MULTIPLICATIVE GROUPS OF DIVISION RINGS

R. HAZRAT, M. MAHDAVI-HEZAVEHI, AND M. MOTIEE

Abstract. Exactly 170 years ago, the construction of the real quaternion algebra by William Hamilton was announced in the Proceedings of the Royal Irish Academy. It became the first example of non-commutative division rings and a major turning point of algebra. To this day, the multiplicative group structure of quaternion algebras have not completely been understood. This article is a long survey of the recent developments on the multiplicative group structure of division rings.

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1. Introduction and Review of Old Results

Let $D$ be a division ring, $D^* = D \setminus \{0\}$ its unit group and $Z(D)$ its center. Evidently, the first departure to connect the algebraic properties of $D$ to that of $D^*$ was done by Wedderburn in 1905. By his theorem, if $D$ has a finite cardinal number then $D^*$ is abelian and hence $D$ is a field (See [44, p. 179]). Since Wedderburn's result classifies a class of division rings in terms of a group theoretic (more exactly set theoretic) property, one way to generalize this observation is to attribute a group theoretic property to $D^*$ and then to explore which division rings enjoy this property. This point of view also includes the detection of the exclusivity of subgroups and quotients of $D^*$. Namely, from Wedderburn's theorem one can easily conclude that if $\text{char}(D) \neq 0$ then every finite subgroup of $D^*$ is a cyclic group. In 1953, Herstein determined some finite groups that can be embedded as a subgroup in a division ring [32]. But, the characterization of all finite groups that can occur as a subgroup of a division ring was done by Amitsur in [6]. According to Amitsur's results, a finite group $G$ sits in the unit group of a division ring if and only if $G$ is one of the following types:

(A1) A group that all of its Sylow subgroups are cyclic.
(A2) $C_m \times Q_{2n+1}$, where $C_m$ is a cyclic group of odd order $m$, an element of order $2^n$ of $Q_{2n+1}$ centralizes $C_m$ and an element of order 4 inverts $C_m$.
(A3) $Q_8 \times M$, where $M$ is a group of type (A1).
(A4) The binary octahedral group of order 48.
(A5) $\text{SL}(2,3) \times M$, where $M$ is a group of type (A1).
(A6) The group $\text{SL}(2,5)$.

where $Q_{2n+1}$ is the generalized quaternion group of order $2^{n+1}$ for each $n \in \mathbb{N}$. Moreover, in [16] and [90, §2] successful efforts have been made to study the structure of locally finite subgroups of a division ring. We recall that a group $G$ is called locally finite if every finitely generated subgroup of $G$ is finite. A remarkable result concerning the quotients of $D^*$ is due to Kaplansky which asserts that if $D^*$ is center-by-periodic then it is commutative [43]. This contains Noether-Jacobson Theorem as a special case. (Recall that by Noether-Jacobson Theorem, if $D$ is an algebraic division algebra over some subfield $F$, then $D^*$ contains an element $a$ not in $F$ that is separable over $F$ [51, p. 244].) Before 1950, an interesting problem concerning the group structure of $D$ was to figure out how far $D^*$ is from commutativity. Thus, it was reasonable for one to explore what happens when $D^*$ is nilpotent or more specifically is soluble. Guided by this viewpoint, the roles of the multiplicative commutators in the structure of $D$ were made prominent. In this direction in [37] Hua has proved that if for a natural number $r \geq 2$, all $r$-mixed commutators (elements of $D^*$ that are defined inductively by $[a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2$ and $[a_1, a_2, \ldots, a_r] = [a_1, [a_2, \ldots, a_r]]$ of $D^*$ lie in some division subring, then $D$ is a field. Moreover, he proved that if an element of $D^*$ commutes with all $r$-mixed commutators, then it is central. Finally, in 1950 he proved that if $D^*$ is soluble, then $D$ is a field [38]. At the other extreme, the roles of additive and multiplicative commutators in the structure of $D^*$ were extensively studied in 1970s by some authors. In [36] Herstien, Procesi and Schacher showed that if every additive commutator (an element of the form $xy - yx$ for some $x, y \in D^*$) is torsion modulo $Z(D)$ then $[D : Z(D)] \leq 4$. 

\[ \text{Introduction and review of old results} \]

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Putcha and Yaqub also proved that if every multiplicative commutator is killed by a power of 2 in $D^*/Z(D^*)$ then $D^*$ is commutative \cite{81}. However, the general case of this problem was investigated by Herstein in \cite{34} by proving that if either every $[a, b]$ has a finite order in $D^*$ or $D$ is of finite dimension over $Z(D)$ and every $[a, b]$ is torsion modulo $Z(D)$ then $D$ is a field. An alternative interesting result of \cite{34} asserts that every subnormal periodic subgroup of $D^*$ is central. Afterwards, in \cite{35} he showed that whenever $Z(D)$ is countable, then (i) if $[a, b]^n \in Z(D)$ for all $a, b \in D^*$, for some $n \in \mathbb{N}$ then $D$ is a field, and (ii) If $N$ is a subnormal subgroup of $D^*$ that is periodic modulo $Z(D)$ then $N$ is central. The last result is a special type of another absorbing old problem determining how much subnormal subgroups of $D^*$ reflect the multiplicative structure of $D^*$. In other words, how “big” subnormal subgroups are in $D^*$. One of the earlier known results depending on this subject had been obtained by Herstein in \cite{33} by showing that every noncentral element of $D^*$ possesses infinitely many conjugates. Then, in 1957 Scott proved that $D^*/Z(D^*)$ has no nontrivial abelian normal subgroup or equivalently every abelian normal subgroup of $D^*$ is central \cite{87}. He also extended the results of \cite{33} by demonstrating that the number of conjugates of every noncentral element in $D^*$ is equal to $|D|$, the cardinal of $D$. In 1960, in \cite{39} Huzurbazar provided a generalization of Scott’s theorem. His generalization states that every locally nilpotent subnormal subgroup of $D^*$ is central. But, the most important result concerning the structure of subnormal subgroups was obtained by Stuth in 1964 asserting that (i) If $G$ is a noncentral subnormal subgroup of $D^*$ and $x^G$ is the conjugacy class of the noncentral element $x \in D^*$ in $G$, then the division subring generated by $x^G$ is $D$, (ii) Every soluble subnormal subgroup of $D^*$ is central \cite{91}. Another aspect in the study of the unit groups of division rings is to achieve analogous properties of general linear groups over fields. Namely, Bachmuth asked whether Tits’ Alternative is valid for general linear groups over division rings. In \cite{53} Lichtman answered it negatively by showing that every division ring can be embedded into a division ring $K$ so that $K^*$ contains a finitely generated subgroup that is not soluble-by-finite and does not contains a noncyclic free subgroup. Then he asked whether $D^*$ has a noncyclic free subgroup. A more general question posed as a conjecture by Goncalves and Mandel in \cite{18}. More exactly, they conjectured that every noncentral subnormal subgroup of a division ring contains a noncyclic free subgroup. They also proved that the conjecture is true in several cases (see \cite[p. 208]{44}). Also, in a preceding work, Goncalves had shown that the conjecture has a positive answer whenever $D$ is of finite dimension over its center. Moreover, an earlier result of Lichtman asserts that if a subnormal subgroup $G$ of $D^*$ contains a nonabelian nilpotent subgroup then $G$ has a noncyclic free subgroup \cite{54}. Note that an affirmative answer to this conjecture implies a great part of the above mentioned results of Hua, Herstein and Stuth. However, in this article our aim is to review and give the proofs of some recent works on the structure of the unit groups of division rings. We would like to point out that our viewpoint is focused on the application of the above results in discovering the role of some special subgroups in the structure of a division ring.

1.1. Conventions. We now collect some of the terminology and notation that will be used throughout the paper:
If $R$ is a ring, we write $Z(R)$ for the center of $R$; $R^*$ for the group of units of $R$; $M_n(R)$ for all $n \times n$ matrices over $R$; and $\text{GL}_n(R)$ for the group of all invertible $n \times n$ matrices over $R$. For a subgroup $G$ of $R^*$, the $R$-linear hull of $G$ is always indicated by $R[G]$, i.e.,

$$R[G] := \{ \sum_{j=1}^{m} r_j g_j | r_j \in R, g_j \in G \text{ and } m \in \mathbb{N} \}.$$ 

Moreover, if $D$ is a division ring with center $F$ and $G$ a subgroup of $D^*$ then we denote the division subring generated by $F$ and $G$ by $F(G)$. It is not hard to see that $F(G) = F[G]$ whenever all the elements of $G$ are algebraic over $F$. Let $F$ be a field. We use $F_{\text{alg}}$ for an algebraic closure of $F$. For an $F$-algebra $A$, we write $[A : F]$ for the dimension of $A$ over $F$ as an $F$-vector space. By an $F$-central simple algebra we mean an $F$-algebra $A$ where $Z(A) = F$ and $[A : F] < \infty$. Moreover, an $F$-central division algebra is an $F$-central simple algebra that is also a division ring. Moreover, sometimes we use the phrase “division algebra” for division rings that are finite dimensional over their centers. Now, let $A$ be an $F$-central simple algebra and $K$ be a maximal subfield of $A$. It is known that if $L$ is an algebraic extension of $K$ then $A \otimes_F L$ is isomorphic to a matrix algebra over $L$. This immediately implies that $[A : F]$ is a square. We refer to $\sqrt{[A : F]}$ as the degree of $A$ and denote it by $\text{deg}(A)$. Also, by Wedderburn’s Theorem $A$ is isomorphic to a matrix ring over an $F$-central division algebra $D$ where $D$ is unique up to isomorphism. The (Schur) index of $A$ is $\text{ind}(A) = \text{deg}(D)$. Let $A$ be an $F$-central simple algebra and $L$ be a splitting field of $A$ which means that there exists an isomorphism $\phi : A \otimes_F L \to M_n(L)$ (Evidently $n = \text{deg}(A)$). From some classical results in the theory of central simple algebras, it can be seen that $\det(\phi(a \otimes 1)) \in F^*$ for each $a \in A^*$. Moreover, the value of $\det(\phi(a \otimes 1))$ is independent of the choice of the splitting field $L$ and the isomorphism $\phi$ (cf. [11, p. 145]). Thus, we can define a map

$$\text{Nrd}_{A/F} : A^* \to F^*$$

$$a \mapsto \det(\phi(a \otimes 1)),$$

without ambiguity. This map is called the “reduced norm”. For a field $F$, we use $\text{Br}(F)$ for the Brauer group of $F$. Also, for an $F$-central simple algebra $A$ we denote its equivalence class in $\text{Br}(F)$ by $[A]$. If $K/F$ is a Galois extension of fields, $\text{Gal}(K/F)$ stands for the Galois group of $K$ over $F$. If $K/F$ is a Galois extension that is cyclic and $\sigma$ is the generator of $\text{Gal}(K/F)$, and if $b \in F^*$ we write $(K/F, \sigma, b)$ for the cyclic $F$-algebra generated by $K$ and $x$ with relations $xc = \sigma(c)x$ for all $c \in K$ and $x^n = b$. Note that such an algebra is an $F$-central simple algebra of degree $n$ and also we have $(K/F, \sigma, b) = \bigoplus_{i=1}^{n-1} Kx^i$. For every natural number $n$ the notation $\mu_n(F)$ is frequently used for the group of $n$-th roots of unity in the field $F$. Now, let $G$ be a group. For each subset $X$ of $G$ we use the symbol $\langle X \rangle$ for the subgroup generated by $X$. Also, for a subgroup $H$ of $G$ we write $N_G(H)$ and $C_H(G)$ for the normalizer and centralizer of $H$ in $G$, respectively. If $a, b \in G$, we denote the commutator $a^{-1}b^{-1}ab$ by $[a, b]$. Also, we put $G'$ for the commutator subgroup of $G$, i.e., $G' = \langle [a, b] | a, b \in G \rangle$. The commutator subgroups of $G$ of higher orders that are defined inductively are also denoted by $G^{(n)} = (G^{(n-1)})'$. $G$ is called “soluble” if $G^{(n)} = \{1\}$ for some natural number $n$. The notation $a^b$ also stands for $b^{-1}ab$. If $H$ and $K$ are two subgroups of
2. Soluble-by-finite subgroups of division algebras

This section deals with the properties of soluble-by-finite subgroups of division algebras. We will see that the subgroups of this type arise naturally in some problems concerning the structure of $D^*$. 

2.1. Crossed product division algebras. As is well known, all of the incipient examples of (finite dimensional) division algebras had a particular common property; all of them contained a maximal subfield that was Galois over the center. Such a division algebra is called a crossed product. The existence of a non-crossed product division algebra remained as an open problem till 1972 when Amitsur presented the first example of a non-crossed product division algebra in [7]. Also, more examples of non-crossed product division algebras are produced by some other authors. For a survey on this subject see [95, §5]. But, perhaps the first important result connecting the crossed product problem to the multiplicative group of a division algebra, is the following remarkable theorem of Albert (cf. Proposition in [79, p. 286] or [11, p. 87]):

**Theorem 2.1 (Albert’s Criterion).** Let $D$ be an $F$-central division algebra of prime degree $p$. If $D^*/F^*$ contains a non-trivial element of order $p$, then $D$ is a cyclic division algebra, i.e., $D$ has a maximal subfield $K$, such that $K/F$ is a cyclic extension.

Motivated by the above result, in [70] the authors investigate the relation between the structure of the unit group of a division algebra $D$ and the cyclicity conditions when the index of $D$ is a prime number. The next theorem, is the main result of [70].

**Theorem 2.2.** Let $D$ be an $F$-central division algebra of prime degree $p$. Then the following conditions are equivalent:

1. $D$ is a cyclic division algebra;
2. $D^*$ contains a non-abelian soluble subgroup;
3. $D^*/F^*$ contains a non-trivial torsion subgroup;
4. $D^*/F^*$ contains an element of order $p$.

To prove Theorem 2.2, we need the following lemma.

**Lemma 2.3.** Let $D$ be a finite dimensional division algebra over its center $F$. If $G$ is a soluble subgroup of $D^*$, then $G$ contains an abelian normal subgroup $A$ of finite index.

**Proof.** We know that $D \otimes_F F_{\text{alg}} \cong M_n(F_{\text{alg}})$ where $n = \deg(D)$. Thus, we may view $D^*$ as a subgroup of $\text{GL}_n(F_{\text{alg}})$. Now, by Lie-Kochlin-Mal’cev Theorem [84, p. 434], $G$ contains
a normal subgroup \( A \) such that \( A' \) is unipotent, i.e., for each \( a \in A' \) the element \( 1 - a \) is nilpotent. But, since the only unipotent subgroup of \( D^* \) is \( \{1\} \), we must have \( A' = \{1\} \). This means that \( A \) is abelian and the proof is complete.

Lemma 2.3 is our major tool in the subsequent works. Now, we are going to present a proof for Theorem 2.2.

**Proof of Theorem 2.** (1) \( \Rightarrow \) (2). Let \( K \) be the maximal subfield of \( D \) where \( K/F \) is a cyclic extension. Thus, by the Skolem-Noether Theorem, there is an element \( z \in D^* \) such that \( D = \bigoplus_{j=1}^{p-1} Kz^j \) and for each \( k \in K \), \( kz = z\sigma(k) \), where \( \sigma \) is a generator of \( \text{Gal}(K/F) \). Put \( S = K^*(z) \). \( S \) is non-abelian, because \( \sigma \) is non-trivial. On the other hand, \( z^p \in K^* \) as \( \sigma^p = 1_K \). Therefore, \( S/K^* \) is a cyclic group and hence \( S \) is non-abelian soluble.

(2) \( \Rightarrow \) (3). Let \( G \) be a non-abelian soluble subgroup of \( D^* \). By Lemma 2.3, we can choose a normal abelian subgroup \( A \) of \( G \) with \( |G:A| < \infty \). Take \( A \) maximal and put \( H = C_G(A) \). If \( H \neq A \), then \( H \) is non-abelian, because \( A \) is maximal. Thus, \( F[H] = D \) and so \( A \subseteq F^* \). Therefore, \( F^*H/F^* \) is a non-trivial torsion subgroup of \( D^*/F^* \). Now, let \( H = A \). Thus, \( A \) is non-central in \( G \) and hence \( K = F[A] \) is a maximal subfield of \( D \). If \( g \in G \setminus A \), then \( g \) induces a non-trivial \( F \)-automorphism \( \sigma \) of \( K \) by \( k \mapsto gkg^{-1} \). Since \( \sigma \) has a finite order, then \( g^m \in K^* \) for some \( m \neq 1 \). But, \( \sigma(g^m) = ggg^{-1} = g^m \). So \( g^m \in F \), because \( F \) is the fixed field of \( \sigma \). Now, \( F^*(g)/F^* \) is a non-trivial torsion subgroup of \( G \).

(3) \( \Rightarrow \) (4). Let \( T \) be a subgroup of \( D^* \) such that \( T/F^* \) is torsion. Denote by \( D^1 \) the kernel of the reduced norm map \( \text{Nrd}_D : D^* \to F^* \). Consider the two following cases:

**Case 1.** \( T \cap D^1 \subseteq F^* \). If we choose an element \( u \in T \setminus F^* \), then \( u^p \text{Nrd}_D(u)^{-1} \in T \cap D^1 \subseteq F^* \) and hence \( u^p \in F^* \).

**Case 2.** \( T \cap D^1 \not\subseteq F^* \). If this is the case, then there is an element \( 1 \neq u \in T \cap D^1 \) such that \( u \not\in F^* \). But, since \( u \in T \), we have \( u^m \in F^* \) for some \( m \) as \( T/F^* \) is torsion. Therefore, \( u^{mp} = \text{Nrd}_D(u)^m = 1 \). Since \( u \neq 1 \), we conclude that \( K = F(u) \) is a cyclic extension of \( F \) with \( [D : K] = p \). Now, by the Skolem-Noether Theorem, there is a \( z \in D^* \) such that the inner automorphism \( d \mapsto zdz^{-1} \) restricts to a non-trivial \( F \)-automorphism of \( K \). Now, a similar argument as in the proof of (2) \( \Rightarrow \) (3) implies that \( z^p \in F^* \).

(4) \( \Rightarrow \) (1). This follows from Albert’s Criterion. \( \square \)

Let \( F \) be a field containing a primitive \( n \)-th root of unity \( \omega_n \) for some natural number \( n \). Recall that a symbol algebra is an algebra of the form \( A = \bigoplus_{i,j=0}^{n-1} Fu^iv^j \) where \( u, v \) are symbols with the requirements \( u^n, v^n \in F \) and \( uv = \omega_n vu \) (cf. [11, §11]). When \( n \) is prime and \( A \) is not a split algebra, one can easily see that \( A \) is a cyclic division algebra. Thus, over a given field \( F \), the class of symbol algebras of prime degree \( p \) is a subclass of cyclic division algebras of degree \( p \). So, it can be of interest to realize when a division algebra of prime degree is a symbol algebra.

**Theorem 2.4** ([69]). Let \( D \) be an \( F \)-central division algebra of prime degree \( p \). Then \( D \) is a symbol algebra if and only if \( D^* \) contains a non-abelian nilpotent subgroup.

**Proof.** If \( D \) is a symbol algebra, then there are elements \( u, v \in D^* \) such that \( u^p, v^p \in F^* \), \( uv = \omega_p vu \) and \( D = \bigoplus_{i,j=1}^{p-1} Fu^iv^j \). Put \( G = F^*(u,v) \). Since \( G/F^* \) is abelian, we conclude
that \( G' \subseteq F^* \subseteq Z(G) \). Hence, \( G \) is nilpotent. Conversely, let \( G \) be a non-trivial nilpotent subgroup of \( D^* \). By Theorem 2.2, we conclude that \( D \) is a cyclic division algebra. Thus, if we show that \( F^* \) contains a primitive \( p \)-th root of unity, then we are done. To prove this, suppose that the length of the lower central series of \( G \) is equal to \( t \), i.e.,

\[
\zeta_0 G \supseteq \zeta_1 G \supseteq \ldots \supseteq \zeta_{t-1} G \supseteq \zeta_t G = 1,
\]

where \( \zeta_0 G = G \) and \( \zeta_i G = [G, \zeta_{i-1} G] \) for \( 1 \leq i \leq t \). Therefore, \( 1 \neq \zeta_{t-1} G \subseteq G' \cap Z(G) \subseteq G' \cap F^* \). Thus, for each \( 1 \neq a \in \zeta_{t-1} G \) we must have \( a^p = \text{Nrd}_D(a) = 1 \), as desired.

Now, let us look more closely at the group theoretic structure of a crossed product division algebra. Suppose \( D \) is a crossed product division algebra with maximal Galois subfield \( K \). By Skolem-Noether Theorem, for every \( \sigma \in \Gamma = \text{Gal}(K/F) \) there is an \( e_\sigma \in D^* \) such that

(i) \( A = \bigoplus_{\sigma \in \Gamma} K e_\sigma \) and \( \sigma(k) = e_\sigma k e_\sigma^{-1} \) (\( k \in K \) and \( \sigma \in \Gamma \));

(ii) There is a 2-cocycle \( f \in Z^2(\Gamma, K^*) \) such that \( e_\sigma e_\tau = f(\sigma, \tau)e_{\sigma\tau} \) (\( \sigma, \tau \in \Gamma \)).

Now, put \( N = N_{D^*}(K^*) \). From (i), we have \( e_\sigma \in N \) for each \( \sigma \in \Gamma \). Thus \( F[N] = D \), i.e., \( N \) is an absolutely irreducible subgroup of \( D^* \). On the other hand, every element of \( N \) induces an \( F \)-automorphism of \( K \) by conjugation. Therefore, \( N/C_{D^*}(K^*) \cong \Gamma \). But, \( C_{D^*}(K) = K \) as \( K \) is a maximal subfield. This implies that \( N/K^* \cong \Gamma \). Consequently, \( N \) is an absolutely irreducible abelian-by-finite (equivalently soluble-by-finite) subgroup of \( D^* \).

Now, the following question arises naturally:

**Question 2.5.** Let \( D \) be a division algebra such that its unit group has an absolutely irreducible soluble-by-finite subgroup \( G \). Is \( D \) a crossed product?

The first significant result concerning Question 2.5, was obtained in [60] where the author showed that if \( G \) is a soluble maximal subgroup (this means that \( G \) is a maximal subgroup of \( D^* \) that is soluble), then the question has a positive answer. Afterwards, this question became the subject of a series of papers including [13], [14], [47] and [45]. In [13], it was proved that if \( G \) is soluble, then \( D \) is a quasi-crossed product division algebra in the sense that it contains a tower of subfields \( F = Z(D) \subsetneq K \subseteq L \) such that \( K/F \) is Galois, \( L \) is a maximal subfield and \( L/K \) is an abelian Galois extension. Also, the results of [14] and [47] assert that \( D \) is a crossed product division algebra if one of the following conditions holds:

(i) \( G \) is either supersoluble or nilpotent;

(ii) \( G \) has no subgroups isomorphic to \( \text{SL}(2,5) \) and \( D \) has a prime power degree.

However, a natural question related to (ii) is what happens when \( G \) has a subgroup isomorphic to \( \text{SL}(2,5) \)? Answering this question requires suitable information concerning the structure of soluble-by-finite subgroups of division algebras. The most important systematic investigation of the structure of soluble-by-finite subgroups in the general case was fulfilled in an interesting paper of Shirvani ([89]), where he has proved that:

**Theorem 2.6.** Let \( D \) be a division algebra of degree \( m \). Then:

1. Every soluble-by-finite subgroup of \( D^* \) is contained in a maximal soluble-by-finite subgroup of \( D^* \);
(2) If \( G \) is a soluble-by-finite subgroup of \( D^* \), then \( G \) contains an abelian normal subgroup \( A \) with \( |G : A| = cmn \), where \( c = 1, 6, 12 \) or \( 30 \), and \( t \) is a proper divisor of \( m \). Moreover, if \( \text{char}(D) > 0 \), or if \( m \) is odd, or if \( G \) is torsion-free, then \( c = 1 \).

Consequently, every soluble-by-finite subgroup of \( D^* \) has a normal abelian subgroup of index dividing \( 60\deg(D)^2 \).

Using this information he showed that if \( \deg(D) \) is such that every group of order \( \deg(D)^2 \) is nilpotent, or if \( G \) is either metabelian or torsion-free, then \( D \) is a crossed product division algebra. The proofs of Shirvani’s results are very technical and include heavy computations. But, in [105], Wehrfritz gave an alternative proof of part (2) of Theorem 2.6 based on the contents of his earlier results in [102] and[104]. Thereby, he also slightly improved Shirvani’s bound for \( |G : A| \). In fact, Wehrfritz results, show that one can replace the bound 60 \( \deg(D)^2 \) by 30 \( \deg(D)^2 \). In the sequel, we try to simplify Shirvani’s idea to achieve the major part of the above mentioned results.

**Lemma 2.7.** Let \( G \) be an irreducible subgroup of \( D^* \). If \( A \) is a normal abelian subgroup of \( G \), then:

1. \( [D : F[C_G(A)]] = |G : C_G(A)|; \)
2. \( C_D(F[A]) = F[C_G(A)]; \)
3. \( F[A]/F \) is Galois with \( \text{Gal}(F[A]/F) \cong G/C_G(A). \)

In particular, if \( C_G(A) = A \) then \( F[A] \) is a Galois maximal subfield of \( D \) and hence \( D \) is a crossed product division algebra.

**Proof.** (1) Put \( H = C_G(A) \). Let \( T \) be a left transversal of \( H \) in \( G \). Clearly, \( T \) is a set of generators of \( D \) over \( F[H] \). Thus, it is enough to show that the elements of \( T \) are linearly independent over \( F[H] \). Let \( \sum_{j=1}^{k} a_j t_j = 0 \) be a relation of minimal length where \( a_j \in F[H] \) and \( t_j \in T \). We may assume that \( t_1 = 1 \). Now, for every \( a \in A \) we have \( 0 = a^{-1}(\sum_{j=1}^{k} a_j t_j)a = \sum_{j=1}^{k} a_j a^{-1}t_j^{-1}a^{-1}a t_j = \sum_{j=1}^{k} a_j t_j^{-1}a^{-1}t_j \). If we subtract this relation from \( \sum_{j=1}^{k} a_j t_j = 0 \), we obtain \( \sum_{j=2}^{k} (a_j - a_j t_j^{-1}a^{-1})t_j = 0 \). Since this is a shorter relation than \( \sum_{j=1}^{k} a_j t_j = 0 \), we must have \( a_j - a_j t_j^{-1}a^{-1} = 0 \) for all \( 2 \leq j \leq k \). Thus, \( t_j^{-1}a^{-1} = 1 \) and hence \( t_j \in C_G(A) = H \) for all \( j \), which is a contradiction.

(2) It is clear that \( F[C_G(A)] \subseteq C_D(F[A]) \). Conversely, let \( \{g_1, \ldots, g_n\} \subseteq G \) be a \( F \)-basis for \( D \). Suppose \( \sum u_{jk} g_{jk} \in C_D(F[A]) \) with \( 0 \neq u_{jk} \in F \). For each \( a \in A \), we have \( \sum au_{jk}g_{jk} = \sum u_{jk}g_{jk}a = \sum au_{jk}[g_{jk}^{-1}, a^{-1}]g_{jk} \). Therefore, \( au_{jk} = au_{jk}[g_{jk}^{-1}, a^{-1}] \) for every \( jk \).

But, \( u_{jk} \neq 0 \) and hence \( [g_{jk}^{-1}, a^{-1}] = 1 \). Thus every \( g_{jk} \) lies in \( C_G(A) \). This implies that \( C_D(F[A]) \subseteq F[C_G(A)] \) and the result follows.

(3) Consider the map \( \sigma : G/H \to \text{Gal}(F[A]/F) \), \( xH \mapsto (u \mapsto x^{-1}ux) \).

Clearly, \( \sigma \) is injective. On the other hand, by the equality \( C_D(F[A]) = F[C_G(A)] \) and the Centralizer Theorem (see [11, p. 42]), we have \( [F[A] : F] = [D : F[H]] = |G/H| \). So, \( F[A]/F \) is Galois and \( \sigma \) is an isomorphism.

The last statement follows from (3). \( \square \)
Corollary 2.8. Let $D$ be an $F$-central division algebra and $G$ be an absolutely irreducible subgroup of $D^*$. If one of the following conditions holds, then $D$ is a crossed product division algebra.

1. $G$ is abelian-by-nilpotent;
2. $G$ is metabelian.

Proof. (1) Let $A$ be a normal abelian subgroup of $G$ such that $G/A$ is nilpotent. Take a maximal abelian normal subgroup $B \supseteq A$. Since $G/B$ is nilpotent, $C_G(B)/B \cap Z(G/B) \neq 1$ unless $C_G(B) = B$. However, if $C_G(B) \neq B$, then we can choose an element $x \in C_G(B) \setminus B$ such that $\langle x, B \rangle/B \subseteq Z(G/B)$. Therefore, $\langle x, B \rangle \supseteq B$ is a normal abelian subgroup of $G$ which is a contradiction, because $B$ is maximal. Thus, $C_G(B) = B$ and the result follows by Lemma 2.7.

(2) If $G$ is metabelian, then $G'$ and $G/G'$ are abelian. Thus, $G$ is abelian-by-nilpotent and the result follows from (1).

Lemma 2.9. Let $D$ be a division algebra and $K$ be a subfield of $D$. Suppose that $G$ is a subgroup of $D^*$ centralizing $K$. If $G'Z(G) \subseteq K^*$, then $[K[G] : K] = |G : G \cap K^*|$.

Proof. Put $A = G \cap K^*$. Let $T$ be a left transversal of $A$ in $G$. Clearly, $T$ generates $K[G]$ as an $K$-algebra. Thus, if we show that the elements of $T$ are linearly independent over $K$, then we are done. Let $\sum_{j=1}^k a_j t_j = 0$ be a relation of minimal length where $0 \neq a_j \in K$ and $t_j \in T$. We may assume that $t_1 = 1$. Now, since $G' \subseteq K \subseteq C_D(G)$, for every $g \in G$ we have $0 = g^{-1}(\sum_{j=1}^k a_j t_j)g = \sum_{j=1}^k a_j t_j g = \sum_{j=1}^k a_j g^{-1}(t_j^{-1}, g^{-1})gt_j = \sum_{j=1}^k a_j [t_j^{-1}, g^{-1}]t_j$. Subtraction from $\sum_{j=1}^k a_j t_j = 0$, yields $\sum_{j=2}^k (a_j - a_j [t_j^{-1}, g^{-1}])t_j = 0$. Since this is a shorter relation than $\sum_{j=1}^k a_j t_j = 0$ (note that since $G' \subseteq K^*$, each $a_j [t_j^{-1}, g^{-1}]$ lies in $K$), we must have $a_j - a_j [t_j^{-1}, g^{-1}] = 0$ for all $2 \leq j \leq k$. Therefore, $[t_j^{-1}, g^{-1}] = 1$. So, $t_j \in Z(G) \subseteq K^*$ and hence $t_j \in A$ for all $2 \leq j \leq k$, which is a contradiction.

Proposition 2.10. Suppose that $D$ is an $F$-central division algebra and that $X$ and $Y$ are subgroups of $D^*$. If $Z(F[X]) = F$ and $Y \subseteq C_D(X)$, then $F[XY] = F[X] \otimes_F F[Y]$.

Proof. Clearly $F[XY]$ is an $F$-central division algebra. Thus, we have $F[XY] = F[X] \otimes_F C_{F[XY]}(F[X])$. But, $F[Y] \subseteq C_{F[XY]}(F[X])$, because $Y \subseteq C_D(X)$. Hence $F[X] \otimes_F F[Y]$ is a $F$-subalgebra of $F[XY]$. Now, let $\{x_1 \ldots x_r\}$ be a $F$-basis for $F[X]$ and $\{y_1 \ldots y_s\}$ be a $F[Y]$-basis for $F[XY]$. One may easily check that $\{x_i y_j | 1 \leq i \leq r, 1 \leq j \leq s\}$ is a $F$-basis for $F[XY]$. So, $\dim_F F[XY] = rs$. But, $y_j$'s are linearly independent over $F$. Therefore, $\dim_F F[Y] \geq s$. Thus, we must have $\dim_F (F[X] \otimes_F F[Y]) = \dim_F F[X] \dim_F F[Y] \geq rs = \dim_F F[XY]$. This implies that $F[XY] = F[X] \otimes_F F[Y]$.

The following result is a special case of [89, Th. 2.6].

Theorem 2.11. Let $D$ be an $F$-central division algebra and $G$ be an irreducible subgroup of $D^*$. Suppose that $N$ is a normal subgroup of $G$ and that $Z(F[N]) = F$. Then there exist subgroups $G_1$ and $G_2$ of $D^*$ such that:

1. $F^* \subseteq G_1$, $N \subseteq G_1$, $F[N] = F[G_1]$ and $G \subseteq G_1 G_2$;
(2) \( D = F[G_1] \otimes_F F[G_2] \);
(3) For every \( i \), there exists an epimorphism \( \phi_i : G \to G_i/F^* \). Moreover, \( N \subseteq \ker \phi_2 \).

Proof. (1) Since \( N \leq G \), every \( g \in G \) induces an \( F \)-automorphism \( u \mapsto g^{-1}ug \) of \( F[N] \). Thus, by the Skolem-Noether Theorem, there is an element \( f_1(g) \in F[N] \) that has the same effect as \( g \) on \( F[N] \) (and of course on \( N \)). So, for every \( u \in F[N] \), we have \( g^{-1}ug = f_1(g)^{-1}uf_1(g) \) and hence \( g f_1(g)^{-1} \in C_D(F[N]) \). Put \( f_2(g) = g f_1(g)^{-1} \). But, it is not hard to check that \( f_1(g) \) and \( f_2(g) \) are unique modulo \( F^* \). Thus, for all \( g, h \in G \) and \( i = 1, 2 \), \( f_i(gh) \in f_i(g) f_i(h) F^* \). Now, set \( G_i = F^*(f_i(g) : g \in G) \). If \( g \in G \), from the equality \( g = f_1(g) f_2(g) \) we conclude that \( G \subseteq G_1 G_2 \). Also, if \( u \in N \), then \( u = f_1(u) k \) for some \( k \in F^* \). Therefore \( N \subseteq G_1 \). Moreover, since \( f_1(g) \) has the same action as \( g \) on \( N \), we have \( N \subseteq G_1 \). Finally, because \( N \subseteq G_1 \subseteq F[N] \), we have \( F[N] \subseteq F[G_1] \subseteq F[N] \). Hence \( F[N] = F[G_1] \).

(2) By construction we have \( G_2 \subseteq C_D(G_1) \). Now, from Proposition 2.10, we conclude that \( F[G] = F[G_1 G_2] = F[G_1] \otimes_F F[G_2] \).

(3) For every \( i = 1, 2 \), the map \( \phi_i : G \to G_i/F^*, g \mapsto f_i(g) F^* \) is the required epimorphism. Furthermore, since \( f_2(n) = n f_1(n)^{-1} \in F^* \) for all \( n \in N \), we have \( N \subseteq \ker \phi_2 \). \( \square \)

Recall that by the work of Amitsur [6, Th. 2], if \( G \) is a finite subgroup of the unit group of a division ring, then \( G \) is one of the following type:

(i) All Sylow subgroups of \( G \) are cyclic;
(ii) All of odd Sylow subgroups of \( G \) are cyclic and the Sylow 2-subgroup of \( G \) is a generalized quaternion group \( Q_{2n+1} \) of order \( 2^{n+1} \) for some natural number \( n \geq 2 \).

Here, we must point out that the generalized quaternion group of order \( 2^{n+1} \) is a group generated by two element \( x, y \) satisfying \( y^{2n+1} = x^2, x^4 = 1 \) and \( xyx^{-1} = y^{-1} \). The above classification is all the material about finite subgroups of a division ring that we need in our approach.

Lemma 2.12. Let \( D \) be a division algebra and \( N \) be a finite subgroup of \( D^* \). If \( N \) is nilpotent of class at most 2, then \( N \) is either cyclic or \( N \cong Q_8 \times C \) where \( C \) is a cyclic group.

Proof. Since \( N \) is nilpotent, it is isomorphic to the direct product of its all Sylow subgroups. Here, two cases may occur:

Case 1. All Sylow subgroups of \( N \) are cyclic. Clearly, in this case \( N \) is cyclic.

Case 2. The Sylow 2-subgroup of \( N \) is \( Q_{2n+1} \) and other Sylow subgroups are cyclic. If this is the case, then \( N \cong Q_{2n+1} \times C \) where \( C \) is a cyclic group. Thus, we must prove that \( n = 2 \). Suppose that \( x, y \) are the generators of \( Q_{2n+1} \). Since \( Q_{2n+1} \) is nilpotent of class 2, we have \( Q_{2n+1} \subseteq Z(Q_{2n+1}) \). Therefore, \( y^2 = [y^{-1}, x^{-1}] \in Z(Q_{2n+1}) \) and so \( [x^{-1}, y^2] = 1 \). But, \( [x^{-1}, y^2] = y^4 \). Thus, we must have \( y^4 = 1 \). This implies that \( n = 2 \), as desired. \( \square \)

To proceed, we need also the following theorem which is a direct consequence of [90, Th. 2.1.11].

Theorem 2.13. If \( D \) is an \( F \)-central division algebra containing a subgroup \( S \cong SL(2, 5) \), then \( Z(F[S]) = F \) if \( \sqrt{5} \in F \) and \( Z(F[S]) = F(\sqrt{5}) \) if \( \sqrt{5} \notin F \). Therefore, \( [Z(F[S]) : F] \leq 2 \).
Now, we can prove the following important result.

**Theorem 2.14.** Let $D$ be an $F$-central division algebra and $G$ be a soluble-by-finite absolutely irreducible subgroup of $D^*$. Then:

1. If $G$ contains a normal subgroup $S \cong SL(2, 5)$, then there are soluble-by-finite subgroups $G_1, G_2$ of $D^*$ such that $F[G_1] = F[Q]$ (where $Q \cong Q_8$), $D = F[Q] \otimes_F F[G_2]$ and $G_2$ has no normal subgroup isomorphic to $Q_8$ or $SL(2, 5)$. Moreover, if $Z(F[S]) = F$ then $S \subseteq G_1$ and if $[Z(F[S]) : F] = 2$, then $Q \subseteq G_1$.

2. If $G$ has a normal subgroup $Q \cong Q_8$, then there are soluble-by-finite subgroups $G_1, G_2$ of $D^*$ such that $F[G_1] = F[Q]$, $D = F[Q] \otimes_F F[G_2]$ and $G_2$ has no normal subgroup isomorphic to $Q_8$ or $SL(2, 5)$. Moreover, $Q \subseteq G_1$.

3. If $G$ has no normal subgroup isomorphic to $Q_8$ or $SL(2, 5)$, then it contains a maximal abelian normal subgroup $A$ of index dividing $\deg(D)^2$. Moreover, $F[A]/F$ is Galois with $G = C_G(A)$, $C_G(A)' \subseteq A$ and so $C_G(A)$ is nilpotent of class at most $2$.

**Proof.** (1) First suppose that $G$ has a normal subgroup $S \cong SL(2, 5)$. Put $L = Z(F[S])$. Then, by Theorem 2.13, two cases may occur:

**Case 1.** $L = F$. So $F[S] = F[Q]$ where $Q$ is the $Q_8$-subgroup of $S$. Therefore, $\deg(F[S]) = 2$. Now, by Theorem 2.11, there exist subgroups $G_1$ and $G_2$ of $D^*$ such that $D = F[Q] \otimes_F F[G_2]$. Also, $G \subseteq G_1G_2$ and $S \subseteq G_1$. If $G_2$ contains a normal subgroup isomorphic to $Q_8$ or $SL(2, 5)$, then by Theorem 2.11, we can decompose $F[G_2]$ as $F[G_2] = F[Q] \otimes_F F[G_3]$ for some subgroup $G_3$. But, this decomposition implies that $M_4(F) = F[Q] \otimes_F F[Q]$ is a subalgebra of $D$ which is a contradiction. So, we conclude that $G_2$ contains no normal subgroup isomorphic to $Q_8$ or $SL(2, 5)$. Moreover, since $G_1, G_2$ are homomorphic images of $G$, they are soluble-by-finite.

**Case 2.** $[L : F] = 2$. Since $G$ normalizes $F[S]$, it also normalizes $L^*$. Therefore, $M = L^*G$ is a soluble-by-finite subgroup of $D^*$ and of course it is absolutely irreducible. Moreover, $S \triangleleft M$. Let $H = C_M(L)$. It is not hard to see that $SC_M(S) \subseteq H$. But, $M = M/C_M(S)$ is isomorphic to a subgroup of $\Aut(S) \cong \Sym(5)$. On the other hand, $SC_M(S)/C_M(S) \cong S/Z(S)$ and hence $|SC_M(S) : C_M(S)| = 60$. Moreover, by Lemma 2.7, we have $|M : H| = 2$. Thus, $\overline{M} = |M : H||H : C_M(S)| \geq |M : H||SC_M(S) : C_M(S)| = 120$. Therefore, $\overline{M} \cong \Sym(5)$ and $H = SC_M(S)$. Let $Q$ be the $Q_8$-subgroup of $S$ and let $T \supseteq C_M(S)$ be such that $T$ is a Sylow 2-subgroup of $\overline{M}$ containing $Q$. Thus $QC_M(S) \subseteq T$ and so $F[S] = L[Q] \subseteq F[T]$. On the other hand $T$ properly contains $C_M(S)$ and $|M : SC_M(S)| = 2$, Thus, we conclude that $M = ST$. Then $D = F[M] = F[ST] = F[S][T] = F[T]$. Also, since $[T : Q] = 2$, we conclude that $Q \triangleleft T$. Now, by Theorem 2.11 we can find a decomposition of $D$ as $D = F[G_1] \otimes_F F[G_2]$ where $M \subseteq G_1G_2$, $F[G_1] = F[Q]$ and $Q \triangleleft G_1$. Moreover, a similar argument as in Case 1, shows that $G_2$ has no normal subgroup isomorphic to $Q_8$ or $SL(2, 5)$.

(2) This follows from Theorem 2.11 and a similar argument as in Case 1 of (1).

(3) By Lemma 2.3, $G$ contains a normal abelian subgroup $A$ of finite index. Take $A$ maximal and put $H = C_G(A)$. If $H = A$, then $F[A]$ is a maximal subfield and hence by
Lemma 2.7, we have \(|G : A| = [D : F[A]] = \text{deg}(D)\) and the result follows. Thus, we assume that \(H \neq A\). Clearly, \(Z(H) = A\). Therefore, \(H\) is center-by-finite and so \(H'\) is finite. But, by our assumption \(G\) has no normal subgroup isomorphic to \(SL(2,5)\). This forces that \(H'\) is soluble. Now, we claim that \(H'\) is abelian. Suppose that \(H'\) is non-abelian. Let \(H^j\) denote the \(j\)-th term of the derived series of \(H\) and \(t\) be the smallest integer such that \(H^t = 1\). So, \(H^{t-2}\) is non-abelian metabelian. Also, \(H^{t-1}\) is abelian and normal in \(G\). Since \(H^{t-1} \subseteq C_G(A)\), \(AH^{t-1}\) is a normal abelian subgroup of \(G\) and hence \(AH^{t-1} = A\), because \(A\) is maximal. Therefore, \(H^{t-1}\) is central in \(H\) and so \(H^{t-2}\) is nilpotent of class 2. Now, by Lemma 2.12, \(H^{t-2}\) is either cyclic or \(Q_8 \times C\) where \(C\) is cyclic. But, \(H^{t-2}\) is non-abelian, so it can not be cyclic. Moreover, if \(H^{t-1} = Q_8 \times C\), then \(Q_8 \triangleleft G\) which is impossible. This contradiction shows that \(H'\) is abelian and hence \(H' \subseteq A\), because \(A\) is maximal. Now, put \(K = F[A]\). By Lemma 2.9 and Lemma 2.7, we have \(|H : K^* \cap H| = [K[H] : K] = \text{deg}(F[H])^2\). On the other hand, \(A \subseteq K^* \cap H \triangleleft G\). Since \(A\) is maximal, we conclude that \(A = K^* \cap H\). Thus, \(|H : A| = \text{deg}(F[H])^2\). Now, Lemma 2.7 implies that \(|G : A| = |G : H[H : A] = [F[G] : F[H]] \text{deg}(F[H])^2\). Finally, the Double Centralizer Theorem gives us \([F[G] : F[H]] \text{deg}(F[H]) = \text{deg}(F[G])\) and consequently \(|G : A|\) divides \(\text{deg}(D)^2\).

**Corollary 2.15.** Let \(D\) be an \(F\)-central division algebra and \(G\) be a soluble-by-finite subgroup of \(D^*\). If \(G\) has no normal subgroup isomorphic to \(SL(2,5)\) (in particular, if \(G\) is soluble), then \(G\) contains a normal abelian subgroup of index dividing \(6 \text{deg}(D)^2\).

**Proof.** Without loss of generality, we may assume that \(G\) is absolutely irreducible. If \(G\) has no normal \(Q_8\)-subgroup, the result follows immediately by (3) of Theorem 2.14. Now, suppose that \(G\) has a normal subgroup \(Q \cong Q_8\). Using (2) of Theorem 2.14, we can find a decomposition \(D = F[G_1] \otimes_F F[G_2]\) for some soluble-by-finite subgroups \(G_1, G_2\) where \(G \subseteq G_1G_2\), \(Q \triangleleft G_1\) and \(G_2\) has no normal \(SL(2,5)\) or \(Q_8\)-subgroup. Now, by the former case, \(G_2\) contains a normal abelian subgroup \(A\) of index dividing \(\text{deg}(D)^2/4\) (note that \(\text{deg}(F[G_2]) = 2\)). Also, \(G_1 \subseteq N = N_{F[Q]}(Q)\). Since \(Q\) is an absolutely irreducible subgroup of \(F[Q]\), we have \(C_N(Q) = F^*\). Therefore, \(N/F^*\) is isomorphic to a subgroup of \(\text{Sym}(4)\) and so \(|G_1 : F^*|\) divides 24 (recall that \(F^* \subseteq G_1\)). Hence, \(|G_1G_2 : F^*A|\) divides 24 \(\text{deg}(D)^2/4 = 6 \text{deg}(D)^2\). So, \(|G : G \cap F^*A|\) divides \(6 \text{deg}(D)^2\), as desired.

**Theorem 2.16.** Let \(D\) be an \(F\)-central division algebra such that \(D^*\) contains an absolutely irreducible soluble-by-finite subgroup \(G\). Then:

1. \(D\) is a quasi-crossed product division algebra;
2. If \(\text{deg}(D)\) is such that every finite group of order \(\text{deg}(D)^2\) is nilpotent, or if \(G\) has no element of order dividing \(\text{deg}(D)\), then \(D\) is a crossed product division algebra over a maximal subfield.

**Proof.** (1) First, suppose that \(G\) has no subgroup isomorphic to \(Q_8\) or \(SL(2,5)\). By Theorem 2.14, \(G\) contains an abelian normal subgroup \(A\) of finite index such that \(F[A]/F\) is Galois. Also, \(H = C_G(A)\) is nilpotent. Thus, if \(B \supseteq A\) is a maximal normal abelian subgroup of \(C_G(A)\) of finite index, then by Corollary 2.8, we have \(C_H(B) = B\) and \(F[B]/F[A]\) is Galois with \(\text{Gal}(F[B]/F[A]) \cong H/C_H(B)\) (note that by Lemma 2.7, \(Z(F[H]) = F[A]\)).
Also, $H/C_H(B)$ is abelian, because $H' \subseteq A \subseteq B$. Furthermore, by the Double Centralizer Theorem $F[B]$ is a maximal subfield of $D$. So, we have a tower of subfields $F \subseteq F[A] \subseteq F[B]$ such that $F[B]$ is a maximal subfield of $D$, $F[A]/F$ is Galois and $F[B]/F[A]$ is abelian Galois, as desired.

Now, if $G$ contains a normal subgroup isomorphic to $Q_8$ or $SL(2,5)$, Then by Theorem 2.14, $D = F[Q'] \otimes_F F[G_2]$ where $G_2$ is soluble-by-finite that has no subgroups isomorphic to $Q_8$ or $SL(2,5)$. But, as above, there exists a tower of subfields $F \subseteq K \subseteq L$ of $F[G_2]$ such that $L$ is a maximal subfield of $F[G_2]$, $K/F$ is Galois and $L/K$ is abelian Galois. Now, one can easily check that $F \subseteq K \otimes_F F(\sqrt{-1}) \subseteq L \otimes_F F(\sqrt{-1})$ is the required tower of subfields for $D$.

(2) Suppose that every finite group of order $\deg(D)^2$ is nilpotent. Let $G$ contain no subgroups isomorphic to $Q_8$ or $SL(2,5)$ and let $A$ be a maximal normal abelian subgroup of $G$ such that $|G:A|$ divides $\deg(D)^2$ (Theorem 2.14). Then, $G$ is abelian-by-nilpotent and hence by Corollary 2.8, $D$ is a crossed product division algebra. Now, suppose that $G$ contains a copy of $Q_8$ or $SL(2,5)$ as a normal subgroup. By Theorem 2.14, $D$ is decomposed as $D = F[Q'] \otimes_F F[G_2]$ where $G_2$ has no normal subgroups $Q_8$ or $SL(2,5)$-subgroup. Thus, as above $F[G_2]$ is a crossed product division algebra. Hence, $D$ is a crossed product division algebra.

Finally, consider the case where $G$ has no elements of order dividing $\deg(D)$. Clearly, $G$ has no $Q_8$ or $SL(2,5)$-subgroup. Let $A$ be a maximal normal abelian subgroup of $G$ with $|G:A| < \infty$. Put $H = C_G(A)$. by Lemma 2.7, $F[H]$ is an $F[A]$-central division algebra. Moreover, Theorem 2.14 implies that $H' \subseteq F[H]' \cap F[A]$. So, if $a \in H'$ then $a^{\deg(F[H])} = \text{Nrd}_{F[H]}(a) = 1$. Therefore, $\text{ord}(a)$ divides $\deg(D)$ which leads to $a = 1$. Thus, $H' = 1$ and so $H$ is abelian. Since $A$ is maximal, we conclude that $H = A$ and the result follows.

As we mentioned before, by the works of Shirvani and Wehrfritz, if $G$ is a soluble-by-finite subgroup of a division algebra $D$, then $G$ has a normal abelian subgroup of index dividing $30 \deg(D)^2$. But in Corollary 2.15, the case in which $G$ has no normal $SL(2,5)$-subgroups is considered. Thus, if we would like to achieve the above bound, we must consider the case $SL(2,5) \leqslant G$. If $Z(F[S]) = F$, a similar method as in Corollary 2.15 is enough to find the required bound. For, since $Z(F[S]) = F$, Theorem 2.14 gives a decomposition $D = F[G_1] \otimes_F F[G_2]$. Since $Z(F[S]) = F$, we must have $C_N(S) = F^*$ where $N = N_{D^*}(S)$. Thus $N/F^* \cong \text{Aut}(S) \cong \text{Sym}(5)$. But, since $G_1 \subseteq N$ we conclude that $|G_1 : F^*|$ divides 120. On the other hand, by Corollary 2.15, $G_2$ has a normal abelian subgroup $A$ of index dividing $\deg(D)^2/4$. Therefore, $|G : G \cap F^*A|$ divides $120 \deg(D)^2/4 = 30 \deg(D)^2$. But, for the case $[Z(F[S]) : F] = 2$ we need a little more work. In fact we need the following stronger version of Theorem 2.14.

**Theorem 2.17.** Let $D$ be a division algebra and $G$ be a soluble-by-finite absolutely irreducible subgroup of $D^*$. If $G$ has no normal subgroups isomorphic to $Q_8$ or $SL(2,5)$, then it contains a characteristic abelian subgroup of index dividing $\deg(D)^2$. 

To prove Theorem 2.17, we need to recall the concept of a general crossed product algebra and a useful lemma of [103]. Let $R$ be a ring, $S$ a subring of $R$ and $G$ a subgroup of the group of units of $R$ normalizing $S$. If $R = S[G]$ and if $N = G \cap S$ is a normal subgroup of $G$ with $R = \bigoplus_{t \in T} St$ for some (and hence any) transversal $T$ of $N$ to $G$, we say that $R$ is a crossed product algebra of $S$ by $G/N$. Note that Lemma 2.7, essentially says that if $D$ is a division algebra and $A$ is a normal abelian subgroup of an absolutely irreducible subgroup $G$ of $D^*$, then $D$ is a crossed product division algebra of $F[C_G(A)]$ by $G/C_G(A)$.

**Lemma 2.18.** Let $R = S[G]$ be a ring where $S$ is a subring of $R$ and $G$ is a subgroup of $R^*$ normalizing $S$. Suppose that $R$ is a crossed product of $S[N_i]$ by $G/N_i$ for normal subgroups $N_i$ of $G$. Set $N = \bigcap N_i$. Then $R$ is a crossed product of $S[N]$ by $G/N$.

**Proof.** See [103, p. 213].

Now, Lemma 2.18 helps us prove Theorem 2.17:

**Proof of Theorem 2.17.** By Theorem 2.14 and Lemma 2.7, $G$ has a maximal abelian normal subgroup $A$ of finite index such that $H = C_G(A)$ is nilpotent of class at most 2 and $D$ is a crossed product of $F[H]$ by $G/H$. Now, it is not hard to see that for each $\phi \in \text{Aut}(G)$, $D$ is a crossed product of $F[H^\phi]$ by $G/H^\phi$. Put $\mathcal{H} = \bigcap\phi H^\phi$ where $\phi$ runs through $\text{Aut}(G)$. Clearly $\mathcal{H}$ is a characteristic subgroup of $G$ that is nilpotent of class at most 2. From Lemma 2.18, it follows that $D$ is a crossed product of $F[\mathcal{H}]$ by $G/\mathcal{H}$. Now, $A = Z(\mathcal{H})$ is a characteristic subgroup of $G$ with $\mathcal{H}' \subseteq A$. But, $|\mathcal{H} : A| = |F[\mathcal{H}] : F[A]|$ by Lemma 2.9. Thus $|G : A| = |G : \mathcal{H}||\mathcal{H} : A| = [D : F[\mathcal{H}]][F[\mathcal{H}] : F[A]] = [D : F[A]]$, as required.

Now, let $G$ be an absolutely irreducible soluble-by-finite subgroup of $D^*$. Suppose that $G$ has a normal $\text{SL}(2,5)$-subgroup $S$ and that $[Z(F[S]) : F] = 2$. In the proof of Theorem 2.14(1) we saw that $G$ is contained in a soluble-by-finite subgroup $M$ such that $S \lhd M$. Moreover, $|M : SC_M(S)| = 2$ and $|SC_M(S) : C_M(S)| = 60$. It is not hard to see that $C_M(S)$ has no normal $\text{SL}(2,5)$ or $Q_8$-subgroup and $\text{deg}(F[C_M(S)])$ divides $\text{deg}(D)/2$. Thus, $C_M(S)$ has a characteristic abelian subgroup $A$ of index dividing $\text{deg}(D)^2/4$. Now, since $A$ is characteristic in $C_M(S)$ and $C_M(S) \lhd M$, we conclude that $A \lhd M$. Finally, $|M : A| = |M : SC_M(S)||SC_M(S) : C_M(S)||C_M(S) : A| = 2 \times 60 \times \text{deg}(D)^2/4 = 30 \text{deg}(D)^2$. Therefore, $|G : G \cap A|$ divides $30 \text{deg}(D)^2$. Thus, we have proved that

**Theorem 2.19.** Let $D$ be a division algebra and $G$ be a soluble-by-finite subgroup of $D^*$. Then $G$ has a normal abelian subgroup of index dividing $30 \text{deg}(D)^2$.

Here, we should point out that the above results concerning the structure of soluble-by-finite subgroups of division algebras were nicely obtained by Wehrfritz for the soluble-by-finite subgroups of $\text{GL}_n(D)$, where $D$ is a division algebra in [107]. His results demonstrates that if $G$ is a soluble-by-finite subgroup of $\text{GL}_n(D)$, then $G$ has an abelian normal subgroup of finite index dividing $b(n)(\text{deg}(D)^2)^n$, where $b(n)$ is an integer valued function which depends only on $n$. By an interesting result of [45], the existence of a finite absolutely irreducible subgroup in a division algebra $D$ guarantees that $D$ is a crossed product division algebra. Here, we give a very short proof for this fact.
Theorem 2.20. Let $D$ be an $F$-central division algebra such that $D^*$ contains an irreducible finite subgroup $G$. Then $D$ is a crossed product division algebra.

Proof. From the first section, recall that $G$ is one of the groups of types (A1)-(A6). We consider all the above cases separately.

(A1) Since $G$ contains no subgroup isomorphic to $Q_8$ or $SL(2,5)$, by Theorem 2.14, it contains a maximal normal abelian subgroup $A$ such that $C_G(A)$ is nilpotent of class at most 2. Therefore, $C_G(A)$ is abelian by Lemma 2.12. But, $A$ is maximal. So, $C_G(A) = A$ and the result follows by Lemma 2.7.

(A2) Let $x \in Q_{2n+1}$ have order $2^n$. Since $x$ centralizes $C_m$, the subgroup $A = C_m \times \langle x \rangle$ is abelian and of course $|G : A| = 2$. Thus $A < G$ and clearly $A$ is non-central. Therefore, $C_G(A) = A$, as desired.

(A3) By Proposition 2.10, we have $D = F[Q_8] \otimes_F F[M]$. Now, by the case (A1), $F[M]$ is a crossed product division algebra which implies that $D$ is a crossed product division algebra.

(A4) Let $Q$ be the normal $Q_8$-subgroup of $G$. By Theorem 2.11, $D$ has a decomposition as $D = F[Q] \otimes_F F[G_2]$, where $G_2/F^*$ is a homomorphic image of $G/Q$. Thus, $|G_2 : F^*|$ divides 6. Therefore, if $G_2$ is non-abelian, then we must have $G_2/F^* \cong \text{Sym}(3)$. Put $A = F^*(a)$, where $a \in G_2$ has order 3 in $G_2/F^*$. Clearly, $A$ is an abelian subgroup of $G_2$ with $|G_2 : A| = 2$. Thus, $A < G_2$ and $C_G(A) = A$ and the result follows.

(A5) Recall that $SL(2,3) \cong \langle i, j, -(1+i+j+ij)/2 \rangle$ where $i, j$ are the usual generators of $Q_8$. Thus, we can easily see that $F[SL(2,3)] = F[Q]$ (where $Q$ is the $Q_8$-subgroup of $SL(2,3)$). Now, a similar argument as used in (A3) shows that $D$ is a crossed product division algebra.

(A6) From [90, Th. 2.1.11] we know that in this case $D$ is the ordinary quaternion division algebra. Thus, it is a crossed product division algebra. \hfill \Box

2.2. Tits’ Alternative for subgroups of a division algebra. As we have seen in the introduction, there are several group theoretic properties whose occurrence in a subnormal subgroup $G$ of $D^*$ lead to the commutativity of $G$. For example, if $G$ is periodic or soluble then by known results of Herstein [34] and Stuth [91] $G$ is contained in $F^*$. Also, whenever $\dim_F D < \infty$ and $G$ satisfies a group identity, then combining the above results and a theorem of Platonov [109, p. 149] entails that $G \subset F^*$. But, many of such conditions are special cases of a more general condition:

"$G$ has no non-cyclic free subgroups."

The existence of a non-cyclic free subgroup in $D^*$ has been posed as a conjecture by Lichtman in [53] and then in subnormal subgroups of $D^*$ by Goncalves and Mandel in [18]. However, it is known that by the work of Goncalves if $D$ is of finite dimension over its center, then these problems have positive answers (cf. [44, p. 207]). On the other hand, in the finite dimensional setting, a stronger result obtained by Wehrfritz asserts that if deg($D$) is a prime power then every subgroup of $D^*$ either contains a non-cyclic free subgroup or possesses an abelian normal subgroup of a finite index dividing $60 \cdot \text{deg}(D)^2$ (see [104]). Here, using the
results of the previous section, we give a similar characterization without any condition on \( \deg(D) \). The main theorem of this section is:

**Theorem 2.21.** Let \( D \) be a finite dimensional division algebra over its center \( F \) and let \( G \) be a subgroup of \( D^* \). Then the following statements are equivalent:

1. \( G \) contains no non-cyclic free subgroups;
2. \( G \) is soluble-by-finite;
3. \( G \) is abelian-by-finite;
4. \( G \) satisfies a group identity.

Moreover, if \( \deg(D) \) is odd then we can replace (2) by

2'. \( G \) is soluble.

Furthermore, if one of the above conditions holds then \( G \) contains an abelian normal subgroup of index dividing \( 30 \deg(D)^2 \).

To prove Theorem 2.21, we need to recall

**Theorem 2.22** (Tits’ Alternative [94]). If \( F \) is a field then

1. If \( \text{char}(F) = 0 \) then every subgroup of \( \text{GL}_n(F) \) either is soluble-by-finite or contains a noncyclic free subgroup;
2. If \( H \) is a finitely generated subgroup of \( \text{GL}_n(F) \), then either \( H \) is soluble-by-finite or \( H \) contains a non-abelian free subgroup.

Note that Theorem 2.21 essentially says that a same conclusion as in Tits’ Alternative is valid for every subgroup \( G \) of \( D^* \) without any extra condition on the set of generators of \( G \) or \( \text{char} \, D \). Thus, it can be considered as a version of Tits’ Alternative for subgroups of division algebras.

We need also the following lemma which is a special case of [101, Lem. 1] combining with Schur’s Theorem (cf. [51, Th. 9.9]).

**Lemma 2.23.** Let \( G \) be a subgroup of \( \text{GL}_n(F) \) such that every finitely generated subgroup of \( G \) is abelian-by-finite. Then \( G \) has an abelian normal subgroup \( A \) such that \( G/A \) is a periodic linear group over \( F \), so that \( G/A \) is a locally finite group.

Now, we are in a position to prove Theorem 2.21.

**Proof of Theorem 2.21.** Lemma 2.3 yields that (2) and (3) are equivalent. If \( A \) is a normal abelian subgroup of \( G \) with \( |G : A| = e \) then clearly \( [x^e, y^e] = 1 \) is a group identity for \( G \). This proves (3)\( \Rightarrow \) (4). Also, (4)\( \Rightarrow \) (1) is trivial. So, it remains to prove (1)\( \Rightarrow \) (3). View \( G \) as a subgroup of \( \text{GL}_n(F) \) where \( n = \dim_F D \). If \( \text{char}(F) = 0 \) then by (1) of Theorem 2.22 \( G \) is soluble-by-finite and the result follows from Lemma 2.23 in this case. Thus, we may assume that \( \text{char}(F) > 0 \). If this is the case, then from Tits’ Alternative, it follows that every finitely generated subgroup of \( G \) is soluble-by-finite. Combining this with Lemma 2.3 implies that every finitely generated subgroup of \( G \) is abelian-by-finite. Now, by Lemma
we conclude that $G$ contains an abelian normal subgroup $A$ such that $G/A$ is locally finite. Put

$$S = \{ N \triangleleft G | N \text{ is abelian and } G/N \text{ is locally finite} \}.$$ 

Since $A \in S$, it follows that $S \neq \emptyset$. So we can choose a maximal non-trivial element $B \in S$. Let $H = C_G(B)$. Since $B$ is maximal in $S$ it can be seen that $Z(H) = B$ and so $H$ is center-by-(locally finite). This forces that $H'$, the derived subgroup of $H$, is locally finite. Now, given $h, k \in H'$ the subgroup $\langle h, k \rangle$ is finite and so is cyclic as $\text{char}(F) > 0$. Thus $hk = kh$ which means that $H'$ is abelian. But, $H' \triangleleft G$ and centralizes $B$. Therefore $BH'$ is a normal abelian subgroup of $G$ and thus $BH' = B$, because $B$ is maximal. Hence $H' \subseteq B$. So, if we put $K = F[B]$ then from Lemma 2.9 it follows that $[K[H] : K] = |H : B|$. On the other hand from Lemma 2.7 we know that $[F[G] : F[H]] = |G : H|$. This gives $|G : A| = [F[G] : F[H]][F[H] : K]$ (note that $F[H] = K[H]$) which immediately implies that $|G : A|$ divides $\text{deg}(D)^2$. Thus, $G$ is abelian-by-finite.

Now, if $\text{deg}(D)$ is odd, then $G$ contains an abelian normal subgroup $A$ of index dividing $\text{deg}(D)^2$ by Theorem 2.14. Now, since $G/A$ has an odd order, by Feit-Thompson theorem about the solubility of the groups of odd order ([17]), it is soluble and hence in this case we have $(2) \Rightarrow (2)'$.

The rest follows from Theorem 2.19. 

3. FINITELY GENERATED SUBNORMAL SUBGROUPS OF DIVISION RINGS

As we have seen in the first section, by a theorem of Herstien every periodic subnormal subgroup of $D^*$ is central. Trivially, this implies that every finite subnormal subgroup in a division ring is contained in the center. However, a natural question is to ask what happens when we replace the phrase “finite” by “finitely generated”. This problem was initially investigated in [2], where Akbari and Mahdavi-Hezavehi proved that if $D$ is a division algebra and its center is an algebraic extension of $\mathbb{Q}$ then every finitely generated normal subgroup of $D^*$ is central. They also showed that for every division algebra $D$ and every $n \geq 2$ the group $\text{GL}_n(D)$ has no noncentral finitely generated normal subgroups. Another notable result in this direction obtained in [4] imparts that every finitely generated normal subgroup of a division algebra is central. Finally, the structure of finitely generated normal subgroups of general linear groups over division algebras was completely determined in [66]. The main result of this paper guarantees that for every division algebra $D$ and each $n \in \mathbb{N}$, the group $\text{GL}_n(D)$ has no noncentral finitely generated subnormal subgroups. This section is mainly devoted to give this observation and some peripheral consequences. Before presenting the main result, it is beneficial to list some needful old theorems.

**Theorem 3.1** (Stuth [91]). Let $D$ be a division ring. Then every subnormal soluble subgroup of $D^*$ is central.

**Theorem 3.2** (Herstein [34]). All periodic subnormal subgroups of a division ring are central.

**Corollary 3.3.** Let $D$ be a division ring. Then every subnormal soluble-by-finite subgroup of $D^*$ is central.
Proof. Let $H$ be a subnormal soluble-by-finite subgroup of $D^*$ and $S$ be a soluble normal subgroup of $H$ of finite index. Now, subnormality of $S$ in $D^*$ implies that $S$ is central (Theorem 3.1). This forces that $H$ is center-by-finite and so $H'$ is a finite subnormal subgroup of $D^*$. Now, from Theorem 3.2 it follows that $H'$ is central and so it is abelian. Therefore, $H$ is soluble and hence it is central. □

Theorem 3.4. Let $F$ be a field and $H$ be a finitely generated subgroup of $G = GL_n(F)$. If \( \{a + xI_n \mid a \in H, x \in F\} \subseteq N_G(H) \), then $H$ is soluble-by-finite.

Proof. If $F$ is finite, there is nothing to prove. Thus, we may assume that $F$ is infinite. Let $H = \langle a_1, a_2, \ldots, a_k \rangle$ and $R$ be the subring of $M_n(F)$ generated by the elements of the set \( \{a_j, a_j^{-1}\}_{j=1}^k \). Now, one can easily check that $R \subseteq M_n(P(\Lambda))$ where $P$ is the prime subfield of $F$ and $\Lambda$ is the set of elements in $F$ occurring as the entries of $a_j$ and $a_j^{-1}$, $j = 1, \ldots, k$. Thus, we have $H \subseteq GL_n(P(\Lambda))$. Now, the Noether Normalization Lemma implies that $P(\Lambda)$ contains a subfield $L$ of finite codimension such that $L = Q$ or $L = K[y]$ for some subfield $K$ and $y$ is transcendental over $K$ (note that since $H$ is infinite, if char($F$) > 0, then $P(\Lambda)$ is not algebraic over $P$). Put $m = [P(\Lambda) : L]$ and consider the following sequence of mappings

\[
M_n(P(\Lambda)) \xrightarrow{\text{nat}} M_n(L) \otimes_L P(\Lambda) \xrightarrow{1 \otimes u} M_n(L) \otimes_L M_m(L) \xrightarrow{\text{nat}} M_m(L),
\]

where $\otimes$ is the regular representation. Since all the above maps are injective, $H$ is embedded in $GL_m(L)$. Also, since $\otimes : u \mapsto uI_m$ we have \( \{a + uI_{nm} \mid a \in H, u \in L\} \subseteq N_{GL_m(L)}(H) \). Now, since $L$ is the field of fractions of $Z$ or $K[y]$ and $H$ is finitely generated, a similar argument as above implies that $H \subseteq GL_m(S[\Delta])$ where $S$ is either $Z$ or $K[y]$ and $\Delta = \{u_1/v_1, \ldots, u_t/v_t\}$ is a finite subset of $L$. Now, let $H$ have a free subgroup $\mathcal{F}$ and $a, b \in \mathcal{F}$ be arbitrary. Suppose at least one of the entries of $b + xI_{nm}$ is a polynomial in $x$ of degree $n$. Since \( \text{deg}(b + xI_{nm})^{-1}aba(b + xI_{nm}) \in H \) depends on $x$. Since \( \text{det}(b + xI_{nm}) \) is a polynomial in $x$ of degree $nm$, we conclude that for each $1 \leq i, j \leq nm$, the $(i, j)$th entry of $(b + xI_{nm})^{-1}$ is of the form $p_{ij}(x)/q(x) \in L(x)$ where $\text{deg} q(x) = nm$ and $\text{deg} p_{ij}(x) \leq nm - 1$. So, the $(i, j)$th entry of $(b + xI_{nm})^{-1}aba(b + xI_{nm})$ is of the form $p_{ij}(x)/q(x) \in L(x)$ where $\text{deg} q(x) = nm$ and $\text{deg} p_{ij}(x) \leq nm$. Let the $(r, s)$th entry of $(b + xI_{nm})^{-1}aba(b + xI_{nm})$ depend on $x$. Put $p_r(x) = \sum_{i=0}^{nm} a_i x^i$ and $q(x) = x^{nm+1} + \sum_{i=0}^{nm-1} b_i x^i$. If $a_0 = u_{r+1}/v_{r+1}$, then $p_r(x)/q(x) = p_{rs}(x)/q(x) - a_0 \in S[\Delta \cup \{u_i/v_i\}]$ for each $x \in L$. Also, it is clear that $\text{deg} p(x) \leq nm - 1$. Multiplying $p(x)$ and $q(x)$ by suitable scalers, we may assume that $p(x), q(x) \in S[x]$. Put $p(x) = \sum_{i=0}^{nm-1} a_i x^i$ and $q(x) = \sum_{i=0}^{nm} b_i x^i$. But $b_0 \neq 0$, because $\text{det}(b) \neq 0$. Now, changing the variable $x$ to $b_0 x$ gives $p_1(x), q_1(x) \in S[x]$ such that $\text{deg} q_1 = nm$ and $\text{deg} p_1 \leq nm - 1$, where the constant term of $q_1(x)$ is 1 and for each $x \in L$, we have $p_1(x)/q_1(x) \in S[\Delta \cup \{u_i/v_i\}]$. Let $\mathcal{P} = \{p_1, \ldots, p_l\}$ be the set of all primes occurring in the factorizations of $\{v_1, \ldots, v_{l+1}\}$ into prime factors in $S$. For each natural number $r$, put $x_r = (p_1 \ldots p_l)^r$. Since $\text{deg} p_1 < \text{deg} q_1$, for a large enough number $r$, we have $p_1(x_r)/q_1(x_r) < 1$ if $S = Z$ and the degree of the denominator of $p_1(x_r)/q_1(x_r)$ with respect to $y$ is greater than the numerator if $S = K[y]$. Thus, there is a $p_j \in \mathcal{P}$ such that $p_j$ divides $q_1(x_r)$. But, since the constant term of $q_1(x)$ is 1, we have $\gcd(q_1(x_r), p_j) = 1$ for each $r \geq 1$ and each $1 \leq j \leq l$ which is a contradiction. Thus $(b + xI_{nm})^{-1}aba(b + xI_{nm})$ does not depend on $x$. Hence, we must have $babab^{-1} = (b + 1)aba(b + 1)^{-1}$. Consequently, $baba = abab$. But, this gives a nontrivial relation between $a$ and $b$ which is a contradiction,
because $a,b$ are arbitrary in the free group $\mathcal{F}$. This shows that $H$ has no free subgroup. Now, the result follows by Tits’ Alternative.

Corollary 3.5. Let $D$ be an $F$-division algebra. If $H$ is a finitely generated normal subgroup of $D^*$, then $H$ is central.

Proof. View $D^*$ as a subgroup of $\text{GL}_m(F)$ where $m = [D : F]$. Since $\{a + xI_m | a \in H, x \in F\}$ is contained in $N_{\text{GL}_m(F)}(D^*)$, Theorem 3.4 shows that $H$ must be soluble-by-finite. Now, the result follows from Corollary 3.3.

There is an elegant proof of Corollary 3.5 in [61] based on some known results in the theory of PI-rings. However, the argument used in Theorem 3.4 has this flexibility that can be modified to give similar results for subnormal subgroups instead of normal subgroups. To achieve this, let $H$ be a finitely generated subgroup of $\text{GL}_n(F)$. Let there be a finite class $\{H_j\}_{j=1}^r$ of subgroups of $\text{GL}_n(F)$ such that $H = H_r \triangleright H_{r-1} \triangleright \ldots \triangleright H_1$ where $\{a + xI_n | a \in H, x \in F\} \subseteq N_G(H_1)$. Keep the notations of the proof of Theorem 3.4 and recall that in the course of the proof of this theorem we had viewed $H$ as a subgroup of $\text{GL}_m(L)$. Now, for each pair $a,b \in H$ and $x \in L$, set $c_i(a,b,x) = (b + xI)a(b + xI)^{-1}$, and for $i > 1$ define $c_i$ inductively by $c_{i-1}bc_{i-1}$. Thus $c_i \in H_1$ and by induction $c_r \in H$. Here, we claim that for each $i$ we have $c_i = (b + xI)w_i(a,b)(b + xI)^{-1}$ where $w_i(a,b)$ is a reduced word in $a,a^{-1},b,b^{-1}$, the first and last letters of which are $a$ or $a^{-1}$, respectively. For $i = 1$ there is nothing to prove. Also, if $c_i = (b + xI)w_i(a,b)(b + xI)^{-1}$ then by induction we conclude that $c_{i+1} = c_i bc_i^{-1} = [(b + xI)w_i(a,b)(b + xI)^{-1}]b[(b + xI)w_i(a,b)^{-1}(b + xI)^{-1}] = c_{i+1} = c_i bc_i^{-1} = (b + xI)[w_i(a,b)bw_i(a,b)^{-1}](b + xI)^{-1}$ and since the first and last alphabets of $w_i(a,b)bw_i(a,b)^{-1}$ are $a$ and $a^{-1}$, the claim is established. Now, a similar argument as in the proof of Theorem 3.4 by using $c_r = (b + xI)w_r(a,b)(b + xI)^{-1}$ instead of $(b + xI)aba(b + xI)^{-1}$ implies that $H$ is soluble-by-finite. Now, as in Corollary 3.5 we can directly obtain

Theorem 3.6. Let $D$ be an $F$-central division algebra. Then every finitely generated subnormal subgroup of $D^*$ is central.

Here is a good place to exhibit some additional results related to the structure of subnormal subgroups of general linear groups over division rings.

Theorem 3.7 ([64]). Let $D$ be a division ring with center $F$. If either $n \geq 3$ or $n = 2$ but $D$ contains at least four elements, then for every subnormal subgroup $N$ of $\text{GL}_n(D)$ we have either $N \subseteq F$ or $\text{SL}_n(D) \subseteq N$.

Also, in [2], it is proved that if $n \geq 2$ and $N$ is an infinite noncentral normal subgroup of $\text{GL}_n(D)$ then $N$ is not finitely generated provided that $D$ is of finite dimension over its center. Combining this with Theorems 3.6 and 3.7 one can easily prove that

Theorem 3.8. Let $D$ be an infinite $F$-central division algebra. If $N$ is a finitely generated subnormal subgroup of $\text{GL}_n(D)$ ($n \geq 1$) then $N \subseteq F^*$.

We close this section by recalling a useful result of [4] which will be applied in the subsequent sections.
Corollary 3.9. Let $D$ be a division ring with center $F$ and let $M$ be a maximal subgroup of $D^*$. If $|M : M \cap F^*| < \infty$, then $D = F$.

Proof. Put $A = M \cap F^*$. Let $T = \{u_1, \ldots, u_k\}$ be a left transversal of $A$ in $M$ and $E = \{\sum_{j=1}^k f_j u_j | f_j \in F\}$. Clearly $E$ is a finite dimensional division algebra and $M$ is a maximal subgroup of $E^*$ (not necessarily proper maximal subgroup). Choose an element $u \in E^*$ such that $u = 1$ if $M = E^*$ and $u \in E^* \setminus M$ if $M \neq E^*$. From the maximality of $M$, it follows that $E^* = \langle T, u \rangle F^*$. So $\langle T, u \rangle$ is a finitely generated normal subgroup of $E^*$. Thus, $\langle T, u \rangle$ is central in $E^*$ by Corollary 3.5. This forces that $E$ is a field. Now, if $M \neq E^*$ then $E = D$ and the result follows. Thus we may assume that $M = E^*$. This implies that $|E^* : F^*| < \infty$ and hence we have either $M^* = E^* = F^*$ or $E$ is a finite field properly containing $F$. If $M = F^*$ then $|D^* : F^*| = p$ for some prime $p$ as $F^* \triangleleft D^*$. This implies that $D^* / F^*$ is cyclic and hence $D$ is commutative. Now, consider the case in which $E$ is a finite field. In this case, there is an element $a \in D^*$ such that $E^* = \langle a \rangle$. Since $a$ has a finite order and is noncentral, Herstein’s Lemma ([51, p. 206]) implies that there is a $b \in D^*$ such that $a^b = a^i \neq a$. Thus, $b \in N_{D^*}(E^*)$ and so $\langle M, b \rangle \subseteq N_{D^*}(E^*)$. Now, since $M$ is maximal we conclude that $N_{D^*}(E^*) = D^*$ and thus $E^* \triangleleft D^*$. Finally, the Cartan-Brauer-Hua [51, p. 211] Theorem implies that $D$ is commutative.

4. Maximal subgroups

Some old results presented in Section 1 show that in a division ring $D$ subnormal subgroups behave similar to $D^*$ in several manners. For example they are very far from being commutative as well as they are not periodic. These phenomena imply that the subnormal subgroups are “big”. But, one would like to know that how big maximal subgroups are in $D^*$? This section is devoted to presenting some recent results concerning this question.

4.1. Existence of free subgroups in maximal subgroups. As we have seen, by a result of Goncalves every subnormal subgroup of a division algebra contains a noncyclic free subgroup. Following this outcome, in [60] the author studied the existence of noncyclic free subgroups in a maximal subgroup of a division algebra. The main result of this study asserts that in a noncrossed product division algebra $D$ every maximal subgroup of $D^*$ contains a noncyclic free subgroup, that is a similar result which holds for subnormal subgroups of $D^*$. Here, we are going to achieve this interesting result. We begin by

Proposition 4.1. Let $D$ be an $F$-central division algebra. If $D^*$ has a noncommutative soluble-by-finite maximal subgroup $M$, then $M$ is absolutely irreducible and contains an abelian normal subgroup $A$ such that $C_M(A) = A$, hence $D$ is a crossed product division algebra.

Proof. Since $M$ is maximal we have either $F[M] = D$ or $F[M]^* = M$. But the latter case implies that the division algebra $F[M]$ is commutative (Theorem 3.1). This yields the commutativity of $M$ which is a contradiction. So $F[M] = D$, i.e., $M$ is an absolutely irreducible subgroup of $D^*$. Now, since $M$ is soluble-by-finite, it contains an abelian normal subgroup of finite index $A$. Take $A$ maximal in $M$. However, since $M$ is maximal and
\[ \langle M, C_D^*(A) \rangle \subseteq N_D^*(A) \] we have either \( N_D^*(A) = D \) or \( C_D^*(A) \subseteq M \). But, if \( N_D^*(A) = D \) then by Theorem 3.1 we obtain \( A \subseteq F^* \) and so \( M/M \cap F^* \) is finite. Here, Corollary 3.9 yields \( D = F \) which is absurd. Thus \( C_D^*(A) \subseteq M \) and consequently \( C_D^*(A) \) is soluble-by-finite. On the other hand, by the Double Centralizer Theorem \( C_D^*(A) = C_D^*(F[A]) \) is a division algebra. This forces that \( C_D^*(A) \) is commutative, because its unit group is soluble-by-finite. Thus \( C_M(A) \) is abelian. However \( A \subseteq C_M(A) \triangleleft M \) and \( A \) is a maximal abelian normal subgroup of \( M \). This gives \( C_M(A) = A \) and the result follows. \( \Box \)

Combining Theorem 2.21 and Proposition 4.1 implies that:

**Theorem 4.2.** Let \( D \) be a finite dimensional division algebra over its center \( F \). If \( M \) is a non-abelian maximal subgroup of \( D^* \) then the following statements are equivalent:

1. \( M \) does not contain a non-cyclic free subgroup;
2. \( M \) contains an abelian normal subgroup \( A \) such that \( C_M(A) = A \) and \( M/A \cong \text{Gal}(F[A]/F) \);
3. \( M \) is soluble-by-finite;
4. \( M \) is abelian-by-finite;
5. \( M \) satisfies a group identity.

Moreover, if either of the above conditions holds, then \( D \) is a crossed product division algebra. In particular, if \( D \) is a noncrossed product division algebra then every maximal subgroup of \( D^* \) contains a noncyclic free subgroup.

To proceed, we have to recall the following theorem which characterizes the field extensions \( K/F \), where \( K \) is radical over \( F \), i.e., \( K^*/F^* \) is periodic. For a proof see [51, p. 245].

**Theorem 4.3.** Let \( K/F \) be a proper field extension and let \( P \) be the prime subfield of \( F \). If \( K \) is radical over \( F \) then \( \text{char}(F) > 0 \), and either \( K \) is purely inseparable over \( F \) or \( K \) is algebraic over \( P \).

**Corollary 4.4.** Let \( D \) be a noncommutative \( F \)-central division algebra. If \( M \) is a soluble-by-finite maximal subgroup of \( D^* \). Then

1. \( F^* \subseteq M \);
2. If \( M/F^* \) is periodic, then \( M \) is commutative.

**Proof.** (1) Since \( M \) is maximal we have either \( F^*M = M \) or \( F^*M = D^* \). But the latter case implies that \( D^* \) is soluble-by-finite which is impossible. Thus \( F^*M = M \) and the result follows.

(2) Suppose that \( M \) is noncommutative. By Proposition 4.1, \( M \) has a proper normal abelian subgroup such that \( C_M(A) = A \). Thus \( F[A]/F \) is Galois by Lemma 2.7. On the other hand since \( M \) is maximal, as in (1) we conclude that \( F[A] \subseteq M \) and hence \( F[A]^*/F^* \) is torsion. Therefore, from Theorem 4.3 it follows that \( F[A]/F \) is separable, \( \text{char} F = p > 0 \) and \( F[A] \) is algebraic over the prime subfield of \( F \). Consequently, \( F \) is algebraic over the prime subfield \( \mathbb{F}_p \). This forces that \( D \) is commutative which is a contradiction. \( \Box \)

Corollary 4.4 makes a facility to improve the result of Corollary 3.9 as follows (of course in the finite dimensional case).
Theorem 4.5. Let $D$ be a noncommutative $F$-central division algebra. If $M$ is a maximal subgroup of $D^*$ such that $M/M \cap F^*$ is periodic then $M$ is abelian.

Proof. Since $M/M \cap F^*$ is torsion, it follows that $M$ contains no non-cyclic free subgroup. Now, the result follows from Theorem 2.21 and Corollary 4.4. □

Corollary 4.6. Let $D$ be an $F$-central division algebra and $M$ be a maximal subgroup of $D^*$. If $M$ is nilpotent then $M$ is abelian.

Proof. As in the proof of Proposition 4.1 one can conclude that $F[M] = D$. Moreover, it is not hard in this case to see that $Z(M) = F^*$. But $M$ is center-by-finite (It is well-known, however for a reference one can see [106, Th. 1]) and so the result follows from Theorem 4.5. □

Corollary 4.2 has been nicely extended in [62] by describing some properties of maximal subgroups of $GL_n(D)$ where $D$ is a division algebra. The results appeared in [62] specify that every maximal subgroup $M$ of $GL_n(D)$ either contains a non-cyclic free subgroup or contains a (subnormal) subgroup $A$ that is a direct product of the multiplicative groups of finitely many proper field extensions of the center $F$ of $D$, such that $A$ has a finite index in $M$ when $F$ has characteristic zero, or $M/A$ is locally finite when $F$ has characteristic $p \neq 0$. Moreover, a more general result was presented in [48] concerning the existence of free subgroups in the maximal subgroups of a subnormal subgroup $N$ in $GL_n(D)$ which we are going to display here without proof.

Theorem 4.7. Let $D$ be a noncommutative division algebra, and $N$ a subnormal subgroup of $GL_n(D)$ ($n \geq 1$). Given a maximal subgroup $M$ of $N$, then either $M$ contains a noncyclic free subgroup or there exists an abelian subgroup $A$ and a finite family $\{K_j\}_{j=1}^r$ of fields properly containing $F$ with $K_j^* \subset M$ for all $1 \leq j \leq r$ such that $M/A$ is finite if $\text{char}(F) = 0$ and $M/A$ is locally finite if $\text{char}(F) \neq 0$, where $A \subseteq K_1^* \times \ldots K_r^*$.

As we have seen, proving the above results were strongly dependent on the conclusion of Corollary 3.9. This corollary was also applied to determine the structure of maximal subgroups of $GL_n(D)$ with a finite conjugacy class in [49] by proving

Theorem 4.8. Let $D$ be an infinite division ring with center $F$ and $M$ a maximal subgroup of $GL_n(D)$ ($n \geq 1$). If $M$ is a finite conjugacy class group then it is abelian.

The proof of Theorem 4.8 requires using some known results from the theory of PI-rings. However, we are going to present the proof for the case $n = 1$. The following theorem is the main needed material at the beginning of this direction.

Theorem 4.9. A linear group with finite conjugacy class is center-by-finite.

Proof of Theorem 4.8 for $n = 1$. At first, let $D$ be of finite dimension over $F$. Since $M$ is maximal in $D^*$ we have either $F[M]^* = M$ or $F[M] = D$. But, in the former case Theorem 4.9 implies that the finite dimensional division algebra $F[M]$ has a center-by-finite unit group. Thus $F[M]$ is abelian and the result follows. Also, in the latter case one can easily
deduce that $Z(M) = F^*$ and again by Theorem 4.9 we conclude that $|M : F^*|$ is finite and the result follows from Corollary 3.9.

Now, consider the general case. As above we can conclude that $F^* \subseteq M$. Let $x$ be a noncentral element of $M$. Since $x$ has finitely many conjugates, it follows that $|M : C_M(x)| < \infty$. Thus $M$ has a normal subgroup $N$ of finite index which is contained in $C_M(x)$. But $M \subseteq N_{D^*}(F(N))$ as $N \lhd M$. However, since $M$ is maximal we have either $N_{D^*}(F(N)) = M$ or $N_{D^*}(F(N)) = D^*$. Let $N_{D^*}(F(N)) = M$. If this is the case, then $F(N)^* \subseteq M$ is a finite conjugacy class group. So its derived group is finite (see [88, p.442]) and hence is central. This immediately implies that $F(N)^*$ is solvable. Thus by Hua’s Theorem $F(N)$ is a field with $|M : F(N)^*| < \infty$. Now, we claim that $D$ is of finite dimension over $F$. For, let $F[M]^* \neq M$. Since $M$ is maximal we have $F[M] = D$. This forces that $[D : F(N)]_i < \infty$ as $|M : F(N)^*|$ is finite. This establishes our claim in this case (cf. [51, Th. 15.8]). But, if $F[M]^* = M$, $M \cup \{0\}$ is a division ring and then combining this with the same argument as above establish our claim. However, to prove that $M \cup \{0\}$ is a division ring, it is enough to show that for every pair $a, b \in M$ with $a \neq b$ we have $a - b \in M$. To achieve this, put $u = a^{-1}b$. Since $u \neq 1$ and $|M : F(N)^*| < \infty$ we have $u^m \in F(N)$ for some $m > 1$. If $u^m = 1$ then $u - 1$ is algebraic over $F$ and so $(u - 1)^{-1} \in F[M]$. This yields $u - 1 \in F[M]^* = M$. Also if $u^m \neq 1$ then $(u - 1)(u^{m-1} + \ldots + 1) = u^m - 1 \in F(N)^*$. This again implies that $u - 1 \in M$. Therefore, $a - b = (u - 1)b \in M$ and so our claim. Here, by the first paragraph we deduce that $M$ is abelian. Now, we are left with the case in which $N_{D^*}(F(N)) = D^*$. Here, the Cartan-Brauer-Hua Theorem guarantees that $F(N) = D$ or $F(N) \subseteq F$. In the former case, every element of $D$ commutes with $x$ as $N \subseteq C_M(x)$. This forces that $x \in F$ which contradicts the choice of $x$. In the latter case we also have $N \subseteq F^*$ and since $|M : N| < \infty$ we conclude that $|M : F^*| < \infty$. This implies that $D$ is commutative which is absurd. \qed

4.2. Nilpotent and soluble maximal subgroups. As we have seen in Theorem 4.2, if $D$ is a noncrossed product division algebra then every maximal subgroup of $D^*$ contains a noncyclic free subgroup. But, there is no substantial information for maximal subgroups of an infinite dimensional division ring. However, some special types of groups that have no noncyclic free subgroup are nilpotent and soluble groups. Thus it may be of interest to explore the nature of such groups when they appear as a maximal subgroup of a division ring. This was the main theme in a series of recent works including [1], [4] and [12]. The most successful effort to specify what types of nilpotent groups can become visible as a maximal subgroup of a division ring has been made in [12] which asserts that such a nilpotent group is abelian. To present the proof of this fact, at first we must give the following

Lemma 4.10. Let $D$ be a division ring with center $F$. If $M$ is a nilpotent maximal subgroup of $D^*$ then:

(1) $M'$ is abelian;
(2) $M$ has a normal maximal abelian subgroup $A$ containing $M'$ such that $K = A \cup \{0\}$ is a subfield of $D$;
(3) If $u \in M \setminus A$ then $F(N) = D$ where $N = \langle A, u \rangle$. 


Proof. (1) If $M$ is abelian then there is nothing to prove. So we assume that $M$ is nonabelian. Since $M$ is maximal and nilpotent we have $F^* = \mathcal{Z}(M) \subsetneq M$ and hence $M/F^*$ has a nontrivial center. Let $F^* u \in \mathcal{Z}(M/F^*)$ and $u \notin F^*$. Now, consider the homomorphism $\phi : M \rightarrow F^*, x \mapsto xux^{-1}u^{-1}$. Since ker $\phi = C_M(u)$ we conclude that $M/C_M(u)$ is abelian and hence $M' \subseteq C_M(u)$. But, $F(M') \neq D$ because $u \notin F^*$. At the other extreme, $M \subseteq N_{D^*}(F(M')^*)$ yields either $N_{D^*}(F(M')^*) = D$ or $F(M')^* \lhd M$. However, in the former case the Cartan-Brauer-Hua Theorem implies that $F(M') \subseteq F$ because $F(M') \neq D$. Also, in the latter case $F(M')$ is abelian by Theorem 3.1. Therefore, in either case we conclude that $M'$ is abelian, as desired.

(2) By Zorn’s Lemma $M$ has a maximal abelian normal subgroup $A$ containing $M'$. Now, since $M$ is maximal in $D^*$ we have either $N_{D^*}(F(A)) = D$ or $F(A)^* \subseteq M$. But, if $N_{D^*}(F(A)) = D$ then by Theorem 3.1 we should have $F(A)^* \subseteq F^* \subseteq M$. Thus, in either case $F(A)^*$ is an abelian normal subgroup of $M$ containing $A$. However, since $A$ is a maximal abelian normal subgroup of $M$ we conclude that $F(A)^* = A$.

(3) First note that since $A \nsubseteq N$, $N$ is nonabelian. Now, since $M' \subseteq N$ we have $N \lhd M$. Therefore $M \subseteq N_{D^*}(F(N)^*)$. Thus we conclude that $F(N)^* \lhd D^*$ or $F(N)^* \subseteq M$. But by Theorem 3.1 the second case can not occur. So $F(N)^* \lhd D^*$. Now, by the Cartan-Brauer-Hua Theorem $F(N) = D$ because $N$ is nonabelian.

Recall that a domain $R$ is called a right Ore domain if for every pair of nonzero elements $a, b \in R$ one has $aR \cap bR \neq 0$. Also, a left Ore domain is defined in a similar manner. A domain that is both right and left Ore domain is called an Ore domain. By a theorem of Ore, a ring $R$ has a classical ring of quotients if and only if it is an Ore domain (see [78, p. 146]). The following proposition is also helpful.

**Proposition 4.11** ([90, p. 26]). Let $R = E[G]$ be a ring where $E$ is division subring of $R$ and $G$ is a subgroup of $R^*$. If $G/E \cap G$ is an infinite cyclic group and $R$ is a crossed product of $E$ by $G/E \cap G$ then $R$ is an Ore domain.

**Theorem 4.12.** Let $D$ be a noncommutative division ring with center $F$. If $M$ is a nilpotent maximal subgroup of $D^*$, then $D$ is of finite dimension over $F$ and $M$ is abelian.

**Proof.** By Lemma 4.10, $M$ contains a normal abelian subgroup $A$ such that $K = A \cup \{0\}$ is a field. First suppose that no elements of $M \setminus K^*$ is algebraic over $K$ and let $v \in M \setminus K^*$. If we set $G = K^* \langle v^2 \rangle$ then by the fact that $v$ is not algebraic over $K$ one can easily show that the ring $F[G] = \oplus_{j \in Z}Kv^{2j}$ is a crossed product of $K$ by $G/K^* \cong \langle v^2 \rangle$. But, since $\langle v^2 \rangle$ is an infinite cyclic group from Proposition 4.11 it follows that $F[G]$ is an Ore domain. Hence the ring of quotients of $F[G]$ is $F(G) = D$ (note that $F(G) = D$ by Lemma 4.10). Therefore, every element of $D$ is of the form $d_1d_2^{-1}$ where $d_1, d_2 \in F[G]$ and $d_2 \neq 0$. So $v = s_1s_2^{-1}$ for $s_1, s_2 \in F[G]^*$. On the other hand $s_1 = \sum_{j=1}^{m}k_jv^{2j}$ and $s_2 = \sum_{j=1}^{m}k'_jv^{2j}$ where $k_j, k'_j \in K$. Hence $\sum_{j=1}^{m}v^{2j} = \sum_{j=1}^{m}k_jv^{2j}$. Now, if we let $l_j = vk_jv^{-1}$, for any $0 \leq j \leq m$, then $l_j$’s lie in $K$ and we have $\sum_{j=1}^{m}l_jv^{2j+1} = \sum_{j=1}^{m}k_jv^{2j}$ which implies that $v$ is algebraic over $K$, which is a contradiction. Thus $M \setminus K^*$ has an element $u$ algebraic over $K$. Assume that $u$ satisfies an equation of the form $x^n + \sum_{j=0}^{n-1}k_jx^j = 0$, where $k_j \in K$ for every $0 \leq i \leq n - 1$. It is not hard to see that $R = \sum_{j=0}^{n}Ku^j$ is a division subring of $D$ with $[R : K]_l < \infty$. Therefore
\[ F(N) = R \text{ where } N = \langle K^*, u \rangle. \] On the other hand Lemma 4.10 implies that \( F(N) = D. \) So \([D : K] < \infty\) and hence \( D \) is finite dimensional over \( F \) (cf. [51, Th. 15.8]). Now, by Corollary 4.6 we conclude that \( M \) is abelian. \( \square \)

In fact, a more general result about the structure of maximal nilpotent subgroups of \( \text{GL}_n(D) \) was provided in [12] which we are going to present here without proof.

**Theorem 4.13.** Let \( D \) be an infinite division ring and \( M \) be a maximal nilpotent subgroup of \( \text{GL}_n(D) \) \((n \geq 1)\). Then \( M \) is abelian.

Here we must point out that the properties of a locally nilpotent maximal subgroup in a division ring are also investigated in [21] where Hai and Thin show that if \( D \) is algebraic over its center then every locally nilpotent maximal subgroup of \( D^* \) is abelian. Moreover, in another work, Hai has proved that if \( M \) is a locally nilpotent maximal subgroup of a division ring that is algebraic over the center then it is either the multiplicative group of some maximal subfield or it is center-by-(locally finite) [19].

Now, for the case in which the division ring is of finite dimension over its center the following result is also proved.

**Theorem 4.14.** Let \( D \) be a noncommutative \( F \)-central division algebra. Let \( M \) be a non-abelian soluble maximal subgroup of \( D^* \). Then \( D \) contains a maximal subfield \( K \) such that \( K^* \triangleleft M, K/F \) is a cyclic extension of degree \( p \) where \( p \) is a prime and \( M/K^* \cong \text{Gal}(K/F) \), i.e., \( D \) is a cyclic division algebra of prime degree.

**Proof.** By Proposition 4.1 we know that \( M \) is absolutely irreducible in \( D^* \) and has an abelian normal subgroup \( A \) such that \( F[A] \) is a maximal subfield of \( D \) that is Galois over \( F \) and \( M/A \cong \text{Gal}(F[A]/F) \). Put \( K = F[A] \). Since \( K^*M \) is soluble and \( M \) is maximal, we can easily conclude that \( K^*M = M \) and so \( K^* \triangleleft M \). Also, by the maximality of \( A \) in \( M \) we deduce that \( K^* = A \). Hence \( M/K^* \cong \text{Gal}(K/F) \). Now, it remains to prove that \( |\text{Gal}(K/F)| \) is a prime number. Otherwise, let \( |\text{Gal}(K/F)| \) is not a prime number. Thus \( \text{Gal}(K/F) \) has a nontrivial normal subgroup \( N \). Let \( E \) be the fixed field of \( N \). So we have a chain \( F \subset E \subset K \) of subfields. From field theory we know that \( E \) is invariant under each \( \sigma \in \text{Gal}(K/F) \). This forces that \( M \) lies in \( N_{D^*}(E^*) \). But, if \( M \neq N_{D^*}(E^*) \) by the maximality of \( M \) we must have \( E^* \triangleleft D^* \) and by the Cartan-Brauer-Hua Theorem \( E \subset F \) which is a contradiction. Thus \( M = N_{D^*}(E^*) \) and \( N_{D^*}(E^*) \) is soluble. On the other hand \( C_{D^*}(E^*) \subseteq N_{D^*}(E^*) \). Therefore \( C_D(E) \) is a division subalgebra with a soluble unit group. It follows from Hua’s Theorem that \( C_D(E) \) is a field. Finally, by the Centralizer Theorem we conclude that \( C_D(E) = E \), i.e., \( E \) is a maximal subfield and so \( E = K \) which is a contradiction. \( \square \)

Let \( M \) be a maximal subgroup of \( D^* \) where \( D \) is a finite dimensional division algebra. Also, suppose that \( M \) contains no non-cyclic free subgroup. If \( \text{deg}(D) \) is odd, then from Theorem 2.21 it follows that \( M \) is soluble. Now, suppose that \( \text{deg}(D) = 2^m \) for some \( m \). By Theorem 4.2, \( M \) contains an abelian normal subgroup \( A \) such that \( M/A \) has order \( \text{deg}(D) \) and hence \( M \) is soluble. Here, using Theorem 4.14 yields that \( \text{deg}(D) \) is a prime number. Thus, the following corollary is immediate:
Corollary 4.15. Let $D$ be an $F$-central division algebra and $M$ be a maximal subgroup of $D^*$. If either $\deg(D)$ is odd or $\deg(D) = 2^m$ for some $m$, then the following statements are equivalent:

1. $M$ has no non-cyclic free subgroups;
2. $M$ is soluble and $\deg(D)$ is a prime number.

Therefore, whenever $\deg(D)$ is not a prime and either it is an odd number or is equal to $2^m$ for some $m$, then every maximal subgroup of $D^*$ contains a non-cyclic free subgroup.

A perfect description of the structure of a division ring with an algebraic nonabelian locally soluble maximal subgroup is also accessible that we shall provide at this place, for the proof see [1, Th. 6].

Theorem 4.16. Let $D$ be a division ring with center $F$ and $M$ a nonabelian locally soluble maximal subgroup of $D^*$ that is algebraic over $F$. Then $[D : F] = p^2$, where $p$ is a prime number, and there exists a maximal subfield $K$ of $D$ such that $K/F$ is a cyclic extension of degree $p$, $K^* \triangleleft M$ and $M/K^* \cong \text{Gal}(K/F)$.

We close this section by remarking some additional results. By a work of Hai and Phuong Ha, the same conclusion as in Theorem 4.16 is also valid if we replace the requirement “$M$ is algebraic over $F$” by “$M$’ is algebraic over $F$” [20]. Also, in another work, Dorbidi, Fallah-Moghaddam and Mahdavi-Hezavehi proved that if for a division ring $D$ the group $\text{GL}_n(D)$ has a nonabelian soluble maximal subgroup then $n = 1$, $[D : F] < \infty$ and hence a similar result as in Theorem 4.16 holds. They also proved that if either $F$ is an algebraically closed field or a real closed field then $\text{GL}_2(F)$ contains no soluble maximal subgroups [10].

5. Descending Central Series of Division Rings and Extending of Valuations

As we have mentioned before in the introduction, one aspect in the study of the structure of the unit group of a division ring is to explore the roles of multiplicative commutators in the structure of $D$. This point of view was followed in [57], [65], [63], [67] and [58]. Also, a survey on this topic was provided in [59]. Motivated by various results appeared in the above papers, in [28] the structure of subgroup that become visible in the descending central series of a division ring was investigated and some analogous results for this kind of subgroups were successfully made. At this stage our aim is to provide some interesting results of the above works that are natural generalizations of known old results. Here, before giving our theorems we must recall

Theorem 5.1 (Wedderburn’s Factorization). Let $D$ be a division ring with center $F$ and $u \in D^*$. If $u$ is algebraic over $F$ with minimal polynomial $f(t) = t^m + a_{m-1}t^{m-1} + \ldots + a_1 t + a_0$, then there are $d_1, \ldots, d_m \in D$ such that $f(t) = (t - d_1 ud_1^{-1}) \ldots (t - d_m ud_m^{-1})$.

Surely, the following lemma is the most important tool in our subsequent work.

Lemma 5.2. Let $D$ be a division ring with center $F$ and $N$ be a normal subgroup of $D^*$. If $u \in N$ is algebraic over $F$ and the degree of the minimal polynomial of $u$ over $F$ is $m$, then $u^m = N_{F(u)/F(u)c}$ for some $c \in [D^*, N]$. Thus $u^m \in Z(N)[D^*, N]$. 

Proof. Let $f(t)$ be the minimal polynomial of $u$ over $F$. From field theory, we have
\[ f(t) = t^n - Tr_{F(u)/F}(u)t^{n-1} + \ldots + (-1)^m N_{F(u)/F}(u). \] (5.1)
Now, using Wedderburn's Factorization Theorem and (5.1), one obtains
\[ N_{F(u)/F}(u) = d_1ud_1^{-1} \ldots d_mud_m^{-1} \]
where $d_i \in D$. But
\[ d_1ud_1^{-1} \ldots d_mud_m^{-1} = [d_1^{-1}, u^{-1}]u^{-1} \ldots [d_m^{-1}, u^{-1}]u^{-1} = u^mc \]
for some $c \in [D^*, N]$. Thus we have $u^m = N_{F(u)/F}(u)c$. Now, since $N \triangleleft D^*$, we have $c \in N$ and so $N_{F(u)/F}(u) \in F^* \cap N \subseteq Z(N)$, as desired. \hfill \Box

**Corollary 5.3.** If $D$ is a division ring with center $F$ and $N$ is a subgroup of $D^*$, then:

1. If $N$ contains some $\zeta_i D^*$ and $a \in D$ is algebraic over $F$, then $u$ is algebraic over $F^* N$;
2. If $D$ is finite dimensional over $F$ with $\deg(D) = n$ and if $N$ is normal in $D^*$, then $N^n \subseteq \text{Nrd}_D(N)[D^*, N] \subseteq Z(N)[D^*, N]$.

**Proof.** (1) Clearly we can replace $N$ by $\zeta_i D^*$. The proof will be by induction on $i$. For $i = 0$, there is nothing to prove. Let $D^*/F^* \zeta_{i-1} D^*$ be torsion. Thus, if $a \in D^*$ then there is a nonnegative integer $r$ such that $a^r = fb$ for some $f \in F$ and $b \in \zeta_{i-1} D^*$. Now, by Lemma 5.2 we have $b^m \in F^* \zeta_i D^*$ for some $m$. Thus $a^{rm} = f^m b^m \in F^* \zeta_i D^*$, as desired.

(2) Let $a \in N$ and $K$ be a maximal subfield of $D$ containing $a$. Put $[K : F(a)] = k$. By Lemma 5.2 we conclude that $a^{[F(a):F]} = N_{F(a)/F}(a)c$ for some $c \in [D^*, N]$. Now, we have
\[
a^n = a^{k[F(a):F]} = N_{F(a)/F}(a)^k c^k = N_{F(a)/F}(a^k)c^k = N_{F/K}(a)c^k = \text{Nrd}_D(a)c^k \subseteq \text{Nrd}_D(N)[D^*, N]. \]

**Corollary 5.4.** Let $D$ be a division algebra of index $n$. For any $i > 0$, the quotient $\zeta_i D^*/\zeta_{i+1} D^*$ is a torsion abelian group of bounded exponent $n$.

**Proof.** Since $D'$ is contained in the kernel of $\text{Nrd}_D$, if in Corollary 5.3 we put $N = \zeta_i D^*$ then we have $(\zeta_i D^*)^n \subseteq \zeta_{i+1} D^*$. \hfill \Box

**Theorem 5.5.** Let $D$ be a division ring with center $F$. If $\zeta_i D^*$ is algebraic over $F$ for some $i$, then $D$ is algebraic over $F$.

**Proof.** If $D = F$ then there is nothing to prove. Thus we may assume that $D \neq F$. Put $H = F^* \zeta_i D^*$. Since $D^*$ is not nilpotent we have $\zeta_i D^* \not\subseteq F^*$ and so $F^* \not\subseteq H$. Now, consider the ring $R$ generated by the elements of $H$. First we claim that $R$ is algebraic over $F$. For, if $a \in \zeta_i D^*$ and $b \in D$ is algebraic over $F$, then $ab = ba$ in the group $D^*/F^* \zeta_{i+1} D^*$. On the other hand by Corollary 5.3, $\overline{a}$ and $\overline{b}$ are torsion and so $\overline{ab}$ is torsion. Therefore, $(ab)^k \in H$.
for some $k$ and hence $ab$ is algebraic over $F$. Now, if $u, v \in H$ we have $u + v = u(1 + u^{-1}v)$. But $u$ and $1 + u^{-1}v$ are algebraic over $F$. It follows that $u + v$ is algebraic over $F$. Hence $R$ is algebraic over $F$ and the claim is established. Now, since $R$ is algebraic over $F$ we conclude that $R$ is a division subring of $D$. Clearly, $R^* \subset D^*$ and so, by the Cartan-Brauer-Hua Theorem, $R = D$, as desired.

**Corollary 5.6.** Let $D$ be a division ring with center $F$. If $\zeta_iD^*$ is radical over $F$ for some $i$, then $D$ is commutative.

**Proof.** From Theorem 5.5 it follows that $D$ is algebraic over $F$. Therefore, by Corollary 5.3, $D$ is radical over $F^*\zeta_iD^*$ and so $D^*$ is radical over $F^*$. Now, the result follows by Kaplansky’s Theorem [43].

**Corollary 5.7.** If $D$ is a noncommutative division ring with center $F$, then every $\zeta_iD^*$ contains an element separable over $F$.

**Proof.** If every element of $\zeta_iD^*$ is purely inseparable over $F$, then $\zeta_iD^*$ is radical over $F$ and hence $D$ is necessarily commutative, which is a contradiction.

Let $D$ be a division ring and $K$ be a proper division subring of $D$. By a result of Hua ([37]) if $\zeta_iD^* \subseteq K$ then $D$ is commutative. The following theorem generalizes this phenomenon.

**Theorem 5.8.** Let $D$ be a division ring with center $F$ and let $K$ be a proper division subring of $D$. If $\zeta_iD^*$ is radical over $K$ for some $i$, then $D$ is a field.

**Proof.** Let $\zeta_iD^*$ be radical over $K$ for some $i$. If $\zeta_iD^* = 1$ then $D^*$ is nilpotent and thus there is nothing to prove. So we may consider the case in which $\zeta_iD^* \neq 1$. We prove the theorem in three steps.

**Step 1.** We show that if $a, b \in \zeta_iD^*$, $1 \neq a \in K$ and $b \notin K$ then there is a positive integer $n$ such that $ba^n = a^n b$. To see this, consider the elements $u_1 = (b + a)^{-1}a(b + a)$ and $u_2 = (b + 1)^{-1}a(b + 1)$. Clearly $u_1, u_2$ are in $\zeta_iD^*$. Now, since $\zeta_iD^*$ is radical over $K$, there is a positive integer $n$ such that $u_1^n = (b + a)^{-1}a^n(b + a) \in K$ and $u_2^n = (b + 1)^{-1}a^n(b + 1) \in K$. But a simple calculation shows that $b(u_1^n - u_2^n) = a^n(a - 1) + u_2^n - au_1^n \in K$. Therefore, if $u_1^n - u_2^n \neq 0$ then $b \in K$ which is absurd. So we have $u_1^n - u_2^n = 0$ and hence $(a - 1)a^n = (a - 1)u$ where $u = u_1^n = u_2^n$. However, $a \neq 1$ implies that $u = a^n$, $(b + 1)a^n = a^n(b + 1)$ and so $ba^n = a^n b$.

**Step 2.** Here we claim that for every pair $u, v \in \zeta_iD^* \cap K$, there is some positive integer $m$ such that $v^m u = uv^m$. If $\zeta_iD^* \subseteq K$, then Cartan-Brauer-Hua Theorem implies that $D = F(\zeta_iD^*) = K$ which is impossible. So $\zeta_iD^* \notin K$ and thus we can choose an element $a \in \zeta_iD^*$ such that $a \notin K$. Hence $au \in \zeta_iD^*$ and $au \notin K$. Now, by Step 1 there are positive integers $r, s$ such that $v^r a = au^r$ and $v^s au = au^s$. This forces that $v^{rs} u = uv^{rs}$ and the claim is established.

**Step 3.** Here, we prove the theorem. Let $u, v \in \zeta_iD^*$ and $L = F(u, v)$ denote the division subring generated by $u, v$ and $F$. Suppose that $a \in \zeta_iL^*$ is arbitrary. Since $\zeta_iL^* \subseteq \zeta_iD^*$, we have $a^t \in K$ for some positive integer $t$. Now, by Step 1 and Step 2 we conclude that there is a positive integer $n$ such that $a^n u = ua^n$ and $a^n v = va^n$. This shows that $a^n \in Z(L)$. So
$\zeta_i L^*$ is center-by-periodic and hence $L$ is commutative by Corollary 5.6. Therefore, $uv = vu$ and thus $\zeta_i D^*$ is commutative. Now, from Corollary 5.6 the result follows. \qed

Let $\Gamma$ be an additive totally ordered abelian group. Let $\Delta$ be a set properly containing $\Gamma$ and $\infty \in \Delta \setminus \Gamma$. Extend the operation of $\Gamma$ to $\Gamma \cup \{\infty\}$ by

$$x + \infty = \infty + x = \infty$$

for all $x \in \Gamma \cup \{\infty\}$. Also, extend the ordering of $\Gamma$ to $\Gamma \cup \{\infty\}$ by $x < \infty$ for all $x \in \Gamma$. Let $D$ be a division algebra over its center $F$, of index $n$. By a valuation on $D$ with values in $\Gamma$, we mean a map $v : D \to \Gamma \cup \{\infty\}$ satisfying, for all $a, b \in D$,

(V1) $v(a) = \infty$ iff $a = 0$;
(V2) $v(ab) = v(a) + v(b)$;
(V3) $v(a + b) \geq \min\{v(a), v(b)\}$.

The standard reference for noncommutative valuation theory is Schilling’s book [86]. Let $K \subseteq D$ be a division ring extension. Let $w$ and $v$ are valuations on $K$ and $D$ with values in $\Gamma \cup \{\infty\}$, respectively. We say that $v$ is an extension of $w$ whenever $v|_K = w$. It is well-known that if $F \subseteq K$ is a field extension, then every valuation on $F$ has an extension to $K$ (possibly many different extensions). On the other hand, in the setting of noncommutative division rings this property does not always hold. However, a criterion for when a valuation can be extended from $F = Z(D)$ to $D$ has been given independently by Wadsworth ([97]) and Ershov ([15]) for the case in which $[D : F] < \infty$. More precisely, they showed that if $D$ is an $F$-central division algebra, then the following are equivalent:

(i) a valuation $v$ on $F$ extends to $D$;
(ii) $v$ has a unique extension to each subfield $K$ of $D$ with $F \subseteq K \subseteq D$.

But, a careful reading of Wadsworth’s proof shows that $(1) \Rightarrow (2)$ is valid if $D$ is algebraic over $F$. Motivated by this observation, in [56] the following generalization of the above mentioned result was provided.

**Theorem 5.9.** Let $D$ be a division ring algebraic over its center $F$, and let $v$ be a valuation on $F$. Then the following are equivalent:

1. $v$ extends to a valuation on $D$;
2. $v$ has a unique extension to each subfield $K$ of $D$ with $[K : F] < \infty$ and $Z(D') \subseteq \{u \in F|v(u) = 0\}$.

To prove Theorem 5.9 we need the following lemma which is a direct outcome of Lemma 5.2.

**Lemma 5.10.** Let $D$ be a division ring with center $F$. Let $u, v$ and $uv$ in $D^*$ be algebraic over $F$ and denote by $K_u, K_v$ and $K_{uv}$ subfields of $D$ containing $u, v$ and $uv$ with $[K_u : F] = r$, $[K_v : F] = s$ and $[K_{uv} : F] = t$, respectively. Then

$$N_{K_u/F}(u)^{d/r} N_{K_v/F}(v)^{d/s} c$$

for some $c \in Z(D')$, where $d$ is the least common multiple of $r, s$ and $t$. 
Proof. By Lemma 5.2 we have $N_{K_a/F}(u) = \bar{w}'$, $N_{K_a/F}(v) = \bar{v}'$ and $N_{K_a/F}(uv) = \bar{u}\bar{v} = \bar{w}'\bar{v}'$ in the Whitehead group $K_1(D) = D^*/D'$. Thus $N_{K_a/F}(uv)^{t/s} = N_{K_a/F}(u)^{t/s} N_{K_a/F}(v)^{t/s}$ and the result follows.

Proof of Theorem 5.9. (1)$\Rightarrow$(2). By the remark before the theorem, $v$ has a unique extension to each subfield $K$. Let $w$ be the valuation on $D$ such that $w|_F = v$. Since $D$ is algebraic over $F$, one can easily conclude that the value group $\Gamma_D := w(D^*)$ is abelian. Thus $w(a) = 0$ for every $a \in D'$.

(2)$\Rightarrow$(1). Let $\Gamma_F := v(F^*)$ be the value group of $v$. Let $\Delta \cong \Gamma_F \otimes_\mathbb{Z} \mathbb{Q}$ be the divisible hull of $\Gamma_F$. The total ordering on $\Gamma_F$ extends uniquely to a total ordering on $\Delta$, and for each algebraic extension $L/F$ and each extension $w$ of $v$, we may view $w(L^*)$ as a subgroup of $\Delta$. Now, for each $a \in D^*$ put $K_a = F(a)$ and define the function $w : D^* \to \Delta$ by

$$w(a) = \frac{1}{n}v(N_{K_a/F}(a)),$$

where $n = [K_a : F]$. First we verify that $w|_{K_a}$ is the valuation on $K_a$ extending $v$. Let $N$ be the normal closure of $K_a$ over $F$ and $u : N \to \Delta$ be any valuation on $N$. If $a_1, \ldots, a_n$ are the roots of the minimal polynomial of $a$ over $F$, we know that all of them lie in $N$ and $N_{K_a/F}(a) = a_1 \ldots a_n$. For each $j$ there is an $F$-automorphism $\sigma_j$ of $N$ sending $a$ to $a_j$. Since $u|_{K_a}$ and $(u \circ \sigma_j)|_{K_a}$ are each valuations on $K_a$ extending $v$, by hypothesis they are the same. So, $u(b_j) = u(\sigma_j(b)) = u(b)$. Thus we have

$$w(a) = \frac{1}{n}v(N_{K_a/F}(a)) = \frac{1}{n}v(a_1 \ldots a_n) = \frac{1}{n}(u(b_1) + \ldots + u(b_n)) = u(b).$$

Thus, $w|_{K_a} = u|_{K_a}$ which is the valuation on $K_a$ extending $v$. To show that $w$ is a valuation on all of $D$, take any $a, b \in D^*$. Now, by Lemma 5.10 we have

$$w(ab) = \frac{1}{k}v(N_{F(ab)/F}(ab)) = \frac{1}{d}v(N_{F(ab)/F}(ab)^{d/\ell}) = \frac{1}{d}v(N_{F(a)/F}(a)^{d/r} N_{F(b)/F}(b)^{d/s} c) = \frac{1}{r}v(N_{F(a)/F}(a)) + \frac{1}{s}v(N_{F(b)/F}(b)) + \frac{1}{d}v(c) = w(a) + w(b).$$

Finally, assume that $b \neq -a$. Since $w|_{F(ab)}$ is a valuation we have $w(1 + a^{-1}b) \geq \min\{w(1), w(a^{-1}b)\}$. Thus, using the multiplicative property for $w$,

$$w(a + b) = w(a) + w(1 + a^{-1}b) \geq w(a) + \min\{w(1), w(a^{-1}b)\} = \min\{w(a), w(b)\},$$

as desired.

Remark 5.11. If in Theorem 5.9, $D$ is of finite dimensional over $F$ with index $n$ then one can easily check that for every $a \in D^*$,

$$\frac{1}{[F(a) : F]}v(N_{F(a)/F}(a)) = \frac{1}{n}v(N_{K/F}(a)).$$
where $K$ is any maximal subfield containing $a$. Thus, in this case we have

$$w(a) = \frac{1}{n}v(\text{Nrd}_D(a)).$$

This is exactly the Wadsworth-Ershov formula for the extended valuation.

Here, we need to recall some definitions from the theory of valued division algebras. Let $D$ be a valued $F$-central division algebra with value group $\Gamma_D$. We denote the valuation ring of $D$ by $V_D = \{d \in D^* | v(d) \geq 0\} \cup \{0\}$, its unique maximal ideal by $M_D = \{d \in D^* | v(d) > 0\} \cup \{0\}$, and its residue class division ring by $\overline{D}$. Set $U_D = \{d \in D^* | v(d) = 0\}$ so that $U_D = V_D^*$. The restriction of $v$ to $F^*$ is denoted by $w$ which defines a valuation with value group $w(F^*) = \Gamma_F$. Also $V_F$, $M_F$, $\overline{F}$, and $U_F$ are defined similarly. Since $V_F \cap M_D = M_F$, we can consider the residue class field $\overline{F}$ as a subalgebra of $\overline{D}$. The valuation $w$ over $F$ is called Henselian if $w$ has a unique extension to any algebraic extension field of $F$. So by Theorem 5.9 if $F$ is Henselian then the valuation of $F$ has a unique extension to $D$. In this setting, $D$ is called a tame division algebra if $Z(\overline{D})/\overline{F}$ is separable and $\text{char}(\overline{F}) \nmid n$.

Now, we return to our study of descending central series of a division algebra by applying Lemma 5.2 to find a short and elementary proof for Platonov’s Congruence Theorem that is the key step in reduced $K$-Theory connecting $\text{SK}_1(D)$ to $\text{SK}_1(\overline{D})$ (See Section 6). The method that we are going to follow was given in [28].

**Theorem 5.12 (Congruence Theorem).** Let $D$ be a tame division algebra over a Henselian field $F = Z(D)$, of index $n$. Then $1 + M_D = (1 + M_F)[D^*, 1 + M_D]$ and $(1 + M_D) \cap D^{(1)} = [D^*, 1 + M_D]$, where $D^{(1)}$ is the kernel of the reduced norm map.

**Proof.** First we claim that $(1 + M_F) \cap D^{(1)} = 1$. To prove our claim let $1 + f \in (1 + M_F) \cap D^{(1)}$. Thus we have $(1 + f)^n = 1$. But

$$v(0) = v((1 + f)^n - 1) = v(nf + (n(n - 1)/2)f^2 + \ldots + f^{n-1})$$

$$= v(f) + v(n + (n(n - 1)/2)f + \ldots + f^{n-1}) = 0.$$ (5.2)

Now, since $D$ is tame $\overline{n} \neq 0$ in $\overline{F}$ and so $n \in U_F$, that is $v(n) = 0$. On the other hand, since $v(f) > 0$, we have $v((n(n - 1)/2)f + \ldots + f^{n-1}) > 0$. Therefore

$$v(n + (n(n - 1)/2)f + \ldots + f^{n-1}) = \min\{v(n), v((n(n - 1)/2)f + \ldots + f^{n-1})\} = 0.$$ Hence by (5.2) we have $v(0) = v(f)$ and from (V1) it follows that $f = 0$ and so our claim. Now, in Corollary 5.3 if we take $N = 1 + M_D$ then we obtain

$$(1 + M_D)^n \subseteq ((1 + M_D) \cap F^*)[D^*, 1 + M_D].$$

Since the valuation is tame and Henselian, the Hensel’s Lemma implies that $(1 + M_D)$ is $n$-divisible. Therefore $1 + M_D = (1 + M_F)[D^*, 1 + M_D]$. Now, using the fact that $(1 + M_F) \cap D^{(1)} = 1$, the theorem follows. \qed

Using the Congruence Theorem and results of [15] and the fact that if $D$ is totally ramified then $\overline{D} \cong \mu_e(\overline{F})$ where $e = \exp(\Gamma_D/\Gamma_F)$ (see Proposition 3.1 of [98]). In [52], Tignol and Lewis presented an explicit formula for $\text{SK}_1(D)$. In fact they have shown that if $D$ is a tame
division algebra over its center $F$ of degree $n$, then $\text{SK}_1(D) \cong \mu_n(F)/\mu_e(F)$. We will obtain this result in the next section as a particular case of a more general setting.

**Corollary 5.13.** Let $D$ be a tame and Henselian division algebra. Then $[D^*, [D^*, 1 + M_D]] = [D^*, 1 + M_D]$, i.e., $[D^*, 1 + M_D]$ is $D^*$-perfect. In particular, $[D^*, 1 + M_D] \subseteq \zeta_iD^*$ for all $i > 0$.

**Theorem 5.14.** Let $F$ be a Henselian field and $D$ be a tame and totally ramified $F$-central division algebra of index $n$. Then

1. $\zeta_iD^* = \zeta_{i+1}D^*$ for all $i \geq 2$;
2. $\zeta_iD^*/\zeta_{i+1}D^* \cong \mathbb{Z}_e$, where $e = \exp(\Gamma_D/\Gamma_F)$.

**Proof.** (1) Since $D$ is totally ramified, we have $\overline{D} = F$ and so $D_U = U_F(1 + M_D)$. But $D' \subseteq D_U$ and hence $D' \subseteq U_F(1 + M_D)$. This yields $\zeta_2D^* \subseteq [D^*, 1 + M_D]$. Now, using Corollary 5.13 implies that $\zeta_2D^* = [D^*, 1 + M_D]$ and thus $\zeta_2D^*$ is $D^*$-perfect. Therefore $\zeta_2D^* = \zeta_1D^*$ for all $i \geq 2$.

(2) Consider the reduction map

$$U_D/1 + M_D \to \overline{D}' = D'/1 + M_D,$$

where $\overline{D}'$ is an isomorphism.

The restriction of this map to $D'$ gives an isomorphism

$$\frac{D'}{D' \cap (1 + M_D)} \cong \overline{D}'$$

Now, from the Congruence Theorem, the equality $\zeta_2D^* = [D^*, 1 + M_D]$ and the fact that $\overline{D}' \cong \mathbb{Z}_e$ it follows that $\zeta_1D^*/\zeta_2D^* \cong \mathbb{Z}_e$. □

**Theorem 5.15.** Let $F$ be a Henselian field and $D$ be a tame and unramified $F$-central division algebra of index $n$. Then

1. $\zeta_iD^*/\zeta_{i+1}D^* \cong \zeta_iD^*/\zeta_{i+1}D^*$, for all $i \geq 1$.
2. $[D^*, 1 + M_D] \subseteq \zeta_iD^*$, for any $i \geq 1$.

**Proof.** (1) We first show that $\overline{\zeta_iD^*} = \zeta_i\overline{D}^*$. To do so, consider the following exact sequence:

$$1 \rightarrow F^*U_D/F^* \rightarrow D^*/F^* \rightarrow \Gamma_D/\Gamma_F \rightarrow 1.$$

Since $\Gamma_D = \Gamma_F$, the above sequence implies that $D^* = F^*U_D$. Therefore, if $a \in D^*$ and $b \in \zeta_iD^*$ then the element $a^{-1}b^{-1}ab$ may be written in the form $\alpha^{-1}\beta^{-1}\alpha\beta$ where $\alpha, \beta \in U_D$. This shows that $\overline{\zeta_iD^*} = \zeta_i\overline{D}^*$. On the other hand from the Congruence Theorem and Corollary 5.13 we have

$$[D^*, 1 + M_D] \subseteq (1 + M_D) \cap \zeta_iD^* \subseteq [D^*, 1 + M_D],$$

and so $(1 + M_D) \cap \zeta_iD^* = [D^*, 1 + M_D]$. Thus the restriction of the reduction map on the subgroup $\zeta_iD^*$ gives rise to an isomorphism

$$\frac{\zeta_iD^*}{[D^*, 1 + M_D]} \cong \zeta_i\overline{D}^*. \quad (5.3)$$

Therefore $\zeta_iD^*/\zeta_{i+1}D^* \cong \zeta_i\overline{D}'/\zeta_{i+1}\overline{D}'$. \hfill \square
(2) By (5.3) we have an isomorphism
\[ \frac{D'}{[D^*, 1 + M_D]} \cong \overline{D'} \]
Also, by Corollary 5.6, \( \overline{D'} \) is not a torsion group and so \( D'/[D^*, 1 + M_D] \) is not torsion. But, by Corollary 5.4 the group \( D'/\zeta D^* \) is a torsion group and by Corollary 5.13, \([D^*, 1 + M_D] \subseteq \zeta_i D^*\). This implies that \([D^*, 1 + M_D] \subseteq \zeta_i D^*\). \( \Box \)

6. The functor \( \text{CK}_1 \) and Reduced \( K \)-Theory

As is known, some useful tools that are used to explore the structure of \( K_1 \) groups of simple Artinian rings are “Determinant-like” functions. Namely, if \( A = M_n(D) \) is a simple Artinian ring \((D \text{ is a division ring})\) then thanks to the Dieudonné determinant we have \( K_1(A) \cong K_1(D) \). Thus, the computation of \( K_1(A) \) reduces to that of \( K_1(D) \). But in the case that \( D \) is of finite dimension over its center \( F \), another determinant-like function is the reduced norm map \( \text{Nrd}_D : D^* \to F^* \). Now, since \( F^* \) is commutative and \( \text{Nrd}_D \) is a multiplicative map, all multiplicative commutators in \( D^* \) are killed by \( \text{Nrd}_D \) and thus we have the following exact sequence

\[
1 \longrightarrow \text{SK}_1(D) \longrightarrow K_1(D) \longrightarrow \text{Nrd}_D K_1(F)
\]

where \( \text{SK}_1(D) = \{ a \in D^* | \text{Nrd}_D(a) = 1 \}/D' \). In comparison with the properties of usual determinant on matrix groups over commutative fields, it is natural to ask if in general \( \text{SK}_1(D) = 1 \). The question of the triviality of \( \text{SK}_1(D) \) was open for many years and was called the “Tanaka-Artin” problem. Before 1975, the time that Platonov gave the first examples of division algebras with non-trivial \( \text{SK}_1 \), the general feeling was that \( \text{SK}_1 \) is trivial for every division algebra and mostly the interested researchers were taking steps in directions to prove this problem positively. For example, in 1943 Nakayama and Matsushima showed that if \( F \) is a local field then \( \text{SK}_1(D) = 1 \) (cf. [76]). Also, in 1950 [99] Wang proved that if the index of \( D \) is square free or if \( F \) is an algebraic number field, then \( \text{SK}_1(D) = 1 \). However, the methods used by Platonov to compute \( \text{SK}_1(D) \) are called “Reduced \( K \)-Theory” these days. For a survey on \( \text{SK}_1 \) and the results concerning the computation of this group after Platonov see [95, §6] and references therein. Now, let us review some interesting characteristics of the group \( \text{SK}_1 \) below. (See [73] for the complete list of the properties of \( \text{SK}_1 \) and [11] for the proofs.) Let \( F \) be a field and \( D \) be an \( F \)-central division algebra. Then

- For any field extension \( L/F \) one has a homomorphism \( \text{SK}_1(D) \to \text{SK}_1(D \otimes_F L) \). On the other hand \( \text{SK}_1 \) enjoys a transfer map, that is, for each finite extension \( L/F \) there exists a norm homomorphism \( \text{SK}_1(D \otimes_F L) \to \text{SK}_1(D) \). Now, since \( \text{SK}_1(M_n(L)) = 1 \), one can deduce that \( \text{SK}_1(D) \) is a torsion abelian group of bounded exponent \( \text{ind}(D) \) and if the degree \( [L : F] \) is relatively prime to \( \text{ind}(D) \), then \( \text{SK}_1(D) \cong \text{SK}_1(D_1) \times \text{SK}_1(D_2) \).
- If \( D = D_1 \otimes_F D_2 \) and \( \text{ind}(D_1) \) and \( \text{ind}(D_2) \) are relatively prime, then \( \text{SK}_1(D) \cong \text{SK}_1(D_1) \times \text{SK}_1(D_2) \).
- In the case of a valued division algebra \( \text{SK}_1 \) is stable, namely \( \text{SK}_1(D) \cong \text{SK}_1(D) \), where \( D \) is an unramified division algebra. Indeed \( \text{SK}_1(D((x))) \cong \text{SK}_1(D) \).
• SK$_1$ is stable under purely transcendental extensions, i.e., if $L/F$ is a purely transcendental extension then SK$_1(D \otimes_F L) \cong$ SK$_1(D)$. Indeed SK$_1(D(x)) \cong$ SK$_1(D)$.

Now in contrast, for an $F$-central simple algebra $A$ consider the map $K_1(F) \to K_1(A)$ which is induced by the inclusion and consider the group

$$\text{CK}_1(A) = \text{Coker}(K_1(F) \to K_1(A)) \cong A^*/F^*A'.$$

The first systematic investigation of the algebraic properties of CK$_1$ for a division algebra $D$ was initiated in [23] and then was continued in [22]. The main point of view in the above mentioned papers was showing that the algebraic properties of CK$_1(D)$ is closely related to those of SK$_1(D)$ and some functorial properties of SK$_1(D)$ can be carried over CK$_1(D)$.

Also, one can easily see that the following sequence is exact:

$$1 \to \mu_n(F) \to \frac{Z(D')}{Z(D')} \to \text{SK}_1(D) \to \text{CK}_1(D) \to \frac{\text{Nrd}_D(D^*)}{F^{*n}} \to 1,$$

where $n = \text{ind}(D)$, $f, g$ are canonical homomorphisms and $\nu$ is induced by the reduced norm map. Therefore, we have SK$_1(D) \cong \mu_n(F)/Z(D')$ if and only if CK$_1(D) \cong \text{Nrd}_D(D^*)/F^{*n}$. This observation can be used to open a new way for the computation of SK$_1(D)$ as we shall see later. Here, we are going to review some SK$_1$-like properties of CK$_1$. At first, we observe that if $A = M_n(D)$ is a central simple algebra then thanks to the Dieudonné determinant, one has CK$_1(A) \cong D^*/F^{*n}D'$. Thus, the functor CK$_1$ from the category of central simple algebras to the category of abelian groups is not Morita invariant. (Recall that for the functor SK$_1$ we have SK$_1(A) \cong$ SK$_1(D)$.) Now, Corollary 5.3 implies that CK$_1(D)$ is abelian torsion of bounded exponent $\text{ind}(D)$. Therefore, CK$_1(A)$ is abelian torsion of bounded exponent $\text{deg}(A)$.

Let $D$ be an $F$-central division algebra and $A$ be a simple Artinian ring such that $F \subseteq Z(A)$ and $[A : F] < \infty$. The natural embedding of $D$ in $D \otimes_F A$ ($a \mapsto a \otimes 1$) induces a group homomorphism $\mathcal{I} : \text{CK}_1(D) \to \text{CK}_1(D \otimes_F A)$. The following proposition provides a reverse map.

**Proposition 6.1** (Transfer Map). *Let $D$ be an $F$-central division algebra and $A$ be a simple Artinian ring such that $F \subseteq Z(A)$ and $[A : F] < \infty$. Then there is a homomorphism $\mathfrak{P} : \text{CK}_1(D \otimes_F A) \to \text{CK}_1(D)$ such that $\mathfrak{P}\mathcal{I} = \eta_m$, where $m = [A : F]$ and $\eta_m(x) = x^m$.***

**Proof.** Consider the following sequence when we tensor with $D$ over $F$,

$$D \to D \otimes_D A \overset{1 \otimes l}{\longrightarrow} D \otimes_D M_m(F) \overset{\eta}{\longrightarrow} M_m(D), \quad (6.1)$$

where $l : A \to M_m(F)$ is the left regular representation. If $a \in D$ then pushing $a$ along the sequence 6.1 we arrive at $aI_m$ where $I_m$ is the identity matrix. Now, the above maps and the Dieudonné determinant yield the following sequence of abelian group homomorphisms

$$\text{CK}_1(D) \overset{\mathfrak{P}}{\longrightarrow} \text{CK}_1(D \otimes_F A) \overset{\eta}{\longrightarrow} \text{CK}_1(M_m(D)) \overset{\delta}{\longrightarrow} \text{CK}_1(D), \quad (6.2)$$

where $\mathfrak{P}(a) = aI_m$ and $\delta$ is induced by the Dieudonné determinant. Now, one can observe that $\delta \mathfrak{P} = \eta_m$. Thus if we let $\mathfrak{P} = \delta \mathfrak{P}$ then we are done. \qed
The following corollary shows that the analogous result for the behavior of SK₁ under the extension of the ground field holds for CK₁ too.

**Corollary 6.2.** Let D be an F-central division algebra and A be a simple Artinian ring such that F ⊆ Z(A) and [A : F] < ∞. If ind(D) and [A : F] are relatively prime, then the canonical homomorphism J : CK₁(D) → CK₁(D ⊗ₖ A) is injective.

**Proof.** Let ind(D) = n and [A : F] = m. Suppose J(x) = 1 for some x ∈ CK₁(D). By 6.1, xᵐ = J(x) = 1. But CK₁(D) is of bounded exponent n. Hence xⁿ = 1. Now, since m and n are relatively prime, we conclude that x = 1. □

**Theorem 6.3.** Let A and B be F-central division algebras such that ind(A) and ind(B) are relatively prime. Then CK₁(A ⊗ₖ B) ≅ CK₁(A) × CK₁(B).

**Proof.** Since CK₁(A ⊗ₖ B) is abelian of bounded exponent mn where m = ind(A) and n = ind(B) we have that CK₁(A ⊗ₖ B) ≅ ℋ × ℋ where exp(ℋ)|m and exp(ℋ)|n. Consider the following diagram of mappings

\[
\begin{array}{ccc}
CK₁(A) & \xrightarrow{J₁} & CK₁(A ⊗ₖ B) \\
\downarrow{J₂} & & \downarrow{J₃} \\
CK₁(A ⊗ₖ B ⊗ₖ B^{op}) & \xrightarrow{ϕ} & CK₁(Mₙ(Z(A))) & \xrightarrow{δ} & CK₁(A) \\
\downarrow{J₄} & & & & \\
CK₁(A ⊗ₖ B ⊗ₖ B^{op ⊗ₖ F}) & \xrightarrow{J₅} & CK₁(Mₙ(Z(A ⊗ₖ B))) & \xrightarrow{J₆} & CK₁(A ⊗ₖ B),
\end{array}
\]

where J₁, J₂ and J₃ are induced by the natural embedding, ϕ, ̂ϕ are induced by the usual isomorphism and δ, ̂δ are induced by the Dieudonné determinant. Now, by Proposition 6.1 we have δϕO₂J₁ = η₁. But ind(A) and n are coprime. So δϕO₂J₁ is an isomorphism. Also J₁(CK₁(A)) ⊆ ℋ as CK₁(A) is m-torsion. Thus we conclude that δϕO₂|₃₄ is surjective. Now, we show that δϕO₂|₃₄ is injective. To prove this, first note that if 1 ≠ w ∈ ℋ then ̂ϕO₂J₂(w) = η₁(w) = wⁿ ≠ 1 and so J₂|₃₄ is injective. Suppose u ∈ ℋ is such that δϕJ₂(u) = 1. Then ϕJ₂(u) ∈ ker δ and it is not hard to see that ϕJ₂(u)ⁿ = 1. Thus ϕJ₂(uⁿ) = 1 and hence uⁿ = 1 as ϕ and J₂|₃₄ are injective. On the other hand wⁿ = 1. This forces that u = 1, in other words CK₁(A) ⊆ ℋ. Similarly, it can be shown that CK₁(B) ⊆ ℋ. This completes the proof. □

Here, we have to point out that the same functorial property as in Theorem 6.3 also holds for central simple algebras. In other words, if A and B are F-central simple algebras with coprime degrees then CK₁(A ⊗ₖ B) ≅ CK₁(A) × CK₁(B) (cf. [31]). In the next proposition we are going to find a scheme to compute CK₁(D) as well as SK₁(D).

**Proposition 6.4.** Let F be a Henselian field with valuation w and D be a tame F-central division algebra of index n. If \( \frac{D^w}{D} \) is a field then CK₁(D) ≅ Γ₀/D/Γ₉ and SK₁(D) ≅ \( \mu_n(F)/Z(D^w) \).
Proof. Let \( v \) be the extension of \( w \) to \( D \). Set \( NU_F = \text{Nrd}_D(D^*) \cap U_F \) and let \( x \in NU_F \). Then \( w(x) = 0 \) and \( x = \text{Nrd}_D(b) \) for some \( b \in D^* \). Now, by Remark 5.11 we have that \( v(b) = 0 \) and thus \( b \in U_D \). Therefore, the reduced norm map induces an epimorphism

\[
\text{Nrd}_D : \frac{F^*U_D}{F^*D'} \rightarrow \frac{F^*nNU_F}{F^*n}.
\]

Also since \( \Gamma_D \) is torsion free, multiplying by \( n \) induces an isomorphism

\[
\eta_n : \frac{\Gamma_D}{n\Gamma_D} \rightarrow \frac{n\Gamma_D}{n^2\Gamma_D}.
\]

On the other hand by Remark 5.11 we have \( v(\text{Nrd}_D(D^*)) = n\Gamma_D \). Thus, \( v \) induces the two following epimorphisms:

\[
v : \text{CK}_1(D) \rightarrow \frac{\Gamma_D}{n\Gamma_D}, \quad v : \frac{\text{Nrd}_D(D^*)}{F^*n} \rightarrow \frac{n\Gamma_D}{n^2\Gamma_D}.
\]

Now, applying the fact that \( \text{CK}_1(D) \) is of bounded exponent \( n \), one deduces that the following diagram is commutative with exact rows and columns where \( f, g \) are canonical maps.

\[
\begin{array}{cccccc}
1 & \quad 1 & \quad 1 & \quad 1 & \quad 1 & \\
F^*U_D & \text{Nrd}_D & F^*nNU_F & \rightarrow & 1 & \\
F^*D' & & & & & \\
1 & \rightarrow & \mu_n(F) & \rightarrow & \text{SK}_1(D) & \rightarrow & \text{CK}_1(D) & \rightarrow & \text{Nrd}_D(D^*) & \rightarrow & 1 & \\
& & f & \rightarrow & g & \rightarrow & v & \rightarrow & \eta_n & \rightarrow & \eta_n & \rightarrow & 1 & \\
& & & & & & & & & & & & \\
& & 1 & \rightarrow & \Gamma_D & \rightarrow & \Gamma_F & \rightarrow & \eta_n & \rightarrow & \eta_n & \rightarrow & 1 & \\
& & & & & & & & & & & & \\
& & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \\
\end{array}
\]

(6.3)

Now, if we show that \( F^*U_D = F^*D' \) then we are done. To prove this, first note that by the Congruence Theorem we have \( 1 + M_D \subseteq (1 + M_F)D' \). Consider the reduction map \( U_D \rightarrow D' \).

We have the sequence

\[
\overline{D} \xrightarrow{\sim} U_D/1 + M_D \xrightarrow{\text{nat.}} U_D/(1 + M_F)D' \xrightarrow{\text{nat.}} F^*U_D/F^*D'.
\]

Thus \( \theta : \overline{D}/\overline{F^*D'} \rightarrow F^*U_D/F^*D' \) is an isomorphism. Considering the fact that \( \overline{D}/\overline{F^*D'} = 1 \), the theorem follows.

\[\square\]

Corollary 6.5. Let \( F \) be a Henselian field and \( D \) be a tame \( F \)-central division algebra of index \( n \). If \( \overline{D}/\overline{F^*D'} = 1 \) then \( \text{SK}_1(D) \cong \mu_n(F)/\overline{F^*D'} \cap \overline{F} \).
Proof. Since the valuation is tame and Henselian, using Hensel’s Lemma, it is easy to see that \( \mu_n(F) \to \mu_n(\overline{F}) \), \( a \to \overline{a} \) is an isomorphism. Also, it is not difficult to prove that \( Z(D') \cong Z(\overline{D'}) \cong DM \cap \overline{F} \). Therefore, \( SK_1(D) \cong \mu_n(F)/Z(D') \cong \mu_n(\overline{F})/DM \cap \overline{F} \).

To continue our study, we need to recall the following result of [15].

**Theorem 6.6** (Ershov). Let \( F \) be a Henselian field and \( D \) be a tame F-central division algebra of index \( n \). If \( a \in U_D \) then

\[
\overline{\text{Nrd}}_D(a) = n_Z(\overline{D})/\overline{F} \text{Nrd}(\overline{a})^{n/mn'}
\]

where \( m = \text{ind}(\overline{D}) \) and \( mn' = [Z(\overline{D}) : \overline{F}] \).

Let \( D \) be Henselian tame \( F \)-central division algebra of index \( n \). From the Congruence Theorem we have \( (1 + M_D) = (1 + M_F)[D^*, 1 + M_D] \). Taking the reduced norm from both sides we obtain \( \text{Nrd}_D(1 + M_D) = (1 + M_F)^n \). Now, since \( D \) is tame and Henselian \((1 + M_F)\) is \( n \)-divisible. Thus we have the following equality:

\[
\text{Nrd}_D(1 + M_D) = 1 + M_F.
\] (6.4)

We will use this observation in the proof of the next theorem.

**Theorem 6.7.** Let \( F \) be a Henselian field and \( D \) be a tame \( F \)-central division algebra of index \( n \). Then

1. If \( D \) is unramified then \( \text{CK}_1(D) \cong \text{CK}_1(\overline{D}) \) and \( \text{SK}_1(D) \cong \text{SK}_1(\overline{D}) \);
2. If \( D \) is totally ramified then \( \text{CK}_1(D) \cong \Gamma_D/\Gamma_F \) and \( \text{SK}_1(D) \cong \mu_n(\overline{F})/\mu_e(\overline{F}) \) where \( e = \exp(\Gamma_D/\Gamma_F) \);
3. If \( \overline{D} \) is a field, \( \overline{D}/\overline{F} \) is a cyclic extension with \( [\overline{D} : \overline{F}] = k \) and \( N_{\overline{D}/\overline{F}}(\overline{D}) = \overline{F}^k \) then \( \text{CK}_1(D) \cong \Gamma_D/\Gamma_F \) and \( \text{SK}_1(D) \cong \mu_n(\overline{F})/\overline{D} \cap \overline{F} \).

*Proof.* (1) Since \( D \) is unramified \( \Gamma_D/\Gamma_F = 1 \) and thus by (6.3) and a similar argument as used in the proof of Proposition 6.4, we have

\[
\text{CK}_1(D) \cong F^*U_D/F^*D' \cong \overline{D} / \overline{F} \cdot \overline{F}.
\] (6.5)

On the other hand, by [42, Prop. 1.7], \( Z(\overline{D})/\overline{F} \) is normal and \( \text{Gal}(Z(\overline{D})/\overline{F}) \) is a homomorphic image of \( \Gamma_D/\Gamma_F \). However, \( D \) is a tame division algebra. So \( Z(\overline{D})/\overline{F} \) is Galois with \( \text{Gal}(Z(\overline{D})/\overline{F}) = 1 \). Therefore, \( Z(\overline{D}) = \overline{F} \). Also, by the proof of Theorem 5.15 we have \( \overline{D} = \overline{D}' \). So, by (6.5) we conclude that \( \text{CK}_1(D) \cong \text{CK}_1(\overline{D}) \). Moreover, if \( a \in U_D \) then \( \text{Nrd}_D(a) = \text{Nrd}_{\overline{D}}(\overline{a}) \) by Theorem 6.6. (Note that by the fundamental inequality we have \( \text{ind}(\overline{D}) = \text{ind}(D) \).) Therefore, the reduction map induces a monomorphism

\[
D^{(1)}/D^{(1)} \cap (1 + M_D) \to \overline{D}^{(1)}.
\]

This map is in fact surjective and hence an isomorphism, for if \( \overline{a} \in \overline{D}^{(1)} \) then \( \text{Nrd}_D(a) = 1 \). Thus \( \text{Nrd}_D(a) \in 1 + M_F \). So, by (6.4) there exists \( b \in 1 + M_D \) such that \( \text{Nrd}_D(a) = \text{Nrd}_D(b) \). Therefore \( ab^{-1} \in D^{(1)} \) and \( ab^{-1} \mapsto \overline{a} \), as desired. Now, from the Congruence Theorem we obtain \( \text{SK}_1(\overline{D}) \cong \text{SK}_1(D) \).
(2) Since \( D \) is totally ramified, clearly we have \( D^{r}/D \cong \Gamma_{D}/\Gamma_{F} \). So by Proposition 6.4 we obtain \( \text{CK}_{1}(D) \cong \Gamma_{D}/\Gamma_{F} \). Also since in this case \( D^{r} = \mu_{e}(F) \) from Corollary 6.5 we conclude that \( \text{SK}_{1}(D) \cong \mu_{n}(F)/\mu_{e}(F) \).

(3) From Proposition 6.4 it is enough to prove \( N_{\overline{D}/F}(\overline{D}^{1})/\overline{F}^{n} \cong \overline{D}/\overline{F}^{r} \overline{D}^{r} \). To show this, consider the norm function \( N_{\overline{D}/F} : \overline{D}^{r} \to \overline{F}^{n} \). Now, if \( x \in U_{D} \) then by Ershov’s formula it follows that \( \text{Nrd}_{D}(x) = N_{\overline{D}/F}(x) \). This shows that \( D^{r} \subseteq \ker N_{\overline{D}/F} \). Conversely, if \( x \in \ker N_{\overline{D}/F} \) then by Hilbert Theorem 90, there is a \( \overline{b} \) such that \( x = \overline{b}\sigma(\overline{b})^{-1} \) where \( \sigma \) is the generator of \( \text{Gal}(\overline{D}/\overline{F}) \). Now, since the fundamental homomorphism \( D^{*} \to \text{Gal}(\overline{Z}(\overline{D}/\overline{F})) \) is surjective, it follows that \( \sigma \) is of the form \( \sigma(z) = \overline{c}ac^{-1} \), for some \( c \in D^{*} \). This implies that \( x \in \overline{D}^{r} \) and so \( \ker N_{\overline{D}/F} = \overline{D}^{r} \). Therefore

\[
\overline{D}/\overline{F}^{r} \overline{D}^{r} \cong N_{\overline{D}/F}(\overline{D}^{r})/\overline{F}^{n},
\]

because the image of \( \overline{D}/\overline{F}^{r} \overline{D}^{r} \) under \( N_{\overline{D}/F} \) is \( \overline{F}^{n} \).

**Corollary 6.8.** Let \( D \) be an \( F \)-central division algebra of index \( n \). If \( \text{char}(F) \nmid n \) then \( \text{CK}_{1}(D) \cong \text{CK}_{1}(D((x))) \) and \( \text{SK}_{1}(D) \cong \text{SK}_{1}(D((x))) \).

Here, we are going to use the above results in computing of \( \text{CK}_{1} \) and \( \text{SK}_{1} \) for some division algebras.

**Example 6.9.** Recall that a field \( F \) is called real Pythagorean if \(-1 \notin F^{*2} \) and the sum of any two square elements of \( F \) is a square in \( F \). Let \( F \) be a real Pythagorean field and \( \mathcal{Q} \) be the ordinary quaternion algebra over \( F \), i.e.,

\[
\mathcal{Q} = \{ a + bi + c j + d ij | a, b, c, d \in F, i^{2} = j^{2} = -1, ij = -ji \}.
\]

It is easy to see that \( \mathcal{Q} \) is a division algebra. If \( u = a + bi + c j + d ij \in \mathcal{Q}^{*} \) then \( \text{Nrd}_{\mathcal{Q}}(u) = a^{2} + b^{2} + c^{2} + d^{2} \in F^{*2} \) as \( F \) is real Pythagorean. So there is a \( v \in F^{*} \) such that \( v^{2} = \text{Nrd}_{\mathcal{Q}}(u) \). Now we have \( \text{Nrd}_{\mathcal{Q}}(u/v) = \text{Nrd}_{\mathcal{Q}}(u)/v^{2} = 1 \). Thus \( u/v \in D^{(1)} \). But by Wang’s Theorem \( \text{SK}_{1}(D) = 1 \) and hence \( u/v \in D^{1} \). Therefore \( u \in F^{*}D^{1} \). This implies that \( \text{CK}_{1}(\mathcal{Q}) = 1 \). Now, we will show that the converse is also true. Let \( F \) be a field and \( \mathcal{Q} \) be the ordinary quaternion algebra over \( F \). We establish that if \( \mathcal{Q} \) is a division algebra and \( \text{CK}_{1}(\mathcal{Q}) = 1 \) then \( F \) is real Pythagorean. To prove this, first note that since \( \mathcal{Q} \) is a division ring, \(-1 \notin F^{*2} \). Also, since \( \text{CK}_{1}(\mathcal{Q}) = 1 \) we have \( D^{*} = F^{*}D^{1} \). Now, taking the reduced norm from both sides we conclude that \( \text{Nrd}_{\mathcal{Q}}(\mathcal{Q}) = F^{*2} \). Thus if \( u, v \in \mathcal{F} \) then \( u^{2} + v^{2} = \text{Nrd}_{\mathcal{Q}}(u + vi) \in F^{*2} \). This immediately implies that \( F \) is real Pythagorean.

Now, let \( F \) be real Pythagorean and \( \mathcal{Q} \) be the quaternion algebra over \( F \). Using Corollary 6.8 we conclude that \( \text{CK}_{1}(\mathcal{Q}((x))) = 1 \) and so \( F((x)) \) (the center of \( \mathcal{Q}((x))) \) is real Pythagorean. Therefore, we have proved that if \( F \) is real Pythagorean then for each \( n \) the field \( F((x_{1}, \ldots, x_{n})) \) is real Pythagorean.

**Example 6.10.** Let \( \mathbb{R} \) be the field of real numbers and \( F = \mathbb{R}((x_{1}, \ldots, x_{n})) \). The field \( F \) has a natural valuation \( v : F \to \mathbb{Q}_{n}^{+} \mathbb{Z} \) given by

\[
v(\sum_{j_{1}} \ldots \sum_{j_{n}} c_{j_{1}} \ldots c_{j_{n}} x_{j_{1}} \ldots x_{j_{n}}) = \min\{(j_{1}, \ldots, j_{n}) | c_{j_{1}} \ldots c_{j_{n}} \neq 0\},
\]

\[(6.6)\]
where $\oplus_{i=1}^n \mathbb{Z}$ is given right-to-left lexicographical ordering. The above valuation is Henselian and if $D$ is an $F$-central division algebra then $D$ is obviously tame. It is also clear that $\overline{F} = \mathbb{R}$. Because the algebraically closed field $\mathbb{C} = R(\sqrt{-1})$ has a trivial Brauer group, we conclude that $\overline{D} = \mathbb{R}$ or $\overline{D} = \mathbb{C}$ or $\overline{D} = \mathbb{Q}$ where $\mathbb{Q}$ is the ordinary quaternion division algebra over $\mathbb{R}$. Now, the above example shows that in all cases $\overline{D'}/\overline{F'}/\overline{D'} = 1$. Therefore, Corollary 6.5 yields $SK_1(D) \cong \mu_n(\mathbb{R})/\overline{D'} \cap \mathbb{R}$. But from number theory we know that $[D:F]$ is a power of 2. So $\mu_n(\mathbb{R}) = 1$. On the other hand by the work of Wadsworth $-1 \in D'$ (cf. [98]). Thus we have $SK_1(D) = 1$

Now let $F = \mathbb{C}((x_1, x_2, x_3, x_4))$ and $n$ a natural number. For $j = 1, 2$ let $D_j = A_{\omega_n}(x_j, x_{j+1}; F)$ where $\omega_n$ is the primitive $n$-th root of unity and $A_{\omega_n}(x_j, x_{j+1}; F)$ is the symbol algebra generated by symbols $\alpha, \beta$ subject to the relations $\alpha^n = x_j$, $\beta^n = x_{j+1}$ and $\alpha\beta = \omega_n\beta\alpha$. By example 3.6 of [93] we know that $D = D_1 \otimes_F D_2$ is an $F$-central division algebra with $\text{ind}(D) = n^2$ and $\Gamma_D/F \Gamma_F \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Thus $e = \exp(\Gamma_D/F) = n$. On the other hand, $D$ is totally ramified as $\mathbb{C}$ is algebraically closed. Therefore, from Theorem 6.7 we have $SK_1(D) \cong \mathbb{Z}/n\mathbb{Z}$. This implies that there exists a field such that when $D$ runs over all $F$-central division algebras, then $SK_1(D)$ can be any finite cyclic group. Note that if $D$ is an arbitrary division algebra over $F$, then $D$ is totally ramified and $SK_1(D)$ is a finite cyclic group.

In Example 6.10 we have deduced both Theorems of [55], which are obtained by using heavy machinery of reduced $K$-Theory, as natural examples of the above theorems. In the next example, we show that the group $CK_1$ can be any finite cyclic group of odd order.

**Example 6.11.** Let $F = \mathbb{F}_2((x))$. Equipped with its natural valuation, $F$ is local and hence $Br(F) \cong \mathbb{Q}/\mathbb{Z}$. Now, since $\exp([D]) = \text{ind}(D)$ for every $F$-division algebra $D$ ($[D]$ is the equivalence class of $D$ in the Brauer group of $F$), given a natural number $n$, there exists an $F$-central division algebra $D$ such that $\text{ind}(D) = n$. Now, let $n$ be odd. Since $F$ is Henselian, the natural valuation has a unique extension to $D$. Clearly $\overline{F} = \mathbb{F}_2$, $D$ is tame, and $\overline{D}/\overline{F}$ is cyclic (recall that the Brauer group of every finite field is trivial and every finite extension of a finite field is cyclic). At the other extreme, we have $N_{\overline{D}/\overline{F}}(\overline{D}) = \mathbb{F}_2^2 = \{1\}$ which is obviously $n$-divisible. Now, Theorem 6.7 implies that $CK_1(D) \cong \mathbb{Z}/n\mathbb{Z}$. Thus, when $[D]$ ranges over the Brauer group of $F$ then $CK_1(D)$ can be any finite cyclic group of odd order.

**Example 6.12.** Let $\mathbb{C}$ be the field of complex numbers, $n_1, n_2, \ldots, n_r$ an arbitrary sequence of positive integers and $n = n_1n_2\ldots n_r$. If $i$ is even, set $D_{i+1} = D_i((x_{i+1}))$ and if $i$ is odd we define $\sigma_i : D_i \to D_i$ by $\sigma_i(x_i) = \omega_i x_i$, where $\omega_i$ is a primitive $n_{(i+1)}$-th root of unity. Then $D_{i+1} = D_i((x_{i+1}, \sigma_i))$. So, we have $D = D_2 = \mathbb{C}((x_1, x_2, \sigma_1, x_3, \ldots, \sigma_{2r-1}))$. By Hilbert’s construction (cf. [11]), $F = Z(D) = \mathbb{C}((x_1^{n_1}, x_2^{n_2}, \ldots, x_{2r-1}^{n_{2r-1}}, x_{2r}^{n_{2r}}))$ with $[D:F] = n^2$. Now, if we consider the natural valuation on $D$ then $\Gamma_D = \oplus_{j=1}^r (\mathbb{Z} \oplus \mathbb{Z})$ and $\Gamma_F = \oplus_{j=1}^r(n_j \mathbb{Z} \oplus \mathbb{Z})$. In this setting $D$ is a tame and totally ramified as $\mathbb{C}$ is algebraically closed. Therefore, from Theorem 6.7 we have $CK_1(D) \cong \Gamma_D/\Gamma_F \cong \oplus_{j=1}^r(\mathbb{Z}/n_j\mathbb{Z} \oplus \mathbb{Z}/n_j\mathbb{Z})$. This shows that for every abelian group $A$ there exists a division algebra $D$ with $CK_1(D) \cong A \oplus A$.

**Theorem 6.13.** Let $D$ be a tame division algebra over a Henselian field $F = Z(D)$ of index $n$. If $D$ has a maximal cyclic extension $L/F$ such that $\overline{D'}/\overline{F'}/\overline{D'} = 1$ then $SK_1(D) = 1$. 


Proof. As in the proof of Proposition 6.4 one may conclude that the reduction map induces an isomorphism $\overline{D}/\overline{L}D' \rightarrow U_D/U_LD'$. This yields $U_D = U_LD'$. Now, if $x \in D^{(1)}$ then $x = ld$ where $l \in L$ and $d \in D'$. So $\text{Nrd}_D(x) = N_{L/F}(l) = 1$. From the Hilbert Theorem 90, $l$ is of the form $a\sigma(a)^{-1}$ where $\sigma$ is the generator of $\text{Gal}(L/F)$. Now the Skolem-Noether Theorem implies that $\sigma(a) = cac^{-1}$ for some $c \in D^*$. Therefore $l = aca^{-1}c^{-1} \in D'$, as desired.

Example 6.14. Let $L/F$ be a cyclic extension with $\text{Gal}(L/F) = \langle \sigma \rangle$. Let $\text{char}(F) \nmid n$ where $n = [L : F]$ and consider the division algebra $D = L((x, \sigma))$ which is obtained by the Hilbert construction. We know that $Z(D) = F((x^n))$ and $\text{ind}(D) = n$. Considering the natural valuation of $D$ we have $\overline{D} = \overline{L}$. So Theorem 6.13 shows that $\text{SK}_1(D) = 1$.

In the following example we show that in contrast with $\text{SK}_1$, the group $\text{CK}_1$ is not stable under purely transcendental extensions.

Example 6.15. Let $D$ be a division algebra over its center $F$ with index $n$. Let $\mathcal{P}$ runs over all irreducible monic polynomials of $F[x]$ and $n_\mathcal{P}$ be the index of $D \otimes_F F[x]/\mathcal{P}$. By [52, Prop. 7] the sequence

$$1 \rightarrow K_1(D) \rightarrow K_1(D(x)) \rightarrow \bigoplus_{\mathcal{P}} (n_\mathcal{P}/n)\mathbb{Z} \rightarrow 1$$

(6.7)

which is obtained from the localization of exact sequences in algebraic $K$-Theory, is split exact. Thus, since every $(n_\mathcal{P}/n)\mathbb{Z}$ is torsion-free, the sequence (6.7) induces an isomorphism

$$\psi : \tau(K_1(D)) \rightarrow \tau(K_1(D(x))), \quad \overline{a} \mapsto a \otimes 1,$$

where $\tau(K_1(D))$ and $\tau(K_1(D(x)))$ are the torsion subgroups of $K_1(D)$ and $K_1(D(x))$, respectively. Now, if $\overline{a} \in \text{SK}_1(D)$ then $\text{Nrd}_{D(x)}(a \otimes 1) = \text{Nrd}_D(a) = 1$ and so $a \otimes 1 \in \text{SK}_1(D(x))$. Conversely, if $\overline{b} \in \text{SK}_1(D(x))$ then there is an $\overline{a} \in \tau(K_1(D))$ such that $\overline{b} = a \otimes 1$. But $\text{Nrd}_{D(x)}(a \otimes 1) = \text{Nrd}_D(a) = 1$. This implies that the restriction of $\psi$ on $\text{SK}_1(D)$ induces an isomorphism $\text{SK}_1(D) \rightarrow \text{SK}_1(D(x))$. On the other hand since $\text{CK}_1(D)$ is the cokernel of the natural map $K_1(F) \rightarrow K_1(D)$, applying the Snake Lemma to the commutative diagram

$$\begin{array}{ccc}
1 &\rightarrow& K_1(F) \\
\downarrow & & \downarrow \\
1 &\rightarrow& K_1(D) \\
\downarrow & & \downarrow \\
K_1(F(x)) &\rightarrow& \bigoplus_{\mathcal{P}} \mathbb{Z} &\rightarrow& 1 \\
\downarrow & & \downarrow & & \downarrow \\
K_1(D(x)) &\rightarrow& \bigoplus_{\mathcal{P}} (n_\mathcal{P}/n)\mathbb{Z} &\rightarrow& 1 \\
\downarrow & & \downarrow & & \downarrow \\
& & \mathbb{Z} &\rightarrow& 1.
\end{array}$$

yields the following split exact sequence

$$1 \rightarrow \text{CK}_1(D) \rightarrow \text{CK}_1(D(x)) \rightarrow \bigoplus_{\mathcal{P}} (n_\mathcal{P}/n)\mathbb{Z} \rightarrow 1.$$  

(6.8)

Now, consider the special case in which $D$ is the real quaternion $\mathbb{Q}$. Since the degree of every irreducible polynomial over $\mathbb{R}$ is either 1 or 2 we can easily see that $\bigoplus_{\mathcal{P}} (n_\mathcal{P}/n)\mathbb{Z} \cong \bigoplus_{i=1}^{\infty} 2\mathbb{Z}$. So, from Example 6.9 and (6.8) we conclude that $\text{CK}_1(\mathbb{Q}(x)) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$.

Unlike $\text{SK}_1$ it seems that $\text{CK}_1$ is rarely trivial. In fact in [23] the following conjecture was posed.
Conjecture 6.16. \( CK_1(D) \) is trivial if and only if \( D \) is quaternion over a real Pythagorean field.

The first attempt to prove Conjecture 6.16 was fulfilled in [30] where it is proved that if \( D \) is a tensor product of cyclic algebras then the answer to this conjecture is positive. Afterwards in [24], Hazrat and Wadsworth proved that if \( F \) is finitely generated over a subfield \( F_0 \) but not algebraic over \( F_0 \), then \( F \) is NKNT, i.e., for any non-commutative division algebra \( D \) finite dimensional over \( F \) (and not necessary central over \( F \)), then \( NK_1(D) = D^*/F^*D^{(1)} \) is non-trivial and hence \( CK_1(D) \neq 1 \). In what follows we are going to review these results.

Theorem 6.17. Let \( D \) be a noncommutative division algebra similar to a cyclic algebra \( A \). If one of the following conditions holds, then \( NK_1(D) \neq 1 \):

1. \( F \) contains a square root of \(-1\);
2. \( \text{char}(F) = 2 \);
3. The degree of \( A \) is odd.

Proof. It is not hard to see that every primary component of \( D \) is similar to a cyclic algebra. Thus by Theorem 6.3 we may assume that \( \text{ind}(D) = p^n \) where \( p \) is a prime number and \( n \geq 1 \). Put \( A = (E/F, \sigma, a) \). Choose \( A \) of minimal degree. Then \( \text{deg}(A) = p^n \) and \( a \notin F^*p \), because if \( a = b^p \) then \([A] = [(E_0/F, \sigma, b)] \) where \([E : E_0] = p \), contradicting the minimality of \( \text{deg}(A) \). Let \( \alpha \in A^* \) be such that \( \sigma(l) = \alpha^{-1}l\alpha \) and \( \alpha^p = a \). Now, one can easily see that the minimal polynomial of \( \alpha \) over \( F \) is \( x^p - a \). Thus \( \text{Nrd}_A(\alpha) = N_{F(\alpha)/F}(\alpha) = (-1)^{m-1}a \).

But if \( \text{char}(F) = 2 \) or \( p \) is odd then \( \text{Nrd}_A(\alpha) = a \notin F^*p \). Also if \( F \) contains a square root of \(-1\) and \( p = 2 \) then \( \text{Nrd}_A(\alpha) = -a \notin F^{*2} \) as \( a \notin F^{*2} \). Therefore all of the above conditions give \( \text{Nrd}_A(\alpha) \notin F^*p \). However, thanks to the Dieudonné determinant, \( \text{Nrd}_A(A^*) = \text{Nrd}_D(D^*) \).

Thus \( \text{Nrd}_D(D^*) \notin F^*p \). This forces that \( \text{Nrd}_D(D^*) \neq F^*p \) and so \( NK_1(D) \neq 1 \). (Note that \( NK_1(D) \cong D^*/F^*D^{(1)} \) with the isomorphism given by the reduced norm map.)

Recall that by Albert’s Main Theorem [11, p. 110] every \( p \)-algebra is Brauer equivalent to a cyclic \( p \)-algebra. Using this fact and Theorem 6.17 the following corollary is immediate.

Corollary 6.18. If \( D \) is a \( p \)-division algebra then \( NK_1(D) \neq 1 \).

Proposition 6.19. Let \( D \) be a cyclic division algebra. Then \( NK_1(D) = 1 \) if and only if \( D \) is the ordinary quaternion division algebra and \( F \) is real Pythagorean.

Proof. Let \( NK_1(D) = 1 \). From Theorem 6.3 and Theorem 6.17 it follows that \( \text{deg}(D) = 2^m \) for some \( m \) and \( \sqrt{-1} \notin F^* \). Let \( D = (E/F, \sigma, a) \) and put \( n = \text{deg}(D) \). Let \( \alpha \in D^* \) be such that \( D = \bigoplus_{j=0}^{n-1} E\alpha^j \) and \( \alpha^n = a \). As in the proof of Theorem 6.17 we have \( \text{Nrd}_D(\alpha) = -a = -\alpha^n \). But \( \text{Nrd}_D(\alpha) \in F^*n \) because \( NK_1(D) = 1 \). So \( -\alpha^n = f^n \) for some \( f \in F^* \). This yields \( (af^{-1})^n = -1 \). Thus we may replace \( a \) by \(-1\), i.e., \( D = (E/F, \sigma, -1) \). Now, let \( m > 1 \) and let \( L \supset F \) be a subfield of \( E \) with \([L : F] = 2 \). Since \( m > 1 \) then the division algebra \( Z_D(L) \cong (E/L, \sigma^2, -1) \) has index \( 2^{m-1} > 1 \). But if \( L \) contains a square root of \(-1\) then \( Z_D(L) \) is not a division algebra which is a contradiction. So \( \sqrt{-1} \notin L \). On the other hand \( L = F(\beta) \) for some \( \beta \in L \) with \( \beta^2 \in F \). Since the minimal polynomial of \( \beta \) over \( F \) is \( x^2 - \beta^2 \)
we conclude that $N_{L/F}(\beta) = -\beta^2$. Therefore

$$N_{L/F}(\beta) = N_{L/F}(N_{E/L}(\beta)),$$

(6.9)

$$N_{L/F}(\beta^{2m-1}) = N_{L/F}(\beta)^{2m-1} = (-\beta^2)^{2m-1} = \beta^{2m}.$$  

At the other extreme, $\beta = fd$ for some $f \in F$ and $d \in D^{(1)}$ as $NK_1(D) = 1$. Thus $N_{D}(\beta) = f^{2m}$. From (6.9) we obtain $\beta^{2m} = f^{2m}$ and hence $\beta f)^{2m} = 1$. Let $j$ be the smallest nonnegative integer such that $(\beta f)^j = 1$. If $j = 0$ or 1 then $\beta = \pm f$ which is absurd. So $j \geq 2$. Hence $\sqrt{-1} \in L$ and $L = F(\sqrt{-1})$. This contradiction implies that $m = 1$. Thus $D$ is a quaternion division algebra. Now, a similar argument as above shows that every maximal subfield of $D$ is $F$-isomorphic to $F(\sqrt{-1})$ and therefore $D$ is the ordinary quaternion division algebra. Now, from Example 6.9 it follows that $F$ is real Pythagorean.

We must point out that by the work of Hazrat and Vishne, a more general conclusion than Proposition 6.19 holds for cyclic central simple algebras. In [30, Cor. 2.11] they showed that if $A$ is a cyclic $F$-central simple algebra with $NK_1(A) = 1$ then $A$ is a matrix algebra over $F$ or over an ordinary quaternion division algebra. Now we are in a position to prove the following theorem.

**Theorem 6.20.** Let $D = C_1 \otimes_F \ldots \otimes_F C_k$ be an $F$-central division algebra where $C_1, \ldots, C_k$ are cyclic algebras over $F$. If $NK_1(D) = 1$ then $D$ is a cyclic division algebra. Hence $D$ is the ordinary quaternion division algebra and $F$ is real Pythagorean.

**Proof.** By Theorem 6.3 we may assume that $deg(D)$ is a prime power. Let $n = deg(D)$ and $n_i = deg(C_i)$. For each $i$, let $K_i$ be a cyclic maximal subfield of $C_i$, and $e_i \in C_i$ an element inducing an automorphism $\sigma_i$ of order $n_i$ of $K_i/F$. Then $b_i = e_i^{n_i} \in F^*$. Now, $N_{D}(z_i) = N_{D_{C_i}}(z_i)^{n/n_i} = ((-1)^{n_i-1}b_i)^{n/n_i} = b_i^{n/n_i}$ (note that since $n$ is a prime power $(n_i - 1)n_i/n_i$ is even). But since $NK_1(D) = 1$ we have $b_i^{n/n_i} \in F^{*n}$. Thus multiplying $e_i$ by a central element we may assume $b_i$ is an $n/n_i$-th root of unity. Now, let $b$ be the generator of the group $\langle b_1, \ldots, b_k \rangle$. So every $C_i$ is a cyclic algebra of the form $(K_i/F, \sigma_i, b^{g_i})$ for some $g_i$ and $D$ is similar in the Brauer group to cyclic algebra of degree $\text{lcm}(n_1, \ldots, n_k)$. But since $\text{lcm}(n_1, \ldots, n_k) \leq n_1 \ldots n_k (= n)$ and $D$ is a division algebra we conclude that $\text{lcm}(n_1, \ldots, n_k) = n_1 \ldots n_k$. This forces that $k = 1$ (because $n$ is a prime power) and now the result follows from Proposition 6.19. \qed

**Example 6.21.** Since every division algebra over a global field is a cyclic algebra, Theorem 6.20 implies that every global field is NKNT. Likewise, every nonreal local field is NKNT. But the field of real numbers $\mathbb{R}$ is not NKNT as the ordinary $\mathbb{R}$-division algebra has trivial $NK_1$. At the other extreme $\mathbb{R}(x)$ is NKNT. For if $L$ is a finite field extension of $\mathbb{R}(x)$ and $D$ is an $L$-central noncommutative division algebra, then by Tsen’s Theorem, $L(\sqrt{-1})$ splits $D$ and thus $D$ is a quaternion division algebra. Now, since $L$ is not Pythagorean $NK_1(D) \neq 1$ (Theorem 6.20).

As mentioned above, a significant result concerning Conjecture 6.16 was obtained in [24]. Here we present the main result of of that paper without proof.
**Theorem 6.22.** Let $D$ be a noncommutative division algebra with center $F$ which is finitely generated over some subfield $F_0$. If $NK_1(D)$ is trivial, then $[F : F_0] < \infty$.

Theorem 6.22 can be restated as:

**Corollary 6.23.** Let $F$ be field which is finitely generated but not algebraic over some subfield $F_0$. Then $F$ is NKNT.

Also from Theorem 6.22 and Example 6.21 the following corollary is immediate.

**Corollary 6.24.** If $D$ is a noncommutative division algebra whose center is finitely generated over its prime subfield or over an algebraically closed field, then $NK_1(D)$ is nontrivial. Hence $CK_1(D) \neq 1$.

**Remark 6.25.** An inverse problem to the reduced $K$-theory is the problem of triviality of the group $SK_1(D)$. Note that

$$SK_1(D) = \ker \Big( K_1(D) \xrightarrow{\text{Nrd}} K_1(F) \Big),$$

whereas

$$CK_1(D) = \text{coker} \Big( K_1(F) \xrightarrow{id} K_1(D) \Big).$$

In the reduced $K$-theory, the nontriviality of $SK_1$ is the major question, whereas with the group $CK_1$, it is the triviality which has remained an open question. The conjecture is that $CK_1(D)$ is trivial if and only if $D$ is an ordinary quaternion division algebra over Pythagorean field [25]. Recall that this conjecture has a direct application in solving the open problem whether a multiplicative group of a division algebra has a maximal subgroup [25]. Indeed, since $CK_1(D)$ is torsion of bounded exponent, if it is not trivial, it has maximal subgroups and therefore $D^*$ has (normal) maximal subgroups. Thus finding the maximal subgroups in $D^*$ reduces to the case that $CK_1(D)$ is trivial. It was shown in [25] that quaternion division algebras over pythagorean fields do have (non-normal) maximal subgroups. Thus if the above conjecture is settled positively, one concludes that the multiplicative group of a division algebra does have a maximal subgroup.

The groups $SK_1$ and $CK_1$ are both torsion groups of bounded exponent. This is one of the main properties of these groups that make them tractable. With the view of (6.11), for an Azumaya algebra $A$ over its center $R$, the inclusion map $\text{id} : R \to A$ gives the following exact sequence

$$1 \to ZK_i(A) \to K_i(R) \to K_i(A) \to CK_i(A) \to 1.$$  \hspace{1cm} (6.12)

Here $K_i$ for $i \geq 0$ are the Quillen $K$-groups and $ZK_i(A)$, $CK_i(A)$ are the kernel and co-kernel of $K_i(R) \to K_i(A)$ respectively. Clearly when $A$ is a division algebra and $i = 1$, we get $CK_1(D)$ defined in (6.11). The fact that $CK_i(A)$ and $ZK_i(A)$ are torsion of bounded exponent was studied in several papers (see [8, 29]). A consequence of this, is that the $K$-theory of $A$ coincides with the $K$-theory of its base ring up to torsions.

\footnote{Not to be mistaken with the established “inverse problem”, that is to find all the abelian groups which appear as the group $SK_1$.}
7. Graded approach to the theory of division algebras

As previous sections show, valued division algebras have been one of the main sources of producing (counter)examples in the theory of central simple algebras. In fact, valued division algebras have been the key ingredients in Amitsur’s non-crossed product construction and in Platonov’s construction of division algebras \( D \) with a non-trivial reduced Whitehead group. These two constructions settled major and several-decade open problems. A glance at these important works shows the formidable technical calculations required in the presence of valuations to achieve the results.

Starting with a division algebra \( D \) with a valuation, one can construct a graded ring

\[
\text{gr}(D) = \bigoplus_{\gamma \in \Gamma_D} \text{gr}(D)_{\gamma},
\]

where \( \Gamma_D \) is the value group of \( D \) and the summands \( \text{gr}(D)_{\gamma} \) arise from the filtration on \( D \) induced by the valuation (see §7.1 for details). Indeed, \( \text{gr}(D) \) is a graded division ring, i.e., every nonzero homogeneous element of \( \text{gr}(D) \) is a unit. While \( \text{gr}(D) \) has a much simpler structure than \( D \), nonetheless \( \text{gr}(D) \) provides a remarkably good reflection of \( D \) in many ways, particularly when the valuation on the center \( Z(D) \) is Henselian. The approach of making calculations in \( \text{gr}(D) \), then lifting back to get nontrivial information about \( D \) has been remarkably successful. This has provided enough motivation for a systematic study of this correspondence, notably by Boulagouaz, Hazrat, Hwang, Tignol and Wadsworth [9, 27, 26, 40, 41, 92, 96], and to compare certain functors defined on these objects, notably the Brauer groups, Witt groups, and the reduced Whitehead groups.

In the case of reduced Whitehead group that we are concerned in this note, it was proved in [27, Th. 4.8, Th. 5.7] that if a valuation \( v \) on \( Z(D) \) is Henselian and \( D \) is tame over \( Z(D) \), then

\[
\text{SK}_1(D) \cong \text{SK}_1(\text{gr}(D)) \quad (7.1)
\]

and

\[
\text{SK}_1(\text{gr}(D)) \cong \text{SK}_1(q(\text{gr}(D))), \quad (7.2)
\]

where \( q(\text{gr}(D)) \) is the division ring of quotients of \( \text{gr}(D) \). This has allowed the recovery of many of some known calculations of \( \text{SK}_1(D) \) with much easier proofs, as well as leading to the determination of \( \text{SK}_1(D) \) in some new cases. In this section we demonstrate how to obtain (7.1). Most of the material in this section are taken from [27, 74].

7.1. Graded division algebras. Here we establish the notation and recall some fundamental facts about graded division algebras indexed by a totally ordered abelian group, and about their connections with valued division algebras. In addition, we establish some important homomorphisms relating the group structure of a valued division algebra to the group structure of its associated graded division algebra.

Let \( R = \bigoplus_{\gamma \in \Gamma} R_{\gamma} \) be a graded ring, i.e., \( \Gamma \) is an abelian group, and \( R \) is a unital ring such that each \( R_{\gamma} \) is a subgroup of \( (R, +) \) and \( R_{\gamma} \cdot R_{\delta} \subseteq R_{\gamma + \delta} \) for all \( \gamma, \delta \in \Gamma \). Set

\[
\Gamma_R \ = \ \{ \gamma \in \Gamma \mid R_{\gamma} \neq 0 \}, \quad \text{the grade set of } R;
\]

\[
R^h \ = \ \bigcup_{\gamma \in \Gamma_R} R_{\gamma}, \quad \text{the set of homogeneous elements of } R.
\]
For a homogeneous element of $R$ of degree $\gamma$, i.e., an $r \in R_\gamma \setminus 0$, we write $\deg(r) = \gamma$.

Recall that $R_0$ is a subring of $R$ and that for each $\gamma \in \Gamma_R$, the group $R_\gamma$ is a left and right $R_0$-module. A subring $S$ of $R$ is a graded subring if $S = \bigoplus_{\gamma \in \Gamma_R} (S \cap R_\gamma)$. For example, the center of $R$, denoted $Z(R)$, is a graded subring of $R$. If $T = \bigoplus_{\gamma \in \Gamma} T_\gamma$ is another graded ring, a graded ring homomorphism is a ring homomorphism $f : R \to T$ with $f(R_\gamma) \subseteq T_\gamma$ for all $\gamma \in \Gamma$. If $f$ is also bijective, it is called a graded ring isomorphism; we then write $R \cong_{gr} T$.

For a graded ring $R$, a graded left $R$-module $M$ is a left $R$-module with a grading $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$, where the $M_\gamma$ are all abelian groups and $\Gamma'$ is an abelian group containing $\Gamma$, such that $R_\gamma \cdot M_\delta \subseteq M_{\gamma+\delta}$ for all $\gamma \in \Gamma_R, \delta \in \Gamma'$. Then, $\Gamma_M$ and $M^h$ are defined analogously to $\Gamma_R$ and $R^h$. We say that $M$ is a graded free $R$-module if it has a base as a free $R$-module consisting of homogeneous elements.

A graded ring $E = \bigoplus_{\gamma \in \Gamma} E_\gamma$ is called a graded division ring if $\Gamma$ is a torsion-free abelian group and every non-zero homogeneous element of $E$ has a multiplicative inverse. Note that the grade set $\Gamma_E$ is actually a group. Also, $E_0$ is a division ring, and $E_\gamma$ is a 1-dimensional left and right $E_0$ vector space for every $\gamma \in \Gamma_E$. The requirement that $\Gamma$ be torsion-free is made because we are interested in graded division rings arising from valuations on division rings, and all the grade groups appearing there are torsion-free. Recall that every torsion-free abelian group $\Gamma$ admits total orderings compatible with the group structure. (For example, $\Gamma$ embeds in $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ which can be given a lexicographic total ordering using any base of it as a $\mathbb{Q}$-vector space.) By using any total ordering on $\Gamma_E$, it is easy to see that $E$ has no zero divisors and that $E^*$, the multiplicative group of units of $E$, coincides with $E^h \setminus \{0\}$ (cf. [41], p. 78). Furthermore, the degree map

$$\deg : E^* \to \Gamma_E$$

is a group homomorphism with kernel $E_0^*$.

By an easy adaptation of the ungraded arguments, one can see that every graded module $M$ over a graded division ring $E$ is graded free, and every two homogenous bases have the same cardinality. We thus call $M$ a graded vector space over $E$ and write $\dim_E(M)$ for the rank of $M$ as a graded free $E$-module. Let $S \subseteq E$ be a graded subring which is also a graded division ring. Then, we can view $E$ as a graded left $S$-vector space, and we write $[E : S]$ for $\dim_S(E)$. It is easy to check the “Fundamental Equality,”

$$[E : S] = [E_0 : S_0] |\Gamma_E : \Gamma_S|,$$

where $[E_0 : S_0]$ is the dimension of $E_0$ as a left vector space over the division ring $S_0$ and $|\Gamma_E : \Gamma_S|$ denotes the index in the group $\Gamma_E$ of its subgroup $\Gamma_S$.

A graded field $T$ is a commutative graded division ring. Such a $T$ is an integral domain, so it has a quotient field, which we denote $q(T)$. It is known, see [40], Cor. 1.3, that $T$ is integrally closed in $q(T)$. An extensive theory of graded algebraic extensions of graded fields has been developed in [40]. For a graded field $T$, we can define a grading on the polynomial ring $T[x]$ as follows: Let $\Delta$ be a totally ordered abelian group with $\Gamma_T \subseteq \Delta$, and fix $\theta \in \Delta$. We have

$$T[x] = \bigoplus_{\gamma \in \Delta} T[x]_\gamma,$$

where

$$T[x]_\gamma = \{ \sum a_i x^i \mid a_i \in T^h, \deg(a_i) + i\theta = \gamma \}.$$
This makes $T[x]$ a graded ring, which we denote $T[x]^\theta$. Note that $\Gamma_{T[x]^\theta} = \Gamma_T + \langle \theta \rangle$. A homogeneous polynomial in $T[x]^\theta$ is said to be $\theta$-homogenizable. If $E$ is a graded division algebra with center $T$, and $a \in E^h$ is homogeneous of degree $\theta$, then the evaluation homomorphism $\epsilon_a : T[x]^\theta \to T[a]$ given by $f \mapsto f(a)$ is a graded ring homomorphism. Assuming $[T[a] : T] < \infty$, we have $\ker(\epsilon_a)$ is a principal ideal of $T[x]$ whose unique monic generator $h_a$ is called the minimal polynomial of $a$ over $T$. It is known, see [40], Prop. 2.2, that if $\deg(a) = \theta$, then $h_a$ is $\theta$-homogenizable.

If $E$ is a graded division ring, then its center $Z(E)$ is clearly a graded field. The graded division rings considered in this note will always be assumed finite-dimensional over their centers. The finite-dimensionality assures that $E$ has a quotient division ring $q(E)$ obtained by central localization, i.e., $q(E) = E \otimes_T q(T)$ where $T = Z(E)$. Clearly, $Z(q(E)) = q(T)$ and $\text{ind}(E) = \text{ind}(q(E))$, where the index of $E$ is defined by $\text{ind}(E)^2 = [E : T]$. If $S$ is a graded field which is a graded subring of $Z(E)$ and $[E : S] < \infty$, then $E$ is said to be a graded division algebra over $S$.

A graded division algebra $E$ with center $T$ is said to be unramified if $\Gamma_E = \Gamma_T$. From (7.4), it follows then that $[E : S] = [E_0 : T_0]$. At the other extreme, $E$ is said to be totally ramified if $E_0 = T_0$. In a case in the middle, $E$ is said to be semiramified if $E_0$ is a field and $[E_0 : T_0] = [\Gamma_E : \Gamma_T] = \text{ind}(E)$. These definitions are motivated by analogous definitions for valued division algebras ([95]). Indeed, if a valued division algebra is unramified, semiramified, or totally ramified, then so is its associated graded division algebra.

7.2. Commutators of graded division rings. In a division ring, additive and multiplicative commutators play important roles and there are extensive results in the literature known as commutativity theorems. The main theme in these results is that, additive and multiplicative commutators are “dense” in a division ring. For example, if an element commutes with all additive commutators, then it is already a central element. It seems that this trend continues for the additive commutators for a graded division ring as we will see in this section. However the multiplicative commutators are too “isolated” to determine the structure of a graded division ring.

Let $E$ be a graded division ring with graded center $T$. A homogeneous additive commutator of $E$ is an element of the form $ab - ba$ where $a, b \in E^h$. We will use the notation $[a, b] = ab - ba$ for $a, b \in E^h$ and let $[H, K]$ be the additive group generated by $\{dk - kd : d \in H^h, k \in K^h\}$ where $H$ and $K$ are graded subrings of $E$. Parallel to the theory of division rings, one can show that if all the homogenous additive commutators of graded division ring $E$ are central, then $E$ is a graded field. To observe this, one can carry over the non-graded proof, mutatis-mutandis, to the graded setting, see, e.g., [51], Prop. 13.4. Alternatively, let $y \in E^h$ be an element which commutes with homogeneous additive commutators of $E$. Then $y$ commutes with all (non-homogeneous) commutators of $E$. Consider $[x_1, x_2]$ where $x_1, x_2 \in q(E)$. Since $q(E) = E \otimes_T q(T)$, it follows that $y[x_1, x_2] = [x_1, x_2]y$. So $y$ commutes with all commutators of $q(E)$, a division ring, thus $y \in q(T)$. But $E^h \cap q(T) \subseteq T^h$, proving that $y \in T^h$. Thus, $E$ is commutative. Again parallel to the theory of division rings, one can prove that if $K \subseteq E$ are graded division rings, with $[E, K] \subseteq K$ and $\text{char}(K) \neq 2$, then $K \subseteq Z(E)$. However,
for this one it seems there is no shortcut, and one needs to carry out a proof similar to the one for ungraded division rings, as in ([51], Prop. 3.7, see Proposition 7.2).

The paragraph above shows some similar behavior between the Lie algebra structure of division rings and that of graded division rings. However, this analogy often fails for the multiplicative structure of graded division algebras. For example, the Cartan-Brauer-Hua theorem (the multiplicative analogue of the statement above that if $K \subseteq E$ are graded division rings, with $[E, K] \subseteq K$ and char$(K) \neq 2$, then $K \subseteq Z(E)$) is not valid in the graded setting. Also, the multiplicative group $E^*$ of a totally ramified graded division algebra $E$ is nilpotent (since $E' \subseteq E_0^* = T_0^* \subseteq Z(E^*)$), while the multiplicative group of a noncommutative division ring is not even solvable, cf. [91]. Furthermore, a totally ramified graded division algebra $E^*$ is radical over its center $T$ (since $E^*_{\exp(T/E)} \subseteq T^*$), but this is not the case for any noncommutative division ring ([51], Th. 15.15). Nonetheless, one significant theorem involving conjugates that can be extended to the graded setting is the Wedderburn factorization theorem, Theorem 7.3.

We first establish a relation between the support of $E$ and that of the additive commutator subgroup (see Millar’s Thesis [74] for more on this theme).

**Lemma 7.1.** Let $E = \bigoplus_{\gamma \in \Gamma} E_\gamma$ be a graded division algebra over its centre $T$.

1. If $E$ is totally ramified, then $\emptyset \neq \text{Supp}([E, E]) \subseteq \Gamma_E$.
2. If $E$ is not totally ramified, then $\text{Supp}(E) = \text{Supp}([E, E])$.

**Proof.** (1) Clearly $\emptyset \neq \text{Supp}([E, E]) \subseteq \Gamma_E$. Since $E_0 = T_0 = Z(E) \cap E_0$ we have $E_0 \subseteq Z(E)$. Suppose $0 \in \text{Supp}([E, E])$. Then there is an element $\sum_i (x_iy_i - y_ix_i) \in [E, E]$, with $\text{deg}(x_i) + \text{deg}(y_i) = 0$ for all $i$. If $x_iy_i - y_ix_i = 0$ for all $i$, then clearly the sum is also zero. Thus there are non-zero homogeneous elements $x \in E_\gamma$, $y \in E_\delta$ with $0 \neq xy - yx \in E_0$ and $\gamma + \delta = 0$.

Then $(xy - yx)y^{-1} \neq 0$, as $y^{-1} \in E_{-\delta} \setminus 0$ and $xy - yx \in E_0 \setminus 0$, so their product is a non-zero homogeneous element of degree $-\delta$. Since

$$(xy - yx)y^{-1} = xyy^{-1} - yxy^{-1} = y^{-1}yx - yxy^{-1},$$

we have $y^{-1}(yx) \neq (yx)y^{-1}$; that is $yx \notin Z(E)$. Since $yx \in E_0$, this contradicts the fact that $E_0 = T_0$, so $0 \notin \text{Supp}([E, E])$.

(2) It is clear that $\text{Supp}([E, E]) \subseteq \Gamma_E$. For the reverse containment, for $\gamma \in \Gamma_E$ we will show that there is an $x \in E_\gamma$ which does not commute with some $y \in E_\delta$ for some $\delta \in \Gamma_E$. Suppose not, then $E_\gamma \subseteq Z(E)$, so $E_\gamma = T_\gamma$. Let $x \in E_\gamma$, $d \in E_0$, $y \in E_\delta$ be arbitrary non-zero elements. Then

$$x(dy) = (dy)x = d(yx) = d(xy) = (dx)y = y(dx) = (yd)x = x(yd).$$

So for all $d \in E_0$, $y \in E_\delta$ we have $x(dy) = x(yd)$. Since $x$ is a non-zero homogeneous element, it is invertible. This implies $dy = yd$, so $E_0 = T_0$ contradicting the fact that $E$ is not totally ramified. Then there is an $x \in E_\gamma$ which does not commute with $y \in E_\delta$, so $xyy^{-1} - y^{-1}xy \neq 0$ proving $\gamma \in \text{Supp}([E, E])$.

Similarly to the theory of division ring, one can establish the following.
Proposition 7.2. Let $K \subseteq E$ be graded division rings, with $[E, K] \subseteq K$. If $\text{char} \, K \neq 2$, then $K \subseteq Z(E)$.

Proof. First note that the condition $[E, K] \subseteq K$ is equivalent to $[E^h, K^h] \subseteq K^h$. Let $a \in E^h \setminus K^h$ and $c \in K^h$. We will show $ac = ca$. We have $[a, [a, c]] + [a^2, c] = 2a[a, c] \in K$. If $[a, c] \neq 0$, then, since $\text{char} \, K \neq 2$, this implies $a \in K$, contradicting our choice of $a$. Thus $[a, c] = 0$.

Now let $b, c \in K^h$. Consider $a \in E^h \setminus K^h$. Then $a, ab \in E^h \setminus K^h$ and we have shown that $[a, c], [ab, c] = 0$. Then $[b, c] = a^{-1} \cdot [ab, c] = 0$. It follows that $K \subseteq Z(E)$. □

7.3. The graded Wedderburn factorization theorem. Let $D$ be a noncommutative division ring with center $F$, and let $a \in D$ with minimal polynomial $f$ in $F[x]$. Any conjugate of $a$ is also a root of this polynomial. Since the number of conjugates of $a$ is infinite ([51], 13.26), this suggests that $f$ might split completely in $D[x]$. In fact, this is the case, and it is called the Wedderburn factorization theorem. We now carry over this theorem to the setting of graded division algebras. (This is used in proving Th. 7.6).

Theorem 7.3 (Wedderburn Factorization Theorem). Let $E$ be a graded division ring with center $T$ (with $\Gamma_E$ torsion-free abelian). Let $a$ be a homogenous element of $E$ which is algebraic over $T$ with minimal polynomial $h_a \in T[x]$. Then, $h_a$ splits completely in $E$. Furthermore, there exist $n$ conjugates $a_1, \ldots, a_n$ of $a$ such that $h_a = (x-a_n)(x-a_{n-1})\ldots(x-a_1)$ in $E[x]$.

Proof. The proof is similar to Wedderburn’s original proof for a division ring ([100], see also [51] for a nice account of the proof). We sketch the proof for the convenience of the reader. For $f = \sum c_i x^i \in E[x]$ and $a \in E$, our convention is that $f(a)$ means $\sum c_i a^i$. Since $\Gamma_E$ is torsion-free, we have $E^* = E^h \setminus \{0\}$.

I: Let $f \in E[x]$ with factorization $f = gk$ in $E[x]$. If $a \in E$ satisfies $k(a) \in T \cdot E^*$, then $f(a) = g(a')k(a)$, for some conjugate $a'$ of $a$. (Here $E$ could be any ring with $T \subseteq Z(E)$.)

Proof. Let $g = \sum b_i x^i$. Then, $f = \sum b_i k x^i$, so $f(a) = \sum b_i k(a) a^i$. But, $k(a) = te$, where $t \in T$ and $e \in E^*$. Thus, $f(a) = \sum b_i t e a^i = \sum b_i e a^i e^{-1} te = \sum b_i (eae^{-1})^i te = g(eae^{-1})k(a)$. □

II: Let $f \in E[x]$ be a non-zero polynomial. Then $r \in E$ is a root of $f$ if and only if $x-r$ is a right divisor of $f$ in $E[x]$. (Here, $E$ could be any ring.)

Proof. We have $x^i - r^i = (x^{i-1} + x^{i-2}r + \ldots + r^{i-1})(x-r)$ for any $i \geq 1$. Hence,

$$f - f(r) = g \cdot (x-r)$$ (7.6)

for some $g \in E[x]$. So, if $f(r) = 0$, then $f = g \cdot (x-r)$. Conversely, if $x-r$ is a right divisor of $f$, then equation (7.6) shows that $x-r$ is a right divisor of the constant $f(r)$. Since $x-r$ is monic, this implies that $f(r) = 0$. □

III: If a non-zero monic polynomial $f \in E[x]$ vanishes identically on the conjugacy class $A$ of $a$ (i.e., $f(b) = 0$ for all $b \in A$), then $\text{deg}(f) \geq \text{deg}(h_a)$. 
Proof. Consider \( f = x^m + d_1 x^{m-1} + \ldots + d_m \in E[x] \) such that \( f(a) = 0 \) and \( m < \deg(h_a) \) with \( m \) as small as possible. Suppose \( a \in E_\gamma \), so \( A \subseteq E_\gamma \), as the units of \( E \) are all homogeneous. Since the \( E_{\gamma',}\)-component of \( f(b) \) is 0 for each \( b \in A \), we may assume that each \( d_i \in E_{\gamma'} \). Because \( f \notin T[x] \), some \( d_i \notin T \). Choose \( j \) minimal with \( d_j \notin T \), and some \( e \in E^* \) such that \( ed_j \neq d_j e \). For any \( c \in E \), write \( c' := ece^{-1} \). Thus \( d_j' \neq d_j \) but \( d_j' = d_j \) for \( \ell < j \). Let \( f' = x^m + d_j' x^{m-1} + \ldots + d_m' \in E[x] \). Now, for all \( b \in A \), we have \( f'(b') = [f(b)]' = 0' = 0 \). Since \( eAe^{-1} = A \), this shows that \( f'(A) = 0 \). Let \( g = f - f' \), which has degree \( j \) with leading coefficient \( d_j - d_j' \). Then, \( g(A) = 0 \). But, \( d_j - d_j' \in E_{\gamma'} \setminus \{0\} \subseteq E^* \). Thus, \((d_j - d_j')^{-1}g\) is monic of degree \( j < m \) in \( E[x] \), and it vanishes on \( A \). This contradicts the choice of \( f \); hence, \( m \geq \deg(h_a) \).

We now prove the theorem. Since \( h_a(a) = 0 \), by (II), \( h_a \in E[x] \cdot (x - a) \). Take a factorization

\[
h_a = g \cdot (x - a_r) \ldots (x - a_1),
\]

where \( g \in E[x], a_1, \ldots, a_r \in A \) and \( r \) is as large as possible. Let \( k = (x - a_r) \ldots (x - a_1) \in E[x] \). We claim that \( k(A) = 0 \), where \( A \) is the conjugacy class of \( a \). For, suppose there exists \( b \in A \) such that \( k(b) \neq 0 \). Since \( k(b) \) is homogenous, we have \( k(b) \in E^* \). But, \( h_a = gk \), and \( h_a(b) = 0 \), as \( b \in A \); hence, (I) implies that \( g(b') = 0 \) for some conjugate \( b' \) of \( b \). We can then write \( g = g_1 \cdot (x - b') \), by (II). So \( h_a \) has a right factor \( (x - b')k = (x - b')(x - a_r) \ldots (x - a_1) \), contradicting our choice of \( r \). Thus \( k(A) = 0 \), and using (III), we have \( r \geq \deg(h_a) \), which says that \( h_a = (x - a_r) \ldots (x - a_1) \).

Remark 7.4 (Dickson Theorem). One can also see that, with the same assumptions as in Th. 7.3, if \( a, b \in E \) have the same minimal polynomial \( h \in T[x] \), then \( a \) and \( b \) are conjugates. For, \( h = (x - b)k \) where \( k \in T[b][x] \). But then by (III), there exists a conjugate of \( a \), say \( a' \), such that \( k(a') \neq 0 \). Since \( h(a') = 0 \), by (I) some conjugate of \( a' \) is a root of \( x - b \). (This is also deducible using the graded version of the Skolem-Noether theorem, see [41], Prop. 1.6.)

7.4. Graded division ring associated to a valued division algebra. We recall how to associate a graded division algebra to a valued division algebra.

Let \( D \) be a division algebra finite dimensional over its center \( F \), with a valuation \( v : D^* \to \Gamma \). So \( \Gamma \) is a totally ordered abelian group, and \( v \) satisfies the conditions that for all \( a, b \in D^* \),

\[
\begin{align*}
(i) \quad v(ab) &= v(a) + v(b); \\
(ii) \quad v(a + b) &\geq \min\{v(a), v(b)\} \quad (b \neq -a).
\end{align*}
\]

Let

\[
V_D = \{a \in D^* : v(a) \geq 0\} \cup \{0\}, \text{ the valuation ring of } v; \\
M_D = \{a \in D^* : v(a) > 0\} \cup \{0\}, \text{ the unique maximal left (and right) ideal of } V_D; \\
\overline{D} = V_D/M_D, \text{ the residue division ring of } v \text{ on } D; \text{ and} \\
\Gamma_D = \text{im}(v), \text{ the value group of the valuation.}
\]
For background on valued division algebras, see [42] or the survey paper [95]. One associates to $D$ a graded division algebra as follows: For each $\gamma \in \Gamma_D$, let
\[
D^{>\gamma} = \{ d \in D^*: v(d) \geq \gamma \} \cup \{ 0 \}, \text{ an additive subgroup of } D;
\]
\[
D^{\geq \gamma} = \{ d \in D^*: v(d) \geq \gamma \} \cup \{ 0 \}, \text{ a subgroup of } D^{>\gamma}; \text{ and}
\]
\[
\text{gr}(D)_\gamma = D^{>\gamma}/D^{>\gamma}.
\]
Then define
\[
\text{gr}(D) = \bigoplus_{\gamma \in \Gamma_D} \text{gr}(D)_\gamma.
\]

Because $D^{>\gamma} D^{\geq \delta} + D^{\geq \gamma} D^{>\delta} \subseteq D^{>\gamma+\delta}$ for all $\gamma, \delta \in \Gamma_D$, the multiplication on $\text{gr}(D)$ induced by multiplication on $D$ is well-defined, giving that $\text{gr}(D)$ is a graded ring, called the associated graded ring of $D$. The multiplicative property (i) of the valuation $v$ implies that $\text{gr}(D)$ is a graded division ring. Clearly, we have $\text{gr}(D)_0 = \overline{T}$ and $\Gamma_{\text{gr}(D)} = \Gamma_D$. For $d \in D^*$, we write $\tilde{d}$ for the image $d + D^{>v(d)}$ of $d$ in $\text{gr}(D)_{v(d)}$. Thus, the map given by $d \mapsto \tilde{d}$ is a group epimorphism $D^* \rightarrow \text{gr}(D)^*$ with kernel $1 + M_D$.

The restriction $v|_F$ of the valuation on $D$ to its center $F$ is a valuation on $F$, which induces a corresponding graded field $\text{gr}(F)$. Then it is clear that $\text{gr}(D)$ is a graded $\text{gr}(F)$-algebra, and by (7.4) and the Fundamental Inequality for valued division algebras,
\[
[\text{gr}(D) : \text{gr}(F)] = [\overline{T} : \overline{T}]|\Gamma_D : \Gamma_F| \leq [D : F] < \infty.
\]

7.5. **Reduced Whitehead group of a graded division algebra.** A main theme of this part of the note is to study the correspondence between $\text{SK}_1$ of a valued division algebra and that of its associated graded division algebra. Let $A$ be an Azumaya algebra of constant rank $n^2$ over a commutative ring $R$. Then there is a commutative ring $S$ faithfully flat over $R$ which splits $A$, i.e., $A \otimes_R S \cong M_n(S)$. For $a \in A$, considering $a \otimes 1$ as an element of $M_n(S)$, one then defines the reduced characteristic polynomial, the reduced trace, and the reduced norm of $a$ by
\[
\text{char}_A(x, a) = \det(x - (a \otimes 1)) = x^n - \text{Trd}_A(a)x^{n-1} + \ldots + (-1)^n\text{Nrd}_A(a).
\]

Using descent theory, one shows that $\text{char}_A(x, a)$ is independent of $S$ and the choice of isomorphism $A \otimes_R S \cong M_n(S)$, and that $\text{char}_A(x, a)$ lies in $R[x]$; furthermore, the element $a$ is invertible in $A$ if and only if $\text{Nrd}_A(a)$ is invertible in $R$ (see Knus [50], III.1.2, and Saltman [85], Th. 4.3). Let $A^{(1)}$ denote the set of elements of $A$ with the reduced norm 1. One then defines the reduced Whitehead group of $A$ to be $\text{SK}_1(A) = A^{(1)}/A'$, where $A'$ denotes the commutator subgroup of the group $A^*$ of invertible elements of $A$. The reduced norm residue group of $A$ is defined to be $\text{SH}^0(A) = R^*/\text{Nrd}_A(A^*)$. These groups are related by the exact sequence:
\[
1 \rightarrow \text{SK}_1(A) \rightarrow A^*/A' \xrightarrow{\text{Nrd}} R^* \rightarrow \text{SH}^0(A) \rightarrow 1 \tag{7.7}
\]

Now let $E$ be a graded division algebra with center $T$. Since $E$ is an Azumaya algebra over $T$ ([9], Prop. 5.1 or [41], Cor. 1.2), its reduced Whitehead group $\text{SK}_1(E)$ is defined.

We have other tools as well for computing $\text{Nrd}_E$ and $\text{Trd}_E$:
Proposition 7.5. Let $E$ be a graded division ring with center $T$. Let $q(T)$ be the quotient field of $T$, and let $q(E) = E \otimes_T q(T)$, which is the quotient division ring of $E$. We view $E \subseteq q(E)$. Let $n = \text{ind}(E) = \text{ind}(q(E))$. Then for any $a \in E$,

1. $\text{char}_E(x, a) = \text{char}_{q(E)}(x, a)$, so

$$\text{Nrd}_E(a) = \text{Nrd}_{q(E)}(a) \quad \text{and} \quad \text{Trd}_E(a) = \text{Trd}_{q(E)}(a). \quad (7.8)$$

2. If $K$ is any graded subfield of $E$ containing $T$ and $a \in K$, then

$$\text{Nrd}_E(a) = N_{K/T}(a)^{n/[K:T]} \quad \text{and} \quad \text{Trd}_E(a) = \frac{n}{[K:T]} \text{Tr}_{K/T}(a).$$

3. For $\gamma \in \Gamma_E$, if $a \in E_{\gamma}$ then $\text{Nrd}_E(a) \in E_{n\gamma}$ and $\text{Trd}(a) \in E_{\gamma}$. In particular, $E^{(1)} \subseteq E_0$.

(4) Set $\delta = \text{ind}(E)/\left(\text{ind}(E_0)[Z(E_0) : T_0]\right)$. If $a \in E_0$, then,

$$\text{Nrd}_E(a) = N_{Z(E_0)/T_0} \text{Nrd}_{E_0}(a)^{\delta} \in T_0 \quad \text{and} \quad \text{Trd}_E(a) = \delta \text{Tr}_{Z(E_0)/T_0} \text{Trd}_{E_0}(a) \in T_0. \quad (7.9)$$

Proof. (1) The construction of reduced characteristic polynomials described above is clearly compatible with scalar extensions of the ground ring. Hence, $\text{char}_E(x, a) = \text{char}_{q(E)}(x, a)$ (as we are identifying $a \in E$ with $a \otimes 1$ in $E \otimes_T q(T)$). The formulas in (7.8) follow immediately.

(2) Let $h_a = x^m + t_{m-1}x^{m-1} + \ldots + t_0 \in q(T)[x]$ be the minimal polynomial of $a$ over $q(T)$. As noted in [40], Prop. 2.2, since the integral domain $T$ is integrally closed and $E$ is integral over $T$, we have $h_a \in T[x]$. Let $\ell_a = x^k + s_k x^{k-1} + \ldots + s_0 \in T[x]$ be the characteristic polynomial of the $T$-linear function on the free $T$-module $K$ given by $c \mapsto ac$. By definition, $N_{K/T}(a) = (-1)^k s_0$ and $\text{Tr}_{K/T}(a) = -s_{k-1}$. Since $q(K) = K \otimes_T q(T)$, we have $[q(K) : q(T)] = [K : T] = k$ and $\ell_a$ is also the characteristic polynomial for the $q(T)$-linear transformation of $q(K)$ given by $q \mapsto aq$. So, $\ell_a = h_a^{k/m}$. Since $\text{char}_{q(E)}(x, a) = h_a^{n/m}$ (see [83], Ex. 1, p. 124), we have $\text{char}_{q(E)}(x, a) = \ell_a^{n/k}$. Therefore, using (1),

$$\text{Nrd}_E(a) = \text{Nrd}_{q(E)}(a) = \left[(-1)^k s_0\right]^{n/k} = N_{K/T}(a)^{n/k}.$$ 

The formula for $\text{Trd}_E(a)$ in (2) follows analogously.

(3) From the equalities $\text{char}_E(x, a) = \text{char}_{q(E)}(x, a)$, we have $\text{Nrd}_E(a) = \left[(-1)^m t_0\right]^{n/m}$ and $\text{Trd}_E(a) = -\frac{n}{m} t_{m-1}$. As noted in [40], Prop. 2.2, if $a \in E_{\gamma}$, then its minimal polynomial $h_a$ is $\gamma$-homogenizable in $T[x]$ as in (7.5) above. Hence, $t_0 \in E_{m\gamma}$ and $t_{m-1} \in E_{\gamma}$. Therefore, $\text{Nrd}_E(a) \in E_{n\gamma}$ and $\text{Trd}(a) \in E_{\gamma}$. If $a \in E^{(1)}$ then $a$ is homogeneous, since it is a unit of $E$, and since $1 = \text{Nrd}_E(a) \in E_{n \deg(a)}$, necessarily $\deg(a) = 0$.

(4) Suppose $a \in E_0$. Then, $h_a$ is 0-homogenizable in $T[x]$, i.e., $h_a \in T_0[x]$. Hence, $h_a$ is the minimal polynomial of $a$ over the field $T_0$. Therefore, if $L$ is any maximal subfield of $E_0$ containing $a$, we have $N_{L/T_0}(a) = \left[(-1)^m t_0\right]^{L: T_0/m}$. Now,

$$n/m = \delta \text{ind}(E_0)[Z(E_0) : T_0]/m = \delta [L : T_0]/m.$$

Hence,

$$\text{Nrd}_E(a) = \left[(-1)^m t_0\right]^{n/m} = \left[(-1)^m t_0\right]^{\delta[L:T_0]/m} = N_{L/T_0}(a)^{\delta} = N_{Z(E_0)/T_0} N_{L/Z(E_0)}(a)^{\delta} = N_{Z(E_0)/T_0} \text{Nrd}_{E_0}(a)^{\delta}.$$
The formula for Trd_E(a) is proved analogously. \qed

In the rest of this section we study the reduced Whitehead group SK_1 of a graded division algebra. As we mentioned in the introduction, the motif is to show that working in the graded setting is much easier than in the non-graded setting.

The most successful approach to computing SK_1 for division algebras over Henselian fields is due to Ershov in [15], where three linked exact sequences were constructed involving a division algebra D, its residue division algebra \( \overline{D} \), and its group of units \( U_D \) (see also [95], p. 425). From these exact sequences, Ershov recovered Platonov’s examples [80] of division algebras with nontrivial SK_1 and many more examples as well. In this section we will easily prove the graded version of Ershov’s exact sequences (see diagram (7.11)), yielding formulas for SK_1 of unramified, semiramified, and totally ramified graded division algebras. This will be used to show that SK_1 of a tame division algebra over a Henselian field coincides with SK_1 of its associated graded division algebra. We can then readily deduce from the graded results many established formulas in the literature for the reduced Whitehead groups of valued division algebras (see Cor. 7.15). This demonstrates the merit of the graded approach.

If \( N \) is a group, we denote by \( N^n \) the subgroup of \( N \) generated by all \( n \)-th powers of elements of \( N \). A homogeneous multiplicative commutator of \( E \), where \( E \) is a graded division ring, has the form \( aba^{-1}b^{-1} \) where \( a, b \in E^* = E^h \setminus \{0\} \). We will use the notation \( [a, b] = aba^{-1}b^{-1} \) for \( a, b \in E^* \). Since \( a \) and \( b \) are homogeneous, note that \( [a, b] \in E_0 \). If \( H \) and \( K \) are subsets of \( E^* \), then \( [H, K] \) denotes the subgroup of \( E^* \) generated by \( \{ [h, k] : h \in H, k \in K \} \). The group \([E^*, E^*]\) will be denoted by \( E' \).

**Proposition 7.6.** Let \( E = \bigoplus_{a \in \Gamma} E_a \) be a graded division algebra with graded center \( T \), with \( \text{ind}(E) = n \). Then,

1. If \( N \) is a normal subgroup of \( E^* \), then \( N^n \subseteq \text{Nrd}_E(N)[E^*, N] \).
2. \( \text{SK}_1(E) \) is \( n \)-torsion.

**Proof.** Let \( a \in N \) and let \( h_a \in q(T)[x] \) be the minimal polynomial of \( a \) over \( q(T) \), and let \( m = \deg(h_a) \). As noted in the proof of Prop. 7.5, \( h_a \in T[x] \) and \( \text{Nrd}_E(a) = [(-1)^m h_a(0)]^{n/m} \). By the graded Wedderburn Factorization Theorem 7.3, we have \( h_a = (x - d_1 ad_1^{-1}) \ldots (x - d_m ad_m^{-1}) \) where each \( d_i \in E^* \subseteq E^h \). Note that \([E^*, N]\) is a normal subgroup of \( E^* \), since \( N \) is normal in \( E^* \). It follows that

\[
\text{Nrd}_E(a) = (d_1 ad_1^{-1} \ldots d_m ad_m^{-1})^{n/m} = ([d_1, a][d_2, a] \ldots [d_m, a]a)^{n/m} = a^n d_a \quad \text{where } d_a \in [E^*, N].
\]

Therefore, \( a^n = \text{Nrd}_E(a)d_a^{-1} \in \text{Nrd}_E(N)[E^*, N] \), yielding (1). (2) is immediate from (1) by taking \( N = E^{(1)} \). \qed

We recall the definition of the group \( \tilde{H}^{-1}(G, A) \), which will appear in our description of \( \text{SK}_1(E) \). For any finite group \( G \) and any \( G \)-module \( A \), define the norm map \( N_G : A \to A \) as follows: For any \( a \in A \), let \( N_G(a) = \sum_{g \in G} ga. \) Consider the \( G \)-module \( I_G(A) \) generated as
an abelian group by \( \{a - ga : a \in A \text{ and } g \in G \} \). Clearly, \( I_G(A) \subseteq \ker(N_G) \). Then,
\[
\hat{H}^{-1}(G, A) = \ker(N_G)/I_G(A).
\]
(7.10)

Recall also that for an abelian group \( G \),
\[
G \wedge G := G \otimes_{\mathbb{Z}} G/(g \otimes g),
\]
where \( g \in G \).

**Theorem 7.7.** Let \( E \) be any graded division ring finite dimensional over its center \( T \). So, \( Z(E_0) \) is Galois over \( T_0 \); let \( G = \text{Gal}(Z(E_0)/T_0) \). Let \( \delta = \text{ind}(E)/\left( \text{ind}(E_0)[Z(E_0) : T_0] \right) \), and let \( \mu_\delta(T_0) \) be the group of those \( \delta \)-th roots of unity lying in \( T_0 \). Let \( \tilde{N} = \hat{N}_{Z(E_0)/T_0} \circ \text{Nrd}_{E_0} : E_0^* \rightarrow T_0^* \). Then, the rows and column of the following diagram are exact:

\[
\begin{array}{ccc}
1 & \rightarrow & \text{SK}_1(E_0) \rightarrow \ker \tilde{N}/[E_0^*, E^*] \xrightarrow{\text{Nrd}_{E_0}} \hat{H}^{-1}(G, \text{Nrd}_{E_0}(E_0^*)) \rightarrow 1 \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
& \Gamma_E/\Gamma_T \wedge \Gamma_E/\Gamma_T \xrightarrow{\alpha} E^{(1)}/[E_0^*, E^*] \rightarrow \text{SK}_1(E) \rightarrow 1 \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
\mu_\delta(T_0) \cap \tilde{N}(E_0^*) & \rightarrow & \tilde{N} & \rightarrow & 1 \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
& & 1 & & 
\end{array}
\] 

(7.11)

The map \( \alpha \) in (7.11) is given as follows: For \( \gamma, \delta \in \Gamma_E \), take any nonzero \( x_\gamma \in E_\gamma \) and \( x_\delta \in E_\delta \). Then, \( \alpha((\gamma + \Gamma_T) \wedge (\delta + \Gamma_T)) = [x_\gamma, x_\delta] \mod [E_0^*, E^*] \).

**Proof.** By Prop. 2.3 in [41], \( Z(E_0)/T_0 \) is a Galois extension and the map \( \theta : E^* \rightarrow \text{Aut}(E_0) \), given by \( e \mapsto (a \mapsto eae^{-1}) \) for \( a \in E_0 \), induces an epimorphism \( E^* \rightarrow G = \text{Gal}(Z(E_0)/T_0) \). In the notation for (7.10) with \( A = \text{Nrd}_{E_0}(E_0^*) \), we have \( N_G \) coincides with \( \hat{N}_{Z(E_0)/T_0} \) on \( A \). Hence,
\[
\ker(N_G) = \text{Nrd}_{E_0}(\ker(\tilde{N})).
\]
(7.12)

Take any \( e \in E^* \) and let \( \sigma = \theta(e) \in \text{Aut}(E_0) \). For any \( a \in E_0^* \), let \( h_a \in Z(T_0)[x] \) be the minimal polynomial of \( a \) over \( Z(T_0) \). Then \( \sigma(h_a) \in Z(T_0)[x] \) is the minimal polynomial of \( \sigma(a) \) over \( Z(T_0) \). Hence, \( \text{Nrd}_{E_0}(\sigma(a)) = \sigma(\text{Nrd}_{E_0}(a)) \). Since \( \sigma|_{Z(T_0)} \in G \), this yields
\[
\text{Nrd}_{E_0}([a, e]) = \text{Nrd}_{E_0}(\sigma(a^{-1})) = \text{Nrd}_{E_0}(a)\sigma(\text{Nrd}_{E_0}(a))^{-1} \in I_G(A),
\]
(7.13)
hence \( \tilde{N}([a, e]) = 1 \). Thus, we have \( [E_0^*, E^*] \subseteq \ker(\tilde{N}) \subseteq E^{(1)} \) with the latter inclusion from Prop. 7.5(4). The formula in Prop. 7.5(4) also shows that \( \tilde{N}(E^{(1)}) \subseteq \mu_\delta(T_0) \). Thus, the vertical maps in diagram (7.11) are well-defined, and the column in (7.11) is exact. Because \( \text{Nrd}_{E_0} \) maps \( \ker(\tilde{N}) \) onto \( \ker(N_G) \) by (7.12) and it maps \([E_0^*, E^*] \) onto \( I_G(A) \) by (7.13) (as \( \theta(E^*) \) maps onto \( G \)), the map labelled \( \text{Nrd}_{E_0} \) in diagram (7.11) is surjective.
with kernel $E_0^{(1)}[E_0^*, E^*]/[E_0^*, E^*]$. Therefore, the top row of (7.11) is exact. For the lower row, since $[E^*, E^*] \subseteq E_0^*$ and $E^*/(E_0^* E(Z(E^*)) \cong \Gamma_E/\Gamma_T$, the following lemma yields an epimorphism $\Gamma_E/\Gamma_T \wedge \Gamma_E/\Gamma_T \to [E^*, E^*]/[E_0^*, E^*]$. Given this, the lower row in (7.11) is evidently exact. □

We need the following lemma whose proof is left to the reader (see [27]).

**Lemma 7.8.** Let $G$ be a group, and let $H$ be a subgroup of $G$ with $H \supseteq [G, G]$. Let $B = G/(H Z(G))$. Then, there is an epimorphism $B \wedge B \to [G, G]/[H, G]$.

**Corollary 7.9.** Let $E$ be a graded division ring with center $T$.

1. If $E$ is unramified, then $SK_1(E) \cong SK_1(E_0)$.
2. If $E$ is totally ramified, then $SK_1(E) \cong \mu_n(T_0)/\mu_e(T_0)$ where $n = \text{ind}(E)$ and $e$ is the exponent of $\Gamma_E/\Gamma_T$.
3. If $E$ is semiramified, then for $G = \text{Gal}(E_0/T_0) \cong \Gamma_E/\Gamma_T$ there is an exact sequence
   $$G \wedge G \to \tilde{H}^{-1}(G, E_0^*) \to SK_1(E) \to 1.$$  
   (7.14)
4. If $E$ has maximal graded subfields $L$ and $K$ which are respectively unramified and totally ramified over $T$, then $E$ is semiramified and $SK_1(E) \cong \tilde{H}^{-1}(\text{Gal}(E_0/T_0), E_0^*)$.

**Proof.** (1) Since $E$ is unramified over $T$, we have $E_0$ is a central $T_0$-division algebra, $\text{ind}(E_0) = \text{ind}(E)$, and $E^* = E_0^* T^*$. It follows that $G = \text{Gal}(Z(E_0)/T_0)$ is trivial, and thus $\tilde{H}^{-1}(G, \text{Nrd}_E(E_0))$ is trivial; also, $\delta = 1$, and from (7.9), $\text{Nrd}_{E_0}(a) = \text{Nrd}_E(a)$ for all $a \in E_0$. Furthermore, $[E_0^*, E^*] = [E_0^*, E_0^* T^*] = [E_0^*, E_0^*]$ as $T^*$ is central. Plugging this information into the exact top row of diagram (7.11) and noting that the exact sequence extends to the left by $1 \to [E_0^*, E^*]/[E_0^*, E_0^*] \to SK_1(E_0)$, part (1) follows.

2. When $E$ is totally ramified, $E_0 = T_0$, $\delta = n$, $\tilde{N}$ is the identity map on $T_0$, and $[E^*, E_0^*] = [E^*, T_0^*] = 1$.

By plugging all this into the exact column of diagram (7.11), it follows that $E^{(1)} \cong \mu_n(T_0)$. Also by [41] Prop. 2.1, $E' \cong \mu_e(T_0)$ where $e$ is the exponent of the torsion abelian group $\Gamma_E/\Gamma_T$. Part (2) now follows.

3. As recalled at the beginning of the proof of Th. 7.7, for any graded division algebra $E$ with center $T$, we have $Z(E_0)$ is Galois over $T_0$, and there is an epimorphism $\theta: E^* \to \text{Gal}(Z(E_0)/T_0)$. Clearly, $E_0^*$ and $T^*$ lie in $\ker(\theta)$, so $\theta$ induces an epimorphism $\theta': \Gamma_E/\Gamma_T \to \text{Gal}(Z(E_0)/T_0)$. When $E$ is semiramified, by definition $[E_0: T_0] = [\Gamma_E : \Gamma_T] = \text{ind}(E)$ and $E_0$ is a field. Let $G = \text{Gal}(E_0/T_0)$. Because $|G| = [E_0: T_0] = [\Gamma_E : \Gamma_T]$, the map $\theta'$ must be an isomorphism. In diagram (7.11), since $SK_1(E_0) = 1$ and clearly $\delta = 1$, the exact top row and column yield $E^{(1)}/[E_0^*, E^*] \cong \tilde{H}^{-1}(G, E_0^*)$. Therefore, the exact row (7.14) follows from the exact second row of diagram (7.11) and the isomorphism $\Gamma_E/\Gamma_T \cong G$ given by $\theta'$. (4) Since $L$ and $K$ are maximal subfields of $E$, we have $\text{ind}(E) = [L : T] = [L_0 : T_0] \leq [E_0 : T_0]$ and $\text{ind}(E) = [K : T] = [\Gamma_K : \Gamma_T] \leq [\Gamma_E : \Gamma_T]$. It follows from (7.4) that these inequalities are equalities, so $E_0 = L_0$ and $\Gamma_E = \Gamma_K$. Hence, $E$ is semiramified, and (3) applies. Take any $\eta, \nu \in \Gamma_E/\Gamma_T$, and any inverse images $a, b$ of $\eta, \nu$ in $E^*$. The left map in (7.14) sends
\( \eta \wedge \nu \) to \( aba^{-1}b^{-1} \) mod \( I_G(E_0^*) \). Since \( \Gamma_F = \Gamma_K \), these \( a \) and \( b \) can be chosen in \( K^* \), so they commute. Thus, the left map of (7.14) is trivial here, yielding the isomorphism of (4). 

The proof of the following Lemma is in [27].

**Lemma 7.10.** Let \( F \subseteq K \) be fields with \( [K : F] < \infty \). Let \( v \) be a Henselian valuation on \( F \) such that \( K \) is defectless over \( F \). Then, for every \( a \in K^* \), with \( \bar{a} \) its image in \( \mathrm{gr}(K)^* \),

\[
N_{K/F}(a) = N_{\mathrm{gr}(K)/\mathrm{gr}(F)}(\bar{a}).
\]

**Corollary 7.11.** Let \( F \) be a field with Henselian valuation \( v \), and let \( D \) be a tame \( F \)-central division algebra. Then for every \( a \in D^* \), \( \mathrm{Nrd}_{\mathrm{gr}(D)}(\bar{a}) = \mathrm{Nrd}_D(a) \).

**Proof.** Recall from §7.1 that the assumption \( D \) is tame over \( F \) means that \( [D : F] = [\mathrm{gr}(D) : \mathrm{gr}(F)] \) and \( \mathrm{gr}(F) = Z(\mathrm{gr}(D)) \). Take any maximal subfield \( L \) of \( D \) containing \( a \). Then \( L/F \) is defectless as \( D/F \) is defectless, so \( [\mathrm{gr}(L) : \mathrm{gr}(F)] = [L : F] = \mathrm{ind}(D) = \mathrm{ind}(\mathrm{gr}(D)) \). Hence, using Lemma 7.10 and Prop. 7.5(2), we have,

\[
\mathrm{Nrd}_D(a) = \mathrm{N}_{L/F}(a) = N_{\mathrm{gr}(L)/\mathrm{gr}(F)}(\bar{a}) = \mathrm{Nrd}_{\mathrm{gr}(D)}(\bar{a}).
\]

The next proposition will be used several times below. It was proved by Ershov in [15], Prop. 2, who refers to Yanchevskii [108] for part of the argument. One can find a complete proof in [27].

**Proposition 7.12.** Let \( F \subseteq K \) be fields with Henselian valuations \( v \) such that \( [K : F] < \infty \) and \( K \) is tamely ramified over \( F \). Then \( N_{K/F}(1 + M_K) = 1 + M_F \).

**Corollary 7.13.** Let \( F \) be a field with Henselian valuation \( v \), and let \( D \) be an \( F \)-central division algebra which is tame with respect to \( v \). Then, \( \mathrm{Nrd}_D(1 + M_D) = 1 + M_F \).

**Proof.** Take any \( a \in 1 + M_D \) and any maximal subfield \( K \) of \( D \) with \( a \in K \). Then, \( K \) is defectless over \( F \), since \( D \) is defectless over \( F \). So, \( a \in 1 + M_K \), and \( \mathrm{Nrd}_D(a) = N_{K/F}(a) \in 1 + M_F \) by the first part of the proof of Prop. 7.12, which required only defectlessness, not tameness. Thus, \( \mathrm{Nrd}_D(1 + M_D) \subseteq 1 + M_F \). For the reverse inclusion, recall from [41], Prop. 4.3 that as \( D \) is tame over \( F \), it has a maximal subfield \( L \) with \( L \) tamely ramified over \( F \). Then by Prop. 7.12,

\[
1 + M_F = N_{L/F}(1 + M_L) = \mathrm{Nrd}_D(1 + M_L) \subseteq \mathrm{Nrd}_D(1 + M_D) \subseteq 1 + M_F,
\]

so equality holds throughout.

We can now prove the main result of this section.

**Theorem 7.14.** Let \( F \) be a field with Henselian valuation \( v \) and let \( D \) be a tame \( F \)-central division algebra. Then \( \mathrm{SK}_1(D) \cong \mathrm{SK}_1(\mathrm{gr}(D)) \).

**Proof.** Consider the canonical surjective group homomorphism \( \rho : D^* \to \mathrm{gr}(D)^* \) given by \( a \mapsto \bar{a} \). Clearly, \( \ker(\rho) = 1 + M_D \). If \( a \in D^{(1)} \subseteq V_D \) then \( \bar{a} \in \mathrm{gr}(D)_0 \) and by Cor. 7.11,

\[
\mathrm{Nrd}_{\mathrm{gr}(D)}(\bar{a}) = \mathrm{Nrd}_D(a) = 1.
\]
This shows that \( \rho(D^{(1)}) \subseteq \text{gr}(D)^{(1)} \). Now consider the diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & (1 + M_D) \cap D' & \rightarrow & D' & \xrightarrow{\rho} \text{gr}(D)' & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & (1 + M_D) \cap D^{(1)} & \rightarrow & D^{(1)} & \rightarrow & \text{gr}(D)^{(1)} & \rightarrow & 1
\end{array}
\] (7.15)

The top row of the above diagram is clearly exact. The Congruence Theorem implies that the left vertical map in the diagram is an isomorphism. Once we prove that \( \rho(D^{(1)}) = \text{gr}(D)^{(1)} \), we will have the exactness of the second row of diagram (7.15), and the theorem follows by the exact sequence for cokernels.

To prove the needed surjectivity, take any \( b \in \text{gr}(D)^* \) with \( \text{Nrd}_{\text{gr}(D)}(b) = 1 \). Thus \( b \in \text{gr}(D)^0 \) by Th. 7.6. Choose \( a \in V_D \) such that \( \overline{a} = b \). Then we have,

\[
\text{Nrd}_D(a) = \text{Nrd}_{\text{gr}(D)}(a) = \text{Nrd}_{\text{gr}(D)}(b) = 1.
\]

Thus \( \text{Nrd}_D(a) \in 1 + M_F \). By Cor. 7.13, since \( \text{Nrd}_D(1 + M_D) = 1 + M_F \), there is \( c \in 1 + M_D \) such that \( \text{Nrd}_D(c) = \text{Nrd}(a)^{-1} \). Then, \( ac \in D^{(1)} \) and \( \rho(ac) = \rho(a) = b \). \( \square \)

Having now established that the reduced Whitehead group of a division algebra coincides with that of its associated graded division algebra, we can easily deduce stability of \( \text{SK}_1 \) for unramified valued division algebra, due originally to Platonov (Cor. 3.13 in [80]), and also a formula for \( \text{SK}_1 \) for a totally ramified division algebra ([52], p. 363, see also [15], p. 70), and also a formula for \( \text{SK}_1 \) in the nicely semiramified case ([15], p. 69), as natural consequences of Th. 7.14:

**Corollary 7.15.** Let \( F \) be a field with Henselian valuation, and let \( D \) be a tame division algebra with center \( F \).

1. If \( D \) is unramified then \( \text{SK}_1(D) \cong \text{SK}_1(D) \).
2. If \( D \) is totally ramified then \( \text{SK}_1(D) \cong \mu_n(F)/\mu_e(F) \) where \( n = \text{ind}(D) \) and \( e \) is the exponent of \( \Gamma_D/\Gamma_F \).
3. If \( D \) is semiramified, let \( G = \text{Gal}(D/F) \cong \Gamma_D/\Gamma_F \). Then, there is an exact sequence

\[
G \wedge G \rightarrow \widehat{H}^{-1}(G, D') \rightarrow \text{SK}_1(D) \rightarrow 1.
\] (7.16)

4. If \( D \) is nicely semiramified, then \( \text{SK}_1(D) \cong \widehat{H}^{-1}(\text{Gal}(D/F), D') \).

**Proof.** Because \( D \) is tame, \( Z(\text{gr}(D)) = \text{gr}(F) \) and \( \text{ind}(\text{gr}(D)) = \text{ind}(D) \). Therefore, for \( D \) in each case (1)–(4) here, \( \text{gr}(D) \) is in the corresponding case of Cor. 7.9. (In case (3), that \( D \) is semiramified means \( [D:F] = [\Gamma_D: \Gamma_F] = \text{ind}(D) \) and \( D \) is a field. Hence \( \text{gr}(D) \) is semiramified. In case (4), since \( D \) is nicely semiramified, by definition (see [42], p. 149) it contains maximal subfields \( K \) and \( L \), with \( K \) unramified over \( F \) and \( L \) totally ramified over \( F \). (In fact, by [75], Th. 2.4, \( D \) is nicely semiramified if and only if it has such maximal subfields.) Then, \( \text{gr}(K) \) and \( \text{gr}(L) \) are maximal graded subfields of \( \text{gr}(D) \) by dimension count and the graded double centralizer theorem,[41], Prop. 1.5(b), with \( \text{gr}(K) \) unramified over \( \text{gr}(F) \) and \( \text{gr}(L) \) totally ramified over \( \text{gr}(F) \). Then, so is \( \text{gr}(D) \) in the case (4) of
8. The Group $G(D)$ and the Existence of Maximal Subgroups

In previous sections we have seen that the group $CK_1(D)$ is nontrivial in several cases. But the question whether $CK_1(D) \neq 1$ in the general case is still open. Now, consider a division algebra of index $n$ with $CK_1(D) \neq 1$. Since $CK_1(D)$ is abelian of bounded exponent $n$ one can easily conclude that in this case $D^*$ has a (normal) maximal subgroup. In [3] it is conjectured that each finite dimensional division algebra has a maximal (not necessarily normal) subgroup. In [25], in an attempt to prove this conjecture it was shown that if $D^*$ has no normal maximal subgroup then there exists a noncyclic division algebra of a prime index $p > 3$. Also the authors showed that every quaternion division algebra has a maximal subgroup. Thus the existence of maximal subgroups reduces to the nontriviality of $CK_1(D)$. Now, consider the group $G(D) = D^*/Nrd_D(D^*)D'$. By Corollary 5.3, this group is also abelian of bounded exponent $n$. Thus if $G(D) \neq 1$ then we can conclude that $D^*$ has a (normal) maximal subgroup. At the other extreme since $CK_1(D)$ is a homomorphic image of $G(D)$ it seems that showing the nontriviality of $G(D)$ is easier than that of $CK_1(D)$. In [46] Keshavarzipour and Mahdavi-Hezavehi proved that if the deg $(D) = p^m$ for some prime $p$ and $\mu_p \subset Z(D)$ then triviality of $G(D)$ forces that $D$ is the quaternion division algebra. However, a general criterion for the triviality of $G(D)$ was provided in [69]. This section is devoted to presenting this criterion and then applying this criterion to achieve the main result of [25]. We begin our study with the following lemma. Before stating it we recall that a field $F$ is called Euclidean if $F^* = F^{*2} \times \langle -1 \rangle$ and every sum of two squares is again a square. In what follows $C_m$ stands for the cyclic group of order $m$.

**Lemma 8.1.** Let $F$ be a field such that $F^* = F^{*m} \times C_m$ and $p$ a prime dividing $m$. Then, for a cyclic extension $L/F$ of degree $p$ the following statements are equivalent:

1. $N_{L/F}(L^*) \neq F^*$;
2. $L = F(\sqrt{-1})$ and $F$ is Euclidean.

**Proof.** (1)$\Rightarrow$(2) Clearly, $F$ contains a primitive $p$-th root of unity. Thus, by Kummer theory $L = F(\alpha)$ for some $\alpha \in L$ with $\alpha^p - b \in F^* \setminus F^{*p}$. But, from $F^* = F^{*m} \times C_m$ we have $F^{*m} = F^{*m^2}$ and so $F^{*m} = F^{*mp}$ because $p$ divides $m$. Therefore,

$$\frac{F^*}{F^{*p}} \cong \frac{F^{*m} \times C_m}{F^{*mp} \times (C_m)^p} = \frac{F^{*m} \times C_m}{F^{*m} \times (C_m)^p} \cong C_p.$$ 

Hence $F^* = \langle b \rangle F^{*p}$. Now, let $p$ be odd. Since the minimal polynomial of $\alpha$ over $F$ is $x^p - b$, we obtain $N_{L/F}(\alpha) = (-1)^{p+1}b = b$ and hence $b \in N_{L/F}(L^*)$. On the other hand, $F^{*p} \subseteq N_{L/F}(L^*)$ implies $F^* = \langle b \rangle F^{*p} \subseteq N_{L/F}(L^*)$ and so $N_{L/F}(L^*) = F^*$, which is a contradiction. Thus, $p = 2$. Now, consider the case in which $-1 \in F^{*2}$. Then $\sqrt{-1} \in F$. Therefore, $N_{L/F}(\sqrt{-1} \alpha) = b$ and consequently $N_{L/F}(L^*) = F^*$ which is also a contradiction. So we have $-1 \notin F^{*2}$, $F^* = \langle -1 \rangle \times F^{*2}$ and $L = F(\sqrt{-1})$. Since $F^{*2} \subseteq N_{L/F}(L^*)$ and $N_{L/F}(L^*) \neq F^*$ we obtain $N_{L/F}(L^*) = F^{*2}$. Now, given $u, v \in F^*$, from $L = F(\sqrt{-1})$ we
have \( u^2 + v^2 = N_{L/F}(u + v\sqrt{-1}) \in F^s \) which means that \( F \) is Euclidean. The converse \((2) \Rightarrow (1)\) is clear. \(\square\)

To proceed our study we need to recall Merkurjev’s Theorem and Albert’s Main Theorem from the theory of central simple algebras. For the proofs see the Corollary of [72] and [5, p. 109], respectively.

**Theorem 8.2** (Merkurjev). Let \( F \) be a field and \( p \) be a prime number. If \( [F(\mu_p) : F] \leq 3 \) then the \( p\Br(F) \) (the subgroup of \( \Br(F) \) that is formed by all the elements killed by \( p \)) is generated by cyclic algebras of degree \( p \).

**Theorem 8.3** (Albert). Let \( F \) be a field with nonzero characteristic \( p \). Then every \( p \)-algebra is similar to a cyclic \( p \)-algebra.

Now, let \( F \) be a field with \( F^s = F^{sp} \) for some prime number \( p \). If \( [F(\mu_p) : F] \leq 3 \), then by Merkurjev’s Theorem, \( p\Br(F) \) is generated by classes of cyclic \( F \)-algebras of degree \( p \) and thus every \( F \)-central simple algebra splits. So \( \Br(F)_p = 0 \). Also if \( \text{char}(F) = p \) and \([A] \in p\Br(F)\), then by Albert’s Main Theorem \([A] = [(a, L/F, \sigma)]\) for some cyclic field extension \( L/F \), where \([L : F] = p \)-th power. Since \( F^s = F^{sp} \) for each \( e \in \mathbb{N} \), we conclude that \( N_{L/F}(L^s) = F^s \). So \([A] = 0 \) in \( \Br(F) \) and again we obtain \( \Br(F)_p = 0 \).

**Lemma 8.4.** Given a field \( F \), we have the following:

1. Let \( 2\Br(F) \neq 0 \). Then \( F^{s2} = F^{s4} \) if and only if \( F \) is Euclidean;
2. If \( n \) is odd, then \( F^s = F^{sn} \) if and only if \( F^{s2} = F^{s2n} \);
3. Let \( D \) be a non-commutative \( F \)-central division algebra of index \( n \). If \( F^{sn} = F^{s2n} \), then \( F^{s2} = F^{s2n} \).

**Proof.** (1) If \( F \) is Euclidean, we clearly have \( F^{s2} = F^{s4} \). Conversely, let \( F^{s2} = F^{s4} \). If \( \text{char}(F) = 2 \), then \( F^s = F^{s2} \). Thus the above argument yields \( 2\Br(F) = 0 \) which contradicts our assumption. Thus, we may assume that \( \text{char}(F) \neq 2 \) and hence \(-1 \in F \). If \(-1\) is a square, then \( F^s \) is 2-divisible and again \( 2\Br(F) = 0 \) which is absurd. So, suppose \(-1\) is not a square. So that \( F^s = F^{s2} \times (-1) \). Since \( 2\Br(F) \neq 0 \) there is a cyclic algebra of degree 2 over \( F \) which is a division algebra. If \( L \) is its maximal subfield then \( N_{L/F} \) is not surjective. Hence, the result follows from Lemma 8.1.

(2) If \( F^s \) is \( n \)-divisible then, one may easily check that \( F^{s2} = F^{s2n} \). Conversely, suppose that \( F^{s2} = F^{s2n} \). If \( a \in F \) then, there is a \( b \in F^s \) such that \( a^2 = b^{2n} \). Since \( n \) is odd, \( a = (\pm b)^n \in F^{sn} \) and hence \( F^s \subseteq F^{sn} \), as desired.

(3) Firstly, let \( \mu_n(F) = 1 \). Thus \( F^s = F^{sn} \). If \( n \) is even, \( \text{char}(F) = 2 \) and the 2-primary component of \( D \) splits which is a contradiction. So \( n \) is odd and the result follows from (2). Now, it remains to consider the case in which \( \mu_n(F) \neq 1 \). If this is the case, then \( \mu_n(F) \) is a cyclic group \( C_m \) for some \( m \) dividing \( n \). Clearly, we have \( F^s = F^{sn}C_m \). Now, we claim that \( F^{sn} \cap C_m = 1 \). For let \( \mu_p \subseteq F^{sn} \cap C_m \) for some prime \( p \) dividing \( m \). Since \( F^{sn} = F^{sp}p \) the \( p \)-Sylow subgroup \( P \) of \( C_m \) is contained in \( F^{sn} \). But \( C_m = P \triangleleft H \) for some cyclic subgroup \( H \) of \( C_m \) of order \( m/|P| \). Since \( F^{sn} = F^{sn}H = F^{sp}pH \) and \( (p, |H|) = 1 \), we conclude that \( F^s = F^{sn}H = F^{sp}p \). Now, the argument before the lemma yields
\[ Br(F)_p = 0 \] which is a contradiction and the claim is established. Therefore, we obtain
\[ F^* = F^{\ast n} \times C_m \] and hence \( F^* = F^{\ast n} \times C_m \). If \( m \) has an odd prime divisor, then by Lemma 8.1, \( N_{L/F}(L^*) = F^* \) for every cyclic extension \( L/F \) of index \( p \). Hence, by Merkurjev’s Theorem, \( Br(F) = 0 \) which is a contradiction. Therefore, \( m = 1 \) if \( n = \deg(D) \) is odd and \( m = 2^e \) for some \( e \geq 0 \) if \( n \) is even. However, the case that \( n \) is odd yields \( F^* = F^{\ast n} \) and by (2) the result follows. So, we assume that \( n \) is even. If \( e = 0 \), then \( C_m = 1 \) which is a contradiction, because \( \mu_n(F) \neq 1 \). Therefore \( e > 0 \). If \( e > 1 \), then for every quadratic extension \( L/F \), the norm map \( N_{L/F} : L^* \to F^* \) is surjective, by Lemma 8.1. This also gives us the contradiction \( 2Br(F) \neq 0 \). Hence \( e = 1 \) and \( F \) is Euclidean. So \( F^* = F^{n_2} \times C_2 \) and consequently \( F^{*2} = F^{*2n} \), as desired. \( \square \)

Recall that whenever \( F \) is a Euclidean field then \( F^{*2} \) defines an ordering. One may easily check that this statement is equivalent to the definition of a Euclidean field.

We are now in a position to find a criterion for the triviality of the group \( \overline{G}(D) = D^*/Nrd_D(D^*) D^{(1)} \) which is a homomorphic image of \( G(D) \). Note that \( \overline{G}(D) \) is isomorphic to \( Nrd_D(D^*)/N_{D/F}(D^*) \) where \( N_{D/F} : D^* \to F^* \) is the norm map and the required isomorphism is given by the reduced norm.

**Lemma 8.5.** Let \( D \) be an \( F \)-central division algebra of even index \( n \). If \( F^{*2} \) is \( n \)-divisible then

1. the 2-primary component of \( D \) is the ordinary quaternion division algebra over \( F \);
2. The image of \( D^* \) under the reduced norm is \( F^{*2} \).

**Proof.** (1) Since \( n \) is even, \( F \) is Euclidean by Lemma 8.4. Put \( L = F(\sqrt{-1}) \). Given \( a + \sqrt{-1}b \in L \), if we set
\[
v = \frac{a}{\sqrt{a^2 + b^2}}, \quad u = \sqrt{a^2 + b^2}(\sqrt{\frac{1 + v}{2}} + \text{sign}(b) \sqrt{-1}\sqrt{\frac{1 - v}{2}}),
\]
where
\[
\text{sign}(b) = \begin{cases} 1 & b > 1 \\ -1 & b < 1 \end{cases},
\]
then \( v^2 = a + \sqrt{-1}b \). This yields \( L^* = L^{*2} \) and so \( Br(L)_2 = 0 \). Therefore the 2-primary component of \( D \) splits by \( L \). Hence, by Theorem 7 of [11, p. 64] there exists an \( F \)-central simple algebra \( A \) containing a copy of \( L \) as a maximal subfield such that the 2-primary component of \( D \) is Brauer equivalent to \( A \). But, clearly \( A \) is the ordinary quaternion division algebra \( \mathbb{Q} \). This establishes (1).

(2) By (1) \( F \) is Euclidean and the 2-primary component of \( D \) is \( \mathbb{Q} \). Thus if \( a \in D^* \) then \( Nrd_D(a)^m \in Nrd_D(Q^*) = F^{*2} \) (cf. [11, Lem 5, p. 158]). But, \( F^{*2} \) is uniquely \( m \)-divisible as \( F \) is Euclidean. Therefore \( Nrd_D(a) \in F^{*2} \) and so
\[
Nrd_D(D^*) \subseteq F^{*2} = F^{*2m} \subseteq Nrd_D(D^*).
\]
This forces that \( Nrd_D(D^*) = F^{*2} \). \( \square \)

**Theorem 8.6.** Let \( D \) be an \( F \)-central division algebra. Then \( \overline{G}(D) = 1 \) if and only if \( F^{*2} \) is \( n \)-divisible.
Proof. If $\overline{G}(D) = 1$, then

$$\text{Nrd}_D(D^*) = N_{D/F}(D^*) = \text{Nrd}_D(D^*)^n.$$ 

Therefore

$$\text{Nrd}_D(D^*)^n \subseteq F^*n \subseteq \text{Nrd}_D(D^*) = \text{Nrd}_D(D^*)^n$$

and thus $\text{Nrd}_D(D^*) = F^*n$. This gives $F^*n = F^*n^2$ and so from Lemma 8.4 we obtain $F^*n^2 = F^*n$.

To prove the converse, first suppose that $n$ is odd. This assumption combining with Lemma 8.4 gives $F^* = F^*n$. So

$$\text{Nrd}_D(D^*) = F^* = F^*n = N_{D/F}(D^*)$$

which is equivalent to $\overline{G}(D) = 1$. Moreover, if $n$ is even then Lemma 8.5 yields

$$\text{Nrd}_D(D^*) = F^*2 = F^*2n = N_{D/F}(D^*),$$

as desired. $\square$

From Theorem 8.2, Theorem 8.3 and Theorem 8.6 the following corollary is immediate.

**Corollary 8.7.** Let $D$ be an $F$-central division algebra of index $p^m$ where $p$ is a prime number. If $[F(\mu_p) : F] \leq 3$ then $\overline{G}(D) = 1$ if and only if $D$ is the ordinary quaternion division algebra and $F$ is Euclidean. Also, if $\text{char}(F) = p$ then $\overline{G}(D) \neq 1$.

Note that Theorem 8.6 and Lemma 8.5 essentially say that, if we want to produce a division algebra of degree greater than 2 with trivial $G$ then $\text{deg}(D)$ should not be a power of 2. So we must have $pBr(F) \neq 0$ for some odd prime $p$ dividing $\text{deg}(D)$; for each such $p$, $pBr(F)$ should be generated by noncyclic division algebras and so we must have $[F(\mu_p) : F] \geq 4$ (so $p \geq 5$) by Corollary 8.7. The existence of such noncyclic division algebras is one of the oldest and most challenging questions in the theory of division algebras.

Now, we follow our investigation on the existence of maximal subgroups in division algebras. Before that we recall some notions from the theory of groups. Let $\pi$ be a set of prime numbers. A natural number $n$ is said to be a $\pi$-number if every prime divisor of $n$ belongs to $\pi$. Recall that a group $G$ is called $\mathfrak{S}_\pi$-perfect if it does not contain any subgroup of a finite $\pi$-number index. If $\pi$ is the set of all primes, every $\mathfrak{S}_\pi$-perfect group is called $\mathfrak{S}$-perfect. The following theorem provides a criterion for when $D^*$ is $\mathfrak{S}_\pi$-perfect, where $\pi$ is the set of all primes dividing ind$(D)$. It also singles out the relation between the notion of $\mathfrak{S}_\pi$-perfection and the triviality of $G(D)$.

**Theorem 8.8.** Given an $F$-central division algebra $D$ of index $n$, the following conditions are equivalent:

1. $G(D) = 1$;
2. $\text{SK}_1(D) = 1$ and $F^*2$ is $n$-divisible;
3. $\text{CK}_1(D) = 1$ and $F^*2$ is $n$-divisible;
4. $D^*$ is $\mathfrak{S}_\pi$-perfect where $\pi$ is the set of all primes dividing $\text{deg}(D)$.
Proof. Firstly, we observe that a routine calculation shows that the following diagram is commutative with exact rows:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & SK_n(D) & \xrightarrow{f_1} & SK_1(D) & \xrightarrow{g_1} & G(D) & \xrightarrow{\text{Nrd}_D} & \overline{G}(D) & \longrightarrow & 1 \\
& & i_1 & & \downarrow{1_{SK_1(D)}} & & \pi_1 & & \pi_2 & & \\
1 & \longrightarrow & \frac{\mu_n(F)}{Z(D')} & \xrightarrow{f_2} & SK_1(D) & \xrightarrow{g_2} & CK_1(D) & \xrightarrow{\text{Nrd}_D} & NK_1(D) & \longrightarrow & 1, \\
\end{array}
\]  

(8.1)

where the maps \(f_1, f_2, g_1, g_2\) are canonical homomorphisms, \(i_1\) is induced by inclusion, \(\pi_1, \pi_2\) are natural projections, and

\[
SK_n(D) = \frac{\mu_n(F) \cap \text{Nrd}_{D/F}(D^*)}{Z(D') \cap \text{Nrd}_{D/F}(D^*)}.
\]

(Recall that \(Z(D') = \mu_n(F) \cap D'\).)

(1)\(\Rightarrow\)(2) If \(G(D) = 1\) then \(\overline{G}(D) = 1\) and hence \(F^{*2} = F^{*2n}\) by Theorem 8.6. But, from the exactness of the first row of (8.1), we have

\[
SK_1(D) \cong SK_n(D).
\]

(8.2)

Now, if \(n\) is odd then \(F^*\) is \(n\)-divisible by Lemma 8.4. By the Merkurjev Theorem \(F\) contains no \(n\)-th roots of unity and hence by (8.2) we conclude that \(SK_1(D) = 1\). Moreover, if \(n\) is even then by Lemma 8.4, \(F\) is Euclidean and \(\text{Nrd}_{D/F}(D^*) = F^{*2}\) (Lemma 8.5). However, \(F^{*2}\) contains no \(n\)-th roots of unity as \(F\) is Euclidean and again by (8.2) the result follows.

(2)\(\Rightarrow\)(3) Since \(SK_1(D) = 1\) we have

\[
CK_1(D) = NK_1(D).
\]

(8.3)

On the other hand \(\overline{G}(D) = 1\) by Theorem 8.6. Hence \(NK_1(D) = 1\) because \(\pi_2\) is an epimorphism. Therefore, by (8.3) we conclude that \(CK_1(D) = 1\).

(3)\(\Rightarrow\)(1) Theorem 8.6 gives us \(\overline{G}(D) = 1\). Furthermore, from the assumption \(CK_1(D) = 1\) and the exactness of the second row of (8.1) we obtain

\[
SK_1(D) \cong \frac{\mu_n(F)}{Z(D')}.
\]

(8.4)

Thus, if \(n\) is odd then by Lemma 8.4 and the Merkurjev Theorem we conclude that \(\mu_n(F) = 1\). Hence (8.4) yields \(SK_1(D) = 1\). This forces that \(G(D) = 1\) because the first row of (8.1) is exact. Finally, let \(n\) be even. In this case Lemma 8.4 shows that \(F\) is Euclidean and so \(\mu_n(F) = \{1, -1\}\). On the other hand, we know that in every division algebra of even degree \(-1\) belongs to the commutator subgroup \((98)\). So \(\mu_n(F) = Z(D')\) and by (8.4) we conclude that \(SK_1(D) = 1\). Now, from (8.1) it follows that \(G(D) = 1\).

(1)\(\Rightarrow\)(4) If \(G(D) = 1\), then we have \(\overline{G}(D) = 1\). So, \(\text{Nrd}_D(D^*)\) is \(n\)-divisible and hence it is \(p\)-divisible for every prime number \(p\) dividing \(n\). Now, suppose that \(D^*\) is not \(\mathfrak{F}_n\)-perfect. Thus it contains a normal subgroup \(N\) of index \(k\) for some \(\pi\)-number \(k\). By the Main Theorem of [82], \(D^*/N\) is a finite soluble group. Therefore, it contains a normal maximal subgroup \(M\) of index \(p\), where \(p\) divides \(k\) and so does \(n\). Hence \(D^*\) contains a normal maximal subgroup \(M\) of index \(p\) for some prime divisor \(p\) of \(n\). But \(D' \subseteq M\) as \(M \triangleleft D^*\). Now, if
Nrd\(_D(D^*) \subseteq M\), then Nrd\(_D(D^*)D' \subseteq M \varsubsetneq D^*\) and so G(D) \neq 1, which is a contradiction. Moreover, if Nrd\(_D(D^*) \notin M\) then D* = Nrd\(_D(D^*)M\) and hence

\[
\frac{\text{Nrd}_D(D^*)}{\text{Nrd}_D(D^*) \cap M} \cong \frac{\text{Nrd}_D(D^*)M}{M} \cong \frac{D^*}{M} \cong C_p.
\]

This guarantees that Nrd\(_D(D^*)\) has a normal subgroup of index p which contradicts the fact that Nrd\(_D(D^*)\) is p-divisible. Therefore D* is \(\mathfrak{F}_p\)-perfect.

\(4 \Rightarrow 1\) It is clear. \(\square\)

Let \(\mathcal{Q}\) be the real quaternion division algebra. Example 6.9 shows that \(\mathcal{Q}((x))\) has a trivial CK\(_1\). But \(Z(\mathcal{Q}((x))) = \mathbb{R}((x))\) has a natural discrete rank 1 valuation and hence \(\mathbb{R}((x))^2 \neq \mathbb{R}((x))^4\). So \(G(\mathcal{Q}((x))) \neq 1\) by Theorem 8.8. Therefore, even for the case CK\(_1(D) = 1\), D* may have a normal maximal subgroup. This observation shows that to approach the problem of the existence of normal maximal subgroups in D* it is better to work with G(D) instead of CK\(_1(D)\).

Now, let G be a group (possibly nonabelian). We say that G is divisible if G/G' is divisible as an abelian group. Equivalently, G is divisible if for every natural number n and g \(\in G\) there are elements b \(\in G\) and h \(\in G'\) such that g = b^nh. One can easily show that G is divisible if and only if it contains no normal maximal subgroup. In the following theorem we will see that the notions of divisibility and \(\mathfrak{F}_p\)-perfection are the same in the setting of division algebras.

**Theorem 8.9.** Given an F-central division algebra D of degree n, then the following conditions are equivalent:

1. D* is divisible or equivalently D* has no normal maximal subgroup;
2. D* is \(\mathfrak{F}_p\)-perfect;
3. SK\(_1(D) = 1\), F* is divisible if n is odd and F*\(^2\) is divisible if n is even.

**Proof.** (1) \(\Rightarrow\) (2) If D* has a subgroup of finite index, then it contains a normal subgroup of finite index. Now, a similar argument as in the proof of (1) \(\Rightarrow\) (4) in Theorem 8.8 one can find a normal maximal subgroup which is a contradiction. So D* is \(\mathfrak{F}_p\)-perfect.

(2) \(\Rightarrow\) (1) It is clear from the definition.

(1) \(\Rightarrow\) (3) If D* is divisible then from Theorem 8.8 it follows that SK\(_1(D) = 1\). Now, let n be odd. If this is the case, then F* is n-divisible by Theorem 8.8. Thus Nrd\(_D(D^*) = F^*\) and hence F* is divisible as D* is divisible. Moreover, if n is even then Nrd\(_D(D^*) = F^{*\circ 2}\) by Lemma 8.5. This implies that F*\(^2\) is divisible.

(3) \(\Rightarrow\) (1) As in the proof of (1) \(\Rightarrow\) (3) we have Nrd\(_D(D^*) = F^*\) if n is odd and Nrd\(_D(D^*) = F^{*\circ 2}\) if n is even. In either case Nrd\(_D(D^*)\) is divisible. On the other hand by Theorem 8.8 we know that G(D) = 1 and so D* = Nrd\(_D(D^*)D'\). Therefore

\[
\frac{D^*}{D'} \cong \frac{\text{Nrd}_D(D^*)}{\text{Nrd}_D(D^*) \cap D'}
\]

and hence D* is divisible. \(\square\)
Recall that if $D$ is a division algebra then thanks to the Dieudonné determinant the group $\text{GL}_n(D)/\text{SL}_n(D)$ is isomorphic to $D^*/D'$ for every natural number $n$. Therefore, Theorem 8.6 is valid for every central simple algebra.

Let $F$ be a Euclidean field with divisible $F^*2$ (for example a real closed field). If $Q$ is the quaternion division algebra over $F$ then from Theorem 8.9 it follows that $Q^*$ has no normal maximal subgroup. However, in the case that $F = \mathbb{R}$ in [60] it was shown that the subgroup $\mathbb{C} \cup \mathbb{C}j$ is a (nonnormal) maximal subgroup of $Q^*$. Also, a different approach to prove that the real quaternion has a maximal subgroup was considered in [1]. But the existence of a maximal subgroup in a quaternion division algebra, in the general case, was demonstrated in [25] by a refinement of the argument given in [60]. Here we are going to show what the nature of such a subgroup of $Q^*$ can be. Since $F$ is assumed to be Euclidean, then it has a valuation ring $V$ which is determined by the ordering:

$$V = \{b \in F \mid |b| \leq n \text{ for some } n \in \mathbb{N}\},$$

with maximal ideal

$$M = \{b \in F \mid |b| \leq 1/n \text{ for every } n \in \mathbb{N}\}.$$

Thus, $F \setminus V$ is the set of all elements “infinitely large” relative to the rational numbers $\mathbb{Q} \subset F$. Also, $M$ is the set of all elements of $F$ “infinitesimal” relative to $\mathbb{Q}$. Let $P$ be the “purely imaginary part” of $Q$, i.e., $P = \{bi + cj + dk \mid b, c, d \in F\}$ and

$$S(P) = \{\alpha \in P \mid \|\alpha\| = 1\}$$

be the unit sphere in $P$ (note that for $x \in Q$, we have $\|x\| = \sqrt{\text{Nrd}_Q(x)}$. Suppose that

$$\Delta = \{\alpha \in S(P) \mid \|\alpha - i\| \in M\},$$

the set of elements “infinitesimally near” to $i$. In this setting one can observe that

$$G = \{x \in Q^* \mid ix \in \Delta \cup -\Delta\}$$

is a maximal subgroup of $Q^*$ (see [25]).

By combining the above observation, Theorem 8.9, Corollary 8.7 and Lemma 3 of [71] which asserts that every divisible field of characteristic zero contains all primitive roots of unity, we obtain:

**Theorem 8.10.** Let $D$ be an $F$-central division algebra and suppose that $D^*$ has no maximal subgroups. Then,

1. If $\deg(D)$ is even, then $D = Q \otimes_F E$ where $E$ is an $F$-central division algebra of odd degree, and $F$ is Euclidean (so $\text{char}(F) = 0$) with $F^*2$ divisible;
2. If $\deg(D)$ is odd, then $\text{char}(F) > 0$, $\text{char}(F) \nmid \deg(D)$ and $F^*$ is divisible;
3. In either case, there is an odd prime number $p$ dividing $\deg(D)$; for each such $p$ we have $[F(\mu_p) : F] \geq 4$ (so $p \geq 5$) and $p\text{Br}(F)$ is generated by noncyclic algebras of degree $p$.

Finally, we give some additional results concerning the group $\text{G}(D)$ and maximal subgroups of $D^*$. For the proofs of these theorems see [69].
Theorem 8.11. If $A$ and $B$ are $F$-central simple algebras of coprime degrees, then $\overline{G}(A \otimes_F B) \cong \overline{G}(A) \times \overline{G}(B)$.

Theorem 8.12. If $D$ is an $F$-central division algebra with $\overline{G}(D) = 1$, then the following assertions are equivalent:

1. $D$ contains an absolutely irreducible nilpotent subgroup;
2. $D$ is a nilpotent crossed product division algebra whose exponent in $\text{Br}(F)$ is equal to $\deg(D)$;
3. $D$ is the ordinary quaternion division algebra and $F$ is Euclidean.

Theorem 8.13. Let $D$ be a tame division algebra of index $n$ over its Henselian center $F$. If either of the following conditions holds, then we have $\overline{G}(D) \cong \overline{G}(D^*) \cong \Gamma_D/n\Gamma_D$:

1. $D$ is unramified and $\overline{G}(D) = 1$;
2. $D$ is totally ramified and $\overline{F}^*$ is $n$-divisible;
3. $D$ is semiramified, $\overline{D}/\overline{F}$ is a cyclic extension, and $N_{\overline{D}/\overline{F}}(\overline{D}^*)$ is $n$-divisible.

Theorem 8.14. Let $D$ be an $F$-central division algebra of index $n \geq 1$ such that $[D] \neq [\mathcal{Q}]$ in $\text{Br}(F)$, where $\mathcal{Q}$ is the ordinary quaternion division algebra. Moreover, assume that $[D] = \sum_{j=1}^{k}[D_j]$, where $D_j$’s are $F$-central cyclic division algebras. If $D^*$ contains a divisible maximal subgroup, then $D = F$.

9. Radicable division algebras

Let $D$ be a division ring. Recall that $D$ is called right (left) algebraically closed if every polynomial equation with coefficients in $D$ has a right (left) root in $D$. When $D$ is of finite dimension over its center, there is a perfect description of such a division algebra. By Niven-Jacobson Theorem if $Z(D)$ is a real closed field then $D$ is right as well as left algebraically closed (see [51, p. 255] or [77]). Also, by a theorem of Baer if $D$ is right algebraically closed then $D$ is the ordinary quaternion division algebra and $Z(D)$ is a real closed field ([51, p. 255]). Now, combining these results gives

Theorem 9.1. The finite dimensional right algebraically closed division algebras are precisely the division algebras of quaternions over a real closed field. These division algebras are also left algebraically closed.

However, our point of view here is to explore how much the above classification depends on the group theoretic structure of $D$. To setup our problem, for a moment we come back to the commutative case. Let $F$ be an algebraically closed field. Then its unit group is divisible. More exactly, $F^*$ is isomorphic to a group of one of the following types:

1. $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}^\lambda$;
2. $\bigoplus_{p \neq q} \mathbb{Z}_p \oplus \mathbb{Q}^\lambda$ for a prime number $p$.

for an infinite cardinal $\lambda$. Conversely, all groups of the above types (i) and (ii) are isomorphic to the unit group of a suitable algebraically closed field (cf. [44, pp. 106-107]). Moreover, by a theorem in [44, p. 107] the unit group of a field $F$ is isomorphic to the multiplicative group of a real closed field if and only if $F^* \cong \mathbb{Q}^\lambda \times \mathbb{Z}_2$ for some infinite cardinal $\lambda$. 

On the other hand, a natural generalization of the notion of divisibility in the nonabelian setting is the notion of “radicability”. We recall that a (nonabelian) group is called radicable if for each $g \in G$ and each $n \in \mathbb{N}$ there exists an element $h \in G$ such that $h^n = g$. A division ring $D$ is called radicable if $D^*$ is a radicable group. In particular, when $D^*$ is abelian the notions of divisibility and radicability coincide and $D$ is called a divisible (otherwise indivisible) field. From the definition, it follows that the multiplicative group of each right (left) algebraically closed division ring is radicable. So, it can be of interest to explore what happens when the multiplicative group of a division ring is radicable. In this direction, in [68] all division algebras with radicable multiplicative groups with indivisible center have been classified. More precisely, the authors provided a new version of Theorem 9.1 for radicable division algebras. Here, our aim is to present the main theorem of [68] as a consequence of the results appeared in the previous section. We proceed by recalling some definitions and facts from [68].

Let $F$ be a field and $K/F$ be a finite extension. Recall that $K$ is called a radical extension if there are finite intermediate fields $F \subseteq K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n = K$ such that for all $1 \leq i \leq n$, $K_{i+1} = K_i(\alpha_i)$ with $\alpha_i^{m_i} \in K_i$ for some $m_i \in \mathbb{N}$. $F$ is called radically closed if it possesses no radical extensions.

**Proposition 9.2.** Let $F$ be a field. Then there exists a unique, up to isomorphism, extension $F_{\text{rad}}/F$ which is radically closed and contained in any radically closed algebraic extension of $F$.

**Proof.** Let $F_{\text{alg}}$ be a fixed algebraic closure of $F$. Set $L_0 = F$ and for $i > 0$ let $L_i$ be the splitting field of the family $\{x^n - a \mid a \in L_{i-1}, \ n \in \mathbb{N}\}$ in $F_{\text{alg}}$. Put $F_{\text{rad}} := \bigcup_{i=0}^{\infty} L_i$. Clearly, $F_{\text{rad}}$ is radically closed as well as divisible. Now, one can easily check that every radical extension of $L$ in $F_{\text{alg}}$ is contained in $F_{\text{rad}}$. Also, the uniqueness follows from the standard theorems of field theory. \qed 

Proposition 9.2 provides us a facility to introduce the notion of **radical closure** and **radically real closed field**. For every field $F$, we refer to the extension $F_{\text{rad}}/F$ as the radical closure of $F$. Note that Proposition 9.2 ensures that $F_{\text{rad}}$ is independent of the choice of the algebraic closure of $F$. Moreover, we say that $F$ is radically real closed field if $\sqrt{-1} \notin F$ and $F_{\text{rad}} = F(\sqrt{-1})$.

**Example 9.3.** From field theory, recall that there are some polynomial equations in $\mathbb{Q}[x]$ that are not soluble by radicals. Using this fact, one can easily observe that $\mathbb{Q}_{\text{rad}} \neq \mathbb{Q}_{\text{alg}}$. Also, it is not hard to check that if $a \in \mathbb{Q}_{\text{rad}}$, then the complex conjugate of $a$ is contained in $\mathbb{Q}_{\text{rad}}$, i.e., $\overline{a} \in \mathbb{Q}_{\text{rad}}$. Now, consider the automorphism $\sigma : \mathbb{Q}_{\text{rad}} \to \mathbb{Q}_{\text{rad}}$, given by $a \mapsto \overline{a}$. Since $\sqrt{-1} \in \mathbb{Q}_{\text{rad}}$ we conclude that $\sigma \neq 1$. Moreover, it is clear that $\sigma^2 = 1_{\mathbb{Q}_{\text{rad}}}$. Hence $\sigma$ has order 2 which yields $F(\sqrt{-1}) = \mathbb{Q}_{\text{rad}}$ where $F$ is the fixed field of $\sigma$. This shows that $F$ is radically real closed and is not real closed, because $\mathbb{Q}_{\text{rad}}$ is not algebraically closed.

The next theorem can be viewed as an analogue of the classical Frobenius Theorem (cf. [51, p. 208]).
Theorem 9.4. If $K$ is a divisible finite field extension of $F$, then the following statements are equivalent:

1. $F$ is indivisible;
2. $F$ is radically real closed;
3. $\text{Br}(F) \cong \mathbb{Z}_2$.

Moreover, if one of the above conditions holds, then $F^*$ is isomorphic to the multiplicative group of a real closed field and the ordinary $F$-quaternion division algebra $\mathcal{O}$ is the only noncommutative division algebra with center $F$.

Proof. (1)$\Rightarrow$(2). Let $E$ be a minimal extension of $F$ contained in $K$ with divisible unit group. Note that since $F^*$ is not divisible we have $1 < [E : F] < \infty$. Let $L$ be a maximal subfield of $E$ containing $F$. Clearly $L^*$ is indivisible. Now, we claim that $L^* = N_{E/L}(E^*) \times C_m$ for some $m \neq 1$ dividing $[E : L]$. To establish our claim, set $[E : L] = n$. Since $N_{E/L}(E^*)$ is divisible and $L^*$ is indivisible, it follows that $L^* = N_{E/L}(E^*) \times N$ for some non-trivial subgroup $N$ of $F^*$. (Note that since $N_{E/L}(E^*)$ is divisible, it is injective as a $\mathbb{Z}$-module.) On the other hand, we know that $L^*$ is contained in $N_{E/L}(E^*)$. So $N \cong L^*/N_{E/L}(E^*)$ has exponent $n$ and thus every element of $N$ is a root of the polynomial $x^n - 1$ in $L$. Therefore, $N$ is a finite cyclic subgroup of $L^*$ of order dividing $n$, as desired. Now, let $p$ be a prime divisor of $m$ and $P = \langle \alpha \rangle$ be the $p$-Sylow subgroup of $C_m$. Clearly $\alpha^{1/p} \not\in L$ and hence $L \not\subseteq L(\alpha^{1/p}) \subseteq E$. By the maximality of $L$ we obtain $E = L(\alpha^{1/p})$. But $\mu_p \not\subseteq L$ as $p$ divides $m$. Thus, by Kummer Theory $E/L$ is a cyclic extension with $[E : L] = p$. Since $N_{E/L}(E^*) \not\cong L^*$, from Lemma 8.1, we conclude that $L$ is Euclidean. Hence $\sqrt{-1} \not\in L$ and so $\sqrt{-1} \not\in F$. If $F(\sqrt{-1})$ is indivisible, we may apply the above argument to obtain $\sqrt{-1} \not\in F(\sqrt{-1})$, which is a contradiction. Therefore, $F(\sqrt{-1})$ is divisible and hence $E = F(\sqrt{-1})$. Now, since $\text{char}(F) = 0$, by Lemma 3 of [71], $E$ contains all roots of unity. Thus, for every $a \in E$ the polynomial $x^n - a$ splits in $E[x]$. Therefore, $E$ has no proper radical extension. This yields $F \not\subseteq F_{\text{rad}} \subseteq E = F(\sqrt{-1})$ and thus $F_{\text{rad}} = F(\sqrt{-1})$.

(2)$\Rightarrow$(3). By a similar argument as above, we have $F^* = N_{F_{\text{rad}}/F}(F^*_{\text{rad}}) \times C_2$. This clearly implies that $F^* = F^{*2} \times C_2$. Now, Lemma 8.1 shows that $F$ is Euclidean and thus $\text{char}(F) = 0$. Here we observe that since the ordinary quaternion algebra $\mathcal{O}$ over $F$ is a division algebra we have $\text{Br}(F) \neq 0$. At the other extreme, since $F(\sqrt{-1})$ is divisible of characteristic zero, it contains all roots of unity and thus by Merkurjev Theorem $F(\sqrt{-1})$ has a trivial Brauer group. Therefore, each $F$-central simple algebra splits by $F(\sqrt{-1})$. Now, if $0 \neq [D] \in \text{Br}(F)$ then by Theorem 7 of [11, p. 64] there exists an $F$-central simple algebra $A$ such that $[D] = [A]$ and $F(\sqrt{-1})$ is a maximal subfield of $A$. Thus, $A$ is a quaternion division algebra and hence $A = \mathcal{O}$ because the only quaternion division algebra over a Euclidean field is the ordinary one. This implies that $\text{Br}(F) = \{[F], [\mathcal{O}]\} \cong \mathbb{Z}_2$.

(3)$\Rightarrow$(1). Since $\text{Br}(F) \cong \mathbb{Z}_2$, we conclude that $2\text{Br}(F) \neq 0$. But, by Merkurev Theorem $2\text{Br}(F)$ is generated by division algebras of degree 2. This forces that $F^*$ is indivisible, because every division algebra of degree 2 is cyclic.

Finally, as we have seen above, we have $F^* = F^{*2} \times \langle -1 \rangle$. Now, by a theorem of [44, p. 107] it follows that $F^*$ is isomorphic to the unit group of a suitable real closed field. \qed
Theorem 9.5. Let $F$ be an indivisible field. If $D$ is an $F$-central division algebra, then the following statements are equivalent:

1. $D$ contains a divisible subfield $K$ containing $F$;
2. $F$ is radically real closed and $D$ is the ordinary quaternion division algebra $\Omega$;
3. $D$ is radicable.

Furthermore, if one of the above conditions holds, then $F^*$ is isomorphic to the multiplicative group of a real closed field.

Proof. Clearly (1) and (2) are equivalent by Theorem 9.4.

$(2) \implies (3)$. Since $F(\sqrt{-1})^*$ is divisible, for each $a \in F^*$ and $n \in \mathbb{N}$ there exists a $b \in F(\sqrt{-1}) \subseteq D^*$ such that $b^n = a$. Now, if $a \in D^* \setminus F^*$, then $F(a)$ is a quadratic subfield of $D$. But $F(a)$ is separable over $F$ as $\text{char}(F) = 0$ (note that every radically real closed field is Euclidean and thus has characteristic zero). Hence $F(a)/F$ is a cyclic extension. On the other hand, Kummer Theory asserts that quadratic extensions of $F$ are in one to one correspondence with the subgroups of $F^*/F^{\times 2} \cong \langle -1 \rangle$. Therefore, $F(\sqrt{-1})$ is the only quadratic extension of $F$, up to isomorphism. This forces that $F(\sqrt{-1}) \cong F$ and consequently $F(a)$ is divisible. So, for each $n \in \mathbb{N}$, there is a $\beta \in F(\sqrt{-1})$ such that $\beta^n = a$. This shows that $D^*$ is radicable.

$(3) \implies (2)$. By definition $D^*$ is divisible and hence from Theorem 8.10 it follows that $D = Q \otimes_F E$, where $E$ has odd degree and $F$ is Euclidean. So we must prove that $\deg(E) = 1$. Otherwise, let $s = \deg(E) > 1$. First suppose that $F(\sqrt{-1})$ contains an $s$-th root of unity. So $\mu_p \subset F(\sqrt{-1})$ for at least one prime dividing $s$ and for such a prime, $[F(\mu_p) : F]$ is equal or less than 3. On the other hand as in the proof of Theorem 9.4 we can prove that $\text{Nrd}_D(D^*)$ is divisible and $F^* = \text{Nrd}_D(D^*) \times \langle -1 \rangle$. Now, since $\text{Nrd}_D(D^*)$ and $\langle -1 \rangle$ are $p$-divisible we conclude that $F^*$ is $p$-divisible. At this stage, Merkurev Theorem yields $p \text{Br}(F) = 0$ which is a contradiction. Thus, we are left with the case that $F(\sqrt{-1})$ has no $s$-th root of unity. Here, we claim that $F(\sqrt{-1})$ is divisible. To prove our claim, first we note that since $F(\sqrt{-1})^*$ is 2-divisible (see the proof of Lemma 8.5), we must show that $F^* = F^{\times p}$ for every odd prime $p$. Let $u \in F(\sqrt{-1})^* \setminus F^*$ and $p$ be an odd prime number. Put $E = C_D(F(\sqrt{-1}))$. By the Centralizer Theorem, $E$ is an $F(\sqrt{-1})$-central division algebra with $\deg(E) = s \neq 1$. Since $D^*$ is radicable, there is a $w \in D^*$ such that $w^p = u$. Because $w \in C_D(u)$ we have $w \in C_D(F(u)) = C_D(F(\sqrt{-1})) = E$. Taking the reduced norm, we obtain $\text{Nrd}_E(w)^p = u^p$. Since $F(\sqrt{-1})$ contains no $s$-th roots of unity we conclude that $\text{Nrd}_E(w^p = u$. Thus $u \in F(\sqrt{-1})^{\times p}$. Also, a similar argument as in the proof of $(2) \implies (3)$ shows that $F^*$ is $p$-divisible. Thus our claim is established. But, the divisibility of $F(\sqrt{-1})^*$ is in contrast with Lemma 3 of [71] which asserts that $F(\sqrt{-1})$ contains all primitive roots of unity. Thus $s = 1$ and hence $D = Q$, as required.

Finally, for $a$ and $p$ as above there exists $v \in D^*$ such that $v^p = u$. But $v \in C_D(F(\sqrt{-1})) = F(\sqrt{-1})$ as $F(\sqrt{-1})$ is a maximal subfield. Therefore, $F(\sqrt{-1})$ is radically closed and hence $F$ is radically real closed. \qed
Example 9.6. Consider the ordinary quaternion division algebra over $\mathbb{Q}_{\text{rad}}$. By the above theorem this division algebra is radicable. But, it is not algebraically closed, otherwise by Baer’s Theorem $\mathbb{Q}_{\text{rad}}$ would be real closed which is a contradiction.

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