ON MINIMUM COST SPARSEST INPUT-CONNECTIVITY FOR CONTROLLABILITY OF LINEAR SYSTEMS

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Abstract. This article deals with algorithmic techniques to design sparsest input-connectivity while retaining controllability of linear systems. We assume that the input matrix is constrained in the sense that the set of states that each input (if present,) can influence is known a priori, and that each inter-connection between an input and a state is associated with a certain cost. In this setting we determine a set of input-connections that lead to the minimum cost and ensures that the resulting system is structurally controllable. We identify a large class of systems for which these problems are solvable in polynomial time using efficient algorithms. Graph-theoretic tools are employed to reduce the above class of constrained design problems to problems related to maximum matching and maximum flow. Illustrative examples are included to demonstrate the efficacy of the techniques developed here.

1. Introduction

Dynamical networks arise in a wide-range of application scenarios involving systems such as biological, social, economical, industrial, and transportation systems [1, 2, 3, 4], where the states of the systems are updated overtime via its own dynamics. For example, a traffic network [5] can be modelled as a dynamical system where the load on each road get influenced by other nearby roads and therefore needs to be updated frequently, a social network [6] where the state of each person (describing, e.g., his/her opinion on a particular topic,) gets affected by his/her friends and is updated from time to time, a gene regulatory network [7] where expressions of proteins are affected by certain specific genes, etc. Of late, the subject of control of such large-scale systems has been attracting considerable attention due to its important academic and practical significance.

On the one hand, the gigantic sizes of the large-scale systems available today have made the problem of identifying a smallest subset of the inputs to control the systems a very relevant problem. This particular problem is difficult: in fact, it was proved in [8] that the problem of finding a smallest set of actuators to ensure a linear system controllable is NP-hard. On the other hand, in many practical situations, it is often necessary to consider the cost of the interconnection connecting an input to a system state; this cost may depend on various factors including the specific functionality of each control, reliability, installation and maintenance, and even environmental conditions such as humidity and temperature. In such cases it is necessary to minimize the number of connections between the inputs and the states as well as the cost of using those connections. Consider, by way of an example, a multiagent system of robots modelled as a dynamical system, where each robot interacting with others over wireless channels to perform a pre-defined task, and external inputs are connected to a pre-specified set of states. In this situation, it is desirable to have a few controls directly controlling a few robots rather than...
squander resources by distributing controllers to all the states to which they are connected, thus indirectly controlling the rest of the system. Such an architecture is especially common in decentralized control or situations where a central controller may be incapable of simultaneously controlling all agents/component subsystems.

The focus of this article is on identifying a sparsest set of connections between the inputs and the system states along with minimizing the overall cost of using those connections to make the system controllable with a pre-specified input structure. By input structure we mean that the set of states to which an input is directly connected is known a priori. Therefore, the sparsest set of sub-connections must be selected from the available set of connections between the inputs and the states to ensure that the system is controllable. To this end, we employ structural systems theory to address this problem, where a class of system theoretic problems may be treated by employing only the connections between the system states, inputs, and outputs. Several interesting and perhaps non-intuitive assertions can be derived via this theory and it is useful especially for systems whose parameters are not exactly known due to various reasons including ageing of system components, structural alterations, etc. Structural analysis of control systems via structural controllability was introduced in [9], and over the past several years a considerable amount of research has been done in this area, see e.g., [10, 11, 12, 13, 14].

In the context of our problem, there have been several efforts to solve allied problems. The authors of [15, 11] considered the problem of identifying the minimum number of inputs required to guarantee structural controllability by employing ideas from maximum matching, and provided a polynomial time algorithm to solve that problem. [16] addressed the problem of selecting the fewest states to be influenced to achieve structural controllability assuming that every input can directly control only a single state starting with an input matrix that is square and diagonal. The article [17] considered the minimum cost input design problem where the objective is to find a diagonal input matrix which makes the system structurally controllable and incurs minimum cost when each state is associated with a certain cost. It was demonstrated that finding an input matrix with fewest non-zero entries or having the minimum cost has polynomial time complexity. In contrast to these prior investigation, if the input matrix is pre-specified, then the problem of selecting an input set of minimum cardinality to guarantee that the resulting system is structurally controllable becomes NP-hard [18]; known as minimum constrained input selection problem (minCIS). [19] reduced this problem to the minimum cost fixed flow problem and provided a polynomial time approximation algorithm to identify a solution.

Throughout this article we assume that the input matrix and the set of states that each input can influence are known a priori. We exploit the techniques of structural system theory to deal with these three different but related problems corresponding to the controllability of a linear system:

- The first problem deals with identifying a minimal set of connections between the inputs and the states to ensure that the resulting system is structurally controllable.
- We assume that each connection between the input and the state has a non-negative cost associated with it. The objective of the second problem is to determine a minimal set of connections between the inputs and the states that incurs minimum cost while ensuring the structural controllability of the system.
- The focus of the third problem is to obtain a set of connections, where the objective is to minimize the overall cost of using those connections and not the number
of connections from the inputs to the states in order to maintain structural controllability.

The precise statements of the above problems are given in §3. We identify mild conditions under which all the three problems are solvable in polynomial time (in dimension of states and inputs) using efficient algorithmic techniques.

This article unfolds as follows: §2 reviews certain concepts from graph theory that will be needed in this sequel. §3 gives the precise statement of the problems dealt in this article, and §4 provides efficient polynomial time algorithms to obtain solution for the problems in §3 under mild assumptions on the system matrix along with a short discussion. In §5 we demonstrate the effectiveness of our algorithms by providing some illustrative examples.

2. Preliminaries

The notations employed here are standard: We denote the set of real numbers by \( \mathbb{R} \), the set of integers by \( \mathbb{Z} \), the set of non-negative real numbers by \( \mathbb{R}^+ \), the positive integers by \( \mathbb{N}^+ \), and we let \( [n] := \{1, 2, \ldots, n\} \) for \( n \in \mathbb{N}^+ \). We denote by \( |X| \) the cardinality of a finite set \( X \). We denote by \( I_n \) the identity matrix of dimension \( n \). If \( A \in \mathbb{R}^{n \times n} \), then \( A_{ij} \) represents the entry located at \( i \)th row and \( j \)th column. We define a function \( \mathbb{I} \) associated with the \( A_{ij} \) entry as follows:

\[
\mathbb{I}_{(A_{ij} \neq 0)} = \begin{cases} 
1 & \text{if } A_{ij} \neq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Consider a linear time-invariant system

\[
\dot{x}(t) = \tilde{A} x(t) + \tilde{B} u(t), \quad t \in \mathbb{N}^+,
\]

where \( x(t) \in \mathbb{R}^d \) are the states and \( u(t) \in \mathbb{R}^m \) are the inputs at time \( t \), and \( \tilde{A} \in \mathbb{R}^{d \times d} \) and \( \tilde{B} \in \mathbb{R}^{d \times m} \) are the given state and input matrices respectively. Throughout we have assumed that the number of inputs \( m \) is such that \( m = O(d) \), where \( d \) is the number of states in (2.2). The system (2.2) is completely described by the pair \((\tilde{A}, \tilde{B})\), and we shall interchangeably refer to (2.2) and \((\tilde{A}, \tilde{B})\) in this sequel.

In our analysis the precise numerical values of the entries of \( \tilde{A} \) and \( \tilde{B} \) will not matter, but the information about the locations of fixed zeros in \( \tilde{A} \) and \( \tilde{B} \) will be essential. For any matrix \( R \), the sparsity matrix of \( R \) is defined to be matrix of same dimension as \( R \) with each entry either a zero or a symbol, denoted by \( * \). A numerical realisation of \( R \) is obtained by assigning numerical values to the star entries of the sparsity matrix of \( R \). Let \( A \in \{0, *\}^{d \times d} \) and \( B \in \{0, *\}^{d \times m} \) represent the sparsity matrices of the system matrix \( \tilde{A} \) and the input matrix \( \tilde{B} \). With this information, we have the following definition of structural controllability:

**Definition 2.1.** A pair \((A, B)\) is said to be structurally controllable if there exists at least one numerical realization \((A', B')\) of \((A, B)\) such that \((A', B')\) is controllable.\(^1\)

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\(^1\)It is well-known that if a pair \((A, B)\) is structurally controllable, then almost all numerical realizations of \((A, B)\) are controllable [20].
the input vertices to the state vertices in the graph $G(A,B)$. The edges in $E_A$ are referred as state-connections. The edges in $E_B$ are referred as input-connections in this sequel. Sometimes we shall need the digraph $G(A) = (\mathcal{A}, E_A)$ with vertex set $\mathcal{A}$ and edge set $E_A$ considering the edges between only the state vertices.

A digraph $G_s = (V_s, E_s)$ with $V_s \subset \mathcal{A}$ and $E_s \subset E_A$ is called a subgraph of $G(A)$. When $\mathcal{A}' \subset \mathcal{A}$, the induced subgraph consists of $\mathcal{A}'$ and all the edges whose endpoints are contained in $\mathcal{A}'$. A sequence of edges $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$, where each $(u_j, v_j) \in E_A$, is called a directed path from $v_1$ to $v_k$. A state vertex $v_i \in \mathcal{A}$ is said to be accessible if there exists a path from some input $u_j$ to $v_i$; otherwise, it is inaccessible. The digraph $G(A)$ is strongly connected if for each ordered pair of vertices $(v_i, v_j)$, there exists a directed path from $v_i$ to $v_j$. A strongly connected component (SCC) of $G(A)$, usually denoted by $S$, is a maximally strongly connected subgraph of $G(A)$. Thus, the states of the graph $G(A, B)$ are accessible if and only if all the SCCs are accessible. A characterization of the SCCs of the digraph $G(A)$ is given in the following definition.

**Definition 2.2.** An SCC $S$ in the digraph $G(A)$ is said to be source strongly connected component (SSCC) if there is no directed edge from the vertices of other SCCs into any vertex of $S$.

As a consequence of the above definition, all the states are accessible if and only if all the SSCCs are accessible. While accessibility of all states is a necessary condition for structural controllability, it is not sufficient. In addition to the accessibility condition mentioned above the digraph $G(A,B)$ should also satisfy a no-dilation condition:

**Definition 2.3.** For the digraph $G(A,B)$ corresponding to (2.2) and a subset $T \subset \mathcal{A}$, the in-neighbourhood of $T$ is the set

$$N^-(T) = \left\{ v \mid (v, v_j) \in E_A \cup E_B, v_j \in T, v \in \mathcal{A} \cup \mathcal{U} \right\}.$$  

Each vertex in $N^-(T)$ is termed as an in-neighbour of $T$. The directed graph $G(A,B)$ is said to have a dilation if there exists a set $T \subset \mathcal{A}$ such that $|N^-(T)| < |T|$.

The digraph $G(A)$ derived above from (2.2) can also be represented by an undirected bipartite graph in the following standard fashion: $\Gamma(A) := (V_A, V_A^2, E_A)$, where $V_A^1 := \{v_1^1, v_2^1, \ldots, v_d^1\}$, $V_A^2 := \{v_1^2, v_2^2, \ldots, v_d^2\}$, and $E_A = \{(v_i^1, v_i^2) \mid A_{ij} \neq 0\}$. We shall need a few more definitions in the context of graph $\Gamma(A)$. A matching $M$ in $\Gamma(A)$ is a subset of edges that do not share vertices. A maximum matching $M$ in $\Gamma(A)$ is defined as a matching $M$ that has largest number of edges among all possible matchings. An edge $e$ is said to be matched if $e \in M$. A vertex is said to be matched if it belongs to an edge in the matching $M$; otherwise, it is unmatched. A matching $M$ in $\Gamma(A)$ is said to be perfect if all the vertices in $\Gamma(A)$ are matched.

In the similar manner, we can define the undirected bipartite graph $\Gamma(A,B)$ associated with the graph $G(A,B)$ in the following way: $\Gamma(A,B) := (V_A, V_B, V_A^2, E_A \cup E_B)$, where $V_B := \{u_1, u_2, \ldots, u_m\}$, and $E_B = \{(u_j, v_i^2) \mid B_{ij} \neq 0\}$. We say that a system of distinct representatives (SDR) exists for $V_A^2$ in $\Gamma(A,B)$ if there exists a matching $M$ in $\Gamma(A,B)$ that covers or matches all the vertices of $V_A^2$. It is important to note that the presence of dilation in $G(A,B)$ can be easily checked by using a matching condition that relates $\Gamma(A,B)$ and the no-dilation condition.

**Proposition 2.4.** [12, Theorem 2] A digraph $G(A,B)$ has no dilation if and only if there exists an SDR in the bipartite graph $\Gamma(A,B)$. 


A fundamental connection between system theoretic property of structural controllability and certain structural properties of $G(A, B)$ is given by:

**Theorem 2.5.** [9, Theorem 1, p. 207] The following are equivalent:

(a) The pair $(A, B)$ is structurally controllable.

(b) In the digraph $G(A, B)$ derived from (2.2), every state vertex $v_i \in A$ is accessible and $G(A, B)$ is free of dilations.

**Remark 2.6.** Given $G(A) = (A, E_A)$, the SCCCs can be determined in $O(|A| + |E_A|)$ computations. We know that $|A| = d$ and $|E_A| = O(d^2)$. The procedure for finding the SCCCs involves $O(d^2)$ computations and checking for existence of an SDR in $\Gamma(A, B)$ involves $O(d^{2.5})$ computations. Thus, structural controllability of a pair $(A, B)$ can be accurately checked in $O(d^{2.5})$ computations [21].

We define a cost function $c : E_A \cup E_B \rightarrow \mathbb{R}^+$ which assigns non-negative costs to the edges of the bipartite graph $\Gamma(A, B)$, represented by $(\Gamma(A, B); c)$. Subsequently, we introduce minimum cost maximum matching (MCMM) problem. This problem deals with finding a maximum matching of $(\Gamma(A, B); c)$ that incurs the minimum cost-sum of its edges; in other words, determining a maximum matching $M^*$ such that $\sum_{e \in M^*} c(e) \leq \sum_{e \in M} c(e)$, where $M$ is any maximum matching in $(\Gamma(A, B); c)$. The problem of finding a MCMM in $\Gamma(A, B)$ can be efficiently solved using Hungarian algorithm [22] with computation complexity of $O((d + m)^3)$, where $d$ is the number of state vertices and $m$ is the number of input vertices in $G(A, B)$.

We review a classical problem, namely the **maximum flow problem** [23, p. 176], [24], where the objective is to find a maximum flow in a flow network. A flow network is a digraph $F = (V, E)$, where the vertex set $V$ consists of a distinguished source vertex $s$ and sink vertex $t$, respectively. Every edge $e \in E$ is given a non-negative capacity $b(e)$. We define a flow $f$ as a function $f : E \rightarrow \mathbb{R}^+$ to each edge in the network $F$. We say $f^+(v)$ for the total flow on edges leaving $v$ and $f^-(v)$ for the total flow on edges entering $v$.

**Definition 2.7.** In a flow network $F$, a flow is said to be **feasible** if it satisfies the following conditions:

- conservation constraints: for each edge $e \in E$, we have $0 \leq f(e) \leq b(e)$, and
- conservation constraints: for every vertex $v \notin \{s, t\}$, $f^+(v) = f^-(v)$.

The value of the flow $f$, $\text{val}(f)$, is the net flow $f^+(s) - f^-(s)$ from the source $s$ or $f^-(t) - f^+(t)$ into the sink $t$. The objective of maximum flow problem is to find a feasible flow $f^*$ such that $\text{val}(f^*) \geq \text{val}(f)$ for any feasible flow $f$. It is a well studied problem and there exists many algorithms that find a maximum flow $f^*$ in polynomial time.

In this sequel, we also make use of a well-studied flow problem, namely the **minimum cost flow problem.** We define a cost function $c : E \rightarrow \mathbb{R}^+$, which assigns cost to each edge $e \in E$ in the flow network $F = (V, E)$.

The minimum cost flow problem is given by:

\[
\begin{align*}
\text{minimize} & \quad \sum_{e \in E} c(e)f(e) \\
\text{subject to} & \quad \begin{cases} f \text{ is a feasible flow} \\
\text{val}(f) \geq \ell,
\end{cases}
\end{align*}
\]

(2.3)

where $\ell$ is the value of flow required to be sent from source $s$ to sink $t$ in the flow network $F$. There are various polynomial-time algorithm for solving minimum cost
flow problem. The well-known polynomial time algorithms include capacity scaling algorithm, double scaling algorithm, minimum mean cycle-cancelling algorithm and enhanced capacity scaling algorithm—see [25, Chapter 10] for details. We use a (strongly) polynomial algorithms that runs in $O(k^4 \log k)$ in a generic flow network with $k$ vertices given in [26].

3. Problem Formulation

Before formally stating the three optimisation problems, we introduce the various norms needed in this sequel.

For a matrix $N \in \{0, *\}^{n \times k}$ (where $n, k \in \mathbb{N}^*$)

- $\|N\|_0$ denote the number of non-zero entries in the matrix $N$.
- Let each non-zero entry in $N$ is associated with a non-negative cost. For instance, if $N_{ij} \neq 0$ assume that $w_{ij} \geq 0$ is the cost corresponding to it. We define $\|N\|_{w} := \sum_{i=1}^{n} \sum_{j=1}^{k} w_{ij} \mathbb{1}_{\{N_{ij} \neq 0\}}$, where $\mathbb{1}$ is the function, defined in (2.1).

Given $A \in \{0, *\}^{d \times d}$ and $B \in \{0, *\}^{d \times m}$, throughout we assume that the pair $(A, B)$ is structurally controllable. For a matrix $B \in \{0, *\}^{d \times m}$, let the collection of locations of fixed zeros of $B$ be $Z(B)$, i.e., $Z(B) := \{(i, j) \mid B_{ij} = 0\}$. Define $\mathcal{K} := \{B' \mid Z(B) \subset Z(B'), (A, B') \text{ is structurally controllable}\}$.

Since $(A, B)$ is structurally controllable by assumption, $\mathcal{K}$ is always non-empty.

We deal with the following three optimization problems:

**Problems:** Let $A \in \{0, *\}^{d \times d}$ and $B \in \{0, *\}^{d \times m}$ be given such that $(A, B)$ is structurally controllable:

- Determine an input matrix $B^*$ that solves

  $$\underset{B' \in \mathcal{K}}{\text{minimize}} \quad \|B'\|_0.$$  

- Each non-zero entry in $B$ is associated with a non-negative cost. Let $w_{ij} \geq 0$ denote the cost of using the input-connection connecting the input vertex $u_j$ to the state vertex $v_i$ in $G(A, B)$. Determine an input matrix $B^*$ that solves the following optimisation problem:

  $$\underset{B' \in \mathcal{K}}{\text{minimize}} \quad \|B'\|_{w}$$

  subject to $\|B'\|_0 \leq \|B''\|_0$ for all $B'' \in \mathcal{K}$.

- Let $w_{ij} \geq 0$ denote the cost of using the input-connection connecting the input vertex $u_j$ to the state vertex $v_i$ in $G(A, B)$. Determine an input matrix $B^*$ that solves the problem:

  $$\underset{B' \in \mathcal{K}}{\text{minimize}} \quad \|B'\|_{w},$$

  where $\|B'\|_{w} = \sum_{i=1}^{d} \sum_{j=1}^{m} w_{ij} \mathbb{1}_{\{B'_{ij} \neq 0\}}$.

**Remark 3.1.** As part of our premise, we assume throughout that the given pair $(A, B)$ is structurally controllable. Clearly, finding brute-force solutions to Problems (P1)-(P3) requires checking all possible input matrices $B' \in \mathcal{K}$, which is quite an impossible combinatorial problem for even moderately sized pairs $(A, B)$. For example, consider an input matrix $B \in \{0, *\}^{d \times m}$ containing $n$ non-zero entries.
To identify a solution to Problems \((P_1)-(P_3)\) requires testing all possible combinations of the subsets of the \(n\) entries. Therefore, the number of computations needed is exponential. However, the algorithms discussed here identify solutions of these problems efficiently in polynomial time complexity without using brute-force techniques. For example, we use the Ford-Fulkerson algorithm to find a maximum flow \(f^*\) in a flow network \(F = (V, E)\) in \(O(|E| \cdot \text{val}(f^*))\) \cite[p. 658]{21} to solve Problem \((P_1)\).

4. **Main Results**

In this section we address Problems \((P_1)\), \((P_2)\), and \((P_3)\) by imposing mild assumptions on the structure of the digraph \(G(A)\) and its associated bipartite graph \(\Gamma(A)\) for a system matrix \(A\). Our results will be derived under two mutually exclusive sets of assumptions: \S 4.1 contains our solution of Problems \((P_1)-(P_3)\) under a perfect matching assumption on \(\Gamma(A)\), and \S 4.2 contains our solution of Problems \((P_1)-(P_3)\) under strong connectivity hypothesis on \(G(A)\); neither one of the assumptions implies the other, so in this sense our results in \S 4.1 and \S 4.2 are complementary.

4.1. **Systems with perfect matching.**

**Assumption 4.1.** *We stipulate that the system matrix \(A\) is such that the bipartite graph \(\Gamma(A)\) has a perfect matching.*

This is indeed a reasonable assumption since a large class of systems, for instance systems including epidemic dynamics, power grids, multi-agent systems, etc., \cite{27, 28, 29} exhibit this feature. Assumption 4.1 ensures that the bipartite graph \(\Gamma(A)\) has a perfect matching \(M\). Since \(\Gamma(A)\) is an induced subgraph of \(V_A^1 \cup V_A^2\) in \(\Gamma(A, B)\), matching \(M\) covers all the vertices of \(V_A^2\) in \(\Gamma(A, B)\). Thus, the bipartite graph \(\Gamma(A, B)\) has an SDR. Proposition 2.4 implies that \(G(A, B)\) has no-dilation even in the absence of any input vertex. By Theorem 2.5, it suffices to ensure the accessibility criterion for structural controllability of the pair \((A, B)\). This implies that the pair \((A, B)\) is structurally controllable if and only if all the SSCCs of \(G(A)\) are accessible from at least one of the input vertices in \(G(A, B)\). The following Lemma is central to the development of our results:

**Lemma 4.2.** *Suppose that Assumption 4.1 holds. Consider a structurally controllable pair \((A, B)\), and let \(q\) denote the number of SSCCs in \(G(A)\). If \(B^*\) solves Problem \((P_1)\), then \(\|B^*\|_0 = q\).***

**Proof.** We establish the assertion in two steps: In step (i) we prove that \(\|B^*\|_0 \leq q\) and in step (ii) we show that \(\|B^*\|_0 \geq q\).

Step (i). Since Assumption 4.1 holds, the pair \((A, B)\) is structurally controllable if and only if all the SSCCs of \(G(A)\) are accessible from the input vertices. Therefore, every SSCC has at least one state vertex directly connected to one of the input vertices in \(G(A, B)\). Also, for each SCC exactly one input-connection is sufficient to ensure accessibility. This confirms that there exists a \(B^*\) such that \((A, B^*)\) is structurally controllable, i.e., \(B^* \in \mathcal{K}\) and \(\|B^*\|_0 = q\). In other words, \(\|B^*\|_0 \leq q\).

Step (ii). Suppose that there exists a \(B^* \in \mathcal{K}\) such that \(\|B^*\|_0 < q\). We assume that \(\|B^*\|_0 = q - 1\). It means that there are \(q\) SSCCs in \(G(A)\) and only \(q - 1\) input-connections. Since all the SSCCs are vertex-disjoint from each other there exists at least one SSCC not accessible from any input vertex. This contradicts the assumption that \((A, B^*)\) is structurally controllable. Therefore, every \(B'' \in \mathcal{K}\) is such that \(\|B''\|_0 \geq q\), leading to \(\|B^*\|_0 \geq q\). The assertion follows. \(\square\)
To tackle Problem $\mathcal{P}_1$, under Assumption 4.1 we first construct a flow network $\mathcal{F}_1(A,B)$ corresponding to the given pair $(A,B)$ by following the recipe in Algorithm 1. Let $\deg(u_j)$ denote the number of state vertices directly connected to an input vertex $u_j$.

**Algorithm 1:** Procedure for the construction of flow network $\mathcal{F}_1(A,B)$ of the system $(A,B)$ under Assumption 4.1

| Input: $A \in \{0,\star\}^{d \times d}$, $B \in \{0,\star\}^{d \times m}$, and $G(A)$ | Output: The flow network $\mathcal{F}_1(A,B)$ |
|---------------------------------------------------------------|--------------------------------------------------|
| 1 Determine the SSCCs $\mathcal{S} = \{\mathcal{S}_i\}_{i=1}^q$ |
| 2 Construct flow network $\mathcal{F}_1(A,B)$ with vertex set $V_{\mathcal{F}_1}$ and edge set $E_{\mathcal{F}_1}$ as follows |
| 3 $V_{\mathcal{F}_1} \leftarrow \{s\} \cup \{\mathcal{S}_i\}_{i=1}^q \cup \{u_j\}_{j=1}^m$ |
| $e \in E_{\mathcal{F}_1} \leftarrow \begin{cases} (s, \mathcal{S}_i) & \text{for } i \in [q], \\ (\mathcal{S}_i, u_j) & \text{if } B_{rj} = \star \text{ for some } v_r \in \mathcal{S}_i, \\ (u_j, t) & \text{for } j \in [m]. \end{cases}$ |
| $b(e) \leftarrow \begin{cases} \deg(u_j) & \text{for } e = (u_j, t), j \in [m], \\ 1 & \text{otherwise.} \end{cases}$ |

Given a pair $(A,B)$ and its associated digraph $G(A,B)$, we first find the SSCCs of $G(A)$ (Step 1). Then we define the vertex set $V_{\mathcal{F}_1}$ (Step 3) and edge set $E_{\mathcal{F}_1}$ (Step 4), source-sink pair $s$, $t$ and the capacity $b$ (Step 5) as depicted in Algorithm 1.

Note that the construction of the flow network $\mathcal{F}_1(A,B)$ has polynomial time complexity. Indeed, given $G(A) = (\mathcal{A}, E_\mathcal{A})$, the SSCCs can be determined in $O(|\mathcal{A}| + |E_\mathcal{A}|)$ computations. We know that $|\mathcal{A}| = d$ and $|E_\mathcal{A}| = O(d^2)$. So, the SSCCs can be obtained in $O(d^2)$ computations. The rest of the constructions in Algorithm 1 have linear complexity. Therefore, the overall complexity of constructing $\mathcal{F}_1(A,B)$ is $O(d^2)$, where $d$ is the number of state vertices in $\mathcal{A}$.

**Lemma 4.3**. [19, Theorem 3.3, p. 3] Suppose that Assumption 4.1 holds. Consider the pair $(A,B)$ and let $q$ denote the number of SSCCs in $G(A)$. Then $(A,B)$ is structurally controllable if and only if the value of the maximum flow in the flow network $\mathcal{F}_1(A,B)$ in Algorithm 1 is at least $q$.

Since there are exactly $q$ edges, each of capacity 1, originating from source $s$ to every SSCCs $\{S_i\}_{i=1}^q$, the value of any flow $f$ can not exceed $q$, i.e., for any feasible flow $\text{val}(f) \leq q$. Given a structurally controllable pair $(A,B)$, by Lemma 4.3 it follows that there exists at least one feasible flow $f_1$ with $\text{val}(f_1) \geq q$. Therefore, the value of a maximum flow in $\mathcal{F}_1(A,B)$ is $q$.

We use the Ford-Fulkerson algorithm to find a maximum flow in $\mathcal{F}_1(A,B)$. By [21, Theorem 26.11, p. 667] (known as integrality property), we know that if all the capacities of the edges in a flow network are integers, then the maximum flow produced by the Ford-Fulkerson method has a property that the value of the flow is an integer. Moreover, for all the edges $e$, $f(e)$ is an integer, and the maximum flow $f$ in the network $\mathcal{F}_1(A,B)$ can be computed in $O(|E_{\mathcal{F}_1}| \text{val}(f))$. Since $|E_{\mathcal{F}_1}| = O(d^2)$ and $\text{val}(f) = O(d)$, where $d$ is the number of state vertices in $\mathcal{A}$, the overall complexity of finding a maximum flow in $\mathcal{F}_1(A,B)$ is $O(d^3)$. We use Lemma 4.3.
and the integrality property to find a maximum flow in $F_1(A,B)$ to determine a solution of Problem (P1).

**Theorem 4.4.** Let $(A,B)$ be a linear system and suppose that Assumption 4.1 holds. Consider the flow network $F_1(A,B)$ described by Algorithm 1 and a maximum flow $f$ obtained by the Ford-Fulkerson method. Define $H_f := \{(j(S_j), j) \mid f(S_j, u_j) = 1\}$, where $j(S_j)$ represents the index of a state vertex that belongs to $S_j$ and has an input-connection from $u_j$. If $(j(S_j), j) \in H_f$ then $B^*_{j(S_j), j} = \ast$, and $B^* \in \mathcal{K}$ is a solution of Problem (P1).

**Proof.** The result that $B^* \in \mathcal{K}$ follows from Lemma 4.3 and the observation that the value of the maximum flow through $F_1(A,B^*)$ is $q$. Also, the method employed to find a maximum flow in $F_1(A,B)$ ensures that the number of non-zero entries in $B^*$ is $q$. Therefore, by Lemma 4.2, $B^*$ is an optimal solution of Problem (P1). $\square$

We move on to Problem (P2). Recall from §3 that $w_{ij}$ is the cost associated with the input-connection from input vertex $u_j$ to state vertex $v_i$. Let $w_{\max}$ denote the maximum cost assigned to an input-connection (corresponding to a non-zero entry in $B$) among all the input-connections present in $G(A,B)$. In our setting for solving Problem (P2), we impose the condition that if an input vertex $u_k$ does not have an input-connection to a state vertex $v_l$ (determined by the given input matrix $B$), then $w_{lk} = \infty$ (for practical purposes $w_{lk}$ is taken to be $w_{\max} + 1$). We provide the following Algorithm 2 to obtain a solution of Problem (P2).

| Algorithm 2: Algorithm to solve Problem (P2) |
|---------------------------------------------|
| **Input:** $A \in \{0, \ast\}^{d \times d}$, $B \in \{0, \ast\}^{d \times m}$ |
| **Output:** The input matrix $B^*$ |
| 1 Determine the SSCCs $\{S_j\}_{j=1}^d$ |
| 2 $L \leftarrow \emptyset$ |
| 3 for each $S_j$ do |
| (1) for each state vertex $v_i \in S_j$, choose the smallest cost $w_{ik}$ among $\{w_{i1}, w_{i2}, \ldots, w_{im}\}$. |
| (2) choose a state vertex of least cost, say $v_l$, with cost $w_{lk}$ among all the state vertices in $S_j$. |
| 4 $L \leftarrow L \cup (u_k, v_l)$ |
| end for |
| 5 Define: |
| $B^*_{lk} \leftarrow \begin{cases} \ast & \text{if } e = (u_k, v_l) \in L, \\ 0 & \text{otherwise.} \end{cases}$ |

The structural controllability of the given pair $(A,B)$ ensures that no input-connection corresponding to cost $\infty$ ($w_{\max} + 1$) is selected by the algorithm.

**Theorem 4.5.** Let $(A,B)$ be a linear system and suppose that Assumption 4.1 holds. The procedure outlined in Algorithm 2 yields a matrix $B^*$ such that $(A,B^*)$ is structurally controllable and solves Problem (P2).

**Proof.** It follows from the procedure outlined in Algorithm 2 that exactly one state vertex, having an input-connection from some input vertex, is selected for each SCC at each iteration. Therefore, $(A,B^*)$ is structurally controllable, i.e., $B^* \in \mathcal{K}$ and $\|B^*\|_0 = q$. By Lemma 4.2 it follows that the $B^*$ so obtained has minimum
also ensures that \( B^* \) has the least cost among the collection of all sparsest \( B' \in \mathbb{K} \), i.e., \( \| B^* \|_w = \sum_{d=1}^d \sum_{k=1}^m \| B_{ik}^* \|_w \| B_{ik}^* \neq 0 \) has the least value. 

Finding the SSCCs involves \( O(d^2) \) computations, where \( d \) denote the number of state vertices in \( A \). Each iteration of Algorithm 2 has a complexity of \( O(d^2) \). The number of iteration is \( q \), i.e., equal to the number of SSCCs. Since \( q = O(d) \), the overall complexity of Algorithm 2 to identify a solution to Problem (\( P_2 \)) is \( O(d^3) \).

The strategy designed for identifying a solution to Problem (\( P_2 \)) namely, Algorithm 2, also provides a solution to Problem (\( P_3 \)).

**Proposition 4.6.** Let \((A, B)\) be a linear system and suppose that Assumption 4.1 holds. Then \( B^* \) obtained in Algorithm 2 also solves Problem (\( P_3 \)).

**Proof.** Suppose that the assertion is false, and there exists another \( B' \in \mathbb{K} \) such that \( \| B' \|_w < \| B^* \|_w \). If \( \| B' \|_0 = q = \| B^* \|_0 \), then our assumption is false. So, consider the case, where \( \| B' \|_0 > q = \| B^* \|_0 \). Without loss of generality, assume that \( \| B' \|_0 = q + 1 \). Since \( B' \in \mathbb{K} \) implies that at least \( q \) of the input-connections are connected to one state vertex in each SCC. Therefore, it is possible to extract a new input matrix \( B'' \in \mathbb{K} \) from \( B' \) such that \( \| B'' \|_0 = q \) and \( \| B'' \|_w \leq \| B' \|_w < \| B^* \|_w \). This contradicts the optimality of \( B^* \), and completes the proof. □

**Remark 4.7.** Generally, the vertex-variant of a problem is difficult to solve as compared to the edge-variant associated with it. For example, finding an independent set (a set of non-adjacent vertices) of maximum cardinality in an undirected graph is an NP-hard problem. However, the problem finding a maximum matching (a set of non-adjacent edges) in an undirected graph admits many polynomial time algorithms to compute it optimally. In a similar manner, a different but related problem is the minCIS discussed in §3.1. Recall that minCIS deals with identifying an input set of minimum cardinality, when an input matrix is pre-specified, to ensure that the resulting system is structurally controllable. The fact that minCIS is NP-hard is showed in [18], consequently there does not exist any polynomial time algorithm to solve it optimally. The authors of [18] observed that the minCIS is NP-hard when the bipartite graph \( F(A) \) associated with the system matrix \( A \) has a perfect matching. However, we demonstrate in this sequel that it is possible to select a sparsest input matrix \( B^* \) in polynomial time (when the input matrix \( B \) is known a priori,) and the objective is to minimize the number of non-zero entries of \( B \) not the number of inputs.

### 4.2. Strongly connected systems.

**Assumption 4.8.** We stipulate that the system matrix \( A \) is such that the digraph \( G(A) \) is strongly connected.

The preceding condition assures that all the state vertices in \( G(A) \) are accessible by using only one input-connection connecting an input vertex to a state vertex in \( G(A, B) \). Thus, by Theorem 2.5, only the no-dilation criterion has to be satisfied to ensure structural controllability of the pair \((A, B)\). We start with an algorithm to solve Problem (\( P_3 \)):

**Theorem 4.9.** Consider a linear system \((A, B)\) and Suppose Assumption 4.8 holds. The procedure outlined in Algorithm 3 ensures that the input matrix \( B^* \) obtained from it is such that \((A, B^*)\) is structurally controllable and solves Problem (\( P_3 \)). In addition, the complexity of the algorithm is \( O((d + m)^3) \).
Algorithm 3: Algorithm to solve Problem (P₃) under Assumption 4.8

Input: \( A \in \{0, \ast\}^{d \times d} \) and \( B \in \{0, \ast\}^{d \times m} \)

Output: The input matrix \( B^* \)

1. Construct \( \Gamma(A, B) = (V_A^1 \cup V_B, V_A^2, E_A \cup E_B) \).

2. For each edge \( e \in \Gamma(A, B) \) define cost \( c \)

3. Define:

\[
\begin{aligned}
c(e) &= \begin{cases} 
0 & \text{for } e = (v_i^1, v_j^1) \in E_A, \\
w_{kj} & \text{for } e = (u_i, v_j^2) \in E_B.
\end{cases}
\end{aligned}
\]

4. Find a minimum cost maximum matching (MCMM) in \( (\Gamma(A, B); c) \), say \( M^* \)

5. Define:

\[
B^*_{ik} \left\{ \begin{array}{ll}
\ast & \text{if } e = (u_k, v_i^2) \in M^* \cap E_B, \\
0 & \text{otherwise}.
\end{array} \right.
\]

Proof. It is given that pair \((A, B)\) is structurally controllable. Proposition 2.4 implies that there exists an SDR for \( V_A^2 \) in \( \Gamma(A, B) \). The cost structure (see Step 3) of Algorithm 3 ensures that when a MCMM is computed, it uses the input-connections which minimizes the cost and obtains an SDR for \( V_A^2 \) in \( \Gamma(A, B) \). Therefore, the obtained input matrix \( B^* \) (see Step 5 of Algorithm 3) is a solution of Problem (P₃).

Given the pair \((A, B)\), the construction of \( \Gamma(A, B) \) has linear complexity. The complexity associated with finding a MCMM in \( \Gamma(A, B) \) under a cost function \( c \) defined in (4.1) is \( O((d + m)^3) \) [22], where \( d \) and \( m \) are the number of state and input vertices in \( G(A, B) \). Therefore, the overall complexity of Algorithm 3 is \( O((d + m)^3) \), as asserted.

Note that each non-zero entry in \( B^* \) obtained in Algorithm 3 corresponds to an input-connection connecting a distinct input vertex to a state vertex in \( V_A^2 \).

Algorithm 3 is also utilized to provide a solution \( B^* \) for Problem (P₁) when Assumption 4.8 holds. Observe that Algorithm 3 with cost function \( c \) defined in (4.1) minimizes the cost of using those input-connections and not their number. If all the input-connections have uniform cost, then Problem (P₁) is a special case of Problem (P₃). We define the following cost function \( c \) from \( e \in \Gamma(A, B) \) for Algorithm 3 to solve Problem (P₁).

\[
\begin{aligned}
c(e) &= \begin{cases} 
0 & \text{for } e \in E_A, \\
1 & \text{for } e \in E_B.
\end{cases}
\end{aligned}
\]

Now we move to Problem (P₂). To address this problem, under Assumption 4.8, we first construct a flow network \( F_2(A, B) \) corresponding to the given pair \((A, B)\).

In other words, given a pair \((A, B)\), we first identify a solution of Problem (P₁) to determine the number of non-zero entries in the sparsest input matrix necessary to ensure structural controllability (Step 1). Then we define the vertex set \( V_{F_2} \) (Step 3) and edge set \( E_{F_2} \) (Step 4), source-sink pair \( s \), \( t \) and the capacity \( b \) (Step 5) as depicted in Algorithm 4.

It is easy to see that the construction of flow network \( F_2(A, B) \) has polynomial-time complexity. Given \((A, B)\), a solution of Problem (P₁) is computed in \( O((d + m)^3) \) computations. The rest of the constructions in Algorithm 4 has linear complexity. Therefore, the overall complexity of constructing \( F_2(A, B) \) is \( O((d + m)^3) \), where \( d \) and \( m \) are the number of state vertices and input vertices in \( G(A, B) \).
Algorithm 4: Procedure for the construction of flow network $F_2(A, B)$ of the system $(A, B)$ under Assumption 4.8

**Input:** $A \in \{0, \star\}^{d \times d}$ and $B \in \{0, \star\}^{d \times m}$

**Output:** The flow network $F_2(A, B)$

1. Determine the minimum number of non-zero entries in the input matrix which is a solution of Problem $(P_1)$, say $r$.
2. Construct flow network $F_2(A, B)$ with vertex set $V_{F_2}$ and edge set $E_{F_2}$ as follows:
   
   $V_{F_2} \leftarrow \{s, t\} \cup \{v_i\}_{i=1}^d \cup \{v_i^2\}_{i=1}^d \cup \{u_j\}_{j=1}^m \cup \{\}$
   
   $E_{F_2} \leftarrow$
   
   $\begin{cases}$
   $(s, v_i) & \text{for } i \in [d],$
   $(s, \ast)$
   $(\ast, u_j) & \text{for } j \in [m],$
   $(v_i, v_i^2) & \text{if } A_{kk} = \ast,$
   $(u_j, v_i^2) & \text{if } B_{kj} = \ast,$
   $(v_i^2, t) & \text{for } k \in [d].$
   \end{cases}$

3. $b(e) \leftarrow$

   $\begin{cases}$
   $r & \text{for } e = (s, \ast),$
   $1 & \text{otherwise}.$
   \end{cases}$

Since there are exactly $d$ edges, each of capacity 1, from the vertices in $V_{F_2}$ to the sink $t$, the value of any flow $f$ in $F_2(A, B)$ can not exceed $d$, i.e., for any feasible flow $f$ $\text{val}(f) \leq d$. Also, since $(A, B)$ is structurally controllable there exists at least one feasible flow of value equal to $d$ in $F_2(A, B)$. Therefore, the value of a maximum flow in $F_2(A, B)$ is equal to $d$.

We consider the flow network $F_2(A, B)$ augmented with cost vector $c$ (referred to as $(F_2(A, B); c)$) by defining:

$$(4.3) \quad c(e) \leftarrow \begin{cases} w_{kj} & \text{for } e = (u_j, v_i^2) \text{ and } B_{kj} = \ast, \\
0 & \text{otherwise.} \end{cases}$$

Recall minimum cost flow problem discussed in §2. We now demonstrate that we can obtain a solution of Problem $(P_2)$ by solving a minimum cost flow problem in $(F_2(A, B); c)$. We find an optimal flow $f$ which minimizes the cost $\sum_{e \in E_{F_2}} c(e)f(e)$ with $\text{val}(f) = d$ in $(F_2(A, B); c)$, where $d$ is the number of state vertices in $A$.

**Theorem 4.10.** Consider the linear system $(A, B)$, and suppose Assumption 4.8 holds. Consider $(F_2(A, B); c)$, $V_A = \{v_i\}_{i=1}^d$, $V_B = \{u_j\}_{j=1}^m$, and $V_A^2 = \{v_i^2\}_{i=1}^d$. Let $f^*$ be a flow computed by solving minimum cost flow problem in $(F_2(A, B); c)$ with $\text{val}(f^*) = d$, where $d$ is the number of state vertices in $A$. Define $I_{f^*} = \{(j, k) \mid f^*(u_j, v_k^2) = 1\}$. Then the input matrix $B^*$

$$(4.4) \quad B^*_{kj} \leftarrow \begin{cases} * & \text{if } (j, k) \in I_{f^*}, \\
0 & \text{otherwise}, \end{cases}$$

is a solution of Problem $(P_2)$.

**Proof.** First, we show that a solution $B^*$, obtained by running a minimum cost flow $f^*$ such that $\text{val}(f^*) = d$, belongs to $K$. Theorem 9.10 in [25, p. 318] ensures that if all the capacities in a flow network $(F_2(A, B); c)$ are integers, then there is
an optimal integral flow $f^*$, i.e., for every $e \in E_F$, $f^*(e) \in \mathbb{Z}$. If $f^*(e) > 0$, then $f^*(e) \geq 1$. Since the capacity $b(e) = 1$ for all $e = (u_j, v^2_k)$, where $j \in [m]$ and $k \in [d]$, we have $f^*(e) \leq 1$. This shows that $f^*(e)$ should be equal to 1 for all $e = (u_j, v^2_k)$ if $f^*(e) > 0$. This argument is also applicable to the rest of the edges in $F_2(A, B)$ with capacity 1. For any flow $f$, $\text{val}(f) = f^*(t) - f^*(t)$, where $t$ is the sink vertex.

Since there are exactly $d$ edges, each of capacity 1, connecting the vertices $v^2_k$ (where $k \in [d]$) to the sink vertex $t$, each of the edges carries unit flow. By the property of flow conservation at vertices $V^2_A$ and flow integrality, there exists $p_k \in V^2_A \cup V^2_B$ such that $f^*((p_k, v^2_k)) = 1$. Since the capacity of the outgoing edges from $V^2_A \cup V^2_B$ is 1, we have $p_k \neq p_{k_2}$ unless $k_1 = k_2$. Each vertex $v^2_k \in V^2_A$ corresponds to a distinct vertex in $V^2_A \cup V^2_B$ in $f^*$. The set of edges $\{(p_k, v^2_k) | k \in [d]\}$ constitutes an SDR for $V^2_A$ in the induced subgraph on $V^2_A \cup V^2_B$, which is a directed version of $\Gamma(A, B)$. Notice that $I_{f^*}$ contains the indices of only those $(p_k, v^2_k)$ pairs for which $p_k \in V^2_B$. This proves that the $B^*$ so obtained is such that $(A, B^*)$ is structurally controllable, i.e., $B^* \in K$. It is specified that the capacity of the edge $(s, z)$ is $r$, where $r$ is equal to the number of non-zero entries in the solution of Problem ($P_2$). Each vertex of $V_B$ has a directed edge of unit capacity from $z$. Therefore, it follows that at most $r$ vertices of $V_B$ have outgoing edges to $V^2_A$ with unit flow. But at least $r$ vertices of $V_B$ are required to ensure structural controllability of $(A, B^*)$. Thus, $B^*$ is also minimal, i.e., $\|B^*\|_0 = r$.

Second, we prove that $B^*$ incurs the minimum cost. By construction, non-negative costs $w_{kj}$ are assigned to the input-connections $(u_j, v^2_k)$ in the flow network $F_2(A, B)$ as given in (4.3): the other edges in $F_2(A, B)$ have zero cost. Consider $E_{f^*} := \{(u_j, v^2_k) | f^*((u_j, v^2_k)) = 1\}$ and $I_{f^*} := \{(j, k) | f^*((u_j, v^2_k)) = 1\}$. We now have

$$\sum_{e \in E_{f^*}} c(e)f^*(e) = \sum_{e \in E_{f^*}} c(e).1 = \sum_{(j, k) \in I_{f^*}} w_{kj} = \|B^*\|_w.$$  

By property of the minimum cost flow $f^*$, we see that $B^*$ has the least cost among all the input matrices using the minimum number of input-connections.

Since $m = O(d^2)$, the number of vertices in $F_2(A, B)$ is $2d + m + 3 = O(d)$. Therefore, the overall complexity of finding a minimum cost flow is $O(d^2 \log d)$. In other words, we have obtained a solution of Problem ($P_2$) in $O(d^2 \log d)$ computations.

**Remark 4.11.** The non-negative irrational costs assigned to the input-connections available are approximated to rationals. While computing a feasible flow for a minimum cost flow problem we convert these rational costs into integers by multiplying with a large number.

**Remark 4.12.** If the system matrix $A$ is such that both Assumption 4.1 and Assumption 4.8 are satisfied, then Problems ($P_1$)-($P_3$) can be solved in a straightforward manner: Only one input-connection is enough to satisfy the conditions given in Theorem 2.5, i.e., to ensure structural controllability of pair $(A, B^*)$. Thus, $\|B^*\|_0 = 1$. Also, we may pick an input-connection with the least cost among the costs assigned to all the input-connections.

**Remark 4.13.** The set of state vertices $F \subseteq A$ is forbidden if no input is allowed to be directly connected to any vertex in $F$.

If a non-empty forbidden set is present, then the premise of Problems ($P_1$)-($P_3$) is altered by removing the input-connections that connects an input vertex to a state vertex $v_i \in F$. We obtain a new input matrix from $B$, say $\bar{B}$. Verification of whether the pair $(A, \bar{B})$ is

\[2\]
structurally controllable or not can be done in polynomial time (see Remark 2.6), and the solutions of Problems (P1)-(P3) are obtained by using the same techniques as discussed in this article.

4.3. Discussion. The problem of minimal input selection is studied in [16, 17, 30] under various costs assigned to the input-connections. The authors of [16] address the problem of finding the minimum number of state vertices to be actuated to make the system structurally controllable when the given input matrix is identity, i.e., $I_d$. In this direction, [17] finds a minimum cost diagonal input matrix required to ensure structural controllability when each state is associated with a certain cost. [30] deals with determining a sparsest minimum cost input selection problem where each input is multiple-dedicated while incurring different costs. All these problems discussed above admit polynomial time solutions, but assume that the given input matrix has a diagonal structure.

Recently, the authors of [31] considered the problem of identifying a sub-collection of state-connections and input-connections, from the available set of state and input-connections, that leads to a minimum cost and guarantees that the resulting system is structurally controllable. It was observed that this problem is NP-hard by showing that the Hamiltonian path problem\(^4\) is polynomially reducible to an instance of this problem. If zero costs are assigned to the state-connections then this problem reduces to the problem of identifying a sub-collection of input-connections from the available set of input-connections that incurs minimum cost and ensures structural controllability of the resulting system, which is Problem (P3) precisely. This reduction does not show that Problem (P3) is also NP-hard and hence, needs further investigation. We do however provide an algorithm which guarantees a 2-approximate solution to Problem (P3) in the general setup.

\begin{algorithm}
\textbf{Input:} $A \in \{0, \star\}^{d \times d}$ and $B \in \{0, \star\}^{d \times m}$
\textbf{Output:} The input matrix $B^*$
1 Use Algorithm 2 to obtain $B'$.
2 Use Algorithm 3 to obtain $B''$.
3 Let $\{S_j\}_{j=1}^r$ be the SSCCs for which there exists a $v_\ell \in S_j$ s.t. $B''_{\ell k} = \star$, where $r \leq q$.
4 for $i = 1, \ldots, r$
\hspace{1cm} update $B'$ by letting $B'_{\ell k} = 0$ for all $v_\ell \in S_i$
end
5 $B^* = B' \cup B''$.
\end{algorithm}

In Algorithm 5, step 3 collects those SSCCs, say $\{S_j\}_{j=1}^r$ (where $r \leq q$), of $G(A)$ which have at least one state vertex $v_\ell \in S_j$ that has an input-connection from some input $u_k$ in $B''$, i.e., $B''_{\ell k} = \star$. This confirms that the SSCCs $\{S_j\}_{j=1}^r$ are accessible by using the input-connections corresponding to $B''$. Step 4 removes those input-connections corresponding to $B'$ needed to make $\{S_j\}_{j=1}^r$ accessible.

Clearly the obtained $B^* \in K$, since it ensures that all the SSCCs are accessible and an SDR exists in the bipartite graph $\Gamma(A, B^*)$ associated with $G(A, B^*)$. Let $\hat{B}$ be an optimal solution to Problem (P3). The optimal cost for satisfying each

\(^3\)An input is said to be dedicated if it is directly connected to only one state vertex.

\(^4\)A Hamiltonian path is a directed path which visits every vertex exactly once. It is well-known that determining whether a Hamiltonian path exists in a graph is NP-hard [21].
condition in Theorem 2.5 individually is at most $\|\hat{B}\|_w$. Thus, $\|\hat{B}\|_w \geq \|B\|_w$, and $\|\hat{B}\|_w \geq \|B\|_w$ leading to $2\|\hat{B}\|_w \geq \|B\|_w + \|B\|_w = \|B\|_w$.

Again, note that since Algorithm 2 and Algorithm 3 have polynomial time complexity $O(d^3)$ and $O((d + m)^3)$ respectively. It is easy to see that Algorithm 5 has $O((d + m)^3)$ complexity.

5. Illustrative Examples

Example 1: Let the structures of the system and input matrices be

$$A = \begin{bmatrix}
0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
* & 0 & 0 \\
0 & 0 & 0 \\
* & 0 & 0 \\
0 & * & 0 \\
0 & 0 & 0 \\
0 & * & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.$$  

The digraph $G(A, B)$ associated with the pair $(A, B)$ is shown in Fig. 1.

![Illustration of the digraph $G(A, B)$](image)

**Figure 1.** Illustration of the digraph $G(A, B)$. Each vertex in a coloured box represents an input. The black edges denote the state-connections and the cyan coloured dashed edges denote input-connections.

Observe that $(A, B)$ is structurally controllable. Recall from §3 that $w_{ij}$ is the cost of the input-connection between the input $u_j$ and the state $v_i$. Let the costs in the present case be $w_{11} = 15$, $w_{31} = 10$, $w_{41} = 20$, $w_{62} = 15$, $w_{72} = 5$, $w_{83} = 5$, and $w_{10,3} = 10$. Clearly, $\Gamma(A)$ has a perfect matching and the SSCCs of $G(A)$ are: $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_7, v_8\}$, and $S_3 = \{v_9, v_{10}\}$. We construct a flow graph $F_1(A, B)$ in accordance with Algorithm 1 as shown in Fig. 2.

A maximum flow in $F_1(A, B)$ gives us a flow $f^*$ such that $f^*(S_1, u_1) = 1$, $f^*(S_2, u_3) = 1$, and $f^*(S_3, u_3) = 1$. Subsequently, we get an input matrix $B^*$ with $B^{*}_{11} = *$, $B^{*}_{33} = *$, and $B^{*}_{10,3} = *$ that solves Problem $(F_1)$. Observe that the flow network associated with $B^*$, i.e., $F_1(A, B^*)$, has a maximum flow $f^*$ with
val($f^*$) = 3. All the SCCCs $S_1$, $S_2$, and $S_3$ of $G(A)$ are accessible from the input vertices in $G(A,B^*)$; of course, the input vertices associated with each SCCCs may not be distinct from each other.

We move to Algorithm 2 to solve Problem ($P_2$). After we execute Algorithm 2, the solution obtained is $B^*$ with $B_{51}^* = \ast$, $B_{72}^* = \ast$, and $B_{10,3}^* = \ast$, and it has the (minimum) cost of 25. Clearly, $G(A,B^*)$ has all the SCCCs $S_1$, $S_2$, and $S_3$ of $G(A)$ accessible from some input vertex, and the sum of the cost of the input-connections in $B^*$ have the least value among all the input matrices $B' \in K$.

It follows from Proposition 4.6 that the input matrix $B^*$ calculated as above also solves Problem ($P_3$).

**Example 2:** Let

\[
A = \begin{bmatrix}
0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 & * \\
* & 0 & 0 & 0 \\
0 & * & 0 & 0 \\
0 & * & 0 & 0 \\
0 & * & 0 & 0 \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{bmatrix}.
\]

Observe that $(A, B)$ is structurally controllable. Suppose that costs of the input-connections are $w_{21} = 5$, $w_{61} = 1$, $w_{71} = 2$, $w_{81} = 2$, $w_{32} = 5$, $w_{62} = 1$, $w_{72} = 5$, $w_{82} = 6$, $w_{53} = 5$, $w_{63} = 2$, $w_{73} = 1$, $w_{83} = 2$, $w_{14} = 5$, $w_{64} = 5$, $w_{74} = 3$, and $w_{84} = 1$. Notice that $G(A)$ is strongly connected, and therefore we resort to Algorithm 3 to solve Problem ($P_3$). A MCMM $M_1$ of total cost 2 in $(\Gamma(A,B);c)$ (under the cost function $c$ defined in (4.1)) is $\{(v_1, v_4), (v_1, v_5), (v_1, v_6), (v_2, v_4), (v_2, v_5), (v_2, v_6), (u_1, v_5), (u_1, v_6), (u_2, v_4)\}$. Therefore, the solution we obtain for Problem ($P_3$) is $B^*$ with $B_{51}^* = \ast$, and $B_{64}^* = \ast$.

If all the costs are uniform, then the MCMM $M_2$ in $(\Gamma(A,B);c)$ (under the cost function $c$ defined in (4.2)) is $M_2 = \{(v_1, v_4), (v_1, v_5), (v_1, v_6), (v_2, v_4), (v_2, v_5), (v_2, v_6), (v_1, v_5), (v_1, v_6), (u_2, v_4)\}$ and $B^*$ has $\ast$ entries at $B_{51}^* = \ast$, and $B_{64}^* = \ast$, which is a solution of Problem ($P_1$). The digraph $G(A,B^*)$ has an SDR and therefore, $B^* \in K$. Note that the number of input-connections in $M_1$ and $M_2$ is same. In general, a MCMM computed to solve Problem ($P_3$) may have more input-connections as compared to input-connections associated with MCMM computed to solve Problem ($P_1$).

![Illustration of the flow network $F_1(A,B)$ with SCCCs: $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_7, v_8\}$, and $S_3 = \{v_9, v_{10}\}$, and inputs $u_1$, $u_2$ and $u_3$. Each edge has a capacity depicted by the number over it in the figure.](image-url)
We appeal to Algorithm 4 to solve Problem (P_2). The flow graph \( F_2(A, B) \) associated with \((A, B)\) is depicted in Fig. 3. The capacity of the edge \((s, z)\) is 2 computed by solving Problem \((P_1)\).

![Flow Graph](image)

**Figure 3.** Illustration of the flow network \((F_2(A, B); c)\) corresponding to the pair \((A, B)\). The capacity of the edge \((s, z)\) is 2 computed by solving Problem \((P_1)\). For simplicity, the cost \(c\) (defined in \((4.3)\)) assigned to every edge is not depicted in the figure. We obtain a minimum cost flow \(f^*\) in \(F_2(A, B)\), shown by the coloured dashed edges, where the magenta coloured edges represent the state-connections and the cyan coloured edges represent the input-connections associated with \(f^*\).

An optimal minimum cost flow \(f^*\) with \(\text{val}(f^*) = 8\) is computed with \(I_{f^*} = \{(1, 6), (4, 8)\}\) as defined in Theorem 4.10. Then, \(B^*\) is such that \(B_{61}^* = \ast\), and \(B_{84}^* = \ast\) with total cost of 2.

Clearly Fig. 3 depicts that there exists an SDR in the induced subgraph on \(V_A \cup V_B\), which is a directed version of \(\Gamma(A, B)\). Therefore, the obtained \(B^*\) is such that \((A, B^*)\) is structurally controllable, i.e., \(B^* \in K\). The minimum cost flow \(f^*\) not only yields a \(B^* \in K\) but also ensures that it is the sparsest among all \(B' \in K\), i.e., \(\|B^*\|_0 = 2\). Moreover, it guarantees that \(B^*\) so obtained has the least cost.

**References**

[1] G. Orosz, J. Moehlis, and R. M. Murray, “Controlling biological networks by time-delayed signals,” *Philosophical Transactions of the Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences*, vol. 368, no. 1911, pp. 439–454, 2010.

[2] J. Marcelino and M. Kaiser, “Critical Paths in a Metapopulation Method of H1N1: Efficiently Delaying Influenza Spreading through Flight Cancellation,” *PLOS Currents*, 2012.

[3] S. Currarini, M. O. Jackson, and P. Pin, “An economic model of friendship: homophily, minorities, and segregation,” *Econometrica. Journal of the Econometric Society*, vol. 77, no. 4, pp. 1003–1045, 2009.

[4] D. J. Watts and S. H. Strogatz, “Collective Dynamics of Small-World Networks,” *Nature*, vol. 393, pp. 440–442, 1998.

[5] R. Liu, “The Dracula Dynamic Traffic Network Microsimulation Model,” *Simulation Approaches in Transportation Analysis: Recent Advanced and Challenges*, vol. 31, pp. 23–56, 2005.
[6] T. Opsahl and P. Panzarasa, “Clustering in Weighted Networks,” Social Networks, vol. 31, pp. 155–163, 2009.
[7] I. Stewart, “Network opportunity,” Nature, vol. 427, pp. 601–604, 2004.
[8] A. Olshevsky, “Minimal controllability problems,” IEEE Transactions on Control of Network Systems, vol. 1, no. 3, pp. 249–258, 2014.
[9] C. T. Lin, “Structural controllability,” IEEE Transactions on Automatic Control, vol. 19, no. 3, pp. 201–208, 1974.
[10] J. Dion, C. Commault, and J. Woude, “Generic properties and control of linear structured systems: a survey,” Automatica, vol. 39, pp. 1125–1144, 2003.
[11] Y. Y. Liu, J. J. Slotine, and A. L. Barabasi, “Controllability of complex networks,” Nature, vol. 473, no. 7346, p. 167, 2011.
[12] A. Olshevsky, “Minimum input selection for structural controllability,” in 2015 American Control Conference (ACC), 2015, pp. 2218–2223.
[13] U. A. Khan and A. Jadbabaie, “Coordinated networked estimation strategies using structured systems theory,” in 2011 50th IEEE Conference on Decision and Control and European Control Conference, 2011, pp. 2112–2117.
[14] M. Doostmohammadian and U. A. Khan, “On the genericity properties in distributed estimation: Topology design and sensor placement,” IEEE Journal of Selected Topics in Signal Processing, vol. 7, no. 2, pp. 195–204, 2013.
[15] C. Commault, J. Dion, and J. Woude, “Characterization of generic properties of linear structured systems for efficient computations,” Kybernetika (Prague), vol. 38, no. 5, pp. 503–520, 2002, special issue on system structure and control (Prague, 2001).
[16] S. Pequito, S. Kar, and A. P. Aguiar, “A framework for structural input/output and control configuration selection in large-scale systems,” IEEE Transactions on Automatic Control, vol. 61, no. 2, pp. 303–318, 2016.
[17] ——, “Minimum cost input/output design for large-scale linear structural systems,” Automatica J. IFAC, vol. 68, pp. 384–391, 2016.
[18] ——, “On the complexity of the constrained input selection problem for structural linear systems,” Automatica, vol. 62, pp. 193–199, 2015.
[19] S. Moothedath, P. Chaporkar, and M. N. Belur, “A flow-network based polynomial-time approximation algorithm for the minimum constrained input structural controllability problem,” IEEE Transactions on Automatic Control, vol. PP, no. 99, pp. 1–8, 2018.
[20] K. J. Reinschke, Multivariable Control: a Graph-Theoretic Approach, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 1988, vol. 108.
[21] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, Introduction to Algorithms, 3rd ed. MIT Press, Cambridge, MA, 2009.
[22] J. Munkres, “Algorithms for the assignment and transportation problems,” Journal of the Society of Industrial and Applied Mathematics, vol. 5, pp. 32–38, 1957.
[23] D. B. West, Introduction to Graph Theory. Prentice Hall, Inc., Upper Saddle River, New Jersey, 1996.
[24] D. R. Fulkerson and G. B. Dantzig, “Computation of maximal flows in networks,” Naval Research Logistics Quarterly, vol. 2, pp. 277–283 (1956), 1955.
[25] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Network Flows: Theory, Algorithms, and Applications. Prentice Hall Inc., Englewood Cliffs, New Jersey, 1993.
[26] J. B. Orlin, “A faster strongly polynomial minimum cost flow algorithm,” Operations Research, vol. 41, no. 2, pp. 338–350, 1993.
[27] M. D. Ilic, L. Xie, U. A. Khan, and J. M. F. Moura, “Modeling of future cyber-physical energy systems for distributed sensing and control,” IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans, vol. 40, no. 4, pp. 825–838, 2010.
[28] S. Jafarī, A. Ajorlou, and A. G. Aghdam, “Leader localization in multi-agent systems subject to failure: a graph-theoretic approach,” Automatica. A Journal of IFAC, the International Federation of Automatic Control, vol. 47, no. 8, 2011.
[29] M. Newman, A. L. Barabási, and D. J. Watts, Eds., The Structure and Dynamics of Networks, ser. Princeton Studies in Complexity. Princeton University Press, Princeton, NJ, 2006.
[30] S. Pequito, J. Svacha, G. J. Pappas, and V. Kumar, “Sparsest minimum multiple-cost structural leader selection,” in in Proceedings of the 5th IFAC workshop on distributed estimation and control in networked systems, vol. 48, no. 22, 2015, pp. 144–149.
[31] Y. Zhang and T. Zhou, “On the edge insertion/deletion and controllability distance of linear structural systems,” ArXiv e-prints, March 2018. [Online]. Available: http://adsabs.harvard.edu/abs/2018arXiv180307929Z
