A Finer Calibration Analysis for Adversarial Robustness

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Abstract

We present a more general analysis of $\mathcal{H}$-calibration for adversarially robust classification. By adopting a finer definition of calibration, we can cover settings beyond the restricted hypothesis sets studied in previous work. In particular, our results hold for most common hypothesis sets used in machine learning. We both fix some previous calibration results (Bao et al., 2020) and generalize others (Awasthi et al., 2021). Moreover, our calibration results, combined with the previous study of consistency by Awasthi et al. (2021), also lead to more general $\mathcal{H}$-consistency results covering common hypothesis sets.

Keywords: calibration, consistency, adversarial robustness.

1. Introduction

Rich learning models trained on large datasets often achieve a high accuracy in a variety of applications (Sutskever et al., 2014; Krizhevsky et al., 2012). However, such complex models have been shown to be susceptible to imperceptible perturbations (Szegedy et al., 2013): an unnoticeable perturbation can, for example, result in a dog being classified as an electronics device, which could lead to dramatic consequences in practice in many applications.

This has motivated the introduction and analysis of the notion of adversarial loss, which requires a predictor not only to correctly classify an input point $x$ but also to maintain the same classification for all points at a small $\ell_p$ distance of $x$ (Goodfellow et al., 2014; Madry et al., 2017; Tsipras et al., 2018; Carlini and Wagner, 2017).

The problem of designing effective learning algorithms with theoretical guarantees for the adversarial loss has been the topic of a number of recent studies (Bao et al., 2020; Awasthi et al., 2021). In particular, these authors have initiated a theoretical analysis of the $\mathcal{H}$-calibration and $\mathcal{H}$-consistency of surrogate losses for the adversarial 0/1 loss.

Bao et al. (2020) analyzed $\mathcal{H}$-calibration for adversarially robust classification in the special case where $\mathcal{H}$ is the family of linear models. However, several comments are due regarding that work. First, the definition of calibration adopted by the authors does not coincide with the standard definition (Steinwart, 2007) in the case of the linear models they study, although it does match that definition in the case of the family of all measurable functions (Steinwart, 2007, Section 4.1): the minimal inner risk in the definition should be defined for a fixed $x$ and the infimum should be over $f$, instead of an infimum over both $f$ and $x$. Second, and this is crucial, $\mathcal{H}$-calibration, in
general, does not imply $\mathcal{H}$-consistency, unless a property such as $\mathcal{P}$-minimizability holds (Steinwart, 2007, Theorem 2.8). $\mathcal{P}$-minimizability holds for standard binary classification and the family of all measurable functions (Steinwart, 2007, Theorem 3.2). However, it does not hold, in general, for adversarially robust classification and a specific hypothesis set $\mathcal{H}$. As a result, the claim made by the authors that the calibrated surrogates they propose are $\mathcal{H}$-consistent is incorrect, as shown by Awasthi et al. (2021). Third, the authors analyze $\mathcal{H}$-calibration with respect to the loss function $\phi_\gamma(x) \mapsto 1_{y(f(x) \leq \gamma)}$ in the case where $\mathcal{H} \supset [-1, 1]$ is the general family of functions. However, $\phi_\gamma$ only coincides with the adversarial 0/1 loss $\ell_\gamma$ in Equation (10) in the special case where $\mathcal{H}$ is the family of linear models (Bao et al., 2020, Proposition 1).

Awasthi et al. (2021) also recently studied the $\mathcal{H}$-calibration and $\mathcal{H}$-consistency of adversarial surrogate losses. They pointed out the issues just mentioned about the study of Bao et al. (2020) and considered more general hypothesis sets, such as generalized linear models, ReLU-based functions, and one-layer ReLU neural networks. They identified natural conditions under which $\mathcal{H}$-calibrated losses can be $\mathcal{H}$-consistent in the adversarial scenario. They also derived calibration results under the correct definition of the minimal inner risk by analyzing the equivalence of two definitions. However, with this method of calibration analysis, the calibration considered by the authors needs to be a uniform calibration (Steinwart, 2007, Definition 2.15) instead of non-uniform calibration (Steinwart, 2007, Definition 2.7). In view of that, their positive result imposes an extra restriction on the parameters of the hypothesis sets, which can be removed through the analysis presented here.

**Our Contributions.** Building on previous work by Awasthi et al. (2021), we present a more general analysis of $\mathcal{H}$-calibration for adversarially robust classification for more general hypothesis sets. For example, our Theorem 8, Theorem 11 and Theorem 17 apply to most common hypothesis sets. Furthermore, for the specific hypothesis sets considered in previous work, our results either fix existing calibration results (Bao et al., 2020) or generalize them (Awasthi et al., 2021). More precisely, our Theorem 13 is a correction to the main positive result, Theorem 11 in (Bao et al., 2020), where we prove the theorem under the correct calibration definition. Moreover, our Theorem 14 extends the results for linear models to generalized linear models. Our Corollary 9, Theorem 10, Theorem 11 and Corollary 12 are stronger versions of the negative calibration results Theorem 10, Corollary 11, Theorem 12 and Corollary 13 in (Awasthi et al., 2021), since the calibration considered in (Awasthi et al., 2021) is uniform calibration (Steinwart, 2007, Definition 2.15) which is stronger than non-uniform calibration (Steinwart, 2007, Definition 2.7) considered in our paper. Our Theorem 16 and Corollary 18 are generalizations of the positive calibration results of Awasthi et al. (2021), since our results hold without the unboundedness assumptions for parameters of the hypothesis sets.

2. Preliminaries

We adopt much of the notation used in (Awasthi et al., 2021). We will denote vectors as lowercase bold letters (e.g. $\mathbf{x}$). The $d$-dimensional $l_2$-ball with radius $r$ is denoted by $B^d_r = \{ \mathbf{z} \in \mathbb{R}^d \mid \|\mathbf{z}\|_2 \leq r \}$. We denote by $\mathcal{X}$ the set of all possible examples. $\mathcal{X}$ is also sometimes referred to as the input space. The set of all possible labels is denoted by $\mathcal{Y}$. We will limit ourselves to the case of binary classification where $\mathcal{Y} = \{ -1, 1 \}$. Let $\mathcal{H}$ be a family of functions from $\mathbb{R}^d$ to $\mathbb{R}$. Given a fixed but unknown distribution $\mathcal{P}$ over $\mathcal{X} \times \mathcal{Y}$, the binary classification learning problem is then formulated as follows. The learner seeks to select a predictor $f \in \mathcal{H}$ with small generalization error with respect to the distribution $\mathcal{P}$. The generalization error of a classifier $f \in \mathcal{H}$ is defined
by $\mathcal{R}_{\ell_0}(f) = \mathbb{E}_{(x,y) \sim \mathcal{P}}[\ell_0(f,x,y)]$, where $\ell_0(f,x,y) = \mathbb{1}_{yf(x) \leq 0}$ is the standard 0/1 loss. More generally, the $\ell$-risk of a classifier $f$ for a surrogate loss $\ell(f, x, y)$ is defined by

$$\mathcal{R}_\ell(f) = \mathbb{E}_{(x,y) \sim \mathcal{P}}[\ell(f, x, y)].$$  

(1)

Moreover, the minimal $(\ell, \mathcal{H})$-risk, which is also called the Bayes $(\ell, \mathcal{H})$-risk, is defined by $\mathcal{R}_{\ell, \mathcal{H}}^* = \inf_{f \in \mathcal{H}} \mathcal{R}_\ell(f)$. In the standard classification setting, the goal of a consistency analysis is to determine whether the minimization of a surrogate loss $\ell$ can lead to that of the binary loss generalization error. Similarly, in adversarially robust classification, the goal of a consistency analysis is to determine if the minimization of a surrogate loss $\ell$ yields that of the adversarial generalization error defined by $\mathcal{R}_\ell(f) = \mathbb{E}_{(x,y) \sim \mathcal{P}}[\ell_{\gamma}(f, x, y)]$, where

$$\ell_{\gamma}(f, x, y) := \sup_{x' : |x - x'| \leq \gamma} \mathbb{1}_{yf(x') \leq 0}$$  

(2)

is the adversarial 0/1 loss. This motivates the definition of $\mathcal{H}$-consistency (or simply consistency) stated below.

**Definition 1 ($\mathcal{H}$-Consistency)** Given a hypothesis set $\mathcal{H}$, we say that a loss function $\ell_1$ is $\mathcal{H}$-consistent with respect to loss function $\ell_2$, if the following holds:

$$\mathcal{R}_{\ell_1}(f_n) - \mathcal{R}_{\ell_1, \mathcal{H}}^* \xrightarrow{n \to +\infty} 0 \implies \mathcal{R}_{\ell_2}(f_n) - \mathcal{R}_{\ell_2, \mathcal{H}}^* \xrightarrow{n \to +\infty} 0,$$  

(3)

for all probability distributions and sequences of $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$.

For a distribution $\mathcal{P}$ over $\mathcal{X} \times \mathcal{Y}$ with random variables $X$ and $Y$, let $\eta_\mathcal{P} : \mathcal{X} \to [0, 1]$ be a measurable function such that, for any $x \in \mathcal{X}$, $\eta_\mathcal{P}(x) = \mathbb{P}(Y = 1 \mid X = x)$. By the property of conditional expectation, we can rewrite (1) as $\mathcal{R}_\ell(f) = \mathbb{E}_\mathcal{X}[C_\ell(f, x, \eta_\mathcal{P}(x))]$, where $C_\ell(f, x, \eta)$ is the generic conditional $\ell$-risk (or inner $\ell$-risk) defined as followed:

$$\forall x \in \mathcal{X}, \forall \eta \in [0, 1], \quad C_\ell(f, x, \eta) := \eta \ell(f, x, +1) + (1 - \eta)\ell(f, x, -1).$$  

(4)

Moreover, the minimal inner $\ell$-risk on $\mathcal{H}$ is denoted by $C_{\ell_1, \mathcal{H}}^*(x, \eta) := \inf_{f \in \mathcal{H}} C_\ell(f, x, \eta)$. The notion of calibration for the inner risk is often a powerful tool for the analysis of $\mathcal{H}$-consistency (Steinwart, 2007).

**Definition 2 ($\mathcal{H}$-Calibration)** [Definition 2.7 in (Steinwart, 2007)] Given a hypothesis set $\mathcal{H}$, we say that a loss function $\ell_1$ is $\mathcal{H}$-calibrated with respect to a loss function $\ell_2$ if, for any $\epsilon > 0$, $\eta \in [0, 1]$, and $x \in \mathcal{X}$, there exists $\delta > 0$ such that for all $f \in \mathcal{H}$ we have

$$C_{\ell_1}(f, x, \eta) < C_{\ell_1, \mathcal{H}}^*(x, \eta) + \delta \implies C_{\ell_2}(f, x, \eta) < C_{\ell_2, \mathcal{H}}^*(x, \eta) + \epsilon.$$  

(5)

For comparison with previous work, we also introduce the uniform $\mathcal{H}$-calibration in (Steinwart, 2007), which is stronger than Definition 2.

**Definition 3 (Uniform $\mathcal{H}$-Calibration)** [Definition 2.15 in (Steinwart, 2007)] Given a hypothesis set $\mathcal{H}$, we say that a loss function $\ell_1$ is uniform $\mathcal{H}$-calibrated with respect to a loss function $\ell_2$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $\eta \in [0, 1]$, $f \in \mathcal{H}$, $x \in \mathcal{X}$, we have

$$C_{\ell_1}(f, x, \eta) < C_{\ell_1, \mathcal{H}}^*(x, \eta) + \delta \implies C_{\ell_2}(f, x, \eta) < C_{\ell_2, \mathcal{H}}^*(x, \eta) + \epsilon.$$  

(6)
Note that, in the previous work of Awasthi et al. (2021), Definition 3 is adopted, where $\delta$ in (6) is independent of $\eta$ and $x$; the work of Bao et al. (2020) adopts a similar definition. In this paper, we will focus on the non-uniform case, that is Definition 2, where $\delta$ is dependent on $\eta$ and $x$.

There are two advantages to considering non-uniform calibration: it makes it possible to provide stronger negative results on calibration properties of convex surrogates and, it helps us prove more general positive results that hold for most common hypothesis sets $\mathcal{H}$. In contrast, positive results for uniform calibration hold for some restricted hypothesis sets (Awasthi et al., 2021).

Steinwart (2007) showed that if $\ell_1$ is $\mathcal{H}$-calibrated (it suffices to satisfy non-uniform calibration, that is condition (5)) with respect to $\ell_2$, then $\mathcal{H}$-consistency, that is condition (3), holds for any probability distribution verifying the additional condition of $\mathcal{P}$-minimizability (Steinwart, 2007, Definition 2.4). While $\mathcal{P}$-minimizability does not hold in general for adversarially robust classification, Awasthi et al. (2021) showed that the uniform $\mathcal{H}$-calibrated losses are $\mathcal{H}$-consistent under certain conditions. In fact, it also suffices to satisfy non-uniform calibration, that is condition (5) for these results, since their proofs only make use of the weaker non-uniform property.

Next, we introduce the notions of calibration function and an important result characterizing $\mathcal{H}$-calibration from (Steinwart, 2007).

**Definition 4 (Calibration function)** Given a hypothesis set $\mathcal{H}$, we define the calibration function $\delta_{\text{max}}$ for a pair of losses $(\ell_1, \ell_2)$ as follows: for all $x \in \mathcal{X}$, $\eta \in [0, 1]$ and $\epsilon > 0$,

$$\delta_{\text{max}}(\epsilon, x, \eta) = \inf_{f \in \mathcal{H}} \left\{ \left| C_{\ell_1}(f, x, \eta) - C_{\ell_1, \mathcal{H}}^*(x, \eta) \right| \leq \epsilon \right\}.$$

**Proposition 5 (Lemma 2.9 in (Steinwart, 2007))** Given a hypothesis set $\mathcal{H}$, loss $\ell_1$ is $\mathcal{H}$-calibrated with respect to $\ell_2$ if and only if its calibration function $\delta_{\text{max}}$ satisfies $\delta_{\text{max}}(\epsilon, x, \eta) > 0$ for all $x \in \mathcal{X}$, $\eta \in [0, 1]$ and $\epsilon > 0$.

For comparison, Bao et al. (2020, Definition 3) and Awasthi et al. (2021, Definition 2) consider the Uniform Calibration function $\delta(\epsilon)$ and make use of Lemma 2.16 in (Steinwart, 2007) to characterize uniform calibration (Awasthi et al., 2021; Bao et al., 2020, Proposition 4). Note $\delta(\epsilon) > 0$ implies $\delta_{\text{max}}(\epsilon, x, \eta) > 0$ for all $x \in \mathcal{X}$, $\eta \in [0, 1]$, and as a result uniform calibration implies non-uniform calibration. However, the converse does not hold in general.

### 3. Adversarially Robust Classification

In adversarially robust classification, the loss at $(x, y)$ is measured in terms of the worst loss incurred over an adversarial perturbation of $x$ within a ball of a certain radius in a norm. In this work we will consider perturbations in the $l_2$ norm $\| \cdot \|$. We will denote by $\gamma$ the maximum magnitude of the allowed perturbations. Given $\gamma > 0$, a data point $(x, y)$, a function $f \in \mathcal{H}$, and a margin-based loss $\phi: \mathbb{R} \to \mathbb{R}^+$, we define the adversarial loss of $f$ at $(x, y)$ as

$$\tilde{\phi}(f, x, y) = \sup_{x' : \|x - x'\| \leq \gamma} \phi(yf(x')).$$

The above naturally motivates supremum-based surrogate losses that are commonly used to optimize the adversarial 0/1 loss (Goodfellow et al., 2014; Madry et al., 2017; Shafahi et al., 2019; Wong et al., 2020). We say that a surrogate loss $\tilde{\phi}(f, x, y)$ is supremum-based if it is of the form
defined in (8). We say that the supremum-based surrogate is convex if the function \( \phi \) in (8) is convex. When \( \phi \) is non-increasing, the following equality holds (Yin et al., 2019):

\[
\sup_{x':|x-x'| \leq \gamma} \phi(yf(x')) = \phi \left( \inf_{x'|x-x'| \leq \gamma} yf(x') \right).
\]

(9)

The adversarial 0/1 loss defined in (2) is a special kind of adversarial loss (8), where \( \phi \) is the 0/1 loss, that is, \( \phi(yf(x)) = \ell_0(f, x, y) = \mathbb{I}_{yf(x) \leq 0} \). Therefore, the adversarial 0/1 loss has the equivalent form

\[
\ell_0(f, x, y) = \sup_{x'|x-x'| \leq \gamma} \mathbb{I}_{yf(x') \leq 0} = \inf_{x'|x-x'| \leq \gamma} yf(x') \leq 0.
\]

(10)

This alternative equivalent form of adversarial 0/1 loss is more advantageous to analyze than (2) and would be adopted in our proofs. Without loss of generality, let \( \mathcal{X} = B^d_2(1) \) and \( \gamma \in (0, 1) \). In this paper, we aim to characterize surrogate losses \( \ell_1 \) satisfying \( \mathcal{H} \)-calibration (5) with \( \ell_2 = \ell_\gamma \) and for the hypothesis sets \( \mathcal{H} \) which are regular for adversarial calibration.

**Definition 6 (Regularity for Adversarial Calibration)** We say that a hypothesis set \( \mathcal{H} \) is regular for adversarial calibration if there exists a distinguishing \( x \) in \( \mathcal{X} \), that is if there exist \( f, g \in \mathcal{H} \) such that \( \inf_{|x'-x| \leq \gamma} f(x') > 0 \) and \( \sup_{|x'-x| \leq \gamma} g(x') < 0 \).

It suffices to study hypothesis sets \( \mathcal{H} \) that are regular for adversarial calibration not only because all common hypothesis sets admit that property, but also because the following result holds. We say that a hypothesis set \( \mathcal{H} \) is symmetric, if for any \( f \in \mathcal{H} \), \( -f \) is also in \( \mathcal{H} \).

**Theorem 7** Let \( \mathcal{H} \) be a symmetric hypothesis set. If \( \mathcal{H} \) is not regular for adversarial calibration, then any surrogate loss \( \ell \) is \( \mathcal{H} \)-calibrated with respect to \( \ell_\gamma \).

**Proof** Since \( \mathcal{H} \) is symmetric, for any \( x \in \mathcal{X} \), \( f \in \mathcal{H} \), \( \inf_{|x'-x| \leq \gamma} f(x') \leq 0 \) \( \leq \sup_{|x'-x| \leq \gamma} f(x') \). Thus by the definition of inner risk (4) and adversarial 0-1 loss \( \ell_\gamma \) (10), for any \( x \in \mathcal{X} \), \( f \in \mathcal{H} \),

\[
C_{\ell_\gamma, \mathcal{H}}(f, x, \eta) = \eta \mathbb{I}_{\inf_{x'|x-x'| \leq \gamma} f(x') \leq 0} + (1 - \eta) \mathbb{I}_{\sup_{x'|x-x'| \leq \gamma} f(x') \geq 0} = C_{\ell_\gamma, \mathcal{H}}(x, \eta),
\]

which implies any surrogate loss \( \ell \) is \( \mathcal{H} \)-calibrated with respect to \( \ell_\gamma \) by (5).}

Note all the hypothesis sets considered in the previous work (Bao et al., 2020) and (Awasthi et al., 2021) are regular for adversarial calibration. For convenience, we adopt the notation in (Awasthi et al., 2021) to denote these specific hypothesis sets:

- **linear models**: \( \mathcal{H}_{\text{lin}} = \{ x \rightarrow w \cdot x \mid \| w \| = 1 \} \), as in (Bao et al., 2020) and (Awasthi et al., 2021).
- **generalized linear models**: \( \mathcal{H}_g = \{ x \rightarrow g(w \cdot x) + b \mid \| w \| = 1, \| b \| \leq G \} \) where \( g \) is a non-decreasing function, as in (Awasthi et al., 2021); and
- **one-layer ReLU neural networks**: \( \mathcal{H}_{\text{NN}} = \{ x \rightarrow \sum_{j=1}^n u_j (w_j \cdot x) \mid \| u \|_1 \leq \Lambda, \| w_j \| \leq W \} \), where \( (\cdot)_+ = \max(\cdot, 0) \) as in (Awasthi et al., 2021); and
- **all measurable functions**: \( \mathcal{H}_{\text{all}} \) as in (Awasthi et al., 2021).

In the special case of \( g = (\cdot)_+ \), we denote the corresponding ReLU-based hypothesis set as \( \mathcal{H}_{\text{relu}} = \{ x \rightarrow (w \cdot x)_+ + b \mid \| w \| = 1, \| b \| \leq G \} \) as in (Awasthi et al., 2021).
4. $\mathcal{H}$-Calibration Analysis

4.1. Negative results

In this section, we show that the commonly used convex surrogates and supremum-based convex surrogates are not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$, even under the weaker notion of non-uniform calibration. These results can be viewed as a generalization of those given by Awasthi et al. (2021).

4.1.1. Convex losses

We first study convex losses, which are often used for standard binary classification problems.

**Theorem 8** Assume $\mathcal{H}$ satisfies there exists a distinguishing $x_0 \in X$ and $f_0 \in \mathcal{H}$ such that $f_0(x_0) = 0$. If a margin-based loss $\phi : \mathbb{R} \to \mathbb{R}_+$ is convex, then it is not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

In particular, the assumption holds when $\mathcal{H}$ is regular for adversarial calibration and contains 0. The proof of Theorem 8 is included in Appendix A.1. By Theorem 8, we obtain the following corollary, which fixes the main negative result of Bao et al. (2020) and generalizes negative results of Awasthi et al. (2021). Note $\mathcal{H}_{\text{lin}}, \mathcal{H}_{\text{NN}}$ and $\mathcal{H}_{\text{all}}$ all satisfy there exists a distinguishing $x_0 \in X$ and $f_0 \in \mathcal{H}$ such that $f_0(x_0) = 0$. When $g(-\gamma) + G > 0$ and $g(\gamma) - G < 0$, $\mathcal{H}_{g}$ also satisfies this assumption.

**Corollary 9** If a margin-based loss $\phi : \mathbb{R} \to \mathbb{R}_+$ is convex, then,

1. $\phi$ is not $\mathcal{H}_{\text{lin}}$-calibrated with respect to $\ell_\gamma$;
2. Given a non-decreasing and continuous function $g$ such that $g(-\gamma) + G > 0$ and $g(\gamma) - G < 0$. Then $\phi$ is not $\mathcal{H}_{g}$-calibrated with respect to $\ell_\gamma$; Specifically, if $G > \gamma$, then $\phi$ is not $\mathcal{H}_{\text{relu}}$-calibrated with respect to $\ell_\gamma$;
3. $\phi$ is not $\mathcal{H}_{\text{NN}}$-calibrated with respect to $\ell_\gamma$;
4. $\phi$ is not $\mathcal{H}_{\text{all}}$-calibrated with respect to $\ell_\gamma$.

By using the correct calibration Definition 2, 1. of Corollary 9 fixes the main negative result in (Bao et al., 2020).

4.1.2. Supremum-based convex losses

While it is natural to consider convex surrogates for the 0/1 loss, convex supremum-based surrogates are widely used in practice for designing algorithms for the adversarial loss (Madry et al., 2017; Shafahi et al., 2019; Wong et al., 2020). We next present negative results for convex supremum-based surrogates.

**Theorem 10** Let $\phi$ be convex and non-increasing margin-based loss, consider the surrogate loss defined by $\tilde{\phi}(f, x, y) = \sup_{x' \in X, \|x-x'\| \leq \gamma} \phi(yf(x'))$. Then

1. $\tilde{\phi}$ is not $\mathcal{H}_{\text{lin}}$-calibrated with respect to $\ell_\gamma$;
2. Given a non-decreasing and continuous function $g$ such that $g(-\gamma) + G > 0$ and $g(\gamma) - G < 0$. Then $\tilde{\phi}$ is not $\mathcal{H}_{\text{g}}$-calibrated with respect to $\ell_\gamma$; Specifically, if $G > \gamma$, $\tilde{\phi}$ is not $\mathcal{H}_{\text{relu}}$-calibrated with respect to $\ell_\gamma$.
**Theorem 11** Let $\mathcal{H}$ be a hypothesis set containing 0 that is regular for adversarial calibration. If a margin-based loss $\phi$ is convex and non-increasing, then the surrogate loss defined by $\tilde{\phi}(f, x, y) = \sup_{x' : \|x - x'\|_{\leq \gamma}} \phi(y f(x'))$ is not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

The proofs of Theorem 10 and Theorem 11 are also included in Appendix A.1. Since $\mathcal{H}_{NN}$ and $\mathcal{H}_{all}$ both contain 0 and are regular for adversarial calibration, Theorem 11 leads to the following corollary.

**Corollary 12** Let $\phi$ be convex and non-increasing margin-based loss, consider the surrogate loss defined by $\tilde{\phi}(f, x, y) = \sup_{x' : \|x - x'\|_{\leq \gamma}} \phi(y f(x'))$. Then

1. $\tilde{\phi}$ is not $\mathcal{H}_{NN}$-calibrated with respect to $\ell_\gamma$;
2. $\tilde{\phi}$ is not $\mathcal{H}_{all}$-calibrated with respect to $\ell_\gamma$.

Corollary 9, Theorem 10, Theorem 11 and Corollary 12 above are stronger versions of the negative calibration results Theorem 10, Corollary 11, Theorem 12 and Corollary 13 in (Awasthi et al., 2021), since the calibration considered in (Awasthi et al., 2021) is uniform calibration (Steinwart, 2007, Definition 2.15), which is stronger than non-uniform calibration (Steinwart, 2007, Definition 2.7) considered in this work.

**4.2. Positive results**

In this section, we provide alternative surrogate losses that are $\mathcal{H}$-calibrated with respect to $\ell_\gamma$. These results are similar but more general than their counterparts in (Awasthi et al., 2021).

**4.2.1. Margin-based losses**

In light of the negative results of Section 4.1, to find calibrated surrogate losses for adversarially robust classification, we need to consider non-convex ones. One possible candidate is the family of quasi-concave even losses introduced by (Bao et al., 2020, Definition 10). Theorem 13 below is a correction to the main positive result, Theorem 11 in (Bao et al., 2020), where we prove the theorem under the correct calibration definition.

**Theorem 13** Let a margin-based loss $\phi$ be bounded, continuous, non-increasing, and quasi-concave even. Assume that $\phi(-t) > \phi(t)$ for any $\gamma < t \leq 1$. Then $\phi$ is $\mathcal{H}_{lin}$-calibrated with respect to $\ell_\gamma$ if and only if for any $\gamma < t \leq 1$,

$$\phi(\gamma) + \phi(-\gamma) > \phi(t) + \phi(-t).$$

(11)

The proof of Theorem 13 is included in Appendix A.3, where we make use of Lemma 26, which is powerful since it applies to any symmetric hypothesis sets. Note Theorem 11 in (Bao et al., 2020) does not hold any more under the correct calibration Definition 2, since their condition $\phi(\gamma) + \phi(-\gamma) > \phi(1) + \phi(-1)$ is much weaker than (11).

We next extend the above to show that under certain conditions, quasi-concave even surrogate losses are $\mathcal{H}_g$-calibrated for the class of generalized linear models with respect to the adversarial 0/1 loss.
**Theorem 14**  Let \( g \) be a non-decreasing and continuous function such that \( g(1 + \gamma) < G \) and \( g(-1 - \gamma) > -G \) for some \( G \geq 0 \). Let a margin-based loss \( \phi \) be bounded, continuous, non-increasing, and quasi-concave even. Assume that \( \phi(g(-t) - G) > \phi(G - g(-t)) \) and \( g(-t) + g(t) \geq 0 \) for any \( 0 \leq t \leq 1 \). Then \( \phi \) is \( \mathcal{H}_g \)-calibrated with respect to \( \ell_{\gamma} \) if and only if for any \( 0 \leq t \leq 1 \),

\[
\phi(G - g(-t)) + \phi(g(-t) - G) = \phi(g(t) + G) + \phi(-g(t) - G)
\]

and

\[
\min \{ \phi(A(t)) + \phi(-A(t)) \} > \phi(G - g(-t)) + \phi(g(-t) - G),
\]

where \( A(t) = \max_{s \in [-t, t]} g(s) - g(s - \gamma) \) and \( \overline{A}(t) = \min_{s \in [-t, t]} g(s) - g(s + \gamma) \).

The proof of Theorem 14 is included in Appendix A.4. Specifically, when \( g = ()_+ \), by Theorem 14, we obtain the following corollary for \( \mathcal{H}_{relu} \) by using the fact that \( \phi(t) + \phi(-t) \geq \phi(\gamma) + \phi(-\gamma) \) when \( 0 \leq t \leq \gamma \) by Part 2 of Lemma 24. Note when \( g = ()_+ \),

\[
A(t) = \max_{s \in [-t, t]} (s)_+ - (s - \gamma)_+ = \begin{cases} t, & 0 \leq t < \gamma, \\ \gamma, & \gamma \leq t \leq 1. \end{cases}
\]

\[
\overline{A}(t) = \min_{s \in [-t, t]} (s)_+ - g(s + \gamma)_+ = -\gamma.
\]

**Corollary 15**  Assume that \( G > 1 + \gamma \). Let a margin-based loss \( \phi \) be bounded, continuous, non-increasing, and quasi-concave even. Assume that \( \phi(-G) > \phi(G) \). Then \( \phi \) is \( \mathcal{H}_{relu} \)-calibrated with respect to \( \ell_{\gamma} \) if and only if for any \( 0 \leq t \leq 1 \),

\[
\phi(G) + \phi(-G) = \phi(t + G) + \phi(-t - G) \quad \text{and} \quad \phi(\gamma) + \phi(-\gamma) = \phi(G) + \phi(-G).
\]

In order to demonstrate the applicability of Theorem 13, Theorem 14 and Corollary 15, we consider a specific surrogate loss namely the \( \rho \)-margin loss \( \phi_\rho(t) = \min \{1, \max \{0, 1 - \frac{t}{\rho}\} \} \), \( \rho > 0 \), which is a generalization of the ramp loss (see, for example, Mohri et al. (2018)). Using Theorem 13, Theorem 14 and Corollary 15, we can conclude that the \( \rho \)-margin loss is calibrated under reasonable conditions for linear hypothesis sets and non-decreasing \( g \)-based hypothesis sets, since \( \phi_\rho(t) \) is bounded, non-increasing and quasi-concave even. This is stated formally below.

**Theorem 16**  Consider \( \rho \)-margin loss \( \phi_\rho(t) = \min \{1, \max \{0, 1 - \frac{t}{\rho}\} \} \), \( \rho > 0 \). Then,

1. \( \phi_\rho \) is \( \mathcal{H}_{lin} \)-calibrated with respect to \( \ell_{\gamma} \) if and only if \( \rho > 1 \).

2. Given a non-decreasing and continuous function \( g \) such that \( g(1 + \gamma) < G \) and \( g(-1 - \gamma) > -G \) for some \( G \geq 0 \). Assume that \( g(-t) + g(t) \geq 0 \) for any \( 0 \leq t \leq 1 \). Then \( \phi_\rho \) is \( \mathcal{H}_g \)-calibrated with respect to \( \ell_{\gamma} \) if and only if for any \( 0 \leq t \leq 1 \),

\[
\phi_\rho(G - g(-t)) = \phi_\rho(g(t) + G) \quad \text{and} \quad \min \{ \phi_\rho(A(t)), \phi_\rho(-A(t)) \} > \phi_\rho(G - g(-t)),
\]

where \( A(t) = \max_{s \in [-t, t]} g(s) - g(s - \gamma) \) and \( \overline{A}(t) = \min_{s \in [-t, t]} g(s) - g(s + \gamma) \).

3. Assume that \( G > 1 + \gamma \). Then \( \phi_\rho \) is \( \mathcal{H}_{relu} \)-calibrated with respect to \( \ell_{\gamma} \) if and only if \( G \geq \rho > \gamma \).

Theorem 16 is a strict generalization of the positive calibration results in (Awasthi et al., 2021) for \( \mathcal{H}_g \) and \( \mathcal{H}_{relu} \) where the authors require \( G \) to be unbounded. By working with the weaker notion of non-uniform calibration, we avoid such a restriction on \( G \).
4.2.2. SUPREMEUM-BASED MARGIN LOSSES

Recall that in Theorem 11 we ruled out the possibility of finding $\mathcal{H}$-calibrated supremum-based convex surrogate losses with respect to the adversarial 0/1 loss. However, we show that the supremum-based $\rho$-margin loss is indeed $\mathcal{H}$-calibrated. We state the calibration result below and present the proof in Appendix A.3.

**Theorem 17** Consider $\rho$-margin loss $\phi_\rho(t) = \min\{1, \max\{0, 1 - \frac{t}{\rho}\}\}, \rho > 0$. Let $\mathcal{H}$ be a symmetric hypothesis set, then the surrogate loss $\tilde{\phi}_\rho(f, x, y) = \sup_{x' : \|x - x'\|_\gamma} \phi_\rho(yf(x'))$ is $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

By Theorem 17, we obtain the following corollary, since $\mathcal{H}_{\text{lin}}, \mathcal{H}_{\text{NN}}$ and $\mathcal{H}_{\text{all}}$ are all symmetric.

**Corollary 18** Consider $\rho$-margin loss $\phi_\rho(t) = \min\{1, \max\{0, 1 - \frac{t}{\rho}\}\}, \rho > 0$. Let $\tilde{\phi}_\rho(f, x, y) = \sup_{x' : \|x - x'\|_\gamma} \phi_\rho(yf(x'))$ be the surrogate loss. Then,

1. $\tilde{\phi}_\rho$ is $\mathcal{H}_{\text{lin}}$-calibrated with respect to $\ell_\gamma$;
2. $\tilde{\phi}_\rho$ is $\mathcal{H}_{\text{NN}}$-calibrated with respect to $\ell_\gamma$;
3. $\tilde{\phi}_\rho$ is $\mathcal{H}_{\text{all}}$-calibrated with respect to $\ell_\gamma$.

2. of Corollary 18 is a strict generalization of the positive calibration result in (Awasthi et al., 2021) for $\mathcal{H}_{\text{NN}}$ where the authors require $\Lambda$ to be unbounded. By working with the weaker notion of non-uniform calibration, we avoid such a restriction on $\Lambda$.

5. $\mathcal{H}$-consistency

Next, we study the implications of our positive results for non-uniform calibration for establishing $\mathcal{H}$-consistency. As discussed in Section 1, Steinwart (2007) showed that if $\ell_1$ is $\mathcal{H}$-calibrated (it suffices to satisfy non-uniform calibration, that is condition (5)) with respect to $\ell_2$, then $\mathcal{H}$-consistency, that is condition (3), holds for any probability distribution verifying the additional condition of $\mathcal{P}$-minimizability (Steinwart, 2007, Definition 2.4). Although the $\mathcal{P}$-minimizability condition is naturally satisfied and $\mathcal{H}$-calibration often is a sufficient condition for $\mathcal{H}$-consistency in the standard classification setting when considering the family of all measurable functions (Steinwart, 2007, Theorem 3.2), Awasthi et al. (2021) point out that the adversarial loss presents new challenges when dealing with $\mathcal{P}$-minimizability and requires carefully distinguishing among calibration and consistency to avoid drawing false conclusions.

Moreover, Awasthi et al. (2021) show that the $\mathcal{H}$-calibrated losses are $\mathcal{H}$-consistent under certain conditions. Analogously, in this section, we make use of (Awasthi et al., 2021, Theorem 25, Theorem 27) to conclude that the $\mathcal{H}$-calibrated losses studied in previous sections are $\mathcal{H}$-consistent under the same conditions.

**Theorem 19** (Theorem 25 in (Awasthi et al., 2021)) Let $\mathcal{P}$ be a distribution over $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{H}$ a hypothesis set for which $\mathcal{R}_{\ell, \mathcal{H}}^\star = 0$. Let $\phi$ be a margin-based loss. If for $\epsilon \geq 0$, there exists $f^* \in \mathcal{H} \subset \mathcal{H}_{\text{all}}$ such that $\mathcal{R}_\phi(f^*) \leq \mathcal{R}_{\phi, \mathcal{H}_{\text{all}}}^\star + \eta < +\infty$ and $\phi$ is $\mathcal{H}$-calibrated with respect to $\ell_\gamma$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{H}$ we have

$$
\mathcal{R}_\phi(f) + \eta < \mathcal{R}_{\phi, \mathcal{H}}^\star + \delta \quad \Rightarrow \quad \mathcal{R}_{\mathcal{H}_{\ell, \mathcal{H}}}^\star(f) < \mathcal{R}_{\mathcal{H}_{\ell, \mathcal{H}}}^\star + \epsilon.
$$


Theorem 20 (Theorem 27 in \cite{awasthi2021}) Given a distribution $\mathcal{P}$ over $X \times Y$ and a hypothesis set $\mathcal{H}$ such that $\mathcal{R}^*_{\ell_{\gamma}, H}$ $= 0$. Let $\phi$ be a non-increasing margin-based loss. If there exists $f^* \in \mathcal{H} \subset \mathcal{H}_{\text{all}}$ such that $\mathcal{R}_{\phi}(f^*) = R_{\phi, \mathcal{H}_{\text{all}}}^* < \infty$ and $\hat{\phi}(f, x, y) = \sup_{x'}|x-x'| \leq y \phi(yf(x'))$ is $\mathcal{H}$-consistent with respect to $\ell_{\gamma}$, and for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{H}$ we have

$$\mathcal{R}_{\hat{\phi}}(f) < R_{\phi, \mathcal{H}}^* + \delta \implies \mathcal{R}_{\ell_{\gamma}}(f) < R_{\phi, \mathcal{H}}^* + \epsilon.$$ 

Using Theorem 16 in Section 4.2.1 and Theorem 19 above, we conclude that the calibrated $\rho$-margin loss in Section 4.2.1 is consistent with respect to $\ell_{\gamma}$ for all distributions that satisfy the realizability assumption, i.e., $\mathcal{R}^*_{\ell_{\gamma}, \mathcal{H}} = 0$.

Theorem 21 Consider the $\rho$-margin loss $\phi_{\rho}(t) = \min\left\{1, \max\left\{0, 1 - \frac{t}{\rho}\right\}\right\}$, $\rho > 0$. Then,

1. If $\rho > 1$, then $\phi_{\rho}$ is $\mathcal{H}_{\text{lin}}$-consistent wrt $\ell_{\gamma}$ for all distribution $P$ over $X \times Y$ that satisfies $\mathcal{R}^*_{\ell_{\gamma}, \mathcal{H}_{\text{lin}}} = 0$ and there exists $f^* \in \mathcal{H}_{\text{lin}}$ such that $\mathcal{R}_{\phi_{\rho}}(f^*) = R_{\phi_{\rho}, \mathcal{H}_{\text{all}}}^* < \infty$.

2. Given a non-decreasing and continuous function $g$ such that $g(1+\gamma) < G$ and $g(-1+\gamma) > -G$ for some $G \geq 0$. Assume that $g(-t) + g(t) \geq 0$ for any $0 \leq t \leq 1$. Let $\bar{A}(t) = \sup_{[-t,t]} g(s) - g(s-\gamma)$ and $A(t) = \min_{\sup_{[-t,t]} g(s) - g(s+\gamma)}$ for any $0 \leq t \leq 1$. If for any $0 \leq t \leq 1$, $\phi_{\rho}(G-g(-t)) = \phi_{\rho}(g(t)+G)$ and $\min\left\{\phi_{\rho}(A(t)), \phi_{\rho}(-A(t))\right\} > \phi_{\rho}(G-g(-t))$, then $\phi_{\rho}$ is $\mathcal{H}_{\text{relu}}$-consistent wrt $\ell_{\gamma}$ for all distribution $P$ over $X \times Y$ that satisfies $\mathcal{R}^*_{\ell_{\gamma}, \mathcal{H}_{\text{relu}}} = 0$ and there exists $f^* \in \mathcal{H}_{\text{relu}}$ such that $\mathcal{R}_{\phi_{\rho}}(f^*) = \mathcal{R}_{\phi_{\rho}, \mathcal{H}_{\text{all}}}^* < \infty$.

3. If $G > 1+\gamma$ and $G \geq \rho > \gamma$, then $\phi_{\rho}$ is $\mathcal{H}_{\text{relu}}$-consistent wrt $\ell_{\gamma}$ for all distribution $P$ over $X \times Y$ that satisfies $\mathcal{R}^*_{\ell_{\gamma}, \mathcal{H}_{\text{relu}}} = 0$ and there exists $f^* \in \mathcal{H}_{\text{relu}}$ such that $\mathcal{R}_{\phi_{\rho}}(f^*) = \mathcal{R}_{\phi_{\rho}, \mathcal{H}_{\text{all}}}^* < \infty$.

Using Theorem 17 in Section 4.2.2 and Theorem 20, we conclude that the calibrated supremum-based $\rho$-margin loss in Section 4.2.2 is also consistent wrt $\ell_{\gamma}$ for all distributions that satisfy realizability assumptions.

Theorem 22 Consider $\rho$-margin loss $\phi_{\rho}(t) = \min\left\{1, \max\left\{0, 1 - \frac{t}{\rho}\right\}\right\}$, $\rho > 0$. Let $\mathcal{H}$ be a symmetric hypothesis set, then the surrogate loss $\hat{\phi}_{\rho}(f, x, y) = \sup_{x'}|x-x'| \leq y \phi_{\rho}(yf(x'))$ is $\mathcal{H}$-consistent with respect to $\ell_{\gamma}$ for all distributions $\mathcal{P}$ over $X \times Y$ that satisfy: $\mathcal{R}^*_{\ell_{\gamma}, \mathcal{H}} = 0$ and there exists $f^* \in \mathcal{H}$ such that $\mathcal{R}_{\phi_{\rho}}(f^*) = \mathcal{R}_{\phi_{\rho}, \mathcal{H}_{\text{all}}}^* < \infty$.

6. Conclusion

We presented a careful analysis of the $\mathcal{H}$-calibration of surrogate losses, including a series of negative results for surrogate losses commonly used in practice, as well as a number of positive results for surrogate losses that we prove additionally to be $\mathcal{H}$-consistent, provided that some other natural conditions hold. Our results significantly extend previously known results and provide a solid guidance for the design of algorithms for adversarial robustness with theoretical guarantees. Moreover, several of our proof techniques for calibration and consistency can further be relevant to the analysis of other loss functions.
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Appendix A. Deferred Proofs

For convenience, let \( \Delta C_{\ell,\gamma}(f, x, \eta) := C_{\ell}(f, x, \eta) - C^*_\gamma(f, x, \eta) \), \( M(f, x, \gamma) := \inf_{x' : \|x - x'\| \leq \gamma} f(x') \), and \( \overline{M}(f, x, \gamma) := -\inf_{x' : \|x - x'\| \leq \gamma} f(x') = \sup_{x' : \|x - x'\| \leq \gamma} f(x') \).

A.1. Proof of Theorem 8, Theorem 10 and Theorem 11

We first characterize the calibration function \( \delta_{\max}(\epsilon, x, \eta) \) of losses \((\ell, \ell_\gamma)\) at \( \eta = \frac{1}{2}, \epsilon = \frac{1}{2} \) and distinguishing \( x_0 \in X \) given a hypothesis set \( \mathcal{H} \) which is regular for adversarial calibration.

**Lemma 23** Let \( \mathcal{H} \) be a hypothesis set that is regular for adversarial calibration. For distinguishing \( x_0 \in X \), the calibration function \( \delta_{\max}(\epsilon, x, \eta) \) of losses \((\ell, \ell_\gamma)\) satisfies

\[
\delta_{\max}\left(\frac{1}{2}, x_0, \frac{1}{2}\right) = \inf_{\ell \in \mathcal{H}} \Delta C_{\ell,\gamma}(f, x_0, \frac{1}{2}).
\]

**Proof** By the definition of inner risk (4) and adversarial 0-1 loss \( \ell_\gamma \) (10), the inner \( \ell_\gamma \)-risk is

\[
C_{\ell_\gamma}(f, x, \eta) = \eta \mathbb{I}(M(f, x, \gamma) \leq 0) + (1 - \eta) \mathbb{I}(-M(f, x, \gamma)) = \begin{cases} 
1 & \text{if } M(f, x, \gamma) \leq 0 \leq -M(f, x, \gamma), \\
\eta & \text{if } -M(f, x, \gamma) < 0, \\
1 - \eta & \text{if } M(f, x, \gamma) > 0.
\end{cases}
\]

For distinguishing \( x_0 \) and \( \eta \in [0, 1] \), \( \{f \in \mathcal{H} : M(f, x_0, \gamma) < 0\} \) and \( \{f \in \mathcal{H} : M(f, x_0, \gamma) > 0\} \) are not empty sets. Thus

\[
C^*_\gamma(x_0, \eta) = \inf_{f \in \mathcal{H}} C_{\ell_\gamma}(f, x_0, \eta) = \min\{\eta, 1 - \eta\}.
\]

Note for \( f \in \{f \in \mathcal{H} : M(f, x_0, \gamma) \leq 0 \leq -M(f, x_0, \gamma)\} \), \( \Delta C_{\ell_\gamma}(f, x_0, \eta) = \max\{\eta, 1 - \eta\} \); for \( f \in \{f \in \mathcal{H} : M(f, x_0, \gamma) < 0\} \), \( \Delta C_{\ell_\gamma}(f, x_0, \eta) = \eta - \min\{\eta, 1 - \eta\} = \max\{0, \eta - 1\} = |2\eta - 1| \mathbb{I}(2\eta - 1)(M(f, x_0, \gamma)) \geq 0 \) since \( M(f, x_0, \gamma) \leq -M(f, x_0, \gamma) < 0 \); for \( f \in \{f \in \mathcal{H} : M(f, x_0, \gamma) > 0\} \), \( \Delta C_{\ell_\gamma}(f, x_0, \eta) = (1 - \eta) - \min\{\eta, 1 - \eta\} = \max\{0, 1 - 2\eta\} = |2\eta - 1| \mathbb{I}(2\eta - 1)(M(f, x_0, \gamma)) \leq 0 \).

Therefore,

\[
\Delta C_{\ell_\gamma}(f, x_0, \eta) = \begin{cases} 
\max\{\eta, 1 - \eta\} & \text{if } M(f, x_0, \gamma) \leq 0 \leq -M(f, x_0, \gamma), \\
|2\eta - 1| \mathbb{I}(2\eta - 1)(M(f, x_0, \gamma)) \leq 0 & \text{if } M(f, x_0, \gamma) > 0 \text{ or } M(f, x_0, \gamma) < 0.
\end{cases}
\]

By (7), for a fixed \( \eta \in [0, 1] \) and \( x \in X \), the calibration function of losses \((\ell, \ell_\gamma)\) is

\[
\delta_{\max}(\epsilon, x, \eta) = \inf_{f \in \mathcal{H}} \{\Delta C_{\ell,\gamma}(f, x, \eta) \mid \Delta C_{\ell_\gamma}(f, x, \eta) \geq \epsilon\}.
\]

Observe that for all \( \eta \in [0, 1] \),

\[
\max\{\eta, 1 - \eta\} = \frac{1}{2}[(1 - \eta) + \eta + (1 - \eta) - \eta] = \frac{1}{2}[1 + |2\eta - 1|] \geq |2\eta - 1|.
\]
For distinguishing $x_0, \eta = \frac{1}{2}$ and $\epsilon = \frac{1}{2}$, $\Delta C_{\ell, \gamma}(f, x_0, \frac{1}{2}) \geq \frac{1}{2}$ if and only if $\underline{M}(f, x_0, \gamma) \leq \frac{1}{2}$. Therefore,

$$\delta_{\max} \left( \frac{1}{2}, x_0, \frac{1}{2} \right) = \inf_{f \in \mathcal{H}, \underline{M}(f, x_0, \gamma) \leq 0 \leq \overline{M}(f, x_0, \gamma)} \Delta C_{\ell, \gamma}(f, x_0, \frac{1}{2}).$$

Theorem 8 Assume $\mathcal{H}$ satisfies there exists a distinguishing $x_0 \in X$ and $f_0 \in \mathcal{H}$ such that $f_0(x_0) = 0$. If a margin-based loss $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ is convex, then it is not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

Proof By Lemma 23, for distinguishing $x_0 \in X$, the calibration function $\delta_{\max}(\epsilon, x, \eta)$ satisfies

$$\delta_{\max} \left( \frac{1}{2}, x_0, \frac{1}{2} \right) = \inf_{f \in \mathcal{H}, \underline{M}(f, x_0, \gamma) \leq 0 \leq \overline{M}(f, x_0, \gamma)} \Delta C_{\phi, \gamma}(f, x_0, \frac{1}{2}).$$

Suppose that $\phi$ is $\mathcal{H}$-calibrated with respect to $\ell_\gamma$. By Proposition 5, $\phi$ is $\mathcal{H}$-calibrated with respect to $\ell_\gamma$ if and only if its calibration function $\delta_{\max}$ satisfies $\delta_{\max}(\epsilon, x, \eta) > 0$ for all $x \in X, \eta \in [0, 1]$ and $\epsilon > 0$. In particular, the condition requires $\delta_{\max} \left( \frac{1}{2}, x_0, \frac{1}{2} \right) > 0$, that is,

$$\inf_{f \in \mathcal{H}, \underline{M}(f, x_0, \gamma) \leq 0 \leq \overline{M}(f, x_0, \gamma)} \Delta C_{\phi, \gamma}(f, x_0, \frac{1}{2}) > 0,$$

which is equivalent to

$$\inf_{f \in \mathcal{H}, \underline{M}(f, x_0, \gamma) \leq 0 \leq \overline{M}(f, x_0, \gamma)} C_{\phi}(f, x_0, \frac{1}{2}) > \inf_{f \in \mathcal{H}} C_{\phi}(f, x_0, \frac{1}{2}),$$

(13)

By the definition of inner risk (4),

$$C_{\phi}(f, x_0, \frac{1}{2}) = \frac{1}{2} \left( \phi(f(x_0)) + \phi(-f(x_0)) \right).$$

(14)

Since $\phi$ is convex, by Jensen’s inequality, for any $f \in \mathcal{H}$, the following holds:

$$C_{\phi}(f, x_0, \frac{1}{2}) \geq \phi \left( \frac{1}{2} f(x_0) - \frac{1}{2} f(x_0) \right) = \phi(0).$$

For $f = f_0$, we have $f_0(x_0) = 0$ and by (14),

$$C_{\phi}(f_0, x_0, \frac{1}{2}) = \frac{1}{2} \left( \phi(0) + \phi(0) \right) = \phi(0).$$

Moreover, when $f = f_0$, $\underline{M}(f_0, x_0, \gamma) \leq f_0(x_0) = 0 \leq \overline{M}(f_0, x_0, \gamma)$. Thus

$$\inf_{f \in \mathcal{H}, \underline{M}(f, x_0, \gamma) \leq 0 \leq \overline{M}(f, x_0, \gamma)} C_{\phi}(f, x_0, \frac{1}{2}) = \inf_{f \in \mathcal{H}} C_{\phi}(f, x_0, \frac{1}{2}) = \phi(0),$$

where the minimum can be achieved by $f = f_0$, contradicting (13). Therefore, $\phi$ is not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.  

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Theorem 10  Let \( \phi \) be convex and non-increasing margin-based loss, consider the surrogate loss defined by \( \tilde{\phi}(f, x, y) = \sup_{x' \colon \|x-x'\| \leq y} \phi(y f(x')) \). Then

1. \( \tilde{\phi} \) is not \( \mathcal{H}_{\text{lin}} \)-calibrated with respect to \( \ell_y \);

2. Given a non-decreasing and continuous function \( g \) such that \( g(\gamma) + G > 0 \) and \( g(\gamma) - G < 0 \). Then \( \tilde{\phi} \) is not \( \mathcal{H}_y \)-calibrated with respect to \( \ell_y \); Specifically, if \( G > \gamma \), \( \tilde{\phi} \) is not \( \mathcal{H}_{\text{relu}} \)-calibrated with respect to \( \ell_y \).

Proof  By Lemma 23, for distinguishing \( x_0 \in \mathcal{X} \), the calibration function \( \delta_{\max}(\epsilon, x, \eta) \) of losses \((\phi, \ell_y)\) satisfies

\[
\delta_{\max}\left(\frac{1}{2}, x_0, \frac{1}{2}\right) = \inf_{f \in \mathcal{H}_y} \inf_{M(f, x_0, \gamma) \leq 0 \leq M(f, x_0, \gamma)} \Delta C_{\tilde{\phi}, \mathcal{H}_\text{lin}}(f, x_0, \frac{1}{2}).
\]

Next we first consider the case where \( \mathcal{H} = \mathcal{H}_{\text{lin}} \). Take distinguishing \( x_0 \in \mathcal{X} \) and \( f_0 \in \mathcal{H}_{\text{lin}} \) such that \( f_0(x_0) = 0 \). As shown by Awasthi et al. (2020), for \( f \in \mathcal{H}_{\text{lin}} \), \( \mathcal{X} = \{x \to w \cdot x \mid \|w\| = 1\} \),

\[
\begin{align*}
M(f, x, \gamma) &= \inf_{x' \in [x-x'] \leq \gamma} f(x') = \inf_{x' \in [x-x'] \leq \gamma} (w \cdot x') = w \cdot x - \gamma \|w\| = f(x) - \gamma, \\
\overline{M}(f, x, \gamma) &= -\inf_{x' \in [x-x'] \leq \gamma} -f(x') = -\inf_{x' \in [x-x'] \leq \gamma} (-w \cdot x') = w \cdot x + \gamma \|w\| = f(x) + \gamma.
\end{align*}
\]

Suppose that \( \tilde{\phi} \) is \( \mathcal{H}_{\text{lin}} \)-calibrated with respect to \( \ell_y \). By Proposition 5, \( \tilde{\phi} \) is \( \mathcal{H}_{\text{lin}} \)-calibrated with respect to \( \ell_y \) if and only if its calibration function \( \delta_{\max} \) satisfies \( \delta_{\max}(\epsilon, x, \eta) > 0 \) for all \( x \in \mathcal{X}, \eta \in [0, 1] \) and \( \epsilon > 0 \). In particular, the condition requires \( \delta_{\max}\left(\frac{1}{2}, x_0, \frac{1}{2}\right) > 0 \), that is,

\[
\inf_{f \in \mathcal{H}_{\text{lin}}: -\gamma \leq f(x_0) \leq \gamma} \Delta C_{\tilde{\phi}, \mathcal{H}_{\text{lin}}}(f, x_0, \frac{1}{2}) > 0,
\]

which is equivalent to

\[
\inf_{f \in \mathcal{H}_{\text{lin}}: -\gamma \leq f(x_0) \leq \gamma} C_{\tilde{\phi}, \mathcal{H}_{\text{lin}}}(f, x_0, \frac{1}{2}) > \inf_{f \in \mathcal{H}_{\text{lin}}} C_{\tilde{\phi}, \mathcal{H}_{\text{lin}}}(f, x_0, \frac{1}{2}), \tag{15}
\]

By (20), for \( f \in \mathcal{H}_{\text{lin}} \),

\[
C_{\tilde{\phi}, \mathcal{H}_{\text{lin}}}(f, x_0, \frac{1}{2}) = \frac{1}{2} \phi(f(x_0) - \gamma) + \frac{1}{2} \phi(-f(x_0) - \gamma). \tag{16}
\]

Since \( \phi \) is convex, by Jensen’s inequality, for any \( f \in \mathcal{H}_{\text{lin}} \), the following holds:

\[
C_{\tilde{\phi}, \mathcal{H}_{\text{lin}}}(f, x_0, \frac{1}{2}) \geq \phi\left(\frac{1}{2}(f(x_0) - \gamma) - \frac{1}{2}(f(x_0) + \gamma)\right) = \phi(-\gamma).
\]

For \( f = f_0 \), we have \( f_0(x_0) = 0 \) and by (16),

\[
C_{\tilde{\phi}, \mathcal{H}_{\text{lin}}}(f_0, x_0, \frac{1}{2}) = \frac{1}{2} (\phi(-\gamma) + \phi(-\gamma)) = \phi(-\gamma).
\]
Moreover, when \( f = f_0 \), \(-\gamma \leq f_0(x_0) \leq 0 \). Thus
\[
\inf_{f \in \mathcal{C}} C^*_\gamma(f, x_0, 1/2) = \inf_{f \in \mathcal{C}} C_\gamma(f, x_0, 1/2) = \phi(-\gamma),
\]
where the minimum can be achieved by \( f = f_0 \), contradicting (15). Therefore, \( \tilde{\phi} \) is not \( \mathcal{H}_{\text{lin}} \)-calibrated with respect to \( \ell_\gamma \).

Then we consider the case where \( \mathcal{H} = \mathcal{H}_g \). By the assumption on \( g \), \( 0 \) is distinguishing. As shown by Awasthi et al. (2020), for \( f \in \mathcal{H}_g \),
\[
\overline{M}(f, x, \gamma) = g(w \cdot x - \gamma) + b, \quad \underline{M}(f, x, \gamma) = g(w \cdot x + \gamma) + b.
\]
Suppose that \( \tilde{\phi} \) is \( \mathcal{H}_g \)-calibrated with respect to \( \ell_\gamma \). By Proposition 5, \( \tilde{\phi} \) is \( \mathcal{H}_g \)-calibrated with respect to \( \ell_\gamma \) if and only if its calibration function \( \delta_{\max}(\varepsilon, x, \eta) > 0 \) for all \( x \in \mathcal{X}, \eta \in [0, 1] \) and \( \varepsilon > 0 \). In particular, the condition requires \( \delta_{\max}(1/2, 0, 1/2) > 0 \), that is,
\[
\inf_{f \in \mathcal{H}_g} \left( g(-\gamma) + b \right) > 0,
\]
which is equivalent to
\[
\inf_{f \in \mathcal{H}_g} C_\tilde{\phi}(f, 0, 1/2) > 0.
\]
By (20), for \( f \in \mathcal{H}_g \),
\[
C_\tilde{\phi}(f, 0, 1/2) = \frac{1}{2} \phi(g(-\gamma) + b) + \frac{1}{2} \phi(-g(\gamma) - b).
\]
Since \( \phi \) is convex, by Jensen’s inequality, for any \( f \in \mathcal{H}_g \), the following holds:
\[
C_\tilde{\phi}(f, 0, 1/2) \geq \frac{1}{2} \phi\left(\frac{g(-\gamma) + b}{2}\right) = \phi\left(\frac{g(-\gamma) - g(\gamma)}{2}\right).
\]
Take \( f_0 \in \mathcal{H}_g \) with \( b_0 = \frac{g(-\gamma) - g(\gamma)}{2} \), we have \( g(-\gamma) + b_0 = -g(\gamma) - b_0 = \frac{g(-\gamma) - g(\gamma)}{2} \) and by (18),
\[
C_\tilde{\phi}(f_0, 0, 1/2) = \frac{1}{2} \phi(g(-\gamma) + b_0) + \frac{1}{2} \phi(-g(\gamma) - b_0) = \phi\left(\frac{g(-\gamma) - g(\gamma)}{2}\right).
\]
Moreover, when \( f = f_0, g(-\gamma) + b_0 \leq 0 \leq g(\gamma) + b_0 \). Thus
\[
\inf_{f \in \mathcal{H}_g} C_\tilde{\phi}(f, 0, 1/2) = \inf_{f \in \mathcal{H}_g} C_\tilde{\phi}(f, 0, 1/2) = \phi\left(\frac{g(-\gamma) - g(\gamma)}{2}\right),
\]
where the minimum can be achieved by \( f = f_0 \), contradicting (17). Therefore, \( \tilde{\phi} \) is not \( \mathcal{H}_g \)-calibrated with respect to \( \ell_\gamma \).

**Theorem 11** Let \( \mathcal{H} \) be a hypothesis set containing 0 that is regular for adversarial calibration. If a margin-based loss \( \phi \) is convex and non-increasing, then the surrogate loss defined by \( \tilde{\phi}(f, x, y) = \sup_{x' : \|x - x'\| \leq \gamma} \phi(yf(x')) \) is not \( \mathcal{H} \)-calibrated with respect to \( \ell_\gamma \).
Proof By Lemma 23, for distinguishing $x_0 \in X$, the calibration function $\delta_{\max}(\epsilon, x, \eta)$ of losses $(\phi, \ell_\gamma)$ satisfies

$$\delta_{\max} \left( \frac{1}{2}, x_0, \frac{1}{2} \right) = \inf_{f \in \iota; M(f, x_0, \gamma) \leq M(f, x_0, \gamma)} \Delta C_{\tilde{\phi}, \gamma}(f, x_0, \frac{1}{2}).$$

Suppose that $\tilde{\phi}$ is $H$-calibrated with respect to $\ell_\gamma$. By Proposition 5, $\tilde{\phi}$ is $H$-calibrated with respect to $\ell_\gamma$ if and only if its calibration function $\delta_{\max}$ satisfies $\delta_{\max}(\epsilon, x, \eta) > 0$ for all $x \in X$, $\eta \in [0, 1]$ and $\epsilon > 0$. In particular, the condition requires $\delta_{\max} \left( \frac{1}{2}, x_0, \frac{1}{2} \right) > 0$, that is,

$$\inf_{f \in \iota; M(f, x_0, \gamma) \leq M(f, x_0, \gamma)} \Delta C_{\tilde{\phi}, \gamma}(f, x_0, \frac{1}{2}) > 0,$$

which is equivalent to

$$\inf_{f \in \iota; M(f, x_0, \gamma) \leq M(f, x_0, \gamma)} C_{\tilde{\phi}}(f, x_0, \frac{1}{2}) > \inf_{f \in \iota} C_{\tilde{\phi}}(f, x_0, \frac{1}{2}), \quad (19)$$

As shown by Awasthi et al. (2020), $\tilde{\phi}$ has the equivalent form

$$\tilde{\phi}(f, x, y) = \phi \left( \inf_{|x' - x| \leq \gamma} (y f(x')) \right).$$

By the definition of inner risk (4),

$$C_{\tilde{\phi}}(f, x_0, \frac{1}{2}) = \frac{1}{2} \left( \phi \left( M(f, x_0, \gamma) \right) + \phi \left( -M(f, x_0, \gamma) \right) \right). \quad (20)$$

Since $\phi$ is convex, by Jensen’s inequality, for any $f \in \iota$, the following holds:

$$C_{\tilde{\phi}}(f, x_0, \frac{1}{2}) \geq \phi \left( \frac{1}{2} M(f, x_0, \gamma) - \frac{1}{2} M(f, x_0, \gamma) \right) = \phi \left( \frac{1}{2} \left( M(f, x_0, \gamma) - M(f, x_0, \gamma) \right) \right) \geq \phi(0),$$

where the last inequality used the fact that

$$\frac{1}{2} \left( M(f, x_0, \gamma) - M(f, x_0, \gamma) \right) \leq 0$$

and $\phi$ is non-increasing. For $f = 0$, we have $M(f, x_0, \gamma) = \overline{M}(f, x_0, \gamma) = 0$ and by (20),

$$C_{\tilde{\phi}}(f, x_0, \frac{1}{2}) = \frac{1}{2} \left( \phi(0) + \phi(0) \right) = \phi(0).$$

Moreover, when $\overline{M}(f, x_0, \gamma) = \overline{M}(f, x_0, \gamma) = 0$, $M(f, x_0, \gamma) \leq 0 \leq \overline{M}(f, x_0, \gamma)$ is satisfied. Thus

$$\inf_{f \in \iota; M(f, x_0, \gamma) \leq \overline{M}(f, x_0, \gamma)} C_{\tilde{\phi}}(f, x_0, \frac{1}{2}) = \inf_{f \in \iota} C_{\tilde{\phi}}(f, x_0, \frac{1}{2}) = \phi(0),$$

where the minimum can be achieved by $f = 0$, contradicting (19). Therefore, $\tilde{\phi}$ is not $H$-calibrated with respect to $\ell_\gamma$. \[\blacksquare\]
A.2. Property of $\tilde{C}_\phi(t, \eta)$

For a margin-based loss $\phi$, denote $\tilde{C}_\phi(t, \eta) = \eta\phi(t)+(1-\eta)\phi(-t)$ for any $\eta \in [0, 1]$ and $t \in \mathbb{R}$. In this section, we characterize the property of $\tilde{C}_\phi(t, \eta)$ when $\phi$ is bounded, continuous, non-increasing and quasi-concave even, which would be useful in the proof of Theorem 13 and Theorem 14. Without loss of generality, assume that $g$ is continuous, non-decreasing and satisfies $g(-1 - \gamma) + G > 0$, $g(1 + \gamma) - G < 0$.

**Lemma 24** Let $\phi$ be a margin-based loss. If $\phi$ is bounded, continuous, non-increasing, quasi-concave even, then

1. $\tilde{C}_\phi(t, \eta)$ is quasi-concave in $t \in \mathbb{R}$ for all $\eta \in [0, 1]$.

2. $\tilde{C}_\phi(t, \frac{1}{2})$ is even and non-increasing in $t$ when $t \geq 0$.

3. For $l, u \in \mathbb{R}(l \leq u)$, $\inf_{t \in [l, u]} \tilde{C}_\phi(t, \eta) = \min\{\tilde{C}_\phi(l, \eta), \tilde{C}_\phi(u, \eta)\}$ for all $\eta \in [0, 1]$.

4. For all $\eta \in (\frac{1}{2}, 1]$, $\tilde{C}_\phi(t, \eta)$ is non-increasing in $t$ when $t \geq 0$.

5. For all $\eta \in [0, \frac{1}{2})$, $\tilde{C}_\phi(t, \eta)$ is non-decreasing in $t$ when $t \leq 0$.

6. If $\phi(-t) > \phi(t)$ for any $\gamma < t \leq 1$, then, for all $\eta \in (\frac{1}{2}, 1]$ and any $\gamma < t \leq 1$, $\tilde{C}_\phi(-t, \eta) > \tilde{C}_\phi(t, \eta)$.

7. If $\phi(-t) > \phi(t)$ for any $\gamma < t \leq 1$, then, for all $\eta \in [0, \frac{1}{2})$ and any $\gamma < t \leq 1$, $\tilde{C}_\phi(-t, \eta) < \tilde{C}_\phi(t, \eta)$.

8. If $\phi(g(-t) - G) > \phi(G - g(-t))$, $g(-t) + g(t) \geq 0$ for any $0 \leq t \leq 1$, then, for all $\eta \in (\frac{1}{2}, 1]$ and any $0 \leq t \leq 1$, $\tilde{C}_\phi(g(-t) - G, \eta) > \tilde{C}_\phi(g(t) + G, \eta)$.

9. If $\phi(g(-t) - G) > \phi(G - g(-t))$, $g(-t) + g(t) \geq 0$ for any $0 \leq t \leq 1$, then, for all $0 \leq t \leq 1$, $\tilde{C}_\phi(g(-t) - G, \eta) < \tilde{C}_\phi(g(t) + G, \eta)$ for all $\eta \in [0, \frac{1}{2})$ if and only if $\phi(G - g(-t)) + \phi(g(-t) - G) = \phi(g(t) + G) + \phi(-g(t) - G)$.

**Proof** Part 1,2,4 of Lemma 24 are stated in (Bao et al., 2020, Lemma 13). Part 3 is a corollary of Part 1 by the characterization of continuous and quasi-convex functions in (Boyd and Vandenberghe, 2014).

Consider Part 5. For $\eta \in [0, \frac{1}{2})$, and $t_1, t_2 \leq 0$. Suppose that $t_1 < t_2$, then

$$\phi(t_1) - \phi(-t_1) - \phi(t_2) + \phi(-t_2) \geq 0$$
since $\phi$ is non-increasing. By Part 2 of Lemma 24, $\phi(t) + \phi(-t)$ is non-decreasing in $t$ when $t \leq 0$. Therefore, for $\eta \in [0, \frac{1}{2})$,

$$C_\phi(t_1, \eta) - C_\phi(t_2, \eta) = (\phi(t_1) - \phi(t_2) + \phi(-t_1) - \phi(-t_2))/2 + \phi(-t_1) - \phi(-t_2) \leq 0.$$

Consider Part 6. For $\eta \in (\frac{1}{2}, 1]$ and any $\gamma < t \leq 1$,

$$C_\phi(-t, \eta) - C_\phi(t, \eta) = \eta\phi(-t) + (1 - \eta)\phi(t) - \eta\phi(\gamma) - (1 - \eta)\phi(t) = (2\eta - 1) [\phi(-t) - \phi(t)] > 0 \quad \text{since } \eta > \frac{1}{2} \text{ and } \phi(-t) > \phi(t) \text{ for any } \gamma < t \leq 1.$$

Consider Part 7. For $\eta \in [0, \frac{1}{2})$ and any $\gamma < t \leq 1$,

$$C_\phi(t, \eta) - C_\phi(-t, \eta) = (1 - 2\eta) [\phi(-t) - \phi(t)] > 0 \quad \text{since } \eta < \frac{1}{2} \text{ and } \phi(-t) > \phi(t) \text{ for any } \gamma < t \leq 1.$$

Consider Part 8. For $\eta \in (\frac{1}{2}, 1]$ and any $0 \leq t \leq 1$,

$$C_\phi(g(-t) - G, \eta) - C_\phi(g(t) + G, \eta) = (2\eta - 1) [\phi(g(-t) - G) - \phi(G + g(t))] > 0 \quad \text{for any } 0 \leq t \leq 1, \quad \text{(g(t) + G > 0)}$$

Consider Part 9. Since $\phi$ is non-increasing, for any $0 \leq t \leq 1$,

$$\phi(g(-t) - G) - \phi(G - g(-t)) + \phi(-g(t) - G) - \phi(g(t) + G) + \phi(g(t) - G) - \phi(G + g(t)) > 0 \quad \text{(g(t) + G > 0)}$$

$\iff$: Suppose $\phi(G - g(-t)) + \phi(g(-t) - G) = \phi(g(t) + G) + \phi(-g(t) - G)$, then for $\eta \in [0, \frac{1}{2})$,

$$C_\phi(g(-t) - G, \eta) - C_\phi(g(t) + G, \eta) = (2\eta - 1) [\phi(g(-t) - G) - \phi(G + g(t))] > 0 \quad \text{for any } 0 \leq t \leq 1, \quad \text{(g(t) + G > 0)}$$

$$\iff:$$

$$\frac{1}{2}(\phi(G - g(-t)) + \phi(g(-t) - G) - \phi(g(t) + G) - \phi(-g(t) - G)) = 0.$$
Suppose $\tilde{C}_\phi(g(-t) - G, \eta) < \tilde{C}_\phi(g(t) + G, \eta)$ for $\eta \in [0, \frac{1}{2})$, then
\[
\tilde{C}_\phi(g(-t) - G, \eta) - \tilde{C}_\phi(g(t) + G, \eta) \\
= (\phi(g(-t) - G) - \phi(G - g(-t))) + (\phi(-g(t) - G) - \phi(g(t) + G))\eta \\
+ \phi(G - g(-1)) - \phi(-g(1) - G) \\
< 0
\]
for $\eta \in [0, \frac{1}{2})$. By taking $\eta \to \frac{1}{2}$, we have
\[
\frac{1}{2} (\phi(G - g(-t)) + \phi(g(t) - G) - \phi(g(t) + G) - \phi(-g(t) - G)) \\
= (\phi(g(-t) - G) - \phi(G - g(-t))) + (\phi(-g(t) - G) - \phi(g(t) + G))\frac{1}{2} \\
+ \phi(G - g(-t)) - \phi(-g(t) - G) \\
\leq 0.
\]
By Part 2 of Lemma 24, we have
\[
\phi(G - g(-t)) + \phi(g(t) - G) - \phi(g(t) + G) - \phi(-g(t) - G) \\
\geq \phi(g(t) + G) + \phi(-g(t) - G) - \phi(g(t) + G) - \phi(-g(t) - G) \\
(g(-t) + g(t) \geq 0) \\
= 0.
\]
Therefore, $\phi(G - g(-t)) + \phi(g(t) - G) - \phi(g(t) + G) - \phi(-g(t) - G) = 0$, i.e., $\phi(G - g(-t)) + \phi(g(-t) - G) = \phi(g(t) + G) + \phi(-g(t) - G)$.

A.3. Proof of Theorem 13 and Theorem 17

We will make use of general form (10) of the adversarial 0/1 loss:
\[
\ell_\gamma(f, x, y) = \sup_{x' : \|x - x'\| \leq \gamma} \mathbf{1}_{y f(x') \leq 0} = \inf_{x' : \|x - x'\| \leq \gamma} \mathbf{1}_{y f(x') \leq 0}.
\]

Next, we first characterize the calibration function $\delta_{\max}(\epsilon, x, \eta)$ of losses $(\ell, \ell_\gamma)$ given a symmetric hypothesis set $\mathcal{H}$.

**Lemma 25.** Let $\mathcal{H}$ be a symmetric hypothesis set. For a surrogate loss $\ell$, the calibration function $\delta_{\max}(\epsilon, x, \eta)$ of losses $(\ell, \ell_\gamma)$ is
\[
\delta_{\max}(\epsilon, x, \eta) = \begin{cases} 
+\infty & \text{if } x \in \mathcal{X}_1 \text{ or } x \in \mathcal{X}_2, \epsilon > \max\{\eta, 1 - \eta\}, \\
\inf_{f \in \mathcal{H} : M(f, x, \gamma) \leq \eta \Delta C_{\ell, \gamma}(f, x, \eta)} & \text{if } x \in \mathcal{X}_2, \, |2\eta - 1| < \epsilon \leq \max\{\eta, 1 - \eta\}, \\
\inf_{f \in \mathcal{H} : M(f, x, \gamma) \leq (2\eta - 1)\Delta C_{\ell, \gamma}(f, x, \eta)} & \text{if } x \in \mathcal{X}_2, \, \epsilon \leq |2\eta - 1|,
\end{cases}
\]
where $\mathcal{X}_1 = \{x \in \mathcal{X} : M(f, x, \gamma) \leq \eta \leq \Delta C_{\ell, \gamma}(f, x, \eta), \forall f \in \mathcal{H}\}$, $\mathcal{X}_2 = \{x \in \mathcal{X} : \text{there exists } f' \in \mathcal{H} \text{ such that } \Delta C_{\ell, \gamma}(f', x, \gamma) > 0\}$ and $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$. 


Proof By the definition of inner risk (4) and adversarial 0-1 loss $\ell_\gamma (10)$, the inner $\ell_\gamma$-risk is

$$C_{\ell_\gamma} (f, x, \eta) = \eta \mathbb{1}_{\{M(f, x, \gamma) \leq 0\}} + (1 - \eta) \mathbb{1}_{\{M(f, x, \gamma) > 0\}}$$

$$= \begin{cases} 
1 & \text{if } M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma), \\
\eta & \text{if } M(f, x, \gamma) < 0, \\
1 - \eta & \text{if } M(f, x, \gamma) > 0.
\end{cases}$$

Let $X_1 = \{x \in \mathcal{X} : M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma), \forall f \in \mathcal{H}\}$, $X_2 = \{x \in \mathcal{X} : \text{there exists } f' \in \mathcal{H} \text{ such that } M(f', x, \gamma) > 0\}$. It is obvious that $X_1 \cap X_2 = \emptyset$. Since $\mathcal{H}$ is symmetric, for any $x \in \mathcal{X}$, either there exists $f' \in \mathcal{H}$ such that $M(f', x, \gamma) > 0$ and $M(-f', x, \gamma) < 0$, or $M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma)$ for any $f \in \mathcal{H}$. Thus $\mathcal{X} = X_1 \cup X_2$. Note when $x \in X_1$, $\{f \in \mathcal{H} : M(f, x, \gamma) < 0\}$ and $\{f \in \mathcal{H} : M(f, x, \gamma) > 0\}$ are both empty sets. Therefore, the minimal inner $\ell_\gamma$-risk is

$$C^*_\ell_{\gamma, \mathcal{H}} (x, \eta) = \begin{cases} 
1, & x \in X_1, \\
\min\{\eta, 1 - \eta\}, & x \in X_2.
\end{cases}$$

Note when $x \in X_1$, $C_{\ell_\gamma} (f, x, \eta) = 1$ for any $f \in \mathcal{H}$, thus $\Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) = 0$. When $x \in X_2$, for $f \in \{f \in \mathcal{H} : M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma)\}$, $\Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) = 1 - \min\{\eta, 1 - \eta\} = \max\{0, 2\eta - 1\}$; for $f \in \{f \in \mathcal{H} : M(f, x, \gamma) < 0\}$, $\Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) = \eta - \min\{\eta, 1 - \eta\} = \max\{0, 2\eta - 1\}$ since $M(f, x, \gamma) \leq M(f, x, \gamma) < 0$; for $f \in \{f \in \mathcal{H} : M(f, x, \gamma) > 0\}$, $\Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) = 1 - \eta - \min\{\eta, 1 - \eta\} = \max\{0, 1 - 2\eta\} = \max\{0, 1 - 2\eta\}$ since $M(f, x, \gamma) > 0$. Therefore,

$$\Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) = \begin{cases} 
\max\{\eta, 1 - \eta\} & \text{if } x \in X_2, \quad M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma), \\
2\eta - 1 & \text{if } x \in X_2, \quad M(f, x, \gamma) > 0 \text{ or } M(f, x, \gamma) < 0, \\
0 & \text{if } x \in X_1.
\end{cases}$$

By (7), for a fixed $\eta \in [0, 1]$ and $x \in \mathcal{X}$, the calibration function of losses $(\ell, \ell_\gamma)$ is

$$\delta_{\max} (\epsilon, x, \eta) = \inf_{f \in \mathcal{H}} \left\{ \Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) \mid \Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) \geq \epsilon \right\}$$

If $x \in X_1$, then for all $f \in \mathcal{H}$, $\Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) = 0 < \epsilon$, which implies that $\delta_{\max} (\epsilon, x, \eta) = \infty$. Next we consider where $x \in X_2$. By the observation (12), if $\epsilon > \max\{\eta, 1 - \eta\}$, then for all $f \in \mathcal{H}$, $\Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) < \epsilon$, which implies that $\delta_{\max} (\epsilon, x, \eta) = \infty$; if $2\eta - 1 < \epsilon \leq \max\{\eta, 1 - \eta\}$, then $\Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) \geq \epsilon$ if and only if $M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma)$, which leads to

$$\delta_{\max} (\epsilon, x, \eta) = \inf_{f \in \mathcal{H} : M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma)} \Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta);$$

if $\epsilon \leq 2\eta - 1$, then $\Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta) \geq \epsilon$ if and only if $M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma)$ or $(2\eta - 1)M(f, x, \gamma) \leq 0$, which leads to

$$\delta_{\max} (\epsilon, x, \eta) = \inf_{f \in \mathcal{H} : M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma) \text{ or } (2\eta - 1)M(f, x, \gamma) \leq 0} \Delta C_{\ell_\gamma, \mathcal{H}} (f, x, \eta).$$

We then give the equivalent conditions of calibration based on inner $\ell$-risk and $\mathcal{H}$.
Lemma 26  Let $\mathcal{H}$ be a symmetric hypothesis set and $\ell$ be a surrogate loss function. If $\mathcal{X}_2 = \emptyset$, any loss $\ell$ is $\mathcal{H}$-calibrated with respect to $\ell$. If $\mathcal{X}_2 \neq \emptyset$, then $\ell$ is $\mathcal{H}$-calibrated with respect to $\ell$, if and only if for any $x \in \mathcal{X}_2$,

$$
\inf_{f \in \mathcal{H}} \frac{C_{\ell}(f, x, \frac{1}{2})}{\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \frac{1}{2})} > \inf_{f \in \mathcal{H}} \frac{C_{\ell}(f, x, \frac{1}{2})}{\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \frac{1}{2})}, \quad \text{and}
\inf_{f \in \mathcal{H}} \frac{C_{\ell}(f, x, \eta)}{\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta)} \text{ for all } \eta \in (\frac{1}{2}, 1], \quad \text{and}
\inf_{f \in \mathcal{H}} \frac{C_{\ell}(f, x, \eta)}{\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta)} \text{ for all } \eta \in [0, \frac{1}{2}).
$$

where $\mathcal{X}_2 = \{x \in \mathcal{X} : \text{there exists } f' \in \mathcal{H} \text{ such that } M(f', x, \gamma) > 0\}$.

**Proof** Let $\delta_{\max}$ be the calibration function of $(\ell, \ell, \gamma)$ given hypothesis set $\mathcal{H}$. By Lemma 25,

$$
\delta_{\max}(\epsilon, x, \eta) = \left\{ \begin{array}{ll}
+\infty & \text{if } x \in \mathcal{X}_1 \text{ or } x \in \mathcal{X}_2, \epsilon > \max\{1 - \eta, 0\}, \\
\inf_{f \in \mathcal{H}} \frac{\Delta C_{\ell, \mathcal{H}}(f, x, \eta)}{\inf_{f \in \mathcal{H}} \Delta C_{\ell, \mathcal{H}}(f, x, \eta)} & \text{if } x \in \mathcal{X}_2, \lceil 2\eta - 1 \rceil < \epsilon \leq \max\{1 - \eta, 0\}, \\
\inf_{f \in \mathcal{H}} \frac{\Delta C_{\ell, \mathcal{H}}(f, x, \eta)}{\inf_{f \in \mathcal{H}} \Delta C_{\ell, \mathcal{H}}(f, x, \eta)} & \text{if } x \in \mathcal{X}_2, \epsilon < \lceil 2\eta - 1 \rceil,
\end{array} \right.
$$

where $\mathcal{X}_1 = \{x \in \mathcal{X} : M(f, x, \gamma) \leq 0 \leq M(f, x, \gamma), \forall f \in \mathcal{H}\}$, $\mathcal{X}_2 = \{x \in \mathcal{X} : \text{there exists } f' \in \mathcal{H} \text{ such that } M(f', x, \gamma) > 0\}$ and $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$. By Proposition 5, $\ell$ is $\mathcal{H}$-calibrated with respect to $\ell_\gamma$ if and only if its calibration function $\delta_{\max}$ satisfies $\delta_{\max}(\epsilon, x, \eta) > 0$ for all $x \in \mathcal{X}$, $\eta \in [0, 1]$ and $\epsilon > 0$. Since $\delta(\epsilon, x, \eta) = \infty > 0$ when $x \notin \mathcal{X}_2$, any loss $\ell$ is $\mathcal{H}$-calibrated with respect to $\ell_\gamma$ when $\mathcal{X}_2 = \emptyset$. Furthermore, when $\mathcal{X}_2 \neq \emptyset$, we only need to analyze $\delta(\epsilon, x, \eta)$ when $x \in \mathcal{X}_2$.

For $\eta = \frac{1}{2}$, we have for any $x \in \mathcal{X}_2$,

$$
\delta_{\max}(\epsilon, x, \frac{1}{2}) > 0 \text{ for all } \epsilon > 0 \iff \inf_{f \in \mathcal{H}} \frac{C_{\ell}(f, x, \frac{1}{2})}{\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \frac{1}{2})} > \inf_{f \in \mathcal{H}} \frac{C_{\ell}(f, x, \frac{1}{2})}{\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \frac{1}{2})}. \quad (22)
$$

For $1 \geq \eta > \frac{1}{2}$, we have $|2\eta - 1| = 2\eta - 1$, $\max\{1 - \eta, 0\} = \eta$, and

$$
\inf_{f \in \mathcal{H}} \frac{\Delta C_{\ell, \mathcal{H}}(f, x, \eta)}{\inf_{f \in \mathcal{H}} \Delta C_{\ell, \mathcal{H}}(f, x, \eta)} = \inf_{f \in \mathcal{H}} \frac{\Delta C_{\ell, \mathcal{H}}(f, x, \eta)}{\inf_{f \in \mathcal{H}} \Delta C_{\ell, \mathcal{H}}(f, x, \eta)}.
$$

Therefore, $\delta_{\max}(\epsilon, x, \frac{1}{2}) > 0$ for all $x \in \mathcal{X}_2$, $\epsilon > 0$ and $\eta \in (\frac{1}{2}, 1]$ if and only if for all $x \in \mathcal{X}_2$,

$$
\left\{ \begin{array}{ll}
\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta) & \text{for all } \eta \in (\frac{1}{2}, 1] \text{ such that } 2\eta - 1 < \epsilon \leq \eta, \\
\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta) & \text{for all } \eta \in (\frac{1}{2}, 1] \text{ such that } \epsilon \leq 2\eta - 1,
\end{array} \right.
$$

for all $\epsilon > 0$, which is equivalent to for all $x \in \mathcal{X}_2$,

$$
\left\{ \begin{array}{ll}
\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta) & \text{for all } \eta \in (\frac{1}{2}, 1] \text{ such that } \epsilon \leq \eta < \frac{2\eta - 1}{2}, \\
\inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{H}} C_{\ell}(f, x, \eta) & \text{for all } \eta \in (\frac{1}{2}, 1] \text{ such that } \frac{2\eta - 1}{2} \leq \eta.
\end{array} \right. \quad (23)
$$
for all \( \epsilon > 0 \). Observe that
\[
\left\{ \eta \in \left( \frac{1}{2}, 1 \right] \mid \frac{1}{2} \leq \eta < \frac{\epsilon + 1}{2}, \epsilon > 0 \right\} = \left\{ \frac{1}{2} < \eta \leq 1 \right\}, \quad \text{and}
\]
\[
\left\{ \eta \in \left( \frac{1}{2}, 1 \right] \mid \frac{\epsilon + 1}{2} \leq \eta, \epsilon > 0 \right\} = \left\{ \frac{1}{2} < \eta \leq 1 \right\}, \quad \text{and}
\]
\[
\inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \leq 0 \leq \mathcal{M}(f, x, \gamma)} C_{\ell}(f, x, \eta) \geq \inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \leq 0} C_{\ell}(f, x, \eta) \quad \text{for all } \eta.
\]
Therefore, we reduce the above condition (23) as for all \( x \in \mathcal{X}_2 \),
\[
\inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \leq 0} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{F}_n} C_{\ell}(f, x, \eta) \quad \text{for all } \eta \in \left( \frac{1}{2}, 1 \right]. \tag{24}
\]
For \( \frac{1}{2} > \eta \geq 0 \), we have \( |2\eta - 1| = 1 - 2\eta \), \( \max\{\eta, 1 - \eta\} = 1 - \eta \), and
\[
\inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \leq 0 \leq \mathcal{M}(f, x, \gamma) \text{ or } (2\eta - 1)(\mathcal{M}(f, x, \gamma)) \leq 0} \Delta C_{\ell, \eta}(f, x, \eta) = \inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \geq 0} \Delta C_{\ell, \eta}(f, x, \eta).
\]
Therefore, \( \delta_{\max}(\epsilon, x, \frac{1}{2}) > 0 \) for all \( x \in \mathcal{X}_2, \epsilon > 0 \) and \( \eta \in [0, \frac{1}{2}] \) if and only if for all \( x \in \mathcal{X}_2 \),
\[
\left\{ \begin{array}{l}
\inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \leq 0 \leq \mathcal{M}(f, x, \gamma)} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{F}_n} C_{\ell}(f, x, \eta) \quad \text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } 1 - 2\eta < \epsilon \leq 1 - \eta,
\inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \geq 0} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{F}_n} C_{\ell}(f, x, \eta) \quad \text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } \epsilon \leq 1 - 2\eta,
\end{array} \right.
\]
for all \( \epsilon > 0 \), which is equivalent to for all \( x \in \mathcal{X}_2 \),
\[
\left\{ \begin{array}{l}
\inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \leq 0 \leq \mathcal{M}(f, x, \gamma)} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{F}_n} C_{\ell}(f, x, \eta) \quad \text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } \frac{1 - \epsilon}{2} < \eta \leq 1 - \epsilon,
\inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \geq 0} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{F}_n} C_{\ell}(f, x, \eta) \quad \text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } \eta \leq \frac{1 - \epsilon}{2},
\end{array} \right. \tag{25}
\]
for all \( \epsilon > 0 \). Observe that
\[
\left\{ \eta \in [0, \frac{1}{2}) \mid \frac{1 - \epsilon}{2} < \eta \leq 1 - \epsilon, \epsilon > 0 \right\} = \left\{ 0 \leq \eta < \frac{1}{2} \right\}, \quad \text{and}
\]
\[
\left\{ \eta \in [0, \frac{1}{2}) \mid \eta \leq \frac{1 - \epsilon}{2}, \epsilon > 0 \right\} = \left\{ 0 \leq \eta < \frac{1}{2} \right\}, \quad \text{and}
\]
\[
\inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \leq 0 \leq \mathcal{M}(f, x, \gamma)} C_{\ell}(f, x, \eta) \geq \inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \geq 0} C_{\ell}(f, x, \eta) \quad \text{for all } \eta.
\]
Therefore, we reduce the above condition (25) as for all \( x \in \mathcal{X}_2 \),
\[
\inf_{f \in \mathcal{F}_n: \mathcal{M}(f, x, \gamma) \geq 0} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{F}_n} C_{\ell}(f, x, \eta) \quad \text{for all } \eta \in [0, \frac{1}{2}). \tag{26}
\]
To sum up, by (22), (24) and (26), we conclude the proof.

Since \( \mathcal{H}_{lin} \) is a symmetric hypothesis set, we could make use of Lemma 25 and Lemma 26 for proving Theorem 13.
Theorem 13  Let a margin-based loss $\phi$ be bounded, continuous, non-increasing, and quasi-concave even. Assume that $\phi(-t) > \phi(t)$ for any $\gamma < t \leq 1$. Then $\phi$ is $H_{lin}$-calibrated with respect to $\ell, \gamma$ if and only if for any $\gamma < t \leq 1$,

$$\phi(\gamma) + \phi(-\gamma) > \phi(t) + \phi(-t).$$  \hspace{1cm} (11)

Proof As shown by Awasthi et al. (2020), for $f \in H_{lin} = \{ x \to w \cdot x \mid \|w\| = 1 \}$,

$$\overline{M}(f, x, \gamma) = \inf_{x' : |x - x'| \leq \gamma} f(x') = \inf_{x' : \|x - x'\| \leq \gamma} (w \cdot x') = w \cdot x - \gamma \|w\| = f(x) - \gamma,$$

$$\overline{M}(f, x, \gamma) = -\inf_{x' : \|x - x'\| \leq \gamma} -f(x') = -\inf_{x' : \|x - x'\| \leq \gamma} (-w \cdot x') = w \cdot x + \gamma \|w\| = f(x) + \gamma.$$  

Thus for $H_{lin}$, $X_2 = \{ x \in X : \text{there exists } f' \in H_{lin} \text{ such that } M(f', x, \gamma) > 0 \} = \{ x \in X : \text{there exists } f' \in H_{lin} \text{ such that } f'(x) > \gamma \} = \{ x : \gamma < \|x\| \leq 1 \}$ since $f(x) = w \cdot x \in [-\|x\|, \|x\|]$ when $f \in H_{lin}$. Note $H_{lin}$ is a symmetric hypothesis set. Therefore, by Lemma 26, $\phi$ is $H_{lin}$-calibrated with respect to $\ell, \gamma$ if and only if for any $x \in X$ such that $\gamma < \|x\| \leq 1$,

$$\inf_{f \in H_{lin} : f(x) \leq \gamma} C_\phi(f, x, \frac{1}{2}) > \inf_{f \in H_{lin}} C_\phi(f, x, \frac{1}{2}) \text{, and}$$

$$\inf_{f \in H_{lin} : f(x) \leq \gamma} C_\phi(f, x, \eta) > \inf_{f \in H_{lin}} C_\phi(f, x, \eta) \text{ for all } \eta \in \left(\frac{1}{2}, 1\right], \text{ and}$$

$$\inf_{f \in H_{lin} : f(x) \geq -\gamma} C_\phi(f, x, \eta) > \inf_{f \in H_{lin}} C_\phi(f, x, \eta) \text{ for all } \eta \in [0, \frac{1}{2}).$$  \hspace{1cm} (27)

By the definition of inner risk (4), the inner $\phi$-risk is

$$C_\phi(f, x, \eta) = \eta \phi(f(x)) + (1 - \eta) \phi(-f(x)).$$

Note $f(x) = w \cdot x \in [-\|x\|, \|x\|]$ when $f \in H_{lin}$. Therefore, (27) is equivalent to for any $x \in X$ such that $\gamma < \|x\| \leq 1$,

$$\inf_{-\gamma \leq t \leq \gamma} \bar{C}_\phi(t, \frac{1}{2}) > \inf_{-\|x\| \leq t \leq \|x\|} \bar{C}_\phi(t, \frac{1}{2}), \text{ and}$$

$$\inf_{-\|x\| \leq t \leq \gamma} \bar{C}_\phi(t, \eta) > \inf_{-\|x\| \leq t \leq \|x\|} \bar{C}_\phi(t, \eta) \text{ for all } \eta \in \left(\frac{1}{2}, 1\right], \text{ and}$$

$$\inf_{-\|x\| \leq t \leq \|x\|} \bar{C}_\phi(t, \eta) > \inf_{-\|x\| \leq t \leq \|x\|} \bar{C}_\phi(t, \eta) \text{ for all } \eta \in [0, \frac{1}{2}).$$  \hspace{1cm} (28)

Suppose that $\phi$ is $H_{lin}$-calibrated with respect to $\ell, \gamma$. Since by Part 2 of Lemma 24,

$$\inf_{-\gamma \leq t \leq \gamma} \bar{C}_\phi(t, \frac{1}{2}) = \bar{C}_\phi(\gamma, \frac{1}{2}), \text{ and}$$

we obtain $\phi(\gamma) + \phi(-\gamma) = 2\bar{C}_\phi(\gamma, \frac{1}{2}) > 2\bar{C}_\phi(\frac{1}{2}) = \phi(t) + \phi(-t)$ for any $\gamma < t \leq 1$.

Now for the other direction, assume that $\phi(\gamma) + \phi(-\gamma) > \phi(t) + \phi(-t)$ for any $\gamma < t \leq 1$. For $\eta = \frac{1}{2}$, by Part 2 of Lemma 24, we obtain for any $x \in X$ such that $\gamma < \|x\| \leq 1$,

$$\inf_{-\gamma \leq t \leq \gamma} \bar{C}_\phi(t, \frac{1}{2}) = \bar{C}_\phi(\gamma, \frac{1}{2}) > \bar{C}_\phi(\|x\|, \frac{1}{2}) = \inf_{-\|x\| \leq t \leq \|x\|} \bar{C}_\phi(t, \frac{1}{2}).$$
For $\eta \in (\frac{1}{2}, 1]$ and any $x \in X$ such that $\gamma < \|x\| \leq 1$,
\[
\inf_{\|x\| \leq \gamma} \tilde{C}_\phi(t, \eta) = \min\{\tilde{C}_\phi(\gamma, \eta), \tilde{C}_\phi(-\|x\|, \eta)\} \quad \text{(Part 3 of Lemma 24)}
\]
\[
\inf_{\|x\| \leq |x|} \tilde{C}_\phi(t, \eta) = \min\{\tilde{C}_\phi(|x|, \eta), \tilde{C}_\phi(-\|x\|, \eta)\} \quad \text{(Part 3 of Lemma 24)}
\]
\[
= \tilde{C}_\phi(|x|, \eta) \quad \text{(Part 6 of Lemma 24)}
\]
Note for $\eta \in (\frac{1}{2}, 1]$ and any $x \in X$ such that $\gamma < \|x\| \leq 1$, since $\phi$ is non-increasing,
\[
\phi(\gamma) - \phi(-\gamma) + \phi(-\|x\|) \geq \phi(-\|x\|) + \phi(\|x\|) + \phi(-\|x\|) = 0.
\]

Thus
\[
\tilde{C}_\phi(\gamma, \eta) - \tilde{C}_\phi(|x|, \eta) = \eta \phi(\gamma) + (1 - \eta)\phi(-\gamma) - \eta \phi(\|x\|) - (1 - \eta)\phi(-\|x\|)
\]
\[
= (\phi(\gamma) - \phi(-\gamma) - \phi(\|x\|)) \eta + \phi(-\gamma) - \phi(-\|x\|)
\]
\[
\geq (\phi(\gamma) - \phi(-\gamma) - \phi(\|x\|)) \eta + \phi(-\gamma) - \phi(-\|x\|)
\]
\[
= \frac{1}{2} [\phi(\gamma) + \phi(-\gamma) - \phi(\|x\|) - \phi(-\|x\|)]
\]
\[
> 0.
\]

In addition, we have for $\eta \in (\frac{1}{2}, 1]$ and any $x \in X$ such that $\gamma < \|x\| \leq 1$,
\[
\tilde{C}_\phi(-\|x\|, \eta) > \tilde{C}_\phi(|x|, \eta). \quad \text{(Part 6 of Lemma 24)}
\]

Therefore for $\eta \in (\frac{1}{2}, 1]$ and any $x \in X$ such that $\gamma < \|x\| \leq 1$,
\[
\inf_{\|x\| \leq \gamma} \tilde{C}_\phi(t, \eta) = \min\{\tilde{C}_\phi(\gamma, \eta), \tilde{C}_\phi(-\|x\|, \eta)\} > \tilde{C}_\phi(|x|, \eta) = \inf_{\|x\| \leq |x|} \tilde{C}_\phi(t, \eta).
\]

For $\eta \in [0, \frac{1}{2})$ and any $x \in X$ such that $\gamma < \|x\| \leq 1$,
\[
\inf_{\|x\| \leq \gamma} \tilde{C}_\phi(t, \eta) = \min\{\tilde{C}_\phi(-\|x\|, \eta), \tilde{C}_\phi(|x|, \eta)\} \quad \text{(Part 3 of Lemma 24)}
\]
\[
\inf_{\|x\| \leq |x|} \tilde{C}_\phi(t, \eta) = \min\{\tilde{C}_\phi(|x|, \eta), \tilde{C}_\phi(-\|x\|, \eta)\} \quad \text{(Part 3 of Lemma 24)}
\]
\[
= \tilde{C}_\phi(-\|x\|, \eta) \quad \text{(Part 7 of Lemma 24)}
\]
Note for $\eta \in [0, \frac{1}{2})$ and any $x \in X$ such that $\gamma < \|x\| \leq 1$, since $\phi$ is non-increasing,
\[
\phi(-\gamma) - \phi(\gamma) + \phi(\|x\|) \leq \phi(-\|x\|) + \phi(\|x\|) + \phi(-\|x\|) + \phi(\|x\|) = 0.
\]

Thus
\[
\tilde{C}_\phi(-\gamma, \eta) - \tilde{C}_\phi(-\|x\|, \eta) = \eta \phi(-\gamma) + (1 - \eta)\phi(\gamma) - \eta \phi(\|x\|) - (1 - \eta)\phi(-\|x\|)
\]
\[
= (\phi(-\gamma) - \phi(\gamma) - \phi(-\|x\|) + \phi(\|x\|)) \eta + \phi(-\gamma) - \phi(\|x\|)
\]
\[
\geq (\phi(-\gamma) - \phi(\gamma) - \phi(-\|x\|) + \phi(\|x\|)) \eta + \phi(-\gamma) - \phi(\|x\|)
\]
\[
= \frac{1}{2} [\phi(\gamma) + \phi(-\gamma) - \phi(-\|x\|) - \phi(\|x\|)]
\]
\[
> 0.
\]
In addition, we have for \( \eta \in [0, \frac{1}{2}) \) and any \( x \in \mathcal{X} \) such that \( \gamma < \|x\| \leq 1 \),
\[
\tilde{C}_{\phi}(\|x\|, \eta) > \tilde{C}_{\phi}(-\|x\|, \eta). \tag{Part 7 of Lemma 24}
\]
Therefore for \( \eta \in [0, \frac{1}{2}) \) and any \( x \in \mathcal{X} \) such that \( \gamma < \|x\| \leq 1 \),
\[
\inf_{-\gamma \leq t \leq \|x\|} \tilde{C}_{\phi}(t, \eta) = \min \{ \tilde{C}_{\phi}(-\gamma, \eta), \tilde{C}_{\phi}(\|x\|, \eta) \} > \tilde{C}_{\phi}(-\|x\|, \eta) = \inf_{-\|x\| \leq t \leq \|x\|} \tilde{C}_{\phi}(t, \eta) .
\]

**Theorem 17** Consider \( \rho \)-margin loss \( \phi_{\rho}(t) = \min \left\{ 1, \max \left\{ 0, 1 - \frac{t}{\rho} \right\} \right\} , \rho > 0. \) Let \( \mathcal{H} \) be a symmetric hypothesis set, then the surrogate loss \( \tilde{\phi}_{\rho}(f, x, y) = \sup_{x' : \|x-x'\| \leq \gamma} \phi_{\rho}(yf(x')) \) is \( \mathcal{H} \)-calibrated with respect to \( \ell_{\gamma} \).

**Proof** By Lemma 26, if \( \mathcal{X}_2 = \emptyset \), \( \tilde{\phi}_{\rho} \) is \( \mathcal{H} \)-calibrated with respect to \( \ell_{\gamma} \). Next consider the case where \( \mathcal{X}_2 \neq \emptyset \). By Lemma 26, \( \tilde{\phi}_{\rho} \) is \( \mathcal{H} \)-calibrated with respect to \( \ell_{\gamma} \) if and only if for all \( x \in \mathcal{X}_2 \),
\[
\inf_{f \in \mathcal{H} : \mathcal{M}(f, x, \gamma) \leq 0} \mathcal{C}_{\phi_{\rho}}(f, x, \frac{1}{2}) > \inf_{f \in \mathcal{H}} \mathcal{C}_{\phi_{\rho}}(f, x, \frac{1}{2}) ,
\]
and
\[
\inf_{f \in \mathcal{H} : \mathcal{M}(f, x, \gamma) \leq 0} \mathcal{C}_{\phi_{\rho}}(f, x, \eta) > \inf_{f \in \mathcal{H}} \mathcal{C}_{\phi_{\rho}}(f, x, \eta) \text{ for all } \eta \in \left[ \frac{1}{2}, 1 \right] ,
\]
and
\[
\inf_{f \in \mathcal{H} : \mathcal{M}(f, x, \gamma) \geq 0} \mathcal{C}_{\phi_{\rho}}(f, x, \eta) > \inf_{f \in \mathcal{H}} \mathcal{C}_{\phi_{\rho}}(f, x, \eta) \text{ for all } \eta \in \left[ 0, \frac{1}{2} \right] ,
\]
where \( \mathcal{X}_2 = \{ x \in \mathcal{X} : \text{there exists } f' \in \mathcal{H} \text{ such that } \mathcal{M}(f', x, \gamma) > 0 \} \). As shown by Awasthi et al. (2020), \( \tilde{\phi}_{\rho} \) has the equivalent form
\[
\tilde{\phi}_{\rho}(f, x, y) = \phi_{\rho}(\inf_{x' : \|x-x'\| \leq \gamma} (yf(x'))) .
\]
Thus by the definition of inner risk (4), the inner \( \phi_{\rho} \)-risk is
\[
\mathcal{C}_{\phi_{\rho}}(f, x, \eta) = \eta \phi_{\rho}(\mathcal{M}(f, x, \gamma)) + (1 - \eta) \phi_{\rho}(-\mathcal{M}(f, x, \gamma)) .
\]
For any \( x \in \mathcal{X}_2 \), let \( M_x = \sup_{f \in \mathcal{H}} \mathcal{M}(f, x, \gamma) > 0 \). Since \( \mathcal{H} \) is symmetric, we have \( -M_x = \inf_{f \in \mathcal{H}} \mathcal{M}(f, x, \gamma) < 0 \). Since \( \phi_{\rho} \) is continuous, for any \( x \in \mathcal{X}_2 \) and \( \epsilon > 0 \), there exists \( f_x^\epsilon \in \mathcal{H} \) such that \( \phi_{\rho}(\mathcal{M}(f_x^\epsilon, x, \gamma)) < \phi_{\rho}(M_x) + \epsilon \) and \( \mathcal{M}(f_x^\epsilon, x, \gamma) \geq \mathcal{M}(f_x, x, \gamma) > 0, \mathcal{M}(-f_x^\epsilon, x, \gamma) \leq -\mathcal{M}(-f_x^\epsilon, x, \gamma) = -\mathcal{M}(f_x, x, \gamma) < 0 \). Next we analyze three cases:

- **When** \( \eta = \frac{1}{2} \), **since** \( \phi_{\rho} \) is non-increasing,
\[
\inf_{f \in \mathcal{H} : \mathcal{M}(f, x, \gamma) \leq 0} \mathcal{C}_{\phi_{\rho}}(f, x, \frac{1}{2}) = \inf_{f \in \mathcal{H} : \mathcal{M}(f, x, \gamma) \leq 0} \frac{1}{2} \phi_{\rho}(\mathcal{M}(f, x, \gamma)) + \frac{1}{2} \phi_{\rho}(-\mathcal{M}(f, x, \gamma)) \geq \frac{1}{2} \phi_{\rho}(0) + \frac{1}{2} \phi_{\rho}(0) = \phi_{\rho}(0) = 1 .
\]
For any \( x \in X_2 \), there exists \( f' \in \mathcal{H} \) such that \( M(f', x, \gamma) > 0 \) and \( -M(f', x, \gamma) \leq -M(f', x, \gamma) < 0 \), we obtain
\[
C_{\tilde{\phi}_\rho}(f', x, \frac{1}{2}) = \frac{1}{2} \phi_\rho(M(f', x, \gamma)) + \frac{1}{2} \phi_\rho(-M(f', x, \gamma)) = \frac{1}{2} \phi_\rho(M(f', x, \gamma)) + \frac{1}{2} < 1.
\]
Therefore for any \( x \in X_2 \),
\[
\inf_{f \in \mathcal{H}} C_{\tilde{\phi}_\rho}(f, x, \frac{1}{2}) \leq C_{\tilde{\phi}_\rho}(f', x, \frac{1}{2}) < 1 \leq \inf_{f \in \mathcal{H} : M(f, x, \gamma) \leq 0} C_{\tilde{\phi}_\rho}(f, x, \frac{1}{2}). \tag{29}
\]

- When \( \eta \in (\frac{1}{2}, 1] \), since \( \phi_\rho \) is non-increasing, for any \( x \in X_2 \),
\[
\inf_{f \in \mathcal{H} : M(f, x, \gamma) \leq 0} C_{\tilde{\phi}_\rho}(f, x, \eta) = \inf_{f \in \mathcal{H} : M(f, x, \gamma) \leq 0} \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta) \phi_\rho(-M(f, x, \gamma)) \leq \eta \phi_\rho(M_X) + \epsilon + (1 - \eta).
\]
Since \( \eta > \frac{1}{2} \) and \( M_X > 0 \), we have
\[
\inf_{f \in \mathcal{H} : M(f, x, \gamma) \leq 0} C_{\tilde{\phi}_\rho}(f, x, \eta) - C_{\tilde{\phi}_\rho}(f', x, \eta) > (2 \eta - 1)(1 - \phi_\rho(M_X)) - (\eta \phi_\rho(M_X) + \epsilon + (1 - \eta)) = 0,
\]
where we take \( 0 < \epsilon < (2 \eta - 1)(1 - \phi_\rho(M_X)) \).
Therefore for any \( \eta \in (\frac{1}{2}, 1] \) and \( x \in X_2 \), there exists \( 0 < \epsilon < (2 \eta - 1)(1 - \phi_\rho(M_X)) \) such that
\[
\inf_{f \in \mathcal{H}} C_{\tilde{\phi}_\rho}(f, x, \eta) \leq C_{\tilde{\phi}_\rho}(f', x, \eta) < \inf_{f \in \mathcal{H} : M(f, x, \gamma) \leq 0} C_{\tilde{\phi}_\rho}(f, x, \eta). \tag{30}
\]

- When \( \eta \in [0, \frac{1}{2}] \), since \( \phi_\rho \) is non-increasing, for any \( x \in X_2 \),
\[
\inf_{f \in \mathcal{H} : M(f, x, \gamma) \geq 0} C_{\tilde{\phi}_\rho}(f, x, \eta) = \inf_{f \in \mathcal{H} : M(f, x, \gamma) \geq 0} \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta) \phi_\rho(-M(f, x, \gamma)) = 1 - \eta + \inf_{f \in \mathcal{H} : M(f, x, \gamma) \geq 0} \eta \phi_\rho(M(f, x, \gamma)) \geq 1 - \eta + \phi_\rho(M_X)
\]
On the other hand, for any \( x \in X_2 \) and \( \epsilon > 0 \),
\[
C_{\tilde{\phi}_\rho}(f', x, \eta) = \eta \phi_\rho(M(f', x, \gamma)) + (1 - \eta) \phi_\rho(-M(f', x, \gamma)) = \eta + (1 - \eta) \phi_\rho(M(f', x, \gamma)) < \eta + (1 - \eta) \phi_\rho(M_X) + \epsilon
\]
Since \( \eta < \frac{1}{2} \) and \( M_\mathcal{X} > 0 \), we have

\[
\inf_{f \in \mathcal{F}} \mathcal{C}_{\tilde{\phi}_\rho}(f, x, \eta) - \mathcal{C}_{\tilde{\phi}_\rho}(-f^*_x, x, \eta) \geq [1 - \eta + \eta \phi_\rho(M_\mathcal{X})] - [\eta + (1 - \eta) \phi_\rho(M_\mathcal{X}) + \epsilon] = (1 - 2\eta)(1 - \phi_\rho(M_\mathcal{X})) - \epsilon > 0
\]

where we take \( 0 < \epsilon < (1 - 2\eta)(1 - \phi_\rho(M_\mathcal{X})) \).

Therefore for any \( \eta \in (0, \frac{1}{2}) \) and \( x \in \mathcal{X}_2 \), there exists \( 0 < \epsilon < (1 - 2\eta)(1 - \phi_\rho(M_\mathcal{X})) \) such that

\[
\inf_{f \in \mathcal{F}} \mathcal{C}_{\tilde{\phi}_\rho}(f, x, \eta) \leq \mathcal{C}_{\tilde{\phi}_\rho}(-f^*_x, x, \eta) < \inf_{f \in \mathcal{F}} \mathcal{C}_{\tilde{\phi}_\rho}(f, x, \eta).
\]

To sum up, by (29), (30) and (31), we conclude that \( \tilde{\phi}_\rho \) is \( \mathcal{H} \)-calibrated with respect to \( \ell_\gamma \).

**A.4. Proof of Theorem 14**

As shown by Awasthi et al. (2020), for \( f \in \mathcal{H}_g \), the adversarial 0/1 loss has the equivalent form

\[
\ell_\gamma(f, x, y) = \mathbb{I}_{x' : |x' - x| \leq \gamma} \inf_{y' : |y' - y| \leq \gamma} (y_g(w \cdot x') + b)y \leq y_g(w \cdot x - y) + b y \leq 0.
\]

The proofs of Theorem 14 will closely follow the proofs of Theorem 13 and Theorem 17. We will first prove Lemma 27 and Lemma 28 analogous to Lemma 25 and Lemma 26 respectively. Without loss of generality, assume that \( g \) is continuous and satisfies \( g(-1 - \gamma) + G > 0 \), \( g(1 + \gamma) - G < 0 \). Then observe that \( g(-\gamma) + G > 0 \), \( g(\gamma) - G < 0 \) since \( g \) is non-decreasing.

**Lemma 27** For a surrogate loss \( \ell \) and hypothesis set \( \mathcal{H}_g \), the calibration function of losses (\( \ell, \ell_\gamma \)) is

\[
\delta_{\max}(\epsilon, x, \eta) = \begin{cases} +\infty & \text{if } \epsilon > \max\{\eta, 1 - \eta\}, \\ \inf_{f \in \mathcal{F}_{\mathcal{H}_g}} g(w \cdot x - \gamma) + b \leq g(w \cdot x + \gamma) + b \Delta G_{\ell, \mathcal{H}_g}(f, x, \eta) & \text{if } [2\eta - 1] < \epsilon \leq \max\{\eta, 1 - \eta\}, \\ \inf_{f \in \mathcal{F}_{\mathcal{H}_g}} g(w \cdot x - \gamma) + b \leq g(w \cdot x + \gamma) + b \text{ or } (2\eta - 1)g(w \cdot x - \gamma) + b \leq 0 \Delta G_{\ell, \mathcal{H}_g}(f, x, \eta) & \text{if } \epsilon \leq [2\eta - 1]. \end{cases}
\]

**Proof** As with the proof of Lemma 25, we first characterize the inner \( \ell \)-risk and minimal inner \( \ell_\gamma \)-risk for \( \mathcal{H}_g \). By the definition of inner risk (4) and equivalent form of adversarial 0-1 loss \( \ell_\gamma \) for \( \mathcal{H}_g \) (32), the inner \( \ell_\gamma \)-risk is

\[
\mathcal{C}_{\ell_\gamma}(f, x, \eta) = \eta \mathbb{I}_{g(w \cdot x - \gamma) + b \leq 0}(1 - \eta) \mathbb{I}_{g(w \cdot x + \gamma) + b \geq 0} = \begin{cases} 1 & \text{if } g(w \cdot x - \gamma) + b \leq 0 \leq g(w \cdot x + \gamma) + b, \\ \eta & \text{if } g(w \cdot x + \gamma) + b < 0, \\ 1 - \eta & \text{if } g(w \cdot x - \gamma) + b > 0. \end{cases}
\]

where we used the fact that \( g \) is non-decreasing and \( g(w \cdot x - \gamma) \leq g(w \cdot x + \gamma) \). Note for any \( x \in \mathcal{X} \), \( w \cdot x \in [-||x||, ||x||] \). Thus we have \( g(w \cdot x - \gamma) + b \in [g(-||x|| - \gamma) - G, g(||x|| - \gamma) + G] \) and
\[ g(w \cdot x + \gamma) + b \in [g(\|x\| + \gamma) - G, g(\|x\| + \gamma) + G] \] since \( g \) is non-decreasing. By the fact that \( g(-\gamma) + G > 0 \) and \( g(\gamma) - G < 0 \), we obtain the minimal inner \( \ell_\gamma \)-risk, which is for any \( x \in X \),

\[
C^+_{\ell_\gamma}(\mathcal{X}) = \min\{\eta, 1 - \eta\}.
\]

As with the derivation of \( \Delta C_{\ell_\gamma}(f, x, \eta) (21) \), we derive \( \Delta C_{\ell_\gamma}(f, x, \eta) \) as follows. By the observation (12), for any \( x \in X \), for \( f \in \mathcal{H}_g \) such that \( g(w \cdot x - \gamma) + b \leq 0 \leq g(w \cdot x + \gamma) + b \), \( \Delta C_{\ell_\gamma}(f, x, \eta) = 1 - \min\{\eta, 1 - \eta\} = \max\{\eta, 1 - \eta\} \); for \( f \in \mathcal{H}_g \) such that \( g(w \cdot x + \gamma) + b < 0 \), \( \Delta C_{\ell_\gamma}(f, x, \eta) = \eta - \min\{\eta, 1 - \eta\} = \max\{0, 2\eta - 1\} = |2\eta - 1|_{g(w \cdot x + \gamma) + b} \leq 0 \) since \( g(w \cdot x - \gamma) + b \leq g(w \cdot x + \gamma) + b < 0 \); for \( f \in \mathcal{H}_g \) such that \( g(w \cdot x - \gamma) + b > 0 \), \( \Delta C_{\ell_\gamma}(f, x, \eta) = 1 - \eta - \min\{\eta, 1 - \eta\} = \max\{0, 1 - 2\eta\} = |2\eta - 1|_{g(w \cdot x - \gamma) + b} \leq 0 \) since \( g(w \cdot x - \gamma) + b > 0 \).

Therefore,

\[
\Delta C_{\ell_\gamma}(f, x, \eta) = \begin{cases} \max\{\eta, 1 - \eta\} & \text{if } g(w \cdot x - \gamma) + b \leq 0 \leq g(w \cdot x + \gamma) + b, \\ |2\eta - 1|_{g(w \cdot x - \gamma) + b} \leq 0 & \text{if } g(w \cdot x + \gamma) + b < 0 \text{ or } g(w \cdot x - \gamma) + b > 0. \end{cases}
\]

By (7), for a fixed \( \eta \in [0, 1] \) and \( x \in X \), the calibration function of losses (\( \ell, \ell_\gamma \)) given \( \mathcal{I}_g \) is

\[
\delta_{\max}(\epsilon, x, \eta) = \inf_{f \in \mathcal{I}_g} \{\Delta C_{\ell_\gamma}(f, x, \eta) | \Delta C_{\ell_\gamma}(f, x, \eta) \geq \epsilon\}.
\]

As with the proof of Lemma 25, we then make use of the observation (12) for deriving the the calibration function. By the observation (12), if \( \epsilon > \max\{\eta, 1 - \eta\} \), then for all \( f \in \mathcal{I}_g \), \( \Delta C_{\ell_\gamma}(f, x, \eta) < \epsilon \), which implies that \( \delta_{\max}(\epsilon, x, \eta) = \infty \); if \( |2\eta - 1| < \epsilon \leq \max\{\eta, 1 - \eta\} \), then \( \Delta C_{\ell_\gamma}(f, x, \eta) \geq \epsilon \) if and only if \( g(w \cdot x - \gamma) + b \leq 0 \leq g(w \cdot x + \gamma) + b \), which leads to

\[
\delta_{\max}(\epsilon, x, \eta) = \inf_{f \in \mathcal{I}_g} \{\Delta C_{\ell_\gamma}(f, x, \eta) | g(w \cdot x + \gamma) + b \}
\]

if \( \epsilon \leq |2\eta - 1| \), then \( \Delta C_{\ell_\gamma}(f, x, \eta) \geq \epsilon \) if and only if \( g(w \cdot x - \gamma) + b \leq 0 \leq g(w \cdot x + \gamma) + b \) or \( (2\eta - 1)[g(w \cdot x - \gamma) + b] \leq 0 \), which leads to

\[
\delta_{\max}(\epsilon, x, \eta) = \inf_{f \in \mathcal{I}_g} \{\Delta C_{\ell_\gamma}(f, x, \eta) | g(w \cdot x + \gamma) + b \}
\]

Lemma 28  Let \( \ell \) be a surrogate loss function. Then \( \ell \) is \( \mathcal{H}_g \)-calibrated with respect to \( \ell_\gamma \) if and only if for any \( x \in X \),

\[
\inf_{f \in \mathcal{H}_g} \{\Delta C_{\ell}(f, x, \eta) | g(w \cdot x + \gamma) + b \} > \inf_{f \in \mathcal{H}_g} \{\Delta C_{\ell}(f, x, \eta) | g(w \cdot x + \gamma) + b \},\text{ and}
\]

\[
\inf_{f \in \mathcal{H}_g} \{\Delta C_{\ell}(f, x, \eta) | g(w \cdot x + \gamma) + b \} > \inf_{f \in \mathcal{H}_g} \{\Delta C_{\ell}(f, x, \eta) | g(w \cdot x + \gamma) + b \} \text{ for all } \eta \in \left(\frac{1}{2}, 1\right],\text{ and}
\]

\[
\inf_{f \in \mathcal{H}_g} \{\Delta C_{\ell}(f, x, \eta) | g(w \cdot x + \gamma) + b \} > \inf_{f \in \mathcal{H}_g} \{\Delta C_{\ell}(f, x, \eta) | g(w \cdot x + \gamma) + b \} \text{ for all } \eta \in \left[0, \frac{1}{2}\right).
\]
Proof As the proof of Lemma 26 first makes use of Lemma 25 and Proposition 5, we also first make use of Lemma 27 and Proposition 5 in the following proof. Let $\delta_{\text{max}}$ be the calibration function of $(\ell, \ell_\gamma)$ for hypothesis set $\mathcal{H}_g$. By Lemma 27,

$$
\delta_{\text{max}}(\epsilon, x, \eta) = \left\{ \begin{array}{ll}
+\infty & \text{if } \epsilon > \max\{\eta, 1-\eta\},
\inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} \Delta C_{\ell, \mathcal{H}_g}(f, x, \eta) & \text{if } |2\eta - 1| < \epsilon \leq \max\{\eta, 1-\eta\},
\inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} \Delta C_{\ell, \mathcal{H}_g}(f, x, \eta) & \text{if } \epsilon \leq |2\eta - 1|.
\end{array} \right.
$$

By Proposition 5, $\ell$ is $\mathcal{H}_g$-calibrated with respect to $\ell_\gamma$ if and only if its calibration function $\delta_{\text{max}}$ satisfies $\delta_{\text{max}}(\epsilon, x, \eta) > 0$ for all $x \in \mathcal{X}$, $\eta \in [0, 1]$ and $\epsilon > 0$. The following steps are similar to the steps in the proof of Lemma 26, where we analyze by considering three cases.

For $\eta = \frac{1}{2}$, we have for any $x \in \mathcal{X}$,

$$
\delta_{\text{max}}(\epsilon, x, \frac{1}{2}) > 0 \quad \text{for all } \epsilon > 0 \iff \inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} C_{\ell}(f, x, \frac{1}{2}) > \inf_{f \in \mathcal{H}_g^x} C_{\ell}(f, x, \frac{1}{2}). \tag{33}
$$

For $1 \geq \eta > \frac{1}{2}$, we have $|2\eta - 1| = 2\eta - 1$, $\max\{\eta, 1-\eta\} = \eta$, and

$$
\inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} \Delta C_{\ell, \mathcal{H}_g}(f, x, \eta) = \inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} \Delta C_{\ell, \mathcal{H}_g}(f, x, \eta) .
$$

Therefore, $\delta_{\text{max}}(\epsilon, x, \frac{1}{2}) > 0$ for any $x \in \mathcal{X}$ if and only if for any $x \in \mathcal{X}$,

$$
\left\{ \begin{array}{ll}
\inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{H}_g^x} C_{\ell}(f, x, \eta) & \text{for all } \eta \in \left(\frac{1}{2}, 1\right] \text{ such that } 2\eta - 1 < \epsilon \leq \eta,
\inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{H}_g^x} C_{\ell}(f, x, \eta) & \text{for all } \eta \in \left(\frac{1}{2}, 1\right] \text{ such that } \epsilon \leq 2\eta - 1,
\end{array} \right.
$$

for all $\epsilon > 0$, which is equivalent to for any $x \in \mathcal{X}$,

$$
\left\{ \begin{array}{ll}
\inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{H}_g^x} C_{\ell}(f, x, \eta) & \text{for all } \eta \in \left(\frac{1}{2}, 1\right] \text{ such that } \epsilon \leq \eta < \frac{\epsilon + 1}{2},
\inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{H}_g^x} C_{\ell}(f, x, \eta) & \text{for all } \eta \in \left(\frac{1}{2}, 1\right] \text{ such that } \frac{\epsilon + 1}{2} \leq \eta,
\end{array} \right. \tag{34}
$$

for all $\epsilon > 0$. Observe that

$$
\left\{ \eta \in \left(\frac{1}{2}, 1\right] : \frac{1}{2} \leq \eta \leq \frac{\epsilon + 1}{2} \right\} = \left\{ \frac{1}{2} < \eta \leq 1 \right\}, \quad \text{and}
$$

$$
\left\{ \eta \in \left(\frac{1}{2}, 1\right] : \frac{1}{2} \leq \eta \leq \frac{\epsilon + 1}{2} \right\} = \left\{ \frac{1}{2} < \eta \leq 1 \right\}, \quad \text{and}
$$

$$
\left\{ \eta \in \left(\frac{1}{2}, 1\right] : \frac{1}{2} \leq \eta \leq \frac{\epsilon + 1}{2} \right\} = \left\{ \frac{1}{2} < \eta \leq 1 \right\},
$$

Therefore, we reduce the above condition (34) as for any $x \in \mathcal{X}$,

$$
\inf_{f \in \mathcal{H}_g^x : g(w \cdot x - ) + b \leq g(w \cdot x + ) + b} C_{\ell}(f, x, \eta) > \inf_{f \in \mathcal{H}_g^x} C_{\ell}(f, x, \eta) \quad \text{for all } \eta \in \left(\frac{1}{2}, 1\right] . \tag{35}
$$
For $\frac{1}{2} > \eta \geq 0$, we have $|2\eta - 1| = 1 - 2\eta$, $\max\{\eta, 1 - \eta\} = 1 - \eta$, and
\[
\inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} \Delta C_{\ell, H_1}(f, x, \eta) = \inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} \Delta C_{\ell, H_1}(f, x, \eta).
\]
Therefore, $\delta_{\text{max}}(\epsilon, x, \frac{1}{2}) > 0$ for any $x \in X$, $\epsilon > 0$ and $\eta \in (0, \frac{1}{2})$ if and only if for any $x \in X$,
\[
\begin{aligned}
\inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} C_{\ell}(f, x, \eta) > \inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} C_{\ell}(f, x, \eta) \quad &\text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } 1 - 2\eta < \epsilon \leq 1 - \eta, \\
\inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} C_{\ell}(f, x, \eta) > \inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} C_{\ell}(f, x, \eta) \quad &\text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } \epsilon \leq 1 - 2\eta,
\end{aligned}
\]
for all $\epsilon > 0$, which is equivalent to for any $x \in X$,
\[
\begin{aligned}
\inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} C_{\ell}(f, x, \eta) > \inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} C_{\ell}(f, x, \eta) \quad &\text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } \frac{1 - \epsilon}{2} < \eta \leq 1 - \epsilon, \\
\inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} C_{\ell}(f, x, \eta) > \inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} C_{\ell}(f, x, \eta) \quad &\text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } \eta \leq \frac{1 - \epsilon}{2}.
\end{aligned}
\tag{36}
\]
for all $\epsilon > 0$. Observe that
\[
\begin{cases}
\eta \in [0, \frac{1}{2}) : \frac{1 - \epsilon}{2} < \eta \leq 1 - \epsilon, \epsilon > 0 \bigg] = \left\{ 0 \leq \eta < \frac{1}{2} \right\}, \text{ and} \\
\eta \in [0, \frac{1}{2}) : \eta \leq \frac{1 - \epsilon}{2}, \epsilon > 0 \bigg] = \left\{ 0 \leq \eta < \frac{1}{2} \right\}, \text{ and}
\end{cases}
\]
\[
\inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} C_{\ell}(f, x, \eta) \geq \inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} C_{\ell}(f, x, \eta) \text{ for all } \eta.
\]
Therefore we reduce the above condition (36) as for any $x \in X$,
\[
\inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} C_{\ell}(f, x, \eta) > \inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} C_{\ell}(f, x, \eta) \text{ for all } \eta \in [0, \frac{1}{2}). \tag{37}
\]
To sum up, by (33), (35) and (37), we conclude the proof. }

\textbf{Theorem 14} Let $g$ be a non-decreasing and continuous function such that $g(1 + \gamma) < G$ and $g(-1 - \gamma) > -G$ for some $G \geq 0$. Let a margin-based loss $\phi$ be bounded, continuous, non-increasing, and quasi-concave even. Assume that $\phi(g(-t) - G) > \phi(G - g(t))$ and $g(-t) + g(t) \geq 0$ for any $0 \leq t \leq 1$. Then $\phi$ is $H_1$-calibrated with respect to $\ell$ if and only if for any $0 \leq t \leq 1$,
\[
\phi(G - g(t)) + \phi(g(t) - G) = \phi(g(t) + G) + \phi(-g(t) - G)
\]
and
\[
\min\{\phi(\overline{A}(t)) + \phi(-\overline{A}(t)), \phi(\overline{A}(t)) + \phi(-\overline{A}(t))\} > \phi(G - g(t)) + \phi(g(t) - G),
\]
where $\overline{A}(t) = \max_{s \in [-t,t]} g(s) - g(s + \gamma)$ and $\overline{A}(t) = \min_{s \in [-t,t]} g(s) - g(s + \gamma)$.

\textbf{Proof} By Lemma 28, $\phi$ is $H_1$-calibrated with respect to $\ell$ if and only if for any $x \in X$,
\[
\inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} C_{\phi}(f, x, \frac{1}{2}) > \inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} C_{\phi}(f, x, \frac{1}{2}), \text{ and}
\]
\[
\begin{aligned}
\inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} C_{\phi}(f, x, \eta) > \inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} C_{\phi}(f, x, \eta) \text{ for all } \eta \in \left(\frac{1}{2}, 1\right], \text{ and} \\
\inf_{f \in H_1: g(w - x - \gamma) + b \leq g(\gamma + x + \gamma) + b} C_{\phi}(f, x, \eta) > \inf_{f \in H_1: g(w - x + \gamma) + b \geq 0} C_{\phi}(f, x, \eta) \text{ for all } \eta \in [0, \frac{1}{2}). \tag{38}
\end{aligned}
\]
By the definition of inner risk (4), the inner $\phi$-risk is
\[ C_\phi(f, x, \eta) = \eta \phi(f(x)) + (1 - \eta) \phi(-f(x)). \]
and $f(x) = g(w \cdot x) + b \in [g(-\|x\|) - G, g(\|x\|) + G]$ when $f \in \mathcal{H}_g$ since $g$ is continuous and non-decreasing. Specifically, by the assumption that $g(-1 - \gamma) + G > 0$, $g(1 + \gamma) - G < 0$, when $f \in \{ f \in \mathcal{H}_g : g(w \cdot x - \gamma) + b \leq 0 \leq g(w \cdot x + \gamma) + b \}$, $f(x) = g(w \cdot x) + b \in [\min_{\|x\| \leq \|s\|} g(s) - g(\|x\| + G) s, \max_{\|x\| \leq \|s\|} g(s) - g(\|x\| - G)]$; when $f \in \{ f \in \mathcal{H}_g : g(w \cdot x - \gamma) + b \geq 0 \geq g(w \cdot x + \gamma) + b \}$, $f(x) = g(w \cdot x) + b \in [\min_{\|x\| \leq \|s\|} g(s) - g(\|x\|) + G, g(\|x\|) + G]$. For convenience, we denote $\overline{A}(t) = \max_{-t \leq s \leq t} g(s) - g(s - \gamma) \geq 0$ and $\underline{A}(t) = \min_{-t \leq s \leq t} g(s) - g(s + \gamma) \leq 0$ for any $0 \leq t \leq 1$. Therefore, for any $x \in \mathcal{X}$, (38) is equivalent to
\[
\inf_{g(-\|x\|) - G \leq s \leq g(\|x\|) + G} \overline{A}(\|x\|) \geq \frac{1}{2} \overline{C}_\phi(t, \frac{1}{2}) \quad \text{and} \quad \inf_{g(-\|x\|) - G \leq s \leq g(\|x\|) + G} \overline{C}_\phi(t, \frac{1}{2}),
\]
\[
\inf_{g(-\|x\|) - G \leq s \leq g(\|x\|) + G} \overline{C}_\phi(t, \frac{1}{2}) \quad \text{for all } \eta \in \left(\frac{1}{2}, 1\right], \text{ and}
\]
\[
\inf_{g(-\|x\|) - G \leq s \leq g(\|x\|) + G} \overline{C}_\phi(t, \frac{1}{2}) \quad \text{for all } \eta \in \left[0, \frac{1}{2}\right).
\]
Suppose that $\phi$ is $\mathcal{H}_g$-calibrated with respect to $\ell_\gamma$. Since for $\eta \in \left[0, \frac{1}{2}\right)$,
\[
\inf_{g(-\|x\|) - G \leq s \leq g(\|x\|) + G} \overline{C}_\phi(t, \eta) = \min \left\{ \overline{C}_\phi(\|x\|), \eta, \overline{C}_\phi(g(\|x\|) + G, \eta) \right\}
\]
(39)
\[
\text{(Part 3 of Lemma 24)}
\]
\[
\overline{C}_\phi(t, \eta) = \min \left\{ \overline{C}_\phi(g(-\|x\|) - G, \eta), \overline{C}_\phi(g(\|x\|) + G, \eta) \right\}
\]
(39)
\[
\text{(Part 3 of Lemma 24)}
\]
we have $\overline{C}_\phi(g(-\|x\|) - G, \eta) < \overline{C}_\phi(g(\|x\|) + G, \eta)$ for any $x \in \mathcal{X}$, otherwise
\[
\inf_{\|x\| \leq s \leq \|x\| + G} \overline{C}_\phi(t, \eta) \leq \overline{C}_\phi(g(\|x\|) + G, \eta) = \inf_{g(-\|x\|) - G \leq s \leq g(\|x\|) + G} \overline{C}_\phi(t, \eta).
\]
By Part 9 of Lemma 24, $\phi(G - g(-t)) + \phi(g(-t) - G) = \phi(g(t) + G) + \phi(\overline{A}(t))$ for all $0 \leq t \leq 1$.

Also, for any $0 \leq t \leq 1$,
\[
\frac{1}{2} \min \left\{ \phi(\overline{A}(t)) + \phi(-\overline{A}(t)), \phi(\underline{A}(t)) + \phi(-\underline{A}(t)) \right\}
\]
\[
= \inf_{\underline{A}(t) \leq \overline{A}(t)} \overline{C}_\phi(t, \frac{1}{2}) \quad \text{(Part 3 of Lemma 24)}
\]
\[
\geq \frac{1}{2} \min \left\{ \phi(G - g(-t)) + \phi(g(t) - G), \phi(g(t) + G) + \phi(\overline{A}(t)) \right\}
\]
(39)
\[
= \frac{1}{2} \left( \phi(G - g(-t)) + \phi(g(t) - G) \right)
\]
(39)
\[
\text{(Part 3 of Lemma 24)}
\]
\[
\text{(Part 3 of Lemma 24)}
\]
Now for the other direction, assume that for any $0 \leq t \leq 1$,
\[
\phi(G - g(-t)) + \phi(g(-t) - G) = \phi(g(t) + G) + \phi(\overline{A}(t))
\]
and
\[
\min \left\{ \phi(\overline{A}(t)) + \phi(-\overline{A}(t)), \phi(\underline{A}(t)) + \phi(-\underline{A}(t)) \right\} > \phi(G - g(-t)) + \phi(g(t) - G).
\]

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Then for $\eta = \frac{1}{2}$ and any $x \in \mathcal{X}$,

$$\inf_{A(\|x\|) \leq t \leq \overline{A}(\|x\|)} \bar{C}_\phi(t, \frac{1}{2})$$

$$= \frac{1}{2} \min \{ \phi(\overline{A}(\|x\|)) + \phi(-\overline{A}(\|x\|)), \phi(\overline{A}(\|x\|)) + \phi(-\overline{A}(\|x\|)) \} \quad \text{(Part 3 of Lemma 24)}$$

$$\geq \frac{1}{2} (\phi(G - g(-\|x\|)) + \phi(-g(-\|x\|))) + \phi(G, \eta)$$

$$= \frac{1}{2} \min \{ \phi(G - g(-\|x\|)) + \phi(-g(-\|x\|)) \} + \phi(G, \eta) = \bar{C}_\phi(g(-\|x\|) + G, \eta) \quad \text{(by assumption)}$$

$$= \bar{C}_\phi(g(\|x\|) + G, \eta)$$

For $\eta \in (\frac{1}{2}, 1]$ and any $x \in \mathcal{X}$,

$$\inf_{g(-\|x\|) - G \leq \overline{A}(\|x\|)} \bar{C}_\phi(t, \eta) = \min \{ \bar{C}_\phi(g(-\|x\|) - G, \eta), \bar{C}_\phi(\overline{A}(\|x\|), \eta) \} \quad \text{(Part 3 of Lemma 24)}$$

$$\inf_{g(-\|x\|) - G \leq \overline{A}(\|x\|) + G} \bar{C}_\phi(t, \eta) = \min \{ \bar{C}_\phi(g(-\|x\|) - G, \eta), \bar{C}_\phi(g(\|x\|) + G, \eta) \} \quad \text{(Part 3 of Lemma 24)}$$

Since $\phi$ is non-increasing, we have for any $x \in \mathcal{X}$,

$$\phi(-g(\|x\|) - G) - \phi(g(\|x\|) + G) + \phi(\overline{A}(\|x\|)) - \phi(-\overline{A}(\|x\|))$$

$$\geq \phi(-g(\|x\|) - G) - \phi(g(\|x\|) + G) + \phi(g(\|x\|) + G) - \phi(-g(\|x\|) - G)$$

$$= 0.$$

Then for $\eta \in (\frac{1}{2}, 1]$ and any $x \in \mathcal{X}$,

$$\bar{C}_\phi(\overline{A}(\|x\|), \eta) - \bar{C}_\phi(g(\|x\|) + G, \eta)$$

$$= \phi(\overline{A}(\|x\|)) - \phi(-\overline{A}(\|x\|)) + \phi(-g(\|x\|) - G) - \phi(g(\|x\|) + G) \eta + \phi(-\overline{A}(\|x\|)) - \phi(-g(\|x\|) - G)$$

$$\geq \phi(\overline{A}(\|x\|)) - \phi(-\overline{A}(\|x\|)) + \phi(-g(\|x\|) - G) - \phi(g(\|x\|) + G) \frac{1}{2} + \phi(-\overline{A}(\|x\|)) - \phi(-g(\|x\|) - G)$$

$$= \frac{1}{2} (\phi(\overline{A}(\|x\|)) - \phi(-\overline{A}(\|x\|))) - \phi(-g(\|x\|) - G) - \phi(g(\|x\|) + G)) > 0.$$

In addition, by Part 8 of Lemma 24, for all $\eta \in (\frac{1}{2}, 1]$ and any $x \in \mathcal{X}$, $\bar{C}_\phi(g(-\|x\|) - G, \eta) - \bar{C}_\phi(g(\|x\|) + G, \eta) > 0$. As a result, for $\eta \in (\frac{1}{2}, 1]$ and any $x \in \mathcal{X}$,

$$\inf_{g(-\|x\|) - G \leq \overline{A}(\|x\|)} \bar{C}_\phi(t, \eta) - \inf_{g(-\|x\|) - G \leq \overline{A}(\|x\|) + G} \bar{C}_\phi(t, \eta)$$

$$= \min \{ \bar{C}_\phi(g(-\|x\|) - G, \eta) - \bar{C}_\phi(g(\|x\|) + G, \eta), \bar{C}_\phi(\overline{A}(\|x\|), \eta) - \bar{C}_\phi(g(\|x\|) + G, \eta) \}$$

$$> 0.$$

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Finally, for \( \eta \in [0, \frac{1}{2}] \), by Part 9 of Lemma 24, we have \( \bar{C}_{\phi}(g(-\|x\|) - G, \eta) < \bar{C}_{\phi}(g(\|x\|) + G, \eta) \) and

\[
\inf_{\mathcal{A}(\|x\|) \leq t \leq g(\|x\|) + G} \bar{C}_{\phi}(t, \eta) = \min\{\bar{C}_{\phi}(\mathcal{A}(\|x\|), \eta), \bar{C}_{\phi}(g(\|x\|) + G, \eta)\} \quad \text{(Part 3 of Lemma 24)}
\]

\[
\inf_{g(-\|x\|) - G \leq t \leq g(\|x\|) + G} \bar{C}_{\phi}(t, \eta) = \min\{\bar{C}_{\phi}(g(-\|x\|) - G, \eta), \bar{C}_{\phi}(g(\|x\|) + G, \eta)\} = \bar{C}_{\phi}(g(-\|x\|) - G, \eta) \quad \text{(Part 9 of Lemma 24)}
\]

Since \( \phi(\mathcal{A}(\|x\|)) + \phi(-\mathcal{A}(\|x\|)) > \phi(G - g(-\|x\|)) + \phi(g(-\|x\|) - G) \) and \( \phi \) is non-increasing, we have for any \( x \in \mathcal{X} \),

\[
\phi(G - g(-\|x\|)) - \phi(g(-\|x\|) - G) + \phi(\mathcal{A}(\|x\|)) - \phi(-\mathcal{A}(\|x\|)) = \phi(G - g(-\|x\|)) - \phi(-\mathcal{A}(\|x\|)) + \phi(\mathcal{A}(\|x\|)) - \phi(g(-\|x\|) - G)
\]

\[
< \phi(\mathcal{A}(\|x\|)) - \phi(g(-\|x\|) - G) + \phi(\mathcal{A}(\|x\|)) - \phi(g(-\|x\|) - G) = 2[\phi(\mathcal{A}(\|x\|)) - \phi(g(-\|x\|) - G)]
\]

\[
\leq 0.
\]

Then for \( \eta \in [0, \frac{1}{2}] \) and any \( x \in \mathcal{X} \),

\[
\bar{C}_{\phi}(\mathcal{A}(\|x\|), \eta) - \bar{C}_{\phi}(g(-\|x\|) - G, \eta) = [\phi(G - g(-\|x\|)) - \phi(g(-\|x\|) - G) + \phi(\mathcal{A}(\|x\|)) - \phi(-\mathcal{A}(\|x\|))] \eta + \phi(-\mathcal{A}(\|x\|)) - \phi(G - g(-\|x\|)) \geq [\phi(G - g(-\|x\|)) - \phi(g(-\|x\|) - G) + \phi(\mathcal{A}(\|x\|)) - \phi(-\mathcal{A}(\|x\|))] \frac{1}{2} + \phi(-\mathcal{A}(\|x\|)) - \phi(G - g(-\|x\|)) = \frac{1}{2}[\phi(\mathcal{A}(\|x\|)) + \phi(-\mathcal{A}(\|x\|))] - \phi(g(-\|x\|) - G) - \phi(G - g(-\|x\|)) > 0.
\]

In addition, by Part 9 of Lemma 24, for all \( \eta \in [0, \frac{1}{2}] \) and any \( x \in \mathcal{X} \), \( \bar{C}_{\phi}(g(\|x\|) + G, \eta) - \bar{C}_{\phi}(g(-\|x\|) - G, \eta) > 0 \). As a result, for \( \eta \in [0, \frac{1}{2}] \) and any \( x \in \mathcal{X} \),

\[
\inf_{\mathcal{A}(\|x\|) \leq t \leq g(\|x\|) + G} \bar{C}_{\phi}(t, \eta) - \inf_{g(-\|x\|) - G \leq t \leq g(\|x\|) + G} \bar{C}_{\phi}(t, \eta) = \min\{\bar{C}_{\phi}(g(\|x\|) + G, \eta) - \bar{C}_{\phi}(g(-\|x\|) - G, \eta), \bar{C}_{\phi}(\mathcal{A}(\|x\|), \eta) - \bar{C}_{\phi}(g(-\|x\|) - G, \eta)\} > 0.
\]