We examine interacting bosonic higher spin gauge fields in the BRST-antifield formalism. Assuming that an interacting action $S$ is a deformation of the free action with a deformation parameter $g$, we solve the master equation $(S, S) = 0$ from the lower orders in $g$. It is shown that choosing a certain cubic interaction as the first order deformation, we can solve the master equation and obtain an action containing all orders in $g$. The antighost number of the obtained action is less than or equal to two. Furthermore, we show that the obtained action is lifted to that of interacting bosonic higher spin gauge fields on anti-de Sitter spaces.

1. Introduction

Higher spin gauge theories have been studied since 1930s from various interests. For example, they are expected to reveal characteristic aspects of string theory in the high energy limit. String theory may be regarded as a spontaneous symmetry breaking phase of the higher spin gauge theories [1]. Free higher spin gauge theories are well understood by now. There are obstacles to construct consistent interactions. One of them is the no-go theorem [2]. To avoid it, the number of derivatives contained in interaction vertices should be restricted so that higher spin gauge fields are not included in the asymptotic states. For cubic vertices, the allowed number of derivatives are clarified by using a light-cone formulation in [4][5]. For bosonic gauge fields, vertices are constructed explicitly by using Noether’s procedure in [6][7]. In constructing vertices, a generalized curvature tensor [8] is frequently used as a building block. It is gauge-invariant and defined as $s$-curl of a totally symmetric spin-$s$ bosonic gauge field. It is not easy task to construct a gauge-invariant generalized curvature tensor on a general backgrounds including anti-de Sitter (AdS) spaces. In contrast we will employ the Fronsdal tensor [9] as a building block, and look for vertices on AdS spaces. It is also difficult to construct full interactions including not only a cubic vertex but also higher order vertices. We usually construct an interacting theory as a small deformation of the free...
theory including a few lower orders in the deformation parameter $g$. It may not be consistent to construct vertices beyond the cubic order. Even if we could construct interactions to any order, the full interacting action may not be written in a closed form.

In this paper, we will construct actions of interacting bosonic higher spin gauge fields on $D$-dimensional spacetimes in the BRST-antifield formalism. The BRST-antifield formalism is known to be very powerful in constructing interactions systematically [10]. Employing this cohomological method, interaction terms are constructed systematically in [11]. In the present paper, at first, we will construct actions on a flat spacetime using this method, and then lift them to those on AdS spaces. For this we will use the Fronsdal tensor as a building block. To avoid the no-go theorem, spins of gauge fields are restricted appropriately. We will comment on this point in the last section. In section 4, two gauge fields of spin-$s$ and spin-2$s$ are examined, and three gauge fields of spin-$s_1$, spin-$s_2$ and spin-$(s_1 + s_2)$ are examined in section 5. In either cases, we construct a full action $S$ as a deformation of the free action $S^0$, $S = S^0 + g S^1 + g^2 S^2 + \cdots$. It is shown that choosing a certain cubic vertex as $S^1$, the master equation $(S, S) = 0$ can be solved from the lower order in $g$. It is worth noting that the obtained action $S$ contains all orders in $g$ and is written in a closed form by using a geometric series. The action can be rearranged as $S = S_0 + S_1 + S_2$ where the antighost number of $S_a$ is $a$. We extract the gauge symmetry of the interacting action $S_0$ from the BRST symmetry $\delta g B X = (X, S)$. Furthermore, we will write down actions of interacting bosonic higher spin gauge fields on AdS spaces from those on a flat spacetime.

The organization of the present paper is as follows. In the next section, clarifying our notations we present the free action in the BRST-antifield formalism. The BRST deformation scheme is explained in section 3. In section 4, two interacting gauge fields with spin-$s$ and spin-2$s$ are examined. The master equation is solved from the lower orders in $g$ and the full action containing all orders in $g$ is derived. In section 5, the full action for three interacting gauge fields with spin-$s_1$, spin-$s_2$ and spin-$(s_1 + s_2)$ is derived (a derivation of low order deformations is given in appendix B). In section 6, we will write down actions of higher spin bosonic gauge fields on AdS spaces. The last section is devoted to a summary and discussions. In appendix A, we explain how a $\Gamma$-exact term results in a BRST-exact term.

2. Free higher spin bosonic gauge theory

We will explain our notations used in this paper and present the free action in the BRST-antifield formalism.

We introduce a totally symmetric rank-$s$ bosonic tensor field $\phi_{\mu_1 \cdots \mu_s}$ in a $D$-dimensional flat spacetime. The Fronsdal tensor [9] for $\phi_{\mu_1 \cdots \mu_s}$ is defined by

$$F_{\mu_1 \cdots \mu_s}(\phi) = \Box \phi_{\mu_1 \cdots \mu_s} - \partial_{(\mu_1} \partial \cdot \phi_{\mu_2 \cdots \mu_s)} + \partial_{(\mu_1} \partial_{\mu_2} \phi_{\mu_3 \cdots \mu_s)} ,$$

(1)

\footnote{Our convention for symmetrization of indices is as follows

$$\partial_{(\mu_1 \cdots \partial_{\mu_r \phi_{\mu_{r+1} \cdots \mu_n})v_1 \cdots v_m} = \frac{1}{r!(n-r)!} \sum_{\{\mu_1, \cdots, \mu_n\}} \partial_{\mu_1} \cdots \partial_{\mu_r} \phi_{\mu_{r+1} \cdots \mu_n v_1 \cdots v_m} ,$$

where $\{\mu_1, \cdots, \mu_n\}$ indicates that the sum is taken over the permutation of $\mu_1, \cdots, \mu_n$.}
where $\square = \eta^{\mu\nu}\partial_\mu\partial_\nu$ with $\eta_{\mu\nu} = \text{diag}(-1,+1,\cdots,+1)$. A prime on $\phi$ represents the trace of $\phi$, namely $\phi'_{\mu_1\cdots\mu_s} = \phi_{\mu_1\cdots\mu_s}^\rho \rho$, and similarly $\phi''_{\mu_1\cdots\mu_s} = \phi_{\mu_1\cdots\mu_s}^\rho \rho \rho \sigma$. The divergence of $\phi$ is expressed as $\partial \cdot \phi_{\mu_1\cdots\mu_s} = \partial^{\mu_1}\phi_{\mu_1\mu_2\cdots\mu_s}$.

The Fronsdal equation $F_{\mu_1\cdots\mu_s} = 0$ means, when $s = 1$ the Maxwell equation for the gauge field $A_\mu$, $\Box A_\mu - \partial_\mu \partial^\rho A_\rho = 0$, while when $s = 2$ the linealized Einstein equation for the graviton $h_{\mu\nu} - \partial_\mu \partial_\nu h - \partial_\mu \partial_\rho h_\rho = 0$. The gauge transformation of $\phi_{\mu_1\cdots\mu_s}$ which is a natural generalization of $\delta A_\mu = \partial_\mu \xi$ for $s = 1$ and $\delta h_{\mu\nu} = \partial_{(\mu} \xi_{\nu)}$ for $s = 2$, is given by

$$\delta \phi_{\mu_1\cdots\mu_s} = \partial_{(\mu_1} \xi_{\mu_2\cdots\mu_s)} ,$$

(2)

where $\xi$ is the rank-$(s-1)$ gauge parameter. It is straightforward to see that

$$\delta F_{\mu_1\cdots\mu_s}(\phi) = 3\partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} \xi_{\mu_4\cdots\mu_s)} .$$

(3)

Here and hereafter, we require that the gauge parameter is traceless for $s \geq 3$

$$\xi_{\mu_1\cdots\mu_s} = 0 ,$$

(4)

which implies the gauge invariance of the Fronsdal tensor. Furthermore we impose a double traceless constraint on $\phi$

$$\phi''_{\mu_1\cdots\mu_s} = 0$$

(5)

for $s \geq 4$, so that the Fronsdal equation should describe the propagation of a massless spin-$s$ gauge field. The action which leads to the Fronsdal equation is given by [8]

$$S_{dWF} = \frac{1}{2} \int d^D x \phi_{\mu_1\cdots\mu_s} G_{\mu_1\cdots\mu_s}(\phi) .$$

(6)

We have introduced $G_{\mu_1\cdots\mu_s}$ as

$$G_{\mu_1\cdots\mu_s}(\phi) \equiv F_{\mu_1\cdots\mu_s} - \frac{1}{2} \eta_{\mu_1\mu_2} F'_{\mu_3\cdots\mu_s} ,$$

(7)

where $F$ is the Fronsdal tensor defined in (1). A useful and important relation we frequently use in this paper is

$$\int d^D x \varphi_{\mu_1\cdots\mu_s} \tilde{G}_{\mu_1\cdots\mu_s}(\psi) = \int d^D x \tilde{G}_{\mu_1\cdots\mu_s}(\varphi) \psi_{\mu_1\cdots\mu_s} ,$$

(8)

where $\psi$ and $\varphi$ are arbitrary totally symmetric rank-$s$ fields. We have defined $\tilde{G}(A)$ by

$$\tilde{G}_{\mu_1\cdots\mu_s}(A) \equiv G_{\mu_1\cdots\mu_s}(A) + \frac{1}{2} \eta_{\mu_1\mu_2} \partial_{\mu_3} A_{\mu_4\cdots\mu_s}^\rho \rho \rho \sigma ,$$

(9)

where $A$ is an arbitrary totally symmetric rank-$s$ field. When the double traceless constraints on $\psi$ and $\varphi$ would be implemented, one obtains $\tilde{G}(\psi) = G(\psi)$ and $\tilde{G}(\varphi) = G(\varphi)$, so that

$$\int d^D x \varphi_{\mu_1\cdots\mu_s} G_{\mu_1\cdots\mu_s}(\psi) = \int d^D x G_{\mu_1\cdots\mu_s}(\varphi) \psi_{\mu_1\cdots\mu_s} .$$

It follows that varying the action $S_{dWF}$ with respect to $\phi_{\mu_1\cdots\mu_s}$, we obtain the equation of motion $G_{\mu_1\cdots\mu_s}(\phi) = 0$. Taking the trace of $G_{\mu_1\cdots\mu_s}(\phi) = 0$, we obtain $F'_{\mu_1\cdots\mu_s} = 0$ by using $F''_{\mu_1\cdots\mu_s} = 0$ which follows from $\phi'' = 0$. As a result, the equation of motion implies the Fronsdal equation $F_{\mu_1\cdots\mu_s} = 0$. 

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2.1. Free action in the BRST-antifield formalism

Corresponding to the gauge parameter $\xi_{\mu_1 \cdots \mu_{s-1}}$, we introduce a rank-$(s-1)$ Grassmann-odd ghost field $c_{\mu_1 \cdots \mu_{s-1}}$ with the same algebraic symmetry. The $c_{\mu_1 \cdots \mu_{s-1}}$ must be traceless $c'_{\mu_3 \cdots \mu_{s-1}} = 0$ just like $\xi'_{\mu_3 \cdots \mu_{s-1}} = 0$. The gauge invariance of $S_{dWF}$ in (6) is encoded to the BRST invariance under the BRST transformations

$$\delta_B \phi_{\mu_1 \cdots \mu_s} = \partial(\mu_1 c_{\mu_2 \cdots \mu_s}) ,$$
$$\delta_B c_{\mu_2 \cdots \mu_s} = 0 .$$

The gauge field $\phi$ and the ghost field $c$ are collectively called “fields” and denoted as $\Phi^A$. We further introduce “antifields” $\Phi^*_A = \{\phi^*_{\mu_1 \cdots \mu_s} , c^*_{\mu_1 \cdots \mu_{s-1}}\}$ which have the same algebraic symmetries but opposite Grassmann parity. Two gradings are introduced. One is the pure ghost number $pgh$, and the other is the antighost number $agh$. The ghost number $gh$ is defined as $gh \equiv pgh - agh$.

The grading properties are summarized in the Table.1. The antibracket for two functionals, $X(\Phi^A, \Phi^*_A)$ and $Y(\Phi^A, \Phi^*_A)$, is defined by

$$\{X, Y\} \equiv X \frac{\delta}{\delta \Phi^A} \frac{\delta}{\delta \Phi^*_A} Y - X \frac{\delta}{\delta \Phi^*_A} \frac{\delta}{\delta \Phi^A} Y .$$

The action $S_{dWF}$ in (6) can be extended to $S^0[\Phi, \Phi^*]$ such that the BRST transformation of a functional $X(\Phi^A, \Phi^*_A)$ is expressed as

$$\delta_B X = (X, S^0) .$$

Note that $\delta_B$ acts from the right. The nilpotency $\delta_B^2 = 0$ requires the master equation $(S^0, S^0) = 0$. In the present case, the free action is found to be

$$S^0[\Phi, \Phi^*] = \int d^D x \left( \frac{1}{2} \phi^*_{\mu_1 \cdots \mu_s} G_{\mu_1 \cdots \mu_s}(\phi) + \phi^*_{\mu_2 \cdots \mu_s} \partial(\mu_1 c_{\mu_2 \cdots \mu_s}) \right) ,$$

which leads to (10), (11) and

$$\delta_B \phi^*_{\mu_1 \cdots \mu_s} = -G_{\mu_1 \cdots \mu_s}(\phi) ,$$
$$\delta_B c^*_{\mu_2 \cdots \mu_s} = -s \partial \cdot \phi^*_{\mu_2 \cdots \mu_s} + \frac{s}{D+2s-6} \eta(\mu_2 \mu_3 \partial \cdot \phi^*_{\mu_4 \cdots \mu_s}) .$$

Note that the second term in the right-hand side of (16) is required for the nilpotency $\delta_B^2 c^* = 0$. It is straightforward to see that this term leaves the action (14) unchanged.

3. BRST deformation

The BRST deformation scheme is very useful in constructing interactions systematically [10]. We will explain relevant aspects of it in this section.

Suppose that $S$ is a deformation of $S^0$ expanded in a deformation parameter $g$

$$S = S^0 + gS^1 + g^2S^2 + \cdots .$$

When $S$ solves the master equation $(S, S) = 0$, $S$ is invariant under the BRST transformation generated by $S$: $\delta_B^g S \equiv (S, S) = 0$. Here we put a letter $g$ on $\delta_B$ in order to distinguish it...
from $\delta_B$ generated by $S^0$ considered in the preceding section. The master equation $(S, S) = 0$ means at the order of $g^n$

$$\sum_{k=0}^{n} (S^k, S^{n-k}) = 0 .$$  \hspace{1cm} (18)

The equation for $n = 0$, $(S^0, S^0) = 0$, is satisfied by definition. It is worth noting that $S^n$ is determined by all of the lower order terms, $S^k$ ($k < n$). To obtain higher order terms, we must start examining $S^1$ first of all.

The $S^1$ is determined by the equation (18) for $n = 1$

$$(S^1, S^0) = 0 ,$$  \hspace{1cm} (19)

which means that $S^1$ is BRST invariant\footnote{Let $a$ be a $D$-form defined by $S^1 = \int a$, and then this is equivalent to $\delta_B a + db = 0$ . Since a BRST exact part of $a$ corresponds to a trivial field redefinition, we consider $H^0(\delta_B |d)$: the cohomology of the BRST differential $\delta_B$ modulo $d$ at $gh = 0$.}

$$\delta_B S^1 = 0 .$$  \hspace{1cm} (20)

If $S^1$ is BRST-exact, namely $S^1 = (X, S^0) = X \frac{\delta}{\delta \Phi_A} \frac{\delta S^0}{\delta \Phi_A} - X \frac{\delta}{\delta \Phi_A^*} \frac{\delta S^0}{\delta \Phi_A^*}$, $S^1$ can be eliminated by a field redefinition. In fact, the field redefinition $\Phi_A \rightarrow \Phi_A + g u_A$ and $\Phi_A^* \rightarrow \Phi_A^* + g v_A$ causes the change $S^0[\Phi_A, \Phi_A^*] \rightarrow S^0[\Phi_A + g u_A, \Phi_A^* + g v_A] = S^0[\Phi_A, \Phi_A^*] + g \left( \frac{\delta S^0}{\delta \Phi_A} u_A + \frac{\delta S^0}{\delta \Phi_A^*} v_A \right) + \cdots$, which can absorb $(X, S^0)$ by choosing $u_A$ and $v_A$ appropriately.

In this paper, we will choose a cubic interaction as $S^1$. In this case, $S^1$ can be expanded in the antighost number as

$$S^1 = a_2 + a_1 + a_0$$  \hspace{1cm} (21)

where $agh(a_i) = i$. Let us explain the reason why $a_n$ ($n > 2$) are excluded. Since the action $S$ has ghost number 0, $gh(a_i) = 0$, so that $pgh(a_i) = agh(a_i) = i$. As $pgh(a_3) = 3$, $a_3$ must be composed of three $c$’s. But such an $a_3$ has $gh(a_3) \neq 0$. This is because we exclude $a_3$ in the expansion. For $a_n$ ($n > 3$), $pgh(a_n) = n$. It is impossible to construct such a term as a cubic interaction.

Similarly we expand $\delta_B$ with respect to the $agh$ as

$$\delta_B = \Delta + \Gamma ,$$  \hspace{1cm} (22)

where $agh(\Delta) = -1$ and $agh(\Gamma) = 0$. The $\Delta$ and $\Gamma$ are nilpotent and anticommute each other. These differentials act on fields and antifields as summarized in the Table.2. It is easy to see that (20) is expanded with respect to $agh$ as

$$\Gamma a_2 = 0 ,$$  \hspace{1cm} (23)

$$\Delta a_2 + \Gamma a_1 = 0 ,$$  \hspace{1cm} (24)

$$\Delta a_1 + \Gamma a_0 = 0 ,$$  \hspace{1cm} (25)

$$\Delta a_0 = 0 .$$  \hspace{1cm} (26)

Since $agh(a_0) = 0$, $a_0$ does not contain any antifields $\Phi_A^*$. It implies (26).
Table 2: Action of $\Delta$ and $\Gamma$

| $Z$          | $\Delta(Z)$ | $\Gamma(Z)$ |
|--------------|-------------|-------------|
| $\phi_{\mu_1\ldots\mu_s}$ | 0           | $\partial(\mu_1 c_{\mu_2\ldots\mu_s})$ |
| $c_{\mu_1\ldots\mu_{s-1}}$ | 0           | 0           |
| $\phi^*_{\mu_1\ldots\mu_s}$ | $-G_{\mu_1\ldots\mu_s}(\phi)$ | 0           |
| $c^*_{\mu_1\ldots\mu_{s-1}}$ | $-s\partial \cdot \phi^*_{\mu_1\ldots\mu_{s-1}} + \frac{s}{D+2s-1}\eta(\mu_1\mu_2 \partial \cdot \phi^{*t}_{\mu_3\ldots\mu_{s+1}})$ | 0           |

First we consider (23). Since $a_2$ has $agh = pgh = 2$, it is composed of two ghosts $c$'s and one antighost $c^*$. For a non-trivial $a_1$, $\Delta a_2$ must to be $\Gamma$-exact as seen from (24). It implies that $\partial c$ must be included in $a_2$ such as $\int d^D x c^*_{\mu_1\ldots\mu_p 1\ldots p_{r-1}} c_{\nu_1\ldots\nu_q} \rho_{\rho_1\ldots\rho_p} \delta^{(\mu_1\ldots\mu_{r+q})}$. This means that $a_2$ is $\Gamma$-exact. As is explained in appendix A, such a $\Gamma$-exact $a_2$ leads to a BRST-trivial $S^1$. So we may set

$$a_2 = 0.$$  (27)

We construct $a_1$ satisfying $\Gamma a_1 = 0$. Since $a_1$ has $agh = pgh = 1$, it is composed of a ghost $c$, an antifield $\phi^*$ and a gauge field $\phi$. For $\Gamma a_1 = 0$, $\phi$ should appear as a gauge invariant form such as $G(\phi)$, since $\Gamma G(\phi) = 0$ which follows from $\delta_B F = 0$. In addition, noting that $\partial(\mu_1 c_{\mu_2\ldots\mu_{n+1}}) = \Gamma \phi$ and $\partial \cdot c_{\mu_2\ldots\mu_{n+1}} = \frac{1}{2} \Gamma \phi^*_{\mu_2\ldots\mu_{n+1}}$, we choose

$$\int d^D x G^{\mu_1\ldots\mu_p\mu_{1}\ldots\mu_{r}}(\phi) \partial(\mu_1 c_{\mu_2\ldots\mu_{p+n}}) \phi^{*\nu_{1}\ldots\nu_{q}} \rho_{\rho_1\ldots\rho_p}$$  (28)

as $a_1$, which is not $\Gamma$-exact. This is the $a_1$ from which all interaction terms are shown to be constructed successively. In this paper, we concern a special case with $r = 0$. Even for the case with $r \neq 0$, we have found that the $S^n$ can be constructed successively. We hope that we will report this result in another place [12].

In the followings, we will construct a BRST-invariant action $S$ containing all orders in $g$. In the next section, we consider two interacting gauge fields with spin-$s$ and spin-$2s$. In section 5, we generalize it further to the case with three interacting gauge fields with spin-$s_1$, spin-$s_2$ and spin-$(s_1 + s_2)$.

4. Interaction of two gauge fields with spin-$s$ and spin-$2s$

In this section, we introduce two gauge fields with spin-$s$ and spin-$2s$. The free action for the spin-$s$ gauge field is given in (14). For notational simplicity, rank-$s$ fields are denoted as column vectors, such as $\phi_{\mu_1\ldots\mu_s} = \phi$, $\phi^{*}_{\mu_1\ldots\mu_s} = \phi^*$ and $\partial(\mu_1 c_{\mu_2\ldots\mu_s}) = \partial c$. The number of components of these vectors is $d(s) = \frac{(D-1+s)!}{(D-1)!}$. In this notation, the free action is written

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as

\[ S^0(s) = \int d^Dx \left( \frac{1}{2} \phi^T G(\phi) + \phi^* T \partial c \right). \] (29)

The rank-2s fields are represented as \( d(s) \times d(s) \) square matrices, such as \( \phi_{\mu_1 \cdots \mu_2s} = \phi, \phi^*_{\mu_1 \cdots \mu_2s} = \phi^* \) and \( \partial_{(\mu_1 \cdots \mu_2s)} = \partial c \), so that the free action is expressed as

\[ S^0(2s) = \int d^Dx tr \left( \frac{1}{2} \phi G(\phi) + \phi^* \partial c \right). \] (30)

The total free action is a sum of (29) and (30).

4.1. Three-point interaction

As explained in section 3, we choose the \( \Gamma \)-nontrivial \( a_1 \) as

\[ a_1 = \int d^Dx G(\phi)^T \partial c \phi^*, \] (31)

which is (28) with \( r = 0 \). We will show that all higher order terms can be constructed from this \( a_1 \). Here we have introduced \( \partial c = \partial_{(\mu_1 \cdots \mu_2s)} \). We note \( \partial c \) in (30) is a symmetric matrix but \( \partial c \) is not. In fact, we note that \( G(\phi)^T \partial c \phi^* = G(\phi)^T (\partial c + (\partial c)^T) \phi^* \neq G(\phi)^T \partial c \phi^* \). One may choose as \( a_1 \)

\[ \int d^Dx \phi^* T \partial c G(\phi) = - \int d^Dx G(\phi)^T (\partial c)^T \phi^* \] (32)

instead of (31), so that the obtained action \( S \) should be the same subject to a field redefinition. This is because the difference between (31) and (32) is a \( \Gamma \)-exact term, \( \int d^Dx G(\phi)^T \partial c \phi^* = - \Gamma \int d^Dx G(\phi)^T \phi \phi^* \).

It is obvious that (31) satisfies \( \Gamma a_1 = 0 \) as \( \Gamma G(\phi) = 0 \) which follows from the gauge invariance of \( F(\phi) \). To solve (25) for \( a_0 \), we derive

\[ - \Delta a_1 = \int d^Dx G(\phi)^T \partial c G(\phi) = \int d^Dx \frac{1}{2} G(\phi)^T \partial c G(\phi) = \Gamma \int d^Dx \frac{1}{2} G(\phi)^T \phi G(\phi). \] (33)

It implies that we may determine \( a_0 \) as

\[ a_0 = \int d^Dx \frac{1}{2} G(\phi)^T \phi G(\phi). \] (34)

As a result, we have obtained \( S^1 \) in (21) composed of (27), (31) and (34).

4.2. Four-point interaction

We will proceed to derive \( S^2 \). The \( S^2 \) is determined by the master equation (18) for \( n = 2 \)

\[ 2(S^2, S^0) + (S^1, S^1) = 0 \quad \leftrightarrow \quad \delta_B S^2 + \frac{1}{2} (S^1, S^1) = 0. \] (35)

It implies that \( S^2 \) is composed of four fields because \( S^1 \) is composed of three fields. Expanding \( S^2 \) with respect to \( agh \) as

\[ S^2 = b_2 + b_1 + b_0 \] (36)
where \( agh(b_i) = i \), we find that (35) reduces to

\[
\Gamma b_2 = 0 ,
\]

(37)

\[
\Delta b_2 + \Gamma b_1 + \frac{1}{2}(a_1, a_1) = 0 ,
\]

(38)

\[
\Delta b_1 + \Gamma b_0 + (a_1, a_0) = 0 .
\]

(39)

We have not included \( b_n \) \((n > 2)\) in the expansion (36) simply because we can solve (35) without them.

First we will solve (38) for \( b_2 \) and \( b_1 \). One derives using \( a_1 \) in (31)

\[
-\frac{1}{2}(a_1, a_1) = \int d^Dx G(\phi)^T \delta c G(\delta c \phi^*)
\]

\[
= \int d^Dx G(\phi)^T \delta c G(\delta c \phi^*) - \int d^Dx G(\phi)^T \tilde{\delta} c G(\delta c \phi^*) .
\]

(40)

In the second equality, we have used \( \delta c = \tilde{\delta} c + (\tilde{\delta} c)^T \). We find that the first term in the last line is \( \Gamma \)-exact

\[
\Gamma \int d^Dx G(\phi)^T \phi \tilde{G}(\delta c \phi^*) ,
\]

(41)

while the second term is \( \Delta \)-exact

\[
-\int d^Dx (\tilde{\delta} c G(\phi))^T \tilde{G}(\delta c \phi^*) = \Delta \int d^Dx \frac{1}{2}(\delta c \phi^*)^T \tilde{G}(\delta c \phi^*) ,
\]

(42)

where we have used (8). As a result, we obtain

\[
b_2 = \int d^Dx \frac{1}{2}(\delta c \phi^*)^T \tilde{G}(\delta c \phi^*) ,
\]

(43)

\[
b_1 = \int d^Dx G(\phi)^T \phi \tilde{G}(\delta c \phi^*) .
\]

(44)

It is obvious to see that \( b_2 \) above satisfies (37).

Next we will solve (39) for \( b_0 \). Noting that

\[
-(a_1, a_0) = \int d^Dx G(\phi)^T \delta c G(\phi G(\phi))
\]

\[
= \int d^Dx \tilde{G}(\phi G(\phi))^T (\tilde{\delta} c)^T G(\phi)
\]

\[
= \int d^Dx (\phi G(\phi))^T \tilde{G}(\phi G(\phi)) ,
\]

(45)

where (8) is used in the last equality, we find that

\[
-\Delta b_1 - (a_1, a_0) = \int d^Dx G(\phi)^T \phi \tilde{G}(\delta c G(\phi)) + \int d^Dx (\phi G(\phi))^T \tilde{G}(\phi G(\phi))
\]

\[
= \int d^Dx G(\phi)^T \phi \tilde{G}(\phi G(\phi))
\]

\[
= \Gamma \int d^Dx \frac{1}{2}(\phi G(\phi))^T \tilde{G}(\phi G(\phi)) .
\]

(46)

This implies that

\[
b_0 = \int d^Dx \frac{1}{2}(\phi G(\phi))^T \tilde{G}(\phi G(\phi)) .
\]

(47)

As a result, we found that \( S^2 \) in (36) is composed of (43), (44) and (47).
4.3. Five-point interaction

In order to guess the form of \( S^n \), let us proceed to derive \( S^3 \). The \( S^3 \) is determined by the master equation (18) for \( n = 3 \)

\[
(S^3, S^0) + (S^2, S^1) = 0 \quad \leftrightarrow \quad \delta_B S^3 + (S^1, S^2) = 0 .
\]  

(48)

Expanding \( S^3 \) with respect to \( agh \) as

\[
S^3 = c_2 + c_1 + c_0
\]

(49)

where \( agh(c_i) = i \), we find that (48) reduces to

\[
\Gamma c_2 + (a_1, b_2) = 0 , \quad \Delta c_2 + \Gamma c_1 + (a_1, b_1) + (a_0, b_2) = 0 , \quad \Delta c_1 + \Gamma c_0 + (a_1, b_0) + (a_0, b_1) = 0 .
\]

(50) \( \quad \) (51) \( \quad \) (52)

As seen below, we can solve (48) without \( c_n \) \( (n > 2) \), so that they are not included in (49). First we will solve (50) for \( c_2 \). One derives using (31) and (43)

\[
-(a_1, b_2) = \int d^Dx \, \bar{G}(\partial c \phi^*)^T \partial c \bar{G}(\partial c \phi^*) = \int d^Dx \frac{1}{2} \bar{G}(\partial c \phi^*)^T \partial c \bar{G}(\partial c \phi^*) = \Gamma \int d^Dx \frac{1}{2} \bar{G}(\partial c \phi^*)^T \phi \bar{G}(\partial c \phi^*)
\]

(53)

where in the second equality, \( \partial c + (\partial c)^T = \partial c \) is used. This implies that

\[
c_2 = \int d^Dx \frac{1}{2} \bar{G}(\partial c \phi^*)^T \phi \bar{G}(\partial c \phi^*) .
\]

(54)

Next we solve (51) for \( c_1 \). For this purpose we derive

\[
-\Delta c_1 = \int d^Dx \, \bar{G}(\partial c \phi^*)^T \phi \bar{G}(\partial c G(\phi)) , \quad -(a_1, b_1) = \int d^Dx \, [G(\phi)^T \partial c \bar{G}(\phi G(\partial c \phi^*)) + \bar{G}(\phi G(\phi))^T \partial c \bar{G}(\partial c \phi^*)] ,
\]

(55) \( \quad \) (56)

\[
-\Delta c_2 = \int d^Dx \, \bar{G}(\partial c \phi^*)^T \phi \bar{G}(\partial c G(\phi)) ,
\]

(57)

Combining the second term in the right hand side of (56) with the right hand side of (57), we obtain

\[
\int d^Dx \, \bar{G}(\phi G(\phi))^T \partial c \bar{G}(\partial c \phi^*)
\]

\[
= \Gamma \int d^Dx \, \bar{G}(\phi G(\phi))^T \phi \bar{G}(\partial c \phi^*) - \int d^Dx \, \bar{G}(\partial c G(\phi))^T \phi \bar{G}(\partial c \phi^*) .
\]

(58)

On the other hand, the first term in the right hand side of (56) may be rewritten as

\[
\int d^Dx \, \bar{G}(\phi G(\partial c \phi^*))^T (\partial c)^T G(\phi) = \int d^Dx \, \bar{G}(\partial c \phi^*)^T \phi \bar{G}((\partial c)^T G(\phi)) ,
\]

(59)

so that this term and the right hand side of (55) make

\[
\int d^Dx \, \bar{G}(\partial c \phi^*)^T \phi \bar{G}(\partial c G(\phi)) .
\]

(60)
Because (60) cancels out the last term in (58), we conclude that
\[-\Delta c_2 - (a_1, b_1) - (a_0, b_2) = \Gamma \int d^D x \tilde{G}(\phi G(\phi))^T \phi \tilde{G}(\tilde{\partial}c\phi^*) ,\]  
which implies that
\[c_1 = \int d^D x \tilde{G}(\phi G(\phi))^T \phi \tilde{G}(\tilde{\partial}c\phi^*) .\]  
Finally, (52) is examined to determine $c_0$. It is straightforward to see that
\[-\Delta c_1 - (a_1, b_0) = \int d^D x \tilde{G}(\phi G(\phi))^T \phi \tilde{G}(\partial c G(\phi)) + \int d^D x G(\phi)^T \partial c \tilde{G}(\phi G(\phi))
= \int d^D x \tilde{G}(\phi G(\phi))^T \phi \tilde{G}(\partial c G(\phi))
= \Gamma \int d^D x \frac{1}{2} \tilde{G}(\phi G(\phi))^T \phi \tilde{G}(\phi G(\phi)) - \int d^D x \frac{1}{2} \tilde{G}(\phi G(\phi))^T \partial c \tilde{G}(\phi G(\phi))\]  
where in the second equality we have used (8) and transposed the integrand, and that
\[-(a_0, b_1) = \int d^D x \tilde{G}(\phi G(\phi))^T \tilde{\partial} c G(\phi)\]
= \int d^D x \frac{1}{2} \tilde{G}(\phi G(\phi))^T \tilde{\partial} c G(\phi)(\phi) .\]  
Because (64) cancels out the last term in the most right hand side of (63), we conclude that
\[-\Delta c_1 - (a_1, b_0) - (a_0, b_1) = \Gamma \int d^D x \frac{1}{2} \tilde{G}(\phi G(\phi))^T \phi \tilde{G}(\phi G(\phi)) .\]
This implies that
\[c_0 = \int d^D x \frac{1}{2} \tilde{G}(\phi G(\phi))^T \phi \tilde{G}(\phi G(\phi)) .\]
As a result, $S^3$ in (49) is found to be composed of (54), (62) and (66).

4.4. $n$-point interaction

From the results obtained in the previous subsections 4.1, 4.2 and 4.3, we can guess the form of $S^n$ as follows
\[S^n = \alpha^n_2 + \alpha^n_1 + \alpha^n_0 ,\]  
\[\alpha^n_2 = \int d^D x \frac{1}{2} (\tilde{\partial}c\phi^*)^T \Phi^{n-2}[\tilde{G}(\tilde{\partial}c\phi^*)] = \int d^D x \frac{1}{2} \Phi^{n-2}[\tilde{G}(\tilde{\partial}c\phi^*)]^T \tilde{\partial}c\phi^* ,\]  
\[\alpha^n_1 = \int d^D x (\tilde{\partial}c\phi^*)^T \Phi^{n-1}[G(\phi)] = \int d^D x \Phi^T \Phi^{n-1}[\tilde{G}(\tilde{\partial}c\phi^*)] ,\]  
\[\alpha^n_0 = \int d^D x \frac{1}{2} \Phi^T \Phi^n [G(\phi)] .\]  
Here we have introduced $\Phi$ by $\Phi^0[A] = A$, $\Phi^1[A] \equiv \tilde{G}(\phi A)$, $\Phi^2[A] \equiv \tilde{G}(\phi\tilde{G}(\phi A))$, and so on. Since $\alpha^1_2 = \alpha^1_0 = 0$ for $S^1$, $\Phi^{-1}[A] = 0$ is understood. Observe that $S^1$, $S^2$ and $S^3$ derived in the previous subsections, coincide with $S^n$ in (67)-(70) for $n = 1, 2, 3$, respectively.
We shall show that $S^n$ ($n \geq 1$) above solves the master equation (18). Now, suppose that $S^k$ ($k < n$) solves the master equation at the order of $g^k$ ($k < n$). We will solve the master equation (18) for $\alpha^2_n$, $\alpha^1_n$ and $\alpha^0_n$ and show that they coincide with (68), (69) and (70).

The master equation (18) at the order of $g^n$ is expanded with respect to $agh$ as

$$
\Gamma \alpha^2_n + \frac{1}{2} \sum_{k=2}^{n-1} (\alpha^k_2, \alpha^0_{n-k}) + \frac{1}{2} \sum_{k=1}^{n-2} (\alpha^k_1, \alpha^2_{n-k}) = 0 , \quad (71)
$$

$$
\Delta \alpha^2_n + \Gamma \alpha^1_n + \frac{1}{2} \sum_{k=2}^{n-1} (\alpha^k_2, \alpha^0_{n-k}) + \frac{1}{2} \sum_{k=1}^{n-1} (\alpha^k_1, \alpha^1_{n-k}) + \frac{1}{2} \sum_{k=1}^{n-2} (\alpha^k_0, \alpha^2_{n-k}) = 0 , \quad (72)
$$

$$
\Delta \alpha^1_n + \Gamma \alpha^0_n + \frac{1}{2} \sum_{k=1}^{n-1} (\alpha^k_1, \alpha^0_{n-k}) + \frac{1}{2} \sum_{k=1}^{n-1} (\alpha^k_0, \alpha^1_{n-k}) = 0 . \quad (73)
$$

First we will solve (71) for $\alpha^2_n$. Since

$$
(\alpha^k_2, \alpha^0_{n-k}) = - \int d^Dx \Phi^{k-2}[\tilde{G}(\partial c\phi^*)]^T \partial c \Phi^{n-k-1}[\tilde{G}(\partial c\phi^*)] , \quad (74)
$$

$$
(\alpha^k_1, \alpha^0_{n-k}) = - \int d^Dx \Phi^{n-k-2}[\tilde{G}(\partial c\phi^*)]^T \partial c \Phi^{k-1}[\tilde{G}(\partial c\phi^*)] , \quad (75)
$$

we obtain

$$
- \frac{1}{2} \sum_{k=2}^{n-1} (\alpha^k_2, \alpha^0_{n-k}) - \frac{1}{2} \sum_{k=1}^{n-2} (\alpha^k_1, \alpha^2_{n-k}) = \frac{1}{2} \int d^Dx \sum_{l=0}^{n-3} \Phi^l[\tilde{G}(\partial c\phi^*)]^T \partial c \Phi^{n-3-l}[\tilde{G}(\partial c\phi^*)]
$$

$$
= \Gamma \int d^Dx \frac{1}{2} (\partial c\phi^*)^T \Phi^{n-2}[\tilde{G}(\partial c\phi^*)] . \quad (76)
$$

This implies that $\alpha^2_n$ is given as (68).

Next we will derive $\alpha^1_n$ from (72). We observe that

$$
- \Delta \alpha^1_n - \frac{1}{2} \sum_{k=2}^{n-1} (\alpha^k_2, \alpha^0_{n-k}) - \frac{1}{2} \sum_{k=1}^{n-2} (\alpha^k_0, \alpha^2_{n-k})
$$

$$
= \int d^Dx (\partial c\phi^*)^T \Phi^{n-2}[\tilde{G}(\partial c\phi)] + \int d^Dx \sum_{l=0}^{n-3} \Phi^l[\tilde{G}(\partial c\phi^*)]^T \partial c \Phi^{n-2-l}[G(\phi)]
$$

$$
= \int d^Dx \sum_{l=0}^{n-2} \Phi^l[\tilde{G}(\partial c\phi^*)]^T \partial c \Phi^{n-2-l}[G(\phi)] \quad (77)
$$

where in the second equality we have used a useful relation

$$
\int d^Dx A^T \Phi^m[\tilde{G}(B)] = \int d^Dx \Phi^m[\tilde{G}(A)]^T B \quad (78)
$$

which follows from (8). Furthermore deriving

$$
- \frac{1}{2} \sum_{k=1}^{n-1} (\alpha^k_1, \alpha^0_{n-k}) = \int d^Dx \sum_{l=0}^{n-2} \Phi^l[G(\phi)]^T \partial c \Phi^{n-2-l}[\tilde{G}(\partial c\phi^*)] , \quad (79)
$$
we find that
\[
- \Delta \alpha_n^2 - \frac{1}{2} \sum_{k=2}^{n-1} (\alpha_k, \alpha_0^{n-k}) - \frac{1}{2} \sum_{k=1}^{n-1} (\alpha_1^k, \alpha_0^{n-k}) - \frac{1}{2} \sum_{k=1}^{n-2} (\alpha_1^k, \alpha_2^{n-k})
\]
\[
= \int d^D x \sum_{l=0}^{n-2} \Phi_l^T [\hat{G} (\tilde{\partial} c \phi^*)]^T \tilde{\partial} c \Phi^{n-2-l} [G(\phi)]
\]
\[
= \Gamma \int d^D x (\tilde{\partial} c \phi^*)^T \Phi^{n-1} [G(\phi)] .
\]
This implies that \( \alpha_n^0 \) is given as in (69).
Finally, we derive \( \alpha_0^0 \) from (73). It is straightforward to derive
\[
- \Delta \alpha_1^0 - \frac{1}{2} \sum_{k=1}^{n-1} (\alpha_0^k, \alpha_0^{n-k}) - \frac{1}{2} \sum_{k=1}^{n-1} (\alpha_1^k, \alpha_1^{n-k})
\]
\[
= \int d^D x \Phi^{n-1} [G(\phi)]^T \tilde{\partial} c G(\phi) + \int d^D x \sum_{l=0}^{n-2} \Phi_l^T [G(\phi)]^T \tilde{\partial} c \Phi^{n-2-l} [G(\phi)]
\]
\[
= \int d^D x \sum_{l=0}^{n-1} \Phi_l^T [G(\phi)]^T \tilde{\partial} c \Phi^{n-2-l} [G(\phi)]
\]
\[
= \int d^D x \frac{1}{2} \sum_{l=0}^{n-1} \Phi_l^T [G(\phi)]^T \tilde{\partial} c \Phi^{n-2-l} [G(\phi)]
\]
\[
= \Gamma \int d^D x \frac{1}{2} \phi^T \Phi^0 [G(\phi)] .
\]
This implies that \( \alpha_0^0 \) is given as in (70).
Summarizing the above results, we have shown that \( S_n \) is surely given in (67)-(70).

4.5. BRST-invariant action of interacting spin-s and spin-2s gauge fields
We will examine the total action \( S \). In the above we have derived \( S^k \) \((k = 1, 2, \cdots)\). The free action is a sum of \( S^0(s) \) and \( S^0(2s) \) given in (29) and (30), respectively. By gathering these results together, the total action is given as
\[
S = S^0(s) + S^0(2s) + \sum_{k=1}^{\infty} g^k S^k
\]
\[
= \int d^D x \left[ \frac{1}{2} \phi^T G(\phi) + \phi^* T \tilde{\partial} c + \text{tr} \left( \frac{1}{2} \phi G(\phi) + \phi^* \tilde{\partial} c \right) + \frac{1}{2} \phi^T \sum_{k=1}^{\infty} g^k \Phi^k [G(\phi)]
\]
\[
+ \phi^T \sum_{k=1}^{\infty} g^k \Phi^{k-1} [\hat{G} (\tilde{\partial} c \phi^*)] + \frac{1}{2} (\tilde{\partial} c \phi^*)^T \sum_{k=2}^{\infty} g^k \Phi^{k-2} [\hat{G} (\tilde{\partial} c \phi^*)] \right] .
\]
We find that the action turns to the form expanded in $agh$ as

$$S = S_0 + S_1 + S_2,$$

$$S_0 = \int d^D x \left[ \frac{1}{2} \phi^T \frac{1}{1 - g\Phi} G(\phi) + \text{tr} \left( \frac{1}{2} \phi G(\phi) \right) \right],$$

$$S_1 = \int d^D x \left[ \phi^* T \partial c + \text{tr} \left( \phi^* \partial c \right) + \phi^T \frac{g}{1 - g\Phi} \tilde{G}(\tilde{\partial c}^*) \right],$$

$$S_2 = \int d^D x \frac{1}{2} (\tilde{\partial c}^*)^T \frac{g^2}{1 - g\Phi} \tilde{G}^T (\tilde{\partial c}^*),$$

where $agh(S_i) = i$. This is one of our main results in this paper. The obtained action $S$ contains all orders in $g$, and BRST-invariant $\delta_g B S = (S, S) = 0$. It is not likely that $S$ and $S^0$ are related each other by a field redefinition as long as we examined.

We have obtained BRST-invariant action of interacting gauge fields by using the BRST-antifield formalism. Here we examine the gauge invariance of the action $S_0$. The gauge transformation can be extracted from the BRST transformation $\delta_B X = (X, S)$. We find that the gauge transformation with a rank-$(s-1)$ parameter $\xi$ remains unchanged

$$\delta\phi = \partial\xi, \quad \delta\phi = 0,$$

while the gauge transformation with a rank-$(2s-1)$ parameter $\xi$ turns to

$$\delta\phi = - \left( \left[ \frac{g}{1 - g\Phi} G(\phi) \right]^T \partial\xi \right)^T, \quad \delta\phi = \partial\xi.$$

We note that higher order interactions make $\delta\phi$ non-trivial. It is instructive to check the gauge invariance of $S_0$ in (84) under (88). The gauge variation of $S_0$ is

$$\delta S_0 = \int d^D x \left[ \frac{1}{2} \phi^T \delta\left( \frac{1}{1 - g\Phi} G(\phi) \right) - \left( \frac{g}{1 - g\Phi} G(\phi) \right)^T \partial\xi \frac{1}{1 - g\Phi} G(\phi) \right].$$

The gauge variation of the second term in the right hand side of (84) is eliminated by $G(\partial\xi) = 0$. We find that the first term in the right hand side of (89) turns to

$$\sum_{k=0}^{\infty} \int d^D x \frac{1}{2} \sum_{l=0}^{k-1} (g\Phi)^l [G(\phi)]^T (g\partial\xi)(g\Phi)^{k-l-1} [G(\phi)]$$

$$= \int d^D x \frac{1}{2} \sum_{l=0}^{\infty} (g\Phi)^l [G(\phi)]^T (g\partial\xi) \sum_{k=0}^{\infty} (g\Phi)^k [G(\phi)]$$

$$= \int d^D x \frac{1}{2} \left( \frac{1}{1 - g\Phi} G(\phi) \right)^T (g\partial\xi) \frac{1}{1 - g\Phi} G(\phi)$$

$$= \int d^D x \left( \frac{1}{1 - g\Phi} G(\phi) \right)^T (g\partial\xi) \frac{1}{1 - g\Phi} G(\phi).$$

It follows that the right hand side of (89) cancels out. Summarizing, we find that the action $S_0$ in (84) is invariant under the gauge transformations (87) and (88).

In section 6, we will show that the action $S$ obtained in this section can be easily generalized to the action on AdS spaces.
5. Interaction of three gauge fields with spin-$s_1$, spin-$s_2$ and spin-$(s_1 + s_2)$

In this section, we will slightly generalize the model examined in the previous section. We introduce three gauge fields with spin-$s_1$, spin-$s_2$ and spin-$(s_1 + s_2)$. The spin-$s_I$ ($I = 1, 2$) fields are denoted as $d(s_I)$-component column vectors, such as $\phi_{\mu_1 \cdots \mu_{s_I}} = \phi^{(I)}$, $\phi^{*}_{\mu_1 \cdots \mu_{s_I}} = \phi^{*(I)}$ and $\partial(\mu_1 c_{\mu_2 \cdots \mu_{s_I}}) = \partial c^{(I)}$, while the spin-$(s_1 + s_2)$ fields are as rectangular matrices denoted as $\phi_{\mu_1 \cdots \mu_{s_1+s_2}} = \phi$, $\phi^{*}_{\mu_1 \cdots \mu_{s_1+s_2}} = \phi^*$ and $\partial(\mu_1 c_{\mu_2 \cdots \mu_{s_1+s_2}}) = \partial c$. In this notation, the free action is written as

$$S^0 = \int d^D x \left[ \frac{1}{2} \phi^{(I)} T G(\phi^{(I)}) + \phi^{*(I)} T \partial c^{(I)} + \text{tr} \left( \frac{1}{2} \phi G(\phi) + \phi^* \partial c \right) \right]$$

(91)

where the summation over $I = 1, 2$ is understood.

As explained above, $S^1$ is expanded as in (21) with $a_2 = 0$. In the present case, we choose

$$a_1 = \int d^D x \left[ G(\phi^{(I)})^{\mu_1 \cdots \mu_{s_I}} \partial(\mu_1 c_{\mu_2 \cdots \mu_{s_I}})_{\nu_1 \cdots \nu_{s_2}} \phi^{(2)}{\nu_1 \cdots \nu_{s_2}} \right.$$

$$G(\phi^{(2)})^{\nu_1 \cdots \nu_{s_2}} \partial(\nu_1 c_{\nu_2 \cdots \nu_{s_2}})_{\mu_1 \cdots \mu_{s_I}} \phi^{*(1)}{\mu_1 \cdots \mu_{s_I}}]$$

(92)

as $a_1$. We will show that this $a_1$ leads to an action of interacting three gauge fields. For notational simplicity, we write (92) as

$$a_1 = \int d^D x \left[ s_{IJ} G(\phi^{(I)})^T \tilde{\partial} c \phi^{*(J)} \right],$$

(93)

where $s_{IJ}$ denotes a $2 \times 2$ matrix of $s_{12} = s_{21} = 1$ and $s_{11} = s_{22} = 0$. In this notation, the index of the derivative in $\tilde{\partial} c$ is always contracted with one of the indices of the field sitting to its immediate left. In other words, $\tilde{\partial} c$ in (93) is a $d(s_I) \times d(s_J)$ matrix.

We can derive $S^1$, $S^2$ and $S^3$ in a similar way to the one presented in the preceding section. For the present paper to be self-contained, we give a brief derivation of them in appendix B. From the results obtained there, we can guess the form of $S^n$ ($n \geq 1$) as follows

$$S^n = a^n_2 + a^n_1 + \alpha^n_0,$$  

(94)

$$a^n_2 = \int d^D x \frac{1}{2} s_{IJ}^2 \Phi^{n-2} G(\tilde{\partial} c \phi^{*(I)})^T \tilde{\partial} c \phi^{*(J)},$$  

(95)

$$a^n_1 = \int d^D x s_{IJ}^1 \phi^{(I)} T \Phi^{n-1} G(\tilde{\partial} c \phi^{*(J)}),$$  

(96)

$$a^n_0 = \int d^D x \frac{1}{2} s_{IJ}^0 \phi^{(I)} T \Phi^n G(\phi^{(J)}),$$  

(97)

where $\Phi^{-1} = 0$ is understood because $a_2^2 = a_2 = 0$. We note $s_{IJ}^2 = \delta_{IJ}$, $s_{IJ}^3 = \delta_{IJ}$, and so on.

It is easy to see that $S^n$ for $n = 1, 2, 3$ coincide with those obtained in appendix B. We will derive $S^n$ supposing that $S^k$ ($k < n$) solve the master equation at the order of $g^k$ ($k < n$).

The master equation (18) at the order of $g^n$ is expanded as (71), (72) and (73). First, we shall solve (71) for $a_2^n$. It is straightforward to see that

$$- \frac{1}{2} \sum_{k=2}^{n-1} (a_2^k, a_1^{n-k}) - \frac{1}{2} \sum_{k=1}^{n-2} (a_2^k, a_2^{n-k}) = \int d^D x \frac{1}{2} s_{IJ}^1 \sum_{l=0}^{n-3} \Phi^{t_1} G(\tilde{\partial} c \phi^{*(I)})^T \tilde{\partial} c \Phi^{n-3-l} G(\tilde{\partial} c \phi^{*(J)})$$

$$= \Gamma \int d^D x \frac{1}{2} s_{IJ}^1 \Phi^{n-2} G(\tilde{\partial} c \phi^{*(I)})^T \tilde{\partial} c \phi^{*(J)}.$$

(98)
This implies that

$$\alpha_2^n = \int d^D x \frac{1}{2} s^n_{IJ} \Phi^{n-2}[\tilde{\partial} c c \Phi^*(I)] \tilde{\partial} c c \Phi^*(J),$$  \hspace{1cm} (99)

which coincides with (95).

Next we will derive $\alpha_1^n$ from (72). For this purpose, we derive

$$-\Delta \alpha_2^n - \frac{1}{2} \sum_{k=2}^{n-1} (\alpha_k^n, \alpha_{0-k}^n) - \frac{1}{2} \sum_{k=1}^{n-2} (\alpha_k^n, \alpha_{2-k}^n)$$

$$= \int d^D x s^n_{IJ} \sum_{l=0}^{n-2} \Phi^l[\tilde{G}(\tilde{\partial} c c \Phi^*(I))] \tilde{\partial} c c \Phi^{n-2-l}[G(\phi^J)],$$  \hspace{1cm} (100)

and

$$-\frac{1}{2} \sum_{k=1}^{n-1} (\alpha_k^n, \alpha_{1-k}^n) = \int d^D x s^n_{IJ} \sum_{l=0}^{n-2} \Phi^l[G(\phi^I)] \tilde{\partial} c c \Phi^{n-2-l}[\tilde{G}(\tilde{\partial} c c \Phi^*(J))].$$  \hspace{1cm} (101)

Gathering these results together, we find

$$-\Delta \alpha_2^n - \frac{1}{2} \sum_{k=2}^{n-1} (\alpha_k^n, \alpha_{0-k}^n) - \frac{1}{2} \sum_{k=1}^{n-2} (\alpha_k^n, \alpha_{1-k}^n) - \frac{1}{2} \sum_{k=1}^{n-2} (\alpha_k^n, \alpha_{2-k}^n)$$

$$= \int d^D x s^n_{IJ} \sum_{l=0}^{n-2} \Phi^l[\tilde{G}(\tilde{\partial} c c \Phi^*(I))] \tilde{\partial} c c \Phi^{n-2-l}[G(\phi^J)]$$

$$= \Gamma \int d^D x s^n_{IJ}(\tilde{\partial} c c \Phi^*(I))^T \Phi^{n-1}[G(\phi^J)].$$  \hspace{1cm} (102)

This implies that $\alpha_1^n$ is given as in (96).

Finally, we solve (73) for $\alpha_0^n$. It is now easy to derive

$$-\Delta \alpha_1^n - \frac{1}{2} \sum_{k=1}^{n-1} (\alpha_k^n, \alpha_{0-k}^n) - \frac{1}{2} \sum_{k=1}^{n-1} (\alpha_k^n, \alpha_{1-k}^n)$$

$$= \int d^D x s^n_{IJ} \sum_{l=0}^{n-1} \Phi^l[G(\phi^I)] \tilde{\partial} c c \Phi^{n-k}[G(\phi^J)]$$

$$= \int d^D x \frac{1}{2} \sum_{l=0}^{n-1} \Phi^l[G(\phi^I)] \tilde{\partial} c c \Phi^{n-k}[G(\phi^J)]$$

$$= \Gamma \int d^D x \frac{1}{2} \sum_{l=0}^{n-1} \Phi[I, J][G(\phi^J)].$$  \hspace{1cm} (103)

This implies that $\alpha_0^n$ is given as in (97).

Summarizing the above results, we have shown that $S^n$ is surely given as (94)-(97).
5.1. BRST-invariant action of interacting gauge fields of spin-$s_1$, $s_2$ and $s_1 + s_2$

We will examine the total action $S$. In the above we have derived $S^k$ $(k = 1, 2, \cdots)$. The free action $S^0$ is given in (91). Gathering these together, we obtain the total action

$$S = S^0 + \sum_{k=1}^{\infty} g^k S^k$$

$$= \int d^D x \left[ \frac{1}{2} \phi^{(I)T} G(\phi^{(I)}) + \phi^{* (I)T} \partial c^{(I)} + \text{tr} \left( \frac{1}{2} \phi G(\phi) + \phi^* \partial c \right) \right.$$  

$$+ \frac{1}{2} \phi^{(I)T} \sum_{k=1}^{\infty} g^k s^k_I J \Phi^k [G(\phi^{(I)})] + \phi^{(I)T} \sum_{k=1}^{\infty} g^k s^k_I J \Phi^{k-1} [G(\tilde{\partial} c \Phi^{* (J)})]$$  

$$+ \frac{1}{2} (\tilde{\partial} c \phi^{* (I)})^T \sum_{k=2}^{\infty} g^k s^k_I J \Phi^{k-2} [G(\tilde{\partial} c \phi^{* (J)})] \right]. \quad (104)$$

It is straightforward to see that the action turns to the form expanded in $agh$ as

$$S = S_0 + S_1 + S_2 \quad (105)$$

$$S_0 = \int d^D x \left[ \frac{1}{2} \phi^{(I)T} \left( \frac{1}{1 - g s \Phi} \right)_{IJ} G(\phi^{(J)}) + \text{tr} \left( \frac{1}{2} \phi G(\phi) \right) \right], \quad (106)$$

$$S_1 = \int d^D x \left[ \phi^{* (I)T} \partial c^{(I)} + \text{tr} (\phi^* \partial c) + \phi^{(I)T} \left( \frac{g s}{1 - g s \Phi} \right)_{IJ} \tilde{G}(\tilde{\partial} c \phi^{* (J)}) \right], \quad (107)$$

$$S_2 = \int d^D x \frac{1}{2} \left[ \left( \frac{g^2}{1 - g s \Phi} \right)_{IJ} \tilde{G}(\tilde{\partial} c \phi^{* (J)}) \right]^T \tilde{\partial} c \phi^{* (I)}, \quad (108)$$

where $agh(S_i) = i$. This action $S$ contains all orders in $g$, and invariant under BRST transformation $\delta_B^P S = (S, S) = 0$. This is one of our main results in this paper.

We shall examine the gauge invariance of the gauge invariant action $S_0$. The gauge transformation can be extracted from the BRST transformation $\delta_B^P X = (X, S)$. We find the gauge transformation with a rank-$(s_I - 1)$ parameter $\xi^{(I)}$ remains unchanged

$$\delta \phi^{(I)} = \partial \xi^{(I)}, \quad \delta \phi = 0, \quad (109)$$

while the gauge transformation with a rank-$(s_1 + s_2 - 1)$ parameter $\xi$ turns to

$$\delta \phi^{(I)} = - \left( \left[ \frac{g}{1 - g s \Phi} G(\phi) \right]^T \tilde{\partial} \xi \right)^T, \quad \delta \phi = \partial \xi. \quad (110)$$

It is instructive to check the gauge invariance of $S_0$ under (110). The gauge variation of $S_0$ is

$$\delta S_0 = \int d^D x \left[ \frac{1}{2} \phi^{(I)T} T \delta \left( \sum_{l=0}^{\infty} (g s l J \Phi)^I \right) G(\phi^{(J)}) - \left[ \frac{1}{1 - g s \Phi} G(\phi^{(I)}) \right]^T \tilde{\partial} \xi \frac{g s}{1 - g s \Phi} G(\phi^{(J)}) \right]. \quad (111)$$

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We find that the first term in the right hand side of (111) turns to
\[
\sum_{k=0}^{\infty} \int d^Dx \frac{1}{2} \sum_{l=0}^{k-1} (gs\Phi)^l [G(\phi^{(l)})] T \frac{g_s}{1-gs\Phi} \sum_{k=0}^{\infty} (gs\Phi)^k [G(\phi^{(J)})] \\
= \int d^Dx \frac{1}{2} \sum_{l=0}^{\infty} (gs\Phi)^l [G(\phi^{(l)})] T \frac{g_s}{1-gs\Phi} \sum_{k=0}^{\infty} (gs\Phi)^k [G(\phi^{(J)})] \\
= \int d^Dx \left[ \frac{1}{1-gs\Phi} G(\phi^{(l)}) \right] T \frac{g_s}{1-gs\Phi} G(\phi^{(J)}). \tag{112}
\]

It follows that the right hand side of (111) cancels out. As a result, we find that the action \( S_0 \) in (106) is invariant under the gauge transformations (109) and (110).

In the next section, we will write down an action of interacting three gauge fields on AdS spaces from the action obtained in this section.

6. Interacting higher spin gauge fields on AdS spaces

We have obtained actions of interacting gauge fields in sections 4 and 5. One of advantages in our approach is that it is easy to generalize the action on a flat spacetime to that on AdS spaces. In this section, we will write down actions of interacting gauge fields on AdS spaces from those obtained in sections 4 and 5.

First of all, we will introduce objects on AdS spaces. The Fronsdal tensor on AdS spaces is given as
\[
F_{\mu_1...\mu_s}(\phi) = \Box \phi_{\mu_1...\mu_s} - \nabla_{(\mu_1} \nabla^{\sigma} \phi_{\mu_2...\mu_s)\sigma} + \frac{1}{2} \nabla_{(\mu_1} \nabla_{\mu_2} \phi_{\mu_3...\mu_s)} - \frac{m^2}{l^2} \phi_{\mu_1...\mu_s} - \frac{2}{l^2} g(\phi_{\mu_1} \phi'_{\mu_2...\mu_s}), \tag{113}
\]
where \( \Box = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \) and \( l \) denotes the radius of the AdS space. The Fronsdal tensor in (113) reduces to (1) in the flat limit \( l \rightarrow \infty \). The \( m^2 \) means
\[
m^2 = s^2 + s(D - 6) - 2(D - 3). \tag{114}
\]

Under the gauge transformation
\[
\delta \phi_{\mu_1...\mu_s} = \nabla_{(\mu_1} \xi_{\mu_2...\mu_s)}, \tag{115}
\]
we obtain
\[
\delta F_{\mu_1...\mu_s}(\phi) = \frac{1}{2} \nabla_{(\mu_1} \nabla_{\mu_2} \nabla_{\mu_3} \xi'_{\mu_4...\mu_s)} - \frac{4}{l^2} g(\phi_{\mu_1} \phi'_{\mu_2} \nabla_{\mu_3} \xi'_{\mu_4...\mu_s}). \tag{116}
\]
This reduces to (3) in the flat limit. It follows that the Fronsdal tensor is gauge invariant if \( \xi' = 0 \). To derive this relation, we have used the fact on AdS spaces
\[
[\nabla_{\mu_1}, \nabla_{\nu}] \phi_{\alpha_2...\alpha_s} = R_{\mu\nu(\alpha_1 \lambda \phi_{\alpha_2...\alpha_s), \lambda}}, \quad R_{\mu\nu\alpha\beta} = -\frac{1}{l^2} (g_{\mu\alpha} g_{\nu\beta} - g_{\nu\alpha} g_{\mu\beta}). \tag{117}
\]

\*Note that the third term in the right hand side contains a factor 1/2. This is needed for (113) to reproduce (1) in the flat limit. In fact, the term \( \frac{1}{2} (\nabla_{\mu_1} \phi'_{\mu_2} + \nabla_{\mu_2} \nabla_{\mu_1}) \phi' \) contained in the right hand side of (113) becomes \( \partial_{\mu_1} \partial_{\mu_2} \phi' \) in the flat limit. On the other hand, the right hand side of (1) contains \( \partial_{\mu_1} \partial_{\mu_2} \phi' \), which coincides with the flat limit of (113).
A useful and important relation is

\[
\int d^D x \sqrt{-g} \phi^\mu_1 \cdots \phi^\mu_s \tilde{G}_{\mu_1 \cdots \mu_s} (\psi) = \int d^D x \sqrt{-g} \tilde{G}_{\mu_1 \cdots \mu_s} (\varphi) \psi^\mu_1 \cdots \psi^\mu_s ,
\]

where

\[
\tilde{G}_{\mu_1 \cdots \mu_s} (A) = G_{\mu_1 \cdots \mu_s} (A) + \frac{1}{4} g_{\mu_1 \nu_1} \nabla_{\nu_1} \nabla_{\mu_s} A_{\mu_2 \cdots \mu_s} ,
\]

\[
G_{\mu_1 \cdots \mu_s} (A) = F_{\mu_1 \cdots \mu_s} (A) - \frac{1}{2} g_{(\mu_1 \nu_2} F'_{\mu_3 \cdots \mu_s)} (A) .
\]

Observe that (119) surely reduces to (8) in the flat limit.

Now we will write down the action of interacting gauge fields on AdS spaces. It is straightforward to obtain the action on AdS spaces from the action obtained in section 4. One obtains

\[
S = S_0 + S_1 + S_2
\]

\[
S_0 = \int d^D x \sqrt{-g} \left[ \frac{1}{2} \phi^T \frac{1}{1 - g \Phi} G (\phi) + \text{tr} \left( \frac{1}{2} \phi \phi^\phi \right) \right] ,
\]

\[
S_1 = \int d^D x \sqrt{-g} \left[ \phi^* T \nabla c + \text{tr} (\phi^* \nabla c) + \phi^T \frac{g}{1 - g \Phi} \tilde{G} (\nabla c \phi^*) \right] ,
\]

\[
S_2 = \int d^D x \sqrt{-g} \frac{1}{2} (\nabla c \phi^*)^T \frac{g^2}{1 - g \Phi} \tilde{G} (\nabla c \phi^*) .
\]

On the other hand, from the action obtained in section 5 one obtains

\[
S = S_0 + S_1 + S_2
\]

\[
S_0 = \int d^D x \sqrt{-g} \left[ \frac{1}{2} \phi^{(I)T} \frac{1}{1 - g s \Phi} I J G (\phi^{(J)}) + \text{tr} \left( \frac{1}{2} \phi \phi^\phi \right) \right] ,
\]

\[
S_1 = \int d^D x \sqrt{-g} \left[ \phi^{* (I)T} \nabla c^{(I)} + \text{tr} (\phi^* \nabla c) + \phi^{(I)T} \frac{g s}{1 - g s \Phi} I J \tilde{G} (\nabla c \phi^{* (J)}) \right] ,
\]

\[
S_2 = \int d^D x \sqrt{-g} \frac{1}{2} \left[ \frac{g^2}{1 - g s \Phi} I J \tilde{G} (\nabla c \phi^{* (J)}) \right]^T \nabla c \phi^{* (I)} .
\]

These are the actions of interacting gauge fields on AdS spaces. These contain all orders in \( g \), and are invariant under the BRST transformation \( \delta_B S = (S, S) = 0 \).

7. Summary and Discussion

We have constructed actions of interacting bosonic higher spin gauge fields in the BRST-antifield formalism. In section 4, two gauge fields of spin-2s and spin-s are examined, and three gauge fields of spin-s_1, spin-s_2 and spin-(s_1 + s_2) are examined in section 5. In both cases, we constructed an action \( S \) as a deformation of the free action \( S^0 \), \( S = S^0 + g S^1 + g^2 S^2 + \cdots \). Choosing a cubic interaction as \( S^1 \), we solved the master equation \( (S, S) = 0 \) from the lower orders in \( g \). We note that our obtained action \( S \) contains all orders in \( g \) and is BRST-invariant \( \delta_B S = (S, S) = 0 \), while the free action \( S^0 \) is BRST-invariant \( \delta_B S^0 = (S^0, S^0) = 0 \). Our action is composed of terms with \( a g h \leq 2 \), and may be rearranged as \( S = S_0 + S_1 + S_2 \) where \( a g h (S_i) = i \). From the BRST transformation \( \delta_B X = (X, S) \), we have extracted the gauge transformation and showed the gauge invariance of \( S_0 \). In section 6, we
have written down actions of interacting bosonic higher spin gauge fields on AdS spaces from those obtained in sections 4 and 5.

The obtained action is composed of \( n + 2 \) fields and \( 2n + 2 \) derivatives at the order of \( g^n \). The cubic interaction term contains four derivatives. This is compared with the models composed of the generalized curvature tensor. These models avoid the no-go theorem by restricting the number of derivatives. Our models suffer from the no-go theorem as well. So the spin of the fields should be restricted appropriately [4] as follows: \( s = 1, 2 \) for the \((s, 2s, s)\) cubic coupling while \((s_1, s_2, s_3) = (1, 1, 2), (1, 2, 3)\) and \((2, 2, 4)\) for the \((s_1, s_2, s_1 + s_2)\) cubic coupling.

Higher spin gauge theories have attracted renewd interests in the study of the AdS/CFT correspondence [13]. Higher spin gauge theories have been conjectured to be dual to simple conformal field theories.\(^\parallel\) Our models on AdS spaces are much simpler than them and actions are given in this paper. It is interesting to explore the CFT duals to our models.

In this paper, we have examined a special kind of interaction terms, in which each interaction term in \( S \) forms an open chain of fields. This enabled us to employ a matrix notation for higher spin fields. It is possible to use this notation even if each interaction term forms a closed chain of fields. In fact, we have found that even in this case we can construct an action of interacting bosonic higher spin gauge theory. We hope to report this result in another place [12]. The most general interaction term may not form a chain of fields but a complete graph (or a simplex), for example a complete graph with four-vertices \( K_4 \) (or a tetrahedron) for a four-point interaction. Each vertex represents a field and a edge represents a contraction between two fields. In this case, we must employ a tensor-like notation. This is left for a future investigation.

Another interesting issue to pursue is to include fermionic and bosonic gauge fields with mixed indices. There are some interesting works examining a deformation of the free action with fermions. In [18][19] a systematic analysis including fermionic gauge fields was performed in the BRST-antifield formalism\(^\ast\ast\). It is interesting to examine actions of interacting bosonic and fermionic fields including higher orders in \( g \), and to generalize them to those on AdS spaces.

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\(^\parallel\) A bosonic higher spin gauge theory on a three-dimensional AdS space (AdS\(_3\)) [14] has been conjectured [15] to be dual to the ’t Hooft limit of the \( W_N \) minimal model, while a Vasiliev’s higher spin gauge theory on AdS\(_4\) [16] is conjectured [17] to be dual to three-dimensional \( O(N) \) sigma-model.

\(^\ast\ast\) The results were shown to be consistent with those obtained in the light-cone formulation [5] and with one obtained in the tensionless limit of string theory [20].
center, Tsukuba, 3-6 December 2019, for their kind hospitality. In these workshops, H.S. reported preliminary results summarized in this paper.

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M. Taronno, “Higher Spins and String Interactions,” [arXiv:1005.3061 [hep-th]].
A. Γ trivial $a_2$ results in BRST trivial $S^1$

When $S^1$ is BRST trivial, $S^1 = \delta B m$, $a_i$ may be expressed as

$$a_2 = \Gamma m_2 \quad \text{(A1)}$$
$$a_1 = \Delta m_2 + \Gamma m_1 \quad \text{(A2)}$$
$$a_0 = \Delta m_1 + \Gamma m_0 \quad \text{(A3)}$$

The reason why $\Delta m_3$ is not included in (A1) is as follows. As $\Delta m_3$ has $agh = 2$ and $gh = 0$, $\Delta m_3$ is composed of a $c^*$ and two $c$’s, But it is not possible to construct $m_3$ from such a $\Delta m_3$.

Suppose that $a_2$ is Γ-exact as is given in (A1). Substituting this into (24), we derive

$$\Gamma a_1 = -\Delta (\Gamma m_2) = \Gamma (\Delta m_2) \quad \text{(A4)}$$

so that $a_1$ turns to the form of (A2). Similarly, when $a_1$ takes the form (A2), substituting it into (25) one sees that $a_0$ takes the form (A3). We conclude that a Γ-trivial $a_2$ leads to a BRST-trivial $S^1$. So we may set $a_2 = 0$. Similarly a Γ-trivial $a_1$ leads to a BRST-trivial $S^1$.

B. Derivation of three-, four- and five-point interaction of three gauge fields

We will derive $S^1$, $S^2$ and $S^3$ from (93).

**Three-point interaction**

It is obvious that $a_1$ in (93) satisfies $\Gamma a_1 = 0$ since $\Gamma G(\phi^{(I)}) = 0$. Noting that

$$-\Delta a_1 = \int d^D x \, s_{IJ} G(\phi^{(I)}) T \bar{\phi} G(\phi^{(J)}) \quad \text{(B1)}$$

and that

$$\Gamma \int d^D x \, \frac{1}{2} s_{IJ} G(\phi^{(I)}) T \phi G(\phi^{(J)})$$

$$= \int d^D x \, \frac{1}{2} s_{IJ} G(\phi^{(I)}) \mu_1 \cdots \mu_s \delta(\mu_1 \cdots \mu_s, \nu_1 \cdots \nu_s) G(\phi^{(J)}) \nu_1 \cdots \nu_s$$

$$= \int d^D x \, s_{IJ} G(\phi^{(I)}) T \bar{\phi} G(\phi^{(J)}) \quad \text{(B2)}$$

we conclude that from (25)

$$a_0 = \int d^D x \, \frac{1}{2} s_{IJ} G(\phi^{(I)}) T \phi G(\phi^{(J)}) \quad \text{(B3)}$$

As a result, we obtained $S^1$ in (21) composed of (27), (93) and (B3).
Four-point interaction

Next we derive $S^2$ in (36). First we will solve (38) for $b_2$ and $b_1$. One derives

$$ -\frac{1}{2}(a_1, a_1) = \int d^D x \left[ G(\phi(I))^T \partial c G(\partial c \phi^*(I)) \right] , \hspace{1cm} (B4) $$

as $s_{IJK} = \delta_{IK}$. Noting that the right hand side of the above equation can be expressed as

$$ \Gamma \int d^D x G(\phi(I))^T \phi \tilde{G}(\partial c \phi^*(I)) + \Delta \int d^D x \frac{1}{2} \tilde{G}(\partial c \phi^*(I))^T \partial c \phi^*(I) , \hspace{1cm} (B5) $$

we obtain

$$ b_2 = \int d^D x \frac{1}{2} \tilde{G}(\partial c \phi^*(I))^T \partial c \phi^*(I) , \hspace{1cm} (B6) $$

$$ b_1 = \int d^D x G(\phi(I))^T \phi \tilde{G}(\partial c \phi^*(I)) . \hspace{1cm} (B7) $$

It is obvious that (37) is satisfied. Next we will solve (39) for $b_0$. It is straightforward to see that

$$ -\Delta b_1 - (a_1, a_0) = \Gamma \int d^D x \frac{1}{2} G(\phi(I))^T \phi \tilde{G}(\phi G(\phi(I))) . \hspace{1cm} (B8) $$

This implies that

$$ b_0 = \frac{1}{2} G(\phi(I))^T \phi \tilde{G}(\phi G(\phi(I))) . \hspace{1cm} (B9) $$

We conclude that $S^2$ in (36) is found to be composed of (B6), (B7) and (B9).

Five-point interaction

The $S^3$ is expanded as in (49). First we solve (50) for $c_2$. Deriving

$$ -(a_1, b_2) = \Gamma \int d^D x \frac{1}{2} s_{IJ} \tilde{G}(\partial c \phi^*(I))^T \partial c \tilde{G}(\partial c \phi^*(J)) , \hspace{1cm} (B10) $$

we obtain

$$ c_2 = \int d^D x \frac{1}{2} s_{IJ} \tilde{G}(\partial c \phi^*(I))^T \partial c \tilde{G}(\partial c \phi^*(J)) . \hspace{1cm} (B11) $$

Next we examine (51). It is easy to derive

$$ -\Delta c_2 = \int d^D x s_{IJ} \tilde{G}(\partial c \phi^*(I))^T \partial c \tilde{G}(\phi G(\phi(J))) , \hspace{1cm} (B12) $$

$$ -(a_1, b_1) = \int d^D x s_{IJ} \left[ G(\phi(I))^T \partial c \tilde{G}(\phi \tilde{G}(\partial c \phi^*(J))) + \tilde{G}(\phi G(\phi(I)))^T \partial c \tilde{G}(\partial c \phi^*(J)) \right] , \hspace{1cm} (B13) $$

$$ -(a_0, b_2) = \int d^D x s_{IJ} \tilde{G}(\partial c \phi^*(I))^T \partial c \tilde{G}(\phi G(\phi(J))) . \hspace{1cm} (B14) $$

The sum of the right hand side of (B12) and the first term in the right hand side of (B13) turns to

$$ \int d^D x s_{IJ} \tilde{G}(\phi \tilde{G}(\partial c \phi^*(I)))^T \partial c G(\phi(J)) , \hspace{1cm} (B15) $$

while the second term in the right hand side of (B13) plus the right hand side of (B14) becomes

$$ \Gamma \int d^D x s_{IJ} \tilde{G}(\phi G(\phi(I)))^T \partial c \tilde{G}(\partial c \phi^*(J)) - \int d^D x s_{IJ} \tilde{G}(\partial c G(\phi(I)))^T \partial c G(\partial c \phi^*(J)) . \hspace{1cm} (B16) $$
It follows from these results that
\[
-\Delta c_2 - (a_1, b_1) - (a_0, b_2) = \Gamma \int d^Dx \, s_{IJ} \tilde{G}(\phi G(\phi^{(I)}))^T \tilde{\phi} \tilde{G}(\tilde{\partial} c \phi^{(J)}) ,
\]
so that we may obtain
\[
c_1 = \int d^Dx \, s_{IJ} \tilde{G}(\phi G(\phi^{(I)}))^T \tilde{\phi} \tilde{G}(\tilde{\partial} c \phi^{(J)}) .
\]

Finally, (52) is examined. It is straightforward to see that
\[
-\Delta c_1 - (a_1, b_0) = \Gamma \int d^Dx \, \frac{1}{2} s_{IJ} \tilde{G}(\phi G(\phi^{(I)}))^T \phi \tilde{G}(\phi G(\phi^{(J)})) - \int d^Dx \, \frac{1}{2} s_{IJ} \tilde{G}(\phi G(\phi^{(I)}))^T \tilde{\partial} c \tilde{G}(\phi G(\phi^{(J)})) ,
\]
and that
\[
-(a_0, b_1) = \int d^Dx \, s_{IJ} \tilde{G}(\phi G(\phi^{(I)}))^T \tilde{\partial} c \tilde{G}(\phi G(\phi^{(J)}))
\]
which cancels out the second term in the right hand side of (B19). As a result we obtain
\[
-\Delta c_1 - (a_1, b_0) - (a_0, b_1) = \Gamma \int d^Dx \, s_{IJ} \frac{1}{2} \tilde{G}(\phi G(\phi^{(I)}))^T \phi \tilde{G}(\phi G(\phi^{(J)})) ,
\]
which implies that
\[
c_0 = \int d^Dx \, s_{IJ} \frac{1}{2} \tilde{G}(\phi G(\phi^{(I)}))^T \phi \tilde{G}(\phi G(\phi^{(J)})) .
\]
Summarizing the results, we find the $S^3$ in (49) is composed of (B11), (B18) and (B22).