Nonuniform Bribery*

Piotr Faliszewski
Department of Computer Science
University of Rochester
Rochester, NY 14627

November 30, 2007

Abstract

We study the concept of bribery in the situation where voters are willing to change their votes as we ask them, but where their prices depend on the nature of the change we request. Our model is an extension of the one of Faliszewski et al. [FHH06], where each voter has a single price for any change we may ask for. We show polynomial-time algorithms for our version of bribery for a broad range of voting protocols, including plurality, veto, approval, and utility based voting. In addition to our polynomial-time algorithms we provide NP-completeness results for a couple of our nonuniform bribery problems for weighted voters, and a couple of approximation algorithms for NP-complete bribery problems defined in [FHH06] (in particular, an FPTAS for plurality-weighted-$b$ bribery problem).

1 Introduction

Multiagent systems can often be viewed as artificial societies of autonomous agents, with each agent having his/her own set of goals, desires, and plans. Within such artificial societies, just like within natural ones, it often becomes necessary for a group of agents to arrive at a common decision (e.g., in a planning environment when no single agent can solve his/her own problems, but where they are capable of solving problems cooperatively). A very natural approach to handling such situations is to hold an election.

Unfortunately, as is well known due to theorems of Arrow [Arr63], Gibbard and Satterthwaite [Gib73, Sat75], and Duggan and Schwartz [DS00] neither there are ideal election systems nor there are ones that avoid giving agents incentive to vote strategically. Bartholdi, Tovey, Trick, and Orlin [BTT89, BO91, BTT92], brilliantly observed that computational complexity of figuring out voter’s strategic behavior might be so high, as to perhaps be enough of a barrier to prevent agents from attempting strategic actions.

Several scenarios of strategic behavior are considered in the literature. In control we assume that the organizer of the election attempts to modify their structure (e.g., via partitioning the voters into districts) in order to obtain a result most desirable for him/herself, see, e.g., [BTT92, FHHR07, HHR07, PRZ07]. In the case of manipulation, a coalition of voters calculates what vote each member of the coalition should cast in order to obtain the result the coalition desires, see, e.g., [BTT89, BO91, CSL07, CS03, CS06, HH07, PR07, PRZ07].

*Supported in part by grant NSF-CCF-0426761. A version of this paper appears as URCS-TR-2007-922.
In the case of bribery, an external agent tries to ensure a victory of one of the candidates via bribing some voters to change their votes. Bribery was introduced in [FHH06] and further studied in [FHHR07].

All of these problems can be studied both in the constructive case (where the goal is to ensure our favorite candidate’s victory) and in the destructive case (where we try to prevent a hated candidate from winning). Also, in many situations it is natural to assume that different voters have different weights (e.g., consider the stockholders of a company or various parts of a multicriteria decision-making system that, internally, performs an election between its components, weighted based on components’ confidence.)

In this paper we focus on the problem of bribery, introduced by Faliszewski et al. [FHH06]. The authors of that paper studied bribery in several scenarios, depending on the voting system used and whether the voters were weighted and/or were assigned price-tags for changing their votes. In particular, in the priced cases they assumed that each voter is willing to change his/her vote arbitrarily, provided that the briber pays the fixed-per-voter price. Such an assumption is fairly unrealistic. For example, let us consider an election with four candidates, \(a\), \(b\), \(c\), and \(d\). It is easy to imagine a voter who, e.g., prefers option \(a\) to \(b\) to \(c\) to \(d\), but who actually really wants either \(a\) or \(b\) to win, with a slight preference towards \(a\), and who absolutely does not want either of the remaining options to be chosen. Such a voter might be willing, at a small price, to change his/her vote to rank \(b\) first, but would never, regardless of the money offered, change the vote to rank either \(c\) or \(d\) first.

A different example where it is useful to model voters as having such nonuniform prices is best seen from the point of view of the briber. A briber that wants some candidate \(p\) to win, might want to follow a certain policy in his/her bribing. For example, he/she might not want to bribe anyone to vote for \(p\) in order not to cast “bad light” on \(p\). Such a briber would have to make \(p\) a winner via bribes that redistribute other candidate’s support. Using the nonuniform model of bribery one could express this policy via setting the prices for voting for \(p\) so high as to be outside of the allowed budget.

The above policy was studied in [FHH06] but, naturally, different policies can easily be devised. (E.g., policies preventing the briber from affecting some voters, or only allowing him/her to bribe voters to change their votes in a limited way.)

Yet another scenario where nonuniform bribery model is useful regards the issue of coalition formation. Consider an election where one of the voters realizes that his/her option is very unlikely to win, but where there are many agents (voters) that support options similar, but slightly different. Such an agent might want to find out which of the others, but as few as possible, he/she would have to convince to form a coalition with him/her in order to have enough voting power as to choose an option that all of them would be reasonably satisfied with. One way to compute this set would be to: (a) find a group of voters that currently vote for options similar to the agent’s, (b) form nonuniform bribery instance where those voters can be bribed at a relatively low price to vote for the agent’s option and all other briberies are either very expensive or impossible (beyond budget), (c) compute a minimum-cost nonuniform bribery that ensures that the agent’s favorite option wins. The voters involved in this bribery would be candidates for the coalition.

In view of the above discussion we see that the issue of nonuniform bribery is both important and useful, even in the cases where we are not really “bribing” anyone, but simply are trying to strategically plan our behavior. In this paper we give a number of results that show that the problem

\(^1\)The following discussion is not very technical; we defer technical issues to further sections.
of nonuniform bribery can often be solved in polynomial time in the case of unweighted voters, yet easily becomes NP-complete if the voters are weighted. We also give several approximation algorithms for previously studied NP-complete bribery problems.

2 Preliminaries

In this section we provide our notation, model of elections, and briefly describe our main algorithmic tool, flow networks.

We view an election as a pair \((C, V)\), where \(C\) is a set of candidates, \(C = \{c_1, \ldots, c_m\}\), and \(V\) is a multiset of voters, each represented via his/her preference over \(C\). In this paper we represent each voter’s preference as an \(m\)-dimensional vector of nonnegative integers, indicating voter’s perceived utility from electing each candidate. (Our model is inspired by the discussion of a similar notion in the paper by Elkind and Lipmaa [EL05].)

**Definition 2.1** Let \(k\) and \(b\) be two positive integers. By a \((k, b)\)-election we mean an election over some candidate set \(C = \{c_1, \ldots, c_m\}\), where each voter from the voter set \(V\) distributes \(k\) integral points among the candidates, never assigning more than \(b\) points to a single candidate.\(^2\) In the end, the candidates with most points are the winners.

A free-form \((k, b)\)-election is a \((k, b)\)-election where voters can choose not to use all of their points.

The standard preference model studied in computational social choice literature assumes that each voter has a strict linear order of preference over all candidates. Our model is more appropriate for utility-based voting, and is powerful enough to capture such voting rules as plurality (as \((1, 1)\)-elections; in plurality elections each voter assigns one point to his/her favorite candidate), veto (as \((m - 1, 1)\)-elections; in veto election each voter gives a single point to everyone except for his/her most hated option), approval (as free-form \((m, 1)\)-elections; in approval voting each voter either approves or disapproves of each of the candidates and the candidates with most approvals win), and \(t\)-approval (as \((t, 1)\)-elections; as approval, but each candidate has to approve of exactly \(t\) candidates). Utility-based voting is a very attractive concept and we believe it is interesting to study it in computational context.

We use \((k, b)\)-elections as our model of voting. It is particularly nice for our nonuniform bribery as one can devise a reasonable pricing scheme for voters; we assume that for each ordered pair of candidates each voter has a, possibly different, price for moving a single point from one candidate to the other. Similar schemes can be devised for preference-order-based voting but those we came up with were neither as convincing nor as elegant.

The main result of this paper, Theorem 3.1, follows via an algorithm that uses min-cost flow problem as a crucial subroutine. We now briefly describe the min-cost flow problem and define appropriate notation.

A flow network is defined via a set of nodes \(N = \{s, t, n_1, \ldots, n_m\}\), where \(s\) is the source and \(t\) is the sink, a capacity function \(cpc : N \times N \rightarrow \mathbb{N}\), and a cost function \(prc : N \times N \rightarrow \mathbb{N}\). We say that two nodes, call them \(u\) and \(v\), are connected if \(cpc(u, v) > 0\). Note that \(cpc(u, v)\) does not need to equal \(cpc(v, u)\); the channels connecting two nodes are unidirectional. A function \(f : N \times N \rightarrow \mathbb{Z}\) is a flow in such a network if it satisfies the following constraints: (a) \((\forall u,v \in N)\) \(|f(u,v)| \leq\)

\(^2\)Note that if either \(k\) is too big, or there aren’t enough candidates then such an election is impossible.
Though, it would still be legal to move to some candidate \( q \) twice formally means that briberies are only legal if for each voter of course, points are anonymous, unnamed, entities; our requirement of not moving “the same” point twice formally means that briberies are only legal if for each voter \( v \) they move, within the

\[ \text{cpc}(u,v) \] (i.e., we do not send flow beyond capacity), (b) \( \forall u,v \in N \) \[ f(u,v) = -f(v,u) \], and (c) \( \forall u \in N - \{s,t\} \sum_{v \in N - \{u\}} f(u,v) = 0 \). We interpret \( f(u,v) = t, t > 0 \), as \( t \) units of flow traveling on the edge from \( u \) to \( v \). A negative flow value indicates reversed direction of travel. Note that units of flow can travel in both directions at the same time, but it is unnecessary. The value of a flow is defined as \( \sum_{u \in N - \{s\}} f(s,u) \) and its cost is \( \sum_{u,v \in N} \text{prc}(u,v) f(u,v) \). That is, for each two nodes \( u,v \in N \) we pay \( \text{prc}(u,v) \) for sending each single unit of flow from \( u \) to \( v \).

In the min-cost flow problem we are given a nonnegative integer \( K \), a flow network, i.e., a set of nodes with source and sink nodes, a capacity function, and a cost function, and our goal is to find a minimum-cost flow of value \( K \) (or to indicate that such a flow doesn’t exist). It is well known that this problem is solvable in polynomial-time and we point interested readers to an excellent monograph of Ahuja et al. [AMO93].

Finally, we explain what we mean by a fully polynomial-time approximation scheme (FPTAS). Due to space constraints the definition is very brief. An algorithm \( A \) is an FPTAS for a given minimization problem (e.g., a problem of finding minimum-cost bribery) if (a) algorithm \( A \) on input \( (I, \varepsilon) \), where \( I \) is an instance of the problem and \( \varepsilon \) is a positive real number, \( 0 < \varepsilon < 1 \), runs in time polynomial in the size of \( I \) and \( \frac{1}{\varepsilon} \), and (b) \( A \) has the property that if \( O \) is the value of the optimal solution for \( I \) then \( A(I, \varepsilon) \) produces a solution with value \( S \) such that \( S \leq (1+\varepsilon)O \). We stress that algorithm \( A \) has to output a correct solution for the problem, only that the cost of this solution may be somewhat above the optimum.

3 Nonuniform Bribery

In this section we define our nonuniform bribery problem for \( (k,b) \)-elections and present our results.

Let \( k \) and \( b \) be two positive integer values (possibly dependent on the number of candidates and voters in \( E \); see the next sentence). We define \( (k,b) \)-bribery problem as follows: The input is a \( (k,b) \)-election \( E \) with candidate set \( C = \{c_1, \ldots, c_m\} \) and a multiset \( V \) of voters \( v_1, \ldots, v_n \), where each voter \( v_i \) is represented via an \( m \)-dimensional integer vector describing in an obvious way how many points \( v_i \) assigns to which candidates, a nonnegative integer \( B \) (the budget), and for each voter \( v_i \) a price function

\[ \pi_i : C \times C \rightarrow \mathbb{N}. \]

A unit bribery involves asking some voter \( v_\ell \in V \) to move a single point that \( v_\ell \) currently assigns to some candidate \( c_i \) to another candidate \( c_j \). The cost of such a unit bribery is \( \pi_\ell(c_i, c_j) \). (Naturally, for each \( \ell \in \{1, \ldots, n\} \) and each candidate \( c_i \) we have \( \pi_\ell(c_i, c_i) = 0 \).) The question is if given the budget \( B \), this \( (k,b) \)-election \( E \), and functions \( \pi_1, \ldots, \pi_n \) (one for each voter), it is possible to perform a set of unit briberies of total cost at most \( B \), such that (a) preferred candidate \( p = c_1 \) becomes a winner, and, (b) each voter assigns at most \( b \) points to each candidate (i.e., the rules of a \( (k,b) \)-election are not broken). We explicitly require that all the unit briberies are executed “in parallel,” that is, a briber cannot first bribe voter \( v_\ell \) to move a point from some candidate \( c_i \) to another candidate \( c_j \) and afterward move that same point from \( c_j \) to yet another candidate \( c_q \) (Though, it would still be legal to move to \( c_q \) a point that \( v_\ell \) had assigned to \( c_j \) before the bribery. Of course, points are anonymous, unnamed, entities; our requirement of not moving “the same” point twice formally means that briberies are only legal if for each voter \( v_\ell \) they move, within the

\footnote{We could allow such sequential operations and all our results would hold via proofs similar to those presented here, but we feel that the model of parallel unit briberies is more appropriate.}
unmentioned edges have capacity 0. For each voter $v_\ell$ before bribery.

We define free-form $(k,b)$-bribery problem analogously, only that each voter can choose to not assign some of his/her points to any candidate and we can bribe voters to either give those unassigned points to some candidate, or to remove points from a given candidate and not assign them to anyone else. In order to accommodate this possibility we extend the price functions $\pi_\ell$ to be mappings from $\hat{C} \times \hat{C}$ to $\mathbb{N}$, where $\hat{C} = C \cup \{\varepsilon\}$. $\varepsilon$ represents the slot for unassigned points. Naturally, each voter can have as many unassigned points as he/she wishes to (i.e., the $b$-bound does not apply to the unassigned points).

We now give our main result regarding $(k,b)$-bribery.

**Theorem 3.1** There is an algorithm that solves both $(k,b)$-bribery instances and free-form $(k,b)$-bribery instances in time polynomial in $k$ and the size of the instance.

We devote a large chunk of the remainder of this section to proving Theorem 3.1. However, some discussion is in order before we jump into the proof. In particular, note that the running time of our algorithm is polynomial not in the size of $(k,b)$-bribery instance, but in the size of the instance and the value $k$. This means that if the value of $k$ is large then the algorithm runs very slowly. However, for many interesting cases (all that we have mentioned, e.g., plurality, veto, utility based voting where $k$ is bounded by some polynomial in the number of candidates) our result guarantees polynomial-time solution for this nonuniform bribery problem.

**Proof of Theorem 3.1** Due to length restrictions, we only provide a proof sketch for the case of $(k,b)$-bribery, and not for the free-form variant.

Our input is a $(k,b)$-election $E$, with candidate set $C = \{c_1, \ldots, c_m\}$ and voter multiset $V = \{v_1, \ldots, v_n\}$, a nonnegative integer $B$ (the budget), and voters’ price functions $\pi_1, \ldots, \pi_n$. Our goal is to ensure that our distinguished candidate $p = c_1$ is a winner of the election via a bribery of cost at most $B$.

Our proof follows via constructing a series of flow networks and computing min-cost solutions for them. The intuition here is that the points that voters assign to candidates are modeled via the units of flow traveling in the network. We design our networks in such a way that minimizing the cost of the flow, in essence, maximizes the number of points our designated candidate $p = c_1$ via a minimum-cost bribery.

We know that each candidate can at most receive $kn$ points. For each nonnegative integer $K$ between 1 and $kn$ our algorithm tests if there is a bribery of cost at most $B$ that ensures that $p$ receives exactly $K$ points and every other candidate receives at most $K$ points. Let us now fix a value of $K$ and show how such a test can be executed.

We form a network flow with node set $N = \{s, t\} \cup \bigcup_{i=1}^{n} C_i \cup \bigcup_{i=1}^{n} C'_i \cup F$, where $F = \{f_1, \ldots, f_m\}$ and for each $i \in \{1, \ldots, n\}$ we have $C_i = \{c_{i1}, \ldots, c_{im}\}$, $C'_i = \{c'_{i1}, \ldots, c'_{im}\}$. For each $i \in \{1, \ldots, n\}$, nodes in the sets $C_i$ represent point distribution of voter $v_i$ before bribery, nodes in $C'_i$ represent point distribution of voter $v_i$ after the bribery, and nodes in $F$ are used to enforce the rules of $(k,b)$-election and to sum points for all candidates.

We introduce the following capacities and costs for edges in our network. Assume that all unmentioned edges have capacity 0. For each voter $v_\ell$ and candidate $c_i$ we have $cpc(s, c_\ell)$ equal to the number of points $v_\ell$ assigns to $c_i$ before bribery and $prc(s, c_\ell) = 0$. These edges model delivering appropriate number of points to nodes from sets $C_1, \ldots, C_n$. 

Naturally, each voter can have as many unassigned points as he/she wishes to (i.e., the $b$-bound does not apply to the unassigned points).
For each node $c_{i}$ and each candidate $c_{j}$ we have $\text{cpc}(c_{i}, c_{j}) = k$ and $\text{prc}(c_{i}, c_{j}) = \pi_{i}(i, j)$. These edges model unit briberies. The briber can ask each voter to move his points as the briber likes, but pays appropriate price for moving each point.

For each node $c'_{i}$ we set $\text{cpc}(c'_{i}, f_{i}) = b$ and $\text{prc}(c'_{i}, f_{i}) = 0$. These edges enforce that no candidate can receive more than $b$ points from a single voter. Finally, for each node $f_{i}$, we have $\text{cpc}(f_{i}, t) = K$, $\text{prc}(f_{i}, t) = 0$, and for all $i \in \{2, \ldots, m\}$ we have $\text{prc}(f_{i}, t) = T$, where $T$ is an integer higher than the cost of any possible bribery (e.g., take $T = 1 + \max_{\ell, i, j} \pi_{\ell}(c_{i}, c_{j})$).

To perform our test we compute minimum-cost flow of value $kn$ in this network. If such a flow doesn’t exist then it means that there is no legal way of distributing points via bribery in such a way that each voter has score at most $K$. In such a case we disregard this value of $K$ and continue with the next one. We can interpret our minimum-cost flow as follows: Each set of nodes $C_{1}, \ldots, C_{n}$ receives units of flow corresponding to the distribution of points of each voter (because flow has value $kn$ and because of the capacities of edges connecting the source with nodes in $C_{1}, \ldots, C_{n}$) and units of flow travel from nodes in sets $C_{i}$ to nodes in sets $C'_{i}$, respectively, modeling our bribery. Finally, they all accumulate in nodes from set $F$, from where they all reach the sink. The nature of edges between nodes in $C'_{i}$s and nodes in $F$ guarantees that we arrive with a legal distribution of points for each voter. The cost of this flow can be expressed as $T \cdot (kn - p’s\ score\ after\ bribery) + \text{cost-of-bribery}$. Since $T$ is chosen to be larger than any possible cost of bribery, we know that minimum cost enforces that node $f_{1}$, corresponding to $p = c_{1}$, receives as many units of flow as legally possible. If $p$ can legally receive $K$ units of flow then minimum-cost flow delivers this many units; the capacity of the edge linking $p = c_{1}$ with the sink is $K$ so the flow cannot deliver more. If the flow cannot legally deliver $K$ units of flow to $c_{1}$ then we can safely disregard this network (because we have already handled this flow when analyzing smaller values of $K$). Interpretation of our flow gives that our bribery guarantees that $p$ gets exactly $K$ points and all the other candidates receive at most $K$ points each (as each node $f_{2}, \ldots, f_{n}$ can only deliver $K$ units of flow to the sink). This is achieved via a minimum cost bribery as the cost-of-bribery is the remaining part of the cost of our flow, after we consider the payment for delivering units of flow to the sink.

This way we can test in polynomial-time for each $K$ if there is a nonuniform bribery of cost at most $B$ that ensures that $p$ receives exactly $K$ points and all other candidates receive at most $K$ points. Thus, in total, the running time of our algorithm is polynomial in the size of our election and $k$.

Since we have shown that by proper choice of the values $k$ and $b$ we can express plurality, veto, approval, $t$-approval, and utility-based voting (where the number of points to distribute is bounded by a polynomial in the number of candidates), we have the following corollary.

**Corollary 3.2** Nonuniform bribery is solvable in polynomial time for the following election systems: plurality, veto, approval, $t$-approval, utility-based voting where the number of points each voter can distribute is polynomial in the number of candidates.

It is natural to ask if the above two results hold for the case of weighted voters. Let $(k, b)$-weighted-bribery and free-form $(k, b)$-weighted bribery be the analogs of $(k, b)$-bribery and free-form $(k, b)$-bribery for the case when voters have weights. Unfortunately, these problems are NP-complete even for very restricted values of $k$ and $b$. In particular, the following result holds.

**Theorem 3.3** $(1, 1)$-weighted-bribery is NP-complete.
Proof. Faliszewski et al. [FHH06] showed that what they call plurality-weighted-negative-bribery problem is NP-complete. This problem is defined as follows: Given a weighted plurality election \( E = (C, V) \) with a distinguished candidate \( p \) and a nonnegative integer \( B \), is it possible to pick up to \( B \) voters in \( V \) and change their votes so that neither of them ranks \( p \) first yet \( p \) is a winner of the election? It is easy to see that this problem reduces to \((1,1)\)-weighted-bribery. Given an instance of plurality-weighted-negative-bribery we form an instance of \((1,1)\)-weighted-bribery with the same election \( E \) (it is trivial to convert preferences expressed as linear orders to appropriate vectors in this case), the same budget \( B \), and where each unit bribery has cost 1, except for unit briberies that would give a point to \( p \). Those cost \( B + 1 \). The reader should convince him/herself that this reduction works in polynomial time and is correct.

On the other hand, \((1,1)\)-weighted-bribery is trivially in NP. □

We could similarly show that \((m-1,1)\)-weighted-bribery, where \( m \) is the number of candidates in the election, is NP-complete simply via invoking the fact that veto-weighted-bribery is NP-complete [FHH06].

Together with Theorem 3.3 this shows that in general we should not expect efficient algorithms for \((k,b)\)-weighted-bribery that work for all values of \( k \) and \( b \). However, there might be interesting algorithms for special cases.

In particular, we focus on the restriction of \((1,1)\)-weighted-bribery to the case where each voter has a single price for moving his/her point to any of the candidates. This problem is called plurality-weighted-$\textit{bribery$ in \[FHH06\] and we will use this name. We also consider a restriction of free-form \((m,1)\)-bribery, where \( m \) is the number of candidates, to a situation where each voter has for each candidate a price of flipping the support for that candidate. This restriction is equivalent to what in \[FHH06\] is called approval-weighted-$\textit{bribery$′. Again, we will use the name from \[FHH06\].

The following theorem shows that these two special cases of (free-form) \((k,b)\)-weighted-bribery can be solved efficiently using approximation algorithms.

**Theorem 3.4** plurality-weighted-$\textit{bribery$ and approval-weighted-$\textit{bribery$′ both have fully polynomial-time approximation schemes (FPTAS).

Proof. Faliszewski et al. [FHH06] gave polynomial-time algorithms for both plurality-weighted-$\textit{bribery$ and approval-weighted-$\textit{bribery$′ for the case when prices within those problems are encoded in unary. Running times of these algorithms are polynomial in the size of the problem and the value of the largest price. We use this fact here to give an FPTAS for both problems. In both cases our results follow by an extension of the scaling argument used, e.g., in an FPTAS for Knapsack.

Let \( L \) be either plurality-weighted-$\textit{bribery$ or approval-weighted-$\textit{bribery$′, let \( I \) be an instance of \( L \) and let \( \varepsilon \) be a positive real number, \( 0 < \varepsilon < 1 \). By \( T \) we mean the highest price occurring in \( I \) and we assume that \( T \) is at least 1. Otherwise, any solution is optimal. Also, we let \( N \) be the total number of prices that occur in \( I \). For plurality, \( N \) is the number of voters and for approval, \( N \) is the number of voters times the number of candidates. Thus, an upper bound on the cost of any bribery in \( I \) is \( TN \). We give an FPTAS for \( L \).

The idea of our algorithm is to scale down the prices so that they are polynomially bounded in \( N \) and \( \frac{1}{\varepsilon} \) and to run the polynomial-time algorithm of [FHH06] on thus modified instance. However, instance \( I \) might include some voters with very high prices that are not needed in the optimal solution. By choosing a scaling factor appropriate for a high price range we might essentially lose all the information regarding the smaller prices, those that actually participate in the optimal
solution. Thus, instead of performing one scaling, we perform polynomially many of them. We start with a fairly small scaling factor and keep on increasing it. With a small scaling factor some of the prices within $I$ will be too large, and so we assume that the voters with those prices cannot be bribed.

Our algorithm executes $\lceil \log T \rceil$ iterations. In each iteration variable $t$ contains our current guess of an upper bound on the largest price used within the optimal solution. We start with $t = 1$ and double it after every iteration.

Given a value of $t$, an iteration is executed as follows. Set $K = \frac{t}{N}$ and construct an instance $I'$ that is identical to $I$, only that: (a) each price $p$ such that $p \leq t$, is replaced by $\lceil \frac{p}{t} \rceil$, and (b) any higher price is replaced by $\frac{1+2\varepsilon}{\varepsilon} N^2 + 1$. Note that due to this transformation $I'$ has all prices polynomially bounded in $\frac{1}{\varepsilon}$ and $N$. We find an optimal solution for $I'$ using the polynomial-time algorithm from \cite{FTIH06}. If the solution has cost $\frac{1+2\varepsilon}{\varepsilon} N^2 + 1$ or higher then we discard it and otherwise, we store it for future use.

After all the iterations are finished, we return a stored solution with the lowest cost. Clearly, we have at least one solution as in the last iteration $t \geq T$.

We claim that this algorithm finds a solution with cost within $2\varepsilon$ of the optimal one. Let $O$ be an optimal solution and let $u$ be the highest price used within solution $O$. Let us consider an iteration of our algorithm with $t$ such that $\frac{t}{2} \leq u \leq t$. Let $S$ be our optimal solution to instance $I'$ in this iteration. Since $I$ and $I'$ vary only in voter’s prices, $S$ and $O$ are valid solutions for both $I$ and $I'$. By $p(S)$ and $p(O)$ we mean the prices of briberies specified in $S$ and in $O$, respectively, expressed using prices from instance $I$. By $p'(S)$ and $p'(O)$ we mean analogous values, but with respect to prices in $I'$. It holds that

$$ p(O) \leq p(S) \leq K p'(S) \leq K p'(O). $$

The first inequality holds because $O$ is an optimal solution to $I$ and the second one follows because instance $I'$ has prices rounded up. The last inequality is due to the fact that $S$ is an optimal solution for $I'$. For any price $p$ in $I$, we have a corresponding price $p' = \lceil \frac{p}{K} \rceil$ in $I'$ and, due to rounding, it is easy to see that $p \leq K p' \leq p + K$. Since any bribery involves paying at most $N$ prices we have that

$$ K p'(O) \leq p(O) + N K. $$

Since $NK = \varepsilon t$, and in this iteration we have $\frac{t}{2} \leq u \leq t$, naturally we have that $\frac{t}{2} \leq u \leq p(O)$. Thus, we have

$$ p(S) \leq K p'(S) \leq p(O) + 2\varepsilon p(O) = (1 + 2\varepsilon) p(O). $$

It remains to see that our algorithm does not discard solution $S$.

We have that the highest price in $O$ is $u$, and so $p(O) \leq Nu \leq N t$. Via the above estimate we have that $K p'(S) \leq p(O)(1 + 2\varepsilon) \leq (1 + 2\varepsilon) N t$, and so $p'(S) \leq \frac{N t}{K} (1 + 2\varepsilon) \leq \frac{1+2\varepsilon}{\varepsilon} N^2$. Thus, solution $S$ is stored within this iteration. Other iterations may only improve our solution and thus the final solution we output is within $2\varepsilon$ of the optimal one. \hfill \square

Unfortunately, it is unlikely that, in general, for polynomially bounded values of $k$ and $b$, there is an FPTAS for $(k,b)$-bribery. The reason for this is hidden in the proof of Theorem 3.3. If there was an FPTAS for $(1,1)$-weighted-bribery then, via using appropriately good approximation, one could solve plurality-weighted-negative-bribery exactly. Thus, we have the following corollary.

**Corollary 3.5** There is no FPTAS that solves $(k,b)$-weighted-bribery instances for any given $k, b$.  

8
We point readers interested in the issue of approximation with respect to control and bribery to the work of Brelsford [Bre07].

We conclude this section on a somewhat different note. Copeland rule, which here we will call Copeland$^{0.5}$, is the following: For each pair of candidates $c_i$ and $c_j$ we ask each voter which one among the two he/she prefers. The one preferred by majority receives one point. In case of a tie both candidates receive half a point. In the end, the candidates with most points are winners.

Faliszewski et al. [FHHR07], among others, studied similar elections where voters represent their votes via so called irrational preference tables. An irrational preference table is a symmetric function that given two candidates returns the one that the voter prefers. Faliszewski et al. [FHHR07] defined a very natural bribery model for thus represented Copeland elections, where flipping a value of each irrational preference table entry comes at unit cost; they called this problem microbribery. We note that microbribery in the case where flipping each entry of each preference table may have a different price is a natural example of what in this paper we call nonuniform bribery. Faliszewski et al. gave polynomial-time algorithms for microbribery for Copeland$^0$ (a system just like Copeland$^{0.5}$, but where no points are granted in case of a tie in a head-to-head contest) and for Copeland$^1$ (defined, analogously, as giving one point to each candidate in a tied head-to-head contest). Their algorithms naturally extend to a situation where each preference-table-entry-flip comes with its own price. Here we report that via fairly simple modifications their algorithm can also be extended to handle Copeland$^{0.5}$.

**Theorem 3.6** There is a polynomial-time algorithm for microbribery in Copeland$^{0.5}$, even if each preference-table-entry-flip comes with its own, potentially distinct, price.

We omit the proof, which we expect to include in the full version of [FHHR07]. However, we mention that the proof again uses the flow-network technique we used before and that the key difference as compared to [FHHR07] is that we include a gadget that allows us to “split” a point that travels in the network in case we need to model a tie between two candidates.

We mention that it does not seem to be easy to generalize our algorithm to tie-handling values other than 0, $\frac{1}{2}$, and 1. The cases of 0 and 1 are special because we can manage to pump appropriate number of points through our flow network, and the case of $\frac{1}{2}$ is easy to handle as then both candidates in a tie are handled symmetrically in the network. We suspect that microbribery for other tie-handling values might be NP-complete.

## 4 Discussion and Open Problems

In this paper we have introduced and studied the concept of nonuniform bribery, where briber’s payment to each particular voter depends on the nature of the bribery performed. We have argued that this generalization of the problem is interesting as it both models the reality more precisely (voters may not be willing to modify their votes in certain ways, or may require higher payments for such changes), and it can be useful for coalition-formation tasks and to model briber’s policies. We have shown efficient algorithms for nonuniform bribery for the case of plurality, veto, variants of approval, variants of utility-based voting, and Copeland. However, our nonuniform bribery problem is so expressive that as soon as we consider weighted voters we can easily find special cases that are NP-complete.

---

4We are following the naming scheme of [FHHR07] here.
We have also shown that some of the known NP-complete bribery problems in fact have fully polynomial-time approximation schemes.

We conclude this paper with several open questions. In particular we are interested in the complexity of unweighted manipulation of Borda count (an election system somewhat similar to our utility-based voting, but where the amounts of points that the voters have to assign to candidates are set very rigidly). To the best of our knowledge, unweighted coalitional manipulation has not been studied for Borda. Similarly, we are interested in the complexity of unweighted bribery for Borda. We also mention that the complexity of microbribery for Copeland and tie-handling values other than 0, $\frac{1}{2}$ and 1 is unknown and poses an interesting challenge.

**Acknowledgements**

I would like to thank Edith Hemaspaandra, Lane Hemaspaandra, Jörg Rothe and Henning Schnoor for very interesting and useful discussions.

**References**

[AMO93] R. Ahuja, T. Magnanti, and J. Orlin. *Network Flows: Theory, Algorithms, and Applications*. Prentice-Hall, 1993.

[Arr63] K. Arrow. *Social Choice and Individual Values*. John Wiley and Sons, 1951 (revised edition, 1963).

[BO91] J. Bartholdi, III and J. Orlin. Single transferable vote resists strategic voting. *Social Choice and Welfare*, 8(4):341–354, 1991.

[Bre07] E. Brelfsford. Approximation and elections. Master’s thesis, Rochester Institute of Technology, Rochester, NY, May 2007.

[BTT89] J. Bartholdi, III, C. Tovey, and M. Trick. The computational difficulty of manipulating an election. *Social Choice and Welfare*, 6(3):227–241, 1989.

[BTT92] J. Bartholdi, III, C. Tovey, and M. Trick. How hard is it to control an election? *Mathematical and Computer Modeling*, 16(8/9):27–40, 1992.

[CS03] V. Conitzer and T. Sandholm. Universal voting protocol tweaks to make manipulation hard. In *Proceedings of the 18th International Joint Conference on Artificial Intelligence*, pages 781–788. Morgan Kaufmann, August 2003.

[CS06] V. Conitzer and T. Sandholm. Nonexistence of voting rules that are usually hard to manipulate. In *Proceedings of the 21st National Conference on Artificial Intelligence*. AAAI Press, July/August 2006.

[CSL07] V. Conitzer, T. Sandholm, , and J. Lang. When are elections with few candidates hard to manipulate? *Journal of the ACM*, 54(3):1–33, 2007.

[DS00] J. Duggan and T. Schwartz. Strategic manipulability without resoluteness or shared beliefs: Gibbard–Satterthwaite generalized. *Social Choice and Welfare*, 17(1):85–93, 2000.
[EL05] E. Elkind and H. Lipmaa. Hybrid voting protocols and hardness of manipulation. In The 16th Annual International Symposium on Algorithms and Computation, ISAAC 2005, pages 206–215. Springer-Verlag Lecture Notes in Computer Science #3872, December 2005.

[FHH06] P. Faliszewski, E. Hemaspaandra, and L. Hemaspaandra. The complexity of bribery in elections. In Proceedings of the 21st National Conference on Artificial Intelligence, pages 641–646. AAAI Press, July 2006.

[FHHR07] P. Faliszewski, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Llull and Copeland voting broadly resist bribery and control. In Proceedings of the 22nd National Conference on Artificial Intelligence, pages 724–730. AAAI Press, July 2007.

[Gib73] A. Gibbard. Manipulation of voting schemes. Econometrica, 41(4):587–601, 1973.

[HH07] E. Hemaspaandra and L. Hemaspaandra. Dichotomy for voting systems. Journal of Computer and System Sciences, 73(1):73–83, 2007.

[HHR07] E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Anyone but him: The complexity of precluding an alternative. Artificial Intelligence, 171(5-6):255–285, April 2007.

[PR07] A. Procaccia and J. Rosenschein. Junta distributions and the average-case complexity of manipulating elections. Journal of Artificial Intelligence Research, 28:157–181, February 2007.

[PRZ07] A. Procaccia, J. Rosenschein, and A. Zohar. Multi-winner elections: Complexity of manipulation, control, and winner-determination. In Proceedings of the 20th International Joint Conference on Artificial Intelligence, pages 1476–1481. AAAI Press, January 2007.

[Sat75] M. Satterthwaite. Strategy-proofness and Arrow’s conditions: Existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory, 10(2):187–217, 1975.