ON THE FUJITA-ZARISKI DECOMPOSITION ON THREEFOLDS.

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ABSTRACT. We prove that, on a smooth threefold, pseudoeffective divisors with closed and one-dimensional diminished base locus have birationally a Fujita-Zariski decomposition.

1. Introduction

Let $S$ be a smooth projective surface defined over $\mathbb{C}$. Let $D$ be an effective divisor on $S$. In 1962 O. Zariski proved in [17] the existence of two divisors $P, N$ such that

(1) $N = \sum a_i N_i$ is effective, $P$ is nef and $D = P + N$;
(2) either $N = 0$ or the matrix $(N_i \cdot N_j)$ is negative definite;
(3) $(P \cdot N_i) = 0$ for all $i$.

Such a decomposition is called Zariski decomposition of $D$ and it is unique.

Fujita in [7] generalized the statement to pseudoeffective divisors. Moreover he noticed in [8] that the divisor $P$ is the unique divisor that satisfies the following property:

(α): for any birational model $f : X' \to X$ and any nef divisor $L$ on $X'$ such that $f_* L \leq D$ we have $f_* L \leq P$.

Due to the importance of the Zariski decomposition on surfaces, several generalizations to higher dimensional varieties have been studied. A very nice survey that collects the different definitions and their main properties is [16]. The property (α) gives rise to the following generalization.

Definition 1.1 (Definition 6.1, [16]). Let $X$ be a smooth complex projective variety and $D$ a pseudoeffective divisor. A decomposition $D = P_f + N_f$ is called a Zariski decomposition in the Fujita sense (or simply Fujita-Zariski decomposition) if

(1) $N_f \geq 0$;
(2) $P_f$ is nef;
(3) for any birational model $f : X' \to X$ and any nef divisor $L$ on $X'$ such that $f_* L \leq D$ we have $f_* L \leq P$.

It follows from the definition that, if a Fujita-Zariski decomposition exists, then it is unique (see Remark 2.2).

The importance of the Fujita-Zariski decomposition is very well illustrated by the results by Birkar [2] and Birkar-Hu [3] that proved the equivalence between the existence of log minimal
model for pairs and the existence of the Fujita-Zariski decomposition for log canonical divisors. We refer to [2, Theorem 1.5] and [3, Theorem 1.2] for the precise statements.

A measure of the failure of nefness of a pseudoeffective divisor is the diminished base locus. If \( D \) is a pseudoeffective divisor, we define its diminished base locus as follows

\[
\mathbb{B}_-(D) = \bigcup_{A \text{ ample}} \mathbb{B}(D + A)
\]

where \( \mathbb{B}(D + A) = \bigcap \{\text{Supp}(D + A) | \Delta \geq 0, \Delta \sim_{\mathbb{R}} D + A\} \).

The diminished base locus depends only on the numerical equivalence class of \( D \) by [6, Proposition 1.19].

The behavior in many examples shows that, instead of looking for a decomposition of a pseudoeffective divisor \( D \) on \( X \), it is more natural to look for a decomposition of \( f^*D \) on a suitable birational model \( f: Y \to X \) (see for instance [16, Example 6.7]). Even in this setting the existence of such a decomposition fails. Indeed Nakayama found an example [14, Theorem IV.2.10] of a four-dimensional manifold and a pseudoeffective divisor \( D \) such that for any \( f: Y \to X \) birational morphism \( f^*D \) does not admit a Nakayama-Zariski decomposition. Moreover, very recently, J. Lesieutre in [11] found an example of a three-dimensional smooth variety \( X \) and a pseudoeffective divisor \( D \) on \( X \) such that its diminished base locus is dense in \( X \) and that cannot admit a Fujita-Zariski decomposition on any birational model of \( X \).

In this work, in order to prove the existence of the Fujita-Zariski decomposition, we will use another decomposition, introduced by Nakayama [14, Definition III.1] and called Nakayama-Zariski decomposition. Such a decomposition has the following properties (for a precise definition and a proof of the properties see Section 2).

- \( D = P_{\sigma}(D) + N_{\sigma}(D) \) with \( P_{\sigma}(D) \) nef and \( N_{\sigma}(D) \) effective.
- A Nakayama-Zariski decomposition of \( D \) is also a Fujita-Zariski decomposition of \( D \) ([14, Proposition III.1.14, Remark III.1.17(2)])).
- If \( X \) is smooth, the support of \( N_{\sigma}(D) \) is the divisorial part of the diminished base locus (it follows easily from the definitions and from [6, Proposition 1.19]).
- If \( D \) admits birationally a Nakayama-Zariski decomposition, then \( \mathbb{B}_-(D) \) is closed (see Remark 2.7).

Moreover, in order to prove that \( D \) admits a Nakayama-Zariski decomposition, it is not restrictive to assume that \( D \) is nef in codimension one (cf. Remark 2.7). The above facts prove that, if \( D \) is nef in codimension one and admits birationally a Nakayama-Zariski decomposition, then \( \mathbb{B}_-(D) \) is closed and of codimension greater than or equal to 2. In this work we prove that in the three-dimensional case this condition is also sufficient.

**Theorem 1.2.** Let \( X \) be a complex projective smooth variety of dimension 3. Let \( D \) be a pseudoeffective divisor such that its diminished base locus is the union of a finite number of curves. Then there exists a birational morphism \( \mu: \tilde{X} \to X \) such that \( \mu^*D \) has a Nakayama-Zariski decomposition on \( \tilde{X} \).

By [14, Proposition III.1.14, Remark III.1.17(2)] we have the following.
Theorem 1.3. Let $X$ be a complex projective smooth variety of dimension 3. Let $D$ be a pseudoeffective divisor such that its diminished base locus is the union of a finite number of curves. Then there exists a birational morphism $\mu: \tilde{X} \to X$ such that $\mu^*D$ has a Fujita-Zariski decomposition on $\tilde{X}$.

In [14] several partial results were proved on the Zariski decomposition in dimension 3 that correlated the existence of the Fujita-Zariski decomposition over a curve $C$ (see [14, III.4] for a complete definition) to the stability of the conormal bundle of $C$. More precisely, let $D$ be a pseudoeffective divisor on $X$ and $C$ a curve such that $D \cdot C < 0$. Let $I_C$ be the ideal defining $C$ in $X$. If the conormal bundle $I_C/I_C^2$ is semistable, then, as remarked by Nakayama (cf. Lemma 2.14 below), the divisor $P_\sigma(\varphi^*D)$ has positive intersection with every curve of the exceptional divisor of

$$\varphi: \text{Bl}_C X \to X.$$  

If $I_C/I_C^2$ is unstable then, again by Nakayama (cf. Lemma 2.11), there exists a short exact sequence

$$0 \to \mathcal{L} \to I_C/I_C^2 \to \mathcal{M} \to 0$$

such that $\deg \mathcal{L} > \deg \mathcal{M}$. By [14, Lemma III.4.6] if the conormal bundle is not “too much unstable”, namely if $2 \deg \mathcal{M} \geq \deg \mathcal{L}$, then there exists a birational model $\varphi: X' \to X$ such that $\varphi^*D$ has a Fujita-Zariski decomposition over $C$. Therefore the study of the semistability properties of the conormal bundle of a curve in a threefold plays a very important role in the proof of our result. With this respect a key intermediate technical step in the proof of Theorem 1.2 is the following.

Theorem 1.4. Let $X$ be a smooth complex projective variety of dimension 3. Let $\Sigma \subseteq X$ be a smooth curve and assume that the conormal bundle

$$I_\Sigma/I_\Sigma^2$$

is not semistable as a vector bundle of rank two on $\Sigma$. Then there exists a sequence of blow-ups $\varphi: \tilde{X} \to X$ along smooth curves not contained in $\Sigma$ such that, if $\tilde{\Sigma}$ is the strict transform of $\Sigma$ in $\tilde{X}$, then $I_{\tilde{\Sigma}}/I_{\tilde{\Sigma}}^2$ is semistable.

Actually, in order to prove Theorem 1.2 we will need a statement that is much more precise than Theorem 1.4, namely Theorem 3.3, that gives also a control of the degree of the conormal bundle.

The strategy of the proof of Theorem 1.2 will be the following: we will choose a curve $\Sigma_1$ that has negative intersection with $D$ and we will construct a model $\varphi: \tilde{X} \to X$ where $\Sigma_1$ has semistable conormal bundle and where the degree of the conormal bundle is “big enough”. The threefold $\tilde{X}$ is obtained by blowing-up curves that meet $\Sigma_1$ transversally and such that their tangent bundle has a suitable direction. We prove that the diminished base locus $\mathbb{B}_-(\varphi^*D)$ is the union of the strict transforms of the curves of $\mathbb{B}_-(D)$ and two chains of rational curves. Then we blow-up the strict transform of $\Sigma_1$ and the two chains of curves. We obtain $\mu_1: X^{(1)} \to X$. In $X^{(1)}$ we pick a curve $\Sigma_2$ such that $P_\sigma(\mu_1^*D) \cdot \Sigma_2 < 0$ and we repeat the same procedure. We end up with $\mu: \tilde{X} \to X$ such that $\mathbb{B}_-(\mu^*D)$ has pure codimension
one and is contained in Exc(\(\mu\)). Then we prove that \(P_\sigma(\mu^*D)\) is nef. By [4], the curves that have negative intersection with \(P_\sigma(\mu^*D)\) are contained in \(B_-(\mu^*D)\). We verify the nefness with an explicit computation, using the fact that the irreducible components of \(B_-(\mu^*D)\) are birational to ruled surfaces.

We hope that this work can be of some help to find conditions, in dimension \(\geq 3\), insuring the existence of a Fujita-Zariski decomposition. Another application could be the study of foliations in varieties of dimension 3. Indeed in the Brunella-McQuillan classification of foliations on surfaces ([5] [12]) a key role is played by the Zariski decomposition of the canonical bundle of the foliation.

This work is organized as follows: Section 2 collects some preliminary definitions and results about the \(\nu\)-decomposition and the semistability of vector bundles on curves. Section 3 is devoted to the proof of Theorem 3.3, with which we make the conormal bundle of a curve semistable and of degree arbitrarily big. Section 4 contains the proof of Theorem 1.2.

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2. Preliminaries

In this section we collect some definitions and basic facts about the Fujita-Zariski decomposition, the \(\sigma\)-decomposition and the \(\nu\)-decomposition. Moreover we state various results on curves that will be used later.

2.1. Fujita-Zariski decomposition.

**Definition 2.1** (Definition 6.1, [16]). Let \(X\) be a smooth complex projective variety and \(D\) a pseudoeffective divisor. A decomposition \(D = P_f + N_f\) is called a Zariski decomposition in the Fujita sense (or simply Fujita-Zariski decomposition) if

\[(1)\quad N_f \geq 0;\]
\[(2)\quad P_f \text{ is nef};\]
\[(3)\quad \text{for any birational model } f : X' \to X \text{ and any nef divisor } L \text{ on } X' \text{ such that } f_*L \leq D \text{ we have } f_*L \leq P.\]

**Remark 2.2.** It follows from the definition that, if a Fujita-Zariski decomposition exists, then it is unique. Indeed, if \(D = P'_f + N'_f\) is another Fujita-Zariski decomposition, then, from the property (3) of the definition applied to the two decompositions \(D = P'_f + N'_f\) and \(D = P_f + N_f\), we obtain \(P_f \leq P'_f\) and \(P'_f \leq P_f\).
2.2. $\sigma$-decomposition. In [14] we have the following definitions. Let us denote by $|B|_{\text{num}}$ the set of effective $\mathbb{R}$-divisors $\Delta$ numerically equivalent to $B$.

**Definition 2.3** (Definition III.1.1, [14]). Let $D$ be a pseudoeffective divisor of a nonsingular projective variety. Let $\Gamma$ be a prime divisor and $A$ an ample divisor. We define

$$
\sigma_{\Gamma}(D) = \lim_{\varepsilon \to 0} \inf_{\varepsilon > 0} \{\text{mult}_{\Gamma}\Delta \mid \Delta \in |D + \varepsilon A|_{\text{num}}\}.
$$

The limit does not depend on the choice of the ample divisor $A$ by [14, Lemma III.1.5] and thus it depends only on the numerical equivalence class of $D$. Moreover, by [14, Corollary III.1.11] there is only a finite number of prime divisors $\Gamma$ satisfying $\sigma_{\Gamma}(D) > 0$. Thus the expression

$$
\sum_{\Gamma} \sigma_{\Gamma}(D)\Gamma
$$

defines a divisor.

**Definition 2.4** (Definition III.1.12, [14]). Let $D$ be a pseudoeffective divisor of a nonsingular projective variety. We define

$$
N_{\sigma}(D) = \sum \sigma_{\Gamma}(D)\Gamma \quad \text{and} \quad P_{\sigma}(D) = D - N_{\sigma}(D).
$$

The decomposition $D = P_{\sigma}(D) + N_{\sigma}(D)$ is called the $\sigma$-decomposition of $D$.

**Definition 2.5** (Definition III.1.16, [14]). The $\sigma$-decomposition $D = P_{\sigma}(D) + N_{\sigma}(D)$ for a pseudoeffective $\mathbb{R}$-divisor is called Zariski decomposition in Nakayama’s sense (or simply the Nakayama-Zariski decomposition) if $P_{\sigma}(D)$ is nef.

**Remark 2.6.** By [14, Proposition III.1.14, Remark III.1.17(2)], if the Nakayama-Zariski decomposition exists then it is the Fujita-Zariski decomposition. The converse is not known.

**Remark 2.7.** If $D$ is a pseudoeffective divisor that has birationally a Nakayama-Zariski decomposition then its diminished base locus is closed. Indeed let $f: Y \to X$ be a birational model such that $f^*D = P_{\sigma}(f^*D) + N_{\sigma}(f^*D)$ is a Nakayama-Zariski decomposition. Then

$$
\mathbb{B}_-(f^*D) = \bigcup_{A} \mathbb{B}(N_{\sigma}(f^*D) + P_{\sigma}(f^*D) + A)
$$

and, since $P_{\sigma}(f^*D) + A$ is ample,

$$
\mathbb{B}(N_{\sigma}(f^*D) + P_{\sigma}(f^*D) + A) \subseteq \text{Supp}N_{\sigma}(f^*D)
$$

for any $A$, showing that $\mathbb{B}_-(f^*D) \subseteq \text{Supp}N_{\sigma}(f^*D)$. The other containment follows from the definitions of $\sigma$-decomposition and diminished base locus. Then $\mathbb{B}_-(f^*D) = \text{Supp}N_{\sigma}(f^*D)$ is closed and so is $\mathbb{B}_-(D)$ because by e.g. [10, Proposition 2.5]

$$
\mathbb{B}_-(f^*D) = f^{-1}\mathbb{B}_-(D).
$$
Remark 2.8. If $D$ admits birationally a Nakayama-Zariski decomposition, then the diminished base locus of $P_\sigma(D)$ is the union of a finite number of subvarieties of codimension at least two. Indeed it follows easily from the definitions and from [6, Proposition 1.19] that the diminished base locus of the positive part $B_-(P_\sigma(D))$ does not have any component of codimension one. Moreover, if there exists a birational model $\mu: \tilde{X} \to X$ such that $\mu^*P_\sigma(D)$ has a Fujita-Zariski decomposition on $\tilde{X}$

\[ \mu^*P_\sigma(D) = \tilde{P} + \tilde{N}, \]

then the decomposition

\[ \mu^*D = \tilde{P} + \tilde{N} + \mu^*N_\sigma(D) \]

gives a Fujita-Zariski decomposition for $\mu^*D$ on $\tilde{X}$.

2.3. $\nu$-decomposition. Let $D$ be a pseudoeffective divisor. We define the set

\[ S(D) = \{ \Delta \geq 0, \mathbb{R} \text{-divisor} : (D - \Delta)|_\Gamma \text{ is pseudoeffective } \forall \Gamma \text{ prime divisor} \}. \]

For the definition of pseudoeffectivity of $(D - \Delta)|_\Gamma$ we refer to [14, Remark II.5.8].

Definition 2.9 (Definition III.3.2, [14]). Let $D$ be a pseudoeffective divisor. We set

\[ N_\nu(D) = \sum \inf \{ \mathrm{mult}_\Gamma \Delta | \Delta \in S(D) \} \Gamma \]

where the sum runs over all the prime divisors $\Gamma$ in $X$ and set $P_\nu(D) = D - N_\nu(D)$. The decomposition $D = P_\nu(D) + N_\nu(D)$ is called $\nu$-decomposition.

By [14, Proposition III.1.14, Remark III.3.10] if $P_\nu(D)$ is movable, that is, it belongs to the closure of the cone in $N^1(X)$ spanned by classes of fixed part free divisors, then $P_\nu(D) = P_\sigma(D)$.

2.4. Useful results on curves.

Definition 2.10. A vector bundle $E$ on a smooth compact curve is said to be semistable if for any sub vector bundle $0 \neq F \subseteq E$ the following inequality is true

\[ \frac{\deg \det F}{\rank F} \leq \frac{\deg \det E}{\rank E}. \]

Lemma 2.11 (Lemma 1.1, [15]). Let $E$ be a locally free sheaf of rank two on a smooth compact curve $C$.

1. If $E$ is a semistable vector bundle then there exist no curves $\Gamma$ on the ruled surface $\mathbb{P}_C(E)$ with $\Gamma^2 < 0$.

2. If $E$ is unstable, then there exists a unique (up to isomorphisms) exact sequence

\[ 0 \to L \to E \to M \to 0 \quad (2.1) \]

which satisfies the following two conditions:

- $L$ and $M$ are invertible sheaves on $C$;
- $\deg L > \deg M$. 

Remark 2.12. The sequence (2.1) is the Harder-Narashiman filtration and \( L \) is the maximal destabilizing subsheaf.

Definition 2.13. The sequence (2.1) is called characteristic exact sequence of \( E \). We set \( \delta(E) = \deg L - \deg M \). If \( E = I_C/I_C^2 \) is the conormal bundle of a curve in a threefold, then we adopt the notation \( \delta(C) = \delta(I_C/I_C^2) \).

The following lemma plays an important role in the proof of Theorem 1.2.

Lemma 2.14 (Lemma III.4.5, [14]). Let \( D \) be a pseudoeffective divisor on \( X \), let \( C \) be a smooth curve such that \( D \cdot C < 0 \). Let \( \mu: X' \rightarrow X \) be the blowing-up of \( C \) and let \( E \) be the exceptional divisor. If the conormal bundle \( I_C/I_C^2 \) is semistable then the coefficient of \( N_\nu(\mu^*D) \) along \( E \) is

\[
-2D \cdot C \quad \text{deg } I_C/I_C^2
\]

and the positive part \( P_\nu(\mu^*D) \) is nef on \( E \), that is, \( P_\nu(\mu^*D)|_E \) is nef.

The following lemma is probably well known to experts. Since we could not find a reference in the literature, we put a proof here for the reader’s convenience.

Lemma 2.15. Let \( X \) be a smooth variety of dimension 3.

1. Let \( C \subseteq X \) be a curve. Then there exists a birational morphism \( \eta: W \rightarrow X \), composition of blow-ups along smooth curves, such that

\[
\eta^{-1}C = \tilde{C} \cup \bigcup_i G_i
\]

where \( \tilde{C} \) is the strict transform of \( C \) and it is smooth and \( G_i \) is a smooth curve for all \( i \).

2. Let \( C_j \), for \( j = 1, \ldots, l \), be a smooth curve in \( X \). Then there exists a birational morphism \( \eta: W \rightarrow X \), composition of blow-ups along smooth curves, such that

\[
\eta^{-1}(C_1 \cup \ldots \cup C_l) = \bigcup_i G_i
\]

where \( G_i \) is a smooth curve for all \( i \) and for all \( j_1 \neq j_2 \) the curves \( G_{j_1} \cap G_{j_2} \) intersect transversally in at most one point.

Proof. (1) If \( C \) is smooth there is nothing to prove. Then assume that \( C \) is singular. Let \( p \in C \) be a singular point. In a local analytic neighborhood \( U \) of \( p \) we can write \( C \) as a union of irreducible components

\[
C = C_1 \cup \ldots \cup C_k.
\]

We first reduce to the case where \( C_i \) is smooth at \( p \) for every \( i \). Let \( C' \) be one of the \( C_i \)'s. Modulo shrinking \( U \), we can assume that it is isomorphic to an open neighborhood of the origin in \( \mathbb{C}^3 \) and by [9, Theorem 2.26] we can find a map

\[
\gamma: \mathbb{C} \rightarrow C' \\
0 \mapsto p
\]
that is injective and such that the derivative of \( \gamma \) is non zero for any \( t \neq 0 \). If we write the development of each component of \( \gamma \) as a Laurent series we have

\[
\gamma(t) = \left( t^l, \sum a_i t^{m_i}, \sum b_i t^{n_i} \right).
\]

We can assume that the first component is monomial by composing with a suitable biholomorphism of the source \( \mathbb{C} \). We can also assume that

\[
l \leq m_1 \leq n_1.
\]

The injectivity of \( \gamma \) implies that \( l \), the \( m_i \)'s and the \( n_i \)'s are coprime. The order of \( \gamma \) in zero is the minimum of the orders of the three components. We prove by induction on the order that we can desingularize \( C' \) with blow-ups of smooth curves.

If \( l = 1 \) then \( C' \) is smooth.

Assume that \( l > 1 \). Since \( l \), the \( m_i \)'s and the \( n_i \)'s are coprime, there exists an exponent \( m_i \) or \( n_i \) that is not divisible by \( l \). Without loss of generality we can assume that the smaller of such exponents is one of the \( m_i \)'s. Then there exists a biholomorphism of the target \( \mathbb{C}^3 \) of the form

\[
\Psi(x, y, z) = (x, y - p_1(x), z)
\]

with the following properties: \( p_1 \) is a polynomial and

\[
\Psi \circ \gamma(t) = \left( t^l, \sum a_i t^{m_i}, \sum b_i t^{n_i} \right) = (t^l, t^{m'_1} u(t), t^{n_1} v(t))
\]

where \( l \) does not divide \( m'_1 \) and \( u \) and \( v \) are invertible functions. Let

\[
\tilde{X} \to X
\]

be the blowing up of a smooth curve \( \Gamma \) such that

\[
U \cap \Gamma = \{ x = y = 0 \}.
\]

Then a parametrization for the strict transform of \( C' \) is

\[
\tilde{\gamma}(t) = (t^l, t^{m'_1 - l} u(t), t^{n_1} v(t)).
\]

Let

\[
m'_1 = l \cdot q + r
\]

be the result of the euclidean division of \( m'_1 \) by \( l \). If we blow-up \( q \) times a curve of local equation

\[
\{ x = y = 0 \},
\]

a parametrization for the strict transform of \( C' \) is

\[
\tilde{\gamma}(t) = (t^l, t^r u(t), t^{n_1} v(t)).
\]

The order of \( \tilde{\gamma} \) at the singular point is thus \( r < l \). Then we apply the inductive hypothesis and we conclude.

We separate the irreducible components. Now we can assume that \( C_i \) is smooth in \( p \) for every \( i \). Let \( C_1 \) and \( C_2 \) be two irreducible components and let \( \tau_i \) be the tangent of \( C_i \) at \( p \).
If $\tau_1$ and $\tau_2$ are not colinear then we blow-up along a curve $\Gamma$ whose tangent does not lie in the plane generated by $\tau_1$ and $\tau_2$. If $\tilde{C}_1$ is the strict transform of $C_1$ then

$$\tilde{C}_1 \cap \tilde{C}_2 = \emptyset.$$ 

If $C_1$ and $C_2$ have the same tangent direction then we can find two parametrizations of the following form:

$$\gamma_1: \mathbb{C} \to C_1 \quad t \mapsto (t, 0, 0)$$

of $C_1$ and

$$\gamma_2: \mathbb{C} \to C_1 \quad t \mapsto (tw_1(t), t^m w_2(t), t^n w_3(t))$$

of $C_2$ where $w_i$ is an invertible function and $1 \leq m \leq n$. Since $C_2$ has the same tangent direction as $C_1$, we have $m > 1$. We prove by induction on $m$ that we can separate $C_1$ and $C_2$. We blow-up along a curve whose local equation in $U$ is

$$\{x = y = 0\}$$

and we obtain

$$\tilde{X} \to X.$$ 

A parametrization for the strict transforms $\tilde{C}_1$ and $\tilde{C}_2$ are

$$\tilde{\gamma}_1: \mathbb{C} \to C_1 \quad t \mapsto (t, 0, 0)$$

for $\tilde{C}_1$ and

$$\tilde{\gamma}_2: \mathbb{C} \to C_1 \quad t \mapsto (tw_1(t), t^{m-1} w'_2(t), t^n w_3(t))$$

for $\tilde{C}_2$ where $w'_2 = w_2/w_1$. Then we conclude by inductive hypothesis. We notice that the preimage of the singular point $p$ in $\tilde{X}$ is a curve of local equation $\{x = z = 0\}$. Thus it meets $\tilde{C}_1$ and $\tilde{C}_2$ transversally.

(2) The proof of this second item follows the same line as the proof of the first. The statement is proved by blowing-up generic smooth curves through $C_{j_1} \cap C_{j_2}$.

3. Making the conormal bundle semistable

**Lemma 3.1.** Let $X$ be a smooth complex projective variety of dimension 3. Let $\Sigma \subseteq X$ be an irreducible smooth curve. Let

$$0 \to A \to I_{\Sigma}/I_{\Sigma}^2 \to B \to 0$$

be a presentation of the conormal bundle of $\Sigma$. Let $\Gamma$ be a smooth curve that meets $\Sigma$ transversally in one point $p$ and such that the composition

$$A_p \to (I_{\Sigma}/I_{\Sigma}^2)_p \to \Omega_{X,p} \to \Omega_{\Gamma,p}$$
is nonzero. Let $\varphi: X_1 \to X$ be the blow-up of $\Gamma$ and let $\Sigma_1$ be the strict transform of $\Sigma$. Then the conormal bundle of $\Sigma_1$ has a presentation

$$0 \to \tilde{A} \to I_{\Sigma_1}/I_{\Sigma_1}^2 \to \tilde{B} \to 0$$

where $\tilde{A} = \varphi^*A$, $\deg I_{\Sigma_1}/I_{\Sigma_1}^2 = \deg I_{\Sigma}/I_{\Sigma_1}^2 + 1$ and $\deg \tilde{B} = \deg B + 1$.

**Proof.** We have the following short exact sequence of sheaves on $X_1$

$$0 \to \varphi^*\Omega_X \xrightarrow{\Phi} \Omega_{X_1} \xrightarrow{\alpha} \Omega_{X_1/X} \xrightarrow{\beta} 0.$$ 

The morphism of sheaves $\Phi$ is an isomorphism over $X \setminus \Gamma$.

Since $\Sigma \subseteq X$ is a smooth curve, we have the following exact sequence

$$0 \to I_{\Sigma}/I_{\Sigma}^2 \to \Omega_X \otimes \mathcal{O}_{\Sigma} \to \Omega_{\Sigma} \to 0.$$ (3.2)

Analogously for $\Sigma_1 \subseteq X_1$, the strict transform of $\Sigma$, we have

$$0 \to I_{\Sigma_1}/I_{\Sigma_1}^2 \to \Omega_{X_1} \otimes \mathcal{O}_{\Sigma_1} \to \Omega_{\Sigma_1} \to 0.$$ (3.3)

The restriction of the blow-up $\varphi: \Sigma_1 \to \Sigma$ is an isomorphism. Then, sequence (3.2) pulls back to an exact sequence of vector bundles on $\Sigma_1$

$$0 \to \varphi^*I_{\Sigma}/I_{\Sigma}^2 \to \varphi^*\Omega_X \otimes \mathcal{O}_{\Sigma_1} \to \varphi^*\Omega_{\Sigma} \to 0.$$ (3.4)

We claim that

$$\Phi(\varphi^*(I_{\Sigma}/I_{\Sigma}^2)) \subseteq I_{\Sigma_1}/I_{\Sigma_1}^2.$$ 

Indeed we have the following commutative diagram with exact columns

$$
\begin{array}{ccc}
0 & \to & \varphi^*\Omega_X \otimes \mathcal{O}_{\Sigma_1} \\
\varphi^*(I_{\Sigma}/I_{\Sigma}^2) & \xrightarrow{\Phi} & \Omega_{X_1} \otimes \mathcal{O}_{\Sigma_1} \\
0 & \to & \varphi^*\Omega_{\Sigma} \xrightarrow{\alpha} \Omega_{X_1/X} \otimes \mathcal{O}_{\Sigma_1} \\
\varphi^*\Omega_{\Sigma} & \xrightarrow{\cong} & \Omega_{\Sigma_1} \\
0 & \to & 0 \\
\end{array}
$$

Since $\Phi$ induces an isomorphism between $\varphi^*\Omega_{\Sigma}$ and $\Omega_{\Sigma_1}$, we have $\Phi(\ker \alpha) \subseteq \ker \beta$, and the claim is proved.

Moreover the sheaf $\Omega_{X_1/X} \otimes \mathcal{O}_{\Sigma_1}$ is the skyscraper sheaf supported on $p$,

$$\Omega_{X_1/X} \otimes \mathcal{O}_{\Sigma_1} \cong \mathbb{C}_p.$$
Thus we have
\[(3.5) \quad 0 \to \varphi^*(I_{\Sigma}/I_{\Sigma}^2) \xrightarrow{\Phi} I_{\Sigma_1}/I_{\Sigma_1}^2 \to \mathbb{C}_p \to 0.\]
By sequence (3.5) we have
\[(3.6) \quad \text{deg } I_{\Sigma_1}/I_{\Sigma_1}^2 = \text{deg } I_{\Sigma}/I_{\Sigma}^2 + 1.\]
The morphism $\Phi$ has the property that
\[(3.7) \quad \Phi|_{\varphi^*A} \text{ is injective}.\]
Indeed $\Phi$ is an isomorphism over $\Sigma \setminus \{p\}$ and, if we consider the stalk over $p$, on $\varphi^*A_p$ it is nonzero by hypothesis. The sheaf defined by $\tilde{A} := \Phi(\varphi^*A)$ is a sub vector bundle of rank one of $I_{\Sigma_1}/I_{\Sigma_1}^2$. Set $\tilde{B}$ for the quotient, so that we have
\[(3.8) \quad 0 \to \tilde{A} \to I_{\Sigma_1}/I_{\Sigma_1}^2 \to \tilde{B} \to 0.\]
The condition on the degree of $\tilde{B}$ follows from the choice of $\tilde{A}$ and from (3.6). \qed

**Lemma 3.2.** Let $X$ be a smooth complex projective variety of dimension 3. Let $\Sigma \subseteq X$ be an irreducible smooth curve. Assume that the conormal bundle of $\Sigma$ is unstable and that $\delta(\Sigma) = 1$. Let
\[(3.9) \quad 0 \to \mathcal{L} \to I_{\Sigma}/I_{\Sigma}^2 \to \mathcal{M} \to 0\]
be the characteristic exact sequence. Let $\varphi: X_1 \to X$ be the blow-up of a smooth curve $\Gamma$ as in Lemma 3.1 for the sequence (3.8). Let $\Sigma_1$ be the strict transform of $\Sigma$ in $X_1$. Then the conormal bundle of $\Sigma_1$ is semistable.

**Proof.** Let
\[(3.10) \quad 0 \to \tilde{\mathcal{L}} \to I_{\Sigma_1}/I_{\Sigma_1}^2 \to \tilde{\mathcal{M}} \to 0\]
be the sequence given by Lemma 3.1. Then $\text{deg } \tilde{\mathcal{L}} = \text{deg } \tilde{\mathcal{M}}$. Assume that that $I_{\Sigma_1}/I_{\Sigma_1}^2$ is unstable, so by Lemma 2.11(2) we have the characteristic sequence
\[(3.11) \quad 0 \to \mathcal{L}' \to I_{\Sigma_1}/I_{\Sigma_1}^2 \to \mathcal{M}' \to 0\]
and, by definition of characteristic sequence, $\text{deg } \mathcal{L}' > \text{deg } \mathcal{M}'$. Consider now the morphism of sheaves $\chi: \tilde{\mathcal{L}} \to \mathcal{M}'$ given by the composition of the injective arrow of (3.11) and the surjective arrow of (3.9). If $\chi$ is identically zero, then $\tilde{\mathcal{L}} \cong \mathcal{L}'$, which is a contradiction because then also $\tilde{\mathcal{M}} \cong \mathcal{M}'$, but $\text{deg } \mathcal{L} = \text{deg } \mathcal{M}$ and $\text{deg } \mathcal{L}' > \text{deg } \mathcal{M}'$. Then $\chi$ is non-zero, which implies the inequalities $$\text{deg } \mathcal{L}' > \text{deg } \mathcal{M}' \geq \text{deg } \tilde{\mathcal{L}} = \text{deg } \tilde{\mathcal{M}}.$$ But this leads again to a contradiction because $$\text{deg } \mathcal{L}' + \text{deg } \mathcal{M}' = \text{deg } (\det I_{\Sigma_1}/I_{\Sigma_1}^2) = \text{deg } \tilde{\mathcal{L}} + \text{deg } \tilde{\mathcal{M}}.$$ Therefore, if $\delta(\Sigma) = 1$, the conormal bundle $I_{\Sigma_1}/I_{\Sigma_1}^2$ is semistable. \qed
Theorem 3.3. Let $X$ be a smooth complex projective variety of dimension 3. Let $\Sigma \subseteq X$ be an irreducible smooth curve and $N$ an integer number. Then there exists a birational model $\varphi: \hat{X} \to X$ given by a sequence of blow-ups along smooth curves not contained in $\Sigma$ with the following properties. Let $\hat{\Sigma}$ be the strict transform of $\Sigma$ in $\hat{X}$.

1. The degree of $I_{\hat{\Sigma}}/I_{\hat{\Sigma}}^2$ is at least $N$.
2. The vector bundle $I_{\hat{\Sigma}}/I_{\hat{\Sigma}}^2$ is semistable.
3. $\varphi^{-1}\Sigma = \hat{\Sigma} \cup \bigcup_{i=1}^{n} F_s^i \cup \bigcup_{i=1}^{m} F_d^i$ is a chain of curves and $\bigcup_{i=1}^{n} F_s^i$ and $\bigcup_{i=1}^{m} F_d^i$ are chains of smooth rational curves intersecting $\hat{\Sigma}$ in one point.

Proof. (1) Write the conormal bundle as extension of two vector bundles of rank one.

$$0 \to A \to I_{\hat{\Sigma}}/I_{\hat{\Sigma}}^2 \to B \to 0.$$ 

Let $\Gamma_1$ be a smooth curve as in Lemma 3.1. Let $\psi_1: X_1 \to X$ be the blow up of $\Gamma_1$. If $\Sigma_1$ is the strict transform of $\Sigma$, then $\deg I_{\Sigma_1}/I_{\Sigma_1}^2 = I_{\Sigma}/I_{\Sigma}^2 + 1$. Let $E_1$ be the exceptional divisor of $\psi_1$. It is easy to verify that a section $\Gamma_2$ of $E_1 \to \Sigma_1$ passing through $\Sigma_1$ verifies the hypothesis of Lemma 3.1. Then we blow-up $\Gamma_2$. We continue this process until we reach degree $N$.

(2) Assume that we already have the condition on the degree of the conormal bundle of $\Sigma$. If $I_{\Sigma}/I_{\Sigma}^2$ is unstable, consider its characteristic sequence

$$0 \to L \to I_{\Sigma}/I_{\Sigma}^2 \to M \to 0. \tag{3.10}$$

We apply Lemma 3.1 to sequence (3.10): we blow-up a curve $\Gamma_1$ and we obtain $\varphi_1: X_1 \to X$. Let $E_1$ be the exceptional divisor of $\varphi_1$. Then, as in item (1), we blow-up a section $\Gamma_2$ of $E_1 \to \Gamma_1$ meeting $\Sigma$ and we repeat this process $n = \delta(\Sigma)$ times. We obtain $\hat{X} \to X$. Let $\hat{\Sigma}$ be the strict transform of $\Sigma$ in $\hat{X}$. We prove that the conormal bundle of $\hat{\Sigma}$ is semistable by induction on $n$. Let us first suppose that $\delta(\Sigma) = 1$. Then it follows from Lemma 3.2.

Assume that $\delta(\Sigma) > 1$. It is sufficient to prove that $\delta(\Sigma_1) = \delta(\Sigma) - 1$. By Lemma 2.11(2), the sequence

$$0 \to \varphi_1^*L \to I_{\Sigma_1}/I_{\Sigma_1}^2 \to \tilde{M} \to 0$$

given by Lemma 3.1 is the characteristic exact sequence of $\Sigma_1$. Since, again by Lemma 3.1, $\deg \tilde{M} = \deg M + 1$, we have that $\delta(\Sigma_1) = \delta(\Sigma) - 1$. We remark that at each step the degree of the conormal sheaf grows by one:

$$\deg I_{\Sigma_1}/I_{\Sigma_1}^2 = \deg I_{\Sigma}/I_{\Sigma}^2 + 1$$

so item (1) is preserved.

(3) Let $F_d^i$ be the intersection of the preimage of $\Sigma$ with the $i$-th exceptional divisor of the blow-ups made in order to reach degree $N$. Let $F_s^i$ be the intersection of the preimage of $\Sigma$ with the $i$-th exceptional divisor of the blow-ups made in order to reach semistability (see Figure 1). Then both $F_d^i$ and $F_s^i$ are rational curves because they are contained in fibers of the respective blow-ups. It follows from the construction that the $F_s^i$ and the $F_d^i$ form two chains of rational curves. 

\[\square\]
Lemma 3.4. Let \( X \) be a smooth variety of dimension three, let \( F \subseteq X \) be a rational curve. Let \( I_{F_1}/I_{F_1}^2 = \mathcal{O}(a) \oplus \mathcal{O}(b) \) be the conormal bundle of \( F \) in \( X \) and suppose that \( a > b \). Let \( \varphi : X_1 \to X \) be a blow-up given by Lemma 3.1 for the sequence
\[
0 \to \mathcal{O}(a) \to I_{F_1}/I_{F_1}^2 \to \mathcal{O}(b) \to 0.
\]
Let \( F_1 \) be the strict transform of \( F \) in \( X_1 \). Then
\[
I_{F_1}/I_{F_1}^2 = \mathcal{O}(a) \oplus \mathcal{O}(b + 1).
\]

Proof. Since \( F_1 \) is a rational curve, we have
\[
I_{F_1}/I_{F_1}^2 = \mathcal{O}(a') \oplus \mathcal{O}(b')
\]
for some integers \( a', b' \). By Lemma 3.1 \( a' + b' = a + b + 1 \). By the proof of Theorem 3.3 we know that \( a' - b' = a - b - 1 \), leaving as the only possibility \( a' = a \) and \( b' = b + 1 \). \( \square \)
Proposition 3.5. Notation as in Theorem 3.3. The conormal bundle of $F^s_i$, and respectively
of the $F^d_i$, is isomorphic to

$$I_{F^s_i}/I_{F^s_i}^2 = \begin{cases} \mathcal{O} \oplus \mathcal{O}(1) & i = n \\ \mathcal{O} \oplus \mathcal{O}(2) & i < n \end{cases}$$

and

$$I_{F^d_i}/I_{F^d_i}^2 = \begin{cases} \mathcal{O} \oplus \mathcal{O}(1) & i = m \\ \mathcal{O} \oplus \mathcal{O}(2) & i < m \end{cases}$$

Proof. We prove the statement for the curves $F^s_i$, the proof for the $F^d_i$’s being completely
analogous. Let

$$X_n \xrightarrow{\varphi_n} X_{n-1} \rightarrow \ldots \rightarrow X_1 \xrightarrow{\varphi_1} X$$

be the sequence of blow-ups performed in order to achieve semistability. By abuse of notation
we denote with $F^s_i$ the curve in $X_i$ as well as its strict transform in $X_j$ for any $j > i$ and in
$\hat{X}$. If $i = n$, then $F^s_n$ is the fiber of a blow-up and the statement a well-known fact. If $i < n$,
then $F^s_i \subseteq X_i$ has conormal bundle $\mathcal{O} \oplus \mathcal{O}(1)$. By Lemma 3.4, its strict transform in $X_{i+1}$
has conormal bundle $\mathcal{O} \oplus \mathcal{O}(2)$. Then the statement follows because $\varphi_j$ is an isomorphism on
$F^s_i$ for any $j > i + 1$. \hfill $\square$

Remark 3.6. Theorem 1.4 is exactly Theorem 3.3(2).

Remark 3.7. Notice that we cannot perform the blow-ups needed to achieve Theorem 3.3(2)
before those needed to achieve item (1). Indeed, after item (1) the conormal bundle could
not be semistable anymore, even if it had this property before starting the process. On the
other hand, the “semistabilization” naturally increases the degree of the conormal bundle.

4. Proof of the main result

Proof of Theorem 1.2. By Lemma 2.15 we can assume that every irreducible component of
$\mathbb{B}_-(D)$ is a smooth curve. Moreover we can assume that if $\Sigma$, $\Sigma'$ are two irreducible components
of $\mathbb{B}_-(D)$ then the intersection $\Sigma \cap \Sigma'$ is at most one point.

If $D \cdot \Sigma \geq 0$ for any curve $\Sigma \subseteq \mathbb{B}_-(D)$ then $D$ is nef, $\mathbb{B}_-(D) = \emptyset$ and there is nothing to
prove. Then we can assume that there exist curves in $\mathbb{B}_-(D)$ that intersect $D$ negatively.

STEP 1.1 Semistabilization of $\Sigma_1$ and construction of $X^{(1)}$.

Let $\Sigma_1$ be a curve such that $\Sigma_1 \cdot D < 0$. Let $\varphi : \hat{X} \to X$ be a morphism given by Theorem 3.3
applied to $\Sigma_1$. Let $\hat{\Sigma}_1$ be the strict transform of $\Sigma_1$ in $\hat{X}$. Then the conormal bundle of
$\Sigma_1$ is semistable and

$$\varphi^{-1}\Sigma_1 = \hat{\Sigma}_1 \cup \bigcup_{i=1}^n F^s_{1,i} \cup \bigcup_{i=1}^m F^d_{1,i}$$

where $\bigcup F^d_{1,i}$ arose when increasing the degree of the conormal bundle and $\bigcup F^s_{1,i}$ is the chain
arisen from the process of semistabilization of $\Sigma_1$. By abuse of notation we denote by $n$ and
Let $d_1$ be the degree of the conormal bundle of $\hat{\Sigma}_1$. We can take $\varphi$ such that $d_1 > 4$. Set

\[ c_1 = \text{card} \left( \Sigma_1 \cap \bigcup_{i > 1} \Sigma_i \right). \]

Let $\pi: X' \to \hat{X}$ be the blow-up of $\hat{\Sigma}_1$ and $E_1$ the exceptional divisor. By abuse of notation we still denote by $F_{1,i}^s$ the strict transform of $F_{1,i}^s$ in the higher birational models of $\hat{X}$. Then, by Proposition 3.5 and Lemma 3.4, $F_{1,n}^s$ has conormal bundle $O(1) \oplus O(1)$ in $X'$. Let $\nu_n: Y_n \to X''$ be the blow-up of $F_{1,n}^s$. Then, again by Proposition 3.5 and Lemma 3.4, $F_{1,n-1}^s$ has conormal bundle $I_{F_{1,n-1}^s} = O(1) \oplus O(2)$ in $Y_n$. Then $\delta(F_{1,n-1}^s) = 1$ and we are in the situation of Lemma 3.2. Let $\Gamma_{1,n-1}$ be as in Lemma 3.2 and let $\nu'_{n-1}: Y_{n-1} \to Y_n$ be its blow-up, so that $F_{1,n-1}^{ss}$ has semistable conormal bundle in $Y_{n-1}$ and

\[ (\nu'_{n-1})^{-1} F_{1,n-1}^s = F_{1,n-1}^{ss} \cup F_{1,n-1}^s. \]

The curve $F_{1,n-1}^{ss}$ has conormal bundle $I_{F_{1,n-1}^{ss}}/I_{F_{1,n-1}^s}^2 = O \oplus O(1)$ by Proposition 3.5. Let $\nu_1: Y_1 \to Y''$ be the blow-up of $F_{1,n-1}^{ss}$. Then the strict transform of $F_{1,n-1}^{ss}$ in $Y''$ has conormal bundle semistable and isomorphic to $O(1) \oplus O(1)$.

Let $\nu_{n-1}: Y_{n-1} \to Y_{n-1}'$ the blow-up of (the strict transform of) $F_{1,n-2}^s$. Then the conormal bundle of $F_{1,n-2}^s$ in $Y_{n-1}$ is

\[ I_{F_{1,n-2}^s}/I_{F_{1,n-2}^s}^2 = O(1) \oplus O(2) \]

by Proposition 3.5 and Lemma 3.4. We apply again Lemma 3.2 and we perform the same sequence of blow-ups in order to construct $Y_{n-2} \to Y_{n-1}$. Thus we obtain a sequence

\[ Y_1 \xrightarrow{\nu_1} Y_1' \xrightarrow{\nu'_1} Y_1'' \xrightarrow{\nu''_1} \ldots \]

\[ \ldots \xrightarrow{\nu_{n-1}} Y_{n-1} \xrightarrow{\nu'_{n-1}} Y_{n-1}' \xrightarrow{\nu''_{n-1}} Y_{n-1}'' \xrightarrow{\nu_{n}} Y_n \xrightarrow{\nu_a} X' \xrightarrow{\pi} \hat{X} \xrightarrow{\varphi} X. \]

Then we do the same operations on the curves $F_{1,j}^d$. The curve $F_{1,m}^d$ has conormal bundle $O(1) \oplus O(1)$ in $Y_1$. We blow-up $F_{1,m}^d$ and denote the morphism $\xi_m: Z_m \to Y_1$. As before we consider $\xi''_{m-1}: Z''_{m-1} \to Z_m$ so that $F_{1,m-1}^d$ has semistable conormal bundle in $Z''_{m-1}$ and

\[ (\xi''_{m-1})^{-1} F_{1,m-1}^d = F_{1,m-1}^d \cup F_{1,m-1}^{ds} \].
Figure 2.

Let $\xi'_{h-1} : Z'_{m-1} \rightarrow Z''_{m-1}$ be the blow-up of $F_{1,m-1}^d$ and $\xi_{m-1} : Z_{m-1} \rightarrow Z'_{m-1}$ the blow-up of $F_{1,m-1}^{ds}$.

Let $\mu_1 : X^{(1)} \rightarrow X$ be the composition of $\varphi$, $\pi$ the $\nu_i$, $\nu'_i$, $\nu''_i$ and the $\xi_i$, $\xi'_i$, $\xi''_i$.

$$\mu_1 : X^{(1)} = Z_1 \xrightarrow{\xi_1} Z'_1 \xrightarrow{\xi'_1} Z''_1 \xrightarrow{\xi''_1} \ldots$$

$$\ldots \xrightarrow{\xi'_{m-1}} Z_m \xrightarrow{\xi_m} Y_1 \xrightarrow{\nu_1} Y'_1 \xrightarrow{\nu'_1} Y''_1 \xrightarrow{\nu''_1} \ldots$$

$$\ldots \xrightarrow{\nu_{n-1}} Y'_{n-1} \xrightarrow{\nu'_{n-1}} Y''_{n-1} \xrightarrow{\nu''_{n-1}} Y_n \xrightarrow{\nu_n} X' \xrightarrow{\pi} \hat{X} \xrightarrow{\varphi} X.$$  

**STEP 1.2** Coefficients of $P_\nu(\mu_1^*D)$.

Set $E_1$ for the exceptional divisor of $\pi$ and its birational transforms; set $E_{1,j}^s$ for the exceptional divisor of $\nu'_j$, $E_{1,j}^{ss}$ for the exceptional divisor of $\nu_j$, $E_{1,j}^d$ for the exceptional divisor of $\xi'_j$ and $E_{1,j}^{ds}$ for the exceptional divisor of $\xi_j$. Since $\mathbb{B}_-(\mu_1^*D) = \mu_1^{-1} \mathbb{B}_-(D)$, the support of $N_\nu(\mu_1^*D)$
is contained in
\[ E_1 \cup \bigcup_{j=1}^{m} (E_{1,j}^s \cup E_{1,j}^{ss}) \cup \bigcup_{j=1}^{n} (E_{1,j}^d \cup E_{1,j}^{ds}). \]

Then
\[ P_\nu(\mu_1^* D) = \mu_1^* D - \sigma_1 E_1 - \sum_j \sigma_j^s E_{1,j}^s + \sigma_j^{ss} E_{1,j}^{ss} - \sum_j \sigma_j^d E_{1,j}^d + \sigma_j^{ds} E_{1,j}^{ds}. \]

By Lemma 2.14
\[ \sigma_1 = \frac{-2\varphi^* D \cdot \hat{\Sigma}_1}{\deg(I_{\hat{\Sigma}_1}/I_{\hat{\Sigma}_2})} \]
which is positive thanks to the hypothesis \( D \cdot \Sigma_1 < 0 \) and to Theorem 3.3(1). Again by Lemma 2.14
\[ \sigma_j^s = \frac{-2(\pi^* \varphi^* D - \sigma_1 E_1) \cdot F_{1,n}}{2} = \sigma_1. \]

We prove by induction that
\[ \sigma_j^s = \frac{\sigma_1}{2^{n-j}}. \]

By Lemma 2.14 and by inductive hypothesis
\[ \sigma_j^s = \frac{-2((\nu_j'')^* \ldots \nu_j'^* \varphi^* D - \sigma_1 (E_1 + \sum_{i>j} \frac{1}{2^{n-i}} (E_{1,i}^s + E_{1,i}^{ss}))) F_{1,j}^s}{\deg I_{F_j^s}/I_{F_j^s}^s} = \frac{2\sigma_1 1/2^{n-j-1} E_{1,j+1}^s F_{1,j}^s}{4} = \frac{\sigma_1}{2^{n-j}}. \]

With computations analogous to the previous ones we also prove that \( \sigma_j^{ss} = \sigma_j^s \) for every \( j \) and that \( \sigma_j^{ds} = \sigma_j^d = \sigma_j/2^{n-j}. \)

**STEP 1.3 Nefness of \( P_\nu(\mu_1^* D) \).**

We want to prove that the divisor \( P_\nu(\mu_1^* D) \) is nef on
\[ \bigcup_{j=1}^{m} (E_{1,j}^s \cup E_{1,j}^{ss}) \cup \bigcup_{j=1}^{n} (E_{1,j}^d \cup E_{1,j}^{ds}). \]

We prove it for the \( E_{1,j}^s \) and \( E_{1,j}^{ss} \). The other cases are analogous.

In order to prove the claim we need to check that \( P_\nu(\mu_1^* D) \cdot C \geq 0 \) where \( C \) is

1. the generic fiber \( f_j^s \) of \( \nu'_j \);
2. the generic fiber \( f_j^{ss} \) of \( \nu'_j \);
3. the strict transform \( f_j' \) of the fiber of \( \nu'_j \) over \( F_{1,j}^s \cap F_{1,j-1}^s \);
4. the strict transform \( f_j'' \) of the fiber of \( \nu'_j \) over \( F_{1,j}^s \cap F_{1,j}^{ss} \);
5. any section \( h_j^{ss} \) of \( \nu_j \);
6. any section \( h_j^s \) of \( \nu'_j \).

We will write \( P_\nu \) for short. See Figure 3. In case (1) we have \( P_\nu \cdot f_j^s = \sigma_1/2^{n-j} \); in case (2) \( P_\nu \cdot f_j^{ss} = \sigma_1/2^{n-j} \). For case (3) we obtain
\[ P_\nu \cdot f_j' = -\sigma_1 1/2^{n-j} f_{1,j}^s \cdot f_j' - \frac{\sigma_1}{2^{n-j+1}} E_{1,j-1}^s \cdot f_j' = \sigma_1 \left( \frac{1}{2^{n-j}} - \frac{1}{2^{n-j+1}} \right) > 0. \]
For case (4) we obtain
\[ P_{\nu} \cdot f''_j = -\frac{\sigma_1}{2^{n-j}} (E_{1,j}^s \cdot f''_j + E_{1,j}^{ss} \cdot f''_j) = 0. \]

If, as in (5), \( h_{j}^{ss} \) is any section of \( \nu_j \), we have
\[ P_{\nu} \cdot h_j^{ss} = -\frac{\sigma_1}{2^{n-j}} (E_{1,j}^{ss} \cdot h_j^{ss} + E_{1,j}^{s} \cdot h_j^{ss}) \geq -\frac{\sigma_1}{2^{n-j}} (-2 + 1) = \frac{\sigma_1}{2^{n-j}}. \]

Finally, in case (6) we have
\[ P_{\nu} \cdot h_j^s = \begin{cases} -\sigma_1 (E_1 + E_{1,n}^s + 1/2E_{1,n-1}^s) \cdot h_j^s \geq 2\sigma_1 - \sigma_1 (E_1 + 1/2E_{1,n-1}^s) \cdot h_h^s \geq 0 & \text{if } j = n \\ -\sigma_1 (\frac{1}{2^{n-j+1}} E_{1,j-1}^s + \frac{1}{2^{n-j}} E_{1,j}^s + \frac{1}{2^{n-j-1}} E_{1,j+1}^s + \frac{1}{2^{n-j}} E_{1,j}^{ss}) \cdot h_j^s \geq \frac{\sigma_1}{2^{n-j+1}} & \text{if } j < n. \end{cases} \]
STEP k

Let $l$ be the number of irreducible components of $B_-(D)$. Assume that we have $\mu_{k-1} \circ \ldots \circ \mu_1 : X^{(k-1)} \to X$ and that the following are satisfied.

1.a: The image of the exceptional locus of $\mu_{k-1} \circ \ldots \circ \mu_1$ in $X$ is $\Sigma_1 \cup \ldots \cup \Sigma_{k-1}$.

1.b: The diminished base locus has the following form

$$B_-(\mu_{k-1}^* \ldots \mu_1^* D) = \bigcup_{i=1}^{k-1} E_i \cup \bigcup_{i=1}^{k-1} \left( E_{i,j}^s \cup E_{i,j}^{ss} \cup E_{i,j}^d \cup E_{i,j}^{ds} \right) \cup \{l - k + 1 \text{ smooth curves}\}$$

where $E_i$ is the unique divisor such that $(\mu_{k-1} \circ \ldots \circ \mu_1)(E_i) = \Sigma_i$.

2: The divisor $P_\nu(\mu_{k-1}^* \ldots \mu_1^* D)$ has the following form

$$P_\nu(\mu_{k-1}^* \ldots \mu_1^* D) = \mu_{k-1}^* \ldots \mu_1^* D - \sum_{i=1}^{k-1} \sigma_i \left( E_i + \sum_{j} \frac{1}{2^{n_j}} (E_{i,j}^s + E_{i,j}^{ss}) + \sum_{j} \frac{1}{2^{m_j}} (E_{i,j}^d + E_{i,j}^{ds}) \right)$$

where by abuse of notation we denote $n_i$ and $m_i$ by $n$ and $m$. For any $i$ the coefficient $\sigma_i$ is

$$\sigma_i = \frac{-2P_\nu(\mu_1^* \ldots \mu_i^* D) \Sigma_i}{d_i} > 0$$

for some integer number $d_i$. If $j < i$ and $\Sigma_i \cap \Sigma_j \neq \emptyset$ then the following inequalities hold

$$\frac{d_i}{c_j} \left( -P_\nu(\mu_{j-1}^* \ldots \mu_1^* D) \cdot \Sigma_j \right) \left( 1 - \frac{4}{d_j} \right) + 2P_\nu(\mu_{j-1}^* \ldots \mu_1^* D) \cdot \Sigma_i \geq 0,$$

$$\frac{d_j}{d_i} \sigma_j = \frac{-2P_\nu(\mu_{j-1}^* \ldots \mu_1^* D) \cdot \Sigma_j}{d_j} \geq \frac{-2P_\nu(\mu_{j-1}^* \ldots \mu_1^* D) \cdot \Sigma_i}{d_i} = \sigma_i.$$  \hfill (4.1)

3: The divisor $P_\nu(\mu_{k-1}^* \ldots \mu_1^* D)$ is nef on

$$\bigcup_{i=1}^{k-1} \bigcup_{j} \left( E_{i,j}^s \cup E_{i,j}^{ss} \cup E_{i,j}^d \cup E_{i,j}^{ds} \right).$$

Let $\Sigma_k$ be a curve such that $P_\nu(\mu_{k-1}^* \ldots \mu_1^* D) \cdot \Sigma_k < 0$. Set

$$c_k = \text{card} \left( \Sigma_k \cap \bigcup_{j > k} \Sigma_j \right).$$

STEP k.1 Semistabilization of $\Sigma_k$ and construction of $X^{(k)}$.

Let $\varphi_k : \hat{X}_k \to X^{(k-1)}$ be a sequence of blow-ups as in Theorem 3.3 If we denote by $\hat{\Sigma}_k$ the strict transform of $\Sigma_k$, then we can assume that

- $\hat{\Sigma}_k$ has semistable conormal bundle;
\* if \(d_k\) is the degree of the conormal bundle of \(\hat{\Sigma}_k\) then \(d_k > 4\) and if \(h < k\) is such that \(\Sigma_h \cap \Sigma_k \neq \emptyset\) then the following inequalities hold

\[
\frac{d_k}{c_h} (-P_v(\mu_{h-1}^* \ldots \mu_1^* D) \cdot \Sigma_h) \left(1 - \frac{4}{d_h}\right) + 2P_v(\mu_{k-1}^* \ldots \mu_1^* D) \cdot \Sigma_k \geq 0,
\]

\[
\sigma_h = \frac{-2P_v(\mu_{h-1}^* \ldots \mu_1^* D) \cdot \Sigma_h}{d_h} \geq \frac{-2P_v(\mu_{k-1}^* \ldots \mu_1^* D) \cdot \Sigma_k}{d_k} = \sigma_k.
\]

Notice that \(d_k\) such that \(4.3\) and \(4.4\) hold exist thanks to Theorem 3.3(1). Conditions \(4.3\) and \(4.4\) will insure that \(P_v(\mu^* D) \cdot C \geq 0\) where \(C\) is respectively a section of \(E_h \to \Sigma_h\) and the strict transform of the fiber of \(E_n \to \Sigma_h\) over \(\Sigma_k \cap \Sigma_h\).

Thus

\[
\varphi_k^{-1}\Sigma_k = \hat{\Sigma}_k \cup \bigcup_j (F_{k,j}^s \cup F_{k,j}^{ss} \cup F_{k,j}^d \cup F_{k,j}^{ds}).
\]

Let \(\mu_k : X^{(k)} \to X^{(k-1)}\) be the composition of the blow-up of \(\Sigma_k\) and of the blow-up and “stabilization” of the curves \(F_{k,j}^s\) and \(F_{k,j}^d\). We call \(E_k\) the exceptional divisor of the blow-up of \(\hat{\Sigma}_k\) and \(E_{k,j}^s\) (resp. \(E_{k,j}^d\), \(E_{k,j}^{ss}\), \(E_{k,j}^{sd}\)) the exceptional divisor of the blow-ups of \(F_{k,j}^s\) (resp. \(F_{k,j}^d\), \(F_{k,j}^{ss}\), \(F_{k,j}^{sd}\)).

**STEP k.2** Coefficients of \(P_v(\mu_k^* \ldots \mu_1^* D)\).

Exactly as in **STEP 1.2** we can prove that the coefficient of \(E_{k,j}^s\) and \(E_{k,j}^{ss}\) (resp. \(E_{k,j}^d\), \(E_{k,j}^{sd}\)) is \(-\sigma_k/2^{n-j}\) (resp. \(-\sigma_k/2^{n-j}\)). By abuse of notation we write \(n\) and \(m\) instead of \(n_k\) and \(m_k\).

**STEP k.3** Nefness of \(P_v(\mu_k^* \ldots \mu_1^* D)\).

We prove exactly as in **STEP 1.3** that \(P_v(\mu_k^* \ldots \mu_1^* D)\) is nef on

\[
\bigcup_{i=1}^k \bigcup_j (E_{i,j}^s \cup E_{i,j}^{ss} \cup E_{i,j}^d \cup E_{i,j}^{ds}).
\]

**CONCLUSION** Let \(l\) be the number of irreducible components of \(\mathbb{B}_(D)\). After \(l\) steps the construction terminates. Set \(\tilde{X} = X^{(l)}\) and

\[
\mu = \mu_l \circ \ldots \circ \mu_1 : \tilde{X} \to X.
\]

By construction

\[
\mathbb{B}_-(\mu^* D) = \bigcup_{i=1}^l E_i \cup \bigcup_{i=1}^l \bigcup_j (E_{i,j}^s \cup E_{i,j}^{ss} \cup E_{i,j}^d \cup E_{i,j}^{ds}).
\]

We claim that \(P_v(\mu^* D)\) is nef on \(E_1 \cup \ldots \cup E_l\) where \(E_i\) is the exceptional divisor over \(\Sigma_i\). This will conclude the proof. Indeed, if we prove that \(P_v(\mu^* D)\) is nef on \(E_1 \cup \ldots \cup E_l\), then by **STEP k.3** the divisor \(P_v(\mu^* D)\) is nef on \(\mathbb{B}_-(\mu^* D)\) and thus it is nef on \(\tilde{X}\). Since nef divisors are movable, by [14] Proposition III.1.14, Remark III.3.10] the divisor \(P_v(\mu^* D)\) is equal to \(P_\sigma(\mu^* D)\).

In order to prove the nefness of \(P_v(\mu^* D)\) we need to prove that \(P_v(\mu^* D) \cdot C \geq 0\) where \(C\) is
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Figure 4.

(1) the generic fiber $f_i$ of $E_i \to \hat{\Sigma}_i$;
(2) the strict transform $f^s_i$ (resp. $f^s_{i,n}$) of the fiber over $\hat{\Sigma}_i \cap F^s_{i,n}$ (resp. $\hat{\Sigma}_i \cap F^d_{i,m}$);
(3) the strict transform $f'_i$ of the fiber over $\hat{\Sigma}_i \cap \Sigma_j$ for some $j > i$;
(4) any section $h_i$ of $E_i \to \hat{\Sigma}_i$.

See Figure 4. We will write $P_\nu$ for short.

In case (1) we have $P_\nu \cdot f_i = \sigma_i$; in case (2) $P_\nu \cdot f^s_i = -\sigma_i(E_i + E^s_{i,n}) \cdot f^s_i = 0$ and $P_\nu \cdot f^d_i = -\sigma_i(E_i + E^d_{i,m}) \cdot f^d_i = 0$; in case (3) $P_\nu \cdot f'_i = (-\sigma_i E_i - \sigma_j E_j) \cdot f'_i = \sigma_i - \sigma_j \geq 0$ by condition (4.3). We are left with case (4).

\[
P_\nu \cdot h_i = (\mu_1^* \cdots \mu_i^* D - \sum_{j=1}^l \sigma_j E_j - \sigma_i E^l_{i,h} - \sigma_i E^l_{i,h}) \cdot h_i \\
\geq (\mu_{i-1}^* \cdots \mu_1^* D - \sum_{j=1}^{i-1} \sigma_j E_j) \cdot h_i \\
\geq -\sigma_i (\mu_i^* \cdots \mu_1^* D - \sum_{j=1}^{i-1} \sigma_j E_j) \cdot h_i \\
\geq -\sigma_i \left(1 - \frac{4}{d_i}\right) \sum_{j=1}^{i} \{ \sigma_j | \Sigma_j \cap \Sigma_i \neq \emptyset, j > i \}
\]

and the latter term is non negative by condition (4.4). \qed
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