INTEGRAL TRANSFORMS AND DEFORMATIONS OF K3 SURFACES

E. MARKMAN AND S. MEHROTRA

ABSTRACT. Let $X$ be a $K3$ surface and $M$ a smooth and projective moduli space of stable sheaves on $X$ of Mukai vector $v$. A universal sheaf $\mathcal{U}$ over $X \times M$ induces an integral transform $\Phi_{\mathcal{U}} : D^b(X) \rightarrow D^b(M)$ from the derived category of coherent sheaves on $X$ to that on $M$.

1. We prove that $\Phi_{\mathcal{U}}$ is faithful. $\Phi_{\mathcal{U}}$ is not full if the dimension of $M$ is $\geq 4$.

2. We exhibit the full subcategory of $D^b(M)$, consisting of objects in the image of $\Phi_{\mathcal{U}}$, as the quotient of a category, explicitly constructed from $D^b(X)$, by a natural congruence relation defined in terms of the Mukai vector $v$.

3. Let $\mathfrak{M}^0$ be a component of the moduli space of isomorphism classes of marked irreducible holomorphic symplectic manifolds deformation equivalent to the Hilbert scheme $X^{[n]}$ of $n$ points on a $K3$ surface $X$, $n \geq 2$. $\mathfrak{M}^0$ is $21$-dimensional, while the moduli of Kähler $K3$ surfaces is $20$-dimensional. We construct a geometric deformation of the derived categories of $K3$ surfaces over a Zariski dense open subset of $\mathfrak{M}^0$, which coincides with $D^b(X)$ whenever the marked manifold is a moduli space of sheaves on $X$ satisfying a technical condition.

Statement 3 assumes Conjecture 1.3 asserting that the dimension, of the first sheaf cohomology of a reflexive hyperholomorphic sheaf with an isolated singularity, remains constant along twistor deformations. The conjecture is known to hold for hyperholomorphic vector bundles.

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1. Introduction

1.1. A faithful functor. Let $X$ be a projective $K3$ surface, $H$ an ample line bundle on $X$, and $M := M_H(v)$ the (coarse) moduli space of Gieseker-Simpson $H$-stable sheaves $\mathcal{E}$ on $X$ whose Chern classes are specified by the vector $v = v(\mathcal{E}) := \text{ch}(\mathcal{E})\sqrt{td_X}$ in the integral cohomology of $X$. $M$ is a smooth holomorphic-symplectic variety, by [Mu1]. We shall assume here that there aren’t any strict semi-stable sheaves of class $v$, so that $M$ is projective. There is a numerical criterion which guarantees this: it amounts to choosing $H$ to be $v$-generic, that is, in a region complementary to a countable union of suitable hyperplanes depending on $v$. While $X \times M$ need not carry a universal sheaf, there always exists a quasi-universal sheaf on it (see the appendix of [Mu3]). In other words, there is a twisted sheaf $\mathcal{U}$ on $X \times M$ which is universal locally on $M$ in the etale or analytic topology (see [C1]).

Write $\pi_X$ and $\pi_M$ for the two projections from $X \times M$, and let $\Phi_{\mathcal{U}} : D^b(X) \to D^b(M, \theta)$ be the integral transform

$$\Phi_{\mathcal{U}}(\_):=\pi_M \ast (\pi_X^\ast(\_)) \otimes \mathcal{U}.\tag{1.1}$$

Here $\theta$ is the class of $\mathcal{U}$ in the Brauer group of $M$, $D^b(M, \theta)$ stands for the bounded derived category of $\theta$-twisted coherent sheaves on $M$, and $\pi_{M,\ast}$ and $\pi_X^\ast$ are derived functors appropriately defined in this context. We refer the reader to [C1] for a careful discussion of this formalism. Suffice it to say that most of the familiar properties of duality and the usual functorial calculus for derived categories of coherent sheaves continue to hold here. In particular, it follows by Serre duality for twisted sheaves that $\Phi_{\mathcal{U}}$ has a right adjoint $\Psi_{\mathcal{U}}(\_):=\pi_{X,\ast}(\pi_M^\ast(\_)) \otimes \mathcal{U}^\vee)[2].$

The composition $\Psi_{\mathcal{U}} \circ \Phi_{\mathcal{U}} : D^b(X) \to D^b(X)$ is the integral transform with kernel

$$\mathcal{A}:=\pi_{13,\ast}(\pi_1^\ast(\omega_X) \otimes \pi_{12}^\ast(\mathcal{U})^\vee \otimes \pi_{23}^\ast(\mathcal{U}))[2] \in D^b(X \times X),\tag{1.2}$$

where $\pi_{ij}$ is the projection from $X \times M \times X$ onto the product of the $i$-th and $j$-th factors.

The Mukai lattice $\tilde{H}(X, \mathbb{Z})$ of the $K3$ surface $X$ is the integral cohomology $H^\ast(X, \mathbb{Z})$ endowed with the Mukai pairing $(u, v) := -\int_X u^\vee \cup v$, where the duality operator $u \mapsto u^\vee$ changes the sign of the direct summand in $H^2(X, \mathbb{Z})$. If $u = \text{ch}(E)\sqrt{td_X}$ and $v = \text{ch}(F)\sqrt{td_X}$ are the Mukai vectors of two coherent sheaves $E$ and $F$ on $X$, then $(u, v) = -\chi(E^\ast \otimes F)$, by
Hirzebruch-Riemann-Roch, where the dual $E^*$ and the tensor product are taken in the derived category, and $\chi$ is the Euler characteristic. The dimension of the moduli space $M := M_H(v)$ discussed above is $2 + (v, v)$. Denote by $\Delta : X \to X \times X$ the diagonal embedding.

**Theorem 1.1.** (Theorem 2.2). Let $v \in \tilde{H}(X, \mathbb{Z})$ be a primitive class with $(v, v) = 2n - 2$, $n \geq 2$, and $H$ a $v$-generic polarization. One has a natural morphism

$$\alpha : \bigoplus_{i=0}^{n-1} \Delta_* \mathcal{O}_X \otimes \mathbb{C} \operatorname{Ext}^{2i}(\mathcal{O}_M, \mathcal{O}_M)[-2i] \to \mathcal{A}.$$

1. When $M := M_H(v)$ is the Hilbert scheme of $n$ points on $X$ and $\mathcal{U}$ is the universal ideal sheaf the morphism $\alpha$ is an isomorphism. In particular, a choice of a non-zero element of the one-dimensional vector space $\operatorname{Ext}^2(\mathcal{O}_M, \mathcal{O}_M)$ yields an isomorphism $\mathcal{A} \cong \oplus_{i=0}^{n-1} \Delta_* \mathcal{O}_X[-2i]$.

2. In general, for $v$ arbitrary, the structure sheaf of the diagonal $\Delta_* \mathcal{O}_X$ is a direct summand of $\mathcal{A}$ in $D^b(X \times X)$. In particular, the integral transform $\Phi_{\psi} : D^b(X) \to D^b(M, \theta)$ is faithful.

Part (1) of Theorem 2.2 was independently proven by Nick Addington [AD]. The isomorphism $\mathcal{A} \cong \oplus_{i=0}^{n-1} \Delta_* \mathcal{O}_X[-2i]$ was established for some moduli spaces of torsion sheaves on $K3$ surfaces in [ADM]. Further cases where $\alpha$ is an isomorphism are provided in Lemmas 6.9 and 6.16. The proof of the first part of Theorem 1.1 relies heavily on Haiman’s work [Ha1, Ha2] on the $n!$-Conjecture on one hand, and the derived McKay correspondence of Bridgeland-King-Reid on the other [BKR]. The second part makes use of results of Mukai, O’Grady, and Yoshioka, which show that there is an isomorphism of Hodge structures between $H^2(M_H(v), \mathbb{Z})$ and the orthogonal complement $v^\perp$ of the Mukai vector $v$ in the Mukai lattice of $X$ (Muk, OG, Y, see also Theorem 2.16 below).

### 1.2. The full subcategory of $D^b(M, \theta)$ with objects coming from the surface.

Denote by $D^b(X)_{\mathcal{T}}$ the full subcategory of $D^b(M, \theta)$ with objects of the form $\Phi_{\psi}(x)$, for some object $x$ in $D^b(X)$. We provide next an explicit computation of $D^b(X)_{\mathcal{T}}$. We will use the following standard construction in category theory.

**Definition 1.2.** [Mac, Section II.8]. Let $\mathcal{C}$ be a category.

1. A congruence relation $\mathcal{R}$ on $\mathcal{C}$ consists of an equivalence relation $\mathcal{R}_{x_1, x_2}$ on $\operatorname{Hom}(x_1, x_2)$, for every pair of objects $x_1, x_2$ in $\mathcal{C}$, satisfying the following property. Given morphisms $f_1, f_2 : x_1 \to x_2$ related by $\mathcal{R}_{x_1, x_2}$, objects $x_0, x_3$ of $\mathcal{C}$, and morphisms $e : x_0 \to x_1$ and $g : x_2 \to x_3$, the morphisms $gf_1e$ and $gf_2e$ are related by $\mathcal{R}_{x_0, x_3}$ in $\operatorname{Hom}(x_0, x_3)$.

2. Let $\mathcal{R}$ be a congruence relation on $\mathcal{C}$. The quotient category $\mathcal{C}/\mathcal{R}$ is the category whose objects are those of $\mathcal{C}$ and such that $\operatorname{Hom}_{\mathcal{C}/\mathcal{R}}(x_1, x_2) := \operatorname{Hom}_{\mathcal{C}}(x_1, x_2)/\mathcal{R}_{x_1, x_2}$. The natural functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{R}$ is called the quotient functor.

An explicit computation of $D^b(X)_{\mathcal{T}}$ as a quotient category requires an explicit category $\mathcal{C}$ and an explicit relation $\mathcal{R}$. We take $\mathcal{C}$ to be the full subcategory $D^b(X)_{\mathcal{Y}}$ of $D^b(X \times M)$ with objects of the form $\pi_X^*(x)$, for some object $x$ in $D^b(X)$. Set $\operatorname{Hom}_{D^b(X)_{\mathcal{Y}}}(x, y) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{D^b(X)}(x, y[i])$. The category $D^b(X)_{\mathcal{Y}}$ is explicit, since

$$\operatorname{Hom}_{D^b(X)_{\mathcal{Y}}}(x, y) := \operatorname{Hom}_{D^b(X \times M)}(\pi_X^* x, \pi_Y^* y) \cong \operatorname{Hom}_{D^b(M)}(\mathcal{O}_M, \mathcal{O}_M).$$
We describe next a congruence relation on $D^b(X)$. Set $pt := \text{Spec}(\mathbb{C})$ and let $c : M \to pt$ be the constant morphism. We get the object $Y(\mathcal{O}_M) := R_c(\mathcal{O}_M)$ in $D^b(pt)$. As a graded vector space $Y(\mathcal{O}_M)$ is $\bigoplus_{i=0}^{n} H^{2i}(M, \mathcal{O}_M)[-2i]$, where $n = \dim_{\mathbb{C}}(M)/2$. Given a graded vector space $V$, let $1_{D^b(X)} \otimes_{\mathbb{C}} V$ be the endofunctor of $D^b(X)$ sending an object $x$ to $x \otimes_{\mathbb{C}} V$. Set

$$\Upsilon := 1_{D^b(X)} \otimes_{\mathbb{C}} Y(\mathcal{O}_M),$$

$$R := 1_{D^b(X)} \otimes_{\mathbb{C}} H^{2n}(M, \mathcal{O}_M)[-2n].$$

These two endofunctors are integral transforms with kernels $\mathcal{Y} := \Delta_* \mathcal{O}_X \otimes_{\mathbb{C}} Y(\mathcal{O}_M)$ and $\mathcal{R} := \Delta_* \mathcal{O}_X \otimes_{\mathbb{C}} H^{2n}(M, \mathcal{O}_M)[-2n]$ in $D^b(X \times X)$. Let $\pi_X : X \times M \to X$ be the projection. Note that the endofunctor $\Upsilon$ is naturally isomorphic to $R\pi_X \circ \pi_X^*$. We thus have the adjunction isomorphism

$$(1.3) \quad \text{Hom}_{D^b(X)}(x, y) := \text{Hom}_{D^b(X \times M)}(\pi_X^* x, \pi_X^* y) \cong \text{Hom}_{D^b(X)}(x, \Upsilon(y)).$$

The congruence relation $\mathcal{R}$ is defined in terms of a natural transformation

$$h : R \to \Upsilon,$$

which we define next. The natural transformation $h$ is induced by a morphism $h : \mathcal{R} \to \mathcal{Y}$. Write $h = \sum_{i=0}^{n} h_{2i}$ according to the direct sum decomposition of $Y(\mathcal{O}_M)$, so that

$$h_{2i} : \Delta_* (\mathcal{O}_X) \otimes_{\mathbb{C}} H^{2n}(M, \mathcal{O}_M)[-2n] \to \Delta_* (\mathcal{O}_X) \otimes_{\mathbb{C}} H^{2n-2i}(M, \mathcal{O}_M)[2i - 2n].$$

Mukai’s Hodge isometry (2.18) provides a canonical identification of $H^2(X, \mathcal{O}_X)$ with $H^2(M, \mathcal{O}_M)$. It follows that the morphism $h_{2i}$ is naturally a class in $\text{Hom}_X(\Delta_* (\mathcal{O}_X), \Delta_* (\omega_X^{\otimes 2i})[2i])$, as we carefully check in Section 3. In particular, $h_2$ is a class in the Hochschild homology $HH_0(X)$. The Hochschild-Kostant-Rosenberg isomorphism (reviewed in Section 2.4) maps the Mukai vector $v$ to a class in $HH_0(X)$. We set $h_2$ to be the image of $v$, $h_0 := -1$, and $h_4 := -(h_2)^2$. We necessarily have $h_{2i} = 0$, for $i > 2$, for dimension reasons.

Given an object $x$ in $D^b(X)$, let

$$h_x : x \otimes_{\mathbb{C}} H^{2n}(M, \mathcal{O}_M)[-2n] \to x \otimes_{\mathbb{C}} Y(\mathcal{O}_M)$$

be the morphism induced by the natural transformation $h$. Consider the relation $\mathcal{R}$ on $D^b(X)$ given as follows. The morphisms $f_1, f_2$ in $\text{Hom}_{D^b(X)}(x_1, x_2)$ are related by $\mathcal{R}_{x_1, x_2}$, if and only if $f_1 - f_2$ belongs to the image of the homomorphism

$$(1.4) \quad (h_{x_2})_* : \text{Hom}_{D^b(X)}(x_1, x_2) \otimes H^{2n}(M, \mathcal{O}_M)[-2n] \to \text{Hom}_{D^b(X)}(x_1, x_2)$$

induced by composition with $h_{x_2}$. It is easy to check that $\mathcal{R}$ is a congruence relation, as we do in the proof of Theorem 3.3.

We describe next the functor inducing the equivalence between $D^b(X)$ and the quotient category $D^b(X)/\mathcal{R}$. Let $\Xi_{/\mathcal{R}} : D^b(X \times M) \to D^b(M, \theta)$ be the composition of tensorization by $\mathcal{R}$ followed by $R\pi_M$. Then $\Phi_{/\mathcal{R}} = \Xi_{/\mathcal{R}} \circ \pi_X^*$. Let $Q : D^b(X) \to D^b(X)$ be the restriction of the functor $\Xi_{/\mathcal{R}}$.

**Theorem 1.3. (Theorem 3.3)** Assume that the morphism $\alpha$ in Theorem 1.1 is an isomorphism.

1. $\mathcal{R}$ is a congruence relation.
2. The kernel of $Q : \text{Hom}_{D^b(X)}(x, y) \to \text{Hom}_{D^b(M, \theta)}(\Phi_{/\mathcal{R}}(x), \Phi_{/\mathcal{R}}(y))$ is identified with the image of $(h_y)_* : \text{Hom}_{D^b(X)}(x, R(y)) \to \text{Hom}_{D^b(X)}(x, \Upsilon(y))$ via the adjunction isomorphism (1.3).
The functor $Q$ is full. Consequently, $Q$ factors as the composition of the quotient functor $D^b(X) \sim \rightarrow D^b(X)\tau / \mathcal{R} \cong D^b(X)\tau$.

As a consequence we obtain the description of the Yoneda algebra of $\Phi_{\mathcal{M}}(F)$, for a simple sheaf $F$ on $X$, as the quotient of the tensor product of the Yoneda algebras of $F$ and $\mathcal{O}_M$ by the principal ideal generated by an explicit relation (Theorem 3.21). Related results were obtained by A. Krug for the Hilbert scheme of points on a projective surface [K].

Let us describe the key ingredient in the proof of parts (2) and (3) of Theorem 1.3. The equality $\Phi_{\mathcal{M}} = \Xi_{\mathcal{M}} \circ \pi^*$ gives rise to a natural transformation $q : \mathcal{Y} \rightarrow \Psi_{\mathcal{M}} \circ \Phi_{\mathcal{M}}$ given by $q := R\pi_* \tilde{\eta} \pi^*$, where $\tilde{\eta} : 1_{D^b(X \times M)} \rightarrow \Xi_{\mathcal{M}}$ is the unit for the adjunction $\Xi_{\mathcal{M}} \dashv \Xi_{\mathcal{M}}^\dagger$. The natural transformation $q$, in turn, is induced by a morphism of kernels $q : \mathcal{Y} \rightarrow \mathcal{A}$. We get an exact triangle in $D^b(X \times X)$, which admits a splitting

$$Rh \rightarrow \mathcal{Y} q \rightarrow \mathcal{A} \rightarrow R[1],$$

by Proposition 3.9 and Theorem 3.10. The functor $Q$ of Theorem 1.3 is induced by the natural transformation $q : \mathcal{Y} \rightarrow \Psi_{\mathcal{M}} \circ \Phi_{\mathcal{M}}$ in the sense that the following diagram commutes

$$\begin{align*}
\text{Hom}_{D^b(X\times X)}(x, \mathcal{Y}(y)) &\xrightarrow{q} \text{Hom}_{D^b(X)}(x, \Psi_{\mathcal{M}} \Phi_{\mathcal{M}}(y)) \\
\cong &\xrightarrow{\cong} \text{Hom}_{D^b(M\times M)}(\pi^*_x x, \pi^*_y y) \xrightarrow{\Xi_{\mathcal{M}}} \text{Hom}_{D^b(M)}(\Phi_{\mathcal{M}}(x), \Phi_{\mathcal{M}}(y)),
\end{align*}$$

where the vertical arrows are the adjunction isomorphisms (Theorem 3.2). Parts (2) and (3) of Theorem 1.3 follow immediately from the splitting of the exact triangle (1.5).

1.3. A reconstruction of $D^b(X)$ as a category of comodules of a comonad. We shall be using the categorical device of (co)monads and (co)modules over them in order to recover the category $D^b(X)$ in terms of data in $D^b(M \times M)$. The latter data will then be deformed with $M$ yielding non-commutative deformations of $D^b(X)$. Let us briefly recall the necessary notions here. A detailed presentation can be found in Chapter VI of Mac Lane’s text [Mac].

A comonad $\mathbb{L}$ on a category $A$ is simply a comonoid object in the functor category $\text{End}(A)$. Explicitly, $\mathbb{L}$ is a triple $\langle L, \epsilon, \delta \rangle$ where $L : A \rightarrow A$ is an endofunctor, and the counit $\epsilon$, and comultiplication $\delta$ are natural transformations

$$\begin{align*}
\epsilon : L &\rightarrow I, \\
\delta : L &\rightarrow L^2,
\end{align*}$$

satisfying coassociativity:

$$\begin{align*}
L &\xrightarrow{\delta} L^2 \\
\delta &\xrightarrow{\delta \delta} L^3,
\end{align*}$$

and the left and right counit laws:

$$\begin{align*}
L &\xleftarrow{\epsilon L} L^2 \\
\delta &\xrightarrow{\delta \epsilon} L^3.
\end{align*}$$
Associated to any adjunction $F : X \to A$, $G : A \to X$, with $F \dashv G$, there is a natural comonad with $L = FG : A \to A$, $\epsilon$ the counit of the adjunction, and $\delta = F\eta G : L \to L^2$, where $\eta$ is the unit of the adjunction.

A comodule for a comonad $\langle L, \eta, \delta \rangle$ is a pair $(a, h)$ consisting of an object $a \in A$ and an arrow $h : a \to La$ such that $\delta \circ h = Lh \circ h$ and $\epsilon \circ h = id$. A morphism $f : (a, h) \to (a', h')$ is an arrow $f \in \text{Hom}_A(a, a')$ which renders commutative the diagram

$$
\begin{array}{ccc}
a & \xrightarrow{h} & La \\
\downarrow{f} & & \downarrow{Lf} \\
a' & \xrightarrow{h'} & La'.
\end{array}
$$

The set of all $L$-comodules together with their morphisms form a category $A^L$. Finally, monads, and modules over monads are simply the notions dual to those defined above.

Adjoint pairs of functors naturally give rise to (co)monads. In our situation, let $L$ be the composition $\Phi_\mathcal{W} \circ \Psi_\mathcal{W} : D^b(M_H(v), \theta) \to D^b(M_H(v), \theta)$. The adjunction $\Phi_\mathcal{W} \dashv \Psi_\mathcal{W}$ yields the unit $\eta : \Psi_\mathcal{W} \circ \Phi_\mathcal{W} \to id$, the counit $\epsilon : \Phi_\mathcal{W} \circ \Psi_\mathcal{W} \to id$, and the comultiplication

$$
\delta = \Phi_\mathcal{W} \eta \Psi_\mathcal{W} : L \to L^2.
$$

These define a comonad $\mathbb{L} := \langle L, \epsilon, \delta \rangle$ in $D^b(M_H(v), \theta)$. We get the category $D^b(M_H(v), \theta)\mathbb{L}$ of comodules of $\mathbb{L}$. Denote by $\mathbb{T}_\mathcal{W} : D^b(M_H(v), \theta)\mathbb{L} \to D^b(M_H(v), \theta)$ the forgetful functor $\mathbb{T}_\mathcal{W}(\mathcal{G}, h) = \mathcal{G}$. For (co)monads given by adjunctions, there is a natural comparison functor from the source category to the category of (co)modules, which in our case will be denoted as $\hat{\Phi} : D^b(X) \to D^b(M_H(v), \theta)\mathbb{L}$. It takes $\mathcal{H} \in D^b(X)$ to the comodule $(\Phi_\mathcal{W}(\mathcal{H}'), \Phi_\mathcal{W} \eta \mathcal{H}')$.

The unit $\eta : id \to \Psi_\mathcal{W} \circ \Phi_\mathcal{W}$ is split by Theorem 1.1. Hence the triangulated version of the Barr-Beck Theorem [E, MS] (see also [Bal2]) immediately yields:

**Proposition 1.4.** The comparison functor $\hat{\Phi} : D^b(X) \xrightarrow{\cong} D^b(M_H(v), \theta)\mathbb{L}$ is an equivalence of categories. In particular, it induces the structure of a triangulated category on $D^b(M_H(v), \theta)\mathbb{L}$ such that the forgetful functor $\mathbb{T}_\mathcal{W}$ is exact.

This captures $D^b(X)$ in terms of structures defined only on $M_H(v)$.

### 1.4. A deformation of $K3$-categories.

The deformations of $M_H(v)$ arising from those of the underlying K3 surface $X$ form a 20-dimensional locus in the 21-dimensional Kuranishi deformation space of $M_H(v)$. Thus the generic deformation of $M_H(v)$ is not a moduli space of sheaves. The next theorem interprets these as arising from “non-commutative” perturbations of $X$. More precisely, we construct deformations of $D^b(X)$ which correspond to those of $M_H(v)$ away from the moduli space locus.

Let $\mathcal{F} \in D^b(M_H(v) \times M_H(v), \pi^*_1 \theta^{-1} \pi^*_2 \theta)$ be the kernel of the endofunctor $L$. The object $\mathcal{F}$ plays a prominent role in the study of the geometry and cohomology of the moduli spaces $M_H(v)$ (see [Mu3, KLS, Ma1]). Let $\pi_{ij}$ be the projection from $M_H(v) \times X \times M_H(v)$ onto the product of the $i$-th and $j$-th factors. $\mathcal{F}$ is a complex with cohomology concentrated in degrees $-1$ and $0$:

$$
\mathcal{H}^i(\mathcal{F}) = \mathcal{H} \text{om}_{\bar{M}_H(v)}(\pi^*_1 \mathcal{W}, \pi^*_2 \mathcal{W}[i]) \cong \begin{cases} E_{\Delta \bar{M}_H(v)} & \text{if } i = 0, \\
E & \text{if } i = -1,
\end{cases}
$$

where $E$ is a twisted reflexive sheaf of rank $(\dim M_H(v) - 2)$. 


Definition 1.5. We shall refer to $\mathcal{F}$ as the *modular complex* of $M_H(v)$, and to $\mathcal{E}$ as the *modular sheaf* of $M_H(v)$ to emphasize their origin. The sheaf of Azumaya algebras $\mathcal{E}_{nd}(\mathcal{E})$ will be referred to as the *modular Azumaya algebra* of $M_H(v)$.

Let $X$ be a K3 surface with a trivial Picard group and let $\mathcal{F} \in D^b(X^{[n]} \times X^{[n]})$ be the kernel of the endofunctor $L$, so that its square $L^2$ has kernel the convolution $\mathcal{F} \circ \mathcal{F}$. By Proposition 5.1 of [C2], the counit $\epsilon$ and the comultiplication $\delta$ of $L$ correspond to morphisms of objects $\mathcal{F} \to \mathcal{O}_{\Delta}$ and $\mathcal{F} \to \mathcal{F} \circ \mathcal{F}$, respectively; denote these by $\epsilon$ and $\delta$ also. Thus we have the comonad object

$$\langle \mathcal{F}, \epsilon, \delta \rangle$$

in $D^b(X^{[n]} \times X^{[n]})$ representing the comonad $\mathcal{I}$.

We recall here some of the results obtained in [MM2] which are required for the statement of the deformability of the comonad object given in Equation (1.10) (Theorem 1.8). A holomorphic symplectic compact Kähler manifold $M$ is said to be of K3$^{[n]}$-type, if it is deformation equivalent to the Hilbert scheme $X^{[n]}$ of length $n$ subschemes of a K3 surface $X$. The second integral cohomology of $M$ is endowed with a non-degenerate integral symmetric bilinear pairing of signature $(3, 20)$ called the *Beauville-Bogomolov-Fujiki pairing*. Fix a lattice $\Lambda$ isometric to $H^2(X^{[n]}, \mathbb{Z})$. A marking for such $M$ is an isometry $\eta : H^2(M, \mathbb{Z}) \to \Lambda$. Let $\mathfrak{A}$ be a reflexive sheaf of Azumaya algebras on $M \times M$ that is infinitesimally rigid, slope-stable with respect to some Kähler class $\omega$ on $M$, and having the same numerical invariants as the modular Azumaya algebra $\mathcal{E}_{nd}(\mathcal{E})$ above (see [MM2] Section 1). In particular, $c_2(\mathfrak{A})$ is monodromy invariant, under the diagonal action of the monodromy group of $M$ on $H^4(M \times M, \mathbb{Z})$, and hence remains of Hodge type under every smooth Kähler deformation of $M$. Associated to a Kähler class $\omega$ on $M$ is a twistor deformation $\pi : \mathcal{X} \to \mathbb{P}^1_\omega$, where $\mathbb{P}^1_\omega$ is the smooth conic defined in the complex projective plane $\mathbb{P}(H^{2,0}(M) \oplus H^{0,2}(M) \oplus \mathbb{C})$ via the Beauville-Bogomolov-Fujiki pairing. The $\omega$-slope-stability of $\mathfrak{A}$ and the invariance of $c_2(\mathfrak{A})$ imply that the sheaf $\mathfrak{A}$ is $\omega$-stable hyperholomorphic in the sense of Verbitsky [V1], which means that it deforms to a reflexive sheaf of Azumaya algebras $\widetilde{\mathfrak{A}}$ over the fiber square $\mathcal{X} \times_{\mathbb{P}^1_\omega} \mathcal{X}$ of the twistor family. We shall be only interested in reflexive sheaves $\mathfrak{A}$ which additionally satisfy a technical condition on their singularities spelled out in [MM2] Condition 1.6, but which we do not state in full here.

Conjecture 1.6. [MM2, Conj. 1.12] Let $M$ be an irreducible holomorphic symplectic manifold, $\omega$ a Kähler class on $M$, and $E$ a reflexive $\omega$-slope-stable hyperholomorphic sheaf on $M$ with an isolated singularity. Assume that $H^1(X, E) = 0$. Denote by $(\mathcal{X}_t, E_t)$, $t \in \mathbb{P}^1_\omega$, the twistor deformation of $(M, E)$. Then $H^1(\mathcal{X}_t, E_t) = 0$, for all $t \in \mathbb{P}^1_\omega$.

The above conjecture is a theorem of Verbitsky when $E$ is locally free [V3, Cor. 8.1].

Theorem 1.7. Assume that Conjecture 1.6 holds.

1. [MM2, Theorem 1.8] There exists a coarse moduli space $\widetilde{\mathcal{M}}_\Lambda$ of triples $(M, \eta, \mathfrak{A})$ as above which is a non-Huasdorff complex manifold. The period map $\widetilde{P} : \widetilde{\mathcal{M}}_\Lambda \to \{ x \in \mathbb{P}[\Lambda \otimes \mathbb{C}] : (x, x, \mathcal{F}) > 0 \}$ given by $(X, \eta, \mathfrak{A}) \mapsto (\eta(H^{2,0}(M)))$ is a surjective local analytic isomorphism.

2. [MM2, Theorem 1.9] The restriction of the period map to each connected component $\widetilde{\mathcal{M}}_\Lambda^0$ of $\widetilde{\mathcal{M}}_\Lambda$ is generically injective in the following sense. When the Picard group of $M$ is trivial, or cyclic generated by a class of non-negative Beauville-Bogomolov-Fujiki
degree, then a point \((M, \eta, \mathfrak{A})\) of \(\widehat{\mathcal{M}}^0_{\mathfrak{A}}\) is the unique point of \(\widehat{\mathcal{M}}^0_{\Lambda}\) in the corresponding fiber of \(\widehat{P}\).

Fix a component \(\widehat{\mathcal{M}}^0_{\mathfrak{A}}\) of \(\widehat{\mathcal{M}}_{\Lambda}\) containing a point of the form \((X^0_0, \eta_0, \mathfrak{A}_0)\), where \(X_0\) is a K3 surface with a trivial Picard group and \(\mathfrak{A}_0\) is the modular Azumaya algebra of the Hilbert scheme \(X^0_0\). The modular Azumaya algebra \(\mathfrak{A}_0\) is \(\omega\)-slope-stable, with respect to every Kähler class \(\omega\) on \(X^0_0 \times X^0_0\), by \([\text{Ma7}]\). Denote by

\[(1.11) \quad \mathcal{H} \cap \widehat{\mathcal{M}}^0_{\Lambda} \subset \widehat{\mathcal{M}}^0_{\mathfrak{A}}\]

the subset consisting of triples \((X^n, \eta, \mathfrak{A})\), where \(X\) is a K3 surface with a trivial Picard group, \(X^n\) is its Hilbert scheme, and \(\mathfrak{A}\) is the modular Azumaya algebra. By a Zariski open subset of an analytic space we mean the complement of a closed analytic subset. In view of Proposition 1.1.4 the following theorem is the deformation result mentioned above.

**Theorem 1.8.** Assume that Conjecture 1.6 holds. There exists a Zariski dense open subset \(U \subset \widehat{\mathcal{M}}^0_{\mathfrak{A}}\), containing \(\mathcal{H}\), a universal family \(\pi : \mathcal{M} \to U\) of irreducible holomorphic symplectic manifolds, and a Brauer class \(\Theta\) of order \(2n - 2\) over the fiber square \(M^2 := \mathcal{M} \times_U \mathcal{M}\) with the following properties.

1. The triple \((\mathcal{F}, \epsilon, \delta)\), given in Equation (1.10), deforms to a comonad object \(\mathcal{F} := (\mathcal{F}, \pi, \delta)\) in \(D^b(M^2, \Theta)\).
2. Given an open subset \(V\) of \(U\), denote by \(\mathcal{M}^2_V\) the restriction of \(\mathcal{M}^2\) to \(V\) and by \(\Theta_V\) the restriction of \(\Theta\) to \(\mathcal{M}^2_V\). Define \(\mathcal{M}_V\) and \(\mathcal{F}_V\) similarly. When \(V\) is contractible and Stein, there exists a Brauer class \(\theta\) over \(\mathcal{M}_V\), such that \(\Theta_V = \pi^*_1\theta^{-1}\pi^*_2\theta\), so that \(\mathcal{F}_V\) induces an endo-functor of \(D^b(M^2, \Theta)\).
3. The category of comodules \(D^b(M_V, \theta)_\Lambda\) carries a 2-triangulated structure such that the forgetful functor \(D^b(M_V, \theta)_\Lambda \to D^b(M^2, \Theta)\) is 2-exact. The same holds for the category \(D^b(M_u, \theta_u)_{\Lambda_u}\) of comodules in \(D^b(M_u, \theta_u)\) for the fiber \(M_u\) of \(\pi\) over each point \(u\) in \(U\).
4. \(D^b(M_u, \theta_u)_{\Lambda_u}\) is a K3-category in the sense that the shift \([2]\) is a Serre functor.
5. The open set \(U\) is large in the following sense. Let \(M\) be an irreducible holomorphic symplectic manifold of K3\(^{(n)}\)-type, whose Picard group has rank \(\leq 20\). Then there exists an Azumaya algebra \(\mathfrak{A}\) over \(M \times M\), such that the triple \((M, \eta, \mathfrak{A})\) belongs to \(U\).

Part 1 of the theorem is proven in section 6.1, part 2 in Remark 6.5, part 3 in section 6.2, part 4 in section 6.3, and part 5 in section 6.3.

Let \(X\) be an algebraic K3-surface with Picard rank less than 20. In Section 7 we show that the deformations of \(X\) constructed in the above theorem via those of the Hilbert scheme \(X^n\) may be interpreted infinitesimally as deformations of the category of coherent sheaves \(\text{Coh}(X)\) (see [10]). In fact, this family of deformations is the maximal family of generalized (non-commutative and gerby) deformations along which ideal sheaves of length \(n\) subschemes deform as objects of \(\text{Coh}(X)\). Similar statements are true of deformations coming from those of \(M_H(v)\) provided the triple \(y = (M_H(v), \eta, \mathfrak{A})\), with \(\mathfrak{A}\) the modular Azumaya algebra of \(M_H(v)\) (of Definition 1.5), belongs to \(\widehat{\mathcal{M}}^0_{\Lambda}\).

---

1. By a comonad object we mean that the convolutions \(\mathcal{F} \circ \mathcal{F} \circ \cdots \circ \mathcal{F}\) are well defined and are all objects of \(D^b(M^2, \Theta)\). Furthermore, the counit \(\epsilon : \mathcal{F} \to \text{Id}_{\mathcal{M}}\) and comultiplication \(\delta : \mathcal{F} \to \mathcal{F} \circ \mathcal{F}\) satisfy the axioms of a comonad.

2. A 2-triangulated category is an additive category satisfying all the axioms of a Verdier triangulated category, except the octahedral axiom. See [6.2].
There are natural homomorphisms
\[ \text{HH}^2(X) \xrightarrow{\phi} \text{Hom}_{X \times M}(\mathcal{U}, \mathcal{U}[2]) \xrightarrow{\phi^H} \text{HH}^2(M) \]
from the Hochschild cohomologies of $X$ and $M$. The left homomorphism is an injection, while the right is an isomorphism. Inverting the right arrow and composing defines a homomorphism:
\[ \phi^H : \text{HH}^2(X) \rightarrow \text{HH}^2(M). \]
Set $\text{HT}^2(X) := H^0(\wedge^2 TX) \oplus H^1(TX) \oplus H^2(\Theta_X)$ and similarly for $M$. Conjugating $\phi^H$ with the Hochschild-Kostant-Rosenberg isomorphisms yields a map $\phi^T : \text{HT}^2(X) \rightarrow \text{HT}^2(M)$. For any class $t \in \text{HT}^2(X)$, let $\text{Coh}(X,t)$ denote the first-order deformation of the category of coherent sheaves of $X$ in the direction $t$ (see the general construction by Toda [To]). Given a tangent vector $\xi$ at a point $u$ in the open subset $U$ of Theorem 1.8 denote by $\mathcal{M}_\xi$ the first order deformation of the fiber $\mathcal{M}_u$ over the length 2 subscheme of $U$ corresponding to $\xi$. Let $\mathcal{L}_\xi$ be the restriction of the comonad data $\mathcal{L}$ to $\mathcal{M}_\xi$. Recall that Mukai vectors of objects of $\text{D}^b(X)$ are naturally elements of $\text{HH}_0(X)$ and that the Hochschild homology $\text{HH}_*(X)$ is an $\text{HH}^*(X)$-module [C2]. Let $\text{ann}(v^\vee) \subset \text{HT}^2(X)$ be the image via the HKR-isomorphism of the subspace of $\text{HH}^2(X)$ annihilating the dual of the Mukai vector $v$.

**Theorem 1.9.** Keep the notation of Theorem 1.8 and assume that the morphism $\alpha$ in Theorem 1.1 is an isomorphism.

1. The image $\tilde{\phi}^T(\text{ann}(v^\vee))$ is the following subspace of $\text{HT}^2(M)$:
   \[ \{ (\xi,\theta) : \xi \in H^1(TM), \theta \in H^2(\Theta_M), \text{ and } \xi \cdot c_1(\alpha) + (2-2n)\theta = 0 \}. \]

2. Let $\widetilde{\phi}^T : \text{HT}^2(X) \rightarrow H^1(TM)$ be the composition of $\phi^T$ with the projection $\text{HT}^2(M) \rightarrow H^1(TM)$. Then $\widetilde{\phi}^T$ restricts to $\text{ann}(v^\vee)$ as an isomorphism onto $H^1(TM)$. Fix a class $t \in \text{ann}(v^\vee)$ and set $\xi := \tilde{\phi}^T(t)$. The comonad category $\text{D}^b(\mathcal{M}_\xi)_{\text{mc}}$ of Theorem 1.8 (3) is a triangulated category equivalent to the derived category $\text{D}^b(\text{Coh}(X,t))$.

1.5. **Notational conventions.** We shall be working throughout over the complex numbers. The spaces that we deal with will be denoted by roman letters, while their deformations will be denoted by the corresponding calligraphic letters. For instance, if $X$ is a K3 surface, then $\mathcal{X}$ will stand for a flat family with $X$ as its central fiber. Azumaya algebras will be denoted by fraktur letters, such as $\mathfrak{A}$. Sheaves, and more generally, complexes of sheaves in the derived category will be denoted by script letters, while their deformations will be denoted by the same letter decorated with an over-line: for example, $\mathcal{E}$ and $\widetilde{\mathcal{E}}$. The same notational convention for deformations will be followed for Azumaya algebras and for morphisms between complexes.

Given schemes or analytic spaces $X$ and $Y$, and a morphism $f : X \rightarrow Y$, we denote by $f^* : \text{D}^b(Y) \rightarrow \text{D}^b(X)$ the left derived functor of the pullback functor from $\text{Coh}(Y)$ to $\text{Coh}(X)$. When $f$ is proper $f_* : \text{D}^b(X) \rightarrow \text{D}^b(Y)$ will denote the right derived functor of the direct image functor. Occasionally, we will use the notation $Lf^*$ and $Rf_*$ for the same functors to emphasize their derived nature.

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2. A universal monad in \( D^b(X \times X) \)

2.1. The monad associated to a morphism of varieties. The following basic construction will be used repeatedly in the proof of the main result of this section.

Construction 2.1. Let \( f : T \to S \) be a morphism of smooth and projective varieties. We get the endofunctor \( f_* f^* \) of \( D^b(S) \), where the pullback and push forward are in the derived sense. Denote by \( u : 1_S \to f_* f^* \) the unit for the adjunction, and let \( \epsilon : f^* f_* \to 1_T \) be the counit. Set \( \mu := f_* \epsilon f^* : f_* f^* f_* f^* \to f_* f^* \) and consider the monad \( \Upsilon := (f_* f^*, u, \mu) \) in \( D^b(S) \). Any object, which is isomorphic to \( f_* \mathcal{G} \) for some \( \mathcal{G} \in D^b(T) \), admits an action

\[
(2.1) \quad f_* f^* f_* \mathcal{G} \xrightarrow{f_* \epsilon} f_* \mathcal{G},
\]

so that the pair \( (f_* \mathcal{G}, f_* \epsilon) \) is an object of the category \( D^b(S)^{\Upsilon} \) of modules for the monad. We get the functor

\[
\tilde{f}_* : D^b(T) \to D^b(S)^{\Upsilon},
\]

sending an object \( \mathcal{G} \) of \( D^b(T) \) to \( \tilde{f}_*(\mathcal{G}) := (f_* \mathcal{G}, f_* \epsilon) \). What is more, under the isomorphism \( f_* f^* f_* \mathcal{G} \to f_* \mathcal{G} \otimes f_* \mathcal{O}_T \), this monadic action can be seen as an action of the algebra object \( f_* \mathcal{O}_T \). Indeed, recall that the product on \( f_* \mathcal{O}_X \) is given by the composition

\[
f_* \mathcal{O}_T \otimes f_* \mathcal{O}_T \xrightarrow{\cong} f_* f^* f_* f^* \mathcal{O}_S \xrightarrow{\mu_{\mathcal{O}_S}} f_* f^* \mathcal{O}_S \cong f_* \mathcal{O}_T.
\]

The desired compatibility between the the product on \( f_* \mathcal{O}_X \) and its action on an object \( \tilde{f}_*(\mathcal{G}):=(f_* \mathcal{G}, f_* \epsilon) \) follows now from the axioms for a module for the monad \( \Upsilon \).

2.2. A splitting of the monad. Keep the notation of Theorem 17.1. Set \( \text{pt} := \text{Spec}(\mathbb{C}) \). Let \( c : M \to \text{pt} \) be the constant map and set \( Y(\mathcal{O}_M) := \mathcal{R}c_*(\mathcal{O}_M) \), as an object in \( D^b(\text{pt}) \). Then \( Y(\mathcal{O}_M) \) is naturally isomorphic to \( \bigoplus_{i=0}^n \text{Ext}^2(\mathcal{O}_M, \mathcal{O}_M)[-2i] \), thought of as the Yoneda algebra of \( \mathcal{O}_M \).

Let \( u : \pi_{13}, \pi_{13} \to \mathbb{1}_{X \times X} \) be the unit for the adjunction \( \pi_{13} \dashv \pi_{13}, \epsilon : \pi_{13} \pi_{13} \to \mathbb{1}_{X \times M \times X} \) the counit, and \( \mu := \pi_{13}, \mu : (\pi_{13}, \pi_{13})^2 \to \pi_{13}, \pi_{13} \) the multiplication natural transformation. Denote by

\[
(2.2) \quad Y := (\pi_{13}, \pi_{13}, u, \mu)
\]

the monad in \( D^b(X \times X) \). We get the category \( D^b(X \times X)^{Y} \) of modules for the monad \( Y \) and the functor

\[
\pi_{13}^*: D^b(X \times M \times X) \to D^b(X \times X)^{Y},
\]

as a special case of the construction in section 2.1.

\( \mathcal{A} \) is the push-forward of the object \( \mathcal{A} := \pi_{13}^* \omega_X \otimes \pi_{12}^* (\mathcal{U}) \otimes \pi_{23}^* (\mathcal{U})[2] \) of \( D^b(X \times M \times X) \) by \( \pi_{13} \). We get a natural morphism

\[
(2.3) \quad m : \mathcal{A} \otimes_{\mathcal{C}} Y(\mathcal{O}_M) \to \mathcal{A},
\]

so that the object \( (\mathcal{A}, m) \) of \( D^b(X \times X)^{Y} \) is the \( Y \)-module corresponding to the object \( \mathcal{A} \) via the functor \( \pi_{13}^* \). Note that the algebra structure on \( \pi_{13}, \mathcal{O}_{X \times M \times X} \) is now identified with cup product on \( H^*(\mathcal{O}_M) \), or the composition product on \( Y(\mathcal{O}_M) = \bigoplus_{i=0}^n \text{Ext}^2(\mathcal{O}_M, \mathcal{O}_M)[-2i] \).

Let \( \eta : \Delta_*, \mathcal{O}_X \to \mathcal{A} \) be the morphism corresponding to the unit of the adjunction \( \Phi_{\mathcal{A}} \dashv \Psi_{\mathcal{A}} \). We get the composite morphism

\[
(2.4) \quad \Delta_*, \mathcal{O}_X \otimes_{\mathcal{C}} Y(\mathcal{O}_M) \xrightarrow{\eta \otimes \text{id}} \mathcal{A} \otimes_{\mathcal{C}} Y(\mathcal{O}_M) \xrightarrow{m} \mathcal{A}.
\]
Let $\lambda_n$ be the object $\oplus_{i=0}^{n-1} \text{Ext}^2(\mathcal{O}_M, \mathcal{O}_M)[-2i]$ in $D^b(\text{pt})$. We have a natural morphism $\iota : \lambda_n \to Y(\mathcal{O}_M)$. The object $Y(\mathcal{O}_M)$ is naturally the direct sum of $\lambda_n$ and $\text{Ext}^{2n}(\mathcal{O}_M, \mathcal{O}_M)[-2n]$. Set $\alpha := \iota \circ \iota : \Delta_\ast \mathcal{O}_X \otimes_{\mathcal{C}} \lambda_n \to \mathcal{A}$. So $\alpha$ is the composition

$$\Delta_\ast \mathcal{O}_X \otimes_{\mathcal{C}} \lambda_n \overset{\iota}{\longrightarrow} \Delta_\ast \mathcal{O}_X \otimes_{\mathcal{C}} Y(\mathcal{O}_M) \overset{\eta \otimes \text{id}}{\longrightarrow} \mathcal{A} \otimes_{\mathcal{C}} Y(\mathcal{O}_M) \overset{m}{\longrightarrow} \mathcal{A}.$$  

The main result of this section is

**Theorem 2.2.** Let $v \in K(X)$ be a primitive class with $(v, v) = 2n - 2$, $n \geq 2$, and $H$ a $v$-generic polarization.

1. When $v = (1, 0, 1 - n)$, that is when $M := M_H(v)$ is the Hilbert scheme of $n$ points on $X$, then the morphism $\alpha$, displayed in equation (2.5), is an isomorphism. In particular, a choice of a non-zero element $t_M$ of $\text{Ext}^2(\mathcal{O}_M, \mathcal{O}_M)$ determines an isomorphism

   $$\mathcal{A} \cong \oplus_{i=0}^{n-1} \Delta_\ast \mathcal{O}_X [-2i].$$

2. In general, for $v$ arbitrary, the structure sheaf of the diagonal $\Delta_\ast \mathcal{O}_X$ is a direct summand of $\mathcal{A}$ in $D^b(X \times X)$. In particular, the integral transform $\Phi_\mathcal{A} : D^b(X) \to D^b(M, \theta)$ is faithful.

Part (1) of the theorem is proven in Section 2.3 and part (2) in Section 2.5.

The splitting of the monad object $\mathcal{A}$ in Theorem 2.2 (1) extends over a Zariski open subset of the base of a family in the following sense. Let $\pi : \mathcal{X} \to B$ be a smooth and proper family of $K3$ surfaces over an analytic space $B$ and $v$ a continuous primitive section of the local system $R\pi_\ast \mathcal{Z}$ of Mukai lattices. Let $p : \mathcal{M} \to B$ be a smooth and proper family of irreducible holomorphic symplectic manifolds, such that each fiber $\mathcal{M}_b$ of $p$ is isomorphic to the moduli space $M_{H_b}(v_b)$ of $H_b$-stable sheaves with Mukai vector $v_b$ over the fiber $\mathcal{X}_b$ of $\pi$ for some polarization $H_b$ over $\mathcal{X}_b$. We do not assume that $H_b$ varies continuously (see for example [Y1, Prop. 5.1]). There exists a twisted sheaf $\mathcal{W}$ over $\mathcal{X} \times_B \mathcal{M}$, flat over $B$, such that its restriction $\mathcal{W}_b$ to $\mathcal{X}_b \times_B \mathcal{M}_b$ is a universal sheaf for the coarse moduli space $M_{H_b}(v_b)$, by the appendix in [Mu5]. Denote by $\mathcal{A}_b$ the monad object over $\mathcal{X}_b \times_B \mathcal{M}_b$ associated to $\mathcal{W}_b$. The construction of the morphism $\alpha$, given in Equation (2.5) above, goes through in this relative setting to yield a family of morphisms $\alpha_b : \Delta_\ast \mathcal{O}_{\mathcal{X}_b} \otimes_{\mathcal{C}} \lambda_n \to \mathcal{A}_b$, $b \in B$, corresponding to a global morphism

$$\alpha : \Delta_\ast \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{C}} \mathcal{B} \left( \oplus_{i=0}^{n-1} R^2 p_\ast \mathcal{O}_{\mathcal{M}} [-2i] \right) \to \mathcal{A}$$

of monad objects over $\mathcal{X} \times_B \mathcal{X}$.

**Lemma 2.3.** The locus $B_0$ in $B$, where $\alpha_b$ is an isomorphism, is a Zariski open subset of $B$ in the analytic topology. The open subset $B_0$ is non-empty, whenever there exists a point $b_0 \in B$, a $K3$ surface $X$, and an equivalence of derived categories $D^b(\mathcal{X}_{b_0}) \to D^b(X)$, which maps the Mukai vector $v_{b_0}$ to that of the ideal sheaf of a length $n$ subscheme of $X$, and which maps $H_{b_0}$-stable sheaves on $\mathcal{X}_{b_0}$ to ideal sheaves on $X$.

**Proof.** $B_0$ is the complement of the image of the diagonal in $B$ of the support of the object $C_\alpha$ in $D^b(\mathcal{X} \times_B \mathcal{X})$ representing a cone of the morphism $\alpha$. The support of $C_\alpha$ is the union of the support of the sheaf cohomologies of $C_\alpha$. The support of $C_\alpha$ intersects the fiber $\mathcal{X}_b \times \mathcal{X}_b$ if and only if $\alpha_b$ is not an isomorphism, since a point $x \in \mathcal{X}_b \times \mathcal{X}_b$ belongs to the support of $C_\alpha$ if and only if one of the the cohomologies of $C_\alpha \otimes \mathcal{O}_x$ is non-zero, by [BM, Lemma 5.3]. The non-emptiness statement follows from Theorem 2.2 (1). \qed

A more explicit extension of the splitting Lemma 2.3 is given in Lemmas 6.9 and 6.16.
2.3. Splitting of the monad in the Hilbert scheme case. Write $X^{[n]}$ for the Hilbert scheme of $n$ points on $X$ and $S^n X$ for the $n$-th symmetric product of $X$. Denote by $\mu$ the Hilbert-Chow morphism from $X^{[n]}$ to $S^n X$, and let $\pi : X^n \to S^n X$ be the quotient by $\mathfrak{S}_n$. The following result is the main point of the proof of Theorem 2.2 for $M = X^{[n]}$; it allows us to transport calculations from the derived category of $X^{[n]}$ to the more combinatorial $\mathfrak{S}_n$-equivariant derived category of $X^n$. The latter is denoted by $D^b_{\mathfrak{S}_n}(X^n)$ below.

**Theorem 2.4.** ([Ha2], Corollary 5.1) Let $X$ be a smooth quasi-projective surface, and denote by $B_n$ the reduced fiber-product of $X^{[n]}$ and $X^n$ over $S^n X$:

$$
\begin{array}{ccc}
X^{[n]} & \xrightarrow{\mu} & S^n X \\
\downarrow p & & \downarrow \pi \\
X^n & \xrightarrow{q} & \end{array}
$$

Then, the map $q$ is flat, and $Rp_*q^* : D^b(X^{[n]}) \xrightarrow{\cong} D^b_{\mathfrak{S}_n}(X^n)$ is an equivalence of derived categories.

Let us give a quick word of explanation here. Denote by Hilb$^{\mathfrak{S}_n}(X^n)$ the $\mathfrak{S}_n$-Hilbert scheme of $X^n$, the fine moduli space whose closed points parametrize the $\mathfrak{S}_n$ orbits in $X^n$ with structure sheaves isomorphic to the regular representation of $\mathfrak{S}_n$. The flatness of the map $q$ above says that $B_n \subset X^n \times X^{[n]}$ is a family of such $\mathfrak{S}_n$ orbits parametrized by $X^{[n]}$. This yields a morphism from $X^{[n]}$ to Hilb$^{\mathfrak{S}_n}(X^n)$, which is in fact seen to be an isomorphism, identifying $B_n \to X^{[n]}$ with the universal family of Hilb$^{\mathfrak{S}_n}(X^n)$.

On the other hand, given a finite group $G$ acting nicely on a smooth projective variety $M$, the derived McKay correspondence of [BKR] relates the derived category of the $G$-Hilbert scheme $D^b(\text{Hilb}^G(M))$ and the $G$-equivariant derived category $D^b_G(M)$: whenever the map $\text{Hilb}^G(M) \to M/G$ is semismall, it says that the structure sheaf of the universal family gives a Fourier-Mukai type equivalence between these two categories. The map $\mu : X^{[n]} \to S^n X$ satisfies the semismallness hypothesis. This establishes the second statement of Theorem 2.4.

The following is a special case of a vanishing theorem proved in [Ha2]. Let $Z_n \subset X \times B_n$ be the pullback of the universal family $U_n \subset X \times X^{[n]}$ of $n$ points on $X$, and $D_n \subset X \times X^n$ the union of graphs of the $n$ projections to the $i$th factor, $\pi_i : X^n \to X$. Consider the following diagram

$$
\begin{array}{ccc}
& & Z_n \ar[ld] & \ar[r]^t & B_n \\
U_n \ar[r] & X \times B_n & \ar[l]^{\text{id} \times \pi = \hat{q}} & D_n \ar[r]^{\pi} & X^n \\
X \times X^{[n]} \ar[ru]^t & & & \ar[l]^{\text{id} \times q = \hat{q}}
\end{array}
$$

**Theorem 2.5** ([Ha2], Proposition 5.1). $R\hat{p}_*\mathcal{O}_{Z_n} \cong \mathcal{O}_{D_n}$.

---

3Nicely here means that the canonical bundle of $M$ is locally trivial as a $G$-sheaf.
Proposition 2.6. \(R\hat{p}_*\hat{q}^*(\mathcal{I}_{U_n}) \cong \mathcal{I}_{D_n}\), where \(\mathcal{I}_{U_n}\) is the ideal sheaf of \(U_n \subset X \times X^n\), and \(\mathcal{I}_{D_n}\) that of \(D_n \subset X \times X^n\).

Proof. Applying \(R\hat{p}_*\hat{q}^*\) to the sequence
\[
0 \rightarrow \mathcal{I}_{U_n} \rightarrow \mathcal{O}_{X \times X^n} \rightarrow \mathcal{O}_{U_n} \rightarrow 0,
\]
and using the previous result, we obtain the exact triangle
\[
R\hat{p}_*\mathcal{O}_{Z_n} \rightarrow R\hat{p}_*\mathcal{O}_{X \times B_n} \rightarrow \mathcal{O}_{D_n}.
\]
Thus, it suffices to show that \(R\hat{p}_*\mathcal{O}_{X \times B_n} \cong \mathcal{O}_{X \times X^n}\), or even that \(R\hat{p}_*\mathcal{O}_{B_n} \cong \mathcal{O}_{X^n}\). Now, \(Z_n \overset{t}{\rightarrow} B_n\) is flat and finite, being the pullback of \(U_n \rightarrow X^n\). Therefore, \(t_*\mathcal{O}_{Z_n}\) contains \(\mathcal{O}_{B_n}\) as a direct factor. Further, \(D_n \overset{s}{\rightarrow} X^n\) is finite, and \(s \circ \hat{p} = p \circ t\), so that \(s_*R\hat{p}_*\mathcal{O}_{Z_n} \cong s_*\mathcal{O}_{D_n} \cong R\hat{p}_*\mathcal{O}_{B_n}\) is concentrated in degree 0. Consequently \(R\hat{p}_*\mathcal{O}_{B_n}\) is concentrated in degree 0 also, and because \(p\) has connected fibers, we are done. (See also Prop. 1.3.3, [SC].) \(\square\)

Lemma 2.7. (i) Suppose the schemes and morphisms in the commutative diagram

\[
\begin{array}{ccc}
X & \overset{q}{\longrightarrow} & Z \\
\downarrow{p} & & \downarrow{\pi} \\
Y & \underset{\mu}{\longrightarrow} & W
\end{array}
\]

are such that \(Lq^*\) and \(p^!\) are defined between bounded derived categories (for example, if \(Y, Z\) are smooth and projective, and \(X\) is a closed subscheme of \(Y \times Z\)). If \(\Phi := R\hat{p}_*Lq^* : D^b(Y) \cong D^b(Z)\) is an equivalence, given \(\mathcal{M}, \mathcal{N} \in D^b(Y)\), we have a bifunctorial isomorphism
\[
\mu_*R\mathcal{H}om_{D^b(Y)}(\mathcal{M}, \mathcal{N}) \cong \pi_*R\mathcal{H}om_{D^b(Z)}(\Phi(\mathcal{M}), \Phi(\mathcal{N})�).
\]

(ii) Suppose \(G\) is a finite group, and the morphism \(p : X \rightarrow Z\) in part (i) is \(G\)-equivariant. Further, let \(Y = X/G, W = Z/G, q\) and \(\pi\) the quotient morphisms, and \(\mu : X/G \rightarrow Z/G\) the morphism induced by \(p\). Denote by \(D^b_G(Z)\) the derived category of equivariant coherent sheaves. Then, if \(\Phi := R\hat{p}_*Lq^* : D^b(Y) \cong D^b_G(Z)\) is an equivalence, there is a bifunctorial isomorphism
\[
\mu_*R\mathcal{H}om_{D^b(Y)}(\mathcal{M}, \mathcal{N}) \cong [\pi_*R\mathcal{H}om_{D^b_G(Z)}(\Phi(\mathcal{M}), \Phi(\mathcal{N}�))]^G.
\]

Proof. Both parts follow from essentially the same formal calculation using Grothendieck Duality. We provide a proof of part (ii). Denote by \(\pi^*_G\) the composition \([-]^G \circ \pi_*\); the symbols \(q^*_G, \mu^*_G\) are defined similarly. Then,
\[
\pi^*_G R\mathcal{H}om_{D^b_G(Z)}(p_*q^* \mathcal{M}, p_*q^* \mathcal{N}) \cong \pi^*_G p_*R\mathcal{H}om_{D^b_G(Z)}(q^* \mathcal{M}, p^!p_*q^* \mathcal{N})
\]
\[
\cong \mu_*q^*_G R\mathcal{H}om_{D^b_G(Z)}(q^* \mathcal{M}, p^!p_*q^* \mathcal{N})
\]
\[
\cong \mu_*[R\mathcal{H}om_{D^b(Y)}(\mathcal{M}, q^*_G p^! p_* q^* \mathcal{N})]^G
\]
\[
\cong \mu_* R\mathcal{H}om_{D^b(Y)}(\mathcal{M}, (q^*_G p^!)(p_* q^* \mathcal{N})�)
\]
\[
\cong \mu_* R\mathcal{H}om_{D^b(Y)}(\mathcal{M}, \mathcal{N})�.
\]

The first isomorphism is Grothendieck Duality, the second follows from the commutativity of the diagram above and the \(G\)-invariance of \(\mu\), the third is adjunction, and the fourth follows
from the $G$-invariance of $\mathcal{M}$ (see \cite{Sc}, Prop. 1.3.2). The last isomorphism follows from the fact that $q^* p^!$ is the right adjoint of $\Phi = p_* q^*$, and so also its quasi-inverse.

![Diagram](image)

We shall now apply this lemma to the diagram above: Let $p_{ij}$ stand for the projection from the product $X \times X^n \times X$ to the $(i,j)$-th factor. Consider the object $\mathcal{A}$, given in equation (1.2), in the case $M = X^{[n]}$ and $\mathcal{U} = \mathcal{U}_n$. In view of Proposition 2.6, we immediately have the isomorphism

\[ \mathcal{A} \cong \{ Rp_{13*}(p_{12}^* \mathcal{I}_{D_2} \otimes p_{23}^* \mathcal{I}_{D_3})[2]\}^\mathbb{S}_n. \]

Furthermore, the above is an isomorphism of $Y(\mathcal{O}_{X^{[n]}})$-modules, under the natural identification of $Y(\mathcal{O}_{X^{[n]}})$ with $Y(\mathcal{O}_{X^n})^\mathbb{S}_n$.

Denote by $\Delta_i \subset X \times X^n$, the graph of the $i$-th projection $\pi_i : X^n \to X$, and by $\Delta_I$, for $I \subset 1,\ldots,n$, the partial diagonal $\cap_{i \in I} \Delta_i$. By Corollary A.4 of \cite{Sc}, there is a Čech-type $\mathbb{S}_n$-equivariant resolution of $\mathcal{I}_{D_n}$ as follows:

\[ 0 \to \mathcal{I}_{\mathcal{D}_n} \to \mathcal{O}_{X \times X^n} \to \bigoplus_{i=1}^n \mathcal{O}_{\Delta_i} \to \cdots \to \bigoplus_{|I|=k} \mathcal{O}_{\Delta_I} \to \cdots \to \mathcal{O}_{\Delta_{(1,\ldots,n)}} \to 0. \]

As the diagonals $\Delta_i$ intersect transversally, it is easy to see that, in fact, this resolution is the tensor product of complexes

\[ \otimes_{i=1}^n \{ \mathcal{O}_{X \times X^n} \to \mathcal{O}_{\Delta_i} \}, \]

or alternatively,

\[ \mathcal{I}_{D_n} \cong \mathcal{I}_{\Delta_1} \otimes \mathcal{I}_{\Delta_2} \otimes \cdots \otimes \mathcal{I}_{\Delta_n} \]

in $D^b_{\mathbb{S}_n}(X \times X^n)$, where the $\mathcal{I}_{\Delta_i}$ are the ideal sheaves of the indicated diagonals.

Remark 2.8. The $\mathbb{S}_n$-linearization of the component $\bigoplus_{|I|=k} \mathcal{O}_{\Delta_I}$ in the complex above consists simply of permuting the factors $\mathcal{O}_{\Delta_I}$ according to the action of $\mathbb{S}_n$ on the indexing sets $I$. Tracing through the above calculation, it is easily seen that the $\mathbb{S}_n$-linearization of $(\mathcal{I}_{\Delta_1} \otimes \mathcal{I}_{\Delta_2} \otimes \cdots \otimes \mathcal{I}_{\Delta_n})$ is also the expected one, namely, permutation of factors.

The following calculation is due to Mukai. We present the details for the convenience of the reader

Lemma 2.9. (\cite{Ma}, Prop. 4.10). Let $p_{ij}$ be the $(i,j)$-th projection from $X \times X \times X$. Then, there is a natural isomorphism $\mathcal{B} := p_{13*}(p_{12}^* \mathcal{I}_{\Delta} \otimes p_{23}^* \mathcal{I}_{\Delta}) \cong H^2(X, \mathcal{O}_X) \otimes_{\mathcal{O}_{\Delta}} \mathcal{O}_{\Delta[-2]}$, where $\Delta$ is the diagonal in $X \times X$. Equivalently, the relative extension sheaves $\mathcal{E}_{xt}^j (p_{12}^* \mathcal{I}_{\Delta}, p_{23}^* \mathcal{I}_{\Delta})$ vanish, for $j \neq 2$, and $\mathcal{E}_{xt}^2 (p_{12}^* \mathcal{I}_{\Delta}, p_{23}^* \mathcal{I}_{\Delta}) \cong H^2(X, \mathcal{O}_X) \otimes_{\mathcal{O}_{\Delta}} \mathcal{O}_{\Delta}$.
Proof. The convolution of the exact triangle
\[ \mathcal{O}_\Delta \rightarrow \mathcal{O}_{X \times X} \rightarrow \mathcal{I}_\Delta \]
with \( \mathcal{I}_\Delta \) yields the following exact triangle on \( X \times X \):
\[ p_{13} \circ R \mathcal{H} \text{om}(p_{12}^* \mathcal{O}_\Delta, p_{23}^* \mathcal{I}_\Delta) \rightarrow p_{13} \circ p_{23}^* \mathcal{I}_\Delta \rightarrow \mathcal{B}. \]
By the use of flat base-change for the Cartesian diagram
\[
\begin{array}{ccc}
X \times X \times X & \xrightarrow{p_{12}} & X \times X \\
& \searrow \downarrow p_{13} \swarrow & \downarrow p_1 \\
& & X \times X \\
& \nearrow \uparrow p_1 & \rightarrow \uparrow \downarrow \end{array}
\]
the second term in the triangle is isomorphic to \( p_{13}^! p_1^* \mathcal{I}_\Delta \). It is then simple to conclude from the short exact sequence
\[ 0 \rightarrow \mathcal{I}_\Delta \rightarrow \mathcal{O}_{X \times X} \rightarrow \mathcal{O}_\Delta \rightarrow 0 \]
that \( p_{13} \circ p_{23}^* \mathcal{I}_\Delta \cong p_{13}^! p_1^* \mathcal{I}_\Delta \cong H^2(X, \mathcal{O}_X) \otimes_{\mathcal{O}_{X \times X}} \mathcal{O}_{X \times X}[-2] \). The first term in the triangle is computed as follows:
\[
p_{13} \circ R \mathcal{H} \text{om}(p_{12}^* \mathcal{O}_\Delta, p_{23}^* \mathcal{I}_\Delta) \cong p_{13} \circ R \mathcal{H} \text{om}(\mathcal{O}_{X \times X} \otimes \mathcal{O}_\Delta, \mathcal{O}_{X \times X}, p_{23}^* \mathcal{I}_\Delta) \]
\[
\cong p_{13} \circ \mathcal{O}_{X \times X} \circ R \mathcal{H} \text{om}(\mathcal{O}_\Delta, \mathcal{O}_{X \times X}, p_{23}^* \mathcal{I}_\Delta) \]
\[
\cong p_{13} \circ \mathcal{O}_{X \times X} \circ R \mathcal{H} \text{om}(\mathcal{O}_\Delta, \mathcal{O}_{X \times X}, p_{23}^* \mathcal{I}_\Delta) \otimes \mathcal{O}_{X \times X} \otimes \Delta_{12} \mathcal{O}_X[1] \]
\[
\cong p_{13} \circ \mathcal{O}_{X \times X} \circ R \mathcal{H} \text{om}(\mathcal{O}_\Delta, \mathcal{O}_{X \times X}, p_{23}^* \mathcal{I}_\Delta) \otimes \mathcal{O}_{X \times X} \otimes \Delta_{12} \mathcal{O}_X[1] \otimes \mathcal{O}_{X \times X} \otimes \mathcal{O}_{X \times X}[1] \]
\[
\cong p_{13}^! \mathcal{I}_\Delta \otimes \mathcal{O}_{X \times X} \otimes \Delta_{12} \mathcal{O}_X[-2]. \]

The first isomorphism is flat base-change, while the second is Grothendieck duality. As \( \text{Hom}(\mathcal{I}_\Delta, \mathcal{O}_{X \times X}) \cong \mathbb{C} \) and \( \mathcal{B} \) is supported on the diagonal, it follows that \( \mathcal{B} \) is isomorphic to \( H^2(X, \mathcal{O}_X) \otimes \mathcal{O}_{X \times X}[1] \).

Let us now calculate \( Rp_{13}^!(p_{12}^! \mathcal{O}_{D_2} \otimes p_{23}^* \mathcal{J}_n) \) when \( n = 2 \); the answer for general \( n \) will be apparent from this. Set \( X_1 = \cdots \rightarrow X_4 = X \), and for \( I \subset \{1, \ldots, 4\} \), let \( X_I := \times_{i \in I} X_i \). Denote by \( u_I : X_{\{1, \ldots, 4\}} \rightarrow X_I \) the projection to the \( I \)-th factor; similarly, for \( I \subset J \subset \{1, \ldots, 4\} \), \( |I| = 2, |J| = 3 \), let \( v_I : X_J \rightarrow X_I \) be the obvious projection.
We then have
\[ Rp_{13*}(p_{12}^*\mathcal{I}_{D_2} \otimes p_{23}^*\mathcal{I}_{D_2}) \cong u_{144}\{u_{12}^*\mathcal{I}_{\Delta}^\vee \otimes u_{13}^*\mathcal{I}_{\Delta}^\vee \otimes u_{24}^*\mathcal{I}_{\Delta} \otimes u_{34}^*\mathcal{I}_{\Delta}\} \]
\[ \cong v_{144}\{u_{12}^*v_{13}^*\mathcal{I}_{\Delta}^\vee \otimes v_{13}^*\mathcal{I}_{\Delta}^\vee \otimes v_{14}^*\mathcal{I}_{\Delta} \otimes (u_{12}^*\mathcal{I}_{\Delta}^\vee \otimes u_{24}^*\mathcal{I}_{\Delta})\} \]
\[ \cong v_{144}\{(v_{13}^*\mathcal{I}_{\Delta}^\vee \otimes v_{14}^*\mathcal{I}_{\Delta}) \otimes u_{134}(u_{12}^*\mathcal{I}_{\Delta}^\vee \otimes v_{14}^*\mathcal{I}_{\Delta})\} \]
\[ \cong v_{144}\{(v_{13}^*\mathcal{I}_{\Delta}^\vee \otimes v_{14}^*\mathcal{I}_{\Delta}) \otimes v_{14}^*(u_{12}^*\mathcal{I}_{\Delta}^\vee \otimes v_{24}^*\mathcal{I}_{\Delta})\} \]
\[ \cong v_{144}\{(v_{13}^*\mathcal{I}_{\Delta}^\vee \otimes v_{14}^*\mathcal{I}_{\Delta}) \otimes v_{14}^*(v_{12}^*\mathcal{I}_{\Delta}^\vee \otimes v_{24}^*\mathcal{I}_{\Delta})\} \]
\[ \cong (H^2(X, \Theta_X) \otimes \Theta_{\Delta}[−2]) \otimes (H^2(X, \Theta_X) \otimes \Theta_{\Delta}[−2]), \]
where the third isomorphism is the projection formula, the fifth is flat base-change and the last one is the isomorphism of the previous lemma. Clearly, the same working proves the

**Proposition 2.10.** \( Rp_{13*}(p_{12}^*\mathcal{I}_{D_n}^\vee \otimes p_{23}^*\mathcal{I}_{D_n}) \cong (H^2(X, \Theta_X) \otimes \Theta_{\Delta}[−2])^n \otimes \) as objects in the category \( D^b(S_n(X \times X)) \) (where \( S_n \)-acts trivially on \( X \times X \)). Moreover, the \( S_n \)-linearization of the tensor product on the right hand side is simply permutation of factors.

**Proof.** The linearization is clear from the calculation above and Remark 2.8. \( \square \)

**Lemma 2.11.** Let \( S \) be a scheme and \( F \) an object in \( D^b(S) \). Assume that the cohomology sheaves \( H^j(F) \) satisfy the condition: \( \text{Ext}^{k+1}(H^j(F), H^{−k}(F)) = 0, \forall j \) and \( \forall k > 0 \). Then \( F \) is isomorphic to \( \bigoplus_j H^j(F)[−j] \).

**Proof.** Set \( J_F := \{ j : H^j(F) \neq 0 \} \). The proof is by induction on the cardinality \( \sharp(J_F) \) of \( J_F \). Set \( b := \max(J_F) \) and \( a := \min(J_F) \). Then \( F \) is represented by a complex of coherent \( \Theta_S \)-modules
\[ F_a → F_{a+1} → \cdots → F_b, \]
with \( F_i \) in degree \( i \). If \( \sharp(J_F) = 1 \), then \( a = b \) and the statement holds trivially. Assume that the statement holds for every object \( F' \) satisfying \( \sharp(J_{F'}) < n \), where \( n := \sharp(J_F) \). Set \( A := H^a(F) \).
Let \( F' \) be the cone of the natural morphism \( H^a(F)[−a] → F \). Then \( \sharp(J_{F'}) = n−1, H^a(F') = 0, \) and \( H^i(F') = H^i(F) \), for \( i \neq a \). The equivalence \( F' = \bigoplus_j H^j(F')[−j] \) follows, by the induction hypothesis. We get the exact triangle
\[ A[−a] → F → F' → A[1−a]. \]

The morphism \( δ \) decomposes as the sum of the morphisms
\[ δ_j \in \text{Hom}(H^j(F')[−j], A[1−a]) = \text{Ext}^{1+j−a}(H^j(F'), H^a(F)), \]
for \( j > a \). These groups vanish, by assumption. Hence, \( δ = 0 \), and \( F \) is isomorphic to the direct sum of \( A[−a] \) and \( F' \). \( \square \)

**Proof of Theorem 2.2 part 7.**

Step 1: To compute the \( S_n \)-invariants in \( Rp_{13*}(p_{12}^*\mathcal{I}_{D_n}^\vee \otimes p_{23}^*\mathcal{I}_{D_n}) \), we first calculate the \( S_n \)-invariant parts of the the \( n \)-fold multi-tors \( Tor^q_n(\Theta_{\Delta}) := Tor_q(\Theta_{\Delta}, \Theta_{\Delta}, \cdots, \Theta_{\Delta}) \). This is precisely the sort of calculation that is carried out in Corollary B.7 in [12]. In our particular situation this says that \( Tor^q_n(\Theta_{\Delta}) \) has nonzero \( S_n \)-invariants if and only if \( q = 2h, 0 \leq h \leq n−1 \), in which case
\[ Tor^q_{2h}(\Theta_{\Delta}) \cong (\Lambda N_{\Delta\times X \times X})^{\otimes h} \cong H^0(X, \omega_X)^{\otimes h} \otimes \Theta_{\Delta}. \]
Equivalently, the relative extension sheaves $\mathcal{E}xt^j(\pi^*_U \mathcal{I}_U, \pi^*_V \mathcal{I}_U)$ are isomorphic to $\Delta_*\mathcal{O}_X$, for $j$ even in the range $2 \leq j \leq 2n$, and vanish for all other values of $j$.

The hypotheses of Lemma 2.11 applied to the object $R\pi_{13*}(\pi^*_U \mathcal{I}_U \otimes \pi^*_V \mathcal{I}_U)$, are satisfied, since the odd cohomologies of this object vanish, and $\text{Ext}^i_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = 0$ for odd $i$. Lemma 2.11 implies the existence of an isomorphism

$$R\pi_{13*}(\pi^*_U \mathcal{I}_U \otimes \pi^*_V \mathcal{I}_U) \cong \bigoplus_{i=0}^n H^2(X, \mathcal{O}_X)^{\otimes i} \otimes \mathcal{O}_\Delta \mathcal{O}_X[-2i].$$

Note that the $i$-th summand on the right hand side is naturally isomorphic to the sheaf cohomology of degree $2i$ of the left hand side, namely the relative extension sheaf $\mathcal{E}xt^i_{\pi^{13*}_U} (\pi^*_U \mathcal{I}_U, \pi^*_V \mathcal{I}_U)$. The isomorphism $\mathcal{A} \cong \bigoplus_{i=0}^n H^2(X, \mathcal{O}_X)^{\otimes i} \otimes \mathcal{O}_\Delta \mathcal{O}_X[-2i]$ follows immediately from Equation (2.6). In particular, the functor $\Phi_\gamma$ is faithful.

Step 2: We prove in this step the following claim.

**Claim 2.12.** For $i$ even, we have the canonical short exact sequence

$$0 \to \mathcal{H}^i(\mathcal{A}) \overset{\xi^i}{\to} \Delta_* \left[ \mathcal{O}_X \otimes \mathcal{E}xt^i_{\pi^*_U} (\mathcal{I}_U, \mathcal{I}_U) \right] \overset{\xi^i}{\to} \mathcal{H}^{i+2}(\mathcal{A}) \otimes \pi^*_1 \mathcal{O}_X \to 0.$$

For $i$ odd, we have the following natural isomorphism:

$$\Delta_* \left[ \mathcal{O}_X \otimes \mathcal{E}xt^i_{\pi^*_U} (\mathcal{I}_U, \mathcal{I}_U) \right] \overset{\xi^i}{\to} \mathcal{H}^{i+1}(\mathcal{A}) \otimes \pi^*_1 (T^*X).$$

**Proof.** Let $\Delta_{13} : X \times M \to X \times M \times X$ be the diagonal embedding. Consider the short exact sequence:

$$0 \to \pi^*_U \mathcal{I}_{\Delta X} \to \mathcal{O}_{X \times M \times X} \to \Delta_{13*} \mathcal{O}_{X \times M} \to 0.$$

Tensoring with $\tilde{\mathcal{A}} := \pi^*_1 \mathcal{O}_X [2] \otimes \pi^*_2 \mathcal{I}_U \otimes \pi^*_3 \mathcal{I}_U$ we get the exact triangle:

$$\tilde{\mathcal{A}} \otimes \pi^*_{13} \mathcal{I}_{\Delta X} \overset{\xi}{\to} \tilde{\mathcal{A}} \otimes \pi^*_1 \mathcal{O}_X [2] \otimes \Delta_{13*} (\mathcal{I}_U \otimes \mathcal{I}_U) \overset{\xi}{\to} \tilde{\mathcal{A}} \otimes \pi^*_{13} \mathcal{I}_{\Delta X} [1]$$

Applying the derived push-forward $\pi_{13*}$ to the exact triangle (2.7) we get the following exact triangle in $D^b(X \times X)$.

$$\mathcal{A} \otimes \mathcal{I}_{\Delta X} \overset{\xi}{\to} \mathcal{A} \otimes \Delta_* \left[ \mathcal{O}_X [2] \otimes \pi^*_X (\mathcal{I}_U \otimes \mathcal{I}_U) \right] \overset{\xi}{\to} \mathcal{A} \otimes \mathcal{I}_{\Delta X} [1].$$

Let $\iota : \mathcal{I}_{\Delta X} \to \mathcal{O}_{X \times X}$ be the natural inclusion. The object $\mathcal{A}$ has been shown to be a direct sum of sheaves supported on the diagonal, and the morphism $g$ is the composition $\mathcal{A} \otimes \mathcal{I}_{\Delta X} \xrightarrow{\xi\iota} \mathcal{A} \otimes \mathcal{O}_{X \times X} \cong \mathcal{A}$. Hence, the morphism $g$ vanishes and we get a short exact sequence, for every integer $i$.

$$0 \to \mathcal{H}^i(\mathcal{A}) \overset{\xi^i}{\to} \Delta_* \left[ \mathcal{O}_X [2] \otimes \pi^*_X (\mathcal{I}_U \otimes \mathcal{I}_U) \right] \overset{\xi^i}{\to} \mathcal{H}^{i+1}(\mathcal{A} \otimes \mathcal{I}_{\Delta X}) \to 0.$$

It remains to calculate the sheaf cohomologies $\mathcal{H}^{i+1}(\mathcal{A} \otimes \mathcal{I}_{\Delta X} [1])$. Note first the isomorphisms

$$\text{Tor}_{0}^{X \times X}(\mathcal{O}_{\Delta X}, \mathcal{I}_{\Delta X}) \cong \Delta_* T^* X,$$

$$\text{Tor}_{1}^{X \times X}(\mathcal{O}_{\Delta X}, \mathcal{I}_{\Delta X}) \cong \Delta_* \mathcal{O}_X,$$

and $\text{Tor}_i^{X \times X}(\mathcal{O}_{\Delta X}, \mathcal{I}_{\Delta X}) = 0$, if $i$ is not equal to 0 or $-1$. We have already shown that $\mathcal{A}$ is isomorphic to $\bigoplus_{i=0}^{n-1} \mathcal{H}^{2i}(\mathcal{A})$. For even $i$ we get the isomorphism

$$\mathcal{H}^{i+1}(\mathcal{A} \otimes \mathcal{I}_{\Delta X}) \cong \text{Tor}_{i+1}^{X \times X}(\mathcal{H}^{i+2}(\mathcal{A}), \mathcal{I}_{\Delta X}) \cong \mathcal{H}^{i+2}(\mathcal{A}) \otimes \pi^*_1 \mathcal{O}_X.$$

For odd $i$ we get:

$$\mathcal{H}^{i+1}(\mathcal{A} \otimes \mathcal{I}_{\Delta X}) \cong \text{Tor}_{i}^{X \times X}(\mathcal{H}^{i+1}(\mathcal{A}), \mathcal{I}_{\Delta X}) \cong \mathcal{H}^{i+1}(\mathcal{A}) \otimes \pi^*_1 (T^* X).$$

\[\square\]
Step 3: We have the isomorphisms $H^2(\mathcal{I}) \cong \pi_1^* \omega_X \otimes E xt^{2i+2}_{\pi_1^*}(\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U)$, by definition of $\mathcal{I}$. The homomorphisms
\[ e^{j-2} : E xt^j_{\pi_1^*} (\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U) \rightarrow \Delta_* \left( E xt^j_{\pi_X^*} (\mathcal{I}_U, \mathcal{I}_U) \right) \]
are injective, for all $j$, and $e^{2n-2}$ is an isomorphism, by Claim [2.12]. Let $t_M$ be a non-zero element of $H^2(M, \mathcal{O}_M)$, where $M = X^{[n]}$. Then $t_M$ induces a morphism from $\pi_{13, *}(\pi_{12}^* \mathcal{I}_U^\vee \otimes \pi_{23}^* \mathcal{I}_U)$ to $\pi_{13, *}(\pi_{12}^* \mathcal{I}_U^\vee \otimes \pi_{23}^* \mathcal{I}_U)$ [2]. We get an induces homomorphism of relative extension sheaves
\[ t_M : E xt^i_{\pi_1^*} (\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U) \rightarrow E xt^{i+2}_{\pi_1^*} (\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U), \]
via the morphism [2.3]. We prove in this step the following statement.

**Claim 2.13. The sheaf homomorphism**
\[ (2.9) \quad t_M^i : E xt^i_{\pi_1^*} (\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U) \rightarrow E xt^{i+2}_{\pi_1^*} (\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U) \]
is an isomorphism, for $0 \leq i \leq n - 1$.

**Proof.** We have the homomorphism
\[ \mu_{\mathcal{I}_U} (t_M) : E xt^i_{\pi_X^*} (\mathcal{I}_U, \mathcal{I}_U) \rightarrow E xt^{i+2}_{\pi_X^*} (\mathcal{I}_U, \mathcal{I}_U). \]
The composition $\mu_{\mathcal{I}_U} : E xt^i_{\pi_X^*} (\mathcal{I}_U, \mathcal{I}_U) \rightarrow E xt^{i+2}_{\pi_X^*} (\mathcal{I}_U, \mathcal{I}_U)$ is the identity, since rank$(\mathcal{I}_U) = 1$. Thus, the composition
\[ R^0_{\pi_X^*} \mu_{\mathcal{I}_U} \rightarrow E xt^0_{\pi_X^*} (\mathcal{I}_U, \mathcal{I}_U) \rightarrow E xt^2_{\pi_X^*} (\mathcal{I}_U, \mathcal{I}_U) \rightarrow R^2_{\pi_X^*} \mu_{\mathcal{I}_U} \rightarrow \cdots \]
is multiplication by $t_M^n$. Hence, the following is an isomorphism
\[ \mu_{\mathcal{I}_U} (t_M) : E xt^0_{\pi_X^*} (\mathcal{I}_U, \mathcal{I}_U) \rightarrow E xt^{2n}_{\pi_X^*} (\mathcal{I}_U, \mathcal{I}_U). \]
The exact triangle [2.8] is the image of one in $D^b(X \times M \times X)$ and lifts to an exact triangle in the category $D^b(X \times X)$ for the monad $\mathcal{Y}$ given in [2.2]. Taking cohomology we get the following commutative diagram:

We are ready to prove that the homomorphism (2.9) is an isomorphism. The proof is by contradiction. Assume otherwise and let $i$ be the minimal integer in the range $1 \leq i \leq n - 1$, such that
\[ (2.10) \quad t_M^i : E xt^i_{\pi_1^*} (\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U) \rightarrow E xt^{i+2}_{\pi_1^*} (\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U) \]
is not an isomorphism. Then the above homomorphism must vanish. Hence,
\[ (2.11) \quad t_M^i \otimes \pi_1^* \omega_X : E xt^i_{\pi_1^*} (\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U) \otimes \pi_1^* \omega_X \rightarrow E xt^{i+2}_{\pi_1^*} (\pi_{12}^* \mathcal{I}_U, \pi_{23}^* \mathcal{I}_U) \otimes \pi_1^* \omega_X \]
vanishes as well. Consider the following (abbreviated) commutative diagram with short exact columns.

\[
\begin{array}{cccccccc}
0 & \to & \mathcal{E}xt^2_{\pi_13} & \to & t_M & \to & \mathcal{E}xt^2_{\pi_13} & \to & 0 \\
\varepsilon^{-2} & & \varepsilon^0 & & e^0 & & e^{2i-2} & & e^{2i} \\
\Delta_*\mathcal{E}xt^0_{\pi X} & \xrightarrow{\mu_{\mathcal{S}_U}(t_M)} & \Delta_*\mathcal{E}xt^2_{\pi X} & \to & \Delta_*\mathcal{E}xt^2_{\pi X} & \to & \Delta_*\mathcal{E}xt^2_{\pi X} & \to & \Delta_*\mathcal{E}xt^2_{\pi X} \\
f^{-2} & \cong & f^0 & & \eta^{2i-2} & & \eta^{2i} & & \eta^{2i} \\
\mathcal{E}xt^2_{\pi_13} \otimes \pi_1^*\omega_X & \xrightarrow{t_M \otimes \pi_1^*1_{\omega_X}} & \mathcal{E}xt^4_{\pi_13} \otimes \pi_1^*\omega_X & \to & \mathcal{E}xt^2_{\pi_13} \otimes \pi_1^*\omega_X & \to & \mathcal{E}xt^2_{\pi_13} \otimes \pi_1^*\omega_X \\
\end{array}
\]

The vanishing of (2.11), the fact that \( f^{-2} \) is an isomorphism, and the exactness of the columns in the above diagram, combine to imply that the image of

\[
\Delta_*, (\mu_{\mathcal{S}_U}(t_M))^{i+1} : \Delta_*\mathcal{E}xt^0_{\pi X} (\mathcal{I}_U, \mathcal{I}_U) \to \Delta_*\mathcal{E}xt^2_{\pi X} (\mathcal{I}_U, \mathcal{I}_U)
\]

is contained in the image of the injective homomorphism

\[
e^{2i-2} : \mathcal{E}xt^2_{\pi_13} (\pi_1^*\mathcal{I}_U, \pi_2^*\mathcal{I}_U) \to \Delta_*\mathcal{E}xt^2_{\pi X} (\mathcal{I}_U, \mathcal{I}_U).
\]

Hence, the injective homomorphism

\[
\Delta_*, (\mu_{\mathcal{S}_U}(t_M))^{i+1} : \Delta_*\mathcal{E}xt^0_{\pi X} (\mathcal{I}_U, \mathcal{I}_U) \to \Delta_*\mathcal{E}xt^2_{\pi_13} (\mathcal{I}_U, \mathcal{I}_U)
\]

factors through

\[
t_M : \mathcal{E}xt^2_{\pi_13} (\pi_1^*\mathcal{I}_U, \pi_2^*\mathcal{I}_U) \to \mathcal{E}xt^2_{\pi_13} (\pi_1^*\mathcal{I}_U, \pi_2^*\mathcal{I}_U).
\]

In particular, the above homomorphism does not vanish. Hence, the above homomorphism is an isomorphism. The minimality of \( i \) implies that

\[
i^{i-1}_M : \mathcal{E}xt^2_{\pi_13} (\pi_1^*\mathcal{I}_U, \pi_2^*\mathcal{I}_U) \to \mathcal{E}xt^2_{\pi_13} (\pi_1^*\mathcal{I}_U, \pi_2^*\mathcal{I}_U)
\]

is an isomorphism. It follows that the homomorphism (2.10) is an isomorphism. A contradiction. This complete the proof of Claim 2.13.

Step 4: Let \( \iota_2 : \mathcal{E}xt^2_{\pi_13} (\pi_1^*\mathcal{I}_U, \pi_2^*\mathcal{I}_U) \to R\pi_{13}\left(\pi_1^*\mathcal{I}_U \otimes \pi_2^*\mathcal{I}_U\right) \) be the natural morphism from the first non-vanishing sheaf cohomology of the object to the object itself. Let

\[
m : R\pi_{13}\left(\pi_1^*\mathcal{I}_U \otimes \pi_2^*\mathcal{I}_U\right) \otimes \bigcirc Y(\mathcal{O}_M) \to R\pi_{13}\left(\pi_1^*\mathcal{I}_U \otimes \pi_2^*\mathcal{I}_U\right)
\]

be the natural morphism. The morphism of \( Y(\mathcal{O}_M) \)-modules

\[
m \circ (\iota_2 \otimes id) : \mathcal{E}xt^2_{\pi_13} (\pi_1^*\mathcal{I}_U, \pi_2^*\mathcal{I}_U) \otimes \bigcirc Y(\mathcal{O}_M) \to R\pi_{13}\left(\pi_1^*\mathcal{I}_U \otimes \pi_2^*\mathcal{I}_U\right)
\]

induces an isomorphism of the sheaf cohomologies for degrees between 2 and \( 2n \), by Claim 2.13. Thus, the composite morphism \( \alpha \) in equation (2.5) induces an isomorphism of all sheaf cohomologies, i.e., for degrees in the range from 0 to \( 2n - 2 \).

2.4. Hochschild (co)homology. We present a brief review here of some concepts and definitions on this topic which will be used in the proof of part 2 of Theorem 2.2, in the proof of Proposition 3.16, and also later in Section 7. We follow the presentation in [C2, CW]. Those familiar with the Hochschild (co)homology of varieties should skip to Section 2.5.

Let \( T \) be a smooth, projective variety over \( \mathbb{C} \). Write \( \Delta_T : T \to T \times T \) for the diagonal, and \( \pi_i : T \times T \to T, \ i = 1, 2 \), for the projections; denote by \( S_{\Delta_T} \) the kernel of the Serre functor \( \Delta_*\omega_T[\dim T] \), and by \( S_{\Delta_T}^{-1} \) the object \( \Delta_*\omega_T^{-1}[-\dim T] \). Let \( \Delta_T ! : D^b(T) \to D^b(T \times T) \) be the left adjoint of \( \Delta_T^* \).
Both the ring and module structures are defined by Yoneda composition in harmonic structure (ii) The homomorphism and a graded left module $HH$ and a graded module $H$ via convolution; define (2.13) We shall also have occasion to work with the following modified isomorphisms (2.12) Exterior product and contraction define the ring and module structures, respectively. These two structures are related by the Hochschild-Kostant-Rosenberg isomorphism (2.14) where $\eta$ and $\epsilon$ are the natural unit and the counit, respectively. Notice that we have the isomorphism (2.17) 20 E. MARKMAN AND S. MEHROTRA
by Serre duality on $T \times T$, and that $\text{Hom}_{Y \times Y}(\mathcal{O}_{\Delta_Y}, S_{\Delta_Y}[i]) \cong HH_i(Y)^\vee$ in the same way. The above construction defines a homomorphism

$$\Phi^*: HH_i(Y)^\vee \to HH_i(T)^\vee,$$

$$\nu \mapsto \Phi^* \nu.$$

Then the desired map $\Phi_*$ is the transpose of $\Phi^*$ under these identifications.

Recall that any integral transform $\Phi$ induces a map $\varphi: H^*(T, \mathbb{C}) \to H^*(Y, \mathbb{C})$ on singular cohomology. We have the following result stating the compatibility of this map with $\Phi_*$ under the HKR isomorphism:

**Theorem 2.15** ([MSM]). Let $\Phi: D^b(T) \to D^b(Y)$ be an integral transform. Then, the following diagram commutes.

$$
\begin{array}{ccc}
HH_\ast(T) & \xrightarrow{\Phi_*} & HH_\ast(Y) \\
\phi^T & & \phi^T \\
H\Omega_\ast(T) & \xrightarrow{\varphi} & H\Omega_\ast(Y)
\end{array}
$$

2.5. Splitting of the monad for a general moduli space. Let $v \in H^*(X)$ be a primitive and effective class with $\langle v, v \rangle = 2n - 2$, $n \geq 2$, and, as above, let $H$ be a $\nu$-generic polarization. We recall the fundamental result on the second cohomology of the moduli space $M_H(v)$.

Write $v^\perp \subset H^*(X, \mathbb{Z})$ for the sublattice orthogonal to $v$. Mukai introduced the natural homomorphism

$$(2.18) \quad \theta_v: v^\perp \to H^2(M_H(v), \mathbb{Z})$$

$$\theta_v(x) \mapsto \frac{1}{\rho} [\pi_{M,\ast}(v(\mathcal{E}^\vee) \cdot \pi_X^\ast(x))]_2$$

where $\mathcal{E}$ is a quasi-universal family of similitude $\rho$ (see [Mu3, Mu4]).

**Theorem 2.16** ([Hu2] [OG1] [Y1] [Y2]).

1. The moduli space $M_H(v)$ is an irreducible projective holomorphic symplectic manifold deformation equivalent to the Hilbert scheme of $n$ points on $X$.

2. The homomorphism $(2.18)$ is a Hodge isometry between $v^\perp$ and $H^2(M_H(v), \mathbb{Z})$, where the lattice structure on the latter is given by the Beauville-Bogomolov form.

**Proof of Theorem 2.2 part 2.** We treat first the case where an untwisted universal sheaf $\mathcal{U}$ exists over $X \times M_H(v)$. Let $\eta: \mathcal{O}_{\Delta_X} \to \mathcal{U} = \mathcal{U} [2] \circ \mathcal{U}$ be the morphism in $D^b(X \times X)$ corresponding to the unit of the adjunction $\Phi_{\mathcal{U}} \dashv \Psi_{\mathcal{U}}$ (see [C2], Prop. 5.1). It suffices to show that pre-composition induces a surjection:

$$\text{Hom}_{X \times X}(\mathcal{U} [2] \circ \mathcal{U}, \mathcal{O}_{\Delta_X}) \xrightarrow{\text{on}} \text{Hom}_{X \times X}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}).$$

Note that since $S_{\Delta_X} = \mathcal{O}_{\Delta_X} [2]$, by (2.17), we may interpret this as a map

$$\text{Hom}(\mathcal{U} [2] \circ \mathcal{U}, S_{\Delta_X} [-2]) \to HH_{-2}(X)^\vee.$$

Thus the construction (2.16) gives homomorphisms

$$HH_{-2}(M_H(v))^\vee = \text{Hom}(\mathcal{O}_{\Delta_M(v)}, S_{\Delta_M(v)} [-2]) \to \text{Hom}(\mathcal{U} [2] \circ \mathcal{U}, S_{\Delta_X} [-2]) \xrightarrow{\text{on}} HH_{-2}(X)^\vee$$

whose transpose is the natural map in Hochschild homology induced by $\Phi_{\mathcal{U}}$. So it is enough to show that $\Phi_{\mathcal{U}, \ast}: HH_{-2}(X) \to HH_{-2}(M_H(v))$ is injective. Now, note that $I_{\ast}^X(HH_{-2}(X)) \subset$
$H\Omega_{-2}(X) = H^2(\mathcal{O}_X)$ as $\tilde{I}^X_*$ is a graded map. Therefore, by Theorem 2.15,

$$\Phi_{\mathcal{F}}|_{H\Omega_{-2}(X)} = (\tilde{I}^X_*)^{-1} \circ [\varphi_{\mathcal{F}}]_2 \circ \tilde{I}^X_*,$$

where $[\varphi_{\mathcal{F}}]_2$ is the degree 2 part of the map on singular cohomology induced by $\Phi_{\mathcal{F}}$. But observe that by the formula (2.18), $\theta_v = -[\varphi_{\mathcal{F}}]_2$, whence, by Theorem 2.16 $[\varphi_{\mathcal{F}}]_2$ is injective. This proves the result for fine moduli spaces.

We sketch next the proof in the case where the universal sheaf $\mathcal{U}$ is twisted with respect to a Brauer class $\alpha \in H^2_{et}(M_H(v), \mathcal{O}^*)$. Let $F$ be an $H$-stable sheaf of class $v$ and denote by $[F]$ the corresponding point of $M_H(v)$. Let $\beta : \hat{M} \to M_H(v)$ be the blow-up centered at $[F]$. The sheaf cohomology $\mathcal{H}^1(\mathcal{F}_\mathcal{U}(F^\vee))$ is an $\alpha$-twisted reflexive sheaf of rank $2n - 2$ over $M_H(v)$, which is locally free away from the point $[F]$, and the quotient $V$ of $\beta^*\mathcal{H}^1(\mathcal{F}_\mathcal{U}(F^\vee))$ by its torsion subsheaf is a locally free $\beta^*\alpha$-twisted sheaf [Ma5, Prop. 4.5]. Let $p : \mathbb{P}(V) \to M$ be the associated projective bundle. Set $\tilde{\mathcal{U}} := (1 \times \beta p)^*\mathcal{U}$, and let $\pi_{ij}$ be the projection from $X \times \mathbb{P} V \times X$ to the product of the $i$-th and $j$-th factors. Set

$$\mathcal{B} := R\pi_{13*} \left[ \pi_{12*}^* (\tilde{\mathcal{U}})^{\vee} \otimes \pi_{23*}^* \tilde{\mathcal{U}} \otimes \pi_{11*}^* \omega_X [2] \right]$$

Then we have the natural isomorphism $\mathcal{A} := R\pi_{13*} \left[ \pi_{12*}^* \mathcal{U}^{\vee} \otimes \pi_{23*} \mathcal{U} \otimes \pi_{11*}^* \omega_X [2] \right] \cong \mathcal{B}$, where the latter isomorphism follows from the projection formula and the isomorphism $R(\beta p)_* \mathcal{O}_{\mathbb{P} V} \cong \mathcal{O}_{M_H(v)}$.

The Brauer class $p^*\beta^*\alpha$ is trivial, by [Ma2, Lemma 29(4)]. We thus have an equivalence of triangulated categories $D^b(X \times \mathbb{P} V, p^*\beta^*\alpha) \cong D^b(X \times \mathbb{P} V)$ and the image of $\mathcal{U}$ is represented by an untwisted coherent sheaf $\mathcal{G}$. We get an induced isomorphism

$$\mathcal{B} \cong R\pi_{13*} \left[ \pi_{12*}^* \mathcal{G}^{\vee} \otimes \pi_{23*}^* \mathcal{G} \otimes \pi_{11*}^* \omega_X [2] \right].$$

Replacing $M_H(v)$ by $\mathbb{P} V$ and $\mathcal{G}$ by $\mathcal{G}$ in Equation (2.18) we get an analogue $\tilde{\theta}_v : v^\perp \to H^2(\mathbb{P} V, \mathbb{Z})$ of the Mukai homomorphism. The homomorphism $\tilde{\theta}_v$ is the composition of $p^*\beta^* : H^2(M_H(v), \mathbb{Z}) \to H^2(\mathbb{P} V, \mathbb{Z})$ with the Mukai homomorphism (2.18). Indeed, the argument which shows that the Mukai homomorphism is independent of the choice of a quasi-universal sheaf shows also that using either $(1 \times \beta p)^* \mathcal{E}$ or the direct sum $\mathcal{G}^\oplus \rho$ of $\rho$ copies of $\mathcal{G}$ in Equation (2.18) results in the same homomorphism.

Theorem 2.15 applies now to the integral transform $\Phi_\mathcal{G} : D^b(X) \to D^b(\mathbb{P} V)$ with kernel $\mathcal{G}$ and the argument in the case of untwisted universal sheaf goes through to show that $\mathcal{O}_{\Delta X}$ is a direct summand of $\mathcal{B}$, and hence of $\mathcal{A}$ as well.

3. YONEDA ALGEBRAS

We prove Theorem 1.3 in this section. The reader interested only in the results on generalized deformations (Theorems 1.8 and 1.9) may skip this section. Let $X$ be a projective $K3$ surface, $M := M_H(v)$ a moduli space of $H$-stable sheaves with Mukai vector $v$ satisfying the hypothesis of Theorem 2.2 and $\Phi_{\mathcal{U}} : D^b(X) \to D^b(M, \theta)$ the faithful functor in Theorem 2.2. Assume that $(v, v) \geq 2$.

**Definition 3.1.** We say that the monad object $\mathcal{A}$ given in Equation (1.2) is totally split, if the composition $\alpha$ given in Equation (2.5) is an isomorphism.

Assume for the rest of section 3 that the monad object $\mathcal{A}$ is totally split. This is the case if $M$ is the Hilbert scheme $X^{[n]}$ and $\mathcal{U}$ is the universal ideal sheaf, by Theorem 2.2. For a more general sufficient condition for $\alpha$ to be an isomorphism, see Lemmas 6.3 and 6.10.
Set \( pt := \text{Spec}(\mathbb{C}) \) and let \( c : M \to pt \) be the constant morphism. We get the object \( Y(\mathcal{O}_M) := R\pi_* \mathcal{O}_M \) in \( D^b(pt) \). As a graded vector space \( Y(\mathcal{O}_M) \) is \( \bigoplus_{i=0}^n H^{2i}(M, \mathcal{O}_M)[-2i] \), where \( n = \dim_{\mathbb{C}}(M)/2 \). Given a graded vector space \( V \), let \( 1_{D^b(X)} \otimes V \) be the endofunctor of \( D^b(X) \) sending an object \( x \) to \( x \otimes_{\mathbb{C}} V \). Set
\[
\Upsilon := 1_{D^b(X)} \otimes_{\mathbb{C}} Y(\mathcal{O}_M), \\
R := 1_{D^b(X)} \otimes_{\mathbb{C}} H^{2n}(M, \mathcal{O}_M)[-2n].
\]

We define next a natural transformation
\[
h : R \to \Upsilon.
\]
Write \( h = \sum_{i=0}^n h_{2i} \) according to the direct sum decomposition of \( Y(\mathcal{O}_M) \), so that
\[
h_{2i} : 1_{D^b(X)} \otimes_{\mathbb{C}} H^{2n}(M, \mathcal{O}_M)[-2n] \to 1_{D^b(X)} \otimes_{\mathbb{C}} H^{2n-2i}(M, \mathcal{O}_M)[2i-2n].
\]

Let \( t_X \) be a non-zero element of \( H^2(X, \mathcal{O}_X) \), considered as the subspace \( H^{0,2}(X) \) of the complexified Mukai lattice, and let \( t_M \) be its image in \( H^2(M, \mathcal{O}_M) \) via Mukai’s Hodge isometry (2.13). Denote by \( t^*_M : Y(\mathcal{O}_M) \to Y(\mathcal{O}_M) \) the homomorphism, which sends \( t^i_M \) to \( t^{i-1}_M \), \( 1 \leq j \leq n \), and sends 1 to 0. The choice of \( t_M \) identifies \( h_{2i} \) as an element of the Hochschild cohomology \( HH^{3i}(X) \). Explicitly, \( h_{2i} = \tilde{h}_{2i} \otimes (t^*_M)^i \), where \( \tilde{h}_{2i} \) belongs to \( \text{Ext}^{2i}_{X,\mathcal{O}_X}(\mathcal{O}_\Delta_X, \mathcal{O}_\Delta_X) \). Let \( \sigma_X \) be the class in \( H^0(X, \omega_X) \) dual to the class \( t_X \) with respect to Serre’s duality. Given \( h_2 \), the class \( \tilde{h}_2 \otimes \sigma_X \) in \( \text{Hom}_{X,\mathcal{O}_X}(\Delta_{X,*}(\mathcal{O}_X), \Delta_{X,*}(\omega_X)[2]) \) is a class in \( HH_0(X) \), which depends canonically on \( h_2 \) and is independent of the choice of the class \( t_X \), since \( t_M \) depends on \( t_X \) linearly. Hence, the choice of \( h_2 \) corresponds to a choice of a class in \( HH_0(X) \), which we denote by \( h_2 \) as well. Let \( h_2 \) be the class in \( HH_0(X) \) which is mapped to the Chern character \( ch(v) \) in \( H\Omega_0(X) \) of sheaves with Mukai vector \( v \) via the Hochschild-Kostant-Rosenberg isomorphism \( I^X_* : HH_0(X) \to H\Omega_0(X) \).
\[
I^X_* (h_2) := ch(v).
\]

Set \( h_{2i} := (-1)^{i+1}(h_2)^i \). Explicitly,
\[
\begin{align*}
h_0 &= -1, \\
h_4 &= -(h_2)^2 \otimes (t^*_M)^2, \\
h_k &= 0, \text{ for } k > 4.
\end{align*}
\]

Given an object \( x \) in \( D^b(X) \), let
\[
h_x : x \otimes_{\mathbb{C}} H^{2n}(M, \mathcal{O}_M)[-2n] \to x \otimes_{\mathbb{C}} Y(\mathcal{O}_M)
\]
be the morphism induced by the natural transformation \( h \). Let \( \pi_X : X \times M \to X \) be the projection. Note that the endofunctor \( \Upsilon \) is naturally isomorphic to \( R\pi_* \circ \pi^*_X \). Denote\footnote{\( D^b(X)_{\tau} \) is the \textit{Kleisli category} associated to the adjunction \( \Phi_{\mathcal{W}} \dashv \Psi_{\mathcal{W}} \). The subscript \( \tau \) will later denote the monad associated to this adjunction, so that our notation is the standard one [Mac Sec. VI.5].} by \( D^b(X)_{\tau} \) the full subcategory of \( D^b(M) \) with objects of the form \( \Phi_{\mathcal{W}}(x) \), for some object \( x \) in \( D^b(M) \). Let \( \Xi_{\mathcal{W}} : D^b(X \times M) \to D^b(M) \) be the composition of tensorization by \( \mathcal{W} \) followed by \( R\pi_{M,*} \). Then \( \Phi_{\mathcal{W}} = \Xi_{\mathcal{W}} \circ \pi^*_X \). Denote by \( D^b(X)_{\Xi} \) the full subcategory of \( D^b(X \times M) \) with objects of the form \( \pi^*_X(x) \), for some object \( x \) in \( D^b(X) \). Let \( Q : D^b(X)_{\Xi} \to D^b(X)_{\tau} \) be the restriction of the functor \( \Xi_{\mathcal{W}} \).

**Theorem 3.2.** The natural transformation \( q : \Upsilon \to \Psi_{\mathcal{W}} \Phi_{\mathcal{W}} \), given in Equation (2.4) above, has the following properties.
(1) For every pair of objects \( x_1 \) and \( x_2 \) in \( D^b(X) \), the first row below is a short exact sequence.

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Hom}(x_1, Rx_2) & \xrightarrow{(h_{x_2})_*} & \text{Hom}(x_1, x_2) & \xrightarrow{(q_{x_2})_*} & \text{Hom}(x_1, \Psi_y \Phi_y(x_2)) & \rightarrow & 0 \\
\end{array}
\]

where \( \text{Hom}_{D^b(X \times M)}(\pi_X^*(x_1), \pi_X^*(x_2)) \rightarrow \text{Hom}_{D^b(M)}(\Phi_y(x_1), \Phi_y(x_2)). \)

(2) The vertical adjunction isomorphisms above conjugate \((q_{x_2})_*\) to the homomorphism induced by the functor \( Q : D^b(X) \rightarrow D^b(X) \). The functor \( Q \) is full.

The theorem is proven in section \[3.3\] Given objects \( x \) and \( y \) in a bounded triangulated category, set \( \text{Hom}^\bullet(x, y) := \oplus_{k \in \mathbb{Z}} \text{Hom}(x, y[k])[-k] \). Note that \( \text{Hom}^\bullet_{D^b(X)}(x_1, x_2) \) is simply \( \text{Hom}^\bullet(x_1, x_2) \otimes_Y Y(\mathcal{E}_M) \). An explicit calculation of the algebra \( \text{Hom}^\bullet(\Phi_y(x), \Phi_y(x)) \) as a quotient of \( \text{Hom}^\bullet(x, x) \otimes_Y Y(\mathcal{E}_M) \) is carried out in Theorem \[3.21\] for any object \( x \) represented by a simple sheaf.

Section \[3\] is organized as follows. In subsection \[3.1\] we interpret Theorem \[3.2\] in terms of the standard construction of a quotient category by a congruence relation (Definition \[1.2\]). The natural transformation \( h \) gives rise to such a relation and Theorem \[3.2\] expresses the full subcategory \( D^b(X)_\tau \) of \( D^b(M) \) as the quotient category of the category \( D^b(X) \), whose objects are the same as those of \( D^b(X) \), and such that

\[
\text{Hom}^\bullet_{D^b(X)}(x_1, x_2) = \text{Hom}^\bullet_{D^b(X)}(x_1, x_2) \otimes_Y \mathbb{C}[t]/(t^{n+1}),
\]

where \( t \) has degree 2. In subsection \[3.2\] we show that the natural transformation \( q : x \rightarrow T \) in Theorem \[3.2\] induces a monad map between two monads in \( D^b(X) \). Under the analogy between rings and monads, the statement that \( q \) is a monad map says that \( q \) is analogous to a ring homomorphism. The functor \( Q \) in Theorem \[3.2\] is an example of the general construction of a Kleisli lifting of a monad map \[MaMu\] Theorem 2.2.2 and Def. 2.2.3. In subsection \[3.3\] we reduce the proof of Theorem \[3.2\] to the computation of the component \( h_2 \) of the natural transformation \( h \).

Subsections \[3.5\] to \[3.7\] are dedicated to the proof that \( h_2 \) is the inverse image of \( ch(v) \) via the Hochschild-Kostant-Rosenberg isomorphism. The class \( t_X \) in \( H^2(X, \mathcal{O}_X) \) gives rise to a natural transformation \( 1_{D^b(X)} \rightarrow 1_{D^b(X)}[2] \) and hence a morphism \( x \rightarrow x[2] \), for every object \( x \) of \( D^b(X) \). We get two morphisms from \( \Phi_y(x) \rightarrow \Phi_y(x)[2] \) in \( D^b(M) \), one is the image of the former via the functor \( \Phi_y \). The second is induced by the natural transformation \( 1_{D^b(M)} \rightarrow 1_{D^b(M)}[2] \) associated to the image \( t_M \) in \( H^2(M, \mathcal{E}_M) \) of \( t_X \) via Mukai’s Hodge isometry. These two morphisms in \( \text{Hom}_{D^b(M)}(\Phi_y(x), \Phi_y(x)[2]) \) are linearly independent in general, but their traces belong to the one-dimensional space \( H^2(M, \mathcal{E}_M) \). In subsection \[3.5\] we calculate the ratio of the two traces using Hochschild cohomology techniques. When \( x \) is a simple object, the compositions of each of the two morphisms in \( \text{Hom}_{D^b(M)}(\Phi_y(x), \Phi_y(x)[2]) \) with \( t_M^{n-1} \) are linearly dependent in the one-dimensional space \( \text{Hom}_{D^b(M)}(\Phi_y(x), \Phi_y(x)[2n]) \).

In subsection \[3.6\] we relate the ratio of the latter pair to the ratio of traces computed earlier. We get a relation in the Yoneda algebra of \( \Phi_y(x) \), for every simple object \( x \). In subsection \[3.7\] we use that relation to determine the component \( h_2 \) of the natural transformation \( h \) and complete the proof of Theorem \[3.2\].

In subsection \[3.8\] we relate moduli spaces of stable sheaves on \( X \) to certain moduli spaces of sheaves over \( M \).
3.1. A congruence relation associated to the natural transformation $h$. We describe in this subsection how Theorem 3.2 reconstructs the category $D^b(X)$ as a quotient category in the sense of Definition 1.2. Let $D^b(X)_\sim$ be the category whose objects are the same as those of $D^b(X)$, and such that

$$\text{Hom}_{D^b(X)_{\sim}}(x_1, x_2) := \text{Hom}_{D^b(X)}(x_1, Y x_2) = \bigoplus_{k=0}^{2n} \text{Hom}_{D^b(X)}(x_1, x_2[-2k]) \otimes \mathbb{C} H^{2k}(M, \mathcal{O}_M).$$

Note that $\text{Hom}_{D^b(X)_{\sim}}^\bullet(x_1, x_2) = \text{Hom}_{D^b(X)}^\bullet(x_1, x_2) \otimes_{\mathbb{C} Y} Y(\mathcal{O}_M)$. Given morphisms $g$ in $\text{Hom}_{D^b(X)}(x_1, x_2)$ and $f$ in $\text{Hom}_{D^b(X)}(x_2, x_3)$, and elements $a, b$ in $Y(\mathcal{O}_M)$, the composition $(f \otimes a) \circ (g \otimes b)$ is $fg \otimes ab$, and is extended by linearity to all morphisms. Note that $D^b(X)_\sim$ is equivalent to the full subcategory of $D^b(X \times M)$ whose objects are of the form $\pi_X^*(x)$, for some object $x$ in $D^b(X)$. The above composition rule corresponds to composition in $D^b(X \times M)$ via the adjunction isomorphism $\text{Hom}_{D^b(X)}(x_1, Y x_2) \cong \text{Hom}_{D^b(X \times M)}(\pi_X^* x_1, \pi_X^* x_2)$. Let $\pi_X^*: D^b(X) \rightarrow D^b(X)_\sim$ be the functor sending each object to itself and inducing the natural inclusion $\text{Hom}_{D^b(X)}(x_1, x_2) \rightarrow \text{Hom}_{D^b(X)_\sim}(x_1, x_2)$.

Definition 1.2 recalls the notion of a congruence relation on a category. Consider the relation $\mathcal{R}$ on $D^b(X)_\sim$ given in Equation (1.4). Following is a restatement of Theorem 3.2 in the language of quotient categories.

**Theorem 3.3.**

1. $\mathcal{R}$ is a congruence relation.
2. The natural transformation $q$ induces a fully faithful functor $\Sigma: D^b(X)_\sim/\mathcal{R} \rightarrow D^b(M)$.
3. The functor $\Phi_{\mathcal{R}}$ factors through the quotient functor $Q: D^b(X)_\sim \rightarrow D^b(X)_\sim/\mathcal{R}$ as the composition $\Phi_{\mathcal{R}} = \Sigma Q \pi_X^*: D^b(X) \rightarrow D^b(M)$.

**Proof.** The statement follows from Theorem 3.2. It is however instructive to see how the defining Equations (3.1) of $h$ formally imply that $\mathcal{R}$ is a congruence relation, so we will prove part (1) of Theorem 3.3 independently.

The image of $(h_{x_2})_*$, given in (1.4), is mapped under post-composition with elements $g$ of $\pi_X^* [\text{Hom}_{D^b(X)}(x_2, x_3)]$ to the image of $(h_{x_3})_*: \text{Hom}_{D^b(X)}(x_1, Rx_3) \rightarrow \text{Hom}(x_1, Y x_3)$, by the naturality of the transformation $h$. Indeed, if $g = \pi_X^*(\tilde{g})$ and $f$ belongs to $\text{Hom}_{D^b(X)}(x_1, Y x_2)$, then $g \circ f = \tilde{g} \circ f$ and naturality of $h$ yields the following commutative diagram:

$$\begin{array}{ccc}
\text{Hom}_{D^b(X)}(x_1, Rx_2) & \xrightarrow{(h_{x_2})_*} & \text{Hom}_{D^b(X)}(x_1, Y x_2) \\
\downarrow R(\tilde{g})_* & & \downarrow \tau(\tilde{g})_* \\
\text{Hom}_{D^b(X)}(x_1, Rx_3) & \xrightarrow{(h_{x_3})_*} & \text{Hom}_{D^b(X)}(x_1, Y x_3).
\end{array}$$

The algebra $Y(\mathcal{O}_M)$ is generated by the element $t_M$. Let

$$\tau : Y \rightarrow Y[2]$$

be the natural transformation corresponding to multiplication by $t_M$. We may regard the natural transformation $\tilde{h}_2 : \mathbb{1}_{D^b(X)} \rightarrow \mathbb{1}_{D^b(X)[2]}$ as a natural transformation $\tilde{h}_2 : R \rightarrow R[2]$ as well. Then we have the equality

$$\tau \circ h = -h \circ \tilde{h}_2,$$
which follows from the defining Equations (3.1) of $h$. Indeed,

$$h = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\tilde{h}_2^2 \otimes (t_M^*)^2 \\ \tilde{h}_2^2 \otimes t_M^* \\ -1 \end{pmatrix}, \quad \tau \circ h = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ -\tilde{h}_2^2 \otimes t_M^* \\ \tilde{h}_2^2 \end{pmatrix} = -h \circ \tilde{h}_2.
$$

It follows that the image of $(h_{x_2})_*$, given in (1.4), is mapped under post-composition with $1 \otimes t_M \in \text{Hom}_{D^b(X)^\gamma}(x_2, x_2[2])$ to the image of $(h_{x_2[2]})_* : \text{Hom}_{D^b(X)}(x_1, Rx_2[2]) \to \text{Hom}(x_1, \Upsilon x_2[2])$.

The analogous statement holds for powers of $1 \otimes t_M$, by induction. Hence, post-composition with every element $g \in \text{Hom}_{D^b(X)^\gamma}(x_2, x_3)$ maps the image of $(h_{x_2})_*$ to the image of $(h_{x_3})_*$.

Let $e$ be an element of $\text{Hom}_{D^b(X)^\gamma}(x_0, x_1)$. We need to show that pre-composition with $e$ maps the image of $(h_{x_2})_* : \text{Hom}_{D^b(X)}(x_1, Rx_2) \to \text{Hom}_{D^b(X)}(x_1, \Upsilon x_2)$ to the image of $(h_{x_2})_* : \text{Hom}_{D^b(X)}(x_0, Rx_2) \to \text{Hom}_{D^b(X)}(x_0, \Upsilon x_2)$. If $e = \pi_X^*(\tilde{e})$, where $\tilde{e}$ belongs to $\text{Hom}_{D^b(X)}(x_0, x_1)$, then for $a \in \text{Hom}_{D^b(X)}(x_1, Rx_2)$ we have

$$(h_{x_2} \circ a) \circ e = h_{x_2} \circ (a \circ \tilde{e}).$$

The right hand side above belongs to the image of $(h_{x_2})_*$. It remains to prove the statement in case $x_0 = x_1[-2]$ and $e = 1 \otimes t_M$. Now $\pi_M^*(t_M) : \mathbb{1}_{D^b(X \times M)} \to \mathbb{1}_{D^b(X \times M)[2]}$ is a natural transformation. It follows that for every pair of objects $y_1, y_2$ in $D^b(X \times M)$ and for every morphism $f : y_1 \to y_2$ we have the commutative diagram

$$
\begin{array}{ccc}
  y_1 & \xrightarrow{f} & y_2 \\
  \pi_M^*(t_M) \downarrow & & \downarrow \pi_M^*(t_M) \\
  y_1[2] & \xrightarrow{[2](f)} & y_2[2].
\end{array}
$$

This holds in particular for objects $y_i$ of the form $\pi_X^*(x_i)$ and for $f := h_{x_2} \circ a$. We get the equality

$$(h_{x_2} \circ a) \circ (1 \otimes t_M) = (1 \otimes t_M) \circ [2](h_{x_2} \circ a),$$

for all $a \in \text{Hom}_{D^b(X)}(x_1, Rx_2)$. Post-compositions were shown already to preserve the relation $\mathfrak{R}$. \hfill \Box

### 3.2. The monad $\mathcal{A}$ is a quotient of a constant monad.

Set

$$T := \Psi_{\mathcal{A}} \Phi_{D^b(X)} : D^b(X) \to D^b(X).$$

Denote by $\eta : \mathbb{1} \to T$ the unit for the adjunction $\Phi_{D^b(X)} \dashv \Psi_{\mathcal{A}}$, by $\epsilon : \Phi_{D^b(X)} \Psi_{\mathcal{A}} \to \mathbb{1}$ the counit, and set $m := \Psi_{\mathcal{A}} \epsilon \Phi_{D^b(X)} : T^2 \to T$ the multiplication natural transformation. We get the monad

$$\mathbb{T} := (T, \eta, m).$$

Let $\mathcal{A}$ be the object in $D^b(X \times X)$ given in Equation (1.2). Recall that $\mathcal{A}$ is the kernel of the integral transform $T$. We have the natural morphism

$$m : \mathcal{A} \otimes_{\mathbb{C}} Y(\mathcal{O}_M) \to \mathcal{A},$$

given in Equation (2.3). Let $\eta : \mathcal{O}_{\Delta X} \to \mathcal{A}$ be the morphism corresponding to the unit of the adjunction $\Psi_{\mathcal{A}} \dashv \Phi_{D^b(X)}$. Set

$$q := m \circ (\eta \otimes \text{id}) : \mathcal{O}_{\Delta X} \otimes_{\mathbb{C}} Y(\mathcal{O}_M) \to \mathcal{A}.$$
The Yoneda algebra $Y(\mathcal{O}_M)$, considered as an object of $D^b(\text{pt})$, is a monad in an obvious way. Denote by $\tilde{\eta} : 1 \to Y(\mathcal{O}_M)$ its unit and by $\tilde{m} : Y(\mathcal{O}_M) \otimes_Y Y(\mathcal{O}_M) \to Y(\mathcal{O}_M)$ its multiplication. The endo-functor $Y(\mathcal{O}_M) \otimes_Y (\bullet) : D^b(\mathcal{X}) \to D^b(\mathcal{X})$, of tensorization by $Y(\mathcal{O}_M)$ over $\mathcal{C}$, has kernel $\mathcal{Y} := \mathcal{O}_\Delta \otimes_Y Y(\mathcal{O}_M)$. We get a monad
\begin{equation}
\mathcal{Y} := (1, \tilde{\eta}, \tilde{m})
\end{equation}
in $D^b(\mathcal{X})$ as follows. Denote by $\pi_X : X \times M \to X$ the projection. We have the adjunction $\pi_X^* \dashv R\pi_X$, $\mathcal{Y}$ is the kernel of the functor $\mathcal{Y} := R\pi_X \circ \pi_X^*$, and $\mathcal{Y}$ is the monad for that adjoint pair. We denote again by $q$ the natural transformation from $\mathcal{Y}$ to $T$ induced by the homomorphism of kernels (3.3).

**Remark 3.4.** The above definition of $q$ is the one we used earlier in Equation (2.4). The natural transformation $q$ admits a second functorial expression, which will be needed below (in Lemma 3.7). Let $\Xi$ be the kernel of $\mathcal{Y} := R\pi_X \circ \pi_X^*$, so that $\mathcal{Y} = \mathcal{O}_\Delta \otimes_Y \mathcal{Y}$. Denote by $G$ the right adjoint of $\Xi$. Set $F := R\pi_X$, so that $\mathcal{Y} = \mathcal{O}_\Delta \otimes_Y \mathcal{Y}$ and $T = FG\Xi \pi_X^*$. Let $\eta : 1_{D^b(X \times M)} \to G\Xi$ be the unit for the adjunction. Then the natural transformation $q : \mathcal{Y} \to T$ is equal to $F \eta \pi_X^*$. Similarly, the homomorphism $q$ in (3.3) admits an analogous description provided below.

Let $\Delta_2, 3 : X \times M \to (X \times M) \times M$ be the diagonal map. The kernel of the integral functor $\Xi$ is $\Delta_2, 3(\mathcal{Y})$. The kernel of $G$ is $\Delta_2, 3(\mathcal{Y} \otimes \pi_X^* \omega_X [2])$. The kernel $\mathcal{K}$ of $G\Xi \pi_X^*$ is their convolution and is identified as an object in $D^b((X \times M) \times (X \times M))$ as follows. Let $\Delta_2, 4 : X \times M \times X \to (X \times M) \times (X \times M)$ be the diagonal map. Let $\pi_{i,j}$ be the projection from $X \times M \times X$ onto the product of the $i$-th and $j$-th factors. Then
\[ \mathcal{K} = \Delta_2, 4(\pi_{1,2}^*(\mathcal{Y} \otimes \pi_X^* \omega_X [2]) \otimes \pi_{2,3}^*(\mathcal{Y})). \]
Let $p_{i_1 \ldots i_k}$ be the projection from $X \times M \times X \times M$ onto the product of the $i_1, \ldots, i_k$ factors. Let $\eta : \Delta_{X \times M}, \mathcal{O}_{X \times M} \to \mathcal{K}$ be the unit morphism for the adjunction $\Xi \dashv G$. Then
\[ R\pi_{13} : D^b((X \times M) \times (X \times M)) \to D^b(X \times X) \]
maps $\Delta_{X \times M}, \mathcal{O}_{X \times M}$ to the object $\mathcal{Y}$, maps $\mathcal{K}$ to the object $\mathcal{A}$, and maps the morphism $\eta$ to the morphism $q$ given in Equation (3.3). The latter equality is proven in the following Lemma.

**Lemma 3.5.** The equality $q = R\pi_{13}(\eta)$ holds.

**Proof.** Set $q' := R\pi_{13}(\eta)$. Let $\gamma : \pi_{13}^* R\pi_{13} \to 1_{D^b(X \times M \times X)}$ be the counit for the adjunction $\pi_{13}^* \dashv R\pi_{13}$. Let $u : 1_{D^b(X \times X)} \to R\pi_{13}, \pi_{13}^*$ be the unit natural transformation for this adjunction. Let $\tilde{\eta} : \mathcal{O}_\Delta \to R\pi_{13}, \pi_{13}^* \mathcal{O}_\Delta$ be the morphism associated by $u$ to the object $\mathcal{O}_\Delta$. The morphism $\tilde{\eta}$ is itself the unit morphism for the adjunction $\mathcal{O}_\Delta \dashv R\pi_{13}$. Set $\mathcal{A} := R\pi_{13}(\mathcal{K})$, so that $R\pi_{13}, \mathcal{A} \cong \mathcal{A}$. We have
\[ q := m \circ (\eta \otimes \text{id}) = (R\pi_{13}, (\gamma \mathcal{A})) \circ (R\pi_{13}, \pi_{13}^*(\eta)) = R\pi_{13}, (\gamma \mathcal{A} \circ \pi_{13}^* \eta). \]
On the other hand, $q' = R\pi_{13}(\eta) = R\pi_{13}, (R\pi_{13}, (\eta))$. Hence, it suffices to prove the equality
\begin{equation}
R\pi_{13}(\eta) = \gamma \mathcal{A} \circ \pi_{13}^* \eta.
\end{equation}
We have the equality \( \eta = Rp_{13}(\tilde{\eta}) \circ \tilde{\eta} \), since \( \eta \) is the unit morphism for the adjunction \( \Phi \dashv \Psi \), where \( \Phi \dashv = \Xi \circ \pi_X^* \). We get
\[
\pi_{13}^* \eta = \pi_{13}^* R \pi_{13} \left( Rp_{123}(\tilde{\eta}) \right) \circ \pi_{13}^* (\tilde{\eta}),
\]
and Equation (3.5) becomes
\[
Rp_{123}(\tilde{\eta}) = \gamma_{\tilde{\eta}} \circ \pi_{13}^* R \pi_{13} \left( Rp_{123}(\tilde{\eta}) \right) \circ \pi_{13}^* (\tilde{\eta}).
\]
The latter is a special case of Lemma 3.6 applied with \( F = \pi_{13}^* \), \( G = R \pi_{13} \), \( f = Rp_{123}(\tilde{\eta}) \), \( A = \partial_{\Delta_X} \), and \( B = \tilde{\mathcal{A}} \).

Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor, \( F \dashv G \) an adjunction, \( u : 1_{\mathcal{C}} \to GF \) the unit, and \( \gamma : FG \to 1_{\mathcal{D}} \) the counit for the adjunction. Let \( A \) be an object of \( \mathcal{C} \), let \( B \) be an object of \( \mathcal{D} \), and let \( f : F(A) \to B \) be a morphism.

**Lemma 3.6.** \( f = \gamma_B \circ FG(f) \circ F(u_A) \).

**Proof.** The composition \( F(A) \xrightarrow{F(u_A)} F((GF)(A)) \xrightarrow{\gamma_{F(A)}} F(A) \) is the identity, by [Mac Theorem 1]. Hence, it suffices to prove the equality \( \gamma_B \circ FG(f) = f \circ \gamma_{F(A)} \). The latter equality follows from the commutativity of the following diagram. Set \( A' := F(A) \).

\[
\begin{array}{ccc}
\text{Hom}(A', B) & \xrightarrow{G} & \text{Hom}(G(A'), G(B)) \\
\text{Hom}(FG(A'), F(G(B)) & \xrightarrow{F} & \text{Hom}(FG(A'), FG(B)) \\
& \xrightarrow{\gamma_{A'}} & \text{Hom}(FG(A'), B).
\end{array}
\]
The left triangle above commutes, by [Hu1 Lemma 1.21]. The proof of the commutativity of the right triangle is similar. \( \Box \)

**Lemma 3.7.** The natural transformation \( q \) is a monad map\(^5\) in the sense that \( q\tilde{\eta} = \eta \) and the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{Y} \circ \mathcal{A} & \xrightarrow{q\circ \mathcal{A}} & \mathcal{A} \circ \mathcal{A} \\
\mathcal{Y} \circ \mathcal{A} & \xrightarrow{\mathcal{Y} \circ q} & \mathcal{A} \circ \mathcal{A} \\
\mathcal{Y} \circ \mathcal{A} & \xrightarrow{m} & \mathcal{A} \circ \mathcal{A}
\end{array}
\]

**Proof.** The equality \( q\tilde{\eta} = \eta \) is clear. We prove only the commutativity of the above diagram. Let \( \mathcal{B}, \mathcal{C}, \mathcal{D} \) be categories, let \( G : \mathcal{B} \to \mathcal{C} \) and \( F : \mathcal{C} \to \mathcal{D} \) be functors. Assume given adjunctions \( G^* \dashv G \) and \( F^* \dashv F \). Let \( \eta \) and \( \epsilon \) be the unit and counit for \( G^* \dashv G \). Define \( \tilde{\eta}, \tilde{\epsilon} \) similarly for \( F^* \dashv F \), and let \( \tilde{m} := F\tilde{\epsilon}F^* \) be the multiplication for the corresponding monad. Set \( \Psi := FG, \Phi := G^*F^*, T := \Psi \Phi = FGG^*F^*, \) and \( Y := FF^* \). Let the natural transformation \( q : Y \to T \)

\(^5\)We use the term *monad map* following [MaMu] Def. 2.2.3. A monad map is a special case of a *monad functor* between two monads in different categories [St].
be given by \( q := F\eta F^* : FF \rightarrow FGG^*F^* \). We claim that \( q \) is a monad map, in the sense that the following diagram commutes

\[
\begin{array}{ccc}
YT & \xrightarrow{q_T} & TT \\
\downarrow & & \downarrow m \\
YY & \xrightarrow{\tilde{m}} & Y T.
\end{array}
\]

The above diagram is obtained by applying \( F \) on the left and \( F^* \) on the right to the circumference of the following diagram

\[
\begin{array}{ccc}
F^*FGG^* & \xrightarrow{\eta F^* FGG^*} & F^* FGG^* \\
\downarrow \iota G^* & & \downarrow G^* \iota G^* \\
\eta G^* & \xrightarrow{\eta G^*} & G^* G^*
\end{array}
\]

The left triangle and the two right squares in the above diagram evidently commute.

Apply the above argument with \( F := R\pi_X : D^b(X \times M) \rightarrow D^b(X) \) and with the functor \( G : D^b(M) \rightarrow D^b(X \times M) \) given by the composition of \( \pi^*_M \) with tensorization by the object \( \mathcal{V} \otimes [2] \) in \( D^b(X \times M) \). (\( G \) is the right adjoint of \( \Xi_{\mathcal{V}} \)). This establishes that \( q : R\pi_X \pi_X^* \rightarrow T \) is a monad map. Every step in the above argument admits an evident translation to the case of integral functors. Note that we used above the description of \( q \) given in Lemma 3.5. \( \Box \)

**Remark 3.8.** The monad map \( q \), induced by the morphism \( q : \mathcal{V} \rightarrow \mathcal{A} \) of Fourier-Mukai kernels, induces a functor

\[
P : D^b(X) \rightarrow D^b(X)_{Y}
\]

between the categories of modules for the monads \( \mathcal{A} := (T, \eta, m) \) and \( \mathcal{Y} := (Y, \tilde{\eta}, \tilde{m}) \) in \( D^b(X) \) (\( J \) Lemma 1]. The functor \( P \) takes the \( \mathcal{A} \)-module \((x, a)\) to

\[
P(x, a) = (x, a \circ q_x) \in D^b(X)_{Y}.
\]

The functor \( P \) is faithful, as the homomorphisms spaces are both subspaces of those of \( D^b(X) \). The functor \( P \) is an example of an *Eilenberg-Moore lifting* of a monad functor, where the monad functor in our case is \((1_{D^b(X)}, q) \) [MaMu, Def. 2.2.1]. Under the analogy between the monad map \( q \) and an algebra homomorphism, the functor \( P \) corresponds to the change of scalars functor, or to push-forward. The functor \( Q \) in Theorem 3.2 is analogous to a pull-back functor and goes in the opposite direction. For that reason the functor \( P \) will not play a role below.

3.3. A universal relation “ideal”. The following proposition introduces a “universal relation ideal” \( \mathcal{R} \). Consider the object \( \mathcal{R} := \mathcal{O}_{\Delta_X}[-2n] \otimes_{\mathcal{O}_M} \text{Ext}^{2n}(\mathcal{O}_M, \mathcal{O}_M) \) in \( D^b(X \times X) \).

Proposition 3.9. Assume that the monad \( \mathcal{A} \) is totally split (Definition [J.]). There exists a morphism \( h : \mathcal{R} \rightarrow \mathcal{Y} \), unique up to a scalar factor, such that the following is an exact
Proof. There exists an object $\mathcal{R}'$ in $D^b(X \times X)$ and a morphism $h : \mathcal{R}' \to \mathcal{Y}$ such that $\mathcal{R}' \xrightarrow{h} \mathcal{Y} \xrightarrow{q} \mathcal{A} \to \mathcal{R}[1]$ is an exact triangle, by the axioms of a triangulated category. The following composition $\alpha$

\[
\mathcal{O}_{\Delta_X} \otimes_{\mathbb{C}} \lambda_n \xrightarrow{\iota} \mathcal{Y} \xrightarrow{q} \mathcal{A},
\]

given in Equation (2.5), is an isomorphism by the assumption that the monad is totally split. Using the long exact sequence in sheaf cohomology coming from the exact triangle given in Equation (2.5), $h$ is an isomorphism by the assumption that the monad is totally split. Using the long exact sequence in sheaf cohomology coming from the exact triangle $\mathcal{R}' \xrightarrow{h} \mathcal{Y} \xrightarrow{q} \mathcal{A}$, one immediately obtains $\mathcal{R}' \cong \mathcal{O}_{\Delta_X}[-2n] \otimes_{\mathbb{C}} \text{Ext}^{2n}(\mathcal{O}_M, \mathcal{O}_M)$.

The triangle is split as there are no odd-degree self-extensions of $\mathcal{O}_{\Delta_X}$. Finally, $h$ is determined up to scalars as the automorphism group of the object $\mathcal{R}' \cong \mathcal{O}_{\Delta_X}[-2n]$ is $\mathbb{C}^*$.

The morphism $h : \mathcal{R} \to \mathcal{O}_{\Delta_X} \otimes Y(\mathcal{O}_M)$ is naturally an element of

\[
\bigoplus_{j=0}^2 \text{Ext}^{2j}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \otimes_{\mathbb{C}} \text{Hom}(H^{2n}(M, \mathcal{O}_M), H^{2n-2j}(M, \mathcal{O}_M)).
\]

Let $t_X$ be a non-zero element of $H^2(X, \mathcal{O}_X)$, considered as a subspace of the complexified Mukai lattice, and let $t_M$ be its image in $H^2(M, \mathcal{O}_M)$ via Mukai’s Hodge isometry (2.18). Denote by $t^*_M : Y(\mathcal{O}_M) \to Y(\mathcal{O}_M)$ the homomorphism, which sends $t_M^j$ to $t_M^{j-1}$, $1 \leq j \leq n$, and sends 1 to 0. The choice of $t_M$ identifies $h$ as an element of the Hochschild cohomology $HH^*(X)$. Explicitly, $h = \bar{h}_0 \otimes 1 + \bar{h}_2 \otimes t_M^2 + \bar{h}_4 \otimes (t_M^1)^2$, where $\bar{h}_2$ belongs to $\text{Ext}^2(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})$.

Let $\sigma_X$ be the class in $H^0(X, \omega_X)$ dual to the class $t_X$ with respect to Serre’s duality. The class $h_2 := h^2 \otimes \sigma_X$ in $\text{Hom}_{\text{vert}}(\Delta_{X, <}(\mathcal{O}_X), \Delta_{X, >}(\omega_X)[2])$ is a class in $HH_0(X)$, independent of the choice of the class $t_X$, since $t_M$ depends on $t_X$ linearly.

**Theorem 3.10.**

(1) The class $\bar{h}_0$ does not vanish and $\bar{h}_0 \bar{h}_4 = (\bar{h}_2)^2$.

(2) Rescale the morphism $h$, so that $\bar{h}_0 = -1$. Then the class $I_X^*(\bar{h}_2)$ in $HH_0(X)$ is equal to the Chern character $ch(v)$ of the Mukai vector $v$ of sheaves parametrized by $M$.

**Proof.** Part 2 of the theorem is proven in section 3.7. We include here the proof of part 1 which follows formally from Lemma 8.7. The class $\bar{h}_0$ does not vanish, since the sheaf cohomology $H^{2n}(\mathcal{O})$ does not vanish, while $H^{2n-2}(\mathcal{O})$ vanishes. It remains to compute $\bar{h}_4$.

Identify $\mathcal{A}$ with $\Delta_+ \mathcal{O}_X \otimes_{\mathbb{C}} \lambda_n$ via the isomorphism $\alpha$ given in Equation (2.5). The morphism $q : \mathcal{Y} \to \mathcal{A}$ decomposes $q = (q_{i,j})$, $0 \leq i \leq n - 1$, $0 \leq j \leq n$, where $q_{i,j}$ is a morphism

\[
g_{i,j} : \Delta_+ \mathcal{O}_X \otimes_{\mathbb{C}} H^{2j}(M, \mathcal{O}_M)[-2j] \to \Delta_+ \mathcal{O}_X \otimes_{\mathbb{C}} H^{2i}(M, \mathcal{O}_M)[-2i].
\]

Then $q_{i,i}$ is the identity, for $0 \leq i \leq n - 1$, and $q_{i,j} = 0$ for $i \neq j$ and $0 \leq j \leq n - 1$, by construction of $\alpha$. Note that we use here the equality of the two descriptions of $q$ in Lemma 3.5 as $\alpha$ was constructed in terms of the earlier description, while Lemma 3.7 soon to be applied, uses the second description. The morphism $h : \mathcal{R} \to \mathcal{Y}$ decomposes as a column $h = (h_{i,n})$, where $n$ is fixed, $0 \leq i \leq n$, and $h_{i,n}$ is a morphism

\[
h_{i,n} : \Delta_+ \mathcal{O}_X \otimes_{\mathbb{C}} H^{2n}(M, \mathcal{O}_M)[-2n] \to \Delta_+ \mathcal{O}_X \otimes_{\mathbb{C}} H^{2i}(M, \mathcal{O}_M)[-2i].
\]
Clearly, \( h_{i,n} = \begin{cases} \tilde{h}_{2(n-i)} \otimes (t^*_M)^{n-i} & \text{if } 0 \leq i \leq n - 3 \\ \tilde{h}_{n-2,i} \otimes (t^*_M)^{n} & \text{if } n - 2 \leq i \leq n. \end{cases} \)

\[
(q_{i,j}) = \begin{pmatrix} 0 & q_{0,n} & \cdots & q_{n-1,n} \\ I_{\lambda n} & \vdots & & \vdots \\ & & 0 & \vdots \\ & & & q_{n-1,n} \end{pmatrix}, \quad h = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h_{n-2,n} \end{pmatrix}.
\]

We have the equality

\[
0 = qh = \begin{pmatrix} q_{0,n}h_{n,n} \\ \vdots \\ q_{n-3,n}h_{n,n} \\ q_{n-2,n-2}h_{n,n} + q_{n-2,n}h_{n,n} \\ q_{n-1,n-1}h_{n,n} + q_{n-1,n}h_{n,n} \end{pmatrix} = \begin{pmatrix} q_{0,n}\tilde{h}_0 \\ \vdots \\ q_{n-3,n}\tilde{h}_0 \\ \tilde{h}_4 \otimes (t^*_M)^2 + q_{n-2,n}\tilde{h}_0 \\ \tilde{h}_2 \otimes t^*_M + q_{n-1,n}\tilde{h}_0 \end{pmatrix}.
\]

We get the equalities:

\[
(q_{i,j}) = 0, \text{ for } 0 \leq i \leq n - 3,
\]

\[
q_{n-2,n} = -\frac{\tilde{h}_4}{\tilde{h}_0} \otimes (t^*_M)^2,
\]

\[
q_{n-1,n} = -\frac{\tilde{h}_2}{\tilde{h}_0} \otimes t^*_M.
\]

The class \( t_M \) yields a morphism \( t : \Delta_* \mathcal{O}_X[-2] \to \mathcal{V} \), which is an embedding of \( \Delta_* \mathcal{O}_X[-2] \) as a direct summand of \( \mathcal{V} \). We get the commutative diagram

\[
\begin{array}{ccc}
\mathcal{V}[-2] & \xrightarrow{q} & \mathcal{A}[-2] \\
\tau \mathcal{V} & \downarrow & \tau \mathcal{A} \\
\mathcal{V} \circ \mathcal{V} & \xrightarrow{\tau q} & \mathcal{V} \circ \mathcal{A}.
\end{array}
\]

The morphism \( \mathcal{V}q \) in the above diagram appears also in the commutative diagram in the statement of Lemma \([3,7]\). Glue the two diagrams along the arrow \( \mathcal{V}q \). Set \( \tau := m(q\mathcal{A})(\tau \mathcal{A}) : \mathcal{A}[-2] \to \mathcal{A} \) and \( \tilde{\tau} := m(t\mathcal{V}) : \mathcal{V}[-2] \to \mathcal{V} \). The commutativity of these two diagrams yields the equality

\[
\tau q = q\tilde{\tau} : \mathcal{V}[-2] \to \mathcal{A}.
\]

The matrix of \( \tilde{\tau} \) is \((\tilde{\tau}_{i,j}) = \begin{pmatrix} 0 & 0 \\ I_n \otimes t_M & 0 \\ 0 & 0 \end{pmatrix}\), where \( I_n \) is the \( n \times n \) identity matrix and \( t_M : H^{2j}(M, \mathcal{O}_M) \to H^{2j+2}(M, \mathcal{O}_M) \) is the isomorphism obtained by multiplication by the class \( t_M \), for \( 0 \leq j \leq n - 1 \). Hence,

\[
\tilde{\tau}h = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{h}_4 \otimes t^*_M \end{pmatrix}
\]

and we get \( 0 = \tau(qh) = q(\tilde{\tau}h) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{h}_4 \otimes (t^*_M)^2 \end{pmatrix} \).
where the last equality follows from equations (3.8), (3.9), and (3.10). The equality \( \tilde{h}_0 \tilde{h}_4 = (\tilde{h}_2)^2 \) follows.

3.4. Computation of the full subcategory \( D^b(X)_\tau \) of \( D^b(M) \). Let \( D^b(X)_\tau \) be the full subcategory of \( D^b(X \times M) \) consisting of objects of the form \( \pi_X^*(x) \), for some object \( x \) in \( D^b(X) \). We get a natural full and faithful functor \( \Sigma : D^b(X)_\tau \to D^b(X \times M) \). Let \( \Xi_U : D^b(X \times M) \to D^b(M) \) be the composition of tensorization by \( \otimes \) followed by \( R\pi_M \). Then \( \Phi_U = \Xi_U \circ \pi_X^* \). We get the following commutative diagram, where the functor \( Q \) is the restriction of \( \Xi_U \).

\[
\begin{array}{ccc}
D^b(X)_\tau \otimes & \overset{\Sigma}{\longrightarrow} & D^b(X \times M) \\
\pi_X^* \downarrow & & \downarrow \Xi_U \\
D^b(X) \downarrow \Phi_U & & \downarrow \\
D^b(X)_\tau \otimes & \overset{\Sigma}{\longrightarrow} & D^b(M)
\end{array}
\]

Let \( q_{x_1} : \Upsilon(x_1) \to T(x_1) \) be the morphism induced by the natural transformation \( q \), which in turn is induced by the morphism of kernels given in Equation (3.11). We get the homomorphism

\[
(q_{x_2})_* : \text{Hom}_{D^b(X)}(x_1, \Upsilon x_2) \to \text{Hom}_{D^b(X)}(x_1, T x_2)
\]

\[
f \mapsto q_{x_2} \circ f,
\]

and the diagram:

\[
\begin{array}{ccc}
\text{Hom}_{D^b(X \times M)}(\pi_X^*(x_1), \pi_X^*(x_2)) & \cong & \text{Hom}_{D^b(X)}(x_1, \Upsilon x_2) \\
\downarrow \Xi_U & & \downarrow (q_{x_2})_* \\
\text{Hom}_{D^b(X)}(\Phi_U(x_1), \Phi_U(x_2)) & \cong & \text{Hom}_{D^b(X)}(x_1, T x_2)
\end{array}
\]

where the horizontal isomorphisms are due to the adjunctions.

**Lemma 3.11.** The above diagram is commutative.

**Proof.** The commutativity is a special case of the following Lemma applied with \( F := R\pi_X, \ G^* := \Xi_U, \ B := D^b(M), \ C := D^b(X \times M), \ D := D^b(X) \).

Let \( B, C, \) and \( D \) be categories, let \( G : B \to C \) and \( F : C \to D \) be functors, and let \( G^* \dashv G \) and \( F^* \dashv F \) be adjunctions. Let \( \eta : 1_C \to GG^* \) be the unit for the adjunction. Set \( q := F\eta F^* : FF^* \to FGG^*F^* \).

**Lemma 3.12.** The following diagram is commutative for every pair of objects \( x_1, x_2 \) in \( D \).

\[
\begin{array}{ccc}
\text{Hom}(F^* x_1, F^* x_2) & \overset{\cong}{\longrightarrow} & \text{Hom}(x_1, FF^* x_2) \\
\downarrow \text{Hom}(F^* x_1, GG^* F^* x_2) & & \downarrow \text{Hom}(x_1, FGG^* F^* x_2) \\
\text{Hom}(G^* F^* x_1, G^* F^* x_2) & \overset{\cong}{\longrightarrow} & \text{Hom}(x_1, FGG^* F^* x_2)
\end{array}
\]
Proof. All the arrows labeled as isomorphisms correspond to the adjunction isomorphisms. Hence, the lower middle triangle commutes. The left triangle commutes, by [Hil1] Lemma 1.21. The upper right triangle commutes, by definition of $q$ and the naturality of the adjunction isomorphisms.

**Proof of Theorem 3.2.** Theorem 3.10 yields the exact sequence with the natural transformation $h$ satisfying Equations (3.1). The sequence in the statement of Theorem 3.2 is short exact, by the splitting of the exact triangle (3.6). The diagram in Theorem 3.2 is commutative, by Lemma 3.11. The functor $Q$ is full, by the surjectivity of $(q_{xz})_*$ in the diagram in Theorem 3.2 and the commutativity of that diagram.

3.5. **Traces.** Subsections 3.5 to 3.7 are dedicated to the proof of part 2 of Theorem 3.10. Given two smooth projective varieties $X$ and $M$, an integral functor $\Phi : D^b(X) \to D^b(M)$, and an object $x \in D^b(X)$, we have a natural composite homomorphism

$$H^i(X, \mathcal{O}_X) \xrightarrow{\mu_x} \text{Hom}(x, x[i]) \xrightarrow{\Phi} \text{Hom}(\Phi(x), \Phi(x)[i]) \xrightarrow{tr} H^i(Y, \mathcal{O}_Y).$$

The definition of the natural transformations $\mu$ and $tr$ are recalled below. In this subsection we use known results about the functoriality of Hochschild homology in order to provide a topological formula for the homomorphism displayed above in a special case (see Proposition 3.16).

The following lemma will be needed in the proof of Proposition 3.16 below. Let $X$ and $Y$ be smooth projective varieties, $f : X \to Y$ a morphism, $x$ an object of $D^b(X)$ and $y$ an object of $D^b(Y)$. We will use the notation $f_*$ and $f^*$ for the right and left derived functors $Rf_*$ and $Lf^*$ for brevity. Assume given morphisms $t : x \to f^*y$ and $\phi : f^*y \to x \otimes \omega_X[\dim X]$. Let $\eta : 1 \to f_*f^*$ be the unit for the adjunction $f^* \dashv f_*$. Let $f_!$ be the left adjoint of $f^*$ and let $\eta_y(t) \in \text{Hom}(f_!(x), f_*f^*(y))$ be the image of $t$ via the composition

$$\text{Hom}(x, f^*(y)) \xrightarrow{\phi((\eta_y))} \text{Hom}(x, f^*f_*f^*(y)) \cong \text{Hom}(f_!(x), f_*f^*(y)).$$

Note that $\phi \circ t$ belongs to $\text{Hom}(x, x \otimes \omega_X[\dim X])$. Using the isomorphism $f_*(x \otimes \omega_X[\dim X]) \cong f_!(x) \otimes \omega_Y[\dim Y]$ we see that $f_*(\phi) \circ \eta_y(t)$ belongs to $\text{Hom}(f_!(x), f_!(x) \otimes \omega_Y[\dim Y])$. Let

$$Tr_X : \text{Hom}(x, x \otimes \omega_X[\dim X]) \to \mathbb{C}$$

be the composition of the isomorphism $\text{Hom}(x, x \otimes \omega_X[\dim X]) \cong \text{Hom}(x, x)^*$ induced by Serre duality, followed by evaluation $\text{Hom}(x, x)^* \to \mathbb{C}$ on the identity morphism in $\text{Hom}(x, x)$.

**Lemma 3.13.** The following equality holds.

$$Tr_X(\phi \circ t) = Tr_Y(f_*(\phi) \circ \eta_y(t)).$$

**Proof.** Given $z \in D^b(Y)$, we get the adjunction isomorphism

$$\text{Hom}(x, f^*z) \xrightarrow{\cong} \text{Hom}(f_!x, z).$$

Serre duality yields the dual isomorphism:

$$\text{Hom}(f^*z, S_X x) \xleftarrow{\cong} \text{Hom}(z, S_Y f_!x).$$

Thus given $t' \in \text{Hom}(x, f^*z)$ and $\phi' \in \text{Hom}(f^*z, S_X x)$, we have an equality

$$Tr_X(\phi' \circ t') = Tr_Y(\phi' \circ \epsilon \circ f_!(t'))$$

where $\phi' \in \text{Hom}(z, S_Y f_!x)$ is the preimage of $\phi'$, while $\epsilon(f_!(t')) \in \text{Hom}(f_!x, z)$ is the adjoint map to $t'$, where $\epsilon : f_!f^* \to 1$ is the counit.
Replace $z$ by $f_*f^*y$ in (3.15), and pre-compose with the unit to get the diagram (3.13) whose Serre dual is

\[ (3.17) \]

\[ \text{Hom}(f^*y, S_X x) \xrightarrow{\epsilon^{f^*y}} \text{Hom}(f^*f_*f^*y, S_X x) \xrightarrow{\sim} \text{Hom}(f_*f^*y, S_Y f_*x) = \text{Hom}(f_*f^*y, f_*S_X x). \]

Given $t \in \text{Hom}(x, f^*y)$, its image in $\text{Hom}(f_*x, f_*f^*y)$ via (3.13) is nothing but $\eta_y(t)$. Similarly, for $\phi \in \text{Hom}(f^*y, S_X x)$, the preimage of $\phi \circ \epsilon_{f^*y}$ in $\text{Hom}(f^*f_*f^*y, S_X x)$ in $\text{Hom}(f_*f^*y, f_*S_X x)$ via the middle isomorphism in (3.17) is easily verified to be $f_*(\phi)$. Thus, equation (3.16) implies that

\[ T r_X ((\phi \circ \epsilon_{f^*y}) \circ (f^*\eta_y \circ t)) = T r_Y (f_*(\phi) \circ \eta_y(t)). \]

Finally, since $\epsilon_{f^*y} \circ f^*\eta_y = \text{id}_{f^*y}$, we have that $(\phi \circ \epsilon_{f^*y}) \circ (f^*\eta_y \circ t) = \phi \circ t$, and the result follows. \qed

Let $\nu$ be a class in $HH_0(X) \cong \text{Hom}(\Delta_{X,*}\Theta_X, \Delta_{X,*}\omega_X[\dim X])$ and let $x$ be an object of $D^b(X)$. Regarding $\nu$ as a natural transformation from $1_{D^b(X)}$ to the Serre functor we get a morphism $\nu_x : x : x \to x \otimes \omega_x[\dim X]$. Given a class $c$ in $H\Omega_*(X)$, denote by $c_{p,q}$ the direct summand in $H^p(X, \Omega^q_X)$. Set $d := \dim(X)$.

**Lemma 3.14.** The following equality holds: $T r_X(\nu_x) = T r_X \left( \left[ F^X_*(\nu) \sqrt{td_X ch(x^\vee)} \right]_{d,d} \right)$.

Note that the right hand side is the Mukai pairing of $\nu$ and the class $(I^X)_X^{-1}(ch(x))$ in $HH_0(X)$ as defined in \cite{C2} Def. 6.1. Mukai’s sign convention, which we will follow, is different and we would regard the right hand side as *minus* the Mukai pairing.

**Proof.** The statement is essentially the definition of the Chern character as a class in $HH_0(X)$ (see \cite{C2} Sec. 6.2 and \cite{C3} Theorem 4.5)). \qed

Given a scheme $S$ and an object $x \in D^b(S)$, denote by $x^\vee := R\text{Hom}(x, \omega_S) \in D^b(S)$ its dual object. Let

\[ \mu_x : \omega_S \to x^\vee \otimes x \]

be the natural morphism and

\[ \text{tr} : x^\vee \otimes x \to \omega_S \]

the trace morphism (\cite{Mull}, page 114). The following identity holds.

\[ (3.18) \]

\[ \text{tr} \circ \mu_x = \text{rank}(x) \cdot 1. \]

Assume next that $S$ is smooth and projective. Consider the trace pairing

\[ \text{Hom}(x, x[i]) \otimes \text{Hom}(x[i], x \otimes \omega_S[\dim S]) \xrightarrow{\sim} \text{Hom}(x, x \otimes \omega_s[\dim S]) \xrightarrow{\text{tr}} H^{\dim S}(S, \omega_S) \cong \mathbb{C}, \]

where the left arrow is composition. Mukai shows that the above pairing is a perfect pairing, for $0 \leq i \leq \dim S$ (\cite{Mull}, page 114). Mukai’s trace pairing is induced by Serre’s duality as follows. Set $y := x^\vee \otimes x$. We can rewrite Mukai’s pairing as

\[ (3.19) \]

\[ H^i(y) \otimes H^{\dim S - i}(y \otimes \omega_S) \to H^{\dim S}(S, \omega_S), \]

while Serre’s duality yields a pairing

\[ (3.20) \]

\[ H^i(y^\vee) \otimes H^{\dim S - i}(y \otimes \omega_S) \to H^{\dim S}(S, \omega_S). \]

Mukai interprets the composition

\[ R\text{Hom}(R\text{Hom}(x, x)^\vee, R\text{Hom}(x, x)) \cong R\text{Hom}(x, x) \otimes R\text{Hom}(x, x) \xrightarrow{\sim} R\text{Hom}(x, x) \xrightarrow{\text{tr}} \omega_S \]
as an isomorphism \( \psi : RHom(x, x)^\vee \to RHom(x, x) \), or equivalently, \( \psi : y^\vee \to y \). Relating the leftmost factors in (3.19) and (3.20) via \( \psi \) relates Mukai’s trace pairing to Serre’s duality.

Let \( G : D^b(S) \to D^b(S) \) be the functor of tensorization by the object \( x \). The right and left adjoints of \( G \) are both isomorphic to the functor \( G^\dagger : D^b(S) \to D^b(S) \), of tensorization with \( x^\vee \). Let \( \Delta_S : S \to S \times S \) be the diagonal morphism. Then \( \Delta_S(\mu_x) : \Delta_S(\sigma_S) \to \Delta_S(x^\vee \otimes x) \)

induces the unit natural transformation \( \mu_x : id \to G^\dagger G \) for the adjunction. The morphism \( \Delta_S(tr) : \Delta_S(x^\vee \otimes x) \to \Delta_S(\sigma_S) \)

induces the counit natural transformation \( tr : G^\dagger G \to id \). The morphisms

\[
(3.21) \quad \mu_x : \text{Hom}(\sigma_S, \sigma_S[i]) \to \text{Hom}(\sigma_S, x^\vee \otimes x[i]) \cong \text{Hom}(x, x[i]),
\]

\[
G : \text{Hom}(\sigma_S, \sigma_S[i]) \to \text{Hom}(G(\sigma_S), G(\sigma_S)[i])
\]

are equal under the identification \( x = G(\sigma_S) \), by [Hu1] Lemma 1.21.

**Remark 3.15.** If \( \text{Hom}(x, x) \) is one-dimensional, then

\[
(3.22) \quad \text{Hom}(x, x \otimes \omega_S[\dim S]) \xrightarrow{tr} H^{\dim S}(S, \omega_S)
\]

is an isomorphism. Indeed, both spaces are one dimensional and the statement reduces to the non-vanishing of \( tr \) : \( \text{Hom}(x, x) \otimes \text{Hom}(x, x \otimes \omega_S[\dim S]) \to H^{\dim S}(S, \omega_S) \), which holds it being a perfect pairing. Now let \( t \) be an element of \( H^{\dim S}(S, \omega_S) \). As a consequence of the above isomorphism, we see that the element \( \mu_x(t) \) of \( \text{Hom}(x, x \otimes \omega_S[\dim S]) \) vanishes, if \( \text{rank}(x) = 0 \) and \( \text{Hom}(x, x) \) is one-dimensional. Indeed, \( tr(\mu_x(t)) = \text{rank}(x) \cdot t = 0 \) in this case.

Let \( X \) be a projective \( K^3 \) surface, \( M := M_H(v) \) a moduli space of \( H \)-stable sheaves with Mukai vector \( v \) satisfying the hypothesis of Theorem 2.2 and \( \Phi_\Upsilon : D^b(X) \to D^b(M) \) the faithful functor in Theorem 2.2. Assume that \( (v, \nu) \geq 2 \). Given objects \( x \) and \( y \) in \( D^b(X) \), set \( \text{Hom}^\bullet(x, y) := \oplus \text{Hom}(x, y[i])[-i] \), as an object of the derived category of a point. Let \( Y(x) := \text{Hom}^\bullet(x, x) \) be the Yoneda algebra. The morphism \( \mu_x \) induces the natural algebra homomorphism

\[
\mu_x : Y(\sigma_X) \longrightarrow Y(x),
\]

for every object \( x \) of \( D^b(X) \). Define

\[
\mu_{\Phi_\Upsilon}(x) : Y(\sigma_M) \longrightarrow Y(\Phi_\Upsilon(x))
\]

similarly.

Let \( t_X \in H^2(X, \sigma_X) \) be a non-zero element. Let

\[
\varphi_\Upsilon : H^*(X, \mathbb{C}) \longrightarrow H^*(M, \mathbb{C})
\]

be the homomorphism induced by the correspondence \( \sqrt{\text{td}_X ch(\Upsilon)} \sqrt{\text{td}_M} \in H^*(X \times M, \mathbb{Q}) \). Given \( a \in H^*(X, \mathbb{C}) \), denote by \( [\varphi_\Upsilon(a)]_2 \) the graded summand in \( H^2(M, \mathbb{C}) \).

**Proposition 3.16.** For every object \( x \) of \( D^b(X) \), the following equality holds:

\[
(3.23) \quad \text{tr}(\Phi_\Upsilon(\mu_x(t_X))) = \text{rank}(x)[\varphi_\Upsilon(t_X)]_2,
\]

where \( t_X \) on the right hand side is considered as an element of the summand \( H^{0,2}(X) \) of \( H^2(X, \mathbb{C}) \), via the Hodge decomposition, and the left hand side is similarly considered as an element of the subspace \( H^{0,2}(M) \) of \( H^2(M, \mathbb{C}) \).

**Proof.** Let \( \Phi : D^b(X) \to D^b(M) \) be the integral functor with kernel \( \Upsilon \otimes \pi_X^*(x) \). So \( \Phi(a) = \Phi_\Upsilon(x \otimes a) \). Then \( \Phi_\Upsilon(\mu_x(t_X)) = \Phi(t_X) \), by the equality of the two homomorphisms displayed in Equation (3.21). Let

\[
\varphi : H^*(X, \mathbb{C}) \longrightarrow H^*(M, \mathbb{C})
\]
be the homomorphism induced by the correspondence $\pi_{X}^{*}\sqrt{\det} \, ch(\mathcal{W} \otimes \pi_{X}^{*}x) \pi_{M}^{*}\sqrt{\det M}$. Note that $ch(x)t_{X} = v(x)t_{X} = \text{rank}(x)t_{X}$ in $H^{r}(X, \mathbb{C})$. So we get the equality

$$\varphi(t_{X}) = \varphi_{\mathcal{W}}(ch(x)t_{X}) = \text{rank}(x)\varphi_{\mathcal{W}}(t_{X}).$$

It remains to prove the equality

$$\text{tr} \left( \Phi(t_{X}) \right) = [\varphi(t_{X})]_{2}.$$  

Let $\eta$ and $\epsilon$ be the unit and counit for the adjunction $\Delta_{M}^{*} \dashv \Delta_{M}$. We get the following morphisms

$$\Omega_{M} = \Delta_{M}^{*} \mathcal{O}_{M} \times M \xrightarrow{\Delta_{M}^{*}} \Delta_{M}^{*} \Delta_{M} \mathcal{O}_{M} \times M \xrightarrow{\epsilon_{[\Delta_{M}^{*} \mathcal{O}_{M} \times M]}} \Delta_{M}^{*} \mathcal{O}_{M} \times M = \mathcal{O}_{M},$$

which compose to the identity. Let $\eta_{M} : H^{2}(M, \mathcal{O}_{M}) \to H\Omega_{-2}(M)$ be the composition

$$H^{2}(M, \mathcal{O}_{M}) \cong \text{Hom}(\mathcal{O}_{M}, \mathcal{O}_{M}[2]) \xrightarrow{\Delta_{M}^{*}(\eta_{\mathcal{O}_{M} \times M})} \text{Hom}(\mathcal{O}_{M}[-2], \Delta_{M}^{*} \mathcal{O}_{M} \mathcal{O}_{M}, \Delta_{M}^{*} \mathcal{O}_{M} \times M) = \text{Hom}(\mathcal{O}_{M}[-2], \Delta_{M}^{*} \mathcal{O}_{M} \times M) = H\Omega_{-2}(M).$$

The analogous homomorphism $\eta_{X} : H^{2}(X, \mathcal{O}_{X}) \to H\Omega_{-2}(X)$, for the K3 surfaces $X$, is an isomorphism. Let $\epsilon_{M} : H\Omega_{-2}(M) \to H^{2}(M, \mathcal{O}_{M})$ be the morphism induced by $\epsilon_{[\Delta_{M}^{*} \mathcal{O}_{M} \times M]}$.

We have the following diagram, where the middle square commutes by Theorem 2.15.

$$
\begin{array}{ccc}
H\Omega_{-2}(X) & \xrightarrow{\Phi_{*}} & H\Omega_{-2}(M) \\
\downarrow \scriptstyle{\eta_{X}} & & \downarrow \scriptstyle{\eta_{M}} \\
H\Omega_{-2}(X) & \xrightarrow{\varphi} & H\Omega_{-2}(M) \\
\end{array}
$$

Here $\Phi_{*}$ is the homomorphism on Hochschild homology recalled in Equation (2.14). The triangle on the right (with arrow $\epsilon_{M}$) commutes as well. Indeed, the composition

$$\Delta_{M}^{*} \Delta_{M} \mathcal{O}_{M} \times M \xrightarrow{\overline{I}_{M}} \Omega^{0}[i] \xrightarrow{p^{0}} \Delta_{M} \mathcal{O}_{M},$$

where $p^{0}$ is the projection onto the component in degree 0 and $\overline{I}_{M}$ is given in equation (2.13), is nothing but $\epsilon_{[\Delta_{M} \mathcal{O}_{M} \times M]}$. Furthermore, $\epsilon_{M} \circ \eta_{M}$ is the identity, since $\epsilon_{[\Delta_{M} \mathcal{O}_{M} \times M]} \circ \Delta_{M}^{*} \eta_{\mathcal{O}_{M} \times M} = id_{\mathcal{O}_{M}}$

Equality (3.24) reduces to the equality

$$\text{tr} \left( \Phi_{*}(t_{X}) \right) = \pi^{0,2} \left[ \overline{I}_{M}^{*} \left( \Phi_{*}(\eta_{X}(t_{X})) \right) \right].$$

Equivalenty, it suffices to prove that $\epsilon_{M}$ maps $\eta_{M} \left[ \text{tr} \left( \Phi_{*}(t_{X}) \right) \right]$ and $\Phi_{*}(\eta_{X}(t_{X}))$ to the same element of $H^{2}(M)$. Let $(\Phi_{*})^{\dagger}$ be the adjoint of $\Phi_{*}$ with respect to the Serre Duality pairing, given in equation (2.16). We will verify equality (3.25) by establishing the equality

$$\langle \nu, \eta_{M} \left[ \text{tr} \left( \Phi_{*}(t_{X}) \right) \right] \rangle = \langle \nu, (\Phi_{*})^{\dagger} \nu, \eta_{X}(t_{X}) \rangle,$$

for every element $\nu \in H\Omega_{-2}(M)^{\vee}$, which is in the image of the following composition

$$H^{2n}(M, \omega_{M}) \cong \text{Hom}(\mathcal{O}_{M}, \omega_{M}[2n - 2]) \xrightarrow{\Delta_{M}^{*}} \text{Hom}(\mathcal{O}_{M}, \omega_{M}[2n - 2]) \cong H\Omega_{-2}(M)^{\vee},$$

where $\omega_{M}$ is the canonical bundle of $M$. The analogous composition with $X$ is $H^{2n}(X, \omega_{X}) \cong H\Omega_{-2}(X)$. Moreover, $\eta_{X} \circ \epsilon_{M}$ is the identity.
where the right isomorphism is Serre’s duality. This suffices because \( \langle \Delta_M, \bar{\nu} \rangle = \langle \bar{\nu}, \epsilon_M(\lambda) \rangle \)
for every \( \lambda \in HH_{-2}(M) \) and \( \bar{\nu} \in H^{2n-2}(M, \omega_M) \). The following three observations explain the latter equality.

(i) The right isomorphism in the displayed composition above is given also by the Mukai pairing under the identification \( HH_2(M) \cong \text{Hom}(\theta_{\Delta_M}, \omega_{\Delta_M}[2n-2]) \), by [C2, Subsection 4.11].

(ii) The modified HKR isomorphism \( \tilde{I}^M \) is an isometry with respect to the Mukai pairings on \( HH_*(M) \) and \( \Omega_*(M) \) (see the conjecture in [C3, Sec. 1.8] and its proof in [HN, Theorem 0.5]).

(iii) The composition of the map \( \tilde{I}^M \circ \Delta_{M,*} : H^i(M, \omega_M) \to H\Omega_{2n-i}(M) \) with the projection \( H\Omega_{2n-i}(M) \to H^i(M, \omega_M) \) is the identity (see the proof of [HN, Prop. 2.1]).

Let \( \bar{\nu} \) be an element of \( H^{2n-2}(M, \omega_M) \) mapping to \( \nu \). The morphisms \( \Phi(t_X) : \Phi(\mathcal{O}_X) \to \Phi(\mathcal{O}_X)[2] \) and \( \mu_{\Phi(\mathcal{O}_X)[2]}(\bar{\nu}) : \Phi(\mathcal{O}_X)[2] \to \Phi(\mathcal{O}_X) \otimes \omega_M[2n] \) compose to yield the morphism \( \mu_{\Phi(\mathcal{O}_X)[2]}(\bar{\nu}) \circ \Phi(t_X) : \Phi(\mathcal{O}_X) \to \Phi(\mathcal{O}_X) \otimes \omega_M[2n] \).

For any two objects \( x_1, x_2 \) of \( D^b(X) \), we have the homomorphisms

\[
\Phi : \text{Hom}(x_1, x_2) \to \text{Hom}(\Phi(x_1), \Phi(x_2))
\]

and its left adjoint with respect to the Serre Duality pairing

\[
\Phi^\dagger_L : \text{Hom}(\Phi(x_2), \Phi(x_1) \otimes \omega_M[2n]) \to \text{Hom}(x_2, x_1 \otimes \omega_X[2]).
\]

The morphism \( \mu_{\Phi(\mathcal{O}_X)[2]}(\bar{\nu}) \) belongs to \( \text{Hom}(\Phi(x_2), \Phi(x_1) \otimes \omega_M[2n]) \), for \( x_1 = \mathcal{O}_X \) and \( x_2 = \mathcal{O}_X[2] \). Hence, \( \Phi^\dagger_L [\mu_{\Phi(\mathcal{O}_X)[2]}(\bar{\nu})] \) belongs to \( \text{Hom}(\mathcal{O}_X, \mathcal{O}_X \otimes \omega_X) \). The equality

\[
(3.27) \quad Tr_M (\text{tr} [\mu_{\Phi(\mathcal{O}_X)[2]}(\bar{\nu}) \circ \Phi(t_X)]) = Tr_X \left( \Phi^\dagger_L [\mu_{\Phi(\mathcal{O}_X)[2]}(\bar{\nu}) \circ t_X] \right)
\]

is established in [C2, Prop. 3.1].

We prove next that the left hand sides of equations (3.26) and (3.27) are equal. The equality

\[
\text{tr} [\mu_{\Phi(\mathcal{O}_X)[2]}(\bar{\nu}) \circ \Phi(t_X)] = \bar{\nu} \circ \text{tr} (\Phi(t_X))
\]

holds, by [C2, Lemma 2.4]. Now

\[
Tr_M (\bar{\nu} \circ \text{tr} (\Phi(t_X))) = Tr_{M \times M} (\nu \circ \eta_M [\text{tr} (\Phi(t_X))]),
\]

by Lemma [3.13] applied with \( X = M, Y = M \times M, f = \Delta_M, x = \mathcal{O}_M[-2], y = \mathcal{O}_{M \times M}, t = \text{tr} (\Phi(t_X)), \) and \( \phi = \bar{\nu} \). Hence, the left hand sides of equations (3.26) and (3.27) are equal.

It remains to prove that the right hand sides of equations (3.26) and (3.27) are equal. The following relation between \( \Phi^\dagger_L \) and \( (\Phi_*)^\dagger \) holds, for every object \( F \) of \( D^b(X) \)

\[
\Phi^\dagger_L (\mu_{\Phi(F)}(\bar{\nu})) = \left[ (\Phi_*)^\dagger (\Delta_M(\bar{\nu})) \right]_F,
\]

by [C2, Prop. 3.1]. Taking \( F = \mathcal{O}_X \), we get

\[
(3.28) \quad \Phi^\dagger_L (\mu_{\Phi(\mathcal{O}_X)}(\bar{\nu})) = \left[ (\Phi_*)^\dagger (\Delta_M(\bar{\nu})) \right]_{\mathcal{O}_X} = \left[ (\Phi_*)^\dagger (\bar{\nu}) \right]_{\mathcal{O}_X}.
\]

In addition, we have the following relation between the Serre Duality pairing for morphisms in \( D^b(X) \) and the Serre Duality pairing for morphisms in \( D^b(X \times X) \), or for natural transformation.

\[
(3.29) \quad \langle (\Phi_*)^\dagger (\nu), \eta_X(t_X) \rangle = \langle \left[ (\Phi_*)^\dagger (\nu) \right]_{\mathcal{O}_X}, t_X \rangle.
\]
Indeed, the morphism $(\Phi_*)^!(\nu)$ belongs to $\text{Hom}(\Delta_X, \mathcal{O}_X, \Delta_X, \omega_X)$. Hence, $(\Phi_*)^!(\nu) = \Delta_X(\phi)$, for a morphism $\phi$ in $\text{Hom}(\mathcal{O}_X, \omega_X)$. Now apply Lemma 3.13 with $Y = X \times X$, $f = \Delta_X$, $x = \mathcal{O}_X[-2]$, $y = \mathcal{O}_X\times X$, $t = tx$, and $\phi$ as above, to obtain Equation (3.29).

Combining the last two equations above we get:

\[
\langle (\Phi_*)^!(\nu), \eta_X(tx) \rangle = \langle \left( (\Phi_*)^!(\nu) \right)_{\mathcal{O}_X}, tx \rangle = \langle \Phi_L^* (\mu_\Phi(\mathcal{O}_X)(\nu)), tx \rangle.
\]

This is precisely the desired equality of the two right hand sides of equations (3.26) and (3.27).

**Example 3.17.** Let us verify Equation (3.29) in a simple case. Take $n = 1$, identify $X$ with $M := X^{[1]}$, and set $\mathcal{V} := \mathcal{O}_X$ to be the ideal sheaf of the diagonal in $X \times X^{[1]}$. Choose a sheaf $F$ on $X$ satisfying $h^i(F) = 0$, for $i > 0$. Consider the short exact sequence

\[
0 \to \mathcal{V} \to \mathcal{O}_{X \times X} \to \mathcal{O}_{\Delta_X} \to 0.
\]

Then $\text{rank}(\Phi_\mathcal{V}(F)) = \chi(F) - \text{rank}(F)$ and $\Phi_\mathcal{V}(F)$ fits in the exact triangle

\[
F[-1] \to \Phi_\mathcal{V}(F) \to H^0(F) \otimes_{\mathbb{C}} \mathcal{O}_{\Delta_X} \to F.
\]

Furthermore, $\varphi_\mathcal{V}(tx) = -tx$, under the identification of $X$ with $M := X^{[1]}$. The Fourier-Mukai transform $\Phi_\mathcal{O}_{X \times X}$ with respect to the structure sheaf $\mathcal{O}_{X \times X}$ sends $\mu_F(tx) : F \to F^{[2]}$ to zero. Indeed, consider the cartesian diagram

\[
\begin{array}{ccc}
X \times X & \xrightarrow{\pi_2} & X \\
\pi_1 \downarrow & & \kappa \downarrow \\
X & \xrightarrow{\kappa} & \{\text{pt}\}.
\end{array}
\]

Then $R\pi_2^*(\mu_F(tx)) = \kappa^* R\kappa_*(\mu_F(tx)) = \kappa^* \kappa_*(\mu_F(tx))$ and the morphism $\kappa_*(\mu_F(tx)) : H^0(F) \to H^0(F)[2]$ in $D^b(\{\text{pt}\})$ vanishes. We get the commutative diagram

\[
\begin{array}{ccc}
\Phi_{\mathcal{V}}(F) & \xrightarrow{\Phi_{\mathcal{V}}(\mu_F(tx))} & H^0(F) \otimes_{\mathbb{C}} \mathcal{O}_{\Delta_X} \\
\Phi_{\mathcal{V}}(\mu_F(tx)) \downarrow & & \mu_F(tx) \downarrow \\
\Phi_{\mathcal{V}}(F)[2] & \xrightarrow{H^0(F) \otimes_{\mathbb{C}} \mathcal{O}_{\Delta_X}[2]} & F[2].
\end{array}
\]

We see that indeed $\text{tr}(\Phi_{\mathcal{V}}(\mu_F(tx))) = -\text{rank}(F)tx = \text{rank}(F)[\varphi_{\mathcal{V}}(tx)]_2$. Note, by the way, that $\Phi_{\mathcal{V}} : \text{Hom}(F, F[2]) \to \text{Hom}(\Phi_{\mathcal{V}}(F), \Phi_{\mathcal{V}}(F)[2])$ is an isomorphism. If the sheaf $F$ is simple, then $\text{Hom}(\Phi_{\mathcal{V}}(F), \Phi_{\mathcal{V}}(F)[2])$ is one-dimensional. In that case we get the equality

\[
\text{rank}(\Phi_{\mathcal{V}}(F)) \Phi_{\mathcal{V}}(\mu_F(tx)) = \text{rank}(F) \mu_{\Phi_{\mathcal{V}}(F)}(\varphi_{\mathcal{V}}(tx))_2,
\]

since both sides above have the same trace.

**3.6. A relation in the Yoneda algebra of $\Phi_\mathcal{V}(x)$**. Let $t_x$ be a non-zero element of $H^2(X, \mathcal{O}_X)$ and $\varphi_{\mathcal{V}} : H^*(X, \mathbb{C}) \to H^*(M, \mathbb{C})$ the homomorphism in Proposition 3.16. Set

\[
t_M := [\varphi_{\mathcal{V}}(tx)]_2.
\]

**Lemma 3.18.** The class $t_M$ spans $H^{2,0}(M)$. Furthermore, the following equality holds, for every object $x \in D^b(X)$.

\[
\text{tr} (\Phi_{\mathcal{V}}(\mu_x(tx))) = \text{rank}(x)t_M.
\]

**Proof.** The statement follows immediately from Theorem 2.16 and Proposition 3.16. \qed
We regard \( t_X \) also as an element of the subspace \( \text{Ext}^2(\mathcal{O}_X, \mathcal{O}_X) \) of \( Y(\mathcal{O}_X) \). Given an object \( x \) of \( D^b(X) \), set
\[
t_x := \mu_x(t_X) \\
t_{\Phi_x}(x) := \mu_{\Phi_x}(t_M).
\]

Let \( v(x) := ch(x)\sqrt{td_X} \) be the Mukai vector of \( x \). Then \( t_{\Phi_x}(x) \) is an element of \( \text{Ext}^2(\Phi_x, \Phi_x) \) satisfying
\[
(3.32) \quad \text{tr} \left[ t_{\Phi_x}(x) \right] = \text{tr} \left[ \mu_{\Phi_x}(t_M) \right] = \text{rank}(\Phi_x) \cdot t_M = -(v, v(x)) \cdot t_M.
\]

**Lemma 3.19.** Let \( x \) be an object of \( D^b(X) \) satisfying \( \text{Hom}(x, x) \cong \mathbb{C} \). The following equation holds in \( \text{Ext}^{2n}(\Phi_x, \Phi_x) \).
\[
(3.33) \quad -(v, v(x)) \cdot (t_{\Phi_x}(x))^n \Phi_x(t_x) = \text{rank}(x) (t_{\Phi_x}(x))^n.
\]

**Proof.** The vector space \( \text{Ext}^{2n}(\Phi_x, \Phi_x) \) is dual to \( \text{Hom}(\Phi_x, \Phi_x) \), which is isomorphic to \( \text{Hom}(x, x) \), by Theorem 2.2, and is thus one-dimensional. Consequently, the two sides of Equation (3.33) are linearly dependent. Equation (3.33) would thus follow from Equations (3.31) and (3.32), once we prove that the Yoneda product
\[
(t_{\Phi_x}(x))^{n-1} : \text{Ext}^2(\Phi_x, \Phi_x) \to \text{Ext}^{2n}(\Phi_x, \Phi_x)
\]
factors through \( \text{tr} : \text{Ext}^2(\Phi_x, \Phi_x) \to H^2(M, \mathcal{O}_M) \). We prove this factorization next. Note that the morphism \( \mu_{\Phi_x} : \mathcal{O}_M \to \Phi_x^{\vee} \otimes \Phi_x \) is compatible with the Yoneda product. Hence, \( (t_{\Phi_x}(x))^{n-1} = \mu_{\Phi_x}(t_M^{n-1}) \). For every object \( y \) of \( D^b(M) \), and for every integers \( i \) and \( j \), the outer square of the following diagram is commutative.

\[
\begin{array}{ccc}
\text{Ext}^i(y, y) \otimes \text{Ext}^j(y, y) & \longrightarrow & \text{Ext}^{i+j}(y, y) \\
\downarrow id \otimes \mu & & \downarrow \alpha \\
\text{Ext}^i(y, y) \otimes H^j(\mathcal{O}_M) & \longrightarrow & \text{Ext}^{i+j}(y, y) \\
\downarrow \text{tr} \otimes id & & \downarrow \text{tr} \\
H^i(\mathcal{O}_M) \otimes H^j(\mathcal{O}_M) & \longrightarrow & H^{i+j}(\mathcal{O}_M).
\end{array}
\]

The homomorphism \( \alpha \), defined to make the diagram commutative, factors through the bottom left vertical homomorphism \( \text{tr} \otimes id \), whenever the right vertical trace homomorphism is an isomorphism. Apply it with \( y = \Phi_x(x) \), \( i = 2 \), \( j = 2n-2 \), the element \( t_M^{n-1} \) of \( H^{2n-2}(\mathcal{O}_M) \), and observe that the trace homomorphism \( \text{tr} : \text{Ext}^{2n}(\Phi_x, \Phi_x) \to H^{2n}(\mathcal{O}_M) \) is an isomorphism, by Remark 3.15, and the fact that \( \text{Hom}(\Phi_x, \Phi_x) \cong \text{Hom}(x, x) \cong \mathbb{C} \). The Equality (3.33) now follows, where the coefficient on its left hand side is explained by the equality \( \text{rank}(\Phi_x(x)) = -(v, v(x)) \).

Both sides of Equation (3.33) vanish, whenever \( \text{rank}(x) = 0 \) or \( \text{rank}(\Phi_x(x)) = 0 \) (that is \(-(v, v(x)) = 0\)). This can be seen directly, without using the above lemma, as follows. The left hand side vanishes if \( \text{rank}(x) = 0 \), since \( t_x := \mu_x(t_X) \) vanishes, by Remark 3.15. The right hand side vanishes if \( \text{rank}(\Phi_x(x)) = 0 \), since \( (t_{\Phi_x}(x))^n = (\mu_{\Phi_x}(t_M))^n = \mu_{\Phi_x}(t_M^n) \) vanishes, by Remark 3.15 again. Assume next that \( \text{Hom}(x, x) \) is one-dimensional. Let \( tr_x^{-1} \) be the inverse of the isomorphism given in Equation (3.22). If \( \text{rank}(x) \) and \( \text{rank}(\Phi_x(x)) \) do not vanish, Equation (3.33) is equivalent to the following equation:
\[
(3.34) \quad (t_{\Phi_x}(x))^{n-1} \Phi_x(tr_x^{-1}(t_X)) = t_{\Phi_x}(x) (t_M^n).
\]
We will verify Equation (3.34) assuming only that \( \text{rank}(\Phi_Y(x)) \) does not vanish (Theorem 3.21). We expect the above equation to hold even if \( \text{rank}(\Phi_Y(x)) \) vanishes.

3.7. The natural transformation \( h_2 \) is the Mukai vector. Set \( R(M) := \text{Ext}^{2n}(\mathcal{O}_M, \mathcal{O}_M)[-2n] \), regarded as an object of \( D^b(\text{pt}) \). The objects of the exact triangle displayed in Equation (3.6) correspond to kernels of integral endo-functors of \( D^b(X) \). The object \( \mathcal{R} \) corresponds to the functor of tensorization by \( R(M) \) over \( C \). The object \( \Psi := \mathcal{O}_{\Delta_X} \otimes_C Y(\mathcal{O}_M) \) corresponds to the functor of tensorization by \( Y(\mathcal{O}_M) \) over \( C \). The object \( \mathcal{A} \) is the kernel of the functor \( \Psi_Y \Phi_Y \). The morphisms of the exact triangle (3.30) correspond to natural transformations between these endo-functors.

Given objects \( F_1, F_2 \) of \( D^b(X) \), we get the short exact sequence

\[
0 \to \text{Hom}(F_1, F_2 \otimes_C R(M)) \xrightarrow{h_{F_2}} \text{Hom}(F_1, F_2 \otimes_C Y(\mathcal{O}_M)) \xrightarrow{q_{F_2}} \text{Hom}(F_1, \Psi_Y \Phi_Y(F_2)) \to 0.
\]

Exactness of the above sequence follows from the splitting of the exact triangle (3.6). Equivalently, we have the short exact sequence

\[
0 \to \text{Hom}(F_1, F_2[-2n]) \otimes \text{Ext}^{2n}(\mathcal{O}_M, \mathcal{O}_M) \xrightarrow{h_{F_2}} \oplus_{i=0}^{n} \text{Hom}(F_1, F_2[-2i]) \otimes \text{Ext}^{2i}(\mathcal{O}_M, \mathcal{O}_M) \xrightarrow{Q} \text{Hom}(\Phi_Y(F_1), \Phi_Y(F_2)) \to 0.
\]

Set \( Y^{2k} := H^{2k}(\mathcal{O}_M) \), so that \( Y^{2k}[-2k] \) is the graded summand of \( Y(\mathcal{O}_M) \) of degree \( 2k \). Let \( t_X \in H^2(X, \mathcal{O}_X) \) be a non-zero class, \( t_M \in H^2(M, \mathcal{O}_M) \) the class associated to \( t_X \) in Equation (3.30), and let \( t_F \) be the class \( \mu_F(t_X) \) in \( \text{Ext}^2(F, F) \). Write \( h = \tilde{h}_0 \otimes 1 + \tilde{h}_2 \otimes t_M^* + \tilde{h}_4 \otimes (t_M^*)^2 \), using the notation of Theorem 3.10. Above, \( h_{2j} \) is a natural transformation from the identity functor \( 1 \) of \( D^b(X) \) to \( 1[2j] \).

**Theorem 3.20.** Let \( F_1 \) and \( F_2 \) be objects of \( D^b(X) \).

1. The following is a short exact sequence

\[
0 \to \text{Hom}^\bullet(F_1, F_2[-2n]) \otimes Y^{2n} \xrightarrow{h} \text{Hom}^\bullet(F_1, F_2 \otimes Y(\mathcal{O}_M)) \xrightarrow{Q} \text{Hom}^\bullet(\Phi_Y(F_1), \Phi_Y(F_2)) \to 0,
\]

where \( h \) and \( Q \) are homomorphisms of degree 0, and \( Q(g \otimes y) = \mu_{\Phi_Y(F_2)}(y)\Phi_Y(g) \).

2. If \( F_1 \) and \( F_2 \) are sheaves on \( X \), then \( \text{Hom}(\Phi_Y(F_1), \Phi_Y(F_2)[k]) = 0 \), for \( k > 2n \) and for \( k < 0 \). The homomorphism \( Q \) restricts to an isomorphism for degrees in the range \( 0 \leq k \leq 2n - 1 \). In degree \( 2n \) we get the short exact sequence

\[
0 \to \text{Hom}(F_1, F_2) \otimes Y^{2n} \xrightarrow{h} \left[ \text{Ext}^2(F_1, F_2) \otimes Y^{2n-2} \oplus \text{Hom}(F_1, F_2) \otimes Y^{2n} \right] \xrightarrow{Q} \text{Hom}(\Phi_Y(F_1), \Phi_Y(F_2)[2n]) \to 0,
\]

where \( h \) is given by the equality

\[
h(f \otimes t_M^n) = (\tilde{h}_{2F_2} \circ f) \otimes t_M^{n-1} + (\tilde{h}_{0F_2} \circ f) \otimes t_M^n,
\]

for all \( f \in \text{Hom}(F_1, F_2) \). Consequently, if in addition \( \text{Hom}(F_1, F_2) = 0 \), then \( Q \) induces an isomorphism in degree \( 2n \) as well.

3. When \( F \) is a simple sheaf on \( X \), the kernel of

\[
Q : \left[ \text{Ext}^2(F, F) \otimes Y^{2n-2} \oplus \text{Hom}(F, F) \otimes Y^{2n} \right] \to \text{Hom}(\Phi_Y(F), \Phi_Y(F)[2n])
\]

is spanned by the element

\[
(3.36) \quad - (v, v(F^\vee))tr_F^{-1}(t_X) \otimes t_M^{n-1} - 1 \otimes t_M^n,
\]
where $tr_F : \text{Ext}^2(F,F) \to H^2(X,\mathcal{O}_X)$ is the isomorphism given in Equation (3.22).

Proof. 1. The short exact sequence in formula (3.35) establishes the statement in degree zero. For degree $k$, replace $F_2$ by the object $F_2[k]$. We have the equalities

$$(q_{F_2})(g \otimes y) = \Xi_{\mathcal{Y}}(\mu_{\pi_{F_2}}(\pi_M^*(y)) \circ \pi_Y^*(g)) = \mu_{\Xi_{\mathcal{Y}}(\pi_{F_2})}(y) \circ \Xi_{\mathcal{Y}}(g) = \mu_{\Phi_{\mathcal{Y}}(F_2)}(y) \circ \Phi_{\mathcal{Y}}(g),$$

where the first equality follows by Lemma 3.11, the second is due to the fact that the kernel $\Delta_{23}$ is the transcendental subspace. Both components of $h$ is an injective homomorphism from a finite dimensional vector space to itself, by part 1, hence an isomorphism. If $k$ is in the range $0 \leq k \leq 2n-1$, the space $\text{Hom}(F_1,F_2[k-2n]) \otimes Y^{2n}$ vanishes, hence the degree $k$ component of $Q$ is an isomorphism. The statement in degree 2n follows easily from part 1.

2. Note first that $\text{Ext}^j(F_1,F_2)$ vanishes, for $j \notin \{0,1\}$, since $F_1$ and $F_2$ are sheaves. When $F_1$ and $F_2$ are sheaves, and $k > 2n$ or $k < 0$, then the degree $k$ component of $h$ is an injective homomorphism from a finite dimensional vector space to itself, by part 1 hence an isomorphism. If $k$ is in the range $0 \leq k \leq 2n-1$, the space $\text{Hom}(F_1,F_2[k-2n]) \otimes Y^{2n}$ vanishes, hence the degree $k$ component of $Q$ is an isomorphism. The statement in degree 2n follows easily from part 1.

3. If rank($F$) $\neq 0$, then the element given in (3.36) belongs to the kernel of $m_2$, by Equations (3.18) and (3.33). This element must span the kernel, as the kernel is one-dimensional, by part 2. The kernel is spanned by

$$h(1_F \otimes t_M^n) = \tilde{h}_0 \otimes t_M^n + \tilde{h}_2 \otimes t_M^{n-1}$$

as well. Hence, $\tilde{h}_2 = \tilde{h}_0(v,v(F'))tr_F^{-1}(t_M)$, for every simple sheaf $F$ satisfying the inequality rank($F$) $\neq 0$. We conclude that $\tilde{h}_0$ does not vanish and we normalize the natural transformation $h$ by rescaling it, so that $\tilde{h}_0 = -1$. Then

$$(3.37) \quad \tilde{h}_2 = -(v,v(F'))tr_F^{-1}(t_M),$$

for every simple object $F$ satisfying the inequality rank($F$) $\neq 0$. Let $\sigma_X \in H^0(X,\omega_X)$ be the class, such that $Tr_X(t_M \otimes \sigma_X) = 1$. The class $h_2 := \tilde{h}_2 \otimes \sigma_X$ belongs to $HH_0(X)$ and the above equation translates to $Tr_X(h_2) = -(v,v(F'))$. The latter equality holds for all simple sheaves $F$ satisfying rank($F$) $\neq 0$. This suffices to determines the algebraic part of the class $\tilde{h}_2$ in $H\Omega_0(X)$ as $ch(v)\sqrt{Det_X}$, by Lemma 3.14. Hence, Equality (3.37) holds regardless of the vanishing of rank($F$). Part 3 of the Theorem follows. □

Proof of part 3 of Theorem 3.70 Write $H^{1,1}(X,\mathbb{C}) = [H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}] \oplus \Theta(X)$, where $\Theta(X)$ is the transcendental subspace. Both $ch(v)$ and $I^X_*(h_2)$ are elements of $H^0(X,\mathbb{C}) \oplus H^{1,1}(X,\mathbb{C}) \oplus H^4(X,\mathbb{C})$.

Now $ch_1(v)$ belongs to $H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ and Equation (3.37) in the proof of Theorem 3.20 implies that the projection of $I^X_*(h_2)$ to $H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ is equal to $ch_1(v)$. Varying the surface $X$ in a codimension one family in moduli keeping $ch_1(v)$ of Hodge type $(1,1)$, the classes $ch(v)$ and $I^X_*(h_2)$ define two continuous sections of the Hodge bundle $\mathcal{H}^{1,1}$ with fiber $H^{1,1}(X,\mathbb{C})$, the difference of which is purely transcendental. But a purely transcendental continuous section of $\mathcal{H}^{1,1}$ over such a family must vanish, by the density of Hodge structures with trivial transcendental subspace. □

Let $F$ be a simple sheaf on $X$ and $Y(F) := \oplus_{i \in \mathbb{Z}} \text{Ext}^i(F,F)[-i]$ its Yoneda algebra. Let $Y(\Phi_{\mathcal{Y}}(F)) := \oplus_{i \in \mathbb{Z}} \text{Hom}(\Phi_{\mathcal{Y}}(F),\Phi_{\mathcal{Y}}(F)[i][-i]$ be the Yoneda algebra of $\Phi_{\mathcal{Y}}(F)$. Let

$$f : Y(F) \otimes_{\mathbb{C}} \mathbb{C}[z] \to Y(\Phi_{\mathcal{Y}}(F))$$

be the algebra homomorphism restricting to $Y(F) \otimes 1$ as $\Phi_{\mathcal{Y}}$ and sending the indeterminate $z$ to $\mu_{\Phi_{\mathcal{Y}}(F)}(t_M)$ in $\text{Hom}(\Phi_{\mathcal{Y}}(F),\Phi_{\mathcal{Y}}(F)[2])$. The homomorphism $f$ is well defined, since
\(\mu_{\Phi_{\mathcal{V}}(F)}(t_M)\) belongs to the center of \(Y(\Phi_{\mathcal{V}}(F))\), as \(t_M\) is a natural transformation from \(\mathbb{I}_{D^b(M)}\) to \(\mathbb{I}_{D^b(M)[2]}\).

**Theorem 3.21.** The homomorphism \(f\) is surjective and its kernel is the ideal generated by (3.38)

\[
1 \otimes z^n + (v, v(F')) tr^{-1}_F(t_X) \otimes z^{n-1}.
\]

Note that the equality \(f(1 \otimes z^n) = -f((v, v(F')) tr^{-1}_F(t_X) \otimes z^{n-1})\) implies the vanishing of \(f(1 \otimes z^{n+1})\), since \((tr^{-1}_F(t_X))^2 = 0\). Hence, \(f\) factors through \(Y(F) \otimes Y(\mathcal{O}_M)\).

**Proof.** The algebra \(Y(F) \otimes_\mathbb{C} \mathbb{C}[z]\) is graded, where \(z\) has degree 2. Let \(Y'\) be its quotient by the ideal generated by the homogeneous element (3.38). The homomorphism \(f\) factors through a homomorphism \(\tilde{f} : Y' \rightarrow Y(\Phi_{\mathcal{V}}(F))\), by Theorem 3.20 (3). The algebra \(Y'\) is graded. Denote by \(Y'_d\) its graded summand of degree \(d\). Then

\[
\dim(Y'_d) = \begin{cases} \dim \operatorname{Ext}^1(F, F) & \text{if } d \text{ is odd and } 0 < d < 2n, \\ 2 & \text{if } d \text{ is even and } 0 < d < 2n, \\ 1 & \text{if } d = 0 \text{ or } d = 2n, \\ 0 & \text{otherwise}. \end{cases}
\]

These are precisely the dimensions of the graded summands of \(Y(\Phi_{\mathcal{V}}(F))\), by Theorem 3.20 (2). Hence, it suffices to prove that the homomorphism \(\tilde{f}\) is surjective. Surjectivity follows from Theorem 3.20 (2). \(\square\)

### 3.8. Moduli spaces of sheaves over the moduli space \(M\)

We continue to assume that \(M := M_H(v)\) and the modular object \(\mathcal{A}\) over \(X \times X\) is totally split as in Definition 3.1. Let \(S\) be a scheme of finite type over \(\mathbb{C}\). A coherent \(\mathcal{O}_S\)-module \(F\) is simple, if \(\operatorname{End}_S(F, F)\) is spanned by the identity. Let \(\operatorname{Spl}(S)\) be the moduli space of simple sheaves over \(S\) [AK]. Denote by \([F]\) the point of \(\operatorname{Spl}(S)\) corresponding to \(F\).

**Corollary 3.22.** Let \(F\) be a simple coherent \(\mathcal{O}_X\)-module and assume that \(\Phi_{\mathcal{V}}(F)[i]\) is equivalent to a coherent \(\mathcal{O}_M\)-module \(F_M\), for some integer \(i\). Then \(F_M\) is simple. The functor \(\Phi_{\mathcal{V}}\) induces an isomorphism, from the open subset

\[
U := \{[F] : \Phi_{\mathcal{V}}(F)[i] \text{ is equivalent to a coherent } \mathcal{O}_M\text{-module}\}
\]

of \(\operatorname{Spl}(X)\), onto an open subset of \(\operatorname{Spl}(M)\).

**Proof.** If \(F\) is simple, then it is a smooth point of the moduli space \(\operatorname{Spl}(X)\) of simple sheaves [MN1]. The functor \(\Phi_{\mathcal{V}}\) induces isomorphisms \(\Phi_{\mathcal{V}} : \operatorname{Ext}^i_X(F, F) \rightarrow \operatorname{Ext}^i_M(F_M, F_M)\), for \(i = 0, 1\), by Theorem 3.20 (1). Hence, \(F_M\) is simple, if it is a sheaf. The functor \(\Phi_{\mathcal{V}}\) defines a morphism \(\phi : U \rightarrow \operatorname{Spl}(M)\). This follows from a flatness result for Fourier-Mukai functors ([Mu2], Theorem 1.6). The differential of \(\phi\) at \([F]\) is the isomorphism of Zariski tangent spaces \(\Phi_{\mathcal{V}} : \operatorname{Ext}^1_X(F, F) \rightarrow \operatorname{Ext}^1_M(F_M, F_M)\). Combining the injectivity of the differential with the smoothness of \(\operatorname{Spl}(X)\), we conclude that the dimension \(D\) of \(\operatorname{Spl}(M)\) at \([F_M]\) is larger than or equal to the dimension \(d\) of \(\operatorname{Spl}(X)\) at \([F]\). The surjectivity of the differential implies that \(D \leq d\). Thus, \(D = d = \dim(\operatorname{Ext}^1(F_M, F_M))\), and \(\operatorname{Spl}(M)\) is smooth at \([F_M]\). The homomorphism \(\Phi_{\mathcal{V}} : \operatorname{Hom}(F_1, F_2) \rightarrow \operatorname{Hom}(\Phi(F_1), \Phi(F_2))\) is an isomorphism, for any two sheaves \(F_1, F_2\) on \(X\), by Theorem 3.20 (2). It follows that \(\phi\) is injective and hence an isomorphism onto its image. \(\square\)

**Example 3.23.** Choosing \(F\) to be a sky-scraper sheaf of a point of the K3 surface \(X\) we see that \(X\) is a connected component of \(\operatorname{Spl}(M)\). The twisted universal sheaf \(\mathcal{V}\) over \(X \times M\) for the coarse moduli space \(M := M_H(v)\) of \(H\)-stable sheaves over the K3 surface \(X\) is also a universal sheaf of \(X\) as a moduli space of sheaves over \(M\).
Example 3.24. Let $\Phi_{\mathcal{U}'} : D^b(X) \to D^b(M)$ be the integral functor defined by replacing the kernel $\mathcal{U}$ with $\mathcal{U}'$ in equation (1.1). Define $\Psi_{\mathcal{U}'} : D^b(M) \to D^b(X)$ similarly. Note that the kernel of the integral functor $\Psi_{\mathcal{U}'} \circ \Phi_{\mathcal{U}'}$ is isomorphic to $\oplus_{n=1}^{\infty} \mathcal{O}_X[-2i]$, by Theorem 2.2 as its kernel is the pullback of the kernel $\mathcal{A}'$ of $\Psi_{\mathcal{U}'} \circ \Phi_{\mathcal{U}'}$ via the transposition $\tau$ of the two factors of $X \times X$. Now $\mathcal{A}'$ is $\tau$-invariant. Theorem 3.20 applies to the functor $\Phi_{\mathcal{U}'}$ as well. Let $w \in K(X)$ be another class and $H'$ a $w$-generic polarization, such that $\mathcal{M}_H(w)$ is smooth and projective. Assume that both $\mathcal{U}$ and $\mathcal{U}'$ are locally free. Denote by $w_n$ the class of $w \otimes [H^\otimes n]$. Then $\mathcal{M}_H(w)$ is isomorphic to $\mathcal{M}_H(w_n)$. Assume that both $v$ and $w$ have positive rank. Fix $n$ sufficiently negative. Then $\text{Ext}^i(F, F')$ is isomorphic to $H^i(F^* \otimes F')$ if $F$ is locally free and to $H^{2-i}(F \otimes (F')^*)^*$ if $F'$ is locally free, and it thus vanishes, for $i \neq 2$, for all $H$-stable sheaves $F$ of class $v$, and for all $H'$-stable sheaves $F'$ of class $w$. Thus, $\Phi_{\mathcal{U}'}(F')[2]$ is equivalent to a coherent sheaf, for all such $F'$. We get a component of $\text{Spl}(\mathcal{M}_H(v))$ isomorphic to $\mathcal{M}_H(w)$, via Corollary 3.22.

4. The transposition of the factors of $M \times M$

Let $X$ be a K3 surface. If $X$ is projective we assume given a primitive Mukai vector $v$ with $c_1(v)$ of Hodge-type $(1,1)$ and a $v$-generic polarization $H$ and we set $M := M_H(v)$. We also consider the case where $X$ is a Kähler non-projective K3 surface and $M = X^{[n]}$. Let $\mathcal{F}$ be the modular object and $\mathcal{E}$ the modular sheaf over $X \times X$ (Definition 1.5). We calculate the extension sheaves $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{M \times M})$ and $\mathcal{E}xt^1(\mathcal{E}, \mathcal{E})$ and the torsion sheaves $\mathcal{T}or_i(\mathcal{E}^*, \mathcal{E})$ in this section. Let $\tau : M \times M \to M \times M$ be the transposition of the two factors.

Lemma 4.1. The following two objects of $D^b(M \times M)$ are isomorphic.

(4.1) $\mathcal{F} \cong \tau^* \mathcal{F}'[2]$.

Proof. The object $\mathcal{F}$ is defined as $R\pi_{13*} (\pi_{12}^* \mathcal{U} \otimes \pi_{23}^* \mathcal{U}[2])$. We have the isomorphisms:

$$
\mathcal{F}' \cong R\mathcal{H}om (R\pi_{13*} (\pi_{12}^* \mathcal{U} \otimes \pi_{23}^* \mathcal{U}[2]), \mathcal{O}_{M \times M})
\cong R\pi_{13*} \left\{ R\mathcal{H}om \left( \pi_{12}^* \mathcal{U} \otimes \pi_{23}^* \mathcal{U}[2], \pi_{13}^* \mathcal{O}_{M \times M} \right) \right\}
\cong R\pi_{13*} \left( \pi_{12}^* \mathcal{U} \otimes \pi_{23}^* \mathcal{U}^* \right) \cong \tau^* \mathcal{F}[-2],
$$

where the first isomorphism is clear, the second is Grothendieck-Verdier Duality, the third uses the triviality of $\omega_{\pi_{13}}$, and the last is clear.

Let $\Delta \subset M \times M$ be the diagonal. Pushforward via the inclusion morphism embeds the category of coherent sheaves on $\Delta$ in that of $M \times M$. We suppress the pushforward notation and regard the former as a subcategory of the latter. Recall that $\mathcal{E} := \mathcal{H}^{-1} (\mathcal{F})$, $\mathcal{H}^0(\mathcal{F}) \cong \mathcal{O}_\Delta$, and all other sheaf cohomologies of $\mathcal{F}$ vanish. The following is a more precise description of $\mathcal{E}$ due to the first author.

Proposition 4.2 (Ma5, Proposition 4.5). The reflexive sheaf $\mathcal{E}$ is locally free away from $\Delta$. Its restriction to $\Delta$ is untwisted, and is described as follows:

(i) $\mathcal{E} \otimes \mathcal{O}_\Delta \cong \Omega_\Delta^2 / \mathcal{O}_\Delta \cdot \sigma$, where $\sigma$ is the symplectic form;
(ii) $\mathcal{T}or_i(\mathcal{E}, \mathcal{O}_\Delta) \cong \Omega_\Delta^{i+2}$, for $i > 0$.

Set $\mathcal{E}' := R\mathcal{H}om(\mathcal{E}, \mathcal{O}_{M \times M})$ and $\mathcal{E}^* := \mathcal{H}om(\mathcal{E}, \mathcal{O}_{M \times M})$, so that $\mathcal{E}^* := \mathcal{H}^0(\mathcal{E}')$.

---

6 If $v = (r, c, s)$ is primitive, $H$ is $v$-generic, $r \geq 2$, $(r, c, s + 1) = ku$, where $k$ is an integer and $u$ is a primitive Mukai vector satisfying $(u, u) \leq -4$, then all sheaves parametrized by $M_H(v)$ are locally free.
Lemma 4.3. The following sheaves isomorphisms exist:
\[ \mathcal{E}^* \cong \tau^* \mathcal{E}, \]
\[ \mathcal{E}xt^1(\mathcal{E}, \Theta_{M \times M}) \cong \mathcal{O}_\Delta, \]
\[ \mathcal{E}xt^{2n-2}(\mathcal{E}, \Theta_{M \times M}) \cong \mathcal{O}_\Delta, \]
and \( \mathcal{E}xt^i(\mathcal{E}, \Theta_{M \times M}) \cong 0 \), if \( i \) does not belong to \( \{0, 1, 2n - 2\} \).

Proof. Dualizing the exact triangle \( \mathcal{E}[1] \to \mathcal{F} \to \Theta_\Delta \to \mathcal{E}[2] \), we get the exact triangle
\[ \mathcal{E}^\vee[-2] \to \omega_\Delta[-2n] \to \mathcal{F}^\vee \to \mathcal{E}^\vee[-1]. \]
Trivializing \( \omega_\Delta \), shifting by 2, and applying the isomorphism \( (1) \), we get the following exact triangle.
\[ \mathcal{E}^\vee \to \Theta_\Delta[2-2n] \to \tau^* \mathcal{F} \to \mathcal{E}^\vee[1]. \]
The sheaf homomorphisms \( \mathcal{H}^i(\alpha) : \mathcal{H}^i(\Theta_\Delta[2-2n]) \to \mathcal{H}^i(\tau^* \mathcal{F}) \) vanish for all \( i \), since \( n \geq 2 \). The long exact sequence of sheaf cohomology breaks into \( \mathcal{H}^{-1}(\tau^* \mathcal{F}) \cong \mathcal{H}^0(\mathcal{E}^\vee) \), \( \mathcal{H}^0(\tau^* \mathcal{F}) \cong \mathcal{H}^0(\mathcal{E}^\vee) \), and \( \mathcal{H}^{2n-2}(\mathcal{E}^\vee) \cong \mathcal{O}_\Delta \), and \( \mathcal{H}^i(\mathcal{E}^\vee) = 0 \), if \( i \) does not belong to \( \{0, 1, 2n - 2\} \).

The following proposition is used in the proof of [MM2, Theorem 1.11].

Proposition 4.4. The sheaf \( \mathcal{E}xt^i(\mathcal{E}, \mathcal{E}) \) comes with a decreasing filtration
\[ F^0 \mathcal{E}xt^i(\mathcal{E}, \mathcal{E}) \subset F^{-1} \mathcal{E}xt^i(\mathcal{E}, \mathcal{E}) \subset \cdots \subset F^{2-2n} \mathcal{E}xt^i(\mathcal{E}, \mathcal{E}) = \mathcal{E}xt^i(\mathcal{E}, \mathcal{E}) \]
with graded pieces \( E_{pq}^i := F^p \mathcal{E}xt^{p+q}(\mathcal{E}, \mathcal{E})/F^{p+1} \mathcal{E}xt^{p+q}(\mathcal{E}, \mathcal{E}) \) satisfying
1. \( E_{0,0}^i = F^0 \mathcal{E}xt^0(\mathcal{E}, \mathcal{E}) \cong \mathcal{E}^* \otimes \mathcal{E} / \text{torsion}, \)
\[ E_{i,-1}^i \cong \Omega_\Delta^1, \]
\[ E_{2n-2,2-2n}^i \cong \Omega_\Delta^{2n}, \]
and all other graded pieces of the filtration of \( \mathcal{E}xt^0(\mathcal{E}, \mathcal{E}) \) vanish.
2. We have the short exact sequence
\[ 0 \to \Omega_\Delta^1/\sigma \to \mathcal{E}xt^i(\mathcal{E}, \mathcal{E}) \to \Omega_\Delta^{2n-1} \to 0, \]
where the subsheaf is \( E_{1,0}^i \) and the quotient is \( E_{2n-2,2-2n}^i \) and all other graded pieces of the filtration of \( \mathcal{E}xt^1(\mathcal{E}, \mathcal{E}) \) vanish.
3. \( \mathcal{E}xt^i(\mathcal{E}, \mathcal{E}) \cong \Omega_\Delta^{2n-i}, \) for \( 2 \leq i \leq 2n - 3 \).
\[ \mathcal{E}xt^{2n-2}(\mathcal{E}, \mathcal{E}) \cong \Omega_\Delta^1/\sigma, \]
and \( \mathcal{E}xt^i(\mathcal{E}, \mathcal{E}) = 0, \) for \( i > 2n - 2 \).
4. The torsion subsheaf of \( \mathcal{E}^* \otimes \mathcal{E} \) is isomorphic to \( \Omega_\Delta^4 \).
\[ \mathcal{F}or_j(\mathcal{E}^*, \mathcal{E}) \cong \Omega_\Delta^{j+4}, \text{ for } j \geq 1. \]
In particular, \( \mathcal{F}or_j(\mathcal{E}^*, \mathcal{E}) \) vanishes for \( j > 2n - 4 \).

Proof. Let \( W_j \to W_{j+1} \to \cdots \to W_{-1} \to W_0 \to \mathcal{E} \) be a locally free resolution of \( \mathcal{E} \). Denote the locally free complex by \( (W_\bullet, d) \) and let \( (W_\bullet^*, d^*) \) be the dual complex. We get the double complex
\[ W^{p,q} := W^p_{-p} \otimes W_q, \quad p \geq 0, q \leq 0 \]
and the associated single complex \( K^k := \oplus_{p+q=k} W^{p,q} \) with differential \( D \), which restricts to \( W^{p,q} \) as \( (-1)^p (d^* \otimes 1) + (1 \otimes d) : W^{p,q} \to W^{p+1,q} \oplus W^{p,q+1} \) [GH, Ch. 4, Sec. 5]. Note that \( (W_\bullet, d) \) is quasi-isomorphic to \( \mathcal{E}, (W_\bullet^*, d^*) \) represents \( \mathcal{E}^\vee = R^\infty \mathcal{H}om(\mathcal{E}, \Theta_{M \times M}) \), and \( (K^\bullet, D) \) represents \( R^\infty \mathcal{H}om(\mathcal{E}, \mathcal{E}) \) in \( D^0(M \times M) \). In particular, we have the isomorphism
\[ \mathcal{H}^4(K^\bullet, D) \cong \mathcal{E}xt^4(\mathcal{E}, \mathcal{E}). \]
Consider the decreasing filtration \( F^q K^i := \oplus_{p+q=i, q \geq 0} W_{p,q} \) and denote the induced filtration on sheaf cohomology by \( F^q \mathcal{E} x i^i (\mathcal{E}, \mathcal{E}) \). Set \( E^q_{\mathcal{H}} := F^q \mathcal{E} x i^{p+q} (\mathcal{E}, \mathcal{E}) / F^{q+1} \mathcal{E} x i^{p+q} (\mathcal{E}, \mathcal{E}) \). We have a spectral sequence converging to \( \mathcal{H} \times (K^i, D) \), with

\[
E_1^{0,q} := \mathcal{H}^0((W^* \times d^* \otimes W_q) / \mathcal{E} x i^i (\mathcal{E}, \mathcal{E}) / \mathcal{E} x i^i (\mathcal{E}, \mathcal{E}) \otimes W_q, \\
E_2^{p,q} := E_1^{p,q} \otimes \mathcal{E} x i^i (\mathcal{E}, \mathcal{E} / \mathcal{E} x i^i (\mathcal{E}, \mathcal{E} \otimes W_q, 1 \otimes d) \simeq \mathcal{T} or_{-q} (\mathcal{E} x i^i (\mathcal{E}, \mathcal{E} \otimes W_q, \mathcal{E})).
\]

We have seen that \( \mathcal{E} x i^i (\mathcal{E}, \mathcal{E} \otimes W_q) \) is isomorphic \( \mathcal{E} \Delta \), for \( p = 1 \) and \( p = 2n - 2 \), and it vanishes for all other positive values of \( p \), by Lemma 4.3. Furthermore, we have the isomorphisms \( \mathcal{T} or_{-q} (\mathcal{E} \Delta, \mathcal{E}) \simeq \Omega_{\mathcal{E}}^{2-q} \), for \( q < 0 \), and \( \mathcal{T} or_{0} (\mathcal{E} \Delta, \mathcal{E}) \simeq \Omega_{\mathcal{E}}^{2} / (\sigma) \), by Proposition 4.2. We get the following table for the \( E_2 \) term of the spectral sequence.

| \( q \backslash p \) | 0 | 1 | \cdots | 2n-2 |
|---|---|---|---|---|
| 0 | \( \mathcal{E} \ast \otimes \mathcal{E} \) | \( \Omega_{\mathcal{E}}^{2} / (\sigma) \) | \( \Omega_{\mathcal{E}}^{2} / (\sigma) \) |
| -1 | \( \mathcal{T} or_{1} (\mathcal{E} \ast, \mathcal{E}) \) | \( \Omega_{\mathcal{E}}^{2} \) | \( \Omega_{\mathcal{E}}^{2} \) |
| -2 | \( \mathcal{T} or_{2} (\mathcal{E} \ast, \mathcal{E}) \) | \( \Omega_{\mathcal{E}}^{2} \) | \( \Omega_{\mathcal{E}}^{2} \) |
| \vdots | \vdots | \vdots | \vdots |
| 2\( n \) | \( \mathcal{T} or_{2n-2} (\mathcal{E} \ast, \mathcal{E}) \) | \( \Omega_{\mathcal{E}}^{2n} \) | \( \Omega_{\mathcal{E}}^{2n} \) |

We show first that \( E_{\infty}^{2n-2,q} = E_{\infty}^{2n-2,q} \), for all \( q \). For \( j \geq 2 \), the differential of the spectral sequence is \( d_j : E_{j-1}^{2n-2,q} \to E_{j}^{2n-j+1,q+j} \). The vanishing of the columns other than for \( p = 0, 1, 2n - 2 \) implies that \( E_{\infty}^{2n-2,q} = E_{\infty}^{2n-2,q} \). The differential \( d_{2n-2} : E_{2n-2}^{2n-2,q} \to E_{2n-2}^{1,q+2n-2} \) vanishes for all \( q \). Indeed, for \( q = 2 - 2n \) the homomorphism is a section of \( H^0 (\Omega_{\mathcal{E}}^{2} / (\sigma)) \), which vanishes. For all other values of \( q \) the differential vanishes since its target vanishes.

The above vanishing of \( d_j : E_{j}^{p,q} \to E_{j}^{p-j+1,q+j} \), for \( j \geq 2 \) and \( p \geq 2 \), implies also that \( E_{\infty}^{1,q} = E_{2}^{1,q} \) and \( E_{\infty}^{0,q} = E_{2}^{0,q} \).

The sheaves \( E_{\infty} \) vanish for \( p + q < 0 \), since \( \mathcal{E} x i^{p+q} (\mathcal{E}, \mathcal{E}) \) vanishes for these values. Hence,

\[
d_2 : E_{2}^{1,q} \to E_{2}^{0,q+2}
\]

is injective for \( q \leq -2 \) and surjective for \( q < -2 \). In particular, the homomorphism

\[
d_2 : E_{2}^{1,-2} = \Omega_{\mathcal{E}}^{2} \to \mathcal{E} \ast \otimes \mathcal{E}
\]

is injective and the sheaves \( \mathcal{T} or_{-q} (\mathcal{E} \ast, \mathcal{E}) \) are as claimed. \( \square \)

**Remark 4.5.** Let \( \beta : Y \to M \times M \) be the blow-up centered along the diagonal. We claim that the sheaf \( \mathcal{E} \ast \mathcal{E} \) has a non-trivial torsion subsheaf. This is seen as follows. The sheaf \( \mathcal{E} \ast \mathcal{E} \) is isomorphic to \( \beta \ast \mathcal{V} \), for a locally free sheaf \( \mathcal{V} \) over \( Y \), by [Ma5], Prop. 4.5. \( \mathcal{E} \ast \otimes \mathcal{E} \mathcal{V} \) has a non-trivial torsion subsheaf, by Proposition 4.4 [4]. On the other hand, \( \mathcal{E} \ast \otimes \mathcal{E} \simeq \beta \ast (\mathcal{V} \otimes \beta \ast \mathcal{E}) \), by the projection formula. Hence, \( \mathcal{V} \otimes \beta \ast \mathcal{E} \) has a non-trivial torsion subsheaf. We conclude that so does \( \beta \ast \mathcal{E} \), as claimed.

5. **A simple and rigid comonad in \( D^b (M \times M) \)**

5.1. **\( \mathcal{F} \) is simple and rigid.** We keep the notation of section 4.1 so \( (X, H) \) is a polarized \( K3 \) surface, \( M := M_H(v) \) is a smooth and projective moduli space of \( H \)-stable sheaves on \( X \), of dimension \( 2n \), \( n \geq 2 \), \( \mathcal{U} \) is a universal sheaf over \( X \times M \), \( \Phi_{\mathcal{U}} : D^b (X) \to D^b (M, \theta) \) is the integral functor with kernel \( \mathcal{U} \), and \( \Psi_{\mathcal{U}} : D^b (M, \theta) \to D^b (X) \) is its right adjoint. Consider the following object in \( D^b (X \times M, \pi_M^* \theta^{-1}) \).

\[
\mathcal{V} := \mathcal{U} \otimes \pi_X^* \omega_X [2].
\]
Let $\pi_{ij}$ be the projection from $M \times X \times M$ onto the product of the $i$-th and $j$-th factors. The kernel of the endo-functor $\mathbb{L} := \Phi_{\mathcal{U}} \Psi_{\mathcal{U}} : D^b(M, \theta) \to D^b(M, \theta)$ is the object

$$\mathcal{F} = \pi_{13} (\pi_{12}^* \pi_{23}^* \mathcal{U})$$

of $D^b(M \times M, \pi_1^* \theta^{-1} \pi_2^* \theta)$. We prove in this section the following statement.

**Lemma 5.1.** Assume that $M = X^{[n]}$ and $\mathcal{U}$ is the universal ideal sheaf or, more generally, that the morphism $\alpha$ of Theorem 1.1 is an isomorphism.

1. The following isomorphisms hold:
   
   $$\text{Hom}(\mathcal{F}, \mathcal{F}) \cong \mathbb{C},$$
   
   $$\text{Hom}(\mathcal{F}, \mathcal{F}[k]) = 0,$$

   for odd $k$.

2. More generally, $\text{Hom}(\mathcal{F}, \mathcal{F}[k])$ is isomorphic to

   $$\oplus_{i=0}^{n-1} \left[ HH^{k-2i}(X) \right]^{\oplus i+1} \oplus \oplus_{j=0}^{n-2} \left[ HH^{k+2j+4-4n}(X) \right]^{\oplus j+1},$$

   for all integers $k$.

**Proof.** Part (1): Note that the Hochschild cohomology $HH^i(X) := \text{Hom}_{X \times X}(\Theta_{\Delta X}, \Theta_{\Delta X}[i])$ vanishes for $i < 0$, for $i > 4$, and for odd $i$. Part (1) follows from part (2) and the isomorphism $HH^0(X) := \text{Hom}_{X \times X}(\Theta_{\Delta X}, \Theta_{\Delta X}) \cong \mathbb{C}$.

Part (2): Let $\Phi_{\mathcal{U}} : D^b(X) \to D^b(M, \theta^{-1})$ be the integral functor with kernel $\mathcal{U}$ and let $\Psi_{\mathcal{U}} : D^b(M, \theta^{-1}) \to D^b(X)$ be its right adjoint. We get the object $\mathcal{U}^\vee \otimes \mathcal{U} := \pi_{13}^* \mathcal{U} \otimes \pi_{24}^* \mathcal{U}$.

in $D^b(X \times X \times M, \pi_1^* \theta^{-1} \pi_2^* \theta)$. Let

$$\Gamma : D^b(X \times X) \longrightarrow D^b(M \times M, \pi_1^* \theta^{-1} \pi_2^* \theta)$$

be the integral functor with kernel $\mathcal{U}^\vee \otimes \mathcal{U}$ and let $\Gamma^\dagger_R : D^b(M \times M, \pi_1^* \theta^{-1} \pi_2^* \theta) \to D^b(X \times X)$ be its right adjoint. Then $\Gamma^\dagger_R$ is the integral transform with kernel

$$(\mathcal{U}^\vee \otimes \mathcal{U})^\vee \otimes \pi_1^* \omega_X \otimes \pi_2^* \omega_X[4] \cong \mathcal{U} \otimes \mathcal{U}^\vee.$$

Note that $\Gamma$ is the cartesian product of the functors $\Phi_{\mathcal{U}}$ and $\Phi_{\mathcal{U}}$. Similarly, $\Gamma^\dagger_R$ is the cartesian product of their right adjoints $\Psi_{\mathcal{U}}$ and $\Psi_{\mathcal{U}}$.

Let $\mathcal{A}$ be the object of $D^b(X \times X)$ given in Equation (1.2). We have the following isomorphisms.

$$\Gamma(\Theta_{\Delta X}) \cong \mathcal{F},$$

$$\Gamma^\dagger_R(\Theta_{\Delta M}) \cong \mathcal{A}.$$

The composition $\Psi_{\mathcal{U}} \Phi_{\mathcal{U}} : D^b(X) \to D^b(X)$ has kernel $\mathcal{A}$. The kernel of the composition $\Psi_{\mathcal{U}} \Phi_{\mathcal{U}} : D^b(X) \to D^b(X)$ is the pull-back of $\mathcal{A}$ via the involution of $X \times X$ interchanging the two factors. $\mathcal{A}$ is invariant under this pull-back, and so $\mathcal{A}$ is the kernel of $\Psi_{\mathcal{U}} \Phi_{\mathcal{U}}$ as well. We conclude that the composite endo-functor

$$\Gamma^\dagger_R \Gamma : D^b(X \times X) \longrightarrow D^b(X \times X)$$

has kernel $\pi_{13}^* \mathcal{A} \otimes \pi_{24}^* \mathcal{A}$ in $D^b(X \times X \times X \times X)$. We get the following isomorphism.

$$\Gamma^\dagger_R(\Theta_{\Delta X}) \cong \mathcal{A} \circ \mathcal{A}.$$
The object \( \mathcal{A} \) is isomorphic to \( \mathcal{O}_{\Delta X} \otimes_{\mathcal{C}} \lambda_n \), where \( \lambda_n \) is an object of \( D^b(pt) \), by Theorem \( \ref{thm:rigidity} \) or by the assumption that \( \alpha \) is an isomorphism. Hence, the endo-functor \( \Gamma^*_R \Gamma \) is isomorphic to the functor of tensorization over \( \mathcal{C} \) with the object \((\lambda_n)^{\otimes 2}\) of \( D^b(pt) \). Now
\[
(\lambda_n)^{\otimes 2} \cong \oplus_{i=0}^{n-1} (\mathcal{O}_{pt}[-2i])^{\otimes i+1} \oplus \oplus_{j=0}^{n-2} (\mathcal{O}_{pt}[2j + 4 - 4n])^{\otimes j+1}.
\]
We get the isomorphisms:
\[
\text{Hom}_{M \times M}(\mathcal{F}, \mathcal{F}[k]) \cong \text{Hom}_{M \times M}(\Gamma(\mathcal{O}_{\Delta X}), \Gamma(\mathcal{O}_{\Delta X})[k])
\cong \text{Hom}_{X \times X} (\mathcal{O}_{\Delta X}, \Gamma^*_R \Gamma(\mathcal{O}_{\Delta X})[k])
\cong \text{Hom}_{X \times X} (\mathcal{O}_{\Delta X}, \mathcal{O}_{\Delta X} \otimes_{\mathcal{C}} (\lambda_n)^{\otimes 2}[k]).
\]
Part (2) follows immediately from the isomorphisms above. \( \square \)

**5.2. \( \mathcal{E} \) is simple and rigid.** Let \( \mathcal{E} \) be the sheaf cohomology \( \mathcal{H}^{-1}(\mathcal{F}) \), where \( \mathcal{F} \) is the object of \( D^b(M \times M) \) given in Equation \( \ref{eq:5.1} \). We get the exact triangle
\[
(5.4) \quad \mathcal{E}[1] \xrightarrow{\beta} \mathcal{F} \xrightarrow{\gamma} \mathcal{O}_{\Delta M} \xrightarrow{\epsilon} \mathcal{E}[2],
\]
where \( \epsilon \) is the morphism inducing the counit for the adjunction \( \Phi \dashv \Psi \). The following rigidity result is a crucial ingredient in the proof of Theorem \( \ref{thm:rigidity} \).

**Lemma 5.2.** Assume that \( M = X[n] \) and \( \mathcal{U} \) is the universal ideal sheaf. Then the sheaf \( \mathcal{E} \) is simple and rigid. In other words, \( \text{Hom}(\mathcal{E}, \mathcal{E}) \) is one-dimensional and \( \text{Ext}^1(\mathcal{E}, \mathcal{E}) \) vanishes.

**Proof.** Step 1: The integral functor \( \Gamma^*_R : D^b(M \times M) \to D^b(X \times X) \) takes the exact triangle \( \ref{eq:5.4} \) to the exact triangle
\[
(5.5) \quad \Gamma^*_R(\mathcal{E}[1]) \xrightarrow{\Gamma^*_R(\beta)} \mathcal{A} \circ \mathcal{A} \xrightarrow{m} \mathcal{A} \xrightarrow{\Gamma^*_R(\gamma)} \Gamma^*_R(\mathcal{E}[1]),
\]
where \( \mathcal{A} \circ \mathcal{A} \) is the convolution and \( m := \Gamma^*_R(\epsilon) \) is the multiplication, by Equations \( \ref{eq:5.2} \), \( \ref{eq:5.3} \), and the proof of Lemma \( \ref{lem:isomorphism} \). Now \( m \) has a right inverse given by
\[
\mathcal{A} \cong \mathcal{A} \circ \mathcal{O}_{\Delta X} \xrightarrow{n^{-1}} \mathcal{A} \circ \mathcal{A},
\]
where \( \eta : \mathcal{O}_{\Delta X} \to \mathcal{A} \) induces the unit for the adjunction \( \Phi \dashv \Psi \). Thus, \( \Gamma^*_R(\gamma) = 0 \) and the exact triangle \( \ref{eq:5.5} \) splits. The left adjoint \( \Gamma^*_L \) of \( \Gamma \) is isomorphic to a shift of the right adjoint \( \Gamma^*_R \), since \( X \) and \( M \) have trivial canonical line bundles. We conclude that \( \Gamma^*_L(\gamma) = 0 \).

In particular, the homomorphism
\[
\Gamma^*_L(\beta)^* : \text{Hom}(\Gamma^*_L(\mathcal{F}), x) \to \text{Hom}(\Gamma^*_L(\mathcal{E}[1]), x)
\]
is surjective, for all objects \( x \) of \( D^b(X \times X) \). Take \( x = \mathcal{O}_{\Delta X}[k] \), apply the adjunction \( \Gamma^*_L \dashv \Gamma \), and use the isomorphism \( \Gamma(\mathcal{O}_{\Delta X}) \cong \mathcal{F} \), to conclude that the homomorphism
\[
\text{Hom}(\mathcal{F}, \mathcal{F}[k]) \to \text{Hom}(\mathcal{E}[1], \mathcal{F}[k])
\]
is surjective, for all \( k \). We get the inequality
\[
\text{dim} \text{Hom}(\mathcal{E}[1], \mathcal{F}[k]) \leq \text{dim} \text{Hom}(\mathcal{F}, \mathcal{F}[k]),
\]
for all \( k \).

**Step 2:** Apply the functor \( \text{Hom}(\mathcal{E}, \bullet) \) to the exact triangle \( \ref{eq:5.4} \). We get the long exact sequence
\[
\text{Hom}(\mathcal{E}, \mathcal{O}_{\Delta M}[k - 1]) \to \text{Hom}(\mathcal{E}, \mathcal{E}[k + 1]) \to \text{Hom}(\mathcal{E}, \mathcal{F}[k]) \to \text{Hom}(\mathcal{E}, \mathcal{O}_{\Delta M}[k]).
\]
When \( k = -1 \), the first and fourth terms vanish and so \( \text{Hom}(\mathcal{E}, \mathcal{E}) \to \text{Hom}(\mathcal{E}, \mathcal{F}[1]) \) is an isomorphism. Now \( \dim \text{Hom}(\mathcal{E}, \mathcal{F}[1]) \leq \dim \text{Hom}(\mathcal{F}, \mathcal{F}), \) by the inequality \( \text{(5.6)} \), and \( \dim \text{Hom}(\mathcal{F}, \mathcal{F}) = 1 \), by Lemma 5.1 We conclude that \( \dim \text{Hom}(\mathcal{E}, \mathcal{E}) = 1. \)

When \( k = 0 \), the first term in the long exact sequence above vanishes. The third term vanishes, by the inequality \( \text{(5.6)} \) and the vanishing of \( \text{Hom}(\mathcal{F}, \mathcal{F}[1]) \) established in Lemma 5.1 We conclude that \( \text{Hom}(\mathcal{E}, \mathcal{E}[1]) = 0. \)

Assume that \( M = X^{[n]} \) as above. Let \( \mathcal{E} : D^b(M) \to D^b(M) \) be the endo-functor with kernel \( \mathcal{E} \) in \( D^b(M \times M). \) Given a point \( m \in M, \) denote by \( C_m \) the corresponding sky-scraper sheaf.

The following Lemma is used in [MM2].

**Lemma 5.3.** The homomorphism \( \kappa : \text{Hom}(C_m, C_m[1]) \to \text{Hom}(\mathcal{E}(C_m), \mathcal{E}(C_m)[1]), \) induced by \( \mathcal{E}, \) is surjective, for all \( m \in M. \)

**Proof.** The homomorphism \( \tilde{\kappa} : \text{Hom}(C_m, C_m[1]) \to \text{Hom}(\mathcal{L}(C_m), \mathcal{L}(C_m)[1]), \) induced by \( \mathcal{L}, \) is an isomorphism, by Theorem 5.20 Set \( \mathcal{E}_m := \mathcal{E}(C_m) \) and \( \mathcal{F}_m := \mathcal{L}(C_m). \) The exact triangle

\[
\mathcal{E}[1] \to \mathcal{F} \to \mathcal{O}_\Delta \to \mathcal{E}[2]
\]

gives rise to the exact triangle

\[
C_m[-1] \to \mathcal{E}_m[1] \xrightarrow{\beta_m} \mathcal{F}_m \to C_m.
\]

The pullback homomorphism \( \beta_m^* : \text{Hom}(\mathcal{F}_m, \mathcal{F}_m[1]) \to \text{Hom}(\mathcal{E}_m[1], \mathcal{F}_m[1]) \) is surjective, by an argument analogous to that in step 1 of the proof of Lemma 5.2. The kernel of the push-forward homomorphism \( \beta_m_* : \text{Hom}(\mathcal{E}_m[1], \mathcal{E}_m[2]) \to \text{Hom}(\mathcal{E}_m[1], \mathcal{F}_m[1]) \) is the image of \( \text{Hom}(\mathcal{E}_m[1], C_m) \to \text{Hom}(\mathcal{E}_m[1], \mathcal{E}_m[2]). \) Now \( \text{Hom}(\mathcal{E}_m[1], C_m) \) clearly vanishes. Hence, \( \beta_{m,*} \) is injective.

Given an element \( \xi \in \text{Hom}(C_m, C_m[1]), \) we have the equality \( \beta_m \circ \mathcal{E}(\xi) = \mathcal{L}(\xi) \circ \beta_m, \) since the homomorphism of kernels induces a natural transformation \( \beta : \mathcal{E}[1] \to \mathcal{L}. \) We get the equality

\[
\beta_m^* \circ \tilde{\kappa} = \beta_{m,*} \circ \kappa : \text{Hom}(C_m, C_m[1]) \to \text{Hom}(\mathcal{E}_m[1], \mathcal{F}_m[1]).
\]

The homomorphism \( \beta_m^* \circ \tilde{\kappa} \) is surjective, being a composition of such. Thus, \( \beta_{m,*} \circ \kappa \) is surjective. Hence \( \beta_{m,*} \) is an isomorphism. Thus, \( \kappa \) is surjective. \( \square \)

### 6. A Deformation of the Derived Category \( D^b(X) \)

We prove Theorem 1.8 in this section. Although we talk of deforming categories, as will be seen presently, we shall only need to work with deformations of certain fixed objects and morphisms in a derived category. These are defined as follows.

**Definition 6.1.** Let \( Y \to S \) be a flat family of spaces (varieties or analytic spaces), and \( s \) a point of \( S. \)

1. A deformation of a perfect complex \( \mathcal{E} \in D^b(Y_s) \) is an \( S \)-perfect complex \( \mathcal{E} \in D^b(Y) \) together with an isomorphism \( \varphi : i^* \mathcal{E} \xrightarrow{\cong} \mathcal{E}, \) where \( i : Y_s \to Y \) is the closed immersion.
2. Given a morphism \( f : \mathcal{E}_1 \to \mathcal{E}_2 \) between perfect complexes and deformations \( (\mathcal{E}_i, \varphi_i), \) a compatible deformation of \( f \) is a morphism \( \mathcal{F} : \mathcal{E}_1 \to \mathcal{E}_2 \) such that \( f \circ \varphi_1 = \varphi_2 \circ i^* \mathcal{F}. \)
6.1. Deformability of $\mathcal{F}$. Denote by $\mathcal{M}_A$ the moduli space parametrizing marked pairs $(M, \eta)$ consisting of a holomorphic symplectic manifold $M$, and an isometry $\eta : H^2(M, \mathbb{Z}) \rightarrow \Lambda$. The moduli space $\mathcal{M}_A$ is constructed in [Hu2]. Let $\phi : \mathcal{M}_A \rightarrow \mathcal{M}_A$ be the forgetful map given by $(M, \eta, \mathfrak{A}) \mapsto (M, \eta)$. Let $\mathcal{M}_A^1$ be the connected component of $\mathcal{M}_A$ in Theorem 1.8. Let $\mathcal{M}_A^0$ be the connected component containing $\phi(\mathcal{M}_A^1)$. Then $\phi : \mathcal{M}_A^0 \rightarrow \mathcal{M}_A^1$ is surjective, by [MM2, Theorem 4.14]. The following is the key result from [MM2] for the sequel; it is a direct consequence of Theorems 1.9(1), and 1.11(1), and Lemma 5.3 in that article.

**Theorem 6.2.** Let $X$ be a K3 surface with trivial Picard group. Then, the fiber of the morphism $\phi : \mathcal{M}_A^0 \rightarrow \mathcal{M}_A^0$ over $(X^{[n]}, \eta, \mathfrak{A})$ consists of the single point $(X^{[n]}, \eta, \mathfrak{A}) \in \mathcal{M}_A^0$, where $\mathfrak{A}$ or $\mathfrak{A}^*$ is the modular Azumaya algebra of the Hilbert scheme $X^{[n]}$ (Def. 1.5).

The following is an immediate corollary of Theorems 1.7 and 6.2 and the density of Hilbert schemes in $\mathcal{M}_A^0$ [MM1].

**Corollary 6.3.** The subset $\text{Hilb}$ of $\mathcal{M}_A^0$, given in Equation (1.11), consisting of triples $(X^{[n]}, \eta, \mathfrak{A})$ where $X$ is a K3 surface with a trivial Pic(X), is dense in $\mathcal{M}_A^0$. For each such triple, $\mathfrak{A}$ or $\mathfrak{A}^*$ is the modular Azumaya algebra over $X^{[n]}$.

Let $\mathcal{M}^0$ be the universal family over the moduli space of marked pairs $\mathcal{M}_A^0$. Set $\tilde{\mathcal{M}}^0 := \mathcal{M}^0 \times_{\mathcal{M}_A^0} \tilde{\mathcal{M}}_A$. There exists a universal Azumaya algebra $\mathfrak{A}$ over $\tilde{\mathcal{M}}^0 \times_{\mathcal{M}_A^0} \mathcal{M}^0$ (see [MM2, Section 4]).

**Proposition 6.4.** There exists a Zariski dense open subset $U \subset \mathcal{M}_A^0$, containing Hilb, a universal twisted sheaf $\mathcal{E}$ over

$$\mathcal{M}^2 := \left(\tilde{\mathcal{M}}^0 \times_{\mathcal{M}_A^0} \tilde{\mathcal{M}}^0\right) \times_{\mathcal{M}_A^0} U,$$

satisfying $\text{End}(\mathcal{E}) \cong \mathfrak{A}$, and an extension

$$\mathcal{E}[1] \rightarrow \mathcal{F} \rightarrow E_{\Delta_M}$$

of twisted sheaves over $\mathcal{M}^2$, where $\mathcal{M} := \tilde{\mathcal{M}}^0 \times_{\mathcal{M}_A^0} U$, and $\Delta_M$ is the image of $\mathcal{M}$ via the diagonal embedding $\Delta : \mathcal{M} \rightarrow \mathcal{M}^2$. This extension is non-split along every fiber of the projection $\Pi : \mathcal{M}^3 \rightarrow U$. Let $\tilde{\Pi} : \mathcal{M}^3 \rightarrow \mathcal{M}^2$, $1 \leq i < j \leq 3$, the natural projections. The Brauer class $\Theta$ of $\mathcal{E}$ satisfies the equality

$$\Pi_{12}^*(\Theta)\Pi_{23}^*(\Theta) = \Pi_{13}^*(\Theta).$$

Consequently, both $\mathcal{F}$ and the convolution $\mathcal{F} \circ \mathcal{F}$ are objects of $D^b(\mathcal{M}^2, \Theta)$.

**Proof.** Step 1: We show first that the Brauer class of $\mathfrak{A}$ restricts as a trivial class to the diagonal $\Delta_{\mathcal{M}^0} \subset \mathcal{M}^0 \times_{\mathcal{M}_A^0} \mathcal{M}^0$ over a Zariski dense open subset of $\mathcal{M}_A^0$. Let $\mu_{2n-2} \subset \mathbb{C}^*$ be the group of roots of unity of order dividing $2n - 2$. The exponential map

$$\exp\left(\frac{2\pi i (\bullet)}{2n - 2}\right) : \mathbb{Z} \rightarrow \mu_{2n-2}$$

While a universal family need not exist over the moduli space of marked pairs of a general class of holomorphic symplectic manifolds, such a family does exist over $\mathcal{M}_A^0$, as manifolds of $K3^{(m)}$-type have no nontrivial automorphisms which act trivially on cohomology in degree 2. This follows for Hilbert schemes by a result of Beauville [E2], and consequently also for their deformations by [Hu1].
factors through an isomorphism $\mathbb{Z}/(2n-2)\mathbb{Z} \cong \mu_{2n-2}$. Given a marked pair $(M, \eta)$ in $\mathfrak{M}_\Lambda^0$ we get an isomorphism

$$\tilde{\eta} : H^2(\mathcal{M} \times M, \mu_{2n-2}) \cong \Lambda/(2n-2)\Lambda$$

induced by the marking $\eta$. The component $\tilde{\mathfrak{M}}^0_\Lambda$ comes with a coset

$$\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2),$$

with $\tilde{\theta}_1 = -\tilde{\theta}_2$ and $\tilde{\theta}_1$ of order $2n-2$ in $\Lambda/(2n-2)\Lambda$, such that the Brauer class of the Azumaya Algebra $\mathfrak{A}$ of a triple $(M, \eta, \mathfrak{A})$ in this component is the image of $\tilde{\eta}^{-1}(\tilde{\theta})$ in $H^2(\mathcal{M} \times M, \mathcal{O}^*)$, by construction [MM2, Eq. (7.6)]. The Brauer class $\Theta$ of $\mathfrak{M}_\Lambda$ in $H^2(\mathfrak{M}_\Lambda^0 \times \mathfrak{M}_\Lambda^0, \mathcal{O}^*)$, having order $2n-2$, is the image of a topological class $\tilde{\Theta}$ in $H^2(\mathcal{M}^0 \times \mathfrak{M}_\Lambda^0, \mu_{2n-2})$. The restriction of $\tilde{\Theta}$ to the fibers $M \times M$ of the family over a marked pair $(M, \eta)$ is $\tilde{\eta}^{-1}(\tilde{\theta})$.

Given an open analytic subset $U'$ of $\tilde{\mathfrak{M}}^0_\Lambda$, over which the differential fibration $\pi : \mathcal{M}^0 \to \tilde{\mathfrak{M}}^0_\Lambda$ restricts as a topologically trivial fibration, we get that the pullback $\Delta^*\tilde{\Theta}$ of the Brauer class to $\mathcal{M}^0$ is trivial over $U'$, by the vanishing of $\tilde{\theta}_1 + \tilde{\theta}_2$ mentioned above.

There exists a $\Theta$-twisted sheaf $\mathcal{E}$ representing $\tilde{\mathfrak{M}}$ for some Čech 2-cocycle $\theta$ of $\Theta^*\mathcal{M}^0 \times \mathfrak{M}_\Lambda^0 \mathcal{M}^0$.

[MM2 Section 2]. Over the above open subset $U'$, $\mathcal{E}$ restricts to the diagonal with a trivial Brauer class. Hence, we may adjust the pull-back $\Delta^*(\Theta)$ by a 2-coboundary, and adjust the gluing of $\Delta^*\mathcal{E}$ over $U'$ to get an untwisted sheaf $W$. For every triple $(X^{[n]}, \eta, \mathfrak{A})$ in $U'$, where $\text{Pic}(X)$ is trivial, the restriction of $\mathfrak{A}$ to $X^{[n]} \times X^{[n]}$ is the modular Azumaya algebra or its dual, by Theorem 6.2, and so $W|_{X^{[n]}}$ is isomorphic to $L \otimes ((\Lambda^2 T^* X^{[n]})/(\Theta_X^{[n]} \cdot \sigma))$, for some line-bundle $L$, by Proposition 4.2 (use the isomorphism $\Theta^* \cong \tau^* \Theta$ of Lemma 4.3 for the dual of the modular Azumaya algebra). Set $\mathcal{W} := \Lambda^2 T^*_X/(L_0 \pi^* R_0 \pi^* \Lambda^2 T^*_X)$. We conclude the isomorphism

$$\Theta nd(\mathcal{W}) \cong \Theta nd(\Delta^*\mathcal{E}),$$

of Azumaya algebras over a Zariski dense open subset of $U'$, by the density of Hilbert schemes of $K3$ surfaces with trivial Picard group (Corollary 6.3) and the upper-semi-continuity theorem. Both sides of the isomorphism (6.1) are defined globally over $\mathcal{M}^0$. Hence, the isomorphism holds over a Zariski dense open subset $U''$ of $\tilde{\mathfrak{M}}^0_\Lambda$ containing the dense subset $\text{Hilb}$ given in Equation (1.11).

The sheaf $\Delta^*\mathcal{E}$ is $\Delta^*\Theta$-twisted. The left hand side of (6.4) is an Azumaya algebra with a trivial Brauer class. The isomorphism (6.1) implies that the cocycle $\Delta^*\Theta$ is a coboundary over $U''$, $\Delta^*\Theta = \delta(\zeta)$.

Step 2: Let $\pi_1 : \mathcal{M}^0 \times \tilde{\mathfrak{M}}^0 \to \mathcal{M}^0$ be the projection to the first factor. After a refinement of the open covering of $\mathcal{M}^0 \times \tilde{\mathfrak{M}}^0 \to \mathcal{M}^0$ we may change the gluing of $\mathcal{E}$ over $U''$ via the cochain $\pi_1^*(\zeta)$, so that $\mathcal{E}$ is a twisted sheaf with respect to a new cocycle, denoted again by $\Theta$, which restricts to the trivial cocycle along the diagonal (with value 1 along every triple intersection). Then $\Delta^*\mathcal{E}$ is an untwisted coherent sheaf, which is isomorphic to $\mathcal{W} \otimes \mathcal{L}$, for some line bundle $\mathcal{L}$ over the subset $U''$ of $\tilde{\mathfrak{M}}^0_\Lambda$.

Step 3: Let $\pi : \mathcal{M} \to U''$ be the restriction of the universal family from $\tilde{\mathfrak{M}}^0_\Lambda$ to $U''$. Denote by $\Pi : \mathcal{M} \times \mathcal{M} \to U''$ the projection from the fiber square. We show next that the relative homomorphism sheaf

$$\mathcal{H}_j := \mathcal{H}om_\Pi(\Theta_{\Delta\mathcal{M}}, \mathcal{E}[j])$$
is a line bundle over a Zariski dense open subset $U$ of $U''$ containing $\mathcal{H}ilb$ for $j = 2$, and it vanishes over $U$ for $j = 0, 1$. This $U$ would be also Zariski dense and open in $\mathfrak{M}_*$. Local Grothendieck-Verdier duality yields the isomorphism

$$R\mathcal{H}om(R\Delta_{\mathcal{M}}, \mathcal{O}_\mathcal{M}, \mathcal{E}[j]) \cong R\Delta_* \left[ R\mathcal{H}om(\mathcal{O}_\mathcal{M}, \Delta \mathcal{E}[j]) \right].$$

Applying the functor $R(\Pi)_*$ on both sides, using the vanishing of $R^i\Delta_* \mathcal{O}_\mathcal{M}$ for $i > 0$, we get the isomorphism

$$R\mathcal{H}om(\mathcal{O}_\mathcal{M}, \mathcal{E}[j]) \cong R\pi_*(\omega^*_\pi \otimes L\Delta_{\mathcal{M}} \mathcal{E}[j - 2n]).$$

The relative dualizing sheaf $\omega^*_\pi$ of the morphism $\pi$ is trivial along each fiber and is hence the pullback of a line bundle over $U''$. Thus, the sheaf $R\mathcal{H}om(\mathcal{O}_\mathcal{M}, \mathcal{E}[j])$ is isomorphic to the 0-th direct image of the complex $L\Delta_{\mathcal{M}} \mathcal{E}[j - 2n]$, tensored by a line bundle. For $x \in U''$, denote by $\mathcal{E}[x]$ the restriction of $\mathcal{E}$ to the Cartesian square of the fiber over $x$. The function $\phi_j : U'' \to \mathbb{Z}$ given by

$$x \mapsto \dim(\mathcal{H}^0(\mathcal{M}_x, L\Delta_{\mathcal{M}} \mathcal{E}[x][j - 2n]))$$

is upper semi-continuous by [Hart] Prop. 6.4. The value of $\phi_2$ on triples in $\mathcal{H}ilb$ is 1, by the calculation of the torsion sheaf $\mathcal{F}or_{2n-2}(\mathcal{E}, \Delta, \mathcal{O}_X[0])$ in Proposition 4.2. The set $\mathcal{H}ilb$ is dense in $\mathfrak{M}_*$, by Corollary 6.3. It follows that $\phi \equiv 1$ on a dense open subset $U_2$ of $U''$. The sheaf $R\mathcal{H}om(\mathcal{O}_\mathcal{M}, \mathcal{E}[2])$ is a line bundle over $U_2$, since $U''$ is integral. The vanishing of $\mathcal{F}or_{2n-2}(\mathcal{E}, \Delta, \mathcal{O}_X[0])$ in Proposition 4.2 for $j < 2$, implies the claimed vanishing of $\mathcal{H}_j$ over a Zariski dense subset $U_j$ of $U''$ containing $\mathcal{H}ilb$, for $j = 0, 1$. Set $U := U_0 \cap U_1 \cap U_2$.

Step 4: Set $\mathcal{H} := \mathcal{H}_2$, where $\mathcal{H}_2$ is the line-bundle given in Equation (6.5). The vanishing of $\mathcal{H}_j$, for $j = 0, 1$, and a standard spectral sequence argument yield an isomorphism $\text{Hom}(\mathcal{O}_\mathcal{M}, \Pi^* (\mathcal{H}^{-1}) \otimes \mathcal{E}[2]) \cong H^0 (\mathcal{H}om(\mathcal{O}_\mathcal{M}, \Pi^* (\mathcal{H}^{-1}) \otimes \mathcal{E}[2]))$. The right hand space is $H^0(U, \mathcal{O}_U)$, by the projection formula and the definition of $\mathcal{H}$. Choosing the constant section 1 of the latter we get a tautological extension

$$\Pi^* (\mathcal{H}^{-1}) \otimes \mathcal{E}[1] \to \mathcal{F} \to \mathcal{O}_\mathcal{M}.$$

Replacing the twisted sheaf $\mathcal{E}$ by $\pi^*(\mathcal{H}^{-1}) \otimes \mathcal{E}$ we get the desired extension (6.1).

Step 5: We prove in this step the equality (6.2). Set $r := 2n - 2$. Let $F^pH^k(\mathcal{M}_2^*, \mu_r)$ be the decreasing Leray filtration associated to the morphism $\Pi : \mathcal{M}_2 \to U$. Set $E^{2,q}_{\infty} := F^pH^{p+q}(\mathcal{M}_2^*, \mu_r) / F^{p+1}H^{p+q}(\mathcal{M}_2^*, \mu_r)$. We have the Leray spectral sequence converging to $E^{2,q}_{\infty}$ with $E_2$ terms of the form $E^{2,q}_{2} := H^q(U, R^2\Pi_* \mu_r)$ and differential $d_2 : E^{2,q}_{2} \to E^{2,q}_{2} \otimes \mathcal{O}_U$.

The sheaf $R^2\Pi_* \mu_r$ is the trivial local system $\mu_r$ and the sheaf $R^1\Pi_* \mu_r$ vanishes, since the fibers of $\Pi$ are simply connected. The sheaf $R^2\Pi_* \mu_r$ is the direct sum $[\Lambda/r\Lambda]^{\oplus 2}$ of two copies of the trivial local system $[\Lambda/r\Lambda]$, since the markings provide such a trivialization. We conclude the following:

$$
E^{2,0}_{\infty} = E^{2,0} = H^2(U, \mu_r),
E^{1,1}_{\infty} = 0,
E^{0,2}_{3} = E^{0,2} = [\Lambda/r\Lambda]^{\oplus 2},
E^{3,0}_{3} = E^{3,0} = H^3(U, \mu_r),
E^{0,2}_{\infty} = \ker \left[ d_3 : E^{0,2}_{3} \to E^{3,0}_{3} \right].
$$

The description of $E^{2,0}_{\infty}$ implies that the homomorphism $\Pi^* : H^2(U, \mu_r) \to H^2(\mathcal{M}_2^*, \mu_r)$ is injective. The analogous description of the graded summands of the Leray filtrations of $H^2(\mathcal{M}_2^*, \mu_r)$
holds for the $n$-th fiber self-products $\mathcal{M}^n$, $n \geq 1$, where the $E^{0,2}_\infty$ is naturally a subgroup of $[\Lambda/r\Lambda]^{\geq 0}$.

We have seen in Step 2 that $\Theta$ belongs to the kernel of $\Delta^*: H^2(\mathcal{M}^3, \Theta^*) \to H^2(\mathcal{M}, \Theta^*)$. We have seen, in addition, that there exists a lift $\tilde{\Theta} \in H^2(\mathcal{M}^2, \mu_r)$ of $\Theta$. Next we normalize this lift so that it restricts trivially to the diagonal. The composition

$$H^2(\mathcal{M}, \mu_r) \to H^2(\mathcal{M}, \mu_r) / E^1 \to H^2(\mathcal{M}, \mu_r) \to H^0(0, \mathcal{O}^2, \mu_r) \to H^0(0, \mathcal{O}_\mathcal{M}^2, \Theta^*),$$

maps the class $\Delta^*(\tilde{\Theta})$ to the trivial class. The rightmost homomorphism is injective, since the Picard group of a generic fiber of $\pi: \mathcal{M} \to U$ is trivial. Consequently, the image of $\Delta^*(\tilde{\Theta})$ in $H^0(0, \mathcal{O}_\mathcal{M}^2, \Theta^*)$ is trivial and $\Delta^*(\tilde{\Theta})$ belongs to $E^{2,0}_\infty$ with respect to the Leray filtration of $H^2(\mathcal{M}, \mu_r)$. Hence, there exists a class $\alpha$ in $H^2(U, \mu_r)$, such that $\Delta^*(\tilde{\Theta}) = \pi^*(\alpha)$. The image of $\pi^*(\alpha)$ in $H^2(\mathcal{M}, \Theta^*)$ is trivial, since such is the image of $\Delta^*(\tilde{\Theta})$. The image of $\Pi^*(\alpha)$ in $H^2(\mathcal{M}^3, \Theta^*)$ is trivial as well, since $\Pi$ factors through $\pi$. Let $\beta$ be the class $\Theta \Pi^*(\alpha^{-1})$ in $H^2(\mathcal{M}^2, \mu_r)$. Then $\Delta^*(\beta)$ is trivial in $H^2(\mathcal{M}, \mu_r)$ and $\beta$ maps to the Brauer class $\Theta$ in $H^2(\mathcal{M}^2, \Theta^*)$.

The $E^{2,0}_\infty$ graded summands of the Leray filtrations of $H^2(\mathcal{M}^n, \mu_r)$, $n \geq 1$, are all equal to $H^2(U, \mu_r)$ and the $E^{3,1}_\infty$ terms all vanish. Hence, the kernel of $\Delta^*: H^2(\mathcal{M}^2, \mu_r) \to H^2(\mathcal{M}, \mu_r)$ maps injectively into the quotient $E^{0,2}_\infty$, which is naturally a subgroup of $[\Lambda/r\Lambda]^{\geq 2}$. Classes in the kernel map to classes of the form $(-\lambda, \lambda)$. Choose a class $\lambda \in \Lambda^r$. The latter class maps to $(-\lambda, \lambda)$. Similarly, the kernel of the pullback $H^2(\mathcal{M}^3, \mu_r) \to H^2(\mathcal{M}, \mu_r)$, via the diagonal embedding, maps injectively into $[\Lambda/r\Lambda]^{\geq 3}$. The class $\Pi_{12}(\beta) \Pi_{23}(\beta) \Pi_{13}(\beta^{-1})$ restricts trivially to the diagonal and maps to the class

$$(-\lambda, \lambda, 0) + (0, -\lambda, \lambda) + (\lambda, 0, -\lambda) = (0, 0, 0)$$

in $[\Lambda/r\Lambda]^{\geq 3}$. Hence, the class $\Pi_{12}(\beta) \Pi_{23}(\beta) \Pi_{13}(\beta^{-1})$ is trivial in $H^2(\mathcal{M}^3, \mu_r)$. The latter class maps to the class $\Pi_{12}(\Theta) \Pi_{23}(\Theta) \Pi_{13}(\Theta^{-1})$ in $H^2(\mathcal{M}^3, \Theta^*)$. Equality (6.2) follows. This completes the proof of Proposition 6.4.$\square$

Remark 6.5. The description of the graded pieces of the Leray filtration of $H^2(\mathcal{M}^2, \mu_r)$ is Step 5 of the proof above applies to the restriction $\mathcal{M}^2_V$ of $\mathcal{M}^2$ to a contractible open subset $V$ of $U$. In that case we get that $E^{2,0}_{\infty} \cong H^2(V, \mu_r)$ vanishes as well, so does $E^{3,0}_{\infty}$, and $H^2(\mathcal{M}^2_V, \mu_r) = E^{0,2}_{\infty} \cong [\Lambda/r\Lambda]^{\geq 2}$. Similarly, $H^2(\mathcal{M}_V, \mu_r) \cong \Lambda^r$. Consequently, the restriction of $\Theta$ to $\mathcal{M}^2_V$ is equal to $\pi_1^*(\theta^{-1}) \pi_2^*(\theta)$ for some class $\theta \in H^2(\mathcal{M}_V, \mu_r)$. Part 2 of Theorem 1.8 follows.

Let $X$ be a $K3$ surface, $M := X^{[n]}$, and $\delta: \mathcal{F} \to \mathcal{F} \circ \mathcal{F}$ the comultiplication of the comonad object (1.10). Denote by $\pi_{ij}: M \times M \times M \to M \times M$ the projection onto the $ij$-th factor. The adjunction $\pi_1^* \dashv \pi_1_*$ yields the second isomorphism below.

$$\text{Hom}(\mathcal{F}, \mathcal{F} \circ \mathcal{F}) = \text{Hom}(\mathcal{F}, \pi_{13} \cdot [\pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{F}]) \cong \text{Hom}(\pi_{13}^* \mathcal{F}, \pi_{12}^* \mathcal{F} \otimes \pi_{23}^* \mathcal{F}).$$

Denote the image of $\delta$ by $\tilde{\delta}: \pi_{13} \mathcal{F} \to \pi_{12} \mathcal{F} \otimes \pi_{23} \mathcal{F}$. We get the natural morphism

$$\pi_{13} \cdot (\tilde{\delta}) : \pi_{13}^* \mathcal{F} \cong \mathcal{F} \otimes Y(\mathcal{O}_M) \to \mathcal{F} \circ \mathcal{F},$$

where $Y(\mathcal{O}_M) \in D^b(pt)$ is the Yoneda algebra of $M$. Composing $\pi_{13} \cdot (\tilde{\delta})$ with the morphism $1 \otimes t: \mathcal{F} \otimes \mathcal{F} \to Y(\mathcal{O}_M)$ (see (2.5)) we get the natural morphism

$$m: \mathcal{F} \otimes \lambda_n \to \mathcal{F} \circ \mathcal{F}. \quad (6.7)$$

**Lemma 6.6.** The morphism $m$, given in Equation (6.7), is an isomorphism, in the case where $M$ is the Hilbert scheme $X^{[n]}$, the universal sheaf $\mathcal{H} \in D^b(X \times X^{[n]})$ is the ideal sheaf of the universal subscheme, and $\mathcal{F}$ is the modular complex $\mathcal{H} \circ \mathcal{H}^{[2]}$. 

Proof. The morphism \( m \) is obtained from the sequence of morphisms (2.5)

\[
\Delta_* \mathcal{O}_X \otimes \mathcal{O}_{Y(\mathcal{O}_M)} \xrightarrow{\eta \otimes \text{id}} \mathcal{A} \otimes \mathcal{O}_{Y(\mathcal{O}_M)} \xrightarrow{m} \mathcal{A}.
\]

by pre-convolution with \( \mathcal{U}^\vee[2] \), and post-convolution with \( \mathcal{U} \). Here we use the fact discussed at the end of Construction 2.1 that for a morphism \( f : T \to S \), the monadic action of \( f_* f^* \) on objects of the form \( f_* (\mathcal{F}) \) is compatible with the action of the algebra \( f_*(\mathcal{O}_T) \). The statement then follows immediately from Theorem 2.2 Part (1), which says that the composition of the morphisms in the display above is an isomorphism.

Let \( M \) and \( U \) be as in Theorem 1.8 and let \( M^n \) denote the \( n \)-th fiber self-product of \( M \) over \( U \). The convolution \( \mathcal{F} \circ \mathcal{F} \) is defined relative to \( U \), i.e., the tensor product is taken over \( M^3 \). Let \( \Pi : M^2 \to U \) be the natural morphism.

Lemma 6.7. (1) The relative endomorphism sheaf \( \mathcal{H}om_H(\mathcal{F}, \mathcal{F}) \) is canonically isomorphic to the structure sheaf over a dense open subset of \( U \) containing the locus \( \text{Hilb} \).

(2) The relative homomorphism sheaves

\[
\{ \mathcal{H}om_H(\mathcal{F}, \mathcal{F} \circ \mathcal{F}), \mathcal{H}om_H(\mathcal{F}, \mathcal{F} \circ \mathcal{F} \circ \mathcal{F}) \}
\]

are both line bundles on a dense open subset of \( U \).

Proof. We shall consider only the sheaf \( \mathcal{H}om_H(\mathcal{F}, \mathcal{F} \circ \mathcal{F}) \) of part (2). The other sheaves are shown to be line-bundles in the same way, and the sheaf \( \mathcal{H}om_H(\mathcal{F}, \mathcal{F} \circ \mathcal{F}) \) clearly has a global non-vanishing identity section establishing its triviality. For \( y := (M, \eta, \mathfrak{A}) \in U \), denote the restriction \( \mathcal{F}_y |_{M \times M} \) more simply as \( \mathcal{F}_y \). When \( y \) belongs to \( \text{Hilb} \), i.e., when \( M = X \) where \( X \) is a \( K3 \) surface with trivial Pic\( (X) \), we have

\[
\mathcal{F}_y \circ \mathcal{F}_y \cong \mathcal{F}_y \oplus \mathcal{F}_y[-2] \oplus \cdots \oplus \mathcal{F}_y[2 - 2n],
\]

by Theorem 6.2 and Lemma 6.6 (note that when \( \mathfrak{A} \) is the dual of the modular Azumaya algebra \( \mathcal{F}_y \) is the pullback of the modular complex by the automorphism \( \tau \) of \( M \times M \) interchanging the two factors, by lemma 4.3). Furthermore, \( (\mathcal{F} \circ \mathcal{F})_y \) is isomorphic to \( \mathcal{F}_y \circ \mathcal{F}_y \), by the base change theorem [Hart, Proposition 6.3]. The locus \( U' \) where the fiber-wise homomorphisms \( \text{Hom}(\mathcal{F}_y, \mathcal{F}_y \circ \mathcal{F}_y) \) are one-dimensional is locally closed by semi-continuity [Hart, Proposition 6.4]. \( U' \) contains every \( y \in \text{Hilb} \), since \( \text{Hom}(\mathcal{F}_y, \mathcal{F}_y[k]) \) vanishes for the modular \( \mathcal{F}_y \) of a Hilbert scheme and for \( k < 0 \), by Lemma 5.1. \( U' \) is a dense open subset of \( U \), by Corollary 6.3.

We will denote again by \( U \) the open subset where the statement of Lemma 6.7 holds. Similarly, we will continue to denote the universal family by \( \pi : M \to U \) and its fiber square by \( \Pi : M^2 \to U \). Denote the restriction to \( D^b(M^2, \Theta) \) of the extension constructed in Proposition 6.4 by the same symbol \( \mathcal{F} \). To define the structure of a monad object on \( \mathcal{F} \) extending the modular one on the Hilbert schemes, we need to produce a counit \( \eta \) and a comultiplication \( \delta \). The former structure map is in fact given by the very definition of \( \mathcal{F} \) in triangle (6.1):

\[
\mathcal{E}[1] \rightarrow \mathcal{F} \rightarrow \mathcal{O}_{\Delta_M}.
\]

We shall presently produce the other, and verify that the structure maps satisfy the necessary compatibilities.
Any extension $\delta : F \to F \circ F$ of the comultiplication is required to give a section of the two morphisms

$$
(6.8)
\begin{align*}
F \circ \tau : F \circ F & \to F, \\
\tau \circ F : F \circ F & \to F,
\end{align*}
$$

that is, $(F \circ \tau) \circ \delta = (\tau \circ F) \circ \delta = 1_F$. Conversely, the morphisms $(6.8)$ admit sections over a dense open subset of $U$ extending the modular comultiplication; we shall call them $\delta_1$ and $\delta_2$, respectively. Indeed, consider the following pair of morphisms of sheaves over $U$ obtained from $(6.8)$:

$$
\begin{align*}
&\mathcal{H}om_{\Pi}(F, F \circ F) \\
\cong &\mathcal{H}om_{\Pi}(F, F) \circ \mathcal{H}om_{\Pi}(F, F) \\
\cong &\mathcal{O}_U.
\end{align*}
$$

These morphisms are isomorphisms over a dense open subset $U^0$ of $U$ containing $\text{Hilb}$, by Corollary 6.3 and Lemma 5.1. Define $\delta_1$ and $\delta_2$ to be the pre-images of $1_F$.

**Proof of Theorem 1.8 (1).** First suppose that there exists a comultiplication $\delta_x$ for $F|_x$ at some point $x \in U^0$. Then, $\delta_1|_x = \delta_2|_x = \delta_x$ since $\delta_x$ is the unique section of both morphisms in Equation $(6.8)$, by the discussion above. To complete the proof, first note that the equality $\delta_1 = \delta_2$ holds over $U^0$ by the fact that we have established it over a dense subset of it. Second, note that the coassociativity $(1.6)$ and counit laws $(1.7)$ amount to equalities of two sections of the line bundles $\mathcal{H}om_{\Pi}(F, F \circ F)$, respectively $\mathcal{H}om_{\Pi}(F \circ F, F)$, over $U^0$. These equalities hold, again because they have been established for a dense subset of $U^0$. Finally we rename $U^0$ as $U$.

**Remark 6.8.** Let $\Pi_{13} : M^3 \to M^2$ be the projection on the first and third factors. Let $\overline{\lambda}_n$ be the object fitting in the exact triangle

$$
(6.9)
\overline{\lambda}_n \to R\Pi_{13}, \mathcal{O}_{M^3} \to R^{2n-13}, \mathcal{O}_{M^3}
$$

in $D^b(M^2)$, where the right morphism is well defined due to the vanishing of $R^i\Pi_{13}, \mathcal{O}_{M^3}$ for $i > 2n$. The construction of the comultiplication $\delta : F \to F \circ F$ enables us to extend the isomorphism $\mathfrak{m}$ in Equation $(6.7)$ to a morphism

$$
(6.10)
\overline{\lambda}_n \to F \circ \mathcal{O}_U, \overline{\lambda}_n \to F \circ F
$$

of objects over $M^2$, which is an isomorphism over a dense open subset $W$ of $U$ containing $\text{Hilb}$, by Lemma 6.6 and the argument in the proof of Lemma 2.3. Again we rename $W$ as $U$.

**Lemma 6.9.** Let $w := (M, \eta, \mathfrak{A})$ be a point of the open set $W$ of Remark 6.3, where $M \cong M_H(v)$ is isomorphic to a moduli space of $H$-stable sheaves over some $K3$ surfaces $X$, $\mathfrak{S}_w$ is the modular sheaf over $M \times M$ associated to a twisted universal sheaf $\mathcal{V}$ over $X \times M_H(v)$, and $\mathfrak{A}$ is the modular Azumaya algebra $\mathcal{E}nd(\mathfrak{S}_w)$. Let $\mathcal{A}$ be the monad object over $X \times X$ associated to $\mathcal{V}$. Then the morphism $\alpha : \Delta_+ \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{A}$, given in Equation $(2.5)$, is an isomorphism as well.

**Proof.** This follows from the fact that $\mathfrak{m}_w$ is the image of $\alpha$ via the functor $\Gamma$, given in $(5.2)$, and the composition $\Gamma_R^1 \circ \Gamma$ of $\Gamma$ with its right adjoint admits the identity endo-functor of $D^b(X \times X)$ as a direct summand, by Theorem 2.2 $(2)$. Denote by $\Sigma$ the endo-functor of $D^b(X \times X)$, such that $\Gamma_R^1 \circ \Gamma \cong 1_{D^b(X \times X)} \oplus \Sigma$. We have the equalities

$$
\alpha \oplus \Sigma(\alpha) = (\Gamma_R^1 \circ \Gamma)(\alpha) = \Gamma_R^1(\mathfrak{m}_w).
$$
Then \( \alpha \oplus \Sigma(\alpha) \) is an isomorphism, since \( \overline{w} \) is. Hence, each of \( \alpha \) and \( \Sigma(\alpha) \) must be an isomorphism as well. \( \square \)

6.2. The triangulated structure on the category of comodules. We give a proof of part 3 of Theorem 1.8 here, namely, that the category \( D^b(M_V, \theta) \) constructed above carries a 2-triangulated structure. This amounts to verifying Balmer's separability criterion [Bal] for the comonad \( \mathbb{T} \). We start by briefly recalling some of the necessary background.

Let \( \Psi : \mathcal{C} \to \mathcal{D} \) be a functor with left adjoint \( \Phi \). We say that \( \Psi \) is separable if the unit \( \eta : 1_D \to \Psi \Phi \) has a natural retraction \( \xi : \Psi \Phi \to 1_C \).

An additive category \( \mathcal{C} \) is said to be suspended if it is equipped with an auto-equivalence, called suspension, \([1] : \mathcal{C} \to \mathcal{C} \) (which, for simplicity, is considered an isomorphism, \([1]\)^{-1}[1] = \( 1_{\mathcal{C}} = [1][1]^{-1} \)). For example, any triangulated category is a suspended category. A functor between suspended categories is called a suspended functor if it commutes with suspensions. In general, a property \( P \) of suspended categories or functors is stably \( P \) if its definition respects the suspended structures involved. Thus, a suspended functor \( \Phi : \mathcal{D} \to \mathcal{C} \) is stably separable if it is separable, and the splitting \( \xi \) commutes with suspension.

Let \( \mathcal{C} \) be a category, and \( \Lambda = (\Lambda, \epsilon, \delta) \) a comonad on \( \mathcal{C} \). Denote by \( \mathcal{C} \) the category of comodules of \( \Lambda \). The forgetful functor \( F : \mathcal{C} \to \mathcal{C} \) has a right adjoint, the free comodule functor, \( G : \mathcal{C} \to \mathcal{C} \), which is defined as \( G(a) := (\Lambda a, \delta a) \). It is easy to see that if \( \mathcal{C} \) and \( \Lambda \) are suspended, then so is \( \mathcal{C} \), and the pair of functors \( F \) and \( G \) commute with suspension.

Definition 6.10. A comonad \( \mathcal{C} \) on a category \( \mathcal{C} \) is said to be a separable comonad if there exists a natural retraction \( \hat{\delta} : \Lambda^2 \to \Lambda \) of the comultiplication \( \delta : \Lambda \to \Lambda^2 \) such that

\[
\hat{\delta} \Lambda \circ \Lambda \delta = \delta \circ \hat{\delta} = \Lambda \hat{\delta} \circ \delta \Lambda \tag{6.11}
\]

[Ball] Def. 3.5. If \( \mathcal{C} \) is suspended, \( \Lambda \) is said to be stably separable if the various functors and natural morphisms in question respect the suspension.

The following abridged form of Balmer's Main Theorem [Ball] is all we need for our purposes here:

Theorem 6.11 [Ball, Theorem 5.17]. Let \( \mathcal{C} \) be an idempotent-complete category with a triangulation of order \( N \geq 2 \), and let \( \Lambda \) be a stably separable co-monad on \( \mathcal{C} \), such that \( \Lambda : \mathcal{C} \to \mathcal{C} \) is exact up to order \( N \). Then, the category of \( \Lambda \)-comodules \( \mathcal{C} \) admits a triangulation of order \( N \) such that, both the free comodule functor \( G : \mathcal{C} \to \mathcal{C} \) and the forgetful functor \( F : \mathcal{C} \to \mathcal{C} \) are exact up to order \( N \). In fact, each of these properties characterizes the triangulation on the category \( \mathcal{C} \).

Remark 6.12. Rather than go into the specifics of \( N \)-triangulations, we simply observe what is relevant for us (see [Ball, Section 5]): A 2-triangulated category satisfies all the axioms of a triangulated category, except the octahedral axiom, while a 3-triangulated category is also a triangulated category in the sense of Verdier (but not vice-versa). In our intended application of the above result, \( \mathcal{C} \) will be \( D^b(M_V, \theta) \), the derived category of an abelian category, which, as such, is in fact canonically \( N \)-triangulable for all \( N \geq 2 \) [Malt Corollaire, p. 18]. While the comonad \( \mathbb{T} \) is certainly 2-exact, it is not clear at all that it is \( N \)-exact for this structure when \( N > 2 \). We expect this is to be true, at least over algebraic fibers \( M_u \) of \( M_V \), but are unable to prove this at the moment.

\[\text{Note: Recall that } \xi \text{ is a retraction of } \eta \text{ if it is a left inverse of } \eta, \text{ i.e., if } \xi \eta : 1_D \to 1_D \text{ is the identity natural transformation. In this case } \eta \text{ is a section of } \xi.\]
Lemma 6.13. Let \( \Lambda := (\Lambda, \epsilon, \delta) \) be a stable comonad on a suspended category \( \mathcal{C} \) realized by a stable adjunction \( \Phi: \mathcal{D} \rightleftarrows \mathcal{C}: \Psi \). If the functor \( \Psi \) is stably separable, then the comonad \( \Lambda \) is stably separable.

Proof. Note first that \( \Lambda := \Phi \Psi, \epsilon : \Lambda \rightarrow \mathbb{1}_\mathcal{C} \) is the counit, and \( \delta = \Phi \eta \Psi \). The statement essentially follows from the proof in [BBW, §2.9]. We recall the easy details for the sake of completeness: Given a retraction \( \xi : \Psi \Phi \rightarrow \mathbb{1}_\mathcal{D} \) of \( \Psi \), we obtain a retraction of \( \delta = \Phi \eta \Psi \) by setting \( \hat{\delta} = \Phi \xi \Psi \). We will prove only the condition \( \hat{\delta} \Lambda \circ \Lambda \delta = \delta \circ \hat{\delta} \), as the other equality in \((6.11)\) follows in exactly the same way.

The equality \( \hat{\delta} \Lambda \circ \Lambda \delta = \delta \circ \hat{\delta} \) translates to \( (\Phi \xi \Phi \Phi) \circ (\Phi \Phi \Psi \eta) = (\Phi \eta \Psi) \circ (\Phi \xi \Psi) \), which in turn would follow from \( (\xi \Phi) \circ (\Psi \Phi) = \eta \circ \xi \). The latter states, for every object \( x \) of \( \mathcal{D} \), the commutativity of the diagram:

\[
\begin{array}{ccc}
(\Psi \Phi)(x) & \xrightarrow{(\Psi \Phi)\eta_x} & (\Psi \Phi)(\Psi \Phi)(x) \\
\downarrow \xi_x & & \downarrow (\xi(\Psi \Phi))_x \\
x & \xrightarrow{\eta_x} & (\Psi \Phi)(x),
\end{array}
\]

which follows from the naturality of \( \xi \) for the arrows \( \eta_x \).

\( \square \)

Proof of Theorem 1.8(3). Fix a contractible Stein open subset \( \mathcal{V} \) of \( U \) as in the statement of the Theorem. We shall work over \( \mathcal{V} \), but will continue to use the same notation as above for the restrictions of the various morphisms appearing in parts (1), and (2) of this result.

To show that \( D^b(\mathcal{M}_V, \theta)\mathcal{T} \) carries a natural 2-triangulated structure as in the statement, it suffices to produce a retraction \( \hat{\delta} : F_V \circ F_V \rightarrow F_V \) of the comultiplication \( \delta : F_V \rightarrow F_V \circ F_V \) by Theorem 6.11 such that the following two diagrams commute:

\[
\begin{array}{ccc}
F \circ F & \xrightarrow{\delta} & F \\
\downarrow \delta_{\circ F} & & \downarrow \delta \\
F \circ F & \xrightarrow{\delta_{\circ F}} & F \circ F \\
\downarrow \delta_{\circ F} & & \downarrow \delta \\
F \circ F & \xrightarrow{\delta_{\circ F}} & F \circ F
\end{array}
\]

Consider the triangle \((6.9)\). The object \( R\Pi_{13,\mathcal{M}_V} \) is canonically isomorphic to \( \Pi^* R\pi_* \mathcal{O}_{\mathcal{M}_V} \) by base-change, where \( \pi \) and \( \Pi \) are the structure morphisms \( \mathcal{M}_V \rightarrow V \) and \( \mathcal{M}^2_V \rightarrow V \), respectively. As all extensions of line bundles vanish over \( \mathcal{V} \), this object is canonically split: \( R\Pi_{13,\mathcal{M}_V} \cong \mathcal{O}_{\mathcal{M}_V} \otimes F_V Y(\mathcal{O}_{\mathcal{M}_V}) \), where \( Y(\mathcal{O}_{\mathcal{M}_V}) := \oplus_{i=0}^n R^{2i} \pi_* \mathcal{O}_{\mathcal{M}_V} \). Define \( \bar{\delta} \) to be the composition \( \bar{\delta} : F_V \circ F_V \rightarrow F_V \otimes F_V \), where \( \bar{m} \) is given in Equation \((6.10)\) and the second arrow is induced by the splitting above. We first observe that with this definition of \( \bar{\delta} \), the diagrams \((6.12)\) commute when restricted to points corresponding to Hilbert schemes with their modular complexes. Indeed, in this case, the morphism \( \mathcal{M} \) is nothing but the morphism on kernels corresponding to the following retraction of the comultiplication

\[
\Phi \Psi \Phi = \Lambda^2 \xrightarrow{\Phi \Psi} \Phi \Psi = \Lambda.
\]

Here \( (\Phi, \Psi) \) is the adjoint pair \( \Phi : D^b(X) \rightleftarrows D^b(X^{[n]}): \Psi \) realizing our comonad \( \Lambda \), and \( \xi : \Psi \Phi \rightarrow \mathbb{1}_{D^b(X)} \) is the retraction of \( \eta : \mathbb{1}_{D^b(X)} \rightarrow \Psi \Phi \) given by Theorem 2.2 Part (1). Thus (the restrictions of) the two diagrams \((6.12)\) commute by Lemma 6.13.
Consider the sheaf $\mathcal{H}om_H(\mathcal{F} \circ \mathcal{F}, \mathcal{F} \circ \mathcal{F})$. It follows from Lemma 5.1 and semi-continuity of $\mathcal{F}$, Proposition 6.4 [Hart] that this sheaf is a vector bundle on a dense open subset of $U$ containing $\mathcal{H}ilb$. As above, we denote this set by $U$ also. The latter is the open subset of Theorem 1.8.

Set $d = \delta \circ \tilde{\delta} - (\mathcal{F} \circ \tilde{\mathcal{F}} \circ \mathcal{F})$; we note that $d_x = 0$ at points $x \in \mathcal{H}ilb \cap V$. Therefore, since this locus is dense, $d = 0$ on $V$. The commutativity of the left diagram above follows. The argument establishing the commutativity of the right diagram is similar.

6.3. **Monodromy invariance.** We prove part 5 of Theorem 1.8 in this section. We first recall the monodromy action on the moduli space $M^{[n]}_\Lambda$, and observe that the open subset $U$ in Theorem 1.8 may be chosen to be monodromy invariant. We then use a density theorem of Verbitsky to deduce the stated property of $U$.

The isometry group $O(\Lambda)$ acts on the moduli space of marked pairs $M^{[n]}_\Lambda$ as follows. An element $g \in O(\Lambda)$ acts on a marked pair $(M, \eta)$ by $g(M, \eta) = (M, g\eta)$. Let $M^{[n]}_\Lambda$ be a connected component of $M^{[n]}_\Lambda$ of marked pairs of $K^{[n]}$-type. Denote by $G \subset O(\Lambda)$ the subgroup which send $M^{[n]}_\Lambda$ to itself. The subgroup $G$ is related to the monodromy subgroup $Mon^2(M)$ of the isometry group of the second integral cohomology a manifold $M$ of $K^{[n]}$-type as follows. Given a pair $(M, \eta)$ in $M^{[n]}_\Lambda$, we have the equality

$$G = \eta \circ Mon^2(M) \circ \eta^{-1}.$$ 

Given an element $u \in \Lambda$ satisfying $(u, u) = \pm 2$, let $R_u$ be the reflection given by $R_u(x) := x - \frac{2(u, x)}{(u, u)} u$. Set

$$\rho_u := \begin{cases} R_u & \text{if } (u, u) = -2 \\ -R_u & \text{if } (u, u) = 2. \end{cases}$$

The group $G$ is the subgroup of $O(\Lambda)$ generated by $\{\rho_u : (u, u) = \pm 2\}$, by [Ma4, Theorem 1.2]. There exists a character

$$\text{cov} : G \to \mathbb{Z}/2\mathbb{Z}$$

satisfying $\text{cov}(\rho_u) = \begin{cases} 0 & \text{if } (u, u) = -2 \\ 1 & \text{if } (u, u) = 2. \end{cases}$ (see [Ma3, Sec. 4.1]). The unordered pair of cosets $\{\tilde{\theta}_0, -\tilde{\theta}_1\}$ in $\Lambda/(2n - 2)\Lambda$, given in Equation (6.3), is $G$-invariant and $g(\tilde{\theta}_1) = (-1)^{\text{cov}(g)} \tilde{\theta}_1$, by [Ma5] Lemma 7.3. Let $M^{[n]}_\Lambda$ be a connected component of $M^{[n]}_\Lambda$ containing a triple $(X^{[n]}_0, \eta_0, \mathfrak{A}_0)$, where $\mathfrak{A}_0$ is the modular Azumaya algebra over the cartesian square $X^{[n]}_0 \times X^{[n]}_0$ of the Hilbert scheme of a $K3$ surface $X_0$.

**Theorem 6.14.**

1. The $G$-action on $M^{[n]}_\Lambda$, given by

$$g(M, \eta, \mathfrak{A}) = (M, g\eta, \mathfrak{A}^{(s\text{cov}(g))})$$

maps the connected component $M^{[n]}_\Lambda$ to itself, where $\mathfrak{A}^{(s\text{cov}(g))}$ is $\mathfrak{A}$, if $\text{cov}(g) = 0$, and $\mathfrak{A}^*$, if $\text{cov}(g) = 1$.

2. The open subset $U$ of $M^{[n]}_\Lambda$, in Theorem 1.8 may be chosen to be invariant with respect to the above action of $G$.

**Proof.** 1. This statement is a version of [MM2, Theorem 1.11] and its proof is included in the proof of that Theorem.

2. All the properties that points of $U$ were required to satisfy depend on the isomorphism class of the Azumaya algebra. Hence, $U$ may be enlarged replacing it by the union of all translates $g(U)$, for all $g$ in the kernel of $\text{cov}$. We may thus assume that $U$ is $\ker(\text{cov})$-invariant.
Choose an element $g \in G$ with $\text{cov}(g) = 1$. The dense subset $\text{Hilb}$ is $G$-invariant, by definition. Hence, $\text{Hilb}$ is contained in $U \cap g(U)$. The latter is $G$-invariant.

Example 6.15. Let $D \subset X^{[n]}$ be the divisor of non-reduced subschemes, $d \in H^2(X^{[n]}, \mathbb{Z})$ the class of $D$, and $R_d : H^2(X^{[n]}, \mathbb{Z}) \to H^2(X^{[n]}, \mathbb{Z})$ the reflection given by $R_d(x) = x - \frac{2(x,d)}{(d,d)}$. Then $R_d$ is a monodromy operator, by [Ma3, Cor. 1.8], and a Hodge isometry. We have $\text{cov}(R_d) = 1$, by [Ma3, Lemma 4.10(4)] ($R_d$ is the image of the duality operator $v \mapsto v^\vee$ via the homomorphism $f$ in that Lemma). Let $\mathcal{A}$ be the modular Azumaya algebra over $X^{[n] \times X^{[n]}}$ and $\eta$ a marking for $X^{[n]}$. Then the triples $(X^{[n]}, \eta, \mathcal{A})$ and $(X^{[n]}, \eta R_d, \mathcal{A}^*)$ belong to the same connected component $\widetilde{\mathcal{M}}^0_\Lambda$ and have the same period. Hence, the two triples are inseparable points in moduli.

Proof of part 2 of Theorem 1.8. $U$ is a non-empty open $G$-invariant subset of $\widetilde{\mathcal{M}}^0_\Lambda$, by Theorem 6.14. Hence, its image $\phi(U)$ in $\mathcal{M}_\Lambda$ is a $G$-invariant non-empty open subset. The main result of [Y4, Theorem 4.11] states that the $G$-orbit of a marked hyperkahler manifold $(M, \eta)$ with second Betti number $b_2(M) \geq 5$ and Picard rank $\leq b_2(M) - 3$ is dense in its connected component $\mathcal{M}_\Lambda^0$. Hence, $\phi(U)$ contains $(M, \eta)$, for every marking $\eta$, such that $(M, \eta)$ belongs to the image $\mathcal{M}_\Lambda^0$ of $\mathcal{M}_\Lambda^0$ in the moduli space of marked pairs.

The following is a sufficient condition for a triple to belong to the open subset $U$ mentioned in Theorem 1.8.

Lemma 6.16. The $G$-orbit of a point $(M, \eta, \mathcal{A})$ of $\widetilde{\mathcal{M}}^0_\Lambda$ is dense in $\mathcal{M}_\Lambda^0$, and is thus contained in the open subset $U$ of Theorem 1.8 supporting the deformation $(\mathcal{F}, \zeta, \theta)$ of comonad objects, as well as in the open subset $W$ of Remark 6.8 where the the square $\mathcal{F} \circ \mathcal{F}$ is isomorphic to a direct sum of shifts of $\mathcal{F}$, provided the following conditions are satisfied.

1. The rank of $\text{Pic}(M)$ is $\leq 20$.
2. The order of the Brauer class of $\mathcal{A}$, which is the image in $H^2(M \times M, \Theta^*)$ of $\eta^{-1}(\tilde{\theta})$, is $2n - 2$. Here $\tilde{\theta}$ is the class given in (6.5).

Proof. Condition (1) implies that the fiber $\tilde{\phi}^{-1}(\tilde{\phi}(M, \eta, \mathcal{A}))$ intersects $U$, by Theorem 1.8 (5). Condition (2) implies that the fiber $\phi^{-1}(\phi(M, \eta, \mathcal{A}))$ consists of a single point. Indeed, the condition implies that the Azumaya algebra $\mathcal{A}'$ of a triple $(M, \eta, \mathcal{A}')$ in this fiber is slope stable with respect to every Kähler class on $M \times M$, by [Ma5, Prop. 7.8]. It follows that $\mathcal{A}$ is isomorphic to $\mathcal{A}'$, by [MM2, Lemma 5.3].

Remark 6.17. Let $X$ be a $K3$ surface, $v$ a primitive algebraic Mukai vector, and $H$ a $v$-generic polarization, such that the dimension of $M := M_H(v)$ is $\geq 4$. Denote by $\mathcal{A}$ the modular Azumaya algebra over $M \times M$ (Definition 1.5). There exists a marking $\eta$ for $M$, such that $(M, \eta, \mathcal{A})$ belongs to $\widetilde{\mathcal{M}}^0_\Lambda$, if and only if $\mathcal{A}$ is $\pi_1^*\omega + \pi_2^*\omega$ slope-stable, as an Azumaya algebra, with respect to some Kähler class $\omega$ on $M$. The above slope-stability of $\mathcal{A}$ with respect to every Kähler class on $M$ is known when the order of the Brauer class of $\mathcal{A}$ is equal to the rank $2n - 2$ of $\mathcal{A}$, by [Ma5, Prop. 7.8], as well as when Pic($X$) is trivial and $v = (1, 0, 1 - n)$ is the Mukai vector of the ideal sheaf of a length $n$ subscheme, so that $M = X^{[n]}$, by [Ma7]. Stability of the modular Azumaya algebra over $X^{[n] \times X^{[n]}}$ with respect to some Kähler class on $X^{[n]}$ is known whenever the rank of Pic($X$) is less than $20$, by [Ma7]. Stability being a Zariski open condition, membership of $(M_H(v), \eta, \mathcal{A})$ in $\mathcal{M}_\Lambda^0$ follows for the generic member of a family of such moduli spaces over an irreducible base, once known for some fiber.
Corollary 6.18. Let $X$ be a $K3$ surface of Picard rank $\leq 19$, $v$ a primitive algebraic Mukai vector, $H$ a $v$-generic polarization, such that the dimension of $M_H(v)$ is $\geq 4$. Assume further that

$$\text{gcd}(u, v) : u \in H^*(X, \mathbb{Z}) \text{ and } c_1(u) \in H^{1,1}(X) = (v, v).$$

Then the morphism $a$ given in Equation (6.14) is an isomorphism (so the monad object $\mathcal{A}$ is totally split).

Proof. The Picard rank of $M_H(v)$ is $\leq 20$. The order of the Brauer class of the modular Azumaya algebra $\mathfrak{A}$ over $M_H(v) \times M_H(v)$ is equal to the right hand side of Equation (6.13), by [Ma5, Lemma 7.3]. The rank of the Azumaya algebra $\mathfrak{A}$ is equal to the right hand side of Equation (6.13). Thus $\mathfrak{A}$ is slope-stable, as an Azumaya algebra, with respect to every Kähler class, by [Ma5, Prop. 7.8]. Hence, there exists a marking $\eta$, such that the triple $(M_H(v), \eta, \mathfrak{A})$ corresponds to a point in the open set $U$ of Theorem 1.8 by Lemma 6.9. The assertion now follows from Lemma 6.9 $\square$

6.4. A $K3$ category. Let $M$ be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type admitting a deformed comonad structure $\mathfrak{l} := (L, \epsilon, \delta)$ constructed above.

Proposition 6.19. The shift by $[2]$ is a Serre functor for the category $D^b(M, \theta)^\perp_{\mathcal{F}}$ over a dense $G$-invariant open subset, containing $\text{Hilb}$, of the moduli space $\mathcal{M}_\Lambda$ of triples. In other words, given objects $a$ and $b$ of $D^b(M, \theta)^\perp_{\mathcal{F}}$, there exists a natural isomorphism

$$\text{Hom}(a, b) \cong \text{Hom}(b, a[2]^*).$$

Proof. Let $\tilde{L} : D^b(M, \theta) \to D^b(M, \theta)^\perp_{\mathcal{F}}$ be the natural functor taking an object $x$ of $D^b(M, \theta)$ to $(L(x), \delta_x : L(x) \to L^2(x))$. The full subcategory of $D^b(M, \theta)^\perp_{\mathcal{F}}$ with objects of the form $\tilde{L}(x)$, for an object $x$ of $D^b(M, \theta)$, will be denoted by $D^b(M, \theta)^\perp_{\mathcal{F}}$. We consider first the case $a, b \in D^b(M, \theta)^\perp_{\mathcal{F}}$, with $a = \tilde{L}(x)$ and $b = \tilde{L}(y)$ for objects $x, y$ of $D^b(M)$. Denote by $F : D^b(M, \theta)^\perp_{\mathcal{F}} \to D^b(M, \theta)$ the forgetful functor. Then $\tilde{L}$ is the right adjoint of $F$. The right adjoint of the functor $L$ is isomorphic to $L[2n - 2]$ over the dense subset of the moduli space of triples consisting of Hilbert schemes, by Lemma 4.1. The kernel $\mathcal{F}$ of the functor $L$ has a one dimensional space $\text{Hom}(\mathcal{F}, \mathcal{F})$ if $M$ is a Hilbert scheme, and so the set over which the isomorphism of Lemma 4.1 holds is a dense open subset. We omit the proof of the former statement, which is similar to the proof of Lemma 6.7, followed by that of Lemma 2.3. Over this open set we get

$$\text{Hom}(x, F\tilde{L}(y)[2n - 2]) = \text{Hom}(x, L(y)[2n - 2]) \cong \text{Hom}(L(x), y) = \text{Hom}(F\tilde{L}(x), y) \cong \text{Hom}(\tilde{L}(x), \tilde{L}(y)),$$

where the equalities above follow from the equality $F\tilde{L} = L$ and the isomorphisms are due to the adjunctions $L \dashv L[2n - 2]$ and $F \dashv \tilde{L}$. We conclude that $L[2n - 2]$ is a left adjoint to the restriction $F' : D^b(M, \theta)^\perp_{\mathcal{F}} \to D^b(M, \theta)$ of the forgetful functor $F$ to the subcategory $D^b(M, \theta)^\perp_{\mathcal{F}}$ of $D^b(M, \theta)^\perp_{\mathcal{F}}$.

We get the bi-functorial isomorphisms

$$\text{Hom}(\tilde{L}(x), \tilde{L}(y)) \cong \text{Hom}(L(x), y) \cong \text{Hom}(y, L(x)[2n])^* \cong \text{Hom}(\tilde{L}(y), \tilde{L}(x)[2])^*.$$

The first isomorphism is due to the adjunction $F \dashv \tilde{L}$ and the equality $F\tilde{L} = L$. The second is Serre Duality for $D^b(M, \theta)$. The last is due to the adjunction $\tilde{L}[2n - 2] \dashv F'$. Thus, $D^b(M, \theta)^{\perp}_{\mathcal{F}}$ is a $K3$ category.
Denote by \( \sigma_{x,y} : \text{Hom}(\hat{L}(x), \hat{L}(y)) \to \text{Hom}(\hat{L}(y), \hat{L}(x)[2])^* \) the isomorphism given in Equation (6.15). Let \( e : \hat{L}(w) \to \hat{L}(x) \) be a morphism. Bi-functoriality yields the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(\hat{L}(x), \hat{L}(y)) & \xrightarrow{\sigma_{x,y}} & \text{Hom}(\hat{L}(y), \hat{L}(x)[2])^* \\
\sigma_e & & \sigma_e^* \\
\text{Hom}(\hat{L}(w), \hat{L}(y)) & \xrightarrow{\sigma_{w,y}} & \text{Hom}(\hat{L}(y), \hat{L}(w)[2])^*,
\end{array}
\]

and the equality \( \sigma_{w,y} e^* = (e_+)^* \sigma_{x,y} \). Similarly, given a morphism \( f : \hat{L}(y) \to \hat{L}(z) \), we get the analogous equality

\[
\sigma_{x,y} (e_* f^*) = (e_+)^* (f^*)^* \sigma_{x,y}.
\]

The comonad category \( D^b(M, \theta)^I \) is the idempotent completion of the category of free comodules \( D^b(M, \theta)_{T^I} \). This follows from [Bal1] Prop. 2.10 and triangulated Barr-Beck [MS]. Objects of the idempotent completion are pairs \((\hat{L}(x), e)\), where \( e \in \text{Hom}(\hat{L}(x), \hat{L}(x))\) is an idempotent. A morphism in \( \text{Hom}(\hat{L}(x), e), (\hat{L}(y), f)\) is a morphism \( g : \hat{L}(x) \to \hat{L}(y) \) satisfying \( fg = g = ge \). In other words, \( \text{Hom}(\hat{L}(x), e), (\hat{L}(y), f)\) is the image of the idempotent endomorphism \( e_+ f_* \) of \( \text{Hom}(\hat{L}(x), \hat{L}(y))\).

Set \( e_1 := e_+ f_* \), \( e_2 := e_+(1 - f)_* \), \( e_3 := (1 - e_+) f_* \), and \( e_4 := (1 - e_+)(1 - f)_* \). These are commuting idempotent endomorphisms of \( \text{Hom}(\hat{L}(x), \hat{L}(y)) \) satisfying \( e_i e_j = 0 \), if \( i \neq j \), and \( \sum_{i=1}^4 e_i = 1 \). Set \( \tilde{e}_1 := (e_+)(f^*)_*, \tilde{e}_2 := (e_+)((1 - f)^*)_*, \tilde{e}_3 := ((1 - e_+)(f^*)_*, \tilde{e}_4 := ((1 - e_+)((1 - f)^*)_*. \) These are commuting idempotent endomorphisms of \( \text{Hom}(\hat{L}(y), \hat{L}(x)[2])^* \) satisfying \( \tilde{e}_i \tilde{e}_j = 0 \), if \( i \neq j \), and \( \sum_{i=1}^4 \tilde{e}_i = 1 \). We get the decomposition \( \sigma_{x,y} = \sum_{i=1}^4 \sum_{j=1}^4 \tilde{e}_j \sigma_{x,y} e_i \). Equation (6.16) implies that \( \sigma_{x,y} e_i = \tilde{e}_i \sigma_{x,y} \), \( 1 \leq i \leq 4 \) (note that the equation holds with \( e \) replaced by \( 1 - e \) or \( f \) replaced by \( 1 - f \)). Hence, \( \tilde{e}_j \sigma_{x,y} e_i = 0 \), if \( i \neq j \). Consequently, \( \sigma_{x,y} \) maps the image of \( e_i \) isomorphically onto the image of \( \tilde{e}_i \), for \( 1 \leq i \leq 4 \). Considering the case \( i = 1 \) we conclude that \( \sigma_{x,y} \) maps \( \text{Hom}(\hat{L}(x), e), (\hat{L}(y), f) = Im(e_1) \) isomorphically onto \( Im(\tilde{e}_1) \). Now \( Im(\tilde{e}_1) \) maps isomorphically onto \( \text{Hom}(\hat{L}(y), f), (\hat{L}(x)[2], e)^* \) via the natural homomorphism \( \text{Hom}(\hat{L}(y), \hat{L}(x)[2])^* \to \text{Hom}(\hat{L}(y), f), (\hat{L}(x)[2], e)^* \). We thus obtain the desired isomorphism.

7. Comparison with Toda’s Category

The first order deformations of the category of coherent sheaves \( \text{Coh}(S) \) on a smooth, projective variety \( S \) are parametrized by its degree 2 Hochschild cohomology \( HH^2(S) \). This has an interesting interpretation via the HKR-isomorphism,

\[
I^* : HH^2(S) = H^0(\wedge^2 T_S) \oplus H^1(T_S) \oplus H^2(\mathcal{O}_S) \xrightarrow{\cong} HH^2(S),
\]

namely, the general deformation may be understood as composed of non-commutative, complex and “gerby” parts corresponding to the three summands.

Given a class \( \eta \in HH^2(S) \), Toda gave an explicit construction [To] of the corresponding infinitesimal deformation \( \text{Coh}(S, \eta) \): Let \( \eta_{0,2} \) be the component of \( \eta \) in \( HH^2(\mathcal{O}_S) \). Then \( \text{Coh}(S, \eta) \) is the \( \mathbb{C}[\varepsilon]/(\varepsilon^2) \)-linear abelian category of \( \eta_{0,2} \)-twisted coherent sheaves of modules
Theorem 7.1 ([10]). Let $S$ and $Y$ be smooth, projective varieties, and suppose that there is a Fourier-Mukai equivalence $\Phi : D^b(S) \to D^b(Y)$. If $\phi : HH^i(S) \simto HH^i(Y)$ is the induced map on cohomology, there is an equivalence $\Phi^! : D^b(\text{Coh}(S, \eta)) \to D^b(\text{Coh}(Y, \phi(\eta)))$ such that the following diagram commutes (up to natural isomorphisms of functors):

$$
\begin{array}{ccc}
D^b(S) & \xrightarrow{i_*} & D^b(\text{Coh}(S, \eta)) \xrightarrow{\phi^!} & D^-(S) \\
\Phi & | & \downarrow \Phi^! & | & \downarrow \Phi^-
\end{array}
$$

The notation $D^-$ refers to the derived category of bounded above complexes of coherent sheaves, while $i_*$ and $i^*$ stand for restriction and extension of scalars (see [10, Section 4]), respectively.

In this section, $M$ will denote a moduli space $M_H(v)$. Our goal is to prove that the category of comodules constructed in the previous section via deformations of $M$ agrees infinitesimally with Toda’s category. In other words, for every direction in the 21-parameter space that $\mathcal{F}$ deforms, there is a class $\eta \in HT^2(X)$ for our K3 surface $X$ such that the comodule category over the dual numbers is exact equivalent to $D^b(\text{Coh}(X, \eta))$. This is Theorem 7.15. It is proven under the following assumption, which will remain in force sections 7.2 and 7.3.

Assumption 7.2. The moduli space $M = M_H(v)$ is a fine moduli space on an algebraic K3 surface $X$, supporting a universal family $\mathcal{F}$, such that the kernel $\mathcal{V} = [2] \circ \mathcal{W}$ is totally-split (as in Definition 7.1). Further, there exists a triple $(M, \eta, \mathcal{A}) \in \overline{\mathbb{M}}^0 \Lambda$ such that $\mathcal{A}$ is the modular Azumaya algebra of the moduli space $M_H(v)$ (see Remark 6.17).

Remark 7.3. The assumption of algebraicity on $X$ is in order to use the results of [10], which are stated in the context of smooth and projective varieties. One expects these results to hold over proper analytic manifolds also, but this generalization does not appear in the literature.

7.1. Hochschild cohomology and deformations. Let $\Phi : D^b(S) \to D^b(Y)$ be a Fourier-Mukai equivalence; write $\mathcal{P} \in D^b(S \times Y)$ for its kernel and $\mathcal{Q} \in D^b(Y \times S)$ for that of its inverse. We have an isomorphism $\phi : HH^i(S) \to HH^i(Y)$ defined as

$$(\Theta_S \xrightarrow{\eta} \Theta_S[i] \xrightarrow{} (\Theta_Y = \mathcal{P} \circ \Theta_S \circ \mathcal{Q}) \xrightarrow{\mathcal{Q} \circ \Theta_S[i] \circ \mathcal{Q}} \mathcal{P} \circ \Theta_S[i] \circ \mathcal{Q} = \Theta_Y[i])$$

In particular, for $i = 2$, this defines a bijective correspondence between the infinitesimal deformations of $\text{Coh}(S)$ and those of $\text{Coh}(Y)$. Note, however, that this makes use of the fact that $\Phi$ is invertible. In fact, one does not have functoriality for Hochschild cohomology under general integral transforms.

For later use, we state a criterion of Toda and Lowen for when an object in the derived category can be deformed to first order. First recall the construction of the Atiyah class $a(\mathcal{G}) \in \text{Hom}(\mathcal{G}, \mathcal{G} \otimes \Omega_S[1])$ for any object $\mathcal{G} \in D^b(S)$: Regarding the sequence

$$0 \to \mathcal{J}_S / \mathcal{I}_S^2 \to \Theta_S \otimes \mathcal{J}_S / \mathcal{I}_S^2 \to \Theta_S \to 0$$

as a sequence of Fourier-Mukai kernels, and taking integral transforms of $\mathcal{G}$ accordingly, we obtain the triangle

$$(\mathcal{G} \otimes \Omega_S \to \pi_2, (\pi_1^* \otimes \Theta_S) \otimes \mathcal{J}_S / \mathcal{I}_S^2 \to \mathcal{G})$$

Then, $a(\mathcal{G})$ is the extension class of 7.2.
Theorem 7.4. ([Toa Prop. 6.1; [Loa Thm. 1.1]) Let $\mathcal{G} \in D^b(S)$ and $u \in HT^2(S)$. There exists a perfect object $\mathcal{F}_1 \in D^b(S,u)$ such that $i^*\mathcal{F}_1 \cong \mathcal{G}$ if and only if $u \cdot \exp a(\mathcal{G}) = 0$ in $\Hom(\mathcal{G}, \mathcal{G}[2])$.

Here $\exp a(\mathcal{G}) = \sum a^i(\mathcal{G})$, the summand $a^i(\mathcal{G}) \in \Hom(\mathcal{G}, \mathcal{G} \otimes \Omega^i[u])$ being the $i$-fold composition of $a(\mathcal{G})$ with itself, followed by anti-symmetrization.

Remark 7.5. The sufficiency of the vanishing of $u \cdot \exp a(\mathcal{G})$ for the deformability of $\mathcal{G}$ was first proven by Toda, op. cit. The necessity of this condition in this criterion follows from Lowen’s work, who proved that the obstruction to the deformability of $\mathcal{G}$ is precisely the image of $u$ under the characteristic morphism:

$$HH^2(S) \xrightarrow{\chi_{\mathcal{G}}} \Hom_S(\mathcal{G}, \mathcal{G}[2]).$$

Regard the elements of $HH^2(S)$ as natural transformations between the functors $1_{D^b(S)}$ and $[2]$. Evaluating them on $\mathcal{G}$ defines $\chi_{\mathcal{G}}$. The statement of the criterion can then be deduced from the commutativity of the following diagram (see [C3 Proposition 4.5]):

$$\begin{array}{ccc}
HT^2(S) & \xrightarrow{\exp a(\mathcal{G})} & \Hom_S(\mathcal{G}, \mathcal{G}[2]) \\
\downarrow{\gamma'} & & \downarrow{\chi_{\mathcal{G}}} \\
HH^2(S) & & \\
\end{array}$$

Remark 7.6. Theorem 7.4 is the main component of the proof Theorem 7.1. Using it, [Toa] proves that that the kernel of the Fourier-Mukai transform $\Phi : D^b(S) \to D^b(Y)$ deforms to the derived category of the first-order deformation of $\text{Coh}(S \times Y)$ in the direction $p_S^*(\eta) + p_Y^*(\phi(\eta))$. Here $(\subset)$ denotes the action of transposition of factors on Hochschild cohomology.

Example 7.7. The identity functor $D^b(S) \to D^b(S)$ is interesting in light of Theorem 7.1. Its kernel $\mathcal{O}_\Delta$ must deform along every direction $p_1^*(-\eta) + p_2^*(\eta)$ by Remark 7.6. It is possible to explicitly define a canonical deformation, the “structure sheaf of the deformed diagonal” $\mathcal{O}_\Delta$. In keeping with the following sections, rather than $\eta$, we prefer to work with its image $u = I^2(\eta) \in HT^2(S)$ under the HKR isomorphism. Also, we assume that $u = (\gamma, 0, 0)$, with $\gamma \in H^0(\Lambda^2 TM)$, which is the only somewhat subtle case.

The theory of quasi-coherent sheaves on first-order noncommutative deformations mirrors that for schemes [Toa], [Loa]. In particular, over any affine $U$, such a sheaf $\mathcal{M}$ is determined by its sections $\mathcal{M}(U)$, and its sections over a principal affine $U_f$ are precisely the localization $\mathcal{M}(U)_f$ [Toa] Lemma 3.1, Def. 4.1.

Fix an ample line bundle $\mathcal{L}$ on $S$; for any section $f \in \Gamma(S, \mathcal{L}^n)$, $n \in \mathbb{Z}$, let $S_f$ be the affine open where $f$ does not vanish. Consider the affine open covering $\mathcal{C}'$ of $S$ given by $\{S_f : f \in \Gamma(S, \mathcal{L}^n), n \in \mathbb{Z}\}$. Note that $\mathcal{C}'$ is closed under intersection, and that given any two elements of $\mathcal{C}'$, their intersection is a principal affine in each of them. Consider the affine open covering $\mathcal{C} := \{U \times V : U, V \in \mathcal{C}'\}$ of $S \times S$.

Let $D$ stand for the dual numbers over $\mathbb{C}$. Denote the class $-p_1^*(\gamma) + p_2^*(\gamma)$ by $-\gamma \boxplus \gamma$. Note that given $U \times V \in \mathcal{C}$, $\mathcal{O}_{S \times S}^{-\gamma \boxplus \gamma}(U \times V) = \mathcal{O}_{S}^{-\gamma}(U) \otimes_D \mathcal{O}_{S}^{\gamma}(V)$. For each $U \times V \in \mathcal{C}$, let $\mathcal{M}_{U \times V}$ be the $D$-flat coherent $\mathcal{O}_{U \times V}^{-\gamma \boxplus \gamma}$ module whose sections over $(U \times V)$ are $\mathcal{O}_{S}^{-\gamma}(U \cap V)$. The right $\mathcal{O}_{S}^{-\gamma}(U) \otimes_D \mathcal{O}_{S}^{\gamma}(V)$-module structure of $\mathcal{O}_{S}^{-\gamma}(U \cap V)$ is given by restriction to $U \times V$, followed by left and right multiplication in the ring $\mathcal{O}_{S}^{-\gamma}(U \cap V)$. We claim that the coherent sheaves $\mathcal{M}_{U \times V}$ glue to give a coherent sheaf $\mathcal{M}_\Delta$ over $S \times S$. It suffices to check that for any $U \times V \in \mathcal{C}$, and principal affine subsets $U_g \subset U$, $V_h \subset V$, there is a canonical identification between the
modules $\mathcal{A}_{U \times V}(U_g \times V_h)$ and $\mathcal{A}_{U \times V}(U_g \times V_h)$. The former is the localization with respect to the left $\mathcal{O}_S^\ell(U \cap V)$, right $\mathcal{O}_S^r(U \cap V)$ bimodule structure, $\mathcal{O}_S^\ell(U \cap V)_{gh}$. The latter is $\mathcal{O}_S^r(U \cap V)_{gh}$, the localization being with respect to the right $\mathcal{O}_S^r(U \cap V)$-module structure. Given any two local sections $a, b$ of $\mathcal{O}_S \otimes \mathcal{D}$, $a \ast \gamma b = b \ast \gamma a$, so these two localizations are equal.

### 7.2. A map on tangent spaces

We wish to compare the Hochschild cohomologies of $X$ and $M_H(v)$, but as the functor $\Phi_\mathcal{Y} : D^b(X) \to D^b(M_H(v))$ is not an equivalence, we do not a priori have a homomorphism $\phi : HH^s(X) \to HH^s(M)$. Nevertheless, the natural construction \((7.1)\) can be modified to give a workable map between the degree 2 components of these groups:

**Construction 7.8.** Write $\mathcal{Y} \in D^b(M \times X)$ for $\mathcal{Y}[2]$, the kernel of the adjoint to $\Phi_\mathcal{Y}$. As in (7.1), given $\eta \in HH^2(X)$, one obtains a map:

$$\mathcal{F} = \mathcal{Y} \circ \mathcal{O}_{\Delta_X} \circ \mathcal{Y} \circ \mathcal{O}_{\Delta_M}[2] \circ \mathcal{Y} = \mathcal{F}[2].$$

Applying the functor $\text{Hom}_{M^2(v)}(\_ , \mathcal{O}_{\Delta_M}[2])$ to the triangle $\mathcal{F}[1] \to \mathcal{F} \to \mathcal{O}_{\Delta_M}$, we obtain the sequence

$$\text{Hom}(\mathcal{O}_{\Delta_M}, \mathcal{O}_{\Delta_M}[2]) \to \text{Hom}(\mathcal{F}, \mathcal{O}_{\Delta_M}[2]) \to \text{Hom}(\mathcal{F}, \mathcal{O}_{\Delta_M}[1]).$$

The first and last groups are (0). Assuming this for the moment, set $\phi^{HH}(\eta)$ to be the unique lift of $\epsilon[2](\mathcal{Y} \circ \eta \circ \mathcal{Y}) \in \text{Hom}_{M^2(v)}(\mathcal{F}, \mathcal{O}_{\Delta_M}[2])$ in $\text{Hom}(\mathcal{O}_{\Delta_M}, \mathcal{O}_{\Delta_M}[2])$. This defines the desired map

$$\phi^{HH} : HH^2(X) \to HH^2(M).$$

We prove the claim using the spectral sequence

$$E_2^{p,q} = \text{Hom}^p(H^{-q}(\mathcal{O}_{\Delta_M}), \mathcal{O}_M) \implies \text{Hom}^{p+q}(\mathcal{O}_{\Delta_M}, \mathcal{O}_M)$$

and Proposition 4.2. Observe that $H^{-1}(\mathcal{O}_{\Delta_M}) \cong 0$ or $1$ vanishes by the latter result. Hence, $E_2^{0,1}$ vanishes. The question reduces to proving the vanishing of the terms

$$E_2^{i,0} := \text{Hom}^i(M/O_M \cdot \sigma, \mathcal{O}_M)$$

for $i = 0, 1$. This follows from the facts that the sheaf $(\Omega^2_M/O_M \cdot \sigma)$ is a self-dual direct summand of $\Omega^2_M$, and that $H^i(\Omega^2_M/O_M \cdot \sigma)$ vanishes for $i = 0, 1$.

We can say this slightly differently: There are natural maps

$$HH^2(X) \xrightarrow{\mathcal{Y}} \text{Hom}_{X \times M}(\mathcal{Y}, \mathcal{Y}[2]) \xrightarrow{\mathcal{Y} \circ \mathcal{Y}} HH^2(M),$$

the left map is an injection, while the right is an isomorphism. The map $\phi^{HH}$ is the one obtained by composing the first with the inverse of the second.

Hochschild homology is naturally a module over Hochschild cohomology, the action being composition. This structure gives rise to the homomorphisms

$$m_X : HH^2(X) \to \text{Hom}(HH_0(X), HH_{-2}(X))$$

$$m_M : HH^2(M) \to \text{Hom}(HH_0(M), HH_{-2}(M)).$$

We note that $m_X$ is an isomorphism and $m_M$ is injective. By functoriality of Hochschild homology, we also have the maps

$$\Phi_* := \Phi_{\mathcal{Y}} : HH_*(X) \to HH_*(M), \quad \text{and}$$

$$\Psi_* := \Psi_{\mathcal{Y}} : HH_*(M) \to HH_*(X).$$

The map $\phi^{HH}$ intertwines the Hochschild module structures via $\Phi_*$ and $\Psi_*$. 
Lemma 7.9. The following diagram is commutative for every \( \lambda \in HH^2(X) \).

\[
\begin{array}{ccc}
HH_1(M) & \xrightarrow{\Psi_*} & HH_0(X) \\
m_M(\phi^{HH}(\lambda)) & & m_X(\lambda) \\
HH_{i-2}(M) & \xrightarrow{\Psi_*} & HH_{i-2}(X)
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{\Phi_*} & \\
& m_M(\phi^{HH}(\lambda)) & \\
& \Phi_* & HH_{i-2}(M)
\end{array}
\]

Proof. Let \( \lambda_X \) be a class in \( HH^2(X) \) and \( \lambda_M \) a class in \( HH^2(M) \) satisfying the equality \( \Psi \circ \lambda_X = \lambda_M \circ \Psi \) in \( \text{Hom}_{X \times M}(\oplus \mathcal{O}, \mathcal{O}'[2]) \). Then the following diagram commutes

\[
\begin{array}{ccc}
HH_0(X) & \xrightarrow{\Phi_*} & HH_1(M) \\
m_X(\lambda_X) & & m_M(\lambda_M) \\
HH_{i-2}(X) & \xrightarrow{\Phi_*} & HH_{i-2}(M),
\end{array}
\]

by the proof of [AT, Prop. 6.1]. The above statement holds and its proof applies without the assumption that the right map in diagram \([7.4]\) is an isomorphism. The statement thus applies also to the functor \( \Psi_\mathcal{O} \). Apply the statement with \( \lambda_X = \lambda \) and \( \lambda_M = \phi^{HH}(\lambda) \) to verify the commutativity of both squares in the statement of the Lemma.

The endomorphism \( \Phi_* \Psi_* \) of \( HH_*(M) \) is self-adjoint with respect to the Mukai pairing. It satisfies \( (\Phi_* \Psi_*)^2 = n\Phi_* \Psi_* \). Indeed, the kernel of \( \Psi_\mathcal{O} \circ \Phi_\mathcal{O} \) is the direct sum \( \oplus_{i=0}^{n-1} \mathcal{O}_\Delta[-2i] \) by Assumption \([7.2]\) and \( \Psi_\mathcal{O}(\Phi_\mathcal{O}(x)) = x \oplus x[-2] \oplus \cdots \oplus x[2-2n] \) for any object \( x \) in \( D^b(X) \). Hence, \( \Psi_* \Phi_* \) is multiplication by \( n \). The subspace \( \text{Im}(\Phi_\Psi) \) is the eigenspace of \( \Phi_* \Psi_* \) with eigenvalue \( n \) and the subspace \( \ker(\Psi_* \Phi_\Psi) \) is the eigenspace with eigenvalue \( 0 \). We get the orthogonal direct sum decomposition

\[
HH_*(M) = \text{Im}(\Phi_* \Psi_\Phi) \oplus \ker(\Psi_*),
\]

Let \( E \) be an \( H \)-stable sheaf on \( X \) with Mukai vector \( v \in HH_0(X) \). Let \( \alpha \in HH_0(M) \) be the Mukai vector of \( \Phi_\mathcal{O}(E^\vee[2]) \) and let \( \beta \in HH_0(M) \) be the Mukai vector of the sky-scraper sheaf of a point. Given a subset \( \Sigma \) of \( HH_0(M) \), denote by \( \text{ann}(\Sigma) \) the subspace of \( HH^2(M) \) consisting of classes \( \xi \), such that \( m_M(\xi)(c) = 0 \), for all \( c \in \Sigma \). We use the analogous notation for subsets of \( HH_0(X) \).

Lemma 7.10. (1) The image of \( \phi^{HH} \) is equal to the subspace of \( HH^2(M) \) consisting of classes \( \xi \), such that \( m_M(\xi) \) commutes with \( \Phi_* \Psi_* \).

(2) The following equality of subspaces of \( HH^2(M) \) holds:

\[
\phi^{HH}(\text{ann}(v^\vee)) = \text{ann}(\alpha, \beta).
\]

(3) The normalized HKR isomorphism maps the subspace \( \phi^{HH}(\text{ann}(v^\vee)) \) of \( HH^2(M) \) into the direct sum \( H^1(TM) \oplus H^2(\mathcal{O}_M) \) in \( HT^2(M) \). The image is equal to the subspace

\[
\mathfrak{O} := \{ (\xi, \theta) : \xi \in H^1(TM), \theta \in H^2(\mathcal{O}_M), \text{ and } \xi \cdot c_1(\alpha) + (2-2n)\theta = 0 \}.
\]

Proof. (1) \( \Phi_* \Psi_* \) commutes with \( m_M(\phi^{HH}(\lambda)) \), for all \( \lambda \in HH^2(X) \), by Lemma \([7.9]\). We know that \( \phi^{HH} \) is injective and that its image has codimension \( 1 \). Hence it remains to exhibit classes \( \xi \) of \( HH^2(M) \), which do not commute with \( \Phi_* \Psi_* \).

\( \Psi_\mathcal{O} \) maps the sky-scraper sheaf of the point \([E]\) corresponding to the isomorphism class of the sheaf \( E \) to the object \( E^\vee[2] \) in \( D^b(X) \). Let \( e \) be the class of \( E^\vee[2] \) in \( HH_0(X) \). Then \( \alpha = \Phi_\Psi(e), \Psi_\Psi(\alpha) = ne, \text{ and } \Psi_*(\beta) = e \). Hence,

\[
\Psi_* (\alpha - ne) = 0.
\]
The class $c_1(\alpha)$ does not vanish \cite[Lemma 7.2]{Ma5}. Hence, there exists a class $\xi$ of $H^1(TM)$, such that $\xi \cdot c_1(\alpha)$ is a non-zero class in $H^2(M, \mathcal{O}_M) \subset H\Omega_{-2}(M)$. Considering $\xi$ as a class in $H\Omega^2(M)$, via the HKR isomorphism, then $\xi$ belongs to $\text{ann}(\beta)$, but it does not belong to $\text{ann}(\alpha)$. Hence, $m_M(\xi)(\alpha - n\beta) \neq 0$. Assume that $m_M(\xi)$ commutes with $\Phi_*\Psi_*$. Then $\text{Im}(\Phi_*\Psi_*)$ and $\ker(\Psi_*)$ are invariant with respect to $m_M(\xi)$. In addition, we have

$$m_M(\xi)(\alpha - n\beta) = m_M(\xi)(\alpha)$$

The left hand side belongs to $\ker(\Psi_*)$, since $\alpha - n\beta$ does, and the right hand side belongs to the image of $\Phi_*$, since $\alpha$ does. Hence, $m_M(\xi)(\alpha)$ vanishes and $\xi$ belongs to $\text{ann}(\alpha)$. A contradiction. Hence, $m_M(\xi)$ does not commute with $\Phi_*\Psi_*$.

(2) Let $\lambda \in H\Omega^2(X)$ be a class, such that $\phi^{HH}(\lambda)$ belongs to $\text{ann}(\beta)$. Then $\phi^{HH}(\lambda)$ commutes with $\Phi_*\Psi_*$ and so $\phi^{HH}(\lambda)$ belongs also to $\text{ann}(\alpha)$, as shown in the proof of part (1). Hence, the intersection $\text{Im}(\phi^{HH}) \cap \text{ann}(\beta)$ is contained $\text{ann}\{\alpha, \beta\}$. Note that both $\text{ann}(\beta)$ and $\text{Im}(\phi^{HH})$ are hyperplanes in $H\Omega^2(M)$. We have seen above that $\text{ann}(\beta)$ is not contained in $\text{ann}(\alpha)$. We conclude the equality

$$\text{Im}(\phi^{HH}) \cap \text{ann}(\beta) = \text{ann}\{\alpha, \beta\}.$$ 

Furthermore, the hyperplanes $\text{Im}(\phi^{HH})$ and $\text{ann}(\beta)$ are distinct and $\text{ann}\{\alpha, \beta\}$ has codimension 2 in $H\Omega^2(M)$. The equality

$$\phi^{HH}(\text{ann}(\nu')) = \text{Im}(\phi^{HH}) \cap \text{ann}(\alpha),$$

follows from the commutativity of the right square in Lemma 7.9 and the injectivity of $\Phi_*$. Hence, if $\text{ann}(\alpha)$ is a hyperplane in $H\Omega^2(M)$, then the hyperplanes $\text{ann}(\alpha)$ and $\text{Im}(\phi^{HH})$ are distinct. We conclude that either $\text{ann}(\alpha)$ is contained in $\text{ann}(\beta)$, or $\text{ann}(\alpha)$ is a hyperplane and the three hyperplanes $\text{ann}(\alpha)$, $\text{ann}(\beta)$, and $\text{Im}(\phi^{HH})$ are distinct.

If $H_1, H_2, H_3$ are three distinct hyperplanes in a vector space, such that $H_1 \cap H_2$ and $H_2 \cap H_3$ are equal to the same subspace $W$, then $H_1 \cap H_3 = W$. The equality

$$\text{Im}(\phi^{HH}) \cap \text{ann}(\alpha) = \text{ann}\{\alpha, \beta\}$$

thus follows from (7.7). The above equality is clear if $\text{ann}(\alpha)$ is contained in $\text{ann}(\beta)$. The equality (7.5) follows from the last two displayed equalities.

(3) The HKR isomorphism maps $\text{ann}(\alpha)$ into the hyperplane in $HT^2(M)$, consisting of classes $(\pi, \xi, \theta), \pi \in H^0(\wedge^2 TM), \xi \in H^1(TM), \theta \in H^2(\mathcal{O}_M)$, satisfying

$$\pi \cdot \alpha_2 + \xi \cdot c_1(\alpha) + (2 - 2n)\theta = 0,$$

where $\alpha_2$ is the graded summand in $H^2(\Omega^2_M)$ of the image via the HKR isomorphism of the Mukai vector $\alpha$, since the rank of $\alpha$ is $2 - 2n$. Indeed the latter hyperplane is the kernel of the composition $HT^2(M) \rightarrow H\Omega_{-2}(M) \rightarrow H^2(\mathcal{O}_M)$ of pairing with $\alpha$ followed by projection on the direct summand $H^2(\mathcal{O}_M)$. (We are using here the proof of Caldararu’s conjecture about the isomorphism of harmonic and Hochschild structures \cite{CRVdB}.) The HKR isomorphism maps $\text{ann}(\beta)$ onto the direct sum $H^1(M, TM) \oplus H^2(M, \mathcal{O}_M)$. The statement now follows from part (2). 

\[7.3. \text{The comparison.}\] Given $u \in HT^2(X)$, it will be convenient to adopt the simpler notation $D^h(X, u)$ for the deformed category $D^h(Coh(X, u))$; a similar notation is used for the corresponding categories on $M$. Let $\phi^T : HT^2(X) \rightarrow HT^2(M)$ denote the conjugate $(I^3_M)^{-1} \circ \phi^{HH} \circ I^2_X$. Also let $\tilde{z} : HT^*(X) \rightarrow HT^*(X)$ be the operation which on a homogeneous $t \in H^p(\wedge^q T_X)$ is defined as $\tilde{z} := (-1)^q t$. For $u \in HT^*(X)$, and $w \in HT^2(M)$, the class $\pi_X u + \pi_M w \in HT^2(X \times M)$ will be denoted by $u \boxplus w$; the same notation will be followed for the other Cartesian products, such as $M \times M$, that appear.
Lemma 7.11. Let \( u \in HT^2(X) \), and \( w = \phi^T(u) \in HT^2(M) \). There is a perfect object \( \mathcal{W}_1 \in D^b(X \times M, -\check{u} \boxplus w) \) whose derived restriction \( i^* (\mathcal{W}_1) \) is isomorphic to \( \mathcal{W} \).

Proof. By Theorem 7.4 it suffices to show that the degree 2 piece of \( (\check{u} \boxplus w) \cdot \exp(a(\mathcal{W})) \) is 0. We repeat Toda’s calculation of this obstruction in our setting. According to [15, Lemma 5.8] (see Remark 7.12 below), the following diagrams commute:

\[
\begin{array}{ccc}
HT^*(X \times M) & \times \exp(a(\mathcal{W})) & \text{Hom}^*_\mathcal{X}M(\mathcal{W}, \mathcal{W}) \\
\pi_X & & \downarrow \mathcal{W} \\
HT^*(X) & \tau_* I_X^* & HH^*(X) \\
\pi_M & & \downarrow \mathcal{W} \\
HT^*(M) & I_M^* & HH^*(M)
\end{array}
\]

The symbol \( \tau_* \) in the first diagram is the involution of \( HH^*(X) \) arising from the interchange of the factors of \( X \times X \). One sees that \( \tau_* I_X^*(t) = I_X^*(t) \) (see the discussion preceding Proposition 6.1 of [10]), so that

\[
(-\pi_X^\ddag \check{u} + \pi_M^\ddag w) \cdot \exp(a(\mathcal{W})) = -\mathcal{W} \circ \tau_* I_X^*(\check{u}) + I_M^*(w) \circ \mathcal{W}
\]

(7.8)

The description (7.4) of the map \( \phi^{HH} \) yields that \( \phi^{HH}(I_X^*(u)) \circ \mathcal{W} = \mathcal{W} \circ I_X^*(u) \). The result follows immediately.

Remark 7.12. Although it is assumed everywhere in [15] that the functor \( \Phi_{\mathcal{W}} \) is an equivalence, Lemma 5.8 of that article holds without this requirement. Indeed, Toda only works with the compositions \( \exp(a(\mathcal{W}))_X \) and \( \exp(a(\mathcal{W}))_M \) rather than the morphisms \( \exp(a)^X_\mathcal{W} \) and \( \exp(a)^M_\mathcal{W} \) (see p. 212, op. cit.), and while the latter morphisms may not exist without the assumption that \( \Phi_{\mathcal{W}} \) is an equivalence, the former always do.

Let \( \mathcal{O}^{-\check{u},w}_{X \times M} \) be the deformed structure sheaf of the product \( X \times M_H(v) \), and write \( \overline{\mathcal{W}}_1 \) for \( R\mathcal{H}om_*(\mathcal{W}_1, \mathcal{O}^{-\check{u},w}_{X \times M})[2] \). Note that derived dualization is defined on the full subcategory of perfect complexes of \( D^b(X \times M, -\check{u} \boxplus w) \), and sends it into the subcategory of perfect complexes of \( D^b(M \times X, -\check{u} \boxplus u) \):

\[
R\mathcal{H}om_*(\_, \mathcal{O}^{-\check{u},w}_{X \times M}) : D_{\text{perf}}(X \times M, -\check{u} \boxplus w) \to D_{\text{perf}}(M \times X, -\check{u} \boxplus u)
\]

The functors corresponding to restriction and extension of scalars between various categories will be simply denoted \( i_* \) and \( i^* \), without reference to the underlying spaces.

Consider the convolution \( \overline{\mathcal{W}}_1 \circ \overline{\mathcal{W}}_1 \in D^b(M \times M, -\check{w} \boxplus w) \). Lemma A.5 of [MSM] implies that \( i^*(\overline{\mathcal{W}}_1 \circ \overline{\mathcal{W}}_1) \cong \mathcal{F} \). We conclude that \( \mathcal{F} \) deforms along the direction \( -\check{w} \boxplus w \in HT^2(M \times M) \), \( w = \phi^T(u) \), for any \( u \in HT^2(X) \). Denote this infinitesimal deformation \( \overline{\mathcal{W}}_1 \circ \overline{\mathcal{W}}_1 \) by \( \overline{\mathcal{F}}_u \).

The object \( \mathcal{F} \) is the restriction to a fiber of the family \( \mathcal{F}_u \mathcal{F}_u \mathcal{F}_u \mathcal{F}_u \mathcal{F}_u \mathcal{F}_u \mathcal{F}_u \) constructed in [6.1] by Assumption 7.2 that is, there exists a triple \( (M, \eta, \mathcal{A}) \in U \subset \mathcal{M}^0_{\mathcal{A}} \), such that \( \mathcal{F} \cong \mathcal{F}|_{M \times M} \). Given a class \( \xi \in H^1(TM) \), let \( \mathcal{M}_\xi \) denote the first-order infinitesimal deformation of \( M \) in the direction of \( \xi \). Let \( \mathcal{F}_\xi \) be the restriction of \( \mathcal{F} \) to the fiber square \( \mathcal{M}_\xi^2 \) of \( \mathcal{M}_\xi \) over the length 2 subscheme of \( \mathcal{M}^0_{\mathcal{A}} \) determined by \( \xi \).
Lemma 7.13.  (1) Let ξ ∈ H^1(TM). The infinitesimal deformation \( \mathcal{F}_ξ \) of \( \mathcal{F} \) is an object of the derived category \( D^b(M × M, -\hat{w} ⊕ w) \), where \( w = (0, ξ, θ) \) is a class in the subspace \( \mathcal{U} ⊂ H^2(M) \) defined in Lemma 7.10. (2) Write \( u \) for the class \( (φ^T)^{-1}(w) ∈ H^2(X) \). There is an isomorphism \( \mathcal{F}_u ≅ \mathcal{F}_ξ \).

Proof.  (1) Fix a point \([E] ∈ M\). The restriction \( \mathcal{F}|_{M \times [E]} \) is isomorphic to \( Φ_E(E^ν[2]) \). Making use of the fact proven in Lemma 7.14 that \( τ^* \mathcal{F} ≅ \mathcal{F}^ν[2] \), we see that the Chern character of \( \mathcal{F} \) has the form:

\[
ch(\mathcal{F}) = (2 - 2n) + (-π_1^*(c_1(α)) + π_2^*(c_1(α))) + ch_2(\mathcal{F}) + ... ∈ H^i(Ω^i_{M×M}).
\]

The product \((-\hat{w} ⊕ w) · ch(\mathcal{F}) \) in \( π_1^*(H^2(\mathcal{O}_M)) ⊕ π_2^*(H^2(\mathcal{O}_M)) ≅ H^0(\mathcal{O}_M) \) vanishes if and only if \( w ∈ \mathcal{U} \). This product is nothing but the trace of the obstruction class \((-\hat{w} ⊕ w) · \exp a(\mathcal{F}) ∈ \text{Hom}(\mathcal{F}, \mathcal{F}[2]) \) [HuL, 10.1.6]. As \( \mathcal{F} \) deforms, Theorem 7.4 implies that this class must indeed vanish, which completes the proof.

(2) This should follow immediately from the infinitesimal rigidity of \( \mathcal{F} \) from known results. However, we were not able to locate a precise reference, so we sketch a short argument specific to our case. Consider the object \( \mathcal{F}_u \). Being a deformation of an object with cohomology sheaves in degrees contained in the interval \([-1,0]\), it is perfect with cohomology in degrees contained in \([-1,0]\). So, \( H^0(\mathcal{F}_u) \) is perfect, hence flat over the dual numbers; using the observation of the previous sentence again, we get that \( H^{-1}(\mathcal{F}_u) \) is flat over the dual numbers. Therefore,

\[
i^*(H^0(\mathcal{F}_u)) ≅ (H^0(i^*\mathcal{F}_u)) ≅ O_{Δ^M}.
\]

Similarly, \( i^*(H^{-1}(\mathcal{F}_u)) ≅ E \). The (twisted) sheaves \( O_{Δ^M} \) and \( E \) are infinitesimally rigid (see Lemma 5.2). This implies that \( H^0(\mathcal{F}_u) ≅ O_{Δ^M_ξ} \) and \( H^{-1}(\mathcal{F}_u) ≅ O_{ξ} \), where \( O_{ξ} \) is the restriction of \( E \) to \( M^2_ξ \) [MM2, Lemma 4.11]. It follows that the object \( \mathcal{F}_u \) is an extension of \( O_{Δ^M_ξ} \) by \( O_{ξ}[1] \). Step 3 of Proposition 6.3 says that, up to scalars, there is a unique such extension. The claimed isomorphism follows from this.

Lemma 7.14. The functor \( Ψ_{Ψ_1} : D^b(M, w) → D^b(X, u) \) is right adjoint to the functor \( Φ_{Ψ_1} : D^b(X, u) → D^b(M_H(v), w) \). Moreover, the unit \( η : 1_{D^b(X, u)} → Ψ_{Ψ_1} Φ_{Ψ_1} \) of this adjunction is split, that is, there exists a natural transformation \( ζ : Ψ_{Ψ_1} Φ_{Ψ_1} → 1_{D^b(X, u)} \) such that \( ζ η ≅ 1_{D^b(X, u)} \).

Proof. The exact sequence \( 0 → C → C[ε]/(ε^2) → C → 0 \) yields the triangle

\[
i_∗i^∗\mathcal{F}_1 ∪ \mathcal{O}_1 → \mathcal{F}_1 ∪ \mathcal{O}_1 → i_∗i^∗\mathcal{F}_1 ∪ \mathcal{O}_1
\]

where \( \mathcal{F}_1 ∪ \mathcal{O}_1 ∈ D^b(X × X, -\hat{u} ⊕ u) \) is the convolution of \( \mathcal{F}_1 \) with \( \mathcal{O}_1 \). We have the isomorphism \( i_∗i^∗\mathcal{F}_1 ∪ \mathcal{O}_1 ≅ i_*i^*\mathcal{F}_1 ∪ \mathcal{O}_1 \) by base-change [MSM, Lemma A.5].

Write \( \mathcal{O}_{Δ^X} ∈ D^b(X × X, -\hat{u} ⊕ u) \) for the structure sheaf of the diagonal. This is defined above when the gerby part \( θ ∈ H^2(\mathcal{O}_X) \) of \( u \) is 0. In general, set \( \mathcal{O}_{Δ^X} = Δ^X \mathcal{O}_X ^ν \), with the \( \mathcal{O}_X ^ν \)-module structure defined in Example 7.7. This makes sense as a \(( -θ ⊕ θ) \)-twisted sheaf because the class \(( -θ ⊕ θ) \) restricts to the trivial class along the diagonal in \( S × S \).

---

This content is based on the work of [Lieb1, Lieb2], and the original paper referenced may need to be consulted for the full context and details. The provided text is a representative excerpt intended to convey the essence of the mathematical arguments and results presented.
Consider the following sequence arising from applying the functor \( \text{Hom}_{D^b(X \times X, - \oplus u)}(\mathcal{O}_{\Sigma X}, \cdot) \) to (7.9):

\[
0 \to \text{Hom}_{X^2}(\mathcal{O}_{\Sigma X}, \mathcal{V} \circ \mathcal{U}) \to \text{Hom}_{D^b(X^2, - \oplus u)}(\mathcal{O}_{\Sigma X}, \mathcal{V} \circ \mathcal{U})_1 \to \text{Hom}_{X^2}(\mathcal{O}_{\Sigma X}, \mathcal{V} \circ \mathcal{U}) \to 0
\]

As \( \mathcal{V} \circ \mathcal{U} \cong \bigoplus_{i=0}^{n-1} \mathcal{O}_{\Sigma X}[-2i] \), we see that

\[
\text{Hom}_{X^2}(\mathcal{O}_{\Sigma X}, \mathcal{V} \circ \mathcal{U}) \cong \mathbb{C}, \quad \text{Hom}_{X^2}(\mathcal{O}_{\Sigma X}, \mathcal{V} \circ \mathcal{U}[1]) \cong HH^1(X) = 0,
\]

from which it follows that the sequence above is exact. Let \( \eta \) be the isomorphism. As this isomorphism is bi-functorial in \( \mathcal{A}, \mathcal{B} \), by Proposition 1.1 of [Ha-RD] that the composition of the arrows in the central column is a lift of the unit \( \eta \). We claim that \( \eta_1 : \mathcal{O}_{\Sigma X} \to \mathcal{V}_1 \circ \mathcal{U}_1 \) be a lift of the unit \( \eta \). The first and third columns can be identified with the composition

\[
\text{Hom}(\Phi_{\mathcal{W}}, \mathcal{A}, i_\ast \mathcal{B}) \to \text{Hom}(\mathcal{V}_1 \circ \mathcal{U}_1, \mathcal{B}) \to \text{Hom}(\mathcal{A}, i_\ast \mathcal{B})
\]

by flat base-change and the adjunction \( i_\ast \dashv i_\ast \), which is clearly an isomorphism. It now follows by Proposition 1.1 of [Ha-RD] that the composition of the arrows in the central column is an isomorphism. As this isomorphism is bi-functorial in \( \mathcal{A} \) and \( \mathcal{B} \), the claim is proved.

Let \( \mathcal{H}_1 \) be the cone of the map \( \eta_1 \):

\[
\mathcal{O}_{\Sigma X} \xrightarrow{\eta_1} \mathcal{V}_1 \circ \mathcal{U}_1 \to \mathcal{H}_1.
\]

As above, we can compute the extension group \( \text{Hom}_{D^b(X^2, - \oplus u)}(\mathcal{H}_1, \mathcal{O}_{\Sigma X}[1]) \) by applying the functor \( \text{Hom}_{D^b(X^2, - \oplus u)}(\mathcal{H}_1, \cdot) \) to the triangle \( i_\ast \mathcal{O}_{\Sigma X} \to \mathcal{O}_{\Sigma X} \to i_\ast \mathcal{O}_{\Sigma X} \) and chasing the resulting long exact sequence. Using the fact that \( \text{Hom}_{X^2}(\mathcal{H}_1, \mathcal{O}_{\Sigma X}[1]) \subset HH^{\text{odd}}(X) = 0 \), the result is that this group is \( (0) \). In particular, triangle (7.10) is split, from which the second statement of the lemma follows immediately.

Fix \( u \in (I_X^2)^{-1}(\text{ann}(v')) \), and set \( w = \phi^T(u) \in HT^2(M) \); the class \( w \) has the form \((0, \xi, \theta)\) by Lemma 7.10. As above, let \( \mathcal{F}_1 \) the restriction of \( \mathcal{F} \) to \( \mathcal{M}_1 \). Note that \( \mathcal{F}_1 \), together with the structure maps \( \epsilon_1 \) and \( \delta_1 \), defines a comonad \( \langle l_1, \epsilon_1, \delta_1 \rangle \) on \( D^b(M, u) \) by Theorem 1.8.

**Theorem 7.15.** (Theorem 1.3 (2)) There is an exact equivalence of triangulated categories between \( D^b(X, u) \) and the category of comodules \( D^b(M, w)^{l_1} \).

**Proof.** Let \( \mathcal{W}_1 \) be the deformation of \( \mathcal{W} \) corresponding to the class \( -u \oplus w \) constructed in Lemma 7.11. Denote the comonad arising from the adjoint pair \( \Phi_{\mathcal{W}_1}^{-1} \approx \psi_{\mathcal{W}_1} \) by \( \langle L'_1, \epsilon'_1, \delta'_1 \rangle \).

The unit \( \eta : 1 \to \psi_{\mathcal{W}_1} \psi_{\mathcal{W}_1}^{-1} \) is split by the previous lemma. Therefore, the categories \( D^b(M, u)^{l_1} \) and \( D^b(X, u) \) are equivalent by the Barr-Beck Theorem for triangulated categories [E, MS, Ba2], which says that for a split adjunction the comparison functor is an equivalence (see the discussion preceding the statement of Proposition 1.4). To prove the result, it only remains to show that there is an isomorphism of comodons between \( l_1 \) and \( l'_1 \).
We have seen above that the kernel $\mathcal{F}_u$ of the functor $L'_1$ is isomorphic to $\mathcal{F}_\xi$ (Lemma 7.13); fix an isomorphism $\mu : \mathcal{F}_u \to \mathcal{F}_\xi$. Note that $\text{Hom}(\mathcal{F}_u, \mathcal{F}_\xi) \cong \mathbb{C}[\varepsilon]/(\varepsilon^2)$ by Lemma 6.7. Let $\mathcal{E}$ be the universal twisted sheaf. Step 3 of the proof of (6.4) shows that $\mathcal{O}'_{\Delta^\wedge, \mathcal{E}[1]}(\mathcal{O}_{\Delta^\wedge})$ vanishes over $U$. Applying the functor $\mathcal{R}[\mathcal{E}_{\Delta^\wedge}]$ to the exact triangle

$$\mathcal{E} \to \mathcal{F} \to \mathcal{O}_{\Delta^\wedge}$$

allows one to conclude that $\text{Hom}(\mathcal{F}_u, \mathcal{O}_{\Delta^\wedge}) \cong \text{Hom}(\mathcal{F}_\xi, \mathcal{O}_{\Delta^\wedge}) \cong \mathbb{C}[\varepsilon]/(\varepsilon^2)$. So we may modify $\mu$ by a scalar in order that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}_u & \xrightarrow{\epsilon'_1} & \mathcal{O}_{\Delta^\wedge} \\
\mu \downarrow & & \downarrow \\
\mathcal{F}_\xi & \xrightarrow{\epsilon_1} & \mathcal{O}_{\Delta^\wedge}
\end{array}$$

It only remains to check that the isomorphism $\mu$ is compatible with the comultiplication maps $\delta'_1$ and $\delta_1$, that is, the left square in the diagram below commutes. Use the equalities $(\epsilon'_1 \circ 1_{\mathcal{F}_u}) \circ \delta'_1 = 1_{\mathcal{F}_u}$ and $(\epsilon_1 \circ 1_{\mathcal{F}_\xi}) \circ \delta_1 = 1_{\mathcal{F}_\xi}$, and the commutativity of the right square to conclude that the composition $(\epsilon_1 \circ 1_{\mathcal{F}_\xi}) \circ (\mu \circ \mu) \circ \delta'_1 = \mu$.

$$\begin{array}{ccc}
\mathcal{F}_u & \xrightarrow{\delta'_1} & \mathcal{F}_u \circ \mathcal{F}_u \\
\mu \downarrow & & \downarrow \\
\mathcal{F}_\xi & \xrightarrow{\delta_1} & \mathcal{F}_\xi \circ \mathcal{F}_\xi
\end{array}$$

Thus, $(\mu \circ \mu) \circ \delta'_1 \circ \mu^{-1}$ is a section of $\epsilon_1 \circ 1_{\mathcal{F}_\xi}$. The section $\delta_1$ is unique as composition by $\epsilon_1 \circ 1_{\mathcal{F}_\xi}$ gives an isomorphism $\text{Hom}(\mathcal{F}_\xi, \mathcal{F}_\xi \circ \mathcal{F}_\xi) \cong \text{Hom}(\mathcal{F}_\xi, \mathcal{F}_\xi)$ by Assumption 7.2 and Lemma 6.7. We conclude the equality $\delta_1 = (\mu \circ \mu) \circ \delta'_1 \circ \mu^{-1}$, proving the commutativity of the left square. \qed

8. Variations of Hodge structures

Let $X$ be a $K3$ surface, $M := M_H(v)$ a moduli space of sheaves on $X$, $\mathfrak{A}$ the modular Azumaya algebra over $M \times M$, and assume that $(M, \eta, \mathfrak{A})$ belongs to the open set $U$ of $\mathfrak{M}_\lambda^\wedge$ in Theorem 1.8 for some marking $\eta$ (see Remark 6.17). The Hodge structure of the Mukai lattice of $X$ can be deformed, as we deform the category $D^b(M, \theta)^L$, of comodules for the counit $(L, \epsilon, \delta)$, along deformations of $(M, \mathfrak{A}, L, \epsilon, \delta)$. These deformations of the Hodge structure are defined in [Ma4] Theorem 1.10. The deformed Mukai lattice should be related to the Hochschild homology of the category $D^b(M, \theta)^L$, and the type (1, 1) sublattice should be related to the numerical lattice of the $K$-group of $D^b(M, \theta)^L$. The Mukai lattice of $M_H(v)$, as defined in [Ma4] Theorem 1.10, is that of $X$, by [Ma4] Theorem 1.14. As we deform the Mukai lattice of $M_H(v)$, the class $v$ remains of Hodge-type (1, 1), while its orthogonal complement remains isometric to $H^2(M, \mathbb{Z})$, by [Ma4] Theorem 1.10. In particular, the class $v$ spans the sublattice of integral classes of Hodge type (1, 1), for the Mukai lattice of a generic deformation of $M$. This agrees with Theorem 1.9 in the current paper, which suggests that the family of deformation we get is the complete family of deformations of $D^b(X)$, which preserve the Hodge type of the class $v$. Such deformations include deformations of $D^b(X)$,
which are equivalent to derived categories of coherent sheaves in non-commutative and gerby deformations of the $K3$ surface $X$ [10, MSM].

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Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003

E-mail address: markman@math.umass.edu

Facultad de Matemáticas, PUC Chile, Av. Vicuña Mackenna 4860, Santiago, Chile; Chennai Mathematical Institute, H1, SIPCOT IT Park, Siruseri, Kelambakkam 603103, India

E-mail address: smehrotra@mat.uc.cl