Topological order of the Rys F-model and its breakdown in realistic square spin ice: Topological sectors of Faraday loops

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Abstract – Both the Rys F-model and antiferromagnetic square ice possess the same ordered, antiferromagnetic ground state, but the ordering transition is of second order in the latter, and of infinite order in the former. To tie this difference to topological properties and their breakdown, we introduce a Faraday lines representation where loops carry the energy and magnetization of the system. Because the F-model does not admit monopoles, its Faraday loops have distinct topological properties, absent in square ice, and which allow for a natural partition of its phase space into topological sectors. Then, the Néel temperature corresponds to a transition from topologically trivial to non-trivial Faraday loops. Because magnetization is a homotopy invariant of the Faraday loops, and it is zero for topologically trivial ones, the susceptibility is zero below a critical field. In square spin ice, instead, monopoles destroy the homotopy invariance and the parity distinction among loops, thus erasing this rich topological structure. Consequently, even trivial loops can be magnetized in square ice, and their susceptibility is never zero.

Introduction. – In 1967, Lieb solved [1] the Rys F-model [2] demonstrating an antiferromagnetic transition with rather unusual features: it is of infinite order and yet there is an order parameter, which is also infinitely smooth [3]; moreover, a critical field is needed to elicit magnetization below \( T_c \). Lieb’s work predated by five years the results of Kosterlitz and Thouless (KT) [4], which tied an infinitely continuous transition to topological properties. But apparently, and regrettably, the importance of the infinite continuity was not immediately recognized in the F-model, nor was it associated to anything topological.

In this work we seek to make explicit the topological nature of the system, and explain why similar systems (non-degenerate square ice [5–7]) lack the same topological properties. We do so by mapping it to intuitive yet rigorous “Faraday lines”, and then use them to deduce heuristically the transition and the properties of the model. Line representations are often used in vertex models [1,8,9] and Lieb’s line representation, while immensely clever in allowing for an exact solution via transfer matrix, is not particularly conducive to physical intuition and does not make the topological features of the system explicit. Our Faraday lines, instead, carry all the relevant observables: energy, magnetization, parity, and \( \mathbb{Z}_2 \) symmetry breaking. In the F-model, we show, Faraday lines are always directed loops, and magnetization is the homotopy invariant of topologically non-trivial loops, explaining the critical field for susceptibility. We show that ordering corresponds to a transition between topological sectors of trivial and non-trivial Faraday loops.

Various reasons motivate our conceptualization. Firstly, we wish to elucidate how infinitely continuous transitions are related to topological sectors in a well-known system. Secondly, vertex models enjoy wide applicability and are currently studied [9–19]. Thirdly, and generally, one wonders if the very features that make many topological models compelling also make them physically unrealistic.

The F-model, introduced for ferroelectrics [2,20], approximates the low-energy physics of nanomagnetic artificial square ice [5–7] and of monolayer spin ice [21,22]. And yet, these real systems possess none of its special properties [23–27]. Their transitions are innocuously second order [28–31], and their susceptibility is never zero.
Our framework explains those differences: in realistic spin ice, monopoles are sink and sources of the Faraday lines, thus destroying the topological structure of the model.

**Six-vertex models, F-model.** – A *six-vertex model* [9–18] is a set of binary spins placed on the edges of a square lattice (fig. 1) such that only the six ice rule obeying vertices (two spins pointing in, two pointing out [32]) are allowed, denoted t-I and t-II of energies $\epsilon_I, \epsilon_{II}$.

The Rys F-model is a particular six-vertex model whose energies are $0 = \epsilon_I < \epsilon_{II}$. A spin configuration has energy $H = \epsilon_{II}N_{II}$, where $N_{II}$ is the number of t-II vertices. The F-model is invariant under the $\mathbb{Z}_2$ time reversal symmetry, parity symmetry $A \leftrightarrow B$ ($A, B$ are the alternating sublattices), and discretized translations. Its two ordered ground states are antiferromagnetic tessellations of t-I, have opposite staggered [3] order parameter $\psi = \pm 1$, and thus break the $\mathbb{Z}_2$ symmetry. Hence, one expects a continuous transition. In fact we know [1] that the transition is *infinitely* continuous with algebraic correlations for $T > T_c = \epsilon_{II}/\hbar^2$.

In the two-dimensional ice model, instead, $\epsilon_I = \epsilon_{II}$ and there is no energy scale and thus no transition. It mimics the degeneracy of water ice in two dimensions, also solved by Lieb [33], describes the ground state of degenerate square ice [34], a case of classical topological order [35,36], and is the infinite $T$ limit of the F-model.

**Faraday loops.** – The crux of our approach consists in choosing the proper description for the magnetic texture. In the antiferromagnetic ground state, the coarse graining of magnetization over a vertex (or more) is zero. Therefore, instead of considering the elementary spins $\vec{S}_i$, we describe magnetization by assigning it to the vertices $\vec{v}_i$, such that $\sum_i \vec{S}_i = \sum_i \vec{M}_i$.

Only t-IIs carry magnetization and they can be represented by an arrow connecting the centers of the plaquettes whose spins converge head-to-head or toe-to-toe in the vertex, thus assigning to it the magnetization $M_x = \pm 1$, $M_y = \pm 1$ as in fig. 1(a). Because of the topological constraints, the magnetic moments of t-II vertices are always joined into Faraday lines that carry the magnetization and energy of the system. On a torus, they are always *directed loops*, distinguished by topological triviality, parity, and chirality. All of this we show below.

Consider $L_x \times L_y$ vertices on a torus ($L_x, L_y$ even). The following propositions can be verified directly:

i) *Any spin configuration can be mapped into a set of non-intersecting, directed Faraday loops*; indeed, a square plaquette can support 0, 2, or 4 t-IIs on its vertices. If 2, they can always be connected unambiguously. If 4, which we call a *pinch*, they can be joined in two directed lines in two ways. For a configuration with $P$ pinches, there are $2^P$ loop-representations (fig. 1(b)–(d)).

ii) *Loops have a defined parity*: with the usual alternating $A/B$ assignment of plaquette parity, a loop will only cross either $A$ or $B$ plaquettes.

iii) *A and B loops* (red and blue in fig. 1) *cannot cross*.

iv) *The direction of loops* is assigned thus: two nearby loops have the same (respectively, opposite) direction if and only if they have same (respectively, opposite) parity. If a loop is directly contained into another loop, the two have the same (respectively, opposite) direction if and only if they have opposite (respectively, same) parity.

Modulo the pinches, the spins-loops correspondence is bijective. Any set of directed loops drawn on the square lattice such that i)–iv) hold true corresponds to a spin configuration for the six-vertex model.

Finally, Faraday loops are also *domain walls* separating anti-ferromagnetic domains with opposite sign of the staggered order parameter $\psi$. For completeness, we show in the Supplementary Material *Supplementary-material.pdf* (SM) how the Faraday lines relate to the height formalism.

A trivial loop of the torus is one that can be contracted to zero. We call a spin configuration *topologically non-trivial* if at least one of its representations contains at least one (and therefore an even number of) non-trivial loop(s).
Faraday loops are the elementary excitations of the system, and the F-model is a loop gas. Crucially, while only (and all) loops carry local magnetization, only topologically non-trivial loops carry net magnetization. Given a loop \( \gamma \) made of vertices \( v \), its total magnetization is
\[
\vec{M}_\gamma = \sum_{v \in \gamma} \vec{M}_v.
\]
(1)

Then \( \vec{M}_\gamma = 0 \) if and only if \( \gamma \) is topologically trivial. If, e.g., \( \gamma \) wraps around the \( x \) direction once, the net magnetization of the loops is \( M_y = 0 \), \( M_x = \pm L_x \varepsilon_x \) regardless of the length or shape of the loop\(^1\).

We have reached a crucial result: in the six-vertex model, magnetization is a homotopy invariant of the Faraday lines description. Therefore, topologically trivial spin configurations have zero net magnetization and do not couple with an external field.

**Topological sectors.** – To partition the phase space \( \mathcal{P} \) (the set of all spin configurations) into topological sectors (subsets of \( \mathcal{P} \) of defined topology) for Faraday loops, call \( \mathcal{T} \) the sector of all topologically trivial configurations, and \( W \) its complementary. From eq. (1), only configurations in \( W \) can have magnetization and we can further partition it accordingly.

We call a trivial (respectively, non-trivial) elementary update of a spin configuration the flip of a trivial (respectively, non-trivial) loop of spins that are all head to toe. Consider \( n_x \) pairs of alternating \( A \) and \( B \) non-trivial loops in the direction \( x \) (fig. 2(c), (d) shows \( n = 2 \), with \( 0 < n_x \leq L_y/2 \)), from iv), their magnetization has the same direction. Call \( \mathcal{M}_{\pm n_x,0} \) the set of all topologically trivial updates of such configurations. Because of homotopy invariance, trivial updates do not alter magnetization: from eq. (1), each configuration in \( \mathcal{M}_{n_x,0} \) carries magnetization \( M_y = 0, M_x = 2n_x L_x \) and magnetization density \( m_y = 0, m_x = 2n_x/L_y \). The same can be done to generate the sector \( \mathcal{M}_{0,n_y} \), and \( \mathcal{M}_{n_x,n_y} \), as the reader can verify via pairs of parallel helices.

Crucially, the union (which we call \( \mathcal{M} \)) of these magnetic sectors does not exhaust \( W \). Call \( \mathcal{W}_0 \) the set of all non-trivial configurations that have zero net magnetization. We can partition \( \mathcal{W}_0 \) into: \( \mathcal{W}_{A_xB_x} \) (respectively, \( \mathcal{W}_{A_xB_y} \)), the sets of all configurations representable via non-trivial loops of type \( A \) and \( B \) in the \( x \) (respectively, \( y \)) direction; and \( \mathcal{W}_{A_xA_y} \) (respectively, \( \mathcal{W}_{B_xB_y} \)) the sets of all configurations representable via non-trivial loops of parity \( A \) (respectively, \( B \)) wrapping in both \( x \) and \( y \) directions (fig. 2 (bottom)). Proposition iii) forbids other sectors.

In summary, \( \mathcal{P} \) is partitioned into \( \mathcal{T} \) (trivial) and \( \mathcal{W} \) (winding). \( \mathcal{W} \) is partitioned into \( \mathcal{M} \) (winding, magnetic) and \( \mathcal{W}_0 \) (winding, non-magnetic). \( \mathcal{W}_0 \) is partitioned into \( \mathcal{W}_{A_xA_y}, \mathcal{W}_{B_xB_y}, \mathcal{W}_{A_xB_y}, \) and \( \mathcal{W}_{A_xB_y} \).

\(^1\)Naturally, a Faraday line can wrap around the torus more than once, forming a helix of indices \( n_x, n_y \). Then its magnetization is \( M_y = \pm n_y L_y \varepsilon_y, M_x = \pm n_x L_x \varepsilon_x \).

Fig. 2: (a) Venn diagrams of the partition of the phase space into topological sectors and schematics for elements of \( \mathcal{T} \) (b), \( \mathcal{M} \) (c), \( \mathcal{W}_0 \) (d), and \( \mathcal{W} \) (e), (f)) represented on the torus. \( \mathcal{P} \) is partitioned into \( \mathcal{T} \) and \( \mathcal{W} \), corresponding to trivial and non-trivial loops. \( \mathcal{W} \) is partitioned into \( \mathcal{M} \) (corresponding to magnetized configurations) and \( \mathcal{W}_0 \) (configurations that are topologically non-trivial but have zero magnetization). \( \mathcal{W}_0 \) is partitioned into sectors corresponding to loops of different parity.

We can introduce the winding topological order parameters, \( w_A \) and \( w_B \), for each parity. For a configuration \( C \) and its (possibly multiple) loop representation(s) \( R \) made of loops \( \gamma \) we define
\[
w_A(C) = \sup_{R \in C} \sum_{\gamma \in R} \left| \frac{1}{L_x L_y} \sum_{v \in \gamma} \vec{M}_v \right|
\]
as the density of winding number of \( A \) loops of the configuration \( C \). Then, \( w^+ = w_A + w_B, w^- = w_A - w_B \).

**Temperature transition.** – When the phase space is partitioned into sectors \( \mathcal{D} \subset \mathcal{P} \), we can call
\[
Z_D(T) = \sum_{\mathcal{C} \in \mathcal{D}} \exp[-\mathcal{H}(\mathcal{C})/T],
\]
whose sum is restricted to configurations in \( \mathcal{D} \), the partition function of \( \mathcal{D} \), \( P_D = Z_D/Z \) is then the probability of finding the system in a configuration of the sector \( \mathcal{D} \). Any observable is said to be limited to the sector \( \mathcal{D} \) if obtained from \( F_D = -T \ln Z_D \). The total partition function is the sum of the partition functions of the sectors.

If \( P_D \to 1^- \) in the thermodynamic limit (and thus \( F \to F_D \)), we say that the system is asymptotically confined to the sector \( \mathcal{D} \). In this language, a phase transition
corresponds to the system “switching” between different sectors of the phase space, to which it is confined in the thermodynamic limit. When those sectors are topologically distinct, we say that the transition is topological. (We discuss in the SM how this concept does not exclude so-called “topological sector fluctuations” [37,38].)

First we show that in absence of a field, the system is asymptotically confined to $\mathcal{U} \cup \mathcal{W}_0$, that is the complementary of $\mathcal{M}$. Indeed, consider $f(m, T)$, the density of free energy limited to the sector $\mathcal{M}_{mL_y/2.0}$. Then $H_x = \partial_m f(m, T)$ is the magnetic field. $f(m)$ must be convex and even in $m$, and therefore has minimum at $m = 0$. Note that the thermal average of $\psi$ in $\mathcal{W}$ is zero. To prove it, consider, e.g., $\mathcal{M}_{1.0}$. Its lowest energy state is degenerate, corresponding to one $A$ and one $B$ non-trivial loops each of length $L_x$, variously assigned, subdividing the torus in two domains of opposite $\psi$. Averaging $\psi$ over all those configurations then returns zero. The same argument can be replicated for any sector in $\mathcal{W}$. That only $\mathcal{T}$ can exhibit long range order and thus $\psi \neq 0$ only in $\mathcal{T}$ can also be understood intuitively: the symmetry breaking that leads to $\psi \neq 0$ is driven by the contraction of the domain walls (Faraday loops) because of their tensile strength. But outside of $\mathcal{T}$, by definition, there are always some non-contractible loops.

Thus, antiferromagnetic ordering in the absence of a field must correspond to a transition between the topological sectors $\mathcal{T}$ and $\mathcal{W}_c$. Consider a non-trivial loop of lowest energy winding around the $x$-axis. It has length $L_x$ and degeneracy $\left( \frac{L_x}{L_x/2} \right) \sim 2^{L_x}$, for large $L_x$. Its free energy is then

$$\Delta F = L_x (11 - T \ln 2), \quad (4)$$

and goes to $-\infty$ ($+\infty$) in the thermodynamic limit for $T > T_c$ ($T < T_c$) with $T_c = \epsilon_{11}/\ln 2$. Crucially, it corresponds to the Néel temperature in Lieb’s solution.

As in the heuristic argument of Kosterlitz and Thouless, eq. (4) suggests that above $T_c$ the system is asymptotically confined to the topologically non-trivial sector $\mathcal{W}_c$, where $\psi = 0$, and below $T_c$ to the trivial $\mathcal{T}$ where $\psi \neq 0$. Therefore, $T_c = \epsilon_{11}/\ln 2$ is the Néel temperature. Then, low $T$ configurations correspond to an antiferromagnetic background decorated by Faraday loops (domain walls, fig. 1(b)). As $T$ increases, loops grow and coalesce forming at $T_c$ a topologically non-trivial network (fig. 1(c)), in a classical analogue to a string-net condensation [39].

There is more. We know that below $T_c$, $\mathcal{T}$ hosts a $\mathbb{Z}_2$ symmetry breaking in the sign of $\psi$. But a topological symmetry breaking also exists in $\mathcal{W}$, above $T_c$, between the topological sectors $\mathcal{W}_{A_{c}A_{c}}$ and $\mathcal{W}_{B_{c}B_{c}}$, as they have the same free energy but different loop parity. At $T > T_c$, the systems must choose whether the network of winding loops has parity $A$ or $B$, because loops of different parity cannot cross. This leads to a breaking of the $A \leftrightarrow B$ parity symmetry of the topologically non-trivial loops and thus in the sign of $w^-$.

Thus, at $T < T_c$ we have $w_A = w_B = 0$, $\psi > 0$. At $T > T_c$ we have $\psi = 0$, $w^+ > 0$ and $w^- = \pm w^+$. One suspects that $w^+, w^-$ reach zero infinitely continuously as $T \rightarrow T_c^+$ just like $\psi$ [3] does for $T \rightarrow T_\psi$, though we are incapable of predicting it within our framework.

Field induced transitions. – We now consider transitions under field between the demagnetized $\mathcal{T}$ and magnetized $\mathcal{M}$, for which $\psi$ is discontinuous except at $T = T_c$.

Consider $f(m)$. At $T = 0$, the free energy is trivially $f(m) = \epsilon_{11}m$, and the curve of magnetization $m = m(H)$ is a step function ($m = 0$ for $H < \epsilon_{11}$, $m = 1$ for $H > \epsilon_{11}$). Moreover, $f(\pm 1, T) = \epsilon_{11}$ at any temperature. Indeed, the sector of saturated magnetization $\mathcal{W}_{L_y/2.0}$ ($m = 1$) contains configurations for which all the horizontal spins are pointing to the right, whereas half of the vertical rows point up and half down, and therefore they are all of the same energy, while their entropy is subextensive.

For $m \simeq 0$, at $0 < T < T_c$, entropy favors configurations in which $mL_y$ horizontal, non-trivial loops of alternating $A/B$ parity and of magnetization aligned to the right are maximally spaced at a distance $1/|m|$, arbitrarily large for small $m$. We make therefore the ansatz that the free energy can be approximated by a trivial term from the bulk plus a non-trivial contribution from the loops, or from eq. (4),

$$f(m) \approx f_T + |m|(\epsilon_{11} - T \ln 2). \quad (5)$$

The weak singularity in $m = 0$ implies a critical field

$$|H_c(T)| = \epsilon_{11} - T \ln 2 \quad (6)$$

doing magnetization. Indeed, for $|H| < H_c$, $f(m) - Hm$ has a minimum in $m = 0$, the system is asymptotically confined to $\mathcal{T}$ and there is no magnetization nor susceptibility. Instead, no critical field exists when $T > T_c$. In such case, there are topologically non-trivial loops even at $m = 0$ and they can be biased even by weak fields to be of the proper alternation of parity leading to susceptibility, and thus to a curvature in $f(m)$ for $m = 0$. Note that the ansatz holds away from criticality. Close to $T_c$, Faraday lines cannot be considered independent and non-interacting, as all the loops in the system coalesce. Thus, the form of $H(T)$ can deviate from eq. (6). Indeed, while this letter was under review, a preprint has appeared [40] in which, via the cavity method, eq. (6) is verified at small $T$ but with deviations as $T$ approaches $T_c$.

Summary of results for the Rys F-model. – Figure 3 summarizes our results. The top panel, left, shows the phase diagram expressed in terms of topological sectors. Top panel, right, sketches $f(m)$ at different temperatures, from which curves for $m$ and $\psi$ can be obtained qualitatively (bottom panels).

When $0 < H < H_c(T)$, $0 < T < T_c$ the system is asymptotically confined to $\mathcal{T}$, and the magnetic field has no effect on the free energy. Thus, $\psi$ drops discontinuously to zero across the critical line (red) as the system switches to the sectors in $\mathcal{M}$ and magnetization develops. The entire line $H = 0$, $T \geq T_c$ corresponds to the system being
Finally, the reader might have noticed that for $T > 0$ we have sketched $f(m)$ such that $\lim_{m \to +1} \frac{df}{dm} = +\infty$. This vertical asymptote implies that magnetic saturation is only reached in the limit of infinite field, and that a Kasteleyn transition—often seen in spin ice systems $[8,41,42]$—is absent in the Rys F-model. We explain in the SM how this is understood in terms of Faraday lines.

**Monopoles and Faraday lines in spin ice.**—Any kinetics of the F-model must involve topologically trivial updates only, at least in the thermodynamic limit. But sectors in $\mathcal{M}$ differ by non-trivial updates. This assures ergodicity breaking $[43]$: the system will persist in a magnetized, high-energy state forever after the field is removed. Also, its magnetization remains independent of changes of temperature (though its free energy would change). This is, of course, unphysical and does not happen in real systems such as square spin ice $[5,6]$. Indeed, square ice evolves via individual spin flips, which are forbidden in the F-model as they lead to ice rule violations. These violations are monopoles and, as we show now, they destroy the topological structure.

At nearest neighbors, square spin ice is described by the sixteen-vertex model $[28]$, which contains all the possible vertex configurations. Figure 4 shows the ten extra, ice rule violating vertices, called t-III and t-IV (which we somehow improperly call monopoles, but see below), endowed with a topological charge $(\pm 2, \pm 4)$ defined as the difference of spins pointing in and out, and of energies $\epsilon_1 < \epsilon_{II} < \epsilon_{IV}$. This energy hierarchy describes the most common magnetic realizations $[5,23,24,29–31,44,45]$ and also particle-based ices $[25–27,46]$ via a proper mapping at equilibrium $[47,48]$, though different hierarchies are obtained through various clever methods $[34,49–52]$. A Faraday lines description proves useful in all these cases.

Figure 4 shows that monopoles too can be incorporated into a Faraday picture, but they modify it essentially. They allow for the mixing of $A$ and $B$ lines, and thus there is no longer parity distinction for the loops. Thus, Faraday lines can now be just lines and not necessarily loops, with monopoles as their sinks and sources. The total flux of the Faraday lines into a monopole is equal to its topological charge, in a geometric version of Gauss’ law.

Domain walls are still made of Faraday lines but can contain an even number of ±2 monopoles, whose charge alternates in sign along the loop. Therefore, magnetization is no longer an homotopy invariant of non-contractible loops: unlike in the F-model, domain walls are themselves magnetizable (fig. 5, rightmost panel) and can couple to an external field even in the antiferromagnetic phase. There is therefore no critical field for magnetization and susceptibility is never zero, even below $T_c$, unlike in the F-model.

There is still a critical field $H_c(T)$ for the disappearance of $\psi$ (and clearly $H_c(0) = \epsilon_{II}$, $H_c(T_c) = 0$) but the system can be magnetized for $H < H_c(T)$ via the paramagnetism of the domain wall loops (fig. 5), proportional to the density of t-II.
In sum, the previous partition of the phase space in topological sectors breaks down, and indeed, the antiferromagnetic transition in square ice is known to be of second order [28].

Finally, the kinetics of real spin ice loses the topological ergodicity breaking of the Rys F-model and becomes monopole kinetics: a single spin flip corresponds to either creation-annihilation of a monopole pair or to its motion. In turn, monopole creation controls the nucleation of domain walls, monopole motion controls domain wall growth, contraction, or fluctuation, as will be shown in future work.

**Faraday Lines and Dirac strings.** – For an antiferromagnetic square ice [5] the language of Dirac strings can be unsuitable. For instance, it is often said that monopoles in square ice are “linearly confined” by the tensile strength (or energy cost per unit length) of the Dirac strings. That is in general not true. Indeed, looking at fig. 4 the reader can verify that two monopoles sitting on a domain wall, and connected by Faraday lines, can feel no force bringing them together—or indeed no force at all: it is generally not true that a spin always exists, impinging on a given monopole, whose flipping reduces the overall energy by moving the monopole closer to another one.

Generally speaking, monopoles reside on domain walls made of Faraday lines. We can talk about Dirac strings connecting monopoles and exerting an attraction among them only in the case in which a monopole pair can be annihilated on a t-I antiferromagnetic tessellation by flipping a single directed line of spins. While interesting, such case does not describe all the possible situations, nor in fact the more common ones in a thermal ensemble. Furthermore, such a case is still describable by two Faraday lines running parallel to the claimed Dirac string, both starting from the negative monopoles and ending in the positive one (many examples are visible in fig. 4, circled by green lines), as Faraday lines unequivocally carry energy and magnetization.

The Faraday representation is also useful when the square ice is degenerate [34,49–51], or $\epsilon_I = \epsilon_II < \epsilon_III < \epsilon_IV$, as we will show elsewhere. In future work, we will treat more in depth the relationship between Faraday lines, the so-called Dirac strings, the gauge freedom associated to the height function representation, and magnetic fragmentation [53–55].

**Conclusion.** – We have introduced a Faraday line picture for square ice in general, and the six-vertex model in particular. When monopoles are absent, such as in the case of the six-vertex model, Faraday lines allow for a partition of the phase space into topological sectors, thus bringing physical insight on the topological nature of the transitions of the Rys F-model. In spin ice materials this representation survives, but its topological features break down because of monopoles. Our picture of topological non-trivial Faraday loops can be employed in finite-size realizations of artificial spin ice on a cylinder, which are now possible [56], or in studying problems for the six-vertex model with fixed boundary conditions [9–18].

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**Fig. 4:** Top: the ten extra vertices (monopoles) included in the sixteen-vertex model can also be represented by arrows separating frustrated spins and whose sum represents the total magnetization. The t-IIIIs (left of dotted line) have topological charges $\pm 2$, magnetizations $M = (\pm 1,0), (0, \pm 1)$, and energy $\epsilon_{III} > \epsilon_{II}$. t-IV vertices have topological charges $\pm 4$, zero magnetization, and energy $\epsilon_{IV} > \epsilon_{III}$. For all, $\psi = 0$. In the resulting loops representation (bottom left panel), domain wall loops mix parity and invert magnetization at the monopoles, (red and blue dots for negative and positive monopoles, respectively). Hence, topologically trivial loops can carry net magnetization (the bottom right panel shows horizontal net magnetization).

**Fig. 5:** Sketches of $m$ and $\psi$ in a square ice. At $T = 0$ we have the usual step function as a pointwise limit. The case below $T_c$ differs considerably from the F-model. There is small but non-zero susceptibility at low field, and the maximum susceptibility corresponds to a left neighborhood of $\epsilon_{II}$ where $\psi = 0$. We thank C. CASTELNOVO (Cambridge) and C. BATISTA (Tennessee) for useful feedback, and also D. M. ARROO and S. BRAMWELL (University College London) for useful discussions on topological sector.

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