Fibonacci and Lucas Identities the Golden Way

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Abstract
By expressing Fibonacci and Lucas numbers in terms of the powers of the golden ratio, \( \alpha = \frac{1 + \sqrt{5}}{2} \) and its inverse, \( \beta = -\frac{1}{\alpha} = \frac{1 - \sqrt{5}}{2} \), a multitude of Fibonacci and Lucas identities have been developed in the literature. In this paper, we follow the reverse course: we derive numerous Fibonacci and Lucas identities by making use of the well-known expressions for the powers of \( \alpha \) and \( \beta \) in terms of Fibonacci and Lucas numbers.

1 Introduction
The Fibonacci numbers, \( F_n \), and the Lucas numbers, \( L_n \), are defined, for \( n \in \mathbb{Z} \), through the recurrence relations

\[
F_n = F_{n-1} + F_{n-2}, \quad (n \geq 2), \quad F_0 = 0, \quad F_1 = 1;
\]

and

\[
L_n = L_{n-1} + L_{n-2}, \quad (n \geq 2), \quad L_0 = 2, \quad L_1 = 1;
\]

with

\[
F_{-n} = (-1)^{n-1}F_n, \quad L_{-n} = (-1)^nL_n.
\]

Throughout this paper, we denote the golden ratio, \((1 + \sqrt{5})/2\), by \( \alpha \) and write \( \beta = (1 - \sqrt{5})/2 = -1/\alpha \), so that \( \alpha\beta = -1 \) and \( \alpha + \beta = 1 \). The following well-known algebraic properties of \( \alpha \) and \( \beta \) can be proved directly from Binet’s formula for the \( n \)th Fibonacci number or by induction:

\[
\alpha^n = \alpha^{n-1} + \alpha^{n-2},
\]

\[
\beta^n = \beta^{n-1} + \beta^{n-2},
\]

\[
\alpha^n = \alpha F_n + F_{n-1},
\]

\[
\alpha^n \sqrt{5} = \alpha^n(\alpha - \beta) = \alpha L_n + L_{n-1},
\]

\[
\beta^n = \beta F_n + F_{n-1},
\]

\[
\beta^n \sqrt{5} = \beta^n(\alpha - \beta) = \beta L_n - L_{n-1},
\]

\[
\beta^n = -\alpha F_n + F_{n+1}.
\]
\( \beta^n \sqrt{5} = \alpha L_n - L_{n+1} \), \hspace{1cm} (1.11)
\[ \alpha^{-n} = (-1)^{n-1} \alpha F_n + (-1)^n F_{n+1} \] \hspace{1cm} (1.12)

and
\[ \beta^{-n} = (-1)^n \alpha F_n + (-1)^n F_{n-1} . \] \hspace{1cm} (1.13)

Hoggatt et. al. [4] derived, among other results, the identity
\[ F_{k+t} = \alpha^k F_t + \beta^t F_k , \]
which, upon multiplication by \( \alpha^s \), can be put in the form
\[ \alpha^s F_{k+t} = \alpha^{s+k} F_t + (-1)^t \alpha^{s-t} F_k . \] \hspace{1cm} (1.14)

Identity (1.14) is unchanged under interchange of \( k \) and \( t \), \( k \) and \( s \) and interchange of \( t \) and \( -s \) and \( s \) and \( -t \); we therefore have three additional identities:
\[ \alpha^s F_{k+t} = \alpha^{s+t} F_k + (-1)^k \alpha^{s-k} F_t , \] \hspace{1cm} (1.15)
\[ \alpha^k F_{s+t} = \alpha^{s+k} F_t + (-1)^t \alpha^{k-t} F_s , \] \hspace{1cm} (1.16)
and
\[ \alpha^{s-t} F_k = \alpha^{k-t} F_s + (-1)^s \alpha^{-t} F_{k-s} . \] \hspace{1cm} (1.17)

As Koshy [7, p.79] noted, the two Binet formulas
\[ F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} , \quad L_n = \alpha^n + \beta^n , \] \hspace{1cm} (1.18)
expressing \( F_n \) and \( L_n \) in terms of \( \alpha^n \) and \( \beta^n \), can be used in tandem to derive an array of identities.

Our aim in writing this paper is to derive numerous Fibonacci and Lucas identities by emphasizing identities (1.6) – (1.11), expressing \( \alpha^n \) and \( \beta^n \) in terms of \( F_n \) and \( L_n \). Our method relies on the fact that \( \alpha \) and \( \beta \) are irrational numbers. We will make frequent use of the fact that if \( a, b, c \) and \( d \) are rational numbers and \( \gamma \) is an irrational number, then \( a \gamma + b = c \gamma + d \) implies that \( a = c \) and \( b = d \); an observation that was used to advantage by Griffiths [3].

As a quick illustration of our method, take \( x = \alpha F_p \) and \( y = F_{p-1} \) in the binomial identity
\[ \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} = (x + y)^n , \]
to obtain
\[ \sum_{j=0}^{n} \binom{n}{j} \alpha^j F_p^j F_{p-1}^{n-j} = \alpha^{np} , \] \hspace{1cm} (1.19)
which, by multiplying both sides by \( \alpha^q \), can be written
\[ \sum_{j=0}^{n} \binom{n}{j} \alpha^{j+q} F_p^j F_{p-1}^{n-j} = \alpha^{np+q} , \] \hspace{1cm} (1.20)
which, by identity (1.6), evaluates to
\[
\alpha \sum_{j=0}^{n} \binom{n}{j} F_p^j F_{p-1}^{n-j} F_{j+q} + \sum_{j=0}^{n} \binom{n}{j} F_p^j F_{p-1}^{n-j} F_{j+q-1} = \alpha F_{np+q} + F_{np+q-1}. \quad (1.21)
\]

Comparing the coefficients of \(\alpha\) in (1.21), we find
\[
\sum_{j=0}^{n} \binom{n}{j} F_p^j F_{p-1}^{n-j} F_{j+q} = F_{np+q}; \quad (1.22)
\]
valid for non-negative integer \(n\) and arbitrary integers \(p\) and \(q\).

Identity (1.22) contains many known identities as special cases.

If we write (1.20) as
\[
\sum_{j=0}^{n} \binom{n}{j} \alpha^{j+q} \sqrt{5} F_p^j F_{p-1}^{n-j} = \alpha^{np+q} \sqrt{5} \quad (1.23)
\]
and apply identity (1.7), we obtain the Lucas version of (1.22), namely,
\[
\sum_{j=0}^{n} \binom{n}{j} F_p^j F_{p-1}^{n-j} L_{j+q} = L_{np+q}. \quad (1.24)
\]

**Lemma 1.** The following properties hold for \(a, b, c\) and \(d\) rational numbers:

\[
\begin{align*}
  a\alpha + b &= c\alpha + d \iff a = c, \quad b = d, \quad \text{P1} \\
  a\beta + b &= c\beta + d \iff a = c, \quad b = d, \quad \text{P2} \\
  \frac{1}{c\alpha + d} &= \left( \frac{c}{c^2 - d^2 - cd} \right) \alpha - \left( \frac{c + d}{c^2 - d^2 - cd} \right), \quad \text{P3} \\
  \frac{1}{c\beta + d} &= \left( \frac{c}{c^2 - d^2 - cd} \right) \beta - \left( \frac{c + d}{c^2 - d^2 - cd} \right), \quad \text{P4} \\
  \frac{a\alpha + b}{c\alpha + d} &= \frac{cb - da}{c^2 - d^2 - cd} \alpha + \frac{ca - db - cb}{c^2 - cd - d^2}, \quad \text{P5} \\
\end{align*}
\]

and
\[
\frac{a\beta + b}{c\beta + d} = \frac{cb - da}{c^2 - d^2 - cd} \beta + \frac{ca - db - cb}{c^2 - cd - d^2}. \quad \text{P6}
\]

Properties P3 to P6 follow from properties P1 and P2.

The rest of this section is devoted to using our method to re-discover known results or to discover results that may be easily deduced from known ones. In establishing some of the identities we require the fundamental relations \(F_{2n} = F_n L_n\), \(L_n = F_{n-1} + F_{n+1}\) and \(5F_n = L_{n-1} + L_{n+1}\). Presumably new results will be developed in section 2.
Fibonacci and Lucas addition formulas

To derive the Fibonacci addition formula, use (1.6) to write the identity
\[ \alpha^{p+q} = \alpha^p \alpha^q \] (1.25)
as
\[ \alpha F_{p+q} + F_{p+q-1} = (\alpha F_p + F_{p-1})(\alpha F_q + F_{q-1}), \]
from which, by multiplying out the right side, making use of (1.6) again, we find
\[ \alpha F_{p+q} + F_{p+q-1} = \alpha(F_pF_q + F_{p-1}F_q) + F_pF_q + F_{p-1}F_{q-1} \]
\[ = \alpha(F_pF_{q+1} + F_{p-1}F_q) + F_pF_q + F_{p-1}F_{q-1}. \] (1.26)
Equating coefficients of \( \alpha \) (property P1) from both sides of (1.26) establishes the well-known Fibonacci addition formula:
\[ F_{p+q} = F_pF_{q+1} + F_{p-1}F_q. \] (1.27)
A similar calculation using the identity
\[ \alpha^p \beta^q = (-1)^q \alpha^{p-q} \] (1.28)
produces the subtraction formula
\[ (-1)^q F_{p-q} = F_pF_{q-1} - F_{p-1}F_q, \] (1.29)
which may, of course, be obtained from (1.27) by changing \( q \) to \(-q\).
The Lucas counterpart of identity (1.27) is obtained by applying (1.6) and (1.7) to the identity
\[ \alpha^{p+q} \sqrt{5} = (\alpha^p \sqrt{5}) \alpha^q, \] (1.30)
and proceeding as in the Fibonacci case, giving
\[ L_{p+q} = F_pL_{q+1} + F_{p-1}L_q. \] (1.31)
Application of (1.6) to the right side and (1.7) to the left side of the identity
\[ 5\alpha^{p+q} = (\alpha^p \sqrt{5})(\alpha^q \sqrt{5}) \] (1.32)
produces
\[ 5F_{p+q} = L_pL_{q+1} + L_{p-1}L_q. \] (1.33)
Fibonacci and Lucas multiplication formulas

Subtracting identity (1.14) from identity (1.15) gives
\[ F_t \left( \alpha^{s+k} - (-1)^k \alpha^{s-k} \right) = F_k \left( \alpha^{s+t} - (-1)^t \alpha^{s-t} \right). \] (1.34)
Applying identity (1.6) to identity (1.34) and equating coefficients of \( \alpha \), we obtain
\[ F_t (F_{s+k} - (-1)^k F_{s-k}) = F_k (F_{s+t} - (-1)^t F_{s-t}) \] (1.35)
which, upon setting \( k = 1 \), gives
\[ F_t L_s = F_{s+t} - (-1)^t F_{s-t}. \] (1.36)
Writing identity (1.34) as
\[ F_t \left( \alpha^{s+k} \sqrt{5} - (-1)^k \alpha^{s-k} \sqrt{5} \right) = F_k \left( \alpha^{s+t} \sqrt{5} - (-1)^t \alpha^{s-t} \sqrt{5} \right), \] (1.37)
applying identity (1.7) and equating coefficients of \( \alpha \) produces
\[ F_t \left( L_{s+k} - (-1)^k L_{s-k} \right) = F_k \left( L_{s+t} - (-1)^t L_{s-t} \right), \] (1.38)
which, upon setting \( k = 1 \), gives
\[ 5F_t F_s = L_{s+t} - (-1)^t L_{s-t}. \] (1.39)

Adding identity (1.14) and identity (1.15), making use of identity (1.6) and equating the coefficients of \( \alpha \), we obtain
\[ 2F_{k+t} F_s = F_t \left( F_{s+k} + (-1)^k F_{s-k} \right) + F_k \left( F_{s+t} + (-1)^t F_{s-t} \right), \] (1.40)
which, at \( k = t \) reduces to
\[ L_t F_s = F_{s+t} + (-1)^t F_{s-t}. \] (1.41)

Similarly, adding identity (1.14) and identity (1.15), multiplying through by \( \sqrt{5} \), making use of identity (1.7) and equating the coefficients of \( \alpha \), we have
\[ 2F_{k+t} L_s = F_t \left( L_{s+k} + (-1)^k L_{s-k} \right) + F_k \left( L_{s+t} + (-1)^t L_{s-t} \right), \] (1.42)
which, at \( k = t \) reduces to
\[ L_t L_s = L_{s+t} + (-1)^t L_{s-t}. \] (1.43)

**Cassini’s identity**

Since
\[ \alpha^n \beta^n = (\alpha \beta)^n = (-1)^n; \] (1.44)
applying identities (1.6) and (1.10) to the left hand side of the above identity gives
\[ \alpha^n \beta^n = (\alpha F_n + F_{n-1})(-\alpha F_n + F_{n+1}) \]
\[ = -\alpha^2 F_n^2 + \alpha(\alpha F_n F_{n+1} - F_n F_{n-1}) + F_{n-1} F_{n+1} \]
\[ = \alpha(-F_n^2 + F_n^2) - F_n^2 + F_{n-1} F_{n+1}. \]

Thus, according to (1.44), we have
\[ \alpha(-F_n^2 + F_n^2) - F_n^2 + F_{n-1} F_{n+1} = (-1)^n. \]
Comparing coefficients of \( \alpha^0 \) from both sides gives Cassini’s identity:
\[ F_{n-1} F_{n+1} = F_n^2 + (-1)^n. \] (1.45)

To derive the Lucas version of (1.45), write
\[ (\alpha^n \sqrt{5})(\beta^n \sqrt{5}) = (-1)^n 5; \] (1.46)
apply (1.7) and (1.11) to the left hand side, multiply out and equate coefficients, obtaining
\[ L_{n-1} L_{n+1} - L_n^2 = (-1)^{n-1} 5. \] (1.47)
General Fibonacci and Lucas addition formulas and Catalan’s identity

From identity (1.14), we can derive an addition formula that includes identity (1.27) as a particular case.

Using identity (1.6) to write the left hand side (lhs) and the right hand side (rhs) of identity (1.14), we have

\[
\text{lhs of (1.14)} = \alpha F_s F_{k+t} + F_{s-1} F_{k+t}
\]

and

\[
\text{rhs of (1.14)} = \alpha F_{s+k} F_t + F_{s+k-1} F_t + \alpha (-1)^t F_{s-t} F_k + (-1)^t F_{s-t-1} F_k
\]

Comparing the coefficients of \(\alpha\) from (1.48) and (1.49), we find

\[
F_s F_{k+t} = F_{s+k} F_t + (-1)^t F_{s-t} F_k
\]

of which identity (1.27) is a particular case.

Setting \(t = s - k\) in (1.50) produces Catalan’s identity:

\[
F_s^2 = F_{s+k} F_{s-k} + (-1)^{s+k} F_k^2.
\]

Multiplying through identity (1.14) by \(\sqrt{5}\) and performing similar calculations to above produces

\[
L_s F_{k+t} = L_{s+k} F_t + (-1)^t L_{s-t} F_k,
\]

which at \(t = s - k\) gives

\[
F_{2s} = L_{s+k} F_{s-k} + (-1)^{s+k} F_{2k}.
\]

Sums of Fibonacci and Lucas numbers with subscripts in arithmetic progression

Setting \(x = \alpha^p\) in the geometric sum identity

\[
\sum_{j=0}^{n} x^j = \frac{1 - x^{n+1}}{1 - x}
\]

and multiplying through by \(\alpha^q\) gives

\[
\sum_{j=0}^{n} \alpha^{pj+q} = \frac{\alpha^q - \alpha^{pn+p+q}}{1 - \alpha^p}.
\]

Thus, we have

\[
\sum_{j=0}^{n} \alpha^{pj+q} = \frac{\alpha(F_{pn+p+q} - F_q) + (F_{pn+p+q-1} - F_q-1)}{\alpha F_p + (F_{p-1} - 1)},
\]

from which, with the use of identity (1.6) and property P5, we find

\[
\sum_{j=0}^{n} F_{pj+q} = \frac{F_p(F_{pn+p+q-1} - F_{q-1}) - (F_{p-1} - 1)(F_{pn+p+q} - F_q)}{L_p - 1 + (-1)^{p-1}},
\]

valid for all integers \(p, q\) and \(n\). The derivation here is considerably simpler than in the direct use of Binet’s formula as done, for example, in Koshy [7, p. 86] and by Freitag [2].
Multiplying through identity (1.55) by \( \sqrt{5} \) gives

\[
\sum_{j=0}^{n} \alpha^{p_j+q} \sqrt{5} = \frac{\alpha^q \sqrt{5} - \alpha^{pn+p+q} \sqrt{5}}{1 - \alpha^p},
\]

(1.58)

from which, by identities (1.7) and (1.6), and properties P5 and P1, we find

\[
\sum_{j=0}^{n} L_{pj+q} = \frac{F_p (L_{pm+p+q-1} - L_{q-1}) - (F_{p-1} - 1)(L_{pn+p+q} - L_q)}{L_p - 1 + (-1)^{p-1}}.
\]

(1.59)

Generating functions of Fibonacci and Lucas numbers with indices in arithmetic progression

Setting \( x = y \alpha^p \) in the identity

\[
\sum_{j=0}^{\infty} x^j = \frac{1}{1 - x}
\]

(1.60)

and multiplying through by \( \alpha^q \) gives

\[
\sum_{j=0}^{\infty} \alpha^{pj+q} y^j = \frac{\alpha^q}{1 - \alpha^p y} = \frac{F_q \alpha + F_{q-1}}{-y F_{p} \alpha + 1 - y F_{p-1}}.
\]

(1.61)

Application of identity (1.6) and properties P5 and P1 then produces

\[
\sum_{j=0}^{\infty} F_{pj+q} y^j = \frac{F_q + (-1)^q F_{p-q} y}{1 - L_p y + (-1)^p y^2}.
\]

(1.62)

To find the corresponding Lucas result, we write

\[
\sum_{j=0}^{\infty} \alpha^{pj+q} \sqrt{5} y^j = \frac{\alpha^q \sqrt{5}}{1 - \alpha^p y}
\]

(1.63)

and use identity (1.7) and properties P5 and P1, obtaining

\[
\sum_{j=0}^{\infty} L_{pj+q} y^j = \frac{L_q - (-1)^q L_{p-q} y}{1 - L_p y + (-1)^p y^2}.
\]

(1.64)

Identity (1.62), but not (1.64), was reported in Koshy [7, identity 18, p.230]. The case \( q = 0 \) in (1.62) was first obtained by Hoggatt [5] while the case \( p = 1 \) in (1.64) is also found in Koshy [7, identity (19.2), p.231].

2 Main results

2.1 Recurrence relations

Theorem 1. The following identities hold for integers \( p, q \) and \( r \):

\[
F_{p+q+r} = F_{q-1} F_r F_{p-1} + (F_{q+1} F_r + F_{q-1} F_{r-1}) F_p + F_q F_{r+1} F_{p+1},
\]

(2.1)
and

\[ F_{p+q+r} = F_{p-1}F_rF_{q-1} + (F_{p+1}F_r + F_{p-1}F_{r-1})F_q + F_pF_{r+1}F_{q+1}, \]  
(2.2)

\[ F_{p+q+r} = F_{q-1}F_pF_{r-1} + (F_{q+1}F_p + F_qF_{p-1})F_r + F_qF_{p+1}F_{r+1}, \]  
(2.3)

\[ F_{p+q+r} = F_{r-1}F_qF_{p-1} + (F_{r+1}F_q + F_{r-1}F_{q-1})F_p + F_rF_{q+1}F_{p+1}, \]  
(2.4)

\[ L_{p+q+r} = F_{q-1}F_rL_{p-1} + (F_{q+1}F_r + F_{q-1}F_{r-1})L_p + F_qF_{r+1}L_{p+1}, \]  
(2.5)

\[ L_{p+q+r} = F_{p-1}F_rL_{q-1} + (F_{p+1}F_r + F_{p-1}F_{r-1})L_q + F_pF_{r+1}L_{q+1}, \]  
(2.6)

\[ L_{p+q+r} = F_{q-1}F_pL_{r-1} + (F_{q+1}F_p + F_qF_{p-1})L_r + F_qF_{p+1}L_{r+1}, \]  
(2.7)

\[ L_{p+q+r} = F_{r-1}F_qL_{p-1} + (F_{r+1}F_q + F_{r-1}F_{q-1})L_p + F_rF_{q+1}L_{p+1}, \]  
(2.8)

\[ 5F_{p+q+r} = L_{q-1}L_rF_{p-1} + (L_{q+1}L_r + L_{q-1}L_{r-1})F_p + L_qL_{r+1}F_{p+1}, \]  
(2.9)

\[ 5F_{p+q+r} = L_{p-1}L_qF_{r-1} + (L_{p+1}L_q + L_{p-1}L_{q-1})F_q + L_pL_{q+1}F_{r+1}, \]  
(2.10)

\[ 5F_{p+q+r} = L_{q-1}L_pF_{r-1} + (L_{q+1}L_p + L_{q-1}L_{p-1})F_r + L_qL_{p+1}F_{r+1}, \]  
(2.11)

\[ 5F_{p+q+r} = L_{r-1}L_qF_{p-1} + (L_{r+1}L_q + L_{r-1}L_{q-1})F_p + L_rL_{q+1}F_{p+1}, \]  
(2.12)

\[ 5L_{p+q+r} = L_{q-1}L_rL_{p-1} + (L_{q+1}L_r + L_{q-1}L_{r-1})L_p + L_qL_{r+1}L_{p+1}, \]  
(2.13)

\[ 5L_{p+q+r} = L_{p-1}L_qL_{r-1} + (L_{p+1}L_q + L_{p-1}L_{q-1})L_q + L_pL_{q+1}L_{r+1}, \]  
(2.14)

\[ 5L_{p+q+r} = L_{q-1}L_pL_{r-1} + (L_{q+1}L_p + L_{q-1}L_{p-1})L_r + L_qL_{p+1}L_{r+1}, \]  
(2.15)

\[ 5L_{p+q+r} = L_{r-1}L_qL_{p-1} + (L_{r+1}L_q + L_{r-1}L_{q-1})L_p + L_rL_{q+1}L_{p+1}. \]  
(2.16)

**Proof.** Identities (2.1) – (2.16) are derived by applying (1.6) and (1.7) to the following identities:

\[ \alpha^{p+q+r} = \alpha^p \alpha^q \alpha^r, \]

\[ \alpha^{p+q+r} \sqrt{5} = (\alpha^p \sqrt{5}) \alpha^q \alpha^r, \]

\[ 5\alpha^{p+q+r} = (\alpha^p \sqrt{5})(\alpha^q \sqrt{5}) \alpha^r \]

and

\[ 5\alpha^{p+q+r} \sqrt{5} = (\alpha^p \sqrt{5})(\alpha^q \sqrt{5})(\alpha^r \sqrt{5}). \]

Note that identities (2.2) – (2.4) are obtained from identity (2.1) by interchanging two indices from \( p, q \) and \( r \). \( \square \)

**Theorem 2.** The following identities hold for integers \( p, q, r, s \) and \( t \):

\[ F_{p+q-r}F_{t-s+r} + F_{p+q-r-1}F_{t-s+r-1} = F_{p-s}F_{t+q} + F_{p-s-1}F_{t+q-1}, \]  
(2.17)

\[ F_{p+q-r}L_{t-s+r} + F_{p+q-r-1}L_{t-s+r-1} = F_{p-s}L_{t+q} + F_{p-s-1}L_{t+q-1}, \]  
(2.18)

\[ L_{p+q-r}F_{t-s+r} + L_{p+q-r-1}F_{t-s+r-1} = L_{p-s}F_{t+q} + L_{p-s-1}F_{t+q-1}, \]  
(2.19)

and

\[ L_{p+q-r}L_{t-s+r} + L_{p+q-r-1}L_{t-s+r-1} = L_{p-s}L_{t+q} + L_{p-s-1}L_{t+q-1}. \]  
(2.20)
Proof. Identity (2.17) is proved by applying identity (1.6) to the identity

\[ \alpha^{p+q-r}\alpha^{t-s+r} = \alpha^{p-s}\alpha^{t+q}, \]

multiplying out the products and applying property P1. Identities (2.18) and (2.19) are derived by writing

\[ \alpha^{p+q-r}(\alpha^{t-s+r}\sqrt{5}) = \alpha^{p-s}(\alpha^{t+q}\sqrt{5}) \]

and

\[ (\alpha^{p+q-r}\sqrt{5})\alpha^{t-s+r} = (\alpha^{p-s}\sqrt{5})\alpha^{t+q}, \]

and applying identities (1.6) and (1.7) and property P1. Finally identity (2.20) is derived from

\[ (\alpha^{p+q-r}\sqrt{5})(\alpha^{t-s+r}\sqrt{5}) = (\alpha^{p-s}\sqrt{5})(\alpha^{t+q}\sqrt{5}). \]

\[ \Box \]

2.2 Summation identities

2.2.1 Binomial summation identities

Lemma 2. The following identities hold for positive integer \( n \) and arbitrary \( x \) and \( y \):

\[ \sum_{j=0}^{n} \binom{n}{j} y^j x^{n-j} = (x + y)^n, \quad (2.21) \]

\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x + y)^j x^{n-j} = (-1)^n y^n, \quad (2.22) \]

\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} y^j (x + y)^{n-j} = x^n, \quad (2.23) \]

\[ \sum_{j=0}^{n} \binom{n}{j} j y^{j-1} x^{n-j} = n(x + y)^{n-1}, \quad (2.24) \]

\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} j(x + y)^{j-1} x^{n-j} = (-1)^n n y^{n-1} \quad (2.25) \]

and

\[ \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} y^{j-1} j(x + y)^{n-j} = nx^{n-1}. \quad (2.26) \]

Proof. Setting \( z = 0 \) in the binomial identity

\[ \sum_{j=0}^{n} \binom{n}{j} y^j e^{jz} x^{n-j} = (x + ye^z)^n \quad (2.27) \]
The following identities hold for integers $k, t, s$ and positive integer $n$:

\[
\sum_{j=0}^{n} (-1)^{tj} \binom{n}{j} F_{k}^{j} F_{t}^{n-j} F_{(s+k)n-(t+k)j} = F_{t+k}^{n} F_{sn}^{n}, \tag{2.29}
\]

\[
\sum_{j=0}^{n} (-1)^{tj} \binom{n}{j} F_{k}^{j} F_{t}^{n-j} L_{(s+k)n-(t+k)j} = F_{t+k}^{n} L_{sn}^{n}, \tag{2.30}
\]

\[
\sum_{j=0}^{n} (-1)^{tj} \binom{n}{j} F_{k+t}^{j} F_{t}^{n-j} F_{(s+k)n-(t+k)j} = (-1)^{n(t+1)} F_{k}^{n} F_{n(s-t)}, \tag{2.31}
\]

\[
\sum_{j=0}^{n} (-1)^{tj} \binom{n}{j} F_{k+t}^{j} F_{t}^{n-j} L_{(s+k)n-(t+k)j} = (-1)^{n(t+1)} F_{k}^{n} L_{n(s-t)}, \tag{2.32}
\]

\[
\sum_{j=0}^{n} (-1)^{(t+1)j} \binom{n}{j} F_{k}^{j} F_{k+t}^{n-j} F_{sn-tj} = F_{t}^{n} F_{n(s+k)}, \tag{2.33}
\]

\[
\sum_{j=0}^{n} (-1)^{(t+1)j} \binom{n}{j} F_{k}^{j} F_{k+t}^{n-j} L_{sn-tj} = F_{t}^{n} L_{n(s+k)}, \tag{2.34}
\]

\[
(-1)^{t} \sum_{j=1}^{n} (-1)^{tj} \binom{n}{j} j F_{k}^{j-1} F_{t}^{n-j} F_{(k+s)n+t-s-(k+t)j} = n F_{k+t}^{n-1} F_{s(n-1)}, \tag{2.35}
\]

\[
(-1)^{t} \sum_{j=1}^{n} (-1)^{tj} \binom{n}{j} j F_{k}^{j-1} F_{t}^{n-j} L_{(k+s)n+t-s-(k+t)j} = n F_{k+t}^{n-1} L_{s(n-1)}, \tag{2.36}
\]

\[
\sum_{j=1}^{n} (-1)^{j} \binom{n}{j} j F_{k+t}^{j-1} F_{t}^{n-j} F_{(s+k)n-s-kj} = (-1)^{n(t+1)+t} n F_{k}^{n-1} F_{(s-t)(n-1)}, \tag{2.37}
\]

\[
\sum_{j=1}^{n} (-1)^{j} \binom{n}{j} j F_{k+t}^{j-1} F_{t}^{n-j} L_{(k+s)n-s-kj} = (-1)^{n(t+1)+t} n F_{k}^{n-1} L_{(s-t)(n-1)}, \tag{2.38}
\]

\[
(-1)^{(t+1)} \sum_{j=1}^{n} (-1)^{(t+1)j} \binom{n}{j} j F_{k+t}^{j-1} F_{t}^{n-j} F_{(n-1)+t-tj} = n F_{t}^{n-1} F_{s(k)(n-1)}, \tag{2.39}
\]

\[\text{and}\]

\[
(-1)^{(t+1)} \sum_{j=1}^{n} (-1)^{(t+1)j} \binom{n}{j} j F_{k+t}^{j-1} F_{t}^{n-j} L_{(n-1)+t-tj} = n F_{t}^{n-1} L_{s(k)(n-1)}. \tag{2.40}
\]
Proof. Choosing $x = \alpha^{s+k}F_t$ and $y = (-1)^t\alpha^{s-t}F_k$ in identity (2.21) and taking note of identity (1.14), we have

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} F_k^j F_t^{n-j} \alpha^{(s+k)n-(t+k)j} = F_{k+t}^n \alpha^{ns}.
\] (2.41)

Application of identity (1.6) and property P1 to (2.41) produces identity (2.29). To prove (2.30), multiply through identity (2.41) by $\sqrt{5}$ to obtain

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} F_k^j F_t^{n-j} \{\alpha^{(s+k)n-(t+k)j} \sqrt{5}\} = F_{k+t}^n \{\alpha^{ns} \sqrt{5}\}.
\] (2.42)

Use of identity (1.7) and property P1 in identity (2.42) gives identity (2.30). Identities (2.31) – (2.40) are derived in a similar fashion.

Lemma 3. The following identities hold true for integer $r$, non-negative integer $n$, and arbitrary $x$ and $y$:

\[
\sum_{j=0}^{n} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} (xy)^{r-j} (x-y)^{2j+1} = x^{r+n+1}y^{r-n} - y^{r+n+1}x^{r-n},
\] (2.43)

\[
\sum_{j=0}^{n} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} (x(x-y))^{r-j} y^{2j+1} = x^{r+n+1}(x-y)^{r-n} - (x-y)^{r+n+1}x^{r-n}
\] (2.44)

and

\[
\sum_{j=0}^{n} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} (y(y-x))^{r-j} x^{2j+1} = y^{r+n+1}(y-x)^{r-n} - (y-x)^{r+n+1}y^{r-n}.
\] (2.45)

Proof. Jennings [6, Lemma (i)] derived an identity equivalent to the following:

\[
\sum_{j=0}^{n} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} \left(\frac{z^2 - 1}{z}\right)^{2j} = \frac{z^2z^{2n} - z^{-2n}}{z^2 - 1}.
\] (2.46)

Setting $z^2 = x/y$ in the above identity and clearing fractions gives identity (2.43). Identity (2.44) is obtained by replacing $y$ with $x-y$ in identity (2.43). Identity (2.45) is obtained by interchanging $x$ with $y$ in identity (2.44).

Theorem 4. The following identities hold for non-negative integer $n$ and integers $s$, $k$, $r$ and $t$:

\[
(-1)^t \sum_{j=0}^{n} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} (F_{k+t}F_t)^{r-j} F_k^{2j+1} F_{(2s+k)+s-(t+k)j} = F_{k+t}^{r+n+1} F_n^{r-n} F_{s(r+n+1)+(s+k)(r-n)} - F_t^{r+n+1} F_{k+t}^{r-n} F_{(s+k)(r+n+1)+s(r-n)}.
\] (2.47)

\[
(-1)^t \sum_{j=0}^{n} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} (F_{k+t}F_t)^{r-j} F_k^{2j+1} L_{r(2s+k)+s-(t+k)j} = F_{k+t}^{r+n+1} F_n^{r-n} L_{s(r+n+1)+(s+k)(r-n)} - F_t^{r+n+1} F_{k+t}^{r-n} L_{(s+k)(r+n+1)+s(r-n)}.
\] (2.48)
\[
\sum_{j=0}^{n} (-1)^{tj} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} (F_{k+t}F_{t})^{r-j} F_{t}^{2j+1} F_{r(2s-t)+s+k+(2k+t)j} \\
= (-1)^{tn} F_{k+t}^{r+n+1} F_{s(2r+1)-(r-n)} - (-1)^{tn+t} F_{k+t}^{r+n+1} F_{s(2r+1)-(r+n+1)} \\
\tag{2.49}
\]

\[
\sum_{j=0}^{n} (-1)^{tj} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} (F_{k+t}F_{t})^{r-j} F_{k}^{2j+1} L_{r(2s-t)+s+k+(2k+t)j} \\
= (-1)^{tn} F_{k+t}^{r+n+1} F_{s(2r+1)-(r-n)} - (-1)^{tn+t} F_{k+t}^{r-n} L_{s(2r+1)-(r+n+1)} \\
\tag{2.50}
\]

\[
\sum_{j=0}^{n} (-1)^{(t-1)j} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} (F_{t}F_{k})^{r-j} F_{k+t}^{2j+1} F_{s(2r+1)+(k-t)r-(k-t)} \\
= (-1)^{(t-1)n} F_{t}^{r+n+1} F_{k}^{r-n} F_{s(2r+1)+k(r+n+1)+t(n-r)} \\
- (-1)^{(t-1)(n+1)} F_{t}^{r+n+1} F_{k}^{r-n} F_{s(2r+1)-(k(n-r)-(r+n+1)} \\
\tag{2.51}
\]

and

\[
\sum_{j=0}^{n} (-1)^{(t-1)j} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} (F_{t}F_{k})^{r-j} F_{k+t}^{2j+1} L_{s(2r+1)+(k-t)r-(k-t)} \\
= (-1)^{(t-1)n} F_{t}^{r+n+1} F_{k}^{r-n} L_{s(2r+1)+k(r+n+1)+t(n-r)} \\
- (-1)^{(t-1)(n+1)} F_{t}^{r+n+1} F_{k}^{r-n} L_{s(2r+1)-(k(n-r)-(r+n+1)} \\
\tag{2.52}
\]

**Proof.** Each of identities (2.47) – (2.52) is proved by setting \( x = \alpha^{s+t}F_{t} \) and \( y = (-1)^{t} \alpha^{s-t}F_{k} \) in the identities of Lemma 3 and taking note of identity (1.14) while making use of identities (1.6) and (1.7) and property P1. \( \square \)

### 2.2.2 Summation identities not involving binomial coefficients

**Lemma 4** ([1, Lemma 1]). Let \((X_t)\) and \((Y_t)\) be any two sequences such that \(X_t\) and \(Y_t\), \(t \in \mathbb{Z}\), are connected by a three-term recurrence relation \(hX_t = f_1X_{t-a} + f_2Y_{t-b}\), where \(h\), \(f_1\) and \(f_2\) are arbitrary non-vanishing complex functions, not dependent on \(t\), and \(a\) and \(b\) are integers. Then, the following identity holds for integer \(n\):

\[
f_2 \sum_{j=0}^{n} f_1^{n-j} h^j Y_{t-na-b+aj} = h^{n+1} X_t - f_1^{n+1} X_{t-(n+1)a}.
\]

**Theorem 5.** The following identities hold for integers \(n, k, s\) and \(t\):

\[
(-1)^{nk+t-1} F_k \sum_{j=0}^{n} (-1)^{kj} F_{n(s-k)+s-t+2kj} = F_k F_{s(k)(n+1)} - F_{t(n+1)k} F_{s(n+1)} \\
\tag{2.53}
\]
\begin{align}
(-1)^{nk+t-1} F_k \sum_{j=0}^{n} (-1)^{kj} L_{n-k+s-t+2kj} &= F_t L_{(s+k)(n+1)} - F_{t+(n+1)k} L_{s(n+1)}; \quad (2.54) \\
(-1)^{nt+k-1} F_t \sum_{j=0}^{n} (-1)^{tj} F_{n(s-t)+s-k+2tj} &= F_k F_{(s+t)(n+1)} - F_{k+(n+1)t} F_{s(n+1)}; \quad (2.55) \\
(-1)^{nt+k-1} F_t \sum_{j=0}^{n} (-1)^{tj} L_{n(s-t)+s-k+2tj} &= F_k L_{(s+t)(n+1)} - F_{k+(n+1)t} L_{s(n+1)}; \quad (2.56) \\
(-1)^{ns+t-1} F_s \sum_{j=0}^{n} (-1)^{sj} F_{n(k-s)+k-t+2sj} &= F_t F_{(k+s)(n+1)} - F_{t+(n+1)s} F_{k(n+1)} \quad (2.57) \\
\text{and} \\
(-1)^{ns+t-1} F_s \sum_{j=0}^{n} (-1)^{sj} L_{n(k-s)+k-t+2sj} &= F_t L_{(k+s)(n+1)} - F_{t+(n+1)s} L_{k(n+1)}. \quad (2.58)
\end{align}

Proof. Write identity (1.14) as \( \alpha^{s+k} F_t = \alpha^s F_{t+k} + (-1)^{t-1} \alpha^{s-t} F_k \) and identify \( h = \alpha^{s+k} \), \( f_1 = \alpha^s \), \( f_2 = F_k \), \( a = -k \), \( b = 0 \), \( X_t = F_t \) and \( Y_t = (-1)^{t-1} \alpha^{s-t} \) in Lemma 4. Application of identity (1.6) to the resulting summation identity yields identity (2.53). Identity (2.54) is obtained by multiplying the \( \alpha \)-sum by \( \sqrt{5} \) and using identity (1.7). Identities (2.55) – (2.58) are obtained from identities (2.53) and (2.54) by interchanging \( k \) and \( t \); and \( s \) and \( k \), in turn, since identity (1.14) remains unchanged under these operations.

Lemma 5. The following identities hold for integers \( r \) and \( n \) and arbitrary \( x \) and \( y \):

\begin{align}
(x - y) \sum_{j=0}^{n} y^{-j} x^j &= y^{-n} x^{n+1} - y^{r+1}, \quad (2.59) \\
x \sum_{j=0}^{n} y^{-j} (x + y)^j &= y^{-n} (x + y)^{n+1} - y^{r+1} \quad (2.60) \\
\text{and} \\
(x - y) \sum_{j=0}^{n} x^{-j} y^j &= x^{r+1} - x^{-n} y^{n+1}. \quad (2.61)
\end{align}

Proof. Identity (2.59) is obtained by replacing \( x \) with \( x/y \) in identity (1.54). Identity (2.60) is obtained by replacing \( x \) with \( x + y \) in identity (2.59). Finally, identity (2.61) is obtained by interchanging \( x \) and \( y \) in identity (2.59).

Theorem 6. The following identities hold for integers \( r \), \( n \), \( s \), \( k \) and \( t \):

\begin{align}
(-1)^t F_k \sum_{j=0}^{n} F^{r-j}_{k+t} F^{-j}_{r(s+k)+s-t-kj} &= F_{r-n} F_{k+t}^{n+1} F_{s(r+1)+k(r-n)} - F_{r+1} F_{s(k)(r+1)}; \quad (2.62) \\
(-1)^t F_k \sum_{j=0}^{n} F^{r-j}_{k+t} L^{r-j}_{r(s+k)+s-t-kj} &= F_{r-n} F_{k+t}^{n+1} L_{s(r+1)+k(r-n)} - F_{r+1} L_{s(k)(r+1)}. \quad (2.63)
\end{align}
\begin{equation}
(-1)^t F_r \sum_{j=0}^{n} (-1)^j F_k^{r-j} F_r^{j} F_{r-s-t+j} = F_{k+1}^{r-1} F_{r+s-k+1} - (-1)^{t(r+1)} F_{k+1}^{r+1} F_{r+s-k+1}.
\end{equation}

(2.64)

\begin{equation}
(-1)^t F_t \sum_{j=0}^{n} (-1)^j F_k^{r-j} F_r^{j} F_{r-s-t+j} = F_{k+1}^{r-1} F_{r+s-k+1} - (-1)^{t(r+1)} F_{k+1}^{r+1} F_{r+s-k+1}.
\end{equation}

(2.65)

\begin{equation}
(-1)^t F_k \sum_{j=0}^{n} F_k^{r-j} F_r^{j} F_{s(t-s-t+k-j)} = F_{k+1}^{r+1} F_{s(t-s-t+k-j)} - F_{k+1}^{r-n} F_{s(t-s-t+k-j)}.
\end{equation}

(2.66)

and

\begin{equation}
(-1)^t F_k \sum_{j=0}^{n} F_k^{r-j} F_r^{j} F_{s(t-s-t+k-j)} = F_{k+1}^{r+1} F_{s(t-s-t+k-j)} - F_{k+1}^{r-n} F_{s(t-s-t+k-j)}.
\end{equation}

(2.67)

\textbf{Proof.} Identities (2.62) and (2.63) and identities (2.66) and (2.67) are obtained by setting \( x = \alpha^p F_{k+t} \) and \( y = \alpha^q F_{k} \) in identities (2.59) and (2.61) while taking note of identity (1.14). Identities (2.64) and (2.65) are derived by setting \( x = \alpha^{s+k} F_{t} \) and \( y = (-1)^t \alpha^{s-t} F_{k} \) in identity (2.66).

\textbf{Theorem 7.} The following identities hold for integers \( p, q \) and \( n \):

\[
\sum_{j=0}^{n} j F_{p}^{j} F_{p(n+1)+q-1} - (F_{k+1}^{r-1}) F_{p(n+1)+q} = \frac{L_{p} \alpha - (F_{p} - 1) F_{p(n+1)+q}}{L_{p} - 1 + (-1)^{p-1}} + \frac{(F_{p} - 2 F_{p})(F_{p(n+2)+q-1} - F_{p+q-1})}{(F_{p} - 2 F_{p}) (F_{p+1} - 2 F_{p+1} + 1) - (F_{p} - 2 F_{p})^2} \tag{2.68}
\]

\[
\sum_{j=0}^{n} j F_{p}^{j} F_{p(n+1)+q-1} - (F_{p} - 1) F_{p(n+1)+q} = \frac{L_{p} \alpha - (F_{p} - 1) F_{p(n+1)+q}}{L_{p} - 1 + (-1)^{p-1}} + \frac{(F_{p} - 2 F_{p})(L_{p}(n+2)+q-1 - L_{p+q-1})}{(F_{p} - 2 F_{p}) (F_{p+1} - 2 F_{p+1} + 1) - (F_{p} - 2 F_{p})^2} - \frac{(F_{p} - 2 F_{p}) (F_{p+1} - 2 F_{p+1} + 1) - (F_{p} - 2 F_{p})^2}{L_{p}(n+2)+q - L_{p+q}} \tag{2.69}
\]

\textbf{Proof.} Differentiating identity (1.54) with respect to \( x \) and multiplying through by \( x \) gives

\[
\sum_{j=0}^{n} j x^j = (n + 1) \frac{x^{n+1}}{x-1} - \frac{x^{n+2} - x}{(x-1)^2}. \tag{2.70}
\]

Setting \( x = \alpha^p \) and multiplying through by \( \alpha^q \) produces

\[
\sum_{j=0}^{n} j \alpha^{p(j+q)} = \frac{(n + 1) \alpha^{p(n+1)+q}}{\alpha^p - 1} + \frac{\alpha^{p(n+2)+q} - \alpha^{p+q}}{2 \alpha^p - \alpha^{2p} - 1}, \tag{2.71}
\]

from which the results follow through the use of identities (1.6) and (1.7) and properties P1 and P5.
3 Concluding remarks

A Fibonacci-like sequence \((G_k)_{k \in \mathbb{Z}}\) is one whose initial terms \(G_0\) and \(G_1\) are given integers, not both zero, and

\[
G_k = G_{k-1} + G_{k-2}, \quad G_0 = G_{-1} + G_{-2} - G_{-3}.
\]  

\begin{equation}
G_k = G_{k-1} + G_{k-2}, \quad G_0 = G_{-1} + G_{-2} - G_{-3}.
\end{equation}

\(3.1\)

Identities (1.4) and (1.5) show that the sequences \((A_k)_{k \in \mathbb{Z}}\) and \((B_k)_{k \in \mathbb{Z}}\), where \(A_k = \alpha^k\) and \(B_k = \beta^k\) are Fibonacci-like, with respective initial terms \(A_0 = \alpha^0 = 1, A_1 = \alpha\) and \(B_0 = \beta^0 = 1, B_1 = \beta = -1/\alpha\). The identity

\[
F_{s-t}G_{k+m} = F_{m-t}G_{k+s} + (-1)^{s+t+1}F_{m-s}G_{k+t},
\]

\(3.2\)

can therefore be written for the golden ratio and its inverse as

\[
F_{s-t}\alpha^{k+m} = F_{m-t}\alpha^{k+s} + (-1)^{s+t+1}F_{m-s}\alpha^{k+t}
\]

\begin{equation}
F_{s-t}\alpha^{k+m} = F_{m-t}\alpha^{k+s} + (-1)^{s+t+1}F_{m-s}\alpha^{k+t}
\end{equation}

\(3.3\)

and

\[
F_{s-t}\beta^{k+m} = F_{m-t}\beta^{k+s} + (-1)^{s+t+1}F_{m-s}\beta^{k+t}.
\]

\begin{equation}
F_{s-t}\beta^{k+m} = F_{m-t}\beta^{k+s} + (-1)^{s+t+1}F_{m-s}\beta^{k+t}.
\end{equation}

\(3.4\)

Proceeding as in previous calculations, many identities can be derived using identities (3.3) and (3.4). For example, setting \(x = F_{m-t}\alpha^{k+s}\) and \(y = (-1)^{s+t}F_{m-s}\alpha^{k+t}\) in the binomial identity

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} y^j x^{n-j} = (x - y)^n
\]

and multiplying through by \(\alpha^p\) produces

\[
\sum_{j=0}^{n} (-1)^{(s+t+1)j} \binom{n}{j} F_{m-s}^j F_{m-t}^{n-j} \alpha^{(k+t)j + (k+s)(n-j)+p} = F_{s-t}^n \alpha^{(k+m)n+p},
\]

from which we find

\[
\sum_{j=0}^{n} (-1)^{(s+t+1)j} \binom{n}{j} F_{m-s}^j F_{m-t}^{n-j} F_{(s+k)n+(t-s)} = F_{s-t}^n F_{(k+m)n+p}
\]

and

\[
\sum_{j=0}^{n} (-1)^{(s+t+1)j} \binom{n}{j} F_{m-s}^j F_{m-t}^{n-j} L_{(s+k)n+(t-s)} = F_{s-t}^n L_{(k+m)n+p}.
\]

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