Analysis of a new hyperchaotic system with two large positive Lyapunov exponents

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Abstract. The Qi hyperchaotic system is further investigated for some basic properties. Both 1-D and 2-D Lyapunov exponents (LE) spectra show that the system has two large positive LEs over very large parameter regions. A histogram shows the stochastic properties to be very similar to that of Gaussian white noise, verifying and indicating its great potential for some relevant engineering applications such as secure communications.

1. Introduction
In general, hyperchaos with more than one positive Lyapunov exponent (LE) is more disordered than ordinary chaos. Hyperchaotic systems have more complicated topological structures and dynamics than ordinary chaotic ones [1]. Furthermore, hyperchaos has attracted more attention to applications such as communication encryption [2].

Hyperchaos was first reported by Rössler [3], and the first circuit implementation was realized by Matsumoto et al. [4]. Lately, Nikolov and Clodong [5] modified the Rössler hyperchaotic system [4] and Cafagna, et al. [6] produced a new hyperchaotic attractors using coupling. By modifying Chua’s circuit, Thamilmaran et al. [7] presented a new hyperchaotic system. Wang, et al. proposed a new hyperchaotic system [8].

One positive LE qualitatively distinguishes chaos from other dynamics and more than one positive LE definitely determines the existence of hyperchaos. The values of positive LEs can quantitatively be a good measure of how chaotic the attractor is and differentiate their degrees of disorder since they are able to exponentially demonstrate the rates of stretching, folding and dynamical complexity of a hyperchaotic system [9]. In general, there exist some common problems in most existing hyperchaotic systems. Firstly, two positive LEs simultaneously appear in a very narrow parameter range and their values are relatively small [4-8]. Furthermore, the high-magnitude frequency bandwidths of these hyperchaotic signals are not broad enough for many engineering applications, especially for communication encryption [10].

Recently, we proposed a 4-dimensional (4-D) chaotic system [11]. Some complicated dynamical behaviors, bifurcation analysis, chaos control, circuit implementation and a very complicated four-wing chaotic attractor of the 4-D chaotic system were investigated [12]. We also found a new...
asymmetric chaotic system and showed the system has more complicated dynamics and much broader frequency bandwidths of high magnitude than the commonly known ordinary chaotic systems and some commonly known hyperchaotic systems [10], due to the largest positive LE of about 12. Very recently, we found a new hyperchaotic system with extremely two large positive LEs about 13 and 3, respectively [1] and demonstrate its circuit implementation. The system has an extremely broad high-magnitude frequency bandwidth which is more than 20 times wider than for ordinary chaotic systems and most existing hyperchaotic system, which is very desirable for engineering applications such as secure communications.

In this paper, we further investigate the basic dynamics and properties of the new proposed hyperchaotic system. More precisely, we will demonstrate that the system has two extremely large positive LEs over very wide parameter regions by using 1-D and 2-D LE spectra. A histogram statistically show the stochastic properties to be very similar to that of Gaussian white noise, which shows that the system has extremely rich dynamics and a large degree of disorder and randomness.

2. The Qi hyperchaotic system
The Qi hyperchaotic system proposed in [1] is described as

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_2x_3, \\
\dot{x}_2 &= b(x_1 + x_2) - x_1x_3, \\
\dot{x}_3 &= -cx_1 - ex_2 + x_1x_2, \\
\dot{x}_4 &= -dx_1 + fx_3 + x_1x_2.
\end{align*}
\]

(1)

Here, \( x_i (i = 1, 2, 3, 4) \) are state variables and \( a, b, c, d, e, f \) are positive constant parameters.

The equilibria of system (1) can be found by solving the following algebraic equations:

\[
\begin{align*}
0 &= a(x_3 - x_1) + x_2x_3, \\
0 &= b(x_1 + x_2) - x_1x_3, \\
0 &= -cx_1 - ex_2 + x_1x_2, \\
0 &= -dx_1 + fx_3 + x_1x_2.
\end{align*}
\]

Let

\[
p = \sqrt{a^2 + 6ab + b^2}, \quad g = (d - e)/(cd + ef), \quad q = \sqrt{a^2 + a^2b^2 + 6a^2b}, \quad m_1 = a^2 + 3ab + q/b, \\
m_2 = (a^2 + 3ab - q)/b, \quad x_{11} = \sqrt{2m_1/g} a/(2a - m_1), \quad x_{12} = \sqrt{2m_2/g} a/(2a - m_2), \quad x_{21} = \sqrt{m_1/2g}, \\
x_{22} = \sqrt{m_2/2g}, \quad x_{31} = gx_{11}x_{21}, \quad x_{32} = gx_{12}x_{21}, \quad x_{41} = (c + f)/(cd + ef)x_{11}x_{21}, \quad x_{42} = (c + f)/(cd + ef)x_{12}x_{22}.
\]

It is easy to prove that \( q, m_1, m_2 > 0, \ 2a - m_1 \neq 0 \) and \( 2a - m_2 \neq 0 \), when the parameters \( a, b, c, d, e, f \) are positive constants.

Remark 1: When \( d < e \), i.e. \( g < 0 \), system (1) has only one equilibrium, which is the origin \( S_0 = [0, 0, 0, 0]^T \); when \( d > e \), i.e. \( g > 0 \), system (1) has 5 real equilibria, which are denoted by

\[
S_0, \quad S_1 = [x_{10}, x_{20}, x_{30}, x_{40}]^T, \quad S_2 = [-x_{10}, -x_{20}, x_{30}, x_{40}]^T, \quad S_3 = [x_{10}, x_{20}, x_{30}, x_{40}]^T, \quad S_4 = [-x_{10}, -x_{20}, x_{30}, x_{40}]^T.
\]

(2)

3. Analysis of Lyapunov spectra and bifurcation analysis
Fig. 1 indicates the LE spectra over the parameter range \( b \in [20, 24] \) with fixed \( a = 50, c = 13, d = 8, e = 33, f = 30 \), sampling time \( \tau = 2 \times 10^{-4} \). One can observe that there are two positive LEs over quite a wide range of parameters, which implies that the system is hyperchaotic over a broad range. The first LE is very large with \( l_1 \in [10.7741, 12.9798] \), the second LE is considerable large with \( l_2 \in [0.4145, 2.6669] \), the third LE is approximately zero, and the forth LE is negative with \( l_4 < -60 \). As will be clearly seen below, when the parameter \( b \) is varying, the second LE will become even more prominently positive. It is noticeable that the system has only one equilibrium, i.e. the origin, under these parameters setting according to Remark 1.
From the LE spectra shown above, one can see that the system has hyperchaos over a large range of parameters. However, only the 1-D LE spectra are drawn. To clearly display the parameter scope of hyperchaos, consider two parameters, \( a, b \), in the region \( R = \{(a, b) \mid 49 \leq a \leq 55, 20 \leq b \leq 24\} \), with fixed \( c = 13, d = 8, e = 33, f = 30 \).

From the first and second LEs, we can confirm that there are no sinks, sources and periodic orbits in the region \( R \) (there are in other regions). As shown in Fig. 2, in which the black grid section is at zero level, all the first LE and the second LE are positive, i.e. \( l_1, l_2 > 0 \), and their values are prominently large, i.e. \( 0 < l_1 < 3 \) and \( 8 < l_2 < 13 \). One can see that the first and second LEs gradually increases with increasing \( b \) and decreasing \( a \). This implies that the hyperchaos of system (1) is gradually strengthened.

Simulations and LE computations show that the system can be hyperchaotic in different combined regions of surfaces, cubic subspaces, hypercubic subspaces etc., determined by the 6-D parameters.

![Fig. 1. The Lyapunov exponent spectra versus \( b \).](image1)
![Fig. 2. The first LE and second LEs in 2-D region](image2)

The leading LE is less than 2 for most existing hyperchaotic systems, for example \( l_1 \in [0.11, 1.7] \) in [4-8]. The second largest LE is typically also relatively small with \( l_2 \in [0.02, 0.18] \) in [4-7] and with \( l_2 < 0.42 \) in [8]. However, the leading LE \( l_1 \) of the Qi hyperchaotic system here is notably equal to 12.9798. Similarly, the second largest LE is considerably large and positive, with a value of 2.8924. The exponential expansions of an attractor with positive LEs are incompatible with motions on a bounded space unless there are many foldings. Obviously, the values of the leading LE and the second LE imply an exponential rate of stretching and folding, i.e. \( e^{l_1} \) and \( e^{l_2} \), in two different directions, as the orbit moves. This can be a good measure of how chaotic the attractor is [9]. For comparison, consider a typically hyperchaotic system with two positive LEs, \( l_1 = 0.6, l_2 = 0.1 \). Then, the rates of stretching and folding of the Qi hyperchaotic system, which has \( l_1 = 12.98 \) and \( l_2 = 2.89 \), will be \( n \) times and \( m \) times faster than that of the assumed hyperchaotic system, where

\[
\left. n = e^{12.9798} e^{0.6} \right|_{l_1} = 2.3795 \times 10^{7}, \quad m = e^{2.8924} e^{0.1} \left|_{l_2} = 16.3201. \right.
\]

(3)

Fix the parameters \( a = 49, b = 25, c = 13, d = 8, e = 33, f = 30 \). Then, the LEs are \( [l_1, l_2, l_3, l_4] = [13.505, 2.868, 0.001, 60.362], \) and the Lyapunov dimension is 3.27. A 3-D view and a phase portrait of the system orbits are shown in Figs. 3(a, b), respectively, where the running time is 25 (s). It can be seen that:
1) the orbits are more disordered than the ordinary chaotic and the hyperchaotic ones;
2) the attractors have very irregular forms, neither butterfly nor scroll ones, which are visually more complex than for ordinary chaotic attractors.
3) The orbits are extremely abundant and dense when compared with the orbits of the Lorenz system, as shown in Figs. 4 (a, b) with the same running time $T=25$ (s).

![3-D view in $x_1$-$x_2$-$x_3$ space plane](image1)

**Fig. 3.** The orbits of system (1) with parameters $a=49, b=25, c=13, d=8, e=33, f=30$.

![Projection on the $x_1$-$x_2$ plane](image2)

![The Time response of the Lorenz chaotic system with $a=10, b=8/3, c=28$](image3)

![The time response of the Qi hyperchaotic system with $a=50, b=25, c=13, d=8, e=33, f=30$](image4)

**Fig. 4.** The time responses of the Lorenz system and the Qi hyperchaotic system.

A histogram can statistically show the stochastic properties of a signal generated by a system. Figs. 5(a, b) are the estimated probability densities of the Lorenz system and system (1), respectively. It is clear that the stochastic properties of the hyperchaotic orbits are very similar to that of Gaussian white noises, but that of the Lorenz system has no such property. The data number of samples is $Y=15000$ and the number of histogram bins is $M=1000$ for both systems.

A basic and prominent property of a chaotic system is its seemingly erratic behavior, where a key element is the sensitivity of a trajectory to initial conditions. Trajectories are usually predictable for a short period of time. But, over a long period, chaotic trajectories become unpredictable. Different
chaotic systems have different degrees of sensitivity to initial conditions, which differentiate their degrees of disorder. Figs. 6(a) and (b) show the time responses from two initial conditions of the hyperchaotic system in [8] and of the new hyperchaotic system (1), respectively. For clarity here, the period is taken to be only 1.5 (s); otherwise, the time response of the new hyperchaotic system will be too dense to view. Here, the solid lines and the dotted ones denote the trajectories at two initial conditions from the two hyperchaotic systems, respectively. The two orbits of the proposed hyperchaotic system oscillate drastically and almost immediately diverge. However, the two orbits generated from the hyperchaotic system in [8] follow each other for a long periodic time before diverging. It should be noted that we only changed one initial element, i.e., in Fig. 6(b), but two initial elements, i.e. $x_{10}$ and $x_{20}$ in Fig. 6(a), otherwise the two orbits do not diverge during $t \in [0, 1.5]$. Therefore, the orbit of the proposed hyperchaotic system shows a much stronger sensitivity to initial conditions and is more unpredictable over the same period of time than those of the hyperchaotic system in [8]. It can be well explained in terms of LE definition that the distance between the trajectories from the two initial conditions exponentially increases on average. The rates of divergence are measured by the positive LEs. The positive LEs of the hyperchaotic system in [8] are $l_1 = 1.0181$ and $l_2 = 0.4180$, respectively; but they are much bigger, i.e. $l_1 = 13.505$ and $l_2 = 2.868$, in the new hyperchaotic system (1).

![Fig. 6(a)](image) ![Fig. 6(b)](image)

(a) The state $x$ of the Lorenz system with $a = 10, b = 8/3, c = 28$
(b) The state $x_1$ of the Qi hyperchaotic system with $a = 49, b = 25, c = 13, d = 8, e = 33, f = 30$

Fig. 5. A comparison of the probability densities between the Lorenz system and the Qi hyperchaotic system.

4. Conclusions

In this paper, we mathematically analyzed the basic properties of the Qi hyperchaotic system. It has two large positive LEs over large parameters regions, which implies the system exhibits very strong randomness and a high degree of disorder, as demonstrated by the histogram analysis.

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Fig. 6. A comparison of sensitivities to initial conditions for two hyperchaotic systems.

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