On the Comparison of the Distinguishing Coloring and the Locating Coloring of Graphs

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Abstract. Let $G$ be a simple connected graph. Then, $\chi_L(G)$ and $\chi_D(G)$ will denote the locating chromatic number and the distinguishing chromatic number of $G$, respectively. In this paper, we investigate a comparison between $\chi_L(G)$ and $\chi_D(G)$. We prove that $\chi_D(G) \leq \chi_L(G)$. Moreover, we determine some types of graphs whose locating and distinguishing chromatic numbers are equal. Specially, we characterize all graphs $G$ of order $n$ with property that $\chi_D(G) = \chi_L(G) = k$, where $k = 3, n-2$ or $n-1$. In addition, we construct graphs $G$ with $\chi_D(G) = n$ and $\chi_L(G) = m$ for every $4 \leq n \leq m$.

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1. Introduction

All graphs mentioned in this paper are assumed to be simple and undirected. All terminologies are standard and one can refer to [6].

One of the important and applicable concepts in graph theory is graph coloring. The subject graph coloring is one of the best known, popular, and extensively researched subjects in the field of graph theory, having many conjectures, which are still open and studied by various mathematicians and computer scientists. In this article, we will discuss the following two types of graph coloring.

Let $G$ be a graph and $c$ be a proper $k$-coloring of $G$, and let $\pi = (V_1, V_2, \ldots, V_k)$ where $V_i$ is the set of all vertices colored by $i$ for $1 \leq i \leq k$. The color code of vertex $v$ with respect to $\pi$, denoted by $c_{\pi}(v)$, is defined as the ordered $k$-tuple $(d(v, V_1), d(v, V_2), \ldots, d(v, V_k))$ such that $d(v, V_i)$ is the minimum distance from $v$ to each vertex in $V_i$ for $1 \leq i \leq k$. If all vertices of $G$ have distinct color codes, then $c$ is called a locating coloring of $G$. The
locating chromatic number, $\chi_L(G)$, is the minimum number of colors needed in a locating coloring of $G$.

The concept of locating coloring was introduced by Chartrand et al. [9] in 2002. Since then, the locating coloring number has been the subject of many researchers, for more details see [2,3,5,8].

A $k$-coloring of a graph $G$ is said to be a distinguishing $k$-coloring of $G$ if it is a proper $k$-coloring of $G$ and the identity automorphism is the only color-preserving automorphism of $G$. The distinguishing chromatic number $\chi_D(G)$ of $G$ is the least $k$ such that $G$ has a distinguishing $k$-coloring.

Collins and Trenk [11] introduced the distinguishing chromatic number of a graph. Many authors obtained more results on the distinguishing chromatic number and related subjects, see [10,13].

The locating coloring of a graph is dealing with the distance between vertices of a graph. The distinguishing coloring is discussing about vertices and automorphisms which are distance-preserving. Therefore, it would be interesting to give a relation between the above two kinds of coloring. In this article, we will compare the locating and distinguishing colorings. In fact, we prove that any locating coloring of a connected graph is a distinguishing coloring. Note that in some cases, for example complete multipartite graphs, the above two colorings are the same, but in many cases a graph $G$ has $\chi_D(G) < \chi_L(G)$. It is our main interest to see when $\chi_D(G) = \chi_L(G)$ or $\chi_D(G) < \chi_L(G)$. In this paper, we state some results for sufficient conditions that a distinguishing coloring is a locating coloring. In addition, we characterize all graphs $G$ of order $n$ with the property that $\chi_D(G) = \chi_L(G) = k$, where $k = 3, n - 2$ or $n - 1$.

We finish this introduction with an open problem related to all graphs with the same locating and distinguishing chromatic numbers.

Open Problem. Characterize the class $\mathcal{A}$ of graphs such that $G \in \mathcal{A}$ if and only if $\chi_D(G) = \chi_L(G)$.

2. General Relations

In this section, we are going to compare the locating chromatic number and the distinguishing chromatic number of a graph. First, let us state the following two theorems from [9,11] which give necessary and sufficient conditions for a graph $G$ to have $\chi_L(G) = |V(G)|$ or similarly when $\chi_D(G) = |V(G)|$.

**Theorem 2.1.** [9] Let $G$ be a connected graph of order $n \geq 3$. Then, $\chi_L(G) = n$ if and only if $G$ is a complete multipartite graph.

**Theorem 2.2.** [11] Let $G$ be a graph. Then, $\chi_D(G) = |V(G)|$ if and only if $G$ is a complete multipartite graph.

Now, we compare the locating and distinguishing coloring of a graph in the following theorem.

**Theorem 2.3.** Let $G$ be a connected graph. Then, any locating coloring of $G$ is a distinguishing coloring of $G$. 
Proof. Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be the color partition of a locating coloring of $G$. If $|V_i| = 1$ for all $i$, $1 \leq i \leq k$, then $\chi_L(G) = |V(G)|$ by Theorem 2.1. Hence, $G$ is a complete multipartite graph. Therefore, by Theorem 2.2 $\chi_D(G) = |V(G)|$ and $\mathcal{P}$ is a distinguishing coloring of $G$.

Thus, let $|V_j| \geq 2$ for some $j$, $1 \leq j \leq k$. We claim that $\mathcal{P}$ is a distinguishing coloring of $G$. Suppose, to the contrary, that $\mathcal{P}$ is not a distinguishing coloring of $G$. Hence, there exists a non-identity automorphism $f$ that preserves the color classes. Without loss of generality, we may assume that $V_j = \{a_1, a_2, \ldots, a_m\}$ with $m \geq 2$ and $f(a_1) = a_2$. Let for each $i$, $1 \leq i \leq k$, $d(a_1, V_i) = d(a_1, b_i)$ and $d(a_2, V_i) = d(a_2, c_i)$ for some $b_i, c_i \in V_i$. Consider an $a_1 - b_i$ path and an $a_2 - c_i$ path in $G$. Since the image of a path under an automorphism is also a path, and $f$ is an automorphism that preserves colors, the distance of $a_1$ from the color class $V_i$ is not larger than the distance of $a_2$ from that class, and vice versa. Therefore, we have $d(a_1, V_i) = d(a_2, V_i)$ for all $1 \leq i \leq k$. It means that the color codes of $a_1$ and $a_2$ are the same and it is a contradiction. Hence, the proof is completed. 

By Theorem 2.3, the following result is obtained directly.

**Corollary 2.4.** For any connected graph $G$, $\chi_D(G) \leq \chi_L(G)$.

Next, we quote the definition of the metric dimension of a graph $G$. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in a connected graph $G$ and a vertex $v$ of $G$, the ordered $k$-tuple $r_W(v)$ of $v$ with respect to $W$ is defined by

$$r_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)),$$

where $d(v, w_i)$ is the distance between $v$ and $w_i$, $1 \leq i \leq k$. The set $W$ is a resolving set for $G$ if the $k$-tuples $r_W(v)$, $v \in V(G)$, are distinct. The metric dimension of $G$ is the minimum cardinality of a resolving set for $G$ and is denoted by $\dim(G)$. These concepts were introduced independently in Refs. [14,15].

In Ref. [9], Chartrand et al. gave an upper bound for $\chi_L(G)$ in terms of $\chi(G)$ and $\dim(G)$. They proved that $\chi_L(G) \leq \chi(G) + \dim(G)$. It is obvious that the above upper bound is sharp. For instance, if $P_{2k+1}$ is a path of length $2k$, then $\chi_L(P_{2k+1}) = 3$, $\chi(P_{2k+1}) = 2$ and $\dim(P_{2k+1}) = 1$. Now, by Theorem 2.3, we get the above upper bound for $\chi_D(G)$.

**Corollary 2.5.** For any connected graph $G$, $\chi_D(G) \leq \chi(G) + \dim(G)$.

One can see that any upper bound for $\chi_L(G)$ will be an upper bound for $\chi_D(G)$ as well, similarly, any lower bound for $\chi_D(G)$ will be a lower bound for $\chi_L(G)$. The following corollary is coming from this point of view. We can refer to [9].

**Corollary 2.6.** If $G$ is a connected graph of order $n \geq 3$ with diameter $\text{diam}(G) \geq 2$, then $\chi_D(G) \leq n - \text{diam}(G) + 2$.

Notice that a distinguishing coloring of $G$ is not necessarily a locating coloring of $G$. For example, let $\{\{a_1, a_5\}, \{a_2, a_4, a_6\}, \{a_3, a_7\}\}$ be distinguishing coloring classes of $P_7$. This coloring of $P_7$ is distinguishing but not
Proof. Assume first that \( \chi_{\pi}(a_2) = \chi_{\pi}(a_4) = (1, 0, 1) \). In addition, it can be seen that 
\( \chi_D(P_7) = \chi_L(P_7) = 3 \) (Fig. 1).

It is interesting to see the difference between \( \chi_D(G) \) and \( \chi_L(G) \). In the following we shall prove that these two graph invariants are independent of each other, except that \( \chi_D(G) \) can never exceed \( \chi_L(G) \). To do so, let us consider the result of Behtoei and Anbarloei [4] regarding the locating chromatic number of a wheel \( W_n := K_1 + C_n \).

**Theorem 2.7** [4]. Let \( n_o = \min\{t : t \in \mathbb{N}, n \leq \frac{1}{2}(t^2(t - 1))\} \). If \( n \geq 9 \), then

\[
\chi_L(W_n) = \begin{cases} 
    n_0 + 1, & n \neq \frac{1}{2}(n_0^2(n_0 - 1)) - 1, \\
    n_0 + 2, & n = \frac{1}{2}(n_0^2(n_0 - 1)) - 1.
\end{cases}
\]

Now, let us prove the following theorem.

**Theorem 2.8.** For every \( 4 \leq n \leq m \), there exists a graph \( G \) having locating chromatic number \( m \) and distinguishing chromatic number \( n \).

**Proof.** Assume first that \( n = 4 \). Let \( G \) be a wheel \( W_k := K_1 + C_k \), for \( k \geq 9 \). Let \( v \) be the center of the wheel and \( v_1, v_2, \ldots, v_k \) be the vertices of the cycle in the wheel (in a cyclic order). Define a vertex coloring \( f \) on \( V(W_k) \rightarrow \{1, 2, 3, 4\} \) with \( f(v) = 1 \) and satisfying the following property.

\[
f(v_i) = \begin{cases} 
    4, & i = 1, 4 \\
    3, & i (\neq 1) \text{ is odd} \\
    2, & i (\neq 4) \text{ is even}.
\end{cases}
\]

It is easy to see that \( f \) is a minimum distinguishing coloring on \( W_k \). Therefore, we have \( \chi_D(W_k) = 4 \) for all \( k \geq 9 \). Suppose that \( k \) has the property that

\[
\frac{1}{2}(m - 2)^2(m - 3) < k \leq \frac{1}{2}(m - 1)^2(m - 2) \quad \text{and} \quad k \neq \frac{1}{2}(m - 1)^2(m - 2) - 1.
\]

Now, by Theorem 2.7, we have \( \chi_L(W_k) = m \). Note that if \( n = m = 4 \) then the unique possible value is \( k = 9 \).

Let \( n > 4 \). Let \( G_{k,t} \) denote the graph obtained by joining the center vertex of a wheel \( W_k \) to the center vertex of a star graph \( S_t := K_{1,1} \), by an edge. Assume that \( G = G_{k,n-1} \), where \( \frac{1}{2}(m - 2)^2(m - 3) < k \leq \frac{1}{2}(m - 1)^2(m - 2) \) and \( k \neq \frac{1}{2}(m - 1)^2(m - 2) - 1 \). Thus, \( \chi_D(G) = n \). By Theorem 2.7, we can consider an \( m \)-locating coloring for \( W_k \) in \( G \). Without loss of generality, we may suppose that the centers of \( W_k \) and \( S_{n-1} \) are colored by colors 1 and 2, respectively. In addition, color the non-center vertices of \( S_{n-1} \) with distinct colors used in \( W_k \). Let \( V_1, V_2, \ldots, V_m \) be the color classes of the above coloring of \( G \). Let \( v \in V(W_k) \) and \( u \in V(S_{n-1}) \) be two vertices with the same color. If \( v, u \in V_1 \), \( 2 \leq i \leq m \), then \( v \) and \( u \) have the different distance from \( V_1 \). (Note that in this step the condition \( n > 4 \) has been used since if \( n = 4 \).
Theorem 2.10. Let \( d \) be the distinguishing chromatic number means that \( \{v_1, v_2, \ldots, v_m\} \) is a locating coloring for \( G \) and so \( \chi_L(G) = m \).

**Proof.** We first present a locating \( n \)-coloring of \( G \). For this purpose, give color 1 to vertex \( a \). Let us partition \( N(a) \) into sets \( A_i \), for \( 2 \leq i \leq n \), where the size of each \( A_i \) is \( n - 1 \). We assign color \( i \) to all the vertices of \( A_i \), \( 2 \leq i \leq n \). Now, if we color all the vertices of \( N(A_i) \setminus \{a\} \) with colors \( 1, 2, \ldots, i - 1, i + 1, \ldots, n \), \( 2 \leq i \leq n \), then we have a locating \( n \)-coloring of \( G \). Therefore, \( \chi_L(G) \leq n \).

We claim that \( \chi_D(G) = n \). For a contradiction, consider a distinguishing \((n - 1)\)-coloring of \( G \). There exists at least one color \( i \), \( (1 \leq i \leq n - 1) \), such that more than \( n - 1 \) vertices of \( N(a) \) are colored by \( i \). Let \( A_i \) denote the subset of vertices of \( N(a) \) which are colored by \( i \). Thus, \( |N(A_i) \setminus \{a\}| > n - 1 \). This implies that there are two vertices \( x \) and \( y \) of \( N(A_i) \setminus \{a\} \) having the same color. Let \( x_1, x_2 \) be two pendant vertices where \( x \sim x_1 \) and \( y \sim x_2 \). Now, define an automorphism \( \alpha \) that swaps \( x \) and \( y \), \( x_1 \) and \( x_2 \), and fix the other vertices. Then, \( \alpha \) preserves the coloring, a contradiction. Therefore, Corollary 2.4 concludes that \( \chi_D(G) = \chi_L(G) = n \).

For the other implication, consider the vertices of the graph without names. Since no two paths of length 2 starting from \( a \) can have the same ordered pair of colors, the locating \( n \)-coloring presented above is the only way to color the graph. \( \square \)

In the following theorem, we show that all graphs \( G \) of order \( n \) with distinguishing chromatic number \( n - 1 \) have locating chromatic number \( n - 1 \).

**Theorem 2.10.** Let \( G \) be a connected graph with \( \chi_D(G) = |V(G)| - 1 \). Then, \( \chi_L(G) = |V(G)| - 1 \).

**Proof.** Let \( c \) be a distinguishing \( \chi_D(G) \)-coloring which induces the partition \( \pi = (\{v_1\}, \{v_2\}, \ldots, \{v_{n-2}\}, \{v_{n-1}, v_n\}) \). The only vertices in \( G \) with the same color are \( v_{n-1} \) and \( v_n \). Therefore, it suffices to show that the color codes of these two vertices are distinct. Of course, \( v_{n-1} \) and \( v_n \) are not adjacent. Let \( A = N(v_{n-1}) - N(v_n) \) and \( B = N(v_n) - N(v_{n-1}) \). Since under coloring \( c \)
there is no non-identity automorphism preserving the above color class, then at least one of $A$ or $B$ is not empty, say $|A| \geq 1$ and $v_i \in A$ for some $i$. Therefore, the $i^{th}$ component of $c_{\pi}(v_{n-1})$ is 1, but the one of $c_{\pi}(v_n)$ is $\geq 2$. This means that the color codes of these two vertices are different. Thus, $c$ is a locating coloring of $G$. By Corollary 2.4, we obtain $\chi_L(G) = |V(G)| - 1$.

Note that Chartrand et al. [8] have characterized all connected graphs of order $n \geq 4$ with locating chromatic number $n - 1$.

3. Graphs with $\chi_D(G) = \chi_L(G) = 3$

In this section, we will further address graphs $G$ with $\chi_L(G) = \chi_D(G)$. In particular, we characterize all graphs $G$ with $\chi_L(G) = \chi_D(G) = 3$. From now on, let $T$ be a set of all trees $T$ with locating chromatic number 3. Baskoro and Asmiati [2] characterized all trees on $n$ vertices ($n \geq 3$) with locating chromatic number 3 as follows.

**Theorem 3.1** [2]. A tree $T$ is in $T$ if and only if $T$ is any subtree of one of the trees $(A)$, $(B)$ or $(C)$ in Fig. 3 containing vertices $X$, $Y$ and $Z$, with non-negative $a, b, c, d, e, f, i, j, k, p$; $g \geq 1, h \geq 1$; $i = j$ and $k = p$.

Let $x$ be a vertex in a tree $T$. A branch $B$ of $T$ at $x$ is defined to be a maximal subtree containing $x$ as an endpoint. That is, a branch of $T$ at $x$ is the subgraph induced by $x$ and one of the components of $T - \{x\}$. The following theorem characterizes all trees $T$ with $\chi_D(T) = \chi_L(T) = 3$. 

Theorem 3.2. A tree $T$ in $\mathcal{T}$ has $\chi_D(T) = 3$ if and only if there are two distinct branches $A$ and $B$ of $T$ at a vertex $x$ such that there exists an isomorphism $f : A \rightarrow B$ with $f(x) = x$.

Proof. Let $T$ be a tree in $\mathcal{T}$. Then, $\chi_L(T) = 3$, and $T$ must be isomorphic to one of the trees characterized in Theorem 3.1. However, $\chi_D(T) = 2$ or 3. If there are two distinct branches $A$ and $B$ of $T$ at vertex $x$ such that there exists an isomorphism $f : A \rightarrow B$ with $f(x) = x$, then the proper 2-coloring $\alpha$ of $T$ will be not a distinguishing coloring of $T$. This is true since the isomorphism $f$ can be extended to be an automorphism $f'$ on $T$ with $f'(A) = B$, $f'(B) = A$, $f'(x) = x$ and $f'(v) = v$ for any other vertex in $T$ so that $f'$ preserves the coloring $\alpha$ on $T$. But now, since $\chi_L(T) = 3$, by using a locating coloring of such a tree, we have $\chi_D(T) = 3$.

Conversely, let $\chi_D(T) = 3$. This means that there is a non-identity automorphism $f$ on $T$ preserving the 2-coloring $\alpha$ on $T$. By $f$, there must be two vertices of degree one of $T$, say $u$ and $v$, such that $f(u) = v$. This means that $\alpha(u) = \alpha(v)$, hence the distance between $u$ and $v$ is even. Then, there exists a vertex $x$ that is a midpoint of the path connecting $u$ and $v$ in $T$. Thus, we have that $f(A) = B$, $f(x) = x$ with $A$ and $B$ are the branches of $T$ at $x$ containing $u$ and $v$, respectively. Therefore, we complete the proof. \(\square\)

Next, we characterize all graphs $G$ other than trees with $\chi_D(G) = \chi_L(G) = 3$. Asmiati and Baskoro [1] have characterized all graphs $G$ containing a cycle with $\chi_L(G) = 3$. Such graphs are stated in the following theorem.

Theorem 3.3 [1]. Let $G$ be a connected graph containing a cycle with $\chi_L(G) = 3$. Then,

1. If $G$ is bipartite then $G$ is isomorphic to a subgraph of the graph in Fig. 4 containing at least edges $XZ'$, $YZ$ and all edges in the cycle $X \sim X_1 \sim X_2 \sim \cdots \sim X_p \sim X' \sim Y' \sim Y_p \sim \cdots \sim Y_1 \sim Y \sim X$.

2. If $G$ is not bipartite then $G$ is isomorphic to a subgraph of either of the graph (A), (B), (C) or (D) in Fig 5 containing a smallest odd cycle $C_m$ with edges $e, e'$ and $e''$.

Let $\mathcal{G}$ be the set of all graphs $G$ containing a cycle with $\chi_L(G) = 3$, characterized in Theorem 3.3. Then, we have the following characterization of all graphs $G$ with $\chi_D(G) = \chi_L(G) = 3$.

Theorem 3.4. Let $G$ be a connected graph containing a cycle with $\chi_D(G) = \chi_L(G) = 3$. Then, $G$ is isomorphic to one of the following:

1. Any subgraph $H$ of the graph in Fig. 4 containing at least edges $XZ'$, $YZ$ and all edges in the cycle
   
   $X \sim X_1 \sim X_2 \sim \cdots \sim X_p \sim X' \sim Y' \sim Y_p \sim \cdots \sim Y_1 \sim Y \sim X$,
   
   such that
   
   (a) $\deg(Z) = 3$ and $h = r - 1$. 
   


Figure 4. All bipartite graphs $G$ other than trees with $\chi_L(G) = 3$ and with a minimum locating coloring

(b) $H$ containing edges $X'X''$ and $Y'Y''$, $\deg(Z) \leq 2$, $\deg(X_i) = \deg(X_j)$ when $d(X_i, X) = d(X_j, X')$, $(1 \leq i, j \leq p)$; and $H$ satisfies one of the following conditions:

(i) $k = k'$, $r = r'$ and $d(X, X')$ is odd.

(ii) $k = r'$, $r = k'$ and $d(X, X')$ is even.

2. Any subgraph of the graph (A), (B), (C) or (D) in Fig. 5 containing a smallest odd cycle $C_m$ with edges $e, e'$ and $e''$.

Proof. Let $G$ be a graph containing a cycle with $\chi_L(G) = 3$. Then, $G$ must be isomorphic to one of the graphs characterized in Theorem 3.3. However, not all members $G$ of $G$ will have $\chi_D(G) = 3$. We divide the proof into two cases.

Case 1. $G$ is bipartite.

Since $G$ is a connected bipartite graph, $G$ has a unique proper 2-coloring. Hence, if there exists an automorphism that preserves the proper coloring of $G$ we must have $\chi_D(T) = 3$. For this, we apply necessary and sufficient conditions to the graph in Fig. 4. If $\deg(Z) = 3$, then the only non-trivial automorphism is the automorphism that interchanges two vertices $w$ and $w'$ and fixes other vertices of $G$ that are not on the path $w - w'$. This automorphism is available if $h = r - 1$. Clearly, this automorphism preserves the proper 2-coloring of $G$. Let $\deg(Z) \leq 2$. Without loss of generality, we may assume that $h = 0$. In this case, there are two possible non-trivial automorphisms that preserve the proper coloring. Assume that $\alpha_1$ and $\alpha_2$ are those automorphisms such that $\alpha_1(X) = X'$ and $\alpha_2(X) = Y'$. Therefore, by Observation 16 [12], automorphisms $\alpha_1$ and $\alpha_2$ conclude the conditions (b) and also (b)(i) and (b)(ii), respectively.

Case 2. $G$ is not bipartite.

Since $\chi(G) \geq 3$, then using a locating coloring of $G$ in Fig. 5, we obtain $\chi_D(G) = 3$. Therefore, all non-bipartite graphs $G$ in $G$ have $\chi_D(G) = 3$. 
Therefore, we complete the proof. □

4. Graphs with $\chi_L(G) = \chi_D(G) = |V(G)| - 2$

In this section, we investigate graphs $G$ such that $\chi_D(G) = \chi_L(G) = |V(G)| - 2$. To do this, we begin with Theorem 4.1, in which all graphs of order $n$ with distinguishing chromatic number $n - 2$ were characterized in Ref. [7]. The 17 types of graphs $G$ with $\chi_D(G) = n - 2$ are described in Theorem 4.1. There is one among them which has a subtype with $\chi_L(G) = n - 1$, the others have $\chi_L(G) = n - 2$ as it will be proved in this section. We first state some preliminaries that are needed.

For any graph $G$ with vertices $(v_1, \ldots, v_n)$ and for any collection of vertex-disjoint graphs $H_1, \ldots, H_n$, let $G(H_1, \ldots, H_n)$ denote the graph obtained from $G$ by replacing each $v_i$ with a copy of $H_i$ and replacing each edge $v_i v_j$ by $H_i \vee H_j$. If an $H_i$ is vacuous, i.e., $H_i = \emptyset$, then replacing $v_i$ by $\emptyset$ refers to deleting $v_i$ and all edges incident to it. Note that the substituted $H$ can be an independent set; i.e., both the empty set and independent sets are viewed as "complete multipartite" graphs. We present three defined graphs $\hat{G}_5$, $\hat{G}_6$ and $\hat{G}_7$ along with a class of labeled graphs $\mathcal{G}_3$ consisting of two non-isomorphic graphs. The labeled graphs $\hat{G}_5$, $\hat{G}_6$ and $\hat{G}_7$ have vertices $(v_1, v_2, v_3, v_4)$, $(v_1, v_2, v_3, v_4, v_5)$ and $(v_1, v_2, v_3, v_4, v_5, v_6)$ respectively, while a labeled graph $\hat{G}_3$ belonging to the class $\mathcal{G}_3$ has vertices $(v_1, v_2, v_3, v_4, v_5)$, see Figs. 6 and 7. Furthermore, define $\hat{K}_2$ and $\hat{K}_3$ to be the labeled complete
graphs of orders two and three respectively, where $\hat{K}_2(v_1, v_2)$ has vertices $(v_1, v_2)$ and $\hat{K}_3(v_1, v_2, v_3)$ has vertices $(v_1, v_2, v_3)$. In particular, if $H_1, H_2$ are nonvacuous complete multipartite graphs, then $\hat{K}_2(H_1, H_2)$ represents a complete multipartite graph with at least two parts.

**Theorem 4.1** [7, Theorem 3.5]. Let $G$ be a graph of order $n > 4$. Then, $\chi_D(G) = n - 2$ if and only if $G$ is the join of a complete multipartite graph (possibly vacuous) with one of the following:

(a) $P_5$
(b) $C_5$
(c) $C_6$
(d) $2K_3$
(e) $\overline{K}_r \cup K_2$, for $r \geq 2$
(f) $\hat{K}_2(\overline{K}_r, H_1) \cup K_2$, for $r \geq 2$
(g) $\hat{K}_3(H_1, H_2, H_3) \cup K_2$
(h) $\hat{K}_2(H_1, H_2) \cup \overline{K}_2$
(i) $2K_2 \vee 2K_2$
(j) $2K_2 \vee (\hat{K}_2(H_1, H_2) \cup K_1)$
(k) $2K_2 \cup K_1$
(l) $(2K_2 \vee H_1) \cup K_1$
(m) $\hat{G}_3(H_1, H_2, H_3, K_1, K_1)$, for $\hat{G}_3 \in \mathcal{G}_3$
(n) $\hat{G}_5(H_1, H_2, K_1, K_1)$
(o) $\hat{G}_5(K_1, H_1, H_2, K_1)$
(p) $\hat{G}_6(K_1, H_1, H_2, H_3, K_1)$
(q) $\hat{G}_7(H_1, H_2, H_3, H_4, K_1, K_1)$

where each of $H_1, H_2, H_3, H_4$ is a nonvacuous complete multipartite graph.

In the following, we will compute the locating chromatic number of the graphs characterized in Theorem 4.1. For this, we first present the exceptional case where the distinguishing chromatic number is not equal to the locating chromatic number in a subcase.

**Lemma 4.2.** Let $G$ be a graph of order $n > 4$. Let $G$ be the join of a complete multipartite graph (possibly vacuous) with $\hat{G}_5(K_1, H_1, H_2, K_1)$, where each of $H_1, H_2$ is a nonvacuous complete multipartite graph. If at least one of $H_1$ or $H_2$ is edgeless, then $\chi_L(G) = n - 1$. Otherwise, $\chi_L(G) = n - 2$.

**Proof.** Since $G \setminus \{v_1, v_4\}$ is a complete multipartite graph, then in any locating coloring of $G$ all vertices of $G$ other than $v_1$ and $v_4$ must receive distinct colors. Thus, $\chi_L(G) \geq n - 2$. Note that, $\chi_L(G) \leq n - 1$ always is an upper bound because $v_1$ and $v_4$ can get the same color distinct from the colors in $H_1 \cup H_2$.

Assume first that, $H_1$ is edgeless. However, $\chi(G) \neq n - 2$, since otherwise, the color of $v_1$ will be the same as the one of some vertex $a$ in $H_2$, and the color of $v_4$ will be the same as the one of some vertex $b$ in $H_1$. But, this implies that the color codes of $v_4$ and $b$ are the same, a contradiction. Therefore, $\chi_L(G) = n - 1$.

Now, suppose that $H_1$ and $H_2$ are not edgeless, and so $|V(H_1)| \geq 2$ and $|V(H_2)| \geq 2$. Color the vertices $v_1$ and $v_4$ by the color of a vertex of $H_2$ (as $x_1$) and a vertex of $H_1$ (as $y_1$), respectively. For a locating $(n - 2)$-coloring, we should investigate the color codes of $v_1$ and $x_1$, and also $v_4$ and $y_1$. Since $H_2$ does not have isolated vertices, there exist a vertex $x_2 \in V(H_2)$ such that $1 = d(x_1, x_2) \neq d(v_1, x_2) = 2$. Therefore, the color codes of $v_1$ and $x_1$ are distinct in the coordinate of the color $x_2$. Similarity, the vertices $v_4$ and $y_1$ have distinct color codes. Therefore this coloring is a locating coloring and $\chi_L(G) = n - 2$. \[\Box\]

Note that, since in Lemma 4.2, we investigate graph $\hat{G}_5(K_1, H_2, K_1, K_1)$, we remove it in the next lemma.

**Lemma 4.3.** Let $G$ be a graph of order $n > 4$. Let $G$ be the join of a complete multipartite graph (possibly vacuous) with one of the following:

(i) $\hat{G}_3(H_1, H_2, H_3, K_1, K_1)$, for $\hat{G}_3 \in \mathcal{G}_3$
(ii) $\hat{G}_5(H_1, H_2, K_1, K_1)$, $H_1 \not\cong K_1$
(iii) $\hat{G}_7(H_1, H_2, H_3, H_4, K_1, K_1)$

where each of $H_1, H_2, H_3, H_4$ is a nonvacuous complete multipartite graph. Then, $\chi_L(G) = n - 2$.

**Proof.** (i) Graph $G[V(G) \setminus \{v_4, v_5\}]$ is a complete multipartite graph. Therefore, color all vertices in $V(G) \setminus \{v_4, v_5\}$ with distinct colors. Now, color vertex $v_4$ with a color of some vertex $a$ in $H_3$ and color vertex $v_5$ with a color of some vertex $b$ in $H_2$. In both cases: either $v_4 \sim v_5$ or $v_4 \sim v_5$, the color codes of all vertices in $G$ are different. Therefore, we obtain a locating $(n - 2)$-coloring of $G$. Since $\chi_D(G) \leq \chi_L(G)$, then $\chi_L(G) = n - 2$. 

(ii) In this case, \(G \setminus \{v_3, v_4\}\) is a complete multipartite graph. Therefore, color all vertices in \(G\) other than \(v_3\) and \(v_4\) with distinct colors. If \(H_1\) is edgeless, color vertex \(v_3\) and \(v_4\) by two distinct colors of some vertex in \(H_1\). Otherwise, color vertex \(v_3\) by a color of some vertex in \(H_1\) and \(v_4\) by a color of some vertex in \(H_2\). Then, all the color codes of vertices in \(G\) will be distinct. Since \(\chi_D(G) \leq \chi_L(G)\), then \(\chi_L(G) = n - 2\).

(iii) Similarly, we give distinct colors to all vertices other than \(v_5\) and \(v_6\). Color vertex \(v_5\) by a color of some vertex of \(H_1\) and \(v_6\) by a color of some vertex of \(H_4\). Then, we obtain a locating \((n - 2)\)-coloring of \(G\) and thus \(\chi_L(G) = n - 2\).

\[\square\]

**Lemma 4.4.** Let \(G\) be a graph of order \(n > 4\). Let \(G\) be the join of a complete nonvacuous multipartite graph with one of the following graphs:

(i) \(2K_3\)
(ii) \(\bar{K}_2(K_r, H_1) \cup K_2\), for \(r \geq 2\)
(iii) \(\bar{K}_r \cup K_2\), for \(r \geq 2\)
(iv) \(\bar{K}_2(H_1, H_2) \cup \bar{K}_2\)
(v) \(\bar{K}_3(H_1, H_2, H_3) \cup K_2\)
(vi) \(2K_2 \vee (\bar{K}_2(H_1, H_2) \cup K_1)\)
(vii) \(2K_2 \vee 2K_2\)
(viii) \((2K_2 \vee H_1) \cup K_1\)
(ix) \(2K_2 \cup K_1\)
(x) \(\bar{G}_6(K_1, H_1, H_2, H_3, K_1)\)

where each of \(H_1, H_2, H_3, H_4\) is a nonvacuous complete multipartite graph. Then, \(\chi_L(G) = n - 2\).

**Proof.** It is easily to verify that the graph \(G\) is connected. In each case we will select distinct vertices \(x_1, x_2, x_3, x_4\) of \(G\) and define a \((n - 2)\)-coloring \(c\) such that all vertices other than \(x_1, x_2, x_3, x_4\) receive distinct colors and \(c(x_1) = c(x_3), c(x_2) = c(x_4)\).

(i) Let \(x_1\) and \(x_2\) be two vertices in the first copy of \(K_3\), \(x_3\) and \(x_4\) be two vertices in the second copy of \(K_3\).

(ii), (iii) Let \(x_1\) and \(x_2\) be the two vertices of \(K_2\), \(x_3\) and \(x_4\) be two vertices of \(\bar{K}_r\).

(iv) Let \(x_1\) and \(x_2\) be the two vertices of \(\bar{K}_2\), \(x_3 \in H_1\) and \(x_4 \in H_2\).

(v) Let \(x_1\) and \(x_2\) be the two vertices of \(\bar{K}_2\), \(x_3 \in H_1\) and \(x_4 \in H_2\).

(vi) Let \(x_1\) and \(x_3\) be two nonadjacent vertices in \(2K_2\), \(x_2 \in H_1\) and \(x_4 \in K_1\).

(vii) Let \(x_1\) and \(x_3\) be two nonadjacent vertices in the first copy of \(2K_2\), \(x_2\) and \(x_4\) be two nonadjacent vertices in the second copy of \(2K_2\).

(viii) Let \(x_1\) and \(x_3\) be two nonadjacent vertices in \(2K_2\), \(x_2 \in K_1\) and \(x_4 \in H_1\).

(ix) Let \(x_1\) and \(x_3\) be two nonadjacent vertices in \(2K_2\), \(x_2 \in K_1\) and \(x_4 \in 2K_2\).

(x) Let \(x_1 \in K_1, x_3 \in H_2, x_2 \in H_3\) and \(x_4\) be in the second copy of \(K_1\).
By selecting four vertices $x_1, x_2, x_3, x_4$ above in each case, it can be verified that all color codes of the vertices of $G$ are distinct. Therefore, $c$ is locating coloring of $G$. Since $\chi_D(G) \leq \chi_L(G)$ and $\chi_D(G) = n - 2$, thus $\chi_L(G) = n - 2$.

Lemma 4.5. Let $G$ be a graph of order $n > 4$. Let $G$ be the join of a complete (possibly vacuous) multipartite graph with one of $\{P_5, C_5, C_6\}$. Then, $\chi_L(G) = n - 2$.

Proof. The graph $G$ is always connected. Color the vertices $P_5$ and $C_5$ by colors $1, 2, 3, 1, 2$ in (linear or cyclic) ordering; color $C_6$ by $1, 2, 3, 4, 1, 2$ in cyclic ordering. Next, color all other vertices of $G$ with distinct colors. Such a coloring is locating with $n - 2$ colors. Therefore, $\chi_L(G) = n - 2$ for each case.

Now, we are in a position to use the above results for classifying all graphs $G$ with $\chi_L(G) = \chi_D(G) = n - 2$, as follows.

Theorem 4.6. Let $G$ be a connected graph of order $n > 4$. Then, $\chi_L(G) = \chi_D(G) = n - 2$ if and only if $G$ is the join of a complete multipartite graph (possibly vacuous) with one of the following:

(a) $P_5$
(b) $C_5$
(c) $C_6$
(d) $2K_3$
(e) $K_r \cup K_2$, for $r \geq 2$
(f) $\hat{K}_2(K_r, H_1) \cup K_2$, for $r \geq 2$
(g) $\hat{K}_2(H_1, H_2, H_3) \cup K_2$
(h) $2K_2 \lor 2K_2$
(i) $2K_2 \lor (\hat{K}_2(H_1, H_2) \cup K_1)$
(j) $2K_2 \cup K_1$
(k) $(2K_2 \lor H_1) \cup K_1$
(l) $\hat{G}_3(H_1, H_2, H_3, K_1, K_1)$, for $\hat{G}_3 \in \mathcal{G}_3$
(m) $\hat{G}_5(H_1, H_2, H_3, K_1, K_1), H_1 \not\cong K_1$
(n) $\hat{G}_7(H_1, H_2, H_3, H_4, K_1, K_1)$
(o) $\hat{G}_6(K_1, H_1, H_2, H_3, K_1)$
(p) $\hat{G}_5(K_1, H_1, H_2, K_1)$, where none of $H_1$ and $H_2$ is edgeless.

Proof. The result is immediate, by Lemmas 4.2–4.5 and Theorem 4.1.

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