EQUATIONS OF 2-LINEAR IDEALS AND ARITHMETICAL RANK

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Abstract. In this paper we consider reduced homogeneous ideals \( J \subset S \) of a polynomial ring \( S \), having a 2-linear resolution.

1. We study systems of generators of \( J \subset S \).
2. We compute the arithmetical rank for a large class of projective curves having a 2-linear resolution.
3. We show that the fiber cone \( \text{proj} \mathcal{F}(I_L) \) of a lattice ideal \( I_L \) of codimension two is a set theoretical complete intersection.

1 Introduction

In this paper we work on reduced algebraic sets on the projective space \( X \subset \mathbb{P}^n \). At the end of the XIX century, it was known by the italians geometers that if \( X \) is irreducible then the degree of \( X \) is always greater than the codimension of \( X + 1 \). They classified the varieties with minimal degree by geometric means.

Theorem 1 (Bertini, Castelnuovo, Del Pezzo) Let \( X \subset \mathbb{P}^n \) be any irreducible nondegenerate variety of dimension \( k \) having (minimal) degree \( n - k + 1 \). Then \( X \) is either

1. A quadric hypersurface;
2. a cone over the Veronese surface in \( \mathbb{P}^5 \); or
3. a rational normal scroll.

Later Jo Harris has given a complete proof of the above results, in particular we quote the following result from [Ha], page 108:

The ideal defining a rational normal scroll is generated by the \( 2 \times 2 \) minors of the following scroll matrix:

\[
\mathcal{B} = \begin{pmatrix}
L_{1,0} & L_{1,1} & \cdots & L_{1,c_1} & \cdots & L_{r,0} & L_{r,1} & \cdots & L_{r,c_r} \\
L_{1,1} & L_{1,2} & \cdots & L_{1,c_1+1} & \cdots & L_{r,1} & L_{r,2} & \cdots & L_{r,c_r+1}
\end{pmatrix},
\]

where \( L_{1,0}, L_{1,1}, \ldots, L_{1,c_1+1}, \ldots, L_{r,1}, L_{r,2}, \ldots, L_{r,c_r+1} \) are all linearly independent forms.

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We consider this matrix as formed by blocks

\[ B_i = \begin{pmatrix} L_{i,0} & L_{i,1} & \cdots & L_{i,c_i} \\ L_{i,1} & L_{i,2} & \cdots & L_{i,c_i+1} \end{pmatrix}. \]

If \( c_i = 0 \) we will say that \( B_i \) is a generic block, and if \( c_i > 0 \) that \( B_i \) is a non generic block.

It then follows that in the irreducible case, Harris has described a system of generators for the ideal of a variety having minimal degree. We can also consider the problem to describe set theoretically the varieties of minimal degree, and in particular determine the arithmetical rank, that is the minimal number of elements which generates up to the radical the ideal of a variety having minimal degree. There are very few results in this direction we cite the followings:

**Theorem 2** (Verdi) \[ \text{[V]} \] Let consider a rational normal scroll \( C \) which defines a projective curve, its defining ideal \( M \) is generated by the the \( 2 \times 2 \) minors of the scroll matrix with only one non generic block:

\[ B = \begin{pmatrix} L_0 & L_1 & \cdots & L_c \\ L_1 & L_2 & \cdots & L_{c+1} \end{pmatrix}. \]

Where \( c > 0 \). For \( j = 1, \ldots, c \) let

\[ F_j := \sum_{k=0}^{j} (-1)^k \binom{j}{k} (L_{j+1})^{j-k} L_k(L_j)^k \]

then

\[ M = \text{rad}(F_1, \ldots, F_c). \]

Since the codimension of \( C \) is \( c \) this implies that \( C \) (or \( M \)) is a set theoretical complete intersection.

On the other hand we have the following result which follows from the works of Hochster, Bruns-Vetta-Schwanzl:

**Theorem 3** Let \( B \) be a scroll matrix with \( r \geq 2 \) columns and only generic blocks, then

- If \( \text{char} K = 0, \) \( \text{cd}(M) = \text{ara}(M) = 2r - 3; \) \( \text{projdim} (S/M) = r - 1. \)

- If \( \text{char} K = p > 0, \) then \( \text{ara}(M) = 2r - 3, \) and since \( R/M \) is a Cohen-Macaulay ring, we have that \( \text{projdim} (S/M) = \text{cd}(M) = r - 1. \)

When the algebraic set is not irreducible, Xambo \[ \text{[X]} \] has described from the geometric point of view the algebraic sets being connected in codimension one and having minimal degree. The problem of describing generators of their ideals is much more complicated (see for example \[ \text{[CEP]} \]), and a first work in this direction was done by Barile and Morales \[ \text{[BM2]}, \text{[BM4]} \]. Later in \[ \text{[EGHP]} \] it was introduced and developed the notion of linearly joined varieties. In my recent work \[ \text{[Mo1]} \], I study the case of linearly joined linear spaces, there I have computed the projective dimension, the cohomological dimension and the arithmetical rank.
The purpose of this paper (see section 2), is to describe more precisely the generators of ideals having a 2-linear resolution. For ideals having 2-linear resolution we refer the reader to [EG], [BM2], [BM4], [EGHP], [Mo1].

In section 3 we consider a large class of ideals having a 2-linear resolution (in fact projective curves) and we compute its arithmetical rank. This work extends all the works in this direction in the litterature, in particular those very interesting of Margherita Barile [B1].

Our results apply to the special case where $X$ is the fiber cone $\text{proj} \mathcal{F}(I_L)$ of a lattice ideal $I_L$ of codimension two. Let recall that by the successively works [G], [GMS1], [GMS2], [BM1] and [HM], the fiber cone is a reduced arithmetically Cohen-Macaulay algebraic set of minimal degree. See section 4.

2 Equations of 2-linear ideals.

We recall the following result which follows from [BM4] and [EGHP]

Theorem 4 The following conditions are equivalent:

1. The reduced ideal $\mathcal{J} \subset S := K[\mathbf{V}]$ has 2-linear resolution,

2. Let $\mathcal{J} = \mathcal{J}_1 \cap ... \cap \mathcal{J}_l$ be the primary decomposition of $\mathcal{J} \subset S := K[\mathbf{V}]$. For all $i = 1,...,l$, there exist sublinear spaces $\mathcal{D}_i, \mathcal{P}_i \subset \mathbf{V}$, with $\mathcal{D}_i = 0, \mathcal{P}_1 = 0$, and ideals $\mathcal{M}_i \subset K[\mathbf{V}]$ such that

   • a) for all $i = 1,...,l$, $\mathcal{J}_i = (\mathcal{M}_i, \mathcal{Q}_i)$, $\mathcal{M}_i$ is the ideal of the $2 \times 2$ minors of a scroll matrix,
   • b) $\mathcal{Q}_i = \mathcal{D}_i \oplus \mathcal{P}_i$,
   • c) $\mathcal{D}_i = \Delta_{i+1} \oplus ... \oplus \Delta_i$,
   • d) $\mathcal{M}_i \subseteq (\Delta_i)$ for all $i = 2,...,l$,
   • e) $\mathcal{M}_i \subseteq (\mathcal{P}_j)$ for all $1 \leq i < j \leq l$,
   • f) $\bigcap_{k=1}^{i-1} (\mathcal{Q}_j) \subseteq (\mathcal{P}_k, \mathcal{D}_{k-1})$ for all $k = 2,...,l$.

In the following result we will try to have more information on the ideals $\mathcal{M}_i$:

Lemma 1 Let $\mathcal{B}_i$ be a non generic block of a scroll matrix, with $c_i + 1$ columns,

$$\mathcal{B}_i = \begin{pmatrix}
L_{i,0} & L_{i,1} & \cdots & L_{i,c_i} \\
L_{i,1} & L_{i,2} & \cdots & L_{i,c_i+1}
\end{pmatrix},$$

suppose that the ideal $\mathcal{M}_i$ generated by all the $2 \times 2$ minors of $\mathcal{B}_i$, is contained in a linear ideal $(\Delta)$. Then for all $i = 1,...,r$ we have either

1. $\{L_{i,0}, L_{i,1}, ..., L_{i,c_i}\} \subset \Delta$ or $\{L_{i,1}, L_{i,2}, ..., L_{i,c_i+1}\} \subset \Delta$

2. there exists a linear form $H \notin \Delta$ and a non zero constant $\alpha_i$ such that $\{L_{i,0} - H, L_{i,2} - \alpha_i H, ..., L_{i,c_i+1} - \alpha_i^{c_i+1} H\} \subset \Delta$
Proof Let $U \subset V$ be a subvector space such that $V = U \oplus \Delta$, we can write $L_{i,j} = L'_{i,j} + H_{i,j}$ with $L'_{i,j} \in \Delta, H_{i,j} \in U$. Since for all $j \geq 0$, $L_{i,j}L_{i,j+2} - L^2_{i,j+1} \in (\Delta)$, we have $H_{i,j}H_{i,j+2} - H^2_{i,j+1} = 0$ We have to consider two cases:

1. $H_{i,j} = 0$ for some $1 \leq j \leq c_i$, this implies $H_{i,1} = ... = H_{i,c_i} = 0$, and from the relation $L_{i,0}L_{i,c_i+1} - L_{i,1}L_{i,c_i} \in (\Delta)$ we have that $H_{i,0}H_{i,c_i+1} = 0$ so that we have either $H_{i,0} = 0$ or $H_{i,c_i+1} = 0$.

2. $H_{i,j} \neq 0$ for all $1 \leq j \leq c_i$, the relations $H_{i,j}H_{i,j+2} - H^2_{i,j+1} = 0$ imply that there exist a non zero constant $\alpha$, such that $H_{i,j} = \alpha_1^2H_{i,0}$, and so $L_{i,j} = L'_{i,j} + \alpha_1^2H_{i,0}$ with $L'_{i,j} \in \Delta, H_{i,0} \not\in \Delta$.

Lemma 2 Let

$$B = \begin{pmatrix} L_{1,0} & L_{2,0} & ... & L_{r,0} \\ L_{1,1} & L_{2,1} & ... & L_{r,1} \end{pmatrix}$$

be a $2 \times r$ generic matrix (with $r \geq 2$), suppose that the ideal $M$ generated by all the $2 \times 2$ minors of $B$, is contained in a linear ideal $(\Delta)$. Then we have either

1. Up to permutation of the lines of $B$, there exists some $i$ such that $L_{i,0} \in \Delta, L_{i,1} \not\in \Delta$; in this case the elements of the first line of $B$ are in $\Delta$,

2. there exists some $i$ such that $L_{i,0}, L_{i,1} \in \Delta$; in this case the matrix obtained from $B$ by deleting the column $i$, $B'$ is still a generic matrix with $r - 1$ columns, and if $r - 1 \geq 2$ then we apply recursively this lemma to $B'$,

3. for any $i = 1, ..., r$ there exists a nonzero linear form $H_i \not\in \Delta$, and a non zero constant $\alpha$ such that $\{L_{i,0} - H_i, L_{i,1} - \alpha H_i\} \subset \Delta$,

4. for any $i = 1, ..., r$ there exists linear forms $H_1, H_2 \not\in \Delta$, and a non zero constant $\alpha_i$ such that $\{L_{i,0} - \alpha_i H_1, L_{i,1} - \alpha_i H_2\} \subset \Delta$.

Proof Let $U$ be a subvector space of $V$ such that $V = U \oplus \Delta$, so that we can write $L_{i,j} = L'_{i,j} + H_{i,j}$, with $L'_{i,j} \in \Delta, H_{i,j} \in U$. Since $L_{1,0}L_{i,1} - L_{1,1}L_{i,0} \in (\Delta)$ we have that $H_{1,0}H_{i,1} - H_{1,1}H_{i,0} = 0$, we have two choices:

1. $H_{i,j} = 0$ for some $i,j$, then up to a permutation of the lines and columns we can assume that $H_{1,j} = 0$. If $H_{1,1} \neq 0$ the above relation implies that $H_{1,0} = 0$ and then all the entries of the first line of $B$ are in $\Delta$. If both $H_{1,0} = 0, H_{1,1} = 0$, then $L_{1,0}, L_{1,1} \in \Delta$, in this case the matrix obtained from $B$ by deleting the first column, $B'$ is still a generic matrix with $r - 1$ columns,

2. $H_{i,j} \neq 0$ for all $i,j$, then

   - there exist a nonzero constant $\alpha$ such that $H_{1,1} = \alpha H_{0,0}$, and this implies $H_{i,1} = \alpha H_{i,0}$, for all $i$

   - there exist nonzero constants $\alpha_i$ such that $H_{i,1} = \alpha_i H_{0,0}$, and this implies $H_{i,1} = \alpha_i H_{0,1}$, for all $i$. 

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**Proposition 1** Let $\mathcal{B}$ be a scroll matrix, suppose that the ideal $\mathcal{M}$ generated by the $2 \times 2$ minors of $\mathcal{B}$ is contained in a linear ideal $(\Delta)$. We have either:

1. All the blocks of $\mathcal{B}$ are generic, so Lemma 2 applies.
2. The entries of exactly one line of $\mathcal{B}$ are in $\Delta$.
3. there exist one block $\mathcal{B}_i$ of $\mathcal{B}$ which is non generic, and all the entries of $\mathcal{B}_i$ are in $\Delta$ then the matrix $\mathcal{B}'$ obtained from $\mathcal{B}$ by deleting $\mathcal{B}_i$ is a scroll matrix with less columns than $\mathcal{B}$ and we must apply recursively this proposition to $\mathcal{B}'$.
4. there exist one $\mathcal{B}$ which is non generic, and any entry of a non generic block $\mathcal{B}_i$ is not in $\Delta$. In this case there exists a non zero constant $\alpha$, and

   - for any non generic block $\mathcal{B}_i$ there exists a linear form $H_i \notin \Delta$, such that $\{L_{i,0} - H_i, L_{i,1} - \alpha H_i, ..., L_{i,c_i+1} - \alpha c_i+1 H_i\} \subset \Delta$.
   - for any generic block $\mathcal{B}_i$, there exists a linear form $H_i$, null or $H_i/\in \Delta$, such that $\{L_{i,0} - H_i, L_{i,1} - \alpha H_i\} \subset \Delta$.

**Proof** The proof follows easily from the above lemmas.

**3 Arithmetical rank of some 2-linear ideals.**

From now on, we suppose that $\mathcal{J}$ is a reduced ideal having a $2-$linear resolution, we know the primary decomposition

$$\mathcal{J} = (\mathcal{M}_1, \mathcal{Q}_1) \cap ... \cap (\mathcal{M}_l, \mathcal{Q}_l)$$

where $\mathcal{Q}_i = \mathcal{D}_i \oplus \mathcal{P}_i$, $\mathcal{D}_i = \Delta_2 \oplus ... \oplus \Delta_{i+1}$, $\mathcal{M}_i \subseteq (\Delta_i)$ for all $i = 2, ..., l$, $\mathcal{M}_i \subseteq (\mathcal{P}_j)$ for all $i = 1, ..., l - 1$ and $j = i + 1, ..., l$. We recall the following theorem from [Mo1]

**Theorem 5** Let $\mathcal{J}$ be a reduced ideal having a $2-$linear resolution, with the above notations

1. Let $\mathcal{Q} := \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l$, then $\text{projdim} (S/\mathcal{J}) = \text{projdim} (S/\mathcal{Q}) = \max_{i=2,...,l} \{ \dim_K (\mathcal{P}_i + \mathcal{D}_{i-1}) - 1 \}$.
2. With the assumption that for all $i$, $\mathcal{M}_i$ is a set theoretical complete intersection, we have $\text{cd} \mathcal{J} = \text{projdim} (S/\mathcal{J}) = \max_{i=2,...,l} \{ \dim_K (\mathcal{P}_i + \mathcal{D}_{i-1}) - 1 \}$.

**Theorem 6** Let $\mathcal{J}$ be a reduced ideal having a $2-$linear resolution, assume that for all $i$ the matrix $\mathcal{B}_i$ is either the matrix null or has at most one block and one line of $\mathcal{M}_i$ is included in $(\Delta_j)$ for all $i = 2, ..., l$, and one line of $\mathcal{M}_i$ is included in $(\mathcal{P}_j)$ for all $i = 1, ..., l - 1$ and $j = i + 1, ..., l$. Then $\text{ara}(\mathcal{J}) = \text{cd}(\mathcal{J}) = \text{projdim} (S/\mathcal{J})$. 

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Proof Let $B_i = \begin{pmatrix} L_{i,0} & L_{i,1} & \cdots & L_{i,c_i} \\ L_{i,1} & L_{i,2} & \cdots & L_{i,c_i+1} \end{pmatrix}$, and $M_i$ the ideal generated by all the $2 \times 2$ minors of $B_i$. We call $L_{i,0}, L_{i,c_i+1}$, "corner" entries of $B_i$ and the other "inner" entries of $B_i$. We know that $J$ is generated by all the ideals $M_1, ..., M_l$ and $J = \bigcup_{j=1}^k (\Delta_j \times \mathcal{P}_j)$, where $(\Delta_j \times \mathcal{P}_j)$ is the ideal generated by all the products $f g$, with $f \in \Delta_j, g \in \mathcal{P}_j$. We have also the property that $(\Delta_i \times \mathcal{P}_j) \subseteq \mathcal{P}_j$ when $j < i$.

First we study the intersection of the two primary components of $J$. We have that $(M_1, Q_1) \cap (M_2, Q_2) = (D_2, (M_1, M_2, (\Delta_2) \times \mathcal{P}_2)$ since $M_1 \subset \mathcal{P}_2$, by our hypothesis we can suppose that the first line of $B_1$, $L_1 := \{L_{1,0}, L_{1,1}, ..., L_{1,c_1}\}$ is included in $\mathcal{P}_2$, and we can find a vector space $\tilde{\mathcal{P}}_2$ containing a corner element of $B_1$ such that $\tilde{\mathcal{P}}_2 = \tilde{\mathcal{P}}_2 \oplus \tilde{\mathcal{L}}_1$, where $\tilde{\mathcal{L}}_1$ is the vector space generated by all the inner entries of $B_1$. By the same reasons we can find a vector space $\tilde{\Delta}_2$ containing a corner element of $B_2$ such that $\Delta_2 = \tilde{\Delta}_2 \oplus \tilde{\mathcal{L}}_2$, where $\tilde{\mathcal{L}}_2$ is the vector space generated by all the inner entries of $B_2$.

Suppose that $L_{1,0} \in \tilde{\mathcal{P}}_2$, working modulo $M_1$ we have the relations $L_{1,0}L_{1,2} - L_{1,1}^2 = 0$ so for any element $g \in \Delta_2$ we will have $(L_{1,0}g)(L_{1,2}g) = (L_{1,1}g)^2$, so that $(L_{1,1}g)^2 \in (M_1, (L_{1,0}g))$, since we have also $(L_{1,1}g)(L_{1,3}g) = (L_{1,2}g)^2$, then $(L_{1,2}g)^2 \in (M_1, (L_{1,1}g))$, by repeating this argument we have that there exists some powers $n_i$ such that $(L_{1,i}g)^{n_i} \in (M_1, (L_{1,0}g))$. Let $L_{2,0} \in \tilde{\Delta}_2$ a "corner" entry of $B_2$, we use the relations in the matrix $B_2$, then for all "inner" element $L$ of $B_2$ and every element $h \in \mathcal{P}_2$ there exists some powers $m_i$ such that $(Lg)^{m_i} \in (M_2, (L_{2,0}h))$.

In other words, let $\tilde{\mathcal{L}}_i$ the vector space generated by the inner entries of the matrix $B_i$, We have found vector spaces $\tilde{\mathcal{P}}_2, \tilde{\Delta}_2$, such that

$$\mathcal{P}_2 = \tilde{\mathcal{P}}_2 \oplus \tilde{\mathcal{L}}_1, \Delta_2 = \tilde{\Delta}_2 \oplus \tilde{\mathcal{L}}_1,$$

and if $I_2$ is the ideal generated by $M_1, M_2$ and $\Delta_2 \times \tilde{\mathcal{P}}_2$ then up to radical the following ideals $(I_2, \mathcal{P}_2), (M_1, Q_1) \cap (M_2, Q_2)$ are equal. Let remark that $\dim \tilde{\mathcal{P}}_2 = \dim \mathcal{P}_2 - c_1$, $\dim \tilde{\Delta}_2 = \dim \Delta_2 - c_2$.

We will prove the following statement:

For any $i \geq 2$, there exists a decomposition

$$\mathcal{P}_i = \tilde{\mathcal{P}}_i \oplus \tilde{\mathcal{L}}_1 \oplus ... \oplus \tilde{\mathcal{L}}_{i-1}, \Delta_i = \tilde{\Delta}_i \oplus \tilde{\mathcal{L}}_i,$$

such that

(H1,i) for any $j < k \leq i$ we have $\tilde{\mathcal{P}}_j \times \tilde{\Delta}_j \subseteq (\tilde{\mathcal{P}}_k)$.

(H2,i) Let $I_i$ be the ideal generated by $M_1, ..., M_i$ and $\bigcup_{j=2}^i (\Delta_j \times \tilde{\mathcal{P}}_j)$. Up to radical the ideals $(I_i, \mathcal{D}_i)$, and $\bigcap_{j=2}^i (M_1, ..., M_i, Q_1)$ are equal.

We suppose that this is true for some $i \geq 2$ and prove it for $i+1$. Since $\Delta_j \times \mathcal{P}_j \subseteq (\mathcal{P}_{i+1})$ for any $j < i+1$, and since $(\mathcal{P}_{i+1})$ is a prime ideal, for any $j < i+1$ we have either $\Delta_j \subseteq \mathcal{P}_{i+1}$ or $\mathcal{P}_j \subseteq \mathcal{P}_{i+1}$ (equality like vector spaces), we have two cases:
1. \( \Delta_j \subset \mathcal{P}_{i+1} \) for all \( j = 2, \ldots, i \)

2. there exists \( 2 \leq i_0 \leq i \) such that \( \mathcal{P}_{i_0} \subset \mathcal{P}_{i+1} \) and \( \Delta_j \subset \mathcal{P}_{i+1} \) for all \( j = i_0 + 1, \ldots, i \)

Since \( \mathcal{M}_{i+1} \subset (\Delta_{i+1}) \), then by our hypothesis a line of \( \mathcal{B}_{i+1} \) is included in \( \Delta_{i+1} \), let \( \widetilde{\Delta_{i+1}} \) be a subvector space containing the corner element in this line such that \( \Delta_{i+1} = \Delta_{i+1} \oplus \mathcal{L}_{i+1} \).

In the first case, it follows that \( \mathcal{P}_{i+1} \) contains \( \bigoplus_{j=2}^{\Delta_j} \) and since \( \mathcal{M}_{i} \subset (\mathcal{P}_{i+1}) \), by our hypothesis \( \mathcal{P}_{i+1} \) contains a line in the matrix \( \mathcal{B}_{i} \), formed by a corner element \( \mathcal{L}_{i} \) and the inner elements, \( \widetilde{\Delta_{i}} \subset \mathcal{P}_{i+1} \), so \( \mathcal{P}_{i+1} \supset (\bigoplus_{j=2}^{\Delta_j} \mathcal{L}_{i} \oplus K\mathcal{L}_{i}) \oplus \mathcal{P}_{i_0} \oplus \bigoplus_{j=2}^{\Delta_j} \mathcal{L}_{j} \). Let \( \widetilde{\mathcal{P}_{i+1}} \) a vector space containing \( \bigoplus_{j=2}^{\Delta_j} \mathcal{L}_{j} \oplus K\mathcal{L}_{1} \) such that \( \mathcal{P}_{i+1} = \mathcal{P}_{i+1} \oplus \bigoplus_{j=2}^{\Delta_j} \mathcal{L}_{j} \). It is clear that \((H_{i+1,1})\), \((H_{i+1,2})\) are satisfied in this case.

In the second case, it follows that \( \mathcal{P}_{i+1} \) contains \( \bigoplus_{j=2}^{\Delta_j} \) and since \( \mathcal{M}_{i_0} \subset (\mathcal{P}_{i+1}) \), by our hypothesis \( \mathcal{P}_{i+1} \) contains a line in the matrix \( \mathcal{B}_{i_0} \), formed by a corner element \( L_{i_0} \) and the inner elements, \( \widetilde{\Delta_{i_0}} \subset \mathcal{P}_{i+1} \), so \( \mathcal{P}_{i+1} \supset (\bigoplus_{j=2}^{\Delta_j} \mathcal{L}_{i_0} \oplus \mathcal{P}_{i_0} \oplus \bigoplus_{j=2}^{\Delta_j} \mathcal{L}_{j} \). Let \( \widetilde{\mathcal{P}_{i+1}} \) a vector space containing \( \bigoplus_{j=2}^{\Delta_j} \mathcal{L}_{j} \oplus K\mathcal{L}_{1} \) such that \( \mathcal{P}_{i+1} = \mathcal{P}_{i+1} \oplus \bigoplus_{j=2}^{\Delta_j} \mathcal{L}_{j} \). It is clear that \((H_{i+1,1})\), \((H_{i+1,2})\) are satisfied in this case.

We have used the property that for any \( k = 1, \ldots, l \) the entries of the matrix \( \mathcal{B}_{k} \) are linearly independent and also linearly independent with \( \bigoplus_{j=k+1}^{\Delta_j} \mathcal{B}_{k} \).

As a conclusion, \( \mathcal{J} \) up to the radical equal to the ideal \( \mathcal{I} \) generated by \( \mathcal{M}_{1}, \ldots, \mathcal{M}_{i} \) and \( \bigcup_{j=2}^{\Delta_j} (\mathcal{D}_{j} \times \mathcal{P}_{j}) \). On the other hand the ideal \( \mathcal{K} := \bigcap_{j=1}^{\Delta_j} (\mathcal{D}_{j} \times \mathcal{P}_{j}) \) has a 2-resolution, with primary decomposition : \( \mathcal{K} = \bigcap_{j=1}^{\Delta_j} (\mathcal{D}_{j} \times \mathcal{P}_{j}) \), where \( \mathcal{D}_{j} := \bigoplus_{k=j+1}^{\Delta_k} \mathcal{D}_{k} \) and so \( \text{ara} (\mathcal{K}) = \max_{2 \leq j \leq l} \{ \dim \mathcal{D}_{j-1} + \dim \mathcal{P}_{j} \} - 1 \) by a quick computation we have that

\[
\dim \mathcal{D}_{j-1} + \dim \mathcal{P}_{j} = \dim \mathcal{D}_{j-1} + \dim \mathcal{P}_{j} - (c_1 + \ldots + c_l)
\]

so that

\[
\text{ara} (\mathcal{J}) \leq \text{ara} (\mathcal{K}) + (c_1 + \ldots + c_l) = \max_{2 \leq j \leq l} \{ \dim \mathcal{D}_{j-1} + \dim \mathcal{P}_{j} \} - 1 = \text{projdim} (S/\mathcal{J}).
\]

On the other hand by [Mo1] we have that \( \text{cd} \mathcal{J} = \text{projdim} (S/\mathcal{J}) \leq \text{ara} (\mathcal{J}) \), the equality follows.

**Remark 1** We can apply the method of the proof of the above theorem in order to get an upper bound for the arithmetical rank of any 2-linear ideal in terms of the arithmetical rank of rational normal scrolls. Note that the problem to compute the arithmetical rank of rational normal scrolls is still open in general.

**Theorem 7** Let \( \mathcal{J} \) be a reduced ideal having a 2-linear resolution,

\[
\mathcal{J} = (\mathcal{M}_1, \mathcal{Q}_1) \cap \ldots \cap (\mathcal{M}_l, \mathcal{Q}_l),
\]

where \( \mathcal{M}_i \) is either the ideal null or the ideal generated by all the 2 \times 2 minors of a scroll matrix \( \mathcal{B}^{(i)} \). Assume that \( \mathcal{B}^{(i)} \) has \( j_i \) non-generic blocks, and the \( k \)-non-generic block in
$B^{(i)}$ has $c_{i,k} + 1$ columns. Moreover assume that one line of $B^{(i)}$ is included in $(\Delta_i)$ for all $i = 2, ..., l$, and one line of $B^{(i)}$ is included in $(P_j)$ for all $1 \leq i < j \leq l$. Setting $\tau_i = c_{i,1} + ... + c_{i,j}$, we have that

$$\text{ara} (\mathcal{J}) \leq \text{projdim} (S/\mathcal{J}) + \sum_{i=1}^{l} (\text{ara} \mathcal{M}_i - \tau_i).$$

**Example 1** Consider $S = K[a, b, c, x, y, z, u, v, w]$ the ring of polynomials, and

$\mathcal{J}_1 = (uv-w^2, a, b, c); \mathcal{J}_2 = (y, z, v, w, a, b); \mathcal{J}_3 = (x, z-u, v, w, c, b); \mathcal{J}_4 = (x-u, y-u, a, c, v, w).$

The sequence $\mathcal{J}_1, ..., \mathcal{J}_4$ satisfies the condition 2 of Theorem 4, so $\mathcal{J} := \cap_{i=1}^{4} \mathcal{J}_i$ has a 2-linear resolution, $\text{projdim} (S/\mathcal{J}_i) = 6$ and $\mathcal{J} = \text{rad} (uv-w^2, cb, ca + ab, cy + ax + b(x-u), cz + a(z-u) + b(y-u), cv + av + bv)$.

**Example 2** Consider $S = K[a, b, c, x, y, z, u, v, w]$ the ring of polynomials, and

$\mathcal{J}_1 = (a, b, c, x); \mathcal{J}_2 = (x(x-u) - c^2, a, b, y, z); \mathcal{J}_3 = (b, x, z-u, c); \mathcal{J}_4 = (x, y-u, a, c).$

The sequence $\mathcal{J}_1, ..., \mathcal{J}_4$ satisfies the condition 2 of Theorem 4, so $\mathcal{J} := \cap_{i=1}^{4} \mathcal{J}_i$ has a 2-linear resolution. $\mathcal{J}$ is generated by the polynomial $x(x-u) - c^2$ and the elements in the following tableau:

```
bc
ba ac
b(y-u) ax cy
b(z-u) xy cz
xz,
```

$\text{projdim} (S/\cap_{i=1}^{4} \mathcal{J}_i) = 5$, up to the radical $\mathcal{J}$ is generated by the polynomial $x(x-u) - c^2$ and the elements in the following tableau:

```
ba ax
b(y-u) a(z-u) xy
xz
```

Thus $\text{ara} \mathcal{J} = 5$, and $\mathcal{J} = \text{rad} (x(x-u) - c^2, bx, ab + ax, b(y-u) + a(z-u) + xy, xz)$.
Example 3 In this example we can compute the arithmetical rank by using a general theorem due to Eisenbud and Evens. Consider $S = K[x_1, \ldots, x_{2r}, \Delta]$ a ring of polynomials where $\Delta$ is a set of variables, $K$ is a field of characteristic zero, let $0 \leq \alpha \leq r$,

$$J_1 = (| x_1 \ldots x_r, \Delta); J_2 = (x_1, \ldots, x_{2r-\alpha});$$

and $J := J_1 \cap J_2$, then $J$ has a 2-linear resolution. We have the following exact sequence:

$$0 \to S/J_1 \cap J_2 \to S/J_1 \oplus S/J_2 \to S/(J_1 + J_2) \to 0,$$

which gives rise to the long exact sequence:

$$\to H^{h-1}_{J_1 \cap J_2}(S) \to H^h_{J_1 + J_2}(S) \to H^h_{J_1}(S) \oplus H^h_{J_2}(S) \to H^h_{J_1 \cap J_2}(S) \to H^{h+1}_{J_1 + J_2}(S) \to .$$

By using the Bruns-Vetta Schwanzl’s theorem and Proposition 2 of [Ha-Mo], we get $\text{projdim}(S/J) = \text{dim} S - \alpha = \text{projdim}(S/J_1 + J_2)$, and by [EE] we get $\text{Eisen}(S/J) = \text{Eisen}(S/J_1 + J_2)$, and finally $\dim S - \alpha - 1 \leq \text{ara} J \leq \dim S - 1$, if $\alpha = 0$ we have $\text{ara} J = \dim S - 1$.

Example 4 More generally, consider a ring of polynomials $S = K[\Delta, P]$, where $\Delta, P$ are disjoints sets of variables, $M \subset (P)$ be any ideal, let

$$J_1 = (M, \Delta); J_2 = (P); J := J_1 \cap J_2,$$

then by the argument developed in the above example we have that $\text{cd} J = \text{dim} S - 1$, and by [EE] $\text{ara} J \leq \text{dim} S - 1$, so $\text{ara} J = \text{dim} S - 1$.

We give now some simple open cases.

Example 5 Consider $S = K[a, b, c, d, e, f]$ the ring of polynomials, $B = \begin{pmatrix} c - f & d - f \\ d - f & d + c - f \end{pmatrix}$ and $F$ the determinant of $B$, then the ideal $J = \bigcap_{i=1}^3 J_i$, where

$$J_1 = (F, a, b); J_2 = (b, c, d); J_3 = (c, d);$$

has a 2-linear resolution, $\text{cd} (J) = \text{projdim} (S/J) = 3$ and $J = \text{rad} (F, bc, ac + bd, ad + be)$, so $3 \leq \text{ara} (J) \leq 4$. We guess that $\text{ara} (J) = 4$.

Example 6 Consider $S = K[a, b, c, d, \Delta]$ a ring of polynomials, $B = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, then the ideal $J = J_1 \cap J_2$, where

$$J_1 = (ad - bc, \Delta); J_2 = (b, d)$$

has a 2-linear resolution, $\text{cd} (J) = \text{projdim} (S/J) = \text{card} (\Delta) + 1$. In [BT] it is proved that $\text{ara} (J) = \text{card} (\Delta) + 1$. 

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Example 7 Consider $S = K[a, b, c, d, e, f, g]$ the ring of polynomials, $B = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $F = ad - bc$, then the ideal $\mathcal{J} = \bigcap_{i=1}^{3} \mathcal{J}_i$, where

$$\mathcal{J}_1 = (F, e, f); \mathcal{J}_2 = (b, d, f); \mathcal{J}_3 = (b, d, g);$$

has a 2-linear resolution, $\text{cd}(\mathcal{J}) = \text{projdim}(S/\mathcal{J}) = 3$. We will prove that $\text{ara}(\mathcal{J}) = 3$. Set $q_1 := aF + be$; and $q'_1 := a^2q_1 + bf, q'_2 := cF + de + fg; q'_3 := (ac - e)q_1 + df$; we assert that $\mathcal{J} = \text{rad}(q'_1, q'_2, q'_3)$, so $\text{ara}(\mathcal{J}) = 3$. First we have that $dq'_1 - bq'_2 = q'_1$ so $q_1 \in \text{rad}(q'_1, q'_2, q'_3)$, and it follows that $bf, df \in \text{rad}(q'_1, q'_2, q'_3)$. We have also that $dq'_1 - bq'_2 = F^2 - bfg$ which implies that $F \in \text{rad}(q'_1, q'_2, q'_3)$, and then $be \in \text{rad}(q'_1, q'_2, q'_3)$, now $deq'_3 = cdeF + (de)^2 + dfeg, fggq'_2 = (cfg)F + (df)eg + (fg)^2$, and we have that $de, f g \in \text{rad}(q'_1, q'_2, q'_3)$. The proof is over.

Example 8 Consider $S = K[a, b, c, d, e, f]$, the ring of polynomials, $B = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $F = ad - bc$, then the ideal $\mathcal{J} = \bigcap_{i=1}^{3} \mathcal{J}_i$, where

$$\mathcal{J}_1 = (F, e, f); \mathcal{J}_2 = (b, d, f); \mathcal{J}_3 = (a, c, e);$$

has a 2-linear resolution, $\text{cd}(\mathcal{J}) = \text{projdim}(S/\mathcal{J}) = 3$. It is not difficult to see that $3 \leq \text{ara}(\mathcal{J}) \leq 5$, we guess that $3 = \text{ara}(\mathcal{J})$.

4 Fiber cone of codimension two lattices ideals.

Let $\mathcal{L} \subset \mathbb{Z}^r$ be a lattice which contains no nonnegative vectors. Any vector $v \in \mathcal{L}$ can be written as $v = v_+ - v_-$, where both vectors $v_+, v_-$ have non-negative coordinates. The lattice ideal $I_{\mathcal{L}} \subset S := K[x_1, \ldots, x_r]$ is the ideal generated by all binomials $f_v = x_+^v - x_-^v$, where $v$ runs in $\mathcal{L}$. Prime lattice ideals are called Toric ideals, and a variety defined by a lattice ideal have a Torus action. By definition the Fiber cone of $I_{\mathcal{L}}$ is the ring $\mathcal{F}(I_{\mathcal{L}}) = \bigoplus_{n \geq 0} \mathcal{L}^n/m_{\mathcal{L}}^{a+1}$.

We have the following result from [HM]

Theorem 8 If $I_{\mathcal{L}}$ is a radical ideal with $\mu$ generators and $K$ is infinite, then $\mathcal{F}(I_{\mathcal{L}})$ has dimension three, is reduced, arithmetically Cohen-Macaulay, of minimal degree. Moreover we have a description

$$\mathcal{F}(I) = K[T_1, \ldots, T_\mu]/\mathcal{A}, \text{ and } \mathcal{A} = (\mathcal{M}_1, \mathcal{Q}_1) \cap \ldots \cap (\mathcal{M}_l, \mathcal{Q}_l),$$

where for all $k = 1, \ldots, l \ Q_i$ is generated by a subset of $\{T_1, \ldots, T_\mu\}$ and $\mathcal{M}_i$ is either 0 or the ideal generated by all the $2 \times 2$ minors of a scroll matrix with only one block, which entries are subsets of $\{T_1, \ldots, T_\mu\}$.

Corollary 1 If $I_{\mathcal{L}}$ is a radical ideal with $\mu$ generators and $K$ is infinite, then $\mathcal{F}(I_{\mathcal{L}})$ is a set theoretical complete intersection.

The corollary is an immediate consequence of the above theorem and the theorem [6]
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