Fermion pair production in QED and the backreaction problem in (1+1)-dimensional boost-invariant coordinates revisited

Bogdan Mihaila,1 John F. Dawson,2 and Fred Cooper3,4

1Los Alamos National Laboratory, Los Alamos, NM 87545
2Department of Physics, University of New Hampshire, Durham, NH 03824
3National Science Foundation, 4201 Wilson Blvd., Arlington, VA 22230
4Santa Fe Institute, Santa Fe, NM 87501

We study two different initial conditions for fermions for the problem of pair production of fermions coupled to a classical electromagnetic field with backreaction in (1+1) boost-invariant coordinates. Both of these conditions are consistent with fermions initially in a vacuum state. We present results for the proper time evolution of the electromagnetic field with backreaction in (1+1) boost-invariant coordinates revisited in Cartesian case. In those, we consider a slightly different initial state, namely approximate free fields for the fermions, which automatically leads to zero current. We study the evolution of the problem for both initial conditions and show that at short to moderate times they are equivalent and are slightly different at very late times.

This semi-classical approximation to the initial value problem describes the fermions as a quantum field but treats the electric field classically. The current used in Maxwell’s equation is calculated using the vacuum expectation value of the quantum Dirac current. As discussed in previous papers, this approximation is equivalent to the first term in a large-N approximation to N-QED where there are N flavors of fermions present.

We study here the problem in (1+1) boost-invariant coordinates. This kinematic situation is related to the kinematics of the early phase of plasma evolution following a relativistic heavy ion collision with the electric field a simplification for the semiclassical chromoelectric field expected to be produced in that situation. We study (1+1) dimensions for simplicity here, where charge renor-
The current is given by: \[ \partial_a V^\mu (x) = \frac{\partial}{\partial x^a} ( - \sinh \eta, - \sinh \eta / \tau, \cosh \eta / \tau ) . \] The γ-matrices in this frame are denoted by a tilde: \[ \hat{\gamma}^\mu (x) = \tilde{V}^\mu (x) \gamma^\mu , \] and its inverse:

\[ dx^a = V^\mu_a (x) dx^\mu , \quad \partial_a = V^\mu_a (x) \partial_\mu , \]
where we have introduced the notation \([dk] = dk/(2\pi)\). Here, \(A^{(\lambda)}_k\) are mode operators and \(\phi^{(\lambda)}_k(\tau)\) are two independent mode functions satisfying the equation:

\[
[i\gamma^0 \partial_\tau - \gamma^3\pi_k(\tau) - m] \phi^{(\lambda)}_k(\tau) = 0, \tag{2.19}
\]

where

\[
\pi_k(\tau) = \frac{1}{\tau} \left[ k - eA(\tau) \right]. \tag{2.20}
\]

It is now useful to add and subtract the upper and lower components of the spinor \(\phi^{(\lambda)}_k\) by writing:

\[
\phi^{(\lambda)}_k(\tau) = U F^{(\lambda)}_k(\tau), \quad \text{with} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.21}
\]

Here \(U^\dagger = U^{-1} = U^T\). Then \(F^{(\lambda)}_k(\tau)\) satisfies an equation of Hamiltonian form:

\[
i \partial_\tau F^{(\lambda)}_k(\tau) = H(\tau) F^{(\lambda)}_k(\tau), \tag{2.22}
\]

with

\[
H(\tau) = \begin{pmatrix} \pi_k(\tau) & m \\ m & -\pi_k(\tau) \end{pmatrix} = K_k(\tau) \cdot \bm{\sigma}. \tag{2.23}
\]

Here \(K_k(\tau)\) is a vector defined in an abstract space \(\mathcal{R}\) with unit vectors \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\) and given by:

\[
K_k(\tau) = m \hat{e}_1 + \pi_k(\tau) \hat{e}_3. \tag{2.24}
\]

We can introduce the \(2 \times 2\) dimensional density matrix \(\rho_k(\tau)\) and a “polarization” vector \(P_k(\tau)\) in \(\mathcal{R}\) with the definitions:

\[
\rho^{(\lambda)}_k(\tau) = F^{(\lambda)}_k(\tau) F^{(\lambda)*}_k(\tau) = \frac{1}{2} \left[ (1 + P^{(\lambda)}_k(\tau)) \cdot \bm{\sigma} \right]. \tag{2.25}
\]

Then from (2.22), the polarization vector \(P^{(\lambda)}_k(\tau)\) obeys the vector equation of motion:

\[
\partial_\tau P^{(\lambda)}_k(\tau) = 2 K_k(\tau) \times P^{(\lambda)}_k(\tau). \tag{2.26}
\]

Since \(H(\tau)\) in Eq. (2.23) is hermitian, \(F^{(\lambda)}_k(\tau)\) satisfies a conservation equation:

\[
\partial_\tau \left[ F^{(\lambda)}_k(\tau) F^{(\lambda)*}_k(\tau) \right] = 0. \tag{2.27}
\]

So if we choose the two spinors to be orthonormal at \(\tau = \tau_0\), they remain orthonormal for all \(\tau\). In Sec. VI we show how to do this. So we can assume that these spinors are orthonormal and complete for all \(\tau\):

\[
F^{(\lambda)}_k(\tau) F^{(\lambda)*}_k(\tau) = \delta_{\lambda,\lambda'}, \tag{2.28a}
\]

\[
\sum_{\lambda=\pm} F^{(\lambda)}_k(\tau) F^{(\lambda)*}_k(\tau) = 1. \tag{2.28b}
\]

Probability conservation also requires that the polarization vector \(P^{(\lambda)}_k(\tau)\) for both of these solutions to remain on the unit sphere for all time \(\tau\). So to summarize, the fermi field can be written as:

\[
\hat{\psi}(\tau, \eta) = S(\eta) U \hat{F}(\tau, \eta)/\sqrt{\tau}, \tag{2.29}
\]

where the field \(\hat{F}(\tau, \eta)\) obeys the anti-commutation relation:

\[
\{ \hat{F}_\alpha(\tau, \eta), \hat{F}_{\alpha'}(\tau, \eta') \} = \delta_{\alpha,\alpha'} \delta(\eta - \eta'). \tag{2.30}
\]

and is expanded in terms of the spinors \(F^{(\lambda)}_k(\tau)\) which satisfy (2.22):

\[
\hat{F}(\tau, \eta) = \int^{-\infty}_{-\infty} [dk] \sum_{\lambda=\pm} \hat{A}^{(\lambda)}_k e^{ik\eta} F^{(\lambda)}_k(\tau). \tag{2.31}
\]

We can use this orthogonality toinvert (2.31) to get:

\[
\hat{A}^{(\lambda)}_k = \int^{-\infty}_{-\infty} d\eta e^{-ik\eta} F^{(\lambda)*}_k(\tau) \hat{F}(\tau, \eta), \tag{2.32}
\]

for any time \(\tau\). Using (2.30), we then find that the mode operators \(\hat{A}^{(\lambda)}_k\) obey the anti-commutation relation:

\[
\{ \hat{A}^{(\lambda)}_k, \hat{A}^{(\lambda')}_{k'} \} = (2\pi) \delta_{\lambda,\lambda'} \delta(k - k'). \tag{2.33}
\]

It is traditional to define separate positive and negative energy operators by setting:

\[
\hat{a}^{(+)k} = \hat{a}_k, \quad \text{and} \quad \hat{a}^{(-)k} = \hat{a}^\dagger_k. \tag{2.34}
\]

We choose our initial state to be the vacuum with no particle or anti-particle present. Then:

\[
\hat{a}_k | 0 \rangle = 0, \quad \text{and} \quad \hat{b}_k | 0 \rangle = 0. \tag{2.35}
\]

This means that:

\[
( \{ \hat{A}^{(+)k}, \hat{A}^{(+)k'} \} ) = -(2\pi) \lambda \delta_{\lambda,\lambda'} \delta(k - k'), \tag{2.36}
\]

a result we will use in the next section.

### III. Maxwell’s Equation

Maxwell’s equation is given in Cartesian coordinates in Eq. (2.34) with the current given in Eq. (2.31). For our boost-invariant coordinates, Maxwell’s equation reads:

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} F^{\mu\nu}(x) \right] = J^\nu(x), \tag{3.1}
\]

where \(\sqrt{-g} = \tau\). Now \(A_\mu = (0, -A(\tau))\), so the only non-vanishing elements of the field tensor are:

\[
F_{\tau,\eta}(x) = -F_{\eta,\tau}(x) = -\partial_\tau A(\tau) \equiv \tau E(\tau) \tag{3.2}
\]

This last equation defines what we call the electric field \(E(\tau) \equiv (\partial_\tau A(\tau))/\tau\). Then using the metric \(g^{\mu\nu}(x) = \text{diag}(1, 1/\tau^2)\), we get:

\[
F^{\tau,\eta}(\tau) = -F^{\eta,\tau}(\tau) = -E(\tau)/\tau, \tag{3.3}
\]
and Maxwell’s equation becomes:

$$\partial_\tau E(\tau) = -J(\tau).$$  \(3.4\)

Here we have defined a “reduced” current \(J(\tau)\) by:

$$J(\tau) = \frac{e}{2\tau} \int_{-\infty}^{\infty} \langle [\hat{\psi}(\eta, \tau), \gamma^5(\tau) \hat{\psi}(\eta, \tau)] \rangle = \frac{e}{2\tau} \langle [\hat{\phi}(\eta, \tau), \gamma^5 \hat{\phi}(\eta, \tau)] \rangle, \quad (3.5)$$

Using the field expansion (2.18) and the expectation value (2.30) of the mode operators, we find for the reduced current:

$$J(\tau) = \frac{e}{2\tau} \int_{-\infty}^{\infty} [dk] \sum_{\lambda = \pm 1} \int_{-\infty}^{\infty} [dk'] \sum_{\lambda' = \pm 1} \times e^{i(k-k')\eta} \left[ \phi_k^{(\lambda)}(\tau) \gamma^5 \phi_{k'}^{(\lambda)}(\tau) \right] \left[ [\hat{A}_k^{(\lambda)}, \hat{A}_k'^{(\lambda')}] \right] \nonumber$$

$$= -\frac{e}{2\tau} \int_{-\infty}^{\infty} [dk] \sum_{\lambda = \pm 1} \lambda \left[ k \omega \right] \left[ \int_{-\infty}^{\infty} [dk'] \sum_{\lambda' = \pm 1} \frac{\partial}{\partial k'} \right] P_3^{(+)}(\pi_k, \tau).$$ \(3.6\)

Here we have used the completeness statement (2.28b) to write the current in terms of positive energy solutions only. In the last line, we changed integration variables from \(k\) to \(\pi_k(\tau)\), using \(d\pi_k = dk/\tau\), and defined \(P(\pi_k, \tau) \equiv P_k(\tau)\). Maxwell’s equation (3.3) becomes:

$$\partial_\tau E(\tau) = e \int_{-\infty}^{\infty} [d\pi_k] \, P_3^{(+)}(\pi_k, \tau).$$ \(3.7\)

Recall that \(P_3\) is the third component of the polarization vector in the space \(\mathcal{R}\).

\section*{IV. ADIABATIC EXPANSION}

The large momentum behavior of the solutions of the Dirac equation can be obtained by looking at the adiabatic expansion of these solutions. Perhaps the simplest way to do this is from the polarization equation (2.20). In order to count powers of time derivatives, we put: \(\partial_\tau \rightarrow \epsilon \partial_\tau\), and set:

$$P_k(\tau) = P_k^{(0)}(\tau) + \epsilon P_k^{(1)}(\tau) + \epsilon^2 P_k^{(2)}(\tau) + \cdots \quad (4.1)$$

Substitution of this into Eq. (2.20) and equating powers of \(\epsilon\) give the results:

$$P_k^{(0)} = \frac{K_k}{\omega},$$ \(4.2a\)

$$P_k^{(1)} = \frac{K_k \times K_k}{2\omega^3},$$ \(4.2b\)

$$P_k^{(2)} = \frac{3(K_k \cdot K_k)K_k - \omega^2 K_k}{4\omega^5} + \mathcal{N}_k K_k,$$ \(4.2c\)

where \(\omega = \sqrt{\pi_k^2 + m^2}\) and

$$\mathcal{N}_k = -\frac{1}{8} \pi_k^2 + \frac{1}{4} \pi_k \pi_k' - \frac{5}{8} \omega^2 \frac{\pi_k^2}{\omega^2}. \quad (4.3)$$

We have suppressed the \(\tau\) dependence here of these quantities. The dot denotes a partial derivative with respect to \(\tau\). Explicitly, we find:

$$P_1 = \frac{m^2}{\omega} + c^2 m \left( -\frac{1}{8} \frac{\pi_k^2}{\omega^5} + \frac{1}{4} \frac{\pi_k \pi_k'}{\omega^3} - \frac{5}{8} \frac{\pi_k^2}{\omega^2} \right) + \cdots$$

$$P_2 = \epsilon m \frac{\pi_k}{\omega} + \cdots$$

$$P_3 = \frac{\pi_k}{\omega} - c^2 m^2 \left( \frac{1}{8} \frac{\pi_k}{\omega^5} - \frac{5}{8} \frac{\pi_k^2}{\omega^2} \right) + \cdots \quad (4.4)$$

So setting \(\epsilon \rightarrow 1\), Maxwell’s equation (3.3) becomes:

$$\mathcal{E}(\tau) = e \int_{-\infty}^{\infty} \langle [\pi_k, -m \pi_k + 5 \pi_k^2] \rangle \quad (4.5)$$

All terms odd in \(\pi_k\) vanish by symmetric integration. After integration, Eq. (4.5) becomes:

$$\mathcal{E}(\tau) = -\frac{e^2}{6\pi m^2} \mathcal{E}(\tau) + J_{\text{sub}}(\tau). \quad (4.6)$$

Here, the first term corresponds to finite charge renormalization in (1+1) dimensions and can be brought over to the left hand side of the equation. The current \(J_{\text{sub}}(\tau)\) is explicitly finite by power counting and is initially zero.

An adiabatic expansion of the Dirac equation can also be carried out from solutions of the second-order form of the Dirac equation. In Section VIB below, we show that this gives the same result as in Eqs. (4.4).

\section*{V. ENERGY-MOMENTUM TENSOR}

In the boost-invariant coordinate system, the average value of the total energy-momentum tensor is given by Eqs. (4.1) and (4.2) of Ref. 6, and is the sum of two terms:

$$T_{\mu\nu} = T_{\mu\nu}^{\text{matter}} + T_{\mu\nu}^{\text{field}} = \text{diag}(E, \tau^2 P), \quad (5.1)$$

where

$$T_{\mu\nu}^{\text{matter}} = \frac{1}{4} \langle [\hat{\psi}(x), \gamma_{\mu}(x) (i D_{\nu}) \hat{\psi}(x)] \rangle + \text{h.c.} \quad (5.2a)$$

$$T_{\mu\nu}^{\text{field}} = g_{\mu\nu} \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + F_{\mu\alpha} g^{\alpha\beta} F_{\beta\nu}. \quad (5.2b)$$

Here \(D_\nu = \partial_\nu + i e A_\mu(x)\) and the subscript notation \((\mu, \nu)\) means to symmetrize the term. From our definitions in Section III and Eq. (3.2), the field part of the energy-momentum tensor is given by:

$$T_{\mu\nu}^{\text{field}} = \text{diag}(E^2/2, -\tau^2 E^2/2) + \mathcal{N}_k K_k. \quad (5.3)$$

We denote the matter part of the energy-momentum tensor as:

$$T_{\mu\nu}^{\text{matter}} = \text{diag}(\varepsilon, \tau^2 P). \quad (5.4)$$
For the matter field, we first note that \( D_\nu \hat{\psi}(x) = S(x) \nabla_\nu \hat{\phi}(x)/\sqrt{\tau} \), where \( \nabla_\nu \) is the covariant derivative defined below Eq. (2.11). For the \( T_{\tau \tau} = \varepsilon(\tau) \) component, \( \nabla_\tau = \partial_\tau \), and using the field expansion (2.19), Eqs. (2.21), (2.22), and (2.23), we find:

\[
\varepsilon(\tau) = -\frac{1}{2\tau} \int_{-\infty}^{+\infty} \left[ \frac{d\pi_k}{d\pi_k} \right] \sum_\lambda \lambda \text{Tr}[\rho_k^{(\lambda)}(\tau) H(\tau)]
\]

\[
+ \frac{1}{\tau} \int_{-\infty}^{+\infty} \sum_\lambda \lambda \pi_k(\tau) \text{Tr}[\rho_k^{(+)}(\tau) \sigma_3]
\]

\[
= -\frac{1}{\tau} \int_{-\infty}^{+\infty} \sum_\lambda \lambda \pi_k(\tau) \text{Tr}[\rho_k^{(+)}(\tau) \sigma_3] + \frac{E^2}{2}.
\]

(5.5)

So from (5.1),

\[
E = -\frac{1}{2\tau} \int_{-\infty}^{+\infty} [d\pi_k] K(\pi_k) \cdot P(\pi_k, \tau) + \frac{E^2}{2},
\]

(5.7a)

\[
P = -\int_{-\infty}^{+\infty} [d\pi_k] \pi_k P_3^{(+)}(\pi_k, \tau) - \frac{E^2}{2}.
\]

(5.7b)

The covariant derivative of the energy-momentum tensor in boost-invariant coordinates is conserved:

\[
T^{\mu \nu} = \partial_\mu T^{\mu \nu} + \Gamma^{\mu \rho \sigma} T^{\rho \sigma \nu} + \Gamma^{\mu \sigma \nu} T^{\rho \sigma} = 0.
\]

(5.8)

The Christoffel symbols are defined by: \( \Gamma^{\lambda}_{\mu \nu}(x) = V^\lambda a(x) (\partial_\mu V^a_\nu(x)) \). In our case, the non-vanishing symbols are given by:

\[
\Gamma^{\tau}_{\eta \eta} = \tau, \quad \Gamma^{\eta}_{\tau \eta} = \Gamma^{\eta}_{\eta \tau} = 1/\tau.
\]

(5.9)

So we find that

\[
\partial_\tau T^{\tau \tau} + T^{\tau \tau} / \tau + T^{\eta \eta} = 0,
\]

(5.10)

or

\[
\partial_\tau (\tau E) + P = 0.
\]

(5.11)

Using the equation of motion (2.20) and Maxwell’s equation (1.3), one can show that Eq. (5.11) is automatically satisfied.

Using Eqs. (4.9), the adiabatic expansion for the energy is given by:

\[
E = \left(1 + \frac{e^2}{6\pi m^2}\right) \frac{E^2}{2} + \frac{1}{24\pi^2} \tau^2 - \int_{-\infty}^{+\infty} [d\pi_k] 2 \omega + \cdots
\]

(5.12)

We recognize the first term as a finite renormalization of the charge, the second term as a renormalization of the cosmological constant, and the third term as a sum of the zero point energies of pairs of particles and anti-particles with energy \( \omega(\pi_k) \). We subtract these terms from the calculation of the energy and arrive at a finite energy \( E_{\text{sub}} \) given by:

\[
E_{\text{sub}} = \frac{E^2}{2} + \int_{-\infty}^{+\infty} [d\pi_k] \left[ -K(\pi_k) \cdot \mathbf{p}(\pi_k, \tau) + \omega - \frac{\pi_k^2}{\omega^0} \right]
\]

(5.13)

For the pressure, the adiabatic expansion gives:

\[
P_{\text{sub}} = -\frac{E^2}{2} + \int_{-\infty}^{+\infty} [d\pi_k] \left[ -\pi_k p_3(\pi_k, \tau) + \frac{\pi_k^2}{\omega} - m^2 \left( \frac{1}{4} \omega^0 - \frac{5}{8} \frac{\pi_k^2}{\omega^0} \right) \right]
\]

(5.14)

Eqs. (5.13) and (5.15) are now finite.

VI. INITIAL CONDITIONS

The simplest choice of initial conditions is to find approximate free-field solutions of Eq. (6.1) near \( \tau = \tau_0 \). This strategy was used in Ref. [8], and automatically provides a zero current at \( \tau = \tau_0 \). We call this the “one-field” method, and is discussed in Section VI A below. In previous studies of the backreaction problem by Cooper et al. [8] adiabatic initial conditions were used which required averaging over two different solutions to the Dirac equation to obtain an zero current at initial proper time \( \tau_0 \). We call this the “two-field” method. We discuss this method in Section VI B.

A. One-field method

At \( \tau = \tau_0 \equiv 1/m, A(\tau_0) = 0 \) and \( H(\tau_0) \) is given by:

\[
H(\tau_0) = m \left( \begin{array}{cc} k & 1 \\ 1 & -k \end{array} \right)
\]

(6.1)

So at \( \tau \approx \tau_0 \), \( F_k(\tau) \) obeys the approximate equation of motion:

\[
i \partial_\tau F_{0;k}(\tau) = H(\tau_0) F_{0;k}(\tau).
\]

(6.2)

Writing

\[
F_{0;k}(\tau) = \tilde{F}_{0;k} e^{-i\omega(\tau-\tau_0)},
\]

(6.3)
We find that \( \omega(\tau_0) = \pm \omega_0 \), where \( \omega_0 = m \sqrt{k^2 + 1} \). Positive frequency solutions given by:

\[
\hat{F}_{0;k}^{(+)}(\tau) = \sqrt{\frac{\omega_0 + mk}{2\omega_0}} \begin{pmatrix} 1 \\ \cos(\theta_k/2) \\ \sin(\theta_k/2) \end{pmatrix},
\]

and negative frequency solutions by:

\[
\hat{F}_{0;k}^{(-)}(\tau) = \sqrt{\frac{\omega_0 + mk}{2\omega_0}} \begin{pmatrix} -\zeta \\ -\sin(\theta_k/2) \\ \cos(\theta_k/2) \end{pmatrix},
\]

with \( \zeta = m/(\omega_0 + mk) \). Here \( \sin \theta_k = 1/\sqrt{k^2 + 1} \) and \( \cos \theta_k = k/\sqrt{k^2 + 1} \), with \( 0 \leq \theta_k \leq \pi \). Density matrices for these solutions are given by:

\[
\rho_k^{(+)} = F_{0;k}^{(+)}(\tau) F_{0;k}^{(+*)}(\tau) \quad (6.6a)
\]

\[
= \begin{pmatrix} \cos^2(\theta_k/2) & \sin(\theta_k/2) \cos(\theta_k/2) \\ \sin(\theta_k/2) \cos(\theta_k/2) & \sin^2(\theta_k/2) \end{pmatrix},
\]

\[
\rho_k^{(-)} = F_{0;k}^{(-)}(\tau) F_{0;k}^{(-*)}(\tau) \quad (6.6b)
\]

\[
= \begin{pmatrix} \sin^2(\theta_k/2) & -\sin(\theta_k/2) \cos(\theta_k/2) \\ -\sin(\theta_k/2) \cos(\theta_k/2) & \cos^2(\theta_k/2) \end{pmatrix},
\]

and are independent of \( \tau \). The corresponding polarization vectors are also independent of \( \tau \) and are given by:

\[
\mathbf{F}_{0;k}^{(+)} = \sin \theta_k \mathbf{e}_1 + \cos \theta_k \mathbf{e}_3 = \frac{K_k(\tau_0)}{\omega_0} = -\mathbf{F}_{0;k}^{(-)} \quad (6.7)
\]

The initial spinors are orthogonal and complete:

\[
F_{0;k}^{(\lambda)}(\tau) F_{0;k}^{(\lambda')\dagger}(\tau) = \delta_{\lambda,\lambda'}, \quad (6.8a)
\]

\[
\sum_{\lambda = \pm} F_{0;k}^{(\lambda)}(\tau) F_{0;k}^{(\lambda)\dagger}(\tau) = 1. \quad (6.8b)
\]

So if we set \( F_{0;k}^{(\lambda)}(\tau_0) = \hat{F}_{0;k}^{(\lambda)} \) at \( \tau = \tau_0 \), then the exact solutions remain orthogonal and complete for all \( \tau \) and Eq. (2.25) is satisfied. As we have seen in Section III, only the positive energy solutions are needed for the backreaction calculation.

The initial spinors can serve to define a particle number operator. Since these initial mode functions form a complete set, we can expand the quantum field in terms of them:

\[
\hat{F}_\alpha(\tau, \eta) = \int_{-\infty}^{+\infty} [dk] \sum_{\lambda} \hat{A}_{0;k}^{(\lambda)}(\tau) e^{ik\eta} F_{0;\alpha;k}^{(\lambda)}(\tau), \quad (6.9)
\]

from which we obtain the equal time anti-commutation relation:

\[
\{ \hat{A}_{0;k}^{(\lambda)}(\tau), \hat{A}_{0;k'}^{(\lambda')}(\tau) \} = 2(\pi) \delta_{\lambda,\lambda'} \delta(k - k'). \quad (6.11)
\]

Inserting the expansion (2.31) into the right-hand-side of Eq. (6.10), we can relate the \( \hat{A}_{0;k}^{(\lambda)}(\tau) \) mode operators to the \( \hat{A}_{k}^{(\lambda)} \) mode operators. We find:

\[
\hat{A}_{0;k}^{(\lambda)}(\tau) = \int_{-\infty}^{+\infty} dz \sum_{\alpha} e^{-i\eta z} F_{0;\alpha;k}^{(\lambda)*}(\tau) \hat{F}_\alpha(\tau, \eta), \quad (6.10)
\]

\[
\hat{A}_{k}^{(\lambda)}(\tau) \quad ( \lambda = \pm) \quad (6.12)
\]

where

\[
C_{k}^{(\lambda,\lambda')}(\tau) = F_{0;k}^{(\lambda)}(\tau) F_{k}^{(\lambda')}(\tau). \quad (6.13)
\]
Particles are defined in reference to these initial states where a clear distinction between particles and anti-particles can be made. We define an average phase space number density $n_k(\tau)$ by:

$$ n_k(\tau) = \frac{d^2 N(\tau)}{dk d\eta}, \quad (6.14) $$

and is computed using the relation:

$$ n_k(\tau) (2\pi) \delta(k - k') = \langle \hat{A}_{\lambda}^{(+)}(\tau) \hat{A}_{\lambda'}(\tau) \rangle. \quad (6.15) $$

Inserting (6.12) into (6.15), and using

$$ \langle \hat{A}_{k}^{(\lambda)}(\tau) \hat{A}_{k'}^{(\lambda')} \rangle = \delta_{\lambda,\lambda'} (2\pi) \delta(k - k'), \quad (6.16) $$

we find:

$$ n_k(\tau) = \vert C_k^{(+)}(\tau) \vert^2 = \vert F_{0:k}^{(+)\dagger}(\tau) F_{k}^{(-)}(\tau) \vert^2 $$

$$ = 1 - \vert F_{0:k}^{(+)}(\tau) F_{k}^{(+)}(\tau) \vert^2 $$

$$ = 1 - \text{Tr}[\rho_{0:k}(\tau) \rho_{k}(\tau) \tau] $$

$$ = \frac{1}{2} \left[ 1 - \mathbf{P}_{0:k}^{(+)} \mathbf{P}_{k}^{(+)}(\tau) \right]. \quad (6.17) $$

We see immediately that $n_k(\tau_0) = 0$ at $\tau = \tau_0$.

We note that in the one-field method the current is automatically zero at $\tau = \tau_0$: Eq. (3.7) with

$$ P_{\frac{3}{2}}^{(+)}(\tau_0) = \frac{K_{3}(\tau_0)}{\omega_0} = \frac{\pi_k}{\omega_0}, \quad (6.18) $$

leads to a zero current because the integrand in Eq. (3.7) is odd in $\pi_k$. Furthermore, one of the subtleties of the one-field method is that the zero-current point is an unstable equilibrium point. This is most easily seen from the equation of motion, Eq. (2.26), of the polarization vector. For $\tau = \tau_0$, we find that

$$ \frac{\partial}{\partial \tau} \mathbf{P}_{k}^{(+)}(\tau_0) = 2 \mathbf{K}_k(\tau_0) \times \mathbf{P}_{k}^{(+)}(\tau_0) $$

$$ = 2 \mathbf{K}_k(\tau_0) \times \mathbf{K}_k(\tau_0)/\omega_0 = 0. \quad (6.19) $$

However the second derivative is not zero:

$$ \frac{\partial^2}{\partial \tau^2} \mathbf{P}_{k}^{(+)}(\tau_0) = \frac{2m^2}{\sqrt{k^2 + 1}} \left( k - eE_0/m^2 \right) \hat{e}_y. \quad (6.20) $$

**B. Two field method**

Here we start from solutions of the second-order Dirac equation. Writing the spinor $\tilde{F}_k(\tau)$ in the form:

$$ F_k^{(+)}(\tau) = \begin{pmatrix} f_{k,s}^{(+)}(\tau) \\ f_{k,-s}^{(+)}(\tau) \end{pmatrix}, \quad (6.21) $$

from Dirac’s Eq. (2.22), we can find a second-order equation for either the upper or lower component:

$$ \{ \partial^2_\tau + \omega^2(\tau) - i s \hat{\pi}_k(\tau) \} f_{k,s}^{(+)}(\tau) = 0, \quad (6.22) $$

where $s = \pm 1$ designates the upper or lower component. A parametrization of these mode functions of the form:

$$ f_{k,s}^{(+)}(\tau) = \frac{\tilde{A}_{k,s}^{(+)}(\tau)}{\sqrt{2 \Omega_{k,s}^{(+)}(\tau)}} \times \exp \left\{ -i \int_{\tau_0}^{\tau} \left[ \Omega_{k,s}^{(+)}(\tau') - s \frac{i \hat{\pi}_{k}(\tau')}{2 \Omega_{k,s}^{(+)}(\tau')} \right] d\tau' \right\}, \quad (6.23) $$

leads to a second-order nonlinear equation for $\Omega_{k,s}^{(+)}(\tau)$ given by:

$$ \frac{1}{2} \Omega_{k,s}^{(-)} - \frac{3}{4} \left[ \Omega_{k,s}^{(-)} \right]^2 + \frac{1}{2} \frac{s \hat{\pi}_{k}(\tau)}{\Omega_{s}^{(-)}} - \frac{1}{4} \left[ \frac{s \hat{\pi}_{k}(\tau)}{\Omega_{s}^{(-)}} \right]^2 - \frac{s \hat{\pi}_{k}(\tau)}{\Omega_{s}^{(-)}} \Omega_{s}^{(-)} \Omega_{s}^{(-)} + \Omega_{s}^{(-)} = \omega^2. \quad (6.24) $$
Here, and in the following, we suppress the dependencies on $\tau$, $k$, and the positive energy superscript. Solutions of the nonlinear equation (6.23) for $\Omega_s$, subject to initial conditions given below, completely determine $f_s$. Once we find $f_s$, we can get the other Dirac component from Dirac’s equation:

$$f_{-s} = \frac{1}{m} \left( i \partial_\tau + s \pi_k \right) f_s = \frac{Z_s}{m} f_s,$$  

(6.25)

where $Z_{k,s}^{(+)}(\tau)$ is given by:

$$Z_s = X_s + i Y_s = \Omega_s + s \pi_k - i \frac{\dot{\Omega}_s + s \pi^s_k}{2 \Omega_s},$$  

(6.26)

The normalization requirement: $\sum_s |f_s|^2 = 1$ means that:

$$|f_s|^2 = \frac{m^2}{m^2 + |Z_s|^2}, \quad |f_{-s}|^2 = \frac{|Z_s|^2}{m^2 + |Z_s|^2},$$  

(6.27)

which fixes the normalization factor $A_s$. It is an easy matter now to get all the terms of the density matrix $\rho_s$, and we find:

$$P_{1,s} = \frac{2 X_s}{m^2 + |Z_s|^2},$$  

(6.28a)

$$P_{2,s} = \frac{2 Y_s}{m^2 + |Z_s|^2},$$  

(6.28b)

$$P_{3,s} = \frac{m^2 - |Z_s|^2}{m^2 + |Z_s|^2}.$$  

(6.28c)

We are now in a position to carry out an adiabatic expansion of the nonlinear equation (6.24). We again count derivatives with respect to $\tau$ by putting: $\partial_\tau \leftrightarrow \epsilon \partial_\tau$, and expand

$$\Omega_s = \Omega_s^{(0)} + \epsilon \Omega_s^{(1)} + \epsilon^2 \Omega_s^{(2)} + \cdots$$  

(6.29)

Inserting this into (6.24) and inverting the equation gives $\Omega_s^{(0)} = \omega$ and $\Omega_s^{(1)} = 0$, from which we find:

$$\Omega_s^{(2)} = \frac{2}{\epsilon^2 \omega^2} \left[ \frac{1}{2} \frac{\pi^2_k}{\omega^2} + \frac{5}{8} \frac{\pi^2_k}{\omega^7} \right].$$  

(6.30)

From this we find that:

$$Z_s = X_s + i Y_s = \left( \omega - s \pi_k \right) \left[ 1 + i \epsilon \frac{s \pi_k}{2 \omega^2} + \epsilon^2 \frac{\Omega_s^{(2)}}{(\omega - s \pi_k)} + \cdots \right].$$  

(6.31)

So from our general expressions (6.28), it is easy to show that:

$$P_{1,s} = \frac{m}{\omega} + \epsilon^2 m \left( - \frac{1}{2} \frac{\pi^2_k}{\omega^5} + \frac{5}{8} \frac{\pi^2_k}{\omega^7} \right) + \cdots,$$

$$P_{2,s} = \epsilon m \frac{s \pi_k}{2 \omega^3} + \cdots,$$

$$P_{3,s} = \frac{s \pi_k}{\omega} - \epsilon^2 s m^2 \left( \frac{1}{4} \frac{\pi^2_k}{\omega^5} - \frac{5}{8} \frac{\pi^2_k}{\omega^7} \right) + \cdots.$$  

(6.32)

For $s = 1$, Eqs. (6.32) are in agreement with Eqs. (6.14). So to second adiabatic order $P_{1,s}$ is independent of $s$, but $P_{2,s}$ and $P_{3,s}$ change sign with $s$.

To specify the initial conditions for second-order nonlinear Eq. (6.24) at $\tau = \tau_0 = 1/m$ one needs two initial conditions. Since the vacuum state is not unique when particles are being produced, one usually chooses some approximate adiabatic vacuum state of given order as discussed in Ref. 11. The authors in Ref. 6 chose the first-order adiabatic conditions as:

$$\Omega_{k,s}^{(+)}(\tau_0) = \omega_m = m \sqrt{k^2 + 1},$$  

(6.33a)

$$\dot{\Omega}_{k,s}^{(+)}(\tau_0) = \omega_{m} = m^2 \frac{k (E_0 - k)}{\sqrt{k^2 + 1}},$$  

(6.33b)

where $E_0 = \epsilon E_0/m^2$. The initial conditions are independent of $s$.

Using Eq. (6.32), we obtain that for each value of $s$ this choice of initial conditions at $\tau = \tau_0$ will lead to a non-vanishing current, $J_s(\tau_0)$. However, if we average over the two sets of solutions $s = \pm$ and choose for the Maxwell equation:

$$\partial_\tau E(\tau) = \frac{e}{2} \int_{-\infty}^{+\infty} [d\pi_k] \left( P_{3,s}^{(+)}(\pi_k, \tau) + P_{3,s}^{(-)}(\pi_k, \tau) \right),$$  

(6.34)

then the renormalized Maxwell equation will start with a zero value for the current.
FIG. 4: (Color online) Proper-time evolution of the momentum dependent particle density distribution, \(n_{\pi_k}\) defined in Eq. (6.14), showing the oscillation of the centroid of the particle-density distribution between positive and negative values of \(\pi_k\). Here, we show results for the one-field method. Results for the two-field initial conditions scenario (not shown) are very similar, as it is to be expected from the results illustrated in Fig. 3.

VII. NUMERICAL RESULTS

We have performed numerical calculations for both sets of initial conditions described above. We employed a fourth-order Runge-Kutta method to solve the coupled Dirac equation and backreaction problem. The \(k\)-momentum variable, which is dimensionless, was discretized on a nonuniform piece-wise momentum grid with a cutoff at \(k = \Lambda_k\). We found that a value of \(\Lambda_k \approx 200\) was necessary to obtain numerical results insensitive with respect to the cutoff. For the purpose of calculating the subtracted values of the current \(J(\tau)\), matter energy \(\varepsilon(\tau)\), matter pressure \(p(\tau)\), and fermion particle density \(dN(\tau)/d\eta\), we needed to compute the momentum integrals with respect to the variable, \(\pi_k\) rather than \(k\). The corresponding momentum cutoff in \(\pi_k\)-space was chosen to be 20\% greater than \(\tau_{\text{max}}\Lambda_k\) to allow for possible very large values of \(A(\tau)\), which is unknown at the beginning of the calculation. The momentum integrals in \(\pi_k\)-space were performed using a Chebyshev integration method with spectral convergence \[12\]. Using the procedure outlined here, we found that approximately 8000 mode functions were necessary to obtain a converged numerical result. The conservation of the energy-momentum tensor, see Eq. (5.8), served as a numerical test: we found that the renormalized energy-momentum tensor was conserved within machine precision.

For the purpose of this comparison, we took: \(m = 1\), \(e = 1\), \(\tau_0 = 1/m = 1\), \(A(\tau_0) = 0\), and \(E(\tau_0) = 4\). These strong-field initial conditions have been shown to produce sufficient fermion pairs at \(\tau = \tau_0\) for plasma oscillations to take place. In Fig. 4, we show the proper-time evolution of the electromagnetic field, \(A(\tau)\), electric field, \(E(\tau)\), and current, \(J(\tau)\), for the one-field and two-field methods described in text. The components of the matter part of the energy-momentum tensor, \(\varepsilon(\tau)\) and \(p(\tau)\),
for the two simulations are shown in Fig. 2. Finally, the proper-time evolution of the particle density, $dN/d\eta$, defined in Eq. (6.14), is given in Fig. 3. For both methods, the ratio $\tau\varepsilon(\tau)/[dN/d\eta]$ is seen to oscillate around the numerical value of 1, consistent with the hydrodynamical picture, as explained in Ref. 6. We notice that the two sets of solutions are almost identical at short and intermediate times. The two solutions become out of phase at late times due to the slightly different initial conditions. However, in the real problem we expect that interactions between the fermions would eliminate these oscillations.

The proper-time evolution of the momentum-dependent particle-density distribution, $n_{\pi_k}$, corresponding to the choice of initial conditions in the one-field method, is shown in Fig. 4. We note that the centroid of the particle-density distribution oscillates between positive and negative values of $\pi_k$. The oscillation of the number density is a result of the current oscillating in sign, the current in momentum space being related to the number density times the velocity of light. This effect is also seen classically when two infinite oppositely charged parallel plates initially a finite distance apart are released and allowed to pass through one another. In that case both the current and electric field oscillate in an analytically derivable manner [12]. Results for the case of the two-field method (not shown) are very similar, as it is to be expected from the results depicted in Fig. 3 (see also Ref. [14]).

VIII. CONCLUSIONS

To conclude, in this paper we report an initial-conditions sensitivity study for the problem of pair production of fermions coupled to a “classical” electromagnetic field with backreaction in (1+1) boost-invariant coordinates. We discuss two methods of choosing the initial conditions which are consistent with having the fermions in a “vacuum state.” We conclude that the two methods of starting out the calculation produce essentially the same answer. Based on our numerical simulations, there seems to be little reason theoretically or otherwise to use the two-field method discussed previously in Ref. 6, as it doubles the storage requirements and computational time. This is important for our forthcoming studies of fermion particle production with backreaction in QED and QCD.

We emphasize here that in the case of the squared Dirac equation (two-field method) there are two independent solutions of the second-order differential equation for the mode functions, each of which provides a basis for two different fermi fields. In order to make compatible the physical requirement that the initial current is zero with the initial choice that the fermions were initially chosen to be a first-order adiabatic vacuum state, the authors of Ref. 6 simply averaged these two solutions to produce a current which was zero at $\tau = \tau_0$. So doubling the number of fermi fields allows one to produce consistent initial conditions if we define the current by averaging over the two sets of solutions. By staying with the original first-order Dirac equation, in the one-field method we were able to satisfy the initial condition of zero current by choosing a slightly cruder initial state for the fermion fields. This choice, however, reduces by half the size and duration of the calculation.

Acknowledgments

This work was performed in part under the auspices of the United States Department of Energy. The authors would like to thank the Santa Fe Institute for its hospitality during the completion of this work.

[1] J. Schwinger, Phys. Rev. 82, 664 (1951).
[2] G. V. Dunne, in From Fields to Strings: Circumnavigating Theoretical Physics, edited by M. Shifman, A. Vainshtein, and J. Wheater (Singapore; River Edge, N.J., 2005), pp. 445–522, hep-th/0406216.
[3] F. Cooper, J. F. Dawson, and B. Mihaila (2008), arXiv:0806.1249 [hep-ph].
[4] F. Cooper, G. Frye, and E. Schonberg, Phys. Rev. D 11, 192 (1975).
[5] J. D. Bjorken, Phys. Rev. D 27, 140 (1983).
[6] F. Cooper, J. M. Eisenberg, Y. Kluger, E. Mottola, and B. Svetitsky, Phys. Rev. D 48, 190 (1993), hep-ph/9212206.
[7] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper, and E. Mottola, Phys. Rev. D 45, 4659 (1992).
[8] F. Cooper and V. M. Savage, Phys. Lett. B 545, 307 (2002), hep-ph/0208057.
[9] F. Cooper, S. Habib, Y. Kluger, E. Mottola, J. P. Paz, and P. R. Anderson, Phys. Rev. D 50, 2848 (1994), hep-ph/9405352.
[10] B. Mihaia, J. F. Dawson, and F. Cooper, Phys. Rev. D 74, 036006 (2006), hep-ph/0608156.
[11] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge University Press, Cambridge, England, 1984).
[12] B. Mihaia and I. Mihaia, J. Phys. A: Math. Gen. 35, 731 (2002), physics/9901005.
[13] C. M. Bender and F. Cooper (1989), IA-UR-89-2693 (available from http://www.slac.stanford.edu/spires).
[14] Animations of the proper-time evolution of the momentum-dependent particle-density distribution from simulations using the one-field and two-field methods, respectively, can be found at http://einstein.unh.edu/BogdanMihaila/qed-1+1/.