Measurement of modular values and connection to the annihilation operator

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Abstract – Modular value of an observable of a pre- and postselected quantum system is a concept similar to weak value, but in contrast to describing only weak coupling, it can describe the coupling of any strength but only for qubit meters, so it may be used more widely in some scenarios. Besides, modular value has been proved to have advantages on measuring weak values of nonlocal joint observables and nonlocal entanglement states. In this paper, we extend the work of J. S. Lundeen and K. J. Reschb on connection between the measured values of multiparticle observables of a pre- and postselected quantum system and annihilation operator by modular value and give their relation. Based on the above, we propose a new method different from tomography to measure modular values, and we find that our annihilation operator measurement (AOM) method can significantly reduce the number of measurements from \(O(4^n)\) to \(O(2^n)\) compared with the original tomography method.

Introduction. – Since von Neumann measurement was proposed in 1950s, in 1988 Aharonov, Albert, and Vaidman (AAV) \([1]\) proposed the “weak measurement” as an extension of standard von Neumann measurement, which is a “strong” measurement. Weak measurement has been extensively studied since it was proposed both in theory and in experiment \([2]\), and there have been still many important developments in recent years. Weak values, which are measured values of observables by weak measurement, can play a unique and important role in small signal amplification and parameter estimation \([2–5]\), direct measurement of quantum states \([6–11]\), nonclassical features of quantum mechanics, such as nonlocality \([11,12]\), paradoxes in quantum mechanics \([13,14]\), uncertainty principle \([15,16]\), etc.

The weak measurement can be performed by coupling a variable of a pointer weakly to the observable of the measured system, and then projecting the system to a post-selected state so to read out the measured value of the observable on the pointer. However, the weak measurement technology always requires a small coupling strength, so this greatly limits its application range. Besides, weak measurement also has difficulties in measuring nonlocal joint observables because it requires that the interaction term constructed with nonlocal joint observables appears in the interaction Hamiltonian \([17,18]\). In 2010, a new concept of “modular value” \([19]\) of an observable of a pre- and postselected quantum system was proposed by Y. Kedem and L. Vaidman as an extension of the weak value, and then was generalized to “generalized modular value” by Ho and Imoto in 2017 \([20]\). After that, the relation between the modular value and the weak value has also been obtained \([21]\). The main differences between modular values and weak values are reflected in three
aspects: first, instead of coupling the measured observable
to a continuous variable in von Neumann measurement
and weak measurement, the modular values are obtained
by coupling the measured observable to a discrete vari-
able; second, the former has no limitation on the coupling
strength; last, the modular values contain a parameter of
coupling strength. Therefore, the modular value measure-
ment can overcome some traditional disadvantages of weak
measurement, specifically, it is not limited by the coupling
strength, and is proved to have great advantages on mea-
suring nonlocal joint observables and nonlocal entangled
states directly [22]. In addition, the physical meaning of
modular values has also been studied in recent years [26–28].

The practical measurement of joint weak values and
their connection to the annihilation operator has been
proposed by Lundeen and Reschb in 2005 [29], however,
the connection between modular values and annihilation
operator is still unclear. In this paper, we extend the
connection between the measured value of multiparticle
observables of a pre- and postselected quantum system
and annihilation operator by modular value, and obtain
their relation. Based on the above, we also give a new
method different from tomography to measure modular
values. Besides, we find that the computation process in
the relation between the measured values of observables
of a pre- and postselected quantum system and the
annihilation operator [29] can be greatly simplified with
the method of modular value. We organize this paper as
follows: we introduce the von Neumann measurement at
first, and then, based on that, we introduce the modular
value and weak value of the observable briefly. Then
the relation between modular values and annihilation
operator is given, and we find that the N-observable
modular values also have a simple connection with the
N-particle joint annihilation operator. Finally, with the
relation, we propose a new method different from
tomography to measure modular values, and we find that
our annihilation operator measurement (AOM) method
can significantly reduce the number of measurements from
$O(4^n)$ to $O(2^n)$ compared with the original tomography
method.

**Modular value of a single-particle observable.**—
The von Neumann interaction was originally proposed to
model standard quantum measurement, which was real-
ized by coupling the observable $A$ of measured system to
the momentum $M$ of the pointer variable,

$$ H = g(t)AM, \quad \int g(t)dt = k, \quad k \in (0, +\infty), \quad (1) $$

where $k$ is usually referred to as coupling strength. In no
special case, when several quantities are defined in differ-
ent Hilbert spaces, the tensor product symbol \(\otimes\) is
often omitted to be concise, such as $AM \equiv A \otimes M, AP \equiv A \otimes P$, etc.

Suppose that the initial system-meter state is $|\psi(0)\rangle = |\psi_i\rangle|\phi_i\rangle$ ($|\psi_i\rangle$ is the initial state of the pointer with wave
function $\phi(q)$, $|\psi_i\rangle$ the initial state of the system), when
$k \ll 1$, after the meter interacting with the system,

$$ |\psi(t)\rangle = \exp \left( -i \frac{H dt}{\hbar} \right) |\psi_i\rangle|\phi_i\rangle $$

$$ = \exp \left( -i kAM \right) |\psi_i\rangle|\phi_i\rangle $$

$$ \approx \left( I - i kAM \right) |\psi_i\rangle|\phi_i\rangle. \quad (2) $$

If we project the system state to a postselected state $|\psi_f\rangle$
after the measurement interaction (1), then

$$ \langle \psi_f | \left( I - i kAM \right) |\psi_i\rangle|\phi_i\rangle \approx $$

$$ = \langle \psi_f | \psi_i \rangle \exp \left( -i kA_wM \right) |\phi_i\rangle $$

$$ = \langle \psi_f | \psi_i \rangle \phi(q - kA_w), \quad (3) $$

where

$$ A_w \equiv \frac{\langle \psi_i | A |\psi_f\rangle}{\langle \psi_i | \psi_f\rangle}. \quad (4) $$

Here $A_w$ is defined as the weak value of observable $A$
and can be read out by observing the shift of the pointer.
From eq. (4) we can know that the range of weak values
can exceed the eigenvalue spectrum of $A$ and $A_u$ is usually
complex. Especially, when we set $|\psi_f\rangle = |\psi\rangle = |\psi\rangle$, then

$$ A_w = \langle \psi | A |\psi\rangle, $$

which means the value of the von Neumann measurement can be considered as a special case of
weak value.

In eq. (1), the momentum $M$ is a continuous variable,
similarly, an observable $A$ of the system can also couple
to a discrete variable of a qubit meter [19],

$$ H = g(t)AP, \quad (5) $$

where $P = |1\rangle\langle 1|$ is a projection operator.

Suppose that the initial system-meter state is $|\psi(0)\rangle = |\psi_i\rangle|\phi_i\rangle$ ($|\phi_i\rangle = \alpha|0\rangle + \beta|1\rangle$, $|0\rangle$ and $|1\rangle$ can be thought of
as ground state and the first excited state, respectively),
after the meter interacting with the system,

$$ |\psi(t)\rangle = \exp \left( -i \frac{H dt}{\hbar} \right) |\psi_i\rangle|\phi_i\rangle $$

$$ = \exp \left( -i kAP \right) |\psi_i\rangle (\alpha|0\rangle + \beta|1\rangle) $$

$$ = (\alpha \exp \left( -i kAP \right) |0\rangle + \beta \exp \left( -i kAP \right) |1\rangle) |\psi_i\rangle $$

$$ = (\alpha|0\rangle + \beta \exp \left( -i kA \right) |1\rangle) |\psi_i\rangle. \quad (6) $$

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Then the system is projected to the postselected state $|\psi_f\rangle$, 
\[
\alpha|\psi_f\rangle|\psi_i\rangle + \beta|\psi_f\rangle|\psi_i\rangle \exp \left( -\frac{i k A}{\hbar} \right) = 
\langle \psi_f | \psi_i \rangle \left( \alpha |0\rangle + \beta |\psi_i\rangle \right) = 
\langle \psi_f | \psi_i \rangle (\alpha|0\rangle + \beta A_m|1\rangle),
\]
where 
\[
A_m = \langle \psi_f | \exp \left( -\frac{i k A}{\hbar} \right) |\psi_i\rangle,
\]
which is defined as the modular value of observable $A$ and usually also a complex number. This modular value $A_m$ describes the effect of the pre- and postselected system on the qubit meter.

It is worth noting that when $k \to 0$, 
\[
A_m \approx \frac{\langle \psi_f | 1 - \frac{ik A}{\hbar} |\psi_i\rangle}{\langle \psi_f | \psi_i \rangle} = 1 - \frac{k}{\hbar} A_w = 1 - \frac{k}{\hbar} A_m \approx \exp \left( -\frac{i k A_m}{\hbar} \right).
\]

However, the weak measurement condition requires that $k \ll 1$ and even $k \to 0$, which seriously weakens the application scope of the weak measurement technology. Comparing eq. (8) with (4), we can know that the modular measurement has no limit to the coupling strength $k$, so it avoids the disadvantages of the weak measurement. With normalization, eq. (7) will become 
\[
|\phi_{f_i}\rangle = N (\alpha|0\rangle + \beta A_m|1\rangle),
\]
where $N = 1/\sqrt{|\alpha|^2 + |\beta A_m|^2}$ is the normalization factor. By observing eq. (10), we found that 
\[
A_m = \frac{1}{|N|^2} \langle a^\dagger \rangle_{f_i},
\]
where $\langle a \rangle_{f_i} \equiv \langle \phi_{f_i}|a|\phi_{f_i}\rangle$ and $a$ is the annihilation operator. This is our final expression for the modular value of the single-particle observable. Equation (11) reflects the relation between the modular value on the system and the average value of the annihilation operator on the meter. Comparing with the computation process of the relation between joint weak values and the annihilation operator [29], we greatly simplify it with the method of the modular value without any approximation because there is no limitation to the coupling strength.

Besides, on the one hand, as we all know, in the harmonic oscillator, 
\[
a = \sqrt{\frac{m\omega}{2\hbar}} X + i \sqrt{\frac{1}{2m\omega h}} M,
\]
\[
a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} X - i \sqrt{\frac{1}{2m\omega h}} M,
\]
and then according to eq. (11), $\langle a \rangle_{f_i}$ will also be obtained.

Modular values of multiparticle observables. –
We now consider a derivation of modular values of multiparticle observables on the $N$-particle system. We

\[
X = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger),
\]
\[
M = -i \sqrt{\frac{m\omega h}{2}} (a - a^\dagger),
\]
where $\omega$ and $m$ are the angular frequency and quality of the harmonic oscillator, respectively. So eq. (11) will be 
\[
A_m = \frac{1}{|N|^2} \beta\alpha^* \left( \sqrt{\frac{m\omega}{2\hbar}} \langle X \rangle_{f_i} + i \sqrt{\frac{1}{2m\omega h}} \langle M \rangle_{f_i} \right),
\]
and if we set $\beta\alpha^* = \beta^*\alpha$, i.e., $\beta\alpha^*$ real (this means $\phi_i$ is a real one without the imaginary parts, which is a common practice for experimenters to make an experiment easier [13,19,22]), then eq. (15) will be 
\[
\langle X \rangle_{f_i} = 2|N|^2 \sqrt{\frac{\hbar}{2m\omega}} \beta\alpha^* \text{Re} A_m,
\]
\[
\langle M \rangle_{f_i} = 2|N|^2 \sqrt{\frac{m\omega h}{2\hbar}} \beta\alpha^* \text{Im} A_m.
\]

According to eq. (14), we can measure the modular value $A_m$ by measuring $\langle X \rangle_{f_i}, \langle M \rangle_{f_i}$ on the meter (more technical details can be seen in fig. 1 of ref. [2] and fig. 1 of [9]). This can be a new method different from tomography to measure the modular value. On the other hand, in general, $A_m$ can be obtained by tomography on the final state of the meter [19]. Specifically, through state tomography [30], if the final state is found to be $\alpha'|0\rangle + \beta'|1\rangle$, then
\[
A_m = \alpha\beta',
\]
suppose \( A_j \) is the observable of the \( j \)-th particle, then \( A_j \) is coupled to its own meter (the schematic diagram is shown in fig. 1),

\[
H = g(t) \sum_{j=1}^{n} A_j P_j, \tag{18}
\]

where \( P_j = |1\rangle \langle 1| \) is the corresponding projection operator on the \( j \)-th meter.

If we prepare the initial state of the system-meter to be \(|\psi(0)\rangle = |\psi_i\rangle |\phi_i\rangle \) (\( |\phi_i\rangle = \alpha|0\rangle^\otimes n + |1\rangle^\otimes n \), it is worth noting that \( |\phi_i\rangle \) is an entanglement state there), similar to eq. (6), we can get

\[
|\psi(t)\rangle = \exp\left(-i \frac{H dt}{\hbar}\right) |\psi_i\rangle |\phi_i\rangle \\
= \exp\left(-i k \sum_{j=1}^{n} A_j P_j \frac{\hbar}{h}\right) |\psi_i\rangle |\alpha|0\rangle^\otimes n + |1\rangle^\otimes n \rangle \\
= \prod_{j=1}^{n} \exp\left(-i k A_j P_j \frac{h}{\hbar}\right) |\psi_i\rangle |\alpha|0\rangle^\otimes n + |1\rangle^\otimes n \rangle \\
= |\psi_i\rangle |\alpha|0\rangle^\otimes n + \beta \exp\left(-i k \sum_{j=1}^{n} A_j \frac{h}{\hbar}\right) |1\rangle^\otimes n \rangle. \tag{19}
\]

Then the system is projected to the postselected state \(|\psi_f\rangle\) and we normalize the whole state

\[
|\phi\rangle_{fs} = \mathcal{N} |\alpha|0\rangle^\otimes n + \beta \left( \sum_{j=1}^{m} A_j \right) |1\rangle^\otimes n, \tag{20}
\]

where \( \sum_{j=1}^{m} A_j \rangle \rangle m = (|\psi_f\rangle \exp\left(-i k \sum_{j=1}^{m} A_j \right) |\psi_i\rangle /|\psi_f| |\psi_i\rangle, \) and \( \mathcal{N} = 1/\sqrt{|\alpha|^2 + |\beta|^2 (|\sum_{j=1}^{m} A_j |^2)}, \) By observing eq. (20), we found that

\[
\left( \sum_{j=1}^{m} A_j \right) = \frac{1}{|\mathcal{N}|^{2\alpha^\ast \beta}} \left( \prod_{j=1}^{m} a_j \right) \tag{21}
\]

This is our final expression for the modular value of multiparticle observables with entanglement state meter. Equation (21) reflects the relation between the modular value on the \( N \)-particle system and the average value of the \( N \)-joint annihilation operator on the meter. Similarly to eq. (14), we can get

\[
\left( \sum_{j=1}^{m} A_j \right) = \frac{1}{|\mathcal{N}|^{2\alpha^\ast \beta}} \left( \prod_{j=1}^{m} \left( \frac{\hbar}{2|\mu_0|} X_j + i \frac{1}{2m_0 \hbar} M_j \right) \right) \tag{22}
\]

According to eq. (22) we can find that \( \sum_{j=1}^{m} A_j \rangle \rangle m \) can be obtained when all operator product permutations constructed by \( \{X_j, M_j\} \) are measured. In general, \( \sum_{j=1}^{m} A_j \rangle \rangle m \neq \sum_{j=1}^{m} A_j \rangle \rangle m \). It is worth noting that in order to obtain modular values, for a \( N \)-qubit system, \( 2^N - 1 \) \( \mathcal{O}(4^N) \) measurements are needed by the method of quantum state tomography used by kedem and vaidman in their initial paper about “modular value” [19], however, only \( 2^N \) \( \mathcal{O}(2^N) \) measurements are needed by our method of annihilation operator measurement (AOM) in eq. (11) and eq. (21), which can be converted to the measurement of position and momentum in eq. (14) and eq. (22), respectively. Our AOM method significantly simplifies the measurement of modular values.

If we make the initial state of system-meter be a separable state \(|\psi(0)\rangle = |\psi_i\rangle |\phi_i\rangle \), we now set the initial state of meter \( |\phi_j\rangle = \prod_{j=1}^{n} (|\alpha\rangle_j + |\beta\rangle_j) \rangle \), which is a separate state there. Similarly to eq. (19), we can get

\[
|\psi(t)\rangle = \exp\left(-i \frac{H dt}{\hbar}\right) |\psi_i\rangle |\phi_i\rangle \\
= \exp\left(-i k \sum_{j=1}^{n} A_j P_j \frac{\hbar}{h}\right) |\psi_i\rangle \prod_{j=1}^{n} (|\alpha\rangle_j + |\beta\rangle_j) \rangle \\
= \prod_{j=1}^{n} \exp\left(-i k A_j P_j \frac{h}{\hbar}\right) |\psi_i\rangle \prod_{j=1}^{n} (|\alpha\rangle_j + |\beta\rangle_j) \rangle \\
= |\psi_i\rangle \prod_{j=1}^{n} (|\alpha\rangle_j + \beta \exp\left(-i k A_j \frac{h}{\hbar}\right) |1\rangle_j \rangle. \tag{23}
\]

Then the system is projected to the postselected state \(|\psi_f\rangle\) like the above and we normalize the whole state

\[
|\phi\rangle_{fs} = \mathcal{N} \prod_{j=1}^{n} (|\alpha\rangle_j + \beta |A_j \rangle \rangle m |1\rangle_j \rangle, \tag{24}
\]

where \( |A_j \rangle \rangle m = (|\psi_f\rangle \exp\left(-i k A_j \right) |\psi_i\rangle /|\psi_f| |\psi_i\rangle, \) and \( \mathcal{N} = 1/\sqrt{\prod_{j=1}^{n} (|\alpha|^2 + |\beta|^2 (|A_j |^2)}, \) is the normalization factor. Unlike the meter state in eq. (20), the meter state in eq. (24) is a separable state. Therefore, by observing eq. (24), we can find that

\[
\sum_{q \subset [1, n]} |A_q \rangle \rangle m = \frac{1}{\mathcal{N}^{2\alpha^\ast \beta}} \sum_{q \subset [1, n]} \langle a_q |_{fi}, \tag{25}
\]
which reflects the relation between the modular value of the observable for an arbitrary single particle and the average value of the annihilation operator on the corresponding single meter. Equation (25) is equivalent to

$$\sum_{q \in [1, n]} (A_q)_m = \frac{1}{|N|^2(\beta \alpha^*)^n} \left( \sqrt{\frac{m \omega}{2\hbar}} \sum_{q \in [1, n]} (X_q)_{fi} + i \sqrt{\frac{1}{2m \omega \hbar}} \sum_{q \in [1, n]} (M_q)_{fi} \right).$$

(26)

So \((A_j)_m\) can be measured by measuring each \(X_j\) and \(M_j\) on each single meter. At the same time, by observing eq. (24), we also found that

$$\prod_{j=1}^{n} (A_j)_m = \frac{1}{|N|^2(\beta \alpha^*)^n} \left( \prod_{j=1}^{n} a_j \right)_{fi}. \quad(27)$$

Equation (25) and eq. (27) are our final expressions for the modular of multiparticle observables with separate state meter. Equation (27) reflects the relation between the product of the modular value of the observable for each single particle and the average value of the joint annihilation operator on the meter. Because each \((A_j)_m\) can be obtained, similarly to eq. (22), we can get

$$\prod_{j=1}^{n} (A_j)_m = \frac{1}{|N|^2(\beta \alpha^*)^n} \left( \prod_{j=1}^{n} \left( \sqrt{\frac{m \omega}{2\hbar}} X_j + i \sqrt{\frac{1}{2m \omega \hbar}} M_j \right) \right)_{fi} = \frac{1}{|N|^2(\beta \alpha^*)^n} \prod_{j=1}^{n} \left( \sqrt{\frac{m \omega}{2\hbar}} (X_j)_{fi} + i \sqrt{\frac{1}{2m \omega \hbar}} (M_j)_{fi} \right).$$

(28)

So we can also measure \(\prod_{j=1}^{n} (A_j)_m\) by measuring each \(X_j\) and \(M_j\) on each single meter.

Conclusions. – In this paper, we study the connection between the modular value of a single observable of a pre- and postselected quantum system and annihilation operator at first, and obtain their relation, following which, we propose a new method of AOM different from quantum state tomography used by Y. Kedem and L. Vaidman in their initial paper about “modular value” to measure the modular value of a single particle. Then we give the relation between modular values of multiparticle observables and annihilation operator, and find that the \(N\)-observable modular values also have a simple connection with the joint annihilation operator. Then we also give their AOM method of modular values for multiparticle observables, and it is worth noting that our new AOM method can significantly reduce the number of measurements from \(O(4^n)\) to \(O(2^n)\) compared with the original tomography method. Besides, we find the computation process in the relation between the measured values of observables of a pre- and postselected quantum system and the annihilation operator can be greatly simplified with the method of modular value.

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