Reformulation of Laplacian-$b$ motion in terms of stochastic Komatu–Loewner evolution in the chordal case

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Abstract

We investigate the relation between the Laplacian-$b$ motion and stochastic Komatu–Loewner evolution (SKLE) on multiply connected subdomains of the upper half-plane, both of which are analogues to SLE. In particular, we show that, if the driving function of an SKLE is given by a certain stochastic differential equation, then this SKLE is the same as a time-changed Laplacian-$b$ motion. As an application, we prove the finite time explosion of SKLE corresponding to Laplacian-0 motion, or SLE$_6$, in the sense that the solution to the Komatu–Loewner equation for the slits blows up.

Keywords: SLE, stochastic Komatu–Loewner evolution, Laplacian-$b$ motion, explosion, $H$-excursion
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1 Introduction

Ever since Schramm [12] introduced the stochastic Loewner evolution with parameter $\kappa > 0$ (abbreviated as SLE$_\kappa$), numerous studies have been conducted for identifying the scaling limits of several two-dimensional lattice models in statistical physics, such as loop-erased random walk and percolation. Most of these results are established on simply connected planar domains such as the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C}; \text{Im} z > 0 \}$, since SLE$_\kappa$ is defined via the theory of conformal maps on simply connected domains, especially via the chordal Loewner equation. It is thus a non-trivial problem to extend SLE$_\kappa$ to multiply connected domains, and the way of extension is not unique. For example, Lawler [8] introduced the Laplacian-$b$ motion $LM_b$ with $b = (6 - \kappa)/(2\kappa)$ as a candidate of the scaling limit of Laplacian-$b$ random walk. Its definition is based on the chordal Loewner equation and the Maruyama–Girsanov transform of the driving process associated with SLE$_\kappa$. On the other hand, Chen and Fukushima [2] introduced the stochastic Komatu–Loewner evolution SKLE$_{\alpha,\beta}$ as a process satisfying the domain Markov property and invariance under linear conformal maps, both of which are typical properties of SLE$_\kappa$. (Here, we use $\beta$ instead of $b$ in [2, Eq. (3.32)] to avoid a conflict with the exponent $b$ above.) Its definition is motivated by [1] and based on the chordal Komatu–Loewner equations.

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1Zhan [13] considered a similar object independently, where he called it the harmonic random Loewner chain and stood in a viewpoint different from that of Lawler.
In this paper, we investigate the relation between the two processes above by means of Chen, Fukushima and Suzuki \[4\] and the author \[9\]. After the definitions of \(\text{SKLE}_{\alpha,\beta}\) and \(\text{LM}_b\) are reviewed in Sections 2.1 and 2.2, respectively, we show in Section 3.1 that \(\text{LM}_b\) can be regarded as \(\text{SKLE}_{\alpha,\beta}\) with appropriate \(\alpha\) and \(\beta\) modulo time-change. In particular, the relation between \(\text{SKLE}_{\alpha,\beta}\) and \(\text{LM}_b\) becomes rather simple in the case \(b = 0\) (\(\kappa = 6\)). By using this identification, we prove in Section 3.2 that the finite time explosion of the solution to the Komatu–Loewner equation for the slits \[2, \text{Eq. (3.32) and (3.33)}\] corresponding to \(\text{LM}_0\). We note that this provides a non-trivial example that satisfies the assumption of \[10, \text{Theorem 3.2}\].

2 Preliminaries

2.1 Stochastic Komatu–Loewner evolution

First of all, we fix some notations. Let \(N\) be a positive integer.

- \(\text{Slit}\) stands for the set of all the elements
  
  \[s = (s_j)_{j=1}^{3N} = (y_1, \ldots, y_N, x_1, \ldots, x_N, x_1', \ldots, x_N') \in (0, \infty)^N \times \mathbb{R}^{2N}\]
  
  with \(x_j < x'_j\) for each \(j = 1, \ldots, N\) such that either \(x_j < x_k\) or \(x'_k < x_j\) holds if \(y_j = y_k\) for two distinct numbers \(j, k \in \{1, \ldots, N\}\).

- \(C_j(s)\) denotes the segment whose endpoints are \(z_j = x_j + iy_j\) and \(z'_j = x'_j + iy_j\) for \(s \in \text{Slit}\) and \(1 \leq j \leq N\).

- \(D(s)\) denotes the \(\text{standard slit domain } \mathbb{H} \setminus \bigcup_{j=1}^{3N} C_j(s)\) for \(s \in \text{Slit}\).

- The functions \(b_l: \mathbb{R} \times \text{Slit} \to \mathbb{R}, 1 \leq l \leq 3N\), are defined by
  
  \[b_l(\xi_0, s) = \begin{cases} -2\pi \Re \Psi_{D(s)}(z_l, \xi_0) & (1 \leq l \leq N) \\ -2\pi \Im \Psi_{D(s)}(z_{l-N}, \xi_0) & (N + 1 \leq l \leq 2N) \\ -2\pi \Psi_{D(s)}(z_{l-2N}, \xi_0) & (2N + 1 \leq l \leq 3N) \end{cases}\]

  where \(\Psi_D(s)\) stands for the \(\text{complex Poisson kernel of Brownian motion with darning (BMD)}\) for \(D(s)\) \[3, \text{Lemma 4.1}\].

Also, we recall that a set \(F \subset \mathbb{H}\) is called a \((\mathbb{H})\)-hull if \(F\) is bounded and relatively closed in \(\mathbb{H}\) and if \(\mathbb{H} \setminus F\) is still simply connected. For any hull \(F\) in a standard slit domain \(D\), there exists a unique conformal map \(f_F\) from \(D \setminus F\) onto another standard slit domain \(\tilde{D}\), which is called the \(\text{canonical map}\), such that the \(\text{hydrodynamic normalization}\) \(f_F(z) = z + a/z + o(z^{-1})\) as \(z \to \infty\) holds. The positive constant hcap\(^D\)(\(F\)) := \(a\) is called the \(\text{half-plane capacity relative to} D\). See \[3, \text{Proposition 2.3}\].

Let \(t_0 \in (0, \infty), a_t\) be a strictly increasing differentiable function in \(t \in [0, t_0]\) with \(a_0 = 0\), and \(\xi(t)\) be an \(\mathbb{R}\)-valued continuous function on this interval. We consider the following ordinary differential equations:

\[
\frac{d}{dt}s_l(t) = \frac{\dot{a}_t}{2}b_l(\xi(t), s(t)), \quad 1 \leq l \leq 3N, \tag{2.1}
\]

\[
\frac{d}{dt}g_0(z) = -\pi \dot{a}_t \Psi_{D(s(t))}(g_0(z), \xi(t)), \quad g_0(z) = z \in D(s(0)). \tag{2.2}
\]
Here, the dot stands for the $t$-derivative. We call (2.2) the chordal Komatu–Loewner equation \([1, 3]\) and (2.1) the Komatu–Loewner equation for the slits \([1, 2]\). Let \(\zeta\) be the explosion time of the solution \(s(t)\) to (2.1). We put \(t_z := \sup\{t > 0; |g_t(z) - \zeta(t)|\}\) for \(z \in D = D(s(0))\) and \(F_t := \{z \in D; t_z \leq t\}\) for \(t \in [0, \zeta)\). Then \(\{F_t\}_{t \in [0, \zeta)}\) is a family of continuously growing hulls \([9, \text{Definition 4.2}]\) with \(\text{hcp}^D(F_t) = a_t\). \(g_t\) is the canonical map from \(D \setminus F_t\) onto \(D(s(t))\) for each \(t \in [0, \zeta)\), and it holds that

\[
\bigcap_{\delta > 0} g_t(F_{t+\delta} \setminus F_t) = \{\xi(t)\}. \tag{2.3}
\]

We call a function \(\xi(t)\) satisfying (2.3) the driving function of \(\{F_t\}\). See \([2, \text{Section 5}]\) for the proof of these facts. We remark that they are valid also in the case \(N = 0\), except that (2.1) does not appear. Hence we may put \(\zeta = \infty\). In this case, the complex Poisson kernel is

\[
\Psi_{\mathbb{H}}(z, \xi_0) = -\frac{1}{\pi} \frac{1}{z - \xi_0},
\]

and (2.2) is called the chordal Loewner equation.

Let \(\alpha\) be a non-negative function on \(\text{Slit}\) homogeneous with degree 0. Here, a function \(f(s)\) is said to be homogeneous with degree \(\delta \in \mathbb{R}\) if \(f(cs) = c^\delta f(s)\) holds for all \(c > 0\) and \(s \in \text{Slit}\). Let \(\beta\) be a function on \(\text{Slit}\) homogeneous with degree \(-1\). We further suppose that both \(\alpha\) and \(\beta\) are locally Lipschitz continuous. \(\text{SLE}_{\alpha, \beta}\) on a standard slit domain \(D\) \([2, \text{Section 5}]\) is defined as the random continuously growing hulls \(F_t\) in \(D\) that are obtained via the procedure above with \(a_t = 2t\) and \(\xi(t)\) given by the following stochastic differential equation (SDE)

\[
d\tilde{\xi}(t) = \alpha(s(t) - \tilde{\xi}(t)) \, dB_t + \beta(s(t) - \tilde{\xi}(t)) \, dt. \tag{2.4}
\]

Here, \(\tilde{\xi}(t)\) stands for the 3N-dimensional vector whose first \(N\) entries are zero and last \(2N\) entries are \(\xi(t)\). In this case, we should regard (2.1) and (2.3) together as a system of SDEs, and \(\zeta\) above is replaced by the explosion time of the solution \((\xi(t), s(t))\) to this system. \(\text{SLE}_{\alpha}\) is a special case of this definition where \(N = 0\), \(\alpha = \sqrt{\kappa}\) and \(\beta = 0\).

Since it is sometimes convenient to regard \(\alpha\) and \(\beta\) in (2.4) as functions on \(\mathbb{R} \times \text{Slit}\), we introduce the notation \(f(\xi_0, s) := f(s - \xi_0)\) for a function \(f\) on \(\text{Slit}\). The function \(f(\xi_0, s)\) so defined has the invariance under horizontal translation \(f(\xi_0, s) = f(0, s - \tilde{\xi}_0)\). Conversely, we define \(\tilde{f}(s) := \tilde{f}(0, s)\) if a function \(\tilde{f}\) on \(\mathbb{R} \times \text{Slit}\) has this invariance.

### 2.2 Laplacian-\(b\) motion

Let \(Z^\mathbb{H} = (Z^\mathbb{H}, P^Z_{z_0})\) be an absorbing Brownian motion in \(\mathbb{H}\) starting at \(z_0 \in \mathbb{H}\) and \(P^{Z^\mathbb{H}}(t, z, dw)\) be the transition probability of \(Z^\mathbb{H}\). Doob’s \(h\)-transform \(\hat{Z} = (\hat{Z}_t, \hat{P}^Z_{z_0})\) with harmonic function \(h(z) = \mathbb{H} z\) is called an \(\mathbb{H}\)-excursion \([7, \text{Section 5.3}], \[8, \text{Section 3.5}]\). In other words, \(\hat{Z}\) is a strong Markov process with transition probability

\[
\hat{P}(t, z, dw) := \frac{\mathbb{H} w}{\mathbb{H} z} P^{Z^\mathbb{H}}(t, z, dw), \quad t \geq 0, \ z \in \mathbb{H}. \tag{2.5}
\]
Let \( \kappa \) follows: Let \( \kappa (\cdot) \) be an \( \mathbb{Q} \)-martingale, then by the Maruyama–Girsanov theorem, \( \mathbb{Q} \) measure is then bounded and thus a uniformly integrable martingale. If we define the \( \mathbb{Q} \)-motion and that \( \mathcal{I} \) is the \( \mathbb{Q} \)-martingale. Therefore, we can define an \( \mathbb{H} \)-excursion \( \hat{Z} \) starting at a boundary point \( z_0 \in \partial \mathbb{H} \) as well, and the lifetime of \( \hat{Z} \) is infinite \( \hat{P}_{z_0} \)-almost surely.

Let \( D' := \mathbb{H} \setminus \bigcup_{j=1}^{N} A_j \), where \( A_j \subset \mathbb{H}, 1 \leq j \leq N \), are mutually disjoint, compact continua with smooth boundaries. Put

\[
Q(z, D') := \hat{P}_z \left( \hat{Z}(0, \infty) \subset D' \right), \quad z \in D' \cup \partial \mathbb{H}.
\]

It then follows from (2.5) that

\[
Q(z, D') = 1 - \hat{P}_z (\hat{\tau}_{D'} < \infty) = 1 - \frac{\mathbb{E}_z^\mathbb{H} \left[ \mathfrak{Z}_{\hat{\tau}_{D'}}^\mathbb{H} : \tau_{D'} < \infty \right]}{\mathbb{P}^z}, \quad z \in D', \quad (2.6)
\]

where \( \hat{\tau}_{D'} \) and \( \tau_{D'} \) are the exiting times from \( D' \) of \( \hat{Z} \) and \( Z^\mathbb{H} \), respectively. \( Q(\xi_0, D') \) for \( \xi_0 \in \partial \mathbb{H} \) is the limit of (2.4) as \( z \to \xi_0 \). We see later that \( Q(\xi_0, D') \) is a differentiable function of \( \xi_0 \in \mathbb{R} \). In particular, we can consider \((\partial \log Q/\partial \xi_0)(\xi_0, D')\).

By using the function \( Q \), the construction of \( \text{LM}_b \) for \( b > -1/2 \) is done as follows: Let \( \kappa := 6/(2b-1) \) and \((g^b_t, F^b_t)_{t \geq 0} \) be an \( \text{SLE}_\kappa \) driven by \( U(t) = \sqrt{t}B_t \), where \( B = (B_t)_{t \geq 0} \) is a standard Brownian motion on a filtered probability space \((\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with the usual conditions. We put \( T_{D'} := \inf \{ t > 0 ; F^b_t \not\subset D' \} \) and \( D'_t := g^b_t (D' \setminus F^b_t) \), and define the processes \( X \) and \( X^{(n)} \) for \( n \in \mathbb{N} \) by

\[
X_t := \frac{6 - \kappa}{2\sqrt{\kappa}} \frac{\partial \log Q}{\partial \xi_0} (U(t), D'_t) \mathbf{1}_{\{ t < T_{D'} \}} \quad \text{and} \quad X_t^{(n)} := X_t \mathbf{1}_{\{ t < T_n \}},
\]

where \( T_n := T_{D'} \wedge \inf \{ t > 0 ; |X_t| \geq n \} \). We further define the exponential local martingale

\[
M_t^{(n)} := \exp \left( \int_0^t X_s^{(n)} dB_s - \frac{1}{2} \int_0^t |X_s^{(n)}|^2 ds \right)
\]

and the stopping times \( S_{n,m} := T_n \wedge \inf \{ t > 0 ; |M_t^{(n)}| \geq m \}, \ m \in \mathbb{N} \). The stopped local martingale

\[
M_t^{(n)} \mathbf{1}_{\{ t \wedge S_{n,m} \}} := \exp \left( \int_0^{t \wedge S_{n,m}} X_s dB_s - \frac{1}{2} \int_0^{t \wedge S_{n,m}} |X_s|^2 ds \right)
\]

is then bounded and thus a uniformly integrable martingale. If we define the measure \( Q^{(n,m)} \) on \( \mathcal{F}_\infty \) by

\[
Q^{(n,m)}(A) := \mathbb{E}^\mathbb{P} \left[ \mathbf{1}_A \lim_{t \to \infty} M_t^{(n)} \mathbf{1}_{t \wedge S_{n,m}} \right], \quad A \in \mathcal{F}_\infty,
\]

then by the Maruyama–Girsanov theorem

\[
B_t^{(n,m)} := B_t - \int_0^{t \wedge S_{n,m}} X_s ds, \quad t \in [0, \infty),
\]
is a standard Brownian motion under the measure $Q^{(n,m)}$. In other words, the driving function $U(t)$ of $\{F_t^0\}$ satisfies the following SDE under $Q^{(n,m)}$:

$$dU(t) = \frac{6 - \kappa}{2} \frac{\partial \log Q}{\partial \xi_0}(U(t), D'_t) dt + \sqrt{\kappa} dB^{(n,m)}_t, \quad 0 \leq t \leq S_{n,m}. \tag{2.7}$$

(The law of $\{F_t^0\}_{0 \leq t \leq S_{n,m}}$ under $Q^{(n,m)}$ is regarded as $\text{LM}_b$ stopped by $S_{n,m}$. Motivated by (2.7), Lawler [8] defined $\text{LM}_b$ as the random Loewner evolution driven by a solution to (2.7) with $B^{(n,m)}$ replaced by a Brownian motion independent of $n$ and $m$.

**Remark 2.1.** The exponent $b$ implicitly appears in (2.7) in the sense that $(6 - \kappa)/2 = kb$. Since the chordal Loewner equation that we consider in this paper is the linear time-change of [8, Eq. (4.16)], the SDE (2.7) is the time-change of the original one given in [8, Section 4.4]. The exponential martingale $M^{(n)}_t$ above is also expressed in a different fashion there.

Although we do not explain the reason why the random evolution defined in the above manner is a candidate of the scaling limit of Laplacian-$b$ random walk, the argument in which Lawler derived the SDE (2.7) is remarkable in that we can extend $\text{SLE}_\kappa$ to multiply connected domains by the Maruyama–Girsanov transformation.

### 3 Relation between SKLE and LM

#### 3.1 Reformulation of Laplacian-$b$ motion as SKLE

Let $D'$ and $(g_0', F'_0)$ be as in Section 2.2 and $h$ be a conformal map hydrodynamically normalized from $D'$ to a standard slit domain $D$, whose existence and uniqueness are ensured by [9, Proposition 2.3]. For each $t$, we denote the canonical map of the hull $F_t := h(F'_t)$ by $g_t: D \setminus F_t \to D_t$ and put $h_t := g_t \circ h \circ (g_0')^{-1}$. Then by [9, Theorem 4.8], $(F_t)_{0 \leq t \leq S_{n,m}}$ is a family of continuously growing hulls in $D$, the map $g_t$ satisfies (2.2) with $\dot{a}_t = 2h'_i(U(t))^2$ and $\xi(t) = h_t(U(t))$, and the solution $s(t)$ to (2.1) with these $a_t$ and $\xi(t)$ enjoys $D(s(t)) = D_t$. Moreover, it follows from [9, Eq. (4.7)] that

$$d\xi(t) = \left\{ \frac{6 - \kappa}{2} \left( h'_i(U(t)) \frac{\partial \log Q}{\partial \xi_0}(U(t), D'_t) - h''_i(U(t)) \right) \right.$$

$$- h'_i(U(t))^2 b_{\text{BMD}}(\xi(t), s(t)) \right\} dt + \sqrt{\kappa} h'_i(U(t)) dB^{(n,m)}_t \tag{3.1}$$

for $t \leq S_{n,m}$ under the measure $Q^{(n,m)}$. Here, $b_{\text{BMD}}$ is the BMD domain constant, a locally Lipschitz function on $\mathbb{R} \times \text{Slit}$ which is invariant under horizontal translation and homogeneous with degree $-1$. See Eq. (6.1) and Lemma 6.1 of [2]. The aim of this subsection is to rewrite (3.1) in the form of (2.4) and to check that its coefficients satisfy the conditions that are required in Section 2.1 to define $\text{SKLE}_{\alpha,\beta}$.

To deform the expression (3.1) into a form independent of $D'_t$, we utilize the conformal transformation rule [8, Eq. (3.12)] for $\partial \log Q / \partial \xi_0$:

$$\frac{\partial \log Q}{\partial \xi_0}(U(t), D'_t) = h'_i(U(t)) \frac{\partial \log Q}{\partial \xi_0}(\xi(t), D_t) + \frac{h''_i(U(t))}{h'_i(U(t))} \tag{3.2}$$
Substituting (3.2) into (3.1) yields
\[ d\xi(t) = \left\{ \frac{6 - \kappa}{2} \frac{\partial \log Q(\xi(t), s(t))}{\partial \xi_0} - b_{\text{BMD}}(\xi(t), s(t)) \right\} h'_t(U(t))^2 dt + \sqrt{\kappa} h'_t(U(t)) dB^{(n,m)}_t, \quad t \leq S_{n,m}, \]
where \( Q(\xi_0, s) := Q(\xi_0, D(s)) \) for \( \xi_0 \in \mathbb{R} \) and \( s \in \text{Slit} \).

We now reparametrize \( \{F_t\} \) by the half-plane capacity relative to \( D \), that is, \( \tilde{F}_t := F_{\alpha^{-1}(2t(S_{n,m}))} \) with \( \tilde{S}_{n,m} := a(S_{n,m})/2 \). By this time-change, we have hcap\(D(\tilde{F}_t) = 2t \) for all \( t \in [0, \tilde{S}_{n,m}] \). All the other quantities that are reparametrized in the same manner are indicated by adding check mark symbol as well. Brownian motion \( (\tilde{B}^{(n,m)}_t)_{t \geq 0} \) then exists on some enlargement of the filtered probability space \( (\Omega, (\tilde{F}_t)_{t \geq 0}, Q^{(n,m)}) \), satisfying
\[ d\tilde{\xi}(t) = \left\{ \frac{6 - \kappa}{2} \frac{\partial \log Q}{\partial \xi_0}(\tilde{\xi}(t), \tilde{s}(t)) - b_{\text{BMD}}(\tilde{\xi}(t), \tilde{s}(t)) \right\} dt + \sqrt{\kappa} d\tilde{B}^{(n,m)}_t, \quad t < \tilde{S}_{n,m}, \]
by [11, Theorem V.1.7].

The expression (3.3) suggests the relation between LM\(b \) and SKLE\(\alpha, \beta \). Namely, if we can define SKLE\(\alpha, \beta \) on \( D \) with
\[ \alpha(s) := \sqrt{\kappa} \quad \text{and} \quad \beta(s) := \frac{6 - \kappa}{2} \frac{\partial \log Q}{\partial \xi_0}(0, s) - b_{\text{BMD}}(0, s), \quad (3.4) \]
and pull it back to \( D' \) by the map \( h: D' \rightarrow D \), then the resulting random evolution is a time-changed LM\(b \) on \( D' \). Indeed, we can do the reverse procedure, that is, start at (3.3) to get (2.7) by a similar computation and enlargement of the underlying probability space using the inverse map \( h^{-1} \). The remaining thing is thus to check \( \alpha \) and \( \beta \) of (3.4) satisfies the conditions in Section 2.1.

**Proposition 3.1.** (\( \partial \log Q/\partial \xi_0 \)(\( \xi_0, s \)) is invariant under horizontal translations, homogeneous with degree \( -1 \) and locally Lipschitz continuous.

**Proof.** The translation invariance and homogeneity with degree \( -1 \) are obvious from the conformal transformation rule (3.2). We therefore prove only the local Lipschitz continuity.

We put \( D := D(s) \) and \( C_j := C_j(s) \) for \( s \in \text{Slit} \), and denote by \( K_D \) the Poisson kernel of absorbing Brownian motion in \( D \). We also denote by \( \varphi^{(j)}_s(z) \) the harmonic measure of \( C_j \). In other words, \( \varphi^{(j)}_s(z) \) is a unique bounded harmonic function on \( D \) with boundary values 1 on \( C_j \) and 0 on \( \partial \mathbb{H} \cup \bigcup_{k \neq j} C_k \). The quantity \( Q(\xi_0, s) \) for \( \xi_0 \in \partial \mathbb{H} \) is computed by using (2.4) and \( K_D \) as follows:
\[ Q(\xi_0, s) = \lim_{z \rightarrow \xi_0} Q(z, D) = 1 - \lim_{z \rightarrow \xi_0} \frac{\mathbb{P}_z^1[3wK_D(\xi_0, w) \wedge \tau_D < \infty]}{3z}, \]
\[ = 1 + \frac{\partial}{\partial n_{\xi_0}} \sum_{j=1}^N \int_{\partial D} \varphi^{(j)}_s(\xi_0) dw, \]
\[ = 1 + \sum_{j=1}^N y_j \frac{\partial}{\partial n_{\xi_0}} \varphi^{(j)}_s(\xi_0). \quad (3.5) \]
Here, $n_0$ stands for the outward unit normal vector at $\xi_0$, and $\partial \xi_j$ represents the boundary of $\mathbb{H} \setminus C_j$ in the path distance topology. Namely, it consists of the left and right endpoints $z_j$ and $z_j'$, the upper side $C_j^+$ of the slit and the lower one $C_j^-$. It follows from (3.4) that

$$\frac{\partial \log Q}{\partial \xi_0}(\xi_0, s) = \left(1 + \sum_{j=1}^N y_j \frac{\partial}{\partial n_\xi_0} \varphi_{s^j}(\xi_0)\right)^{-1} \sum_{j=1}^N y_j \frac{\partial}{\partial \xi_0} \frac{\partial}{\partial n_\xi_0} \varphi_{s^j}(\xi_0).$$

The function $\partial \n_0 \varphi_{s^j}(\xi_0)$ is locally Lipschitz by [3, Eq. (9.24)]. We can mimic the argument of [3, Section 9], in which [3, Eq. (9.24)] was derived from [3, Eq. (9.12)], to show that $\partial \xi_0 \partial \n_0 \varphi_{s^j}(\xi_0)$ is also locally Lipschitz. We thus reach the desired conclusion. \qed

By Proposition 3.1 we can define SKLE with $\alpha$ and $\beta$ given by (3.4) and thus obtain a time-changed LM on pulling it back to $D'$. If $D'$ itself is a standard slit domain, then SKLE is exactly the same as the time-changed LM on $D'$.

### 3.2 Explosion time of SKLE corresponding to LM

Lawler mentioned in [8, Section 4.6] that LM with $b = (6 - \kappa)(2\kappa)$ for $0 < \kappa \leq 4$ is not likely to exit the domain $D'$, that is, $T_{D'}$ in Section 2.2 should be infinite. This observation is partly based on the fact that SLE with $0 < \kappa \leq 4$ is a simple curve with probability one [7, Proposition 6.9]. However, the proof of the property $T_{D'} = \infty$ has not been known so far. One of the difficulties is that there may not exist a single probability measure $Q$ on $(\Omega, \mathcal{F}, \mathbb{P})$ under which (2.7) holds for all $t \in [0, T_{D'})$ with $B^{(n,m)}$ replaced by a Brownian motion independent of $n$ and $m$. Therefore, the problem itself may be ‘ill-posed’ unless such a measure $Q$ exists. Note that, in general, we cannot solve the SDE (2.7) without any change of measures, because this equation is not closed under the unknown variable $U(t)$ but contains another unknown variable $D'_t$. Thus it seems difficult to obtain the measure $Q$ from weak solutions to this SDE.

Instead of the original exit problem of LM, we consider the explosion problem of SKLE with $\alpha$ and $\beta$ given by (3.4). As mentioned in Section 2.1, SKLE is defined only up to the explosion time $\zeta$ of the solution $W_t = (\xi(t), s(t))$ to the system of SDEs (2.1) and (2.3). This $\zeta$ is considered to be closely related to the exit time of SKLE from the standard slit domain $D$ as described in [10, Section 1], and the asymptotic behavior of the solution to (2.1) around $\zeta$ is investigated in [10, Section 3]. In this subsection, we prove that $\alpha$ and $\beta$ of (3.4) satisfy the assumption of [10, Theorem 3.2]. Recall that

$$R(\xi_0, s) := \min_{1 \leq j \leq N} \text{dist}(C_j(s), \xi_0)$$

is a function on $\mathbb{R} \times \text{Slit}$ having the invariance under horizontal translation. We say that a function $f$ on Slit enjoys Condition (B') if $f(s)$ is bounded on the set $(s \in \text{Slit}; R(s) = R(0,s) > r)$ for every $r > 0$. Condition (B') on the coefficients in (2.3) is the assumption of [10, Theorem 3.2]. Since $\alpha(s) = \sqrt{\kappa}$ clearly satisfies this condition, we only have to prove it for $\beta$ of (3.4):

**Proposition 3.2.** $(\partial \log Q/\partial \xi_0)(0,s)$ satisfies Condition (B').
Proof. By the scaling property that is derived from \cite{22}, it suffices to prove that 
$(\partial \log Q/\partial \xi_0)(0, s)$ is bounded when $R(s) > 1$. By the strong Markov property of the $\mathbb{H}$-excursion $\hat{Z}_t$, we have

$$Q(z, D) = \mathbb{E}_z \left[ Q(\hat{Z}_{\hat{\sigma}_{D^+}}, D) \right] = \int_0^\pi Q(e^{i\theta}, D) K_{D^+}(z, e^{i\theta}) \frac{\sin \theta}{32} d\theta$$

for $|z| < 1$. Here, $\hat{\sigma}_{D^+}$ denotes the hitting time of $\hat{Z}$ to $D^+ = \mathbb{D} \cap \mathbb{H}$. Letting $z \to \xi_0 \in \partial \mathbb{H}$ yields that

$$Q(\xi_0, s) = \int_0^\pi Q(e^{i\theta}, D) \frac{\partial}{\partial n_{\xi_0}} K_{D^+}(\xi_0, e^{i\theta}) \sin \theta d\theta.$$ 

Hence we have

$$\frac{\partial \log Q}{\partial \xi_0} (\xi_0, s) = \frac{\partial}{\partial \xi_0} Q(\xi_0, s) = \frac{\int_0^\pi Q(e^{i\theta}, D) \frac{\partial}{\partial n_{\xi_0}} K_{D^+}(\xi_0, e^{i\theta}) \sin \theta d\theta}{\int_0^\pi Q(e^{i\theta}, D) \frac{\partial}{\partial n_{\xi_0}} K_{D^+}(\xi_0, e^{i\theta}) \sin \theta d\theta}.$$ 

Computing the Poisson kernel $K_{D^+}$ gives the following expression:

$$\frac{\partial \log Q}{\partial \xi_0} (0, s) = \frac{2 \int_0^{\pi/2} Q(e^{i\theta}, D) \sin^2 \theta \cos \theta d\theta}{\int_0^{\pi} Q(e^{i\theta}, D) \sin^2 \theta d\theta}.$$ 

The right-hand side is bounded by two.

We expect that the explosion time $\zeta$ can be regarded as the exit time of SKLE$_{\kappa, \beta}$ under Condition (B’), though we have not shown it yet, and that the exit problem of LM$_0$ can be interpreted as follows: $\zeta = \infty$ holds almost surely if and only if $0 < \kappa \leq 4$. This assertion is difficult to prove for general $\kappa$, and thus we look only at the case $\kappa = 6$ ($b = 0$) in this article. The situation becomes simpler in this case because the drift term of (2.7) vanishes. Hence LM$_0$ is just SLE$_6$, and especially any change of measures is not needed to obtain LM$_0$. The reconstruction procedure in Section \cite{4} therefore provides another proof of \cite[Theorem 4.2]{4}, which shows that the law of SLE$_6$ coincides with SKLE$_{\sqrt{6}, -\text{bMD}}$ modulo time-change until it exits $D$. The following result follows from the fact that the SLE$_6$ hull is space-filling with probability one [7, Proposition 6.10]:

**Theorem 3.3.** The explosion time $\zeta$ of the solution $W_t = (\xi(t), s(t))$ to (2.1) and \cite{22} whose coefficients are $\alpha(s) = \sqrt{6}$ and $\beta(s) = -\text{bMD}(s)$ is finite with probability one.

Proof. Let $\{F_t\}_{t \in [0, \infty)}$ be an SLE$_6$ hull defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{Q})$ with the usual conditions. \cite[Proposition 6.10]{7} asserts that

$$Q\left( \bigcup_{t \in [0, \infty)} F_t = \mathbb{H} \right) = 1,$$

and hence $T_D := \inf\{t > 0; F_t \not\subset D\} < \infty$ $\mathbb{Q}$-a.s. We define

$$a_t := \text{hcap}^D(F_t), \quad \hat{T}_D := \frac{a(T_D)}{2} \text{ and } \hat{F}_t := F_{a_t^{-1}(2(t \wedge \hat{T}_D))}$$

$\{\hat{F}_t\}_{t \leq \hat{T}_D}$ is then the SKLE$_{\sqrt{6}, -\text{bMD}}$ hull in $D$ defined on an enlargement of $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{Q})$. The pair of associated driving function $\hat{\xi}(t)$ and slit vector $\hat{s}(t)$ is thus a weak solution of the SDEs (2.1) and (2.2) whose coefficients are the
above $\alpha$ and $\beta$ up to $\tilde{T}_D$. The stopping time $\tilde{T}_D$ is its explosion time, because if the solution could be continued, then $F_{\tilde{T}_D}$ were included by $D$, which contradicts to the definition of $T_D$.

Because of the uniqueness in law of solutions to the SDEs (2.1) and (2.4), it suffices to show that $\tilde{T}_D$ is finite $Q$-a.s. for completing the proof. We now take a sample $\{F_t\}$ such that $T_D < \infty$. Since $F_{\tilde{T}_D}$ is then bounded, there exists $R > 0$ such that $F_{\tilde{T}_D} \subset B(0, R) \cap D$. By [2, Eq. (A.20)], we have

$$2\tilde{T}_D = h\text{cap}^D(F_{\tilde{T}_D}) \leq \frac{2R}{\pi} \int_0^\pi \mathbb{E}_{\text{Re} e^i \theta} \left[ Z_{\sigma_{B(0, R) \cap D}}; \sigma_{B(0, R) \cap D} < \infty \right] \sin \theta \, d\theta \leq \frac{4R^2}{\pi} < \infty.$$ 

As $T_D < \infty$ holds $Q$-a.s., we reach the desired conclusion.

For $\kappa$ other than 6, the proof of Theorem 3.3 does not work. We have to deal with the quantity $\partial \log Q / \partial \xi_0$ and reveal how much repulsive force is exerted between the slit vector $s(t)$ and driving function $\xi(t)$. This is yet to be investigated.

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