Elastic $\alpha$-$^{12}$C scattering at low energies with the bound states of $^{16}$O in effective field theory

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The elastic $\alpha$-$^{12}$C scattering for $l = 0, 1, 2, 3$ channels at low energies is studied, including the energies of excited bound states of $^{16}$O, in effective field theory. We introduce a new renormalization method due to the large suppression factor produced by the Coulomb interaction when fitting the effective range parameters to the phase shift data. After fitting the parameters, we calculate asymptotic normalization constants of the $0^+_2, 1^-_1, 2^+_1, 3^-_1$ states of $^{16}$O. We also discuss the uncertainties of the present study when the amplitudes are interpolated to the stellar energy region of the $^{12}$C($\alpha,\gamma$)$^{16}$O reaction.

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1. Introduction

The radiative $\alpha$ capture on carbon-12, $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$, is one of the fundamental reactions in nuclear-astrophysics, which determines the $C/O$ ratio synthesized in the stars [1]. The reaction rate of the process at the Gamow peak energy, $T_G = 0.3$ MeV, however, cannot be determined in experiment due to the Coulomb barrier. It is necessary to employ a theoretical model to extrapolate the reaction rate down to $T_G$ by fitting model parameters to experimental data typically measured at a few MeV. During a last half century, a lot of experimental and theoretical studies for the reaction have been carried out. See Refs. [2, 3, 4, 5] for review.

The elastic $\alpha^{-^{12}\text{C}}$ scattering at low energies is an important reaction to fix some parameters of a model for the study. Accurate measurements of the elastic scattering have been reported in Refs. [6, 7], and those data provide indispensable input for the parameter fittings. Elastic scattering data at low energies in general can be used for deducing an asymptotic normalization constant (ANC), which determines an overall strength of a nuclear reaction involving bound states [8, 9, 10].

The ANC of deuteron, for example, where the deuteron is a simple system consisting of loosely bound proton and neutron, leads to an overall factor of the reactions at low energies, such as radiative neutron capture on a proton at BBN energies [11, 12] and proton-proton fusion in the Sun [13, 14, 15, 16]. The ANC of deuteron is accurately determined by two effective range parameters: the deuteron binding momentum and effective range [17, 18], which are accurately fixed from the deuteron binding energy and elastic $NN$ scattering at low energies. On the other hand, to deduce ANCs for nuclear reactions relevant in nuclear-astrophysics is not so simple: the Coulomb interaction between heavier nuclei plays a negative role by preventing ones from obtaining elastic scattering data at very low energies, which makes the deduction of ANCs in terms of effective range expansion difficult [19, 20]. Recently, a new method of the parameterization for deducing the ANCs of nuclear reactions is suggested by Ramirez Suarez and Sparenberg [21], and new results of the ANCs by using the new method are reported in Refs. [22, 23].

Effective field theories (EFTs) provide us a model independent and systematic method for theoretical calculations. An EFT for a system in question can be built by introducing a scale which separates relevant degrees of freedom at low energies from irrelevant degrees of freedom at high energies. An effective Lagrangian is written down in terms of the relevant degrees of freedom and perturbatively expanded by counting the number of derivatives order by order. The irrelevant degrees of freedom are integrated out and their effect is embedded in coefficients appearing in the Lagrangian. Thus, a transition amplitude is systematically calculated by writing down Feynman diagrams, while the coefficients appearing in the Lagrangian are fixed by experiment. For review, one may refer to Refs. [26, 27, 28, 29]. For last two decades, various processes essential in nuclear-astrophysics have been investigated by constructing EFTs, which are $p(n, \gamma)d$ at BBN energies [11, 12] and $pp$ fusion [13, 14, 15, 16], $^3\text{He}(\alpha, \gamma)^7\text{Be}$ [30] and $^7\text{Be}(p, \gamma)^7\text{B}$ [31, 32] in the Sun.

In our previous work [24], we have constructed an EFT of the radiative capture reaction, $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$, obtained the counting rules for the reaction at $T_G$, and fitted some
parameters of the theory to the phase shift data of the elastic scattering. (We briefly review the counting rules for the radiative capture and elastic scattering reactions in the following sections.) In the parameter fitting to the phase shift data, we have introduced resonance energies of $^{16}$O as a large scale of the theory. As suggested by Teichmann [25], below the resonance energies, the Breit-Wigner-type parameterization for resonances can be expanded in powers of the energy, and one can obtain an expression of the amplitude in terms of the effective range expansion. We have determined three effective range parameters of the elastic scattering for $l = 0, 1, 2$ channels by fitting them to the phase shift data, but not included the excited bound states of $^{16}$O in the study. Though the phase shift data below the resonance energies can be reproduced very well by using the fitted parameters, we find that significant uncertainties in the elastic amplitudes are remained when extrapolating them to $T_G$.

In the present work, we incorporate the excited binding energies for $0_2^+, 1_1^-, 2_1^+, 3_1^-$ ($l_{\pi-i}^{th}$) states of $^{16}$O in the parameter fitting to the phase shift data of the elastic scattering for $l = 0, 1, 2, 3$ channels. Our assumption for the parameter fitting is that fitted curves which interpolate the amplitude between the phase shift data and the excited binding energies can be represented by several terms of a polynomial function. As will be discussed in detail below, however, we find a mismatch between the strength of the amplitudes estimated from the phase shift data and the first few terms of a polynomial function obtained from the Coulomb self-energy term in the dressed $^{16}$O propagator. Because those terms from the Coulomb self-energy are larger, at most by two order of magnitude, than the term estimated by the phase shift data, we introduce a new renormalization method; we assume that those large terms should be renormalized by counter terms, the role of which we assign to the effective range terms. Thus we include the effective range parameters up to third order ($n = 3$ in powers of $k^{2n}$) for the $l = 0, 1, 2$ channels and up to fourth order ($n = 4$) for the $l = 3$ channel. After fitting the parameters to the phase shift data, we calculate the ANC of the states of $^{16}$O and compare our results to the existing ones.

This paper is organized as the following: In Sec. 2, the approach based on an EFT for the radiative capture reaction is briefly reviewed, and the expression of equations related to the elastic scattering amplitudes, the phase shifts, and the effective range parameters are displayed. In Sec. 3, we introduce a new renormalization method and describe the details of the numerical fitting to the elastic scattering data. In Sec. 4, the numerical results obtained in this work are exhibited, and finally in Sec. 5, the results and discussion of the work are presented. In Appendix, the structure of the UV divergence and the counter terms of the elastic scattering amplitudes in the conventional renormalization method are summarized.

2. EFT for the radiative capture and elastic scattering at low energies

In the study of the radiative capture process, $^{12}$C($\alpha, \gamma$)$^{16}$O, at $T_G = 0.3$ MeV employing an EFT, at such a low energy, we regard the ground states of $\alpha$ and $^{12}$C as point-like particles whereas the first excited state energies of $\alpha$ and $^{12}$C are chosen as irrelevant degrees of freedom, by which a large scale of the theory is determined. The effective
Figure 1: Diagrams for dressed $^{16}$O propagator. A thick (thin) dashed line represents a propagator of $^{12}$C ($\alpha$), and a thick and thin double dashed line with and without a filled blob represent a dressed and bare $^{16}$O propagator, respectively. A shaded blob represents a set of diagrams consisting of all possible one-potential-photon-exchange diagrams up to infinite order and no potential-photon-exchange one.

Figure 2: Diagram of the scattering amplitude. See the caption of Fig. 1 as well.

Lagrangian for the process is constructed in terms of two spinless scalar fields for $\alpha$ and $^{12}$C, and the terms of the Lagrangian are expanded in terms of the number of derivatives. An expression of the effective Lagrangian has been obtained in Eq. (1) in Ref. [24]. The expansion parameter of the theory is $Q/\Lambda_H \sim 1/3$ where $Q$ denotes a typical momentum scale $Q \sim k_G$: $k_G$ is the Gamow peak momentum, $k_G = \sqrt{2\mu T_G} \simeq 41$ MeV, where $\mu$ is the reduced mass of $\alpha$ and $^{12}$C. $\Lambda_H$ denotes a large momentum scale $\Lambda_H \sim \sqrt{2\mu_4 T_{(4)}}$ or $\sqrt{2\mu_{12} T_{(12)}} \sim 150$ MeV where $\mu_4$ is the reduced mass of one and three-nucleon system and $\mu_{12}$ is that of four and eight-nucleon system. $T_{(4)}$ and $T_{(12)}$ are the first excited energies of $\alpha$ and $^{12}$C, respectively. By including the terms up to next-to-next-to-leading order, therefore, one may obtain about 10% theoretical uncertainty for the process.

The amplitudes of the elastic scattering are calculated from diagrams depicted in Figs. 1 and 2. In our previous works, we have obtained the scattering amplitudes for $l$-th partial wave states as [24, 33, 34]

$$A_l = \frac{2\pi (2l+1)P_l(\cos \theta)e^{2i\sigma_l} W_l(\eta) C_\eta^2}{\mu K_l(k) - 2\kappa H_l(k)},$$  \hspace{1cm} (1)

where $k$ is the magnitude of relative momentum between $\alpha$ and $^{12}$C and $\theta$ is the scattering angle in the C.M. frame. In addition, $\eta$ is the Sommerfeld parameter, $\eta = \kappa/k$, where $\kappa$ is the inverse of the Bohr radius, $\kappa = Z_2 Z_6 \mu \alpha$, and

$$C_\eta^2 = \frac{2\pi \eta}{e^{2\pi \eta} - 1}, \quad W_l(\eta) = \frac{\kappa^{2l}}{(l)!^2} \prod_{n=0}^{l} \left(1 + \frac{n^2}{\eta^2}\right), \quad H_l(k) = W_l(\eta) H(\eta),$$  \hspace{1cm} (2)
with

\[ H(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta). \]  

(3)

\( \psi(z) \) is the digamma function, \( P_l(x) \) are the Legendre polynomials, and \( \sigma_l \) are the Coulomb phase shifts. When \( \eta \) goes to zero, the factor \( C^2_\eta \) is normalized to one whereas, when \( \eta \) becomes large, the Gamow factor, \( P = \exp(-2\pi\eta) \), appears from the factor \( C^2_\eta \propto P \). We note that the function, \( e^{2i\eta}W_\eta(\eta)C^2_\eta \), in the numerator is from the initial and final state Coulomb interactions between \( \alpha \) and \( ^{12}\text{C} \) in Fig. 2.

The function \( K_l(k) \) represents the interaction due to the short range nuclear force (compared with the long range Coulomb force), which is obtained in terms of the effective range parameters as

\[ K_l(k) = -\frac{1}{a_l} + \frac{1}{2} \gamma^2_l k^2 - \frac{1}{4} P_l k^4 + Q_l k^6 - R_l k^8 + \cdots. \]  

(4)

Because the UV divergence comes out of the loop integrals in the Coulomb self-energy terms (from the diagrams in Fig. 1), we need to introduce counter terms for renormalization. We employ the dimensional regularization and the structure of the UV divergence from the Coulomb self-energy terms is given in Appendix. Thus, we need to include one counter term, \(-1/a_0\) term in the effective range expansion for \( l = 0 \), two counter terms, \(-1/a_1\) and \( r_1 \) for \( l = 1 \), three counter terms, \(-1/a_2, r_2, \) and \( P_2 \) for \( l = 2 \), and four counter terms, \(-1/a_3, r_3, P_3, \) and \( Q_3 \) for \( l = 3 \), to remove the UV divergence and make the terms finite. The other higher order terms in the effective range expansion are introduced as finite terms in the conventional renormalization method and supposed to obey counting rules in which higher order terms are less important than lower order terms. We find that the expression obtained in Eq. (1) reproduces well the previous results reported in Refs. [19, 37, 38, 39].

At the binding energies of excited states of \(^{16}\text{O} \), the amplitudes should have a pole at \( k_b \) where \( \gamma_l \) are the binding momenta \(^3\), \( \gamma_l = \sqrt{2\mu B_l} \); \( B_l \) denote the binding energies of excited states of \(^{16}\text{O} \). Thus the denominator of the scattering amplitude, \( D_l(k) \), should vanish at \( k_b \):

\[ D_l(k_b) = K_l(k_b) - 2\kappa H_l(k_b) = 0. \]  

(5)

Using this condition, the first effective range parameter, \( a_l \), is related to other effective range parameters as

\[ -\frac{1}{a_l} = \frac{1}{2} \gamma^2_l + \frac{1}{4} P_l \gamma^4_l + Q_l \gamma^6_l + R_l \gamma^8_l + \cdots + 2\kappa H_l(k_b), \]  

(6)

\(^2\)In this work, we employ a modified representation for effective range parameters from that presented in Ref. [35]. Here we use the effective volume-like parameter \( P_l \) rather than the shape parameter \( P_l \) represented as \(-r^2 P_l k^4\). In addition, we introduced an opposite sign for the \( R_l \) term so as to have positive sign in the bounding energy in Eq. (6). We had employed another parameterization (\( v \) parameterization) for the effective range parameters in Ref. [36].

\(^3\)The quantity \( \gamma_l \) is also referred to the bound-state wave number in the low energy scattering theory.
and we remove the \( a_t \) dependence from the amplitude. Thus, we have \( D_t(k) \) as

\[
D_t(k) = \frac{1}{2} \gamma_t \left( k^2 + \gamma_t^2 \right) - \frac{1}{4} P_t \left( k^4 - \gamma_t^4 \right) + Q_t \left( k^6 + \gamma_t^6 \right) - R_t \left( k^8 - \gamma_t^8 \right) + \cdots \\
-2 \kappa \left[ H_t(k) - H_t(k_b) \right].
\]  

(7)

The remaining effective range parameters are fixed by using the phase shift data of the elastic scattering.

The differential cross section of the elastic scattering is represented in terms of the pure Coulomb scattering part and the Coulomb modified nuclear scattering part, as presented in Eq. (3) in Ref. [24], where the scattering function \( U \) is \( U = \exp[2i(\delta_t + \omega_t)] \) where \( \delta_t \) are the phase shifts for \( l \)-th partial waves and \( \omega_t = \sigma_t - \sigma_0 \) with \( \sigma_t = \arg \Gamma(1 + l + i\eta) \). Thus the scattering amplitudes are represented in terms of \( \delta_t \) as [40]

\[
A_t = \frac{2\pi (2l + 1) P_t(\cos \theta)e^{2i\sigma_t}}{\mu k \cot \delta_t - ik}.
\]  

(8)

By comparing two expressions of the amplitudes \( A_t \) in Eqs. (1) and (8), one has a relation between the phase shift and the effective range parameters in \( D_t(k) \) as

\[
W_t(\eta)C^2_\eta k \cot \delta_t = \text{Re}D_t(k).
\]  

(9)

To estimate the ANC, \( |C_b| \), for the \( 0^+_2, 1^-_1, 2^+_1, 3^-_1 \) states of \( ^{16}\text{O} \), we employ the definition of \( |C_b| \) from Eq. (14) in Ref. [39]:

\[
|C_b| = \gamma_t \Gamma(l + 1 + |\eta_b|) \left( \frac{dD_t(k)}{dk^2} \bigg|_{k^2 = -\gamma_t^2} \right)^{-\frac{1}{2}} \text{ (fm}^{-1/2}\text{)},
\]  

(10)

where \( \eta_b = \kappa/k_b \).

3. **Fitting the parameters to phase shifts data**

Four excited states of \( ^{16}\text{O} \) exist below the \( \alpha-^{12}\text{C} \) threshold, which we include in the parameter fitting in the present study. The binding energies, \( B_i(l^\pi) \), of the \( i \)-th excited bound states of \( ^{16}\text{O} \) in \( l^\pi \) states from the \( \alpha-^{12}\text{C} \) threshold energy are \( B_1(0^+) = 1.113 \), \( B_2(3^-) = 1.032 \), \( B_3(2^+) = 0.245 \), \( B_4(1^-) = 0.045 \) MeV. Thus, the binding momenta, \( \gamma_t = \sqrt{2\mu B_i(l^\pi)} \), are

\[
\gamma_t = 79.843, \ 15.860, \ 37.007, \ 75.954 \ \text{(MeV)},
\]  

(11)

for the \( 0^+_2, 1^-_1, 2^+_1, 3^-_1 \) states, respectively, where \( \mu = m_\alpha m_C/(m_\alpha + m_C) = 2795.079 \) MeV with \( m_\alpha = 3727.379 \) MeV and \( m_C = 11174.862 \) MeV. As mentioned above, the first effective range term, \( a_t \), is constrained by using the binding momenta.

To fix the other effective range parameters, the phase shift data for each \( l \)-th partial wave state are used. In the present work, we employ the phase shift data from the Tischhauser et al.’s paper [7]. The reported energies of the \( \alpha \) particle in the lab. frame are
\( T_\alpha = 2.6-6.6 \text{ MeV, and corresponding momenta in the } C.M. \text{ frame are } k = 105-166 \text{ MeV (i.e., } k = \sqrt{1.5\mu T_\alpha}). \) Because our large momentum scale of the theory is \( \Lambda_H \sim 150 \text{ MeV,} \) the convergence of the expansion series should be carefully examined when the elastic scattering data are used for the parameter fitting. In addition, because we do not explicitly include the resonance states of \(^{16}\text{O}\) in the theory, we restrict data sets for the parameter fitting below the resonance energies, \( T_\alpha = 6.52, 3.23, 3.57, 5.09 \text{ MeV for } 0^+, 1^+, 2^+, 3^+ \) states, respectively; the corresponding momenta are \( k = 166, 117, 123, 146 \text{ MeV for the } l = 0, 1, 2, 3, \text{ channels, respectively. (We will mention data sets we choose for the parameter fitting in detail below.)} \)

We are now in position to discuss a new renormalization method. The effective range parameters in \( K_i(k) \) are expanded in powers of \( k^2 \) whereas the real part of the function \( H_i(k) \) can be expanded in powers of \( k^2 \) as well. For the function \( H(\eta) \) in \( H_i(k) \), one has

\[
ReH(\eta) = \frac{1}{12\kappa^2}k^2 + \frac{1}{120\kappa^4}k^4 + \frac{1}{252\kappa^6}k^6 + \frac{1}{240\kappa^8}k^8 + \cdots, \tag{12}
\]

where \( \eta = \kappa/k; \kappa \) is the inverse of the Bohr radius, \( \kappa \approx 245 \text{ MeV,} \) and is regarded as another large scale of the theory. This expansion is reliable in our study for the elastic scattering at low energies along with the effective range expansion in \( K_i(k) \). Thus, the right-hand-side of equation, \( ReD_i(k) \), in Eq. (9) can be expanded as a power series of \( k^2 \) for both \( K_i(k) \) and \( 2\kappa ReH_i(k) \). Meanwhile, the left-hand-side of Eq. (9) is suppressed by the factor \( C_\eta^2 \), due to the Gamow factor \( P = \exp(-2\pi\eta) \).

In the case of the s-wave, for example, the reported phase shift at the smallest energy, \( T_\alpha = 2.6 \text{ MeV,} \) is \( \delta_0 = -1.893^\circ \) [7]. The factor \( C_\eta^2 \) becomes \( C_\eta^2 \approx 6 \times 10^{-6} \) at \( k = 104 \text{ MeV} \) which corresponds to \( T_\alpha = 2.6 \text{ MeV,} \) and the left-hand-side of Eq. (9) numerically becomes \( C_\eta^2 k \cot \delta_0 = -0.019 \text{ MeV.} \) The function \( 2\kappa ReH_0(k) \) is expanded as

\[
2\kappa ReH_0(k) = \frac{1}{6\kappa}k^2 + \frac{1}{60\kappa^3}k^4 + \frac{1}{126\kappa^5}k^6 + \frac{1}{120\kappa^7}k^8 + \cdots
\]

\[
= 7.441 + 0.136 + 0.012 + 0.002 + \cdots \text{ (MeV),} \tag{13}
\]

at \( k = 104 \text{ MeV.} \) The numerical values in the second line of Eq. (13) correspond to the terms appearing in the first line of the equation in order. One can see that the power series converges well, but the first and second terms are two and one order of magnitude larger compared with the value estimated by using the experimental data in the left-hand-side of Eq. (9), \( -0.019 \text{ MeV.} \) Thus, we regard those terms unnaturally large, and it is necessary to introduce a new renormalization method, in which the counter terms remove the unnaturally large terms and make the terms in a natural size. In other words, we assume that fitting polynomial functions are represented as a natural power series at the low energy region, and to maintain such polynomial functions, large cancellations for the first and second terms with the \( r_l \) and \( P_l \) effective terms, respectively, are expected. So we include the three effective range parameters, \( r_l, P_l, \) and \( Q_l, \) for the \( l = 0 \) channel, as the counter terms. The same tendency can be seen in the \( l = 1, 2 \) channels whereas one needs four effective range parameters for the \( l = 3 \) channel. Thus, we employ the three effective range parameters, \( r_l, P_l, Q_l, \) for the \( l = 0, 1, 2 \) channels and the four effective...
range parameters, \( r_1, P_1, Q_1, R_1 \) for the \( l = 3 \) channel when fitting the parameters to the phase shift data below.\(^4\)

4. Numerical results

As discussed above, we fit the three effective range parameters, \( r_1, P_1, Q_1, R_1 \) to the phase shift data for the \( l = 0, 1, 2 \) channels and the four effective range parameters, \( r_1, P_1, Q_1, R_1 \) to those for the \( l = 3 \) channel, while \( a_l \) are constrained by using the relation in Eq. (6) with the binding momenta \( \gamma_l \). To examine the sensitivity to the choice of data sets, we employ three sets of the phase shift data \([7]\) below the resonance energy for each partial wave, which have different energy ranges: three data sets for \( l = 0 \) denoted by \( S_0, S_1, S_2 \) have the data at \( T_o = 2.6-3.6, 2.6-3.8, 2.6-4.0 \) MeV, respectively, those for \( l = 1(2) \) denoted by \( P_0, P_1, P_2, (D_0, D_1, D_2) \) have the data at \( T_o = 2.6-3.0, 2.6-3.1, 2.6-3.2 \) MeV, respectively, and those for \( l = 3 \) denoted by \( F_0, F_1, F_2 \) have the data at \( T_o = 2.6-4.6, 2.6-4.8, 2.6-5.0 \) MeV, respectively.

When the parameters are fitted to the data, large cancellations between the terms in powers of \( k^2 \) appearing from the \( K_i(k) \) and \( 2\kappa H_i(k) \) functions, the \( r_1, P_1, Q_1, R_1 \) effective range terms and from the \( 2\kappa H_i(k) \) function, are expected. We denote the terms from the \( 2\kappa H_2(k) \) function corresponding to the effective range terms as \( \tilde{r}_1, \tilde{P}_1, \tilde{Q}_1, \tilde{R}_1 \), and we have

\[
\tilde{r}_0 = \frac{1}{3\kappa}, \quad \tilde{P}_0 = -\frac{1}{15\kappa^3}, \quad \tilde{Q}_0 = \frac{1}{126\kappa^5}, \quad (14)
\]

\[
\tilde{r}_1 = \frac{1}{3\kappa}, \quad \tilde{P}_1 = -\frac{11}{15\kappa}, \quad \tilde{Q}_1 = \frac{31}{1260\kappa^3}, \quad (15)
\]

\[
\tilde{r}_2 = \frac{1}{12\kappa^3}, \quad \tilde{P}_2 = -\frac{51}{60\kappa}, \quad \tilde{Q}_2 = \frac{191}{1008\kappa}, \quad (16)
\]

\[
\tilde{r}_3 = \frac{1}{108\kappa^5}, \quad \tilde{P}_3 = -\frac{47}{180\kappa^3}, \quad \tilde{Q}_3 = \frac{5297}{22680\kappa}, \quad \tilde{R}_3 = -\frac{17101}{90720\kappa}. \quad (17)
\]

Those values (and variations from them for some terms) are used as the initial input of the effective range parameters for the parameter fitting.\(^5\)

In Tables 1, 2, 3, 4, fitted values and errors of the effective range parameters, \( r_1, P_1, Q_1, R_1 \) for \( l = 0, 1, 2, 3 \) channels to the data sets \( \{S_0, S_1, S_2\}, \{P_0, P_1, P_2\}, \{D_0, D_1, D_2\}, \{F_0, F_1, F_2\} \), respectively, are presented. The errors of the fitted parameters stem from those of the phase shift data. Numerical values of the \( \tilde{r}_1, \tilde{P}_1, \tilde{Q}_1, \tilde{R}_1 \) terms for \( l = 0, 1, 2, 3 \) are also shown in the tables. The values of \( a_l \) are calculated by using the fitted effective range parameters in Eq. (6). The values in the second last column are the real part of the denominator \( ReD_i(k) \) of the scattering amplitude at the energy corresponding to \( T_G \)\(^6\) (i.e., at \( k = k_G \)), and those in the last column are the ANC, \( |C_0| \), for the \( 0^+_2, 1^-_1, 2^+_1, 3^-_1 \) states.

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\(^4\)In the new parameterization of elastic scattering suggested by Ramirez Suarez and Sparenberg, the \( K_i(k) \) and \( 2\kappa H_i(k) \) functions are merged, and a new function for the parameterization is defined as \( \Delta_i(E) = C^2_0\kappa \cot \delta_i \), which is parameterized by using the Pade approximation [21].

\(^5\)We employ a SciPy module, curve_fit, in optimization package when fitting the effective range parameters to the phase shift data.

\(^6\)The \( \alpha \) energy in the lab. frame corresponding to \( T_G \) is \( T_\alpha = \frac{4}{3}T_G \simeq 0.4 \) MeV.
Table 1: Effective range parameters, \( r_0, P_0, Q_0 \), fitted by using the data sets, \( S0, S1, S2 \); values of \( \tilde{r}_0, \tilde{P}_0, \tilde{Q}_0 \) are included in last raw. The values of \( a_0, ReD_{0G} \), and \( |C_b| \) for the \( 0^+_2 \) state are calculated by using \( r_0, P_0, Q_0 \). For details, see the text.

| \( a_0 \) (fm) | \( r_0 \) (fm) | \( P_0 \) (fm\(^3\)) | \( Q_0 \) (fm\(^3\)) | \( ReD_{0G} \) (MeV) | \( |C_b| \) (fm\(^{-1/2}\)) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( S0 \)        | \( 6.2 \times 10^4 \) | 0.268514(3)    | -0.0343(4)      | 0.0019(2)       | 4.2(7) \times 10^{-3} | 6.8(16) \times 10^{2} |
| \( S1 \)        | \( 6.6 \times 10^4 \) | 0.268514(3)    | -0.0342(3)      | 0.0020(3)       | 4.0(5) \times 10^{-3} | 7.4(15) \times 10^{2} |
| \( S2 \)        | \( 5.8 \times 10^4 \) | 0.268513(3)    | -0.0345(2)      | 0.0018(1)       | 4.4(4) \times 10^{-3} | 6.4(7) \times 10^{2} |

Table 2: Effective range parameters, \( r_1, P_1, Q_1 \), fitted by using the data sets, \( P0, P1, P2 \); values of \( \tilde{r}_1, \tilde{P}_1, \tilde{Q}_1 \) are included in last raw. The values of \( a_1, ReD_{1G} \), and \( |C_b| \) for the \( 1^-_1 \) state are calculated by using \( r_1, P_1, Q_1 \). For details, see the text.

| \( a_1 \) (fm\(^3\)) | \( r_1 \) (fm\(^{-1}\)) | \( P_1 \) (fm) | \( Q_1 \) (fm\(^3\)) | \( ReD_{1G} \) (MeV\(^3\)) | \( |C_b| \) (fm\(^{-1/2}\)) |
|-----------------------|-----------------------|-----------------|-----------------|-----------------|-----------------|
| \( P0 \)              | -1.8 \times 10^5      | 0.4150(6)       | -0.577(8)       | 0.019(3)        | 2.7(8) \times 10^{2} | 1.9(4) \times 10^{14} |
| \( P1 \)              | -1.6 \times 10^5      | 0.4153(2)       | -0.574(2)       | 0.020(1)        | 3.0(3) \times 10^{2} | 1.8(1) \times 10^{14} |
| \( P2 \)              | -1.3 \times 10^5      | 0.4157(2)       | -0.569(2)       | 0.023(1)        | 3.5(3) \times 10^{2} | 1.6(1) \times 10^{14} |

Table 3: Effective range parameters, \( r_2, P_2, Q_2 \), fitted by using the data sets, \( D0, D1, D2 \); values of \( \tilde{r}_2, \tilde{P}_2, \tilde{Q}_2 \) are included in the last raw. The values of \( a_2, ReD_{2G} \), and \( |C_b| \) for the \( 2^+_1 \) state are calculated by using \( r_2, P_2, Q_2 \). For details, see the text.

| \( a_2 \) (fm\(^3\)) | \( r_2 \) (fm\(^{-3}\)) | \( P_2 \) (fm\(^{-1}\)) | \( Q_2 \) (fm) | \( ReD_{2G} \) (fm\(^{-2}\)) | \( |C_b| \) (fm\(^{-1/2}\)) |
|-----------------------|-----------------------|-----------------|-----------------|-----------------|-----------------|
| \( D0 \)              | 10.3 \times 10^3      | 0.155(4)        | -1.12(7)        | 0.11(3)         | -1.66(156) \times 10^{-4} | 2.4(3) \times 10^{4} |
| \( D1 \)              | 6.5 \times 10^3       | 0.152(2)        | -1.16(4)        | 0.08(2)         | -2.6(9) \times 10^{-4} | 2.3(2) \times 10^{4} |
| \( D2 \)              | 4.3 \times 10^3       | 0.149(2)        | -1.21(3)        | 0.06(1)         | -3.8(6) \times 10^{-4} | 2.1(1) \times 10^{4} |

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are related to the estimate of the but the fitted effective range parameters are largely canceled with the corresponding \( \tilde{P}_l\) and \( \tilde{Q}_l\).

while those for the \( 0^-\) with theoretical estimates, (2) evaluated earlier, which basically agree with the experimental values mentioned above, respectively, and in underestimates for the other theoretical estimates, (14).

In the values of the real part of denominator, \( l\), 3 states, which are also important to estimate the cascade transitions, are recently studied in experiment and reported first time.

For the ANC, \( |C_b|\), for the \( 1^-\) state, we find that our result, \( |C_b| = (1.6-1.9) \times 10^{14} \) (fm\(^{-1/2}\)), is in good agreement with experimental values, \( (2.10\pm0.14)\times10^{14}, (2.00\pm0.35)\times10^{14}, (2.08\pm0.20)\times10^{14} \), obtained by Avila et al. [41], Oulebsir et al. [42], Brune et al. [43], respectively, and underestimates for other experimental ones, \( (5.1\pm0.6) \times 10^{14} \) [44] and \( (17.4-26.4) \times 10^{14} \) [45]. We also find good agreement with theoretical estimates, \( (2.22-2.24)\times10^{14}, \) obtained from a potential model calculation by Katsuma [46], and \( 2.14(6) \times 10^{14} \) and \( 2.073 \times 10^{14} \) from the new method of the parameterization by Ramirez Suarez and Sparenberg [21] and by Orlov et al. [23], respectively.

For the ANC, \( |C_b|\), for the \( 2^+\) state, our result, \( |C_b| = (2.1-2.4) \times 10^{4} \) (fm\(^{-1/2}\)), is in underestimates to experimental values, \( (12.2 \pm 0.7) \times 10^{4} \) [41], \( (14.4 \pm 2.8) \times 10^{4} \) [42], \( (11 \pm 1) \times 10^{4} \) [43], \( (34.5 \pm 0.5) \times 10^{4} \) [44], \( (12.2-18.2) \times 10^{4} \) [45]. Other experimental estimates evaluated earlier, which basically agree with the experimental values mentioned above, can be found in Table VI in Ref. [47]. On the other hand, our result is in good agreement with theoretical estimates, \( (2.41 \pm 0.38) \times 10^{4} \) and \( 2.106 \times 10^{4} \), from the effective range analysis up to the \( r_2\) term by Konig et al. [19] and up to the \( P_2\) term by Orlov et al. [20], respectively, and in underestimates for the other theoretical estimates, \( (14.45 \pm 0.85) \times 10^{4} \) from the supersymmetric potential model by Sparenberg [48] and \( (12.6 \pm 0.5) \times 10^{4} \) from the \( R\) matrix analysis with a microscopic cluster model by Dufour and Descouvemont [47] and \( 5.050 \times 10^{4} \) from the new method of the parameterization by Orlov et al. [23].

For the ANCs, \( |C_b|\), for the \( 0^-\) and \( 3^-\) states, our result, \( |C_b| = (6.4-7.4) \times 10^{2} \) (fm\(^{-1/2}\))

| \( a_3(\text{fm}^7) \) | \( r_3(\text{fm}^{-5}) \) | \( P_3(\text{fm}^{-3}) \) | \( Q_3(\text{fm}^{-1}) \) | \( R_3(\text{fm}) \) | \( ReD_{3G} \) (fm\(^{-7}\)) | \( |C_b|\) (fm\(^{-1/2}\)) |
|---|---|---|---|---|---|---|
| \( F0 \) | \(-1.4 \times 10^4\) | \(0.0319(1)\) | \(-0.453(11)\) | \(0.317(9)\) | \(-0.141(8)\) | \(7.8(8) \times 10^{-4}\) | \(1.2(1) \times 10^2\) |
| \( F1 \) | \(-1.5 \times 10^3\) | \(0.0320(1)\) | \(-0.459(9)\) | \(0.311(7)\) | \(-0.146(6)\) | \(7.4(7) \times 10^{-4}\) | \(1.3(1) \times 10^2\) |
| \( F2 \) | \(-1.8 \times 10^3\) | \(0.0322(1)\) | \(-0.472(7)\) | \(0.301(6)\) | \(-0.156(5)\) | \(6.4(6) \times 10^{-4}\) | \(1.5(1) \times 10^2\) |

Table 4: Effective range parameters, \( r_3, P_3, Q_3, R_3\), fitted by using the data sets, \( F0, F1, F2\); values of \( \tilde{r}_3, \tilde{P}_3, \tilde{Q}_3, \tilde{R}_3\) are included in the last row. The values of \( a_3, ReD_{3G}\), and \(|C_b|\) for the \( 3^-\) state are calculated by using \( r_3, P_3, Q_3, R_3\). For details, see the text.
for the $0^+_2$ state is in underestimate to an experimental value, $(15.6 \pm 1.0) \times 10^2$ [41] and in overestimate to a theoretical value, $4.057 \times 10^2$ [23]. Meanwhile, our result, $|C_4| = (1.2-1.5) \times 10^2$ (fm$^{-1/2}$) for the $3^-_1$ state is in very good agreement to the experimental value, $(1.39 \pm 0.09) \times 10^2$, recently reported by Avila et al. [41].

To examine the convergence of the power series in terms of $k^2$ at $k = k_G$, we add the effective range terms and those from the $2\kappa \text{Re} H_l(k)$ functions together. In Table 5, we show the ratios of the terms after normalizing those terms by $-1/a_l$ for $l = 0, 2, 3$ and by $\frac{1}{2}(r_1 - \tilde{r}_1)k_G^2$, for $l = 1$ because of their dominance where the effective range parameters fitted by using the data sets, $S2$, $P2$, $D2$, $F2$, are used. As discussed above, the expansion parameter at $T_G$ is $Q \sim 1/3$, so the $k_G^2$, $k_G^4$, $k_G^6$ terms are expected to be a few tenth, a few hundredth, a few thousandth to the leading order terms, respectively. We find good convergence of the power series for $l = 0, 3$ at $k = k_G$. On the other hand, the $1/a_1$ term is small compared to the $\frac{1}{2}(r_1 - \tilde{r}_1)k_G^2$ term for $l = 1$ and the $1/a_2$ and $\frac{1}{2}(r_2 - \tilde{r}_2)k_G^2$ terms are comparable for $l = 2$, but the higher order terms are well converged for $l = 1, 2$ as expected by the counting rules of the theory.

Table 5: Ratios of the terms in the power series to $-1/a_l$ for $l = 0, 2, 3$ and to $\frac{1}{2}(r_1 - \tilde{r}_1)k_G^2$ for $l = 1$ at $k = k_G$ where the effective range parameters fitted by using the data sets, $S2$, $P2$, $D2$, $F2$, have been used.

| $l$ | $\frac{1}{a_l}$ | $\frac{1}{2}(r_1 - \tilde{r}_1)k_G^2$ | $- \frac{1}{2}(P_l - \tilde{P}_l)k_G^2$ | $\frac{1}{2}(Q_l - \tilde{Q}_l)k_G^2$ |
|-----|---------------|-----------------|-----------------|-----------------|
| $S2$ | 1             | 0.276           | 0.012           | 0.004           |
| $P2$ | 0.154         | 1               | 0.215           | 0.016           |
| $D2$ | 1             | 0.946           | 0.316           | 0.031           |
| $F2$ | 1             | 0.195           | 0.023           | 0.002           |

In Figs. 3, 4, 5, 6, the curves of phase shift $\delta_l$ (left panels) and the real part of denominator, $\text{Re} D_l(k)$, of the scattering amplitude (right panels) for $l = 0, 1, 2, 3$, respectively, are plotted as functions of $T_\alpha$ by using the values of $r_l$, $P_l$, $Q_l$, $R_l$, which are fitted by using the data sets denoted by $\{S0, S1, S2\}$, $\{P0, P1, P2\}$, $\{D0, D1, D2\}$, $\{F0, F1, F2\}$. Experimental data of the phase shift are also included in the figures. In addition, a filled box in the right panel represents the binding energy of the excited $0^+_2$, $1^-_1$, $2^+_1$, or $3^-_1$ state in each of the figures.

We find that the curves of $\delta_l$ plotted by using the different sets of the fitted parameters are in good agreement with each other and reproduce the experimental data within the errors, except for the large energy region, $T_\alpha = 3.0-3.2$ MeV for $l = 1$. One can see the significant separations of the curves of $\text{Re} D_l(k)$ at the interpolated energy region where the experimental data do not exist for the $l = 1, 2, 3$ channels, but the values of $\text{Re} D_G$ at $k = k_G$ (i.e., $T_\alpha = 0.4$ MeV) for the different sets of the parameters are still in good agreement within the errors, as we have seen in the tables.

5. Results and discussion
Figure 3: Phase shift, $\delta_0$, (left panel) and the real part of denominator, $ReD_0(k)$, of the amplitude (right panel) as functions of $T_\alpha$ (where $k = \sqrt{1.5\mu T_\alpha}$). Curves are plotted by using the effective range parameters, fitted from the $S_0$, $S1$, $S2$ data sets, presented in Table 1. Exp. phase shift data are also included in the figure. A filled box in the right panel represents the excited binding energy of the $0^+_2$ state.

Figure 4: Phase shift, $\delta_1$, (left panel) and the real part of denominator, $ReD_1(k)$, of the amplitude (right panel) as functions of $T_\alpha$. Curves are plotted by using the effective range parameters, fitted from the $P0$, $P1$, $P2$ data sets, presented in Table 2. Exp. phase shift data are also included in the figure. A filled box in the right panel represents the binding energy of the $1^+_1$ state.
Figure 5: Phase shift, $\delta_2$, (left panel) and the real part of denominator, $ReD_2(k)$, of the amplitude (right panel) as functions of $T_\alpha$. Curves are plotted by using the effective range parameters, fitted from the $D0$, $D1$, $D2$ data sets, presented in Table 3. Exp. phase shift data are also included in the figure. A filled box in the right panel represents the binding energy of the $2_{1}^{+}$ state.

Figure 6: Phase shift, $\delta_3$, (left panel) and the real part of denominator, $ReD_3(k)$, of the amplitude (right panel) as functions of $T_\alpha$. Curves are plotted by using the effective range parameters, fitted from the $F0$, $F1$, $F2$ data sets, presented in Table 4. Exp. phase shift data are also included in the figure. A filled box in the right panel represents the binding energy of the $3_{1}^{-}$ state.
In the present work, we have fitted the effective range parameters to the phase shift data of the elastic scattering for \( l = 0, 1, 2, 3 \) below the resonance energies of \(^{16}\text{O}\) in EFT. The excited binding energies of the \( 0^+_2, 1^-_1, 2^+_1, 3^-_1 \) states of \(^{16}\text{O}\) are also included in the parameter fitting. Because of a mismatch between the terms from the \( 2\kappa H_l(k) \) functions and a term obtained from the phase shift data, we have introduced a new renormalization method: we assign the effective range terms as a role of the counter terms so as to obtain a natural power series for the fitting polynomial functions at the low energy region. Thus, we have fitted three effective range parameters, \( r_l, P_l, Q_l \), for \( l = 0 \), and four effective range parameters, \( r_l, P_l, Q_l, R_l \), for \( l = 3 \) to the phase shift data. (Those fitted values of the effective range parameters are used when we study the radiative capture reaction of \( \alpha \) and \(^{12}\text{C}\) in EFT in the future.) After fitting the effective range parameters, we have calculated the real part of the denominator, \( \text{Re}D_lG \), of the scattering amplitude at the energy corresponding to \( T_G \) and the ANCs for the \( 0^+_2, 1^-_1, 2^+_1, 3^-_1 \) states. In addition, we have interpolated and plotted the real part of the denominator of the scattering amplitude between the binding energy and the phase shift data.

In fitting the effective range parameters for the all partial waves, we find that the errors of the fitted effective range parameters are tiny whereas the effective range terms are almost exactly canceled with the terms from the \( 2\kappa H_l(k) \) function. Thus, we obtain 9-94% errors in \( \text{Re}D_lG \) depending on the choice of the input data sets and partial waves, while the power series in terms of \( k^2_G \) in \( \text{Re}D_l(k) \) well converges at the energy corresponding to \( T_G \), as expected by the counting rules of the theory. In the figures, though the curves of the phase shifts plotted by using the different sets of the parameters are in good agreement, those of \( \text{Re}D_l(k) \) are significantly different in the interpolated region between the binding energies and the phase shift data where no experimental data are available. Nonetheless, \( \text{Re}D_lG \) from the different sets of the parameters are still in good agreement within the error bars.

For the ANC, \( |C_b| \), for the \( 1^-_1 \) state, we find our result is in good agreement with the other theoretical estimates and the recent experimental estimates. Thus, the estimates of \( |C_b| \) for the \( 1^-_1 \) state converge both theoretically and experimentally. For the ANC, \( |C_b| \), for the \( 2^+_1 \) state, our result is in good agreement with the theoretical estimates based on the effective range expansion but in underestimates, more than by the factor of 5, compared with those of the other theories and the experiments. As seen in Eq. (10), such a large \( |C_b| \) can be obtained by a very small slope of \( \text{Re}D_2(k) \) at the binding energy of the \( 2^+_1 \) state. That indicates a very large scattering length and a drastic cancelation between the \( r_2 \) term and the \( \tilde{r}_2 \) term. Meanwhile, as seen in the right panel of Fig. 5, the phase shift data are quite distant from the bound state energy, while the higher order terms involve in the fitting. Thus, it is hard to discriminate which curve is better than the others in the present approach. To have accurate experimental data of the phase shift down to, e.g., \( T_\alpha = 1 \) or 1.5 MeV could improve the situation. For the ANCs, \( |C_b| \), for the \( 0^+_2 \) and \( 3^-_1 \) states, the first experimental result is recently reported by Avila et al. [41]. We find that our result for the \( 0^+_2 \) state is about a half compared to the experimental estimate, while our result for the \( 3^-_1 \) state is in good agreement to the experimental value. It may be necessary to wait for a further confirmation experimentally and theoretically.
for the $0^+_2$ and $3_1^-$ states.

In the present work, we have introduced a new renormalization method from an observation of a mismatch between the terms from the Coulomb self-energy term, the $2\kappa H_l(k)$ function, and the term obtained from the phase shift data, by assuming to have a natural power series of the fitting polynomial functions at the low energies. Our conjecture about the observation is, on one hand, that it may be caused simply due to the severe suppression factor, the Gamow factor, at the low energies. On the other hand, it may stem from our assumption that the $\alpha$ and $^{12}\text{C}$ states are point-like. That implies that the interaction length scale between the $\alpha$ and $^{12}\text{C}$ vanishes, and thus the short range effect should be renormalized by introducing the counter terms. A more systematic study about the issue, indeed, would be necessary in the future.

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Appendix

In this appendix, we display the UV divergent terms from the Coulomb self-energy in terms of the $J$ functions and discuss the counter terms in the conventional renormalization method. The Coulomb self-energy terms for $l = 0,1,2,3$ are calculated from the $J$ functions defined below:

\[
J^{l=0}_0(p) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{d^3\vec{q}'}{(2\pi)^3} \langle \vec{q}' | \hat{G}^{(+)}_C | \vec{q} \rangle ,
\]

\[
J^{l=1}_{2,i,x}(p) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{d^3\vec{q}'}{(2\pi)^3} q_i' \langle \vec{q}' | \hat{G}^{(+)}_C | \vec{q} \rangle q_x ,
\]

\[
J^{l=2}_{4;i,j,x,y}(p) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{d^3\vec{q}'}{(2\pi)^3} \left( q_i' q_j' - \frac{1}{3} \delta_{ij} q'^2 \right) \langle \vec{q}' | \hat{G}^{(+)}_C | \vec{q} \rangle \left( q_x q_y - \frac{1}{3} \delta_{xy} q'^2 \right) ,
\]

\[
J^{l=3}_{6,i,j,k,x,y,z}(p) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{d^3\vec{q}'}{(2\pi)^3} \left[ q_i' q_j' q_k' - \frac{1}{5} \left( \delta_{ij} q_k' + \delta_{ik} q_j' + \delta_{jk} q_i' \right) q'^2 \right] \langle \vec{q}' | \hat{G}^{(+)}_C | \vec{q} \rangle
\times \left[ q_x q_y q_z - \frac{1}{5} (\delta_{xy} q_z + \delta_{xz} q_y + \delta_{yz} q_x) q'^2 \right] ,
\]

where $\hat{G}^{(+)}_C$ is the Coulomb propagator, $G^{(\pm)}_C = 1/(E - \hat{H}_0 - \hat{V}_C \pm i\epsilon)$, with $H_0 = p^2/(2\mu)$ is the free two particle Hamiltonian: $\mu$ is a reduced mass, and $V_C = \alpha Z_1 Z_2/r$ is the repulsive Coulomb force.

The $J$ functions become infinity due to the loop integrals, and we employ the dimensional regularization in $d = 4 - 2\epsilon$ space-time dimensions. The expression of the $J^{l=0}_0(p)$ function is well known and one has [13]

\[
J^{l=0}_0(p) = J^{l=0,\text{div}}_0 - \frac{\kappa \mu}{\pi} H(\eta) ,
\]
where \( J_0^{l=0,\text{div}} \) is the divergent part of the \( J_0^{l=0}(p) \) function,

\[
J_0^{l=0,\text{div}} = \frac{\kappa \mu}{2\pi} \left[ \frac{1}{\epsilon} - 3C_E + 2 + \ln \left( \frac{\pi \mu^2_{DR}}{4\kappa^2} \right) \right],
\]

(23)

where \( C_E = 0.577 \cdots \) and \( \mu_{DR} \) is a scale parameter from the dimensional regularization. In addition, \( \kappa = \alpha Z_1 Z_2 \mu \), and the \( H(\eta) \) function has been presented in Eq. (3). We note that \( J_0^{l=0,\text{div}} \) does not depend on the momentum \( p \). Thus the divergent term is renormalized a counter term, \(-1/\alpha_0 \) term in the effective range expansion for \( l = 0 \).

The UV divergent terms from the other \( J \) functions are obtained as

\[
J_{2,i,x}^{l=1,\text{div}}(p) = O_{i,x}^{l=1} (\kappa^2 + p^2) J_0^{l=0,\text{div}},
\]

(24)

\[
J_{i,j}^{l=2,\text{div}}(p) = O_{i,j}^{l=2} \frac{1}{4} (\kappa^2 + p^2) (\kappa^2 + 4p^2) J_0^{l=0,\text{div}},
\]

(25)

\[
J_{i,k,x,y}^{l=3,\text{div}}(p) = O_{i,k,x,y}^{l=3} \frac{1}{36} (\kappa^2 + p^2) (\kappa^2 + 4p^2) (\kappa^2 + 9p^2) J_0^{l=0,\text{div}},
\]

(26)

with

\[
O_{i,x}^{l=1} = \int \frac{d\Omega}{4\pi} \hat{l}_i \hat{l}_x = \frac{1}{3} \delta_{ix},
\]

(27)

\[
O_{i,j}^{l=2} = \int \frac{d\Omega}{4\pi} \left( \hat{l}_i \hat{l}_j - \frac{1}{3} \delta_{ij} \right) \left( \hat{l}_x \hat{l}_y - \frac{1}{3} \delta_{xy} \right) = \frac{1}{15} \left( \delta_{ix} \delta_{jy} + \delta_{iy} \delta_{jax} - \frac{2}{3} \delta_{ij} \delta_{xy} \right),
\]

(28)

\[
O_{i,k,x,y}^{l=3} = \int \frac{d\Omega}{4\pi} \left[ \hat{l}_i \hat{l}_j \hat{l}_k - \frac{1}{5} \left( \delta_{ij} \hat{l}_k + \delta_{ik} \hat{l}_j + \delta_{jk} \hat{l}_i \right) \right] \left[ \hat{l}_x \hat{l}_y \hat{l}_z - \frac{1}{5} \left( \delta_{xy} \hat{l}_z + \delta_{xz} \hat{l}_y + \delta_{yz} \hat{l}_x \right) \right]
\]

\[
= \frac{1}{105} \left[ \delta_{ix} \delta_{jy} \delta_{kz} + 5 \text{ terms} - \frac{2}{5} \left( \delta_{ij} \delta_{kx} \delta_{yz} + 8 \text{ terms} \right) \right].
\]

(29)

To renormalize the divergent terms from the Coulomb self-energy, which now depend on the powers of \( p^2 \), one needs two counter terms, \(-1/\alpha_1 \) and \( r_1 \) in the effective range expansion for \( l = 1 \), three counter terms, \(-1/\alpha_2, r_2, P_2 \) for \( l = 2 \), and four counter terms, \(-1/\alpha_3, r_3, P_3, Q_3 \) for \( l = 3 \).

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