Comment on “Fluctuations in Extractable Work Bound the Charging Power of Quantum Batteries”

In Ref. [1] the connection between the charging power of a quantum battery and the fluctuations of a “free energy operator” whose expectation value characterizes the maximum extractable work of the battery is studied using both closed- and open-system analyses. Recently, it has been shown [2] that the conclusions of Ref. [1] do not hold for open-system dynamics. Since the two analyses are physically equivalent approaches to studying the dynamics of the battery, in light of the findings of Ref. [2] we critically examine whether the conclusions of Ref. [1] hold for closed-system dynamics. In doing so, we find a few mistakes in the analysis of Ref. [1] and obtain the correct bound on the charging power. As a result, for closed-system dynamics the conclusions of Ref. [1] are in general not correct.

The starting point of our analysis is the charging power

\[ P(t) = -i \text{Tr}(\rho F \otimes I_{\text{SBA}} |V|) , \]

where \( \rho \) is the full state of the closed system \( \text{SBAW} \), \( F \) is a Hermitian operator, known as the free energy operator of the battery \( W \), and \( V \) is the interaction Hamiltonian between the battery and the bath \( S \), system \( A \), and ancilla \( A \). Following Ref. [1], we define \( \delta F = F - \langle F \rangle_W \) and \( \delta V = V - \langle V \rangle \), where \( \langle F \rangle_W = \text{Tr}(\rho_W F) \) and \( \langle V \rangle = \text{Tr}(\rho V) \).

After some algebra, we obtain

\[ |P(t)|^2 = |\text{Tr}(\rho |\delta F \delta V|)|^2 , \]

where for notational simplicity we will henceforth use the shorthand notation \( \delta F = \delta F \otimes I_{\text{SBA}} \). Note that Eq. (2) is an equality instead of an inequality given by Eq. (9) of Ref. [1]. It is convenient to rewrite Eq. (2) as

\[ |P(t)|^2 = |\text{Tr}(\sqrt{\rho} \delta F \delta V \sqrt{\rho} - \sqrt{\rho} \delta V \delta F \sqrt{\rho})|^2 , \]

where we have used the fact that \( \rho \) is a positive operator. We note that since \( \sqrt{\rho} \delta F \delta V \sqrt{\rho} \) and \( \sqrt{\rho} \delta V \delta F \sqrt{\rho} \) are Hermitian conjugates of each other, it follows that \( \text{Tr}(\sqrt{\rho} \delta F \delta V \sqrt{\rho}) \) and \( \text{Tr}(\sqrt{\rho} \delta V \delta F \sqrt{\rho}) \) are complex conjugates of each other. As a matter of fact, this is the utmost important point that is missed in the analysis of Ref. [1]. With this point in mind, we can rewrite Eq. (3) as

\[ |P(t)|^2 = |\text{Tr}(\sqrt{\rho} \delta F \delta V \sqrt{\rho})|^2 + |\text{Tr}(\sqrt{\rho} \delta V \delta F \sqrt{\rho})|^2 - 2 \text{Re}[\text{Tr}(\rho |\delta F \delta V|)] , \]

where \( \text{Re} \) denotes the real part.

To find the upper bound on \( |P(t)|^2 \), again following Ref. [1], we use the fact that for a positive operator \( A \) and Hermitian operators \( B \) and \( C \), the Cauchy-Schwarz inequality implies \( |\text{Tr}(\sqrt{ABC} \sqrt{A})|^2 \leq |\text{Tr}(AB^2)||\text{Tr}(AC^2)| \). Equation (4) then leads to

\[ |P(t)|^2 \leq 2(|\text{Tr}(\rho |\delta F|^2)| \text{Tr}(\rho |\delta V|^2) - \text{Re}[\text{Tr}(\rho |\delta F \delta V|)]) = 2(\sigma_F^2 \sigma_V^2 - \text{Re} [\text{Cov}(F, V)]) . \]

Here, \( \sigma_F^2 \) is the variance of \( F \) in the battery state \( \rho_W \), \( \sigma_V^2 \) is the variance of \( V \) in the full state \( \rho \), and \( \text{Cov}(F, V) \) is the covariance between \( F \) and \( V \) in the full state \( \rho \).

Specifically, we have

\[ \sigma_F^2 = \langle F^2 \rangle_W - \langle F \rangle_W^2 , \quad \sigma_V^2 = \langle V^2 \rangle - \langle V \rangle^2 , \quad \text{Cov}(F, V) = \langle (F \otimes I_{\text{SBA}}) \rangle - \langle F \rangle_W \langle V \rangle \] .

Moreover, the Cauchy-Schwarz inequality \( \sigma_F^2 \sigma_V^2 \geq |\text{Cov}(F, V)|^2 \) implies \( \sigma_F^2 \sigma_V^2 - \text{Re} [\text{Cov}(F, V)]^2 \geq 0 \) as it should be. Evidently, Eq. (5) is the corrected expression for Eqs. (9) and (12) of Ref. [1].

The last step of our analysis is to verify the case in which the battery state is an eigenstate of the free energy operator. Suppose \( \rho_W = |j \rangle \langle j | \) and \( F|j \rangle = w_j |j \rangle \) with \( w_j \) the real eigenvalue; we obtain \( \sigma_F^2 = \text{Cov}(F, V) = 0 \), which implies \( P(t) = 0 \). However, under the assumption of a very general charging process with \( \sigma_V^2 \neq 0 \), we stress that in our analysis \( \rho_W = |j \rangle \langle j | \) is only a sufficient condition for the battery to have a nonzero charging power, as opposed to a sufficient and necessary condition in the original incorrect analysis of Ref. [1]. Moreover, even though the total system is initially in a product state with the battery in an eigenstate of \( F \), the interaction \( V \) will make the battery entangled with the other subsystems, giving rise to a mixed battery state.

It is conceivable that there exist entangled full states \( \rho \) and mixed battery states \( \rho_W = \text{Tr}_{\text{SBA}}(\rho) \) with nonzero \( \sigma_F^2 \) and \( \text{Cov}(F, V) \) but \( P(t) = 0 \).

Since there is still no consensus on the notion of work in the quantum regime, the discrepancy between the conclusions of Ref. [2] and of the present work seems to suggest that the free energy operator \( F \) introduced in Ref. [1] does not properly quantify the work content of the battery in a physically consistent manner. This is certainly an important issue that deserves further investigation.

This work was supported in part by the MOST of Taiwan under grant 109-2112-M-032-009.

Shang-Yung Wang
Department of Physics
Tamkang University
New Taipei City 25137, Taiwan

[1] L. P. García-Pintos, A. Hamma, and A. del Campo, Phys. Rev. Lett. 125, 040601 (2020).
[2] S. Cusumano and L. Rudnicki, Phys. Rev. Lett. 127, 028901 (2021).