Symbolic Dynamics of Homoclinic Orbits in a Symmetric Map

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Abstract

Symbolic dynamics for homoclinic orbits in the two-dimensional symmetric map, $x_{n+1} + cx_n + x_{n-1} = 3x_n^3$, is discussed. Above a critical $c^*$, the system exhibits a fully-developed horse-shoe so that its global behavior is described by a complete ternary symbolic dynamics. The relative location of homoclinic orbits is determined by their sequences according to a simple rule, which can be used to numerically locate orbits in phase space. With the decrease of $c$, more and more pairs of homoclinic orbits collide and disappear. Forbidden zone in the symbolic space induced by the collision is discussed.
1 Introduction

This paper concerns locating homoclinic orbits (HOs) for maps. The topic received some recent interests because it is closely related to finding spatially localized oscillatory solutions in one-dimensional lattices (see [1] and references therein). Numerical methods usually involve a shooting strategy or an assumption of the initial topology of HOs [2, 3, 4, 5]. In both cases, a deep understanding of the qualitative behavior of maps is important. As a coarse-grained description of dynamics, symbolic dynamics (SD) has been productively applied to one- and two-dimensional maps [8]. The effectiveness of SD relies on the hyperbolic nature of the underlying dynamics. Therefore, SD should be a powerful tool to describe the qualitative behavior of HOs when the system is dominated by unstable motion. In this paper, we shall discuss this through a specific map.

The system we consider is the map with its orbits \( \{x_n\} \) obeying the relation

\[ x_{n+1} + cx_n + x_{n-1} = 3x_n^3, \]  

which has been studied in [2, 3]. The coefficient 3 can be re-scaled to an arbitrary positive number, so only \( c \) is the parameter of dynamical significance. Let \( (a_n, b_n) = (x_n, x_{n-1}) \), Eq.(1) can be rewritten as a two-dimensional conservative map,

\[ (a_{n+1}, b_{n+1}) = F(a_n, b_n) = (3a_n^3 - ca_n - b_n, a_n). \]  

The system contains a fixed point \( O = (0, 0) \), which is unstable when \( |c| > 2 \). As in Ref. [7], we consider the orbits that are homoclinic to point \( O \) at \( c > 2 \). The paper is organized as follows. Section 2 gives a preliminary discussion of the map. Section 3 contains a general description of its SD. In Sec. 4, symmetric HOs are discussed in detail. This is followed by a brief summary.

2 Preliminary

2.1 Symmetries

Map (1.2) is invariant under the spatial reversion \((a, b) \to (-a, -b)\). Moreover, it has two inverse symmetries, \( I_1 : (a, b) \to (b, a) \) and \( I_2 : (a, b) \to (-b, -a) \). \( I_1 \) and \( I_2 \) satisfy \( I_1 F I_1 = I_2 F I_2 = F^{-1} \) and \( I_1^2 = I_2^2 = I \). The inverse symmetries determine four fundamental symmetry lines,
\[
\begin{align*}
sl_1 &: I_1(a, b) = F(a, b) \quad \text{or} \quad 3a^3 - ca - 2b = 0 \\
sl_2 &: I_1(a, b) = (a, b) \quad a - b = 0 \\
sl_3 &: I_2(a, b) = F(a, b) \quad a = 0 \\
sl_4 &: I_2(a, b) = (a, b) \quad a + b = 0
\end{align*}
\]

For points on these lines, their forward and backward trajectories are related by simple rules. Specifically, \(F^k(X)\) coincides with \(I_1 F^{1-k}(X), I_1 F^{-k}(X), I_2 F^{1-k}(X)\) or \(I_2 F^{-k}(X)\) if \(X\) belongs to \(sl_1, sl_2, sl_3\) or \(sl_4\), respectively. Therefore, if \(X\) further belongs to the unstable manifold of a fixed point, it must be a homoclinic or heteroclinic point. This property was used to locate HOs with symmetry \([7]\).

### 2.2 Normal form for stable and unstable manifolds

Numerical investigation of HOs relies essentially on a precise computation of the stable and unstable manifolds. For canonic systems, the invariant manifold of an unstable fixed point \(X_0 = (a_0, b_0)\) can be determined by the Birkhoff normal form \([9]\). The manifold is given by the parameterized curve \(X(t) = (a(t), b(t))\) with

\[
\begin{align*}
a(t) &= \sum_{i=0}^{\infty} a_i t^i \\
b(t) &= \sum_{i=0}^{\infty} b_i t^i,
\end{align*}
\]

where coefficients \(\{a_i, b_i\}\) are determined by requiring

\[
X(\lambda t) = F(X(t)).
\]

Here \(\lambda\) is one of the eigenvalues of \(F\) at \(X_0\), specifically, \(|\lambda| > 1\) for the unstable manifold and \(|\lambda| < 1\) for the stable one. Take \(X_0 = O\) as an example. Its unstable manifold can be written as

\[
O_u(t) = (G(\lambda t), G(t)),
\]

where \(\lambda = -(c + \sqrt{c^2 - 4})/2\) and \(G(t) = g_1 t + g_2 t^2 + g_3 t^3 \ldots\) with \(g_1 = 1\) and

\[
g_k = \frac{3}{\lambda^k + \lambda^{-k} + c} \sum_{q+r+s=k} g_q g_r g_s
\]

for \(k > 1\). When \(k \to \infty\), \(g_k\) drops very fast if \(c\) is not too small (e.g. \(c > 3\)). In our numerical study, \(O_u(t)\) at \(t \sim 1\) is computed by the series expansion with a reasonable cutoff. Applying \(F\) on a precisely determined piece of \(O_u(t)\) (2.4), we may obtain \(O_u(t)\) at large \(t\) with high precision.
3 Symbolic dynamics: general description

3.1 Fully-developed horse-shoe and complete symbolic dynamics

The global dynamics of (1.2) at large $c$ is most conveniently accounted for based on the relation between the stable and unstable manifolds of the fixed points $A = (x_d, x_d)$ and $B = (-x_d, -x_d)$, where $x_d = \sqrt{(c + 2)/3}$. Figure 1a shows these invariant manifolds at $c = 5.7$, denoted by $A_s, A_u, B_s$ and $B_u$, respectively. The first part of $A_s$, $ACE$, is approximately a horizontal line. $F^{-1}$ maps $AC$ to $ACE$ and two additional nearly horizontal segments, $EG$ and $GJ$, forming a s-turn. The first three parts of $A_u, B_s$ and $B_u$ can be generated from $A_s$ by taking advantage of the symmetry.

Under the action of $F^{-1}$, the area confined by the first segments of $A_s, A_u, B_s$ and $B_u$ (the diamond-shaped $ACBD$ in Fig. 1a) is transformed to the s-shaped region $ACEGJBDFHK$, indicating that the global dynamics exhibits a well-developed horse-shoe. This is schematically shown in Fig.1b. After taking into consideration of the symmetry, we can see that this geometric structure is guaranteed by the intersection between $EGJ$ consisting of the second and third segments of $A_s$ and $AD$, the first part of $A_u$. The points of intersection correspond to two orbits that are homoclinic to point $A$. In fact, the two points lie on $sl_1$ and hence correspond to HOs with reflection symmetry. With the decrease of $c$, the two points move toward each other and collide at $c = c^* \approx 5.453254835$, leading a tangency of $A_s, A_u$ and $sl_1$.

As a prototype of chaotic dynamics, the global behavior of horse-shoe is well-known. In our case, it is conveniently described by a complete ternary SD.

Symbolic representation of points All stable (or unstable) manifolds, that generally refer to invariant vector field and shrink under the action of $F$ (or $F^{-1}$)[8], can be naturally divided into three groups and coded with $+, 0$ and $-$, respectively (see Fig. 1). Correspondingly, a point $X$ in phase space is represented by a bilateral symbolic sequence

$$S(X) = \ldots s_{-2}s_{-1} \bullet s_0s_1s_2\ldots \equiv S_-(X) \bullet S_+(X), \quad (3.1)$$

where $s_k \in \{+, 0, -\}$ is the code for the stable (or unstable) manifold on which $F^k(X)$ (or $F^{k+1}(X)$ ) is located. For example, the symbolic sequence for points $C, J, D$ and $K$ in Fig.1a are $-\infty \bullet +\infty$, $-\infty \bullet -\infty$, $+\infty \bullet -\infty$ and $+\infty \bullet +\infty$, respectively.

The completeness of SD means that each sequence is realized by at least one phase space.
point. By definition we have
\[ S \circ F(X) = F \circ S(X), \]  
where \( F \) is the right shift of “•” with respect to the sequence. Therefore, a bilateral symbolic sequence without “•” can be used to represent an orbit without its initial point specified. For example, the symbolic sequences of the fixed points \( A, B \) and \( O \) are +\( \infty \), −\( \infty \) and 0\( \infty \), respectively.

Order of symbolic sequences
A point is indicated by its forward sequence \( S_+ \) (hence stable manifold) and backward sequence \( S_- \) (unstable manifold). The location of stable manifolds in the phase space induce a nature order to forward symbolic sequences, \( \bullet s_0 s_1 s_2 ... > \bullet s'_0 s'_1 s'_2 ... \) if the stable manifold of \( \bullet s_0 s_1 s_2 ... \) is above that of \( \bullet s'_0 s'_1 s'_2 ... \). The assignment of letters implies
\[ \bullet + > \bullet 0 > \bullet - , \]
i.e., any sequence beginning with “+” is larger than any sequence beginning with “0”, which is in turn larger than any sequence beginning with “−”. The comparison of two arbitrary sequences relies on the fact that + and − preserve the order while those of point \( O \) are negative.) Assume that \( S = s_1 s_2 ... s_n s_{n+1} ... \) and \( S' = s_1 s_2 ... s_n s'_{n+1} ... \) and \( \bullet s_{n+1} > \bullet s'_{n+1} \), then \( \bullet S > \bullet S' \) (or \( \bullet S < \bullet S' \) if the number of “0” in \( s_1 s_2 ... s_n \) is even (or odd). For example, the first three segments of \( A_s \) correspond to \( +\infty, 0+\infty \) and \( - +\infty \), respectively. It can be easily verified that the first among the three is the largest forward sequence, and that there is no sequence which is smaller than the second and at the same time larger than the third. Similarly, backward sequences can be ordered in accordance with the location of their unstable manifolds in the phase space. We have
\[ ... s_3 s_2 s_1 \bullet > ... s'_3 s'_2 s'_1 \bullet \Leftrightarrow \bullet s_1 s_2 s_3 ... > \bullet s'_1 s'_2 s'_3 ... . \]
We shall indicate a point, an orbit or a piece of invariant manifold with the same symbolic sequence if there is no danger of causing confusion.

3.2 Under-developed horse-shoe and truncated symbolic dynamics

When \( c < c^* \), the second and third segments of \( A_s \) do not cross the first segment of \( A_u \). In this case, there are some “forbidden” sequences, e.g. \( +\infty 0+\infty \) and \( +\infty - +\infty \), which cannot be realized by any real orbits. Consider a sequence \( S = S_- S_+ \). When \( c > c^* \), it corresponds to a real orbit on which there is a point being the intersection of stable manifold \( \bullet S_+ \) and unstable manifold \( S_- \). When we reduce \( c \), the two segments of manifold will vary continuously with \( c \) and they must be tangent at a critical value of \( c = c_b \) before \( S \) is forbidden. The tangency can
be viewed as a collision of two orbits. Since the two colliding orbits are infinitely close when 
\( c \to c^* \), generally their corresponding sequences differ by only one symbol, say, one is \( S_10S_2 \) and 
the other is \( S_1 - S_2 \). (A logically more meaningful statement is that, to maintain the simple 
order of invariant manifolds, the partition line that separates \( S_10\cdot \) and \( S_1 - \cdot \) should pass the 
tangent point\([3]\). The tangency of stable manifold \( \cdot S_2 \) with unstable manifolds \( S_10\cdot \) and \( S_1 - \cdot \) 
implies a forbidden zone in the space of sequences or symbolic space\([3]\). Specifically speaking, 
sequence \( S' = S'_1 \cdot S'_2 \) with 
\[
S_1 - \cdot \leq S'_1 \cdot \leq S_10\cdot \quad \text{and} \quad \cdot S'_2 \geq \cdot S_2
\]
(3.5a) 
corresponds to no point in the phase space (Fig.2a). Similarly, if the two colliding orbits are 
\( S_10S_2 \) and \( S_1 + S_2 \), the forbidden zone is given by 
\[
S_1 + \cdot \geq S'_1 \cdot \geq S_10\cdot \quad \text{and} \quad \cdot S'_2 \leq \cdot S_2.
\]
(3.5b) 
Of course, any sequence \( S' \) with certain \( F^k(S') \) satisfies (3.5) is also forbidden.

Due to symmetry, it is not exceptional in our case that the sequences of the two colliding 
orbits differ by two symbols. The forbidden zone due to such collision will be also discussed.

4 Symmetric homoclinic orbits

We now focus on the orbits that are homoclinic to the fixed point \( O \). The unstable manifold 
of \( O \) is given by (2.4). From the symmetry, the stable manifold can be written as \( O_s(t) = (G(t),G(\lambda t)) \). By definition, HOs correspond to the solutions of equations 
\( G(\lambda t) = G(t') \) and 
\( G(t) = G(\lambda t') \). However, the equations have little practical use because \( G(t) \) changes violently at 
large \( t \). The more useful knowledge is the global location of the stable and unstable manifolds.

When \( c > c^* \), each sequence \( 0^\infty 0^\infty \) corresponds to a HO. To locate the orbit in the phase 

space, we can scan initial point \( X \) on \( 0^\infty \cdot \), the first segment of \( O_u \), searching for the target 
\( S_+(X) = 0^\infty \). For a symmetric HO, the target can be replaced by \( F^k(X) \) or alternatively 
the scan can be performed on a symmetry line. In all cases, the order of forward sequences can 
be used for an effective searching strategy. Having a HO at large \( c \), we can trace it to small \( c \) 
until it collides with another HO and disappears. We shall not go into details about the locating 
of HOs. We shall focus on the compatibility between HOs at \( c < c^* \).

In the following discussion, the stable and unstable manifolds are always referred to as \( O_s \) 
and \( O_u \). We first give another description of the forbidden zone induced from the tangency of
stable and unstable manifolds, or the collision of two HOs. Let \( X \) and \( X' \) be two homoclinic points which collide each other when \( c \to c_b \). For \( c > c^* \), the stable and unstable manifolds connecting \( X \) and \( X' \) form a closed curve \( O(X, X') \) (see Fig. 2), which shrinks to a single point when \( c = c_b \). Note that both \( O_s \) and \( O_u \) are continuous and self-avoiding curves. Consider a homoclinic point enclosed by \( O(X, X') \) or lying on \( O(X, X') \). Being the intersection of stable and unstable manifolds, the point cannot cross \( O(X, X') \) when \( c \) is continuously varied. The area enclosed by \( O(X, X') \) at \( c > c^* \) vanish before \( c = c_b \). Therefore, \( O(X, X') \) defines the forbidden zone corresponds to the collision of \( X \) and \( X' \).

It can be seen that the forbidden zone defined by \( O(X, X') \) coincides with that defined by (3.5) if \( S(X) \) and \( S(X') \) differ by only one symbol (see Fig.2b). To study the evolution of symmetric HOs, we need also consider the the case that \( S(X) \) and \( S(X') \) differ by two symbols. For this purpose, the structure of stable and unstable manifolds should be studied in detail. Consider the \( 3^k \) segments of stable manifold of the form \( \bullet S0^\infty \), where \( S \) exhausts all the ternary strings with length \( |S| = k \). The relative locations of these segments of manifold in the phase space are determined by the order of \( \bullet S \). We ask that how the \( 3^k \) segments are connected to form a continuous curve. All these segments are generated from \( \bullet 0^\infty \) by \( F^{-k} \), i.e., \( F^{-k}(\bullet S0^\infty) = \bullet S0^\infty \). We may cut the segment \( \bullet 0^\infty \), according to backward sequences, into \( 3^k \) pieces, each of which are coded with \( \bullet S \). The way to joint these \( 3^k \) pieces into the whole segment \( \bullet 0^\infty \) is determined by the order of the \( 3^k \) strings \( S \). After expanding under \( F^{-k} \), the segment \( \bullet 0^\infty \) becomes the \( 3^k \) segments of \( \bullet S0^\infty \), so they should be connected also according to \( \bullet S \). In the simplest case of \( k = 1 \), three segments of \( O_s \) from top to bottom are \( \bullet + 0^\infty \), \( \bullet 0^\infty \) and \( \bullet - 0^\infty \). Forming an s-turn, \( \bullet + 0^\infty \) and \( \bullet - 0^\infty \) joint with \( \bullet 0^\infty \) at its left and right ends, respectively. For \( k = 2 \), the nine segments of \( O_s \) are ordered from top to bottom as

\[ ++, +0, +-, 0-, 00, 0+, -+, -0 \text{ and } --, \]

as shown in Fig. 3 schematically by horizontal lines. In the figure \( S \) is also taken as the abbreviation for \( \bullet S0^\infty \). These segments are connected end to end according to the following order of backward \( \bullet S \):

\[ ++, 0+, --, -0, 00, 0+, --, 0- \text{ and } --. \]

The arrangement is shown in Fig.3. We see that pair \((00, +0)\) joint at their left ends, \((+0, +-)\) joint at their right ends, \((+-, 0-)\) joint at their left ends, and so on. In general, the sequence pair of connected segments of stable manifold take one of the following forms,

\[
\begin{align*}
(i) & \quad S = -^m0S_0 \quad \text{and} \quad S' = -^m + S_0, \\
(ii) & \quad S = +^m0S_0 \quad \text{and} \quad S' = +^m - S_0,
\end{align*}
\]
where \( m \geq 0 \). Segment pair of type (i) are connected at their left ends while pair of type (ii) are connected at their right ends. By taking advantage of the symmetry, a parallel description can be given to the unstable manifold.

Now we study the forbidden zone implied by the collision of two symmetric HOs. Let us first consider two examples. In the first case, \( S(X) = 0^\infty + + + 0^\infty \) and \( S(X') = 0^\infty - + + + - 0^\infty \). From the above discussion, \( 0^\infty \bullet \) and \( 0^\infty - \bullet \) are connected at their upper ends while \( \bullet + + + 0^\infty \) and \( \bullet + + + - 0^\infty \) are connected at their right ends. The forbidden zone enclosed by \( \mathcal{O}(X, X') \) is shown in Fig. 2c. We see that it is the union of two forbidden zones, one is defined by \( \sigma O \) enclosed by \( 0^\infty \bullet \) and the other by \( \sigma O \) enclosed by \( 0^\infty - \bullet \). They are connected at the bottom. On the other hand, \( \bullet 00 - 0^\infty \) and \( \bullet 0 - 0^\infty \) are connected at the right ends. The forbidden zone enclosed by \( \mathcal{O}(X', X'') \) is shown in Fig. 2d, which again turns out to be the union of two regions enclosed by \( \mathcal{O}(X, X'') \) and \( \mathcal{O}(X', X'') \) with \( S(X'') = 0^\infty - 0 \bullet 0 + - 0^\infty \) (or \( 0^\infty - + \bullet 00 - 0^\infty \)).

The next example is \( S(X) = 0^\infty - 0 \bullet 00 - 0^\infty \) and \( S(X') = 0^\infty - + \bullet 0 + - 0^\infty \). Now \( 0^\infty - 0 \bullet \) and \( 0^\infty - + \bullet \) are connected at the bottom. On the other hand, \( \bullet 00 - 0^\infty \) and \( \bullet 0 + - 0^\infty \) are joint neighboring segments on \( O_s \). They are connected by \( \bullet - 0 - 0^\infty \) and \( \bullet - + - 0^\infty \). The area enclosed by \( \mathcal{O}(X, X') \) is shown in Fig. 2d, which again turns out to be the union of two regions enclosed by \( \mathcal{O}(X, X'') \) and \( \mathcal{O}(X', X'') \) with \( S(X'') = 0^\infty - 0 \bullet 0 + - 0^\infty \) (or \( 0^\infty - + \bullet 00 - 0^\infty \)).

In general, suppose \( S(X) = 0^\infty S_1 0 \bullet S_2 0 S_3 0^\infty \) and \( S(X') = 0^\infty S_1 0 \bullet S_2 \sigma_2 S_3 0^\infty \), where \( \sigma_{1,2} \in \{+, -\} \). This covers all the colliding symmetric HOs with two different letters. As seen in the two examples, it can be shown that the forbidden zone enclosed by \( \mathcal{O}(X, X') \) is the union of that defined by \( \mathcal{O}(X, X'') \) and \( \mathcal{O}(X', X'') \), where \( S(X'') = 0^\infty S_1 0 \bullet S_2 \sigma_2 S_3 0^\infty \) (or \( 0^\infty S_1 0 \bullet S_2 0 S_3 0^\infty \)). In other words, when forming the forbidden zone, the collision of \( X \) and \( X' \) can be regarded as the triple collision involving \( X, X' \) and \( X'' \). Since either the pair of \( S(X) \) and \( S(X'') \) or the pair of \( S(X') \) and \( S(X'') \) differ by one symbol, the corresponding forbidden zone is given by \( (3.5) \). Thus, the rule can be used to construct the forbidden zone for the above general case.

We have numerically calculated all symmetric HOs \( 0^\infty S0^\infty \) with \( |S| \leq 7 \) and traced them from \( c = 6 \) to collision parameters \( c_b \). Among the 160 orbits, only \( 0^\infty \pm 0^\infty \) and \( 0^\infty \pm \mp 0^\infty \) persist till \( c = 2 \). We obtained 94 collisions, each of them corresponds to a tangency of \( O_s \) and a symmetry line (see Fig.4 for some examples). Due to symmetry, only half of the collisions are independent. They are ordered with descending \( c_b \) and listed in Tab. 1. We further examine the compatibility between symmetric HOs. Collision of a pair of symmetric HOs may implies the elimination of some others. For example, it can be verified that the forbidden zone of the colliding HO pair \( (S, S') = (+ - - -, +00-) \) includes the HO sequence pair \( (- + - -, -0-) \). Therefore, the collision of HO pair \( (- + - -, -0-) \) should occurs at a larger \( c \) than that for \( (+ - - -, +00-) \). In fact, the former appears as the 9-th collision in Tab. 1 while the latter is the 16-th. We find that all such
restrictions in the 47 pairs of sequences in Tab.1 are in accordance with the order of $c_0$ (Fig.5).

5 Summary

In the above we have discussed the symbolic dynamics of system (1.2) with a focus on the orbits that are homoclinic to point $O = (0,0)$. When $c > c^*$, the global dynamics is described by a complete ternary symbolic dynamics, so that each sequence $0^\infty S 0^\infty$ corresponds a homoclinic orbit. The location of these orbits are qualitatively determined by their corresponding sequences according to simple rules. With the decrease of $c$, more and more symmetric homoclinic orbits collide in pair and disappear. Based on the geometrical constraint that any homoclinic point cannot cross a region enclosed by the stable and unstable manifolds, the compatibility between homoclinic orbits is discussed.

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Figure Captions

Fig. 1 (a) First three segments of $A_s$, $A_u$, $B_s$ and $B_u$ at $c = 5.7$. The dotted lines $(a, b = \pm \sqrt{c}/3)$ give a convenient partition of manifolds. (b) Schematic diagram of hose-shoe. The square $ACBD$ is mapped by $F^{-1}$ to the shaded area.

Fig. 2 Schematic diagrams to show forbidden sequences due to tangency between the stable and unstable manifolds. Figure (a) explains rule (3.5a). The shadowed areas in (b-d) represent the forbidden zones when $X$ and $X'$ collide.

Fig. 3 Arrangement of the first 9 segments of $O_s$.

Fig. 4 Typical tangencies of $O_u$ (thick line) and symmetry line (thin line). Each tangency represents a collision of symmetric homoclinic orbit pair.

Fig. 5 Compatibility between symmetric homoclinic orbit pairs listed in Tab. 1. A dot at $(i, j)$ means that the collision of $i$-th pair implies the elimination the $j$-th pair. Note that all dots are located within the region $i > j$. 
| order | \( c_b \) | \( S \) | \( S' \) | order | \( c_b \) | \( S \) | \( S' \) |
|-------|---------|-------|-------|-------|---------|-------|-------|
| 1     | 5.451097 | + + + + + + + + | + + 0 + + + | 25    | 3.3139443 | + + 0 + + | + + 0 + + + |
| 2     | 5.4158176 | + + + + + + 0 + + | + + 0 + | 26    | 3.3128838 | + + 0 + + | + + 0 + + |
| 3     | 5.3937939 | + + 0 + + + + + + | + + 0 + + + | 27    | 3.3064455 | + + 0 + + | + + 0 + + |
| 4     | 5.106014 | + + + + + + + + + | + + 0 + + + | 28    | 3.3028984 | + + 000 + + | + + 000 + + |
| 5     | 5.0779902 | + + + + + + + + | + + 0 + | 29    | 3.2975459 | + + 0 + + | + + 0 + + |
| 6     | 4.9161551 | + + + + + + + + + | + + 0 + + + | 30    | 3.2907686 | + + 0 + + | + + 0 + + |
| 7     | 4.8549163 | + + + + + + + + | + + 0 + | 31    | 3.2821047 | + + + + + + | + + + + + + |
| 8     | 4.7677 | + + + + + + + + | + + 0 + | 32    | 3.2817081 | + + + + + + | + + + + + + |
| 9     | 4.6777145 | + + + + + + + + | + + 0 + | 33    | 3.2807520 | + + + + + + | + + + + + + |
| 10    | 4.525223 | + + + + + + + + | + + 0 + | 34    | 3.1883361 | + + 0 + + | + + 000 + + |
| 11    | 4.443742 | + + + + + + + | + + + | 35    | 3.9640007 | + + 0 + + | + + 0 + + |
| 12    | 4.3722249 | + + + + + + + + + | + + 0 + | 36    | 3.9614903 | + + 0 + + | + + 0 + + |
| 13    | 4.2904264 | + + + + + + + + | + + + | 37    | 3.9381982 | + + 0 + + | + + 0 + + |
| 14    | 4.1671612 | + + + + + + + + | + + 0 + | 38    | 3.9373534 | + + 0 + + | + + 0 + + |
| 15    | 3.6530814 | + + 0 + + + + + | + + 000 + | 39    | 3.9205856 | + + 0 + + | + + 0 + + |
| 16    | 3.6477729 | + + + + + + + + | + + 0 + | 40    | 3.9083662 | + + 0 + 0 + | + + 0 + + |
| 17    | 3.5589054 | + + 0 + + + + + | + + 000 + | 41    | 3.9066760 | + + 000 + + | + + 000 + + |
| 18    | 3.3422180 | + + 0 + + + + + + + + + | + + 000 + | 42    | 3.7863722 | + + 0 + + + | + + 000 + + |
| 19    | 3.3422164 | + + + + + + + + + | + + 0 + | 43    | 3.7747446 | + + 0 + + + | + + 000 + + |
| 20    | 3.3419908 | + + + + + + + + | + + 0 + | 44    | 3.7457804 | + + 000 + + | + + 0000 + + |
| 21    | 3.3419494 | + + + + + + + + | + + 0 + | 45    | 3.6839843 | + + 000 + | + + 000 + + |
| 22    | 3.3409187 | + + + + + + + + | + + 0 + | 46    | 3.6170320 | + + 000 + | + + 000 + + |
| 23    | 3.3381777 | + + + + + + + + | + + 0 + | 47    | 2.5628482 | + + 0000 + | + + 0000 + + |
| 24    | 3.3375984 | + + + + + + + + | + + 0 + |
