In thermal amorphous systems, the first peak of the pair correlation function \( g(r) \) shows a maximum height \( g_1^{\text{max}} \) at a volume fraction \( \phi = \phi_c \) that increases with the temperature. \( g_1^{\text{max}} \) diverges at the \( T = 0 \) jamming transition at \( \phi = \phi_c \). Molecular dynamics simulations show that some typical quantities, such as the pressure, bulk modulus, shear modulus, and boson peak frequency that behave power law scalings with \( \phi - \phi_c \) in marginally jammed solids at \( T = 0 \), all show scalings with \( \phi - \phi_c \) when \( \phi > \phi_c \), while the scalings break down when \( \phi < \phi_c \). The presence of \( g_1^{\text{max}} \) is thus not only a thermal vestige of the \( T = 0 \) jamming transition, but more importantly the structural signature of the jamming transition.
respectively. Particles $i$ and $j$ interact via a short range repulsive potential: $V(r_{ij}) = \epsilon(1 - r_{ij}/\sigma_{ij})^\alpha/\alpha$ when their separation $r_{ij}$ is less than the sum of their radii $\sigma_{ij} = (\sigma_i + \sigma_j)/2$, and zero otherwise. We use $\alpha = 2$ (harmonic) and $5/2$ (Hertzian) to study the potential dependence. The length, time, and energy are in the units of $\sigma, \sqrt{m\sigma^2/\epsilon}$, and $\epsilon$. The Boltzmann constant $k_B$ is set to unity.

We measure the pair correlation function $g(r)$ for large particles. As shown in Fig. 2(a-b), for both harmonic and Hertzian repulsions, $g_1$, the height of the first peak of $g(r)$, is maximal at $\phi = \phi_c$. In the range of temperatures studied, Fig. 2(a) shows that

$$T \propto (\phi - \phi_c)^\alpha,$$

which is independent on the particle interactions. The dependence of $g_1^{\text{max}}$ on $\phi_c$ is the same as $g_1(\phi)$ in marginally jammed solids at $T = 0$ [10], which hints that $g_1^{\text{max}}$ comes along with a typical material property change presented in jammed solids. The value of $g_1^{\text{max}}$ does not show any apparent system size or history dependence [3].

At $T = 0$, multiple quantities such as the pressure, bulk modulus, shear modulus, boson peak frequency, and coordination number (number of interacting neighbors per particle) show power law scalings with $\phi - \phi_c$ in marginally jammed solids [3, 8]. The pressure $P$ is scaled with $(\phi - \phi_c)^{-1}$ [5]. At $T > 0$, we calculate the pressure from the virial expression $P = (Nk_B T + \sum_{ij} r_{ij} F_{ij})/L^3$, where $F_{ij} = -\nabla V(r_{ij})/\nabla r_{ij}$ is the force between particles $i$ and $j$ and the sum is over all pairs of interacting particles. When the kinetic energy is much lower than the potential energy, $r_{ij}$ fluctuates around its equilibrium $T = 0$ value $r_{ij}^0$. The fluctuations are so small that they only contribute a tiny correction to the $T = 0$ pressure. We thus expect that $P$ is still proportional to $(\phi - \phi_c)^{-1}$ at $T > 0$ if the systems are jammed. Fig. 2(c-d) show that when $\phi > \phi_c$,

$$P = A_P (\phi - \phi_c)^{-1},$$

where $A_P$ is nearly insensitive to the temperature; when $\phi < \phi_c$, Eq. 3 stops working. Accordingly, when $\phi > \phi_c$, there is a temperature independent bulk modulus $B = \phi B_0 / \sigma_c \propto (\phi - \phi_c)^{-2}$, while the bulk modulus varies with both volume fraction and temperature when $\phi < \phi_c$.

At $T = 0$, the shear modulus $G$ is scaled with $(\phi - \phi_c)^{-3/2}$ in marginally jammed solids [5]. The ratio of the moduli $G/B \propto (\phi - \phi_c)^{1/2}$ that decays to zero at Point $J$, while in normal solids like crystals, $G/B$ does not show strong volume fraction dependence. At $T > 0$, we measure the storage shear modulus $G'$ at various frequencies $\omega$ by applying an oscillatory shear strain $\gamma \sin(\omega t)$ in the $x$–direction (shear gradient is in the $y$–direction) and measuring the shear stress $\sigma_{xy} = -\sum_i p_{x,i} \phi y_{i,j}/m + \sum_{ij} x_{ij} F_{x,i}/L^3$, where $p_{x,i}$ and $p_{y,i}$ are the $x$– and $y$–components of the momentum of particle $i$, $x_{ij}$ is the $x$–component of $r_{ij}$, and $F_{y,i,j}$ is the $y$–component of $F_{ij}$. We measure $G'$ at sufficiently small $\gamma$, so that linear response is guaranteed. Again, the ratio of moduli is independent of the potential energy, the shear stress is determined by the particle interactions. In Fig. 2(e-f), we show $G'$ measured over two decades of low frequencies. In the range of frequencies and volume fractions studied, $G'$ does not
and Hertzian (red squares) repulsions, where \( \phi \) value of

\[
\phi > \phi_c \quad \text{and} \quad 5/0 \quad \text{in marginally jammed solids at} \quad T > 0. \quad \text{When} \quad \phi > \phi_c \quad \text{dependence of} \quad A \quad \text{on the frequency} \quad \text{when}\]

\[
G' = A_G(T) (\phi - \phi_c)^{\alpha - 1}, \quad (4)
\]

where \( A_G \) increases with decreasing the temperature. When \( \phi < \phi_c \), Eq. (4) no longer holds. The temperature dependence of \( A_G \) is not surprising. At a fixed volume fraction \( \phi > \phi_c \), if we increase the temperature of a jammed solid approaching unjamming transition, the shear modulus \( G'(T) \) must decrease. \( A_G = G'(T)/(\phi - \phi_c)^{\alpha - 1} \) thus decreases with increasing the temperature.

The scaling of \( G' \) with \( \phi - \phi_c \) is different from that in marginally jammed solids at \( T = 0 \). This discrepancy should arise from the thermal motion of particles around their equilibrium positions. Unlike in crystals in which the shear modulus \( G \) is determined by the response of the inter-particle forces to the strain, e.g. \( G \) is almost constant upon compression for harmonic repulsions \( (\alpha = 2) \), in jammed systems the heterogeneous inter-particle bond orientation and forces are coupled together, which lead to a much more complicated volume fraction dependence of the shear modulus. The thermal motion of particles increases the heterogeneity, so that the volume fraction dependence of the shear modulus at \( T > 0 \) is pushed further away from crystals. The ratio of the moduli \( G/B \propto (\phi - \phi_c) \) shows a stronger volume fraction dependence than that in marginally jammed solids.

The vibrational properties of marginally jammed solids have shown some anomalies. For instance, in marginally jammed solids, there is a plateau in the density of vibrational states \( D(\omega) \) at low frequencies [8]. The plateau extends to lower frequencies approaching the unjamming transition. Excess low-frequency vibrational modes beyond Debye’s prediction aggregate in the vicinity of a characteristic frequency \( \omega^* \) and build up the boson peak [20, 21]. The boson peak frequency \( \omega^* \) is scaled with \( (\phi - \phi_c)^{(\alpha - 1)/2} \) [6, 7, 14]. At \( T > 0 \), the density of vibrational states can be calculated from the Fourier transform of the velocity correlation \( C(t) = (\vec{v}(t) \cdot \vec{v}(0)) \) [22]:

\[
D(\omega) = \frac{1}{3T} \int_0^\infty C(t) \cos(\omega t) dt, \quad (5)
\]

where \( \langle \cdot \rangle \) denotes the average over particles and ensembles. Fig. 4(a) shows the density of vibrational states of systems with harmonic repulsions measured at \( T = 10^{-5} \) and various \( \phi \). When \( \phi > \phi_c \), a plateau is developed in \( D(\omega) \). The plateau shifts to lower frequencies and grows with decreasing \( \phi \). When \( \phi < \phi_c \), the plateau disappears and is replaced with a peak that captures the general feature of the density of states of liquids [22].
The disappearance of the plateau near \( \phi_c \) is a sign of the unjamming transition.

In Fig. (1b) we highlight the boson peak by plotting the reduced density of vibrational states \( D(\omega)/\omega^2 \). The boson peak shifts to lower frequencies, while its magnitude increases, with decreasing \( \phi \). As shown in Fig. (2g-h), \( g_1^{\text{max}} \) signifies the crossover between two regimes of different volume fraction dependence of \( \omega^* \). When \( \phi < \phi_c \),

\[
\omega^* = A_v(T)/(\phi - \phi_c)^{(\alpha - 1)/2},
\]

where \( A_v \) decreases with increasing the temperature. Eq. (6) starts to break down when \( \phi < \phi_c \). The temperature dependence of \( A_v \) is understandable from the fact that the boson peak in glasses shifts to lower frequencies with increasing the temperature \([20, 21]\). At a fixed \( \phi > \phi_c \), \( A_v = \omega^*/(\phi - \phi_c)^{(\alpha - 1)/2} \) thus decreases with increasing the temperature.

In marginally jammed solids, another important quantity that shows the power law scaling with \( \phi - \phi_c \) is the coordination number \( z : z = z_c \propto (\phi - \phi_c)^{1/2} \). At Point J, \( z = z_c = 2d \) is the isostatic value, the minimum constraint requirement to maintain mechanical stability, where \( d \) is the dimension of space. In thermal systems, \( z \) fluctuates in time, which is no longer a valid criterion of mechanical stability. Although the time averaged coordination number drops quickly when \( \phi < \phi_c \), the coordination number does not show any well-defined scalings with \( \phi - \phi_c \).

The scalings shown in Eqs. (1)-(4) and (6) demonstrate that thermal amorphous systems at \( \phi > \phi_c \) are jammed, since similar scalings exist in marginally jammed solids at \( T = 0 \). The temperature dependent crossover volume fraction \( \phi_c \) separates two regimes with apparently different material properties. The occurrence of the maximum height of the first peak of \( g(r) \) is thus a structural signature of the jamming transition in general. For systems studied, the jamming transition in the \( T - \phi \) plane of the jamming phase diagram shown in Fig. (1) is well-defined by Eq. (1), which converges to Point J at \( T = 0 \).

Eqs. (1)-(4) and (6) also highlight the key role of Point J in controlling the jamming transition and material properties of jammed solids. Point J is the only critical point in the \( T - \phi \) plane of the jamming phase diagram, at which diverging length scales are able to exist. During the jamming transition of thermal amorphous systems we do not expect to observe diverging length scales because the critical scalings are all controlled by \( \phi_c \).

For thermal systems as shown in Fig. (2) the relaxation time exceeds the measurable time window and shows rapid divergence at a much lower volume fraction than \( \phi_c \). The jamming transition signified by \( g_1^{\text{max}} \) is thus deeply inside the glass transition at which the system is already a glass. It is ambiguous whether there exists the thermodynamic glass transition and whether the colloidal glass transition in the hard sphere limit happens at Point J. The jamming transition interpreted by the glass transition is thus not well-defined. Our study propose an unambiguous liquid-solid transition of amorphous systems determined by a robust structural signature, which opens a new point of view to understand the phase transition of amorphous systems and inspires new experiments \([22]\).

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