Energy Contents of Some Well-Known Solutions in Teleparallel Gravity

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Abstract
In the context of teleparallel equivalent to General Relativity, we study energy and its relevant quantities for some well-known black hole solutions. For this purpose, we use the Hamiltonian approach which gives reasonable and interesting results. We find that our results of energy exactly coincide with several prescriptions in General Relativity. This supports the claim that different energy-momentum prescriptions can give identical results for a given spacetime. We also evaluate energy-momentum flux of these solutions.

Keywords: Teleparallel Gravity; Black holes; Energy-Momentum.
PACS: 04.20.Cv; 04.20.Dw

1 Introduction
One of the most interesting but challenging problems in Einstein theory of General Relativity (GR) is the localization of energy. This problem still needs a definite answer due to its unusual nature and various viewpoints. Many renowned researchers have devoted much attention to this intricate issue. The foremost energy-momentum prescription was proposed by Einstein

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himself. Following this, a large number of different energy-momentum prescriptions \[2\]-\[6\] have been derived. However, many of these are coordinate dependent, i.e., results will be meaningful only if the calculations are performed in quasi-Cartesian coordinates. Møller \[7\] and Komar \[8\] tried to overcome this weakness and proposed a coordinate independent energy-momentum prescription.

Penrose \[9\] introduced coordinate independent prescription of quasi-local mass. However, Bergqvist \[10\] showed that no two quasi-local mass definitions agreed for the Reissner-Nordström and Kerr spacetimes. Bernstein and Tod \[11\] explored the shortcomings of Penrose quasi-local mass definition in handling the Kerr metric. Afterwards, Virbhadra \[12, 13\] introduced the coincidence concept of different energy-momentum prescriptions and gave a new spirit to this problem. It has been investigated by many people \[14\] that different energy-momentum complexes demonstrate a high degree of consistency by yielding the same and reasonable result for a given spacetime. Virbhadra and his colleagues \[15, 16\] explored that Einstein, Landau-Lifshitz, Papapetrou and Weinberg (ELLPW) prescriptions provide the same results for any spacetime of Kerr-Schild class and more general spacetimes if calculations are performed in Kerr-Schild Cartesian coordinates. However, Sharif and Fatima \[17\] found results which did not support this viewpoint.

It has been found \[18, 19\] that telleparallel equivalent to General Relativity (TEGR) provides more satisfactory solution of the energy-momentum problem than does GR. This theory is based on tetrad field initiated by Møller \[20\]. Mikhail et al. \[18\] re-formulated Møller energy-momentum complex in this alternative theory. Sharif and Amir \[21\] found that energy for the closed Friedmann model is consistent with GR. However, it was concluded that energy-momentum prescriptions did not necessarily provide consistent results in two theories, i.e., TEGR and GR.

Recent literature \[22\]-\[25\] shows keen interest in the evaluation of energy-momentum by Lagrangian framework in TEGR. Maluf et al. \[26\] defined gravitational energy, momentum and angular momentum by using this formulation \[27\]. After that, many people \[28\] have used this procedure to evaluate energy and its contents for different solutions. In a recent paper \[29\], we have discussed energy and its related quantities for a class of regular black holes coupled with non-linear electrodynamics source.

In this paper, this study is extended to evaluate energy and its contents for some well-known black hole solutions by using the Hamiltonian approach. The results will be compared to those found by using different prescriptions in
The paper is organized as follows: In section 2, we present the formulation to evaluate gravitational energy, momentum, angular momentum, gravitational and matter energy-momentum fluxes. Section 3 contains brief discussion of some black hole solutions. In section 4, we evaluate energy, momentum and angular momentum for these solutions. Section 5 is devoted to study gravitational and matter energy-momentum fluxes. In the last section, we present summary and discussion of the results obtained.

Following conventions are considered throughout the paper: Spacetime indices \((\mu, \nu, \rho, \ldots)\) and tangent space indices \((a, b, c, \ldots)\) run from 0 to 3. Here \(\mu = 0\), \(i\) and \(a = (0), (i)\) denote time and space indices respectively.

## 2 Energy-Momentum and Hamiltonian Approach in Teleparallel Theory

The Weitzenböck connection \([34]\) is defined in terms of tetrad field \(e^a_\mu\) as

\[
\Gamma^\lambda_{\mu\nu} = e^a_\lambda \partial_\nu e^a_\mu
\]

and the torsion tensor

\[
T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu.
\]

The Lagrangian density of the gravitational field in the presence of matter \([27]\) is

\[
L = -\kappa e \Sigma^{abc} T_{abc} - L_M,
\]

where \(\kappa = 1/16\pi\), \(e = \det(e^a_\mu)\) and the tensor \(\Sigma^{abc}\) is given by

\[
\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^{b} - \eta^{ab}T^{c}).
\]

Consequently, the field equations are

\[
e_{a\lambda}e_{b\mu}\partial_\nu(e\Sigma^{b\lambda\nu}) - e(\Sigma^{b\nu}_{\ a} T_{ba\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}) = \frac{1}{4\kappa}eT_{a\mu}, \quad \delta L_M = eT_{a\mu}.
\]

The total Hamiltonian density is defined as \([35]\)

\[
H(e_{ai}, \Pi_{ai}) = e_{a0}C^a + \alpha_{ik}\Gamma^{ik} + \beta_k\Gamma^k + \partial_k(e_{a0}\Pi^{ak}),
\]

where \(C^a\), \(\Gamma^{ik}\), \(\Gamma^k\) and \(\alpha_{ik}\), \(\beta_k\) express primary constraints and Lagrangian multipliers respectively.
The gravitational energy-momentum over an arbitrary volume \( V \) is

\[
P^a = - \int_V d^3x \partial_i \Pi^{ai},
\]

where \(-\partial_i \Pi^{ai} = \partial_i (4\kappa e \Sigma^{a0i})\) represents the energy-momentum density \(^{26}\). The above expression can be written in terms of fluxes as

\[
\frac{dP^a}{dt} = -\Phi^a_g - \Phi^a_m,
\]

where

\[
\Phi^a_g = \int_S dS_j \phi^{aj}, \quad \Phi^a_m = \int_S dS_j (\epsilon \epsilon^{a}_{\mu} T^{j\mu})
\]

represent the a component of the gravitational energy-momentum flux and matter energy-momentum flux \(^{36}\) while \( S \) is the spatial boundary of the volume \( V \). The quantity \( \phi^{aj} \) describe the a component of the gravitational energy-momentum flux density in \( j \) direction and its expression is

\[
\phi^{aj} = \kappa \epsilon^{aj}_{\mu} (4T^{bcj} T_{b\mu} - \delta^{aj}_{\mu} \delta^{bcd} T_{bed}).
\]

The total angular momentum \(^{37}\) is

\[
M^{ik} = 2\kappa \int_V d^3x \epsilon [ -g^{im} g^{kj} T^{0m}_j + (g^{im} \delta^{jk} - \delta^{jk} g^{0i}) T^{mj}_j ].
\]

### 3 Black Hole Solutions

The generalized form of black hole solutions is given by

\[
ds^2 = -F dt^2 + F^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]

where \( F = (1 - 2\frac{M(r)}{r}) \). This metric can be reduced to some well-known black holes under special choices of \( M(r) \). Some of them are given below.

#### 3.1 de Sitter-Schwarzschild Black Hole

Dymnikova \(^{38}\) found a regular black hole solution in de Sitter form which exhibits Schwarzschild like behavior by replacing its singularity with de Sitter core. The line element is found for

\[
M_1(r) = m(1 - \exp(-\frac{r^3}{r^*_3})),
\]

where \( r^3_3 = 2mr^2_0, \ r^2_0 = \frac{3}{\Lambda} \). It behaves asymptotically Schwarzschild and de Sitter solutions when \( r \to \infty \) and \( r \to 0 \) respectively. Here \( \Xi_1 = 0 \).
3.2 Regular Black Hole with Cosmological Constant

Mo Wen-Juan et al. \cite{39} introduced a class of regular black hole solution with cosmological constant \( \Lambda \) in non-linear electrodynamics. The solution is given in the metric form with

\[
M_2(r) = \left( \frac{m r^3}{(r^2 + q^2)^{3/2}} - \frac{q^2 r^3}{2(r^2 + q^2)^2} + \frac{\Lambda r^3}{6} \right),
\]

where \( m, q \) and \( \Lambda \) represent mass, electric charge and cosmological constant respectively. For \( \Lambda = 0 \), the solution reduces to the case discussed by Ayon-Beato and Garcia \cite{40} in which they have shown that the field strength and all curvature invariants are regular everywhere. The appearance of cosmological constant does not destroy the regularity of the solution. The associated electric field strength \( \Xi_2 \) is given by

\[
\Xi_2 = qr^4 \left( \frac{r^2 - 5q^2}{(r^2 + q^2)^4} + \frac{15m}{2(r^2 + q^2)^{7/2}} \right).
\]

For \( m = 0 = q \), the solution represents the dS (\( \Lambda > 0 \)) or AdS (\( \Lambda < 0 \)) space. The solution behaves asymptotically Reissner-Nordström black hole in dS/AdS space depending on the sign of the cosmological constant \( \Lambda \) as

\[
F = 1 - \frac{2m}{r} + \frac{q^2}{r^2} + O\left( \frac{1}{r^3} \right) - \frac{\Lambda r^2}{3}.
\]

This class reduces to the Schwarzschild solution for \( q = 0 = \Lambda \).

3.3 Bardeen Regular Black Hole

Ayon-Beato and Garcia \cite{41} gave physical interpretation of Bardeen model \cite{42} by showing that charge associated with it acts as a magnetic monopole charge. This is described by the metric with

\[
M_3(r) = \frac{mr^3}{(r^2 + e^2)^{3/2}}
\]

and the associated magnetic field strength is given by

\[
\Xi_3 = \frac{e^2}{2r^4}.
\]
Here \( m \) and \( e \) stand for mass and monopole charge of a self-gravitating magnetic field of non-linear electrodynamics source respectively. This solution exhibits black hole behavior for \( e^2 \leq (16/27)m^2 \). The curvature invariants corresponding to this solution are regular everywhere. It behaves asymptotically as

\[
F = 1 - \frac{2m}{r} + \frac{3me^2}{r^3} + O\left(\frac{1}{r^5}\right).
\]

For \( e = 0 \), the solution reduces to the Schwarzschild spacetime.

### 3.4 Dyadosphere of a Reissner-Nordström Black Hole

According to Ruffini \[43\], the dyadosphere is defined as the region outside the horizon of an electromagnetic black hole where the electromagnetic field is stronger than the well-known Heisenbeg-Euler critical value for electron-positron pair production

\[
\varepsilon_{cr} = \frac{m_c^2 c^3}{\tilde{e} \hbar}.
\]

Here \( m_\varepsilon \) and \( \tilde{e} \) play the role of mass and charge of an electron. This concept was introduced to explain gamma ray bursts. The dyadosphere region for Reissner-Nordström black hole is described by the radial interval \( r_+ \leq r \leq r_{ds} \), where \( r_+ \) (called horizon of black hole) and \( r_{ds} \) represent the inner and outer radii of the dyadosphere. The expressions for \( r_+ \) and \( r_{ds} \) are given by

\[
r_+ = \frac{Gm}{c^2} \left(1 \pm \sqrt{1 - \frac{q^2}{Gm^2}}\right),
\]

\[
r_{ds} = \sqrt{\frac{\hbar}{m_\varepsilon c}} \left(\frac{Gm}{c^2}\right) \left(\frac{mpl}{m_\varepsilon}\right) \left(\frac{\tilde{e}}{q_{pl}}\right) \left(\frac{q}{\sqrt{Gm}}\right)
\]

respectively. Here \( m, q, mpl = \sqrt{\frac{\hbar c}{G}} \) and \( q_{pl} = \sqrt{\hbar c} \) denote mass, charge, Planck mass and Planck charge respectively. The electron-positron pair production processes occur over the whole region of dyadosphere and hence the total energy confined in the dyadosphere is given by \[43\]

\[
E_{dya} = \frac{q^2}{2r_+} (1 - \frac{r_+}{r_{ds}})(1 - \frac{r_+^2}{r_{ds}^2}).
\]

De Lorenci et al. \[45\] obtained Reissner-Nordström black hole in the dyadosphere form with

\[
M_4(r) = m - \frac{q^2}{2r} + \frac{\sigma q^4}{10r^5}.
\]
The Reissner-Nordström solution arises for the vanishing of $\sigma$. The corresponding electric field source is
\[ \Xi_4 = -\frac{q}{r^2} + 4\frac{\sigma q^3}{r^6}. \] (24)

4 Energy, Momentum and Angular Momentum

The tetrad components associated to (12) can be obtained by using the procedure of [46] as
\[
e^a_{\mu}(r, \theta, \phi) = \begin{pmatrix}
\sqrt{F} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{F}} \cos \phi \sin \theta & r \cos \phi \cos \theta & -r \sin \phi \sin \theta \\
0 & \frac{1}{\sqrt{F}} \sin \phi \sin \theta & r \sin \phi \cos \theta & r \cos \phi \sin \theta \\
0 & \frac{1}{\sqrt{F}} \cos \theta & -r \sin \theta & 0
\end{pmatrix}
\] (25)
with $e = \det(e^a_{\mu}) = \frac{1}{2} r^2 \sin \theta$. The non-vanishing components of torsion tensor are found by substituting the tetrad components in Eq. (2), i.e.
\[
T_{(0)01} = \sqrt{F}, \quad T_{(1)12} = (1 - \frac{1}{\sqrt{F}}) \cos \theta \cos \phi,
\]
\[
T_{(1)13} = -(1 - \frac{1}{\sqrt{F}}) \sin \theta \sin \phi, \quad T_{(2)12} = (1 - \frac{1}{\sqrt{F}}) \cos \theta \sin \phi,
\]
\[
T_{(2)13} = (1 - \frac{1}{\sqrt{F}}) \sin \theta \cos \phi, \quad T_{(3)12} = -(1 - \frac{1}{\sqrt{F}}) \sin \theta.
\] (26)

The corresponding non-zero components of $T_{\lambda\mu\nu} = e^a_{\lambda} T_{a\mu\nu}$ will become
\[
T_{001} = \sqrt{F} \sqrt{F}, \quad T_{212} = r(1 - \frac{1}{\sqrt{F}}), \quad T_{313} = r \sin^2 \theta(1 - \frac{1}{\sqrt{F}}),
\] (27)
where dot indicates derivative with respect to $r$. In view of Eq. (4), the energy density corresponding to (12) is
\[
- \partial_t \Pi^{(0)i} = 4k r \partial_1 (\sin \theta(1 - \sqrt{F})).
\] (28)

Consequently, the energy turns out to be
\[
\bar{P}^{(0)} = E = r[1 - \sqrt{F}],
\]
\[
E = r[1 - \sqrt{1 - \frac{2M(r)}{r}}],
\] (29)
Making use of the binomial expansion with \( r \gg M(r) \), it takes the form

\[
E \approx M(r).
\]  

For different black hole solutions, we have the following form of energy:

- For de Sitter-Schwarzschild black hole, inserting Eq. (13) in the above equation, it follows that

\[
E_1 = m(1 - \exp(-\frac{r^3}{r_*^2})).
\]  

This energy distribution is the same as calculated by Dymnikova [38] using the standard formula for mass of the de Sitter-Schwarzschild solution, i.e.

\[
m(r) = \int_0^r T_0^0 d^3x = R_g(r)/2.
\]  

This also coincides with the results of Yang and Radinschi [30] evaluated by using Einstein, Weinberg and Tolman prescriptions respectively, i.e.

\[
E_1 = E_E = E_W = E_T = m(r) = R_g(r)/2 = m(1 - \exp(-\frac{r^3}{r_*^2})).
\]  

We see that the energy \( E_1 \) vanishes at \( r = 0 \) while for \( r \to \infty \) it is \( m \). Thus the total energy is given by the parameter \( m \) which is the same as the ADM mass for this spacetime. Also, \( E(r) > 0 \) when \( 0 \leq r < \infty \).

- For the Regular black hole with cosmological constant, we have

\[
E_2 = \frac{mr^3}{(r^2 + q^2)^{3/2}} - \frac{q^2r^3}{2(r^2 + q^2)^2} + \frac{\Lambda r^3}{6}.
\]  

When there is no cosmological constant, it turns out to be

\[
E_2 = \frac{m}{(1 + \frac{q^2}{r^2})^{3/2}} - \frac{q^2}{2r(1 + \frac{q^2}{r^2})^2}.
\]  

This expression is exactly the same as found by Yang et al. [31] in GR using Einstein, Weinberg energy-momentum prescriptions but different
from Møller complex. It is also discussed by the same authors \[47\] as a special case. After applying power series expansion, the above expression will become

\[
E_2 = m - \frac{q^2}{2r} - \frac{3mq^2}{2r^2} + \frac{q^4}{r^3} + \frac{15mq^4}{8r^4} - \frac{3q^6}{2r^5} + O\left(\frac{1}{r^6}\right). \tag{36}
\]

It can also be written as

\[
E_2 = E_{Tod} - \frac{3mq^2}{2r^2} + \frac{q^4}{r^3} + \frac{15mq^4}{8r^4} - \frac{3q^6}{2r^5} + O\left(\frac{1}{r^6}\right), \tag{37}
\]

where \(E_{Tod}\) represents the energy computed by Tod \[48\] for Reissner-Nordstrøm solution by using Penrose quasi-local mass definition.

- The energy of **Bardeen regular black hole** turn out to be

\[
E_3 = \frac{mr^3}{(r^2 + e^2)^{3/2}}. \tag{38}
\]

This is the same as found by Sharif \[32\]

\[
E_{ELLPW} = \frac{mr^3}{(r^2 + e^2)^{3/2}}. \tag{39}
\]

He also employed the Møller energy-momentum prescription and concluded that it also coincides with ELLPW prescriptions at large distances. This reduces to the energy of Schwarzschild solution for \(e = 0\).

- The **dyadosphere of a Reissner-Nordström** black hole has the following energy distribution

\[
E_4 = m - \frac{q^2}{2r} + \frac{\sigma q^4}{10r^5}. \tag{40}
\]

This result exactly agrees with ELLPW energy-momentum prescriptions obtained by Xulu \[33\] and slightly different from Møller prescription \[49\]. For \(\sigma = 0\), this corresponds to Reissner-Nordström black hole. The energy-momentum prescriptions ELLPWM give consistent result for this special case and reduces to the mass of Schwarzschild solution for \(q = 0\).

We would like to mention here that the momentum and angular momentum turn out to be constant for all these solutions.
5 Energy-Momentum Flux

The gravitational energy flux becomes constant due to vanishing of all the components of gravitational energy flux density $\phi^{(0)j}$, i.e. for $a = 0$, we have $\Phi_g^{(0)} = \text{constant}$. We carry out the calculations for momentum flux density and obtain the following components

$$
\begin{align*}
\phi^{(1)1} &= 2\kappa \sin^2 \theta \cos \phi (\sqrt{F}(\sqrt{F} - 1)^2), \\
\phi^{(1)2} &= 2\kappa \sin \theta \cos \phi (\sqrt{F})(\sqrt{F} - 1), \\
\phi^{(1)3} &= -2\kappa \sin \phi (\sqrt{F})(\sqrt{F} - 1), \\
\phi^{(2)1} &= 2\kappa \sin^2 \theta \sin \phi (\sqrt{F}(\sqrt{F} - 1)^2), \\
\phi^{(2)2} &= 2\kappa \sin \theta \cos \theta \sin \phi (\sqrt{F})(\sqrt{F} - 1), \\
\phi^{(2)3} &= 2\kappa \cos \phi (\sqrt{F})(\sqrt{F} - 1), \\
\phi^{(3)1} &= 2\kappa \sin \theta \cos \phi (\sqrt{F}(\sqrt{F} - 1)^2), \\
\phi^{(3)2} &= -2\kappa \sin^2 \theta (\sqrt{F})(\sqrt{F} - 1), \\
\phi^{(3)3} &= 0.
\end{align*}
$$

By replacing $a = i = 1, 2, 3$ in Eq. (9), we get the momentum flux as follows

$$
\begin{align*}
\Phi_g^{(1)} &= -2\kappa \pi \sin \phi (\frac{F}{2} - \sqrt{F}) + \text{const}, \\
\Phi_g^{(2)} &= 2\kappa \pi \cos \phi (\frac{F}{2} - \sqrt{F}) + \text{const}, \\
\Phi_g^{(3)} &= -4\kappa \pi \sin^2 \theta (\frac{F}{2} - \sqrt{F}) + \text{const}.
\end{align*}
$$

These turn out to be constant when we apply the condition $r \gg M(r)$, i.e.

$$
\frac{F}{2} - \sqrt{F} = (1/2 - M(r)/r) - \sqrt{1 - 2M(r)/r} \approx -\frac{1}{2}.
$$

Consequently, the above expressions reduce to

$$
\begin{align*}
\Phi_g^{(1)} &= \kappa \pi \sin \phi + \text{const}, \\
\Phi_g^{(2)} &= -\kappa \pi \cos \phi + \text{const}, \\
\Phi_g^{(3)} &= 2\kappa \pi \sin^2 \theta + \text{const}.
\end{align*}
$$

Thus the components of gravitational momentum flux are free of parameters like $m$, $q$ and $\Lambda$. They only depend upon spherical coordinates $\theta$ and $\phi$. 

10
Now we calculate matter energy-momentum flux for de Sitter-Schwarzschild black hole which requires the non-zero components of the energy-momentum tensor

\[
T^{00} = \frac{\Lambda e^{-\frac{\Lambda r^3}{6m}}}{\kappa(1 - \frac{2m}{r} + \frac{2me^{-\frac{\Lambda r^3}{6m}}}{r})},
\]

\[
T^{11} = \frac{\Lambda}{\kappa e^{\frac{\Lambda r^3}{6m}}} (1 - \frac{2m}{r} + \frac{2me^{-\frac{\Lambda r^3}{6m}}}{r}),
\]

\[
T^{22} = \frac{\Lambda}{2kr^2} (2 - \frac{r^3}{2m}) e^{-\frac{\Lambda r^3}{6m}},
\]

\[
T^{33} = \frac{\Lambda}{2kr^2 \sin^2 \theta} (2 - \frac{r^3}{2m}) e^{-\frac{\Lambda r^3}{6m}}.
\]

(44)

The matter energy-momentum flux turns out to be

\[
\Phi^{(0)}_m = \text{const},
\]

\[
\Phi^{(1)}_m = -\frac{\pi r^2 \Lambda}{2\kappa} e^{\frac{\Lambda r^3}{6m}} \cos \phi + \text{const},
\]

\[
\Phi^{(2)}_m = \frac{\pi r^2 \Lambda}{2\kappa} e^{\frac{\Lambda r^3}{6m}} \sin \phi + \text{const},
\]

\[
\Phi^{(3)}_m = -\frac{1}{\kappa} \pi r^2 \Lambda \sin^2 \theta e^{\frac{\Lambda r^3}{6m}} + \text{const}.
\]

(45)

The matter energy-momentum flux corresponding to charged black hole solutions can be evaluated by using the electromagnetic energy-momentum tensor. Its non-zero components are

\[
T^{00} = \frac{-\Xi^2}{8\pi F}, \quad T^{11} = \frac{F\Xi^2}{8\pi},
\]

\[
T^{22} = \frac{-\Xi^2}{8\pi r^2}, \quad T^{33} = \frac{-\Xi^2}{8\pi r^2 \sin^2 \theta}.
\]

(46)

Here \( \Xi \) is the electric field related to each charged black hole solution. The energy flux of matter becomes constant and the matter momentum flux for the charged black hole solutions are give as follows:

- The matter flux for regular black hole solution with cosmological con-
stant is evaluated by substituting the electric field $\Xi_2$ in Eq.(46), i.e.

$$\begin{align*}
\Phi_m^{(1)} &= \frac{1}{8} \sin \phi \int (r \Xi_2^2) dr + \text{const}, \\
\Phi_m^{(2)} &= -\frac{1}{8} \cos \phi \int (r \Xi_2^2) dr + \text{const}, \\
\Phi_m^{(3)} &= \frac{1}{4} \sin^2 \theta \int (r \Xi_2^2) dr + \text{const}. \quad (47)
\end{align*}$$

• The matter flux for Bardeen regular black hole takes the following form

$$\begin{align*}
\Phi_m^{(1)} &= -\frac{7 e^4}{32 r^8} \sin \phi + \text{const}, \\
\Phi_m^{(2)} &= \frac{7 e^4}{32 r^8} \cos \phi + \text{const}, \\
\Phi_m^{(3)} &= -\frac{7 e^4}{16 r^8} \sin^2 \theta + \text{const}. \quad (48)
\end{align*}$$

• The dyadosphere of a Reissner-Nordström black hole has the matter flux

$$\begin{align*}
\Phi_m^{(1)} &= -\frac{q^2}{8 r^4} \sin \phi [3 + \frac{176 \sigma^2 q^4}{r^8} - \frac{56 \sigma^2 q^2}{r^4}] + \text{const}, \\
\Phi_m^{(2)} &= \frac{q^2}{8 r^4} \cos \phi [3 + \frac{176 \sigma^2 q^4}{r^8} - \frac{56 \sigma^2 q^2}{r^4}] + \text{const}, \\
\Phi_m^{(3)} &= -\frac{q^2}{4 r^4} \sin^2 \theta [3 + \frac{176 \sigma^2 q^4}{r^8} - \frac{56 \sigma^2 q^2}{r^4}] + \text{const}. \quad (49)
\end{align*}$$

The values of $\Phi_g$ and $\Phi_m$ represent the transfer of gravitational and matter energy-momentum respectively.

6 Summary and Discussion

The debate of energy localization has generated a great deal of interest for a number of scientists in GR and TEGR, but could not provide a unique answer. In the current work, we have computed the gravitational energy and its relevant quantities like momentum, angular momentum, gravitational and matter energy-momentum fluxes. We have investigated these quantities for
four well-known black hole solutions, i.e., de Sitter-Schwarzschild black hole, regular black hole solution with cosmological constant, Bardeen regular black hole and dyadosphere of a charged black hole. For this purpose, we have used Hamiltonian approach in the realm of TEGR. It is worthwhile to mention here that our results for energy distribution exactly coincide with those evaluated by different authors using different prescriptions in GR. These are given by Eqs.(31), (34), (38) and (40). It is also interesting to note that these expressions reduce to ADM mass and this result also corresponds to the Schwarzschild solution. Our results support the idea that the energy-momentum complexes can give the same result for a given spacetime. The momentum and angular momentum for these solutions become constant.

The gravitational and matter energy-momentum flux have also been evaluated for these black hole solutions. We find that the gravitational and matter energy flux vanish while the components of gravitational momentum flux become constant in the asymptotic region. This indicates that the flow is uniform in that region and occurs in the $\theta$ and $\phi$ directions. Moreover, the components of matter flux associated to the de Sitter-schwarzschild black hole depends on $\Lambda$, $\theta$ and $\phi$. The matter flow shows inward and outward falling for the variation of these parameters. The components of matter flux corresponding to the charged black holes are given in Eqs.(47)-(49). The matter flux of the de Sitter-Schwarzschild and charged black holes vanish for $\Lambda = 0$ and $q = 0$ respectively.

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