Sparse Regularization: Convergence Of Iterative Jumping Thresholding Algorithm

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Abstract—In recent studies on sparse modeling, non-convex penalties have been received considerable attentions due to their superiorities on sparsity-inducing over the convex counterparts. Compared with the convex optimization approaches, however, the non-convex approaches have more challenging convergence analysis. In this paper, we study the convergence of a non-convex iterative thresholding algorithm for solving sparse recovery problems with a certain class of non-convex penalties, whose corresponding thresholding functions are discontinuous with jump discontinuities. Therefore, we call the algorithm the iterative jumping thresholding (IJT) algorithm. Our main result states that the IJT algorithm converges to a stationary point for almost all possibilities of the regularization parameter except some “bad” choices, while the number of such “bad” choices is less than the dimension of the unknown variable vector. Furthermore, we demonstrate that the IJT algorithm converges to a local minimizer at an asymptotically linear rate under certain conditions. In addition, we derive a posteriori computational error estimate, which can be used to design practical terminal rules for the algorithm. It should be pointed out that the $l_q$-norm ($0 < q < 1$) is an important subclass of the class of non-convex penalties studied in this paper. The established convergence analysis provides a theoretical guarantee for a wide range of applications of these non-convex penalties.

Index Terms—Sparse Regularization, Non-convex optimization, iterative thresholding algorithm, $l_q$ regularization ($0 < q < 1$)

I. INTRODUCTION

The sparse vector recovery problems emerging in many areas of scientific research and engineering practice have attracted considerable attention in recent years ([1]-[4]). Typically, applications include variable selection [5], visual coding [6], signal processing [7], compressed sensing [1], [2], machine learning [8], and microwave imaging [9], [10]. These problems can be mathematically modeled as the following $l_0$-norm regularized optimization problems

$$\min_{x \in \mathbb{R}^N} \{ F(x) + \lambda \|x\|_0 \},$$

where $F : \mathbb{R}^N \rightarrow [0, \infty)$ is a proper lower-semicontinuous function, $\|x\|_0$, formally called the $l_0$ quasi-norm, denotes the number of nonzero components of $x$ and $\lambda > 0$ is a regularization parameter. However, the optimization problem (1) is generally intractable to solve, due to the discontinuity and non-convexity of the $l_0$ quasi-norm.

In order to overcome such difficulty, many continuous penalties were proposed to substitute the $l_0$-norm in the following optimization problem

$$\min_{x \in \mathbb{R}^N} \{ F(x) + \lambda \Phi(x) \},$$

where $\Phi(x)$ is a certain separable, continuous penalty with $\Phi(x) = \sum_{i=1}^N \phi(|x_i|)$, and $x = (x_1, \cdots, x_N)^T$. One of the most important cases is the $l_1$-norm with $\Phi(x) = \|x\|_1 = \sum_{i=1}^N |x_i|$. The $l_1$-norm is convex and thus, the corresponding $l_1$-norm regularized optimization problem can be efficiently solved. Because of this, the $l_1$-norm gets its popularity and has been accepted as a very useful tool for the modeling of the sparsity problems. Nevertheless, the $l_1$-norm may not induce further sparsity when applied to certain applications (say, compressed sensing) [11], [12], [13], [14], [15]. Alternatively, many non-convex penalties were proposed as relaxations of the $l_0$-norm. Some typical non-convex examples are the $l_q$-norm ($0 < q < 1$) ([2], [3], [4], [5], Smoothly Clipped Absolute Deviation (SCAD) [16], Minimax Concave Penalty (MCP) [17] and Log-Sum Penalty (LSP) [11]. Compared with the $l_1$-norm, the non-convex penalties can usually induce better sparsity while the corresponding non-convex regularized optimization problems are generally more difficult to solve.

There are mainly four classes of algorithms to solve the non-convex regularized optimization problem (2). The first class is the half-quadratic (HQ) algorithm [18], [19]. The basic idea of HQ algorithm is to formulate the original objective function as an infimum of a family of augmented functions via introducing a dual variable, and then minimize the augmented function along the primal and dual variables in an alternate fashion. Fixing the dual variable, the augmented function is quadratic with respect to the primal variable, thus getting the terminology “Half-Quadratic”. However, HQ algorithms can be efficient only when both subproblems are easily solved (particularly, when both subproblems have the closed-form solutions). The second class is the iteratively reweighted algorithm. The iteratively reweighted least squares (IRLS) minimization [20], [21], [22] and iteratively reweighted $l_1$-minimization (IRL1) [11] are two of the most important algorithms of this class. More specifically, the IRLS algorithm solves a sequence of weighted least squares problems, which
can be viewed as some approximate problems to the original optimization problem. Similarly, the IRL1 algorithm solves a sequence of non-smooth weighted $l_1$-minimization problems, and hence can be seen as the non-smooth counterpart to the IRLS algorithm. Nevertheless, the iteratively reweighted algorithms can be only efficient when applied to such non-convex regularization problems, of which the corresponding non-convex penalty can be well approximated via the quadratic function or the weighted $l_1$-norm function.

The third class is the difference of convex functions algorithm (DC programming) [23], which is also called Multi-Stage (MS) convex relaxation [24]. The key idea of DC programming is to consider a proper decomposition of the objective function. More specifically, the DC programming converts the non-convex penalized problem into a convex reweighted $l_1$ minimization problem (called primal problem) and another convex problem (called dual problem), and then iteratively optimizes the primal and dual problems [23]. Hence, it can be only applied to certain a family of non-convex penalties that can be decomposed as a difference of convex functions. The last class is the iterative thresholding algorithm, which fits the framework of the forward-backward splitting algorithm [32] and the generalized gradient projection algorithm [29] when applied to a separable non-convex penalty. Intuitively, the iterative thresholding algorithm can be seen as a procedure of Landweber iteration projected by a certain thresholding operator. Thus, the thresholding operator plays a key role in the iterative thresholding algorithm. For some special non-convex penalties such as SCAD, MCP, LSP and $l_q$-norms with $q = 1/2, 2/3$, the associated thresholding operators can be expressed analytically [15], [25], [26]. Compared to the other types of non-convex algorithms such as HQ, IRLS, IRL1 and DC programming algorithms, the iterative thresholding algorithm can be implemented fast and have almost the least computational complexity for large scale problems [9], [10], [27]. Consequently, the iterative thresholding algorithm gets its popularity and is accepted as an efficient algorithm for sparsity problems.

One of the significant differences between the convex and non-convex algorithms is that the convergence analysis of a non-convex algorithm is in general tricky. Although the effectiveness of the iterative thresholding algorithms for the non-convex regularized optimization problems has been verified in many applications, its convergence has not be thoroughly investigated. There are still three mainly open questions as follows:

1) When does the algorithm converge? So far, for most of non-convex penalties, only subsequential convergence of the iterative thresholding algorithm can be claimed.
2) Where does the algorithm converge? Does the algorithm converge to a global minimizer or more practically, a local minimizer due to the non-convexity of the optimization problem?
3) How fast does the algorithm converge?

In this paper, we perform the convergence analysis for the iterative jumping thresholding algorithm (called IJT algorithm henceforth) for solving a certain class of non-convex regularized optimization problems. One of the most significant features of such non-convex problems is that the corresponding thresholding functions are discontinuous with jump discontinuities (see Fig. 1). Moreover, the corresponding thresholding functions are not nonexpansive in general. Among these non-convex penalties, the well-know $l_q$ quasi-norm with $0 < q < 1$ is one of the most typical cases. We show that IJT algorithm can converge for almost all possibilities of the regularization parameter except some “bad” choices, while the number of such “bad” choices is less than the dimension of the unknown variable vector, $N$. Furthermore, when $\lambda$ is less than a positive constant (only depending on the convergent point and functions $F$ and $\Phi$), we demonstrate that IJT algorithm can converge to a local minimizer of the non-convex optimization problem (see Theorem 2), and the convergence speed of the IJT algorithm is asymptotically linear (see Theorem 3). In other words, when the iterative vector is sufficiently close to the convergent point, the rate of convergence of IJT algorithm is linear. It implies that given a good initialization, the IJT algorithm will converge very fast. In addition, we derive a posteriori error bound, which can be used to construct a practical stopping rule of the IJT algorithm. As a typical case, we apply the developed convergence results to the $l_q$ ($0 < q < 1$) regularization.

The reminder of this paper is organized as follows. In section II, we briefly review some general methods for the convergence analysis of the non-convex algorithms, and then discuss some related work. In section III, we give the problem settings and then introduce the iterative jumping thresholding (IJT) algorithm with some basic properties. In section IV, we give the established convergence results of the IJT algorithm. In section V, we apply the established theoretical analysis to the $l_q$ ($0 < q < 1$) regularization. We conclude this paper in section VI.

II. RELATED WORK

Let \{x^n\} be the sequence generated by a non-convex algorithm for minimizing a non-convex objective function $f$. A general method (labeled as general method 1) for the convergence analysis of a non-convex algorithm in a finite-dimensional real space can be described as follows :

1) Show that the sequence \{f(x^n)\} is monotonically decreasing and bounded from below.
2) Under additional conditions show that the existence of an accumulation point of the sequence \{x^n\}.
3) Under additional assumptions show that the accumulation point is isolated.

The first step of the general method 1 implies that the sequence of function values is convergent. Commonly, together with the boundeness of \{x^n\}, the subsequential convergence of the non-convex algorithm can be claimed. Furthermore, if the accumulation point is isolated, then the convergence of the whole sequence \{x^n\} can be claimed.

It should be noted that the monotonically decreasing property of the sequence of the function values will hold naturally for most descent methods when the stepsize of the descent direction is sufficiently small [32]. Moreover, for
some other typical non-convex algorithms including the half-quadratic, FOCUSS, IRL1 and DC Programming algorithms, the same properties of the corresponding function value sequences have also been verified respectively in [30], [20], [39], [23]. The boundedness of the iterative sequence \( \{x^n\} \) can be guaranteed by some mild assumptions such as the coercivity of the objective function \( f \) (that is, \( f(x) \to \infty \) whenever \( \|x\|_2 \to \infty \)). However, it is generally very hard (even impossible) to show that the accumulation points are isolated since the set of the accumulation points of a non-convex algorithm may be a continuum. Therefore, for most non-convex algorithms, subsequential convergence can only be claimed without further assumptions on the isolatism of accumulation points. In this paper, we will not follow this direction to prove the convergence of the IJT algorithm.

Besides general method 1, there is another general method (labeled as **general method 2**) to establish the convergence of non-convex algorithms by the establishment of

\[
\sum_{n=0}^{+\infty} \|x^{n+1} - x^n\|_2 < +\infty
\]  

(3)

under certain conditions. A classical way to show a sequence satisfying (3) is to construct a monotonically increasing and convergent sequence \( \{t^n\} \), commonly called the majorizing sequence for \( \{x^n\} \) (see, Chapter 12.4 in [31]), such that

\[
\|x^{n+1} - x^n\|_2 \leq t^{n+1} - t^n
\]

for each \( n \in \mathbb{N} \). Compared with general method 1, general method 2 can be applied to the case that the accumulation points are not isolated. Therefore, we will follow this direction to justify the convergence of the IJT algorithm for non-convex regularized optimization problems.

Recently, Attouch et al. [32] have justified the convergence of a family of descent methods according to general method 2 by assuming the objective function has the so-called Kurdyka-Łojasiewicz (KL) property [33], [34] and the sufficient decrease property, i.e.,

\[
f(x^n) - f(x^{n+1}) \geq a\|x^{n+1} - x^n\|_2^2,
\]

where \( a \) is a positive constant. More specifically, the KL property can be briefly described as the following definition, which is taken from [32].

**Definition 1** (Kurdyka-Łojasiewicz property).

(a) The function \( f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \) is said to have the Kurdyka-Łojasiewicz property at \( x^* \in \text{dom} \, \partial f \) if there exist \( \eta \in (0, +\infty] \), a neighborhood \( U \) of \( x^* \) and a continuous concave function \( \varphi : [0, \eta) \to \mathbb{R}_+ \) such that:

(i) \( \varphi(0) = 0 \);
(ii) \( \varphi \) is \( C^1 \) on \( (0, \eta) \);
(iii) for all \( s \in (0, \eta) \), \( \varphi'(s) > 0 \);
(iv) for all \( x \) in \( U \cup \{x\} \), \( \varphi(f(x)) < f(x) < f(x^*) + \eta \), the Kurdyka-Łojasiewicz inequality holds

\[
\varphi'(f(x) - f(x^*)) \text{dist}(0, \partial f(x)) \geq 1.
\]

(4)

(b) Proper lower semi-continuous functions which satisfy the Kurdyka-Łojasiewicz inequality at each point of dom \( \partial f \) are called KL functions.

There are many kinds of functions satisfying the KL inequality such as real analytic functions, semi-algebra functions and locally strongly convex functions.

Instead of the KL inequality, we give a condition much clearer and easier to check for the convergence of IJT algorithm. More specifically, we find that IJT algorithm converges for almost all possibilities of the regularization parameter \( \lambda \) except some “bad” values. In the other words, assume that \( x^* \) is an accumulation point of the sequence \( \{x^n\} \). Let \( I \) be the support set of \( x^* \) and \( x^n_I \) be a subvector of \( x^n \) restricted to \( I \). We demonstrate that if the principle matrix \( \nabla^2_I f(x^*) + \lambda \phi_2(x^*) \) is nonsingular, then \( \{x^n\} \) converges to \( x^* \) (where \( \nabla^2_I f(x^*) \) and \( \phi_2(x_I) \) are defined as in (19) and (22) respectively). Compared to the other non-convex algorithms including half-quadratic [30], FOCUSS [20], IRL1 [39] and DC programming [23] algorithms, we derive a clearly sufficient condition instead of the direct assumption on the isolatism of the accumulation points for the convergence of IJT algorithm. Furthermore, the convergence speed of the IJT algorithm is also demonstrated in this paper.

Besides the aforementioned non-convex algorithms, there are some other very related algorithms and work. In the following, we will compare the obtained theoretical results on the iterative thresholding algorithm with these algorithms and work.

The first class of tightly related algorithms are the iterative shrinkage and thresholding (IST) algorithms, which mainly refer to two generic algorithms and some specific algorithms. The first generic algorithm related to the IJT algorithm is the generalized gradient projection (called GGP for short) algorithm [28], [29]. In [28], the GGP algorithm was proposed for the \( l_1 \) regularization problem. In such a convex setting, the finite support convergence and eventual linear convergence rate was given in [28]. In [29], Bredies and Lorenz extended the GGP algorithm to solve the following general non-convex optimization model in the infinite-dimensional Hilbert space

\[
\min_{x \in X} \left\{ F(x) + \lambda \Phi(x) \right\},
\]

(5)

where \( X \) is an infinite-dimensional Hilbert space, \( F : X \to [0, \infty) \) is assumed to be a proper lower-semicontinuous (l.s.c.) function with gradient \( \nabla F(x) \) being Lipschitz continuous and \( \Phi : X \to [0, \infty) \) is weakly lower-semicontinuous (possibly non-smooth and non-convex). Furthermore, the iterative form of the GGP algorithm was specified as

\[
x^{n+1} \in \text{Prox}_{\lambda \Phi, \mu}(x^n - \mu \nabla F(x^n)),
\]

where \( \text{Prox}_{\lambda \Phi, \mu} \) represents the proximity operator of \( \Phi \) as defined in (11). It can be observed that the IJT algorithm is a special case of GGP algorithm when applied to a separable function with gradient \( \nabla F(x) \) being Lipschitz continuous and \( \Phi \). We recently extended GGP algorithm to converge subsequentially to a stationary point [29] (that is, there is a subsequence of the algorithm that converges to a stationary point). However, as a specific case of GGP algorithm, we have justified that the IJT algorithm can assuredly converge to a local minimizer with an asymptotically linear convergence rate.

Another related generic algorithm is the general iterative shrinkage and thresholding (GIST) algorithm suggested in
The GIST algorithm is proposed for the following general non-convex regularized optimization problem

$$\min_{x \in \mathbb{R}^N} \{ F(x) + \lambda R(x) \}, \tag{6}$$

where $F$ is assumed to be continuously differentiable with Lipschitz continuous derivative, and $R(x)$ is a continuous function and can be rewritten as the difference of two different convex functions. As compared with the Assumption 2, we can find that the optimization model considered in this paper is distinguished from the model (6) studied in [26]. Moreover, only the subsequential convergence of the GIST algorithm can be justified in [26], while the convergence of the whole sequence and further the asymptotically linear convergence rate of IJT algorithm are demonstrated in this paper.

Besides these two generic algorithms, there are some other specific iterative thresholding algorithms related to IJT algorithm. Among them, the hard algorithm and the soft algorithm are two representatives, which respectively solves the $l_0$ regularization and $l_1$ regularization [36], [37]. It was demonstrated in [36], [37] that when $\mu = 1$ both hard and soft algorithms can converge to a stationary point whenever $\|A\|_2 < 2$. These classical convergence results can be generalized when a stepsize parameter $\mu$ is incorporated with the IST procedures, and in this case, the convergence condition becomes

$$0 < \mu < \|A\|_2^{-2}. \tag{7}$$

It can be seen from Corollary 1 that (7) is the exact condition of the convergence of IJT algorithm when applied to the $l_q$ regularization with $0 < q < 1$, which then supports that the classical convergence results of IST have been extended to the non-convex $l_q$ ($0 < q < 1$) regularization case. Furthermore, it was shown in [30] that when the measurement matrix $A$ satisfies the so-called finite basis injective (FBI) property and the stationary point possesses a strict sparsity pattern, the soft algorithm can converge to a global minimizer of $l_1$ regularization with a linear convergence rate. Such result is not surprising because of the convexity of $l_1$ regularization. As for convergence speed of the hard algorithm, it was demonstrated in [36] that under the condition $\mu = 1$ and $\|A\|_2 < 2$, hard algorithm will converge to a local minimizer with an asymptotically linear convergence rate (as far as we know, no result was given however for the case when step size $\mu$ is taken into consideration). However, both as algorithms for solving non-convex models, Corollary 2 reveals that IJT algorithm shares the same asymptotic convergence speed with the hard algorithm.

III. ITERATIVE JUMPING THRESHOLDING ALGORITHM

In this section, we first present the basic settings of the considered non-convex regularized optimization problems, then introduce the iterative jumping thresholding (IJT) algorithm for these problems. In the end of this section, we briefly review some basic properties of IJT algorithm obtained in [29], which then serve as the basis of further analysis in the next sections.

A. Problem Settings

We consider the following composite optimization problem

$$\min_{x \in \mathbb{R}^N} \{ T(x) = F(x) + \lambda \Phi(x) \}, \tag{8}$$

where $\Phi(x)$ is assumed to be separable with $\Phi(x) = \sum_{i=1}^N \phi(|x_i|)$. Moreover, we make several assumptions on the problem (8).

Assumption 1. $F : \mathbb{R}^N \to [0, \infty)$ is a proper lower semi-continuous function and also satisfies the following assumptions:

(a) $F$ has Lipschitz continuous derivative, that is, it holds

$$\| \nabla F(u) - \nabla F(v) \|_2 \leq L \| u - v \|_2, \forall u, v \in \mathbb{R}^N,$$

where $L > 0$ is a Lipschitz constant.

(b) $F$ has continuous twice derivative $\nabla^2 F(x)$ for any $x \in \mathbb{R}^N$.

It should be noted that Assumption 1 is a general assumption for $F$. Many formulations in machine learning satisfy Assumption 1. For example, the following least square and logistic loss functions are two commonly used ones which satisfy Assumption 1:

$$F(x) = \frac{1}{2m} \| U x - y \|_2^2 \text{ or } \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-y_i u_i^T x)),$$

where $u_i \in \mathbb{R}^N$ for $i = 1, 2, \ldots, m$, $U = [u_1, \ldots, u_m]^T \in \mathbb{R}^{m \times N}$ is a data matrix and $y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m$ is a target vector. Moreover, in both signal and image processing, $F$ is commonly taken as the least square of the observation model, that is,

$$F(x) = \| A x - y \|_2^2,$$

where $y \in \mathbb{R}^m$ is an observation vector and $A \in \mathbb{R}^{m \times N}$ is an observation matrix. It can be easily verified that such $F$ also satisfies Assumption 1.

In the following, we give some basic assumptions on $\phi$, most of which were considered in [29].

Assumption 2. $\phi : [0, \infty) \to [0, \infty)$ is continuous and satisfies the following assumptions:

(a) $\phi$ is non-decreasing with $\phi(0) = 0$ and $\phi(z) \to \infty$ when $z \to \infty$.

(b) For each $b > 0$, there exists an $a > 0$ such that $\phi(z) \geq a z^2$ for $z \in [0, b]$.

(c) $\phi$ is differentiable on $(0, \infty)$ and the derivative $\phi'$ is strictly convex with $\phi''(z) \to \infty$ for $z \to 0$ and $\phi''(z) / z \to 0$ for $z \to \infty$.

(d) $\phi$ has continuous twice derivative $\phi''$ on $(0, \infty)$.

It can be observed that Assumption 2(a) ensures the coercivity of $\phi$, and thus the existence of the minimizer of the optimization problem (8). Assumption 2(b) guarantees the weak sequential lower semi-continuity of $\phi$ in $l^2$, and Assumption 2(c) induces the sparsity of the penalty $\Phi$. In practice, there are many non-convex functions satisfying Assumption 2. Two of the most typical subclasses are $\phi(z) = z^q$ and $\phi(z) = \log(1 + z^q)$ with $q \in (0, 1)$ as shown in Fig. 1.

Remark 1. In [32], Attouch et al. have justified the convergence of the forward-backward splitting algorithm for the...
non-convex optimization problems under the assumption that the objective functions are KL functions. It can be observed that many functions satisfying Assumption 1 and 2 are also KL functions (say, $l_q$ regularization with $0 < q < 1$ as studied in Section V), however, it is not always true. In the following, we give a specific one-dimensional function that satisfies Assumption 1 and 2, but not the assumption of KL function. Given any function $\phi$ satisfying Assumption 2, let $g = f + \phi$ with $f$ being defined as follows

$$f(z) = \begin{cases} a_1(z - b_1)^2 + c_1, & \text{for } z \leq 1/2 \\ \exp\left(-\frac{1}{(z-1)^2}\right) - \phi(z) + C, & \text{for } 1/2 < z < 1 \\ C - \phi(1), & \text{for } z = 1 \\ \exp\left(-\frac{1}{(z-1)^2}\right) - \phi(z) + C, & \text{for } 1 < z < 3/2 \\ a_2(z - b_2)^2 + c_1, & \text{for } z \geq 3/2 \end{cases},$$

where $e = \exp(1)$, $a_1 = 80e^{-4} - \frac{1}{2}\phi''(1/2)$, $b_1 = \frac{1}{2} + \frac{16e^{-4} - \phi'(1/2)}{\phi'(1/2)}$, $a_2 = 80e^{-4} - \frac{1}{2}\phi''(3/2)$, $b_2 = \frac{3}{2} - \frac{16e^{-4} - \phi'(3/2)}{\phi'(3/2)}$, $C = \phi(3/2) + \max\{\phi(1/2), a_1(1/2-b_1)^2, \phi(3/2) + a_2(3/2-b_2)^2\}$, $c_1 = C + e^{-4} - \phi(1/2) - a_1(1/2-b_1)^2$, and $c_2 = C + e^{-4} - \phi(3/2) - a_2(3/2-b_2)^2$. Thus,

$$g(z) = \begin{cases} a_1(z - b_1)^2 + c_1 + \phi(|z|), & \text{for } z \leq 1/2 \\ \exp\left(-\frac{1}{(z-1)^2}\right) + C, & \text{for } 1/2 < z < 1 \\ C, & \text{for } z = 1 \\ \exp\left(-\frac{1}{(z-1)^2}\right) + C, & \text{for } 1 < z < 3/2 \\ a_2(z - b_2)^2 + c_1 + \phi(z), & \text{for } z \geq 3/2 \end{cases}.$$  

When $1/2 < z < 3/2$, we define a function $h(z)$ as

$$h(z) = \begin{cases} \exp\left(-\frac{1}{(z-1)^2}\right), & \text{for } 1/2 < z < 1 \\ 0, & \text{for } z = 1 \\ \exp\left(-\frac{1}{(z-1)^2}\right), & \text{for } 1 < z < 3/2 \end{cases}.$$  

It can be easily checked that $f$ satisfies Assumption 1 due to the function $h$ is $C^\infty$ and $\phi$ is $C^2$ in the interval $(1/2, 3/2)$. However, according to [33], it shows that $h$ fails to satisfy the KL inequality (4) at $z = 1$. Therefore, $g$ must be not a KL function. We show the figures of $f$ and $g$ in Fig. 2 with $\phi(|z|) = |z|^{1/2}$.

In order to describe IJT algorithm, we need to generalize the proximity operator from convex case to a non-convex penalty $\Phi$, that is,

$$\text{Prox}_{\lambda \mu, \phi}(x) = \arg \min_{u \in \mathbb{R}^N} \left\{ \frac{\|x - u\|_2^2}{2} + \lambda \mu \Phi(u) \right\},$$  

where $\mu > 0$ is a parameter. Since $\Phi$ is separable, thus computing $\text{Prox}_{\lambda \mu, \phi}$ is reduced to solve a one-dimensional minimization problem, that is,

$$\text{prox}_{\lambda \mu, \phi}(z) = \arg \min_{v \in \mathbb{R}} \left\{ \frac{|z - v|^2}{2} + \lambda \mu \phi(|v|) \right\}.$$  

Therefore,

$$\text{Prox}_{\lambda \mu, \phi}(x) = (\text{prox}_{\lambda \mu, \phi}(x_1), \ldots, \text{prox}_{\lambda \mu, \phi}(x_N))^T.$$
then \( \text{prox}_{\lambda \mu, \phi} \) is well defined and can be specified as
\[
\text{prox}_{\lambda \mu, \phi}(z) = \begin{cases} 
\text{sign}(z) \rho_{\mu}^{-1}(|z|), & \text{for } |z| \geq \tau_{\mu} \\
0, & \text{for } |z| \leq \tau_{\mu}
\end{cases}
\] (14)
for any \( z \in \mathbb{R} \) with
\[
\tau_{\mu} = \rho_{\mu}(\eta_{\mu})
\] (15)
and
\[
\eta_{\mu} = \psi^{-1}((\lambda \mu)^{-1}).
\] (16)
Moreover, the range domain of \( \text{prox}_{\lambda \mu, \phi} \) is \( \{0\} \cup [\eta_{\mu}, \infty) \).

It can be observed that the proximity operator is discontinuous with jump discontinuities, which is one of the most significant features of such a class of non-convex penalties studied in this paper. Moreover, it can be easily checked that the proximity operator is not nonexpansive in general. Due to these, the convergence analysis of the corresponding non-convex algorithm gets challenging. (Some specific proximity operators are shown in Fig. 1(b).)

With the definition of the proximity operator, the iterative jumping thresholding (IJT) algorithm can be proposed to solve the non-convex regularized optimization problem (9) (when applied to the separable case, the IJT algorithm fits the framework of the generalized gradient projection algorithm studied in [29]). Formally, the iterative form of IJT algorithm can be expressed as follows
\[
x^{n+1} \in \text{prox}_{\lambda \mu, \phi}(x^n - \mu \nabla F(x^n)),
\] (17)
where \( \mu > 0 \) is a stepsize parameter. For simplicity, we define
\[
G_{\lambda \mu, \phi}(x) = \text{prox}_{\lambda \mu, \phi}(x - \mu \nabla F(x))
\]
for any \( x \in \mathbb{R}^N \). For the iterative jumping thresholding algorithm, we call \( \text{prox}_{\lambda \mu, \phi} \) the jumping thresholding function.

**Remark 2.** For some specific \( l_q \)-norm (say, \( q = 1/2, 2/3 \)), the proximity operator can be expressed analytically [15], [25] (as shown in Fig. 1(b)).

**Remark 3.** Although the \( l_0 \)-norm does not satisfy Assumption 2, the hard thresholding function is also discontinuous with jump discontinuities. Due to such discontinuity of the hard thresholding function, we will show that the convergence of the hard algorithm can be easily given according to a similar analysis of the IJT algorithm in Section V.

### IV. Convergence Analysis

In the last section, it can be only claimed that any sequence \( \{x^n\} \) generated by IJT algorithm subsequentially converges to a fixed point \( x^* \) of \( G_{\lambda \mu, \phi} \). In this section, we will answer such three open questions of the IJT algorithm presented in the introduction, i.e., when, where and how fast does the algorithm converge? More specifically, we first prove that the IJT algorithm converges to a stationary point for almost all possibilities of the regularization parameter \( \lambda \) except some “bad” choices, and then show that the stationary point is also a local minimizer of the optimization problem with some additional assumptions, and further demonstrate that the asymptotic convergence rate of IJT algorithm is linear.

#### A. Convergence To A Stationary Point

In order to justify the convergence of the whole sequence \( \{x^n\} \), we first show the finite support and sign convergence of the sequence, that is, the support sets and signs of \( \{x^n\} \) will remain stable after finite number of iterations. We denote \( I = \text{Supp}(x^*) \) and \( I^n = \text{Supp}(x^n) \) for each \( n \in \mathbb{N} \).

**Lemma 3.** Let \( \{x^n\} \) be a sequence generated by IJT algorithm. Assume that \( 0 < \mu < \frac{1}{2} \) and \( x^* \) is a limit point of \( \{x^n\} \), then there exists a sufficiently large positive integer \( n^* \) such that when \( n > n^* \), it holds
(a) \( I^n = I \);
(b) \( \text{sign}(x^n) = \text{sign}(x^*) \).

**Proof:** By the assumption of Lemma 3, there exits a subsequence \( \{x^{n_j}\} \) converges to \( x^* \), i.e.,
\[
x^{n_j} \rightarrow x^* \quad \text{as} \quad j \rightarrow \infty.
\] (18)
Thus, there exists a sufficiently large positive integer $j_0$ such that $\|x^{n_j} - x^*\|_2 < \eta_\mu$ when $j \geq j_0$. Moreover, by Property 1(b), there also exists a sufficiently large positive integer $n^*$ such that $\|x^n - x^{n+1}\|_2 < \eta_\mu$ when $n > n^*$. Without loss of generality, we let $n^* = n_{j_0}$. In the following, we first prove that $I^n = I$ and then $\text{sign}(x^n) = \text{sign}(x^*)$ whenever $n > n^*$.

In order to prove $I^n = I$, we first show that $I^{n_j} = I$ when $j \geq j_0$ and then verify that $I^{n+1} = I^n$ when $n > n^*$. We now prove by contradiction that $I^{n_j} = I$ whenever $j \geq j_0$. Assume this is not the case, namely, that $I^{n_j} \neq I$. Then we easily derive a contradiction through distinguishing the following two possible cases:

Case 1: $I^{n_j} \neq I$ and $I^{n_j} \cap I \subset I^{n_j}$. In this case, then there exists an $i_{n_j}$ such that $i_{n_j} \in I^{n_j} \setminus I$. By Lemma 2, it then implies

$$\|x^{n_j} - x^*\|_2 \geq |x^{n_j}_{i_{n_j}}| \geq \min_{i \in I} |x_{i_j}^*| \geq \eta_\mu,$$

which contradicts to $\|x^{n_j} - x^*\|_2 < \eta_\mu$.

Case 2: $I^{n_j} \neq I$ and $I^{n_j} \cap I = I^n$. In this case, it is obvious that $I^{n_j} \subset I$. Thus, there exists an $i^*$ such that $i^* \in I \setminus I^{n_j}$. By Lemma 2, we still have

$$\|x^{i^*} - x^*\|_2 \geq |x_{i^*}^*| \geq \min_{i \in I} |x^*_i| \geq \eta_\mu,$$

and it contradicts to $\|x^{i^*} - x^*\|_2 < \eta_\mu$.

Thus, we have justified that $I^{n_j} = I$ when $j \geq j_0$. Similarly, it can be also claimed that $I^{n+1} = I^n$ whenever $n > n^*$. Therefore, whenever $n > n^*$, it holds $I^n = I$.

As $I^n = I$ when $n > n^*$, it suffices to test that $\text{sign}(x^n_{i}) = \text{sign}(x^*_{i})$ for any $i \in I$. Similar to the first part of proof, we will first check that $\text{sign}(x^{n_j}_{i}) = \text{sign}(x^*_{i})$, and then $\text{sign}(x^{n+1}_{i}) = \text{sign}(x^*_{i})$ for any $i \in I$ by contradiction. We now prove $\text{sign}(x^{n+1}_{i}) = \text{sign}(x^*_{i})$ for any $i \in I$. Assume this is not the case. Then there exists an $i^* \in I$ such that $\text{sign}(x^{n_j}_{i^*}) \neq \text{sign}(x^*_{i^*})$, and hence,

$$\text{sign}(x^{n_j}_{i^*})\text{sign}(x^*_{i^*}) = -1.$$

From Lemma 2, it then implies

$$\|x^{n_j} - x^*\|_2 \geq |x^{n_j}_{i^*} - x^*_{i^*}| \geq |x^{n_j}_{i^*}| + |x^*_{i^*}| \geq \min_{i \in I} |x^{n_j}_{i^*}| + |x^*_{i^*}| \geq 2\eta_\mu,$$

contradicting again to $\|x^{n_j} - x^*\|_2 < \eta_\mu$. This contradiction shows $\text{sign}(x^{n+1}) = \text{sign}(x^*)$. Similarly, we can also show that $\text{sign}(x^{n+1}) = \text{sign}(x^n)$ whenever $n > n^*$. Therefore, $\text{sign}(x^n) = \text{sign}(x^*)$ when $n > n^*$.

With this, the proof of Lemma 3 is completed.

As shown by Property 2, we have known that $\{x^n\}$ converges subsequentially to $x^*$. In order to ensure the convergence of the whole sequence, we may need some additional conditions. For better describing these conditions, we first give some notations. For any vector $x \in \mathbb{R}^N$ and any index set $I \subset I_N = \{1, \cdots, N\}$, $x_I$ denotes the subvector of $x$ restricted to $I$. $I^c$ represents the complementary set of $I$, that is, $I^c = I_N \setminus I$. Denote $\mathcal{X}^*$ as the set of limit points of $\{x^n\}$ and $I = \text{supp}(x^*)$ for $x^* \in \mathcal{X}^*$. For $x \in \mathbb{R}^N$, $[\nabla F(x)]_I$ represents the subvector of $\nabla F(x)$ restricted to $I$. For any $x^* \in \mathbb{R}^N$, we denote

$$\nabla^2 F(x^*)\big|_{x=x*} \nabla^2 F(x^*) = \frac{\partial^2 F(x^*)}{\partial x_I^T \partial x_I},$$

(19)

For any symmetric matrix $A \in \mathbb{R}^{K \times K}$, we denote $\sigma_i(A)$ as the $i$-th eigenvalue of $A$, and $\sigma_{\text{min}}(A)$, $\sigma_{\text{max}}(A)$ as the minimal and maximal eigenvalues of $A$ in magnitude, respectively. Let $K$ be the number of the components of the set $I$. We define a projection mapping associated with $I$, that is,

$$P_I : \mathbb{R}^N \to \mathbb{R}^K, P_Ix = x_I, \forall x \in \mathbb{R}^N.$$

We denote $P_I^T$ as the transpose of $P_I$, i.e.,

$$P_I^T : \mathbb{R}^K \to \mathbb{R}^N, (P_I^T z)_I = z \text{ and } (P_I^T z)_{I^c} = 0, \forall z \in \mathbb{R}^K.$$

For any $\epsilon > 0$, we define a one-dimensional real space

$$\mathcal{R}_\epsilon = \mathbb{R} \setminus (-\epsilon, \epsilon).$$

Particularly, let $\mathcal{R}_0 = \mathbb{R} \setminus \{0\}$. Let $\mathcal{Z}^* = P_I X^*$, that is,

$$\mathcal{Z}^* = \{P_I x^* \mid x^* \in \mathcal{X}^*\}.$$

We define two new functions $T : \mathbb{R}^K_{\eta_0/2} \to \mathbb{R}$ and $f : \mathbb{R}^K_{\eta_0/2} \to \mathbb{R}$ with

$$T(z) = T_\lambda(P_I^T z),$$

and

$$f(z) = F(P_I^T z),$$

for any $z \in \mathcal{R}_{\eta_0/2}$, respectively. For any $z^* \in \mathcal{Z}^*$, it can be observed that $z^* \in \mathbb{R}^K_{\eta_0}$ by Lemma 2, and $z^*$ is indeed a critical points of $T$ from Property 3 (a). Moreover, we denote $C_{ns,z^*}$ as

$$C_{ns,z^*} = \{\sigma_i(\sqrt{2} f(z^*)) \mid i = 1, 2, \cdots, K\},$$

(20)

where $\sigma_i(\sqrt{2} f(z^*))$ represents the $i$-th eigenvalue of $\sqrt{2} f(z^*)$. By Lemma 1, it shows that $\phi'(u) < 0$ when $u > 0$, then $C_{ns,z^*}$ makes sense. Furthermore, we define a series of mappings, that is,

$$\phi_{1,m} : \mathbb{R}^m_0 \to \mathbb{R}^m, \phi_{1,m}(x) = (\text{sign}(x_1)\phi'(|x_1|), \cdots, \text{sign}(x_m)\phi'(|x_m|))^T,$$

(21)

and

$$\phi_{2,m} : \mathbb{R}^m_0 \to \mathbb{R}^{m \times m}, \phi_{2,m}(x) = \text{diag}(\phi''(|x_1|), \cdots, \phi''(|x_m|)),$$

(22)

$m = 1, \cdots, N$, where $\text{diag}(x)$ represents the diagonal matrix generated by $x$. For brevity, we will denote $\phi_{1,m}$ and $\phi_{2,m}$ as $\phi_1$ and $\phi_2$ respectively when $m$ is fixed and there is no confusion.

With these notations, we present the convergence results of the whole sequence $\{x^n\}$ as follows.

**Theorem 1.** Assume that $F$ and $\phi$ satisfy Assumption 1 and 2, respectively. Suppose that $0 < \mu < \frac{1}{L}$, and $x^*$ is a limit point of the sequence $\{x^n\}$ generated by IT algorithm.
Let $I = \text{Supp}(x^*)$ and $z^* = P_I x^*$. Then the whole sequence converges to $x^*$ if

$$\lambda > 0 \text{ and } \lambda \not\in C_{ns,z^*},$$

where $C_{ns,z^*}$ is specified as in (20).

As shown by Theorem 1, we can claim that IJT algorithm converges “almost sure” with respect to the regularization parameter $\lambda$, that is, the IJT algorithm can converge for almost all possibilities of $\lambda$ except some “bad” values. While the number of such “bad” values is less than the dimension of $x$, i.e., $N$. To prove Theorem 1, we also need the following several lemmas.

**Lemma 4.** For any symmetric matrix $A \in \mathbb{R}^{K \times K}$, if $A$ is nonsingular, then for any $z \in \mathbb{R}^K$, it holds

$$\|Az\|_2 \geq \sigma_{\min}(A)\|z\|_2,$$

where $\sigma_{\min}(A)$ represents the minimal eigenvalue of $A$ in magnitude.

**Proof:** Let $\{\sigma_i(A), i = 1, 2, \cdots, K\}$ be the set of eigenvalues of $A$. Without loss of generality, we assume that $|\sigma_1(A)| \geq |\sigma_2(A)| \geq \cdots \geq |\sigma_K(A)|$. Since $A$ is symmetric and nonsingular, then $|\sigma_{\min}(A)| = |\sigma_K(A)| > 0$. Let $U\Lambda U^T$ be a singular value decomposition of $A$, i.e., $A = U\Lambda U^T$, where $\Lambda$ is a diagonal matrix with $\Lambda(i,i) = |\sigma_i(A)|$ for $i = 1, 2, \cdots, K$, and $U$ is an orthogonal matrix containing the corresponding eigenvectors. Then

$$\|Az\|_2 = \|U\Lambda U^T z\|_2 = \|\Lambda U^T z\|_2 \\
\geq |\sigma_{\min}(A)|\|U^T z\|_2 = |\sigma_{\min}(A)|\|z\|_2.$$

Thus, we end the proof of this lemma.

In the following, we will give some lemmas on the local behaviors of $T$ including the nonsingularity of $\nabla^2 T$ and the Lipschitz continuity of $\nabla T$ at a small neighborhood of $z^*$.

**Lemma 5.** Assume that $F$ and $\phi$ satisfy Assumption 1 and 2, respectively. For any $z^* \in \mathcal{Z}^*$, if $\lambda \not\in C_{ns,z^*}$, then there exists a constant $0 < \epsilon_0 < \eta_{\mu}/2$ such that $\nabla^2 T(z)$ is nonsingular for any $z \in B(z^*, \epsilon_0) := \{z||z - z^*||z < \epsilon_0\}$. Moreover, for any $0 < \epsilon < \epsilon_0$, let

$$\sigma_{\epsilon,z^*} = \inf_{z \in B(z^*, \epsilon)} |\sigma_{\min}(\nabla^2 T(z))|,$$

then $\sigma_{\epsilon,z^*} > 0$.

**Proof:** Note that for each $1 \leq i \leq K$, the $i$-th eigenvalue of $\nabla^2 T(z)$ satisfies

$$\sigma_i(\nabla^2 T(z^*)) = \sigma_i(\nabla^2 f(z^*)) + \lambda \phi''(|z_i^*|).$$

Thus, if $\lambda \not\in C_{ns,z^*}$, then $\nabla^2 T(z^*)$ is nonsingular. Moreover, since $\nabla^2 F$ is continuous on $\mathbb{R}^N$ and $\phi''$ is also continuous on $(0, \infty)$, it can be easily checked that both $\nabla^2 f$ and $\nabla^2 T$ are continuous on $\mathbb{R}^{K_{\eta_{\mu}}/2}$. Therefore, there exists a sufficiently small constant $0 < \epsilon_0 < \eta_{\mu}/2$ such that $B(z^*, \epsilon_0) \subseteq \mathbb{R}^{K_{\eta_{\mu}}/2}$ and $\nabla^2 T(z)$ is nonsingular with

$$|\sigma_{\min}(\nabla^2 T(z))| > 0$$

for any $z \in B(z^*, \epsilon_0)$. In addition, given an $0 < \epsilon < \epsilon_0$,

$$\sigma_{\epsilon,z^*} = \inf_{z \in B(z^*, \epsilon)} |\sigma_{\min}(\nabla^2 T(z))| \\
\geq \min_{z \in B(z^*, \epsilon)} |\sigma_{\min}(\nabla^2 T(z))| > 0,$$

where $\overline{B(z^*, \epsilon)}$ represents the closure of $B(z^*, \epsilon)$, that is, $\overline{B(z^*, \epsilon)} = \{z||z - z^*||2 \leq \epsilon\}$.

**Lemma 6.** Assume that $F$ and $\phi$ satisfy Assumption 1 and 2, respectively. Suppose that $z^* \in \mathcal{Z}^*$. Given a constant $0 < \epsilon_0 < \eta_{\mu}/2$ (as specified in Lemma 5). Then for any $0 < \epsilon < \epsilon_0$ there exist a positive constant $L_T$ (depending on $z^*$ and $\epsilon$) such that

$$\|\nabla T(u) - \nabla T(v)\|_2 \leq L_T \|u - v\|_2, \forall u, v \in B(z^*, \epsilon).$$

**Proof:** Note that for any $z \in B(z^*, \epsilon_0)$

$$\nabla T(z) = [\nabla F(P_{I}^T z)]_I + \lambda \phi_1(z).$$

For any $u, v \in B(z^*, \epsilon)$, it holds

$$\|\nabla T(u) - \nabla T(v)\|_2 \leq \|\nabla F(P_{I}^T u) - \nabla F(P_{I}^T v)\|_2 + \lambda \|\phi(u) - \phi_1(v)\|_2$$

$$\leq \|\nabla F(P_{I}^T u) - \nabla F(P_{I}^T v)\|_2 + \lambda \sum_{i=1}^K |\phi'(|u_i|) - \phi'(|v_i|)|^2. \tag{23}$$

By Assumption 1, $\nabla F$ is Lipschitz continuous with the Lipschitz constant $L$, then

$$\|\nabla F(P_{I}^T u) - \nabla F(P_{I}^T v)\|_2 \leq L \|P_{I}^T u - P_{I}^T v\|_2 = L \|u - v\|_2. \tag{24}$$

Moreover, we define a closed interval $C_{\epsilon}$ as

$$C_{\epsilon} = [\min_i |z_i^*| - \epsilon, \max_i |z_i^*| + \epsilon].$$

Since $0 < \epsilon < \epsilon_0 < \eta_{\mu}/2$, then

$$\min_i |z_i^*| - \epsilon > \frac{\eta_{\mu}}{2}$$

and for any $z \in B(z^*, \epsilon)$,

$$|z_i| \in C_{\epsilon}, \text{ for } i = 1, 2, \cdots, K.$$

By Assumption 2, $\phi''$ is continuous on $(0, \infty)$. Thus, $\phi''$ is uniformly continuous and bounded on $C_{\epsilon}$. Let

$$L_\phi = \max_{w \in C_{\epsilon}} |\phi''(w)|.$$

For any $w_1, w_2 \in C_{\epsilon}$, by the mean-value theorem, there exists a constant $t \in (0, 1)$ such that

$$|\phi'(w_1) - \phi'(w_2)| = |\phi''(w_2 + t(w_1 - w_2))(w_1 - w_2)| \leq L_\phi |w_1 - w_2|.$$

Thus, for each $1 \leq i \leq K$,

$$|\phi'(|u_i|) - \phi'(|v_i|)| \leq L_\phi |u_i - v_i| \leq L_\phi |u_i - v_i|.$$

Consequently, for any $u, v \in B(z^*, \epsilon)$,

$$\|\phi_1(u) - \phi_1(v)\|_2 \leq L_\phi \|u - v\|_2. \tag{25}$$

Let

$$L_T = L + \lambda L_\phi.$$

Plugging (24) and (25) into (23), it implies

$$\|\nabla T(u) - \nabla T(v)\|_2 \leq L_T \|u - v\|_2$$
for any \( u, v \in B(z^*, \epsilon) \).

**Lemma 7.** Assume that \( F \) and \( \phi \) satisfy Assumption 1 and 2, respectively. Given a constant \( 0 < \epsilon_0 < \eta_0 / 2 \) (as specified in Lemma 5). Suppose that \( z^* \in \mathcal{Z}^* \) and \( \lambda \not\in C_n, z^* \). Then for any \( 0 < \epsilon < \epsilon_0 \), there exists a positive constant \( C^* \) (depending on \( L_T, \epsilon \)) such that

\[ |T(z) - T(z^*)| \leq C^* \| \nabla T(z) \|_2 \]

for any \( z \in B(z^*, \epsilon) \).

**Proof:** Note that \( z^* \) is a critical point of \( T \), i.e., \( \nabla T(z^*) = 0 \), then

\[ |T(z) - T(z^*)| = |T(z) - T(z^*) - \nabla T(z^*)(z - z^*)| = \int_{t=0}^{1} \langle \nabla T(z^* + t(z - z^*)), - \nabla T(z^*) \rangle dt \leq \int_{t=0}^{1} \| \nabla T(z^* + t(z - z^*)) - \nabla T(z^*) \|_2 \| z - z^* \|_2 dt. \]

By Lemma 5, the matrix \( \nabla^2 T(z^* + t_0(z - z^*)) \) is nonsingular and

\[ |\sigma_{\min}(\nabla^2 T(z^* + t_0(z - z^*)))| \geq \sigma_{e,z^*} > 0. \]

Furthermore, by Lemma 4,

\[ \| \nabla^2 T(z^* + t_0(z - z^*)) \|_2 \geq |\sigma_{\min}(\nabla^2 T(z^* + t_0(z - z^*)))| \| (z - z^*) \|_2 \geq \sigma_{e,z^*} \| (z - z^*) \|_2. \]

Plugging (30) into (29),

\[ \| \nabla T(z) \|_2 \geq \sigma_{e,z^*} \| (z - z^*) \|_2. \]

Let

\[ C^* = \frac{L_T}{2\sigma_{e,z^*}}. \]

Combining (28) and (31), it implies

\[ |T(z) - T(z^*)| \leq C^* \| \nabla T(z) \|_2. \]

Thus, we complete the proof of the lemma.

With these lemmas, we give the proof of Theorem 1 as follows.

**Proof:** Let

\[ a = \frac{1}{2} \left( \frac{1}{\mu} - L \right) \quad \text{and} \quad b = \frac{1}{\mu} + L. \]

By Lemma 3, there exists a sufficiently large integer \( n^* > 0 \) such that when \( n > n^* \),

\[ I^n = I \quad \text{and} \quad \text{sign}(x^n) = \text{sign}(x^*). \]

Therefore, we can claim that \( \{ x^n \} \) converges to \( x^* \) if the new sequence \( \{ x^{i+n^*} \}_{i \in \mathbb{N}} \) converges to \( x^* \), which is also equivalent to the convergence of the sequence \( \{ z^{i+n^*} \}_{i \in \mathbb{N}} \), i.e.,

\[ z^{i+n^*} \to z^* \quad \text{as} \quad i \to \infty \] (32)

with \( z^{i+n^*} = P_{T} x^{i+n^*} \) and \( z^* = P_{T} x^* \).

By Property 1 and 2, we have known the following facts:

(i) \( \| x^{n+1} - x^n \|_2 \to 0 \) as \( n \to \infty \);

(ii) \( \{ T_{\lambda}(x^n) \} \) is monotonically decreasing and converges to \( T_{\lambda}(x^*) \);

(iii) there exists a subsequence \( \{ x^{n_j} \} \) converges to \( x^* \), that is,

\[ x^{n_j} \to x^* \quad \text{as} \quad j \to \infty. \]

Therefore, for a give constant \( 0 < \epsilon < \epsilon_0 \) (\( \epsilon_0 \) is specified as in Lemma 5), there exists a sufficiently large integer \( j^* > 0 \) such that \( n_{j^*} > n^* \), and when \( n \geq n_{j^*} \),

\[ \| x^{n+1} - x^n \|_2 < \epsilon \quad \text{and} \quad 0 < T_{\lambda}(x^n) - T_{\lambda}(x^*) < \epsilon, \] (33)

and when \( j \geq j^* \),

\[ \| x^{n_j} - x^* \|_2 < \epsilon, \] (34)

and further

\[ \| x^* - x^{n_{j^*}} \|_2 + 2 \left( \frac{b}{\sqrt{a}} + \frac{C^* b}{a} \right) \sqrt{T_{\lambda}(x^{n_{j^*}}) - T_{\lambda}(x^*)} < \epsilon. \] (35)

We redefine a new sequence \( \{ \hat{z}^n \} \) for \( n \in \mathbb{N} \) with

\[ \hat{z}^n = z^{n+n_{j^*}} = P_{T} x^{n+n_{j^*}}. \]

Then the following inequalities hold naturally for each \( n \in \mathbb{N} \),

\[ \| \hat{z}^{n+1} - \hat{z}^n \|_2 < \epsilon \quad \text{and} \quad 0 < T(\hat{z}^n) - T(\hat{z}^*) < \epsilon, \]

and

\[ \| \hat{z}^* - \hat{z}^0 \|_2 + 2 \left( \frac{b}{\sqrt{a}} + \frac{C^* b}{a} \right) \sqrt{T(\hat{z}^0) - T(\hat{z}^*)} < \epsilon. \] (36)

Therefore, the convergence of \( \{ x^n \} \) is equivalent to the convergence of \( \{ \hat{z}^n \} \).

The key point to prove the convergence of \( \{ \hat{z}^n \} \) is to justify the following claim: for \( n = 1, 2, \cdots \)

\[ \hat{z}^n \in B(z^*, \epsilon) \] (37)

and

\[ \sum_{i=1}^{n} \| \hat{z}^{i+1} - \hat{z}^i \|_2 + \| \hat{z}^{n+1} - \hat{z}^n \|_2 \leq \| \hat{z}^0 - \hat{z}^1 \|_2 + \frac{2\sqrt{C^* b}}{a} \left( \sqrt{T(\hat{z}^1) - T(z^*)} - \sqrt{T(\hat{z}^{n+1}) - T(z^*)} \right). \] (38)

By Property 1 (a), it can be observed that

\[ a \| \hat{z}^{n+1} - \hat{z}^n \|_2^2 \leq T(\hat{z}^n) - T(\hat{z}^{n+1}) \] (39)
for any \( n \in \mathbb{N} \). Fix \( n \geq 1 \), we claim that if \( \hat{z}^n \in B(z^*, \epsilon) \), then
\[
2\|\hat{z}^{n+1} - \hat{z}^n\|_2 \leq \|\hat{z}^n - \hat{z}^{n-1}\|_2 + \frac{2\sqrt{C^*b}}{a} \left( \sqrt{T(\hat{z}^n) - T(z^*)} - \sqrt{T(\hat{z}^{n+1}) - T(z^*)} \right). \tag{40}
\]
By Lemma 7,
\[
|T(\hat{z}^n) - T(z^*)| \leq C\|\nabla T(\hat{z}^n)\|_2^2. \tag{41}
\]
Moreover, by Property 3 (b), it can be easily checked that
\[
\hat{z}^n + \lambda\phi_1(\hat{z}^n) = \hat{z}^{n-1} - \mu\nabla f(\hat{z}^{n-1}),
\]
which implies
\[
\mu(\nabla f(\hat{z}^n) + \lambda\phi_1(\hat{z}^n)) = (\hat{z}^{n-1} - \hat{z}^n) + \mu(\nabla f(\hat{z}^n) - \nabla f(\hat{z}^{n-1})),
\]
and thus,
\[
\|\nabla T(\hat{z}^n)\|_2 = \frac{1}{\mu}(\|\hat{z}^{n-1} - \hat{z}^n\| + \mu(\nabla f(\hat{z}^n) - \nabla f(\hat{z}^{n-1}))_2).
\]
By Assumption 1, \( \nabla F \) is Lipschitz continuous with the Lipschitz constant \( L \), then
\[
\|\nabla f(\hat{z}^n) - \nabla f(\hat{z}^{n-1})\|_2 = \|\nabla F(P^T_f \hat{z}^n) - \nabla F(P^T_f \hat{z}^{n-1})\|_2 \\
\leq \|\nabla F(P^T_f \hat{z}^n) - \nabla F(P^T_f \hat{z}^{n-1})\|_2 \\
\leq L\|P^T_f \hat{z}^n - P^T_f \hat{z}^{n-1}\|_2 \\
= L\|\hat{z}^n - \hat{z}^{n-1}\|_2.
\]
Therefore,
\[
\|\nabla T(\hat{z}^n)\|_2 \leq \frac{1}{\mu} + L\|\hat{z}^n - \hat{z}^{n-1}\| = b\|\hat{z}^n - \hat{z}^{n-1}\|. \tag{42}
\]
Furthermore, by the concavity of the function \( \sqrt{x} \) for \( s > 0 \),
\[
\sqrt{T(\hat{z}^n) - T(z^*)} - \sqrt{T(\hat{z}^{n+1}) - T(z^*)} \geq \frac{T(\hat{z}^n) - T(\hat{z}^{n+1})}{2\sqrt{T(\hat{z}^n) - T(z^*)}} \tag{43}
\]
Plugging the inequalities (39), (41) and (42) into (43) and after some simplifications,
\[
\|\hat{z}^n - \hat{z}^{n+1}\|_2^2 \leq \frac{2\sqrt{C^*b}}{a} \|\hat{z}^n - \hat{z}^{n-1}\|_2 \\
\times \left( \sqrt{T(\hat{z}^n) - T(z^*)} - \sqrt{T(\hat{z}^{n+1}) - T(z^*)} \right).
\]
Using the inequality \( \sqrt{x} \leq \frac{\alpha + \beta}{2} \) for any \( \alpha, \beta \geq 0 \), we conclude that inequality (40) is satisfied. Thus, the claim (38) can be easily derived from (40).

In the following, we will prove the claim (37) by induction. First, by (36), it implies
\[
\hat{z}^0 \in B(z^*, \epsilon).
\]
Second, it can be observed that
\[
\|\hat{z}^1 - z^*\|_2 \leq \|\hat{z}^0 - z^*\|_2 + \|\hat{z}^0 - \hat{z}^1\|_2 \\
\leq \sqrt{T(\hat{z}^0) - T(z^*)} + \|\hat{z}^0 - z^*\|_2 \\
\leq \sqrt{T(\hat{z}^0) - T(z^*)} + \|\hat{z}^0 - z^*\|_2 < \epsilon,
\]
where the second inequality holds for (39), the third inequality holds due to \( T(z^*) \leq T(\hat{z}^1) \leq T(\hat{z}^0) \) and the last inequality holds for (39). Therefore, \( \hat{z}^1 \in B(z^*, \epsilon) \).

Third, suppose that \( \hat{z}^n \in B(z^*, \epsilon) \) for \( n \geq 1 \), then
\[
\|\hat{z}^{n+1} - z^*\|_2 \leq \|z^* - \hat{z}^0\|_2 + \|\hat{z}^0 - \hat{z}^1\|_2 + \sum_{i=1}^{n} \|\hat{z}^{i+1} - \hat{z}^i\|_2 \\
\leq \|z^* - \hat{z}^0\|_2 + 2\|\hat{z}^0 - \hat{z}^1\|_2 \\
+ \frac{2\sqrt{C^*b}}{a} \left( \sqrt{T(\hat{z}^1) - T(z^*)} - \sqrt{T(\hat{z}^{n+1}) - T(z^*)} \right),
\]
where the second inequality holds for (38). Moreover,
\[
\|\hat{z}^0 - \hat{z}^1\|_2 \leq \sqrt{T(\hat{z}^0) - T(z^*)} \leq \sqrt{T(\hat{z}^0) - T(z^*)} \tag{45}
\]
where the first inequality holds for (39) and the second inequality holds for \( T(z^*) \leq T(\hat{z}^1) \leq T(\hat{z}^0) \). Also, it is obvious that
\[
\sqrt{T(\hat{z}^1) - T(z^*)} - \sqrt{T(\hat{z}^{n+1}) - T(z^*)} \leq \sqrt{T(\hat{z}^0) - T(z^*)}. \tag{46}
\]
Plugging (45) and (46) into (44) and using (36), we can claim that
\[
\hat{z}^{n+1} \in B(z^*, \epsilon).
\]
By (38), it shows that
\[
\sum_{i=1}^{\infty} \|\hat{z}^{i+1} - \hat{z}^i\|_2 < +\infty,
\]
which implies that the sequence \{ \hat{z}^n \} converges to some \( z^{**} \). While we have assumed that there exists a subsequence \{ \hat{z}^{n_i} \} converges to \( z^* \), then it must hold
\[
z^{**} = z^*.
\]
Consequently, we can claim that \{ \hat{z}^n \} converges to \( z^* \).

With these, we end the proof of Theorem 1.

\[ \blacksquare \]

\section*{B. Convergence To A Local Minimizer}

As shown in Theorem 1, under some mild conditions, the IJIT algorithm can converge to a stationary point \( x^* \). In this subsection, we will justify that \( x^* \) is also a local minimizer of the optimization problem under some further assumptions.

\textbf{Theorem 2.} Suppose that \( F \) and \( \phi \) satisfy Assumption 1 and 2, respectively. Assume that \( 0 < \mu < \frac{1}{2} \), and the sequence
\[ \{ x^* \} \text{ generated by ITT algorithm converges to } x^*. \text{ Let } I = \text{Supp}(x^*). \text{ Then } x^* \text{ is a local minimizer of } T_\lambda \text{ if the following conditions hold:} \]

(a) \( \sigma_{\min} (\nabla^2_{II} F(x^*)) > 0; \)
(b) \( 0 < \lambda < \min_{i \in I} \left\{ \frac{\sigma_i (\nabla^2_{II} F(x^*))}{\phi_i(x_i^*)} \right\} \)

Intuitively, under the condition (b) in Theorem 2, for any \( i \in I, \) it holds

\[ \sigma_i (\nabla^2_{II} F(x^*)) + \lambda \phi'' (|x_i^*|) > 0. \]

This implies that the principle submatrix restricted to the index set \( I \) of the Hessian matrix of \( T_\lambda \) at \( x^* \) is positively definite. Thus, the convexity of the objective function can be guaranteed in a neighborhood of \( x^* \). As a consequence, \( x^* \) should be a local minimizer.

**Proof:** Let

\[ \epsilon = \frac{1}{2} \min_{i \in I} \{ \sigma_i (\nabla^2_{II} F(x^*)) + \lambda \phi'' (|x_i^*|) \}. \quad (47) \]

By the assumption of Theorem 2, it holds \( \epsilon > 0 \). Furthermore, we define two constants

\[ C_e = \max \left\{ \| \nabla^2_{II} F(x^*) \|_1 + 2 \epsilon, \| \nabla_{II}^2 F(x^*) + (\nabla_{II}^2 F(x^*))^T \|_1 \right\}, \quad (48) \]

and

\[ C = \frac{\sigma_{\mu}}{\lambda \mu} + \frac{\sqrt{N e}}{\lambda} C_e, \]

where \( e = \min_{i \in I} |x_i^*| \). In the following, we will show that there exists a constant \( 0 < c < 1 \), it holds

\[ T_\lambda (x^* + h) - T_\lambda (x^*) \geq 0, \]

whenever \( \| h \|_2 < ce \).

Actually, we have

\[ T_\lambda (x^* + h) - T_\lambda (x^*) = F(x^* + h) - F(x^*) + \lambda \left( \sum_{i \in I} \phi(|x_i^* + h_i|) - \phi(|x_i^*|) + \sum_{i \in I^c} \phi(|h_i|) \right). \quad (49) \]

By Taylor expansion, it holds

\[ F(x^* + h) - F(x^*) = h^T \nabla^2 F(x^*) h + \frac{1}{2} h^T \nabla^2_{II} F(x^*) h + \frac{1}{2} h^T \nabla^2_{I^c} F(x^*) h_{I^c} + \lambda \phi(h_1) + o(\| h \|_2^2), \]

and

\[ \sum_{i \in I} \phi(|x_i^* + h_i|) - \phi(|x_i^*|) = h^T \phi_1(x_1^*) + \frac{1}{2} h^T \phi_2(x_1^*) h_{I^c} + o(\| h \|_2^2). \quad (50) \]

By Property 3(a), it implies

\[ |\nabla F(x^*)|_I + \lambda \phi_1(x_1^*) = 0. \quad (52) \]

Plugging (51), (51) and (52) into (49), it becomes

\[ T_\lambda (x^* + h) - T_\lambda (x^*) = \frac{1}{2} h^T \nabla^2_{II} F(x^*) h_{I^c} + \frac{1}{2} h^T \nabla^2_{I^c} F(x^*) h_{I^c} + \frac{1}{2} h^T \nabla^2_{II} F(x^*) h_{I^c} + \frac{1}{2} h^T \nabla^2_{I^c} F(x^*) h_{I^c} \]

\[ + h^T \nabla_{II}^2 F(x^*) h_{I^c} - \lambda \sum_{i \in I^c} \phi(h_i) + o(\| h \|_2^2). \]

Moreover, by the definition of \( o(\| h \|_2^2) \), there exists a constant \( 0 < c_e < 1 \) (depending on \( e \)) such that \( |o(\| h \|_2^2)| \leq c_e \| h \|_2^2 \) whenever \( \| h \|_2 < c_e e \). Therefore

\[ T_\lambda (x^* + h) - T_\lambda (x^*) \leq \frac{1}{2} h^T \nabla^2_{II} F(x^*) h_{I^c} + \frac{1}{2} h^T \nabla^2_{I^c} F(x^*) h_{I^c} - \epsilon \| h \|_2^2 + \frac{1}{2} h^T \nabla^2_{II} F(x^*) h_{I^c} + \frac{1}{2} h^T \nabla^2_{I^c} F(x^*) h_{I^c} \]

\[ - \epsilon \| h \|_2^2 + h^T \nabla_{II}^2 F(x^*) h_{I^c} + \lambda \sum_{i \in I^c} \phi(h_i). \quad (54) \]

Furthermore, we divide the right side of the inequality (53) into three parts, that is, \( E_1, E_2 \) and \( E_3 \) with

\[ E_1 = \frac{1}{2} h^T \nabla^2_{II} F(x^*) + \lambda \phi_2(x_1^*) h_{I^c} - \epsilon \| h \|_2^2, \]

\[ E_2 = \frac{1}{2} h^T \nabla^2_{II} F(x^*) - 2 \epsilon h_{I^c}, \]

\[ E_3 = h^T \nabla_{II}^2 F(x^*) h_{I^c} + \lambda \sum_{i \in I^c} \phi(h_i). \quad (57) \]

By the definition of \( \epsilon \) as in (47), it can be observed that

\[ E_1 \geq \left( \frac{1}{2} \min_{i \in I} \{ \sigma_i (\nabla^2_{II} F(x^*)) + \lambda \phi'' (|x_i^*|) \} - \epsilon \right) \| h \|_2^2 \geq 0. \quad (58) \]

By Assumption 2(c), for any \( z \in R_+ \), it holds \( \phi'(z) \to \infty \) as \( z \to 0^+ \), which implies that there exists a sufficiently small constant \( 0 < c < \frac{1}{\sqrt{\mu}} \) such that \( \phi'(z) > C \) for any \( 0 < z < ce \). By Taylor expansion and \( \phi(0) = 0 \), we have

\[ \phi(z) \geq Cz, \]

when \( 0 < z < ce \). Moreover, by Property 3(a), for any \( i \in I^c \),

\[ \phi(h_i) + \frac{1}{\lambda} \phi_2(x_1^*) h_i \geq \phi(h_i) - \frac{\sigma_i}{\lambda \mu} |h_i| \geq (C - \frac{\sigma_i}{\lambda \mu}) |h_i|. \]

Thus,

\[ E_3 \geq \lambda \left( C - \frac{\sigma_i}{\lambda \mu} \right) \| h \|_1 = \sqrt{N e} C_e \| h \|_1. \]

Moreover, since \( \| h \|_2 < c_e e \) with \( 0 < c_e < 1 \), then \( \| h \|_1 < \sqrt{N e} \), \( \| h \|_1 < \sqrt{N e} \) and \( \| h \|_1 < \sqrt{N e} \). It is easy to check that

\[ |E_2| \leq \frac{1}{2} \| h \|_1 (\| \nabla^2_{II} F(x^*) \|_1 + 2 \epsilon) \| h \|_1 \]

\[ + \frac{1}{2} \| h \|_1 (\| \nabla^2_{II} F(x^*) \|_1 + 2 \epsilon) \| h \|_1 \leq \| h \|_1 C_e \| h \|_1 \leq \sqrt{N e} C_e \| h \|_1 \leq E_3, \]
where the second inequality holds for \( \|h_{1,:}\|_1 \leq \|h\|_1, \) \( \|h_{1,:}\|_1 \leq \|h\|_1 \) and the definition of \( C_\varepsilon \) as specified in (48). It implies that
\[
E_2 + E_3 \geq 0. \tag{59}
\]
By (58) and (59), it holds
\[
T_\lambda(x^* + h) - T_\lambda(x^*) \geq E_1 + E_2 + E_3 \geq 0,
\]
for any sufficiently small \( h \). Therefore, \( x^* \) is a local minimizer of \( T_\lambda \).

C. Asymptotically Linear Convergence Rate

In the last subsection, we have shown that IJT algorithm converges to a local minimizer under certain conditions. In this subsection, we will study the rate of convergence of IJT algorithm.

In order to derive the asymptotic convergence rate of IJT algorithm, we first give some assumptions on \( \nabla F \) and \( \phi' \) in the neighborhood of \( x^* \). For any \( 0 < \varepsilon < \eta_\mu \), we define a neighborhood of \( x^* \) as follows
\[
\mathcal{N}(x^* , \varepsilon) = \{ x \in \mathbb{R}^N : \| x - x^* \|_2 < \varepsilon, x I = 0 \}.
\]

Assumption 3. For any \( x \in \mathcal{N}(x^* , \varepsilon) \), assume that there exist two positive constants \( \alpha_{F,\varepsilon} \) and \( \alpha_{\phi,\varepsilon} \) (both constants depending on \( \varepsilon \)) such that
(a) \( \langle \nabla F(x), x - x^* \rangle \geq \alpha_{F,\varepsilon} \| x - x^* \|_2^2 \);
(b) \( \langle \phi(x) - \phi(x^*), x - x^* \rangle \geq -\alpha_{\phi,\varepsilon} \| x - x^* \|_2^2 \),
where \( \phi \) is defined as in (21).

Generally speaking, Assumption 3(a) states that \( \nabla F \) should be uniformly monotone in \( \mathcal{N}(x^* , \varepsilon) \), and Assumption 3(b) requires essentially the locally Lipschitz continuity of \( -\phi \) in \( \mathcal{N}(x^* , \varepsilon) \). It is obvious that \( \alpha_{F,\varepsilon} \leq L \), where \( L > 0 \) is the Lipschitz constant of \( \nabla F \). Under the Assumption 3, we present an important lemma on the rate of convergence of IJT algorithm.

Lemma 8. Assume that the sequence \( \{ x^n \} \) generated by IJT algorithm converges to \( x^* \). Let \( I = Supp(x^n) \) and \( e = \min_{i \in I} |x_i^*| \). Moreover, for any \( 0 < \varepsilon < \eta_\mu \), suppose that \( F \) and \( \phi \) satisfy Assumption 3 with the corresponding constants \( \alpha_{F,\varepsilon} \) and \( \alpha_{\phi,\varepsilon} \). If the following conditions hold
(a) \( 0 < \lambda < \frac{\alpha_{F,\varepsilon}}{\alpha_{\phi,\varepsilon}} \)
(b) \( 0 < \mu < \min\left(\frac{2(\alpha_{F,\varepsilon} - \lambda \alpha_{\phi,\varepsilon})}{L^2}, \frac{1}{\lambda} \right) \)
then there exists a sufficiently large positive integer \( n_0 \) and a constant \( \rho \in (0, 1) \) such that when \( n > n_0 \),
\[
\| x^{n+1} - x^* \|_2 \leq \rho \| x^n - x^* \|_2,
\]
and
\[
\| x^{n+1} - x^* \|_2 \leq \frac{\rho}{1 - \rho} \| x^{n+1} - x^* \|.
\]

Proof: Since \( \{ x^n \} \) converges to \( x^* \), then for any \( 0 < \varepsilon < \eta_\mu \), there exists a sufficiently large integer \( n_0 > n^* \) (where \( n^* \) is specified as in Lemma 3) such that
\[
\| x^n - x^* \|_2 < \varepsilon
\]
when \( n > n_0 \). Let \( I^n = Supp(x^n) \). By Lemma 3, it holds \( I^n = I \) and \( sign(x^n) = sign(x^*) \) when \( n > n_0 \). Furthermore, by Property 3, for any \( i \in I \),
\[
x_i^n + \lambda \mu \text{sign}(x_i^n) \phi'(x_i^n) = x_i^n - \mu \langle \nabla F(x^*), x_i^n \rangle,
\]
and
\[
x_i^{n+1} + \lambda \mu \text{sign}(x_i^{n+1}) \phi'(x_i^{n+1}) = x_i^n - \mu \langle \nabla F(x^*), x_i^n \rangle,
\]
when \( n > n_0 \). Consequently,
\[
\langle x^{n+1}_i - x^*_i, x^n_i - x^*_i \rangle + \lambda \mu (\phi(x_i^{n+1}) - \phi(x_i^n)) = (x^n_i - x^*_i) - \mu (\| \nabla F(x^n) \|_I - \| \nabla F(x^*) \|_I),
\]
and then
\[
\| x^{n+1}_i - x^*_i \|_2^2 + \lambda \mu (\phi(x_i^{n+1}) - \phi(x_i^n)) = (x^{n+1}_i - x^*_i, x^n_i - x^*_i) - \mu (\| \nabla F(x^n) \|_I - \| \nabla F(x^*) \|_I).
\]
By Assumption 3, the left side of (60) satisfies
\[
\| x^{n+1}_i - x^*_i \|_2^2 + \lambda \mu (\phi(x_i^{n+1}) - \phi(x_i^n)) \geq (1 - \lambda \mu \alpha_{\phi,\varepsilon}) \| x^{n+1}_i - x^*_i \|_2^2,
\]
and the right side of (60) satisfies
\[
\langle x^{n+1}_i - x^*_i, x^n_i - x^*_i \rangle - \mu (\| \nabla F(x^n) \|_I - \| \nabla F(x^*) \|_I) \leq \| x^{n+1}_i - x^*_i \|_2 (\| x^n_i - x^*_i \|_2 - \mu (\| \nabla F(x^n) \|_I - \| \nabla F(x^*) \|_I)).
\]
Without loss of generality, we assume that \( \| x^{n+1}_i - x^*_i \|_2 > 0 \), otherwise, it demonstrates that IJT algorithm converges to \( x^* \) in finite iterations. Thus, it becomes
\[
(1 - \lambda \mu \alpha_{\phi,\varepsilon}) \| x^{n+1}_i - x^*_i \|_2^2 \leq \| x^{n+1}_i - x^*_i \|_2^2 \leq (1 - \lambda \mu \alpha_{\phi,\varepsilon}) \| x^{n+1}_i - x^*_i \|_2^2.
\]
Since \( 0 < \lambda < \frac{\alpha_{F,\varepsilon}}{\alpha_{\phi,\varepsilon}} \), \( \alpha_{F,\varepsilon} < L \) and \( 0 < \mu < \frac{1}{L} \), then
\[
1 - \lambda \mu \alpha_{\phi,\varepsilon} > 1 - \mu L > 0.
\]
Moreover,
\[
\| x^n_i - x^*_i \|_2 - \mu (\| \nabla F(x^n) \|_I - \| \nabla F(x^*) \|_I) \leq \| x^n_i - x^*_i \|_2^2 + \lambda \mu^2 (\| \nabla F(x^n) \|_I - \| \nabla F(x^*) \|_I) \leq 2 \mu \| x^n_i - x^*_i \|_2 (\| \nabla F(x^n) \|_I - \| \nabla F(x^*) \|_I) \leq \frac{1}{2} (1 - \mu L^2 \| x^n_i - x^*_i \|_2).
\]
Combining (61) and (62), it implies
\[
\| x^{n+1}_i - x^*_i \|_2 \leq \frac{\sqrt{1 - 2 \mu L^2}}{1 - \lambda \mu \alpha_{\phi,\varepsilon}} \| x^n_i - x^*_i \|_2.
\]
Let
\[
\rho = \frac{\sqrt{1 - 2 \mu L^2}}{1 - \lambda \mu \alpha_{\phi,\varepsilon}}.
\]
Since \( 0 < \mu < \frac{2(\alpha_{F,\varepsilon} - \lambda \alpha_{\phi,\varepsilon})}{L^2} \), it is easy to check that
\[
0 < \rho < 1.
\]
Thus, when \( n > n_0 \)
\[
\| x^{n+1}_i - x^*_i \|_2 = \| x^{n+1}_i - x^*_i \|_2 \leq \rho \| x^n_i - x^*_i \|_2 = \rho \| x^n - x^* \|_2.
\]
(63)
Consequently, the asymptotic convergence rate of IJT algorithm is linear.

The error bound can be easily derived by the triangle inequality

$$
\|x^n - x^*\|_2 \leq \|x^{n+1} - x^*\|_2 + \|x^{n+1} - x^n\|_2
$$

and \((63)\).

From Lemma 8, it demonstrates that if \(F\) and \(\phi\) satisfy Assumption 3, then IJT algorithm converges exponentially fast when the number of iteration is sufficiently large. In the following, we will show that Assumption 3 is satisfied naturally under Assumption 1 and 2. If \(\sigma_{\min}(\nabla_{II}^2 F(x^*)) > 0\), then by Taylor expansion, it is easy to check that for any \(0 < \varepsilon < \eta\), \(x \in \mathcal{N}(x^*, \varepsilon)\), there exist two sufficiently small positive constants \(c_F\) and \(c_\phi\) depending on \(x\) with \(c_F \to 0\) and \(c_\phi \to 0\) as \(\varepsilon \to 0\) such that

\[
\langle (\nabla^2 F(x^*)_I - \nabla^2 F(x^*)_I, x_I - x_I^* \rangle \geq (\sigma_{\min}(\nabla_{II}^2 F(x^*)) - c_F)\|x_I - x_I^*\|_2^2,
\]

and

\[
\langle \phi_1(x_I) - \phi_1(x_I^*), x_I - x_I^* \rangle \geq (\phi''(\varepsilon) - c_\phi)\|x_I - x_I^*\|_2^2,
\]

(64)

and

\[
\langle \phi_1(x_I) - \phi_1(x_I^*), x_I - x_I^* \rangle \geq (\phi''(\varepsilon) - c_\phi)\|x_I - x_I^*\|_2^2,
\]

(65)

where \((65)\) holds for \(\phi'\) being strictly convex on \((0, \infty)\), and thus \(\phi''\) being nondecreasing on \((0, \infty)\), consequently, \(\sigma_{\min}(\nabla_{II}^2 F(x^*)) > 0\), \(\phi''\) being nondecreasing on \((0, \infty)\), and consequently, \(\sigma_{\min}(\nabla_{II}^2 F(x^*)) > 0\).

Theorem 3. Suppose that \(F\) and \(\phi\) satisfy Assumption 1 and 2, respectively. Assume that the sequence \(\{x^n\}\) generated by IJT algorithm converges to \(x^*\). Let \(I = \text{Supp}(x^*)\) and \(e = \min_{i \in I} |x_i^*|\). Moreover, if the following conditions hold

(a) \(\sigma_{\min}(\nabla_{II}^2 F(x^*)) > 0\),

(b) \(0 < \lambda < -\frac{\sigma_{\min}(\nabla_{II}^2 F(x^*))}{\phi''(x^*)}\),

(c) \(0 < \mu < \min\{\frac{2\sigma_{\min}(\nabla_{II}^2 F(x^*)) + \lambda \phi''(x^*)}{\lambda^2 - (\lambda \phi''(x^*))^2}, 1\}\),

then there exists a sufficiently large positive integer \(n_0\) and a constant \(\rho^* \in (0, 1)\) such that when \(n > n_0\),

$$
\|x^{n+1} - x^*\|_2 \leq \rho^*\|x^n - x^*\|_2,
$$

and

$$
\|x^{n+1} - x^*\|_2 \leq \frac{\rho^*}{\rho^*}\|x^{n+1} - x^n\|_2.
$$

By Assumption 2(c), \(\phi'\) is strictly convex on \((0, \infty)\), then \(\phi''\) is nondecreasing on \((0, \infty)\). Thus,

$$
-\frac{\sigma_{\min}(\nabla_{II}^2 F(x^*))}{\phi''(x^*)} \leq \min_{i \in I} \left\{ -\frac{\sigma_i(\nabla_{II}^2 F(x^*))}{\phi''(x_i^*)} \right\}.
$$

It implies that \(x^*\) is also a local minimizer under the conditions of Theorem 3.

Proof: Let

\[
C_1 = 1 + \lambda \mu \phi''(x^*),
\]

and

\[
C_2 = \sqrt{1 - 2\mu \sigma_{\min}(\nabla_{II}^2 F(x^*)) + \mu^2 L^2}.
\]

By the assumption of Theorem 3, it is easy to check that

\[
C_1 > C_2 > 0.
\]

Since both \(c_F\) and \(c_\phi\) approach to zero as \(\varepsilon\) approaching to zero, then we can take a sufficiently small \(0 < \varepsilon < \eta_\mu\) such that

\[
0 < c_F < \frac{(C_1 - C_2)(C_1 + 3C_2)}{8\mu} \quad \text{and} \quad 0 < c_\phi < \frac{C_1 - C_2}{2\lambda \mu}.
\]

Furthermore, let

\[
\alpha_{F, \varepsilon} = \sigma_{\min}(\nabla_{II}^2 F(x^*)) - c_F,
\]

\[
\alpha_{\phi, \varepsilon} = -\phi''(\varepsilon) + c_\phi,
\]

and

\[
\rho^* = \frac{\sqrt{1 - 2\mu \alpha_{F, \varepsilon} + \mu^2 L^2}}{1 - \lambda \mu \alpha_{F, \varepsilon}}.
\]

By the assumption that \(\{x^n\}\) converges to \(x^*\), thus, for such \(\varepsilon > 0\), there exists a sufficiently large integer \(n_0 > n^*\) (where \(n^*\) is specified as in Lemma 3) such that when \(n > n_0\), \(x^n \in \mathcal{N}(x^*, \varepsilon)\). According to the proof procedure of Lemma 8, in order to prove Theorem 3, it suffices to check the following inequalities:

\[
0 < 1 - \lambda \mu \alpha_{F, \varepsilon} > 0,
\]

\[
1 - 2\mu \alpha_{F, \varepsilon} + \mu^2 L^2 \geq 0,
\]

and

\[
0 < \rho^* < 1.
\]

Actually, we have

\[
1 - \lambda \mu \alpha_{F, \varepsilon} = 1 + \lambda \mu \phi''(x^*) - \lambda \mu c_\phi > \frac{C_1 + C_2}{2} > 0, \quad (66)
\]

\[
1 - 2\mu \alpha_{F, \varepsilon} + \mu^2 L^2 \geq 1 - 2\mu \alpha_{F, \varepsilon} + \mu^2 \alpha_{F, \varepsilon}^2 \geq 0, \quad (67)
\]

\[
1 - 2\mu \alpha_{F, \varepsilon} + \mu^2 L^2 = C_2^2 + 2\mu c_F,
\]

\[
< C_2^2 + \frac{(C_1 - C_2)(C_1 + 3C_2)}{4} = \left(\frac{C_1 + C_2}{2}\right)^2, \quad (68)
\]

and thus

\[
0 < \rho^* < 1.
\]

Moreover, the posteriori error bound can be directly derived by \(\|x^{n+1} - x^n\|_2 \leq \rho^*\|x^n - x^*\|_2\) and the triangle inequality \(\|x^n - x^*\|_2 \leq \|x^{n+1} - x^n\|_2 + \|x^n - x^*\|_2\). Therefore, we have completed the proof of Theorem 3.

It can be observed that the conditions of Theorem 3 are slightly stricter than those of Theorem 2. In the following, we will show that the condition on \(\mu\) in Theorem 3 can be extended to \(0 < \mu < 1/L\) if we add some additional assumptions on the higher order differentiability of \(\phi\) in the neighborhood of the convergent point \(x^*\). We state this as the following Theorem 4.

Theorem 4. Assume that \(0 < \mu < \frac{1}{L}\). Let \(\{x^n\}\) be a sequence generated by IJT algorithm and converge to \(x^*\). Let \(I = \text{Supp}(x^*)\) and \(e = \min_{i \in I} |x_i^*|\). Moreover, if the following conditions hold

(a) \(\sigma_{\min}(\nabla_{II}^2 F(x^*)) > 0\);

(b) \(0 < \lambda < -\frac{\sigma_{\min}(\nabla_{II}^2 F(x^*))}{\phi''(x^*)}\);

(c) \(\varepsilon > 0\), \(\phi''(\varepsilon)\) is well-defined and bounded on the set \(\cup_{i \in I} B(x_i^*, \varepsilon)\), where \(B(x_i^*, \varepsilon) := (x_i^* - \varepsilon, x_i^* + \varepsilon)\).
then there exists a sufficiently large positive integer $n_0 > 0$ and a constant $\rho \in (0, 1)$ such that when $n > n_0$,
\[ \|x^{n+1} - x^*\|_2 \leq \rho \|x^n - x^*\|_2, \]
and
\[ \|x^{n+1} - x^*\|_2 \leq \frac{\rho}{1 - \rho} \|x^{n+1} - x^n\|_2. \]

Proof: Let
\[ c_1 = \frac{1 - \mu \sigma_{\min}(I^T FI(x^*))}{1 + \lambda \mu \phi''(e)}. \]
By the assumption of Theorem 4, it holds $0 < c_1 < 1$. Furthermore, we can take a positive constant $\epsilon$ such that
\[ 0 < \epsilon < 1 - c_1. \]
For any $0 < c < 1$, let
\[ g(c) = \max_{i \in I} \max_{\{x_i : x_i - x_i^* < \epsilon \}} \{ \frac{\mu \lambda \phi''(|x_i|)}{2(1 + \lambda \mu \phi''(x_i^*))} \} \]
and
\[ c_\epsilon(c) = \frac{1 - c_1 - \epsilon}{g(c)} \eta_\mu. \]
Since $g(c)$ is non-decreasing with respect to $c$, and thus $c_\epsilon(c)$ is non-increasing with respect to $c$. Therefore, there exists a positive constant $c^*$ such that
\[ c^* < c_\epsilon(c^*). \]

Since $\{x^n\}$ converges to $x^*$, then there exists an $n^* > n^*$ (where $n^*$ is specified as in Lemma 3), when $n > n^*$, it holds
\[ \|x^n - x^*\|_2 \leq c^* \eta_\mu. \]

By Lemma 3, when $n > n^*$, it holds $\text{Supp}(x^n) = \text{Supp}(x^*)$ and $\text{sign}(x^n) = \text{sign}(x^*)$, and thus $\|x^n - x^*\|_2 = \|x^n - x^*\|_2$. Let $u^n = x^n - \mu \nabla F(x^n)$ and $u^* = x^* - \mu \nabla F(x^*)$.

On one hand, by Property 3, for any $i \in I$, $u^n_i - u^*_i = (x^n_i - x^*_i) + \lambda \mu \phi''(x^n_i) - \phi''(x^*_i)$. By Taylor expansion, for any $i \in I$, there exists an $\xi_i \in (0, 1)$, such that
\[ \phi''(x^n_i) - \phi''(x^*_i) = \phi'((x^n_i + \xi_i(x^n_i - x^*_i))) - \phi'(x^*_i) \]
\[ = \frac{1}{2} \phi'''(x^n_i + \xi_i(x^n_i - x^*_i))(x^n_i - x^*_i)^2, \]
where $x^n_i - x^*_i = (x^n_i - x^*_i) + \xi_i(x^n_i - x^*_i)$. Thus, we have
\[ u^n_i - u^*_i = (1 + \lambda \mu \phi''(x^n_i))(x^n_i - x^*_i) \]
\[ + \frac{1}{2} \phi''(x^n_i) \lambda \mu \phi''(x^n_i)(x^n_i - x^*_i)^2. \]

On the other hand, for any $i \in I$, $u^n_i - u^*_i = (x^n_i - x^*_i) - \mu(\nabla F(x^n)_i - \nabla F(x^*)_i)$. Therefore, for any $i \in I$, we have
\[ (1 + \lambda \mu \phi''(x^n_i))(x^n_i - x^*_i) \]
\[ + \frac{1}{2} \phi''(x^n_i) \lambda \mu \phi''(x^n_i)(x^n_i - x^*_i)^2 \]
\[ = (x^n_i - x^*_i) - \mu(\nabla F(x^n)_i - \nabla F(x^*)_i). \]

Let $h^n = x^n - x^*$. Let $\Lambda_1$ and $\Lambda_2$ be two different diagonal matrices with
\[ \Lambda_1(i, i) = 1 + \lambda \mu \phi''(x^n_i), \Lambda_2(i, i) = \frac{1}{2} \phi''(x^n_i) \lambda \mu \phi''(x^n_i) \]
for $i \in I$. By the assumptions of Theorem 4, for any $i \in I$,
\[ \Lambda_1(i, i) = 1 + \lambda \mu \phi''(x^n_i) \geq 1 + \lambda \mu \phi''(e) \]
\[ > 1 - \mu \sigma_{\min}(\nabla_{II} F(x^*)) \geq 0, \]
thus, $\Lambda_1$ is invertible. With the definitions of $\Lambda_1$ and $\Lambda_2$, then (74) becomes
\[ \Lambda_1 h^{n+1}_i + \Lambda_2(h^{n+1}_i \circ h^{n+1}_i) = h^n_i - \mu(\nabla F(x^n)_i - \nabla F(x^*)_i), \]
where $\circ$ denotes the Hadamard product or elementwise product. Moreover, by Taylor expansion,
\[ \nabla F(x^n) - \nabla F(x^*) = \nabla^2 F(x^*)(x^n - x^*) + o(\|x^n - x^*\|^2). \]
Consequently,
\[ \nabla F(x^n)_i - \nabla F(x^*)_i = \nabla^2 F(x^*)(x^n_i - x^*_i) + o(h^n_i). \]
Plugging (77) into (76), it becomes
\[ \Lambda_1 h^{n+1}_i + \Lambda_2(h^{n+1}_i \circ h^{n+1}_i) = h^n_i - \mu(\nabla F(x^n)_i - \nabla F(x^*)_i), \]
where $I$ denotes the identity matrix with the size $|I| \times |I|$ with $|I|$ being the cardinality of the set $I$. Furthermore, since $\Lambda_1$ is invertible, then
\[ h^{n+1}_i = \Lambda_1^{-1}(I - \mu \nabla_{II} F(x^*))h^n_i - \Lambda_1^{-1} \Lambda_2(h^{n+1}_i \circ h^{n+1}_i) + o(\|h^n_i\|^2). \]
By the definition of $o(\|h^n_i\|_2)$, there exists a constant $c^*$ (depending on $e$) such that
\[ |o(\|h^n_i\|_2)| \leq \epsilon \|h^n_i\|_2 \]
when $\|h^n_i\|_2 < c^* \eta_\mu$. Thus, we can take $c_0 = \min\{c^*, c^*_\epsilon\}$ and $n_0 > n^*$ such that when $n > n_0$,
\[ \|x^n - x^*\|_2 < c^* \eta_\mu. \]
Then (78) implies that
\[ \|h^{n+1}_i\|_2 \leq \|\Lambda_1^{-1}(I - \mu \nabla_{II} F(x^*))h^n_i\|_2 \]
\[ + \epsilon \|h^n_i\|_2 + \|\Lambda_1^{-1} \Lambda_2(h^{n+1}_i \circ h^{n+1}_i)\|_2 \]
\[ \leq \|\Lambda_1^{-1}(I - \mu \nabla_{II} F(x^*))\|_2 \|h^n_i\|_2 \]
\[ + \epsilon \|h^n_i\|_2 + g(e) \|h^{n+1}_i\|^2 \]
\[ \leq \left( 1 - \mu \sigma_{\min}(\nabla_{II} F(x^*)) + \epsilon \right) \|h^n_i\|_2 \]
\[ + g(e) \|h^{n+1}_i\|^2 \]
\[ \leq (c_1 + \epsilon) \|h^n_i\|_2 + g(e) c^* \eta_\mu \|h^{n+1}_i\|^2, \]
where the second inequality holds for the definition of $g(e)$ as specified in (71) and $c^* \geq c_0$, the third inequality holds for $\sigma_{\max}(I - \mu \nabla_{II} F(x^*)) \leq 1 - \mu \sigma_{\min}(\nabla_{II} F(x^*))$ and $\min_{i \in I} |\Lambda_1(i, i)| \geq 1 + \lambda \mu \phi''(e) > 0$, the last inequality holds for $\|h^{n+1}_i\|_2 < c^*_\epsilon \eta_\mu$ and the definition of $c_1$ as specified in (69). Furthermore, by (72) and (73), it holds
\[ 1 - c^* g(c^*_\epsilon) \eta_\mu > c_1 + \epsilon > 0. \]
Therefore, it implies that
\[ \| h_{l+1} \|_2 \leq \frac{c_1 + \epsilon}{1 - c^* g(c^*) \eta_{l+1}} \| h_{l} \|_2, \]
and then
\[ \| x_{n+1} - x^* \|_2 \leq \frac{c_1 + \epsilon}{1 - c^* g(c^*) \eta_{l+1}} \| x_{n} - x^* \|_2. \]

Let
\[ \rho = \frac{c_1 + \epsilon}{1 - c^* g(c^*) \eta_{l+1}}. \]
Then
\[ 0 < \rho < 1. \]

Thus, the asymptotic convergence rate of IJT algorithm is linear.

Moreover, the error bound can be easily derived by the asymptotic convergence rate and the triangle inequality.

V. APPLICATION TO \( l_q \) REGULARIZATION (\( 0 < q < 1 \))

In this section, we apply the established theoretical results to a typical case, \( l_q \) regularization with \( 0 < q < 1 \).

Mathematically, \( l_q \) (\( 0 < q < 1 \)) regularization can be formulated as follows
\[ \min_{x \in \mathbb{R}^N} \left\{ T_h(x) = \frac{1}{2} \| Ax - y \|_2^2 + \lambda \| x \|_q \right\}, \]
where \( A \in \mathbb{R}^{M \times N} \) (commonly, \( M < N \)) is usually called the sensing matrix, \( y \in \mathbb{R}^M \) is called the measurements, \( x \) is commonly assumed to be sparse, i.e., \( \| x \|_0 \leq N \), and \( \| x \|_q = \sum_{i=1}^{N} |x_i|^q \) is commonly called the \( l_q \)-norm. Thus, in such a special case, \( F(x) = \frac{1}{2} \| Ax - y \|_2^2 \) and \( \Phi(x) = \| x \|_q \) with \( \phi(x) = x^q \) defined on \((0, \infty)\). In [29], Bredies and Lorenz demonstrated that the one-dimensional proximity operator \( \text{prox}_{\lambda \mu, q} \) of \( l_q \)-norm can be expressed as
\[ \text{prox}_{\lambda \mu, q}(z) = \begin{cases} (\cdot + \lambda \mu q \text{sign}(\cdot) \cdot |q-1|^{-1})(z), & \text{for } |z| \geq \tau_{\mu, q} \\ 0, & \text{for } |z| \leq \tau_{\mu, q} \end{cases} \]
for any \( z \in \mathbb{R} \) with
\[ \tau_{\mu, q} = \frac{2 - q}{2 - q} (2 \lambda \mu (1 - q))^{\frac{1}{2-q}}, \]
\[ \eta_{\mu, q} = (2 \lambda (1 - q))^{\frac{1}{2-q}}, \]
and the range domain of \( \text{prox}_{\lambda \mu, q} \) is \( \{0\} \cup [\eta_{\mu, q}, \infty) \). Furthermore, for some special \( q \) (say, \( q = 1/2, 2/3 \)), the corresponding proximity operators can be expressed analytically [15], [25].

The convergence of the iterative jumping thresholding algorithm for \( l_q \) regularization (also fits the framework of the inexact forward-backward splitting algorithm when applied to \( l_q \) regularization) was justified in [32] based on the Kurdyka-Łojasiewicz (KL) inequality [33], [34]. As a special case of Theorem 5.1 in [32], we state the convergence of IJT algorithm for \( l_q \) regularization as the following corollary.

**Corollary 1.** Let \( \{x^n\} \) be a sequence generated by IJT algorithm for \( l_q \) regularization with \( q \in (0, 1) \). Assume that
\[ 0 < \mu < \frac{1}{\| A \|_2^2}, \]
then \( \{x^n\} \) converges to a stationary point of \( l_q \) regularization.

Furthermore, it is easy to check that \( F(x) = \frac{1}{2} \| Ax - y \|_2^2 \) and \( \phi(z) = z^q \) satisfy the Assumption 1 and 2, respectively. In addition, \( \phi(z) = z^q \) also satisfies the condition (c) in Theorem 4 naturally. Therefore, we present the following corollary to demonstrate the asymptotically linear convergence rate of IJT algorithm for \( l_q \) regularization.

**Corollary 2.** Assume that \( 0 < \mu < \| A \|_2^{-2} \). Let \( \{x^n\} \) be a sequence generated by IJT algorithm for \( l_q \) (\( 0 < q < 1 \)) regularization and converge to \( x^* \). Let \( I = \text{Supp}(x^*) \) and \( e = \min_{i \in I} |x_i^*| \). Moreover, if the following conditions hold:
(a) \( \sigma_{\min}(A_i^q A) > 0 \);
(b) \( 0 < \lambda < \frac{\sigma_{\min}(A_i^q A)^{2-q}}{q(1-q)} \),
then there exists a sufficiently large positive integer \( n_0 \) and a constant \( \rho \in (0, 1) \) such that when \( n > n_0 \),
\[ \| x_{n+1} - x^* \|_2 \leq \rho \| x_n - x^* \|_2, \]
and
\[ \| x_{n+1} - x^* \|_2 \leq \frac{\rho}{1 - \rho} \| x_{n+1} - x^* \|_2. \]

In addition, \( x^* \) is also a local minimizer of \( l_q \) regularization.

**Remark 4.** In a recent paper [35], Zeng et al. have studied the convergence of a specific iterative thresholding algorithm called the iterative half thresholding algorithm for \( l_{1/2} \) regularization. It can be observed that the convergence rate of the iterative half thresholding algorithm obtained in [35] is just a special case of the results presented in this section.

**Remark 5.** Although the \( l_q \)-norm does not satisfy Assumption 2, it can be observed that the finite support and sign convergence property (i.e., Lemma 3) holds naturally for hard algorithm due to the hard thresholding function possesses the similar discontinuity of the jumping thresholding function. Furthermore, once the support of the sequence generated by hard algorithm converges, the iterative form of hard algorithm is equal to the simple Landweber iteration, and thus the convergence and asymptotically linear convergence rate of hard algorithm can be directly claimed.

VI. CONCLUSION

We have conducted a study of the convergence of the iterative jumping thresholding (IJT) algorithm for certain a class of non-convex regularized optimization problems. One of the most significant features of such a class of iterative thresholding algorithms is that the associated thresholding functions are discontinuous with jump discontinuities. Moreover, the corresponding thresholding functions are not nonexpansive due to the nonconvexity of the penalty functions. Among such class of non-convex optimization problems, the \( l_q \) (\( 0 < q < 1 \)) regularization problem is one of the most typical subclass considered in this paper.

The main contribution of this paper is the establishment of the convergence and rate-of-convergence results of IJT algorithm for certain class of non-convex optimization problems. In summary, we have verified that
(a) the IJT algorithm converges to a stationary point for almost all the possibilities of the regularization parameter.
λ except some “bad” choices. The number of such “bad” choices is less than $N$.

(b) the IJT algorithm converges to a local minimizer of the optimization problem under certain conditions;

(c) the convergence speed of IJT algorithm is eventually linear;

(d) when applied to the $l_q$ ($0 < q < 1$) regularization, the IJT algorithm can converge to a local minimizer with an asymptotically linear convergence rate. In a loose circumstance, i.e., $\mu \in (0, \|A\|^{-2}_2)$.

The obtained convergence results to a local minimizer generalize those known for the soft and hard algorithms.

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