On a Generalized $\beta H$ – Trirecurrent Finsler Space

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DOI: https://doi.org/10.47372/uajnas.2019.n2.a16

Abstract

In this paper, we introduced a Finsler space for which the $h$ – curvature tensor $H_{jkh}$ (curvature tensor of Berwald) satisfies the condition

$$
\beta_i \beta_m \beta_n H_{jkh} = c_{\ell mn} H_{jkh} + d_{\ell mn} (\delta^i_k g_{jhn} - \delta^i_h g_{jnk}) - 2y^\tau b_{mn} \beta_r (\delta^i_k C_{jhr} - \delta^i_h C_{jkr})
$$

where $c_{jkm}$ is (h) hv – torsion tensor, $\beta_i \beta_m \beta_n$ is Berwald's covariant differential operator of the third order with respect to $x^n$, $x^m$ and $x^\ell$, successively, $\beta_i \beta_r$ is Berwald's covariant differential operator of the second order with respect to $x^\ell$ and $x^r$, successively, $\beta_r$ is Berwald's covariant differential operator of the first order with respect to $x^r$, $c_{\ell mn}$ and $d_{\ell mn}$ are non – zero covariant tensors field of third order, $b_{mn}$ and $w_{mn}$ are non – zero covariant tensors field of second order and $\mu_r$ is non – zero covariant vector field. We called this space a generalized $\beta H$ – trirecurrent space. The aim of this paper is to develop some properties of a generalized $\beta H$ – trirecurrent space by obtaining Berwald's covariant derivative of the third order for the (h) $v$ – torsion tensor $H_{jkh}^i$ and the deviation tensor $H^i_k$, the curvature vector $H_k$ and the scalar curvature $H$ are investigated.

Key words: Finsler space, generalized $\beta H$ – trirecurrent space, Ricci tensor.

1. Introduction

Pandey P.N., Saxena S.S. and Goswami A. [3] introduced and studied a generalized H– recurrent Finsler space. F.Y.A.Qasem [4] introduced and discussed generalized H– birecurrent curvature tensor and W.H.A. Hadi [1] studied the generalized – birecurrent for some tensors and studied some special spaces in this space. Let $F_n$ be an $n$ – dimensional Finsler spaces equipped with the metric function $F$ satisfies condition [5], the tensor $C_{ijk}$ is positively homogeneous of degree $-1$ in $y^i$ and symmetric in all its indices and is called (h) $v$ – torsion tensor [2]. According to Euler's theorem on homogeneous functions, this tensor satisfies the following:

$$
(1.1) \quad C_{ijk} y^i = C_{jki} y^i = C_{kij} y^i = 0.
$$

Berwald's covariant derivative $B_{k}T_{j}^i$ of an arbitrary tensor field $T_{j}^i$ with respect to $x^k$ is given by [5]

$$
(1.2) \quad B_{k}T_{j}^i = \partial_k T_{j}^i - (\partial_r T_{j}^i) G_{kr}^i + T_{j}^\ell G_{rk}^i - T_{r}^i G_{jk}^r.
$$

Berwald's covariant derivative of $y^i$ vanish as identically [5], i.e.

$$
(1.3) \quad B_{k}y^i = 0.
$$

In view of (1.2), the second covariant derivative of an arbitrary vector field $X^i$ with respect to $x^h$ in the sense of Berwald [5].

$$
(1.4) \quad B_{h}B_{k}X^i = \partial_k (B_{h}X^i) - (\partial_r B_{h}X^i) G_{kr}^i - (B_{r}X^i) G_{hk}^r + (B_{h}X^r) G_{rk}^i.
$$

Using (1.4) and taking skew – symmetric part, with respect to the indices $k$ and $h$, we get the commutation formula for Berwald's covariant differentiation as follows [5]:
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(1.5) $B_h B_k X^i - B_k B_h X^i = X^r H^i_{rkh} - \left( \partial_r X^i \right) H^i_{kh}$

where

(1.6) (a) $H^i_{jkh} := \partial_h G^i_{jk} + G^r_{jk} G^i_{rh} + G^r_{rjh} G_k^i - h/k$ and (b) $H^i_{kh} := \partial_h G^i_k + G^r_{rk} G^i_{rh} - h/jk$.

**Remark 1.1.** $h/k$ means the subtraction from the former term by interchanging the indices $h$ and $k$.

The tensors $H^i_{jkh}$ and $H^i_{kh}$, as defined above, are called $h$ – curvature tensor ($h$ – curvature tensor of Berwald) and $h(\nu)$ – torsion tensor are positively homogeneous of degree zero and one in $y^i$, respectively.

Berwald constructed the tensors $H^i_{jkh}$ and $H^i_{kh}$ from the tensor $H^i_k$ called by him as deviation tensor, according to

(1.7) $H^i_k := 2 \partial_h G^i + \partial_\nu G^i y^s + 2 G^r_{hs} G^s - G^i_s G^s_h$.

The $h(\nu)$ – torsion tensor and the deviation tensor satisfy the following [5]:

(1.8) (a) $H^i_{jkh} y^j = H^i_{kh}$, (b) $H^i_{jkh} y^j = H^i_k$ and (c) $\partial_j H^i_{k} = H^i_{jk}$.

The $H$ – Ricci tensor, the curvature vector and the scalar curvature satisfy the following [5]:

(1.9) (a) $H^i_{jki} = H_{jki}$, (b) $H^i_{kki} = H_k$ and (c) $H^i_{i} = H$.

**2. Generalized $\beta H$ – Trirecurrent Space**

A Finsler space for which Berwald curvature tensor $H^i_{jkh}$ satisfies the generalized recurrence property, i.e. characterized by the equation [3]

(2.1) $\beta_n H^i_{jkh} = \lambda_n H^i_{jkh} + \mu_n (\delta^i_k g_{jh} - \delta^i_{h} g_{jk})$, $H^i_{jkh} = 0$,

where $\beta_n$ is the differential operator with respect to $x^n$ in the sense of Berwald, $\lambda_n$ and $\mu_n$ are non – zero covariant vectors field and is called the recurrence vectors field, such space known as generalized $H$ – recurrent Finsler space.

Taking the $\beta$ – covariant derivative for (2.1) with respect to $x^m$, we get

$$\beta_m \beta_n H^i_{jkh} = (\beta_m \lambda_n) H^i_{jkh} + \lambda_n (\beta_m H^i_{jkh}) + (\beta_m \mu_n) (\delta^i_k g_{jh} - \delta^i_{h} g_{jk}) + \mu_n \beta_m (\delta^i_k g_{jh} - \delta^i_{h} g_{jk}).$$

In view of the equation (2.1), we can write the above equation as

$$\beta_m \beta_n H^i_{jkh} = (\beta_m \lambda_n + \lambda_n \beta_m) H^i_{jkh} + (\lambda_n \mu_m + \beta_m \mu_n) (\delta^i_k g_{jh} - \delta^i_{h} g_{jk}) - 2y^r \mu_n \beta_r (\delta^i_k C_{jhm} - \delta^i_{h} C_{jkm}),$$

which can be written as [4]

(2.2) $\beta_m \beta_n H^i_{jkh} = a_{mn} H^i_{jkh} + b_{mn} (\delta^i_k g_{jh} - \delta^i_{h} g_{jk}) - 2y^r \mu_n \beta_r (\delta^i_k C_{jhm} - \delta^i_{h} C_{jkm}),$

where $a_{mn} = \beta_m \lambda_n + \lambda_n \beta_m$ and $b_{mn} = \lambda_n \mu_m + \beta_m \mu_n$ are non – zero covariant tensors field of second order and $\beta_m \beta_n$ is the differential operator with respect to $x^n$ and $x^m$, successively, such space known as generalized $\beta H$ – birecurrent space.

**Remark 2.1.** The expression $\beta$ – covariant derivative stands the covariant derivative in the sense of Berwald.

Taking the $\beta$ – covariant derivative for (2.2) with respect to $x^\ell$, we get

$$\beta_{\ell} \beta_m \beta_n H^i_{jkh} = (\beta_{\ell} a_{mn}) H^i_{jkh} + a_{mn} (\beta_{\ell} H^i_{jkh}) + (\beta_{\ell} b_{mn}) (\delta^i_k g_{jh} - \delta^i_{h} g_{jk})$$

$$+ b_{mn} \beta_{\ell} (\delta^i_k g_{jh} - \delta^i_{h} g_{jk}) - 2y^r (\beta_{\ell} \mu_n) \beta_r (\delta^i_k C_{jhm} - \delta^i_{h} C_{jkm}) - 2y^r \mu_n \beta_{\ell} \beta_r (\delta^i_k C_{jhm} - \delta^i_{h} C_{jkm}).$$

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In view of the equation (2.1), we can write the above equation as

\[ \beta_i \beta_m b_n H^i_{kh} = (\beta_i a_{mn} + a_{mn} \lambda_i) H^i_{jk} + (\beta_i b_{mn} + a_{mn} \mu_i) (\delta^i_k g_{jh} - \delta^i_h g_{jk}) \\
- 2y^r b_{mn} \beta_r (\delta^i_k C_{jht} - \delta^i_h C_{jkt} - 2y^r (\beta_i \mu_n) \beta_r (\delta^i_k g_{jh} - \delta^i_h g_{jk}) \\
- 2y^r \mu_n \beta_i \beta_r (\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) \]

which can be written as

\[ (2.3) \quad \beta_i \beta_m b_n H^i_{kh} = c_{lmn} H^i_{jk} + d_{lmn} (\delta^i_k g_{jh} - \delta^i_h g_{jk}) - 2y^r b_{mn} \beta_r (\delta^i_k C_{jht} - \delta^i_h C_{jkt}) \\
- 2y^r w_{tn} \beta_r (\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) - 2y^r \mu_n \beta_i \beta_r (\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) , \]

where \( c_{lmn} = \beta_i a_{mn} + a_{mn} \lambda_i \) and \( d_{lmn} = \beta_i b_{mn} + a_{mn} \mu_i \) are non-zero covariant tensors field of third order \( w_{tn} = \beta_i \mu_n \) is non-zero covariant tensor field of second order and \( \beta_i \beta_m b_n \) is the differential operator with respect to \( x^n, x^m \) and \( x^\ell \), successively.

**Definition 2.1.** A Finsler space \( F_n \) for which Berwald curvature tensor \( H^i_{jk} \) satisfies the condition (2.3) and called a generalized \( \beta H \) – tricurrent space and the tensor a generalized \( \beta \) – tricurrent. We shall denote them briefly as \( G \beta H \) – TR and \( G\beta \) – TR, respectively.

Now, transvecting the condition (2.3) by \( y^j \), in view of (1.3), and by using (1.8a) and (1.1), we get

\[ (2.4) \quad \beta_i \beta_m b_n H^i_{kh} = c_{lmn} H^i_{jk} + d_{lmn} (\delta^i_k y_h - \delta^i_h y_k) \]

Transvecting (2.4) by \( y^h \), in view of (1.3) and by using (1.8b), we get

\[ (2.5) \quad \beta_i \beta_m b_n H^i_{kh} = c_{lmn} H^i_{lk} + d_{lmn} (\delta^i_k F^2 - y^l y_k) . \]

Thus, we may conclude

**Theorem 2.1.** In \( G \beta H \) – TR \( F_n \), Berwald’s covariant derivative of the third order for the \( h(v) \) – torsion tensor \( H^i_{jk} \) and the deviation tensor \( H^i_{k} \) given by the equations (2.4) and (2.5), respectively.

Contracting the indices \( i \) and \( k \) in the condition (2.3) and using (1.9a), we get

\[ (2.6) \quad \beta_i \beta_m b_n H^i_{kh} = c_{lmn} H^i_{jh} + d_{lmn} (n-1) g_{jh} - 2b_{mn} (n-1) \beta_i C_{jht} - 2y^r w_{tn} (n-1) \beta_r C_{jhm} \\
- 2y^r \mu_n (n-1) \beta_i \beta_r C_{jhm} . \]

Thus, we may conclude

**Theorem 2.2.** In \( G \beta H \) – TR \( F_n \), Berwald’s covariant derivative of the third order for the \( H \) – Ricci tensor \( H^i_{jk} \) given by the equation (2.6).

Contracting the indices \( i \) and \( k \) in the equations (2.4) and (2.5), using(1.9b) and (1.9c), we get

\[ (2.7) \quad \beta_i \beta_m b_n H^i = c_{lmn} H^i_n + d_{lmn} (n-1) y^h \]

and

\[ (2.8) \quad \beta_i \beta_m b_n H = c_{lmn} H_n + d_{lmn} F^2 . \]

The equations (2.7) and (2.8) show that the curvature vector \( H^i \) and the scalar curvature \( H \) can't vanish because the vanishing of them would imply \( d_{lmn} = 0 \), a contradiction.

Thus, we may conclude

**Theorem 2.3.** In \( G \beta H \) – TR \( F_n \), the curvature vector \( H^i \) and the curvature scalar \( H \) are non - vanishing.

We know that [5]

\[ (2.9) \quad H^i_{jk} = H_{hk} - H_{kh} . \]

Taking the \( \beta \) – covariant derivative of the third order with respect to \( x^n, x^m \) and \( x^\ell \), successively,for the equation (2.9), we get
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$$\beta_i \beta_m \beta_n H_{kh}^i = \beta_i \beta_m \beta_n H_{hk} - \beta_i \beta_m \beta_n H_{kh}.$$  
By using (2.6), the above equation can be written
$$\beta_i \beta_m \beta_n H_{kh}^i = c_{\ell mn} H_{hk} + d_{\ell mn} (n-1) g_{hk} - 2y^r b_{mn} (n-1) \beta_r C_{hk\ell} - 2y^r w_{\ell mn} (n-1) \beta_r C_{kh\ell}.$$  
Then, the above equation can be written as
$$\beta_i \beta_m \beta_n H_{kh}^i = c_{\ell mn} (H_{hk} - H_{kh}).$$  
By using (2.9) in the above equation, we get
$$\beta_i \beta_m \beta_n (H_{hk} - H_{kh}) = c_{\ell mn} (H_{hk} - H_{kh}).$$

Thus, we may conclude

**Theorem 2.4.** In $G \beta H$ – TR $F_n$, the tensor $(H_{hk} - H_{kh})$ behaves as trirecurrent.

We know that [5]

$$H_{kh}^i = \frac{1}{3} (\delta_k H_h^i - \delta_h H_k^i).$$

Taking the $\beta$ – covariant derivative of third order with respect to $x^n$, $x^m$ and $x^\ell$, successively, for the equation (2.10), we get
$$\beta_i \beta_m \beta_n H_{kh}^i = \frac{1}{3} \beta_i \beta_m \beta_n (\delta_k H_h^i - \delta_h H_k^i).$$

In view of (1.8c), the above equation can be written as
$$\beta_i \beta_m \beta_n H_{kh}^i = \frac{1}{3} (\beta_i \beta_m \beta_n H_{kh}^i - \beta_i \beta_m \beta_n H_{hk}^i).$$

By using (2.4) in the above equation, we get
$$\beta_i \beta_m \beta_n H_{kh}^i = \frac{1}{3} [ c_{\ell mn} H_{hk} + d_{\ell mn} (\delta_k y_h - \delta_h y_k) - c_{\ell mn} H_{hk} + d_{\ell mn} (\delta_k y_h - \delta_h y_k) ]$$

which can be written as
$$\beta_i \beta_m \beta_n H_{kh}^i = \frac{1}{3} [ c_{\ell mn} (H_{hk} - H_{hk}^i) + d_{\ell mn} (\delta_k y_h - \delta_h y_k)$$
or
$$\beta_i \beta_m \beta_n H_{kh}^i = v_{\ell mn} (H_{hk} - H_{hk}^i) + w_{\ell mn} (\delta_k y_h - \delta_h y_k),$$

where $v_{\ell mn} = \frac{1}{3} c_{\ell mn}$ and $w_{\ell mn} = \frac{2}{3} d_{\ell mn}$.

Thus, we may conclude

**Theorem 2.5.** In $G \beta H$ – TR $F_n$, Berwald torsion tensor $H_{kh}^i$ is generalized – trirecurrent tensor.

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حول تعميم فضاء فنسلر $\beta H$ - ثلاثي المعاودة
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DOI: https://doi.org/10.47372/uajnas.2019.n2.a16

المُلخص
تم تقديم تعريف هذا الفضاء من خلال اشتقاق المعادلة المميزة له لتمييز فضاء $\beta H$ - ثنائي المعاودة $\beta H$، وأطلقتنا عليه الفضاء المعمم $G\beta H$ - ثلاثي المعاودة ورمزنا له بالرمز $\mathcal{G}_{\beta H}$، كما تم الوصول إلى $H_h$ والمشتقة التفوقية $H_{/ H}$ وراقي $H_{/ H}$ - ريشي.

بعض المبرهنة المختصرة بهذا الخصوص. وقد تم إثبات أن الموتير $H_{/ H}$ - ريشي هي موثرة لا تنتمي في هذا الفضاء.

الكلمات المفتاحية: فضاء فنسلر، مشتقة متحدة الاختلاف بمفهوم بروالد من المرتبة الثالثة - موتير $H_{/ H}$ - ريشي.