An observational criterion to look for an inspiral in a non-Kerr spacetime

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Abstract. In this short article we present a useful observational tool for gravitational wave detectors. More specifically, if we are looking for extreme-mass-ratio inspiraling objects in a non-Kerr spacetime, we could exploit the consequences of the KAM and the Poincaré-Birkhoff theorem which predicts plateaus in the ratio of frequencies \(f_\rho/f_z\), that are related to a generic geodesic orbit in such a spacetime, as a function of the initial conditions of the orbit itself. While both these frequencies are changing under radiation reaction, their ratio is expected to stay stationary if it passes through such a plateau. Therefore, if detectors are able to discern the fundamental frequencies due to \(\rho\) and \(z\) oscillations of the orbit, they could in principle detect the non-Kerr-ness of the spacetime involved, just by monitoring the ratio of these two frequencies.

1. Introduction

Gair et al. [1] explored the geodesic bound orbits in a specific member of the family of spacetimes, constructed by Manko and Novikov [2] in 1990, and have shown that they lie either in a region which is not very near the center of spacetime and the corresponding trajectories in phase space look very much like the ones in a Kerr spacetime (with respect to their frequency spectra), or in a region near the center, where the orbits there exhibit chaotic behavior.

We have thoroughly analyzed the same kind of orbits in the same background and have shown that although the general picture is the one described by Gair et al, there are some details about the orbital characteristics that could be suitably exploited in a data analysis scheme to distinguish this spacetime from a Kerr spacetime. More specifically, we know from KAM (Kolmogorov-Arnold-Moser) theorem about dynamical systems, that a perturbed integrable dynamical system has most of its phase-space tori deformed but not destroyed. As a consequence the corresponding orbits are quasi-periodic and on a surface of section they define simple closed KAM curves. On the other hand, the resonant tori, that is the ones that are characterized by commensurate ratio of frequencies, disintegrate and the corresponding sections on a surface of section form a chain of islands as predicted by the Poincaré-Birkhoff theorem.

The geodesic orbits in a stationary axisymmetric metric are characterized by three fundamental frequencies \(f_\phi, f_\rho\), and \(f_z\); the azimuthal frequency of the orbit and the frequencies at which the orbit oscillates with respect to \(\rho\) and \(z\) coordinates. These frequencies will, in
principle, show up in the corresponding gravitational wave signal emitted when a low mass body orbits around a massive central object with a gravitational field described by such a metric. The Manko-Novikov spacetime, which can be considered as a perturbed Kerr axisymmetric spacetime (actually that was the idea behind the construction of such a spacetime) since by suitable tuning of its parameters it can be turned into a Kerr metric, has most of its 4-dimensional tori deformed with respect to the corresponding Kerr ones leading to simple closed invariant curves (KAM curves), while the resonant ones form Birkhoff chains of islands. The sequence of KAM curves correspond to continuously different irrational ratios of frequencies $f_\rho$ and $f_z$, while the whole region of initial conditions inside a specific Birkhoff chain of islands corresponds to a specific commensurate ratio of frequencies.

In a real inspiraling binary system with extreme ratio of masses, that happens to be described by a Kerr-like, but not exactly Kerr, spacetime background (the Manko-Novikov spacetime is merely an example of such a generic case), the orbits will evolve adiabatically under radiation reaction following non-geodesic trajectories and therefore both fundamental frequencies of $\rho$ and $z$ (namely $f_\rho$ and $f_z$) will evolve continuously. As an orbit evolves, the corresponding trajectory in phase space may pass through a Birkhoff island. Then the ratio between $f_\rho$ and $f_z$ remains constant, while the frequencies continue to change adiabatically. This will be a clear “smoking gun” for a non-Kerr metric that could be then correlated with other orbital characteristics related to a non-Kerr spacetime, like the ones suggested in [1], to make sure that the spacetime we are monitoring through gravitational waves can not be described by a Kerr black-hole metric. Furthermore the quantitative characteristics from the suggested non-Kerr criterion will offer complementary information for more accurate determination of the specific metric itself.

In the next section we present the case that we have analyzed in order to exhibit our idea, and we give further quantitative details of the observational non-Kerr criterion. In a forthcoming paper [3] we plan to go to a deeper investigation, by exploring the quantitative details of the plateaus of the frequency ratios in a realistic orbital evolution.

2. The Manko-Novikov metric: an example of a Non-Kerr case

We have chosen to scrutinize the same non-Kerr ‘bumpy’ spacetime that Gair et al [1] had used to explore the geodesic orbital characteristics when they tried to find observational signatures that distinguish such a spacetime from a Kerr spacetime. Namely, we have used the Manko-Novikov spacetime [2] which has been built as a non-linear superposition of Weyl solutions, with the Kerr solution. More specifically, we have used a member of the family of Manko-Novikov solutions, which is described by one more parameter than the corresponding Kerr metric, the $q$ parameter which describes the deviation of the quadrupole moment of the new solution from the Kerr metric. Thus, this metric turns into an exact Kerr metric when the $q$ parameter becomes zero. It should be noted though, that apart from the quadrupole moment, the specific Manko-Novikov solution has all higher order moments different from the corresponding Kerr moments as well, when $q \neq 0$. A detailed description of the metric can be found in [1].

We have integrated a wide range of geodesic orbits in such a spacetime numerically, by following a Runge-Kutta 7th-8th order numerical scheme. In our thorough analysis we have observed that within the outer allowed region of motion (see [1]) besides the KAM curves on the surface of section, there are chain-like regions of narrow islands between specific KAM curves. These islands are characterized by a constant ratio of fundamental frequencies that are related to the oscillations along the $\rho$ and $z$ axes, while this ratio is continuously and monotonically changing as we move from one KAM curve to the next one (see Fig. 2) away from the main island’s center (see Fig. 1), which is the central point of the KAM curves on the surface of sections and corresponds to the degenerate curve where $\rho = \text{const}$. The existence of such chains of islands is expected from the Poincaré-Birkhoff theorem in perturbed integrable dynamical
systems. They rise from the commensurate tori which are destroyed when one departs from an integral system (in our case the Kerr metric).

![Figure 1](image_url)

**Figure 1.** A sequence of invariant curves on the surface of section determined by intersections of the plane $z = 0$ by phase-space trajectories of geodesic orbits in a Manko-Novikov metric with $q = 0.95$, $a_n = 0$ for all $n > 2$, and $\chi = 0.9$; these are actually the parameters used by [1] in their plots. All curves correspond to orbits with the same physical parameters $E = 0.95$ and $L_z = 3M$, but they differ with respect to their initial conditions. Most of the curves formed are closed KAM curves, similar to the corresponding ones in a Kerr metric. On the other hand there is a number of islands formed between closed KAM curves, that correspond to resonant tori; for example, the inner triplet of islands corresponds to a resonance 1:3, while the outer doublet of islands corresponds to a resonance 1:2. The $\rho$ coordinate is measured in units of $M$, while the $\dot{\rho}$ coordinate is dimensionless.

If a geodesic orbit happens to lie in such a chain of islands, it will incorporate in its spectrum two fundamental frequencies which form a ratio of integers. Moreover this will be true for a finite range of initial conditions, even though the fundamental frequencies themselves vary continuously from one set of initial conditions to another. The locking of the ratio of frequencies does not hold for geodesics that belong on KAM curves. Thus, by monitoring the gravitational waves emitted by a binary with an extreme ratio of masses we could in principle discern the fundamental frequencies $f_\rho, f_z$, in the spectrum of its gravitational wave signal and by following their adiabatic evolution through gravitational radiation, look for plateaus in their ratio. These plateaus would then be a true “smoking gun” for observing a non-Kerr metric.

Practically, in order to locate a chain of islands on a surface of section we have used the
so-called rotation number (for a review see [4]) which is defined by the following relation

$$\nu = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\Delta \theta_i}{2\pi},$$  \hspace{1cm} (1)

where $\Delta \theta_i$ is the angle formed by the $i$-th and $(i+1)$-th successive cross-sections of the trajectory in phase space on the surface of section, with respect to the center of the main island. The rotation number is nothing but the ratio of fundamental frequencies $(f_\rho/f_z)_{\text{mod}1}$. Thus, by acquiring a large number of sections we may be able to measure sufficiently accurately the ratio of frequencies and determine if we are close to commensurate ratio by approaching a fraction of small integers like the 1:3. This gives us a very useful numerical tool to search for a chain of islands. The rotation number remains stationary as long as our initial conditions cross an island and it varies when we cross its border and enter into the region of KAM curves.

Figure 2. This is the rotation number $\nu_\theta = 1 - \nu$, where $\nu$ was defined in (1), as a function of $\rho(0)$ (the rest of initial conditions are $\dot{\rho}(0) = 0$, $z(0) = 0$). The plateau showing up at $\nu_\theta = 2/3$ corresponds to the chain of islands with ratio of frequencies $(f_\rho/f_z) = 1/3$ (see Figure 1). A smaller one at ratio $(f_\rho/f_z) = 1/2$ is not visible. The inlet shows a magnification of the plateau.

In order to exploit quantitatively the time interval that the ratio of frequencies remains constant as the orbit spirals inwards due to radiation reaction to gain information about the spacetime on which the orbit evolves, we need to evolve the orbit adiabatically and measure the effective time interval during which the orbit stays within an island of a disintegrated commensurate torus. This will be analyzed in a forthcoming paper [3].
3. General remarks and future analysis

There is a number of issues related with the potentiality of applying our proposed criterion in real physical extreme-mass-ratio binary systems. Firstly, not all orbits are expected to evolve in a way that passing through an island of constant ratio of frequencies will take place. Then, even if it passes through such a chain of islands it may happen that it misses the plateau of constant frequencies due to fast evolution of the orbit. Finally, there is no way to evolve accurately the orbit by means of radiation reaction, and the only possibility of analyzing the situation is to appeal on approximate calculations. Such an approach is expected to give us only a crude estimate of the duration of an effective plateau. All these issues will be further explored in a forthcoming paper [3].

Although our analysis is based on a specific analytic axisymmetric solution of the vacuum Einstein equations, the existence of a plateau in the evolution of the ratio of frequencies should be a generic property of all spacetimes that deviate slightly from a Kerr metric. Therefore the observational criterion of a non-Kerr metric that we suggest could be widely used.

In order to be helpful observationally, the analysis of the subject discussed here should explore as well the accuracy of discerning the existence of a plateau in the ratio of frequencies, while all frequencies are drifting slowly. This is a technical issue which we plan to investigate in the future. In [3], we present a crude quantitative analysis of the time interval of the effective plateau as a function of the masses of the binary, for a few interesting and indicative cases of orbits.

Acknowledgments

Acknowledgments T. A. Apostolatos would like to thank Kip Thorne for hospitality in Caltech during the spring of 2006 where he had the first contact with the strange Kerr-like behavior of orbits in a non-Kerr metric, as well as Chao Li and Ilya Mandel for useful discussions about the subject. This work was supported by the research funding program “Kapodistrias” with Grant No 70/4/7672. G. Lukes-Gerakopoulos was supported by the Greek Foundation of State Scholarships (IKY).

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