Dynamic models of production planning with continuous time in projects of new products development

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Abstract. Dynamic models of production volume planning in the projects of development of new products at an industrial enterprise are considered. In the process of new products development, the learning curve effect manifests itself, which leads to a reduction in labor intensity, depending on the cumulative volume of production. The dynamic planning problem of the production is formalized mathematically as a task of optimal control of the production system. The system change dynamics is described by standard differential equation. Using the Pontryagin maximum principle, an optimal strategy has been found: the optimal production volumes should be inversely proportional to unit costs of production and directly proportional to the discount factor. Analytical formulas for determining the optimal volumes and cumulative volumes of production are obtained for the power-law model of the learning curve. Also, the study of the influence of the learning index and the discount rate on the optimal production volumes and cumulative production volumes was carried out.

1. Introduction
In new products development projects at industrial enterprises the learning curve effect manifests itself, which means reduction of the time spent by workers on performing repetitive manufacturing operations. Initially the learning effect was discovered by engineer T. Wright in the US aircraft industry [1]. Further on, the most complete reviews and comparison of learning curve effect in production activity were presented in the scientific literature [2-4].

Dynamic change in unit costs with an increase in the cumulative volume of production makes the tasks of dynamic optimization pivotal. The goal is to find the optimal volumes of production at each point in time with given time, production and financial constraints in order to achieve the extremum of the chosen economic criterion. To search for analytical solutions, it is proper to consider the dynamic optimization problems of productive activity with continuous time.

2. Dynamic models of production planning with continuous time

2.1. Statement of the dynamic production planning problem with continuous time
The dynamics of the enterprise production activity is described by a standard differential equation:

$$\frac{dx(t)}{dt} = u(t),$$

where $x(t)$ - the cumulative production volume at time $t$, $u(t)$ - the production volume. The choice of the production volume $u(t)$ is the management of the enterprise.
The new products development project at an industrial enterprise is considered at a fixed time period:

\[ 0 \leq t \leq T, \]

where \( T \) - the horizon of project planning.

In the initial moment of time, the number of products produced is as follows:

\[ x(0) = x_0 \]  \hspace{1cm} (2)

In the final point in time, the cumulative volume of finished products should be equal to the specified volume:

\[ x(T) = x_0 + R. \]  \hspace{1cm} (3)

where \( R \) – specified number of finished products.

The production volume is imposed by the following restrictions:

\[ 0 < u(t) \leq x_0 + R - x(t), \]  \hspace{1cm} (4)

The instant cost function for the production of products (costs at time \( t \)) is defined as the multiplication of the product unit costs \( c(t) \) and production volume \( u(t) \):

\[ C(t) = c(x(t))u(t). \]  \hspace{1cm} (5)

The change dynamics in the unit costs of products from the cumulative production volume is described by different models of the learning curve. The most typical models are power, exponential and logistic ones, which are described in the scientific literature [1-4].

The degree model of the learning curve has the following form:

\[ c(x(t)) = ax(t)^{-b}. \]  \hspace{1cm} (6)

where \( a \) – costs for the production of the first product, \( b \) – learning index.

The learning index characterizes the speed of decrease in the unit costs of product with an increase in the cumulative production volume.

Exponential model of learning curve:

\[ c(x(t)) = k + \beta e^{-\alpha x(t)} \]  \hspace{1cm} (7)

where \( \alpha \) - learning index \( k \), \( \beta \) - parameters of the exponential model.

Logistic model of learning curve:

\[ c(x(t)) = c_{\text{min}} + (c_{\text{max}} - c_{\text{min}}) \left( \frac{1}{1 + e^{\alpha x(t)}} \right), \]  \hspace{1cm} (8)

where \( c_{\text{min}} \), \( c_{\text{max}} \) - minimum and maximum values of unit costs for product manufacturing, \( \alpha \) - learning index, \( \beta \) - logistic model parameter.

As a criterion of optimality, we consider the minimization of the total discounted growth rate of the instant cost function:

\[ \tilde{J} = \int_0^T e^{-\delta t} \frac{\dot{C}(t)}{C(t)} dt \to \text{min}. \]  \hspace{1cm} (9)

where \( \delta \) - is the discount rate, \( \frac{\dot{C}(t)}{C(t)} = [\ln C(t)]' \) - is the logarithmic derivative of the instant cost function, which has an economic sense of the growth rate of the instant cost function.

3. Statement

For a positive and absolutely continuous function \( C(t) \), the maximization (minimization) of the following functional

\[ \tilde{J} = \int_0^T e^{-\delta t} \frac{\dot{C}(t)}{C(t)} dt \]  \hspace{1cm} (10)

is equivalent to the functional maximizing (minimizing):
\[ J = \int_{0}^{T} e^{-\delta t} \ln C(t) dt. \]  

The proof of the statement is given in the Appendix.

Taking into account the statement, as the criterion of optimality, we take minimization of the total discounted logarithmic function of instant costs (11). We substitute the expression for the instant cost function (5) into the functional (11):

\[ J = \int_{0}^{T} e^{-\delta t} \ln[c(x(t)) u(t)] dt. \]  

The task of dynamic production planning is to find the optimal production volumes \( u(t)^{\text{opt}} \) satisfying the constraint (4), which transfer the production process (1) from the initial state (2) to the final state (3) and minimize the total discounted logarithmic function of instant costs (12).

3.1. Analytical solution of the dynamic production planning problem with continuous time

To solve the formulated optimal control problem with continuous time (1)-(4), (12), we apply the Pontryagin maximum principle [5]. Hamiltonian function is stated below:

\[ H(t, x, \psi, u) = \psi(t) u(t) - e^{-\delta t} \ln[c(x(t))] - e^{-\delta t} \ln[u(t)], \]  

where \( \psi(t) \) - is an auxiliary variable that satisfies the following conjugate equation:

\[ \frac{d\psi}{dt} = -\frac{\partial H}{\partial x} = e^{-\delta t} \frac{\partial}{\partial x} \ln[c(x(t))]. \]

In accordance with the Pontryagin maximum principle, at each point of the optimal trajectory the Hamiltonian function reaches its maximum with respect to the control parameters. The maximum of the control Hamiltonian is found from the condition:

\[ \frac{\partial H}{\partial u} = 0. \]  

We define the optimal control from the condition (14):

\[ u(t)^{\text{opt}} = e^{-\delta t}. \]  

The system of conjugate equations can be written as follows:

\[ \begin{cases} \frac{dx}{dt} = -\frac{e^{-\delta}}{\psi} \\ \frac{d\psi}{dt} = e^{-\delta} \frac{\partial}{\partial x} \ln[c(x(t))] \end{cases} \]  

From the equations of system (16) follows:

\[ dt = e^{\delta} \psi dx. \]  

\[ dt = e^{\delta} \left( \frac{\partial}{\partial x} \ln[c(x(t))] \right)^{-1} d\psi. \]

The symmetric form of the system (16) taking into account equations (17), (18) will have the form:

\[ dt = \psi dx = \left( \frac{\partial}{\partial x} \ln[c(x(t))] \right)^{-1} d\psi. \]  

Let us perform the separation of variables in the second differential equation (19):

\[ \frac{d\psi}{\psi} = \frac{\partial}{\partial x} \ln[c(x(t))] dx. \]  

Let us find the general solution of the differential equation (20):

\[ \psi = C_0 c(x(t)), \]  

where \( C_0 \) - the integration constant.
The optimal control \( (15) \) taking into account \( (21) \) takes the following form:

\[
\begin{align*}
    u(t)_{\text{opt}} &= \frac{e^{-\delta t}}{C_0 \psi(x(t))}. \\
    \psi &= C_1 x(t)^{-b}.
\end{align*}
\]  

From the obtained condition for optimal control \( (22) \) it follows that the optimal production volumes should be inversely proportional to the unit costs of production and directly proportional to the discount factor.

Let us find the optimal control and optimal trajectory for the power model of the learning curve. The formula \( (6) \) can be substituted in the resulting expression for the conjugate variable \( (21) \):

\[
    \psi = C_1 x(t)^{-b}. 
\]

where \( C_1 = C_0 a \) - the integration constant.

We substitute formula \( (23) \) into the differential equation \( (17) \):

\[
    dt = e^{\delta t} C_1 x^{-b} dx. 
\]

The general solution of equation \( (24) \) will have the form:

\[
    t = \frac{1}{\delta} \ln \left[ \frac{C_2 - C_1 \delta x^{1-b}}{1-b} \right]. 
\]

We define the integration constants \( C_1 \) and \( C_2 \) from the boundary conditions \( (2) \) and \( (3) \):

\[
    C_1 = \frac{(1-e^{-\delta T})(1-b)}{\delta[(x_0 + R)^{1-b} - x_0^{1-b}]}, \\
    C_2 = 1 + \frac{(1-e^{-\delta T})x_0^{1-b}}{(x_0 + R)^{1-b} - x_0^{1-b}}. 
\]

Substituting the constants of integration \( (26), (27) \) into formula \( (25) \), we find the equation of the optimal trajectory of the cumulative production volume:

\[
    x(t)_{\text{opt}} = \left( x_0^{1-b} + \frac{(1-e^{-\delta t})(x_0 + R)^{1-b} - x_0^{1-b}}{(1-e^{-\delta T})(x_0 + R)^{1-b} - x_0^{1-b}} \right)^{\frac{1}{1-b}}. 
\]

We define the optimal control by substituting the formula \( (23) \) into the condition \( (15) \) with the found expression for \( (26) \):

\[
    u(t)_{\text{opt}} = \frac{\delta e^{-\delta t}}{1-e^{-\delta T}} \left( x_0^{1-b} + \frac{(1-e^{-\delta t})(x_0 + R)^{1-b} - x_0^{1-b}}{(1-e^{-\delta T})(x_0 + R)^{1-b} - x_0^{1-b}} \right)^{\frac{1}{1-b}} (x_0 + R)^{1-b} - x_0^{1-b}. 
\]

Let us find the instant cost function \( (5) \), taking into account formulas \( (28) \) and \( (29) \) on the optimal trajectory with optimal control:

\[
    C(t, x^{\text{opt}}, u^{\text{opt}}) = a \frac{\delta e^{-\delta t}}{1-e^{-\delta T}} (x_0 + R)^{1-b} - x_0^{1-b}. 
\]

Analyzing \( (30) \) we come to the conclusion that under optimal control, the change in the instant costs function depends only on the discount factor \( e^{-\delta t} \).

The dependence of the optimal trajectory of the cumulative production volume and the optimal production volume on the learning index was studied. For the calculations, the following data were used: the given cumulative volume of production of items \( R=240 \) pieces, the number of time periods \( T = 12 \) months, the volume of production in the initial period \( x_0 = 1 \) piece.

Figure 1 and Figure 2 show the trajectories of the optimal cumulative volume and volume of production of a new product for a constant discount rate \( \delta = 10\% \) when learning speed \( b \) changes.

Figure 3 and Figure 4 show the trajectories of the optimal cumulative volume and volume of production of a new product for a constant learning speed \( b = 0.5 \) when discount rate \( \delta \) changes.
Figure 1. The trajectory of the optimal cumulative volume of production, $\delta=10\%$.

Figure 2. The trajectory of the optimal volume of production, $\delta=10\%$.

Figure 3. The trajectory of the optimal cumulative volume of production, $b=0.5$. 
4. Conclusion

Dynamic models of production planning with continuous time are considered in the paper in the projects of development of new products at an industrial enterprise. As a result of the dynamic planning problem solvation with the Pontryagin maximum principle, an optimal strategy was found: the optimal production volumes should be inversely proportional to the unit costs of production and directly proportional to the discount factor.

For the power-law model of the learning curve, analytical formulas of optimal production volumes at each moment of time and an optimal trajectory of the cumulative production volume are obtained. An analytical formula is found for the instant costs function on the optimal trajectory under optimal control.

The study carried out in this paper allowed us to draw the following conclusions:

1. The optimal strategy of enterprise management is the gradual increase in production volumes from the minimum in the initial periods to the maximum in the final periods.
2. With the increase in learning speed, the optimal trajectory of the cumulative production volume becomes more "convex".
3. With the increase in discount rate, the optimal trajectory of the cumulative production volume becomes less "convex".
4. При оптимальном управлении на оптимальной траектории изменение функции мгновенных затрат зависит только от коэффицента дисконтирования.
5. With optimal control on the optimal trajectory, the change in the instantaneous cost function depends only on the discount factor.

5. Appendix

Proof of the statement.

We integrate the functional (10) by parts:

\[
\int_0^T e^{-\delta t} \frac{\dot{C}(t)}{C(t)} dt = e^{-\delta T} \ln C(T) - \ln C(0) + \delta \int_0^T e^{-\delta t} \ln C(t) dt. \tag{31}
\]

We introduce the function \( g(t) = e^{\delta t} \ln C(t) \).

Then values of the function at the initial and final moment of time are \( g(0) = \ln C(0) \) and \( g(T) = e^{\delta T} \ln C(T) \). Expression (31) takes the form:

\[
\int_0^T e^{-\delta t} \frac{\dot{C}(t)}{C(t)} dt = g(T) - g(0) + \delta \int_0^T g(t) dt. \tag{32}
\]
5.1. The case of maximizing of the functional

In the case of maximization of the functional (32), the function \( g(t) \) is an increasing one, while \( g(T) > g(0) \). The geometric interpretation of the integral \( S_g = \int_0^T g(t) \, dt \) is the area of the curvilinear trapezium, bounded above by the positive function \( g(t) \), below by the abscissa axis and by the straight lines \( t=0 \) and \( t=T \). The rectangle area bounded above by the straight line \( g(t) = g(T) \), below by the abscissa axis and by the straight lines \( t = 0 \) and \( t = T \) can be defined on the one hand through the integral, and on the other hand as the multiplication of length by height:

\[
S_T = \int_0^T g(T) \, dt = Tg(T).
\]  
(33)

Similarly, the rectangle area bounded above by the line \( g(t) = g(0) \), below by the abscissa axis and by the straight lines \( t = 0 \) and \( t = T \) can be found:

\[
S_0 = \int_0^T g(0) \, dt = T g(0).
\]  
(34)

From the formulas (33) and (34) follows that

\[
g(T) = \frac{1}{T} \int_0^T g(T) \, dt.
\]  
(35)

\[
g(0) = \frac{1}{T} \int_0^T g(0) \, dt.
\]  
(36)

Then the functional (32), taking into account formulas (35) and (36), can be written:

\[
\int_0^T e^{-a} \frac{C(t)}{C(t)} \, dt = g(T) - g(0) + \delta \int_0^T g(t) \, dt = \frac{1}{T} \int_0^T [g(T) - g(0)] \, dt + \delta \int_0^T g(t) \, dt.
\]  
(37)

The integral \( \int_0^T [g(T) - g(0)] \, dt = S_{T0} \) defines the rectangle area, bounded above by a straight line \( g(t) = g(T) \), below by a straight line \( g(t) = g(0) \) and by straight lines \( t = 0 \) and \( t = T \). The formula \( \delta \int_0^T g(t) \, dt \) geometrically can be interpreted as the area of a squeezed curvilinear trapezium \( \delta S_g \), since \( \delta < 1 \). The expression \( \frac{1}{T} \int_0^T [g(T) - g(0)] \, dt = \frac{1}{T} S_{T0} \) calculates the area of the squared rectangle \( S_{T0} \).

The sum of the areas of the transformed curvilinear trapezium \( \partial S_g \) and the rectangle \( \frac{1}{T} S_{T0} \) can be defined as the area of the curvilinear trapezium, bounded above by the positive function \( \lambda_1 g(t) \) (\( \lambda_1 \) - the constant factor), below the abscissa axis and the straight lines \( t = 0 \) and \( t = T \):

\[
\int_0^T e^{-a} \frac{C(t)}{C(t)} \, dt = \frac{1}{T} \int_0^T [g(T) - g(0)] \, dt + \delta \int_0^T g(t) \, dt = \int_0^T \lambda_1 g(t) \, dt.
\]

Since \( \lambda_1 \) is a constant factor, the maximization of the functional \( \int_0^T \lambda_1 g(t) \, dt \) will be equivalent to maximizing of the functional \( \int_0^T g(t) \, dt = \frac{1}{T} \int_0^T e^{-a} \ln C(t) \, dt \). Thus, the statement is proved.

5.2. The case of minimization of the functional

In the case of minimization of the functional (32), the function \( g(t) \) is decreasing, in this case \( g(T) < g(0) \). The formula (37) will have the form:
If $\delta > \frac{1}{T}$, then the difference of the areas of the transformed curvilinear trapezium $\delta S_g$ and the rectangle $\frac{1}{T} S_{T_0}$ can be defined as the area of the curvilinear trapezium, bounded above by the positive function $\lambda_2 g(t)$ ($\lambda_2$ - the constant factor), below by the abscissa axis and the straight lines $t = 0$ and $t = T$:

$$\int_0^T e^{-\delta} \frac{C(t)}{C_0(t)} dt = -\frac{1}{T} \int_0^T [g(0) - g(T)] dt + \delta \int_0^T g(t) dt = \int_0^T \lambda_2 g(t) dt. \quad (38)$$

Minimization of the functional $\int_0^T \lambda_2 g(t) dt$ will be equivalent to minimizing of the functional $\int_0^T g(t) dt = \int_0^T e^{-\delta} \ln C(t) dt$. The statement is proved.

If $\delta < \frac{1}{T}$, then the difference of the areas of the transformed curvilinear trapezium $\delta S_g$ and the rectangle $\frac{1}{T} S_{T_0}$ can be defined as the area of the inverted curvilinear trapezium $S_p$, bounded above by a straight line $g(t) = g(T)$, below by a function $\delta_0 g(t)$, and by lines $t = 0$ and $t = T$. Since the area difference will be negative, the problem of minimizing of the functional (38) will be equivalent to the problem of the area maximizing of the inverted curvilinear trapezium. Maximizing the area of the inverted curvilinear trapezium $S_p$ is equivalent to minimizing the area of the compressed curvilinear trapezium $\delta S_g = \delta \int_0^T g(t) dt$. Minimizing of the functional $\delta \int_0^T g(t) dt$ will be equivalent to minimizing of the functional $\int_0^T g(t) dt = \int_0^T e^{-\delta} \ln C(t) dt$. The statement is proved.

6. References
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Acknowledgments
The reported study was funded by RFBR and Samara region according to the research project № 17-46-630606.