Vanishing of $L^2$-Betti numbers and failure of acylindrical hyperbolicity of matrix groups over rings

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Let $R$ be an infinite commutative ring with identity and $n \geq 2$ be an integer. We prove that for each integer $i = 0, 1, \ldots, n-2$, the $L^2$-Betti number $b^{(2)}_i(G) = 0$, when $G = \text{GL}_n(R)$ the general linear group, $\text{SL}_n(R)$ the special linear group, $E_n(R)$ the group generated by elementary matrices. When $R$ is an infinite principal ideal domain, similar results are obtained for $\text{Sp}_{2n}(R)$ the symplectic group, $\text{ESp}_{2n}(R)$ the elementary symplectic group, $\text{O}(n, n)(R)$ the split orthogonal group or $\text{EO}(n, n)(R)$ the elementary orthogonal group. Furthermore, we prove that $G$ is not acylindrically hyperbolic if $n \geq 4$. We also prove similar results for a class of noncommutative rings. The proofs are based on a notion of $n$-rigid rings.

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1 Introduction

In this article, we study the $s$-normality of subgroups of matrix groups over rings together with two applications. Firstly, the low-dimensional $L^2$-Betti numbers of matrix groups are proved to be zero. Secondly, the matrix groups are proved to be not acylindrically hyperbolic in the sense of Dahmani–Guirardel–Osin [6] and Osin [17]. Let us briefly review the relevant background.

Let $G$ be a discrete group. Denote by

$$\hat{L}(G) = \{ f : G \to \mathbb{C} \mid \sum_{g \in G} \| f(g) \|^2 < +\infty \}$$

the Hilbert space with inner product $\langle f_1, f_2 \rangle = \sum_{x \in G} f_1(x) \overline{f_2(x)}$. Let $B(\hat{L}(G))$ be the set of all bounded linear operators on the Hilbert space $\hat{L}(G)$. By definition, the group von Neumann algebra $\mathcal{N}G$ is the completion of the complex group ring $\mathbb{C}[G]$ in $B(\hat{L}(G))$ with respect to the weak operator topology. There is a continuous, additive von Neumann dimension that assigns to every right $\mathcal{N}G$-module $M$ a value $\text{dim}_{\mathcal{N}G}(M) \in [0, \infty]$ (see Definition 6.20 of [14]). For a group $G$, let $EG$ be the universal covering space of its classifying space $BG$. Denote by $c^{\text{sing}}_{\ast}(EG)$ the singular
chain complex of $EG$ with the induced $\mathbb{Z}G$-structure. The $L^2$-homology is the singular homology $H^G_*(EG;N^G)$ with coefficients $N^G$, i.e. the homology of the $N^G$-chain complex $N^G \otimes_{\mathbb{Z}G} C^\text{sing}_*(EG)$. The $i$-th $L^2$-Betti number of $G$ is defined by

$$b_i^{(2)}(G) := \dim_{N^G}(H^G_i(EG;N^G)) \in [0, \infty].$$

The $L^2$-homology and $L^2$-Betti numbers are important invariants of spaces and groups. They have many applications to geometry and $K$-theory. For more details, see the book [14].

It has been proved that the $L^2$-Betti numbers are (almost) zero for several class of groups, including amenable groups, Thompson’s group (cf. [14], Theorem 7.20), Baumslag-Solitar group (cf. [7, 1]), mapping class group of a closed surface with genus $g \geq 2$ except $b_{3g-3}^{(2)}$ (cf. [12], Corollary D.15) and so on (for more information, see [14], Chapter 7). Let $R$ be an associative ring with identity and $n \geq 2$ be an integer. The general linear group $\text{GL}_n(R)$ is the group of all $n \times n$ invertible matrices with entries in $R$. For an element $r \in R$ and any integers $i,j$ such that $1 \leq i \neq j \leq n$, denote by $e_{ij}(r)$ the elementary $n \times n$ matrix with 1s in the diagonal positions and $r$ in the $(i,j)$-th position and zeros elsewhere. The group $\text{E}_n(R)$ is generated by all such $e_{ij}(r)$, i.e.

$$E_n(R) = \langle e_{ij}(r) | 1 \leq i \neq j \leq n, r \in R \rangle.$$

When $R$ is commutative, we define the special linear group $\text{SL}_n(R)$ as the subgroup of $\text{GL}_n(R)$ consisting of matrices with determinants 1. For example in the case $R = \mathbb{Z}$, the integers, we have that $\text{SL}_n(\mathbb{Z}) = E_n(\mathbb{R})$. The groups $\text{GL}_n(R)$ and $E_n(R)$ are important in algebraic $K$-theory.

In this article, we prove the vanishing of lower $L^2$-Betti numbers for matrix groups over a large class of rings, including all infinite commutative rings. For this, we introduce the notion of $n$-rigid rings (for details, see Definition 3.1). Examples of $n$-rigid (for any $n \geq 1$) rings contain the following (cf. Section 3):

- infinite integral domains;
- $\mathbb{Z}$-torsionfree infinite noetherian ring (may be non-commutative);
- infinite commutative noetherian rings (moreover, any infinite commutative ring is 2-rigid);
- Finite-dimensional algebras over $n$-rigid rings.

We prove the following results.
Theorem 1.1 Suppose \( n \geq 2 \). Let \( R \) be an infinite \((n-1)\)-rigid ring and \( E_n(R) \) the group generated by elementary matrices. For each \( i \in \{0, \cdots, n-2\} \), the \( L^2 \)-Betti number \( b^{(2)}_{i}(E_n(R)) = 0 \).

Since \( b^{(2)}_{1}(E_2(\mathbb{Z})) \neq 0 \), the above result does not hold for \( i = n - 1 \) in general.

Corollary 1.2 Let \( R \) be any infinite commutative ring and \( n \geq 2 \). For each \( i \in \{0, \cdots, n-2\} \), the \( L^2 \)-Betti number

\[
b^{(2)}_{i}(\text{GL}_n(R)) = b^{(2)}_{i}(\text{SL}_n(R)) = b^{(2)}_{i}(E_n(R)) = 0.
\]

Let \( \text{SL}_n(R) \) be a lattice in a semisimple Lie group, e.g. when \( R = \mathbb{Z} \) or a subring of algebraic integers. It follows from results of Borel, which rely on global analysis on the associated symmetric space, that the \( L^2 \)-Betti numbers of \( \text{SL}_n(R) \) vanish except possibly in the middle dimension of the symmetric space (cf. [5, 16]). In particular, all the \( L^2 \)-Betti numbers of \( \text{SL}_n(\mathbb{Z}) \) (\( n \geq 3 \)) are zero (cf. [8], Example 2.5). For any infinite integral domain \( R \) and any \( i \in \{0, \cdots, n-2\} \), Bader-Furman-Sauer [1] proves that the \( L^2 \)-Betti number \( b^{(2)}_{i}(\text{SL}_n(R)) = 0 \). M. Ershov and A. Jaikin-Zapirain [9] prove that the noncommutative universal lattices \( E_n(\mathbb{Z}[x_1, \cdots, x_k]) \) (and therefore \( E_n(R) \) for any finitely generated associative ring \( R \)) has Kazhdan’s property (T) for \( n \geq 3 \). This implies that for any finitely generated associative ring \( R \), the first \( L^2 \)-Betti number of \( E_n(R) \) vanishes (cf. [4]).

We consider more matrix groups as follows. Let \( R \) be a commutative ring with identity. The symplectic group is defined as

\[
\text{Sp}_{2n}(R) = \{ A \in \text{GL}_{2n}(R) \mid A^T \varphi_n A = \varphi_n \},
\]

where \( A^T \) is the transpose of \( A \) and

\[
\varphi_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.
\]

Similarly, the split orthogonal group is defined as

\[
O(n, n)(R) = \{ A \in \text{GL}_{2n}(R) \mid A^T \psi_n A = \psi_n \}
\]

where

\[
\psi_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.
\]

For symplectic and orthogonal groups, we obtain the following.
**Theorem 1.3**  Let $R$ be an infinite principal ideal domain (PID) and $\text{Sp}_{2n}(R)$ the symplectic group with its elementary subgroup $E\text{Sp}_{2n}(R)$ (resp. $\text{O}(n,n)(R)$ the orthogonal group and its elementary subgroup $E\text{O}(n,n)(R)$). We have the following.

(i) For each $i = 0, \cdots, n-2$ ($n \geq 2$), the $L^2$-Betti number

$$b_i^{(2)}(\text{Sp}_{2n}(R)) = b_i^{(2)}(E\text{Sp}_{2n}(R)) = 0.$$

(ii) For each $i = 0, \cdots, n-2$ ($n \geq 2$), the $L^2$-Betti number

$$b_i^{(2)}(\text{O}(n,n)(R)) = b_i^{(2)}(E\text{O}(n,n)(R)) = 0.$$

The proofs of Theorem 1.1 and Theorem 1.3 are based on a study of the notion of weak normality of particular subgroups in matrix groups, introduced in [1] and [18]. We present another application of the weak normality of subgroups in matrix groups as follows.

Acylindrically hyperbolic groups are defined by Dahmani–Guirardel–Osin [6] and Osin [17]. Let $G$ be a group. An isometric $G$-action on a metric space $S$ is said to be acylindrical if for every $\varepsilon > 0$, there exist $R, N > 0$ such that for every two points $x, y \in S$ with $d(x, y) \geq R$, there are at most $N$ elements $g \in G$ which satisfy $d(x, gx) \leq \varepsilon$ and $d(y, gy) \leq \varepsilon$. A $G$-action by isometries on a hyperbolic geodesic space $S$ is said to be elementary if the limit set of $G$ on the Gromov boundary $\partial S$ contains at most 2 points. A group $G$ is called *acylindrically hyperbolic* if $G$ admits a non-elementary acylindrical action by isometries on a (Gromov-$\delta$) hyperbolic geodesic space. The class of acylindrically hyperbolic groups includes non-elementary hyperbolic and relatively hyperbolic groups, mapping class groups of closed surface $\Sigma_g$ of genus $g \geq 1$, outer automorphism group $\text{Out}(F_n)$ ($n \geq 2$) of free groups, directly indecomposable right angled Artin groups, 1-relator groups with at least 3 generators, most 3-manifold groups, and many other examples.

Although there are many analogies among matrix groups, mapping class groups and outer automorphism groups of free groups, we prove that they are different on acylindrical hyperbolicity, as follows.

**Theorem 1.4**  Suppose that $n$ is an integer.

(i) Let $R$ be a 2-rigid (eg. commutative) ring. The group $E_n(R)$ ($n \geq 3$) is not acylindrically hyperbolic.
(ii) Let \( R \) be a commutative ring. The group \( G \) is not acylindrically hyperbolic, if \( G = \text{GL}_n(R) \) (\( n \geq 3 \)) the general linear group, \( \text{SL}_n(R) \) (\( n \geq 3 \)) the special linear group, \( \text{Sp}_{2n}(R) \) (\( n \geq 2 \)) the symplectic group, \( \text{ESp}_{2n}(R) \) (\( n \geq 2 \)) the elementary symplectic group, \( \text{O}(n,n)(R) \) (\( n \geq 4 \)) the orthogonal group, or \( \text{EO}(n,n)(R) \) (\( n \geq 4 \)) the elementary orthogonal group.

When \( R \) is commutative, the failure of acylindrical hyperbolicity of the elementary groups \( E_n(R) \), \( \text{ESp}_{2n}(R) \) and \( \text{EO}(n,n)(R) \) is already known to Mimura [15] by studying property TT for weakly mixing representations. But our approach is different and Theorem 1.4 is more general, even for elementary subgroups. Explicitly, for noncommutative rings we have the following.

**Corollary 1.5** Let \( R \) be a noncommutative \( \mathbb{Z} \)-torsionfree infinite noetherian ring, integral group ring over a polycyclic-by-finite group or their finite-dimensional algebra. For each nonnegative integer \( i \leq n - 2 \), we have
\[
b_i^{(2)}(E_n(R)) = 0.
\]
Furthermore, the group \( E_n(R) \) (\( n \geq 3 \)) is not acylindrically hyperbolic.

## 2 \( s \)-normality

Recall from [1] that the \( n \)-step \( s \)-normality is defined as follows.

**Definition 2.1** Let \( n \geq 1 \) be an integer. A subgroup \( H \) of a group \( G \) is called \( n \)-step \( s \)-normal if for any \( (n + 1) \)-tuple \( \omega = (g_0, g_1, \ldots, g_n) \in G^{n+1} \), the intersection
\[
H^{\omega} := \cap_{i=0}^{n} g_i H g_i^{-1}
\]
is infinite. A 1-step \( s \)-normal group is simply called \( s \)-normal.

The following result is proved by Bader, Furman and Sauer (cf. [1], Theorem 1.3).

**Lemma 2.2** Let \( H \) be a subgroup of \( G \). Assume that
\[
b_i^{(2)}(H^{\omega}) = 0
\]
for all integers \( i, k \geq 0 \) with \( i + k \leq n \) and every \( \omega \in G^{k+1} \). In particular, \( H \) is an \( n \)-step \( s \)-normal subgroup of \( G \). Then
\[
b_i^{(2)}(G) = 0
\]
for every \( i \in \{0, \ldots, n\} \).
The following result is important for our later arguments (cf. [14], Theorem 7.2, (1-2), p.294).

**Lemma 2.3** Let $n$ be any non-negative integers. Then

(i) For any infinite amenable group $G$, the $L^2$-Betti numbers $b^{(2)}_n(G) = 0$.

(ii) Let $H$ be a normal subgroup of a group $G$ with vanishing $b^{(2)}_i(H) = 0$ for each $i \in \{0, 1, \ldots, n\}$. Then for each $i \in \{0, 1, \ldots, n\}$, we have $b^{(2)}_i(G) = 0$.

We will also need the following fact (cf. [17], Corollary 1.5, Corollary 7.3).

**Lemma 2.4** The class of acylindrically hyperbolic groups is closed under taking $s$-normal subgroups. Furthermore, the center of an acylindrically hyperbolic group is finite.

### 3 Rigidity of rings

We introduce the notion of $n$-rigidity of rings. For a ring, all $R$-modules are right modules and homomorphisms are right $R$-module homomorphisms.

**Definition 3.1** For a positive integer $n$, an infinite ring $R$ is called $n$-rigid if every $R$-homomorphism $R^n \to R^{n-1}$ of the free modules has an infinite kernel.

A related concept is the strong rank condition: a ring $R$ satisfies the strong rank condition if there is no injection $R^n \to R^{n-1}$ for any $n$ (see Lam ([13], p.12). Clearly, $n$-rigidity for any $n$ implies the strong rank condition for a ring. Fixing the standard basis of both $R^n$ and $R^{n-1}$, the kernel of an $R$-homomorphism $\phi : R^n \to R^{n-1}$ corresponds to a system of $n - 1$ linear equations with $n$ unknowns over $R$:

$$S: \sum_{1 \leq i \leq n} a_{ij}x_i = 0, \quad 1 \leq j \leq n - 1,$$

with $a_{ij} \in R$, $1 \leq i \leq n$, $1 \leq j \leq n - 1$. Therefore, the strong rank condition asserts that the system $S$ has non-trivial solutions over $R$, while the $n$-rigidity property requires that $S$ has infinitely many solutions.

Many rings are $n$-rigid. For example, infinite integral rings are $n$-rigid for any $n$ by considering the dimensions over quotient fields. Moreover, let $A$ be a ring satisfying the strong rank condition (e.g. noetherian ring, cf. Theorem 3.15 of [13]). Suppose that $A$ is a torsion-free $\mathbb{Z}$-module, where $\mathbb{Z}$ acts on $A$ via $\mathbb{Z} \cdot 1_A$. Since the kernel $A^n \to A^{n-1}$ is a nontrivial $\mathbb{Z}$-module, the ring $A$ is $n$-rigid for any $n$.

We present several basic facts on $n$-rigid rings as follows.
Lemma 3.2 \( n \)-rigid implies \((n - 1)\)-rigid.

Proof For any \( R \)-homomorphism \( f : R^{n-1} \to R^{n-2} \), we could add a copy of \( R \) as direct summand to get a map \( f \oplus \text{id} : R^{n-1} \oplus R \to R^{n-2} \oplus R \). The two maps have the same kernel.

Lemma 3.3 Let \( R \) be an \( n \)-rigid ring for any \( n \geq 1 \). Suppose that an associative ring \( A \) is a finite-dimensional \( R \)-algebra (i.e. \( A \) is a free \( R \)-module of finite rank with compatible multiplications in \( A \) and \( R \)). Then \( A \) is \( n \)-rigid for any \( n \geq 1 \).

Proof Let \( f : A^n \to A^{n-1} \) be an \( A \)-homomorphism. Viewing \( A \) as a finite-dimensional \( R \)-module, we see that \( f \) is also an \( R \)-homomorphism. Embed the target \( A^{n-1} \) into \( R \cdot \text{rank}_R(A) - 1 \). The kernel \( \ker f \) is infinite by the assumption that \( R \) is \( n \cdot \text{rank}_R(A) \)-rigid.

Proposition 3.4 Let \( R \) be an \( n \)-rigid ring and \( u_1, u_2, \cdots, u_{n-1} \in R^m \) (\( m \geq n \)) be arbitrary \( n - 1 \) elements. Then the set
\[
\{ \phi \in \text{Hom}_R(R^m, R) \mid \phi(u_i) = 0, i = 1, 2, \cdots, n - 1 \}
\]
is infinite.

Proof When \( m = n \), we define an \( R \)-homomorphism
\[
\text{Hom}_R(R^n, R) \to R^{n-1}
\]
\[
f \mapsto (f(u_1), f(u_2), \cdots, f(u_{n-1})).
\]
Since \( \text{Hom}_R(R^n, R) \) is isomorphic to \( R^n \), such an \( R \)-homomorphism has an infinite kernel. When \( m > n \), we may project \( R^m \) to its last \( n \)-components and apply a similar proof.

Lemma 3.5 An infinite commutative ring \( R \) is 2-rigid.

Proof Let \( f : R^2 \to R \) be any \( R \)-homomorphism. Denote by
\[
I = \langle xR + yR \mid (x, y) \in \ker f \rangle \trianglelefteq R.
\]
Suppose that \( \ker f \) is finite. When \( (x, y) \in \ker f \), the set \( xR \) and \( yR \) are also finite. Thus \( I \) is finite. Denote by \( a = f((1, 0)) \) and \( b = f((0, 1)) \). Note that \(( -b, a ) \in \ker f \).
For any \( (x, y) \in R^2 \), we have \( ax + by \in I \). Since the set of right cosets \( R/I \) is infinite, we may choose \( (x, x) \) and \( (y, y) \) with \( x, y \) from distinct cosets such that
\[
ax + bx = ay + by.
\]
However, \( (x - y, x - y) \in \ker f \) and thus \( x - y \in I \). This is a contradiction.
To state our result in the most general form, we introduce the following notion.

**Definition 3.6** A ring $R$ is called size-balanced if any finite right ideal of $R$ generates a finite two sided ideal of $R$.

It is immediate that any commutative ring is size-balanced.

**Proposition 3.7** A size-balanced infinite noetherian ring is $n$-rigid for any $n$.

**Proof** Let $f : R^n \to R^{n-1}$ be any $R$-homomorphism. Let $A = (a_{ij})_{(n-1) \times n}$ be the matrix representation of $f$ with respect to the standard basis. Denote by

$$I' = \langle x_1R + x_2R + \cdots + x_nR \mid (x_1, x_2, \cdots, x_n) \in \ker f \rangle \subseteq R.$$ 

First we notice that $I'$ is non-trivial by the strong rank condition of noetherian rings (cf. Theorem 3.15 of [13]). Suppose that $\ker f$ is finite. For any $(x_1, x_2, \cdots, x_n) \in \ker f$ and $r \in R$, each $(x_1r, x_2r, \cdots, x_nr) \in \ker f$. As $\ker f$ is finite, each right ideal $x_iR$ is finite; and hence so is $I'$. Let $I$ be the two sided ideal generated by the finite right ideal $I'$. It is finite as $R$ is assumed to be size-balanced. Therefore, the quotient ring $R/I$ is infinite and noetherian.

Let $\tilde{f} : R/I \to R/I$ be the $R/I$-homomorphism induced by the matrix $\tilde{A} = (\tilde{a}_{ij})$, where $\tilde{a}_{ij}$ is the image of $a_{ij}$. If $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in \ker \tilde{f}$ and $x_i$ is any pre-image of $\tilde{x}_i$, we have

$$A(x_1, x_2, \cdots, x_n)^T \in I^{n-1}.$$ 

As $I$ is finite, so is $I^{n-1}$. If $\ker \tilde{f}$ is infinite, there are two distinct elements in $\ker \tilde{f}$ with pre-image $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ in $R^n$ such that

$$A(x_1, x_2, \cdots, x_n)^T = A(y_1, y_2, \cdots, y_n)^T \in I^{n-1}.$$ 

However, this implies that

$$(x_1, x_2, \cdots, x_n) - (y_1, y_2, \cdots, y_n) \in \ker f.$$ 

We have a contradiction as $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ are distinct in $(R/I)^n$. Therefore, $\ker \tilde{f}$ is finite. Moreover, $\ker \tilde{f}$ is non-trivial by the strong rank condition of noetherian rings. Let $I'_1$ be the pre-image of the right ideal generated by components of elements in $\ker \tilde{f}$ in $R$, which is a finite right ideal by a similar argument as above. It generates a finite two sided ideal $I_1$ of $R$, and it properly contains $I$.

Repeating the argument, we get an infinite ascending sequence

$$I \leq I_1 \leq I_2 \leq \cdots$$

of finite ideals of $R$. This is a contradiction to the assumption that $R$ is noetherian. □
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**Corollary 3.8** Any commutative ring $R$ containing an infinite noetherian subring is $n$-rigid for each $n$.

**Proof** Let $R_0$ be an infinite noetherian subring of $R$. Let

$$S: \sum_{1 \leq i \leq n} a_{ij}x_i = 0, 1 \leq j \leq m$$

be a system of linear equations with $a_{ij} \in R$. Form the infinite commutative subring $R'$ of $R$: $R' = R_0[a_{ij}, 1 \leq i \leq n, 1 \leq j \leq m]$. By the Hilbert basis theorem, $R'$ is infinite noetherian. Proposition 3.7 asserts that the system $S$ has infinitely many solutions in $R'$, and hence in $R$. \qed

**Example 3.9** Let $G$ be a polycyclic-by-finite group and $R = \mathbb{Z}[G]$ be its integral group ring. It is known that $R$ is infinite noetherian (see [11]). Moreover, $R$ is size-balanced by the trivial reason that there are no non-trivial finite right ideals. According to Proposition 3.7, the ring $R$ is $n$-rigid for any $n$.

**Example 3.10** Let $F$ be a nonabelian free group and $\mathbb{Z}[F]$ the group ring. Since $\mathbb{Z}[F]$ does not satisfy the strong rank condition (cf. [13], Exercise 29, p.21.), the ring $\mathbb{Z}[F]$ is not $n$-rigid for any $n \geq 2$.

## 4 Proofs

Let

$$Q = \left\{ \begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} \mid x \in R^{n-1}, A \in \text{GL}_{n-1}(R), \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in E_n(R) \right\}.$$ 

It is straightforward that $Q$ contains the normal subgroup

$$S = \left\{ \begin{pmatrix} 1 & x \\ 0 & I_{n-1} \end{pmatrix} \mid x \in R^{n-1} \right\},$$

an abelian group. Therefore, all the $L^2$-Betti numbers of $S$ and $Q$ are zero when the ring $R$ is infinite.

**Lemma 4.1** Let $k < n$ ($n \geq 3$) be two positive integers. Suppose that $R$ is an infinite $k$-rigid ring. The subgroup $Q$ is $(k - 1)$-step $s$-normal in $E_n(R)$. In particular, $Q$ is $s$-normal if $R$ is infinite 2-rigid.
By Proposition 3.4, $\Phi$ is infinite and thus $T$ is infinite. The proof is finished.

Lemma 4.2 In the proof of Lemma 4.1, the subgroup $T$ is normal in $Q \cap_{i=1}^{n-2} g_i Q g_i^{-1}$.

Proof For any $\phi$, write $e_\phi = (\phi(e_2), \ldots, \phi(e_n))$. With respect to the standard basis, the representation matrix of the transformation $T_\phi$ is $\begin{pmatrix} 1 & e_\phi \\ 0 & I_{n-1} \end{pmatrix}$. For any $\begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} \in Q \cap_{i=1}^{n-2} g_i Q g_i^{-1}$, the conjugate

$$
\begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix}^{-1} \begin{pmatrix} 1 & e_\phi \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & e_\phi A \\ 0 & I_{n-1} \end{pmatrix}.
$$

Define $\psi : U = R^{n-1} \to R$ by $\psi(x) = e_\phi Ax$. For each $i = 1, \ldots, n-2$, we have that

$$
\begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} g_i q_i g_i^{-1} = g_i q_i e_i
$$

for some $q_i \in Q$. Therefore,

$$
\begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} g_i e_i = g_i q_i e_i
$$

and $Au_i = u_i$. This implies that $\psi(u_i) = e_\phi u_i = 0$ for each $i$ and thus $\psi \in \Phi$.

Therefore, the conjugate $\begin{pmatrix} 1 & e_\phi A \\ 0 & I_{n-1} \end{pmatrix}$ lies in $T$, which proves that $T$ is normal.
Proof of Theorem 1.1  By Lemma 4.2, any intersection \( Q \cap_{i=1}^{n-2} g_i Q g_i^{-1} \) contains an infinite normal amenable subgroup \( T \). Therefore, all the \( L^2 \)-Betti numbers of any intersection \( Q \cap_{i=1}^{k} g_i Q g_i^{-1} \) are vanishing for \( k \leq n - 2 \) considering Lemma 3.2. We have that \( b_i(E_n(R)) = 0 \) for any \( 0 \leq i \leq n - 2 \) by Lemma 2.2.

\[ \]

Proof of Corollary 1.2 When \( n = 2 \), it is clear that both \( \text{GL}_n(R) \) and \( \text{SL}_n(R) \) are infinite, since \( E_2(R) \) is an infinite subgroup. Thus \( b_0^{(2)}(\text{GL}_2(R)) = b_0^{(2)}(\text{SL}_2(R)) = 0 \). We have already proved that \( b_1^{(2)}(E_n(S)) = 0 \) for infinite commutative noetherian ring \( S \) and \( 0 \leq i \leq n - 2 \), since the ring \( S \) would be \( k \)-rigid for any integer \( k \) by Proposition 3.7. If \( S \) is a finite subring of \( R \), the group \( E_n(S) \) is also finite. Therefore, we still have \( b_i^{(2)}(E_n(S)) = 0 \) for \( 1 \leq i \leq n - 2 \). Note that every commutative ring \( R \) is the directed colimit of its subrings \( S \) that are finitely generated as \( \mathbb{Z} \)-algebras (noetherian rings by Hilbert basis theorem). Since the group \( E_n(R) \) is the union of the directed system of subgroups \( E_n(S) \), we get that

\[ b_1^{(2)}(E_n(R)) = 0 \]

for \( 0 \leq i \leq n - 2 \) (cf. [14], Theorem 7.2 (3) and its proof). When \( R \) is commutative and \( n \geq 3 \), a result of Suslin says that the group \( E_n(R) \) is a normal subgroup of \( \text{GL}_n(R) \) and \( \text{SL}_n(R) \) (cf. [19]). Lemma 2.3 implies that \( b_i^{(2)}(\text{GL}_n(R)) = b_i^{(2)}(\text{SL}_n(R)) = 0 \) for each \( i \in \{0, \ldots, n-2\} \). 

We follow [2] to define the elementary subgroups of symplectic groups and orthogonal groups. Let \( E_{ij} \) denote the \( n \times n \) matrix with 1 in the \((i,j)\)-th position and zeros elsewhere. Then for \( i \neq j \), the matrix \( e_{ij}(a) = I_n + aE_{ij} \) is an elementary matrix, where \( I_n \) is the identity matrix of size \( n \). With \( n \) fixed, for any integer \( 1 \leq k \leq 2n \), set \( \sigma k = k + n \) if \( k \leq n \) and \( \sigma k = k - n \) if \( k > n \). For \( a \in R \) and \( 1 \leq i \neq j \leq 2n \), we define the elementary unitary matrices \( \rho_{i,\sigma i}(a) \) and \( \rho_{j}(a) \) with \( j \neq \sigma i \) as follows:

- \( \rho_{i,\sigma i}(a) = I_{2n} + aE_{i,\sigma i} \) with \( a \in R \);
- Fix \( \varepsilon = \pm 1 \). We define \( \rho_{j}(a) = \rho_{\sigma j,\sigma i}(-a') = I_{2n} + aE_{ij} - a'E_{\sigma j,\sigma i} \) with \( a' = a \) when \( i,j \leq n \); \( a' = \varepsilon a \) when \( i \leq n < j \); \( a' = a\varepsilon \) when \( j \leq n < i \); and \( a' = a \) when \( n + 1 \leq i,j \).

When \( \varepsilon = -1 \), we have the elementary symplectic group

\[ \text{ESp}_{2n}(R) = \langle \rho_{i,\sigma i}(a), \rho_{j}(a) \mid a \in R, i \neq j, i \neq \sigma j \rangle. \]
When $\varepsilon = 1$, we have the elementary orthogonal group

$$\text{EO}(n, n)(R) = \langle \rho_{ij}(a) \mid a \in R, i \neq j, i \neq \sigma j \rangle.$$  

Note that for the orthogonal group, each matrix $\rho_{i,\sigma j}(a)$ is not in $\text{EO}(n, n)(R)$.

There is an obvious embedding

$$\text{Sp}_{2n}(R) \to \text{Sp}_{2n+2}(R),$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}.$$  

Denote the image of $A \in \text{Sp}_{2n}(R)$ by $I \oplus A \in \text{Sp}_{2n+2}(R)$. Let

$$Q_1 = \langle (I \oplus A) \cdot \Pi_{i=1}^{2n} \rho_{i1}(a_i) \mid a_i \in R, A \in \text{Sp}_{2n-2}(R), I \oplus A \in \text{ESp}_{2n}(R) \rangle;$$

and

$$S_1 = \langle \Pi_{i=1}^{2n} \rho_{i1}(a_i) \mid a_i \in R \rangle.$$  

Similarly, we can define

$$Q_2 = \langle (I \oplus A) \cdot \Pi_{i=1, i \neq n+1}^{2n} \rho_{i1}(a_i) \mid a_i \in R, A \in \text{O}(2n-2, 2n-2)(R),$$

$$I \oplus A \in \text{EO}(n, n)(R) \rangle;$$

and

$$S_2 = \langle \Pi_{i=1, i \neq n+1}^{2n} \rho_{i1}(a_i) \mid a_i \in R \rangle.$$  

Since $S_i$ is abelian and normal in $Q_i$, all the $L^2$-Betti numbers of $Q_i$ vanish for $i = 1, 2$.

**Proof of Theorem 1.3** We prove the theorem by induction on $n$. When $n = 2$, both $\text{Sp}_{2n}(R)$ and $\text{O}(n, n)(R)$ are infinite and therefore we have

$$b^{(2)}_0(\text{Sp}_4(R)) = b^{(2)}_0(\text{O}(4, 4)(R)) = 0.$$  

The subgroup $\text{ESp}_{2n}(R)$ is normal in $\text{Sp}_{2n}(R)$ when $n \geq 2$ and the subgroup $\text{EO}(n, n)(R)$ is normal in $\text{O}(n, n)(R)$ when $n \geq 3$ (cf. [3], Cor. 3.10). It suffices to prove the vanishing of Betti numbers for $G = \text{ESp}_{2n}(R)$ and $\text{EO}(n, n)(R)$.

We check the condition of Lemma 2.2 for $Q = Q_1$ (resp. $Q_2$) as follows. Note that

$$Q = \{ g \in G \mid ge_1 = e_1 \}.$$
Let $g_1, g_2, \ldots, g_k$ ($g_0 = 1_{2n}, k \leq n - 2$) be any $k$-elements in $G$ and
\[K = \langle g_0 e_1, g_1 e_1, \ldots, g_k e_1 \rangle\]
the submodule in $R^{2n}$ generated by all $g_i e_1$. Recall that the symplectic (resp. orthogonal) form $\langle -, - \rangle : R^{2n} \times R^{2n} \to R$ is defined by $\langle x, y \rangle = x^T \varphi_n y$ (resp. $\langle x, y \rangle = x^T \psi_n y$). Denote
\[C := \{ v \in R^{2n} | \langle v, g_i e_1 \rangle = 0 \text{ for each } i = 0, \ldots, k - 1 \} .\]
Let $\epsilon = -1$ for $\text{ESp}_{2n}(R)$ and 1 for $\text{EO}(n, n)(R)$. For each $r \in R$, set $\delta^r_\epsilon = r$ if $\epsilon = -1$ and $\delta^r_\epsilon = 0$ if $\epsilon = 1$. For each $u, v \in C$ with $\langle u, u \rangle = \langle u, v \rangle = \langle v, v \rangle = 0$, define the the transvections in $G$ (cf. [20], p.287, Eichler transformations in [10], p.214, p.223-224)
\[
\tau(u, v) : R^{2n} \to R^{2n} \text{ by } x \mapsto x + \epsilon u \langle v, x \rangle - v \langle u, x \rangle ,
\]
\[
\tau_{v, r} : R^{2n} \to R^{2n} \text{ by } x \mapsto x - \delta^r_\epsilon v \langle v, x \rangle .
\]
Note that $\tau_{v, r}$ is non-identity only in $\text{ESp}_{2n}(R)$. We have
\[\tau(u, v)(g_i e_1) = \tau_{v, r}(g_i e_1) = g_i e_1\]
for each $i$. Therefore, the transvections $\tau(u, v), \tau_{v, r} \in \cap_{i=0}^{k} g_i Q g_{i-1}^{-1}$. Let
\[T = \{ \tau(u, v), \tau_{v, r} | u, v \in C, \langle u, u \rangle = \langle u, v \rangle = \langle v, v \rangle = 0, r \in R \} ,\]
the subgroup generated by the transvections in $G$. For any $g \in \cap_{i=0}^{k} g_i Q g_{i-1}^{-1}$, we have $gg_i e_1 = g_i e_1$ and thus
\[\langle gu, g_i e_1 \rangle = \langle gu, gg_i e_1 \rangle = \langle u, g_i e_1 \rangle = 0 .\]
This implies that $g \tau(u, v) g^{-1} = \tau(gu, gv) \in T$ and $g \tau_{v, r} g^{-1} = \tau_{gv, r} \in T$. Therefore, the subgroup $T$ is a normal subgroup in $\cap_{i=0}^{k} g_i Q g_{i-1}^{-1}$.

When $R$ is a PID, both the submodule $K$ and the complement $C$ are free of smaller ranks.

**Case (i)** $K \cap C = 0$.

Since $R^{2n} = K \bigoplus C$ (note that each $g_i e_1$ is unimodular), the symplectic (resp. orthogonal) form on $R^{2n}$ restricts to a non-degenerate symplectic (resp. orthogonal) form on $C$. Let $T < G$ as defined before. It is known that the transvections generate the elementary subgroups (cf. [10], p.223-224) and thus $T \cong \text{ESp}_{2n}(R)$ (resp. $\text{EO}(m, m)(R)$) for $m = \text{rank}(C) \leq n - 2$. Since $k \leq n - 2$, we have $m \geq 4$. By induction,
\[b_2^{(2)}(\cap_{i=0}^{k} g_i Q g_{i-1}^{-1}) = b_2^{(2)}(T) = 0 .\]
implies that $1.3$ would hold for any general infinite commutative ring by a similar argument as the proof of Corollary 1.2.

Case (ii) $K \cap C \neq 0$.

For any $u, v \in K \cap C$ and any $g \in \cap_{i=0}^{k} Qg_{i}^{-1}$, we have that $gu = u, gv = v$ and

$$g\tau(u, v)g^{-1} = \tau(gu, gv) = \tau(u, v).$$

This implies that $\tau(u, v)$ lies in the center of $\cap_{i=0}^{k} Qg_{i}^{-1}$. Note that when $G = \text{ESp}_{2n}(R)$, the transvection $\tau(u, u)$ is not trivial for any $u \in K \cap C$. When $G = \text{EO}(n, n)(R)$ and rank$(K \cap C) \geq 2$, the transvection $\tau(u, v)$ is not trivial for any linearly independent $u, v \in K \cap C$. Moreover, for two elements $r, s$ with $r^2 \neq s^2$, we have $\tau(rs, rs) = \tau(su, sv)$ when $\tau(u, v) \neq I_{2n}$ (take note that for $G = \text{ESp}_{2n}(R)$, we can just let $u = v$ from above). The infinite PID $R$ contains infinitely many square elements. In summary, as $K \cap C$ is a free $R$-module, the subgroup

$$T' = \langle \tau(u, v) \mid v \in K \cap C \rangle < G$$

is an infinite abelian normal subgroup of $\cap_{i=0}^{k} Qg_{i}^{-1}$. Therefore,

$$b_{i}^{(2)}(\cap_{i=0}^{k} Qg_{i}^{-1}) = b_{i}^{(2)}(T') = 0$$

for each integer $s \geq 0$. Therefore, for any $i \leq n - 2$, we have that $b_{i}^{(2)}(G) = 0$ by Lemma 2.2.

The remaining situation is that $G = \text{EO}(n, n)(R)$ and rank$(K \cap C) = 1$. Choose the decomposition $C = (K \cap C) \bigoplus C_{1}$. The orthogonal form restricts to a non-degenerate orthogonal form on $C_{1}$ (suppose that for some $x \in C_{1}$ we have $\langle x, y \rangle = 0$ for any $y \in C_{1}$. Since $\langle x, k \rangle = 0$ for any $k \in K$, we know that $\langle x, y \rangle = 0$ for any $y \in C$. This implies $x \in K$, which gives $x = 0$). Since $k \leq n - 2$, the even number rank$(C_{1}) \geq 4$. A similar argument as case (i) finishes the proof.

\[ \square \]

**Remark 4.3** Let $T$ be the normal subgroup of $\cap_{i=0}^{k} Qg_{i}^{-1}$ constructed in the proof of Theorem 1.3. We do not know whether the $L^{2}$-Betti numbers $b_{i}^{(2)}(T) = 0$ for a general infinite $(2n - 1)$-rigid commutative ring $R$ when $i \leq n - 2 - k$. If yes, Theorem 1.3 would hold for any general infinite commutative ring by a similar argument as the proof of Corollary 1.2.
Proof of Theorem 1.4  Note that when $R$ is commutative, the elementary subgroups $E_n(R)$, $\text{ESP}_{2n}(R)$ and $\text{EO}(n, n)(R)$ are normal in $\text{SL}_n(R)$, $\text{Sp}_{2n}(R)$ and $O(n, n)(R)$, respectively (cf. [19], [3] Cor. 3.10). Therefore, it is enough to prove the failure of acylindrically hyperbolicity for elementary subgroups. We prove (i) first. If $R$ is finite, all the groups will be finite and thus not acylindrically hyperbolic. If $R$ is infinite, then it is 2-rigid and the subgroup $Q$ is $s$-normal by Lemma 4.1. Suppose that $E_n(R)$ is acylindrically hyperbolic. Lemma 2.4 implies that both $Q$ and $S$ are acylindrically hyperbolic. However, the subgroup $S$ is infinite abelian, which is a contradiction to the second part of Lemma 2.4.

For (ii), we may also assume that $R$ is infinite since any finite group is not acylindrically hyperbolic. It suffices to prove that $Q_1$ (resp. $Q_2$) is $s$-normal in $\text{ESP}_{2n}(R)$ (resp. $\text{EO}(n, n)(R)$). (Note that $Q_1$ and $Q_2$ contain the infinite normal subgroups $S_1$ and $S_2$, respectively. If $G$ is acylindrically hyperbolic, the infinite abelian subgroup $S_1$ or $S_2$ would be acylindrically hyperbolic. This is a contradiction to the second part of Lemma 2.4.) By definition, this is to prove that for any $g \in G$, the intersection $Q \cap g^{-1}Qg$ is infinite for $Q = Q_1$ and $Q_2$. Denote by $ge_1 = (x_1, \cdots, x_n, y_1, \cdots, y_n)^T$.

Let

$$t_A = \prod_{1 \leq i < j \leq n} \rho_{i, n+j}(a_{ij}) = \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \in G,$$

where $A = (a_{ij})$ is an $n \times n$ matrices with entries in $R$. Note that $a_{ij} = a_{ij}$ or $-a_{ij}$ depending on $G = \text{ESP}_{2n}(R)$ or $\text{EO}(n, n)(R)$. Moreover, we have $\rho_{i, n+j}(a) \notin \text{EO}(n, n)(R)$ and $\rho_{i, n+j}(a) \in \text{ESP}_{2n}(R)$ for any $a \in R$. Direct calculation shows that $t_A(ge_1) - ge_1 = ((y_1, \cdots, y_n)A^T, 0, \cdots, 0)^T$. When $n \geq 4$ and $G = \text{EO}(n, n)(R)$, the map $f : R^{\frac{n(n-1)}{2}} \rightarrow R^n$ defined by

$$(a_{ij})_{1 \leq i < j \leq n} \mapsto A(y_1, \cdots, y_n)^T$$

has an infinite kernel $\ker f$ by 2-rigidity of infinite commutative rings. This implies that $\langle t_A \mid (a_{ij})_{1 \leq i < j \leq n} \in \ker f \rangle < Q \cap g^{-1}Qg$ is infinite. When $n \geq 2$ and $G = \text{ESP}_{2n}(R)$, the map $f : R^{\frac{n(n-1)}{2}} \rightarrow R^n$ defined by $(a_{ij})_{1 \leq i < j \leq n} \mapsto A(y_1, \cdots, y_n)^T$ has an infinite kernel $\ker f$ and $Q \cap g^{-1}Qg$ is infinite by a similar argument. The proof is finished.

Proof of Corollary 1.5  By Proposition 3.7, Example 3.9 and Lemma 3.3, all these rings are $n$-rigid for any $n \geq 1$. The corollary follows Theorem 1.1 and Theorem 1.4.

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