Ultraproducts of continuous posets

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We dedicate this paper to Ervin Fried, teacher and friend.

It is known that nontrivial ultraproducts of complete partially ordered sets (posets) are almost never complete. We show that complete additivity of functions is preserved in ultraproducts of posets.

An \( n \)-ary function \( f \) in a poset is said to be \textit{completely additive} if \( f(s_1, \ldots, s_n) \) is the supremum of \( \{ f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n \} \) whenever \( s_1, \ldots, s_n \) are the suprema of \( X_1, \ldots, X_n \) respectively. Completely additive functions are also called sup-preserving, or continuous.

\textbf{Theorem 1} Assume that \( \mathfrak{B}_i \) are posets with completely additive operations, for \( i \in I \). The operations remain completely additive in \( \mathfrak{B} = P(\mathfrak{B}_i : i \in I)/F \), the ultraproduct of the \( \mathfrak{B}_i \)'s modulo any ultrafilter \( F \) on \( I \).

\textbf{Proof.} First we consider the case of a unary operation \( f \), and then we will reduce the general case to this one. Assume that \( f_i \) is sup-preserving in \( \mathfrak{B}_i \) for all \( i \in I \) and let \( f \) denote their ultraproduct.

Case I: \( f \) is unary.

Assume that \( X \subseteq B \) and \( s \) is the supremum of \( X \) in \( \mathfrak{B} \). We want to show that \( f(s) \) is the supremum of \( \{ f(x) : x \in X \} \) in \( \mathfrak{B} \). Let \( f(X) \) denote \( \{ f(x) : x \in X \} \). It is clear that \( f(s) \) is an upper bound of \( f(X) \), since a sup-preserving function is always monotonic. Assume that \( y \) is any upper bound for \( f(X) \), we want to show that \( y \geq f(s) \).

Consider the formula \( \alpha(x, s, y) \) defined as \( x \leq s \land f(x) \leq y \). Let \( z = \sup A \) denote that \( z \) is the supremum of \( A \). We show that

\begin{equation}
(1) \quad s = \sup A, \text{ where } A = \{ x : \alpha(x, s, y) \}.
\end{equation}
Indeed, $s$ is an upper bound for $A$ since $\alpha(x, s, y) \rightarrow x \leq s$. Assume $z$ is any upper bound for $A$. By our assumptions that $s = \sup X$ and $y$ is an upper bound for $f(X)$ we have $X \subseteq A$. Thus $z$ is an upper bound for $X$ (since it is so for $A$), hence $z \geq s$ since $s = \sup X$, as was desired.

Now we will use Los Lemma, the Fundamental Theorem of Ultraproducts, to show the existence of the analogous suprema in almost all of the $\mathcal{B}_i$. Since $A$ is defined by the formula $\alpha$, we get that (1) is expressible by a first-order logic formula $\sigma(s, y)$, namely the following formula will do:

$$\forall z (\forall x (\alpha(x, s, y) \rightarrow x \leq z) \rightarrow s \leq z).$$

(We omitted the part $\forall x (\alpha(x, s, y) \rightarrow s \leq x)$ since this follows directly from the definition of $\alpha$.) Let $\bar{s} \in s$, $\bar{y} \in y$ be arbitrary (i.e., $s = \bar{s}/F$, $y = \bar{y}/F$), by the Los Lemma then we have $\{i \in I : \mathcal{B}_i \models \sigma(\bar{s}_i, \bar{y}_i)\} \in F$. Let $J$ denote this set and let $i \in J$. Let $A_i = \{x \in B_i : \alpha(x, \bar{s}_i, \bar{y}_i)\}$, then $\mathcal{B}_i \models \sigma(\bar{s}_i, \bar{y}_i)$ implies that $\bar{s}_i = \sup A_i$. By continuity of $f_i$ we get that

$$f_i(\bar{s}_i) = \sup f_i(A_i).$$

This last statement can also be expressed by a first-order logic formula, namely by the following $\varphi(s, y)$:

$$\forall z (\forall x (\alpha(x, s, y) \rightarrow f(x) \leq z) \rightarrow f(s) \leq z).$$

(We used $\alpha(x, s, y) \rightarrow f(x) \leq y$.) Thus, by (2) we have that $\mathcal{B}_i \models \varphi(\bar{s}_i, \bar{z}_i)$ for all $i \in J \in F$. By the Los Lemma we get $\mathcal{B} \models \varphi(s, y)$, i.e.,

$$f(s) = \sup f(A).$$

Since $y$ is an upper bound of $f(A)$ (by $\alpha(x, s, y) \rightarrow f(x) \leq y$), we get $f(s) \leq y$ as was desired.

Case II: $f$ is $n$-ary for some $n \geq 2$.

We reduce this case to Case I by using a straightforward generalization of Theorem 1.6 (i’), (ii’) of [2] that states that an operation is sup-preserving iff it is sup-preserving in each of its coordinates. We write out the simple proof because we are in a slightly different setting (in [2], the poset is assumed to be a Boolean algebra).

Assume that $\mathfrak{A}$ is any poset in which $f$ is an $n$-ary function. For any elements $a_1, \ldots, a_n \in A$ and $j \in \{1, \ldots, n\}$ let $f(a_1, \ldots, a_{j-1}, -, a_{j+1}, \ldots, a_n)$
denote the unary function that takes any \( z \in A \) to \( f(a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_n) \). (When \( j = 1 \) use \( f(-, a_2, \ldots, a_n) \) and when \( j = n \) use \( f(a_1, \ldots, a_{n-1}, -) \) in place of \( f(a_1, \ldots, a_{j-1}, -, a_{j+1}, \ldots, a_n) \)). Let’s call \( f(a_1, \ldots, a_{j-1}, -, a_{j+1}, \ldots, a_n) \) a unary instance of \( f \).

**Lemma 1** (a version of Thm.1.6 in [2]) Assume that \( A \) is any poset in which \( f \) is an \( n \)-ary operation. Statements (i) and (ii) below are equivalent:

(i) \( f \) is sup-preserving

(ii) all the unary instances of \( f \) are sup-preserving

**Proof of Lemma 1** Assume (i), then (ii) holds by using \( X_i = \{a_i\} \) when \( i \neq j \) in the definition of \( f \) being sup-preserving.

For simplicity and transparency, we write out the proof of (ii) \( \Rightarrow \) (i) for the case \( n = 2 \). Assume (ii) and let \( X, Y \subseteq A \) with \( s, z \) being the suprema of \( X, Y \) respectively. We have to show that \( f(s, z) \) is the supremum of \( Z = \{f(x, y) : x \in X, y \in Y\} \). By our assumption (ii) we have that \( f(x, z) \) is the supremum of \( Z_x = \{f(x, y) : y \in Y\} \), for all \( x \in X \). Also, \( f(s, z) \) is the supremum of \( V = \{f(x, z) : x \in X\} \). Thus, it is enough to show that the supremum of \( V \) is the same as the supremum of \( Z \). Clearly, any upper bound \( b \) of \( V \) is an upper bound of \( Z \), because \( f(x, y) \leq f(x, z) \leq f(s, z) \leq b \).

Assume that \( b \) is an upper bound of \( V \), we want to show that it is an upper bound of \( V \). Indeed, \( f(x, z) \) is the supremum of \( Z_x \subseteq Z \), so \( f(x, z) \leq b \) since \( b \) is an upper bound of \( Z \supseteq Z_x \), and we are done with proving the equivalence of (i) and (ii). Lemma 1 has been proved.

Assume now that \( f_i \) is a sup-preserving operation in each \( \mathfrak{B}_i \). We want to show that all the unary instances of their ultraproduct \( f \) are sup-preserving in \( \mathfrak{B} \). For transparency, we write out the proof for \( n = 2 \). Let \( y \in B \) and \( \bar{y} \in \mathfrak{B} \). Then \( f(y, -) \) is the ultraproduct of the unary functions \( f_i(y_i, -) \) of \( \mathfrak{B}_i \) (for \( i \in I \)). All these are sup-preserving by Lemma 1 and our assumption that \( f_i \) of \( \mathfrak{B}_i \) are sup-preserving. By the already proven Case I then we get that their ultraproduct, \( f(y, -) \), is sup-preserving, too. Similarly for \( f(-, y) \) for any \( y \in B \). By Lemma 1 then \( f \) is sup-preserving in \( \mathfrak{B} \). \( \square \)

We close the paper with some remarks.

a) Let’s call a function *quasi-complete* if it preserves the suprema of nonempty sets. Thus, quasi-complete functions may not take the smallest element, which when it exists is the supremum of the empty set, to itself,
in other words, they are not necessarily normal. The proof of Thm.1 goes through for quasi-complete operations, too, as follows. Since we want to show quasi-completeness of \( f \), we assume \( X \neq \emptyset \). Then \( A \neq \emptyset \) by \( X \subseteq A \), and hence \( A_i \neq \emptyset \) for almost all \( i \in I \). So we can use quasi-completeness of \( f_i \) to infer \( f(\bar{s}_i) = \sup f(A_i) \) from \( \bar{s}_i = \sup A_i \), and this is the only place in the proof of Thm.1 where we used completeness of \( f_i \).

b) Our theorem has an application in considering ultraproducts of complex algebras of relational structures. Let us note that complex algebras are always complete Boolean algebras with sup-preserving operations. Suppose one starts with a system of relational structures, then forms an ultraprodut of their complex algebras. Now, this ultraproduct most likely is not complete. Givant has shown in [1, Thm.1.35, pp.56-60] that the completion of this ultraproduct is canonically isomorphic to the complex algebra of the corresponding ultraproduct of the original relational structures. As Givant pointed out to us, for his proof to be valid, one must know that the ultraproduct of the complex algebras is a Boolean algebra with quasi-complete operators in order to be able to form the completion of this algebra, and this part of the argument was missing from [1]. The present Thm.1 makes the proof complete.

References

[1] Givant, S. R., Duality theories for Boolean algebras with operators. Springer Monographs in Mathematics, Springer, 2014. xiv+233pp.

[2] Jónsson, B., Tarski, A., Boolean Algebras with operators. Part I. Amer. J. Math. 73 (1951), pp.891-939.