General Doubly Stochastic Maximum Principle and Its
Applications to Optimal Control of Stochastic Partial
Differential Equations

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Abstract

In this paper, we prove the necessary and sufficient maximum principles (NSMPs in short) for the optimal control of systems described by a quasilinear stochastic heat equation within convex control domains, which all the coefficients contain control variables. For that, the optimal control problem of fully coupled forward-backward doubly stochastic system is studied. We apply our NSMPs to treat a kind of forward-backward doubly stochastic linear quadratic optimal control problems and an example of optimal control of stochastic partial differential equations (SPDEs in short) as well.

Key words: Maximum principle, forward-backward doubly stochastic differential equation, convex perturbation, stochastic partial differential equation, Malliavin calculus.

1 Introduction

In order to provide a probabilistic interpretation for the solutions of a class of quasilinear stochastic partial differential equations (SPDEs in short), Pardoux and Peng [14] introduced

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the following backward doubly stochastic differential equation (BDSDE in short):
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overline{B}_s - \int_t^T Z_s d\overline{W}_s. 
\] (1.1)

Note that the integral with respect to \( \{B_t\} \) is a “backward Itô integral” and the integral with respect to \( \{W_t\} \) is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral (for more details see [10] and [14]). Pardoux and Peng [14] have obtained the relationship between BDSDEs and a certain quasilinear stochastic partial differential equations (SPDEs in short). More precisely
\[
\begin{aligned}
\left\{ 
\begin{array}{l}
u(t, x) = \phi(x) + \int_t^T \left[Lu(s, x) + f(s, x, \nu(s, x), (\nabla \nu)(s, x))\right] ds \\
+ \int_t^T g(s, x, \nu(s, x), (\nabla \nu)(s, x))d\overline{B}_s, \\
\end{array}
\right.
\end{aligned}
\]

where \( \nu : [0, T] \times \mathbb{R}^d \to \mathbb{R}^k \) where \( d, k \in \mathbb{N} \), and \( \nabla \nu(s, x) \) denotes the first order derivative of \( \nu(s, x) \) with respect to \( x \), and
\[
L = \begin{pmatrix} Lu_1 \\ \vdots \\ Lu_k \end{pmatrix},
\]
with
\[
L\phi(x) = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij}(x) \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial \phi(x)}{\partial x_i}
\]
(for more details see [14]).

In 2003, Peng and Shi [17] introduced a type of time-symmetric forward-backward stochastic differential equations, i.e., so-called fully coupled forward-backward doubly stochastic differential equations (FBDSDE in short):
\[
\begin{aligned}
\left\{ 
\begin{array}{l}
y_t = x + \int_0^t f(s, y_s, Y_s, z_s, Z_s)ds + \int_0^t g(s, y_s, Y_s, z_s, Z_s)d\overline{W}_s - \int_0^t z_s d\overline{B}_s, \\
Y_t = \phi(y_T) + \int_t^T F(s, y_s, Y_s, z_s, Z_s)ds + \int_t^T G(s, y_s, Y_s, z_s, Z_s)d\overline{B}_s + \int_t^T Z_s d\overline{W}_s,
\end{array}
\right.
\] (1.2)

In FBDSDEs (1.2), the forward equation is “forward” with respect to a standard stochastic integral \( d\overline{W}_t \), as well as “backward” with respect to a backward stochastic integral \( d\overline{B}_t \); the coupled “backward equation” is “forward” under the backward stochastic integral \( d\overline{B}_t \) and “backward” under the forward one. In other words, both the forward equation and the backward one are types of BDSDE (1.1) with different directions of stochastic integrals. So (1.2) provides a very general framework of fully coupled forward-backward stochastic systems. Peng and Shi [17] proved the existence and uniqueness of solutions to FBDSDE
(1.2) with arbitrarily fixed time duration under some monotone assumptions. FBDSDE (1.2) can provide a probabilistic interpretation for the solutions of a general class of quasilinear SPDEs.

In this paper, we consider the following quasilinear SPDEs with control variable:

\[
\begin{align*}
\left\{ \begin{array}{l}
u(t,x) = \varphi(x) + \int_t^T [L^v u(s,x) + F(s,x,u(s,x),(\nabla u g)(s,x,u),v(s))] \, ds \\
+ \int_t^T G(s,x,u(s,x),(\nabla u g)(s,x,u),v(s)) \, dB_s, \quad 0 \leq t \leq T,
\end{array} \right.
\end{align*}
\tag{1.3}
\]

where \( u : [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k \) and \( \nabla u(s,x) \) denotes the first order derivative of \( u(s,x) \) with respect to \( x \), and

\[
L^v u = \begin{pmatrix}
L^v u_1 \\
\vdots \\
L^v u_k
\end{pmatrix},
\]

with

\[
L^v \phi(x) = \frac{1}{2} \sum_{i,j=1}^d (gg^*)_{ij} (x,u,v) \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d f_i(x,u,v) \frac{\partial \phi(x)}{\partial x_i}.
\]

It is worth to pointing out that all the coefficients contain the control variable. (For more details see Section 5).

Let us describe the problem solved in this paper. Set \( U_{ad} \) be an admissible control set. The definitions of notations used here can be found in Section 2. The optimal control problem of SPDEs (1.3) is to find an optimal control \( v^* (\cdot) \in U_{ad} \), such that

\[
J(v^*(\cdot)) = \inf_{v(\cdot) \in U_{ad}} J(v(\cdot)),
\]

where \( J(\cdot) \) is its cost function as follows:

\[
J(v(\cdot)) = \mathbb{E} \left[ \int_0^T l(s,x,u(s,x),(\nabla u \sigma)(s,x,u),v(s)) \, ds + \gamma(u(0,x)) \right]. \tag{1.4}
\]

As we have known, stochastic control problem of the SPDEs arising from partial observation control has been studied by Mortensen [9], using a dynamic programming approach, and subsequently by Bensoussan [2], [3], using a maximum principle method. See [4], [15] and the references therein for more information. Our approach differs from the one of Bensoussan. More precisely, we relate the FBDSDE to one kind of SPDEs with control variables where the control systems of SPDEs can be transformed to the relevant control systems of FBDSDE. To our knowledge, this is the first time to treat the optimal control problems of SPDEs from a new perspective of FBDSDE. It is worth mentioning that the quasilinear SPDEs in [12] Øksendal considered can just be related to our partially coupled FBDSDE. Recently, Zhang and Shi [25], obtained the similar results, however, in their paper, the coefficients \( \sigma \) and \( g \) do not contain the control variable, respectively.

This paper is organized as follows. Section 2 is devoting to stating the problems and some assumptions. In Section 3 and Section 4, we give the necessary and sufficient maximum
principles for fully couple forward-backward doubly stochastic control systems, respectively, in global form. As an application, we study the optimal control of SPDEs in Section 5. For simplicity of notations, we consider the one-dimensional case. It is necessary to point out that all the results can be extended to multi-dimensional cases. Finally, in Section 6 our results are further illustrated by solving optimal controls of LQ problem and a special SPDEs using the Malliavin calculus, respectively.

## 2 Statement of the problems

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space, $\{W_t\}_{t \geq 0}$ and $\{B_t\}_{t \geq 0}$ be two mutually independent standard Brownian motions, with value respectively in $\mathbb{R}^d$ and $\mathbb{R}^t$, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of $P$-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$
\mathcal{F}_t^W = \sigma \{W_r; \ 0 \leq r \leq t\} \bigvee \mathcal{N}, \quad \mathcal{F}_{t,T}^B = \sigma \{B_r - B_t; \ t \leq r \leq T\} \bigvee \mathcal{N},
$$

and

$$
\mathcal{F}_t = \mathcal{F}_t^W \bigvee \mathcal{F}_{t,T}^B, \ \forall t \in [0, T].
$$

Note that $\{\mathcal{F}_t^W; t \in [0, T]\}$ is an increasing filtration and $\{\mathcal{F}_{t,T}^B; t \in [0, T]\}$ is a decreasing filtration, and the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing.

We denote $M^2(0, T; \mathbb{R}^n)$ the space of (class of $dP \otimes dt$ a.e equal) all $\{\mathcal{F}_t\}$-measurable $n$-dimensional processes $v$ with norm of $\|v\|_M = \left[ \mathbb{E} \int_0^T |v(s)|^2 ds \right]^{\frac{1}{2}} < \infty$. Obviously $M^2(0, T; \mathbb{R}^n)$ is a Hilbert space. For any given $u \in M^2(0, T; \mathbb{R}^n)$ and $v \in M^2(0, T; \mathbb{R}^n)$, one can define the (standard) forward Itô’s integral $\int_0^T u_s d\bar{W}_s$ and backward Itô’s integral $\int_T^0 v_s d\bar{B}_s$. They are both in $M^2(0, T; \mathbb{R}^n)$, (see [14] for details).

Let $L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ denote the space of all $\{\mathcal{F}_T\}$-measurable $\mathbb{R}^n$-valued random variable $\xi$ satisfying $\mathbb{E}|\xi|^2 < \infty$.

**Definition 1.** A stochastic process $X = \{X_t; t \geq 0\}$ is called $\mathcal{F}_t$-progressively measurable, if for any $t \geq 0$, $X$ on $\Omega \times [0, t]$ is measurable with respect to $(\mathcal{F}_t^W \times \mathcal{B}([0, t])) \lor (\mathcal{F}_{t,T}^B \times \mathcal{B}([t, T]))$.

For simplicity of notations, hereafter we consider the one-dimensional case. In the present paper all the results can be extended to multi-dimensional cases. Under this framework, we consider the following forward-backward doubly stochastic control system

$$
\begin{align*}
& \text{d}y(t) = f(t, y(t), Y(t), z(t), Z(t), v(t)) \text{d}t + g(t, y(t), Y(t), z(t), Z(t), v(t)) \text{d}\bar{W}_t - z(t) \text{d}\bar{B}_t, \\
& \text{d}Y(t) = -F(t, y(t), Y(t), z(t), Z(t), v(t)) \text{d}t - G(t, y(t), Y(t), z(t), Z(t), v(t)) \text{d}B_t + Z(t) \text{d}\bar{W}_t, \\
& y(0) = x_0, \quad Y(T) = \varphi(y(T)) ,
\end{align*}
$$

(2.1)
where \((y(t), Y(t), z(t), Z(t), v(t)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, x_0 \in \mathbb{R}\), is a given constant, \(t > 0\) and \(T > 0\),

\[
F : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},
\]

\[
f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},
\]

\[
G : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},
\]

\[
g : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},
\]

\[
\varphi : \mathbb{R} \to \mathbb{R}.
\]

Let \(\mathcal{U}\) be a nonempty convex subset of \(\mathbb{R}\). We define the admissible control set

\[
\mathcal{U}_{ad} \doteq \{v(\cdot) \in M^2(0, T; \mathbb{R}); v(t) \in \mathcal{U}, 0 \leq t \leq T, \text{ a.e., a.s.}\}.
\]

Our optimal control problem is to minimize the cost function:

\[
J(v(\cdot)) = \mathbb{E} \left[ \int_0^T l(t, y(t), Y(t), z(t), Z(t), v(t)) \, dt + \Phi(y(T)) + \gamma(Y(0)) \right]
\] (2.2)

over \(\mathcal{U}_{ad}\), where

\[
l : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},
\]

\[
\Phi : \mathbb{R} \to \mathbb{R},
\]

\[
\gamma : \mathbb{R} \to \mathbb{R}.
\]

An admissible control \(u(\cdot)\) is called an optimal control if it attains the minimum over \(\mathcal{U}_{ad}\). That is to say, we want to find a \(u(\cdot)\), such that

\[
J(u(\cdot)) \doteq \inf_{v(\cdot) \in \mathcal{U}_{ad}} J(v(\cdot)).
\]

(2.1) is called the state equation, the solution \((y(\cdot), Y(\cdot), z(\cdot), Z(\cdot))\) corresponding to \(u(\cdot)\) is called the optimal trajectory. Next we will give some notations:

\[
\zeta = \begin{pmatrix} Y \\ Z \end{pmatrix}, \quad A(t, \zeta) = \begin{pmatrix} -F & -G \\ f & g \end{pmatrix}(t, \zeta).
\]

We use the usual inner product \(\langle \cdot, \cdot \rangle\) and Euclidean norm \(|\cdot|\) in \(\mathbb{R}, \mathbb{R}^l,\) and \(\mathbb{R}^d\). All the equalities and inequalities mentioned in this paper are in the sense of \(dt \otimes dP\) almost surely on \([0, T] \times \Omega\). We assume that

(H1) \(\left\{\begin{array}{l}
\text{For each } \zeta \in \mathbb{R}^{1+1+1+1}, A(\cdot, \zeta) \text{ is an } \mathcal{F}_t\text{-measurable process defined on } [0, T] \\
\text{with } A(\cdot, 0) \in M^2(0, T; \mathbb{R}^{1+1+1+1}).
\end{array}\right.\)

(H2) \(A(t, \zeta)\) and \(\varphi(y)\) satisfy Lipschitz conditions: there exists a constant \(k > 0\), such that

\[
\left\{\begin{array}{l}
|A(t, \zeta) - A(t, \tilde{\zeta})| \leq k |\zeta - \tilde{\zeta}|, \quad \forall \zeta, \tilde{\zeta} \in \mathbb{R}^{1+1+1+1}, \forall t \in [0, T], \\
|\varphi(y) - \varphi(\tilde{y})| \leq k |y - \tilde{y}|, \quad \forall y, \tilde{y} \in \mathbb{R}.
\end{array}\right.
\]
The following monotonic conditions introduced in [17], are the main assumptions in this paper.

\[ \langle A(t, \zeta) - A(t, \bar{\zeta}), \zeta - \bar{\zeta} \rangle \leq -\mu |\zeta - \bar{\zeta}|^2, \]

\[ \forall \zeta = (y, Y, z, Z)^T, \bar{\zeta} = (\bar{y}, \bar{Y}, \bar{z}, \bar{Z})^T \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \forall t \in [0,T]. \]

or

\[ \langle A(t, \zeta) - A(t, \bar{\zeta}), \zeta - \bar{\zeta} \rangle \geq \mu |\zeta - \bar{\zeta}|^2, \]

\[ \forall \zeta = (y, Y, z, Z)^T, \bar{\zeta} = (\bar{y}, \bar{Y}, \bar{z}, \bar{Z})^T \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \forall t \in [0,T]. \]

Proposition 2. For any given admissible control \( v(\cdot) \), we assume (H1), (H2) and (H3) (or (H1), (H2) and (H3)' ) hold. Then FBDSDE (2.1) has the unique solution \( (y(\cdot), Y(\cdot), z(\cdot), Z(\cdot)) \in M^2(0, T; \mathbb{R}^{1+1+1+1+1}). \)

The proof of Proposition 2 can be seen in [17] and [26]. We assume:

1) \( F, f, G, g, \varphi, l, \Phi, \gamma \) are continuously differentiable with respect to \( (y, Y, z, v) \), \( y \) and \( Y \);
2) The derivatives of \( F, f, G, g, \varphi \) are bounded;
3) The derivatives of \( l \) are bounded by \( C (1 + |y| + |Y| + |z| + |Z| + |v|) \);
4) The derivatives of \( \Phi \) and \( \gamma \) with respect to \( y, Y \) are bounded by \( C (1 + |y|) \) and \( C (1 + |Y|) \), respectively.

Lastly, we need the following extension of Itô’s formula (for more details see [14]).

Proposition 3. Let \( \alpha \in S^2 ([0,T]; \mathbb{R}^k), \beta \in M^2 ([0,T]; \mathbb{R}^k), \gamma \in M^2 ([0,T]; \mathbb{R}^{k \times t}), \delta \in S^2 ([0,T]; \mathbb{R}^{k \times d}) \) satisfy: \( \alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \delta_s dW_s, \quad 0 \leq t \leq T. \) Then

\[ |\alpha_t|^2 = |\alpha_0|^2 + 2 \int_0^t \langle \alpha_s, \beta_s \rangle ds + 2 \int_0^t \langle \alpha_s, \gamma_s dB_s \rangle + 2 \int_0^t \langle \alpha_s, \delta_s dW_s \rangle - \int_0^t |\gamma_s|^2 ds + \int_0^t |\delta_s|^2 ds, \]

\[ \mathbb{E} |\alpha_t|^2 = \mathbb{E} |\alpha_0|^2 + 2 \mathbb{E} \int_0^t \langle \alpha_s, \beta_s \rangle ds - \mathbb{E} \int_0^t |\gamma_s|^2 ds + \mathbb{E} \int_0^t |\delta_s|^2 ds. \]

More generally, if \( \phi \in C^2 (\mathbb{R}^k) \),

\[ \phi (\alpha_t) = \phi (\alpha_0) + \int_0^t \langle \phi' (\alpha_s), \beta_s \rangle ds + \int_0^t \langle \phi' (\alpha_s), \gamma_s dB_s \rangle + \int_0^t \langle \phi' (\alpha_s), \delta_s dW_s \rangle - \frac{1}{2} \int_0^t \text{Tr} \left[ \phi'' (\alpha_s) \gamma_s \gamma_s^* \right] ds + \frac{1}{2} \int_0^t \text{Tr} \left[ \phi'' (\alpha_s) \delta_s \delta_s^* \right] ds. \]
S^2 (0, T; \mathbb{R}^k) denotes the space of (classes of \, dt \otimes dP \text{ a.e.}) all \mathcal{F}_t\text{-progressively measurable } k\text{-dimensional processes } v \text{ with}
\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |v(t)|^2 \right) < \infty. \]

3 \quad \text{A necessary maximum principle for optimal controls of forward-backward doubly stochastic control systems}

We consider the forward-backward doubly stochastic control system (2.1) and the cost function (2.2). Let \( u(\cdot) \) be an optimal control and \( (y(\cdot), Y(\cdot), z(\cdot), Z(\cdot)) \) be the corresponding trajectory. Let \( v(\cdot) \) be any given admissible control such that \( u(\cdot) + v(\cdot) \in \mathcal{U}_{ad} \). Since \( \mathcal{U}_{ad} \) is convex, then for any \( 0 \leq \rho \leq 1 \), \( u_\rho(\cdot) = u(\cdot) + \rho v(\cdot) \) is also in \( \mathcal{U}_{ad} \).

We introduce the following variational equation of FBDSDE (2.1):

\[
\begin{align*}
\text{d}y^1(t) &= [f_y(t, y(t), Y(t), z(t), Z(t), u(t)) y^1(t) \\
& \quad + f_Y(t, y(t), Y(t), z(t), Z(t), u(t)) Y^1(t) \\
& \quad + f_z(t, y(t), Y(t), z(t), Z(t), u(t)) z^1(t) \\
& \quad + f_Z(t, y(t), Y(t), z(t), Z(t), u(t)) Z^1(t) \\
& \quad + f_v(t, y(t), Y(t), z(t), Z(t), u(t)) v(t)] \text{d}t \\
& \quad + [g_y(t, y(t), Y(t), z(t), Z(t), u(t)) y^1(t) \\
& \quad + g_Y(t, y(t), Y(t), z(t), Z(t), u(t)) Y^1(t) \\
& \quad + g_z(t, y(t), Y(t), z(t), Z(t), u(t)) z^1(t) \\
& \quad + g_Z(t, y(t), Y(t), z(t), Z(t), u(t)) Z^1(t) \\
& \quad + g_v(t, y(t), Y(t), z(t), Z(t), u(t)) v(t)] \text{d}W_t - z^1(t) \text{d}\bar{B}_t,
\end{align*}
\]

\[
\begin{align*}
\text{d}Y^1(t) &= -[F_y(t, y(t), Y(t), z(t), Z(t), u(t)) y^1(t) \\
& \quad + F_Y(t, y(t), Y(t), z(t), Z(t), u(t)) Y^1(t) \\
& \quad + F_z(t, y(t), Y(t), z(t), Z(t), u(t)) z^1(t) \\
& \quad + F_Z(t, y(t), Y(t), z(t), Z(t), u(t)) Z^1(t) \\
& \quad + F_v(t, y(t), Y(t), z(t), Z(t), u(t)) v(t)] \text{d}t \\
& \quad - [G_y(t, y(t), Y(t), z(t), Z(t), u(t)) y^1(t) \\
& \quad + G_Y(t, y(t), Y(t), z(t), Z(t), u(t)) Y^1(t) \\
& \quad + G_z(t, y(t), Y(t), z(t), Z(t), u(t)) z^1(t) \\
& \quad + G_Z(t, y(t), Y(t), z(t), Z(t), u(t)) Z^1(t) \\
& \quad + G_v(t, y(t), Y(t), z(t), Z(t), u(t)) v(t)] \text{d}\bar{B}_t + Z^1(t) \text{d}\bar{W}_t,
\end{align*}
\]

\[ y^1(0) = 0, \quad Y^1(t) = \varphi_y(y(T)) y^1(T). \]

From (H3), (H4) and Proposition 2, it is easy to check that (3.1) satisfies (H1), (H2) and (H3). Then there exists a unique quadruple of \((y^1(t), Y^1(t), z^1(t), Z^1(t))\) in \(M^2(0, T)\) satisfying FBDSDE (3.1). We denote by \((y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t))\) the trajectory of FBDSDE (2.1)
corresponding to $u_\rho (\cdot)$ as follows.

$$
\begin{align*}
\text{d}y_\rho (t) &= f(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u_\rho (t)) \, \text{d}t \\
&\quad + g(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u_\rho (t)) \, \text{d}\overrightarrow{W}_t - z_\rho (t) \, \text{d}\overleftarrow{B}_t, \\
\text{d}Y_\rho (t) &= -F(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u_\rho (t)) \, \text{d}t \\
&\quad - G(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u_\rho (t)) \, \text{d}\overleftarrow{B}_t \\
y_\rho (0) &= x_0, \quad Y_\rho (T) = \varphi (y_\rho (T)),
\end{align*}
$$

Then we will study the solutions to forward-backward doubly stochastic control systems with parameter.

**Lemma 4.** Assume that $(H1)$-$\text{(H4)}$ hold. Then we have

$$
\begin{align*}
\lim_{\rho \to 0} \frac{y_\rho (t) - y(t)}{\rho} &= y^1(t), \\
\lim_{\rho \to 0} \frac{Y_\rho (t) - Y(t)}{\rho} &= Y^1(t), \\
\lim_{\rho \to 0} \frac{z_\rho (t) - z(t)}{\rho} &= z^1(t), \\
\lim_{\rho \to 0} \frac{Z_\rho (t) - Z(t)}{\rho} &= Z^1(t),
\end{align*}
$$

where the limits are in $M^2(0, T)$.

**Proof.** Firstly, we show the continuous dependence of solutions with respect to the parameter $\rho$. Let

$$
\begin{align*}
\hat{y}(t) &= y_\rho (t) - y(t), \\
\hat{Y}(t) &= Y_\rho (t) - Y(t), \\
\hat{z}(t) &= z_\rho (t) - z(t), \\
\hat{Z}(t) &= Z_\rho (t) - Z(t).
\end{align*}
$$
We have

\[
\begin{align*}
\mathrm{d}\dot{y}(t) &= \left[f(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t) + \rho v (t))
- f(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t))
+ f(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t)) \right] \mathrm{d}t \\
\quad &+ \left[g(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t) + \rho v (t))
- g(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t)) \right] \mathrm{d}B_t, \\
\end{align*}
\]

\[
\begin{align*}
\mathrm{d}\dot{Y}(t) &= -\left[F(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t) + \rho v (t))
- F(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t)) \right] \mathrm{d}t \\
&+ \left[G(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t) + \rho v (t))
- G(t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t)) \right] \mathrm{d}B_t \\
&+ \left[\varphi (y_\rho (T)) - \varphi (y (T)) \right] \mathrm{d}W_t, \\
\end{align*}
\]

\[
\begin{align*}
\dot{y}(0) = 0, \quad \dot{Y}(T) = \varphi (y_\rho (T)) - \varphi (y (T)). \\
\end{align*}
\]

We will prove \(\left(\dot{y}(t), \dot{Y}(t), \dot{z}(t), \dot{Z}(t)\right)\) converge to 0 in \(M^2(0, T)\) as \(\rho \to 0\). Applying Itô’s
Thus we get \( \mathbb{E} \langle \dot{y}(T), \varphi(y_{\rho}(T)) - \varphi(y(T)) \rangle \)
\[
E \int_0^T \langle A(t, \xi_{\rho}) - A(t, \xi), \xi_{\rho} - \xi \rangle \, dt 
- E \int_0^T \dot{y}(t) [F(t, y_{\rho}(t), Y_{\rho}(t), z_{\rho}(t), Z_{\rho}(t), u(t) + \rho v(t)] 
- F(t, y_{\rho}(t), Y_{\rho}(t), z_{\rho}(t), Z_{\rho}(t), u(t)] \, dt 
+ E \int_0^T \dot{Y}(t) [f(t, y_{\rho}(t), Y_{\rho}(t), z_{\rho}(t), Z_{\rho}(t), u(t) + \rho v(t)] 
- f(t, y_{\rho}(t), Y_{\rho}(t), z_{\rho}(t), Z_{\rho}(t), u(t)] \, dt 
- E \int_0^T \dot{z}(t) [G(t, y_{\rho}(t), Y_{\rho}(t), z_{\rho}(t), Z_{\rho}(t), u(t) + \rho v(t)] 
- G(t, y_{\rho}(t), Y_{\rho}(t), z_{\rho}(t), Z_{\rho}(t), u(t)] \, dt 
+ E \int_0^T \dot{Z}(t) [g(t, y_{\rho}(t), Y_{\rho}(t), z_{\rho}(t), Z_{\rho}(t), u(t) + \rho v(t)] 
- g(t, y_{\rho}(t), Y_{\rho}(t), z_{\rho}(t), Z_{\rho}(t), u(t)] \, dt 
\leq \ - \mu E \int_0^T \left[ |\dot{y}(t)|^2 + |\dot{Y}(t)|^2 + |\dot{z}(t)|^2 + |\dot{Z}(t)|^2 \right] \, dt 
+ \frac{\mu}{4} \frac{E}{4} \int_0^T \left[ |\dot{y}(t)|^2 + |\dot{Y}(t)|^2 + |\dot{z}(t)|^2 + |\dot{Z}(t)|^2 \right] \, dt 
+ \frac{1}{\mu} \rho^2 C E \int_0^T |v(t)|^2 \, dt,
\]
where
\[
\xi_{\rho}(t) = (y_{\rho}(t), Y_{\rho}(t), z_{\rho}(t), Z_{\rho}(t), u(t))^T, \\
\xi(t) = (y(t), Y(t), z(t), Z(t), u(t))^T, \\
A(t, \xi) = \begin{pmatrix} - F(t, \xi) \\ f(t, \xi) \\ - G(t, \xi) \\ g(t, \xi) \end{pmatrix}, \quad A(t, \xi_{\rho}) = \begin{pmatrix} - F(t, \xi_{\rho}) \\ f(t, \xi_{\rho}) \\ - G(t, \xi_{\rho}) \\ g(t, \xi_{\rho}) \end{pmatrix}.
\]
Thus we get
\[
E \int_0^T \left[ |\dot{y}(t)|^2 + |\dot{Y}(t)|^2 + |\dot{z}(t)|^2 + |\dot{Z}(t)|^2 \right] \, dt \leq \rho^2 C E \int_0^T |v(t)|^2 \, dt.
\]
Then it follows that \((\tilde{y}(t), \tilde{Y}(t), \tilde{z}(t), \tilde{Z}(t))\) converge to 0 in \(M^2(0,T)\) as \(\rho\) tends to 0. Set

\[
\begin{align*}
\triangle y(t) &= \frac{y_\rho(t) - y(t)}{\rho}, \\
\triangle Y(t) &= \frac{Y_\rho(t) - Y(t)}{\rho}, \\
\triangle z(t) &= \frac{z_\rho(t) - z(t)}{\rho}, \\
\triangle Z(t) &= \frac{Z_\rho(t) - Z(t)}{\rho},
\end{align*}
\]

then

\[
\begin{cases}
    d\triangle y(t) = \frac{f(t,y_\rho(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-f(t,y(t),Y(t),z(t),Z(t),u(t))}{\rho} \, dt + \frac{g(t,y_\rho(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-g(t,y(t),Y(t),z(t),Z(t),u(t))}{\rho} \, d\overrightarrow{B_t}, \\
    -\triangle z(t) \, d\overrightarrow{W_t}, \\
    -d\triangle Y(t) = \frac{F(t,y_\rho(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-F(t,y(t),Y(t),z(t),Z(t),u(t))}{\rho} \, dt + \frac{G(t,y_\rho(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-G(t,y(t),Y(t),z(t),Z(t),u(t))}{\rho} \, d\overrightarrow{B_t}, \\
    -\triangle Z(t) \, d\overrightarrow{W_t}, \\
\end{cases}
\]

\[
\triangle y(0) = 0, \quad \triangle Y(T) = \frac{\varphi(y_\rho(T)) - \varphi(y(T))}{\rho}.
\]

The above equations can be expressed as follows

\[
\begin{cases}
    d\tilde{y}(t) = \tilde{f}(t, \triangle y(t), \triangle Y(t), \triangle z(t), \triangle Z(t), \varrho(t)) \, dt + \tilde{g}(t, \triangle y(t), \triangle Y(t), \triangle z(t), \triangle Z(t), \varrho(t)) \, d\overrightarrow{W_t}, \\
    -\tilde{z}(t) \, d\overrightarrow{B_t}, \\
    -d\tilde{Y}(t) = \tilde{F}(t, \triangle y(t), \triangle Y(t), \triangle z(t), \triangle Z(t), \varrho(t)) \, dt + \tilde{G}(t, \triangle y(t), \triangle Y(t), \triangle z(t), \triangle Z(t), \varrho(t)) \, d\overrightarrow{B_t}, \\
    -\tilde{Z}(t) \, d\overrightarrow{W_t}, \\
\end{cases}
\]

\[
\triangle y(0) = 0, \quad \triangle Y(T) = \frac{\varphi(y_\rho(T)) - \varphi(y(T))}{\rho},
\]

where \(\tilde{\varrho} = \tilde{f}, \tilde{F}, \tilde{g}, \tilde{G}, \) respectively,

\[
\tilde{\varrho}(t, \triangle y, \triangle Y, \triangle z, \triangle Z, v) = A^\varrho(t) \triangle y + B^\varrho(t) \triangle Y + C^\varrho(t) \triangle z + D^\varrho(t) \triangle Z + E^\varrho(t) v,
\]
and

$$A^\theta(t) = \begin{cases} \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)) - \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)+\Delta \rho) \over \Delta \rho - \rho(t), & \rho(t) - \rho(t) \neq 0, \\ 0, & \text{otherwise}; \end{cases}$$

$$B^\theta(t) = \begin{cases} \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)) - \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)+\Delta \rho) \over \Delta \rho - \rho(t), & \rho(t) - \rho(t) \neq 0, \\ 0, & \text{otherwise}; \end{cases}$$

$$C^\theta(t) = \begin{cases} \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)) - \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)+\Delta \rho) \over \Delta \rho - \rho(t), & \rho(t) - \rho(t) \neq 0, \\ 0, & \text{otherwise}; \end{cases}$$

$$D^\theta(t) = \begin{cases} \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)) - \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)+\Delta \rho) \over \Delta \rho - \rho(t), & \rho(t) - \rho(t) \neq 0, \\ 0, & \text{otherwise}; \end{cases}$$

$$E^\theta(t) = \begin{cases} \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)) - \theta(t,y(t),Y(t),z(t),Z(t),\rho(t)+\Delta \rho) \over \Delta \rho - \rho(t), & \rho(t) - \rho(t) \neq 0, \\ 0, & \text{otherwise}. \end{cases}$$

From the continuous dependence of solutions with respect to the parameter $\rho$, it follows that

$$\lim_{\rho \to 0} A^\theta(t) = \theta_y(t,y(t),Y(t),z(t),Z(t),u(t)), $$

$$\lim_{\rho \to 0} B^\theta(t) = \theta_Y(t,y(t),Y(t),z(t),Z(t),u(t)), $$

$$\lim_{\rho \to 0} C^\theta(t) = \theta_z(t,y(t),Y(t),z(t),Z(t),u(t)), $$

$$\lim_{\rho \to 0} D^\theta(t) = \theta_Z(t,y(t),Y(t),z(t),Z(t),u(t)), $$

$$\lim_{\rho \to 0} E^\theta(t) = \theta_v(t,y(t),Y(t),z(t),Z(t),u(t)).$$

According to the continuous dependence of solutions with respect to the parameter and the uniqueness of solutions of FBDSDDE (3.1), the solutions ($\Delta y(t), \Delta Y(t), \Delta z(t), \Delta Z(t)$) converge to ($y^1(t), Y^1(t), z^1(t), Z^1(t)$) in $M^2(0,T;R^{1+1+1+1})$ as $\rho \to 0$. The proof is completed. □

Now we give the variational inequality.

**Lemma 5.** Assume that (H1)-(H4) hold. Then we have

$$E \Phi_y(y(T))y^1(T) + E \gamma_Y(Y(0))Y^1(0)$$

$$+ E \int_0^T [I_y(t,y(t),Y(t),z(t),Z(t),u(t))y^1(t)$$

$$+ I_Y(t,y(t),Y(t),z(t),Z(t),u(t))Y^1(t)$$

$$+ I_z(t,y(t),Y(t),z(t),Z(t),u(t))z^1(t)$$

$$+ I_Z(t,y(t),Y(t),z(t),Z(t),u(t))Z^1(t)$$

$$+ I_v(t,y(t),Y(t),z(t),Z(t),u(t))v(t)]dt$$

$$\geq 0.$$
Proof. From Lemma 4 and (H4), we can get
\[
\lim_{\rho \to 0} E \left[ \Phi (y_\rho (T)) - \Phi (y (T)) \right] = \Phi (y (T)) y^1 (T),
\]
\[
\lim_{\rho \to 0} E \left[ \gamma (Y_\rho (0)) - \gamma (Y (0)) \right] = \gamma (Y (0)) Y^1 (0),
\]
and
\[
\lim_{\rho \to 0} \rho^{-1} E \int_0^T \left[ l (t, y_\rho (t), Y_\rho (t), z_\rho (t), Z_\rho (t), u (t) + \rho v (t))
-l (t, y (t), Y (t), z (t), Z (t), u (t)) \right] dt
= E \int_0^T \left[ l_y (t, y (t), Y (t), z (t), Z (t), u (t)) y^1 (t)
+l_Y (t, y (t), Y (t), z (t), Z (t), u (t)) Y^1 (t)
+l_z (t, y (t), Y (t), z (t), Z (t), u (t)) z^1 (t)
+l_Z (t, y (t), Y (t), z (t), Z (t), u (t)) Z^1 (t)
+l_v (t, y (t), Y (t), z (t), Z (t), u (t)) v (t) \right] dt.
\]
On the other hand, since \( u (\cdot) \) is an optimal control, it follows that
\[
\rho^{-1} \left[ J (u (\cdot) + \rho v (\cdot)) - J (u (\cdot)) \right] \geq 0.
\]
Therefore the desired result is obtained. \( \Box \)

Now we introduce the adjoint equation by virtue of dual technique and Hamilton function for our problem. From the variational inequality obtained in Lemma 5, the maximum
principle can be proved by using Itô’s formula. The adjoint equations are

\[
\begin{align*}
\dot{p}(t) &= [F_Y(t, y(t), Y(t), z(t), Z(t), u(t)) \, p(t) \\
&\quad - f_Y(t, y(t), Y(t), z(t), Z(t), u(t)) \, q(t) \\
&\quad + G_Y(t, y(t), Y(t), z(t), Z(t), u(t)) \, k(t) \\
&\quad - g_Y(t, y(t), Y(t), z(t), Z(t), u(t)) \, h(t) \\
&\quad - l_Y(t, y(t), Y(t), z(t), Z(t), u(t))] \, dt \\
&\quad + [F_Z(t, y(t), Y(t), z(t), Z(t), u(t)) \, p(t) \\
&\quad - f_Z(t, y(t), Y(t), z(t), Z(t), u(t)) \, q(t) \\
&\quad + G_Z(t, y(t), Y(t), z(t), Z(t), u(t)) \, k(t) \\
&\quad - g_Z(t, y(t), Y(t), z(t), Z(t), u(t)) \, h(t) \\
&\quad - l_Z(t, y(t), Y(t), z(t), Z(t), u(t))] \, dB_t, \\
\dot{q}(t) &= [F_y(t, y(t), Y(t), z(t), Z(t), u(t)) \, p(t) \\
&\quad - f_y(t, y(t), Y(t), z(t), Z(t), u(t)) \, q(t) \\
&\quad + G_y(t, y(t), Y(t), z(t), Z(t), u(t)) \, k(t) \\
&\quad - g_y(t, y(t), Y(t), z(t), Z(t), u(t)) \, h(t) \\
&\quad - l_y(t, y(t), Y(t), z(t), Z(t), u(t))] \, dt \\
&\quad + [F_z(t, y(t), Y(t), z(t), Z(t), u(t)) \, p(t) \\
&\quad - f_z(t, y(t), Y(t), z(t), Z(t), u(t)) \, q(t) \\
&\quad + G_z(t, y(t), Y(t), z(t), Z(t), u(t)) \, k(t) \\
&\quad - g_z(t, y(t), Y(t), z(t), Z(t), u(t)) \, h(t) \\
&\quad - l_z(t, y(t), Y(t), z(t), Z(t), u(t))] \, d\hat{B}_t + h(t) \, d\hat{W}_t, \\
p(0) &= -\gamma_Y(Y(0)), \quad q(T) = \varphi_y(y(T)) \, p(T) + \Phi_y(y(T)).
\end{align*}
\] (3.2)

It is easy to check that FBDSDE (3.2) satisfies (H1), (H2) and (H'3), so it has a unique solution \((p(t), q(t), k(t), h(t)) \in M^2(0, T; \mathbb{R}^{1+1+1+1})\).

We define the Hamiltonian function \(H\) as follows:

\[
H(t, y(t), Y(t), z(t), Z(t), v(t), p(t), q(t), k(t), h(t)) = \\
\langle q(t), f(t, y(t), Y(t), z(t), Z(t), v(t)) \rangle \\
- \langle p(t), F(t, y(t), Y(t), z(t), Z(t), v(t)) \rangle \\
- \langle k(t), G(t, y(t), Y(t), z(t), Z(t), v(t)) \rangle \\
+ \langle h(t), g(t, y(t), Y(t), z(t), Z(t), v(t)) \rangle \\
+ l(t, y(t), Y(t), z(t), Z(t), v(t)).
\] (3.3)

FBDSDE (3.2) can be rewritten as

\[
\begin{align*}
\dot{p}(t) &= -H_Y dt - H_Z d\hat{W}_t - k(t) \, d\hat{B}_t, \\
\dot{q}(t) &= -H_y dt - H_z d\hat{B}_t + h(t) \, d\hat{W}_t, \\
q(T) &= \varphi_y(y(T)) \, p(T) + \Phi_y(y(T)), \\
p(0) &= -\gamma_Y(Y(0)), \quad 0 \leq t \leq T.
\end{align*}
\] (3.4)
Proof. Applying Itô’s formula to \( \langle y^1(t), q(t) \rangle + \langle Y^1(t), p(t) \rangle \) on \([0, T]\), we have

\[
\begin{align*}
    &\mathbb{E} \left[ \langle y^1(T), q(T) \rangle + \langle Y^1(T), p(T) \rangle - \langle y^1(0), q(0) \rangle - \langle Y^1(0), p(0) \rangle \right] \\
    &+ \mathbb{E} \int_0^T \left[ l_y(t, y(t), Y(t), z(t), Z(t), u(t)) y^1(t) \\
    &+ l_Y(t, y(t), Y(t), z(t), Z(t), u(t)) Y^1(t) \\
    &+ l_z(t, y(t), Y(t), z(t), Z(t), u(t)) z^1(t) \\
    &+ l_Z(t, y(t), Y(t), z(t), Z(t), u(t)) Z^1(t) \\
    &+ l_v(t, y(t), Y(t), z(t), Z(t), u(t)) v(t) \right] dt \\
    &= \mathbb{E} \int_0^T \left[ \langle q(t), f_v(t, y(t), Y(t), z(t), Z(t), u(t)) v(t) \rangle - \langle p(t), F_v(t, y(t), Y(t), z(t), Z(t), u(t)) v(t) \rangle - \langle k(t), G_v(t, y(t), Y(t), z(t), Z(t), u(t)) v(t) \rangle \\
    &+ \langle h(t), g_v(t, y(t), Y(t), z(t), Z(t), u(t)) v(t) \rangle + \langle v(t), l_v(t, y(t), Y(t), z(t), Z(t), u(t)) \rangle \right] dt.
\end{align*}
\]

From the variational inequality in Lemma 5 and noting (3.3), for any \( v(\cdot) \in \mathcal{U}_{ad} \) such that \( u(\cdot) + v(\cdot) \in \mathcal{U}_{ad} \), we have

\[
\mathbb{E} \int_0^T \langle H_v(t, y(t), Y(t), z(t), Z(t), u(t), p(t), q(t), k(t), h(t)), v(t) \rangle dt \geq 0.
\]

For \( \forall v \in \mathcal{U} \), we set

\[
v(t) = \begin{cases} 
0, & t \in [0, t], \\
v, & t \in [t, t + \epsilon), \\
0, & t \in [t + \epsilon, T].
\end{cases}
\]

Then we have

\[
\mathbb{E} \int_t^{t+\epsilon} \langle H_v(t, y(t), Y(t), z(t), Z(t), u(t), p(t), q(t), k(t), h(t)), v \rangle dt \geq 0.
\]
Notice the fact that
\[ E \int_t^{t+\epsilon} \langle H_v(t, y(t), Y(t), z(t), Z(t), u(t), p(t), q(t), k(t), h(t)), u(t) \rangle \, dt = 0. \]

Differentiating with respect to \( \epsilon \) at \( \epsilon = 0 \) gives
\[ E \langle H_v(t, y(t), Y(t), z(t), Z(t), u(t), p(t), q(t), k(t), h(t)), v - u(t) \rangle \geq 0, \]
a.e., a.s. \( t \in [0, T] \).

The proof is completed. \( \square \)

4 A sufficient maximum principle for optimal controls of forward-backward doubly stochastic control systems

In this section, we investigate a sufficient maximum principle for the optimal control problem stated in Section 2. For simplicity of notations, we use the subscript label.

**Theorem 7. (Sufficient maximum principle).** Let \( \left( \tilde{u}_t; \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t \right)_{0 \leq t \leq T} \) be a quintuple and suppose there exist a solution \( \left( \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right)_{0 \leq t \leq T} \) of the corresponding adjoint forward-backward doubly stochastic equation (3.2) such that for arbitrary admissible control \( v(\cdot) \in \mathcal{U}_{ad} \), we have

\[
E \int_0^T \left\langle \tilde{k}_t, \left( Y_t - \tilde{Y}_t \right) \right\rangle^2 \, dt < \infty, \tag{4.1}
\]

\[
E \int_0^T \left\langle \tilde{p}_t, \left( Z_t - \tilde{Z}_t \right) \right\rangle^2 \, dt < \infty, \tag{4.2}
\]

\[
E \int_0^T \left\langle \tilde{h}_t, (y_t - \tilde{y}_t) \right\rangle^2 \, dt < \infty, \tag{4.3}
\]

\[
E \int_0^T \left\langle \tilde{q}_t, (z_t - \tilde{z}_t) \right\rangle^2 \, dt < \infty, \tag{4.4}
\]

\[
E \int_0^T \left\langle Y_t - \tilde{Y}_t, H_Z \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right\rangle^2 \, dt < \infty, \tag{4.5}
\]

\[
E \int_0^T \left\langle \tilde{p}_t, \left( G \left( t, y_t, Y_t, z_t, Z_t \right) - G \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t \right) \right) \right\rangle^2 \, dt < \infty, \tag{4.6}
\]

\[
E \int_0^T \left\langle (y_t - \tilde{y}_t), H_z \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right\rangle^2 \, dt < \infty, \tag{4.7}
\]

\[
E \int_0^T \left\langle \tilde{q}_t, \left( g \left( t, y_t, Y_t, z_t, Z_t \right) - g \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t \right) \right) \right\rangle^2 \, dt < \infty. \tag{4.8}
\]
Further, suppose that for all $t \in [0, T]$, $H(t, y, z, Z, v, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t)$ is convex in $(y, Y, z, Z, v)$, and $\gamma(Y)$ is convex in $Y$ and $\Phi$ is convex in $y$, moreover the following conditions holds
\[
E[H(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, v, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t)] = \inf_{v \in U} E[H(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, v, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t)].
\] (4.9)

Then $\tilde{u}_t$ is an optimal control.

Proof. Let $(y_t, Y_t, z_t, Z_t, v_t) = (y_t^{(v)}, Y_t^{(v)}, z_t^{(v)}, Z_t^{(v)}, v_t)$ be an arbitrary quintuple satisfying the control system (2.1). According to the definition of the cost function (2.2), we have
\[
J(v(\cdot)) - J(\tilde{u}(\cdot)) = E \int_0^T \left[ l(t, y_t, Y_t, z_t, Z_t, v_t) - l(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t) \right] dt + E \Phi(y_T) - \Phi(\tilde{y}_T) + E \gamma(Y_0) - \gamma(\tilde{Y}_0)
\]
where
\[
I_1 = E \int_0^T \left[ l(t, y_t, Y_t, z_t, Z_t, v_t) - l(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t) \right] dt,
I_2 = E \Phi(y_T) - \Phi(\tilde{y}_T),
I_3 = E \gamma(Y_0) - \gamma(\tilde{Y}_0).
\]
Now applying Itô’s formula to $\langle \tilde{p}_t, Y_t - \tilde{Y}_t \rangle + \langle \tilde{q}_t, y_t - \tilde{y}_t \rangle$ on $[0, T]$, we get

\[
\begin{align*}
\langle \tilde{p}_T, Y_T - \tilde{Y}_T \rangle + \langle \tilde{q}_T, y_T - \tilde{y}_T \rangle - \langle \tilde{p}_0, Y_0 - \tilde{Y}_0 \rangle - \langle \tilde{q}_0, y_0 - \tilde{y}_0 \rangle \\
= \langle \Phi_y (\tilde{y}_T) , y_T - \tilde{y}_T \rangle + \langle \gamma_Y (\tilde{Y}_0) , Y_0 - \tilde{Y}_0 \rangle \\
= \int_0^T \left\{ \left( Z_t - \tilde{Z}_t \right), \left( -H_Z \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right) \right\} dt \\
- \int_0^T \left\{ \tilde{k}_t, \left( G \left( t, y_t, Y_t, z_t, Z_t, v_t \right) - G \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right) \right\} dt \\
+ \int_0^T \left\{ \left( z_t - \tilde{z}_t \right), \left( -H_z \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right) \right\} dt \\
+ \int_0^T \left\{ \tilde{h}_t, \left( g \left( t, y_t, Y_t, z_t, Z_t, v_t \right) - g \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right) \right\} dt \\
+ \int_0^T \left\{ \left( Y_t - \tilde{Y}_t \right), \left( -H_Y \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right) \right\} dt \\
+ \int_0^T \left\{ \left( Y_t - \tilde{Y}_t \right), \left( -H_Z \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right) \right\} d\tilde{W}_t \\
- \int_0^T \left\{ \tilde{k}_t, \left( Y_t - \tilde{Y}_t \right) \right\} d\tilde{B}_t \\
- \int_0^T \left\{ \tilde{p}_t, \left( F \left( t, y_t, Y_t, z_t, Z_t, v_t \right) - F \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right) \right\} dt \\
- \int_0^T \left\{ \tilde{p}_t, \left( G \left( t, y_t, Y_t, z_t, Z_t, v_t \right) - G \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right) \right\} d\tilde{B}_t \\
+ \int_0^T \left\{ \tilde{p}_t, \left( Z_t - \tilde{Z}_t \right) \right\} d\tilde{W}_t \\
+ \int_0^T \left\{ \left( y_t - \tilde{y}_t \right), \left( -H_y \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right) \right\} dt \\
+ \int_0^T \left\{ \left( y_t - \tilde{y}_t \right), \left( -H_z \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right) \right\} d\tilde{B}_t \\
+ \int_0^T \left\{ \left( y_t - \tilde{y}_t \right), \tilde{h}_t dW_t \right\} \\
+ \int_0^T \tilde{q}_t \left( f \left( t, y_t, Y_t, z_t, Z_t, v_t \right) - f \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right) dt
\end{align*}
\]
where we claim that
\[
\begin{align*}
Y_T - \tilde{Y}_T &= \varphi(y_T) - \varphi(\tilde{y}_T) = \varphi_y(\tilde{y}_T)(y(T) - \tilde{y}(T)), \\
y_0 - \tilde{y}_0 &= x_0 - x_0 = 0, \\
\tilde{p}_0 &= -\gamma Y_0, \\
\tilde{q}_T &= \Phi_y(\tilde{y}_T) - \varphi_y(\tilde{y}(T)) \tilde{p}(T).
\end{align*}
\]
By Davis inequality, under the conditions (4.1)-(4.8), we can ensure that the stochastic integrals with respect to the Brownian motion have zero expectations. Moreover, by virtue of convexity of \(\Phi\) and \(\gamma\), it follows instantly that
\[
I_2 + I_3 = E[\Phi(y_T) - \Phi(\tilde{y}_T)] + E\left[\gamma (Y_0) - \gamma \left(\tilde{Y}_0\right)\right] \\
\geq E\left<\Phi_y(y_T), y_T - \tilde{y}_T\right> + E\left<\gamma_Y(Y_0), Y_0 - \tilde{Y}_0\right> \\
= -E \int_0^T \left<\left(\begin{array}{c} Y_t - \tilde{Y}_t \\ \tilde{p}_t \end{array}\right), H_Y \left(\begin{array}{c} t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \end{array}\right)\right> dt \\
- E \int_0^T \left<\tilde{p}_t, \left(F(t, y_t, Y_t, z_t, Z_t, v_t) - F(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t)\right)\right> dt \\
- E \int_0^T \left<\left(y_t - \tilde{y}_t\right), H_y \left(\begin{array}{c} t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \end{array}\right)\right> dt \\
+ E \int_0^T \left<\tilde{q}_t, \left(g(t, y_t, Y_t, z_t, Z_t, v_t) - g(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t)\right)\right> dt \\
- E \int_0^T \left<\left(Z_t - \tilde{Z}_t\right), H_Z \left(\begin{array}{c} t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \end{array}\right)\right> dt \\
- E \int_0^T \left<\tilde{k}_t, \left(G(t, y_t, Y_t, z_t, Z_t, v_t) - G(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t)\right)\right> dt \\
- E \int_0^T \left<\left(z_t - \tilde{z}_t\right), H_z \left(\begin{array}{c} t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \end{array}\right)\right> dt \\
+ E \int_0^T \left<\tilde{h}_t, \left(g(t, y_t, Y_t, z_t, Z_t, v_t) - g(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t)\right)\right> dt \\
= -\Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5,
\]
where

\[ 
\begin{align*}
\Xi_1 &= E \int_0^T \left< H_y \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right), (y_t - \tilde{y}_t) \right> dt \\
&\quad + E \int_0^T \left< H_Y \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right), \left( Y_t - \tilde{Y}_t \right) \right> dt \\
&\quad + E \int_0^T \left< H_z \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right), (z_t - \tilde{z}_t) \right> dt \\
&\quad + E \int_0^T \left< H_Z \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right), (Z_t - \tilde{Z}_t) \right> dt \\
\Xi_2 &= -E \int_0^T \left< \tilde{p}_t, F \left( t, y_t, Y_t, z_t, Z_t, \nu_t \right) - F \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right> dt \\
\Xi_3 &= E \int_0^T \left< \tilde{q}_t, g \left( t, y_t, Y_t, z_t, Z_t, \nu_t \right) - g \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right> dt \\
\Xi_4 &= -E \int_0^T \left< \tilde{k}_t, G \left( t, y_t, Y_t, z_t, Z_t, \nu_t \right) - G \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right> dt \\
\Xi_5 &= E \int_0^T \left< \tilde{h}_t, g \left( t, y_t, Y_t, z_t, Z_t, \nu_t \right) - g \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right> dt.
\end{align*}
\]

Noting the definition of $H$ and $I_1$, we have

\[ 
\begin{align*}
I_1 &= E \int_0^T \left[ l \left( t, y_t, Y_t, z_t, Z_t, \nu_t \right) - l \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right] dt \\
&= E \int_0^T \left[ H \left( t, y_t, Y_t, z_t, Z_t, \nu_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) - H \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right] dt \\
&\quad - E \int_0^T \left[ \left< \tilde{q}_t, f \left( t, y_t, Y_t, z_t, Z_t, \nu_t \right) - f \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right> \right] dt \\
&\quad + E \int_0^T \left[ \left< \tilde{p}_t, F \left( t, y_t, Y_t, z_t, Z_t, \nu_t \right) - F \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right> \right] dt \\
&\quad + E \int_0^T \left[ \left< \tilde{k}_t, G \left( t, y_t, Y_t, z_t, Z_t, \nu_t \right) - G \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right> \right] dt \\
&\quad - E \int_0^T \left[ \left< \tilde{h}_t, g \left( t, y_t, Y_t, z_t, Z_t, \nu_t \right) - g \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t \right) \right> \right] dt \\
&= \Xi_6 - \Xi_2 - \Xi_3 - \Xi_4 - \Xi_5,
\end{align*}
\]

where

\[ \Xi_6 = E \int_0^T \left[ H \left( t, y_t, Y_t, z_t, Z_t, \nu_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) - H \left( t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t \right) \right] dt. \]
On the one hand, by the virtue of convexity of \( H(t, y, z, v, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) \) with respect to \((y, Y, z, Z, v)\), we obtain

\[
H(t, y_t, Y_t, z_t, Z_t, v_t, p_t, q_t, k_t, h_t) - H(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) \\
\geq H_y(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (y_t - \bar{y}_t) \\
+ H_Y(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (Y_t - \bar{Y}_t) \\
+ H_z(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (z_t - \bar{z}_t) \\
+ H_Z(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (Z_t - \bar{Z}_t) \\
+ H_u(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (v_t - \bar{u}_t)
\]

\[(4.10)\]

On the other hand, we know

\[
E \left[ H_u(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (v_t - \bar{u}_t) \right] \geq 0.
\]

Consequently, associating with \((4.10)\), we claim that

\[
\Xi_6 = E \int_0^T \left[ H(t, y_t, Y_t, z_t, Z_t, v_t, p_t, q_t, k_t, h_t) - H(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) \right] dt \\
\geq E \int_0^T H_y(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (y_t - \bar{y}_t) dt \\
+ E \int_0^T H_Y(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (Y_t - \bar{Y}_t) dt \\
+ E \int_0^T H_z(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (z_t - \bar{z}_t) dt \\
E \int_0^T H_Z(t, \bar{y}_t, \bar{Y}_t, \bar{z}_t, \bar{Z}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \bar{k}_t, \bar{h}_t) (Z_t - \bar{Z}_t) dt \\
= \Xi_1.
\]

Then, it follows that

\[
J(v(\cdot)) - J(u(\cdot)) = I_1 + I_2 + I_3 \\
= \Xi_6 - \Xi_2 - \Xi_3 - \Xi_4 - \Xi_5 \\
- \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 \\
\geq \Xi_1 - \Xi_2 - \Xi_3 - \Xi_4 - \Xi_5 \\
- \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 \\
= 0.
\]
Since $v(\cdot) \in U_{ad}$ is arbitrary, we say that $\tilde{u}(\cdot)$ is an optimal control. The proof is completed.

\section{Applications to optimal control problems of stochastic partial differential equations}

In this section, we will give necessary and sufficient maximum principles for optimal control of SPDEs. Let us first give some notations from [14]. For convenience, all the variables in this section are one-dimensional. It is necessary to point out that all the results in this section can be extended to multi-dimensional cases, but we use the notations in general case. From now on $C^k(R; R)$, $C^k_{l,b}(R; R)$, $C^k_p(R; R)$ will denote respectively the set of functions of class $C^k$ from $R$ into $R$, the set of those functions whose partial derivatives of order less than or equal to $k$ are bounded (and hence the function itself grows at most linearly at infinity), and the set of those functions which, together with all their partial derivatives of order less than or equal to $k$, grow at most like a polynomial function of the variable $x$ at infinity. We consider the following quasilinear SPDEs with control variable:

\begin{equation}
\begin{aligned}
\left\{ u(t, x) &= \varphi(x) + \int_t^T [L^v u(s, x) + F(s, x, u(s, x), (\nabla u\sigma)(s, x, u), v(s))] \, ds \\
&+ \int_t^T G(x, u(s, x), (\nabla u\sigma)(s, x, u), v(s)) \, dB_s, \quad 0 \leq t \leq T,
\end{aligned}
\end{equation}

(5.1)

where $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$ and $\nabla u(s, x)$ denotes the first order derivative of $u(s, x)$ with respect to $x$, and

\[ L^v u = \begin{pmatrix} L^v u_1 \\ \vdots \\ L^v u_k \end{pmatrix}, \]

with

\[ L\varphi(x) = \frac{1}{2} \sum_{i,j=1}^d (gg^*)_{ij}(x, u, v) \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d f_i(x, u, v) \frac{\partial \varphi(x)}{\partial x_i}. \]

In this section, for simplicity, we also set $d = k = 1$, and

\[ f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \]
\[ g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \]
\[ F : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \]
\[ G : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \]
\[ \varphi : \mathbb{R} \rightarrow \mathbb{R}. \]

In order to assure the existence and uniqueness of solutions for (5.1) and (5.3) below, we give the following assumptions for sake of completeness (see [14] for more details).
(A1) \[
\begin{aligned}
&f \in C^3_{t,b}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathbb{R}), \\
g \in C^3_{t,b}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathbb{R}), \\
&\varphi \in C^3_p(\mathbb{R}; \mathbb{R}), \\
&F (t, \cdot, \cdot, \cdot, v) \in C^3_{t,b}(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathbb{R}), \\
&F (\cdot, x, y, z, v) \in M^2 (0, T; \mathbb{R}), \\
&G (t, \cdot, \cdot, \cdot, v) \in C^3_{t,b}(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathbb{R}), \\
&G (\cdot, x, y, z, v) \in M^2 (0, T; \mathbb{R}), \\
&\forall t \in [0, T], x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}, v \in \mathbb{R}.
\end{aligned}
\]

(A2) Assume that (H1), (H2) and (H3) hold.

Let \( \mathcal{U}_{ad} \) be an admissible control set. The optimal control problem of SPDE (5.1) is to find an optimal control \( v^* (\cdot) \in \mathcal{U}_{ad} \), such that

\[
J (v^* (\cdot)) = \inf_{v (\cdot) \in \mathcal{U}_{ad}} J (v (\cdot)),
\]

where \( J (\cdot) \) is the cost function as follows:

\[
J (v (\cdot)) = \mathbb{E} \left[ \int_0^T l (s, \bar{x}, u (s, \bar{x}), (\nabla u g) (s, \bar{x}, u (s, \bar{x})), v (s)) d\bar{x} = X^{0,x} (s) ds + \gamma (u (0, x)) \right].
\]

(5.2)

Here we assume \( l \) and \( \gamma \) satisfy (H4) and \( X^{0,x} (s) \) defined below. We can transform the optimal control problem of SPDE (5.1) into one of the following FBDSDE with control variable \( v (\cdot) \):

\[
\begin{aligned}
X^{t,x} (s) &= x + \int_t^s f (X^{t,x} (r), Y^{t,x} (r), v (r)) dr + \int_t^s g (X^{t,x} (r), Y^{t,x} (r), v (r)) d\bar{W}_r, \\
Y^{t,x} (s) &= \varphi (X^{t,x} (T)) + \int_t^T F (r, X^{t,x} (r), Y^{t,x} (r), Z^{t,x} (r), v (r)) dr \\
&\quad + \int_s^T G (r, X^{t,x} (r), Y^{t,x} (r), Z^{t,x} (r), v (r)) d\bar{B}_r \\
&\quad - \int_s^T Z^{t,x} (r) d\bar{W}_r, \\
&0 \leq t \leq s \leq T,
\end{aligned}
\]

(5.3)

where \( \{X^{t,x} (\cdot), Y^{t,x} (\cdot), Z^{t,x} (\cdot), v (\cdot)\} \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, x \in \mathbb{R} \). The corresponding optimal control problem of FBDSDEs (5.3) is to find an optimal control \( v^* (\cdot) \in \mathcal{U}_{ad} \), such that

\[
J (v^* (\cdot)) = \inf_{v (\cdot) \in \mathcal{U}_{ad}} J (v (\cdot)),
\]

where \( J (v (\cdot)) \) is the cost function the same as (5.2):

\[
J (v (\cdot)) = \mathbb{E} \left[ \int_0^T l (s, X (s), Y (s), Z (s), v (s)) ds + \gamma (Y (0)) \right].
\]

Now we consider the following adjoint FBDSDEs involving the four unknown processes \((p (t), q (t), k (t), h (t))\):

\[
\begin{aligned}
dp (t) &= (F_Y p (t) + G_Y k (t) - f_Y q (t) - g_Y h (t) - l_Y) dt \\
&\quad + (F_{Zp} p (t) - G_{Zk} k (t) - l_Z) d\bar{W}_t - k (t) d\bar{B}_t, \\
\dq (t) &= (F_X p (t) - f_X q (t) + G_X k (t) - g_X h (t) - l_X) dt + h (t) d\bar{W}_t, \\
p (0) &= -\gamma_Y (Y (0)), \\
q (T) &= -\varphi_X (X (T)) p (T), \\
&0 \leq t \leq T.
\end{aligned}
\]

(5.4)
It is easy to see that (5.4) satisfies (H1), (H2) and (H’3), so it is uniquely solvable by virtue of Proposition 2. Therefore we know that (5.4) has a unique solution $(p(\cdot), q(\cdot), k(\cdot), h(\cdot)) \in M^2(0, T; \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$. Define the Hamilton function as follows:

$$
\bar{H}(t, X, Y, Z, v, p, q, k, h) = H(t, X, Y, 0, Z, v, p, q, k, h) = l(t, X, Y, Z, v) - k \cdot G(t, X, Y, Z, v) + q \cdot f(X, Y, v) - p \cdot F(t, X, Y, Z, v) + h \cdot g(X, Y, v).
$$

(5.5)

We now formulate a maximum principle for the optimal control system of (5.3).

**Theorem 8.** Suppose (A1)-(A2) hold. Let $(X(\cdot), Y(\cdot), Z(\cdot), v^*(\cdot))$ be an optimal control and its corresponding trajectory of (5.3), $(p(\cdot), q(\cdot), k(\cdot), h(\cdot))$ be the solution of (5.4). Then the maximum principle holds, that is, for $t \in [0, T], \forall v \in U$,

$$
\langle \bar{H}(t, X(t), Y(t), Z(t), v^*(t), p(t), q(t), k(t), h(t)), v - v^*(t) \rangle \geq 0, \text{ a.e., a.s.}
$$

*Proof.* By Theorem 6 in Section 3, we get the desired result. \qed

For relationship between (5.1) and (5.3), we have

**Lemma 9.** For any given admissible control $v(\cdot)$, we assume (A1) and (A2) hold. Then (5.3) has a unique solution $(X^{t,x}(\cdot), Y^{t,x}(\cdot), Z^{t,x}(\cdot)) \in M^2(0, T; \mathbb{R} \times \mathbb{R} \times \mathbb{R})$.

**Lemma 10.** For any given admissible control $v(\cdot)$, we assume (A1) and (A2) hold. Let $\{u(t, x); 0 \leq t \leq T, x \in \mathbb{R}\}$ be a random field such that $u(t, x)$ is $\mathcal{F}^B_t$-measurable for each $(t, x), u \in C^{0,2}([0, T] \times \mathbb{R}; \mathbb{R})$ a.s., and $u$ satisfies SPDE (5.1). Then $u(t, x) = Y^{t,x}(t)$.

**Lemma 11.** For any given admissible control $v(\cdot)$, we assume (A1) and (A2) hold. Then $\{u(t, x) = Y^{t,x}(t); 0 \leq t \leq T, x \in \mathbb{R}\}$ is a unique classical solution of SPDE (5.1).

The proofs are classical, we omit it. Now set the Hamilton function as follows:

$$
\bar{H}(t, x, u, \nabla u \sigma, v, p, q, k, h) = l(t, x, u, \nabla u \sigma, v) - k \cdot G(t, x, u, \nabla u \sigma) + q \cdot f(x, u, v) - p \cdot F(t, x, u, \nabla u \sigma, v) + h \cdot g(x, u, v).
$$

We can state the maximum principle for the optimal control problem of SPDE (5.1).

**Theorem 12.** *(Necessary maximum principle)* Suppose $u(t, x)$ is the optimal solution of SPDE (5.1) corresponding to the optimal control $v^*(\cdot)$ of (5.1). Then we have, for any $v \in U$ and $t \in [0, T], x \in \mathbb{R}$,

$$
\langle \bar{H}_v(t, x, u(t, x), \nabla u \sigma)(t, x), v^*(t), p(t), q(t), k(t), h(t)), v - v^*(t) \rangle \geq 0, \text{ a.e., a.s.}
$$

*Proof.* By virtue of lemma 9,10,11, the optimal control problem of SPDE (5.1) can be transformed into the one of FBDSDE (5.3). Hence, from Theorem 8, the desired result is easily obtained. \qed

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Next we apply our sufficient maximum principle to get the following result.

**Theorem 13. (Sufficient maximum principle)** For \( \forall t \in [0,T] \), let \( \hat{v} = \hat{v} (t) \in \mathcal{U}_{ad} \) with corresponding solution \( \hat{u} (t,x) \) of (5.1) and let \( \left( \hat{X} (t) , \hat{Y} (t) , \hat{Z} (t) , \hat{v} (t) \right) \) be quadruple and \( \left( \hat{p} (t) , \hat{q} (t) , \hat{k} (t) , \hat{h} (t) \right) \) be a solution of the associated adjoint FBDSDEs (5.3) and (5.4), respectively. Assume that \( \bar{H} \left( t, X, Y, Z, v, \hat{p} (t) , \hat{q} (t) , \hat{k} (t) , \hat{h} (t) \right) \) is convex in \( (X,Y,Z,v) \), and \( \gamma (Y) \) is convex in \( Y \), moreover the following condition holds

\[
\mathbb{E} \left[ \bar{H} \left( t, \hat{X} (t) , \hat{Y} (t) , \hat{Z} (t) , \hat{v} (t) , \hat{p} (t) , \hat{q} (t) , \hat{k} (t) , \hat{h} (t) \right) \right] \\
= \inf_{v \in \mathcal{U}} \mathbb{E} \left[ \bar{H} \left( t, \hat{X} (t) , \hat{Y} (t) , \hat{Z} (t) , v, \hat{p} (t) , \hat{q} (t) , \hat{k} (t) , \hat{h} (t) \right) \right].
\]

Then \( \hat{v} (t) \) is an optimal control for the problem (5.2).

*Proof.* Noting the above assumptions, by Theorem 7, it is easy to get desired result. \( \square \)

**Remark 14.** In [12], Bernt Øksendal proved a sufficient maximum principle for the optimal control of system described by a quasilinear stochastic heat equation, that is

\[
dY (t,x) = \left\{ \begin{array}{l}
[LY (t,x) + b (t,x,Y (t,x),v (t))] dt \\
+\sigma (t,x,Y (t,x),v (t)) dW_t;
\end{array} \right.
\]

\( (t,x) \in [0,T] \times G. \) \hspace{1cm} (5.6)

\[
Y (0,x) = \xi (x); \quad x \in \overline{G} \hspace{1cm} (5.7)
\]

\[
Y (t,x) = \eta (t,x); \quad (t,x) \in (0,T) \times \partial G. \hspace{1cm} (5.8)
\]

Here \( G \) is an open set in \( \mathbb{R}^n \) with \( C^1 \) boundary \( \partial G \) and

\[
L \phi (x) = \sum_{i,j=1}^{n} a_{ij} (x) \frac{\partial^2}{\partial x_i \partial x_j} \phi + \sum_{i=1}^{n} b_i (x) \frac{\partial}{\partial x_i} \phi, \quad \phi \in C^2 (\mathbb{R}^n)
\]

where \( a (x) = [a_{ij} (x)]_{1 \leq i,j \leq n} \) is a given symmetric definite symmetric \( n \times n \) matrix with entries \( a_{ij} (x) \in C^2 (G) \cap C (\overline{G}) \) for all \( i, j = 1,2, \cdots, n \) and \( b_i (x) \in C^2 (G) \cap C (\overline{G}) \) for all \( i, j = 1,2, \cdots, n \). It is worth to pointing out that our method to get the sufficient maximum principle is completely different from his, and the most important thing is that in our SPDEs, the coefficients of the elliptic operator contain control variables and \( Y \) (for more information see Theorem 2.1-Theorem 2.3 in [12]).
6 Applications

We now illustrate the results of Section 3 by looking at some examples. Theoretically, the maximum principles presented in Section 3 and Section 4 characterizes the optimal control through some necessary and sufficient conditions. However, it is not immediately feasible to implement such principles directly, partially due to the difficulty of computing fully coupled forward-backward doubly stochastic system. In this section, we give two special examples and show how to explicitly solve them using our maximum principle.

6.1 Example 1

We provide a concrete example of forward-backward doubly stochastic LQ problems and give the explicit optimal control and validate our major theoretical results in Theorem 6. (Necessary maximum principle). First let the control domain be $\mathcal{U} = [-1, 1]$. Consider the following linear forward-backward doubly stochastic control system. We assume that $l = d = 1$.

\[
\begin{aligned}
\frac{dy(t)}{dt} &= (z(t) - Z(t) + v(t)) \, d\bar{W}_t - z(t) \, d\bar{B}_t, \\
\frac{dY(t)}{dt} &= - (z(t) + Z(t) + v(t)) \, d\bar{B}_t + Z(t) \, d\bar{W}_t, \\
y(0) &= 0, \quad Y(T) = 0, \quad t \in [0, T],
\end{aligned}
\]

where $T > 0$ is a given constant and the cost function is

\[
J(v(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T \left( y^2(t) + Y^2(t) + z^2(t) + Z^2(t) + v^2(t) \right) dt + \frac{1}{2} \mathbb{E} y^2(0) + \frac{1}{2} \mathbb{E} y^2(T).
\]

(6.2)

Note that (6.1) are linear control system. According to the existence and uniqueness of (6.1), it is straightforward to know the optimal control is $u(\cdot) \equiv 0$, with the optimal state trajectory $(y(t), Y(t), z(t), Z(t)) \equiv 0$, $t \in [0, T]$. Notice that the adjoint equation associated with the optimal quadruple $(y(t), Y(t), z(t), Z(t)) \equiv 0$ are

\[
\begin{aligned}
\frac{dp(t)}{dt} &= -Y(t) \, dt - (k(t) - h(t) - Z(t)) \, d\bar{W}_t - k(t) \, d\bar{B}_t, \\
\frac{dq(t)}{dt} &= -y(t) \, dt - (k(t) - h(t) - z(t)) \, d\bar{B}_t + h(t) \, d\bar{W}_t, \\
p(0) &= 0, \quad q(T) = 0, \quad t \in [0, T].
\end{aligned}
\]

(6.3)
Obviously, \((p(t), q(t), k(t), h(t))\equiv 0\) is the unique solution of (6.3). Instantly, we give the Hamiltonian function is 
\[
H(t, y(t), Y(t), z(t), Z(t), v, p(t), q(t), k(t), h(t))
\]
\[
= \frac{1}{2} \left( y^2(t) + Y^2(t) + z^2(t) + Z^2(t) + v^2 \right) - k(t)(z(t) + Z(t) + v) + h(t)(z(t) - Z(t) + v)
\]
\[
= \frac{1}{2}v^2.
\]
It is clear that, for any \(v \in \mathcal{U}\), we always have
\[
E \langle H_v(t, y(t), Y(t), z(t), u(t), p(t), q(t), k(t), h(t)), v - u(t) \rangle = 0.
\]

### 6.2 Example 2

In this subsection we will provide a special optimal control of SPDEs by Theorem 13. (Sufficient maximum principle). We now introduce some notations. For any random variable \(F\) of the form
\[
F = f(W(h_1), \ldots W(h_n); B(k_1), \ldots B(k_p))
\]
with \(f \in C^\infty_b(R^{n+p})\), \(h_1, \ldots h_n \in L^2([0, T], R^d)\), \(k_1, \ldots k_p \in L^2([0, T], R^d)\), where
\[
W(h_i) = \int_0^T h_i(t) \, dW_t, \quad B(h_i) = \int_0^T k_i(t) \, dB_t,
\]
we let
\[
D_tF = \sum_{i=1}^n f'_i(W(h_1), \ldots W(h_n); B(k_1), \ldots B(k_p)) h_i(t), \quad 0 \leq t \leq T.
\]
For such an \(F\), we define its 1,2-norm as:
\[
\|F\|_{1,2} = \left( E \left[ F^2 + \int_0^T |D_tF|^2 \, dt \right] \right)^{\frac{1}{2}}.
\]
\(S\) denotes the set of random variable of the above form. We define the Sobolev space:
\[
\mathbb{D}^{1,2} = \overline{S^{1,2}}.
\]
The "derivation operator" \(D\) extends as an operator from \(\mathbb{D}^{1,2}\) into \(L^2(\Omega; L^2([0, T], R^n))\).

Now we modify the stochastic reaction-diffusion equation considered in [12] which can be described the density of a population at time \(t \in [0, T]\) and at the point \(x \in R\) as follows.
\[
\begin{cases}
    u(t, x) = x + \int_t^T \left[ u^2(s) \triangle u(s, x) + u(s, x) + \nabla u(s, x) v(s) \right] ds \\
    + \int_t^T u(s, x) \, dB_s, \quad 0 \leq t \leq T,
\end{cases}
\]
(6.4)
and $x \in R, v \in U_{ad}$. The two Brownian motions $W$ and $B$ are one-dimensional. Suppose we want to minimize the following performance criterion

$$J (v) = \mathbb{E} \left[ \int_0^T \frac{v^\gamma (s)}{\gamma} ds + u (0, x) \right],$$

where $\gamma \geq 1$. In this case the Hamiltonian gets the form

$$H (t, X, Y, Z, v, p, q, k, h) = \frac{v^\gamma}{\gamma} - k (Y + Z) - pY + hv.$$

Obviously, it is convex in $(Y, Z, v)$. The corresponding FBDSDEs are

$$\left\{ \begin{array}{l}
X^{t,x} (s) = x + \int_t^s v (r) dW_r,
Y^{t,x} (s) = X^{t,x} (T) + \int_t^T (Y^{t,x} (r) + Z^{t,x} (r)) dr \\
+ \int_s^T Y (r) d\bar{B}_r - \int_s^T Z^{t,x} (r) d\bar{W}_r, 0 \leq t \leq s \leq T,
\end{array} \right. \quad (6.5)$$

It is easy to obtain the solutions of (6.5) are

$$Y^{t,x} (s) = \mathbb{E} \left[ X^{t,x} (T) \exp \{ W_T - W_t + B_T - B_s \} | \mathcal{F}_s \right]. \quad (6.6)$$

Besides, the adjoint processes are

$$\left\{ \begin{array}{l}
dp (s) = (p (s) + k (s)) ds + p (t) dW_t - k (s) d\bar{B}_s,
\dq (s) = h (s) dW_s,
p (t) = -1, \quad q (T) = -p (T), \quad t \leq s \leq T.
\end{array} \right. \quad (6.7)$$

The solutions of (6.7) are

$$\begin{array}{l}
p (s) = \mathbb{E} \left[ - \exp \{ W_s + W_t + B_s - B_t \} | \mathcal{F}_s \right],
\dq (s) = \mathbb{E} \left[ -p (T) | \mathcal{F}_W \right],
h (s) = D_s q (s), \quad \text{a.e., } 0 \leq t \leq s \leq T.
\end{array} \quad (6.8)$$

The function

$$v \rightarrow H (t, X, Y, Z, v, p, q, k, h) = \frac{v^\gamma}{\gamma} - kY - pY + hv.$$

is minimum when

$$v (t) = (h (t))^{\frac{1}{\gamma - 1}}, \quad 0 \leq t \leq T.$$

where $h (t)$ are given by (6.8).

Acknowledgments.
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