Singular perturbations and Lindblad-Kossakowski differential equations

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Abstract—We consider an ensemble of quantum systems whose average evolution is described by a density matrix, solution of a Lindblad-Kossakowski differential equation. We focus on the special case where the decoherence is only due to a highly unstable excited state and where the spontaneously emitted photons are measured by a photo-detector. We propose a systematic method to eliminate the fast and asymptotically stable dynamics associated to the excited state in order to obtain another differential equation for the slow part. We show that this slow differential equation is still of Lindblad-Kossakowski type, that the decoherence terms and the measured output depend explicitly on the amplitudes of quasi-resonant applied field, i.e., the control. Beside a rigorous proof of the slow/fast (adiabatic) reduction based on singular perturbation theory, we also provide a physical interpretation of the result in the context of coherence population trapping via dark states and decoherence-free subspaces. Numerical simulations illustrate the accuracy of the proposed approximation for a 5-level systems.

Index Terms—Quantum systems, Lindblad-Kossakowski master equation, singular perturbations, optical pumping, coherent population trapping, adiabatic approximation.

I. INTRODUCTION

Under the usual assumptions of optical pumping and/or coherent population trapping, the Lindblad-Kossakowski master equation describing the dynamics of the density operator admits multiple time-scales. In this paper, we are studying the fast/slow structure resulting from a separation between

• the life-times of the excited and unstable states assumed to be short.
• the oscillation periods, assumed to be long, associated to the energies of the other stable states and to the Rabi pulsation generated by the control, coupling the unstable and stable states in a quasi-resonant way.

Usually, the elimination of the fast dynamics is performed in terms of coherence vector gathering in a single column the coefficients of the density matrix. In this form, the system is not written in a standard form [7, chapter 9] (also called Tikhonov normal form) with a clear splitting of the coherence vector into two sub-vectors: a fast sub-vector and a slow one. Some tedious linear algebra and changes of variables are then needed to put the system into the standard form in order to perform the adiabatic (quasi-static approximation) elimination of the fast dynamics. Moreover with such coherence vector we loose the physical interpretation of the master equation in terms of Hamiltonian and jump operators, explained in [6, chapter 4].

The main contribution of this note is to propose a more intrinsic elimination of the fast part of the dynamics by using only matrix manipulations for systems with a structure sketched on figure 1. The main theoretical guide is the geometric theory of singularly perturbed differential systems initiated in [5] and center manifold techniques to approximate the invariant slow manifold [7, 8]. Such theoretical guides have been already used in [4] in the context of reduction of kinetics combustion schemes. These guides avoid here the use of the coherence vector and provide a slow dynamics that is also a Markovian master equation of Lindblad-Kossakowski type with a slow Hamiltonian and slow jump operators. This slow master equation describes the dynamics of the density matrix of the open quantum system that lives in the Hilbert space spanned by the stable states. As far as we know, such formulation of the slow dynamics is new, even in the physicist community, and could be of some interest for the control. In particular, the controls appear explicitly in the decoherence terms and the output map.

The note is organized as follows. Section II is devoted to the modeling of systems depicted on figure 1 to the three time-scales structure and to the elimination (by averaging, i.e., by the rotating wave approximation usually used by physicists) of the fastest time-scale attached to the transition frequencies between the stable and unstable states. The resulting model, equation (8) with complex value controls $\Omega_k$ and measured output $y$, still admits two time-scales, an asymptotically stable fast part and a slow part. Extraction of the slow part is the object of section III where the approximation Theorem 1 is proved. For readers not interested by these technical developments, we have summarized at the end of this section III the main formula for deriving the slow master equation (23) from the original slow/fast one. In section IV we compute the slow approximation when the Hamiltonian $H$ corresponds to (8) and provides physical interpretations in terms of slow Hamiltonian and slow jump operator depending directly on the control amplitudes $\Omega_k$. In section V we compare, numerically on a five-level system, the slow/fast master equation with the slow one.

A preliminary version of these results can be found in [8]. The authors thank Guilhem Dubois from LKB for several discussions on the physics underlying coherence population trapping.
II. THE THREE TIME-SCALE MASTER EQUATION

Such master equations typically describe coherent population trapping when a laser irradiates an \((N + 1)\)-level system \[2\]. The system is composed of \(N\) (fine or hyperfine) ground states \((|g_k\rangle)\) having energy separations in the radio-frequency or microwave region, and an excited state \(|e\rangle\) coupled to the lower ones by optical transitions at frequencies \((\omega_k)\) (see Figure 1). Naturally, the decay times for the optical coherences are assumed to be much shorter than those corresponding to the ground state transitions (here assumed to be metastable).

The quantum Markovian master equation of Lindblad-Kossakowski type, modeling the evolution of a statistical ensemble of identical systems given by Figure 1, reads (see [6, chapter 4] for a tutorial and up-to-date exposure on such master equations):

\[
\frac{d}{dt}\rho = -i\frac{\hbar}{\delta} [H_0 + u H_1, \rho] + \frac{1}{2} \sum_{k=1}^{N} \Gamma_k (2Q_k \rho Q_k^\dagger - Q_k^\dagger Q_k \rho - \rho Q_k^\dagger Q_k)
\]

\[
y = \sum_{k=1}^{N} \Gamma_k \text{Tr} (Q_k^\dagger Q_k \rho)
\]

where \(u \in \mathbb{R}\) is the controlled input (laser field) and \(y \geq 0\) is the measured output (number of photons per time unit spontaneously emitted from the excited state \(|e\rangle\)). Here the Hermitian operators \(H_0\) and \(H_1\) are, respectively, the free Hamiltonian and the interaction Hamiltonian with a coherent source of photons \(u(t) \in \mathbb{R}\):

\[
H_0 = \lambda |e\rangle \langle e| + \sum_{k=1}^{N} \lambda_k |g_k\rangle \langle g_k|
\]

\[
H_1 = \frac{\mu_k}{\hbar} \langle g_k | \langle e| + |e\rangle \langle g_k|
\]

The \(\mu_k\) being coupling and constant parameters. Moreover, the quantum jump operators \(Q_k\) corresponding to the spontaneous emission from the state \(|e\rangle\) towards \(|g_k\rangle\) are given as follows:

\[
Q_k = |g_k\rangle \langle e|.
\]

One easily has the following relations

\[
Q_k Q_l = 0, \quad Q_k^\dagger Q_k = P = |e\rangle \langle e| \quad \forall k \neq l \in \{1, \ldots, N\}.
\]

The transition frequencies, \(\omega_k = \lambda - \lambda_k\) (where \(\lambda\) and \(\lambda_k\) are the eigenvalues of \(\frac{\hbar}{\delta}\) corresponding to the energy levels \(|e\rangle\) and \(|g_k\rangle\), respectively), are supposed to be much larger than the decoherence rates \(\Gamma_k\). The control field \(u(t)\) is assumed to be in the quasi-resonant regime with respect to the natural frequencies of the system:

\[
u(t) = \sum_{k=1}^{N} u_k e^{i(\omega_k - \delta_k)t} + u_k^* e^{-i(\omega_k - \delta_k)t}
\]

where the complex amplitudes \(u_k\) are varying slowly and where \(\delta_k\) are the small de-tuning frequencies. We have thus three different time scales:

1) The very fast time-scale associated to the optical frequencies \(\Omega_k\).
2) The fast time-scale associated to the life times of the excited state \(|e\rangle\), \(\Gamma_k\).
3) The slow time-scale associated to the laser amplitude \(|\mu_k u_k|\) and to the other atomic transition frequencies \(\omega_{kl} = \omega_k - \omega_l\), \(k \neq l\).

We are interested here by the slow time-scale of system \([1]\) where the control \(u(t)\) is given by (4) with the following timescales separation:

\[
|\omega_{kl}|, |\mu_k u_k| \ll \Gamma_k, \ll \omega_{kl}, \quad \text{and} \quad \left| \frac{\partial}{\partial t} u_k \right| \ll \Gamma_k |u_k|
\]

with \(k, l, k', k'' \in \{1, \ldots, N\}, k \neq l\).

Elimination of the fastest time-scales is standard. It corresponds to the so-called rotating wave approximation and can be justified by averaging techniques. This is not the object of this note and thus we just recall here the resulting master equation:

\[
\frac{d}{dt}\rho = -i\frac{\hbar}{\delta} [H, \rho] + \frac{1}{2} \sum_{k=1}^{N} \Gamma_k (2Q_k \rho Q_k^\dagger - Q_k^\dagger Q_k \rho - \rho Q_k^\dagger Q_k).
\]

Calculating the secular terms of \(ue^{-i\frac{\mu_k}{\hbar}t}H_1 e^{i\frac{\mu_k}{\hbar}t}\), the effective Hamiltonian \(H\) is given as follows:

\[
H = \sum_{k=1}^{N} \delta_k |g_k\rangle \langle g_k| + \Omega_k |g_k\rangle \langle e| + \Omega_k^* |e\rangle \langle g_k|.
\]

with \(\Omega_k = \mu_k u_k\). Note that the measured output remains unchanged

\[
y = \sum_k \Gamma_k \text{Tr} (P \rho) = \sum_k \Gamma_k \text{Tr} (|e\rangle \langle e| \rho).
\]
We are led to the following master equation:

\[
\frac{d}{dt}\rho = -i\left[\sum_{k=1}^{N}\delta_{k}^{g}\langle g_{k}\rangle + \Omega_{k}\langle g_{k}\rangle + \Omega_{k}^{\dagger}\langle e\rangle + \Omega_{k}^{\dagger}\langle g_{k}\rangle, \rho\right] + 2\sum_{k=1}^{N}\Gamma_{k}\left(\langle e\rangle|\rho|e\rangle - \langle e\rangle|\rho - \rho|e\rangle\langle e\rangle\right),
\]

\[
y = \left(\sum_{k=1}^{N}\Gamma_{k}\right)\text{Tr}\left(|e\rangle\langle e|\rho\right)
\]

where the \(\Omega_{k}\)'s are the laser de-tunings and where the two time-scales separation results from:

\[
|\delta_{k}|,|\Omega_{k}| \ll \Gamma_{k}\quad \text{and} \quad \left|\frac{d}{dt}\Omega_{k}\right| \ll \Gamma_{k}'|\Omega_{k}|
\]

for \(k, k' \in \{1, \ldots, N\}\).

### III. Slow/fast reduction

We can therefore take \(\Gamma_{k} = \Gamma_{k}/\epsilon\) where \(\epsilon\) is a small positive parameter. Thus we have a master equation with the following structure:

\[
\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho] + \sum_{k=1}^{N}\frac{\Gamma_{k}}{2\epsilon}\left(2\langle Q_{k}\rho Q_{k}^{\dagger}\rangle - Q_{k}^{\dagger}Q_{k}\rho - \rho Q_{k}^{\dagger}Q_{k}\right),
\]

where \(\Gamma_{k}\)'s and \(H\) (given by (6) for example) have the same orders of magnitude but where \(\epsilon > 0\) is a small parameter.

Define, with \(P = |e\rangle\langle e|\),

\[
\rho_{f} = P\rho + P\rho P - P\rho P
\]

\[
\rho_{s} = (1 - P)\rho(1 - P) + \frac{\epsilon}{\hbar}\sum_{k=1}^{N}\Gamma_{k}Q_{k}\rho Q_{k}^{\dagger}
\]

We have

\[
\rho = \rho_{s} + \rho_{f} - \frac{1}{\epsilon}\left(\sum_{k=1}^{N}\Gamma_{k}\right)\sum_{k=1}^{N}\Gamma_{k}Q_{k}\rho_{f}Q_{k}^{\dagger}
\]

and therefore \(\rho \mapsto (\rho_{f}, \rho_{s})\) is a bijective map. This map is a sort of “change of variables” decoupling the slow part from the fast part of the dynamics. Note that, in the slow part, \(\rho_{s}\), we have somehow removed the fast dynamics associated to the optical state \(|e\rangle\). Indeed, this change of variable leads to a standard form:

\[
\frac{d}{dt}\rho_{f} = -\left(\sum_{k=1}^{N}\Gamma_{k}\right)(\rho_{f} + P\rho_{f}P)
\]

\[
-\frac{i}{\hbar}(P[H, \rho] + [H, \rho]P - P[H, \rho]P),
\]

\[
\frac{d}{dt}\rho_{s} = (1 - P)[H, \rho](1 - P) + \frac{1}{\epsilon}\left(\sum_{k=1}^{N}\Gamma_{k}\right)\sum_{k=1}^{N}\Gamma_{k}Q_{k}[H, \rho]Q_{k}^{\dagger}
\]

where \(\frac{1}{\epsilon}\) only appears in first equation defining \(\frac{d}{dt}\rho_{f}\). Therefore \(\rho_{f}\) is associated to the fast part of the dynamics and \(\rho_{s}\) represents the slow part. The fast part is asymptotically stable because \(-\left(\sum_{k=1}^{N}\Gamma_{k}\right)(\rho_{f} + P\rho_{f}P)\) defines a negative definite super-operator on the space of Hermitian operators:

\[
\text{Tr}\left(-\left(\rho_{f} + P\rho_{f}P\right)\rho_{f}\right) = -\|\rho_{f}\|^{2} + \|P\rho_{f}P\|^{2}.
\]

Moreover its inverse is given by:

\[
X \mapsto X - \frac{1}{2}PXP.
\]

Here we can apply the slow manifold approximation described in the Appendix A. Computing the first order terms, we find the following approximation for \(\rho_{f}\) with respect to \(\rho_{s}\):

\[
\rho_{f} = \frac{-2\epsilon}{\hbar}\left(\sum_{k=1}^{N}\Gamma_{k}\right)(PH\rho_{s} - \rho_{s}PH) + O(\epsilon^{2}).
\]

Inserting now the equations (11) into the equation (13), we have:

\[
\frac{d}{dt}\rho_{s} = \frac{-2\epsilon}{\hbar}\left(\sum_{k=1}^{N}\Gamma_{k}\right)(PH\rho_{s} - \rho_{s}PH) + O(\epsilon^{2}).
\]

Applying now the first order approximation (15), and after some simple but tedious computations, we have

\[
\frac{d}{dt}\rho_{s} = \frac{2\epsilon}{\hbar^{2}}\left(1 - P\right)[H, \rho_{s}](1 - P)
\]

\[
-\frac{4\epsilon}{\hbar^{2}}\sum_{k=1}^{N}\Gamma_{k}Q_{k}H\rho_{s}Q_{k}^{\dagger} + O(\epsilon^{2}).
\]

Here, we have in particular applied (16) as well as the fact that \(Q_{k}P = Q_{k}\). Continuing the computations, we have

\[
\frac{d}{dt}\rho_{s} = \frac{-2\epsilon}{\hbar^{2}}\left(1 - P\right)[H, \rho_{s}]
\]

\[
+\frac{2\epsilon}{\hbar^{2}}\sum_{k=1}^{N}\Gamma_{k}Q_{k}^{\dagger}[H, \rho_{s}]Q_{k}^{\dagger} - Q_{k}^{\dagger}\rho_{s}Q_{k}^{\dagger}Q_{k}^{\dagger}Q_{k}^{\dagger}Q_{k}^{\dagger}Q_{k}^{\dagger}Q_{k}^{\dagger}
\]

where we have defined

\[
\overline{\Gamma} = (1 - P)[H, \rho_{s}]
\]

and

\[
\overline{Q}_{k} = \frac{1}{\hbar}\left(\sum_{k=1}^{N}\Gamma_{k}\right)(1 - P)Q_{k}H(1 - P).
\]
Note that
\[ \overrightarrow{Q^\dagger_k Q_k} = \frac{1}{\hbar^2 \left( \sum_{k=1}^{N} \Gamma_k \right)} T (1 - P) H P H (1 - P), \]

which, in particular, allows us passing from (17) to (18).

The situation is different for the measured output \( y \). We have:
\[ y(t) = \frac{1}{\epsilon} \left( \sum_{k=1}^{N} \Gamma_k \right) \text{Tr} (P \rho) = \frac{1}{\epsilon} \left( \sum_{k=1}^{N} \Gamma_k \right) \text{Tr} (P \rho_f) \]
\[ = \frac{-2i}{\hbar} \text{Tr} (P (PH \rho_s - \rho_s HP)) + O(\epsilon). \]

But \( \text{Tr} (P (PH \rho_s - \rho_s HP)) = 0 \). We should therefore consider the second order terms otherwise the first order approximation yields \( y = 0 \). Using the Appendix A simple but tedious computations end up by the following natural approximation:
\[ y(t) = 4\epsilon \left( \sum_{k=1}^{N} \Gamma_k \right) \text{Tr} (P \rho_s) + O(\epsilon^2), \tag{21} \]

where we have defined
\[ P = \overrightarrow{Q^\dagger_k Q_k} \frac{1}{\hbar^2 \left( \sum_{k=1}^{N} \Gamma_k \right)} T (1 - P) H P H (1 - P). \]

In order to derive (21), we only need to apply (26) with the appropriate values of the functions given in (12) and (13) and the inverse map given in (14).

We can therefore prove the following theorem:

**Theorem 1.** Consider \( \rho \) the solution of the Markovian master equation (9) and \( \rho_s \) the solution of the slow master equation (18) with (19) and (20). Assume for the initial states \( |\rho(0) - \rho_s(0)\rangle = \sqrt{\text{Tr}} ((|\rho(0) - \rho_s(0)\rangle \langle \rho(0) - \rho_s(0)|)) = O(\epsilon) \). Then
\[ |\rho(t) - \rho_s(t)\rangle = \sqrt{\text{Tr}} ((|\rho(t) - \rho_s(t)\rangle \langle \rho(t) - \rho_s(t)|)) = O(\epsilon) \]
on a time scale \( t \sim 1/\epsilon \).

Moreover the output \( y(t) \) of the system (given by (7)) may be written as in (21).

**Remark 1.** Note that, the approximation of this theorem is stronger than the usual one only ensuring an error of order \( O(\epsilon) \) on a finite time \( T \) rather than on a time scale of \( t \sim T/\epsilon \). This stronger result is due to the Hamiltonian structure of the dominant part of the dynamics.

**Proof:** Applying (11) and the singular perturbation theory of the appendix A we have \( \rho(t) = \tilde{\rho}_s(t) + O(\epsilon) \) where
\[ \frac{d}{dt} \tilde{\rho}_s = \frac{i}{\hbar} \left[ \mathcal{H}, \tilde{\rho}_s \right] + 2\epsilon \left( \sum_{k=1}^{N} \Gamma_k \right) \left( 2Q_k \tilde{\rho}_s f_k^\dagger - Q_k^\dagger \tilde{\rho}_s f_k - \tilde{\rho}_s Q_k^\dagger f_k^\dagger \right) + O(\epsilon^2), \]
\[ \tilde{\rho}_s(0) = \rho(0). \tag{22} \]

Denoting by \( \tilde{\rho}_s = \rho_s - \rho_s \), we have
\[ \frac{d}{dt} \text{Tr} \left( \tilde{\rho}_s^2 \right) \leq \]
\[ 8\epsilon \left( \sum_{k=1}^{N} \Gamma_k \right) \left( \text{Tr} \left( Q_k \tilde{\rho}_s f_k Q_k^\dagger \tilde{\rho}_s f_k \right) - \text{Tr} \left( Q_k \tilde{\rho}_s f_k Q_k^\dagger \tilde{\rho}_s f_k \right) \right) + \text{Tr} \left( O(\epsilon^2) \delta \rho_s \right). \]

This, together with Cauchy-Schwarz inequality, implies
\[ \text{Tr} \left( \tilde{\rho}_s^2 (t) \right) \leq \text{Tr} \left( \tilde{\rho}_s^2 (0) \right) + \epsilon L \int_0^t \text{Tr} \left( \delta \rho_s \delta \rho_s \right) \frac{(\lambda)}{2} \text{d}t \]
\[ + \epsilon^2 \int_0^t \text{Tr} \left( \delta \rho_s \delta \rho_s \right) \frac{(\lambda)}{2} \text{d}t, \]

where \( L \) and \( C \) are positive constants. Note that, \( \delta \rho_s^2 \) being definite positive, we have
\[ \text{Tr} \left( \delta \rho_s \delta \rho_s \right) \leq \text{Tr} \left( \tilde{\rho}_s \tilde{\rho}_s \right). \]

Therefore, noting \( \xi = \sqrt{\text{Tr} \left( \tilde{\rho}_s \tilde{\rho}_s \right) } \), we have
\[ \xi^2(t) \leq \xi^2(0) + \epsilon L \int_0^t \xi^2(\tau) \frac{(\lambda)}{2} \text{d}t + \epsilon^2 C \int_0^t \xi(\tau) \frac{(\lambda)}{2} \text{d}t. \]

Denoting \( \zeta = \xi(t) + \frac{C}{2\epsilon} \), some simple computations lead to
\[ \zeta^2(t) \leq 2\xi^2(0) + 2\epsilon L \int_0^t \xi^2(\tau) \frac{(\lambda)}{2} \text{d}t - \frac{C^2}{2L^2} t + \frac{C^2}{2L^2} \epsilon^2 \]
\[ \leq \xi^2(0) + \frac{C^2}{2L^2} \epsilon^2 + 2\epsilon L \int_0^t \xi^2(\tau) \frac{(\lambda)}{2} \text{d}t. \]

Applying the Gronwall lemma, we have
\[ \zeta^2(t) \leq \left[ \xi^2(0) + \frac{C^2}{2L^2} \epsilon^2 \right] e^{2\epsilon L t}. \]

Noting that, by the Theorem’s assumption, \( \xi(t) = O(\epsilon) \), we have \( \xi(t) = O(\epsilon) \) on a time scale of \( t \sim 1/\epsilon \). As \( \xi(t) = \zeta(t) + O(\epsilon) \), this trivially finishes the proof.

From a practical point of view, the main result of this section is as follows. The correct slow approximation (also called by physicists adiabatic approximation) of the system described by
\[ \frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] \]
\[ + \sum_{k=1}^{N} \frac{\Gamma_k}{2} \left( 2Q_k \rho Q_k^\dagger - Q_k^\dagger \rho Q_k - \rho Q_k^\dagger Q_k \right) \]
with \( Q_k = \langle \gamma_k | \rho \rangle \) and where the \( \Gamma_k \)’s are much larger than \( \frac{\hbar}{\mathcal{T}} \) and where the output reads \( y = \sum_k \Gamma_k \text{Tr} \left( Q_k \rho \right) \), is given by
\[ \frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] \]
\[ + \sum_{k=1}^{N} \frac{4\Gamma_k}{2} \left( 2Q_{s,k} \rho Q_{s,k}^\dagger - Q_{s,k}^\dagger \rho Q_{s,k} - \rho Q_{s,k}^\dagger Q_{s,k} \right) \]
\[ - \sum_{k=1}^{N} \frac{4\Gamma_k}{2} \left( 2Q_{s,k} \rho Q_{s,k}^\dagger - Q_{s,k}^\dagger \rho Q_{s,k} - \rho Q_{s,k}^\dagger Q_{s,k} \right) \]
\[ \tag{23} \]
where \( \rho_s \) is the density operator associated to the space spanned by the \( |g_k\rangle \)'s, where the slow Hamiltonian is

\[
H_s = (1 - P)H(1 - P)
\]

and the slow jump operators are

\[
Q_{s,k} = Q_k \frac{H}{\hbar}(1 - P).
\]

Here, we have set \( \Gamma = \sum_k \Gamma_k \) and \( P = |\epsilon\rangle \langle \epsilon| \). The slow approximation of the measured output is still given by the standard formula

\[
y_s = \sum_{k=1}^{n} 4\Gamma_k \text{Tr} \left( Q_{s,k}^\dagger Q_{s,k} \rho_s \right).
\]

IV. PHYSICAL INTERPRETATION

In this section, we provide a physical interpretation of the last section’s result for the particular Hamiltonian of the system \( \mathcal{H} \). We get

\[
H_s = \sum_k \delta_k |g_k\rangle \langle g_k|
\]

and

\[
Q_{s,k} = \sqrt{\sum_l |\Omega_l|^2} |g_k\rangle \langle b_{\Omega_l}| \quad \text{with} \quad |b_{\Omega_l}| = \frac{\sum_l \Omega_l |g_l\rangle}{\sqrt{\sum_l |\Omega_l|^2}}.
\]

Let us set

\[
\gamma_k = \frac{\sum_l |\Omega_l|^2}{(\sum_l \Gamma_l)^2} 4\Gamma_k
\]

the slow master equation reads

\[
\frac{d}{dt} \rho_s = -i \left[ \sum_k \delta_k |g_k\rangle \langle g_k|, \rho_s \right] + \sum_k \frac{\gamma_k}{2} \left( 2 |b_{\Omega_l}\rangle \langle b_{\Omega_l}| |g_k\rangle \langle g_k| - |b_{\Omega_l}\rangle \langle b_{\Omega_l}| \rho |b_{\Omega_l}\rangle \langle b_{\Omega_l}| \right)
\]

with \( y_s = (\sum_k \gamma_k) |b_{\Omega_l}\rangle \rho_s |b_{\Omega_l}\rangle \). Thus, whenever all the detunings \( \delta_k \) vanish, the unitary state \( |b_{\Omega_l}\rangle \) is the bright state and the vector-space orthogonal to \( |b_{\Omega_l}\rangle \) is a decoherence free space since on this sub-space, the Lindblad-Kossakowski terms identically vanish and the output \( y \) is null. Notice that the controls \( \Omega_k \) appear only in the decoherence terms and have disappeared form the slow Hamiltonian.

If we restrict ourselves to the case of a 3-level \( \Lambda \)-system \( (N = 2) \), we have \( H_s = \frac{\Gamma}{2} (|g_1\rangle \langle g_1| - |g_2\rangle \langle g_2|) \),

\[
|b_{\Omega_l}\rangle = \frac{\Omega_1}{\sqrt{\Omega_1^2 + \Omega_2^2}} |g_1\rangle + \frac{\Omega_2}{\sqrt{\Omega_1^2 + \Omega_2^2}} |g_2\rangle
\]

is the bright state of the \( \Lambda \)-system, as in the context of the coherent population trapping. As it can be seen easily, whenever no de-tuning is admitted \( (\delta = 0) \), the dark state

\[
|d\rangle = |b\rangle^\perp = \frac{\Omega_2^*}{\sqrt{\Omega_1^2 + \Omega_2^2}} |g_1\rangle - \frac{\Omega_1^*}{\sqrt{\Omega_1^2 + \Omega_2^2}} |g_2\rangle
\]

is the only equilibrium state of the slow dynamics.

V. NUMERICAL SIMULATIONS

Finally let us check the relevance of the adiabatic reduction result of the Section [IV] in a simulation. Here, we consider a \((4+1)\)-level system given by the following parameters

\[
\begin{align*}
\delta_1 &= 0.5 \\
\delta_2 &= 1.2 \\
\delta_3 &= 0.7 \\
\delta_4 &= 1.0 \\
\Omega_1 &= 1.0 \\
\Omega_2 &= 1.2 \\
\Omega_3 &= 1.1 \\
\Omega_4 &= 1.3 \\
\Gamma_1 &= 5.0 \\
\Gamma_2 &= 4.0 \\
\Gamma_3 &= 7.0 \\
\Gamma_4 &= 5.0
\end{align*}
\]

The simulations of Figure 2 illustrate the output signal derived from the reduced slow dynamics, \( y_s \), versus the slow/fast dynamics, \( y \). The simulation time is taken to be \( 2.5T_s \), where \( T_s \) is the time scale of the slow system determined by

\[
T_s = \frac{\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4}{\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2}.
\]

The initial conditions are identical \( \rho(0) = \rho_s(0) = \sum_{g_k} |g_k\rangle \langle g_k| \). We observe, for the slow/fast dynamics, a first fast transient corresponding to the relaxation time of the fast dynamics for \( t \) between 0 and \( 1/\Gamma_k \), i.e. \( t \in [0, 1/4] \), and then two slow transients very similar for both master equations: the slow approximation is clearly valid only for time-scale much larger than \( 1/\Gamma_k \).

![Fig. 2. The output of the reduced slow dynamics (solid line) versus the output for the (4+1)-level system (dashed line). The parameters are listed in (24). Even if the time-scale separation is not so large with an \( \epsilon \) around \( \frac{\Gamma}{\hbar} \), such adiabatic approximation captures quite precisely the slow part of the dynamics, as stated by theorem (4).](image)

CONCLUSION

We observed that for an ensemble of independent and identical quantum systems, and whenever the decoherence dynamics due to the measurement is much faster than the other dynamics, the adiabatic approximation helps us to find the slow dynamics as well as the measurement result with respect to the slow dynamics. Note that in this new system, the decoherence term can be removed in a first order approximation. We obtain therefore a system of the form

\[
\frac{d}{dt} \rho_s = -i \hbar [H_s, \rho_s],
\]

where the control appears linearly in the reduced Hamiltonian \( H_s \). This system corresponds to a bilinear system with the wavefunctions as state variables. Furthermore, we have access to a measurement \( y \) given by the reduced slow evolution. We
can henceforth consider a control problem with continuous measurement associated to this system.

Notice that, by some simple but tedious computations, one can extend the result of this paper to the more general case of an $N + M$-level system with $N$ metastable ground states and $M$ highly unstable excited states.

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APPENDIX

This appendix has for goal to remind an approximation technique that can be perfectly justified using the geometrical tools of singular perturbation and the center manifold [7], [3], [4].

Consider the slow/fast system ($x$ and $y$ are of arbitrary dimensions, $f$ and $g$ are regular functions)

$$\frac{d}{dt} x = f(x, y), \quad \frac{d}{dt} y = \frac{1}{\epsilon} Ay + g(x, y)$$

where $x$ and $y$ are respectively the slow and fast states (Tikhonov coordinates), all the eigenvalues of the matrix $A$ have strictly positive real parts, and $\epsilon$ is small strictly positive parameter. Therefore the invariant attractive manifold admits for equation

$$y = \epsilon A^{-1} g(x, 0) + O(\epsilon^2)$$

and the restriction of the dynamics on this slow invariant manifold reads

$$\frac{d}{dt} x = f(x, \epsilon A^{-1} g(x, 0)) + O(\epsilon^2)$$

$$= f(x, 0) + \epsilon \frac{\partial f}{\partial y}|_{(x,0)} A^{-1} g(x, 0) + O(\epsilon^2).$$

The Taylor expansion of $g$ can be used to find the higher order terms. For example, the second order term in the expansion of $y$ is given by:

$$y = \epsilon A^{-1} g(x, 0) +$$

$$\epsilon^2 A^{-1} \left( \frac{\partial g}{\partial y}|_{(x,0)} A^{-1} g(x, 0) - A^{-1} \frac{\partial g}{\partial x}|_{(x,0)} f(x, 0) \right) + O(\epsilon^3),$$

and so on.