JUCYS–MURPHY ELEMENTS AND WEINGARTEN MATRICES

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ABSTRACT. We provide a compact proof of the recent formula of Collins and Matsumoto for the Weingarten matrix of the orthogonal group using Jucys–Murphy elements.

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1. Introduction

1.1. Motivation. In this note, we discuss the calculation of the so-called Weingarten matrix for unitary and orthogonal groups. Let us recall its origin. The initial problem is the computation of the integral over certain compact groups of matrices, of polynomials of the entries of these matrices. More abstractly, the problem amounts to finding the projector from a tensor product of representations onto its invariant subspace. As will be described in the next section, the result is expressed in terms of a certain matrix $W$, the Weingarten matrix. Such computations are useful in various areas of theoretical physics and mathematics: it seems that they were originally considered by Weingarten in the context of gauge theories [10] and then reappeared in matrix models and in free probability. In particular Collins et al [5, 3] studied this Weingarten matrix using representation theory of the symmetric group and derived some explicit expressions for $W$. It is the goal of this note to provide elementary proofs of these results using Jucys–Murphy elements. Note that the only case where this procedure is new is the case of the orthogonal group, where the original proof by Collins and Matsumoto [4] involved fairly involved machinery (see also [1]); use of Jucys–Murphy for the unitary case had already been considered in [9, 7].

1.2. Weingarten matrix as a pseudo-inverse. Let $V$ be a (real, or complex) vector space endowed with a symmetric non-degenerate bilinear form $⟨⋅|⋅⟩$. We are interested in an explicit expression for the orthogonal projector $Π$ onto a given subspace $V_0$. The strategy is then to (i) identify a set of elements $|u_i⟩ ∈ V$ which generate $V_0$; and (ii) try the Ansatz $Π = ∑_{i,j} |u_i⟩ W_{i,j} ⟨u_j|$ where the matrix $W$ is to be determined. For $Π$ to be orthogonal, $W$ must be a symmetric matrix. Furthermore, writing that $⟨u_i| Π |u_j⟩ = ⟨u_i| u_j⟩$ leads to the following matrix identity: $GWG = G$, where $G$ is the Gram matrix: $G_{i,j} = ⟨u_i| u_j⟩$. In fact it is easy to see that these are all the conditions on $W$. If the $|u_i⟩$ are independent, $G$ is invertible and we conclude directly that $W = G^{-1}$. If $G$ is not invertible, $W$ is not uniquely

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determined, but a convenient choice\footnote{We will explain this later.}, which we make here, is to require $W$ to be the \textit{pseudo-inverse} of $G$, that is the symmetric matrix such that $GWG = G$ and $WGW = W$.

In what follows, $V$ will be equipped with a representation of some group, and $V_0$ will be the invariant subspace with respect to that action. In this case $W$ will be called the \textit{Weingarten matrix}.

1.3. \textbf{Jucys–Murphy elements.} In this note we only use standard methods due to Young, Jucys\footnote{Jucys, private communication.} and Murphy\footnote{Murphy, private communication.}. Let $n$ be a positive integer. We consider in what follows the natural embedding $\mathbb{C}[S_1] \subset \mathbb{C}[S_2] \subset \cdots \subset \mathbb{C}[S_n]$, where elements of $\mathbb{C}[S_k]$ act trivially on numbers greater than $k$. Denote the transposition of $i$ and $j$ by $s_{i,j}$. We recall that to a partition $\lambda$ with $|\lambda| = n$ parts, represented by a Young diagram, one can associate the set $\text{SYT}(\lambda)$ of standard Young tableaux $T$ of shape $\lambda$ i.e. fillings of each box $(i,j) \in \lambda$ with numbers $T(i,j) \in \{1, \ldots, n\}$ which are increasing along rows and columns.

We now define the Jucys–Murphy elements. $m_k$ is an element of $\mathbb{C}[S_k]$ given by:

$$m_1 = 0, \quad m_k = \sum_{i=1}^{k-1} s_{i,k} \quad k = 2, \ldots, n$$

(the definition of $m_1$ is a convenient convention). Note that $m_k$ ($k \geq 2$) commutes with $\mathbb{C}[S_{k-1}]$; this implies in particular that the $m_k$, $k = 2, \ldots, n$, form a commutative subalgebra of $\mathbb{C}[S_n]$. It is in fact a maximal commutative subalgebra.

Next we introduce Young’s \textit{orthogonal} idempotents $e_T$, $T$ standard Young tableau with $n$ boxes. They are characterized by the following properties: first they are a complete set of orthogonal idempotents:

$$e_T e_{T'} = \delta_{TT'} \sum_{T' : |T'| = n} e_{T'} = 1$$

and secondly they “diagonalize” the Jucys–Murphy elements:

$$m_k e_T = e_T m_k = c(T_k)e_T \quad k = 1, \ldots, n$$

where $c(T_k)$ is the content of the box labelled $k$ in $T$: $c(T_k) = j - i$ if $T(i,j) = k$.

\textit{Example:} there are two tableaux of shape $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}$:

$$m_1 \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} = 0 \quad m_2 \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} \quad m_3 \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$$

Finally define

$$P_\lambda = \sum_{T \in \text{SYT}(\lambda)} e_T$$
\[ P_\lambda = \frac{\chi_\lambda(1)}{|S_n|} \sum_{\sigma \in S_n} \chi_\lambda(\sigma^{-1}\sigma) \]

though we shall not need this expression here. For the sake of completeness, let us also remark that since \( e_T \) commutes with all the \( m_k \), it should be expressible as a polynomial of them; and indeed, we have the following (Lagrange interpolation type) inductive definition:
\[ e_T = e_{\bar{T}} \prod_{T' : \bar{T} = T, T' \neq T} \frac{m_{n-c(T')}}{c(T') - c(T)} \]
where \( \bar{T} \) denotes the tableau \( T \) with its last box removed.

\[ \text{Proposition 1 (Jucys).} \]
(2.1)
\[ \prod_{k=1}^{n} (\tau + m_k) = \sum_{\sigma \in S_n} \sigma \tau^{\text{number of cycles of } \sigma} \]

There are many ways to prove this identity. We provide an elementary one now, based on a standard inductive construction of permutations.

**Proof.** First, note that there are as many terms in the r.h.s. as in the l.h.s. after expanding the product \((\tau + m_1) \cdots (\tau + m_n)\). Our proof will consist in an identification term by term. We proceed by induction on \( n \). Consider a permutation \( \sigma \in S_n \). Either (i) \( n \) is a fixed point of \( \sigma \), in which case we can apply the induction hypothesis to \( \sigma_{\{1, \ldots, n-1\}} \in S_{n-1} \) and we know that it corresponds to one term in \((\tau + m_1) \cdots (\tau + m_{n-1})\) with coefficient \( \tau^{\text{number of cycles of } \sigma_{\{1, \ldots, n-1\}}} \); furthermore, \( \sigma \) has one more cycle that \( \sigma_{\{1, \ldots, n-1\}} \) and therefore we identify \( \sigma \) with the term in \((\tau + m_1) \cdots (\tau + m_n)\) with the same choice in the first \( n - 1 \) factors, and in which we further pick the multiplication by \( \tau \) inside \( \tau + m_n \); or (ii) \( n \) is not a fixed point in which case we can similarly apply the induction hypothesis to \( \sigma s_{\sigma^{-1}(n), n_{\{1, \ldots, n-1\}}} \in S_{n-1} \):

\[ \sigma = (\text{other cycles. . . }) \quad \sigma s_{\sigma^{-1}(n), n_{\{1, \ldots, n-1\}}} = (\text{other cycles. . . }) \]

Let \( \tau \) be a positive integer. As warming up, we consider the simple and well-studied case [3, 9, 7] of the action of \( U(\tau) \) (or its complexification \( GL(\tau) \)) on \( V = (\mathbb{C}^\tau)^{\otimes n} \otimes (\mathbb{C}^\tau)^{\otimes n} \simeq Hom((\mathbb{C}^\tau)^{\otimes n}) \) where the first \( n \) factors of the tensor product are in the fundamental representation of \( U(\tau) \) and the last \( n \) in its dual representation. Schur–Weyl duality provides us with a set of invariants indexed by permutations in \( S_n \). The corresponding Gram matrix is easily computed:
\[ G_{\sigma, \sigma'} = \tau^{\text{number of cycles of } \sigma^{-1}\sigma'} \quad \sigma, \sigma' \in S_n \]

Next we need the classical identity: [6]
Noting that $\sigma$ has as many cycles as $\sigma s_{\sigma^{-1}(n),n}$, we conclude that it is identified with the term in $(\tau+m_1)\cdots(\tau+m_n)$ with the same choice in the first $n-1$ factors, and in which we pick the transposition $s_{\sigma^{-1}(n),n}$ inside $\tau+m_n$.

The induction is concluded by noting that the statement at $n=1$ is trivial. \hfill \square

One observes that the r.h.s. of (2.1) looks very similar to the entries of the Gram matrix $G$. In fact, if we call $G$ the quantity in (2.1), it is easy to see that $G$ is the matrix of $G \in \mathbb{C}[S_n]$ acting in either the left or right regular representation of $\mathbb{C}[S_n]$, with standard basis $S_n$.

Now, by inserting $1 = \sum_{T;|T|=n} e_T$ in the formula (2.1) above and applying (1.3), one finds that $e_T G$ depends on $T$ only via its shape $\lambda$; therefore, one can write

$$G = \sum_{\lambda \vdash n} c_{\lambda} P_{\lambda}$$

Thus, $G^{-1} = \sum_{\lambda \vdash n} c_{\lambda}^{-1} P_{\lambda}$ when $G$ is invertible, and more generally, noting that the matrix of $P_{\lambda}$ acting in left and right representation is symmetric (cf (1.5)), we conclude:

**Proposition 2** (Collins). **If one defines**

$$W = \sum_{\lambda \vdash n} c_{\lambda}^{-1} P_{\lambda}$$

**then the Weingarten matrix $W$ is the matrix of $W$ in both left or right regular representation.**

Theorem 2.1 of [3] is recovered by replacing $P_{\lambda}$ with its expression (1.5) and by noting that $c_{\lambda}$ is up to a constant factor the dimension of the $GL(\tau)$ irreducible representation associated to $\lambda$, that is the Schur function with partition $\lambda$ and parameters $1, \ldots, 1_{\tau}$.

### 3. Weingarten matrix for the orthogonal group

We now consider the case of $O(\tau)$ (either real or complex orthogonal group) acting on $V = ((\mathbb{C}^{\tau})^{\otimes 2n} \simeq Hom((\mathbb{C}^{\tau})^{\otimes n})$ where $\mathbb{C}^{\tau}$ is in the fundamental representation of $O(\tau)$. The Weingarten matrix in this case was considered in [5, 11, 4]. Once again Schur–Weyl duality provides us with a set of generators of the invariant subspace, as elements of the Brauer algebra of size $n$, which for our purposes are conveniently defined as follows: they are involutions without fixed points of $\{1, \ldots, 2n\}$. Let us denote by $B_n$ their set. The Gram matrix turns out to be:

$$G_{\pi,\pi'} = \tau^{\frac{1}{2}\text{number of cycles of } \pi\pi'}$$

Graphically, one half of the number of cycles of $\pi\pi'$ is simply the number of loops produced by pasting together $\pi$ and $\pi'$ viewed as pairing of points $\{1, \ldots, 2n\}$:
$S_{2n}$ acts by conjugation on $B_n$, and this action is transitive. Let us pick a particular element $\beta_n \in B_n$:

$$\beta_n = s_1,2 \cdots s_{2n-1},2n = \begin{array}{cccccc}
1 & 2 & 3 & \cdots & 2n-1 & 2n
\end{array}$$

Then $B_n \simeq S_{2n}/H_n$, where $H_n$ is the stabilizer of $\beta_n$. Explicitly, $H_n$ is the subgroup of $S_{2n}$ that is generated by the transpositions $s_{2i-1},2i$, $i = 1, \ldots, n$, and by the “double elementary transpositions” $s_{2i-1},2i+1 \cdot s_{2i},2i+2$, $i = 1, \ldots, n-1$, making it isomorphic to the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ (or wreath product $\mathbb{Z}/2\mathbb{Z} \wr S_n$) a.k.a. the hyperoctahedral group.

It is convenient to define $\mathbb{C}[B_n]$ as the subspace of $\mathbb{C}[S_{2n}]$ consisting of vectors which are stable by right multiplication by $H_n$. This way we can work entirely inside the group algebra $\mathbb{C}[S_{2n}]$ (this means that we use the definitions of section 1.3 with $n$ replaced with $2n$). Define

$$P_{H_n} = \frac{1}{|H_n|} \sum_{h \in H_n} h$$

$P_{H_n}$ averages over the group $H_n$. Its image by multiplication on the right on $\mathbb{C}[S_{2n}]$ is exactly $\mathbb{C}[B_n]$. $\mathbb{C}[B_n]$ possesses a standard basis indexed by $B_n$: to $\pi \in B_n$ one associates $\sigma P_{H_n}$, where $\sigma$ is any permutation such that $\pi = \sigma \beta_n \sigma^{-1}$.

Below we shall use the property that in order to average over $H_n$ one can first average over a subgroup $\Gamma$ of it, i.e. with obvious notations $P_{H_n} = P_{\Gamma} P_{H_n} = P_{H_n} P_{\Gamma}$.

We now have the following remarkable identity, similar to Prop. 1:

**Proposition 3.** There exists a choice of representatives of cosets of $S_{2n}/H_n$, that is of $\sigma_\pi$ satisfying $\sigma_\pi \beta_n \sigma_\pi^{-1} = \pi$ for each $\pi \in B_n$, such that

$$\prod_{k=1}^n (\tau + m_{2k-1}) = \sum_{\pi \in B_n} \sigma_\pi \frac{1}{2} \text{number of cycles of } \beta_n \pi$$

**Proof.** The proof is extremely similar to that of Prop. 1. It is convenient to order the product in the l.h.s. as $(\tau + m_{n-1}) \cdots (\tau + m_1)$. Note that there are as many terms in the two sides of the equation, namely $(2n-1)!!$. Once again we shall provide a term by term identification, and proceed by induction. Start with a pairing $\pi \in B_n$. There are two possibilities. Either (i) $\pi(2n) = 2n - 1$, in which case one applies the induction hypothesis to $\pi|_{\{1, \ldots, 2n-2\}}$, and note that $\pi \beta_n$ has two more cycles than their restriction to $\{1, \ldots, 2n-2\}$, namely the two cycles coming from the extra loop $2n-1 \rightarrow 2n$. Then $\pi$ corresponds to the same term in $(\tau + m_{2n-3}) \cdots (\tau + m_1)$ and to the term $\tau$ in $\tau + m_{2n-1}$, with the same representative $\sigma_\pi = \sigma_{\pi|_{\{1, \ldots, 2n-2\}}}$. Or (ii) $\pi(2n) < 2n - 1$, in which case one conjugates $\pi$ by $s_{\pi(2n),2n-1}$, with the following effect:

$$\pi = \begin{array}{cccccc}
\pi(2n) & \pi(2n-1) & \cdots & \pi(1) & 2n-1 & 2n
\end{array}$$

Now apply the induction hypothesis to $\pi' := s_{\pi(2n),2n-1} \pi s_{\pi(2n),2n-1}^{-1}$, noting that the number of cycles of $\pi' \beta_{n-1}$ is the same as that of $\pi \beta_n$ (the loop passing through $\pi(2n-1)$ and $\pi(2n)$ has simply shrunk). Define $\sigma_\pi = s_{\pi(2n),2n-1} \sigma_{\pi'}$. Then $\sigma_\pi \pi_\beta_n \sigma_\pi^{-1} = \pi$, and $\pi$ corresponds
to the same term in $(\tau + m_{2n-3}) \cdots (\tau + m_1)$ as $\pi'$, and to the transposition $s_{\pi(2n),2n-1}$ in $\tau + m_{2n-1}$.

The statement at $n = 1$ is trivial ($\sigma_{\beta_1} = 1$). \quad \square

Note that contrary to the unitary case, the l.h.s. of (3.1) is a nonsymmetric polynomial of the Jucys–Murphy elements and is therefore not central. Its introduction is justified by the

**Lemma 1.** Let $G$ be the quantity in (3.1):

\begin{equation}
G = \prod_{k=1}^{n}(\tau + m_{2k-1})
\end{equation}

Then $GP_{H_n} = P_{H_n}G$, that is $G$ acting by multiplication on the right leaves the subspace $\mathbb{C}[B_n]$ stable. Furthermore, its matrix in the standard basis of $\mathbb{C}[B_n]$ is $G$.

**Proof.** Proposition 3 implies that

$$GP_{H_n} = \frac{1}{|H_n|} \sum_{\sigma \in S_{2n}} \sigma \cdot \tau^{\frac{1}{2} \text{number of cycles of } \sigma_{\beta_n}^{-1}}$$

Apply to both sides the anti-isomorphism of $\mathbb{C}[S_{2n}]$ which sends a permutation to its inverse. Clearly the r.h.s. is invariant, as are $G$ and $P_{H_n}$; thus $GP_{H_n} = P_{H_n}G$. Since $\mathbb{C}[B_n]$ is by definition the image of $P_{H_n}$ acting on the right, we have the first part of the lemma.

The second part consists in rewriting the equality above (with $GP_{H_n}$ replaced with $P_{H_n}G$) in the standard basis of $\mathbb{C}[B_n]$, the $\sigma_{\pi}P_{H_n}$ ($\pi \in B_n$):

$$\sigma_{\pi}P_{H_n}G = \frac{1}{|H_n|} \sum_{\sigma \in S_{2n}} \sigma_{\pi} \sigma \cdot \tau^{\frac{1}{2} \text{number of cycles of } \sigma_{\beta_n}^{-1}}$$

$$= \frac{1}{|H_n|} \sum_{\sigma' \in S_{2n}} \sigma' \cdot \tau^{\frac{1}{2} \text{number of cycles of } \sigma'_{\beta_n}^{-1} \sigma_{\beta_n}^{-1}} \quad (\sigma' = \sigma_{\pi})$$

$$= \sum_{\pi' \in B_n} G_{\pi',\pi} \sigma_{\pi'}P_{H_n} \quad (\pi' = \sigma'_{\beta_n}^{-1})$$

\quad \square

As an amusing corollary, this implies that the matrices $G$ commute for distinct values of the parameter $\tau$.

In order to obtain an explicit formula for $G$, we need the following

**Proposition 4.** Let $T$ be a tableau with $2n$ boxes. Then $P_{H_n}e_T \neq 0$ implies that the numbers $(2k - 1, 2k)$ are on the same line of $T$ (i.e. $\begin{array}{c} 2k-1 \ 2k \end{array}$) for all $k = 1, \ldots, n$.

This can actually be found in [2].

**Proof.** Let $k$ be an integer between 1 and $n$. The key of the proof is the following identity:

\begin{equation}
P_{H_n}(m_{2k} - m_{2k-1} - 1) = 0
\end{equation}

Writing $P_{H_n}(m_{2k} - m_{2k-1} - 1) = P_{H_n}(s_{2k-1,2k-1} - 1 + \sum_{i=1}^{k-1}(s_{2i-1,2k-1} - s_{2i-1,2k-1} + s_{2i,2k} - s_{2i,2k-1}))$, one notes that the first term is annihilated by $1 + s_{2k-1,2k}$, that is averaging over a subgroup
\[ \mathbb{Z}/2\mathbb{Z} of \mathcal{H}_n, \text{ while the } i^{th} \text{ term in the sum is annihilated by averaging over the subgroup } \mathcal{H}_2 \text{ of } \mathcal{H}_n \text{ which acts on } \{2i - 1, 2i, 2k - 1, 2k\} \text{ (the latter calculation being simply the special case } n = 2 \text{ of the formula), which results in the equality (3.3).} \]

Now multiply on the right (3.3) by \( e_T \) for some standard Young tableau \( T \). We find
\[ P_{\mathcal{H}_n} e_T (c(T_{2k}) - c(T_{2k-1}) - 1) = 0 \]
If \( P_{\mathcal{H}_n} e_T \neq 0 \), then \( c(T_{2k}) - c(T_{2k-1}) - 1 = 0 \), which by inspection implies that the box \( 2k \) is directly to the right of the box \( 2k - 1 \) in \( T \). \( \square \)

As a consequence, the tableaux \( T \) for which \( P_{\mathcal{H}_n} e_T \neq 0 \) have a very special structure; they are in bijection with tableaux of \( n \) boxes by the “doubling” procedure in which each box \( k \) is replaced with two boxes \( 2k - 1 \) \( 2k \). E.g.
\[
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \rightarrow \begin{array}{cccc}
1 & 2 & 5 & 6 \\
3 & 4 \\
\end{array}
\]

In particular, the only allowed shapes have even lengths of rows; let us denote them by \( 2\lambda \) where \( \lambda \vdash n \).

We can now proceed similarly to the calculation of the previous section, that is insert say on the left \( 1 = \sum_T e_T \) in (3.2) and then project using \( P_{\mathcal{H}_n} \). We find that \( P_{\mathcal{H}_n} e_T G \) is either zero or depends on \( T \) only via its shape \( 2\lambda \). We conclude by direct computation that
\[ P_{\mathcal{H}_n} G = P_{\mathcal{H}_n} \sum_{\lambda \vdash n} c_\lambda P_{2\lambda} \]
\[ c_\lambda := \prod_{(i,j) \in \lambda} (\tau + 2j - 1 - i) \]

In other words, multiplication on the right by \( G \) on \( \mathbb{C} [\mathcal{B}_n] \) is the same as multiplying by \( \sum_{\lambda \vdash n} c_\lambda P_{2\lambda} \) (on the left or the right, since the \( P_{2\lambda} \) are central). Thus, we finally have the

**Proposition 5.** Define
\[
W = \sum_{\lambda \vdash n, c_\lambda \neq 0} c_\lambda^{-1} P_{2\lambda}
\]

Then \( W \) is the matrix of \( W \) acting by multiplication on the left or the right on \( \mathbb{C} [\mathcal{B}_n] \).

Note that \( P_{2\lambda} \) plays here the role of projector onto isotypic components of \( \mathbb{C} [\mathcal{B}_n] \) as a \( \mathbb{C} [S_{2n}] \)-module. The main result of [4] (Thm 3.1) is formula (3.4) in which these projectors have been rewritten more explicitly using (1.5), i.e. \( (P_{2\lambda})_{\pi,\pi'} = \frac{\chi_{2\lambda}(1)}{|S_{2n}|} \sum_{\sigma \in S_{2n} : \sigma \pi' = \pi \sigma} \chi_{2\lambda}(\sigma^{-1}) \).

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