Strong transitivity properties for operators

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Abstract

Given a Furstenberg family $\mathcal{F}$ of subsets of $\mathbb{N}$, an operator $T$ on a topological vector space $X$ is called $\mathcal{F}$-transitive provided for each non-empty open subsets $U, V$ of $X$ the set $\{n \in \mathbb{Z}_+ : T^n(U) \cap V \neq \emptyset\}$ belongs to $\mathcal{F}$. We classify the topologically transitive operators with a hierarchy of $\mathcal{F}$-transitive subclasses by considering families $\mathcal{F}$ that are determined by various notions of largeness and density in $\mathbb{Z}_+$.

1 Introduction

Throughout this paper $X$ denotes a topological space and $\mathcal{U}(X)$ the set of non-empty open subsets of $X$. When $X$ is a topological vector space, $\mathcal{L}(X)$ stands for the set of operators (i.e., linear and continuous self-maps) on $X$. An operator $T \in \mathcal{L}(X)$ is called hypercyclic if there exists a vector $x \in X$ such that for each $V$ in $\mathcal{U}(X)$ the time return set

$$N_T(x, V) = N(x, V) := \{ n \geq 0 : T^n x \in V \}$$

is non-empty, or equivalently (since $X$ has no isolated points) an infinite set. When $X$ is an $F$-space (that is, a complete and metrizable topological vector space), we know thanks to Birkhoff’s transitivity theorem that $T$ is hypercyclic if and only if it is topologically transitive, that is, provided

$$N_T(U, V) = N(U, V) := \{ n \geq 0 : T^n(U) \cap V \neq \emptyset \}$$

is infinite for every $U, V \in \mathcal{U}(X)$.

Since 2004, several refined notions of hypercyclicity based on the properties of time return sets $N(x, V)$ have been investigated: frequent hypercyclicity [3, 2], $\mathcal{U}$-frequent hypercyclicity [21, 9], reiterative hypercyclicity

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More recently a general notion called $\mathcal{A}$-hypercyclicity, which generalizes the abovementioned notions of hypercyclicity, has been used to investigate the different types of hypercyclic operators, see [7, 9].

Our aim here is to investigate refined notions of topological transitivity based on properties satisfied by the return sets $N(U, V)$. Some of these are already well-known, such as the topological notions of mixing, weak-mixing, and ergodicity, say. Recall that a continuous self-map $T$ on $X$ is called mixing provided $N(U, V)$ is cofinite for each $U, V \in \mathcal{U}(X)$. Also, $T$ is called weakly mixing whenever $T \times T$ is topologically transitive on $X \times X$, and this occurs precisely when the return set $N(U, V)$ is thick (i.e. contains arbitrarily long intervals) for each $U, V \in \mathcal{U}(X)$ [19]. Finally, $T$ is topologically ergodic provided $N(U, V)$ is syndetic (i.e. has bounded gaps) for each $U, V \in \mathcal{U}(X)$. It is known that topologically ergodic operators are weakly mixing [14]. The above mentioned notions may be stated through the concept of a (Furstenberg) family. The symbols $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the sets of integers and of positive integers, respectively.

**Definition 1.1.** We say that a non-empty collection $\mathcal{F}$ of subsets of $\mathbb{Z}_+$ is a family provided that each set $A \in \mathcal{F}$ is infinite and that $\mathcal{F}$ is hereditarily upward (i.e. for any $A \in \mathcal{F}$, if $B \supset A$ then $B \in \mathcal{F}$). The dual family $\mathcal{F}^*$ of $\mathcal{F}$ is defined as the collection of subsets $A$ of $\mathbb{Z}_+$ such that $A \cap B \neq \emptyset$ for every $B \in \mathcal{F}$.

Some standard families are the following: The family $\mathcal{I}$ of infinite sets, whose dual family $\mathcal{I}^*$ coincides with the family of cofinite sets. The family $\mathcal{T}$ of thick sets, whose dual family is $\mathcal{S} = \mathcal{T}^*$, the family of syndetic sets. For a topologically transitive map $T$ a distinguished family is

$$N_T := \{ A \subset \mathbb{Z}_+ : N_T(U, V) \subseteq A \text{ for some } U, V \in \mathcal{U}(X) \}.$$

From now on the symbol $\mathcal{F}$ will always denote a family.

**Definition 1.2.** We say that a continuous map $T$ on $X$ is $\mathcal{F}$-transitive (or an $\mathcal{F}$-map, for short) provided $N_T \subset \mathcal{F}$, that is, provided $N(U, V) \in \mathcal{F}$ for each $U, V \in \mathcal{U}(X)$. If in addition $X$ is a topological vector space and $T \in \mathcal{L}(X)$ we call $T$ an $\mathcal{F}$-transitive operator (or $\mathcal{F}$-operator for short).

Hence the $\mathcal{I}$-operators are precisely those operators which are topologically transitive, and the $\mathcal{T}$-operators and $\mathcal{S}$-operators are precisely those which are mixing and weak mixing, respectively. The $\mathcal{T}^* = \mathcal{S}$-operators, that is, the topologically ergodic operators.

We present here some new classes of topologically transitive operators by considering families $\mathcal{F}$ defined in terms of various notions of density and largeness in $\mathbb{Z}_+$. A hierarchy of fourteen classes (which include the earlier mentioned classes defined by properties of return sets $N(x, V)$) appears in Figure 2 and summarizes our findings. We stress that while trivially any $\mathcal{F}_1$-map is an $\mathcal{F}_2$-map when $\mathcal{F}_1 \subset \mathcal{F}_2$, it is possible that the classes of
The paper is organized as follows. In Section 2 we describe some general facts about families $\mathcal{F}$ and their corresponding $\mathcal{F}$-transitive maps and operators. In Theorem 2.4 we provide an extension of the Hypercyclicity Criterion that ensures an operator to be $\mathcal{F}$-transitive. We apply this criterion in Section 3 to characterize $\mathcal{F}$-transitivity among unilateral and bilateral weighted backward shift operators on $c_0$ and $\ell_p$ ($1 \leq p < \infty$) spaces. To illustrate, we establish in Corollary 3.4 that a unilateral backward shift $B_w$ is topologically ergodic precisely when its weight sequence $w = (w_n)_n$ satisfies that each set

$$A_M = \{ n : \prod_{j=1}^{n} w_j > M \} \quad (M > 0)$$

is syndetic. Section 4 is dedicated to $\mathcal{F}$-operators induced by families $\mathcal{F}$ given by sets of positive or full (lower or upper) asymptotic density or Banach density. In Section 5, we look at $\mathcal{F}$-operators induced by families $\mathcal{F}$ commonly used in Ramsey theory, and we compare the classes that we obtain with the class of reiteratively hypercyclic operators (Subsection 5.1).

Some natural questions conclude the paper.

2 $\mathcal{F}$-Transitivity

In this section we introduce a sufficient condition for an operator to be an $\mathcal{F}$-operator, which we call the $\mathcal{F}$-Transitivity Criterion, and it is in the same vein of the Hypercyclicity Criterion. Moreover, we will study the notion of hereditarily $\mathcal{F}$-operator.

We will be interested in the following three special properties a family $\mathcal{F}$ can have: being a filter, being partition-regular, and being shift-invariant. We use the following notation: given two families $\mathcal{F}_1$ and $\mathcal{F}_2$

$$\mathcal{F}_1 \cdot \mathcal{F}_2 := \{ A \cap B : A \in \mathcal{F}_1, \ B \in \mathcal{F}_2 \}.$$ 

Obviously, $\mathcal{F}_1 \subseteq \mathcal{F}_1 \cdot \mathcal{F}_2$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1 \cdot \mathcal{F}_2$. A family $\mathcal{F}$ is a filter provided it is invariant under finite intersections (i.e., provided $\mathcal{F} \cdot \mathcal{F} \subseteq \mathcal{F}$). Say, the family $\mathcal{I}^*$ of cofinite sets is a filter while the families $\mathcal{I}$ and $\mathcal{S}$ of infinite sets and of syndetic sets are not.

The second property, that of being partition regular, will be useful for us to identify filters. A family $\mathcal{F}$ on $\mathbb{Z}_+$ is said to be partition regular if for every $A \in \mathcal{F}$ and any finite partition $\{ A_1, \ldots, A_n \}$ of $A$, there exists some $i = 1, \ldots, n$ such that $A_i \in \mathcal{F}$. The family $\mathcal{I}$ is an example of partition regular family, while the families $\mathcal{I}^*$, $\mathcal{T}$ and $\mathcal{S}$ are not. Later we will see other examples of partition regular families: the family of piecewise syndetic sets (see Remark 2.5), the family of sets with positive upper (Banach) density (see Section 4), the families of $\Delta$-sets and of $\mathcal{I}\mathcal{P}$-sets (see Section 5).
Lemma 2.1. Given a family $\mathcal{F}$, the following are equivalent:

(I) $\mathcal{F}$ is partition regular,

(II) $A \cap A' \in \mathcal{F}$ for every $A \in \mathcal{F}$ and $A' \in \mathcal{F}^+$ (i.e., $\mathcal{F} \cdot \mathcal{F}^+ \subset \mathcal{F}$),

(III) $\mathcal{F}^*$ is a filter.

Proof. (I) $\implies$ (II): Given $A \in \mathcal{F}$ and $A' \in \mathcal{F}^*$ it is clear that $A \cap A' \neq \emptyset$ by definition of dual family. Since $(A \cap A') \cup (A \setminus A') = A$, either $A \cap A' \in \mathcal{F}$ or $A \setminus A' \in \mathcal{F}$ by (I). Since $(A \setminus A') \cap A' = \emptyset$, by definition of dual family we necessarily have $A \cap A' \in \mathcal{F}$.

(II) $\implies$ (III): For arbitrary $A', B' \in \mathcal{F}^*$ and $A \in \mathcal{F}$, by applying (II) and the definition of dual family we have $A \cap (A' \cap B') = (A \cap A') \cap B' \neq \emptyset$, which yields that $\mathcal{F}^*$ is a filter.

(III) $\implies$ (I): We will just show that, given $A \in \mathcal{F}$ and $A_1, A_2 \subset \mathbb{Z}_+$ such that $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$, then either $A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$. The general case can be deduced by an inductive process. Since $\mathcal{F} = \mathcal{F}^{**}$, we need to show that $A_i \cap A' \neq \emptyset$ for every $A' \in \mathcal{F}^*$, for $i = 1$ or $i = 2$. Suppose that there exist $A', B' \in \mathcal{F}^*$ with $A_1 \cap A' = \emptyset$ and $A_2 \cap B' = \emptyset$. Since $\mathcal{F}^*$ is a filter, then $C' := A' \cap B' \in \mathcal{F}^*$. Thus,

$$\emptyset \neq A \cap C' \subset (A_1 \cap A') \cup (A_2 \cap B') = \emptyset,$$

which is a contradiction. \qed

Notice that $(\mathcal{F}^*)^* = \mathcal{F}$ for any family $\mathcal{F}$: the inclusion $\mathcal{F} \subset (\mathcal{F}^*)^*$ is immediate. Conversely, if $A \in (\mathcal{F}^*)^*$, then $\mathbb{Z}_+ \setminus A \notin \mathcal{F}^*$ by the definition of a dual family. This means that there exists $B \in \mathcal{F}$ such that $B \cap (\mathbb{Z}_+ \setminus A) = \emptyset$. That is, $B \subset A$, which gives $A \in \mathcal{F}$.

Thus any family is a dual family, and Lemma 2.1 also gives that a family $\mathcal{F}$ is a filter if and only if $\mathcal{F}^* \cdot \mathcal{F} \subset \mathcal{F}^*$ and if and only if $\mathcal{F}^*$ is partition regular. Another consequence of Lemma 2.1 is that any family $\mathcal{F}$ that is both a filter and partition regular (called an ultrafilter) must satisfy $\mathcal{F} = \mathcal{F}^*$.

Finally, our third property: A family $\mathcal{F}$ on $\mathbb{Z}_+$ is said to be shift-invariant provided for every $i \in \mathbb{Z}_+$ and each $A \in \mathcal{F}$, we have $(A - i) \cap \mathbb{Z}_+ \in \mathcal{F}$. We say that $\mathcal{F}$ is called shift+$\cdot$-invariant if for every $i \in \mathbb{Z}_+$ and each $A \in \mathcal{F}$, we have $A + i \in \mathcal{F}$. When $\mathcal{F}$ is both, shift$\cup$-invariant and shift+$\cdot$-invariant, we simply call it shift invariant. For instance, the families of infinite sets, cofinite sets, thick sets and syndetic sets are shift invariant.

We may gain shift invariance by reducing a family. Given a family $\mathcal{F}$, we define

$$\mathcal{F}_+ = \{ A \subset \mathbb{Z}_+ : \forall N \in \mathbb{Z}_+ \exists B \in \mathcal{F} \text{ such that } A \supset B + [0,N] \},$$

$$\mathcal{F}_- = \{ A \subset \mathbb{Z}_+ : \forall N \in \mathbb{Z}_+ \exists B \in \mathcal{F} \text{ such that } A \supset (B + [-N,0]) \cap \mathbb{Z}_+ \},$$

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\[ \mathcal{F} = \{ A \subset \mathbb{Z}_+ \ : \ \forall N \in \mathbb{Z}_+ \exists B \in \mathcal{F} \text{ such that } A \supset (B + [-N, N]) \cap \mathbb{Z}_+ \}. \]

So for any family \( \mathcal{F} \) we have the inclusions \( \mathcal{F} \subset \mathcal{F}_+ \subset \mathcal{F} \) and \( \mathcal{F}_- \subset \mathcal{F} \), and that \( \mathcal{F}_- \) is shift\(_-\)-invariant, \( \mathcal{F}_+ \) is shift\(_+\)-invariant, and \( \mathcal{F} \) is shift invariant.

**Lemma 2.2.** If \( \mathcal{F} \) is a filter on \( \mathbb{Z}_+ \), so is \( \mathcal{F}_- \). Moreover, for any family \( \mathcal{F} \) satisfying \( \mathcal{F} \cdot \mathcal{F} \subset \mathcal{F} \) the subfamily \( \mathcal{F} \) is a filter.

**Proof.** Let \( A_1, A_2 \in \mathcal{F}_- \). We have to show that \( A_1 \cap A_2 \in \mathcal{F}_- \). Given \( N \in \mathbb{N} \), there are \( B_1(N), B_2(N) \in \mathcal{F} \) such that \( (B_1(N) + [-2N, 2N]) \cap \mathbb{Z}_+ \subset A_1 \) and \( (B_2(N) + [-2N, 2N]) \cap \mathbb{Z}_+ \subset A_2 \). For \( i = 1, 2 \) we define

\[ \bar{A}_i(N) := \bigcup_{J \geq N} (B_i(J) + [-J, J]) \cap \mathbb{Z}_+. \]

Clearly \( \bar{A}_1(N), \bar{A}_2(N) \in \mathcal{F}_- \) for each \( N \in \mathbb{N} \). By hypothesis, \( B(N) := \bar{A}_1(N) \cap \bar{A}_2(N) \in \mathcal{F} \), \( N \in \mathbb{N} \). To prove that \( A_1 \cap A_2 \in \mathcal{F}_- \) we just need to show that \( (B(N) + [-N, N]) \cap \mathbb{Z}_+ \subset \bar{A}_1 \cap \bar{A}_2 \) for every \( N \in \mathbb{N} \). Indeed, given \( N \in \mathbb{N} \) and \( m \in (B(N) + [-N, N]) \cap \mathbb{Z}_+ \), we write \( m = k(N) + l(N) \) with \( k(N) \in B(N) \) and \( l(N) \in [-N, N] \). By definition of \( B(N) \) we have

\[ k(N) = k_1(J_1) + l_1(J_1) = k_2(J_2) + l_2(J_2) \]

for some \( k_1(J_i) \in B_i(J_i), \ l_i(J_i) \in [-J_i, J_i], \ J_i \geq N, \ i = 1, 2 \).

Thus

\[ m = k_1(J_1) + l_1(J_1) + l(N) \in (B_1(J_1) + [-2J_1, 2J_1]) \cap \mathbb{Z}_+ \subset A_1, \]

and, analogously, \( m \in A_2 \), which yields the result.

The rest of the section is dedicated to \( \mathcal{F} \)-maps and \( \mathcal{F} \)-operators. Every \( \mathcal{F}_- \)-map is an \( \mathcal{F} \)-map, since \( \mathcal{F} \subset \mathcal{F}_- \). The next lemma gives conditions for the converse, and is used in Proposition 3.1

**Lemma 2.3.** Let \( \mathcal{F} \) be a family on \( \mathbb{Z}_+ \) and let \( T \) be a \( \mathcal{F} \)-map. The following are equivalent.

(i) \( T \) is weakly mixing,

(ii) \( T \) is an \( \mathcal{F}_- \)-map.

**Proof.** (i) implies (ii): Given \( N \in \mathbb{N} \) and \( U, V \in \mathcal{U}(X) \), since \( T \) is weakly mixing, by Furstenberg result we know that \( \mathcal{N}_T \) is a filter, so there are \( U', V' \in \mathcal{U}(X) \) such that

\[ N(U', V') \subset N(T^{-m}(U), V) \cap N(U, T^{-m}(V)), \]

for \( m = 0, \ldots, N \). By \( \mathcal{F} \)-transitivity we have \( N(U', V') \in \mathcal{F} \). We then conclude that \( (N(U', V') + [-N, N]) \cap \mathbb{Z}_+ \subset N(U, V) \), and \( T \) is \( \mathcal{F}_- \)-transitive.

(ii) implies (i): If \( T \) is an \( \mathcal{F}_- \)-map, since every element of \( \mathcal{F}_- \) is thick, we have that \( \mathcal{N}_T \) consists of thick sets and, as we already recalled in the introduction, this means that \( T \) is weakly mixing. \( \square \)
To state the $\mathcal{F}$.Transitivity Criterion, we recall the notion of limit along a family $\mathcal{F}$: Given a sequence $\{x_n\}_n$ in $X$ and $x \in X$, we say that

$$\mathcal{F} \lim_n x_n = x,$$

provided $\{n \in \mathbb{Z}_+ : x_n \in U\} \in \mathcal{F}$ for each neighbourhood $U$ of $x$.

**Theorem 2.4.** *(\mathcal{F}.Transitivity Criterion)*  Let $T$ be an operator on a topological vector space $X$ and let $\mathcal{F}$ be a family on $\mathbb{Z}_+$ such that $\mathcal{F}$ is a filter. Suppose there exist $D_1, D_2$ dense sets in $X$, and (possibly discontinuous) mappings $S_n : D_2 \to X$, $n \in \mathbb{N}$ satisfying

(a) $\mathcal{F} \lim_n T^n(x) = 0$ for every $x \in D_1$

(b) $\mathcal{F} \lim_n (S_n(y), T^n S_n(y)) = (0, y)$ for every $y \in D_2$.

Then $T$ is an $\mathcal{F}$-operator.

**Proof.** Let $U, V \in \mathcal{U}(X)$. We fix $U', V' \in \mathcal{U}(X)$ and a 0-neighbourhood $W$ such that $U' + W \subset U$ and $V' + W \subset V$. Given $N \in \mathbb{N}$, pick $x \in D_1 \cap T^{-N} U'$ and $y \in D_2 \cap T^{-N} V'$. By continuity of $T$ we easily get

$$\mathcal{F}_+ \lim_n T^n x = 0,$$

which yields $N(T^{-N} U', W) \in \mathcal{F}_+$. That is, there is $A \in \mathcal{F}$ such that $A + [0, 2N] \subset N(T^{-N} U', W)$. Therefore,

$$(A + [-N, N]) \cap \mathbb{Z}_+ \subset (N(T^{-N} U', W) - N) \cap \mathbb{Z}_+ \subset N(U', W),$$

and, since $N$ was arbitrary, we have that $N(U', W) \in \mathcal{F}$.

Also, we find a 0-neighbourhood $W' \subset W$ with $T^m(W') \subset W$ and $y + W' \subset T^{-N} V'$, $m = 0, \ldots, 2N$. There is $A \in \mathcal{F}$ such that $S_n y \in W'$ and $T^n S_n(y) \in y + W'$ for all $n \in A$. Thus,

$$(T^{(n-m)}(T^m S_n(y)), T^m S_n(y)) \in (y + W', T^m(W')) \subset (T^{-N} V', W'),$$

for $m = 0, \ldots, 2N$ and for every $n \in A$. In particular, $(A + [-N, N]) \cap \mathbb{Z}_+ \subset N(W, V')$. Since $N$ was arbitrary, we obtain that $N(W, V') \in \mathcal{F}$. Therefore,

$$N(U, V) \supset N(U' + W, V' + W) \supset N(U', W) \cap N(W, V') \in \mathcal{F} \cdot \mathcal{F} \subset \mathcal{F},$$

that is, $T$ is an $\mathcal{F}$-operator. \hfill $\square$

**Remark 2.5.** 1. By Lemma 2.2, the assumption that $\mathcal{F}$ be a filter is trivially satisfied in the case that $\mathcal{F}$ is a filter, but Theorem 2.4 applies beyond this case. For instance, the family $\mathcal{F} = \mathcal{S}$ of syndetic sets is not a filter, and $\mathcal{S} = \mathcal{F} \mathcal{S}$ is the family of thickly syndetic sets, which is
a filter. So every operator that satisfies the $S$-Transitivity Criterion is a $TS$-operator.

In contrast, if we consider the family of piecewise syndetic sets $\mathcal{PS} = \mathcal{T} \cdot S$ (i.e., $A$ is piecewise syndetic if, and only if, it is the intersection of a thick set with a syndetic set), then $\mathcal{PS} = \mathcal{T}$, and $\emptyset \in \mathcal{T} \cdot \mathcal{T}$. Thus the hypotheses of Theorem 2.4 are not satisfied. Actually, it is not hard to construct an operator $T$ such that conditions (a) and (b) in Theorem 2.4 are satisfied for $F = \mathcal{PS}$, with $T$ not even transitive.

2. Another remarkable case is provided by, given a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$, considering the filter

$$\mathcal{F} := \{ A \subset \mathbb{N} : \exists j \in \mathbb{N} \text{ with } A \supset \{ n_k : k \geq j \} \}.$$ 

In this case Theorem 2.4 turns out to coincide with the classical Hypercyclicity Criterion. Moreover, since the Hypercyclicity Criterion characterizes the weakly mixing operators on separable $F$-spaces [8], we have that every weakly mixing operator $T$ on a separable $F$-space $X$ supports a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that $T$ is an $\mathcal{F}$-operator, where

$$\mathcal{F} := \{ A \subset \mathbb{N} : \forall N \in \mathbb{N} \exists j \in \mathbb{N} \text{ with } A \supset \{ n_k : k \geq j \} + [-N, N] \}.$$ 

3. We note that for an $\mathcal{F}$-operator $T$ with $\mathcal{F}$ a filter it is not true in general that $T$ must satisfy the $S$-Transitivity Criterion for some filter $\mathcal{G} \subset \mathcal{F}$: just consider the family $\mathcal{F} = \mathcal{I}^r$ of cofinite sets and the fact that there exist mixing operators not satisfying Kitai’s Criterion [12, Theorem 2.5].

4. Recall that for the case $\mathcal{F} = \mathcal{I}$, Furstenberg [10, Proposition II.3] showed that once $T \oplus T$ is an $\mathcal{I}$-map on $X^2$, every direct sum $\oplus_{j=1}^r T$ on $X^r$ is an $\mathcal{I}$-map too ($r \in \mathbb{N}$). The assumptions of the $\mathcal{F}$-Transitivity Criterion on an operator $T$ clearly ensure that (any direct sum $\oplus_{j=1}^r T$ will satisfy the $\mathcal{F}$-Transitivity Criterion on the space $X^r$ and thus that) $\oplus_{j=1}^r T$ is an $\mathcal{F}$-operator on $X^r$, for every $r \in \mathbb{N}$.

We next introduce the concept of a hereditarily $\mathcal{F}$-operator, and we establish links with that of an $\mathcal{F}$-operator.

**Definition 2.6.** We say that a continuous map $T$ is a *hereditarily $\mathcal{F}$-map* if $N(U, V) \cap A \in \mathcal{F}$ for every $U, V \in \mathcal{U}(X)$ and every $A \in \mathcal{F}$ (that is, $N_T \cdot \mathcal{F} \subset \mathcal{F}$). In addition, if $X$ is a topological vector space and $T \in \mathcal{L}(X)$, we say that $T$ is a *hereditarily $\mathcal{F}$-operator*.

Clearly, hereditarily $\mathcal{F}$-maps are $\mathcal{F}$-maps. Moreover, they are automatically $\mathcal{F}^*$-maps since $N_T \cdot \mathcal{F} \subset \mathcal{F} \not\supset \emptyset$. Also, for a filter $\mathcal{F}$ the concepts of $\mathcal{F}$-map and hereditarily $\mathcal{F}$-map are equivalent. More generally, we have:
Proposition 2.7. Let $T$ be a continuous map on a complete separable metric space $X$ without isolated points.

(A) Let $\mathcal{F}$ be a partition regular family. Then the following are equivalent:

1. $T$ is an $\mathcal{F}^*$-map;
2. $T$ is a hereditarily $\mathcal{F}^*$-map;
3. $T$ is a hereditarily $\mathcal{F}$-map;
4. $hcA := \{x \in X : \{T^n x : n \in A\} = X\}$ is a dense $(G^\delta)$ set in $X$ for any $A \in \mathcal{F}$.

(B) Let $\mathcal{F}$ be a filter. Then the following are equivalent:

i. $T$ is an $\mathcal{F}$-map;
ii. $T$ is a hereditarily $\mathcal{F}$-map;
iii. $T$ is a hereditarily $\mathcal{F}^*$-map;
iv. $hcA := \{x \in X : \{T^n x : n \in A\} = X\}$ is a dense $(G^\delta)$ set in $X$ for any $A \in \mathcal{F}^*$.

Proof. We will just show (A) since (B) follows by taking duals and Lemma 2.1. Indeed, condition (1) is equivalent to (2) because $\mathcal{F}^*$ is a filter. The fact that (1) implies (3) is a consequence of Lemma 2.1 too, while the converse was already noticed before for general families. Finally the equivalence between (1) and (4) can be shown in a similar way as Birkhoff’s transitivity theorem [15].

Note that when considering the family $\mathcal{F} = I$ of infinite sets in Proposition 2.7 (A) we obtain the known equivalences for mixing maps.

Remark 2.8. By the same argument for an operator $T$ on a separable topological vector space $X$, the first three equivalences of statements (A) and (B) still hold. We also point out that as with the hypercyclic case we have the following comparison principle for $\mathcal{F}$-maps and transference principle for $\mathcal{F}$-operators, see [15, Chapter 12].

1. (Comparison Principle) Any quasifactor of an $\mathcal{F}$-map is an $\mathcal{F}$-map. Indeed, let $T : X \to X$ be an $\mathcal{F}$-map and let $S : Y \to Y$ and $\phi : X \to Y$ be maps so that $\phi \circ T = S \circ \phi$, where $\phi$ has dense range. Then for any non-empty open subsets $U$ and $V$ of $Y$ we have $N_S(U, V) = N_T(\phi^{-1}(U), \phi^{-1}(V)) \in \mathcal{F}$.

2. (Transference Principle) Let $\mathcal{F}$ be a family and let $T$ be an operator on a topological vector space $X$ so that each operator $S$ on an $F$-space that is quasi-conjugate to $T$ via an operator (that is, it supports a dense range operator $J : X \to Y$ with $JT = SJ$) is an $\mathcal{F}$-map. Then $T$ is an $\mathcal{F}$-map.
3 \( \mathcal{F} \)-transitive weighted shift operators

Each bounded bilateral weight sequence \( w = (w_k)_{k \in \mathbb{Z}} \), induces a bilateral weighted backward shift operator \( B_w \) on \( X = c_0(\mathbb{Z}) \) or \( \ell^p(\mathbb{Z}) \) (1 \( \leq p < \infty \)) given by \( B_w e_k := w_k e_{k-1} \), where \( (e_k)_{k \in \mathbb{Z}} \) denotes the canonical basis of \( X \).

Similarly, each bounded sequence \( w = (w_n)_{n \in \mathbb{N}} \) induces a unilateral weighted backward shift operator \( B_w \) on \( X = c_0(\mathbb{Z}_+) \) or \( \ell^p(\mathbb{Z}_+) \) (1 \( \leq p < \infty \)), given by \( B_w e_n := w_n e_{n-1}, n \geq 1 \) and \( B_w e_0 := 0 \), where \( (e_n)_{n \in \mathbb{Z}_+} \) denotes the canonical basis of \( X \).

Our characterization of \( \mathcal{F} \)-transitive weighted backward shifts will rely on the properties of the sets \( A_{M,j} \) and \( \tilde{A}_{M,j} \) defined as

\[
A_{M,j} := \left\{ n \in \mathbb{N} : \prod_{i=j+1}^{j+n} |w_i| > M \right\}
\]

\[
\tilde{A}_{M,j} := \left\{ n \in \mathbb{N} : \frac{1}{\prod_{i=j-n+1}^{j} |w_i|} > M \right\},
\]

where \( M > 0 \) and \( j \in \mathbb{Z} \). In the case \( j = 0 \), we just write \( A_M, \tilde{A}_M \) instead of \( A_{M,0}, \tilde{A}_{M,0} \) respectively. We note that Salas' \[20\] characterization of hypercyclic (i.e., transitive) bilateral weighted shifts on the above sequence spaces may be formulated as

\[ B_w \text{ is hypercyclic } \iff \forall M > 0 \ \forall N \in \mathbb{N} \ \bigcap_{j=-N}^{N} (A_{M,j} \cap \tilde{A}_{M,j}) \neq \emptyset. \]

In other words, since \( A_{M',j} \subset A_{M,j} \) and \( \tilde{A}_{M',j} \subset \tilde{A}_{M,j} \) whenever \( M' > M > 0 \), the collection of subsets \( \{A_{M,j}, \tilde{A}_{M,j}\}_{M>0,j \in \mathbb{Z}} \) should form a filter subbase for the hypercyclicity of \( B_w \). In that case, we denote by \( \mathcal{A}_w \) the generated filter. Therefore, for the characterization of weighted shifts \( B_w \) that are \( \mathcal{F} \)-operators for a certain family \( \mathcal{F} \) we need to assume that \( \mathcal{A}_w \) is a filter.

When \( B_w \) is hypercyclic (i.e., when \( \mathcal{A}_w \) is a filter), we can describe a filter base of \( \mathcal{A}_w \), which will be very useful in the characterization of weighted shifts that are \( \mathcal{F} \)-operators, and it is given by the collection of sets

\[ \{A_{M,j} \cap \tilde{A}_{M,j} : M > 0 \text{ and } j \in \mathbb{N}\}. \]

Actually, this is a consequence of the observation that, if \( M_1, M_2 > 0 \) and \( j_1, j_2, j_3 \in \mathbb{Z} \) with \( j_3 > \max\{|j_1|,|j_2|\} \), then there is \( M_3 > 0 \) such that

\[ A_{M_3,j_3} \subset A_{M_1,j_1} \cap A_{M_2,j_2} \quad \text{and} \quad \tilde{A}_{M_3,j_3} \subset \tilde{A}_{M_1,j_1} \cap \tilde{A}_{M_2,j_2}. \]

Indeed, let \( M := \sup_{i \in \mathbb{Z}} |w_i| \). We fix \( M_3 > K(M_1 + M_2)(1 + M)^{2j_3} \), where

\[ K := 1 + \max_{-j_3 \leq m_1 \leq m_2 \leq j_3} \prod_{i=m_1}^{m_2} |w_i|^{-1}. \]
If \( n \in A_{M_3,j_3} \) then
\[
\prod_{i=j_3+1}^{j_3+n} |w_i| = \left( \prod_{i=j_3+1}^{j_3+n} |w_i| \right) \prod_{i=j_3+1}^{j_3+n} |w_i| > M_3 \prod_{i=j_3+1}^{j_3+n} |w_i| > M_1.
\]
That is, \( n \in A_{M_1,j_1} \). The same argument shows \( n \in A_{M_2,j_2} \). Analogously, we also have \( \bar{A}_{M_3,j_3} \subset \bar{A}_{M_1,j_1} \cap \bar{A}_{M_2,j_2} \).

**Proposition 3.1.** Let \( B_w \) be a bilateral weighted backward shift on \( X = c_0(\mathbb{Z}) \) or \( \ell^p(\mathbb{Z}) \), \( 1 \leq p < \infty \). Then the following are equivalent:

1. \( B_w \) is an \( \widetilde{\mathcal{F}} \)-operator;
2. \( B_w \) is an \( \mathcal{F} \)-operator;
3. for every \( j \in \mathbb{N} \) and \( M > 0 \), \( A_{M,j} \cap \bar{A}_{M,j} \in \mathcal{F} \);
4. \( B_w \) is hypercyclic, \( A_w \subset \mathcal{F} \), and \( B_w \) satisfies the \( A_w \)-Criterion.

In addition, if \( \widetilde{\mathcal{F}} \) is a filter, then the above conditions are equivalent to

5. for every \( j \in \mathbb{N} \) and \( M > 0 \) we have \( A_{M,j} \in \mathcal{F} \) and \( \bar{A}_{M,j} \in \mathcal{F} \).

**Proof.** Obviously, (1) implies (2). The reverse implication is a consequence of Lemma 2.3 since transitive weighted shifts are weakly mixing. Also, (4) implies (2). To show that (2) implies (3), given \( N,j \in \mathbb{N} \) arbitrary, we must find nonempty open sets \( U,V \subset X \) such that
\[
N(U,V) \subset A_{N,j} \cap \bar{A}_{N,j}.
\]
Indeed, we fix \( R > N \),
\[
U := \{ x \in X : |x_j| > \frac{1}{R} \} \cap \{ x \in X : \|x\| < 1 \},
\]
and we set
\[
V = \{ x \in X : \|x - (N+1)e_j\| < \frac{1}{R^2} \}.
\]
If \( m \in N(U,V) \) and \( x \in U \) is such that \( B_w^m x \in V \), then
\[
\left| \left( \prod_{i=j+1}^{j+m} w_i \right) x_{j+m} - (N + 1) \right| < \frac{1}{R^2} < 1 \quad \text{and} \quad (3.2)
\]
\[
\left| \left( \prod_{i=l+1}^{l+m} w_i \right) x_{l+m} \right| < \frac{1}{R^2} \quad \text{if } l \neq j.
\]
Since \( x \in U \), we deduce from (3.2) that
\[
\prod_{i=j+1}^{j+m} |w_i| > \left( \prod_{i=j+1}^{j+m} |w_i| \right) |x_{j+m}| > N,
\]
which implies that $m \in \mathcal{A}_{N,j}$.

On the other hand, $B_w^m x \in V$ forces $m > 0$ since $U$ and $V$ do not intersect. Thus, $l := j - m \neq j$, and (3.3) implies

\[ \left( \prod_{i=j-m+1}^j |w_i| \right) < \left( \prod_{i=j-m+1}^j |w_i| \right) R |x_j| < \frac{1}{R} < \frac{1}{N}, \]

that yields $m \in \bar{A}_{N,j}$. Thus the inclusion (3.1) is satisfied, and property (3) holds.

To prove that (3) implies (4), since $B_w$ is hypercyclic (i.e., $\mathcal{A}_w$ is a filter) and $\mathcal{A}_w \subset \mathcal{F}$ because $\mathcal{F}$ contains a basis of $\mathcal{A}_w$, we just need to show that $B_w$ satisfies the $\mathcal{A}_w$-criterion.

Let $D$ be the set of all finitely supported vectors in $X$ and let $S_w$ be the weighted forward shift defined on $D$ by

\[ S_w e_i := \frac{1}{w_{i+1}} e_{i+1}. \]

If we consider $S_n := S_w^n$ then we have $B_w^n S_n x = x$ for every $x \in D$. It suffices to show that

- $\mathcal{A}_w$-lim$_n B_w^n x = 0$ for every $x \in D$;
- $\mathcal{A}_w$-lim$_n S_n x = 0$ for every $x \in D$.

For the rest of the proof we assume that $X = \ell^p(\mathbb{Z})$ with $1 \leq p < \infty$. The proof is similar if $X = c_0(\mathbb{Z})$. Let $x \in D$, $\varepsilon > 0$ and $V_\varepsilon := \{ x \in \ell^p(\mathbb{Z}) : \|x\| < \varepsilon \}$. First, we show that $\{n \in \mathbb{N} : B_w^n x \in V_\varepsilon \} \in \mathcal{A}_w$. Since $x \in D$, we can write $x = \sum_{j=-m}^m x_j e_j$ for some $m \in \mathbb{N}$ and we then have

\[ B_w^n x = \sum_{j=-m-n}^m \left( \prod_{i=j+1}^{j+n} w_i \right) x_{j+n} e_j. \]

Let $M = \|x\|_\infty 2m/\varepsilon$ and $n \in \bigcap_{j=-m}^m \bar{A}_{M,j} \in \mathcal{A}_w$. We have

\[ \|B_w^n x\|^p = \sum_{j=-m}^m \left( \prod_{i=j-n+1}^j w_i \right) \left| x_j \right|^p < \sum_{j=-m}^m \left( \frac{\varepsilon}{\|x\|_\infty 2m} \right)^p |x_j|^p < \varepsilon^p, \]

which implies

\[ \bigcap_{j=-m}^m \bar{A}_{M,j} \subseteq \{ n \in \mathbb{N} : B_w^n y \in V_\varepsilon \}, \]

thus $\{ n \in \mathbb{N} : B_w^n y \in V_\varepsilon \} \in \mathcal{A}_w$. It remains to show that $\{ n \in \mathbb{N} : S_n x \in V_\varepsilon \} \in \mathcal{A}_w$. Indeed, we have

\[ S_n x = S_w^n x = \sum_{j=-m}^m \frac{x_j}{\prod_{i=j+1}^{j+n} w_i} e_{j+n}. \]
Let $M = \|x\|_\infty 2m/\varepsilon$ and $n \in \bigcap_{j=-m}^{m} A_{M,j}$. We then have

$$\|S_nx\|^p = \sum_{j=-m}^{m} \left| \frac{x_j}{\prod_{i=j+1}^{j+n} w_i} \right|^p < \frac{2m\varepsilon^p}{(2m)^p} \leq \varepsilon^p,$$

which implies

$$\bigcap_{j=-m}^{m} A_{M,j} \subseteq \{ n \in \mathbb{N} : S_ny \in V_\varepsilon \}.$$  

Consequently, $\{ n \in \mathbb{N} : S_ny \in V_\varepsilon \} \in \mathcal{A}_w$, and $B_w$ is an $\mathcal{F}$-operator.

Certainly, condition (3) implies (5). If (5) holds, the argument preceding this Proposition yields that, for each $j \in \mathbb{N}$ and for every $M > 0$, the sets $A_{M,j}$ and $\tilde{A}_{M,j}$ belong to $\tilde{\mathcal{F}}$, which gives (3) since $\tilde{\mathcal{F}}$ is a filter.

When $\mathcal{F} = \mathcal{I}^*$ is the filter of cofinite sets, we obtain as a consequence the well known characterization of mixing bilateral weighted shifts. On the other hand, the case $\mathcal{F} = \mathcal{S}$ offers again an interesting result.

**Corollary 3.2.** Let $B_w$ be a bilateral weighted backward shift on $X = c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$. Then the following are equivalent:

1. $B_w$ is a topologically ergodic operator;
2. for every $j \in \mathbb{N}$ and $M > 0$, $A_{M,j}$ and $\tilde{A}_{M,j}$ are syndetic sets.

The unilateral version of Proposition 3.1 we provide next relies only on the sets $A_{M,j}$. Notice that for a hypercyclic unilateral weighted shift $B_w$ the collection of sets $\{A_{M,j} : M > 0 \text{ and } j \in \mathbb{N}\}$ forms a base of a filter (which we call again $\mathcal{A}_w$) since, as before, if $M_1, M_2 > 0$ and $j_1, j_2, j_3 \in \mathbb{N}$ with $j_3 > \max\{j_1, j_2\}$, then there is $M_3 > 0$ such that

$$A_{M_3,j_3} \subset A_{M_1,j_1} \cap A_{M_2,j_2}.$$  

This fact yields a simplification of the corresponding characterization of unilateral weighted shifts that are $\mathcal{F}$-operators, which can be further simplified if $\mathcal{F}$ is a shift-invariant family. The unilateral version of Proposition 3.1 can be stated as follows.

**Proposition 3.3.** Let $B_w$ be an unilateral weighted backward shift on $c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+)$ ($1 \leq p < \infty$). The following are equivalent:

1. $B_w$ is an $\tilde{\mathcal{F}}$-operator;
2. $B_w$ is an $\mathcal{F}$-operator;
3. for every $j \in \mathbb{N}$ and $M > 0$, the set $A_{M,j} \in \mathcal{F}$;
4. $B_w$ is hypercyclic, $A_w \subset \mathcal{F}$, and $B_w$ satisfies the $A_w$-Criterion.
If in addition $\mathcal{F}$ is shift-invariant, the above conditions are equivalent to

(5) for every $M > 0$ the set $A_M \in \mathcal{F}$.

Proof. We only prove that if $\mathcal{F}$ is shift-invariant then condition (5) implies (3). Let $M > 0$ and $j \in \mathbb{N}$. We fix $M' > M (\sup_{i \in \mathbb{N}} |w_i|)^j$ such that $A_{M'} \subset [j + 1, +\infty]$. Given $n \in A_{M'}$, we have

$$\prod_{s=j+1}^{n} |w_s| = \frac{\prod_{s=1}^{n} |w_s|}{\prod_{s=1}^{j} |w_s|} > \frac{M'}{(\sup_{i \in \mathbb{N}} |w_i|)^j} > M.$$  

This implies that $A_{M'} - j \subset A_{M,j}$. Since $\mathcal{F}$ is a shift-invariant family, we conclude that $A_{M,j} \in \mathcal{F}$. \hfill \Box

In consequence we have the following characterization of topologically ergodic unilateral backward weighted shifts.

Corollary 3.4. Let $B_w$ be an unilateral weighted backward shift on $X = c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+)$, $1 \leq p < \infty$, then the following are equivalent:

(1) $B_w$ is topologically ergodic;

(2) for every $M > 0$ the set $A_M$ is syndetic.

We conclude this section by considering finite products of $\mathcal{F}$-maps.

Proposition 3.5. Let $T_1, \ldots, T_m$ be continuous maps on $X$, then

(1) for $n \geq 1$, $T_n^1$ is an $\mathcal{F}$-map on $X$ if and only if $T_1$ is an $\mathcal{F}_n$-map where $\mathcal{F}_n := \{A \subset \mathbb{Z}_+ : \text{$\frac{1}{n}(A \cap n\mathbb{Z}_+) \in \mathcal{F}$}\}$. In other words, $T_n^1$ is an $\mathcal{F}$-map on $X$ if and only if for every $U, V \in \mathcal{U}(X)$, $N_{T_n^1}(U, V) \cap n\mathbb{Z}_+ \in n\mathcal{F}$.

(2) If $\mathcal{F}$ is a filter then $T_1 \times T_2 \times \cdots \times T_m$ is an $\mathcal{F}$-map on $X^m$ if and only if $T_1$ is an $\mathcal{F}$-map on $X$ for every $1 \leq l \leq m$.

Proof. (1) If $n \geq 1$, then $T_n^1$ is an $\mathcal{F}$-map on $X$ if and only if $N_{T_n^1}(U, V) \in \mathcal{F}$ for every $U, V \in \mathcal{U}(X)$. We remark that $N_{T_n^1}(U, V) = \frac{1}{n}(N_{T_1}(U, V) \cap n\mathbb{Z}_+)$. Therefore, $N_{T_n^1}(U, V) \in \mathcal{F}$ if and only if $N_{T_1}(U, V) \in \mathcal{F}_n$.

(2) Note that $T_1 \times T_2 \times \cdots \times T_m$ is an $\mathcal{F}$-map on $X^m$ if and only if $\bigcap_{i=1}^{m} N_{T_i}(U, V_i) \in \mathcal{F}$, for any $(U_i, V_i)_{i=1}^{m} \in (\mathcal{U}(X) \times \mathcal{U}(X))^m$. The conclusion follows since $\mathcal{F}$ is a filter. \hfill \Box

Hence by Proposition 3.1 and Proposition 3.5 we have the following corollary.

Corollary 3.6. Let $\mathcal{F}$ be a filter and $B_w$ be a bilateral weighted backward shift on $X = \ell^p(\mathbb{Z})$ or $c_0(\mathbb{Z})$. Then, for every $m \in \mathbb{N}$, the following are equivalent:

(1) $B_w \oplus B_w^2 \oplus \cdots \oplus B_w^m$ is an $\mathcal{F}$-operator on $X^m$;

(2) For every $1 \leq l \leq m$, $M > 0$ and $j \in \mathbb{Z}$, $A_{M,j} \cap l\mathbb{Z}_+ \in l\mathcal{F}$ and $\bar{A}_{M,j} \cap l\mathbb{Z}_+ \in l\mathcal{F}$. 

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4 Return sets and densities

The purpose of this section is to analyze which kind of density properties the sets \( N(U, V) \) can have for a given hypercyclic operator, and classify the hypercyclic operators accordingly. We first recall the definitions of the asymptotic densities and the Banach densities in \( \mathbb{Z}_+ \).

**Definition 4.1.** Let \( A \subseteq \mathbb{Z}_+ \) be given. The *upper and lower asymptotic density* of \( A \) are defined respectively by

\[
\overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} \quad \text{and} \quad \underline{d}(A) = \liminf_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}.
\]

The *upper and lower Banach density* of \( A \) are defined by

\[
\overline{Bd}(A) = \lim_{s \to \infty} \frac{\alpha^s}{s} \quad \text{and} \quad Bd(A) = \lim_{s \to \infty} \frac{\alpha_s}{s},
\]

where for each \( s \in \mathbb{Z}_+ \)

\[
\alpha^s = \limsup_{k \to \infty} |A \cap [k+1, k+s]| \quad \text{and} \quad \alpha_s = \liminf_{k \to \infty} |A \cap [k+1, k+s]|.
\]

In general we have \( Bd(A) \leq \underline{d}(A) \leq \overline{d}(A) \leq \overline{Bd}(A) \) and

\[
d(A) + \overline{d}(\mathbb{Z}_+ \setminus A) = 1 \quad \text{and} \quad Bd(A) + \overline{Bd}(\mathbb{Z}_+ \setminus A) = 1. \tag{4.1}
\]

We will consider the following families.

\[
\overline{D} = \{A \subseteq \mathbb{Z}_+: \overline{d}(A) > 0\}, \quad \underline{D} = \{A \subseteq \mathbb{Z}_+: \underline{d}(A) > 0\},
\]

\[
\overline{BD} = \{A \subseteq \mathbb{Z}_+: \overline{Bd}(A) > 0\}, \quad \underline{BD} = \{A \subseteq \mathbb{Z}_+: Bd(A) > 0\},
\]

\[
\overline{D}_1 = \{A \subseteq \mathbb{Z}_+: \overline{d}(A) = 1\}, \quad \underline{D}_1 = \{A \subseteq \mathbb{Z}_+: \underline{d}(A) = 1\},
\]

\[
\overline{BD}_1 = \{A \subseteq \mathbb{Z}_+: \overline{Bd}(A) = 1\}, \quad \underline{BD}_1 = \{A \subseteq \mathbb{Z}_+: Bd(A) = 1\}.
\]

Notice that each of these families is shift invariant, and that \( \underline{D}_1 \) and \( \overline{BD}_1 \) are filters. Moreover,

1. \( \overline{BD}_1 = \mathcal{T} \), the family of thick sets,
2. \( \overline{BD} = \mathcal{S} \), the family of syndetic sets,
3. \( \overline{BD} \supset \mathcal{PS} \), the family of piecewise syndetic sets,
4. \( \overline{BD}_1 \subset \mathcal{TS} \), the family of thickly syndetic sets,
5. \( \overline{BD}_1 = \overline{BD}_1^* = \overline{D}_1 = \underline{D}_1 = \overline{BD}_1^* = \underline{BD}_1^* \) by (4.1).

In consequence, \( T \) is weakly mixing if and only if \( T \) is a \( \overline{BD}_1 \)-map.

Weighted shift operators and Proposition 3.3 help us to provide some counterexamples which allow us to distinguish the different notions of \( \mathcal{F} \)-operators.
Proposition 4.2. Let $X = c_{0}(\mathbb{Z}_{+})$, then

1. there exists a $\mathcal{BD}_{1}$-operator which is not $\mathcal{D}$-operator.
2. there exists a $\mathcal{D}_{1}$-operator which is not $\mathcal{D}$-operator.
3. there exists a $\mathcal{D}_{1}$-operator which is not $\mathcal{BD}_{1}$-operator.

Proof. (1) Consider the weight sequence

$$w = (1,\ldots,1,2,2^{-1},1,\ldots,1,2,2,2^{-2},1,\ldots,1,2,2,2^{-3},1,\ldots,1,\ldots)$$

We first observe that $\sup_{n} \prod_{i=1}^{n} w_{i}$ is infinite, hence $B_{w}$ is weakly mixing, see Chapter 4 in [15]. In other words, $B_{w}$ is $\mathcal{BD}_{1}$-operator.

On the other hand, by Proposition 3.3 we know that it suffices to show that $\bar{d}(A_{1}) = 0$ in order to deduce that $B_{w}$ is not a $\mathcal{D}$-operator. In other words, it suffices to show that $\bar{d}\left(\left\{ n \in \mathbb{N} : \prod_{i=1}^{n} w_{i} > 1 \right\}\right) = 0$ and this holds if $(m_{k})$ grows sufficiently rapidly.

(2) Consider the weight

$$w = (1,\ldots,1,2,\ldots,2,2^{-m_{0}},1,\ldots,1,2,\ldots,2,2^{-m_{1}},1,\ldots,1,2,\ldots,2,\ldots).$$

Thanks to Proposition 3.3 it suffices to find sequences $(m_{k})_{k}, (n_{k})_{k}$ such that

- $\bar{d}\left(\left\{ n : \prod_{i=1}^{n} w_{i} = 1 \right\}\right) = 1$
- $\bar{d}(A_{M}) = \bar{d}\left(\left\{ n : \prod_{i=1}^{n} w_{i} > M \right\}\right) = 1$, for every $M > 0$.

Indeed, if $\bar{d}\left(\left\{ n : \prod_{i=1}^{n} w_{i} = 1 \right\}\right) = 1$ then

$$\bar{d}\left(\left\{ n : \prod_{i=1}^{n} w_{i} > 1 \right\}\right) = 1 - \bar{d}\left(\left\{ n : \prod_{i=1}^{n} w_{i} \leq 1 \right\}\right) = 0.$$

Define sequences of intervals in the following way: $A_{k} = [10^{2k+1}, 10^{2k+2}[ and $B_{k} = [10^{2k+2}, 10^{2k+3}[ for every $k \in \mathbb{Z}_{+}$.

So $\mathcal{A} = \bigcup_{k \in \mathbb{N}} A_{k}$ and $\mathcal{B} = \bigcup_{k \in \mathbb{N}} B_{k}$ are disjoint with $\bar{d}(\mathcal{A}) = \bar{d}(\mathcal{B}) = 1$.

Hence, setting $m_{k} = |A_{k}|, n_{k} = |B_{k}|$ for every $k$, we are done.

(3) Let $m_{k} = 10^{2k}$ for every $k \in \mathbb{Z}_{+}$. We consider the weight

$$w = (1,2,2^{-1},1,1,2,\ldots,2,2^{-m_{0}},1,1,2,\ldots,2,2^{-m_{1}},1,1,1,1,$$

$$1,\ldots,1,2,\ldots,2,2^{-m_{2}},\ldots).$$

The set $A_{1} = \left\{ n : \prod_{i=1}^{n} w_{i} > 1 \right\}$ has arbitrarily large gaps, hence $B_{w}$ is not an $\mathcal{BD}$-operator by Proposition 3.3. On the other hand, we have for every $M > 1$

$$\bar{d}(A_{M}) = \bar{d}\left(\left\{ n : \prod_{i=1}^{n} w_{i} > M \right\}\right) = 1.$$

Hence, $B_{w}$ is $\mathcal{D}_{1}$-operator by Proposition 3.3. \hfill \square
Mixing operators obviously are $\mathcal{BD}_1$-operators, but the converse is false, this is the argument of the next result.

**Proposition 4.3.** There exists a $\mathcal{BD}_1$-operator on $c_0(\mathbb{Z}_+)$ which is not mixing.

**Proof.** Consider the weight $w = (w_n)_{n=1}^\infty$ defined by

$$w = (1, 2, 2^{-1}, 2, 2^{-2}, \ldots, 2, 2^{-n}, \ldots).$$

The weighted shift $B_w$ is not mixing since $\prod_{i=1}^n w_i$ does not tend to infinity as $n$ tends to infinity (see, e.g., Chapter 4 in [15]). It remains to show that $Bd(A_M) = 1$ for every $M \geq 1$. Let $M > 1$ and $n \in \mathbb{N}$ such that $2^{n-1} < M \leq 2^n$. If $k > n(n+1)/2$ and $s \geq (n+1) + (n+2) + \cdots + 2n = n(3n+1)/2$, then there is $l_s > 1$ such that $(l_s - 1)n(l_s(3n+1)/2 + 1)/2 \leq s < (l_s)n((l_s+2)n+1)/2$. An easy computation shows that we have $|A_M \cap [k, k+s]| \geq s - l_s(n^2 + n) > (l_s^2/2 - l_s - 1)n^2 - l_s n$. Therefore,

$$\alpha_s := \lim_{k \to \infty} |A_M \cap [k, k+s]| \geq (l_s^2/2 - l_s - 1)n^2 - l_s n,$$

and thus

$$Bd(A_M) = \lim_{s \to \infty} \frac{\alpha_s}{s} \geq \lim_{s \to \infty} \frac{(l_s^2/2 - l_s - 1)n^2 - l_s n}{(l_s^2/2 + l_s)n^2 + l_s n} = 1.$$

We conclude by Proposition 3.3.

**Proposition 4.4.** There exists a $\mathcal{BD}$-operator on $\ell^1(\mathbb{Z}_+)$ which is not a $\mathcal{BD}_1$-operator.

**Proof.** Let $A_n = \underbrace{2, \ldots, 2}_n$, $B_1 = A_1$, $B_n = [B_{n-1}, A_n, B_{n-1}]$, and consider the weight sequence

$$w = (A_1, A_2, A_1, A_3, A_1, A_2, A_1, A_4, A_1, A_2, A_1, A_3, A_1, A_2, A_1, \ldots).$$

Since $A_M$ has bounded gaps for every $M > 0$, we have from Corollary 3.4 that $B_w$ is topologically ergodic, i.e., it is a $\mathcal{BD}$-operator.

In view of Proposition 3.3, it now suffices to show that

$$\bar{d}\left(\{k \in \mathbb{N} : \prod_{i=1}^k |w_i| > 1\}\right) < 1.$$

We first notice that

$$|B_n| = 3 \cdot 2^n - n - 3 \quad \text{and} \quad \beta_n := \left|\left\{k \leq |B_n| : \prod_{i=1}^k |w_i| = 1\right\}\right| = 2^n - 1.$$
Now we observe that $\prod_{i=1}^{k} |w_i| \geq 1$ for all $k \in \mathbb{N}$. Therefore, we have

$$d\left(\left\{ k \in \mathbb{N} : \prod_{i=1}^{k} |w_i| > 1 \right\}\right) = \limsup_n \frac{\left\{ k \in [1, n] : \prod_{i=1}^{k} |w_i| > 1 \right\}}{n}$$

$$= \limsup_n \frac{\left\{ k \leq |\mathcal{B}_n| + n + 1 : \prod_{i=1}^{k} |w_i| > 1 \right\}}{|\mathcal{B}_n| + n + 1}$$

$$= \lim_n \frac{|\mathcal{B}_n| - \beta_n + n + 2}{|\mathcal{B}_n| + n + 1} = \lim_n \frac{2 \cdot 2^n}{3 \cdot 2^n - 2} = \frac{2}{3} < 1.$$

Figure 1 below summarizes the results of this section. We remark that:

- by Proposition 4.2 (1), there exists a $\overline{BD}_1$-operator which is not a $\overline{D}_1$-operator and a $\overline{BD}$-operator which is not a $\overline{D}$-operator;
- by Proposition 4.2 (2), there exists a $\overline{D}_1$-operator which is not a $\overline{D}_1$-operator and a $\overline{D}$-operator which is not a $\overline{D}$-operator;
- by Proposition 4.2 (3), there exists a $\overline{D}_1$-operator which is not a $\overline{BD}_1$ and a $\overline{D}$-operator which is not a $\overline{BD}$-operator.

On the other hand, by Proposition 4.4, there exists a $\overline{BD}$-operator which is not a $\overline{BD}_1$-operator;

- $\overline{BD}$-operator which is not a $\overline{D}_1$-operator;
- $\overline{D}$-operator which is not a $\overline{D}_1$-operator;
- $\overline{D}$-operator which is not a $\overline{D}_1$-operator.
5 Some special families

In this section we study new classes of $\mathcal{F}$-transitive operators given by families commonly used in Ramsey Theory. For a rich source on these families see [16]. For instance, we will consider the families of $\Delta$-sets and of $\mathcal{IP}$-sets, as well as their dual families.

$\Delta = \{ A \subseteq \mathbb{Z}^+ : (B - B) \cap \mathbb{Z}^+ \subseteq A, \text{ for some infinite subset } B \text{ of } \mathbb{Z}^+ \}$

$\mathcal{IP} = \{ A \subseteq \mathbb{Z}^+ : \exists (x_n)_n \subseteq \mathbb{N} \text{ with } \sum_{n \in F} x_n \in A, \forall F \subset \mathbb{Z}^+ \text{ finite} \}$.

The families $\Delta^*$ and $\mathcal{IP}^*$ are filters since $\Delta$ and $\mathcal{IP}$ are partition regular. In addition, we have

\[ \mathcal{J}^* \subseteq \Delta^* \subseteq \mathcal{IP}^* \subseteq \mathcal{S} \]

\[ \mathcal{J}^* \subseteq \mathcal{IP}^* \subseteq \mathcal{PS}^* \subseteq \mathcal{S}, \]  \hspace{1cm} (5.1)

see [6] for details. In linear dynamics, some of the widely studied classes are the mixing and weakly mixing operators. As we already mentioned, an operator $T$ is mixing if and only if it is an $\mathcal{J}^*$-operator and $T$ is weakly mixing.
if and only if $T$ is a $T$-operator. We recall that the class of $\mathcal{T}_S$-operators coincides with the class of topologically ergodic operators by Lemma 2.3 (see also the exercises in [15, Chapter 2]). Moreover, since $\mathcal{T}_S = \mathcal{P}_S^*$ and $\mathcal{T}_S$ is a filter, we know that $\mathcal{P}_S^*$ is partition regular (Lemma 2.1). With the help of Proposition 2.7 applied to $\mathcal{F} = \mathcal{P}_S$ we can therefore complete the picture.

**Proposition 5.1.** Let $T \in \mathcal{L}(X)$, where $X$ is a separable F-space. The following are equivalent:

1. $T$ is a topologically ergodic operator;
2. $T$ is a hereditarily $\mathcal{T}_S$-operator;
3. $T$ is a $\mathcal{T}_S$-operator;
4. $T$ is a hereditarily $\mathcal{P}_S$-operator;
5. $hcA := \{ x \in X : \{ T^nx : n \in A \} = X \}$ is a dense $(G_δ)$ set in $X$ for any $A \in \mathcal{P}_S$.

We will distinguish different classes of $\mathcal{F}$-operators by means of Proposition 3.3. Given a family $\mathcal{F}$, the following are two standard ways to induce shift-invariant families

$$\mathcal{F}_+ := \bigcup_{k \in \mathbb{Z}} (\mathcal{F} + k),$$

$$\mathcal{F}^* := \bigcap_{k \in \mathbb{Z}} (\mathcal{F} + k),$$

where $\mathcal{F} + k := \{ A \subset \mathbb{Z}_+ : \exists B \in \mathcal{F} \text{ with } (B + k) \cap \mathbb{Z}_+ \subset A \}$, $k \in \mathbb{Z}$. We have

$$\widetilde{\mathcal{F}} \subseteq \mathcal{F} \subseteq \mathcal{F} \subseteq \mathcal{F}_+.$$

Moreover, for any $A \subseteq \mathbb{Z}_+$ we have

$$A \in (\mathcal{F}_+)^*, \text{ if and only if } A \in (\mathcal{F}_{+})^*. \quad (5.2)$$

It is well-known that $\Delta$ and $\mathcal{I}P$ are not shift invariant, while $\mathcal{P}S$ is. Also, if $\mathcal{F} = \Delta, \mathcal{I}P$ or $\mathcal{P}S$ and $\mathcal{G} = \mathcal{F}$ or $\mathcal{F}_+$ then

$$A \in \mathcal{G}^* \text{ if and only if } \mathbb{Z}_+ \setminus A \notin \mathcal{G}, \quad (5.3)$$

since $\mathcal{G}$ is partition regular.

**Proposition 5.2.** Every $\mathcal{F}$-operator is an $\mathcal{F}_+$-operator.

**Proof.** Let $U, V \in \mathcal{U}(X)$ and $k \geq 0$. We have $N(U, T^{-k}V) + k \subset N(U, V)$. Moreover, since $X$ has no isolated points, by transitivity we can find non-empty open sets $U' \subset U$ and $V' \subset V$ such that $N(T^{-k}U', V') \subset [k, +\infty[$. Thus we have

$$N(T^{-k}U', V')) - k \subset N(U, V).$$

We can conclude that every $\mathcal{F}$-operator is an $\mathcal{F}_+$-operator. $\square$
We next compare the notions of mixing operator, $\Delta^*$-operator, $\mathcal{I}P^*$-operator and topologically ergodic operator.

**Proposition 5.3.** There exists a topologically ergodic weighted backward shift on $X = c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+), 1 \leq p < \infty$, which is not an $\mathcal{I}P^*$-operator.

**Proof.** Consider the set

$$B = \left\{ \sum_{n \in F} 2^n : F \text{ finite set of } \mathbb{N} \right\}.$$  

Clearly $B \in \mathcal{I}P$ and thus $\mathbb{Z}_+ \setminus B \notin \mathcal{I}P^*$ by (5.3). Let $(b_n)$ be the increasing enumeration of $B$. We define the weight $w = (w_m)_{m=1}^\infty$ as follows

$$w = (2, \ldots, 2, \underbrace{\frac{1}{w_{b_1}}}_{w_{b_1}} \ldots, 2, \underbrace{\frac{1}{w_{b_2}}}_{w_{b_2}} 2, \ldots, 2, \underbrace{\frac{1}{w_{b_3}}}_{w_{b_3}} \ldots). \quad (5.4)$$

Now, $A_1 := \{n \geq 1 : \prod_{i=1}^n w_i > 1\} = \mathbb{Z}_+ \setminus B$, hence $B_w$ is not an $\mathcal{I}P^*$-operator by Proposition 3.3. On the other hand, it is easy to see that $B \notin \mathcal{P}S$. Then $(B+i) \notin \mathcal{P}S$ for every $i \geq 0$, since $\mathcal{P}S$ is shift invariant. Hence, by (5.3) the set $\mathbb{Z}_+ \setminus (B+i) \in \mathcal{P}S^*$ for every $i \geq 0$. Now observe that $A_2 := \{n \geq 1 : \prod_{i=1}^n w_i > 2^i\} = \mathbb{Z}_+ \setminus \left( \bigcup_{i=0}^i (B+i) \right) = \bigcap_{i=0}^i (\mathbb{Z}_+ \setminus (B+i)) \in \mathcal{P}S^*$, since $\mathcal{P}S^*$ is a filter. Hence $B_w$ is a $\mathcal{P}S^*$-operator, or equivalently a topologically ergodic operator, by Proposition 3.3.  

**Proposition 5.4.** There exists a weighted backward shift operator on $X = c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+), 1 \leq p < \infty$, which is an $\mathcal{I}P^*$-operator but not a $\Delta^*$-operator.

**Proof.** Let $B$ be an infinite subset of $\mathbb{N}$ with unbounded gaps and let $(b_n)_n$ be an increasing enumeration of $B$. So there exists an increasing sequence $(n_k)$ such that

$$b_{n_k+1} - b_{n_k} \to \infty. \quad (5.5)$$

Consider the weight sequence $w = (w_m)_{m=1}^\infty$ given by (5.4). As before $\{n \geq 1 : \prod_{i=1}^n w_i > 1\} = \mathbb{Z}_+ \setminus B$, so it would be desirable that $B \in \Delta$ and thus that $\mathbb{Z}_+ \setminus B \notin \Delta^*$ since this would imply that $B_w$ is not a $\Delta^*$-operator.

On the other hand, it can be verified that for every $M > 0$ and $j \in \mathbb{N}$ there exists a finite subset $F$ of $\mathbb{Z}$ such that $A_{M,j} = \mathbb{Z}_+ \setminus (\cup_{i \in F} B + i)$. Hence, in order to conclude that $B_w$ is an $\mathcal{I}P^*$-operator, by Proposition 3.3 and condition (5.3) we need to verify

$$\bigcup_{i \in F} (B+i) \notin \mathcal{I}P \quad (5.6)$$

for any finite subset $F$ of $\mathbb{Z}$. Now, since $\mathcal{I}P$ is partition regular, condition (5.6) is obtained if $B \notin \mathcal{I}P_+$ and this in turn is equivalent to $\mathbb{Z}_+ \setminus B \in \mathcal{P}S_+$.
if there exists (see a detailed study in [7]).

T was introduced by Bayart and Grivaux in [3], [2]. When

\[ x \in \mathbb{R} \]

is called an \( A \)-hypercyclic vector. Such a \( A \)-hypercyclic vector is a \( \mathcal{A} \)-hypercyclic vector for \( T \).

\[ \text{Evidently, every mixing operator is a } \mathcal{A} \text{-operator but the converse is not true.} \]

Proposition 5.5. There exists a \( \mathcal{A} \)-weighted backward shift on \( c_0(\mathbb{Z}_+) \) or \( \ell^p(\mathbb{Z}_+), 1 \leq p < \infty \), which is not mixing.

Proof. Let \( B = \{ b_i : b_1 = 2, b_{i+1} = b_i + 2, i \in \mathbb{N} \} \). Consider the weight sequence \( w = (w_m)_{m=1}^{\infty} \) given by (5.4), so we have

\[ w = (2, 2^{-1}, 2, 2^{-2}, 2, 2, 2^{-3}, \ldots) \]

We know that \( B_w \) is not mixing since \( \prod_{i=1}^{n} w_i \) does not tend to infinity as \( n \) tends to infinity. On the other hand, it can be verified that for every \( M > 0 \) and \( j \in \mathbb{N} \) there exists a finite subset \( F \) of \( \mathbb{Z} \) such that \( A_{M,j} = \mathbb{Z}_+ \setminus (\bigcup_{i \in F} B + i) \). Hence, in order to conclude that \( B_w \) is a \( \mathcal{A} \)-operator, by Proposition 3.3 and condition (5.3) we need to verify \( \bigcup_{i \in F} B + i \notin \Delta \), for every finite subset \( F \) of \( \mathbb{Z} \).

So, let \( F \) be a finite subset of \( \mathbb{Z} \) with \( N = \max_{a \in F} |a - b| \). Suppose that \( \bigcup_{i \in F} B + i \) is a \( \Delta \)-set. Then, there exists an increasing sequence \((d_m)_m\) such that \( \bigcup_{i \in F} B + i = \Delta((d_m)_m) \), where \( \Delta((d_m)_m) \) denotes the set of differences of \((d_m)_m\) defined by \( \Delta((d_m)_m) = \{ d_j - d_i : 1 \leq i < j \} \). Fix \( d_{j_1}, d_{j_2}(j_1 < j_2) \) such that \( |d_{j_2} - d_{j_1}| > N \). Then for each \( m \in \mathbb{N} \) we have

\[ |d_{j_2} - d_{j_1}| = |(d_{j_m} - d_{j_1}) - (d_{j_m} - d_{j_2})|, \]

which means that the distance \( |d_{j_2} - d_{j_1}| \) between elements of \( \bigcup_{i \in F} B + i \) is attained infinitely many times, which is not the case taking into account the way in which \( B \) was defined. We thus conclude that \( \bigcup_{i \in F} B + i \notin \Delta \). \( \square \)

5.1 Connection with \( \mathcal{A} \)-hypercyclicity

In this subsection we investigate the connection between the classes of hypercyclic operators considered throughout this work and the notion of \( \mathcal{A} \)-hypercyclicity studied in [7].

Given a family \( \mathcal{A} \) on \( \mathbb{Z}_+ \), an operator \( T \in \mathcal{L}(X) \) is called \( \mathcal{A} \)-hypercyclic if there exists \( x \in X \) such that \( N(x,V) \in \mathcal{A} \) for each \( V \in \mathcal{U}(X) \). Such a vector \( x \) is called an \( \mathcal{A} \)-hypercyclic vector for \( T \).

When \( \mathcal{A} = \mathcal{D} \), the operator \( T \) is called frequently hypercyclic. This class was introduced by Bayart and Grivaux in [3], [2]. When \( \mathcal{A} = \mathcal{D}^* \), the operator \( T \) is called \( \mathcal{U} \)-frequently hypercyclic; this class was introduced by Shkarin [21]. When \( \mathcal{A} = \mathcal{B}^* \mathcal{D} \), the operator \( T \) is called reiteratively hypercyclic [18] (see a detailed study in [7]).
The frequently hypercyclic operators constitute by far the most extensively studied class of operators amongst the three classes mentioned above. Clearly any frequently hypercyclic operator is an U-frequently hypercyclic operator, which in turn is reiteratively hypercyclic. The hierarchy between frequently hypercyclic and U-frequently hypercyclic operators as well as a full characterization for weighted shift operators have been established by Bayart and Ruzsa [5]. A complementary study of this kind, taking into account reiterative hypercyclicity can be found in [7].

In particular, we already know that there exists a mixing weighted shift which is not reiteratively hypercyclic as shown in [7]. On the other hand, there exists a frequently hypercyclic (hence reiteratively hypercyclic) operator which is not mixing, see [1]. Reiteratively hypercyclic operators are topologically ergodic [7, 13]. One can therefore wonder whether any reiteratively hypercyclic operator is a $\Delta^*$-operator or an IP$^*$-operator.

**Proposition 5.6.** Let $T \in \mathcal{L}(X)$ be a reiteratively hypercyclic operator. Then

$$N(U, V) \in \bigcap_{t \in N(U, V)} \left( \Delta^* + t \right),$$

for every $U, V$ non-empty open sets in $X$.

**Proof.** Let $U, V \in \mathcal{U}(X)$ and $n \in N(U, V)$. The set $U_n = U \cap T^{-n}V$ is a non-empty open set. Since $T$ is reiteratively hypercyclic, there exists $x \in X$ such that $\overline{\text{Bd}}(N(x, U_n)) > 0$.

Let $s_1, s_2 \in N(x, U_n)$. We have

$$T^{s_1 - s_2 + n}(T^{s_2}x) = T^n(T^{s_1}x) \in V.$$ 

In other words,

$$N(x, U_n) - N(x, U_n) + n \subseteq N(U, V). \quad (5.7)$$

The desired result then follows from Theorem 3.18 in [11], which implies that $A - A \in \Delta^*$ whenever $A \in \overline{\text{Bd}}$.

The family $\Delta^*$ is not shift invariant ($2\mathbb{N} := \{2n : n \in \mathbb{N}\} \notin \Delta^*$ while $2\mathbb{N} + 1 \notin \Delta^*$). Hence, we cannot deduce from Proposition 5.6 that every reiteratively hypercyclic operator is a $\Delta^*$-operator. In fact, we are not able to answer in general the following question: is any reiteratively hypercyclic operator either a $\Delta^*$-operator or an IP$^*$-operator? However we can show that the answer is yes if we consider bilateral or unilateral weighted shifts.

**Proposition 5.7.** If $B_w$ is reiteratively hypercyclic on $X = \ell^p(\mathbb{Z}), 1 \leq p < \infty$, or $X = c_0(\mathbb{Z})$, then $B_w$ is an $\Delta^*$-operator.

In order to prove Proposition 5.7, we first state two lemmas. The first one directly follows from Proposition 5.6.
Lemma 5.8. Let $U, V$ non-empty open sets in $X$ such that $U \cap V \neq \emptyset$, if $T$ is reiteratively hypercyclic on $X$ then $N(U, V) \in \Delta^*$.

Let $X = \ell^p(\mathbb{Z})$, $1 \leq p < \infty$, or $c_0(\mathbb{Z})$. The second lemma will rely on the non-empty open sets $U_{R,j}$ defined for every $R > 1$ and every $j \in \mathbb{Z}$ by

$$U_{R,j} = \{ U \in \mathcal{U}(X) : |x_j| > \frac{1}{R}, \forall x \in U \}.$$ 

In particular, we remark that if $MR > 1$ then $B((M + 1)e_j; \frac{1}{MR}) \in U_{R,j}$, where $B(y;\epsilon)$ stands for the open ball centered at $y$ with radius $\epsilon$.

Lemma 5.9. Let $M > 0$, $j \in \mathbb{Z}$ and $R > 1$ such that $MR > 1$. Suppose there exists $U \in U_{R,j}$ such that for any non-empty open subset $\bar{U}$ of $U$ it holds $N(\bar{U}, B((M + 1)e_j; \frac{1}{MR})) \in \Delta^*$. Then $A_{M,j} \in \Delta^*$ and $\bar{A}_{M,j} \in \Delta^*$.

Proof. Let $(z(m))_m$ be a dense set in $X$ such that

$$z(m) = (z(m)_1, \ldots, z(m)_m, 0 \ldots)$$

and $U_m = B(z(m); 1/m)$. Let $U \in U_{R,j}$ such that for any non-empty open subset $\bar{U}$ of $U$ we have $N(\bar{U}, B((M + 1)e_j; \frac{1}{MR})) \in \Delta^*$. Then there exists $m$ such that $U_m \subset U$ and hence $N(U_m, B((M + 1)e_j; \frac{1}{MR})) \in \Delta^*$. Pick $r \in N(U_m, B((M + 1)e_j; \frac{1}{MR}))$ with $r > m$ and $x \in U_m$ such that $B_r^x \in B((M + 1)e_j; \frac{1}{MR})$.

Then, we have

$$\left| \left( \prod_{i=j+1}^{j+r} w_i \right) x_{j+r} - (M + 1) \right| < \frac{1}{MR} \quad (5.8)$$

and for every $t \neq j$

$$\left| \left( \prod_{i=t+1}^{t+r} w_i \right) x_{t+r} \right| < \frac{1}{MR} \quad (5.9)$$

By (5.8) we get,

$$\prod_{i=1}^{r} |w_{i+j}| > \prod_{i=1}^{r} |w_{i+j}x_{r+j}| > M,$$

where the first inequality follows since $r > m$. We conclude that $N(U_m, B((M + 1)e_j; \frac{1}{MR})) \setminus \{1 \ldots m\} \subseteq A_{M,j}$ and thus $A_{M,j} \in \Delta^*$.

On the other hand, by (5.9), we get $\prod_{i=j-r+1}^{j} |w_i x_j| < \frac{1}{MR}$, hence

$$\prod_{i=j-r+1}^{j} |w_i| \frac{1}{R} < \prod_{i=j-r+1}^{j} |w_i x_j| \frac{1}{MR}.$$

We deduce that $\prod_{i=j-r+1}^{j} |w_i| < \frac{1}{M}$ and thus $\bar{A}_{M,j} \in \Delta^*$.

$\square$
Proof of Proposition 5.7

Suppose $B_w$ is not a $\Delta^*$-operator on $X$, then by Proposition 3.1, there exists $M > 0$ and $j \in \mathbb{Z}$ such that $A_{M,j} \notin \Delta^*$ or $\bar{A}_{M,j} \notin \Delta^*$. Let $R > 1$ such that $MR > 1$. By Lemma 5.9, it follows that

$$\forall U \in U_{R,j} \quad \exists \tilde{U} \subseteq U : N(\tilde{U}, B((M+1)e_j; \frac{1}{MR})) \notin \Delta^*.$$ 

Since $B((M+1)e_j; \frac{1}{MR}) \in U_{R,j}$, we can consider $U = B((M+1)e_j; \frac{1}{MR})$ and there thus exists $\tilde{U} \subseteq U$ such that $N(\tilde{U}, U) \notin \Delta^*$, which by Lemma 5.8 is not possible if $B_w$ is reiteratively hypercyclic. This concludes the proof of Proposition 5.7.

Analogously, we have the following result for unilateral weighted shifts.

**Proposition 5.10.** If $B_w$ is reiteratively hypercyclic on $X = \ell^p(\mathbb{Z}^+)$, $1 \leq p < \infty$, or on $X = c_0(\mathbb{Z}^+)$, then $B_w$ is a $\Delta^*$-operator.

**Proposition 5.11.** There exists a reiteratively hypercyclic operator on $c_0(\mathbb{Z}^+)$ which is not a $D_1$-operator.

**Proof.** Let $B_w$ be the weighted shift on $c_0(\mathbb{Z}^+)$ given by

$$w_k = \begin{cases} 2 & \text{if } k \in S \\ \prod_{\nu=1}^{k-1} w_{\nu}^{-1} & \text{if } k \in (S+1) \setminus S \\ 1 & \text{otherwise.} \end{cases}$$

where $S := \bigcup_{l \geq 1} [l 10^j - j, l 10^j + j]$. It is shown in [7, Theorem 17] that $B_w$ is reiteratively hypercyclic and that

$$\overline{d}(\{k \in \mathbb{N} : \prod_{i=1}^{k} |w_i| \geq 2^j\}) \to 0.$$ 

In particular, we deduce that there exists $j \geq 1$ such that $\overline{d}(\{k \in \mathbb{N} : \prod_{i=1}^{k} |w_i| \geq 2^j\}) < 1$ and in view of Proposition 3.3, we can conclude that $B_w$ is not a $D_1$-operator. \[\square\]

Figure 2 summarizes what we know after this work.
We recall the following questions that remain open.

**Question 5.12.** Does there exist a $\mathcal{D}$-operator which is not a $\mathcal{BD}_1$-operator? In other words, does there exist $T \in \mathcal{L}(X)$ being a $\mathcal{D}$-operator but not weakly mixing?

Note that if it were the case, then such operator $T$ must not be weighted shift.

**Question 5.13.** Is any reiteratively hypercyclic operator an $\Delta^*$-operator or an $\mathcal{IP}^*$-operator?

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