(0,2) Landau-Ginzburg Theory*

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A large class of (0,2) Calabi-Yau $\sigma$-models and Landau-Ginzburg orbifolds are shown to arise as different “phases” of supersymmetric gauge theories. We find a phenomenon in the Landau-Ginzburg phase which may enable one to understand which Calabi-Yau $\sigma$-models evade destabilization by worldsheet instantons. Examples of (0,2) Landau-Ginzburg vacua are analyzed in detail, and several novel features of (0,2) models are discussed. In particular, we find that (0,2) models can have different quantum symmetries from the (2,2) models built on the same Calabi-Yau manifold, and that a new kind of topology change can occur in (0,2) models of string theory.

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1. Introduction

The relationship between Calabi-Yau $\sigma$-models with the gauge connection set equal to the spin connection and $(2,2)$ superconformal field theories has been a rich source of conjectures and insights for physicists (and mathematicians) in the past several years. These models are of particular interest to string theorists because they provide compactifications of the heterotic string which lead to $N = 1$ supersymmetric models of elementary particle physics [1].

In the course of studying these phenomenologically promising string vacua, a wealth of “stringy” phenomena which provide windows into how string theory modifies physics at short distance scales have been uncovered. These string-theoretic modifications of our classical notions of geometry are likely to provide important clues in uncovering a more satisfactory formulation of the theory.

Of course, from the point of view of trying to understand the space of ground states of classical string theory, or even trying to find phenomenologically promising vacua, there is little reason to restrict oneself to the study of $(2,2)$ theories. Indeed, $(0,2)$ Calabi-Yau theories were introduced precisely because they offer improved phenomenological prospects, compared to their $(2,2)$ counterparts [2]. In particular, while $(2,2)$ theories lead to an observable $E_6$ unification group in spacetime, $(0,2)$ theories with $SO(10)$ or $SU(5)$ gauge group can be constructed by choosing the vacuum gauge bundle to have rank four or five.

Beyond a few preliminary explorations of $(0,2)$ Calabi-Yau models and orbifolds [3,4,5], however, little is known about $(0,2)$ theories. This is largely because of the assertion in [6] that the generic $(0,2)$ $\sigma$-model is destabilized by worldsheet instantons, topologically non-trivial maps from the string worldsheet to rational curves in the Calabi-Yau target space. Although it was pointed out in [6] that vacuum gauge bundles which split non-trivially over all of the rational curves in the Calabi-Yau manifold would evade this destabilization, explicit verification of that condition in non-trivial examples is prohibitively difficult. Therefore, without any classes of explicit and tractable $(0,2)$ vacua to study, this vast region of the moduli space of classical string vacua has remained largely unexplored.

Recently, Witten pointed out [7] that one can quite explicitly construct certain massive $(0,2)$ $\sigma$-models which are $(0,2)$ deformations of the more conventional $(2,2)$ Landau-Ginzburg orbifolds [8]; he further argues that there are no instantons which could destabilize these $(0,2)$ Landau-Ginzburg theories. As in the $(2,2)$ case, $(0,2)$ Landau-Ginzburg orbifolds are continuously connected to Calabi-Yau models in a non-singular “phase diagram,” so it seems likely that by studying Landau-Ginzburg theory, one can learn about
Calabi-Yau vacua again in the (0,2) case. Because (0,2) Landau-Ginzburg theories can be constructed quite explicitly using the methods of [9], one finally has a large class of explicit (0,2) vacua available for detailed examination.

In the present work, we undertake the detailed study of (0,2) Landau-Ginzburg vacua, including (0,2) models which are not merely deformations of (2,2) theories. Our results indicate that the study of (0,2) string vacua is likely to reveal many surprises unanticipated from the (2,2) case.

In §2, we proceed in the style of [7] to examine the full phase diagram of the massive theories which flow, in the infrared, to a class of (0,2) Calabi-Yau σ-models much more general than that discussed in [7]. We find several surprises in the (0,2) case; for example, (0,2) models built on complete intersection Calabi-Yau manifolds admit a Landau-Ginzburg representation, while the corresponding (2,2) models did not.

In §3, we discuss the Landau-Ginzburg phase of generic (0,2) theories in more detail, generalizing the results of [9]. We find that the Landau-Ginzburg theories provide a natural and physical criterion which might indicate which (0,2) theories are destabilized by worldsheet instantons and which are not. Some (0,2) models seem to exist only in highly restricted subspaces of their purported moduli space, while others (including non-deformation (0,2) theories with phenomenologically interesting gauge groups) appear to make sense generically.

In §4, we present an example of each type of (0,2) model. First, we present an exhaustive analysis of a complete intersection model with $SO(10)$ gauge group. The Landau-Ginzburg phase seems to be sensible for generic choices of its defining data. We then present another example which cannot be sensibly embedded in a heterotic string theory for generic choices of its defining data; for non-generic choices of a special type, however, the model still makes sense, and defines a consistent (0,2) theory. This model is therefore constrained to lie on a small subspace of its naive moduli space. Presumably, this is the Landau-Ginzburg symptom of the worldsheet instanton disease found in the Calabi-Yau phase.

§5 contains a discussion of some very interesting novel features exhibited by the (0,2) vacua we have studied. In particular, we find that such vacua exhibit new quantum symmetries, distinct from their (2,2) cousins defined in the same $\mathbb{CP}^n$, and we provide an example which suggests that a new kind of topology change occurs in (0,2) models.
2. Microscopic Lagrangians, and the Phase Structure of (0,2) Models

To obtain a heterotic string theory in 4 dimensions with SO(10) or SU(5) gauge group, we need as the “internal” part of the theory a (0,2) superconformal field theory with central charge \((c, \bar{c}) = (6 + \tilde{r}, 9)\), where \(\tilde{r}\) is 4 for \(SO(10)\), and 5 for \(SU(5)\). In addition to the right-moving \(N = 2\) algebra, there should also be a left-moving \(U(1)\) current algebra with

\[
J(z)J(w) = \frac{\tilde{r}}{(z - w)^2}
\]

so that the linearly realized part of the gauge group is \(SO(8) \times U(1)\) or \(SO(6) \times U(1)\), respectively. This left-moving \(U(1)\) is used to implement the GSO projection for the left-moving (“gauge”) fermions, just as \(\bar{J}\), the \(U(1)\) in the right-moving \(N = 2\) algebra is used to implement the GSO projection on the right-moving (“Lorentz”) fermions. By the analogue of spectral flow, the representations of the linearly-realized part of the gauge group assemble themselves into representations of the full gauge group (table (1)).

The most familiar example of such an SCFT is a \((0,2)\) \(\sigma\)-model on a Calabi-Yau manifold \(M\). The “new” data in a \((0,2)\) \(\sigma\)-model is a choice of a rank \(\tilde{r}\) stable holomorphic vector bundle \(E \to M\), with \(c_1(E) = 0\) and \(c_2(E) = c_2(T)\), where \(T\) is the holomorphic tangent bundle of \(M\) \[2\].

| \(\tilde{r} = 4\) | Rep. of \(SO(10)\) | Rep. of \(SO(8) \times U(1)\) | Cohomology Group |
|------------------|--------------------|-----------------------------|-----------------|
| 45               | \(8_{-2}' \oplus 28_0 \oplus 1_0 \oplus 8_2'\) | \(H^*(M, \mathcal{O})\) |
| 16               | \(8_{-1}' \oplus 8_{1}'\)                       | \(H^*(M, E)\)         |
| 10               | \(1_{-2} \oplus 8_0' \oplus 1_2\)               | \(H^*(M, \wedge^2 E)\) |
| 1                | \(1_0\)                                             | \(H^*(M, \text{End } E)\) |

| \(\tilde{r} = 5\) | Rep. of \(SU(5)\) | Rep. of \(SO(6) \times U(1)\) | Cohomology Group |
|------------------|--------------------|-----------------------------|-----------------|
| 24               | \(4_{-5/2} \oplus 15_0 \oplus 1_0 \oplus 4_{5/2}\) | \(H^*(M, \mathcal{O})\) |
| 10               | \(4_{-3/2} \oplus 6_1\)                           | \(H^*(M, E)\)         |
| 5                | \(4_{-1/2} \oplus 1_2\)                           | \(H^*(M, \wedge^2 E)\) |
| 1                | \(1_0\)                                             | \(H^*(M, \text{End } E)\) |

**Table 1:** Representations of the linearly realized part of the gauge group and how they assemble themselves.
In a very beautiful paper [7], Witten showed how one can construct a (2,2) Calabi-Yau \( \sigma \)-model as the infrared limit of a (2,2) supersymmetric quantum field theory. Moreover, as one varies the parameters of this “microscopic” theory, the theory exhibits different phases. In the simplest case, there are two phases. One is the Calabi-Yau \( \sigma \)-model; the other is a (deformed) Landau-Ginzburg theory.

We would like to follow in the same spirit and construct the microscopic (0,2) theory whose IR limit is the desired (0,2) Calabi-Yau \( \sigma \)-model, and then examine its phase structure.

The first step in this construction is to specify the stable vector bundle \( E \to M \) which couples to the left-moving worldsheet fermions. In the (2,2) case, this was just \( T \), the tangent bundle of \( M \). One method of defining \( E \), which has a simple field-theoretic realization, is the following. For simplicity, we will consider the simple case of a hypersurface in \( W \mathbb{P}^4 \), with homogeneous coordinates \( \phi_i \). Consider the exact sequence

\[
0 \to E \to \bigoplus_{a=1}^{\tilde{r}+1} \mathcal{O}(n_a) \otimes F_a(\phi) \to \mathcal{O}(m) \to 0 \tag{2.1}
\]

where the \( n_a \) are positive integers, \( m = \sum n_a \), and the \( F_a \) are homogeneous polynomials of the appropriate degrees. We demand that the \( F_a \) be such that at no point in \( M \) do they all vanish simultaneously. If the \( F_a \) are chosen sufficiently generically, then the kernel \( E \) is a stable rank \( \tilde{r} \) vector bundle with vanishing first Chern class, \( c_1(E) = 0 \). The “Maruyama construction” (of the dual bundle \( E^* \)) discussed in [3] is a special case of the above, in which \( \tilde{r} \) of the \( n_a \) are chosen to be equal to 1.

If the \( F_a \) are chosen “badly”, then \( E \) is in general a semistable torsion-free sheaf. However, just as “bad” choices of defining polynomial can lead to a singular variety, which nonetheless is perfectly fine as a string theory, here too the string theory that results from a nongeneric choice of \( F_a \)’s is well-behaved.

As in the (2,2) case [7], we introduce (our (0,2) conventions are found in the appendix) a set of chiral superfields \( \Phi_i \) with \( U(1) \) gauge charges equal to \( w_i \), the weights of the corresponding homogeneous coordinates in \( W \mathbb{P}^4 \). The hypersurface is the zero locus of a degree \( d \) polynomial \( W(\phi) \) where the Calabi-Yau condition (i.e. \( c_1(T) = 0 \)) is \( d = \sum w_i \).

To enforce \( W = 0 \), we will need a Fermi superfield \( \Sigma \) of charge \( -d \). To construct the bundle \( E \) above, we introduce some Fermi superfields \( \Lambda^a \) with charges \( n_a \), and a chiral superfield

\[ 1 \] The \( n_a \) must be strictly positive, otherwise \( E \) is never stable.
$P$ with charge $-m$. Finally, we have the $(0,2)$ gauge multiplets, $V$ and $A$. In Wess-Zumino gauge, $A$ contains the left-moving gauge field $a$, a pair of left-moving gauginos $\alpha, \bar{\alpha}$, and the auxiliary field $D$. $V$ contains the right-moving component of the gauge field $\bar{a}$. The $(0,2)$ superpotential is

$$S_W = \int d^2z d\theta \sum W(\Phi) + P \Lambda^a F_a(\Phi)$$

(2.2)

The first term will ensure that we lie on the hypersurface $W(\phi) = 0$, and the second term will insure that the $\lambda^a$ (the lowest components of the $\Lambda^a$) are sections of the bundle $E$.

Of course, the bundle $E$ must satisfy a constraint on its second Chern class:

$$c_2(E) = c_2(T)$$

(2.3)

In the present context, this is the equation

$$m^2 - \sum n_a^2 = d^2 - \sum w_i^2$$

(2.4)

which is none other than the condition for the cancellation of the U(1) gauge anomaly for the fields in our theory.

This is a Diophantine equation for the positive integers $n_a$, and in general, we won’t find many solutions. We are, however, always guaranteed at least one. Let $r = 4$, and set $n_i = w_i$, then the equation is satisfied identically. This solution for $E$ is simply a stable deformation of $T \oplus \mathcal{O}$ (more precisely, of an extension of $T$ by $\mathcal{O}$). Indeed, $E$ is isomorphic to such an extension precisely when $F_i(\phi) = \frac{\partial W}{\partial \phi_i}$.

To see this, we need only examine the definition of the tangent bundle $T$ of a Calabi-Yau hypersurface. Consider the sequence

$$0 \to \mathcal{O} \xrightarrow{u} \bigoplus_{i=1}^5 \mathcal{O}(w_i) \xrightarrow{v} \mathcal{O}(d) \to 0$$

(2.5)

where

$$u(f) = (w_1 f, w_2 f, \ldots, w_5 f), \quad v(s_1, s_2, \ldots, s_5) = \sum_i s_i \frac{\partial W}{\partial \phi_i}$$

This sequence is clearly not exact (the dimensions don’t add up), but nonetheless, the image of $u$ is contained in the kernel of $v$, since we have $\sum_i w_i \phi_i \partial_i W(\phi) = d W(\phi) = 0$ on $M$. The cohomology of this sequence, $\ker(v)/\text{im}(u)$ is the tangent bundle $T$. For this
very special defining data, we thus have an exact sequence $0 \to O \to \ker(v) \to T \to 0$.\footnote{Although we have only shown that $\ker(v)$ is an extension of $T$ by $O$, one can show using the methods of \S 3 that the physical spectrum of the corresponding Landau-Ginzburg theory is isomorphic to that based on $T \oplus O$ (i.e. the corresponding (2,2) theory). The point is that the $\bar{Q}_+$ operator differs from that of the $T \oplus O$ theory simply by the presence of a $\sigma W(\phi)$ term in $\bar{Q}_{+\cdot L}$. However, when $F_i = \frac{\partial W}{\partial \phi_i}$, this term is $\bar{Q}_{+\cdot L}$ trivial. Thus the $\bar{Q}_+$ cohomology is the same as when this term is absent, i.e. isomorphic to that of the (2,2) theory.}

The definition of $E$ is simply to replace $\partial_i W$ in the definition of the map $v$ by generic polynomials $F_i$ of the same degree, so that, for generic $F_i$, $E$ is stable. Thus the (0,2) theory discussed in [3] and many of the (0,2) theories in [5] are simply deformations of the corresponding (2,2) theories.

The net number of generations of matter fields (16s of $SO(10)$ or $(\bar{5} + 10)$s of $SU(5)$) is $\frac{1}{2}c_3(E)$. If $M$ is smooth (which is all we will consider in this paper), $c_3(E)$ is given by

$$c_3(E) = \frac{1}{3}(m^3 - \sum \alpha^3)J^3.$$  

(2.6)

For these deformations of $T \oplus O$, the number of generations is, of course, exactly the same as that of the corresponding (2,2) model. For this reason, we will mostly concern ourselves with (0,2) theories which are not deformations of (2,2) theories.

Integrating out the the $D$ auxiliary field in the gauge multiplet and the auxiliary fields in the Fermi multiplets, we obtain the scalar potential

$$U = |W(\phi)|^2 + |p|^2|F_a(\phi)|^2 + \frac{\epsilon^2}{2} \left( \sum w_i |\phi_i|^2 - m|p|^2 - r \right)^2$$  

(2.7)

where $r$ is the coefficient of the Fayet-Iliopoulos D-term.

In general $r$ gets additively renormalized at one loop [4]. The counterterm is proportional to the sum of the $U(1)$ charges of the scalar fields in the theory

$$\Delta r_{1\text{-loop}} \propto \sum_{\text{bosons}} Q_i$$  

(2.8)

For the model we have been discussing, this sum is equal to $(d - m)$. In general, this might not be zero, in which case, $r$ runs as a function of RG scale. This is inconvenient, since we were hoping to use $r$ as an RG-invariant parameter to label the infrared fixed points that we obtain. In particular, $t = r + i\theta$ was to parametrize the Kähler moduli space of the model.
The situation, however, is easily saved. Simply introduce a chiral superfield $X$ of charge $(m - d)$, and a Fermi superfield $\Omega$ of charge $(d - m)$ and add the term

$$\Delta S_W = \int d^2z d\theta \Omega X$$

(2.9) to the superpotential. The addition of these new “spectator” fields cancels the 1-loop contribution to the $D$ term, restoring $t$ to its status as an RG invariant.

The scalar potential is, of course modified by the addition of the scalar $x$. However, minimizing $U$ always uniquely fixes $x = 0$, and the spectators, being massive, drop out of the low-energy physics. For the most part, we will simply ignore the effects of the spectators, save for their making $r$ an RG-invariant.

2.1. The Calabi-Yau phase

For $r \gg 0$, the semiclassical approximation is a good guide to determine the ground state of the theory. To find the ground state, we simply need to minimize the classical scalar potential. The minimum is obtained by setting $p = 0, \sum w_i |\phi_i|^2 = r$, and $W(\phi) = 0$. This spontaneously breaks the gauge symmetry and one component of $\phi$ gets eaten by the gauge field. Three more pick up masses from (2.7). The remaining three (complex) massless $\phi$’s parametrize the Calabi-Yau hypersurface $M$.

The right-moving fermion $\pi$, the superpartner of $p$, picks up a mass with one linear combination of the left-moving fermions $\lambda^a$, through the Yukawa coupling

$$\pi \lambda^a F_a(\phi)$$

which follows from the superpotential (2.2). The remaining massless $\lambda^a$’s transform as local sections of the bundle $E$ defined by (2.1). The left-moving fermion $\sigma$ in $\Sigma$ picks up a mass with one linear combination of the right-moving fermions $\psi_i$ in $\Phi_i$ through the Yukawa coupling

$$\sigma \psi_i \frac{\partial W}{\partial \phi_i}$$

The remaining $\psi$’s are thus in the kernel of the map $v$ in (2.7). The Yukawa coupling with the gaugino

$$- \sum_i w_i^a \psi_i \bar{\phi}_i$$

(2.10) gives mass to another linear combination of the $\psi$’s, getting rid of the image of the map $u$ in (2.3). The remaining massless right-moving fermions thus transform as local sections of the tangent bundle $T$. Thus the massless fields form a (0,2) Calabi-Yau $\sigma$-model in which the scalars $\phi$ parametrize a Calabi-Yau hypersurface $M$, the right-moving fermions $\psi$ are sections of the tangent bundle $T$, and the left-moving fermions $\lambda$ are sections of the bundle $E$. 

7
2.2. The Landau-Ginzburg phase

For $r \ll 0$, we again have to minimize the scalar potential. Now the minimum is obtained for $|p|^2 = |r|/m$, $\phi = 0$. The gauge symmetry is again spontaneously broken. Again, $p$ becomes massive and drops out of the low energy theory, as does its superpartner $\pi$, this time through its Yukawa coupling with the gaugino

$$m\alpha_\pi \tilde{p}$$

What remains is a Landau-Ginzburg theory with the superpotential (after a trivial rescaling of the fields)

$$S_W = \int d^2 z d\theta \: \Sigma W(\Phi) + \Lambda^a F_a(\phi) \quad (2.11)$$

Actually, since the field $p$ which got the VEV has charge $m$, a $Z_m$ subgroup of the gauge group remains unbroken, so we, in fact, have a Landau-Ginzburg orbifold. This will be the subject of study in §3.

2.3. The point $r = \theta = 0$

At $r = 0$, the classical potential is minimized for $p = \phi^i = 0$, corresponding to unbroken gauge symmetry. In the (2,2) case, the analysis of the behaviour of the theory in the neighbourhood of this point is quite delicate. At $r = 0$, the classical potential for the scalar superpartner of the gauge field (called $\sigma$ in [7]) becomes flat, and semiclassical reasoning becomes invalid. A somewhat delicate analysis (involving working at nonzero $\theta$) is required to show that spectrum of low-lying states (when the theory is quantized on a circle) is discrete, and varies continuously as one passes through $r = 0$ [7].

In our (0,2) theories, showing this is a triviality. There is no scalar superpartner of the gauge field whose spectrum might become continuous. All of the scalar fields have potentials that grow at infinity, so the low-lying spectrum is always discrete, even at $r = 0$.

2.4. Generalizations

There are lots of generalizations of the basic construction we have outlined. Let us look at a few of them.

The first obvious generalization is to look at Calabi-Yau manifolds realized as complete intersections in weighted projective space. Introduce several Fermi superfields $\Sigma_j$, with
charges $-d_j$, where $d_j$ are the degrees of the polynomials $W_j(\phi)$. Instead of (2.2), consider the more general superpotential

$$S_W = \int d^2z d\theta \sum_j W_j(\Phi) + P \Lambda^a F_a(\Phi)$$

(2.12)

The Calabi-Yau condition is now $\sum d_j = \sum w_i$, and the condition that $c_2(E) = c_2(T)$ now reads

$$m^2 - \sum n_a^2 = \sum d_j^2 - \sum w_i^2$$

(2.13)

Again, this is just the condition that the gauge symmetry be nonanomalous. As before, for $r \gg 0$, we clearly have a (0,2) Calabi-Yau $\sigma$-model on the Calabi-Yau manifold which is the simultaneous vanishing locus of all the $W_j$’s.

Our first surprise comes when we look at $r \ll 0$. In the corresponding (2,2) case, this is a “hybrid phase” about which little concrete can at present be said. Here, however, $r \ll 0$ is again simply a (0,2) Landau-Ginzburg orbifold. The point is that, whereas in the (2,2) theory, there are several negatively-charged scalars which could in principle pick up a VEV, here there is still only one – $p$ – and so we have a Landau-Ginzburg phase, rather than a hybrid. This is but the first hint that the Kähler moduli space of (0,2) theories is in general very different from that of the corresponding (2,2) theory. We will see this more explicitly in §5.

We can also consider complete intersections in products of weighted projective spaces. Introduce two $U(1)$ gauge groups and two sets of chiral multiplets: $\Phi_i$ with charges $(w_i, 0)$ and $\Upsilon_i$ with charges $(0, \tilde{w}_i)$. Then let the superpotential be

$$S_W = \int d^2z d\theta \sum_j W_j(\Phi, \Upsilon) + P \Lambda^a F_a(\Phi, \Upsilon)$$

(2.14)

Again, the condition $c_2(E) = c_2(T)$ is simply the requirement that the gauge anomalies vanish. There are now two Fayet-Iliopoulos D-terms and, correspondingly, four phases depending on the signs of the coefficients of these D-terms. One phase, when both of the $r_i$ are positive, is the familiar Calabi-Yau phase. When both of the $r_i$ are negative, it is $p$ that develops an expectation value. This leaves a $U(1)$ subgroup of the gauge group unbroken. Thus this phase is a gauged Landau-Ginzburg theory. The remaining phases are hybrids.

There is one more generalization that has no analogue in the (2,2) case. We can have two (or more) factors in the vacuum gauge bundle $E$, $E = E_1 \oplus E_2$, and if we wish, we
can embed $E_1$ in the first $E_8$ and $E_2$ in the second $E_8$ of the $E_8 \times E_8$ heterotic string. To represent this situation, we can generalize (2.14) still further:

$$S_W = \int d^2 z d\theta \sum_j W_j(\Phi, \Upsilon) + P_1 \Lambda_1^a F_a(\Phi, \Upsilon) + P_2 \Lambda_2^a G_a(\Phi, \Upsilon)$$  \hspace{1cm} (2.15)$$

where the $F_a$ are the polynomials which define the bundle $E_1$ and $G_a$ are the polynomials which define $E_2$. A complete intersection in a product of two weighted projective spaces, with two factors in the vacuum gauge bundle again has a Landau-Ginzburg phase.

Finally, we can consider generalizations of our basic construction of the vector bundle $E$. For instance, we can generalize (2.1) to

$$0 \rightarrow E \rightarrow \bigoplus_{a=1}^{\tilde{r}+2} \mathcal{O}(n_a) \rightarrow \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \rightarrow 0$$ \hspace{1cm} (2.16)$$

This is accomplished by generalizing (2.2) to

$$S_W = \int d^2 z d\theta \sum W(\Phi) + P_1 \Lambda^a F_a(\Phi) + P_2 \Lambda^a \tilde{F}_a(\Phi)$$ \hspace{1cm} (2.17)$$

where $P_1$ has charge $-m_1$ and $P_2$ has charge $-m_2$.

Clearly, the variations on this theme are nearly endless. However, aside from being able to write down the expected phase diagram, one cannot, in most of these models, say much more. In particular, one would like to show that the models do indeed have $(0,2)$ SCFT’s as their IR limits. In the context of Calabi-Yau $\sigma$-models, doubt has been cast on the existence of the infrared fixed point due to worldsheet instanton effects [6].

In order to bolster our confidence that at least a subclass of the microscopic quantum field theories we have been studying really do have $(0,2)$ SCFT’s as their IR limits, we will focus on those that have a Landau-Ginzburg phase. The Landau-Ginzburg theory can be constructed quite explicitly using the methods of [9,10,11]. At least for these theories, we can know with confidence that the LG phase has as its IR limit a $(0,2)$ SCFT. Since at nonzero $\theta$, the LG phase interpolates smoothly into the Calabi-Yau phase (the only singularity is at the point $r = \theta = 0$), we can be confident that the Calabi-Yau phase exists as well. In addition, we will be able to compute the massless spectrum of the LG theory, and compare the results with those computed in the $\sigma$-model.

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3 Very similar techniques were first developed in the context of $(2,2)$ theories in [8,9,10,11].
3. The Landau-Ginzburg Phase

Consider the Landau-Ginzburg theory with superpotential

\[ W(\Phi^i, \Lambda^a, \Sigma^j) = \Lambda^a F_a(\Phi_i) + \Sigma^j W_j(\Phi_i) \]  

where \( a = 1, \ldots, \bar{r} + 1, i = 1, \ldots, N + 1, j = 1, \ldots, N - 3 \). This is the effective superpotential that remains in the Landau-Ginzburg phase of (2.12), after setting \( P \) equal to its vacuum expectation value. For simplicity, we have not included the “spectator multiplets” \( \Omega, X \), though we clearly could do so with little extra effort. Including them changes none of the subsequent analysis. Their effect is solely to cancel the renormalization of the Fayet-Iliopoulos D-term.

Transversality of the defining equations \( W_j(\phi) = 0 \) and our conditions on the \( F_a(\phi) \) which define the bundle \( E \) in the Calabi-Yau phase are sufficient to insure that in the Landau-Ginzburg phase, the superpotential (3.1) is a quasi-homogeneous function of the fields with an isolated singularity at the origin of field space. In particular, the \( U(1) \) gauge charges \( w_i, n_a, -d_j \) and \( m \) are integers which satisfy

\[ W(\epsilon^{w_i} \Phi_i, \epsilon^{n_a} \Lambda^a, \epsilon^{-d_j + m} \Sigma^j) = \epsilon^m W(\phi_i, \Lambda^a, \Sigma^j) \].  

The quasi-homogeneity (3.2) of the superpotential guarantees the existence of a right-moving \( U(1) \) R-symmetry which plays an important role in (0,2) models.

Actually, the models we are interested in have an additional global left-moving \( U(1) \) symmetry. Let us define

\[ q_i = \frac{w_i}{m}, \quad q_a = \frac{n_a}{m}, \quad q_j = 1 - \frac{d_j}{m} \].  

Then the left and right \( U(1) \) charges of the various fields are given in table (2).

| Field | Left \( U(1) \) charge \( q \) | Right \( U(1) \) charge \( \bar{q} \) |
|-------|--------------------------------|----------------------------------|
| \( \phi^i \) | \( q_i \) | \( q_i \) |
| \( \psi^i \) | \( q_i \) | \( q_i - 1 \) |
| \( \lambda^a \) | \( q_a - 1 \) | \( q_a \) |
| \( \sigma^j \) | \( q_j - 1 \) | \( q_j \) |

**Table 2:** Left and right \( U(1) \) charges of fields in Landau-Ginzburg theory.
We are only interested in describing Landau-Ginzburg theories which are the $r \to -\infty$ limit of some “high energy theory” with a Calabi-Yau description at $r \to \infty$. Hence, as in §2, we require that the charges satisfy various quadratic conditions so that the gauge and global $U(1)$ symmetries are non-anomalous, and of course, we impose

$$\sum_i w_i = \sum_j d_j$$

so that at $r \to \infty$, the equations $W_j(\phi_i) = 0$ describe a Calabi-Yau manifold in $\mathbb{P}^N$.

For string theory, we are interested in the conformal limit of the theory defined by (3.1). It is, as we have emphasized, only an assumption that the quantum field theory we are discussing has as its IR limit a (0,2) SCFT (the same might be said about the (2,2) quantum field theories in [7]). Nevertheless, making this assumption, we will be able to say quite a lot about the properties of that SCFT. That is the goal of this section. In §3.3, we will see that there are some nontrivial conditions to be satisfied for this SCFT to be embedded in a string theory.

Because the left-moving stress-tensor $T(z)$ is an element of the chiral algebra formed by the $\bar{Q}_+$ cohomology (see [12]), the conformal anomaly of the left-moving CFT is reliably computed in the massive theory and is given by

$$c = \sum_a (1 - 3q_a^2) + \sum_j (1 - 3q_j^2) + \sum_i (2 + 3q_i^2 - 6q_i).$$

Using the quadratic condition

$$\sum_a q_a^2 + \sum_j q_j^2 = \sum_i q_i^2 + 1$$

and the condition (3.4), a few steps of algebra reveal that for the theory (3.1) the left central charge (3.5) simplifies to

$$c = 6 + \tilde{r}$$

as expected.

Similarly, the left-moving $U(1)$ current, $J_z$, is in the chiral algebra formed by the $\bar{Q}_+$ cohomology, and so we can accurately compute, in the massive theory, the central term in the $J \cdot J$ operator product expansion. It is given by

$$\sum_a (q_a - 1)^2 + \sum_j (q_j - 1)^2 - \sum_i q_i^2 = \tilde{r}$$

12
as expected.

In the standard way, we supplement the Landau-Ginzburg theory with $16 - 2\tilde{r}$ extra left-moving free fermions

$$L' = \int \sum_{I=1}^{16-2\tilde{r}} \lambda^I \bar{\partial} \lambda^I$$  \hspace{1cm} (3.9)

which generate an $SO(16 - 2\tilde{r})$ current algebra (which combines with the left-moving $U(1)$ of the Landau-Ginzburg theory to yield a maximal subgroup of the visible spacetime gauge group, which is $E_6$ for $\tilde{r} = 3$, $SO(10)$ for $\tilde{r} = 4$, and $SU(5)$ for $\tilde{r} = 5$). There are also 16 left-moving free fermions which generate the “hidden” $E_8$ spacetime gauge symmetry; we shall leave them in their NS sector for the duration of this paper, and only consider states which are singlets under the “hidden” gauge group.

What about the right-moving central charge? We cannot reliably compute $\bar{c}$ from the $\bar{T}(\bar{z}) T(0)$ operator product in the massive theory. However, $\bar{c}$ appears in other places in the N=2 superconformal algebra; in particular, the central term in the $\bar{J} \cdot \bar{J}$ operator product is $\bar{c}/3$. This is nothing other than the anomaly (of a fictitious gauge field coupled to $\bar{J}$). This can be computed reliably by a one-loop computation in the massive theory\footnote{This trick, of using the $U(1)$ anomaly to compute the $N = 2$ central charge, has been exploited by Fendley and Intriligator in a somewhat different context for perturbed $N = 2$ theories \cite{Fendley:1993vm}.}

The result is as expected:

$$\frac{\bar{c}}{3} = \sum_i (q_i - 1)^2 - \sum_a q_a^2 = 3$$  \hspace{1cm} (3.10)

which again follows from using the conditions (3.4) and (3.6).

3.1. Twisted Sectors and Quantum Numbers

Recall that in the passage from the high energy theory with superpotential (2.12) to the low energy theory with superpotential (3.1) at $r \ll 0$, the $P$ field develops an expectation value

$$|P| = \sqrt{\frac{-r}{m}}$$  \hspace{1cm} (3.11)

and spontaneously breaks the $U(1)$ gauge symmetry to a residual $\mathbb{Z}_m$ subgroup. This is the reflection in the formalism of §2 that Calabi-Yau sigma models are actually related not to the Landau-Ginzburg theory of (3.1), but to an orbifold of (3.1) by

$$e^{-2\pi i \oint J(z)}.$$  \hspace{1cm} (3.12)
So in computing the spectrum of the theory (3.1), we will also be interesting in finding states from various twisted sectors.

In fact, the Landau-Ginzburg orbifold of interest is not quite the orbifold obtained by projecting onto integral values of (3.12). We are interested in considering both the (R,R) and the (NS,R) sectors, which yield spacetime fermions (the (R,NS) and (NS,NS) sectors, which give the spacetime bosons, then follow by supersymmetry). Taking account of the extra free-fermions \( \lambda^I \) of (3.9), we find that we are actually interested in projecting onto states with \( g = 1 \) where

\[
g = \exp(-i\pi J_0) \times (-1)^{\lambda}
\]

and \( \lambda \) denotes the number of \( \lambda^I \) excitations. There are then sectors twisted by \( g^k \) for \( k = 0, \ldots, 2m \); for even \( k \), we call them (R,R) sectors (and in particular put the fermions (3.9) in their Ramond sector) and for odd \( k \), we call them (NS,R) sectors (and give anti-periodic boundary conditions to (3.9)).

Similarly, the GSO-projection for right-moving Ramond sector ends up correlating states with \( \bar{q} + 3/2 \) even with left-handed spacetime fermions and states with \( \bar{q} + 3/2 \) odd with right-handed spacetime fermions. The right-moving \( U(1) \) charge also indicates the type of spacetime supermultiplet a state represents; states with \( \bar{q} = \pm \frac{1}{2} \) are parts of chiral or antichiral superfields in spacetime, while states with \( \bar{q} = \pm \frac{3}{2} \) are parts of vector superfields.

The effect of the GSO-projection in left-moving Ramond sectors is best summarized separately for models with \( E_6 \), \( SO(10) \) and \( SU(5) \) gauge group. The \( E_6 \) case has been discussed extensively in [9], so here we limit ourselves to discussing \( SO(10) \) and \( SU(5) \) theories.

For \( SO(10) \) theories, in Ramond sectors one must account for the zero modes of the free fermions (3.9) which realize the \( SO(8) \) current algebra. If we assemble these pairwise into four complex fermions, their zero modes make the ground state a 16 dimensional representation of \( SO(8) \). This is in fact a reducible representation; it splits into the spinor \( \oplus \) spinor' representation of \( SO(8) \), \( 16 = 8^s \oplus 8^{s'} \). Then the GSO projection associates states with \( q = 2 \) odd with \( 8^s \)'s of \( SO(8) \) and states with \( q = 2 \) even with \( 8^{s'} \)'s of \( SO(8) \). The \( SO(8) \times U(1) \) quantum numbers of the various multiplets which arise in \( SO(10) \) theories were summarized in table (1)

For \( SU(5) \) theories, the same discussion applies almost verbatim, the only changes being that the 3 complex fermions provide a reducible 8 dimensional representation of
SO(6) which splits as $4 \oplus \bar{4}$. Then the GSO projection associates states with $q - \frac{5}{2}$ odd with $\bar{4}$s of $SO(6)$ and states with $q - \frac{5}{2}$ even with $4$s of $SO(6)$. The $SO(6) \times U(1)$ quantum numbers of the multiplets which arise in $SU(5)$ theories were also summarized in table (1).

In order to determine the spectrum of states from twisted sectors, we must also know the quantum numbers of the ground states. Because the fermions have twisted boundary conditions, the ground state of a twisted sector generically has fractional fermion numbers and non-vanishing $U(1)$ charges. Obvious modification of the formulae in [9] yields the following for the $U(1)$ charges of the ground state of the $g^k$ twisted sector:

$$\bar{q}_k = \sum_i (q_i - 1) \left( \frac{k(q_i - 1)}{2} + \left[ \frac{k(1 - q_i)}{2} \right] + \frac{1}{2} \right) + \sum_a q_a \left( -\frac{kq_a}{2} + \left[ \frac{kq_a}{2} \right] + \frac{1}{2} \right)$$

$$+ \sum_j q_j \left( -\frac{kq_j}{2} + \left[ \frac{kq_j}{2} \right] + \frac{1}{2} \right)$$

$$q_k = \sum_i q_i \left( \frac{k(q_i - 1)}{2} + \left[ \frac{k(1 - q_i)}{2} \right] + \frac{1}{2} \right) + \sum_a (q_a - 1) \left( -\frac{kq_a}{2} + \left[ \frac{kq_a}{2} \right] + \frac{1}{2} \right)$$

$$+ \sum_j (q_j - 1) \left( -\frac{kq_j}{2} + \left[ \frac{kq_j}{2} \right] + \frac{1}{2} \right) .$$

(3.14)

(3.15)

Defining the $\theta_{i,k}$, $\theta_{a,k}$ and $\theta_{j,k}$ to be the phases present in (3.13) and (3.14) (basically, the twist on the fields relative to being anti-periodic), these formulae simplify to

$$\bar{q}_k = \sum_i (q_i - 1) \theta_{i,k} + \sum_a q_a \theta_{a,k} + \sum_j q_j \theta_{j,k}$$

(3.16)

$$q_k = \sum_i q_i \theta_{i,k} + \sum_a (q_a - 1) \theta_{a,k} + \sum_j (q_j - 1) \theta_{j,k} .$$

(3.17)

We also need to determine the ground state eigenvalues of $L_0$ ($\tilde{L}_0$ vanishes by right-moving supersymmetry). These are easily determined using the normal formulas for the energy of a free twisted boson or fermion (and remembering to include the extra free $E_8$ and $SO(16 - 2\tilde{r})$ fermions). It turns out that the vacuum energy in the $k$th sector for even $k$ is given by:

$$E_R = -\frac{\bar{r} - 3}{8} + \sum_a \frac{\theta_{a,k}^2}{2} + \sum_j \frac{\theta_{j,k}^2}{2} - \sum_i \frac{\theta_{i,k}^2}{2} .$$

(3.18)

For odd $k$, the formula becomes:

$$E_{NS} = -\frac{5}{8} + \sum_a \frac{\theta_{a,k}^2}{2} + \sum_j \frac{\theta_{j,k}^2}{2} - \sum_i \frac{\theta_{i,k}^2}{2} .$$

(3.19)

One notes that even for the (R,R) sectors, the vacuum energy will generically no longer vanish; in order to find even the spectrum of generations and anti-generations, one must keep the lowest excited modes of the various fields.
3.2. The Born-Oppenheimer Approximation and $\bar{Q}_{+,L}$ cohomology

As in [9], because we are looking for massless states in spacetime, we can employ a Born-Oppenheimer approximation and truncate the quantum fields to their lowest excited modes. The right-moving N=2 supersymmetry algebra is

$$\{\bar{Q}_-,\bar{Q}_+\} = L_0, \quad \bar{Q}_-^2 = \bar{Q}_+^2 = 0 \, .$$  \hspace{1cm} (3.20)

We are interested in finding massless states, which are annihilated by $\bar{L}_0$; by (3.20), finding such states is equivalent to calculating the cohomology of the $\bar{Q}_+$ operator. Actually, we are interested in the subspace of the $\bar{Q}_+$ cohomology which is also annihilated by $L_0$; since in twisted sectors $L_0$ generally does not annihilate the vacuum (see (3.18), (3.19)), we generally need to consider excitations over the vacuum by the lowest modes of the various fields.

Fortunately, an explicit and useful expression for the generators of the N=2 algebra of Landau-Ginzburg models, and in particular the $\bar{Q}_+$ operator, has recently been provided in [12] (see also [14], where a closely related construction was given):

$$\bar{Q}_+ = i \oint (i\bar{\psi}^i\bar{\partial}\phi_i + W|_{\theta=0}) \, .$$  \hspace{1cm} (3.21)

For purposes of computing the cohomology of (3.21), it is convenient to divide it further into

$$\bar{Q}_{+,L} = i \oint W|_{\theta=0}, \quad \bar{Q}_{+,R} = i \oint i\bar{\psi}^i\bar{\partial}\phi_i \, .$$  \hspace{1cm} (3.22)

Then, because rescaling the superpotential $W \rightarrow \epsilon W$ amounts to a modification of the kinetic energy, which is $\bar{Q}_+$ exact and cannot affect the $\bar{Q}_+$ cohomology, one may compute the cohomology of (3.21) treating $\bar{Q}_{+,L}$ as a perturbation. The formal justification of this argument is given in [9]. The result is that the cohomology of $\bar{Q}_+$ is equal to the cohomology of $\bar{Q}_{+,L}$ in the cohomology of $\bar{Q}_{+,R}$. The cohomology of $\bar{Q}_{+,R}$ is just the space of states with holomorphic dependence on bosonic zero modes and no right-moving excitations; furthermore, the $\psi^i$ and $\bar{\psi}^i$ zero modes can be omitted. So in evaluating the $\bar{Q}_+$ cohomology of models, we will in practice evaluate the cohomology of $\bar{Q}_{+,L}$ in the truncated Hilbert space of states which form the $\bar{Q}_{+,R}$ cohomology.

---

5 Sectors with positive vacuum eigenvalue obviously contribute no massless states, and do occasionally occur.
3.3. Stability, World Sheet Instantons, and Landau-Ginzburg Theory

It has long been believed that the generic (0,2) theory based on a stable bundle $E$ does not exist because of worldsheet instanton effects [6], but the criterion which distinguished those theories which do exist from those which don’t has proven elusive. Can Landau-Ginzburg theory help?

First let us consider (0,2) theories which are deformations of (2,2) theories. In the class of models we have been discussing, these are rank $\tilde{r} = 4$ bundles $E$ on hypersurfaces in $WP^4$, where \( \{n_i\} = \{w_i\} \). When

$$F_i(\phi) = \partial_i W(\phi)$$

we have the original (2,2) theory. There are extra gluinos (states with $\bar{q} = -3/2$) which fill out the adjoint of $E_6$. In $SO(8)$ language, these are an $8^*_1$, $8^*_1$ from the R sector, and $8^*_1$, $8^*_1$, 10 from the NS sector. Clearly, we can deform this theory in a way that breaks (2,2) supersymmetry, but leaves $E_6$ unbroken, by generalizing (3.23) to have $W$ be in the ideal generated by the $F_i$s.

Can we deform further and actually break $E_6 \rightarrow SO(10)$? Clearly, this is possible only if we can give masses to the extra gluinos through the Higgs mechanism. In the current language, we want the corresponding states to cease to be in the $\bar{Q}_{+,L}$ cohomology by pairing up with states at $\bar{q} = -1/2$. These are matter fields. For example, the $8^*_1$ should pair up with an $8^*_1$ at $\bar{q} = -1/2$ which is part of a $27$ of $E_6$, and the $8^*_1$ gluino should pair up with an $8^*_1$ which is part of a $\bar{27}$.

In the case of the $8^*_1$, which occurs in the untwisted ($k = 0$) sector, this is never a problem. However, for the $8^*_1$, we now see that there can be an obstruction. For though there may indeed be an $8^*_1$ matter field lurking around, it need not occur in the same twisted sector as the $8^*_1$ gluino. It is then impossible for them to pair up. This would spoil the quantum symmetry of the LG theory. Thus further (0,2) deformations which break $E_6$ are forbidden.

This is not an idle curiosity. It occurs, for example for the quintic in $\mathbb{CP}^4$. There the $8^*_1$ gluino occurs in the $k = 2$ sector, but the $8^*_1$ matter field occurs in the $k = 4$ sector. Thus, contrary to expectations, the (0,2) $SO(10)$ deformation of the (2,2) model on the quintic described in [3] does not exist.\(^6\)

---

\(^6\) This is not to say that the $E_6$-breaking flat directions found in [15] do not exist. Rather, if they do exist, the corresponding (0,2) SCFTs are not describable as the low-energy limit of the microscopic lagrangians in §2. It would be very interesting to find the worldsheet SCFTs which do describe these deformations.
A more recherché example is provided by the hypersurface \( Y_{W;4;10} \) in \( W\mathbb{P}^4_{1,1,1,2,5} \), and a rank 4 bundle defined by \( \{n_a\} = \{1, 1, 1, 1, 7\} \). This example will be discussed in more detail in §4.2. If one takes \( W \) to be of width 1 in the ideal generated by the \( F_a s \), this is actually an \( E_6 \) theory which is not a deformation of a (2,2) theory. There are 165 \( 27s \), whose \( 8_{-1} \) components lie in the \( k = 0 \) sector and a \( \overline{27} \), whose \( 8_1 \) components is the ground state of the \( k = 6 \) sector. This is in agreement with the index, \( \frac{1}{2}c_3(E) = 164 \). Unfortunately, the \( 8_1 \) gluino is the state \( \overline{\phi^3} |0\rangle \) in the \( k = 4 \) sector. It cannot pair up with the \( \overline{27} \) without spoiling the quantum symmetry, so we are not allowed to deform this theory to generic values of the data, where it would be an \( SO(10) \) theory.

Thus we have a simple criterion for determining whether a would-be \( SO(10) \) theory is really an \( E_6 \) theory. Simply look for extra gluinos coming from twisted sectors (it suffices to look for \( 8_1 \) gluinos coming from \( k \) even sectors). If we find one, and there isn’t an \( 8_1 \) matter field from the same twisted sector for it to pair up with, then we must choose the defining data in such a way that we in fact get an \( E_6 \) theory.

Note that though this argument was phrased at the Landau-Ginzburg point, where the quantum symmetry is unbroken, it still holds when we move away from the LG point by turning on a VEV for the twist field (the Kähler modulus). Though the quantum symmetry is spontaneously broken (by the VEV of the twist field), the Higgs mass term should still transform as some definite representation of the quantum symmetry. But this is impossible, since the part that comes from the \( k = 0 \) sector is a singlet, whereas the part which pairs up the \( 8_1 \)'s transforms nontrivially under the quantum symmetry.

It would be very interesting to understand the manifestation of this phenomenon on the Calabi-Yau side. Presumably, the \( E_6 \) theories for special values of the defining data are fine, but when one deforms them to generic values of the defining data, the resulting \( SO(10) \) theories are destabilized by worldsheet instantons.

The statement, which seems quite remarkable, is that for the special values of the defining data, the \( F_a s \) are such that, when restricted to any rational curve \( C \) on \( M \), they become linearly dependent (more precisely, one of them ends up in the ideal generated by the other 4, viewed now as elements of the polynomial ring \( \mathbb{C}[H^0(\mathcal{O}_C(1))] \)).

---

7 The width of \( W \) is the minimum degree of the polynomial \( Q(\phi) \) such that \( QW \) is in the ideal generated by the \( F_a s \).

8 Note that there is a mistake in the table in [5] listing the number of generations for this model.
is precisely the condition that the bundle $E$ defined by (2.1) splits nontrivially when restricted to $C$. When we deform the $F_a$s, however, the resulting bundle $E$ has trivial splitting type and the model is destabilized by worldsheet instantons [3,6].

Perhaps even more remarkable from the worldsheet instanton point of view, is that there are models in which generic choices of the $F_a$s lead to stable theories. In the Landau-Ginzburg description, however, there is nothing remarkable about them at all. There are simply no extra gluinos coming from twisted sectors to mess things up. We will look in detail at one such model in §4.1.

4. Examples

In this section, we present some examples of (0,2) Landau-Ginzburg vacua in greater detail. In §4.1 we provide a detailed discussion of an $SO(10)$ theory on a complete intersection in $W\mathbb{P}^5$. In somewhat less detail, we discuss in §4.2 the example in $W\mathbb{P}^{4,1,1,2,5}$ which was mentioned in §3.3.

4.1. The Complete Intersection $Y_{W5;4,4}$

Consider the (0,2) model which consists of a complete intersection Calabi-Yau manifold in $W\mathbb{P}^{5,1,1,1,2,2}$ defined by two equations of degree four (so $d_1 = d_2 = 4$), and a rank four vacuum gauge bundle with $\{n_a\} = \{1, 1, 1, 1, 1\}$ (so $m = 5$). This is the $Y_{W5;4,4}$ (0,2) model listed in [5].

Following the formalism of §2,3 we see that there should be a Landau-Ginzburg phase with field content and $U(1)$ charges given in table (3).

$x$ and $\omega$ are the lowest components of the “spectator” superfields; since they appear quadratically in $\bar{Q}_+$, they drop out in the computation of the massless spectrum and we ignore them henceforth.

Since $m = 5$, this model is a $\mathbb{Z}_5$ orbifold and there are sectors twisted by $g^k$ for $k = 0, \ldots, 9$. The vacuum energies and $U(1)$ charges of sectors $k = 0, \ldots, 5$ are given in table (4). The vacuum quantum numbers and spectra arising from higher $k$ sectors can be inferred from those of the $k = 0, \ldots, 5$ sectors by CPT.

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9 Equivalently, it is the condition that $H^0(E|_C \otimes \mathcal{O}_C(-1)) \neq 0$. 

19
The sectors of origin and $SO(8) \times U(1)$ quantum numbers of the the left-handed gauginos of $SO(10)$ (assuming a “generic” choice of defining data) are shown in table (5). The right-handed gauginos come from sectors $k = 0, 8, 9$ and are related to the above by CPT. For “non-generic” choices of the defining data one can obtain extra $U(1)$ gauginos from the $k = 1$ sector; these are similar to the extra $U(1)$ gauge symmetries that arise in Gepner models. Such gauginos are always accompanied by extra massless higgsinos, which give them a mass as one moves away from the point of enhanced symmetry.

$$SO(8) \times U(1) \text{ rep.} \quad k \quad \text{State}$$

| $8^s_{-2}$ | 0 | $|0\rangle$ |
| 280 | 1 | $\lambda^I_{-\frac{1}{2}} \lambda^J_{-\frac{1}{2}} |0\rangle$ |
| $1_0$ | 1 | $\left(\sum_i w_i \phi^i_{-\frac{3}{2}} \bar{\phi}^i_{-1+\frac{3}{2}} + 4 \sum \lambda^a_{-\frac{3}{2}} \bar{\lambda}^a_{-\frac{3}{2}} \right. \left. + 4 \sum \sigma^j_{-\frac{3}{2}} \bar{\sigma}^j_{-\frac{3}{2}} \right) |0\rangle$ |
| $8^s_{2}'$ | 2 | $|0\rangle$ |

**Table 5:** The left-handed $45$ ($\bar{q} = -3/2$) in the Landau-Ginzburg phase of $Y_{W5;4,4}$.
The generations, which are 16s of \( SO(10) \), arise as shown in table (6). There are generically no anti-generations. The degree 5 polynomial \( P(\phi) \) is subject to the equivalence relation

\[
P_5(\phi) \sim P_5(\phi) + c^a F_a(\phi) + s^j(\phi) W_j(\phi) .
\] (4.1)

There are 80 independent degree five polynomials modulo these relations, in agreement with the expected number of generations \( \frac{1}{2} c_3(E) \).

| \( SO(8) \times U(1) \) rep. | \( k \) | State |
|-----------------------------|-----|------|
| \( 8^a_{-1} \)           | 0   | \( P_5(\phi_i)|0\) |
| \( 8^a_1 \)               | 1   | \( \lambda^I_{-\frac{1}{4}} P_5(\phi_i)|0\) |

**Table 6**: Right-handed 16s (\( \bar{q} = -1/2 \)) in the Landau-Ginzburg phase of \( Y_{W5:4,4} \).

The left-handed 16’s which are the antiparticles of these arise in a slightly unobvious fashion. The \( 8^a_1 \) component arises as 80 states in the \( k = 0 \) sector of the form \( P_{11}(\phi) \bar{\lambda}^a|0\), \( P_{11}(\phi) \bar{\sigma}^j|0\) which are in the cohomology of \( \bar{Q}_{+L} \). The \( 8^a_{-1} \) component arises as 80 states in the \( k = 9 \) sector of the form \( \lambda^I P_5(\bar{\phi})|0\).

The left- and right-handed 10s of \( SO(10) \) arise as shown in table (7). The degree 10 polynomial \( P_{10}(\phi) \) is subject to the equivalence relation

\[
P_{10}(\phi) \sim P_{10}(\phi) + c^a(\phi) F_a(\phi) + s^j(\phi) W_j(\phi) .
\] (4.2)

Generically, there are 72 independent degree 10 polynomials \( P_{10}(\phi) \) modulo these relations. Two more 10s arise from twisted sectors. Slightly subtle are the \( 1_2 \) components of the left-handed 10s, which arise from the \( k = 5 \) sector. Though this is a NS sector, the left-moving fermions are untwisted. There are 42 states of the form \( \phi_0^{1,2} \times a \) fifth order polynomial in \{\( \bar{\lambda}_o^1, \ldots, \bar{\lambda}_o^5, \bar{\sigma}_o^1, \bar{\sigma}_o^2 \)\}. Of these, generically, two are in the cohomology of \( \bar{Q}_{+L} \). All in all, we find 74 left-handed and 74 right-handed 10s of \( SO(10) \).
SO(8) × U(1) rep. & Left-handed ($\bar{q} = \frac{1}{2}$) & Right-handed ($\bar{q} = -\frac{1}{2}$) \\
\hline
$S_0'$ & $k$ & State & $k$ & State \\
\hline
0 & $P_{10}(\phi^i_0)|0\rangle$ & 0 & $P_{6}(\phi^i_0)\lambda^a_0|0\rangle$, $P_{6}(\phi^i_0)\sigma^j_0|0\rangle$ \\
6 & $\phi^{1,2}_{-\frac{3}{5}}|0\rangle$ & 4 & $\bar{\phi}^{1,2}_{-\frac{3}{5}}|0\rangle$ \\
1 & $P_{10}(\phi^i_{-\frac{1}{2}})|0\rangle$ & 1 & $P_{6}(\phi^i_{-\frac{1}{2}})\lambda^a_{-\frac{3}{5}}|0\rangle$, $P_{6}(\phi^i_{-\frac{1}{2}})\sigma^j_{-\frac{3}{5}}|0\rangle$ \\
7 & $\phi^{1,2}_{-\frac{3}{5}}|0\rangle$ & 5 & $\phi^{1,2}_{-\frac{3}{5}}\prod_{i=5}^{47}\{\lambda^a_i,\sigma^j_i\}|0\rangle$ \\
9 & $P_{6}(\phi^i_{-\frac{1}{2}})\lambda^a_{-\frac{3}{5}}|0\rangle$, $P_{6}(\phi^i_{-\frac{1}{2}})\sigma^j_{-\frac{3}{5}}|0\rangle$ & 9 & $P_{10}(\phi^i_{-\frac{1}{2}})|0\rangle$ \\
5 & $\phi^0_{1,2}|0\rangle$ & 3 & $\bar{\phi}^{1,2}_{-\frac{3}{5}}|0\rangle$ \\
\hline

Table 7: 10s in the Landau-Ginzburg phase of $Y_{W5,4,4}$.

We can also find the spectrum of gauge singlet matter fields, which always originate in the (NS,R) sectors. The right-handed singlets are shown in table (8). The states in the $k = 1$ sector should be considered modulo the equivalence relations

\[
F_a(\phi^i_{-\frac{1}{2}})\lambda^a_{-\frac{3}{5}}|0\rangle \sim F_a(\phi^i_{-\frac{1}{2}})\sigma^j_{-\frac{3}{5}}|0\rangle \sim 0
\]
\[
W_j(\phi^i_{-\frac{1}{2}})\lambda^a_{-\frac{3}{5}}|0\rangle \sim W_j(\phi^i_{-\frac{1}{2}})\sigma^j_{-\frac{3}{5}}|0\rangle \sim 0
\]
\[
P_2(\phi^i_{-\frac{1}{2}}) \left[ \frac{\partial F_a}{\partial \phi^{1,2}_{-\frac{1}{5}}} \lambda^a_{-\frac{3}{5}} + \frac{\partial W_j}{\partial \phi^{1,2}_{-\frac{1}{5}}} \sigma^j_{-\frac{3}{5}} \right] |0\rangle \sim 0
\]
\[
P_1(\phi^{3,...6}_{-\frac{1}{10}}) \left[ \frac{\partial F_a}{\partial \phi^{3,...6}_{-\frac{1}{10}}} \lambda^a_{-\frac{3}{5}} + \frac{\partial W_j}{\partial \phi^{3,...6}_{-\frac{1}{10}}} \sigma^j_{-\frac{3}{5}} \right] |0\rangle \sim 0.
\]

Subjecting the states in the $k = 1$ sector to the equivalence relations (4.3), we find 318 gauge singlets. Adding to them the 21 obtained from quadratic, antisymmetric combinations of the $\lambda^a_{-\frac{1}{5}}$ and $\sigma^j_{-\frac{1}{5}}$ oscillators in the $k = 3$ sector, we find a total of 339 right-handed gauge singlets in this model.
$$SO(8) \times U(1) \text{ rep.} \quad k \quad \text{State}$$

|       |       |                                                                 |
|-------|-------|-----------------------------------------------------------------|
| $1_0$ | 1     | $P_4(\phi^i_{-\frac{1}{2}})\lambda^a_{-\frac{1}{2}}|0\rangle$, $P_4(\phi^i_{-\frac{3}{4}})\sigma^j_{-\frac{3}{4}}|0\rangle$ |
| $1_0$ | 3     | $\lambda^a_{-\frac{3}{4}}\lambda^b_{-\frac{1}{2}}|0\rangle$, $\lambda^a_{-\frac{1}{2}}\sigma^j_{-\frac{3}{4}}|0\rangle$, $\sigma^1_{-\frac{3}{4}}\sigma^2_{-\frac{1}{2}}|0\rangle$ |

Table 8: Right-handed singlets ($\bar{q} = -1/2$) in the Landau-Ginzburg phase of $Y_{W5;4,4}$.

**Comparison with Calabi-Yau expectations**

How does the spectrum we have found compare with that expected from the Calabi-Yau side? The $16$s agree both in number (80), and in deformation-theoretic representation\[3\]. From the definition of $E$ (2.1), it is clear that $H^1(E)$ has a deformation-theoretic representation as degree 5 polynomials modulo $\{F_a(\phi), W_j(\phi)\}$.

Of course, the net number of generations ($16$s minus $16'$s) is protected by an index theorem, and must be the same in the two phases. Not so the number of $10$s. A mass term for $10$s is not forbidden in the spacetime superpotential, and we might expect their number to jump as we move about in the moduli space.

Taking the second exterior power of (2.1), one finds that $H^2(\wedge^2 E)$ (which is Serre-dual to $H^1(\wedge^2 E)$) has a deformation-theoretic representation as degree 10 polynomials modulo $\{F_a(\phi), W_j(\phi)\}$\[12\]. This accounts for 72 of the 74 $10$s we have found. The other two (from the $k = 5, 6, 7$ sectors) have no analogues in the Calabi-Yau analysis. It would be interesting to see them pick up a mass as we move away from the Landau-Ginzburg point.

Finally, we turn to the singlets. In the Landau-Ginzburg phase of $(2,2)$ theories, one is able to further subdivide the gauge singlets into deformations of complex structure (elements of $H^1(T)$), deformations of Kähler structure (elements of $H^1(T^*)$) and deformations of the bundle $E$ (elements of $H^1(End E)$, where $E$ is taken to be the tangent bundle for a $(2,2)$ model)\[3\]. Some of the states we have found clearly cry out for such an interpretation. For instance, the states $P_4(\phi^i)\sigma^j|0\rangle$ in the $k = 1$ sector are naturally associated

---

10 One also finds that $H^1(\wedge^2 E)$ is isomorphic to the cohomology of the sequence $H^0(O(2)^{\oplus 10}) \rightarrow H^0(O(6)^{\oplus 5}) \rightarrow H^0(O(10))$, where $u(s_{abc}) = \epsilon^{abcde}s_{abc}F_d$ and $v(t^a) = t^aF_a$. After some work, one finds that this gives precisely the expressions that we found for 72 of the right-moving $10$s in terms of degree six polynomials in $\phi$. 23
with deformations of the corresponding polynomial $W_j(\phi)$. However, not all of the allowed deformations of $W_j$ are accounted for. In particular, we have modded out by deformations proportional to the $F_a(\phi)$. But the states $\lambda^a\sigma^j|0\rangle$ from the $k = 3$ sector precisely compensate for this error.

Similarly, the Calabi-Yau analysis (using the methods of [16]) tells us that an element of $H^1(\text{End } E)$ is specified by giving five degree 4 polynomials, modulo \{\$F_a, W_j$\}. Since there are 51 independent such polynomials, we expect to find 255 states corresponding to elements of $H^1(\text{End } E)$. Natural candidates are the states $P_4(\phi^i)\lambda^a|0\rangle$ in the $k = 1$ sector.

Since $\chi(M) = -144$, the Calabi-Yau analysis has accounted for 73 complex structure moduli, one Kähler modulus, and 255 elements of $H^1(\text{End } E)$, or 329 of the 339 singlets that we have found.

There was no guarantee that we would come even close to agreement, but we have done surprisingly well. There are ten extra singlet states in the $k = 3$ sector over what was expected from field theory, and there were two extra 10s.

Of course, it is probably wise not to take too seriously the distinction between moduli which deform the complex structure of $M$, and those which deform the holomorphic structure of $E$. In (0,2) Landau-Ginzburg theory, there is really no invariant distinction between them. As we shall see in §5, this opens up the possibility of topology-changing processes in (0,2) theories which are impossible in (2,2) theories.
4.2. The Hypersurface $Y_{W;4;10}$

Now we discuss, in somewhat less detail than the previous example, the (0,2) model which consists of a Calabi-Yau hypersurface in $W\mathbb{P}^4_{1,1,1,2,5}$ with a gauge bundle defined by $\{n_a\} = \{1,1,1,1,7\}$. This model serves as an illustration of the phenomenon described in §3.3; it only seems to make sense for highly restricted choices of the defining data, and is then an $E_6$ theory instead of an $SO(10)$ theory (which is what one would expect generically, since it has a rank four vacuum gauge bundle).

The Landau-Ginzburg phase has field content and $U(1)$ charges given in table (9) (we have omitted the “spectators” $x$ and $\omega$ of §2; they do not enter in any of the computations). Since $m = 11$, this model is a $\mathbb{Z}_{11}$ orbifold and there are sectors twisted by $g^k$ for $k = 0, \ldots 21$. The vacuum energies and $U(1)$ charges of sectors $k = 0, \ldots 11$ are given in table (10). The content of the other sectors can be inferred by CPT.

| Field    | Left $U(1)$ charge $q$ | Right $U(1)$ charge $\bar{q}$ |
|----------|------------------------|-------------------------------|
| $\phi_{1,2,3}$ | $\frac{1}{11}$ | $\frac{1}{11}$ |
| $\phi_4$ | $\frac{2}{11}$ | $\frac{2}{11}$ |
| $\phi_5$ | $\frac{5}{11}$ | $\frac{5}{11}$ |
| $\psi_{1,2,3}$ | $-\frac{10}{11}$ | $\frac{1}{11}$ |
| $\psi_4$ | $-\frac{9}{11}$ | $\frac{2}{11}$ |
| $\psi_5$ | $-\frac{6}{11}$ | $\frac{5}{11}$ |
| $\lambda_{1,\ldots,4}$ | $-\frac{10}{11}$ | $\frac{1}{11}$ |
| $\lambda_5$ | $-\frac{4}{11}$ | $\frac{7}{11}$ |
| $\sigma$ | $-\frac{10}{11}$ | $\frac{1}{11}$ |

**Table 9:** Left and right $U(1)$ charges of fields in Landau-Ginzburg phase of $Y_{W;4;10}$.

| $k$ | Vacuum $U(1)$ Charges $(q, \bar{q})$ | Vacuum Energy |
|-----|-------------------------------------|---------------|
| 0   | $(-2, -\frac{3}{2})$               | 0             |
| 1   | $(0, -\frac{3}{2})$                | -1            |
| 2   | $(2, -\frac{3}{2})$                | 0             |
| 3   | $(-\frac{6}{11}, -\frac{23}{22})$  | $-\frac{9}{11}$ |
| 4   | $(\frac{16}{11}, -\frac{23}{22})$  | $-\frac{1}{11}$ |
| 5   | $(-\frac{7}{11}, -\frac{23}{22})$  | $-\frac{13}{22}$ |
| 6   | $(1, -\frac{1}{2})$                | 0             |
| 7   | $(-\frac{17}{11}, -\frac{1}{2})$   | $-\frac{9}{22}$ |
| 8   | $(\frac{5}{11}, -\frac{1}{22})$    | $-\frac{2}{11}$ |
| 9   | $(-\frac{18}{11}, -\frac{3}{22})$  | $-\frac{4}{11}$ |
| 10  | $(\frac{4}{11}, -\frac{3}{22})$    | $-\frac{2}{11}$ |
| 11  | $(\frac{26}{11}, \frac{3}{22})$    | 0             |

**Table 10:** Quantum numbers of twisted sector vacua in Landau-Ginzburg phase of $Y_{W;4;10}$.

The Landau-Ginzburg superpotential is given explicitly by

$$W = \Lambda^a F_a(\phi) + \Sigma W(\phi)$$  \hspace{1cm} (4.4)
and as explained in §3.3, this model does not define a sensible string theory for “generic” choices of the defining data. (If it did, it would be an $SO(10)$ theory.) However, it does make sense and yields an $E_6$ theory on a subspace of the set of choices for $W$ and the $F_a$ where $W$ is taken to be of width one in the ideal generated by the $F_a$. Equivalently, we can choose $F_1$ to be of width one in the ideal generated by $W, F_2, \ldots, F_5$. A simple choice of data defining such a sensible model is given by

$$
F_1(\phi) = \phi_1\phi_2^0, \quad F_2(\phi) = \phi_2^{10},
$$

$$
F_3(\phi) = \phi_3^{10}, \quad F_4(\phi) = \phi_5^2,
$$

$$
F_5(\phi) = \phi_4^2
$$

with $W(\phi)$ a generic degree 10 polynomial chosen such that \{ $W(\phi) = 0$ \} is smooth and does not contain the point $\phi_1 \neq 0, \phi_2 = \ldots = \phi_5 = 0$.

The adjoint of $E_6$ decomposes under $SO(10)$ as $78 = 16' \oplus 45 \oplus 1 \oplus 16$. The states of this model are, however, more naturally arranged by their quantum numbers under the $SO(8) \times U(1)$ subgroup of $SO(10)$ which is linearly realized. In table (11) we list the left-handed gauginos of $E_6$ which come from the R ($k$ even) sectors. States from NS sectors fill out the rest of the $78$ of $E_6$. These are the $28_0 \oplus 1_0 \oplus 1_0 \oplus 8_v^\text{10}$ from the $k = 1$ sector and the $8_v^{\text{11}}$ from the $k = 3$ sector.

| $SO(8) \times U(1)$ rep. | $k$ | State |
|--------------------------|-----|-------|
| $8_{-2}'$               | 0   | $|0\rangle$ |
| $8_{-1}'$               | 0   | $(\phi_0^1 \lambda_0^2 - \phi_0^2 \lambda_0^1)|0\rangle$ |
| $8_{2}'$               | 2   | $|0\rangle$ |
| $8_{1}'$               | 4   | $\bar{\phi}_{-1}^5 | 0\rangle$ |

**Table 11:** Ramond sector states which form the left-handed $78$ ($\vec{q} = -3/2$) in the Landau-Ginzburg phase of $Y_{W4;10}$.

The generations of $E_6$ decompose under $SO(10)$ as $27 = 1 \oplus 16 \oplus 10$, which can be further decomposed under $SO(8) \times U(1)$. In table (12), we display the components of the right-handed $27$s which come from the $k$ even sectors. The polynomials $P_{11}(\phi)$ and $P_{22}(\phi)$ are always defined modulo the ideal generated by the $F_a(\phi)$ and $W(\phi)$. For $F_1$ of width 1 in the ideal generated by $F_2, \ldots, F_5, W$, there are exactly 165 states of each type when mod out by the ideal. So there are 165 $27$s of $E_6$.  

26
There is, in this model, also one right-handed anti-generation, a $\mathbf{27}$ whose components from $k$ even sectors are displayed in table (13).

| $SO(8) \times U(1)$ rep. | $k$ | State         | $SO(8) \times U(1)$ rep. | $k$ | State         |
|---------------------------|-----|---------------|---------------------------|-----|---------------|
| $8^s_{-1} (\subset 16)$  | 0   | $p_{11}(\phi_0^i)|0\rangle$ | $8^s_{1} (\subset 16')$ | 6   | $|0\rangle$  |
| $8^s_0' (\subset 10)$    | 0   | $p_{22}(\phi_0^i)|0\rangle$ | $8^s_0 (\subset 10)$     | 8   | $\bar{\phi}_{-\frac{5}{2}}^5|0\rangle$ |

**Table 12:** Right-handed generations in the Landau-Ginzburg phase of $Y_{W4;10}$.

**Table 13:** Right-handed anti-generation in the Landau-Ginzburg phase of $Y_{W4;10}$.

In particular, the Landau-Ginzburg theory produces a net of $165 - 1 = 164$ generations, in agreement with $\frac{1}{2} c_3(E)$ in the Calabi-Yau phase.

For “generic” choices of the data (4.5), the extra $8^s_{-1}$ gaugino from the $k = 0$ sector and the extra $8^v_1$ and $1_0$ gauginos from the $k = 1$ sector would be able to pair up with a matter field (the corresponding components of a $\mathbf{27}$) at $\bar{q} = -1/2$ and gain masses via the Higgs mechanism; this is no surprise, the $k = 0, 1$ sectors represent exactly the part of the Landau-Ginzburg theory corresponding to the field theory limit. But even for generic choices of the data, it is impossible for the extra gauginos from the $k = 3, 4$ sectors to become massive, because there are no available $\mathbf{27}$s in the same twisted sector. Thus, though we appear to have a sensible $E_6$ theory with this choice of the defining data, we cannot deform it to an $SO(10)$ theory.

5. Discussion

We have seen in §4 that the (0,2) Landau-Ginzburg theories have perfectly reasonable spectra, central charges, etc. This is strong evidence that the Landau-Ginzburg phase indeed describes fully-fledged (0,2) SCFTs. Since the Calabi-Yau phase is continuously connected to the LG phase, our conviction that the CY phase is also well-defined is strengthened. We have also seen that certain Landau-Ginzburg theories can be embedded sensibly in a string theory only for special values of their defining data because there is a quantum obstruction to breaking $E_6 \to SO(10)$. It remains to be seen how this restriction is recovered from the consideration of worldsheet instantons in the Calabi-Yau phase.

The microscopic Lagrangians of §2 are a powerful tool for exploring the phase structure of both (2,2) and (0,2) theories. In many cases, knowledge of the microscopic Lagrangian is enough to determine the global structure of the Kähler moduli space.
The simplest case is the one parameter Kähler moduli space with two phases, a Calabi-Yau phase for \( r \gg 0 \) and a Landau-Ginzburg phase for \( r \ll 0 \). For a (2,2) model defined on a degree \( d \) hypersurface in (weighted) \( \mathbb{P}^4 \), the global structure of the moduli space is that of the complex \( a \) plane, minus the points \( a^d = 1 \), modded out by \( \mathbb{Z}_d \) generated by \( a \rightarrow e^{2\pi i/d}a \). The point \( a = 0 \) corresponds to \( r = -\infty \), the Landau-Ginzburg point (which has a \( \mathbb{Z}_d \) quantum symmetry [17]). The points \( a^d = 1 \) all get mapped into the phase transition point \( r = \theta = 0 \), and \( a = \infty \) into \( r = \infty \). This picture can be verified explicitly using mirror symmetry [18,19].

For (0,2) theories of the sort we have been discussing, the only difference in the structure of the Kähler moduli space, as predicted by the microscopic Lagrangian, is that \( d \) is replaced by \( m \). For hypersurfaces in \( W\mathbb{P}^4 \), we always have those (0,2) theories which are deformations of the corresponding (2,2) theory, and hence have \( m = d \). But we also have theories, such as the example discussed in §4.2, for which \( m \neq d \). In these models, the global structure of the (0,2) Kähler moduli space is different from that of the corresponding (2,2) Kähler moduli space; in the example of §4.2, it is a \( \mathbb{Z}_{11} \), rather than a \( \mathbb{Z}_{10} \) orbifold.

If one generalizes only slightly to complete intersections in weighted projective space, the differences become even more pronounced. As we have already seen, the microscopic theory in the (2,2) case is in a hybrid phase for \( r \ll 0 \), whereas the (0,2) theory is in an ordinary Landau-Ginzburg phase.

Since we don’t have a terribly good handle on these hybrid theories, the microscopic Lagrangian is not much use in determining the structure of the Kähler moduli space for (2,2) complete intersections. Mirror symmetry, however, has yielded results, both for the \( Y_{W5;4,4} \) complete intersection that we discussed in §4.1, and for the closely related manifold \( Y_{W5;6,6} \), which is the complete intersection of two degree 6 polynomials in \( W\mathbb{P}^{5,3,3,2,2,1,1} \). Klemm and Theisen, in studying mirror symmetry for these two manifolds, found the global structure of the corresponding (2,2) Kähler moduli spaces [20]. The result is that the Kähler moduli space again has the general form

\[
\mathbb{C} - \{a_1^d + a_2^d = 1\} \quad \mod \mathbb{Z}_{d_1 + d_2} 
\]

(5.1)

where \( d_1 + d_2 = 8 \) for \( Y_{W5;4,4} \) and 12 for \( Y_{W5;6,6} \).

We have studied the (0,2) models on both of these manifolds. Details of the former were given in §4.1, the latter has \( \{n_a\} = \{1,1,1,1,4\} \) [3]. The Landau-Ginzburg points have, respectively, \( \mathbb{Z}_5 \) and \( \mathbb{Z}_8 \) quantum symmetries. Thus, the structure of the Kähler
moduli space of these (0,2) theories is as in (5.1), but with \( d_1 + d_2 \) replaced by \( m \), where \( m = 5, 8 \) respectively.

At large radius, the identifications in question simply correspond to shifts of the \( B_{\mu \nu} \) field by \( 2\pi \). What differs between the (2,2) theories and (0,2) theories is how this obvious symmetry continues down to small (negative infinite) radius.

The algebraic geometers, who through the study of mirror symmetry have only recently become accustomed to thinking about this “enlarged” Kähler moduli space are in for a shock. There is not one Kähler moduli space, but several. There is the (2,2) moduli space, with which they have become familiar, but there are also the Kähler moduli spaces which arise from (0,2) models and which, in general, have quite different global structure.

Perhaps the most exciting feature of the enlarged Kähler moduli space of (2,2) theories to emerge recently is that it may contain several CY phases corresponding to topologically distinct Calabi-Yau manifolds [7,21]. These different manifolds correspond to different ways of resolving the singularities of a single singular Calabi-Yau space.

Though we have, in this paper, confined ourselves to (0,2) models on smooth Calabi-Yau manifolds, nothing restricted us to this case. The Landau-Ginzburg theories make perfect sense, even when the corresponding Calabi-Yau manifold \( M \) is singular. In order to compare with the Calabi-Yau side, however, we need to specify both how the singularities of \( M \) are resolved and how the sheaf \( E \) is lifted to a bundle on the resolved space. If there are several different resolutions of \( M \), this leads to the phenomenon of [21,22]. There may also, however, be distinct liftings of \( E \) to the resolved space [23], which leads to the possibility of topologically distinct (0,2) models leading to the same Landau-Ginzburg theory.

An even wilder manifestation of topology-change is possible in (0,2) theories. Recall that at the Landau-Ginzburg point, there is no invariant distinction between the fermionic superfields \( \Sigma^j \) which multiply the \( W_j \) defining the manifold \( M \) and the fermionic superfields \( \Lambda^a \) which multiply the \( F_a \) defining the bundle \( E \). Therefore, we can contemplate an automorphism of the (0,2) Landau-Ginzburg theory which exchanges the roles of the \( \Sigma^j \)'s and the \( \Lambda^a \), and hence of the \( W_j \)'s and the \( F_a \)'s. Thus topologically distinct (0,2) models, defined on different manifolds, may be described by the same Landau-Ginzburg theory.

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11 And, indeed, between different (2,2) theories as well.
12 Enlarged, both because it contains the moduli of the \( B_{\mu \nu} \) field and because it describes, in addition to the Calabi-Yau phase, other phases which have no geometrical interpretation. The phrase “enlarged Kähler moduli space” was coined in [21,22].
An example of this is provided by complete intersections in $\mathbb{P}^5_{1,2,2,3,3}$. Consider the following two theories

1) $M_1$ is the complete intersection of a degree $d_1 = 5$ and a $d_2 = 8$ hypersurface. $E_1$ is the rank 3 bundle on $M_1$ defined by $\{n_a\} = \{1, 1, 2, 6\}$, $m = 10$. So the polynomials $\{W_1(\phi), W_2(\phi), F_1(\phi), F_2(\phi), F_3(\phi), F_4(\phi)\}$ which define $(M_1, E_1)$ have degrees $\{5, 8, 9, 9, 8, 4\}$. The Landau-Ginzburg superpotential for this theory is

$$W = \Sigma^j W_j(\Phi) + \Lambda^a F_a(\Phi)$$

2) $M_2$ is the complete intersection of a degree $d_1 = 4$ and a degree $d_2 = 9$ hypersurface. $E_2$ is the rank 3 bundle on $M_2$ defined by $\{n_a\} = \{1, 2, 2, 5\}$, $m = 10$. It is described by exactly the same Landau-Ginzburg superpotential, but with $\Sigma^1 \leftrightarrow \Lambda^4$, $\Sigma^2 \leftrightarrow \Lambda^2$.

As Landau-Ginzburg theories, $(M_1, E_1)$ and $(M_2, E_2)$ are isomorphic. They have exactly the same spectrum of states, and give rise to identical string theories. Geometrically, however, they are quite different. Following [24], we can calculate the orbifold Euler characteristics of $M_{1,2}$. They are different; $\chi(M_1) = -104$, and $\chi(M_2) = -108$. Here we have (0,2) theories on different manifolds, with different Euler characteristics, which are nonetheless isomorphic as Landau-Ginzburg theories.\footnote{The number of generations, of course, is the same in these two theories, $\frac{1}{2} c_3(E_1) = \frac{1}{2} c_3(E_2)$.}

From our brief encounter with (0,2) theories, it is clear that we have entered a rich and largely unexplored territory. It is likely that even greater surprises await us.

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Appendix : (0,2) Conventions

Our (0,2) superspace has coordinates \((z, \bar{z}, \theta^+, \theta^-)\). The spinor derivatives are
\[
\bar{D} = \frac{\partial}{\partial \theta^\pm} + \theta^\mp \partial \bar{z}.
\]
A chiral superfield \(\Phi\) has, as its lowest component a scalar \(\phi\), and satisfies the chiral constraint
\[
\bar{D} \Phi = 0
\]
In components, we have
\[
\Phi = \phi + \theta^- \psi + \theta^+ \partial \bar{z} \phi
\]
A Fermi superfield \(\Lambda\) satisfies the same chiral constraint \(\bar{D} \Lambda = 0\), but has as its lowest component a left-moving fermion \(\lambda\). Its component expansion is
\[
\Lambda = \lambda + \theta^- l + \theta^+ \partial \bar{z} \lambda
\]
where \(l\) is an auxiliary field.

We also need to introduce the (0,2) gauge multiplets. These consist of a real superfield \(V\), whose lowest component is a scalar, and a superfield \(\mathcal{A}\), whose lowest component is the left-moving component of the gauge field, \(a\). A super-gauge transformation is
\[
V \rightarrow V - i(\chi - \bar{\chi}), \quad \mathcal{A} \rightarrow \mathcal{A} - i(\chi + \bar{\chi})
\]
with \(\chi\) a chiral superfield \(\bar{D} \chi = \bar{D} \bar{\chi} = 0\). In Wess-Zumino gauge, the nonzero components are
\[
V = \theta^- \theta^+ \bar{a}
\]
\[
\mathcal{A} = a + \theta^+ \alpha - \theta^- \bar{\alpha} + \theta^- \theta^+ D
\]
where \(\bar{a}\) is the right-moving component of the gauge field, \(\alpha, \bar{\alpha}\) are left-moving gauginos, and \(D\) is an auxiliary field.

Under a super-gauge transformation,
\[
\Phi \rightarrow e^{2iQx} \Phi, \quad \bar{\Phi} \rightarrow e^{-2iQ \bar{x}} \bar{\Phi},
\]
so a gauge-invariant kinetic term for \(\Phi\) is
\[
S_{\Phi} = \frac{1}{2} \int d^2z d^2\theta \left( \partial - QA \right) \bar{\Phi} \Phi - \bar{\Phi} \left( \partial + QA \right) \Phi
\]
\[
= \int d^2z \left( \partial - Qa \right) \bar{\phi} (\partial + Q\bar{a}) \phi + \left( \partial - Q\bar{a} \right) \bar{\phi} (\partial + Qa) \phi + \bar{\psi} (\partial + Qa) \psi
\]
\[
+ Q(\bar{\alpha} \bar{\psi} - \alpha \psi \bar{\phi}) - Q \bar{\phi} \phi D
\]
where
\[ \tilde{\Phi} = e^{QV} \Phi, \quad \bar{\tilde{\Phi}} = e^{QV} \bar{\Phi}. \]

An invariant kinetic term for \( \Lambda \) is
\[
S_\Lambda = \frac{1}{2} \int d^2z d^2\theta \, \bar{\Lambda} e^{2QV} \Lambda \\
= \int d^2z \, \bar{\lambda}(\partial z + Q\bar{a}) \lambda + \frac{1}{2} \bar{\ell} \ell
\]

If we define the gauge-covariant spinor derivatives
\[ \mathcal{D}_\pm = e^{\pm V} D_\pm e^{\mp V}, \quad D = \partial + A, \]
we can write the gauge-invariant field strengths
\[
\mathcal{F} = [\mathcal{D}, \bar{\mathcal{D}}_+] = -\alpha + \theta^-(f + D) - \theta^+ \partial z \alpha \\
\bar{\mathcal{F}} = [\mathcal{D}, \bar{\mathcal{D}}_-] = \bar{\alpha} + \theta^+(f - D) - \theta^- \partial z \bar{\alpha}
\]
where \( f = \partial z \bar{a} - \partial z \alpha \). The gauge kinetic term is
\[
S_\mathcal{F} = -\frac{1}{2e^2} \int d^2z d^2\theta \, \mathcal{F} \bar{\mathcal{F}} \\
= \frac{1}{e^2} \int d^2z \, \left( -\frac{1}{2} f^2 + \frac{1}{2} D^2 - \alpha \partial z \bar{\alpha} \right).
\]

The Fayet-Iliopoulos D-term and the \( U(1) \) \( \theta \)-term can be written as
\[
S_D = \frac{t}{2} \int d^2z d\theta^- \mathcal{F} - \frac{\bar{t}}{2} \int d^2z d\theta^+ \bar{\mathcal{F}} \\
= r \int d^2z \, D + i\theta \int d^2z f
\]
where \( t = r + i\theta \).

Finally, a \((0,2)\) superpotential has the form
\[
S_W = \int d^2z d\theta^- \Lambda F(\Phi) + \int d^2z d\theta^+ \bar{\Lambda} \bar{F}(\Phi) \\
= \int d^2z \, \left( lF(\phi) - \lambda \frac{\partial F}{\partial \phi} \psi \right) + \text{h.c.}
\]
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