THE ANALYTIC LATTICE COHOMOLOGY OF ISOLATED SINGULARITIES

TAMÁS ÁGOSTON AND ANDRÁS NÉMETHI

ABSTRACT. We associate (under a minor assumption) to any analytic isolated singularity of dimension \( n \geq 2 \) the ‘analytic lattice cohomology’ \( \mathcal{H}^*_\text{an} = \oplus_{q \geq 0} \mathcal{H}^q_\text{an} \). Each \( \mathcal{H}^q_\text{an} \) is a graded \( \mathbb{Z}[U] \)-module. It is the extension to higher dimension of the ‘analytic lattice cohomology’ defined for a normal surface singularity with a rational homology sphere link. This latest one is the analytic analogue of the ‘topological lattice cohomology’ of the link of the normal surface singularity, which conjecturally is isomorphic to the Heegaard Floer cohomology of the link.

The definition uses a good resolution \( \tilde{X} \) of the singularity \((X, o)\). Then we prove the independence of the choice of the resolution, and we show that the Euler characteristic of \( \mathcal{H}^*_\text{an} \) is \( h^{n-1}(O_{\tilde{X}}) \). In the case of a hypersurface weighted homogeneous singularity we relate it to the Hodge spectral numbers of the first interval.

1. INTRODUCTION

1.1. In the classification of singular germs one can proceed in many different directions. The first level is the topological classification of the singularity links using topological (smooth) invariants. Then one continues with the much harder analytic classification with the help of different analytic invariants. In this process one usually uses sheaf cohomologies associated with different analytic sheaves. However, if we wish to keep certain deep interference with recent developments in topology then we might naturally ask:

what are the analytic analogs of the celebrated cohomology theories produced by the low-dimensional topology in the last decades (e.g. of the Heegaard Floer theory)?

The Heegaard Floer theory, defined by Ozsváth and Szabó, associates to any oriented compact 3-manifold a graded \( \mathbb{Z}[U] \)-module, see e.g. [36, 37, 38]. Its Euler characteristic is the Seiberg–Witten invariant of the link. It is equivalent with several other cohomology theories: the Monopole Floer Homology of Kronheimer and Mrowka, the Seiberg–Witten version of Floer homology presented by Marcolli and Wang, or Hutchings’ Embedded Contact Homology. They produce extremely strong results in low dimensional topology. Our task is to develop an analytic analogue.

1.1.1. The first bridge between the Heegaard Floer theory and the analytic theory of singularities is realized by the topological lattice cohomology \( \mathbb{H}^*_{top} = \oplus_{q \geq 0} \mathbb{H}^q_{top} \) introduced by the second author in [29]. It is associated with the link of a normal surface singularity (a special plumbed 3-manifold), whenever the link is a rational homology sphere. Each \( \mathbb{H}^q_{top} \) is a graded \( \mathbb{Z}[U] \)-module. An improvement of \( \mathbb{H}^0_{top} \) is a graded root, a special tree with \( \mathbb{Z} \)-graded vertices (where the edges correspond to the \( U \)-action). They were defined using a good resolution. Some of their key properties are the following:

(a) \( \mathbb{H}^*_{top} \) is independent of the choice of the resolution, it depends only on the link \( M \),
(b) \( \mathbb{H}^*_{top}(M) = \oplus_{\sigma \in \text{Spin}^c(M)} \mathbb{H}^*_{top}(M, \sigma) \),
(c) the Euler characteristic is the (normalized) Seiberg–Witten invariant indexed by \( \text{Spin}^c(M) \),
(d) \( \mathbb{H}^*_{top}(M) \) satisfies several exact sequences (analogues of the exact triangles of \( HF^+ \)) [14, 31].
In fact, we obtain more than the definition of the ‘lattice homotopy type’ of (finite cubical topological) spaces \( \oplus \{ S_n \} \). More precisely, one expects \( H^\text{top}_{\text{odd/even}} \cong \oplus_{q \text{ odd/even}} \mathbb{Z} \oplus \mathbb{Z}^q \) as graded \( \mathbb{Z}[U] \)-modules (up to a shift).

This conjecture has been affirmatively answered for a number of important families of singularities \( \{ \mathcal{S} \} \), including those links which are Seifert fibered three-manifolds. \( \{ \mathcal{S} \} \) provides a spectral sequence from the lattice cohomology to the \( HF \)-cohomology, whose degeneration is equivalent with the conjecture.

For several properties and applications in singularity theory see \( \{ \mathcal{S} \} \). For its connection with the classification of projective rational plane cuspidal curves (via superisolated surface singularities) see \( \{ \mathcal{S} \} \). Furthermore, by its construction and key properties, \( \mathbb{H}^\text{top}_{\text{an}} \) realizes several deep connections with analytic invariants of the germ as well (e.g. it provides sharp topological bounds for analytic invariants, see e.g. \( \{ \mathcal{S} \} \)).

**1.1.2.** Recently, in \( \{ \mathcal{S} \} \) we introduced their analytic analogues, the analytic lattice cohomology \( \mathbb{H}^\text{an}_{\text{top}} = \oplus_{q \geq 0} \mathbb{Z}^q \), associated with a normal surface singularity with a rational homology sphere link. It is constructed from analytic invariants of a good resolution, however it turns out that it is independent of the choice of the resolution. Formally it has a very similar structure as its topological analogue, e.g. the analogue of \( \{ \mathcal{S} \} \) is valid and each \( \mathbb{H}^\text{an}_{\text{top}}(X, \sigma) \) is a graded \( \mathbb{Z}[U] \)-module. The Euler characteristic of \( \mathbb{H}^\text{an}_{\text{top}}(X) \) is the equivariant geometric genus.

Additionally, we succeeded in constructing even a morphism of graded \( \mathbb{Z}[U] \)-modules \( \mathcal{F}^\text{an} : \mathbb{H}^\text{an}_{\text{top}} \to \mathbb{H}^\text{top}_{\text{an}} \). This is an isomorphism for some ‘nice’ analytic structures. In such cases we have the identity of the Euler characteristics as well, namely of the Seiberg–Witten invariant with the geometric genus. In fact, historically, this identity (The Seiberg–Witten Invariant Conjecture of the second author and Nicolaescu \( \{ \mathcal{S} \} \)), valid for ‘nice’ analytic structures) led to the discovery of \( \mathbb{H}^\text{an}_{\text{top}} \). However, if we fix a topological type, and we move the possible analytic structure supported on this topological type, then the analytic lattice cohomologies reflect the modification of the analytic structures, for several examples see \( \{ \mathcal{S} \} \).

**1.1.3.** What is very surprising is that \( \mathbb{H}^\text{an}_{\text{top}} \) can be extended to other dimensions too, to isolated singularities of dimension \( n \geq 2 \), but even to the case of curves. In this note we present this extension to the higher dimensional singular germs.

Again, in the definition we use a good resolution \( \tilde{X} \to X \) of the singular germ \( (X, o) \), with exceptional curve \( E \). In the definition the multivariable divisorial filtration associated with the irreducible components of \( E \) has a key role. We verify that the newly defined \( \mathbb{H}^\text{an}_{\text{top}} \) is independent of the resolution whenever \( h^{n-1}(\mathcal{E}) = 0 \), and its Euler characteristic is \( h^{n-1}(\mathcal{E}) \).

As an example, for isolated weighted homogeneous hypersurface singularities, by the Reduction Theorem \( \{ \mathcal{S} \} \) the lattice cohomology can be computed via the divisorial filtration of a unique exceptional divisor (the exceptional divisor of the weighted blow up). Since this \( Z \)-filtration can be identified with the corresponding Newton filtration (and the number of lattice points below the Newton diagram is \( p \), \( h^{n-1}(\mathcal{E}) \)), and the Hilbert function of the Newton filtration can be identified by the Hodge spectrum in the interval \( (0, 1) \), we get a characterization of \( \mathbb{H}^\text{an}_{\text{top}} \) by the Hodge spectrum in \( (0, 1) \).

**1.1.4.** In fact, we obtain more than the definition of the ‘lattice cohomology’. Indeed, we define a sequence of (finite cubical topological) spaces \( \{ S_n \} \) with inclusions \( \cdots \subset S_n \subset S_{n+1} \subset \cdots \) such that \( \mathbb{H}^\text{an}_{\text{top}} = \oplus_{n \geq 0} H^*(S_n, \mathbb{Z}) \), and the homotopy type of the tower of spaces depends only on the analytic type of \( (X, o) \). Therefore, in the spirit of the constructions of ‘Khovanov homotopy type’ of R. Lipshitz and S. Sarkar, or of ‘Knot Floer stable homotopy type’ of C. Manolescu and S. Sarkar, in fact we have constructed the ‘(analytic) lattice homotopy type’ of \( (X, o) \) via the tower \( \{ S_n \} \).
1.1.5. Note that in the case $n = 2$ the analytic lattice cohomology was defined as model the topological lattice cohomology (and it was motivated by the topological cohomologies of the low-dimensional topology).

The higher dimensional case has the interesting aspect that we define the analytic lattice cohomology without having any parallel topological model. In fact, the definition of the topological $H^{\ast}_{top}$ is obstructed very seriously, since in the dimensions $n > 2$ the link $M$ contains much less information from the singularity, e.g. $M$ can even be the standard sphere $S^{2n-1}$ for rather non-trivial analytic types $(X, o)$.

However, we expect the existence of a parallel theory in this higher dimensional case too: our expectation is that it should be the higher dimensional version of the Embedded Contact Homology (ECH), where the contact structure (induced by the analytic structure of $(X, o)$) on $M$ really plays a role. (Recall that in the $n = 2$ case this contact structure can topologically be identified [26], a fact which does not hold in higher dimensions [46].) Research in finding ECH in higher dimension was initiated by Colin–Honda [10].

1.2. The structure of the paper. In section 2 we review the general definition of the lattice cohomology, of the path lattice cohomology and of the graded root associated with a weight function. In section 3 we review some statements regarding the Euler characteristic of a lattice cohomology (in a combinatorial setup).

In section 4, after we review certain analytic results regarding singularities and resolutions, we define the analytic lattice cohomology and graded root (via a good resolution). In Theorem 4.3.1 we prove their independence of the resolution. Using results of section 3 we identify the Euler characteristic as well. Subsection 4.7 proves a ‘Reduction Theorem’. Using this we can reduce the rank of the lattice (used in the basic construction). This new lattice is identified by a set of ‘bad’ vertices.

In order to define the new objects, and also to prove their independence of the resolution, we need to impose an assumption, namely the vanishing of $h^{n-1}(\mathcal{E}_{E})$. In section 5 we relate this vanishing with some mixed Hodge theoretical invariants. E.g. in the case of isolated hypersurface singularities it is equivalent with the non-existence of spectral numbers equal to one. (Hence, if the link is rational homology sphere, then this condition is automatically satisfied.) In section 6 we discuss the case of weighted homogeneous hypersurface singularities.

For more examples in the case $n = 2$ see [1].

2. Preliminaries. Basic properties of lattice cohomology

2.1. The lattice cohomology associated with a weight function. [26, 29]

2.1.1. Weight function. We consider a free $\mathbb{Z}$-module, with a fixed basis $\{E_{\nu}\}_{\nu \in \nu'}$, denoted by $\mathbb{Z}^\nu$, $s := |\nu'|$. Additionally, we consider a weight function $w_0 : \mathbb{Z}^\nu \to \mathbb{Z}$ with the property

\[(2.1.2) \quad \text{for any integer } n \in \mathbb{Z}, \text{ the set } w_0^{-1}((-\infty, n]) \text{ is finite.}\]

2.1.3. The weighted cubes. The space $\mathbb{Z}^\nu \otimes \mathbb{R}$ has a natural cellular decomposition into cubes. The set of zero-dimensional cubes is provided by the lattice points $\mathbb{Z}^\nu$. Any $l \in \mathbb{Z}^\nu$ and subset $I \subset \nu'$ of cardinality $q$ defines a $q$-dimensional cube $\Box_q = (l, I)$, which has its vertices in the lattice points $(l + \sum_{\nu \in I'} E_{\nu})_{\nu'}$, where $I'$ runs over all subsets of $I$. The set of $q$-dimensional cubes is denoted by $\mathcal{D}_q \ (0 \leq q \leq s)$.

Using $w_0$ define $w_q : \mathcal{D}_q \to \mathbb{Z} \ (0 \leq q \leq s)$ by $w_q(\Box_q) := \max\{w_0(l) : l \text{ is a vertex of } \Box_q\}$.

For each $n \in \mathbb{Z}$ we define $S_n = S_n(w) \subset \mathbb{R}^\nu$ as the union of all the cubes $\Box_q$ (of any dimension) with $w(\Box_q) \leq n$. Clearly, $S_n = \emptyset$, whenever $n < m_0 := \min\{w_0\}$. For any $q \geq 0$, set

\[H^q(\mathbb{R}^\nu, w) := \oplus_{n \geq m_0} H^q(S_n, \mathbb{Z}) \quad \text{and} \quad H^q_{\text{red}}(\mathbb{R}^\nu, w) := \oplus_{n \geq m_0} H^q(S_n, \mathbb{Z}).\]

Then $H^q$ is $\mathbb{Z}$ (in fact, $2\mathbb{Z}$)-graded, the $2n$-homogeneous elements $H^q_{2n}$ consist of $H^q(S_n, \mathbb{Z})$. Also, $H^q$ is a $\mathbb{Z}[U]$-module; the $U$-action is given by the restriction map $r_{n+1} : H^q(S_{n+1}, \mathbb{Z}) \to H^q(S_n, \mathbb{Z})$. The same is true for $H^q_{\text{red}}$. Moreover, for $q = 0$, a fixed base-point $l_w \in S_{m_0}$ provides an augmentation (splitting)
2.2.1. Fix path lattice cohomology. (2.1.7) It turns out that (2.2.4) quadrant (gets the reduced path lattice cohomology q)

\[ \mathbb{H}^0 \simeq \mathcal{J}^+_{2m} \oplus \mathbb{H}^0_{\text{red}} = (\oplus_{n \geq m} \mathbb{Z}) \oplus (\oplus_{n \geq m} \mathcal{J}^0(S_n, \mathbb{Z})) \] and \[ \mathbb{H}^* \simeq \mathcal{J}^+_{2m} \oplus \mathbb{H}^*_{\text{red}}. \]

Though \[ \mathbb{H}^*_{\text{red}}(\mathbb{R}^4, w) \] has finite \( Z \)-rank in any fixed homogeneous degree, in general, without certain additional properties of \( w_0 \), it is not finitely generated over \( \mathbb{Z} \), in fact, not even over \( \mathbb{Z}[U] \).

2.1.4. Restrictions. Assume that \( T \subset \mathbb{R}^4 \) is a subspace of \( \mathbb{R}^4 \) consisting of a union of some cubes (from \( \mathcal{Q} \)). For any \( q \geq 0 \) define \( \mathbb{H}^q(T, w) = \oplus_{n \geq \min w_0} \mathbb{H}^q(S_n \cap T, Z) \). It has a natural graded \( \mathbb{Z}[U] \)-module structure. The restriction map induces a natural graded \( \mathbb{Z}[U] \)-module homogeneous homomorphism \[ r^* : \mathbb{H}^*(\mathbb{R}^4, w) \to \mathbb{H}^*(T, w) \] (of degree zero).

In our applications to follow, \( T \) (besides the trivial \( T = \mathbb{R}^4 \) case) will be one of the following: (i) the first quadrant \( (\mathbb{R}_{>0})^4 \), (ii) the rectangle \( [0, c] = \{ x \in \mathbb{R}^4 : 0 \leq x \leq c \} \) for some lattice point \( c \geq 0 \), or (iii) a path of composed edges in the lattice, cf. 2.2.

2.1.5. The ‘Euler characteristic’ of \( \mathbb{H}^* \). Fix \( T \) as in 2.1.4 and we will assume that each \( \mathbb{H}^*_{\text{red}}(T, w) \) has finite \( Z \)-rank. The Euler characteristic of \( \mathbb{H}^*_{\text{red}}(T, w) \) is defined as \[ \text{eu}(\mathbb{H}^*_{\text{red}}(T, w)) := -\min \{ w(l) : l \in T \cap \mathbb{Z}^4 \} + \sum_q (-1)^q \text{rank}_Z(\mathbb{H}^q_{\text{red}}(T, w)). \]

Lemma 2.1.6. [30] If \( T = [0, c] \) for a lattice point \( c \geq 0 \), then

\[ \sum_{\square_q \subset T} (-1)^{q+1}w_\square(\square_q) = \text{eu}(\mathbb{H}^*(T, w)). \]

2.2. Path lattice cohomology. [29]

2.2.1. Fix \( Z' \) as in 2.1 and fix also a compatible weight functions \( \{ w_q \}_q \) as in 2.1.2. Consider also a sequence \( \gamma := \{ x_i \}_{i=0}^t \) so that \( x_0 = 0, x_i \neq x_j \) for \( i \neq j \), and \( x_{i+1} = x_i + E_{\gamma(i)} \) for \( 0 \leq i < t \). We write \( T \) for the union of 0-cubes marked by the points \( \{ x_i \} \), and of the segments of type \( \{ x_i, x_{i+1} \} \). Then, by 2.1.4 we get a graded \( \mathbb{Z}[U] \)-module \( \mathbb{H}^*(T, w) \), which is called the path lattice cohomology associated with the ‘path’ \( \gamma \) and weights \( \{ w_q \}_{q=0}^1 \). It is denoted by \( \mathbb{H}^*(\gamma, w) \). It has an augmentation with \( \mathcal{J}^+_{2m_{\gamma}} \), where \( m_{\gamma} := \min \{ w_0(x_i) \} \), and one gets the reduced path lattice cohomology \( \mathbb{H}^0_{\text{red}}(\gamma, w) \) with

\[ \mathbb{H}^0(\gamma, w) \simeq \mathcal{J}^+_{2m_{\gamma}} \oplus \mathbb{H}^0_{\text{red}}(\gamma, w). \]

It turns out that \( \mathbb{H}^q(\gamma, w) = 0 \) for \( q \geq 1 \), hence its ‘Euler characteristic’ can be defined as (cf. 2.1.5)

\[ \text{eu}(\mathbb{H}^*(\gamma, w)) := -m_{\gamma} + \text{rank}_Z(\mathbb{H}^0_{\text{red}}(\gamma, w)). \]

Lemma 2.2.3. One has the following expression of \( \text{eu}(\mathbb{H}^*(\gamma, w)) \) in terms of the values of \( w \):

\[ \text{eu}(\mathbb{H}^*(\gamma, w)) = -w_0(0) + \sum_{i=0}^{t-1} \max \{ 0, w_0(x_i) - w_0(x_{i+1}) \}. \]

2.3. Graded roots and their cohomologies. [26, 28]

Definition 2.3.1. Let \( \mathcal{R} \) be an infinite tree with vertices \( \gamma \) and edges \( \delta \). We denote by \( [u, v] \) the edge with end-vertices \( u \) and \( v \). We say that \( \mathcal{R} \) is a graded root with grading \( \tau : \gamma \to \mathbb{Z} \) if

(a) \( \tau(u) - \tau(v) = \pm 1 \) for any \( [u, v] \in \delta \);

(b) \( \tau(u) > \min \{ \tau(v), \tau(w) \} \) for any \( [u, v], [u, w] \in \delta, v \neq w \);

(c) \( \tau \) is bounded from below, \( \tau^{-1}(n) \) is finite for any \( n \in \mathbb{Z} \), and \( |\tau^{-1}(n)| = 1 \) if \( n \gg 0 \).

An isomorphism of graded roots is a graph isomorphism, which preserves the gradings.
Definition 2.3.2. The $\mathbb{Z}[U]$-modules associated with a graded root. Let us identify a graded root $(\mathcal{R}, \tau)$ with its topological realization provided by vertices ($0$-cubes) and segments ($1$-cubes). Define $w_0(v) = \tau(v)$, and $w_1([u, v]) = \max\{\tau(u), \tau(v)\}$ and let $S_n$ be the union of all cubes with weight $\leq n$. Then we might set (as above) $H^i(\mathcal{R}, \mathcal{X}) = \oplus_{n \geq \text{max}} H^i(S_n, \mathcal{Z})$. However, at this time $H^2(\mathcal{R}, \mathcal{R}) = 0$; we set $H(\mathcal{R}, \mathcal{R}) := H^0(\mathcal{R}, \mathcal{R})$. Similarly, one defines $H_{\text{red}}(\mathcal{R}, \mathcal{R})$ using the reduced cohomology, hence $H(\mathcal{R}, \mathcal{R}) \simeq \mathcal{Z} \oplus H_{\text{red}}(\mathcal{R}, \mathcal{R})$.

2.3.3. The graded root associated with a weight function. Fix a $\mathbb{Z}$-module and a weight function $w_0$. Consider the sequence of topological spaces (finite cubical complexes) $\{S_n\}_{n \geq m_w}$ with $S_n \subset S_{n+1}$, cf. 2.1.3. Let $\pi_0(S_n) = \{\xi^0_n, \ldots, \xi^m_n\}$ be the set of connected components of $S_n$.

Then we define the graded graph $(\mathcal{R}_{w}, \tau_w)$ as follows. The vertex set $\mathcal{V}(\mathcal{R}_{w})$ is $\cup_{n \in \mathbb{Z}} \pi_0(S_n)$. The grading $\tau_w : \mathcal{V}(\mathcal{R}_{w}) \rightarrow \mathbb{Z}$ is $\tau_w(\xi^0_n) = n$, that is, $\tau_w|_{\pi_0(S_n)} = n$. Furthermore, if $\xi^i_n \subset \xi^{i+1}_n$, for some $i$, then we introduce an edge $[\xi^i_n, \xi^{i+1}_n]$. All the edges of $\mathcal{R}_{w}$ are obtained in this way.

One verifies that $(\mathcal{R}_{w}, \tau_w)$ satisfies all the required properties of the definition of a graded root, except possibly the last one: $|\tau_w^{-1}(n)| = 1$ whenever $n \gg 0$.

The property $|\tau_w^{-1}(n)| = 1$ for $n \gg 0$ is not always satisfied. However, the graded roots associated with connected definite negative plumbing graphs (see below) satisfy this condition as well.

Proposition 2.3.4. If $\mathcal{R}$ is a graded root associated with $(T, w)$ and $|\tau_w^{-1}(n)| = 1$ for all $n \gg 0$ then $H(\mathcal{R}) = H^0(T, w)$.

3. COMBINATORIAL LATTICE COHOMOLOGY

3.1. In this section we review several combinatorial statements regarding the lattice cohomology associated with any weight function with certain combinatorial properties. We follow [1].

3.1.1. Fix $\mathbb{Z}^s$ with a fixed basis $\{E_v\}_{v \in \mathcal{V}}$. Write $E_l = \sum_{v \in l} E_v$ for $l \subset \mathcal{V}$ and $E = E_{\mathcal{V}}$. Fix also an element $c \in \mathbb{Z}^s$, $c \geq E$. Consider the lattice points $R = R(0, c) := \{l \in \mathbb{Z}^s : 0 \leq l \leq c\}$, and assume that to each $l \in R$ we assign

(i) an integer $\gamma(l)$ such that $h(0) = 0$ and $h(l) \geq h(l)$ for any $v$,

(ii) an integer $\gamma^c(l)$ such that $h^c(l + E_v) \leq h^c(l)$ for any $v$.

Once $h$ is fixed with (i), a possible choice for $h^c$ is $h^{ym}$, where $h^{ym}(l) = h(c - l)$. Clearly, it depends on $c$.

3.1.2. We say that the $h$-function satisfies the ‘matroid rank inequality’ if

\[
h(l_1) + h(l_2) \geq h(\min\{l_1, l_2\}) + h(\max\{l_1, l_2\}), \quad l_1, l_2 \in R.
\]

This implies the ‘stability property’, valid for any $\bar{l} \geq 0$ with $\bar{l} \notin E_v$, namely

\[
h(l) = h(l + E_v) \Rightarrow h(l + \bar{l}) = h(l + E_v).
\]

If $h$ is given by a filtration (see below) then it automatically satisfies the matroid rank inequality.

3.1.5. We consider the set of cubes $\{Q_q\}_{q \geq 0}$ of $R$ as in 2.1.3 and the weight function

\[
w_0 : Q_0 \rightarrow \mathbb{Z} \text{ by } w_0(l) := h(l) + h^c(l) - h^0(0).
\]

Clearly $w_0(0) = 0$. Furthermore, we define $w_q : Q_q \rightarrow \mathbb{Z}$ by $w_q(\square_q) = \max\{w_0(l) : l \text{ is a vertex of } \square_q\}$. We will use the symbol $w$ for the system $\{w_q\}_q$. It defines the lattice cohomology $H^*(R, w)$. Moreover, for any increasing path $\gamma$ connecting $0$ and $c$ we also have a path lattice cohomology $H^0(\gamma, w)$ as in 2.2.1. Accordingly, we have the numerical Euler characteristics $eu(\mathbb{H}^n(R, w)), eu(\mathbb{H}^0(\gamma, w))$ and $\min_{\gamma} eu(\mathbb{H}^0(\gamma, w))$.

Lemma 3.1.6. [1] We have $0 \leq eu(\mathbb{H}^0(\gamma, w)) \leq h^0(0) - h^c(c)$ for any increasing path $\gamma$ connecting $0$ to $c$. The equality $eu(\mathbb{H}^0(\gamma, w)) = h^0(0) - h^c(c)$ holds if and only if for any $i$ the differences $h(x_i + 1) - h(x_i)$ and $h^c(x_i) - h^c(x_{i+1})$ simultaneously are not nonzero.
**Definition 3.1.7.** Fix \( (h, h^p, R) \) as in 3.1.1. We say that the pair \( h \) and \( h^p \) satisfy the ‘Combinatorial Duality Property’ (CDP) if \( h(l + E_v) - h(l) \) and \( h^p(l + E_v) - h^p(l) \) simultaneously cannot be nonzero for \( l, l + E_v \in R \). Furthermore, we say that \( h \) satisfies the CDP if the pair \( (h, h^{p_{\text{sym}}}) \) satisfies it.

**Definition 3.1.8.** We say that the pair \( (h, h^p) \) satisfy the

(a) ‘path eu-coincidence’ if \( ec(\overline{\mathbb{H}}^p(\gamma, w)) = h^p(0) - h^p(c) \) for any increasing path \( \gamma \).

(b) ‘eu-coincidence’ if \( ec(\mathbb{H}^p(R, w)) = h^p(0) - h^p(c) \).

**Remark 3.1.9.** Example 4.3.3 of 11 shows the following two facts.

Even if \( h \) satisfies the path eu-coincidence (and \( h^p = h^{p_{\text{sym}}} \)), in general it is not true that \( \overline{\mathbb{H}}^p(\gamma, w) \) is independent of the choice of the increasing path. (This statement remains valid even if we consider only the symmetric increasing paths, where a path \( \gamma = \{ s_i \}_{i=0}^t \) is symmetric if \( x_{i+1} - x_i = x_i - x_{i-1} \) for any \( i \).

Even if \( h \) satisfies both the path eu-coincidence and the eu-coincidence, in general it is not true that \( \mathbb{H}^p(R, w) \) equals any of the path lattice cohomologies \( \overline{\mathbb{H}}^p(\gamma, w) \) associated with a certain increasing path. (E.g., in the mentioned Example 4.3.3 we have \( \overline{\mathbb{H}}^1(R, w) \neq 0 \), a fact which does not hold for any path lattice cohomology.) However, amazingly, all the Euler characteristics agree.

**Theorem 3.1.10.** Assume that \( h \) satisfies the stability property, and the pair \( (h, h^p) \) satisfies the Combinatorial Duality Property. Then the following facts hold.

(a) \( (h, h^p) \) satisfies both the path eu- and the eu-coincidence properties: for any increasing \( \gamma \) we have

\[
eu(\overline{\mathbb{H}}^p(\gamma, w)) = eu(\mathbb{H}^p(R, w)) = h^p(0) - h^p(c).
\]

(b) \[
\sum_{l \geq 0} \sum_{I} (-1)^{|I|+1} w(I, \gamma) t^I = \sum_{l \geq 0} \sum_{I} (-1)^{|I|+1} h(l + E_I) t^I.
\]

4. Analytic lattice cohomology of isolated singularities

4.1. Some analytic properties of isolated singularities.

4.1.1. Let \( (X, o) \) be an irreducible isolated singularity of dimension \( n \geq 2 \). Usually we fix a (small) representative \( X \) such that it is a contractible Stein space. We fix a good resolution \( \phi : \tilde{X} \to X \). Set \( E = \phi^{-1}(o) \) for the irreducible set, let \( E = \cup_{v \in \mathcal{E}} E_v \) be its irreducible decomposition.

**Theorem 4.1.2.** 15 (Grauert–Riemenschneider Theorem) \( R^i \phi \ast \Omega^n_X = 0 \) for \( i > 0 \).

If \( N \subseteq \tilde{X} \) is a (conveniently small) strictly Levi pseudoconvex neighborhood of \( E \) then \( H^{n-1}(N, \mathcal{O}_N) \) is finite dimensional by 15 Th. IX.B.6]. Furthermore, the restriction \( H^{n-1}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^{n-1}(N, \mathcal{O}_N) \) is an isomorphism 23 Lemma 3.1. In particular, \( H^{n-1}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \) is finite dimensional, and we can assume that \( \tilde{X} \) is a strictly Levi pseudoconvex neighborhood of \( E \) (as \( N \) above).

**Theorem 4.1.3.** 23 50 \( R^{n-1} \phi \ast (\mathcal{O}_{\tilde{X}})_0 \simeq H^{n-1}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \) is dual as a \( \mathbb{C} \)-vector space with \( H^0(\tilde{X} \setminus E, \Omega^n_X)/H^0(\tilde{X}, \Omega^n_X) \).

We write \( K_\tilde{X} \) for the canonical divisor, that is, \( \Omega^n_\tilde{X} \simeq \mathcal{O}_{\tilde{X}}(K_\tilde{X}) \).

We set \( L = H_{2n-2}(\tilde{X}, \mathbb{Z}) = H_{2n-2}(E, \mathbb{Z}) \). It is a free \( \mathbb{Z} \)-module generated by the classes of \( \{ E_v \}_v \). We identify it with the group of Weil divisors supported on \( E \), hence any \( l \in L \) has the form \( l = \sum n_v E_v \) with \( n_v \in \mathbb{Z} \). We write \( l \in L_{\geq 0} \) if \( l \) is effective (\( n_v \geq 0 \) for all \( v \)), and \( l \in L_{>0} \) if \( l \) is non-zero effective.

**Theorem 4.1.4.** One has the Serre Duality isomorphism: \( H^0(l, \mathcal{O}_{\tilde{X}}(K_\tilde{X} + l)) \rangle = H^{n-1}(\mathcal{O}_E)_0^\ast \) for any \( l \in L_{>0} \).
If $c \in L_{>0}$ with $c \gg 0$ (i.e., $n_v \gg 0$ for all $v$), then $H^i(c, \mathcal{O}_c) \simeq H^i(\bar{X}, \mathcal{O}_\bar{X})$ for $i > 0$ by Formal Function Theorem [17]. Similarly, for $c \gg 0$ we also have $H^0(\bar{X}, \Omega^n_{\bar{X}}(c)) = H^0(\bar{X} \setminus E, \Omega^n_{\bar{X}})$, and

\begin{equation}
H^{n-1}(\bar{X}, \mathcal{O}_\bar{X})^* \simeq H^0(\bar{X}, \Omega^n_{\bar{X}}(c))/H^0(\bar{X}, \Omega^n_{\bar{X}}).
\end{equation}

More generally, for any $l > 0$ we have

\begin{equation}
H^{n-1}(l, \mathcal{O}_l)^* \simeq H^0(\bar{X}, \Omega^n_{\bar{X}}(l))/H^0(\bar{X}, \Omega^n_{\bar{X}}).
\end{equation}

Indeed, using the exact sequence of sheaves $0 \to \Omega^n_X \to \Omega^n_X(l) \to \Omega^n_X(l)|_l \to 0$ and the Grauert–Riemenschneider vanishing we obtain that $H^0(\bar{X}, \Omega^n_{\bar{X}}(l))/H^0(\bar{X}, \Omega^n_{\bar{X}}) = H^0(l, \Omega^n_{\bar{X}}(l))$, which is Serre dual with $H^{n-1}(\mathcal{O}_l)$.

Next, we define the Hilbert function associated with the divisorial filtration of $\mathcal{O}_{X,o}$ (or of $H^0(\bar{X}, \mathcal{O}_{\bar{X}})$): for any $l \in L_{\geq 0}$ set

\begin{equation}
h(l) = \dim \frac{H^0(\bar{X}, \mathcal{O}_{\bar{X}})}{H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-l))}.
\end{equation}

Then $h$ is increasing (that is, $h(l_1) \geq h(l_2)$ whenever $l_1 \geq l_2$ and $h(0) = 0$.

We also define another numerical invariant for any $l \geq 0$, namely

\begin{equation}
h^\circ(l) = \dim \frac{H^0(\bar{X} \setminus E, \Omega^n_{\bar{X}})}{H^0(\bar{X}, \Omega^n_{\bar{X}}(l))}.
\end{equation}

Then $h^\circ$ is decreasing, $h^\circ(0) = h^{n-1}(\mathcal{O}_{\bar{X}})$ and $h^\circ(c) = 0$ for $c \gg 0$.

Since $h$ is induced by a filtration, it satisfies the matroid rank inequality

\begin{equation}
h(l_1) + h(l_2) \geq h(\bar{l}) + h(l),
\end{equation}

where $l = \min\{l_1, l_2\}$ and $\bar{l} = \max\{l_1, l_2\}$. By the very same reason $l \mapsto h^\circ(-l)$ (hence $l \mapsto h^\circ(l)$ too) satisfies the matroid rank inequality

\begin{equation}
h^\circ(l_1) + h^\circ(l_2) \geq h^\circ(\bar{l}) + h^\circ(l).
\end{equation}

From (4.1.6) we obtain that

\begin{equation}
h^\circ(l) = h^{n-1}(\mathcal{O}_{\bar{X}}) - h^{n-1}(\mathcal{O}_l).
\end{equation}

This shows that $l \mapsto h^{n-1}(\mathcal{O}_l)$ satisfies the ‘opposite’ matroid rank inequality

\begin{equation}
h^{n-1}(\mathcal{O}_l_1) + h^{n-1}(\mathcal{O}_l_2) \leq h^{n-1}(\mathcal{O}_\bar{l}) + h^{n-1}(\mathcal{O}_l).
\end{equation}

Recall that $l \mapsto h^{n-1}(\mathcal{O}_l)$ is increasing and its stabilized value (for $l \gg 0$) is $h^{n-1}(\mathcal{O}_{\bar{X}})$.

**Proposition 4.1.13. (Existence of the cohomology cycle)** Assume that $h^{n-1}(\mathcal{O}_{\bar{X}}) \neq 0$. Then there exists a unique minimal cycle $Z_{coh} > 0$ such that $h^{n-1}(\mathcal{O}_{\bar{X}}) = h^{n-1}(\mathcal{O}_{\bar{X}})$. The cycle $Z_{coh}$ has the property that for any $l \not\geq Z_{coh}$ one has $h^{n-1}(\mathcal{O}_l) < h^{n-1}(\mathcal{O}_{\bar{X}})$.

**Proof.** Use (4.1.12). In fact, the proof of the existence of the cohomology cycle for surface singularities from [40] can also be adapted (which proves the opposite matroid inequality as well). \hfill \square

If $h^{n-1}(\mathcal{O}_{\bar{X}}) = 0$ then we define $Z_{coh}$ as the zero cycle.

**Corollary 4.1.14.** For any $l > 0$ one has $h^{n-1}(\mathcal{O}_l) = h^{n-1}(\mathcal{O}_{\min(l, Z_{coh})})$.

**Proof.** Use the monotonicity of $h^{n-1}(\mathcal{O}_l)$ and the opposite matroid inequality for $l$ and $Z_{coh}$. \hfill \square

4.1.15. It is well-known that both $h^{n-1}(\mathcal{O}_{\bar{X}})$ and $h^{n-1}(\mathcal{O}_E)$ are independent of the choice of the resolution, they depend only on $(X, o)$. Moreover, the natural map $H^{n-1}(\mathcal{O}_{\bar{X}}) \to H^{n-1}(\mathcal{O}_E)$ is surjective [45] (2.14).
Example 4.1.6. Assume that \((X, o)\) is Gorenstein. Then, for any good resolution \(\tilde{X} \to X\) there exists \(Z_K \in L\) such that \(\Omega^e_X = \mathcal{O}_{\tilde{X}}(-Z_K)\). Let us write \(Z_K\) as \(Z_{K,+} + Z_{K,-}\), where \(Z_{K,+}, -Z_{K,-} \in L_{\geq 0}\), and in their support there is no common \(E_v\). E.g., if \((X, o)\) is rational then \(Z_{K,+} = 0\). Ishii in [19, 3.7] proved that \(Z_{K,+} \geq Z_{coh}\).

A good resolution is called ‘essential’ if \(Z_{K,-} = 0\). For surface singularities essential good resolutions exist (e.g. the minimal good resolution is such). However, in higher dimensions there are singularities without any essential good resolutions.

Recall that in general \(h^o(l) = \dim H^0(\Omega^e_X(c)/H^0(\tilde{X}, \Omega^n_X(l)))\), valid for any \(c \geq Z_{coh}\). Now, in the Gorenstein case, if \(\tilde{X}\) is an essential good resolution (i.e. \(Z_K \in L_{\geq 0}\)), then the previous expression for \(h^o\) transforms for \(c = Z_K\) into \(h^o(l) = \dim H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-Z_K + l))\) for \(0 \leq l \leq Z_K\). Hence \(h^o(l) = h(Z_K - l)\). That is, \(h^o\) is the symmetrized \(h\) with respect to \(Z_K = 0\).

4.2. The analytic lattice cohomology associated with \(\phi\). Let us fix some \(c \geq Z_{coh}\) and we consider the rectangle \(R(0, c)\). We also define the weight function on the lattice points of \(R(0, c)\) by

\[
w_0(l) := h(l) + h^o(l) - h^o(0) = h(l) - h^{n-1}(\mathcal{O}_l).
\]

By the above discussions we obtain that \(w_0\) satisfies the matroid rank inequality.

Lemma 4.2.1. Consider the case \(c = \infty\) and \(w_0 : L_{\geq 0} \to \mathbb{Z}\) defined as in 4.2. Then \(w_0\) satisfies (2.1.2), namely \(w_0^{-1}((-\infty, n])\) is finite for any \(n \in \mathbb{Z}\).

Proof. Assume the opposite. Then there exists an infinite sequence of cycles \(\{l_i\}_{i \geq 1}\) such that \(h(l_i) \leq n\) for any \(i\), and for a certain \(v \in V\) the \(v\)-coordinates \(\{l_{i,v}\}_{i}\) tend to infinity. Then, choose another sequence \(\{\tilde{l}_i\}\) with \(\tilde{l}_{i,v} = l_{i,v}\) but all the other coordinates are bounded. For this again \(h(\tilde{l}_i) \leq n\). Then \(\{\tilde{l}_i\}\) admits an increasing subsequence \(\{x_j\}\) such that \(\lim_{j \to \infty} x_j = \infty\) and the sequence \(\{x_{j,v}\}_{j}\) is constant for any other \(w \neq v\). Since \(h(x_j) \leq n\), the sequence of ideal \(H^0(\mathcal{O}_{\tilde{X}}(-x_j))\) must stabilize for \(j\) large. Let us choose some \(f\) from this stabilized vector space, and let \(m_v\) be its multiplicity along \(E_v\). Then for any \(j\) sufficiently large \(x_{j,v} > m_v\), hence \(f \notin H^0(\mathcal{O}_{\tilde{X}}(-x_j))\), which is a contradiction. □

Furthermore, we define \(w_q : \mathbb{Q}_q \to \mathbb{Z}\) by \(w_q(\square_q) = \max\{w_0(l) : l\) is any vertex of \(\square_q\}\). In the sequel we write \(w\) for the system \(\{w_q\}_q\) if there is no confusion.

The compatible weight functions \(\{w_q\}_q\) for any \(c \geq Z_{coh}\) (finite or infinite) define the lattice cohomology \(\mathbb{H}^\ast(R(0, c), w)\) and a graded root \(\mathcal{R}(R(0, c), w)\).

Lemma 4.2.2. \(\mathbb{H}^\ast(R(0, c), w)\) and \(\mathcal{R}(R(0, c), w)\) are independent of the choice of \(c\) \((Z_{coh} \leq c \leq \infty)\).

Proof. Fix some \(c \geq Z_{coh}\) and choose \(E_v\) in the support of \(c - Z_{coh}\). Then for any \(l \in R(0, c)\) with \(l_v = c_v\) we have \(\min\{l, Z_{coh}\} = \min\{l - E_v, Z_{coh}\}\). Therefore, by Corollary 4.1.4 \(h^{n-1}(\mathcal{O}_{l - E_v}) = h^{n-1}(\mathcal{O}_l)\), thus \(w_0(l - E_v) \leq w_0(l)\). Then for any \(n \in \mathbb{Z}\), a strong deformation retraction in the direction \(E_v\) realizes a homotopy equivalence between the spaces \(S_n \cap R(0, c)\) and \(S_n \cap R(0, c - E_v)\). A natural retraction \(r : S_n \cap R(0, c) \to S_n \cap R(0, c - E_v)\) can be defined as follows (for notation see 2.1.3). If \(\square = (l, I)\) belongs to \(S_n \cap R(0, c - E_v)\) then \(r\) on \(\square\) is defined as the identity. If \((l, I) \cap R(0, c - E_v) = \emptyset\), then \(l_v = c_v\), and we set \(r(x) = x - E_v\). Else, \(\square = (l, I)\) satisfies \(v \in I\) and \(l_v = c_v - 1\). Then we retract \((l, I)\) to \((l, I \setminus v)\) in the \(v\)-direction. The strong deformation retraction is defined similarly. □

Corollary 4.2.3. (a) The graded root \(\mathcal{R}(R(0, c), w)\) satisfies \(|v^{-1}(n)| = 1\) for any \(n \gg 0\).

(b) \(\mathbb{H}^\ast_{red}(R(0, c), w)\) is a finitely generated \(\mathbb{Z}\)-module (for any finite or infinite \(c \geq Z_{coh}\)).

Proof. For any \(n \gg 0\) we have \(R(0, c) = S_n\), hence \(S_n\) is contractible for such \(n\). □
In the sequel we rewrite the $c$-independent $\mathbb{H}^*(R(0,c),w)$ and $\mathfrak{R}(R(0,c),w)$ as $\mathbb{H}^*_{an}(\phi)$ and $\mathfrak{R}_{an}(\phi)$ respectively.

4.3. The analytic lattice cohomology of $(X,o)$, independence of $\phi$.

Fix some $c \geq Z_{coh}$ and consider $R = R(0,c)$ and $\mathbb{H}^*_{an}(\phi)$ as above.

**Theorem 4.3.1.** Assume that $h^{n-1}(\mathcal{O}_E) = 0$. Then $\mathbb{H}^*_{an}(\phi)$ and $\mathfrak{R}_{an}(\phi)$ are independent of the choice of the resolution $\phi$.

**Proof.** By the Weak Factorization Theorem [49] it is enough to show that for a fixed resolution $\phi$ blowing up a smooth subvariety of $E$ does not change $\mathbb{H}^*_{an}(\phi)$. Indeed, any two good resolutions are connected by a sequence of such blowups and blowdowns.

So let us fix a good resolution $\phi$ with exceptional set $E = \cup_{v \in \mathcal{V}} E_v$, and blow up a compact smooth irreducible subvariety $F$ on $E$. Let $\pi$ be this blowup, and write $\phi' := \phi \circ \pi$. Let $E' = (\phi')^{-1}(o), E'_{new} = \pi^{-1}(F)$. We set $E'_v$ for the strict transform of $E_v$, hence $E' = (\cup E'_v) \cup E'_{new}$.

Let $r \geq 2$ be the codimension of $F$. Furthermore, let $\mathcal{F} := \{v \in \mathcal{V} : F \subset E_v\}$. Since $F \subset E$ and $F$ is irreducible, necessarily $\mathcal{F} \neq \emptyset$. Furthermore, since $E$ is a normal crossing divisor, $|\mathcal{F}| \leq r$.

Let $L$ and $L'$ be the corresponding free $\mathbb{Z}$-modules. Associated with $\phi$, let $\mathfrak{h}$ be the Hilbert function, $w_0$ the analytic weight and $S_0(\phi) = \bigcup \{[\square] : w([\square]) \leq n\}$. We use similar notations $\mathfrak{h}'$, $w'_0$ and $S_0(\phi')$.

We have the following natural morphisms: $\pi_* : L' \to L$ defined by $\pi_*(\sum x_i E'_i + x_{new} E'_{new}) = \sum x_i E_i$, and $\pi^* : L \to L'$ defined by $\pi^*(\sum x_i E_i) = \sum x_i E'_i + (\sum_{v \in \mathcal{F}} x_v) \cdot E'_{new}$.

The following lemma will be used several times.

**Lemma 4.3.2.** $H^0(\tilde{X}', \pi^* L'(aE'_{new})) = H^0(\tilde{X}, \mathcal{L})$ for any $a \geq 0$ and line bundle $\mathcal{L}$ on $\tilde{X}$.

**Proof.** The composition $H^0(\tilde{X}, \mathcal{L}) \overset{\pi^*}{\to} H^0(\tilde{X}', \pi^* L'(aE'_{new})) \to H^0(\tilde{X}', E'_{new}, \pi^* L'(aE'_{new})) \cong H^0(\tilde{X} \setminus F, L')$ is injective, and the inclusion $H^0(\tilde{X}, \mathcal{L}) \to H^0(\tilde{X} \setminus F, L')$ is an isomorphism since $r \geq 2$. 

For any $x \in R$, Lemma [4.3.2] applied for $\mathcal{L} = \mathcal{O}_{\tilde{X}}(-x)$ gives

$$h^0(\pi^* x + aE'_{new}) = h^0(x) \text{ for any } a \leq 0$$

is increasing for $a \geq 0$.

We wish a similar fact for $h^0$. First note that $K_{\tilde{X}} = \pi^* K_{\tilde{X}} + (r-1)E'_{new}$ (see [17] Ex. II.8.5). Then,

$$h^{n-1}(\mathcal{O}_{\pi^* x + aE'_{new}}) = \dim \frac{H^0(\tilde{X}', \Omega^*_{\tilde{X}}(\pi^* x + aE'_{new}))}{H^0(\tilde{X}', \Omega^*_{\tilde{X}}(x))}$$

= \dim \frac{H^0(\tilde{X}', \mathcal{O}_{\tilde{X}}(\pi^* K_{\tilde{X}} + (r-1)E'_{new} + \pi^* x + aE'_{new}))}{H^0(\tilde{X}', \mathcal{O}_{\tilde{X}}(\pi^* K_{\tilde{X}} + (r-1)E'_{new})))}

By Lemma [4.3.2], $H^0(\tilde{X}', \mathcal{O}_{\tilde{X}}(\pi^* K_{\tilde{X}} + (r-1)E'_{new})) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}})) = H^0(\tilde{X}, \Omega^*_{\tilde{X}})$, while

$$H^0(\tilde{X}', \mathcal{O}_{\tilde{X}}(\pi^* K_{\tilde{X}} + (r-1)E'_{new} + \pi^* x + aE'_{new})) = H^0(\tilde{X}, \Omega^*_{\tilde{X}}(x))$$

whenever $r - 1 + a \geq 0$. Therefore,

$$h^{n-1}(\mathcal{O}_{\pi^* x + aE'_{new}}) \text{ is increasing for } a \leq 1 - r$$

= $h^{n-1}(\mathcal{O}_x)$ for any $a \geq 1 - r$.

In particular, (4.3.4) applied for $a = 1 - r$ we obtain that

$$h^{n-1}(\mathcal{O}_{\pi^* x + (r-1)E'_{new}}) = h^{n-1}(\mathcal{O}_{\pi^* (r-1)E'_{new}})$$

Indeed, $h^{n-1}(\mathcal{O}_{\pi^* c - (r-1)E'_{new}}) = h^{n-1}(\mathcal{O}_{\pi^* c})$. 

In the sequel we rewrite the $c$-independent $H^*(R(0,c),w)$ and $\mathfrak{R}(R(0,c),w)$ as $H^*_{an}(\phi)$ and $\mathfrak{R}_{an}(\phi)$ respectively.
Next we compute the lattice cohomology of the rectangles $R = R(0,c)$ and $R' = R(0,\pi^*c)$ associated with $w$ and $w'$ respectively.

If $w'_0(\pi^*x + aE'_{new}) \leq n$, then $w_0(x) \leq n$ too. In particular, the projection $\pi_{\mathbb{R}}$ in the direction of $E'_{new}$ induces a well-defined map $\pi_{\mathbb{R}} : S_n(\phi') \to S_n(\phi)$. We claim that this is a homotopy equivalence (with all fibers non-empty and contractible).

4.3.6. Recall that $|\mathcal{F}| \leq r$. In the first case we assume that $|\mathcal{F}| \leq r - 1$.

Our goal is to prove that $\pi_{\mathbb{R}} : S_n(\phi') \to S_n(\phi)$ is a homotopy equivalence.

Let us first verify that $\pi_{\mathbb{R}} : S_n(\phi') \to S_n(\phi)$ is onto.

Consider a lattice point $x \in S_n(\phi)$. Then $w_0(x) \leq n$. But then $w'_0(\pi^*x) = w_0(x) \leq n$ too, hence $\pi^*(x) \in S_n(\phi')$ and $x = \pi_{\mathbb{R}}(\pi^*x) \in \text{im}(\pi_{\mathbb{R}})$.

Next take a cube $(x, I) \subset S_n(\phi)$ ($I \subset \mathcal{F}$). This means that $w_0(x + E_F) \leq n$ for any $I' \subset I$. But then

$$\pi^*(x + E_F) = \pi^*x + E_F + \varepsilon \cdot E'_{new},$$

where $\varepsilon = |I \cap \mathcal{F}|$. In particular, by our assumption, $\varepsilon \in \{0, \ldots, r-1\}$. Hence

$$w'_0(\pi^*x + E'_F) = w'_0(\pi^*(x + E_F) - \varepsilon E'_{new}) \overset{4.3.5}{=} w_0(x + E_F) \leq n.$$

Therefore $(\pi^*x, I) \in S_n(\phi')$ and $\pi_{\mathbb{R}}$ projects $(\pi^*x, I)$ isomorphically onto $(x, I)$.

Next, we show that $\pi_{\mathbb{R}}$ is in fact a homotopy equivalence. In order to prove this fact it is enough to verify that if $\Box \in S_n(\phi)$ and $\Box^\circ$ denotes its relative interior, then $\pi_{\mathbb{R}}^{-1}(\Box^\circ) \cap S_n(\phi')$ is contractible.

Let us start again with a lattice point $x \in S_n(\phi)$, then $\pi_{\mathbb{R}}^{-1}(x) \cap S_n(\phi')$ is a real interval (whose end-points are lattice points, considered in the real line of the $E_{new}$ coordinate). Let us denote it by $\mathcal{F}(x)$. Now, if $\Box = (x, I)$, then we have to show that all the intervals $\mathcal{F}(x + E_F)$ associated with all the subsets $I' \subset I$ have a common lattice point. But this is exactly what we verified above: the $E_{new}$-coordinate of $\pi^*(x)$ is such a common point. Therefore, $\pi_{\mathbb{R}}^{-1}(\Box^\circ) \cap S_n(\phi')$ has a strong deformation retraction (in the $E_{new}$ direction) to the contractible space $(\pi^*x, I)^\circ$.

For any $l \in L$ let $N(l) \subset \mathbb{R}^4$ denote the union of all cubes which have $l$ as one of their vertices. Let $U(l)$ be its interior. Write $U_n(l) := U(l) \cap S_n(\phi)$. If $l \in S_n(\phi)$ then $U_n(l)$ is a contractible neighbourhood of $l$ in $S_n(\phi)$. Also, $S_n(\phi)$ is covered by $\{U_n(l)\}_l$. Moreover, $\pi_{\mathbb{R}}^{-1}(U_n(l))$ has the homotopy type of $\pi_{\mathbb{R}}^{-1}(l)$, hence it is contractible. More generally, for any cube $\Box$,

$$\pi_{\mathbb{R}}^{-1}( \text{vertex of } \Box U_n(l)) \sim \pi_{\mathbb{R}}^{-1}(\Box^\circ)$$

which is contractible by the above discussion. Since all the intersections of $U_n(l)$’s are of this type, we get that the inverse image of any intersection is contractible. Hence by Čech covering (or Leray spectral sequence) argument, $\pi_{\mathbb{R}}$ induces an isomorphism $H^*(S_n(\phi'), \mathbb{Z}) = H^*(S_n(\phi), \mathbb{Z})$. In fact, this already shows that $\mathbb{E}_{an}^*(\phi') = \mathbb{E}_{an}^*(\phi)$. In order to prove the homotopy equivalence, one can use quasifibration, defined in [13]; see also [12], e.g. the relevant Theorem 6.1.5. Since $\pi_{\mathbb{R}} : S_n(\phi') \to S_n(\phi)$ is a quasifibration, and all the fibers are contractible, the homotopy equivalence follows.

4.3.9. Assume now that $|\mathcal{F}| = r$. The proof starts very similarly. Indeed, as above, for any lattice point $x \in S_n(\phi)$ we have $\pi^*(x) \in S_n(\phi')$ and $x = \pi_{\mathbb{R}}(\pi^*x) \in \text{im}(\pi_{\mathbb{R}})$.
If we take a cube \((x,I) \subset S_n(\phi)\) \((I \subset \mathcal{F})\), then \(w_0(x + E_{I'}) \leq n\) for any \(I' \subset I\). We consider the identity (4.3.7) as above. If \(|I \cap \mathcal{F}| \leq r - 1\) then the proof from 4.3.6 works unmodified, hence \(\pi_{n,n} : S_n(\phi') \to S_n(\phi)\) is a homotopy equivalence.

Assume next that \(|I \cap \mathcal{F}| = r\), i.e. \(\mathcal{F} \subset I\). Write \(J := I \setminus \mathcal{F}\).

4.3.10. Case 1. Let us analyse the cube \((\pi^*x,I)\) as a possible cover of \((x,I)\).

Using (4.3.5) we obtain that for any \(I' \subset I\) such that \(|I' \cap \mathcal{F}| \leq r - 1\) we have \(\pi^*x + E_{I'} \in S_n(\phi')\). Indeed, \[w_0(\pi^*x + E_{I'}) = w_0(\pi^*x + E_I) - |I'|E_{I'}^{\text{new}} \leq w_0(x + E_I) \leq n.\]

But the vertices \(\pi^*x + E_{I'}\), with \(|I' \cap \mathcal{F}| = r\), are not necessarily in \(S_n(\phi')\).

However, let us assume that \(w_0(\pi^*x + E_{I'}) = w_0(\pi^*x + E_I + E_{I'}^{\text{new}})\), or \(w_0(\pi^*x + E_I - E_{I'}^{\text{new}}) = w_0(\pi^*(x + E_I) - (r - 1)E_{I'}^{\text{new}})\). Then by (4.3.3) and (4.3.4) we obtain that \(h^{n-1}(\mathcal{O}_{\pi^*x + E_{I'}}) = h^{n-1}(\mathcal{O}_{\pi^*x + E_I + E_{I'}^{\text{new}}})\). By the opposite matroid rank inequality of \(h^{n-1}\) and (4.3.3) and (4.3.4) again we obtain that \(w_0(\pi^*x + E_I - E_{I'}) = w_0(\pi^*(x + E_I) + E_{I'}^{\text{new}})\) for any \(J' \subset J\). In particular, \[w_0(\pi^*x + E_{I'} - E_I) = w_0(\pi^*(x + E_I + E_{I'}^{\text{new}}) - E_I) = w_0(x + E_I - E_{I'}) \leq w_0(x + E_I - E_{I'}) \leq n.\]

That is, the vertices of type \(\pi^*x + E_J + E_{I'}\) of \((\pi^*x,I)\) are in \(S_n(\phi')\). For all other vertices we already know this fact (see above, or use (4.3.5)). Hence \((\pi^*x,I)\) is in \(S_n(\phi')\) and it projects via \(\pi_{n,n}\) bijectively to \((x,I)\). Furthermore, \(\pi_{n-1}^{-1}(x,I) \cap S_n(\phi')\) admits a strong deformation retraction to \((\pi^*x,I)^{\pi}\), hence it is contractible.

4.3.11. Case 2. Let us analyse the second candidate, the cube \((\pi^*x + E_{I'}^{\text{new}} + E_I,I)\), as a possible cover of \((x,I)\).

Using (4.3.5) we obtain that for any \(I' \subset I\) \(|I' \cap \mathcal{F}| \neq 0\) we have \(\pi^*x + E_{I'}^{\text{new}} + E_I \in S_n(\phi')\). Indeed, \[w_0(\pi^*x + E_{I'}^{\text{new}} + E_I) = w_0(\pi^*(x + E_I) + E_{I'}^{\text{new}} - |I' \cap \mathcal{F}|E_{I'}^{\text{new}}) \leq w_0(x + E_I) \leq n,\]

since \(0 \leq |I' \cap \mathcal{F}| - 1 \leq r - 1\). But in this case, the vertices \(\pi^*x + E_{I'}^{\text{new}} + E_I\) with \(I' \cap \mathcal{F} = \emptyset\) are not necessarily in \(S_n(\phi')\).

However, let us assume at this time that \(w_0(\pi^*x) = w_0(\pi^*x + E_{I'}^{\text{new}})\). Then by (4.3.3) and (4.3.4) we obtain that \(b'(\pi^*x) = b'(\pi^*x + E_{I'}^{\text{new}})\). By the matroid rank inequality of \(b'\) we get that \(b'(\pi^*x + E_{I'}^{\text{new}}) = b'(\pi^*x + E_{I'}^{\text{new}} + E_{I'}^{\text{old}})\) for any \(J' \subset J\). This again via (4.3.3) and (4.3.4) shows that \(w_0(\pi^*x + E_{I'}^{\text{new}}) = w_0(\pi^*(x + E_{I')}^{\text{new}})\). In particular, \[w_0(\pi^*x + E_{I'}^{\text{new}} + E_I) = w_0(\pi^*(x + E_I) + E_{I'}^{\text{new}} - E_I) = w_0(x + E_I) \leq n.\]

That is, the vertices of type \(\pi^*x + E_{I'}^{\text{new}} + E_I\) of \((\pi^*x + E_{I'}^{\text{new}},I)\) are in \(S_n(\phi')\). For all other vertices we already know this fact (see above). Hence \((\pi^*x + E_{I'}^{\text{new}},I)\) is in \(S_n(\phi')\) and it projects via \(\pi_{n,n}\) bijectively to \((x,I)\).

Furthermore, \(\pi_{n-1}^{-1}(x,I) \cap S_n(\phi')\) admits a deformation retraction to \((\pi^*x + E_{I'}^{\text{new}},I)^{\pi}\), hence it is contractible.

4.3.12. Case 3. If \(w_0(\pi^*x + E_{I'}^{\text{new}}) = w_0(\pi^*x + E_{I'}^{\text{new}} + E_{I'}^{\text{old}})\) then by Case 1 we cover \((x,I)\) by a cube. If \(w_0(\pi^*x) = w_0(\pi^*x + E_{I'}^{\text{new}})\) then the same happens by Case 2. Here in this case we assume that neither of these is satisfied, that is \(w_0(\pi^*x + E_{I'}) > w_0(\pi^*x + E_{I'}^{\text{new}})\) and \(w_0(\pi^*x) < w_0(\pi^*x + E_{I'}^{\text{new}})\).

If \(w_0(\pi^*x + E_{I'}) > w_0(\pi^*x + E_{I'}^{\text{new}})\) then \(h^{n-1}(\mathcal{O}_{\pi^*x + E_{I'}}) < h^{n-1}(\mathcal{O}_{\pi^*x + E_{I'}^{\text{new}}})\).

If \(w_0(\pi^*x) < w_0(\pi^*x + E_{I'}^{\text{new}})\) then \(b'(\pi^*x) < b'(\pi^*x + E_{I'}^{\text{new}})\).

These two conditions imply (use (4.1.6)):

\[
\begin{cases} 
(a) & H^0(\mathcal{O}_X(-\pi^*x - E_{I'}^{\text{new}})) \subseteq H^0(\mathcal{O}_X(-\pi^*x)), \\
(b) & H^0(\tilde{\mathbf{X}}', \gamma_X^*(\pi^*x + E_{I'}^{\text{new}})) \subseteq H^0(\tilde{\mathbf{X}}', \beta_X^*(\pi^*x + E_{I'}^{\text{new}})). 
\end{cases}
\]

By part (a) there exists a function \(f \in H^0(\tilde{\mathbf{X}}', \mathcal{O}_X)\) such that \(\text{div}_{E_{I'}}(f) \geq \pi^*x\), and in this inequality the \(E_{I'}^{\text{new}}\)-coordinate entries are equal. By part (b), there exists a global \(n\)-form \(\omega\) such that \(\text{div}_{E_{I'}}(\omega) \geq -\pi^*x - E_{I'}^{\text{new}}\), and the \(E_{I'}^{\text{new}}\)-coordinate entries are equal. Therefore, the form \(f \omega \in H^0(\tilde{\mathbf{X}}' \setminus E_{I'}, \Omega^2_X)\) has the property...
that $\text{div}_V(f \omega) \geq -E'_1 - E''_{\text{new}}$ with equality at the $E''_{\text{new}}$ coordinate. In particular, again by duality (4.1.6), we obtain that in $X'$ the following strict inequality holds:

$$(4.3.13) \quad h^{n-1}(\mathcal{O}_{E'_1 + E''_{\text{new}}}) > h^{n-1}(\mathcal{O}_{E'_1}) (\forall' = \forall \cup \{\text{new}\}, I \subset \forall').$$

But by our assumption $h^{n-1}(\mathcal{O}_{E'_1}) = 0$ (a condition independent of resolution), hence $h^{n-1}(\mathcal{O}_{E''_{\text{new}}}) = 0$ for any $V \subset \forall'$. In particular (4.3.13) cannot happen since $h^{n-1}(\mathcal{O}_{E'_1 + E''_{\text{new}}}) = h^{n-1}(\mathcal{O}_{E'_1}) = 0.$

**Notation 4.3.15.** In the sequel we will use for $\mathbb{H}^n_{\text{an}}(\phi)$ the notation $\mathbb{H}^n(X, o)$, and for $\mathcal{R}_{\text{an}}(X, o)$ the notation $\mathcal{R}_{\text{an}}(X, o)$. They are called the analytic lattice cohomology and the analytic graded root of $(X, o)$ respectively. They are invariants of the germ $(X, o)$.

**Remark 4.3.16.** The resolution $\tilde{X} \to X$ can be factorized through the normalization $(\tilde{X}, o)$ of $(X, o)$. In particular, $\mathbb{H}^n_{\text{an}}(X, o) = \mathbb{H}^n_{\text{an}}(\tilde{X}, o)$ and $\mathcal{R}_{\text{an}}(X, o) = \mathcal{R}_{\text{an}}(\tilde{X}, o)$.

4.4. The ‘Combinatorial Duality Property’ of the pair $(\mathfrak{h}, h^n)$.

4.4.1. Next, we wish to apply Theorem 4.1.10. Note that $\mathfrak{h}$ satisfies the stability property, since it satisfies the matroid rank inequality (being induced by a filtration). Next we verify the CDP condition.

**Lemma 4.4.2.** Assume that $h^{n-1}(\mathcal{O}_E) = 0$. Then there exists no $l \in L_{>0}$ and $v \in \forall'$ such that the differences $\mathfrak{h}(l + E_v) - \mathfrak{h}(l) \text{ and } \mathfrak{h}^v(l + E_v)$ are simultaneously strictly positive.

**Proof.** If $\mathfrak{h}(l + E_v) > \mathfrak{h}(l)$ then there exists a global function $f \in H^0(\mathcal{O}_{\tilde{X}})$ with $\text{div}_E f \geq l$, where the $E_v$-coordinate is $\text{div}_E f = l_v$. Similarly, if $\mathfrak{h}^v(l) > \mathfrak{h}(l + E_v)$ then there exists a global $n$-form $\omega$ with possible poles along $E$, with $\text{div}_E \omega \geq -l - E_v$, and $(\text{div}_E \omega)_v = -l_v - 1$. In particular, the form $f \omega$ satisfies $\text{div}_E f \omega \geq -E_v$, and $(\text{div}_E f \omega)_v = -1$. This implies $H^0(\Omega^v_X(E_v))/H^0(\Omega^v_{\tilde{X}}) \neq 0$, or, by (4.1.6), $h^{n-1}(\mathcal{O}_{E_v}) \neq 0$. This last fact contradicts $h^{n-1}(\mathcal{O}_E) = 0.$

4.5. The Euler characteristic $\text{eu}(\mathbb{H}^n_{\text{an}}(X, o))$.

4.5.1. Now we can apply Theorem 4.1.10.

Let us consider any increasing path $\gamma$ connecting 0 and $c$, that is, $\gamma = \{x_i\}_{i=0}^c, x_{i+1} = x_i + c'_{i+1} - 1$, $x_0 = 0$ and $x_i = c, c \geq Z_{\text{coh}}$, and let $\mathbb{H}^0(\gamma, w)$ be the path lattice cohomology as in 2.2.1. Accordingly, we have the numerical Euler characteristic $\text{eu}(\mathbb{H}^0(\gamma, w))$ as well.

**Theorem 4.5.2.** $\text{eu}(\mathbb{H}^n_{\text{an}}(X, o)) = h^{n-1}(\mathcal{O}_{\tilde{X}})$. Furthermore, for any increasing path $\gamma$ connecting 0 and $c$ (where $c \geq Z_{\text{coh}}$) we also have $\text{eu}(\mathbb{H}^n_{\text{an}}(\gamma, w)) = h^{n-1}(\mathcal{O}_{\tilde{X}})$.

This means that $\mathbb{H}^n_{\text{an}}(X, o)$ is a categorification of $h^{n-1}(\mathcal{O}_{\tilde{X}})$, that is, it is a graded cohomology $\mathbb{Z}[U]$-module whose Euler characteristic is $h^{n-1}(\mathcal{O}_{\tilde{X}})$.

**Lemma 4.5.3.** Assume that $h^{n-1}(\mathcal{O}_{\tilde{X}}) = 0$. Then $h^{n-1}(\mathcal{O}_E) = 0$ too (hence the analytic lattice cohomology and the graded root are well-defined). Furthermore, $\mathbb{H}^n_{\text{an}}(X, o) = \mathcal{S}^+_0$. In particular, $\mathbb{H}^n_{\text{an}, \text{red}}(X, o) = 0$ and the graded root $\mathcal{R}_{\text{an}}(X, o)$ is the ‘bamboo’ $\mathcal{R}_{(0)}$: $|\mathfrak{m}| = 0$ and $|\mathfrak{m}^{-1}(n)| = 1$ for any $n \geq 0$.

Conversely, if $h^{n-1}(\mathcal{O}_E) = 0$ (i.e. the lattice cohomology is well-defined) and $\mathbb{H}^n_{\text{an}}(X, o) = \mathcal{S}^+_0$ then $h^{n-1}(\mathcal{O}_{\tilde{X}}) = 0$ too.

**Proof.** The first statement follows from the surjectivity of $H^{n-1}(\mathcal{O}_{\tilde{X}}) \to H^{n-1}(\mathcal{O}_E)$. Next, we have to show that $S_n = \emptyset$ for any $n < 0$ and $S_n$ is contractible for any $n \geq 0$. Since $Z_{\text{coh}} = 0$ the rectangle $R(0, Z_{\text{coh}})$ has a single lattice point $l = 0$ with $w_0(0) = 0$. Hence $S_n \sim S_n \cap R(0, Z_{\text{coh}}) = \{0\}$. Conversely, $\mathbb{H}^n_{\text{an}}(X, o) = \mathcal{S}^+_0$ implies that $\min(w_0) = 0$ and $\text{eu}(\mathbb{H}^n_{\text{an}}) = 0$, hence $h^{n-1}(\mathcal{O}_{\tilde{X}}) = 0$ by Theorem 4.5.2. $\square$
4.6. **Weighted cubes and the Poincaré series** \( P(t) \). Assume that \( c = \infty \), i.e. \( R(0, c) = L_{\geq 0} \).

Recall that \( h(l) \) was defined for any \( l \in L \geq 0 \). Let us extend this definition: define \( h(l) \) for any \( l \in L \) by \( h(l) = h(\max\{0, l\}) \). This is compatible with the fact that \( H^0(\mathcal{O}_X(l)) = H^0(\mathcal{O}_X(\max\{0, l\})) \).

Then the multivariable Hilbert series \( H(t) = \sum_{l \in L} h(l) t^l \) determines the multivariable analytic Poincaré series \( P(t) = \sum_{l \in L} p(l) t^l \) (cf. [8][11][30]) by

\[
P(t) = -H(t) \prod_v (1 - t_v^{-1}), \quad \text{or} \quad p(l') = \sum_{t \in \{1, \ldots, d\}} (-1)^{l' + 1} h(l' + E_l).
\]

Then one verifies (using \( h(l) = h(\max\{0, l\}) \)) that \( P(t) \) is supported on \( L_{\geq 0} \). Furthermore, Theorem 3.1.10 and (4.6.1) combined show that the analytic Poincaré series associated with the divisorial filtration of the local ring \( \mathcal{O}_{X, o} \) has the following interpretation in terms of the (analytic) weighted cubes \( \square = (l, l) \):

\[
P(t) = \sum_{l \geq 0} \sum_{l} (-1)^{|l|+1} w_{an}(l, l) t^l
\]

whenever \( h^{n-1}(\mathcal{O}_E) = 0 \).

**Remark 4.6.2.** Let \( u \) be a vertex of \( \mathcal{R}_{an}(X, o) \) of valency one. This means that it is a local minimum of \( \tau \) with respect to the natural partial ordering given by the edges and \( \tau \). Set \( n := \tau(u) \) and let \( \mathcal{C} = \mathcal{C}_{\tau}^n \) be the connected component of \( S_n \) which represents \( u \), cf. [2][3][3]. Let \( l_m \in L \) be the maximal element of \( \mathcal{C} \) with respect to \( \leq \). (In fact, using the matroid rank inequality of \( w_0 \), cf. [4][2], one shows that \( l_m \) is unique.) Then \( w_0(l_m + E_v) > w_0(l_m) \) for any \( v \in \mathcal{V}' \). By CDP we also obtain \( h(l_m + E_v) > h(l_m) \) for any \( v \in \mathcal{V}' \). In particular, there exists a function \( f : (X, o) \to (\mathbb{C}, 0) \) such that the restriction to \( E \) of the divisor of \( f \circ \phi \) is \( l_m \). In other word, \( l_m \) is in the analytic semigroup \( \mathcal{S}_{an} \) associated with \( \phi \). Hence, local minimums of \( \mathcal{R}_{an}(X, o) \) represent elements of \( \mathcal{S}_{an} \). In this way we cannot represent all the elements of \( \mathcal{S}_{an} \), since by Lemma [4][2][2] \( \mathcal{C} \) must also contain a lattice point in \( R(0, Z_{coh}) \) too.

4.7. **Analytic Reduction Theorem.**

4.7.1. Our next goal is to prove a ‘Reduction Theorem’. Via such a result, the rectangle \( R = R(0, c) \) can be replaced by another rectangle sitting in a lattice of smaller rank. The procedure starts with identification of a set of ‘bad’ vertices. More precisely, we decompose \( \mathcal{V}' \) as a disjoint union \( \overline{\mathcal{V}} \sqcup \mathcal{V}^* \), where the vertices \( \overline{\mathcal{V}} \) are the ‘essential’ ones, the ones which dominate the others, and the coordinates \( \mathcal{V}^* \) are those which ‘can be eliminated’. The goal is to replace the rectangle \( R(0, Z^s_{coh}) \) with a rectangle of \( Z^s \), with \( s = |\overline{\mathcal{V}}| \).

In the topological case of surface singularities the possible choice of \( \overline{\mathcal{V}} \) was dictated by combinatorial properties of the Riemann–Roch expression \( \chi \) (the topological weight function), with a special focus on the topological characterization of rational germs [26][22]. In the analytical case of surface singularities we used certain analytic properties of 2–forms [11]. The present high–dimensional case is a direct generalization of this.

4.7.2. Let \( (X, o) \) be an isolated singularity of dimension \( n \geq 2 \), and we fix a good resolution \( \phi \) as above.

**Definition 4.7.3.** We say that the subset \( \overline{\mathcal{V}} \) of \( \mathcal{V}' \) is a \( B_{an} \)-set if it satisfy the following property: if some differential form \( \omega \in H^0(\overline{X} \setminus E, \Omega^1_{\overline{X}}) \) satisfies (div\( \omega \))\( |_{\overline{\mathcal{V}}} \geq -E_{\overline{\mathcal{V}}} \) then necessarily \( \omega \in H^0(\overline{X}, \Omega^1_{\overline{X}}) \). By (4.1.6) this is equivalent with the vanishing \( h^1(\mathcal{O}_Z) = 0 \) for any \( Z = E_{\overline{\mathcal{V}}} + l^s \), where \( l^s \geq 0 \) and it is supported on \( \mathcal{V}^s \).

4.7.4. Associated with a disjoint decomposition \( \mathcal{V}' = \overline{\mathcal{V}} \sqcup \mathcal{V}^* \), we write any \( l \in L \) as \( l = l^s + l^s \), or \( (l, l^s) \), where \( \overline{l} \) and \( l^s \) are supported on \( \overline{\mathcal{V}} \) and \( \mathcal{V}^* \) respectively. We also write \( \overline{R} \) for the rectangle \( R(0, \overline{l}) \), the \( \overline{\mathcal{V}} \)-projection of \( R(0, c) \) with \( c \geq Z_{coh} \).

For any \( \overline{l} \in \overline{R} \) define the weight function

\[
\overline{w}_0(\overline{l}) = h(\overline{l}) + h^c(\overline{l} + c^s) - h^{n-1}(\mathcal{O}_{\overline{X}}) = h(\overline{l}) - h^1(\mathcal{O}_{\overline{X} + c^s}).
\]
Consider all the cubes of $\mathcal{R}$ and the weight function $\varpi_q : \mathcal{Q}(\mathcal{R}) \to \mathbb{Z}$ by $\varpi_q(\Box_q) = \max \{ w_0(\tilde{l}) : \tilde{l} \text{ is any vertex of } \Box_q \}$.

**Theorem 4.7.5. Reduction theorem for the analytic lattice cohomology.** If $\overline{\mathcal{F}}$ is an $B_{an}$-set then

$$\mathbb{H}_{an}^s(R, w) = \mathbb{H}_{an}^s(\overline{\mathcal{R}}, \overline{w}).$$

**Proof.** For any $\mathcal{J} \subset \mathcal{V}$ write $c_\mathcal{J}$ for the $\mathcal{J}$-projection of $c = Z_{coh}$. We proceed by induction, the proof will be given in $|\mathcal{V}^+|$ steps. For any $\overline{\mathcal{F}} \subset \mathcal{J} \subset \mathcal{V}$ we create the inductive setup. We write $\mathcal{J}^* = \mathcal{V} \setminus \mathcal{J}$, and according to the disjoint union $\mathcal{J} \sqcup \mathcal{J}^* = \mathcal{V}$ we consider the coordinate decomposition $l = (l_\mathcal{J}, l_{\mathcal{J}^*})$. We also set $R_\mathcal{J} = R(0, c_\mathcal{J})$ and the weight function

$$w_\mathcal{J}(l_\mathcal{J}) = h(I_\mathcal{J}) + c_\mathcal{J} + h_{\mathcal{J}^*}(l_{\mathcal{J}^*}) - h^{n-1}(O_X).$$

Then for $\overline{\mathcal{F}} \subset \mathcal{J} \subset \mathcal{V}$, $\mathcal{J} = \mathcal{J} \cup \{ v_0 \} (v_0 \not\in \mathcal{J})$, we wish to prove that $\mathbb{H}_{an}^s(R_\mathcal{J}, w_\mathcal{J}) = \mathbb{H}_{an}^s(R_\mathcal{J}, w_\mathcal{J})$. For this consider the projection $\pi_\mathcal{J} : R_\mathcal{J} \to R_\mathcal{J}$.

For any fixed $y \in R_\mathcal{J}$ consider the fiber $\{ (y \pm E_{w_\mathcal{J}})_{0 \leq t \leq c_{v_0}, t \in \mathbb{Z}} \}$.

Note that $t \mapsto h(y \pm tE_{w_\mathcal{J}})$ is increasing. Let $t_0 = t_0(y)$ be the smallest value for which $h(y \pm tE_{w_\mathcal{J}}) < h(y \pm (t + 1)E_{w_\mathcal{J}})$. If $t \mapsto h(y \pm tE_{w_\mathcal{J}})$ is constant then we take $t_0 = c_{v_0}$. If $t_0 < c_{v_0}$, then $t_0$ is characterized by the existence of a function

$$f \in H^0(\mathcal{X}, E_{\mathcal{O}_X}^\omega) \text{ with } (\text{div}_E f)|_{\mathcal{J}} \geq y, \quad (\text{div}_E f)|_{v_0} = t_0.$$  

Symmetrically, $t \mapsto h_\mathcal{J}^*(y + c_\mathcal{J} + tE_{w_\mathcal{J}})$ is decreasing. Let $t_0^\mathcal{J} = t_0(y)$ be the smallest value for which $h_\mathcal{J}^*(y + c_\mathcal{J} + tE_{w_\mathcal{J}}) = h_\mathcal{J}^*(y + c_\mathcal{J} + (t + 1)E_{w_\mathcal{J}})$. The value $t_0^\mathcal{J}$ is characterized by the existence of a form

$$\omega \in H^0(\mathcal{X}, E_{\mathcal{O}_X}^\omega) \text{ with } (\text{div}_E \omega)|_{\mathcal{J}} \leq -y, \quad (\text{div}_E \omega)|_{v_0} = -t_0^\mathcal{J}.$$  

This shows that there exists a form $f \omega \in H^0(\mathcal{X}, E_{\mathcal{O}_X}^\omega)$ such that $(\text{div}_E f \omega)|_{\mathcal{J}} \geq 0$ and $(\text{div}_E f \omega)|_{v_0} = t_0 - t_0^\mathcal{J}$. By the $B_{an}$ property we necessarily must have $t_0 - t_0^\mathcal{J} \geq 0$. Therefore, the weight $t \mapsto w_\mathcal{J}(y \pm tE_{w_\mathcal{J}}) = h_\mathcal{J}^*(y \pm tE_{w_\mathcal{J}}) + h_\mathcal{J}^*(y \pm tE_{w_\mathcal{J}} + c_\mathcal{J} + tE_{w_\mathcal{J}}) - h^{n-1}(O_X)$ is decreasing for $t \leq t_0^\mathcal{J}$, is increasing for $t \geq t_0$. Moreover, for $t_0^\mathcal{J} \leq t \leq t_0$ it takes the constant value $h_\mathcal{J}^*(y + h_\mathcal{J}^*(y + c_\mathcal{J} + tE_{w_\mathcal{J}} + c_\mathcal{J} + tE_{w_\mathcal{J}}) - h^{n-1}(O_X) = w_\mathcal{J}(y)$.

Next we fix $y \in R_\mathcal{J}$ and some $l \in \mathcal{J}$ (hence a cube $(y, l)$ in $R_\mathcal{J}$). We wish to compare the intervals $[t_0^\mathcal{J}(y \pm E_{w_{\mathcal{J}}}), t_0(y \pm E_{w_{\mathcal{J}}})]$ for all subsets $l' \subset l$. We claim that they have at least one common element (in fact, it turns out that $t_0(y)$ works).

Note that $h_\mathcal{J}^*(y \pm tE_{w_\mathcal{J}}) = h_\mathcal{J}^*(y \pm (t + 1)E_{w_\mathcal{J}})$ implies $h_\mathcal{J}^*(y \pm tE_{w_\mathcal{J}} + E_{w_{\mathcal{J}}}) = h_\mathcal{J}^*(y \pm (t + 1)E_{w_\mathcal{J}} + E_{w_{\mathcal{J}}})$ for any $l'$, hence $t_0(y) \leq t_0(y \pm E_{w_{\mathcal{J}}})$. In particular, we need to prove that $t_0(y) \geq t_0(y \pm E_{w_{\mathcal{J}}})$. Similarly as above, the value $t_0(y \pm E_{w_{\mathcal{J}}})$ is characterized by the existence of a form

$$\omega_{w_{\mathcal{J}}} \in H^0(\mathcal{X}, E_{\mathcal{O}_X}^\omega) \text{ with } (\text{div}_E \omega_{w_{\mathcal{J}}})|_{\mathcal{J}} \leq -y - E_{w_{\mathcal{J}}}, \quad (\text{div}_E \omega_{w_{\mathcal{J}}})|_{v_0} = -t_0(y \pm E_{w_{\mathcal{J}}}).$$  

Hence the from $f \omega_{w_{\mathcal{J}}} \in H^0(\mathcal{X}, E_{\mathcal{O}_X}^\omega)$ satisfies $\text{div}_E f \omega_{w_{\mathcal{J}}}|_{\mathcal{J}} \leq -E_{w_{\mathcal{J}}}$ and $(\text{div}_E f \omega_{w_{\mathcal{J}}})|_{v_0} = t_0(y) - t_0(y \pm E_{w_{\mathcal{J}}})$. By the $B_{an}$ property we must have $t_0(y) - t_0(y \pm E_{w_{\mathcal{J}}}) \geq 0$.

Set $S_{\mathcal{J}, n}$ and $S_{\mathcal{J}, n}$ for the lattice spaces defined by $w_\mathcal{J}$ and $w_{\mathcal{J}}$. If $y \pm tE_{w_\mathcal{J}} \in S_{\mathcal{J}, n}$ then $w_\mathcal{J}(y \pm tE_{w_\mathcal{J}}) \leq n$, hence by the above discussion $w_\mathcal{J}(y) \leq n$ too. In particular, the projection $\pi_\mathcal{J} : R_\mathcal{J} \to R_\mathcal{J}$ induces a map $S_{\mathcal{J}, n} \to S_{\mathcal{J}, n}$. We claim that it is a homotopy equivalence. The argument is similar to the proof from 3.3.1 via the above preparations. 

□

**Corollary 4.7.9.** If $(X, o)$ admits a resolution $\phi$ with a $B_{an}$-set of cardinality $\mathcal{J}$, then $\mathbb{H}_{an}^\bullet((X, o), \mathcal{J}) = 0$.

**Example 4.7.10.** Assume that $(X, o)$ is Gorenstein, cf. Example 4.1.16. Let $\overline{\mathcal{F}}$ be a $B_{an}$-set and assume that $Z_K_{\overline{\mathcal{F}}} \geq 0$. Since $Z_{coh} \leq Z_K_{\overline{\mathcal{F}}}$ (cf. (19)), we can take $c = Z_K_{\overline{\mathcal{F}}}$. Then, for any $\overline{I} \in R(0, Z_K_{\overline{\mathcal{F}}})$, the
Remark 4.7.15. If
\[ h^0(\bar{X}, \mathcal{O}_{\bar{X}}(a)) = h^0(\bar{X}, \mathcal{O}_{\bar{X}}). \]
Indeed, using the exact sequence 0 \to \mathcal{O}_{\bar{X}} \to \mathcal{O}_{\bar{X}}(a) \to \mathcal{O}_{a}(a) \to 0 it is enough to prove that
\[ h^0(\mathcal{O}_{a}(a)) = 0. \]
But this follows by Serre duality and Grauer–Riemenschneider vanishing. Next, note that for \( a \geq 0 \), both supported on \( E \) but without common \( E_c \)–term in their supports, one has
\[ h^0(\bar{X}, \mathcal{O}_{\bar{X}}(a-b)) = h^0(\mathcal{O}_{\bar{X}}(b)) \cap h^0(\mathcal{O}_{\bar{X}}(a)) = h^0(\mathcal{O}_{\bar{X}}(b)) \cap h^0(\mathcal{O}_{\bar{X}}) = h^0(\mathcal{O}_{\bar{X}}(b)). \]
In particular,
\[ h^0(\bar{X}, \mathcal{O}_{\bar{X}}(a-b)) = h^0(\mathcal{O}_{\bar{X}}(b)) \cap h^0(\mathcal{O}_{\bar{X}}(a)) = h^0(\mathcal{O}_{\bar{X}}(b)) \cap h^0(\mathcal{O}_{\bar{X}}) = h^0(\mathcal{O}_{\bar{X}}(b)). \]
Therefore, the weight function \( \bar{w}_0 \) on \( R(0, \bar{Z}_K) \) is
\[ \bar{w}_0(\bar{t}) = h(\bar{t}) + h(\bar{Z}_K - \bar{t}) - h^{n-1}(\mathcal{O}_{\bar{X}}). \]
That is, \( \bar{w}_0 \) is obtained by the symmetrization of the restriction of \( h \) to \( R(0, \bar{Z}_K) \).

4.7.13. Under the assumption \( h^{n-1}(\mathcal{O}_{E}) = 0 \), if \( (X, o) \) is Gorenstein, then \( B_{\text{an}} \)–sets \( \bar{\mathcal{T}} \) with \( Z_K|_{\bar{\mathcal{T}}} \geq 0 \) exist for any good resolution \( \bar{X} \to X \). Indeed, we have the following fact.

Lemma 4.7.14. If \( (X, o) \) is Gorenstein and \( h^{n-1}(\mathcal{O}_{E}) = 0 \) then the support of \( Z_{K,+} \) is a \( B_{\text{an}} \)–set.

Proof. Denote the support of \( Z_{K,+} \) by \( l \). Assume that \( h^{n-1}(\mathcal{O}_{E + l^*}) \neq 0 \), where \( l^* \geq 0 \) and it is supported on \( \mathcal{V}' \setminus I \). But by Corollary 4.1.14 we also have \( h^{n-1}(\mathcal{O}_{E + l^*}) = h^{n-1}(\mathcal{O}_{\min[E + l^*, Z_{\text{coh}}(\ell)]}) \). Since \( Z_{\text{coh}} \leq Z_{K,+} \) [19], we get that \( h^{n-1}(\mathcal{O}_{E}) \neq 0 \). But this contradicts \( h^{n-1}(\mathcal{O}_{E}) = 0 \).

Remark 4.7.15. Assume that \( n = 2 \), \( h^{n-1}(\mathcal{O}_{E}) = 0 \), and \( (X, o) \) is normal but not necessarily Gorenstein. Then the rational cycle \( Z_K \in L \otimes \mathbb{Q} \) can be defined, it is the (anti)canonical cycle, numerically equivalent with \( K_{\bar{X}} \) (see e.g. [1]). Then again \( Z_{\text{coh}} \leq Z_{K,+} \) (see e.g. [1]), hence the very same proof gives the following: if \( h^{n-1}(\mathcal{O}_{E}) = 0 \) then the support \( \{ Z_{K,+} \} \) is a \( B_{\text{an}} \)–set.

If we choose \( \bar{\mathcal{T}} \) as the support of \( \{ Z_{K,+} \} \), and we also take \( c = [Z_{K,+}] \), then in (4.7.6) \( c^* = 0 \) and
\[ \bar{w}_0(\bar{t}) = h(\bar{t}) + h(\bar{Z}_K - \bar{t}) - h^{n-1}(\mathcal{O}_{\bar{X}}). \]
for any \( \bar{t} \in R(0, c) \).

5. \( h^{n-1}(\mathcal{O}_{E}) \) and the cohomology of the link

5.1. \( h^{n-1}(\mathcal{O}_{\bar{X}}) \) and the geometric genus.

Let \( (X, o) \) be an isolated complex singularity and let \( \phi : \bar{X} \to X \) be a good resolution as above. Let \( \mathcal{O}_{X,o} \) be the normalization of \( \mathcal{O}_{X,o} \) and let \( \delta(X, o) \) be the delta invariant \( \dim(\mathcal{O}_{X,o}/\mathcal{O}_{(X, o)}) \) of \( (X, o) \). Note that \( \mathcal{O}_{X,o} \) can also be identified with \( \mathcal{O}_{X,o} \). As usual, we define the geometric genus by
\[ (-1)^{n+1} p_h(X, o) := \delta(X, o) + \sum_{i \geq 1} (-1)^i h^i(X, \mathcal{O}_{X}). \]

The following facts are well–known [50, 20]

(i) \( h^i(X, \mathcal{O}_{\bar{X}}) = 0 \) for \( i \geq n \);
(ii) \( h^{n-1}(\bar{X}, \mathcal{O}_{\bar{X}}) = \dim(\mathcal{H}^0(\bar{X} \setminus E, \Omega^n_{\bar{X}})/\mathcal{H}^0(\bar{X}, \Omega^n_{\bar{X}})) \), cf. (4.1.6);
(iii) $h^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \dim H^i_{(\tilde{X})}(\tilde{X}, \mathcal{O}_{\tilde{X}})$ for $1 \leq i \leq n - 2$;

(iv) If $(X, o)$ is Cohen–Macaulay then $(\tilde{X}, o)$ is normal and $h^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for $1 \leq i \leq n - 2$. In particular, $p_\mathcal{E}(X, o) = h^{n-1}(\tilde{X}, \mathcal{O}_{\tilde{X}})$. (For this use the characterization of the Cohen–Macaulay property in terms of the local cohomology, cf. part (iii).)

5.1.1. Recall that $(X, o)$ is called rational if $(X, o)$ is normal and $h^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for $i \geq 1$. If $(X, o)$ is Cohen–Macaulay and $h^{n-1}(\mathcal{O}_E) = 0$ then $(X, o)$ is rational if and only if $\mathbb{H}^n_{\mathcal{E}}(X, o) = \mathcal{O}_0^+$, cf. Lemma 4.5.3.

5.2. $h^{n-1}(\mathcal{O}_E)$ and the link of $(X, o)$.

Recall that all the (local) cohomology groups $H^k_{(\mathcal{E})}(X)$, $H^k_{(\tilde{X})}(\tilde{X})$, and $H^k(E)$ admit mixed Hodge structures. The Hodge filtration will be denoted by $F^\cdot$. By [45, Corollary 1.2] we have the following short exact sequence

$$0 \to \text{Gr}^F_0 H^{n+1}(X) \to \text{Gr}^F_1 H^{n+1}(\tilde{X}) \to \text{Gr}^F_1 H^{n+1}(E, C) \to 0.$$  

Let $M$ denote the link of $(X, o)$, an oriented $(2n - 1)$–dimensional compact manifold. By [45, Corollary 1.15] we have an isomorphism $H^k_{(\mathcal{E})}(X) = H^k(M, C)$ for any $1 \leq k \leq 2n - 2$, which is compatible with the mixed Hodge structures. Furthermore, by [45] page 516 we also have an isomorphism $H^{n-1}(E, \mathcal{O}_E) = \text{Gr}^0 F^n H^{n-1}(E, C)$, which is dual to $\text{Gr}^0 F^n H^{n+1}(\tilde{X})$. Therefore, we have an exact sequence

$$0 \to \text{Gr}^F_0 H^n(M) \to H^{n-1}(\mathcal{O}_E) \to \text{Gr}^F_0 H^{n+1}(E, C) \to 0.$$  

Hence, the vanishing $h^{n-1}(\mathcal{O}_E) = 0$ implies the vanishing $\dim \text{Gr}^F_0 H^n(M) = 0$.

5.3. $h^{n-1}(\mathcal{O}_E)$ and a smoothing of $(X, o)$. Assume that $(\mathcal{X}, o)$ is a complex space of dimension $n + 1$ with at most an isolated singularity at $o$. Let $f : (\mathcal{X}, o) \to (\mathbb{C}, 0)$ be a flat map such that $(f^{-1}(0), o) \simeq (X, o)$. We also assume that both $\mathcal{X}$ and $X$ are contractible Stein spaces and $f$ induces a differentiable locally trivial fibration over a small punctured disc. Let $X_\infty$ be the nearby (Milnor) fiber $f^{-1}(i)$ for $i \neq 0$ sufficiently small. It is an $n$–dimensional complex Stein manifold. Note that $\mathcal{O}_X \simeq M$.

Then $H^k(X_\infty)$ carries a mixed Hodge structure. Then by [45, Proposition 2.13] $\dim \text{Gr}^F_0 H^n(X_\infty) = p_\mathcal{E}(X, o)$. Furthermore, by [45] Proposition 1.15

$$h^{n-1}(\mathcal{O}_E) = p_\mathcal{E}(X, o) - \dim \text{Gr}^0 F^k H^n(X_\infty) + \dim \text{Gr}^0 F^{n-1}(X_\infty).$$

If $\tilde{\mathcal{X}} \to \mathcal{X}$ is an embedded good resolution of the pair $(\mathcal{X}, X)$ with exceptional space $\mathcal{E} \subset \tilde{\mathcal{X}}$ then by [45, page 526] we also have $\dim \text{Gr}^0 F^k H^n(X_\infty) \simeq H^k(\mathcal{O}_E)$. Hence we have

$$h^{n-1}(\mathcal{O}_E) = \dim \text{Gr}^0 F^k H^n(X_\infty) - \dim \text{Gr}^0 F^k H^n(X_\infty) + h^{n-1}(\mathcal{O}_E).$$

E.g., if $(X, o)$ is Cohen–Macaulay (e.g. if it is complete intersection), then $h^{n-1}(\mathcal{O}_E) = 0$ (cf. [45, Corollary 2.16]), hence we have a description of $h^{n-1}(\mathcal{O}_E)$ in terms of the Hodge filtration of $H^n(X_\infty)$.

5.4. $h^{n-1}(\mathcal{O}_E)$ and $h^{n-1}(\mathcal{O}_{\tilde{X}})$ for hypersurface singularities.

Assume that $(X, o)$ is an isolated hypersurface singularity with Milnor number $\mu$. Then the (Hodge) spectrum consists of $\mu$ rational numbers in the interval $(0, n + 1)$. Their position is symmetric with respect to $(n + 1)/2$. The number of spectral numbers in $(0, 1)$ (or, symmetrically, in $[n, n + 1]$) is $p_\mathcal{E}(X, o) = h^{n-1}(\mathcal{O}_{\tilde{X}})$. More generally, the dimension of $\text{Gr}^F_0 H^n(X_\infty)$ is the number of spectral numbers in the interval $[k, k + 1)$. Hence, $\dim \text{Gr}^F_0 H^n(X_\infty)$ is the number of spectral numbers in the interval $(0, 1)$. Since in this case $h^{n-1}(\mathcal{O}_E) = 0$, (5.3.2) gives that

$$h^{n-1}(\mathcal{O}_E) = \{\text{number of spectral numbers} = 1\} = \dim \text{Gr}^F_0 H^n(M).$$

In particular, if $M$ is a rational homology sphere then $h^{n-1}(\mathcal{O}_E) = 0$. 

6. Examples

6.1. Isolated weighted homogeneous hypersurface singularities.

6.1.1. Assume that \((X,o) \subset (\mathbb{C}^{n+1},0)\) is defined by a weighted homogeneous polynomial \(f(z_0,\ldots,z_n)\) of weights \((w_0,\ldots,w_n)\) and degree \(d\). Here \(w_i \in \mathbb{Z}_{>0}\), \(\gcd(w_0,\ldots,w_n) = 1\) and all the nontrivial monomials of \(f\) have the form \(c_k z^k = c_k z_0^{k_0} \cdots z_n^{k_n}\), where \(c_k \in \mathbb{C}^+\) and \(\sum k w_i = d\). We assume that \((X,o)\) has an isolated singularity.

A partial resolution of \(X\) can be obtained by a weighted blow up of \(o \in X\). This creates an exceptional set \(X_\infty\). Then we can continue the resolution procedure and we construct a good resolution \(\tilde{X} \rightarrow X\). Let \(\mathcal{V}\) be the index set of irreducible exceptional sets, as above. We denote the strict transform of \(X_\infty\) by \(E_\infty\). It is irreducible (see below), let it be indexed by \(v_\infty \in \mathcal{V}\). We wish to show that \(\{v_\infty\} \subset \mathcal{V}\) is a \(B_\infty\) set.

Let us provide more details about the weighted blow up. Consider the weighted projective \(n\)-space \(\mathbb{P}^n_w\) of weights \((w_0,\ldots,w_n)\) and the hypersurface \(X_\infty \subset \mathbb{P}^n_w\) given by the equation \(f(z_0,\ldots,z_n) = 0\). Since \(X\) has an isolated singularity, \(X_\infty\) is necessarily irreducible. Next, take the incidence variety

\[
\mathcal{I} := \{(v,[u]) \in X \times \mathbb{P}^n_w, v \in X \setminus \{0\}, [v]_w = [u]_w\}
\]

where \([u]_w\) denotes the class of \(u\) in \(\mathbb{P}^n_w\). Then its closure with the restriction of the first projection is the needed weighted blow up with exceptional set \(X_\infty\).

The space \(X_\infty\) is a \(V\)-manifold \([44]\). This means that its singularities locally are quotients of type \(\mathbb{C}^n/G\), where \(G\) is a finite small subgroup of \(\text{GL}(n,\mathbb{C})\). Since quotient singularities are Cohen–Macaulay \([18]\) and rational \([43]\), we have to resolve in continuation only ‘mild’ singularities. In particular, we can expect that the only ‘significant’ irreducible exceptional set is \(E_\infty\).

Lemma 6.1.2. If \(h^{n-1}(\mathcal{O}_E) = 0\) then \(\{v_\infty\}\) is a \(B_\infty\) subset of \(\mathcal{V}\).

Proof. We need to show that \(h^{n-1}(\mathcal{O}_{E_\infty+1}) = 0\) for any \(l^* \geq 0\) supported on \(\mathcal{V}_* = \mathcal{V} \setminus \{v_\infty\}\). By \([41,1.0]\) this means that there exists no differential form \(\alpha \in H^0(\tilde{X} \setminus E, \Omega^2_{\tilde{X}})\) such that its pole has order one along \(E_\infty\).

The point is that we know all the candidate differential forms. Indeed, since \((X,o)\) is Gorenstein, it admits a Gorenstein form \(\alpha_G\) (unique up to a non–zero constant). In fact, it is the restriction of \(dx_0 \wedge \cdots \wedge dx_l/(\partial f/\partial x_l)\) to \(X \setminus \{o\}\). Then consider all the monomials of type \(z^k\) with \(\sum (k_i + 1) w_i \leq d\). These correspond to the lattice points \(k + (1,\ldots,1)\) with strictly positive coordinates not above the Newton boundary. Then the classes of the differential forms \(z^k \alpha_G\) form a basis of \(H^0(\tilde{X} \setminus E, \Omega^2_{\tilde{X}})/H^0(\tilde{X}, \Omega^2_{\tilde{X}})\) \([25]\).

Note also that the divisorial filtration associated with \(E_\infty\) agrees with the combinatorial filtration associated with the Newton diagram \([24,42]\). Hence, the pole order of any linear combination \(\sum_k c_k z^k = \{c_k \in \mathbb{C}\}\) is \(\max\{\text{pole order } z^k : c_k \neq 0\}\). Therefore, the pole order of any linear combination \(\sum c_k z^k \alpha_G = \{c_k \in \mathbb{C}\}\) is \(\max\{\text{pole order } z^k \alpha_G : c_k \neq 0\}\). Moreover, since the pole order of \(\alpha_G\) along \(E_\infty\) is \(d + 1 - \sum w_i\) (cf. \([25]\)), the pole order of \(z^k \alpha_G\) is \(d + 1 - \sum w_i\). Hence, this is 1 exactly when \(\sum (k_i + 1) w_i = d\), i.e., if the corresponding lattice point is on the Newton boundary. But such lattice points from the Newton boundary produce spectral numbers 1 \([43,41]\). Since there exists no spectral number equal to 1 (cf. \([5,4,1]\)), there exists no such differential form either.

6.1.3. In particular, we can apply the Analytic Reduction Theorem for this vertex. In the next paragraphs we show that the weight function \(w_0\) of the reduction is determined by the set of spectral numbers from the interval \((0,1)\) (recall that their number is exactly \(p_\infty\)).

By the Reduction Theorem we have to determine \(\mathcal{H}(\tilde{f})\) for \(\tilde{f} \in \mathcal{R}(0,\mathbb{Z}_K) = \mathbb{Z} \cap [0,d+1-\sum w_i]\).

Note that both \(\mathcal{O}_{C^{n+1},0}\) and \(\mathcal{O}_{X,o}\) are graded local algebras, graded by \(\deg(z^k) = \sum k w_i\). For \(\mathcal{H}(\tilde{f})\) we need to know \(\deg_{\deg<d+1-\sum w_i}\). Since \(\deg(f) = d\), the homogeneous components of \(\mathcal{O}_{C^{n+1},0}\) and \(\mathcal{O}_{X,o}\) in degrees
deg < d + 1 − \sum w_i are the same. They are determined by the lattice points \( k \in \mathbb{Z}^{n+1}_{\geq 0} \) with \( \sum_i k_i w_i \leq d - \sum w_i \). These points correspond bijectively to the lattice points \( k + (1, \ldots, 1) \in \mathbb{Z}^{n+1}_{\geq 0} \) with \( (k + 1) w_i \leq d \). These lattice points are those which are not above the Newton boundary. But there are no lattice points on the boundary, hence these lattice points are all under the boundary, and they provide the spectral numbers. Each such lattice point contributed in the spectrum by \( \alpha = \sum_i (k_i + 1) w_i / d \).

Let \( P(t) = \sum_{\ell \geq 0} p(\ell) t^\ell \) be the Poincaré series of the graded algebra \( \mathcal{O}_{X,o} \), and \( P(t)_{< d + 1 - \sum w_i} \) be the Poincaré polynomial counting the dimensions of the homogeneous components of degree < \( d + 1 - \sum w_i \). Then \( h(\ell) = \sum_{\ell < T} p(\ell) \).

If \( \{ \alpha_1, \ldots, \alpha_l \} \) are the spectral numbers of \( (X,o) \) then we write \( \text{Spec}(t) = \sum_i t^{\alpha_i} \). Let \( \text{Spec}_{(0,1)}(t) \) be \( \sum_{\alpha_i < 1} t^{\alpha_i} \). Then it is known that [33, 5.11]

\[
\text{Spec}(t) = \prod_i \frac{t^{w_i/d} - t}{1 - t^{w_i/d}}.
\]

From above we also have

\[
\text{Spec}_{(0,1)}(t) = \sum_{k_i \geq 0, \sum(k_i + 1) w_i < d} t^{\sum(k_i + 1) w_i / d}, \quad \text{Spec}_{(0,1)}(t^d) = \sum t^{w_i} P(t)_{< d + 1 - \sum w_i}.
\]

Hence, for any \( 0 \leq \ell \leq d + 1 - \sum w_i \) we have

\[
h(\ell) = \# \{ \alpha \text{ spectral number with } \alpha < (\ell + \sum w_i) / d \},
\]

and \( \mathfrak{w}(\ell) = h(\ell) + h(d + 1 - \sum w_i - \ell) - p_{\bar{\ell}} \).

**Remark 6.1.4.** From the above discussion \( \mathcal{H}^{\geq 1}_{an}(X,o) = 0 \) and \( \mathfrak{R}_{an}(X,o) \) is completely determined by the Hodge spectrum \( \text{Spec}_{(0,1)} \). (Recall that \( \mathfrak{R}_{an}(X,o) \) determines \( \mathcal{H}^0_{an}(X,o) \) as well as its \( \mathfrak{H} \).)

On the other hand, from \( \mathfrak{R}_{an}(X,o) \) we cannot recover the precise values of the spectral numbers. Indeed, e.g. for the Brieskorn (normal, minimally elliptic surface) singularities of type \((2,3,7) \) and \((2,3,11) \) have the same graded root, but their spectrum in \((0,1) \) are different. Both have only one spectral number in \((0,1) \), they are \( 41 \) and \( 61 \) respectively. On the other hand, the graded root certainly provides interesting information about the mutual position of the spectral numbers.

**Remark 6.1.5.** It would be very interesting to generalize this Hodge theoretical connection to all hypersurfaces, and to find other reinterpretations of \( \mathcal{H}^{\geq 1}_{an} \) in terms of other classical analytic invariants (e.g. the multiplicity).

### 6.2. Newton nondegenerate hypersurface singularities.

Assume that \( f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, o) \) is an isolated hypersurface singularity which is nondegenerate with respect to a convenient Newton diagram [21, 47]. We also assume that there is no lattice point on the Newton boundary with all coordinates strictly positive. This is equivalent to the fact that there exists no spectral number equal to 1 [41].

The normal vectors of the top faces of the Newton diagram determine a dual fan. A regular subdivision of this fan determines a toric resolution \( \tilde{X} \to X \) of \((X,o) = (\{ f = 0 \}, 0) \) (even an embedded resolution, but that one is not needed here). Then by the toric correspondence, the normal directions of the top faces of the Newton diagram determine exceptional divisors in \( \tilde{X} \), they are irreducible by the nondegeneracy assumption. Let their collection be \( \mathcal{V} \). A very same proof as in the weighted homogeneous case shows that \( \mathcal{V} \) is a \( B_{an} \)-set. One has again to consider the lattice points below the diagram, their number is \( p_{\mathcal{V}} \), and they index both the spectral numbers in the interval \((0,1) \) and also differential forms forming a basis in \( H^0(\tilde{X} \setminus E, \Omega^n_{\tilde{X}}) \) and \( H^0(\tilde{X}, \Omega^n_{\tilde{X}}) \), [25, 41]. The details are left to the reader.

However, with respect to the complete discussion from [6, 1] the parallelism with the weighted homogeneous germs breaks at some point: in the Newton nondegenerate case the combinatorial Newton filtration
of the lattice points below the Newton diagram usually does not coincide with the corresponding divisorial filtration. The description of $\mathbb{H}_{an}^*$ and of $\mathcal{R}_{an}$ is the subject of a forthcoming paper.

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ALFRED RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, RÉÁLTANODA UTCA 13-15, H-1053, BUDAPEST, HUNGARY

ELTE - UNIVERSITY OF BUDAPEST, DEPT. OF GEOMETRY, BUDAPEST, HUNGARY

Email address: agoston.tamas@renyi.hu

ALFRED RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, RÉÁLTANODA UTCA 13-15, H-1053, BUDAPEST, HUNGARY

BCAM - BASQUE CENTER FOR APPLIED MATH., MAZARREDO, 14 E48009 BILBAO, BASQUE COUNTRY – SPAIN

Email address: nemethi.andras@renyi.hu