Optimization of Heated Structures for Several Load Cases

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Abstract. Optimization methods providing maximum strength and rigidity of the thin-walled structure are considered. A critical overview of the works related to the optimization including the heated design is done. To optimize for several cases of loading, a minimization of functional is applied, which is the sum of the total potential strain energies for each case of loading. The optimization process begins with the initial basic structure, which includes all possible variants of the location of the power elements. In the optimization process, inefficient elements are eliminated by minimizing their rigidity. Numerical optimization studies on the example of truss construction are carried out.

1. Introduction
The task of designing and obtaining the load-carrying structure of the minimum mass is paramount at all stages of the creation of the flight vehicle (FV). Typically, this process is called optimization, and it means the structure must be optimal. Under the term "optimal" in this paper we will understand the design of maximum strength and stiffness (minimum compliance) within a given mass or Vice versa – the minimization of the mass at a given strength and stiffness. Obviously, the design can be optimal only for one case of loading. In the process of functioning of the structure, it can be affected by various loads both in magnitude and in the nature of application. Moreover, these loads can be both mechanical and thermal. In this regard, the obtaining a rational design in the optimization process is very important problem.

The optimal solution of the problem is achieved primarily by choosing the optimal topology of the structure (that is, the location in a given design area of the load-bearing elements and their relationship) – this process is usually called topological optimization. Further refinement and selection of structural parameters of load-bearing elements is carried out within the selected structure (parametric optimization). Historically, these two stages of load-bearing structure design have always been carried out separately by different departments of design organizations and different specialists: designers and strength specialists. The emergence of computational and numerical methods, in particular FEM, allowed to bring these two processes closer together. However, the leading role in the process of topological optimization is left to the person with experience and intuition, which is quite difficult to formalize.

The presence of temperature effects due to significant heating or cooling can significantly complicate the process of finding the optimal structure. In the conditions of rapid development of super-and hypersonic FV, this direction requires careful research.
2. Overview of optimization methods

The methods of nonlinear mathematical programming which called the direct optimization methods are the most common to solve the problem of parametric optimization [1]. For the design variables within the specified types of FE most often accept parameters that describe the dimensions of the cross sections of finite elements (plate thickness or cross-sectional area of the rod). In this case, the optimization problem is usually described by a large number of design variables and functional constraints, estimated for real structures in hundreds of thousands, which complicates a detailed study of the search area and, consequently, obtaining accurate results. This circumstance is complicated by the fact that the methods of nonlinear mathematical programming use a very time-consuming procedure for analyzing the sensitivity of stress constraints for the set of design points for all design variables. In this regard, the use of indirect methods of optimization of elastic systems, which can have higher efficiency due to the purposeful use of the characteristic properties of the design object, is of practical interest.

As part of the indirect approach, the necessary conditions to be met by the desired design are recorded, and the procedure for finding this design is constructed. Optimality conditions are either derived from the mathematical formulation of the problem, or based on the behaviour characteristics of some classes of elastic systems [2, 3, 4]. For solving the above problem are most often used two criteria: of full-stress structure or of the greatest rigidity.

In most cases, at the initial stages of power design of thin-walled structures such as the wing, tail, body of the aircraft is most often used FEM with two types of finite elements. One-dimensional elements describe the work of shelves spars, stringers, belts ribs and frames. Two-dimensional membrane are used to describe the work of the skin, walls of longerons and ribs. The construction is called full-stress (equal-strength), if in all its elements, in which the cross section is greater than the minimum allowable value, the limiting stress state is realized in at least one of the cases of loading [4]. To search for full-stress project uses the classic formula of the stresses relationship:

\[
x^{j+1}_i = x^j_i \max_c \left( \frac{\sigma_{eq,c}}{\sigma_i} \right), \quad c = 1, 2, ..., s
\]

where \( x^{j+1}_i \) is design parameter (area for one-dimensional element \( A \), thickness for two-dimensional element \( t \) – figure 1), \( \sigma_{eq,c} \) is the equivalent stress in the \( i^{th} \) element in the \( c^{th} \) case of loading, \( \sigma_i \) is the allowable equivalent stress for the \( i^{th} \) element, \( s \) is the number of loading cases, \( j \) is the iteration number.

![Figure 1. Structure, its fragment, and two types load-carrying elements with main parameters (one dimensional element: area \( A \) and length \( S=l \), two dimensional element: thickness \( t \) and plane area \( S \))](image)

Since the 1990s, the scientific direction of topological design began to dominate approaches that use the distribution of material in a fixed reference area in the format of the so-called "homogenization method" [5]. This representation of the structure is similar to that of a gray scale in discrete form, corresponding to the raster representation of the geometry of the supporting scheme. Although the concept of homogenization is very popular, it has a number of disadvantages. One is the question of the existence of a solution. Another drawback is related to the method of solution. In many applications, the optimal structure topology must consist solely of macroscopic variation of a single material and void. This means that the density of the structure is given by the integer parameterization "0-1" (often called black-and-white construction).

Currently, the most widely used methods of structural topology optimization based on "variable density". Like the homogenization methods, these methods work on a fixed finite element domain.
(FE). However, instead of a set of properties, microstructure, each FE contains only one of the design variables. This variable is often understood as the density of the element material.

When tracing back the idea of "body of variable density", the vast majority of authors refer to the paper by [5] published in 1988. However, this idea was suggested a little earlier in a paper by Komarov [6] published in Russian in 1984.

The critical aspect of the methods based on the structures of variable density is the choice of an appropriate interpolation and a method for connection of physical characteristics with the continuously calculated values of density. Most often, the distributed function for topological design is interpreted as the density $\rho_i$ of each FE in the form

$$E(\rho_i) = \rho_i^T \bar{E}$$  \hspace{1cm} (2)

Here, $E(\rho_i)$ is the scaled modulus of elasticity of the material, $\bar{E}$ is the modulus of the solid material, $p$ is the penalty parameter. For $0 \leq \rho_{\min} \leq \rho_i \leq 1$, the modulus of elasticity $E$ falls between zero at zero density (the absence of the body) and its "solid" value $\bar{E}$ at $\rho_i = 1$. A minimum density value is required to prevent difficulties associated with a possible singularity in FE matrices and with the inability of the material to appear in a region with zero density. With the choice of such parameterization, there is a need to direct the problem to a solid/empty solution. This is usually achieved by optimizing the topology based on implicit penalization methods, the most common of which is a solid isotropic material with the penalization method “Solid Isotropic Material with Penalization” (SIMP) [7, 8].

3. Optimization of head structure

3.1. Full-stress structures

As is known, the main influence of heating is associated with changes in mechanical characteristics, redistribution of stresses and increase in deformations. Formula (1) allows you to easily take into account the first factor. And as shown by the computational studies in the case of the presence of low level heating recurrent formula (3) is quite workable for the receiving of full-stress structures. However, for a high level temperature field is more efficient recursive formula proposed by Adelman [9]. For multiply load cases it can be written as:

$$x_i^{(i+1)} = x_i^{(i)} \frac{\max_c \{\sigma_{eq_{i,c,M}} - \sigma_{eq_{i,c,T}}\}}{\sigma_{eq_{i,c,M}}} \gamma_i^{(c-1)}, c = 1, 2, ..., s$$ \hspace{1cm} (3)

where $\sigma_{eq_{i,c,M}}$, $\sigma_{eq_{i,c,T}}$ are the equivalent stresses only from mechanical and from thermal loads on the $j^{th}$ iteration.

The formula (3) is based on the assumption that when the structural elements parameters change, the temperature stresses change slightly or do not change at all.

For highly heated aircraft it is very important to provide the necessary rigidity due to special requirements for aerodynamics. In some cases, rigid deformation requirements can be crucial for spacecraft. In this regard, we will focus more on the method of optimization of the structure, aimed at obtaining the structure of the maximum rigidity.

The works [10] give examples showing that a temperature change of only a few degrees radically changes the optimal structure. This once again confirms the idea that to ensure maximum rigidity of the structure at the same time from mechanical and thermal loads is quite difficult.

For the structural optimization of load-carrying elements with heating made of dissimilar materials for one load case, the following formulation is proposed basing on the optimality-criteria method:

$$\min \sigma : \quad F(x) = \int e^T \sigma dV = \sum_{i=1}^{n} \frac{e_i^T \sigma}{e_i} V = \sum_{i=1}^{n} \frac{N_i^2 S_i}{e_i} e_i$$ \hspace{1cm} (4)

under
\[
M(x) = \frac{\sum_{i=1}^{n} \rho_i x_i S_i}{M_0} = 1
\]

(5)

\[
\mathbf{K}(x) U = \mathbf{F}_M + \mathbf{F}_T(x)
\]

(6)

\[0 \leq x_{\min} \leq x \leq x_{\max}.
\]

(7)

where \( F \) is the target function (in this case, it is the compliance), \( x \) is the vector of the design variables, \( x_{\min} \) is the minimum of \( x \) introduced to avoid singularity when solving a system of algebraic equations, \( x_{\max} \) is the maximum of \( x \), \( \mathbf{F}_M \) is the load vector from mechanical load, \( \mathbf{F}_T \) is the load vector from thermal load, \( U \) is the global displacement vector, \( \mathbf{K} \) is the global stiffness matrix, \( M \) is the current mass, \( M_0 \) is the initial mass of the computational domain, \( \rho \) is the density of each \( \text{FE} \), \( V_i \) is the volume of the \( i^{th} \) \( \text{FE} \), which is for every type of \( \text{FE} \). \( V_i = x_i S_i \), \( e_i \) is the the pseudo-elastic module (generalized module), introduced in [11].

For the example of one-dimensional element, the physical meaning of \( e_i \) is explained in figure 2. The value of \( e_i \) does not depend on the sequence of application of the mechanical and thermal loads. The value \( e_i \) also does not depend on this sequence. The value of \( \sum e_i \sigma_i \) in equation (4) is the sum of the specific potential strain energy and specific additional potential strain energy. It is obvious that the expression of \( F \) will be equal to the sum of the potential strain energy (PSE) and the additional PSE.

**Figure 2.** Geometric interpretation of a pseudo-elastic module \( e_i \) on the stress-strain diagram depending on the action sequence of loads: (a) mechanical load is followed by heating, (b) mechanical load follows heating.

The Lagrange multipliers method is used in order to find the optimal values of \( x_i \) satisfying equations (4) – (7). The iterative procedure of this method assumes that the equivalent loads of \( N_{eq} \) and the pseudo-elasticity modules \( e_i \) do not change in the \( \text{FE} \) in the transition from one iteration to another. All changes occur only due to the design parameters \( x_i \). Thus, the Lagrangian can be given in the form

\[
L(x, \lambda) = \sum_{i=1}^{n} \frac{N_{eq}^2 S_i}{e_i x_i} + \lambda \left( \sum_{i=1}^{n} \rho_i x_i S_i - M_0 \right),
\]

(8)

which yields the following system of equations

\[
\frac{\partial L}{\partial x_i} = \frac{N_{eq}^2 S_i}{e_i x_i} + \lambda \rho_i S_i = 0, \quad (i = 1, 2, ..., n),
\]

(9)

\[
\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} \rho_i x_i S_i - M_0 = 0
\]

(10)

where \( \lambda \) is the Lagrange multiplier.

From (9) follows that

\[
x_i = N_{eq}(\lambda \rho_i e_i)^{-1/2}, \quad (i = 1, 2, ..., n).
\]

(11)
It is important that in some heated FE, the generalized module $e_i$ can be (or become) negative at a current iteration. For example, a clamped heated rod has $e_i = -\infty$. Obviously, as the design parameters, such items are to be as small as possible $x_i = x_{\min_i} (i = p + 1, p + 2, n)$, and the elements themselves should be excluded from the list of active elements whose parameters are calculated by the final recurrent formula. But this question calls for additional studies, which will be discussed later.

The Lagrange multiplier is calculated from (10) and (11), which is then substituted into (11). The final expression for finding the structural parameters, taking into account several cases of loading, can be written as

$$x_i^{(n+1)} = \frac{\sum_i N_{e_{i,c}}^j (\rho_x e_{i,c}^j)^{1/2} [M_0 - \sum_i M_{i,c}^j (\sum_i N_{e_{i,c}}^j (\rho_x e_{i,c}^j)^{1/2} S_j)^{-1}] (i = 1, 2, ..., p)},$$

where $M_{i,c}^j$ is the mass of the FE with design parameters of negative modulus $e_i$ in the $c^{th}$ load case.

4. Testing of method to structural optimization

By the one load case, the criteria of rigidity and strength of the structure lead to the same design parameters, and the structure degenerates into a statically definable one. Studies show that in several load cases of the parameters the obtained by the criteria of stiffness and strength do not always coincide. We can check this fact and the reliability of the proposed optimization algorithm is tested on the well-known example of Razani – a three-rod truss which is loaded with either from force $F_1$ or $F_2$ without heating (figure 3, a). The analytical solution of this is given in [12]. For the numerical solution of this problem, we assume that $a = 1$ m, $b = 2$ m, $F_1 = F_2 = 2 \cdot 10^3$ N, the mass of the structure is $M_0 = 23.7$ kg, modulus of elasticity is $E = 2 \cdot 10^{11}$ Pa, and material density is $p = 7800$ kg/m$^3$. The result of optimization by the criterion of full stress structure is the statically determinate truss, in which the first rod is fully functional only at load $F_2$, and the third rod – at load $F_1$ (figure 3, b). The second rod is completely turned off from work.

In the optimal design with the criterion of maximum stiffness, the second rod is preserved with an area approximately two times smaller than in rods 1 and 3, while its ultimate stress state is not realized in any of the cases of loading (figure 3, c). But this option allows you to get a more rigid design. It should be noted that a similar distribution of design parameters is obtained if the forces act along the x and y axes (figure 3, d).

![Figure 3](image)

**Figure 3.** Razan’s trust (a), optimal decision according to criterion of full stress structure (b), optimal decision according to criterion of maximum stiffness (c, d)

It is obvious the presence of an underloaded element in the previous example is due to the fact that the original structure was not optimal. To obtain better results, we need to perform a topological optimization. As an initial scheme, we take a variant of a symmetric form with a uniform arrangement of 25 elements (figure 4, a). As the result of the optimization for two load cases we receive the statically definable two-rod truss with angle $\beta = 35^\circ$ (figure 4, b).

Next, we consider the loading of the same 25 rod truss first by force, then by the action of a uniform temperature field with $\Delta T = 50^\circ$ (figure 5, a). The design of the maximum stiffness will be the
same as in the previous example - the two-rod truss with an angle $\beta \approx 35^\circ$ (figure 5, b). It is noted that with the increase in the temperature level, the convergence deteriorates markedly.

5. Conclusion
A criterion method of topological optimization of heated structures with elements from different materials for several cases of loading based on parametric synthesis by eliminating inefficient elements from the most common initial structure is proposed.

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