Causality and the effective range expansion

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Abstract

We derive the generalization of Wigner’s causality bounds and Bethe’s integral formula for the effective range parameter to arbitrary dimension and arbitrary angular momentum. We also discuss the impact of these constraints on the separation of low- and high-momentum scales and universality in low-energy scattering. Some of our results were summarized earlier in a letter publication. In this work, we present full derivations and several detailed examples.

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I. INTRODUCTION

Causality plays a fundamental role in physics. The principle that no action can be observed before its cause puts important constraints on physical theories. In classical electrodynamics, causality leads to the Kramers-Kronig relations which relate the real and imaginary parts of the dielectric constant. In quantum mechanics, causality requires that no scattered wave is produced before the incident wave first reaches the scatterer. For finite-range interactions the constraints of causality on elastic phase shifts were first investigated by Wigner [1]. To illustrate the underlying physics, we consider a wavepacket of outgoing spherical waves in $d$ spatial dimensions,

$$ f_{\text{out}}(r) = \int_{0}^{\infty} dp \ e^{ipr} \tilde{f}_{\text{out}}(p), \quad (1) $$

where normalization factors and the $r^{-(d-1)/2}$ radial dependence are absorbed into the definition of $f_{\text{out}}(r)$. When this wavepacket is scattered, the $S$-matrix multiplies asymptotic outgoing states by a phase factor $e^{2i\delta(p)}$, where $\delta(p)$ is the elastic phase shift. We assume the momentum distribution $\tilde{f}_{\text{out}}(p)$ is sharply peaked around some nonzero value $\bar{p}$. If $f_{\text{out}}^\delta(r)$ is the scattered wavepacket, then

$$ f_{\text{out}}^\delta(r) = \int_{0}^{\infty} dp \ e^{ipr} e^{2i\delta(p)} \tilde{f}_{\text{out}}(p) 
\approx e^{2i\delta(\bar{p})} e^{-2i\delta'(\bar{p})\bar{p}} f_{\text{out}}[r + 2\delta'(\bar{p})]. \quad (2) $$

As a consequence, the wavepacket is shifted forward by $\Delta r = -2\delta'(\bar{p})$ relative to the wavepacket with no scattering. If we consider the wavepacket as a function of time, the same shift can be interpreted as a time shift or delay for the scattered wavepacket [1],

$$ \Delta t = 2 \frac{d\delta}{dE} \biggr|_{\tilde{E}}, \quad (3) $$

where $\tilde{E}$ is the energy corresponding with $\bar{p}$. The radius shift and time delay are sketched in Fig. [1]. A classical analysis of particle trajectories suggests that if the interactions have a finite range $R$, then causality requires $-\delta'(\bar{p}) \leq R$. While this argument is qualitatively correct, it ignores the quantum mechanical spread of the wavepacket in space. The precise causality bound and the consequences of the bound are the subject of the present analysis.

In a previous letter, we have explored the impact of causality constraints on low-energy universality [2]. Low-energy universality can appear when there is a large separation between the short-distance scale of the interaction and the long-distance scales relevant to the physical system. One example of low-energy universality is the unitarity limit. The strict unitarity limit refers to an idealized system where the range of the interaction is zero and the $S$-wave scattering length is infinite. In experiment, the strict unitarity limit can not be reached because real systems always have finite range interactions. Since finite range effects are suppressed at low energies, one also refers to systems in the unitarity limit if they have infinite scattering length but finite range. In nuclear physics, cold dilute neutron matter is close to the unitarity limit since the neutron-neutron scattering length is much larger than the range. Stable many-body systems with infinite scattering length can be created in experiments with two types of ultracold fermionic atoms using Feshbach resonances.
The implications of causality for universality in $S$-wave scattering are well understood \[\text{Ref. [3]}.\] In this case, the unitarity limit can always be reached by tuning the scattering length to infinity. For reviews of recent cold atom experiments exploring physics of the unitarity limit, see Refs. [4, 5]. Theoretical overviews of ultracold Fermi gases close to the unitarity limit and their numerical simulations are given in [6, 7]. A general review of universality at large scattering length can be found in [8].

The implications of causality for large scattering length physics in higher partial waves are more intricate. Because of causality, it is not always possible to reach the unitarity limit by tuning external parameters \[\text{Ref. [2, 9]}.\] Several experiments have investigated strongly-interacting $P$-wave Feshbach resonances in $^6\text{Li}$ and $^{40}\text{K}$ [10–14]. A key question is whether the physics of these strongly-interacting $P$-wave systems is universal, and if so, what are the relevant low-energy parameters. A positive answer to this question would provide a connection between the atomic physics of $P$-wave Feshbach resonances and the nuclear physics of $P$-wave alpha-neutron interactions in halo nuclei. Some progress in addressing these questions has been made with low-energy models of $P$-wave atomic interactions [15–20] and $P$-wave alpha-neutron interactions [21–24]. A renormalization group study showed that scattering should be weak in higher partial waves unless there is a fine tuning of multiple parameters [9]. Complementary work was carried out by Ruiz Arriola and collaborators. A discussion of the Wigner bound in the context of chiral two-pion exchange can be found in [25] while correlations between the scattering length and effective range related to the Wigner bound were discussed in [26].

In Ref. [2], we have addressed the question of universality and provided expressions for the causality constraints in arbitrary dimension $d$ and for arbitrary angular momentum $L$. Our analysis applies to any finite-range interaction that is energy independent, non-singular, and spin independent. Our results can be viewed as a generalization of the analysis of Phillips and Cohen [3], who derived a Wigner bound for the $S$-wave effective range for short-range interactions in three dimensions. In the current paper, we present full derivations of Wigner's causality bounds [1] and Bethe's integral formula [27] for the effective range parameter to...
arbitrary dimension $d$ and angular momentum $L$. The extension of Bethe’s integral formula for $d = 3$ and $L > 0$ was first derived by Madsen [28]. We discuss the impact of these constraints on the separation of low- and high-momentum scales and universality in low-energy scattering using several detailed examples.

The paper is organized as follows. In Sec. II we set the stage by defining angular momentum, scattering phase shifts, and the effective range expansion for the general case of $d$ spatial dimensions. The equation for the radial wave function and the Wronskian identity for two solutions with different energies are derived in Sec. III. This identity is used in Sec. IV to derive the causality bound on the effective range for zero energy and the general bound for finite energies. In Sec. V we discuss the impact of these bounds on low-energy universality. In particular, we address the question of the unitarity limit. The impact of the causality bounds is illustrated in detail in Sec. VI using three different examples: a spherical step potential in $d$ dimensions, alpha-neutron scattering which corresponds to exponentially bounded interactions, and the long-range van der Waals interaction. We end with a summary and outlook in Sec. VII. Finally, in the Appendices we explicitly check the Wronskian identities for all possible values of $d$ and $L$ and demonstrate the equivalence of our causality bound with Wigner’s original bound on the energy derivative of the phase shift.

II. PRELIMINARIES

Our goal is to generalize Bethe’s integral formula for the effective range parameter and the related causality bound for arbitrary $d$ and $L$. We start with some general definitions.

We consider two non-relativistic spinless particles in $d$ dimensions with a rotationally-invariant two-body interaction. The interaction is assumed to be energy independent and have a finite range $R$ beyond which the particles are non-interacting. Let $\mu$ be the reduced mass and $p^2/(2\mu)$ be the total energy. For $d > 1$ angular momentum is specified by $d - 1$ integer quantum numbers [29, 30]

$$L = \{M_1, \ldots, M_{d-1}\},$$

satisfying

$$|M_1| \leq M_2 \leq \cdots \leq M_{d-2} \leq M_{d-1}.$$  \hspace{1cm} (5)

We let $L$ label the absolute value of the top-level quantum number, $|M_{d-1}|$. For example when $d = 3$, $M_1$ is the angular momentum projection $M = -L, \ldots, L$ and $M_2 = L = 0, 1, 2, \ldots$ is the total angular momentum. In $d = 2$, there is only one angular momentum quantum number $|M_1| = L = 0, 1, 2, \ldots$. The case of one spatial dimension is special since continuous rotations do not exist. Instead of rotational invariance, the key symmetry in one dimension is invariance under parity. We assume a parity-symmetric interaction and write $L = L = 0$ for even parity and $L = L = 1$ for odd parity. In the following all results we derive for rotationally-invariant interactions in $d > 1$ are also valid for parity-symmetric interactions in $d = 1$.

We analyze the two-body system in the center-of-mass frame using units with $\hbar = 1$ for convenience. The full wave function for reduced mass $\mu$ and energy $p^2/(2\mu)$, can be separated into a radial part and an angular part via

$$\Psi_{L,d}^{(p)}(r) = R_{L,d}(\hat{r})Y_L(\hat{r}),$$

\hspace{1cm} (6)
where the \( Y_L(\hat{r}) \) are hyperspherical harmonics. The hyperspherical harmonics in \( d \) dimensions satisfy the orthogonality condition,

\[
\int Y_L^*(\hat{r}) Y_{L'}(\hat{r}) \, d\Omega_d = \delta_{L',L},
\]

where \( \Omega_d \) is the solid angle and the sum rule \[29\],

\[
\sum_{L, \, L \, \text{fixed}} Y_L^*(\hat{r}) Y_L(\hat{r'}) = \frac{(d + 2L - 2)(d - 4)!!}{N_d} C_L^{d/2-1}(\hat{r} \cdot \hat{r'}).
\]

Here \( C_L^{d/2-1} \) is a Gegenbauer polynomial, and the normalization factor \( N_d \) is given by,

\[
N_d = \frac{(d - 2)!! 2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}.
\]

For \( d = 2 \) the expressions are defined in the limit \( d \to 2 \).

We can check that these give the expected sum rules for \( d \leq 3 \). In one dimension we get even and odd parity functions of \( \hat{r} \cdot \hat{r'} \),

\[
(d + 2L - 2)(d - 4)!! C_L^{d/2-1}(\hat{r} \cdot \hat{r'}) = \begin{cases} 1 & \text{for } L = 0 \\ \hat{r} \cdot \hat{r'} & \text{for } L = 1. \end{cases}
\]

For two dimensions, we have a sum of modes \( e^{\pm i L \theta} \) for \( \theta = \cos^{-1}(\hat{r} \cdot \hat{r'}) \),

\[
\lim_{d \to 2} \frac{(d + 2L - 2)(d - 4)!! C_L^{d/2-1}(\hat{r} \cdot \hat{r'})}{(d + 2L - 2)(d - 4)!! C_L^{d/2-1}(\hat{r} \cdot \hat{r'})} = \begin{cases} 1 & \text{for } L = 0 \\ 2 \cos\left[L \cos^{-1}(\hat{r} \cdot \hat{r'})\right] & \text{for } L > 0. \end{cases}
\]

In three dimensions, we recover the Legendre polynomials,

\[
(d + 2L - 2)(d - 4)!! C_L^{d/2-1}(\hat{r} \cdot \hat{r'}) = (2L + 1) P_L(\hat{r} \cdot \hat{r'}).
\]

The scattering phase shifts are directly related to the elastic scattering amplitude \( f_{L,d}(p) \), where

\[
f_{L,d}(p) \propto \frac{p^{2L}}{p^{2L+d-2} \cot \delta_{L,d}(p) - i p^{2L+d-2}}.
\]

In addition to having finite range, we assume also that \( p^{2L+d-2} \cot \delta_{L,d}(p) \) does not diverge at \( p = 0 \) and the interaction is not too singular at short distances. Specifically, we require that the effective range expansion defined below in Eq. (15) converges for sufficiently small \( p \). Moreover, the reduced radial wave function \( u_{L,d}^{(p)} \) defined in Eq. (21) must satisfy that \( \frac{d}{dr} u_{L,d}^{(p)} \) is finite and \( u_{L,d}^{(p)} \) vanishes as \( r \to 0 \). In \[30\], these short-distance regularity conditions are shown to be fulfilled for interactions arising from a static potential,

\[
W(r, r') = V(r) \delta(r - r'),
\]

provided that \( V(r) = O(r^{-2+\epsilon}) \) as \( r \to 0 \) for positive \( \epsilon \). In our discussion, however, we make no assumption that the interactions correspond to a local potential.
The effective range expansion for general dimension \(d\) and angular momentum \(L\) is

\[
p^{2L+d−2} \left[ \cot \delta_{L,d}(p) − \delta_{(d \mod 2),0} \frac{2}{\pi} \ln (p \rho_{L,d}) \right] = −\frac{1}{a_{L,d}} + \frac{1}{2} r_{L,d} p^2 + \sum_{n=0}^{\infty} (−1)^{n+1} \mathcal{P}_{L,d}^{(n)} p^{2n+4}.
\]

(15)

The term \(\delta_{(d \mod 2),0}\) is 0 for odd \(d\) and 1 for even \(d\). \(a_{L,d}\) is the scattering parameter,\(^1\) \(r_{L,d}\) is the effective range parameter, and \(\mathcal{P}_{L,d}^{(n)}\) are the \(n\)th-order shape parameters. \(\rho_{L,d}\) is an arbitrary length scale that can be scaled to any nonzero value. The rescaling results in a shift of the dimensionless coefficient of \(p^{2L+d−2}\) on the right-hand side of Eq. (15), and we define \(\bar{\rho}_{L,d}\) as the special value for \(\rho_{L,d}\) where this coefficient is zero.

Throughout our discussion we use the examples of \(S\)-wave and \(P\)-wave scattering in three dimensions to illustrate general formulas. For \(d = 3\) and \(L = 0\) the scattering amplitude is

\[
f_{0,3}(p) \propto \frac{1}{p \cot \delta_{0,3}(p) − ip},
\]

(16)

and the effective range expansion is

\[
p \cot \delta_{0,3}(p) = −\frac{1}{a_{0,3}} + \frac{1}{2} r_{0,3} p^2 + \sum_{n=0}^{\infty} (−1)^{n+1} \mathcal{P}_{0,3}^{(n)} p^{2n+4}.
\]

(17)

For \(d = 3\) and \(L = 1\) we have

\[
f_{1,3}(p) \propto \frac{p^2}{p^3 \cot \delta_{1,3}(p) − ip^3}
\]

(18)

and

\[
p^3 \cot \delta_{1,3}(p) = −\frac{1}{a_{1,3}} + \frac{1}{2} r_{1,3} p^2 + \sum_{n=0}^{\infty} (−1)^{n+1} \mathcal{P}_{1,3}^{(n)} p^{2n+4}.
\]

(19)

### III. RADIAL EQUATION AND WRONSKIAN IDENTITY

The next step is the derivation of the Wronskian identity for the solutions of the radial Schrödinger equation. The causality bound then follows directly from this identity.

The interaction is assumed to have finite range \(R\) beyond which the particles are non-interacting. With the interaction written as a real symmetric operator with kernel \(W(r, r')\), the radial Schrödinger equation is

\[
p^2 R_{L,d}^{(p)}(r) = −\frac{1}{r^{d−1}} \frac{d}{dr} \left[ r^{d−1} \frac{d}{dr} R_{L,d}^{(p)}(r) \right] + \frac{L(L + d − 2)}{r^2} R_{L,d}^{(p)}(r) + 2\mu \int_0^R dr' W(r, r') u_{L,d}^{(p)}(r').
\]

(20)

\(^1\) For \(S\)-wave scattering in three spatial dimensions, \(a_{0,3}\) has dimensions of length and is usually called scattering length.
We rescale the radial wave function $R_{L,d}^{(p)}(r)$ as
\[ u_{L,d}^{(p)}(r) = (pr)^{(d-1)/2} R_{L,d}^{(p)}(r), \] (21)
and obtain
\[ p^2 u_{L,d}^{(p)}(r) = -\frac{d^2}{dr^2} u_{L,d}^{(p)}(r) + \frac{(2L + d - 1)(2L + d - 3)}{4r^2} u_{L,d}^{(p)}(r) \]
\[ + 2\mu \int_0^R dr' W(r, r') u_{L,d}^{(p)}(r'). \] (22)

The normalization of $u_{L,d}^{(p)}(r)$ is chosen so that for $r \geq R$,
\[ u_{L,d}^{(p)}(r) = \sqrt{\frac{pr\pi}{2}} \left[ \cot \delta_{L,d}(p) J_{L,d/2-1}(pr) - Y_{L,d/2-1}(pr) \right] \]
\[ = p^{L+d/2-3/2} \left[ \cot \delta_{L,d}(p) \times S_{L+d/2-3/2}(pr) + C_{L+d/2-3/2}(pr) \right]. \] (23)
Here $J_\alpha$ and $Y_\alpha$ are the Bessel functions of the first and second kind, $S_\alpha$ and $C_\alpha$ are the Riccati-Bessel functions of the first and second kind, and $\delta_{L,d}(p)$ is the phase shift for partial wave $L$. Our conventions for the Bessel functions and Riccati-Bessel functions are given in Appendix A. In the following, we use the notation $u_{L,d}^{(0)}(r)$ as shorthand for the limit $p \to 0$,
\[ u_{L,d}^{(0)}(r) = \lim_{p \to 0} u_{L,d}^{(p)}(r). \] (24)

For our first example, $d = 3$ and $L = 0$, the rescaled radial wave function for $r \geq R$ is
\[ u_{0,3}^{(p)}(r) = \sqrt{\frac{pr\pi}{2}} \left[ \cot \delta_{0,3}(p) J_{1/2}(pr) - Y_{1/2}(pr) \right] \]
\[ = \frac{\sin [pr + \delta_{0,3}(p)]}{\sin [\delta_{0,3}(p)]}. \] (25)
This satisfies the radial equation,
\[ p^2 u_{0,3}^{(p)}(r) = -\frac{d^2}{dr^2} u_{0,3}^{(p)}(r). \] (26)

For $d = 3$ and $L = 1$, we find
\[ u_{1,3}^{(p)}(r) = p \sqrt{\frac{pr\pi}{2}} \left[ \cot \delta_{1,3}(p) J_{3/2}(pr) - Y_{3/2}(pr) \right] \]
\[ = \frac{\sin [pr + \delta_{1,3}(p)] - pr \cos [pr + \delta_{1,3}(p)]}{r \sin [\delta_{1,3}(p)]}. \] (27)
In this case
\[ p^2 u_{1,3}^{(p)}(r) = \left( -\frac{d^2}{dr^2} + \frac{2}{r^2} \right) u_{1,3}^{(p)}(r). \] (28)
We now multiply Eq. (29) by $\rho$. Integrating from radius $r$, and therefore

$$p_A^2 u_A(r) = -u''_A(r) + \frac{(2L + d - 1)(2L + d - 3)}{4r^2} u_A(r) + 2\mu \int_0^R dr' W(r, r') u_A(r'), \quad (29)$$

$$p_B^2 u_B(r) = -u''_B(r) + \frac{(2L + d - 1)(2L + d - 3)}{4r^2} u_B(r) + 2\mu \int_0^R dr' W(r, r') u_B(r'). \quad (30)$$

We now multiply Eq. (29) by $u_B$, multiply Eq. (30) by $u_A$, and subtract the two,

$$(p_A^2 - p_B^2) u_A(r) u_B(r)$$

$$= -u_B(r) u''_A(r) + u_A(r) u''_B(r) + 2\mu \int_0^R dr' [u_B(r) W(r, r') u_A(r') - u_A(r) W(r, r') u_B(r')] . \quad (31)$$

Integrating from radius $\rho$ to some radius $r \geq R$, we get

$$(p_A^2 - p_B^2) \int_\rho^r dr' u_A(r') u_B(r')$$

$$= - \int_\rho^r dr' u_B(r') u''_A(r') + \int_\rho^r dr' u_A(r') u''_B(r') + 2\mu \int_\rho^R dr \int_\rho^R dr' [u_B(r) W(r, r') u_A(r') - u_A(r) W(r, r') u_B(r')] , \quad (32)$$

and therefore

$$(p_B^2 - p_A^2) \int_\rho^r dr' u_A(r') u_B(r')$$

$$= (u_B u'_A - u_A u'_B)|_\rho^r$$

$$- 2\mu \int_\rho^R dr \int_\rho^R dr' [u_B(r) W(r, r') u_A(r') - u_A(r) W(r, r') u_B(r')] . \quad (33)$$

By assumption the interaction is sufficiently well-behaved at the origin and admits a regular solution. In particular,

$$\lim_{\rho \to 0^+} u_B(\rho) u'_A(\rho) = \lim_{\rho \to 0^+} u_A(\rho) u'_B(\rho) = 0. \quad (34)$$

So we have

$$u_B(r) u'_A(r) - u_A(r) u'_B(r) = (p_B^2 - p_A^2) \int_0^r dr' u_A(r') u_B(r'). \quad (35)$$

The left-hand side of Eq. (35) corresponds with the Wronskian of $u_B$ and $u_A$, $W[u_B, u_A](r)$, and so we have

$$W[u_B, u_A](r) = (p_B^2 - p_A^2) \int_0^r dr' u_A(r') u_B(r'). \quad (36)$$

8
In the following, it will be useful to rearrange Eq. (23) and express $u^{(p)}_{L,d}(r)$ for $r \geq R$ in terms of functions $s(p, r)$ and $c(p, r)$ so that

$$u^{(p)}_{L,d}(r) = p^{2L+d-2} \left[ \cot \delta_{L,d}(p) - \delta_{d \text{mod} 2},0 \frac{2}{\pi} \ln (p \rho_{L,d}) \right] s(p, r) + c(p, r). \quad (37)$$

Later in our discussion we derive $s(p, r)$ and $c(p, r)$ for each of the possible cases for $2L + d$, and show that both functions are analytic in $p^2$. The first two series coefficients will be useful,

$$s(p, r) = s_0(r) + s_2(r)p^2 + O(p^4), \quad (38)$$

$$c(p, r) = c_0(r) + c_2(r)p^2 + O(p^4). \quad (39)$$

Combining with the effective range expansion, we find that

$$u^{(p)}_{L,d}(r) = -\frac{1}{a_{L,d}} s_0(r) + c_0(r) + \left[ \frac{1}{2} r_{L,d}s_0(r) - \frac{1}{a_{L,d}} s_2(r) + c_2(r) \right] p^2 + O(p^4). \quad (40)$$

The Wronskian $W[u_B, u_A](r)$ for $r \geq R$ is then

$$W[u_B, u_A](r) = (p_B^2 - p_A^2) \left\{ \frac{1}{2} r_{L,d}W[s_0, c_0](r) + \left( \frac{1}{a_{L,d}} \right)^2 W[s_2, s_0](r) - \frac{1}{a_{L,d}} W[s_2, c_0](r) - \frac{1}{a_{L,d}} W[c_2, s_0](r) + W[c_2, c_0](r) \right\} + O(p_A^4) + O(p_B^4). \quad (41)$$

Starting from the Wronskian integral formula Eq. (36), we set $p_A = 0$ and take the limit $p = p_B \to 0$. When combined with the expansion in Eq. (41) we find that for $r \geq R$,

$$- r_{L,d}W[s_0, c_0](r) = b_{L,d}(r) - 2 \int_0^r dr' \left[ u^{(0)}_{L,d}(r') \right]^2, \quad (42)$$

where

$$b_{L,d}(r) = 2W[c_2, c_0](r) - \frac{2}{a_{L,d}} \left\{ W[s_2, c_0](r) + W[c_2, s_0](r) \right\} + \frac{2}{a_{L,d}^2} W[s_2, s_0](r). \quad (43)$$

A fundamental result for second-order differential equations known as Abel’s differential equation identity [31] implies that the Wronskian of $s(p, r)$ and $c(p, r)$ is independent of $r$. Given our choice of normalization, the Wronskian of $s(p, r)$ and $c(p, r)$ is $-1$ for all $p$. This implies that

$$W[s_0, c_0](r) = -1, \quad (44)$$

$$W[s_2, c_0](r) = W[c_2, s_0](r). \quad (45)$$

In Appendix A we check explicitly that these identities hold in all cases and derive explicit expressions for $b_{L,d}(r)$.

For our first example, $d = 3$ and $L = 0$, the functions $s(p, r)$ and $c(p, r)$ are

$$s(p, r) = \frac{\sin pr}{p}, \quad (46)$$

$$c(p, r) = \frac{\sin r}{r}.$$
\[ c(p, r) = \cos pr. \]  
\begin{equation}
(47)
\end{equation}

The low-momentum expansions of these functions are

\[ s(p, r) = r - \frac{r^3}{6} p^2 + O(p^4), \]  
\begin{equation}
(48)
\end{equation}

\[ c(p, r) = 1 - \frac{r^2}{2} p^2 + O(p^4). \]  
\begin{equation}
(49)
\end{equation}

From Eq. (43) the Wronskians of the expansion coefficients give

\[ b_{0,3}(r) = 2r - \frac{2r^2}{a_{0,3}} + \frac{2r^3}{3a_{0,3}^2}. \]  
\begin{equation}
(50)
\end{equation}

For our second example, \( d = 3 \) and \( L = 1 \), the functions \( s(p, r) \) and \( c(p, r) \) are

\[ s(p, r) = \frac{1}{p^2} \left( \frac{\sin pr}{pr} - \cos pr \right), \]  
\begin{equation}
(51)
\end{equation}

\[ c(p, r) = p \left( \frac{\cos pr}{pr} + \sin pr \right). \]  
\begin{equation}
(52)
\end{equation}

In this case the low-momentum expansions are

\[ s(p, r) = \frac{r^2}{3} - \frac{r^4}{30} p^2 + O(p^4), \]  
\begin{equation}
(53)
\end{equation}

\[ c(p, r) = \frac{1}{r} + \frac{r}{2} p^2 + O(p^4). \]  
\begin{equation}
(54)
\end{equation}

The Wronksians of the coefficients lead to the result

\[ b_{1,3}(r) = -\frac{2}{r} - \frac{2r^2}{3a_{1,3}} + \frac{2r^5}{45a_{1,3}^2}. \]  
\begin{equation}
(55)
\end{equation}

\section{Causality Bounds}

We are now in the position to write down the causality bound for the effective range \( r_{L,d} \). Using the Wronskian identities, Eqs. (44, 45), we can simplify Eqs. (42, 43) to

\[ r_{L,d} = b_{L,d}(r) - 2 \int_0^r dt' \left[ u^{(0)}_{L,d}(t') \right]^2, \]  
\begin{equation}
(56)
\end{equation}

\[ b_{L,d}(r) = 2W[c_2, c_0](r) - \frac{4}{a_{L,d}} W[s_2, c_0](r) + \frac{2}{a_{L,d}^2} W[s_2, s_0](r). \]  
\begin{equation}
(57)
\end{equation}

These equations hold for any \( r \geq R \). In Appendix A, we derive explicit expressions for the quantity \( b_{L,d}(r) \) for all relevant combinations of \( d \) and \( L \). In particular, we find for \( 2L + d = 2 \):

\[ b_{L,d}(r) = \frac{2r^2}{\pi} \left\{ \ln \left( \frac{r}{2\rho_{L,d}} \right) + \gamma - \frac{1}{2} + \frac{\pi}{2a_{L,d}} \right\}^2 \left( \frac{1}{4} \right), \]  
\begin{equation}
(58)
\end{equation}
for $2L + d = 4$:

$$b_{L,d}(r) = \frac{4}{\pi} \left[ \ln \left( \frac{r}{2\rho_{L,d}} \right) + \gamma \right] - \frac{4}{a_{L,d}} \left( \frac{r}{2} \right)^2 + \frac{\pi}{a_{L,d}} \left( \frac{r}{2} \right)^4,$$

(59)

and when $2L + d$ is any positive odd integer or any even integer $\geq 6$:

$$b_{L,d}(r) = -\frac{2\Gamma(L + \frac{d}{2} - 2)\Gamma(L + \frac{d}{2} - 1)}{\Gamma(L + \frac{d}{2})\Gamma(L + \frac{d}{2} + 1)} \frac{1}{a_{L,d}} \left( \frac{r}{2} \right)^2 \frac{2\pi}{\Gamma(L + \frac{d}{2}) \Gamma(L + \frac{d}{2} + 1)} \frac{1}{a_{L,d}} \left( \frac{r}{2} \right)^{2L+d}.$$

(60)

Since the integrand on the right-hand side of Eq. (56) is positive semi-definite, $r_{L,d}$ satisfies the upper bound

$$r_{L,d} \leq b_{L,d}(r)$$

(61)

for any $r \geq R$. Eq. (61) together with Eqs. (58, 59, 60) constitutes the generalization of the causality bound on the effective range for arbitrary dimension $d$ and angular momentum $L$.

We can extend our causality bound to nonzero values of $p$. Consider any $p \neq 0$. If $\sin \delta_{L,d}(p) \neq 0$, then $W[u_B, u_A](r)$ for $r \geq R$ is an analytic function of $p_A$ and $p_B$ in a neighborhood of $p$. In this neighborhood, we can also consider $W[u_B, u_A](r)$ as an analytic function of the variables $p_B^2 - p_A^2$ and $p_B^2 + p_A^2$. Since $W[u_B, u_A](r)$ is antisymmetric with respect to $p_A$ and $p_B$, it is an odd function of $p_B^2 - p_A^2$. Hence

$$\frac{W[u_B, u_A](r)}{p_B^2 - p_A^2}$$

(62)

is also analytic with respect to $p_B^2 - p_A^2$ and $p_B^2 + p_A^2$. Taking the limit $p_B \to p_A$, we find that

$$\lim_{p_B \to p_A} \frac{W[u_B, u_A](r)}{p_B^2 - p_A^2} = \int_0^r dr' u_A^2(r').$$

(63)

Since the right-hand side is non-negative, we conclude that for any $p_A \neq 0$ such that $\sin \delta_{L,d}(p_A) \neq 0$,

$$\lim_{p_B \to p_A} \frac{W[u_B, u_A](r)}{p_B^2 - p_A^2} \geq 0.$$

(64)

This is the generalization of the causality bound for nonzero $p$. The equivalence of our causality bound with Wigner’s original bound on the energy derivative of the phase shift [1] is demonstrated in Appendix B.

V. CAUSALITY CONSTRAINTS ON LOW-ENERGY UNIVERSALITY

We now discuss the impact of the causality constraints from Eqs. (61, 58, 59, 60) on low-energy universality.
We consider the scattering amplitude in the low-energy limit $p \to 0$ while keeping the interaction range $R$ fixed. Let $\alpha_{L,d}(p)$ describe an effective scattering parameter,

$$\alpha_{L,d}(p) = -\frac{1}{p^{2L+d-2} \cot \delta_{L,d}(p) - i p^{2L+d-2}}.$$  

(65)

The scattering amplitude is proportional to $\alpha_{L,d}(p)$ times a factor of $p^{2L}$ from the angular momentum projection,

$$f_{L,d}(p) \propto p^{2L} \alpha_{L,d}(p).$$  

(66)

In the limit $p \to 0$, the hierarchy of terms in the effective range expansion depends on the value of $2L + d$. This hierarchy is sketched in Fig. 2. In particular, the effective range parameter is as important at low energies as the unitarity contribution for $2L + d = 4$ and becomes more important for $2L + d \geq 5$. This implies that the scale-invariant unitarity limit cannot be reached in those cases because the Wigner bound prevents the effective range from being tuned to zero. In the following, we discuss the various cases in detail.

For $2L + d = 1$, we find that

$$\alpha_{L,d}(p) = -i p + a_{L,d}^{-1} p^2 + O(p^4).$$  

(67)

As $p \to 0$, the effective scattering parameter has a scale-invariant weak-coupling limit

$$\alpha_{L,d}(p) \approx -i p \to 0,$$  

(68)

with the dimensionful parameter, $a_{L,d}^{-1}$, determining the leading correction to scale-invariant physics.

For $2L + d = 2$, we find

$$\alpha_{L,d}(p) = -\frac{\pi}{2 \ln (-i p \rho_{L,d})} + O(p^2/\ln^2 p),$$  

(69)
where $\bar{\rho}_{L,d}$ denotes the special value for $\rho_{L,d}$ that makes the inverse scattering parameter $1/a_{L,d}$ on the right-hand side of Eq. (15) equal to zero. As $p \to 0$, the effective scattering parameter has a logarithmic weak-coupling limit

$$\alpha_{L,d}(p) \approx -\frac{\pi}{2\ln(-ip\bar{\rho}_{L,d})} \to 0,$$  \hspace{1cm} (70)

parameterized by the length parameter $\bar{\rho}_{L,d}$.

For $2L + d \geq 3$, it is more convenient to consider the inverse effective scattering parameter. For $2L + d = 3$,

$$\alpha_{L,d}^{-1}(p) = a_{L,d}^{-1} + ip + O(p^2).$$  \hspace{1cm} (71)

If we fine-tune the interaction so that $a_{L,d}^{-1} = 0$, then as $p \to 0$ the effective scattering parameter has a scale-invariant strong-coupling limit

$$\alpha_{L,d}(p) \approx ip \to 0.$$  \hspace{1cm} (72)

For $d = 3$ and $L = 0$, this is the physics of the unitarity limit in three dimensions.

For $2L + d = 4$, we have

$$\alpha_{L,d}^{-1}(p) = a_{L,d}^{-1} - \frac{2}{\pi}p^2 \ln(-ip\bar{\rho}_{L,d}) + O(p^4),$$  \hspace{1cm} (73)

where $\bar{\rho}_{L,d}$ denotes the special value for $\rho_{L,d}$ that makes the dimensionless effective range parameter $r_{L,d}$ on the right-hand side of Eq. (15) equal to zero. After fine-tuning $a_{L,d}^{-1} = 0$, then as $p \to 0$ the effective scattering parameter has a logarithmic strong-coupling limit

$$\alpha_{L,d}(p) \approx -\frac{2}{\pi}p^2 \ln(-ip\bar{\rho}_{L,d}) \to 0.$$  \hspace{1cm} (74)

We note the emergence of a second relevant dimensionful parameter, $\bar{\rho}_{L,d}$. In the limit $|a_{L,d}| \to \infty$, our causality bound for $2L + d = 4$ places an upper bound on $\bar{\rho}_{L,d}$,

$$\bar{\rho}_{L,d} \leq \frac{R}{2} e^{\gamma}.$$  \hspace{1cm} (75)

For $2L + d \geq 5$, we have

$$\alpha_{L,d}^{-1}(p) = a_{L,d}^{-1} - \frac{1}{2}r_{L,d}p^2 + \begin{cases}  
O(p^3) & \text{for } 2L + d = 5, \\
O(p^4 \ln p) & \text{for } 2L + d = 6, \\
O(p^4) & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (76)

Again we first fine-tune $a_{L,d}^{-1} = 0$. This produces a power-law strong-coupling limit proportional to the dimensionful parameter, $r_{L,d}$,

$$\alpha_{L,d}^{-1}(p) \approx -\frac{1}{2}r_{L,d}p^2 \to 0.$$  \hspace{1cm} (77)

Since this is not scale invariant we might consider a second fine-tuning where $r_{L,d}$ is also tuned to zero. However, this is not allowed by causality. In the limit $|a_{L,d}| \to \infty$ our causality bound for $2L + d \geq 5$ places an upper bound on $r_{L,d}$,

$$r_{L,d} \leq -\frac{2\Gamma(L + \frac{d}{2} - 2)\Gamma(L + \frac{d}{2} - 1)}{\pi} \left(\frac{R}{2}\right)^{-2L-d+4}.$$  \hspace{1cm} (78)
Since the expression on the right-hand side is a fixed negative number, the value \( r_{L,d} = 0 \) is not allowed.

We see that for \( 2L + d \geq 4 \) we are left with two relevant parameters which parametrize the strong-coupling low-energy limit. This corresponds to two relevant directions near a fixed point of the renormalization group, and the universal behavior is characterized by two low-energy parameters. For the case of \( P \)-wave neutron-alpha scattering in three dimensions, this issue was already discussed in [21]. Proper renormalization of an effective field theory for \( P \)-wave scattering requires the inclusion of field operators for the scattering volume and the effective range at leading order. In the renormalization group study of [9], the emergence of two relevant directions around a fixed point was observed for various model potentials.

VI. IMPACT OF CAUSALITY BOUNDS

In the following, we illustrate the impact of the causality bounds for three examples. We start with a spherical step potential in \( d \) dimensions. This corresponds to a purely short-range interaction and our causality bounds apply strictly. As a second example, we consider the neutron-alpha interaction which is characterized by resonant \( P \)-wave interactions. The interaction is mediated by pion exchange which corresponds to an exponentially-bounded interaction of \( O(e^{-r/R}) \) at large distances. In this case, the results should still be accurate with only exponentially small corrections. Finally, we consider the long-range van der Waals interaction where our general bounds do not apply. We show how our treatment must be modified in this case relevant to ultracold atoms.

A. Spherical step potential in \( d \) dimensions

As an example of the results discussed we consider a spherical step potential with radius \( R \) and depth \( V_{\text{step}} \).

\[
W(r, r') = V_{\text{step}} \theta(R - r) \delta(r - r').
\]

(79)

We define \( \kappa \) so that \( \kappa^2 = -2\mu V_{\text{step}} \) and \( p' = \sqrt{p^2 + \kappa^2} \). A repulsive step corresponds with positive imaginary \( \kappa \) and an attractive step corresponds with a positive real \( \kappa \). For the exterior region, \( r \geq R \), the wave function \( u_{L,d}^{(p)}(r) \) is given by Eq. (23). For the interior region, \( r < R \), the wave function is

\[
u_{L,d}^{(p)}(r) \propto \sqrt{\frac{p'r\pi}{2}} J_{L+d/2-1}(p'r).
\]

(80)

Matching at the boundary \( r = R \) we find

\[
\cot \delta_{L,d}(p)
= \frac{pR J_\alpha(p'R) [Y_{\alpha+1}(pR) - Y_{\alpha-1}(pR)] - p'RY_\alpha(pR) [J_{\alpha+1}(p'R) - J_{\alpha-1}(p'R)]}{pR J_\alpha(p'R) [J_{\alpha+1}(pR) - J_{\alpha-1}(pR)] - p'RY_\alpha(pR) [J_{\alpha+1}(p'R) - J_{\alpha-1}(p'R)]},
\]

(81)

where \( \alpha = L + d/2 - 1 \).

We use Eq. (81) to generate the effective range expansion for

\[
p^{2L+d-2} \left[ \cot \delta_{L,d}(p) - \delta_{d \mod 2,0} \frac{2}{\pi} \ln \left( p \rho_{L,d} \right) \right].
\]

(82)
For $2L + d = 1$, the lowest two coefficients in the effective range expansion are

$$a_{L,d}^{-1} R^{-1} = - \frac{\kappa R + \cot (\kappa R)}{\kappa R},$$  \hspace{1cm} (83)

$$r_{L,d} R^{-3} = \frac{\kappa R (2\kappa^2 R^2 - 3) + 3(\kappa R^2 - 1) \cot (\kappa R) + 3\kappa R \cot^2 (\kappa R)}{3\kappa^3 R^3}. \hspace{1cm} (84)$$

In each case we multiply by powers of $R$ to render the quantity dimensionless. For $2L + d = 2$, we use the convention $\rho_{L,d} = \frac{R}{2}$. In this case the dimensionless coefficients are

$$a_{L,d}^{-1} R = \frac{\kappa R \cos (\kappa R)}{\kappa R \cos (\kappa R) - \sin (\kappa R)},$$ \hspace{1cm} (85)

$$r_{L,d} R^{-2} = \frac{1}{\pi \kappa^2 R^2} \left[ \kappa^2 R^2 - 2 + 2\kappa R J_0(\kappa R) J_1(\kappa R) \right]. \hspace{1cm} (86)$$

For $2L + d = 3$,

$$a_{L,d}^{-1} R^2 = - \frac{4}{\pi} \frac{J_0(\kappa R)}{J_2(\kappa R)}, \hspace{1cm} (87)$$

$$r_{L,d} R^{-1} = \frac{2\kappa^3 R^3 + 2\kappa R (\kappa^2 R^2 - 3) \cos (2\kappa R) + 3(-2\kappa^2 R^2 + 1) \sin (2\kappa R)}{6\kappa R [-\kappa R \cos (\kappa R) + \sin (\kappa R)]^2}. \hspace{1cm} (88)$$

For $2L + d = 4$, we again use the convention $\rho_{L,d} = \frac{R}{2}$,

$$a_{L,d}^{-1} R^2 = \frac{3\kappa^2 R^2}{-3 + \kappa^2 R^2 + 3\kappa R \cot (\kappa R)}, \hspace{1cm} (89)$$

$$r_{L,d} = \left\{ 8(2\gamma - 1) \kappa R J_1^2(\kappa R) + \kappa R J_0^2(\kappa R) \left[ (4\gamma - 3) \kappa^2 R^2 - 4 \right] + 4\kappa R J_0(\kappa R) J_2(\kappa R) + 8 J_0(\kappa R) J_1(\kappa R) \left[ (-2\gamma + 1) \kappa^2 R^2 + 1 \right] \right\} \times \frac{1}{\pi \kappa R \left[ \kappa R J_0(\kappa R) - 2 J_1(\kappa R) \right]^2}. \hspace{1cm} (90)$$

For $2L + d = 5$,

$$a_{L,d}^{-1} R^3 = \frac{3\kappa^2 R^2}{-3 + \kappa^2 R^2 + 3\kappa R \cot (\kappa R)}, \hspace{1cm} (91)$$

$$r_{L,d} R = \left\{ -18\kappa R \left( \kappa^2 R^2 + 5 \right) + 18\kappa R \left( \kappa^2 R^2 - 10 \right) \cos (2\kappa R) + 45 \left( -2\kappa^2 R^2 + 3 \right) \sin (2\kappa R) \right\} \times \frac{\kappa R}{10 \left[ 3\kappa R \cos (\kappa R) + (\kappa^2 R^2 - 3) \sin (\kappa R) \right]^2}. \hspace{1cm} (92)$$

In Figs. 3 and 4, we plot the function $[b_{L,d}(r) - r_{L,d} R^{2L+d-4}]$ as a function of $r/R$ for the sample values $2L + d = 2, 3$. We see that the function is non-negative for $r/R \geq 1$, as required by causality. As $\kappa^2 R^2 \rightarrow -\infty$, the potential becomes a hard spherical barrier with the wave function $u_{L,d}^{(p)}(r)$ vanishing in the interior region, $r \leq R$. Since

$$\int_0^R dr' \left[ u_{L,d}^{(0)}(r') \right]^2 \rightarrow 0^+, \hspace{1cm} (93)$$

the causality bound is saturated in the hard barrier limit for $r = R$.

$$b_{L,d}(R) - r_{L,d} \rightarrow 0^+. \hspace{1cm} (94)$$

For the other values of $2L + d$ not shown in Figs. 3 and 4, the behavior is qualitatively the same.
Our results are exact only for the case where the interaction vanishes for $r \geq R$. For exponentially-bounded interactions of $O(e^{-r/R})$ at large distances, the results should still be accurate with only exponentially small corrections. For an exponentially-bounded but otherwise unknown interaction, the non-negativity condition for $b_{L,d}(r) - r_{L,d}$ can be used to determine the minimum value for $R$ consistent with causality. One example of an exponentially-bounded interaction is the three-dimensional scattering of an alpha particle...
and neutron. We consider $S$-wave and $P$-wave alpha-neutron scattering.

For $S$-wave scattering,

$$b_{0,3}(r) = 2r - \frac{2r^2}{a_{0,3}} + \frac{2r^3}{3a_{0,3}^2},$$

and for $P$-wave scattering,

$$b_{1,3}(r) = -\frac{2}{r} - \frac{2r^2}{3a_{1,3}} + \frac{2r^3}{45a_{1,3}^2}.$$  

(95)
(96)

In Fig. 5 we plot $b_{L,3}(r) - r_{L,3}$ for the $S_{1/2}$, $P_{1/2}$, and $P_{3/2}$ channels. A qualitatively similar plot was introduced for nucleon-nucleon scattering in the $S$-wave spin-singlet channel [32]. We use the values $a_{0,3} = 2.464(4)$ fm and $r_{0,3} = 1.39(4)$ fm for $S_{1/2}$; $a_{1,3} = -13.82(7)$ fm$^3$ and $r_{1,3} = -0.42(2)$ fm$^{-1}$ for $P_{1/2}$; and $a_{1,3} = -62.951(3)$ fm$^3$ and $r_{1,3} = -0.882(1)$ fm$^{-1}$ for $P_{3/2}$ [33]. The non-negativity condition for $b_{L,3}(r) - r_{L,3}$ gives $R \geq 1.1$ fm for $S_{1/2}$, $R \geq 2.6$ fm for $P_{1/2}$, and $R \geq 2.1$ fm for $P_{3/2}$. For comparison, the alpha root-mean-square radius and pion Compton wavelength are both about 1.5 fm. Since the minimum values for $R$ are not small when compared with these, some caution is required when choosing the cutoff scale for an effective theory of alpha-neutron interactions.

We can use the results for the spherical step potential to reproduce the scattering parameter and effective range parameter for alpha-neutron scattering. For the $S$-wave, $r_{0,3}a_{0,3}^{-1} = 0.564$. This gives $\kappa^2 R^2 = -5.56$. In turn this implies $r_{0,3}/R = 0.329$ and therefore $R = 4.23$ fm. For the $P_{1/2}$-channel, $r_{1,3}^{-3}a_{1,3}^{-1} = 0.977$. This gives $\kappa^2 R^2 = 5.034$, $r_{1,3}R = -1.67$, and so $R = 3.97$ fm. For the $P_{3/2}$-channel, $r_{1,3}^{-3}a_{1,3}^{-1} = 0.0232$. This gives $\kappa^2 R^2 = 8.90$, $r_{1,3}R = -2.84$, and $R = 3.22$ fm.

FIG. 5: Plot of $b_{L,3}(r) - r_{L,3}$ as a function of $r$ for alpha-neutron scattering in the $S_{1/2}$, $P_{1/2}$, and $P_{3/2}$ channels.
C. Van der Waals interaction

The physics of long-range interactions must be treated separately since each long-range behavior determines its own low-energy universality class. For cold alkali atoms our analysis must be modified to take into account long-range van der Waals interactions of the type

\[ W(r, r') = -C_6 r^{-6} \delta(r - r') \]  

for \( r, r' \geq R \). It is convenient to reexpress \( C_6 \) in terms of the length scale \( \beta_6 = (2\mu C_6)^{1/4} \). In the following, we set \( d = 3 \) and drop the \( d \) subscript. Instead of free Bessel functions, scattering states should be compared with exact solutions of the attractive \( r^{-6} \) potential \(^{[34, 35]}\). The effect of the interactions for \( r < R \) are described by a finite-range \( K \)-matrix \( K_L(p^2) \) that is analytic in \( p^2 \) \(^{[36]}\),

\[ K_L(p^2) = \sum_{n=0,1,\ldots} K_L^{(2n)} p^{2n}. \]

When phase shifts are measured relative to free spherical Bessel functions, the effective range expansion is no longer analytic in \( p^2 \). For \( L = 0 \), the leading non-analytic term is proportional to \( p^3 \),

\[ p \cot \delta_0(p) = -\frac{[\Gamma(1/4)]^2 K_0^{(0)}}{2\pi \beta_6 [K_0^{(0)} - 1]} + \frac{[\Gamma(1/4)]^2 \beta_6^2}{6\pi \beta_6} \left[ \frac{K_0^{(0)}}{K_0^{(0)} - 1} \right]^2 p^2 \]

\[ -\frac{[\Gamma(1/4)]^4}{60\pi} \frac{\beta_6^2}{[K_0^{(0)} - 1]^2} p^3 + O(p^4 \ln p). \]

For \( L = 1 \) the leading non-analytic term is proportional to \( p^1 \),

\[ p^3 \cot \delta_1(p) = \frac{18 [\Gamma(3/4)]^2 K_1^{(0)}}{\pi \beta_6^3 \left[ K_1^{(0)} + 1 \right]} + \frac{324 [\Gamma(3/4)]^4}{35\pi \beta_6^2 \left[ K_1^{(0)} + 1 \right]^2} p \]

\[ -4410 [\Gamma(3/4)]^2 \left[ K_1^{(0)} + 1 \right] \left\{ \beta_6^2 \left[ K_1^{(0)} + 1 \right] - 5K_1^{(2)} \right\} + 5832 [\Gamma(3/4)]^6 \beta_6^2 \left[ K_1^{(0)} \right]^3 \]

\[ + \frac{1225\pi \beta_6^3 \left[ K_1^{(0)} + 1 \right]^3}{p^2}. \]

This term voids the standard definition of the effective range parameter for \( P \)-waves.

However, one can still obtain useful information. The zero-energy resonance limit is reached by tuning the lowest-order \( K \)-matrix coefficient \( K_L^{(0)} \) to zero. In this limit the leading non-analytic terms in the effective range expansion vanishes, and one can define an
effective range parameter for both $S$- and $P$-waves \[^{35, 37}\],

$$r_0 = \frac{[\Gamma(1/4)^2] \left[ \beta_6^2 + 3K_0^{(2)} \right]}{3\pi \beta_6},$$  \hfill (101)

$$r_1 = \frac{-36 \left[ \Gamma(3/4)^2 \right] \left[ \beta_6^2 - 5K_1^{(2)} \right]}{5\pi \beta_6^3}.$$  \hfill (102)

For the case of single-channel scattering of alkali atoms, the coefficients $K_L^{(2)}$ are negligible compared with $\beta_6^2$. This is also true for some multi-channel Feshbach resonance systems \[^{38}\]. In these cases we observe that the upper bounds for $r_L$ in Eq. (60) are satisfied for $L = 0$ and $L = 1$ when we naively take $R \sim \beta_6$. In general, there may be multi-channel systems where the coefficients $K_L^{(2)}$ cannot be neglected. Nevertheless, the coefficients $K_L^{(2)}$ should satisfy causality bounds similar to those derived here for the effective range parameter.

\section{VII. SUMMARY AND OUTLOOK}

In this paper, we have addressed the question of universality and the constraints of causality on quantum scattering processes for arbitrary dimension $d$ and arbitrary angular momentum $L$. We have derived the Wronskian identity for two solutions of the radial Schrödinger equation with different energy and used this identity to generalize the causality bound on the effective range to arbitrary $d$ and $L$. Moreover, we have derived a general causality bound for energies away from threshold.

For finite-range interactions, we have shown that causal wave propagation can have significant consequences for low-energy universality and scale invariance. For $2L + d \geq 4$, two relevant low-energy parameters are required in the strong-coupling low-energy limit. In the language of the renormalization group, this corresponds to two relevant directions in the vicinity of a fixed point. In particular, we confirm earlier findings for three-dimensional $P$-wave scattering \[^{21}\] based on renormalization arguments and for higher partial waves in general \[^{9}\] in the framework of the renormalization group.

In the low-energy limit, the hierarchy of terms in the effective range expansion depends on the value of $2L + d$ (cf. Fig. \[^{2}\]). In particular, the effective range parameter is as important at low energies as the unitarity contribution for $2L + d = 4$ and becomes more important for $2L + d \geq 5$. Our results imply that the scale-invariant unitarity limit can not be reached in this case because the causality bound prevents the effective range from being tuned to zero. This has important consequences for the universal properties of systems with $P$-wave and higher partial wave interactions. Causality also constrains the wave functions and the probability to find particles at large spatial separation \[^{2}\]. This issue will be discussed in a separate publication.

We stress that our results strictly apply only to energy-independent interactions and single-channel systems. For energy-dependent interactions it is possible to generate any energy dependence for the elastic phase shifts even when the interaction $W(r, r'; E)$ vanishes beyond some finite radius $R$ for all $E$. Under these more general conditions, there are no longer any Wigner bounds and the constraints of causality seem to disappear. However, it is misleading to regard interactions of this more general type as having finite range. As noted in the introduction, the scattering time delay is given by the energy derivative of the
phase shift. The energy dependence of the interaction can by itself generate large negative
time delays and thereby reproduce the scattering of long-range interactions. In this sense
the range of the interaction as observed in scattering is set by the dependence of $W(r, r'; E)$
on the radial coordinates $r,r'$ as well as the energy $E$. In this case our bound can be
viewed as an estimate for the minimum value of this interaction range. For coupled-channel
dynamics without partial wave mixing the analysis can proceed by first integrating out
higher-energy contributions to produce a single-channel effective interaction. In order to
satisfy our condition of energy-independent interactions, this should be carried out using a
technique such as the method of unitary transformation described in Ref. [39–41].

Our analysis concerns only the question of universality in two-body scattering. Universality
for higher few-body systems requires a detailed analysis for each system under
consideration. For resonant $S$-wave interactions, the question of universality has already
been explored for three and more particles [8]. In two dimensions, the properties of $N$-
A boson droplets are universal for $N$ large, but below some critical value [12 [44]. In three
dimensions, the Efimov effect generates a universal spectrum of shallow three-body bound
states [45] and two universal four-body states are attached to each three-body Efimov state
[46, 47]. Effective field theory and renormalization group methods may provide a useful
starting point to extend these studies to more particles and higher angular momentum [48–
51]. Our results help to clarify some of the conceptual and calculational issues relevant to
few-body systems for general dimension and angular momentum and their simulation using
short-range interactions.

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Appendix A: Expressions for $b_{L,d}(r)$

In this Appendix, we check explicitly that the identities Eqs. (44, 45) hold for all combi-
nations of $d$ and $L$ and derive explicit expressions for $b_{L,d}(r)$. Moreover, our conventions for
the Riccati-Bessel and Bessel functions are given in Eqs. (A4 A5 A14 A15).

1. Positive odd integer $2L + d$

For positive odd integer $2L + d$ it is convenient to use the second line of Eq. (23). For
$r \geq R,$

$$v_{L,d}^{(p)}(r) = p^{L+d/2-3/2} \left[ \cot \delta_{L,d}(p) S_{L+d/2-3/2}(pr) + C_{L+d/2-3/2}(pr) \right].$$  \hspace{1cm} (A1)

Writing this in the form dictated in Eq. (37), we have

$$s(p, r) = p^{-L-d/2+1/2} S_{L+d/2-3/2}(pr),$$  \hspace{1cm} (A2)

$$c(p, r) = p^{L+d/2-3/2} C_{L+d/2-3/2}(pr).$$  \hspace{1cm} (A3)
When $2L + d$ is a positive odd integer, $L + d/2 - 3/2$ is an integer greater than or equal to $-1$. For integer $n$, the Riccati-Bessel functions are given by

$$S_n(x) = \sqrt{\pi} x^{n+1} \sum_{m=0}^{\infty} \frac{i^{2m} \Gamma(m + n + \frac{3}{2})}{\Gamma(m + 1)} x^{2m}$$  \hspace{1cm} (A4)

$$C_n(x) = \frac{1}{\sqrt{\pi}} x^{-n} \sum_{m=0}^{\infty} \frac{i^{2m+n} \Gamma(n + \frac{3}{2})}{\Gamma(m + 1)} x^{2m}$$  \hspace{1cm} (A5)

The first two series coefficients for $s(p, r)$ and $c(p, r)$ in powers of $p^2$ are

$$s_0(r) = \frac{\sqrt{\pi}}{\Gamma(L + \frac{d}{2})} \left( \frac{r}{2} \right)^{L+d/2-1/2}, \quad s_2(r) = -\frac{\sqrt{\pi}}{\Gamma(L + \frac{d}{2} + 1)} \left( \frac{r}{2} \right)^{L+d/2+3/2}$$  \hspace{1cm} (A6)

$$c_0(r) = \frac{\Gamma(L + \frac{d}{2} - 1)}{\sqrt{\pi}} \left( \frac{r}{2} \right)^{-L-d/2+3/2}, \quad c_2(r) = \frac{\Gamma(L + \frac{d}{2} - 2)}{\sqrt{\pi}} \left( \frac{r}{2} \right)^{-L-d/2+7/2}$$  \hspace{1cm} (A7)

The corresponding Wronskians are

$$W[s_0, c_0](r) = -1$$  \hspace{1cm} (A8)

$$W[s_2, s_0](r) = \frac{\pi}{\Gamma(L + \frac{d}{2}) \Gamma(L + \frac{d}{2} + 1)} \left( \frac{r}{2} \right)^{2L+d}$$  \hspace{1cm} (A9)

$$W[s_2, c_0](r) = W[c_2, s_0](r) = \frac{r^2}{2(2L + d - 2)}$$  \hspace{1cm} (A10)

$$W[c_2, c_0](r) = -\frac{\Gamma(L + \frac{d}{2} - 2) \Gamma(L + \frac{d}{2} - 1)}{\pi} \left( \frac{r}{2} \right)^{2L-d+4}$$  \hspace{1cm} (A11)

We conclude that for $r \geq R$,

$$b_{L,d}(r) = -2\frac{\Gamma(L + \frac{d}{2} - 2) \Gamma(L + \frac{d}{2} - 1)}{\pi} \left( \frac{r}{2} \right)^{2L-d+4}$$

$$-\frac{4}{L + \frac{d}{2} - 1} a_{L,d} \left( \frac{r}{2} \right)^2$$

$$+\frac{2\pi}{\Gamma(L + \frac{d}{2}) \Gamma(L + \frac{d}{2} + 1)} \frac{1}{a_{L,d}^2} \left( \frac{r}{2} \right)^{2L+d}$$  \hspace{1cm} (A12)

2. Positive even integer $2L + d$

For positive even integer $2L + d$ it is convenient to use the first line of Eq. (23). For $r \geq R$,

$$u_{L,d}^{(p)}(r) = \sqrt{\frac{pr\pi}{2}} p^{L+d/2-3/2} \left[ \cot \delta_{L,d}(p) J_{L+d/2-1}(pr) - Y_{L+d/2-1}(pr) \right]$$  \hspace{1cm} (A13)
When \(2L+d\) is a positive even integer, \(L+d/2-1\) is a non-negative integer. For non-negative integer \(n\) we have

\[
J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n},
\]

and

\[
Y_n(x) = \frac{2}{\pi} \left(\ln \frac{x}{2} + \gamma\right) J_n(x) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{\Gamma(n-k)}{\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k-n} - \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (H_m + H_{n+m})}{\Gamma(m+1)\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}
\]  

(A15)

\(\gamma\) is the Euler-Mascheroni constant, and \(H_k\) is the \(k\)th harmonic number,

\[
H_k = \sum_{m=1}^{k} \frac{1}{m}.
\]

(A16)

We can write \(u_{L,d}^{(p)}(r)\) in the form dictated in Eq. (37), if we let

\[
s(p, r) = \sqrt{\frac{pr\pi}{2}} p^{-L-d/2+1/2} J_{L+d/2-1}(pr)
\]

(A17)

\[
c(p, r) = -\sqrt{\frac{pr\pi}{2}} p^{L+d/2-3/2} \left[Y_{L+d/2-1}(pr) - \frac{2}{\pi} \ln (pp_{L,d}) J_{L+d/2-1}(pr)\right].
\]

(A18)

We now consider each of the possible cases for positive even integer \(2L + d\).

\[a. \text{ Case } 2L + d = 2\]

When \(2L + d = 2\) we have

\[
s(p, r) = \sqrt{\frac{r\pi}{2}} J_0(pr)
\]

(A19)

\[
c(p, r) = -\sqrt{\frac{r\pi}{2}} \left[Y_0(pr) - \frac{2}{\pi} \ln (pp_{L,d}) J_0(pr)\right].
\]

(A20)

Both functions are analytic in \(p^2\). The first two series coefficients are

\[
s_0(r) = \sqrt{\pi} \left(\frac{r}{2}\right)^{1/2}, \quad s_2(r) = -\sqrt{\pi} \left(\frac{r}{2}\right)^{5/2}
\]

(A21)

\[
c_0(r) = -\frac{2}{\sqrt{\pi}} \left(\frac{r}{2}\right)^{1/2} \left[\ln \left(\frac{r}{2p_{L,d}}\right) + \gamma\right], \quad c_2(r) = \frac{2}{\sqrt{\pi}} \left(\frac{r}{2}\right)^{5/2} \left[\ln \left(\frac{r}{2p_{L,d}}\right) + \gamma - 1\right].
\]

(A22)

The corresponding Wronskians are

\[
W[s_0, c_0](r) = -1,
\]

(A23)
\[ W[s_2, s_0](r) = \pi \left( \frac{r}{2} \right)^2, \quad (A24) \]

\[ W[s_2, c_0](r) = W[c_2, s_0](r) = -2 \left[ \ln \left( \frac{r}{2\rho_{L,d}} \right) + \gamma - \frac{1}{2} \right] \left( \frac{r}{2} \right)^2, \quad (A25) \]

\[ W[c_2, c_0](r) = \frac{4}{\pi} \left\{ \left[ \ln \left( \frac{r}{2\rho_{L,d}} \right) + \gamma - \frac{1}{2} \right]^2 + \frac{1}{4} \right\} \left( \frac{r}{2} \right)^2. \quad (A26) \]

The function \( b_{L,d}(r) \) for this case is

\[ b_{L,d}(r) = \frac{2r^2}{\pi} \left\{ \left[ \ln \left( \frac{r}{2\rho_{L,d}} \right) + \gamma - \frac{1}{2} + \frac{\pi}{2a_{L,d}} \right]^2 + \frac{1}{4} \right\}. \quad (A27) \]

b. Case 2L + d = 4

For 2L + d = 4,

\[ s(p, r) = \sqrt{\frac{\pi r}{2}} p^{-1} J_1(pr), \quad (A28) \]

\[ c(p, r) = -\sqrt{\frac{\pi r}{2}} p \left[ Y_1(pr) - \frac{2}{\pi} \ln (pp_{L,d}) J_1(pr) \right]. \quad (A29) \]

The series coefficients are

\[ s_0(r) = \sqrt{\pi} \left( \frac{r}{2} \right)^{3/2}, \quad s_2(r) = -\sqrt{\frac{\pi}{2}} \left( \frac{r}{2} \right)^{7/2}, \quad (A30) \]

\[ c_0(r) = \frac{1}{\sqrt{\pi}} \left( \frac{r}{2} \right)^{-1/2}, \quad c_2(r) = -\frac{1}{\sqrt{\pi}} \left( \frac{r}{2} \right)^{3/2} \left[ 2 \ln \left( \frac{r}{2\rho_{L,d}} \right) + 2\gamma - 1 \right], \quad (A31) \]

and the Wronskians are

\[ W[s_0, c_0](r) = -1, \quad (A32) \]

\[ W[s_2, s_0](r) = \frac{\pi}{2} \left( \frac{r}{2} \right)^4, \quad (A33) \]

\[ W[s_2, c_0](r) = W[c_2, s_0](r) = \left( \frac{r}{2} \right)^2, \quad (A34) \]

\[ W[c_2, c_0](r) = \frac{2}{\pi} \ln \left( \frac{r}{2\rho_{L,d}} \right) + \gamma \]. \quad (A35) \]

The function \( b_{L,d}(r) \) is

\[ b_{L,d}(r) = \frac{4}{\pi} \left[ \ln \left( \frac{r}{2\rho_{L,d}} \right) + \gamma \right] - \frac{4}{a_{L,d}} \left( \frac{r}{2} \right)^2 + \frac{\pi}{a_{L,d}^2} \left( \frac{r}{2} \right)^4. \quad (A36) \]
c. Case $2L + d \geq 6$

The last case we consider is when $2L + d$ is an even integer greater than or equal to 6. Here we have

$$s(p, r) = \sqrt{\frac{r^2}{2}p^{L-d/2+1}} J_{L+d/2-1}(pr)$$

$$c(p, r) = -\sqrt{\frac{r^2}{2}}p^{L+d/2-1} \left[ Y_{L+d/2-1}(pr) - \frac{2}{\pi} \ln(p\rho_{L,d}) J_{L+d/2-1}(pr) \right].$$

The first two series coefficients have exactly the same form as in the case for positive odd integer $2L + d$,

$$s_0(r) = \frac{\sqrt{\pi}}{\Gamma(L + \frac{d}{2})} \left( \frac{r}{2} \right)^{L+d/2-1/2}, \quad s_2(r) = -\frac{\sqrt{\pi}}{\Gamma(L + \frac{d}{2} + 1)} \left( \frac{r}{2} \right)^{L+d/2+3/2},$$

$$c_0(r) = \frac{\Gamma(L + \frac{d}{2} - 1)}{\sqrt{\pi}} \left( \frac{r}{2} \right)^{-L-d/2+3/2}, \quad c_2(r) = \frac{\Gamma(L + \frac{d}{2} - 2)}{\sqrt{\pi}} \left( \frac{r}{2} \right)^{-L-d/2+7/2}.$$ (A40)

We conclude the same result for $b_{L,d}(r)$ as for odd $2L + d$ as written in Eq. (A12).

**Appendix B: Equivalence with Wigner’s bound**

In this Appendix, we demonstrate the equivalence of our causality bound with Wigner’s original bound on the energy derivative of the phase shift $\frac{\partial \phi}{\partial E}$.

Let $I_A(r)$ be a free incoming radial wave for momentum $p_A$,

$$I_A(r) = \sqrt{\frac{p_A^2 \pi}{2}} p_A^{L+d/2-3/2} \left[ -i \cdot J_{L+d/2-1}(p_A r) - Y_{L+d/2-1}(p_A r) \right].$$ (B1)

We are using the same phase convention as Wigner but a different normalization. From Abel’s differential equation identity, the Wronskian of $I_A$ and $I_A^*$ is independent of $r$. For our chosen normalization of the incoming wave,

$$I_A(r) I_A^*(r) - I_A^*(r) I_A(r) = 2ip_A^{2L+d-2}.$$ (B2)

For $r \geq R$,

$$u_A(r) = \frac{ie^{-i\delta_A}}{2 \sin \delta_L(p)} \left[ I_A(r) - e^{2i\delta_A} I_A^*(r) \right].$$ (B3)

We define $\alpha_A(r)$ as the reciprocal logarithmic derivative of $u_A(r)$,

$$\alpha_A(r) = \frac{u_A'(r)}{u_A(r)} = \frac{I_A(r) - e^{2i\delta_A} I_A^*(r)}{I_A^*(r) - e^{2i\delta_A} I_A^*(r)}. (B4)$$

Then

$$e^{2i\delta_A} = \frac{I_A(r) - \alpha_A(r) I_A'(r)}{I_A^*(r) - \alpha_A(r) I_A^*(r)},$$ (B5)

$$e^{2i\delta_A} [ I_A^*(r) - \alpha_A(r) I_A^*(r) ] = I_A(r) - \alpha_A(r) I_A(r).$$ (B6)
We place a dot on top of a function to indicate the derivative with respect to $p_A$. Differentiating Eq. (B6) with respect to $p_A$, we get

$$2i e^{2i\delta A} \dot{\delta}_A [I_A^*(r) - \alpha_A(r)I_A^*(r)] + e^{2i\delta A} \left[ \dot{I}_A^*(r) - \alpha_A(r)\dot{I}_A^*(r) - \dot{\alpha}_A(r)I_A^*(r) \right] = \dot{I}_A(r) - \alpha_A(r)\dot{I}_A^*(r) - \dot{\alpha}_A(r)I_A^*(r). \tag{B7}$$

Solving for $\dot{\delta}_A$ gives

$$\dot{\delta}_A = F_A(r) + G_A(r)\dot{\alpha}_A(r), \tag{B8}$$

where

$$F_A(r) = -\frac{1}{2i} \frac{\dot{I}_A^*(r) - \alpha_A(r)\dot{I}_A^*(r) - e^{-2i\delta A} \left[ \dot{I}_A(r) - \alpha_A(r)\dot{I}_A(r) \right]}{I_A^*(r) - \alpha_A(r)I_A^*(r)}. \tag{B9}$$

and

$$G_A(r) = \frac{1}{2i} \frac{I_A(r)I_A^*(r) - I_A^*(r)I_A^*(r)}{|I_A(r) - \alpha_A(r)I_A^*(r)|^2}. \tag{B10}$$

Both $F_A$ and $G_A$ can be simplified further. We replace $\alpha_A$ using Eq. (B4) and find

$$F_A(r) = -\frac{1}{i} \text{Re} \left\{ \dot{I}_A(r) \left[ I_A^*(r) - e^{-2i\delta A} I_A^*(r) \right] - \dot{I}_A^*(r) \left[ I_A^*(r) - e^{-2i\delta A} I_A(r) \right] \right\}. \tag{B11}$$

The Wronskian identity, Eq. (B2), leads to

$$F_A(r) = -\frac{1}{2p_A^{2L+d-2}} \text{Re} \left\{ \dot{I}_A(r)I_A^*(r) - \dot{I}_A(r)I_A^*(r) - e^{-2i\delta A} \left[ \dot{I}_A(r)I_A^*(r) - \dot{I}_A(r)I_A(r) \right] \right\}. \tag{B12}$$

With the same Wronskian identity, $G_A$ simplifies to

$$G_A(r) = \frac{p_A^{2L+d-2}}{|I_A(r) - \alpha_A(r)I_A^*(r)|^2}. \tag{B13}$$

We note that

$$\dot{\alpha}_A(r) = \lim_{p_B \to p_A} \frac{u_B(r)}{u_B(r)} - \frac{u_A(r)}{u_A(r)} = \lim_{p_B \to p_A} \frac{W[u_B,u_A](r)}{(p_B - p_A)u_B(r)u_A(r)}. \tag{B14}$$

For any $p_A \neq 0$, we use Eq. (63) to get

$$\dot{\alpha}_A(r) = \frac{2p_A}{[u_A'(r)]^2} \int_0^r dr' \left[ u_A(r') \right]^2. \tag{B15}$$

We see that both $G_A(r)$ and $\dot{\alpha}_A(r)$ are non-negative and so

$$\dot{\delta}_A = F_A(r) + G_A(r)\dot{\alpha}_A(r) \geq F_A(r). \tag{B16}$$

This inequality holds for all $p_A \neq 0$, and therefore also holds in the limit $p_A \to 0$. This is Wigner’s causality bound generalized to arbitrary dimension $d$. 

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Away from threshold, the equivalence between Wigner’s bound and Eq. (64) is clear from the biconditional statement,
\[ \dot{\delta}_A \geq F_A(r) \Leftrightarrow \dot{\alpha}_A(r) = \lim_{p_B \rightarrow p_A} \frac{W[u_B, u_A](r)}{(p_B - p_A) u_B'(r) u_A'(r)} \geq 0. \]  
(B17)

For \( p_A = 0 \), the equivalence with our causality bounds follows from the low-energy expansion for \( W[u_B, u_A] \),
\[ W[u_B, u_A](r) = p_B^2 \left\{ \frac{1}{2} r_{L,d} W[s_0, c_0](r) + \left( \frac{1}{a_{L,d}} \right)^2 W[s_2, s_0](r) - \frac{1}{a_{L,d}} W[c_2, c_0](r) + W[c_2, s_0](r) \right\} + O(p_B^4). \]  
(B18)

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