Abstract

The likelihood function plays a pivotal role in statistical inference; it is adaptable to a wide range of models and the resultant estimators are known to have good properties. However, these results hinge on correct specification of the data generating mechanism. Many modern problems involve extremely complicated distribution functions, which may be difficult – if not impossible – to express explicitly. This is a serious barrier to the likelihood approach, which requires not only the specification of a distribution, but the correct distribution. Non-parametric methods are one way to avoid the problem of having to specify a particular data generating mechanism, but can be computationally intensive, reducing their accessibility for large data problems. We propose a new approach that combines multiple non-parametric likelihood-type components to build a data-driven approximation of the true function. The new construct builds on empirical and composite likelihood, taking advantage of the strengths of each. Specifically, from empirical likelihood we borrow the ability to avoid a parametric specification, and from composite likelihood we utilize multiple likelihood components. We will examine the theoretical properties of this composite empirical likelihood, both for purposes of application and to compare properties to other established likelihood methods.

1 Introduction

Likelihood functions are a very flexible and powerful approach for estimation of model parameters. The asymptotic properties of likelihood functions, as shown by Wilks (1938), allow us to perform inferential tests even when the distribution of the statistics describing the parameters is unknown. The classical likelihood function (which we will refer to as the Fisher likelihood) requires exact specification of the probability function $f(Z; \theta)$. In most applications the true data distribution is unknown, so an assumption must be made. In some cases where the data distribution can be described, the likelihood function is still impossible to express mathematically due to the complexity of the probability density function. There are many alternatives to the Fisher likelihood; here we focus on two such methods: the composite likelihood and the empirical likelihood.

Composite likelihood (Lindsay, 1988) appropriately combines conditional and marginal densities in order to construct an approximation to the Fisher likelihood. To demonstrate a basic example of the approach let $X$ and $Y$ be random vectors in $\mathbb{R}^1$ from some joint distribution $F$. Denote the density function as $f$ and the likelihood function as $L$. We can construct the following composite
likelihood objects:

\[ \mathcal{L}_{CM} = f_X(\theta; x)f_Y|X(\theta; y, x) = \mathcal{L}_X(\theta)\mathcal{L}_Y|X(\theta), \]
\[ \mathcal{L}_{MM} = f_X(\theta; x)f_Y(\theta; y) = \mathcal{L}_X(\theta)\mathcal{L}_Y(\theta), \]
\[ \mathcal{L}_{CC} = f_{X|Y}(\theta; x,y)f_{Y|X}(\theta; y,x) = \mathcal{L}_{X|Y}(\theta)\mathcal{L}_{Y|X}(\theta). \]

Each composite likelihood is the product of proper likelihoods derived from the data. \( \mathcal{L}_{CM} \) rewrites the true likelihood as the product of a conditional and marginal. \( \mathcal{L}_{MM} \) and \( \mathcal{L}_{CC} \), however, are only equivalent to the true likelihood if \( X \) and \( Y \) are independent. The theoretical properties and justifications for specific forms are explored in Lindsay (1988). Besag (1974) introduces a multidimensional form of \( \mathcal{L}_{CC} \), which he calls the pseudolikelihood, to create an approximation to a likelihood for spatial lattice data. \( \mathcal{L}_{MM} \) is commonly referred to as the independence likelihood and only permits inference on marginal parameters. The theoretical properties of independence likelihoods have been explored by Cox & Reid (2004) and Varin (2008). A general overview on composite likelihoods can be found in Varin et. al. (2011).

Empirical likelihood is a nonparametric approach for the estimation of the likelihood function from the data. The general form and first-order asymptotics of the empirical likelihood are explored by Qin & Lawless (1994). Under mild regularity conditions empirical likelihoods inherit the asymptotic properties of the Fisher likelihood (Wilks, 1938; Qin & Lawless, 1994). Empirical likelihood generally permits a Bartlett correction (DiCiccio et. al., 1991); an exception is explored in Lazar & Mykland (1999). Empirical likelihood has been applied to univariate data (Owen, 1988), bivariate data (Owen, 1990), generalized linear models (Kolaczyk, 1994) and many other settings. The main drawback of empirical likelihood is computational; in all but the simplest cases (such as inference on a single mean parameter), empirical likelihood does not result in a closed form solution, hence computation of estimators and confidence intervals is non-trivial and time consuming.

Desirable features of any statistical method are robustness, flexibility and computational simplicity. A defining property of any likelihood method is its wide applicability. Our proposed method, which combines the composite and empirical approaches, does not require any distributional assumptions like the empirical likelihood and maintains the flexibility of construction seen with composite likelihoods. We call this construct composite empirical likelihood since we are building a composite likelihood using empirical likelihoods.

We show that under mild conditions the asymptotic distribution of this construction inherits the asymptotic properties seen in both empirical likelihood and parametric likelihood. These results place the composite empirical likelihood as a general case of many existing likelihood methods.

2 Definition of Composite Empirical Likelihood

Before defining the composite empirical likelihood, we first formally define composite likelihood and empirical likelihood.

*Definition 1.* Let \( z_1, \ldots, z_n \) be a \( k \)-variate sample from some distribution \( F_0 \). We define conditional or marginal events for which the likelihood \( \mathcal{L}_j(\theta) \) can be written for \( j = 1, \ldots, J \), with the
The composite likelihood is
\[
L_C(\theta) = \prod_{j=1}^{J} \{L_j(\theta)\}^{w_j}
\]
where \(w_j\) is a predetermined weight.

If \(w_j\) is equal for all \(j\) the weights can be ignored for purposes of maximization. The Fisher likelihood can also be viewed as a specific case of \(L_C(\theta)\). If the true probability function is \(f_0\), we then have
\[
L_j(\theta) = L_0(\theta) = f_0(z, \theta).
\]

**Definition 2.** Let \(z_1, \ldots, z_n\) be \(k\)-variate independent identically distributed observations from some distribution \(F_0\). The empirical likelihood function is
\[
L(F) = \prod_{i=1}^{n} dF(z_i) = \prod_{i=1}^{n} Pr(Z = z_i) = \prod_{i=1}^{n} u_i.
\]

The empirical likelihood function is maximized by the empirical distribution function
\[
L(F_n) = \prod_{i=1}^{n} n^{-1},
\]
so the empirical likelihood ratio function \(R(F) = L(F)/L(F_n)\) can be written as
\[
R(F) = \prod_{i=1}^{n} nu_i.
\]

Suppose now we are interested in the estimation of a \(p \times 1\) parameter \(\theta\). We add additional constraints in the form of \(r \geq p\) unbiased estimating equations \(g_j(z, \theta)\) for \(j = 1, \ldots, r\). The profile empirical likelihood ratio function is
\[
R_E(\theta) = \sup_u \left( \prod_{i=1}^{n} nu_i \mid u_i \geq 0, \sum_{i=1}^{n} u_i = 1, \sum_{i=1}^{n} u_i g(z_i, \theta) = 0 \right).
\]

Provided that \(\theta\) is inside the convex hull of the points \(z_1, \ldots, z_n\) a unique value of Equation (1) exists (Owen, 1988). By definition \(R_E(\theta) = 0\) for all \(\theta\) not inside the convex hull.

The estimators using both composite likelihood and empirical likelihood are the parameter values which maximize the respective likelihood functions, and both estimators are asymptotically normal (Varin et. al., 2011; Qin & Lawless, 1994). Furthermore the asymptotic distribution of the test statistic for hypothesis testing is \(\chi^2\) using empirical likelihood and a weighted \(\chi^2\) using composite likelihood (see Qin & Lawless, 1994 for empirical likelihood and Varin et. al., 2011 for composite likelihood).

To define the composite empirical likelihood let \(Z \in \mathbb{R}^k\) be from some distribution \(F_0\). Define \(Z_{.j}\) as a (univariate or multivariate) subset of \(Z\) for \(j = 1, \ldots, J\). We will assume that all \(Z_{.j}\) come from some distribution \(F_j\). For simplicity we will assume the sample sizes of each likelihood component are equal so \(n_j = n\) for all \(j\).
Define $g_j$ for $j = 1, \ldots, J$ as the estimating equations for each subset and define the parameters as $\theta_j$ having dimension $p_j \times 1$. The dimension of the parameter $\theta$ is $p = \sum_{j=1}^{J} p_j$, and finally assume $r_j \geq p_j$ for all $j$, where $r_j$ is the dimension of $g_j$. For each subset $j$ the component empirical likelihood is

$$L^{(j)}_E(\theta) = \left( \sup_{u_j} \prod_{i=1}^{n} u_{i,j} \left| \sum_{i=1}^{n} u_{i,j} g_j(z_{i,j}, \theta) = 0, u_{i,j} \geq 0, \sum_{i=1}^{n} u_{i,j} = 1 \right. \right),$$

and the composite empirical likelihood function is

$$L_{CE}(\theta) = \prod_{j=1}^{J} L^{(j)}_E(\theta) = \prod_{j=1}^{J} \left( \sup_{u_j} \prod_{i=1}^{n} u_{i,j} \left| \sum_{i=1}^{n} u_{i,j} g_j(z_{i,j}, \theta) = 0, u_{i,j} \geq 0, \sum_{i=1}^{n} u_{i,j} = 1 \right. \right).$$

(2)

Note that the construction of the composite empirical likelihood consists of proper empirical likelihood components multiplied together. The composite likelihood allows for the addition of a weight on each component, but we do not explore that option here.

**Example 1.** Let $Z = [Z_1, Z_2, Z_3, Z_4]$ where $Z_k$ for $k = 1, \ldots, 4$ are each univariate all with a sample size of $n$. If we assume that $Z_1$ and $Z_2$ are correlated, $E(Z_1) = E(Z_3) = \mu_1$ and $E(Z_2) = E(Z_4) = \mu_2$ we build a composite empirical likelihood where the first likelihood component consists of $Z_1$ and $Z_2$, the second likelihood component consists of $Z_3$ and the third component consists of $Z_4$ ($J = 3$). The estimating equations are

$$g_1(z_{i1}, z_{i2}, \mu_1, \mu_2) = [z_{i1} - \mu_1, z_{i2} - \mu_2]^T$$

$$g_2(z_{i3}, \mu_1, \mu_2) = z_{i3} - \mu_1$$

$$g_3(z_{i4}, \mu_1, \mu_2) = z_{i4} - \mu_2$$

so the composite empirical likelihood for this example is

$$L_{CE}(\mu_1, \mu_2) = L^{(1)}_E(\mu_1, \mu_2)L^{(2)}_E(\mu_1, \mu_2)L^{(3)}_E(\mu_1, \mu_2)$$

$$= \left( \sup_{u_1} \prod_{i=1}^{n} u_{i,1} \left| \sum_{i=1}^{n} u_{i,1}[z_{i,1}, z_{i,2}]^T = [\mu_1, \mu_2]^T, u_{i,1} \geq 0, \sum_{i=1}^{n} u_{i,1} = 1 \right. \right)$$

$$\times \left( \sup_{u_2} \prod_{i=1}^{n} u_{i,2} \left| \sum_{i=1}^{n} u_{i,2} z_{i,3} = \mu_1, u_{i,2} \geq 0, \sum_{i=1}^{n} u_{i,2} = 1 \right. \right)$$

$$\times \left( \sup_{u_3} \prod_{i=1}^{n} u_{i,3} \left| \sum_{i=1}^{n} u_{i,3} z_{i,4} = \mu_2, u_{i,3} \geq 0, \sum_{i=1}^{n} u_{i,3} = 1 \right. \right).$$
Following the derivation in Owen (1988, 1990) and Qin & Lawless (1994) we express $u_{i,j}$ from Equation 2 separately for each $j$ in terms of the $r_j \times 1$ Lagrange multipliers $t_j$ as

$$u_{i,j} = n^{-1} \left\{ 1 + t_j^T g_j(z_{i,j}, \theta) \right\}^{-1}$$

with the following restrictions

$$0 = \sum_{i=1}^{n} u_{i,j} g_j(z_{i,j}, \theta) = \sum_{i=1}^{n} \left\{ 1 + t_j^T g_j(z_{i,j}, \theta) \right\}^{-1} g_j(z_{i,j}, \theta)$$

where the values of $t_j$ are determined based on the value of $\theta$. Furthermore $t_j$ for all $j$ and $\theta$ must satisfy $1 + t_j^T g_j(z_{i,j}, \theta) \geq 1/n$ since $0 \leq u_{i,j} \leq 1$. The $t_j$ are differentiable functions of $\theta$ (Qin & Lawless, 1994).

The maximum composite empirical likelihood estimator is the value of $\theta$ which maximizes $L_{CE}(\theta)$. We denote this by $\hat{\theta}_{CE}$.

3 Main Results

Our derivations and results follow from Qin & Lawless (1994). We derive the asymptotic distribution of the maximum composite empirical likelihood estimator along with the asymptotic distribution of the log composite empirical likelihood ratio.

First define

$$\ell^{(j)}_{CE}(\theta) = -\log(L^{(j)}_E(\theta));$$

then the negative log composite empirical likelihood function is

$$\ell_{CE}(\theta) = -\log \left( \prod_{j=1}^{J} L^{(j)}_E(\theta) \right) = \sum_{j=1}^{J} \ell^{(j)}_{E}(\theta).$$

Assumption 1. Let $\theta_0$ be the true value of $\theta$. Then for all $j$

(a) $E\{g_j(Z_j, \theta_0)g_j^T(Z_j, \theta_0)\}$ is positive definite.
(b) $\partial g_j(Z_j, \theta)/\partial \theta$ is continuous in a neighborhood of the true value $\theta_0$.
(c) $\|\partial g_j(Z_j, \theta)/\partial \theta\|$ and $\|g_j(Z_j, \theta)\|^3$ are both bounded by some integrable function in the same neighborhood of $\theta_0$.
(d) The rank of $E\{\partial g_j(Z_j, \theta)/\partial \theta\}$ is $p_j$.

Lemma 1. Under Assumption 1 $\ell_{CE}(\theta)$ attains its minimum value at some point $\hat{\theta}_{CE}$ in the interior of the ball $\|\theta - \theta_0\| \leq n^{-1/3}$ with probability 1 as $n \to \infty$. 

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Furthermore $\hat{\theta}_{CE}$ and $\hat{t}_j = t_j(\hat{\theta}_{CE})$ for all $j$ satisfy

$$Q_{1j}(\hat{\theta}_{CE}, \hat{t}_j) = 0$$

where

$$Q_{1j}(\theta, \hat{t}_j) = \frac{1}{n} \sum_{i=1}^{n} \{1 + \hat{t}_j^T g_j(z_{i,j}, \theta)\}^{-1} g_j(z_{i,j}, \theta)$$

and

$$Q_2(\hat{\theta}_{CE}, \hat{t}_1, \ldots, \hat{t}_J) = 0$$

where

$$Q_2(\theta, t_1, \ldots, t_J) = \sum_{j=1}^{J} \left\{ \frac{1}{n} \sum_{i=1}^{n} \{1 + \hat{t}_j^T g_j(z_{i,j}, \theta)\}^{-1} \left( \frac{\partial g_j(z_{i,j}, \theta)}{\partial \theta} \right)^T t_j \right\}.$$ 

The proof of Lemma (1) follows directly from Qin & Lawless (1994, Lemma 1).

**Assumption 2.** Let $\theta_0$ be the true value of $\theta$. Then for all $j$

(a) The second derivative $\partial^2 g_j(Z_j, \theta) / \partial \theta \partial \theta^T$ is continuous in $\theta$ in a neighborhood of the true value $\theta_0$.

(b) $\|\partial^2 g_j(Z_j, \theta) / \partial \theta \partial \theta^T\|$ can be bounded by some integrable function in the neighborhood of $\theta_0$.

For simplicity we will denote $g_j(Z_j, \theta)$ as $g_j$.

**Theorem 1.** Under Assumptions 1 and 2

$$\sqrt{n}(\hat{\theta}_{CE} - \theta_0) \rightarrow N(0, W^{-1}_\theta V W^{-1}_\theta)$$

where

$$W_\theta = \sum_{j=1}^{J} E \left( \frac{\partial g_j}{\partial \theta} \right)^T E \left( \{E(g_j g_j^T)\}^{-1} E \left( \frac{\partial g_j}{\partial \theta} \right) \right)$$

and

$$V = \sum_{j=1}^{J} \sum_{k=1}^{J} E \left( \frac{\partial g_j}{\partial \theta} \right)^T E \left( \{E(g_j g_j^T)\}^{-1} E(g_j g_k^T) \{E(g_k g_k^T)\}^{-1} E \left( \frac{\partial g_k}{\partial \theta} \right) \right).$$

The covariance of $\hat{\theta}_{CE}$ shows similarities to the estimators of both empirical and composite likelihood. The matrix $W_\theta$ is identical to the covariance matrix shown in Qin & Lawless (1994) for empirical likelihood, while $V$ operates as a nonparametric equivalent of the variability matrix seen
with composite likelihood (Varin et. al., 2011). Also note that if $E(g_j, g_k) = 0$ for all $j \neq k$ then $V = W_\theta$ and the covariance matrix reduces to $W_\theta^{-1}$.

Analogous to other likelihood functions, we develop a composite empirical likelihood ratio statistic in order to find efficient estimators, and by extension confidence intervals and test statistics. The remaining results hold under Assumptions 1 and 2.

**Theorem 2.** Let $\theta = [\phi^T, \nu^T]^T$ be a $p$ dimensional vector where $\phi$ is a $q \times 1$ vector and $\nu$ is a $(p - q) \times 1$ vector. The profile composite empirical likelihood ratio test statistic for $H_0 : \phi = \phi_0$ is

$$T = 2\ell_{CE}(\phi_0, \hat{\nu}_{CE}(\phi_0)) - 2\ell_{CE}(\hat{\phi}_{CE}, \hat{\nu}_{CE})$$

where $\hat{\nu}_{CE}(\phi_0)$ minimizes $\ell_{CE}(\phi, \nu)$ with respect to $\phi_0$. Under $H_0$

$$T \to Q(\lambda)$$

as $n \to \infty$. $Q(\lambda) = \sum_{i=1}^l \lambda_i \chi^2(1)$ is a weighted $\chi^2$ random variable where $\lambda_i$ for $i = 1, \ldots, l$ is the set of all non zero eigenvalues of

$$A\Gamma = \begin{bmatrix} A_{11} & \cdots & A_{1J} \\ \vdots & \ddots & \vdots \\ A_{J1} & \cdots & A_{JJ} \end{bmatrix} \begin{bmatrix} E(g_1 g_1^T) & \cdots & E(g_1 g_J^T) \\ \vdots & \ddots & \vdots \\ E(g_J g_1^T) & \cdots & E(g_J g_J^T) \end{bmatrix}$$

where

$$A_{jk} = \left\{ E(g_j g_j^T) \right\}^{-1} \times \left\{ E \left( \frac{\partial g_j}{\partial \theta} \right)^T W_\theta^{-1} E \left( \frac{\partial g_k}{\partial \theta} \right)^T - E \left( \frac{\partial g_j}{\partial \nu} \right)^T W_\nu^{-1} E \left( \frac{\partial g_k}{\partial \nu} \right) \right\} \times \left\{ E(g_k g_k^T) \right\}^{-1},$$

$$W_\nu = \sum_{j=1}^J E \left( \frac{\partial g_j}{\partial \nu} \right)^T \left\{ E(g_j g_j^T) \right\}^{-1} E \left( \frac{\partial g_j}{\partial \nu} \right)$$

and $W_\theta$ is as defined in Theorem (1).

The result of Theorem (2) highlights that the composite empirical likelihood function (like the composite likelihood) is not inherently asymptotically equivalent to the true likelihood. As a consequence the asymptotic distribution of $T$ is not the $\chi^2$ seen in ordinary empirical likelihood (Owen, 1988; Owen, 1990; Qin & Lawless, 1994) but rather the weighted $\chi^2$ derived from composite likelihood (Varin et. al., 2011).

If we assume (or know) that $cov(g_j, g_k) = 0$ for all $j \neq k$ then $Q_{1j}(\theta_0, 0)$ and $Q_{1k}(\theta_0, 0)$ are asymptotically independent for all $j \neq k$. The following corollary shows that given this additional assumption the asymptotic distribution of the test statistic using the composite empirical likelihood reduces to the standard $\chi^2$.
Corollary 1. In addition to Assumptions 1 and 2 let \( \text{cov}(g_j, g_k) = 0 \) for all \( j \neq k \). Then

\[
T \rightarrow \chi^2(q)
\]
as \( n \rightarrow \infty \). \( T \) is defined in Theorem (2).

We give proofs of Theorem (1), (2) and Corollary (1) in the appendix. The next two corollaries give the asymptotic distribution of the test statistic when there are no nuisance parameters.

Corollary 2. The composite empirical likelihood ratio statistic for testing \( H_0 : \theta = \theta_0 \) is

\[
T = 2\ell_{CE}(\theta_0) - 2\ell_{CE}(\hat{\theta}_{CE})
\]
where \( \ell_{CE} \) is given by Equation (3). Under \( H_0 \)

\[
T \rightarrow Q(\lambda)
\]
as \( n \rightarrow \infty \). \( Q(\lambda) = \sum_{i=1}^l \lambda_i \chi^2(1) \) is a weighted \( \chi^2 \) random variable where \( \lambda_i \) for \( i = 1, \ldots, l \) is the set of all non zero eigenvalues of

\[
A \Gamma = \begin{bmatrix}
A_{11} & \cdots & A_{1J} \\
\vdots & \ddots & \vdots \\
A_{J1} & \cdots & A_{JJ}
\end{bmatrix}
\begin{bmatrix}
E(g_1g_1^T) & \cdots & E(g_1g_J^T) \\
\vdots & \ddots & \vdots \\
E(g_Jg_1^T) & \cdots & E(g_Jg_J^T)
\end{bmatrix}
\]

where

\[
A_{jk} = \{E(g_jg_j^T)\}^{-1} E\left(\frac{\partial g_j}{\partial \theta}\right)^\tau W^{-1} E\left(\frac{\partial g_k}{\partial \theta}\right)^\tau \{E(g_kg_k^T)\}^{-1}
\]
and \( W_\theta \) is as defined in Theorem (1).

Corollary 3. Assume that \( \text{cov}(g_j, g_k) = 0 \) for all \( j \neq k \). Then

\[
T \rightarrow \chi^2(p)
\]
as \( n \rightarrow \infty \) when \( H_0 \) is true. \( T \) is as defined in Corollary (2).

We do not provide proofs for Corollaries (2) and (3) as they follow from the proofs for Theorem (2) and Corollary (1) by noting that \( \nu = \emptyset \).

The distributions of the test statistics from Theorem (2) and Corollary (2) require that we know the \( A \) and \( \Gamma \) matrices in order to determine the weighted \( \chi^2 \). Since these are generally unknown, we can estimate the matrices for a given value of \( \tilde{\theta} \) using the sample data \( z \). For confidence intervals we use \( \hat{\nu}_{CE} \) as the value of \( \nu \). For hypothesis testing we can use the null hypothesis values \( \phi_0 \) and \( \hat{\nu}_{CE}(\phi_0) \), or for computational efficiency replace \( \hat{\nu}_{CE}(\phi_0) \) with \( \hat{\nu}_{CE} \) since the maximum composite empirical likelihood is a consistent estimator of \( \nu \) (Barndorff-Nielsen & Cox, 1994, page 91).
The covariance between each $g_j$ and $g_k$ is estimated using

$$\hat{\Gamma}_{jk}(\tilde{\theta}) = \frac{1}{n} \sum_{i=1}^{n} g_j(z_{ij}, \tilde{\theta}) g_k^T(z_{ik}, \tilde{\theta}),$$

each $j, k$th block of $A$ is

$$\hat{A}_{jk}(\tilde{\theta}) = \left( \frac{1}{n} \sum_{i=1}^{n} g_j(z_{ij}, \tilde{\theta}) g_j^T(z_{ij}, \tilde{\theta}) \right)^{-1} \times \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_j(z_{ij}, \tilde{\theta})}{\partial \theta} \right) \hat{W}_{\theta}(\tilde{\theta})^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_k(z_{ik}, \tilde{\theta})}{\partial \theta} \right)^T - \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_j(z_{ij}, \tilde{\theta})}{\partial \nu} \right) \hat{W}_{\nu}(\tilde{\theta})^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_k(z_{ik}, \tilde{\theta})}{\partial \nu} \right)^T \right\} \times \left( \frac{1}{n} \sum_{i=1}^{n} g_k(z_{ik}, \tilde{\theta}) g_k^T(z_{ik}, \tilde{\theta}) \right)^{-1},$$

and

$$\hat{W}_{\theta}(\tilde{\theta}) = \sum_{j=1}^{J} \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_j(z_{ij}, \tilde{\theta})}{\partial \theta} \right)^T \left( \frac{1}{n} \sum_{i=1}^{n} g_j^T(z_{ij}, \tilde{\theta}) g_j(z_{ij}, \tilde{\theta}) \right)^{-1} \right. \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_j(z_{ij}, \tilde{\theta})}{\partial \theta} \right) \right],$$

$$\hat{W}_{\nu}(\tilde{\theta}) = \sum_{j=1}^{J} \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_j(z_{ij}, \tilde{\theta})}{\partial \nu} \right)^T \left( \frac{1}{n} \sum_{i=1}^{n} g_j^T(z_{ij}, \tilde{\theta}) g_j(z_{ij}, \tilde{\theta}) \right)^{-1} \right. \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_j(z_{ij}, \tilde{\theta})}{\partial \nu} \right) \right].$$

## 4 Numerical Studies

### 4.1 Simulation of Theoretical Results

In order to examine the performance of the proposed method given several different data distributions we generate data from bivariate normal, bivariate chi square, and bivariate uniform ($J = 2$) distributions. We use 500 replicates of varying sample size $n$. The bivariate normal random variables are generated with parameter values of $\mu = 1$ and $\sigma^2 = 2$. The bivariate chi square random variables are generated with $df = 1$. The bivariate uniform random variables are generated with a lower bound of $a = 1 - \sqrt{6}$ and an upper bound of $b = 1 + \sqrt{6}$. We vary the values of the correlation $\rho$. All data distributions have an expected value of 1, variance of 2 and correlation of $\rho$. The mean
is the parameter of interest and we are assuming that the mean is the same for both variables, so
\( g_x(x_i, \theta) = x_i - \mu \) and \( g_y(y_i, \theta) = y_i - \mu \).

Using the result from Theorem (1) our estimator will have an asymptotic mean of 1 and an asymptotic variance of \( n^{-1}(1 + \rho) \). We use the sample mean and sample variance to empirically estimate these values, and we show the percentage of false rejections out of the 500 simulations at \( \alpha = 0.10, 0.05, 0.01 \) for the two sided test of \( H_0 : \mu = 1 \). Table 1 shows the asymptotic variances at specified sample sizes and correlation.

Table 1: Theoretical variance of \( \hat{\theta}_{CE} \) based on sample size and correlation.

| n  | \( \rho \) |
|----|-----------|
| 10 | 0.100     |
| 10 | 0.110     |
| 10 | 0.150     |
| 10 | 0.190     |
| 50 | 0.020     |
| 50 | 0.022     |
| 50 | 0.030     |
| 50 | 0.038     |
| 100| 0.010     |
| 100| 0.011     |
| 100| 0.015     |
| 100| 0.019     |

The weighted \( \chi^2 \) distribution for the test statistic is estimated using the approach proposed by Welch (1938). Since we know the underlying moments we can show

\[ T \to (1 + \rho) \chi^2(1) \]

as \( n \to \infty \). We will use the true value of \( \rho \) for the distribution of the test statistic.

Table 2: Mean and Variance of Parameter Estimates and Cumulative Distribution of \( T \) at \( \alpha = 0.10, 0.05, 0.01 \) when data are bivariate normal.

| \( \rho \) | \( n \) | Obs. Mean | Obs. Variance | Prop. \( p < 0.10 \) | Prop. \( p < 0.05 \) | Prop. \( p < 0.01 \) |
|---------|------|----------|--------------|-----------------|-----------------|-----------------|
| 0.00    | 10   | 1.010    | 0.115        | 0.208           | 0.144           | 0.068           |
| 0.00    | 50   | 1.012    | 0.022        | 0.124           | 0.070           | 0.016           |
| 0.00    | 100  | 1.002    | 0.011        | 0.120           | 0.064           | 0.012           |
| 0.10    | 10   | 1.009    | 0.124        | 0.202           | 0.140           | 0.066           |
| 0.10    | 50   | 1.013    | 0.024        | 0.126           | 0.066           | 0.016           |
| 0.10    | 100  | 1.002    | 0.012        | 0.122           | 0.060           | 0.014           |
| 0.50    | 10   | 1.006    | 0.155        | 0.194           | 0.124           | 0.052           |
| 0.50    | 50   | 1.016    | 0.031        | 0.110           | 0.056           | 0.014           |
| 0.50    | 100  | 1.001    | 0.016        | 0.106           | 0.062           | 0.016           |
| 0.90    | 10   | 1.005    | 0.186        | 0.170           | 0.112           | 0.048           |
| 0.90    | 50   | 1.018    | 0.038        | 0.116           | 0.050           | 0.016           |
| 0.90    | 100  | 1.001    | 0.020        | 0.110           | 0.056           | 0.012           |
Table 3: Mean and Variance of Parameter Estimates and Cumulative Distribution of $T$ at $\alpha = 0.10, 0.05, 0.01$ when data are bivariate chi square.

| $\rho$ | $n$ | Obs. Mean | Obs. Variance | Prop. $p < 0.10$ | Prop. $p < 0.05$ | Prop. $p < 0.01$ |
|--------|-----|-----------|---------------|------------------|------------------|------------------|
| 0.00   | 10  | 0.923     | 0.102         | 0.314            | 0.252            | 0.156            |
|        | 50  | 0.978     | 0.018         | 0.124            | 0.078            | 0.022            |
|        | 100 | 0.998     | 0.009         | 0.092            | 0.060            | 0.020            |
| 0.10   | 10  | 0.931     | 0.114         | 0.316            | 0.254            | 0.160            |
|        | 50  | 0.981     | 0.021         | 0.148            | 0.088            | 0.024            |
|        | 100 | 0.997     | 0.011         | 0.102            | 0.062            | 0.018            |
| 0.50   | 10  | 0.962     | 0.148         | 0.268            | 0.204            | 0.126            |
|        | 50  | 0.988     | 0.030         | 0.150            | 0.080            | 0.026            |
|        | 100 | 0.997     | 0.015         | 0.102            | 0.056            | 0.014            |
| 0.90   | 10  | 0.990     | 0.174         | 0.250            | 0.198            | 0.106            |
|        | 50  | 0.991     | 0.037         | 0.132            | 0.080            | 0.028            |
|        | 100 | 0.996     | 0.018         | 0.102            | 0.050            | 0.014            |

Table 4: Mean and Variance of Parameter Estimates and Cumulative Distribution of $T$ at $\alpha = 0.10, 0.05, 0.01$ when data are bivariate uniform.

| $\rho$ | $n$ | Obs. Mean | Obs. Variance | Prop. $p < 0.10$ | Prop. $p < 0.05$ | Prop. $p < 0.01$ |
|--------|-----|-----------|---------------|------------------|------------------|------------------|
| 0.00   | 10  | 1.037     | 0.113         | 0.162            | 0.110            | 0.044            |
|        | 50  | 1.016     | 0.020         | 0.100            | 0.058            | 0.022            |
|        | 100 | 1.010     | 0.010         | 0.086            | 0.054            | 0.014            |
| 0.10   | 10  | 1.038     | 0.119         | 0.158            | 0.106            | 0.044            |
|        | 50  | 1.017     | 0.022         | 0.104            | 0.052            | 0.022            |
|        | 100 | 1.010     | 0.011         | 0.086            | 0.052            | 0.014            |
| 0.50   | 10  | 1.038     | 0.144         | 0.140            | 0.082            | 0.030            |
|        | 50  | 1.020     | 0.029         | 0.088            | 0.040            | 0.014            |
|        | 100 | 1.010     | 0.014         | 0.086            | 0.058            | 0.018            |
| 0.90   | 10  | 1.033     | 0.174         | 0.124            | 0.060            | 0.020            |
|        | 50  | 1.022     | 0.035         | 0.080            | 0.034            | 0.012            |
|        | 100 | 1.008     | 0.019         | 0.106            | 0.058            | 0.008            |

Table 2 shows that when the data are normal the observed mean and variance of $\hat{\theta}_{CE}$ match very closely with the theoretical values regardless of sample size. We also see that as the sample size increases the numbers of false rejections approach the theoretical values. Table 3 indicates that when the data are chi square the false rejection percent is much higher than expected with small sample sizes, but as the sample size increases the false rejection rates approach the theoretical values. The poor performance with smaller sample sizes is not unexpected given the skewness of the chi square distribution. The results in Table 4 are very similar to those seen in Table 2 most.
likely due to the symmetry of the uniform distribution, and along with the results from Table 3, confirm that the asymptotic properties hold with nonnormally distributed data.

4.2 Comparison to Empirical Likelihood

We now examine two cases where we have to estimate $A\Gamma$. In the first set of simulations we repeat the settings of Section 4.1. The second set of simulations examines inference on the variance parameter. We compare the composite approach to empirical likelihood setups.

Since we do not assume knowledge of the covariance and expected value of the derivatives of $g_x$ and $g_y$ we estimate $A$ and $\Gamma$ using the sample versions (see Qin & Lawless, 1994 for details). We approximate the weighted $\chi^2$ distribution by $\hat{a}\chi^2(\hat{b})$ where

$$\hat{a} = \frac{\text{tr} \{ \hat{A}(\hat{\theta})\hat{\Gamma}(\hat{\theta}) \}^2}{\text{tr} \{ \hat{A}(\hat{\theta})\hat{\Gamma}(\hat{\theta}) \}}$$

and

$$\hat{b} = \frac{\text{tr} \{ \hat{A}(\hat{\theta})\hat{\Gamma}(\hat{\theta}) \}^2}{\text{tr} \{ \hat{A}(\hat{\theta})\hat{\Gamma}(\hat{\theta}) \}}.$$

The estimators $\hat{a}$ and $\hat{b}$ are functions of $\tilde{\theta}$, so similar to parametric tests we use $\bar{\tilde{\theta}} = \hat{\theta}$ when working with confidence intervals and $\tilde{\theta} = \theta_0$ for tests of hypothesis. To examine how the three functions compare we examine the mean and variance of $\hat{\theta}_{CE}$ (which is the estimator of the single parameter $\mu$), the length of the 95% confidence interval, the proportion of times the lower and upper endpoints do not cover the true value $\mu = 1$ and how many times the test of $H_0 : \mu = 1$ is rejected at $\alpha = 0.05$.

For the first simulation we consider the following three functions

$$L_{CE}(\theta) = \left\{ \sup_u \left( \prod_{i=1}^n nu_i \mid \sum_{i=1}^n u_ix_i = \mu, u_i \geq 0, \sum_{i=1}^n u_i = 1 \right) \times \sup_v \left( \prod_{i=1}^n nv_i \mid \sum_{i=1}^n v_iy_i = \mu, v_i \geq 0, \sum_{i=1}^n v_i = 1 \right) \right\}.$$

$$L_{E1}(\theta) = \sup_p \left( \prod_{i=1}^{2n} 2np_i \mid \sum_{i=1}^n p_ix_i + \sum_{i=1}^n p_{n+i}y_i = \mu, p_i \geq 0, \sum_{i=1}^n p_i = 1 \right).$$

$$L_{E2}(\theta) = \sup_p \left( \prod_{i=1}^n np_i \mid \sum_{i=1}^n p_ix_i = \mu, \sum_{i=1}^n p_iy_i = \mu, p_i \geq 0, \sum_{i=1}^n p_i = 1 \right).$$

$L_{E1}(\theta)$ is the empirical likelihood form used by Owen (1988), which combines the two data vectors $x$ and $y$ into a single vector. $L_{E2}(\theta)$ is the bivariate case explored in Owen (2001), except we are assuming a common mean. $L_{E1}(\theta)$ is what we would use if we assume that $X$ and $Y$ are $i.i.d.$, while $L_{E2}(\theta)$ accounts for correlation between $X$ and $Y$. 

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Table 5: Comparison of $L_{CE}(\theta)$, $L_{E1}(\theta)$ and $L_{E2}(\theta)$ using bivariate normal with 500 replicates.

| $\rho$ | $n$ | Method | Obs. Mean | Obs. Variance | Avg. Length | Prop. $L > 1$ | Prop. $U < 1$ | Prop. $p < 0.05$ |
|--------|-----|--------|-----------|---------------|-------------|---------------|---------------|-----------------|
| 0.00   |     | $L_{CE}(\theta)$ | 1.005 | 0.044 | 0.761 | 0.046 | 0.044 | 0.080 |
|        |     | $L_{E1}(\theta)$ | 1.006 | 0.043 | 0.786 | 0.044 | 0.024 | 0.068 |
|        |     | $L_{E2}(\theta)$ | 1.005 | 0.045 | 1.481 | 0.040 | 0.028 | 0.092 |
| 100    |     | $L_{CE}(\theta)$ | 0.993 | 0.011 | 0.390 | 0.036 | 0.034 | 0.066 |
|        |     | $L_{E1}(\theta)$ | 0.993 | 0.011 | 0.392 | 0.032 | 0.040 | 0.072 |
|        |     | $L_{E2}(\theta)$ | 0.993 | 0.011 | 2.193 | 0.026 | 0.020 | 0.070 |
| 0.10   |     | $L_{CE}(\theta)$ | 0.995 | 0.047 | 0.791 | 0.038 | 0.044 | 0.080 |
|        |     | $L_{E1}(\theta)$ | 0.997 | 0.046 | 0.783 | 0.040 | 0.036 | 0.076 |
|        |     | $L_{E2}(\theta)$ | 0.995 | 0.048 | 1.376 | 0.040 | 0.038 | 0.086 |
| 0.50   |     | $L_{CE}(\theta)$ | 1.000 | 0.011 | 0.409 | 0.032 | 0.028 | 0.058 |
|        |     | $L_{E1}(\theta)$ | 1.000 | 0.011 | 0.392 | 0.040 | 0.028 | 0.068 |
|        |     | $L_{E2}(\theta)$ | 1.000 | 0.011 | 1.815 | 0.016 | 0.016 | 0.056 |
| 0.90   |     | $L_{CE}(\theta)$ | 1.006 | 0.062 | 0.921 | 0.036 | 0.038 | 0.068 |
|        |     | $L_{E1}(\theta)$ | 1.010 | 0.061 | 0.772 | 0.066 | 0.058 | 0.124 |
|        |     | $L_{E2}(\theta)$ | 1.001 | 0.067 | 1.017 | 0.032 | 0.042 | 0.074 |
| 100    |     | $L_{CE}(\theta)$ | 0.995 | 0.015 | 0.479 | 0.030 | 0.022 | 0.052 |
|        |     | $L_{E1}(\theta)$ | 0.995 | 0.015 | 0.392 | 0.054 | 0.054 | 0.108 |
|        |     | $L_{E2}(\theta)$ | 0.995 | 0.015 | 0.739 | 0.030 | 0.022 | 0.054 |
| 0.90   |     | $L_{CE}(\theta)$ | 0.982 | 0.084 | 1.058 | 0.042 | 0.038 | 0.080 |
|        |     | $L_{E1}(\theta)$ | 0.981 | 0.084 | 0.767 | 0.080 | 0.106 | 0.186 |
|        |     | $L_{E2}(\theta)$ | 0.984 | 0.087 | 1.099 | 0.044 | 0.042 | 0.088 |
| 0.90   |     | $L_{CE}(\theta)$ | 1.005 | 0.018 | 0.537 | 0.028 | 0.022 | 0.050 |
|        |     | $L_{E1}(\theta)$ | 1.005 | 0.018 | 0.388 | 0.070 | 0.066 | 0.136 |
|        |     | $L_{E2}(\theta)$ | 1.005 | 0.018 | 0.566 | 0.026 | 0.022 | 0.048 |
Table 6: Comparison of $L_{CE}(\theta)$, $L_{E1}(\theta)$ and $L_{E2}(\theta)$ using bivariate chi square with 500 replicates.

| $\rho$ | $n$ | Method     | Obs. Mean | Obs. Variance | Avg. Length | Prop. $L > 1$ | Prop. $U < 1$ | Prop. $p < 0.05$ |
|--------|-----|------------|-----------|---------------|-------------|--------------|--------------|----------------|
|        |     | $L_{CE}(\theta)$ | 0.976     | 0.044         | 0.684       | 0.028        | 0.098        | 0.122          |
| 25     |     | $L_{E1}(\theta)$ | 0.999     | 0.041         | 0.783       | 0.020        | 0.052        | 0.072          |
|        |     | $L_{E2}(\theta)$ | 0.974     | 0.044         | 1.981       | 0.026        | 0.078        | 0.132          |
| 0.00   |     | $L_{CE}(\theta)$ | 0.997     | 0.009         | 0.382       | 0.018        | 0.050        | 0.066          |
| 100    |     | $L_{E1}(\theta)$ | 1.001     | 0.009         | 0.394       | 0.014        | 0.038        | 0.052          |
|        |     | $L_{E2}(\theta)$ | 0.997     | 0.009         | 4.123       | 0.004        | 0.016        | 0.064          |
| 0.10   |     | $L_{CE}(\theta)$ | 1.000     | 0.012         | 0.403       | 0.026        | 0.044        | 0.066          |
| 100    |     | $L_{E1}(\theta)$ | 1.002     | 0.012         | 0.395       | 0.032        | 0.042        | 0.074          |
|        |     | $L_{E2}(\theta)$ | 1.000     | 0.012         | 3.644       | 0.030        | 0.016        | 0.072          |
| 0.50   |     | $L_{CE}(\theta)$ | 1.012     | 0.060         | 0.883       | 0.024        | 0.072        | 0.090          |
| 100    |     | $L_{E1}(\theta)$ | 1.022     | 0.061         | 0.782       | 0.048        | 0.080        | 0.128          |
|        |     | $L_{E2}(\theta)$ | 0.998     | 0.060         | 1.601       | 0.022        | 0.074        | 0.102          |
| 0.90   |     | $L_{CE}(\theta)$ | 1.006     | 0.076         | 1.029       | 0.020        | 0.080        | 0.096          |
| 100    |     | $L_{E1}(\theta)$ | 1.009     | 0.076         | 0.757       | 0.082        | 0.110        | 0.192          |
|        |     | $L_{E2}(\theta)$ | 0.986     | 0.082         | 1.886       | 0.026        | 0.088        | 0.142          |
| 0.95   |     | $L_{CE}(\theta)$ | 0.989     | 0.018         | 0.535       | 0.024        | 0.040        | 0.062          |
| 100    |     | $L_{E1}(\theta)$ | 0.990     | 0.018         | 0.386       | 0.058        | 0.110        | 0.168          |
|        |     | $L_{E2}(\theta)$ | 0.984     | 0.018         | 1.998       | 0.024        | 0.046        | 0.078          |
Table 7: Comparison of $L_{CE}(\theta)$, $L_{E1}(\theta)$ and $L_{E2}(\theta)$ using bivariate uniform with 500 replicates.

| $\rho$ | $n$ | Method | Obs. Mean | Obs. Variance | Avg. Length | Prop. $L > 1$ | Prop. $U < 1$ | Prop. $p < 0.05$ |
|--------|-----|--------|-----------|--------------|-------------|---------------|---------------|----------------|
|        | 0.00| $L_{CE}(\theta)$ | 0.995 | 0.009 | 0.388 | 0.016 | 0.028 | 0.042 |
|        | 25  | $L_{E1}(\theta)$ | 1.005 | 0.009 | 0.390 | 0.020 | 0.024 | 0.044 |
| 0.10   |     | $L_{E2}(\theta)$ | 1.005 | 0.009 | 0.390 | 0.016 | 0.026 | 0.042 |
| 0.50   |     | $L_{CE}(\theta)$ | 0.983 | 0.057 | 0.916 | 0.018 | 0.032 | 0.040 |
| 0.90   |     | $L_{E1}(\theta)$ | 0.982 | 0.056 | 0.768 | 0.048 | 0.066 | 0.114 |
|        | 25  | $L_{E2}(\theta)$ | 0.982 | 0.058 | 0.922 | 0.014 | 0.034 | 0.048 |
|        | 100 | $L_{CE}(\theta)$ | 0.985 | 0.070 | 1.051 | 0.018 | 0.022 | 0.040 |
|        |     | $L_{E1}(\theta)$ | 0.985 | 0.071 | 0.769 | 0.064 | 0.078 | 0.142 |
|        |     | $L_{E2}(\theta)$ | 0.985 | 0.072 | 1.050 | 0.020 | 0.026 | 0.046 |
|        | 100 | $L_{CE}(\theta)$ | 1.003 | 0.021 | 0.536 | 0.042 | 0.016 | 0.058 |
|        |     | $L_{E1}(\theta)$ | 1.003 | 0.021 | 0.390 | 0.094 | 0.082 | 0.176 |
|        |     | $L_{E2}(\theta)$ | 1.003 | 0.021 | 0.537 | 0.042 | 0.014 | 0.056 |

Tables 5, 6 and 7 show that when there is no correlation $L_{E1}(\theta)$ performs best both in terms of standard deviation of the estimate, the number of false rejections for the hypothesis test, and the percentage of time the confidence intervals fail to capture the true parameter value. For small sample sizes $L_{CE}(\theta)$ performs better than $L_{E2}(\theta)$ when the correlation is 0, indicating that our method is better accounting for both variables being independent. As the correlation increases $L_{CE}(\theta)$ performs better than $L_{E1}(\theta)$ in terms of false rejections of the hypothesis tests and capture of the true parameter value of the confidence intervals. The confidence interval of $L_{CE}(\theta)$ has a shorter average length than $L_{E2}(\theta)$ in many cases, and a comparable false rejection rate. The variance of $\hat{\theta}$, as expected, is comparable for all three methods. Overall this simulation shows that $L_{E1}(\theta)$ is optimal if the variables are not correlated, but $L_{CE}(\theta)$ is advantageous in that no assumption about the correlation needs to be made since it performs as well as $L_{E2}(\theta)$ when there is no correlation and better than $L_{E1}(\theta)$ if there is correlation.
We now examine a case where there are four likelihood components \((J = 4)\), nuisance parameters, and we have to estimate \(A\Gamma\). We compare the composite approach to the standard empirical likelihood. We will examine inference on a common variance parameter but assume that the variable means are different.

We consider the following two functions

\[
L_{CE}(\theta) = \prod_{j=1}^{4} \left\{ \sup_{u_j} \left( \prod_{i=1}^{n} u_{ij} \left| \sum_{i=1}^{n} u_{ij} g(z_{ij}, \mu_j, \sigma^2), u_{ij} \geq 0, \sum_{i=1}^{n} u_{ij} = 1 \right| \right) \right\}
\]

\[
L_E(\theta) = \sup_{u} \left( \prod_{i=1}^{n} u_i \left| \sum_{i=1}^{n} u_i g(z_i, \mu_1, \mu_2, \mu_3, \mu_4, \sigma^2), u_i \geq 0, \sum_{i=1}^{n} u_i = 1 \right| \right)
\]

where

\[
g_j(z_{ij}, \mu_j, \sigma^2) = \left[ \frac{z_{ij} - \mu_j}{(z_{ij} - \hat{\mu}_j)^2 - \frac{n-1}{n} \sigma^2} \right]
\]

where \(\hat{\mu}_j\) is the solution to

\[
\sum_{i=1}^{n} u_{ij}(z_{ij} - \mu_j) = 0
\]

and

\[
g(z_i, \mu_1, \mu_2, \mu_3, \mu_4, \sigma^2) = \begin{bmatrix}
g_1(z_{i1}, \mu_1, \sigma^2) \\
g_2(z_{i2}, \mu_2, \sigma^2) \\
g_3(z_{i3}, \mu_3, \sigma^2) \\
g_4(z_{i4}, \mu_4, \sigma^2)
\end{bmatrix}
\]

Note that the factor \(\frac{n-1}{n}\) is to remove the bias of \(\hat{\sigma}^2\).

To examine how the two functions compare we generate 500 realizations from the bivariate normal distribution with

\[
\mu = [-3, 1, 2, 0]^T
\]

(to demonstrate a case with unequal means) and

\[
\Sigma_{ij} = \rho^{|i-j|}
\]

for \(i, j = 1, 2, 3, 4\).

We compare the mean and variance of \(\hat{\sigma}^2\), the length of the 95% confidence interval, how many times the lower and upper endpoints do not cover the true value \(\sigma^2 = 1\) and how many times the test of \(H_0 : \sigma_0^2 = 1\) is rejected at \(\alpha = 0.05\).
Table 8: Comparison of $L_{CE}(\theta)$ and $L_E(\theta)$ using bivariate normal with 100 replicates.

| $\rho$ | $n$ | Method | Obs. Mean | Obs. Variance | Avg. Length | Prop. $L > 1$ | Prop. $U < 1$ | Prop. $p < 0.05$ |
|--------|-----|--------|-----------|---------------|-------------|---------------|---------------|----------------|
| 25     | 0.00| $L_{CE}(\theta)$ | 0.944 | 0.020 | 0.559 | 0.020 | 0.068 | 0.166 |
|        |     | $L_E(\theta)$     | 1.000 | 0.614 | 1.181 | 0.030 | 0.002 | 0.232 |
| 100    |     | $L_{CE}(\theta)$ | 0.994 | 0.005 | 0.296 | 0.016 | 0.030 | 0.060 |
|        |     | $L_E(\theta)$     | 0.994 | 0.005 | 0.272 | 0.024 | 0.052 | 0.076 |
| 25     | 0.10| $L_{CE}(\theta)$ | 0.965 | 0.025 | 0.558 | 0.022 | 0.084 | 0.162 |
|        |     | $L_E(\theta)$     | 1.129 | 3.178 | 1.194 | 0.058 | 0.000 | 0.214 |
| 100    |     | $L_{CE}(\theta)$ | 0.993 | 0.005 | 0.296 | 0.008 | 0.034 | 0.056 |
|        |     | $L_E(\theta)$     | 0.992 | 0.005 | 0.265 | 0.020 | 0.052 | 0.064 |
| 25     | 0.50| $L_{CE}(\theta)$ | 0.952 | 0.029 | 0.604 | 0.010 | 0.100 | 0.156 |
|        |     | $L_E(\theta)$     | 1.065 | 2.096 | 1.142 | 0.042 | 0.006 | 0.240 |
| 100    |     | $L_{CE}(\theta)$ | 0.994 | 0.008 | 0.346 | 0.018 | 0.056 | 0.090 |
|        |     | $L_E(\theta)$     | 0.991 | 0.009 | 0.300 | 0.028 | 0.082 | 0.098 |
| 25     | 0.90| $L_{CE}(\theta)$ | 0.967 | 0.059 | 0.867 | 0.018 | 0.090 | 0.124 |
|        |     | $L_E(\theta)$     | 1.009 | 1.479 | 1.016 | 0.044 | 0.016 | 0.272 |
| 100    |     | $L_{CE}(\theta)$ | 1.002 | 0.015 | 0.512 | 0.014 | 0.030 | 0.052 |
|        |     | $L_E(\theta)$     | 0.990 | 0.016 | 0.435 | 0.026 | 0.062 | 0.080 |

Table 8 shows that the average value of the maximum composite empirical likelihood estimator is very close to that of the empirical likelihood estimator, indicating that our method performs comparably in terms of the expected value of the estimator. Our method in some cases results in a much smaller average length of the confidence interval compared to empirical likelihood. This is due to several unusually large upper bound solutions for $L_E(\theta)$, whereas $L_{CE}(\theta)$ is less prone to this issue if the sample sizes are small.

### 4.3 Computation Time

To demonstrate the computational gains using composite empirical likelihood we divide a standard univariate empirical likelihood into multiple pieces (making it a composite empirical likelihood), compute each piece in parallel, and compare to the time required to compute the standard empirical likelihood. As we will show, this results in a substantial decrease in the time needed to optimize the function. The estimating equation is $z_{ij} - \mu$ (so $\theta$ is $\mu$) making this the same setup from Section 4.1. We measure the time (in seconds) required to compute the empirical likelihood and composite empirical likelihood when $\mu = 1$ over 500 replications.

Since we have univariate data and randomly split the data into $J$ equal pieces the composite empirical likelihood is
\[ L_{CE}(\mu) = \prod_{j=1}^{J} \left( \sup_{u_j} \prod_{i=1}^{n_j} u_{i,j} \left| \sum_{i=1}^{n_j} u_{i,j} z_{i,j} = \mu, u_{i,j} \geq 0, \sum_{i=1}^{n_j} u_{i,j} = 1 \right. \right) \].

We divide the data \( z = [z_1, \ldots, z_J] \) into equal sizes so \( n_j = n/J \). Since this is a univariate sample each subset of the data is independent of all other subsets, hence the likelihood components are also independent. Finally note that when \( J = 1 \) this is the standard empirical likelihood.

![Figure 1](image1.png)

Figure 1: Distribution of computation times with \( J = 1, 2, 4 \) for \( n = 20, 60, 100 \) using uncorrelated univariate chi square with 500 replicates.

Figure 1 shows that as the sample size increases the computation times when \( J = 1 \) also significantly increases, and there is an increase in the variability of the computation time. As we increase the number of likelihood components (which allows us to take advantage of the parallel computing environment) the decrease in computation times is substantial. In this example splitting the likelihood into two pieces with a sample size of 100 cuts the median computation time down from 3.42 seconds to 0.864 seconds, while using four likelihood components brings the median computation time down to 0.373 seconds.

5 Discussion

An immediate question that arises from the construction of the composite empirical likelihood is how many components should one use. One principle is that variables that are correlated will be combined into a single component while independent variables can be separate components (as in Example 1). This idea can also be applied to create a pairwise composite empirical likelihood.
where each component consists of two variables, allowing for inference on correlation between pairs. If all components are independent then the test statistic is a standard \( \chi^2 \) eliminating the need to compute the weights. A critical aspect to note is that independence of the likelihood components is based on \( \text{cov}(g_j, g_k) = 0 \), so there are cases where the variables themselves may be correlated but the likelihood components are not.

The simplest application of composite empirical likelihood is separation of large sample univariate data. In this case the balance between the number of components and the number of observations in each component is a matter of user preference and computing resources. For minimizing run time and reducing the possibility of an optimization algorithm failing to converge we suggest (based on our own experiences) having at least as many components as parallel pools (which determines how many components can be computed simultaneously) provided there is a minimum of 10 observations per parameter for each likelihood component. Also the approach shown in Section 4.3 can be applied to any empirical likelihood. Since one of the assumptions of the empirical likelihood is the data are \( i.i.d. \) the test statistic will still be \( \chi^2 \) since each likelihood piece will also be independent.

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**Appendix**

*Proof of Theorem 1.* First by Taylor expansion of \( Q_{1j}(\hat{\theta}_{CE}, \hat{t}_j) \) around \( \theta_0 \) and 0

\[
0 = Q_{1j}(\hat{\theta}_{CE}, \hat{t}_j) = Q_{1j}(\theta_0, 0) + \frac{\partial}{\partial \theta} Q_{1j}(\theta_0, 0)(\hat{\theta}_{CE} - \theta_0) + \frac{\partial}{\partial \hat{t}_j} Q_{1j}(\theta_0, 0)(\hat{t}_j - 0) + o_p(\delta_j)
\]

where \( \delta_j = ||\hat{\theta}_{CE} - \theta_0|| + ||\hat{t}_j|| \). Solving for \( \hat{t}_j \) yields

\[
(\hat{t}_j - 0) = \left(-\frac{\partial Q_{1j}(\theta_0, 0)}{\partial \hat{t}_j}\right)^{-1} \left( Q_{1j}(\theta_0, 0) + \frac{\partial}{\partial \theta} Q_{1j}(\theta_0, 0)(\hat{\theta}_{CE} - \theta_0) + o_p(\delta_j) \right)
\]

for all \( j \). Now by Taylor expansion of \( Q_2(\hat{\theta}_{CE}, \hat{t}_1, \ldots, \hat{t}_J) \) around \( \theta_0, 0, \ldots, 0 \)

\[
0 = Q_2(\hat{\theta}_{CE}, \hat{t}_1, \ldots, \hat{t}_J)
\]

\[
= Q_2(\theta_0, 0, \ldots, 0) + \frac{\partial}{\partial \theta} Q_2(\theta_0, 0, \ldots, 0)(\hat{\theta}_{CE} - \theta_0)
\]

\[
+ \sum_{j=1}^J \frac{\partial}{\partial \hat{t}_j} Q_2(\theta_0, 0, \ldots, 0)(\hat{t}_j - 0) + o_p(\delta)
\]
where $\delta = \| \hat{\theta}_{CE} - \theta_0 \| + \sum \| \hat{t}_j \|$.

Since $Q_2(\theta_0, 0, \ldots, 0) = 0$ and $\partial Q_2(\theta_0, 0, \ldots, 0)/\partial \theta = 0$

$$0 = \sum_{j=1}^{J} \frac{\partial}{\partial t_j} Q_2(\theta_0, 0, \ldots, 0)(\hat{t}_j - 0) + o_p(\delta). \quad (5)$$

Substituting Equation 4 into Equation 5 gives

$$0 = J \sum_{j=1}^{J} \left( \frac{\partial}{\partial t_j^T} Q_2(\theta_0, 0, \ldots, 0) \right) \left\{ Q_1_j(\theta_0, 0) + \frac{\partial}{\partial \theta} Q_1_j(\theta_0, 0)(\hat{\theta}_{CE} - \theta_0) \right\} + o_p(\max(\delta_j, \delta)).$$

The derivatives of $Q_1_j$ and $Q_2$ with respect to $\theta$ and $t_j$ are

$$\frac{\partial Q_{1j}(\theta_0, 0)}{\partial \theta} = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_j(z_{i,j}, \theta_0)}{\partial \theta} \rightarrow E \left( \frac{\partial g_j}{\partial \theta} \right)$$

$$\frac{\partial Q_{1j}(\theta_0, 0)}{\partial t_j} = -\frac{1}{n} \sum_{i=1}^{n} g_j(z_{i,j}, \theta_0) g_j^T(z_{i,j}, \theta_0) \rightarrow -E(g_j g_j^T)$$

$$\frac{\partial Q_2(\theta_0, 0, \ldots, 0)}{\partial t_j} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\partial g_j(x_i, \theta_0)}{\partial \theta} \right)^T \rightarrow E \left( \frac{\partial g_j}{\partial \theta} \right)^T$$

so

$$0 = \sum_{j=1}^{J} \left( E \left( \frac{\partial g_j}{\partial \theta} \right)^T \left\{ E(g_j g_j^T) \right\}^{-1} \left\{ -Q_{1j}(\theta_0, 0) \right\} \right)$$

$$- \sum_{j=1}^{J} \left( E \left( \frac{\partial g_j}{\partial \theta} \right)^T \left\{ E(g_j g_j^T) \right\}^{-1} E \left( \frac{\partial g_j}{\partial \theta} \right)(\hat{\theta}_{CE} - \theta_0) \right)$$

$$+ o_p(1) \Rightarrow$$

$$(\hat{\theta}_{CE} - \theta_0) = W_{\theta}^{-1} \sum_{j=1}^{J} \left( E \left( \frac{\partial g_j}{\partial \theta} \right)^T \left\{ E(g_j g_j^T) \right\}^{-1} \left\{ -Q_{1j}(\theta_0, 0) \right\} \right) + o_p(1)$$

where

$$W_{\theta} = \sum_{j=1}^{J} \left( E \left( \frac{\partial g_j}{\partial \theta} \right)^T \left\{ E(g_j g_j^T) \right\}^{-1} E \left( \frac{\partial g_j}{\partial \theta} \right) \right).$$

For all $j$

$$-\sqrt{n}Q_{1j}(\theta_0, 0) \rightarrow N(0, E(g_j g_j^T)),$$
so the variance of $\sqrt{n}(\hat{\theta}_{CE} - \theta_0)$ is

$$
\begin{bmatrix}
W_\theta^{-1} E \left( \frac{\partial g_1}{\partial \theta} \right)^T \{E(g_1 g_1^T)\}^{-1} & \cdots & W_\theta^{-1} E \left( \frac{\partial g_J}{\partial \theta} \right)^T \{E(g_J g_J^T)\}^{-1}
\end{bmatrix} \times
\begin{bmatrix}
E(g_1 g_1^T) & \cdots & E(g_1 g_J^T) \\
\vdots & \ddots & \vdots \\
E(g_J g_1^T) & \cdots & E(g_J g_J^T)
\end{bmatrix}

\times
\begin{bmatrix}
\{E(g_1 g_1^T)\}^{-1} E \left( \frac{\partial g_1}{\partial \theta} \right) W_\theta^{-1} \\
\vdots \\
\{E(g_J g_J^T)\}^{-1} E \left( \frac{\partial g_J}{\partial \theta} \right) W_\theta^{-1}
\end{bmatrix}

= \sum_{k=1}^J \sum_{j=1}^J \left( W_\theta^{-1} E \left( \frac{\partial g_j}{\partial \theta} \right)^T \{E(g_j g_j^T)\}^{-1} E(g_j g_k^T) \{E(g_k g_k^T)\}^{-1} E \left( \frac{\partial g_k}{\partial \theta} \right) W_\theta^{-1} \right)

= W_\theta^{-1} V W_\theta^{-1}

which completes the proof.

Proof of Theorem 2. Following Qin & Lawless (1994) for a given value of $\theta$

$$
\ell_{CE}(\theta) = \sum_{j=1}^J \left( \frac{n}{2} \left\{ \frac{1}{n} \sum_{i=1}^n g_j^T(z_{i,j}, \theta) \right\} \left\{ \frac{1}{n} \sum_{i=1}^n g_j(z_{i,j}, \theta) g_j^T(z_{i,j}, \theta) \right\}^{-1} \times
\left\{ \frac{1}{n} \sum_{i=1}^n g_j(z_{i,j}, \theta) \right\} \right) + o_p(1)

so using the notation from Lemma (1)

$$
\ell_{CE}(\theta_0) = \sum_{j=1}^J \left( \frac{n}{2} Q_{1j}^T(\theta_0, 0) \{E(g_j g_j^T)\}^{-1} Q_{1j}(\theta_0, 0) \right) + o_p(1).

Using the derivatives shown in the proof of Lemma (1) we have by Taylor expansion of $Q_{1j}(\theta_0, 0)$

$$
\ell_{CE}(\hat{\theta}_{CE}) = \ell_{CE}(\theta_0) -
\frac{n}{2} \sum_{j=1}^J \sum_{k=1}^J \left( Q_{1j}^T(\theta_0, 0) \{E(g_j g_j^T)\}^{-1} E \left( \frac{\partial g_j}{\partial \theta} \right) W_\theta^{-1} \times
E \left( \frac{\partial g_j}{\partial \theta} \right)^T \{E(g_j g_j^T)\}^{-1} Q_{1j}(\theta_0, 0) \right) + o_p(1),

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and

\[
\ell_{CE}(\phi_0, \hat{\nu}_{CE}(\phi_0)) = \ell_{CE}(\theta_0) - \frac{n}{2} \sum_{j=1}^{J} \sum_{k=1}^{K} \left( Q_{ij}^T(\theta_0, 0) \left\{ E(g_j g_j^T) \right\}^{-1} E \left( \frac{\partial g_j}{\partial \nu} \right) \right) \times E \left( \frac{\partial g_j}{\partial \theta} \right)^T \left\{ E(g_j g_j^T) \right\}^{-1} Q_{ij}(\theta_0, 0) + o_p(1).
\]

Now

\[
T = 2\ell_{CE}(\hat{\theta}_{CE}) - 2\ell_{CE}(\phi_0, \hat{\nu}_{CE}(\phi_0))
\]

\[
= \sum_{j=1}^{J} \sum_{k=1}^{K} \left( \sqrt{n} Q_{ij}^T(\theta_0, 0) \left\{ E(g_j g_j^T) \right\}^{-1} E \left( \frac{\partial g_j}{\partial \theta} \right) \right) \times E \left( \frac{\partial g_k}{\partial \theta} \right)^T \left\{ E(g_k g_k^T) \right\}^{-1} Q_{1k}(\theta_0, 0)
\]

\[
- \sum_{j=1}^{J} \sum_{k=1}^{K} \left( \sqrt{n} Q_{ij}^T(\theta_0, 0) \left\{ E(g_j g_j^T) \right\}^{-1} E \left( \frac{\partial g_j}{\partial \nu} \right) \right) \times E \left( \frac{\partial g_k}{\partial \nu} \right)^T \left\{ E(g_k g_k^T) \right\}^{-1} Q_{1k}(\theta_0, 0)
\]

which can be written as

\[
\begin{bmatrix}
\sqrt{n} Q_{11}^T(\theta_0, 0) & \cdots & \sqrt{n} Q_{1J}^T(\theta_0, 0)
\end{bmatrix} \times A \times \begin{bmatrix}
\sqrt{n} Q_{11}(\theta_0, 0) \\
\vdots \\
\sqrt{n} Q_{1J}(\theta_0, 0)
\end{bmatrix}
\]

where A is defined in Theorem (2). \( \sqrt{n} Q_{1j}(\theta_0, 0) \) converges to a normal with mean 0 and covariance \( E(g_j g_j^T) \) which completes the proof.

**Proof of Corollary 1.** T can be written as

\[
T = 2\ell_{CE}(\hat{\theta}_{CE}) - 2\ell_{CE}(\phi_0, \hat{\nu}_{CE}(\phi_0))
\]

\[
= \sum_{j=1}^{J} \sum_{k=1}^{K} \left( \sqrt{n} Q_{ij}^T(\theta_0, 0) \left\{ E(g_j g_j^T) \right\}^{-1/2} \left\{ E(g_j g_j^T) \right\}^{-1/2} E \left( \frac{\partial g_j}{\partial \theta} \right) \right) \times W_{\theta}^{-1} E \left( \frac{\partial g_k}{\partial \theta} \right)^T \left\{ E(g_k g_k^T) \right\}^{-1/2} \left\{ E(g_k g_k^T) \right\}^{-1/2} Q_{1k}(\theta_0, 0)
\]

\[
- \sum_{j=1}^{J} \sum_{k=1}^{K} \left( \sqrt{n} Q_{ij}^T(\theta_0, 0) \left\{ E(g_j g_j^T) \right\}^{-1/2} \left\{ E(g_j g_j^T) \right\}^{-1/2} E \left( \frac{\partial g_j}{\partial \nu} \right) \right) \times W_{\nu}^{-1} E \left( \frac{\partial g_k}{\partial \nu} \right)^T \left\{ E(g_k g_k^T) \right\}^{-1/2} \left\{ E(g_k g_k^T) \right\}^{-1/2} Q_{1k}(\theta_0, 0)
\].

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\[
\sqrt{n} \left\{ E(g_j g_j^T) \right\}^{-1/2} Q_{ij} (\theta_0, 0) \text{ is asymptotically standard multivariate normal and } \text{cov}(Q_{ij}(\theta_0, 0), Q_{ik}(\theta_0, 0)) = 0 \text{ for all } j \neq k. \text{ So we need only show that } A \text{ is idempotent. We can rewrite } A \text{ as }
\]
\[
A = A^\theta - A^\nu
\]
where the \( j, k \)th entries of each matrix are
\[
A^\theta_{j,k} = \left\{ E(g_j g_j^T) \right\}^{-1/2} E \left( \frac{\partial g_j}{\partial \theta} \right) W_{\theta}^{-1} E \left( \frac{\partial g_k}{\partial \theta} \right)^T \left\{ E(g_k g_k^T) \right\}^{-1/2}
\]
and
\[
A^\nu_{j,k} = \left\{ E(g_j g_j^T) \right\}^{-1/2} E \left( \frac{\partial g_j}{\partial \nu} \right) W_{\nu}^{-1} E \left( \frac{\partial g_k}{\partial \nu} \right)^T \left\{ E(g_k g_k^T) \right\}^{-1/2}.
\]
Both \( A^\theta \) and \( A^\nu \) are idempotent, with ranks \( p \) and \( q \) respectively. To establish that \( T \) is \( \chi^2 \) with \( q \) degrees of freedom we only need to show that \( A \) is non-negative definite (see Rao, 1973, page 187). We have
\[
\left( A^\theta \right)_{jk} \propto E \left( \frac{\partial g_j}{\partial \theta} \right) W_{\theta}^{-1} E \left( \frac{\partial g_k}{\partial \theta} \right)^T \geq \left[ E \left( \frac{\partial g_j}{\partial \phi} \right), E \left( \frac{\partial g_j}{\partial \nu} \right) \right] \left[ \begin{array}{cc} 0 & 0 \\ 0 & W_{\nu}^{-1} \end{array} \right] \left[ \begin{array}{c} E \left( \frac{\partial g_k}{\partial \phi} \right)^T \\ E \left( \frac{\partial g_k}{\partial \nu} \right)^T \end{array} \right] = E \left( \frac{\partial g_j}{\partial \nu} \right) W_{\nu}^{-1} E \left( \frac{\partial g_j}{\partial \nu} \right)^T \propto (A^\nu)_{jk}
\]
so \( A \) is non negative definite with rank \( p - (p - q) = q \).

\[\square\]

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