Topological modes protected by chiral and two-fold rotational symmetry in a spring-mass model with a Lieb lattice structure

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We propose how to realize the topological modes protected by chiral and rotation symmetry for a mechanical system. Specifically, we show the emergence of topological modes protected by chiral and two-fold rotational symmetry by a spring-mass system with a Lieb lattice structure and dents on the floor. Moreover, comparing the results of a tight-binding model, we have found the additional topological modes for our spring-mass model due to the extra degrees of freedoms. Our approach to realize the topological modes can be applied to other cases with rotation symmetry, e.g., a system of a honeycomb lattice with three-fold rotational symmetry.

Introduction: In this decade, topological phases have been extensively analyzed as new quantum states which host boundary modes protected by topological properties in the bulk.¹⁻¹⁹ Remarkably, it turned out that these topological phenomena can be observed even beyond the quantum systems²⁰⁻³⁴ (e.g., photonic crystals,²⁴,3⁰,3¹ mechanical systems²⁰⁻²³,2⁵,2⁶,2⁹,3²,3⁴ etc.). These topological phenomena beyond the quantum systems originate from the fact that these classical systems are described by an eigenvalue equation. One of the advantages of these classical systems is high controllability of their parameters. So far thanks to their high controllability, a variety of topological phases has been proposed for spring-mass systems (SMMs) which are periodic arrangements of the mass points and springs.²¹,2²,2⁶,2⁹,3²,3⁴ For instance, Chern insulators²¹,2² and higher-order topological phases³⁴ have been realized for SMMs.

Along with the above development, the notion of the topological protection has also been extended to gapless quantum systems. A representative example is a Weyl semi-metal showing the gapless modes in the bulk. These gapless modes are protected by a finite value of the Chern number which is the topological invariant for two-dimensional systems in class A (no symmetry).³⁵,3⁶ Similar topological gapless modes can also be found for other local symmetry (e.g. time-reversal symmetry, particle-hole symmetry, and chiral symmetry).³⁷ Furthermore, the analysis of topological gapless states are further developed by taking into account the spatial symmetry. In particular, the coexistence of chiral symmetry and spatial symmetry gives topologically stable zero energy modes protected by these symmetry.¹⁷

Unfortunately, to our knowledge, the above topological modes protected by chiral and spatial symmetry has not been experimentally observed so far. One of the reasons is that for quantum systems such as metals and insulators, long range hoppings break the chiral symmetry which is essential for the above topological modes.

Under this background, in this paper, we theoretically propose how to realize the above topological modes by taking advantage of the high controllability of the classical systems. Specifically, we realize the topological modes for the SMM with a Lieb lattice structure. We note that the simple preparation of the SMM does not preserve chiral symmetry; by tuning the on-site potential arising from the dents on the floor, we have an SMM preserving chiral symmetry. In addition, by comparing the results of a tight-binding model (TBM), we also find that the extra degrees of freedoms of the SMM increases the number of topological modes, which is an unique phenomenon of SMMs. Our approach for the realization can be applied to a general two-dimensional tight-binding model (TBM).

The topological modes protected by chiral and spatial symmetry: Firstly, we briefly review the general arguments of the topological modes protected by chiral and spatial symmetry for quantum systems.¹⁷ It is well-known that the Schrödinger equation is reduced to the eigenvalue problem. We set a matrix \( H \) for the eigenvalue equation. Additionally, we consider a chiral operator \( \mathcal{T} \) and a spatial operator \( \mathcal{R} \). Chiral symmetry and spatial symmetry of \( H \) are written as

\[
\{ H, \mathcal{T} \} = 0, \quad \{ H, \mathcal{R} \} = 0. \tag{1a}
\]

From (1a), chiral symmetry makes the pair of positive and negative eigenvalues of \( H \).

Let us assume that chirality of lattices is not changed by the operators \( \mathcal{R} \),

\[
\{ \mathcal{T}, \mathcal{R} \} = 0. \tag{2}
\]

From (1b) and (2), one can see that the matrices \( H \), \( \mathcal{T} \) and \( \mathcal{R} \) are block-diagonalized by the eigenvectors of \( \mathcal{R} \),

\[
H = \begin{pmatrix}
H_{R_1} & & \\
& H_{R_2} & \\
& & \ddots
\end{pmatrix}, \tag{3a}
\]

\[
\mathcal{T} = \begin{pmatrix}
\mathcal{T}_{R_1} & & \\
& \mathcal{T}_{R_2} & \\
& & \ddots
\end{pmatrix}, \tag{3b}
\]

\[
\mathcal{R} = \begin{pmatrix}
R_1 & & \\
& R_2 & \\
& & \ddots
\end{pmatrix}. \tag{3c}
\]
\[ R_i = r_i \hat{1}_n, \quad \text{for } i = 1, 2, \cdots, \tag{3d} \]

where \( R_i, r_i (r_i \neq r_j \text{ for } i \neq j) \), and \( n_i \) denote an eigenspace of \( R_i \), an eigenvalue, and the number of degeneracy of \( r_i \), respectively. Here, the indices of \( H \) and \( \Gamma \) denote the eigenspace of \( R_i \). Additionally, \( \hat{1}_n \) is the \( n \times n \) identity matrix. We note that chiral symmetry and spatial symmetry are preserved at each eigenspace, \( \{ H_R, \Gamma_R \} = 0 \) and \( [H_R, R_i] = 0 \).

The topological index is defined as

\[
\nu_{R_i} = \text{Tr} \Gamma_{R_i}, \tag{4}
\]

for each eigenspace.\textsuperscript{17,38} This index denotes the difference between the number of \( u^+_n \) and \( u^-_n \) where \( H_R u^+_n = 0 \) and \( \Gamma_R u^-_n = \pm u^+_n \). Therefore, this index guarantees the \( |\nu_{R_i}| \) topological modes protected by chiral symmetry in the eigenspace \( R_i \) at least. For the total system, the topological index is written as

\[
\nu = \sum R_i |\nu_{R_i}|. \tag{5}
\]

This index guarantees that at least there exist \( \nu \) topological modes protected by the operators \( R \) and \( \Gamma \) in the total system. In the rest of this paper, by the topological modes, we denote topological modes protected by chiral symmetry and rotation symmetry.

**TBM with a Lieb lattice structure:** Based on the above generic argument, we analyze the nearest-neighbor TBM with a Lieb lattice structure under the periodic boundary condition [see Fig. 1 (a)]. Our analysis for the TBM elucidates that the three-fold degeneracy at \( M \) point is protected by chiral symmetry and two-fold rotational symmetry. For a later use, we set unit vectors as \( \vec{a}_1 = (1, 0)^T \) and \( \vec{a}_2 = (0, 1)^T \). High symmetry points \( \Gamma, M, \) and \( X \) in the first Brillouin zone are located at \( (k_x, k_y) = (0, 0), (\pi, \pi), \) and \( (\pi, 0) \), respectively [see Fig. 1 (b)].

For the TBM, the problem is reduced to the eigenvalue equation, \( h(\vec{k})\psi_{\vec{k}}^p = \epsilon\psi_{\vec{k}}^p \) where \( h(\vec{k}) \), \( \psi_{\vec{k}}^p \), and \( \epsilon \) denote the bulk Hamiltonian, an eigenstate function, and an eigenenergy, respectively. Here, \( p = 1, 2, 3 \) represents the sublattice. \( \psi_{\vec{k}}^p \) denotes the element of the eigenstate function \( \psi_{\vec{k}} \).

Under the basis \( (\psi_{\vec{k}}^1, \psi_{\vec{k}}^2, \psi_{\vec{k}}^3) \), the explicit form of the bulk Hamiltonian is written by

\[
h(\vec{k}) = \begin{pmatrix}
0 & -\epsilon & -t - \epsilon e^{i k_x} \\
-t - \epsilon e^{i k_x} & 0 & 0 \\
-t - \epsilon e^{i k_x} & 0 & 0
\end{pmatrix}. \tag{6}
\]

where \( t \) denotes the nearest-neighbor hopping parameter taking a real value. By solving the eigenvalue equation of \( h(\vec{k}) \), we obtain the dispersion relation. Figure 1 (c) shows the band structure for \( t = 1 \) in the first Brillouin zone. Here, the horizontal axis denotes the high-symmetry lines in the Brillouin zone connecting \( \Gamma, M, \) and \( X \) points [see Fig. 1 (b)].

Let us focus on chiral symmetry and two-fold rotational symmetry of this system. We introduce a chiral operator \( \Gamma_T \) and a two-fold rotational operator \( R_T \) around the point denoted by \( X \) in Fig. 1 (a),

\[
\Gamma_T = \text{diag}(1, -1, -1), \tag{7a}
\]

\[
R_T(\vec{k}) = \text{diag}(1, e^{-i k_x}, e^{-i k_y}). \tag{7b}
\]

Two-fold rotational symmetry is written as \( R_T(\vec{k})h(\vec{k})R_T^\dagger(\vec{k}) = h(-\vec{k}) \). From Eq. (6) and Eq. (7), one can confirm two equations \( h(\vec{k}_0), \nu_T = 0 \) and \( h(\vec{k}_0), R_T(\vec{k}_0) = 0 \) with \( \vec{k}_0 \) denoting momentum at high symmetry points \( \Gamma, M, \) and \( X \). This indicates the preservation of chiral and two-fold rotational symmetry at the high symmetry points \( \Gamma, M, \) and \( X \).

Here, we discuss topologically protected band touching at high-symmetry points. From the Eq. (7a) and Eq. (7b), one can see that the equation \( [\Gamma_T, R_T(\vec{k})] = 0 \) holds. This denotes that chirality of the lattices is not changed by \( R_T \). Therefore, \( \Gamma_T \) can be block-diagonalized by eigenvectors of \( R_T \). Thus, the number of the topological modes protected by two-fold rotational symmetry at the high-symmetry points is

\[
\nu = \begin{cases}
1 & \text{at } \Gamma \\
3 & \text{at } M \\
1 & \text{at } X
\end{cases}. \tag{8}
\]

The above results can be obtained from \( \Gamma_{R,1} = 1 \) and \( \Gamma_{R,-1} = -1 \).

For example, at \( M \) point, we have \( \Gamma_{R,1} = 1 \) and \( \Gamma_{R,-1} = -1 \), which result in \( \nu_{R,1} = 1 \) and \( \nu_{R,-1} = -2 \). These modes can be observed as one flat band at \( \Gamma \) and \( X \) points in the band structure and as one flat band and one Dirac cone at \( M \) point. These topological modes are stable as long as \( t \) is large. For small \( t \), they are replaced by Dirac cones.

In the above, we have discussed topological modes protected by chiral symmetry and two-fold symmetry for the TBM. We note however, that chiral symmetry for metals and insulators is preserved only within an approximation where the long range hoppings are discarded. This chiral symmetry breaking with long-range hoppings indicates the difficulty in realizing the topologically protected modes for metals and insulators.

**SMM with a Lieb lattice structure:** In contrast to the TBM, an SMM preserves chiral symmetry without any approximation since the coupling appears between the bases connected by springs. Therefore, we realize the topological modes for an SMM with a Lieb lattice structure [see Fig. 2 (a)] to overcome the difficulty in realizing the preservation of chiral symmetry for the TBM.

Let us see the motion of the mass points in the SMM with a Lieb lattice structure under the periodic boundary condi-
For the simplicity, we set the mass of mass points as unit, and unit vectors as \( \vec{a}_1 = (1, 0)^T \) and \( \vec{a}_2 = (0, 1)^T \). High-symmetry points in the first Brillouin zone are the same as the points for the TBM. For a later use, we define a spring constant as \( K \), and the displacements of the mass points from the equilibrium point as \( \phi_{k,i}^\mu \). Here, \( p = 1, 2, 3 \) denotes a sublattice and \( \mu = x, y \) denotes directions in the two-dimensional space. Additionally, we define the ratio of a natural length of the spring is extended by the tension.

The SMM of the Lieb lattice structure. For a later use, we define a spring constant as \( K \), points for the TBM. For a later use, we define a spring constant as \( K \), and the displacements of the mass points from the equilibrium point as \( \phi_{k,i}^\mu \). Here, \( p = 1, 2, 3 \) denotes a sublattice and \( \mu = x, y \) denotes directions in the two-dimensional space. Additionally, we define the ratio of a natural length of the spring is extended by the tension.

The equation of motion describing the mass points under the periodic boundary condition is written as

\[
\frac{d^2}{dt^2} \vec{\phi} = -D(\vec{k}) \vec{\phi},
\]

where \( D(\vec{k}) \) is the positive semi-definite matrix called a momentum-space dynamical matrix whose dimension is six since there are three sublattices and two spatial coordinates. This matrix can be divided into two terms, \( D(\vec{k}) = D^\mu(\vec{k}) + D^\phi \). \( D^\mu(\vec{k}) \) and \( D^\phi \) are positive semi-definite and positive definite matrices, respectively. \( D^\mu(\vec{k}) \) denotes the contribution of the springs. However, \( D^\phi(\vec{k}) \) has the non-uniform diagonal elements. Therefore, this system does not preserve chiral symmetry because \( D^\mu(\vec{k}) \) does not anticommute with a chiral operator.

Thus, in order to maintain chiral symmetry, we introduce the on-site potential \( D^\phi \) arising from dents on the floor. Mass points on the dents oscillate because of the gravity. Therefore, this on-site potential only affects the diagonal elements of \( D(\vec{k}) \), and makes diagonal elements of \( D(\vec{k}) \) constant. Moreover, this on-site potential has high controllability since the potential depends on the shape of the dents.

Specifically, the explicit form of \( D^\mu(\vec{k}) \) and \( D^\phi \) in this system is written as

\[
D^\mu(\vec{k}) = \begin{pmatrix}
D_{11}^\mu & D_{12}^\mu & D_{13}^\mu \\
(D_{12}^\mu)^T & D_{22}^\mu & D_{23}^\mu \\
(D_{13}^\mu)^T & D_{23}^\mu & D_{33}^\mu
\end{pmatrix},
\]

with

\[
D_{11}^\mu = \text{diag}(2K\eta, 2K\eta, 2K, 2K(1+\eta), 2K(1+\eta), 2K),
\]

\[
D_{12}^\mu = \text{diag}(4K - 2K\eta, 4K - 2K\eta),
\]

\[
D_{13}^\mu = \text{diag}(2K(1-\eta)),
\]

\[
D_{12}^\mu = \text{diag}(2K(1-\eta)),
\]

\[
D_{13}^\mu = \text{diag}(2K, 2K(1-\eta)),
\]

\[
D_{12}^\mu = \text{diag}(2K, 2K(1-\eta)),
\]

\[
D_{13}^\mu = \text{diag}(2K(1+\eta), 2K(1+\eta), 2K).
\]

Here, \( \hat{0}_n \) denotes the \( n \times n \) zero matrix. For a later use, we choose \( D_\eta \) so that the diagonal elements of \( D(\vec{k}) \) become \( 4K \).

We note that there does not exist the coupling between the displacements along \( x \) and \( y \) direction since the springs are aligned horizontally along the \( x \) and \( y \) direction.

Assuming that the mass points oscillate with a frequency \( \omega \), one can write \( d_{k,i}^\mu = e^{-i\omega t} \hat{a}_{k,i}^\mu \). Substituting this into Eq. (9), we obtain

\[
-\omega^2 \vec{\xi} + D(\vec{k}) \vec{\xi} = 0,
\]

where \( \vec{\xi} = (\phi_{k,1}^x, \phi_{k,1}^y, \phi_{k,2}^x, \phi_{k,2}^y, \phi_{k,3}^x, \phi_{k,3}^y)^T \) is the eigenvector of \( D(\vec{k}) \).

As a result, the problem is reduced to the eigenvalue equation analogous to the Schrödinger equation. By solving Eq. (13), we obtain the dispersion relation. Figure 2 (b) shows the band structure for \( K = 1, \eta = 0.7 \). We note that the momentum space dynamical matrix corresponds the Hamiltonian of the TBM since the SMM is reduced to two copies of the TBM for \( \eta = 0 \), i.e., the tension is infinitely strong.

Let us discuss the symmetry for this mechanical system.

We define a chiral operator \( \mathcal{T}_S \) as

\[
\mathcal{T}_S = \mathcal{T}_R \otimes \hat{1}_2 = \text{diag}(1, 1, -1, -1, -1, -1).
\]

Additionally, we define a two-fold rotational operator \( \mathcal{R}_2 \) around the point \( X \) shown in Fig. 2 (a) in this mechanical system as

\[
\mathcal{R}_2(\vec{k}) = R_T(\vec{k}) \otimes (-\hat{1}_2),
\]

\[
= \text{diag}(-1,-1,-e^{-ik},-e^{-ik},-e^{-ik},-e^{-ik}).
\]

These operators correspond to the ones for the TBM, respectively. However, dimensions of these matrix are doubled since the operations are applied not only to the sublattice degrees of freedoms, but also to directions of the displacements. Two-fold rotational operator is written as \( \mathcal{R}_2(\vec{k}) \text{diag}(\vec{k}) \mathcal{R}_2(\vec{k}) \).

The diagonal elements of the momentum-space dynamical matrix just shift the band structure, \( \omega^2(\vec{k}) \). Therefore, this system preserves “chiral symmetry” since \( D(\vec{k}) \) subtracted the diagonal elements anticommute with \( \mathcal{T}_S \), \( \{D(\vec{k}) - 4K\mathcal{1}_6, \mathcal{T}_S\} = 0 \). We note that the topological modes for an SMM are not zero modes due to the constant contribution of the momentum dynamical matrix. In addition, this system preserves two-fold rotational symmetry at the high-symmetry points \( \Gamma \), \( M \) and \( X \): \( [D(\vec{k}), \mathcal{R}_2(\vec{k})] = 0 \).

From Eq. (14a) and Eq. (14b), one can confirm that chiral-
ity of the lattices is not changed by the operator $R_S(\vec{k})$ since $[\Gamma_S, R_S(\vec{k})] = 0$. $\Gamma_S$ can be block-diagonalized by the eigenvectors of $R_S(\vec{k})$. Then, the number of the topological modes protected by two-fold rotational symmetry for this system is

$$
\nu = \begin{cases} 
2 & \text{at } \Gamma \\ 
6 & \text{at } M \\ 
2 & \text{at } X
\end{cases}
$$

(15)

at least. The above results can be obtained by noticing $\Gamma_S = \Gamma_T \otimes I_2$. We note that the number of topological modes for the SMM is doubled compared to that for the TBM, which is unique properties of the Lieb lattice structure. These topological modes are observed as two flat bands at $\Gamma$ and $X$ points, and two flat bands and two Dirac cones at $M$ point in the band structure [see Fig. 2 (b)]. We note that these topological modes are stable for any $\eta$ since $\Gamma_S$ and $R_S(\vec{k})$ do not depend on $\eta$. Therefore, these modes for the SMM are stable even when the longitudinal and the transverse waves are coupled.

Additionally, the extra topological modes protected by rotational symmetry are unique topological phenomena for the mechanical system. These additional topological modes originate from the internal degrees of freedoms (i.e., dispacements along the $x$- and $y$-direction) which may also shift the eigenspace of $R$ due to the angular momentum.

We finish this part with a remark on generality of our approach. Our approach to realize the topological modes protected by chiral symmetry and rotation symmetry can be applied to generic two-dimensional SMMs. Indeed, we can realize the topological modes protected by chiral symmetry and three-fold rotational symmetry for the SMM with a honeycomb structure.

**Summary:** We have proposed how to realize the topological modes protected by chiral and rotation symmetry for two-dimensional systems. Specifically, by taking advantage of the high controllability of SMMs, we have realized topological modes protected by chiral symmetry and rotation symmetry for the mechanical system. These additional topological modes originate from the internal degrees of freedoms. Our approach by the SMM and the dents on the floor can be applied to a general two-dimensional TBM.

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1) Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993).
2) S. Ryu, and Y. Hatsugai, Phys. Rev. Lett. 89, 077002 (2002).
3) C. L. Kane, and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).
4) A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B 78, 195125 (2008).
5) M. Z. Hasan, and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
6) L. Fidkowski and A. Kitaev, Phys. Rev. B 81, 134509 (2010).
7) S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, N. J. Phys. 12, 065010 (2010).
8) X.-L. Qi, and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
9) L. Fu, Phys. Rev. Lett. 106, 106802 (2011).
10) R. Slager, A. Mesaros, V. Juričić, and J. Zaanen, Nat. Phys. 9, 98 (2011).
11) Y. C. Hu, and T. L. Hughes, Phys. Rev. B 84, 153101 (2011).
12) M. Levin, and A. Stern, Phys. Rev. B 86, 115131 (2012).
13) Y. Hatsugai, T. Morimoto, T. Kawarabayashi, Y. Yamamoto, and H. Aoki, N. J. Phys. 15, 035023 (2013).
14) Y. Ando, J. Phys. Soc. Jpn. 82, 102001 (2013).
15) Y. Z. Zhao, and Z. D. Wang, Phys. Rev. Lett. 110, 240404 (2013).
16) C.-K. Chiu, H. Yao, and S. Ryu, Phys. Rev. B 88, 075142 (2013).
17) M. Koshino, T. Morimoto, and M. Sato, Phys. Rev. B 90, 115207 (2014).
18) C. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. 88, 035005 (2016).
19) M. Geier, L. Trifunovic, M. Hoskam, and P. W. Brouwer, Phys. Rev. B 97 205135 (2018).
20) C. L. Kane and T. Lubensky, Nat. Phys. 10 39 (2014).
21) Y.-T. Wang, P.-G. Luan, and S. Zhang, N. J. Phys. 17, 073031 (2015).
22) T. Kariyado, and Y. Hatsugai, Sci. Rep. 5, 18107 (2015).
23) R. Susstrunk and S. D. Huber, Proc. Natl. Acad. Sci. U.S.A. 113, E4767 (2016).
24) S. Oono, T. Kariyado, and Y. Hatsugai, Phys. Rev. B 94, 125125 (2016).
25) K. Bertoldi, V. Vitelli, J. Christensen, and M. van Hecke, Nat. Rev. Mater. 2, 17066 (2017).
26) Y. Takahashi, T. Kariyado, and Y. Hatsugai, N. J. Phys. 19, 035003 (2017).
27) C. H. Lee, S. Imhof, C. Bayer, F. Bayer, J. Brehm, L. W. Molenkamp, T. Kessling, and R. Thomale, Commun. Phys. 1, 39 (2018).
28) S. Imhof, C. Bayer, F. Bayer, J. Brehm, L. W. Molenkamp, T. Kessling, F. Schindler, C. H. Lee, M. Greiter, T. Neupert, and R. Thomale, Nat. Phys. 14, 925 (2018).
29) Y. Takahashi, T. Kariyado, and Y. Hatsugai, Phys. Rev. B 99, 024102 (2019).
30) T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schniter, J. Simon, O. Zilberberg, and I. Carusotto, Rev. Mod. Phys. 91, 015006 (2019).
31) Y. Ota, F. Liu, R. Katasumuri, K. Watanabe, K. Wakahayashi, Y. Arakawa, adn S. Iwamoto, Optica 6, 786 (2019).
32) T. Yoshida and Y. Hatsugai, Phys. Rev. B 100, 054109 (2019).
33) T. Yoshida, T. Mizoguchi, and Y. Hatsugai, arXiv:1912.12022 (2019).
34) H. Wakao, T. Yoshida, H. Araki, T. Mizoguchi, and Y. Hatsugai, Phys. Rev. B 101, 094107 (2020).
35) H. Weng, C. Fang, Z. Fang, B. A. Bernevig, and X. Dai, Phys. Rev. X 5, 011029 (2015).
36) S. Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian,C. Zhang, R. Sankar, G. Chang, Z. Yuan, C.-C. Lee, S.-M. Huang, H. Zheng, J. Ma, D. S. Sanchez, B. Wang, A. Bansil, F. Chou, P. P. Slabayev, H. Lin, S. Jia, and M. Z. Hasan, Science 349, 613 (2015).
37) T. BzdúÅek, and M. Sigrist, Phys. Rev. B 96, 155105 (2017).
38) B. Sutherland, Phys. Rev. B 34, 5208 (1986).