phi = \partial \cdot \theta'
\]

thus the functional delta function of \( \Phi \) becomes the gauge fixing for \( \theta' \).

The relevant bosonic field will be \( \theta_\mu \), since by virtue of the previous definitions \( \theta_\mu' \) is decoupled and will consequently be ignored. The resulting generating functional for \( \theta_\mu \) and the decoupled fermions is
\[ Z(J) = \int \mathcal{D}\chi \mathcal{D}\chi \mathcal{D}\theta \exp \left\{ - \int d^3 x [\chi^\dagger(\partial + M)\chi + \frac{\lambda}{2} (\partial \cdot \theta)^2 \\
+ \frac{1}{2} \theta_\mu \left[ \frac{\partial^2}{C_+} (g C_+ - \partial^2) P_+ + \frac{\partial^2}{C_-} (g C_- - \partial^2) P_- \right]_{\mu\nu} \theta_\nu + iJ_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \theta_\lambda \right\} \right. \] (71)

We can also write it more explicitly, in the notation of ref. [9]:

\[ Z(J) = \int \mathcal{D}\chi \mathcal{D}\chi \mathcal{D}\theta \exp \left\{ - \int d^3 x [\chi^\dagger(\partial + M)\chi + \frac{\lambda}{2} (\partial \cdot \theta)^2 \\
- \frac{1}{4} F_{\mu\nu}(\theta) C_1 F_{\mu\nu}(\theta) + i\frac{\theta_\mu C_2 \epsilon_{\mu\nu\lambda} \partial_\nu \theta_\lambda - \frac{q}{4} F_{\mu\nu}(\theta) + iJ_\mu \epsilon_{\mu\nu\lambda} \partial_\nu \theta_\lambda] \right\} \] (72)

where

\[ C_1 = \frac{F}{(-\partial^2)F^2 + G^2} \quad C_2 = \frac{G}{(-\partial^2)F^2 + G^2} . \] (73)

and \( F = F(-\partial^2), G = G(-\partial^2) \) are given in terms of their Fourier transforms \( \tilde{F} \) and \( \tilde{G} \) by

\[ \tilde{F}(k^2) = \frac{\left| \frac{m}{4\pi k^2} \right|}{4\pi} \left[ 1 - \frac{1 - \frac{k^2}{4m^2}}{\arcsin(1 + \frac{4m^2}{k^2})^{-\frac{1}{2}}} \right] , \] (74)

and

\[ \tilde{G}(k^2) = \frac{q}{4\pi} + \frac{m}{2\pi |k|} \arcsin(1 + \frac{4m^2}{k^2}^{-\frac{1}{2}}) , \] (75)

where \( q \) is an arbitrary integer.

Since \( \chi \) and \( \chi^\dagger \) are decoupled from the relevant bosonic field \( \theta \), one recovers from (72) the bosonization of the Thirring model in 3 dimensions, with a bosonic action given, in the quadratic approximation of \( J(\Phi) \), by

\[ S_{bos} = \int d^3 x \left[ -\frac{1}{4} F_{\mu\nu}(\theta) C_1 F_{\mu\nu}(\theta) + i\frac{\theta_\mu C_2 \epsilon_{\mu\nu\lambda} \partial_\nu \theta_\lambda - \frac{q}{4} F_{\mu\nu}(\theta) + \frac{\lambda}{2} (\partial \cdot \theta)^2 \right] \] (76)

and with the correspondence between the currents

\[ \Psi^\dagger(x) \gamma_\mu \Psi(x) \rightarrow i\epsilon_{\mu\nu\lambda} \partial_\nu \theta_\lambda(x) . \] (77)
But the important new point is that, with our method, we are able to express the initial fermion fields in terms of the bosonic ones and free fermionic fields, which was not the case in the usual approach to bosonization in 3 dimensions:

$$\Psi = e^{-i\varphi} \left\{ 1 - (\bar{\varphi} + M)^{-1} \left( \bar{\varphi}[\gamma_{\mu} C_{1} \mathcal{P}_{\mu\nu} \theta_{\nu} - i\gamma_{\mu} (-\partial^{2})^{-1} C_{2}\epsilon_{\mu\nu\lambda} \partial_{\nu} \theta_{\lambda}] + \bar{\varphi}' \right) \right\}^{-\frac{1}{2}} \chi.$$  

(78)

We end up this section by remarking that this redefinition of the fermionic fields yields the proper expectation value for the current (77), as can be verified by an application of (48) to this case.
4 The two ways of decoupling the fermions in 1 + 1 dimensions

We shall here compare the usual way of decoupling massless fermions in $d = 2$, which relies upon the anomalous Jacobian method, and ours, which involves non-anomalous Jacobians, and moreover holds true in any number of dimensions (in particular $d = 2$). We perform the consistency check that both must lead to the same answer when applied to the two dimensional Abelian case.

Let us begin by reviewing the anomalous Jacobian method. We start from the Euclidean generating functional

$$Z(A) = \int D\bar{\Psi} D\Psi \exp[-\int d^2x \bar{\Psi}(\partial + ie A)\Psi] .$$

(79)

The Dirac operator $\partial + ie A$ may be written as

$$\partial + ie A = \partial + ie \partial^{-1} A$$

(80)

where we understand $\partial^{-1}$ as acting on $A$ only, and not on the fermionic field that may eventually appear on the right. Similarly, $\partial$ acts on $(\partial^{-1} A)$.

Now, $(\partial^{-1} A)$ in $1 + 1$ dimensions, leads in a natural way to the decomposition of the vector field $A_\mu$ into its longitudinal and transverse parts, since

$$\partial^{-1} A = \partial^{-2} \partial A = \partial^{-2}(\partial \cdot A + i\gamma_5 e_{\mu\nu} \partial_\mu A_\nu)$$

(81)

or

$$\partial^{-1} A = \frac{1}{e} \partial(i\gamma_5 \Phi_1 + \Phi_2)$$

(82)

where $\Phi_1 = e \partial^{-2} e_{\mu\nu} \partial_\mu A_\nu$ and $\Phi_2 = e \partial^{-2} \partial \cdot A$.

Then one may write the Dirac operator as

$$\partial + ie A = e^{-i\Phi_2 + i\gamma_5 \Phi_1} \partial e^{i\Phi_2 + i\gamma_5 \Phi_1}$$

(83)

and achieve decoupling by performing the transformation

$$\Psi(x) = e^{-i\Phi_2(x) - i\gamma_5 \Phi_1(x)} \chi(x)$$

$$\bar{\Psi}(x) = \bar{\chi}(x)e^{i\Phi_2(x) - i\gamma_5 \Phi_1(x)} .$$

(84)
After the transformation (84) is performed, the action for the new fermions is of course free

\[ S_{F}(\bar{\Psi}, \Psi; A) = S_{F}(\bar{\chi}, \chi; 0) \]  

(85)

and there appears, due to the change of variables, an anomalous Jacobian

\[ J(\Phi_1) = \exp \left[ -\frac{1}{2\pi} \int d^2 x (\partial_{\mu} \Phi_1)^2 \right] = \exp \left[ \frac{e^2}{4\pi} \int d^2 x F_{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu} \right]. \]  

(86)

Thus, we have for \( Z(A) \)

\[ Z(A) = \exp \left[ \frac{e^2}{4\pi} \int d^2 x F_{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu} \right] \int \mathcal{D} \bar{\chi} \mathcal{D} \chi \exp \left[ \int d^2 x \bar{\chi} \partial \chi \right]. \]  

(87)

Let us now show that our way of decoupling yields to an identical result. In Euclidean space, we must redefine the fields according to

\[ \Psi(x) = (1 + i e \bar{\partial}^{-1} A)^{-\frac{1}{2}} \chi(x) \]
\[ \Psi^\dagger(x) = \chi^\dagger(x)(1 + i e A \partial^{-1})^{-\frac{1}{2}}, \]  

(88)

after which the action becomes also free, as in the previous case. The difference is that now we need to evaluate a (non anomalous) Jacobian

\[ J(A) = \frac{\det(\bar{\partial})}{\det(\partial)} \]  

(89)

Again, due to the fact that we are considering the special case on two spacetime dimensions, we may use the important result that (89) receives a non-vanishing contribution only from the second order vacuum polarisation, which therefore is the exact result for the logarithm of the Jacobian,

\[ J(A) = \exp \left[ \frac{e^2}{4\pi} \int d^2 x F_{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu} \right], \]  

(90)

and we have

\[ Z(A) = \exp \left[ \frac{e^2}{4\pi} \int d^2 x F_{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu} \right] \int \mathcal{D} \chi^\dagger \mathcal{D} \chi \exp \left[ \int d^2 x \bar{\chi} \partial \chi \right], \]  

(91)

to be compared with (87).
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