DOUBLING AND POINCARÉ INEQUALITIES FOR UNIFORMIZED MEASURES ON GROMOV HYPERBOLIC SPACES

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Abstract. We generalize the recent results of Björn-Björn-Shanmugalingam [2] concerning how measures transform under the uniformization procedure of Bonk-Heinonen-Koskela for Gromov hyperbolic spaces [4] by showing that these results also hold in the setting of uniformizing Gromov hyperbolic spaces by Busemann functions that we introduced in [9]. In particular uniformly local doubling and uniformly local Poincaré inequalities for the starting measure transform into global doubling and global Poincaré inequalities for the uniformized measure. We then show in the setting of uniformizations of universal covers of closed negatively curved Riemannian manifolds equipped with the Riemannian measure that one can obtain sharp ranges of exponents for the uniformized measure to be doubling and satisfy a 1-Poincaré inequality. Lastly we introduce the procedure of uniform inversion for uniform metric spaces, and show that both the doubling property and the $p$-Poincaré inequality are preserved by uniform inversion for any $p \geq 1$.

1. Introduction

Broadly speaking our work in this paper has three closely related objectives. The primary objective is to generalize the recent results of Björn-Björn-Shanmugalingam [2] concerning how measures transform under the uniformization procedure of Bonk-Heinonen-Koskela for Gromov hyperbolic spaces [4]. We will consider how measures transform under the generalization of this uniformization procedure that we introduced in [9]. As in [2], we will show that uniformizing these measures upgrades uniformly local doubling properties and uniformly local Poincaré inequalities to global doubling and global Poincaré inequalities for the uniformized space. Our results allow us to construct a number of interesting new unbounded metric measure spaces supporting Poincaré inequalities. A particularly important example is uniformizations of hyperbolic fillings of unbounded metric spaces, which play a key role in our followup work [7] concerning extension and trace theorems for Besov spaces on noncompact doubling metric measure spaces.

Our second objective is to show that in the presence of a cocompact isometric discrete group action on the Gromov hyperbolic space we start with, it is often possible to apply the theorems of [2] to a much wider range of exponents than the ones considered there, leading in several cases to ranges that we can verify are sharp due to well-known results on Patterson-Sullivan measures in these contexts. In Theorem 1.4 we tie this threshold to the volume growth entropy of universal covers of closed negatively curved Riemannian manifolds. In Remark 3.10 we briefly explain how Patterson-Sullivan measures on the Gromov boundary arise as renormalized limits of the uniformized measures considered here and in [2].

The final topic that we consider is procedures for transforming metric measure spaces that preserve the doubling property and $p$-Poincaré inequalities for a given $p \geq 1$. Oftentimes in analysis on metric spaces it is preferable to work on either a bounded or an unbounded space depending on the nature of the question under consideration. Thus there has been a significant amount of interest in procedures for passing back and forth between bounded
and unbounded spaces while retaining as much information as possible. In the abstract metric space setting transformations between bounded and unbounded spaces can be realized through inversions [6], which generalize the classical notion of Möbius inversions in the complex plane. It was shown by Li-Shanmugalingam [16] that measures can be transformed under these inversions in such a way that a number of desirable properties can be preserved. However they were not able to obtain unconditional invariance of Poincaré inequalities under inversions, and they in fact showed that Poincaré inequalities cannot always be preserved [16, Example 3.3.13]. In the final section of this paper we introduce an alternative inversion operation based on uniformization that is specialized to uniform metric spaces. With the additional assistance of some results from [2] we will show that this operation preserves Poincaré inequalities. A more in-depth discussion of this is given in Section 5.

We now introduce the setting of [9] and [2] in order to state our main theorems. For precise definitions regarding general notions in Gromov hyperbolic spaces we refer to Section ??, while for a more detailed treatment of the uniformization procedure discussed in this introduction we refer to Section 2. We begin with a proper geodesic \( \rho \) denoting the line integral of \( \gamma \) along \( \rho \) joining \( x, y \). For a curve \( \rho \) denote the line integral of \( \gamma \) along \( \rho \) if and only if for all functions \( b \) of \( \gamma \) from one another. For a given geodesic ray \( \gamma : [0, \infty) \to X \) up to the equivalence relation that two geodesic rays are equivalent if they are at bounded distance from one another. For a given geodesic ray \( \gamma : [0, \infty) \to X \), the Busemann function \( b_\gamma : X \to \mathbb{R} \) associated to \( \gamma \) is defined by the limit
\[
(1.1) \quad b_\gamma(x) = \lim_{t \to \infty} d(\gamma(t), x) - t.
\]
We then define
\[
(1.2) \quad B(X) = \{b_\gamma + s : \gamma \text{ a geodesic ray in } X, s \in \mathbb{R}\},
\]
and refer to any function \( b \in B(X) \) as a Busemann function on \( X \). See [9, (1.4-1.5)] for further details. The Busemann functions \( b \in B(X) \) are all 1-Lipschitz functions on \( X \). For a Busemann function \( b \) of the form \( b = b_\gamma + s \) for some \( s \in \mathbb{R} \), we define the endpoint \( \omega \in \partial X \) of \( \gamma \) to be the basepoint of \( b \) and say that \( b \) is based at \( \omega \).

For \( z \in X \) we define \( b_\gamma(z) = d(x, z) \) to be the distance from \( z \). We augment the set of Busemann functions with the set of translates of distance functions on \( X \),
\[
(1.3) \quad D(X) = \{b_\gamma + s : z \in X, s \in \mathbb{R}\}.
\]
For \( b \in D(X) \) with \( b = b_z + s \) for some \( z \in X \) and \( s \in \mathbb{R} \) we then refer to \( z \) as the basepoint of \( b \), in analogy to the case of Busemann functions. We write \( \mathcal{B}(X) = D(X) \cup B(X) \). Then all functions \( b \in \mathcal{B}(X) \) are 1-Lipschitz.

For each \( b \in \mathcal{B}(X) \) and each \( \varepsilon > 0 \) we define a positive density \( \rho_{\varepsilon, b} \) on \( X \) by
\[
\rho_{\varepsilon, b}(x) = e^{-\varepsilon b(x)}.
\]
For a curve \( \gamma \) in \( X \) we let
\[
\ell_{\varepsilon, b}(\gamma) = \int_\gamma \rho_{\varepsilon, b} ds,
\]
denote the line integral of \( \rho_{\varepsilon, b} \) along \( \gamma \). We let \( (X_{\varepsilon, b}, d_{\varepsilon, b}) \) denote the metric space obtained by conformally deforming \( X \) by the density \( \rho_{\varepsilon, b} \), i.e., defining the new distance \( d_{\varepsilon, b} \) for \( x, y \in X \) by
\[
(1.4) \quad d_{\varepsilon, b}(x, y) = \inf \ell_{\varepsilon, b}(\gamma),
\]
with the infimum taken over all curves \( \gamma \) joining \( x \) to \( y \). The metric space \( X_{\varepsilon, b} \) is bounded if and only if \( b \in D(X) \) [9, Proposition 4.4].
Our main results concern the corresponding effect of this conformal deformation on measures on \( X \). We will require the following key definition. For the entirety of this paper a metric measure space is a triple \( (X, d, \mu) \) consisting of a metric space \( (X, d) \) equipped with a Borel measure \( \mu \).

**Definition 1.1.** Let \( (X, d, \mu) \) be a metric measure space and let \( B_X(x, r) \) denote the open ball of radius \( r > 0 \) centered at \( x \in X \). The measure \( \mu \) is doubling if there is a constant \( C_\mu \geq 1 \) such that for every \( x \in X \) and \( r > 0 \) we have
\[
\mu(B_X(x, 2r)) \leq C_\mu \mu(B_X(x, r)).
\]
If the inequality (1.5) only holds for balls of radius at most \( R_0 \) then we will say that \( \mu \) is doubling on balls of radius at most \( R_0 \). We will alternatively say that \( \mu \) is uniformly locally doubling if there is an \( R_0 > 0 \) such that \( \mu \) is doubling on balls of radius at most \( R_0 \).

We let \( \mu \) be a given Borel measure on \( X \) that is doubling on balls of radius at most \( R_0 \) with constant \( C_\mu \). For each \( \beta > 0 \) we define a measure \( \mu_{\beta,b} \) on \( X \) by
\[
d\mu_{\beta,b}(x) = \rho_{\beta,b}(x) d\mu(x) = e^{-\beta b(x)} d\mu(x),
\]
for \( x \in X \). We will consider \( \mu_{\beta,b} \) as a measure on \( X_{\varepsilon,b} \) and extend it to the completion \( \bar{X}_{\varepsilon,b} \) by setting \( \mu_{\beta,b}(\partial X_{\varepsilon,b}) = 0 \), where \( \partial X_{\varepsilon,b} = \bar{X}_{\varepsilon,b} \setminus X_{\varepsilon,b} \) denotes the complement of \( X_{\varepsilon,b} \) inside its completion. Our first theorem shows that there is a threshold \( \beta_0 \) depending only on \( R_0 \) and \( C_\mu \) such that if \( \beta \geq \beta_0 \) then the uniformly locally doubling measure \( \mu \) on \( X \) transforms into a measure \( \mu_{\beta,b} \) on \( X_{\varepsilon,b} \) that is doubling at all scales.

Our theorem requires two additional hypotheses on \( X \) and the density \( \rho_{\varepsilon,b} \), which we briefly summarize here. We recall that we are assuming that \( X \) is a proper geodesic \( \delta \)-hyperbolic space and that \( b \in \mathcal{B}(X) \). The first is that \( X \) is \( K \)-roughly starlike from the basepoint \( \omega \in X \cup \partial X \) of the chosen function \( b \in \mathcal{B}(X) \) for a given constant \( K \geq 0 \). Roughly speaking this condition requires that each point \( x \in X \) lies within distance \( K \) of a geodesic ray or line starting from \( \omega \). We defer a precise definition to Section 2 as one must distinguish the cases \( \omega \in X \) and \( \omega \in \partial X \).

The second requirement is that \( \rho_{\varepsilon,b} \) is a Gehring-Hayman density for \( X \) with constant \( M \geq 1 \) (abbreviated as GH-density). This means that for each \( x, y \in X \) and each geodesic \( \gamma \) joining \( x \) to \( y \) we have
\[
d_{\varepsilon,b}(x, y) \leq M t_{\varepsilon,b}(\gamma).
\]
In other words, \( \rho_{\varepsilon,b} \) is a GH-density if geodesics in \( X \) minimize distance in \( X_{\varepsilon,b} \) up to a universal multiplicative constant. This requirement is not as stringent as it first appears; by the work of Bonk-Heinonen-Koskela [4 Theorem 5.1] there is an \( \varepsilon_0 = \varepsilon_0(\delta) > 0 \) depending only on \( \delta \) such that \( \rho_{\varepsilon,b} \) is a GH-density with constant \( M = 20 \) for any \( b \in \mathcal{B}(X) \) and any \( 0 < \varepsilon \leq \varepsilon_0 \). For CAT(\(-1\)) spaces one may use a threshold \( \varepsilon_0 = 1 \) instead [9 Theorem 1.10].

The rough starlikeness hypothesis and the GH-density hypothesis together guarantee that the conformal deformation \( X_{\varepsilon,b} \) has a number of nice properties by our previous results in [9] that are summarized in Section 2 and are used heavily in the proofs of our theorems.

We then have the following theorem; below “the data” refers to the collection of parameters \( \delta, K, \varepsilon, M, \beta, R_0, \) and \( C_\mu \).

**Theorem 1.2.** There is \( \beta_0 = \beta_0(R_0, C_\mu) > 0 \) such that if \( \beta \geq \beta_0 \) then the measure \( \mu_{\beta,b} \) on \( X_{\varepsilon,b} \) is doubling with constant \( C_{\mu,\beta} \) depending only on the data.

This theorem generalizes the main result of Björn-Björn-Shanmugalingam [22 Theorem 1.1] to the setting in which the uniformization \( X_{\varepsilon,b} \) is potentially unbounded, i.e., the
case $b \in B(X)$. The case $b \in D(X)$ is more or less already contained in [2, Theorem 1.1], with the exception that they only consider the original parameter range $0 < \varepsilon \leq \varepsilon_0$ of Bonk-Heinonen-Koskela. An explicit value $\beta_0 = \frac{17 \log C_{\text{ext}}}{3R_0}$ is given in [2, Theorem 1.1]; a similar explicit estimate for $\beta_0$ can be extracted from our proof. In the process of proving Theorem 1.2 we formulate a useful criterion (Proposition 3.3) for checking that $\mu_{\beta,b}$ is doubling on $\bar{X}_{\varepsilon,b}$. This criterion will be used to verify Theorem 1.4 below as well as some key claims in our followup work [7].

The doubling property for $\mu_{\beta,b}$ on $\bar{X}_{\varepsilon,b}$ is the key property needed to transform uniformly local $p$-Poincaré inequalities on $X$ into global $p$-Poincaré inequalities on $X_{\varepsilon,b}$. The following theorem makes this claim precise. We refer to Section 3 for the precise definitions of uniformly local $p$-Poincaré inequalities and global $p$-Poincaré inequalities. We retain the same hypotheses regarding rough starlikeness and the GH-density property that we assumed in Theorem 1.2. Below “the data” refers to the parameters $\delta, K, \varepsilon, M, \beta$, the doubling constant $C_{\mu_{\beta,b}}$ for $\mu_{\beta,b}$, the power $p$, the radius and the constants $C_{\text{PI}}$ and $\lambda$ appearing in the uniformly local $p$-Poincaré inequality (4.2) as well as the local doubling radius and constant $C_{\mu}$ for $\mu$.

**Theorem 1.3.** Suppose that the metric measure space $(X, d, \mu)$ is uniformly locally doubling and supports a uniformly local $p$-Poincaré inequality for some $p \geq 1$. Suppose further that for a given $\beta > 0$ we have that $\mu_{\beta,b}$ is doubling on $X_{\varepsilon,b}$ with constant $C_{\mu_{\beta,b}}$. Then the metric measure spaces $(X_{\varepsilon,b}, d_{\varepsilon,b}, \mu_{\beta,b})$ and $(\bar{X}_{\varepsilon,b}, d_{\varepsilon,b}, \mu_{\beta,b})$ each support a $p$-Poincaré inequality with constant $C_{\text{PI}}^*$ depending only on the data.

By combining Theorems 1.2 and 1.3 we see that if we assume $(X, d, \mu)$ is uniformly locally doubling and supports a uniformly local $p$-Poincaré inequality then for $\beta \geq \beta_0$ we always have that $(X_{\varepsilon,b}, d_{\varepsilon,b}, \mu_{\beta,b})$ is doubling and supports a $p$-Poincaré inequality, and the same is true with $\bar{X}_{\varepsilon,b}$ replacing $X_{\varepsilon,b}$. For $b \in D(X)$ Theorem 1.3 essentially follows directly from [2, Theorem 1.1] and its proof. For Busemann functions $b \in B(X)$ our uniformization construction in [9] is designed such that minimal modifications to the proofs in [2] are required. We emphasize that Theorem 1.3 does not require us to restrict to the range $\beta \geq \beta_0$ considered in Theorem 3.3; it only requires that $\mu_{\beta,b}$ is doubling on $X_{\varepsilon,b}$.

As indicated previously, when $X$ comes equipped with a cocompact discrete isometric group action it is possible to significantly improve Theorem 1.2 by obtaining a better, often sharp threshold $\beta_0$ for $\mu_{\beta,b}$ to be doubling. This is the content of Theorem 3.9 in Section 3. We highlight here an interesting corollary of this theorem that illustrates the power of this method.

We consider a complete simply connected $n$-dimensional Riemannian manifold $X$ with sectional curvatures $\leq -1$ together with a cocompact discrete isometric action of a group $\Gamma$ on $X$. We let $\mu$ denote the Riemannian volume on $X$, which is $\Gamma$-invariant. The *volume growth entropy* of $X$ is given by the limit for any $x \in X$,

$$h_X = \lim_{R \to \infty} \frac{\log \mu(B_X(x,R))}{R}.$$  

For the existence of this limit see [17]. The quantity $h_X$ shows up in many places, for instance it is also equal to the topological entropy of the geodesic flow on the unit tangent bundle of the quotient of $X$ by $\Gamma$ [17]. The constants in Theorem 1.4 are uniform in the sense that they do not depend on the choice of function $b \in B(X)$.

**Theorem 1.4.** For each $\beta > h_X$ the metric measure spaces $(X_{1,b}, d_{1,b}, \mu_{\beta,b})$ and $(\bar{X}_{1,b}, d_{1,b}, \mu_{\beta,b})$ for $b \in B(X)$ are doubling and support a $1$-Poincaré inequality with uniform constants.
In Remark 5.10 we explain why the threshold $h_X$ is sharp. The constants in Theorem 1.4 are uniform in the sense that they do not depend on the choice of $b \in \bar{B}(X)$, although they will depend on the choice of exponent $\beta > h_X$.

Our results regarding preservation of Poincaré inequalities for uniform metric spaces are proved in Section 5. These results follow formally by combining Theorems 1.2 and 1.3 above with results in [8] and [2]; since the initial setup is quite different from that of our theorems above we have isolated the discussion of those results to Section 5. The rest of the paper is structured as follows: in Section 2 we review some results from our previous work regarding uniformizing Gromov hyperbolic spaces and extend some results from [2] to the setting of uniformizing by Busemann functions. In Section 3 we analyze the doubling properties of the uniformized measure 1.6 and prove Theorem 1.2. Lastly in Section 4 we prove Theorems 1.3 and 1.4.

2. Uniformization

2.1. Definitions. Let $X$ be a set and let $f$, $g$ be real-valued functions defined on $X$. For $c \geq 0$ we will write $f \doteq_c g$ if

$$|f(x) - g(x)| \leq c,$$

for all $x \in X$. If the exact value of the constant $c$ is not important or implied by context we will often just write $f \doteq g$. The relation $f \doteq g$ will sometimes be referred to as a rough equality between $f$ and $g$. Similarly for $C \geq 1$ and functions $f, g : X \to (0, \infty)$, we will write $f \asymp_C g$ if for all $x \in X$,

$$C^{-1}g(x) \leq f(x) \leq Cg(x).$$

We will write $f \asymp g$ if the value of $C$ is implied by context. We will write $f \lesssim_C g$ if $f(x) \leq Cg(x)$ for all $x \in X$ and $f \gtrsim_C g$ if $f(x) \geq C^{-1}g(x)$ for $x \in X$. Thus $f \asymp_C g$ if and only if $f \lesssim_C g$ and $f \gtrsim_C g$. As with the other notation, we will drop the constant $C$ and just write $f \lesssim g$ or $f \gtrsim g$ if the value of $C$ is implied by context. We will generally stick to the convention of using $c \geq 0$ for additive constants and $C \geq 1$ for multiplicative constants. To indicate on what parameters – such as $\delta$ – the constants depend on we will write $c = c(\delta)$, etc. At the beginning of each section we will indicate on what parameters the implied constants of the inequalities $\lesssim$ and $\gtrsim$, the comparisons $\asymp$, and the rough equalities $\doteq$ are allowed to depend. We will often reiterate these conditions for emphasis.

For a metric space $(X, d)$ we will write $B_X(x, r) = \{y \in X : d(x, y) < r\}$ for the open ball of radius $r > 0$ centered at a point $x \in X$. We will write $\bar{B}_X(x, r) = \{y \in X : d(x, y) \leq r\}$ for the closed ball of radius $r > 0$ centered at $x$. We note that the inclusion $\bar{B}_X(x, r) \subset B_X(x, r)$ of the closure of the open ball into the closed ball can be strict in general. By convention all balls $B \subset X$ are considered to have a fixed center and radius, even though it may be the case that we have $B_X(x, r) = B_X(x', r')$ as sets for some $x \neq x'$, $r \neq r'$. All balls $B \subset X$ are also considered to be open balls unless otherwise specified. We will write $r(B)$ for the radius of a ball $B$. For a ball $B = B_X(x, r)$ in $X$ and a constant $c > 0$ we write $cB = B_X(x, cr)$ for the corresponding ball with radius scaled by $c$. For a subset $E \subset X$ we write $\text{diam}(E) = \sup\{d(x, y) : x, y \in E\}$ for the diameter of $E$ and write $\text{dist}(x, E) = \inf\{d(x, y) : y \in E\}$ for the infimal distance of a point $x \in X$ to $E$.

Let $f : (X, d) \to (X', d')$ be a map between metric spaces. We say that $f$ is isometric if $d'(f(x), f(y)) = d(x, y)$ for $x, y \in X$. We recall that a curve $\gamma : I \to X$ is a geodesic if it is an isometric mapping of the interval $I \subset \mathbb{R}$ into $X$. We say that $X$ is geodesic if any two points in $X$ can be joined by a geodesic. A geodesic triangle $\Delta$ in $X$ consists of three points $x, y, z \in X$ together with geodesics joining these points to one another. Writing
\[ \Delta = \gamma_1 \cup \gamma_2 \cup \gamma_3 \] as a union of its edges, we say that \( \Delta \) is \( \delta \)-thin for a given \( \delta \geq 0 \) if for each point \( p \in \gamma_i \), \( i = 1, 2, 3 \), there is a point \( q \in \gamma_j \) with \( d(p, q) \leq \delta \) and \( i \neq j \). A geodesic metric space \( X \) is Gromov hyperbolic if there is a \( \delta \geq 0 \) such that all geodesic triangles in \( X \) are \( \delta \)-thin; in this case we will also say that \( X \) is \( \delta \)-hyperbolic. When considering Gromov hyperbolic spaces \( X \) we will usually use the generic distance notation \( |xy| := d(x, y) \) for the distance between \( x \) and \( y \) in \( X \) and the generic notation \( xy \) for a geodesic connecting two points \( x, y \in X \), even when this geodesic is not unique.

A metric space \((X, d)\) is proper if its closed balls are compact. The Gromov boundary \( \partial X \) of a proper geodesic \( \delta \)-hyperbolic space \( X \) is defined to be the collection of all geodesic rays \( \gamma : [0, \infty) \to X \) up to the equivalence relation of two rays being equivalent if they are at a bounded distance from one another. We will often refer to the point \( \omega \in \partial X \) corresponding to a geodesic ray \( \gamma \) as the endpoint of \( \gamma \). Using the Arzela-Ascoli theorem it is easy to see in a proper geodesic \( \delta \)-hyperbolic space that for any points \( x, y, z \in X \cup \partial X \) there is a geodesic \( \gamma \) joining \( x \) to \( y \). We will continue to write \( xy \) for any such choice of geodesic joining \( x \) to \( y \). We will allow our geodesic triangles \( \Delta \) to have vertices on \( \partial X \), in which case we will still write \( \Delta = xyz \) if \( \Delta \) has vertices \( x, y, z \).

As in our previous work \cite{10}, we will use the notation \( \partial X \) for the Gromov boundary of \( X \) even though it conflicts with the notation \( \partial \Omega = \Omega \setminus \Omega \) for the metric boundary of a metric space \((\Omega, d)\) inside its completion \( \Omega \). Since we always assume that \( X \) is proper we will always have \( \bar{X} = X \), so the metric boundary of \( X \) will always be trivial. Thus there will be no ambiguity in using \( \partial X \) for the Gromov boundary as well.

For \( x, y, z \in X \) the Gromov product of \( x \) and \( y \) based at \( z \) is defined by
\[
(2.1) \quad (x|y)_z = \frac{1}{2}(|xz| + |yz| - |xy|).
\]

We can also take the basepoint of the Gromov product to be any function \( b \in \mathcal{B}(X) \). For \( b \in \mathcal{B}(X) \) the Gromov product based at \( b \) is defined by
\[
(2.2) \quad (x|y)_b = \frac{1}{2}(b(x) + b(y) - |xy|).
\]

For \( b \in \mathcal{D}(X) \), \( b(x) = d(x, z) + s \) this reduces to the notion of Gromov product in \((2.1)\), as we have \( (x|y)_b = (x|y)_z + s \).

We now consider an incomplete metric space \((\Omega, d)\) and write \( \partial \Omega = \overline{\Omega} \setminus \Omega \) for the metric boundary of \( \Omega \) in its completion \( \overline{\Omega} \). We write \( d_\Omega(x) := \text{dist}(x, \partial \Omega) \) for the distance of a point \( x \in \Omega \) to the boundary \( \partial \Omega \). An important observation that we will use without comment is that \( d_\Omega \) defines a 1-Lipschitz function on \( \Omega \), i.e., for \( x, y \in \Omega \) we have
\[
|d_\Omega(x) - d_\Omega(y)| \leq d(x, y).
\]

For a curve \( \gamma : I \to \Omega \) we write \( \ell(\gamma) \) for the length of \( \gamma \) and say that \( \gamma \) is rectifiable if \( \ell(\gamma) < \infty \). For an interval \( I \subset \mathbb{R} \) and \( t \in I \) we write \( I_{\leq t} = \{ s \in I : s \leq t \} \) and \( I_{\geq t} = \{ s \in I : s \geq t \} \). For a rectifiable curve \( \gamma : I \to \Omega \) we write \( \gamma_- , \gamma_+ \in \Omega \) for the endpoints of \( \gamma \). Writing \( t_- \in (-\infty, \infty) \) and \( t_+ \in (-\infty, \infty) \) for the endpoints of \( I \), these are defined by the limits \( \gamma(t_-) = \lim_{t \to t_-} \gamma(t) \) and \( \gamma(t_+) = \lim_{t \to t_+} \gamma(t) \) in \( \Omega \) which exist because \( \ell(\gamma) < \infty \).

**Definition 2.1.** For a constant \( A \geq 1 \) and an interval \( I \subset \mathbb{R} \), a curve \( \gamma : I \to \Omega \) is \( A \)-uniform if
\[
(2.3) \quad \ell(\gamma) \leq Ad(\gamma_- , \gamma_+),
\]
and if for every $t \in I$ we have
\[
(2.4) \quad \min\{\ell(\gamma|_{I_{\varepsilon,t}}), \ell(\gamma|_{I_{\varepsilon,t}^c})\} \leq Ad_{\Omega}(\gamma(t)).
\]

The metric space $\Omega$ is $A$-uniform if it is locally compact and if any two points in $\Omega$ can be joined by an $A$-uniform curve.

We extend Definition (2.1) to the case of non-rectifiable curves $\gamma : I \to \Omega$ by replacing \([2.3]\) with the condition that $d(\gamma(s), \gamma(t)) \to \infty$ as $s \to t_-$ and $t \to t_+$. We keep the requirement \([2.4]\) the same. Observe that with this extended definition the inequality \([2.4]\) implies that an $A$-uniform curve $\gamma$ is always locally rectifiable, meaning that each compact subcurve of $\gamma$ is rectifiable. We note that it is easily verified from the definitions that the property of a curve $\gamma$ being $A$-uniform is independent of the choice of parametrization of $\gamma$.

Now let $X$ be a proper geodesic $\delta$-hyperbolic space. We define $X$ to be $K$-roughly starlike from a point $z \in X$ if for each $x \in X$ there is a geodesic ray $\gamma : [0, \infty) \to X$ such that $\operatorname{dist}(x, \gamma) \leq K$. Similarly for $\omega \in \partial X$ we define $X$ to be $K$-roughly starlike from $\omega$ if for each $x \in X$ there is a geodesic line $\gamma : \mathbb{R} \to X$ with $\gamma|_{(-\infty, 0]} \in \omega$ and $\operatorname{dist}(x, \gamma) \leq K$. When $\partial X$ contains at least two points $K$-rough starlikeness from any point $x \in X \cup \partial X$ implies $K'$-rough starlikeness from all points of $X \cup \partial X$ for a constant $K' \geq 0$ by [8] Proposition 1.13. We also note that rough starlikeness from $\omega$ immediately implies that $\partial_{\omega}X \neq \emptyset$.

We fix a function $b \in B(X)$ with basepoint $\omega \in X \cup \partial X$ and let $\varepsilon > 0$ be such that the density $\rho_{\varepsilon}(x) = e^{-\varepsilon b(x)}$ is a GH-density on $X$ with constant $M$. Since $b$ is 1-Lipschitz we have the Harnack type inequality for $x, y \in X$,
\[
(2.5) \quad e^{-\varepsilon|xy|} \leq \frac{\rho_{\varepsilon}(x)}{\rho_{\varepsilon}(y)} \leq e^{\varepsilon|xy|}.
\]

We write $X_{\varepsilon} = X_{\varepsilon,b}$ for the conformal deformation of $X$ with conformal factor $\rho$ and write $d_{\varepsilon} = d_{\varepsilon,b}$ for the resulting distance on $X_{\varepsilon}$. We write $\ell_{\varepsilon}(\gamma) := \ell_{\varepsilon,b}(\gamma)$ for the lengths of curves measured in the metric $d_{\varepsilon}$ and $\ell(\gamma)$ for the lengths of curves measured in $X$. The properness of $X$ implies that $X_{\varepsilon}$ is locally compact. By [9] Theorem 1.4] the metric space $X_{\varepsilon}$ is incomplete and geodesics in $X$ are $A$-uniform curves in $X_{\varepsilon}$. In particular the metric space $(X_{\varepsilon}, d_{\varepsilon})$ is $A$-uniform. Furthermore the space $X_{\varepsilon}$ is bounded if and only if $b \in D(X)$. The proof of [9] Theorem 1.4] shows that when $b \in D(X)$ all geodesics in $X$ have finite length in $X_{\varepsilon}$, while in the case $b \in B(X)$ geodesics have finite length if and only if they do not have the basepoint $\omega$ of $b$ as an endpoint. For $x \in X_{\varepsilon}$ we write $B_{\varepsilon}(x, r)$ for the open ball of radius $r > 0$ centered at $x$ in the metric $d_{\varepsilon}$ on $X_{\varepsilon}$, and for $x \in X$ we write $B_{X}(x, r)$ for the open ball of radius $r$ centered at $x$ in $X$.

For $x \in X_{\varepsilon}$ write $d_{\varepsilon}(x) = d_{X_{\varepsilon}}(x)$ for the distance to the metric boundary $\partial X_{\varepsilon}$ of $X_{\varepsilon}$. By [9] Theorem 1.6] there is a canonical identification $\varphi_{\varepsilon} : \partial_{\varepsilon}X \to \partial X_{\varepsilon}$ of the Gromov boundary of $X$ relative to $\omega$ and the metric boundary $\partial X_{\varepsilon}$ of $X_{\varepsilon}$; we recall that $\partial_{\varepsilon}X = \partial X$ if $\omega \in X$ and $\partial_{\varepsilon}X = \partial X \setminus \{\omega\}$ if $\omega \in \partial X$. The correspondence is given by showing that any sequence $\{x_n\}$ in $X$ converging to a point $\xi \in \partial_{\varepsilon}X$ is a Cauchy sequence in $X_{\varepsilon}$ converging to a point of $\partial X_{\varepsilon}$.

The local compactness of $X_{\varepsilon}$ implies by the Arzela-Ascoli theorem that, for a given $x, y \in X$, a minimizing curve $\gamma$ for the right side of \([1.4]\) always exists. It is easy to see that such a curve must be a geodesic in $X_{\varepsilon}$, from which we conclude that $X_{\varepsilon}$ is always geodesic. By [1] Proposition 2.20] the completion $\hat{X}_{\varepsilon}$ of $X_{\varepsilon}$ is proper, and in particular is also locally compact. A second application of Arzela-Ascoli then shows that $\hat{X}_{\varepsilon}$ is also geodesic.

We collect here two important quantitative results regarding the uniformization $X_{\varepsilon}$ from our previous work [9]. The standing assumptions for the rest of this section are that $X$ is a
Lemma 2.2. [9, Lemma 4.7] For $x, y \in X$ we have
\begin{equation}
\label{eq:2.6}
d_\varepsilon(x, y) \asymp e^{-\varepsilon |xy|} \min\{1, |xy|\}.
\end{equation}

Lemma 2.3. [9, Lemma 4.15] For $x \in X$ we have
\begin{equation}
\label{eq:2.7}
d_\varepsilon(x) \asymp \rho_\varepsilon(x).
\end{equation}

Lemmas 2.2 and 2.3 are stated for $b \in \mathcal{B}(X)$ in [9], however as noted in [9, Remark 4.24] the estimates for $b \in \mathcal{D}(X)$ can be deduced from the estimates for $b \in \mathcal{B}(X)$ by attaching a ray to $X$ at the basepoint of a given $b \in \mathcal{D}(X)$.

We conclude this section by adapting two key claims from [2] to our setting. The first claim adapts [2, Theorem 2.10]. The proof is essentially the same.

Lemma 2.4. There is a constant $C_* = C_*(\delta, K, \varepsilon, M) \geq 1$ such that for any $x \in X$ and any $0 < r \leq \frac{1}{2}d_\varepsilon(x)$ we have the inclusions,
\begin{equation}
\label{eq:2.8}
B_X \left(x, \frac{C_*^{-1}r}{\rho_\varepsilon(x)}\right) \subset B_\varepsilon(x, r) \subset B_X \left(x, \frac{C_* r}{\rho_\varepsilon(x)}\right).
\end{equation}

Proof. Let $y \in B_X(x, C_*^{-1}r/\rho_\varepsilon(x))$, for a constant $C_* \geq 1$ to be determined. Let $\gamma$ be a geodesic in $X$ joining $x$ to $y$ and let $z \in \gamma$. Then, since $r \leq \frac{1}{2}d_\varepsilon(x)$, we have by Lemma 2.3,
\begin{equation}
|xz| \leq \frac{C_*^{-1}d_\varepsilon(x)}{2\rho_\varepsilon(x)} \leq C_*^{-1}C,
\end{equation}
with $C = C(\delta, K, \varepsilon, M) \geq 1$. This then implies by the Harnack inequality (2.5),
\begin{equation}
\rho_\varepsilon(z) \asymp e^{C_*^{-1}C_*} \rho_\varepsilon(x).
\end{equation}
Choosing $C_*$ large enough that $e^{C_*^{-1}C_*} \leq 2$, we then obtain that
\begin{equation}
\rho_\varepsilon(z) \asymp_2 \rho_\varepsilon(x),
\end{equation}
for $z \in \gamma$. We conclude that
\begin{equation}
d_\varepsilon(x, y) \leq \int_\gamma \rho_\varepsilon \, ds
\leq 2\rho_\varepsilon(x)|xy|
\leq 2C_*^{-1}r
< r,
\end{equation}
provided we take $C_* > 2$. This gives the inclusion on the left side of (2.8).

For the inclusion on the right side of (2.8), let $y \in B_\varepsilon(x, r)$ and let $\gamma_\varepsilon$ be a geodesic in $X_\varepsilon$ connecting $x$ to $y$. For $z \in \gamma_\varepsilon$ we then have $z \in B_\varepsilon(x, r)$ and therefore $d_\varepsilon(z) \geq \frac{1}{2}d_\varepsilon(x)$ by the triangle inequality since $r \leq \frac{1}{2}d_\varepsilon(x)$. Applying Lemma 2.3 we then have
\begin{equation}
\rho_\varepsilon(z) \geq C^{-1}d_\varepsilon(z)
\geq \frac{1}{2} C^{-1}d_\varepsilon(x)
\geq C^{-1}\rho_\varepsilon(x),
\end{equation}
for $z \in \gamma_\varepsilon$. This gives the inclusion on the right side of (2.8).
for a constant $C = C(\delta, K, \varepsilon, M) \geq 1$. Using this we conclude that
\[ r > d_\varepsilon(x, y) = \int_{\gamma_\varepsilon} \rho_\varepsilon \, ds \geq C^{-1} \rho_\varepsilon(x)|xy|, \]

since $\ell(\gamma_\varepsilon) \geq |xy|$. Choosing $C_\varepsilon$ to be greater than the constant $C$ on the right side of this inequality, we then conclude that
\[ |xy| < \frac{C_\varepsilon r}{\rho_\varepsilon(x)}, \]

which gives the right side inclusion in (2.8). \hfill \Box

Following [2], the balls $B_\varepsilon(x, r)$ for $x \in X_\varepsilon$, $0 < r \leq \frac{1}{3}d_\varepsilon(x)$ will often be referred to as subWhitney balls.

The second claim adapts [2] Lemma 4.8] to our setting. The proof given in [2] strongly relies on the uniformization $X_\varepsilon$ being bounded in their setting, so when $b \in \mathcal{B}(X)$ we will have to take an approach that is somewhat different.

**Lemma 2.5.** There is a constant $\kappa_0 = \kappa_0(\delta, K, \varepsilon, M)$ such that for every $x \in X_\varepsilon$ and every $0 < r \leq 2 \text{diam } X_\varepsilon$ we can find a ball $B_\varepsilon(z, \kappa_0 r) \subset B_\varepsilon(x, r)$ with $d_\varepsilon(z) \geq 2\kappa_0 r$.

**Proof.** The claim for $b \in \mathcal{D}(X)$ of the form $b_\varepsilon(x) = d(x, z)$ for some $z \in X$ follows from repeating the proof of [2] Lemma 4.8] in our setting. For $b \in \mathcal{D}(X)$ of the form $b_\varepsilon(x) = d(x, z) + s$ for some $z \in X$, $s \in \mathbb{R}$, the claim then follows by observing that $X_\varepsilon = e^{-s}X_{\varepsilon, z}$, i.e., $X_\varepsilon$ is obtained by scaling by a factor of $e^{-s}$ the metric on the conformal deformation of $X$ by $\rho_{\varepsilon, z}(x) = e^{-|xz|}$. We can thus assume that $b \in \mathcal{B}(X)$ with basepoint $\omega \in \partial X$, which implies that $\text{diam } X_\varepsilon = \infty$.

Let $x \in X_\varepsilon$ and $r > 0$ be given. The function $y \to d_\varepsilon(y)$ on $X_\varepsilon$ is continuous, positive, unbounded (since $X_\varepsilon$ is unbounded) and takes values arbitrarily close to 0 since $d_\varepsilon(x_n) \to 0$ for any sequence of points $\{x_n\}$ in $X_\varepsilon$ converging to a point of $\partial X_\varepsilon$. Since $X_\varepsilon$ is connected we can then conclude by the intermediate value theorem that $d_\varepsilon(X_\varepsilon) = (0, \infty)$, i.e., for any $r > 0$ we can find a point $z_0 \in X_\varepsilon$ such that $d_\varepsilon(z_0) = r$. For our given $r > 0$ we fix such a point $z_0$ and let $\sigma$ be a geodesic in $X$ joining $x$ to $z_0$; recall that we can consider points $x \in \partial X_\varepsilon$ as points of $\partial_0 X$ through the identification $\partial X_\varepsilon \cong \partial_0 X$. Then $\sigma$ is an $A$-uniform curve in $X_\varepsilon$ with $A = A(\delta, K, \varepsilon, M) \geq 1$. Since $\sigma$ does not have $\omega$ as an endpoint, it has finite length $\ell_\varepsilon(\sigma) \leq Ad_\varepsilon(x, z_0)$ in $X_\varepsilon$. We parametrize $\sigma$ by $d_\varepsilon$-arclength and orient it from $x$ to $z_0$.

We first assume that $\ell_\varepsilon(\sigma) \geq \frac{2}{3}r$. In this case we set $z = \sigma(\frac{1}{4}r)$. Then since $\sigma$ is $A$-uniform we have $d_\varepsilon(z) \geq \frac{3}{4}r$ and
\[ B_\varepsilon \left( z, \frac{r}{6A} \right) \subset B_\varepsilon \left( x, \frac{r}{3} + \frac{r}{6A} \right) \subset B_\varepsilon(x, r). \]

So in this case we can use any $\kappa \leq \frac{1}{6A}$.

Now consider the case in which $\ell_\varepsilon(\sigma) < \frac{2}{3}r$. We then set $z = z_0$ and observe that
\[ B_\varepsilon \left( z_0, \frac{r}{3} \right) \subset B_\varepsilon \left( x, \ell_\varepsilon(\sigma) + \frac{r}{3} \right) \subset B_\varepsilon(x, r). \]

By construction we have $d_\varepsilon(z_0) = r$. Thus in this case any $\kappa \leq \frac{1}{3}$ will work. By combining these two cases we can then set $\kappa_0 = \frac{1}{6A}$, noting that $A \geq 1$. \hfill \Box

The conclusion of Lemma 2.5 is closely related to the corkscrew condition for domains in metric spaces. See [3] Definition 2.4].
3. Doubling for Uniformized Measures

In this section we will prove Theorem 1.2 and lay some of the groundwork for proving our other theorems. We will frequently make use of the following consequence of the doubling estimate (1.6) for a metric measure space \((X, d, \mu)\): if \(\mu\) is doubling on balls of radius at most \(R_0\) with constant \(C_\mu\) and \(0 < r \leq R \leq R_0\) then

\[
\mu(B_X(x, R)) \lesssim_C \mu(B_X(x, r)),
\]

with constant \(C\) depending only on \(C_\mu\) and the ratio \(R/r\). This estimate follows by iterating the estimate (1.6) and noting that \(\mu(B_X(x, R)) \geq \mu(B_X(x, r))\) since \(B_X(x, r) \subseteq B_X(x, R)\).

We will require the following proposition from [2], which is stated there in a more general form.

**Proposition 3.1.** [2, Proposition 3.2] Let \((X, d)\) be a geodesic metric space and let \(\mu\) be a Borel measure on \(X\) that is doubling on balls of radius at most \(R_0\) with doubling constant \(C_\mu\). Then for any \(R_1 > 0\) the measure \(\mu\) is doubling on balls of radius at most \(R_1\), with doubling constant depending only on \(R_1/R_0\) and \(C_\mu\).

Thus if \(\mu\) is doubling on balls of radius at most \(R_0\) then given any \(R_1 > 0\) we can assume that \(\mu\) is also doubling on balls of radius at most \(R_1\), at the cost of increasing the uniform local doubling constant of \(\mu\) by an amount depending only on \(R_1/R_0\) and \(C_\mu\).

We now describe the setting of this section. We begin with a proper geodesic \(\delta\)-hyperbolic \(X\) together with a function \(b \in B(X)\) with basepoint \(\omega\) such that \(X\) is \(K\)-roughly starlike from \(\omega\). We let \(\varepsilon > 0\) be such that the associated density \(\rho_\varepsilon\) is a GH-density for \(X\) with constant \(M\). As in the previous section we write \(X_\varepsilon\) for the uniformization of \(X\), \(d_\varepsilon\) for the distance on \(X_\varepsilon\), etc. We let \(\mu\) be a Borel regular measure on \(X\) such that there is an \(R_0 > 0\) for which \(\mu\) is doubling on balls of radius at most \(R_0\) with doubling constant \(C_\mu\). For a given \(\beta > 0\) we then define the uniformized measure \(\mu_\beta = \mu_{\beta, b}\) on \(X_\varepsilon\) as in (3.1).

In the claims in the rest of this section all implicit constants will depend only on \(\delta, K, \varepsilon, M, \beta, R_0\), and \(C_\mu\). We will refer to this collection of seven parameters as the data. We will refer to the specific parameters \(\delta, K, \varepsilon, M, \beta\) as the uniformization data and say that a constant depends only on the uniformization data if it depends only on these five parameters. At several points we will need to increase \(R_0\) by an amount depending only on the uniformization data in order to ensure that \(\mu\) is doubling at a larger scale using Proposition 3.1. When we do this we will also need to increase \(C_\mu\) by a corresponding amount depending only on the uniformization data and the local doubling constant \(C_\mu\) for \(\mu\).

**Remark 3.2.** We will also often refer to just the four parameters \(\delta, K, \varepsilon, M\) as the uniformization data. It will be clear from context when \(\beta\) can and cannot be excluded from the list.

The first part of this section will be devoted to proving the following technical criterion for \(\mu_\beta\) to be doubling on \(X_\varepsilon\). Throughout this section we let \(\kappa_0 = \kappa_0(\delta, K, \varepsilon, M)\) be defined as in Lemma 2.5 and set \(\kappa_1 = \kappa_0/10\).

**Proposition 3.3.** Suppose that there is a constant \(C_0 \geq 1\) such that for any \(\xi \in \partial X_\varepsilon\), \(r > 0\), and \(z \in X\) we have that whenever \(B_\varepsilon(z, \kappa_1 r) \subseteq B_\varepsilon(\xi, r)\) and \(d_\varepsilon(z) \geq 2\kappa_1 r\),

\[
\mu_\beta(B_\varepsilon(\xi, r)) \leq C_0 r^{\beta/\varepsilon} \mu(B_X(z, R_0)).
\]

Then \(\mu_\beta\) is doubling on \(X_\varepsilon\) with doubling constant \(C_{\mu_\beta}\) depending only on the data and \(C_0\).
We have formulated Proposition 3.3 in the manner that is most convenient for us to verify in practice, however this comes at the cost of obscuring the connection of the inequality (3.2) to the doubling property for $\mu_{\beta}$. In order to prove Proposition 3.3 we will need a series of lemmas that establish this connection. Our first claim corresponds to [2, Lemma 4.5]. It provides an estimate on the measure of subWhitney balls in $X_{\varepsilon}$.

**Lemma 3.4.** Let $x \in X$ and $0 < r \leq \frac{1}{2} d_{\varepsilon}(x)$. Then

$$\mu_{\beta}(B_{\varepsilon}(x, r)) \asymp \rho_{\beta}(x) \mu \left( B_{X} \left( x, \frac{r}{\rho_{\varepsilon}(x)} \right) \right),$$

with comparison constant depending only on the data.

**Proof.** By Lemma 2.3 we have for all $y \in B_{\varepsilon}(x, r)$,

$$\rho_{\beta}(y) = \rho_{\varepsilon}(y)^{3/\varepsilon} \asymp d_{\varepsilon}(y)^{3/\varepsilon} \asymp d_{\varepsilon}(x)^{3/\varepsilon} \asymp \rho_{\beta}(x),$$

with the comparison $d_{\varepsilon}(y) \approx_{2} d_{\varepsilon}(x)$ following from the condition on $r$. Applying Lemma 2.4 and the chain of comparisons (3.3), we conclude that

$$\mu_{\beta}(B_{\varepsilon}(x, r)) \asymp \rho_{\beta}(x) \mu(B_{\varepsilon}(x, r)) \lesssim \rho_{\beta}(x) \mu \left( B_{X} \left( x, \frac{C_{\varepsilon} r}{\rho_{\varepsilon}(x)} \right) \right),$$

with $C_{\varepsilon} = C_{\varepsilon}(\delta, K, \varepsilon, M)$ being the constant from Lemma 2.4. A similar argument using the other inclusion from Lemma 2.4 shows that

$$\mu_{\beta}(B_{\varepsilon}(x, r)) \gtrsim \rho_{\beta}(x) \mu \left( B_{X} \left( x, \frac{C_{\varepsilon}^{-1} r}{\rho_{\varepsilon}(x)} \right) \right).$$

We thus conclude that

$$\rho_{\beta}(x) \mu \left( B_{X} \left( x, \frac{C_{\varepsilon}^{-1} r}{\rho_{\varepsilon}(x)} \right) \right) \lesssim \mu_{\beta}(B_{\varepsilon}(x, r)) \lesssim \rho_{\beta}(x) \mu \left( B_{X} \left( x, \frac{C_{\varepsilon} r}{\rho_{\varepsilon}(x)} \right) \right)$$

The condition on $r$ implies that

$$\frac{r}{\rho_{\varepsilon}(x)} \leq \frac{1}{2} \frac{d_{\varepsilon}(x)}{\rho_{\varepsilon}(x)} \leq C,$$

with $C$ depending only on the uniformization data by Lemma 2.3. By Proposition 3.1 we can, at the cost of increasing the local doubling constant $C_{\mu}$ of $\mu$ by an amount depending only on the data, assume that $R_{0} > C C_{\varepsilon}$ for the constant $C$ in inequality (3.3) and the constant $C_{\varepsilon}$ in Lemma 2.4. Then the comparison (3.1) allows us to conclude that

$$\mu \left( B_{X} \left( x, \frac{C_{\varepsilon}^{-1} r}{\rho_{\varepsilon}(x)} \right) \right) \asymp \mu \left( B_{X} \left( x, \frac{r}{\rho_{\varepsilon}(x)} \right) \right) \asymp \mu \left( B_{X} \left( x, \frac{C_{\varepsilon} r}{\rho_{\varepsilon}(x)} \right) \right).$$

Combining this comparison with inequality (3.4) proves the lemma. \qed

By combining Lemma 3.4 with Lemma 2.5, we obtain the following estimate for $\mu_{\beta}(B_{\varepsilon}(x, r))$ when $0 < r \leq \frac{1}{2} d_{\varepsilon}(x)$. We recall that $\kappa_{1} = \kappa_{0}/10$, where $\kappa_{0}$ is defined as in Lemma 2.5.

**Lemma 3.5.** Let $x \in X$ and $0 < r \leq \frac{1}{2} d_{\varepsilon}(x)$. Let $z \in X$ be given such that $B_{\varepsilon}(z, \kappa_{1} r) \subset B_{\varepsilon}(x, r)$ and $d_{\varepsilon}(z) \geq 2 \kappa_{1} r$. Then

$$\mu_{\beta}(B_{\varepsilon}(x, r)) \asymp \mu_{\beta}(B_{\varepsilon}(z, \kappa_{1} r)),$$

with comparison constants depending only on the data.
Proof. By Lemmas 2.3 and 2.4 we have

$$|xz| \leq \frac{C_z r}{\rho_z(x)} \leq \frac{C_z d_z(x)}{2\rho_z(x)} \lessapprox 1,$$

with implied constant depending only on the uniformization data, where $C_z$ is the constant from Lemma 2.3. Since $z \in B_z(x, r)$ and $r \leq \frac{1}{2} d_z(x)$, we conclude that we have $d_z(z) \approx_2 d_z(x)$. We thus obtain from Lemma 2.3 that $\rho_z(z) \approx \rho_z(x)$ with comparison constant depending only on the uniformization data. Since $d_z(z) \geq 2\kappa_1 r$, we have by Lemma 2.3 that

$$1 \gtrapprox \frac{\kappa_1 r}{\rho_z(z)} \equiv \frac{r}{\rho_z(z)} \equiv \frac{r}{\rho_z(x)} \equiv \frac{C_z r}{\rho_z(x)},$$

with all implied constants depending only on the uniformization data, since $\kappa_1$ depends only on the uniformization data. We can thus apply Proposition 3.1 to conclude that we can assume that $\mu$ is doubling on balls of radius at most any of the terms appearing in (3.7), at the cost of increasing the local doubling constant of $\mu$ by an amount depending only on the data. It follows that

$$\mu \left( B_X \left( z, \frac{\kappa_1 r}{\rho_z(z)} \right) \right) \lessapprox \mu \left( B_X \left( z, \frac{r}{\rho_z(z)} \right) \right) \lessapprox \mu \left( B_X \left( z, \frac{r}{\rho_z(x)} \right) \right) \lessapprox \mu \left( B_X \left( x, \frac{C_z r}{\rho_z(x)} \right) \right) \lessapprox \mu \left( B_X \left( x, \frac{r}{\rho_z(x)} \right) \right)$$

with implied constants depending only on the data. The third comparison above follows from the fact that $z \in B_X \left( x, \frac{C_z r}{\rho_z(x)} \right)$ by (3.6). Since the comparison $\rho_\beta(z) \approx \rho_\beta(x)$ follows from the comparison $\rho_z(z) \approx \rho_z(x)$ (with comparison constants depending only on the uniformization data), applying Lemma 2.4 to $B_z(z, \kappa r)$ and $B_z(x, r)$ (note that $\kappa_1 r \leq \frac{1}{2} d_z(z)$ by assumption) then gives

$$\mu_\beta(B_z(z, \kappa_1 r)) \lessapprox \mu_\beta(B_z(x, r)),$$

with comparison constants depending only on the data. \qed

Our final lemma estimates the right side of inequality (3.2) in terms of $\mu_\beta(B_z(z, \kappa_1 r))$. The reason for choosing $5r$ as the upper bound for $d_z(z)$ will be clear in the proof of Proposition 3.3

Lemma 3.6. Let $z \in X$ and $r > 0$ be such that $2\kappa_1 r \leq d_z(z) < 5r$. Then

$$\mu_\beta(B_z(z, \kappa_1 r)) \approx r^{\beta/\epsilon} \mu(B_X(z, R_0)),$$

with comparison constant depending only on the data.

Proof. The assumptions imply that $d_z(z) \approx r$, hence $\rho_\beta(z) \approx r^{\beta/\epsilon}$ by Lemma 2.3, with comparison constants depending only on the uniformization data since $\kappa_1$ depends only on the uniformization data. Thus by Lemma 2.4 we have

$$\mu_\beta(B_z(z, \kappa_1 r)) \approx r^{\beta/\epsilon} \mu \left( B_X \left( z, \frac{\kappa_1 r}{\rho_z(z)} \right) \right),$$
with comparison constant depending only on the data. Since \( \rho(z) \approx d_X(z) \approx r \), we have \( \frac{\rho(z)}{d_X(z)} \approx C' \) for a constant \( C' \) depending only on the uniformization data. Using Proposition 3.1 we can assume that \( \mu \) is doubling on balls of radius at most \( C'R_0 \), at the cost of increasing the doubling constant by an amount depending only on the data. From this we conclude that the comparison \( (3.8) \) holds.

We can now prove Proposition 3.3.

**Proof of Proposition 3.3** We split the proof of the doubling property for \( \mu_\beta \) into two cases depending on the center \( x \in X_\varepsilon \) of the ball. The first case is that in which \( 0 < r \leq \frac{1}{2}d_X(x) \), which implies in particular that \( x \in X_\varepsilon \). Then we can apply Lemma 3.4 to both \( B_\varepsilon(x, r) \) and \( B_\varepsilon(x, 2r) \). We conclude that

\[
\mu_\beta(B_\varepsilon(x, r)) \approx \mu \left( B_X \left( x, \frac{r}{\rho_\varepsilon(x)} \right) \right) \approx \mu \left( B_X \left( x, \frac{2r}{\rho_\varepsilon(x)} \right) \right) \approx \mu_\beta(B_\varepsilon(x, 2r)),
\]

with comparison constants depending only on the data. To justify the middle comparison in (3.9), we observe that since \( 2r \leq d_X(x) \) we have by Lemma 2.3 that each of the middle two balls in \( X \) in (3.9) on the right side of this inequality have radius at most \( C' \) for some constant \( C' \) depending only on the uniformization data. By Proposition 3.1 we can assume that \( \mu \) is doubling on balls of radius at most \( C' \), at the cost of increasing the doubling constant of \( \mu \) by an amount depending only on the data. This gives the desired doubling estimate for the right side of (3.9). We note that this first case does not require the use of the assumed inequality \( (3.2) \).

The second case is that in which \( d_X(x) < 4r \). We can then find a point \( \xi \in \partial X_\varepsilon \) such that \( B_\varepsilon(x, r) \subset B_\varepsilon(\xi, 5r) \). We then use Lemma 2.5 to choose a point \( z \in X_\varepsilon \) such that \( B_\varepsilon(z, 6r) \subset B_\varepsilon(x, r) \) and \( d_X(z) \geq 2\kappa_0 r \). Then we must have \( d_X(z) < 5r \) since \( z \in B_\varepsilon(\xi, 5r) \). Since \( B_\varepsilon(z, \kappa_0 r) \subset B_\varepsilon(\xi, 5r) \) and \( \kappa_0 > 5\kappa_1 \), we conclude from Lemma 3.6 and the assumed inequality \( (3.2) \) that

\[
\mu_\beta(B_\varepsilon(x, r)) \approx \mu_\beta(B_\varepsilon(z, \kappa_1 r)),
\]

with comparison constant depending only on the data and \( C_0 \). Since we also have \( B_\varepsilon(x, 2r) \subset B_\varepsilon(\xi, 10r) \) and \( \kappa_0 = 10\kappa_1 \), the same combination of Lemma 3.6 and (3.2) also shows that

\[
\mu_\beta(B_\varepsilon(x, 2r)) \approx \mu_\beta(B_\varepsilon(z, \kappa_1 r)),
\]

with comparison constant depending only on the data and \( C_0 \). Combining (3.10) and (3.11) gives the desired doubling estimate in this second case.

We will now prove Theorem 1.2 by showing, in analogy to [2 Proposition 4.7], that \( \mu_\beta \) is always doubling on \( X_\varepsilon \) for \( \beta \) sufficiently large. We will need the following refinement of Proposition 3.1.

**Lemma 3.7.** [2 Lemma 3.5] Let \((X, d)\) be a geodesic metric space and let \( \mu \) be a measure on \( X \) that is doubling on balls of radius at most \( R_0 \) with constant \( C_\mu \). Let \( n \in \mathbb{N} \) be a given integer.

1. For \( x, y \in X \) and \( 0 < r \leq R_0 \) satisfying \( d(x, y) < nr \), we have
   \[
   \mu(B_X(x, r)) \leq C_\mu^n \mu(B_X(y, r)).
   \]

2. For \( 0 < r \leq \frac{1}{4}R_0 \), every ball \( B \subset X \) of radius \( nr \) can be covered by at most \( C_\mu^{7(n+4)/6} \) balls of radius \( r \).
Proof of Theorem 1.2. We will prove this theorem using the criterion of Proposition 3.3. All implied constants will depend only on the data, meaning only on the parameters $\delta$, $K$, $\epsilon$, $M$, $\beta$, $R_0$ and $C_{\mu}$, unless otherwise noted. Let $\xi \in \partial X_\epsilon$, $z \in X_\epsilon$, and $r > 0$ be given such that $B_\epsilon(z, \kappa_1 r) \subset B_\epsilon(\xi, r)$ and $d_\epsilon(z) \geq 2\kappa_1 r$, where we recall that $\kappa_1 = \kappa_0/10$ depends only on the data. We then have $\rho_\beta(z) \asymp r^{\beta/\epsilon}$ by the proof of Lemma 3.6. We define for $n \geq 1$,

$$A_n = \{ x \in B_\epsilon(\xi, r) \cap X_\epsilon : e^{-\epsilon n} r \leq d_\epsilon(x) < e^{-(\epsilon - 1) n} r \}.$$ 

Since $x \in B_\epsilon(\xi, r)$ implies that $d_\epsilon(x) < r$, we have $B_\epsilon(\xi, r) \cap X_\epsilon = \bigcup_{n=1}^\infty A_n$. Since $\mu_\beta$ is extended to $\partial X_\epsilon$ by setting $\mu_\beta(\partial X_\epsilon) = 0$, we conclude that

$$\mu_\beta(B_\epsilon(\xi, r)) = \sum_{n=1}^\infty \mu_\beta(A_n).$$

For any given $x \in A_n$ we either have $|xz| < 1$ or $|xz| \geq 1$. In the second case we use Lemma 2.2 to obtain

$$e^{\epsilon |xz|} = \frac{e^{-2\epsilon |xz| \nu}}{\rho_\epsilon(x) \rho_\epsilon(z)} \asymp \frac{d_\epsilon(x, z)^2}{d_\epsilon(x) d_\epsilon(z)} \leq \frac{(d_\epsilon(x, \xi) + d_\epsilon(\xi, z))^2}{2\kappa_1 e^{-\epsilon n} r^2} \leq \frac{2e^{\epsilon n}}{\kappa_1} \lesssim e^{\epsilon n},$$

with implied constant depending only on $\delta$, $K$, $\epsilon$, and $M$. We then conclude that $|xz| \leq n + c_0$, with $c_0 = c_0(\delta, K, \epsilon, M) \geq 0$. Since this inequality trivially holds with $c_0 = 0$ when $|xz| < 1$, we in fact obtain the inequality $|xz| \leq n + c_0$ in both cases. We then choose $n_0 = n_0(\delta, K, \epsilon, M)$ to be the minimal integer such that $n_0 \geq c_0$. Then for $n \geq n_0$ we have $|xz| \leq 2n$ for $x \in A_n$. For $1 \leq n \leq n_0$ we then have $|xz| \leq 2n_0$ and therefore $A_n \subset B_X(z, 2n_0)$. By Proposition 3.1 we can then assume that $\mu$ is doubling on balls of radius at most $2n_0$ in $X$, at the cost of increasing the doubling constant by an amount depending only on $\delta, K, \epsilon$, and $M$. We conclude that

$$\mu(B_X(z, 2n_0)) \asymp \mu(B_X(z, R_0)),$$

with comparison constant depending only on the data. By the Harnack inequality (2.3) for $\rho_\beta$ we conclude for $x \in B_X(z, 2n_0)$ that $\rho_\beta(x) \asymp \rho_\beta(z) \asymp r^{\beta/\epsilon}$. Putting all of this together, we conclude that

$$\mu_\beta \left( \bigcup_{n=1}^{n_0} A_n \right) \leq \mu_\beta(B_X(z, 2n_0)) \lesssim r^{\beta/\epsilon} \mu(B_X(z, R_0)).$$

We now consider the case $n > n_0$, for which we have $|xz| \leq 2n$ whenever $x \in A_n$. We apply Proposition 3.1 to ensure that $\mu$ is doubling on balls of radius at most $R_1 = \max\{4R_0, R_0 + 2\}$. The doubling constant $C_{\mu}'$ for $\mu$ on balls of radius at most $R_1$ then depends only on $R_0$ and $C_{\mu}$. In particular $C_{\mu}'$ does not depend on $\beta$. Applying (2) of
Lemma 3.7, we cover $A_n \subset B_X(z, 2n)$ with $N_n \lesssim e^{\alpha n}$ many balls $B_{n,j}$ of radius $R_0$, where

$$\alpha = \alpha(R_0, C_\mu) = \frac{7}{6} \log C'_\mu.$$ 

We set $\beta_0 := 3\alpha$ and assume that $\beta \geq \beta_0$. Note that $\beta_0 = \beta_0(R_0, C_\mu)$ depends only on $R_0$ and $C_\mu$.

We can clearly assume that each ball $B_{n,j}$ intersects $A_n$, from which we conclude that the centers $x_{n,j}$ of the balls $B_{n,j}$ satisfy

$$|x_{n,j}| \leq R_0 + 2n < R_1 n,$$

since $n \geq 1$ and $R_1 > R_0 + 2$. Applying (1) of Lemma 3.7 then gives that

$$\mu(B_{n,j}) \leq (C'_\mu)^n \mu(B_X(z, R_1)) \leq e^{\alpha n} \mu(B_X(z, R_1)).$$

For $x \in A_n$, we have,

(3.12) 

$$\rho_\beta(x) \asymp d_\varepsilon(x)^{\beta/\varepsilon} \asymp (e^{-\varepsilon n \rho})^{\beta/\varepsilon}.$$ 

The Harnack inequality (2.5) implies that $\rho_\beta(y) \asymp \rho_\beta(x_{n,j})$ for each $y \in B_{n,j}$ (since each ball $B_{n,j}$ has radius $R_0$). Furthermore, since there is some point $y \in A_n$ such that $|x_{n,j}, y| \leq R_0$, it follows from the comparison (3.12) that $\rho_\beta(x_{n,j}) \asymp (e^{-\varepsilon n \rho})^{\beta/\varepsilon}$. Thus we conclude that

$$\mu_\beta(B_{n,j}) \asymp \rho_\beta(x_{n,j}) \mu(B_{n,j})$$

$$\lesssim (e^{-\varepsilon n \rho})^{\beta/\varepsilon} \mu(B_{n,j})$$

$$\lesssim e^{-\beta n \rho^{\beta/\varepsilon}} e^{\alpha n} \mu(B_X(z, R_1)).$$

By our restriction $\beta \geq \beta_0 = 3\alpha$, we conclude that

$$\mu_\beta(B_{n,j}) \lesssim e^{-2\alpha n \rho^{\beta/\varepsilon}} \mu(B_X(z, R_1)).$$

It then follows from this inequality and the bound $N_n \lesssim e^{\alpha n}$ that

$$\mu_\beta \left( \bigcup_{n=n_0+1}^{\infty} A_n \right) \leq \sum_{n=n_0+1}^{\infty} \sum_{j=1}^{N_n} \mu_\beta(B_{n,j})$$

$$\lesssim r^{\beta/\varepsilon} \mu(B_X(z, R_1)) \sum_{n=n_0+1}^{\infty} N_n e^{-2\alpha n}$$

$$\lesssim r^{\beta/\varepsilon} \mu(B_X(z, R_1)) \sum_{n=n_0+1}^{\infty} e^{-\alpha n}$$

$$\lesssim r^{\beta/\varepsilon} \mu(B_X(z, R_1)),$$

with the final inequality following by summing the geometric series. By combining the cases $1 \leq n \leq n_0$ and $n > n_0$ we conclude that

$$\mu_\beta(B_\varepsilon(x, r)) \lesssim r^{\beta/\varepsilon} \mu(B_X(z, R_1)).$$

Since $\mu$ is doubling up to the radius $R_1$ with doubling constant depending only on the data, we conclude by Proposition 3.3 that $\mu_\beta$ is doubling on $X_\varepsilon$ with constant depending only on the data. \qed

We now discuss a setting in which it is possible to obtain sharper estimates for the threshold $\beta_0$ above which $\mu_\beta$ is doubling. In particular this will allow us to prove the doubling claim in Theorem 1.4. We will keep the setting of Theorem 1.2 and then assume in addition that we have a cocompact discrete isometric action of a group $\Gamma$ on $X$. Briefly
recalling the definitions, the action by \( \Gamma \) is \emph{isometric} if each element \( g \in \Gamma \) defines an isometry of \( X \). It is \emph{cocompact} if there is a compact set \( E \subset X \) such that \( X = \bigcup_{g \in \Gamma} g(E) \), i.e., if \( X \) is covered by the translates of a compact subset under the action of \( \Gamma \). It is \emph{discrete} if for each compact subset \( E \subset X \) the number of \( g \in \Gamma \) such that \( g(E) \cap E \neq \emptyset \) is finite.

We will assume in addition that the uniformly locally doubling measure \( \mu \) is \( \Gamma \)-invariant, meaning that \( \mu(g^{-1}(E)) = \mu(E) \) for each measurable subset \( E \subset X \) and each \( g \in \Gamma \). Such measures often arise naturally in the context of the geometry of \( X \); for instance if \( X \) is a tree with bounded vertex degree and edges of unit length then we can take \( \mu \) to be the measure on \( X \) induced from the 1-dimensional Lebesgue measure on the edges. Another case is the setting of Theorem 1.4 when \( X \) is the universal cover of a closed Riemannian manifold \( M \) with sectional curvatures \( \leq -1 \), in which case we can take \( \mu \) to be the Riemannian volume on \( X \). We can assume by Proposition 3.1 and the cocompactness of the action of \( \Gamma \) that the doubling radius \( R_0 \) for \( \mu \) is large enough that for each \( x \in X \) the translates of \( B_X(x, R_0) \) by \( \Gamma \) cover \( X \).

For \( x \in X \) and \( R > 0 \) we set

\[
N_\Gamma(x, R) = \#\{ g \in \Gamma : |xg(x)| \leq R \},
\]

with \( \#E \) denoting the cardinality of a set \( E \). We consider the critical exponent \( h_X \) defined by the following limit for a fixed \( x \in X \),

\[
(3.13) \quad h_X = \limsup_{R \to \infty} \frac{\log N_\Gamma(x, R)}{R}.
\]

Standard arguments using the cocompactness of the action of \( \Gamma \) show that \( h_X \) does not depend on the choice of point \( x \in X \). We observe that \( h_X \) can equivalently be thought of as the limit

\[
(3.14) \quad h_X = \limsup_{R \to \infty} \frac{\log \mu(B_X(x, R))}{R},
\]

by observing that the translates \( g(B_X(x, R_0)) \) for \( g \in \Gamma \) will cover \( X \) with bounded overlap by the uniformly local doubling property of \( \mu \) and the discreteness of the action of \( \Gamma \); for this we can always enlarge the doubling radius to \( 2R_0 \) using Proposition 3.1 to obtain the bounded overlap property. Consequently \( \mu(B_X(x, R)) \) will be comparable to \( N_\Gamma(x, R) \) when \( R \) is large, which shows that the limits (3.13) and (3.14) are the same. The volume growth entropy (1.3) considered in Theorem 1.4 is a special case of the limit (3.14).

It’s clear from applying (2) of Lemma 3.7 to the limit (3.14) that we have \( h_X < \infty \). Thus for each \( h > h_X \) and \( x \in X \) we have a constant \( C_{h,x} \geq 1 \) such that for all \( R > 0 \),

\[
(3.15) \quad N_\Gamma(x, R) \leq C_{h,x} e^{hR}.
\]

The lemma below shows that we can take the constant \( C_{h,x} \) to be independent of \( x \).

**Lemma 3.8.** For each \( h > h_X \) there is a constant \( C_h \) such that we have for all \( x \in X \) and \( R > 0 \),

\[
(3.16) \quad N_\Gamma(x, R) \leq C_h e^{hR}.
\]

**Proof.** Fix \( x \in X \) and let \( C_{h,x} \) be the constant in (3.15). Recall that \( R_0 > 0 \) was chosen such that the translates of \( B_X(x, R_0) \) by \( \Gamma \) cover \( X \). For each \( y \in B_X(x, R_0) \) we have

\[
N_\Gamma(y, R) \leq N_\Gamma(x, R + R_0) \leq C_{h,x} e^{h(R + R_0)}.
\]
This implies that for $y \in B_X(x, R_0)$ we can take $C_h = C_{h,x}e^{2hR_0}$. It then follows that if $y \in g(B_X(x, R_0))$ for some $g \in \Gamma$ then

$$N_\Gamma(y, R) = N_\Gamma(g^{-1}(y), R) \leq C_h e^{hR}.$$ 

**Proposition 3.9.** For each $\beta > h_X$ the measure $\mu_\beta$ on $\bar{X}_e$ is doubling with doubling constant $C_{\mu_\beta}$ depending only on the data and the constant $C_h$ in (3.11) with $h = (\beta + h_X)/2$.

**Proof.** We follow the outline of the proof of Theorem 1.2 above for a given $\beta > h_X$, but using the estimate (3.16) in place of the use of Lemma 3.7. As in that proof, we let $\xi \in \partial X_e$, $z \in X_e$, and $r > 0$ be given such that $B_e(z, \kappa_1 r) \subset B_e(\xi, r)$ and $d_e(z) \geq 2\kappa_1 r$, where we recall that $\kappa_1 = \kappa_0/10$ depends only on the uniformization data, and we will take all implied constants to depend only on the data and the constant $C_h$ in Lemma 3.3 with $h = (\beta + h_X)/2$, except where otherwise noted. We define for $n \geq 1$,

$$A_n = \{x \in B_e(\xi, r) \cap X_e : e^{-\varepsilon n}r \leq d_e(x) < e^{-\varepsilon(n-1)}r\},$$

and note as before that we have

$$\mu_\beta(B_e(\xi, r)) = \sum_{n=1}^{\infty} \mu_\beta(A_n).$$

The estimates of the proof of Theorem 1.2 then show that we have a constant $c_0 = c_0(\delta, K, \varepsilon, M) \geq 0$ such that for each $n \geq 1$ and $x \in A_n$ we have $|xz| \leq n + c_0$.

Recall that $R_0 > 0$ was chosen such that the translates of the ball $B_0 := B_X(z, R_0)$ by $\Gamma$ cover $X$. For each $n \geq 1$ we let $\{g_{n,j}\}_{j=1}^{s_n} \subset \Gamma$ be a minimal collection of group elements such that the balls $g_{n,j}(B_0)$ cover $A_n$ for $1 \leq j \leq s_n$. By minimality we can assume that each of these balls intersects $A_n$. Setting $c_* = 2R_0 + c_0$, we then have $g_{n,j}(B_0) \subset B_X(z, n + c_*)$ for $1 \leq j \leq s_n$. In particular $g_{n,j}(y) \in B_X(z, n + c_*)$ for each $n$ and $j$. It follows from (3.16) with $h = (\beta + h_X)/2$ that

$$s_n \leq C_l e^{h(n+c_*)} \approx e^{hn},$$

since $h > h_X$, with the second comparison making the constants implicit. On the other hand, letting $x_{n,j} \in g_{n,j}(B_0) \cap A_n$ be a point in this intersection, we have $d_e(x_{n,j}) \approx r e^{-\varepsilon n}$ and therefore $\mu_\beta(x_{n,j}) \approx r^{-\beta/\varepsilon} e^{-\beta n}$ by Lemma 2.3. Hence $\mu_\beta(x_{n,j}) \approx r^{-\beta/\varepsilon} e^{-\beta n}$. Since all of the balls $g_{n,j}(B_0)$ have radius $R_0$ and since $\mu$ is $\Gamma$-invariant, the Harnack inequality (2.5) implies that

$$\mu_\beta(g_{n,j}(B_0)) \approx r^{\beta/\varepsilon} e^{-\beta n} \mu(g_{n,j}(B_0)) = r^{\beta/\varepsilon} e^{-\beta n} \mu(B_0).$$

Thus we conclude that

$$\mu_\beta(A_n) \lesssim \sum_{j=1}^{s_n} \mu_\beta(g_{n,j}(B_0)) \lesssim r^{\beta/\varepsilon} s_n e^{-\beta n} \mu(B_0) \lesssim r^{\beta/\varepsilon} e^{(h-\beta)n} \mu(B_0).$$

Since $h < \beta$, we obtain by summing the geometric series that

$$\mu_\beta(B_e(\xi, r)) \lesssim \sum_{n=1}^{\infty} r^{\beta/\varepsilon} e^{(h-\beta)n} \mu(B_0) \lesssim r^{\beta/\varepsilon} \mu(B_0).$$

Thus the hypotheses of Proposition 3.3 hold, so we can conclude the desired doubling estimate for $\mu_\beta$. \qed

Proposition 3.9 proves the doubling claim of Theorem 1.4 as we will see in the next section. As Remark 3.10 below indicates, this range for the measure to be doubling is generally sharp.
Remark 3.10. For this remark we assume that we are in the setting of Theorem 1.4, we let $X$ be a complete simply connected negatively curved Riemannian manifold with sectional curvatures $\leq -1$ and assume that we have a cocompact isometric discrete action by a group $\Gamma$ on $X$. We denote the $\Gamma$-invariant Riemannian volume on $X$ by $\mu$, fix a point $z \in X$, and consider the measure $\mu_{\beta,z}$ on $X$ defined for each $\beta > 0$ by

$$d\mu_{\beta,z}(x) = e^{-\beta|x|}d\mu(x).$$

By the theory of Patterson-Sullivan measures (see for instance [18, Théoréme 1.7]) we have $\mu_{\beta,z}(X) < \infty$ if and only if $\beta > h_X$. Since the conformal deformation $X_{1,z}$ of $X$ with conformal factor $\rho_{1,z}(x) = e^{-|x|}$ is bounded, this implies that $\mu_{\beta,z}$ is not doubling on $X_{1,z}$ when $\beta \leq h_X$. If we consider the renormalizations $\tilde{\mu}_{\beta,z} = \mu_{\beta,z}(X)^{-1}\mu_{\beta,z}$ of $\mu_{\beta,z}$ for $\beta > h_X$ as a measure on $X_{1,z} \cong X \cup \partial X$ and take the limit as $\beta \to h_X$ then these measures converge in the weak* topology to a measure $\nu_z$ on $X \cup \partial X$ that is supported on $\partial X$; here we are using that the induced topology on $X \cup \partial X$ from the identification $X_{1,z} \cong X \cup \partial X$ coincides with the standard cone topology on $X \cup \partial X$, see [4] Remark 4.14(b). This measure $\nu_z$ will be uniformly comparable with the Patterson-Sullivan measure on $\partial X$ based at $z$.

4. Poincaré inequalities for uniformized measures

We begin this section by formally introducing Poincaré inequalities. We let $(X, d, \mu)$ be a metric measure space with the property that $0 < \mu(B) < \infty$ for all balls $B \subset X$. For a measurable subset $E \subset X$ satisfying $0 < \mu(E) < \infty$ and a function $u$ that is $\mu$-integrable over $E$ we write

$$u_E = \int_E u \, d\mu = \frac{1}{\mu(E)} \int_E u \, d\mu,$$

for the mean value of $u$ over $E$. Let $u : X \to \mathbb{R}$ be given. A Borel function $g : X \to [0, \infty]$ is an upper gradient for $u$ if for each rectifiable curve $\gamma$ joining two points $x, y \in X$ we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds.$$

A measurable function $u : X \to \mathbb{R}$ is integrable on balls if for each ball $B \subset X$ we have that $u$ is integrable over $B$. For a given $p \geq 1$ we say that $X$ supports a $p$-Poincaré inequality if there are constants $\lambda \geq 1$ and $C_p > 0$ such that for each measurable function $u : X \to \mathbb{R}$ that is integrable on balls, for each ball $B \subset X$, and each upper gradient $g$ of $u$ we have

$$\int_B |u - u_B| \, d\mu \leq C_p \lambda \left( \int_B g^p \, d\mu \right)^{1/p},$$

for a constant $C_p > 0$. The constant $\lambda$ is called the dilation constant. If there is a constant $R_0 > 0$ such that (4.2) only holds on balls of radius at most $R_0$ then we will say that $X$ supports a $p$-Poincaré inequality on balls of radius at most $R_0$. We will also say that $X$ supports a uniformly local $p$-Poincaré inequality. By Hölder’s inequality a metric measure space that supports a $p$-Poincaré inequality also supports a $q$-Poincaré inequality for each $q \geq p$, and the same is true in regards to supporting a uniformly local $p$-Poincaré inequality.

For this section we carry over the same standing hypotheses and notation as discussed at the start of Section 3. We will assume in addition that we are given $p \geq 1$ such that the Gromov hyperbolic space $X$ is equipped with a uniformly locally doubling measure $\mu$ that supports a $p$-Poincaré inequality on balls of radius at most $R_0$, where $R_0$ is the same radius up to which $\mu$ is doubling on $X$. We note that Proposition 3.1 implies that there is no loss of generality in assuming that these two radii are the same. We will also assume that $\mu_{\beta}$...
is doubling on $\tilde{X}_\epsilon$ for some constant $C_{\mu_\beta}$. We will show under these hypotheses that the
metric measure space $(\tilde{X}_\epsilon, d_\epsilon, \mu_\beta)$ supports a $p$-Poincaré inequality with dilation constant
$\lambda = 1$ and constant $C_{\mu_\beta}$ depending only on the uniformization data and the constants $R_0$, $C_\mu$, $C_\beta$, $p$, $\lambda$, and $C_{\mu_\beta}$ associated to the uniformly local doubling property of $\mu$, the global
doubling of $\mu_\beta$, and the uniformly local $p$-Poincaré inequality on $X$. In particular this proves Theorem 1.3.

The proof splits into two steps. In the first step we show that the $p$-Poincaré inequality (4.2) holds on sufficiently small subWhitney balls in the metric measure space $(X_\epsilon, d_\epsilon, \mu_\beta)$.
The proof is essentially identical to [2, Lemma 6.1]. In the statement and proof of Lemma 4.2 “the data” refers to the uniformization data and the constants $R_0$, $C_\mu$, $p$, $\lambda$, and $C_{\mu_\beta}$.

For Lemma 4.2 we do not need to assume that $\mu_\beta$ is doubling. We will require the following easy lemma.

**Lemma 4.1.** [1, Lemma 4.17] Let $u : X \to \mathbb{R}$ be integrable, let $p \geq 1$, let $\alpha \in \mathbb{R}$, and let $E \subset X$ be a measurable set with $0 < \mu(E) < \infty$. Then

$$
\left( \int_E |u - u_E|^p \, d\mu \right)^{1/p} \leq 2 \left( \int_E |u - \alpha|^p \, d\mu \right)^{1/p}
$$

**Lemma 4.2.** There exists $c_0 > 0$ depending only on the uniformization data and $R_0$ such that for all $x \in X_\epsilon$ and all $0 < r \leq c_0 d_\epsilon(x)$ the $p$-Poincaré inequality (4.2) for $\mu_\beta$ holds on the ball $B_\epsilon(x, r)$ with dilation constant $\hat{\lambda}$ and constant $C_{\mu_\beta}$ depending only on the data.

**Proof.** Put $B_\epsilon = B_\epsilon(x, r)$ with $0 < r \leq c_0 d_\epsilon(x)$, where $0 < c_0 \leq \frac{1}{2}$ is a constant to be determined. Let $C_\star$ be the constant of Lemma 2.4. We choose $c_0 > 0$ small enough that $c_0 C_\star^2 \leq \frac{1}{2}$. We conclude by applying Lemma 2.4 twice that

$$
B_\epsilon \subset B := B_X \left( x, \frac{C_\star r}{\rho_\epsilon(x)} \right) \subset B_\epsilon \left( x, C_\star^2 r \right) = \hat{\lambda} B_\epsilon,
$$

with $\hat{\lambda} = C_\star^2$, since

$$
C_\star^2 r \leq c_0 C_\star^2 d_\epsilon(x) \leq \frac{1}{2} d_\epsilon(x).
$$

Moreover by (5.3) we see that for all $y \in \hat{\lambda} B_\epsilon$ we have $\rho_\beta(y) \asymp \rho_\beta(x)$ with comparison constant depending only on the uniformization data.

Now let $u$ be a function on $X_\epsilon$ that is integrable on balls and let $g_\epsilon$ be an upper gradient of $u$ on $X_\epsilon$. By the same basic calculation as in [2, (6.3)] we have that $g := g_\rho, \rho_\epsilon$ is an upper gradient of $u$ on $X$. For $c_0$ sufficiently small (depending only on the uniformization data and $R_0$) we will have by Lemma 2.3 that

$$
\frac{C_\star r}{\rho_\epsilon(x)} \leq \frac{c_0 c_0 d_\epsilon(x)}{\rho_\epsilon(x)} \leq R_0.
$$

Thus the $p$-Poincaré inequality (4.2) (for $\mu$) holds on $B$. Since $\rho_\beta(y) \asymp \rho_\beta(x)$ on $\hat{\lambda} B_\epsilon$ with comparison constant depending only on the uniformization data (by (5.3)) we have that

$$
\mu_\beta(B) \asymp \rho_\beta(x) \mu(B),
$$

with comparison constant depending only on the uniformization data, and the same comparison holds with either $B_\epsilon$ or $\hat{\lambda} B_\epsilon$, replacing $B$. Writing $u_{B, \mu} = \int_B u \, d\mu$, we conclude by using the inclusions of (4.3), the measure comparison (4.4), and the $p$-Poincaré inequality
for \( \mu \) on \( B \),
\[
\int_{B_x} |u - u_{B,\mu}| \, d\mu_\beta \lesssim \int_{B} |u - u_{B,\mu}| \, d\mu \\
\leq \frac{2C_{PI} C \rho_\varepsilon (x)}{\rho_\varepsilon (x)} \left( \int_{B} g_\rho \, d\mu \right)^{1/p} \\
\lesssim \frac{r}{\rho_\varepsilon (x)} \left( \int_{B} (g_\rho \rho_\varepsilon)^p \, d\mu_\beta \right)^{1/p} \\
\lesssim r \left( \int_{\lambda B_x} g_\rho \, d\mu_\beta \right)^{1/p},
\]
where all implied constants depend only on the data. By Lemma [14] we can replace \( u_{B,\mu} \) with \( u_{B,\mu_\beta} = \int_{B} u \, d\mu_\beta \) on the left to conclude the proof of the lemma. \( \square \)

The second part of the proof is the following key proposition.

**Proposition 4.3.** [2] Proposition 6.3 | Let \( \Omega \) be an \( A \)-uniform metric space equipped with a doubling measure \( \nu \) such that there is a constant \( 0 < c_0 < 1 \) for which the \( p \)-Poincaré inequality holds for fixed constants \( C_{PI} \) and \( \lambda \) on all subWhitney balls \( B \) of the form \( B = B_\Omega (x, r) \) with \( x \in \Omega \) and \( 0 < r \leq c_0 d_\Omega (x) \). Then the metric measure space \( (\Omega, d, \nu) \) supports a \( p \)-Poincaré inequality with dilation constant \( A \) and constant \( C_{PI} \) depending only on \( A, c_0, p, C_{PI}, \lambda \), and the doubling constant \( C_\nu \) for \( \nu \).

This proposition is stated for bounded \( A \)-uniform metric spaces in [2] but the proof works without modification for unbounded \( A \)-uniform metric spaces provided that the doubling property of \( \nu \) holds at all scales and the \( p \)-Poincaré inequality on subWhitney balls hold at all appropriate scales.

We can now verify the global \( p \)-Poincaré inequality on \( X_\varepsilon \), which proves Theorem 1.3. Below “the data” includes all the constants from Lemma 4.2 as well as the doubling constant \( C_{\mu_\beta} \) for \( \mu_\beta \).

**Proof of Theorem 1.3.** By Lemma 4.2 there is a \( c_0 > 0 \) determined only by the data such that the \( p \)-Poincaré inequality holds on subWhitney balls of the form \( B_\varepsilon (x, r) \) with \( x \in X_\varepsilon \), with uniform constants \( C_{PI} \) and \( \bar{\lambda} \). Since \( (X_\varepsilon, d_\varepsilon) \) is an \( A \)-uniform metric space with \( A = A(\delta, K, \varepsilon, M) \) and we assumed \( \mu_\beta \) is globally doubling on \( X_\varepsilon \) with constant \( \mu_\beta \), it follows from Proposition 4.3 that the metric measure space \( (X_\varepsilon, d_\varepsilon, \mu_\beta) \) supports a \( p \)-Poincaré inequality with constant \( C_{PI} \) depending only on the data and dilation constant \( A \). Since \( X_\varepsilon \) is geodesic it follows that the \( p \)-Poincaré inequality \( (4.2) \) in fact holds with dilation constant 1, with constant \( C_{PI} \) depending only on the data [13, Theorem 4.18].

By [14] Lemma 8.2.3 we conclude that the completion \( (X_\varepsilon, d_\varepsilon, \mu_\beta) \) (with \( \mu_\beta (\partial X_\varepsilon) = 0 \)) also supports a \( p \)-Poincaré inequality with constants depending only on the constants for the \( p \)-Poincaré inequality on \( X_\varepsilon \) and the doubling constant of \( \mu_\beta \). Since \( X_\varepsilon \) is also geodesic it follows by the same reasoning [13, Theorem 4.18] that we can take the dilation constant to be 1 in this case as well. \( \square \)

We can now prove Theorem 1.4 as well.

**Proof of Theorem 1.4.** Let \( X \) be a complete simply connected \( n \)-dimensional Riemannian manifold \( X \) with sectional curvatures \( \leq -1 \) that is equipped with a cocompact discrete isometric action of a group \( \Gamma \). Then \( X \) is \( \delta \)-hyperbolic with \( \delta = \delta (\mathbb{H}^2) \) being the same as that of the hyperbolic plane \( \mathbb{H}^2 \) of constant negative curvature \( -1 \) [5, p. 169]. Let \( \mu \) be the
The space $X$ is 0-roughly starlike from any point of $X \cup \partial X$ since any geodesic $\gamma : I \to X$ defined on any interval $I \subset \mathbb{R}$ can be uniquely extended to a full geodesic line $\gamma : \mathbb{R} \to X$. By [9, Theorem 1.10] the densities $\rho_{1,b}$ for $b \in \mathcal{B}(X)$ are GH-densities with a uniform constant $M$. Thus we can apply the results of the previous sections here with this constant $M$ and $\delta = \delta(\mathbb{H}^2)$, $K = 0$, and $\varepsilon = 1$.

Choose $R_0 > 0$ large enough that for each $x \in X$ the translates of the ball $B_X(x, R_0)$ by $\Gamma$ cover $X$. On each such ball $B_X(x, R_0)$ the Riemannian metric on $X$ is biLipschitz to the standard Euclidean metric on the unit ball in $\mathbb{R}^n$ with biLipschitz constant independent of $x$ (by the cocompactness of $\Gamma$) and the Riemannian volume is uniformly comparable to the standard $n$-dimensional Lebesgue measure. Since $\mathbb{R}^n$ equipped with the $n$-dimensional Lebesgue measure is a doubling metric measure space that supports a 1-Poincaré inequality [13, Chapter 4], it follows that $X$ equipped with $\mu$ is uniformly globally doubling and supports a uniformly locally 1-Poincaré inequality. We remark that all of the parameters considered so far are independent of the choice of $b \in \mathcal{B}(X)$.

We conclude by Proposition 3.9 that for each $\beta > h_X$ the metric measure space $(\bar{X}_{1,b}, d_{1,b}, \mu_{\beta,b})$ is doubling with a uniform doubling constant $C_{\mu,b}$ independent of the choice of $b \in \mathcal{B}(X)$. The 1-Poincaré inequality on $(X_{1,b}, d_{1,b}, \mu_{\beta,b})$ and $(\bar{X}_{1,b}, d_{1,b}, \mu_{\beta,b})$ then follows from Theorem [13].

5. Uniform inversion

In this section we consider a procedure that we will call uniform inversion that can be used to convert bounded uniform metric spaces into unbounded uniform metric spaces and vice versa. This procedure can be thought of as a variation of the inversion procedure considered in [15] that is specialized to the context of uniform metric spaces. We show that this procedure can be extended to measures in such a way that it preserves the doubling property and $p$-Poincaré inequalities for a given $p \geq 1$. For general metric measure spaces it was shown by Li and Shanmugalingam [16] that the doubling property can be preserved under sphericalization and inversion, however they were only able to obtain preservation of $p$-Poincaré inequalities under the additional assumption that the space was annularly quasiconvex. This condition excludes many uniform metric spaces such as those that are obtained by uniformizing trees. With a weaker assumption Durand-Cartagena and Li [11] showed that $p$-Poincaré inequalities can be preserved once $p$ is sufficiently large. Using the results of the previous sections we will show that uniform inversion preserves $p$-Poincaré inequalities for all $p \geq 1$.

Let $(\Omega, d)$ be an $A$-uniform metric space, $A \geq 1$. We denote the distance to the metric boundary of $\Omega$ by $d(x) := d_{\Omega}(x)$ for $x \in \Omega$. The quasihyperbolic metric on $\Omega$ is defined by, for $x, y \in \Omega$,

\begin{equation}
(5.1) \quad k(x, y) = \inf \int_0^1 \frac{ds}{d(\gamma(s))},
\end{equation}

where the infimum is taken over all rectifiable curves joining $x$ to $y$. The metric space $Y = (\Omega, k)$ is called the quasihyperbolization of the metric space $(\Omega, d)$. We note that $Y$ can equivalently be thought of as the conformal deformation of $\Omega$ with conformal factor $\rho(x) = d(x)^{-1}$. The quasihyperbolization $Y$ is a proper geodesic $\delta$-hyperbolic space by [4, Theorem 3.6] with $\delta = \delta(A)$ depending only on $A$.

To precisely state our claims below we introduce the following ratio when $\Omega$ is bounded,

\begin{equation}
(5.2) \quad \phi(\Omega) := \frac{\text{diam } \Omega}{\text{diam } \partial \Omega},
\end{equation}
where we define \( \phi(\Omega) = \infty \) if \( \partial \Omega \) contains only one point. Until the end of this section we will always assume that \( \phi(\Omega) < \infty \) if \( \Omega \) is bounded, i.e., that \( \partial \Omega \) contains at least two points.

**Remark 5.1.** The case of bounded \( \Omega \) with \( \partial \Omega \) containing only one point is rather degenerate so we will not discuss it here. For instance if \( \Omega = [0,1] \) then its quasihyperbolization \( Y \) is isometric to \([0,\infty)\), and a Busemann function \( b \) on \([0,\infty)\) based at the only point \( \infty \) in the Gromov boundary of \([0,\infty)\) is given by \( b(t) = -t \) for \( t \in [0,\infty) \). By direct calculation we then see for every \( \varepsilon > 0 \) that \( Y_{\varepsilon,b} \) is also isometric to \([0,\infty)\). In particular \( Y_{\varepsilon,b} \) is actually a complete metric space, so it can’t be a uniform metric space.

When \( \Omega \) is unbounded there is a constant \( K = K(A) \) depending only on \( A \) such that \( Y \) is \( K \)-roughly starlike from any point of \( Y \cup \partial Y \), while when \( \Omega \) is bounded there is a constant \( K = K(\Omega) \) such that \( Y \) is \( K \)-roughly starlike from any \( z \in \Omega \) such that \( d(z) = \sup_{x \in \Omega} d(x) \), and a constant \( K' = K'(A,\phi(\Omega)) \) depending only on \( A \) and the ratio \( \phi(\Omega) \) such that \( Y \) is \( K' \)-roughly starlike from any point of \( Y \cup \partial Y \) [8, Proposition 3.3]. The dependence of \( K' \) on \( \phi(\Omega) \) in the bounded case is necessary by [8, Example 3.4].

From the discussion after inequality (1.7) we can find an \( \varepsilon = \varepsilon(A) > 0 \) depending only on \( A \) (since \( Y \) is \( \delta \)-hyperbolic with \( \delta = \delta(A) \)) such that for any \( b \in B(Y) \) the density \( \rho_{\varepsilon,b}(x) = e^{-\varepsilon(\varepsilon(x))} \) on \( Y \) is a GH-density with constant \( M = 20 \). We will fix such an \( \varepsilon \) for each value of \( A \) for the rest of this section.

**Definition 5.2.** For a given \( b \in B(Y) \) we let \( \Omega_b = Y_{\varepsilon,b} \) denote the conformal deformation of \( Y \) with conformal factor \( \rho_{\varepsilon,b} \). We will refer to the metric space \( \Omega_b \) as the **uniform inversion** of \( \Omega \) based at \( b \).

The next proposition shows that uniform inversions of \( \Omega \) have the properties suggested by their name.

**Proposition 5.3.** Let \( \Omega \) be an \( A \)-uniform metric space. Let \( Y = (\Omega,k) \) be the quasihyperbolization of \( \Omega \). Then \( \Omega_b \) is an \( A' \)-uniform metric space for each \( b \in B(Y) \) with \( A' = A'(A) \) if \( \Omega \) is unbounded and \( A' = A'(A,\phi(\Omega)) \) if \( A' \) is unbounded. Furthermore \( \Omega_b \) is bounded if and only if \( b \in D(Y) \).

All of the claims of Proposition 5.3 follow from applying [9, Theorem 1.1] to \( Y \), since \( Y \) is \( \delta = \delta(A) \)-hyperbolic, \( K = K(A) \)-roughly starlike from any point of \( Y \cup \partial Y \) if \( \Omega \) is unbounded (with \( K = K(A,\phi(\Omega)) \) instead if \( \Omega \) is bounded) and \( \rho_{\varepsilon,b} \) is a GH-density with constant \( M = 20 \) (and \( \varepsilon = \varepsilon(A) \)). In particular uniform inversion can be used to produce an unbounded uniform metric space \( \Omega_b \) from a bounded uniform metric space \( \Omega \) by choosing \( b \in B(Y) \), and similarly can be used to produce a bounded uniform metric space \( \Omega_b \) from an unbounded uniform metric space \( \Omega \) by choosing \( b \in D(Y) \).

The primary reason to consider uniform sphericalization and inversion is that these operations can be extended to measures in such a way as to preserve the doubling property and the \( p \)-Poincaré inequality for all \( p \geq 1 \). This comes at a price of increased complexity of these operations as opposed to the standard inversion operation, with the loss of several nice features of the latter that are obtained in [6]. We remark that one can show that the identity map \( \Omega \to \Omega_b \) is always quasimöbius for any \( b \in B(Y) \), as is true of ordinary inversion; see [8, Proposition 4.4].

Now suppose in addition that \( \Omega \) is equipped with a Borel measure \( \nu \) that is doubling and satisfies \( 0 < \nu(B) < \infty \) for all balls \( B \subset \Omega \). We write \( C_\nu \) for the doubling constant of \( \nu \).
For each $\alpha > 0$ we define a measure $\mu^\alpha$ on $\Omega$ by
\[ d\mu^\alpha(x) = d(x)^{-\alpha}d\nu(x), \]
and consider $\mu$ as a measure on $Y$. Then [2] Proposition 7.3] shows for each $\alpha > 0$ that $\mu$ is doubling on balls of radius at most $R_0 = 1$ with local doubling constant $C_\mu^\alpha$ depending only on $A$ and $\alpha$. We let $\beta_0 > 0$ be the exponent determined by applying Theorem 1.2 to the quasihyperbolization $\hat{Y}$ equipped with the measure $\mu^\alpha$ in relation to its uniformization $\Omega_b = Y_{\varepsilon,b}$ with $b \in \hat{B}(Y)$ and $\varepsilon = \varepsilon(A)$. We then choose $\beta \geq \beta_0$ and set $\nu_{\alpha,\beta,b} = (\mu^\alpha)_{\beta,b}$ to be the measure obtained from $\mu^\alpha$ by applying the formula (1.6) with our chosen $b \in \hat{B}(Y)$. We consider $\nu_{\alpha,\beta,b}$ as defining a two parameter family of measures on $\Omega_b$ and write $d_b$ for the metric on $\Omega_b$. Applying Theorems 1.2 and 1.3 to this family yields the following theorem.

**Theorem 5.4.** Let $(\Omega, d, \nu)$ be a doubling metric measure space with doubling constant $C_\nu$ such that $(\Omega, d)$ is an $A$-uniform metric space with $\phi(\Omega) < \infty$ if $\Omega$ is bounded. Let $Y = (\Omega, k)$ be the quasihyperbolization of $\Omega$. Then for each $\alpha > 0$ there is a constant $\beta_0 = \beta_0(\alpha, A, C_\nu)$ (if $\Omega$ is unbounded) or $\beta_0 = \beta_0(\alpha, A, C_\nu, \phi(\Omega))$ (if $\Omega$ is bounded) such that for any $b \in \hat{B}(Y)$ and any $\beta \geq \beta_0$ we have that $\nu_{\alpha,\beta,b}$ is doubling on $\Omega_b$ with doubling constant $C_{\nu_{\alpha,\beta,b}}$ depending only on $\alpha$, $\beta$, $A$, and $C_\nu$ (and $\phi(\Omega)$ if $\Omega$ is bounded).

If furthermore the metric measure space $(\Omega, d, \nu)$ supports a $p$-Poincaré inequality for a given $p \geq 1$ then the metric measure space $(\Omega_b, d_b, \nu_{\alpha,\beta,b})$ supports a $p$-Poincaré inequality with constants depending only on $\alpha$, $\beta$, $A$, $C_\nu$, $p$, and the constants in (4.2) (and $\phi(\Omega)$ if $\Omega$ is bounded).

The final claim follows from the fact that under the hypotheses of the proposition the metric measure space $(Y, k, \mu^\alpha)$ supports a uniformly local $p$-Poincaré inequality by [2] Proposition 7.4 with radius and constants depending only on $\alpha$, $A$, $C_\nu$, $p$, and the constants in the $p$-Poincaré inequality for $(\Omega, d, \nu)$. Hence we can directly apply Theorem 1.3 to $(Y_{\varepsilon,b}, k_{\varepsilon,b}, (\mu^\alpha)_{\beta,b}) = (\Omega_b, d_b, \nu_{\alpha,\beta,b})$ in this case.

We remark that it is not immediately clear what choices of $\alpha$ and $\beta$ are natural in the context of Proposition 5.4, which is why we have left them as free parameters. Theorem 5.4 shows that $(\Omega_b, d_b, \nu_{\beta,\beta,b})$ is always a doubling metric measure space once $\beta$ is large enough in relation to $\alpha$, and that $p$-Poincaré inequalities transfer over to this space from $(\Omega, d, \nu)$.

**References**

[1] A. Björn and J. Björn. *Nonlinear potential theory on metric spaces*, volume 17 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2011.

[2] A. Björn, J. Björn, and N. Shanmugalingam. Bounded geometry and $p$-harmonic functions under uniformization and hyperbolization. *J. Geom. Anal.* To appear.

[3] J. Björn and N. Shanmugalingam. Poincaré inequalities, uniform domains and extension properties for Newton-Sobolev functions in metric spaces. *J. Math. Anal. Appl.*, 332(1):190–208, 2007.

[4] M. Bonk, J. Heinonen, and P. Koskela. Uniformizing Gromov hyperbolic spaces. *Astérisque*, (270):viii+99, 2001.

[5] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999.

[6] Stephen M. Buckley, David A. Herron, and Xiangdong Xie. Metric space inversions, quasihyperbolic distance, and uniform spaces. *Indiana Univ. Math. J.*, 57(2):837–890, 2008.

[7] C. Butler. Extension and trace theorems for noncompact doubling spaces. 2020. arXiv:2009.10168.

[8] C. Butler. Uniformization, $\beta$-bilipschitz maps, sphericalization, and inversion. 2020. arXiv:2008.06806.

[9] C. Butler. Uniformizing Gromov hyperbolic spaces with Busemann functions. 2020. arXiv:2007.11143.

[10] Sergei Buyalo and Viktor Schroeder. *Elements of asymptotic geometry*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007.
[11] Estibalitz Durand-Cartagena and Xining Li. Preservation of p-Poincaré inequality for large p under sphericalization and flattening. Illinois J. Math., 59(4):1043–1069, 2015.

[12] É. Ghys and P. de la Harpe, editors. Sur les groupes hyperboliques d’après Mikhael Gromov, volume 83 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[13] J. Heinonen. Lectures on analysis on metric spaces. Universitext. Springer-Verlag, New York, 2001.

[14] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson. Sobolev spaces on metric measure spaces, volume 27 of New Mathematical Monographs. Cambridge University Press, Cambridge, 2015. An approach based on upper gradients.

[15] David Herron, Nageswari Shanmugalingam, and Xiangdong Xie. Uniformity from Gromov hyperbolicity. Illinois J. Math., 52(4):1065–1109, 2008.

[16] Xining Li and Nageswari Shanmugalingam. Preservation of bounded geometry under sphericalization and flattening. Indiana Univ. Math. J., 64(5):1303–1341, 2015.

[17] Anthony Manning. Topological entropy for geodesic flows. Ann. of Math. (2), 110(3):567–573, 1979.

[18] Thomas Roblin. Ergodicité et équidistribution en courbure négative. Mém. Soc. Math. Fr. (N.S.), (95):vi+96, 2003.