BOUNDED WEAK SOLUTION AND LONG TIME BEHAVIOR OF A DEGENERATE PARTICLE FLOW MODEL

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Abstract. In this paper it is proved that a strongly degenerate parabolic equation with a given critical density $\rho_{\text{cr}}$, derived from a density dependent particle flow model, has a global bounded weak solution. The solution is shown to converge towards a steady state as $t \to \infty$; when the average density $\rho_{\text{av}}$ is larger than $\rho_{\text{cr}}$ the steady state coincides with $\rho_{\text{av}}$ and the convergence rate is exponential with respect to the $L^2$ norm, while in the case that $\rho_{\text{av}} < \rho_{\text{cr}}$ the steady state is unknown and the convergence is algebraic with respect to the 2-Wasserstein metric. A formal reformulation of the system as a nondegenerate free boundary problem is also shown under suitable assumptions on the initial data. Finally numerical experiments in the 2-dimensional case are presented, which show that segregation phenomena can appear when the initial average density is smaller than the critical density.

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1. Introduction

Motivated by a newly developed density dependent particle flow model in [15] for material flow and swarming, in this article, we consider the strongly degenerate parabolic equation with non-flux boundary condition

$$\begin{align*}
\partial_t \rho &= \Delta f(\rho) \quad \text{on } Q_T \equiv \Omega \times (0, T), \\
\nabla f(\rho) \cdot \nu &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
\rho(\cdot, 0) &= \rho_0 \quad \text{on } \Omega,
\end{align*}$$

where $f : [0, \infty) \to [0, \infty)$ satisfies:

$$\begin{align*}
f &\in C^1([0, \infty)), \quad f' > 0 \quad \text{on } (\rho_{\text{cr}}, \infty), \quad f = 0 \quad \text{on } [0, \rho_{\text{cr}}],
\end{align*}$$

and $\rho_{\text{cr}} > 0$ is the given critical density and $\Omega$ is an open, bounded domain. We also use the notation $Q \equiv \Omega \times (0, \infty)$.

Strongly degenerate parabolic equations and systems appear in the literature of sedimentation related modelling with different boundary conditions [2][7]. A classical scalar degenerate parabolic equation has the following form

$$\begin{align*}
\partial_t \rho + \nabla \cdot (\Phi(\rho)) &= \Delta f(\rho),
\end{align*}$$

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where \( f \) is a given nondecreasing Lipschitz continuous function with \( f(0) = 0 \) and \( \Phi \) is a given density dependent flux function. It is well known that due to the strong degeneracy, this type of equations exhibit many hyperbolic properties such as the formation of shocks, nonuniqueness of the weak solutions. For example, it is proved in [14] that weak solution is unique if it satisfies the entropy condition. Another result for entropy solution in multi-dimension has been obtained in [3]. For more results on entropy solution we refer to [1, 5, 6, 17]. The well-posedness theory of weak entropy solution to strongly degenerate parabolic equation on Riemannian manifold is established in [11]. For general theory of weak solution of degenerate parabolic equation we refer to [16, 18], and furthermore to [13, Section 5] for a short review of the development of the mathematical theory in this direction. Recently, a density dependent particle flow model is formally derived through mean field limit in [15]

\[
\partial_t \rho + \nabla \cdot (\rho \cdot v_T - k(\rho) \nabla \rho) = 0, 
\]

where \( k(\rho) = \rho H(\rho - \rho_{cr}) \) with \( \rho_{cr} > 0 \) the critical density and \( H \) is the Heaviside function. Originally, eq. (5) has been employed as a material flow model to describe the transport of identical homogeneous particles with velocity \( v_T \) subject to a spring-damper interaction force [9]. It has also been used to describe the dynamical behaviour of pedestrian crowds [10, 12] as well as biological swarms subject to both attractive and repulsive interaction forces [4]. Unlike the parabolic-hyperbolic equation (4), the transport part of equation (5) is linear. Therefore the particle flow model has no hyperbolic structure. For simplicity, we study this equation, as a first step research, without transport term. We also give the regularity assumption on \( f \) in (3), instead of taking the Heaviside function, for technical simplicity.

Due to the parabolic structure of this equation, the results for existence and uniqueness of weak solution is not surprising, but also not trivial, given the presence of a strong degeneracy.

**Theorem 1** (Existence and uniqueness of the weak solution). Assume that (3) holds and the initial data \( \rho_0 \in L^\infty(\Omega) \). Then the initial boundary value problem (1), (2) admits a unique global-in-time weak solution \( \rho \) satisfying

\[
\begin{align*}
\rho &\in L^\infty(\Omega), & \nabla f(\rho) &\in L^2(\Omega), & \partial_t \rho &\in L^2(0, \infty; H^1(\Omega)'), \\
- \int_0^T \int_\Omega \rho \partial_t \phi dx dt - \int_\Omega \rho_0 \phi(0) dx + \int_0^T \int_\Omega \nabla f(\rho) \cdot \nabla \phi dx dt &= 0 & \forall \phi &\in C_c^\infty(\Omega \times [0, T]), & \forall T > 0.
\end{align*}
\]

and such that, for a.e. \( x \in \Omega \), the mapping \( t \in (0, \infty) \mapsto \min(\rho(x, t), \rho_{cr}) \in \mathbb{R} \) is nondecreasing.

The existence of global-in-time weak solutions is shown by regularising the equation with a Laplacian term and employing a compactness argument based on div-curl lemma, while the uniqueness of weak solutions is proved via a duality argument which exploits the monotonicity of \( f \).
The main contribution of this paper is the investigation of the long time behaviour. We obtain the following result and show some numerical experiments, mostly concerned with the subcritical case.

Let us preliminarily define the (relative) free energy density

\begin{align}
F(s|\rho_\infty) := F(s) - F(\rho_\infty) - f(\rho_\infty)(s - \rho_\infty), \\
F(s) := \int_0^s f(s')ds', \\
s \geq 0.
\end{align}

In some situations we will make the following additional assumption on \( f \):

\begin{align}
\exists \kappa > 1 : \forall R > 0, \exists C_R, c_R > 0 : \\
c_R(s - \rho_{cr})^\kappa \leq f(s) \leq C_R(s - \rho_{cr})^\kappa, \\
\rho_{cr} \leq s \leq R.
\end{align}

**Theorem 2** (Long-time-behaviour). Assume that (3) holds and that \( \Omega \) is connected. Let \( \rho \) be a solution of (1), (2) and define \( \rho_\infty \equiv \frac{1}{|\Omega|} \int_{\Omega} \rho_0 dx \).

If \( \rho_\infty > \rho_{cr} \) then

\begin{align}
\rho(t) \to \rho_\infty \quad \text{as } t \to \infty \text{ strongly in } L^p(\Omega), \quad \forall 1 \leq p < \infty,
\end{align}

\begin{align}
\exists \lambda > 0 : \int_\Omega F(\rho(t)|\rho_\infty)dx \leq e^{-\lambda t} \int_\Omega F(\rho_0|\rho_\infty)dx \quad t > 0.
\end{align}

If \( \rho_\infty = \rho_{cr} \) then (10) holds.

If \( \rho_\infty < \rho_{cr} \) then

(i) A function \( \hat{\rho} \in L^\infty(\Omega) \) exists such that

\begin{align}
\rho(t) \to \hat{\rho} \quad \text{as } t \to \infty \text{ strongly in } L^p(\Omega), \quad \forall 1 \leq p < \infty,
\end{align}

and it holds \( \hat{\rho} \leq \rho_{cr} \) a.e. in \( \Omega \).

(ii) Assume that (9) is fulfilled. Then

\begin{align}
\|(\rho(t) - \rho_{cr})_+\|_{L^1(\Omega)} + W_2(\rho(t), \hat{\rho}) \leq C t^{-\frac{1}{\kappa - 1}}
\end{align}

for \( t \to \infty \), where \( W_2 \) is the 2-Wasserstein distance.

(iii) If \( \text{meas}(\{|\rho_0 > \rho_{cr}\}) > 0 \), then there exists \( T^* > 0 \) such that \( \int_0^{T^*} \int_\Omega |\nabla \rho|^2 dx dt = \infty \).

The long-time behaviour of the solutions in the supercritical case \( \rho_\infty > \rho_{cr} \) is obtained from a free energy identity and a Poincaré-Wirtinger-type inequality (see Lemma 7 in the Appendix). In the subcritical regime \( \rho_\infty < \rho_{cr} \) the aforementioned tools only provide us with a partial information on the long-time behaviour of the solution, and are therefore complemented by a nonconstructive argument, based on a monotonicity property of the solution, which yields the convergence of the solution towards a steady state. The algebraic rate of convergence with respect to the 2-Wasserstein distance follows from a duality argument and Assumption (9), which describes the behaviour of \( f \) near the critical value. We wish to point out that the obtained algebraic decay rate resembles closely the result holding for the porous medium equation in the whole space [8], which is in agreement with the intuition that the dynamics of (1) under Assumption (9) should resemble a porous medium equation in the region close to the critical value. Finally,
in the critical case $\rho_\infty = \rho_{cr}$ we are not able to show a suitable Poincaré-Wirtinger-type inequality and thus we cannot derive a decay estimate for the solution, but on the other hand we can deduce strong $L^p(\Omega)$ convergence towards the steady state for any $p < \infty$ by exploiting the free energy estimate and mass conservation.

The arrangement of the paper is the following. In Section 2 we prove Thr. 1. In Section 3 we prove Thr. 2. In Section 4 we show that (1)–(2) can be formally equivalently reformulated as a free boundary problem at least in the case of radially symmetric initial data which are decreasing in the radial coordinate. Finally, in Section 5 several numerical experiments for the solution behaviour are presented, showing the convergence towards the constant steady state in the supercritical case (which is expected from the analysis), as well as a segregation phenomenon for initial data with subcritical mass.

2. Existence and uniqueness analysis (Proof of theorem 1)

In this section, we prove Theorem 1. The proof of the existence starts from the construction of a family of approximated solutions depending on two approximation parameters $\varepsilon$ and $\delta$. We will first show that the approximated solutions satisfy estimates that are uniform with respect to the approximation parameters, then we will invoke the Div-Curl Lemma to deduce the existence of a strongly convergent subsequence of approximated solutions and show that the limit of such subsequence is a global weak solution to (1) satisfying (6), (7). In the end, we prove the uniqueness by using the monotonicity property of $f$ together with the $H^{-1}$ method.

Approximated solutions. We introduce two approximation parameters: $\delta > 0$ (lower order regularization) and $\varepsilon > 0$ (higher order regularization).

For $0 < \delta < \min\{\|\rho_0\|^{-1}_\infty, \rho_{cr}^{-1}\}$ define

$$f_\delta(r) = \begin{cases} f(r) & r < \delta^{-1}; \\ f(\delta^{-1}) + f'(\delta^{-1})(r - \delta^{-1}) & r \geq \delta^{-1}. \end{cases}$$

Let us consider the approximated problem (with $m > d/2$ a given integer)

\[\int_0^T \langle \partial_t \rho^{(\varepsilon, \delta)}, \phi \rangle dt + \varepsilon \int_0^T (\rho^{(\varepsilon, \delta)}, \phi)_{H^m(\Omega)} dt + \delta \int_0^T (\rho^{(\varepsilon, \delta)}, \phi)_{H^1} dt = -\int_0^T \int_\Omega \nabla f_\delta(\rho^{(\varepsilon, \delta)}) \cdot \nabla \phi dx dt \quad \forall \phi \in L^2(0, T; H^m(\Omega)),\]

(13)

\[\rho^{(\varepsilon, \delta)}(0) = \rho_0 \quad \text{in } \Omega,\]

(14)

to be solved for $\rho^{(\varepsilon, \delta)} \in L^2(0, T; H^m(\Omega))$. We reformulate (13)(14) as a fix point problem for the mapping $T : (u, \sigma) \in L^2(0, T; H^1(\Omega)) \times [0, 1] \mapsto \rho \in L^2(0, T; H^1(\Omega))$,

\[\int_0^T \langle \partial_t \rho, \phi \rangle dt + \varepsilon \int_0^T (\rho, \phi)_{H^m(\Omega)} dt + \delta \int_0^T (\rho, \phi)_{H^1} dt = -\sigma \int_0^T \int_\Omega \nabla f_\delta(u) \cdot \nabla \phi dx dt \quad \forall \phi \in L^2(0, T; H^m(\Omega)),\]

(15)
The mapping $\mathcal{T}$ is clearly well defined given the assumptions on $f$ and the definition of $f_\delta$. Furthermore $\mathcal{T}(\cdot,0)$ is constant. Also choosing $\phi = \rho$ in (15) yields easily the bound
\begin{equation}
\|\rho\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t \rho\|_{L^2(0,T;H^m(\Omega))} + \|\rho\|_{L^2(0,T;H^m(\Omega))} \leq C(\varepsilon, \delta)(\|\rho_0\|_2 + \|u\|_{L^2(0,T;H^m(\Omega))}).
\end{equation}
This means that $\mathcal{T}(L^2(0,T;H^1(\Omega)) \times [0,1])$ is bounded in the space
\[ \{ \rho \in L^2(0,T;H^m(\Omega)) \mid \partial_t \rho \in L^2(0,T;H^m(\Omega)') \} \]
which, due to Aubin-Lions Lemma, embeds compactly in $L^2(Q_T)$ and via interpolation also in $L^2(0,T;H^1(\Omega))$. This means that $\mathcal{T}$ is also compact. Standard arguments yield the continuity of $\mathcal{T}$ in a straightforward way. Finally, a $\sigma$–uniform bound on the elements of the set
\[ \{ \rho \in L^2(0,T;H^1(\Omega)) \mid \mathcal{T}(\rho, \sigma) = \rho \} \]
follows immediately from (17). Therefore Leray-Schauder fix point theorem yields the existence of a fix point $\rho = \rho^{(\varepsilon,\delta)}$ for $\mathcal{T}(\cdot,1)$, that is, a solution $\rho^{(\varepsilon,\delta)} \in L^2(0,T;H^m(\Omega))$ to (13), (14).

**Uniform estimates and compactness argument.** Now we aim at taking the limit $\varepsilon \to 0$ in (13), (14). First notice that testing (13) against $\rho^{(\varepsilon,\delta)} \in L^2(0,T;H^m(\Omega))$ and exploiting the nonnegativity of $f_\delta'$ lead easily to the bounds
\begin{equation}
\|\rho^{(\varepsilon,\delta)}\|_{L^\infty(0,T;L^2(\Omega))} + \delta^{1/2}\|\rho^{(\varepsilon,\delta)}\|_{L^2(0,T;H^1(\Omega))} + \varepsilon^{1/2}\|\rho^{(\varepsilon,\delta)}\|_{L^2(0,T;H^m(\Omega))} \leq C\|\rho_0\|_2.
\end{equation}
A uniform bound for the time derivative $\partial_t \rho^{(\varepsilon,\delta)}$ in $L^2(0,T;H^m(\Omega)')$ is also derived in a straightforward way from (13), (18). From Aubin-Lions Lemma it follows that (up to subsequences) $\rho^{(\varepsilon,\delta)}$ is strongly convergent as $\varepsilon \to 0$ in $L^2(Q_T)$ towards $\rho^{(0)} \in L^2(0,T;H^1(\Omega))$, which is the solution to
\begin{equation}
\int_0^T \langle \partial_t \rho^{(0)}, \phi \rangle dt + \delta \int_0^T (\rho^{(0)}, \phi)_{H^1} dt = -\int_0^T \int_\Omega \nabla f_\delta(\rho^{(0)}) \cdot \nabla \phi dx dt \quad \forall \phi \in L^2(0,T;H^1(\Omega)),
\end{equation}
\begin{equation}
\rho^{(0)}(0) = \rho_0 \quad \text{in } \Omega.
\end{equation}
The next step is taking the limit $\delta \to 0$ in (19), (20). Choosing $\phi = f_\delta(\rho_0) \in L^2(0,T;H^1(\Omega))$ in (19) leads to
\begin{align*}
\int_\Omega f_\delta(\rho^{(0)}(T)) dx + \delta \int_0^T \int_\Omega \rho^{(0)} f_\delta(\rho^{(0)}) dx dt + \delta \int_0^T \int_\Omega |\nabla \rho^{(0)}|^2 f_\delta'(|\rho^{(0)}) dx dt \\
+ \int_0^T \int_\Omega |\nabla f_\delta(\rho^{(0)})|^2 dx dt = \int_\Omega F_\delta(\rho_0) dx, \quad t > 0,
\end{align*}
where the approximated free energy is given by
\begin{equation}
F_\delta(\rho) \equiv \int_0^\rho f_\delta(u) du, \quad \rho > 0.
\end{equation}
Therefore the Div-Curl Lemma allows us to deduce (the bar denotes weak limit)
\[ \bar{f}_\delta(\rho^{(0)}) = \bar{f}_\delta(\rho^{(0)}) \rho^{(0)} \quad \text{a.e. in } Q_T \]
assuming that \( f_\delta(r) = f(r) \) for \( r \leq \delta^{-1} \), which means, given the uniform \( L^\infty \) bound for \( \rho^{(0)} \), that for \( \delta > 0 \) small enough \( f_\delta(\rho^{(0)}) = f(\rho^{(0)}) \) a.e. in \( Q_T \). Therefore
\[ f(\rho^{(0)}) \rho^{(0)} = f(\rho^{(0)}) \rho^{(0)} \quad \text{a.e. in } Q_T. \]
Being $f$ monotone, we deduce
\[ f(\rho^{(0)}) = f(\overline{\rho}) \quad \text{a.e. in } Q_T \]

Let $\rho = \overline{\rho}$. We just proved that $f_0(\rho^{(0)}) \to^* f(\rho)$ weakly* in $L^\infty(Q_T)$ as $\delta \to 0$ (up to subsequences). Furthermore (22) implies the $L^2(Q_T)$–weak convergence of $\nabla f_0(\rho^{(0)})$, meaning that
\[ \nabla f_0(\rho^{(0)}) \to \nabla f(\rho) \quad \text{weakly in } L^2(Q_T). \]

A uniform bound for $\partial_t \rho^{(0)}$ in $L^2(0, T; H^1(\Omega'))$ can be easily derived from (19), (22) implying that
\[ \partial_t \rho^{(0)} \to \partial_t \rho \quad \text{weakly in } L^2(0, T; H^1(\Omega')). \]

We are therefore able to take the limit $\delta \to 0$ in (19) and obtain a weak solution to (1), (2).

**Uniqueness.** Let $\rho_1, \rho_2$ be weak solutions to (1) satisfying
\[ \rho_1, \rho_2 \in L^\infty(Q), \quad \nabla f(\rho_1), \nabla f(\rho_2) \in L^2(Q), \quad \partial_t \rho_1, \partial_t \rho_2 \in L^2(0, \infty; H^1(\Omega')), \]
with the same initial datum: $\rho_1(\cdot, 0) = \rho_2(\cdot, 0) = \rho^\text{in}$ in $\Omega$. Therefore it holds
\[
\begin{cases}
\partial_t (\rho_1 - \rho_2) = \Delta (f(\rho_1) - f(\rho_2)) & \text{in } Q, \\
\nu \cdot \nabla (f(\rho_1) - f(\rho_2)) = 0 & \text{on } \partial \Omega \times (0, T), \\
(\rho_1 - \rho_2)(\cdot, 0) = 0 & \text{on } \Omega.
\end{cases}
\]

Define the function $u : Q \to \mathbb{R}$ as the unique solution to
\[
\begin{cases}
-\Delta u = \rho_1 - \rho_2 & \text{in } \Omega, \ t > 0, \\
\nu \cdot \nabla u = 0 & \text{on } \partial \Omega, \ t > 0, \\
\int_{\Omega} u \, dx = 0 & t > 0.
\end{cases}
\]

Clearly $u \in L^\infty(0, T; H^1(\Omega))$. Therefore we can use $u$ as test function in (24). We obtain
\[
\int_{\Omega} |\nabla u(t)|^2 \, dx + \int_0^t \int_{\Omega} (\rho_1 - \rho_2)(f(\rho_1) - f(\rho_2)) \, dx \, d\tau = 0, \quad t \in (0, T).
\]

Being $f$ nondecreasing, it follows that $u \equiv 0$ in $Q$ and so $\rho_1 \equiv \rho_2$ on $Q$. The weak solution is therefore unique.

**Monotonicity property.** We now prove that, for a.e. $x \in \Omega$, the mapping $t \in (0, \infty) \mapsto \min(\rho(x, t), \rho_{cr}) \in \mathbb{R}$ is nondecreasing. Let $\psi \in C_c^\infty(\Omega), \ \psi \geq 0$ a.e. in $\Omega$ arbitrary. Since $\rho^{(0)} \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q_T)$ we can test (19) against $(\rho^{(0)} - \rho_{cr})^2 \psi$ and obtain
\[
\frac{1}{4} \int_{\Omega} (\rho^{(0)}(t) - \rho_{cr})^2 \psi \, dx + \frac{1}{2} \int_{0}^{t} \int_{\Omega} (\rho^{(0)}(\tau) - \rho_{cr})^2 \psi \, dx \, d\tau + 3\delta \int_{0}^{t} \int_{\Omega} (\rho - \rho_{cr})^2 |\nabla \rho^{(0)}|^2 \psi \, dx \, d\tau
\]
\[
+ \delta \int_{0}^{t} \int_{\Omega} (\rho^{(0)} - \rho_{cr})^2 \nabla \psi \cdot \nabla \rho^{(0)} \, dx = \frac{1}{4} \int_{\Omega} (\rho_0 - \rho_{cr})^2 \psi \, dx
\]
\[- \int_0^t \int_\Omega f_3'(\rho^0)(\rho^0 - \rho_{cr})^3 \nabla \rho^0 \cdot \nabla \psi dx dt' - 3 \int_0^t \int_\Omega f_3''(\rho^0)(\rho^0 - \rho_{cr})^2 |\nabla \rho^0|^2 \psi dx dt'.\]

However, since \( f_3(s) = f(s) = 0 \) for \( s \leq \rho_{cr} \), the last two integrals on the right-hand side of the above identity vanish. Given that the third integral on the left-hand side is nonnegative (since \( \psi \geq 0 \)), we get
\[
\frac{1}{4} \int_\Omega (\rho^0(t) - \rho_{cr})^4 \psi dx + \delta \int_0^t \int_\Omega (\rho^0(t) - \rho_{cr})^3 \psi dx dt'
+ \delta \int_0^t \int_\Omega (\rho^0(t) - \rho_{cr})^2 \nabla \psi \cdot \nabla \rho^0 dx \leq \frac{1}{4} \int_\Omega (\rho_0 - \rho_{cr})^4 \psi dx.
\]

An integration by parts in the third integral on the left-hand side of the above inequality yields
\[
\frac{1}{4} \int_\Omega (\rho^0(t) - \rho_{cr})^4 \psi dx + \delta \int_0^t \int_\Omega (\rho^0(t) - \rho_{cr})^3 \psi - \frac{1}{4} (\rho^0(t) - \rho_{cr})^4 \Delta \psi dx dt'
\leq \frac{1}{4} \int_\Omega (\rho_0 - \rho_{cr})^4 \psi dx.
\]

Given the choice of \( \psi \) and the uniform \( L^\infty(Q_T) \) bounds for \( \rho^0(t) \), the second integral on the left-hand side of the above inequality vanishes in the limit \( \delta \to 0 \), yielding
\[
\int_\Omega (\rho(t) - \rho_{cr})^4 \psi dx \leq \int_\Omega (\rho_0 - \rho_{cr})^4 \psi dx, \quad t > 0.
\]

Since \( \psi \) is an arbitrary nonnegative test function, we infer
\[
(\rho(t) - \rho_{cr})^4 \leq (\rho_0 - \rho_{cr})^4 \quad \text{a.e. in } \Omega, \quad t > 0.
\]

Since \( (x)_- = -(-x)_+ \) for every \( x \in \mathbb{R} \), it follows
\[
(\rho_{cr} - \rho(t))_+ \leq (\rho_{cr} - \rho_0)_+ \quad \text{a.e. in } \Omega, \quad t > 0.
\]

Given that \( \min(s, \rho_{cr}) = \rho_{cr} + (s - \rho_{cr})_- = \rho_{cr} - (\rho_{cr} - s)_+ \) for \( s \in \mathbb{R} \), we conclude that, for a.e. \( x \in \Omega \), the mapping \( t \in (0, \infty) \mapsto \min(\rho(x, t), \rho_{cr}) \in \mathbb{R} \) is nondecreasing. This finishes the proof of Thr. [1]

**Remark 1.** The monotonicity property showed in Thr. [1] implies in particular that
\[
\inf_{\Omega \times (0, \infty)} \rho \geq \inf_{\Omega} \rho_0.
\]

As a consequence, if \( \rho_0 \) is uniformly positive in \( \Omega \), then \( \rho \) is uniformly positive in \( \Omega \times (0, \infty) \).

### 3. Long-time behaviour (proof of theorem [2])

In this section, we concentrate in proving Theorem [2]. The proof is divided into several lemmas.

The following Lemma yields the exponential convergent rate for the solution in the supercritical case.
Lemma 1. Let $F(\rho|\rho_\infty)$ as in (8). If $\rho_\infty > \rho_{cr}$ then a constant $\lambda > 0$ exists such that
\[
\int_\Omega F(\rho(t)|\rho_\infty)dx \leq e^{-\lambda t} \int_\Omega F(\rho_0|\rho_\infty)dx \quad t > 0.
\]

Proof. Testing (1) against $f(\rho) - f(\rho_\infty)$ yields
\[
\frac{d}{dt} \int_\Omega F(\rho(t)|\rho_\infty)dx = -\int_\Omega |\nabla f(\rho(t))|^2 dx.
\]
Lemma 7 implies
\[
\frac{d}{dt} \int_\Omega F(\rho(t)|\rho_\infty)dx \leq -c \int_\Omega |f(\rho(t)) - f(\rho_\infty)|^2 dx.
\]
On the other hand, since $\rho_\infty > \rho_{cr}$, it follows
\[
F(\rho(t)|\rho_\infty) \leq C |f(\rho(t)) - f(\rho_\infty)|^2, \quad t > 0.
\]
Indeed, $\rho \in L^\infty(\Omega \times (0,\infty))$, so the above inequality will follow provided that
\[
\frac{F(s|\rho_\infty)}{|f(s) - f(\rho_\infty)|^2} \leq C \quad \text{as } s \to \rho_\infty,
\]
which is easily verified by employing De L'Hospital's theorem. We deduce
\[
\frac{d}{dt} \int_\Omega F(\rho(t)|\rho_\infty)dx \leq -\lambda \int_\Omega F(\rho(t)|\rho_\infty)dx, \quad t > 0.
\]
Gronwall’s Lemma yields the statement. This finishes the proof of the Lemma. \(\square\)

Lemma 2. Let $\rho$ be the weak solution to (1)-(2) according to Thr. 1.

If $\rho_\infty \geq \rho_{cr}$ then
\[
(27) \quad \rho(t) \to \rho_\infty \quad \text{as } t \to \infty \text{ strongly in } L^p(\Omega), \quad \forall 1 \leq p < \infty.
\]

If $\rho_\infty < \rho_{cr}$ then
\[
(28) \quad (\rho(t) - \rho_{cr})_+ \to 0 \quad \text{as } t \to \infty \text{ strongly in } L^p(\Omega), \quad \forall 1 \leq p < \infty.
\]

Proof. We distinguish three cases.

Case 1: $\rho_\infty > \rho_{cr}$. Relation (27) follows immediately from Lemma 1 and the $L^\infty(\Omega \times (0,\infty))$ bounds for $\rho$.

Case 2: $\rho_\infty < \rho_{cr}$. Integrating (26) in time yields
\[
\int_\Omega F(\rho(t))dx + \int_0^t \int_\Omega |\nabla f(\rho)|^2 dx d\tau = \int_\Omega F(\rho_0)dx, \quad t > 0.
\]
Due to the above free energy identity, we have a sequence $t_n \to \infty$ such that
\[
\int_\Omega |\nabla f(\rho(t_n))|^2 dx \to 0 \quad \text{as } n \to \infty.
\]
Poincaré’s Lemma yields
\[ f(\rho(t_n)) - \frac{1}{|\Omega|} \int_{\Omega} f(\rho(t_n)) \, dx \rightarrow 0 \quad \text{strongly in } L^2(\Omega). \]

On the other hand, the time-uniform \( L^\infty \) bounds for \( \rho \) imply that \( \frac{1}{|\Omega|} \int_{\Omega} f(\rho(t_n)) \, dx \) is bounded, from which we obtain a subsequence \((t_n_k)\) of \((t_n)\) such that
\[ \frac{1}{|\Omega|} \int_{\Omega} f(\rho(t_n_k)) \, dx \rightarrow \lambda \quad \text{as } k \rightarrow \infty. \]

Therefore,
\[ (29) \quad f(\rho(t_n_k)) \rightarrow \lambda \geq 0 \quad \text{strongly in } L^2(\Omega). \]

We claim that the number \( \lambda \) appearing in \( (29) \) is zero. In fact, assume by contradiction that \( \lambda > 0 \). From \( (3) \) it follows that \( \rho(t_n_k) \rightarrow f^{-1}(\lambda) > \rho_{cr} \text{ a.e. in } \Omega \), implying thanks to the \( L^\infty(\Omega) \) bounds for \( \rho(t_n_k) \) that \( \rho(t_n_k) \rightarrow f^{-1}(\lambda) > \rho_{cr} \text{ strongly in } L^1(\Omega) \), against mass conservation. So \( \lambda = 0 \), meaning that \( f(\rho(t_n_k)) \rightarrow 0 \text{ a.e. in } \Omega \). Since \( (3) \) holds, it follows that
\[ \limsup_{k \rightarrow \infty} \rho(t_n_k) \leq \rho_{cr} \quad \text{a.e. in } \Omega. \]

From the definition of \( F \) and \( (3) \) we deduce
\[ \limsup_{k \rightarrow \infty} F(\rho(t_n_k)) \leq F(\limsup_{k \rightarrow \infty} \rho(t_n_k)) \leq F(\rho_{cr}) = 0 \quad \text{a.e. in } \Omega, \]
which implies (given that \( F \) is nonnegative)
\[ F(\rho(t_n_k)) \rightarrow 0 \quad \text{a.e. in } \Omega. \]

Given the uniform \( L^\infty \) bounds for \( \rho \), it follows that
\[ \lim_{k \rightarrow \infty} \int_{\Omega} F(\rho(t_n_k)) \, dx = 0. \]

However, since
\[ \frac{d}{dt} \int_{\Omega} F(\rho(t)) \, dx = - \int_{\Omega} |\nabla f(\rho(t))|^2 \, dx \leq 0, \]
the function \( t \mapsto \int_{\Omega} F(\rho(t)) \, dx \) is nonincreasing, meaning that
\[ \lim_{t \rightarrow \infty} \int_{\Omega} F(\rho(t)) \, dx = 0. \]

It follows that \( F(\rho(t)) \rightarrow 0 \text{ a.e. in } \Omega \), easily implying that
\[ \limsup_{t \rightarrow \infty} \rho(t) \leq \rho_{cr} \quad \text{a.e. in } \Omega. \]

This in turn yields that \( (\rho(t) - \rho_{cr})_+ \rightarrow 0 \text{ a.e. in } \Omega \). This fact, together with the \( L^\infty \) bounds for \( \rho \), yields \( (28) \).
Case 3: $\rho_\infty = \rho_{cr}$. It is straightforward to see that the arguments in Case 2 apply also to this case, implying that (28) holds. However, mass conservation implies

$$\int_\Omega (\rho(t) - \rho_{cr})^- dx = - \int_\Omega (\rho(t) - \rho_\infty)^- dx = \int_\Omega (\rho(t) - \rho_\infty)^+ dx - \int_\Omega (\rho(t) - \rho_\infty) dx$$

$$= \int_\Omega (\rho(t) - \rho_{cr})^+ dx \to 0 \quad \text{as } t \to \infty,$$

meaning that $\rho(t) - \rho_{cr} \to 0$ strongly in $L^1(\Omega)$ as $t \to \infty$. This fact, together with the $L^\infty$ bounds for $\rho$, yields (10). This finishes the proof of the Lemma. \( \square \)

The results in Thr. 2 related to the supercritical case $\rho_\infty \geq \rho_{cr}$ have now been proven. We focus now on the subcritical case $\rho_\infty < \rho_{cr}$.

**Lemma 3.** Let $\rho$ be the weak solution to (1) according to Thr. 1 and let $\rho_\infty < \rho_{cr}$. The following statements hold.

1. A function $\hat{\rho} \in L^\infty(\Omega)$ exists such that $\rho(t) \to \hat{\rho}$ strongly in $L^p(\Omega)$ for every $p < \infty$ as $t \to \infty$. Furthermore it holds $\hat{\rho} \leq \rho_{cr}$ a.e. in $\Omega$.

2. If $\text{meas}([\rho_0 > \rho_{cr}) > 0$, then there exists $T^* > 0$ such that $\int_0^{T^*} \int_\Omega |\nabla \rho|^2 dx dt = \infty$.

**Proof.** Let us now show that the statement 1 is true. From Thr. 1 it follows that $\min(\rho(t), \rho_{cr})$ is a.e. convergent in $\Omega$ as $t \to \infty$. Furthermore, from Lemma 2 it follows that $(\rho(t) - \rho_{cr})_+$ is a.e. convergent in $\Omega$ as $t \to \infty$. Since $\rho(t) = \min(\rho(t), \rho_{cr}) + (\rho(t) - \rho_{cr})_+$, we deduce that $\rho(t)$ is a.e. convergent in $\Omega$ as $t \to \infty$ towards some limit function $\hat{\rho}$. However, given the uniform $L^\infty$ bounds for $\rho$, we conclude that $\rho(t) \to \hat{\rho}$ strongly in $L^p(\Omega)$ for every $p < \infty$. The upper bound on $\hat{\rho}$ follows immediately from Thr. 2. Thus Statement 1 holds.

Let us now prove the second statement. We proceed by contradiction. Assume that $\text{meas}([\rho_0 > \rho_{cr}) > 0$ and that $\nabla \rho \in L^2(0,T;L^2(\Omega))$ for every $T > 0$. It follows that, given any $\psi \in C^\infty_c(\Omega)$, we can employ $(\rho - \rho_{cr})^2 \psi$ as a test function in (1), obtaining

$$\frac{1}{4} \int_\Omega (\rho(t) - \rho_{cr})_+^4 \psi dx = \frac{1}{4} \int_\Omega (\rho_0 - \rho_{cr})^4 \psi dx$$

$$- \int_0^t \int_\Omega (\rho - \rho_{cr})^3 \nabla f(\rho) \cdot \nabla \psi dx dt' - \int_0^t \int_\Omega \nabla [(\rho - \rho_{cr})^3] \cdot \nabla f(\rho) \psi dx dt',$$

for $t \in (0,T)$. Since $\nabla \rho \in L^2(0,T;L^2(\Omega))$ we can rewrite the above identity as

$$\frac{1}{4} \int_\Omega (\rho(t) - \rho_{cr})_+^4 \psi dx = \frac{1}{4} \int_\Omega (\rho_0 - \rho_{cr})_+^4 \psi dx$$

$$- \int_0^t \int_\Omega (\rho - \rho_{cr})^3 f'(\rho) \nabla \rho \cdot \nabla \psi dx dt' - 3 \int_0^t \int_\Omega (\rho - \rho_{cr})^2 f'(\rho) |\nabla \rho|^2 \psi dx dt'.$$
Clearly the last two integrals on the right-hand side of the above inequality vanish due to the properties of \( f \). We deduce
\[
\frac{1}{4} \int_{\Omega} (\rho(t) - \rho_{cr})^4 \psi dx = \frac{1}{4} \int_{\Omega} (\rho_0 - \rho_{cr})^4 \psi dx
\]
for every \( \psi \in C_c^\infty(\Omega) \), yielding \( (\rho(t) - \rho_{cr})^4 = (\rho_0 - \rho_{cr})^4 \) a.e. in \( \Omega \), and so
\[
\min(\rho(t), \rho_{cr}) = \rho_{cr} + (\rho(t) - \rho_{cr})_+ = (\rho_0 - \rho_{cr})_+ + \rho_{cr} = \min(\rho_0, \rho_{cr}) \quad \text{a.e. in } \Omega, \quad t \in (0, T).
\]
Being \( T \) arbitrary, it means that the above identity holds for every \( t > 0 \). As a consequence,
\[
\rho(t) = \min(\rho(t), \rho_{cr}) + (\rho(t) - \rho_{cr})_+ = \min(\rho_0, \rho_{cr}) + (\rho(t) - \rho_{cr})_+, \quad t > 0.
\]
Taking the limit \( t \to \infty \) on both sides of the above equality and employing Lemma 2 as well as statement 1 lead to
\[
\hat{\rho} = \min(\rho_0, \rho_{cr}) \quad \text{a.e. in } \Omega.
\]
However, the above identity together with the assumption on the initial datum yield
\[
\int_{\Omega} (\rho_0 - \hat{\rho}) dx = \int_{\Omega} (\rho_0 - \min(\rho_0, \rho_{cr})) dx = \int_{\rho_0 > \rho_{cr}} (\rho_0 - \rho_{cr}) dx > 0,
\]
against mass conservation. Therefore Statement 2 holds. This finishes the proof of the lemma.

**Lemma 4.** Let \( \Omega \) connected, \( F \) as in (8), \( \rho_\infty < \rho_{cr} \) and assume (9) holds. Then
\[
(30) \quad \int_{\Omega} F(\rho(t)) dx \leq C t^{-\frac{k+1}{k}} \quad \text{as } t \to \infty.
\]

**Proof.** Putting (26) and Lemma 7 in the Appendix together yields
\[
\frac{d}{dt} \int_{\Omega} F(\rho(t)) dx \leq -c \int_{\Omega} |f(\rho(t))|^2 dx.
\]
From assumption (9) it follows
\[
F(s) \leq C_R \int_{\rho_{cr}}^{s} (r - \rho_{cr})^\kappa dr = \frac{C_R}{1 + \kappa} (s - \rho_{cr})^{\kappa+1} \leq \frac{C_R c_R^{-(\kappa+1)/\kappa}}{1 + \kappa} f(s)^{1+1/\kappa} \quad s \leq R,
\]
and given the \( L^\infty(\Omega \times (0, \infty)) \) bounds for \( \rho \) we obtain
\[
\frac{d}{dt} \int_{\Omega} F(\rho(t)) dx \leq -c \int_{\Omega} F(\rho(t))^{\frac{2\kappa}{\kappa+1}} dx \leq -c \left( \int_{\Omega} F(\rho(t)) dx \right)^{\frac{2\kappa}{\kappa+1}},
\]
where the last inequality follows from Jensen’s inequality. Gronwall’s Lemma yields the statement. This finishes the proof of the Lemma. 

□
Lemma 5. Assume that $\Omega$ is connected, $\rho_\infty < \rho_{cr}$ and (9) holds. Then
\[\|\rho(t) - \rho_{cr}\|_{L^1(\Omega)} \leq C t^{-\frac{1}{\kappa-1}}, \quad W_2(\rho(t), \rho) \leq C t^{-\frac{1}{\kappa-1}},\]
for $t \to \infty$, where $W_2$ is the 2-Wasserstein distance.

Proof of Lemma 5. Lemma 4 and Assumption 9 lead to
\[\int_{\Omega} (\rho - \rho_{cr}) \kappa + 1 dx \leq C \int_{\Omega} F(\rho) dx \leq C t^{-\frac{1}{\kappa-1}} \kappa^{-1} \kappa^{-1} \quad \text{as } t \to \infty.\]
Jensen’s inequality yields
\[\int_{\Omega} (\rho - \rho_{cr}) dx \leq C t^{-\frac{1}{\kappa-1}} \kappa^{-1} \kappa^{-1} \quad \text{as } t \to \infty.\]
We now prove the statement related to the 2-Wasserstein metric. It is known that this latter is equivalent to a negative homogeneous Sobolev norm:
\[W_2^2(\rho(t), \hat{\rho}) \leq C \|\rho(t) - \hat{\rho}\|_{H^1(\Omega)},\]
where
\[\|\varphi\|_{H^1(\Omega)} = \|\nabla \varphi\|_{L^2(\Omega)} \quad \forall \varphi \in H^1(\Omega).\]
Define for $t \geq 0$
\[
\begin{cases}
-\Delta V(t) = \rho(t) - \langle \rho(t) \rangle \quad \text{in } \Omega \\
\nu \cdot \nabla V(t) = 0 \quad \text{on } \partial \Omega,
\end{cases}
\begin{cases}
-\Delta \hat{V} = \hat{\rho} - \langle \hat{\rho} \rangle \quad \text{in } \Omega \\
\nu \cdot \nabla \hat{V} = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]
Since $\langle \rho(t) \rangle = \langle \hat{\rho} \rangle$, we deduce
\[\|\rho(t) - \hat{\rho}\|_{H^1(\Omega)} \leq \|\nabla (V(t) - \hat{V})\|_{L^2(\Omega)}.\]
Let us now compute
\[
\frac{d}{dt} \frac{1}{2} \|\nabla (V(t) - \hat{V})\|_{L^2(\Omega)}^2 = \int_{\Omega} \nabla(V(t) - \hat{V}) \cdot \nabla \partial_t V(t) dx
\]
\[= \int_{\Omega} (V(t) - \hat{V}) \partial_t \rho(t) dx
\]
\[= \int_{\Omega} (V(t) - \hat{V}) \Delta f(\rho(t)) dx.
\]
Integrating by parts twice leads to
\[-\frac{d}{dt} \frac{1}{2} \|\nabla (V(t) - \hat{V})\|_{L^2(\Omega)}^2 = \int_{\Omega} (\rho(t) - \hat{\rho}) f(\rho(t)) dx.
\]
The uniform $L^\infty$ bounds for $\rho$ and Assumption 9 imply
\[-\frac{d}{dt} \frac{1}{2} \|\nabla (V(t) - \hat{V})\|_{L^2(\Omega)}^2 \leq C \int_{\Omega} f(\rho(t)) dx
\]
We point out that (31) uniquely determines $R_0$ since $\rho(t) \rightarrow \check{\rho}$ (thanks to Lemma 3), we obtain

$$\|\nabla (V(t) - \hat{V})\|_{L^2(\Omega)} \leq C t^{-\frac{s}{2n}} \text{ as } t \rightarrow \infty.$$ 

Integrating the above inequality between $t$ and $\infty$ and noticing that $\lim_{t \rightarrow \infty} \|\nabla (V(t) - \hat{V})\|_{L^2(\Omega)} = 0$ since $\rho(t) \rightarrow \check{\rho}$ (thanks to Lemma 3), we obtain

$$\|\nabla (V(t) - \hat{V})\|_{L^2(\Omega)}^2 \leq C t^{-\frac{s}{n}} \text{ as } t \rightarrow \infty.$$ 

This finishes the proof of the Lemma.

This finishes the proof of Theorem 2.

Remark 2. We point out that for $\rho_\infty > \rho_{cr}$ it holds $F(s|\rho_\infty|) \geq c|s - \rho_\infty|^2$ for $s \geq 0$, so (11) implies that $\rho(t) \rightarrow \rho_\infty$ in $L^2(\Omega)$ with exponential rate.

4. Formal reformulation as a free boundary problem.

Here we show that (11)–(2), in the case of radially symmetric initial data which are decreasing in the radial coordinate, can be formally equivalently reformulated as a free boundary problem which only degenerates on the free boundary.

We assume that $\rho_0 \in C^0(\overline{\Omega})$ satisfies

$$\rho_0(x) = \check{\rho}_0(|x|) \quad x \in \Omega, \quad \check{\rho}_0 \text{ decreasing in } (0, \infty), \quad \int_{\Omega} \rho_0 dx < \rho_{cr} < \rho_0(0).$$ 

Define

$$R_0 := \check{\rho}_0^{-1}(\rho_{cr}) > 0, \quad E_0 := \{x \in \Omega : \rho_0(x) > \rho_{cr}\} = \{x \in \Omega : |x| < R_0\} \neq \emptyset,$$

$$E_\infty = \{x \in \Omega : |x| < R_\infty\}, \quad \frac{1}{|E_\infty|} \int_{E_\infty} \rho_0 dx = \rho_{cr}.$$ 

We point out that (31) uniquely determines $R_\infty$ since, being $\check{\rho}_0$ decreasing, so is the mapping $r \in (0, \infty) \mapsto \int_{|x| < cr} \rho_0 dx \in \mathbb{R}$. Also, $E_\infty \subset \Omega$ since $\int_{\Omega} \rho_0 dx < \rho_{cr}$ by assumption.

Let us consider the following free boundary problem

$$\begin{align*}
\partial_t u &= \Delta f(u) & x \in E(t), & t > 0, \\
u &= \rho_{cr}, & (\rho_{cr} - \rho_0) \partial_t R(t) &= \|\nabla f(u)\| & x \in \partial E(t), & t > 0, \\
u(x, 0) &= \rho_0(x), & R(0) = R_0 & x \in E_0, \\
E(t) &= \{x \in \Omega : |x| < R(t)\} & t > 0.
\end{align*}$$
We point out that the unknown of (32)–(34) are the functions \( u : G \to [0, \infty) \) and \( R : [0, \infty) \to [0, \infty) \), where we defined
\[
G := \{(x, t) \in \Omega \times (0, T) : x \in E(t)\} \subset Q_T.
\]
We say that \( u \in C^{1,1}(G) \) if (and only if) \( \partial_i u, \partial^2 \partial_{x_i x_j} u \in C^0(G) \) for \( i, j = 1, \ldots, d \). We are going to prove the following

**Lemma 6.** Assume that (32)–(34) has a (strong) solution \((u, R)\) such that
\[
u \in C^0(\overline{G}) \cap C^{2,1}(G), \quad R \in C^0([0, \infty)) \cap C^1((0, \infty)),
\]
\( u \) is radially symmetric, \( u > \rho_{cr} \) on \( G \), \( \partial_t R \geq 0 \) on \( (0, \infty) \).

Then the function \( \rho : \Omega \times (0, T) \to [0, \infty) \) defined as
\[
(36) \quad \rho(t) = u(t)\mathcal{X}_{E(t)} + \rho_0\mathcal{X}_{\Omega \setminus E(t)} \quad t > 0
\]
is a weak solution to (1)–(2) according to Thr. 1. Furthermore \( E(t) = \{\rho(t) > \rho_{cr}\} \) for \( t \geq 0 \) and the steady state \( \hat{\rho} = \lim_{t \to \infty} \rho(t) \) is given by
\[
\hat{\rho} = \rho_{cr}\mathcal{X}_{E_{\infty}} + \rho_0\mathcal{X}_{\Omega \backslash E_{\infty}},
\]
where the set \( E_{\infty} \) is like in (31), with \( R_{\infty} = \lim_{t \to \infty} R(t) \).

**Proof.** Notice that from the fact that \( \partial_t R \geq 0 \) on \((0, \infty)\) it follows that \( E_0 \subset E(t) \) for \( t \geq 0 \) and therefore \( \rho_0 < \rho_{cr} \) on \( \Omega \setminus \overline{E(t)} \). It follows that \( E(t) = \{\rho(t) > \rho_{cr}\} \) for \( t \geq 0 \). The assumptions on \( u \) imply that (6) holds. We wish to prove that also (7) is fulfilled. Let \( \phi \in C_c^{\infty}(\Omega \times [0, T)) \) be a test function. It holds
\[
(37) \quad - \int_0^T \int_{\Omega} \rho \partial_t \phi dx dt - \int_{\Omega} \rho_0 \phi(0) dx
= - \int_0^T \int_{E(t)} u \partial_t \phi dx dt - \int_0^T \int_{\Omega \setminus E(t)} \rho_0 \partial_t \phi dx dt - \int_{\Omega} \rho_0 \phi(0) dx
= - \int_0^T \int_{E(t)} (u - \rho_0) \partial_t \phi dx dt - \int_0^T \int_{\Omega} \rho_0 \partial_t \phi dx dt - \int_{\Omega} \rho_0 \phi(0) dx
= - \int_0^T \int_{E(t)} \partial_t[(u - \rho_0)\phi] dx dt + \int_0^T \int_{E(t)} \phi \partial_t u dx dt.
\]
From the definition of \( E(t) \) it follows
\[
\frac{d}{dt} \int_{E(t)} (u - \rho_0) \phi dx = \frac{d}{dt} \int_{\partial B_1} \int_0^{R(t)} (u - \rho_0) \rho^{d-1} dr d\sigma
\]
where \((r, \sigma) \in [0, \infty) \times \partial B_1\) are the spherical coordinates on \( \mathbb{R}^d \). It follows
\[
\frac{d}{dt} \int_{E(t)} (u - \rho_0) \phi dx
\]
\[ \int_{\partial B_1} [(u - \rho_0)\phi]_{r=R(t)} R(t)^{d-1} \partial_t R(t) d\sigma + \int_{\partial B_1} \int_{0}^{R(t)} \partial_t [(u - \rho_0)\phi] r^{d-1} dr d\sigma \]

\[ = \int_{\partial B_1} [(u - \rho_0)\phi]_{r=R(t)} R(t)^{d-1} \partial_t R(t) d\sigma + \int_{E(t)} \partial_t [(u - \rho_0)\phi] dx \]

so from (37) it follows

\[- \int_{0}^{T} \int_{\Omega} \rho \partial_t \phi dx dt - \int_{\Omega} \rho_0 \phi(0) dx \]

\[ = - \int_{0}^{T} \frac{d}{dt} \int_{E(t)} (u - \rho_0)\phi dx dt \]

\[ + \int_{0}^{T} \int_{\partial B_1} [(u - \rho_0)\phi]_{r=R(t)} R(t)^{d-1} \partial_t R(t) d\sigma dt + \int_{0}^{T} \int_{E(t)} \phi \partial_t u dx dt \]

\[ = \int_{0}^{T} \int_{\partial B_1} [(u - \rho_0)\phi]_{r=R(t)} R(t)^{d-1} \partial_t R(t) d\sigma dt + \int_{0}^{T} \int_{E(t)} \phi \partial_t u dx dt. \]

The boundary conditions (33) imply

(38)

\[- \int_{0}^{T} \int_{\Omega} \rho \partial_t \phi dx dt - \int_{\Omega} \rho_0 \phi(0) dx \]

\[ = \int_{0}^{T} \int_{\partial E(t)} \phi |\nabla f(u)| d\Sigma dt + \int_{0}^{T} \int_{E(t)} \phi \partial_t u dx dt. \]

Let us now compute

\[ \int_{0}^{T} \int_{\Omega} \nabla \phi \cdot \nabla f(\rho) dx dt = - \int_{0}^{T} \int_{\Omega} \Delta \phi f(\rho) dx dt = - \int_{0}^{T} \int_{E(t)} \Delta \phi f(u) dx dt \]

where the last identity follows from the fact that \( \rho_0 < \rho_{cr} \) on \( \Omega \setminus E(t) \) for every \( t \geq 0 \).

Employing the boundary conditions (33) yields

\[ \int_{0}^{T} \int_{\Omega} \nabla \phi \cdot \nabla f(\rho) dx dt = \int_{0}^{T} \int_{E(t)} \nabla \phi \cdot \nabla f(u) dx dt \]

\[ = - \int_{0}^{T} \int_{E(t)} \phi \nabla f(u) \cdot \nu d\Sigma dt - \int_{0}^{T} \int_{E(t)} \phi \Delta f(u) dx dt. \]

It holds that \( u(t) > \rho_{cr} \) in \( E(t) \) and \( u(t) = \rho_{cr} \) on \( \partial E(t) \), meaning that \( f(u(t)) > 0 \) in \( E(t) \) and \( f(u(t)) = 0 \) on \( \partial E(t) \). As a consequence

\[ \nu(t) = - \frac{\nabla f(u(t))}{|\nabla f(u(t))|} \text{ on } \partial E(t) \cap \{|\nabla f(u(t))| > 0\}. \]

Thus \( \nabla f(u(t)) \cdot \nu(t) = -|\nabla f(u(t))| \text{ on } \partial E(t) \cap \{|\nabla f(u(t))| > 0\}. \) Since \( \nabla f(u(t)) \cdot \nu(t) = 0 = -|\nabla f(u(t))| \text{ on } \partial E(t) \cap \{|\nabla f(u(t))| = 0\} \), we conclude that \( \nabla f(u(t)) \cdot \nu(t) = -|\nabla f(u(t))| \text{ on } \partial E(t) \cap \{|\nabla f(u(t))| > 0\} \).
derivatives are discretized via centered finite di

\[
\partial E(t), \text{ implying }
\int_0^T \int_\Omega \nabla \phi \cdot \nabla f(\rho) dx dt = - \int_0^T \int_{\partial E(t)} \phi |\nabla f(u)| d\Sigma dt - \int_0^T \int_{E(t)} \phi \Delta f(u) dx dt.
\]

Summing the above identity and (38) and employing (32) yields (7). This means that \( \rho \)

is a weak solution to (1)–(2) in the sense of Thr. 1. Finally, the expressions for \( \dot{\rho} \) and \( R_\infty \)

follow from Thr. 2 and mass conservation, respectively. This finishes the proof of the

Lemma.

□

A remarkable aspect of the (formal) result contained in Lemma 6 is the fact that the

solution \( \rho \) to (1)–(2) is discontinuous along the critical set \( \partial E(t) = \{ x \in \Omega : |x| = R(t) \} \),

as the density remains above the critical value \( \rho_{cr} \) inside \( E(t) \) while it stays below \( \rho_{cr} \)

outside \( E(t) \). In fact, the evolution of \( \rho(t) \) is driven by diffusion inside \( E(t) \), which pushes \( \rho(t) \)

downwards towards the critical value \( \rho_{cr} \); as a consequence of mass conservation,

the domain \( E(t) \) expands (i.e. \( R(t) \) grows with time). On the other hand, outside \( E(t) \) the

solution \( \rho(t) \) coincides with the initial datum since the equation (1) reduces to \( \partial_t \rho(t) = 0 \)

in \( \Omega \setminus E(t) \). As a consequence of these combined phenomena, the border of \( E(t) \) is a
discontinuity (hyper)surface for every time \( t > 0 \). Notice the agreement between these

observations and Statement (iii) for the case \( \rho_\infty < \rho_{cr} \) in Thr. 2; as a matter of fact, the

hypothesis that the set \( \{ \rho_0 > \rho_{cr} \} \) has positive measure was also assumed in the proof of

Lemma 6.

5. Numerical results

In this section, we discuss several numerical tests. The main goal is to observe the

long time behaviour of solutions as indicated in Theorem 2 and segregation phenomena

that are not covered by our theoretical investigations but also observed in 15. Since

our intention is not the design of a better numerical scheme, we simply use the standard

backward Euler and central finite difference scheme to discretize (1).

In the numerical tests, we take \( \Omega = (-1, 1)^2 \in \mathbb{R}^2 \) and assume \( f \) to be

\[
f(r) = (r - 1)^2, \quad r \geq 0.
\]

Therefore \( \rho_{cr} = 1 \) in this setting.

5.1. Numerical scheme. The numerical scheme reads as follows. An implicit Euler time

discretization is employed:

\[
\rho^{(n)} - \tau \text{div}(f'(\rho^{(n)}) \nabla \rho^{(n)}) = \rho^{(n-1)}, \quad n \geq 1,
\]

where \( \rho^{(0)} \) is the initial datum and \( \rho^{(n)} \) is the solution at time \( t_n = n\tau \). The spatial

derivatives are discretized via centered finite differences:

\[
\begin{align*}
\rho_{ij}^{(n)} &= \frac{1}{h_x} (f_{i+1/2,j}^{(1,n)} - f_{i-1/2,j}^{(1,n)}) - \frac{h_x}{h_y} (f_{i,j+1/2}^{(2,n)} - f_{i,j-1/2}^{(2,n)}) = \rho_{ij}^{(n-1)}, \\
\rho_{ij}^{(1,n)} &= \frac{1}{h_y} f'(\rho_{ij}^{(n)})(\rho_{i+1/2,j}^{(n)} - \rho_{i-1/2,j}^{(n)}), \\
\rho_{ij}^{(2,n)} &= \frac{1}{h_x} f'(\rho_{ij}^{(n)})(\rho_{i,j+1/2}^{(n)} - \rho_{i,j-1/2}^{(n)}),
\end{align*}
\]
where $\rho_{ij}^{(n)}$ is the value of $\rho^{(n)}$ at the point $(x_i, y_j)$ of the spatial grid, namely

$$(x_i, y_j) = (-1 + ih_{x}, -1 + jh_{y}), \quad i = 0, \ldots, N_x, \quad j = 0, \ldots, N_y, \quad h_x = \frac{2}{N_x}, \quad h_y = \frac{2}{N_y}.$$ 

At every time step, (40) is solved via the following fix point scheme:

$$\rho_{ij}^{(n,0)} := \rho_{ij}^{(n-1)},$$

$$k \geq 1 : \begin{cases}
\rho_{ij}^{(n,k)} = \frac{1}{h_{x}} f'(\rho_{ij}^{(n,k-1)})(\rho_{i+1/2,j}^{(n,k)} - \rho_{i-1/2,j}^{(n,k)}), \\
J_{ij}^{(2,n,k)} = \frac{1}{h_{y}} f'(\rho_{ij}^{(n,k-1)})(\rho_{i,j+1/2}^{(n,k)} - \rho_{i,j-1/2}^{(n,k)}), \\
\rho_{ij}^{(n,k+1)} = \frac{1}{h_{x}} f'(\rho_{ij}^{(n,k)})(\rho_{i+1/2,j}^{(n,k)} - \rho_{i-1/2,j}^{(n,k)}), \\
J_{ij}^{(2,n,k)} = \frac{1}{h_{y}} f'(\rho_{ij}^{(n,k)})(\rho_{i,j+1/2}^{(n,k)} - \rho_{i,j-1/2}^{(n,k)}),
\end{cases}$$

for $k \geq 1$. Notice that (41) is just a linear system.

The computation of the sequence $(\rho_{ij}^{(n,k)})_{k \geq 0}$ is stopped once the relative difference between two subsequent iterates is lower than a certain tolerance. The final iterate is defined as the approximate solution $\rho_{ij}^{(n)}$ to (40). Otherwise, in case the number of fixed point iterations exceed a certain threshold without achieving convergence, the iteration is stopped and an error flag is returned, which is then used in the time iteration (see following part).

While the spatial grid is uniform and the number of intervals $N_x, N_y$ are fixed, the timestep $\tau$ is chosen at every time iteration in an adaptive way. Precisely, if the relative difference between the new and the old iterates is larger than a certain tolerance, or the fixed point procedure returned an error flag (see previous part), the timestep is reduced by a factor 1/2, while if the relative difference between the new and the old iterates and the number of fixed point iterations are smaller than certain tolerances, then the timestep is increased by a factor 11/10.

5.2. Results. We present here some results employed with the scheme illustrated in the previous subsection. Three sets of results are presented, the first corresponding to the supercritical case (that is, the spatial average of the initial datum is larger than the critical value), the second and third corresponding to the subcritical case (that is, the spatial average of the initial datum is smaller than the critical value). In both cases we observe that the evolution of the system is driven by the diffusion in the supercritical region (i.e. the region of $\Omega$ where the solution is larger than $\rho_{cr}$) together with mass conservation. Indeed, while clearly the diffusion in the supercritical region pushes the mass downwards towards the critical density, this process drives the evolution of the solution in the whole spatial domain thanks to mass conservation: the mass distribution in the subcritical region (where the solution is smaller than $\rho_{cr}$) is pushed aside and upwards by the mass coming down from the supercritical region. We also observe that in the supercritical case these two processes are sufficiently strong to bring the whole mass distribution to the constant steady state $\rho_{cr}$, while in the subcritical case the evolution of the system stops before achieving this convergence: the solution is
simply pushed by the diffusion towards a nonconstant profile which lies entirely in the subcritical region, and after such point the solution do not change any more. We remark that these phenomena can be already observed after a rather short simulation time.

In what follows $\int_{\Omega} = |\Omega|^{-1} \int_{\Omega}$

**Supercritical case.** We choose here as initial datum

$$\rho^{(0)}(x) = \rho_{\infty} \frac{e^{-3|x|^2}}{\int_{\Omega} e^{-3|y|^2} dy} \quad x \in \Omega, \quad \rho_{\infty} = \frac{3}{2} > 1 = \rho_{cr}.$$  

The spatial domain discretization is a uniform grid of 401 × 401 points.

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**Figure 1.** Supercritical case. We observe convergence to the constant steady state $= \rho_{\infty}$ of average mass.

In this case we observe that the solution converges towards the constant steady state given by the average mass $\rho_{\infty}$, see Figure 1. The convergence is faster around the center of the mass distribution, where the solution takes the largest values, while it is slowest close to the vertexes of the rectangular domain $\Omega = [-1, 1]^2$. While the solution remains very small around such points for some time, eventually the combination of diffusion in the supercritical region and mass conservation pushes the solution everywhere towards the constant steady state. Such a behaviour is to be expected from the analysis result in Theorem 2.

**Subcritical case.** This is the most interesting case, as the analysis provides us with weaker results in this regime (namely, the steady state $\hat{\rho} = \lim_{t \to \infty} \rho(t)$ is unknown in general). We discretize now the spatial domain via a uniform grid with 501 × 501 points (the increased number of grid points with respect to the supercritical case aims at obtaining a more accurate result for the more interesting subcritical case). We present two examples.
Example 1. We choose as initial datum the sum of two Gaussians with the same variance:

\[ \rho_{\text{in}}(x) = \rho_{\infty} \frac{e^{-16|x-x^{(0)}|^2} + e^{-16|y-x^{(0)}|^2}}{\int_{\Omega} (e^{-16|y-x^{(0)}|^2} + e^{-16|y+x^{(0)}|^2})dy} \quad x \in \Omega, \quad x^{(0)} = \left(-\frac{7}{20}, \frac{7}{20}\right), \quad \rho_{\infty} = \frac{3}{4}. \]

In Figure 2 we observe that the two mass distributions, which are quite separate at initial time, are pushed towards a joint mass distribution which roughly equals the critical density \( \rho_{\text{cr}} = 1 \) on a large part of \( \Omega \), while vanishes near the border of \( \Omega \).

Example 2. We choose again as initial datum the sum of two Gaussians with the same variance, but differently from the previous example, in this case the variance and distance between the centers are larger and the average density is smaller:

\[ \rho_{\text{in}}(x) = \rho_{\infty} \frac{e^{-8|x-x^{(0)}|^2} + e^{-8|y-x^{(0)}|^2}}{\int_{\Omega} (e^{-8|y-x^{(0)}|^2} + e^{-8|y+x^{(0)}|^2})dy} \quad x \in \Omega, \quad x^{(0)} = \left(\frac{3}{4}, -\frac{3}{4}\right), \quad \rho_{\infty} = \frac{3}{10}. \]

In Figure 3 we observe that this time the two mass distribution remains separated until the solution falls entirely underneath the critical value and the solution stops evolving; the final configuration of the system is given by two disjoint mass distributions separated by a valley.

Figure 2. Subcritical case, example 1. In the final configuration the two initial mass distributions merge into a single one.

6. Conclusion

In this paper we have proved the existence of a bounded unique global-in-time solution for a strongly degenerate parabolic problem modeling material flow and swarming. Furthermore we have proved that the solution converges to a steady state, which is generally expected in diffusion equations. Such steady state is equal to the average of the initial datum in the supercritical case, while in the subcritical case it is not known for general initial data, although we showed that it is a.e. not larger than the critical density. The convergence is proved to be exponential in the \( L^2(\Omega) \) norm in the supercritical case,
while in the subcritical case it is shown to be algebraic in the Wasserstein metric. We also presented a formal argument according to which the system can be equivalently reformulated as a free boundary problem in the case of radially symmetric initial data which are decreasing in the radial coordinate. Numerical experiments for both cases supercritical and subcritical are carried out. In the subcritical case, a segregation effect is observed when the distance between two high density regions is large enough, while no segregation happens when such distance is relatively small. This motivates future analytical projects aiming at determining criteria for initial data yielding the onset of segregation phenomena, and the investigation of how the solution generates segregation behaviour on a broader viewpoint. The issue of determining the value of the steady state in the subcritical case is also worth investigating.

Appendix

Auxiliary results. We prove here a generalized Poincaré’s Lemma.

Lemma 7. Let \( \Omega \) be connected, \( f \) as in (3) and \( 0 \leq \rho_\infty \neq \rho_{cr} \). For every \( R > 0 \) a constant \( C_R > 0 \) exists such that

\[
\| f(\rho) - f(\rho_\infty) \|_{L^2(\Omega)} \leq C_R \| \nabla f(\rho) \|_{L^2(\Omega)}
\]

for every nonnegative \( \rho \in L^\infty(\Omega) \) such that \( \nabla f(\rho) \in L^2(\Omega), \| \rho \|_{L^\infty(\Omega)} \leq R \) and \( \int_\Omega \rho \, dx = \rho_\infty \).

Proof. By contradiction. Let \( (\rho_n)_{n \in \mathbb{N}} \) be a sequence of nonnegative bounded functions \( \Omega \to \mathbb{R} \) such that

\[
\| \rho_n \|_{L^\infty(\Omega)} \leq R, \quad \langle \rho_n \rangle = \rho_\infty, \quad n \| \nabla f(\rho_n) \|_{L^2(\Omega)} < \| f(\rho_n) - f(\rho_\infty) \|_{L^2(\Omega)}, \quad n \in \mathbb{N}.
\]

As a consequence \( \| f(\rho_n) - f(\rho_\infty) \|_{L^2(\Omega)} > 0 \) for every \( n \), so we can define \( u_n = (f(\rho_n) - f(\rho_\infty))/\| f(\rho_n) - f(\rho_\infty) \|_{L^2(\Omega)} \). It holds that \( \| \nabla u_n \|_{L^2(\Omega)} < 1/n \), so \( \nabla u_n \to 0 \) strongly in \( L^2(\Omega) \).
Since $\|u_n\|_{L^2(\Omega)} = 1$, $u_n$ is bounded in $H^1(\Omega)$ and therefore via compact Sobolev embedding it is relatively compact in $L^2(\Omega)$, so up to subsequences $u_n \to u$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. Given that $\forall u_n \to 0$ strongly in $L^2(\Omega)$ and $\Omega$ is connected, we deduce that $u$ is constant in $\Omega$. Since $\|u_n\|_{L^2(\Omega)} = 1$ it follows that $u \neq 0$. As a consequence (up to subsequences)
\[
\frac{f(\rho_n) - f(\rho_\infty)}{\|f(\rho_n) - f(\rho_\infty)\|_{L^2(\Omega)}} \to u \quad \text{a.e. in } \Omega \text{ and strongly in } L^2(\Omega).
\]
We distinguish two cases.

**Case 1:** $\rho_\infty > \rho_{cr}$. Relation (42) implies that, for every $\varepsilon > 0$ there exists $\Omega_\varepsilon \subset \Omega$ such that $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ and
\[
\frac{f(\rho_n) - f(\rho_\infty)}{\|f(\rho_n) - f(\rho_\infty)\|_{L^2(\Omega)}} \to u \quad \text{strongly in } L^\infty(\Omega_\varepsilon).
\]
We show now that $\|f(\rho_n) - f(\rho_\infty)\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. Indeed, if (by contradiction) there exists a subsequence (not relabeled) of $\rho_n$ such that $\|f(\rho_n) - f(\rho_\infty)\|_{L^2(\Omega)} \geq c > 0$ for every $n \in \mathbb{N}$, then (43) and the fact that $u$ is a nonzero constant imply the existence of $N_\varepsilon > 0$ such that
\[
f(\rho_n) \geq f(\rho_\infty) + \frac{1}{2}cu \quad \text{if } u > 0, \quad f(\rho_n) \leq f(\rho_\infty) + \frac{1}{2}cu \quad \text{if } u < 0,
\]
a.e. in $\Omega_\varepsilon$, for $n > N_\varepsilon$. It follows that either $\rho_n \leq \rho_\infty - K$ a.e. in $\Omega_\varepsilon$, $n > N_\varepsilon$, or $\rho_n \geq \rho_\infty + K$ a.e. in $\Omega_\varepsilon$, $n > N_\varepsilon$, for some $\varepsilon$–independent constant $K > 0$, which clearly violates the constraint $\int_{\Omega} \rho_n \, dx = \rho_\infty$. Therefore
\[
\|f(\rho_n) - f(\rho_\infty)\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty.
\]
From (43), (44) we deduce
\[
f(\rho_n) - f(\rho_\infty) \to 0 \quad \text{strongly in } L^\infty(\Omega_\varepsilon).
\]
Since $\rho_\infty > \rho_{cr}$ and $f$ is a one-to-one mapping on $(\rho_{cr}, \infty)$, we easily deduce that
\[
\rho_n \to \rho_\infty \quad \text{strongly in } L^\infty(\Omega_\varepsilon),
\]
Being $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ and $\rho_n$ bounded in $L^\infty(\Omega)$ we obtain that
\[
\rho_n \to \rho_\infty \quad \text{strongly in } L^p(\Omega) \text{ for every } p < \infty,
\]
and therefore
\[
\frac{\rho_n - \rho_\infty}{f(\rho_n) - f(\rho_\infty)} \to \frac{1}{f'(\rho_\infty)} > 0 \quad \text{strongly in } L^2(\Omega).
\]
From the above relation and (42) it follows
\[
\frac{\rho_n - \rho_\infty}{\|f(\rho_n) - f(\rho_\infty)\|_{L^2(\Omega)}} \to \frac{u}{f'(\rho_\infty)} \neq 0 \quad \text{strongly in } L^1(\Omega),
\]
which contradicts the constraint $\int_{\Omega} \rho_n \, dx = \rho_\infty$. 

Case 2: \( \rho_\infty < \rho_{cr} \). In this case \( f(\rho_\infty) = 0 \) and \( u > 0 \). We deduce from (42) for a.e. \( x \in \Omega \) \( \exists n^*(x) \in \mathbb{N} : f(\rho_n(x)) > 0 \) for \( n \geq n^*(x) \).

Given the definition of \( f \), it follows for a.e. \( x \in \Omega \) \( \exists n^*(x) \in \mathbb{N} : \rho_n(x) > \rho_{cr} \) for \( n \geq n^*(x) \).

We deduce
\[
\liminf_{n \to \infty} \rho_n \geq \rho_{cr} \quad \text{a.e. in } \Omega.
\]

Fatou’s Lemma and the assumption on the mass of \( \rho_n \) yield
\[
\rho_{cr} = \int_\Omega \rho_{cr} \, dx \leq \int_\Omega \liminf_{n \to \infty} \rho_n \, dx \leq \liminf_{n \to \infty} \int_\Omega \rho_n \, dx = \rho_\infty
\]
against the assumption \( \rho_\infty < \rho_{cr} \). This finishes the proof of the Lemma. \( \square \)

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