Spin Hall Conductivity on the Anisotropic Triangular Lattice

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We present a detailed study of the spin Hall conductivity on a two-dimensional triangular lattice in the presence of Rashba spin-orbit coupling. In particular, we focus part of our attention on the effect of the anisotropy of the nearest neighbor hopping amplitude. It is found that the presence of anisotropy has drastic effects on the spin Hall conductivity, especially in the hole doped regime where a significant increase or reversed sign of the spin Hall conductivity has been obtained. We also provide a systematic analysis of the numerical results in terms of Berry phases. The changes of signs observed at particular density of carriers appear to be a consequence of both Fermi surface topology and change of sign of electron velocity. In addition, in contrast to the two-dimensional square lattice, it is shown that the tight binding spin-orbit Hamiltonian should be derived carefully from the continuous model on the triangular lattice.

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I. INTRODUCTION

The incorporation of electron spin degree of freedom and its associated magnetic moment into electronic devices has initiated a rapidly growing field of spin electronics, called spintronics. One of the first ideas of spintronics is attributed to Datta and Das in 1990. The authors have proposed a spin field-effect transistor with the innovation of manipulating a pure spin current. Such a manipulation is possible due to the relativistic effect of the coupling between spin and orbital degrees of freedom which can be manipulated e.g. by means of a gate voltage. This spin-orbit coupling (SOC) can emerge in various ways in semiconductors. One source is the lack of inversion symmetry of the confining potential, which defines the 2D electron gas, generated by the heterostructure of the semiconductor. This Bychkov-Rashba type SOC has the advantage that it can be directly tuned by an applied gate voltage. Another way to introduce SOC in the sample is by choosing a noncentrosymmetric material where the lack of inversion center creates a SOC called Dresselhaus SOC. Developing further the correspondence between (charge) electronics and spintronics with the aim of creating pure spin currents, D'yakonov and Perel proposed already in 1971 the spin Hall effect (SHE). This mechanism requires spin dependent impurity scattering such as skew scattering or side jump mechanism. In today’s terminology, it is known as extrinsic SHE. It was experimentally confirmed in 2004/2005 by angle-resolved optical detection of spin polarization at the edges of a two-dimensional layer.

In this paper, we will focus on the SHE which arises even in the absence of impurities, the so called intrinsic SHE, which is due to SOC. The latter is assumed in this paper to be of Bychkov-Rashba type. It lifts, without breaking time reversal symmetry, the spin degeneracy of the eigenstates. The theory of the SHE has been developed in the last ten years, as reviewed in Refs. 6,9, but only recently experimental evidence has been reported for electrical manipulation and detection of intrinsic SHE in ballistic HgTe/HgCdTe quantum wells. Starting at low electron fillings in the conduction band where both Rashba bands are occupied, one finds the “universal” spin Hall conductivity (SHC) \( \sigma_{\text{SH}} = e/8\pi \), which is independent of the SOC strength for sufficiently small couplings. Going beyond the long wavelength approximation, the underlying lattice geometry becomes important. On a square lattice using a single-orbital model (spin-split) the SHC \( \sigma_{\text{SH}} \) shows electron-hole symmetry: \( \sigma_{\text{SH}} \) is an odd function of the Fermi energy \( E_F \) which thus vanishes at half-filling. However both, shape and sign of SHC are lattice structure dependent. This can, for instance, nicely be illustrated with the appearance of several plateaus as seen in the SHC calculations on honeycomb and kagome lattice by Liu et al., Ref. 13,14. Both of these lattices contain several atoms per unit cell respectively two and three. The kagome lattice can be seen as a triangular lattice with 3 atoms per unit cell. The question which arises is whether similar differences as seen between square and kagome SHC are expected already in the case of the simple triangular lattice. In contrast to the square lattice, the triangular lattice brings along the property
of geometrical frustration which is at the origin of exotic properties. In addition, electron-hole symmetry is broken on the triangular lattice. Among exciting properties, the triangular based lattices are known to favor in some cases ferromagnetic groundstates when the electron-electron interaction is switched on.\textsuperscript{15–17} This is also known as flat band ferromagnetism.\textsuperscript{18–21} In these systems, the intrinsic frustrating nature of the lattice is a crucial ingredient in the stabilization of ferromagnetism. The importance of geometrical frustration has also been shown to play an important role in superfluid-Mott insulator quantum phase transitions in ultracold quantum gases.\textsuperscript{22} It has been suggested that the supersolid phase reported results from the competition between Mott localization and frustration.\textsuperscript{23}

Among other interesting features found on triangular lattices one can also mention superconductivity in CoO layered compounds\textsuperscript{24} or various anisotropies in organic bisethylendithio conductors.\textsuperscript{25}

Back to the SHE, it is interesting to mention that there are already experimental realizations of surfaces with triangular geometry. Indeed, giant Rashba SOC has been recently reported on surfaces of Au(111), Bi(111) and Bi/Ag(111).\textsuperscript{20} In this manuscript we present a detailed analysis of the SHC on the anisotropic triangular lattice as a function of the carrier concentration. In addition, the obtained numerical results will be discussed in the context of Berry phases.\textsuperscript{27} One of the aims of including anisotropy in the hopping amplitudes is to allow us to tune in a continuous way the position of the Van Hove singularity and in the same time change the topology of the Fermi surface. Experimentally, the anisotropy in the hopping integrals could be achieved by introducing strains or growing samples on surfaces which have small differences in the lattice parameters.

II. TIGHT BINDING HAMILTONIAN ON TRIANGULAR LATTICE WITH SOC

We consider a 2D electron gas on the triangular lattice as sketched in Fig. 1.

Including the Rashba SOC with coupling strength $\alpha_2$ the Hamiltonian reads

$$H = \frac{1}{2m^*} \hat{k}^2 + \alpha_2 (\hat{k}_y \sigma_x - \hat{k}_x \sigma_y)$$

$$= \frac{1}{2m^*} (\hat{k} + eA_{SO})^2 - m^* \alpha_2^2,$$

where,

$$A_{SO} = \frac{2m^*}{e} \hat{d}S,$$

$$\hat{d} \equiv \begin{pmatrix} 0 & -\alpha_2 \\ \alpha_2 & 0 \end{pmatrix},$$

and $\sigma_i, i = x, y$ are the Pauli matrices, $S$ is the spin-1/2 operator and $m^*$ the electron effective mass.

As it will become clear in the following, it is convenient to introduce the lattice unitary vectors,

$$\mathbf{e}_x = \left( \begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right), \quad \mathbf{e}_4 = \left( \begin{array}{c} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{array} \right), \quad \mathbf{e}_5 = \left( \begin{array}{c} \frac{-1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{array} \right).$$

The Hamiltonian can be recast in the following form,

$$H = \frac{2}{32m^*} ((\mathbf{e}_x \cdot \hat{k})^2 + (\mathbf{e}_4 \cdot \hat{k})^2 + (\mathbf{e}_5 \cdot \hat{k})^2) - m^* \alpha_2^2$$

$$= \frac{2}{32m^*} (\hat{k}_x^2 + \hat{k}_4^2 + \hat{k}_5^2) - \frac{\alpha_2}{3} (\hat{k}_4 - \hat{k}_5 + 2\hat{k}_x) \sigma_y$$

$$+ \frac{\alpha_2}{\sqrt{3}} (\hat{k}_4 + \hat{k}_5) \sigma_x,$$

where the canonical moment is given by

$$\hat{\mathbf{k}} = \mathbf{k} + \alpha_2 m^* \left( -\sigma_y \quad \sigma_x \right).$$

The discretization of the Hamiltonian leads to the Tight Binding Hamiltonian (TBH) whose nearest neighbor hopping amplitudes are listed in Tab. II. This Hamiltonian is identical to that one would obtain directly using the well known tight-binding expression of the SO part and that reads,

$$H^{(R)} = \frac{2}{32m^*} \sum_{(ij)_{\sigma\sigma'}} \left( \sigma \times \hat{e}_{ij} \right)_{\sigma \sigma'} c_{i\sigma} \bar{c}_{j\sigma'},$$

where $\hat{e}_{ij}$ is the unitary vector pointing from site $i$ and $j$. Note the presence of the 2/3 coefficient in both Eq. (7) and in Eq. (9). We could absorb this coefficient in the hopping definition but we have chosen to keep them in order to facilitate the direct comparison to existing calculations. The comparison between our TBH and that derived in Ref. 28 reveals differences in the hopping amplitudes. The difference in the derived TBH results from the discretization procedure. By using the relation between

![Figure 1: (Color online) Two dimensional triangular lattice in the x-y plane. $t$ and $t'$ are the hopping amplitudes, $a$ is the lattice constant. The directions are numbered.](image)
\[ (i, j) \] \[ H^{(0)}_{(i, j)} \] \[ H^{(R)}_{(i, j)} \]

\begin{tabular}{|c|c|c|}
\hline
(i, j) & \[ H^{(0)}_{(i, j)} \] & \[ H^{(R)}_{(i, j)} \] \\
\hline
(0, 1) & \(-\frac{2}{3}t_0 \) & \( i\frac{2}{3}\sigma_x - i\frac{2}{3}\sigma_y \) \\
(0, 2) & \(-\frac{2}{3}t_0 \) & \( \frac{1}{\sqrt{3}}\sigma_x + i\frac{1}{\sqrt{3}}\sigma_y \) \\
(0, 3) & \(-\frac{2}{3}t_0 \) & \( \frac{2}{3}\sigma_x \) \\
(0, 4) & \(-\frac{2}{3}t_0 \) & \( -i\frac{2}{3}\sigma_x + i\frac{1}{3}\sigma_y \) \\
(0, 5) & \(-\frac{2}{3}t_0 \) & \( -i\frac{1}{3}\sigma_x \) \\
(0, 6) & \(-\frac{2}{3}t_0 \) & \( -i\frac{1}{3}\sigma_y \) \\
\hline
\end{tabular}

Table I: Matrix elements in the tight-binding model for the kinetic part \( H^{(0)} \) and the Rashba part \( H^{(R)} \), where \( i^R = \alpha_z/(2a) \), \( t_0 = 1/(2m^*a^2) \) and \( a \) denotes the lattice constant.

\( k_x, k_y \) and \( k_4, k_5 \) one can generate an infinite number of rigorously equivalent Hamiltonians in the continuous picture with different coefficients for \( \sigma_x \) and \( \sigma_y \). However, the discretization will lead to an infinite number of inequivalent TBHs. Thus this crucial step has to be done carefully. Note that this problem will not occur on the square lattice because the connecting vectors \( e_x \) and \( e_y \) are orthogonal to each other and the discretization can be done unambiguously. Then, how to proceed in order to derive the correct TBH? There are two procedures. The most straightforward is to use directly Eq. (9) as often done in the literature. The second procedure consist in re-expressing the continuous Hamiltonian in term of the canonical momentum as done in Eq. (7) and then only discretize it by properly defining the real space derivative.

**Spectrum and Density of States**

Let us now proceed with the diagonalization of the anisotropic TBH. We define the hopping amplitude in the \( x \) and \( 0 - t \) direction as \( t = (2/3)t_0 \) and that in the \( 0 - 5 \) direction as \( \tilde{t} = (2/3)t_0 \). From now on, the lattice constant \( a \) will be set to one, \( a \equiv 1 \).

The spectrum of the TBH is given by

\[
E_{\pm}(k) = -\frac{2}{3}(2t_0 \cos(k_x) + \cos(k_z)) + 2t_0 \cos(k_z) \pm \sqrt{2}t^R F_1(k), \quad (10)
\]

where \( F_1(k) \) is defined as

\[
F_1(k) \equiv \left( 3 + \cos(k_x) - \cos(2k_x) - (1 + 2 \cos(k_x)) \cos(\sqrt{3}k_y) + 8 \cos\left(\frac{k_x}{2}\right) \cos\left(\frac{\sqrt{3}k_y}{2}\right) \sin\left(\frac{k_x}{2}\right)^2 \right)^{\frac{1}{2}}. \quad (11)
\]

The corresponding eigenvectors are

\[
|\lambda^{\pm}(k)\rangle = \frac{1}{\sqrt{2}} \left( \pm e^{i\phi(k)} \right), \quad (12)
\]

with \( \phi(k) \) given by

\[
e^{i\phi(k)} = \frac{i}{\sqrt{2}} F_1(k) \left( \cos\left(\frac{\sqrt{3}k_y}{2}\right) \sin\left(\frac{k_x}{2}\right) + \sin(k_x) + i\sqrt{3} \cos\left(\frac{k_x}{2}\right) \sin\left(\frac{\sqrt{3}k_y}{2}\right) \right)^{-1}. \quad (13)
\]

Before proceeding with the calculation of the SHC, let us first discuss some features of the energy spectrum which, as will be seen, will appear to be important and helpful to understand the characteristics of this dynamical quantity. As it is well known, the SHC can be directly related to the \( k \)-space curvature,\(^{29}\) this will be presented in detail in what follows, Sec. III. In this context, the singular values of the Berry connection at degeneracy points of the energy spectrum are of crucial importance. To prepare the insight into the topological nature of this problem, we first calculate the degeneracies by solving directly \( F_1(k) = 0 \). The result is plotted in Fig. 2.

In the absence of SOC (\( t^R = 0 \)) and in the isotropic case (\( t_0 = \tilde{t}_0 \)), the density of states (DOS) exhibits a Van Hove singularity at Fermi energy \( E = 4/3t_0 \). It corresponds to a straight line which goes through \( P_1 \) and \( P_3 \). For finite values of SOC strength, in the isotropic case, this peak is now split as clearly seen in Fig. 4. In addition, one has two additional peaks at the boundary of the energy spectrum.\(^{30}\) The energy dispersion is plotted as a function of momentum in Fig. 3(d). As seen, the degeneracy

![Figure 2:](image-url) (Color online) Degeneracy points in the energy spectrum of \( H \) in \( k \)-space (for the 1st Brillouin zone shown):

\( M \equiv P_1 = \{0, -2\pi/\sqrt{3}\}^T, P_2 = \{2\pi/3, -2\pi/\sqrt{3}\}^T, P_3 = \{\pi, -\pi/\sqrt{3}\}^T, K \equiv P_4 = \{4\pi/3, 0\}^T, P_5 = \{\pi, \pi/\sqrt{3}\}^T \). The blue dots indicate the saddle points in case of vanishing SOC.
points correspond to the points $P_i$ ($i = 1, 2, \ldots, 5$) and $\Gamma$. One finds $E(P_1) = E(P_2) = E(P_3)$ and $E(P_2) = E(P_3)$. In Fig. 4 these particular energies are indicated by dashed lines. When we switch on the anisotropy the singularities in the DOS change significantly as expected. In Fig. 3 one now sees that one degeneracy is lifted $E(P_1) = E(P_3) \neq E(P_3)$, and we still have $E(P_2) = E(P_2)$. Thus in the presence of anisotropy we have four relevant energies that we denote,

$$E(\Gamma) \equiv E_{d1} = - (4/3)(2 + t_0/t_0),$$
$$E(P_2) \equiv E_{d2} = (4/3)(2 - t_0/t_0),$$
$$E(P_1) = E(P_3) \equiv E_{s1} = (4/3)t_0,t_0,$n
$$E(P_2) = E(P_2) \equiv E_{s2} = (4/3) + (2/3)t_0/t_0,$n

with the energy expressed in values of $t_0$. Note that, two different sets of labels $(d_1,d_2)$ and $(S_1, S_2)$ are introduced. The meaning of this separation will become clear in what follows.

### III. SPIN HALL CONDUCTIVITY

#### Numerical Results of SHC

Having analyzed the energy spectrum, we now proceed further with the calculation of dynamical properties, more precisely the spin Hall conductivity $\sigma_{SH}$. The SHC is calculated within linear response theory using Kubo formula,

$$\sigma_{SH}(E_F) = \frac{1}{V} \sum_{E_m < E_F < E_n} \frac{\Im \langle \psi_m | J^x | \psi_n \rangle \langle \psi_n | v_y | \psi_m \rangle}{(E_n - E_m)^2 + \eta^2},$$

(where $V$ is the total number of lattice sites, $E_i$ the eigenenergies, $|\psi_i\rangle$ the corresponding eigenstates, $E_F$ the Fermi energy and $\eta$ a positive infinitesimal. The spin current operator is

$$J^z = \frac{\hbar}{4} \{s_z, v\},$$

and the velocity is defined with the usual expression $v = i[H, r]$. To calculate the SHC we first compute the matrix element density function $j(x,y)$ defined by

$$j(x,y) = \sum_{m,n} M_{mn} \delta(x - E_m) \delta(y - E_n),$$

where the matrix element $M_{mn}$ is

$$M_{mn} = \Im \langle \psi_m | J^x | \psi_n \rangle \langle \psi_n | v_y | \psi_m \rangle.$$ और

Note that the function $j(x,y)$ is a particularly useful and convenient quantity for calculations carried out by Kernel Polynomial Method in the presence of disorder.

We can thus rewrite $\sigma_{SH}(E_F)$ in terms of $j(x,y)$, leading to the following expression:

$$\sigma_{SH}(E_F) = \frac{e}{V} \int_{E_{min}}^{E_{max}} \int_{E_{min}}^{E_{max}} dx \ dy \frac{f_{E_F}(x) - f_{E_F}(y)}{(y - x)^2 + \eta^2} j(x,y),$$

where $f_{E_F}$ is the Fermi-Dirac distribution.

In the present case the only non-zero matrix elements are

$$M_{\pm}(k) = \pm M_{\mp}(k)$$

and

$$\sigma_{SH}(E_F) = \Im \{\langle \lambda^+(k) | J^x | \lambda^-(k) \rangle \langle \lambda^-(k) | v_y | \lambda^+(k) \rangle\},$$

where $\lambda^\pm(k)$ are the spin eigenstates.

Figure 4: (Color online) DOS $\rho(E)$ for a system of size $L = 300$ (cutoff in the Lorenzian was chosen to be $\eta = 0.03$) and $t_R = 0.3t_0$, for different values of $t_0/t_0 = 0, 0.2, \ldots, 1$. Dashed lines indicate the degeneracy points energy, see Fig. 3.
where

\[
M_\pm(k) = 4\sqrt{2}tR \sin(k_x) \left( 1 + 2 \cos \left( \frac{k_x}{2} \right) \cos \left( \frac{\sqrt{3}k_y}{2} \right) \right) \\
\times \left( 2t_0 \sin(k_x) + t_0 \sin(k_y) - \tilde{t}_0 \sin(\tilde{k}_5) \right) / (3F_1(k)).
\] (22)

The function \(j(x, y)\) is plotted in Fig. 5 for the isotropic case. Notice the symmetry due to Eq. 20. One can clearly see a sign change for \(j\) values at \((x, y > x)\) resp. \((x, y < x)\). This gives a clear indication for a sign-change in \(\sigma_{SH}\) at a particular Fermi level. In fact, this energy coincides with \(E_{S1}\). The resulting SHC is plotted in Fig. 6. At low electron density we observe a sharp increase to \(\sigma_{SH} = e/(8\pi)\) as expected since the lattice structure is irrelevant. As we increase the Fermi energy further, \(\sigma_{SH}\) remains almost flat up to \(E/t_0 \approx 1\) (extended plateau). Then at \(E = E_{S1}\) the sign of SHC changes in agreement with the discussion on \(j(x, y)\). More remarkable is the change of amplitude of the SHC in the region of negative conductivity. Indeed, it increases by 60% with respect to the case of low electron density.

Interestingly, the calculations indicate that the finite size effects are almost negligible up to \(E_{S1}\), but not in the hole doped regime. As mentioned before, one has to be careful in the derivation of TBH. We now compare the calculated SHC with that obtained using the TBH of Ref. 28 (filled circles). The system contains \(70 \times 70\) sites. The vertical dashed line corresponds to \(E_{S1}\).

Figure 6: (Color online) (a) SHC for different system sizes \(L = 50, \ldots, 300\) for the isotropic case. The SOC strength is \(t^R = 0.3t_0\). (b) Comparison between the \(\sigma_{SH}\) calculated within our model (continuous line) and that of Ref. 28 (filled circles). The system contains \(70 \times 70\) sites. The vertical dashed line corresponds to \(E_{S1}\).

Figure 7: As we discussed it previously a change of sign occurs at \(E_{S1}\). As we reduce \(t_0\) we observe a shift of the crossing energy towards \(E_F = 0\). One also notices drastic changes in the shape of \(\sigma_{SH}\) beyond \(E_{S1}\). In particular, we observe two minima and a significant reduction of the amplitude of SHC. This is especially pronounced for the case \(t_0 = 0.7t_0\). As the anisotropy is further decreased, the second minimum disappears and a second change of sign of the SHC is found. This occurs at the
value of $\tilde{t}_0 = (2/3)t_0$. Thus, below this value one has two changes of sign, one at $E_S1$ and the second at $E_S2$. One natural question which arises is why beyond this value SHC changes its sign a single time? This will be answered and understood when discussing the SHC in the framework of Berry phases. Remark that the second change of sign could not be anticipated from the DOS. Indeed, in contrast to the peak seen at $E_S1$ no peak, no singularity or pronounced feature is visible at $E_S2$, see for instance the case $t_0 = (1/3)t_0$ in Fig. 4. The limiting case $\tilde{t}_0 = 0$ requires also some additional attention. In the absence of SOC, the system is topologically equivalent to the square lattice. As seen in Fig. 7 the results for $\sigma_{SH}$ are similar to that of the square lattice for $E_F < 0$ (see e.g. Ref. 12,36,37). However, for $E_F > 0$ (below $E_F/t_0 < 1$), the amplitude is reduced to half of that of the square lattice. The spin flip term in the $(2 - 5)$ direction at finite SOC is at the origin of these crucial differences in the hole doped regime. It is surprising that the drastic changes take place only for $E_F > 0$, we will shed light on it in what follows.

Geometric Interpretation of SHC

In our case the Berry phases are explicitly given by the integral
\[ \gamma^\pm = \oint_{C^\pm} A^\pm(k) \cdot dk, \]  
along a loop $C^\pm = \{k_x, k_y\}$ in momentum space with the condition $E^\pm(k_x, k_y) = E_F$, the Berry connection $A^\pm(k)$ is
\[ A^\pm(k) = \left\langle \lambda^\pm(k) \left| \frac{\partial}{\partial k} \right| \lambda^\pm(k) \right\rangle. \]  

In our case we have found that $A^+(k) = A^-(k)$. Indeed, the eigenvectors, Eq. 12, do not depend on the hopping anisotropy and therefore the Berry connection is independent of $\tilde{t}_0/t_0$. The Berry connection is shown as a stream plot in Fig. 8. Examining it in the vicinity of the degeneracy points of the energy spectrum, $\Gamma$ and $P_i$ ($i = 1, \ldots, 5$), we find different winding directions for $A$. It diverges exactly at these points. In order to relate our findings to the Berry connection we reformulate the SHC, Eq. (15), in terms of it,
\[ \sigma_{SH}(E_F) = \frac{e}{2} \sum_{k,m=\pm} \frac{f_{E_F}(E_m(k))}{E_m(k) - E_{m}(k)} \cdot [A^m(k) \times v^0(k)]_z, \]
where $v^{(0)}(k)$ is the velocity in absence of SOC. Let us show that the SHC sign is connected to that of $[\mathcal{A}^m(k) \times v^{(0)}(k)]_z$. For that purpose we first consider the isotropic case. Notice that below $E_{S1}$, as the term $(E_+(k) - E_-(k))$ is always positive, the sign change of SHC is associated to that of $[\mathcal{A}^m(k) \times v^{(0)}(k)]_z$. For $E_F < E_{S1}$ the winding direction is always clockwise for the Berry connection and the velocity is always pointing outside (see Fig. 9 (a)), resulting then in a positive SHC. Beyond $E_{S1}$ the Fermi surface topology is completely different, it consists now of closed loops around the $P_2$ and $P_4$ as illustrated in Fig. 9 (b). Furthermore, the direction of the velocity is now directed inside (towards $P_2$ and $P_4$). However, the winding directions (see Fig. 8) of the Berry connection is unchanged, clockwise as around the $\Gamma$ point. This results in a sign change of the spin Hall conductivity. We now proceed further and discuss the presence of anisotropy. Since as mentioned before, $A$ is independent of $t_0$ thus the changes in SHC are associated with the Fermi surface topology only. Let us start with the extreme case of a fully anisotropic system, e.g. $t_0 = 0$. As seen before, the SHC exhibits two changes of sign, one at $E_{S1}$ and the other at $E_{S2}$. Thus, we focus our attention on these two Fermi energies. The situation at $E_{S1}$, Fig. 10 (a) and (b), is very similar to that of the isotropic case, except the fact that the Fermi surface at $E_F = E_{S1} + \epsilon$ is now also surrounding the point $P_2$ for which the winding direction of Berry connection is now negative (anti-clockwise). As the velocity direction is still pointing inwards, the negative winding around $P_2$ can lead to a sign change in $[\mathcal{A}^m(k) \times v^{(0)}(k)]_z$, thus, to the appearance of terms in Eq. 25 which reduce the absolute value of $\sigma_{SH}$. This explains why $\sigma_{SH}$ varies from $\sim -1.6e/(8\pi)$ in the isotropic case to $\sim -0.5e/(8\pi)$ for $t_0 = 0$. Concerning the second sign change, we find at $E_F = E_{S2} \pm \epsilon$ the same velocity direction. However, the Fermi surface at $E_F = E_{S2} + \epsilon$ is, in contrast to Fig. 10 (b) and (c), surrounding only $P_5$ which as mentioned before has the opposite winding direction with respect to that of $P_2$ and $P_4$. This yields the topological explanation of the change of sign at $E_{S2}$.

\section{IV. CONCLUSION}

In this paper we have performed a detailed theoretical study of the spin Hall conductivity on the anisotropic triangular lattice in the presence of Rashba spin-orbit...
coupling. Note that experimentally anisotropy could be e.g. tuned by taking advantage of lattice mismatches in semiconductor heterostructures. To conclude, it has been shown that on such a lattice the tight binding model should be carefully derived from the continuous Hamiltonian. In addition, our calculations have revealed that anisotropy in the hopping term has drastic effects on the SHC especially in the hole doped regime. Furthermore, depending on the anisotropy strength, one finds either a single or two changes of sign in the SHC at particular carrier densities. The origin of the observed changes in the SHC has been interpreted and understood geometrically in terms of Berry phases. The high symmetry points which have different winding directions of the Berry connection control the rapid variation of the SHC. More specifically, the changes of sign and amplitude of the SHC have been associated to the topology of the Fermi surface, the number of high symmetry points enclosed in it and to the sign of the velocity in the absence of SO coupling.

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