On the quantum Horn problem

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Abstract

Let $K$ be a compact, connected, simply-connected simple Lie group. Given two conjugacy classes $O_1$ and $O_2$ in $K$, we consider the multiplicative Horn question: What conjugacy classes are contained in $O_1 \cdot O_2$? It is known that answering this question remains to describe a convex polytope $P_K$. In 2003, Teleman-Woodward gave a complete list of inequalities for $P_K$. Their list contains redundant inequalities. In this paper, we describe $P_K$ by a smaller list of inequalities.

Warning. During the redaction of this paper, Belkale-Kumar independently obtained in [BK13] similar results. Moreover, it is proved in [BK13] that the list of inequalities obtained in [BK13] and here is irredundant.

1 Introduction

1.1 The additive Horn problem

Let $K$ be a compact, connected, simply-connected simple Lie group and let $\mathfrak{k}$ denote its Lie algebra. Let $O_1$ and $O_2$ be two adjoint $K$-orbit in $\mathfrak{k}$. Then the sum $O_1 + O_2 = \{\xi_1 + \xi_2 : \xi_1 \in O_1 \text{ and } \xi_2 \in O_2\}$ is $K$-stable. The so called Horn question is:

What adjoint $K$-orbits are contained in $O_1 + O_2$?

Parametrization of adjoint orbits. Let $G$ denote the complexification of $K$. Fix a maximal torus $T$ of $G$ such that $T_K := T \cap K$ is a maximal torus of $K$. Any root $\alpha$ of $(G, T)$ induces (by derivation) a linear form (still denoted by $\alpha$) on the Lie algebra Lie$(T)$ of $T$. The Lie algebra Lie$(T_K)$ of $T_K$ identifies with the real Lie subalgebra of $\xi \in$ Lie$(T)$ such that $\alpha(\xi) \in \mathbb{R}$ for any root $\alpha$.

Let $X_*(T)$ denote the group of one parameter subgroups of $T$. It identifies with a sublattice of Lie$(T)$. Moreover, the spanned real vector space $X_*(T)_\mathbb{R} := X_*(T) \otimes \mathbb{R}$ is the real Lie subalgebra of $\xi \in$ Lie$(T)$ such that $\alpha(\xi) \in \mathbb{R}$ for any root $\alpha$.

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Choose a Borel subgroup $B$ of $G$ containing $T$. Let $\Delta$ denote the associated set of simple roots. The dominant chamber in $X_*(T)_{\mathbb{R}}$ is

$$X_*(T)_{\mathbb{R}}^+ = \{ \tau \in X_*(T)_{\mathbb{R}} : \langle \tau, \alpha \rangle \geq 0 \ \forall \alpha \in \Delta \}.$$ 

Any adjoint $K$-orbit in $\text{Lie}(K)$ contains a unique element of the form $\sqrt{-1}\tau$ for some $\tau \in X_*(T)_{\mathbb{R}}^+$; we denote by $O_{\tau}$ the adjoint $K$-orbit containing $\sqrt{-1}\tau$.

The Horn cone. Answering the Horn question is equivalent of describing the set

$$\mathcal{P}_{\text{Lie}(K)} = \{ (\tau_1, \tau_2, \tau_3) \in X_*(T)_{\mathbb{R}}^3 : O_{\tau_1} + O_{\tau_2} + O_{\tau_3} \succeq 0 \}.$$ 

According to Kirwan’s convexity theorem \cite{Kir84}, $\mathcal{P}_{\text{Lie}(K)}$ is a convex polytope of nonempty interior in $X_*(T)_{\mathbb{R}}$. Belkale-Kumar \cite{BK06} obtained an explicit list of inequalities that characterize $\mathcal{P}_{\text{Lie}(K)}$. Before stating their result, we introduce notation on cohomology.

### 1.2 The Belkale-Kumar cohomology

Let $W$ denote the Weyl group $G$ and let $s_\alpha \in W$ denote the simple reflexion associated to $\alpha \in \Delta$. The simple reflections $s_\alpha$ generated $W$ and determine a length function $l$.

Let $P$ be a standard (that is containing $B$) parabolic subgroup of $G$ and let $W_P$ denote its Weyl group. The set of minimal length representative of $W/W_P$ is denoted by $W^P$. For any $w \in W^P$, let $X_w = \overline{BwP/P} \subset G/P$ denote the Schubert variety. The Poincaré dual class $\sigma_w \in H^{2(\dim(G/P) - l(w))}(G/P, \mathbb{Z})$ of the homology class of $X_w$ is a Schubert class. Let $\sigma_w^\vee$ be the Poincaré dual class of $\sigma_w$.

Recall that $H^*(G/P, \mathbb{Z}) = \bigoplus_{w \in W^P} \mathbb{Z}\sigma_w$. We define the structure constants $c(w_1, w_2, w_3)$ associated to three Schubert classes $\sigma_{w_1}, \sigma_{w_2}$ and $\sigma_{w_3}$ by the identity

$$\sigma_{w_1}\sigma_{w_2} = \sum c(w_1, w_2, w_3)\sigma_{w_3}^\vee,$$

where the sum runs over $w_3 \in W^P$. The cohomology ring of $G/P$ is graded by $\deg(\sigma_w) = 2(\dim(G/P) - l(w))$ for any $w \in W^P$. In particular, $c(w_1, w_2, w_3) \neq 0$ implies that

$$l(w_1) + l(w_2) + l(w_3) = 2 \dim(G/P). \quad (1)$$

Let $\Phi^+$ denote the set of positive roots. Let $R^u(P)$ denote the unipotent radical of $P$ and let $\Phi(G/P)$ denote the set of roots of $R^u(P)$. For $w \in W$, let $\Phi(w) = \{ \alpha \in \Phi^+ : -w\alpha \in \Phi^+ \}$ be the inversion set. Recall that $w \in W^P$ if and only if $\Phi(w) \subset \Phi(G/P)$. Condition (1) can be rewritten like

$$\sharp\Phi(w_1) + \sharp\Phi(w_2) + \sharp\Phi(w_3) = 2\sharp\Phi(G/P). \quad (2)$$

Let $L$ be the Levi subgroup of $P$ containing $T$ and let $Z$ be the neutral component of the center of $L$. For any character $\chi$ of $Z$ we set

$$\Phi(G/P, \chi) = \{ \alpha \in \Phi(G/P) : \alpha|_Z = \chi \}.$$ 

$$\quad (3)$$
For \(w \in W^P\) we also set \(\Phi(w,\chi) = \Phi(w) \cap \Phi(G/P,\chi)\). Now condition (3) is equivalent to

\[
\sum_{\chi \in X^*(Z)} \left( \sharp \Phi(w_1,\chi) + \sharp \Phi(w_2,\chi) + \sharp \Phi(w_3,\chi) \right) = 2 \sum_{\chi \in X^*(Z)} \sharp \Phi(G/P,\chi). \tag{4}
\]

The main theorem of [BK06] combined with [RR11, Proposition \(\star\)] allow to obtain the following result.

**Theorem 1** Let \((\tau_1,\tau_2,\tau_3) \in (X_+(T))_R^3\). Then \((\tau_1,\tau_2,\tau_3) \in \mathcal{P}_{\text{Lie}(K)}\) if and only if

\[
\langle w_1 \varpi_\beta, \tau_1 \rangle + \langle w_2 \varpi_\beta, \tau_2 \rangle + \langle w_3 \varpi_\beta, \tau_3 \rangle \leq 0,
\]

for any simple root \(\beta\), any nonnegative integer \(d\) and any \((w_1,w_2,w_3)\) such that

\[
c(w_1,w_2,w_3) = 1, \tag{5}\]

and for any \(\chi \in X^*(Z)\)

\[
\sharp \Phi(w_1,\chi) + \sharp \Phi(w_2,\chi) + \sharp \Phi(w_3,\chi) = 2 \sharp \Phi(G/P,\chi). \tag{6}\]

### 1.3 The multiplicative Horn problem

**The multiplicative Horn question.** Let \(O_1\) and \(O_2\) be two conjugacy classes in \(K\). Then the product \(O_1 \cdot O_2 = \{k_1k_2 : k_1 \in O_1 \text{ and } k_2 \in O_2\}\) is stable by conjugacy. This article is concerned by the multiplicative Horn question:

What conjugacy classes are contained in \(O_1 \cdot O_2\)?

**Parametrization of the conjugacy classes.** Let \(\theta\) be the longest root of \(G\) relatively to \(T \subset B\). The fundamental alcove in \(X_+(T)\) is

\[
\mathcal{A}_s = \{ \tau \in X_+(T) : \langle \tau, \alpha \rangle \geq 0 \quad \forall \alpha \in \Delta, \langle \tau, \theta \rangle \leq 1 \}.
\]

Consider the exponential map

\[
\exp : \text{Lie}(T_K) \rightarrow T_K, \quad \mu \mapsto \exp(\mu).
\]

Any conjugacy class in \(K\) contains a unique element of the form \(\exp(\sqrt{-1} \tau)\) for some \(\tau \in \mathcal{A}_s\) (see e.g. [Bou05, Chapter IX. §5]); we denote by \(O_\tau\) the conjugacy class containing \(\tau\).

**The multiplicative Horn polytope.** Answering the multiplicative Horn question is equivalent of describing the set

\[
\mathcal{P}_K = \{(\tau_1,\tau_2,\tau_3) \in \mathcal{A}_s^3 : O_{\tau_1} \cdot O_{\tau_2} \cdot O_{\tau_3} \ni e\},
\]
where $e$ is the unit element of $K$. According to the convexity theorem proved by Meinrenken-Woodward [MW98], $\Delta$ is a convex polytope of nonempty interior in $\mathcal{A}$. Teleman-Woodward [TW03] obtained an explicit list of inequalities that characterize $\mathcal{P}_K$. The aim of this article is to determine a smaller list of inequalities that still characterize the polytope. Before stating Teleman-Woodward’s theorem, we introduce notation on quantum cohomology.

### 1.4 Quantum cohomology of $G/P$

Let $\Delta_P$ be the set of simple roots of $(L, T)$. The Picard group $\text{Pic}(G/P)$ identifies with $H^2(G/P, \mathbb{Z}) = \oplus_{\alpha \in \Delta - \Delta_P} \mathbb{Z} \sigma_\alpha$. We denote by $(\sigma_{s_\beta}^*)_{\beta \in \Delta - \Delta_P}$ the $\mathbb{Z}$-basis of $\text{Hom}(H^2(G/P, \mathbb{Z}), \mathbb{Z})$ dual of the $(\sigma_{s_\beta})_{\beta \in \Delta - \Delta_P}$.

Let $\gamma : \mathbb{P}^1 \rightarrow G/P$ be a curve. Identifying the group $\text{Pic}(\mathbb{P}^1)$ to $\mathbb{Z}$ (by mapping ample line bundles on positive integers), the pullback of line bundles induces an element of $\text{Hom}(H^2(G/P, \mathbb{Z}), \mathbb{Z})$ called the degree of $\gamma$ and denoted by $d(\gamma)$. By construction $d(\gamma) \in \sum_{\beta \in \Delta - \Delta_P} \mathbb{Z} \sigma_{s_\beta}^*$.

Let $\rho$ and $\rho^L$ denote the half sum of positive roots of $G$ and $L$ respectively. For any $\beta \in \Delta - \Delta_P$, set

$$n_\beta = (\beta^\vee, 2(\rho - \rho^L)),$$

where $\beta^\vee$ is the simple coroot. Fix $\underline{d} = \sum_{\beta \in \Delta - \Delta_P} d_\beta \sigma_{s_\beta}^*$ for some $d_\beta \in \mathbb{Z}_{\geq 0}$.

Let $\overline{M}_{0,3}(G/P, \underline{d})$ be the moduli space of stable maps of degree $\underline{d}$ with 3 marked points into $G/P$. It is a projective variety of dimension

$$\dim(\overline{M}_{0,3}(G/P, \underline{d})) = \dim(G/P) + \sum_{\beta \in \Delta - \Delta_P} d_\beta n_\beta.$$

It comes equipped with 3 evaluation maps $\text{ev}_i : \overline{M}_{0,3}(G/P, \underline{d}) \rightarrow G/P$. The Gromov-Witten invariant associated to three Schubert classes (corresponding to $w_i \in W^P$) and a degree $\underline{d}$ is then the intersection number

$$GW(w_1, w_2, w_3; \underline{d}) = \int_{\overline{M}_{0,3}(G/P, \underline{d})} \text{ev}_1^*(\sigma_{w_1}) \cdot \text{ev}_2^*(\sigma_{w_2}) \cdot \text{ev}_3^*(\sigma_{w_3}).$$

For any $\alpha \in \Delta - \Delta_P$, we introduce a variable $q_\alpha$. Consider the group

$$QH^*(G/P, \mathbb{Z}) := H^*(G/P, \mathbb{Z}) \otimes \mathbb{Z}[q_\beta : \beta \in \Delta - \Delta_P]$$

$$= \bigoplus_{w \in W^P} \mathbb{Z}[q_\beta : \beta \in \Delta - \Delta_P] \sigma_w.$$ 

The $\mathbb{Z}[q_\beta : \beta \in \Delta - \Delta_P]$-linear quantum product $*$ on $QH^*(G/P, \mathbb{Z})$ is defined by, for any $w_1, w_2 \in W^P$,

$$\sigma_{w_1} * \sigma_{w_2} = \sum GW(w_1, w_2, w_3; \underline{d}) q_\lambda^d \sigma_{w_3}^\vee,$$

where the sum runs over $w_3 \in W^P$ and over $\underline{d} \in \sum_{\beta \in \Delta - \Delta_P} \mathbb{Z}_{\geq 0} \sigma_{s_\beta}^*$. Here, if $\underline{d} = \sum_{\beta \in \Delta - \Delta_P} d_\beta \sigma_{s_\beta}^*$ then $q_\underline{d} = \prod_{\beta \in \Delta - \Delta_P} q_{s_\beta}^{d_\beta}$. 

4
1.5 Teleman-Woodward inequalities

Fix for a moment a simple root $\beta$, the corresponding maximal standard parabolic subgroup $P_\beta$ and the fundamental weight $\varpi_\beta$. Let $w_1$, $w_2$, and $w_3$ in $W^P$. A degree for curves in $G/P_\beta$ is a nonnegative integer $d$. Consider the following linear inequality on points $(\tau_1, \tau_2, \tau_3)$ in $X(T) \otimes \mathbb{R}$:

$$I_\beta(w_1, w_2, w_3; d) = \langle w_1 \varpi_\beta, \tau_1 \rangle + \langle w_2 \varpi_\beta, \tau_2 \rangle + \langle w_3 \varpi_\beta, \tau_3 \rangle \leq d.$$

We can now state Teleman-Woodward’s theorem (see [TW03]).

**Theorem 2 (Teleman-Woodward (see [TW03]))** Let $(\tau_1, \tau_2, \tau_3) \in \mathbb{A}^3$. Then $(\tau_1, \tau_2, \tau_3)$ is in $P_K$ if and only if inequality $I_\beta(w_1, w_2, w_3; d)$ is fulfilled for any simple root $\beta$, any nonnegative integer $d$ and any $(w_1, w_2, w_3)$ such that

$$\text{GW}(w_1, w_2, w_3; d\sigma^*_\beta) = 1$$

in $G/P_\beta$.

1.6 Our main result

Our main result is a refinement of the condition (8).

Here, $P$ is any standard parabolic subgroup of $G$. The grading on $H^*(G/P, \mathbb{Z})$ extends to the quantum setting by setting $\deg(q_\beta) = 2n_\beta$ for any $\beta \in \Delta - \Delta_P$. In particular, for $d = \sum_{\beta \in \Delta - \Delta_P} d_\beta \sigma^*_\beta$, $\text{GW}(w_1, w_2, w_3; d) \neq 0$ implies that

$$l(w_1) + l(w_2) + l(w_3) + \sum_{\beta \in \Delta - \Delta_P} d_\beta n_\beta = 2 \dim(G/P).$$

Condition (11) can be rewritten like

$$2\Phi(w_1) + 2\Phi(w_2) + 2\Phi(w_3) + \sum_{\beta \in \Delta - \Delta_P} d_\beta \beta^\vee = 2\Phi(G/P).$$

Set $h = \sum_{\beta \in \Delta - \Delta_P} d_\beta \beta^\vee$. Since $2(\rho - \rho_L) = \sum_{\alpha \in \Phi(G/P)} \alpha$, condition (11) can be rewritten like

$$2\Phi(w_1) + 2\Phi(w_2) + 2\Phi(w_3) + \sum_{\alpha \in \Phi(G/P)} \langle h, \alpha \rangle = 2\Phi(G/P),$$

or like

$$\sum_{\chi \in X^*(Z)} \left( \sum_{i=1}^3 2\Phi(w_i, \chi) + \sum_{\alpha \in \Phi(G/P, \chi)} \langle h, \alpha \rangle \right) = 2 \sum_{\chi \in X^*(Z)} \Phi(G/P, \chi).$$

**Theorem 3** Let $(\tau_1, \tau_2, \tau_3) \in \mathbb{A}^3$. Then $(\tau_1, \tau_2, \tau_3)$ is in $\Delta(K)$ if and only if inequality $I_\beta(w_1, w_2, w_3; d)$ is fulfilled for any simple root $\beta$, any nonnegative integer $d$ and any $(w_1, w_2, w_3)$ such that, in $QH^*(G/P_\beta)$,

$$\text{GW}(w_1, w_2, w_3; d\sigma^*_\beta) = 1,$$
and for any \( \chi \in X^*(Z) \)

\[
\sharp \Phi(w_1, \chi) + \sharp \Phi(w_2, \chi) + \sharp \Phi(w_3, \chi) + \sum_{\alpha \in \Phi(G/P, \chi)} d(\beta^\vee, \alpha) = 2\sharp \Phi(G/P, \chi). \tag{14}
\]

1.7 Comparaison with Teleman-Woodward theorem

We made some explicit computations using Anders Buch’s qcalc Maple package, SageMath and Normaliz. The used programs, some files containing explicit list of inequalities and additional computation are available on author’s webpage (see [Res13]).

Here, we give some quantitative aspects for the group \( G_2 \) and the groups of type \( B, C \) or \( D \) up to rank 6. More precisely, in the two last column of Table 1.7 appear the numbers of vertices and facets of the polytope \( P_K \). In column “MAX”, all the inequalities corresponding to nonzero GW-invariants are counted. In column “TW”, the number of inequalities given by Theorem 2 is given. The inequalities obtained by combining Theorems 1 and 2 are counted. In column “Th 3”, only the inequalities given by Theorem 3 are counted. All these numbers of inequalities include the \( 3 \ast (\text{rank}+1) \) inequalities of dominancy and alcove. One can observe that Theorem 3 gives a list of inequalities significantly smaller than the combination of Theorems 1 and 2.

It is worthy to observe that in any computed examples the number of facets is equal to the number of inequalities given by our main result. One can conjecture that the list of inequalities given by Theorem 3 is irredudant. The analogous result for the additive Horn problem is proved in [Res10].


2 Notation

In this section, we reintroduce more carefully and complete the notation used in the introduction.

2.1 Notation on the group $G$

Let $G$ be a simple simply connected Lie group and $Z(G)$ its center. Set $G_{ad} = G/Z(G)$ and $T_{ad} = T/Z(G)$. We fix a Borel subgroup $B$ of $G$ and a maximal torus $T$ contained in $B$. Let $\Phi$ and $\Phi^+$ denote the sets of roots and positive roots respectively. If $\alpha$ belongs to $\Phi$, $\alpha^\vee$ denote the corresponding coroot. The set of simple roots is denoted by $\Delta$. For $\alpha \in \Delta$, $\varpi_\alpha \in X^*(T)$ denotes the corresponding fundamental weight and $\varpi_{\alpha^\vee} \in X_+(T_{ad})$ denotes the associated fundamental coweight. Let $\rho$ be the half sum of the positive roots. Recall that $\rho = \sum_{\alpha \in \Delta} \varpi_\alpha$.

Note that $X^*(T) = \oplus_{\alpha \in \Delta} \mathbb{Z} \varpi_\alpha$. Let $Q = \oplus_{\alpha \in \Delta} \mathbb{Z} \alpha \in X^*(T)$ denote the root lattice. Similarly $X_+(T) = \oplus_{\alpha \in \Delta} \mathbb{Z} \alpha^\vee$ and $P^\vee := \oplus_{\alpha \in \Delta} \mathbb{Z} \varpi_{\alpha^\vee}$. Let $W$ be the Weyl group and $w_0$ be its longest element. Let $h \in X_+(T)$. Write $h = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee$. Note that $(\rho, h) = \sum_{\alpha \in \Delta} n_\alpha$. The dominance order on $X_+(T)$ is the partial order $\geq$ defined by $h \geq h' \iff h - h' \in \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \alpha^\vee$.

Let $\theta$ denote the longest root. Set $X^*(T)_\mathbb{R} = X^*(T) \otimes \mathbb{R}$ and its dual space $X_+(T)_\mathbb{R} = X_+(T) \otimes \mathbb{R}$. There exists a $W$-invariant Euclidean scalar product $(\ , \ )$ on $X^*(T)_\mathbb{R}$. Moreover, it is unique modulo positive scalar. We fix a choice by assuming that $(\theta, \theta) = 2$. Using $(\ , \ )$, we identify $X^*(T)_\mathbb{R}$ with $X_+(T)_\mathbb{R}$. The transition relations are, for any $\alpha \in \Delta$:

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)} (\alpha, \square), \quad \varpi_{\alpha^\vee} = \frac{2}{(\alpha, \alpha)} \varpi_{\alpha, \square}.$$ 

Observe that $\frac{2}{(\alpha, \alpha)} = 1, 2$ or $3$, depending if $\alpha$ is not short, short in type $\neq G_2$ and short in type $G_2$. In particular $X^*(T) \supset Q \supset X_+(T) \subset P^\vee \subset X^*(T)$.

A one parameter subgroup $\tau$ of $T$ is said to be dominant if $(\tau, \alpha) \geq 0$ for any $\alpha \in \Delta$. The set of dominant one parameter subgroups is denoted by $X_+(T)^+$. Similarly; $\lambda \in X^*(T)$ is dominant, or belongs to $X^*_+ \subset X^*(T)$ if $(\lambda, \alpha^\vee) \geq 0$. We extend these definitions and notations to $X^*(T)_\mathbb{R}$ and $X_+(T)_\mathbb{R}$.

For $\tau \in X_+(T)$, we denote by $P(\tau)$ the set $g \in G$ such that $\tau(t)g(t^{-1})$ has a limit in $G$ when $t$ goes to $0$. It is a parabolic subgroup of $G$. It contains $B$ if and only if $\tau$ is dominant.

2.2 The affine Kac-Moody Lie algebra

Endow $\hat{L} = g \otimes \mathbb{C}(z) \oplus \mathbb{C}c \oplus \mathbb{C}d$ with the usual Lie bracket (see e.g. [Kum02 Chap XIII]). Set $\hat{\mathfrak{h}} = \text{Lie}(T) \oplus \mathbb{C}c \oplus \mathbb{C}d$. We identify $\text{Lie}(T)^*$ with the orthogonal
of $\mathbb{C}c \oplus \mathbb{C}d$ in $\hat{h}$. Define $\Lambda$ and $\delta$ in $\hat{h}^*$ by

$$
\delta : h \mapsto 0, c \mapsto 0, d \mapsto 1;
\Lambda : h \mapsto 0, c \mapsto 1, d \mapsto 0.
$$

The simple roots of $\hat{L}_g$ are

$$
\alpha_0 = \delta - \theta, \alpha_1, \ldots, \alpha_l.
$$

For any fundamental weight $\varpi$ of $g$, set $\tilde{\varpi} = \varpi + \varpi(\theta^\vee)\Lambda \in \hat{h}^*$. Fix a numbering $\alpha_1, \ldots, \alpha_l$ of the simple roots of $g$. Set $\tilde{\varpi}_0 = \Lambda$. The fundamental weights of $\hat{L}_g$ are $\tilde{\varpi}_0, \tilde{\varpi}_1, \ldots, \tilde{\varpi}_l$.

Set $\hat{\delta}^+_Z = Z\tilde{\varpi}_0 + \cdots + Z\tilde{\varpi}_l + Z\delta$, and

$$
\hat{\delta}^{++}_Z = Z_{\geq 0}\tilde{\varpi}_0 + \cdots + Z_{\geq 0}\tilde{\varpi}_l + Z\delta.
$$

Fix $\lambda = \lambda + \overline{l}\Lambda + z\delta \in \hat{h}^*_Z$ with $\lambda \in \hat{h}^*_Z$, $\overline{l} \in \mathbb{Z}_{\geq 0}$ and $z \in \mathbb{Z}$. If $\lambda \in \hat{h}^{++}_Z$, that is, if $\langle \lambda, \theta^\vee \rangle \leq \overline{l}$, then there exists a simple $\hat{L}_g$-module $\mathcal{H}(\lambda)$ of highest weight $\lambda$. The subspace of $\mathcal{H}(\lambda + \overline{l}\Lambda)$ annihilated by $g \otimes z\mathbb{C}[z]$ is isomorphic as a $g$-module to $V(\lambda)$.

Let $s_0, s_1, \ldots, s_l$ be the set of simple reflections. They generate the affine Weyl group $\tilde{W}$ which is isomorphic to $W \ltimes Q^\vee$. Moreover, $\tilde{W}$ is a Coxeter group and the length is given by

$$
l(\overline{t}^h w) = \sum_{\alpha \in \Phi^+} |\langle h, \alpha \rangle| + \sum_{\alpha \in \Phi^+} |\langle h, \alpha \rangle - 1|.
$$

The group $\tilde{W}$ acts on $\hat{h}^*$. In particular the action of $Q^\vee$ is given by

$$
h \mapsto T_h : \begin{array}{ccc}
\hat{h}^* & \longrightarrow & \hat{h}^* \\
\chi & \longmapsto & \chi + \chi(c)(h, \square) - [\chi(h) + \frac{1}{2}(h, h)\chi(c)]\delta.
\end{array}
$$

### 2.3 The fusion product

If $\mathfrak{a}$ is a Lie algebra and $M$ is a $\mathfrak{a}$-module, we denote by $[M]_\mathfrak{a}$, the biggest quotient of $M$ where $\mathfrak{a}$ acts trivially.

Let $\lambda_1$, $\lambda_2$, and $\lambda_3$ be three dominant weights of $g$ and $\overline{l} \in \mathbb{Z}_{\geq 0}$ such that

$$
\langle \lambda_i, \theta^\vee \rangle \leq \overline{l}, \quad \text{for any } i = 1, 2, 3.
$$

Then $\lambda_i + \overline{l}\Lambda \in \hat{h}^{++}_Z$, and we can consider the $(\hat{L}_g)^3$-module $\mathcal{H}(\lambda_1 + \overline{l}\Lambda) \otimes \mathcal{H}(\lambda_2 + \overline{l}\Lambda) \otimes \mathcal{H}(\lambda_3 + \overline{l}\Lambda)$.

Consider $\mathbb{P}^1$ with three pairwise distinct marked points $p_1, p_2$ and $p_3$. Consider the ring of regular functions $\mathcal{O}(\mathbb{P}^1 - \{p_1, p_2, p_3\})$ and the Lie algebra
For any $p_i$, by fixing a local coordinate $z_i$ around this point of $\mathbb{P}^1$, one gets a morphism $\mathcal{O}(\mathbb{P}^1 - \{p_1, p_2, p_3\}) \to \mathbb{C}(z)$. In particular, we just defined three morphisms $\mathcal{O}(\mathbb{P}^1 - \{p_1, p_2, p_3\}) \to \mathcal{O}(\mathbb{C}(z))$, or one morphism $\mathcal{O}(\mathbb{P}^1 - \{p_1, p_2, p_3\}) \to (\mathcal{O}(\mathbb{C}(z)))^3$. This defines an action of $\mathcal{O}(\mathbb{P}^1 - \{p_1, p_2, p_3\})$ on the $(\mathcal{L}_0)^3$-module $\mathcal{H}(\lambda_1 + \Lambda) \otimes \mathcal{H}(\lambda_2 + \Lambda) \otimes \mathcal{H}(\lambda_3 + \Lambda)$. The Vacua space is defined by

$$V_{p_1}(\lambda_1, \lambda_2, \lambda_3, \ell) = \left( \mathcal{H}(\lambda_1 + \Lambda) \otimes \mathcal{H}(\lambda_2 + \Lambda) \otimes \mathcal{H}(\lambda_3 + \Lambda) \right)_g \otimes \mathcal{O}(\mathbb{P}^1 - \{p_1, p_2, p_3\})$$

It is proved to be finite dimensional (see e.g. [Bea96]). Moreover, the fusion product $\otimes^\ell$ is defined by

$$V(\lambda_1) \otimes^\ell V(\lambda_2) = \sum_{\langle \lambda, \theta' \rangle \leq \ell} \dim(V_{p_1}(\lambda_1, \lambda_2, \lambda_3; \ell)) V(-\delta_0 \lambda_3).$$

The product $\otimes^\ell$ is associative and commutative (see e.g. [Bea96]).

### 2.4 The fusion product polytope

The fundamental alcove in $X^*(T)_Q$ is

$$\Delta^*_Q = \{ \lambda \in X^*(T)_Q : \left\{ \begin{array}{l} \langle \lambda, \alpha' \rangle \geq 0 \quad \forall \alpha \in \Delta \\ \langle \lambda, \theta' \rangle \leq 1 \end{array} \right. \}.$$

For any $\lambda \in \Delta^*_Q$, $\ell \in \mathbb{Z}_{>0}$ if $\ell \lambda + \Lambda \in \mathfrak{h}^*_\mathbb{C}$ then it is dominant. Set

$$P_\otimes = \{ (\lambda_1, \lambda_2, \lambda_3) \in (\Delta^*_Q)^3 : \exists \ell > 0 \quad V(-\ell \delta_0 \lambda_3) \subset V(\ell \lambda_1) \otimes^\ell V(\ell \lambda_2) \}.$$

**Theorem 4** Let $(\lambda_1, \lambda_2, \lambda_3) \in (\Delta^*_Q)^3$. Then $(\lambda_1, \lambda_2, \lambda_3) \in P_\otimes$ if and only if

$$\langle w_1 \omega_{\beta'}, \lambda_1 \rangle + \langle w_2 \omega_{\beta'}, \lambda_2 \rangle + \langle w_3 \omega_{\beta'}, \lambda_3 \rangle \leq \frac{2}{(\beta, \beta)} d. \quad (15)$$

for any simple root $\beta$, any nonnegative integer $d$ and any $(w_1, w_2, w_3) \in (W^P)^3$ such that

$$GW(w_1, w_2, w_3; d\sigma^*_s) = 1, \quad (16)$$

and for any $\chi \in X^*(T)$

$$\sharp \overline{\Phi}(w_1, \chi) + \sharp \overline{\Phi}(w_2, \chi) + \sharp \overline{\Phi}(w_3, \chi) + \sum_{\alpha \in \Phi(G/P, \chi)} \langle h, \alpha \rangle = 2\sharp \overline{\Phi}(G/P, \chi). \quad (17)$$

Let $\lambda \in X^*(T)_Q$. Since $\theta' = (\theta, \Box)$, $\lambda$ belongs to $\Delta^*_Q$ if and only if $\lambda \in \Delta^*_Q$. Theorems 3 and 4 are equivalent knowing the following.

**Theorem 5 (see [TW03])** Let $(\lambda_1, \lambda_2, \lambda_3) \in (\Delta^*_Q)^3$. Then $(\lambda_1, \lambda_2, \lambda_3) \in P_\otimes$ if and only if

$$(\langle (\lambda_1, \Box), (\lambda_2, \Box) \rangle, (\lambda_3, \Box)) \in P_K.$$
3 The affine Grassmannian

In this section we collect some results and notation on the affine Grassmannian $\mathcal{G}$ of $G$. Set $L_{\text{alg}} G = G(\mathbb{C}[z, z^{-1}])$ and $L_{\text{alg}}^> G = G(\mathbb{C}[z])$. Consider the affine Grassmannian $\mathcal{G} = L_{\text{alg}} G / L_{\text{alg}}^> G$.

3.1 Line bundles

Let $\bar{L}_{\text{alg}} G = \mathbb{C}^* \rtimes L_{\text{alg}} G$ and $G$ denote the affine Kac-Moody group associated to $G$; it is a central extension of $\bar{L}_{\text{alg}} G = \mathbb{C}^* \rtimes L_{\text{alg}} G$. The maximal torus of $G$ containing $T$ is denoted by $\hat{T}$; its Lie algebra is $\hat{h}$ and its character group is $\hat{h}^\ast$. The group $G$ acts on $G$.

Let $\tilde{l} \in \mathbb{Z}$. There exists a unique $G$-linearized line bundle $L(\tilde{l} \Lambda)$ on $G$ such that $\hat{h}$ acts on the fiber over the base point of $G$ by the weight $-\tilde{l} \Lambda$ (see e.g. [Kum02, Chap VII]). Moreover, $H^0(\mathcal{G}, L(\tilde{l} \Lambda))$ is zero if $\tilde{l} < 0$ and isomorphic to the dual of $H(\tilde{l} \Lambda)$ if $\tilde{l} \geq 0$.

Recall that $G$ is a central extension of the semidirect product $\mathbb{C}^* \rtimes L_{\text{alg}} G$:

\[
1 \longrightarrow \mathbb{C}^* \longrightarrow G \longrightarrow \mathbb{C}^* \times L_{\text{alg}} G \longrightarrow 1.
\]

This exact sequence splits canonically over $L_{\text{alg}}^> G$. In particular, $L(\tilde{l} \Lambda)$ admits a $L_{\text{alg}}^> G$-linearization.

3.2 The Cartan decomposition

Any one parameter subgroup $h$ of $T$ can be seen as an element of $L_{\text{alg}} G$. Its image in $\mathcal{G}$ is denoted by $L_h$. Then $\{ L_h : h \in X_+(T) \}$ is the set of $T$-fixed points in $\mathcal{G}$. The $L_{\text{alg}}^> G$-orbit of $L_h$ only depends on the $W$-orbit of $h$ in $X_+(T)$; it is denoted by $\mathcal{G}_h$. It is a quasiprojective variety of finite dimension $\langle \rho, h \rangle$ (if $h \in X_+(T)^+$) and the Cartan decomposition asserts that

\[
\mathcal{G} = \bigsqcup_{h \in X_+(T)^+} \mathcal{G}_h.
\]

The closure of $\mathcal{G}_h$ is described by the order $\leq$:

\[
\overline{\mathcal{G}}_h = \bigcup_{h' \in X_+(T)^+ \atop h' \leq h} \mathcal{G}_{h'}.
\]

There exists a unique one parameter subgroup $\delta^\vee$ of $\hat{T}$ such that $\langle \delta^\vee, \delta \rangle = 1$, $\langle \delta^\vee, \Lambda \rangle = 0$, and $\langle \delta^\vee, \hat{h} \rangle = 0$, for any $\chi \in X_+(T)$. The irreducible components
of $G\delta^\vee$ are the $G.L_h$ for $h \in X_\ast(T)$. Moreover, $G.L_h$ is isomorphic to $G/P(h)$. Then

$$G_h = \{ x \in G : \lim_{t \to 0} \delta^\vee(t)x \in G.L_h \}.$$  

**Raffinement.** Consider the evaluation morphism $ev_0 : L_{alg}^> G \to G$ at $z = 0$. Set $B = ev_0^{-1}(B)$. We have

$$G = \bigcup_{h \in X_\ast(T)} B L_h.$$

### 3.3 The Birkhoff decomposition

Consider the action of the group $L_{alg}^\leq G = G(\mathbb{C}[z^\pm])$ on $G$. Its orbits are parametrized by $X_\ast(T)^+$ and setting $G^h = L_{alg}^\leq GL_h$, we have

$$G = \bigcup_{h \in X_\ast(T)^+} G^h.$$  

Moreover

$$\overline{G}^h = \bigcup_{h \in X_\ast(T)^+} G^{h'}.\quad (18)$$

For any $h \in X_\ast(T)^+$, the orbit $G_h$ has codimension $\langle \rho, h \rangle$. Moreover

$$G^h = \{ x \in G : \lim_{t \to \infty} \delta^\vee(t)x \in G.L_h \}.$$  

**Raffinement.** Let $ev_\infty : L_{alg} G \to G$ denote the evaluation at $z^{-1} = 0$. Set $B^- = ev_\infty^{-1}(B^-)$. Then

$$G = \bigcup_{h \in X_\ast(T)} B^- L_h.$$  

For any $h \in X_\ast(T)^+$,

$$G^h = \bigcup_{w \in W} B^- L_{wh}.\quad (18)$$

Consider $\rho^\vee$ the half sum of the positive coroots. It is a dominant and regular one parameter subgroup of $T_{ad}$. Moreover,

$$B^- L_h = \{ \cdot \in \mathbb{X} | \lim_{t \to \infty} (\delta^\vee + \rho^\vee)(t) \cdot \cdot = L_h \}.\quad (19)$$

### 3.4 The Peterson decomposition

Consider the group $L_{alg} U$. 

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11
Theorem 6 We have
\[ L_{\text{alg}}G = \bigcup_{w \in W} BwL_{\text{alg}}U = \bigcup_{w \in W} B^-wL_{\text{alg}}U. \]

For later use, we prove the following lemma due to Peterson [Pet97]. It implies easily Theorem 6.

Lemma 1 Let \( g \in L_{\text{alg}}G \). Assume that \( g \in B^-w_1z^{h_1}L_{\text{alg}}U \) and \( g \in Bw_2z^{h_2}L_{\text{alg}}U \) for \( w_1, w_2 \in W \), and \( h_1, h_2 \in Q' \).

Consider \( \phi : \mathbb{P}^1 \rightarrow G/B \) that extends \( \mathbb{C}^* \rightarrow G/B \).

The map \( \phi \) has degree \( h_2 - h_1 \in \text{Hom}(\text{Pic}(G/B) = X(T), \mathbb{Z}) = \text{Hom}(X(T), \mathbb{Z}) \). Moreover \( \phi(\infty) \in B^-w_1B/B \) and \( \phi(0) \in Bw_2B/B \).

Proof. Write \( g(z) = b(z)\tilde{w}_2z^{h_2}u(z) \) with \( b(z) \in B, \tilde{w}_2 \in N(T) \) a representant of \( w_2 \) and \( u(z) \in L_{\text{alg}}U \). Then \( \phi(z) = b(z)\tilde{w}_2z^{h_2}u(z)B/B = b(z)w_2B/B. \) Since \( b(0) \in B \), we obtain \( \phi(0) \in Bw_2B/B \). Similarly \( \phi(\infty) \in B^-w_1B/B \).

It remains to compute the degree of \( \phi \). Fix a dominant weight \( \lambda \) of \( G \). Consider the irreducible \( G \)-representation \( V(\lambda) \) of highest weight \( \lambda \) and an highest weight vector \( v_\lambda \). Consider the morphism \( G/B \rightarrow \mathbb{P}(V(\lambda)), gB/B \mapsto g[\cdot\cdot\cdot] \) and its composition \( \phi_\lambda : \mathbb{P}^1 \rightarrow \mathbb{P}(V(\lambda)) \) with \( \phi \). It remains to prove that \( \text{deg}(\phi_\lambda) = \langle \lambda, h_1 - h_2 \rangle \).

We reuse the writing \( g(z) = b(z)\tilde{w}_2z^{h_2}u(z) \):

\[ \forall z \in \mathbb{C}^* \quad \phi_\lambda(z) = [b(z)\tilde{w}_2z^{h_2}u(z) \cdot v_\lambda] = [z^{\langle \lambda, h_2 \rangle}b(z)\tilde{w}_2 \cdot v_\lambda]. \]

Since \( b(z) \) is polynomial in \( z \), this implies that the valuation (at zero) of \( z^{\langle \lambda, h_2 \rangle}b(z)\tilde{w}_2 \cdot v_\lambda \) is at least \( \langle \lambda, h_2 \rangle \). Since \( b(z) \) has a limit in \( G \) at \( z = 0 \), \( b(z)\tilde{w}_2 \cdot v_\lambda \) has a nonzero limit in \( V(\lambda) \) at \( z = 0 \). Hence the valuation of \( z^{\langle \lambda, h_2 \rangle}b(z)\tilde{w}_2 \cdot v_\lambda \) is exactly \( \langle \lambda, h_2 \rangle \).

A similar computation with \( B^-w_1z^{h_1}L_{\text{alg}}U \) shows that the degree of \( z \mapsto g(z)v_\lambda \) is exactly \( \langle \lambda, h_1 \rangle \). Finally, the degree of \( \phi_\lambda \) is \( \langle \lambda, h_1 - h_2 \rangle \).

For \( h \in X_*(T) \), set \( S_h = L_{\text{alg}}UL_h. \) Then
\[ \mathcal{G} = \bigcup_{h \in X_*(T)} S_h, \]
and
\[ \overline{S_h} = \bigcup_{h' \in X_*(T)} S_{h'}, \] (20)

The orbit \( S_h \) has neither finite dimension nor finite codimension. The fixed points of \( \rho^\vee \) are the \( L_h \) for \( h \in X_*(T) \) and
\[ S_h = \{ x \in \mathcal{G} : \lim_{t \to 0} \rho^\vee(t)x = L_h \}. \]
Variation. Let \( P \supset B \) be a parabolic subgroup and consider \( L_{\text{alg}}P \). There exists a surjective group morphism

\[
\mathcal{X} : L_{\text{alg}}P \longrightarrow \text{Hom}(X^*(P), \mathbb{Z})
\]
defined as follows. Let \( p \in L_{\text{alg}}P \) considered as a regular map \( p : \mathbb{C}^* \longrightarrow P \) and \( \chi \in X^*(P) \). Then \( \chi \circ p \) is a regular map from \( \mathbb{C}^* \) to \( \mathbb{C}^* \). Hence, there exist \( n \in \mathbb{Z} \) and \( \lambda \in \mathbb{C}^* \) such that \( \chi(p)(z) = \lambda z^n \), for any \( z \in \mathbb{C}^* \). Then \( \mathcal{X}(p)(\chi) \) is defined to be \( n \). The kernel of \( \mathcal{X} \) is denoted by \( (L_{\text{alg}}P)_0 \).

Let \( L \) be the Levi subgroup of \( P \) containing \( T \) and \( L^{ss} \) be its semisimple part. Two orbits \( S_h \) and \( S_{h'} \) are contained in the same \( (L_{\text{alg}}P)_0 \)-orbit if and only if \( h - h' \in X_\ast(T \cap L^{ss}) \). Since \( X_\ast(T) = \oplus_{\alpha \in \Delta} \mathbb{Z} \alpha^\vee \) and \( X_\ast(T \cap L^{ss}) = \oplus_{\alpha \in \Delta_p} \mathbb{Z} \alpha^\vee \), we get

\[
\mathcal{G} = \bigcup_{h \in \oplus_{\alpha \in \Delta - \Delta_p} \mathbb{Z} \alpha^\vee} S_h^P, \tag{21}
\]
where \( S_h^P = (L_{\text{alg}}P)_0 P \). Let \( \tau \in X_\ast(T) \) such that \( P = P(\tau) \). The irreducible components of the fixed point set \( \mathcal{G}^\tau \) are the orbit \( C_h^L := L_{\text{alg}}L^{ss}.L_h \) for \( h \in \oplus_{\alpha \in \Delta - \Delta_p} \mathbb{Z} \alpha^\vee \). Moreover,

\[
S_h^P = \{ x \in \mathcal{G} : \lim_{t \to 0} \tau(t)x \in C_h^L \}. \tag{22}
\]

Observe that \( C_h^L = L_{\text{alg}}L^{ss}.L_h \) is well-defined for any \( h \in X_\ast(T) \), but depends only on the class of \( h \) in \( X_\ast(T)/X_\ast(T \cap L^{ss}) \). Above, we choose \( \oplus_{\alpha \in \Delta - \Delta_p} \mathbb{Z} \alpha^\vee \) as a complete system of representant for this quotient. The following result due to Peterson-Woodward gives another representative (see [Woo05, Lemma 1]):

**Lemma 2** Each class \( h \in X_\ast(T)/X_\ast(T \cap L^{ss}) \) has a unique representative \( h_{PW} \in X_\ast(T) \) such that

\[
(h_{PW}, \alpha) = 0 \text{ or } -1, \text{ for any } \alpha \in \Phi^+ \cap \Phi(L).
\]

It is easy to check that the standard Iwahori subgroup of \( L_{\text{alg}}L^{ss} \) fixes \( h_{PW} \).

4. GIT for \( L_{\text{alg}}^0 G \) acting on the affine Grassmannian

4.1 Fusion product and \( L_{\text{alg}}^{\leqslant 0} G \)-invariant sections

We think about \( z^{-1} \) as a coordinate on \( \mathbb{P}^1 - \{0\} \). Hence, for \( p \in \mathbb{P}^1 - \{0\} \), we have a morphism of evaluation \( ev_p : L_{\text{alg}}^{\leqslant 0} G \longrightarrow G, g(z^{-1}) \mapsto g(p) \).

Let \( X = \mathcal{G} \times (G/B)^3 \).

Recall that we have fixed three pairwise distinct points \( p_1, p_2 \) and \( p_3 \) in \( \mathbb{P}^1 - \{0\} \).
Let \( \tilde{l} \in \mathbb{Z}_{\geq 0} \) and \( \lambda_i \) (for \( i = 1, \ldots, 3 \)) be three dominant characters of \( B \). Let \( \mathcal{L} = \mathcal{L}(\tilde{l}\Lambda) \otimes \mathcal{L}(\lambda_1) \otimes \mathcal{L}(\lambda_2) \otimes \mathcal{L}(\lambda_3) \) be the associated line bundle on \( \mathbb{X} \).

**Lemma 3** The dual of the Vacua space \( V^{\psi_1}(\lambda_1, \lambda_2, \lambda_3, \tilde{l}) \) is isomorphic to the space of \( L_{\text{alg}} \)-invariant sections of \( \mathcal{L} \)

**Proof.** By [Bea96, Corollary 2.5], the Vacua space is isomorphic to

\[
\left( H(\tilde{l}\Lambda) \otimes V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3) \right)_{g \otimes \mathcal{O}(\mathbb{P}^1 - \{0\})}.
\]

Its dual is the set of \( g \otimes \mathcal{O}(\mathbb{P}^1 - \{0\}) \)-invariant vectors in \( \left( H(\tilde{l}\Lambda) \otimes V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3) \right)^* \). Since \( V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3) \) is finite dimensional, \( \left( H(\tilde{l}\Lambda) \otimes V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3) \right)^* = H(\tilde{l}\Lambda)^* \otimes V(\lambda_1)^* \otimes V(\lambda_2)^* \otimes V(\lambda_3)^* \). This space is isomorphic to \( H^0(\mathbb{X}, \mathcal{L}) \). The lemma follows. \( \square \)

4.2 Convex numerical function

Let \( E \) be a finite dimensional real vector space and let \( E^* \) denote its dual space. Let \( \mu : E^* \rightarrow \mathbb{R} \) be a function. It is said to be positively homogeneous if \( \mu(t\varphi) = t\mu(\varphi) \) for any \( \varphi \in E^* \) and any nonnegative real number \( t \). The positively homogeneous function \( \mu \) is said to be convex if

\[
\forall \varphi, \psi \in E^* \quad \mu(\varphi + \psi) \geq \mu(\varphi) + \mu(\psi).
\]

**Remark.** Pay attention to our convention which is nonstandard in convex analysis. Our convention is that of toric geometry.

To any positively homogeneous convex function \( \mu \) is associated the compact convex set

\[
C_\mu = \{ x \in E : \forall \varphi \in E^* \quad \varphi(x) \geq \mu(\varphi) \}.
\]

The correspondence \( \mu \mapsto C_\mu \) is bijective since, by Hahn-Banach’s theorem

\[
\mu(\varphi) = \inf_{x \in C_\mu} \varphi(x).
\]

The function \( \mu \) is said to be piecewise linear if there exists a fan \( \Sigma \) in \( E^* \) such that the restrictions of \( \mu \) to its maximal cones are linear. Observe that \( \mu \) is piecewise linear if and only if \( C_\mu \) is polyhedral. In this case, to any maximal cone \( \sigma \) in \( \Sigma \) we associate \( x_\sigma \) which is the unique point in \( C_\mu \) such that \( \varphi(x_\sigma) = \mu(\varphi) \) for any \( \varphi \in \sigma \). Then, \( C_\mu \) is the convex hull of the points \( x_\sigma \). Dually, \( C_\mu \) is the set of \( x \in E \) such that \( \varphi(x) \geq \mu(\varphi) \) for any \( \varphi \) on a ray of \( \Sigma \).

The point \( 0 \) belongs to \( C_\mu \) if and only if \( \mu(\varphi) \leq 0 \) for any \( \varphi \in E^* \). Fix a scalar product \( (\cdot, \cdot) \) on \( E \) and hence on \( E^* \). We denote by \( \| \| \) the associated
Definition. The point \( m \) is homogeneous map from \( X \) have \( L \) if for any \( \tau \) by \( k > \) exists that is not semistable is said to be unstable.

Let the following properties:

(i) \( \| \varphi_0 \| = 1; \)

(ii) \( \mu(\varphi_0)x_0 \) belongs de \( C_\mu \), where \( x_0 \) is given by \( (x_0, \square) = \varphi. \)

4.3 Numerical semistability

Let \( \tau \in X \). Observe that the closure \( \overline{T.x} \) of the orbit \( T.x \) is a finite dimensional projective variety. Let \( \tau \) be a one parameter subgroup of \( T \). Consider \( F = \lim_{t \to 0} \tau(t) \). Recall from [MFK94] that \( \mu^\tau(x, \tau) \in Z \) is characterized by \( \tau(t), \mu^\tau(x, \tau) \) for any \( t \in C^* \) and any \( F \) in the fiber \( L_\mu \) over \( F \) in \( \mathcal{L} \). The map \( \tau \mapsto \mu^\tau(x, \tau) \) extends uniquely to a continuous, positively homogeneous map from \( X_*^\tau(T)_R \) to \( R \). This extension, still denoted by \( \mu \), is convex.

Definition. The point \( \tau \in X \) is said to be numerically semistable relatively to \( \mathcal{L} \) if for any \( g \in L_a^O G \) and any dominant one parameter subgroup \( \tau \) of \( T \), we have

\[
\mu^\tau(gx, \tau) \leq 0.
\]

Let \( X_{nss}(\mathcal{L}) \) denote the set of numerically semistable points in \( X \). A point that is not semistable is said to be unstable.

Consider the set \( \mathcal{C}_{nss}(X) \) of \( (\lambda_1, \lambda_2, \lambda_3, \hat{l}) \) in \((X^*_\mathcal{Q}(T))^3 \times \mathcal{Q} \) such that there exists \( k > 0 \) satisfying

(i) \( k\lambda_1, k\lambda_2, k\lambda_3 \) are dominant integral weights and \( k\hat{l} \in \mathcal{Z}_{>0}; \)

(ii) \( \frac{\Delta}{1}, \frac{\Delta}{\hat{l}} \) and \( \frac{\Delta}{\hat{l}} \) belong to the alcove \( \mathcal{A}^*; \)

(iii) \( X_{nss}(\mathcal{L}(k\hat{l} \Lambda) \otimes \mathcal{L}(k\lambda_1) \otimes \mathcal{L}(k\lambda_2) \otimes \mathcal{L}(k\lambda_3)) \) is not empty.

Our main statement can be formulated in terms of numerical semistability as follows.

Theorem 7 Let \( (\lambda_1, \lambda_2, \lambda_3) \in (X^*_\mathcal{Q}(T))^3 \) and \( \hat{l} \in \mathcal{Z}_{>0} \) such that \( \frac{\Delta}{1}, \frac{\Delta}{\hat{l}} \) and \( \frac{\Delta}{\hat{l}} \) belong to the alcove \( \mathcal{A}^* \). Then \( (\lambda_1, \lambda_2, \lambda_3, \hat{l}) \in \mathcal{C}_{nss}(X) \) if and only if

\[
<w_1 \omega_\beta, \lambda_1> + <w_2 \omega_\beta, \lambda_2> + <w_3 \omega_\beta, \lambda_3> \leq \frac{2}{(\beta, \beta)} \hat{l}. \tag{23}
\]

for any simple root \( \beta \), any nonnegative integer \( d \) and any \( (w_1, w_2, w_3) \in (W_P^\mathcal{A})^3 \) such that

\[
GW(w_1, w_2, w_3; d\sigma^*_\beta) = 1, \tag{24}
\]

15
and for any $\chi \in X^*(Z)$

$$2\Phi(w_1, \chi) + 2\Phi(w_2, \chi) + 2\Phi(w_3, \chi) + \sum_{\alpha \in \Phi(G/P, \chi)} \langle h, \alpha \rangle = 2\Phi(G/P_3, \chi).$$ (25)

### 4.4 Degree of numerical instability

We first compute explicitly $\mu^C(\cap, \tau)$ in terms of the Peterson decomposition.

**Lemma 4** Recall that $\cap \in X$ and $\tau \in X_*(T)$ is dominant. Let $h \in X_*(T)$ and $w_i \in W$ ($i = 1, 2, 3$) such that $\cap$ belongs to $S_{-h} \times Uw_1^{-1}B/B \times Uw_2^{-1}B/B \times Uw_3^{-1}B/B$. Then

$$\mu^C(\cap, \tau) = \hat{l}(h, \tau) + \sum_{i=1}^3 \langle w_i \tau, \lambda_i \rangle.$$

**Proof.** The group $T$ acts on the fiber over $B/B$ in $L(\lambda_i)$ with weight $-\lambda_i$. It follows that it acts on the fiber over $w_i^{-1}B/B$ in $L(\lambda_i)$ with weight $-w_i^{-1}\lambda_i$.

Similarly, $h$ acts on the fiber over $L_{-h}$ in $L(\bar{\Lambda})$ by the weight $-\bar{\Pi}_{-h}(\Lambda)$ (with notation of Section 2.2). But $-\bar{\Pi}T_{-h}(\Lambda) = \bar{\Lambda} - (\bar{l}(h, \square) - \frac{1}{2}(h, h)\delta$ and $T$ acts on the fiber over $L_{-h}$ in $L(\bar{\Lambda})$ by weight $-\bar{l}(h, \square)$. The lemma follows. \[\square\]

We set

$$M^C(x) = \sup_{\tau \in X_*(T)^+_{\text{nontrivial}}} \frac{\mu^C(gx, \tau)}{||\tau||},$$

for $g \in L_{\text{alg}} G$.

**Proposition 1** Assume that $\cap$ is not numericaly semistable. Then $M^C(\cap)$ is finite and there exist $g \in L_{\text{alg}}^0 G$ and $\tau \in X_*(T)^+_{\text{nontrivial}}$ such that

$$M^C(\cap) = \frac{\mu^C(g\cap, \tau)}{||\tau||}.$$

**Proof.** Let $h_0 \in X_*(T)^+$ such that the projection of $\cap$ on $G$ belongs to $G^{h_0}$. Let $h \in X_*(T)$ such that $S_{-h} \cap G^{h_0} \neq \emptyset$. We claim that

$$h \leq -w_0h_0.$$

Let $y \in S_{-h} \cap G^{h_0}$. By formula (18), there exists $w \in W$ such that $y \in B^-L_{w_0}$. Then, by (19) $\lim_{t \to \infty}(\delta^\gamma + \rho^\gamma)(t)y = L_{w_0}$. Since $S_{-h}$ is $T$-stable, $L_{w_0}$ belongs to $S_{-h}$. By (20), this implies that $w_0 \leq -h'$. Since $h_0$ is dominant $w_0h_0 \leq w_0h$. The claim follows.

Denote by $\Theta$ the set of $(-h, w_1^{-1}, w_2^{-1}, w_3^{-1}) \in X_*(T) \times W^3$ such that

$$L_{\text{alg}}^0 G \cap (S_{-h} \times Uw_1^{-1}B/B \times Uw_2^{-1}B/B \times Uw_3^{-1}B/B) \neq \emptyset.$$
By Lemma 4, we have

$$M^E(\tau) = \sup_{\tau \in X_*(T)_{\mathbb{R}}^+ \text{ s.t. } \parallel\tau\parallel = 1} \hat{L}(h, \tau) + \sum_{i=1}^{3} \langle w_i \tau, \lambda_i \rangle.$$  

For such a $h$, the claim asserts that $h \leq -w_0 h_0$. In particular, for any dominant $\tau \in X_*(T)_{\mathbb{R}}$, we have $(h, \tau) \leq (-w_0 h_0, \tau)$. It follows that $M^E(\tau)$ is finite.

By the above argument, there exists sequences $(-h_n, (w_1^n)^{-1}, (w_2^n)^{-1}, (w_3^n)^{-1}) \in \Theta$ and $\tau_n \in X_*(T)_{\mathbb{R}}$ such that

$$\lim_{n \to \infty} \hat{L}(h_n, \tau) + \sum_{i=1}^{3} \langle w_i \tau_n, \lambda_i \rangle = M^E(\tau) \quad \text{and} \quad \parallel\tau_n\parallel = 1.$$  

By extracting a subsequence, one may assume that each $w_i^n$ is constant (equal to $w_i$) and that $\tau_n$ tends to $\tau_0 \in X_*(T)_{\mathbb{R}}$. Then, $(-h_n, w_1^{-1}, w_2^{-1}, w_3^{-1}) \in \Theta$, $\parallel\tau_0\parallel = 1$ and

$$\lim_{n \to \infty} \hat{L}(h_n, \tau_0) + \sum_{i=1}^{3} \langle w_i \tau_0, \lambda_i \rangle = M^E(\tau).$$

Set $M' = \frac{1}{4}(M^E(\tau) - \sum_{i=1}^{3} \langle w_i \tau_0, \lambda_i \rangle - 1)$. Since $\tau_0$ is dominant, for $h \leq -w_0 h_0$, the function $h \mapsto (h, \tau_0)$ takes only finitely many values greater than $M'$. Hence the sequence $\hat{L}(h_n, \tau_0) + \sum_{i=1}^{3} \langle w_i \tau_0, \lambda_i \rangle$ is stationary and there exists $(-h, w_1^{-1}, w_2^{-1}, w_3^{-1}) \in \Theta$ such that

$$M^E(\tau) = \hat{L}(h, \tau_0) + \sum_{i=1}^{3} \langle w_i \tau_0, \lambda_i \rangle.$$  

Let $F$ be the face of $X_*(T)_{\mathbb{R}}^+$ containing $\tau_0$ in its relative interior. Then

$$M^E(\tau) = \sup_{\tau \in F \text{ s.t. } \parallel\tau\parallel = 1} \hat{L}(h, \tau_0) + \sum_{i=1}^{3} \langle w_i \tau_0, \lambda_i \rangle.$$  

Since the linear form $\tau \mapsto \hat{L}(h, \tau_0) + \sum_{i=1}^{3} \langle w_i \tau_0, \lambda_i \rangle$ and the cone $F$ are rational, this supremum is reached on a rational half line. It follows that $\tau_0 = \frac{\tau'_0}{\parallel\tau'_0\parallel}$ for some rational $\tau'_0$.  

**4.5 Relation with parabolic bundles**

A **flagged bundle** $(E, \xi_1, \xi_2, \xi_3)$ on $\mathbb{P}^1$ at the three marked points $p_1, p_2$ and $p_3$ is the given of a principal $G$-bundle $E$ on $\mathbb{P}^1$ and three parabolic reductions $\xi_i \in \mathcal{E}_{p_i}/B$ at the three points $p_i$. Let us recall how to associate to any point
of \( X \) a flagged bundle on \( \mathbb{P}^1 \). Assume that \( \{p_1, p_2, p_3\} \cap \{0, \infty\} \) is empty. Let \( i_0 : \mathbb{C} \to \mathbb{P}^1, z \mapsto [z, 1] \) and \( i_\infty : \mathbb{C} \to \mathbb{P}^1, z \mapsto [1, z] \) denote the two open embeddings. Their images cover \( \mathbb{P}^1 \), and for any \( z \in \mathbb{C}^* \), \( i_0(z) = i_\infty(z^{-1}) \). Let \( g \in L_{\text{alg}}G \). Thinking about \( g \) as a transition function on \( i_0(\mathbb{C}) \cap i_\infty(\mathbb{C}) \), we get a principal \( G \)-bundle \( \mathcal{E} \) with two trivializations \( i_0 : \mathbb{C} \times G \to \mathcal{E} \) over \( \mathbb{P}^1 - \{\infty\} \) and \( i_\infty : \mathbb{C} \times G \to \mathcal{E} \) over \( \mathbb{P}^1 \). Moreover, for any \( z \in \mathbb{C}^* \) and \( h \in G \), we have

\[ i_0(z, h) = i_\infty(z^{-1}, g(z)h). \]

Consider also the two sections \( \sigma_0 \) and \( \sigma_\infty \) defined respectively on \( \mathbb{P}^1 - \{\infty\} \) and \( \mathbb{P}^1 - \{0\} \) by

\[ \sigma_0(i_0(z)) = i_0(z, e) \quad \text{and} \quad \sigma_\infty(i_\infty(z)) = i_\infty(z, e), \]

for any \( z \in \mathbb{C} \). The map \( g \mapsto (\mathcal{E}, \sigma_0, \sigma_\infty) \) is a bijection from \( L_{\text{alg}}G \) to the set of principal bundles on \( \mathbb{P}^1 \) endowed with two sections.

Let \( g_1 \in L_{\text{alg}}^0G \) and \( g_2 \in L_{\text{alg}}G \). Let \( \mathcal{E}' \), \( i_0' \), \( i_\infty' \), \( \sigma_0' \) and \( \sigma_\infty' \) be as above when \( g \) is replaced by \( g_1g_2^{-1} \). Then, there exits an isomorphism \( \Theta : \mathcal{E} \to \mathcal{E}' \) of principal \( G \)-bundles such that

\[ \Theta(i_0(z, h)) = i_0'(z, g_2(z)h) \quad \text{and} \quad \Theta(i_\infty(z, h)) = i_\infty'(z, g_1(z^{-1})h), \]

for any \( z \in \mathbb{C} \) and \( h \in G \). Moreover

\[ \Theta \circ \sigma_0 \circ i_0 = \sigma_0' \circ i_0 \circ g_2 \quad \text{and} \quad \Theta \circ \sigma_\infty \circ i_\infty = \sigma_\infty' \circ i_\infty \circ g_1. \]

In other words, \( g_1g_2^{-1} \) corresponds to \( (\mathcal{E}, \sigma_0g_2^{-1}, \sigma_\infty g_1^{-1}) \).

Now \( \mathcal{G} = L_{\text{alg}}G/L_{\text{alg}}^0G \) corresponds to the set of pairs \((\mathcal{E}, \sigma_\infty)\). Let

\[ \shuffle = (gL_{\text{alg}}^0G/L_{\text{alg}}G, g_1B/B, g_2B/B, g_3B/B) \in X. \]

Let \((\mathcal{E}, \sigma_\infty)\) corresponding to \(gL_{\text{alg}}^0G/L_{\text{alg}}G\). Consider, for any \( i = 1, 2, 3 \), the point \( \sigma_\infty(p_i)g_iB/B \) in \( \mathcal{E}_{p_i}/B \); it is a parabolic reduction \( \xi_i \) at \( p_i \). One checks that two points \( \shuffle \) and \( \shuffle' \) in \( X \) induces the same flagged bundle \((\mathcal{E}, \xi_1, \xi_2, \xi_3)\) if and only if they belong to the same \( L_{\text{alg}}^0G \)-orbit.

The given of \( \shuffle \in X \) also determines a section \( \sigma_\infty : \mathbb{P}^1 - \{0\} \to \mathcal{E} \). This section induces a section \( \mathbb{P}^1 - \{0\} \to \mathcal{E}/B \) that extends to a parabolic reduction \( \bar{\sigma} : \mathbb{P}^1 \to \mathcal{E}/B \).

**Parabolic degree.** Recall that \( (\lambda_1, \lambda_2, \lambda_3) \in (X^*(T))^3, \ell \in \mathbb{Z}_{\geq 0} \) and \( \mathcal{L} = \mathcal{L}(\ell \Lambda) \otimes \mathcal{L}(\lambda_1) \otimes \mathcal{L}(\lambda_2) \otimes \mathcal{L}(\lambda_3) \) on \( X \). Let us explain how \( \mu^\mathcal{L}(\shuffle, \tau) \) can be expressed in terms of the flagged bundle \((\mathcal{E}, \xi_1, \xi_2, \xi_3)\) endowed with the parabolic reduction \( \bar{\sigma} \).

Fix \( i = 1, 2 \) or 3. Both \( \bar{\sigma}(p_i) \) and \( \xi_i \) belong to \( \mathcal{E}_{p_i}/B \). Fixing an identification \( \mathcal{E}_{p_i} \simeq G \) (which is equivariant for the right \( G \)-actions), the pair \((\bar{\sigma}(p_i), \xi_i)\) gives a point in \( G/B \times G/B \). The \( G \)-orbit of this point does not depend on the chosen identification \( \mathcal{E}_{p_i} \simeq G \); in particular, it belongs to \( G.(B/B, w_i^{-1}B/B) \) for some well defined \( w_i \in W \).
Consider \((\tau, \square) \in X^*(T)\). The parabolic reduction \(\bar{\sigma}\) induces a principal \(B\)-bundle \(E_B\). We denote by \(\mathbb{C}_\tau\) the one-dimensional representation of \(B\) associated to the character \((\tau, \square)\) of \(B\). We can define the line bundle \(E_B \times_B \mathbb{C}_\tau\) on \(\mathbb{P}^1\). Its degree \(\text{deg}(E_B \times_B \mathbb{C}_\tau)\) belongs to \(\mathbb{Z}\).

The parabolic degree relatively to \(L\) is defined by

\[
\text{pardeg}(E, \xi_1, \xi_2, \xi_3, \bar{\sigma}, \tau) = \text{deg}(E_B \times_B \mathbb{C}_\tau) + \sum_{i=1}^{3} (w^{-1}_i \lambda_i, \tau). \tag{26}
\]

**Lemma 5** With above notation, we have

\[
\mu^L(\tau, \tau) = \text{pardeg}(E, \xi_1, \xi_2, \xi_3, \bar{\sigma}, \tau).
\]

**Proof.** Let \(h \in X_*(T)\) and \(v_i \in W\) (for \(i = 1, 2, 3\)) such that \(\tau \in S_h \times Uv^{-1}_1 B/B \times Uv^{-1}_2 B/B \times Uv^{-1}_3 B/B\). With Lemma 4, it is sufficient to prove that \((h, \tau) = \text{deg}(E_B \times_B \mathbb{C}_\tau)\), and that \(v_i W_\tau = v_i W_\tau\), for any \(i = 1, 2, 3\). These are direct verifications. \(\square\)

Recall from e.g. [HS10], that \((E, \xi_1, \xi_2, \xi_3)\) is said to be semistable relatively to \(L\) if and only if for any dominant \(\tau \in X_*(T)\) and any parabolic reduction \(\bar{\sigma} : \mathbb{P}^1 \to E/B\) we have

\[
\text{pardeg}(E, \xi_1, \xi_2, \xi_3, \bar{\sigma}, \tau) \leq 0.
\]

**Corollary 1** Fix \(\tau \in X_\tau\) and the corresponding flagged principal bundle \((E, \xi_1, \xi_2, \xi_3)\). Then \(\tau\) is numerically semistable in the sense of Definition 4.3 if and only if the flagged principal \((E, \xi_1, \xi_2, \xi_3)\) is semistable relatively to \(L\).

### 4.6 Generic toric reduction

Let \((E, \xi_1, \xi_2, \xi_3)\) be a flagged principal bundle. Let \(\Omega\) be a nonempty open subset of \(\mathbb{P}^1\) and \(\eta : \Omega \to E/T\) be a reduction defined on \(\Omega\). Let \(\tau \in X_*(T)\). Since \(T \subset P(\tau)\), we have a quotient map \(E/T \to E/P(\tau)\). Hence \(\eta\) induces a reduction \(\Omega \to E/P(\tau)\) that extends to \(\tilde{\sigma} : \mathbb{P}^1 \to E/P(\tau)\). Consider the map

\[
\mu_\eta : X_*(T) \to \mathbb{Z} \quad \tau \mapsto \text{pardeg}(E, \xi_1, \xi_2, \xi_3, \bar{\sigma}, \tau).
\]

Note that in (26), we replace \(E_B \times_B \mathbb{C}_\tau\) by \(E_{P(\tau)} \times_P \mathbb{C}_\tau\). Since \(P(\tau)\) and \(\tilde{\sigma}\) only depends on the signs of the \(\langle \tau, \alpha \rangle\) for \(\alpha \in \Phi\), the map \(\mu_\eta\) is piecewise linear. In particular, it extends to a positively homogeneous, continuous, piecewise linear function from \(X_*(T) \otimes \mathbb{R}\) to \(\mathbb{R}\). This extension is still denoted by \(\mu_\eta\). The following proposition will play an important role.

**Proposition 2** Assume that \(\frac{\lambda_1}{l}, \frac{\lambda_2}{l}\) and \(\frac{\lambda_3}{l}\) belong to the alcove \(A^\ast\). Then the map \(\mu_\eta\) is convex.
**Proof.** Recall that the cones of the Weyl fan are the subsets of $\tau \in X_*(T) \otimes \mathbb{R}$ such that for each $\alpha \in \Phi$, the product $\langle \tau, \alpha \rangle$ is fixed to be negative, positive or zero. The parabolic subgroup $P(\tau)$ only depends on the cone of the Weyl fan containing $\tau$ in its relative interior. Now, the formula (20) and Lemma 5 show that the restriction of $\mu_\eta$ to any such cone is linear. Then it is sufficient to check convexity when one goes from any chamber to an adjacent one. To simplify notations, we assume that one of these two chambers is the dominant one $X_*(T)_R^1$. The other one is $s_\alpha X_*(T)_R^1$ for some simple root $\alpha$. The minimal parabolic subgroup $P^\alpha$ associated to some simple root $\alpha$ is the closure of $Bs_\alpha Bs_\alpha$. Let $\mu \in \Hom(X_*(T), \mathbb{Q})$ (resp. $\mu' \in \Hom(X_*(T), \mathbb{Q})$) whose the restriction to $X_*(T)_R^1$ (resp. $s_\alpha X_*(T)_R^1$) is equal to $\mu_\eta$ (resp $\mu'_\eta$). Recall that $\alpha^\vee$ is orthogonal to the span of $X_*(T)_R^1 \cap (s_\alpha X_*(T)_R^1)$ and oriented toward $X_*(T)_R^1$. The convexity on the union of these two chambers is equivalent to the following inequality

$$\mu(\alpha^\vee) \geq \mu'(\alpha^\vee).$$

(27)

Let $L^\alpha$ denote the Levi subgroup of $P^\alpha$ containing $T$ and let $R^\alpha(P^\alpha)$ denote the unipotent radical of $P^\alpha$. Consider the reduction $\tilde{\sigma}_{\alpha} : \mathbb{P}^1 \rightarrow \mathcal{E}/P^\alpha$ induced by $\eta$ and $\mathcal{E}_{P^\alpha}$ be the associated principal $P^\alpha$-bundle. Since $L^\alpha$ identifies with $P/R^\alpha(P^\alpha)$, $\mathcal{E}_{P^\alpha}/R^\alpha(P^\alpha)$ is a principal $L^\alpha$-bundle on $\mathbb{P}^1$ denoted by $\mathcal{E}_{L^\alpha}$. We are going to endow $\mathcal{E}_{L^\alpha}$ with a flagged structure and express $\mu(\alpha^\vee) - \mu'(\alpha^\vee)$ in terms of this flagged bundle.

Set $\tilde{\alpha} = \frac{2}{(\alpha, \alpha)} \alpha$. First, observe that $\mathcal{E}_B \times_B \mathbb{C}_{\tilde{\alpha}}$ (resp. $\mathcal{E}_B' \times_{B'} \mathbb{C}_{\tilde{\alpha}}$) identifies with $\mathcal{E}_{B \cap L^\alpha} \times_{B \cap L^\alpha} \mathbb{C}_{\tilde{\alpha}}$ (resp. $\mathcal{E}_{B' \cap L^\alpha} \times_{B' \cap L^\alpha} \mathbb{C}_{\tilde{\alpha}}$). In particular,

$$\deg(\mathcal{E}_B \times_B \mathbb{C}_{\tilde{\alpha}}) - \deg(\mathcal{E}_B' \times_{B'} \mathbb{C}_{\tilde{\alpha}}) = \deg(\mathcal{E}_{B \cap L^\alpha} \times_{B \cap L^\alpha} \mathbb{C}_{\tilde{\alpha}}) - \deg(\mathcal{E}_{B' \cap L^\alpha} \times_{B' \cap L^\alpha} \mathbb{C}_{\tilde{\alpha}}).$$

(28)

Fix $i = 1, 2$ or 3. Choose an identification $\mathcal{E}_{p_i} \simeq G$ such that $\tilde{\sigma}_{\alpha}(p_i)$ corresponds with $P^\alpha$. Let $B_i$ be the Borel subgroup $G$ associated to the flagged structure at $p_i$. Then $(B_i \cap P^\alpha)/R^\alpha(P^\alpha)$ is a Borel subgroup of $L^\alpha$. Let $\xi^\alpha$ be the associated flagged structure at $p_i$ for $\mathcal{E}_{L^\alpha}$. Now $(\mathcal{E}_{L^\alpha}, \xi^\alpha_1, \xi^\alpha_2, \xi^\alpha_3)$ is a flagged principal $L^\alpha$-bundle.

Observe that $\mathcal{E}/B$ and $\mathcal{E}/B'$ identify canonically. Then, $\tilde{\sigma}_B$ and $\tilde{\sigma}_{B'}$ are two sections of this $G/B$-bundle. We have the commutative diagram

$$\begin{array}{ccc}
\tilde{\sigma}_B & \longrightarrow & \mathcal{E}/B \\
\downarrow & & \downarrow \pi \\
\mathbb{P}^1 & \longrightarrow & \mathcal{E}/P^\alpha \\
\tilde{\sigma}_{B'} & \longrightarrow & \mathcal{E}/B' \\
\downarrow & & \downarrow \pi \\
\tilde{\sigma}_{B'} & \longrightarrow & \mathcal{E}/B
\end{array}$$

20
where the vertical maps \( \pi \) are induced by the inclusion \( B \subset P^a \). By construction 
\( \pi(\bar{\sigma}_B(p_1)) = \pi(\bar{\sigma}_B(p_2)) = \sigma_a(p_i) \). Recall that \( w_i \in W \) is characterized by the relation 
\( (\bar{\sigma}_B(p_i), \xi_{p_i}) \in G(B/B, w_1^{-1}B/B) \). Similarly we defined \( w'_i \). Then, \( w'_i \)

is equal to either \( w_i \) or \( w_i s_a \).

Consider the \( L^a \)-irreducible representation \( V_\varpi_a(L^a) \) of highest weight \( \varpi_a \).
It has dimension two, a unique \( B \cap L^a \)-fixed line \( d \) and a unique \( B' \cap L^a \) fixed line \( d' \). Consider the associated short exact sequence

\[
0 \longrightarrow d \longrightarrow V_\varpi_a(L^a) \longrightarrow V_\varpi_a(L^a)/d' \longrightarrow 0
\]

Consider the associated morphisms of vector bundles on \( \mathbb{P}^1 \):

\[
\mathcal{E}_{L^a \cap B} \times \mathcal{E}_{L^a \cap B} d \longrightarrow \mathcal{E}_{L^a} \times \mathcal{E}_{L^a} V_\varpi_a(L^a) \longrightarrow \mathcal{E}_{L^a \cap B'} \times \mathcal{E}_{L^a \cap B'} V_\varpi_a(L^a)/d'
\]

Set \( \lambda_i^a = w_i \lambda_i \), for \( i = 1, 2, 3 \) (HERE \( w_i \in W^{P_a} \)). Let \( v_i = s_a \) or \( e \) denote the relative position of the flag on \( (\mathcal{E}_{L^a})_{p_i} \) and \( \bar{\sigma}_B(p_i) \). Similarly \( v'_i \). Inequality \( (27) \) is equivalent to

\[
\frac{\text{deg}(\mathcal{E}_{L^a \cap B} \times L^a \cap B d) + \frac{1}{2} \sum \langle v_i \lambda_i^a, \alpha^\vee \rangle - \frac{1}{2} \sum \langle v'_i \lambda_i^a, \alpha^\vee \rangle}{\text{deg}(\mathcal{E}_{L^a \cap B'} \times L^a \cap B' V_\varpi_a(L^a)/d) + \frac{1}{2} \sum \langle v'_i \lambda_i^a, \alpha^\vee \rangle} \geq 0.
\]

But \( \langle v_i \lambda_i, \alpha^\vee \rangle - \langle v'_i \lambda_i, \alpha^\vee \rangle \) is either equal to \( 0, \pm 2 \langle \lambda_i^a, \alpha^\vee \rangle \). The two cases \( 0 \) and \( \pm \) are easy. Consider the last case. The point is that in this case the morphism has to vanish at \( p_i \). In particular, if this case occurs \( d \) times (when \( i \) runs over \( \{1, 2, 3\} \)) then \( \text{deg}(\mathcal{E}_{L^a \cap B} \times L^a \cap B d) - \text{deg}(\mathcal{E}_{L^a \cap B'} \times L^a \cap B' V_\varpi_a(L^a)/d) \geq d \). But we have \( 0 \leq \langle \lambda_i^a, \alpha^\vee \rangle /d \leq 1 \). Inequality \( (27) \) follows.

**The polytope \( \mathcal{P}_\eta \).** Consider in \( X_*(T)_R \) the fan \( \Sigma \) whose the maximal cones are the Weyl chambers. By formula \( (20) \) and Lemma \( \ref{lem:mu_eta} \), the restriction of \( \mu_\eta \)
to any Weyl chamber is linear. In particular, \( \mu_\eta \) is piecewise linear. Like in Section \( 4.2 \), consider the associated polytope

\[
\mathcal{P}_\eta = \{ \chi \in X_*(T)_R : \forall \tau \in X_*(T) \quad \langle \tau, \chi \rangle \geq \mu_\eta(\tau) \}.
\]

Let \( B' \) be a Borel subgroup of \( G \) containing \( T \) and let \( C' \) denote the corresponding Weyl chamber in \( X_*(T)_R \). Let \( \chi_{B'} \) be the only point in \( X_*(T)_Q \) such that 
\( \langle \chi_{B'}, \tau \rangle = \mu_\eta(\tau) \) for any \( \tau \) in \( C' \). Then \( \mathcal{P}_\eta \) is the convex hull of the \( \chi_{B'} \) for various Borel subgroups \( B' \) containing \( T \).

Similarly the rays of \( \Sigma \) correspond bijectively with the maximal parabolic subgroups \( P \) containing \( T \). For any such parabolic subgroup, let \( \tau_P \) denote the unique indivisible one parameter subgroup of \( T \) such that \( P = P(\tau_P) \). Then

\[
\mathcal{P}_\eta = \{ \chi \in X_*(T)_R : \forall \text{ maximal } P \supset T \quad \langle \chi, \tau_P \rangle \geq \mu_\eta(\tau_P) \}.
\]
4.7 Canonical reduction

Let \( \land \in \mathbb{X} \) be numerically unstable. Let \( g \in L_{\text{alg}}^0 G \) and let \( \tau_0 \) be an indivisible dominant one-parameter subgroup of \( T \) such that \( M^E(\land) = \frac{\mu^E(g \land, \tau_0)}{\|\tau_0\|} \). To the point \( g \land \) corresponds a flagged bundle \( E \) with a section \( \sigma_\infty \) over \( \mathbb{P}^1 - \{0\} \). This section extends to a parabolic reduction \( \sigma_{g \land} : \mathbb{P}^1 \rightarrow E/P(\tau_0) \).

**Proposition 3** Assume that \( \lambda \), \( \mu \) and \( \nu \) belong to the alcove \( \Lambda^\ast \). Let \( \land \in \mathbb{X} \) and \( (E, \xi_1, \xi_2, \xi_3) \) be the associated flagged bundle. Assume that \( \land \) is unstable relatively to \( L \).

Let \( g_1 \) and \( g_2 \) in \( L_{\text{alg}}^0 G \) and let \( \tau_1 \) and \( \tau_2 \) be two dominant indivisible one parameter subgroups of \( T \) such that

\[
M^E(\land) = \frac{\mu^E(g_1 \land, \tau_1)}{\|\tau_1\|} = \frac{\mu^E(g_2 \land, \tau_2)}{\|\tau_2\|}.
\]

Then

(i) \( \tau_1 = \tau_2 \); set \( P = P(\tau_1) = P(\tau_2) \).

(ii) \( g_2 g_1^{-1} \in L_{\text{alg}}^0 P \).

(iii) The two reductions \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \) from \( \mathbb{P}^1 \) to \( E/P \) associated to \( g_1 \land \) and \( g_2 \land \) respectively coincide. This reduction is called the canonical reduction of \( (E, \xi_1, \xi_2, \xi_3) \).

**Proof.** Let \( \sigma_1^B \) and \( \sigma_2^B \) denote the parabolic reductions associated to \( g_1 \land \) and \( g_2 \land \) respectively. For \( x \in \mathbb{P}^1 \), \( \sigma_B(x) \) and \( \sigma_B'(x) \) belong to \( E_x/B \). Choosing an identification \( E_x/B \simeq G/B \), we get two point in \( G/B \). The element \( w \in W \) such that \( G, (B/B, wB/B) \) does not depend on the identification and is denote by \( \text{rel} (\sigma_B(x), \sigma_B'(x)) \). By the finiteness of the Bruhat decomposition, there exists \( w \in W \) and a nonempty open subset \( \Omega \) of \( \mathbb{P}^1 \) such that \( \text{rel} (\sigma_B(x), \sigma_B'(x)) = w \) for any \( x \in \Omega \). Set \( B_w = wBw^{-1} \). Observe the \( E/B \) and \( E/B_w \) are canonically isomorphic. Let \( p, p_w : E \rightarrow E/B \) be the two projections induced respectively by the inclusions \( T \subset B \) and \( T \subset B_w \). Up to changing \( \Omega \), one may assume that \( E \) is trivial on \( \Omega \). Then, there exists a reduction \( \eta : \Omega \rightarrow E/T \) such that \( \sigma_1^B = p \circ \eta \) and \( \sigma_2^B = p_w \circ \eta \).

By Lemma 5 we have

\[
\mu^E(g_1 \land, \tau_1) = \mu_\eta(\tau_1) \quad \text{and} \quad \mu^E(g_2 \land, \tau_2) = \mu_\eta(w\tau_2w^{-1}).
\]

In particular, \( \sup_{\tau \in X_\ast(T)} \frac{\mu_\eta(\tau)}{\|\tau\|} = M^E(\land) \). By Proposition 2 the function is convex. In particular, it has a unique maximum on the unit sphere. Hence \( \tau_1 = w\tau_2w^{-1} \). Since \( \tau_1 \) and \( \tau_2 \) are assumed to be dominant, \( \tau_1 = \tau_2 \).

As in the proposition set \( P = P(\tau_1) \). We have also proved that \( \tau_2 = w\tau_2w^{-1} \). Hence \( w \) belongs to \( WP \). Then \( p \) and \( p_w \) induce the same map \( q : E/T \rightarrow E/P \).
Therefore, $\sigma_1^B$ and $\sigma_2^B$ induce the same reduction $\sigma : \mathbb{P}^1 \to \mathcal{E}/P$. The last assertion of the proposition follows.

Let $\sigma_\infty$ be the section of $\mathcal{E}$ on $\mathbb{P}^1 - \{0\}$ associated to $\sigma \circ$. Then $g_1 \circ$ and $g_2 \circ$ correspond respectively to $\sigma_\infty g_1^{-1}$ and $\sigma_\infty g_2^{-1}$. But, we just proved these two local trivialisations induce the same section of $\mathcal{E}/P$. Hence $g_1^{-1}(z)P/P = g_2^{-1}(z)P/P$ for any $z$. The second assertion is proved. □

**Definition.** Let $\sigma \in \mathfrak{X}$ be unstable relatively to $\mathcal{L}$. Let $(\mathcal{E}, \xi_1, \xi_2, \xi_3)$ be the associated flagged principal bundle. Let $\tau_0$ denote the dominant one parameter subgroup of $T$ satisfying Proposition 3. Set $P = P(\tau_0)$. Let $\sigma : \mathbb{P}^1 \to \mathcal{E}/P$ be the canonical reduction of $(\mathcal{E}, \xi_1, \xi_2, \xi_3)$ and $\mathcal{E}_P$ the associated principal $P$-bundle. For $i = 1, 2$ and $3$, let $w_i \in W_P$ denote the relative position of $(\xi_i, \sigma(p_i))$. Finally, we define a $\mathbb{Z}$-linear map

$$h : X^*(P) \to \mathbb{Z}, \quad \chi \mapsto \deg(\mathcal{E}_P \times_P \mathcal{G}_\chi).$$

The **Harder-Narashiman type** (HN-type for short) of $\sigma$ (or of $(\mathcal{E}, \xi_1, \xi_2, \xi_3)$) is the tuple $(\tau_0, P, h, w_1, w_2, w_3)$.

**A characterization of the canonical reduction.** Let $P \supset T$ be a parabolic subgroup, let $R^u(P)$ denote its unipotent radical and let $L$ denote its Levi subgroup containing $T$.

Let $(\mathcal{E}, \xi_1, \xi_2, \xi_3)$ be a flagged bundle. Let $\sigma : \mathbb{P}^1 \to \mathcal{E}/P$ be a parabolic reduction. Let $\mathcal{E}_P \subset \mathcal{E}$ denote the principal $P$-subbundle associated to $\sigma$. Then the quotient $\mathcal{E}_P/R^u(P)$ is a principal $L$-bundle. Consider a marked point $p_i$ and choose an identification of the fiber $\mathcal{E}_{p_i}$ with $G$ (as a torsor). Then $\sigma(p_i)$ determines a parabolic subgroup $P'$ of $G$ conjugated to $P$. Similarly $\xi_i$ determines a Borel subgroup $B'$ of $G$. Then $(P' \cap B')/R^u(P')$ is a Borel subgroup of $P'/R^u(P')$. This Borel subgroup (which is independent on the choice) can be chosen as a flag $\xi_i^l$ over $p_i$ in $\mathcal{E}_P/R^u(P)$. Then $(\mathcal{E}_P/R^u(P), \xi_1^l, \xi_2^l, \xi_3^l)$ is a flagged $L$-bundle over $\mathbb{P}^1(p_1, p_2, p_3)$.

Assume now that $(\mathcal{E}, \xi_1, \xi_2, \xi_3)$ and the parabolic reduction $\sigma$ come from $\sigma \in \mathfrak{X}$. Let $\tau \in X_*(T)$ such that $P = P(\tau)$. Set $\tau_0 = \lim_{\gamma \to 0} \tau(\gamma)$ $\circ$. It belongs to the fixed point set $X^\tau$. Each irreducible component of $(G/B)^\tau$ contains a unique $B \cap L$ fixed point and so identifies canonically with $L/(B \cap L)$. On the other hand, $L_{\text{alg}}/L$ acts transitively on $G^\tau$ that identifies with the affine grassmannian $\mathcal{G}(L)$ of the group $L$. Using these identifications, the point $\tau_0$ gives a point $\tau_0^{\circ}$ of $\mathcal{G}(L) \times (L/B \cap L)^3$. Hence $\tau_0$ determines a flagged principal $L$-bundle. This bundle is $(\mathcal{E}_P/R^u(P), \xi_1^l, \xi_2^l, \xi_3^l)$.

Here, and like before, $\tau$ is a one parameter subgroup of $T$, $L$ is the centralizer of the image of $\tau$ and $P$ is the associated parabolic subgroup. Since $Z(L)$ is contained in $T$, $X_*(Z(L))$ is contained in $X_*(T)$. It is $\oplus_{\alpha \in \Delta - \Delta_P} \mathbb{Z} \omega_\alpha^\tau$. In particular $\tau = \sum_{\alpha \in \Delta - \Delta_P} n_\alpha \omega_\alpha^\tau$, for some integers $n_\alpha$. The restriction
map $X^*(L) \to X^*(T)$ is injective, and $X^*(L)$ identifies with $\oplus_{\alpha \in \Delta - \Delta_P} \mathbb{Z} \omega_{\alpha}$. Consider $(\tau, \square) \in X^*(T) \otimes \mathbb{Q}$. We have: $(\tau, \square) = \sum_{\alpha \in \Delta - \Delta_P} n_{\alpha}(\omega_{\alpha}, \square) = \sum_{\alpha \in \Delta - \Delta_P} n_{\alpha} \frac{2}{2\alpha, \alpha} \omega_{\alpha}$. In particular, $(\tau, \square)$ belongs to $X^*(L)$.

**Proposition 4** Assume that $\frac{\Delta^+}{\ell}$, $\frac{\Delta^-}{\ell}$ and $\frac{\Delta}{\ell}$ belong to the alcove $\mathbb{A}^*$. Let $\cap \in \mathbb{X}$ be unstable and $\tau$ be a dominant one parameter subgroup of $T$. Set $P = P(\tau)$, $L = G^*$ and $\sigma : \mathbb{P}^1 \to \mathcal{E}/P$ be the parabolic reduction associated to $\cap$ and $P$. Consider the flagged principal $L$ bundle $(\mathcal{E}_P/R^n(P), \xi_1^L, \xi_2^L, \xi_3^L)$.

The following are equivalent

(i) $M^\mathcal{L}(\cap) = \frac{\mu^\mathcal{L}(\cap, \tau)}{\|\tau\|}$;

(ii) $\mathcal{E}_P/R^n(P)$ is semistable for $L$ relatively to the line bundle $\mathcal{L} \cap - \mu^\mathcal{L}(\cap, \tau)\frac{(30)}{\|\tau\|}$.

**Proof.** Set $\mathcal{L'} = \mathcal{L} \oplus -\mu^\mathcal{L}(\cap, \tau)\frac{(30)}{\|\tau\|}$ and $\chi = \mu^\mathcal{L}(\cap, \tau)\frac{(30)}{\|\tau\|}$. Assume that $M^\mathcal{L}(\cap) = \frac{\mu^\mathcal{L}(\cap, \tau)}{\|\tau\|}$. Let $\zeta \in X_*(T)$ and $l \in L^{>0}_{\text{al}} L$. We have to prove that

$$\mu^\mathcal{L}(l \cap, \zeta) \leq \langle \chi, \tau \rangle.$$ (29)

Consider the parabolic reduction $\sigma_\zeta : \mathbb{P}^1 \to \mathcal{E}/Q$ induced by $l \cap$ and $\zeta$. By construction, there exists an embedding $Q \subset P$ such that if $p : \mathcal{E}/Q \to \mathcal{E}/P$ denotes the corresponding projection, we have $p \circ \sigma_\zeta = \sigma$.

Consider now a generic reduction $\eta$ to $\mathcal{E}/T$ and $q : \mathcal{E}/T \to \mathcal{E}/Q$ such that $\sigma_\zeta = q \circ \eta$. Consider the convex function $\mu_\eta$ and the polytope $\mathcal{P}_\eta$. Since $\frac{\mu_\eta(\tau)}{\|\tau\|} = M^\mathcal{L}(\eta)$, the point $\chi$ belongs to $\mathcal{P}_\eta$. Recall that $\mathcal{F}_\tau$ denote the face of $\mathcal{P}_\tau$ corresponding to the inequality $(\tau, \square) \geq \mu^\mathcal{L}(\cap, \tau)$. This face is the polytope of $\mathcal{E}_P/R^n(P)$ for $\eta$ relatively to $\mathcal{L}$. Hence the polytope of $\mathcal{E}_P/R^n(P)$ for $\eta$ relatively to $\mathcal{L}'$ is $\mathcal{F} - \chi$. It contains 0. Inequality (29) follows.

Conversely, assume that $\mathcal{E}_P/R^n(P)$ is semistable for $L$ relatively to the line bundle $\mathcal{L}'$. Consider a generic reduction $\eta$ to $\mathcal{E}/T$ and $p_P : \mathcal{E}/T \to \mathcal{E}/P$ such that $\sigma = p_P \circ \eta$. By the usual argument it is sufficient to prove that

$$M^\mathcal{L}(\eta) = \frac{\mu_\eta(\tau)}{\|\tau\|}.$$ (30)

The polytope $\mathcal{P}_L$ of $\mathcal{E}/R^n(P)$ for $\eta$ relatively to $\mathcal{L}'$ is the convex hull of the points $\chi_B - \chi$ for various $B$ such that $T \subset B \subset P$. This polytope is contained in $\mathcal{P}_\eta - \chi$ which is the convex hull of the points $\chi_B$ for various $B \supset T$. By assumption, 0 belongs to $\mathcal{P}_L$. Hence $\chi$ belongs to $\mathcal{P}_\eta$. Equality (30) follows. □

The following proposition is analogous to [Nes84, Theorem 9.3] or [RRS84, Proposition 1.9].
Proposition 5. Assume that $\lambda_1$, $\lambda_2$, and $\lambda_3$ belong to the alcove $A^*$. Let $\varpi \in X$ be unstable. Let $g$ in $L_{alg}^< 0 G$ and let $\tau$ be the dominant indivisible one parameter subgroup of $T$ such that

$$M^\mathcal{L}(\varpi) = \frac{\mu^\mathcal{L}(gx, \tau)}{\|\tau\|}.$$ 

Set $\varpi_0 = \lim_{s \to 0} \tau(s)g\varpi$. Then

(i) $M^\mathcal{L}(\varpi_0) = M^\mathcal{L}(\varpi)$;

(ii) $M^\mathcal{L}(\varpi_0) = \frac{\mu^\mathcal{L}(\varpi_0, \tau)}{\|\tau\|}$.

**Proof.** Since $\mu^\mathcal{L}(\varpi, \tau) = \mu^\mathcal{L}(\varpi_0, \tau)$, the second assertion implies the first one. By Proposition 4 applied to $\varpi = \varpi_0$, the second assertion is equivalent to the fact that $E/R^u(\Lambda)$ is semistable relatively to $L \otimes -\mu^\mathcal{L}(\varpi, \tau)\frac{\tau}{\|\tau\|}$. But, by Proposition 4 applied to $\varpi$, this is true. □

4.8 The set of numerically semistable points

For later use let us state the following well known result.

Lemma 6. Assume that $\lambda_1$, $\lambda_2$, and $\lambda_3$ belong to the alcove $A^*$. Then $X_{num}(\mathcal{L}(\Lambda) \otimes \mathcal{L}(\lambda_1) \otimes \mathcal{L}(\lambda_2) \otimes \mathcal{L}(\lambda_3))$ is open in $X$.

4.9 The open stratum

In this section, we assume that no point in $X$ is numerically semistable relatively to $\mathcal{L}$; that is that $M^\mathcal{L}(\varpi) > 0$ for any $\varpi \in X$. Set

$$d_0 = \inf_{\varpi \in X} M^\mathcal{L}(\varpi),$$

and

$$X^o(\mathcal{L}) = \{ \varpi \in X : M^\mathcal{L}(\varpi) = d_0 \}.$$ 

This subset of $X$ is called the open stratum. This term is justified by the following proposition.

Proposition 6. (i) The set $X^o(\mathcal{L})$ is open and nonempty.

(ii) For any $\varpi$ and $\varpi'$ in $X^o(\mathcal{L})$, $M^\mathcal{L}(\varpi) = M^\mathcal{L}(\varpi')$.

(iii) All points in $X^o(\mathcal{L})$ have the same indivisible dominant adapted one parameter subgroup of $T$. Let $\tau^o$ denote this 1-PS.

(iv) Set $P = P(\tau^o)$. There exists an $(L_{alg}P)_0 \times P^3$ orbit in $X$ such that for any $\varpi \in X^o(\mathcal{L})$ and any $g \in L_{alg}^< G$ such that $\mu^\mathcal{L}(g\varpi, \tau^o) = \|\tau^o\|d_0$, we have $g\varpi \in C^+$. 25
Proof. By the valuative criterion of openness, to prove the openness of $X^o(L)$ it is sufficient to prove the following lemma.

**Lemma 7** Let $R$ be a discrete valuation ring and set $S = \text{Spec}(R)$. Let $\eta$ denote the generic point of $S$ and let $0$ denote the special one. Let $E$ be a flagged bundle on $\mathbb{P}^1 \times S$.

Then $M^\mathcal{E}(\mathcal{E}_0) \geq M^\mathcal{E}(\mathcal{E}_\eta)$.

The Behrend’s proof of [Beh91, Proposition 7.1.3] applies here. His proof also shows the end of the proposition.

Another useful reference is [Hei08, Proposition 2]. \qed

**5 Gromov-Witten invariants and affine grassmannian**

**5.1 The homogeneous space $L_{alg}^< G/L_{alg}^< P$**

Let $P$ be a standard parabolic subgroup of $G$. Let $\mathbf{d} \in \text{Hom}(X^*(P), \mathbb{Z})$. We denote by $\text{Mor}(\mathbb{P}^1, G/P, \mathbf{d})$ the set of regular maps from $\mathbb{P}^1$ to $G/P$ of degree $\mathbf{d}$. It is empty or a quasiprojective variety. The disjoint union of the $\text{Mor}(\mathbb{P}^1, G/P, \mathbf{d})$ when $\mathbf{d}$ runs over $\text{Hom}(X^*(P), \mathbb{Z})$ is denoted by $\text{Mor}(\mathbb{P}^1, G/P)$.

Let $g \in L_{alg}^< G$. Then $g \circ \iota^{-1}$ (with notation of Section 4.3) is a regular map from $\mathbb{P}^1 \setminus \{0\}$ to $G$. By composition with the projection $G \to G/P$, one obtains a regular map from $\mathbb{P}^1 \setminus \{0\}$ to $G/P$. Since $G/P$ is proper, this maps extend to $\mathbb{P}^1$. Let $\Theta(g) \in \text{Mor}(\mathbb{P}^1, G/P)$ denotes this map. Observe that, for any $g, g' \in L_{alg}^< G$, $\Theta(g) = \Theta(g')$ is and only if $g^{-1}g \in L_{alg}^< G$. Hence we just construct an injective map

$$\Theta : L_{alg}^< G/L_{alg}^< P \quad gP_{alg}^< \quad \to \quad \text{Mor}(\mathbb{P}^1, G/P) \quad \Theta(g).$$

Fix $\gamma \in \text{Mor}(\mathbb{P}^1, G/P)$. Since any $P$-principal bundle on $\mathbb{P}^1 \setminus \{0\}$ is trivial, the restriction of $\gamma$ raises to $G$. Hence $\gamma$ belongs to the image of $\Theta$ that is surjective.

Recall that

$$L_{alg} G = \bigcup_{h \in \oplus_{\alpha \in \Delta - \Delta_p, \mathbb{Z}^\vee}^{0} L_{alg}^> G \cdot z^h(L_{alg} P)_0.$$

**Lemma 8** Let $g \in L_{alg}^< G$ and $h \in \oplus_{\alpha \in \Delta - \Delta_p, \mathbb{Z}^\vee}^{0} \simeq \text{Hom}(X^*(P), \mathbb{Z})$. Then $\Theta(gL_{alg}^< P)$ has degree $h$ if and only if $g$ belongs to $L_{alg}^> G \cdot z^h(L_{alg} P)_0$.

**Proof.** By the decomposition just before the lemma, it is sufficient to prove that if $g$ belongs to $L_{alg}^> G \cdot z^h(L_{alg} P)_0$ then $\Theta(gL_{alg}^< P)$ has degree $h$.

By Theorem 6 there exist $w' \in W$ and $h' \in X,(T \cap L^w)$ such that $g$ belongs to $B^- w' z^h L_{alg} U$. Similarly, there exist $w'' \in W$ and $h'' \in X,(T \cap L^w)$ such that $g$ belongs to $B w'' z^{h+h'} L_{alg} U$. By Lemma 4 the curve $\Theta(g) : \mathbb{P}^1 \to G/B$ associated to $g$ has degree $h + h' - h''$. Hence $\Theta(gL_{alg}^< P)$ has degree $h$. \qed

26
5.2 Gromov-Witten invariants as “degree”

Fix $w_1, w_2, w_3$ in $W^P$. With the notation of the introduction, fix also $d = \sum_{\beta \in \Delta - \Delta_P} d_\beta \sigma_{s_\beta}$ for some $d_\beta \in \mathbb{Z}_{\geq 0}$. Set $h = \sum_{\beta \in \Delta - \Delta_P} d_\beta \beta^\vee \in X_*^{ss}(T)$. Consider

$$C = L_{alg} L^\infty L_{-h} \times L w_1^{-1} B / B \times L w_2^{-1} B / B \times L w_3^{-1} B / B$$

and

$$C^+ = (L_{alg} P)_0 L_{-h} \times P w_1^{-1} B / B \times P w_2^{-1} B / B \times P w_3^{-1} B / B.$$ 

Observe that $L_{alg}^0 P$ is contained in $(L_{alg} P)_0$. In particular $C^+$ is stable by the action of $L_{alg}^0 P$. Consider on $L_{alg}^0 G \times C^+$ the action of $L_{alg}^0 P$ given by the formula $p.(g, \rho) = (gp^{-1}, p\rho)$. This action is free and the quotient set is denote by $L_{alg}^0 G \times L_{alg}^0 P C^+$. The class of $(g, \rho)$ in $L_{alg}^0 G \times L_{alg}^0 P C^+$ is denoted by $[g : \rho]$. Consider the map

$$\eta : L_{alg}^0 G \times L_{alg}^0 P C^+ \longrightarrow \mathbb{X}$$

$$[g : \rho] \longrightarrow g \rho.$$

Observe that $L_{alg}^0 G / L_{alg}^0 P$ and $L_{alg}^0 G \times L_{alg}^0 P C^+$ have no natural structure of ind-varieties and are considered in this paper as sets.

**Proposition 7** Recall the definition of $n_\beta$ from (7).

(i) If $l(w_1) + l(w_2) + l(w_3) = \dim(G/P) + \sum_{\beta \in \Delta - \Delta_P} d_\beta n_\beta$ then for $\rho \in \mathbb{X}$ sufficiently general, the fiber $\eta^{-1}(\rho)$ has cardinality $GW(\sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3}: \delta)$.

(ii) If $l(w_1) + l(w_2) + l(w_3) = \dim(G/P) + \sum_{\beta \in \Delta - \Delta_P} d_\beta n_\beta$ then for $\rho \in \mathbb{X}$ sufficiently general, the fiber $\eta^{-1}(\rho)$ is either empty or finite.

**Proof.** Let $\rho = (g, L_0, g_i B / B, g_2 B / B, g_3 B / B) \in \mathbb{X}$ with $g \in L_{alg} G$, $g_i \in G$ for $i = 1, 2$ and $3$. Since $\eta$ is $L_{alg}^0 G$-equivariant, $L_{alg}^0 G L_0$ is dense in $G$ and viewed the assumption of genericity in the proposition, we may assume that $g$ is trivial.

If $\gamma$ denote an element of $L_{alg}^0 G / L_{alg} P$ (sometimes viewed as a curve on $G/P$ using $\Theta$), we denote by $\gamma$ a representative in $L_{alg}^0$. Then $\eta^{-1}(\rho)$ identifies with

$$\{ \gamma \in L_{alg}^0 G / L_{alg} P \mid \begin{align*}
\gamma(p_i) &\in g_i B w_i / B \quad \forall i = 1, 2, 3 \\
\gamma &\in (L_{alg} P)_0 L_{-h} L_{alg}^0 G
\end{align*} \},$$

that is with

$$\{ \gamma \in L_{alg}^0 G \mid \begin{align*}
\gamma(p_i) &\in g_i B w_i P, \quad \forall i = 1, 2, 3 \\
\gamma &\in L_{alg}^0 G h(L_{alg} P)_0
\end{align*} \} / L_{alg}^0 P.$$ 

By Lemma 3 this set identifies using $\Theta$ with the set of curves $\gamma \in \text{Mor}(\mathbb{P}^1, G/P, d)$ such that $\gamma(p_i) \in g_i B w_i P / P$, for any $i = 1, 2, 3$. Now, the proposition follows from Kleiman’s theorem. \qed
5.3 Other fibers of \( \eta \)

Let \( \text{g} \) denote the base point of \( L_{\text{alg}} G/(L_{\text{alg}} P)_0 \). Since \( L_{\text{alg}}^0 P \) is contained in \((L_{\text{alg}} P)_0, L_{\text{alg}}^0 G \text{g} \) identifies with \( L_{\text{alg}} G/L_{\text{alg}} P \).

**Proposition 8** Let \( l_1, l_2 \) and \( l_3 \) in \( L \). Set \( \cap = (L_{−h}, l_1 w_1^{-1} B/B, l_2 w_2^{-1} B/B, l_3 w_3^{-1} B/B) \) in \( C \). Then the fiber \( \eta^{-1}(\cap) \) identifies with the set of \( g \text{g} \in L_{\text{alg}}^0 G/L_{\text{alg}}^0 P \) such that

(i) \( g \in z^{-h} L_{\text{alg}}^0 G z^h \cap L_{\text{alg}}^0 G \);  
(ii) \( \Theta(g \text{g})(p_i) \in l_i w_i^{-1} B w_i P/P \) for any \( i = 1, 2, 3 \).

**Proof.** Like in the proof of Proposition 7, we obtain that \( \eta^{-1}(\cap) \) identifies with

\[
\{ \tilde{\gamma} \in L_{\text{alg}}^0 G : \left\{ \tilde{\gamma}(p_i) \in l_i w_i^{-1} B w_i P, \quad \forall i = 1, 2, 3 \right\} \}/L_{\text{alg}}^0 P.
\]

In particular, \( \tilde{\gamma} \text{g} \) belongs to \( L_{\text{alg}}^0 G \text{g} \) and to \( z^{-h} L_{\text{alg}}^0 G z^h \). Since \( L_{\text{alg}}^0 G, z^{-h} L_{\text{alg}}^0 G z^h \), and the stabilizer of \( \text{g} \) contain \( T \) the intersection \( L_{\text{alg}}^0 G \text{g} \cap z^{-h} L_{\text{alg}}^0 G z^h \) is equal to \( (L_{\text{alg}}^0 G \cap z^{-h} L_{\text{alg}}^0 G z^h) \text{g} \). The proposition follows. \( \square \)

6 Description of the GIT-cone

6.1 Satisfied inequalities

**Lemma 9** Let \( (\lambda_1, \lambda_2, \lambda_3) \in (X^*(T)_{\mathbb{Q}})^3 \) and \( \tilde{\lambda} \in \mathbb{Z}_{>0} \) such that \( \frac{\lambda_1}{\tilde{\lambda}}, \frac{\lambda_2}{\tilde{\lambda}} \) and \( \frac{\lambda_3}{\tilde{\lambda}} \) belong to the alcove \( \mathcal{A}^* \). Let \( \tau \) be a dominant one parameter subgroup of \( T \) and set \( P = P(\tau) \). Let \( w_1, w_2, w_3 \) in \( W^P \) and let \( \tilde{d} = \sum_{\beta \in \Delta_{-\Delta}} d_\beta \sigma^\beta_s \) for some \( d_\beta \in \mathbb{Z}_{\geq 0} \). Set \( h = \sum_{\beta \in \Delta_{-\Delta}} d_\beta \beta^\vee \in X_*(T) \). Assume that

\[
GW(w_1, w_2, w_3; d) \neq 0.
\]

If \( (\lambda_1, \lambda_2, \lambda_3, \tilde{\lambda}) \in C^{\text{ss}}(X) \) then

\[
\langle w_1 \tau, \lambda_1 \rangle + \langle w_2 \tau, \lambda_2 \rangle + \langle w_3 \tau, \lambda_3 \rangle \leq \langle \tilde{\lambda} \tau, h \rangle. \tag{31}
\]

**Proof.** Consider

\[
C^+ = (L_{\text{alg}} P)_0 L_{-h} \times P w_1^{-1} B/B \times P w_2^{-1} B/B \times P w_3^{-1} B/B,
\]

and the map

\[
\eta : L_{\text{alg}}^0 G \times L_{\text{alg}}^0 P C^+ \rightarrow X,
\]

\[
[g : \cap] \mapsto g \cap.
\]

By Proposition 7 and Lemma 6 there exists a numerically semistable point in the image of \( \eta \). Then there exists a numerically semistable point \( \cap \) in \( C^+ \). We deduce that \( \mu^C(\cap, \tau) \leq 0 \). By Lemma 4 this inequality is equivalent to the inequality to prove. \( \square \)
6.2 A first description of $\mathcal{C}^{\text{nss}}(X)$

We first reprove Telemann-Woodward’s Theorem 2 in our context.

**Lemma 10** Let $(\lambda_1, \lambda_2, \lambda_3) \in (X^e(T)_Q^+)^3$ and $\hat{l} \in \mathbb{Z}_{\geq 0}$ such that $\frac{1}{\hat{l}} \not\in \mathbb{Z}^+$ and $\frac{1}{\hat{l}^2}$ belong to the alcove $\mathcal{A}^*$. Then $(\lambda_1, \lambda_2, \lambda_3, \hat{l}) \in \mathcal{C}^{\text{nss}}(X)$ if and only if

$$
(w_1 \varpi_{\beta^\vee}, \lambda_1) + (w_2 \varpi_{\beta^\vee}, \lambda_2) + (w_3 \varpi_{\beta^\vee}, \lambda_3) \leq \frac{2}{(\beta, \beta)} \hat{l} d,
$$

(32)

for any simple root $\beta$, any nonnegative integer $d$ and any $(w_1, w_2, w_3) \in (W_{P_{\beta}})^3$ such that

$$
GW(w_1, w_2, w_3; d\sigma_{s_{\beta}}^*) = 1.
$$

(33)

**Proof.** If $(\lambda_1, \lambda_2, \lambda_3, \hat{l}) \in \mathcal{C}^{\text{nss}}(X)$ then the inequalities are satisfied by Lemma 9. Conversely, assume that $(\lambda_1, \lambda_2, \lambda_3, \hat{l}) \not\in \mathcal{C}^{\text{nss}}(X)$ that is that $X^{\text{nss}}(L)$ is empty. Consider the open stratum $X^e(L)$ and $d_0$ the common value of $M^e(\cdot)$ for $\varpi$ in $X^e(L)$. Let $\tau_0$, $P = P(\tau_0)$ and $C^+$ be like in Proposition 6. Write

$$
C = (L_{\text{alg}} L_{\text{ss}}, L_{\text{h}}, L_{\text{w}_1^{-1} B/B}, L_{\text{w}_2^{-1} B/B}, L_{\text{w}_3^{-1} B/B}),
$$

with usual notation. Write $h = \sum_{\beta \in \Delta - \Delta_{P}} d_{\beta} \beta_{\beta^\vee}$ and set $d = \sum_{\beta \in \Delta - \Delta_{P}} d_{\beta} \sigma_{s_{\beta}}^*.$

Consider the map

$$
\eta : L_{\text{alg}}^{\infty} G \times L_{\text{alg}}^{\infty} P C^+ \to X^e \quad [g : \varpi] \mapsto g \varpi.
$$

By Proposition 5 for any $\varpi \in X^e(L)$, the fiber $\eta^{-1}(\varpi)$ is not empty. By Proposition 5 this fiber is reduced to one point. Since $X^e(L)$ is open, Proposition 7 implies that

$$
GW(w_1, w_2, w_3; d) = 1.
$$

Lemma 9 shows that inequality (31) is satisfied for any $\tau$ such that $P = P(\tau)$ and any point in $\mathcal{C}^{\text{nss}}(X)$. By construction the $X^e(L) \cap C^+$ is not empty. Fix $\varpi$ in it. Then $\mu_{P}(\varpi, \tau_0) = d_0 > 0$. Hence Lemma 1 implies that inequality (31) for $\tau = \tau_0$ is not satisfied by $L$.

With Lemma 9 we just proved that a point belongs to $\mathcal{C}^{\text{nss}}(X)$ if and only if it satisfies the inequalities (31) for any $\tau$, $h$ and $w_i$’s such that $GW(w_1, w_2, w_3; d) = 1$. It remains to prove that the inequalities coming from nonmaximal parabolic subgroups are redundant. Consider such an inequality (31) associated to some non-maximal standard parabolic subgroup $P$, some $\tau \in X_*(T)$, and $w_1, w_2, w_3$ and $h$. Dually, we have to prove that this inequality (31) does not generate an extremal ray of the dual cone of $\mathcal{C}^{\text{nss}}(X)$. By Lemma 9, inequality (31) holds for any $\tau' \in X_*(T)$ such that $P = P(\tau')$. But, the set $\tau'$ such that $P = P(\tau')$ generate an open cone of dimension two in $X_*(T)_Q$ and inequality (31) depends linearly on $\tau'$. Hence inequality (31) cannot be extremal.
6.3 End of the proof of Theorem 7

Proof. It remains to prove that if $(\lambda_1, \lambda_2, \lambda_3, \ell) \not\in C^{\text{nor}}(X)$, then there exists an inequality \( \mathcal{V} \) that satisfies condition (25) and that is not fulfilled by this point. Consider the open strata $X^o(\mathcal{L})$. Let $\tau_0, P = P(\tau_0)$ and $C^+$ be like in Proposition 3. Let $(L_{-h}, w_1^{-1}B/B, w_2^{-1}B/B, w_3^{-1}B/B) \in C^+$ with usual notation. Let $h_{PW} \in X_\nu(T)$ be the Peterson-Wooldward lifting of $h$. Let $\rhd \in C^+ \cap X^o(\mathcal{L})$ and set $F = \lim_{t \to 0} \tau(t) \rhd$. By Proposition 3 $F$ is numerically semistable for the group $L_{\text{alg}}^0$ relatively to the line bundle $\mathcal{L} \otimes -\mu^\mathcal{L}(\rhd, \tau_0)_{\mathcal{L}, \tau_0}$. In particular, $C^{\text{nor}}(\mathcal{L} \otimes -\mu^\mathcal{L}(\rhd, \tau_0)_{\mathcal{L}, \tau_0}, L_{\text{alg}}^0 \mathcal{L})$ is not empty and open by Lemma 6. Now, Proposition 4 implies that for general $\rhd$ in $C$, we have $\|\tau_0\|M^\mathcal{L}(\rhd) = \mu^\mathcal{L}(\rhd, \tau_0)$. Then, Proposition 4 shows that $\eta^{-1}(\rhd)$ is one point for general $\rhd$ in $C$.

The Peterson-Wooldward lifting $h_{PW}$ has the property that the $L_{\text{alg}}^0$-orbit of $L_{-h_{PW}}$ is dense in $L_{\text{alg}}L_{-h}$. Since $\eta$ is equivariant, we deduce that for general $l_1, l_2$ and $l_3$ in $L$ the fiber $\eta^{-1}(L_{-h_{PW}}, l_1w_1^{-1}B/B, l_2w_2^{-1}B/B, l_3w_3^{-1}B/B)$ is one point.

Let $P^-$ denote the parabolic subgroup containing $T$ and opposite to $P$. Consider the Lie algebra $\text{Lie}(R^u(P^-))$ of the unipotent radical of $P^-$. It is a $L$-module. Consider its decomposition in weight spaces under the action of $Z$: \[
\text{Lie}(R^u(P^-)) = \bigoplus_{\chi \in X^*(Z)} \text{Lie}(R^u(P^-))_{\chi}. \tag{34}\]

It is known that each $\text{Lie}(R^u(P^-))_{\chi}$ is an irreducible $L$-module.

Consider the open $P^-$-orbit $\Omega$ in $G/P$. It is stable by the action of $L$ and isomorphic as a $L$-variety to $\text{Lie}(R^u(P^-))$. Let us fix such an isomorphism $\zeta : \text{Lie}(R^u(P^-)) \longrightarrow \Omega$. For each $i = 1, 2, 3$, set $V_i = \zeta^{-1}(\Omega \cap w_i^{-1}Bw_iP/P)$. It is well known that $V_i$ is a linear subspace of $\text{Lie}(R^u(P^-))$ stable by $Z$. Then \[
V_i = \bigoplus_{\chi \in X^*(Z)} V_{i,\chi}, \text{ where } V_{i,\chi} = V_i \cap \text{Lie}(R^u(P^-))_{\chi}. \tag{35}\]

Set $K_h = z^{-h_{PW}}L_{\text{alg}}^0Gz^{-h_{PW}} \cap L_{\text{alg}}^0G$. It is a finite dimensional connected algebraic group containing $T$. Consider \[
\mathcal{M}_h = K_h/(L_{\text{alg}}^0P \cap K_h). \]

Moreover, $\mathcal{M}$ contains $\mathcal{M}_h^\circ = K_h \cap L_{\text{alg}}R^u(P^-)$ as an open subset. But $\mathcal{M}_h$ is contained in $L_{\text{alg}}G/L_{\text{alg}}^0P$ and any $m \in \mathcal{M}_h$ can be seen as a regular map $\Theta(m)$ from $\mathbb{P}^1$ to $G/P$. If $m$ belongs to $\mathcal{M}_h^\circ$, then $\Theta(m)(\mathbb{P}^1 - \{0\})$ is contained in $\Omega$. Hence $\mathcal{M}_h^\circ$ can be seen as a set of polynomial functions from $\mathbb{P}^1 - \{0\}$ to $\Omega$. Composing with $\zeta$ and identifying $\mathbb{P}^1 - \{0\}$ with $\mathbb{C}$ (with the coordinate $z^{-1}$), we get an embedding of $\mathcal{M}_h^\circ$ in $\text{Mor}(\mathbb{C}, \text{Lie}(R^u(P^-)))$. For each root $\alpha \in \Phi(G/P)$,
let us fix a nonzero element $\xi_{-\alpha} \in Lie(R^u(P^-))_{-\alpha}$ of weight $-\alpha$. The roots of $z^{-h_{PW}}L_{alg}^uGz^{h_{PW}}$ are the images of the roots of $L_{alg}^uG$ by $-h_{PW}$ viewed as an element of the affine Weyl group (see Section 2.2). The root $\alpha + n\delta$ is a root of $K_h$ if and only if $-\langle h_{PW}, \alpha \rangle \leq n \leq 0$. Hence

$$M_\circ_h := \left\{ \sum_{\alpha \in \Phi(G/P)} P_\alpha \xi_{-\alpha} : P_\alpha \in \mathbb{C}[z^{-1}] \text{ and } \deg(P_\alpha) \leq \langle h_{PW}, \alpha \rangle \right\}$$

has a natural structure of a complex vector space. Note that, by convention, $\deg(0) = -\infty$.

Consider now

$$M_{h,\chi}^\circ = \left\{ \sum_{\alpha \in \Phi(G/P,\chi)} P_\alpha \xi_{-\alpha} : P_\alpha \in \mathbb{C}[z^{-1}] \text{ and } \deg(P_\alpha) \leq \langle h_{PW}, \alpha \rangle \right\}. \quad (36)$$

Then $M^\circ$ is a product:

$$M_h^\circ = \bigoplus_{\chi \in X^*(Z)} M_{h,\chi}^\circ. \quad (37)$$

For any linear subspace $W$ in $Lie(R^u(P^-))$ and any $p \in \mathbb{P}^1 - \{0\}$, set

$$M_h^\circ(p, W) := \{ m \in M_h^\circ : m(p) \in W \}.$$

Since $m \mapsto m(p)$ is linear, $M_h^\circ(p, W)$ is a linear subspace and

$$\dim(M_h^\circ) - \dim(M_h^\circ(p, W)) \leq \dim(Lie(R^u(P^-))) - \dim(W). \quad (38)$$

Similarly, for any $\chi \in X^*(Z)$, for any linear subspace $W$ in $Lie(R^u(P^-))_\chi$ and any $p \in \mathbb{P}^1 - \{0\}$, set

$$M_{h,\chi}^\circ(p, W) := \{ m \in M_{h,\chi}^\circ : m(p) \in W \}.$$

Then

$$\dim(M_{h,\chi}^\circ) - \dim(M_{h,\chi}^\circ(p, W)) \leq \dim(Lie(R^u(P^-))_\chi) - \dim(W). \quad (39)$$

Consider

$$\mathcal{X} = \{(l_1, l_2, l_3, m) \in L^3 \times M_h^\circ : \forall i = 1, 2, 3 \quad m(p_i) \in l_iV_i\}, \quad (40)$$

and the two projections

\[ \begin{array}{ccc}
\mathcal{X} & \xrightarrow{p} & L^3 \\
\downarrow q & & \downarrow M_h^\circ
\end{array} \]
Fix any \( l_1, l_2 \) and \( l_3 \) in \( L \). Then \( m \in q(p^{-1}(l_1, l_2, l_3)) \) if and only if, for any \( i = 1, 2, 3 \), \( m(p_i) \) belongs to \( l_i V_i \). In other words

\[
q(p^{-1}(l_1, l_2, l_3)) = \bigcap_{i=1}^{3} M^0_h(p_i, l_i V_i).
\]

The decomposition \((34)\) is respected by the action of \( L \), the subspaces \( V_i \) (see \((35)\)) and the vector space \( M^0_h \) (see \((37)\)). Hence

\[
q(p^{-1}(l_1, l_2, l_3)) = \bigoplus_{\chi \in X^*(Z)} \bigcap_{i=1}^{3} M^0_{h,\chi}(p_i, l_i V_i,\chi).
\]

Proposition \(8\) implies that for general \( l_1, l_2 \) and \( l_3 \) in \( L \), \( p^{-1}(l_1, l_2, l_3) \) is one point. Then, for any \( \chi \in X^*(Z) \),

\[
\sum_{i=1}^{3} \dim(M^0_{h,\chi}) - \dim(M^0_{h,\chi}(p_i, l_i V_i,\chi)) \geq \dim(M^0_{h,\chi}).
\]

Combining with \((39)\), we obtain

\[
3 \dim(Lie(R^a(P^\chi))) - \sum_{i=1}^{3} \dim(V_i,\chi) \geq \dim(M^0_{h,\chi}). \tag{41}
\]

From \((36)\), we deduce

\[
\dim(M^0_{h,\chi}) \geq \sum_{\alpha \in \Phi(G/P,\chi)} \left( \langle h_{PW}, \alpha \rangle + 1 \right), \tag{42}
\]

and

\[
3 \dim(Lie(R^a(P^\chi))) - \sum_{i=1}^{3} \dim(V_i,\chi) \geq \sum_{\alpha \in \Phi(G/P,\chi)} \left( \langle h_{PW}, \alpha \rangle + 1 \right). \tag{43}
\]

By summing inequalities \((43)\), when \( \chi \) runs in \( X^*(Z) \), we find

\[
3 \dim(G/P) - \sum_{i=1}^{3} \dim(V_i) \geq \dim(G/P) + \sum_{\alpha \in \Phi(G/P)} \langle h_{PW}, \alpha \rangle.
\]

Since \( GW(w_1, w_2, w_3; d) = 1 \), this inequality is actually an equality. Hence each inequalities \((43)\) is an equality. These equalities are readily equivalent to condition \((25)\).

**Remark.** The above proof shows that inequality \((42)\) is an equality, in the setting of the theorem. With \((36)\) this implies that

\[
\forall \alpha \in \Phi(G/P,\chi) \quad \langle h_{PW}, \alpha \rangle \geq -1. \tag{44}
\]

It is a natural question to ask if inequality \((44)\) is satisfied for any maximal \( P \) (associated to the simple root \( \beta \)) and any \( h \in \mathbb{Z}_{\geq 0}^{\beta^\vee} \).

32
References

[Bea96] Arnaud Beauville, *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993) (Ramat Gan), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., 1996, pp. 75–96.

[Beh91] Kai A. Behrend, *The lefschetz trace formula for the moduli stack of principal bundles stacks*, Ph.D. thesis, University of California, Berkeley, 1991.

[BK06] Prakash Belkale and Shrawan Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. 166 (2006), no. 1, 185–228.

[BK13] ———, *The multiplicative eigenvalue problem and deformed quantum cohomology*, arXiv:1310.3191, October 2013.

[Bou05] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 7–9*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2005, Translated from the 1975 and 1982 French originals by Andrew Pressley.

[Hei08] Jochen Heinloth, *Semistable reduction for G-bundles on curves*, J. Algebraic Geom. 17 (2008), no. 1, 167–183.

[HS10] Jochen Heinloth and Alexander H. W. Schmitt, *The cohomology rings of moduli stacks of principal bundles over curves*, Doc. Math. 15 (2010), 423–488.

[Kir84] Frances Clare Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, Princeton, NJ, 1984.

[Kum02] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[MFK94] David Mumford, John Fogarty, and Frances Kirwan, *Geometric invariant theory*, 3d ed., Springer Verlag, New York, 1994.

[MW98] E. Meinrenken and C. Woodward, *Hamiltonian loop group actions and Verlinde factorization*, J. Differential Geom. 50 (1998), no. 3, 417–469.

[Nes84] Linda Ness, *A stratification of the null cone via the moment map*, Amer. Jour. of Math. 106 (1984), 1281–1325.

[Pet97] Dale Peterson, *Quantum cohomology of G/P*, Lecture Course, M.I.T. (Sprint Terms, 1997).
[Res10] Nicolas Ressayre, *Geometric invariant theory and generalized eigenvalue problem*, Invent. Math. 180 (2010), 389–441.

[Res13] ———, *Homepage*, October 2013.

[RR84] S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. J. (2) 36 (1984), no. 2, 269–291.

[RR11] Nicolas Ressayre and Edward Richmond, *Branching Schubert calculus and the Belkale-Kumar product on cohomology*, Proc. Amer. Math. Soc. 139 (2011), 835–848.

[TW03] Constantin Teleman and Christopher Woodward, *Parabolic bundles, products of conjugacy classes and Gromov-Witten invariants*, Ann. Inst. Fourier 53 (2003), no. 3, 713–748.

[Woo05] Christopher T. Woodward, *On D. Peterson's comparison formula for Gromov-Witten invariants of G/P*, Proc. Amer. Math. Soc. 133 (2005), no. 6, 1601–1609 (electronic).