A class of gradings of simple Lie algebras

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Abstract. In this paper we give a classification of parabolic subalgebras of simple Lie algebras over $\mathbb{C}$ that satisfy two properties. The first property is Lynch’s sufficient condition for the vanishing of certain Lie algebra cohomology spaces for generalized Whittaker modules associated with the parabolic subalgebra and the second is that the moment map of the cotangent bundle of the corresponding generalized flag variety be birational onto its image. We will call this condition the moment map condition.

1. Introduction

The purpose of this paper is to give a classification of parabolic subalgebras of simple Lie algebras over $\mathbb{C}$ that satisfy two properties. The first property is Lynch’s sufficient condition for the vanishing of certain Lie algebra cohomology spaces for generalized Whittaker modules associated with the parabolic subalgebra and the second is that the moment map of the cotangent bundle of the corresponding generalized flag variety be birational onto its image. We will call this condition the moment map condition. Associated to each parabolic subalgebra of a simple Lie algebra $\mathfrak{g}$ is a $\mathbb{Z}$-grading and to each $\mathbb{Z}$-grading of $\mathfrak{g}$ corresponds a parabolic subalgebra. The first condition is that the parabolic subalgebra has a Richardson element in the first graded part (where the grading is the one associated to the parabolic subalgebra).

If $G$ is a semi-simple Lie group over $\mathbb{R}$ and if $P$ is a parabolic subgroup of $G$ such that the intersection of the complexification of its Lie algebra intersected with each simple factor of the complexification of $\text{Lie}(G)$ satisfies the two conditions then one can prove holomorphic continuation of Jacquet integrals and a variant of a multiplicity one theorem for degenerate principal series associated with $P$ ([14], [15]). The full classification corresponding to the first condition was the subject of our joint paper [3] where we classified the so-called “nice” parabolic subalgebras of simple Lie algebras over $\mathbb{C}$. Thus the point of this paper is to list the elements of the list in [3] that satisfy the second condition. The second condition is just the assertion that the stabilizer of a Richardson element in the nilradical of the parabolic subgroup corresponding to the parabolic subalgebra of the adjoint group is the same as the stabilizer in the adjoint group. In the case of type $A_n$ this

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condition is automatic (cf. [6]) thus nice implies both conditions. For the other classical groups the moment map condition is not automatically satisfied. One especially striking aspect of this classification is that the exceptional groups are “better behaved” than the classical groups. Indeed, except for the case of $E_7$ every nice parabolic subalgebra of an exceptional simple Lie algebra also satisfies the moment map condition. In the case of $E_7$ there are 3 (out of the 29) that do not.

One can argue that the results of this paper are for all practical purposes in the literature. Indeed, most of our work involves the classical groups and amounts to explaining how certain results of Hesselink [6] (which he basically attributes to [13]) apply. However, if one is not an expert in the subject then the determination of whether a parabolic subgroup satisfies the conditions of the holomorphic continuation and multiplicity one theorem would involve a serious effort in a field only peripherally related to the desired application. We therefore felt that there was value in placing the necessary results in a short paper that could be understood with only the knowledge of the essentials of algebraic group theory and to give an easily applied listing of the pertinent parabolic subalgebras.

The paper is organized as follows: The full classification is described in the next section. In section 3 we discuss the moment map condition and consider the stabilizer subgroup in $P$ of a Richardson element. Section 4 treats the proof of the theorem in the classical case. Following that, the proofs for the exceptional cases are found in section 5. In the appendix we list all parabolic subalgebras of the exceptional Lie algebras where there is a Richardson element in the first graded part and such that the moment map condition is satisfied.

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2. Statements of the results

If not specified otherwise, $g$ will denote a simple Lie algebra over the complex numbers. Fix a Borel subalgebra $b$ in $g$, let $\mathfrak{h} \subset b$ be a Cartan subalgebra of $g$. We always use the Bourbaki-numbering of simple roots.

Let $p \subset g$ be a parabolic subalgebra, $p = m \oplus u$ (where $m$ is a Levi factor and $u$ the corresponding nilpotent radical of $p$). After conjugation we can assume that $p$ contains the chosen Borel subalgebra and $m \supset \mathfrak{h}$. If $b$ has been fixed then we will say that $p$ is standard if $p \supset b$ from now on. In particular, if $p$ is standard then it is given by a subset of $\Delta$, namely the simple roots such that both root spaces $g_{\pm \alpha}$ belong to the Levi factor of $p$.

Thus such a parabolic subalgebra is described by an $n$-tuple, $(u_1, ..., u_n)$ in $\{0,1\}^n$: ones correspond to simple roots with root spaces not in $m$. Equivalently, a parabolic subalgebra is given by a coloring of the Dynkin diagram of the Lie algebra: a black (colored) node corresponds to a simple root whose root space belongs to $m$. Here, one has to be very careful since there exist different notations. Our choice was motivated by the coloring for Satake diagrams. Let $(u_1, ..., u_n)$ define the parabolic subalgebra $p$ and and $H \in \mathfrak{h}$ be defined by $\alpha_i(H) = u_i$. If we set $g_i = \{x \in g | [H,x] = ix\}$ then $p = \sum_{i \geq 0} g_i$.

2.1. Results in the classical cases. As is usual, we will refer to the simple Lie algebras of type $A_n, B_n, C_n, D_n$ as the classical Lie algebras and the remaining five simple Lie algebras will be called exceptional. We realize the classical Lie
algebras as subalgebras of $\mathfrak{gl}_N$ for $N = n + 1, 2n + 1, 2n, 2n$ respectively. With $\mathfrak{A}_n$ the trace zero matrices, $\mathfrak{B}_n, \mathfrak{D}_n$ the orthogonal Lie algebra of the symmetric form with matrix with all entries 0 except for those on the skew diagonal which are 1 and $\mathfrak{C}_n$ the symplectic Lie algebra for the symplectic form with matrix whose only nonzero entries are skew diagonal and the first $n$ are 1 and the last $n$ are $-1$. With this realization we take as our choice of Borel subalgebra the intersection of the corresponding Lie algebra with the upper triangular matrices in $\mathfrak{gl}_N$. We will call a parabolic subalgebra that contains this Borel subalgebra standard. If $\mathfrak{p}$ is a standard parabolic subalgebra then we refer to the Levi factor that contains the diagonal Cartan subalgebra, $\mathfrak{h}$, by $\mathfrak{m}$ and call it the standard Levi factor. One has to be careful in the case of $\mathfrak{D}_n$ since the two parabolic subalgebras with exactly one of the last two simple roots $\alpha_n, \alpha_{n-1}$ as roots of the Levi factor are conjugate under an outer automorphism. To avoid ambiguity for $n \geq 5$, we will always assume in this case that $\alpha_{n-1}$ is the root of the Levi factor.

Thus for all classical Lie algebras the standard Levi factor is then in diagonal block form given by a sequence of square matrices on the diagonal. We denote it by $\underline{d} = (d_1, \ldots, d_m)$. For the orthogonal and symplectic Lie algebras, these sequences are palindromic. In that case we write $\underline{d} = (d_1, \ldots, d_r, d_r, \ldots, d_1)$ or $\underline{d} = (d_1, \ldots, d_r, d_{r+1}, d_r, \ldots, d_1)$. If $\mathfrak{p}$ is a parabolic subalgebra for one of these Lie algebras then if $\mathfrak{m}$ is the standard Levi factor of the parabolic subalgebra to which it is conjugate then we will say that $\mathfrak{m}$ is the standard Levi factor.

We describe now the parabolic subgroups in the classical cases that have the desired properties.

**THEOREM 2.1.** Let $P \subset G$ be a parabolic subgroup of a classical group. Let $\mathfrak{m}$ be the standard Levi factor of the Lie algebra of $P$, let $\underline{d}$ be the vector describing the lengths of the blocks in $\mathfrak{m}$. Then there is a Richardson element $x \in \mathfrak{g}_1$ with $G_x = P_\xi$ if and only if $P \subset G$ is one of the following:

(i) $G = \text{SL}_{n+1}$ and the entries of $\underline{d}$ satisfy $d_1 \leq \cdots \leq d_s \geq \cdots \geq d_r$ (unimodality of $\underline{d}$).

(ii) $G = \text{Sp}_{2n}$; $\underline{d}$ is unimodal and if $\mathfrak{m}$ has an odd number of blocks, all $d_i$ are even.

(iii) $G = \text{SO}_N$; $\underline{d}$ is unimodal and if $\mathfrak{m}$ has an even number of blocks then there is at most one odd $d_i$ and in this case, $i < r$ and $d_i \leq d_r - 3$.

As a consequence, any such $P$ is given by an $\mathfrak{sl}_2$-triple (see the next section for the definition) if $G = \text{Sp}_{2n}$, $G = \text{SO}_{2n+1}$ or in the case of $G = \text{SO}_{2n}$ with $m$ having an odd number of blocks. If $G = \text{SO}_{2n}$ with an even number of blocks, the only such $P$ that are not given by an $\mathfrak{sl}_2$-triple are the ones with one odd $d_i$ (and $d_i \leq d_r - 3$).

It is often useful to know when the orbit of Richardson elements has normal closure. We keep the notation of Theorem 2.1 above. By 9, any orbit closure is normal for $G = \text{SL}_n$. Using 8 (for all cases except for part of the very even orbits in $SO_{4n}$) and 12 (for the remaining very even orbits), one can check whether the orbit of Richardson elements has normal closure.

**REMARK 2.2.** Let $P = P(d) \subset G$ with $G = \text{Sp}_{2n}$ or $\text{SO}_N$ be one of the parabolic subgroups of Theorem 2.1. Let $X \in \mathfrak{n}$ be a Richardson element for $P$. Then the orbit $GX$ has normal closure exactly in the following cases.
(i) $G = \text{Sp}_{2n}$
- $m$ has an even number of blocks.
- $m$ has an odd number of blocks and $d_1 = d_2 = \cdots = d_r$.

(ii) $G = \text{SO}_N$
- $m$ has an even number of blocks.
- $m$ has an odd number of blocks, and
  - $d_1 = \cdots = d_r$ (even),
  - $d_1 = \cdots = d_s$, $d_{s+1} = d_s + 2$, $d_{s+2} = \cdots = d_r$ (even),
  - there is one odd entry $d_i$, $d_i \leq d_r - 3$ with $i = 1$
  and $d_2 = \cdots = d_r$ or $i > 1$, $d_1 = \cdots = d_{i-1}$, $d_i - d_{i-1} = 1$
  and $d_{i+1} = \cdots = d_r$.

2.2. Results in the exceptional cases. In this subsection we will state the classification of nice parabolic subalgebras for the exceptional simple Lie algebras.

The parabolic subalgebra will be given by an $n$-tuple where $n$ is the rank and the entries are $\alpha_i(H)$ where $H$ is the element that gives the grade corresponding to the parabolic subalgebra and the $\alpha_i$ are the simple roots in the Bourbaki order.

We give an explicit list of all the parabolic subgroups with a Richardson element in $\mathfrak{g}_1$ and with $P_x = G_x$ for Richardson elements in the appendix.

We first recall that for $G = G_2$ or $F_4$ all parabolic subalgebras with a Richardson element in $\mathfrak{g}_1$ are given by an $\mathfrak{sl}_2$-triple. (cf. [3]). Since all such $P$ satisfy $P_x = G_x$ as we will see later, the list of parabolic subgroups with a Richardson element in $\mathfrak{g}_1$ and with $P_x = G_x$ is just the list of parabolic subgroups given by an $\mathfrak{sl}_2$-triple.

If $G$ is of type $E_n$ than the picture is the following:

**Theorem 2.3.** The parabolic subgroups $P$ of $E_n$ that have a Richardson element $x$ in $\mathfrak{g}_1$ and $P_x = G_x$ are the parabolic subgroups that have a Richardson element in $\mathfrak{g}_1$ except for the following three subgroups of $E_7$:

- $(1, 1, 0, 0, 0, 1, 1)$
- $(0, 0, 1, 0, 0, 1)$
- $(0, 0, 0, 1, 0, 1)$

3. Birationality of the moment map

Let $G$ be a connected semisimple linear algebraic group over the complex numbers, $B$ a Borel subgroup containing the maximal torus $T$. We denote the Lie algebras with corresponding gothic letters. Let $P \supset B$ be a parabolic subgroup of $G$, $P = M \cdot N$ with $M$ a Levi factor and $N$ the corresponding unipotent radical.

The dual of the cotangent bundle of $G/P$, $T^*(G/P)$ is a $G$-manifold with a natural symplectic form. In particular, there exists a moment map

$$\mu_P : T^*(G/P) \longrightarrow \mathfrak{g}^*$$

After identifying $T^*(G/P)$ with the homogeneous $G$-vector bundle $G \times^P (\mathfrak{g}/\mathfrak{p})^*$ and dualizing (use $G \times^P (\mathfrak{g}/\mathfrak{p})^* \cong G \times^P \mathfrak{n}$) we obtain the map

$$\psi_P : G \times^P \mathfrak{n} \longrightarrow \mathfrak{g}$$

$$(g, X) \mapsto Ad(g)X$$

where $\mathfrak{n} = \text{Lie } N$ is the nilradical of $P$. Its image is $G \mathfrak{x}$ for $x$ a Richardson element in $\mathfrak{n}$. In case $P = B$, a Richardson element is a regular nilpotent element and so the image is equal to the nullcone $G \mathfrak{n} = \mathcal{N}$ which is normal.
Of special interest is the case where $\psi_P$ is birational onto its image. In that case, a result of the second named author shows that for the space of generalized Whittaker vectors (certain linear maps on a representation induced from a Fréchet representation space of $P_R$) a multiplicity one theorem holds if $P$ has a Richardson element in $g_1$ (cf. [14]).

From the discussion above we find that the following holds:

**Lemma 3.1.** Let $P$ be a parabolic subgroup of $G$. Then the induced map $\psi_P$ is birational onto its image if and only if for any Richardson element $x$ for $P$ the stabilizer $G_x$ is contained in $P$.

To describe the main class of parabolic subgroups for which there is a Richardson element in $g_1$ and such that the stabilizers of Richardson elements in $G$ resp. in $P$ agree, we introduce the following notion:

**Definition 3.2.** Let $p = \oplus_{j \geq 0} g_j$ be a parabolic subalgebra where the grading of $g$ is given by $H \in h$. We say that $P$ (or $p$) is given by an $\mathfrak{sl}_2$-triple if there exists a nonzero $x \in g_1$ such that the Jacobson-Morozov triple through $x$ is \{x, 2H, y\} (for some $y \in g$).

By construction, such an $x$ is a Richardson element for $P$ that belongs to $g_1$. Furthermore, we recall the following standard result (cf. [10], a simple proof can be found in [14]).

**Theorem 3.3.** Let $P \subset G$ be a parabolic subgroup. If $P$ is given by an $\mathfrak{sl}_2$-triple then $P_x = G_x$ for Richardson elements.

We will see that if $P$ has a Richardson element $x$ in $g_1$ and $P_x = G_x$ then $P$ is given by an $\mathfrak{sl}_2$-triple for types $B_n, C_n, G_2$ and $F_4$ and for $D_n$ if the number of blocks in $m$ is odd.

The following observations proves to be very useful for the exceptional cases.

**Lemma 3.4.** If $P$ is a parabolic subgroup that has a Richardson $x$ element in $g_1$ then the number of components of $P_x$ is the same as the number of components of $M_x$.

**Proof.** Let $p = n \cdot m \in P_x$, $n \in N$ and $m \in M$. Set $y = Ad(m)x$. This lies in $g_1$ since $Ad(m)$ preserves every component $g_j$. Now $n$ is unipotent, so $x = Ad(p)x = Ad(n)(Ad(m)x) = Ad(n)y = y \mod \sum_{j > 1} g_j$. So $y = x$, i.e. $m \in M_x$ and then $n \in N_x$. This gives $P_x = M_x N_x$, hence $P_x/P_x^o = M_x/M_x^o$ ($N_x$ is connected).

**Lemma 3.5.** Let $P_i \subset G$, $i = 1, 2$, be parabolic subgroups with Richardson elements $x_i$ in $g_1$. Assume that the corresponding Levi factors are conjugate. Then $x_1$ and $x_2$ are conjugate.

**Proof.** This is Corollary 5.18 in [4].

**Corollary 3.6.** In the situation above, the stabilizer subgroups $(M_1)_{x_1}$ and $(M_2)_{x_2}$ have the same number of components.
4. Proof of the theorem, classical case

We prove the statements of Theorem 4.1 by a case by case check.

Recall that if $G$ is classical case, $P \subset G$ is given by the sequence of block lengths in the standard Levi factor, $d = (d_1, \ldots, d_m)$ for $A_n$ and $d_{even} = (d_1, \ldots, d_r, d_r, \ldots, d_1)$ or $d_{odd} = (d_1, \ldots, d_r, d_{r+1}, d_r, \ldots, d_1)$ (with $d_{r+1} > 0$) for $B_n$, $C_n$, $D_n$. If $G$ is of type $A_n$, then we always have $P_x = G_x$ (a nilpotent), since $G_x = \mathfrak{gl}(n+1, \mathbb{C})_x \cap G$ or as Hesselink observed in [6 Section 3.1]:

**Lemma 4.1.** Let $x \in \mathfrak{g}_1$ be a nilpotent element. Then the centralizer $G_x$ of $x$ in $G$ is connected.

In our earlier paper [3] we have described the parabolic subgroups of the classical groups that are given by an $\mathfrak{sl}_2$-triple if and only if $d$ is unimodal and palindromic. On the other hand, any parabolic subgroup of $A_n$ with unimodal sequence of block lengths in the standard Levi factor has a Richardson element in $\mathfrak{g}_1$. So there are many parabolic subgroups that are not given by an $\mathfrak{sl}_2$-triple but still have a Richardson element in $\mathfrak{g}_1$ and birational $\psi_P$.

From a more general statement of Hesselink for the number of components of $G_x/P_x$ for types $B_n$, $C_n$ and $D_n$ we obtain the characterization of parabolic subgroups of these with $G_x = P_x$ ($x$ a Richardson element).

To formulate the statement we introduce some notation: As is customary, set $\epsilon = 1$ for $G = \text{Sp}_{2n}$ and $\epsilon = 0$ for $G = \text{SO}_{2n}$. If $x$ is a Richardson element and $\lambda$ its partition, ordered as $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots$, we define $N_{odd}(\lambda)$ to be the number of odd parts in $\lambda$ and $B(\lambda) := \{ j \in \mathbb{N} \mid \lambda_j > \lambda_{j+1}, \lambda_j \not\equiv \epsilon \mod 2 \}$.

**Theorem 4.2.** Let $G$ be of type $B_n$, $C_n$ or $D_n$, $P \subset G$. Let $\lambda$ be the partition of the Richardson element for $P$ corresponding to its Jordan form as an element of $M_n(\mathbb{C})$. Then the induced map $\psi_P$ is birational onto its image exactly in the following cases:

\begin{align*}
(i) & \quad N_{odd}(\lambda) = d_{r+1} & \text{if } d = d_{odd} \\
(ii) & \quad N_{odd}(\lambda) = 0 & \text{if } d = d_{even}, \ G = \text{Sp}_{2n}; \\
& \quad \text{if } d = d_{even}, \ G = \text{SO}_{2n}, \ B(\lambda) = \emptyset; \\
(iii) & \quad N_{odd}(\lambda) = 2 & \text{if } d = d_{even}, \ G = \text{SO}_{2n}, \ B(\lambda) \neq \emptyset.
\end{align*}

**Proof.** This is a special case of Theorem 7.1 in [6].

As an immediate consequence, we obtain:

**Corollary 4.3.** Let $P$ be a parabolic subgroup of $\text{Sp}_{2n}$, given by $d = d_{even}$. Then $P$ has a Richardson element $x \in \mathfrak{g}_1$ and $P_x = G_x$ if and only if $P$ has a Richardson element in $\mathfrak{g}_1$ if and only if $d_1 \leq \cdots \leq d_r$.

**Proof.** For any $P \subset \text{Sp}_{2n}$ given by $d_{even}$ the partition of a Richardson element has only even parts. Explicitly, $\lambda = (2r)^{d_1}, (2r-2)^{d_2-d_1}, \ldots, 4^{d_{r-1}-d_{r-2}}, 2^{d_r-d_{r-1}}$ ([2 Lemma 4.7]). So by part (i) of Theorem 4.2 $G_x = P_x$ for Richardson elements. The statement then follows since $P$ has a Richardson element in $\mathfrak{g}_1$ if and only if $d_{even}$ is unimodal ([3 Theorem 1.3]).

**Corollary 4.4.** Let $P \subset G = \text{Sp}_{2n}$ be given by $d_{odd}$. Then $P$ has a Richardson element $x$ in $\mathfrak{g}_1$ and $P_x = G_x$ if and only if all $d_i$ are even.
PROOF. From [3] Theorem 1.3] we know that \( P \) has a Richardson element in \( g_1 \) if and only if \( d_{\text{odd}} \) is unimodal and if any odd \( d_i \) appears only once among \( d_1, \ldots, d_r \). From [2] Lemma 4.7 the dual of the partition of a Richardson element is \( d_{r+1} \cup \{\sum_{i \in D_o} d_i \} \cup \{\sum_{i \in D_o} d_i - 1, d_i + 1 \} \) where \( D_o := \{d_i \mid d_i \equiv 1 \} \) is the set of odd entries of \( d_{\text{odd}} \). In any case, the partition of a Richardson element has \( d_{r+1} \) parts. If all \( d_i \) are even, the partition of a Richardson element is

\[
\lambda : (2r + 1)^{d_1}, (2r - 1)^{d_2-d_1}, \ldots, 3^{d_r-d_{r-1}}, 1^{d_{r+1}-d_r}
\]

In particular, its parts are all odd and by (i) of Theorem 4.2 we get \( G_x = P_x \) for Richardson elements.

If we assume that \( D_o \) is not empty (i.e. that there are odd \( d_i \) then the partition \( \lambda \) of a Richardson element still has \( d_{r+1} \) parts. As one can check, \( \lambda \) contains 2 even parts for every odd \( d_i \). So by (ii) of Theorem 4.2 \( \psi_P \) is not birational (it is in fact a k-fold covering map where \( k = 2^{d_{r+1}-2-|D_o|} \), using the full version of Theorem 7.1 of [3]).

COROLLARY 4.5. Let \( P \) be a parabolic subgroup of \( \text{SO}_{2n} \) that is given by \( d = d_{\text{even}} \). Then \( P \) has a Richardson element \( x \) in \( g_1 \) and \( G_x = P_x \) if and only there is at most one odd \( d_i \) and in that case, \( d_i \leq d_r - 3 \).

PROOF. We recall Theorem 1.4 of [3]: \( P \) has a Richardson element in \( g_1 \) exactly in the following cases: \( d_1 \leq \cdots \leq d_r \) or \( d_1 \leq \cdots \leq d_{t-1} < d_t \) and \( d_{t+1} = \cdots = d_r \) are equal to \( d_t - 1 \) (for \( t = 1 \) the condition \( d_{t-1} < d_t \) means \( d_t > 1 \)). Furthermore, in both cases, odd \( d_i \) appear only once among \( d_1, \ldots, d_r \).

In the second case we may reorder \( \lambda \) to have \( d_1 \leq \cdots \leq d_r \). The corresponding parabolic subalgebra still has a Richardson element in \( g_1 \). Since the two Levi factors are conjugate, we can apply Corollary 3.5 to see that the stabilizers of the corresponding Richardson elements have the same number of components. Furthermore, the corresponding Richardson elements are conjugate. So using Lemma 3.4 we see that \( G_x = P_x \) can be checked for the reordered \( \lambda \).

So from now on we assume \( d_1 \leq \cdots \leq d_r \). The dual of the partition of a Richardson element is \( \{\sum_{i \in D_o} d_i \} \cup \{\sum_{i \in D_o} d_i - 1, d_i + 1 \} \), where \( D_o \) is the set of odd \( d_i \) among \( d_1, \ldots, d_r \) ([2] Lemma 4.7]). The corresponding partition has no odd parts if all \( d_i \) are even, in that case, it is

\[
2^{d_1}, 2(r-1)^{d_2-d_1}, \ldots, 2^{d_r-d_{r-1}}.
\]

In particular, we then have \( B(\lambda) = \emptyset \), and by (ii) of Theorem 4.2 \( G_x = P_x \). Now let \( D_o \neq \emptyset \). For every element in \( D_o \), the partition of a Richardson element has two odd entries, i.e. \( N_{\text{odd}}(\lambda) = 2|D_o| \) and by (ii) and (iii), \( |D_o| \leq 1 \). It remains to consider the case \( D_o = \{d_i \} \). One checks that the corresponding partition is

\[
2^{d_1}, 2(r-i)^{d_{i-1}-d_{i-2}}, 2(r-(i-1))^{d_{i-2}-d_{i-3}}, (2(r-(i-1)) - 1)^2, \]

\[
2(r-i)^{d_{i+1}-d_{i-1}}, 2(r-(i+1))^{d_{i+2}-d_{i+1}}, \ldots, 2^{d_r-d_{r-2}}.
\]

It has two odd entries. If \( d_i = d_r - 1 \) then \( d_i + 1 = d_{i+1} = \cdots = d_r \) and the smallest nonzero entries of the partition are \( 2(r-(i-1)) - 1 \), i.e. odd. So \( B(\lambda) \) is empty and hence by (ii) of Theorem 4.2 \( G_x \neq P_x \). Otherwise, \( d_i \leq d_r - 3 \) and the smallest entries are even, so \( B(\lambda) \neq \emptyset \) and \( G_x = P_x \).

The unimodality now follows with the observation that there is no odd \( d_i \) with \( \max_j \{d_j \} - d_i = 1 \).
It remains to deal with the case of the special orthogonal groups in the case where the standard Levi factor has an odd number of blocks. By Theorem 4.2 the induced map \( \psi_p \) is birational onto its image if and only if the partition of a Richardson element has exactly \( d_{r+1} \) odd parts.

**Corollary 4.6.** Let \( P \subset G = SO_N \) be given by \( d = d_{\text{odd}} \). Then \( P \) has a Richardson element \( x \) in \( g_1 \) and \( G_x = P_x \) if and only if \( d_1 \leq \cdots \leq d_r \leq d_{r+1} \).

**Proof.** We recall that \( P \) has a Richardson element in \( g_1 \) in exactly the following cases: either \( d_1 \leq \cdots \leq d_{r+1} \) or there is a \( t \leq r \) such that \( d_1 \leq \cdots \leq d_{t-1} < d_t \) and \( d_{t+1} = \cdots = d_{r+1} \) is equal to \( d_t + 1 \).

We first consider the case where \( d_{\text{odd}} \) is unimodal (so \( d_{r+1} \) is maximal). By Lemma 4.7 of [2], the dual of the partition of a Richardson element has just the entries \( d_1, d_1, \ldots, d_r, d_r, d_{r+1} \) and so the partition \( \lambda \) of a Richardson element is

\[
(2r+1)^{d_1}, (2r-1)^{d_2-d_1}, \ldots, 3^{d_{r-1} - d_r}, 1^{d_{r+1} - d_r},
\]

and has \( d_{r+1} \) odd parts. Thus by (i) of Theorem 4.2 the map \( \psi_p \) is birational onto its image.

Now we consider the case where \( d_{\text{odd}} \) is not unimodal. As in the proof of Corollary 4.5 we can reorder \( d \) since the corresponding parabolic subalgebra also has a Richardson element in \( g_1 \). So we may assume \( d_1 \leq \cdots \leq d_r \), \( d_{r+1} = d_r + 1 \).

By the Lemma 4.7 below, \( \lambda \) has \( d_{r+1} + 2 \) entries. If we show that all of these are odd, then by (i) of Theorem 4.2 the induced map \( \psi_p \) is not birational onto its image. In fact, it will be a two-fold cover. Let \( d' = d_{\text{odd}} \) be the dimension vector obtained by replacing \( d_r \) by \( d_{r} - 1 \). Then \( d' \) is unimodal. Let \( X(d') \) resp. \( X(d') \) be the corresponding Richardson elements. It is easy to see that the maps \( X(d)^k \) and \( X(d')^k \) have the same rank for every \( k \in \mathbb{N} \) (for details we refer the reader to [3, Section 3]). Hence \( \dim \ker X(d)^k = \dim \ker X(d')^k + 2 \) for all \( k \). From that one can compute the partitions \( \lambda \) resp. \( \lambda' \) obtained from \( \lambda' \) by adding \( \{1, 1\} \). We already know that the partition \( \lambda' \) has \( d_{r+1} \) odd entries. Hence \( \lambda \) has \( d_{r+1} + 2 \) odd entries. \( \square \)

**Lemma 4.7.** Let \( P \subset G \) be a parabolic subgroup of \( SO_N \) or \( Sp_{2m} \), \( P \) be given by \( d = d_{\text{odd}} \). If \( \max d_i = d_{r+1} + 1 \) then the partition \( \lambda \) of a Richardson element has \( d_{r+1} + 2 \) parts.

**Proof.** For \( X \in \mathfrak{g}_1 \) is a nilpotent element with partition \( \lambda \), the number of parts of \( \lambda \) is equal to the dimension of the kernel of the map \( X \). This follows immediately from the formula for the partition for nilpotent matrices, as given in [3, Section 3]:

\[
\lambda : m^{a_m}, (m-1)^{a_{m-1}}, \ldots, 1^{a_1}
\]

with \( a_j := 2 \dim \ker X^j - \dim \ker X^{j-1} - \dim \ker X^{j+1} \), \( m \) the maximal number such that \( \ker X^{m-1} \subseteq \ker X^m = \mathbb{C}^N \) (with \( \dim \ker X^{m+1} = \mathbb{C}^N \)). In particular, \( \lambda \) has \( \sum_{k=1}^m a_k = \dim \ker X \) parts.

Now if \( X \) is a Richardson element for \( P \) it is in particular a generic element of the nilradical. W.l.o.g. we can assume \( d_1 \leq \cdots \leq d_r \) (by Lemma 3.5). So the rank of \( X \) is

\[
\text{rk} X = 2 \sum_{i=1}^{r-1} \min \{d_i, d_{i+1}\} + 2 \min \{d_r, d_{r+1}\} = 2(d_1 + \cdots + d_{r-1}) + 2d_{r+1}
\]
Hence its kernel has dimension $2d_r - d_{r+1} = d_{r+1} + 2$. \hfill \Box

From the description of parabolic subalgebras given by an $\mathfrak{sl}_2$-triple that is in our earlier paper [3] we can now deduce:

**Corollary 4.8.** For types $B_n$, $C_n$ and $D_n$, the parabolic subgroups with a Richardson element in $g_1$ and with $P_x = G_x$ are those given by an $\mathfrak{sl}_2$-triple except in the case of $D_n$ with an even number of blocks. There, the extra family of examples are the ones with one odd entry $d_i$ (with $d_i \leq d_r - 3$).

**Remark 4.9.** The proofs of the statements Corollary 4.3, 4.4, 4.5 and 4.6 all directly used Theorem 4.2 via the partition of the corresponding Richardson element. The statements could have also been proved indirectly using Corollary 7.7 of [6]. Since this corollary uses very intricate notation and is stated without proof, we decided to include the proofs for our statements.

**5. Result for exceptional groups**

The parabolic subgroups of $G_2$, $F_4$ and $E_6$ with a Richardson element $x$ in $g_1$ all satisfy $G_x = P_x$. In the case of $G_2$ and $F_4$ this follows from the fact that all parabolic subgroups with a Richardson element in $g_1$ are given by an $\mathfrak{sl}_2$-triple. Hence the claim follows by Theorem 3.3.

For $E_7$ we can use a list of Sommers. In [11], Sommers classified the nilpotent elements of the exceptional Lie groups whose centralizers are not connected. For $E_6$ all these are the nilpositive element of an $\mathfrak{sl}_2$-triple. In particular, they all have the property $G_x = P_x$ for Richardson elements.

It remains to understand $E_7$ and $E_8$. From the classification of parabolic subgroups with a Richardson element in $g_1$ we know that there are five parabolic subgroups in $E_7$ resp. one in $E_8$ that are not given by an $\mathfrak{sl}_2$-triple but still have a Richardson element in $g_1$. We list them here together with the dimension of the Richardson orbit $O_x$ (obtained from the formula $\dim m = \dim g^x$ that holds for Richardson elements) and its Bala-Carter label. Recall that an entry 1 stands for a simple root that is not a root of the standard Levi factor.

|   | Label |   |
|---|-------|---|
| $E_7$ |  
| a) | $(1,1,0,0,1,0,1)$ | 118 | $D_6$ |
| b) | $(1,1,0,0,0,0,1)$ | 106 | $D_5(a_1)$ |
| c) | $(0,1,1,0,0,1,1)$ | 118 | $D_6$ |
| d) | $(0,0,1,0,0,0,1)$ | 104 | $A_4 + A_1$ |
| e) | $(0,0,0,0,1,0,1)$ | 104 | $A_4 + A_1$ |
| $E_8$ |  
| f) | $(0,0,1,0,0,1,0)$ | 216 | $D_6$ |

Note that the Levi factors of a) and c) are the same as well as the Levi factors of e) and d). Since the condition $P_x = G_x$ only depends on the Levi factor (cf. Lemma 3.3), it is enough to understand one of e) and d) resp. one of a) and c). Using Sommers list one checks that the groups $G_x$ are connected for a), c) and f). So in these cases, the parabolic subgroup has the property $G_x = P_x$.

**Lemma 5.1.** Let $P$ be one of the remaining parabolic subgroups b), d), e), with $x$ a Richardson element for $P$. Then $G_x \geq P_x$. 
We discuss b) and d) to prove this claim.

**Example 5.2.** Let $P$ be the parabolic subalgebra $(0,0,1,0,0,1)$ of $E_7$. Set $X$ to be the following element of $\frak{g}_1$:

$$X = x_3 + x_{24567} + x_{134} + x_{23456} + x_{12345}$$

where $x_I$ denotes a non-zero element of the root space of $\alpha_I := \sum_{i \in I} \alpha_i$. Using the program GAP for the Lie algebra $L := E_7$ one computes that the dimension of the centralizer of $X$ in $L$ is 29, the corresponding command is

$$\text{LieCentralizer}(L, \text{Subalgebra}(L, [X]));$$

But this is just the dimension of the Levi factor, so $X$ is a Richardson element for this $P$.

From Sommers list \[\text{[11]}\] we know that $G_X$ is disconnected. We now show that $P_X$ is connected. Lemma \[\text{[40]}\] implies that it is enough to prove that $M_x$ is connected. We will now calculate the stabilizer of $X$ in the Levi factor. If $M$ is the Levi factor of $P$ then $M_0 := (M, M) = GL_2 \times GL_5$. The first graded part $\frak{g}_1$ corresponds to the representation $(\mathbb{C}^2)^* \otimes \Lambda^2(\mathbb{C}^5) \oplus (\mathbb{C}^5)^*$. Under this correspondence, the Richardson element has the form

$$V := v_2 \otimes (e_2 \wedge e_4 + e_3 \wedge e_5) + v_1 \otimes (e_4 \wedge e_5 + e_1 \wedge e_2) + e_5.$$

(The $e_i$ form a basis of $\mathbb{C}^5$, the $v_i$ of $\mathbb{C}^*$ and $e_i^*$ of $(\mathbb{C}^5)^*$). Elements of $M_0$ stabilizing $V$ are pairs $(h, g) \in GL_2 \times GL_5$ that satisfy in particular $ge_5^* = e_5^*$. Hence for $g = (g_{ij})_{ij}$ we have $g_{ii} = 0$ ($1 \leq 4$) and $g_{55} = 1$. Let $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $GL_2$.

Writing out the equations given by $(h, g)V = V$ one uses $\det h \neq 0$ to obtain $g_{13} = g_{23} = g_{14} = g_{24} = 0$. With these simplifications, the stabilizer in $M_0$ can be computed with mathematica; it is the subgroup

$$\left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \frac{d}{a} & 0 & 0 & 0 & 0 \\ \frac{g_{21}}{a} & 0 & 0 & -\frac{g_{22}}{a} & 0 \\ -\frac{e^* c}{d^2} & 0 & \frac{1}{d} & -\frac{e^* c}{d^2} & 0 \\ \frac{e^* d}{c^2} & 0 & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_2 \times GL_5.$$

But this is just $(\mathbb{C}^*)^2 \times \mathbb{C}^2$ which is connected.

**Example 5.3.** Let $P$ be the parabolic subalgebra $(1,1,0,0,0,1)$ of $E_7$. As in the previous example, one can use GAP to check that the following element

$$X = x_7 + x_{13456} + x_{23456} + x_{23456} - x_{234} - x_{245}$$

of $\frak{g}_1$ satisfies $\dim \frak{g}^X = \dim \frak{m} = 27$, so it is a Richardson element for $P$.

By Sommers list, we know that $G_X$ is not connected. To see whether $P_x$ is connected we calculate the stabilizer of $X$ in the standard Levi factor $M$ of $P$. We have $M_0 = (M, M) = GL_5$ and $\frak{g}_1$ corresponds to the representation $\mathbb{C}^5 \otimes \Lambda^2(\mathbb{C}^5) \oplus (\mathbb{C}^5)^*$, and the Richardson element has the form

$$V := e_1 \wedge e_3 + e_5^* + e_1 + e_3 \wedge e_5 + e_2 \wedge e_4.$$

So we have to determine whether the set $\{g \in GL_5 \mid gV = V\}$ is connected. First we observe that $ge_1 = e_1$ and $ge_5^* = e_5^*$ imply $g_{11} = g_{55} = 1$ whereas the other
entries of the first column and last row are zero. Using mathematica one obtains

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & g_{22} & 0 & g_{24} & g_{25} \\
0 & g_{32} & 1 & g_{34} & g_{35} \\
0 & g_{42} & 0 & g_{44} & g_{45} \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

together with the equations

\[
\begin{align*}
g_{22}g_{34} - g_{32}g_{24} - g_{25} &= 0 \\
g_{22}g_{44} - g_{42}g_{24} - 1 &= 0 \\
g_{32}g_{44} - g_{42}g_{34} + g_{45} &= 0
\end{align*}
\]

The second equation implies \( g \in \text{SL}_5 \). We use mathematica to compute a Gröbner basis for these equations in the nine variables and obtain

\[
\begin{align*}
-g_{34}g_{42} + g_{32}g_{44} + g_{45}, \\
g_{34} - g_{25}g_{44} + g_{24}g_{45}, \\
g_{32} - g_{25}g_{42} + g_{22}g_{45}, \\
1 - g_{24}g_{42} + g_{22}g_{44}, \\
-g_{25} - g_{24}g_{32} + g_{22}g_{34}
\end{align*}
\]

We set \( x_1 = g_{22}, x_2 = g_{44}, x_3 = g_{42}, x_4 = g_{24} \) and \( y_1 = g_{25}, y_2 = g_{45}, y_3 = g_{32}, y_4 = g_{34} \). We also set

\[
A = \begin{bmatrix}
-x_4 & x_1 \\
-x_2 & x_3
\end{bmatrix}
\]

then the fourth equation says that \( \det A = 1 \). Hence

\[
A^{-1} = \begin{bmatrix}
x_3 & -x_1 \\
x_2 & -x_4
\end{bmatrix}
\]

The other four equations say that

\[
\begin{bmatrix}
0 & A \\
A^{-1} & 0
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}
= \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}
\]

This is equivalent to

\[
A \begin{bmatrix}
y_3 \\
y_4
\end{bmatrix}
= \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
\]

This implies that the isotropy group of \( V \) is isomorphic with \( \text{SL}_2 \times \mathbb{C}^3 \) as a variety (here the \( \mathbb{C}^3 \) is given by the free variables \( y_3, y_4 \) and \( g_{35} \)) which is connected.

**Appendix**

Here we include the list of all parabolic subgroups in the exceptional Lie algebras where there is a Richardson element in the first graded part \( g_1 \) and where \( G_x = P_x \) holds for Richardson elements. We list them as in our article 3. In the case of \( \text{E}_7 \) we omit the three parabolic subgroups with \( P_x \not\subseteq G_x \) (examples 5, 20, 25).
| $G_2$ | $F_4$ |
|-------|-------|
| (1, 1) | (1, 1, 1, 1) |
| (1, 0) | (1, 1, 0, 1) |
| (0, 0) | (1, 1, 0, 0) |
|       | (1, 0, 0, 1) |
|       | (0, 1, 0, 1) |
|       | (0, 1, 0, 0) |
|       | (0, 0, 0, 1) |
|       | (0, 0, 0, 0) |

| $E_6$ | $E_7$ | $E_8$ |
|-------|-------|-------|
| 1 (1, 1, 1, 1, 1, 1) | (1, 1, 1, 1, 1, 1, 1) | (1, 1, 1, 1, 1, 1, 1) |
| 2 (1, 1, 1, 0, 1, 1) | (1, 1, 1, 0, 1, 1, 1) | (1, 1, 1, 0, 1, 1, 1) |
| 3 (1, 1, 0, 1, 1, 0) | (1, 1, 0, 1, 1, 0, 1) | (1, 1, 0, 1, 1, 0, 1) |
| 4 (1, 1, 0, 1, 0, 1) | (1, 1, 0, 1, 0, 1, 1) | (1, 0, 0, 1, 0, 1, 1) |
| 5 (1, 0, 0, 1, 1, 0) | (1, 0, 0, 1, 1, 0, 1) | (1, 0, 0, 1, 1, 0, 1) |
| 6 (1, 0, 0, 0, 1, 0) | (1, 0, 0, 0, 1, 0, 1) | (1, 0, 0, 0, 1, 0, 1) |
| 7 (1, 0, 0, 0, 0, 1) | (1, 0, 0, 0, 0, 1, 1) | (1, 0, 0, 0, 0, 1, 1) |
| 8 (1, 0, 0, 0, 0, 0) | (1, 0, 0, 0, 0, 0, 1) | (1, 0, 0, 0, 0, 0, 1) |
| 9 (1, 0, 0, 0, 0, 0) | (0, 1, 1, 0, 0, 1, 1) | (0, 1, 1, 0, 0, 1, 1) |
| 10 (0, 1, 1, 0, 0, 0) | (0, 1, 1, 0, 0, 0, 0) | (0, 1, 1, 0, 0, 0, 0) |
| 11 (0, 1, 1, 0, 0, 0) | (0, 1, 1, 0, 0, 0, 0) | (0, 1, 1, 0, 0, 0, 0) |
| 12 (0, 1, 1, 0, 0, 0) | (0, 1, 1, 0, 0, 0, 0) | (0, 1, 1, 0, 0, 0, 0) |
| 13 (0, 1, 1, 0, 0, 0) | (0, 1, 1, 0, 0, 0, 0) | (0, 1, 1, 0, 0, 0, 0) |

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