Transverse bending of waveguide by gravity load

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Abstract. The paper considers the problem of gravitational loading of an extended thin-walled structure of a waveguide with a rectangular cross-section. The waveguide is modeled by a composite structure of 4 plates. The solution was based on the theory of plates and shells for each plate constituting the cross-section of the waveguide. The analytical solution is based on 5th-degree polynomials using the Airy function and the Saint Venant semi-inverse method. Calculations according to the proposed method were compared with data obtained according to beam theory. The comparison confirmed the correctness of the technique and showed that the proposed solution predicts large values of maximum stresses and deformations by 10-14% due to taking into account the fineness of the structure. Also, the proposed approach revealed tangent stresses at the angles of the rectangular cross-section, which arise due to the inhomogeneity of deformation of the elements of this cross-section.

1. Introduction
Gravitational load on structural elements is a widespread type of loading and is caused by various reasons [1-4]: accelerated movement, vibrations, Earth, etc. Accounting for gravitational loading usually greatly complicates the calculation of the stressed-deformed state of the body and, if possible, they try to neglect it. Such neglect is often correct since the stresses and deformations resulting from it are usually much less than from workloads. However, in some cases, for example, in aerospace engineering, inertial loading plays a significant role. In this case, neglecting gravity or using simplified calculation methods can lead to large calculation errors. For example, using beam theory to calculate thin-walled structures for the effect of the gravitational load is a very rough approach and leads to calculation errors.

This paper proposes an approach based on the theory of plates and shells [5-14], which makes it possible to take into account as much as possible all the features of the thin-walled design of the waveguide. The resulting expressions are rather complex, but the equations for maximum stresses and strains can be used in engineering calculations to evaluate the strength and stiffness of a waveguide or similar thin-walled structure with a rectangular cross-section.

2. Waveguide construction model
Consider the straight section of the waveguide, which is loaded by gravitational acceleration in the transverse direction (figure 1). Let the waveguide have a hinged support, and its geometry is formed by connecting four plates at right angles to each other.
The thin-walled design of the waveguide can be modeled according to shell theory, but the section's right angles make it difficult to use since the radius of curvature of the section will be a breaking function. Therefore, consider the geometry of a waveguide consisting of 4 separate plates [15]. For each plate, according to plate theory, it is necessary to compose and solve two differential equations [16,17] together with boundary conditions:

\[
\begin{align*}
\frac{\partial^2 \varphi_i}{\partial \alpha_i^2} + 2 \frac{\partial^2 \varphi_i}{\partial \alpha_i^2 \partial \beta_i^2} + \frac{\partial^4 \varphi_i}{\partial \beta_i^4} &= 0; \\
\frac{\partial^2 \omega_i}{\partial \alpha_i^2} + 2 \frac{\partial^2 \omega_i}{\partial \alpha_i^2 \partial \beta_i^2} + \frac{\partial^4 \omega_i}{\partial \beta_i^4} &= -\frac{\rho t}{D} \frac{\partial^2 \omega_i}{\partial \tau^2} \\
\frac{\partial^2 \varphi_{i+1}}{\partial \alpha_i^2} = 0; & \quad \frac{\partial^2 \varphi_i}{\partial \alpha_i^2} = 0; \quad \frac{\partial^2 \varphi_{i+1}}{\partial \alpha_i^2 \partial \beta_i^2} = 0; \quad \frac{\partial^2 \varphi_i}{\partial \alpha_i^2 \partial \beta_i^2} = 0; \quad \frac{\partial^2 \omega_{i+1}}{\partial \alpha_i^2} = 0; \quad \frac{\partial^2 \omega_i}{\partial \alpha_i^2} = 0; \quad \frac{\partial^2 \omega_{i+1}}{\partial \alpha_i^2 \partial \beta_i^2} = 0; \quad \frac{\partial^2 \omega_i}{\partial \alpha_i^2 \partial \beta_i^2} = 0.
\end{align*}
\]

where \(i = 1,2,3,4\) is the number of plates; \(\alpha, \beta, z_i\) is the local coordinate system of \(i\)th plate; \(\omega_i = \omega_i(\alpha, \beta)\) is the deflection function of \(i\)th plate; \(\varphi_i = \varphi_i(\alpha, \beta)\) is the Airy stress function of \(i\)th plate; \(E\) is the Young’s module; \(\tau\) is the time; \(\rho\) is the density of plate material; \(D\) is the bending stiffness of \(i\)th plate.

The solution of the problem will begin with the solution for the side walls 2 and 4, since they have a significantly higher bending stiffness and strength compared to the plates 1 and 3 at a given load direction (figure 1). The solution obtained for the plates 2 and 4 will be a boundary condition for the plates 1 and 3, which are stretched and compressed in combination with bending along the curve formed by the deformed edges of the side plates.

2.1. Side plates solution

Figure 2 shows the design for plate 2, taking into account the interaction with other waveguide plates.

**Figure 1.** Waveguide geometry and load.

**Figure 2.** Stresses in the side plate 2.
The Airy stress function \( \varphi_i (\alpha_i, \beta_i) \) or plate 2 is taken as a polynomial [16,17] in the form:

\[
\varphi_2 (\alpha_2, \beta_2) = \frac{d_2}{6} \alpha_2^3 \beta_2^3 - \frac{d_3}{30} \beta_2^5 + \frac{d_1}{6} \beta_2^3 + \frac{b_2}{2} \alpha_2^2 \beta_2 + \frac{a_2}{2} \alpha_2^2.
\] (3)

In this case, the expressions for the stresses determining the stress state of plate 2 will be based on known Airy dependencies, which, taking into account the inertia forces, will take the form:

\[
\sigma_{a2} = \frac{\partial^2 \varphi_2}{\partial \beta_2^2} = d_3 \alpha_2^2 \beta_2^2 - \frac{2d_5 \beta_2^2}{3} + (d_3 + \mu \gamma) \beta_2; \quad \sigma_{b2} = \frac{\partial^2 \varphi_2}{\partial \alpha_2^2} = \frac{d_5 \beta_2^2}{3} + (b_3 + \gamma) \beta_2 + a_2; \quad \tau_{a2b2} = -\frac{\partial^2 \varphi_2}{\partial \alpha_2 \partial \beta_2} = -d_5 \alpha_2 \beta_2^2 - b_3 \alpha_2. \] (4)

The values of the coefficients in equation (7) will be found after substituting this expression into boundary conditions on the sides of plate 2:

\[
\sigma_{a2} \left( \beta_2 = \pm \frac{H}{2} \right) = \tau_{a2b2} \left( \beta_2 = \pm \frac{H}{2} \right) = \mp \sigma_{\kappa}, \quad \int_{-\frac{H}{2}}^{\frac{H}{2}} \beta_2 \cdot \sigma_{a2} \, d\beta_2 = M_{z0} \left( \alpha_2 = \pm \frac{L}{2} \right). \] (5)

After substituting expressions (3) into boundary conditions (6), we obtain the required coefficients in the Airy function (6):

\[
a_2 = 0; \quad b_3 = -\sigma_{\kappa} \frac{3}{h} \left( \frac{\gamma}{2} - \frac{\tau_{a2b2}}{L} \right); \quad d_3 = M_{z0} \frac{12}{h^3} - \mu \gamma - d_5 \frac{5L^2 - 2h^2}{20}; \quad d_5 = \frac{6}{h^3} \left[ \sigma_{\kappa} \frac{2}{h} + \gamma + \frac{\tau_{a2b2}}{L} \right].
\]

Equations for normal and shear stresses are derived from the Airy function:

\[
\sigma_{a2} = d_3 \alpha_2^2 \beta_2^2 - \frac{2d_5 \beta_2^2}{3} + (d_3 + \mu \gamma) \beta_2; \quad \sigma_{b2} = \frac{d_5 \beta_2^2}{3} + (b_3 + \gamma) \beta_2 + a_2; \quad \tau_{a2b2} = -d_5 \alpha_2 \beta_2^2 - b_3 \alpha_2. \] (7)

The deformed state of the plates is determined on the basis of the Airy function (7) according to the method of Papkovich P.F. [16], which is described by the equations:

\[
u_2 (\alpha_2, \beta_2) = \frac{1}{6G} \left[ -A_1 \alpha_2 \beta_2^3 + B_1 \alpha_2 \beta_2 + C_1 \alpha_2 \beta_2 \right],
\] (8)

\[
u_2 (\alpha_2, \beta_2) = A_2 \cdot \beta_2^2 + \frac{1}{2AG} \left[ B_2 \beta_2^4 - C_2 \cdot \alpha_2 \beta_2^2 + D_2 \beta_2^2 + E_2 \cdot \alpha_2^4 + F_2 \cdot \alpha_2^2 \right],
\] (9)

where \( A_1 = d_5 (2-\mu); \quad B_1 = d_3 (1-\mu); \quad C_1 = 3(d_3 - \mu d_3 - \mu b_3); \quad A_2 = \frac{0.5-\mu}{1-\mu} \frac{\gamma}{2G}; \quad B_2 = d_5 (1+\mu); \quad C_2 = 6\mu d_3; \quad D_2 = 2(3b_3 - 3\mu b_3 + d_3 - 3\mu d_3); \quad E_2 = B_1; \quad F_2 = 6(\mu b_3 - 2b_3 - d_3 + d_3 \mu).

Expressions (9,10) determine the stressed and deformed state of the side plate during gravitational loading. Due to the symmetry of the problem, this solution will be valid for side plate 4.

2.2. Top and bottom plate solution
The load for top plate 1 is determined by several external influences:
1) stretching along the length from bending of side plates conjugated with it;
2) bending by changing the curvature of the side plates attached to it;
3) inertial loading across its entire plane.
Consider these load components individually. The tension of plate 1 is defined as:
\[
\sigma_{x1} = \sigma_{h1} + \frac{h}{2} = -\left(\frac{2h^3 + 3h'L^2}{2h^2} - \frac{3h'}{5}\right) \left(\sigma_{h1} + \frac{\gamma - q}{L}\right).
\] (10)

Further, plate 1 is bent by changing the curvature of the side plates attached thereto by an amount of:
\[
\rho_{x2}(\alpha_2, \beta_2) = \frac{1}{2G} \left(\mu - 1\right)(b_3 + d_3) - b_3 - \frac{\mu h'^2}{4} d_3. \]
(11)

Then the normal bending stresses can be defined as [18]:
\[
\sigma_{Mx1}(z_1) = M_{\beta_1} \rho_{x2} = \rho_{x2}Ez_1 = \left(\mu - 1\right)(b_3 + d_3) - b_3 - \frac{\mu h'^2}{4} d_3 \right) z_i.
\] (12)

where \(-\frac{t}{2} \leq z_i \leq \frac{t}{2}.

Finally, plate 1 undergoes transverse bending from inertia forces distributed throughout its area. The decision for this case can be made according to [13,18] as:
\[
\sigma_{Mx1}(z_1) = M_{\beta_1} \rho_{x2} = \frac{0.817 \cdot \rho_{x2}t^2b^2}{J_{\beta_1}} z_i. \]
(13)

The sum of the stress components (12,14,15) makes it possible to determine the stress state of plate 1. The solution for the deformed state of plate 1 is very difficult, but the calculations show that in real cases of waveguide operation it can be neglected due to the small values.

The lower plate 2 will have similar stressed and deformed states. Thus, an analytical solution was obtained for the stressed and deformed state of the entire waveguide structure.

3. Comparative calculation
To check the obtained equations describing the stressed and deformed state of the waveguide, comparative calculations were made with the beam model. The restraint conditions were adopted according to figure 1, waveguide cross-sectional dimensions: \(b\times h=15\times 32 \text{ mm}\), length: \(l=0.4 \text{ m}\), wall thickness \(t=1.2 \text{ mm}\), \(P=7\times 10^{10} \text{ Pa}\), Poisson’s coefficient 0.3, density 2770 kg/m³, gravitational load was taken equal to 200 m/s², moment \(M_{\alpha} = 0\). Maximum normal and shear stresses and maximum deflection were compared. The results of the comparison are shown in table 1.

| Table 1. Results of comparative calculation of waveguide. |
|-----------------------------------------------------------|
| Beam theory | Proposed method |
| \(\sigma_{x, MAX} \) (Pa) | \(\tau_{y, MAX} \) (Pa) | \(\tau_{2, MAX} \) (Pa) | \(u_{y, MAX} \) (m) |
| 5 149 887.9 | 1 141 844.7 | 963 029.1 | -0.000149 |
| 5 593 864.7 | 1 278 866.1 | - | -0.00013 |
| Deviation | 7.9 % | 12 % | - | 14 % |

The obtained results of the waveguide calculations confirmed the correctness of the proposed method and made it possible to identify the features of the stress state.
4. Discussion
The proposed solution on plate theory makes it possible to more accurately take into account the features of the stress state of a thin-walled structure. In the case of a waveguide, the developed technique revealed the presence of tangent stresses $\tau_2'$ on the connection line of the side with the upper and lower plates. These stresses are not taken into account by beam theory because the section is assumed to be solid or thick-walled. Taking into account the thin-walled section according to the theory of plates allows you to calculate this stress and take it into account when evaluating the strength and rigidity of the structure. Also, the plates are characterized by a violation of the assumption of non-injection of fibers, which leads to higher values of the remaining components of stresses and deformations compared to the beam (table 1).

5. Conclusion
In this paper, an analytical solution was obtained for the stressed and deformed state of the waveguide according to plate theory, which made it possible to take into account the features of the work of its thin-walled cross-section. The proposed approach and resolving equations can be used to calculate the strength and stiffness of any extended thin-walled structures with a composite cross-section.

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