EXTENDED CONNECTION IN YANG-MILLS THEORY

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Abstract. The three fundamental geometric components of Yang-Mills theory -gauge field, gauge fixing and ghost field- are unified in a new object: an extended connection in a properly chosen principal fiber bundle. To do this, it is necessary to generalize the notion of gauge fixing by using a gauge fixing connection instead of a section. From the equations for the extended connections curvature, we derive the relevant BRST transformations without imposing the usual horizontality conditions. We show that the gauge fields standard BRST transformation is only valid in a local trivialization and we obtain the corresponding global generalization. By using the Faddeev-Popov method, we apply the generalized gauge fixing to the path integral quantization of Yang-Mills theory. We show that the proposed gauge fixing can be used even in the presence of a Gribovs obstruction.
I. Introduction

It is our objective in the present work to show how the main geometric structures of Yang-Mills theory can be unified in a single geometric object, namely a connection in an infinite dimensional principal fiber bundle. We will show how this geometric formalism can be useful to the path integral quantization of Yang-Mills theory. Some of the historical motivations for the study of an extended connection in Yang-Mills theory are the following. In the beginning of the 80's Yang Mills theory was at the center of important mathematical developments, especially Donaldson's theory of four manifolds' invariants \[9\] and Witten's interpretation of this theory in terms of a topological quantum field theory \[23\] (for a general review on topological field theories see Refs.\[5,8\]). A central aspect of these theories is the study of the topological properties of the space \(\mathcal{A}/\mathcal{G}\), where \(\mathcal{A}\) is the configuration space of connections in a \(G\)-principal bundle \(P \to M\) and \(\mathcal{G}\) the gauge group of vertical automorphisms of \(P\). It is possible to show that under certain hypotheses one obtains a \(\mathcal{G}\)-principal bundle structure \(\mathcal{A} \to \mathcal{A}/\mathcal{G}\) \[9\]. The non-triviality of many invariants is then intimately linked with the topological non-triviality of this bundle. In Ref.\[3\] Baulieu and Singer showed that Witten's theory can be interpreted in terms of the gauge fixed version of a topological action through a standard BRST procedure. To do so, the authors unify the gauge field \(A\) and the ghost field \(c\) for Yang-Mills symmetry in an extended connection \(\omega = A + c\) defined in a properly chosen principal bundle. The curvature \(\mathcal{F}\) of \(\omega\) splits naturally as \(\mathcal{F} = F + \psi + \phi\), where \(\psi\) and \(\phi\) are the ghost for the topological symmetry and the ghost for ghost respectively (the necessity of a ghost for ghost is due to the dependence of the topological symmetry on the Yang-Mills symmetry). By expanding the expressions for the curvature \(\mathcal{F}\) and the corresponding Bianchi identity, the BRST transformations for this topological gauge theory are elegantly recovered. It is commonly stated that the passage from this topological Yang-Mills theory to the ordinary (i.e. non-topological) case is mediated by the horizontality (or flatness) conditions, i.e. by the conditions \(\psi = \phi = 0\) (see Refs.\[2,4,22\]). In this case, an extended connection \(\omega = A + c\) can also be defined for an ordinary Yang-Mills theory with a horizontal curvature of the form \(\mathcal{F} = F\).

In this work, an extended connection \(A\) for an ordinary Yang-Mills theory will be defined as the sum of two factors. The first factor is a universal family \(A^U\) parameterized by \(\mathcal{A}\) of connections in the \(G\)-principal bundle \(P \to M\). The second factor is an arbitrarily chosen connection \(\eta\) in the \(\mathcal{G}\)-principal fiber bundle \(\mathcal{A} \to \mathcal{A}/\mathcal{G}\). We will show that the connection \(\eta\) encodes the ghost field that generates the BRST complex. In a local trivialization the ghost field can be identified with the canonical vertical part of the connection \(\eta\), which corresponds...
to the Maurer-Cartan form of the gauge group \(G\) \([6]\). Alternatively, the ghost field can be considered a universal connection in the gauge group’s Weil algebra. The connection \(\eta\) can then be defined as the image of this universal connection under a particular Chern-Weil homomorphism. We will then argue that the connection \(\eta\) also defines what might be called \textit{generalized gauge fixing}. In fact, the connection \(\eta\) defines a horizontal subspace at each point of the fiber bundle \(\mathcal{A} \to \mathcal{A}/G\). These subspaces can be considered first order infinitesimal germs of sections. The significant difference is that the connection \(\eta\) induces a global section \(\sigma_\eta\) in a fiber bundle associated to the \textit{space of paths} in \(\mathcal{A}/G\). In fact, the section \(\sigma_\eta\) assigns the horizontal lift defined by \(\eta\) to each path \([\gamma] \subset \mathcal{A}/G\). Since the path integral is not an integral in the \textit{space of fields} \(\mathcal{A}\), but rather an integral in the \textit{space of paths} in \(\mathcal{A}\), the section \(\sigma_\eta\) allows us to eliminate the gauge group’s infinite volume in the corresponding path integral. We will then show that the gauge fixed action corresponding to the generalized gauge fixing defined by \(\eta\) can be obtained by means of the usual Faddeev-Popov method. One of the main advantages over the usual formulation is that the connection \(\eta\) is globally well-defined even when the topology of the fiber bundle \(\mathcal{A} \to \mathcal{A}/G\) is not trivial (Gribov’s obstruction). In this way, the existence of a generalized notion of gauge fixing demonstrates that the Faddeev-Popov method is still valid even in the presence of a Gribov’s obstruction. It is worth stressing that the extended connection \(\mathcal{A}\) encodes not only the gauge and ghost fields (as the connection \(\omega = A + c\) in Ref.\([3]\)), but also the definition of a global gauge fixing of the theory. We will then show that the proposed formalism allows us to recover the BRST transformations of the relevant fields without imposing the usual horizontality conditions (\([2, 4, 22]\)). This implies that the extended connection’s curvature \(\mathcal{F}\) does not necessarily have the horizontal form \(\mathcal{F} = F\). Moreover, we will show that the gauge field’s standard BRST transformation is valid only in a local trivialization of the fiber bundle \(\mathcal{A} \to \mathcal{A}/G\). We will then find the corresponding global generalization.

The paper is organized as follows. In section \(\text{II}\) we define the extended connection. In section \(\text{III}\) we study the curvature of this connection and show how the curvature forms induce the BRST transformations of the different fields. In section \(\text{IV}\) we study the relation between the gauge fixing connection and the ghost field. In section \(\text{V}\) we define the generalized gauge fixing at the level of path integrals. In section \(\text{VI}\) we calculate the gauge fixed action by means of the Faddeev-Popov method. In the final section we summarize the proposed formalism.
II. Extended Connection

Let $\mathcal{M}$ denote space-time. We will suppose that it is possible to define a foliation of $\mathcal{M}$ by spacelike hypersurfaces. This foliation is defined by means of a diffeomorphism $\iota: \mathcal{M} \to \mathbb{R} \times M$, where $M$ is a smooth 3-dimensional Riemannian manifold. We will also assume for the sake of simplicity that $M$ is compact. Let $G$ be a compact Lie group with a fixed invariant inner product in its Lie algebra $\mathfrak{g}$ and let $\mathcal{P} \to \mathcal{M}$ be a fixed $G$-principal bundle. Using the diffeomorphism $\iota$ and the fact that $\mathbb{R}$ is contractible, we can assume that the fiber bundle $\mathcal{P} \to \mathcal{M}$ over the space-time $\mathcal{M}$ is the pullback of a fixed $G$-principal bundle $P \to M$ over the space $M$

$$
\begin{array}{ccc}
\mathcal{P} & \cong & p_{st}^*(P) \\
| & & | \\
\mathcal{M} & \xrightarrow{p_{st}} & M,
\end{array}
$$

where $p_{st}: \mathcal{M} \cong \mathbb{R} \times M \to M$ is the projection onto the second factor.

We will denote by $\text{Ad}(P)$ the fiber bundle $P \times_G G \to M$ associated to the adjoint action of $G$ on itself and by $\text{ad}(P)$ the vector bundle $P \times_G \mathfrak{g} \to M$ associated to the adjoint representation of $G$ on $\mathfrak{g}$. The gauge group $\mathcal{G}$ is the group of vertical automorphisms of $P$. It can be naturally identified with the space of sections of $\text{Ad}(P)$. Its Lie algebra $\text{Lie}(\mathcal{G})$ is the space of sections of $\text{ad}(P)$. Its elements can be identified with $G$-equivariant maps $\mathfrak{g}: P \to \mathfrak{g}$.

In the case of a principal fiber bundle over a finite dimensional manifold with a compact structure group, there are three equivalent definitions of connections [9, Chapter 2]. In what follows, we will also consider connections on infinite dimensional spaces (see Refs. [17], [18]). For the general case, we will use the following as the basic definition [16].

**Definition 1.** Let $K$ be a Lie group with Lie algebra $\mathfrak{k}$ and let $E \xrightarrow{\pi} X$ be a $K$-principal bundle over a manifold $X$ (both $K$ and $X$ can have infinite dimension). A connection on $E$ is an equivariant distribution $H$, i.e. a smooth field of vector spaces $H_p \subset T_p \mathcal{E}$ (with $p \in E$) such that

1. For all $p \in E$ there is a direct sum decomposition

$$TE_p = H_p \oplus \ker d\pi_p.$$  

2. The field is preserved by the induced action of $K$ on $TE$, i.e.

$$H_{pg} = R_g^* H_p,$$

where $R_g^*$ denotes the differential of the right translation by $g \in K$. 
As in the finite dimensional case, we can assign to each connection a $K$-equivariant $\mathfrak{k}$-valued 1-form $\omega$ on $E$ such that $H_p = \ker \omega$. The action of $K$ on $\mathfrak{k}$ is the adjoint action.

A connection can also be considered a $K$-invariant splitting of the exact sequence

$$0 \to \ker d\pi \xrightarrow{\iota} TE \xrightarrow{\pi^*TX} \pi^*TX \to 0,$$

where $\pi^*TX \to E$ is the pullback induced by the projection $\pi : E \to X$ of the bundle $TX \to X$. In this sequence $H_p = \sigma(\pi^*T_{\pi(p)}X)$. Given a connection $\omega$ on $E$ the splitting considered in equation (2) induces an isomorphism $TE \simeq \pi^*TX \oplus \ker d\pi$.

The bundle $T_vE \doteq \ker d\pi$ is called the bundle of vertical tangent vectors. This bundle is intrinsically associated to the definition of principal bundles. Given a connection, the isomorphism $TE \simeq \pi^*TX \oplus \ker d\pi$ induces a projection $\Pi^\omega : TE \to T_vE$. By definition, the vertical cotangent bundle $T_v^*E$ is the annihilator of $\sigma(\pi^*TX)$. In other words, a $k$-form $\alpha$ is vertical if and only if it vanishes whenever one of its arguments is a vector in $\sigma(\pi^*TX)$. It is worth noticing that the definition of the bundle $T_v^*E$ requires a connection. The sections of $\Omega^k_v(E) \doteq \wedge^k T_v^*E$ are called vertical $k$-forms. By definition, the connection form $\omega$ is a vertical form.

Given a connection there is a decomposition of the de Rham differential on $E$

$$d_E = d_H + d_V$$

into a horizontal and a vertical part. The horizontal part corresponds to the covariant derivative. The vertical part is defined by the expression

$$d_V \alpha(X_1, \ldots, X_n) = d\alpha(\Pi^\omega_1 X_1, \ldots, \Pi^\omega_n X_n),$$

where $\alpha$ is a $(n-1)$-form. The vertical forms $\Omega^k_v(E)$ equipped with the vertical differential $d_V$ define the vertical complex.

Let’s now suppose that it is possible to define a global section $\sigma : X \to E$. This section defines a global trivialization $\varphi_\sigma : X \times K \to E$, where $\varphi_\sigma(x, g) = \sigma(x) \cdot g$. This trivialization induces a distinguished connection $\omega_\sigma$ on $E$ such that the pullback connection $\tilde{\omega}_\sigma = \varphi_\sigma^* \omega_\sigma$ coincides with the canonical flat connection on $X \times K$. Roughly speaking, the horizontal distribution defined by $\omega_\sigma$ at $p = \sigma(x)$ is tangent to the section $\sigma$. The vertical complex defined by the connection $\tilde{\omega}_\sigma$ can be naturally identified with the de Rham complex of $K$. This implies that $d_V = d_K$. Since the connection form $\tilde{\omega}_\sigma$ is a $\mathfrak{k}$-valued $K$-invariant vertical form, it can be identified with the Maurer-Cartan form $\theta_{MC}$ of the group $K$. On the contrary, a general connection $\omega$ defines a splitting of $TE$ which does not coincide with the splitting induced by the section $\sigma$. In other words, the horizontal distribution
defined by \( \omega \) is not tangent to \( \sigma \). In fact, the pullback connection \( \varphi^*\omega \) at \((x,g)\) can be written as a sum
\[
\varphi^*\omega = ad_{g^{-1}}\omega + \theta_{MC}, \tag{5}
\]
where \( \omega = \sigma^*\omega \in \Omega^1(X) \otimes \mathfrak{k} \) and \( \theta_{MC} \in \mathfrak{k}^* \otimes \mathfrak{k} \) is the Maurer-Cartan form of \( K \). \(^7\)

We will denote connections on the \( G \)-principal fiber bundle \( P \to M \) by the letter \( A \). In a local trivialization \( P|_U \cong U \times G \) we have an induced local \( \mathfrak{k} \)-valued 1-form \( A_U \). The forms \( A_U \) are the so-called gauge fields \(^1\) The configuration space of all connections is an affine space modelled on the vector space \( \Omega^1(M, \mathfrak{g}) \) consisting of 1-forms with values on the adjoint bundle \( ad(P) \). The gauge group \( \mathcal{G} \) acts on this configuration space by affine transformations. We will fix a metric \( g \) on \( M \) and an invariant scalar product \( tr \) on \( \mathfrak{g} \). These data together with the corresponding Hodge operator \( * \) induce a metric on \( \Omega^k(M, \mathfrak{g}) \). Hence, a metric can be defined in the spaces \( \Omega^k(M, \mathfrak{g}) \), \( k \geq 1 \), by means of the expression
\[
\langle \Omega_1, \Omega_2 \rangle = \int_M tr(\Omega_1 \ast \Omega_2). \tag{6}
\]

Since the action of \( \mathcal{G} \) on the configuration space of connections is not free, the quotient generally is not a manifold. This problem can be solved by using framed connections \(^9\). The letter \( \mathcal{A} \) will denote the space of framed connections of Sobolev class \( L^2_{l-1} \) for the metric defined in \(^6\), where \( l \) is a fixed number bigger than 2. The group \( \mathcal{G} \) is the group of gauge transformations of Sobolev class \( L^2_{l} \). The action of \( \mathcal{G} \) on \( \mathcal{A} \) is free (see Ref.[9, section 5.1.1]). We will denote by \( \mathcal{B} \) the quotient \( \mathcal{A}/\mathcal{G} \). Uhlenbeck's Coulomb gauge fixing theorem (see Ref.[9, section 2.3.3]) implies, for a generic metric \( g \), local triviality. Hence \( \mathcal{A} \to \mathcal{B} \) is a \( \mathcal{G} \)-principal bundle.

The initial geometric arena for our construction is the pullback \( G \)-principal bundle \( p^*(P) \to \mathcal{A} \times M \), which is obtained by taking the pullback of the bundle \( P \to M \) by the projection \( \mathcal{A} \times M \xrightarrow{p} M \). The bundle \( p^*(P) \) can be identified with \( \mathcal{A} \times \mathcal{P} \):

\[
p^*(P) = \mathcal{A} \times P \quad \xrightarrow{p} \quad P \quad \text{and} \quad \mathcal{A} \times M \quad \xrightarrow{p} \quad M.
\]

\(^1\)In a covariant framework, the connection \( A \) can be regarded as the spatial part of a connection \( \mathcal{A} \) on \( P \to M \). In fact, since we can identify \( \mathcal{P} \) with \( \mathbb{R} \times \mathcal{P} \) each connection \( \mathcal{A} \) has a canonical decomposition \( \mathcal{A} = A(t) + A_0(t) dt \), where \( A(t) \) is a time-evolving connection on \( \mathcal{P} \) and \( A_0(t) \) is a time-evolving section of \( ad(\mathcal{P}) = \mathcal{P} \times_G \mathfrak{g} \). The action of the gauge group \( \mathcal{G} \) on \( A_0 \) is induced by the natural action on associated bundles. This action is the restriction of the action of the automorphism group of \( \mathcal{P} \) to \( \{t\} \times M \).
The gauge group $G$ has an action on $\mathcal{A} \times M$ induced by its action on $\mathcal{A}$. This action is covered by the action of $G$ on $p^*(P)$ induced by its action on both $\mathcal{A}$ and $P$.

**Proposition 2.** The bundle $p^*(P) \to \mathcal{A} \times M$ induces a $G$-principal bundle $Q = (\mathcal{A} \times P)/G \to \mathcal{B} \times M$.

We therefore have the following tower of principal bundles:

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\begin{array}{ccc}
G & \longrightarrow & p^*(P) = \mathcal{A} \times P \\
\downarrow & & \downarrow q \\
G & \longrightarrow & Q = (\mathcal{A} \times P)/G \\
\downarrow & & \downarrow p \\
\mathcal{B} \times M & \end{array}
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The fiber bundle $p^*(P) = \mathcal{A} \times P \to \mathcal{A} \times M$ can be considered a universal family (parameterized by the space $\mathcal{A}$) of fiber bundles $P \to M$ with tautological connections $A$. The universal family $\mathbf{A}^U$ of tautological connections is defined in the following way. Let $(A, x)$ be a point of $\mathcal{A} \times M$. Then the elements of the fiber of $\mathcal{A} \times P$ over $(A, x)$ have the form $(A, p)$, with $p \in \pi^{-1}(x)$. Let us fix one of these elements. Let $v \in T(\mathcal{A} \times P)_{(A, p)}$ be a tangent vector such that $\pi_A(v) \in TM \subset T(\mathcal{A} \times M)$ (i.e., $v$ is tangent to a copy of $P$ in $\mathcal{A} \times P$). Then $\mathbf{A}^U(v) = A(v)$. We will write $\mathbf{H}^U$ for the distribution associated to the family $\mathbf{A}^U$. For each element $A \in \mathcal{A}$, the distribution $\mathbf{H}^U$ induces the distribution $H_A$ on $TP$ defined by the connection $A$.

The universal family of connections $\mathbf{A}^U$ allows us to define parallel transports along paths contained in any copy of $M$ inside $\mathcal{A} \times M$.

Let’s now pick a connection in the $\mathcal{G}$-principal bundle $\mathcal{A} \to \mathcal{B}$. This connection will be defined by means of a 1-form $\eta \in \Omega^1(\mathcal{A}) \otimes \text{Lie}(\mathcal{G})$. We will denote by $\mathbf{H}_\eta$ the corresponding equivariant distribution. This connection will define a generalized notion of gauge fixing. In fact, let’s suppose that it is possible to define a global gauge fixing section $\sigma : \mathcal{B} \to \mathcal{A}$. This section defines a global trivialization $\varphi_\sigma : \mathcal{B} \times \mathcal{G} \to \mathcal{A}$. One can then define an induced flat connection $\eta_\sigma$ on $\mathcal{A} \to \mathcal{B}$ such that the corresponding distribution $\mathbf{H}_{\eta_\sigma}$ is always tangent to $\sigma$. In other words, the pullback $\varphi_\sigma^* \eta_\sigma$ coincides with the canonical flat connection on $\mathcal{B} \times \mathcal{G}$ (see Ref.[19]). This shows that a global gauge fixing section $\sigma$ can always be expressed in terms of a flat connection.

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2In Ref.[19] the authors analyze the particular case of the Coulomb connection for a $SU(2)$ Yang-Mills theory on $S^3 \times \mathbb{R}$. The authors point out that in the absence of a global section, the gauge can be consistently fixed by means of such a connection.
On the contrary, a connection $\eta$ can not in general be integrated to a section. A local obstruction is the curvature and a global one the monodromy. Besides, it is always possible to define a global connection $\eta$, even when the topology of the fiber bundle $\mathcal{A} \to \mathcal{B}$ is not trivial (Gribov’s obstruction). In sections V and VI we will use the connection $\eta$ as a generalized gauge fixing for the path integral quantization of Yang-Mills theory. In particular, we will show that the connection $\eta$ does not have to be flat in order to induce a well defined gauge fixing.

Let’s consider now a particular example of a gauge fixing connection. Due to the affine structure of $\mathcal{A}$ there is a canonical diffeomorphism $T\mathcal{A} \simeq \mathcal{A} \times \Omega^1(M, g)$. We will consider $\Omega^1(M, g)$ with the inner product defined by (6). Since $\text{ad}(P)$ is a vector bundle associated to the principal bundle $P$, any connection $A$ on $P$ induces a covariant derivative $d_A : \Omega^k(M, g) \to \Omega^{k+1}(M, g)$. Let

$$d_A^* : \Omega^{k+1}(M, g) \to \Omega^k(M, g)$$

be the adjoint operator. This is a differential operator of first order. Define

$$\mathcal{H}_A : \text{Ker}\{d_A^* : \Omega^1(M, g) \to \Omega^0(M, g)\}. \quad (7)$$

Then $\mathcal{H}_A$ defines a connection on $\mathcal{A}$ called the Coulomb connection (see for example Ref. [9, p. 56]).

The distribution $\mathcal{H}_\eta$ together with $H^U$ define a smooth distribution $\widetilde{\mathcal{H}}$ on $Tp^*(P)$. If $(A, p) \in p^*(P)$, then

$$\widetilde{\mathcal{H}}_{(A, p)} = \mathcal{H}_\eta(A) \oplus H^U(A)(p).$$

**Proposition 3.** The distribution $\widetilde{\mathcal{H}}$ is transversal to the orbits of the action of $\mathcal{G}$ on $\mathcal{A} \times P$ and to the fibers of $p^*(P) \to \mathcal{A} \times P$.

**Proof.** Let $(A, p) \in p^*(P)$. Then we have two homomorphisms of vector spaces $\iota : g \to T_pP$ and $\kappa : \text{Lie}(\mathcal{G}) \to T_A\mathcal{A}$. These homomorphisms are induced by the principal bundle structures of $P$ and $\mathcal{A}$. For each point $p$ there is also a homomorphism of Lie algebras $\tau_p : \text{Lie}(\mathcal{G}) \to g$ given by $\tau_p(g) = g(p)$, where we use the identification of the elements of $\text{Lie}(\mathcal{G})$ with equivariant maps $g : P \to g$. With these definitions the tangent space to the orbit $F_{\mathcal{G}}$ of the action of $\mathcal{G}$ at the point $(A, p)$ is equal to

$$TF_{\mathcal{G}}(A, p) = \{v - v \in T_A\mathcal{A} \oplus T_pP \mid v = \kappa(g) \text{ and } v = \iota(\tau_p(g))\}. \quad (8)$$

The tangent spaces to the orbits are contained in the sum of the tangent spaces to the orbits in $\mathcal{A}$ and $P$. The proposition follows from the fact that the connections are transversal to these spaces. $\square$
Remark 4. The distribution $\tilde{\mathcal{H}}$ does not define a connection on $p^*(P) \to \mathcal{A} \times M$ due to the fact that the tangent space at $(A, p)$ has a decomposition

$$Tp^*(P)_{(A, p)} = TF_g(A, p) \oplus \mathcal{H}_\eta(A) \oplus H^U(A)(p) \oplus \iota(g),$$

being $\iota(g)$ the vertical subspace. Hence, the distribution $\tilde{\mathcal{H}}$ does not define vector spaces complementary to the vertical subspace $\iota(g)$. However there is a reason for the introduction of this distribution which is explained by the following Lemma.

Lemma 5. The distribution $\tilde{\mathcal{H}}$ is $G$-invariant and induces a connection $\mathcal{H}$ on $Q = (\mathcal{A} \times P)/G \to \mathcal{B} \times M$.

This lemma follows from the invariance of the distributions $\mathcal{H}_\eta$ and $H_U$.

Let $\mathcal{E}$ be the connection on the bundle $p^*(P) \to \mathcal{A} \times M$ obtained as the pullback of the connection $\mathcal{H}$ by the projection $\mathcal{A} \times M \to \mathcal{B} \times M$. This pullback can be understood either in the language of distributions or in the language of forms. Let’s consider the diagram

$$\begin{array}{ccc}
p^*(P) = \mathcal{A} \times P & \xrightarrow{q} & Q = \frac{\mathcal{A} \times P}{G} \\
& \mathcal{A} \times M & \xrightarrow{\pi} \mathcal{B} \times M
\end{array}$$

The map $q : p^*(P) \to Q$ induces a map $q_* : Tp^*(P) \to TQ$. At each point $(A, p)$ the subspace of $T_{(A, p)}p^*(P)$ which defines $\mathcal{E}$ is $q_*^{-1}(\mathcal{H}(A, p))$. The $g$-valued 1-form $\mathcal{A} \in \Omega^1(\mathcal{A} \times P) \otimes g$ associated to $\mathcal{E}$ is the pullback by $q$ of the 1-form associated to $\mathcal{H}$. We will now identify this distribution and this 1-form.

The distribution which defines the new connection at each point $(A, p)$ is the direct sum

$$\mathcal{E}_{(A, p)} = TF_g(A, p) \oplus \mathcal{H}_\eta(A) \oplus H^U(A)(p).$$

(9)

If $(A, p) \in \mathcal{A} \times P$ and $v = v_1 + v_2 \in T_A\mathcal{A} \oplus T_pP$, then we define a $g$-valued 1-form $\mathcal{A}$ on $\mathcal{A} \times P$ given by

$$\mathcal{A}(v) = A^U_{(A, p)}(v_2) + \eta(v_1)(p).$$

(10)

We will now show that the horizontal distribution defined by $\mathcal{A}$ is effectively given by (9).
Lemma 6. If $v \in TF_{(A,p)} \oplus \mathcal{H}_{(A,p)}$, then $\mathbb{A}(v) = 0$.

Proof. (i) If $v \in H^U(A) \subset \mathcal{H}_{(A,p)}$, then
\[
\mathbb{A}(v) = A^U_{(A,p)}(v) = A(v) = 0
\]
by definition of the connection $A$.

(ii) If $v \in \mathcal{H}_\eta(A) \subset \mathcal{H}_{(A,p)}$, then
\[
\mathbb{A}(v) = \eta(v)(p) = 0
\]
by definition of the connection $\eta$.

(iii) If $v = \kappa(g) - \iota(\tau_p(g)) \in TF_{(A,p)}$, then
\[
\mathbb{A}(v) = -A^U_{(A,p)}(\iota(\tau_p(g))) + \eta(\kappa(g))(p)
\]
\[
= -A(\iota(\tau_p(g))) + \eta(\kappa(g))(p)
\]
\[
= -\tau_p(g) + g(p)
\]
\[
= 0,
\]
where we have used that by definition of connection $A \circ \iota = id_g$ and $\eta \circ \kappa = id_{Lie(\mathfrak{g})}$. □

Remark 7. It is the gauge fixing connection $\mathcal{H}_\eta$ that allows us to make the decompositions (9) and (10) of the horizontal distribution $\mathcal{E}$ and the corresponding 1-form $\mathbb{A}$. The reason is that these kinds of decompositions require the choice of a complement to a subspace of a vector space.

Remark 8. An important difference with the work of Baulieu and Singer for topological Yang-Mills theory is that in Ref. [3] the connection $\omega$ is a natural connection, which is defined by using the orthogonal complements to the orbits of $G$. In order to define this orthogonal complements one uses the fact that the space $\mathbb{A} \times P$ has a Riemannian metric invariant under $\mathbb{G} \times G$ (see Ref. [1] for details). In our case the connection $\mathbb{A}$, being tautological in the factor $P$, is not natural in the factor $\mathbb{A}$, in the sense that the gauge fixing connection $\eta$ can be freely chosen. This freedom is in fact the freedom to choose the gauge.

III. Extended curvature and the BRST complex

In this section we will begin to consider the rich geometric structure induced by the connection $\mathcal{H}$. We will do this through the pullback form $\mathbb{A}$ and its curvature $\mathbb{F}$. Since we have a diffeomorphism $p^*(P) \simeq \mathbb{A} \times P$, the de Rham complex of $\mathfrak{g}$-valued forms on $p^*(P)$ is the graded tensor product of the de Rham complexes of $\mathbb{A}$ and $P$, i.e.
\[
\Omega^*(p^*(P)) \otimes \mathfrak{g} \simeq \Omega^*(\mathbb{A}) \otimes \Omega^*(P) \otimes \mathfrak{g}.
\]
(11)

This fact has two consequences. Firstly, the forms we are considering are naturally bigraded. Secondly, the exterior derivative $\Delta$ in $p^*(P)$ can be decomposed as $\Delta = \delta + d$, where $\delta$ and $d$ are the exterior derivatives
in $\mathcal{A}$ and $P$ respectively. Since the forms that we are considering are equivariant forms, the right complex to study these forms is

$$\Omega^*(\mathcal{A}) \otimes (\Omega^*(P) \otimes_{G} \mathfrak{g}).$$  \hspace{1cm} (12)

Since $$\Omega^*(\mathcal{A}) \otimes (\Omega^0(P) \otimes_{G} \mathfrak{g}) \simeq \Omega^*(\mathcal{A}) \otimes \mathcal{L}ie(\mathcal{G}),$$ (13)
the $\mathcal{L}ie(\mathcal{G})$-valued equivariant $k$-forms on $\mathcal{A}$ can be considered as elements of bidegree $(k, 0)$ of the complex (12). In particular the connection form $\eta$ defines an element of bidegree $(1, 0)$ of this complex.

Using the splitting of the exterior derivative and the decomposition of forms we obtain the following decomposition of the curvature $F$:

$$F = \Delta A + \frac{1}{2} [A, A] = F^{(2,0)} + F^{(1,1)} + F^{(0,2)},$$

where

$$F^{(2,0)} = \delta \eta + \frac{1}{2} [\eta, \eta] \equiv \phi,$$  \hspace{1cm} (14)

$$F^{(1,1)} = \delta A^U + d\eta + [A^U, \eta] \equiv \psi,$$  \hspace{1cm} (15)

$$F^{(0,2)} = dA^U + \frac{1}{2} [A^U, A^U] \equiv F^U.$$  \hspace{1cm} (16)

The $(0,2)$-form $F^U$ is the universal family of curvature forms corresponding to the universal family of connections $A^U$. In this sense, equation (16) is the extension to families of the usual Cartan’s structure equation. The $(2,0)$-form $\phi$ is the curvature of the connection $\eta$. The $(1,1)$-form $\psi$ is a mixed term which involves both the gauge fields and the gauge fixing connection. This last term shows that this construction mixes in a non-trivial way the geometric structures coming from the fiber bundles $P \rightarrow M$ and $\mathcal{A} \rightarrow \mathcal{B}$.

We will now decompose equations (14) and (15) in order to recover the usual BRST transformations of the gauge and ghost fields. Let $\Omega^k_\mathcal{V}(\mathcal{A})$ for $k > 0$ be the vertical differential forms induced by $\eta$. We define $\Omega^0_\mathcal{V}(\mathcal{A}) = C^\infty(\mathcal{A})$. The differential of the de Rham complex of $\mathcal{A}$ induces a differential $\delta_\mathcal{V}$ on the vertical forms. The complex $\Omega^*_\mathcal{V}(\mathcal{A}) \otimes \Omega^*(P) \otimes_{G} \mathfrak{g}$ will be termed the vertical complex.

This decomposition shows that we can identify the vertical complex with a subcomplex of $\Omega^*(\mathcal{A}) \otimes \Omega^*(P) \otimes_{G} \mathfrak{g}$. Via this identification, the $(1,0)$-form $\eta$ can be identified with a vertical form. In particular, we see that the $(1,1)$-forms have a decomposition in two terms, one of which is the part of degree $(1, 1)$ of the vertical complex.

We will now write the explicit decomposition of both sides of the equation (15). Let $\delta = \delta_\mathcal{V} + \delta_H$ be the decomposition of the de Rham differential on $\mathcal{A}$ induced by $\eta$. On degree $(0, *)$ the decomposition is given by the following definition. Let $p_\mathcal{V}$ and $p_H$ be the projectors onto the factors associated with the decomposition into vertical and horizontal forms. If we think of elements of $\Omega^0(\mathcal{A}) \otimes \Omega^1(P) \otimes_{G} \mathfrak{g}$ as
given on each copy \( \{A\} \times \mathcal{L}(\mathcal{G}) \) by the covariant derivative \( d_A : \mathcal{L}(\mathcal{G}) \rightarrow \Omega^1(P) \otimes \mathcal{L}(\mathcal{G}) \) associated to the connection \( A \). Recall that sections of \( \text{ad}(P) \) can be seen as equivariant functions \( P \rightarrow \mathfrak{g} \) and that the covariant derivative is \( d_A = d \circ \pi_A \), where \( d \) is the exterior derivative in \( P \) and \( \pi_A : T_P \rightarrow T_Q \) is the horizontal projection. These constructions are equivariant, which explains the codomain in equation (17).

The term \( d_{A^U} \eta \) is by definition the extension

\[
1 \otimes d_{A^U} : \Omega^1(\mathcal{A}) \otimes \mathcal{L}(\mathcal{G}) = \Omega^1(\mathcal{A}) \otimes (\Omega^0(P) \otimes_G \mathfrak{g}) \rightarrow \Omega^1(\mathcal{A}) \otimes \Omega^1(P) \otimes_G \mathfrak{g}
\]

applied to \( \eta \). Since the homomorphism \( 1 \otimes d_{A^U} \) acts on the second factor, it preserves vertical forms. It follows that \( d_{A^U} \eta = d \eta + [A^U, \eta] \) is a vertical form. From these remarks we see that the vertical summand of the left hand side of equation (15) is

\[
\delta_V A^U + d \eta + [A^U, \eta].
\]

We will consider now the right hand side of the equation (15). We will demonstrate the following proposition.

**Proposition 9.** The element \( \psi \) is horizontal for the connection \( \eta \).

**Proof.** Since the connection \( A \) is the pullback of a connection on the \( G \)-fiber bundle \( Q = (\mathcal{A} \times P)/\mathcal{G} \rightarrow \mathcal{A} \times \mathcal{G} \times M \) the same is true for the curvature \( \mathcal{F} \). If we denote \( \omega_{\mathcal{H}} \) and \( \mathcal{F}_{\mathcal{H}} \) for the connection and curvature forms of the distribution \( \mathcal{H} \) on \( Q \), then one has

\[
A = q^* \omega_{\mathcal{H}}, \quad \mathcal{F} = q^* \mathcal{F}_{\mathcal{H}},
\]

where \( q \) is the projection \( \mathcal{A} \times P \xrightarrow{\pi} Q = (\mathcal{A} \times P)/\mathcal{G} \).

If \( X \) is a vector tangent to the fibers \( TF_{\mathcal{G}} \) of the action of \( \mathcal{G} \) on \( \mathcal{A} \times P \) then the contraction \( \iota_X \mathcal{F} \) is equal to \( \iota_X q^* \mathcal{F}_{\mathcal{H}} = \iota_{q^*X} \mathcal{F}_{\mathcal{H}} \). Since \( X \) has the form \( X = (v, -v) \in TF_{\mathcal{G}} \) with \( TF_{\mathcal{G}} \) given by (8), then \( q_*X = 0 \). This results from the fact that the vectors tangent to \( \mathcal{G} \) are projected to zero when we take the quotient by the action of \( \mathcal{G} \). Hence, the contraction \( \iota_X \mathcal{F} \) of the curvature \( \mathcal{F} \) with a vector \( X = (v, -v) \) tangent to the fibers given by (8) is zero:
\( \iota_{(\nu, -\nu)} F = \iota_{q, (\nu, -\nu)} F_{\mathcal{H}} = 0. \) \hspace{1cm} (19)

An analysis of the different components of \( F \) shows that

- \( \iota_{(\nu, -\nu)} F^{(2,0)} = \iota_{(\nu)} F^{(2,0)} = 0 \) since \( F^{(2,0)} \) is induced by the connection \( \eta \) and \( \nu \) is vertical for this connection.
- \( \iota_{(\nu, -\nu)} F^{(0,2)} = \iota_{-\nu} F^{(0,2)} = 0 \) since \( F^{(0,2)} \) is induced by the connection \( A \) and \( -\nu \) is tangent to the fibers of \( p^*(P) \rightarrow \mathcal{A} \times P \).
- \( \iota_{\nu} F^{(1,1)} = 0 \) since \( -\nu \) is tangent to the fibers of \( p^*(P) \rightarrow \mathcal{A} \times P \).

From these remarks and equation (19) it follows that

\[
0 = \iota_{(\nu, -\nu)} F = \iota_{(\nu, -\nu)} F^{(1,1)} = \iota_{\nu} F^{(1,1)} = \iota_{\nu} \psi.
\]

The equation (15) then splits in the equations

\[
\begin{align*}
\delta_V A^U &= -d_A \nu - \eta, \hspace{1cm} (20) \\
\delta_H A^U &= \psi. \hspace{1cm} (21)
\end{align*}
\]

The equation for the \((2, 0)\)-form \( \phi \) can also be canonically decomposed in components belonging to the vertical and horizontal complexes. The differential \( \delta \) acting on elements of \( \Omega^1(\mathcal{A}) \otimes \text{Lie}(\mathcal{G}) = \Omega^1(\mathcal{A}) \otimes \Omega^0(P) \otimes G \) also has a decomposition \( \delta = \delta_V + \delta_H \), where \( \delta_V = \delta \circ p_V \) and \( \delta_H = \delta \circ p_H \). The horizontal part \( \delta_H \) corresponds to the covariant derivative with respect to the connection \( \eta \). Therefore we have a splitting

\[
\delta \eta = \delta_V \eta + \delta_H \eta \in \Omega^2(\mathcal{A}) \otimes (\Omega^0(P) \otimes G) = \Omega^2(\mathcal{A}) \otimes \text{Lie}(\mathcal{G}).
\]

By definition of the curvature \( \phi \) associated to the connection \( \eta \) we have

\[
\delta_H \eta = \phi. \hspace{1cm} (22)
\]

The vertical component of the equation is such that \( \delta_V \eta + \delta_H \eta = \delta \eta = \phi - \frac{1}{2} [\eta, \eta] \). We have then

\[
\delta_V \eta = -\frac{1}{2} [\eta, \eta]. \hspace{1cm} (23)
\]

In the next section we will show how the BRST transformations of the gauge and ghost fields can be obtained from equations (20) and (23) respectively.

### IV. The relationship between the gauge fixing connection and the ghost field

The proposed formalism allows us to further clarify the relationship between the gauge fixing and the ghost field. To do so, we shall first work in a local trivialization \( \varphi_{\sigma_i} : U_i \times \mathcal{A} \rightarrow \pi^{-1}(U_i) \) defined by a local gauge fixing section \( \sigma_i : U_i \rightarrow \mathcal{A} \) over an open subset \( U_i \subset \mathcal{A}/\mathcal{G} \). Let \( \eta \) be the pull-back by \( \varphi_{\sigma_i} \) of the connection form \( \eta \) restricted to \( \pi^{-1}(U_i) \).
As we have seen in section II, the connection form $\tilde{\eta}$ at $([A], g) \in U_i \times \mathcal{G}$ takes the form
\[
\tilde{\eta} = \text{ad}_g^{-1} \eta_i + \theta_{MC},
\]
where $\eta_i = \sigma_i^* \eta \in \Omega^1(U_i) \otimes \text{Lie}(\mathcal{G})$ is the local form of the connection $\eta$ and $\theta_{MC} \in \text{Lie}(\mathcal{G})^* \otimes \text{Lie}(\mathcal{G})$ is the Maurer-Cartan form of the gauge group $\mathcal{G}$ [7]. The Maurer-Cartan form satisfies the equation
\[
\delta G \theta_{MC} = -\frac{1}{2} [\theta_{MC}, \theta_{MC}].
\]
(25)

The formal resemblance between this equation and the BRST transformation of the ghost field led in Ref.[6] to the identification of $\delta_{BRST}$ and $c$ with the differential $\delta_g$ and the Maurer-Cartan form $\theta_{MC}$ of $\mathcal{G}$ respectively. Hence, equation (24) shows that the ghost field can be identified with the canonical vertical part of the gauge fixing connection $\eta$ expressed in a local trivialization.

We will now show that it is possible to recover the standard BRST transformation of the gauge field $\delta_{BRST}A = -d_A c$ from equation (20). To do so, we will first suppose that it is possible to define a global gauge fixing section $\sigma : \mathcal{B} \to \mathcal{A}$. As we have seen in section II the associated trivialization $\varphi_\sigma$ induces a distinguished connection $\eta_\sigma$ such that $\varphi_\sigma^* \eta_\sigma = \theta_{MC}$. Therefore, equation (20) yields in this trivialization
\[
\delta_g A^U = -d_A^U \theta_{MC}.
\]
(26)

This equation is an extension to families of the usual BRST transformation of the gauge field $A$. If a global gauge fixing section cannot be defined, then it is possible to show that the usual BRST transformation of $A$ is valid locally. In fact, since $\delta_V A^U$ is a vertical form, the substitution of the local decomposition (24) in equation (20) yields the BRST transformation (26). We can thus conclude that the usual BRST transformation of $A$ given by (26) is only valid in a local trivialization of $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. Therefore, equation (20) can be considered the globally valid BRST transformation of the gauge field $A$. In fact, we will now show that equation (20) plays the same role as the usual BRST transformation of $A$ given by (26).

Let’s now consider equation (20). According to the definition of connections, $\eta(\xi^\sharp) = \xi$, where $\xi^\sharp$ is the fundamental vector field in $T \mathcal{A}$ corresponding to $\xi \in \text{Lie}(\mathcal{G})$. In doing so, the usual gauge transformation of $A$ is recovered
\[
\delta A = (\delta_{BRST} A)(\xi) = -d_A(c(\xi)) = -d_A \xi.
\]
(27)

\[
\delta A^U = (\delta_V A^U)(\xi^\sharp) = -d_A^U (\eta(\xi^\sharp)) = -d_A^U \xi.
\]
This equation is the universal family of infinitesimal gauge transformations defined in (27). Therefore, equation (20) can be consistently considered the globally valid extension to families of the usual BRST transformation of $A$.

The identification of the ghost field with the canonical vertical part of the gauge fixing connection $\eta$ depends on a particular trivialization of the fiber bundle. Nevertheless, we will now show that it is also possible to understand the relationship between $c$ and $\eta$ without using such a trivialization. To do so it is necessary to introduce the Weil algebra of the gauge group $G$ (see Refs. [10], [13], [21]). The Weil algebra of a Lie algebra $\mathfrak{k}$ is the tensor product $W(\mathfrak{k}) = S^*\mathfrak{k} \otimes \Lambda^*\mathfrak{k}$ of the symmetric algebra $S^*\mathfrak{k}$ and the exterior algebra $\Lambda^*\mathfrak{k}$ of $\mathfrak{k}$ (where $\mathfrak{k}$ and $\mathfrak{k}^*$ are dual spaces). Let $T_a$ and $\vartheta^a$ be a base of $\mathfrak{k}$ and $\mathfrak{k}^*$ respectively. The Weil algebra is then generated by the elements $\theta^a = 1 \otimes \vartheta^a$ and $\zeta^a = \vartheta^a \otimes 1$.

The graduation is defined by assigning degree 1 to $\theta^a$ and degree 2 to $\zeta^a$. Let’s define the elements $\theta$ and $\zeta$ in $W(\mathfrak{k}) \otimes \mathfrak{k}$ as $\theta = \theta^a \otimes T_a$ and $\zeta = \zeta^a \otimes T_a$ respectively. In fact, the element $\theta$ is the Maurer-Cartan form $\theta_{MC}$ of the group. The Weil’s differential $\delta_W$ acts on these elements by means of the expressions

$$\delta_W \theta = \zeta - \frac{1}{2} [\theta, \theta],$$
$$\delta_W \zeta = -[\theta, \zeta].$$

These equations reproduce in the Weil algebra the Cartan’s structure equation and the Bianchi identity respectively. What is important to note here is that the connection $\eta$ in the $G$-principal bundle $\mathcal{A} \to \mathcal{A}/G$ can be defined as the image of $\theta_{MC}$ under a particular Chern-Weil homomorphism

$$\omega : (W(\mathfrak{Lie}(\mathcal{G})) \otimes \mathfrak{Lie}(\mathcal{G}), \delta_W) \longrightarrow (\Omega^* (\mathcal{A}) \otimes \mathfrak{Lie}(\mathcal{G}), \delta)$$
$$\theta_{MC} \mapsto \eta.$$

Therefore, the Weil algebra is a universal model for the algebra of a connection and its curvature. In this way, a gauge fixing of the theory by means of a connection $\eta$ can be defined by choosing a particular Chern-Weil’s immersion $\omega$ of the “universal connection” $\theta_{MC}$ into $\Omega^* (\mathcal{A}) \otimes \mathfrak{Lie}(\mathcal{G})$. Hence, the ghost field can be considered a universal connection whose different immersions $\omega$ define different gauge fixings of the theory.

We will now show that the usual BRST transformation of the ghost field can be recovered from equation (23). To do so, it is necessary to restrict the attention to the vertical complexes. Indeed, the connection $\eta \in \Omega^1 (\mathcal{A}) \otimes \mathfrak{Lie}(\mathcal{G})$ defines a homomorphism of differential algebras

$$\omega_V : (\mathfrak{Lie}(\mathcal{G})^* \otimes \mathfrak{Lie}(\mathcal{G}), \delta_\mathcal{G}) \to (\Omega^* V(\mathcal{A}) \otimes \mathfrak{Lie}(\mathcal{G}), \delta_V),$$
with \( \omega_V(\theta_{MC}) = \eta \). This means that \( \omega_V(\delta_\theta \alpha) = \delta_V(\omega_V(\alpha)) \) for \( \alpha \in \mathfrak{Lie}(G)^* \otimes \mathfrak{Lie}(G) \). Therefore, equation (23) yields

\[
\delta_V(\omega_V(\theta_{MC})) = -\frac{1}{2} [\omega_V(\theta_{MC}), \omega_V(\theta_{MC})]
\]

\[
\omega_V(\delta_\theta \theta_{MC}) = \omega_V(-\frac{1}{2} [\theta_{MC}, \theta_{MC}]).
\]

Therefore

\[
\delta_\theta \theta_{MC} = -\frac{1}{2} [\theta_{MC}, \theta_{MC}],
\]

which coincides with the ghost field’s BRST transformation.

**Remark 10.** Contrary to what is commonly done in order to reobtain the BRST transformations for the ordinary (non-topological) Yang-Mills case, it has not been necessary to impose the horizontality conditions \( \phi = \psi = 0 \) on the extended curvature \( \mathcal{F} \) (see for example Refs. [2], [4], [22]).

### V. Path integral gauge fixing

#### V.1. Usual gauge fixing

The central problem in the quantization of Yang-Mills theory is computing the transition amplitudes

\[
\langle [A_0] | [A_1] \rangle = \int_{T^* \mathcal{P}([A_0],[A_1])} \exp\{iS\} \mathcal{D}A \mathcal{D}\pi,
\]

where \( S \) is the canonical action, \( \mathcal{D}A \) is the Feynman measure on the space of paths

\[
\mathcal{P}([A_0],[A_1]) = \{ \gamma : [0,1] \rightarrow \mathcal{A}/\mathcal{G} | \gamma(i) = [A_i], i = 0,1 \},
\]

in \( \mathcal{A}/\mathcal{G} \) and \( \mathcal{D}\pi \) is a Feynmann measure in the space of moments. The canonical Yang-Mills’s action is given by the expression

\[
S = \int dt \int d^3x \left( \dot{A}_a^k \pi_a^k - \mathcal{H}_0 (\pi_a^k, B_a^k) - A_0^a \phi_a \right),
\]

with \( \pi_a^k = F_a^{ko} \) and \( B_a^k = \frac{1}{2} \varepsilon_{kmn} F_m^{an} \) (where \( F_m^{an} \) are the field strengths). The Yang-Mills Hamiltonian \( \mathcal{H}_0 (\pi_a^k, B_a^k) \) is

\[
\mathcal{H}_0 (\pi_a^k, B_a^k) = \frac{1}{2} [\pi_a^k \pi_a^k + B_a^k B_a^k],
\]

and the functions \( \phi_a \) are

\[
\phi_a = -\partial_k \pi_a^k + f_{ab}^c \pi_c^k A_b^k.
\]

The pairs \( (A_a^k, \pi_a^k) \) are the canonical variables of the theory. The temporal component \( A_0^a \) is not a dynamical variable, but the Lagrange multiplier for the generalized Gauss constraint \( \phi_a = 0 \).

The geometry of the quotient space \( \mathcal{A}/\mathcal{G} \) is generally quite complicated. The usual approach is to replace the integral (28) with an integral over the space of paths in the affine space \( \mathcal{A} \). To do so, one must
pick two elements $A_i \in \pi^{-1}[A_i]$ in the fibres $[A_i] \in \mathcal{A}/\mathcal{G}$ ($i = 0, 1$). Then one replaces the integral (28) by
\[
\langle A_0 \mid A_1 \rangle = \int_{T^*P(A_0, A_1)} \exp\{iS\} \mathcal{D}A \mathcal{D}\pi,
\]
where the integral is now defined on the cotangent bundle of the following space of paths in $\mathcal{A}$
\[
P(A_0, A_1) = \{\gamma : [0, 1] \to \mathcal{A} \mid \gamma(0) = A_0, \gamma(1) = A_1\}.
\]

The problem with this approach is that it introduces an infinite volume in the path integral, which corresponds to the integration over unphysical degrees of freedom. The projection $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{G}$ induces a projection
\[
\tilde{\pi} : \mathcal{P}(A_0, A_1) \to \mathcal{P}([A_0], [A_1]).
\]
The path group
\[
\widetilde{\mathcal{G}} = \{g(t) : [0, 1] \to \mathcal{G} \mid g(0) = g(1) = id\}
\]
acts on $\mathcal{P}(A_0, A_1)$ by pointwise multiplication and the fibers of $\tilde{\pi}$ consist of the orbits of the action of $\widetilde{\mathcal{G}}$. Since the action $S$ is invariant under the action of $\widetilde{\mathcal{G}}$, one needs to extract the volume of this group from the integral (32). In order to get rid of this infinite volume the usual approach is to fix the gauge by defining a section $\sigma : \mathcal{A}/\mathcal{G} \to \mathcal{A}$ such that $\sigma([A_i]) = A_i, i = 0, 1$. The gauge fixing section $\sigma$ induces a map $\tilde{\sigma}$
\[
\mathcal{P}(A_0, A_1) \xrightarrow{\tilde{\sigma}} \mathcal{P}([A_0], [A_1]),
\]
defined by $\tilde{\sigma}(\gamma) = \sigma \circ \gamma$, which is a section of $\tilde{\pi}$. This section $\tilde{\sigma}$ induces a trivialization
\[
\mathcal{P}(A_0, A_1) \simeq \mathcal{P}([A_0], [A_1]) \times \widetilde{\mathcal{G}}
\]
and a similar decomposition at the level of cotangent bundles. The ghost field appears when one computes the Jacobian which relates the corresponding measures. One can then use Fubini’s theorem in order to extract the irrelevant and problematic factor.

It is worth noting that the definition of a gauge fixing section $\sigma$ in the $\mathcal{G}$-principal fiber bundle of fields $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ is only an auxiliary step for defining a section $\tilde{\sigma}$ of the $\widetilde{\mathcal{G}}$-projection $\mathcal{P}(A_0, A_1) \xrightarrow{\tilde{\pi}} \mathcal{P}([A_0], [A_1])$ in the space of paths where the path integral is actually defined.

V.2. Generalized gauge fixing. We will now consider in which sense the connection $\eta$ can be used to fix the gauge. This gauge fixing will be globally well-defined, even if there is a Gribov’s obstruction. We shall begin by considering paths in $\mathcal{A}$ such that the initial condition
$A_0$ is fixed and the final condition is defined \textit{only up to a gauge transformation} (see Ref. [19, p. 123]). This means that the final condition can be any element of the final fiber $\pi^{-1}[A_1]$. The corresponding space of paths is

$$\mathcal{P}(A_0, \pi^{-1}[A_1]) = \{\gamma : [0, 1] \to A | \gamma(0) = A_0, \pi(\gamma(1)) = [A_1]\}.$$ 

The relevant path group is now

$$\mathcal{PG} = \{g(t) : [0, 1] \to G | g(0) = \text{id}_G\}.$$ 

This group acts on $\mathcal{P}(A_0, \pi^{-1}[A_1])$. This action defines the projection

$$\pi : \mathcal{P}(A_0, \pi^{-1}[A_1]) \to \mathcal{P}([A_0], [A_1]).$$ 

It is easy to show that the action of the path group $\mathcal{PG}$ on $\mathcal{P}(A_0, \pi^{-1}[A_1])$ is free. We will not need to assume that it is a principal bundle.

The gauge fixing by means of the connection $\eta$ is defined by taking parallel transports along paths in $A/G$ of the initial condition $A_0 \in \pi^{-1}[A_0]$ (as has already been suggested in Ref. [19]). This procedure defines a section $\sigma_\eta$ of the projection $\pi$

$$\mathcal{P}(A_0, \pi^{-1}[A_1]) \xrightarrow{\sigma_\eta} \mathcal{P}([A_0], [A_1]).$$ 

$$\sigma_\eta (\gamma) = \eta(\dot{\gamma}(t)) = 0, \forall t. \quad (34)$$ 

In local bundle coordinates this condition defines a non-linear ordinary equation. The explicit form of this equation is given in (57) (see Ref. [18] for details). In the case of the Coulomb connection defined in (7), equation (34) becomes

$$d^*_\gamma(t) \dot{\gamma}(t) = 0, \forall t. \quad (35)$$ 

This equation can be expressed in local terms on $M$ and $P$ for each $t$.

If $\pi : X \to X/G$ is a quotient space by a principal action, then any section $\sigma : X/G \to X$ induces a global trivialization $\Phi : X/G \times G \cong X$, where $\Phi([x], g) = \sigma([x]) \cdot g$. It can be shown that $\Phi$ is a diffeomorphism. It follows that the space of paths $\mathcal{P}(A_0, \pi^{-1}[A_1])$ can be factorized as

$\sigma_\eta ¥\cdot ¥ ¥\cdot ¥ \cdot ¥ \cdot ¥ ¥\cdot ¥$
Extended connection in Yang-Mills theory

\[ \mathcal{P}([A_0], [A_1]) \times \mathcal{P}\mathcal{G}. \] A similar decomposition is induced at the level of cotangent bundles.

Strictly speaking the path integral is not an integral on the space of fields \( \mathcal{A} \), but rather an integral on the space of paths. Hence, the section \( \sigma_\eta \) induced by the connection \( \eta \) suffices to get rid of the infinite volume of the path group \( \mathcal{P}\mathcal{G} \).

Remark 11. The generalized gauge fixing can be also be defined as the null space of a certain functional as follows. The Lie algebra \( \mathfrak{Lie}(\mathcal{G}) \) of the gauge group \( \mathcal{G} \) can be identified with the sections of the adjoint bundle \( \text{ad}(P) \). The invariant metric on \( \mathfrak{g} \) induces a metric \( <, >_{\text{ad}(P)} \) on \( \text{ad}(P) \). Using this metric we define a \( \mathcal{G} \)-invariant metric on \( \mathfrak{Lie}(\mathcal{G}) = \Gamma(\text{ad}(P)) \) by

\[
\langle \sigma_1, \sigma_2 \rangle_{\mathfrak{Lie}(\mathcal{G})} = \int_M <\sigma_1(p), \sigma_2(p)>_{\text{ad}(P)} \, dx. \tag{36}
\]

Then we define the functional \( \mathcal{F}: \mathcal{P}(A_0, \pi^{-1}[A_1]) \to \mathbb{R} \) by

\[
\mathcal{F}(\gamma) = \int_0^1 \|\eta(\dot{\gamma}(t))\|^2_{\mathfrak{Lie}(\mathcal{G})} \, dt \tag{37}
\]

This is a positive functional and the image of the section \( \sigma_\eta \) is the null space of \( \mathcal{F} \).

VI. Faddeev-Popov method revisited

We will now proceed to implement the proposed generalized gauge fixing at the level of the path integral. To do so, we will show that the usual Faddeev-Popov method can also be used with this generalized gauge fixing. We will then start by introducing our gauge fixing condition at the level of the transition amplitude

\[
\langle A_0 | \pi^{-1}[A_1] \rangle = \int_{T^*\mathcal{P}(A_0, \pi^{-1}[A_1])} \exp\{iS\} \mathcal{D}\mathcal{A}\mathcal{D}\mathcal{\pi}. \tag{38}
\]

The first possible form of the gauge fixing condition is \( \delta(\mathcal{F}(\gamma)) \) where \( \delta \) is the Dirac delta function on \( \mathbb{R} \) and \( \mathcal{F}(\gamma) \) the functional \( \tag{37} \). This form is mathematically consistent and does not require any product of distributions. This gauge fixing condition has the direct exponential representation

\[
\delta(\mathcal{F}(\gamma)) = \int d\lambda e^{i\lambda \mathcal{F}(\gamma)} = \int d\lambda e^{i\lambda \int_0^1 \|\eta(\dot{\gamma}(t))\|^2_{\mathfrak{Lie}(\mathcal{G})} \, dt} = \int d\lambda e^{i\lambda \int_0^1 \|\eta(\dot{\gamma}(t))\|^2_{\mathfrak{g}} \, dt}
\]

The second form is based on the elementary observation that the integral of a continuous positive function is zero if and only if the
function is zero at all points. One can then define the gauge fixing condition

$$\delta(\eta(\dot{\gamma})) = \lim_{N} \prod_{k=1}^{N} \delta_{\text{Lie}}(g)(\eta(\dot{\gamma}(t_k)))$$

$$= \lim_{N,M} \prod_{k=1}^{N} \prod_{j=1}^{M} \delta_{g}(g(\dot{\gamma}(t_k))(x_j)),$$

where $\delta_{\text{Lie}}(g)$ is the delta function on $\text{Lie}(g)$ defined as an infinite product of the Dirac delta $\delta_{g}$ on $g$. If $T_\alpha$ is a fixed basis of $g$, we can write $\delta_{g}(g(\dot{\gamma}(t_k))(x_j))$ in terms of $\delta(\eta(\dot{\gamma}(t_k))(x_j)^\alpha)$, where now the delta function is the usual delta function on $\mathbb{R}^3$.

As usual we define the element $\Delta^{-1}[\gamma]$ as

$$\Delta^{-1}[\gamma] = \int_{g \in g} Dg' \delta(\eta(g'g')),$$  \hspace{1cm} (40)

where $\gamma g'$ denotes the right action of $\mathcal{PG}$ on $\mathcal{P}(A_0, \pi^{-1}[A_1])$. 

**Proposition 12.** The element $\Delta^{-1}[\gamma]$ is $\mathcal{PG}$ invariant.

**Proof.**

$$\Delta^{-1}[\gamma] = \int_{g \in g} Dg' \delta(\eta(g'g')) = \int_{g \in g} D(g'g') \delta(\eta(g'g'))$$

$$= \int_{g \in g} D(g'g') \delta(\eta(g'g')) = \Delta^{-1}[\gamma].$$

We can then express the number one in the following way:

$$1 = \Delta[\gamma] \int_{g \in g} Dg' \delta(\eta(g'g'))$$  \hspace{1cm} (41)

Roughly speaking, the element $\Delta[\gamma]$ corresponds to the determinant of the operator which measures the gauge fixing condition’s variation under infinitesimal gauge transformations.

It can be shown that the element $\Delta[\gamma]$ is never zero. The local gauge fixing condition $\eta(\dot{\gamma}(t)) = 0$ induces a well-defined section $\sigma_\eta$ of the $\mathcal{PG}$-projection $\pi : \mathcal{P}(A_0, \pi^{-1}[A_1]) \rightarrow \mathcal{P}([A_0], [A_1])$ in the space of paths. Since the sub-manifold defined by the image of $\sigma_\eta$ is by definition transversal to the action of $\mathcal{PG}$, an infinitesimal gauge transformation of the gauge fixing condition $\eta(\dot{\gamma}(t)) = 0$ will be always non-trivial. In this way we can argue that the element $\Delta[\gamma]$ is never zero. This fact ensures that the connection $\eta$ induces a well-defined global gauge fixing, even when the topology of the fiber bundle $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}$ is not trivial.
By inserting (41) in (38) we obtain

$$\langle A_0 | \pi^{-1}[A_1] \rangle = \int_{T^*P(A_0, \pi^{-1}[A_1])} \mathcal{D}A \mathcal{D} \pi \Delta[\gamma] \int_{\mathcal{P}g} \mathcal{D}g' \delta(\eta(g')) \exp\{iS\}.$$  

If we now perform in the usual manner a gauge transformation taking $\gamma g'$ to $\gamma$ we obtain

$$\langle A_0 | \pi^{-1}[A_1] \rangle = \int_{\mathcal{P}g} \mathcal{D}g' \int_{T^*P(A_0, \pi^{-1}[A_1])} \mathcal{D}A \mathcal{D} \pi \Delta[\gamma] \delta(\eta(\dot{\gamma})) \exp\{iS\},$$

where we have used that $\mathcal{D}A \mathcal{D} \pi$, $\Delta^{-1}[\gamma]$ and $S$ are gauge invariant. In this way we have isolated the infinite volume of the path group $\mathcal{P}g$.

We will now follow the common procedure for finding the new terms in the action coming from the Dirac delta $\delta(\eta(\dot{\gamma}))$ and the element $\Delta[\gamma]$.

In order to find an integral representation of the gauge condition’s Dirac delta $\delta(\eta(\dot{\gamma}))$ we will start by finding the integral representation of the Dirac delta function $\delta_{\text{Lie}(\mathcal{G})}$ on $\text{Lie}(\mathcal{G})$ used in (39). If $\xi \in \text{Lie}(\mathcal{G})$, the Dirac delta $\delta_{\text{Lie}(\mathcal{G})}(\xi)$ defined as

$$\delta_{\text{Lie}(\mathcal{G})}(\xi) = \lim_M \prod_{j=1}^M \delta_{\mathcal{G}}(\xi(x_j)),$$

can be expressed in terms of the integral representations of the Dirac delta $\delta_{\mathcal{G}}(\xi(x_j))$ on $\mathcal{G}$ as

$$\delta_{\text{Lie}(\mathcal{G})}(\xi) = \lim_M \prod_{j=1}^M \int \mathcal{D}\lambda(x_j) e^{i \sum_{j=1}^M \langle \lambda(x_j), \xi(x_j) \rangle_{\mathcal{G}}},$$

$$= \int \mathcal{D}\lambda e^{i \int dx \langle \lambda(x), \xi(x) \rangle_{\mathcal{G}}},$$

$$= \int \mathcal{D}\lambda e^{i \langle \lambda, \xi \rangle_{\text{Lie}(\mathcal{G})}},$$

where $\lambda$ is a section of $\text{ad}(P) = P \times_G \mathfrak{g}$ and $\mathcal{D}\lambda = \lim_M \prod_{j=1}^M d\lambda(x_j)$. The Dirac delta $\delta(\eta(\dot{\gamma}))$ of the gauge fixing condition can then be
expressed as
\[
\delta(\eta(\dot{\gamma})) = \lim_{N} \prod_{k=1}^{N} \delta_{\mathcal{Lie}(\mathcal{G})}(\eta(\dot{\gamma}(t_k)))
\]
\[
= \lim_{N} \prod_{k=1}^{N} \int \tilde{\mathcal{D}} \lambda_k e^{\int \lambda_k} \sum_{k=1}^{N} \lambda_k \eta(\dot{\gamma}(t_k))(x)_{\mathcal{Lie}(\mathcal{G})}
\]
\[
= \int \mathcal{D} \lambda e^{\int \lambda \eta(\dot{\gamma}(t))} \mathcal{D} \lambda dt = \int \mathcal{D} \lambda e^{\int \lambda \eta(\dot{\gamma}(t))} dx dt,
\]
where \( \lambda \) is a time-evolving section of \( \text{ad}(P) = P \times G \). The final measure \( \mathcal{D} \lambda \) is then
\[
\mathcal{D} \lambda = \lim_{N} \prod_{k=1}^{N} \tilde{\mathcal{D}} \lambda_k = \lim_{N,M} \prod_{k=1}^{N} \prod_{j=1}^{M} d\lambda_k(x_j).
\]

Let’s now compute explicitly the element \( \Delta[\gamma] \). Let \( X \) be an element of the Lie algebra \( \mathcal{L}ie(\mathcal{P}G) \) identified with the tangent space of \( \mathcal{P}G \) at the identity element. Given a path \( \gamma(t) \in \mathcal{P}(A_0, \pi^{-1}[A_1]) \), one must calculate the variation of \( \eta(\dot{\gamma}) \) under an infinitesimal gauge transformation defined by \( X \in \mathcal{L}ie(\mathcal{P}G) \). Let \( u \to k_u \) be the uniparametric subgroup of \( \mathcal{P}G \) generated by \( X \) by means of the exponential map \( \exp : \mathcal{L}ie(\mathcal{P}G) \to \mathcal{P}G \). We have then \( X = \frac{d}{du} k_u \big|_{u=0} \in \mathcal{L}ie(\mathcal{P}G) \).

**Remark 13.** The gauge fixing condition has a natural interpretation in terms of the geometry of \( \mathcal{P}(A_0, \pi^{-1}[A_1]) \). The tangent space \( T_\gamma \mathcal{P}(A_0, \pi^{-1}[A_1]) \) to \( \mathcal{P}(A_0, \pi^{-1}[A_1]) \) at a path \( \gamma \) can be identified with the sections of the pullback \( \gamma^*(TA) \). The connection \( \eta \) induces a map from \( T_\gamma \mathcal{P}(A_0, \pi^{-1}[A_1]) \) to the Lie algebra of \( \mathcal{P}G \). The tangent field \( \dot{\gamma} \) represents a marked point in \( T_\gamma \mathcal{P}(A_0, \pi^{-1}[A_1]) \). The gauge fixing condition means that we will only consider paths such that the connection \( \eta \) vanishes on the marked point \( \dot{\gamma} \).

Using the description of the tangent spaces to path spaces given in the previous remark we can identify \( X \) with a map \( t \to X_t \) with \( 0 \leq t \leq 1 \) and \( X_t \in \mathcal{L}ie(\mathcal{G}) \). In order to find an expression for the vectors \( X_t \) one must take into account that \( k_u \) describes a family of paths in the gauge group \( \mathcal{G} \) parameterized by \( t \). This means that \( k_u = g_u(t) \subset \mathcal{G} \) for \( u \) fixed. To emphasize this, let us write \( k_u(t) \). For a given \( t \), the vector \( X_t \in \mathcal{L}ie(\mathcal{G}) \) is then given by \( X_t = \frac{d}{du} k_u(t) \big|_{u=0} \). If \( \gamma \in \mathcal{P}(A_0, \pi^{-1}[A_1]) \) we must compute
\[
\frac{d}{du} \left( \eta(\gamma(t)) k_u(t) \left( \frac{d}{dt} R_{k_u(t)}(\dot{\gamma}(t)) \right) \right) \bigg|_{u=0}.
\]
At a fixed time $t_0$ the time derivative in (42) is equal to
\begin{align*}
\frac{d}{dt} R_{k_u(t)\gamma}(t) \bigg|_{t=t_0} &= \frac{d}{dt} R_{k_u(t_0)k_u(t_0)^{-1}k_u(t)} \gamma(t) \bigg|_{t=t_0} \\
&= \left( \frac{d}{dt} R_{k_u(t)} \gamma(t) \bigg|_{t=t_0} \right) \\
&\quad + \frac{d}{dt} R_{k_u(t_0)^{-1}k_u(t)} \left( R_{k_u(t)} \gamma(t_0) \right) \bigg|_{t=t_0} \\
&= dR_{k_u(t_0)}(\dot{\gamma}(t_0)) + \iota(\gamma(t_0)k_u(t_0))(X_u),
\end{align*}
where $X_u = k_u(t_0)^{-1} \frac{d}{dt} k_u(t) \bigg|_{t=t_0}$. The first term in the last equation is the differential of the action of $k_u(t)$. We have used that the differential of a function $f$ is defined as $df(X) = \left. \frac{df(\gamma(t))}{dt} \right|_{t=t_0}$ with $X = \left. \frac{d\gamma(t)}{dt} \right|_{t=t_0}$.

The second term is the homomorphism from the Lie algebra of $\mathcal{G}$ to the vertical vector fields (see the appendix [A] for detailed definitions).

Let's apply the connection form $\eta$ to each term in equation (43).

Firstly, we have
\begin{equation}
\eta_{\gamma(t_0)k_u(t_0)}(dR_{k_u(t_0)}(\dot{\gamma}(t_0))) = \text{Ad}(k_u^{-1}(t_0))\eta_{\gamma(t_0)}(\dot{\gamma}(t_0)).
\end{equation}

The equality (44) follows from one of the connection's defining properties (see (53) in the appendix [A]). The second term is
\begin{equation}
\eta_{\gamma(t_0)k_u(t_0)} \left( \iota(\gamma(t_0)k_u(t_0))(X_u) \right) = X_u
\end{equation}
where we have used equation (54). The infinitesimal variation defined by $X \in \text{Lie}(\mathcal{P}\mathcal{G})$ is given by the sum of
\begin{align*}
\frac{d}{dt} \left( \text{Ad}(k_u(t_0)^{-1})\eta_{\gamma(t_0)}(\dot{\gamma}(t_0)) \right) \bigg|_{t=0} &= \text{Ad}(-X_{t_0})\eta_{\gamma(t_0)}(\dot{\gamma}(t_0)) \\
&= [-X_{t_0}, \eta_{\gamma(t_0)}(\dot{\gamma}(t_0))].
\end{align*}
and \( \frac{d}{du}X_u|_{u=0} \). Let’s now calculate this last term:

\[
\frac{d}{du}X_u|_{u=0} = \left. \frac{d}{du} \left( k_u(t_0)^{-1} \frac{dk_u(t)}{dt} \right) \right|_{u=0, t=t_0} = \left. \left( -k_u(t_0)^{-2} \frac{dk_u(t_0)}{du} \frac{dk_u(t)}{dt} + k_u(t_0)^{-1} \frac{d}{du} \frac{dk_u(t)}{dt} \right) \right|_{u=0, t=t_0} = -k_0(t_0)^{-2} \frac{dk_0(t_0)}{du} \left. \frac{dk_0(t)}{dt} \right|_{t=t_0} + k_0(t_0)^{-1} \frac{d}{dt} \left[ \frac{dk_0(t)}{du} \right]_{u=0} \left. \right|_{t=t_0} = \dot{X}_t(t_0),
\]

where in the last step we used that \( k_0(t) = \text{id}_g \forall t \).

The infinitesimal variation of the gauge fixing condition is then

\[
\frac{d}{du} \left( \eta_\gamma(t) k_u(t) \left( \frac{d}{dt} R_{k_u(t)} \gamma(t) \right) \right) \bigg|_{u=0} = [-X_t, \eta_\gamma(t) \langle \dot{\gamma}(t) \rangle] + \dot{X}_t.
\]

This expression defines a linear endomorphism \( M_\gamma \) in \( \text{Lie}(\mathcal{P}\mathcal{G}) \) for each path \( \gamma(t) \). Equivalently, it defines a linear endomorphism \( M_\gamma(t) \) in \( \text{Lie}(\mathcal{G}) \) for each \( t \).

In order to find the exponential representation of the element \( \Delta [\gamma] \), we will introduce a Grassmann algebra generated by the anticommuting variables \( c \) and \( \bar{c} \). By following the common procedure, we can express the element \( \Delta [\gamma] \) as

\[
\Delta [\gamma] = \int \mathcal{D} \bar{c} \mathcal{D} c \left( \frac{\partial}{\partial x} M_\gamma(t) c \right) dx dt,
\]

where

\[
\mathcal{D} c = \lim_{N} \prod_{k=1}^{N} \mathcal{D} c_k = \lim_{N,M} \prod_{k=1}^{N} \prod_{j=1}^{M} dc_k(x_j),
\]

and the same for \( \mathcal{D} \bar{c} \) (see Ref.\cite{24} for a precise definition of \( c \) and \( \bar{c} \)).

By gathering all the pieces together, the path integral takes the form

\[
\langle A_0|\pi^{-1}[A_1] \rangle = \int_{\mathcal{D} \bar{c} \mathcal{D} c} \mathcal{D} \pi \mathcal{D} \lambda \mathcal{D} c \mathcal{D} \bar{c} \exp \{ i S_{gf} \}, \tag{46}
\]

where \( S_{gf} \) is the gauge fixed action

\[
S_{gf} = \int d^4 x \left( \dot{A}_k \pi^k - \mathcal{H}_0 - A_0 \phi + \langle \lambda, \eta(\dot{\gamma}) \rangle_\theta - i \langle \bar{c}, M_\gamma c \rangle_\theta \right). \tag{46}
\]

By explicitly introducing the indices of the Lie algebra \( \mathfrak{g} \), the endomorphisms \( M_\gamma(t)(x) \) can be expressed as

\[
M_\gamma(t)(x)^c_a = -\eta_\gamma(t)(\dot{\gamma}(t))^b f^c_{ab} + \delta^c_a \partial_0,
\]

where \( f^c_{ab} \) are the structure constants of \( \mathfrak{g} \).
where \( f_{ab}^c \) are the structure constants of \( g \). The gauge fixed action takes then the form

\[
S_{gf} = \int d^4x \left( A_a^a \dot{A}_a^k - \mathcal{H}_0 - A_0^a \phi_0 + \lambda_a \eta(\dot{\gamma})^a + i \overline{c}_a \eta(\dot{\gamma})^b f_{ab}^c c_c - i \overline{c}_a \dot{c}_a \right).
\]

The last term can be recast as \( +i \overline{c}_a \dot{c}_a \). Therefore, this term can be interpreted as the kinetic term corresponding to the new pair of canonical variables \((c, \overline{c})\).

**VII. Conclusions**

The principal aim of this work was to study the quantization of Yang-Mills theory by using an extended connection \( \mathcal{A} \) defined in a properly chosen principal bundle. This connection unifies the three fundamental geometric objects of Yang-Mills theory, namely the gauge field, the gauge fixing and the ghost field. This unification is an extension of the known fact that the gauge and ghost fields can be assembled together as \( \omega = A + c \) [3].

The first step in the unification process was to generalize the gauge fixing procedure by replacing the usual gauge fixing section \( \sigma \) with a gauge fixing connection \( \eta \) in the \( G \)-principal bundle \( \mathcal{A} \rightarrow \mathcal{A}/G \). We have then shown that the connection \( \eta \) also encodes the ghost field of the BRST complex. In fact, the ghost field can either be considered the canonical vertical part of \( \eta \) in a local trivialization or the universal connection in the gauge group’s Weil algebra. The unification process continues by demonstrating that the universal family of gauge fields \( A^U \) and the gauge fixing connection \( \eta \) can be unified in the single extended connection \( \mathcal{A} = A^U + \eta \) on the \( G \)-principal bundle \( \mathcal{A} \times \mathcal{P} \rightarrow \mathcal{A}/G \times M \). In this way, we have shown that the extended connection \( \mathcal{A} \) encodes the gauge field, the gauge fixing and the ghost field. A significant result is that it is possible to derive the BRST transformations of the relevant fields without imposing the usual horizontality or flatness conditions on the extended curvature \( \mathcal{F} = \phi + \psi + F^U \) ([2], [4], [22]). In other words, it is not necessary to assume that \( \phi = \psi = 0 \). Moreover, the proposed formalism allows us to show that the standard BRST transformation of the gauge field \( A \) is only valid in a local trivialization of the fiber bundle \( \mathcal{A} \rightarrow \mathcal{A}/G \). In fact, equation (20) can be considered the corresponding global generalization.

We then applied this geometric formalism to the path integral quantization of Yang-Mills theory. Rather than selecting a fixed representative for each \([A] \in \mathcal{A}/G\) by means of a section \( \sigma \), the gauge fixing connection \( \eta \) allows us to parallel transport any initial condition \( A_0 \in \mathcal{A} \) belonging to the orbit \([A_0]\). A significant advantage of this procedure is that one can always define a section \( \sigma_\eta \) of the projection \( \mathcal{P}(A_0, \pi^{-1}[A_1]) \rightarrow \mathcal{P}([A_0], [A_1]) \) in the space of paths, even when it is
It is not possible to define a global section $\sigma : \mathcal{A}/\mathcal{G} \to \mathcal{A}$ in the space of fields. Since the path integral is not an integral in the space of fields $\mathcal{A}$, but rather an integral in the space of paths in $\mathcal{A}$, such a section $\sigma_{\eta}$ suffices for eliminating the infinite volume of the group of paths $\mathcal{P}\mathcal{G}$. Hence, this generalized gauge fixing procedure is globally well-defined even when the topology of the fiber bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ is not trivial (Gribov’s obstruction).

We then used the standard Faddeev-Popov method in order to introduce the generalized gauge fixing defined by $\eta$ at the level of the path integral. The corresponding gauge fixed extended action $S_{gf}$ was thereby obtained. We have thus shown that the Faddeev-Popov method can be used even when there is a Gribov’s obstruction.

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Appendix A. The geometry of $\eta$

In this appendix we will review some geometric properties of the connection $\eta$. The gauge group $\mathcal{G}$ consists of diffeomorphisms $\varphi : P \rightarrow P$ such that $\pi \varphi = \pi$ and $\varphi(pg) = \varphi(p)g$ (with $g \in G$). Each element $\varphi \in \mathcal{G}$ can be associated to a map $g : P \rightarrow G$ by $\varphi(p) = pg(p)$. This map $g$ satisfies $g(ph) = h^{-1}g(p)h = \text{ad}(h^{-1})g(p)$. This description of the elements of $\mathcal{G}$ also allows one to describe the elements of the Lie algebra $\mathfrak{Lie}(\mathcal{G})$. These elements consist of maps $g : P \rightarrow \mathfrak{g}$ such that $g(ph) = \text{Ad}(h^{-1})g(p)$.

The gauge group $G$ acts on the right on $\mathcal{A}$ via the pullback of connections. If $\varphi : P \rightarrow P$ is an element of $\mathcal{G}$ and $A \in \mathcal{A}$, then the action is given by:

$$R_\varphi A = \varphi^* A. \quad (48)$$

This action is commonly described in terms of the function $g$ by the transformation formula

$$R_\varphi A = g^{-1}A g + g^{-1}dg. \quad (49)$$

The first term in the right hand side of equation (49) denotes the composition

$$T_p P \xrightarrow{A_{\varphi_s}} \mathfrak{g} \xrightarrow{\text{ad}_g^{-1}(p)} \mathfrak{g}. \quad (50)$$

The second part $g^{-1}dg$ is the pullback of the Maurer-Cartan form $\omega$ defined by the map $g : P \rightarrow G$.

The action (48) induces two constructions of interest. The first one is the infinitesimal action. This is a morphism of Lie algebras $\iota : \mathfrak{Lie}(\mathcal{G}) \rightarrow \Gamma(TP)$, where $\Gamma(TP)$ is the Lie algebra of the vector fields on $P$. We will identify elements of a Lie algebra with tangent vectors at the identity. If $\zeta \in \mathfrak{Lie}(\mathcal{G})$ and $\varphi_s$ is a curve such that

$$\frac{d}{ds} \varphi_s \bigg|_{s=0} = \zeta,$$

and $A \in \mathcal{A}$, then we define

$$\iota_A \zeta = \frac{d}{ds} R_{\varphi_s} A \bigg|_{s=0}. \quad (51)$$

In terms of the explicit description of equation (49) we obtain

$$\iota_A \zeta = \text{Ad}(\zeta)A + \frac{d}{ds} g^{-1}dg \bigg|_{s=0}. \quad (52)$$

The second action is the differential of the action of $\mathcal{G}$. Since $\mathcal{A}$ is an affine space, there is a natural identification $T\mathcal{A} = \mathcal{A} \oplus \mathcal{A}$. An element $V \in T_A \mathcal{A}$ can then be identified with a connection. Let $A_s$ be a curve in $\mathcal{A}$ such that $\frac{d}{ds}A_s \big|_{s=0} = V$. Then we define

$$dR_\varphi V = \frac{d}{ds} \text{ad}(g)A_s \bigg|_{s=0}. \quad (53)$$
Remark 14. The infinitesimal action and the differential of the action are geometric intrinsic constructions. They only depend on the vectors $\zeta$ and $V$ and not on the particular curves used to compute them.

The connection $\eta$ has the following properties

$$\eta(i(\zeta)) = \zeta, \quad \eta(dR_{\varphi}V) = \text{ad}(g^{-1})\eta(V).$$

Appendix B. The local form of $\eta$

The local form of a connection in a principal bundle is commonly described in terms of Christoffel symbols. This description has been extended to infinite dimensions in Ref.[17, Chapter VIII]. We will give a brief description of the constructions involved. Let $U_\alpha$ be a trivializing covering of $\mathcal{A}/\mathcal{G}$ with trivializing functions $\psi_\alpha : U_\alpha \times \mathcal{G} \rightarrow \mathcal{A}|_{U_\alpha}$. Then the pullback of $\eta$ defines a $\text{Lie}(\mathcal{G})$-valued 1-form on $U_\alpha \times \mathcal{G}$. Let $v \in T_x U_\alpha$ and $w \in T_g \mathcal{G}$ be two tangent vectors. Then

$$\psi_\alpha^*\eta(v, w) =: -\Gamma^\alpha(v, g) + w$$

for a certain 1-form $\Gamma^\alpha$ with values on the vector fields on $\mathcal{G}$. The 1-form $\Gamma^\alpha$ is called the local Christoffel form of $\eta$ in the trivialization $(U_\alpha, \psi_\alpha)$. If $\gamma$ is a path in $U_\alpha$, then any lift $\tilde{\gamma}$ of $\gamma$ to $U_\alpha \times \mathcal{G}$ has the form $\tilde{\gamma}(t) = (\gamma(t), \tau(t))$ for a path $\tau$ in $\mathcal{G}$. Then the gauge fixing condition is given in local coordinates by the equation

$$\frac{d}{dt}\tau(t) - \Gamma^\alpha(\frac{d}{dt}\tau(t), \gamma(t)) = 0.$$

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