Multivariate polynomial interpolation and sampling in Paley–Wiener spaces

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Abstract

In this paper, an equivalence between existence of particular exponential Riesz bases for spaces of multivariate bandlimited functions and existence of certain polynomial interpolants for functions in these spaces is given. Namely, polynomials are constructed which, in the limiting case, interpolate \( \{ (\tau_n, f(\tau_n)) \}_n \) for certain classes of unequally spaced data nodes \( \{ \tau_n \}_n \) and corresponding \( ℓ_2 \) sampled data \( \{ f(\tau_n) \}_n \). Existence of these polynomials allows one to construct a simple sequence of approximants for an arbitrary multivariate bandlimited function \( f \) which demonstrates \( L_2 \) and uniform convergence on \( \mathbb{R}^d \) to \( f \). A simpler computational version of this recovery formula is also given at the cost of replacing \( L_2 \) and uniform convergence on \( \mathbb{R}^d \) with \( L_2 \) and uniform convergence on increasingly large subsets of \( \mathbb{R}^d \). As a special case, the polynomial interpolants of given \( ℓ_2 \) data converge in the same fashion to the multivariate bandlimited interpolant of that same data. Concrete examples of pertinent Riesz bases and unequally spaced data nodes are also given.

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1. Introduction

Approximation of bandlimited functions (see Definition 2.5) as limits of polynomials has a rich pedigree, as historical answers to the following question illustrate: if \( \{ \text{sinc}(· − \tau_n) \}_{n ∈ \mathbb{Z}} \) is a Riesz basis for \( PW_1 \), what are the canonical product expansions of the biorthogonal functions for...
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this Riesz basis? The first results along these lines were given by Paley and Wiener in [12], and improved upon by Levinson in [9], while Levin extends these results to different classes of Riesz bases in [8]. A complete solution is given by Lyubarskii and Seip in [10] and Pavlov in [13].

In particular they prove the following theorem which is the philosophical starting point of this paper.

**Theorem 1.1.** If \( \{\tau_n\}_n \subset \mathbb{R} \) (where \( \tau_n \neq 0 \) when \( n \neq 0 \)) is a sequence such that the family of functions \( \{\text{sinc}(\cdot - \tau_n)\}_n \) is a Riesz basis for \( PW_1 \), then the function

\[
S(z) = \lim_{r \to \infty} (z - \tau_0) \prod_{\{\tau_n : |\tau_n| < r, n \neq 0\}} \left(1 - \frac{z}{\tau_n}\right)
\]

is entire where convergence is uniform on compacta, and the biorthogonal functions \( \{G_n\}_n \) of \( \{\text{sinc}(\cdot - \tau_n)\}_n \) are given by

\[
G_n(z) = \frac{S(z)}{(z - \tau_n)S'(\tau_n)}.
\]

The following is a readily proven corollary of Theorem 1.1.

**Corollary 1.2.** If \( \{\tau_n\}_n \subset \mathbb{R} \) and \( \{G_k\}_k \) are defined as in Theorem 1.1, then for each \( k \), there exists a sequence of polynomials \( \{\Phi_{N,k}\}_N \) such that

1. \( \Phi_{N,k}(\tau_n) = G_k(\tau_n) \) when \( |\tau_n| < N \).
2. \( \lim_{N \to \infty} \Phi_{N,k} = G_k \) uniformly on compacta.

Corollary 1.2 motivates two questions.

1. Let \( \{\tau_n\}_n \subset \mathbb{R}^d \) be a sequence such that \( \{e^{i\langle \cdot, \tau_n \rangle}\}_n \) is a Riesz basis for \( L^2([-\pi, \pi]^d) \). What are sufficient conditions on \( \{e^{i\langle \cdot, \tau_n \rangle}\}_n \) such that every multivariate bandlimited function \( f \) (not just biorthogonal functions associated with a particular exponential Riesz basis), has a corresponding sequence of polynomials which interpolates \( f \) on increasingly large subsets of \( \{\tau_n\}_n \)?
2. If polynomial interpolants (of the type described above) exist for a multivariate bandlimited function, can they be used to approximate the function in some simple and straightforward way?

Let \( \{\tau_n\}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d \) be a sequence such that the family of exponentials \( \{e^{i\langle \cdot, \tau_n \rangle}\}_{n \in \mathbb{Z}^d} \) is a uniformly invertible Riesz basis for \( L^2([-\pi, \pi]^d) \) (defined in Section 4). Under this condition, Theorem 5.4 shows that polynomial interpolants of the type described in question (1) exist, along with bounds on the coordinate degree (not just the total degree) of each polynomial. This theorem also addresses question (2) by demonstrating that multivariate bandlimited functions can be approximated globally in both uniform and \( L^2 \) metrics by a rational function times a multivariate sinc function. Stated informally,

\[
f(t) \simeq \Psi(t) \frac{\text{sinc}(t)}{Q_{d,\ell}(t)}, \quad \ell > 0,
\]

(1)
where \( \{ \psi_\ell \}_{\ell \in \mathbb{N}} \) is the desired sequence of interpolating polynomials and \( \{ Q_{d, \ell} \}_{\ell} \) is a sequence of polynomials which eventually removes the zeros of sinc. The fraction in (1) becomes more computationally complicated as \( \ell \) increases. \textbf{Theorem 6.1} gives a more satisfactory answer to question (2) by using

\[
\exp \left( - \sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{\| t \|^2_{2k}}{(\ell + 1/2)^{2k-1}} \right), \quad \ell > 0
\]

in lieu of the fraction in (1). The exponent in the expression above is now a rational function of \( \ell \). This simplicity necessitates replacing global \( L_2 \) and uniform convergence with a more local (though not totally local) convergence. \textbf{Corollary 6.2} is of particular interest as a multivariate analogue of \textbf{Theorem 1.1}, stated informally as

\[
f(t) \simeq \psi_\ell(t), \quad \ell > 0.
\]

The author is unaware of any other multivariate theorem addressing questions (1) and (2) above which satisfies the following.

(a) The exponential Riesz bases under consideration are not necessarily tensor products of single-variable Riesz bases.

(b) Convergence stronger than “uniform convergence on compacta” is proven.

It should be noted that \textbf{Theorems 5.4} and \textbf{6.1}, and \textbf{Corollary 6.2} do not, at this point, recover \textbf{Corollary 1.2} in its generality of allowable sequences \( \{ \tau_n \}_{n} \subset \mathbb{R} \); however, the comments above show that their value is due primarily to their multidimensional nature and convergence properties.

For a non-overlapping approach to approximation of analytic functions on \textit{bounded} subsets of \( \mathbb{C}^n \) by interpolating polynomials, see [15].

This paper is outlined as follows. Section 2 covers the necessary preliminary and background material regarding bandlimited functions. In Section 3, the notion of a uniformly invertible Riesz basis is presented, and basic properties are established. Relevant examples of uniformly invertible Riesz bases are given in Section 4. \textbf{Theorems 5.4} and \textbf{6.1} are proven in Sections 5 and 6 respectively, along with pertinent corollaries. Section 7 briefly addresses the optimality of growth rates appearing in \textbf{Theorem 6.1}.

2. Preliminary notions

We begin by reviewing the relevant Hilbert space theory.

We use the convention that an isomorphism between two Hilbert spaces is a linear map which is bijective and bi-continuous. To avoid confusion of indices, \( t = (t_1, \ldots, t_n) \) will denote a continuous variable, while \( \tau \) and \( v \) will be reserved for sampling nodes \( \{ \tau_n \}_{n} \) and \( \{ v_n \}_{a} \) where \( \tau_n = (\tau_{n1}, \ldots, \tau_{nd}) \) and \( v_n = (v_{n1}, \ldots, v_{nd}) \).

\textbf{Definition 2.1.} A Riesz basis for a Hilbert space \( H \) is a sequence \( \{ f_n \}_{n \in \mathbb{N}} \) such that the following criteria are satisfied.

1. For all \( f \in H \), there exists a unique sequence of scalars \( \{ a_n \}_{n=1}^{\infty} \in \ell_2(\mathbb{N}) \) such that

\[
f = \sum_{n=1}^{\infty} a_n f_n.
\]
There exist constants \(0 < m < M < \infty\) such that for all \(\{a_n\}_{n=1}^{\infty} \in \ell_2(\mathbb{N})\),

\[
m \sum_{n=1}^{\infty} |a_n|^2 \leq \left\| \sum_{n=1}^{\infty} a_n f_n \right\|^2 \leq M \sum_{n=1}^{\infty} |a_n|^2.
\]

Restated, a Riesz basis is a sequence \(\{f_n\}_{n \in \mathbb{N}} \subset H\) such that for any orthonormal basis \(\{e_n\}_{n=1}^{\infty}\), the map \(L e_n = f_n\) is an isomorphism. If \(\{f_n\}_{n \in \mathbb{N}}\) is a Riesz basis, \(\{a_n\}_{n \in \mathbb{N}} \in \ell_2(\mathbb{N})\), and \(\sigma\) is a permutation of \(\mathbb{N}\), then

\[
\sum_{n=1}^{\infty} a_n f_n = \sum_{n=1}^{\infty} a_{\sigma(n)} f_{\sigma(n)}.
\]

This allows us to index a Riesz basis by any countable set. In subsequent sections, we will index Riesz bases by \(\mathbb{Z}^d\) for convenience.

**Definition 2.2.** If \(A\) is a bounded linear operator on \(H\), then \(A^*\) denotes the adjoint operator of \(A\); that is, \(A^*\) is the unique bounded linear operator on \(H\) such that

\[
\langle Ax, y \rangle = \langle x, A^* y \rangle, \quad x, y \in H.
\]

**Proposition 2.3.** If \(\{f_n\}_{n \in \mathbb{N}}\) is a Riesz basis for a Hilbert space \(H\), then there exists a unique set of functions \(\{f_n^*\}_{n \in \mathbb{N}}\) (the biorthogonal functions of \(\{f_n\}_{n \in \mathbb{N}}\)) such that

1. \(\{f_n^*\}_{n \in \mathbb{N}}\) is a Riesz basis for \(H\), and
2. \(\langle f_n, f_m^* \rangle = \delta_{nm}\) for \(n, m \in \mathbb{N}\).

**Sketch of proof of Proposition 2.3.** Let \(\{e_n\}_{n \in \mathbb{N}}\) be an orthonormal basis for \(H\). The map \(L\) defined by

\[
L e_n = f_n
\]

is an isomorphism. Define \(f_n^* = (L^*)^{-1} e_n\) for \(n \in \mathbb{N}\), then \(\{f_n^*\}_{n \in \mathbb{N}}\) is the desired sequence. □

Note that biorthogonality is preserved under unitary transformations.

If \(\{f_n\}_{n \in \mathbb{N}}\) is a Riesz basis for a Hilbert space \(H\) with corresponding biorthogonal Riesz basis \(\{f_n^*\}_{n \in \mathbb{N}}\), then **Proposition 2.3** implies

\[
f = \sum_{n=1}^{\infty} \langle f, f_n^* \rangle f_n = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n^*, \quad f \in H.
\]

For convenience, the space \(L_2([-\pi, \pi]^d)\) will be referred to as \(\mathcal{H}_d\).

**Definition 2.4.** Define the sequence of functions \(e_n : [-\pi, \pi]^d \to \mathbb{C}\) by

\[
e_n(\cdot) = \frac{1}{(2\pi)^{d/2}} e^{i(\cdot, n)}, \quad n \in \mathbb{Z}^d.
\]

Note that \(\{e_n\}_{n \in \mathbb{Z}^d}\) is an orthonormal basis for \(\mathcal{H}_d\).

We use the \(d\)-dimensional \(L_2\) isometric Fourier transform

\[
\mathcal{F}(f)(\cdot) = \mathrm{P.V.} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi) e^{-i(\cdot, \xi)} d\xi, \quad f \in L_2(\mathbb{R}^d)
\]
where the inverse transform is given by

\[
F^{-1}(f)(\cdot) = \text{P.V.} \int_{\mathbb{R}^d} f(\xi) e^{i\langle \cdot, \xi \rangle} d\xi, \quad f \in L_2(\mathbb{R}^d).
\]

**Definition 2.5.** We define the space of bandlimited (or Paley–Wiener) functions to be the set

\[
PW_d := \{ f \in L_2(\mathbb{R}^d) \mid \text{supp}(F^{-1}(f)) \subset [-\pi, \pi]^d \}
\]

endowed with the \(L_2(\mathbb{R}^d)\) norm.

The definition given above is narrow, but other frequency domains are not considered in this paper. This notation is adopted to avoid unnecessary generality.

**Definition 2.6.** Define the multivariate function sinc : \(\mathbb{R}^d \to \mathbb{R}\) by

\[
sinc(x) := \text{sinc}(x_1) \cdots \text{sinc}(x_d)
\]

where

\[
sinc(t) := \frac{\sin(\pi t)}{\pi t}, \quad t \in \mathbb{R}
\]

is the usual univariate sinc function.

Here are facts concerning \(PW_d\) which will be used frequently.

1. \(PW_d\) is isometric to \(\mathcal{H}_d\) by way of the Fourier transform.
2. Since \(PW_d\) is defined through the Fourier transform, it is readily verified that functions in \(PW_d\) extend to entire functions from \(\mathbb{C}^d\) to \(\mathbb{C}\). In this paper we restrict the domain to \(\mathbb{R}^d\).
3. We have

\[
\mathcal{F}
\left(
\frac{1}{(2\pi)^{d/2}} e^{i(\cdot, \tau)} \chi_{[-\pi, \pi]^d}(\cdot)
\right)(t) = \text{sinc}(t - \tau)
\]

by direct computation.
4. It follows from (1) and (3) above and the definition of \(\{e_n\}_{n \in \mathbb{Z}^d}\) that \(\{\text{sinc}(\cdot - n)\}_{n \in \mathbb{Z}^d}\) is an orthonormal basis for \(PW_d\).
5. If \(f \in PW_d\) and \(t \in \mathbb{R}^d\) then

\[
f(t) = \mathcal{F}(\mathcal{F}^{-1}f)(t) = \left(\mathcal{F}^{-1}f(\cdot), \frac{1}{(2\pi)^{d/2}} e^{i(t, \cdot)}\right)_\mathbb{R}^d.
\]

(3)

6. In \(PW_d\), \(L_2\) convergence implies uniform convergence.

\[
\|f\|_\infty = \sup_{t \in \mathbb{R}^d} \left| \int_{[-\pi, \pi]^d} \left(\mathcal{F}^{-1}f(\xi) e^{-i(t, \xi)} \right) d\xi \right|
\]

\[
\leq \frac{1}{(2\pi)^{d/2}} \int_{[-\pi, \pi]^d} \left|\mathcal{F}^{-1}f(\xi)\right| d\xi
\]

\[
\leq \left( \int_{[-\pi, \pi]^d} \left|\mathcal{F}^{-1}f(\xi)\right|^2 d\xi \right)^{1/2} = \|\mathcal{F}^{-1}f\|_2 = \|f\|_2.
\]
(7) The $d$-dimensional Riemann–Lebesgue Lemma [6, Theorem 8.22, page 249] implies
\[
\lim_{\|x\|_\infty \to \infty} f(x) = 0, \quad f \in PW_d.
\]

Collections of complex exponential functions which are Riesz bases for $H_d$ are of fundamental importance in the study of bandlimited functions. If $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$ is a sequence such that
\[
\{f_n\}_{n \in \mathbb{N}} = \left\{ \frac{1}{(2\pi)^{d/2}} e^{i \langle \cdot, \tau_n \rangle} \right\}_{n \in \mathbb{N}}
\]
is a Riesz basis for $H_d$, then by (3), we have
\[
f(\tau_n) = \left\{ \mathcal{F}^{-1} f_n \right\}.
\]
Applying (2),
\[
\mathcal{F}^{-1} f = \sum_{n=1}^{\infty} f(\tau_n) f^*_n.
\]
Defining $G_n = F f_n^*$ yields
\[
f = \sum_{n=1}^{\infty} f(\tau_n) G_n(t)
\]
where convergence is both in $L_2$ and uniform senses. The following corollary is worth noting.

**Corollary 2.7.** Let \( \left\{ \frac{1}{(2\pi)^{d/2}} e^{i \langle \cdot, \tau_n \rangle} \right\}_{n \in \mathbb{N}} \) be a Riesz basis for $H_d$ with biorthogonal functions $\{f_n^*\}_{n \in \mathbb{N}}$. If $G_n := F f_n^*$ for $n \in \mathbb{N}$, then
\[
G_n(\tau_m) = \delta_{nm}.
\]

In the case that $\{\tau_n\}_{n \in \mathbb{N}} = \mathbb{Z}^d$, (5) yields a $d$-dimensional version of the classical Whittaker–Kotel’nikov–Shannon (WKS) sampling formula:
\[
f(t) = \sum_{n \in \mathbb{Z}^d} f(n) \text{sinc}(t - n), \quad t \in \mathbb{R}^d.
\]

Here are particular examples of exponential Riesz bases for $H_d$ which will be useful later.

**Theorem 2.8 (Kadec).** If $\{\tau_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ is a sequence of real numbers such that
\[
\sup_{n \in \mathbb{Z}} |n - \tau_n| < 1/4,
\]
then the sequence of functions $\left\{ \frac{1}{\sqrt{2\pi}} e^{i \tau_n \cdot (\cdot)} \right\}_{n \in \mathbb{Z}}$ is a Riesz basis for $H_1$.

The following is the scheme that Kadec used to prove Theorem 2.8 in [7] (see [17] for a nice exposition). Define the operator $T$ on $H_1$ by
\[
T \left( \frac{1}{\sqrt{2\pi}} e^{i n \cdot (\cdot)} \right) = \frac{1}{\sqrt{2\pi}} e^{i n \cdot (\cdot)} - \frac{1}{\sqrt{2\pi}} e^{i \tau_n \cdot (\cdot)}, \quad n \in \mathbb{Z}.
\]
Inspired calculation shows that $\|T\| < 1$, so that by Neumann series manipulation, the map

$$I - T : \frac{1}{\sqrt{2\pi}} e^{in(\cdot)} \mapsto \frac{1}{\sqrt{2\pi}} e^{i\tau_n(\cdot)}$$

is an isomorphism.

An impressive generalization of Kadec’s $1/4$ Theorem is Avdonin’s “$1/4$ in the mean” Theorem in [1]. In [16], Sun and Zhou have proven the following multidimensional version of Theorem 2.8 by a direct generalization of Kadec’s original proof.

**Theorem 2.9** (Sun, Zhou). For $d \geq 1$, define

$$D_d(x) = (1 - \cos \pi x + \sin \pi x + \text{sinc}(x))^d - (\text{sinc}(x))^d,$$

and let $x_d$ be the unique number such that $0 < x_d \leq 1/4$ and $D_d(x_d) = 1$. If $\{\tau_n\}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d$ is a sequence such that

$$\sup_{n \in \mathbb{Z}^d} \|n - \tau_n\|_\infty < x_d,$$

then the sequence $\left\{ \frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, \tau_n \rangle} \right\}_{n \in \mathbb{Z}^d}$ is a Riesz basis for $\mathcal{H}_d$.

**Theorem 2.10**, (see [2]), is another generalization of Kadec’s $1/4$ Theorem whose proof, though conceptually similar to that of Theorem 2.9, is technically simpler. The case when $d = 1$ was proven by Duffin and Eachus in [5].

**Theorem 2.10**. If $\{\tau_n\}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d$ is a sequence such that

$$\sup_{n \in \mathbb{Z}^d} \|n - \tau_n\|_\infty < \frac{\ln(2)}{\pi d},$$

then the sequence $\left\{ \frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, \tau_n \rangle} \right\}_{n \in \mathbb{Z}^d}$ is a Riesz basis for $\mathcal{H}_d$.

3. Uniformly invertible Riesz bases

Given an exponential Riesz basis, (5) demonstrates the need to concretely approximate $\{f_n^*\}_{n \in \mathbb{Z}^d}$. The concept of uniform invertibility is introduced to this end, and is precisely defined in Definitions 3.4 and 3.5. Informally speaking, a uniformly invertible Riesz basis is a Riesz basis $\{f_n\}_{n \in \mathbb{Z}^d}$ that satisfies the following two conditions.

1. The sequence $\{f_n\}_{n \in \mathbb{Z}^d}$ can be obtained as the limit (in an appropriate sense) of a sequence of simpler Riesz bases, each one of which (except for finitely many terms) is an orthonormal basis.

2. The set of biorthogonal functions of $\{f_n^*\}_{n \in \mathbb{Z}^d}$ of $\{f_n\}_{n \in \mathbb{Z}^d}$ is also a limit (in an appropriate sense) of the sets of biorthogonal functions of the simpler Riesz bases in (1) above. This is precisely stated in Lemma 3.10.

**Definition 3.1.** Define $C_{\ell,d} = \{-\ell, \ldots, \ell\}^d$.

**Definition 3.2.** For $\ell \in \mathbb{N}$, define $P_\ell : \mathcal{H}_d \to \mathcal{H}_d$ to be the orthogonal projection onto $\text{span}\{e_n\}_{n \in C_{\ell,d}}$. 
Definition 3.3. Let $L : \mathcal{H}_d \to \mathcal{H}_d$ be a bounded linear map. If $P_\ell L P_\ell : P_\ell \mathcal{H}_d \to P_\ell \mathcal{H}_d$ is invertible with inverse mapping $(P_\ell L P_\ell)^{-1}$, then extend $(P_\ell L P_\ell)^{-1}$ to $\mathcal{H}_d$ by defining

$$(P_\ell L P_\ell)^{-1} x = (P_\ell L P_\ell)^{-1} P_\ell x.$$ 

This is a slight abuse of notation, as $P_\ell L P_\ell$ is also a map from $\mathcal{H}_d$ to itself, and is certainly not invertible with that choice of domain and range.

Definition 3.4. Let $L : \mathcal{H}_d \to \mathcal{H}_d$ be an isomorphism. $L$ is uniformly invertible if

1. $P_\ell L P_\ell : P_\ell \mathcal{H}_d \to P_\ell \mathcal{H}_d$ is invertible for sufficiently large $\ell$, and
2. $\limsup_{\ell \to \infty} \| (P_\ell L P_\ell)^{-1} \| < \infty$.

Definition 3.5. A Riesz basis $\{f_n\}_{n \in \mathbb{Z}^d}$ for $\mathcal{H}_d$ is a uniformly invertible Riesz basis (UIRB) if the isomorphism defined by $L_\ell = f_n$ is uniformly invertible.

Definition 3.6. If $L$ is an operator on $\mathcal{H}_d$, define the operator

$$L_\ell = L P_\ell + I - P_\ell.$$ 

We can now state and prove the following lemmas.

Lemma 3.7. Let $\{f_n\}_{n \in \mathbb{Z}^d} \subset \mathcal{H}_d$. Define $L : \text{span}\{e_n\}_{n \in \mathbb{Z}^d} \to \mathcal{H}_d$ by $L e_n = f_n$. For each $\ell > 0$, the following statements are equivalent.

1. $\{f_n\}_{n \in \mathbb{C}_{\ell,d}} \cup \{e_n\}_{n \notin \mathbb{C}_{\ell,d}}$ is a Riesz basis for $\mathcal{H}_d$.
2. $L_\ell$ is an isomorphism.
3. $P_\ell L P_\ell : P_\ell \mathcal{H}_d \to P_\ell \mathcal{H}_d$ is invertible.

Proof of Lemma 3.7. (1) $\Longleftrightarrow$ (2) is immediate.

2. $\implies$ (3): We have $P_\ell L_\ell = P_\ell L P_\ell$, which implies $P_\ell = P_\ell L P_\ell L_\ell^{-1}$, so that

$$P_\ell = (P_\ell L P_\ell)(P_\ell L_\ell^{-1} P_\ell).$$

This shows $P_\ell L P_\ell$ is invertible, and

$$(P_\ell L P_\ell)^{-1} = P_\ell L_\ell^{-1} P_\ell.$$  

(3) $\implies$ (2): Note that $L_\ell$ extends to a continuous map on $\mathcal{H}_d$. First we show that $L_\ell$ is one to one. Say $0 = L_\ell x = L P_\ell x + (I - P_\ell)x$, then $0 = P_\ell L P_\ell x$, so that $0 = (P_\ell L P_\ell)^{-1} P_\ell L P_\ell x = P_\ell x$. We conclude that $x = (I - P_\ell)x$. This implies

$$0 = L_\ell x = L \ell (I - P_\ell)x = (I - P_\ell)x = x.$$ 

Next we show that $L_\ell$ is onto. Note $L_\ell (I - P_\ell)x = (I - P_\ell)x$, so we only need to show that for all $x$, $P_\ell x$ is in the range of $L_\ell$. Given $x \in \mathcal{H}_d$, define

$$y = (P_\ell L P_\ell)^{-1} x + P_\ell x - L(P_\ell L P_\ell)^{-1} x.$$

$$L_\ell y = (L P_\ell + I - P_\ell)((P_\ell L P_\ell)^{-1} x + P_\ell x - L(P_\ell L P_\ell)^{-1} x)$$

$$= (L P_\ell + I - P_\ell)(P_\ell L P_\ell)^{-1} x + (L P_\ell + I - P_\ell) P_\ell x$$

$$- (L P_\ell + I - P_\ell) L(P_\ell L P_\ell)^{-1} x$$

$$= LP_\ell (P_\ell L P_\ell)^{-1} x + LP_\ell x - (P_\ell L P_\ell)(P_\ell L P_\ell)^{-1} x - LP_\ell (P_\ell L P_\ell)^{-1} x$$

$$+ (P_\ell L P_\ell)(P_\ell L P_\ell)^{-1} x$$

$$= P_\ell x.$$ 

In the second and third lines of the equation above we use that $P_\ell (P_\ell L P_\ell)^{-1} = (P_\ell L P_\ell)^{-1}$. 


Now $L_\ell$ is a continuous linear bijection between Hilbert spaces. An application of the Banach Open Mapping Theorem shows that $L_\ell$ is an isomorphism. \qed

**Lemma 3.8.** Define $L$ as in Lemma 3.7. For each $\ell > 0$, $L_\ell$ extends to an isomorphism on $\mathcal{H}_d$ iff it is one to one.

**Proof of Lemma 3.8.** One direction is immediate. Suppose that $L_\ell$ is one to one. It immediately extends to a bounded linear operator on $\mathcal{H}_d$. By Lemma 3.7, we only need to show that $P_\ell L P_\ell : P_\ell \mathcal{H}_d \to P_\ell \mathcal{H}_d$ is invertible. Finite dimensionality of $P_\ell \mathcal{H}_d$ further reduces the problem to showing that $P_\ell L P_\ell : P_\ell \mathcal{H}_d \to P_\ell \mathcal{H}_d$ is one to one. Let $(P_\ell L P_\ell) P_\ell x = 0$. We have

$$L_\ell P_\ell x - (I - P_\ell) L P_\ell x = (L P_\ell + I - P_\ell)(P_\ell x - (I - P_\ell) L P_\ell x) = L_\ell P_\ell x - (I - P_\ell) L P_\ell x = 0.$$ 

Since $L_\ell$ is one to one, we have that $P_\ell x = (I - P_\ell) L P_\ell x$, so that $P_\ell x = 0$. \qed

**Lemma 3.9.** Let $\{f_n\}_{n \in \mathbb{Z}^d}$ be a Riesz basis for $\mathcal{H}_d$ where $Le_n = f_n$. The following are equivalent.

1. $\{f_n\}_{n \in \mathbb{Z}^d}$ is a UIRB.
2. $L_\ell$ is an isomorphism for sufficiently large $\ell$, and

$$\lim_{\ell \to \infty} \| L_\ell^{-1} \| < \infty.$$ 

**Proof of Lemma 3.9.** (1)$\implies$(2): By Lemma 3.7, we only need to show that

$$\lim_{\ell \to \infty} \| L_\ell^{-1} \| < \infty.$$

This follows from the identity

$$L_\ell^{-1} = [I - (I - P_\ell)L](P_\ell L P_\ell)^{-1} + I - P_\ell,$$

when $\ell$ is sufficiently large. This identity is hereby demonstrated.

$$\begin{aligned}
[I - (I - P_\ell)L](P_\ell L P_\ell)^{-1} + I - P_\ell &= [I - (I - P_\ell)L]P_\ell L_\ell^{-1} P_\ell + I - P_\ell \\
&= P_\ell L_\ell^{-1} P_\ell - (I - P_\ell)L P_\ell L_\ell^{-1} P_\ell + I - P_\ell \\
&= P_\ell L_\ell^{-1} P_\ell - L P_\ell L_\ell^{-1} P_\ell + I \\
&= (I - L) P_\ell L_\ell^{-1} P_\ell + I.
\end{aligned}$$

The first equality above follows from (10). We have $(I - L) P_\ell = I - L_\ell$, so

$$\begin{aligned}
[I - (I - P_\ell)L](P_\ell L P_\ell)^{-1} + I - P_\ell &= (I - L_\ell) L_\ell^{-1} P_\ell + I \\
&= L_\ell^{-1} P_\ell - P_\ell + I.
\end{aligned}$$

From the definition of $L_\ell$, we see that $L_\ell(I - P_\ell) = I - P_\ell$. Composing from the left by $L_\ell^{-1}$ yields $I - P_\ell = L_\ell^{-1} (I - P_\ell)$. Rearranging, we have $L_\ell^{-1} P_\ell - P_\ell + I = L_\ell^{-1}$, which proves the identity.

(2) $\implies$ (1): This follows from (10). \qed
Lemma 3.10. If \( \{ f_n \}_{n \in \mathbb{Z}^d} \) is a UIRB for \( \mathcal{H}_d \) and \( L_n = f_n \), then
\[
\lim_{\ell \to \infty} (L_\ell^*)^{-1} f = (L^*)^{-1} f, \quad f \in \mathcal{H}_d.
\] (14)

Proof of Lemma 3.10. Note that
\[
(L_\ell^*)^{-1} - (L^*)^{-1} = (L_\ell^*)^{-1} (L^* - L_\ell^*) (L^*)^{-1}
\]
and
\[
\lim_{\ell \to \infty} L_\ell^* f = L^* f, \quad f \in \mathcal{H}_d.
\]
Combined with Lemma 3.9, we have (14).

Lemma 3.11. Let \( L : \mathcal{H}_d \to \mathcal{H}_d \) given by \( L_n = f_n \), be an isomorphism. The following are equivalent.

(1) \( \{ f_n \}_{n \in \mathbb{Z}^d} \) is a UIRB.

(2) For all \( f \in \mathcal{H}_d \), \( \lim_{\ell \to \infty} (L_\ell^*)^{-1} (I - P_\ell) f = 0. \)

Proof of Lemma 3.11. Applying Lemma 3.9, it is clear that (1) implies (2). For the other direction, note that the equality \( I = (L_\ell^*)^{-1} P_\ell L^* + (L_\ell^*)^{-1} (I - P_\ell) \) implies
\[
I = (L_\ell^*)^{-1} P_\ell L^* + (L_\ell^*)^{-1} (I - P_\ell),
\]
from which \( \{(L_\ell^*)^{-1} P_\ell\}_{\ell > \ell_0} \) is pointwise bounded. Together with the assumption in (2), this implies \( \{(L_\ell^*)^{-1}\}_{\ell > \ell_0} \) is pointwise bounded, hence norm bounded by the uniform boundedness principle. Noting that \( \|(L_\ell^*)^{-1}\| = \|L_\ell^{-1}\| \) yields uniform invertibility of \( L \).

Lemma 3.12. Let \( L : \mathcal{H}_d \to \mathcal{H}_d \) given by \( L_n = f_n \), be an isomorphism. The following are equivalent.

(1) For all \( f \in \mathcal{H}_d \), we have
\[
f = \lim_{\ell \to \infty} (L_\ell^*)^{-1} P_\ell L^* f.
\]

(2) \( \{ f_n \}_{n \in \mathbb{Z}^d} \) is a UIRB.

Proof of Lemma 3.12. Recall (15) and apply Lemma 3.11.

4. Concrete examples of uniformly invertible exponential Riesz bases

Theorems 4.1 and 4.2 show that some earlier examples of exponential Riesz bases (along with simple modifications) are UIRBs.

Theorem 4.1. The Riesz bases described given by Theorems 2.9 and 2.10 are UIRBs.

Theorem 4.2. Let \( \{ \tau_n \}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d \) be a sequence satisfying either
\[
\limsup_{\|n\|_\infty \to \infty} \| \tau_n - n \|_\infty < x_d, \quad D_d(x_d) = 1, \quad 0 < x_d \leq 1/4
\]
where $D_d$ is defined as in Theorem 2.9, or
\[
\limsup_{\|n\|_\infty \to \infty} \|\tau_n - n\|_\infty < \frac{\ln(2)}{\pi d}.
\]
If \( \left\{ \frac{1}{(2\pi)^{d/2}} e^{i\cdot\tau_n} \right\}_{n \in \mathbb{Z}^d} \) is a Riesz basis for $\mathcal{H}_d$, then it is a UIRB.

The following proof of Theorem 4.1 shows that any small norm-perturbation of the identity is a uniformly invertible operator. As a comment, a similar though more involved proof implies that the uniform invertibility of any operator is unchanged under a small norm-perturbation.

**Proof of Theorem 4.1.** Notice that Theorems 2.9 and 2.10 imply that the maps $L_{\ell}$ are isomorphisms for all $\ell > 0$, and that $P_{\ell}LP_{\ell}$ is invertible. The proofs of Theorems 2.9 and 2.10 in [16,2] rely on the fact that the map $L_{\ell}n = fn$ satisfies $\| I - L \| = \delta < 1$. For all $\ell > 0$ this yields $\| P_{\ell} - P_{\ell}LP_{\ell} \| \leq \delta < 1$.

By Neumann series manipulation,
\[
(P_{\ell}LP_{\ell})^{-1} = \sum_{k=0}^{\infty} (P_{\ell} - P_{\ell}LP_{\ell})^k,
\]
which gives
\[
\|(P_{\ell}LP_{\ell})^{-1}\| \leq \frac{1}{1 - \delta}. \quad \Box
\]

In a similar fashion, the proof of Theorem 4.2 (obtained through the auxiliary results below) shows that certain compact perturbations (not necessarily small in norm) of uniformly invertible operators always preserve uniform invertibility.

**Proposition 4.3.** Let $L : \mathcal{H}_d \to \mathcal{H}_d$ be uniformly invertible. If $\Delta : \mathcal{H}_d \to \mathcal{H}_d$ is a compact operator such that $\hat{L} := L + \Delta$ is an isomorphism, then $\hat{L}$ is uniformly invertible.

**Proof of Proposition 4.3.** From the definition of $L_{\ell}$, we have
\[
I = (I - P_{\ell})L_{\ell}^{-1} + LP_{\ell}L_{\ell}^{-1},
\]
so that
\[
L^{-1}(P_{\ell} - I)L_{\ell}^{-1} = P_{\ell}L_{\ell}^{-1} - L^{-1}
\]
for sufficiently large $\ell$. This implies
\[
(L^*_{\ell})^{-1}P_{\ell}L_{\ell} = (L^*_{\ell})^{-1}(P_{\ell} - I)(L^*)^{-1}. \quad (17)
\]
As $\ell \to \infty$, the right hand side of (17) has 0 limit pointwise. Combined with the compactness of $\Delta^*$, we obtain
\[
\lim_{\ell \to \infty} (L^*_{\ell})^{-1}P_{\ell}L_{\ell} = (L^*)^{-1}\Delta^* \quad (18)
\]
where the limit is in the operator norm topology. Taking the adjoint of each term in (18) and adding the identity yields
\[
\lim_{\ell \to \infty} I + \Delta P_{\ell}L_{\ell}^{-1} = I + \Delta L^{-1} = (L + \Delta)L^{-1}, \quad (19)
\]
Lemma 4.4. Given a sequence \( \{ \tau_n \}_{n \in \mathbb{N}} \subset \mathbb{R}^d \), define \( \{ f_n \}_{n \in \mathbb{N}} = \left\{ \frac{1}{(2 \pi)^{d/2}} e^{i \langle \cdot, \tau_n \rangle} \right\}_{n \in \mathbb{N}} \). Suppose there exists a number \( B \) such that
\[
\left\| \sum_{n=1}^{m} a_n f_n \right\| \leq B \left( \sum_{n=1}^{m} |a_n|^2 \right)^{1/2}
\]
for all \( \{ a_n \}_{n=1}^{m} \subset \mathbb{C} \). If \( \{ \nu_n \}_{n \in \mathbb{N}} \subset \mathbb{R}^d \), and
\[
\{ g_n \}_{n \in \mathbb{N}} := \left\{ \frac{1}{(2 \pi)^{d/2}} e^{i \langle \cdot, \nu_n \rangle} \right\}_{n \in \mathbb{N}},
\]
then for all \( r, s \geq 1 \) and any finite sequence \( \{ a_n \}_{n=r}^{s} \), we have
\[
\left\| \sum_{n=r}^{s} a_n (f_n - g_n) \right\| \leq B \left( \exp \left( \pi d \sup_{r \leq n \leq s} \| \tau_n - \nu_n \|_\infty \right) - 1 \right) \left( \sum_{n=r}^{s} |a_n|^2 \right)^{1/2}.
\]

The proof of Lemma 4.4 is very similar to the proof of Lemma 5.3 in [3] and is therefore omitted. A consequence of Lemma 4.4 is the following corollary.

Corollary 4.5. Given two sequences \( \{ \tau_n \}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d \) and \( \{ \nu_n \}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d \), define \( \{ f_n \}_{n \in \mathbb{Z}^d} \) and \( \{ g_n \}_{n \in \mathbb{Z}^d} \) by \( f_n(\cdot) = \frac{1}{(2 \pi)^{d/2}} e^{i \langle \cdot, \tau_n \rangle} \) and \( g_n(\cdot) = \frac{1}{(2 \pi)^{d/2}} e^{i \langle \cdot, \nu_n \rangle} \). If \( \{ f_n \}_{n \in \mathbb{Z}^d} \) is a Riesz basis for \( \mathcal{H}_d \), and
\[
\lim_{\| n \|_\infty \to \infty} \| \tau_n - \nu_n \|_\infty = 0,
\]
then the operator \( K \) defined by \( Ke_n = f_n - g_n \) is compact.
Proof of Theorem 4.2. Apply Theorem 4.1 and Corollary 4.6. Apply Proposition 4.3 and Corollary 4.5.

Proof of Corollary 4.6. Let \( \{\tau_n\}_{n \in \mathbb{Z}^d}, \{v_n\}_{n \in \mathbb{Z}^d}, \{f_n\}_{n \in \mathbb{Z}^d}, \{g_n\}_{n \in \mathbb{Z}^d} \) be defined as in Corollary 4.5. If \( \{f_n\}_{n \in \mathbb{Z}^d} \) is a UIRB, and \( \{g_n\}_{n \in \mathbb{Z}^d} \) a Riesz basis for \( \mathcal{H}_d \), then \( \{g_n\}_{n \in \mathbb{Z}^d} \) is a UIRB.

Proof of Theorem 4.2. Apply Theorem 4.1 and Corollary 4.6.

Simple examples show that in Theorem 4.2, the condition that

\[
\left\{ \frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, \tau_n \rangle} \right\}_{n \in \mathbb{Z}^d}
\]

be a Riesz basis for \( \mathcal{H}_d \) cannot be dropped when \( d \geq 2 \). For example, the standard exponential orthonormal basis \( \{e_n\}_{n \in \mathbb{Z}^d} \) is of course uniformly invertible, but the set

\[
\left\{ \frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, (1,1/2,0,\ldots,0) \rangle} \right\} \cup \{e_n\}_{n \neq 0}
\]

is not a Riesz basis, as

\[
\begin{align*}
\left\{ e^{i\langle \cdot, (1,1/2,0,\ldots,0) \rangle} \right\} \in \text{span} \left\{ e^{i\langle \cdot, (1,n,0,\ldots,0) \rangle} \right\}_{n \in \mathbb{Z}, n \neq 0}
\end{align*}
\]

However, the condition that

\[
\left\{ \frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, \tau_n \rangle} \right\}_{n \in \mathbb{Z}^d}
\]

be a Riesz basis for \( \mathcal{H}_d \) can be dropped when \( d = 1 \). This follows from the following theorem.

Theorem 4.7. Let \( \{v_n\}_{n \in \mathbb{Z}^d} \subset \mathbb{R} \) be a sequence such that \( \{f_n\}_{n \in \mathbb{Z}^d} = \left\{ \frac{1}{\sqrt{2\pi}} e^{i\ell \cdot v_n} \right\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \mathcal{H}_1 \). If \( \{\tau_n\}_{n \in \mathbb{Z}^d} \subset \mathbb{R} \) is a sequence of distinct points such that

\[
\lim_{|n| \to \infty} |v_n - \tau_n| = 0,
\]

then \( \{g_n\}_{n \in \mathbb{Z}^d} = \left\{ \frac{1}{\sqrt{2\pi}} e^{i\tau_n} \right\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \mathcal{H}_1 \).
The proof of Theorem 4.7 relies on Lemma 4.8 below, which appears as Lemma 3.1 in [11].

**Lemma 4.8.** Let \( \{f_n\}_{n \in \mathbb{Z}} \) be a complex exponential Riesz basis for \( \mathcal{H}_1 \). If finitely many terms in \( \{f_n\}_{n \in \mathbb{Z}} \) are replaced by arbitrary complex exponential functions, then the resulting sequence (provided it consists of distinct functions) is a Riesz basis for \( \mathcal{H}_1 \).

**Proof of Theorem 4.7.** Define \( L \epsilon = f_n \) and \( \tilde{L} \epsilon = g_n \). By Corollary 4.5, \( \tilde{L} \) is bounded and linear, and \( \tilde{L} = L + \Delta \) for some compact operator \( \Delta \). Define the operator

\[
R_\ell \epsilon_n = \begin{cases} f_n, & |n| \leq \ell \\ g_n, & |n| > \ell. \end{cases}
\]

Rewritten, we have

\[
R_\ell = LP_\ell + (L + \Delta)(I - P_\ell) = L + \Delta(I - P_\ell).
\]

Compactness of \( \Delta \) implies that \( \lim_{\ell \to \infty} R_\ell = L \) in the operator norm topology. We conclude that \( R_\ell \) is an isomorphism for some \( \ell_0 \) sufficiently large; that is, the set

\[
\{f_n\}_{n \leq \ell_0} \cup \{g_n\}_{n > \ell_0}
\]

is a Riesz basis for \( \mathcal{H}_1 \). If we apply Lemma 4.8 by replacing \( \{f_n\}_{n \leq \ell_0} \) with \( \{g_n\}_{n \leq \ell_0} \) in (21), we have that \( \{g_n\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \mathcal{H}_1 \).

5. The first main result

**Definition 5.1.** If a sequence of points \( \{\tau_n\}_{n \in \mathbb{Z}^d} \) is specified, the sequence \( \{f_n\}_{n \in \mathbb{Z}^d} \) refers to

\[
\left\{ \frac{1}{(2\pi)^{d/2}} e^{i \langle \cdot, \tau_n \rangle} \right\}_{n \in \mathbb{Z}^d}.
\]

**Definition 5.2.** If \( \ell > 0 \), and \( \{\tau_n\}_{n \in \mathbb{Z}^d} \) is specified, the sequence \( \{f_\ell, n\}_{n \in \mathbb{Z}^d} \) refers to

\[
\{f_n\}_{n \in C_\ell, d} \cup \{e_n\}_{n \notin C_\ell, d}.
\]

**Definition 5.3.** If any Riesz basis \( \{f_n\}_{n} \) for \( \mathcal{H}_d \) is specified with biorthogonal functions \( \{f_n^*\}_{n} \), the sequence \( \{G_n\}_{n} \) is defined to be \( G_n = \mathcal{F} f_n^* \) (see Corollary 2.7).

If a UIRB is specified, the integer \( \ell_0 \) will refer to the smallest integer \( \ell \) such that

\[
P_\ell L P_\ell : P_\ell \mathcal{H}_d \to P_\ell \mathcal{H}_d
\]

is invertible.

Inner products are all denoted by \( \langle \cdot, \cdot \rangle \). The underlying Hilbert space, be it \( \mathbb{R}^d \), \( \mathcal{H}_d \), or \( PW_d \) should be clear from context. Unless it is explicitly stated otherwise, all norms are Hilbert space norms.

For \( \ell, d \in \mathbb{N} \), define the multivariate polynomial

\[
Q_{d, \ell}(t) = \prod_{k_1=1}^{\ell} \left( 1 - \frac{t_1^2}{k_1^2} \right) \cdots \prod_{k_d=1}^{\ell} \left( 1 - \frac{t_d^2}{k_d^2} \right), \quad t = (t_1, \ldots, t_d).
\]
The multivariate function \( t \mapsto \frac{\text{sinc}(t)}{Q_{d, \ell}(t)} \) has removable discontinuities which can be computed with the aid of the univariate formula

\[
\lim_{t \to n} \frac{\text{sinc}(t)}{Q_{1, \ell}(t)} = \frac{(\ell)!^2}{(\ell + n)! (\ell - n)!}, \quad n \in \{-\ell, \ldots, \ell\}.
\]

For all \( t \), \( \frac{\text{sinc}(t)}{Q_{d, \ell}(t)} \) is taken to mean \( \lim_{\tau \to t} \frac{\text{sinc}(\tau)}{Q_{d, \ell}(\tau)} \). The same is true for the reciprocal.

Here is the first main result of this paper.

**Theorem 5.4.** Let \( \{\tau_n\}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d \). If \( \{f_n\}_{n \in \mathbb{Z}^d} \) is a UIRB, then for all \( f \in PW_d \), there exists a unique sequence of polynomials \( \{\Psi_\ell \mid \Psi_\ell : \mathbb{R}^d \to \mathbb{R}\}_{\ell \geq \ell_0} \) such that

(a) \( \Psi_\ell \) has coordinate degree at most \( 2\ell \).

(b) \( \Psi_\ell(\tau_n) = f(\tau_n) \) for all \( n \in C_{\ell,d} \).

(c) \( f(t) = \lim_{\ell \to \infty} \Psi_\ell(t) \frac{\text{sinc}(t)}{Q_{d, \ell}(t)} \), where the limit is in both \( L_2 \) and uniform senses.

We now give an overview of the proof of Theorem 5.4. \( \{f_n\}_{n \in \mathbb{Z}^d} \) is a UIRB for \( \mathcal{H}_d \), and hence \( \{f_{\ell,n}\}_{n \in \mathbb{Z}^d} \) is a Riesz basis for \( \mathcal{H}_d \). Using (7) to expand each function in the biorthogonal system \( \{G_{\ell,n}\}_{n \in \mathbb{Z}^d} \), we obtain that \( G_{\ell,n} \) is a rational function times a sinc function. Examination of this rational function shows the existence of polynomials \( p_{\ell,n}(t) \) where the coordinate degree of each polynomial \( p_{\ell,n} \) is at most \( 2\ell \), and \( p_{\ell,n}(\tau_m) = \delta_{nm} \) for \( n, m \in C_{\ell,d} \). The existence of polynomials satisfying (a) and (b) follows. Simple estimates show that for large \( \ell \),

\[
G_{\ell,n}(t) \simeq p_{\ell,n}(t) \frac{\text{sinc}(t)}{Q_{d, \ell}(t)}.
\]

If we expand \( f \in PW_d \) against \( \{G_{\ell,n}\}_n \), we have

\[
f(t) = \sum_{n \in C_{\ell,d}} f(\tau_n)G_{\ell,n}(t) + \sum_{n \notin C_{\ell,d}} f(n)G_{\ell,n}(t).
\]

Uniform invertibility shows that the second sum can always be neglected for large \( \ell \). Combined with (22) we have

\[
f(t) \simeq \sum_{n \in C_{\ell,d}} f(\tau_n)G_{\ell,n}(t) \simeq \left( \sum_{n \in C_{\ell,d}} f(\tau_n)p_{\ell,n}(t) \right) \frac{\text{sinc}(t)}{Q_{d, \ell}(t)},
\]

which is statement (c).

The proof of Theorem 5.4 requires several lemmas, beginning with the following equivalence between the existence of particular Riesz bases and a polynomial interpolation condition.

**Lemma 5.5.** Let \( \{\tau_n\}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d \). The sequence \( \{f_{\ell,n}\}_{n \in \mathbb{Z}^d} \) is a Riesz basis for \( \mathcal{H}_d \) iff the following conditions both hold.

1. For all \( n \in C_{\ell,d} \), \( \tau_n \in (\mathbb{R} \setminus (\mathbb{Z} \setminus \{-\ell, \ldots, \ell\}))^d \).

2. For any sequence \( \{c_k\}_{k \in C_{\ell,d}} \), there exists a unique polynomial \( \Psi_\ell \) with coordinate degree at most \( 2\ell \) such that \( \Psi_\ell(\tau_k) = c_k \) for \( k \in C_{\ell,d} \).

**Proof of Lemma 5.5.** Suppose that the sequence \( \{f_{\ell,n}\}_{n \in \mathbb{Z}^d} \) is a Riesz basis for \( \mathcal{H}_d \). We compute the functions \( \{G_{\ell,n}\}_{n \in \mathbb{Z}^d} \) when \( n \in C_{\ell,d} \) by (6) and (7). This yields

\[
G_{\ell,n}(t) = \sum_{k \in C_{\ell,d}} G_{\ell,n}(k) \text{sinc}(t - k)
\]
\[
\begin{align*}
= & \left( \sum_{k \in C_{d,\ell}} G_{\ell,n}(k)(-1)^{k_1+\cdots+k_d} t_1 \cdots t_d \frac{1}{(t_1-k_1) \cdots (t_d-k_d)} \right) \sinc(t), \quad t \in \mathbb{R}^d. \\
\end{align*}
\]

If the \(k\)th summand in (23) is denoted by \(A_{\ell,n,k}\), then
\[
A_{\ell,n,k} = A_{\ell,n,k} \prod_{1 \leq i \leq d} \left( \frac{1}{\prod_{j_i \in \{-\ell, \ldots, \ell\} \setminus \{k_i\}} (t_i - j_i)} \right)
\]
\[
G_{\ell,n}(k)(-1)^{k_1+\cdots+k_d} t_1 \cdots t_d \prod_{1 \leq i \leq d} \left( \frac{1}{\prod_{j_i \in \{-\ell, \ldots, \ell\} \setminus \{k_i\}} (t_i - j_i)} \right)
\]
\[
= \frac{\prod_{1 \leq i \leq d} \left( \frac{1}{\prod_{j_i \in \{-\ell, \ldots, \ell\} \setminus \{k_i\}} (t_i - j_i)} \right)}{\prod_{j_i = 1}^\ell \left( 1 - \frac{t_i^2}{j_i^2} \right) \cdots \prod_{j_d = 1}^\ell \left( 1 - \frac{t_d^2}{j_d^2} \right)}
\]
\[
= \frac{p_{\ell,n,k}(t)}{Q_{d,\ell}(t)},
\]
where \(p_{\ell,n,k}\) is a polynomial with coordinate degree at most \(2\ell\). Substituting into (23), we obtain
\[
G_{\ell,n}(t) = \left( \sum_{k \in C_{d,\ell}} p_{\ell,n,k}(t) \right) \frac{\sinc(t)}{Q_{d,\ell}(t)} := \phi_{\ell,n}(t) \frac{\sinc(t)}{Q_{d,\ell}(t)},
\]
where \(\phi_{\ell,n}\) is a polynomial having coordinate degree at most \(2\ell\). This yields that
\[
1 = \phi_{\ell,n}(\tau_n) \left. \left( \frac{\sinc(t)}{Q_{d,\ell}(t)} \right) \right|_{\tau_n}.
\]
This shows
\[
\phi_{\ell,n}(\tau_n) \neq 0 \quad \text{and} \quad \frac{\sinc(\tau_n)}{Q_{d,\ell}(\tau_n)} \neq 0.
\]

The fact that
\[
\frac{\sinc(t)}{Q_{1,\ell}(t)} = 0 \quad \text{iff} \quad t \in \mathbb{Z} \setminus \{-\ell, \ldots, \ell\}
\]
implies
\[
\frac{\sinc(\tau_n)}{Q_{d,\ell}(\tau_n)} \neq 0 \quad \text{iff} \quad \tau_n \in (\mathbb{R} \setminus (\mathbb{Z} \setminus \{-\ell, \ldots, \ell\}))^d,
\]
which yields statement 1 of Lemma 5.5.
For \( n, m \in \mathbb{C}_{\ell,d} \), \( n \neq m \),
\[
0 = G_{\ell,n}(\tau_m) = \phi_{\ell,n}(\tau_m) \frac{\text{sinc}(\tau_m)}{Q_{d,\ell}(\tau_m)}.
\]

From (24) and (25) we conclude that
\[
\phi_{\ell,n}(\tau_m) = \begin{cases} 
\frac{Q_{d,\ell}(\tau_n)}{\text{sinc}(\tau_n)} & n = m \\
0 & n \neq m 
\end{cases}
\]
for \( n, m \in \mathbb{C}_{\ell,d} \). From this, the existence part of statement (2) in Lemma 5.5 readily follows. Restated, the evaluation map taking the space of all polynomials of coordinate degree at most \( 2\ell \) to \( \mathbb{R}^{(2\ell + 1)^d} \) is onto. These spaces have the same dimension, hence the evaluation map is a bijection. This completes the proof of statement (2).

Suppose that statements (1) and (2) in Lemma 5.5 hold. For \( n \in \mathbb{C}_{\ell,d} \), let \( p_{\ell,n} \) be the unique polynomial of coordinate degree at most \( 2\ell \) such that \( p_{\ell,n}(\tau_m) = \delta_{nm} \) for \( m \in \mathbb{C}_{\ell,d} \). Define
\[
\Phi_{\ell,n}(t) = \frac{Q_{d,\ell}(\tau_n)\text{sinc}(t)}{Q_{d,\ell}(t)\text{sinc}(\tau_n)} p_{\ell,n}(t)
\]
\[
= \frac{(Q_{d,\ell}(\tau_n)\text{sinc}(t))}{Q_{d,\ell}(t)\text{sinc}(\tau_n)} p_{\ell,n}(t_1, \ldots, t_d)
\]
\[
= \frac{(Q_{d,\ell}(\tau_n))}{Q_{d,\ell}(\tau_n)} \frac{p_{\ell,n}(t_1, \ldots, t_d)}{p_{\ell,n}(t_1, \ldots, t_d)} \text{sinc}(\pi t_1) \cdots \text{sinc}(\pi t_d).
\]

If, in (26), we sequentially apply partial fraction decomposition in each real variable \( t_1, \ldots, t_d \), we see that \( \Phi_{\ell,n}(t) \) is of the form
\[
\Phi_{\ell,n}(\cdot) = \sum_{n \in \mathbb{C}_{\ell,d}} a_n \text{sinc}(\cdot - n) \in PW_d.
\]
By (3), we therefore have
\[
\delta_{n,m} = \Phi_{\ell,n}(\tau_m) = \langle \Phi_{\ell,n}(\cdot), \text{sinc}(\cdot - \tau_m) \rangle = \langle \mathcal{F}^{-1}(\Phi_{\ell,n}), f_m \rangle, \quad n, m \in \mathbb{C}_{\ell,d},
\]
and \( \Phi_{\ell,n}(m) = 0 \) when \( m \notin \mathbb{C}_{\ell,d} \). Define the map \( L_{\ell} \) by \( L_{\ell} e_n = f_{\ell,n} \). Let \( f = \sum_{n \in \mathbb{Z}^d} c_n e_n \) such that \( L_{\ell} f = 0 \), then
\[
0 = \sum_{n \in \mathbb{C}_{\ell,d}} c_n f_n + \sum_{n \notin \mathbb{C}_{\ell,d}} c_n e_n.
\]
If, for each \( n \in \mathbb{C}_{\ell,d} \) we integrate the equation above against \( \mathcal{F}^{-1}(\Phi_{\ell,n}) \), we see that \( c_n = 0 \) for \( n \in \mathbb{C}_{\ell,d} \), so that \( c_n = 0 \) for all \( n \in \mathbb{Z}^d \). \( L_{\ell} \) is one to one, so by Lemma 3.8, it is an isomorphism from \( \mathcal{H}_d \) to itself. \( \square \)

**Proof of (a) and (b) of Theorem 5.4.** Lemmas 3.9 and 5.5 imply the existence of a unique sequence of polynomials satisfying statements (a) and (b) of Theorem 5.4, namely,
\[
\Psi_{\ell}(t) = \sum_{n \in \mathbb{C}_{\ell,d}} f(\tau_n) p_{\ell,n}(t),
\]
where \( p_{\ell,n} \) is defined as in the proof of Lemma 5.5. \( \square \)

It remains to show that this sequence of polynomials satisfies statement (c) of Theorem 5.4. This is accomplished by the following propositions.
Proposition 5.6. Let \( \{\tau_n\}_{n \in \mathbb{Z}^d} \) be any sequence in \( \mathbb{R}^d \). The following statements are true.

1. \( \sup_{x \in \mathbb{R}} \sup_{\ell \in \mathbb{N}} \left| \frac{\text{sinc}(x)}{Q_{1, \ell}(x)} \right| = 1. \)

2. Define \( \Delta_{\ell, d} = \left\{ n \in \mathbb{Z}^d \mid \left\| \frac{\tau_n}{\ell+1} \right\|_\infty < \frac{1}{\ell^{2/3}} \right\} \) for \( \ell \in \mathbb{N} \), then
   \[
   0 \leq 1 - \frac{\text{sinc}(\tau_n)}{Q_{d, \ell}(\tau_n)} < 1 - \exp \left( \frac{-d(\ell + 2)}{\ell^{4/3} - 1} \right), \quad n \in \Delta_{\ell, d}. \tag{27}
   \]

Proof of Proposition 5.6. For statement (1), the identity
\[
\text{sinc}(t) = \prod_{k=1}^{\infty} \left( 1 - \frac{t^2}{k^2} \right), \quad t \in \mathbb{R}^d
\]
implies
\[
\frac{\text{sinc}(t)}{Q_{1, \ell}(t)} = \prod_{k=\ell+1}^{\infty} \left( 1 - \frac{t^2}{k^2} \right), \tag{28}
\]
where convergence is uniform on compact subsets of \( \mathbb{C} \). Fix \( \ell \in \mathbb{N} \). If \( t \in [0, \ell + 1] \), then
\[
\left| \frac{\text{sinc}(t)}{Q_{1, \ell}(t)} \right| \leq 1.
\]
Note that \( |Q_{1, \ell}(t)| = \prod_{k=1}^{\ell} \left( \frac{t^2}{k^2} - 1 \right) \) is increasing on \((\ell+1, \infty)\). If \( t \in (\ell+1, \infty) \), then
\[
\left| \frac{\text{sinc}(t)}{Q_{1, \ell}(t)} \right| = \left| \frac{\text{sin}(\pi t)}{\pi t Q_{1, \ell}(t)} \right| < \frac{1}{\pi(\ell + 1)|Q_{1, \ell}(\ell + 1)|}.\]

Computation yields
\[
|Q_{1, \ell}(\ell + 1)| = \frac{(2\ell + 1)!}{\ell!(\ell + 1)!},
\]
so
\[
\left| \frac{\text{sinc}(t)}{Q_{1, \ell}(t)} \right| < \frac{(\ell)!^2}{\pi(2\ell + 1)!} < 1.
\]

Observing that \( \frac{\text{sinc}(t)}{Q_{1, \ell}(t)} \) is even proves statement (1).

For statement (2), let \( t \in \mathbb{R} \) such that \( \frac{t}{\ell+1} < \frac{1}{\ell^{2/3}} \), then \( 0 < \frac{\text{sinc}(t)}{Q_{1, \ell}(t)} \), and
\[
- \log \left( \frac{\text{sinc}(t)}{Q_{1, \ell}(t)} \right) = - \sum_{k=\ell+1}^{\infty} \log \left( 1 - \frac{t^2}{k^2} \right) = \sum_{k=\ell+1}^{\infty} \sum_{j=1}^{\infty} \frac{t^2}{jk^{2j}}
\]
\[
= \sum_{j=1}^{\infty} \frac{1}{j} \left( \sum_{k=\ell+1}^{\infty} \frac{1}{k^{2j}} \right) t^{2j}. \tag{29}
\]

The function \( 1/x^{2j} \) is decreasing, and basic calculus shows that
\[
\sum_{k=\ell+1}^{\infty} \frac{1}{k^{2j}} < \frac{1}{(\ell + 1)^{2j}} + \frac{1}{(2j - 1)(\ell + 1)^{2j-1}}.
\]
Equality (29) implies
\[- \log \left( \frac{\text{sinc}(t)}{Q_{1, \ell}(t)} \right) < \sum_{j=1}^{\infty} \frac{1}{j} \left( \frac{t}{\ell + 1} \right)^{2j} + (\ell + 1) \sum_{j=1}^{\infty} \frac{1}{j(2j - 1)} \left( \frac{t}{\ell + 1} \right)^{2j} < (\ell + 2) \sum_{j=1}^{\infty} \frac{1}{(\ell + 1)^{2j}} < \frac{\ell + 2}{\ell^{4/3} - 1}. \] (30)

If \( n \in \Delta_{\ell, d} \), then for each \( 1 \leq k \leq d \), \( \frac{t_{nk}}{t_{k+1}} > \frac{\ell + 2}{\ell^{4/3} - 1} \), so that
\[ \log \left( \frac{\text{sinc}(\tau_n)}{Q_{d, \ell}(\tau_n)} \right) = \sum_{k=1}^{d} \log \left( \frac{\text{sinc}(\tau_{nk})}{Q_{1, \ell}(\tau_{nk})} \right) > -d(\ell + 2) \frac{1}{\ell^{4/3} - 1}. \]

Statement (2) of Proposition 5.6 follows readily. □

Proposition 5.7. Statement (c) of Theorem 5.4 is true iff
\[ 0 = \lim_{\ell \to \infty} \sum_{n \in C_{\ell, d}} |f(\tau_n)|^2 \left[ 1 - \frac{\text{sinc}(\tau_n)}{Q_{d, \ell}(\tau_n)} \right]^2 := \lim_{\ell \to \infty} S_{\ell, d}, \quad f \in PW_d. \] (31)

Proof of Proposition 5.7. Note that \( L e_n = f_n \) implies \( f_n^* = (L^*)^{-1} e_n \). Similarly, \( f_{\ell, n}^* = (L_{\ell}^*)^{-1} e_n \). Given \( f \in PW_d \), let \( g = F^{-1}(f) \). Eq. (16) shows
\[ F^{-1}(f) = \lim_{\ell \to \infty} (L_{\ell}^*)^{-1} \sum_{n \in C_{\ell, d}} (L_{\ell}^* g, e_n) e_n = \lim_{\ell \to \infty} (L_{\ell}^*)^{-1} \sum_{n \in C_{\ell, d}} (g, f_{\ell, n}) e_n \]
\[ = \lim_{\ell \to \infty} \sum_{n \in C_{\ell, d}} (g, f_{\ell, n}) f_{\ell, n}^* = \lim_{\ell \to \infty} \sum_{n \in C_{\ell, d}} f(\tau_n) f_{\ell, n}^*. \]
Passing to the Fourier transform, we have
\[ f = \lim_{\ell \to \infty} \sum_{n \in C_{\ell, d}} f(\tau_n) F(f_{\ell, n}^*), \quad f \in PW_d, \] (32)
where the limit exists in both \( L_2 \) and uniform senses. Eq. (5) shows that the values of a function in \( PW_d \) on the set \( \{\tau_n\}_{n \in C_{\ell, d}} \cup \{n\}_{n \not \in C_{\ell, d}} \) uniquely determine the function. This and (26) show that
\[ F(f_{\ell, n}^*)(t) = G_{\ell, n}(t) = \frac{Q_{d, \ell}(\tau_n) \text{sinc}(t)}{Q_{d, \ell}(t) \text{sinc}(\tau_n)}, \quad n \in C_{\ell, d}. \]

This implies
\[ \Psi_{\ell}(t) \text{sinc}(t) = \left( \sum_{n \in C_{\ell, d}} f(\tau_n) p_{\ell, n}(t) \right) \frac{\text{sinc}(t)}{Q_{d, \ell}(t)} = \sum_{n \in C_{\ell, d}} f(\tau_n) \frac{\text{sinc}(\tau_n)}{Q_{d, \ell}(\tau_n)} F(f_{\ell, n}^*)(t). \]

Combined with (32), we see that statement (c) of Theorem 5.4 holds iff
\[ 0 = \lim_{\ell \to \infty} \sum_{n \in C_{\ell, d}} f(\tau_n) \left[ 1 - \frac{\text{sinc}(\tau_n)}{Q_{d, \ell}(\tau_n)} \right] F(f_{\ell, n}^*), \quad f \in PW_d. \]
where the limit is in the $L_2$ sense. Passing to the inverse Fourier transform, the equality above holds iff

$$0 = \lim_{\ell \to \infty} (L_\ell^*)^{-1} \left( \sum_{n \in C_{\ell,d}} f(\tau_n) \left[ 1 - \frac{\text{sinc}(\tau_n)}{Q_{d,\ell}(\tau_n)} \right] e_n \right), \quad f \in PW_d. \quad (33)$$

Now $\{L_\ell\}_{\ell > 0}$ is pointwise bounded, so by the uniform boundedness principle,

$$0 < \sup_{\ell} \|L_\ell^*\| = \sup_{\ell} \|L_\ell\| := C < \infty.$$

Uniform invertibility of $L$ implies $0 < \sup_{\ell \geq \ell_0} \|(L_\ell^*)^{-1}\| = \sup_{\ell \geq \ell_0} \|L_\ell^{-1}\| := c < \infty$. Together we have

$$\frac{1}{C} \|g\| \leq \|(L_\ell^*)^{-1} g\| < c \|g\|, \quad g \in H_d.$$

This inequality combined with (33) proves Proposition 5.7. \qed

**Proof of statement (c) in Theorem 5.4.** Let $S_{\ell,d}$ be as in (31). Proposition 5.6 gives the following.

$$S_{\ell,d} \leq \left( \sum_{n \in \Delta_{\ell,d}} + \sum_{n \in \mathbb{Z}^d \setminus \Delta_{\ell,d}} \right) |f(\tau_n)|^2 \left[ 1 - \frac{\text{sinc}(\tau_n)}{Q_{d,\ell}(\tau_n)} \right]^2 \leq \left( 1 - \exp \left( \frac{-d(\ell + 2)}{\ell^4/3 - 1} \right) \right)^2 \sum_{n \in \mathbb{Z}^d} |f(\tau_n)|^2 + \sum_{n: \frac{\ell + 1}{2\ell^3} \leq \|\tau_n\|} 4|f(\tau_n)|^2. \quad (34)$$

Now $\{f(\tau_n)\}_{n \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d)$ implies $\lim_{\ell \to \infty} S_{\ell,d} = 0$, so by Proposition 5.7, statement (c) in Theorem 5.4 is true. \qed

**6. The second main result**

Theorem 5.4 can be simplified. The function

$$t \mapsto \frac{\text{sinc}(t)}{Q_{d,\ell}(t)}$$

becomes more computationally complicated for large values of $\ell$. If, at the cost of global $L_2$ and uniform convergence, we adopt an approximation

$$\text{sinc}(t) \simeq Q_{d,\ell}(t) \exp \left( -\sum_{k=1}^N \frac{1}{k(2k - 1)} \frac{\|t\|_{2k}^{2k}}{(\ell + 1/2)^{2k-1}} \right), \quad (35)$$

we bypass this difficulty as the exponent of the quantity above is simply a rational function of $\ell > 0$. This is stated precisely in Theorem 6.1, which is the second main result of this paper.

**Theorem 6.1.** Let $\{\tau_n\}_{\mathbb{Z}^d} \subset \mathbb{R}^d$ be a sequence such that $\{f_n\}_{n \in \mathbb{Z}^d}$ is a UIRB. If $N$ is a non-negative integer and $A > 0$, define

$$E_{\ell,N,A} = \left[ -A(\ell + 1/2)^{2N+1} \frac{2N+1}{2N+2}, A(\ell + 1/2)^{2N+1} \frac{2N+1}{2N+2} \right].$$
Let \( f \in PW_d \) where \( \{ \Psi_{\ell} \}_\ell \) is the sequence of interpolating polynomials from Theorem 5.4. Define
\[
I_{f,\ell}(t) = \Psi_{\ell}(t) \exp \left( -\sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{\|t\|^{2k}_{2k}}{(\ell + 1/2)^{2k-1}} \right),
\]
then
\[
\lim_{\ell \to \infty} \| f(t) - I_{f,\ell}(t) \|_{L^2((E_{\ell,N,A})^d)} = 0, \tag{36}
\]
and
\[
\lim_{\ell \to \infty} \| f(t) - I_{f,\ell}(t) \|_{L^\infty((E_{\ell,N,A})^d)} = 0. \tag{37}
\]

If \( N = 0 \) in Theorem 6.1, we have the following analogue of Corollary 1.2 to arbitrary multivariate bandlimited functions.

**Corollary 6.2.** For all \( f \in PW_d \), we have
\[
\lim_{\ell \to \infty} \| f(t) - \Psi_{\ell}(t) \|_{L^2([-A(\ell+1/2)^{1/2},A(\ell+1/2)^{1/2}]^d)} = 0. \tag{38}
\]
and
\[
\lim_{\ell \to \infty} \| f(t) - \Psi_{\ell}(t) \|_{L^\infty([-A(\ell+1/2)^{1/2},A(\ell+1/2)^{1/2}]^d)} = 0. \tag{39}
\]

The following theorem provides a nice interpretation of Corollary 6.2. The proof of Theorem 6.3 when \( d = 1 \) appears in [17, Theorem 9, page 143], and the proof for general \( d \) is identical from a functional analytic point of view.

**Theorem 6.3.** Let \( \{ \tau_n \}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d \). The following are equivalent.

1. The sequence of functions \( \{ f_n \}_{n \in \mathbb{Z}^d} \) is a Riesz basis for \( \mathcal{H}_d \).
2. The map \( f \mapsto \{ f(\tau_n) \}_{n \in \mathbb{Z}^d} \) is a bijection from \( PW_d \) to \( \ell_2(\mathbb{Z}^d) \).

Consider a sequence \( \{ \tau_n \}_{n \in \mathbb{Z}^d} \subset \mathbb{R}^d \) subject to the hypotheses of Theorem 6.1 and sampled data \( \{ (\tau_n, c_n) \}_{n \in \mathbb{Z}^d} \) where \( \{ c_n \}_{n \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d) \). A unique sequence of Lagrangian polynomial interpolants exists, and converges to the unique bandlimited interpolant of the same data in a non-local \( L^2 \) and \( L^\infty \) manner.

When \( N = 1 \), we have a sampling theorem with a Gaussian multiplier.
\[
f(t) \simeq \Psi_{\ell}(t) \exp \left( -\frac{\|t\|^{2}_{2}}{(\ell + 1/2)} \right), \quad f \in PW_d.
\]

Compare Theorem 6.1 with Theorem 2.6 in [4], which is a multivariate sampling theorem with a Gaussian multiplier with global \( L_2 \) and uniform convergence. Also compare Theorem 6.1 with Theorem 2.1 in [14], which when \( d = 1 \) and the data nodes are equally spaced, gives another recovery formula involving a Gaussian multiplier in the context of oversampling.

In order to make the equations in this section more readable, we define the following notation.
\[
\sigma(t, n) = \sum_{k=1}^{d} t_k^{2n}, \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d.
\]
The proof of Theorem 6.1 relies on two lemmas whose proofs will be deferred until the end of this section.

**Lemma 6.4.** Let \( d > 0 \), \( N \) be a non-negative integer, and \( A > 0 \). There exists \( M > 0 \) such that for sufficiently large \( \ell \) and any \( t \in (E_{\ell, N, A})^d \),
\[
\left| Q_{d, \ell}(t) \exp \left( -\sum_{k=1}^{N} \frac{1}{k(2k - 1)} \frac{\sigma(t, k)}{(\ell + 1/2)^{2k-1}} \right) - \exp \left( \frac{\sigma(t, N + 1)}{(\ell + 1/2)^{2N+1}(N + 1)(2N + 1)} \right) \frac{\Psi_{\ell}(t)}{Q_{d, \ell}(t)} \text{sinc}(t) \right| \leq M(\ell + 1/2)^{-\frac{1}{N+1}} |\text{sinc}(t)|.
\]

**Lemma 6.5.** For all \( f \in PW_d \) and any non-negative integer \( N \), we have
\[
\lim_{\ell \to \infty} \sup_{t \in (E_{\ell, N, A})^d} \left| \left( \exp \left( \frac{\sigma(t, N + 1)}{(\ell + 1/2)^{2N+1}(N + 1)(2N + 1)} \right) - 1 \right) f(t) \right| = 0.
\]

**Proof of Theorem 6.1.** If \( f \in PW_d \), Theorem 5.4 states that
\[
f(t) = \frac{\Psi_{\ell}(t)}{Q_{d, \ell}(t)} |\text{sinc}(t)| + \xi_{\ell}(t)
\]
where \( \xi_{\ell} \to 0 \) on \( \mathbb{R}^d \) in both \( L_2 \) and \( L_\infty \) senses. By Lemma 6.4, we have
\[
\sup_{t \in (E_{\ell, N, A})^d} \left| \Psi_{\ell}(t) \exp \left( -\sum_{k=1}^{N} \frac{1}{k(2k - 1)} \frac{\sigma(t, k)}{(\ell + 1/2)^{2k-1}} \right) - \exp \left( \frac{\sigma(t, N + 1)}{(\ell + 1/2)^{2N+1}(N + 1)(2N + 1)} \right) \frac{\Psi_{\ell}(t)}{Q_{d, \ell}(t)} \text{sinc}(t) \right| \leq M(\ell + 1/2)^{-\frac{1}{N+1}} \sup_{t \in (E_{\ell, N, A})^d} (|f(t)| - |\xi_{\ell}(t)|),
\]
the right side of which has zero limit. Also,
\[
\sup_{t \in (E_{\ell, N, A})^d} \left| \left( \exp \left( \frac{\sigma(t, N + 1)}{(\ell + 1/2)^{2N+1}(N + 1)(2N + 1)} \right) - 1 \right) \frac{\Psi_{\ell}(t)}{Q_{d, \ell}(t)} \text{sinc}(t) \right| \leq \sup_{t \in (E_{\ell, N, A})^d} \left| \left( \exp \left( \frac{\sigma(t, N + 1)}{(\ell + 1/2)^{2N+1}(N + 1)(2N + 1)} \right) - 1 \right) f(t) \right|
\]
\[
+ \left( \exp \left( \frac{dA^{2(N+1)}}{(N + 1)(2N + 1)} \right) - 1 \right) \sup_{t \in (E_{\ell, N, A})^d} |\xi_{\ell}(t)|,
\]
whose right hand side has zero limit by Lemma 6.5. Combining (40) and (41), we obtain
\[
\lim_{\ell \to \infty} \left\| \Psi_{\ell}(t) \exp \left( -\sum_{k=1}^{N} \frac{1}{k(2k - 1)} \frac{\sigma(t, k)}{(\ell + 1/2)^{2k-1}} \right) - \frac{\Psi_{\ell}(t)}{Q_{d, \ell}(t)} \text{sinc}(t) \right\|_{L_\infty((E_{\ell, N, A})^d)} = 0.
\]
Eq. (37) follows by a final application of Theorem 5.4.
Now we prove (36). Lemma 6.4 and Theorem 5.4 imply
\[
\left\| \Psi_\ell(t) \exp \left( - \sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{\sigma(t,k)}{(\ell+1/2)^{2k-1}} \right) \right. \\
- \left. \exp \left( \frac{\sigma(t, N+1)}{(\ell+1/2)^{2N+1}(N+1)(2N+1)} \right) \frac{\Psi_\ell(t)}{Q_{d,\ell}(t)} \right\|_{L^2((E_{t,N,A})^d)} \\
\leq M(\ell + 1/2)^{-1} \| f - \xi_\ell \|_{L^2((E_{t,N,A})^d)},
\]
the right hand side of which has zero limit. Also,
\[
\left\| \left( \exp \left( \frac{\sigma(t, N+1)}{(\ell+1/2)^{2N+1}(N+1)(2N+1)} \right) - 1 \right) \frac{\Psi_\ell(t)}{Q_{d,\ell}(t)} \right\|_{L^2((E_{t,N,A})^d)} \\
\leq \left\| \left( \exp \left( \frac{\sigma(t, N+1)}{(\ell+1/2)^{2N+1}(N+1)(2N+1)} \right) - 1 \right) f(t) \right\|_{L^2((E_{t,N,A})^d)} \\
+ \left\| \left( \exp \left( \frac{\sigma(t, N+1)}{(\ell+1/2)^{2N+1}(N+1)(2N+1)} \right) - 1 \right) \xi_\ell(t) \right\|_{L^2((E_{t,N,A})^d)}.
\]
The second term in the right hand side of (43) is bounded from above by
\[
\left( \exp \left( \frac{dA^{2(N+1)}}{(N+1)(2N+1)} \right) - 1 \right) \| \xi_\ell \|_{L^2((E_{t,N,A})^d)},
\]
which has zero limit. The integrand of the first term in the right hand side of (43) (as a function on \( \mathbb{R}^d \)), converges uniformly to zero by Lemma 6.5, and is bounded from above by
\[
\left( \exp \left( \frac{dA^{2(N+1)}}{(N+1)(2N+1)} \right) - 1 \right) |f(t)|^2 \in L^1(\mathbb{R}^d),
\]
so this term has zero limit by the Lebesgue Dominated Convergence Theorem. Combining (42) and (43) yields
\[
\lim_{\ell \to \infty} \left\| \Psi_\ell(t) \exp \left( - \sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{\sigma(t,k)}{(\ell+1/2)^{2k-1}} \right) - \frac{\Psi_\ell(t)}{Q_{d,\ell}(t)} \right\|_{L^2((E_{t,N,A})^d)} = 0.
\]
Eq. (36) follows by a final application of Theorem 5.4. \( \square \)

The proof of Lemma 6.4 relies on the following proposition.

**Proposition 6.6.** If \( f : (0, \infty) \to (0, \infty) \) is convex, decreasing, differentiable, and integrable away from 0, then
\[
\frac{1}{4} f'(\ell + 1/2) \leq \sum_{k=\ell+1}^{\infty} f(k) - \int_{\ell + 1/2}^{\infty} f(x) dx \leq 0, \quad \ell \geq 0.
\]

**Proof of Proposition 6.6.** Geometric considerations show that
\[
(1) \quad f(k) \leq \int_{k-1/2}^{k+1/2} f(x) dx, \quad k \geq 1, \quad \text{and}
\]

[Continues with the rest of the document]
Applying Proposition 6.6 to the function \( f \) for \( k \geq 1 \).

The rightmost inequality in (44) follows from statement (1) by summing over \( k \). From statement (2) we obtain

\[
\int_{\ell+1}^{\infty} f(x) \, dx \leq \frac{1}{2} \sum_{k=\ell+1}^{\infty} f(k) + \frac{1}{2} \sum_{k=\ell+1}^{\infty} f(k + 1),
\]

\[
\frac{1}{2} f(\ell + 1) + \int_{\ell+1}^{\infty} f(x) \, dx \leq \sum_{k=\ell+1}^{\infty} f(k),
\]

\[
\frac{1}{2} f(\ell + 1) - \int_{\ell+\frac{1}{2}}^{\ell+1} f(x) \, dx \leq \sum_{k=\ell+1}^{\infty} f(k) - \int_{\ell+\frac{1}{2}}^{\infty} f(x) \, dx.
\]

(45)

There exists \( \ell + 1/2 < \xi < \ell + 1 \) such that

\[
\frac{1}{4} f'(\ell + 1/2) \leq \frac{1}{4} f'(-1) = \frac{1}{2} f(\ell + 1) - \frac{1}{2} f(\ell + 1/2) \leq \frac{1}{2} f(\ell + 1) - \int_{\ell+\frac{1}{2}}^{\ell+1} f(x) \, dx.
\]

Combining the inequality above with (45) proves the proposition. \( \square \)

**Proof of Lemma 6.4.** Letting \(|t| < \ell + 1/2\) and recalling (29) we see that

\[
- \log \left( \frac{\text{sinc}(t)}{Q_{1,\ell}(t)} \right) = \sum_{k=1}^{\infty} \left[ \sum_{j=\ell+1}^{\infty} \frac{1}{j^{2k}} - \frac{1}{(2k - 1)(\ell + 1/2)^{2k-1}} \right] \frac{t^{2k}}{k}.
\]

(46)

Applying Proposition 6.6 to the function \( f(t) = \frac{1}{t^{2}} \) when \( k \geq 1 \) yields

\[
\frac{-k}{2(\ell + 1/2)^{2k+1}} \leq \sum_{j=\ell+1}^{\infty} \frac{1}{j^{2k}} \frac{1}{(2k - 1)(\ell + 1/2)^{2k-1}} \leq 0.
\]

Eq. (46) becomes

\[
- \frac{1}{2(\ell + 1/2)} \sum_{k=1}^{\infty} \left( \frac{t}{\ell + 1/2} \right)^{2k} \leq - \log \left( \frac{\text{sinc}(t)}{Q_{1,\ell}(t)} \right) = \sum_{k=1}^{\infty} \frac{1}{k(2k - 1)(\ell + 1/2)^{2k-1}} \frac{t^{2k}}{k} \leq 0.
\]

Restated,

\[
- \frac{1}{2(\ell + 1/2)} \left( \frac{t}{\ell + 1/2} \right)^{2} \leq \sum_{k=N+1}^{\infty} \frac{1}{k(2k - 1)(\ell + 1/2)^{2k-1}} \frac{t^{2k}}{k} \leq - \log \left( \frac{\text{sinc}(t)}{Q_{1,\ell}(t)} \right) - \sum_{k=N+1}^{\infty} \frac{1}{k(2k - 1)(\ell + 1/2)^{2k-1}} \frac{t^{2k}}{k} \leq 0.
\]

(47)
Exponentiating,
\[
\exp \left( -\frac{1}{2(\ell + 1/2)} \left( \frac{t}{\ell + 1/2} \right)^2 \right) \exp \left( \sum_{k=N+1}^{\infty} \frac{1}{k(2k - 1) (\ell + 1/2)^{2k-1}} \right) \\
\leq \frac{Q_{1,\ell}(t)}{\text{sinc}(t)} \exp \left( -\sum_{k=1}^{N} \frac{1}{k(2k - 1) (\ell + 1/2)^{2k-1}} t^{2k} \right) \\
\leq \exp \left( \sum_{k=N+1}^{\infty} \frac{1}{k(2k - 1) (\ell + 1/2)^{2k-1}} t^{2k} \right). \tag{48}
\]
Let \( \ell \) be chosen large enough so that
\[
A(\ell + 1/2)^{\frac{2N+1}{4N+2}} < \ell + 1/2.
\]
For any \( t \in E_{\ell,N,A} \),
\[
t = c(\ell + 1/2)^{\frac{2N+1}{4N+2}}
\]
for some \( c \in [-A, A] \). For such \( t \), (48) implies
\[
\exp \left( -\frac{1}{2(\ell + 1/2)} \frac{c^2}{1 - c^2(\ell + 1/2)^{\frac{1}{N+1}}} \right) \exp \left( \sum_{k=N+1}^{\infty} \frac{c^{2k}}{k(2k - 1) (\ell + 1/2)^{1-\frac{k}{N+1}}} \right) \\
\leq \frac{Q_{1,\ell}(t)}{\text{sinc}(t)} \exp \left( -\sum_{k=1}^{N} \frac{1}{k(2k - 1) (\ell + 1/2)^{2k-1}} t^{2k} \right) \\
\leq \exp \left( \sum_{k=N+1}^{\infty} \frac{c^{2k}}{k(2k - 1) (\ell + 1/2)^{1-\frac{k}{N+1}}} \right).
\]
If \( t \in (E_{\ell,N,A})^d \), then
\[
t = c(\ell + 1/2)^{\frac{2N+1}{4N+2}}
\]
for some \( c \in [-A, A]^d \). For any such \( t \), we have
\[
\exp \left( -\frac{d}{2(\ell + 1/2)} \frac{A^2}{1 - A^2(\ell + 1/2)^{\frac{1}{N+1}}} \right) \\
\times \exp \left( \sum_{k=N+1}^{\infty} \frac{\sigma(c, k)}{k(2k - 1) (\ell + 1/2)^{1-\frac{k}{N+1}}} \right) \\
\leq \frac{Q_{d,\ell}(t)}{\text{sinc}(t)} \exp \left( -\sum_{k=1}^{N} \frac{1}{k(2k - 1) (\ell + 1/2)^{2k-1}} \frac{\sigma(t, k)}{k(2k - 1) (\ell + 1/2)^{1-\frac{k}{N+1}}} \right) \\
\leq \exp \left( \sum_{k=N+1}^{\infty} \frac{\sigma(c, k)}{k(2k - 1) (\ell + 1/2)^{1-\frac{k}{N+1}}} \right). \tag{49}
\]
On one hand,
\[
\exp \left( \sum_{k=N+1}^{\infty} \frac{\sigma(c, k)}{k(2k-1)} (\ell + 1/2)^{\left(1 - \frac{k}{N+1}\right)} \right)
\leq \exp \left( \frac{\sigma(c, N + 1)}{(N + 1)(2N + 1)} + O \left( (\ell + 1/2)^{\frac{1}{N+1}} \right) \right)
\]
where the “big O” constant is independent of \(c \in [-A, A]^d\). On the other hand,
\[
\exp \left( \frac{\sigma(c, N + 1)}{(N + 1)(2N + 1)} \right) \leq \exp \left( \sum_{k=N+1}^{\infty} \frac{\sigma(c, k)}{k(2k-1)} (\ell + 1/2)^{\left(1 - \frac{k}{N+1}\right)} \right).
\]
Inequalities (49)–(51) yield
\[
\left( \exp \left( - \frac{d}{2(\ell + 1/2)^{\frac{N+2}{N+1}}} \frac{A^2}{1 - A^2(\ell + 1/2)^{\frac{1}{N+1}}} \right) - 1 \right) \exp \left( \frac{\sigma(c, n + 1)}{(N + 1)(2N + 1)} \right)
\leq \frac{Q_{d, \ell}(t)}{\sin(t)} \exp \left( - \sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{\sigma(c, k)}{(\ell + 1/2)^{2k-1}} \right) - \exp \left( \frac{\sigma(c, N + 1)}{(N + 1)(2N + 1)} \right)
\leq \exp \left( \frac{dA^{2(N+1)}}{(N + 1)(2N + 1)} \left( \exp \left( O \left( \frac{1}{(\ell + 1/2)^{\frac{1}{N+1}}} \right) \right) - 1 \right) \right).
\]
The leftmost side of (52) is of the order \(O \left( (\ell + 1/2)^{-\frac{N+2}{N+1}} \right)\), and the rightmost side of (52) is of the order \(O \left( (\ell + 1/2)^{-\frac{1}{N+1}} \right)\), where the “big O” constants are independent of \(c \in [-A, A]^d\). The lemma follows readily.

**Proof of Lemma 6.5.** Equivalently, we need to show
\[
\lim_{\ell \to \infty} \sup_{c \in [-A, A]^d} \left| \left( \exp \left( \frac{\sigma(c, n + 1)}{(N + 1)(2N + 1)} \right) - 1 \right) f \left( c(\ell + 1/2)^{\frac{2N+1}{2N+2}} \right) \right| = 0.
\]
Suppose the contrary. Let \(c_\ell \in [-A, A]^d\) be a value that maximizes the \(\ell\)-th term in the limit above. There exists \(\{\ell_k\}_{k \in \mathbb{N}}\), and \(\epsilon > 0\) such that for all \(k \in \mathbb{N}\),
\[
\epsilon \leq \sup_{c \in [-A, A]^d} \left| \left( \exp \left( \frac{\sigma(c, N + 1)}{(N + 1)(2N + 1)} \right) - 1 \right) f \left( c(\ell_k + 1/2)^{\frac{2N+1}{2N+2}} \right) \right|
\leq \left( \exp \left( \frac{dA^{2(N+1)}}{(N + 1)(2N + 1)} \right) - 1 \right) \left| f \left( c(\ell_k + 1/2)^{\frac{2N+1}{2N+2}} \right) \right|,
\]
so that the sequence \(\left\{ f \left( c(\ell_k + 1/2)^{\frac{2N+1}{2N+2}} \right) \right\}_{k \in \mathbb{N}}\) is bounded away from 0. Eq. (4) implies there exists \(\delta > 0\) such that
\[
\left\| c(\ell_k + 1/2)^{\frac{2N+1}{2N+2}} \right\|_{2(N+1)} \leq \delta
\]
for \(k \in \mathbb{N}\) so that
\[
\| c\ell_k \|_{2(N+1)} \leq \delta(\ell_k + 1/2)^{\frac{2N+1}{2N+2}}.
\]
This forces
\[
\epsilon \leq \sup_{c \in [-A,A]} \left| \exp \left( \frac{\sigma(c, N + 1)}{(N + 1)(2N + 1)} - 1 \right) f\left( c(\ell_k + 1/2)^{2N+1} \right) \right|
\]
\[
\leq \left( \exp \left( \frac{\delta^{2(N+1)}}{(\ell_k + 1/2)^{2N+1}(N + 1)(2N + 1)} - 1 \right) \right) \|f\|_{\infty}.
\]
The last term in the inequality above has limit 0 as \(\ell \to \infty\) which is a contradiction. \(\square\)

7. Comments regarding the optimality of Theorem 6.1

In the statement of Theorem 6.1, it is not apparent whether or not \(\{E_{\ell,N,A}\}_\ell\) can be replaced with a more rapidly growing sequence of intervals; however, Proposition 7.1 shows that if \(f(t) = \text{sinc}(t)\), then (36) and (37) can hold for sequences of intervals \(\{[-A_{\ell,N}, A_{\ell,N}]\}_\ell\) and \(\{[-B_{\ell,N}, B_{\ell,N}]\}_\ell\) (respectively) which grow faster than \(\{E_{\ell,N,A}\}_\ell\). Proposition 7.2 shows that the growth rates for \(A_{\ell,N}\) and \(B_{\ell,N}\) are optimal for sinc. Thus, these bounds limit the growth of any sequence of intervals such that (36) or (37) hold for general multivariate bandlimited functions. Optimality of the rate of growth of \(\{E_{\ell,N,A}\}_\ell\) has not been proven, but comparison with \(\{[-A_{\ell,N}, A_{\ell,N}]\}_\ell\) and \(\{[-B_{\ell,N}, B_{\ell,N}]\}_\ell\) shows it could be improved by (at most) a multiplication by a root of a logarithm.

Proposition 7.1. Define
\[
A_{\ell,N} = \left( \frac{1}{4} (2N + 1)^2 (\ell + 1/2)^{2N+1} \log(\ell + 1/2) \right)^{\frac{1}{2(N+1)}}, \quad \text{and}
\]
\[
B_{\ell,N} = \left( \frac{1}{2} (2N + 1)^2 (\ell + 1/2)^{2N+1} \log(\ell + 1/2) \right)^{\frac{1}{2(N+1)}},
\]
then
\[
\lim_{\ell \to \infty} \|\text{sinc}(t) - I_{\text{sinc},\ell}(t)\|_{L_2([-A_{\ell,N},A_{\ell,N}]^d)} = 0, \quad (53)
\]
and
\[
\lim_{\ell \to \infty} \|\text{sinc}(t) - I_{\text{sinc},\ell}(t)\|_{L_\infty([-B_{\ell,N},B_{\ell,N}]^d)} = 0. \quad (54)
\]

The proof of Proposition 7.1 is a technical refinement of the proofs of Lemmas 6.4 and 6.5, and is omitted for length considerations. This refinement is possible because we know the exact rate of decay of sinc. In the proof of Lemma 6.5, the obstruction to finding a larger growth rate is usage of (4) which only states that each bandlimited function decays uniformly to zero at infinity.

Proposition 7.2. The growth rates in Proposition 7.1 are optimal for the conclusions of that proposition to hold.

(1) Let \(N \geq 0\). If \(\{M_{\ell,N}\}_\ell\) is a sequence of positive numbers such that (53) holds when \(\{A_{\ell,N}\}_\ell\) is replaced by \(\{M_{\ell,N}\}_\ell\), then
\[
\limsup_{\ell \to \infty} \frac{M_{\ell,N}}{A_{\ell,N}} \leq 1. \quad (55)
\]
(2) Let \( N \geq 0 \). If \( \{M_\ell, N\}_\ell \) is a sequence of positive numbers such that (54) holds when \( \{B_\ell, N\}_\ell \) is replaced by \( \{M_\ell, N\}_\ell \), then

\[
\limsup_{\ell \to \infty} \frac{M_\ell, N}{B_\ell, N} \leq 1.
\]

(56)

The proof of Proposition 7.2 (also omitted for length considerations), essentially amounts to showing that for any \( \epsilon > 0 \),

\[
0 < \liminf_{\ell \to \infty} \left\| \frac{\sin(t) - I_{\sin, \ell}(t)}{L_2([A_\ell, N, (1+\epsilon)A_\ell, N]^d)} \right\|,
\]

0 < \liminf_{\ell \to \infty} \left\| \frac{\sin(t) - I_{\sin, \ell}(t)}{L_\infty([B_\ell, N, (1+\epsilon)B_\ell, N]^d)} \right\|.

(57)

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