Conformal factor dynamics in the $1/N$ expansion

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Abstract

We suggest to consider conformal factor dynamics as applying to composite boundstates, in frames of the $1/N$ expansion. In this way, a new model of effective theory for quantum gravity is obtained. The renormalization group (RG) analysis of this model provides a framework to solve the cosmological constant problem, since the value of this constant turns out to be suppressed, as a result of the RG contributions. The effective potential for the conformal factor is found too.
The investigation of the dynamics of the conformal factor is becoming very fashionable. At the classical level, conformal-factor dynamics describe the conformally-flat solutions of the equations for the gravitation theories. At the quantum level, the dynamics of the conformal factor (induced by the conformal anomaly) was suggested in the first paper of Ref. [1] as a tool for the description of quantum gravity (QG) in the infrared (IR) phase. Further study of such effective theory for QG, of its properties, and of some extensions of it, has been carried out in Refs. [1].

Conformal factor dynamics give rise to the effective potential for the conformal factor [2, 3], which is very useful in QG (for a general review of perturbative QG see [4]). In particular, for the case R^2-gravity (a multiplicatively renormalizable theory) the corresponding theory for the conformal factor has been developed in Ref. [5].

The conformal-factor theory leads naturally to the appearance of a theory of the four-dimensional sigma model type, with a very interesting one-loop renormalization [6]. When interacting with the standard model (SM), the conformal factor appears as the dilaton of the theory [7]. It may naturally emerge also in the study of the SM in the context of non-commutative geometry [8].

There exist a certain number of quantum field theories (for example the four-fermion models [9, 10], for a recent discussion see [11]) which allow for an analytical study of their composite boundstates. Some aspects of the gravitational interaction with the Nambu-Jona-Lasinio (NJL) model have been studied in Refs. [12]-[14] already. It is the purpose of the present letter to investigate specifically conformal factor dynamics as corresponding to composite boundstates, using the 1/N expansion and drawing some analogies from the four-fermion theories [12]-[14].

Our starting point is the two-dimensional theory with action

\[ S = \int d^2 x \sqrt{-g} \left[ \bar{\psi} (i \gamma^\mu (x) \nabla_\mu - m) \psi + R - \frac{\Lambda}{2} \right], \]

where the massive N-component spinor \( \psi \) is considered to be a quantum field. The gravitational field, on the other hand, may be either classical or quantum. We also consider the conformal parametrization of the metric

\[ g_{\mu \nu} = \rho^2 \eta_{\mu \nu}, \]

where \( \rho \) is the conformal factor (in general it is \( \rho = \rho(x) \)) and \( \eta_{\mu \nu} \) is the flat fiducial metric. In the QG case, the choice \( \rho = \rho(x) \) corresponds to the gauge fixing. Substituting \( (2) \) into \( (1) \) one
gets, at the classical level,

$$S = \int d^2 x \left[ \overline{\chi} \left( i \gamma^\mu \partial_\mu - m \rho \right) \chi - \frac{\Lambda}{2} \rho^2 \right],$$  \hspace{1cm} (3)

where $\chi = \rho^{1/2} \psi$. Rescaling $\rho \rightarrow \Lambda^{1/2} \rho$, we get

$$S = \int d^2 x \left[ \overline{\chi} \left( i \gamma^\mu \partial_\mu - h \rho \right) \chi - \frac{1}{2} \rho^2 \right],$$  \hspace{1cm} (4)

where $h = m \Lambda^{-1/2}$. As one can see, action (4) has the form that is typical for the Gross-Neveu (GN) model ($\rho \sim \overline{\chi} \chi$). The dynamics of this model are quite well known [10]: asymptotic freedom in the UF limit

$$h^2(t) = \frac{h^2}{1 + h^2 N t / \pi},$$  \hspace{1cm} (5)

where $t$ is the RG parameter. However, since (4) describes also the dynamics of the conformal factor, the interpretation of the function $h^2(t)$ is now completely different from the original interpretation. The $h^2(t)$ here is a combination of the fermionic mass in (1) and of the two-dimensional cosmological constant $\Lambda$. Using the anomalous scaling dimension one gets the running composite field

$$\rho(x, t) = \rho(x) \left( 1 + h^2 N t / \pi \right)^{-1/2}.$$  \hspace{1cm} (6)

The conformal factor has acquired the classical dimension after the rescaling $\rho \rightarrow \Lambda^{1/2} \rho$. Hence, one may argue that the $t$ dependence is due completely to that of the cosmological (dimensional) constant, i.e., $\Lambda(t) \sim \Lambda (1 + h^2 N t / \pi)^{-1}$. Therefore, there appears to be a screening of the cosmological constant in the $1/N$ expansion in the UF regime.

One can also investigate specific features of the effective potential for the conformal factor, which coincides with the GN effective potential [10]. In particular, the appearance of a minimum, i.e., a non-zero vacuum expectation value (v.e.v.), for the conformal factor

$$\rho = \rho_0 \exp \left( 1 - \frac{\pi}{h^2 N} \right)$$  \hspace{1cm} (7)

is interesting. After having shown the possibility to study the dynamics of the conformal factor in two dimensions as the dynamics of the GN model, we now turn to the four-dimensional theory, which is physically more interesting.

We start from the multiplicatively renormalizable theory [4] with action

$$S = \int d^4 x \sqrt{-g} \left[ \overline{\psi} \left( i \gamma^{\mu}(x) \nabla_\mu - m \right) \psi - \frac{\Lambda}{\kappa^2} - \frac{R}{\kappa^2} + \frac{W}{\lambda_1} - \frac{U R^2}{3 \lambda_1} \right],$$  \hspace{1cm} (8)
where $\psi$ is an $N$-component spinor and $W$ the square of the Weyl tensor. The gravitational field may be chosen to be classical or quantum, and the theory remains multiplicatively renormalizable in both cases.

We shall work again with the conformal parametrization (2) for the four-dimensional metric. In the case of four-dimensional QG this does not fix the gauge, contrary to what happens in two dimensions, but it can still be considered as a convenient background [5]. Rewriting action (8), we get

$$S = \int d^4x \left\{ \chi (i\gamma^\mu \partial_\mu - m\rho) \chi - \frac{\Lambda}{\kappa^2} \rho^4 - \frac{6}{\kappa^2} (\partial \rho)^2 + \frac{12U}{\Lambda_1} \left[ \sigma \Box \sigma + 2(\partial \sigma)^2 \Box \sigma + (\partial \sigma)^2 (\partial \sigma)^2 \right] \right\},$$

(9)

where $\chi = \rho^{3/2} \psi$ and $\sigma = \ln \rho$. In this way we have got the classical theory for the conformal factor. At the quantum level, the theory (9) may be considered as an effective theory for QG (see also [1]-[3] and [5]). If we drop the $\rho$-terms with derivatives from action (9), we obtain a model that is reminiscent of the NJL model (where, of course, owing to the absence of the $M^2 \rho^2$-term, it is $\rho \sim (\chi \chi)^{1/3}$).

Now we are going to study the theory (9) in the large-$N$ limit, while concentrating our attention on the RG and low-derivative terms in (9). The higher-derivative terms are actually of lesser importance, moreover, they simply disappear in the subsequent analysis of the effective potential for the conformal factor. First of all, we rescale $\rho \to \sqrt{12} \rho/\kappa$ and denote $h = m\kappa/\sqrt{12}$ and $\lambda = \Lambda \kappa^2 / 6$. Thus,

$$S = \int d^4x \left[ \chi (i\gamma^\mu \partial_\mu - h\rho) \chi - \frac{\lambda}{24} \rho^4 - \frac{1}{2} (\partial \rho)^2 \right].$$

(10)

By integrating over the fermionic field, we get the effective potential for the conformal factor at large $N$:

$$V(\rho) = \frac{\lambda \rho^4}{24} + iN \text{Tr} \ln (i\gamma^\mu \partial_\mu - h\rho),$$

(11)

where $\rho$ is constant. Supposing that, as usually, for large $N$ $\lambda$ scales as $\lambda \sim \tilde{\lambda}N$, where $\tilde{\lambda}$ does not depend on $N$, and using a finite cut-off $\mu$ (see also [13]) we get (notice that $N$ has been factored out)

$$V(\rho) = \frac{\tilde{\lambda} \rho^4}{24} - \frac{1}{(4\pi)^2} \left[ \rho^2 \mu^2 + \mu^4 \ln \left( 1 + \frac{\rho^2}{\mu^2} \right) - \rho^4 \ln \left( 1 + \frac{\mu^2}{\rho^2} \right) \right].$$

(12)

The v.e.v. of the conformal factor can be found from (12) as the solution of the equation

$$\frac{\partial V(\rho)}{\partial \rho} = \frac{\tilde{\lambda}}{6} \rho^2 - \frac{1}{(2\pi)^2} \left[ \mu^2 - \rho^2 \ln \left( 1 + \frac{\mu^2}{\rho^2} \right) \right] = 0.$$  

(13)
Table 1: Numerical values for the vacuum expectation value of the conformal factor, corresponding to a sample of values of $\tilde{\lambda}$ between 0 and 1. They are obtained as the root of Eq. (13) that lies between 0 and 1 (notice that for $\tilde{\lambda} \leq 0.04643$ there is no root in that interval).

| $\tilde{\lambda}$ | root $\rho^2/\mu^2$ |
|------------------|----------------------|
| $10^{-4}$        | 27.2361              |
| $10^{-3}$        | 8.3933               |
| 0.01             | 2.4517               |
| 0.464            | 1.00043              |
| 0.465            | 0.99912              |
| 0.05             | 0.9558               |
| 0.1              | 0.6166               |
| 0.2              | 0.3853               |
| 0.3              | 0.2880               |
| 0.4              | 0.2326               |
| 0.5              | 0.1962               |
| 0.6              | 0.1702               |
| 0.7              | 0.1506               |
| 0.8              | 0.1353               |
| 0.9              | 0.1229               |
| 0.99             | 0.1137               |
| 1                | 0.11275              |

In Table 1 we present a sample of numerical values of the solution $\rho^2/\mu^2$ (which lie between 0 and 1 for $0.04644 \leq \tilde{\lambda} \leq 1$). Notice that from the point of view of the original theory, a non-zero v.e.v. for $\rho$ is more acceptable physically, because for $\rho = 0$ the conformal parametrization (2) becomes degenerate.

Now we turn to the study of the renormalization structure of theory (10), which is rather non-trivial. There are a few different ways to renormalize this theory, the actual problem being the fact that $\rho$ is dimensionless. Considering (10) in the large-$N$ approximation, one can perform the change $\rho \rightarrow i \rho$, which introduces an overall minus sign in the parametrization (2), producing a kinetic term of the standard form. The factor $i$ in the fermionic sector can
be absorbed in $h$ (imaginary fermion mass). Then, supposing that $\rho$ is also a quantum field (i.e. that gravity is quantized in (18)), the renormalization of such theory, of Yukawa type, can be done in the standard way. The beta functions are calculated to be

$$
\beta_h = \frac{4Nh^4}{(4\pi)^2}, \quad \beta_\lambda = \frac{3\lambda^2 + 8N\lambda h^2 - 48Nh^4}{(4\pi)^2}.
$$

(14)

We see that $h = \lambda = 0$ is the IR stable fixed point, and in the IR regime ($t \to -\infty$), we have

$$
h^2(t) \sim -\frac{4\pi^2}{Nt}, \quad \lambda(t) \sim -\frac{48\pi^2}{Nt}.
$$

(15)

Supposing that the Newton coupling constant is a non-running coupling, then the running of $h^2(t)$ gives the RG screening of the imaginary fermionic mass in the infrared. At the same time, the cosmological constant quickly decays in the IR. Thus, in the considered phase of the effective theory for the conformal factor, we get a solution of the cosmological constant problem, as a result of RG effects in the IR.

We may also consider a different version of (10) —as it stands— which will correspond eventually to another QG phase. Now there are no imaginary parameters. After renormalization of $\rho$ (taking into account the negative sign for $(\partial \rho)^2$), we obtain

$$
\beta_h = -\frac{4Nh^4}{(4\pi)^2}, \quad \beta_\lambda = \frac{3\lambda^2 - 8N\lambda h^2 - 48Nh^4}{(4\pi)^2}.
$$

(16)

The theory has now an UF stable fixed point ($t \to +\infty$), where the behavior of $h^2$ and $\lambda$ is

$$
h^2(t) \sim \frac{4\pi^2}{Nt}, \quad \lambda(t) \sim -\frac{48\pi^2}{Nt}.
$$

(17)

Hence, we now obtain a decrease of the cosmological constant in the UF limit. Notice that in comparing this theory with the standard scalar self-interacting theory, we do not have here physical restrictions on the sign of $\lambda$, and a negative sign is perfectly acceptable.

Finally, let us observe that from the point of view of Eq. (3), where $\rho$ is dimensionless, one can understand (1) as being a kind of four-dimensional $\sigma$ model [6]. Then, it has sense to discuss a renormalization of (9) of $\sigma$-model type, at large $N$. In that case $\lambda_2 = \Lambda^2/(6\kappa^2)$, $\rho$ (rescaled as $\rho \to \sqrt{12}\rho$) and $m$ are not renormalized to leading order in $N$, and

$$
\beta_{\lambda_2} = \frac{3\lambda_2^2 - 48Nm^4}{(4\pi)^2}, \quad \beta_{\kappa^{-2}} = \frac{Nm^2}{4\pi^2}.
$$

(18)

As a result one can see that $\kappa^{-2}(t) \sim Nt$ and $\lambda_2(t) \sim -Nt$ at large $t$. Hence, considering the gravitational constant as the running coupling constant and using the generalized RG we do not obtain a damping of the running cosmological constant in that case.
It is very interesting to observe that if we would have started from the anomaly-induced theory for the cosmological factor [1] and had considered its interaction with the massive fermion, we would have obtained the same theory [1] (only some relative coefficients of the higher-derivative terms would be different). Hence, the preceding discussion can be applied without the least change to this case too.

As next step we will now consider the appearance of the conformal factor in the proper NJL dynamics. Starting point is theory (9), where in the place of the fermionic sector we substitute the action for the NJL model [11, 12]

\[ L = i\bar{\psi}\gamma^\mu(x)D_\mu\psi + (\bar{\psi}_L\gamma^a\psi_RaH + \text{h.c.}) - M_\mu^2H\dagger H, \] (19)

being \( \psi_{L,R} = \frac{1}{2}(1 \mp \gamma_5)\psi \), \( H \) an auxiliary scalar and \( M_\mu \) the scalar field mass of the UF scalar \( \mu_{UF} \). Such theory is a four-fermion one [9], where the four-fermion coupling constant is \( G = M_\mu^{-2} \). Notice that the above theory is non-renormalizable (effective), so that there is no need, for instance, of keeping higher-derivative terms in (18). Working in the conformal parametrization (2), after rescaling \( \chi = \rho^{3/2}\psi \) and \( H \to \rho H \) we get the flat-space NJL model, with the only difference with respect to [11, 12] that the scalar-field mass becomes \( \rho \)-dependent: \( \tilde{M}_\mu^2 = \rho^2M_\mu^2 \) in (19). Performing a block-spin RG transformation (in the \( 1/N \) expansion) we can study the RG here precisely in the same way as it was done in [11, 12], the only difference being that the induced scalar mass will now be

\[ M_{\text{ind}}^2 = \rho^2M_\mu^2 - \frac{N}{8\pi^2}(\mu_{UF}^2 - \mu_{IR}^2), \] (20)

and the dilaton sector is not influenced by renormalization. Two parameters, \( \rho \) and \( M_\mu^2 \), appear in (20), so even the value of the conformal factor will be defined by the minimization of the corresponding effective potential, their combination being used for the approximate cancellation of the corresponding term \( N\mu_{UF}^2/(8\pi^2) \). Finally, let us point out that starting from a massive model of GN type and after using (9), the conformal field dynamics are influenced again by fermionic effects, and the total model includes scalars, as the NJL model. The study of this sort of model, where the conformal factor is in the composite field, can be carried out in analogy with that of the NJL model.

In summary, we have considered in this paper the conformal field dynamics in a framework where the conformal factor becomes naturally a boundstate and the \( 1/N \) expansion can be applied. As a particular result, we have obtained in this way a possible mechanism for the vanishing at the RG fixed point of the cosmological constant (assuming that the Newton
constant is a non-running coupling). The interaction of the conformal factor in the NJL model has been discussed too. Our results show that the $1/N$ expansion can well be applied for the description of effective theories of QG. A synthesis of this approach with the anomaly-induced theory for the conformal factor may be very helpful for the resolution of the drawbacks of previous scenarios [1] (like neglecting the spin-2 gravitational modes), by means of the use of the large-$N$ expansion.

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