High energy scattering in the saturation regime including running coupling and rare fluctuation effects

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The analytic result for the S-matrix in the saturation regime including the running coupling is obtained. To get this result we solve the Balitsky and Kovchegov-Weigert evolution equations in the saturation regime, which include running coupling corrections. We study also the effect of rare fluctuations on top of the running coupling. We find that the rare fluctuations are less important in the running coupling case as compared to the fixed coupling case.

I. INTRODUCTION

The Balitsky-Kovchegov (BK) equation [1, 2] is a non-linear evolution equation which describes the high energy scattering of a q̅q dipole on a target in the case of fixed coupling. An analytic solution to the BK equation in the saturation regime has been found by Levin and Tuchin [3]. The BK equation can be viewed as a mean field version of more complete equation [1] where the higher correlations are neglected: The S-matrix of the scattering of two QCD dipoles on a target is replaced in the BK equation by the product of the S-matrices of the individual dipoles. Such a replacement is legitimate only in the absence of fluctuations in the light cone wavefunction of the target [4]. However, in Ref. 5 was shown that rare fluctuations do change the result for the S-matrix in the saturation region.

Recently, the evolution equations which include running coupling effects have been derived by Balitsky and Kovchegov-Weigert [6, 7]. They found that the running coupling corrections are included in the BK kernel by replacing the fixed coupling $\alpha_s$ in it with a “triumvirate” of the running couplings. A more complete evolution equation has been studied by Albacete and Kovchegov [13], they have calculated in addition to the Balitsky and Kovchegov-Weigert equations also the so called subtraction contributions. A numerical solution of the more complete evolution equations were given in [13].

In this work, we will analytically solve these equations in the saturation region and obtain an analytic result for the S-matrix. We find that the running coupling corrections modify the S-matrix a lot as compared to the fixed coupling case. Moreover, we study the effect of the rare fluctuations on top of the running coupling in the way as it was done in Ref. 5 for the fixed coupling case. We find that the rare fluctuations are less important in the running coupling case as compared to the fixed coupling case.

II. FIXED COUPLING CASE

The BK equation [1, 2] gives the evolution with rapidity $Y = \ln(1/x)$ of the scattering amplitude $S(x_\perp, y_\perp, Y)$ of a q̅q dipole with a target which may be another dipole, a hadron or a nucleus. The BK equation is a simple equation to deal with the onset of unitarity and to study parton saturation phenomena at high energies. The analytic solution to the fixed coupling BK equation for the S-matrix deep in the saturation regime has been derived by Levin and Tuchin [2]. This solution agrees with the one derived by solving the BK equation in the small S limit [8]. In this section we will give a simple derivation of the BK equation and its solution in the saturation regime.

A. The BK equation

In the high-energy scattering of a quark-antiquark dipole on a target, it is convenient to view the scattering process in a frame where the dipole is moving along the negative z-axis and the target is moving along the positive z-axis. Further we assume that almost all of the rapidity of the scattering, $Y$, is taken by the target. We denote the scattering amplitude of a dipole, consisting of a quark at transverse coordinate $x_\perp$ and an antiquark at transverse coordinate $y_\perp$, scattering on a target by $S(x_\perp, y_\perp, Y)$. Now suppose we increase Y by a small amount $dY$. We wish to know how $S(x_\perp, y_\perp, Y)$ changes with the small amount $dY$. If the rapidity of the dipole is increased while that of the

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target is kept fixed, then the dipole has a probability to emit a gluon due to the change \(dY\). We now calculate the probability for producing this quark-antiquark-gluon state. In the large \(N_c\) limit the quark-antiquark-gluon state can be viewed as a system of two dipoles – one of the dipoles consists of the initial quark and the antiquark part of the gluon while the other dipole is given by the quark part of the gluon and the initial antiquark. Using the dipole model the probability for producing the quark-antiquark-gluon state from the initial quark-antiquark state is \(\frac{3}{10}\).

\[
\frac{dP}{d^2z} = \frac{\alpha N_c}{2\pi^2} d^2 z \frac{(x_{\perp} - y_{\perp})^2}{(x_{\perp} - z_{\perp})^2(z_{\perp} - y_{\perp})^2},
\]

where \(z_{\perp}\) is the transverse coordinate of the emitted gluon. The change in the \(S\)-matrix, \(dS\), for a dipole-hadron scattering is given by multiplying the probability \(dP\) with the \(S\)-matrix

\[
\frac{\partial}{\partial Y} S(x_{\perp} - y_{\perp}, Y) = \frac{\alpha N_c}{2\pi^2} \int d^2 z \frac{(x_{\perp} - y_{\perp})^2}{(x_{\perp} - z_{\perp})^2(z_{\perp} - y_{\perp})^2} \left[ S^{(2)}(x_{\perp} - z_{\perp}, z_{\perp} - y_{\perp}, Y) - S(x_{\perp} - y_{\perp}, Y) \right],
\]

where \(S^{(2)}(x_{\perp} - z_{\perp}, z_{\perp} - y_{\perp}, Y)\) stands for a simultaneous scattering of the two produced dipoles on the target (see the first diagram on r.h.s of Fig. 1). The last term in (2) describes the scattering of a single dipole on the target because the gluon is not in the wavefunction of the dipole at the time of the scattering (see the last two diagrams in Fig. 1).

It is hard to directly use Eq. (2) to study problems of parton evolution and parton saturation phenomena at high density and high energy QCD, since \(S^{(2)}\) is not known. Using the mean field approximation for the gluonic fields in the target

\[
S^{(2)}(x_{\perp} - z_{\perp}, z_{\perp} - y_{\perp}, Y) = S(x_{\perp} - z_{\perp}, Y)S(z_{\perp} - y_{\perp}, Y),
\]

one gets the Kovchegov equation \(\frac{\partial}{\partial Y} S(x_{\perp} - y_{\perp}, Y) = \frac{\alpha N_c}{2\pi^2} \int d^2 z \frac{(x_{\perp} - y_{\perp})^2}{(x_{\perp} - z_{\perp})^2(z_{\perp} - y_{\perp})^2} \left[ S(x_{\perp} - z_{\perp}, Y)S(z_{\perp} - y_{\perp}, Y) - S(x_{\perp} - y_{\perp}, Y) \right].
\]

With \(N(x_{\perp} - y_{\perp}, Y) = 1 - S(x_{\perp} - y_{\perp}, Y)\), another useful version of the Kovchegov equation is obtained

\[
\frac{\partial}{\partial Y} N(x_{\perp} - y_{\perp}, Y) = \frac{\alpha N_c}{2\pi^2} \int d^2 z \frac{(x_{\perp} - y_{\perp})^2}{(x_{\perp} - z_{\perp})^2(z_{\perp} - y_{\perp})^2} \left[ N(x_{\perp} - z_{\perp}, Y) + N(z_{\perp} - y_{\perp}, Y) - N(x_{\perp} - y_{\perp}, Y) \right].
\]

Eq. (5) has the following probabilistic interpretation: when evolved in rapidity, the initial quark-antiquark dipole of size \(x_{\perp} - y_{\perp}\) decays into two dipoles of size \(x_{\perp} - z_{\perp}\) and \(z_{\perp} - y_{\perp}\) with the decay probability \((x_{\perp} - y_{\perp})^2/(x_{\perp} - z_{\perp})^2(z_{\perp} - y_{\perp})^2\) which is usually called as BFKL kernel. These two dipoles then interact with the target. The non-linear term takes into account a simultaneous interaction of two produced dipoles with the target. The non-linear term prevents the amplitude from growing boundlessly with rapidity and ensures the unitarity of the scattering amplitude. When the scattering is weak \(N \to 0\), the nonlinear term \(N(x_{\perp} - z_{\perp}, Y)N(z_{\perp} - y_{\perp}, Y)\) can be dropped and the linear equation remaining is the dipole version \(\frac{d}{dy}\) of the BFKL equation \([11, 12]\).

\[
\begin{align*}
\frac{dS}{dy} & = \frac{\alpha N_c}{2\pi^2} \int d^2 z \frac{(x_{\perp} - y_{\perp})^2}{(x_{\perp} - z_{\perp})^2(z_{\perp} - y_{\perp})^2} \left[ S(x_{\perp} - z_{\perp}, z_{\perp} - y_{\perp}, Y) - S(x_{\perp} - y_{\perp}, Y) \right],
\end{align*}
\]

FIG. 1: Diagrams corresponding to terms in the evolution equation \([2]\).
B. Solution to the BK equation in the saturation regime

In the high-energy regime where unitarity corrections become important or $S(x_\perp - y_\perp, Y)$ is small, Eq. (4) is easier to use since the quadratic term $S(x_\perp - z_\perp, Y)S(z_\perp - y_\perp, Y)$ can be neglected in which case one needs only keep the second term on the r.h.s of (4) giving

$$\frac{\partial}{\partial Y}S(x_\perp - y_\perp, Y) = \frac{\alpha N_c}{2\pi^2} \int d^2z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2(z_\perp - y_\perp)^2} S(x_\perp - y_\perp, Y).$$

(6)

In the above equation, we have assumed that $S$ is small which holds only when the dipole size is large compared to $1/Q_s$. Therefore the lower bound of integration in (6) should restrict to the regime $(x_\perp - y_\perp)^2 \gg 1/Q_s^2$ as well as to the regime $(x_\perp - z_\perp)^2 \gg 1/Q_s^2$. In the logarithmic regime of integration one gets

$$\frac{\partial}{\partial Y}S(x_\perp - y_\perp, Y) = -2\frac{\alpha N_c}{2\pi^2} \int_{1/Q_s^2}^{(x_\perp - y_\perp)^2} d(z_\perp - y_\perp)^2 \frac{1}{(z_\perp - y_\perp)^2} S(x_\perp - y_\perp, Y).$$

(7)

Note that the factor 2 in the above equation comes from the symmetry of the two regions dominating the integral, either from $1/Q_s \ll r_1 \ll r$, $r_2 \sim r$ or $1/Q_s \ll r_2 \ll r$, $r_1 \sim r$. Now it is easy to get the solution to Eq. (7)

$$S(x_\perp - y_\perp, Y) = \exp \left[ -\frac{c}{2} \left( \frac{\alpha N_c}{\pi} \right)^2 (Y - Y_0)^2 \right] S(x_\perp - y_\perp, Y_0),$$

(8)

where we have used

$$Q_S^2(Y) = \exp \left[ \frac{\alpha N_c}{c \pi} (Y - Y_0) \right] Q_S^2(Y_0)$$

(9)

and

$$Q_S^2(Y_0)(x_\perp - y_\perp)^2 = 1.\quad (10)$$

Eq. (8) gives the standard result given in the literature [8]. We have gone through such a detailed “derivation” of (8) since one of the main purposes of the present paper is to show how Eq. (8) is modified once running coupling effects and rare fluctuations are included.

III. RUNNING COUPLING CASE

The BK equation only considers the resummation of leading logarithmic (LL) $\alpha_s \ln (1/x_B)$ corrections with a fixed coupling constant $\alpha_s$. The running coupling corrections due to fermion (quark) bubble diagrams, which would bring in a factor of $\alpha_s N_f$, modify the evolution equation, which is not leading logarithms anymore. Once including $\alpha_s N_f$ corrections, the obtained contributions have to divided into two parts, running coupling part and the “subtraction” part. The first part has a form as the leading order BK kernel but with the running coupling replacing the fixed coupling and the second part brings in new structures into the evolution equation.

A. Balitsky and Kovchegov-Weigert equations

The evolution equation including higher order corrections reads [13]

$$\frac{\partial S(x_\perp - y_\perp, Y)}{\partial Y} = \mathcal{R}[S] - S[S].$$

(11)

The first term in r.h.s of (11), $\mathcal{R}$, which is referred to as the ‘running coupling’ contribution and resums all power of $\alpha_s N_f$ corrections to the evolution. The $\mathcal{R}$ has a form as the leading order one but with modified kernel which includes all effects of the running coupling

$$\mathcal{R}[S(x_\perp - y_\perp, Y)] = \int d^2z_\perp \tilde{K}(x_\perp, y_\perp, z_\perp) [S(x_\perp - z_\perp, Y) S(z_\perp - y_\perp, Y) - S(x_\perp - y_\perp, Y)].$$

(12)
The BK kernel is modified because the propagator of the emitted gluon in the original parent dipole is now dressed with quark loops in contrast to leading order or fixed coupling one. This modifies the emission probability of the gluon but doesn’t change the leading order interaction terms (see Fig.2A).

Using \( N(x_\perp - y_\perp, Y) = 1 - S(x_\perp - y_\perp, Y) \), another useful version of (12) is:

\[
\mathcal{R} [N(x_\perp - y_\perp, Y)] = \int d^2 z_\perp \tilde{K}(x_\perp, y_\perp, z_\perp) [N(x_\perp - z_\perp, Y) + N(z_\perp - y_\perp, Y)] - N(x_\perp - y_\perp, Y) - N(x_\perp - z_\perp, Y) N(z_\perp - y_\perp, Y)
\]

with modified kernel \( \tilde{K}(x_\perp, y_\perp, z_\perp) \) which has two kinds of expressions since two different separation schemes of running coupling and subtraction have been used in [6, 7] (see [13] for more discussions on separation schemes). Balitsky took the transverse coordinate of either the quark at \( z_{\perp1} \) or the antiquark at \( z_{\perp2} \) to be the subtraction point. He got the kernel of the running coupling contribution as follows [6]:

\[
\tilde{K}^{\text{Bal}}(r, r_1, r_2) = \frac{N_c \alpha_s(r_1^2)}{2 \pi^2} \left[ \frac{r^2}{r_1^2 r_2^2} + \frac{1}{r_1^2} \left( \frac{\alpha_s(r_1^2)}{\alpha_s(r_2^2)} - 1 \right) + \frac{1}{r_2^2} \left( \frac{\alpha_s(r_2^2)}{\alpha_s(r_1^2)} - 1 \right) \right].
\]

Here we introduce the notation \( r = x_\perp - y_\perp, r_1 = x_\perp - z_\perp \) and \( r_2 = z_\perp - y_\perp \) for the sizes of parent and of the new daughter dipoles produced by the evolution. On the other hand, in the subtraction scheme proposed by Kovchegov-Weigert the subtraction point is fixed at the transverse coordinate of the gluon at \( z_\perp = \eta z_{\perp1} + (1 - \eta) z_{\perp2} \) in which \( \eta \) is the longitudinal momentum fraction of gluon carried by quark. They got the modified kernel of the running coupling contribution [7]:

\[
\tilde{K}^{\text{KW}}(r, r_1, r_2) = \frac{N_c \alpha_s(r_1^2)}{2 \pi^2} \left[ \frac{r^2}{r_1^2 r_2^2} \alpha_s(r_1^2) \frac{\alpha_s(r_2^2)}{\alpha_s(R^2)} + \alpha_s(r_2^2) R^2 \right] + \frac{1}{r_1^2}
\]

with

\[
R^2(r, r_1, r_2) = r_1 r_2 \left( \frac{r_2}{r_1} \right) \left( \frac{r_1 + r_2}{r_1^2} \right) \left( \frac{r_2}{r_1} \right) \left( \frac{r_1 + r_2}{r_2^2} \right).
\]

The second term in r.h.s of (11), \( S \), which is referred to as ‘subtraction’ contribution, is given by

\[
S[S] = \alpha_\mu^2 \int d^2 z_{\perp1} d^2 z_{\perp2} K_\perp(x_\perp, y_\perp; z_{\perp1}, z_{\perp2}) \left[ S(x_\perp - w_\perp, Y) S(w_\perp - y_\perp, Y) - S(x_\perp - z_{\perp1}, Y) S(z_{\perp2} - y_\perp, Y) \right]
\]

with \( \alpha_\mu \) the bare coupling. The interaction structures are modified in the above equation since the quark-antiquark pair is added to the evolved wave function (see Fig.2B). The \( K_\perp(x_\perp, y_\perp; z_{\perp1}, z_{\perp2}) \) is a resummed JIMWLK kernel which can be found in [13].

\[
K_\perp(x_\perp, y_\perp; z_{\perp1}, z_{\perp2}) = C_F \sum_{m,n=0}^{1} (-1)^{m+n} K_\perp(x_\perp, y_\perp; z_{\perp1}, z_{\perp2}).
\]
In terms of Balitsky’ subtraction scheme one substitutes \( w_\perp = z_{\perp 1} \) or \( w_\perp = z_{\perp 2} \) in Eq. (17) and gets the subtraction term

\[
S^{\text{Bal}}[S] = \alpha_s^2 \int d^2 z_{\perp 1} d^2 z_{\perp 2} K_1(x_\perp, y_\perp; z_{\perp 1}, z_{\perp 2}) [S(x_\perp - z_{\perp 1}, Y) S(z_{\perp 1} - y_\perp, Y) - S(x_\perp - z_{\perp 1}, Y) S(z_{\perp 2} - y_\perp, Y)].
\]

According to Kovchegov-Weigert’s subtraction scheme one substitutes \( w_\perp = z_\perp = \eta z_{\perp 1} + (1 - \eta) z_{\perp 2} \) in Eq. (17) and gets

\[
S^{\text{KW}}[S] = \alpha_s^2 \int d^2 z_{\perp 1} d^2 z_{\perp 2} K_1(x_\perp, y_\perp; z_{\perp 1}, z_{\perp 2}) [S(x_\perp - z_{\perp 1}, Y) S(z_{\perp 1} - y_\perp, Y) - S(x_\perp - z_{\perp 1}, Y) S(z_{\perp 2} - y_\perp, Y)].
\]

Eq. (17) shows that the \( S[S] \) is of order \( \alpha_s^2 \) while \( R[S] \) is of order \( \alpha_s \) and all terms of \( S[S] \) are quadratic in \( S, S(x_\perp - w_\perp, Y) S(w_\perp - y_\perp, Y), S(x_\perp - z_{\perp 1}, Y) S(z_{\perp 1} - y_\perp, Y) \). Thus, for high rapidity and small \( S \), the subtraction term is small as compared to the running coupling term, as also shown numerically in [13]. Since this is the kinematic region in which we are interested in this paper, we hereafter will neglect the subtraction term. In this paper we study the evolution equation in the saturation regime where the evolution equation including running coupling corrections can be solved analytically.

### B. Solution to Balitsky and Kovchegov-Weigert equations in the saturation regime

In the saturation regime in which the interaction between partons is very strong, \( S(x_\perp - y_\perp, Y) \to 0 \), and unitarity corrections become important, the quadratic terms in (11) can be neglected in which case one needs only keep the second term on the r.h.s of (12). The evolution equation including running coupling is given by

\[
\frac{\partial S(x_\perp - y_\perp, Y)}{\partial Y} = - \int d^2 z \tilde{K}(r, r_1, r_2) S(x_\perp - y_\perp, Y)
\]

with modified kernel \( \tilde{K}(r, r_1, r_2) \). In the saturation region, \( r Q_s(Y) \gg 1 \), the main contribution to the integration on the r.h.s of (21) comes from either

\[ 1/Q_s \ll r_1 \ll r; \quad r_2 \sim r \]

or

\[ 1/Q_s \ll r_2 \ll r; \quad r_1 \sim r. \]

Let us look at one of them, i.e., when \( 1/Q_s \ll r_1 \ll r \), the \( r_2 \) is approximate equal to \( r \), \( r_2 \sim r \), the \( \tilde{K}^{\text{Bal}}(r, r_1, r_2) \) becomes

\[
\tilde{K}^{\text{Bal}}(r, r_1, r_2) = \frac{N_c}{2 \pi^2} \frac{\alpha_s(r_1^2)}{r_1^2} \left[ \frac{1}{r_1} \frac{\alpha_s(r_1^2)}{\alpha_s(r_2^2)} \right] = \frac{N_c}{2 \pi^2} \frac{\alpha_s(r_1^2)}{r_1^2} \left[ \frac{1}{r_1} \frac{\alpha_s(r_1^2)}{\alpha_s(r_2^2)} \right] (21)
\]

and the \( \tilde{K}^{\text{KW}}(r, r_1, r_2) \) has the form as follows

\[
\tilde{K}^{\text{KW}}(r, r_1, r_2) = \frac{N_c}{2 \pi^2} \left[ \alpha_s(r_1^2) \frac{1}{r_1^2} - 2 \frac{\alpha_s(r_1^2)}{\alpha_s(r_2^2)} \frac{r_1 \cdot r_2}{r_1^2 r_2^2} + \alpha_s(r_2^2) \frac{1}{r_2^2} \right],
\]

here we use \( R^2(r, r_1, r_2) \approx r_1^2 \) which can be obtained via simple calculation in (18), with condition of \( 1/Q_s \ll r_1 \ll r \) and \( r_2 \sim r \). In the \( r_1 \ll r_2 \) limit it is the first term which dominates Eq. (25) and has the running coupling scale given by the size of the smaller dipole

\[
\tilde{K}^{\text{KW}}(r, r_1, r_2) \approx \frac{N_c}{2 \pi^2} \alpha_s(r_1^2) \frac{1}{r_1^2}.
\]

We wish to note that the modified Balitsky and Kovchegov-Weigert kernels including running coupling have the same form in the saturation regime. It is an interesting outcome which means that the evolution equation with running
coupling corrections is independent of the choice of transverse coordinate of subtraction point in the saturation regime. And the modified Balitsky and Kovchegov-Weigert equations with running coupling corrections are equivalent to each other in the saturation region. In other words, the $S$-matrix of the Balitsky and Kovchegov-Weigert equations are exactly the same in the saturation region.

Now let us put the modified kernel \( \frac{24}{5} \) or \( \frac{26}{5} \) into \( \frac{21}{5} \), we can get a simplified evolution equation as follows:

\[
\frac{\partial S(r,Y)}{\partial Y} = -\frac{N_c}{2\pi^2} \int_{1/Q_0^2}^{r^2} d^2 r_1 \alpha_s(r_1^2) \frac{1}{r_1^2} S(r,Y),
\]

with the running coupling at one loop accuracy

\[
\alpha_s(r_1^2) = \frac{\mu}{1 + \mu_1 \ln \left( \frac{1}{r_1^2 \Lambda^2} \right)}
\]

giving

\[
\frac{\partial S(r,Y)}{\partial Y} = -\frac{N_c \mu}{\pi \mu_1} \left[ \ln \left( 1 + \mu_1 \ln \left( \frac{Q^2(Y)}{\Lambda^2} \right) \right) - \ln \left( 1 + \mu_1 \ln \left( \frac{1}{r^2 \Lambda^2} \right) \right) \right] S(r,Y)
\]

whose solution (see also \[14\]) is

\[
S(r,Y) = e^{-\frac{N_c \mu}{\pi \mu_1} \left[ \ln^2 \left( \frac{Q^2(Y)}{\Lambda^2} \right) \ln \left( 1 + \mu_1 \ln \left( \frac{Q^2(Y)}{\Lambda^2} \right) \right) - \ln \left( 1 + \mu_1 \ln \left( \frac{Q^2(Y)}{\Lambda^2} \right) \right) \right]} S(r_0,Y)
\]

with the saturation momentum

\[
\ln \left( \frac{Q^2(Y)}{\Lambda^2} \right) = \sqrt{c(Y - Y_0)} + O(Y^{1/6}).
\]

We wish to note that the analytic result for the $S$-matrix including the running coupling corrections is different as compared to the fixed coupling case. The exponent in Eq. \[31\] is decreasing linearly with rapidity while the exponent in Eq. \[5\] is decreasing quadratically with rapidity, which indicates that the running coupling slows down the evolution of the scattering amplitude with rapidity.

**IV. EFFECTS OF RARE FLUCTUATIONS**

**A. Fixed coupling case**

At very high energy the typical configuration of a dipole’s light-cone wavefunction is a Color Glass Condensate which is a state having high occupancy for all gluonic levels of momentum less than or equal to saturation momentum $Q_S$. In the fixed coupling case, the authors of Ref. \[5\] computed the $S$-matrix of two typical configurations (of condensate type) and of dipole-typical configuration scattering, they found that the typical configurations lead to too small results for the $S$-matrix, being proportional to $\exp\left\{ -c_1 Q_S^2 r_0^2 / \alpha_s^2 \right\}$ and $\exp\left\{ -\frac{1}{2} \ln^2(Q_S^2 r_0^2) \right\}$, respectively. $c_1$ and $c$ are constant which are not important for our purpose. Thus they tried to search for configurations which are more rare in the wavefunction but which dominate very high energy dipole-dipole scattering and lead to a larger $S$-matrix. They found the reason why the typical configurations have given a small $S$-matrix is that the typical configurations contain too many gluons at the time of collision, therefore leading to the $S$-matrix is extremely small. This suggests that the strategy for finding the rare configuration is to minimize the number of gluons by suppressing the evolution (see next section for the details of how to obtain the rare configuration). The rare configuration is a state which has no more than one dipole of size $\kappa r_0$ or larger (with $\kappa$ a constant of order 1 and $r_0$ a size of parent dipole) when the system undertakes BK evolution. In the center of mass frame, the $S$-matrix is then given by the probability of the rare configurations for each of the parent dipoles partaking in the collision, times the $S$-matrix for the scattering of two dipoles separated by a rapidity gap $Y_0$,

\[
S_Y \approx e^{-\frac{1}{2c} \ln^2(Q_S^2 r_0^2)} S_{Y_0}(r_0)
\]

which is significantly larger than the results coming from the condensate-condensate and dipole-condensate scattering.
FIG. 3: The configuration in center of mass frame.

B. Running coupling case

Following the framework of Ref. [5], consider the high-energy scattering of dipoles at zero impact parameter in the center of mass frame where one of dipoles is left-moving and the other is right-moving. In order to obtain rare configuration, we suppose that the wavefunction of the right-moving dipole consists only of the parent dipole with size \( r_0 \) in the rapidity interval \( Y_0/2 < y < Y/2 \), where \( Y_0 \) is the critical value of rapidity for the onset of unitarity corrections, with the similar requirement on the left-moving dipole in the rapidity interval \( -Y/2 < y < -Y_0/2 \). In the rapidity interval \( 0 < y < Y_0/2 \) and \( -Y_0/2 < y < 0 \) the right-moving and left-moving dipoles have normal BFKL evolution, respectively.

However, we can’t require that all evolution of right-moving dipoles are absent in the rapidity interval \( Y_0/2 < y < Y \). What we can do is to only allow that evolution which produces very small dipoles, in order to guarantee the system have no more than one dipole of size \( \kappa r_0 \) or larger, with \( \kappa \) a constant of order one. And we setup constraints to suppress the creation of dipoles much smaller than \( r_0 \) at rapidities \( y > Y_0/2 \) to avoid dipoles emitted at intermediate rapidities evolving into dipoles of size \( r_0 \) or larger at rapidity \( Y/2 \). We require that the gluon emission from the parent dipoles is forbidden if the gluon has \( k_\perp \) and \( y \) in the shaded triangles of Fig.3. The line

\[
\ln(k_\perp r_0^2) = \sqrt{c(y - \frac{Y_0}{2})}
\]

and a similar line for the lower triangle, is determined by the requirement that gluons in the right hand side of that line can’t evolve by normal BFKL evolution into shaded triangles.

Now we compute the probability of rare configurations \( S(x_\perp - y_\perp, Y - Y_0) \) which has the same meaning as the survival probability of the parent dipoles after a BFKL evolution over a rapidities interval \( Y - Y_0 \) [5]. This probability decreases with increasing \( Y \) due to gluon emission and the corresponding rate is the same as the virtual term in [13]:

\[
\frac{\partial}{\partial Y} S(x_\perp - y_\perp, Y - Y_0) = - \int d^2 z_\perp K(x_\perp, y_\perp, z_\perp) S(x_\perp - y_\perp, Y - Y_0)
\]
whose solution is similar to Eq. (30):

\[
S(r, Y - Y_0) = e^{- \frac{N_{\text{el}}}{c r_{1}} \left[ \ln^2 \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \frac{1}{\lambda_{1}} \right] + \frac{1}{1 + \lambda_{1}}} \frac{N_0}{c r_{1}} \left[ \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \frac{1}{\lambda_{1}} \right] - \frac{1}{r_{1}^2} \ln \left( 1 + \lambda_{1} \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \right)} S(r_0, Y_0)
\]

(35)

Let \(S(r_0, (Y - Y_0)/2)\) denote the probability of a parent dipole not given rise to any emission of gluon in the upper triangle of Fig.3. The S-matrix can be obtained by the product of \(S(r_0, (Y - Y_0)/2)\) for each of the parent dipoles participating in the scattering, times \(S(r_0, Y_0)\) which is a S-matrix for the scattering of two elementary dipoles. By using (35), one gets:

\[
S(r, Y) = e^{- \frac{N_{\text{el}}}{c r_{1}} \left[ \ln^2 \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \frac{1}{\lambda_{1}} \right] + \frac{1}{1 + \lambda_{1}}} \frac{N_0}{c r_{1}} \left[ \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \frac{1}{\lambda_{1}} \right] - \frac{1}{r_{1}^2} \ln \left( 1 + \lambda_{1} \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \right)} S(r_0, Y_0)
\]

(36)

which only brings in very small corrections to (30) and indicates that the rare fluctuations are less important in the running coupling case as compared to the fixed coupling case [2], where the rare fluctuations are important and the exponential factor of S-matrix in the saturation regime has twice as large as the result which emerges when fluctuations are taken into account. We also consider the rare fluctuations on top of the running coupling effects in a general frame (please see the Appendix), we find the same result as (30).

V. SHAPE OF DIPOLE CROSS SECTION WITH RUNNING COUPLING

The authors of Ref. [15] computed the scattering amplitude for \(rQ_S < 1\) using BFKL evolution and running coupling. Combining the outcome of Ref. [15] and our result (30) which is valid for \(rQ_S \gg 1\), the shape of dipole cross section with running coupling reads:

\[
N(r, Y) = \left\{ \begin{array}{ll}
\left( \frac{Q_S^2}{\Lambda^2} \right)^{-(1-\lambda_{0})} \ln \left( \frac{Q_S^2}{\Lambda^2} \right) + \frac{1}{1 + \lambda_{0}} N_0 & rQ_S \leq 1 , \\
1 - e^{- \frac{N_{\text{el}}}{c r_{1}} \left[ \ln^2 \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \frac{1}{\lambda_{1}} \right] + \frac{1}{1 + \lambda_{1}}} \frac{N_0}{c r_{1}} \left[ \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \frac{1}{\lambda_{1}} \right] - \frac{1}{r_{1}^2} \ln \left( 1 + \lambda_{1} \ln \left( \frac{Q_S^2}{\Lambda^2} \right) \right)} S_0 & rQ_S > 1 ,
\end{array} \right.
\]

where \(\lambda_{0}\) is the solution to \(\chi'(\lambda_{0})(1-\lambda_{0}) = -\chi(\lambda_{0})\) with \(\chi\) the usual BFKL eigenvalue function, \(N_0\) is a constant but with no control at this moment and \(Q_S\) is the saturation momentum including running coupling corrections.

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APPENDIX A: RARE FLUCTUATIONS IN A GENERAL FRAME

Consider a high energy scattering of a right-moving dipole of size \(r_0\) and rapidity \(Y - Y_2\) on a left-moving dipole of size \(r_1\) and rapidity \(-Y_2\) in an arbitrary frame. The frame and scattering picture are illustrated in Fig.4, where \(Y_0\) is a rapidity gap between two dipoles. For later convenience, we require that \(Y_2 \leq \frac{1}{2}(Y - Y_0)\). We require that no additional dipoles can be created from the gluon emission of left-moving dipole \(r_1\) which would have a strong interaction with the right-moving dipoles. The dipoles which would have such strong interactions would be of size \(r \geq 1/Q_S\) at the scattering time. So we should suppress the emission of those dipoles which could become of size \(1/Q_S\) or larger after a normal evolution over the rapidity interval \(-Y_2 < y < 0\). For the right-moving dipole \(r_0\), we suppress evolution over its \(Y - Y_2 - (Y_1 + Y_0)\) with the region of suppression given by the upper shaded triangle of Fig.4. The line

\[
\ln(k^2 r_{0}^2) = \sqrt{c(y - Y_1 - Y_0)}
\]

(A1)
and a similar line for the lower triangle, is determined by the requirement that gluons locating in the right hand side of that line can’t evolve by normal BFKL evolution into shaded triangles. We will determine \( Y_1 \) by maximizing the \( S \)-matrix later. The unshaded triangle, whose rapidity values go from 0 to \( Y_1 \), is a saturation region where the dipole \( r_0 \) has evolved into a Color Glass Condensate.

After we have a clear scattering picture of dipoles, the \( S \)-matrix can be evaluated at hand

\[
S(r_0, r_1, Y) = S_R(r_0, Y - Y_0 - Y_1 - Y_2)S(r_0, r_1, Y_0 + Y_1)S_L(r_1, Y_2) \tag{A2}
\]

with \( S(r_0, r_1, Y_0 + Y_1) \) is the \( S \)-matrix for scattering of an elementary dipole \( r_1 \) on a Color Glass Condensate state which is evolved from dipole \( r_0 \) and \( S_R(r_0, Y - Y_0 - Y_1 - Y_2) \) and \( S_L(r_1, Y_2) \) are the suppression factor from the no emission requirement for two dipoles, which are given in terms of the area of the upper and lower shaded regions of Fig.4. After using (30), one obtains

\[
S_R(r_0, Y - Y_0 - Y_1 - Y_2) = e^{- \frac{N_c \mu}{c \pi \mu_1} \left[ \text{c} \ln \left( \frac{1 + \mu_1 \sqrt{c(Y + Y_0)}}{1 + \mu_1 \ln \left( \frac{1}{\sqrt{S}} \right)} \right) - \frac{1}{2} \right]} (Y - Y_2 - Y_1 - Y_0) + \frac{\sqrt{c(Y + Y_0)}}{\mu_1} - \frac{1}{\mu_1} \ln \left( 1 + \mu_1 \sqrt{c(Y + Y_0)} \right) \tag{A3}
\]

and

\[
S_L(r_1, Y_2) = \exp \left[ - \frac{N_c \mu}{c \pi \mu_1} \left[ \text{c} \ln \left( \frac{1 + \mu_1 \sqrt{c(Y_1 + Y_2)}}{1 + \mu_1 \ln \left( \frac{1}{\sqrt{S}} \right)} \right) - \frac{1}{2} \right] (Y_1 + Y_2) + \frac{\sqrt{c(Y_1 + Y_2)}}{\mu_1} - \frac{1}{\mu_1} \ln \left( 1 + \mu_1 \sqrt{c(Y_1 + Y_2)} \right) \right]. \tag{A4}
\]

The \( S \) can be computed by using the BK equation with running coupling corrections since the BK equation with running coupling corrections describes correctly the scattering of an elementary dipole on a Color Glass Condensate. By using (30), one gets

\[
S(r_0, r_1, Y_0 + Y_1) = e^{- \frac{N_c \mu}{c \pi \mu_1} \left[ \text{c} \ln \left( \frac{1 + \mu_1 \sqrt{c(Y + Y_1)}}{1 + \mu_1 \ln \left( \frac{1}{\sqrt{S}} \right)} \right) - \frac{1}{2} \right]} (Y_0 + Y_1) + \frac{\sqrt{c(Y + Y_1)}}{\mu_1} - \frac{1}{\mu_1} \ln \left( 1 + \mu_1 \sqrt{c(Y + Y_1)} \right) S(r_0, Y_0). \tag{A5}
\]

Substituting (A3), (A4) and (A5) into (A2), one obtains:

\[
S(r_0, r_1, Y) = \exp \left[ - \frac{N_c \mu}{c \pi \mu_1} \left[ \text{c} \ln \left( \frac{1 + \mu_1 \sqrt{c(Y - Y_2 - Y_1 - Y_0)}}{1 + \mu_1 \ln \left( \frac{1}{\sqrt{S}} \right)} \right) - \frac{1}{2} \right] (Y - Y_2 - Y_1 - Y_0) + \frac{\sqrt{c(Y - Y_2 - Y_1 - Y_0)}}{\mu_1}
- \frac{1}{\mu_1} \ln \left( 1 + \mu_1 \sqrt{c(Y - Y_2 - Y_1 - Y_0)} \right) + c \ln \left( \frac{1 + \mu_1 \sqrt{c(Y_1 + Y_2)}}{1 + \mu_1 \ln \left( \frac{1}{\sqrt{S}} \right)} - \frac{1}{2} \right) (Y_1 + Y_2) + \frac{\sqrt{c(Y_1 + Y_2)}}{\mu_1}
- \frac{1}{\mu_1} \ln \left( 1 + \mu_1 \sqrt{c(Y_1 + Y_2)} \right) \right] S(r_0, Y_0). \tag{A6}
\]

which connects to a set of configurations of the wavefunction described by rapidity \( Y_1 \). The \( S \)-matrix is determined by the values of \( Y_1 \) which maximizes the r.h.s of Eq. (A6) or equivalently minimizes the exponent of the r.h.s of Eq. (A6). We obtain

\[
Y_1 = \frac{1}{2} (Y - Y_0) - Y_2. \tag{A7}
\]

Take this \( Y_1 \) into (A6), finally the \( S \)-matrix is:

\[
S(r, Y) = e^{- \frac{N_c \mu}{c \pi \mu_1} \left[ \text{c} \ln \left( \frac{\sqrt{c(\gamma)}}{\Lambda^2} \right) \ln \left( \frac{1 + \mu_1 \sqrt{c(\gamma)}}{1 + \mu_1 \ln \left( \frac{1}{\sqrt{S}} \right)} - \frac{1}{2} \right) + \sqrt{\mu_1 \ln \left( \frac{\sqrt{c(\gamma)}}{\Lambda^2} \right)} - \frac{1}{\mu_1} \ln \left( 1 + \mu_1 \sqrt{c(\gamma)} \right) \right]} \tag{A8}
\]

which is exactly the same as the corresponding result [30] in the center of mass frame.

[1] I. Balitsky, Nucl. Phys. B463 (1996) 99; Phys. Lett. B518 (2001) 235; “High-energy QCD and Wilson lines”, hep-ph/0101042
FIG. 4: The configuration in a general frame.

[2] Yu.V. Kovchegov, Phys. Rev. D60 (1999) 034008; ibid. D61 (1999) 074018.
[3] E. Levin and K. Tuchin, Nucl. Phys. B573 (2000) 83;
[4] A.H. Mueller, A.I. Shoshi and S.M.H. Wong, Nucl. Phys. B715 (2005) 440;
[5] E. Iancu and A.H. Mueller, Nucl. Phys. A730 (2004) 494;
[6] I.I. Balitsky, Phys. Rev. D75 (2007) 014001;
[7] Yu.V. Kovchegov and H. Weigert, Nucl. Phys. A784 (2007) 188;
[8] A.H. Mueller, hep-ph/0111244;
[9] A.H. Mueller, Nucl. Phys. B415 (1994) 373;
[10] A.H. Mueller and A. Shoshi, Nucl. Phys. B692 (2004) 175;
[11] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, Sov. Phys. JETP 45 (1977) 199;
[12] I.I. Balitsky and L.N. Lipatov, Sov. J. Phys. 28 (1978) 822;
[13] J.L. Albacete and Yu.V. Kovchegov, Phys. Rev. D75 (2007) 125021;
[14] A.H. Mueller, Nucl. Phys. B643 (2002) 501;
[15] A.H. Mueller and D.N. Triantafyllopoulos, Nucl. Phys. B640 (2002) 331.