Application of the 3-space approach to the Bianchi II cosmological model

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Abstract

Einstein used 4-dimensional space time geometry to explain gravity. However, in 1962, Baierlein, Sharp and Wheeler proposed a Jacobi type timeless Lagrangian based on the 3-dimensional geometry of space to reproduce the same physics. In 2002, Barbour et al. further extended this idea and they call it 3-space approach. Here we use Bianchi II cosmological model to demonstrate the 3-space idea. Indeed, we find that this theory is more fundamental and the manipulation is more practical. We recover the known and find a new solutions.

1 Introduction

Almost century ago, Einstein used 4-dimensional space time geometry to explain gravity. However, it seems that 4D concept is not the most basic. As Dirac [1] pointed out in 1958: ‘I am inclined to believe from this that four-dimensional symmetry is not a fundamental property of the physical world.’ Four years later, in 1962, Baierlein, Sharp and Wheeler [2] proposed a Jacobi type timeless Lagrangian, i.e., BSW action. They laid down a very nice and concrete foundation, but it did not attract much attention for 40 years. Until recently in 2002, Barbour, Foster and ´O Murchadha [3] extended this method and they call it 3-space approach. This theory uses 3-space, without time, to describe the same physics as the 4-metric does. The motivation of our work is not finding the Einstein solution for the Bianchi II cosmological model. But, in order to appreciate the 3-space theory, we apply it to this universe model for a simple testing. As a consequence, we have recovered the known solution in 4-metric [4], and also find a new set of solutions. Moreover, we remark that finding these solutions using 3-space theory is more practical.

2 Einstein equations of motion in Bianchi II model

The line element of the Bianchi II cosmological model [5] in 4-spacetime is

\[ ds^2 = -dt^2 + f_1^2 dx^2 + f_2^2 dy^2 - 2xf_2^2 dydz + (f_1^2 + x^2f_2^2)dz^2, \]  

where \( f_1 \) and \( f_2 \) depend only on time \( t \). The metric signature we use is +2, Latin indices indicate spatial and Greek means spacetime. Solving the Einstein equation:

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \]

the equations of motion are

\[ G_{00} = \frac{\ddot{f}_1}{f_1} + \frac{2\dot{f}_1\dot{f}_2}{f_1f_2} - \frac{f_2^2}{4f_1^2}, \quad G_{11} = -f_1^2 \left( \frac{\ddot{f}_1}{f_1} + \frac{\ddot{f}_2}{f_2} + \frac{\dot{f}_1\dot{f}_2}{f_1f_2} + \frac{f_2^2}{4f_1^2} \right), \]

\[ xG_{22} = -xf_2^2 \left( \frac{\ddot{f}_1}{f_1} + \frac{\dot{f}_1\dot{f}_1}{f_1f_1} - \frac{3f_2^2}{4f_1^2} \right) = -G_{23}, \quad G_{33} = G_{11} + x^2G_{22}, \]

where \( \dot{f} \) denotes differentiate w.r.t. time \( t \). Confining source free for Einstein equation in vacuum, we find that there are only three independent equations of motion. These solutions are known [4] by a simple transformation \( dt = f_1^2(T) f_2(T) dT \)

\[ f_1 = a_1 \sqrt{e^T \cosh T}, \quad f_2 = a_2 / \sqrt{\cosh T}, \]

where \( a_1, a_2 \) are arbitrary constants and \( T \) is another time parameter. After a simple checking, we find that \( a_2 = \pm 1 \).
As there are only two unknown functions $f_1$ and $f_2$ in the line element, why we have 3 equations from $G_{\mu\nu}$? One might speculate that it is over determined. However, it is not. Here we give three examples to demonstrate this statement. (i) Suppose $(dt/dT, f_1, f_2) = (1, \sqrt{t}, 1)$. We check that this trial solution satisfy $G_{00}$ and $G_{11}$, but fail to fulfill $G_{22}$ requirement. (ii) Suppose $(dt/dT, f_1, f_2) = (e^{2T}, e^T, 2)$. We find that they satisfy $G_{00}$ and $G_{22}$, but violate $G_{11}$. (iii) Suppose $(dt/dT, f_1f_2) = (1, t, \sqrt{12} t)$. This allows $G_{00}$ vanishes, but $G_{11}$ and $G_{22}$ cannot. Hence $G_{00}$, $G_{11}$ and $G_{22}$ are independent. Moreover, through these three concrete examples, one might deduce that constant $f_2$ is not adequate referring to the first two cases and we examine $G_{\mu\nu}$ that indeed it is forbidden. This argument becomes manifest by using the 3-space method in section 3 (i.e., see (16)).

3 The 3-space approach for Bianchi II universe

Here come to the 3-space approach for Bianchi II cosmological model. The BSW type action [2, 3] has the form

$$I = \int d\lambda \int \sqrt{g} \sqrt{3R} \sqrt{\mathcal{T}} \, d^3x,$$

where $\lambda$ is a parameter, the determinant $\sqrt{g} = f_1^2(\lambda)f_2(\lambda)$, the three dimensional scalar curvature $3R = -f_2^2/(2f_1^4)$, and ‘kinetic energy’

$$\mathcal{T} = G^{abcd}(\dot{g}_{ab} - \nabla \xi g_{ab})(\dot{g}_{cd} - \nabla \xi g_{cd}) = -8 \left( \frac{\dot{f}_1f_1}{f_1f_1} + \frac{2\dot{f}_1f_2}{f_1f_2} \right).$$

Note that $G^{abcd} = g^{ac}g^{bd} - g^{ab}g^{cd}$ is the DeWitt supermetric, $\nabla$ is the Lie derivative, $\xi$ is the space of the fields, $\dot{g}_{ab}$ and $\dot{f}$ mean differentiate w.r.t. $\lambda$. The equations of motion can be obtained through the Euler-Lagrange equation

$$\frac{\partial}{\partial \lambda} \frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}} - \frac{\delta \mathcal{L}}{\delta g_{ij}} = 0,$$

where the Lagrangian density $\mathcal{L} = \sqrt{3RgT}$. The momentum $p^{ij} = \partial \mathcal{L}/\partial \dot{g}_{ij}$ and its corresponding non-vanishing components are

$$p^{11} = -\frac{\sqrt{g}g^{11}}{N} \left( \frac{\dot{f}_1}{f_1} + \frac{\dot{f}_2}{f_2} \right), \quad p^{22} = -\frac{\sqrt{g}}{N} \left( g^{22} \left( \frac{\dot{f}_1}{f_1} + \frac{\dot{f}_2}{f_2} \right) + \frac{1}{f_2^2} \left( \frac{\dot{f}_1}{f_1} - \frac{\dot{f}_2}{f_2} \right) \right),$$

$$p^{23} = -\frac{\sqrt{g}g^{23}}{N} \left( \frac{\dot{f}_1}{f_1} + \frac{\dot{f}_2}{f_2} \right), \quad p^{33} = -\frac{\sqrt{g}g^{33}}{N} \left( \frac{\dot{f}_1}{f_1} + \frac{\dot{f}_2}{f_2} \right).$$

The scalar momentum is

$$p = -\frac{2\sqrt{g}}{N} \left( \frac{2\dot{f}_1}{f_1} + \frac{\dot{f}_2}{f_2} \right),$$

where the lapse $N := \sqrt{T/[4(3R)]} = dt/d\lambda$ [3] and the associate expansion of $N$ is

$$0 = \frac{\dot{f}_1f_1}{f_1f_1} + \frac{2\dot{f}_1f_2}{f_1f_2} - \frac{N^2f_2^2}{4f_1^4}.$$  

Here we emphasize that (11) plays a role to connect the equivalence between the Hamiltonian constraint $\mathcal{H}$ and Einstein equation $G_{00}$, i.e., $\mathcal{H} \sim G_{00}$. Such equivalence








relation is not only for Bianchi II model, but also true for all cases. We will explicate this relation in section 4.

On the other hand, we compute the second part of the Euler-Lagrange equation [3]

$$\frac{\delta \mathcal{L}}{\delta g_{ij}} = -\sqrt{g}N(R^{ij} - g^{ij}R) - \frac{2N}{\sqrt{g}} \left( p^{im}p^j_m - \frac{1}{2}pp^{ij} \right) + \sqrt{g}G^{ijmn}\nabla_m\nabla_nN + \mathcal{L}p^{ij}. \quad (12)$$

One can tune $\lambda$ in such a way that the universe has the same expanding rate such that $\xi = 0$. The corresponding non-zero components are

$$\frac{\delta \mathcal{L}}{\delta g_{11}} = -2p^{11}\frac{f_1}{f_1} - 2\frac{p^{22} - p}{f_1^2} \frac{f_1}{f_1}, \quad (13)$$

$$\frac{\delta \mathcal{L}}{\delta g_{22}} = -2p^{23}\frac{f_1}{f_1} - 2\frac{p^{33}}{f_1}. \quad (14)$$

Consequently, explicitly list out the Euler-Lagrange equations as follows

$$0 = \frac{\partial p^{11}}{\partial \lambda} - \frac{\delta \mathcal{L}}{\delta g_{11}} = -g^{11} \frac{\partial}{\partial \lambda} \left[ \frac{\sqrt{g}}{N} \left( \frac{f_1}{f_1} + \frac{f_2}{f_2} \right) \right], \quad (15)$$

$$0 = \frac{\partial p^{22}}{\partial \lambda} - \frac{\delta \mathcal{L}}{\delta g_{22}} = \frac{1}{f_1} \left( \frac{2f_1}{f_2} - x^2 \right) \frac{\partial}{\partial \lambda} \left[ \frac{\sqrt{g}}{N} \left( \frac{f_1}{f_1} + \frac{f_2}{f_2} \right) \right], \quad (16)$$

$$0 = \frac{\partial p^{23}}{\partial \lambda} - \frac{\delta \mathcal{L}}{\delta g_{23}} = -g^{23} \frac{\partial}{\partial \lambda} \left[ \frac{\sqrt{g}}{N} \left( \frac{f_1}{f_1} + \frac{f_2}{f_2} \right) \right], \quad (17)$$

$$0 = \frac{\partial p^{33}}{\partial \lambda} - \frac{\delta \mathcal{L}}{\delta g_{33}} = -g^{33} \frac{\partial}{\partial \lambda} \left[ \frac{\sqrt{g}}{N} \left( \frac{f_1}{f_1} + \frac{f_2}{f_2} \right) \right]. \quad (18)$$

Thus, we have the general result

$$0 = \frac{\partial}{\partial \lambda} \left[ \frac{\sqrt{g}}{N} \left( \frac{f_1}{f_1} + \frac{f_2}{f_2} \right) \right], \quad (19)$$

provided that $f_2$ cannot be a constant which has already exhibited in (15), i.e., $\dot{f}_2$ at the denominator. Basically, there are only two equations indicated in (11) and (19) which is exactly matching with the two unknown functions. While GR gives 3 equations and 2 unknowns, forming a completeness for finding the solutions, we confused that why we have more equations than we expected empirically. Here we compare the results between 3-space and 4-metric, remembering that $dt = N d\lambda$, rewrite (19) in terms of $G_{\mu\nu}$

$$0 = \frac{d}{d\lambda} \left[ \frac{1}{f_1} \frac{d}{d\lambda} (f_1 f_2) \right] = \sqrt{g} (G_{00} + \dot{f}_1^2 G_{11}), \quad (20)$$

under the circumstance that $f_2$ is not real. This means that, if we know this particular restriction in 4D, we only need $G_{00}$ and $G_{11}$, while $G_{22}$ becomes not necessary. However, without $G_{22}$, constant $f_2$ is allowable for $G_{00}$ and $G_{11}$ as mentioned in section 2, i.e., $(f_1, f_2) = (\sqrt{t}, 1)$. Based on the above discussion, the 3-space approach cannot practise any advantage than 4-metric to treat the same problem, but the importance is that it seems really showing a more fundamental concept as Dirac suggested [1].

Searching a relationship between $f_1$ and $f_2$. Consider (19) and let the function inside the square bracket be a constant $k$, i.e.,

$$f_2 = \frac{1}{f_1} \frac{d}{d\lambda} \int \frac{Nk}{f_1} d\lambda. \quad (21)$$
This shows \((f_1, f_2)\) couple together. Generally, given \(f_2\) and then \(f_1\) could be solved. But the question is how to select \(f_2\) in a systematical way such that both \((f_1, f_2)\) satisfy (11) and (19) simultaneously? We are more favourable the 3-space theory instead of 4-metric. Not only the concept is more basic, but also the mathematical manipulation is a bit easier to solve by selecting a specific \(N\). Consider (19) again

\[
\sqrt{\frac{g}{N}} \left( \frac{\dot{f}_1}{f_1} + \frac{\dot{f}_2}{f_2} \right) = k. \tag{22}
\]

Comparing (11) and (22), one may obtain

\[
\frac{\dot{f}_2 f_2}{f_2 f_2} + \frac{N^2}{4 f_1} \left( f_2^2 - \frac{4k^2}{f_2} \right) = 0. \tag{23}
\]

In order to decouple \((f_1, f_2)\), simply allowing \(k = 1/2\) and seeking a suitable \(N\). The general solutions are

\[
f_2 = \frac{A_M}{(\cosh \lambda)^{1/2M}}, \quad f_1 = \frac{B_M}{f_2} \exp \int \sqrt{1 + f_2^2} d\lambda, \tag{24}
\]

where \(M = 1, 2, 3, \ldots\). We find that \(A_M = \pm 1\) for all \(M\), \(B_M\) are constants and

\[
N = \frac{2\sqrt{g}}{\sqrt{1 + f_2^2}} \prod_{n=1}^M \frac{1}{2} \sqrt{1 + f_2^{2n}}. \tag{25}
\]

For example, when \(M = 1\) which means choosing \(N = \sqrt{g}\), we recover the results \((f_1, f_2)\) in [4] as indicated in (4).

Furthermore, using BSW method one more time, we find another set of solutions

\[
f_2 = A_M (\sin \lambda)^{1/2M}, \quad f_1 = \frac{B_M}{f_2} \exp \int \frac{N}{2\sqrt{g}} d\lambda, \tag{26}
\]

again \(M\) is a positive integer, all \(A_M = \pm 1\), \(B_M\) are real and

\[
N = \frac{2f_1^2}{\sqrt{1 + f_2^2}} \prod_{n=1}^M \sqrt{1 + f_2^{2n}} \frac{1}{2f_2^{2(n-1)}}, \tag{27}
\]

In particular, we explicitly write out the first two solutions

\[
M = 1, \quad f_2 = \pm \sqrt{\sin \lambda}, \quad f_1 = \frac{B_1}{\sqrt{1 + \cos \lambda}}. \tag{28}
\]

\[
M = 2, \quad f_2 = \pm (\sin \lambda)^{1/4}, \quad f_1 = \frac{B_2}{f_2} \left[ \frac{\sqrt{\sin \lambda}}{\sqrt{1 - \sin \lambda} + 1} \right]^{1/2}, \tag{29}
\]

provided that \(N_1 = f_1^2/f_2\), \(N_2 = f_1^2\sqrt{1 + f_2^2}/(2f_2^3)\) and \(\lambda \in (0, \pi/2)\). Footnote: here we remind the reader that \((N, f_1, f_2) = (f_1^2/f_2, 1/\sqrt{1 - \sin \lambda}, \sqrt{\cos \lambda})\) is not a new solution since it is duplicated with (28) by a simple transformation: \(\lambda \rightarrow \lambda - \pi/2\).

4 Hamiltonian initial value constraint

Here we try to reproduce (11) using the Hamiltonian initial value constraint

\[
\mathcal{H} = \frac{1}{\sqrt{g}} G_{abcd} p^a p^b p^c p^d - 3 R \sqrt{g}
\]

\[
= -\frac{2\sqrt{g}}{N^2} \left( \frac{\dot{f}_1}{f_1} + \frac{2\dot{f}_1}{f_1} \right) + \frac{N^2 f_2^2}{4 f_1^4}
\]

\[
= -2\sqrt{g} G_{00}. \tag{30}
\]
where $G_{abcd} = g_{ac}g_{bd} - \frac{1}{2}g_{ab}g_{cd}$. Therefore, under the restriction of the Bianchi II model, we deduce that $H \sim G_{00}$. But, looking at (30) carefully, one may suspect that whether this is a special result for this specific cosmological model? It is not likely that satisfy the other models. Since $H$ only contain the non-spatial derivative up to $\dot{g}_{ab}$, while $G_{00}$ should include second time derivative of $g_{\mu\nu}$ in general. Here we claim that $H \sim G_{00}$ is valid in principle. The verification is follows. Having removed away $\ddot{g}_{\mu\nu}$ by redefine the time coordinate in such a way that $g_{0c} = 0$, we find

$$2G_{00} = -\frac{1}{4}G^{abcd}\dot{g}_{ab}\dot{g}_{cd} - 3\dot{R}g_{00}. \quad (31)$$

On the other hand, using the following identity

$$\frac{\partial g_{ab}}{\partial \lambda} = \frac{2N}{\sqrt{g}}G_{abcd}\dot{\xi}^{cd} + \dot{\xi}g_{ab}. \quad (32)$$

Consider the Hamiltonian constraint once more in different symbols

$$H = \sqrt{g}\left[\frac{1}{4N^2}G^{abcd}(\dot{g}_{ab} - \dot{\xi}g_{ab})(\dot{g}_{cd} - \dot{\xi}g_{cd}) - 3\dot{R}\right]$$

$$= -2\sqrt{g}G_{00}, \quad (33)$$

provided that $g_{00} = -1$ which is legitimate by redefining the time coordinate again.

## 5 Conclusion

Baierlein, Sharp and Wheeler proposed a Jacobi type timeless Lagrangian based on the 3-dimensional geometry of space to reproduce the same physics as the 4-metric does. Barbour et. al. further extended this idea and they name it 3-space approach. We think the concept of 3-space idea is more fundamental than 4-metric GR as Dirac pointed out 55 years ago. Here we use Bianchi II cosmological model as an example to illustrate this 3-space idea. We examine that the Hamiltonian constraint $H$ is equivalent to Einstein equation $G_{00}$ in general. Moreover, We find that it is a bit easier to recover the known result which satisfy the Einstein equation in vacuum, on the other hand, we also find another new set of solutions.

## References

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