Solving fuzzy two-point boundary value problem using fuzzy Laplace transform

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Abstract

A natural way to model dynamic systems under uncertainty is to use fuzzy boundary value problems (FBVPs) and related uncertain systems. In this paper we use fuzzy Laplace transform to find the solution of two-point boundary value under generalized Hukuhara differentiability. We illustrate the method for the solution of the well known two-point boundary value problem Schrödinger equation, and homogeneous boundary value problem. Consequently, we investigate the solutions of FBVPs under as a new application of fuzzy Laplace transform.

Keywords: Fuzzy derivative, fuzzy boundary value problems, fuzzy Laplace transform, fuzzy generalized Hukuhara differentiability.

1 Introduction

"The theory of fuzzy differential equations (FDEs) has attracted much attention in recent years because this theory represents a natural way to model dynamical systems under uncertainty", Jamshidi and Avazpour [1]. The concept

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of fuzzy set was introduced by Zadeh in 1965 [2]. The derivative of fuzzy-valued function was introduced by Chang and Zadeh in 1972 [3]. The integration of fuzzy valued function is presented in [9]. Kaleva and Seikala presented fuzzy differential equations (FDEs) in [4, 5]. Many authors discussed the applications of FDEs in [6, 7, 8]. Two-point boundary value problem is investigated in [10]. In case of Hukuhara derivative the funding Green’s function helps to find the solution of boundary value problem of first order linear fuzzy differential equations with impulses [11]. Wintner-type and superlinear-type results for fuzzy initial value problems (FIVPs) and fuzzy boundary value problems (FBVPs) are presented in [12]. The solution of FBVPs must be a fuzzy-valued function under the Hukuhara derivative [13, 14, 15, 16, 17]. Also two-point boundary value problem (BVP) is equivalent to fuzzy integral equation [18]. Recently in [19, 20, 21] the fuzzy Laplace transform is applied to find the analytical solution of FIVPs. According to [22] the fuzzy solution is different from the crisp solution as presented in [13, 14, 15, 16, 23, 24]. In [22] they solved the Schrödinger equation with fuzzy boundary conditions. Further in [19] it was discussed that under what conditions the fuzzy Laplace transform can be applied to FIVPs. For two-point BVP some of the analytical methods are illustrated in [22, 25, 26] while some of the numerical methods are presented in [1, 27]. But every method has its own advantages and disadvantages for the solution of such types of fuzzy differential equation (FDE). In this paper we are going to apply the FLT on two-point BVP [22]. Moreover we investigate the solution of second order Schrödinger equation and other homogeneous boundary value problems [22]. After applying the FLT to BVP we replace one or more missing terms by any constant and then apply the boundary conditions which eliminates the constants. The crisp solution of fuzzy boundary value problem (FBVP) always lies between the upper and lower solutions. If the lower solution is not monotonically increasing and the upper solution is not monotonically decreasing then the solution of the FDE is not a valid level set.

This paper is organized as follows:
In section 2, we recall some basics definitions and theorems. FLT is defined in section 3 and in this section the FBVP is briefly reviewed. In section 4, constructing solution of FBVP by FLT is explained. To illustrate the method, several examples are given in section 5. Conclusion is given in section 6.

2 Basic concepts

In this section we will recall some basics definitions and theorems needed throughout the paper such as fuzzy number, fuzzy-valued function and the
derivative of the fuzzy-valued functions.

**Definition 2.1.** A fuzzy number is defined in [2] as the mapping such that $u : \mathbb{R} \rightarrow [0, 1]$, which satisfies the following four properties

1. $u$ is upper semi-continuous.
2. $u$ is fuzzy convex that is $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$, $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$.
3. $u$ is normal that is $\exists x_0 \in \mathbb{R}$, where $u(x_0) = 1$.
4. $A = \{x \in \mathbb{R} : u(x) > 0\}$ is compact, where $\overline{A}$ is closure of $A$.

**Definition 2.2.** A fuzzy number in parametric form given in [3, 4, 5] is an order pair of the form $u = (\underline{u}(r), \overline{u}(r))$, where $0 \leq r \leq 1$ satisfying the following conditions.

1. $\underline{u}(r)$ is a bounded left continuous increasing function in the interval $[0, 1]$.
2. $\overline{u}(r)$ is a bounded left continuous decreasing function in the interval $[0, 1]$.
3. $\underline{u}(r) \leq \overline{u}(r)$.

If $\underline{u}(r) = \overline{u}(r) = r$, then $r$ is called crisp number.

Now we recall a triangular fuzzy from [2, 19, 20] number which must be in the form of $u = (l, c, r)$, where $l, c, r \in \mathbb{R}$ and $l \leq c \leq r$, then $\underline{u}(\alpha) = l + (c - r)\alpha$ and $\overline{u}(\alpha) = r - (r - c)\alpha$ are the end points of the $\alpha$ level set. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number if

$$\tilde{y}(t) = \begin{cases} 1, & \text{if } y = t, \\ 0, & \text{if } 1 \neq t. \end{cases}$$

For arbitrary fuzzy numbers $u = (\underline{u}(\alpha), \overline{u}(\alpha))$ and $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ and an arbitrary crisp number $j$, we define addition and scalar multiplication as:

1. $(u + v)(\alpha) = (\underline{u}(\alpha) + \underline{v}(\alpha))$.
2. $(\underline{u} + \overline{v})(\alpha) = (\underline{u}(\alpha) + \overline{v}(\alpha))$.
3. $(j\underline{u})(\alpha) = j\underline{u}(\alpha), \ (j\overline{u})(\alpha) = j\overline{u}(\alpha) \ j \geq 0$.
4. $(ju)(\alpha) = j\overline{u}(\alpha)\alpha, \ (ju)(\alpha) = ju(\alpha)\alpha, \ j < 0$. 

3
Definition 2.3. (See Salahshour & Allahviranloo, and Allahviranloo & Barkhordari [19, 20]) Let us suppose that \( x, y \in E \), if \( \exists z \in E \) such that \( x = y + z \). Then, \( z \) is called the H-difference of \( x \) and \( y \) and is given by \( x \ominus y \).

Remark 2.4. (see Salahshour & Allahviranloo [19]). Let \( X \) be a cartesian product of the universes, \( X_1, X_2, \ldots, X_n \), that is \( X = X_1 \times X_2 \times \cdots \times X_n \) and \( A_1, \ldots, A_n \) be \( n \) fuzzy numbers in \( X_1, \ldots, X_n \) respectively. Then, \( f \) is a mapping from \( X \) to a universe \( Y \), and \( y = f(x_1, x_2, \ldots, x_n) \), then the Zadeh extension principle allows us to define a fuzzy set \( B \) in \( Y \) as:

\[
B = \{(y, u_B(y)) | y = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in X\},
\]

where

\[
u_B(y) = \begin{cases}
\sup_{(x_1, \ldots, x_n) \in f^{-1}(y)} \min \{u_{A_1}(x_1), \ldots, u_{A_n}(x_n)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise},
\end{cases}
\]

where \( f^{-1} \) is the inverse of \( f \).

The extension principle reduces in the case if \( n = 1 \) and is given as follows: \( B = \{(y, u_B(y)) | y = f(x), x \in X\} \), where

\[
u_B(y) = \begin{cases}
\sup_{x \in f^{-1}(y)} \{u_A(x)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise}.
\end{cases}
\]

By Zadeh extension principle the approximation of addition of \( E \) is defined by \((u \oplus v)(x) = \sup_{y \in R} \min(u(y), v(x - y))\), \( x \in R \) and scalar multiplication of a fuzzy number is defined by

\[
(k \odot u)(x) = \begin{cases}
u_{\tilde{0}}(x), & k > 0, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \tilde{0} \in E \).

The Hausdorff distance between the fuzzy numbers [7, 13, 19, 20] defined by

\[
d : E \times E \to R^+ \cup \{0\},
\]

\[
d(u, v) = \sup_{r \in [0,1]} \max \{|\mu(r) - \nu(r)|, |\nu(r) - \mu(r)|\},
\]

where \( u = (\mu(r), \nu(r)) \) and \( v = (\nu(r), \mu(r)) \in R \).

We know that if \( d \) is a metric in \( E \), then it will satisfies the following properties, introduced by Puri and Ralescu [28]:
1. \( d(u + w, v + w) = d(u, v), \forall u, v, w \in E. \)

2. \( (k \odot u, k \odot v) = |k|d(u, v), \forall k \in \mathbb{R}, \text{ and } u, v \in E. \)

3. \( d(u \oplus v, w \oplus e) \leq d(u, w) + d(v, e), \forall u, v, w, e \in E. \)

**Definition 2.7.** (see Song and Wu [3]) If \( f : R \times E \rightarrow E \), then \( f \) is continuous at point \((t_0, x_0) \in R \times E \) provided that for any fixed number \( r \in [0, 1] \) and any \( \epsilon > 0 \), \( \exists \delta(\epsilon, r) \) such that \( d([f(t, x)]', [f(t_0, x_0)]') < \epsilon \) whenever \( |t - t_0| < \delta(\epsilon, r) \) and \( d([x]', [x_0]') < \delta(\epsilon, r) \) \( \forall t \in R, x \in E. \)

**Theorem 2.6.** (see Wu [30]). Let \( f \) be a fuzzy-valued function on \([a, \infty)\) given in the parametric form as \((f(x, r), \int f(x, r)dx)\) for any constant number \( r \in [0, 1] \). Here we assume that \( f(x, r) \) and \( \int f(x, r)dx \) are Riemann-Integral on \([a, b]\) for every \( b \geq a \). Also we assume that \( \mathbb{M}(r) \) and \( \mathbb{M}(r) \) are two positive functions, such that \( \int_a^b [f(x, r)]dx \leq \mathbb{M}(r) \) and \( \int_a^b \int f(x, r)dx \leq \mathbb{M}(r) \) for every \( b \geq a \), then \( f(x) \) is improper integral on \([a, \infty)\). Thus an improper integral will always be a fuzzy number.

In short

\[
\int_a^r f(x)dx = (\int_a^b [f(x, r)]dx, \int_a^b \int f(x, r)dx).
\]

It is well known that Hukuhare differentiability for fuzzy function was introduced by Puri and Ralescu in 1983.

**Definition 2.7.** (see Chalco-Cano and Román-Flores [34]). Let \( f : (a, b) \rightarrow E \) where \( x_0 \in (a, b) \). Then, we say that \( f \) is strongly generalized differentiable at \( x_0 \) (Beds and Gal differentiability). If \( \exists \) an element \( f'(x_0) \in E \) such that

1. \( \forall h > 0 \) sufficiently small \( \exists f(x_0 + h) \odot f(x_0), f(x_0) \odot f(x_0 - h), \) then the following limits hold (in the metric \( d \))

\[
\lim_{h \rightarrow 0} \frac{f(x_0 + h) \odot f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0) \odot f(x_0 - h)}{h} = f'(x_0),
\]

Or

2. \( \forall h > 0 \) sufficiently small, \( \exists f(x_0) \odot f(x_0 + h), f(x_0 - h) \odot f(x_0), \) then the following limits hold (in the metric \( d \))

\[
\lim_{h \rightarrow 0} \frac{f(x_0) \odot f(x_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) \odot f(x_0)}{-h} = f'(x_0),
\]

Or

3. \( \forall h > 0 \) sufficiently small \( \exists f(x_0 + h) \odot f(x_0), f(x_0 - h) \odot f(x_0) \) and the following limits hold (in metric \( d \))

\[
\lim_{h \rightarrow 0} \frac{(x_0 + h) \odot f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 - h) \odot f(x_0)}{-h} = f'(x_0),
\]

Or
4. \( \forall h > 0 \) sufficiently small \( \exists f(x_0) \ominus f(x_0 + h), f(x_0) \ominus f(x_0 - h) \), then the following limits holds\( (\text{in metric } d) \)
\[
\lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 + h)}{h} = \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0).
\]

The denominators \( h \) and \( -h \) denote multiplication by \( \frac{1}{h} \) and \( \frac{-1}{h} \) respectively.

**Theorem 2.8.** (Ses Chalco-Cano and Román-Flores [31]).

Let \( f : R \to E \) be a function denoted by \( f(t) = (f(t,r), f(t,r)) \) for each \( r \in [0,1] \). Then

1. If \( f \) is (i)-differentiable, then \( f(t,r) \) and \( f(t,r) \) are differentiable functions and \( f'(t) = (f'(t,r), f'(t,r)) \).
2. If \( f \) is (ii)-differentiable, then \( f(t,r) \) and \( f(t,r) \) are differentiable functions and \( f'(t) = (f'(t), f'(t,r)) \).

**Lemma 2.9.** (see Bede and Gal [32, 33]). Let \( x_0 \in R \). Then, the FDE \( y' = f(x,y), y(x_0) = y_0 \in R \) and \( f : R \times E \to E \) is supposed to be a continuous and equivalent to one of the following integral equations.

\[
y(x) = y_0 + \int_{x_0}^{x} f(t,y(t))dt \quad \forall \quad x \in [x_0, x_1],
\]

or

\[
y(0) = y^1(x) + (-1) \odot \int_{x_0}^{x} f(t,y(t))dt \quad \forall \quad x \in [x_0, x_1],
\]

on some interval \( (x_0, x_1) \subset R \) depending on the strongly generalized differentiability. Integral equivalency shows that if one solution satisfies the given equation, then the other will also satisfy.

**Remark 2.10.** (see Bede and Gal [32, 33]). In the case of strongly generalized differentiability to the FDE’s \( y' = f(x,y) \) we use two different integral equations. But in the case of differentiability as the definition of H-derivative, we use only one integral. The second integral equation as in Lemma 2.10 will be in the form of \( y^1(t) = y_0^1 \odot (-1) \int_{x_0}^{x} f(t,y(t))dt \). The following theorem related to the existence of solution of FIVP under the generalized differentiability.

**Theorem 2.11.** Let us suppose that the following conditions are satisfied.

1. Let \( R_0 = [x_0, x_0 + s] \times B(y_0, q), s, q > 0, y \in E \), where \( B(y_0, q) = \{ y \in E : B(y,y_0) \leq q \} \) which denotes a closed ball in \( E \) and let \( f : R_0 \to E \) be continuous functions such that \( D(0, f(x,y)) \leq M, \forall (x,y) \in R_0 \) and \( 0 \in E \).
2. Let \( g : [x_0, x_0 + s] \times [0, q] \to \mathbb{R} \) such that \( g(x, 0) \equiv 0 \) and \( 0 \leq g(x, u) \leq M, \forall x \in [x_0, x_0 + s], \exists 0 \leq u \leq q \), such that \( g(x, u) \) is increasing in \( u \), and \( g \) is such that the FIVP \( u'(x) = g(x, u(x)), u(x) \equiv 0 \) on \([x_0, x_0 + s]\).

3. We have \( D[f(x, y), f(x, z)] \leq g(x, D(y, z)), \forall (x, y), (x, z) \in \mathbb{R} \) and \( D(y, z) \leq q \).

4. \( \exists d > 0 \) such that for \( x \in [x_0, x_0 + d] \), the sequence \( y_1^n : [x_0, x_0 + d] \to \mathbb{R} \) given by \( y_1^0(x) = y_0, y_1^{n+1}(x) = y_0 \ominus (-1) \int_{x_0}^{x} f(t, y_n(t))dt \) defined for any \( n \in \mathbb{N} \). Then the FIVP \( y' = f(x, y), y(x_0) = y_0 \) has two solutions that is (1)-differentiable and the other one is (2)-differentiable for \( y \).

\[ y_1^1 = [x_0, x_0 + r] \to B(y_0, q), \quad r = \min \{ s, \frac{q}{M}, \frac{q}{M^2}, d \} \] and the successive iterations \( y_1^n(x) = y_0, y_1^{n+1}(x) = y_0 \ominus (-1) \int_{x_0}^{x} f(t, y_n(t))dt \) and \( y_1^{n+1} = y_0, y_1^{n+1}(x) = y_0 \ominus (-1) \int_{x_0}^{x} f(t, y_1^n(t))dt \) converge to these two solutions respectively. Now according to theorem (2.11), we restrict our attention to function which are (1) or (2)-differentiable on their domain except on a finite number of points as discussed in [33].

3. **Fuzzy Laplace Transform**

Suppose that \( f \) is a fuzzy-valued function and \( p \) is a real parameter, then according to [19, 20] FLT of the function \( f \) is defined as follows:

**Definition 3.1.** The FLT of fuzzy-valued function is [19, 20]

\[
\hat{F}(p) = L[f(t)] = \int_0^\infty e^{-pt} f(t)dt, \tag{3.1}
\]

\[
\hat{F}(p) = L[f(t)] = \lim \tau \to \infty \int_0^\tau e^{-pt} f(t)dt, \tag{3.2}
\]

\[
\hat{F}(p) = \left[ \lim \tau \to \infty \int_0^\tau e^{-pt} f(t)dt, \lim \tau \to \infty \int_0^\tau e^{-pt} \overline{f}(t)dt \right], \tag{3.3}
\]

whenever the limits exist.

**Definition 3.2.** Classical Fuzzy Laplace Transform: Now consider the fuzzy-valued function in which the lower and upper FLT of the function are represented by

\[
\hat{F}(p; r) = L[f(t; r)] = [l(f(t; r)), l(\overline{f}(t; r))], \tag{3.4}
\]
where
\[ l[f(t; r)] = \int_0^\infty e^{-pt} f(t; r) dt = \lim_{\tau \to \infty} \int_0^\tau e^{-pt} f(t; r) dt, \quad (3.5) \]
\[ l[f(t; r)] = \int_0^\infty e^{-pt} f(t; r) dt = \lim_{\tau \to \infty} \int_0^\tau e^{-pt} f(t; r) dt. \quad (3.6) \]

3.1 Fuzzy Boundary Value problem

The concept of fuzzy numbers and fuzzy set was first introduced by Zadeh [2]. Detail information of fuzzy numbers and fuzzy arithmetic can be found in [13, 14, 15]. In this section we review the fuzzy boundary valued problem (FBVP) with crisp linear differential equation but having fuzzy boundary values. For example we consider the second order fuzzy boundary problem as [10, 11, 22, 1].

\[ \psi''(t) + c_1(t) \psi'(t) + c_2(t) \psi(t) = f(t), \]
\[ \psi(0) = \tilde{A}, \]
\[ \psi(l) = \tilde{B}. \quad (3.7) \]

4 Constructing Solutions Via FBVP

In this section we consider the following second order FBVP in general form under generalized H-differentiability proposed in [22]. We define

\[ y''(t) = f(t, y(t), y'(t)), \quad (4.1) \]

subject to two-point boundary conditions

\[ y(0) = (\underline{y}(0; r), \bar{y}(0; r)), \]
\[ y(l) = (\underline{y}(l; r), \bar{y}(l; r)). \]

Taking FLT of (4.1)

\[ L[y''(t)] = L[f(t, y(t), y'(t))], \quad (4.2) \]

which can be written as

\[ p^2L[y(t)] \ominus py(0) \ominus y'(0) = L[f(t, y(t), y'(t))]. \]

The classical form of FLT is given below:

\[ p^2l[y(t; r)] - py(0; r) - y'(0; r) = l[f(t, y(0; r), y'(0; r))], \quad (4.3) \]
\[ p^2 l[y(t; r)] - py(0; r) - y'(0; r) = l[f(t, y(0; r), y'(0; r))]. \]  
(4.4)

Here we have to replace the unknown value \( y'(0, r) \) by constant \( F_1 \) in lower case and by \( F_2 \) in upper case. Then we can find these values by applying the given boundary conditions.

In order to solve equations (4.3) and (4.4) we assume that \( A(p; r) \) and \( B(p; r) \) are the solutions of (4.3) and (4.4) respectively. Then the above system becomes

\[ l[y(t; r)] = A(p; r), \]  
(4.5)

\[ l[y(t; r)] = B(p; r). \]  
(4.6)

Using inverse Laplace transform, we get the upper and lower solution for given problem as:

\[ [y(t; r)] = l^{-1}[A(p; r)], \]  
(4.7)

\[ [y(t; r)] = l^{-1}[B(p; r)]. \]  
(4.8)

5 Examples

In this section first we consider the Schrödinger equation [22] with fuzzy boundary conditions under Hukuhara differentiability.

Example 5.1. The Schrödinger FBVP [22] is as follows:

\[ \left( \frac{\hbar^2}{2m} \right) u''(x) + V(x)u(x) = Eu(x), \]  
(5.1)

where \( V(x) \) is potential and is defined as

\[ V(x) = \begin{cases} 
0, & \text{if } x < 0, \\
l, & \text{if } x > 0, 
\end{cases} \]

subject to the following boundary conditions

\[ u(0) = (1 + r, 3 - r), \]
\[ u(l) = (4 + r, 6 - r). \]

Now let \( a = \frac{\hbar^2}{2m} \), \( b = E \). Then, (5.1) becomes

\[ au''(x) + V(x)u(x) = bu(x). \]  
(5.2)

In (5.1) for \( x < 0 \), we discuss \((1,1)\) and \((2,2)\)-differentiability while in the case \( x > 0 \) we will discuss \((1,2)\) and \((2,1)\)-differentiability.
5.1 Case-I: (1,1) and (2,2)-differentiability

For \( x < 0 \), (5.2) becomes

\[
au'' = bu,
\]

\[
au'' - bu = 0.
\]  

(5.3)

Now applying FLT on both sides of equation (5.3), we get

\[
aL[u''(x)] - bL[u(x)] = 0,
\]

where

\[
L[u''(x)] = p^2L[u(x)] \oplus pu(0) \oplus u'(0).
\]

The classical FLT form of the above equation is

\[
l[u''(x, r)] = p^2l[u(x, r)] - pu(0, r) - u'(0, r),
\]

\[
l[u'(x, r)] = p^2l[u(x, r)] - pu(0, r) - u(0, r).
\]

Solving the above classical equations for lower and upper solutions, we have

\[
a\{p^2l[u(x, r)] - pu(0, r) - u'(0, r)\} - bl[u(x, r)] = 0,
\]

or

\[
(ap^2 - b)l[u(x, r)] = a\{pu(0, r) + u'(0, r)\}.
\]

Applying the boundary conditions, we have

\[
(ap^2 - b)l[u(x, r)] = a\{p(1 + r) + F_1\},
\]

where

\[
F_1 = u'(0, r).
\]

Simplifying and applying inverse Laplace we get

\[
u(x, r) = \left(\frac{1+r}{2}\right)l^{-1}\left\{\frac{p}{p^2 - \frac{b}{a}}\right\} + F_1l^{-1}\left\{\frac{1}{p^2 - \frac{b}{a}}\right\}.
\]

Using partial fraction

\[
u(x, r) = \frac{1}{2}\left\{e^{\sqrt{\frac{b}{a}}x} + e^{-\sqrt{\frac{b}{a}}x}\right\} + \frac{F_1}{2\sqrt{\frac{b}{a}}}\left\{e^{\sqrt{\frac{b}{a}}x} - e^{-\sqrt{\frac{b}{a}}x}\right\}.
\]  

(5.4)
Now applying boundary conditions on (5.4) we get

\[ F_1 = \frac{4 + r - \frac{1+r}{2} \{ e^{\sqrt{\beta} x} + e^{-\sqrt{\beta} x} \}}{\frac{1}{2} \sqrt{\beta} \{ e^{\sqrt{\beta} x} - e^{-\sqrt{\beta} x} \}}. \]

Putting value of \( F_1 \) in (5.4) we get

\[ u(x, r) = \left( \frac{1+r}{2} \right) \{ e^{\sqrt{\beta} x} + e^{-\sqrt{\beta} x} \} + \frac{4 + r - \frac{1+r}{2} \{ e^{\sqrt{\beta} x} + e^{-\sqrt{\beta} x} \}}{\{ e^{\sqrt{\beta} x} - e^{-\sqrt{\beta} x} \}} \{ e^{\sqrt{\beta} x} - e^{-\sqrt{\beta} x} \}. \]

Now solving the classical FLT form for \( \overline{u}(x, r) \), we have

\[ a \{ p^2 l[\overline{u}(x, r)] - p\overline{u}(0, r) - \overline{u}'(0, r) \} - bl[\overline{u}(x, r)] = 0, \]

\[ (ap^2 - b)l[\overline{u}(x, r)] = a \{ p\overline{u}(0, r) + \overline{u}'(0, r) \}. \]

Using the boundary conditions, we have

\[ (ap^2 - b)l[\overline{u}(x, r)] = a \{ p(3 - r) + F_2 \}, \]

where

\[ F_2 = \overline{u}'(0, r). \]

Simplifying and applying inverse laplace we get

\[ \overline{u}(x, r) = \left( \frac{3 - r}{2} \right) l^{-1} \left\{ \frac{-p}{p^2 - \frac{b}{a}} \right\} + F_2 l^{-1} \left\{ \frac{1}{ap^2 - \frac{b}{a}} \right\}. \]

Using partial fraction

\[ \overline{u}(x, r) = \left( \frac{3 - r}{2} \right) \{ e^{\sqrt{\beta} x} + e^{-\sqrt{\beta} x} \} + \frac{F_2}{2 \sqrt{\beta} \{ e^{\sqrt{\beta} x} - e^{-\sqrt{\beta} x} \}} \{ e^{\sqrt{\beta} x} - e^{-\sqrt{\beta} x} \}. \] (5.5)

Now applying boundary conditions on (5.5) we have

\[ F_2 = \frac{6 - r - \frac{3-r}{2} \{ e^{\sqrt{\beta} x} + e^{-\sqrt{\beta} x} \}}{\frac{1}{2} \sqrt{\beta} \{ e^{\sqrt{\beta} x} - e^{-\sqrt{\beta} x} \}}. \]

Putting value of \( F_2 \) in (5.5) we get

\[ \overline{u}(x, r) = \left( \frac{3 - r}{2} \right) \{ e^{\sqrt{\beta} x} + e^{-\sqrt{\beta} x} \} + \frac{6 - r - \frac{3-r}{2} \{ e^{\sqrt{\beta} x} + e^{-\sqrt{\beta} x} \}}{\{ e^{\sqrt{\beta} x} - e^{-\sqrt{\beta} x} \}} \{ e^{\sqrt{\beta} x} - e^{-\sqrt{\beta} x} \}. \]
5.2 Case-II: (1) and (2)-differentiability, (2) and (1)-differentiability

For \( x > 0 \), (5.2) becomes

\[
au'' + (l - b)u = 0. \tag{5.6}
\]

Applying FLT and inverse Laplace transform and then simplifying we get the following lower solution.

\[
\begin{align*}
\underline{u}(x, r) & = \left( \frac{1 + r}{2} \right) \left[ \cos \frac{x\sqrt{b-l}}{\sqrt{a}} + \cosh \frac{x\sqrt{b-l}}{\sqrt{a}} \right] + \frac{H_1\sqrt{a}}{2\sqrt{b-l}} \left[ \sin \frac{x\sqrt{b-l}}{\sqrt{a}} + \sinh \frac{x\sqrt{b-l}}{\sqrt{a}} \right] \\
& \quad - \frac{(3 - r)}{2} \left[ \cos \frac{x\sqrt{b-l}}{\sqrt{a}} - \cosh \frac{x\sqrt{b-l}}{\sqrt{a}} \right] - \frac{H_2\sqrt{a}}{2\sqrt{b-l}} \left[ \sin \frac{x\sqrt{b-l}}{\sqrt{a}} - \sinh \frac{x\sqrt{b-l}}{\sqrt{a}} \right],
\end{align*}
\]

or

\[
\underline{u}(x, r) = \frac{1 + r}{2} (c_1) + \frac{H_1\sqrt{a}}{2\sqrt{b-l}} (c_2) - \frac{3 - r}{2} (c_3) - \frac{H_2\sqrt{a}}{2\sqrt{b-l}} (c_4).
\]

The upper solution will be as follows:

\[
\begin{align*}
\overline{u}(x, r) & = \left( \frac{3 - r}{2} \right) \left[ \cos \frac{x\sqrt{b-l}}{\sqrt{a}} + \cosh \frac{x\sqrt{b-l}}{\sqrt{a}} \right] + \frac{H_2\sqrt{a}}{2\sqrt{b-l}} \left[ \sin \frac{x\sqrt{b-l}}{\sqrt{a}} + \sinh \frac{x\sqrt{b-l}}{\sqrt{a}} \right] \\
& \quad - \frac{(1 + r)}{2} \left[ \cos \frac{x\sqrt{b-l}}{\sqrt{a}} - \cosh \frac{x\sqrt{b-l}}{\sqrt{a}} \right] - \frac{H_1\sqrt{a}}{2\sqrt{b-l}} \left[ \sin \frac{x\sqrt{b-l}}{\sqrt{a}} - \sinh \frac{x\sqrt{b-l}}{\sqrt{a}} \right],
\end{align*}
\]

\[
\overline{u}(x, r) = \frac{3 - r}{2} (c_1) + \frac{H_2\sqrt{a}}{2\sqrt{b-l}} (c_2) - \frac{1 + r}{2} (c_3) - \frac{H_1\sqrt{a}}{2\sqrt{b-l}} (c_4)
\]

where

\[
\begin{align*}
c_1 & = \cos \frac{x\sqrt{b-l}}{\sqrt{a}} + \cosh \frac{x\sqrt{b-l}}{\sqrt{a}}, \\
c_2 & = \sin \frac{x\sqrt{b-l}}{\sqrt{a}} + \sinh \frac{x\sqrt{b-l}}{\sqrt{a}}, \\
c_3 & = \cos \frac{x\sqrt{b-l}}{\sqrt{a}} - \cosh \frac{x\sqrt{b-l}}{\sqrt{a}}, \\
c_4 & = \sin \frac{x\sqrt{b-l}}{\sqrt{a}} - \sinh \frac{x\sqrt{b-l}}{\sqrt{a}}.
\end{align*}
\]

\[
H_1 = \frac{2c_2}{c_2^2 - c_4^2} \left[ 4 + r - \frac{r + 1}{2} c_1 + \frac{3 - r}{2} c_3 \right] + \frac{2c_4}{c_2^2 - c_4^2} \left[ 6 - \frac{3 - r}{2} c_1 + \frac{1 + r}{2} c_3 \right],
\]

and

\[
H_2 = \frac{2c_4}{c_2^2 - c_4^2} \left[ 4 + r - \frac{r + 1}{2} c_1 + \frac{3 - r}{2} c_3 \right] + \frac{2c_2}{c_2^2 - c_4^2} \left[ 6 - \frac{3 - r}{2} c_1 + \frac{1 + r}{2} c_3 \right].
\]
Example 5.2. Consider the following fuzzy homogenous boundary value problem

\[ x''(t) - 3x'(t) + 2x(t) = 0, \quad (5.7) \]

subject to the following boundary conditions

\[ x(0) = (0.5r - 0.5, 1 - r), \]
\[ x(1) = (r - 1, 1 - r). \]

Now applying fuzzy Laplace transform on both sides of equation (5.7), we get

\[ L[x''(t)] = 3L[x'(t)] - 2L[x(t)]. \quad (5.8) \]

We know that

\[ L[x''(t)] = p^2L[x(t)] ⊖ px(0) ⊖ x'(0). \]

The classical FLT form of the above equation is

\[ l[x''(t, r)] = p^2l[x(t, r)] - px(0, r) - x'(0, r), \]
\[ l[x'(t, r)] = p^2l[x(t, r)] - px(0, r) - x'(0, r). \]

Now on putting in (5.8), we have

\[ p^2l[x(t, r)] - px(0, r) - x'(0, r) - 3px(t, r) + 3x(0, r) + 2l[x(t, r)] = 0, \quad (5.9) \]
\[ p^2l[x(t, r)] - px(0, r) - x'(0, r) - 3px(t, r) + 3x(0, r) + 2l[x(t, r)] = 0. \quad (5.10) \]

Solving (5.9) for \( l[x(t, r)] \), we get

\[ (p^2 - 3p + 2)l[x(t, r)] = px(0, r) + x'(0, r) + 3x(0, r). \]

Applying boundary conditions, we have

\[ l[x(t, r)] = \frac{(0.5r - 0.5)p}{p^2 - 3p + 2} + 3\frac{(0.5r - 0.5)}{p^2 - 3p + 2} + \frac{A}{p^2 - 3p + 2}. \]

Using partial fraction and then applying inverse Laplace, we get

\[ x(t, r) = (0.5r - 0.5)[-e^t + 2e^{2t}] - 3(0.5r - 0.5)[-e^t + 2e^{2t}] + A[-e^t + e^{2t}]. \]

Using boundary values, we get

\[ x(1, r) = r - 1 = (0.5r - 0.5)[-e + 2e^2] - 3(0.5r - 0.5)[-e + 2e^2] + A[-e + e^2], \]

13
\[ A = \frac{r - 1 + (0.5r - 0.5)[-2e + e^2]}{e^2 - e}. \]

Finally on putting value of \( A \) we have

\[ x(t, r) = (0.5r - 0.5)(-e^t + 2e^{2t}) - 3(0.5r - 0.5)(-e^t + 2e^{2t}) + \frac{r - 1 + (0.5r - 0.5)(-2e + e^2)}{e^2 - e}(-e + e^2) \]

Now solving (5.10) for \( l[x(t, r)] \), we have

\[ (p^2 - 3p + 2)l[\varphi(t, r)] = p\varphi(0, r) + \varphi'(0, r) + 3[\varphi(0, r)]. \]

Applying boundary condition we get

\[ l[\varphi(t, r)] = \frac{(1 - r)p}{p^2 - 3p + 2} - \frac{3(1 - r)}{p^2 - 3p + 2} + \frac{A}{p^2 - 3p + 2}. \]

Using partial fraction and then applying inverse Laplace

\[ \varphi(t, r) = (1 - r)[-e^t + 2e^{2t}] - 3(1 - r)[-e^t + 2e^{2t}] + A[-e^t + e^{2t}]. \quad (5.11) \]

Using boundary values

\[ \varphi(1, r) = 1 - r = (1 - r)[-e + 2e^2] - 3(1 - r)[-e + 2e^2] + A[-e + e^2], \]

\[ A = \frac{1 - r + (1 - r)[-2e + e^2]}{e^2 - e}. \]

Putting value of \( A \) in (5.11) we get

\[ \varphi(t, r) = (1 - r)[-e^t + 2e^{2t}] - 3(1 - r)[-e^t + 2e^{2t}] + \frac{1 - r + (1 - r)[-2e + e^2]}{e^2 - e}[-e + e^{2t}]. \]

6 Conclusion

In this paper, we applied the fuzzy Laplace transform to solve FBVPs under generalized H-differentiability, in particular, solving Schrödinger FBVP. We also used FLT to solve homogenous FBVP. This is another application of FLT. Thus FLT can also be used to solve FBVPs analytically. The method can be extended for an nth order FBVP. This work is in progress.
References

[1] L. Jamshidi and L. Avazpour, Solution of the fuzzy boundary value differential equations under generalized differentiability by shooting method, *Journal of Fuzzy Set Valued Analysis*, **2012**, 1-19, 2012. [http://dx.doi.org/10.5899/2012/jfsva-00136](http://dx.doi.org/10.5899/2012/jfsva-00136)

[2] L. Zadeh, Fuzzy Sets, *Information and Control* **2**, 338-353, 1965.

[3] S. S. L. Chang and L. Zadeh, On fuzzy mapping and control, *IEEE Trans on Syst Man and Cybern*, **2**, 30-34, 1972.

[4] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, **24**(3), 301-317, 1987.

[5] S. Seikala, On the fuzzy initial value problem, *Fuzzy Sets and Systems*, **24**(3), 319-330, 1987.

[6] A. Bencsik, B. Bede, J. Tar, J. Fodor, Fuzzy Differential Equations in Modeling of Hydraulic Differential Servo Cylinders, in: Third Romanian-Hungarian Join Symposium on Applied Computational Intelligence (SACI), Timisoara, Romania, 2006.

[7] W. Congxin, S. Shiji, Exitance theorem to the Cauchy problem of fuzzy differential equations undecompartance-type conditions, *Information Science* **108**, 123-134, 1993.

[8] O. Kaleva, The Cauchy problem for fuzzy differential equations, *Fuzzy Sets and Systems* **35**, 389-396, 1990.

[9] J. J. Buckley, E. Esfandiar and T. Feuring, Fuzzy Mathematics in Economics and Engineering. Heidelberg: Physica-Verlag, 2002.

[10] A. Khastan and J. J. Nieto. A boundary value problem for second order fuzzy differential equations, *Nonlinear Analysis: Theory, Methods & Applications*, **72**(9), 3583-3593, 2010.

[11] J. J. Nieto and R. Rodríguez-López, Some results on boundary value problems for fuzzy differential equations with functional dependence, *Fuzzy Sets and Systems*, **230**, 92-118, 2013.

[12] D. ÓRegan, V. Lakshmikantham and J. J. Nieto, Initial and boundary value problems for fuzzy differential equations. *Nonlinear Analysis: Theory, Methods & Applications* **54**(3), 405-415, 2003.
[13] R. Goetschel Jr. and W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets and Systems*, **18**(1), 31-43, 1986.

[14] V. Lakshmikantham and R. N. Mohapatra. Theory of fuzzy differential equations and inclusions. Vol. 6. CRC Press, 2003.

[15] P. Diamond and P. Kloeden. Metric spaces of fuzzy sets: theory and applications, Singapore: World Scientific, 1994.

[16] J. J. Buckley, E. Eslami and T. Feuring. Fuzzy Mathematics in Economics and Engineering. Heidelberg: Physica-Verlag, 2002.

[17] B. Bede, I. J. Rudas and A. L. Benssik, First order linear fuzzy differential equations under generalized differentiability, *Information Sciences*, **177**(7), 1648-1662, 2007.

[18] M. Chen, C. Wu, X. Xue and G. Liu, On fuzzy boundary value problems, *Information Sciences*, **178**(7), 1877-1892, 2008.

[19] S. Salahshour and T. Allahviranloo, Applications of fuzzy Laplace Transforms, *Soft Computing*, **17**(1), 145-158, 2013.

[20] T. Allahviranloo and M. Barkhordari Ahmadi, Fuzzy Laplace Transforms, *Soft Computing*, **14**(3), 235-243, 2010.

[21] F. Hawrra and K. H. Amal, On fuzzy Laplace transform for fuzzy differential equations of the third order, *Journal of Kerbala University*, **11**(3), 251-256, 2013.

[22] A. Khalili Golmankhaneh, A. Jafarian, D. Baleanu, On Fuzzy Schrodinger equation, submitted.

[23] R. Goetschel Jr. and W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets and Systems* **18**(1), 31-43, 1986.

[24] P. M. Hukuhara, *Intégration des Applications mMeasurables dont la Valeur est un Compact Convexe*, *Funkcialaj Ekvacioj*, **10**, 205-223, 1967.

[25] K. Khalipour and T. Allahviranloo, An initial-value method for two-point boundary value problems, *World Applied Sciences*, **13**(10), 2148-2155, 2011.

[26] S. Saito, Existence and uniqueness of solutions for fuzzy boundary value problems. *Nonlinear analysis and convex analysis (Japanese) (Kyoto, 2001)*. Surikaisekikenkyusho Kokyuroku No., 1246: 200-209.
[27] J. Lu, Variational iteration method for solving two-point boundary value problems, *Journal of Computational and Applied Mathematics*, **207**, 92-95, 2007. [http://dx.doi.org/10.1016/j.cam.2006.07.014](http://dx.doi.org/10.1016/j.cam.2006.07.014)

[28] M. L. Puri and D. A. Ralescu, Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, **114**(2), 409-422, 1986.

[29] S. Song and C. Wu, Existence and uniqueness of solutions to Cauchy problem of fuzzy differential equations, *Fuzzy Sets and Systems*, **110**, 55-67, 2000.

[30] Hsien-Chung Wu, The improper fuzzy Riemann integral and its numerical integration. *Information Sciences*, **111**(1-4), 109-137, 1998.

[31] Y. Chalco-Cano and H. Román-Flores, On new solutions of fuzzy differential equations, *Chaos, Solitons & Fractals*, **38**(1), 112-119, 2008.

[32] B. Bede and S. G. Gal, Remark on the new solutions of fuzzy differential equations, *Chaos, Solitons & Fractals*, 2006.

[33] B. Bede and S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets and Systems*, **151**, 581-599, 2005.