The Cost of Bounded Curvature

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Abstract

We study the motion-planning problem for a car-like robot whose turning radius is bounded from below by one and which is allowed to move in the forward direction only (Dubins car). For two robot configurations \( \sigma, \sigma' \), let \( \ell(\sigma, \sigma') \) be the shortest bounded-curvature path from \( \sigma \) to \( \sigma' \). For \( d \geq 0 \), let \( \ell(d) \) be the supremum of \( \ell(\sigma, \sigma') \), over all pairs \( (\sigma, \sigma') \) that are at Euclidean distance \( d \). We study the function \( \text{dub}(d) = \ell(d) - d \), which expresses the difference between the bounded-curvature path length and the Euclidean distance of its endpoints. We show that \( \text{dub}(d) \) decreases monotonically from \( \text{dub}(0) = \frac{7\pi}{3} \) to \( \text{dub}(d^*) = 2\pi \), and is constant for \( d \geq d^* \). Here \( d^* \approx 1.5874 \). We describe pairs of configurations that exhibit the worst-case of \( \text{dub}(d) \) for every distance \( d \).

1 Introduction

Motion planning or path planning involves computing a feasible path, possibly optimal for some criterion such as time or length, of a robot moving among obstacles; see the book by Lavalle \cite{lavalle} and book chapters by Halperin et al. \cite{halperin} and Sharir \cite{sharir}. A robot generally comes with physical limitations, such as bounds on its velocity, acceleration or curvature. Such differential constraints restrict the geometry of the paths the robot can follow. In this setting, the goal of motion planning is to find a feasible (or optimal) path satisfying both global (obstacles) and local (differential) constraints if it exists.

In this paper, we study the bounded-curvature motion planning problem which models a car-like robot. A car (with front-wheel steering) is constrained to move in the direction that the rear wheels are pointing, and it has a fixed maximum steering angle. This makes the car travel in a motion with fixed minimum turning radius, which means that the car must follow a curvature-constrained path. More precisely, we have the following robot model:

Robot model (Dubins car). The robot is considered a rigid body that moves in the plane. A configuration of the robot is specified by both its location, a point in \( \mathbb{R}^2 \) (typically, the midpoint of the rear axle), and its orientation, or direction of travel. The robot is constrained to move in the forward direction, and its turning radius is bounded from below by a positive constant, which can be assumed to be equal to one by scaling the space. In this context, the robot follows a bounded-curvature path, that is, a differentiable curve whose curvature is constrained to be at most one almost everywhere.

Planning the motion of a car-like robot has received considerable attention in the literature. In this paper, we consider the cost of this restriction: How much longer is the shortest path made by such a robot compared to the Euclidean distance travelled?

Formally, consider two configurations \( \sigma \) and \( \sigma' \). Let \( \ell(\sigma, \sigma') \) denote the length of a shortest curvature-constrained path from \( \sigma \) to \( \sigma' \), and let \( d(\sigma, \sigma') \) denote the Euclidean distance between \( \sigma \) and \( \sigma' \). We define

\[
\text{dub}(d) = \sup \{ \ell(\sigma, \sigma') - d \mid \sigma, \sigma' \text{ configurations with } d(\sigma, \sigma') = d \}.
\]

Note that the supremum here is not a maximum, as the path length is not a continuous function of the orientations at the two endpoints. Our goal is to understand the function \( \text{dub}: \mathbb{R} \to \mathbb{R} \) in detail. While

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this is a natural and fundamental question related to motion planning with bounded curvature, it is also a relevant question that has repeatedly appeared in the literature, with only partial answers so far.

Dubins [11] showed that the shortest curvature-constrained path between two configurations consists of at most three segments, each of which is either a straight segment or a circular arc of radius one. Using ideas from control theory, Boissonnat et al. [7], in parallel with Sussmann and Tang [20], gave an alternative proof. Sussmann [25] extended the characterization to the 3-dimensional case. Bui et al. [10] discussed how the types of optimal paths partition the configuration space, and also proved that optimal paths for free final orientation have at most two segments [6]. Significant work has been done on the problems of deciding whether a bounded-curvature path exists between given configurations among different kinds of obstacles and finding the shortest such path [12, 5, 21, 15, 3, 2, 1, 9, 8].

At least two interesting problems have been studied where not configurations but only locations for the robot are given. The first problem considers a sequence of points in the plane, and asks for the shortest curvature-constrained path that visits the points in this sequence. In the second problem, the Dubins traveling salesman problem, the input is a set of points in the plane, and asks to find a shortest curvature-constrained path visiting all points. Both problems have been studied by researchers in the robotics community, giving heuristics and experimental results [22, 19, 20]. From a theoretical perspective, Lee et al. [18] gave a linear-time, constant-factor approximation algorithm for the first problem. No general approximation algorithms are known for the Dubins traveling salesman problem (the approximation factor of the known algorithms depends on the smallest distance between points).

All this work depends on some knowledge of the function $\text{dub}(\cdot)$. Lee et al. [15], for instance, prove that the approximation ratio of their algorithms is $\max(A, \pi/2 + B/\pi)$, where $A = 1 + \sup\{\text{dub}(d)/d \mid d \geq 2\}$ and $B = \sup\{\text{dub}(d) + d \mid d \leq 2\}$. They claim without proof that $\text{dub}(d) \leq 2\pi$ for $d \geq 2$ and derive from this that $A = 1 + \pi$ and $B \leq 5\pi/2 + 3$, leading to an approximation ratio of about 5.03. We give the first proof of $A = 1 + \pi$, and improve the second bound to $B = 2 + 2\pi$, improving the approximation ratio of their algorithm to $2 + 2/\pi + \pi/2 \approx 4.21$.

Savla et al. [23] prove that $\text{dub}(d) \leq \kappa \pi$, where $\kappa \in [2.657, 2.658]$, and conjecture based on numerical experiments that the true bound is $7\pi/3$. We show that this is indeed true.

Results. We show that $d \mapsto \text{dub}(d)$ is a decreasing function with two breakpoints, at $\sqrt{2}$ and at $d^* \approx 1.5874$ (see Figure 1). More precisely, we have $\text{dub}(0) = 7\pi/3$ and the two breakpoint values are $\text{dub}(\sqrt{2}) = 5\pi/2 - \sqrt{2}$, and $\text{dub}(d^*) = 2\pi$. The function $\text{dub}(d)$ is constant and equal to $2\pi$ for $d \geq d^*$.

![Figure 1: The graph of the function $\text{dub}(d)$.](image)

For $0 \leq d < \sqrt{2}$ and for $d \geq d^*$, the supremum in $[1]$ is in fact a maximum, and we give configurations $\sigma, \sigma'$ at distance $d$ such that $\ell(\sigma, \sigma') = \text{dub}(d) + d$. Perhaps surprisingly, for $\sqrt{2} \leq d < d^*$, there are no such configurations—the supremum is not a maximum.
2 Preliminaries

Notations. For two points \( P \) and \( Q \), we denote by \( \overline{PQ} \) the line segment with endpoints \( P \) and \( Q \), and by \( \hat{PQ} \) an arc of unit radius with endpoints \( P \) and \( Q \). (If the length of \( \overline{PQ} \) is less than two then there are four such arcs, so unless it is clear from the context, we will specify the supporting circle and the orientation of the arc.) We denote the length of the segment \( \overline{PQ} \) as \( |\overline{PQ}| \) or simply as \( |PQ| \), and the length of the arc \( \hat{PQ} \) as \( |\hat{PQ}| \).

Without loss of generality, we assume that the starting configuration is \((0, 0, \alpha)\)—that is, we start at the origin \( S = (0, 0) \) with orientation \( \alpha \)—and the final configuration is \((d, 0, \beta)\)—that is, we arrive at \( F = (d, 0) \) with orientation \( \beta \). Here, \( \alpha \) and \( \beta \) express the orientation of the robot as an angle with the positive \( x \)-axis, and \( d \geq 0 \) is the Euclidean distance of the two configurations.

The open unit (radius) disks tangent to the starting and final configurations are denoted \( L_S, R_S, L_F, R_F \), where the letters \( L \) or \( R \) depend on whether the disk is located on the left or right side of the direction vector (see Figure 2).

Let \( \ell_S, r_S, \ell_F, r_F \) denote the centers of \( L_S, R_S, L_F, R_F \), respectively. For future reference, we note their coordinates:

\[
\ell_S = (\cos(\alpha + \pi/2), \sin(\alpha + \pi/2)) = (-\sin \alpha, \cos \alpha) \tag{2}
\]
\[
r_S = (\cos(\alpha - \pi/2), \sin(\alpha - \pi/2)) = (\sin \alpha, -\cos \alpha) \tag{3}
\]
\[
\ell_F = (d + \cos(\beta + \pi/2), \sin(\beta + \pi/2)) = (d - \sin \beta, \cos \beta) \tag{4}
\]
\[
r_F = (d + \cos(\beta - \pi/2), \sin(\beta - \pi/2)) = (d + \sin \beta, -\cos \beta). \tag{5}
\]

Distances between centers. The following distances will be frequently used:

\[
d_{\ell \ell} = |\ell_S \ell_F| = \sqrt{(d - \sin \beta + \sin \alpha)^2 + (\cos \beta - \cos \alpha)^2} \tag{6}
\]
\[
d_{rr} = |r_S r_F| = \sqrt{(d + \sin \beta - \sin \alpha)^2 + (-\cos \beta + \cos \alpha)^2} \tag{7}
\]
\[
d_{\ell r} = |\ell_S r_F| = \sqrt{(d + \sin \beta + \sin \alpha)^2 + (-\cos \beta - \cos \alpha)^2} \tag{8}
\]
\[
d_{rl} = |r_S \ell_F| = \sqrt{(d - \sin \beta - \sin \alpha)^2 + (\cos \beta + \cos \alpha)^2}. \tag{9}
\]

It will be convenient to name the following terms:

\[
D_{\ell} = d_{\ell} \sqrt{1 - (d_{\ell}/4)^2} \tag{10}
\]
\[
D_r = d_r \sqrt{1 - (d_r/4)^2} \tag{11}
\]

Dubins paths. Dubins [11] showed that for two given configurations in the plane, shortest bounded-curvature paths consist of arcs of unit radius circles (C-segments) and straight line segments (S-segments);
and so proving the monotonicity of the Dubins cost function. Unfortunately, not all Dubins paths have horizontal tangents with the correct orientation (see Figure 3(b) for an example), and so proving the monotonicity of the Dubins cost function will require much more work. However, we can start with the following lemma:

**Lemma 1.** Let $d_1 < d_2$, and $(\alpha, \beta) \in \Box$. If there is a path of length $\ell$ of type $RSR$, $LSL$, $LSR$, or $RSL$ from $(0, 0, \alpha)$ to $(d_1, 0, \beta)$, then there is a path of length $\ell + (d_2 - d_1)$ from $(0, 0, \alpha)$ to $(d_2, 0, \beta)$.

**Proof.** It suffices to show that any of these path types must have a horizontal tangent oriented in the positive $x$-direction. By symmetry, it suffices to show this for $RSR$- and $RSL$-paths. The topmost point on a $RSR$-path necessarily has the correct tangent, so consider an $RSL$-path. It consists of a right-turning
Symmetries. For a fixed $d \geq 0$, determining $\text{dub}(d)$ essentially amounts to finding $(\alpha, \beta) \in \Box$ maximizing $\ell(\alpha, \beta)$ (“essentially” since the maximum may not actually be assumed). We now observe that the function $\ell(\alpha, \beta)$ has two symmetries.

First, we can mirror a path around the $x$-axis. This maps $\alpha$ to $-\alpha$, $\beta$ to $-\beta$, left disks to right disks, and right disks to left disks. As a result, we have, say, $\text{lsr}(d, \alpha, \beta) = \text{rsl}(d, -\alpha, -\beta)$, and in general we have $\ell(d, \alpha, \beta) = \ell(d, -\alpha, -\beta)$. See Figure 4.

Second, we can mirror a path around the line $x = d/2$ and reverse the direction of the path. If the original path connected $(0, 0, \alpha)$ with $(d, 0, \beta)$, the new path connects $(0, 0, 2\pi - \beta)$ to $(d, 0, 2\pi - \alpha)$. The transformation maps left disks to left disks and right disks to right disks, so we have, for instance $\text{lsr}(d, \alpha, \beta) = \text{lsr}(d, 2\pi - \beta, 2\pi - \alpha)$, and in general $\ell(d, \alpha, \beta) = \ell(d, 2\pi - \beta, 2\pi - \alpha)$. See Figure 5.

Considered as symmetries on $\Box$, the mapping $(\alpha, \beta) \mapsto (-\alpha, -\beta)$ is a point symmetry in $(\pi, \pi)$, while the mapping $(\alpha, \beta) \mapsto (2\pi - \beta, 2\pi - \alpha)$ is a reflection around the line $\beta = 2\pi - \alpha$.

It follows that $\sup_{(\alpha, \beta) \in [0, 2\pi]^2} \ell(d, \alpha, \beta) = \sup_{(\alpha, \beta) \in \Delta} \ell(d, \alpha, \beta)$, where $\Delta$ is the triangle with corners $(0, 0)$, $(\pi, \pi)$, and $(0, 2\pi)$, or in other words the region

$$\Delta: \quad 0 \leq \alpha \leq \pi \quad \text{and} \quad \alpha \leq \beta \leq 2\pi - \alpha.$$

In the following we will thus be able to restrict our considerations to the triangle $\Delta$ (see Figure 6(a)).

A new parameterization. We now introduce a new parameterization of the $(\alpha, \beta)$-plane, which will sometimes be more convenient to work with:

$$\sigma = \frac{\beta + \alpha}{2}, \quad \delta = \frac{\beta - \alpha}{2}.$$

In other words, we have

$$\alpha = \sigma - \delta, \quad \beta = \sigma + \delta.$$
Recall our triangle $\Delta$ from above. In the $(\sigma,\delta)$-representation, the triangle $\Delta$ is the triangle

$$\Delta: \quad 0 \leq \sigma \leq \pi \quad \text{and} \quad 0 \leq \delta \leq \sigma,$$

or the bottom right half of the square $\Gamma = [0, \pi]^2$ (see Figure 6(b)). In this representation, our first symmetry maps $(\sigma, \delta)$ to $(-\sigma, -\delta)$, while the second symmetry maps $(\sigma, \delta)$ to $(-\sigma, \delta)$. We thus have point symmetry in the origin, as well as mirror symmetry around the $\delta$-axis. In addition, $(\sigma + \pi, \delta + \pi)$ represents the same angles as $(\sigma, \delta)$ since $\alpha = \sigma - \delta$ and $\beta = \sigma + \delta$, and so we also have point symmetry in the point $(\pi/2, \pi/2)$, or in other words $\ell(d, \sigma, \delta) = \ell(d, \pi - \sigma, \pi - \delta)$.

**Distances between centers (using $\sigma$ and $\delta$).** The following lemma will allow us to express the center distances in terms of $\sigma$ and $\delta$:

$$\ell(d, \sigma, \delta) = \ell(d, \pi - \sigma, \pi - \delta).$$
For clarity, let us define the parts of the three cases. We define:

\[ A = \{ (\alpha, \beta) \in \Gamma \mid d_{\text{in}}(\alpha, \beta) < 2 \text{ and } d_{\text{out}}(\alpha, \beta) < 2 \}, \]

\[ B = \{ (\alpha, \beta) \in \Gamma \mid d_{\text{in}}(\alpha, \beta) < 2 \text{ and } d_{\text{out}}(\alpha, \beta) \geq 2 \}, \]

\[ C = \{ (\alpha, \beta) \in \Gamma \mid d_{\text{in}}(\alpha, \beta) \geq 2 \}. \]

For clarity, let us define the parts of \( A, B \) and \( C \) that lie inside the triangle \( \Delta \):

\[ A^\Delta = A \cap \Delta, \quad B^\Delta = B \cap \Delta, \quad \text{and} \quad C^\Delta = C \cap \Delta. \]
Note that \((\alpha, \beta) \in A\) if and only if \(L_S \cap R_F \neq \emptyset\) and \(R_S \cap L_F \neq \emptyset\). Since \(|\ell_{SR}| = 2\) and \(|\ell_{FR}| = 2\), the triangle-inequality implies \(d_\alpha < 4\) and \(d_\beta < 4\). So both LR- and RLR-paths exist. We will concentrate entirely on these two path types in case A. Note that case A does not occur for \(d \geq 2\), as we have
\[8 > d_{RL}^2 + d_{SL}^2 = 2d^2 + 8\cos^2 \delta \geq 2d^2.\]

Similarly, we have \((\alpha, \beta) \in B\) if and only if \(L_S \cap R_F \neq \emptyset\) and \(R_S \cap L_F = \emptyset\). As in case A, LRL-paths must exist, and \(d_{RL} \geq 2\) implies that RSL-paths exist as well. We will concentrate on RSL- and LRL-paths in case B. We observe the following:

**Lemma 3.** A point \((\sigma, \delta) \in B\) has \(\delta > \pi/2\), and for \((\alpha, \beta) \in B^\Delta\) we have
\[0 \leq \alpha \leq \pi/2 \quad \text{and} \quad \pi + \alpha < \beta < 2\pi - \alpha.\]

**Proof.** \((\sigma, \delta) \in B\) means \(d_{RL}^2 < 4\) and \(d_{SL}^2 \geq 4\), so \(0 > d_{RL}^2 - d_{SL}^2 = 8d\cos \delta \sin \sigma\). Since \(\sin \sigma \geq 0\), we must have \(\cos \delta < 0\), and thus \(\delta > \pi/2\). So \((\sigma, \delta)\) lies in the top half of \(\Gamma\). This top half intersects the triangle \(\Delta\) in the triangle with corners (in \((\alpha, \beta)-\text{coordinates}\)) \((0, \pi), (\pi/2, 3\pi/2),\) and \((0, 2\pi)\), implying (16). \(\square\)

Finally \((\alpha, \beta) \in C\) if and only if \(L_S \cap R_F = \emptyset\). This implies that LSR-paths exist. We will study LSR- and RSR-paths in case C.

We can now refine our Dubins-function \(\text{dub}(d)\) by defining the following three functions:
\[
\text{dub}_A(d) = \sup_{(\alpha, \beta) \in A^\Delta} \ell(d, \alpha, \beta) - d \\
\text{dub}_B(d) = \sup_{(\alpha, \beta) \in B^\Delta} \ell(d, \alpha, \beta) - d \\
\text{dub}_C(d) = \sup_{(\alpha, \beta) \in C^\Delta} \ell(d, \alpha, \beta) - d
\]
and we have
\[\text{dub}(d) = \max\{ \text{dub}_A(d), \text{dub}_B(d), \text{dub}_C(d) \}.\]

### 3 Case C

Case C is the easiest, and we can handle it immediately—the rest of this paper will be dedicated to the study of case A and case B. The arguments presented here are already in Kim’s master thesis [16].

In case C we have \(L_S \cap R_F = \emptyset\). Since \(L_S\) and \(R_F\) are open disks, they are allowed to touch, but not to overlap. We show that there always exists an RSR- or LSR-path of length less than \(d + 2\pi\).

**Lemma 4.** For \((\alpha, \beta) \in C^\Delta\), we have \(\ell(d, \alpha, \beta) \leq 2\pi + d\).

**Proof.** Since \(L_S \cap R_F = \emptyset\), the starting point \(S = (0, 0)\) does not lie in \(R_F\), and so there is a tangent \(\ell_{sr}\) to \(R_F\) through \(S\) that touches \(R_F\) from above. Let \(\alpha_{sr}\) be the angle made by \(\ell_{sr}\) and the positive x-axis (see Figure 8(a)).

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**Figure 8:** RSR-paths in case C.
We will now describe the regions $d\alpha, \beta$ and observe that for $(\alpha, \beta)$ it has length at least $d\alpha, \beta$. Indeed, any point $(\alpha, \beta) \in \Delta$ satisfies $\alpha \leq \min\{\beta, 2\pi - \beta\}$, which implies that $t$ has a positive $y$-component.

It follows that the length of the RSR-path is $|\overrightarrow{ST}| + |\overrightarrow{TST}| + |\overrightarrow{TF}| = |\overrightarrow{SS'}| + |\overrightarrow{S'F}|$. By the triangle-inequality, $|\overrightarrow{SS'}| \leq |\overrightarrow{SF}| + |\overrightarrow{F'T'}|$, and so the length of the RSR-path is at most $|\overrightarrow{SF}| + 2\pi = d + 2\pi$.

Consider now the case where $\alpha < \alpha_{sr}$. We show that the LSR-path from $S$ to $F$ has length at most $d + 2\pi$. The LSR-path consists of an initial left-turning arc $\hat{ST}_S$, a straight line segment $\overrightarrow{TST}_F$, and a final right-turning arc $\overrightarrow{TTF}_F$, where the segment is tangent to $L_S$ and $R_F$ at points $T_S$ and $T_F$. See Figure 9(a). Here, it suffices to observe that $|\overrightarrow{ST}_S| + |\overrightarrow{TST}_F| \leq |\overrightarrow{SF}| + |\overrightarrow{F'T}_F|$, and so $|\overrightarrow{ST}_S| + |\overrightarrow{TST}_F| + |\overrightarrow{TF}| \leq |\overrightarrow{SF}| + 2\pi$.

It turns out that the bound in Lemma 4 is optimal, and we obtain:

**Lemma 5.** For any $d > 0$ we have $\text{dub}_C(d) = 2\pi$.

*Proof.* Lemma 4 implies that $\text{dub}_C(d) \leq 2\pi$, so it remains to provide a matching lower bound. We will show that $\ell(d, \pi, \pi) = 2\pi + d$, and since $(\pi, \pi) \in C^\Delta$, this proves the claim. Consider a shortest bounded-curvature path $G$ from $(0, 0, \pi)$ to $(d, 0, \pi)$. This path must intersect the line $x = 0$ in a point $p$ and the line $x = d$ in a point $q$. The distance $|pq|$ is at least $d$. If the path from $S$ to $p$ intersects $L_S \cup R_S$, then Ahn et al. [18, Fact1] showed that it has length at least $\pi$. Otherwise the path avoids $L_S \cup R_S$ and hence must have length at least $\pi$. The same argument applies to the path from $q$ to $F$, and so the total length of $G$ is at least $d + 2\pi$.

Since $\text{dub}(d) \geq \text{dub}_C(d)$, this establishes a lower bound for the Dubins cost function.

## 4 Regions of the square $\Gamma$ for $0 < d < 2$

We will now describe the regions $A$, $B$, and $C$ of the square $\Gamma$ geometrically. For our purposes it will be sufficient to do this when $0 < d < 2$, so we assume this throughout this section.

We define the angle
\[
\alpha^* = \arcsin \frac{d}{2}.
\]
and observe that for $(\alpha, \beta) = (\alpha^*, 2\pi - \alpha^*)$ as well as for $(\alpha, \beta) = (\pi - \alpha^*, \pi + \alpha^*)$ we have $R_S = R_F$ (Figure 10).

We now argue that there is a curve $(\sigma, d_{ln}(\sigma))_{0 \leq \sigma \leq \pi}$ in $\Gamma$ that connects the two points $(0, \alpha^*)$ and $(\pi, \alpha^*)$, lies strictly between $\delta = \alpha^*$ and $\delta = \pi/2$, and in such points, that $d_{ln} = 2$ on the curve, $d_{ln} < 2$ between the curve and the line $\delta = \pi/2$, and $d_{ln} > 2$ below the curve. Recall from [14] that $d_{ln}^2 = d^2 + 4d \cos \delta \sin \sigma + 4 \cos^2 \delta$. For $\delta = \alpha^*$, we have $\cos^2 \delta = 1 - \sin^2 \delta = 1 - d^2/4$, and so $d_{ln}^2 = 4 + 4d \cos \alpha^* \sin \sigma$. This is equal to 4 for $\sigma \in \{0, \pi\}$, and otherwise larger than 4. For $\delta = \pi/2$, we have $d_{ln}^2 = d^2 < 4$. Finally, for $\delta \in (0, \pi/2)$, we have $\frac{\partial}{\partial \sigma} d_{ln}^2 = -4d \sin \delta \sin \sigma - 8 \cos \delta \sin \delta < 0$, and so $\delta \mapsto d_{ln}^2$ is a decreasing function for $\sigma \in (0, \pi)$, proving the claim.
Consider now \( d_{nl}^2 = d^2 - 4d \cos \delta \sin \sigma + 4 \cos^2 \delta \) by (15). For \( \alpha^* \leq \delta \leq \pi/2 \), we have \( \cos^2 \delta = 1 - \sin^2 \delta \leq 1 - d^2/4 \), and so \( d_{nl}^2 \leq 4 \), with equality only for \( \sigma \in \{0, \pi\} \) and \( \delta = \alpha^* \). Let

\[
\sigma^* = \arcsin \frac{d}{4},
\]

and consider the interval \( 0 \leq \sigma \leq \sigma^* \). On this interval, we have \( d_{nl}^2 \geq 4 \) for \( \delta = 0 \) (with equality only for \( \sigma = \sigma^* \)), \( d_{nl}^2 \leq 4 \) for \( \delta = \alpha^* \) (with equality only for \( \sigma = 0 \)), and \( d_{nl}^2 \geq 4 \) for \( \delta = \pi \). Since for fixed \( \sigma \), \( d_{nl}^2 = 4 \) is a quadratic polynomial in \( \cos \delta \), it has at most two roots in \( 0 \leq \delta \leq \pi \), and thus there must be a unique value \( 0 \leq \delta_{nl}(\sigma) \leq \alpha^* \) on the interval \( 0 \leq \sigma \leq \sigma^* \) where \( d_{nl} = 2 \), and we have \( d_{nl} > 2 \) below the curve \( \delta_{nl}(\sigma) \). The same argument applies to the interval \( \pi - \sigma^* \leq \sigma \). The function \( \delta_{nl}(\sigma) \) is defined on \( [0, \pi] \) and is strictly increasing for half a period, \( \delta_{nl}(0) = 4 \), the claim follows.

We now exploit the fact that \( d_{nl}(\sigma, \delta) = d_{nl}(\sigma, \pi - \delta) \) to obtain our desired subdivision. See Figure 11.

- \( C_1 \) is the region \( \pi - \delta_{nl}(\sigma) < \sigma < \delta_{nl}(\sigma) \leq \pi \). Inside this region we have \( d_{nl} \geq 2 \).
- \( C_2 \) is the region \( 0 < \sigma < \pi - \delta_{nl}(\sigma) \leq \sigma < \pi \). Here we have \( d_{nl} \geq 2 \) and \( d_{nl} \geq 2 \).
- \( C_3 \) is the region \( 0 < \sigma < \pi - \delta_{nl}(\sigma) \leq \sigma < \pi \). Here we have \( d_{nl} \geq 2 \) and \( d_{nl} \geq 2 \).
- \( A \) is the region \( \delta_{nl}(\sigma) < \sigma < \pi - \delta_{nl}(\sigma) \). In this region we have \( d_{nl} < 2 \) and \( d_{nl} < 2 \).
- Finally, \( B \) is the remaining region, where \( \pi - \delta_{nl}(\sigma) \leq \delta, \) but excluding \( C_2 \cup C_3 \). In this region we have \( d_{nl} < 2 \) and \( d_{nl} \geq 2 \).

It is clear from the description above that the five regions \( A, B, C_1, C_2 \), and \( C_3 \) are \( \sigma \)-monotone, meaning that a line parallel to the \( \sigma \)-axis intersects each region in a single interval. We will also need that the region \( B^\Delta \) is monotone with respect to the \( \alpha \)-direction.

**Lemma 6.** For \( 0 < d < 2 \), there are two continuous functions \( \alpha \mapsto \beta_{nl}(\alpha) \) and \( \alpha \mapsto \beta_{nl}(\alpha) \) defined on the interval \( [0, \alpha^*] \) such that \( \beta_{nl}(\alpha^*) = 2\pi - \alpha^* \), and for \( 0 \leq \alpha < \alpha^* \) we have

- \( \alpha + \pi < \beta_{nl}(\alpha) < 2\pi - \alpha < \beta_{nl}(\alpha) < 2\pi - \alpha \);
- \( d_{nl} < 2 \) for \( \beta < \beta_{nl}(\alpha) \), \( d_{nl} = 2 \) for \( \beta = \beta_{nl}(\alpha) \), and \( d_{nl} > 2 \) for \( \beta > \beta_{nl}(\alpha) \);
- \( d_{nl} < 2 \) for \( \beta < \beta_{nl}(\alpha) \), \( d_{nl} = 2 \) for \( \beta = \beta_{nl}(\alpha) \), and \( d_{nl} > 2 \) for \( \beta > \beta_{nl}(\alpha) \);

The function \( \beta_{nl} \) is a monotonically decreasing function of \( \alpha \), and we have

\[
B^\Delta = \{ (\alpha, \beta) \mid \alpha \in [0, \alpha^*], \beta_{nl}(\alpha) \leq \beta < \beta_{nl}(\alpha) \}.
\]

**Proof.** Let us fix an \( \alpha \in [0, \alpha^*] \), so the points \( \ell_S \) and \( r_S \) are fixed. While \( \beta \) ranges over \( [0, 2\pi] \), the point \( \ell_F \) makes a full circle around \( F \). This means that the distance \( d_{nl} \) is strictly increasing for half a period, and strictly decreasing for the other half period. This implies that in the range \( \alpha + \pi \leq \beta \leq 2\pi - \alpha \) there is at most one extremum of \( d_{nl} \). The same argument shows that \( d_{nl} \) has at most one extremum in the range.
We first observe that for $\alpha_d$ the two functions are clearly continuous, and since we have $\delta > \pi$ we have $\delta lr(\sigma) = \delta rl(\sigma)$. Therefore by (15), we have $d_{LR}^2 = d_{RL}^2 = d^2 + 4 \cos^2 \delta > 4$, and so $d_{LR} = d_{RL} > 2$.

Since $d_{LR} = d_{RL} < 2$ for $\beta = \alpha + \pi$, $d_{LR} = d_{RL} > 2$ for $\beta = 2 \pi - \alpha$, and both functions have only one extremum in this range, both functions must assume the value two exactly once in this range, at values $\alpha$ and $\beta$.

We start by a change of perspective, and consider all configurations where $d_{RL}$ is contained in the triangle $0 \alpha \beta$. By Lemma 3, $B^\Delta$ is contained in the triangle $0 \alpha \beta < \pi/2$, $\alpha + \beta < 2 \pi - \alpha$. Consider Figure 10(a). We first observe that for $\pi - \alpha < \pi/2$, the point $F = (d, 0)$ is contained in the interior of $R_S$, and so $d_{RL} < 2$. It follows that for $(\alpha, \beta) \in B^\Delta$ we must have $\alpha \in [0, \pi)$, and the expression for $B^\Delta$ follows from the above.

5 Explicit expressions for the length of $LRL$- and $RLR$-paths

In this section we develop explicit formulas for the length of $LRL$- and $RLR$-paths.

We start by a change of perspective, and consider all configurations where $d_l$ is fixed. We choose a coordinate system where the line $\ell_S \ell_F$ is horizontal, and $\ell_S$ lies to the left of $\ell_F$, see Figure 12(a). We have drawn the two unit-radius disks $M^t$ and $N^t$ tangent to $L_S$ and $L_F$. The points of tangency are $S^t_2$ and $F^t_2$ for $M^t$, and $S^t_1$ and $F^t_1$ for $N^t$. Dubins [11] showed that the length of the middle circular arc of a $CCC$-path is larger than $\pi$, and so it lies on $M^t$.

So any $LRL$-path first follows a leftwards arc on $L_S$, then switches to $M^t$ at $S^t_2$, follows the rightwards arc on $M^t$ until it reaches $F^t_2$, and finally follows a leftwards arc on $L_F$. We note that the middle arc on $M^t$ does not depend on the specific endpoints $S$ and $F$, it is determined entirely by $S^t_2$ and $F^t_2$, and therefore by $d_l$. Let $\mu_l$ denote half the length of the middle circular arc $S^t_2 F^t_2$. We have $\pi/2 < \mu_l \leq \pi$, and $4 \sin(\pi - \mu_l) = d_{LC}$, so that we have

$$\mu_l = \pi - 4 \arcsin \frac{d_{LC}}{4}.$$ (18)
We note the following angles:

The same considerations apply to RLR−→ vector $d$. If $\alpha$, $\beta$, $\gamma$ are the interior angles of $\triangle SFR$ then

\[ \angle S_1F_1S_2 = \angle F_2F_1R = 2\mu_r - \pi, \]

\[ \angle S_1F_2R = \angle F_2F_1R = 2\mu_r - \pi. \]

It is now important to understand the possible locations of the endpoints $S$ and $F$ on the disks.

**Lemma 7.** We have

- $S$ lies on the counter-clockwise arc $S_1^1S_2^1$ of $L_S$ if and only if $d_{nl} < 2$;
- $F$ lies on the clockwise arc $F_2^1F_1^1$ of $L_F$ if and only if $d_{nl} < 2$;
- $S$ lies on the clockwise arc $S_2^2S_1^2$ of $R_S$ if and only if $d_{ns} < 2$;
- $F$ lies on the counter-clockwise arc $F_1^1F_2^1$ of $L_F$ if and only if $d_{ns} < 2$.

If $d_{nl} > 2$ and in addition $0 \leq \alpha \leq \pi/2$, then $S$ lies on the counter-clockwise arc $S_0^0S_1^1$ of $L_S$.

**Proof.** Consider the position of $R_S$ as $S$ moves once around the fixed circle $L_S$. The center $r_S$ describes a circle of radius two around $L_S$, when $S = S_1^1$, we have $R_S = N^1$, when $S = S_2^2$, we have $R_S = M^1$. If $d_{nl} < 2$, we have $R_S \cap L_S \neq \emptyset$, and so $S$ must lie on the counter-clockwise arc $S_1^1S_2^1$. If $d_{nl} > 2$, we have $R_S \cap L_S = \emptyset$, and $S$ must lie on the complementary arc $S_2^2S_1^1$. The same argument applies to RLR−→ paths to determine the location of $F$. We argue similarly for $d_{ns} < 2$ and $d_{ns} \geq 2$.

We observe next that the starting orientation at $S$, which is the forward tangent to $L_S$ at $S$, and the vector $\vec{SF}$ must make an angle of $\alpha$. Since $F \in L_F$, this is impossible for $0 \leq \alpha \leq \pi/2$ and $S$ on the long counter-clockwise arc $S_1^1S_0^0$, and so $d_{nl} \geq 2$ with $0 \leq \alpha \leq \pi/2$ implies that $S \in S_0^0S_1^1$. \[\square\]

**Lemma 8.** Let $d_1 < d_2$, and $(\alpha, \beta) \in B^\Delta$ (where $B^\Delta$ is defined for $d_1$). If there is an LRL−→ path of length $t$ from $(0, 0, \alpha)$ to $(d_1, 0, \beta)$, then there is a path of length $t + (d_2 - d_1)$ from $(0, 0, \alpha)$ to $(d_2, 0, \beta)$.

**Proof.** We only have to prove that the LRL−→ path has a horizontal tangent oriented in the positive $x$-direction. Assume this is not the case, so there is no point on the path were the orientation is 0 or $2\pi$.
The path starts at orientation $\alpha$, the orientation increases to $\alpha + \gamma_1$, decreases to $\alpha + \gamma_1 - 2\mu_L$, and increases again to $\beta = \alpha + \gamma_1 - 2\mu_L + \gamma_2$, without ever leaving the open range $(0, 2\pi)$. But by Lemma 9, $(\alpha, \beta) \in B^\Delta$ implies $d_{\text{HL}} > 2$ and $0 \leq \alpha \leq \pi/2$ and thus, by Lemma 9 and (20), we have $2\mu_L - \pi \leq \gamma_1 \leq \mu_L$ and $0 \leq \gamma_2 \leq 2\mu_L - \pi$. This implies that $\gamma_1 + \gamma_2 \leq 2\mu_L$, and so $\beta \leq \alpha$, a contradiction to $(\alpha, \beta) \in B^\Delta$. \qed

**Lemma 9.** For any $\sigma, \delta$, we have

$$\text{LRL}(\sigma, \delta) \equiv 4\mu_L + 2\delta \pmod{2\pi},$$

For $(\sigma, \delta) \in A$, we have

$$\text{LRL}(\sigma, \delta) \equiv 4\mu_L - 2\delta \pmod{2\pi}.$$ 

For $(\sigma, \delta) \in B$, we have

$$\text{LRL}(\sigma, \delta) \equiv 4\mu_L + 2\delta - 2\pi,$$

$$\text{LRL}(\sigma, \delta) \equiv 4\mu_L - 2\delta + 2\pi.$$

**Proof.** An LRL-path consists of an initial left-turning arc of length $\gamma_1$ on $L_S$, a right-turning arc of length $2\mu_L$, and a final left-turning arc of length $\gamma_2$ on $L_F$. This means that the total change in orientation is $\gamma_1 - 2\mu_L + \gamma_2$. On the other hand, since the initial orientation is $\alpha$ and the final orientation is $\beta$, this must be equal, up to multiples of $2\pi$, to $\beta - \alpha = 2\delta$. It follows that

$$\text{LRL} = \gamma_1 + \gamma_2 + 2\mu_L = \gamma_1 - 2\mu_L + \gamma_2 + 4\mu_L \equiv 2\delta + 4\mu_L.$$ 

For RLR-paths, we can similarly observe that $-\gamma_1 + 2\mu_L - \gamma_2 \equiv 2\delta \pmod{2\pi}$ (here, $\gamma_1$ and $\gamma_2$ are the right-turning arcs) and obtain

$$\text{RLR} = \gamma_1 + \gamma_2 + 2\mu_L = 4\mu_L - (-\gamma_1 + 2\mu_L - \gamma_2) \equiv 4\mu_L - 2\delta.$$ 

Let us now assume that $(\sigma, \delta) \in A$. We have $2\alpha^* < 2\delta < 2\pi - 2\alpha^*$. On the other hand, $\gamma_1 + \gamma_2 - 2\mu_L \geq -2\mu_L \geq -2\pi$. By Lemma 7, $S \in S_{1, 1}^\pi S_{1, 1}^\pi F^\pi F^\pi F^\pi$, and so we can extend the LRL-path to a complete clockwise loop as in Figure 13(b). The loop uses additional left-turns $\zeta_1$ and $\zeta_2$, and an additional right-turn of length $2\mu_L$. The total turning angle of a clockwise loop is $-2\pi$, and thus $\gamma_1 + \gamma_2 + \delta_1 + \delta_2 - 4\mu_L = -2\pi$. Since $2\mu_L \leq 2\pi$ this implies that $\gamma_1 + \gamma_2 - 2\mu_L \leq 0$. From $-2\pi \leq \gamma_1 + \gamma_2 - 2\mu_L \leq 0$ and $0 \leq 2\alpha^* < 2\delta < 2\pi - 2\alpha^* \leq 2\pi$, we conclude that $\gamma_1 + \gamma_2 - 2\mu_L \equiv 2\delta \pmod{2\pi}$ implies $\gamma_1 + \gamma_2 - 2\mu_L = 2\delta - 2\pi$. This shows that $\text{LRL} = 4\mu_L + 2\delta - 2\pi$.

For RLR-paths, we could argue analogously, or we can simply observe that

$$\text{RLR}(\sigma, \delta) = \text{LRL}(\pi - \sigma, \pi - \delta) = 4\mu_L(\pi - \sigma, \pi - \delta) + 2(\pi - \delta) - 2\pi = 4\mu_L(\sigma, \delta) - 2\delta.$$ 

Assume now that $(\sigma, \delta) \in B$. By Lemma 3 we have $\pi < 2\delta \leq 2\pi$. By Lemma 7 and (20) we have

$$2\mu_L - \pi \leq \gamma_1 < 2\pi,$$

$$0 \leq \gamma_2 \leq 2\mu_L - \pi.$$ 

Together these imply $-\pi \leq \gamma_1 + \gamma_2 - 2\mu_L < \pi$. Since $\gamma_1 + \gamma_2 - 2\mu_L \equiv 2\delta \pmod{2\pi}$, we must have $\gamma + \gamma_2 - 2\mu_L = 2\delta - 2\pi$. It follows that $\text{LRL}(\sigma, \delta) = 4\mu_L + 2\delta - 2\pi$.

We turn to the RLR-path in case B. By Lemma 7 and (21) we have (here, $\gamma_1$ and $\gamma_2$ are the right-turning arcs)

$$0 \leq \gamma_1 \leq 2\mu_L - \pi,$$

$$2\mu_L - \pi \leq \gamma_2 < 2\pi.$$ 

Together we have $-\pi \leq \gamma_1 + \gamma_2 - 2\mu_L \leq \pi$. Since $-\gamma_1 - \gamma_2 + 2\mu_L \equiv 2\delta \pmod{2\pi}$, we must have

$$-\gamma_1 + 2\mu_L - \gamma_2 = 2\delta - 2\pi,$$ 

which shows that $\text{RLR} = \gamma_1 + 2\mu_L + \gamma_2 = 4\mu_L - (2\delta - 2\pi) = 4\mu_L - 2\delta + 2\pi$. \qed
6 CCC-paths for $d < 2$ and case A

When $d < 2$, both LRL- and RLR-paths exist for any $(\sigma, \delta) \in \Gamma$. In this section we analyze the length of these two CCC-paths for $0 < d < 2$.

We define three functions $L$, $r$, and $c$ on $\Gamma$:

\[
L(\sigma, \delta) = 4\mu_L(\sigma, \delta) + 2\delta - 2\pi \tag{22}
\]

\[
r(\sigma, \delta) = 4\mu_R(\sigma, \delta) - 2\delta \tag{23}
\]

\[
c(\sigma, \delta) = \min\{L(\sigma, \delta), R(\sigma, \delta)\}. \tag{24}
\]

While these functions are defined and continuous everywhere on $\Gamma$, we have shown in Lemma 9 only that $L(\sigma, \delta) = L(\sigma, \delta)$ for $(\sigma, \delta) \in A \cup B$, and $R(\sigma, \delta) = R(\sigma, \delta)$ for $(\sigma, \delta) \in A$. It follows that $c(\sigma, \delta)$ is the length of the shortest CCC-path for $(\sigma, \delta) \in A$.

We obtain the derivatives of $L$ and $r$ using $\mu_L = \pi - \arcsin \frac{d}{4}$, $\mu_R = \pi - \arcsin \frac{d}{4}$, and Equations (12)–(15):

\[
\frac{\partial L}{\partial \sigma}(\sigma, \delta) = -2d \sin \delta \sin \sigma \tag{26}
\]

\[
\frac{\partial L}{\partial \delta}(\sigma, \delta) = 2 + \frac{-4 \cos \delta \sin \delta + 2d \cos \delta \cos \sigma}{D_L}, \tag{25}
\]

\[
\frac{\partial r}{\partial \sigma}(\sigma, \delta) = \frac{2d \sin \delta \sin \sigma}{D_n}, \tag{27}
\]

\[
\frac{\partial r}{\partial \delta}(\sigma, \delta) = -2 + \frac{-4 \cos \delta \sin \delta - 2d \cos \delta \cos \sigma}{D_n}. \tag{28}
\]

The derivatives are not defined when $D_L = 0$ or $D_R = 0$. $D_L = 0$ occurs when $d = 0$ or $d = 4$, and $D_R = 0$ occurs when $d = 0$ or $d = 4$.

Recall that $\alpha^* = \arcsin \frac{d}{4}$. We define the following rectangle $\Xi \subset \Gamma$:

\[
\Xi : \quad 0 \leq \sigma, \delta \leq \pi \quad \text{and} \quad \alpha^* \leq \delta, \sigma - \alpha^*.
\]

Note that $A \subset \Xi$ (see Figure 11(a)). In the interior of $\Xi$, we have $\sin \delta > \frac{d}{2}$, which implies $\cos \delta^2 < 1 - \frac{d^2}{4}$. So for $\alpha^* < \delta \leq \frac{\pi}{2}$, we have $d_{RL} < 2$ by (15), and for $\frac{\pi}{2} \leq \delta < \pi - \alpha^*$, we have $d_{RL} < 2$ by (11). By the triangle inequality it follows that $d_L < 4$ and $d_R < 4$. Also, by (12) $d_L^2 = (d - 2 \sin \delta \cos \sigma)^2 + 4 \sin \delta (-\cos \sigma + 1)$. Since $\sin \delta > 0$, $d_L = 0$ can occur only when $\sigma = 0$ and $\delta \in \{\alpha^*, \pi - \alpha^*\}$, which occurs at a corner of $\Xi$. Similar arguments hold for the case where $d_R^2 = 0$. Thus, in the interior of $\Xi$, $D_L \neq 0$ and $D_R \neq 0$. 

Figure 13: Proof of Lemma 9 for case A

(a) $0 \leq \gamma_1, \gamma_2 \leq 2\mu_L - \pi$

(b) Completing to a loop
Lemma 10. For \( (\sigma, \delta) \in \Xi \), the function

\begin{itemize}
  \item \( \sigma \mapsto L(\sigma, \delta) \) is decreasing,
  \item \( \sigma \mapsto R(\sigma, \delta) \) is increasing,
  \item \( \delta \mapsto L(\sigma, \delta) \) is increasing,
  \item \( \delta \mapsto R(\sigma, \delta) \) is decreasing.
\end{itemize}

Proof. Consider Equations (26) and (28). Since \( \sin \delta > 0 \) in \( \Xi \) and \( \sin \sigma \geq 0 \) with equality only for \( \sigma \in \{0, \pi\} \), we have \( \frac{\partial}{\partial \sigma} < 0 \) and \( \frac{\partial}{\partial \sigma} > 0 \) in the interior of \( \Xi \).

It remains to discuss the two functions of \( \delta \). We show that if \( Z = 2 \cos \delta (-\sin \delta + d \cos \sigma) \geq 0 \), this is true. Let us assume that \( Z < 0 \). Since \( \sin \delta \geq d/2 \) implies that \( -2 \sin \delta + d \cos \sigma \leq -2(d/2) + d \cos \sigma = 2(-1 + \cos \sigma) \leq 0 \), we have \( \cos \delta > 0 \) and so \( -2 \sin \delta + d \cos \sigma < 0 \). \( \sin \delta \geq d/2 \) also implies that \( d^2 \leq 4 \sin^2 \delta \), so by (29) \( d^2 = d^2 - 4d \sin \delta \cos \sigma + 4 \sin^2 \delta \leq 8 \sin^2 \delta - 4d \sin \delta \cos \sigma = 4 \sin(2 \sin \delta - d \cos \sigma) \). Since \( d^2 \geq 0 \), we have

\[
d^4 \leq 16 \sin^2 \delta (2 \sin \delta - d \cos \sigma)^2.
\] (29)

On the other hand, we have

\[
d^2 = d^2 - 4d \sin \delta \cos \sigma + 4 \sin^2 \delta
\geq d^2 \cos^2 \sigma - 4d \sin \delta \cos \sigma + 4 \sin^2 \delta = (2 \sin \delta - d \cos \sigma)^2.
\] (30)

Now we want to show that \( 2D_L \geq -Z \), which will imply \( \frac{\partial}{\partial \delta} (\sigma, \delta) \geq 0 \). Let us compare the squared terms:

\[
4D_L^2 - Z^2
= 4D_L^2 - 4 \cos^2 \delta (-2 \sin \delta + d \cos \sigma)^2
= 4 \left( d_L^2 \left( 1 - \frac{d_l^2}{16} \right) - \cos^2 \delta (2 \sin \delta - d \cos \sigma)^2 \right)
\geq 4 \left( d_L^2 - \sin \delta (2 \sin \delta - d \cos \sigma)^2 - \cos^2 \delta (2 \sin \delta - d \cos \sigma)^2 \right) \quad \text{(by (29))}
= 4 \left( d_L^2 - 2 \sin \delta - d \cos \sigma)^2 \right) \geq 0 \quad \text{(by (30)).}
\]

One of the inequalities in above formula is a strict inequality: if (30) is an equality, then \( d_L^2 = (2 \sin \delta - d \cos \sigma)^2 \), which means that \( \cos^2 \sigma = 1 \). This implies that \( d^2 < 4 \sin^2 \delta \) since we have \( -2 \sin \delta + d \cos \sigma < 0 \), so (29) is strict. In the case where (29) is an equality, we can argue similarly that (30) is strict.

Similarly, we prove that \( \frac{\partial}{\partial \sigma} (\sigma, \delta) < 0 \), since \( \sin \delta \geq d/2 \) again implies that \( d^2 = d^2 + 4d \sin \delta \cos \sigma + 4 \sin^2 \delta \leq 4 \sin(2 \sin \delta + d \cos \sigma) \), so

\[
d^4 \leq 16 \sin^2 \delta (2 \sin \delta + d \cos \sigma)^2,
\]
and we have

\[
d^2 \geq d^2 \cos^2 \sigma + 4d \sin \delta \cos \sigma + 4 \sin^2 \delta = (2 \sin \delta + d \cos \sigma)^2.
\]

We also need to show that the length of \( LRL \)-paths is monotone in \( \alpha \) and \( \beta \), at least for the cases of interest to us.

Lemma 11. For \( (\alpha, \beta) \in B^\Delta \), the function \( \alpha \mapsto LRL(\alpha, \beta) \) is decreasing.

Proof. By Lemma 9 we have \( LRL(\alpha, \beta) = 4\mu_L + 2\delta - 2\pi \) in \( B^\Delta \). Since \( 2\delta = \beta - \alpha \), \( \mu_L = \pi - \arcsin \frac{\delta}{N} \), \( d_L^2 = (\sin \beta + \sin \alpha)^2 + (\cos \beta - \cos \alpha)^2 \) from (6), and \( \frac{\partial \arcsin \frac{\delta}{N}}{\partial \alpha} = \frac{1}{\sqrt{1 - (\frac{\delta}{N})^2}} \), we have

\[
\frac{\partial LRL}{\partial \alpha}(\alpha, \beta) = - \frac{(d - \sin \beta + \sin \alpha) \cos \alpha + (\cos \beta - \cos \alpha) \sin \alpha}{D_L} - 1.
\]

We also have

\[
(d - \sin \beta + \sin \alpha) \cos \alpha + (\cos \beta - \cos \alpha) \sin \alpha = d \cos \alpha - \sin \beta \cos \alpha + \cos \beta \sin \alpha
= d \cos \alpha - \sin (\beta - \alpha) \geq 0.
\]

The last inequality holds since \( 0 \leq \alpha \leq \pi/2 \) and \( \beta - \alpha \geq \pi \) in \( B^\Delta \) by Lemma 3. It follows that \( \frac{\partial LRL}{\partial \alpha}(\alpha, \beta) \leq -1 < 0 \).
Lemma 12. For \((\alpha, \beta) \in A^\Delta \cup B^\Delta\), the function \(\beta \mapsto LRL(\alpha, \beta)\) is increasing. Moreover, if \((\alpha, \beta) \in B^\Delta\) and \(\beta \geq \pi/2\), then \(\frac{\partial}{\partial \beta} LRL(\alpha, \beta) \geq 1\).

Proof. We have
\[
\frac{\partial LRL}{\partial \beta}(\alpha, \beta) = \frac{(d - \sin \beta + \sin \alpha) \cos \beta + (\cos \beta - \cos \alpha) \sin \beta}{D_L} + 1.
\]
Note that the derivative is not defined if \(D_L = 0\), but as we noted before, that can happen only on the boundary of the region.

We first prove the second statement. \(\beta \geq \pi/2\) implies that \(\sin \beta \leq 0\) and \(\cos \beta \geq 0\). By (16) we have \(\pi/2 - \alpha \leq \beta \leq 2\pi - \alpha\), which implies that \(\cos \beta \leq \cos \alpha\). It follows that the first term of \(\frac{\partial}{\partial \beta} LRL(\alpha, \beta)\) is nonnegative, proving that \(\frac{\partial}{\partial \beta} LRL(\alpha, \beta) \geq 1\).

Let \(U = d - \sin \beta + \sin \alpha\), and \(V = \cos \beta - \cos \alpha\) (note that \(d_L^2 = U^2 + V^2\)). If \(U \cos \beta + V \sin \beta \geq 0\), we have \(\frac{\partial}{\partial \beta} LRL(\alpha, \beta) > 0\), and \(\beta \mapsto LRL(\alpha, \beta)\) is increasing. Now let us assume that \(U \cos \beta + V \sin \beta\) is negative, and let us compare the squared terms of \(\frac{\partial}{\partial \beta} (\alpha, \beta)\) (we only consider the numerator since the denominator is always positive).

\[
\frac{(U \cos \beta + V \sin \beta)^2 - d_L^2}{16} = \frac{U \cos \beta + V \sin \beta}{U^2 + V^2} = \frac{U^2 \cos^2 \beta + V^2 \sin^2 \beta + 2UV \cos \beta \sin \beta}{U^2 + V^2} = \frac{(U^2 + V^2)^2}{16}.
\]

For \((\alpha, \beta) \in A^\Delta \cup B^\Delta\), we have \(d_L^2 < 2\). From (8), \(d_L^2 = (d + \sin \beta + \sin \alpha)^2 + (\cos \beta + \cos \alpha)^2 < 4\). By substituting \(d + \sin \alpha = U + \sin \beta\) and \(\cos \alpha = \cos \beta - V\), we have
\[
d_L^2 < 4
\]
\[
\Leftrightarrow (U + 2 \sin \beta)^2 + (2 \cos \beta - V)^2 < 4
\]
\[
\Leftrightarrow U^2 + V^2 + 4(U \sin \beta - V \cos \beta) + 4(\sin^2 \beta + \cos^2 \beta) < 4
\]
\[
\Leftrightarrow (U^2 + V^2) + 4(U \sin \beta - V \cos \beta) < 0.
\]

Since \(U^2 + V^2 = d_L^2 \geq 0\), this implies that \(U \sin \beta - V \cos \beta < 0\), and further the squared term \(16(U \sin \beta - V \cos \beta)^2\) is greater than \((U^2 + V^2)^2\), which implies that \((U^2 + V^2)^2 - 16(U \sin \beta - V \cos \beta)^2 < 0\), completing the proof.

\]

Lemma 13. The function \(c(\sigma, \delta)\) has no local extremum in the interior of \(\Xi\) except at \((\pi/2, \pi/2)\).

Proof. By Lemma 10, neither \(L\) nor \(R\) has a local extremum in the interior of \(\Xi\), so any local extremum of \(c(\sigma, \delta)\) must be a point in the set \(\Lambda\) of points \((\sigma, \delta)\) with \(L(\sigma, \delta) = R(\sigma, \delta)\). By Lemma 10, \(\Lambda\) is a \(\delta\)-monotone curve. Since \(L(\sigma, \delta) = R(\pi - \sigma, \pi - \delta)\), the curve \(\Lambda\) passes through the point \((\pi/2, \pi/2)\). By Lemma 10, this implies that \(L(\sigma, \delta) < R(\sigma, \delta)\) for the quadrant \(\pi/2 \leq \sigma \leq \pi\), \(\alpha^* \leq \delta \leq \pi/2\), and that \(R(\sigma, \delta) \leq L(\sigma, \delta)\) for the quadrant \(0 \leq \sigma \leq \pi/2\), \(\pi/2 \leq \delta \leq \pi - \alpha^*\) except at the point \((\pi/2, \pi/2)\). By point symmetry, we can restrict our attention to the range \(\pi/2 < \sigma < \pi\), \(\pi/2 < \delta < \pi - \alpha^*\).

Assume for a contradiction that \((\sigma, \delta) \in \Lambda\) is a local extremum of \(L\), restricted to \(\Lambda\). This implies that the gradient \(\nabla L(\sigma, \delta)\) and the normal of \(\Lambda\) in \((\sigma, \delta)\) are linearly dependent, by the method of Lagrange Multipliers. The normal of \(\Lambda\) is the gradient of \(L(\sigma, \delta) - R(\sigma, \delta)\), so \(\nabla L(\sigma, \delta)\) and \(\nabla R(\sigma, \delta)\) must be linearly dependent. Note that \(\nabla L(\sigma, \delta) = (\frac{\partial}{\partial \sigma}(\sigma, \delta), \frac{\partial}{\partial \delta}(\sigma, \delta))\), and \(\nabla R(\sigma, \delta) = (\frac{\partial}{\partial \sigma}(\sigma, \delta), \frac{\partial}{\partial \delta}(\sigma, \delta))\), which we can obtain from Equations (25)–(28).

For the two vectors to be linearly dependent, we would have to have
\[
D_L \frac{\partial L}{\partial \delta}(\sigma, \delta) + D_R \frac{\partial R}{\partial \delta}(\sigma, \delta) = 0,
\]
which means
\[
2D_L - 2D_R - 8 \cos \delta \sin \delta = 0.
\]
which implies by (22) and (23):

On the interval

Lemma 15. For

occur on the vertical side at

Since

We evaluate

Proof. Lemma 14. The maximum \( \Lambda(d) \) occurs with \( \sigma = \pi \) when \( 0 \leq d \leq \sqrt{2} \), and with \( \delta = \pi - \alpha^* \) when \( \sqrt{2} \leq d < 2 \).

Proof. We evaluate \( \mathrm{L}(\pi, \pi - \alpha^*) \) and \( \mathrm{R}(\pi, \pi - \alpha^*) \). Using (12) and (13), we have

which implies by (22) and (23):

Since \( \alpha^* = \arcsin(d/2) \), we have \( \mathrm{R}(\pi, \pi - \alpha^*) < \mathrm{L}(\pi, \pi - \alpha^*) \) for \( d < \sqrt{2} \), equality for \( d = \sqrt{2} \), and \( \mathrm{R}(\pi, \pi - \alpha^*) > \mathrm{L}(\pi, \pi - \alpha^*) \) for \( d > \sqrt{2} \). In the first case, Lemma 10 implies that the maximum must occur on the vertical side at \( \sigma = \pi \), in the last case it must occur on the horizontal side at \( \delta = \pi - \alpha^* \). For \( d = \sqrt{2} \) the maximum occurs at the corner \( (\sigma, \delta) = (\pi, \pi - \alpha^*) \).

Lemma 15. On the interval \( 0 \leq d \leq \sqrt{2} \), the function

- \( d \mapsto \Lambda(d) \) is monotonically increasing.

\[ l(\sigma, \delta) < r(\sigma, \delta) \]

\[ r(\sigma, \delta) < l(\sigma, \delta) \]

Figure 14: The curve \( \Lambda \) where \( \mathrm{L}(\sigma, \delta) = \mathrm{R}(\sigma, \delta) \) in rectangle \( \Xi \) (shaded region), shown for \( d = 1 \).
• $d \mapsto \lambda(d) - d$ is monotonically decreasing.

\textbf{Proof.} We will show below that the two functions $d \mapsto \lambda(d)$ and $d \mapsto \lambda(d) - d$ have no extremum on the interval $(0, \sqrt{2})$. This will imply the claim if we observe that

\[ \lambda(0) = \frac{7\pi}{3}, \]
\[ \lambda(\sqrt{2}) = \frac{5\pi}{2} > \lambda(0) \]
\[ \lambda(\sqrt{2}) - \sqrt{2} = \frac{5\pi}{2} - \sqrt{2} < \lambda(0) - 0. \]

Again we will employ Lagrange multipliers. Let us first give the necessary derivatives. Setting $\sigma = \pi$ and $\pi/2 < \delta \leq \pi - \alpha^*$ (by Lemma 14), we have:

\[ d_L = d + 2 \sin \delta \quad \text{by (12)} \]
\[ d_R = 2 \sin \delta - d \quad \text{by (13)} \]
\[ L(d, \pi, \delta) = 2\pi - 4 \arcsin(d_L/4) + 2\delta \quad \text{by (22)} \]
\[ r(d, \pi, \delta) = 4\pi - 4 \arcsin(d_R/4) - 2\delta \quad \text{by (23)} \]

Let us introduce $F_L = \sqrt{1 - (d_L/4)^2}$ and $F_R = \sqrt{1 - (d_R/4)^2}$ to obtain:

\[ \frac{\partial}{\partial d} L(d, \pi, \delta) = -\frac{1}{F_L} \quad \frac{\partial}{\partial \delta} L(d, \pi, \delta) = 2 - \frac{2 \cos \delta}{F_L} \]
\[ \frac{\partial}{\partial d} r(d, \pi, \delta) = \frac{1}{F_R} \quad \frac{\partial}{\partial \delta} r(d, \pi, \delta) = -2 - \frac{2 \cos \delta}{F_R} \]

We consider first the function $d \mapsto \lambda(d)$. An extremum of $\lambda(d)$ is an extremum of the two-parameter function $(d, \delta) \mapsto L(d, \pi, \delta)$ under the restriction $L(d, \pi, \delta) = r(d, \pi, \delta)$. Such an extremum would have to satisfy the condition $\nabla L(d, \pi, \delta) = \lambda \nabla r(d, \pi, \delta)$. For this to hold,

\[ \lambda = -\frac{F_R}{F_L} = \frac{2 - 2 \cos \delta}{-2 - 2 \cos \delta/F_R}, \]

which implies

\[ F_R - F_L = -2 \cos \delta. \quad (32) \]

But in the following we show that this is impossible. Since we have

\[ 4 \left( \arcsin \frac{d_L}{4} - \arcsin \frac{d_R}{4} \right) \geq 4 \left( \frac{d_L}{4} - \frac{d_R}{4} \right) = 2d, \]

for the condition $L(d, \pi, \delta) = r(d, \pi, \delta)$ to hold, we need

\[ 2\pi - 4\delta \leq -2d \iff \delta \geq \frac{\pi}{2} + \frac{d}{2} \iff \sin \delta \leq \cos \frac{d}{2} \]
\[ \iff -\cos \delta \geq \sin \frac{d}{2}. \quad (33) \]

We also claim that $F_R + F_L > 1$; indeed since $d_L < 2$, $F_R > \sqrt{3}/2$, and since $d_L < 2 + \sqrt{2}$, $F_L > 1 - \sqrt{3}/2$. Since $F_R^2 - F_L^2 = (F_R - F_L)(F_R + F_L - 1) \geq 0$ and $F_R = F_L$ only if $d = 0$ or $\delta = \pi/2$, this implies that

\[ F_R - F_L < F_R^2 - F_L^2 = d \frac{d}{2} \sin \delta. \quad (35) \]

Now,

\[ F_R - F_L < d \frac{d}{2} \sin \delta \quad \text{by (35)} \]
\[ \leq \frac{d}{2} \cos \frac{d}{2} \quad \text{by (33)} \]
\[ \leq 2 \sin \frac{d}{2} \]
\[ \leq -2 \cos \delta \quad \text{by (34)}. \]
The third inequality holds since
\[
\frac{\partial}{\partial d} \left( \frac{d}{2} \cos \frac{d}{2} - 2 \sin \frac{d}{2} \right) = \frac{1}{2} \cos \frac{d}{2} - \frac{d}{4} \sin \frac{d}{2} - \cos \frac{d}{2} \leq 0,
\]
which implies that \( D(d) = (d/2) \cos(d/2) - 2 \sin(d/2) \) is a decreasing function, and finally note that \( D(0) = 0 \).

Consider next the function \( d \mapsto \Lambda(d) - d \). An extremum of \( \Lambda(d) - d \) is an extremum of the two-parameter function \( (d, \delta) \mapsto L(d, \pi, \delta) - d \) under the restriction \( L(d, \pi, \delta) = R(d, \pi, \delta) \). Such an extremum would have to satisfy the condition
\[
\Lambda \nabla (L(d, \pi, \delta) - d) = \nabla (L(d, \pi, \delta) - R(d, \pi, \delta)),
\]
or
\[
\lambda \left( -1 + \frac{1}{F_L^2} - 1, 2 - \frac{2 \cos \delta}{F_L} \right) = \left( -1 + \frac{1}{F_R^2} - 1, 4 - \frac{2 \cos \delta}{F_L} + \frac{2 \cos \delta}{F_R} \right).
\]
The two components give us the following conditions on \( \lambda \):
\[
\lambda = 1 + \frac{1}{F_R} - \frac{1}{F_L} + 1,
\]
\[
\lambda = 1 + \frac{2 + (2 \cos \delta)}{F_R} - \frac{2 - (2 \cos \delta)}{F_L}.
\]
This is equivalent to
\[
\left( 2 - \frac{2 \cos \delta}{F_L} \right) \left( \frac{1}{F_R} - 1 \right) = \left( 2 + \frac{2 \cos \delta}{F_R} \right) \left( \frac{1}{F_L} + 1 \right).
\]
Multiplying out and rearranging the terms gives
\[
(2 - 2 \cos \delta) \left( \frac{1}{F_R} - 1 \right) = 4 \left( \frac{\cos \delta}{F_R F_L} + 1 \right).
\]
Since \( \cos \delta < 0 \) and \( F_R > F_L \), the left-hand side is negative. We will now show that the right-hand side is positive, a contradiction, and so \( d \mapsto \Lambda(d) - d \) cannot have a local extremum.

It is enough to show that \( \cos^2 \delta \leq (F_R F_L)^2 \):
\[
(F_R F_L)^2 - \cos^2 \delta = \left( 1 - \frac{d_2^2 + \sigma^2}{16} + \frac{(d_\theta \cdot \sigma)^2}{16^2} \right) - \cos^2 \delta
\]
\[
= -\frac{2d^2 + 8 \sin^2 \delta}{16} + \frac{(d_\theta \cdot \sigma)^2}{16^2} + \sin^2 \delta
\]
\[
= -\frac{2d^2 + 8 \sin^2 \delta}{16} + \frac{(d_\theta \cdot \sigma)^2}{16^2} \geq 0.
\]
The last inequality holds since \( \sin \delta \geq d/2 \).

It remains to decide whether \( C(d, \pi/2, \pi/2) \) or \( \Lambda(d) \) is larger.

**Lemma 16.** For \( 0 < d \leq \sqrt{2} \), we have \( \max_{(\sigma, \delta) \in \mathbb{S}} C(d, \sigma, \delta) = \Lambda(d) \).

**Proof.** For \( 0 \leq d \leq \sqrt{2} \) and \((\sigma, \delta) = (\pi/2, \pi/2)\), we have
\[
d_2^2 = d^2 + 4 \quad \text{by (12)}
\]
\[
\mu_\pi = \pi - \arcsin(\sqrt{d^2 + 4/4})
\]
\[
L(d, \pi/2, \pi/2) = 4\mu_\pi + 2\delta - 2\pi = 3\pi - 4 \arcsin(\sqrt{d^2 + 4/4}).
\]
Since \( \arcsin \) is an increasing function, \( d \mapsto L(d, \pi/2, \pi/2) \) is a decreasing function. We therefore have
\[
L(d, \pi/2, \pi/2) < L(0, \pi/2, \pi/2) = 7\pi/3.
\]
On the other hand, by Lemma 15, the function \( d \mapsto \Lambda(d) \) is increasing, and so \( \Lambda(d) \geq \Lambda(0) = 7\pi/3 > L(d, \pi/2, \pi/2) \).
We now have all the tools to discuss case A. In case A, which occurs only for \( d < 2 \), we have \( d_{\text{RL}} < 2 \) and \( d_{\text{RS}} < 2 \). We will now justify that it suffices to study \( \text{CCS} \)-paths in this case, as no other path type can be shorter. Since \( \text{LSR} \)- and \( \text{RSL} \)-paths do not exist, it is enough to show the following lemma:

**Lemma 17.** For \((\sigma, \delta) \in A\), we have

\[
\text{RL}(\sigma, \delta) \leq \text{LS}(\sigma, \delta) \\
\text{RL}(\sigma, \delta) \leq \text{RSL}(\sigma, \delta).
\]

**Proof.** Let \( \gamma_1 \) and \( \gamma_2 \) be the length of the left-turning arcs of an \( \text{RL} \)-path. By Lemma 7, the endpoints \( S \) and \( F \) lie on the counterclockwise arcs \( S_1^2 \) of \( L_S \) and \( F_2^1 \) of \( L_F \). Consider now Figure 12(a). The \( \text{LS} \)-path turns left on \( L_S \) until \( S_0 \), goes along the tangent to \( F_0 \), then turns left on \( L_F \) until it reaches \( F \). Since \( \angle S_2^2 F S_0 = \mu_1 \), and \( |S_0 F_0| = d_L \), we have

\[
\text{LS} - \text{RL} = d_L + 2(2\pi - \mu_1) - 2\mu_1 = d_L + 4(\pi - \mu_1) \geq 0
\]

since \( \mu_1 \leq \pi \). The analogous argument shows that \( \text{RL} \leq \text{RSL} \). \(\square\)

**Lemma 18.** For \( 0 < d < \sqrt{2} \), \( \text{dub}_A(d) = \lambda(d) - d \). In other words, the maximum is realized by the unique point \((\sigma, \delta, \lambda)\) on the segment \( \sigma = \pi, \pi/2 \leq \delta \leq \pi - \alpha^* \) where \( L(d, \sigma, \delta, \lambda) = \text{R}(d, \sigma, \delta, \lambda) \).

**Proof.** By Lemma 16 we have \( \max_{(\sigma, \delta, \lambda) \in \Xi} c(d, \sigma, \delta, \lambda) = \lambda(d) \), and the maximum is assumed at the point \((\sigma, \delta, \lambda) \in A^\Delta \). By Lemma 17 we have \( \ell(d, \sigma, \delta, \lambda) = c(d, \sigma, \delta, \lambda) \) for \((\sigma, \delta, \lambda) \in A^\Delta \). Since \((\sigma, \delta, \lambda) \in A^\Delta \subset \Xi \), this means that \( \text{dub}_A(d) = \sup_{(\sigma, \delta, \lambda) \in A^\Delta} \ell(d, \sigma, \delta, \lambda) - d = c(d, \sigma, \delta, \lambda) - d = \lambda(d) - d \). \(\square\)

**Lemma 19.** For \( \sqrt{2} \leq d < 2 \), we have \( \text{dub}_A(d) \leq \max\{2\pi, \text{dub}_B(d)\} \).

**Proof.** Let \( \hat{A} \) be the closure of \( A^\Delta \). Since \( \hat{A} \) is compact and \( c \) is continuous, there is a point \((\sigma, \delta) \in \hat{A} \) where \( c(d, \sigma, \delta) \) assumes its maximum. By Lemma 14 this is necessarily a point where \( L(\sigma, \delta) = \text{R}(\sigma, \delta) \), and either \((\sigma, \delta) = (\pi/2, \pi/2) \), or \((\sigma, \delta) \) lies on the boundary of \( \hat{A} \). By Lemma 14 it cannot occur on the vertical side of \( \Xi \).

Assume first that \( \delta < \pi/2 \). By Lemma 10 we must then have \( \sigma < \pi/2 \). Using Lemmas 10 and 12 we have

\[
c(d, \sigma, \delta) \leq L(d, \sigma, \delta) \leq L(d, \delta, \delta) \leq L(d, \pi/2, \pi/2).
\]

We observed in the proof of Lemma 16 that \( L(d, \pi/2, \pi/2) \) is a decreasing function of \( d \). For \( d = \sqrt{2} \), we already have \( L(\sqrt{2}, \pi/2, \pi/2) = 3\pi - 4\arcsin(\sqrt{6}/4) < 2\pi + \sqrt{2} \), and so \( c(d, \sigma, \delta) \leq d + 2\pi \). The same argument covers the case where \( (\sigma, \delta) = (\pi/2, \pi/2) \).

It remains to consider the possibility that \( \delta > \pi/2 \). In this case \((\sigma, \delta) \) lies on the common boundary of \( A^\Delta \) and \( B^\Delta \). On this boundary we have \( d_{\text{RS}} = 2 \), so \( R_S \) and \( L_F \) touch. Note that in this case \((\sigma, \delta) \) lies in \( B^\Delta \), not in \( A^\Delta \). It remains to observe that \( \text{RLR} \)-path is identical to the \( \text{RSL} \)-path, so we have \( \text{RSL}(\sigma, \delta) = \text{RL}(\sigma, \delta) \). We will prove in Lemma 23 that in \( B^\Delta \) these two path types are always shortest, and so \( c(d, \sigma, \delta) \leq \text{dub}_B(d) + d \). \(\square\)

### 7 RSL-paths for \( d < 2 \) and case B

It was proven by Goaoc et al. [13] that for any \( \text{CCS} \)-path type (that is, one of the types \( \text{LSR}, \text{RSL}, \text{LRS}, \) or \( \text{RSL} \)), the length of a path of this type from \((0, 0, \alpha)\) to \((d, 0, \beta)\) is differentiable at any point \((\alpha, \beta) \in \square \) where such a path exists and both its circular arcs have non-zero length. For the case of \( \text{RSL} \)-paths, they prove specifically that

\[
\frac{\partial}{\partial \alpha} \text{RSL}(\alpha, \beta) = 1 - \cos \gamma_r, \tag{36}
\]

\[
\frac{\partial}{\partial \beta} \text{RSL}(\alpha, \beta) = 1 - \cos \gamma_l, \tag{37}
\]

where \( \gamma_r \) and \( \gamma_l \) are the lengths of the right-turning and the left-turning circular arc on the path.
We recall that case B is the situation where \( d_{\text{hl}}(\alpha, \beta) < 2 \) and \( d_{\text{rl}}(\alpha, \beta) \geq 2 \). For \( 0 < d < 2 \), Lemma 6 gives an explicit description of the region \( B^\Delta \), using the two functions \( \beta_{\text{hl}}(\alpha) \) and \( \beta_{\text{rl}}(\alpha) \). Let us define two extended regions:

\[
B^o = \{ (\alpha, \beta) \mid 0 \leq \alpha \leq \alpha^*, \beta_{\text{hl}}(\alpha) \leq \beta \leq 2\pi - \alpha \},
\]

\[
\bar{B} = \{ (\alpha, \beta) \mid 0 \leq \alpha \leq \alpha^*, \beta_{\text{hl}}(\alpha) \leq \beta \leq \beta_{\text{rl}}(\alpha) \}.
\]

So \( \bar{B} \) is the closure of \( B^\Delta \), while \( B^o \) is the union \( B^\Delta \cup C^\Delta_{L} \) (see Figure 11(b)).

We now investigate where the three segments of an RSL-path can vanish in \( B^o \): First, the S-segment vanishes exactly if \( d_{\text{hl}} = 2 \). This happens exactly on the lower boundary of \( B^o \). By Lemma 7, \( S \) lies on the arc \( S_0^S S_1^S \) of \( L_S \) (see Figure 12), and so \( 0 \leq \gamma_n \leq 2\alpha^* \). If \( \gamma_n = 0 \), then we must have \( S = S_0^S \) and therefore \( F = F_0^F \). This is the case (\( \alpha, \beta \) = (0, 2\pi)). Finally, if \( \gamma_n = 0 \), then \( F \) must lie on the arc \( F_1^F \), and we have \( 2\pi - \beta \leq \alpha \) (see Figure 15). Equality holds only for \( S = S_1^S \), \( F = F_1^F \), which is the case (\( \alpha, \beta \) = (\( \alpha^* \), 2\pi - \( \alpha^* \))). In all other cases, \( 2\pi - \beta < \alpha \) is a contradiction to \( (\alpha, \beta) \in \Delta \).

The above implies that \( \text{RSL}(\alpha, \beta) \) is differentiable in any point in the interior of \( B^o \). The function is continuous everywhere except at the two points \( (\alpha, \beta) = (0, 2\pi) \) and \( (\alpha, \beta) = (\alpha^*, 2\pi - \alpha^*) \). At the first point, the RSL-path degenerates to the line segment \( SF \) of length \( d \), while the limit of \( \text{RSL}(\alpha, \beta) \) for \( (\alpha, \beta) \rightarrow (0, 2\pi) \) is \( d + 2\pi \). For the second point (\( \alpha^*, 2\pi - \alpha^* \)), consider Figure 10(a). At this point, both the straight segment and the left-turning arc vanish at the same time, and the length of the path is only \( |SF| = 2\alpha^* \). However, for \( (\alpha, \beta) \rightarrow (\alpha^*, 2\pi - \alpha^*) \), the limit of \( \text{RSL}(\alpha, \beta) \) is \( |SF| + 2\pi = 2\alpha^* + 2\pi \). We observe that this is exactly the value of \( \text{RLR}(\alpha^*, 2\pi - \alpha^*) \), as the final right-turning arc of the RLR-path vanishes.

We therefore define the following function on \( B^o \):

\[
\text{RSL}^*(\alpha, \beta) = \begin{cases} 
  d + 2\pi & \text{for } (\alpha, \beta) = (0, 2\pi) \\
  2\alpha^* + 2\pi & \text{for } (\alpha, \beta) = (\alpha^*, 2\pi - \alpha^*) \\
  \text{RSL}(\alpha, \beta) & \text{else}
\end{cases}
\]

We have

\textbf{Lemma 20.} For \( (\alpha, \beta) \in B^o \), the function

- \( \alpha \mapsto \text{RSL}^*(\alpha, \beta) \) is increasing,
- \( \beta \mapsto \text{RSL}^*(\alpha, \beta) \) is increasing,
- \( \alpha \mapsto \text{RSL}^*(\alpha, 2\pi - \alpha) \) is increasing,
- \( \text{RSL}^*(\alpha, \beta) \leq \text{RSL}^*(\alpha^*, 2\pi - \alpha^*) = 2\alpha^* + 2\pi \).

\textbf{Proof.} The two derivatives (36) and (37) are defined and both positive in the interior of \( B^o \), implying the first two claims.

For the third claim, we need to show that

\[
0 \leq \frac{\partial}{\partial \alpha} \text{RSL}(\alpha, \beta) - \frac{\partial}{\partial \beta} \text{RSL}(\alpha, \beta) = - \cos \gamma_n + \cos \gamma_l.
\]

for \( \beta = 2\pi - \alpha \). By Lemma 7, \( S \) lies on the arc \( S_0^S S_1^S \) of \( L_S \). If \( \beta = 2\pi - \alpha \), then \( F \) lies on the counter-clockwise arc \( F_1^F \) of \( L_F \). The two circular arcs of the RSL-path have two components, namely, \( \gamma_n = \alpha + \zeta \) and \( \gamma_l = \beta + \zeta = 2\pi - \alpha + \zeta \), see Figure 16. This implies
Figure 16: An RSL-path for $\beta = 2\pi - \alpha$

Figure 17: If $\gamma_R + \gamma_L \leq 2\pi$ then $\text{rsl} \leq d + 2\pi$

\[ -\cos \gamma_R + \cos \gamma_L = -\cos(\alpha + \zeta) + \cos(\alpha - \zeta) = 2 \sin \alpha \sin \zeta \geq 0. \]

The second and third claim immediately imply the last one. \qed

If the two circular arcs of an RSL-path add up to less than a full circle, then the path cannot be too long:

**Lemma 21.** For $(\alpha, \beta) \in B^\circ$ and $d < 2$, if $\gamma_R + \gamma_L \leq 2\pi$, then $\text{rsl}(d, \alpha, \beta) \leq d + 2\pi$.

**Proof.** Let $T_S$ and $T_F$ denote the points of tangency of the $S$-segment to $R_S$ and $L_F$. We observed above that $\gamma_R \leq \pi$. If we also have $\gamma_L \leq \pi$ then $d \geq |T_S T_F|$, and the claim follows immediately.

If $\gamma_R \geq \pi/2$, then we have $\gamma_L \leq 2\pi - \gamma_R \leq 3\pi/2$. But then $d = |SF| \geq 2$, a contradiction. It follows that we must have $\gamma_R < \pi/2$. See Figure 17. Let $S'$ be the point on $L_F$ such that the counter-clockwise arc $S'T_F$ on $L_F$ has length $\gamma_R$. Let $h_S$ and $h_{S'}$ be lines through $S$ and $S'$ orthogonal to the segment $T_S T_F$.

The distance between $h_S$ and $h_{S'}$ is $|T_S T_F|$, and so we have $|SS'| \geq |T_S T_F|$. By the triangle inequality, we have $d + |FS'| = |SF| + |FS'| \geq |SS'| \geq |T_S T_F|$, so $|T_S T_F| - |FS'| \leq d$. Since $2\pi - |FS'| = \gamma_R + \gamma_L$, we have

\[ \text{rsl}(d, \alpha, \beta) = \gamma_R + |T_S T_F| + \gamma_L = |T_S T_F| + 2\pi - |FS'| \leq d + 2\pi. \] \qed

**Lemma 22.** If $\alpha \mapsto \text{rsl}^*(d, \alpha, \beta_{ln}(\alpha))$ has an extremum for $0 < \alpha < \alpha^*$, then $\text{rsl}^*(d, \alpha, \beta_{ln}(\alpha)) \leq d + 2\pi$.

**Proof.** Let $\alpha_0$ be such an extremum, and consider the RSL-path for $(\alpha_0, \beta_{ln}(\alpha_0))$. If $\gamma_R + \gamma_L \leq 2\pi$, then by Lemma 21 we have $\text{rsl}(d, \alpha_0, \beta_{ln}(\alpha_0)) \leq d + 2\pi$. We can therefore assume $\gamma_R + \gamma_L > 2\pi$. The point
(α₀, βₐₙ(α₀)) is an extremum of the function RSL*(α, β), under the constraint that dₐₙR = 4, and so there must be a constant λ such that ∇ RSL*(α₀, βₐₙ(α₀)) = λ∇ dₐₙR(α₀, βₐₙ(α₀)). Using π/2 < σ, δ ≤ π and (14) we have

\[
\frac{\partial}{\partial \sigma} dₐₙR = 4d \cos \delta \cos \sigma > 0
\]
(41)
\[
\frac{\partial}{\partial \delta} dₐₙR = -4 \sin \delta (d \sin \sigma + 2 \cos \delta).
\]
(42)

Using (36) and (37) we get

\[
2 \frac{\partial}{\partial \sigma} RSL* = \frac{\partial}{\partial \beta} RSL* + \frac{\partial}{\partial \alpha} RSL* = 2 - \cos \gamma_L - \cos \gamma_R \geq 0
\]
(43)
\[
2 \frac{\partial}{\partial \delta} RSL* = \frac{\partial}{\partial \beta} RSL* - \frac{\partial}{\partial \alpha} RSL* = -\cos \gamma_L + \cos \gamma_R.
\]
(44)

Inequalities (41) and (43) imply that λ ≥ 0, so let us consider (42). By Lemma 6, we have σ + δ = βₐₙR(α₀) ≥ βₐₙR(α*) = 2π - α* > 3π/2 for d < 2, and so cos δ < cos(3π/2 - σ) = -sin σ. It follows that 2 cos δ < -2 sin σ < -d sin σ, and so (42) is positive.

We have 0 ≤ γ_R ≤ π and we assumed that γ_R + γ_L > 2π. It follows that γ_L > 2π - γ_R ≥ π. Since γ_L > 2π - γ_R, we have cos γ_R < cos γ_L. This implies that (44) is negative. But this means that λ < 0, a contradiction.

Proof. We first claim that the LRL-path or the RSL-path is shorter than any other path type. Since there is no LSR-path in case B, it suffices to exclude path types LSL, RLR, and RSR.

Lemma 23. For (α, β) ∈ Bₐ we have

\[
\text{LRL}(α, β) \leq \text{LSL}(α, β) \leq \text{RLR}(α, β) \leq \text{RSR}(α, β)
\]

Proof. We first compare the LRL-path and the LSL-path. Let γ₁ and γ₂ be the two arcs on the LRL-path. By Lemma 7, we have 2μ_L - π ≤ γ₁ ≤ μ_L and 0 ≤ γ₂ ≤ 2μ_L - π. The LSL-path has length LSL = γ₁ + γ₂ + 2(2π - μ_L) + dₐₙR. We thus have LSL - LRL = dₐₙR - 4(π - μ_L) ≥ 0.

Consider now the LRL-path and the RLR-path. By (12) and (13), we have dₐₙR - dₐₙR = 8d sin δ cos σ ≤ 0 for (σ, δ) ∈ Bₐ, and so dₐₙR ≤ dₐₙR, implying μₐₙR ≤ μₐₙR. By Lemma 8, we have RLR - LRL = 4(μₐₙR - μₐₙR - δ + π) ≥ 0.

Finally, we compare RSL-path and RSR-path. The RSL-path consist of an initial right-turning arc ST₁, a segment T₁T₂, and a final left-turning arc T₂F. The RSR-path consist of an initial right-turning arc S₁R₁, a segment R₁R₂, and a final right-turning arc R₂F, see Figure 18.

We first claim that the arc ST₁ is common to both paths. Indeed, since dₐₙR < 2, by Lemma 7, the initial arc S₁R₁ of the RSR-path must have length at least π (Figure 12(b)), while the arc ST₁ must be shorter than π (Figure 12(a)).
The second case \( \alpha \) unique value in \( \partial B \), \( \alpha \leq \alpha \). For \( \beta \) close to \( \beta_{\text{LS}}(\alpha) \).

Since \( |F_2^2 F| = 2\pi - |F_2^2 T_F| \) and \( |T_S R_1| = 2\pi - |R_1^2 T_S| \), where \( F_2^2 T_F \) is a left-turning arc on the disk \( L_F \) and \( R_1^2 T_S \) is a right-turning arc on disk \( R_S \), we have

\[
\text{RSL} - \text{RSR} = (|T_S T_F| - |F_2^2 T_F|) - (|R_1^2 T_S| + |R_1 R_2| + |R_2^2 F|) = |R_1^2 T_S \cup T_S T_F| - |R_1 R_2 \cup R_2^2 F \cup F_2^2 T_F|.
\]

The path \( R_1 R_2 \cup R_2^2 F \cup F_2^2 T_F \) is a path connecting \( R_1 \) with \( T_F \) while avoiding the interior of \( R_S \). However, the path \( R_1^2 T_S \cup T_S T_F \) is clearly the shortest path of this kind, and so \( \text{RSL} - \text{RSR} \leq 0 \).

It remains to understand the function \( B(d) \). By definition, there is an \( \alpha_n \in [0, \alpha^\ast] \) such that \( B(d) = B(d, \alpha_n, \beta_{\text{LS}}(\alpha_n)) \).

**Lemma 24.** The function \( \alpha \mapsto \text{LRL}(\alpha, \beta_{\text{LS}}(\alpha)) - \text{RSL}(\alpha, \beta_{\text{LS}}(\alpha)) \) is monotonically decreasing on \( [0, \alpha^\ast] \).

**Proof.** We first observe that the function \( \alpha \mapsto \text{LRL}(\alpha, \beta) - \text{RSL}(\alpha, \beta) \) is decreasing. This follows immediately from Lemmas 20 and 11. We also claim that the function \( \beta \mapsto \text{LRL}(\alpha, \beta) - \text{RSL}(\alpha, \beta) \) is increasing for \( \beta \geq 3\pi/2 \). Together, these facts prove the lemma: consider two values \( 0 \leq \alpha_1 < \alpha_2 \leq \alpha^\ast \). Since \( \beta_{\text{LS}}(\alpha) \) is a decreasing function, we have \( \beta_{\text{LS}}(\alpha_1) > \beta_{\text{LS}}(\alpha_2) \geq 2\pi - \alpha^\ast > 3\pi/2 \), and so

\[
\text{LRL}(\alpha_1, \beta_{\text{LS}}(\alpha_1)) - \text{RSL}(\alpha_1, \beta_{\text{LS}}(\alpha_1)) \geq \text{LRL}(\alpha_1, \beta_{\text{LS}}(\alpha_2)) - \text{RSL}(\alpha_1, \beta_{\text{LS}}(\alpha_2))
\]

\[
\geq \text{LRL}(\alpha_2, \beta_{\text{LS}}(\alpha_2)) - \text{RSL}(\alpha_2, \beta_{\text{LS}}(\alpha_2)).
\]

It remains to show that for \( (\alpha, \beta) \in B^\Delta \) with \( \beta \geq \frac{3\pi}{2} \), the function \( \beta \mapsto \text{LRL}(\alpha, \beta) - \text{RSL}(\alpha, \beta) \) is increasing. Since we are in case \( B^\Delta \), by Lemma 7 the point \( S \) lies on the arc \( S_0^\ast S_1^\ast \) of \( L_S \), while \( F \) lies on the arc \( F_2^2 F_1^2 \) of \( L_F \) (see Figure 12(a)). It follows that \( \gamma_L > \beta \geq 3\pi/2 \), and so \( \cos \gamma_L > 0 \). This implies that \( \frac{\partial}{\partial \beta} \text{RSL}(\alpha, \beta) = 1 - \cos \gamma_L < 1 \). On the other hand, by Lemma 12 \( \frac{\partial}{\partial \beta} \text{LRL}(\alpha, \beta) \geq 1 \) for \( \beta \geq \frac{3\pi}{2} \), and the claim follows. \( \square \)

By Lemmas 22 and 24 if \( B(d) > d + 2\pi \), then \( B(d) = B(d, \alpha_n, \beta_{\text{LS}}(\alpha_n)) \), where either \( \alpha_n = 0 \) or \( \alpha_n = \alpha^\ast \) or \( \alpha_n \in (0, \alpha^\ast) \) is the unique value where \( \text{LRL}(\alpha_n, \beta_{\text{LS}}(\alpha_n)) = \text{RSL}^\ast(\alpha_n, \beta_{\text{LS}}(\alpha_n)) \). The first case \( \alpha_n = 0 \) can never occur, as a simple geometric argument similar to Lemma 4 shows that \( \text{RSL}(0, \beta_{\text{LS}}(0)) \leq d + 2\pi \). The second case \( \alpha_n = \alpha^\ast \) holds only for \( d = \sqrt{2} \), as we have seen in the proof of Lemma 14 that for \( d = \sqrt{2} \) we have \( \text{LRL}(\alpha^\ast, 2\pi - \alpha^\ast) < 2\alpha^\ast + 2\pi = \text{RSL}^\ast(\alpha^\ast, 2\pi - \alpha^\ast) \), and \( \text{LRL} \) does not have a maximum at this point. For \( d = \sqrt{2} \), however, \( \text{LRL}(\alpha^\ast, 2\pi - \alpha^\ast) = 2\alpha^\ast + 2\pi = \text{RSL}^\ast(\alpha^\ast, 2\pi - \alpha^\ast) \), so we have shown:

**Lemma 25.** For \( 0 < d < 2 \) if \( \text{dub}_B(d) > 2\pi \), then \( \text{dub}_B(d) = B(d, \alpha_n, \beta_{\text{LS}}(\alpha_n)) \), where \( \alpha_n \) is the unique value in \( [0, \alpha^\ast] \) where \( \text{RSL}^\ast(d, \alpha_n, \beta_{\text{LS}}(\alpha_n)) = \text{LRL}(d, \alpha_n, \beta_{\text{LS}}(\alpha_n)) \).

It remains to argue the monotonicity of \( d \mapsto \text{dub}_B(d) \).
We choose $d$. We now put all the pieces together.

Proof. Assume the function is not monotone, so there is $\sqrt{2} \leq d_1 < d_2 < 2$ such that $2\pi < \text{dub}_B(d_1) < \text{dub}_B(d_2)$. Since $\text{dub}_B(d)$ is continuous, it assumes its maximum $D := \max_{d \in [d_1, d_2]} \text{dub}_B(d)$ on the closed interval $[d_1, d_2]$. The set $\{d \in [d_1, d_2] \mid \text{dub}_B(d) = D\}$ is compact, and so assumes its supremum, say at $d_3$. So we have $\text{dub}_B(d_3) > \text{dub}_B(d_1) > 2\pi$, and $\text{dub}_B(d) < \text{dub}_B(d_3)$ for $d_1 \leq d < d_3$.

By Lemma 25, we have $\text{dub}_B(d) = b(d_3, \alpha_n, \beta_n) - d_3$, where $0 < \alpha_n \leq \arcsin(d_3/2)$ and $\beta_n = \beta_n(\alpha_n)$.

Let us define $d_2 = \max\{d_1, 2\sin(2\pi - \beta_n)\}$. Then $d_1 \leq d_2 < d_3$, and $(\alpha_n, \beta_n) \in B^{A}$ for $d = d_2$. Then there is an RSL- or LRL-path from $(0, 0, \alpha_n)$ to $(d_0, 0, \beta_n)$ of length at most $\text{dub}_B(d_2) + d_2$. By Lemmas 14 and 5 there is then a path of length $\text{dub}_B(d_2) + d$ from $(0, 0, \alpha_n)$ to $(d, 0, \beta_n)$ for all $d \geq d_2$, implying that $\ell(d, \alpha_n, \beta_n) \leq \text{dub}_B(d_2) + d$, a contradiction to the assumption that $\text{dub}_B(d_3) > \text{dub}_B(d_2)$.

The approximation to the value of $d^*$ has been computed numerically. For a given $d$, we first approximate $\beta_n$ numerically by binary search on the interval $[0, \alpha^*]$ using Lemma 24. We can then compute $\beta_n(\alpha_n)$ and $B(d)$.

8 The Dubins cost function

We now put all the pieces together.

Lemma 27. For $0 < d < \sqrt{2}$, $\text{dub}(d) = \text{dub}_A(d)$. The function decreases monotonically from $\text{dub}(0) = 7\pi/3$ to $\text{dub}(\sqrt{2}) = 5\pi/2 - \sqrt{2}$.

Proof. By Lemma 18, $\text{dub}_B(d) = \lambda(d) - d$ for $0 < d < \sqrt{2}$. By Lemma 15, this function decreases monotonically from $\text{dub}_B(0) = 7\pi/3$ to $\text{dub}_B(\sqrt{2}) = 5\pi/2 - \sqrt{2}$, and so $\text{dub}_B(d) > 2\pi = \text{dub}_C(d)$.

By Lemma 20, for any $(\alpha, \beta) \in B^{A}$ there is an RSL-path with length at most $\text{rsl}(\alpha^* - 2\pi - \alpha^*) = RSL(\alpha^*, 2\pi - \alpha^*)$. In the proof of Lemma 14, we observed that for $0 < d < \sqrt{2}$, we have RLR($d, \alpha^*, 2\pi - \alpha^*) < LRL(d, \alpha^*, 2\pi - \alpha^*) \leq \lambda(d) = \text{dub}_A(d) + d$. It follows that $\text{dub}_B(d) > \text{dub}_A(d)$, and so indeed $\text{dub}(d) = \text{dub}_A(d)$ for $0 < d < \sqrt{2}$.

Lemma 28. For $\sqrt{2} \leq d < d^*$, $\text{dub}(d) = \text{dub}_B(d)$. The function decreases monotonically from $\text{dub}(\sqrt{2}) = 5\pi/2 - \sqrt{2}$ to $\text{dub}(d^*) = 2\pi$. For $d^* \leq d < 2$, $\text{dub}(d) = 2\pi$.

Proof. By Lemma 19, we have $\text{dub}_A(d) \leq \max\{2\pi, \text{dub}_B(d)\} = \max\{\text{dub}_C(d), \text{dub}_B(d)\}$, and so $\text{dub}(d) = \max\{2\pi, \text{dub}_B(d)\}$. By Lemma 26 we have $\text{dub}_B(d) > 2\pi$ for $\sqrt{2} \leq d < d^*$, and the lemma follows.

Lemma 29. For $d \geq 2$, we have $\text{dub}(d) = 2\pi$.

Proof. Let $d \geq 2$ and $(\alpha, \beta) \in A$. We need to show that $\ell(d, \alpha, \beta) \leq 2\pi + d$. If $(\alpha, \beta) \in C^{A}$, then this follows from Lemma 4. Otherwise we must have $(\alpha, \beta) \in B^{A}$, and so $\delta > \pi/2$ by Lemma 3. We choose $d_1 < 2$ such that $d_1 > 2\sin(\pi - \delta)$ and $d_1 > d^*$ and consider the configuration $(\alpha, \beta)$ for distance $d_1$. Since case $A$ occurs only within the $\delta$-range $\arcsin(d_1/2) \leq \delta \leq \pi - \arcsin(d_1/2)$, we must be in either case B or case C, so there is a path of type RSR, LSR, RSL, or LRL of length at most $d_1 + \text{dub}(d_1) = d_1 + 2\pi$. By Lemmas 1 and 5 there is then a path from $(0, 0, \alpha)$ to $(d, 0, \beta)$ of length at most $d_1 + 2\pi = (d - d_1) = d + 2\pi$.

We summarize our results in the following theorem:

Theorem 1. The function $\text{dub}(d)$ has two breakpoints at $\sqrt{2}$ and $d^* \approx 1.5874$. For $d < \sqrt{2}$, $\text{dub}(d) = \text{dub}_A(0) = \frac{7\pi}{4}$. For $\sqrt{2} \leq d < d^*$, $\text{dub}(d) = \text{dub}_B(d) \leq \text{dub}_B(\sqrt{2}) = \frac{5\pi}{2} - \sqrt{2}$. For $d \geq d^*$, we have $\text{dub}(d) = 2\pi$.

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