Recent searches have led to the discovery of dark matter (DM) in the universe. A major unsolved problem in particle physics is the nature of DM. The current theory of particle physics, the Standard Model (SM), cannot account for the observed DM density, $\rho_{DM} \approx 0.3\,\text{GeV/cm}^3$ [10]. Recent experiments have searched for axions, a type of DM candidate, and proposed new experiments to search for axion-like particles (ALPs).

Axions are predicted by string theory and are thought to be responsible for the observed DM density. They are expected to be extremely light with $m_a \lesssim \mu\text{eV}$ and have a very high number density and behave like a coherent field. If produced by the misalignment mechanism [4, 5], the axion is expected to be extremely light with $10^{-14} \lesssim m_a \lesssim 1\,\text{eV}$ (see [7–9] for a recent review). This implies that unlike WIMP DM, which would have a few particles per cubic meter, axion dark matter (aDM) would have a very high number density and behave like a coherent field. In this case, the DM energy density is better thought of as the kinetic and potential energy of a classical field rather than a dilute gas of individual particles.

If produced by the misalignment mechanism [4, 5], the time evolution of the axion field is expected to be given by

$$a(t) = a_0 \cos(\omega_a t - x \cdot \mathbf{k}_D),$$

where the frequency of oscillation is equal to the axion mass $\omega_a = m_a$ and an arbitrary overall phase. If aDM is responsible for the observed DM density, we can relate $a_0 = \sqrt{2\rho_{DM}/m_a}$, where $\rho_{DM}$ is the local DM density of $\sim 0.3\,\text{GeV/cm}^3$ [10]. Though aDM is extremely cold, it is expected to have a very small velocity spread due to gravitational effects. In the potential well of the Milky Way we expect a typical local velocity spread of $v_{DM} \sim 220\,\text{km/s}$. This results in a small spread in oscillation frequency due to Doppler shifting, $\Delta\omega_a/\omega_a \sim v_{DM}^2 \approx 10^{-6}$, as well as small spatial gradients on the scale of the de Broglie wavelength, $\lambda_D = 2\pi/|\mathbf{k}_D|$.

Experiments searching for aDM often leverage the fact that the axion couples to the photon and thus creates a small modification to electromagnetism. The axion – or any axion-like particle (ALP) for that matter – will create a modification to the electromagnetic Lagrangian, that can be written in terms of the Maxwell field tensor $F^{\mu\nu}$, electric current $J^{\mu}$, and axion field $a$:

$$\mathcal{L}_{EM} = J^{\mu}A^{\mu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}g_{a\gamma\gamma}aF^{\mu\nu}\tilde{F}_{\mu\nu}. \quad (2)$$

Where $\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, and $g_{a\gamma\gamma}$ is an unknown, but small, coupling between the axion and photon.

The $aF\tilde{F}$ term can be treated as an axion-to-two-photon coupling which converts photons into axions and vice-versa, as in light shining through wall (LSW) [11] and axion helioscope [12, 13] experiments. However, since aDM would imply a high occupation number for the field $a$, the Lagrangian can also be treated in the classical limit as a modification to Maxwell’s equations [14]:

$$\nabla \cdot \mathbf{E} = \rho_e - g_{a\gamma\gamma}B \cdot \nabla a,$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_e - g_{a\gamma\gamma} \left( \mathbf{E} \times \nabla a - \frac{\partial a}{\partial t} \mathbf{B} \right). \quad (3d)$$

These additional terms can be thought of as source terms that drive $\mathbf{E}$ and $\mathbf{B}$-fields at the oscillation frequency of the axion, $\omega_a$. Axion haloscope experiments typically...
create a strong static $B$-field and look for the small AC fields sourced by the aDM terms.

As we will see below, the exact implications of these additional terms for an experiment will depend strongly on the relative size of the detector to the Compton wavelength of the axion $\lambda_a = 2\pi/\omega_a$. Experiments like ADMX [15–17], HAYSTAC [18], and others [19–21] have built resonant cavities to probe axion masses in the range $m_a \sim 10^{-6} - 10^{-5}$ eV. In this range, the axion has Compton wavelengths of order $6 - 60$ cm and comparable to the physical size of the detector. Practical considerations limit the range of masses that can be probed with detectors comparable in size to $\lambda_a$. At shorter Compton wavelengths, $\sim 1$ mm, experiments like MADMAX propose to manipulate electric fields using arrays of dielectric plates [22] to coherently add effects over many Compton wavelengths within their detector. Recently, several experiments have been proposed to search for aDM with much lower masses of $10^{-14} - 10^{-6}$ eV and therefore Compton wavelengths much larger than the detector. These include experiments like ABRACADABRA [23], DM Radio [24], BEAST [25] and others [26–29].

In the limit of large $\lambda_a$, the typical approach is to build a detector with a strong DC magnetic field and search for an induced AC $B$-field. Experiments like [23, 24, 26], take the magneto-quasistatic (MQS) approximation and assume the displacement current in Eqn. 3d to be small. However, [25] proposes an alternate approach, keeping the displacement currents, to measure an induced AC field $E = -g_{a\gamma\gamma} a B$ in a strong DC $B$-field. This has caused disagreement in the community about whether the axion induced electric field would be large enough to be observable or whether it is significantly suppressed—specifically, whether the electric field is given by $E = -g_{a\gamma\gamma} a B$, or whether it is suppressed by powers $\lambda_a$. This has prompted new interpretations of the effect of the axion field in the presence of electromagnetic fields [30].

In section II, we explicitly solve the modified Maxwell’s equations in the case of an infinite solenoid without assuming the MQS approximation and demonstrate explicitly that the electric field vanishes everywhere in the large $\lambda_a$ limit. In section III, we generalize this conclusion and show that for a broad class of detectors, the MQS approximation is always valid in the large $\lambda_a$ limit and that the vanishing of the electric field is a generic quality. From this we conclude that an experiment with a static $B$-field will always be more sensitive to axiom induced magnetic fields over electric fields. Finally, in section IV, we address an alternate—but completely equivalent—approach outlined in [25, 30] which can mislead one into thinking a measurable electric field is always generated.

As is common, we will assume that the spatial gradients of the axion field are negligible, $\nabla a \approx 0$. This is because the de Broglie wavelength is about three orders of magnitude larger than the Compton wavelength.

\[ \lambda_D \approx 10^3 \lambda_a, \]

and thus spatial gradient terms are suppressed.

### II. AXION DARK MATTER AND THE INFINITE SOLENOID

The simplest geometry to consider is the case of the infinitely tall solenoid. Of course, in practice this geometry is not physically achievable. A physical solenoid will have a finite extent and thus returning fields outside the winds of the solenoid. But in many experimental setups, these fringe fields are small compared to the field inside the solenoid and lead to sub-dominant corrections. An infinite solenoid is a useful example on which to see the major effects. We will comment on this point later.

Assume we have an infinitely tall solenoid of radius $R$ pointing along the $\hat{z}$ direction. The current density along the walls is

\[ J = B_0 \delta (\rho - R) \hat{\phi}, \]

such that the unmodified Maxwell’s equations would lead to the solution

\[ \mathbf{B}_f = \begin{cases} B_0 \hat{z} & \rho < R, \\ 0 & \rho > R. \end{cases} \]

See Fig. 1. Even with the modified form of Maxwell’s equations this solution is correct to zeroth order in $g_{a\gamma\gamma}$. Further, let’s assume that current cannot flow along the solenoid walls in the $\hat{z}$ direction. For instance, we can take this to be a densely packed set of current carrying loops that only carry current in the $\hat{\phi}$ direction.

We can take the time derivative of equation 3d, to get

\[ \nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2} + g_{a\gamma\gamma} \frac{\partial^2 a}{\partial t^2} \mathbf{B} + O(g_{a\gamma\gamma}^2), \]

where we have taken advantage of the fact that

\[ -\nabla \times (\nabla \times \mathbf{E}) = \nabla^2 \mathbf{E} - \nabla (\mathbf{\nabla} \cdot \mathbf{E}) = \nabla^2 \mathbf{E}, \]
with $\nabla \cdot \mathbf{E} = 0$ everywhere, and that $\mathbf{J}$ is constant in time.

Similarly, we could take the time derivative of Eqn. 3c, to get

$$-\nabla^2 \mathbf{B} + g_{aH} \frac{\partial}{\partial t} \mathbf{B} = -\frac{\partial^2 \mathbf{B}}{\partial t^2}. \tag{8}$$

Discarding terms of $O(g_{aH}^2)$, we are left with two wave equations to solve:

$$\nabla^2 \mathbf{E} - \frac{\partial^2 \mathbf{E}}{\partial t^2} = \begin{cases} g_{aH} \frac{\partial^2}{\partial t^2} B_0 \hat{z} & \rho < R, \\ 0 & \rho > R, \end{cases} \tag{9a}$$

$$\nabla^2 \mathbf{B} - \frac{\partial^2 \mathbf{B}}{\partial t^2} = -g_{aH} \frac{\partial}{\partial t} B_0 \delta(\rho - R) \hat{\phi} \tag{9b}.$$  

It is clear from these equations that the only non-trivial solutions will be for $\mathbf{E}$ and $\mathbf{B}_\phi$. The other components are not affected by the axion field at leading order. Since the axion field is nicely decomposable into frequency modes, we will move into frequency space and drop transient solutions. Because of the symmetry, we propose the solutions

$$E_z(\rho, t) = \psi_E(\rho) e^{i \omega_a t}, \quad (10a)$$

$$B_\phi(\rho, t) = \psi_B(\rho) e^{i \omega_a t}. \quad (10b)$$

A. The B-Field Solution

Plugging (10b) into (9b) and performing a change of variables to $\rho' = \omega_a \rho$, we get the Bessel equation with a boundary condition at $\rho = R$:

$$\left( \frac{\partial^2}{\partial \rho'^2} + \frac{1}{\rho'} \frac{\partial}{\partial \rho'} + \left(1 - \frac{1}{\rho'^2}\right) \right) \psi_B = -ig_{aH} a_0 B_0 \delta(\rho' - \omega_a R). \tag{11}$$

The solutions to this are Bessel functions of order 1, with a boundary conditions at $\rho' = 0$ and $\rho' = \omega_a R$:

$$\psi_B(\rho') = \begin{cases} a_B J_1(\rho') & \rho' < \omega_a R, \\ b_B H_1^+(\rho') & \rho' > \omega_a R. \end{cases} \tag{12}$$

Here, we required that for $\rho' < \omega_a R$ the diverging $N_1(\rho')$ solution is suppressed, and for $\rho' > \omega_a R$ an outward traveling wave given by the Hankel function, $H_1^+(\rho')$. (An inward traveling wave, $H_1^-(\rho')$, is also a correct solution, however would imply power flowing into the oscillating axion field from infinity rather than out of it.)

We can now find the full solution, by requiring continuity of $B_\parallel$ across the boundary, and a step discontinuity in $\frac{\partial \psi_B}{\partial \rho'}$ as required by the $\delta$ function. (Remember that we specified that current could not flow along $\hat{z}$.)

$$a_B J_1(\omega_a R) - b_B H_1^+(\omega_a R) = 0, \quad (13a)$$

$$\left( \frac{\partial}{\partial \rho'} H_1^+(\rho') - a_B \frac{\partial}{\partial \rho'} J_1(\rho') \right)_{\rho' = \omega_a R} = -ig_{aH} a_0 B_0. \quad (13b)$$

This can then be solved further to yield

$$a_B = -\frac{\pi}{2} g_{aH} a_0 B_0 \omega_a R H_1^+(\omega_a R), \quad (14a)$$

$$b_B = -\frac{\pi}{2} g_{aH} a_0 B_0 \omega_a R J_1(\omega_a R), \quad (14b)$$

where we have leveraged Abel’s identity to simplify the Wronskian of Bessel functions as

$$W(J_1, H_1^+) = J_1 \frac{\partial H_1^+}{\partial \rho'} - \frac{\partial J_1}{\partial \rho'} H_1^+ = \frac{2i}{\pi \rho'}. \tag{15}$$

This fully specifies the solution of the B-field driven by the axion at leading order in $g_{aH}$. Figure 2 shows the behavior of $B_\phi$ for various values of $R/\lambda_a$.

B. The E-Field Solution

Returning to the $E_z$ component, we can plug (10a) into (9a) and performing a change of variables get another Bessel equation:

$$\left( \frac{\partial^2}{\partial \rho'^2} + \frac{1}{\rho'} \frac{\partial}{\partial \rho'} + 1 \right) \psi_E = \begin{cases} -g_{aH} a_0 B_0 & \rho < \omega_a R, \\ 0 & \rho > \omega_a R. \end{cases} \tag{16}$$

which has solutions

$$\psi_E(\rho') = \begin{cases} a_E J_0(\rho') - g_{aH} a_0 B_0 & \rho' < \omega_a R, \\ b_E H_0^+(\rho') & \rho' > \omega_a R. \end{cases} \tag{17}$$

Again, we have required that $\psi_E(\rho')$ be finite at $\rho' = 0$, and an outward traveling wave for $\rho' > \omega_a R$.

Here, the boundary conditions require that $E_z$ and its derivative be continuous across the boundary. The former condition can be seen by integrating $\nabla \times \mathbf{E}$ around a small contour just inside and outside of the solenoid; the latter can be seen by integrating Eqn. 16 between $[\omega_a R - \epsilon, \omega_a R + \epsilon]$ as $\epsilon \to 0$.

$$a_E J_0(\omega_a R) - g_{aH} a_0 B_0 = b_E H_0^+(\omega_a R), \quad (18a)$$

$$a_E J_1(\omega_a R) = b_E H_0^+(\omega_a R). \quad (18b)$$

We can again simplify this further to

$$a_E = \frac{\pi}{2} g_{aH} a_0 B_0 \omega_a R H_1^+(\omega_a R), \quad (19a)$$

$$b_E = \frac{\pi}{2} g_{aH} a_0 B_0 \omega_a R J_1(\omega_a R). \quad (19b)$$

Where we have taken advantage of the Bessel function property that $2\partial_{\rho'} \Omega_{\nu} = \Omega_{\nu-1} - \Omega_{\nu+1}$ and that $\Omega_{\nu} = (-1)\Omega_{-\nu}$ for $\nu \in [J_{\nu}, H_{\nu}^+]$. These equations fully specify the E-field solution.

Putting these together with the solutions for the B-field yields a nice compact form

$$\begin{pmatrix} a_E \\ b_E \\ a_B \\ b_B \end{pmatrix} = \pi g_{aH} a_0 B_0 \omega_a R \begin{pmatrix} i H_1^+(\omega_a R) \\ i J_1(\omega_a R) \\ -H_1^+(\omega_a R) \\ -J_1(\omega_a R) \end{pmatrix}. \tag{20}$$

The solutions for $E_z$ and $B_\phi$ are plotted together in Fig. 2 for various values of $R/\lambda_a$. 

FIG. 2. Numerical solutions for the field strengths for the infinite solenoid configuration. The $E_z$ and $B_\phi$ field strengths are plotted in units of $g_{a\gamma\gamma}a_0 B_0/2R$ for several values of $R/\lambda_a$. The solid lines are the terms driven by the axion field (in phase for $E_z$ and shifted by $\pi/2$ for $B_\phi$). The only approximation are that these are to first order in $g_{a\gamma\gamma}$.

**C. The Long Compton Wavelength Limit**

The variable $\rho'$, is actually the ratio of the radial coordinate scaled by the Compton wavelength of the axion $\rho' = 2\pi \rho/\lambda_a$. Not surprisingly, this marks the Compton wavelength as the relevant length scale of the problem. If $R \ll \lambda_a$, we will get one type of behavior, as compared to $R \sim \lambda_a$ or $R \gg \lambda_a$. This can be seen in Fig. 2.

In the long Compton wavelength limit, $R \ll \lambda_a$ (or equivalently $\rho' = \omega_a R \ll 1$), both sides the solenoid can be thought of as “oscillating in phase” and the fields add coherently over the relevant distance scales. This is the limit relevant for experiments like ABRACADABRA [23], DM Radio [24], BEAST [25] and other LC-resonator searches [26].

We can take the asymptotic limits of the Bessel functions to see how the field near the solenoid behaves. Equation 12 becomes

$$\psi_B(\rho') \approx \begin{cases} 
\frac{a_B}{2} \rho' & \rho' < \omega_a R, \\
-i \frac{b_B}{\pi \rho'} & \rho' > \omega_a R.
\end{cases}$$

(21)

with the coefficients given by

$$a_B = ig_{a\gamma\gamma}a_0 B_0,$$

(22a)

$$b_B = -\frac{\pi g_{a\gamma\gamma}a_0 B_0 \omega_a^2 R^2}{4},$$

(22b)

inserting this and converting back to $\rho$, yields the radial
behavior

\[ \psi_B(\rho) \approx \begin{cases} 
\frac{i}{2} g_{a\gamma\gamma} \omega_a a_0 B_0 \rho & \rho < R, \\
\frac{i}{2} g_{a\gamma\gamma} \omega_a a_0 B_0 \frac{R^2}{\rho} & \rho > R.
\end{cases} \]  

(23)

The factor of \( i \) simply indicates a \( \frac{\pi}{2} \)-phase shift from the axion field. This is expected since the B-field in Eqn. 9b is driven by \( \frac{\partial}{\partial t} \).

Plugging this back into Eqn. 10b, we have our full solution for the axion induced B-field to first order in \( g_{a\gamma\gamma} \) and in the limit of \( \rho, R \ll \lambda_a \):

\[ B_a \approx \begin{cases} 
\frac{i}{2} g_{a\gamma\gamma} \frac{\partial}{\partial \rho} \rho \hat{\phi} & \rho < R, \\
\frac{i}{2} g_{a\gamma\gamma} \frac{\partial}{\partial \rho} \frac{R^2}{\rho} \hat{\phi} & \rho > R.
\end{cases} \]  

(24)

Here, we have summed over axion frequency modes \( \omega_a \) to convert \( i \omega_a a_0 e^{i\omega_a t} \) back into \( \frac{\partial}{\partial t} \) to make the solution true for arbitrary \( a(t) \).

It should be noted, that this is exactly the result that we would expect from taking the MQS approximation and treating the oscillating axion field in a magnetic field as an “effective current”, \( J_{\text{eff}} = g_{a\gamma\gamma} \frac{\partial}{\partial t} B_0 \hat{z} \) as is done in [23, 24, 26].

Looking at the electric field behavior in the long wavelength limit, Eqn. 17 becomes

\[ \psi_E(\rho') \approx \begin{cases} 
a_E - g_{a\gamma\gamma} a_0 B_0 & \rho' < \omega_a R, \\
b_E \left(1 + \frac{i}{2} \left( \ln \left( \frac{\rho'}{\rho} \right) + \gamma \right) \right) & \rho' > \omega_a R.
\end{cases} \]  

(25)

where \( \gamma \) is the Euler-Mascheroni constant, \( (\gamma \approx 0.5772...) \). With the coefficients given by

\[ a_E = g_{a\gamma\gamma} a_0 B_0, \]  

(26)

\[ b_E = -\frac{\pi g_{a\gamma\gamma} a_0 B_0 \omega_a^2 R^2}{4} \approx 0. \]  

(27)

This implies that, to first order in \( g_{a\gamma\gamma} \) and for \( \rho, R \ll \lambda_a \), we have

\[ E_a = 0. \]  

(28)

Or more precisely, that in the limit of \( \rho, R \ll \lambda_a \), electric fields are suppressed by \( \left( \frac{\rho}{\lambda_a} \right)^2 \). This behavior can be seen in Fig. 2.

This is in direct contrast with the argument set forth in [25], which searches for an axion induced electric field in the long Compton wavelength limit. This conclusion is reached here using a particular geometry, but the conclusion is a lot more general, as we will show in the next section. It is worth noting that the E-field solution proposed in that work, \( E = -g_{a\gamma\gamma} a_0 B \), does appear in the solution to Maxwell’s equations as the \( \rho' \) independent term in Eqn. 17. But in the large \( \lambda_a \) limit it is canceled by the other term in the full solution – given in Eqn. 26.

In the short Compton wavelength limit, the field \( E = -g_{a\gamma\gamma} a_0 B \) appears as an offset to the oscillating Bessel function: \( E_\perp = (a E_\perp(a_0 \rho) - g_{a\gamma\gamma} a_0 B_0) e^{i\omega_a t} \). When the Bessel function has many oscillations within \( 0 < \rho < R \), the spatial average approaches \( -g_{a\gamma\gamma} a_0 B_0 e^{i\omega_a t} \). This can be seen in the lower panel of Fig. 2 as the offset between the solid and dotted red lines.

An experimental setup with a capacitor inside the solenoid (similar to [25]) would in fact see charges displaced by the oscillating axion induced E-field. But this would only be a measurable effect in the \( R \gg \lambda_a \) limit (i.e. for frequencies \( \omega_a/(2\pi) \gtrsim 300 \text{ MHz} \)). This is akin to the microwave cavity designs used by [15–21], but without the resonator cavity. Interestingly, there are other recent proposals for the \( R \sim \lambda_a \) regime using this type of detector, but with all resonant enhancement moved into electronics [31]. At shorter wavelengths still, other experimental techniques have been proposed which rely on manipulating the E-field with dielectric plates.[22]. These latter approaches, where \( R \gg \lambda_a \), are not incompatible with the results presented here.

### III. DEMONSTRATING THE MQS APPROXIMATION FOR A GENERIC DETECTOR

The argument in the previous section can be made much more general by directly demonstrating that the MQS approximation holds in the presence of an oscillating axion field in the large \( \lambda_a \) limit. In the following argument, we will make two assumptions:

1. our detector is composed of a collection of \textit{time-independent} charges and currents, \( \rho_c \) and \( J_c \);

2. our detector fits into some box with a diagonal size \( L \). Thus both the \( \rho_c \) and \( J_c \) used to create our primary fields and whatever apparatus we use to detect axion induced fields are contained within \( |x - x'| < L \).

The precise shape of the box in the second assumption is irrelevant – it only establishes a characteristic size for our detector. We make no assumptions about the configuration of the currents and charges within the box.

We can split our E and B fields into terms of similar order in \( g_{a\gamma\gamma} \)

\[ E = E_0 + E_1 e^{i\omega_a t} + O(g_{a\gamma\gamma}^2), \]  

(29a)

\[ B = B_0 + B_1 e^{i\omega_a t} + O(g_{a\gamma\gamma}^2), \]  

(29b)

where \( E_1 \) and \( B_1 \) will be proportional to \( g_{a\gamma\gamma} \). We rewrite our wave equations as

\[ \nabla^2 E_0 = \nabla \rho_c, \]  

(30a)

\[ \nabla^2 B_0 = \nabla \times J_c, \]  

(30b)

\[ \nabla^2 E_1 + \omega_a^2 E_1 = -g_{a\gamma\gamma} \omega_a^2 a_0 B_0, \]  

(30c)

\[ \nabla^2 B_1 + \omega_a^2 B_1 = i g_{a\gamma\gamma} \omega_a a_0 \nabla \times B_0, \]  

(30d)

Notice that \( E_0 \) and \( B_0 \) are independent of time because of our first assumption above.
Focusing on Eqn. 30c, we can split $E_1$ into $E_1 = E'_1 - g_{a\gamma\gamma}a_0 B_0$, and get an equation for $E'_1$

\[ \nabla^2 E'_1 + \omega_a^2 E'_1 = g_{a\gamma\gamma} a_0 \nabla^2 B_0 \]
\[ = g_{a\gamma\gamma} a_0 \nabla \times J_e. \]  
(31)

At this point, we can use the retarded Green’s functions to solve for our fields.

\[ E_0(x) = \int \frac{\nabla \rho_e - d^3 x'}{|x - x'|}, \]  
(32a)

\[ B_0(x) = \int \frac{\nabla \times J_e(x')}{|x - x'|}, \]  
(32b)

\[ E_1(x) = g_{a\gamma\gamma} a_0 \int \frac{e^{i \omega a |x - x'|} - 1}{|x - x'|} \nabla \times J_e(x') \ d^3 x', \]  
(32c)

\[ B_1(x) = ig_{a\gamma\gamma} \omega a_0 \int \frac{e^{i \omega a |x - x'|}}{|x - x'|} J_e(x') \ d^3 x'. \]  
(32d)

Notice that the $-1$ in the Eqn. 32c came from solving for $E'_1$ and substituting Eqn. 32b in for the offset term, $-g_{a\gamma\gamma}a_0 B_0$.

We point out that $\rho_e$ does not appear in our axion induced fields. This is because in the limit that $\nabla a$ is small, we cannot use static electric fields alone to detect axions – regardless of their shape. This is evident from Eqns. 3.

At this point our solution is very general. Up to now, we have only used the first assumption that our charges and currents are constant in time. We use the second assumption to examine what happens in the limit of $L \ll \lambda_a$. Notice that our solutions are completely in terms of charges and currents, which are completely contained within our box of size $L$ – as opposed to fields, which can extend outside of the box.

If both $x$ and $x'$ are within our box then $|x - x'| \leq L$. And now we examine the behavior of the axion induced electric fields by Taylor expanding Eqn. 32c in the limit of $\omega a L \ll 1$:

\[ E_1(x) \approx g_{a\gamma\gamma} a_0 \int \frac{e^{i \omega a |x - x'|} - 1}{|x - x'|} \nabla \times J_e(x') \ d^3 x'. \]
\[ = g_{a\gamma\gamma} a_0 \omega_a \int \nabla \times J_e(x') \ d^3 x' + O((\omega a L)^2) \]
\[ = g_{a\gamma\gamma} a_0 \omega_a \int S \ n \times J_e(x') dA' + O((\omega a L)^2) \]
\[ = O((\omega a L)^2) \]  
($L \ll \lambda_a$).

(33)

Where the last step takes advantage of the fact that our current is contained and so equal to zero at the surface of the box, $S$. Hence, the electric field is suppressed by $(L/\lambda_a)^2$.

Of course, a similar process can be done for Eqn. 32d, but it is easy to see that the relevant difference between this equation and Eqn. 32c is the $-1$ in the numerator. The leading term remains and the result is not suppressed by an additional powers of $\lambda_a$.

This conclusion is very general and does not depend on the precise details of our detector. We only assumed that 1) the currents and charges that drive our primary fields are constant in time; and 2) our detector is of characteristic size $L \ll \lambda_a$. Under these assumptions we have shown that axion induced electric fields are always suppressed. We have actually just showed that the MQS approximation continues to hold in the presence of an oscillating axion field with large $\lambda_a$.

And interesting thing worth noting is that in this calculation we have neglected terms proportional to $\nabla a$ as they are suppressed factors of $\lambda_a/\lambda_D \sim 10^{-3}$. However, when $L/\lambda_a \lesssim 10^{-3}$, it is possible for electric fields generated by the $\nabla a \cdot B$ term in Eqn. 3a to dominate over the electric fields generated by the $\partial t B$ term in Eqn. 3d.

Finally, it is worth describing the behavior of $E_1$ and $B_1$ in the limit of $\omega a L \gg 1$. In this limit, the exponentials in Eqns. 32c and 32d oscillate very rapidly and, under reasonable assumptions about how quickly $J_e$ varies within $L$, will cause the integrals to average to zero. All that will remain is

\[ E_1(x) \approx -g_{a\gamma\gamma} a_0 \int \frac{\nabla \times J_e(x')}{|x - x'|} d^3 x', \]  
($L \gg \lambda$),

(34)

which is exactly the $-g_{a\gamma\gamma}a_0 B_0$ term. This is completely consistent with what we saw in the case of the infinite solenoid where the coefficients in Eqn. 20 fall off with the Bessel functions at large values of $\omega a L$ and all that remains is the $-g_{a\gamma\gamma}a_0 B_0$ term.

From this, we conclude that if (1) our currents and charges are independent of time and (2) with reasonable assumptions about how rapidly our current distributions vary on length scales $\sim \lambda_a \ll L$, the effect of the axion can be given by $E_0(x, t) = -g_{a\gamma\gamma}a_0 B_0$. However, this is not the limit proposed for axion searches in the mass range $m_a \lesssim 1 \mu eV$.

IV. ALTERNATE APPROACH USING POLARIZATION

In this section, we address the approach laid out in [25, 30] and show the way this approach can naively lead one to expect a large electric field when one is not present. But when boundary conditions are correctly applied, this is a perfectly valid approach.

Following [25, 30], we can reformulate Eqn. 3 in terms of the macroscopic fields $D$ and $H$:

\[ \nabla \cdot D = \rho_f + g_{a\gamma\gamma} B \cdot \nabla a, \]  
(35a)

\[ \nabla \cdot B = 0, \]  
(35b)

\[ \nabla \times E = -\frac{\partial B}{\partial t}, \]  
(35c)

\[ \nabla \times H = J_f + \frac{\partial D}{\partial t} - g_{a\gamma\gamma} \left( E \times \nabla a + \frac{\partial a}{\partial t} B \right) \]  
(35d)

Since we are solving these equations in free space (where there are no bound charges), these equations are completely equivalent to Eqn. 3 and should yield the exact
same solutions. However, in this approach we can rewrite this in terms of a set of modified fields

\[ D_a = D - g_{a\gamma\gamma} (aB) , \]
\[ H_a = H + g_{a\gamma\gamma} (aE) , \]

with which we can write an analogous set of macroscopic Maxwell’s equations with no axion modification terms

\[
\nabla \cdot D_a = \rho_f , \quad (37a) \\
\nabla \cdot B = 0 , \quad (37b) \\
\n\nabla \times E = - \frac{\partial B}{\partial t} , \quad (37c) \\
\n\nabla \times H_a = J_f + \frac{\partial D_a}{\partial t} . \quad (37d)
\]

In four-vector notation, what we have done here is to envelope the axion current \( J_a^\mu = g_{a\gamma\gamma} (B \cdot V_a, -E \times V_a + \partial_a B) \) into a redefinition of the electromagnetic field tensor \( F^{\mu\nu} \to \tilde{F}^{\mu\nu} = F^{\mu\nu} - P^{\mu\nu}_a \), where

\[
\partial_\mu P^{\mu\nu}_a = J^\nu . \quad (38)
\]

We can see from a straightforward application of Noether’s theorem that \( P^{\mu\nu}_a \) should be given by

\[
P^{\mu\nu}_a = g_{a\gamma\gamma} \tilde{F}^{\mu\nu} = g_{a\gamma\gamma} \begin{pmatrix} 0 & -B_z - B_y - B_z & 0 \\
B_x & 0 & E_z - E_y \\
B_y - E_z & 0 & E_x \\
B_z & E_y - E_x & 0 \end{pmatrix} . \quad (39)
\]

And of course, the continuity equation follows trivially from the fact that

\[
\partial_\mu J_a^\mu = g_{a\gamma\gamma} \partial_\mu \partial_\nu a \tilde{F}^{\mu\nu} = 0 , \quad (40)
\]

because the derivatives are symmetric under interchange of \( \mu \) and \( \nu \) and \( \tilde{F}^{\mu\nu} \) is anti-symmetric.

This entire approach is completely analogous to the way the macroscopic form of Maxwell’s equations splits the electric current into \( J_{bound} \) and \( J_{free} \) and attaches the former into a redefinition of \( F^{\mu\nu} \to \tilde{F}^{\mu\nu} = F^{\mu\nu} - P^{\mu\nu}_{bound} \), where

\[
P^{\mu\nu}_{\text{bound}} = \begin{pmatrix} 0 & P_x & P_y & P_z \\
-P_x & 0 & M_z & -P_y \\
-P_y & M_z & 0 & P_z \\
-P_z & M_y & -P_y & 0 \end{pmatrix} , \quad (41)
\]

for a material polarization \( \mathbf{P} \) and magnetization \( \mathbf{M} \), such that \( \partial_\mu P^{\mu\nu}_{\text{bound}} = J_a^\nu \). In each of these steps, our equations of motion remain completely unchanged and the continuity equation is always satisfied. We are simply moving terms around.

\[
\partial_\mu F_a^{\mu\nu} = J_a^\nu , \quad (42a) \\
\partial_\mu \tilde{G}^{\mu\nu} = \partial_\mu P^{\mu\nu}_a = J_a^\nu , \quad (42b) \\
\partial_\mu F^{\mu\nu} - \partial_\mu P^{\mu\nu}_{\text{bound}} = J_a^\nu + J_a^\nu , \quad (42c) \\
\partial_\mu F^{\mu\nu} = J_a^\nu + J_a^\nu + J_a^\nu \text{ bound} . \quad (42d)
\]

This appears to be a tidy reformulation of Eqns. 3, however, it must be emphasized that the physics is completely unchanged. Further, great care has to be taken when using these \( D_a \) and \( H_a \) fields, as the simplicity of Eqns. 37 can be deceptive. The reason is that the Lorentz force has not been changed, \( \mathbf{f} = \rho_c \mathbf{E} + \mathbf{J}_c \times \mathbf{B} \). In other words, charges and currents still rearrange themselves in response to \( \mathbf{E} \) and \( \mathbf{B} \)-fields. Therefore boundary conditions must still be placed on \( \mathbf{E} \) and \( \mathbf{B} \) rather than on \( D_a \) and \( H_a \).

The purpose of this approach, however, is to continue the analogy, and to write a set of axion polarization and magnetization fields:

\[
\mathbf{P}_a = -g_{a\gamma\gamma} (a\mathbf{B}) , \quad (43a) \\
\mathbf{M}_a = g_{a\gamma\gamma} (a\mathbf{E}) . \quad (43b)
\]

But this is where the subtleties become critical. We must keep in mind that

\[
\nabla \cdot \mathbf{P} = -g_{a\gamma\gamma} \nabla \cdot (a\mathbf{B}) = -g_{a\gamma\gamma} (\nabla a \cdot \mathbf{B} + a \nabla \cdot \mathbf{B}) = -g_{a\gamma\gamma} \nabla a \cdot \mathbf{B} \sim O(g_{a\gamma\gamma} \mathcal{O}_{DM}) . \quad (44)
\]

In other words, in the limit of small spatial gradients in \( a \), the axion “bound charge density” is suppressed. Substituting this into Eqn. 3a tells us that \( \mathbf{P}_a \) does not create an \( \mathbf{E} \)-field directly. We often intuitively think that an electrically polarized material has an associated electric field. However, this field comes from bound surface charges at the edge of the polarized material. But no such boundary can ever exist for \( \mathbf{P}_a \). So while it might naively appear that an electric field must be present due to the axion polarization, it is not.

Instead a time-varying \( \mathbf{P}_a \) generates a time varying magnetic field and that time-varying magnetic field can generate time-varying electric fields. Stepping back to our example of the infinite solenoid, we can easily calculate the polarization and magnetization to first order in \( g_{a\gamma\gamma} \) (neglecting terms proportional to \( \nabla a \)):

\[
\mathbf{P}_a = \begin{cases} -g_{a\gamma\gamma} a B_0 \hat{z} , & \rho < R , \\
0 & \rho > R , \end{cases} \quad (45a) \\
\mathbf{M}_a = 0 \quad \text{Everywhere} . \quad (45b)
\]

The intuition would be to view this as a time varying electric field inside our solenoid. But there is no divergence in \( \mathbf{P} \) to generate such an electric field. Instead we note that \( \mathbf{P}_a \) varies with \( a \) and plug these values into Eqn. 37d and recover Eqn. 3d. This will recover the result in Sec. II.

This underlines the fact that an axion polarization with no space-time derivatives cannot have any physical manifestations. This is also evident in the Lagrangian because the \( a\tilde{F} \) terms becomes a complete derivative in the limit that \( \partial_\mu a = 0 \). An analogous argument can be made about magnetization induced magnetic fields.
Despite our intuition underwise, the magnetization, \( \mathbf{M}_a \), alone cannot generate a physically observable magnetic field, only when \( \nabla \times \mathbf{M}_a \neq 0 \).

The approach of calculating axion induced polarization and magnetization is completely equivalent to the approach outlined in the first part of this paper. But great care must be taken when using this approach, because subtleties in the application of boundary conditions and physical intuition can conspire to produce physical effects where they should be suppressed.

V. CONCLUSION

In this work, we have stepped through the calculation of the axion induced \( \mathbf{E} \) and \( \mathbf{B} \)-fields in the presence of a strong magnetic field in an infinite solenoid. We showed that the solution \( \mathbf{E} = -\gamma_0 a \gamma_0 \mathbf{B} \) is part of the full solution of the modified Maxwell's equations, however by itself it does not satisfy the required boundary conditions. In the large \( \lambda_a \) limit, the full solution suppresses the electric fields everywhere by \( \left( \frac{R}{\lambda_a} \right)^2 \).

We then laid out the generic derivation of the MQS approximations in the presence of an axion field. And demonstrated that in any experimental setup with a time-independent charge and current distribution, the axion induced \( \mathbf{E} \)-fields are always suppressed relative to the axion induced \( \mathbf{B} \)-fields in the large \( \lambda_a \) limit. The conclusions of this work directly contradict the arguments outlined in [25, 30], and this implies that the limits shown in [25] are too low by \( \sim 6.5 \) orders of magnitude.

Unfortunately, this highlights the fact that it is extremely difficult to detect axion induced electric fields in the large \( \lambda_a \) limit in any experimental setup with static primary fields. This should be kept in mind for future experimental proposals for axion haloscopes in the ultralight mass regime.

Finally, it should be noted that boundary conditions at the edge of the \( \mathbf{B} \)-field shape the behavior of the fields inside. This also underscores the need for all axion haloscopes to carefully analyze the effect of the boundaries of their fields and not necessarily assume that the approximations of an infinite field are valid.

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