The General Class of String Theories on Orbifolds

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Abstract

We investigate the following three consistency conditions for constructing string theories on orbifolds: i) the invariance of the energy-momentum tensors under twist operators, ii) the duality of amplitudes and iii) modular invariance of partition functions. It is shown that this investigation makes it possible to obtain the general class of consistent orbifold models, which includes a new class of orbifold models.
1. Introduction

In the construction of realistic four-dimensional string models, various approaches have been proposed [1-8]. Among them, the orbifold compactification [1] is probably the most efficient method and is believed to provide a phenomenologically realistic string model. The search for realistic orbifold models has been continued by many authors [9-12]. However, only a very small class of orbifold models has been investigated so far and any satisfactory orbifold models have not yet been found. A more general and systematic investigation should be required.

An orbifold [1] will be obtained by dividing a torus by the action of a discrete symmetry group \( G \) of the torus. A large number of studies have been made on a class of orbifold models in which any group element \( g \) of \( G \) is represented by [1]

\[
g = (U, v), \tag{1-1}
\]

or more generally for asymmetric orbifolds [13]

\[
g = (U_L, v_L; U_R, v_R), \tag{1-2}
\]

where \((U_L, U_R)\) are rotation matrices and \((v_L, v_R)\) are shift vectors. The action of \( g \) on a left- and right-moving string coordinate \((X_L, X_R)\) is given by

\[
g(X_L, X_R)g^\dagger = (U_L(X_L + 2\pi v_L), U_R(X_R - 2\pi v_R)). \tag{1-3}
\]

However, to the best of our knowledge, there have been few discussions about the questions whether there might be any other class of consistent orbifold models and whether the action of \( g \) on the string coordinate might in general be given by eq.(1-3).

The purpose of this paper is to answer the question what is the most general class of consistent bosonic string theories on orbifolds. We shall show that any group element \( g \) of \( G \) can indeed be specified by eq.(1-2) but that the action of \( g \) on the string coordinate given in eq.(1-3) is not, in general, correct.

In section 2, we describe the basic setup and discuss consistency conditions of string theories on orbifolds. In section 3, we investigate the cocycle property of vertex operators and present an explicit operator representation of cocycle operators, which are attached to vertex operators to ensure the duality of amplitudes. In section 4, we
discuss the duality of amplitudes in detail. It is shown that the requirement of the
duality of amplitudes severely restricts the allowed action of $g$ on the string coordinate
and that the transformation (1-3) has to be modified in general. In this analysis, we
see that the representation of the cocycle operator given in section 3 plays an crucial
role. In section 5, we discuss one loop modular invariance of partition functions and
see that this argument justifies our prescription. In section 6, we present an example
of orbifold models, which will give a good illustration of our formalism. Section 7 is
devoted to discussions. In appendix A, we prove a theorem, which will be used in the
text. In appendix B, we prove that any representation of cocycle operators can reduce
to the representation given in section 3 by a suitable unitary transformation (up to a
constant phase).

2. Operator Formalism for Bosonic String Theories on Orbifolds

An orbifold [1] will be obtained by dividing a torus by the action of a suitable
discrete group $G$. In the construction of an orbifold model, we start with a
$D$-dimensional toroidally compactified closed bosonic string theory which is specified
by a $(D + D)$-dimensional lorentzian even self-dual lattice $\Gamma_{D,D}$ [14], on which the
left- and right-moving momentum $(p^I_L, p^I_R) (I = 1, \cdots, D)$ lies. Since an orbifold model
is given by specifying the action of each group element $g$ of $G$ on the left- and right-
moving string coordinate $(X^I_L, X^I_R) (I = 1, \cdots, D)$, our aim of this paper is to answer
the question what is the most general allowed action of $g$ on the string coordinate. To
determine the allowed action of $g$ on the string coordinate, we require the following
three conditions:

(i) The invariance of the energy-momentum tensors under the action of $g$; This condi-
tion guarantees the single-valuedness of the energy-momentum tensors on the
orbifold.

(ii) The duality of amplitudes; This is one of the important properties of string the-
tories [15,16].

(iii) Modular invariance of partition functions; Modular invariance plays an important

\[\dagger\] The generalization to a lorentzian even self-dual lattice with signature $(p,q)$ will
be straightforward and will not be discussed here.
role in the construction of consistent string models [16] and conformally invariant field theories [17]. Modular invariance may ensure the ultraviolet finiteness and the anomaly free condition of superstring theories [16,18]. The space-time unitary also requires modular invariance [19].

Although the first and the third conditions (i) and (iii) have already been considered, little attention has been given to the second condition (ii) so far. As we will see later, our main results will be obtained from the detailed analysis of the second condition (ii).

Let us first consider the condition (i), that is, the energy-momentum tensors have to be invariant under the action of $g$. The energy-momentum tensors of the left- and right-movers are given by

$$T_L(z) = \lim_{w \rightarrow z} \frac{1}{2} P^I_L(w) P^I_L(z) - \frac{D}{(w - z)^2},$$

$$T_R(\bar{z}) = \lim_{\bar{w} \rightarrow \bar{z}} \frac{1}{2} P^I_R(\bar{w}) P^I_R(\bar{z}) - \frac{D}{(\bar{w} - \bar{z})^2}, \quad (2-1)$$

where $P^I_L(z)$ and $P^I_R(\bar{z})$ are the momentum operators of the left- and right-movers defined by

$$P^I_L(z) = i \partial_z X^I_L(z),$$

$$P^I_R(\bar{z}) = i \partial_{\bar{z}} X^I_R(\bar{z}), \quad (I = 1, \ldots, D). \quad (2-2)$$

It follows that the energy-momentum tensors are invariant under the action of $g$ if $g$ acts on $(P^I_L(z), P^I_R(\bar{z}))$ as

$$g(P^I_L(z), P^I_R(\bar{z})) g^\dagger = (U^I_J P^I_L(z), U^I_J P^I_R(\bar{z})), \quad (2-3)$$

where $U_L$ and $U_R$ are suitable elements of the $D$-dimensional orthogonal group $O(D)$. Note that $U_L$ is not necessarily equal to $U_R$ and that orbifolds with $U_L \neq U_R$ are called asymmetric orbifolds [13].

In the untwisted sector, the left- and right-moving string coordinates, $X^I_L(z)$ and $X^I_R(\bar{z})$, are expanded as

$$X^I_L(z) = x^I_L - ip^I_L ln z + i \sum_{n \neq 0} \frac{1}{n} \alpha^I_{L_n} z^{-n},$$

$$X^I_R(\bar{z}) = x^I_R - ip^I_R ln \bar{z} + i \sum_{n \neq 0} \frac{1}{n} \alpha^I_{R_n} \bar{z}^{-n}, \quad (I = 1, \cdots, D). \quad (2-4)$$
where \(x^I_L\) and \(p^I_L\) (\(x^I_R\) and \(p^I_R\)) are the center of mass coordinate and momentum of the left- (right-) mover, respectively. The quantization conditions are given by

\[
[x^I_L, p^J_L] = i\delta^{IJ} = [x^I_R, p^J_R],
\]

\[
[\alpha^I_{Lm}, \alpha^J_{Ln}] = m\delta^{IJ}\delta_{m+n,0} = [\alpha^I_{Rm}, \alpha^J_{Rn}],
\]

otherwise zeros.  \hfill (2 - 5)

The toroidal compactification means that the momentum \((p^I_L, p^I_R)\) lies on a \((D + D)\)-dimensional lorentzian even self-dual lattice \(\Gamma^{D,D}\) \cite{14}. In terms of \((p^I_L, \alpha^I_{Ln})\) and \((p^I_R, \alpha^I_{Rn})\), eq. (2-3) can be rewritten as

\[
g(p^I_L, \alpha^I_{Ln})g^\dagger = U^{IJ}_L(p^J_L, \alpha^J_{Ln}),
\]

\[
g(p^I_R, \alpha^I_{Rn})g^\dagger = U^{IJ}_R(p^J_R, \alpha^J_{Rn}).
\]

Since \((p^I_L, p^I_R)\) lies on the lattice \(\Gamma^{D,D}\), the action of \(g\) on \((p^I_L, p^I_R)\) should be an automorphism of \(\Gamma^{D,D}\), i.e.,

\[
(U^{IJ}_L p^J_L, U^{IJ}_R p^J_R) \in \Gamma^{D,D} \quad \text{for all} \quad (p^I_L, p^I_R) \in \Gamma^{D,D}.
\]

\hfill (2 - 7)

Since \(P^I_L(z)\) and \(P^I_R(\bar{z})\) do not include \(x^I_L\) and \(x^I_R\), the relation (2-3) or (2-6) does not completely determine the action of \(g\) on \((x^I_L, x^I_R)\). In fact, the general action of \(g\) on \((x^I_L, x^I_R)\), which is compatible with the quantization conditions (2-5), may be given by \cite{20}

\[
g(x^I_L, x^I_R)g^\dagger = (U^{IJ}_L x^J_L + \pi \frac{\partial \Phi(p^I_L, p^I_R)}{\partial p^I_L}, U^{IJ}_R x^J_R + \pi \frac{\partial \Phi(p^I_L, p^I_R)}{\partial p^I_R}),
\]

\hfill (2 - 8)

where \(\Phi(p^I_L, p^I_R)\) is an arbitrary function of \(p^I_L\) and \(p^I_R\). Let \(g_U\) be the unitary operator which satisfies

\[
g_U(X^I_L(z), X^I_R(\bar{z}))g_U^\dagger = (U^{IJ}_L X^J_L(z), U^{IJ}_R X^J_R(\bar{z})),
\]

\hfill (2 - 9)

and

\[
g_U|0\rangle = |0\rangle,
\]

\hfill (2 - 10)

where \(|0\rangle\) is the vacuum of the untwisted sector. Then, the twist operator \(g\) which generates the transformations (2-6) and (2-8) will be given by

\[
g = e^{i\pi\Phi(p^I_L, p^I_R)}g_U.
\]

\hfill (2 - 11)
At this stage, $\Phi(p_L, p_R)$ is an arbitrary function of $p^I_L$ and $p^I_R$. In section 4, we will see that the second condition (ii) severely restricts the form of the phase factor in $g$.

It may be worth while making a comment on the path integral formalism [21] here. The action of a closed bosonic string theory will be given by

$$S[X] = \int d\tau \int_0^\pi d\sigma \frac{1}{2\pi} \{\eta^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^I + \varepsilon^{\alpha\beta} B^{IJ} \partial_\alpha X^I \partial_\beta X^J\},$$

(2-12)

where $\tau$ and $\sigma$ correspond to the “time” and “space” variables of the world sheet and $B^{IJ}$ ($I, J = 1, \cdots, D$) is an antisymmetric constant background field [14]. The string coordinate $X^I(\tau, \sigma)$ in the untwisted sector will be expanded as

$$X^I(\tau, \sigma) = x^I + (p^I - B^{IJ} w^J) \tau + w^I \sigma + (\text{oscillators}),$$

(2-13)

where $p^I$ and $w^I$ are the center of mass momentum and the winding number, which are related to the left- and the right-moving momenta, $p^I_L$ and $p^I_R$, as follows:

$$p^I_L = \frac{1}{2} p^I + \frac{1}{2} (1 - B)^{IJ} w^J,$$

$$p^I_R = \frac{1}{2} p^I - \frac{1}{2} (1 + B)^{IJ} w^J.$$

(2-14)

Let us consider a transformation

$$X^I \rightarrow U^{IJ} X^J,$$

(2-15)

where $U^{IJ} \in O(D)$. (This corresponds to a symmetric orbifold, i.e., $U_L = U_R \equiv U$.) Clearly the action (2-12) is not invariant under the transformation (2-15) unless

$$[B, U] = 0.$$

(2-16)

The noncommutativity of $B^{IJ}$ and $U^{IJ}$ might cause a trouble in the path integral formalism because the action (2-12) will not be single-valued on the orbifold. On the other hand, this noncommutativity seems to cause no trouble in the operator formalism because the second term in the action (2-12) is a total divergence and hence the explicit $B^{IJ}$-dependence does not appear in the energy-momentum tensors (2-1) as well as the equation of motion. The $B^{IJ}$-dependence can, however, appear in the zero modes as

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$\dagger$ $\eta^{\alpha\beta} = diag(1, -1)$ and $\varepsilon^{01} = -\varepsilon^{10} = 1$. 

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in eqs. (2-13) and (2-14). As we will see in section 4, the noncommutativity of $B^{IJ}$ and $U^{IJ}$ (more generally see eq. (4-6) for asymmetric orbifolds) might cause a trouble even in the operator formalism, that is, the violation of the duality of amplitudes. Its resolution will be our main concern in section 4. In this paper, we will restrict our considerations only to the operator formalism. We will leave the reinterpretation of our results from the point of view of the path integral formalism for future work.

3. A Representation of Cocycle Operators

In this section, we shall investigate cocycle properties of vertex operators and give an explicit operator representation of cocycle operators. Let us consider a vertex operator which describes the emission of a state with the momentum $(k^I_L, k^I_R) \in \Gamma^{D,D}$,

$$V(k^I_L, k^I_R; z) = e^{ik^I_L \cdot X^I_L(z) + ik^I_R \cdot X^I_R(\bar{z})} C_{k^I_L, k^I_R}, \quad (3-1)$$

where $: :$ denotes the normal ordering and $C_{k^I_L, k^I_R}$ is the cocycle operator, which is attached to the vertex operator to ensure the correct commutation relations and the duality of amplitudes [16,22]. The product of two vertex operators

$$V(k^I_L, k^I_R; z)V(k'^I_L, k'^I_R; z'), \quad (3-2)$$

is well-defined if $|z| > |z'|$. The different ordering of the two vertex operators corresponds to the different “time”-ordering. To obtain scattering amplitudes, we must sum over all possible “time”-ordering for the emission of states. We must then establish that each contribution is independent of the order of the vertex operators to enlarge the regions of integrations over $z$ variables [15]. Thus the product (3-2), with respect to $z$ and $z'$, has to be analytically continued to the region $|z'| > |z|$ and to be identical to

$$V(k'^I_L, k'^I_R; z')V(k^I_L, k^I_R; z), \quad (3-3)$$

for $|z'| > |z|$. In terms of the zero modes, the above statement can be expressed as

$$V_0(k^I_L, k^I_R)V_0(k'^I_L, k'^I_R) = (-1)^{k^I_L \cdot k'^I_L - k^I_R \cdot k'^I_R} V_0(k'^I_L, k'^I_R)V_0(k^I_L, k^I_R), \quad (3-4)$$

where

$$V_0(k^I_L, k^I_R) = e^{ik^I_L \cdot X^I_L + ik^I_R \cdot X^I_R} C_{k^I_L, k^I_R}. \quad (3-5)$$
The factor \((-1)^{k_L'k_R' - k_L k_R}\) in eq.(3-4) appears in reversing the order of the nonzero modes of the vertex operators. This annoying factor is the reason for the necessity of the cocycle operator \(C_{k_L, k_R}\).

The second condition (ii) is now replaced by the statement that the duality relation (3-4) has to be preserved under the action of \(g\). To examine this condition, we need to know an explicit operator representation of the cocycle operator \(C_{k_L, k_R}\). For notational simplicity, we may use the following notations: \(k^A \equiv (k^I_L, k^I_R)\), \(x^A \equiv (x^I_L, x^I_R)\), etc. \((A, B, \ldots \text{ run from } 1 \text{ to } 2D \text{ and } I, J, \ldots \text{ run from } 1 \text{ to } D\).) To obtain an operator representation of the cocycle operator \(C_k\), let us assume \([23,24]\)

\[
C_k = e^{i \pi k^A M^{AB} \bar{p}^B},
\]

where the wedge \(\wedge\) may be attached to operators to distinguish between c-numbers and q-numbers. Then, the matrix \(M^{AB}\) has to satisfy

\[
e^{i \pi k^A (M - M^T)^{AB} k'^B} = (-1)^{k^A \eta^{AB} k'^B} \quad \text{for all } k^A, k'^A \in \Gamma^{D,D},
\]

where

\[
\eta^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

A solution to this equation may be given by

\[
M^{AB} = \begin{pmatrix} -\frac{1}{2} B^{IJ} & -\frac{1}{2} (1 - B)^{IJ} \\ \frac{1}{2} (1 + B)^{IJ} & -\frac{1}{2} B^{IJ} \end{pmatrix},
\]

which satisfies

\[
M^{AB} = -M^{BA}.
\]

The \(B^{IJ}\) is an antisymmetric constant matrix and is defined as follows: Any \((D + D)\)-dimensional lorentzian even self-dual lattice \(\Gamma^{D,D}\) can be parametrized in terms of a \(D\)-dimensional Euclidean lattice \(\Lambda\) and an antisymmetric constant matrix \(B^{IJ}\) \([14]\) as

\[
p^I_L = \frac{1}{2} p^I + \frac{1}{2} (1 - B)^{IJ} w^J,
\]

\[
p^I_R = \frac{1}{2} p^I - \frac{1}{2} (1 + B)^{IJ} w^J,
\]

\[\uparrow\] The variables \(p^I, w^I\) and \(B^{IJ}\) are exactly the same as in eq.(2-14).
where

\[(p_L^I, p_R^I) \in \Gamma^{D,D},\]
\[p^I \in 2\Lambda^*,\]
\[w^I \in \Lambda.\]  \hspace{1cm} (3 − 12)

Here, \(\Lambda^*\) denotes the dual lattice of \(\Lambda\). Physically, \(p^I\) and \(w^I\) correspond to the center of mass momentum and the winding number, respectively.

Although we have obtained a representation of the cocycle operator \(C_k\), its representation is not unique. In fact, there exist infinitely many other representations of \(C_k\). However, as we will see in appendix B, by a suitable unitary transformation any representation of \(C_k\) can be shown to reduce to eq.(3-6) with (3-9) up to a constant phase. Thus, it will be sufficient to consider only the representation (3-6) with (3-9) for our purpose. In the next section, we will see that the representation (3-6) plays a crucial role in investigating the duality of amplitudes.

4. The Duality of Amplitudes

In the previous section, we have obtained a representation of the cocycle operator \(C_k\). To explicitly show the dependence of the cocycle operator in the zero mode part of the vertex operator (3-5), we may write

\[V_0(k; M) \equiv e^{ik \cdot \hat{x}} e^{i\pi k \cdot \hat{p}}.\]  \hspace{1cm} (4 − 1)

Under the action of \(g_U\), \(V_0(k; M)\) transforms as

\[g_U V_0(k; M) g_U^\dagger = V_0(U^T k; U^T MU),\]  \hspace{1cm} (4 − 2)

where

\[U^{AB} = \begin{pmatrix} U_{IJ}^L & 0 \\ 0 & U_{IJ}^R \end{pmatrix} \]  \hspace{1cm} (4 − 3)

It is easy to see that the product of \(V_0(k; M)\) and \(V_0(k'; U^T MU)\) satisfies

\[V_0(k; M)V_0(k'; U^T MU) = \xi(-1)^{k \cdot \eta k'} V_0(k'; U^T MU)V_0(k; M),\]  \hspace{1cm} (4 − 4)
where
\[ \xi = e^{-i\pi k \cdot (M - U^T MU)k'}. \] (4.5)

This relation implies that the duality relation (3-4) cannot be preserved under the action of \( g_U \) unless \( \xi = 1 \) for all \( k^A, k'^A \in \Gamma^{D,D} \). It may be worth while noting that if \( \xi \neq 1 \) it means
\[ [M, U] \neq 0. \] (4.6)

For symmetric orbifolds (i.e., \( U_L = U_R \)), eq.(4-6) means the noncommutativity of \( B^{IJ} \) and \( U^{IJ}_L \) (or \( U^{IJ}_R \)). As mentioned in section 2, this noncommutativity may cause a trouble in the path integral formalism and, as just seen above, also in the operator formalism it causes a trouble, that is, the violation of the duality relation (3-4) under the action of \( g_U \).

We have seen that the duality relation (3-4) cannot be preserved under the action of \( g_U \) unless \( \xi = 1 \) for all \( k^A, k'^A \in \Gamma^{D,D} \). It does not, however, mean the violation of the duality relation under the action of \( g \) because the freedom of \( \Phi(p) \) in \( g \) has not been used yet. Define
\[ V_0'(k; M) = g V_0(Uk; M) g^\dagger = e^{ik \cdot \hat{x}} e^{i\pi k \cdot U^T MU \hat{p} e^{i\pi \Phi(\hat{p} + k) - i\pi \Phi(\hat{p})}. \] (4.7)

It is easy to see that
\[ V_0(k; M) V_0'(k'; M) = e^{-i\pi \Theta (-1)^{k \cdot \eta k'} V_0'(k'; M)V_0(k; M), \] (4.8)

where
\[ \Theta = k^A (M - U^T MU)_{AB} k'^B + \Phi(p - k - k') - \Phi(p - k) - \Phi(p - k') + \Phi(p). \] (4.9)

Thus the duality relation (3-4) requires that
\[ \Theta = 0 \mod 2. \] (4.10)

To solve the equation (4-10), it may be convenient to change the basis of the momentum \( p^A \in \Gamma^{D,D} \). Let \( e_a^A \) \( (a = 1, \ldots, 2D) \) be a basis of \( \Gamma^{D,D} \), i.e.,
\[ \Gamma^{D,D} = \{ p^A = \sum_{a=1}^{2D} p^a e_a^A, p^a \in \mathbb{Z} \}. \] (4.11)
Suppose that $\Phi(p)$ is expanded as

$$\Phi(p) = \Phi_0(p) + \Delta \Phi(p), \quad (4 - 12)$$

where

$$\Phi_0(p) = \phi + 2v_ap^a + \frac{1}{2}C_{ab}p^a p^b, \quad (4 - 13)$$

$$\Delta \Phi(p) = \sum_{n=2}^{N} \frac{1}{n!} \Delta C_{a_1 \ldots a_n}^{(n)} p^{a_1} \cdots p^{a_n}, \quad (p^a \in \mathbb{Z}). \quad (4 - 14)$$

Here, $N \geq 2$ is an arbitrary positive integer and the symmetric matrix $C_{ab}$ is defined through the relation,

$$C_{ab} = -e_a^A (M - U^T MU)^A B e^B_b \mod 2. \quad (4 - 15)$$

At first sight, it seems that there is no solution to eq.(4-15) because $C_{ab}$ is a symmetric matrix but $M^{AB}$ is an antisymmetric one. However, we can always find a symmetric matrix $C_{ab}$ satisfying (4-15) because eq.(3-7) with eq.(3-10) implies that

$$e_a^A (M - U^T MU)^A B e^B_b \in \mathbb{Z}, \quad (4 - 16)$$

which guarantees the existence of a solution to eq.(4-15). Inserting eq. (4-12) into eq.(4-9) and using the relation (4-15), we find that the condition (4-10) reduces to

$$\Delta \Phi(p - k - k') - \Delta \Phi(p - k) - \Delta \Phi(p - k') + \Delta \Phi(p) = 0 \mod 2. \quad (4 - 17)$$

Inserting eq.(4-14) into eq.(4-17) and comparing the $N$th order terms of both sides of eq.(4-17) with respect to $p^a$, $k^a$ and $k'^a$, we have

$$\frac{1}{N!} \sum_{a_1, \ldots, a_N=1}^{2^D} \Delta C_{a_1 \ldots a_N}^{(N)} ((p - k - k')^{a_1} \cdots (p - k - k')^{a_N} - (p - k)^{a_1} \cdots (p - k)^{a_N}$$

$$- (p - k')^{a_1} \cdots (p - k')^{a_N} + p^{a_1} \cdots p^{a_N}) = 0 \mod 2, \quad (4 - 18)$$

for all $p^a, k^a, k'^a \in \mathbb{Z}$. This equation gives various constraints on the coefficient $\Delta C_{a_1 \ldots a_N}^{(N)}$. For example,

$$\Delta C_{a \ldots ab}^{(N)} \in (N - 1)! \ 2\mathbb{Z},$$

$$\Delta C_{a \ldots abb}^{(N)} \in (N - 2)! \ 2\!\!\!2\mathbb{Z},$$

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\[ \Delta C_{a \cdots abc}^{(N)} \in (N - 2)! \ 2\mathbb{Z}, \]
\[ \Delta C_{a \cdots abbb}^{(N)} \in (N - 3)! \ 3! \ 2\mathbb{Z}, \]

\[ \ldots \]

etc. \hspace{1cm} (4 - 19)

Then it is not difficult to show the following equality:

\[ \frac{1}{N!} \sum_{a_1, \ldots, a_N = 1}^{2D} \Delta C_{a_1 \cdots a_N}^{(N)} p^{a_1} \cdots p^{a_N} = \frac{1}{N!} \sum_{a = 1}^{2D} \Delta C_{a_1 \cdots a}^{(N)} (p^{a})^N \mod 2, \hspace{1cm} (4 - 20) \]

with \( \Delta C_{a \cdots a}^{(N)} \in (N - 1)! \ 2\mathbb{Z} \). Let \((m, n)\) be the highest common divisor of \(m\) and \(n\) and \(\varphi(n)\) be the Euler function, which is equal to the number of \(d\) such that \((d, n) = 1\) for \(d = 1, 2, \ldots, n - 1\). In appendix A, we will prove the following theorem: Let \(n\) and \(p\) be arbitrary positive integers. Then

\[ p^n = p^{n - \varphi(n)} \mod n. \hspace{1cm} (4 - 21) \]

Using this theorem and noting \( \Delta C_{a \cdots a}^{(N)} \in (N - 1)! \ 2\mathbb{Z} \), we have

\[ \frac{1}{N!} \sum_{a = 1}^{2D} \Delta C_{a_1 \cdots a}^{(N)} (p^{a})^N = \frac{1}{N!} \sum_{a = 1}^{2D} \Delta C_{a_1 \cdots a}^{(N)} (p^{a})^{N - \varphi(N)} \mod 2. \hspace{1cm} (4 - 22) \]

Thus we have found that the \(N\)th order term of \(\Phi(p)\) (i.e., the left hand side of eq.(4-20)) can reduce to the \((N - \varphi(N))\)th order term (i.e., the right hand side of eq.(4-22)). Therefore, we can put the \(N\)th order term of \(\Phi(p)\) to be equal to zero because the right hand side of eq.(4-22) can be absorbed into the \((N - \varphi(N))\)th order term of \(\Phi(p)\) by suitably redefining the coefficient of the \((N - \varphi(N))\)th order term of \(\Phi(p)\).

Next consider the \((N - 1)\)th order term in eq.(4-17) with respect to \(p^a\), \(k^a\) and \(k'^a\). Comparing both sides of eq.(4-17) and using the theorem (4-21), we can show the following equality:

\[ \frac{1}{(N - 1)!} \sum_{a_1, \ldots, a_{N-1} = 1}^{2D} \Delta C_{a_1 \cdots a_{N-1}}^{(N-1)} p^{a_1} \cdots p^{a_{N-1}} \]

\[ = \frac{1}{(N - 1)!} \sum_{a = 1}^{2D} \Delta C_{a_1 \cdots a}^{(N-1)} (p^{a})^{N - 1 - \varphi(N - 1)} \mod 2, \hspace{1cm} (4 - 23) \]
with $\Delta C_{a\cdots a}^{(N-1)} \in (N-2)! \cdot 2\mathbb{Z}$. Thus the $(N-1)$th order term of $\Phi(p)$ can be absorbed into the $(N-1 - \varphi(N-1))$th order term of $\Phi(p)$ by suitably redefining the coefficient of the $(N-1 - \varphi(N-1))$th order term. Repeating the above argument order by order, we conclude that $\Delta \Phi(p)$ can be put to be equal to zero, i.e.,

$$\Delta \Phi(p) = 0 \mod 2. \quad (4-24)$$

We have observed that the duality relation can be preserved under the action of $g$ if $\Phi(p)$ in $g$ is chosen as

$$\Phi(p) = \phi + 2v_a p^a + \frac{1}{2} p^a C_{ab} p^b,$$

or equivalently,

$$\Phi(p) = \phi + 2v^A \eta^{AB} p^B + \frac{1}{2} p^A C^{AB} p^B, \quad (4-25)$$

where the symmetric matrix $C_{ab}$ is defined through the relation (4-15) and

$$v_a = v^A \eta^{AB} e^B_a, \quad C_{ab} = e^A_a C^{AB} e^B_b. \quad (4-26)$$

We will see in the next section that modular invariance requires $\phi = 0$ and imposes some constraints on $v_a$. The symmetric matrix $C_{ab}$ seems not to be defined uniquely in eq.(4-15). Let $C'_{ab}$ be another choice satisfying eq.(4-15). Then, we find

$$\frac{1}{2} \sum_{a,b} (C'_{ab} - C_{ab}) p^a p^b = \frac{1}{2} \sum_{a,b} (C'_{aa} - C_{aa}) (p^a)^2 \mod 2$$

$$= \frac{1}{2} \sum_a (C'_{aa} - C_{aa}) p^a \mod 2, \quad (4-27)$$

where we have used the fact that $C'_{ab} - C_{ab} \in 2\mathbb{Z}$ and $p^a \in \mathbb{Z}$. Thus the difference between $C'_{ab}$ and $C_{ab}$ can be absorbed into the redefinition of $v_a$ and hence the choice of $C_{ab}$ is essentially unique. Therefore, it is concluded that any twist operator $g$ can always be parametrized by $(U_L, v_L; U_R, v_R)$ and that the action (1-3) of $g$ on the string coordinate $X^A = (X^I_L, X^I_R)$ in the untwisted sector is not in general correct but

$$g X^A g^\dagger = U^{AB} (X^B + 2\pi \eta^{BC} v^C + \pi C^{BC} p^C), \quad (4-28)$$

as announced in the introduction.
Does the third term of $\Phi(p)$ in eq.(4-25) affect the physical spectrum? Any physical state on the orbifold has to be invariant under the action of $g$. The third term in eq.(4-25) contributes to $g$ as a momentum-dependent phase and hence plays an important role in extracting physical states from the Hilbert space. Although we have introduced the third term of $\Phi(p)$ in eq.(4-25) to preserve the duality of amplitudes, we will see in the next section that modular invariance will also require the introduction of the third term of $\Phi(p)$ in eq.(4-25).

5. One Loop Modular Invariance

In this section, we will investigate one loop modular invariance of partition functions. Let $Z(h, g; \tau)$ be the partition function of the $h$-sector twisted by $g$ which is defined, in the operator formalism, by

$$Z(h, g; \tau) = \text{Tr}[g e^{i2\pi\tau(L_0 - \frac{D}{24})} - i2\pi\tau(L_0 - \frac{D}{24})]_{h-\text{sector}},$$

(5-1)

where $L_0(\tilde{L}_0)$ is the Virasoro zero mode operator of the left- (right-) mover. The trace in eq.(5-1) is taken over the Hilbert space of the $h$-sector. Then, the one loop partition function will be of the form,

$$Z(\tau) = \frac{1}{N} \sum_{g, h \in G} Z(h, g; \tau),$$

(5-2)

where $N$ is the order of $G$. In the above summation, only the elements $h$ and $g$ which commute each other contribute to the partition function. This will be explained as follows: To calculate $Z(h, g; \tau)$ in the operator formalism, we need to introduce the string coordinate $(X^I_L(z), X^I_R(\tilde{z}))$ in the $h$-sector, which obeys the boundary condition

$$(X^I_L(e^{2\pi i z}), X^I_R(e^{-2\pi i \tilde{z}})) = h \cdot (X^I_L(z), X^I_R(\tilde{z})),

(5-3)$$

up to torus shifts. Let us consider the action of $g$ on the string coordinate in the $h$-sector. Then it turns out that $g(X^I_L(z), X^I_R(\tilde{z}))g^\dagger$ obeys the boundary condition of the $ghg^{-1}$-sector. Let $|h>$ be any state in the $h$-sector. The above observation implies that the state $g|h>$ belongs to the $ghg^{-1}$-sector but not the $h$-sector (unless $g$ commutes with $h$). Therefore, in the trace formula (5-1), $Z(h, g; \tau)$ will vanish identically unless $g$ commutes with $h$. 

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One loop modular invariance of the partition function is satisfied provided

$$Z(h, g; \tau + 1) = Z(h, hg; \tau), \quad (5-4)$$

$$Z(h, g; \frac{-1}{\tau}) = Z(g^{-1}, h; \tau). \quad (5-5)$$

Let us first consider the partition function of the untwisted sector twisted by $g$, i.e., $Z(1, g; \tau)$. It follows from the discussions of section 4 that in the untwisted sector the twist operator $g$ would be of the form

$$g = e^{i\pi \Phi(p)} g_U, \quad (5-6)$$

where

$$\Phi(p) = \phi + 2v^A \eta^{AB} p^B + \frac{1}{2} p^A C^{AB} p^B. \quad (5-7)$$

The symmetric matrix $C^{AB}$ is defined through the relation (4-15) or

$$k^A C^{AB} k'^B = -k^A (M - U^T M U)^{AB} k'^B \quad \text{mod 2}, \quad (5-8)$$

for $k^A, k'^A \in \Gamma^{D,D}$. Let $n$ be the smallest positive integer such that $g^n = 1$. Then, it follows that

$$U^n = 1, \quad (5-9)$$

$$n\phi + \sum_{\ell=0}^{n-1} \{2v^A (\eta U^\ell)^{AB} p^B + \frac{1}{2} p^A (U^{-\ell} C U^\ell)^{AB} p^B \} = 0 \quad \text{mod 2}, \quad (5-10)$$

for all $p^A \in \Gamma^{D,D}$. The zero mode part of $Z(1, g; \tau)$ can easily be evaluated and the result is

$$Z(1, g; \tau)_{\text{zero mode}} = \sum_{(k_R, k_R) \in \Gamma^{d,d}_{g}} e^{i\pi \Phi(k)} e^{i\pi \tau k^2_L - i\pi \bar{\tau} k^2_R}, \quad (5-11)$$

where $\Gamma^{d,d}_{g}$ is the $g$-invariant sublattice of $\Gamma^{D,D}$, i.e.,

$$\Gamma^{d,d}_{g} = \{(k^I_L, k^I_R) \in \Gamma^{D,D} | (U^I_L k^I_L, U^I_R k^I_R) = (k^I_L, k^I_R) \}. \quad (5-12)$$

Here, $(d, \bar{d})$ denotes singature of the lorentzian lattice $\Gamma^{d,d}_{g}$. We now show that the following relation holds for a suitable constant vector $v^A$:

$$\frac{1}{2} k^A C^{AB} k^B = 2v^A \eta^{AB} k^B \quad \text{mod 2}, \quad (5-13)$$
for all \(k^A \in \Gamma_g^d,\). To show this, define
\[
f(k) \equiv \frac{1}{2} k^A C^{AB} k^B.
\] (5 - 14)

Note that
\[
k^A C^{AB} k'^B = -k^A (M - UT^T M)^{AB} k'^B \mod 2
\]
\[= 0 \mod 2 \quad \text{for all } k^A, k'^A \in \Gamma_g^d,
\] (5 - 15)

where we have used eqs. (5-8) and (5-12). It follows that
\[
f(k + k') = f(k) + f(k') \mod 2,
\] (5 - 16)

for all \(k, k' \in \Gamma_g^d\). This relation ensures the existence of a vector \(v'\) satisfying eq.(5-13). Using the relation (5-13), we can write eq.(5-11) as
\[
Z(1, g; \tau)_{\text{zero mode}} = \sum_{(k_L, k_R) \in \Gamma_g^d} e^{i\pi \phi + i2\pi(v + v')} \eta^k e^{i\pi \tau k^2_L - i\pi \overline{\tau} k^2_R}.
\] (5 - 17)

It will be useful to introduce a projection matrix \(P_U\) defined by
\[
P_U = \frac{1}{n} \sum_{\ell=0}^{n-1} U^\ell.
\] (5 - 18)

Noting that \(P_U k = k\) for all \(k \in \Gamma_g^d\) and using the Poisson resummation formula, we have
\[
Z(1, g; -\frac{1}{\tau})_{\text{zero mode}} = e^{i\pi \phi} \frac{(-i\tau)^{\frac{d}{2}}(i\overline{\tau})^{\frac{d}{2}}}{V_{\Gamma_g^d}} \sum_{(q_L, q_R) \in \Gamma_g^d, q^*_L - v^*-v'^*} e^{i\pi \tau q^2_L - i\pi \overline{\tau} q^2_R},
\] (5 - 19)

where \(v^* + v'^* \equiv P_U(v + v')\), \(V_{\Gamma}\) denotes the unit volume of the lattice \(\Gamma\) and \(\Gamma_g^d\) is the dual lattice of \(\Gamma_g^d\). It follows from eq.(5-19) that we can easily extract information about the zero modes of the \(g^{-1}\)-sector because \(Z(1, g; \tau)\) should be related to \(Z(g^{-1}, 1; \tau)\) through the modular transformation, i.e.,
\[
Z(g^{-1}, 1; \tau) = Z(1, g; -\frac{1}{\tau}).
\] (5 - 20)
The degeneracy of the ground state in the \( g^{-1} \)-sector may be given by [13]

\[
\sqrt{\det'(1 - U)} \frac{V_{\Gamma^d, \bar{d}}}{\bar{d}^g G},
\]

(5 - 21)

where the determinant should be taken over the nonzero eigenvalues of \( 1 - U \) and the factor \( \sqrt{\det'(1 - U)} \) will come from the oscillators. The eigenvalues of the momentum \((q_L, q_R)\) in the \( g^{-1} \)-sector may be given by

\[
(q_L, q_R) \in \Gamma_g^d, \bar{d}^* - v^* - v'^*.
\]

(5 - 22)

It should be noted that the momentum eigenvalues in the \( g^{-1} \)-sector are not given by \( \Gamma_g^d, \bar{d}^* - v^* \), which might naively be expected [13]. The origin of the extra contribution \(-v'^*\) is the third term in eq.(5-7), which has been introduced to ensure the duality relation of vertex operators. As we will see later, this extra contribution to the momentum eigenvalues becomes important in the left-right level matching condition.

Information about the zero modes given above is sufficient to obtain \( Z(g^{-1}, 1; \tau) \) because the oscillator part of \( Z(g^{-1}, 1; \tau) \) can unambiguously be calculated. Then, it turns out that the relation (5-20) puts a constraint on \( \phi \) in eq. (5-7), i.e.,

\[
\phi = 0.
\]

(5 - 23)

This is desirable because otherwise the vacuum in the untwisted sector would not be invariant under the action of \( g \) and hence would be removed from the physical Hilbert space. In the point of view of the conformal field theory, the vacuum in the untwisted sector will correspond to the identity operator, which should be included in the operator algebra.

A necessary condition for modular invariance is the left-right level matching condition [13,25]

\[
Z(g^{-1}, h; \tau + n) = Z(g^{-1}, h; \tau).
\]

(5 - 24)

where \( n \) is the smallest positive integer such that \( g^n = 1 \). It follows from eq.(5-1) that the level matching condition is satisfied only if

\[
2n(L_0 - \bar{L}_0) = 0 \mod 2,
\]

(5 - 25)
where $L_0$ ($\bar{L}_0$) is the Virasoro zero mode operator of the left- (right-) mover in the $g^{-1}$-sector. Since any contribution to $L_0$ and $\bar{L}_0$ from the oscillators is a fraction of $n$, the level matching condition can be written as

$$2n(\varepsilon_{g^{-1}} - \bar{\varepsilon}_{g^{-1}} + \frac{1}{2}q^2_L - \frac{1}{2}q^2_R) = 0 \mod 2, \quad \text{for all} \ (q_L, q_R) \in \Gamma^{d,\bar{d}*}_g - v^* - v'^*,$$

(5 - 26)

where $(\varepsilon_{g^{-1}}, \bar{\varepsilon}_{g^{-1}})$ is the conformal dimension (or the zero point energy) of the ground state in the $g^{-1}$-sector and is explicitly given by [1]

$$\varepsilon_{g^{-1}} = \frac{1}{4} \sum_{a=1}^{D} \rho_a (1 - \rho_a),$$

$$\bar{\varepsilon}_{g^{-1}} = \frac{1}{4} \sum_{a=1}^{D} \bar{\rho}_a (1 - \bar{\rho}_a). \quad (5 - 27)$$

Here, $\exp(i2\pi \rho_a)$ and $\exp(i2\pi \bar{\rho}_a)$ ($a = 1, \cdots, D$) are the eigenvalues of $U_L$ and $U_R$ with $0 \leq \rho_a, \bar{\rho}_a < 1$, respectively.

The condition (5-26) can further be shown to reduce to

$$2n(\varepsilon_{g^{-1}} - \bar{\varepsilon}_{g^{-1}} + \frac{1}{2}(v^*_L + v'^*_L)^2 - \frac{1}{2}(v^*_R + v'^*_R)^2) = 0 \mod 2. \quad (5 - 28)$$

To see this, we first note that $\Gamma^{d,\bar{d}*}_g$ can be expressed as [13]

$$\Gamma^{d,\bar{d}*}_g = \mathcal{P}_U \Gamma^{D,D}$$

$$= \{ q^A = \mathcal{P}_U k^A, \ k^A \in \Gamma^{D,D} \}. \quad (5 - 29)$$

This follows from the property that $\Gamma^{D,D}$ is self-dual. From eq. (5-29), any momentum $q^A \in \Gamma^{d,\bar{d}*}_g - v^* - v'^*$ can be parametrized as

$$q^A = \mathcal{P}_U (k - v - v')^A \quad \text{for some} \ k^A \in \Gamma^{D,D}. \quad (5 - 30)$$

Then, we have

$$n(q^2_L - q^2_R) = nq^A \eta^{AB} q^B$$

$$= nk^A (\eta \mathcal{P}_U)^{AB} k^B - 2n(v + v')^A (\eta \mathcal{P}_U)^{AB} k^B + n(v^* + v'^*)^A \eta^{AB} (v^* + v'^*)^B,$$

(5 - 31)
where we have used the relations

\[ P_U \eta = \eta P_U, \]
\[ P_U^2 = P_U, \]
\[ P_U^T = P_U. \]  

(5 – 32)

Since \( \Gamma^{D,D} \) is an even integral lattice and \( U \) is an orthogonal matrix satisfying \( U^n = 1 \), the first term in the right handed side of eq.(5-31) is easily shown to reduce to

\[ nk^A(\eta P_U)^{AB} kB \begin{cases} 
  k^A(\eta U^n)^{AB} kB & \text{mod 2 if } n = \text{even}, \\
  0 & \text{mod 2 if } n = \text{odd.} 
\end{cases} \]  

(5 – 33)

Using the relation (5-13) and noting that \( nP_U k \in \Gamma^{d,d} \), we can rewrite the second term in the right hand side of eq.(5-31) as

\[ -2n(v + v')^A(\eta P_U)^{AB} kB = -2nv^A(\eta P_U)^{AB} kB - \frac{1}{2} k^A \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-1} (U^{-\ell}CU^m)^{AB} kB \mod 2. \]  

(5 – 34)

Replacing \( p \) by \( p + p' \) in eq.(5-10) with eq.(5-23) and then using eq. (5-10) again, we have

\[ p^A \sum_{\ell=0}^{n-1} (U^{-\ell}CU^\ell)^{AB} p'B = 0 \mod 2, \]  

(5 – 35)

for all \( p, p' \in \Gamma^{D,D} \). For \( n \) odd, it is not difficult to show that

\[ -2n(v + v')^A(\eta P_U)^{AB} kB = 0 \mod 2. \]  

(5 – 36)

To derive eq.(5-36), we will use eqs.(5-10), (5-23), (5-34) and (5-35). For \( n \) even, we will find

\[ -2n(v + v')^A(\eta P_U)^{AB} kB = -k^A \sum_{\ell=0}^{\frac{n}{2}-1} (U^{-\ell}CU^{\ell+\frac{n}{2}})^{AB} kB \mod 2. \]  

(5 – 37)

Remembering the relations (3-7), (3-10) and (5-8), we can finally find that for \( n \) even

\[ -2n(v + v')^A(\eta P_U)^{AB} kB = k^A(\eta U^{\frac{n}{2}})^{AB} kB \mod 2. \]  

(5 – 38)

Combining the results (5-33), (5-36) and (5-38) and using the fact that \( k^A(\eta U^{\frac{n}{2}})^{AB} kB \in \mathbb{Z} \), we have

\[ nk^A(\eta P_U)^{AB} kB - 2n(v + v')^A(\eta P_U)^{AB} kB = 0 \mod 2. \]  

(5 – 39)
This completes the proof of (5-28).

We have shown that the left-right level matching condition (5-24) reduces to the condition (5-28), which puts a restriction on the shift vector \( v = (v_L, v_R) \). It should be noticed that the level matching condition (5-28) is not always satisfied for asymmetric orbifold models but trivially satisfied for symmetric ones because \( \varepsilon_{g^{-1}} = \bar{\varepsilon}_{g^{-1}} \) and \( (v^*_L + v'_L)^2 = (v^*_R + v'_R)^2 \) for symmetric orbifold models. For the case of \( C^{AB} = 0 \) in eq. (5-7), it has been proved, in refs. [13,25], that the level matching condition is a necessary and also sufficient condition for one loop modular invariance. Even for the case of nonzero \( C^{AB} \), the sufficiency can probably be shown by arguments similar to refs. [13,25] although to this end we need to know the action of \( g \) on the string coordinate in every twisted sector.

It should be emphasized that the third term in eq. (5-7) plays an important role in the level matching condition because the relation (5-39) might not hold in general if we put \( v' \) to be zero, that is, \( C_{ab} \) to be zero by hand. In the next section, we will see such an example that the introduction of the third term in eq. (5-7) makes partition functions modular invariant.

6. An Example

In this section, we shall investigate a symmetric \( \mathbb{Z}_2 \)-orbifold model in detail, which will give a good illustration of our formalism. Many other examples can be found in ref. [26].

Let us introduce the root lattice \( \Lambda_R \) and the weight lattice \( \Lambda_W \) of \( SU(3) \) as

\[
\Lambda_R = \{ p^I = \sum_{i=1}^{2} n^i \alpha^I_i, n^i \in \mathbb{Z} \},
\]

\[
\Lambda_W = \{ p^I = \sum_{i=1}^{2} m_i \mu^I_i, m_i \in \mathbb{Z} \},
\]

where \( \alpha_i \) and \( \mu^i \) \((i = 1, 2)\) are a simple root and a fundamental weight satisfying \( \alpha_i \cdot \mu^j = \delta^j_i \). We will take \( \alpha_i \) and \( \mu^i \) to be

\[
\alpha_1 = \left( \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2} \right),
\]

\[
\alpha_2 = \left( -\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2} \right),
\]

\[
\mu^1 = \left( -\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2} \right),
\]

\[
\mu^2 = \left( \frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{2} \right).
\]
\[ \alpha_2 = \left( \frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}} \right) , \]

\[ \mu^1 = \left( \frac{1}{\sqrt{2}} \sqrt{\frac{1}{6}} \right) , \]

\[ \mu^2 = \left( \frac{1}{\sqrt{2}}, -\sqrt{\frac{1}{6}} \right) . \]  

(6-2)

The left- and right-moving momentum \((p_L^I, p_R^I) \) \((I = 1, 2)\) is defined by eq.(3-11), i.e.,

\[ p_L^I = \frac{1}{2} p^I + \frac{1}{2} (1 - B)^{IJ} w^J, \]

\[ p_R^I = \frac{1}{2} p^I - \frac{1}{2} (1 + B)^{IJ} w^J , \] 

(6-3)

where \( p^I \) and \( w^I \) are the center of mass momentum and the winding number, respectively and are assumed to lie on the following lattices:

\[ p^I \in 2\Lambda_W, \]

\[ w^I \in \Lambda_R. \]  

(6-4)

The antisymmetric constant matrix \( B^{IJ} \) is chosen as

\[ B^{IJ} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix} . \] 

(6-5)

Then, it turns out that \((p_L^I, p_R^I)\) lies on the following \((2 + 2)\)-dimensional lorentzian even self-dual lattice \( \Gamma^{2,2} \):

\[ \Gamma^{2,2} = \{ (p_L^I, p_R^I | p_L^I, p_R^I \in \Lambda_W, p_L^I - p_R^I \in \Lambda_R \} . \] 

(6-6)

We consider the following \( \mathbb{Z}_2 \)-transformation:

\[ g_U(X_L^I, X_R^I)g_U^\dagger = (U_L^{IJ} X_L^J, U_R^{IJ} X_R^J) , \]

(6-7)

where

\[ U_L^{IJ} = U_R^{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \] 

(6-8)

This is an automorphism of \( \Gamma^{2,2} \), as it should be. According to our prescription, the \( \mathbb{Z}_2 \)-twist operator \( g \) will be of the form

\[ g = e^{i \pi p^A C^{AB} p^B} g_U, \] 

(6-9)
where $p^A \equiv (p^I_L, p^I_R)$ and the symmetric matrix $C^{AB}$ is defined through the relation

$$p^A C^{AB} p^B = -p^A (M - U^T M) ^{AB} p^B \mod 2, \quad (6 - 10)$$

for $p^A, p'^A \in \Gamma^{2,2}$. Here, we have taken a shift vector to zero for simplicity and $M^{AB}, U^{AB}$ are defined by

$$M^{AB} = \begin{pmatrix} -\frac{1}{2} B^{IJ} & -\frac{1}{2} (1 - B)^{IJ} \\ \frac{1}{2} (1 + B)^{IJ} & -\frac{1}{2} B^{IJ} \end{pmatrix} ^{AB},$$

$$U^{AB} = \begin{pmatrix} U^I_L & 0 \\ 0 & U^I_R \end{pmatrix} ^{AB}. \quad (6 - 11)$$

For symmetric orbifolds ($U_L = U_R$), the defining relation (6-10) of $C^{AB}$ may be replaced by

$$(p_L - p_R)^I C^{IJ} (p'_L - p'_R)^J = \frac{1}{2} (p_L - p_R)^I (B - U^T L B_L) ^{IJ} (p'_L - p'_R)^J \mod 2, \quad (6 - 12)$$

where $C^{AB}$ has been assumed to be of the form

$$C^{AB} = \begin{pmatrix} C^{IJ} & -C^{IJ} \\ -C^{IJ} & C^{IJ} \end{pmatrix} ^{AB}. \quad (6 - 13)$$

Thus, the twist operator (6-9) can be written as

$$g = e^{i \frac{\pi}{2} (p_L - p_R)^I C^{IJ} (p'_L - p'_R)^J} g_U. \quad (6 - 14)$$

Since $p^I_L - p^I_R \in \Lambda_R$, the equation (6-12) may be rewritten as

$$\alpha^I_i C^{IJ} \alpha^J_j = \frac{1}{2} \alpha^I_i (B - U^T L B_L) ^{IJ} \alpha^J_j \mod 2. \quad (6 - 15)$$

The right hand side of eq.(6-15) is found to be

$$\frac{1}{2} \alpha^I_i (B - U^T L B_L) ^{IJ} \alpha^J_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} _{ij}, \quad (6 - 16)$$

and hence $C^{IJ}$ cannot be chosen to be zero. We may choose $C^{IJ}$ as

$$\alpha^I_i C^{IJ} \alpha^J_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} _{ij},$$

or

$$C^{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} ^{IJ}. \quad (6 - 17)$$
This choice turns out to be consistent with $g^2 = 1$.

Let us consider the following momentum and vertex operators of the left-mover:

\[ P^I_L(z) = i\partial_z X^I_L(z), \]
\[ V_L(\alpha; z) = e^{i\alpha \cdot X_L(z)} C_\alpha ; \]

where $\alpha$ is a root vector of $SU(3)$ and $C_\alpha$ denotes a cocycle operator. These operators form level one $SU(3)$ Kač-Moody algebra [22]. Under the action of $g$, they transform as

\[ gP^I_L(z)g^\dagger = U^{IJ}_L P^J_L(z), \]
\[ gV_L(\pm \alpha_1; z)g^\dagger = V_L(\pm \alpha_2; z), \]
\[ gV_L(\pm \alpha_2; z)g^\dagger = V_L(\pm \alpha_1; z), \]
\[ gV_L(\pm (\alpha_1 + \alpha_2); z)g^\dagger = -V_L(\pm (\alpha_1 + \alpha_2); z). \]

Thus, the $\mathbb{Z}_2$-invariant physical generators may be given by

\[ J_3(z) = 2P^I_L(z), \]
\[ J_\pm(z) = \sqrt{2}(V_L(\pm \alpha_1; z) + V_L(\pm \alpha_2; z)). \]

These generators are found to form level four $SU(2)$ Kač-Moody algebra [27]. Note that the vertex operators $V_L(\pm (\alpha_1 + \alpha_2); z)$ are not invariant under the action of $g$ and hence they are removed from the physical generators although the root vector $\alpha_1 + \alpha_2$ is invariant under the action of $g_U$.

We now examine one loop modular invariance of the partition function which will be given by

\[ Z(\tau) = \frac{1}{2} \sum_{\ell,m=0}^{1} Z(g^\ell, g^m; \tau), \]

where

\[ Z(g^\ell, g^m; \tau) = \text{Tr}[g^m e^{i2\pi \tau (L_0 - \frac{c}{24}) - i2\pi \bar{\tau} (\bar{L}_0 - \frac{c}{24})}]_{g^\ell - \text{sector}}. \]

The partition functions of the untwisted sector can easily be evaluated and the result is

\[ Z(1, 1; \tau) = \frac{1}{|\eta(\tau)|^4} \sum_{(k_L, k_R) \in \Gamma_{2,2}} e^{i\pi \tau k^2_L - i2\pi \bar{\tau} k^2_R}, \]
\[ Z(1, g; \tau) = \frac{|\vartheta_3(0|\tau)\vartheta_4(0|\tau)|}{|\eta(\tau)|^4} \sum_{(k_L, k_R) \in \Gamma_{g}^{1,1}} e^{i2\pi(v'_{L}k_L-v'_{R}k_R)} e^{i\pi \tau k_L^2 - i\pi \tau k_R^2}, \quad (6-24) \]

where

\[ v'_{L} = v'_{R} = \frac{1}{2\sqrt{2}}, \]

\[ \Gamma_{g}^{1,1} = \{(k_L', k_R') = (\sqrt{2}n, \sqrt{2}n'), n, n' \in \mathbb{Z}\}. \quad (6-25) \]

Here, the shift vector \((v'_{L}, v'_{R})\) has been introduced through the relation (5-13). The function \(\eta(\tau)\) and \(\vartheta_a(\nu|\tau)\) \((a = 1, \cdots, 4)\) are the Dedekind \(\eta\)-function and the Jacobi theta function:

\[ \eta(\tau) = q^{1/12} \prod_{n=1}^{\infty} (1-q^{2n}), \quad (q = e^{i\pi \tau}), \]

\[ \vartheta_{ab}(\nu|\tau) = \sum_{n=-\infty}^{\infty} \exp \{i\pi (n+a)^2 \tau + i2\pi(n+a)(\nu+b)\}, \]

\[ \vartheta_1(\nu|\tau) = \vartheta_{1\frac{1}{2}}(\nu|\tau), \]

\[ \vartheta_2(\nu|\tau) = \vartheta_{2\frac{1}{2}}(\nu|\tau), \]

\[ \vartheta_3(\nu|\tau) = \vartheta_{3\frac{1}{2}}(\nu|\tau), \]

\[ \vartheta_4(\nu|\tau) = \vartheta_{00}(\nu|\tau). \quad (6-26) \]

It follows from the arguments given in section 5 that the degeneracy of the ground state in the \(g\)-sector is

\[ \sqrt{\text{det}^\prime (1-U)} = 1, \quad (6-27) \]

and that the momentum eigenvalues in the \(g\)-sector are given by

\[ (q_L, q_R) \in \Gamma_{g}^{1,1^*} - (v'_{L}, v'_{R}), \quad (6-28) \]

where

\[ \Gamma_{g}^{1,1^*} = \{(q_L, q_R) = (\frac{1}{\sqrt{2}}n, \frac{1}{\sqrt{2}}n'), n, n' \in \mathbb{Z}\}. \quad (6-29) \]

This information is sufficient to obtain \(Z(g, 1; \tau)\) and \(Z(g, g; \tau)\):

\[ Z(g, 1; \tau) = \frac{|\vartheta_3(0|\tau)\vartheta_4(0|\tau)|}{2|\eta(\tau)|^4} \sum_{(q_L, q_R) \in \Gamma_{g}^{1,1^*} - (v'_{L}, v'_{R})} e^{i\pi \tau q_L^2 - i\pi \tau q_R^2}, \quad (6-30) \]

\[ Z(g, g; \tau) = \frac{|\vartheta_4(0|\tau)\vartheta_4(0|\tau)|}{2|\eta(\tau)|^4} \sum_{(q_L, q_R) \in \Gamma_{g}^{1,1^*} - (v'_{L}, v'_{R})} e^{i\pi (q_L^2 - q_R^2)} e^{i\pi \tau q_L^2 - i\pi \tau q_R^2}. \quad (6-31) \]
It is easily verified that \( Z(g^\ell, g^m; \tau) \) satisfies the following desired relations:

\[
Z(g^\ell, g^m; \tau + 1) = Z(g^\ell, g^{m+\ell}; \tau),
\]

\[
Z(g^\ell, g^m; -\frac{1}{\tau}) = Z(g^{-m}, g^\ell; \tau),
\]

and hence the partition function (6-21) is modular invariant. It should be emphasized that the existence of the shift vector \((v'_L, v'_R)\) ensures modular invariance of the partition function: The level matching condition

\[
Z(g, 1; \tau + 2) = Z(g, 1; \tau),
\]

is satisfied because for all \((q_L, q_R) \in \Gamma_{g,1}^1 - (v'_L, v'_R),\)

\[
4\left(\frac{1}{2}q_L^2 - \frac{1}{2}q_R^2\right) = 0 \mod 2.
\]

If we put the shift vector \((v'_L, v'_R)\) or \(C^{IJ}\) in \( g \) to be zero by hand, the level matching condition might, however, be destroyed because eq.(6-34) does not hold for \((q_L, q_R) \in \Gamma_{g,1}^1.\)

It is interesting to note that in terms of the theta functions the partition function obtained above can be expressed as

\[
Z(1, 1; \tau) = \frac{1}{|\eta(\tau)|^4} \sum_{(k_L, k_R) \in \Gamma_{2,2}} e^{i\pi \tau k_L^2 - i\pi \bar{\tau} k_R^2},
\]

\[
Z(1, g; \tau) = \frac{|\vartheta_3(0|\tau)\vartheta_4(0|\tau)|^2}{|\eta(\tau)|^4},
\]

\[
Z(g, 1; \tau) = \frac{|\vartheta_3(0|\tau)\vartheta_2(0|\tau)|^2}{|\eta(\tau)|^4},
\]

\[
Z(g, g; \tau) = \frac{|\vartheta_4(0|\tau)\vartheta_2(0|\tau)|^2}{|\eta(\tau)|^4}.
\]

This partition function is exactly identical to that of a symmetric \( Z'_2 \)-orbifold model whose \( Z'_2 \)-transformation is defined by

\[
Z'_2 : (X_L^I, X_R^I) \rightarrow (-X_L^I, -X_R^I),
\]

instead of the \( Z_2 \)-transformation (6-7). In this orbifold model, level one \( SU(3) \) Kač-Moody algebra can be shown to “break” to level four \( SU(2) \) Kač-Moody algebra. Thus,
although the two orbifold models are defined by the different $\mathbb{Z}_2$-transformations (6-7) and (6-36), they give the same spectrum and interaction [28].

7. Discussions

In this paper, we have investigated the following three consistency conditions in detail: (i) the invariance of the energy-momentum tensors under the action of the twist operators, (ii) the duality of amplitudes and (iii) modular invariance of partition functions. From the analysis of the second condition (ii), we have obtained various important results. The following two points are probably main results of this paper: The first point is the discovery of the third term in eq.(5-7), which is necessary to preserve the duality of amplitudes under the action of $g$ and which plays an important role in modular invariance of partition functions. The second point is that any twist operator $g$ has been proved to be represented by eq.(1-2). To show this, we have seen that the first condition (i) is not sufficient and that the second condition (ii) is crucial to restrict the allowed form of $\Phi(p_L, p_R)$ to eq.(5-7). It should be emphasized that it is very important to show that by a suitable unitary transformation any representation of cocycle operators can reduce to the representation (3-6) with eq.(3-9) up to a constant phase because our analysis has heavily relied on the representation (3-6).

We have found that the string coordinate $X^A = (X^I_L, X^I_R)$ in the untwisted sector transforms under the action of $g$ as

$$g X^A g^\dagger = U^{AB}(X^B + 2\pi \eta^{BC} v^C + \pi C^{BC} p^C). \quad (7-1)$$

It seems that the third term of eq.(7-1) has no clear geometrical meaning. Although the momentum and vertex operators definitely transform under the action of $g$, why does not the string coordinate transform definitely? The reason is probably that in the point of view of the conformal field theory the string coordinate is not a primary field and it is not a well-defined variable on a torus. Thus, there may be no reason why the string coordinate itself should definitely transform under the action of $g$. On the other hand, since the momentum and vertex operators are primary fields and are well-defined on a torus, they should definitely transform under the action of $g$. In fact,
they transform as

\[ g(P^I_L(z), P^I_R(\bar{z})) g^\dagger = (U^I_J P^J_L(z), U^I_J P^J_R(\bar{z})), \]

\[ gV(k_L, k_R; z) g^\dagger = e^{i2\pi v \cdot \eta U^T k_{+i} \bar{k}_{-i}} U^T V(U^T_L k_L, U^T_R k_R; z). \quad (7 - 2) \]

It may be worth while pointing out that the “center of mass coordinate” \( x^A = (x^I_L, x^I_R) \)
always appears as the following combination:

\[ x'^A \equiv x^A + \pi M^{AB} p^B, \quad (7 - 3) \]
in the vertex operators and that \( x'^A \) definitely transform under the action of \( g \), i.e.,

\[ gx'^A g^\dagger = U^{AB} (x'^B + 2\pi \eta^{BC} v^C), \quad (7 - 4) \]

up to torus shifts although \( x^A \) itself does not. This observation strongly suggests that the variable \( x'^A \) is more fundamental than \( x^A \) [24,26].

We have succeeded to obtain the general class of bosonic orbifold models. The
generalization to superstring theories will be straightforward because fermionic fields
will definitely transform under the action of twist operators.

We have restricted our considerations mainly to the untwisted sector. However,
much information about twisted sectors, in particular, zero modes, can be obtained
through modular transformations. Such information is sufficient to obtain the partition
function of the \( g \)-sector \( Z(g, 1; \tau) \) but not \( Z(g, h; \tau) \) in general because we have
not constructed twist operators in each twisted sector. The twist operator \( g \) in the
\( g \)-sector can, however, be found to be of the form

\[ g = e^{i2\pi (L_0 - \bar{L}_0)}. \quad (7 - 5) \]

This follows from the relation

\[ Z(g, g; \tau) = Z(g, 1; \tau + 1). \quad (7 - 6) \]

To obtain an explicit operator representation of any twist operator in every twisted
sector, we may need to construct vertex operators in every twisted sector as in the
untwisted sector. In the construction of vertex operators in twisted sectors, the most
subtle part is a realization of cocycle operators. In the case of \( \xi = 1 \) in eq.(4-5),
(untwisted state emission) vertex operators in any twisted sector have already been constructed with correct cocycle operators in ref.[29]. In the case of $\xi \neq 1$, the prescription given in ref.[29] will be insufficient to obtain desired vertex operators because the duality relation will not be satisfied. Some attempts [30] have been made but the general construction of correct vertex operators is still an open problem.

As mentioned in section 2, there might appear a trouble in the path integral formalism unless eq.(2-16) is satisfied. Our success in the operator formalism, however, probably means that our results can be reinterpreted from the path integral point of view. Then the geometrical meaning will become clear.
In this appendix, we shall prove the following theorem: Let \( n \) be a positive integer. Then, for any positive integer \( p \),
\[
p^n = p^{n-\varphi(n)} \mod n,
\]
where \( \varphi(n) \) is the Euler function which is equal to the number of \( d \) such that \((d, n) = 1\) for \( d = 1, 2, \ldots, n - 1 \). Here, \((d, n)\) denotes the highest common divisor of \( d \) and \( n \).

The Euler function satisfies the following relation:
\[
\varphi(mn) = \varphi(m)\varphi(n) \text{ if } (m, n) = 1.
\]

To prove the theorem (A-1), we start with the Euler’s theorem:
\[
p^{\varphi(n)} = 1 \mod n \text{ for } (p, n) = 1.
\]

Suppose that \( n \) is decomposed as
\[
n = (q_1)^{\ell_1}(q_2)^{\ell_2} \cdots (q_r)^{\ell_r},
\]
where \( q_i \) (\( i = 1, \ldots, r \)) is a prime number and \( q_i \neq q_j \) if \( i \neq j \). Let \( \ell_{\text{max}} \) be the maximum number in the set of \( \{\ell_i, i = 1, \ldots, r\} \). In terms of \( q_i \), any positive integer \( p \) can be decomposed as
\[
p = (q_1)^{\ell'_1} \cdots (q_r)^{\ell'_r} s,
\]
where \( \ell'_i \geq 0 \) and \( s \) is an integer such that \((s, n) = 1\). Then it is not difficult to show that
\[
(q_i)^{\ell'_i(\varphi(n)+\ell_{\text{max}})} = (q_i)^{\ell'_i\ell_{\text{max}}} \mod n,
\]
\[
s^{\varphi(n)+\ell_{\text{max}}} = s^{\ell_{\text{max}}} \mod n.
\]

It follows that for any positive integer \( p \)
\[
p^{\varphi(n)+\ell_{\text{max}}} = p^{\ell_{\text{max}}} \mod n.
\]

The Euler function \( \varphi(n) \) satisfies
\[
\sum_{1 \leq d \leq n \atop d|n} \varphi(d) = n.
\]
where \(d|n\) means that \(d\) is a divisor of \(n\). For any prime number \(q_i\),

\[
\varphi(q_i^{\ell_i}) = q_i^{\ell_i - 1}(q_i - 1) \\
\geq q_i^{\ell_i - 1} \\
\geq \ell_i. \tag{A-9}
\]

Then, it follows from eqs.(A-8) and (A-9) that

\[
n > \ell_{\text{max}} + \varphi(n). \tag{A-10}
\]

Multiplying (A-7) by \(p^{n-\varphi(n)-\ell_{\text{max}}}\) and noting \(n - \varphi(n) - \ell_{\text{max}} > 0\), we finally have

\[
p^n = p^{n-\varphi(n)} \mod n. \tag{A-11}
\]

### Appendix B

In this appendix, we shall prove that by a suitable unitary transformation any representation of the cocycle operator \(C_k\) can reduce to

\[
C_k = e^{i\pi k^A M^{AB} \hat{p}^B}, \tag{B-1}
\]

with eq.(3-9) up to a constant phase.

We first note that the following factor:

\[
e^{i\pi(\theta(\hat{p}+k)-\theta(\hat{p}))}, \tag{B-2}
\]

can be removed from the cocycle operator by a suitable unitary transformation because the cocycle operator \(C_k\) appears always in the combination \(e^{ik\cdot \hat{x}} C_k\).

The cocycle operator \(C_k\) will consist of the zero modes. Since the vertex operator (3-1) should represent the emission of a state with the momentum \(k^A\), the cocycle operator \(C_k\) will not depend on \(\hat{x}^A\) and be represented in terms of \(\hat{p}^A\) as well as \(k^A\). We may write the cocycle operator \(C_k\) into the form

\[
C_k = e^{i\pi k^A M^{AB} \hat{p}^B + i\pi F_k(\hat{p})}, \tag{B-3}
\]
where $M^{AB}$ is defined by eq. (3-9). We require that the zero mode part of the vertex operator (3-5) satisfies

\[ V_0(k)V_0(k') = \varepsilon(k, k')V_0(k + k') \quad (B - 4) \]

\[ = (-1)^{k \cdot k'}V_0(k')V_0(k), \quad (B - 5) \]

where the phase factor $\varepsilon(k, k')$ is assumed to be c-number. The above two conditions can be replaced by

\[ F_k(\hat{p}) - F_{k'}(\hat{p} + k') - F_{k'}(\hat{p}) = \hat{p} - \text{independent mod 2}, \quad (B - 6) \]

\[ F_k(\hat{p} + k') + F_{k'}(\hat{p}) - F_k(\hat{p} + k) - F_{k'}(\hat{p}) = 0 \mod 2. \quad (B - 7) \]

It will be convenient to use the following basis of the momentum: Let $e_a^A$ $(a = 1, \ldots, 2D)$ be a basis of $\Gamma^{D,D}$, i.e., any momentum $k^A \in \Gamma^{D,D}$ can be expressed as

\[ k^A = \sum_{a=1}^{2D} k_a^a e_a^A, \quad k^a \in \mathbb{Z}. \quad (B - 7) \]

In this basis, we assume that $F_k(\hat{p})$ can be expanded in powers of $k^a$ and $\hat{p}^a$ as follows:

\[ F_k(\hat{p}) = \sum_{n=2}^{N} \sum_{a_1, \ldots, a_n = 1}^{2D} \sum_{j=1}^{n-1} \frac{1}{j!(n-j)!} \Delta M_{a_1 \ldots a_j}^{a_{j+1} \ldots a_n} k_a^{a_1} \ldots k_a^{a_j} \hat{p}^{a_{j+1}} \ldots \hat{p}^{a_n}, \]

\[ (k^a, \hat{p}^a \in \mathbb{Z}), \quad (B - 8) \]

where $N$ is an arbitrary positive integer and the coefficient $\Delta M_{a_1 \ldots a_j}^{a_{j+1} \ldots a_n}$ is totally symmetric with respect to lower indices or upper indices.

Our aim of this appendix is now to show that by a suitable unitary transformation $F_k(\hat{p})$ can always reduce to

\[ F_k(\hat{p}) = 0 \mod 2, \quad (B - 9) \]

up to $\hat{p}$-independent constant terms. Before we prove eq. (B-9) for arbitrary $N$, it may be instructive to examine the case of $N = 3$, i.e.,

\[ F_k(\hat{p}) = \sum_{a,b} \Delta M_a^{b} k_a \hat{p}^b + \sum_{a,b,c} \frac{1}{2!} \{ \Delta M_a^{bc} k_a \hat{p}^b \hat{p}^c + \Delta M_{ab}^{c} k_a \hat{p}^b \hat{p}^c \}. \quad (B - 10) \]
Inserting eq.(B-10) into eqs.(B-6) and comparing the third order terms of both sides of eqs.(B-6) with respect to $k^a$, $k'^a$ and $\hat{p}^a$, we find

$$\sum_{a,b,c} \{ \Delta M_{ab}^b k^a k'^b \hat{p}^c - \Delta M_{c}^a k^a k'^b \hat{p}^c \} = 0 \mod 2,$$

$$\sum_{a,b,c} \frac{1}{2!} \{ \Delta M_{ab}^b k^a k'^b k'^c - \Delta M_{c}^a k^a k'_b k'^c \} = 0 \mod 2,$$

for all $k^a, k'^a, \hat{p}^a \in \mathbb{Z}$. From these equations, we can show the following equality:

$$\sum_{a,b,c} \frac{1}{2!} \{ \Delta M_{ab}^b k^a \hat{p}^b \hat{p}^c + \Delta M_{c}^a k^a k'^b \hat{p}^c \} = \frac{1}{3!} \Delta M_{a}^{aa} \{(\hat{p}^a + k^a)^3 - (\hat{p}^a)^3 - (k^a)^3\}$$

$$+ \left( \sum_{a=b<c} + \sum_{c<a=b} \right) \frac{1}{2!} \Delta M_{ac}^{aa} \{(\hat{p}^a + k^a)^2(\hat{p}^c + k^c) - (\hat{p}^a)^2 \hat{p}^c - (k^a)^2 k^c\}$$

$$+ \sum_{a<b<c} \Delta M_{ab}^c (\hat{p}^a + k^a)(\hat{p}^b + k^b)(\hat{p}^c + k^c) - \hat{p}^a \hat{p}^b \hat{p}^c - k^a k^b k^c \mod 2. \quad (B - 12)$$

Then, it follows that the second and the third terms in the right hand side of eq.(B-10) can be removed by a suitable unitary transformation (up to $\hat{p}$-independent terms). Next comparing the second order terms of both sides of eqs.(B-6) with respect to $k^a, k'^a$ and $p^a$, we find

$$\sum_{a,b} \{ \Delta M_{ab}^b k^a k'^b - \Delta M_{b}^a k^a k'^b \} = 0 \mod 2. \quad (B - 13)$$

Without loss of generality, we can assume that the matrix $\Delta M_{ab}^b$ is antisymmetric because the symmetric part of $\Delta M_{ab}^b$ can be removed by a suitable unitary transformation. Since eq.(B-13) then means that $\Delta M_{ab}^b \in \mathbb{Z}$, we can introduce a symmetric matrix $S_{ab}$ through the relation

$$S_{ab} = \Delta M_{ab}^b \mod 2. \quad (B - 14)$$

In terms of $S_{ab}$, the first term of eq.(B-10) can be written as

$$\sum_{a,b} \Delta M_{ab}^b k^a \hat{p}^b = \sum_{a,b} S_{ab} k^a \hat{p}^b \mod 2$$

$$= \sum_{a,b} \frac{1}{2!} S_{ab} \{(\hat{p}^a + k^a)(\hat{p}^b + k^b) - \hat{p}^a \hat{p}^b - k^a k^b\} \mod 2. \quad (B - 15)$$

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Therefore, we have proved that by suitable unitary transformations $F_k(\hat{p})$ given in eq.(B-10) can reduce to eq.(B-9) up to $\hat{p}$-independent terms.

Let us prove eq.(B-9) for arbitrary $N$. By inserting eq.(B-8) into eqs. (B-6) and by comparing the $N$th order terms of both sides of eqs.(B-6) with respect to $k^a, k'^a$ and $\hat{p}^a$, it is not difficult to show the following equality:

$$
\sum_{a_1, \cdots, a_N} \sum_{j=1}^{N-1} \frac{1}{j!(N-j)!} \Delta M_{a_1 \cdots a_j} a_{j+1 \cdots a_N} k^{a_1} \cdots k^{a_j} \hat{p}^a_{j+1} \cdots \hat{p}^{a_N}
$$

$$
= \sum_{a} \left\{ \frac{1}{2!(N-2)!} (\Delta M_{a a} a^a - \Delta M_{a a} a^a) (k^a)^2 (\hat{p}^a)^{N-2} 
+ \frac{1}{3!(N-3)!} (\Delta M_{a a a} a^a - \Delta M_{a a a} a^a) (k^a)^3 (\hat{p}^a)^{N-3} 
+ \cdots + \frac{1}{(N-2)!2!} (\Delta M_{a \cdots a} a^a - \Delta M_{a \cdots a} a^a) (k^a)^{N-2} (\hat{p}^a)^2 \right\} \mod 2, (B-16)
$$

with

$$
\Delta M_{a \cdots a} a^a \cdots a^a - \Delta M_{a \cdots a} a^a \cdots a^a = 0 \mod 2(j-1)!(N-j)!, \quad (B-17)
$$

where in the right hand side of eq.(B-16) we have omitted $\hat{p}$-independent terms as well as the terms which can be removed by unitary transformations. It follows from the theorem (A-1) that all the terms in the right hand side of eq.(B-16) can reduce to lower order terms with respect to $k^a$ and $\hat{p}^a$ and hence they can be absorbed into lower order terms of $F_k(\hat{p})$ by suitably redefining the coefficients of the lower order terms of $F_k(\hat{p})$. Therefore, we conclude that the $N$th order terms of $F_k(\hat{p})$ can be put to be equal to zero.

Repeating the above arguments order by order, we finally come to the conclusion (B-9).
References

[1] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678; B274 (1986) 285.
[2] H. Kawai, D. Lewellyn and A.H. Tye, Phys. Rev. Lett. 57 (1986) 1832; Nucl. Phys. B288 (1987) 1;
   I. Antoniadis, C. Bachas and C. Kounnas, Nucl. Phys. B289 (1987) 87.
[3] W. Lerche, A.N. Schellenkens and N.P. Warner, Phys. Rep. 177 (1989) 1.
[4] D. Gepner, Phys. Lett. B199 (1987) 380; Nucl. Phys. B296 (1987) 757.
[5] Y. Kazama and H. Suzuki, Nucl. Phys. B321 (1989) 232.
[6] C. Vafa and N.P. Warner, Phys. Lett. B218 (1989) 51;
   W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. B324 (1989) 427;
   P.S. Howe and P.C. West, Phys. Lett. B223 (1989) 377; B244 (1989) 270.
[7] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B158 (1985) 316; Nucl. Phys. B261 (1985) 1;
   C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. B262 (1985) 593;
   C.G. Callan, I.R. Klebanov and M.J. Perry, Nucl. Phys. B278 (1986) 78;
   T. Banks, D. Nemeschansky and A. Sen, Nucl. Phys. B277 (1986) 67.
[8] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46.
[9] A. Font, L.E. Ibáñez, F. Quevedo and A. Sierra, Nucl. Phys. B331 (1990) 421.
[10] J.A. Casas and C. Muñoz, Nucl. Phys. B332 (1990) 189.
[11] Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo, Y. Ono and K. Tanioka,
    Nucl. Phys. B341 (1990) 611.
[12] A. Fujitsu, T. Kitazoe, M. Tabuse and H. Nishimura, Intern. J. Mod. Phys. A5 (1990) 1529.
[13] K.S. Narain, M.H. Sarmadi and C. Vafa, Nucl. Phys. B288 (1987) 551; B356 (1991) 163.
[14] K.S. Narain, Phys. Lett. B169 (1986) 41;
    K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.
[15] J.H. Schwarz, Phys. Rep. 8C (1973) 269; 89 (1982) 223;
    J. Scherk, Rev. Mod. Phys. 47 (1975) 123.
[16] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256 (1985) 253; B267 (1986) 75.

[17] J.L Cardy, Nucl. Phys. B270 (1986) 186.

[18] H. Suzuki and A. Sugamoto, Phys. Rev. Lett. 57 (1986) 1665.

[19] N. Sakai and Y. Taniii, Nucl. Phys. B287 (1987) 457.

[20] K. Inoue, S. Nima and H. Takano, Prog. Theor. Phys. 80 (1988) 881.

[21] A.M. Polyakov, Phys. Lett. B103 (1981) 207; B103 (1981) 211.

[22] I. Frenkel and V. Kač, Invent. Math. 62 (1980) 23;
    G. Segal, Commun. Math. Phys. 80 (1981) 301;
    P. Goddard and D. Olive, Intern. J. Mod. Phys. A1 (1986) 303.

[23] V.A. Kostelecký, O. Lechtenfeld, W. Lerche, S. Samuel and S. Watamura, Nucl. Phys. B288 (1987) 173.

[24] M. Sakamoto, Phys. Lett. B231 (1989) 258.

[25] C. Vafa, Nucl. Phys. B273 (1986) 592.

[26] T. Horiguchi, M. Sakamoto and M. Tabuse, Kobe preprint KOBE-92-03 (1992).

[27] M. Sakamoto and M. Tabuse, Phys. Lett. B260 (1991) 70.

[28] M. Sakamoto, Prog. Theor. Phys. 84 (1990) 351.

[29] K. Itoh, M. Kato, H. Kunitomo and M. Sakamoto, Nucl. Phys. B306 (1988) 362.

[30] J. Erler, D. Jungnickel, J. Lauer and J. Mas, preprint SLAC-PUB-5602 (1991).