Optimal protocols for the most difficult repeated coordination games

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Abstract

This paper investigates repeated win-lose coordination games (WLC-games). We analyse which protocols are optimal for these games covering both the worst case and average case scenarios, i.e., optimizing the guaranteed and expected coordination times. We begin by analysing Choice Matching Games (CM-games) which are a simple yet fundamental type of WLC-games, where the goal of the players is to pick the same choice from a finite set of initially indistinguishable choices. We give a complete classification of optimal expected and guaranteed coordination times in two-player CM-games and show that the corresponding optimal protocols are unique in every case—except in the CM-game with four choices, which we analyse separately.

Our results on CM-games are also essential for proving a more general result on the difficulty of all WLC-games: we provide a complete analysis of least upper bounds for optimal expected coordination times in all two-player WLC-games as a function of game size. We also show that CM-games can be seen as the most difficult games among all two-player WLC-games, as they turn out to have the greatest optimal expected coordination times.

Keywords: Repeated coordination games, optimal strategies, average and worst case analysis, relational structures, reachability objectives
1 Introduction

Pure win-lose coordination games (WLC-games) are simple yet fundamental games where all players receive the same payoffs: 1 (win) or 0 (lose). This paper studies repeated WLC-games, where the players make simultaneous choices in discrete rounds until (if ever) succeeding to coordinate on a winning profile. Choice matching games (CM-games) are the simplest class of such games. The choice matching game CM\(_{m}^{n}\) has \(n\) players with the goal to choose the same choice among \(m\) different indistinguishable choices, with no communication during play. The players can use the history of the game (i.e., the players’ choices in different rounds) for their benefit as the game proceeds. For simplicity, we denote the two-player game CM\(_{m}^{2}\) by CM\(_{m}\).

A paradigmatic real-life scenario with a choice matching game relates to a phenomenon that has humorously been called “pavement tango” or “droitwich” in \[1\]. Here two people try to pass each other but may end up blocking each other by repeatedly moving sideways into the same direction. For another example of a choice matching game, consider CM\(_{3}\), the coordination-based variant of the rock-paper-scissors game, pictured on the right.

Here the two players (i.e., columns) coordinate if they succeed choosing an edge from one of the three rows. The players first choose randomly; suppose they select the nodes in dotted circles. Simply based on symmetries, it then makes sense for both players to choose from the last row (solid circles), as each of the two other choices in each column have a symmetric, non-coordinating choice in the other column. This leads to coordination in the second round.

A general \(n\)-player WLC-game is a generalization of CM\(_{m}^{n}\), where the players do not necessarily have to choose from the same row to coordinate, and it may not even suffice to choose from the same row. In classical matrix form representation, two-player choice matching games have ones on the diagonal and zeroes elsewhere, while general two-player WLC-games have general distributions of ones and zeroes; see Definition 2.1 for the full formal details.

In repeated WLC-games, it is natural to try to coordinate as quickly as possible. There are two main scenarios to be investigated: guaranteeing coordination (with certainty) in as few rounds as possible and minimizing the expected number of rounds for coordination. The former concerns the number of rounds it takes to coordinate in the worst case and is measured in terms of guaranteed coordination times (GCTs). The latter relates to the average case analysis measured in terms of expected coordination times (ECTs).

Our contributions. We provide a comprehensive study of upper bounds for coordination in all two-player repeated WLC-games, including a classification of related optimal strategies (called protocols in this work). CM-games are central to our work, being a fundamental class of games and also the most difficult games for coordination—in a sense made precise below.

Two protocols play a central role in our study. We introduce the so-called loop avoidance protocol LA (cf. Definition 4.1) that essentially tells players to play so that the generated history of choices always reduces the symmetries (e.g., automorphisms) of the game structure. We also use the so-called wait-or-move (WM) protocol (cf. Definition 4.4), essentially telling players to randomly alternate between two choices that both coordinate with at least one of the opponent’s two choices. We show that WM leads to coordination in all WLC-games very fast, the ECT being \(3 - 2p\), where \(p\) is the probability of coordinating in the first round with random choices.

We then provide a complete analysis of the optimal ECTs and GCTs in all choice matching games CM\(_{m}\). We also identify the protocols giving the optimal ECTs and GCTs and show their uniqueness, where possible. The table in Figure 1 summarizes these results. This analysis is complete, as we prove that there exists a continuum of optimal protocols for CM\(_{4}\) and establish that for all even \(m\), no protocol guarantees a win in CM\(_{m}\).

Concerning the more general class of all WLC-games, we provide the following complete characterization of upper bounds for the optimal ECTs in all two-player WLC-games as a func-
tion of game size (a game in a classical matrix form is of size $m$ when the maximum of the number of rows and columns is $m$):

**Theorem.** For any $m$, the greatest optimal ECT among two-player WLC-games of size $m$ is as follows:

| Game size |  \[ m \in \mathbb{Z}_+ \setminus \{3, 5\} \] |  \[ m = 5 \] |  \[ m = 3 \] |
|-----------|----------------------------------|-----------------|-----------------|
| Greatest optimal ECT | $3 - \frac{1}{k}$ | $2 + \frac{1}{k}$ | $\frac{1 + \sqrt{1 + 4k}}{2} \approx 1.925$ |

Also, concerning two-player choice matching games, we establish that CM$_m$ has the strictly greatest optimal ECT out of all two-player WLC-games of size $m \neq 3$, making CM-games the most difficult WLC-games to coordinating in. We give a separate full analysis of the case $m = 3$.

**Related work.** Coordination games (see, e.g., [4], [3]) are a key topic in game theory, with the early foundations laid, inter alia, in the works of Schelling [17] and Lewis [15]. Repeated games are—likewise—a key topic, see for example [11], [2], [16]. For seminal work on repeated coordination games, see for example the articles [6], [5], [14].

However, WLC-games are a simple class of games that have not been extensively studied in the literature. In particular, choice matching games clearly constitute a fundamental class of games, and it is thus surprising that the analysis of the current paper has not been previously carried out. Thus the related analysis is well justified; it closes an obvious gap in the literature.

In general, our study differs from the classical game-theoretic study of repeated games where the focus is on accumulated payoffs. Indeed, our repeated WLC-games are based on reachability objectives. Especially our worst case analysis (but also the average case study) has only superficial overlap with most work on repeated games.

However, similar work exists, the most notable example being the seminal article [6] that studies a generalization of WLC-games in a framework that has some similarities with our setting. They introduce (what is equivalent to) the two-player CM-games in their final section on general examples. They also essentially identify the optimal ways of playing CM$_2$ and CM$_3$, discussed also in this article, although in a technically somewhat different setting of accumulated payoffs. Furthermore, they observe that a protocol essentially equivalent to WM is the best way to play CM$_6$, an observation we also make in our setting. However, optimality of WM in CM$_6$ is not proved in [6]. This would require an extensive analysis proving that the players cannot make beneficial use of asymmetric histories created by non-coordinating choices. Indeed, the main technical difficulty in our corresponding setting is to show uniqueness of the optimal protocol.

Nonetheless, despite the differences, the framework of [6] bears some conceptual similarities to ours, e.g., the authors also identify structural protocols (cf. Definition 3.4 below) as the natural notion of strategy for studying their framework. Furthermore, they make extensive use of focal points [17] in analysing how asymmetric histories can potentially be used for coordination.

Relating to uniqueness of protocols, [10] argues that individual rationality considerations are not sufficient for players to “learn how to coordinate” in the setting of [6]. We agree with [10]...
that some conventions are needed if several protocols lead to the optimal result. However—in our framework—since we can prove uniqueness of the optimal protocols for CM$_n$ (when $m \neq 4$), then arguably rational players should adopt precisely these protocols in CM-games.

Techniques used. Some of our results are of course based on massaging techniques from game theory and mathematical analysis to suit our purposes. This involves the standard things: infinite series, analysis of extrema, et cetera. However, the core of our work relies on an original approach to games based on relational structures, as opposed to using the traditional matrix form representation. This approach enables us to use graph theoretic ideas in our arguments.

Both in the worst-case and in average-case analysis, the main technical work relies heavily on analysis of symmetries—especially the way the groups of automorphisms of games evolve when playing coordination games. The most involved result of the worst-case analysis, Theorem 5.2, is proved by reducing the cardinality of the automorphism group of the WLC-game studied in a maximally fast fashion. In the average-case analysis, Theorems 6.2, 6.3, 6.4 are proved via a combination of analysis of extrema; keeping track of groups of automorphisms; graph theoretic methods; and focal points [17] for breaking symmetry. The most demanding part here is to show uniqueness of the protocols involved. Also in the average-case analysis, Theorem 7.2 relies on earlier theorems and an extensive and exhaustive analysis of certain bipartite graphs.

We first used our approach to games via relational structures in [8], [7]. It has been applied also in the repeated setting in [9] and considered in a more general setting in [13], [12].

2 Preliminaries

We define win-lose coordination games as relational structures the same way as in [8], [7], [9]:

Definition 2.1. An $n$-player win-lose coordination game (WLC-game) is a relational structure $G = (A, C_1, \ldots, C_n, W_G)$ where $A$ is a finite domain of choices, each $C_i$ is a non-empty unary relation (representing the choices of player $i$) such that $C_1 \cup \cdots \cup C_n = A$, and $W_G \subseteq C_1 \times \cdots \times C_n$ is an $n$-ary winning relation. For technical convenience, we assume that the players have pairwise disjoint choice sets, i.e., $C_i \cap C_j = \emptyset$ for every $i, j \leq n$ such that $i \neq j$. An tuple $c \in C_1 \times \cdots \times C_n$ is called a choice profile for $G$ and the choice profiles in $W_G$ are called winning choice profiles. We assume that there are no surely losing choices, i.e., choices $c \in A$ that do not belong to any winning choice profile, as rational players would never select such choices. The complement $\overline{G}$ of $G$ is defined as $\overline{G} := (A, C_1, \ldots, C_n, C_1 \times \cdots \times C_n \setminus W_G)$.

We will use the visual representation of WLC games as hypergraphs from [8]: two-player games become just bipartite graphs under this scheme. The choices of each player are displayed as columns of nodes, starting from the choices of player 1 on the left and ending with the column of choices of player $n$. The winning relation consists of lines that represent the winning choice profiles. Thus winning choice profiles are also called edges. See Example A.1 in Appendix A for an illustration of the drawing scheme.

Consider a WLC-game $G = (A, C_1, \ldots, C_n, W_G)$ with $n$ players and $m$ winning choice profiles that do not intersect, i.e., none of the $m$ winning choice profiles share a choice $c \in A$. Such games form a simple yet fundamental and natural class of games, where the goal of the players is simply to pick the same “choice”, i.e., to simultaneously pick one of the $m$ winning profiles. These games are called choice matching games. We let CM$_m^n$ denote the choice matching game with $n$ players and $m$ choices for each player. In this article, we extensively make use of the two-player choice matching games, CM$_2^m$. For these games, we will omit the superscript “2” and simply denote them by CM$_m$. (Recall here the example CM$_3$ pictured in the introduction.)

Interestingly, out of all $n$-player WLC-games where each of the $n$ players has $m$ choices, the game CM$_m^n$ has the least probability of coordination when each player plays randomly. In this
sense these games can be seen the most difficult for coordination. A fully compelling reason for the maximal difficulty of choice matching games is given later on by Corollary 7.3.

3 Repeated WLC-games

A repeated play of a WLC-game $G$ consists of consecutive (one-step) plays of $G$. The repeated play is continued until the players successfully coordinate, i.e., select their choices from a winning choice profile. This may lead to infinite plays. We assume that each player can remember the full history of the repeated play and use this information when planning the next choice. The history of the play after $k$ rounds is encoded in a sequence $H_k$ defined as follows.

**Definition 3.1.** Let $G$ be an $n$-player WLC-game. A pair $(G, H_k)$ is called a stage $k$ (or $k$th stage) in a repeated play of $G$, where the history $H_k$ is a $k$-sequence of choice profiles in $G$. More precisely, $H_k = (H_i)_{i \in \{1, \ldots, k\}}$ where each $H_i$ is an $n$-ary relation $H_i = \{(c_1, \ldots, c_n)\}$ with a single tuple $(c_1, \ldots, c_n) \in C_1 \times \cdots \times C_n$. In the case $k = 0$, we define $H_0 = \emptyset$. The stage $(G, H_0)$ is the initial stage (or the 0th stage). Like $G$, also $(G, H_k)$ is a relational structure.

A stage $k$ contains a history specifying precisely $k$ choice profiles chosen in a repeated play. A winning profile of $(G, H_k)$ is called a touched edge if it contains some choice $c$ picked in some round $1, \ldots, k$ leading to $(G, H_k)$. As we assume that the players only need to coordinate once, we consider repeated plays only up to the first stage where some winning choice profile is selected. If coordination occurs in the $k$th round, then the $k$th stage is called the final stage of the repeated play. But a play can indeed possibly take infinitely long without coordination.

On the right is a drawing of the stage 2 in a repeated play of CM$_2$, the “coordination game variant” of the matching pennies game (or the “pavement tango” from the introduction). Here the players have failed to coordinate in round 1 (having picked the choices with dotted circles) and then failed again by both swapping their choices in round 2 (solid circles).

We next generalize the definition of protocols from [8]. In the current paper, a protocol describes a mixed strategy for all stages in all WLC-games and for all player roles $i$.

**Definition 3.2.** A protocol $\pi$ is a function outputting a probability distribution $f : C_i \rightarrow [0, 1]$ (where $\sum_{c \in C_i} f(c) = 1$) with the input of a player $i$ and a stage $(G, H_k)$ of a repeated WLC-game.

Since a protocol can depend on the full history of the current stage, it gives a mixed, memory-based strategy for any repeated WLC-game. Thus protocols can informally be regarded as global “behaviour styles” of agents over the class of all repeated WLC-games. It is important note that all players can see (and remember) the previous choices selected by all the other players—and also the order in which the choices have been made.

In the scenario that we study, it is obvious to require that the protocols should act independently of the names of choices and the names (or ordering) of player roles $i$. In [6], this requirement follows from the “assumption of no common language” (for describing the game), and in [8], we say that such protocols are structural. To extend this concept for repeated games, we first need to define the notion of a renaming. The intuitive idea of renamings is to extend isomorphisms between game graphs—including the history—to additionally enable permuting the players $1, \ldots, n$ (see Example A.2 in Appendix A for an illustration of the definition).

**Definition 3.3** (Cf. [8]). A renaming between stages $(G, H_k)$ and $(G', H'_k)$ of $n$-player WLC-games $G$ and $G'$ is a pair $(\beta, \pi)$ where $\beta$ is a permutation of $\{1, \ldots, n\}$ and $\pi$ a bijection from the domain of $G$ to that of $G'$ such that

\[\text{Note that if this assumption is not made, then coordination can trivially be guaranteed in a single round in any WLC game by using a protocol which chooses some winning choice profile with probability 1.}\]
Definition 3.4. A protocol \( \pi \) is structural if it is indifferent with respect to renamings, meaning that if \((G, \mathcal{H}_k)\) and \((G', \mathcal{H}'_k)\) are stages with a renaming \((\beta, \pi)\) between them, then for any \(i\) and any \(c \in C_i\), we have \(f(c) = f'(\pi(c))\), where \(f = \pi((G, \mathcal{H}_k), i)\) and \(f' = \pi((G', \mathcal{H}'_k), \beta(i))\).

Note that a structural protocol may depend on the full history, which records even the order in which the choices have been played. Hereafter we assume all protocols to be structural.

Definition 3.5. Let \( G \) be a WLC-game and let \( S \) and \( S' \) be stages of \( G \). Let \( ~' \) be the structural equivalence relation over \( S \) (respectively, \( S' \)). We say that \( S \) and \( S' \) are automorphism-equivalent if \( ~ = ~' \). The stages \( S \) and \( S' \) are structurally similar if one can be obtained from the other by a chain of renamings and automorphism-equivalences.

A choice \( c \) in a stage \( S \) is a focal point if it is not structurally equivalent to any other choice in that same stage \( S \), with the possible exception of choices \( c' \) belonging to a same edge as \( c \). See Example \[3\] for an illustration of focal points. A focal point breaks symmetry and can be used for winning a repeated coordination game. This requires that the players have some (possibly prenegotiated) way to choose some edge \((u, v)\) such that \( u \) or \( v \) is a focal point.

In repeated coordination games, it is natural to try to coordinate as quickly as possible. There are two principal scenarios related to optimizing coordination times: the average case and the worst case. The former concerns the expected number rounds for coordination and the latter the maximum number in which coordination can be guaranteed with certainty.

Definition 3.6. Let \((G, \mathcal{H}_k)\) a stage and let \( \pi \) be a protocol. The one-shot coordination probability (OSCP) from \((G, \mathcal{H}_k)\) with \( \pi \) is the probability of coordinating in a single round from \((G, \mathcal{H}_k)\) when each player follows \( \pi \). The expected coordination time (ECT) from \((G, \mathcal{H}_k)\) with \( \pi \) is the expected value for the number of rounds until coordination from \((G, \mathcal{H}_k)\) when all players follow \( \pi \). The guaranteed coordination time (GCT) from \((G, \mathcal{H}_k)\) with \( \pi \) is the number \( n \) such that the players are guaranteed to coordinate from \((G, \mathcal{H}_k)\) in \( n \) rounds, but not in \( n - 1 \) rounds, when all players follow \( \pi \), if such a number exists. Else this value is \( \infty \).

The OSCP, ECT and GCT from the initial stage \((G, \emptyset)\) with \( \pi \) are referred to as the OSCP, ECT and GCT in \( G \) with \( \pi \). We say that \( \pi \) is ECT-optimal for \( G \) if \( \pi \) gives the minimum ECT in \( G \), i.e., the ECT given by any protocol \( \pi' \) is at least as large as the one given by \( \pi \). GCT-optimality of \( \pi \) for \( G \) is defined analogously.

It is possible that there are several different protocols giving the optimal ECT (or GCT) for a given WLC-game. If two protocols \( \pi_1 \) and \( \pi_2 \) are both optimal, it may be that the optimal value is nevertheless not obtained when some of the players follow \( \pi_1 \) and the others \( \pi_2 \). This leads to a meta-coordination problem about choosing the same optimal protocol to follow. However, such a problem will be avoided if there exists a unique optimal protocol.

Definition 3.7. Let \( \pi \) be a protocol and \( G \) a WLC-game. We say that \( \pi \) is uniquely ECT-optimal for \( G \) if \( \pi \) is ECT-optimal for \( G \) and the following holds for all other protocols \( \pi' \) that are ECT-optimal for \( G \): for any stage \( S \) in \( G \) that is reachable with \( \pi \), we have \( \pi'(S) = \pi(S) \).

Unique GCT-optimality of \( \pi \) for \( G \) is defined analogously.\(^2\)

\(^2\)Note that if two different protocols are uniquely ECT-optimal for \( G \) (and similarly for unique GCT-optimality), then their behaviour on \( G \) can differ only on stages that are not reachable in the first place by the protocols. Also, their behaviour can of course differ on games other than \( G \).
The next lemma states that two structurally similar stages are essentially the same stage with respect to different ECTs and GCTs. The proof is straightforward.

**Lemma 3.8.** Assume stages $S$ and $S'$ of $G$ are structurally similar. Now, for any protocol $\pi$, there exists a protocol $\pi'$ which gives the same ECT and GCT from $S'$ as $\pi$ gives from $S$.

### 4 Protocols for repeated WLC-games

In this section we introduce two special protocols, the loop avoidance protocol $\text{LA}$ and the wait-or-move protocol $\text{WM}$. Informally, $\text{LA}$ asserts that in every round, every player $i$ should avoid—if possible—all choices $c$ that could possibly make the resulting stage automorphism-equivalent (cf. Def. 3.5) to the current stage, i.e., the stage just before selecting $c$.

**Definition 4.1.** The loop avoidance protocol ($\text{LA}$) asserts that in every round, every player $i$ should avoid—if possible—all choices $c$ for which the following condition holds: if the player $i$ selects $c$, then there exist choices for the other players so that the resulting stage is automorphism-equivalent to the current stage. If this condition holds for all choices of the player $i$, then $i$ makes a random choice. Moreover, uniform probability is used among all the possible choices of $i$.

It is easy to see that $\text{LA}$ avoids, when possible, all such stages that are structurally similar to any earlier stage in the repeated play. As structurally similar stages are essentially identical (cf. Lemma 3.8), repetition of such stages can be seen as a “loop” in the repeated play. When trying to guarantee coordination as quickly as possible, such loops should be avoided. In addition to this heuristic justification, Theorems 5.1 and 5.2 give a fully compelling justification for $\text{LA}$ when considering guaranteed coordination in two-player CM-games. For now, we present the following propositions (see Appendix B for proofs); see also Example A.4 in Appendix A for an illustration of the use of $\text{LA}$.

**Proposition 4.2.** $\text{LA}$ is the uniquely ECT-optimal and uniquely GCT-optimal in $\text{CM}_3$.

**Proposition 4.3.** $\text{LA}$ guarantees coordination in games $\text{CM}_m$ in $\lceil m/2 \rceil$ rounds when $m$ is odd, but $\text{LA}$ does not guarantee coordination in $\text{CM}_m$ for any even $m$.

We next present the wait-or-move protocol $\text{WM}$, which naturally appears in numerous real-life two-player coordination scenarios. Informally, both players alternate (with equal probability) between two choices: the players own initial choice and another choice that coordinates with the initial choice of the other player.

**Definition 4.4.** The wait-or-move protocol ($\text{WM}$) for repeated two-player WLC-games goes as follows: first select randomly any choice $c$, and thereafter choose with equal probability $c$ or a choice $c'$ that coordinates with the initial choice of the other player (thereby never picking other choices than $c$ and $c'$). Definition A.5 in Appendix A specifies $\text{WM}$ in more detail.

The following theorem shows that $\text{WM}$ is very fast in relation to ECTs. This holds for all two-player WLC-games, not only choice matching games $\text{CM}_m$. The proof is given in Appendix B.

**Theorem 4.5.** Let $G$ be a WLC-game with one-shot coordination probability $p$ when both players make their first choice randomly. Then the expected coordination time by $\text{WM}$ is at most $3 - 2p$.

**Corollary 4.6.** The ECT with $\text{WM}$ is strictly less than 3 in every two-player WLC-game.

It follows from the proof of Theorem 4.5 that the ECT with $\text{WM}$ is exactly $3 - \frac{2}{m}$ in all choice matching games $\text{CM}_m$. Thus Corollary 4.6 cannot be improved, as the ECTs of the games $\text{CM}_m$ grow asymptotically closer to the strict upper bound 3 when $m$ is increased. In the particular case of $\text{CM}_2$, the ECT with $\text{WM}$ is $3 - \frac{2}{2} = 2$. Thus the following lemma clearly holds.
**Lemma 4.7.** When $S = (\text{CM}_m, \mathcal{H}_k)$ is a non-final stage with exactly two touched edges, then the ECT from $S$ with WM is exactly 2. Moreover, in any WLC-game $G$, if $S' = (G, \mathcal{H}_k)$ is a non-final stage that is reachable by using WM, then the ECT from $S'$ with WM is at most 2.

Williamson eventually leads to coordination with asymptotic probability 1 in all two-player WLC-games. Nevertheless, it clearly does not guarantee (with certainty) coordination in any number of rounds in WLC-games where the winning relation is not the total relation. In a typical real-life scenario, eternal non-coordination is of course impossible by WM, but it is conceivable, for example, that two computing units using the very same pseudorandom number generator will never coordinate due to being synchronized to swap their choices in precisely the same rounds.

It is easy to show that WM is the unique protocol which gives the optimal ECT (namely, 2 rounds) in the “droitwich-scenario” of the game CM$_2$ (see Appendix B for a proof):

**Proposition 4.8.** WM is uniquely ECT-optimal in CM$_2$.

Next we compare compare the pros and cons of LA and WM in two-player choice matching games CM$_m$. Recall that WM does not guarantee coordination in these games (when $m \neq 1$), while LA does guarantee coordination in CM$_m$ if and only if $m$ is odd. Concerning expected coordination times, it is easy to prove that WM gives a smaller ECT than LA in CM$_m$ for all even $m$ (except for the case $m = 2$, where WM and LA behave identically). Thus we now restrict attention to the games CM$_m$ with odd $m$. Then, the probability of coordinating in the $\ell$-th round of CM$_m$ using LA, with $\ell \leq \lfloor m/2 \rfloor$, can relatively easily be seen to be calculable by the formula $P_{\ell,m}$ defined below (where the product is 1 when $\ell = 1$). And using the formula for $P_{\ell,m}$, we also get a formula for the expected coordination time $E_m$ in CM$_m$ with LA:

$$ P_{\ell,m} = \frac{1}{m - 2\ell + 2} \prod_{k=0}^{\ell-2} \frac{m - 2k - 1}{m - 2k}, \quad E_m = \sum_{\ell=1}^{\lfloor m/2 \rfloor} \ell \cdot P_{\ell,m}. $$

Using this and Theorem 4.5, we can compare the ECTs in CM$_m$ with LA and WM for odd $m$.

| $m$ | ECT in CM$_m$ with WM | ECT in CM$_m$ with LA |
|-----|-----------------------|-----------------------|
| 1   | 1                     | 1                     |
| 3   | 2 + $\frac{1}{9}$    | 1 + $\frac{2}{3}$    |
| 5   | 2 + $\frac{1}{3}$    | 2 + $\frac{1}{3}$    |
| 7   | 2 + $\frac{1}{3}$    | 3                     |
| 9   | 2 + $\frac{7}{9}$    | 3 + $\frac{2}{3}$    |

Especially the case $m = 7$ is interesting, as the ECT with LA is exactly 3 which is precisely the strict upper bound for the ECTs with WM for the class of all two-player choice matching games CM$_m$. Furthermore, $m = 7$ is the case where WM becomes faster than LA in relation to ECTs. Thus WM clearly stays faster than LA for all $m \geq 7$, including even values of $m$.

## 5 Optimizing guaranteed coordination times

In this section we investigate when coordination can be guaranteed in two-player CM-games and which protocols give the optimal GCT for them. We begin with the following result.

**Theorem 5.1.** For all even $m \geq 2$, there is no protocol which guarantees coordination in CM$_m$.

**Proof.** Let $\pi$ be a protocol. As $\pi$ is structural, it is possible that in each round of CM$_m$, the players pick a pair $(c, c')$ of choices that are structurally equivalent. Suppose this indeed happens. Now, in each round, there are two types of choices the players can make: (1) they both pick a choice from a touched edge; or (2) they both pick a choice from an untouched edge. As there is always an even number of untouched edges left in the game, the choice of type (2) will never guarantee coordination. And when the players have failed to coordinate so far, they will never succeed by making a choice of type (1) (due to structural equivalence of the choices).
We next consider choice matching games $\text{CM}_m$ with an odd $m$. Proposition 4.3 showed that the GCT with LA in these games is $\lfloor m/2 \rfloor$. The next theorem (proved in Appendix B) shows that this is the optimal GCT for $\text{CM}_m$, and moreover, LA is the unique protocol giving this GCT.

**Theorem 5.2.** For any odd $m \geq 1$, LA is uniquely GCT-optimal for $\text{CM}_m$.

### 6 Optimizing expected coordination times

In this section we investigate which protocols give the best ECTs for two-player choice matching games. We also investigate when the best ECT is obtained by a unique protocol. We already know by Propositions 4.3 and 1.2 that the optimal ECTs for $\text{CM}_2$ and $\text{CM}_3$ are uniquely given by WM and LA, respectively. Thus it remains to consider the games $\text{CM}_m$ with $m \geq 4$. We first cover the case $m \geq 6$ and show that then WM is the unique protocol giving the best ECT. The remaining special cases $m = 4$ and $m = 5$ will then be examined. The following auxiliary lemma (proven in Appendix B) will be used in the proofs.

**Lemma 6.1.** The ECT from $(\text{CM}_m, \mathcal{H}_k)$ with no focal point is at least $\frac{3}{2}$ with any protocol.

We then present a formula for estimating the best ECTs in cases to be investigated. Let $S := (\text{CM}_m, \mathcal{H}_k)$ be a non-final stage with exactly two touched edges. Thus there are $n := m - 2$ untouched edges. Suppose the players use a protocol $\pi$ behaving as follows in round $k + 1$. Both players pick a choice from some touched edge with probability $p$ and from an untouched edge with probability $(1 - p)$. A uniform distribution is used on choices in both classes: probability $\frac{1}{n}$ for both choices on touched edges (which makes sense by Lemma B.2) and probability $\frac{1 - n}{n}$ for each choice on untouched edges (which is necessary with a structural protocol). If one player selects a choice $c$ from a touched edge and the other one a choice $c'$ from an untouched edge, the players win in the next round by choosing the edge with $c'$. Note that $c'$ is a focal point, so the winning edge can be chosen by a structural protocol with probability $1$. (Also other focal points arise which could alternatively be used; cf. Example A.3 in Appendix A.)

Suppose then that $E_1$ is the ECT with $\pi$ from a stage $(\text{CM}_m, \mathcal{H}_{k+1})$ where both players have chosen a touched edge in round $k + 1$ but failed to coordinate. Two different such stages $(\text{CM}_m, \mathcal{H}_{k+1})$ exist, but they are automorphism-equivalent, so $\pi$ can give the same ECT from both of them by Lemma B.3 (Indeed, if $\pi$ gave two different ECTs, it would make sense to adjust it to give the smaller one.) Similarly, suppose $E_2$ is the ECT with $\pi$ from a stage $(\text{CM}_m, \mathcal{H}_{k+1})$ where both players have chosen an untouched edge in round $k + 1$ but failed to coordinate. Note that all possible such stages $(\text{CM}_m, \mathcal{H}_{k+1})$ are renamings of each other, so $\pi$ must give the same ECT from each one. We next establish that the expected coordination time from $(\text{CM}_m, \mathcal{H}_k)$ with $\pi$ is now given by the following formula (to be called formula (E) below):

$$p^2 \left( \frac{1}{2} + \frac{1}{2} (1 + E_1) \right) + 2p(1 - p) \cdot 2 + (1 - p)^2 \left( \frac{1}{n} + \frac{n - 1}{n} (1 + E_2) \right) \quad (E)$$

Indeed, both players choose a touched edge in round $k + 1$ with probability $p^2$. In that case the ECT from $(\text{CM}_m, \mathcal{H}_k)$ is $\frac{1}{2} + \frac{1}{2} (1 + E_1)$, the first occurrence of $\frac{1}{2}$ corresponding to direct coordination and the remaining term covering the case where coordination fails at first. Both players choose an untouched edge in round $k + 1$ with probability $(1 - p)^2$, and then the ECT from $(\text{CM}_m, \mathcal{H}_k)$ is $\frac{1}{n} + \frac{n - 1}{n} (1 + E_2)$. The remaining term $2p(1 - p) \cdot 2$ is the contribution of the case where one player chooses a touched edge and the other player an untouched one. The probability for this is $2p(1 - p)$, and the remaining factor 2 indicates that coordination immediately happens in the subsequent round $k + 2$ using the focal point created in round $k + 1$.

Now consider the following informal argument sketch. In $\text{CM}_m$ with $m \geq 6$, we may assume that $E_1 \leq 2$ and $E_2 \geq \frac{3}{2}$ by Lemmas 4.7 and 5.1. Figure 2 below illustrates the graph of (E) with $E_1 = 2$, $E_2 = \frac{3}{2}$, $n = 4$, so then (E) has a unique minimum at $p = 1$ when $p \in [0, 1]$. This
summarized in Figure 1. See Appendix D for further discussion on optimal play in stages with exactly two touched edges. Clearly, lowering $E_1$, raising $E_2$ or raising $n$ should make it even more beneficial to choose a touched edge. As we indeed can assume that $E_1 \leq 2$ and $E_2 \geq \frac{3}{2}$ in $CM_m$ for $m \geq 6$, this informally justifies that the following theorem holds.

**Theorem 6.2.** WM is uniquely ECT-optimal for each $CM_m$ with $m \geq 6$.

**Proof.** Let $S := (CM_m, \mathcal{H}_k)$, $m \geq 6$, be a non-final stage with precisely two touched edges and $S'$ a stage extending $S$ by one round where the players both choose an untouched edge but fail to coordinate. Let $r_1$ (respectively, $r_2$) be the infimum of all possible ECTs from $S$ (respectively, $S'$) with different protocols. Note that by Lemma 3.8, $r_1$ and $r_2$ are independent of which particular representative stages we choose, as long as the stages satisfy the given constraints.

Let $\epsilon > 0$ and fix some numbers $E_1$ and $E_2$ such that $|E_1 - r_1| < \epsilon$ and $|E_2 - r_2| < \epsilon$. We assume $E_1 \leq 2$ and $E_2 \geq \frac{3}{2}$ by Lemmas 6.4 and 6.5. It is easy to show that with such $E_1$ and $E_2$, the minimum value of the formula (E) with $p \in [0, 1]$ is obtained at $p = 1$ (for any $n = m - 2 \geq 4$).

Thus, after the necessarily random choice in round one, the above reasoning shows that the players should choose a touched edge with probability $p = 1$ in each round. Indeed, assume the earliest occasion that some protocol $\pi_k$ assigns $p \neq 1$ in some stage is round $k$. Then the above shows that the ECT of $\pi_k$ can be strictly improved by letting $p = 1$ in that round. By Lemma 6.2 in the Appendix, a uniform probability over the touched choices should be used.

We then cover the case for $CM_5$. The argument is similar to the case for $CM_m$ with $m \geq 6$, but this time leads to the use of LA instead of WM.

**Theorem 6.3.** For $CM_5$, LA is uniquely ECT-optimal.

**Proof.** Let $S := (CM_5, \mathcal{H}_k)$ be a non-final stage with precisely two touched edges and $S'$ a stage extending $S$ by one round where the players both choose an untouched edge but fail to coordinate. The ECT-optimal protocol from $S'$ chooses the unique winning pair of focal points in round $k + 2$, so we now have $E_2 = 1$. Let $r_1$ be the infimum of all possible ECTs from $S$ with different protocols. Let $\epsilon > 0$ and fix some real number $E_1$ such that $|E_1 - r_1| < \epsilon$, assuming $E_1 \geq \frac{3}{2}$ (cf. Lemma 6.1). It is straightforward to show that with these values, and with $n = 3$, the minimum of (E) when $p \in [0, 1]$ is obtained at $p = 0$. (See also Figure 2 for the graph of (E) when $E_1 = \frac{3}{2}$; for an illustration. Even then the figure suggests to choose an untouched edge.)

Thus, after the necessarily random choice in round one, the above reasoning shows that the players should choose an untouched edge with probability 1 in the second round, thereby following LA. Coordination is guaranteed (latest) in the third round.

In the last case $m = 4$, WM is ECT-optimal, but not uniquely, as there exist infinitely many other ECT-optimal protocols. The reason for this is that—as shown in Figure 2—the graph of (E) becomes the constant line with the value 2 in special case where $E_1 = E_2 = 2$, and then any $p \in [0, 1]$ gives the optimal value for (E). A complete proof is given in Appendix B.

**Theorem 6.4.** WM is ECT-optimal for $CM_4$, but there are continuum many other protocols that are also ECT-optimal.

We have now given a complete analysis of optimal ECTs and GCTs in two-player CM-games summarized in Figure 1. See Appendix D for further discussion on optimal play in CM-games.
7 The hardest two-player WLC-games

In this section we give an optimal characterization of the upper bounds of ECTs in WLC-games as a function of game size. For any $m \geq 1$, an $m$-choice game refers to any two-player WLC-game $G = (A, C_1, C_2, W_G)$ where $m = \max\{|C_1|, |C_2|\}$. Note that, with the classical matrix representation of an $m$-choice game, the parameter $m$ corresponds to the largest dimension of the matrix. In this section we will also show that CM$_m$ can be seen as the hardest $m$-choice game for all $m \neq 3$ (see Corollary 7.3).

Our first theorem shows that the wait-or-move protocol is reasonably “safe” to use in any $m$-choice game with $m \not\in \{3, 5\}$ as it always guarantees an ECT which is at most equal to the upper bound of optimal ECTs of all $m$-choice games for the particular $m$.

**Theorem 7.1.** Let $m \not\in \{1, 3, 5\}$ and consider an $m$-choice game $G = (A, C_1, C_2, W_G) \neq$ CM$_m$. Then the ECT in $G$ with WM is strictly smaller than the optimal ECT in CM$_m$.

**Proof.** By Theorems 6.2, 6.4 and Proposition 4.8 the optimal ECT in CM$_m$ is given by WM. We saw in Section 4 that the ECT with WM is $3 - \frac{2}{m}$ in CM$_m$ and at most $3 - 2p$ in $G$, where $p$ is the one-shot coordination probability when choosing randomly in $G$. Since $G$ is an $m$-choice game, $|W_G| \geq m$. If $|W_G| > m$, then $p > \frac{n}{m} = \frac{1}{m}$. And if $|W_G| = m$, we have $p = \frac{n}{m} = \frac{1}{m} > \frac{1}{m}$ where $n := \min\{|C_1|, |C_2|\} < m$ since $G \neq$ CM$_m$. In both cases, we have $3 - 2p < 3 - \frac{2}{m}$. $\square$

By the greatest optimal ECT among a class $G$ of WLC-games, we mean a value $r$ such that (1) $r$ is the optimal ECT for some $G \in G$; and (2) for every $G \in G$, there is a protocol which gives it an ECT $\leq r$. By Theorem 7.1 the greatest optimal ECT among $m$-choice games is given by WM in CM$_m$ for $m \not\in \{1, 3, 5\}$. Also the special cases of 1, 3 and 5 are covered below:

**Theorem 7.2.** For any $m$, the greatest optimal ECT among $m$-choice games is given below:

| Game size          | $m \not\in \{3, 5\}$ | $m = 5$ | $m = 3$ |
|--------------------|------------------------|---------|---------|
| Greatest optimal ECT | $3 - \frac{2}{m}$      | $2 + \frac{1}{2}$ | $\frac{1 + \sqrt{1 + 2m}}{2}$ ($\approx 1.925$) |

**Proof.** The case $m = 1$ is trivial and the cases $m \not\in \{3, 5\}$ follow from Theorem 7.1. When $m = 3$ or $m = 5$, we need to systematically cover all $m$-choice games and give estimates for ECTs in them. This is done in Appendix C where we provide an extensive graph theoretic analysis of all 3-choice and 5-choice games. It turns out that the greatest optimal ECT among 5-choice games is realized in CM$_5$ (and no other 5-choice game). For $m = 3$, the greatest optimal ECT is also realized by a single WLC-game. This game is pictured below. $\square$

As the greatest optimal ECT is realized uniquely by CM$_5$, the following holds by Theorem 7.1:

**Corollary 7.3.** For $m \neq 3$, the greatest optimal ECT among $m$-choice games is uniquely realized by CM$_m$.

Hence choice matching games can indeed be seen as the most difficult two-player WLC-games—excluding the interesting special case of 3-choice games as discussed above.

8 Conclusion

In this paper we gave a complete analysis for two-player CM-games with respect to both GCTs and ECTs. We also found optimal upper bounds for optimal ECTs for all two-player WLC-games when determined according to game size only. A highly challenging next step would be to find complete characterizations for optimal ECTs (and GCTs) for all WLC-games when determined by the full structure of the game.
A Appendix: Examples and extra definitions

Example A.1. Here we give two examples of drawings of WLC-games: a two-player game $G_1$ with 3 choices for both players and a total of 6 winning profiles represented as edges; and a three-player WLC-game $G_2$ with 2 choices for each player and 4 winning profiles, each represented as a triple of choices connected by (solid or dotted) lines.

\[
G_1: \quad a_1 \quad a_2 \\
b_1 \quad b_2 \quad c_1 \quad c_2 \\
\]

\[
G_2: \quad a_1 \quad a_2 \quad a_3 \\
b_1 \quad b_2 \quad b_3 \\
\]

We now specify some useful notational conventions from \cite{8} for identifying some special WLC-games (see also the figure below for related examples).

- Let $m_1, \ldots, m_n \in \mathbb{Z}_+$. We write $G(m_1 \times \cdots \times m_n)$ for the $n$-player WLC-game where the player $i$ has $m_i$ choices and the winning relation is the universal relation $C_1 \times \cdots \times C_n$.

- Let $m \geq 2$. We write $G(O_m)$ for the two-player WLC-game in which both players have $m$ choices and the winning relation $W_G$ forms a $2m$-cycle through all the $2m$ choices. (Thus the game graph of this WLC-game corresponds to the cycle graph $C_{2m}$.) Similarly we write $G(Z_m)$ for the two-player WLC-game where both players have $m$ choices and $W_G$ forms a $(2m - 1)$-edge path through all choices. Moreover $G(\Sigma_m)$ denotes a WLC-game where the player 1 has $m - 1$ choices, the player 2 has $m$ choices and $W_G$ forms a $(2m - 2)$-edge path through all the choices; the game obtained by permuting the players in $G(\Sigma_m)$ is denoted by $G(\Sigma_m)$.

- Suppose that $G(A)$ and $G(B)$ have been defined and both have the same number of players. Then $G(A + B)$ is the disjoint union of $G(A)$ and $G(B)$, i.e., the game obtained by assigning to each player a disjoint union of her/his choices in $G(A)$ and $G(B)$, with the winning relation for $G(A + B)$ being the union of the winning relations in $G(A)$ and $G(B)$.

- If $m \in \mathbb{Z}_+$, then $G(mA) := G(A + \cdots + A)$ (with $A$ repeated $m$ times).

\[
\begin{array}{c}
G(2 \times 3) & G(O_3) & G(Z_3) & G(\Sigma_3) & G(1 \times 1 + Z_3) & G(3(1 \times 1 \times 1)) \\
\end{array}
\]

Note that the game $G(m(1 \times 1))$ is the two-player choice matching game $CM_m$.

Example A.2. Below we have two stages $(G, \mathcal{H}_2)$ and $(G', \mathcal{H}_2')$, where the players have selected the choices with dotted circles in round 1 and the choices with solid circles in round 2. There is a renaming between the stages $(G, \mathcal{H}_2)$ and $(G', \mathcal{H}_2')$. This is because if we first swap the players in $(G, \mathcal{H}_2)$, then there will be an isomorphism to $(G', \mathcal{H}_2')$. Also note that the choices $c$ and $d$ are structurally equivalent in the initial stage $(G, \mathcal{H}_0)$, but this equivalence is broken when the player 2 selects $c$ in the first round.

\[
\begin{array}{c}
(G, \mathcal{H}_2): \\
\quad a \quad c \\
\quad b \quad d \\
\hline
(G', \mathcal{H}_2'): \\
\quad r \quad u \\
\quad s \quad v \\
\hline
\end{array}
\]
Example A.3. We consider two concrete examples of focal points. However, before that, note that if choice $c_i$ of player $i$ in stage $S$ is a focal point, then one of the following two scenarios hold by the definition of focal points:

- $c_i$ is not structurally equivalent to any other choice in stage $S$.
- $c_i$ is structurally equivalent to some other choices $d_1, \ldots, d_\ell$ in $S$. In this case all the choices $c_i, d_1, \ldots, d_\ell$ must belong to the same single edge of the winning relation $W_G$ for the following reason: the choice $c_i$ is structurally equivalent to the choice $d_j \in \{d_1, \ldots, d_\ell\}$ of player $j$ but $c_i$ is not structurally equivalent to any other choice of player $i$, so $d_j$ cannot be structurally equivalent to any other choice of player $j$.

Now to the examples. Consider the first two rounds of the game $CM_5$, pictured below, where the players fail to coordinate by first selecting the pair $(a_1, b_2)$ and then fail again by selecting the pair $(b_1, c_2)$.

![CM5 diagram](image)

The structural equivalence classes become modified in this scenario as follows:

- Initially all choices are structurally equivalent.
- After the first round, the equivalence classes are $\{a_1, b_2\}$, $\{b_1, a_2\}$ and $\{c_1, d_1, e_1, c_2, d_2, e_2\}$.
- After the second round, the equivalence classes are $\{a_1\}$, $\{a_2\}$, $\{b_1\}$, $\{b_2\}$, $\{c_1\}$, $\{c_2\}$ and $\{d_1, e_1, d_2, e_2\}$.

There are no focal points in the initial stage $S_0$ and the same is true for the next stage $S_1$. However, in the stage $S_2$, all the choices $a_1, b_1, c_1, a_2, b_2, c_2$ become focal points, and the players can thus immediately guarantee coordination in the third round by selecting any winning pair of focal points, i.e., any of the pairs $(a_1, a_2)$, $(b_1, b_2)$, $(c_1, c_2)$. (We note that, from the point of view of the general study of rational choice, it may not be obvious which of these pairs should be selected, so a convention may be needed to fix which protocol to use.)

Consider then the game $\overline{G(O_5)}$, the complement of the cycle game $G(O_5)$. In the pictures below, we present $\overline{G(O_5)}$ also in the form where the choices are arranged in a cycle and we draw the choices of player 2 in white for clarity.

![G(O_5) diagram](image)

Note that all choices are initially structurally equivalent in $\overline{G(O_5)}$. Suppose then that the players fail to coordinate in the first round. This can happen only if they select choices that are “adjacent in the cycle” (see the picture above). Hence, by symmetry, we may assume that the players choose the pair $(a_1, c_2)$ in the first round. Then the equivalence classes after the first round are $\{a_1, e_2\}$, $\{b_1, d_2\}$, $\{c_1, c_2\}$, $\{d_1, b_2\}$ and $\{e_1, a_2\}$. Hence the players can guarantee coordination in the second round by selecting the winning pair $(b_1, d_2)$ of focal points (or alternatively the pair $(c_1, c_2)$ or $(d_1, b_2)$).
Example A.4. We illustrate the use of the LA protocol in the game CM₅, pictured below. Suppose that coordination fails in the first round. By symmetry, we may assume that the players selected a₁ and b₂. Now, in the resulting stage S₁, the structural equivalence classes are \{a₁, b₂\}, \{b₁, a₂\} and \{c₁, d₁, e₁, c₂, d₂, e₂\}.

If the pair (b₁, a₂) is selected in the next round, then the structural equivalence classes do not change and thus the resulting next stage is automorphism-equivalent to S₁. Hence, by following LA, player 1 should avoid selecting b₁ and player 2 should avoid selecting a₂. For the same reason, the players should also avoid selecting the choices a₁ and b₂.

\[
\text{CM}_5: \quad \begin{array}{c}
 a_1 & \bullet & a_2 \\
 b_1 & \bullet & b_2 \\
 c_1 & \bullet & c_2 \\
 d_1 & \bullet & d_2 \\
 e_1 & \square & e_2 \\
\end{array}
\]

Hence, by following LA in S₁, the players will select among the set \{c₁, d₁, e₁, c₂, d₂, e₂\} with the uniform probability distribution. Supposing that they fail again in coordination, we may assume by symmetry that they selected the pair (c₁, d₂). The equivalence classes in the resulting stage S₂ are \{a₁, b₂\}, \{b₁, a₂\}, \{c₁, d₂\}, \{d₁, c₂\} and \{e₁, e₂\}. Now, selecting any of the pairs (a₁, b₂), (b₁, a₂), (c₁, d₂) and (d₁, c₂) leads to a next stage which is automorphism-equivalent to S₂. Thus, by following LA in S₂, the players will select the pair (e₁, e₂). This leads to guaranteed coordination in the third round.

Definition A.5. The **wait-or-move protocol** (WM) for repeated two-player WLC-games goes as follows. Pick your first choice randomly (with a uniform probability over all choices). Then do the following in all non-final stages.

1. Suppose that both players have selected only a single choice (possibly several times) in the previous rounds. Let c₁ be your earlier choice and c₂ the earlier choice of the other player. Then select your next choice according to the probability distribution f such that
   - f(c₁) = \frac{1}{2}, and
   - each choice that coordinates with c₂ is picked with equal probability, the total probability over such choices being \frac{1}{2}.

2. Suppose that both players have selected exactly two choices (possibly several times). Then select one of your previous choices, each with probability \frac{1}{2}.

3. In any other non-final stage, pick your choice randomly. (Note that such a non-final stage cannot even be reached if both players follow WM.)

B Appendix: Complete proofs and additional lemmas

Proposition 4.2 restated. LA is the uniquely ECT-optimal and uniquely GCT-optimal in CM₃.

Proof. Every structural protocol (and thus LA) must choose a random choice in the first round of CM₃. If the players fail to coordinate in the first round, then the only probability distribution that guarantees a win in the second round selects the unique choice from the only untouched edge (as LA instructs). It is thus clear that LA is both uniquely ECT-optimal and uniquely GCT-optimal in CM₃. \(\square\)
Proposition 4.3 restated. LA guarantees coordination in games CM\(_m\) in \([m/2]\) rounds when \(m\) is odd, but LA does not guarantee coordination in CM\(_m\) for any even \(m\).

Proof. For the sake of completeness, we give here a full proof of the proposition. However, the fact that LA does not guarantee coordination in CM\(_m\) for any even \(m\) will also follow directly from Theorem 5.1, whose proof does not depend in any way of the current proposition.

Consider CM\(_m\) with an odd \(m\). As LA is a structural protocol, the players must pick randomly in the first round. Supposing they do not coordinate, this creates two touched edges. In the next round, the players must pick choices that are not on the touched edges, because the protocol LA instructs to pick—if possible—choices that cannot lead to a stage that is automorphism-equivalent to the current stage. Similarly, in every round where the players have failed to coordinate, they must choose from untouched edges. In the worst case, since \(m\) is odd, the players can fail to coordinate until there is exactly one untouched edge left. Then the players coordinate in the next stage, and clearly this takes \(\lceil m/2 \rceil\) rounds.

The scenario is very similar in the case \(m\) is even, but this time the players may end up in a situation where all edges have become touched, but coordination has failed. Then every choice \(c\) of player 1 is structurally equivalent to a choice \(c'\) of player 2 such that \(c\) and \(c'\) are not on the same edge. Since the players are using a structural protocol, they may end up choosing such a pair of structurally equivalent choices, failing coordination. Moreover, if the players indeed choose the pair \((c, c')\), this leads an automorphism-equivalent stage. Therefore, picking \((c, c')\) leads to the same problem again: coordination can fail due to picking structurally equivalent choices. This way the players may end up forever choosing automorphism-equivalent stages without coordinating.

Theorem 4.5 restated. Let \(G\) be a WLC-game with one-shot coordination probability \(p\) when both players make their first choice randomly. Then the expected coordination time by WM is at most \(3 - 2p\).

Proof. Let \(G\) be a WLC-game. First note that the probability for coordination with WM in the first round is trivially \(p\). After that, the players follow WM and thus, in every round, either repeat their previous choice or swap to another choice with equal probability. If one player repeats and the other one swaps, then they coordinate. Thus, in every round after the first round, the one-shot coordination probability is at least \(1/2\). (Note that this probability can be greater than \(1/2\) as \(G\) is not necessarily a choice matching game.)

Let us first consider the case where \(G\) is such that if coordination fails in the first round, then, in every subsequent round, the probability of coordination is exactly \(1/2\). (This includes, e.g., all choice matching games CM\(_m\) with \(m > 1\).) Now the probability of coordinating in the \(k\)th round (and not earlier) is \((1 - p) \cdot \left(\frac{1}{2}\right)^{k-1}\) for all \(k \geq 2\). Hence the expected value \(E\) for the coordination time with WM is calculated as follows.

\[
E = p + (1 - p) \sum_{k \geq 2} \frac{k}{2^{k-1}} = p + (1 - p) \sum_{k \geq 2} \frac{2k}{2^k}.
\]

It is well known that

\[
\sum_{k \geq 1} \frac{k}{2^k} = 2,
\]

whence

\[
\sum_{k \geq 2} \frac{k}{2^k} = \frac{3}{2} \quad \text{and thus} \quad \sum_{k \geq 2} \frac{2k}{2^k} = 3.
\]

Thus \(E = p + (1 - p) \cdot 3 = 3 - 2p\). Therefore, in the general case where the probability of coordinating is at most \(1/2\) in the rounds after the first one, it is now immediate that \(E \leq 3 - 2p\) and thus \(3 - 2p\) is still an upper bound for the expected coordination time with WM.

\[\square\]
Proposition 4.8 restated. WM is uniquely ECT-optimal in CM₂.

Proof. The claim follows directly from Lemma 3.2 given below; we will first present a technical auxiliary definition (Definition B.1) and then prove Lemma B.2.

Definition B.1. Consider a choice matching game CMₘ and assume a stage (CMₘ, ℋₖ) has edges (u, v) and (u′, v′) such that u ∼ v and u′ ∼ v (recall Definition 5.3). Then we say that the nodes u and u′ are conjugates (of each other), and likewise, the choices v and v′ are conjugates.

The following lemma states that protocols become faster if they are adjusted to assign the same probability to conjugate elements in choice matching games.

Lemma B.2. Let S = (CMₘ, ℋₖ) be a stage and π be a protocol which assigns different probabilities p_u and p_u′ to some conjugate nodes u and u′ of S (see Definition B.1 above). Let π′ be the protocol that is otherwise as π but assigns u and u′ the same probability \( \frac{1}{2}(p_u + p_u′) \) in S. Then the ECT from S with π′ is strictly smaller than the ECT from S with π.

Proof. Let v and v′ denote the choices such that CMₘ has edges (u, v) and (u′, v′). As π is a structural protocol, we must have p_v = p_v′ and p_u = p_u′. To simplify notation, call p_u = x and p_u + p_u′ = c. Thus p_u = p_u′ = c − p_u = c − x.

Under the condition that both players end up choosing one of the edges (u, v), (u′, v′) in the stage S, the probability of winning is

\[
\frac{2 \cdot x(c - x)}{c^2} = -\frac{2x^2}{c^2} + \frac{2x}{c}.
\]

This has its global maximum at \( x = \frac{c}{2} = \frac{1}{2}(p_u + p_u′) \). Since π and π′ agree on all moves other than the one discussed here, the claim follows.

Theorem 5.2 restated. For all odd m ≥ 1, LA is uniquely GCT-optimal in CMₘ.

Proof. Let m be odd. Recall that, by Proposition 1.3, the GCT in CMₘ with LA is \([m/2]\) rounds. We assume, for contradiction, that there is some protocol π ≠ LA that guarantees coordination in CMₘ in at most \([m/2]\) of rounds, possibly less. As π ≠ LA, there exists some play of CMₘ where both players follow π, and in some round, at least one of the players chooses a node on a touched edge. (Recall from the proof of Proposition 1.3 that LA never chooses from a touched edge in CM-game with an odd number of edges.) Now, let Sₖ = (CMₘ, ℋₖ) be the first stage of that play when this happens—so if (c, c′) is the most recently recorded pair of choices in Sₖ, then at least one of c and c′ is part of an edge that has already been touched in some earlier round. And furthermore, in all stages Sₜ with ℓ’ < ℓ, the most recently chosen pair does not contain a choice belonging to an edge that was touched in some yet earlier round ℓ’ < ℓ.

In the stage Sₖ₋₁ it therefore holds that for every choice profile (cᵢ, dᵢ), chosen in some round i ≤ (ℓ − 1), the nodes cᵢ and dᵢ are structurally equivalent. Of course also the nodes of Sₖ₋₁ on so far untouched edges are structurally equivalent to each other. Furthermore, the number of already touched edges in Sₖ₋₁ is the even number \( m' = 2(ℓ - 1) \).

We will now show that π does not guarantee a win in \([m/2] - (ℓ - 1)\) rounds when starting from the stage Sₖ₋₁. This completes the proof, contradicting the assumption that π guarantees a win in CMₘ in at most \([m/2]\) rounds.

Now, recall the stage Sₖ from above where (c, c′) contained a choice from an already touched edge. By symmetry, we may assume that c is such a choice. Starting from the stage Sₖ₋₁, consider a newly defined stage Sₖ′ where the first player again makes the choice c but the other player this time makes a structurally equivalent choice c* ∼ c. This is possible as π is a structural protocol. Now note that the choice profile (c, c*) is not winning since c and c* are structurally equivalent.
equivalent choices from already touched edges, and thus either \((c, c')\) is a choice profile that has already been chosen in some earlier round \(j < \ell\), or the nodes \(d, d'\) adjacent in \(CM_m\) to \(c, c\) (respectively) form a choice profile \((d, d')\) chosen in some earlier round \(j < \ell\).

Therefore, in the freshly defined stage \(S'_\ell\), the players have in every stage (including the stage \(S'_\ell\) itself) selected a choice profile that consists of two structurally equivalent choices. Both choices in the most recently selected choice profile in \(S'_\ell\) have been picked from edges that have become touched even earlier. It now suffices to show that it can still take \([m/2] - (\ell - 1)\) rounds to finish the game. To see that this is the case, we shall next consider a play from the stage \(S'_\ell\) onwards where in each remaining round, the choice profile \((e, e')\) picked by the players consists of structurally equivalent choices; such a play exists since \(\pi\) is structural.

Due to picking only structurally equivalent choices in the remaining play, when choosing a profile from the already touched part, the players will clearly never coordinate. And when choosing from the untouched part, immediate coordination is guaranteed if and only if there is only one untouched edge left. Therefore the players coordinate exactly when they ultimately select from the last untouched edge. As the stage \(S'_\ell\) has precisely \(m - 2(\ell - 1)\) untouched edges, winning in this play takes at least
\[
\left\lfloor \frac{m - 2(\ell - 1)}{2} \right\rfloor = [m/2] - (\ell - 1)
\]
rounds to win from \(S'_\ell\).

\[\text{Lemma } \text{B.3} \text{ restated. The ECT from } (CM_m, \mathcal{H}_k) \text{ with no focal point is at least } \frac{3}{2} \text{ with any protocol.} \]

\[\text{Proof. If } (CM_m, \mathcal{H}_k) \text{ has an even number of edges, then, since } (CM_m, \mathcal{H}_k) \text{ has no focal points, we can partition its edges into doubleton sets, each set containing exactly two edges } (u, v) \text{ and } (u', v') \text{ such that } u \sim v' \text{ and } u' \sim v \text{ (whence } u \text{ and } u' \text{ as well as } v \text{ and } v' \text{ are conjugates in the sense of Definition B.1). If } (CM_m, \mathcal{H}_k) \text{ has an odd number of edges and no focal points, then we can construct a partition consisting of similar doubletons together with one tripleton set with edges } (u, v), (u', v'), (u'', v'') \text{ such that all the choices } u, u', u'', v, v', v'' \text{ are all pairwise structurally equivalent.} \]

To coordinate in the next round \(k + 1\), the players must select from the same (doubleton or tripleton) set \(T\) of edges in the partition, and within \(T\), they must choose the same edge. Now recall that the players use the same protocol, and the protocol determines the same probability for all structurally equivalent choices. Thus the probability of hitting the same edge on the condition that the players have chosen from the same doubleton set \(T\) is at most \(\frac{1}{2}\) (this follows easily from the proof of Lemma B.2). The probability of hitting the same edge on the condition that the players choose from the tripleton set is necessarily \(\frac{1}{3}\), as all the six choices within that tripleton are pairwise structurally equivalent, and thus the protocol assigns them the same probabilities. Therefore, for any protocol, \(\frac{1}{2}\) is an upper bound for the probability of coordinating in the next round \(k + 1\).

Now, suppose that the players coordinate with probability \(\frac{1}{2}\) in round \(k + 1\), and suppose they are guaranteed to coordinate in round \(k + 2\) if they fail in round \(k + 1\). Then the ECT for the remaining game is \(\frac{1}{2} + \frac{1}{2} \cdot 2 = \frac{3}{2}\).

The following lemma will be needed in the proof of Theorem 6.4 below.

\[\text{Lemma B.3. Let } S_k = (CM_4, \mathcal{H}_k) \text{ and } S'_n = (CM_4, \mathcal{H}'_n) \text{ be stages of } CM_4 \text{ with exactly 2 and 4 touched edges, respectively, and no focal points. Assume also, for technical convenience, that } S'_n \text{ does not extend the history of } S_k, \text{ i.e., } S'_n \text{ cannot be reached from } S_k. \text{ Let } \pi \text{ be a protocol whose ECT is } r \text{ when starting from } S'_n. \text{ Then there exists a protocol } \pi' \text{ whose ECT is } s \leq r \text{ when starting from } S'_n \text{ and also when starting from } S_k. \text{ In every stage, } \pi' \text{ assigns the same probability to conjugate nodes (cf. Definition B.1).} \]
Proof. By Lemma \[B.2\] it is easy to see that there exists a protocol $\pi^*$ whose ECT when starting from $S'_n$ is some number $s \leq r$ and the following conditions hold:

1. In the stage $S'_n$ and in stages extending the history of $S'_n$, the protocol $\pi^*$ always assigns the same probability to all nodes that are conjugates (cf. Definition \[B.1\]).

2. Whenever a focal point is created, the protocol $\pi^*$ forces the players to coordinate immediately in the next round.

Due to the first condition above, it is possible to copy the behaviour of $\pi^*$ starting from $S'_n$ to all games starting from $S_k$ in the direct way described next. First note that both $S'_n$ and $S_k$ are based on the same graph $CM_4$ with the same set of nodes. We may assume, by symmetry, that conjugate nodes in $S_k$ are also conjugates in $S'_n$. We copy the behaviour of $\pi^*$ in the games starting from $S'_n$ to the games starting from $S_k$ just by assigning the exact same probabilities chosen in $S'_n$ to the exactly same nodes in the corresponding stage $S_{k+\ell}$ (that extends the history of $S_k$ in the same way as $S'_n$ extends the history of $S'_n$). It is easy to see that this constructs a structural protocol due to the condition 1 above stating that $\pi^*$ gives the same probabilities to conjugate nodes. Clearly the copied protocol gives the same ECT starting from $S_k$ as $\pi^*$ gives when starting from $S'_n$.

Now the ultimate desired protocol $\pi'$ is constructed by combining $\pi^*$ and the constructed copy. The assumption that $S'_n$ does not extend the history of $S_k$ is used in this combination step. Note that thus $\pi'$ clearly assigns the same probabilities to conjugates in all stages reachable from $S_k$ and $S'_n$, and by Lemma \[B.2\] we can ensure that $\pi'$ also assigns the same probability to conjugates in all other stages.

\[\Box\]

**Theorem 6.4 restated.** WM is ECT-optimal for $CM_4$, but there are continuum many other protocols that are also ECT-optimal.

Proof. Consider a stage $S_k = (CM_4, H_k)$ with exactly two touched edges. We will first show that no protocol gives an ECT less than 2 from the stage $S_k$. This is done by establishing that existence of such a protocol would imply existence of a protocol in $CM_2$ with ECT less than 2, contradicting Proposition \[4.8\] and Theorem \[4.5\].

Now, suppose, for contradiction, that $\pi$ gives an ECT less than 2 when starting from $S_k$. By Lemma \[B.2\] we can assume that $\pi$ assigns the same probability to conjugate nodes (cf. Definition \[B.1\]) in $S_k$. Therefore the formula (E) (see Section 6) gives the ECT for $\pi$ from $S_k$, given we plug in the right values for $p$, $E_1$, $E_2$ and $n$. We have $n = 2$ and the other values are determined by $\pi$, with $E_1$ corresponding to the situation with two touched edges and $E_2$ to the situation with four touched edges. We may assume that $E_1 < 2$ because all stages with exactly two touched edges are automorphism-equivalent and the ECT from $S_k$ (which has exactly two touched edges) is less than 2. Using Lemma \[B.3\], we see that there exists a protocol $\pi'$ with $E_2 = E_1 < 2$ that also gives an ECT less or equal to the ECT of $\pi$ from $S_k$. And furthermore, the formula (E) with these fixed values $E_2 = E_1 < 2$ (and with $n = 2$) gives the right value for the ECT of $\pi'$ from $S_k$. It is easy to prove that with these values, the formula (E) has its minimum values at $p = 0$ and $p = 1$ when $p \in [0, 1]$; for an illustration, see the graph of (E) in Figure 3 for $E_1 = E_2 = 2 - \epsilon$ for some (small) $\epsilon > 0$.

![Figure 3](image-url)

Figure 3: Left: Curve of (E) when $n = 2$ and $E_1 = E_2 = 2 - \epsilon$ for some $\epsilon > 0$. Right: Curve of (E) when $n = 2$ and $E_1 = E_2 = 2$. 

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Therefore the protocol $\pi''$ that uses $p = 1$ at $S_k$, but otherwise behaves as $\pi'$, has the following properties:

- $\pi''$ has the same ECT (less than 2) as $\pi'$ when starting from $S_k$.
- $\pi''$ shares the values $E_2 = E_2 < 2$ with $\pi'$.

(We note that of course possibly $\pi'' = \pi'$.) Now, $\pi''$ instructs the players to choose from an already touched edge at $S_k$, so every resulting stage $S_{k+1}$ turns out to be automorphism-equivalent to the stage $S_k$. Thus we can repeat the reasoning above concerning $S_k$, this time beginning from $S_{k+1}$. Iterating the argument repeatedly, it is easy to see that in the limit, we get a protocol that behaves precisely as WM but has an ECT less than 2 when starting from $S_k$. This contradicts the fact that the ECT of WM is 2 when starting from $S_k$ by Lemma 4.7.

Having proved that no protocol has an ECT less than 2 in $S_k$, we then observe by Lemma 4.7 that therefore WM is an ECT-optimal protocol for $S_k$ and therefore trivially also for CM4. We still must find continuum many other optimal protocols for CM4. Clearly it suffices to prove that there are continuum many other optimal protocols when starting from an arbitrary stage $S_k$ where we have exactly two touched edges.

Consider again the formula (E) with $n = 2$ and $E_1 = E_2 = 2$, i.e., the values given by WM which we above identified to be ECT-optimal in CM4 and also when starting from $S_k$. It is easy to show that with these values, the formula (E) becomes equal to the constant 2 for all $p \in [0, 1]$; see Figure 3 for an illustration of the corresponding flat curve and its contrast to the case where $E_1 = E_2 = 2 - \epsilon$. Therefore we can clearly modify WM to give any value of $p \in [0, 1]$ when starting from $S_k$ such that, despite the modification, the resulting protocol is still ECT-optimal in CM4. Thus there exist at least continuum many ECT-optimal protocols for CM4. In fact, it is clear that we can analogously modify these protocols also in other stages in addition to $S_k$ without changing the ECT. However, it is straightforward to establish that the number of all protocols for CM4, whether optimal or not, is limited by the continuum, so there indeed exist precisely continuum many ECT-optimal protocols for CM4.

C Appendix: Analysis of ECTs in 3- and 5-choice games

In this section we will systematically analyse all 3-choice games and 5-choice games and give estimates for ECTs in them (recall Section 7 for the exact definition of an $m$-choice game). This analysis is necessary for the special cases $m = 3$ and $m = 5$ in the proof of Theorem 7.2. We will be using the notations for WLC-games from Example A.1.

We first note that the optimal ECT is 1 for all those WLC-games in which coordination can be guaranteed in a single round. Such games are given a complete characterization in [8]. For example, in the game $G(1 \times 1 + 2 \times 2)$, coordination can be guaranteed in a single round by both players selecting choices of degree 1 (which are indeed focal points), or alternatively, by both selecting choices of degree 2 (which form a “winning focal set”).

C.1 Analysis of 3-choice games

In this section we will show that, among all two-player 3-choice games, the greatest optimal ECT is uniquely realized by the game $G(1 \times 2 + 2 \times 1)$. We also show that the optimal ECT for this game is

$$\frac{1 + \sqrt{4 + \sqrt{17}}}{2} \approx 1.925.$$

We first note that if either of the players has a choice of degree 3 in a 3-choice game $G$, then the optimal ECT in $G$ is 1 (since selecting such a choice trivially guarantees coordination). Thus we can restrict our analysis to those 3-choice games in which the degree of each choice is
at most 2. Note that the game graph of $G$ must thus consist of components which are either cycles or paths (in particular, they are subgraphs of the form $G(n)$, $G(1 \times 1)$, $G(1 \times 2)$, $G(n)$, $G(\Sigma_n)$; recall the notations from Example 4.1). We list here systematically all such 3-choice games $G$ grouped by the number of edges in the winning relation $W_G$. (Note that we must have $3 \leq |W_G| \leq 6$ as $G$ is a 3-choice game and the degree of each choice is at least 1 and at most 2.)

| $|W_G|$ | $G(1 \times 2 + 1 \times 1)$ | $G(3(1 \times 1)) = \text{CM}_3$ | $G(O_3)$ |
|-------|-----------------|-----------------|-----|
| 3     | $G(1 \times 2)$ | $G(O_2 + 1 \times 1)$ | $G(O_3)$ |
| 4     | $G(2 \times 1)$ | $G(Z_2)$ | $G(Z_3)$ |
| 5     | $G(2 \times 2 + 1 \times 1)$ | $G(\Sigma)$ |
| 6     | $G(1 \times 2)$ |

Among these games, the only ones that do not have a focal point are the games $\text{CM}_3$, $G(O_3)$, $G(1 \times 2 + 2 \times 1)$ which we analyse below.

• $\text{CM}_3$ (= $G(O_3)$)
  The optimal ECT here is $1 + \frac{2}{3}$ by Proposition 4.2 (see the table in Section 4).

• $G(O_3)$ (= $\overline{\text{CM}_3}$)
  The one-shot coordination probability here is $\frac{2}{3}$. Suppose that the players simply make a random choice in every round (with uniform probability distribution). The obtained ECT can then be calculated as follows:

$$\sum_{k \geq 0} \frac{2}{3} \left( \frac{1}{3} \right)^k (k + 1) = 2 \cdot \sum_{k \geq 0} \frac{k + 1}{3k+1} = 2 \cdot \sum_{k \geq 1} \frac{k}{3^k} \quad (\star) \quad 2 \cdot \frac{3}{4} = 1 + \frac{1}{2}.$$

$(\star)$ It is easy to shown that $\sum_{k \geq 1} \frac{k}{3^k} = \frac{9}{4}$. (It is relatively easy to see that this ECT will indeed be optimal for $\overline{\text{CM}_3}$, but there is no need for us to prove it here.)

• $G(1 \times 2 + 2 \times 1)$

We will show that the optimal ECT for this game is $1 + \frac{\sqrt{4 + \sqrt{17}}}{2}$, but there are several protocols which give this optimal ECT. See below for a proof.

Consider the following game:

$\begin{align*}
  a_1 & \quad c_2 \\
  b_1 & \quad b_2 \\
  c_1 & \quad a_2
\end{align*}$

Recalling the notion of structural equivalence from Definition 3.3, in the initial stage there are two structural equivalence classes:

1. $\{a_1, a_2\}$ and $\{b_1, c_1, b_2, c_2\}$.

If players fail to coordinate by both selecting a node with degree 2, then the next stage will also be of type (1). However, if they fail to coordinate by selecting choices with degree 1, the equivalence class $\{b_1, c_1, b_2, c_2\}$ is split into two classes with two choices. We may assume by symmetry that the players chose $b_1$ and $b_2$, whence we have the following equivalence classes in the next stage:

2. $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$. 

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If players fail to coordinate by selecting the pair \((a_1, a_2), (b_1, b_2)\) or \((c_1, c_2)\), then the next stage will also be of type (2). But if they fail to coordinate by one of them selecting from \(\{b_1, b_2\}\) and the other one selecting from \(\{c_1, c_2\}\), then all symmetries are broken and every choice turns into a focal point—and thus coordination can be guaranteed in the next round.

We first examine a stage \(S_2\) of the type (2) and find the optimal probability distribution for it. The corresponding optimal ECT will be used later for finding the optimal ECT for a stage of type (1).

We first observe that in order to maximize the possibility of breaking symmetries and creating focal points, it is optimal for the players to have the uniform probability distribution for selecting between the sets \(\{b_1, b_2\}\) and \(\{c_1, c_2\}\) (this can be proven similarly as Lemma \([3, 8]\)). Thus, let \(p_2\) denote the probability for selecting within \(\{b_1, b_2, c_1, c_2\}\). Let \(E_2\) denote ECT for the remaining game if players fail to coordinate and fail to create a focal point in \(S_2\) (There are several ways how this can happen, but since all of the resulting stages are of type (2), we may assume the same ECT for all of them by Lemma \([3, 8]\)).

Under the assumptions above, the ECT from \(S_2\), with parameters \(p_2\) and \(E_2\), is given by the following function:

\[
g(p_2, E_2) = (1 - p_2)^2(1 + E_2) + 2p_2(1 - p_2) + p_2^2\left(\frac{1}{2}(1 + E_2) + \frac{1}{2} \cdot 2\right).
\]

The partial derivate \(g_{p_2} = (1 + 3E_2) - 2E_2\) goes the zero when \(p_2\) has the value \(p^*_2 := \frac{2E_2}{1 + 3E_2}\). Whenever \(E_2 \geq 1\), the smallest value for \(g(p_2, E_2)\) is obtained when \(p_2 = p^*_2\). Because both \(g(p, E_2)\) and \(E_2\) refer to ECT from a stage of type (2), \(E_2\) obtains its smallest possible value when

\[
E_2 = g(p^*_2, E_2).
\]

The only (positive) solution for this equation is \(E_2 = \frac{3 + \sqrt{17}}{4} \approx 1.781\). This is the optimal ECT from any stage of type (2).

Next we will use the value \(E_2\) to determine the optimal ECT from a stage \(S_1\) of type (1). Let \(E_1\) denote the ECT for the remaining game if both players select within the set \(\{a_1, a_2\}\). When \(p_1\) denotes the probability of choosing within the set \(\{b_1, c_1, b_2, c_2\}\), the ECT from \(S_1\) is given by the following function:

\[
f(p_1, E_1, E_2) = (1 - p_1)^2(1 + E_1) + 2p_1(1 - p_1) + p_1^2(1 + E_2) = (E_1 + E_2)p_1^2 - 2E_1p_1 + (1 + E_1).
\]

The partial derivate \(f_{p_1} = (2E_1 + 2E_2)p_1 - 2E_1\) goes the zero when \(p_1\) has the value \(p^*_1 := \frac{E_1}{E_1 + E_2}\). Whenever \(E_1, E_2 \geq 1\), the smallest value for \(f(p_1, E_1, E_2)\) is obtained when \(p_1 = p^*_1\). Because both \(f(p, E_1, E_2)\) and \(E_1\) refer to ECT from a stage of type (1), \(E_1\) obtains its smallest possible value when \(E_2 = \frac{3 + \sqrt{17}}{4}\) and we have

\[
E_1 = f(p^*_1, E_2).
\]

When \(E_2 = \frac{3 + \sqrt{17}}{4}\), the only (positive) solution for the equation above is \(E_1 = \frac{1 + \sqrt{4 + \sqrt{17}}}{4}\). This is the optimal ECT from any stage of type (1), and thus, in particular, it is the optimal ECT for the game \(G(1 \times 2 + 2 \times 1)\). Hence the greatest optimal ECT among 3-choice games is uniquely realized by \(G(1 \times 2 + 2 \times 1)\). This concludes the analysis of 3-choice games.

We digress from the main story to make a few interesting remarks. The optimal ECT for \(G(1 \times 2 + 2 \times 1)\) is given by protocols that use the optimal values for \(E_1\) and \(E_2\) (given above) for calculating the probabilities \(p^*_1 \approx 0.5195\) and \(p^*_2 \approx 0.5616\) and use these probabilities
for selecting within the set \( \{b_1, c_1, b_2, c_2\} \) in stages of type (1) and (2), respectively. However, there is no unique protocol which gives the optimal ECT since there are 3 winning pairs of focal points that are formed if players break the symmetry in a stage of type (2).

Also note that, in \( G(1 \times 2 + 2 \times 1) \), the optimal one-shot coordination probability (OSCP) is \( \frac{1}{2} \) and it is obtained by giving the probability \( \frac{1}{2} \) for selecting a choice within the set \( \{b_1, c_1, b_2, c_2\} \) (proof for this claim is similar to the proof of Lemma 13.2). Since ECT-optimal protocols for \( G(1 \times 2 + 2 \times 1) \) do not give the optimal OSCP, we observe that the “greedy protocol” of always optimizing the chances of winning in the next round is not always ECT-optimal. Another example of this phenomenon is the game CM\(_5\) where LA does not give the optimal OSCP in the second round; WM is there the greedy protocol.

### C.2 Analysis of 5-choice games

In this section we will show that, among all two-player 5-choice games, the greatest optimal ECT is uniquely realized by the choice matching game CM\(_5\). Recall that this ECT is obtained by the protocol LA by Proposition 6.3 and its value is \( 2 + \frac{1}{3} \).

We first analyse 5-choice games \( G \) for which we have \( |W_G| > 8 \). For such games, the one-shot coordination probability \( p \), when players make a random choice in the first round, is

\[
p = \frac{|W_G|}{|C_1||C_2|} \geq \frac{9}{25}.
\]

Thus, by Theorem 14.5 the ECT for \( G \) by following WM is at most

\[
3 - 2p \leq 3 - 2 \cdot \frac{9}{25} = 2 + \frac{7}{25} < 2 + \frac{1}{3}.
\]

Thus \( G \) can be given a smaller ECT than the optimal ECT for CM\(_5\).

Hence we can restrict our analysis to those 5-choice games \( G \) whose winning relation \( W_G \) has at most 8 edges. Moreover, we may also assume that neither of the player has a choice of degree 5 as otherwise the optimal ECT is trivially 1.

Suppose first that at least one of the players has a choice of degree 4. Since \( |W_G| \leq 8 \), neither of the players can have more than two such choices and it is impossible that both players have two such choices. If precisely one of the players has precisely one choice of degree 4 (and the other player zero or two such choices), then it is a focal point and the players can immediately coordinate. If one player has two choices, denoted by \( c \) and \( c' \), of degree 4 and the other player has no such choice, then there are (at least three) choices that are connected to both \( c \) and \( c' \). The players can coordinate immediately by one of them selecting among the choices which are connected to both \( c \) and \( c' \). Finally, suppose that both players have exactly one choice of degree 4; we denote these by \( c_1 \) and \( c_2 \). If there is an edge between \( c_1 \) and \( c_2 \), then both of them are focal points. If there is no edge between \( c_1 \) and \( c_2 \), then we must have \( G = G(1 \times 4 + 4 \times 1) \) as \( |W_G| \leq 8 \). The ECT for this game is analysed later on below.

Suppose then that at least one of the players has a choice of degree 3 and none of the choices have a greater degree. As \( |W_G| \leq 8 \), both players have at most two choices of degree 3. We first show that it is impossible that both players have two choices of degree 3. If player 1 has two choices of degree 3, then (s)he can have at most 4 choices in total as the degree of every choice must be at least one. If also player 2 has two choices of degree 3, then (s)he also has at most 4 choices and thus \( G \) cannot be a 5-choice game.

We observe next that there is a focal point in \( G \) if precisely one of the players has precisely one choice of degree 3 and the other player zero or two choices of degree 3. Suppose next that one player has two choices, \( c \) and \( c' \), of degree 3 and the other one has no such choices. Now there must be at least one choice which coordinates with both \( c \) and \( c' \), and the players can guarantee coordination when one selects among \( \{c, c'\} \) and the other one selects a choice which
is connected to both c and c’. Finally, suppose that both players have exactly one choice of
degree 3; these choices are denoted by c1 and c2. If there is an edge between c1 and c2, then
they are focal points. If there is no edge between c1 and c2, then G must be one of the following
5-choice games where |WG| ≤ 8:

(Note that these games have been obtained by adding 1 or 2 edges and 1 or 2 nodes to the
4-choice game G(1 × 3 + 3 × 1).) All the other games above, except for the leftmost game G*,
have a focal point. The game G* is analysed later on below.

We still need to analyse the case where all of the choices in G have a degree at most 2. The
game graph of G must then consist of components which are either cycles or paths (cf. the
corresponding case in Section C.1). We list here systematically all such 5-choice games G with
|WG| ≤ 8.

| | W | | W | | W | | W |
|---|---|---|---|---|---|---|
| G(2(1 × 2) + 1 × 1) | G(Σ4 + 1 × 2) | G(O2 + 1 × 2 + 1 × 1) | Go + 1 × 2) |
| G(1 × 2 + 3(1 × 1)) | G(Σ4 + 2(1 × 1)) | G(O2 + 3(1 × 1)) | G(O3 + 2(1 × 1)) |
| G(5(1 × 1)) = CM5 | G(Z4 + 1 × 2) | G(Σ4 + 1 × 1) | G(O2 + Σ3) |
| | G(Z4 + 2(1 × 1)) | G(Z4 + 1 × 2) | G(O4 + Z2 + 1 × 1) |
| | G(Z4 + 2(1 × 1)) | G(Z4 + 2(1 × 1)) | G(O4 + Z2 + 1 × 1) |
| | | G(Z4 + 2(1 × 1)) | G(O4 + Z2 + 1 × 1) |
| | | G(Σ4 + Z2) | G(Σ4 + Z2) |
| | | G(Σ4 + 2 × 1 × 1) | G(Σ4 + 2 × 1 × 1) |
| | | G(Σ4 + 2 × 1 × 1) | G(Σ4 + 2 × 1 × 1) |
| | | G(Z4 + 2 × 1 × 1) | G(Z4 + 2 × 1 × 1) |
| | | G(Z4 + 2 × 1 × 1) | G(Z4 + 2 × 1 × 1) |
| | | G(Z4 + 2 × 1 × 1) | G(Z4 + 2 × 1 × 1) |
| | | G(Z4 + Z2) | G(Z4 + Z2) |
| | | G(Z4 + Z2) | G(Z4 + Z2) |
| | | G(Z4 + Z2) | G(Z4 + Z2) |

All of the the games listed above have a focal point—except for the following four games: CM5,
G(1 × 2 + 2 × 1 + 2(1 × 1)), G(O3 + 2(1 × 1)) and G(S3 + Z3).

Next we analyse the ECTs for the above-identified 5-choice games G whose optimal ECT is
greater than 1 and for which |WG| ≤ 8.

- CM5
  The optimal ECT here is 2 + 1/4 by Proposition 6.3 (see the table in Section 1).

- G(1 × 4 + 4 × 1)
  We obtain the ECT of 2 rounds with the following protocol: (1) in the first round, select
  the choice of degree 4 with probability 1/2 and some of the choices of degree 1 with the
total probability 1/2; (2) if coordination does not succeed, then continue with WM. It is clear
that this gives the same ECT as WM gives in the choice matching game CM2, this
ECT being 2.

- G* (see the game graph given above)
  As above, we obtain the ECT of 2 rounds by first assigning the probability 1/2 for selecting
  the choice with degree 3 and the probability 1/2 for selecting the choice with degree 2, and
  by continuing with WM thereafter. Again it is clear that this gives the same ECT of 2
  rounds as WM in CM2.
• \( G(\Sigma_3 + \Sigma_3) \)

Again—for practically the same reasons as above—we obtain the ECT 2 by first assigning the probability \( \frac{1}{2} \) for selecting the choice which is “in the middle of a 5-choice path” and the total probability \( \frac{1}{2} \) for selecting any other choice with degree 2, and by continuing with WM thereafter.

• \( G(1 \times 2 + 2 \times 1 + 2(1 \times 1)) \)

The players can follow an optimal protocol for \( G(1 \times 2 + 2 \times 1) \) in the corresponding subgame and thus obtain the ECT of less than 2 rounds (see Section C.1).

• \( G(O_3 + 2(1 \times 1)) \)

The players can keep selecting choices randomly within the subgame \( G(O_3) = \overline{CM_3} \) to obtain the ECT of \( 1 + \frac{1}{2} \) rounds—as shown in Section C.1.

Hence we conclude that the greatest optimal expected coordination time, among all 5-choice games, is uniquely realized by the choice matching game \( CM_5 \).

D Appendix: Further remarks on choice matching games

In the table below we summarize the results on optimal expected and guaranteed coordination times in choice matching games \( CM_m \). The lines (—) mean that no unique protocol exists.

| \( m \) | Optimal expected coordination time in \( CM_m \) | Unique optimal protocol for expected time | Optimal guaranteed coordination time in \( CM_m \) | Unique optimal protocol for guaranteed time |
|---|---|---|---|---|
| 1 | 1 | (any) | 1 | (any) |
| 2 | 2 | WM | \( \infty \) | — |
| 3 | \( 1 + \frac{1}{2} \) | LA | 2 | LA |
| 4 | \( 2 + \frac{1}{2} \) | LA | \( \infty \) | — |
| 5 | \( 2 + 1 \) | LA | 3 | LA |
| 6 | \( 2 + \frac{1}{2} \) | WM | \( \infty \) | — |
| 7 | \( 2 + \frac{1}{2} \) | WM | 4 | LA |
| \( 2k \) | \( 3 - \frac{1}{2} \) | WM | \( \infty \) | — |
| \( 2k + 1 \) | \( 3 - \frac{1}{2} \) | WM | \( k \) | LA |

First note that—interestingly—the game \( CM_3 \) can be considered much easier than the game \( CM_2 \) since the optimal ECT is much smaller. Moreover, coordination in \( CM_3 \) can be guaranteed in two rounds, while it cannot ever be guaranteed in \( CM_2 \). For similar reasons, \( CM_5 \) can also be considered easier than \( CM_4 \).

In several cases there is a single unique protocol which is optimal in all aspects that we have studied in this article. In such cases one can argue that such a protocol should be followed all rational players even if they cannot communicate in advance or share any conventions.\(^3\) In the cases where no single protocol is optimal in all aspects, it is more problematic for the players to choose their protocol—unless they share some convention.

\(^3\)This relies on the assumption that the list of possible preferences consists of either minimizing ECTs or minimizing GCTs. At least the average case and worst case are by far the most common scenarios considered.
The most clear cases here are the games with 3 and 5 (and trivially 1) choices, where the protocol LA is uniquely optimal with respect to both ECT and GCT. Also all the games with an even number of choices, excluding the case \( m = 4 \), are clear since WM is uniquely ECT-optimal and no protocol can guarantee coordination in any number of rounds.

The game CM\(_4\) is the only game for which no protocol is uniquely ECT-optimal (indeed there are uncountably many different ECT-optimal protocols). Moreover, no protocol can guarantee coordination in this game. Based on the analysis on the other choice matching games with an even number of choices, one could possibly argue that players would naturally follow WM also here since it is uniquely ECT-optimal elsewhere and one of the ECT-optimal protocols here as well. However, there seems to be no obvious and fully compelling reason why WM should be preferred to the other ECT-optimal protocols.

The games CM\(_m\) for odd \( m \geq 7 \) can also be problematic since the optimal values for ECT and GCT are given by different (although uniquely optimal) protocols WM and LA, respectively. If both players do not have the same preference about which of these values to optimize (or this is not common knowledge among them), it is not clear for them whether they should follow WM or LA. In the cases where \( m \) is very large, say \( m = 1001 \), WM seems more justified in practice since it is almost impossible that coordination with WM would take more time than with LA. But in the cases where \( m \) is quite small, especially when \( m = 7 \), LA may seem a more balanced option with respect to the both aspects. Recall here that the ECT in CM\(_7\) with LA is 3 rounds while the ECT with WM is only slightly less than 3, and moreover, LA guarantees coordination in 4 rounds while WM does not guarantee it at all.

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**References**

[1] Douglas Adams and John Lloyd. *The Meaning of Liff*. Crown Pub, 1984.

[2] Robert J. Aumann and Michael B. Maschler. *Repeated Games with Incomplete Information*. MIT Press, 1995.

[3] Gary Biglaiser. Coordination in games: A survey. In James W. Friedman, editor, *Problems of Coordination in Economic Activity*, volume 35, pages 49–65. Springer, Dordrecht, 1994.

[4] Russell Cooper. *Coordination Games*. Cambridge University Press, 1999.

[5] Vincent P. Crawford. Adaptive dynamics in coordination games. *Econometrica*, 63(1):103–43, 1995.

[6] Vincent P. Crawford and Hans Haller. Learning how to cooperate: optimal play in repeated coordination games. *Econometrica*, 58(3):571–595, 1990.

[7] Valentin Goranko, Antti Kuusisto, and Raine Rönholm. Rational coordination in games with enriched representations. In Francesco Belardinelli and Estefania Argente, editors, *Multi-Agent Systems and Agreement Technologies EUMAS 2017*, volume 10767 of *LNCS*, pages 323–338. Springer, 2017.

[8] Valentin Goranko, Antti Kuusisto, and Raine Rönholm. Rational coordination with no communication or conventions. In *Proceedings of LORI VI*, volume 10455 of *LNCS*, pages 33-48. Springer, 2017.
[9] Valentin Goranko, Antti Kuusisto, and Raine Rönholm. Gradual guaranteed coordination in repeated win-lose coordination games. In Proceedings of ECAI 2020, To appear, 2020.

[10] Sanjeev Goyal and Maarten Janssen. Can we rationally learn to coordinate? Theory and Decision, 40:29–49, 1996.

[11] Shmuel Zamir Jean-François Mertens, Sylvain Sorin. Repeated Games. Econometric Society Monographs. Cambridge University Press, 2015.

[12] Antti Kuusisto. A double team semantics for generalized quantifiers. CoRR, abs/arXiv:1310.3032v10, 2015.

[13] Antti Kuusisto. On games and computation. CoRR, abs/arXiv:1910.14603, 2019.

[14] Roger Lagunoff and Akihiko Matsui. Asynchronous choice in repeated coordination games. Econometrica, 65:1467–1477, 1997.

[15] D. Lewis. Convention, A Philosophical Study. Harvard University Press, 1969.

[16] George J. Mailath and Larry Samuelson. Repeated Games and Reputations: Long-Run Relationships. Oxford University Press, 2006.

[17] Thomas Schelling. The Strategy of Conflict. Harvard University Press, 1960.