SCALING VS. DYNAMICS IN THE 3D NSE. A VIGNETTE
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ABSTRACT. Two regularity criteria for solutions to the 3D NSE contained in two supercritical spaces with identical scaling are presented. An interesting feature is that – to prevent (possible) singularity formation – the boundedness in the aforementioned spaces is naturally paired with two opposing dynamic conditions imposed on the Littlewood-Paley blocks which are consistent with the phenomena of direct and inverse energy cascades.

1. INTRODUCTION

3D Navier-Stokes equations (NSE) – governing the evolution of a viscous, incompressible flow’s velocity field $u$ and on the whole space – read

\begin{align*}
\partial_t u - \nu \Delta u &= -u \cdot \nabla u - \nabla p + f \quad \text{in } \mathbb{R}^3 \times (0, T), \\
\nabla \cdot u &= 0 \quad \text{in } \mathbb{R}^3 \times (0, T),
\end{align*}

where $\nu$ is the viscosity coefficient, $p$ is the pressure, and $f$ is the forcing. For convenience we take $f$ to be zero and set $\nu = 1$. The flow evolves from an initial vector field $u_0$ taken in an appropriate function space.

The inhomogeneous and homogeneous Besov spaces, respectively denoted $B^{s}_{p,q}$ and $\dot{B}^{s}_{p,q}$, are defined using the Littlewood-Paley formalism. Let $\lambda_j = 2^j$ be an inverse length and let $B_r$ denote the ball of radius $r$ centered at the origin. Fix a non-negative, radial cut-off function $\chi \in C_0^\infty(B_1)$ so that $\chi(\xi) = 1$ for all $\xi \in B_{1/2}$. Let $\phi(\xi) = \chi(\lambda_1^{-1}\xi) - \chi(\xi)$ and $\phi_j(\xi) = \phi(\lambda_j^{-1})(\xi)$. Suppose that $u$ is a vector field of tempered distributions and let $\Delta_j u = F^{-1}\phi_j * u$ for $j \geq 0$ and $\Delta_{-1} = F^{-1}\chi * u$. Then, $u$ can be written as

$$u = \sum_{j \geq -1} \Delta_j u.$$ 

If $F^{-1}\phi_j * u \to 0$ as $j \to -\infty$ in the space of tempered distributions, then we define $\Delta_j u = F^{-1}\phi_j * u$ and have

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u.$$ 

We are primarily interested in Besov spaces with infinite summability index the norms of which are

$$||u||_{B^{s}_{p,q}} := \sup_{-1 \leq j < \infty} \lambda_j^s ||\Delta_j u||_{L^p_{\mathbb{R}^n}}$$

and

$$||u||_{\dot{B}^{s}_{p,q}} := \sup_{-\infty < j < \infty} \lambda_j^s ||\Delta_j u||_{L^p_{\mathbb{R}^n}}.$$
When $q$ is finite the supremum is replaced with a series. See [2] for more details.

Our main result connects the spatially analytic smoothing introduced by viscous diffusion to the volumetric decay of regions tied to peripheral Littlewood-Paley modes – i.e. extremely high frequencies or extremely low frequencies – in two classes of supercritical flows. In the first class we show that the uniform-in-time inclusion of $u$ in $L^2(\mathbb{R}^3)$ (as is the case for Leray solutions) allows us to discard high frequency modes when considering possible singularity formation. Contrastingly, the uniform-in-time inclusion of $u$ in a Besov space of the form $\dot{B}^{-\delta}_{6/(3-2\delta),r}$, and with the same scaling as $L^2(\mathbb{R}^3)$, allows us to ignore low frequency modes instead.

**Theorem 1.** Suppose that $u$ is a distributional solution to 3D NSE which is regular on $(0,T)$. Fix $\epsilon \in (0,1)$.

If $u$ is in $L^\infty(0,T; L^2(\mathbb{R}^3))$, and for a time $t$, scale $\lambda_{Q^*(t)}$ and all $\tau \in (t,T)$, we have

\[
(1) \quad \sup_{j \leq Q^*(t)} \lambda_j^{-\epsilon} \| \Delta_j u(\tau) \|_{L^\infty} \leq \| u(t) \|_{\dot{B}^{-\epsilon}_{\infty,\infty}},
\]

then $u$ can be smoothly extended beyond time $T$.

Alternatively, if $u$ is in $L^\infty(0,T; \dot{B}^{-\delta}_{6/(3-2\delta),r})$ where $0 < \epsilon < \delta$ and $r > 0$, and for a time $t$, a scale $\lambda_{Q^*(t)}$ and all $\tau \in (t,T)$, we have

\[
(2) \quad \sup_{j \geq Q^*(t)} \lambda_j^{-\epsilon} \| \Delta_j u(\tau) \|_{L^\infty} \leq \| u(t) \|_{\dot{B}^{-\epsilon}_{\infty,\infty}},
\]

then $u$ can be smoothly extended beyond time $T$.

The dynamic cut-off indices $Q^*(t)$ and $Q^*(t)$ are identified in the proof.

As indicated above, $L^\infty(0,T; \dot{B}^{-\delta}_{6/(3-2\delta),r})$ scales identically to $L^\infty(0,T; L^2(\mathbb{R}^3))$. This illustrates the limitations of scaling arguments when considering the problem of global regularity for the 3D NSE model. More precisely – in this case – the identical scaling is naturally paired with two contrasting dynamic conditions, consistent with the phenomena of direct and inverse energy cascades, respectively.

The first statement in the theorem is of physical interest, and is reminiscent of the dyadic-scale-restricted Beale-Kato-Majda-type criterion presented in [3]. (As will be clear from its proof it can be formulated whenever $u \in L^\infty(0,T; \dot{B}^{-\delta}_{6/(3-2\delta),r})$ for $\delta \in (0,\epsilon)$.) Thinking within the realm of Leray solutions, this is consistent with turbulence phenomenology: at high enough frequencies – corresponding to small enough physical scales – the diffusion takes over and dominates the nonlinear (inertial) effects. In our approach, the diffusion is manifested via spatially analytic smoothing. As will be seen in the proof, once $u$ belongs to $\dot{B}^{-\epsilon}_{\infty,\infty}$, it becomes analytic for a short time. On the other hand, the energy inequality implies that the volumes of suitable super-level sets of dyadic blocks decay at high frequencies. Once these sets are sufficiently small, analytic smoothness makes it impossible for high frequencies to experience significant excitation. Rigorously, this is realized via the harmonic measure majorization principle.
The remainder of this paper is broken into two sections. In Section 2 we establish needed analytical properties of solutions in terms of Besov space norms. In Section 3 we present the proof of Theorem 1.

2. Uniform spatial analyticity for Littlewood-Paley blocks

The local-in-time wellposedness of mild solutions to the Cauchy problem with \( u_0 \in B_{\infty, \infty}^\epsilon \) is known when \( \epsilon \in [0, 1) \) and with \( u_0 \in \dot{B}_{\infty, \infty}^\epsilon \) when \( \epsilon \in (0, 1) \) (cf. [7, 9]). The proof of Theorem 1 relies on analytical properties of these solutions which are presented as a proposition.

**Proposition 2.** Let \( \epsilon \in (0, 1) \) and suppose \( u_0 \in \dot{B}_{\infty, \infty}^\epsilon \). Then there exists a mild solution \( u \) to 3D NSE on \([0, T_\ast]\) which is smooth on \((0, T_\ast]\) where

\[
T_\ast \leq \left( \frac{c_0}{\|u_0\|_{\dot{B}_{\infty, \infty}^\epsilon}} \right)^{2/(1-\epsilon)},
\]

for a positive, universal constant \( c_0 \). Furthermore, for every \( t \in (0, T_\ast] \) and \( j \in \mathbb{N} \), \( \Delta_j u(x, t) \) is real analytic and agrees with the restriction to \( \mathbb{R}^3 \) of a function \( U_j(x, y, t) + iV_j(x, y, t) \) which is analytic on the domain

\[
\Omega_t = \left\{ x + iy : x, y \in \mathbb{R}^3 \text{ and } |y| \leq \frac{1}{c_0} \sqrt{t} \right\},
\]

and which satisfies

\[
\sup_{j \in \mathbb{Z}} \left\{ \lambda_j^{-\epsilon} \|U_j(t)\|_{L^\infty(\Omega_t)} + \lambda_j^{-\epsilon} \|V_j(t)\|_{L^\infty(\Omega_t)} \right\} \leq c_0 \|u_0\|_{\dot{B}_{\infty, \infty}^\epsilon},
\]

for all \( t \in (0, T_\ast] \).

An analogous result for \( u_0 \in B_{\infty, \infty}^0 \) is valid and states that the mild solution described in [7] agrees with the restriction of an analytic function to \( \Omega_t \) (defined in terms of some possibly different universal constant \( c_0 \)) for \( t \in (0, T_\ast] \) where \( T_\ast < C_\epsilon \|u_0\|_{B_{\infty, \infty}^0}^{2/(\epsilon - 1)} \) for all sufficiently small \( \epsilon \in (0, 1) \) and constants \( C_\epsilon \) determined by \( \epsilon \). In this case the analytic extension of \( u \), denoted by \( U(x, y, t) + iV(x, y, t) \), satisfies

\[
\|U(\cdot, y, t)\|_{B_{\infty, \infty}^0} + \|V(\cdot, y, t)\|_{B_{\infty, \infty}^0} \leq c_0 \|u_0\|_{B_{\infty, \infty}^0},
\]

whenever \( t \in (0, T_\ast] \) and \( |y| \leq c_0^{-1} \sqrt{t} \).

Our proof of Proposition 2 uses an approach developed to study the spatial analyticity of solutions to 3D NSE in subcritical \( L^p \) (cf. [5, 6]). We take \( u \) to be regular on \( \mathbb{R}^3 \times (0, T) \) and denote its analytic extension by \( u(x, y, t) + iv(x, y, t) \). Then, where defined, \( u \) and \( v \) satisfy the complexified Navier-Stokes equations, i.e.,

\[
\begin{align*}
\partial_t u - \Delta u &= -u \cdot \nabla u + v \cdot \nabla v - \nabla p \\
\partial_t v - \Delta v &= -u \cdot \nabla v - v \cdot \nabla u - \nabla \pi \\
\nabla \cdot u &= \nabla \cdot v = 0 \\
u(0) &= u_0, \quad v(0) = 0
\end{align*}
\]

(3)
where \( p \) and \( \pi \) are the real and imaginary parts of the complexified pressure and obey the kinematic systems

\[
\begin{align*}
-\Delta p &= \partial_i \partial_j (u_i u_j - v_i v_j) \\
-\Delta \pi &= 2 \partial_i \partial_i (u_i v_j)
\end{align*}
\]

at each \( t \in (0, T) \).

Consider the classical real-variable approximation scheme for strong solutions constructed by setting \( u^{(0)} = p^{(0)} = 0 \) and iteratively solving the systems

\[
\begin{align*}
\partial_t u^{(n)} - \Delta u^{(n)} &= -u^{(n-1)} \cdot \nabla u^{(n-1)} - \nabla p^{(n-1)} & \text{in } (0, \infty) \times \mathbb{R}^3 \\
\nabla \cdot u^{(n)} &= 0 & \text{in } (0, \infty) \times \mathbb{R}^3 \\
\u^{(n)}(x, 0) &= u_0(x) & \text{in } \mathbb{R}^3.
\end{align*}
\]

Then, \( u^{(n)} \) and \( p^{(n)} \) are solutions non-homogeneous heat or Poisson equations and thus inherit the analytical properties of the non-homogeneous terms. Indeed, for all \( t \in (0, \infty) \) they extend to entire complex analytic functions which we denote by \( u^{(n)} + i v^{(n)} \). These solve systems mirroring (3) and (4).

Fix \( \alpha \in \mathbb{R}^3 \) and adopt the general notation \( f_\alpha(x, t) = f(x, \alpha t, t) \) where \( f \) is defined on \( \mathbb{C}^3 \times \{ t \} \). Using the Cauchy-Riemann system and Duhamel’s principle we obtain the formulas

\[
\begin{align*}
\u^{(0)}_\alpha(t) &= e^{t\Delta} u_0(t) \\
\v^{(0)}_\alpha(t) &= 0
\end{align*}
\]

and, for \( n > 0 \),

\[
\begin{align*}
\u^{(n+1)}_\alpha(t) &= e^{t\Delta} u_0(t) - \alpha_j \int_0^t e^{(t-\tau)\Delta} \partial_j v^{(n+1)}_\alpha(\tau) \, d\tau \\
&\quad - \int_0^t e^{(t-\tau)\Delta} P(u^{(n)}_\alpha) \cdot \nabla u^{(n)}(\tau) \, d\tau + \int_0^t e^{(t-\tau)\Delta} P(v^{(n)}_\alpha) \cdot \nabla v^{(n)}(\tau) \, d\tau \\
\v^{(n+1)}_\alpha(t) &= \alpha_j \int_0^t e^{(t-\tau)\Delta} \partial_j u^{(n+1)}_\alpha(\tau) \, d\tau \\
&\quad - \int_0^t e^{(t-\tau)\Delta} P(v^{(n)}_\alpha) \cdot \nabla u^{(n)}(\tau) \, d\tau - \int_0^t e^{(t-\tau)\Delta} P(u^{(n)}_\alpha) \cdot \nabla v^{(n)}(\tau) \, d\tau
\end{align*}
\]

where \( e^{t\Delta} \) is the heat semigroup and \( P \) is the Weyl-Helmholtz projection.

The main Besov space technique we need concerns the behavior of the heat semigroup and is presented as a lemma. A proof is contained in [7].

**Lemma 3.** Let \( s_0, s_1 \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \).

i. If \( s_0 \leq s_1 \), then

\[
\| e^{t\Delta} f \|_{B^{s_1}_{p, q}} \leq C t^{-(s_1 - s_0)/2} \| f \|_{B^{s_0}_{p, q}}.
\]

ii. If \( s_0 < s_1 \), then

\[
\| e^{t\Delta} f \|_{B^{s_1}_{p, 1}} \leq C t^{-(s_1 - s_0)/2} \| f \|_{B^{s_0}_{p, \infty}}.
\]
Several continuous embeddings will be used (cf. [2, 3]), namely,
\[ \dot{B}_{\infty,1}^0 \hookrightarrow L^\infty \hookrightarrow BMO \hookrightarrow \dot{B}_{\infty,\infty}^0. \]

We are now ready to prove Proposition 2.

**Proof of Proposition 2.** We proceed inductively. Directly from Lemma 3 we have,
\begin{align*}
(7) & \quad \|u_{\alpha}^{(n+1)}(t)\|_{\dot{B}_{\infty,\infty}^\alpha} \leq C_1\|u_0\|_{\dot{B}_{\infty,\infty}^\alpha} \\
(8) & \quad \|u_{\alpha}^{(n)}(t)\|_{L^\infty} \leq \|e^{t\Delta}u_0\|(t)\|_{\dot{B}_{\infty,1}^0} \leq C_2t^{-\frac{3}{2}}\|u_0\|_{\dot{B}_{\infty,\infty}^\alpha}
\end{align*}
where \( t \in (0, \infty) \). Note the corresponding quantities for \( \dot{v}(0) \) are zero.

Our inductive argument relies on the general estimate
\begin{align*}
(9) & \quad \|u_{\alpha}^{(n+1)}(t)\|_{L^\infty} + \|v_{\alpha}^{(n+1)}(t)\|_{L^\infty} \\
& \quad \leq C_2t^{-\frac{3}{2}}\|u_0\|_{\dot{B}_{\infty,\infty}^\alpha} + C|\alpha|t^{(1-\epsilon)/2} \sup_{0 < \tau \leq t} \left[ \tau^{\epsilon/2} \left( \|u_{\alpha}^{(n+1)}(\tau)\|_{L^\infty} + \|v_{\alpha}^{(n+1)}(\tau)\|_{L^\infty} \right) \right] \\
& \quad + C \int_0^t (t - \tau)^{-1/2} \left( \|u_{\alpha}^{(n)}(\tau)\|_{L^\infty}^2 + \|v_{\alpha}^{(n)}(\tau)\|_{L^\infty}^2 \right) d\tau \\
& \quad = I_1 + I_2 + I_3,
\end{align*}
where \( n \in \mathbb{N} \) and \( t \in (0, \infty) \). To establish this estimate we work with the integral formulas 5 and 6. The genesis of \( I_1 \) is clear. The terms involving \( \alpha_j \) are bounded by \( I_2 \). This follows from the fact that \( P \) is a bounded operator on \( \dot{B}_{\infty,\infty}^0 \) as well as the embeddings listed above. Indeed, using Lemma 3 and integrating by parts, we have
\begin{align*}
& \quad \left\| \alpha_j \int_0^t e^{(t-\tau)\Delta} \partial_j u_{\alpha}^{(n+1)}(t) d\tau \right\|_{L^\infty} + \left\| \alpha_j \int_0^t e^{(t-\tau)\Delta} \partial_j v_{\alpha}^{(n+1)}(t) d\tau \right\|_{L^\infty} \\
& \quad \leq |\alpha| \int_0^t \left\| e^{(t-\tau)\Delta} P(v_{\alpha}^{(n+1)}) \right\|_{\dot{B}_{\infty,1}^1} d\tau + \leq |\alpha| \int_0^t \left\| e^{(t-\tau)\Delta} P(v_{\alpha}^{(n+1)}) \right\|_{\dot{B}_{\infty,1}^1} d\tau \\
& \quad \leq C|\alpha| \int_0^t (t - \tau)^{-1/2} \|v_{\alpha}^{(n+1)}\|_{\dot{B}_{\infty,\infty}^0} d\tau + C|\alpha| \int_0^t (t - \tau)^{-1/2} u_{\alpha}(n+1)\|_{\dot{B}_{\infty,\infty}^0} d\tau \\
& \quad \leq C|\alpha| \sup_{0 < \tau \leq t} \left[ \tau^{\epsilon/2} \left( \|u_{\alpha}^{(n+1)}(\tau)\|_{L^\infty} + \|v_{\alpha}^{(n+1)}(\tau)\|_{L^\infty} \right) \right] \int_0^t (t - \tau)^{-1/2} \tau^{-\epsilon/2} d\tau \\
& \quad \leq I_2.
\end{align*}

A similar procedure (namely the first three steps of the above estimate noting that both \( u^{(n)} \) and \( v^{(n)} \) are divergence free) reveals that \( I_3 \) dominates the \( L^\infty \) norms of the remaining terms from the integral equations and justifies (9).

Fix
\[ T_1 = \left( \frac{1}{4 C C_2 \|u_0\|_{\dot{B}_{\infty,\infty}^\alpha}} \right)^{2/(1-\epsilon)}, \]
and assume for induction that for a fixed \( n \) and for all \( t \in (0, T_1) \) we have
\[ t^{\epsilon/2} \left( \|u_{\alpha}^{(n)}(t)\|_{L^\infty} + \|v_{\alpha}^{(n)}(t)\|_{L^\infty} \right) \leq 4 C_2 \|u_0\|_{\dot{B}_{\infty,\infty}^\alpha}. \]
Using this assumption and multiplying (9) by $t^{\epsilon/2}$ we obtain for $t \in (0, T_1]$ that
\[
t^{\epsilon/2} \left( \| u^{(n+1)}(t) \|_{L^\infty} + \| v^{(n+1)}(t) \|_{L^\infty} \right)
\leq C_2 \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}} + C|\alpha| t^{1/2} \sup_{0 < \tau \leq t} \left[ t^{\epsilon/2} \left( \| u^{(n+1)}(\tau) \|_{L^\infty} + \| v^{(n+1)}(\tau) \|_{L^\infty} \right) \right]
+ t^{\epsilon/2} C \left( 2 C_2 \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \right)^2 \int_0^t (t - \tau)^{1/2 - \epsilon} d\tau
\leq C_2 \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}} + C|\alpha| t^{1/2} \sup_{0 < \tau \leq t} \left[ t^{\epsilon/2} \left( \| u^{(n+1)}(\tau) \|_{L^\infty} + \| v^{(n+1)}(\tau) \|_{L^\infty} \right) \right]
+ t^{(1-\epsilon)/2} C \left( 2 C_2 \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \right)^2
\leq 2 C_2 \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}} + C|\alpha| t^{1/2} \sup_{0 < \tau \leq t} \left[ t^{\epsilon/2} \left( \| u^{(n+1)}(\tau) \|_{L^\infty} + \| v^{(n+1)}(\tau) \|_{L^\infty} \right) \right].
\]

Restrict $\alpha$ so that
\[
|\alpha| \leq \frac{1}{2 C \sqrt{T_1}}.
\]

Then, for $t \in (0, T_1]$, we have
\[
t^{\epsilon/2} \left( \| u^{(n+1)}(t) \|_{L^\infty} + \| v^{(n+1)}(t) \|_{L^\infty} \right) \leq 4 C_2 \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}},
\]
and, noting (8), it follows that this estimate is valid for all $n \in \mathbb{N}$.

We now show that successive approximations are contained in the space $\dot{B}^{-\epsilon}_{\infty,\infty}$. Note that whenever $t \in (0, T_1]$ and $n \in \mathbb{N}$ we have
\[
\left\| \int_0^t e^{(t-\tau)\Delta} P(u^{(n)}_\alpha \cdot \nabla u^{(n)}_\alpha)(\tau) \, d\tau \right\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \leq C \int_0^t \left\| e^{(t-\tau)\Delta} P(u^{(n)} \otimes u^{(n)}_\alpha)(\tau) \right\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \, d\tau
\leq C \int_0^t (t - \tau)^{-(1-\epsilon)/2} \| u^{(n)} \otimes u^{(n)}_\alpha \|_{\dot{B}^0_{\infty,\infty}} \, d\tau
\leq C \int_0^t (t - \tau)^{-(1-\epsilon)/2} \| u^{(n)} \|_{L^\infty}^2 \, d\tau
\leq C \left( 4 C_2 \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \right)^2 \int_0^t \tau^{-\epsilon}(t - \tau)^{-(1-\epsilon)/2} \, d\tau
\leq C t^{(1-\epsilon)/2} \left( 4 C_2 \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \right)^2,
\]
while similar estimates hold for other bilinear terms from the integral equations. Let
\[
T_2 = \min \left\{ T_1, \left( \frac{C_1}{4(4 C_2)^2 \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}}} \right)^{2/(1-\epsilon)} \right\}.
\]

The previous estimate implies that
\[
\left\| \int_0^t e^{(t-\tau)\Delta} P(u^{(n)}_\alpha \cdot \nabla u^{(n)}_\alpha)(\tau) \, d\tau \right\|_{\dot{B}^{-\epsilon}_{\infty,\infty}} \leq \frac{C_1}{4} \| u_0 \|_{\dot{B}^{-\epsilon}_{\infty,\infty}},
\]
whenever $t \in (0, T_2]$. Identical bounds hold for the other bilinear terms from (5) and (6). Working with the integral equations these estimates imply that

$$\left\| u^{(n+1)}_\alpha(t) \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}} + \left\| v^{(n+1)}_\alpha(t) \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}} \leq 2C_1 \left\| u_0 \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}} + \left\| \alpha_j \int_0^t e^{(t-\tau)\Delta} \partial_j (u^{(n+1)}_\alpha - v^{(n+1)}_\alpha) \, d\tau \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}}$$

$$\leq 2C_1 \left\| u_0 \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}} + C |\alpha| \sqrt{T_2} \sup_{0<\tau\leq T_2} \left[ \left\| u^{(n+1)}_\alpha(t) \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}} + \left\| v^{(n+1)}_\alpha(t) \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}} \right].$$

Provided

$$|\alpha| \leq \frac{1}{2C \sqrt{T_2}},$$

it follows for all $n \in \mathbb{N}$ and $t \in (0, T_2]$ that

$$\left\| u^{(n+1)}_\alpha(t) \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}} + \left\| v^{(n+1)}_\alpha(t) \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}} \leq 4C_1 \left\| u_0 \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}}.$$

At this stage we fix a universal constant $c_0$ so that $c_0 \geq 4C_1$ and define

$$T_* = \min \left\{ T_2, \left( \frac{1}{c_0 \left\| u_0 \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}}} \right)^{2/(1-\varepsilon)} \right\}.$$

Note that all of the restrictions on $|\alpha|$ mentioned above follow if $|\alpha| < \left( c_0 \sqrt{T_*} \right)^{-1}$.

Taking for granted the known convergence of the $\mathbb{R}^3$-variable scheme (cf. [9]), we have

$$\left\| \hat{\Delta}_j u^{(n)}(x, 0, t) - \hat{\Delta}_j u(x, 0, t) \right\|_\infty \to 0 \text{ as } n \to \infty.$$

Consequently, using Vitali’s Theorem and the fact that $\hat{\Delta}_j u^{(n)}(x, 0, t)$ is the restriction of the analytic function $\hat{\Delta}_j u^{(n)}(x, y, t) + i\hat{\Delta}_j v^{(n)}(x, y, t)$ to $\mathbb{R}^3$, we can extract a subsequence of $\{ \hat{\Delta}_j u^{(n)}(x, y, t) + i\hat{\Delta}_j v^{(n)}(x, y, t) \}$ which converges to an analytic function the restriction of which to $\mathbb{R}^3$ agrees with the $j$-th dyadic block of the mild solution from [9] (see [5] for more details regarding this argument). Consequently, $\hat{\Delta}_j u(x, y, t) + i\hat{\Delta}_j v(x, y, t)$ is analytic on

$$\Omega_t = \left\{ x + iy : x, y \in \mathbb{R}^3 \text{ and } |y| \leq \frac{\sqrt{t}}{c_0} \right\},$$

and, furthermore,

$$\lambda_j^{-\varepsilon} \left( \hat{\Delta}_j \left\| u(x, y, t) \right\|_{L^\infty(\Omega(t))} + \left\| \hat{\Delta}_j v(x, y, t) \right\|_{L^\infty(\Omega(t))} \right) \leq c_0 \left\| u_0 \right\|_{\dot{B}_{\infty,\infty}^{-\varepsilon}},$$

for all $t \in (0, T_0]$.

\[ \square \]

3. Conditional regularity in supercritical spaces

This section contains the proof of Theorem 1 which relies on the sparseness of certain sets associated with peripheral Littlewood-Paley frequencies. We begin by clarifying what we mean by sparse.
Definition 4. Let $x_0$ be a point in $\mathbb{R}^3$, $r > 0$, $S$ an open subset of $\mathbb{R}^3$ and $\delta \in (0, 1)$. The set $S$ is linearly $\delta$-sparse around $x_0$ at scale $r$ if there exists a unit vector $d$ in $S^2$ such that
\[
\frac{|S \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta.
\]

This definition was introduced in [4] to give a geometric measure-type regularity criterion in terms of sparseness of the regions of high vorticity magnitude. It is worth highlighting its local and linear flavor – i.e. that sparseness need only hold in a single direction and at a single scale both of which are locally determined.

The geometric property of sparseness is connected to the analytic smoothness of a flow through the following interpolative lemma.

Lemma 5. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be real analytic at $x_0$ having local analyticity radius $\rho_0$. Let $B_r$ denote the complex ball centered at $x_0$ of radius $r$ and denote by $F + iG$ the complexification of $f$ on $B_{\rho_0}$. Suppose the set $S_{f,m} = \{x \in \mathbb{R}^3 : |f(x)| \geq m\}$ contains $x_0$ and is linearly $\delta$-sparse at $x_0$ with respect to the length scale $r < \rho_0$ in the direction of the unit vector $d$ for some value $\delta \in (0, 1)$. Then
\[
|f(x_0)| \leq m^h M^{1-h},
\]
where $h = h(\delta) = \frac{2}{\pi} \arcsin \frac{1-\delta}{1+\delta}$ and $M = \|F\|_{L^\infty(B_r)} + \|G\|_{L^\infty(B_r)}$.

Our proof of Lemma 5 follows ideas in [4] and uses a relatively recent result on extremal properties of the harmonic measure in the unit disc $\mathbb{D}$ which we include as a proposition.

Proposition 6 ([10]). Let $K$ be a closed subset of $[-1, 1]$ such that $|K| = 2\gamma$ for some $0 < \gamma < 1$. Suppose further that $0 \in \mathbb{D} \setminus K$. Then
\[
\omega(0, \mathbb{D}, K) \geq \omega(0, \mathbb{D}, K_\gamma) = \frac{2}{\pi} \arcsin \frac{1 - (1 - \gamma)^2}{1 + (1 - \gamma)^2},
\]
where $K_\gamma = [-1, -1 + \gamma] \cup [1 - \gamma, 1]$.

Proof of Lemma 5. Noting that the norms involved are invariant with respect to translations and rotations we may assume that $x_0 = 0$ and $d$ is oriented along the first real axis. Let $K$ denote the complement of $S_{f,m} \cap [-rd, rd]$. Then, $K$ is closed, satisfies $|K| \geq 2r(1 - \delta)$, and does not contain the origin. Let $D_r$ denote the disk of radius $r$ centered at the origin lying on the complexification of the first coordinate axis. Then, $D_r$ is contained in the domain of analyticity of $f$. Applying the harmonic measure maximum principle (cf. [11]) with respect to the sets $D_r$ and $K$ at the origin leads to
\[
|f(0)| \leq m^{\omega(0, D_r, K)} M^{1-\omega(0, D_r, K)}.
\]
Rescaling by a factor of $r^{-1}$ lets us apply Proposition 6. Then, noting that harmonic measure is invariant under conformal mappings and monotonic with respect to $K$, we undo this scaling to obtain
\[
\omega(0, D_r, K) \geq h.
\]
Hence,
\[
|f(0)| \leq m^h M^{1-h},
\]
which complete the proof. \[\square\]
We are now ready to prove Theorem 1.

Proof of Theorem 1. Consider sets of the form

\[ S_j(t_1, t_2) = \{(x, t_2) : \lambda_j^{-\epsilon} |\Delta_j u(x, t_2)| > \frac{1}{d_0^0} \|u(\cdot, t_1)\|_{B^{\epsilon}_{\infty, \infty}} \}, \]

where we are assuming \( u(t_1) \in \dot{B}^{\epsilon}_{\infty, \infty} \). Note that \( L^2 \) coincides with \( \dot{B}^0_{2, 2} \). Then if \( u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \) and \( t_1, t_2 \in (0, T) \), it follows that

\[ |S_j(t_1, t_2)| \leq \sup_{0 < \tau < T} \|u\|^2_{L^2} \left( \frac{d_0^0 \lambda_j^{-\epsilon}}{\|u(t_1)\|_{B^{\epsilon}_{\infty, \infty}}} \right)^2 \to 0 \text{ as } j \to \infty. \]

On the other hand, if \( u \in L^\infty(0, T; \dot{B}^{-\delta}_{\beta, \infty}) \) where \( \epsilon < \delta \) and \( \beta = 6/(3 - 2\delta) \), we have

\[ |S_j(t_1, t_2)| \leq \sup_{0 < \tau < T} \|u(\tau)\|^\beta_{\dot{B}^{-\delta}_{\beta, \infty}} \left( \frac{d_0^0 \lambda_j^{\beta - \epsilon}}{\|u(t_1)\|_{B^{\epsilon}_{\infty, \infty}}} \right)^\beta \to 0 \text{ as } j \to -\infty. \]

We adopt the notations \( E(T) = \sup_{0 < \tau < T} \|u\|^2_{L^2} \) and \( E_\beta(T) = \sup_{0 < \tau < T} \|u(\tau)\|^2_{\dot{B}^{-\delta}_{\beta, \infty}} \). For each time \( t \) in \( (0, T) \), we identify a high frequency cutoff index

\[ Q^*(t) = \frac{1}{\epsilon} \log_2 \left( c_\epsilon E(T)^{1/2} \|u(t)\|^\frac{3 - 2(1 - \epsilon)}{2(1 - \epsilon)}_{B^{\epsilon}_{\infty, \infty}} \right), \]

and a low frequency cutoff index

\[ Q_*(t) = \frac{1}{\delta - \epsilon} \log_2 \left( c_{\epsilon, \delta} E_\beta(T)^{1/2} \|u(t)\|^\frac{3 - \delta(1 - \epsilon)}{\delta(1 - \epsilon)}_{B^{\epsilon}_{\infty, \infty}} \right), \]

where \( c_\epsilon \) and \( c_{\epsilon, \delta} \) represent non-dimensional constants depending only on the indicated parameters, the constant \( c_0 \) from Proposition 2 and \( d_0^0 \).

Consider some time \( t_1 \) and a later time \( t_2 \) contained in \([t_1 + T_*(t_1)/2, t_1 + T_*(t_1)]\) (here \( T_*(t) \) denotes the time scale of analyticity as specified in Proposition 2 using \( u(t) \) as the initial data). Whenever \( j \geq Q^*(t_1) \) it follows that

\[ |S_j(t_1, t_2)|^{1/3} \leq \frac{\rho(t_2)}{6}, \]

where \( \rho(t_2) \) denotes the uniform analyticity radius of \( u(t_2) \) as characterized in Proposition 2. This guarantees that \( S_j(t_1, t_2) \) is linearly 1/3-sparse around all its elements at a scale smaller than \( \rho(t_2)/2 \) in some direction. Fix \( h = 1/3 \) and \( \alpha = 2 \). Applying Lemma 5 at any \( x_0 \in S(t_1, t_2) \) entails that

\[ \lambda_j^{-\epsilon} |\Delta_j u(x_0, t_1)| \leq \|u(t_1)\|_{\dot{B}_{\infty, \infty}^\epsilon}, \]

whenever \( j \geq Q^*(t_1) \). This estimate is clearly valid whenever \( x_0 \notin S(t_1, t_2) \). If (1) is satisfied at \( t_1 \), then the estimate extends to all wavenumbers and we obtain

\[ \|u(t_2)\|_{B^{\epsilon}_{\infty, \infty}^\epsilon} \leq \|u(t_1)\|_{B^{\epsilon}_{\infty, \infty}^\epsilon}. \]

Noting that \( Q^*(t_2) \leq Q^*(t_1) \), we also have for all \( \tau > t_2 \) that

\[ \sup_{j \leq Q^*(\tau)} \|\Delta_j u(\tau)\|_{L^\infty} \leq \|u(\tau)\|_{B^{\epsilon}_{\infty, \infty}^\epsilon}. \]
By repeating this argument we obtain a finite collection of times \( t_1, \ldots, t_k \) satisfying
\[
t_{i+1} - t_i \geq T_s(t_1)/2,
\]
and
\[
T - t_k < T_s(t_1)/2,
\]
as well as the estimates
\[
||u(t_i)||_{B_{\infty,\infty}^s} \leq ||u(t_1)||_{B_{\infty,\infty}^s}.
\]
Then, re-solving at time \( t_k \) we obtain a solution regular on \((0, T + T_s(t_1)/2]\) indicating \( T \) is not a blow-up time.

The conclusion for solutions belonging to \( L^\infty(0, T; \dot{B}_{3,3}^{-\delta}) \) satisfying (2) follows in a similar manner, noting that – based on condition (2) – whenever \( j < Q^\ast(t_1) \), we have
\[
|S_j(t_1, t_2)|^{1/3} \leq \frac{\rho(t_2)}{6},
\]
and we then apply Lemma 5 at all \( x_0 \in S_j(t_1, t_2) \) for \( j < Q^\ast(t_1) \). \( \square \)

A technical comment about this proof is appropriate. The crucial element is the volumetric decay at extreme modes visible in (10) and (11). This reflects the mismatch between the smoothness parameter \(-\epsilon\) and, in the first case, 0, while in the second, \(-\delta\). The decay of volumes for high frequencies will occur whenever the integrability index \(-\delta\) is greater than \(-\epsilon\). This is why we can replace \( u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \) with \( u \in L^\infty(0, T; \dot{B}_{3,3}^{-\delta}) \) as mentioned after the statement of Theorem 1. The high frequency cutoff index we obtain in this case is
\[
Q^\ast(t) = \frac{1}{\epsilon - \delta} \log_2 \left( c_{\epsilon, \delta} E_\beta(T)^{1/2} ||u(t)||_{\dot{B}_{\infty, \infty}^{\frac{3-\beta(1-\epsilon)}{2\beta}}} \right).
\]

This reveals that the shift in the relevance of high versus low frequency modes occurs when \( \delta = \epsilon \) (in which case no volumetric decay is apparent).

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