New Formula for the Eigenvectors of the Gaudin Model in the sl(3) Case.

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Abstract

We propose new formulas for eigenvectors of the Gaudin model in the sl(3) case. The central point of the construction is the explicit form of some operator $P$, which is used for derivation of eigenvalues given by the formula

$$|w_1, w_2⟩ = \sum_{n=0}^{\infty} \frac{P^n}{n!} |w_1, w_2, 0⟩,$$

where $w_1, w_2$ fulfil the standard well-know Bethe Ansatz equations.

1 Introduction

In 1973 M. Gaudin [1–3] proposed a new class of integrable quantum models. These models were first formulated for the algebra $g = sl(2)$ and studied intensively by many authors. Let the generators $e$, $f$ and $h$ form a standard basis in $g = sl(2)$. Let $(\lambda) = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ be a set of dominant integral weight of $sl(2)$. Consider the tensor product $V_{\lambda} = V_{\lambda^{(1)}} \otimes \ldots \otimes V_{\lambda^{(N)}}$ and associate a complex number $z_i$ with each factor $V_{\lambda^{(i)}}$ of this tensor product.

For any element $x \in sl(2)$ denote $x^{(i)}$ the operator $1 \otimes \ldots \otimes x \otimes \ldots 1$, which acts as $x$ on the $i^{th}$ factor of $V_{\lambda}$. The Gaudin hamiltonians are mutually commuting operators

$$H_i = \sum_{j \neq i} \frac{1}{2} h^{(i)} h^{(j)} + e^{(i)} f^{(j)} + e^{(j)} f^{(i)} \frac{z_i - z_j}{z_i - z_j}.$$

One of the main problems in Gaudin’s model is to find eigenvectors and eigenvalues of these operators.
The Bethe Ansatz method \cite{2,4–6} is maybe the most effective method for solving the Gaudin model in the $g = \text{sl}(2)$ case. It is trivially sees that the tensor of the highest vectors of $V_{\lambda(i)}$’s $|0\rangle = v_{\lambda(1)} \otimes v_{\lambda(2)} \otimes \ldots \otimes v_{\lambda(N)}$ is an eigenvector in $V_{\lambda}$. One constructs other eigenvectors by acting on this vector by the operators

$$ F(w_r) = \sum_{i=1}^{N} \frac{f^{(i)}}{w_r - z_i} $$

depending on auxiliary parameters $w_1, w_2, \ldots, w_n$. The vectors obtained in this way are called Bethe vectors. These vectors are eigenvectors of the Gaudin model if

$$ \sum_{i=1}^{N} \chi^{(i)}(w_r - z_i) - \sum_{s \neq r}^{N} \frac{2}{w_r - w_s} = 0 \quad \text{for} \quad r = 1, \ldots, n $$

is fulfilled. These equations are called Bethe Ansatz equations. In Section 2 we will briefly repeat the derivation of this well know calculation.

In the nineties of the last century Feigin, Frenkel and Reshetikhin \cite{7} proved the formulas for eigenvectors of the general semisimple Lie algebra $g$. The eigenvectors should be constructed by applying the operators $f^{(j)}_j, j = 1, \ldots, l$ connected with simple roots to the vacuum $|0\rangle$. If now $F_1(w_1)$ and $F_2(w_2)$ do not commute, we are not able to find Bethe equations for $F_1(w_1)F_2(w_2) |0\rangle$. So it is needed to add some extra terms. The right formula can be extracted from solutions of the KZ equation \cite{8} (and in fact can be obtained as quasi-classical asymptotics of such solutions \cite{9}).

$$ \left| w_1^{i_1}, w_2^{i_2}, \ldots, w_m^{i_m} \right> = \sum_{p=\{i_1, I_2, \ldots, I_N\}} \prod_{j=1}^{N} \frac{f^{(j)}_{i_1} f^{(j)}_{i_2} \ldots f^{(j)}_{i_m}}{(w_{i_1} - w_{i_2})(w_{i_2} - w_{i_3}) \ldots (w_{i_m} - z_j)} |0\rangle. \quad (2) $$

Here the summation is taken over all ordered partitions $I^1 \cup I^2 \cup \ldots \cup I^N$ of the set $\{1, \ldots, m\}$, where $I^j = \{i_1^j, i_2^j, \ldots, i_{\alpha_j}^j\}$.

The main result of our paper is formulated in Section 3. For construction of eigenvectors we use not only the operators $F_j(w)$ which are connected with simple roots but the operators which are connected with non-simple roots.

Explicitly, we will study the case $g = \text{sl}(3)$. In this case, the Lie algebra has two simple roots and the corresponding generators are $f_1$ and $f_2$ will define $F_1(w_1)$ and $F_2(w_2)$. For the non-simple root and the generator $f_3 = [f_2, f_1]$ we will define $F_3(w)$ in a similar way. It should be mentioned that in the above construction such operators were not used. Again, as in the sl(2) case, the tensor of the highest vectors of $V_{\lambda_j}$’s is an eigenvector in $V_{\lambda}$. We will denote it by $|0\rangle$ and by

$$ \left| w_1, w_2, w_3 \right> = F_1(w_{1,1})F_1(w_{1,2}) \ldots F_1(w_{1,k})F_2(w_{2,1}) \ldots F_2(w_{2,\ell})F_3(w_{3,1}) \ldots F_3(w_{3,m}) |0\rangle. $$

We will define by $P$ the linear mapping

$$ P |0, w_2, w_3\rangle = P | w_1, 0, w_3\rangle = 0, $$

$$ P | w_1, w_2, w_3\rangle = \sum_{r,s} \left| \frac{w_1 - w_{1,r}, w_2 - w_{2,s}, w_3 + w_{1,r}}{w_{2,s} - w_{1,r}} \right> \ . $$

Our main theorem shows that the vectors

$$ \left| w_1, w_2 \right> = \sum_{n=0}^{\infty} \frac{P^n}{n!} \left| w_1, w_2, 0 \right> \ . $$

are eigenvectors of the Gaudin model if the Bethe Ansatz equations are fulfilled.
2 The Gaudin model for $\text{sl}(2)$

The $\text{sl}(2)$ Gaudin model was studied by many authors [1–3,10,12–18] from different points of view. In this section, we will concentrate on the well-know Bethe Ansatz method for finding eigenvectors and eigenvalues [2,4–6].

Let the generators $e$, $f$ and $h$ form a standard basis in $\text{sl}(2)$ which fulfil the commutations relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h. \quad (3)$$

For the second order Casimir operator we obtain

$$C = ef + fe + \frac{1}{2} h^2.$$

We will define

$$F(u) = \sum_{i=1}^{N} \frac{f^{(i)}}{u - z_i}, \quad E(u) = \sum_{i=1}^{N} \frac{e^{(i)}}{u - z_i}, \quad H(u) = \sum_{i=1}^{N} \frac{h^{(i)}}{u - z_i}. \quad (4)$$

The central point in the Gaudin model is played by the operator

$$T(u) = \frac{1}{2} \left( E(u)F(u) + F(u)E(u) + \frac{1}{2} H^2(u) \right). \quad (5)$$

It is possible to rewrite operator (5) in the form

$$T(u) = \sum_{i=1}^{N} \frac{H_i}{u - z_i} + \frac{1}{2} \sum_{i=1}^{N} \frac{C^{(i)}}{(u - z_i)^2},$$

where $H_i$ are given by (1) and $C^{(i)}$ is the Casimir operator acting on the $i^{th}$ factor of $V_{\lambda}$.

It is easy to show that from the commutation relations (3) and the definitions (4) we obtain for $u \neq w$

$$[E(u), E(w)] = [F(u), F(w)] = [H(u), H(w)] = 0,$$

$$[E(u), F(w)] = - \frac{H(u) - H(w)}{u - w},$$

$$[H(u), E(w)] = -2 \frac{E(u) - E(w)}{u - w},$$

$$[H(u), F(w)] = 2 \frac{F(u) - F(w)}{u - w}.$$

For construction of the Gaudin model we will use the highest representations

$$e^{(i)} v_{\lambda^{(i)}} = 0, \quad h^{(i)} v_{\lambda^{(i)}} = \lambda^{(i)} v_{\lambda^{(i)}}.$$

It is easy to see that the vector $|0\rangle = v_{\lambda^{(1)}} \otimes v_{\lambda^{(2)}} \otimes \ldots \otimes v_{\lambda^{(N)}}$ is eigenvectors of $T(u)$ and the relations

$$E(u) |0\rangle = 0, \quad H(u) |0\rangle = \lambda(u) |0\rangle = \sum_{i=1}^{N} \frac{\lambda^{(i)}}{u - z_i} |0\rangle, \quad T(u) |0\rangle = \tau(u) |0\rangle,$$

where $\tau(u) = \frac{1}{4} \lambda^2(u) - \frac{1}{2} \lambda'(u)$ are hold.
We fix notation
\[ F(w) = F(w_1)F(w_2) \ldots F(w_n), \]
\[ F(w - w_r) = F(w_1) \ldots F(w_{r-1})F(w_{r+1}) \ldots F(w_n), \]
\[ F(w + u) = F(u)F(w_1) \ldots F(w_n) \]
and we can try to obtain, in accordance with the Bethe Ansatz method, further eigenvalues \(|w⟩ = F(w)|0⟩\).

Direct calculation gives
\[ [T(u), F(w)] = -\sum_{r=1}^{n} \frac{F(w)}{u - w_r} \left( H(u) - \sum_{s \neq r} \frac{1}{u - w_s} \right) + \sum_{r=1}^{n} \frac{F(w + u - w_r)}{u - w_r} \left( H(w_r) - \sum_{s \neq r} \frac{2}{w_r - w_s} \right). \]

Applying this equation to the height vector \(|0⟩\) we obtain
\[ T(u) |w⟩ = T_0(u) |w⟩ + T_1(u) |w⟩, \tag{6} \]
where
\[ T_0(u) |w⟩ = \tau(u) |w⟩ - \sum_{r=1}^{n} \left( \lambda(u) - \sum_{s \neq r} \frac{1}{u - w_s} \right) |w⟩, \]
\[ T_1(u) |w⟩ = \sum_{r=1}^{n} \left( \lambda(w_r) - \sum_{s \neq r} \frac{2}{w_r - w_s} \right) (w + u - w_r) |w⟩. \]

It is evident that \(|w⟩\) is the eigenvector \(T(u)\) for all \(u\) iff
\[ T_1(u) |w⟩ = 0 \quad \text{and} \quad T_0(u) |w⟩ = \tau(u; w) |w⟩. \]

The first equation is equivalent to the Bethe equations
\[ \lambda(w_r) - \sum_{s \neq r} \frac{2}{w_r - w_s} = 0 \quad \text{for all} \quad r = 1, \ldots, n \tag{7} \]

and the second condition gives corresponding eigenvalue
\[ \tau(u; w) = \tau(u) - \sum_{r=1}^{n} \frac{\lambda(u)}{u - w_r} + \sum_{r \neq s} \frac{2}{(u - w_r)(w_r - w_s)}. \tag{8} \]

3 The Gaudin model for \(sl(3)\)

The \(sl(3)\) Gaudin model was studied by many authors [19–21] from different points of view. We will concentrate again on finding eigenvectors and eigenvalues. We use the method analogous to that in section 2 for the case \(sl(2)\). In this chapter, we formulate the main result of our paper.

We will start with a basis in \(gl(3)\), \(e_{ij}\), \(i, j = 1, 2, 3\), where
\[ [e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}. \]
The standard basis in sl(3) is then given by
\[ e_1 = e_{12}, \quad e_2 = e_{23}, \quad e_3 = e_{13}, \quad f_1 = e_{21}, \quad f_2 = e_{32}, \quad f_3 = e_{31}, \]
\[ h_1 = [e_1, f_1] = e_{11} - e_{22} \quad \text{and} \quad h_2 = [e_2, f_2] = e_{22} - e_{33}. \]
The second order Casimir operator is defined by
\[ C_2 = e_1 f_1 + e_2 f_2 + e_3 f_3 + f_1 e_1 + f_2 e_2 + f_3 e_3 + \frac{2}{3} (h_1^2 + h_1 h_2 + h_2^2). \]
So the Gaudin Hamiltonian is defined as
\[ I(u) = \frac{1}{2} (F_1(u) E_1(u) + F_2(u) E_2(u) + F_3(u) E_3(u) +
+ E_1(u) F_1(u) + E_2(u) F_2(u) + E_3(u) F_3(u)) +
+ \frac{1}{3} (H_1^2(u) + H_1(u) H_2(u) + H_2^2(u)), \]
where
\[ X(u) = \sum_{i=1}^{N} \frac{x^{(i)}}{u - z_i}. \]
Direct calculation gives for \( u \neq w \) the relation
\[ [X(u), Y(w)] = -\frac{Z(u) - Z(w)}{u - w} \quad \text{iff} \quad [x, y] = z. \]
To fix the notation we write
\[ F_j(w_j) = F_j(w_{1,1}) F_j(w_{1,2}) \ldots F_j(w_{1,k_j}), \quad F(w_1, w_2, w_3) = F_1(w_1) F_2(w_2) F_3(w_3) \]
and
\[ |w_1, w_2, w_3\rangle = F(w_1, w_2, w_3) |0\rangle. \]
The main idea is to use these vectors for construction of eigenvectors of \( I(u) \). Any long but straightforward calculations give
\[ I(u) |w_1, w_2, w_3\rangle = \left( I_1(u) + I_2(u) + I_3(u) + I_0(u) \right) |w_1, w_2, w_3\rangle, \]
where
\[ I_1(u) = I^0_1(u) + I^-(u), \quad I_2(u) = I^0_2(u) + I^-_2(u) + I^+_2(u), \]
\[ I_3(u) = I^0_3(u) + I^+_3(u), \quad I_0(u) = I^0_0(u) + I^+_0(u) \]
and
\[ I^0_1(u) |w_1, w_2, w_3\rangle = \sum_r \left( \lambda_1(w_{1,r}) - \sum_{\bar{r} \neq r} \frac{2}{w_{1,r} - w_{1,\bar{r}}} + \sum_s \frac{1}{w_{1,r} - w_{2,s}} - \sum_t \frac{1}{w_{1,r} - w_{3,t}} \right) \times
\[ \times \frac{|w_1 + u - w_{1,r}, w_2, w_3\rangle}{u - w_{1,r}} \right) \times
- \sum_{r,s} \frac{|w_1 + u - w_{1,r}, w_2 + w_{1,r} - w_{2,s}, w_3\rangle}{(u - w_{1,r})(w_{1,r} - w_{2,s})} \right) \times
- \sum_{r,t} \frac{|w_1 + u - w_{1,r}, w_2, w_3 + w_{1,r} - w_{3,t}\rangle}{(u - w_{3,t})(w_{3,t} - w_{1,r})} \right) \times
\[ I^-(u) |w_1, w_2, w_3\rangle = -\sum_t \frac{|w_1 + u, w_2 + w_{3,t}, w_3 - w_{3,t}\rangle}{u - w_{3,t}} \]
\[ I_2^0(u) | w_1, w_2, w_3 \] = \sum_s \left( \lambda_2(w_{2,s}) + \sum_r \frac{1}{u - w_{1,r}} - \sum_{s \neq s} \frac{2}{w_{2,s} - w_{2,s}} - \sum_t \frac{1}{u - w_{3,t}} \right) \times \frac{|w_1, w_2 + u - w_{2,s}, w_3|}{u - w_{2,s}}

\[ I_2^-(u) | w_1, w_2, w_3 \] = \sum_t \frac{|w_1 + w_{3,t}, w_2 + u, w_3 - w_{3,t}|}{u - w_{3,t}}

\[ I_2^+(u) | w_1, w_2, w_3 \] = -\sum_{r, s \neq \hat{s}} \frac{|w_1 - w_{1,r}, w_2 + u - w_{2,s} - w_{2,s}, w_3 + w_{1,r}|}{(u - w_{1,r})(u - w_{2,s})(u - w_{2,s})}

\[ I_3^0(u) | w_1, w_2, w_3 \] = \sum_t \left( \lambda_3(w_{3,t}) - \sum_r \frac{1}{w_{3,t} - w_{1,r}} - \sum_s \frac{1}{w_{3,t} - w_{2,s}} - \sum_{t \neq t} \frac{2}{w_{3,t} - w_{3,t}} \right) \times \frac{|w_1, w_2, w_3 + u - w_{3,t}|}{u - w_{3,t}} - \sum_{r, t} \frac{|w_1 + w_{3,t} - w_{1,r}, w_2, w_3 + u - w_{3,t}|}{(u - w_{1,r})(w_{1,r} - w_{3,t})} + \sum_{s, t} \frac{|w_1, w_2 + w_{3,t} - w_{2,s}, w_3 + u - w_{3,t}|}{(u - w_{3,t})(w_{3,t} - w_{2,s})}

\[ I_3^+(u) | w_1, w_2, w_3 \] = \sum_{r, s} \left( \lambda_2(w_{1,r}) - \lambda_2(w_{2,s}) \right) \frac{|w_1 - w_{1,r}, w_2 - w_{2,s}, w_3 + u|}{(u - w_{1,r})(w_{1,r} - w_{2,s})} - 2 \sum_{r, s \neq \hat{s}} \frac{|w_1 - w_{1,r}, w_2 - w_{2,s}, w_3 + u|}{(u - w_{1,r})(w_{1,r} - w_{2,s})(w_{2,s} - w_{2,s})} + \sum_{r, s \neq \hat{s}} \frac{|w_1 - w_{1,r}, w_2 + w_{1,r} - w_{2,s} - w_{2,s}, w_3 + u|}{(u - w_{1,r})(w_{1,r} - w_{2,s})(w_{1,r} - w_{2,s})}

\[ I_0^0(u) | w_1, w_2, w_3 \] = \tau(u) \bigg| w_1, w_2, w_3 \bigg. - \sum_r \left( \lambda_1(u) - \sum_{r \neq r} \frac{2}{w_{1,r} - w_{1,r}} + \sum_s \frac{1}{w_{1,r} - w_{2,s}} - \sum_t \frac{1}{w_{1,r} - w_{3,t}} \right) \times \frac{|w_1, w_2, w_3|}{u - w_{1,r}} - \sum_s \left( \lambda_2(u) + \sum_r \frac{1}{w_{2,s} - w_{1,r}} - \sum_{s \neq s} \frac{2}{w_{2,s} - w_{2,s}} - \sum_t \frac{1}{w_{2,s} - w_{3,t}} \right) \times \frac{|w_1, w_2, w_3|}{u - w_{2,s}} - \sum_t \left( \lambda_3(u) - \sum_r \frac{1}{w_{3,t} - w_{1,r}} - \sum_s \frac{1}{w_{3,t} - w_{2,s}} - \sum_{t \neq t} \frac{2}{w_{3,t} - w_{3,t}} \right) \times \frac{|w_1, w_2, w_3|}{u - w_{3,t}} + \sum_{r, t} \frac{|w_1 + w_{3,t} - w_{1,r}, w_2, w_3 + w_{1,r} - w_{3,t}|}{(u - w_{1,r})(u - w_{3,t})}
In these formulas we use for simplicity $\lambda_3(u) = \lambda_1(u) + \lambda_2(u)$.

Now we are able to construct the eigenvectors and calculate the eigenvalues. We define a linear operator $P$ by

$$P \mid 0, w_2, w_3 \rangle = P \mid w_1, 0, w_3 \rangle = 0,$$

$$P \mid w_1, w_2, w_3 \rangle = \sum_{r,s} \frac{\mid w_1 - w_{1,r}, w_2 - w_{2,s}, w_3 + w_{1,r} \rangle}{w_{2,s} - w_{1,r}}$$

and a vector

$$\mid w_1, w_2 \rangle = \sum_{n=0}^{\infty} \frac{P^n}{n!} \mid w_1, w_2, 0 \rangle$$

for any $w_1$ and $w_2$. We show by the fulfilment of Bethe Ansatz equations for $\lambda_1(w_{1,r})$ and $\lambda_2(w_{2,s})$ that $\mid w_1, w_2 \rangle$ is the eigenvector of $I(u)$ for any $u$. So we can formulate our main theorem.

**Theorem.**

*If the Bethe Ansatz conditions: for any $r$*

$$\lambda_1(w_{1,r}) = \sum_{\tilde{r} \neq r} \frac{2}{w_{1,r} - w_{1,\tilde{r}}} + \sum_{s} \frac{1}{w_{1,r} - w_{2,s}} = 0 \quad (9)$$

*and for any $s$*

$$\lambda_2(w_{2,s}) + \sum_{r} \frac{1}{w_{2,s} - w_{1,r}} - \sum_{\tilde{s} \neq s} \frac{2}{w_{2,s} - w_{2,\tilde{s}}} = 0 \quad (10)$$

*are fulfilled then the vector*

$$\mid w_1, w_2 \rangle = \sum_{n=0}^{\infty} \frac{P^n}{n!} \mid w_1, w_2, 0 \rangle$$

*is the eigenvector of the sl(3) Gaudin model and*

$$I(u) \mid w_1, w_2 \rangle = \tau(u; w_1, w_2) \mid w_1, w_2 \rangle,$$

*where*

$$\tau(u; w_1, w_2) = \tau(u) - \sum_{r} \left( \lambda_1(u) - \sum_{\tilde{r} \neq r} \frac{2}{w_{1,r} - w_{1,\tilde{r}}} + \sum_{s} \frac{1}{w_{1,r} - w_{2,s}} \right) \frac{1}{u - w_{1,r}} - \sum_{s} \left( \lambda_2(u) + \sum_{r} \frac{1}{w_{2,s} - w_{1,r}} - \sum_{\tilde{s} \neq s} \frac{2}{w_{2,s} - w_{2,\tilde{s}}} \right) \frac{1}{u - w_{2,s}} \quad (11)$$

**PROOF:** For $\mid w_1, w_2 \rangle$ to be the eigenvector of $I(u)$ with eigenvalue $\tau(u; w_1, w_2)$, there should be

$$I_1(u) \mid w_1, w_2 \rangle = I_2(u) \mid w_1, w_2 \rangle = I_3(u) \mid w_1, w_2 \rangle = 0 \quad (12)$$
and
\[ I_0(u \mid w_1, w_2) = \tau(u; w_1, w_2) \mid w_1, w_2). \] 
(13)

In the following Lemma 1 we show \( I_1(u) \mid w_1, w_2 \) = 0 in detail. Since the proof of the relations \( I_2(u) \mid w_1, w_2 = I_3(u) \mid w_1, w_2 = 0 \) is very similar, we will skip it. The proof of \( I_0(u) \mid w_1, w_2 \) = \( \tau(u; w_1, w_2) \mid w_1, w_2 \) will be given in Lemma 2.

**Lemma 1.**

If (9) is valid, then \( I_1(u) \mid w_1, w_2 = 0 \).

**Proof:** It is easy to see that
\[ \left( \frac{1}{n+1} I_1^-(u) P^{n+1} + I_1^0(u) P^n \right) \mid w_1, w_2 = 0 \]
valid.

For \( n = 0 \) we have
\[
\begin{aligned}
I_1^0(u) \mid w_1, w_2, 0 &= \\
&= \sum_{r_1} \left( \lambda_1(w_{1,r_1}) - \sum_{r_2 \neq r_1} \frac{2}{w_{1,r_1} - w_{1,r_2}} + \sum_{s_1} \frac{1}{w_{1,r_1} - w_{2,s_1}} \right) \frac{w_1 + u - w_{1,r_1} - w_{2,0}}{u - w_{1,r_1}} - \\
&- \sum_{r_1, s_1} \frac{w_1 + u - w_{1,r_1} - w_{2,s_1}}{(u - w_{1,r_1})(w_{2,r_1} - w_{2,s_1})}.
\end{aligned}
\]

For
\[
I_1^- (u) P \mid w_1, w_2, 0 = - \sum_{r_1, s_1} \left( \frac{w_1 + u - w_{1,r_1} - w_{2,s_1}}{u - w_{1,r_1}} \right) \frac{w_2 + w_{1,r_1} - w_{2,s_1}}{w_{2,s_1} - w_{1,r_1}}.
\]

For
\[
\left( I_1^- (u) P + I_1^0 (u) \right) \mid w_1, w_2 = 0
\]
to be valid we obtain condition (9) for any \( r \)
\[
\lambda_1(w_{1,r}) - \sum_{ \tilde{r} \neq r } \frac{2}{w_{1,r} - w_{1,\tilde{r}}} + \sum_s \frac{1}{w_{1,r} - w_{2,s}} = 0.
\]

Here is the origin of the first Bethe Ansatz equation. The second Bethe Ansatz equation arises in the same manner from the equation \( I_2(u) \mid w_1, w_2 = 0 \).

In order to simplify the notation, we denote by \( R_n \) the ordered set of numbers \( r_1, \ldots, r_n \) where \( r_i \neq r_k \) and similarly \( S_n \). So we can write
\[
P^n \mid w_1, w_2, 0 = \sum_{R_n, S_n} \frac{w_1 - w_{1,r_1} - \ldots - w_{1,r_n}, w_2 - w_{2,s_1} - \ldots - w_{2,s_n}, w_{1,r_1} + \ldots + w_{1,r_n}}{(w_{2,s_1} - w_{1,r_1}) \ldots (w_{2,s_n} - w_{1,r_n})},
\]
where the summation is over all sets \( R_n \) and \( S_n \). We will abbreviate it as
\[
P^n \mid w_1, w_2, 0 = \sum_{R_n, S_n} \frac{w_1 - w_{1,R_n}, w_2 - w_{2,S_n}, w_{1,R_n}}{W_n}, \text{ where } W_n = \prod_{k=1}^{n} (w_{2,s_k} - w_{1,r_k}).
\]
So we have

\[ I_1(w)P^n | w_1, w_2, 0 = \]

\[ = \sum_{R_n, S_n} \sum_{r_{n+1} \notin R_n} \left( \lambda_1(w_{1,r_{n+1}}) - \sum_{r_{n+1} \notin R_n} w_{1,r_{n+1}} - w_{1,r_{n+2}} + \sum_{s_{n+1} \notin S_n} w_{1,r_{n+1}} - w_{2,s_{n+1}} - \right. \]

\[ - \sum_{k=1}^{n} \left( w_{1,r_{n+1}} - w_{1,r_k} \right) \right) \left( \frac{1}{u - w_{1,r_{n+1}}} \right) \left( w_{1} + w_{1,R_n} - w_{1,r_{n+1}}, w_{2} - w_{2,S_n}, w_{1,R_n} \right) \]

\[ - \sum_{R_n, S_n} \sum_{r_{n+1} \notin R_n} \left( w_{1} + u - w_{1,R_n} - w_{1,r_{n+1}}, w_{2} + w_{1,r_{n+1}} - w_{2,S_n} - w_{2,s_{n+1}}, w_{1,R_n} \right) \]

\[ - \sum_{R_n, S_n} \sum_{r_{n+1} \notin R_n} \sum_{s_{n+1} \notin S_n}^{n} \left( \right) \left( u - w_{1,r_{n+1}} \right) \left( w_{1} + u - w_{1,R_n} - w_{1,r_{n+1}}, w_{2} - w_{2,S_n}, w_{1,R_n} - w_{1,r_k} + w_{1,r_{n+1}} \right) \]

If we substitute \( \lambda_1(w_{1,r_{n+1}}) \) from (9) into the above formula, we can adjust it to the form

\[ I_1(w)P^n | w_1, w_2, 0 = \]

\[ = \sum_{R_n, S_n} \sum_{r_{n+1} \notin R_n} \left( \frac{1}{u - w_{1,r_{n+1}}} \right) \left( w_{1} + u - w_{1,R_n} - w_{1,r_{n+1}}, w_{2} - w_{2,S_n}, w_{1,R_n} \right) \]

\[ - \sum_{R_n, S_n} \sum_{r_{n+1} \notin R_n} \sum_{s_{n+1} \notin S_n}^{n} \left( \right) \left( u - w_{1,r_{n+1}} \right) \left( w_{1} + u - w_{1,R_n} - w_{1,r_{n+1}}, w_{2} - w_{2,S_n}, w_{1,R_n} - w_{1,r_k} + w_{1,r_{n+1}} \right) \]

Since in the third element we sum over all possible combinations \( n + 1 \) of the different elements \( r_i \), we can change in the sum \( r_k \) by \( r_{n+1} \) and write the third element as

\[ = \sum_{R_{n+1}, S_{n+1}} \sum_{k=1}^{n} \left( \frac{(w_{2,s_k} - w_{1,r_k})}{u - w_{1,r_{n+1}}} \right) \left( w_{1} + u - w_{1,R_{n+1}}, w_{2} - w_{2,S_n}, w_{1,R_{n+1}} - w_{1,r_k} \right) \]

which cancels the first element. So we obtain

\[ I_1(w)P^n | w_1, w_2, 0 = \]

\[ = \sum_{R_{n+1}, S_{n+1}} \left( w_{1} + u - w_{1,R_{n+1}}, w_{2} + w_{1,r_{n+1}} - w_{2,S_n}, w_{1,R_{n+1}} - w_{1,r_k} \right) \]

On the other hand, we have

\[ I_1(w)P^{n+1} | w_1, w_2, 0 = \]

\[ = - \sum_{R_{n+1}, S_{n+1}} \sum_{k=1}^{n+1} \left( w_{1} + u - w_{1,R_{n+1}}, w_{2} + w_{1,r_{n+1}} - w_{2,S_n}, w_{1,R_{n+1}} - w_{1,r_k} \right) \]

\[ \left( u - w_{1,r_k} \right) W_{n+1} \]
Again we sum over all possible combinations $n + 1$ of the different elements $r_i$, we can change in the sum $r_k$ by $r_{n+1}$ and we obtain

$$I_1^\top(u)P^{n+1} | w_1, w_2, 0) =$$

$$= - \sum_{R_{n+1}, S_{n+1}} \sum_{k=1}^{n+1} \frac{w_1 + u - w_1, r_{n+1}, w_2 + w_1, r_{n+1} - w_2, s_{n+1}, w_1, R_{n+1} - w_1, r_{n+1}}{(u - w_1, r_{n+1})W_{n+1}} =$$

$$= -(n + 1) \sum_{R_{n+1}, S_{n+1}} \frac{|w_1 + u - w_1, r_{n+1}, w_2 + w_1, r_{n+1} - w_2, s_{n+1}, w_1, R_n|}{(u - w_1, r_{n+1})W_{n+1}} =$$

$$= -(n + 1)I_1^\top(u)P^n | w_1, w_2, 0) .$$

**Lemma 2.**

If (2) and (11) is valid, then $I_0^\top(u) | w_1, w_2) = \tau(u; w_1, w_2) | w_1, w_2).$

**PROOF:** It is easy to see that

$$I_0^\top(u) | w_1, w_2, 0) = \tau(u; w_1, w_2) | w_1, w_2),$$

where $\tau(u; w_1, w_2)$ is given in (11).

Now we would like to prove for $n \geq 1$ that

$$\left(\frac{1}{n!} (I_0^\top(u) - \tau(u; w_1, w_2))P^n + \frac{1}{(n-1)!} I_1^\top(u)P^{n-1}) | w_1, w_2, 0) = 0 .$$

If we start with the above expressions we obtain

$$I_0^\top(u)P^{n-1} | w_1, w_2, 0) = - \sum_{R_n, S_n} \left(2 \lambda_2 - \lambda_2(w_{2,s_1})\right)\frac{|w_1 - w_1, R_n, w_2 - w_2, s_1, w_1, R_n|}{(u - w_1, r_n)(u - w_2, s_1)W_{n-1}} +$$

$$+ 2 \sum_{R_n, S_n+1} \frac{|w_1 - w_1, R_n, w_2 - w_2, s_1, w_1, R_n|}{(u - w_1, r_n)(u - w_2, s_1)(w_{2,s_1+1} - w_2, n)W_{n-1}}$$

$$I_0^\top(u)P^n | w_1, w_2, 0) = \tau(u) \sum_{R_n, S_n} \frac{|w_1 - w_1, R_n, w_2 - w_2, s_1, w_1, R_n|}{W_n} -$$

$$- \sum_{R_{n+1}, S_{n+1}} \left(\lambda_1(u) - \sum_{r_{n+2} \notin R_{n+1}} \frac{1}{w_1, r_{n+2} - w_1, r_{n+1}} + \sum_{s_{n+1} \notin S_{n+1}} \frac{1}{w_1, r_{n+1} - w_{2,s_n+1}} + \sum_{k=1}^{n} \frac{1}{w_1, r_{n+1} - w_1, r_k} \right)\frac{|w_1 - w_1, R_n, w_2 - w_2, s_1, w_1, R_n|}{(u - w_1, r_n)W_n} -$$

$$- \sum_{R_{n+1}, S_{n+1}} \left(\lambda_2(u) + \sum_{r_{n+2} \notin R_{n+1}} \frac{1}{w_2, s_1 - w_1, r_{n+1}} - \sum_{s_{n+2} \notin S_{n+1}} \frac{1}{w_2, s_1 - w_2, s_2, s_n + 1} - \sum_{k=1}^{n} \frac{1}{w_2, s_1 - w_1, r_k} \right)\frac{|w_1 - w_1, R_n, w_2 - w_2, s_1, w_1, R_n|}{(u - w_2, s_n)W_n} -$$

$$- \sum_{R_n, S_n} \sum_{k=1}^{n} \left(\lambda_3(u) - \sum_{r_{n+2} \notin R_n} \frac{1}{w_1, r_{n+2} - w_1, r_{n+1}} - \sum_{s_{n+1} \notin S_n} \frac{1}{w_1, r_{n+1} - w_2, s_n} - \sum_{l \neq k} \frac{2}{w_{1, r_{n+1} - w_1, r_k}} \right)\frac{|w_1 - w_1, R_n, w_2 - w_2, s_n, w_1, R_n|}{(u - w_1, r_{n+1})W_n} +$$

$$+ \sum_{R_{n+1}, S_{n}} \sum_{k=1}^{n} \frac{|w_1 - w_1, r_{n+1} + w_1, r_k, w_2 - w_2, s_n, w_1, r_{n+1} - w_1, r_k|}{(u - w_1, r_{n+1})(u - w_1, r_k)W_n}$$
First we arrange the second, third and fourth elements in the expression for $I_0^n(u|P_0^n | w_1, w_2, 0)$ and then change $r_k$ by $r_{k+1}$. So we obtain

$$I_0^n(u|P_0^n | w_1, w_2, 0) = \tau(u) \sum_{S_n, r_n} \left[ \sum_{R_{n+1}, S_n} \left( \frac{1}{w_{1,r_{n+1}} - w_{1,r_k}} - \frac{1}{w_{1,r_{n+1}} - w_{2,s_k}} \right) \frac{w_{1,r_{n+1}} - w_{2,s_k}}{w_{2,s_{n+1}} - w_{2,s_k}} \right] \times$$

$$\times \left[ \sum_{k=1}^{n} \left( \frac{1}{w_{1,r_{n+1}} - w_{2,s_k}} \right) \left( u - w_{1,r_{n+1}} \right) \left( u - w_{2,s_k} \right) \right] \right|_{S_n} \right.$$
Hence, we get
\[
\left( I_0^0(u) - \tau(u; w_1, w_2) \right) P^n | w_1, w_2, 0 = \\
= \sum_{R_n, S_n} \sum_{k=1}^n \left( \lambda_1(u) - \sum_{r_{n+1} \neq r_k} \frac{2}{w_{1,r_k} - w_{1,r_{n+1}}} + \sum_{s_{n+1} \neq s_k} \frac{1}{w_{1,r_k} - w_{2,s_{n+1}}} \right) \times \\
\times \left( u - w_{1,r_k} \right) W_n \right) \\
+ \sum_{R_n, S_n} \sum_{k=1}^n \left( \lambda_2(u) + \sum_{r_{n+1} \neq r_k} \frac{1}{w_{2,s_k} - w_{1,r_k}} - \sum_{s_{n+1} \neq s_k} \frac{2}{w_{2,s_k} - w_{2,s_{n+1}}} \right) \times \\
\times \left( u - w_{2,s_k} \right) W_n \right) \\
- \sum_{R_n, S_n} \sum_{k=1}^n \left( \lambda_3(u) - \sum_{r_{n+1} \neq r_k} \frac{1}{w_{1,r_k} - w_{1,r_{n+1}}} - \sum_{s_{n+1} \neq s_k} \frac{1}{w_{1,r_k} - w_{2,s_{n+1}}} \right) \times \\
\times \left( u - w_{1,r_k} \right) W_n \right) \\
- \sum_{R_n, S_n} \sum_{k, \ell=1}^{n-1} \left( u - w_{1,r_k} \right) \left( u - w_{1,r_{n+1}} \right) \left( u - w_{1,r_\ell} \right) W_{n-1} \right) \\
- \sum_{R_n, S_n} \sum_{r_{n+1} \neq r_k} \left( u - w_{2,s_k} \right) \left( u - w_{2,s_{n+1}} \right) W_{n-1} \right)
\]

In the same way after reordering we can write
\[
\frac{1}{n} \left( I_0^0(u) - \tau(u; w_1, w_2) \right) P^n | w_1, w_2, 0 = \\
= \sum_{R_n, S_n} \left( \lambda_1(u) - \sum_{r_{n+1} \neq r_k} \frac{2}{w_{1,r_k} - w_{1,r_{n+1}}} + \sum_{s_{n+1} \neq s_k} \frac{1}{w_{1,r_k} - w_{2,s_{n+1}}} \right) \times \\
\times \left( u - w_{1,r_k} \right) W_n \right) \\
+ \sum_{R_n, S_n} \left( \lambda_2(u) + \sum_{r_{n+1} \neq r_k} \frac{1}{w_{2,s_n} - w_{1,r_k}} - \sum_{s_{n+1} \neq s_k} \frac{2}{w_{2,s_n} - w_{2,s_{n+1}}} \right) \times \\
\times \left( u - w_{2,s_k} \right) W_n \right) \\
- \sum_{R_n, S_n} \left( \lambda_3(u) - \sum_{r_{n+1} \neq r_k} \frac{1}{w_{1,r_k} - w_{1,r_{n+1}}} - \sum_{s_{n+1} \neq s_k} \frac{1}{w_{1,r_k} - w_{2,s_{n+1}}} \right) \times \\
\times \left( u - w_{1,r_k} \right) W_n \right) \\
+ \sum_{R_n, S_n} \sum_{r_{n+1} \neq r_k} \left( u - w_{1,r_k} \right) \left( u - w_{1,r_{n+1}} \right) \left( u - w_{1,r_\ell} \right) W_{n-1} \right) \\
- \sum_{R_n, S_n} \sum_{r_{n+1} \neq r_k} \left( u - w_{2,s_k} \right) \left( u - w_{2,s_{n+1}} \right) W_{n-1} \right)
\]
If we now use the fact $\lambda_3(u) = \lambda_1(u) + \lambda_2(u)$, we get after an arrangement
\[
\left(\frac{1}{n} (I_0^n(u) - \tau(u; w_1, w_2)) P^n + I_0^+(u) P^{n-1} \right) | w_1, w_2, 0) =
\]
\[
= - \sum_{R_n, S_n} \left( \lambda_2(w_{2,n}) + \sum_{r_{n+1} \in R_n} \frac{2}{w_{1,r_n} - w_{1,r_{n+1}}} - \sum_{s_{n+1} \in S_n} \frac{1}{w_{1,r_n} - w_{2,s_{n+1}}} \right) \times
\]
\[
\left| \frac{w_1 - w_{1,R_n}, w_2 - w_{2,S_n}, w_{1,R_n}}{(u - w_{1,r_n}) W_n} + \right|
\]
\[
+ \sum_{R_n, S_n} \left( \lambda_2(w_{2,n}) + \sum_{r_{n+1} \in R_n} \frac{2}{w_{1,r_n} - w_{1,r_{n+1}}} - \sum_{s_{n+1} \in S_n} \frac{1}{w_{2,s_n} - w_{2,s_{n+1}}} \right) \times
\]
\[
\times \left| \frac{w_1 - w_{1,R_n}, w_2 - w_{2,S_n}, w_{1,R_n}}{(u - w_{2,s_n}) W_n} + \right|
\]
\[
- \sum_{R_n, S_n} \left( \sum_{r_{n+1} \in R_n} \frac{1}{w_{1,r_n} - w_{1,r_{n+1}}} + \sum_{s_{n+1} \in S_n} \frac{1}{w_{1,r_n} - w_{2,s_{n+1}}} \right) \times
\]
\[
\times \left| \frac{w_1 - w_{1,R_n}, w_2 - w_{2,S_n}, w_{1,R_n}}{(u - w_{1,r_n}) W_n} + \right|
\]
\[
+ \sum_{R_n, S_n} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) - \sum_{R_n, S_n} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) W_{n-1}
\]
\[
+ 2 \sum_{R_n, S_n} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) - \sum_{R_n, S_n} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) W_{n-1}
\]
\[
\left. \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) W_{n-1} \right)
\]
We now introduce $\lambda_2(w_{2,n})$ from [10] in the above expression and obtain
\[
\left(\frac{1}{n} (I_0^n(u) - \tau(u; w_1, w_2)) P^n + I_0^+(u) P^{n-1} \right) | w_1, w_2, 0) =
\]
\[
= - \sum_{R_n, S_n} \left[ \sum_{r_{n+1} \in R_n} \left( \frac{1}{w_{1,r_n} - w_{1,r_{n+1}}} - \frac{2}{w_{2,s_n} - w_{1,r_{n+1}}} \right) \right] +
\]
\[
+ \sum_{s_{n+1} \in S_n} \left( \frac{1}{w_{2,s_n} - w_{2,s_{n+1}}} - \frac{2}{w_{1,r_n} - w_{2,s_{n+1}}} \right) \times \left| \frac{w_1 - w_{1,R_n}, w_2 - w_{2,S_n}, w_{1,R_n}}{(u - w_{1,r_n}) W_n} + \right|
\]
\[
+ \sum_{R_n+1, S_n} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) W_{n-1}
\]
\[
+ 2 \sum_{R_n, S_n+1} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) W_{n-1}
\]
\[
\left. \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) W_{n-1} \right)
\]
\[
- \sum_{R_n, S_n} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) W_{n-1}
\]
\[
+ \sum_{R_n+1, S_n} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) \left( w_{2,s_n} - w_{1,r_{n+1}} \right) W_{n-1}
\]
\[
+ 2 \sum_{R_n, S_n+1} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) \left( w_{2,s_n} - w_{1,r_{n+1}} \right) W_{n-1}
\]
\[
\left. \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) \left( w_{2,s_n} - w_{2,s_{n+1}} \right) W_{n-1} \right)
\]
\[
- \sum_{R_n, S_n} \left( u - w_{1,r_n} \right) \left( w_{1,r_n} - w_{2,s_{n+1}} \right) \left( w_{1,r_n} - w_{1,r_{n+1}} \right) W_{n-1}
\]
It can be rewritten in the form

\[
\left( \frac{1}{n} (I_0^0(u) - \tau(u; w_1, w_2)) P^n + I_0^+(u) P^{n-1} \right) | w_1, w_2, 0 \rangle = 0.
\]

Now we see that the elements in the bracket are antisymmetric under the change of \( s_k \) by \( s_n \) and the second term is symmetric. So we have

\[
\left( \frac{1}{n} (I_0^0(u) - \tau(u; w_1, w_2)) P^n + I_0^+(u) P^{n-1} \right) | w_1, w_2, 0 \rangle = 0.
\]

4 Concluding remarks and open problems

In the present paper we have proposed the new formula for the eigenvectors of the Gaudin model obtained by using the Bethe ansatz method in the sl(3) case.

In the the sl(3) case we can use the formula (2) for these eigenvectors from the paper [7]. The first interesting problem is to find an explicit connection. We were able to reduce one to the other only in some simple examples such as:

\[
| w_{1,1}, w_{2,1} \rangle = \left( F_1(w_{1,1})F_2(w_{2,1}) + \frac{F_3(w_{1,1})}{w_{2,1} - w_{1,1}} \right) | 0 \rangle
\]

and

\[
| w_{1,1}, w_{1,2}, w_{2,1}, w_{2,2} \rangle = \left( F_1(w_{1,1})F_1(w_{1,2})F_2(w_{2,1})F_2(w_{2,2}) + F_1(w_{1,1})F_2(w_{2,1})F_3(w_{1,2}) + \frac{F_1(w_{1,1})F_3(w_{1,1})}{w_{2,2} - w_{1,2}} + \frac{F_1(w_{1,1})F_2(w_{2,2})F_3(w_{1,2})}{w_{2,2} - w_{1,2}} + \frac{F_1(w_{1,1})F_3(w_{1,1})F_3(w_{1,2})}{w_{2,2} - w_{1,2}} + \frac{F_1(w_{1,1})F_2(w_{2,2})F_3(w_{1,2})}{w_{2,2} - w_{1,2}} + \frac{F_1(w_{1,1})F_3(w_{1,1})F_3(w_{1,2})}{w_{2,2} - w_{1,2}} \right) | 0 \rangle
\]

but we believe that it is possible generally. We studied the case of the algebra sl(3) explicitly. We believe that similar formulas are possible for the general semisimple Lie algebra. Some calculation for the \( B_2 \) algebra is in progress. So the second open problem is to generalize our method to other Lie algebras.

All proofs in the presented paper are direct calculations. So the last problem is to find some indirect proof which can be useful in the general case.

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