STATE-DOMAIN CHANGE POINT DETECTION FOR NONLINEAR TIME SERIES

YAN CUI, JUN YANG, AND ZHOU ZHOU

Abstract. Change point detection in time series has attracted substantial interest, but most of the existing results have been focused on detecting change points in the time domain. This paper considers the situation where nonlinear time series have potential change points in the state domain. We apply a density-weighted antisymmetric kernel function to the state domain and therefore propose a nonparametric procedure to test the existence of change points. When the existence of change points is affirmative, we further introduce an algorithm to estimate their number together with their locations and show the convergence result on the estimation procedure. Numerical simulations and a real data application are given to illustrate our results.

Key words: Change-point detection; nonlinear time series; nonparametric hypothesis test; state domain.

CONTENTS

1. Introduction 1
2. Model Formulation and Basic Assumptions 4
3. State-domain Change Point Detection and Estimation 6
4. Practical Implementation 9
5. Simulation Study 10
6. Application 14
References 17
A. Proof of Theorem 3.2 20
B. Proof of Theorem 3.5 29

1. INTRODUCTION

Consider the nonlinear autoregressive model

\[ X_i = \mu(X_{i-1}) + \sigma(X_{i-1})\eta_i, \]  

(1)

where \( \mu(\cdot) \) and \( \sigma^2(\cdot) \) are unknown regression function and conditional variance, \( \eta_i \) are unobserved independent and identically distributed (i.i.d.) errors with \( \mathbb{E}\eta_i = 0 \) and \( \mathbb{E}\eta_i^2 = 1 \). Special cases of Eq. (1) include threshold AR models [Ton90], exponential

Department of Statistical Sciences, University of Toronto, Canada
E-mail addresses: \{cui, jun, zhou\}@utstat.toronto.edu.
AR models [HO81] and ARCH models [Eng82], among others. In fact, Eq. (1) can be viewed as a discretized version of the nonparametric one-factor diffusion model
\[ dX_t = \mu(X_t)dt + \sigma(X_t)d\mathbb{B}(t), \] (2)
where \( \mu(\cdot) \) is the instantaneous return or drift function, \( \sigma(\cdot) \) is the volatility function and \( \{\mathbb{B}(t)\} \) is the standard Brownian motion. Recently, this model has been widely discussed to understand and model nonlinear temporal systems in various fields. Stanton [Sta97], Chapman and Pearson [CP00] and Fan and Zhang [FZ03] considered the nonparametric estimation of \( \mu(\cdot) \) and \( \sigma^2(\cdot) \). In particular, Eq. (2) can deal with financial data with \( \{X_t\} \) being interest rates, exchange rates, stock prices or other economic quantities. Among others, Zhao and Wu [ZW08] considered kernel quantile estimates of Eq. (2) with the Federal exchange rates between Pound and USD. Liu and Wu [LW10] constructed a simultaneous confidence band for \( \mu(\cdot) \) and \( \sigma(\cdot) \) with the U.S. Treasury yield curve rates data. See also the latter papers for further references.

To date, most investigations on the nonparametric inference procedure of Eq. (1) are based on the assumption that the underlying regression function \( \mu(x) \) is continuous, which will cause serious restriction in many real applications. In fact, in parametric modeling of nonlinear time series, different choices of \( \mu(\cdot) \) with possible discontinuity have drawn much attention in the literature. One of the most prominent examples is the threshold model proposed by [TL80], in which regime switches are triggered by an observed variable crossing an unknown threshold. Also, AR model with regime-switch controlled by a Markov chain mechanism was then introduced by [Ton90]. Furthermore, there is a large amount of evidence on the existence of change points in the state domain in diverse areas, including physics, environmental science, economics and finance. The most familiar to us is that matter will drastically turn into different states (solid, liquid or gas) at some critical temperature, for example, water will become ice if the temperature is lower than zero degree centigrade. In economics, the expanding phase and contracting phase are not always governed by the same dynamics, see [TT94; DJ95; MPQ00] and other references therein. As a result, the occurrence of abrupt changes in the state domain of real data is very common and detecting and estimating for them is of vital importance. Motivated by this, in the current paper we focus on considering the situation where the regression function \( \mu(\cdot) \) is piecewise smooth on the interval \( T = [l, u] \) with a finite but unknown number of change points. More precisely, there exist \( l = a_0 < a_1 < \cdots < a_M < a_{M+1} = u \) such that \( \mu(\cdot) \) is smooth on each of the intervals \( [a_0, a_1), \cdots, [a_M, a_{M+1}] \), i.e.,
\[ \mu(x) = \sum_{j=0}^{M} \mu_j(x)1(x \leq a_{j+1}), \] (3)
where \( M \) is the total number of potential change points. Throughout this article, we assume \( M \) is fixed. Therefore with Eq. (1), before any subsequent analysis can be
carried out, a principle and important problem is to segment the state domain into smooth subregions.

To our knowledge, there exists no results on change point detection of the state-domain regression function $\mu(\cdot)$ in the literature. The purpose of this paper is twofold. First we want to test $\mu(x)$ is smooth or discontinuous over the interval $[l, u]$, that is to test the null hypothesis $H_0 : M = 0$ of Eq. (3). By sliding a density-weighted anti-symmetric kernel through the state domain, we will suggest a nonparametric test statistic and non-trivially apply the discretized multivariate Gaussian approximation result of [Zai87] to the state domain to establish its asymptotic distribution. Additionally, the Gaussian approximation results also directly suggest a finite sample simulation-based bootstrapping method which improves the convergence rate in practical implementations. Second, if $M \geq 1$, we reject the null hypothesis and next want to locate all the change points. In this case, we propose an estimation procedure and show the corresponding asymptotic theory on the accuracy of estimators. Finally, the above theoretical results are of general interest and can be used for a wider class of state-domain change point detection problems.

There is a long-standing literature in statistics discussing time-domain change point detection, where occasional jumps occur in an otherwise smoothly changing time trend. It is impossible to show a complete reference here and we only list some representative works. Müller [M92] and Eubank and Speckman [ES94] employed a kernel method to estimate jump points in smooth curves. Wang [Wan95] suggested using wavelets and provided an excellent review of jump-point estimation. Two-step method was considered by [MS97] and [GHK99] to study the asymptotic convergence properties of the change points. However, these papers only deal with the simplest classical problem in the time domain that there exists a known number of change points apriori. Later, Gijbels, Lambert, and Qiu [GLQ07] suggested a compromise estimation method which can preserve possible jumps in the curve. Zhang [Zha16] considered the situation where the trend function allows a growing number of jump points. All of the existing results in the aforementioned papers focused on detecting change points in the time domain and they always impose the independence assumption between error process and sampling time points. It is well known that state-domain asymptotic theory is very different from that of the time domain. And in our specific case, uniform asymptotic behaviour of our test statistic over the state domain is much more difficult to establish than the corresponding problem in the time domain. Therefore, we make an effort to give a more sophisticated inference for the detection and estimation with potential state-domain change points. In the current paper, we establish that, unlike time-domain change point methods where the long-run variance of the process is of crucial importance in the asymptotics, state-domain change point asymptotics heavily depends on the conditional variance and density. We also provide an estimation procedure using a simulated critical value to detect and locate all the change points.
We show that, with a preassigned probability $\alpha$, the method will asymptotically detect all the change points with an accuracy $c_n$ which is much smaller than $1/\sqrt{n}$, where $n$ is the length of the time series.

The rest of the paper is organized as follows. In Section 2, we introduce the model framework and some basic assumptions. Section 3 contains our main results, including a nonparametric test for determining the existence of change points and a procedure for estimating the number of change points together with their locations. Practical implementation based on a bootstrapping method and a suitable bandwidth selection are discussed in Section 4. Section 5 reports some simulation studies and a real data application is carried out for change point detection in Section 6. Proof of the results from Section 3 are relegated in Appendix A and Appendix B.

2. Model Formulation and Basic Assumptions

Throughout this paper, we use the following notations. For a random vector $X$, denote $X \in L^p$, $p > 0$ if $\|X\|^p := \left(\mathbb{E}|X|^p\right)^{1/p} < \infty$. $F_{U \mid V}(\cdot)$ is the conditional distribution function of $U$ given $V$ and $f_{U \mid V}(\cdot)$ is the conditional density. For function $g$ with $\mathbb{E}|g(U)| < \infty$, let $\mathbb{E}(g(U) \mid V) := \int g(x) dF_{U \mid V}(x)$ be the conditional expectation of $g(U)$ given $V$. $\mathbb{1}$ stands for the indicator function.

Assume that the process $X_i$ is stationary and causal, then we can write

$$X_i = G(\xi_i),$$

where the function $G$ is a measurable function such that $X_i$ exists and $\xi_i = (\cdots, \eta_{i-1}, \eta_i)$ is a shift process. From Eq. (4), one can interpret the transform $G$ as the underlying physical mechanism with $\xi_i$ and $G(\xi_i)$ being the input and output of the system. To facilitate the main results, we shall first introduce the time series dependence measures that will be used in our theory. Assume $X \in L^p$, let

$$X'_n = G(\xi'_n), \quad \xi'_n := (\xi_{i-1}, \eta'_0, \eta_1, \cdots, \eta_n),$$

where $X'_n$ is a coupled process of $X_n$ with $\eta_0$ replaced by an i.i.d. copy $\eta'_0$. Then, we define the physical dependence measure as

$$\theta_{n,p} = \|X_n - X'_n\|_p.$$  

Let $\theta_{n,p} = 0$ if $n < 0$. Thus for $n \geq 0$, $\theta_{n,p}$ measures the dependence of the output $G(\xi_n)$ on the single input $\eta_0$, refer to [Wu05] for more details on the physical dependence measures.

Recall $H_0 : M = 0$. Our aim is to test the null hypothesis that the regression function does not contain any change point. Here, we introduce a density-weighted anti-symmetric kernel function $\tilde{K}_n$, which can be written as

$$\tilde{K}_n(X, x, b) = \frac{w_n^*(x, b)K\left(x - \frac{x}{b}\right) - w_n(x, b)K^*\left(x - \frac{x}{b}\right)}{w_n(x, b)w_n^*(x, b)},$$

(7)
where \( K(\cdot) \) is a kernel function supported on \( S = [0, 1] \) with \( \int_S K(u)du = 1 \) and \( K^*(u) := K(-u) \). The data-dependent weights \( w_n(x, b) \) and \( w_n^*(x, b) \) are defined by

\[
    w_n(x, b) := \frac{1}{nb} \sum_{i=1}^{n} K \left( \frac{X_i - x}{b} \right), \quad w_n^*(x, b) := \frac{1}{nb} \sum_{i=1}^{n} K^* \left( \frac{X_i - x}{b} \right),
\]

where \( b = b_n \) is the bandwidth satisfying \( b \to 0 \) and \( nb \to \infty \). In fact, \( \tilde{K}_n(X, x, b) \) can be approximated by \( [K(x_i - x) - K^*(x_i - x)]/f(x) \), where \( f(x) \) is the density function of \( X_i \). Observe that \( K(x_i - x) - K^*(x_i - x) \) is an anti-symmetric function, we therefore call \( \tilde{K}_n(X, x, b) \) a density-weighted anti-symmetric kernel function. By sliding this kernel function \( \tilde{K}_n \) through the state domain, we can easily test whether \( \mu(x) \) has change points. More specifically, the kernel estimate \( \sum_{k=1}^{n} \tilde{K}_n(X_k - x, b)X_k/nb \) is a boundary kernel approximation to \( \mu_n(x^+) - \mu_n(x^-) \), where \( \mu_n(x^+) \) and \( \mu_n(x^-) \) are the right and left kernel smoothers of \( \mu(x) \). Thus, if \( x \) is a continuous point of \( \mu(x) \), this quantity will be approximately zero. However if it encounters any change point, the quantity will become large. To establish the main results, we need the following assumptions:

(a) There exist \( 0 < \delta_2 \leq \delta_1 < 1 \) such that \( n^{-\delta_1} = \mathcal{O}(b) \) and \( b = \mathcal{O}(n^{-\delta_2}) \).

(b) Let \( \mathbb{E} |\eta_i|^p < \infty \) where \( p > 2/(1 - \delta_1) \).

(c) Suppose that \( X_i \in \mathcal{L}^p \) and \( \theta_{n,p} = \mathcal{O}(\rho^n) \) for some \( p > 0 \) and \( 0 < \rho < 1 \).

(d) The density function \( f \) of \( X_i \) is positive on \( [l - \epsilon, u + \epsilon] \) for some \( \epsilon > 0 \) and there exists a constant \( B < \infty \) such that

\[
    \sup_x \left[ |f_{X_n|x_{n-1}}(x)| + |f'_{X_n|x_{n-1}}(x)| + |f''_{X_n|x_{n-1}}(x)| \right] \leq B, \text{ a.s.}
\]

(e) \( K(\cdot) \) is differentiable over \( (0, 1) \), the right derivative \( K'(0) \) and the left derivative \( K'(1) \) exist, and \( \sup_{0<\omega<1} |K'(\omega)| < \infty \). The Lebesgue measure of the set \( \{u \in [0, 1] : K(u) = 0\} \) is zero. Further assume \( K(0) = K(1) = 0 \), \( K'(0) > 0 \) and \( \int_0^1 uK(u)du = 0 \).

We now comment on the above regularity conditions. Condition (a) specifies the allowable range of the bandwidth. Condition (b) ensures that \( \eta_i \) has finite moments greater than two, which is quite mild. In Condition (c), the quantity \( \theta_{n,p} \) satisfies the geometric moment contraction (GMC) property, which means the dependence is of exponential decay. It is well known that the GMC property is preserved in many nonlinear time series models, for instance, ARMA models, ARCH and GARCH models, see [SW07] for more discussions. Furthermore, denote \( \Theta_n := \sum_{i=0}^{n} \theta_{i,2} \), which measures the cumulative dependence of \( X_0, ..., X_n \) on \( \eta_0 \). Then if Condition (c) holds, we know that \( \Theta_\infty = 1/(1 - \rho) < \infty \) indicates short-range dependence. With Condition (d), we know that the density and conditional density of \( X_i \) exist. Moreover, \( f \) has bounded derivatives up until the second order. Condition (e) puts some restrictions on the
smoothness and order of the kernel function $K$. In particular, $\int_0^1 uK(u)du = 0$ indicates that $K$ is a second-order kernel which has both positive and negative parts on $[0,1]$.

3. State-domain Change Point Detection and Estimation

In this section, we will discuss the detection of the potential change points and introduce an algorithm to estimate their locations and number.

3.1. Test for the existence of change point. With the foregoing discussion, we shall introduce a nonparametric statistic based on the density-weighted anti-symmetric kernel to test whether model Eq. (1) has change points in the state domain. By proper scaling and centering, our test statistic is defined as

$$t_n(x) := \frac{\sqrt{f(x)}}{\sigma(x)} \frac{1}{nh} \sum_{k=1}^{n} \tilde{K}_n(X_{k-1}, x, b) X_k.$$ (10)

In practice, since the form of $f(\cdot)$ and $\sigma(\cdot)$ are not known, we will use the kernel density estimate $f_n(x)$ and Nadaraya–Watson (NW) estimator $\sigma^2_n(x)$ to replace $f(x)$ and $\sigma^2(x)$. That is

$$f_n(x) = \frac{1}{nh} \sum_{k=1}^{n} W \left( \frac{X_{k-1} - x}{h} \right),$$ (11)

where $W(\cdot)$ is a general kernel function with $W(\cdot) \geq 0$ and $\int W(u)du = 1$, $h = h_n$ is the bandwidth sequence satisfying $h \to 0$ and $nh \to \infty$. Let $\tilde{e}_k^2 = [X_k - \mu_n(X_{k-1})]^2$ be the square of the estimated residuals, where $\mu_n(\cdot)$ is the NW estimator of $\mu(\cdot)$, then

$$\sigma^2_n(x) = \frac{1}{nhf_n(x)} \sum_{k=1}^{n} W \left( \frac{X_{k-1} - x}{h} \right) \tilde{e}_k^2.$$ (12)

Remark 3.1. Under Condition (c), Condition (d), and Condition (e), for the kernel density estimate $f_n(x)$, we have

$$\mathbb{E}f_n(x) - f(x) = f''(x)h^2\psi_W + o(h^2),$$ (13)

where $\psi_W := \int u^2W(u)du/2$ and

$$\sup_x |f_n(x) - f(x)| = \mathcal{O}_p \left( \sqrt{\frac{\log n}{nh}} + \frac{1}{\sqrt{n}} + h^2 \right).$$ (14)

See [LW10, Lemma 4.4] for the proof. For $\sigma_n(x)$, similar results are given in [LW10, Theorem 2.5]. \(\triangledown\)
Remark 3.1 provides the uniform consistency of the estimated density and variance. Then, we have the following theorem about the asymptotic properties of the proposed test statistics. First, let \( f_\eta(\cdot) \) be the density function of \( \eta \) and \( \lambda_K = \int K^2(x)dx \).

**Theorem 3.2.** Let \( l, u \in \mathbb{R} \) be fixed. Recall the piecewise formulation of Eq. (3), let \( T^\varepsilon_j \) and \( T^\varepsilon \) be the \( \varepsilon \)-neighborhood of the interval \( T_j = [a_j, a_{j+1}] \) and \( T = [l, u] \), respectively. Let \( T_a = \{a_j\} \) be the set of all change points, \( T^\varepsilon_a \) be the \( \varepsilon \)-neighborhood of \( T_a \). Assume that Condition (a)-Condition (e) hold with \( f_\eta(\cdot) \), \( \sigma(\cdot) \in C^3(T^\varepsilon) \), \( \mu_j(\cdot) \in C^3(T^\varepsilon_j) \) for some \( \varepsilon > 0 \) and \( b \) satisfies

\[
0 < \delta_1 < 1/3, \quad 0 < \delta_2 \leq 1/4, \quad nb^0 \log n = o(1),
\]

then

\[
P \left( \sqrt{nb} \frac{1}{2\lambda_K} \sup_{x \in T \cap (T^\varepsilon_a)^c} |t_n(x)| - d_n \leq \frac{z}{(2 \log \bar{b}^{-1})^{1/2}} \right) \rightarrow e^{-2e^{-z}},
\]

where \( \bar{b} := b/(u - l) \) and

\[
d_n := (2 \log \bar{b}^{-1})^{1/2} + \frac{1}{(2 \log \bar{b}^{-1})^{1/2}} \log \frac{\sqrt{K_2}}{\sqrt{2\pi}}
\]

with \( K_2 := \int_0^1 (K'(u))^2du/\lambda_K \).

**Proof.** See Appendix A. \qed

Theorem 3.2 is a general result which establishes the asymptotic theory of the test statistic. In practical implementation, we will use the density estimates \( f_n(x) \) and variance estimates \( \sigma_n(x) \) instead of \( f(x) \) and \( \sigma(x) \) to calculate \( t_n(x) \) as discussed before. Therefore, we have the following corollary.

**Corollary 3.3.** Denote \( t^*_n(x) = \sqrt{\frac{f_n(x)}{\sigma_n(x)}} \frac{1}{nb} \sum_{k=1}^n \tilde{K}_n(X_{k-1}, x, b) X_k \). Under the conditions of Theorem 3.2 and further assume the bandwidth \( h \leq b \), then the asymptotic result of Theorem 3.2 holds,

\[
P \left( \sqrt{nb} \frac{1}{2\lambda_K} \sup_{x \in T \cap (T^\varepsilon_a)^c} |t^*_n(x)| - d_n \leq \frac{z}{(2 \log \bar{b}^{-1})^{1/2}} \right) \rightarrow e^{-2e^{-z}}.
\]

Note that we add the assumption \( h \leq b \) with the purpose of ensuring the consistency of \( f_n(x) \) and \( \sigma_n(x) \) on \( T \cap (T^\varepsilon_a)^c \). Now, consider the case that there is no change point on \( \mu(\cdot) \), then we have the following similar conclusion.

**Remark 3.4.** Assume \( H_0 : M = 0 \) holds. We further suppose that \( f(\cdot), \sigma(\cdot) \in C^3(T^\varepsilon) \) and the remaining conditions of Theorem 3.2 hold. Then, \( T_a = \emptyset \), \( T^\varepsilon_a = \emptyset \), which
implies $T \cap (T_b)^c = T$. Therefore, the previous theorem reduces to

$$
\mathbb{P} \left( \sqrt{\frac{nb}{2\lambda_K}} \sup_{x \in T} |t_n(x)| - d_n \leq \frac{z}{(2 \log \bar{b}^{-1})^{1/2}} \right) \to e^{-2e^{-z}}. \qquad (19)
$$

Remark 3.4 shows that under the null hypothesis, after proper scaling and centering, our test statistic has the asymptotic extreme value distribution. The above result also provides a tool for us to detect the change points on $\mu(\cdot)$.

If we have a change point, say $a_i$, denote the jump-size of $\mu(\cdot)$ at $a_i$ as $\Delta_i$. Now consider the alternative hypothesis $H_a : M \geq 1$ with $\Delta_i \geq \delta > 0$. When $H_a$ holds, one can see that the above test has asymptotic power 1 as $n \to \infty$. In other words, with some preassigned level $\alpha \in (0, 1)$ and as $n \to \infty$, we have

$$
\mathbb{P} \left( \sup_{x \in T} |t_n(x)| \geq \sqrt{\frac{2\lambda_K}{nb}} \left( d_n - \frac{\log \{ \log(1 - \alpha)^{-1/2} \}^{1/2}}{(2 \log b^{-1})^{1/2}} \right) \right) \to 1. \qquad (20)
$$

Once the null hypothesis of no change point is rejected, then one would be interested in detecting the number of change points together with their locations, which we shall discuss in Section 3.2. Furthermore, note that $\{X_i\}$ is a stationary stochastic process and the asymptotic limits in Theorem 3.2 are the same for all choices of $X_i$. Therefore, one can use Remark 3.4 to construct a critical value by the simulation-based bootstrapping method, which will be illustrated in Section 4.1.

### 3.2. Change-point Estimation

Suppose there exist fixed number $M$ of change points on $\mu(\cdot)$, which is denoted by $l < a_1 < \cdots < a_M < u$, with jump-size $\min_{1 \leq i \leq M} \Delta_i \geq \delta > 0$. One can naturally estimate the corresponding locations of change points by maximizing the test statistics. To be more specific, we shall in the following present a procedure for change point estimation.

- For a fixed level $\alpha$, perform bootstrap procedure (see Section 4.1) to determine the critical value, say $C_{n, \alpha} > 0$.
- Find the region $T_1 := (l, u)$ where the test statistics $|t_n(x)|$ exceed $C_{n, \alpha}$.
- Starting from the interval $T_1$, find the largest $x$ of $|t_n(x)|$ that exceeds the critical value, denote its location as $\hat{a}(1)$, then rule out the interval $[\hat{a}(1) - b, \hat{a}(1) + b]$ from $T_1$ to get $T_2 := T_1 \cap [\hat{a}(1) - b, \hat{a}(1) + b]^c$.
- Repeat the previous step until all significant local maximizers are found. In other words, $|t_n(x)|$ on the remaining intervals are all below $C_{n, \alpha}$.
- Denote the number of detected change points by $\hat{M}$ and re-order the estimated change points as $l < \hat{a}_1 < \cdots < \hat{a}_{\hat{M}} < u$.

The following theorem provides the convergence result on the estimation of the above procedure for both the number $\hat{M}$ and the locations $\hat{a}_i$ of change points.
Theorem 3.5. Under the conditions of Theorem 3.2, let \( \hat{M} \) be the estimated number of change points and \( \{ \hat{a}_i \} \) be the corresponding estimates by the proposed procedure. Then for any given level \( \alpha \), we have

\[
P \left( \left\{ \hat{M} = M \right\} \cap \left\{ \max_{1 \leq i \leq M} |\hat{a}_i - a_i| < c_n \right\} \right) \to 1 - \alpha,
\]

for any \( c_n \) such that \( 1/c_n = \mathcal{O} \left( \sqrt{b \log n} / n \right) \).

Proof. See Appendix B. \( \square \)

This theorem reveals that for any given small probability \( \alpha \), with asymptotic probability \( 1 - \alpha \), our proposed procedure will correctly detect all the change points within a \( c_n \) range. It is important to mention that the range \( c_n = \mathcal{O}(\sqrt{b \log n} / n) \) is smaller than \( n^{-1/2} \). It can also be seen as a product of \( \sqrt{\log n} \) and the optimal convergence rate \( (\sqrt{b/n}) \) of time-domain change-point estimators, which was established in [M92]. Hence, we conjecture that our rate \( c_n \) is nearly optimal for state-domain change-point detection.

4. Practical Implementation

4.1. The bootstrap procedure. It is well known that the convergence rate of the Gumbel distribution in Theorem 3.2 is slow and a very large sample size would be needed for the approximation to be reasonably accurate. To overcome this problem, we shall consider the following simulation-based bootstrapping procedure that can help improve the finite-sample performance of the proposed test.

- First denote \( \Pi_n = \sup_{x \in T} |t_n^*(x)| \), and generate i.i.d. standard normal random variables \( U_k \), \( k = 0, ..., n \).
- Compute the corresponding quantities as

\[
\Pi_n^* = \sup_{x \in T} \left| \frac{g(x)}{nb} \sum_{k=1}^n K_n(U_{k-1}, x, b) U_k \right|,
\]

where \( g(x) \) is the density of a standard normal random variable.

With proper scaling and centering, we know that \( \Pi_n^* \) and \( \Pi_n \) have the same asymptotic Gumbel distribution. Therefore, the cutoff value \( \gamma_{1-\alpha} \) which is the \((1-\alpha)\)th quantile of \( \Pi_n \), can be estimated by calculating the empirical \((1-\alpha)\)th quantile \( q_{1-\alpha} \) of \( \Pi_n^* \) with a large number of replications by the above method. Clearly, we will reject the null hypothesis at level \( \alpha \in (0, 1) \) if \( \Pi_n > q_{1-\alpha} \). When implementing the procedure described in Section 3.2 for estimating change points, we also suggest using \( q_{1-\alpha} \) as the critical value \( C_{n,\alpha} \) to find the detection region.


4.2. Bandwidth selection. The bandwidth used in $f_n(x)$ can be chosen based on classic bandwidth selectors of kernel density. However, the choice of bandwidth $b$ for test statistic $t^*_n(x)$ and $h$ for the estimated variance $\sigma^2_n(x)$ can be quite nontrivial and are usually of practical interest. In this paper, we adopt the standard leave-one-out cross-validation criterion for bandwidth selection suggested by [RS91]:

$$CV(b) = \frac{1}{n} \sum_{k=1}^{n} \left[ X_{k+1} - \mu_n^{(-k)}(X_k) \right]^2,$$

$$CV(h) = \frac{1}{n} \sum_{k=1}^{n} \left[ (X_{k+1} - \mu_n(X_k))^2 - \sigma^2_n^{(-k)}(X_k) \right]^2 (24)$$

where $\mu_n^{(-k)}(X_k)$ and $\sigma^2_n^{(-k)}(X_k)$ are the kernel estimators of $\mu$ and $\sigma^2$ computed with all measurements with the $k$th subject deleted, respectively. For example, a cross-validation bandwidth $\hat{b}$ can be obtained by minimizing $CV(b)$ with respect to $b$, i.e., $\hat{b} = \arg\min_{b \in B} CV(b)$, where $B$ is the allowable range of $b$. The bandwidth selection for $h$ is similar.

5. Simulation Study

In this section, we carry out Monte Carlo simulations to examine the finite-sample performance of our proposed test. Throughout the numerical experiments, the Epanechnikov kernel $W(x) = 0.75(1 - x^2)1(|x| \leq 1)$ is used for estimating density and variance and results based on other commonly used kernels such as rectangle kernel and tricube kernel are similar. Besides, we adopt the higher-order kernel function with the form $K(x) = b\tilde{W}(x) - a\tilde{W}(\sqrt{a}x)$ in the expression of $\tilde{K}_n$, where $\tilde{W}(x)$ is the kernel function on $[0,1]$ by shifting and scaling $W(x)$. From Theorem 3.2, one can see that the power of our test increases as $\lambda_K$ decreases. As a result, we aim to maximize the quantity $Q(a,b) = \int_{0}^{1} K(x)dx$ with the constraints $\int_{0}^{1} xK(x)dx = 0$ to choose $a$ and $b$. It turns out that $Q(a,b)$ is maximized at $a = 0.34$ and $b = \frac{2}{\sqrt{0.34 - 0.34}}$. Hence, we will use $K(x) = \frac{2}{\sqrt{0.34 - 0.34}}[\tilde{W}(x) - 0.34\tilde{W}(\sqrt{0.34}x)]$ in our simulations and data analysis.

5.1. Accuracy of bootstrap. We will perform Monte Carlo simulations to study the accuracy of the proposed bootstrap procedure for finite samples $n = 500$ and $800$. Here, we aim to test the null hypothesis $H_0$ of no change point in the regression function. The number of replications is fixed at 1000 and the number of bootstrap samples is $B = 2000$ at each replication.

To guarantee the stationarity of the process $\{X_i\}$, we need to restrict the scale coefficient of the regressor $X_i$ less than one, see for [FY08, Section 2.1]. Therefore, we investigate the following three scenarios of our model:
• Model A:

\[
\mu(x) = \begin{cases} 
0.4x^3, & |x| \leq 1, \\
0.4, & x > 1, \\
-0.4, & x < -1.
\end{cases} \tag{25}
\]

\[
\sigma(x) = 1.5e^{-0.5x^2} \tag{26}
\]

• Model B:

\[
\mu(x) = \frac{0.3e^x}{1 + e^x}; \tag{27}
\]

\[
\sigma(x) = \begin{cases} 
0.7(1 + x^2), & |x| \leq 1, \\
1.4, & \text{otherwise.}
\end{cases} \tag{28}
\]

• Model C:

\[
\mu(x) = 0.2e^{-0.5x^2}, \quad \sigma(x) = \frac{1.5e^x}{1 + e^x}. \tag{29}
\]

At nominal significant levels \(\alpha = 0.05\) and 0.1, the simulated Type I error rates are reported below for the null hypothesis \(H_0\). From Table 1, one can see that the performances of our bootstrap are reasonably accurate for different sample sizes. Furthermore, when the sample size increases, the simulated Type I errors are relatively close to nominal levels \(\alpha\).

Table 1. Simulated type I error rates under \(H_0\).

| \(\alpha\) | \(n = 500\) | \(n = 800\) |
|---|---|---|
| 0.05 | 0.072 | 0.041 | 0.036 |
| 0.10 | 0.134 | 0.092 | 0.071 |

5.2. **Power of hypothesis testing.** In this subsection, we consider the statistical power of our test under some given alternatives. Here, we consider the following two types of alternatives with a change point of size \(\delta\):

• Model D:

\[
\mu(x) = \begin{cases} 
0.5e^{-x^2}, & x < 0, \\
0.5e^{-x^2} - \delta, & x \geq 0.
\end{cases} \tag{30}
\]

\[
\sigma(x) = e^{-0.5x^2}. \tag{31}
\]
• Model E:

\[
\mu(x) = \begin{cases} 
0.3 - \delta, & x < 0, \\
0.3, & x \geq 0.
\end{cases}
\]

\[
\sigma(x) = \frac{e^x}{1 + e^x}.
\]

Figure 1. Statistical power for testing change point for Model D.

In the alternatives, we choose the size \( \delta \) of the change point from 0 to 1.6 for model D and from 0 to 1 for model E at location \( x = 0 \). In each model, we focus on testing the statistical power under nominal level 0.05 and 0.1 with the sample size \( n = 800 \) based on 1000 replications. The power curves for the above models are plotted in Fig. 1 and Fig. 2, respectively. From them, we find that our testing procedures are quite robust and have strong statistical power as \( \delta \) increases.

5.3. Accuracy for estimating the locations of change points and their number. According to the algorithm in Section 3.2, we focus on estimating the changepoint number and their corresponding locations based on 1000 realizations with sample sizes \( n = 500 \) and 800. Consider the following two cases:
• Case 1: A single change point.

\[ \mu(x) = \begin{cases} 
0.7e^{-x^2}, & x < 0, \\
0.7e^{-x^2} - 1.6, & x \geq 0.
\end{cases} \]  
\[ \sigma(x) = e^{-0.5x^2}. \]  

• Case 2: Two change points.

\[ \mu(x) = \begin{cases} 
0.8x + 0.8, & x < -0.3, \\
-1, & -0.3 \leq x < 0, \\
-0.2x + 0.5, & x \geq 0
\end{cases} \]  
\[ \sigma(x) = \frac{e^x}{1 + e^x}. \] 

The estimators for the locations of change points are compared in terms of their mean absolute deviation errors (MADE) and mean squared errors (MSE). We also report the simulated percentage of correctly estimating the number of change points. The above results are listed in Table 2. Due to the fairly small values of MADE and MSE, one can see that the estimated locations by our approach are quite accurate. Furthermore, in both two cases, as the sample size increases, the percentage for correctly estimating the number of change points becomes larger.
Table 2. Estimation for change-point locations and correct percentage for change-point number.

| Case 1 | n  | MADE  | MSE  | Percentage |
|--------|----|-------|------|------------|
| \( \vartheta = 0 \) | 500 | 0.0195 | 0.0014 | 93.77% |
| | 800 | 0.0134 | 0.0006 | 94.51% |
| Case 2 | n  | MADE  | MSE  | Percentage |
| \( \vartheta_1 = -0.3 \) | 500 | 0.0508 | 0.0043 | 86.59% |
| \( \vartheta_2 = 0 \) | 800 | 0.0496 | 0.0042 | 89.80% |
| \( \vartheta_1 = -0.3 \) | 500 | 0.0386 | 0.0028 | 89.80% |
| \( \vartheta_2 = 0 \) | 800 | 0.0362 | 0.0024 | 89.80% |

Note: \( \vartheta = 0 \), true change point 0 for Case 1; \( \vartheta_1 = -0.3 \) and \( \vartheta_2 = 0 \), true change points \(-0.3\) and 0 for Case 2; MADE, mean absolute deviation error; MSE, mean squared error.

6. Application

Due to the fact that almost all continuous-time models are observed at discrete times, one often uses the following discretized version of Eq. (2)

\[
Y_i = \mu(X_i) + \sigma(X_i)\eta_i,
\]

(38)

where \( Y_i \) denotes the daily return or log-return and \( \eta_i \) are i.i.d. random variables. There is a huge amount of literature on this model. For example, [FY98] considered the nonparametric estimation of \( \mu(\cdot) \) and \( \sigma^2(\cdot) \). Other contributions on nonparametric estimation of Eq. (38) include [JK97], [BP03] and references therein. Notice that our model Eq. (1) is the autoregressive version of Eq. (38). In the following, we will analyze a treasury bill rates data example by using Eq. (38) for detecting change points over the state domain.

Here, we consider the U.S. six-month treasury yield rates data from January 2nd, 1990 to July 31st, 2009. The data can be downloaded from the U.S. Treasury department’s website http://www.ustreas.gov/. It contains 4900 daily rates and its plot is given in Fig. 3. In particular, [LW10] constructed the simultaneous confidence bands (SCB) for drift and volatility functions for this data. Following the idea of [LW10], let \( X_i \) be the rate at day \( i = 1, ..., 4900 \) and \( Y_i = X_{i+1} - X_i \) be the daily return of the model Eq. (38).

Now, we apply the proposed method to test whether the regression function \( \mu(\cdot) \) contains any change point. Note that our test and estimation procedures will not be accurate at regions where data are sparse; that is regions where the density of \( X_i \) is small. As a result, we restrict the whole range of \( x \) from \([0,14.8,49]\) to the interval with data having relative large densities. We choose \( T = [l, u] = [3.6, 6.4] \), which includes 55.57% of the daily yield rates \( X_i \). According to the leave-one-out cross-validation criterion, the selected bandwidths \( b \) and \( h \) are 0.13 and 0.042, respectively. Through
the practical implementation in Section 3.2, we calculate the empirical 99\% quantile of \( \Pi_n^\ast \) with 10000 bootstraps, which is \( C_{n,\alpha} = 4.38 \). Next, we focus on investigating the behaviour of the test statistics in our data, which is shown in Fig. 4. It is easy to see that there indeed exists a change point whose corresponding absolute value of the test statistic exceeds \( C_{n,\alpha} \). Therefore, it should be treated as a change point as suggested by our analysis. Based on the estimation procedure in Section 3.2, this change point is estimated at \( x = 4.2 \).

For comparison, we also nonparametrically fit \( \mu(x) \) pretending that there is no change point. The corresponding estimated regression functions \( \mu_n(x) \) over \([3.6,6.4]\) are plotted on the left hand side of Fig. 5. The right hand side of Fig. 5 shows the fitted drift function \( \mu_n(x) \) with the knowledge of the change point. It is obvious to see that a large jump exists at the change point \( x = 4.2 \), which clearly shows that the relationship between \( Y_i \) and \( X_i \) changes drastically at this point. Observe that without this change point knowledge, our understanding of the relationship between \( Y_i \) and \( X_i \) will be quite different as shown on the left hand side of Fig. 5. This particular example demonstrates that our state-domain change point detection is of crucial importance.

In addition, our analysis suggest that the SCB for the regression function \( \mu(\cdot) \) in [LW10] may not be accurate on the whole range \([3.6,6.4]\) and it is more reasonable to construct the SCB on \([3.6,4.2]\) and \([4.2,6.4]\) separately.

**Figure 3.** U.S. six-month treasury yield rates data from January 2nd, 1990 to July 31st, 2009.
Furthermore, the detected change point itself may have some economic significance in the financial market of the U.S. From Fig. 1, it seems that the six-month treasury bill rates change frequently in the region $[3, 6]$ and $x = 4.2$ means the yield rate is quite high. Under this circumstance, investors tend to sell stocks to buy treasury bills due to their low risk and high return. However, the stock market falls and the interest rates on consumer and business loans rise gradually, which in turn will limit
the return of enterprises. Therefore, as the market analysts said, high yield rates of treasury bills is likely to indicate a potential financial crisis. In addition, as we all know, the ten-year treasury yield rate is always viewed as a benchmark for many financial instruments and the rate 3% is an important “psychological” level that makes market anxious and even leads to a potential crisis. From CNN business news, we find that if short-term rates move higher than long-term rates, it will create an inverted yield curve that often has happened just ahead of recessions. Hence with the above arguments, we conclude that our change point \( x = 4.2 \) for six-month rates to some extent could mean a potential warning sign for the financial market.

References

[BP03] F. M. Bandi and P. C. Phillips. “Fully nonparametric estimation of scalar diffusion models”. *Econometrica* 71.1 (2003), pp. 241–283.

[BR73] P. J. Bickel and M. Rosenblatt. “On some global measures of the deviations of density function estimates”. *The Annals of Statistics* (1973), pp. 1071–1095.

[CP00] D. A. Chapman and N. D. Pearson. “Is the short rate drift actually nonlinear?” *The Journal of Finance* 55.1 (2000), pp. 355–388.

[DJ95] S. N. Durlauf and P. A. Johnson. “Multiple regimes and cross-country growth behaviour”. *Journal of applied econometrics* 10.4 (1995), pp. 365–384.

[Eng82] R. F. Engle. “Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation”. *Econometrica. Journal of the Econometric Society* 50.4 (1982), pp. 987–1007. issn: 0012-9682.

[ES94] R. Eubank and P. Speckman. “Nonparametric estimation of functions with jump discontinuities”. *Lecture Notes-Monograph Series* (1994), pp. 130–144.

[Fre75] D. A. Freedman. “On tail probabilities for martingales”. *The Annals of Probability* (1975), pp. 100–118.

[FY08] J. Fan and Q. Yao. *Nonlinear time series: nonparametric and parametric methods*. Springer Science & Business Media, 2008.

[FY98] J. Fan and Q. Yao. “Efficient estimation of conditional variance functions in stochastic regression”. *Biometrika* 85.3 (1998), pp. 645–660.

[FZ03] J. Fan and C. Zhang. “A re-examination of diffusion estimators with applications to financial model validation”. *Journal of the American Statistical Association* 98.461 (2003), pp. 118–134.

[GHK99] I. Gijbels, P. Hall, and A. Kneip. “On the estimation of jump points in smooth curves”. *Annals of the Institute of Statistical Mathematics* 51.2 (1999), pp. 231–251. issn: 0020-3157.
REFERENCES

[GLQ07] I. Gijbels, A. Lambert, and P. Qiu. “Jump-preserving regression and smoothing using local linear fitting: a compromise”. Annals of the Institute of Statistical Mathematics 59.2 (2007), pp. 235–272.

[HO81] V Haggan and T Ozaki. “Modelling nonlinear random vibrations using an amplitude-dependent autoregressive time series model”. Biometrika 68.1 (Apr. 1981), pp. 189–196. issn: 0006-3444.

[JK97] G. J. Jiang and J. L. Knight. “A nonparametric approach to the estimation of diffusion processes, with an application to a short-term interest rate model”. Econometric Theory 13.5 (1997), pp. 615–645.

[LW10] W. Liu and W. B. Wu. “Simultaneous nonparametric inference of time series”. The Annals of Statistics 38.4 (2010), pp. 2388–2421.

[M92] H.-G. Müller. “Change-points in nonparametric regression analysis”. The Annals of Statistics (1992), pp. 737–761.

[MPQ00] M. M. McConnell and G. Perez-Quiros. “Output fluctuations in the United States: What has changed since the early 1980’s?” American Economic Review 90.5 (2000), pp. 1464–1476.

[MS97] H.-G. Müller and K.-S. Song. “Two-stage change-point estimators in smooth regression models”. Statistics & Probability Letters 34.4 (1997), pp. 323–335. issn: 0167-7152.

[Ros76] M Rosenblatt. “On the maximal deviation of k-dimensional density estimates”. The Annals of Probability 4.6 (1976), pp. 1009–1015.

[RS91] J. A. Rice and B. W. Silverman. “Estimating the mean and covariance structure nonparametrically when the data are curves”. Journal of the Royal Statistical Society: Series B (Methodological) 53.1 (1991), pp. 233–243.

[Sta97] R. Stanton. “A nonparametric model of term structure dynamics and the market price of Interest rate risk”. The Journal of Finance 52.5 (1997), pp. 1973–2002.

[SW07] X. Shao and W. B. Wu. “Asymptotic spectral theory for nonlinear time series”. The Annals of Statistics 35.4 (2007), pp. 1773–1801.

[TL80] H. Tong and K. S. Lim. “Threshold autoregression, limit cycles and cyclical data (with discussion)”. Journal of the Royal Statistical Society: Series B(Statistical Methodology) 42.3 (1980), pp. 245–292. issn: 1369-7412.

[Ton90] H Tong. Nonlinear Time Series Analysis: A Dynamics Approach. 1990.

[TT94] G. C. Tiao and R. S. Tsay. “Some advances in non-linear and adaptive modelling in time-series”. Journal of forecasting 13.2 (1994), pp. 109–131.

[Wan95] Y. Wang. “Jump and Sharp Cusp Detection By Wavelets”. Biometrika 82 (1995), pp. 385–397.
[Wu05] W. B. Wu. “Nonlinear system theory: another look at dependence”. Proceedings of the National Academy of Sciences of the United States of America 102.40 (2005), pp. 14150–14154.

[Zai87] A. Y. Zaitsev. “On the Gaussian approximation of convolutions under multidimensional analogues of SN Bernstein’s inequality conditions”. Probability theory and related fields 74.4 (1987), pp. 535–566.

[Zha16] T. Zhang. “Testing for jumps in the presence of smooth changes in trends of nonstationary time series”. Electronic Journal of Statistics 10.1 (2016), pp. 706–735. ISSN: 1935-7524.

[ZW08] Z. Zhao and W. B. Wu. “Confidence bands in nonparametric time series regression”. The Annals of Statistics 36.4 (2008), pp. 1854–1878.
A. Proof of Theorem 3.2

First, we substitute $X_i = \mu(X_{i-1}) + \sigma(X_{i-1})\eta_i$ to $t_n(x)$ and separate the terms involving $K$ and $K^*$. We first focus on the term involving $K$ only. That is,

\[
\frac{1}{nbw(x, b)} \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) [\mu(X_{k-1}) + \sigma(X_{k-1})\eta_k],
\]

\[
= \frac{1}{nbw(x, b)} \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) [\mu(X_{k-1}) - \mu(x)]
\]

\[
+ \frac{1}{nbw(x, b)} \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) \mu(x)
\]

\[
+ \frac{1}{nbw(x, b)} \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) \sigma(X_{k-1})\eta_k.
\]

Next it is easy to see that by the definition of $w(x, b)$, the second term of the decomposition on the right hand side of Eq. (39) equals $\mu(x)$. For the first term of the decomposition in Eq. (39), according to [LW10, Lemma 5.2], uniformly over $x$, we have that

\[
\frac{1}{nbw(x, b)} \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) [\mu(X_{k-1}) - \mu(x)]
\]

\[
= b^2 \psi_K [\mu''(x)f(x) + 2\mu'(x)f'(x)]
\]

\[
\cdot \frac{\mathbb{E}[w(x, b)]}{\mathbb{E}[w(x, b)]} + O_p(b^3) + O_p(\tau_n)
\]

\[
= b^2 \psi_K [\mu''(x)f(x) + 2\mu'(x)f'(x)]
\]

\[
\cdot \frac{\mathbb{E}[w(x, b)]}{\mathbb{E}[w(x, b)]} + b^2O_p(\sqrt{\log n/nb})
\]

\[
+ O_p \left( \sqrt{\frac{b \log n}{n}} + b^3 + \frac{b}{n} \sqrt{\sum_{k=-n}^{\infty} (\Theta_{n+k} - \Theta_k)^2} \right)
\]

\[
= b^2 \psi_K [\mu''(x)f(x) + 2\mu'(x)f'(x)]
\]

\[
\cdot \frac{\mathbb{E}[w(x, b)]}{\mathbb{E}[w(x, b)]} + O_p \left( \sqrt{\frac{b \log n}{n}} + b^3 \right),
\]

where $\tau_n := \sqrt{\frac{b \log n}{n}} + b^4 + \frac{b}{n} \sqrt{\sum_{k=-n}^{\infty} (\Theta_{n+k} - \Theta_k)^2}$ comes from [ZW08, Lemma 2(ii)], and in the last equality we have applied the assumptions on $b$ and $\sum_{k=-n}^{\infty} (\Theta_{n+k} - \Theta_k)^2$ to get $\frac{b}{n} \sqrt{\sum_{k=-n}^{\infty} (\Theta_{n+k} - \Theta_k)^2} = O(\sqrt{b \log n/n})$. 
For the third term of the decomposition in Eq. (39), according to [LW10, Lemma 5.3], uniformly over \( x \), we have

\[
\frac{1}{n bw(x, b)} \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) \sigma(X_{k-1}) \eta_k
\]

\[
= \frac{1}{nb} \mathbb{E}[w(x, b)] + O_{\mathbb{P}}(\sqrt{\log n/nb}) \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) \sigma(x) \eta_k + O_{\mathbb{P}} \left( \sqrt{\frac{b \log n}{n}} \right)
\]

\[
= \frac{1}{nb} f(x) + O_{\mathbb{P}}(b^2 + \sqrt{\log n/nb}) \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) \sigma(x) \eta_k + O_{\mathbb{P}} \left( \sqrt{\frac{b \log n}{n}} + b^3 \right).
\]  

(41)

Similarly, we can compute the orders for the decomposition of the term involving \( K^* \) and get \( t_n(x) \) by the differences. Note that many terms such as \( \mu(x) \) in the second term and \( O(b^2) \) term in the first term cancel out. Therefore, overall it can be easily verified that

\[
t_n(x) = \sqrt{\frac{f(x)}{\sigma(x)}} \frac{1}{nb f(x)} \sum_{k=1}^{n} \tilde{K} \left( \frac{X_{k-1} - x}{b} \right) \sigma(x) \eta_k + O_{\mathbb{P}} \left( \sqrt{\frac{b \log n}{n} + b^3} \right)
\]

\[
+ O_{\mathbb{P}}(b^2 + \sqrt{\log n/nb})O_{\mathbb{P}}(\sqrt{\log n}),
\]

where \( \tilde{K}(\cdot) \) is anti-symmetric kernel defined by

\[
\tilde{K}(u) := K(u) - K^*(u).
\]  

(42)

Now it suffices to show

\[
\mathbb{P} \left( \sqrt{\frac{nb}{2 \lambda_K}} \sup_{x \in T} \frac{1}{\sqrt{f(x)}} |M_n(x) - M^*_n(x)| - d_n \leq \frac{z}{(2 \log b^{-1})^{1/2}} \right) \rightarrow e^{-2e^{-z}},
\]  

(44)

where \( M_n(x) := \frac{1}{nb} \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) \eta_k \) and \( M^*_n(x) := \frac{1}{nb} \sum_{k=1}^{n} K^* \left( \frac{X_{k-1} - x}{b} \right) \eta_k \).

Note that we assumed \( \mathbb{E}\eta_i = 0 \) and \( \mathbb{E}\eta_i^2 = 1 \). Next, we define a truncated version of \( \eta_i \) by

\[
\bar{\eta}_i := \eta_i \mathbb{1}\{ |\eta_i| \leq (\log n)^{12/(p-2)} \} - \mathbb{E} \left[ \eta_i \mathbb{1}\{ |\eta_i| \leq (\log n)^{12/(p-2)} \} \right].
\]  

(45)

Recalling that we have defined the notation \( \xi_{k_1, k_2} := (\eta_{k_1}, \ldots, \eta_{k_2}) \), we next define \( \bar{M}_n(x) \) using \( m \)-dependent conditional expectations

\[
\bar{M}_n(x) := \frac{1}{nb} \sum_{k=1}^{n} \bar{\eta}_k \sigma^2 \left\{ \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) \mid \xi_{k-1, k-1} \right] - \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) \mid \xi_{k-2, k-2} \right] \right\}.
\]  

(46)

where \( \sigma^2 := \mathbb{E}\bar{\eta}_i^2 \) and \( m := \lfloor n^\tau \rfloor \) with \( \tau < 1 - \delta_1 \).
Then we show in the following that we can approximate $M_n(x)$ by $\tilde{M}_n(x)$.

**Lemma A.1.**

$$\mathbb{P}\left(\sqrt{nb}\sup_{x \in T}\left|M_n(x) - \tilde{M}_n(x)\right| \geq 8(\log n)^{-2}\right) = o(1).$$  \hspace{1cm} (47)

**Proof.** See Appendix A.1.  \hfill $\Box$

Observing that, since $K(\cdot)$ is supported on $[0, 1]$, one of the following two terms must be zero:

$$\mathbb{E}\left[K\left(\frac{X_{k-1} - x}{b}\right) \mid \xi_{k-1,k-m}\right] - \mathbb{E}\left[K\left(\frac{X_{k-1} - x}{b}\right) \mid \xi_{k-2,k-m}\right],$$

$$\mathbb{E}\left[K^*\left(\frac{X_{k-1} - x}{b}\right) \mid \xi_{k-1,k-m}\right] - \mathbb{E}\left[K^*\left(\frac{X_{k-1} - x}{b}\right) \mid \xi_{k-2,k-m}\right].$$  \hspace{1cm} (48)

Hence, defining $\hat{M}_n^*(x)$ similarly as $\tilde{M}_n^*(x)$ using $K^*(\cdot)$ instead of $K(\cdot)$, we focus on the following term

$$\hat{M}_n(x) := \sqrt{\frac{nb}{2\lambda_f(x)}} \left[\tilde{M}_n(x) - \tilde{M}_n^*(x)\right]$$

$$= \frac{1}{\sqrt{nb\lambda_f(x)}} \sum_{k=1}^{n} \frac{\hat{\eta}_k}{\hat{\sigma}^2} \left\{\mathbb{E}\left[K\left(\frac{X_{k-1} - x}{b}\right) \mid \xi_{k-1,k-m}\right] - \mathbb{E}\left[K\left(\frac{X_{k-1} - x}{b}\right) \mid \xi_{k-2,k-m}\right]\right\}.$$  \hspace{1cm} (49)

Next, we prove a lemma on the covariance structure of $\{\hat{M}_n(x)\}$ which is needed for the results in [BR73].

**Lemma A.2.** Define $r(s) = \int K(x)K(x + s)dx/\lambda_K, \hat{r}(s) := \mathbb{E}\hat{M}_n(x)\hat{M}_n(x + s)$ and $\tilde{r}(s) := \int K(x)\tilde{K}(x + s)dx/\lambda_{\tilde{K}}$, then as $s \to 0$, we have

$$\tilde{r}(s) = 1 - \tilde{K}_2|s|^2 + o(|s|^2), \hspace{0.5cm} \hat{r}(s) = r(s) + o(|s|^2),$$  \hspace{1cm} (50)

where $\tilde{K}_2 := \int_{-1}^{1}(\tilde{K}'(x))^2dx/(2\lambda_{\tilde{K}})$. Finally, we have

$$\hat{r}(s) = \tilde{r}(s) + O(b).$$  \hspace{1cm} (51)

**Proof.** See Appendix A.2.  \hfill $\Box$

Finally, we can complete the proof using similar techniques as in [LW10, Proof of Lemma 4.5] to show the following lemma.

**Lemma A.3.**

$$\mathbb{P}\left(\sup_{x \in T}\left|\tilde{M}_n(x)\right| - d_n \leq \frac{z}{(2\log b^{-1})^{1/2}}\right) \to e^{-2e^{-z}}.$$  \hspace{1cm} (52)

**Proof.** See Appendix A.3.  \hfill $\Box$
A.1. Proof of Lemma A.1. First, according to [LW10, Lemma 5.1], we have

$$\mathbb{P} \left( \sqrt{n} b \sup_{x \in T} \left| \frac{1}{nb} \sum_{k=1}^{n} K \left( \frac{X_{k-1} - x}{b} \right) (\eta_k - \tilde{\eta}_k) \right| \geq 3(\log n)^{-2} \right) = o(1), \quad (53)$$

which implies we can approximate $M_n(x)$ by replacing $\eta_k$ with $\tilde{\eta}_k$ in the definition of $M_n(x)$.

Next, let $m = [n^\tau]$ where $\tau < 1 - \delta_1$, we write $K \left( \frac{X_{k-1} - x}{b} \right)$ as a sum of three terms

$$K \left( \frac{X_{k-1} - x}{b} \right) = \left\{ K \left( \frac{X_{k-1} - x}{b} \right) - \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) | \xi_{k-1,k-m} \right] \right\} + \left\{ \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) | \xi_{k-1,k-m} \right] - \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) | \xi_{k-2,k-m} \right] \right\} + \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) | \xi_{k-2,k-m} \right]. \quad (54)$$

Then our assumptions on physical dependence measure imply that the first term of the right hand side of Eq. (54) becomes very small for large $m$. In order to rigorously prove this fact, defining

$$Z_k(x) = \tilde{\eta}_k \left\{ K \left( \frac{X_{k-1} - x}{b} \right) - \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) | \xi_{k-1,k-m} \right] \right\}, \quad (55)$$

we first approximate $\sum_{k=1}^{n} Z_k(x)$ by the skeleton process $\sum_{k=1}^{n} Z_k(x_j), 1 \leq j \leq q_n$, where $q_n = [n^2/b]$ and $x_j = j/(bq_n)$. Following the same arguments as in [LW10, Proof of Lemma 4.2], we have

$$\sup_{x_{j-1} \leq x \leq x_j} \left| \sum_{k=1}^{n} (Z_k(x) - Z_k(x_j)) \right| = o_\mathbb{P} \left( \sqrt{n} b / (\log b)^{-1} \right). \quad (56)$$

Next, we show $\sup_{x \in T} \mathbb{E} |Z_k(x)|$ exponentially decays with $m$. We consider two cases $|X_{k-1} - \mathbb{E}(X_{k-1} | \xi_{k-1,k-m})| \geq \rho_1^m$ and $|X_{k-1} - \mathbb{E}(X_{k-1} | \xi_{k-1,k-m})| < \rho_1^m$, where $\rho_1 = \frac{1+\rho}{2}$. Using the assumption $\theta_{n,p} = O(\rho^n)$, we have

$$\sup_{x \in \mathbb{R}} \mathbb{E} |Z_k(x)| \leq C \mathbb{P}( |X_{k-1} - \mathbb{E}(X_{k-1} | \xi_{k-1,k-m})| \geq \rho_1^m ) + C \sup_{x \in \mathbb{R}} \mathbb{P} \left( \left| \frac{X_{k-1} - x}{b} \in [-1, 1] \right| \right) \leq O(\rho/\rho_1)^m + O(\rho_1^m/b). \quad (57)$$
Now, we can show the maximum of the skeleton process over \( \{x_j\}, j = 1, \ldots, q_n \) is small. Recall that \( m \) is a polynomial of \( n \), then we have

\[
\mathbb{P} \left( \max_{1 \leq j \leq q_n} \left| \sum_{k=1}^{n} Z_k(x_j) \right| \geq \sqrt{n}b(\log b^{-1})^{-2} \right) \leq q_n \frac{\max_{1 \leq j \leq q_n} \mathbb{E} \left| \sum_{k=1}^{n} Z_k(x_j) \right|}{\sqrt{n}b(\log b^{-1})^2} \leq \frac{nq_n}{\sqrt{n}b(\log b^{-1})^2} \sup_{x \in T} \mathbb{E} |Z_k(x)| = o(1). \tag{58}
\]

Next, we show the third term of the decomposition of \( K \left( \frac{X_{k-1} - x}{b} \right) \) in Eq. (54) can also be ignored. In order to show this, we define

\[
N_n(x) = \frac{1}{nb} \sum_{k=1}^{n} \hat{\eta}_k \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) | \xi_{k-1,k-m} \right]. \tag{59}
\]

Using the same argument as in [LW10, Proof of Lemma 4.2], we can approximate \( N_n(x) \) by its skeleton process, since \( \sup_{x_{j-1} \leq x \leq x_j} |N_n(x) - N_n(x_j)| = o_\mathbb{P}((\log n)^{-2}) \). We first approximate \( \sup_{x} |N_n(x)| \) by the maximum over the skeleton process. Then we have \( \mathbb{P} \left( \max_{1 \leq j \leq q_n} |N_n(x_j)| \geq (\log n)^{-2} \right) = o(1) \) using Freedman’s inequality for martingale differences [Fre75]. Therefore, we can approximate \( M_n(x) \) by

\[
\frac{1}{nb} \sum_{k=1}^{n} \frac{\hat{\eta}_k}{\mathbb{E}\hat{\eta}_k^2} \left\{ \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) | \xi_{k-1,k-m} \right] - \mathbb{E} \left[ K \left( \frac{X_{k-1} - x}{b} \right) | \xi_{k-2,k-m} \right] \right\}. \tag{60}
\]

Finally, since \( |1 - \mathbb{E}\hat{\eta}_k^2/\mathbb{E}\hat{\eta}_k^2| = \mathcal{O}((\log n)^{-12/(p-2)}) \), we can replace \( \hat{\eta}_k/\mathbb{E}\hat{\eta}_k^2 \) by \( \hat{\eta}_k/\hat{\sigma}^2 \), which leads to the definition of \( \tilde{M}_n(x) \).

**A.2. Proof of Lemma A.2.** Note that since \( \tilde{K}'(0) > 0 \), we have \( \int K(u)K^*(u \pm s)du = \mathcal{O}(\int_0^{|s|} x(|s| - x)dx) = \mathcal{O}(|s|^3) = o(|s|^2) \). Then by the definition of \( \tilde{r}(s) \), using \( \lambda_{\tilde{K}} = 2\lambda_K \), we have

\[
\tilde{r}(s) = \int \tilde{K}(v)\tilde{K}(v + s)dv/\lambda_{\tilde{K}} \tag{61}
\]

\[
= \frac{1}{\lambda_{\tilde{K}}} \int [K(v) - K^*(v)] [K(v + s) - K^*(v + s)] dv \tag{62}
\]

\[
= \frac{1}{2\lambda_K} \left[ \int K(v + s)K(v)dv + \int K^*(v + s)K^*(v)dv \right. \tag{63}
\]

\[
- \int K^*(v + s)K(v)dv - \int K(v + s)K^*(v)dv \right] \tag{64}
\]

\[
= r(s) + o(|s|^2). \tag{65}
\]
therefore, we have proved \( r(s) = r(s) + o(|s|^2) \).

Next, according to [BR73, Theorems B1 and B2], we have \( r(s) = 1 - K_2|s|^2 + o(|s|^2) \). Note that

\[
\hat{K}_2 = \int_{-1}^{1} (\hat{K}'(x))^2 dx / (2\lambda_K) = \frac{1}{2} \int_{-1}^{1} (\hat{K}'(x))^2 dx / (2\lambda_K) = \frac{1}{2} (2K_2) = K_2. \quad (66)
\]

This implies \( \tilde{r}(s) = 1 - \hat{K}_2|s|^2 + o(|s|^2) \), which can also be obtained directly from [BR73, Theorems B1 and B2].

Finally, we show \( \hat{r}(s) = \tilde{r}(s) + O(b) \). Since \( \{\eta_k\} \) are independent and \( \mathbb{E}\eta_k = 0 \). Also, we know \( |1 - \hat{\sigma}^2| = |1 - \mathbb{E}\eta_k^2/\mathbb{E}\eta_k^2| = O((\log n)^{-12/(p-2)}) \). Then, using \( |f(v + s) - \sqrt{f(t)f(s)}| = O(b) \) uniformly over \( |s - t| \leq 2b \) and \( |v| \leq 2b \), we have

\[
\mathbb{E}\hat{M}_n(t) \hat{M}_n(s) = \frac{1}{n b \lambda_K \hat{\sigma}^2} \int \frac{1}{\sqrt{f(t)f(s)}} \sum_{k=1}^{n} \left\{ \mathbb{E}\left[ \hat{K} \left( \frac{X_{k-1} - t}{b} \right) \hat{K} \left( \frac{X_{k-1} - s}{b} \right) \right] + O(b^2) \right\} \\
= \frac{1}{b \lambda_K \hat{\sigma}^2} \int \frac{1}{f(v + s) + O(b)} \hat{K} \left( \frac{v - t + s}{b} \right) \hat{K} \left( \frac{v}{b} \right) f(v + s) dv + O(b) \\
= \frac{1}{\lambda_K} \int \hat{K} \left( v - t + s \right) \hat{K} \left( v \right) dv + O(b) = \tilde{r}(t - s) + O(b). \quad (67)
\]

A.3. **Proof of Lemma A.3.** As in [BR73], we split the interval \( T \) into alternating big and small intervals \( W_1, V_1, \ldots, W_N, V_N \), where \( W_i = [a_i, a_i + w] \), \( V_i = [a_i + w, a_{i+1}] \), \( a_i = (i - 1)(w + v) \), and \( N = \lceil (u - l)/(w + v) \rceil \). We let \( w \) be fixed, and \( v \) be small which goes to 0. Since \( u \) and \( l \) are fixed numbers, without loss of generality, we assume \( l = 0 \) and \( u = 1 \) in this proof.

First we approximate \( \Omega^+ := \sup_{0 \leq t \leq 1} \hat{M}_n(t) \) by big blocks \( \{W_k\} \). That is, by \( \Psi^+ := \max_{1 \leq k \leq N} \hat{\gamma}_k^+ \), where \( \hat{\gamma}_k^+ := \sup_{t \in W_k} \hat{M}_n(t) \). Then we further approximate \( \hat{\gamma}_k^+ \) via discretization by \( \hat{\Xi}_k^+ := \max_{1 \leq j \leq \chi} M_n(a_k + jax^{-1}) \), where \( \chi = \lfloor wx/a \rfloor \) with \( a > 0 \). We define \( \Omega^-, \Psi^- \), \( \hat{\gamma}_k^- \), and \( \hat{\Xi}_k^- \) similarly by replacing sup or max by inf or min, respectively. Letting \( \Omega = \max(\Omega^+, -\Omega^-) = \sup_{0 \leq t \leq 1} |\hat{M}_n(t)| \) and \( x_z = a_n + z/(2 \log b^{-1})^{1/2} \), we have

\[
\mathbb{P}(\Omega \geq x_z) - \mathbb{P}(\{\Psi^+ \geq x_z\} \cup \{\Psi^- \leq -x_z\}) \leq R_1 + R_2,
\]

\[
\mathbb{P}(\{\Psi^+ \geq x_z\} \cup \{\Psi^- \leq -x_z\}) - \mathbb{P}\left( \bigcup_{k=1}^{N} \left\{ \Xi_k^+ \geq x_z \right\} \cup \bigcup_{k=1}^{N} \left\{ \Xi_k^- \leq -x_z \right\} \right) \leq R_3 + R_4. \quad (68)
\]
where

\[
R_1 := \mathbb{P}\left( \max_{1 \leq k \leq N} \hat{M}_n(t) \geq x \right), \quad R_2 := \mathbb{P}\left( \min_{1 \leq k \leq N} \hat{M}_n(t) \leq -x \right),
\]

\[
R_3 := \sum_{k=1}^N \left| \mathbb{P}(\Upsilon_k^t \geq x) - \mathbb{P}(\Xi_k^t \geq x) \right|, \quad R_4 := \sum_{k=1}^N \left| \mathbb{P}(\Upsilon_k^t \leq -x) - \mathbb{P}(\Xi_k^t \leq -x) \right|.
\]

\( (69) \)

**Lemma A.4.** Let \( \psi \) be the density function of standard Gaussian, and \( H_2(a) \) be the Pickands constants \([BR73, \text{Theorem A1, Lemma A1, and Lemma A3}] \). Let \( t > 0 \) be such that \( \inf\{s^{-2}(1 - \tilde{r}(s)) : 0 \leq s \leq t\} > 0 \). Then for \( a > 0 \), we have

\[
\mathbb{P}\left( \bigcup_{j=1}^{\lfloor tx/a \rfloor} \left\{ \hat{M}_n(v + jax^{-1}) \geq x \right\} \right) = x\psi(x) \frac{H_2(a)}{a} \tilde{K}_{1/2} t + o(x\psi(x)),
\]

uniformly over \( 0 \leq v \leq 1 \). The limit when \( a \to 0 \) also holds, that is

\[
\mathbb{P}\left( \bigcup_{0 \leq s \leq t} \left\{ \hat{M}_n(v + s) \geq x \right\} \right) = x\psi(x) \tilde{K}_{1/2} t / \sqrt{\pi} + o(x\psi(x)),
\]

where we have used the Pickands constants \( H_2 = \lim_{a \to 0} H_2(a)/a = 1/\sqrt{\pi} \). The left tail version of the tail bounds also hold with \( \geq x \) replaced by \( \leq x \).

**Proof.** See Appendix A.4. \( \square \)

Using Lemma A.4, we can show through elementary calculations that

\[
\lim_{a \to 0} \lim_{v \to 0} \limsup_{n \to \infty} R_j = 0, \quad j = 1, \ldots, 4.
\]

(72)

Finally, we can complete the proof by the following lemma.

**Lemma A.5.** We have

\[
\lim_{a \to 0} \lim_{v \to 0} \limsup_{n \to \infty} |h(x) - (1 - \exp(-2 \exp(-z)))| = 0,
\]

(73)

where \( h(x) := \mathbb{P}\left( \bigcup_{k=1}^n \{\Xi_k^+ \geq x\} \cup \bigcup_{k=1}^n \{\Xi_k^- \leq -x\} \right) \).

**Proof.** See Appendix A.5. \( \square \)

**A.4. Proof of Lemma A.4.** We first use discretization for approximating \( \hat{M}_n(x) \).

Let \( s_j = j/((\log n)^6), 1 \leq j < t_n \), where \( t_n = 1 + \lfloor (\log n)^6 t \rfloor, s_{t_n} = t \). Write \( [s_{j-1}, s_j] = \bigcup_{k=1}^{q_n} [s_{j,k-1}, s_{j,k}] \), where \( q_n = \lfloor (s_j - s_{j-1})n^2 \rfloor = \lfloor n^2/(\log n)^6 \rfloor \) and \( s_{j,k} - s_{j,k-1} = \)}
The main steps of the rest of the proof are as follows. First, we approximate \( \hat{M}_n(v + s) \) by \( Y_n \), where

\[
\mathbb{P} \left( \sup_{0 \leq s \leq t} \hat{M}_n(v + s) \geq x \right) \leq \mathbb{P} \left( \max_{1 \leq j \leq t_n} \hat{M}_n(v + s_j) \geq x - \left( \log n \right)^{-2} \right) + C n^{-Q}. \tag{74}
\]

Next, we apply the multivariate Gaussian approximation by Zaitsev [Zai87]. To this end, we first define

\[
u_j(t) := \frac{n}{\sigma^2} \left\{ \mathbb{E} \left[ \tilde{K} \left( \frac{X_{j-1} - t}{b} \right) \mid \xi_{j-1,j-m} \right] - \mathbb{E} \left[ \tilde{K} \left( \frac{X_{j-1} - t}{b} \right) \mid \xi_{j-2,j-m} \right] \right\}. \tag{75}
\]

Then we define

\[
\tilde{u}_j(t) := \frac{\eta_j}{\sigma^2} \left\{ \nu_j(t) \mathbb{1} \{ |\nu_j(t)| \leq \sqrt{n b \left( \log n \right)^{-2p/(p-2)}} \} - \mathbb{E} \left[ \nu_j(t) \mathbb{1} \{ |\nu_j(t)| \leq \sqrt{n b \left( \log n \right)^{-2p/(p-2)}} \} \right] \right\}. \tag{76}
\]

Now we introduce \( \hat{M}_n(t) := \frac{1}{\sqrt{nbK_f(t)}} \sum_{j=1}^n \tilde{u}_j(t) \). Then using [Zai87, Theorem 1.1] as well as \( \sup_j \max_{1 \leq j \leq n} \| \tilde{u}_j(t) - u_j(t) \| \leq C n^{-Q} \) for large enough \( Q \), we have

\[
\mathbb{P} \left( \max_{1 \leq j \leq t_n} \hat{M}_n(v + s_j) \geq x - \left( \log n \right)^{-2} \right) \leq \mathbb{P} \left( \max_{1 \leq j \leq t_n} \hat{M}_n(v + s_j) \geq x - \left( \log n \right)^{-2} \right) + C n^{-Q} \leq \mathbb{P} \left( \max_{1 \leq j \leq t_n} Y_{n,j} \geq x - 2\left( \log n \right)^{-2} \right) + C t_{n/2}^{5/2} \exp \left( -\frac{C (\log n)^{18p/(p-1)}}{t_n^{5/2}} \right) + C n^{-Q}, \tag{77}
\]

where \( (Y_n(1), \ldots, Y_n(t_n)) \) is a centered Gaussian random vector with covariance matrix \( \hat{\Sigma}_n = \text{Cov}(\hat{M}_n(v + s_1), \ldots, \hat{M}_n(v + s_{t_n})) \). The rest of the proof follows exactly the arguments in [LW10, Proof of Lemma 4.6] to apply [BR73, Lemma A3 and Lemma A4].

### A.5. Proof of Lemma A.5

We first define

\[
B_{k,j} := \{ \hat{M}_n(a_k + jax^{-1}) \geq x \} \cup \{ \hat{M}_n(a_k + jax^{-1}) \leq -x \},
\]

\[
D_{k,j} := \{ N_n(a_k + jax^{-1}) \geq x \} \cup \{ N_n(a_k + jax^{-1}) \leq -x \}, \tag{78}
\]

where \( Y_n(\cdot) \) is a centered Gaussian process with covariance function

\[
\text{Cov}(Y_n(s_1), Y_n(s_2)) = \text{Cov}(\hat{M}_n(s_1), \hat{M}_n(s_2)). \tag{79}
\]

The main steps of the rest of the proof are as follows. First, we approximate \( \hat{M}_n(t) \) by \( Y_n(t) \). Then, we approximate \( Y_n(t) \) by another quantity \( \tilde{M}_n(t) \) which is defined
similarly to \( \hat{M}_n(x) \) but using a sequence of i.i.d. random variables instead of the dependent time series \( \{X_k\} \). Finally, we apply [Ros76, Theorem] to show convergence to Gumbel distribution.

First, the following lemma is for approximation of \( \hat{M}_n(t) \) using \( Y_n(t) \).

**Lemma A.6.** Recall that \( w \) and \( v \) are the lengths of big and small blocks \( W_i \) and \( V_i \). Let \( N = \lfloor 1/(w + v) \rfloor \). For any fixed integer \( l \) that \( 1 \leq l \leq N/2 \), we have

\[
\left| \mathbb{P} \left( \bigcup_{k=1}^{N} A_k \right) - \sum_{d=1}^{2l-1} (-1)^{d-1} \left( \sum_{1 \leq i_1 < \cdots < i_d \leq N} - \sum_{I} \right) \mathbb{P} \left( \bigcap_{j=1}^{d} C_{i_j} \right) \right| \leq \frac{C^{2l}}{(2l)!} + O \left( \frac{1}{\log n} \right),
\]

(80)

where \( A_k := \bigcup_{j=1}^{\lfloor w x/a \rfloor} B_{k,j} \), \( C_k := \bigcup_{j=1}^{\lfloor w x/a \rfloor} D_{k,j} \), \( C \) does not depend on \( l \), and

\[
I := \left\{ 1 \leq i_1 < \cdots < i_d \leq N : \min_{1 \leq j \leq d-1} q_j \leq |2w^{-1} + 2| \right\}.
\]

(81)

**Proof.** Define \( \hat{M}_n(t) := \frac{1}{\sqrt{n b} f(t)} \sum_{j=1}^{n} \hat{u}_j(t) \) for given \( d \), where

\[
\hat{u}_j(t) := u_j(t) \mathbb{1} \{ |u_j(t)| \leq \sqrt{n b} (\log n)^{-20d p/(p-2)} \} - \mathbb{E} \left[ u_j(t) \mathbb{1} \{ |u_j(t)| \leq \sqrt{n b} (\log n)^{-20d p/(p-2)} \} \right].
\]

(82)

Then the rest of the proof follows from [LW10, Proof of Lemma 4.10]. \( \square \)

Now let \( \{\eta_i^{(k)}\}, i \leq k \leq n \), be i.i.d. copies of \( \{\eta_i\} \), and \( \xi_j^{(k)} = (\ldots, \eta_{j-1}^{(k)}, \eta_j^{(k)}, \ldots) \). Let \( X_i^{(k)} = G(\xi_j^{(k)}) \). Note that \( X_i^{(k)}, 1 \leq k \leq n \), are i.i.d. Now define \( A'_k \) the same as \( A_k \) except by replacing \( Y_j \) and \( \{\eta_i\} \) with \( X_k^{(k)} \) and \( \{\eta_i^{(k)}\} \), respectively. Repeat the previous lemma, we have

\[
\left| \mathbb{P} \left( \bigcup_{k=1}^{N} A'_k \right) - \sum_{d=1}^{2l-1} (-1)^{d-1} \left( \sum_{1 \leq i_1 < \cdots < i_d \leq N} - \sum_{I} \right) \mathbb{P} \left( \bigcap_{j=1}^{d} C_{i_j} \right) \right| \leq \frac{C^{2l}}{(2l)!} + O \left( \frac{1}{\log n} \right).
\]

(83)

Letting \( n \to \infty \) then \( l \to \infty \), by triangle inequality, we have

\[
\limsup_{n \to \infty} \left| \mathbb{P} \left( \bigcup_{k=1}^{N} A_k \right) - \mathbb{P} \left( \bigcup_{k=1}^{N} A'_k \right) \right| = 0.
\]

(84)

Now the key observation is that we can deal with \( \{A'_k\} \) now and \( A'_k \) are defined using \( \{X_k^{(k)}\} \) which are i.i.d. Next, we define \( R'_1 \) to \( R'_4 \) the same as \( R_1 \) to \( R_4 \) except using \( \{X_k^{(k)}\} \) and \( \{\eta_i^{(k)}\} \) instead of \( \{X_k\} \) and \( \{\eta_i\} \), then by Lemma A.4 and elementary
Supplemental Material 29

Calculations we have \( \lim_{a \to 0} \lim \sup_{v \to 0} \lim \sup_{n \to \infty} R_j' = 0 \) for \( j = 1, \ldots, 4 \). This implies

\[
\lim_{a \to 0} \lim \sup_{v \to 0} \lim \sup_{n \to \infty} \left| \mathbb{P} \left( \bigcup_{k=1}^{N} A'_k \right) - \mathbb{P} \left( \sup_{0 \leq t \leq 1} |\hat{M}'_n(t)| < x \right) \right| = 0, \quad (85)
\]

where \( \hat{M}'_n(t) \) is defined in the same way as \( \hat{M}_n(t) \) by replacing \( \{X_k\} \) with \( \{X'^{(k)}_k\} \), and \( \{\eta_i\} \) with \( \{\eta'^{(k)}_i\} \). Finally, since \( \{X'^{(k)}_k\} \) are i.i.d. we can apply [Ros76, Theorem], which leads to the convergence of \( \mathbb{P} \left( \sup_{0 \leq t \leq 1} |\hat{M}'_n(t)| < x \right) \) to \( e^{-2x^{-2}} \). This completes the proof.

B. Proof of Theorem 3.5

First, let \( r_n \) and \( s_n \) be positive sequences, then \( r_n = \Omega(s_n) \) if \( s_n = o(r_n) \). On the other hand, \( r_n = \Theta(s_n) \) if both \( s_n = O(r_n) \) and \( r_n = O(s_n) \) hold. Note that

\[
P \left( \hat{M} = M \right) \cap \left\{ \max_{1 \leq i \leq M} |\hat{x}_i - x_i| < c_n \right\}
= P \left( \max_{1 \leq i \leq M} |\hat{x}_i - x_i| < c_n | \hat{M} = M \right) P \left( \hat{M} = M \right). \quad (86)
\]

By the validity of the Theorem 3.2, we have \( P \left( \hat{M} > M \right) \to \alpha \). Also, since the number of change points is finite, as \( n \to \infty \) we have \( P \left( \hat{M} < M \right) \to 0 \). Therefore, we have

\[
P \left( \hat{M} = M \right) \to 1 - \alpha. \quad (87)
\]

Since \( M \) is finite, we only need to focus on one change point. Let \( x_0 \) be any of the true change point and \( \hat{x} \) be its estimate, it suffices to show \( P \left( |\hat{x} - x_0| \geq c_n | \hat{M} = M \right) \to 0 \). Without loss of generality, we assume \( \hat{x} - x_0 = \hat{c}_n = o_P(b) \) and \( t_n(x_0) > 0 \). The case \( t_n(x_0) < 0 \) can be shown using similar arguments. Now we follow similar arguments as in [M92]. Define \( \zeta(c) := t_n(x_0 + c) - t_n(x_0) \), for \( c = o(b) \). Then we can write \( \hat{c}_n = \arg \max \zeta(c) \). Therefore, it suffices to show \( \hat{c}_n = O_P \left( \sqrt{\frac{b \log n}{n}} \right) \). Suppose \( b \) is small enough such that the \( b \)-neighborhood of \( x_0 \) does not include any other change points, then we apply the previous decomposition in Eq. (39). Note that since \( x_0 \) is a change point, without loss of generality, we assume \( \mu(x) \) is left continuous at \( x = x_0 \),
then the following term has the order of \( \Theta(\delta) \):

\[
\frac{1}{nb} \sum_{k=1}^{n} \tilde{K} \left( \frac{X_{k-1} - x_0}{b} \right) \frac{\mu(x_0)}{f(x_0)} - \sum_{k=1}^{n} \tilde{K} \left( \frac{X_{k-1} - (x_0 + c)}{b} \right) \frac{\mu(x_0 + c)}{f(x_0 + c)} \right].
\]

Furthermore, using \( \int_0^s K(x)dx = \Theta(s^2) \) because of \( \tilde{K}'(0) > 0 \), we have

\[
\left| \frac{1}{nb} \int_{x_0 + c}^{x_0} K(b) \frac{x}{b} \right| \Theta_\delta + \left| \frac{1}{b} \int_{x_0 + c}^{x_0} K(b) \frac{x}{b} \right| \Theta_\delta + O_P(b^3 + \sqrt{b \log n/n})
\]

Finally, using

\[
\frac{1}{nbf(x_0 + c)} \sum_{k=1}^{n} \tilde{K} \left( \frac{X_{k-1} - (x_0 + c)}{b} \right) \left[ \mu(X_{k-1}) - \mu(x_0 + c) \right]
\]

where the term \( O_P \left( \sqrt{\log n/nb} \right) \) is uniformly over \( x \), we can conclude that

\[
\zeta(c) = - \left( \frac{c}{b} \right)^2 \Theta_\delta(1) + \left( \frac{c}{b} \right) \Theta_\delta \left( \sqrt{\frac{\log n}{nb}} \right) - O_P(b^3 + \sqrt{b \log n/n})
\]

Recall that the estimated change point \( \hat{x} = x_0 + \hat{c}_n \), where \( \hat{c}_n = \arg \max \zeta(c) \), then we have

\[
\hat{c}_n = O_P \left( b \sqrt{\frac{\log n}{nb}} \right) = O_P \left( \sqrt{\frac{b \log n}{n}} \right)
\]

whenever \( b^4 = o(\log n/nb) \). This is always true since we have assumed \( \delta_2 \leq 1/4 \) which implies \( b = O(n^{-1/4}) \) so \( b^4 = O(1/n) = o(\log n/n) \). Therefore, if we choose
If $c_n > 0$ such that $\hat{c}_n = o(c_n)$, then we have $P(|\hat{c}_n| < c_n) \to 0$, which implies $P\left(|\hat{x} - x_0| \geq c_n \mid \hat{M} = M\right) \to 0$. 