Why do we live in 3+1 dimensions?

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Abstract

Noticing that really the fermions of the Standard Model are best thought of as Weyl - rather than Dirac - particles (relative to fundamental scales located at some presumably very high energies) it becomes interesting that the experimental space-time dimension is singled out by the Weyl equation: It is observed that precisely in the experimentally true space-time dimensionality $4=3+1$ the number of linearly independent matrices $n_{Weyl}^2$ dimensionized as the matrices in the Weyl equation equals the dimension $d$. So just in this dimension (in fact, also in a trivial case $d=1$) do the sigma-matrices of the Weyl-equation form a basis. It is also characteristic for this dimension that there is no degeneracy of helicity states of the Weyl spinor for all nonzero momenta.

We would like to interpret these features to signal a special “form stability” of the Weyl equation in the phenomenologically true dimension of space-time. In an attempt of making this stability to occur in an as large as possible basin of allowed modifications we discuss whether it is possible to define what we could possibly mean by “stability of Natural laws”.

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1 Introduction

Since ancient times [1] it has been a challenging question why we have just three space dimensions. In the light of special relativity one may formulate an even slightly extended question: Why do we have just 3+1 dimensions, one time and three space dimensions?

It is the purpose in this contribution to elaborate on the idea [4] that the number of dimensions 3+1 is connected with properties of the Weyl equation, the latter being especially “stable” in these dimensions (stability under perturbations which do not keep Lorentz invariance, say, a priori).

In this connection we shall however also challenge the concept of “stability of Natural laws” (so much studied by one of us [4], [5]). This (second) part of our article is a preliminary effort where we try to seek some “principal boundaries” for the set of ideas which we call the “random dynamics project”.

Before we drift into a detailed discussion of these issues, i.e. our own points of view, which thus aims at connecting the space-time dimensionality with certain properties of the Weyl equation, we would like to mention some other ideas on how to arrive at the space-time dimensionality 3+1. Ideas which to some extent are competing (cf. final remarks, sec. 4.1.) with our own point of view taken here.

Let us first remark, that one of the purposes of this article is to emphasize the scientific value of reading information from phenomenology, i.e. from theories which have been successfully confronted with experiments. Such inspiration from phenomenology supplement more wild and speculative constructions (theories of “everything” [1]) which, in case they do not a priori operate in 3+1 dimensions, clearly have to offer some explanation for why our space-time is 3+1 dimensional.

Phenomenologically inspired attempts of understanding 3+1 dimensions have indeed appeared, cf. e.g. P. Ehrenfest [2], G.J. Whitrow [3] and the book by J.D. Barrow & F.J. Tipler [1]. The arguments presented in this context seems dominantly to be based on observations involving the Coulomb or Newton potential, connected to considerations dealing with electromagnetic and gravitational interactions, respectively. Assuming a Laplace equation for the (electromagnetic or gravitational) field, it has a power related to the dimension of the space. Thus, it can easily be calculated that the Laplace equation in \( d_s \) independent variables, \( \Delta \Psi(x_1, ..., x_{d_s}) = 0 \), will correspond to an inverse \( (d_s - 1) \)th power law of force (i.e. an inverse \( (d_s - 2) \)th power law of the potential) in a Euclidian space of \( d_s \) dimensions. Moreover, this power in the potential is related to the stability of atoms or planetary systems. If the numerical value of the power is larger than just the unity corresponding to

We claim, however, that in a certain elementary sense of the word “everything” it is not (theoretically) ever possible to arrive at a theory of everything since an “infinite regress” argument very fast sets in - which stops, only, if language itself (which parts of it?) unambiguously could select a unique construction of a “T.O.E.”. Not even then, however, does our search for a “truth behind” stop, since we are then, obviously, forced into a philosophy of the (mysterious) power of our language. H.B.Nielsen and S.E.Rugh, ongoing discussions. See also S.E. Rugh [9].
four dimensions - the atom or planetary system gets unstable against falling into the singularity at \( r = 0 \) (or it goes off to infinity, cf. P. Ehrenfest [2]). This instability concerns an unstable state in the terminology of subsection 3.1 below. The argument being the main study of this article turns out to be based on stability in a different sense: Stability against variations in the dynamics (rather than the state).

Other - but more speculative - attempts have been made to explain that there should be 3+1 dimensions: The following list is, we admit, highly selective and random - influenced by talks and papers which we accidently have come across. G.W. Gibbons [1], try to get 3+1 dimensions out of a membrane-model in a higher dimensional setting; K. Maeda [2] study how to get 3+1 dimensional universes as attractor universes in a space of higher dimensional cosmologies and a group of physicists and astrophysicists in Poland, M. Biesiada, M. Demianski, M. Heller, M. Szydlowski and J. Szczesny [3], pursue the viewpoint of dimensional reduction - i.e. arriving at our 3+1 dimensional universe from higher dimensional cosmologies.

In Copenhagen, pregeometric models have recently been investigated by e.g. F. Antonsen [4]. He arrived at the spatial dimension \( d_{\text{em,eff}} = 42/17 \approx 2.5 \) from a pregeometric toy-model for “quantum gravity” based on random graphs, but according to Antonsen this value is probably too low, since the model was only tested with simplexes of dimension 0 – 4. E. Alvarez, J. Cispedes and E. Verdaguer [5] also arrives at dimensionalities around \( d \approx 1.5 – 2.5 \) in pregeometric toy-models assigned with a (quantum) metric described by random matrices.

D. Hochberg and J.T. Wheeler [6] contemplate whether the dimension may appear from a variational principle. Also Jeff Greensite [7] uses a variational principle (roughly making the signature of the metric a dynamical variable) in order to get especially the splitting 3+1 into space and time.

Of course, any development sensitive to dimension and working in the experimental dimension 3+1 may be considered an explanation of this dimension, because an alternative dimension might not be compatible with the same theory. For example, twistor theory (cf. R. Penrose [8]) is such a theory, suggesting the dimension. In fact, it does it in a way exceptionally close to the “explanation” presented in the present article, since both are based on Weyl-spinors.

**How superstrings arrive at 3+1 dimensions?**

Superstring theory [9] has been much studied for a decade as a candidate for a fundamental theory. Let us therefore finally discuss - more lengthy - how the superstring, by construction living in 10 or 26 dimensions, may arrive at our 3+1 dimensional spacetime.

The space-time dimension \( 3 + 1 \) (on our scales) is in superstring theory singled out by several, to some extent, independent considerations:

1. An argument for the dimensionality \( d \geq 3 + 1 \) (cf. R. Brandenberger & C. Vafa [10]): If one assumes some initial beginning - a big bang, say -
with a lot of superstrings wound around various dimensions of space (think for simplicity of a torus) there should statistically be strings of opposite orientations normally enclosing the same dimensions, and they would “as time passes” compensate each other and disappear provided they manage to hit each other and switch their topology, so that such compensation is made possible. As long as there are less than 3+1 (= 4) large (i.e. extended) dimensions there is a very high chance that strings will hit and shift topology because it is very difficult for strings in 3 or less space-dimensions to avoid hitting each other. It turns out that it is basically the number of large dimensions, that counts for whether “hits” take place easily or not.

As the dimension of the large directions reaches 3+1 the process of unfolding slows drastically down and one may imagine this being the reason for there just being 4 dimensions. We remark, that this argument is truly very “stringy” since it makes the dimension of space-time become just twice that of the time track of the string (i.e. 2 + 2 = 4 = 3 + 1).

(2) A compactification argument (“dimensional reduction”): A Calabi-Yau space has exactly 6 dimensions, and so the 10 = 9 + 1 dimensions needed for the superstring to exist (because of the requirement of cancellation of anomalies, or, say, to get the needed zero-point mass squared - contribution as needed) enforces that using 6 of the 10 dimensions in the compactification leaves just 4 = 3 + 1 for the “large” dimensions But, why should we select a Calabi-Yau space as the compactifying space? It is motivated by assuming that $N = 1$ supersymmetry should survive far below the compactification scale of energy, i.e. by the requirement of not breaking this supersymmetry.

While the argument using Calabi-Yau space and the 10 dimensionality of the superstring can hardly stand alone without other support for the string theory, several of the other arguments are sufficiently primitive to be trusted by itself, since based on phenomenologically supported ideas. Note, for instance, that argument (1) for string theory above was also to some extent phenomenological.

Let us now, for a moment, believe in the compactification mechanism, reducing the dimensionality from 10 to 4=3+1. Then, it becomes a central issue of clarification at which scales this compactification takes place. It is challenging for the superstring that it has been suggested by Antoniadis [21] that already at a few TeV one should expect to see the additional dimensions. If true, this has - most likely - severe cosmological implications. One of us [21] would like to make the arguments of Antoniadis completely rigorous, so that strings can not escape this prediction (modulo very general assumptions about string-constructions).

It is interesting, though, that recent attempts tend to formulate superstring theories from the beginning in four spacetime dimensions, rather than existing in 10 or 26 dimensions, of which all but the four extended dimensions of our spacetime somehow become compactified. (Cf, e.g., the summary talk by Steven Weinberg [23] and references therein).

But in that case the superstring has evidently a somewhat less impressive capacity of power as regards an explanation of why we live in 3+1 dimensions.
1.1 Getting inspiration from phenomenology

Instead of relating the dimension number “3+1” to highly speculative constructions, which could be plain wrong, such as strings, we would rather try to identify structures in the well known and established laws of Nature which point towards that the number “3+1” is distinguished.

You may consider this project a sub-project of a program consisting in “near reading” of the empirically based structure we know today, i.e. the formal structure and the parameters of the Standard Model (describing the electroweak and strong “spin 1” gauge-interactions) and Einstein’s theory for the gravitational “spin 2” interactions.

Among the information we do not understand today (and which we necessarily have to be “inspired” from) the following issues were especially stressed by Steven Weinberg [22] at the recent XXVI Conference on High Energy Physics in Austin (August 1992):

- Why do the parameters of the Standard Model take the observed values?
- Why are there three generations of quarks and leptons?
- Why is the Standard Model group $S(U_2 \times U_3) \simeq U(1) \otimes SU(2) \otimes SU(3)$?

And here we would like to add:

- Why do we live in 3+1 dimensions?

In fact, we have found a special property in the Weyl equation which pervades so strongly the Standard Model of the electroweak and strong forces and the known material constituents. This special property (which we encode by the concept of “stability” and which we shall discuss in this contribution) is a property which distinguishes 4=3+1 and 1=0+1 as standing out relative to all other space-time dimensionalities. The number of linearly independent matrices of the type that appear in the Weyl equation is exactly equal to the number of real parameters of the Standard Model.

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2 The desperation in this project is that all this information can be immensely condensed down (to a page or so) so we have actually only very little structure to “read” (and to be inspired from): The truly observed sector of the Standard Model has only of the order of 19 real parameters, some of which are hardly determinable in practice, even using baryon number generation physics: 12 quark and lepton masses of which the three neutrino masses (in the simplest Standard Model) are understood to be just zero, three fine-structure constants, four parameters in the Kobayshi-Maskawa-Cabibbo matrix and the $\theta$-angles.

In this program of “near reading” information from the Standard Model we have also tried to read off remarkable features of the Standard Model gauge group $S(U_2 \times U_3)$ compared to other groups with up to 12 generators (12 gauge bosons), say, and we have found that the standard model group is remarkable by its “skewness” (remarkably few automorphisms), cf. H.B. Nielsen & N. Brene [7].

3 Note, that even relative to the $W$ mass scales most of the quarks and leptons are light particles, and certainly they are all “massless” relative to the Planck scale. Since also right and left components couple differently to the weak gauge bosons, a description in terms of Weyl particles is strongly suggested, cf. also introductory discussion in H.B. Nielsen & S.E. Rugh [6].
dimensionality \( d \) of space-time if \( d = 4 \) (or, in fact, if \( d = 1 \)).

Are the elementary matter constituents, we know today (i.e. the fermions) not Dirac particles rather than Weyl particles? Well, the Weyl spinors - introduced by Hermann Weyl - were originally rejected as candidates for our constituents of matter because they were incompatible with parity conservation. However, seeking fundamental physics (at the Planck scale, say) we ignore the small masses of a few hundred \( GeV/c^2 \) of the quarks and the weak gauge bosons, rendering the connection between right and left handed components completely negligible. Thus we consider all the material constituents (quarks and leptons) we know today as Weyl particles rather than Dirac particles!

So to read some special feature of dimension 3+1 connecting to the Weyl equation means to read it off from all known material fields of which we are build up! (cf., also, the more detailed introductory discussion in H.B. Nielsen & S.E. Rugh (1992) [3]).

The outline for our contribution is as follows: In the following section (section 2) we remind the reader about the Dirac and Weyl equations in an arbitrary number of dimensions. Especially, we calculate the number of components of the Weyl spinor and the crucial observation of the equality of dimension and the number of elements in a basis for the matrices in the Weyl equation is made. In section 3 we then seek to unravel the message to be learned from this observation: It is connected with “a stability” under perturbing the fundamental “dynamics”. We also discuss the general limitations for postulating such a stability. In section 4 we resume and put forward the concluding remarks and dreams, and it is discussed if the observation could just be an accident?

2 The Dirac and Weyl equations in \( d \) dimensions

In order to note the speciality of the 4=3+1 dimensional Weyl equation we shall see that in the general \( d \) dimensional case the number of components \( n_{Weyl} \) of the spinor \( \psi \) of what is reasonably called the Weyl equation

\[
\sigma^\mu D_\mu \psi = 0
\]

is

\[
n_{Weyl} = \begin{cases} 
2^{d/2-1} & \text{for } d \text{ even}, \\
2^{(d-1)/2} & \text{for } d \text{ odd}.
\end{cases}
\]

In fact, the Weyl equation is at first glance only defined for even dimensions and, thus, we have to agree upon, what we will call a Weyl equation in odd dimensions. We outline shortly how to build up a Weyl equation in even and in odd dimensions.
2.1 Construction of the Weyl-equation (in even and odd dimensions) from the Dirac-equation (in even dimension)

The Weyl-equation in arbitrary dimensions is most easily constructed from the Dirac equation, which we shall in the next section construct in even dimensions via an inductive construction. It has only half as many components and is obtained by “$\gamma^{d+1}$-projection” (think here of the $\gamma^5$-projection in the 4 dimensional case) meaning that we first define

$$\gamma^{d+1} = \gamma^0\gamma^1\gamma^2 \cdots \gamma^{d-1}\gamma^d$$

modulo an optional extra phase factor $i$ so as to be hermitean, then choose $\gamma$-matrix representation so that this $\gamma^{d+1}$ becomes diagonal (Weyl-representation), and then thirdly write only that part of the Dirac equation

$$(i\gamma^\mu D_\mu - m)\psi = 0,$$  

which concerns those components of the Dirac field $\psi$ which correspond to one of the eigenvalues of $\gamma^5$, say $+1$. We now drop the mass-term. Since the only remaining term in the Dirac-equation is thereby the “kinetic” one $i\gamma^\mu D_\mu\psi = 0$ and the gamma-matrices only have matrix elements connecting $(\gamma^{d+1} = 1)$-components to $(\gamma^{d+1} = -1)$-components the Weyl equation can be written using only matrices with half as many components as the corresponding Dirac equation from which it is obtained. Really we find the Weyl matrices as off-diagonal blocks in the Dirac matrices in the Weyl representation. That is to say the Weyl-gamma-matrices which we call $\sigma^\mu$ (or $\sigma^\mu_-$ and $\sigma^\mu_+$) if we want to distinguish if we projected on the $\gamma^5$ equal to minus or plus 1 components) are given by

$$\sigma^\mu = \gamma^\mu \frac{1 - \gamma^5}{2}, \quad \sigma^\mu_+ = \gamma^\mu \frac{1 + \gamma^5}{2}$$

with successive removal of the unnecessary entries in the matrices, or better by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_+ \\ \sigma^\mu_- & 0 \end{pmatrix}. $$

2.2 Inductive construction of the $\gamma$ matrices (by addition of two dimensions at a time)

If we are in $d_t$ time dimensions and $d_s$ space dimensions we denote by relativistic invariance the invariance of the equations (to be constructed right now) under the (generalized Lorentz) group $O(d_t, d_s) \equiv O(d_t, d - d_t)$.

Generalizing the construction by Dirac, we seek a construction in $d = d_t + d_s$ dimensions, which involves a set of $\gamma$ matrices, $\gamma^1, \gamma^2, \ldots, \gamma^d$ shaped to obey a specific anti-commutator algebra

$$\{\gamma^\mu, \gamma^n\} = 2\delta^{\mu^n}_{(d)}$$
(where $g^{\mu\nu}_{(d)}$ denotes the metric tensor, taken just to be a series of 1’s and −1’s along the diagonal).

If equation (2.7) is satisfied for the $\gamma$ matrices, the Dirac equation imply a corresponding Klein-Gordon equation

$$(i\gamma^\mu D_\mu + m)(i\gamma^\nu D_\nu - m^2 - [\gamma^\mu, \gamma^\nu]eF_{\mu\nu})\psi = 0 \quad (2.8)$$

(with magnetic moment term in it, and where we used $[D_\mu, D_\nu] = 2eF_{\mu\nu}$) and that is what is wanted to have the usual relativistic dispersion relation

$$p^2 - m^2 = 0 .$$

It is not difficult to show the existence of such $\gamma$ matrices by induction in steps of two in the dimensions, starting, e.g., from two dimensions, to get the even ones.

**Start of the induction in two dimensions**

In two space-time dimensions one may use two of the three Pauli matrices as gamma-matrices, e.g.

$$\gamma^0 = \sigma_x \text{ and } \gamma^1 = i\sigma_y \quad (2.9)$$

in the Minkowski-space case and simply

$$\gamma^0 = \sigma_x \text{ and } \gamma^1 = \sigma_y \quad (2.10)$$

in the euclidian(ized) two dimensional space. These are the Dirac-matrices and they have $n_{Dirac} = 2$ components.

**Step from $d - 2$ to $d$ dimensions.**

Having the Dirac equation in $d - 2$ dimensions (with $d$ even) we can in an inductive construction make the Dirac-matrices for two dimensions higher by the following construction:

Suppose that we already have constructed the gamma-matrices for $d - 2$ dimensions and denoted them with a tilde in order to distinguish them from the gamma-matrices for $d$ dimensions which we construct successively. They obey the anti-commutation algebra

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2g^{\mu\nu}_{(d-2)} \quad (2.11)$$

with

$$\mu, \nu = 1, 2, 3, ..., d - 3, d - 2. \quad (2.12)$$

Then we define the $d$-dimensional gamma-matrices for the first $d - 2$ values of $\mu$ by:

$$\gamma^\mu = \left( \begin{array}{c} 0 & \tilde{\gamma}^\mu \\ \tilde{\gamma}^\mu & 0 \end{array} \right) \quad (2.13)$$
and the two new gamma-matrices by:

\[ \gamma^{d-1} = \text{sign} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \]

\[ \gamma^d = \text{sign} \cdot \begin{pmatrix} 0 & -i1 \\ i1 & 0 \end{pmatrix} . \]

The signs \( \text{sign} \) (which, without loss of generality, need only to take values 1 or \( i \)) are to be chosen so as to arrange the wanted signature for the anti-commutator algebra

\[ \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu}_{(d)} \] (2.15)

which is easily checked with the presented ansatz, cf. (2.13) and (2.14).

### Number of components of the Weyl-and Dirac spinors

Thus we see that the number of components for a Dirac equation for even dimension \( d \) of space-time is\(^4\)

\[ n_{\text{Dirac}} = 2^{d/2} \text{ for } d \text{ even} \] (2.16)

and thus since the Weyl equation has just half as many components it has:

\[ n = n_{\text{Weyl}} = 2^{d/2-1} \text{ for } d \text{ even} \] (2.17)

Note, that in \( d = 4 \) dimensions the Weyl spinor gets two components (the neutrino), while in 10 dimensions, say, the Weyl spinor gets 16 components (a complicated neutrino, indeed) so we see that the number of components grows very fast, being an exponential function of the dimension \( d \). I.e. whereas a “spin 1/2” particle has 2 spin-states (if it is a massive Dirac spinor) as we are used to in four space-time dimensions, a “spin 1/2” particle in 10 dimensions has 16 spin-states (but 32 Dirac-components).

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\(^4\)In a systematic exposition of this construction one would go into Clifford algebras and all that.

\(^5\)This number of components is only a minimal number in the sense that the Weyl spinors could have additional degrees of freedom (as they indeed have, for instance “color”). When we have - until now - ignored such additional degrees of freedom, this is, mathematically speaking, that we have build up the irreducible representations of the algebra (2.7). The question about additional degrees of freedom of the Weyl spinor turns out to be a major “killing attempt” of the entire approach pursued here, cf. the section “An almost true theorem ...” (subsection 3.6).
2.3 The odd dimensional cases

In odd dimensions one may simply include the $\gamma^{(d-1)+1}$-matrix as the matrix number $d$ and use the $d-1$ even dimensional Dirac equation, which we have already constructed. That is to say, we take the Dirac equation one unit of dimension lower than the odd-dimensional equation we want to construct. Then we have all the $\gamma$-matrices already except for the last one - the $d$th one. For the latter we then use the construction of taking the product of all the first $d-1$ matrices. Since it is well known that such a "$\gamma^5$"-type product will anti-commute with all the other gamma-matrices if constructed for even dimension, and since it can by an optional factor $i$ have the sign of its square as wanted, it obviously will satisfy all the needed properties for joining the set of gamma-matrices to get a set for the odd dimension $d$.

The equation, with gamma-matrices, obtained in this way for the odd dimension $d$ is in two ways best considered the odd-dimensional Weyl equation:

1. It violates parity\footnote{In odd space-time and thus even space dimension the point-reflection is just a rotation and when we talk about parity we mean a reflection in a mirror (true mirror symmetry).}, just like the even-dimensional Weyl equation,
2. It cannot be further reduced by "$\gamma^5$"-projection (really $\gamma^{d+1}$),

since its "gamma-five" would become the unit matrix.

However, this "Weyl equation" in odd dimensions deviate from the Weyl equation properties by allowing a mass term, since there is only a trivial $\gamma^{d+1} = 1$ that can be constructed. There is no "chirality" which protects against generation of mass $m \neq 0$ for the Weyl spinor field $\psi$.

One might then define an equation with the double number of components and consider that the odd dimensional Dirac equation. By composing a couple of odd dimensional Weyl equations, being mirror images of each other one easily obtains a "Dirac equation" in odd dimensions which is both symmetric under parity transformations and can be reduced back to its Weyl-components.

With the suggested notation we easily find the number of components for the odd dimensional case:

$$n_{\text{Dirac}} = 2 \cdot 2^{(d-1)/2} = 2^{\frac{d+1}{2}} \text{ for } d \text{ odd}$$  \hspace{1cm} (2.18)

for the Dirac equation (the doubled one) and for the Weyl equation

$$n_{\text{Weyl}} = 2^{(d-1)/2} = 2^{\frac{d-1}{2}} \text{ for } d \text{ odd.}$$  \hspace{1cm} (2.19)

2.4 An observation

With $n_{\text{Weyl}}$ components the number of linearly independent $\sigma^\mu$ type matrices that can be formed is $n_{\text{Weyl}}^2$. Out of these $n_{\text{Weyl}}^2$ possibilities the Weyl equation makes use of $d$. Our crucial observation is that in the phenomenologically true case $d = 4 = 3 + 1$ it happens that all the $n_{\text{Weyl}}^2$ possible matrices are just used once each since in fact

$$n_{\text{Weyl}}^2 = d.$$  \hspace{1cm} (2.20)
Fig.1. This figure shows the graphical way of solving equation (2.21) by presenting as functions of the dimension $d$ a semilogarithmic plot of:
1) the identity function $d$ (the curved curve), 2) right hand sides of (2.21) interpolated to real numbers separately for even and odd $d$ (the two straight lines). Note, that there is also coincidence for $d = 1$ which is however a completely empty and trivial theory. The one for $d \approx 0.33$ is of course not a true solution.

This equation written as an equation for the dimension $d$ of space-time becomes

$$d = \begin{cases} 
(2^{d/2-1})^2 &= 2^{d-2} \text{ for } d \text{ even}, \\
(2^{(d-1)/2})^2 &= 2^{d-1} \text{ for } d \text{ odd}. 
\end{cases} \quad (2.21)$$

As can be easily seen from the figure the only acceptable solutions to this equation are

$$d = \begin{cases} 
1 \\
4 
\end{cases} \quad (2.22)$$

because the solutions appearing at first also $d = 2$ (if 2 had been odd) and an irrational solution around $d \approx 0.3$ (had it been even) are not acceptable.
3 The equation $n_{Weyl}^2 = d$ as a message about robustness and stability of the Weyl equation in 3+1 dimensions

What could possibly be the reason for this coincidence of numbers $n_{Weyl}^2$ and dimension $d$? (If it is not an accident, cf. the final remarks).

It means that the sigma-matrices $\sigma^\mu$ make up a basis for all the possible matrices that could possibly multiply the Weyl spinor in $d$ dimensions, if $d = 4$ (or in $d = 1$, which is a very trivial case indeed $\square$). In particular, this means that

$$\sigma^\mu D_\mu$$

may be considered as the most general linear operator that can act on $\psi$, provided we could consider the $D_\mu$’s completely general expansion coefficients. Now, in fact, $D_\mu$ is the covariant derivative $D_\mu = \partial_\mu - i e A_\mu$. So is this really the most general operator? Yes, it is indeed the most general form of the first two terms in a Taylor expansion of any reasonable well behaved differential operator

$$\Omega(x, p) \approx \Omega(x, p = 0) + \frac{\partial \Omega(x, p = 0)}{\partial p_\rho} p_\rho + \frac{1}{2} \frac{\partial^2 \Omega(x, p = 0)}{\partial p_\rho \partial p_\sigma} p_\rho p_\sigma + \ldots \quad (3.1)$$

The most general equation, which we suppose to be homogenous and linear in the Weyl-spinor field $\psi$ (see discussion later), is thus

$$\Omega(x, p) \psi(x) = 0 \quad , \quad p = p_\mu = i \partial_\mu \quad (3.2)$$

where now $\Omega$ denotes an $2 \times 2$ matrix operator, where we have considered only operators $\Omega$ which map Weyl-spinor waves $\psi(x) = \psi(\vec{x}, t)$ to Weyl-spinor waves (as in the Weyl equation).

Being a $2 \times 2$ matrix operator, $\Omega$ can be expanded in terms of the basis matrices $\sigma^\mu$ (in the case where the equation (2.20), i.e. $n_{Weyl}^2 = d$, is fulfilled we can use the set of the matrices $\sigma^\mu$ as a basis). Let $\Omega_\mu$ denote the basis coefficients, i.e. define

$$\Omega(x, p) = \Omega_\mu(x, p) \sigma^\mu \quad (3.3)$$

We Taylor expand:

$$\Omega_\mu(x, p) = \Omega_\mu(x, p = 0) + \frac{\partial \Omega_\mu(x, p = 0)}{\partial p_\rho} p_\rho + \ldots \quad (3.4)$$

$$= V_\mu^\lambda e A_\lambda(x) + V_\mu^\rho(x) p_\rho + \ldots \quad (3.5)$$

\footnote{Considering this one dimension a time dimension, there is a zero dimensional space or, in other words, only one point. The Weyl equation has in this case the following trivial form, $D_0 \psi = (\partial_0 - i e A_0) \psi = 0$, where $A_0$ can be gauged away, rendering the single-component Weyl spinor completely constant. We thus have a Hilbert space with only one dimension, and thus there is only one quantum state for the Weyl spinor, leaving no room for developments in that “universe”.
}
In the last equation (which is merely notation) we have defined

\[ V^\lambda_\mu(x) = \frac{\partial}{\partial p^\lambda} \Omega_\mu(x, p = 0) ; \quad eA_\lambda(x) = V^{-1}_\lambda^\mu \Omega_\mu(x, p = 0). \quad (3.6) \]

By identifying \( A_\lambda(x) \) with an electromagnetic field and \( V^\lambda_\mu(x) \) with a vierbein (in a gravitational theory) we have thus made the interpretation of the lowest order terms in the Taylor expansion of the general \( 2 \times 2 \) matrix operator equation (3.2) as being nothing but the usual Weyl equation for our Weyl-spinor \( \psi \) (coupled to an electromagnetic field \( A_\lambda \) and to a gravitational field in the usual manner by the vierbein field \( V^\lambda_\mu \)), see also discussion in section 3.5.

We can summarize that Nature has organized just such a dimension \( d \) of space-time as to make the Weyl equation operator (with couplings to some external electromagnetic and gravitational fields unavoidably appearing in the same go) the most general operator in the Taylor expanded limit of small momenta \( p = p_\mu \) (small derivatives \( \partial_x \psi \ll \psi \)).

Why do we need all these Taylor expansions? Well, that we arrive at the Weyl equation via Taylor expansions (keeping only the lowest order terms) is a very natural thing, bearing in mind that we are living in the “infrared limit” compared to some presumed fundamental scale (e.g., the Planck scale, say) in the sense that we have - even with our best accelerators today - access to very small 4-momenta \( p \) only.

In fact, to use such a Taylor expansion (at experimentally accessible energies) is completely analogous to Taylor expanding away higher order curvature terms in the gravitational action thereby arriving at the Einstein-Hilbert action in the “low energy limit”. And just like higher order curvature terms will blow up in the ultraviolet in the action for the gravitational field we could expect that the Weyl equation will be modified in the ultraviolet (for higher momenta) and blow up terms quadratic in the momenta like \( \bar{\psi} \sigma^\mu \partial_\mu \partial_\nu \psi \xi^\nu \) (here \( \xi^\nu \) denotes some vector field, which do not necessarily, in fact better not, have to be the \( A_\nu \) field) etc. etc.

The above means, especially, that we can make a reinterpretation of any little additive change

\[ \sigma^\mu D_\mu \psi = 0 \quad \rightarrow \quad (\sigma^\mu D_\mu + \delta W) \psi = 0 \quad (3.7) \]

in the operator \( \sigma^\mu D_\mu \) acting on \( \psi \) as a change in the \( D_\mu \)'s which again is just a shift in the “electromagnetic field” \( A_\mu \) and/or the “gravitational” vierbein \( V^\lambda_\mu \).

Here \( \delta W = \delta W(x, p) \) could be any \( 2 \times 2 \) matrix operator, which have a well behaved Taylor expansion in such a way that higher order terms than those which are linear in 4-momenta \( p \) may be neglected when \( p \) is small.

Notice that the type of modification terms \( \delta W(x, p) \) we here consider are not at all Lorentz invariant a priori. That the Weyl equation keeps its form and thus Lorentz invariance is only achieved at the cost of allowing the vierbein \( V^\lambda_\mu \) (and \( A_\mu \)) to be modified, but it is still remarkable.
We would thus like to interpret the dimension coincidence as a signal of a certain kind of “form stability”, which we are going to discuss further in a moment.

In 4 dimensions there is only one helicity state (no degeneracy) of the Weyl spinor:

An alternative way, to observe the “stability” of the Weyl spinor in exactly 4 dimensions is to notice that a Weyl particle (antiparticle not included!) for a given momentum has just the following number of Weyl particle polarization states

\[
\frac{n_{Weyl}}{2} = \begin{cases} 
2^{d/2-2} & \text{for } d \text{ even}, \\
2^{(d-3)/2} & \text{for } d \text{ odd}.
\end{cases}
\]  

(3.8)

which means that it is just for \(d = 3\) or \(d = 4\) that there is just no degeneracy of particle states and just one state per momentum! Several states for one momentum signals an instability under modifications \(\delta W(x,p)\), because in general such a modification will cause level repulsion (of the eigenvalues), as is well known from perturbations of levels in quantum mechanics.

We can (at least) say that \(d = 4\) is the highest dimension without degeneracy!

### 3.1 Is it possible to define “structural stability” for Natural Laws?

Let us for a moment set aside the Weyl equation (we return to that) and digress into a discussion of how we possibly can assign a meaning to the notion of “stability of Natural laws” (and, in particular, the stipulated stability of the Weyl equation).

It is remarkable that there are over fifty generally accepted definitions of stability (which we shall not go into here, cf, e.g., Szebehely (1984) or E. Atlee Jackson, p.41 [25]). These definitions deal, however, with the stability of a certain state, i.e. a specific configuration of the degrees of freedom of a fixed dynamical system, a fixed Hamiltonian, say.

The Weyl equation is a law of Nature (or a dynamics), not a “state”. So let us distinguish the concept of stability of a “state” under perturbations in the space of possible states (with the Natural laws fixed) from that of stability of the laws (the dynamics) under perturbations in the space of possible laws of Nature (possible Hamiltonians, say):

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\[\text{E.g. for a 6-dimensional Weyl equation, which according to the observation above, has degeneracies of states, this degeneracy would disappear under (almost) any modification, and hence the modified (perturbed) equation could not be interpreted as a 6-dimensional Weyl equation.}\]
I. Stability of states (keeping the dynamics, i.e. the Natural laws, fixed)

Taking into account the ubiquitous bath of small perturbations and “noise” which surrounds and “attacks” every object (in some specific “state”), it necessarily has to possess some degree of stability (along the directions of the most likely perturbations which “bath” the object) in order to exist in that “state” for a longer interval of time.

This is the type of stability that distinguishes a needle standing vertically on its tip as unstable versus a stable ball, say, lying in a (little) depression. To make the needle balance for a long interval of time requires enormously accurate fine tuning, although it is in principle possible.

II. Stability of a dynamics (a set of equations)

Rather than dealing with the stability of a “state” we would like to focus on the “stability of a dynamics”. In this case, the questions which we will address are whether certain features of the dynamical development are stable under modifications of the Hamiltonian or action, say.

II(a) “Structural stability” of a dynamical system

In particular, the concept of “structural stability” (first put forward by A. Andronov and L.S. Pontryagin more than fifty years ago) of a dynamical system is defined in the following way:

A given physical model, based on the system \( \dot{\xi} = F(\xi, c) \) is "structurally stable" if a slight change in \( F \),

\[
F \to F + \delta F, \quad ||\delta F(\xi)|| < \epsilon \quad \forall \xi \in \mathbb{R}^n
\]

does not result in "essential" changes in the solutions of this system, whereby is meant that the phase portraits of \( \dot{\xi} = F(\xi) \) and \( \dot{\xi} = F(\xi) + \delta F \) are topologically orbitally equivalent. In other words, there exist homeomorphisms that transform the phase space trajectories of the vector field \( F \) to those of the perturbed one \( F + \delta F \) (Cf., e.g., E. Atlee Jackson [25] Vol.I, p. 102, p.380. See, also, e.g. A.A. Coley and R.K. Tavakol [26]).

II(b) “Stability of Natural laws” (in the “random dynamics” spirit)

Removing the focus from “orbital topological equivalence” we would attempt to consider as the essential feature (the stability of which is to be defined)
some effective laws or regularities *appearing in some limit*\(^9\). We may say that
we have stability in the spirit of “random dynamics”\(^3\) if such effective laws
or regularities are unchanged under perturbations (“small” modifications) of
the (fundamental) dynamics of the system.

By analogy: Just like a “state” of a given Hamiltonian, say, have to
possess a certain degree of stability in order to exist for a longer span of time
(otherwise the perturbations have to be finetuned enormously), we speculate:
Could it be that the “Natural laws” also have to possess some degree of
stability? And how may one pursue this idea?

If it *is* so that the Natural laws\(^9\) depend substantially on finetuned
implementations of structure at the Planck scale, say, then there is very
little “room” for “mis-implementations” and “small errors”, as regards the
structure (the Natural laws) at this fundamental scale.

The belief that this is not the case, i.e. the belief that the Natural laws
indeed will show up to exhibit some degree of stability in order to “exist” -
that is the “random dynamics spirit”! (and the viewpoint have been investigat-
gated and implemented in the context of many different “toy-models”. Cf.,
e.g., C.D. Froggatt & H.B. Nielsen\(^5\) and references therein.

However, the “random dynamics” point of view have inherently some
small difficulties and obstacles which we shall now seek to discuss. (Section
3.2-3.5).

Note, also, that structural stability in the strict sense of \(\text{II(a)}\) demands
complete equivalence between phase portraits whereas “the random dynam-
ics” spirit stability only aspire to arrive at the same “form” in some “limit”,
where the same “form” means that you can interpret it as following the same
law(s). Even though \(\text{II(b)}\) is weaker requirement of stability than \(\text{II(a)}\) in
the sense of only caring for a limit, it is of course much stronger if we take
it, \(\text{II(b)}\), to mean stability of the Natural laws under “completely general”
perturbations (whatever that should mean ?, cf. discussions below).

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\(^9\)Dominantly, we shall have in mind this limit to be an *infrared limit*, i.e. physics at our
“human scales” where all the particle and interaction constituents have small momenta
and energies compared to some “presumed” fundamental scale of the “world machinery”
(this world machinery corresponds to the system \(\xi = F(\xi)\) of \(\text{II(a)}\) above).

But also other limits could be thought of: For instance the isospin symmetry of strong
interactions appears in the “limit” of both the up and the down quarks happening to be
very light compared to the scale given by \(\Lambda_{\text{MS}}\). We have, also, a proposal for understanding
the linearity and to some extend hermiticity of the Schrödinger equation as features
appearing in the limit of waiting very long, i.e. very late times. See, e.g., C.D. Froggatt
& H.B. Nielsen\(^5\)

\(^{10}\)Down to scales of \(\sim 10^{-15}\) meter or so, the Natural laws apparently do not change
significantly at different space-time points, since we seemingly have good conservation
of energy and momentum. It may be a bit hard even to imagine how there could be a
breaking of translation symmetry at even smaller scales - without causing problems by
being observable. But we may, either, e.g. imagine the coupling to long waves to be only
tiny \(^3\) or some sort of quantum fluid \(^{24}\).
3.2 How we have weakened the concept of stability in order for the Weyl equation to be stable?

Note especially, that in order to claim the stability of the Weyl equation we restricted the class of perturbations $\delta W$ (modifications) of that equation considerably: We outline below in which sense this basin of perturbations was restricted and comment for each item how we imagine one could - to some extent - relax the restrictions made.

1. The equation is restricted to remain linear and homogeneous in the Weyl spinor field $\psi$

One may argue that additional terms like $\psi^2, \psi^3, \ldots$ would be suppressed if the $\psi$ field itself is considered small. However there is a problem how to suppress the zeroth order term - i.e. the term not depending on $\psi$ - in the limit of weak $\psi$, unless one somehow argues that attention can be restricted to the homogeneous part of the equation, the inhomogeneous solution being just “background”.

2. The number of degrees of freedom (of the Weyl spinor) remained unchanged

We modified the Weyl equation for the Weyl spinor $\psi$ (with two degrees of freedom) with perturbations $\delta W$ restricted to be of the operator-form of $2 \times 2$ matrices.

“Elementary constituents” (elementary particles) often turns out to be “composite” when they are looked upon at higher energy scales (cf. the “quantum ladder”). I.e. they semi-always turn up to have additional degrees of freedom (“frozen in” in the infrared limit) which gradually “wake up” towards the ultraviolet. That is, the effective number of degrees of freedom of some object depends on the energy by which we “look” at it.

In particular, one could imagine a “random dynamics” project where the number of components of what becomes the Weyl-spinor $\psi$ field itself (in the infrared) did dependent on the energy-scales.

Indeed, the toy-model for a “world machinery” described in detail in section 3 of H.B. Nielsen & S.E. Rugh is to be considered as one possible attempt of relaxation on exactly this point. It is in fact argued that - generically - the Fermi surface would consist mainly of points of the dispersion relation with at most two degenerate states, because the points with more than two cannot separate filled and unfilled states for topological reasons. Very near the Fermi surface there should be at most two relevant components, i.e. effectively $n=2$.

We will return to the issue about additional degrees of freedom for the Weyl-spinor $\psi$ in section 3.6 where we are able to offer “An almost true theorem about the number of gauge bosons and fermion components”. However, this theorem is not fulfilled for the left handed quarks (Weyl spinors) - and there is, in fact, some instability of the left handed Weyl spinors (in the

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11 This remark stems from old unpublished work in collaboration with Sudhir Chadha.
restricted sense we have talked about here) when we take into account the
gauge degrees of freedom of the Weyl spinor. But it is remarkable that all
the other Weyl spinors, i.e. the right handed quarks and the leptons, are
stable (in our sense) - provided you do not allow the basin of perturbations
to allow mixing of one irreducible representation with another one.

3. The basin of equations are chosen to be selected from the class of differential equations.
This restriction could be relaxed in many ways. One often implement physics
on a fine lattice (for instance with difference equations) in the ultraviolet,
which would lead to differential equations only as “effective” equations in the
infrared. But that would really only work if the “lattice” used has enough
structure to be approximately a manifold from a long distance point of view.
Even in the section 3 of H.B. Nielsen & S.E. Rugh [8] where we attempt to
make a speculative model which relaxes many of the properties of the Weyl
equation we only relax the equation-operator to be an analytic one in \( x \)'s
(or \( q \)'s) and \( p \)'s or equivalently differential operators \( p = -i\partial/\partial x \). Since
analytic functions are at least approximated by polynomials it means that,
approximately, we did only allow polynomials in the differential operators
and thus essentially kept the requirement of there being a differential operator in the equation. We approximately kept a differential equation there.
Most important is presumably that we keep a kind of “locality” by keeping a
differential equation. To attempt to relax this assumption might of course be
interesting, but very likely this can only be done by asserting (in the same
go) that in almost any structure you can invent an approximate topology
and thus pretend to see some locality.

4. The space-time dimensionality \( d \) is kept fixed under the perturbation \( \delta W \) of the Weyl equation
Even if you allow a higher dimension of space(time) there would be no \( \sigma \)
matrices to go with the extra coordinates and momenta and one could show
that there would be no motion in the extra directions of the modified Weyl
particle. There would, however, be no obvious reason why a right handed
top quark and a right handed charm quark, say (i.e. fermions which belong
to different irreducible representations), should select the same directions in
which not to move. The story of “no motion” in more than 3 space directions
is also only true as long as the number of components of the Weyl
spinor is fixed to 2. However, we (at least) cite suggestive arguments in H.B.
Nielsen & S.E. Rugh [8] for that generically there will effectively only be two
components (see also [5, 10]).

In particular - provided no mechanism prevents different types of Weyl
particles from selecting different “directions of no motion” - human observers,
say, composed of Weyl particles of these different sorts, would be split up
in subparts (moving in different directions) each of which consists of Weyl
particles of a given type in the sense discussed above! Also, there would be
access to more than 3+1 dimensions using several types of Weyl particles in
this way.

When a higher dimensional Weyl spinor with \( d \geq 4 \) can move “fast” in more than three spatial coordinate directions this is a property that is strongly unstable (even restricting to a very small piece around a Fermi surface).

We remark, that every time we attempt to relax one of the restrictions 1,2,3 and 4 mentioned above, we very fast end up with a class of models which is infinitely much bigger than the restricted basin of the defined perturbations.

3.3 Preliminary discussion of some “principal boundaries” for which “basin of perturbations” we can formulate?

We would like to state the “stability” of the Weyl equation as impressive as possible. I.e. the Weyl equation would be stable under “all” perturbations - conceivable as well as inconceivable. This is, clearly, not possible to claim. We cannot claim that the Weyl equation comes out from *everything* (an elephant, say) - even in some limit? Nevertheless\(^\text{12}\) this is the “spirit in the random dynamics”!

Besides those restrictions of the basin just mentioned above (item 1,2,3 and 4) there may even be restrictions of a kind which we humans may not be fully aware - and which may represent principal boundaries we may therefore neither be able to transcend nor fully explore, even in the distant future (and therefore we cannot easily display such boundaries here).

If one should go so far as to even consider logic a part of physics, and thus make a “random physics” model to be also one which relaxes on the principles of our Aristotelian logic, it would get very hard indeed to formulate the basin or class of models of this type, let alone to prove something about that class.

The boundaries for what can be conceived or not are, surely, very hard to locate. As a simple illustration of a possible perturbation which you would hardly imagine with “old fashioned group theory” is the existence of “quantum groups” (e.g. \( SU(2)_q \)) conceived of as a continuous deformation of a group. If one should suggest a perturbation of the Weyl equation which was “analogous” to such continuous deformation of groups (“quantum groups”) we would have a very hard job to do! For instance, perturbation of the equation into some “relation” among symbols which was not even an equation.

However, it is - in our point of view - an interesting project to identify, and try to formulate in print, such fundamental boundaries and limitations\(^\text{12}\)

\(^{12}\)If we take, say, the limit of extremely low temperature the resulting “deeply frozen” elephant might have left only a very tiny fraction of its originally active degrees of freedom and it is not so easy to exclude totally that the remaining vibrations in the trunk and the still active spins could not, somehow, be interpret cleverly as Weyl equation(s) and/or the Standard Model?
for what can - ever - be thought and accomplished. For example, it is well known that the german philosopher Immanuel Kant and the danish physicist Niels Bohr were - seriously - contemplating such issues.\footnote{Cf, e.g., I. Kant “Prolegomena to Any Future Metaphysics” (a shorter, more easily readable version of “Critique of Pure Reason”) \cite{27}. Niels Bohr has discussed our “Conditions for description of Nature” at very many occasions, cf. e.g. \cite{28}.}

Niels Bohr, in particular, emphasized the important role played by our “daily language” which presents a “boundary” in the sense that it is hard to transcend - since we ultimately have to communicare all our ideas and knowledge in “daily language”.

We might add that any attempt to make “great theories” at a very fundamental level especially has to be accompanied with some interpretation back to the daily language. Any such model is thus necessarily complicated by being provided with such an unavoidable system of interpretation assumptions. Irrespective of how “unified” and how simple some ambitious “T.O.E.” would be it cannot be without some assumptions providing the translation from its own formalism into that of daily language. This fact sets a lower limit for its simplicity:

We cannot transcend that limit, but could we reach it?

You may say that the “random dynamics” project precisely intends to reach it. The goal (in the strongest form of “random dynamics”) is to argue that whatever the basic theory is - almost - we can by appropriate interpretation, i.e. by adding the appropriate “translation assumptions” into daily language, make it become a good and well functioning “T.O.E.” But then it becomes also tempting to really make use of this “interpretation assumptions” to the utmost.

### 3.4 So what could (should) “random dynamics” hope to arrive at?

Rather than claiming - as ideal “random dynamics” (a very strong form of random dynamics indeed \footnote{Both the concept “all” and “almost” are not defined a priori. (Presumably, they could only be specified in a way that would ultimately turn out to be rather arbitrary.)} - that \textit{all} (or almost all \footnote{Cf, e.g. I. Kant “Prolegomena to Any Future Metaphysics” (a shorter, more easily readable version of “Critique of Pure Reason”) \cite{27}. Niels Bohr has discussed our “Conditions for description of Nature” at very many occasions, cf. e.g. \cite{28}.}) models at the fundamental level lead to the well known phenomenology (i.e. the Weyl equation, general relativity, the standard model with its particular “skew” gauge group etc.) we should seek a \textit{large class of models} that will lead to the same infrared phenomenology. The “random dynamics” project intend to locate boundaries \( \partial B \) for this (huge?) class of models and the project (however, at the level of “Natural laws”) resembles somewhat the project of finding “universality classes” in phase transitions or other dynamical behaviors.

It is a natural contemplation, that if these boundaries are not too narrow, it leaves (logically) the possibility of having “chaos in the fundamental laws”...
- i.e. the “fundamental structure” may be selected randomly\footnote{And the structure may thus very well differ - in a random way - in various sub-parts of our “world machinery”. This idea is (in spirit) in some contrast to the idea that there is only one unique and universal pervasive “superstring”, say, (a “T.O.E.”) which - at some fundamental scale - is ubiquitously implemented “everywhere” - and implemented so “precisely” that it does not differ, not even by the tiniest “small error”, in any sub-part of our “world machinery”.} - if it is just selected from the class of models restricted by these boundaries.

Thus, in some sense, there is a no need for finetuning of the fundamental structure, except that it has to lie within the boundaries.

Nevertheless, we note, that these boundaries “$\partial \mathcal{B}$” (if these could be made precise enough to be stated formally, say, by mathematical formulas) in a certain sense themselves have status as “Natural Laws” - and so on, ad infinitum. Therefore, this way, one does not circumvent the concept of some “Natural laws” to be implemented.

It would be interesting if the the “random dynamics” project could lead to definitive conclusions as regards the class of fundamental models which would lead to the Standard Model in the infrared limit. It may turn out to be of small (zero) measure (in which case the fundamental structure have to be finetuned enormously) or - which is more in the random dynamics spirit - it may turn out that the models not leading to the standard model would have small (zero) measure.

“Standard philosophy”, as materialized in the last decades, for instance, in the search of one unique “superstring” is inclined towards the first viewpoint, while many arguments for the second viewpoint have been collected in C.D. Froggatt & H.B. Nielsen\footnote{And the structure may thus very well differ - in a random way - in various sub-parts of our “world machinery”. This idea is (in spirit) in some contrast to the idea that there is only one unique and universal pervasive “superstring”, say, (a “T.O.E.”) which - at some fundamental scale - is ubiquitously implemented “everywhere” - and implemented so “precisely” that it does not differ, not even by the tiniest “small error”, in any sub-part of our “world machinery”.}.

Apparently, it is easier to show “instability” than “stability”, because the latter requires stability towards all conceivable (and inconceivable, as well) perturbations, while the first requires only the unambiguous demonstration of instability in some directions of the conceivable perturbations.

If it turned out that the random dynamics project failed in the sense that it could point towards some regularities that has an instability - and thus these regularities have to be finetuned - this conclusion would in our opinion be a very interesting one, and point towards some essential feature of the entire “world machinery” of which we humans participate (so shortly).

In any case it is thus justified to investigate if a finetuning or not is needed. Working the way that one for every discovered law of Nature (which is based on empirical facts) remembers to investigate its stability could be conceived of as using “random dynamics ” as a scientific method.

3.5 Turning to the Weyl equation again

In order to allow ourselves to call the Weyl equation stable, we have - in addition to restricting the basin of perturbations (as discussed above) - to diminish the ambition of stability by restricting the features considered essential and therefore required to be unchanged under the modifications (the
perturbations).

Thus, when we claimed stability of the Weyl equation, under the considered class of perturbations (3.7), we only achieved this because we claimed that it remained the Weyl equation in spite of the fact that it actually was a completely new equation in the sense that the functional forms of the $A_\mu(x)$ and the $V^\mu_\lambda(x)$ had changed. It is only when we maintain that these functional forms are not essential for it being the Weyl equation that the Weyl equation can be called stable, as we did.

This deemphasis (needed for the stability) of the importance of the functional forms of the $x$-dependent coefficients $A_\mu$ (electromagnetic four potential) and $V^\mu_\lambda$ (vierbein), is achieved, or strengthened at least, by declaring that they are “dynamical”. By declaring them “dynamical” we mean that they depend on some bosonic degrees of freedom, say, so that they belong to initial conditions rather than to the laws of Nature. They should be more like the position or the momentum of a particle or the value of a field than like say the mass or the charge of a particle or some coupling constant. Very luckily for such a postulate of $A_\mu$ and $V^\mu_\lambda$ being “dynamical” is the fact that we happen to know phenomenologically already fields precisely having the type of interaction with the Weyl particles which $A_\mu$ and $V^\mu_\lambda$ have! These fields are for $A_\mu$ the electric 4-potential and for $V^\mu_\lambda$ the vierbein, which is easily identified with a gravitational field. In this sense we may say that we need the electromagnetic and the gravitational fields in order to pretend the “stability” of the Weyl-equation! So instead of considering this need for not considering the functional form essential a weakness of the idea of claiming the Weyl equation exceptionally stable in just 4 dimensions, it turns out to just fit nicely provided we have electromagnetic and gravitational fields (as dynamical fields). Rather than having a trouble for our claim we got a postdiction of a couple of well established forces in nature! 

Is the two dimensional Weyl equation really two dimensional?

Note, that if one imposes the stability requirement (of the restricted kind, we have discussed here), then all higher dimensions than 4 have been completely ruled out, they are not stable! Therefore, if “our creator” had chosen 6 dimensions say, he had to “finetune” the Weyl equation, for example in order to avoid terms like some differential operator or some other quantity multiplying a matrix not belonging to the $\sigma$ matrices of the 6 dimensional Weyl equation!

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16In the “toy model” which we put forward in section 3 of H.B. Nielsen & S.E. Rugh we construct (postulate) a very general and complicated quantum mechanical system (which we call the “bosonic” part of the full model) interacting in a random way with the “fermionic world machineries” which represented the Weyl particles. Since all the effect of the “bosonic part” on the Weyl particles is represented just by $A_\mu$ and $V^\mu_\lambda$ it means that this “bosonic part” provides precisely electromagnetic and gravitational fields. The model, however, has a few “small” troubles: The interaction of the vierbein and electromagnetic fields with themselves are a priori non-local, we still have the dimensions in which motion does not take place and, moreover, one needs very strong input assumptions about the state of the “world machinery” (the latter may, though, in principle be o.k.).
If we like to characterize, however, the dimension 4 by the discussed stability of the Weyl equation we meet the problem that the $d = 2$ dimensional Weyl equation is equally stable - or even more stable in a sense - than the four dimensional one. For $d = 2$ there are namely $n_{\text{Weyl}} = 1$ component only and hence only $1^2 = 1$ linearly independent matrices that can occur in the equation.

From the above formulas for the number of components it is easily seen that the Weyl equation in $d = 2$ dimensions has only $n = n_{\text{Weyl}} = 1$ component and thus we have in this case $d > n_{\text{Weyl}}^2$ and the Weyl equation must even use the same “$\sigma$-matrix” more than once. This means that it is very stable, but if one have to use the same $\sigma$-matrix twice, one can with good reason ask if this equation - the two dimensional Weyl equation - is truly two-dimensional! We would say that it is really not two dimensional, because the Weyl particle in two dimensional Minkowskian theory, say, can only move in one direction and always with the speed of light. It really only makes use of one light-cone variable, e.g. $x^+ = x^0 + x^1$. Thus, it is rather to be considered only 1-dimensional! If you accept this point of view a “truly $d$-dimensional Weyl equation which is stable in the sense described above must have equality between the number of linearly independent $\sigma$ or $\gamma$ matrices $n^2$ and the number of dimensions $d$, i.e. $n_{\text{Weyl}}^2 = d$ for the Weyl case.

3.6 An almost true theorem about the number of gauge bosons and the number of different Weyl particles

So far, we have considered only one single Weyl particle, but we ought to consider (at least) the states related by gauge transformations as various components of a single equation. Taking into account that there can be several gauge components, the number of components of the Weyl spinor increases\textsuperscript{17}. It is well known, that the Weyl spinor, indeed, has such internal degrees of freedom (in Nature) - although this has been neglected in the previous discussion - as the reader may have noticed. For instance we know today that the Weyl spinor has color and other gauge degrees of freedom. With such gauge fields interacting with the Weyl particle the Weyl equation still takes formally the form (2.1) but now the covariant derivative is not simply

$$D_\mu = \frac{1}{i} \Omega(x, p) = \partial_\mu - ieA_\mu(x)$$ (3.9)

(ignoring the vierbein by taking it to unit matrix), but rather, say

$$D_\mu = \frac{1}{i} \Omega(x, p) = \partial_\mu - ie\frac{\lambda^a}{2}A_\mu^a(x)$$ (3.10)

where the $\frac{\lambda^a}{2}$'s make up a basis in the representation $r$ (for the Weyl particle in question) for the representation of the Lie algebra of the Yang Mills gauge group.

\textsuperscript{17} This observation (obstacle) is due to a question by Prof. Dragon at the conference
I.e. there is one $\lambda^a$ matrix for each dimension of the Lie algebra, and that again means one $\lambda^a$ matrix for each “color state” of the gauge particle (each gauge particle we could also say). In the Weyl-equation operator $\sigma^\mu D_\mu$ thus occurs the term

$$\sigma^\mu \frac{\lambda^a}{2} A^a_\mu.$$  \hspace{1cm} (3.11)

in the Yang-Mills case and thus the Weyl field $\psi$ really must have $n_{\text{weyl}} \cdot d_r$ components, where $d_r$ denotes the dimension of the representation $r$ under which the Weyl particle transform under gauge transformations. This representation dimension $d_r$ is

$$d_r = \left( \text{number of different Weyl particles} \right).$$

while, as we remember,

$$n_{\text{weyl}} = \left( \text{number of components of the Weyl spinor} \right).$$

The number of matrices that are needed to form a basis is the square of the total number of components, meaning $(n_{\text{weyl}} \cdot d_r)^2$. The condition for stability of the Weyl equation (now coupled to “color” Yang-Mills degrees of freedom) - if we define it in an analogous manner to the equation (2.20) above - is therefore given by the following:

“Almost true” theorem

$$\left( \text{number of different Weyl particles} \right)^2 \times \left( \text{number of components of the Weyl spinor} \right)^2 = \left( \text{number of gauge bosons} \right) \times \left( \text{space-time dimension} \right).$$ \hspace{1cm} (3.12)

Here we have used that the number of different $A^a(x)$-components is

$$d_g d = \left( \text{number of gauge bosons} \right) \times \left( \text{space-time dimension} \right).$$

This can be seen easily, since there is an $a$-value for each of the $d_g$ dimensions of the gauge group.

If we take into account that (at least empirically) we have already the equation (2.20), our main observation implies

$$\left( \text{number of components of the Weyl spinor} \right)^2 = \left( \text{space-time dimension} \right).$$ \hspace{1cm} (3.13)

and combining this equation with the more general stability condition (“the almost true theorem”) we get:
\[
\left( \text{number of different Weyl particles} \right)^2 = \left( \text{number of gauge bosons} \right)
\]

(3.14)

Note, that this splitting of equation (3.12) into (3.13) and (3.14) can be derived from the fact that otherwise the matrices occurring in (3.11) would not be linearly independent.

So this is a prediction from the extension of the observed stability of the Weyl equation. Is it true empirically?

Had it not been for left handed quarks, this would have been an entirely true prediction provided, though very strongly, that we (as seen by Prof. Dragon) only considers one single irreducible representation at a time. That is to say: For each irreducible representation of Weyl particles - such as the right handed strange quarks, say - we count the number of gauge particles coupling non-trivially to this irreducible representation and use that as “the number of gauge particles”. As the number of different Weyl particles we use the number of different Weyl particles in the irreducible representation in question.

then, indeed, the equation (3.14) is fulfilled for all the irreducible representations - except for the left handed quarks!

For example, for the left handed positron only one linear combination of gauge bosons couples - that of the weak hypercharge - and there is only one left handed positron in the irreducible representation, so our equation is fulfilled with \( 1 = 1 \). Actually, in this case our previous discussion worked (in which we neglected that there is more than one component).

Another example is, say, one of the left handed antiquarks: There are 3 different Weyl particles in the irreducible representation, and to them couple 8 gluons and the weak hypercharge gauge boson (a certain linear combination of \( \gamma \) and \( Z^0 \)). That is \( 8 + 1 = 3^2 \).

Only for an irreducible representation of left handed quark we are off: 12 gauge bosons but 6 Weyl particles. That does not fit since \( 12 \neq 6^2 \).

In fact it is so that for all other irreducible representations than the left handed quarks one finds a true representation of a \( U(N) \) factor group of the standard model group \( SMG = S(U(2) \times U(3)) \). Here e.g. \( N = 1 \) for the left handed positron, \( N = 2 \) for the left handed leptons, and \( N = 3 \) for both types of left handed antiquark representations. In such cases one finds

\[
d_r = \left( \text{number of different Weyl particles} \right) = N
\]

(3.15)

and

\[
d_g = \left( \text{number of gauge bosons} \right) = N^2
\]

(3.16)

so we would have the equality in just 4 dimensions (3 + 1). If we make the very important assumption of looking at only one irreducible representation of the gauge group at a time we find that indeed we have the stability in the
standard model for all of them except for the left handed quark representa-
tions.

So had it not been for left handed quarks we could have claimed that the
stability of the Weyl equation as used by Nature (i.e. the Standard Model)
not only supports the dimensionality of space-time but also the type of gauge
group representations found in the standard model. Unfortunately for the
achievements of this principle of stability we have an exception by the left
handed quarks, or in other words: it would predict that there should not
have been left handed quarks, but otherwise it would have worked well for
each irreducible representation separately.

This latter lack of beauty means that the stability principle only is o.k.
provided there is some rule forbidding that different irreducible representa-
tion of fermions can get mixed in the attempts to modify the Weyl equa-
tion(s).

The fact, that instability occurs when mixings are allowed (here mixings
between irreducible representations) is also what we would see if we thought
of the Dirac equation instead of the Weyl equation. That would namely mean
allowing mixing of left and right handed components, and again no stability.

In conclusion, it appears that “our creator” has made the Weyl equations
as stable as possible! This has been achieved both by choosing the dimension
to be 4 = 3 + 1 (and not, for instance, 8) and by choosing the representa-
tions and the gauge group so that in (at least) most cases each irreducible
representation taken separately is stable.

The exception of the left handed quarks it is, in fact, rather unavoid-
able if there have to be gauge anomaly cancellations and we still want mass pro-
tected particles. So, very likely, the choice of the Standard Model is - in some
way of measuring it - the most “stable” possibility.

So stability may even be a principle behind the representation choice,
too! Not only (as we already saw) behind the dimension and the existence
of gauge and gravitational fields.

It must be admitted, however, that there is the little problem with the
working of this principle now where we have included several components
and non-Abelian gauge fields: We also get the vierbein and thereby the
gravitational field into a nontrivial representation of the gauge group. That
particular feature is not so good from the empirical point of view!
4 Concluding remarks and dreams

In the present contribution we have made an empirical observation (sec. 2.4) connected to the Weyl equation, suggested an interpretation in terms of “form stability” of the Weyl equation (sec. 3), and - in a search of expanding the basin of allowed perturbations (modifications) of the Weyl equation (under which we would like to claim its stability) - we have also challenged the concept of “stability of Natural laws”. The latter part (sec. 3.2-3.5) is a kind of preliminary attempt to locate principal boundaries for the set of ideas which we call the “random dynamics” project, described, e.g. in C.D. Froggatt & H.B. Nielsen [4].

As concerns our empirical observation, we have seen that the dimension \( d = 4 \) obey the equation (2.21),

\[
    d = \begin{cases} 
        (2^{d/2} - 1)^2 = 2^{d-2} & \text{for } d \text{ even}, \\
        (2^{(d-1)/2})^2 = 2^{d-1} & \text{for } d \text{ odd}, 
    \end{cases}
\]

which - via the step of the sigma-matrices forming a basis (sec. 3.1) - leads to the suggestion that just in the phenomenologically true dimension \( d = 4 \) is the Weyl equation especially stable. This is so to say the message from “our creator” brought via the dimension. In order to implement this stability it was at first noticed that at low energy (and momentum) Taylor expansions had to be used and - secondly - that we needed both gravitation and electromagnetism. We may put it the way that in order to make the idea of the Weyl equation being stable, as suggested, one predict in retrospect these two forces of nature. Really we could use the non-Abelian Yang Mills fields, as found in the Standard Model, instead of simple Abelian electrodynamics but there are a couple of severe problems at this point:

1. One must be satisfied with looking only for stability under perturbations in which fields in the same irreducible representation are allowed to mix, i.e. we must require linearity and homogeneity in each irreducible representation of Weyl fermions separately. (In the speculative model put up in section 3 of H.B. Nielsen & S.E. Rugh [5] one might hope that mild continuity requirements will turn out to be enough for implementing such a rule. But, a priori, it complicates matters to need such a separation of the different irreducible representations).

2. Even with the restriction to “no mixing” of different irreducible representations the left handed quarks do not fit into the scheme. The problem is that the entire standard model group has only dimension 12 while the number of left handed quark Weyl fields is 6 times the number of components for a single Weyl two-component field. This means that the number of allowed matrices is enhanced by a factor \( 6^2 = 36 \) due to the color and weak isospin degrees of freedom. The number of Yang-Mills field components is, however, correspondingly only enhanced by 12, and that is less than 36. In order that the “form stability” of the Weyl equation shall work for the left handed quarks it would be appropriate with a \( U(6) \)-gauge theory rather than, just,
the Standard Model. Such a model would, however, have gauge anomalies and not have a consistent gauge symmetry.

We presume, that a strongly related expression for the special “form stability” of the Weyl equation is the fact that just in the dimensions $d = 3$ and $d = 4$ is the number of helicity states of a Weyl particle (antiparticle not included) precisely one, so that there is no degeneracy.

In both ways of looking at our result there is a need for a separate discussion of the two dimensional ($d = 2$) case since this is, at first, even more stable than the experimentally observed case ($d = 4$). However, one may claim that this case is not truly two-dimensional since it does not (truly) make use of both dimensions (cf. section 3.5).

Taking (in spite of the “small” troubles) the special stability of the 4-dimensional Weyl equation as a trace in the search for fundamental principles in physics we have then looked into how much we may expand the basin of allowed perturbations (modifications) and still claim the Weyl equation to remain “form stable”. In a sense, such a search is, precisely, what is done in H.B. Nielsen & S.E. Rugh in which we have put up a very general “toy-model” for a pregeometry (a “world machinery”). The latter type of model may in fact be thought of as representing a very general type of linear homogeneous differential equation conceivable as describing a rather general quantum mechanical “machinery” having partly a classical analogue and obeying the smoothness conditions associated with the smoothness properties of this classical analogue. It must, however, be admitted that e.g. Fermi statistics and the identical particle property is rather put in than derived, so far.

Even if there are several troubles (as we have described, sec. 3.2-3.6) in interpreting the Weyl equation as totally stable against all imaginable perturbations, we would still claim that the choice of just 4 dimensions, indeed, points to some sort of stability as a guiding principle.

“Our creator” has also chosen the representations in the standard model so that the Weyl equations become as stable as possible! (at least very stable compared to what could have happened). When He namely use - as is the case except for the left handed quarks - defining representations of an $SU(N)$ group having also a nontrivial Abelian charge there are $N^2$ gauge fields each with $d = 4$ Lorentz components and also 4 times $N^2$ linearly independent matrices. Only for the left handed quarks for which we get 6 times 2 components there are 144 linearly independent matrices but only 12 times 4 gauge potentials to adjust (i.e. $48 \neq 144$).

It turns out that there is a conflict between making the Weyl equation stable and keeping mass-protection (i.e. unless we satisfy the no anomaly condition trivially). For instance, if one simply postulated some gauge fields in a group $SU(6)$ extending the group $SU(2) \times SU(3) \subseteq SU(6)$ providing all the (special) unitary transformations of an irreducible representation corre-

\textsuperscript{18}The “no degeneracy” principle leaves also $d = 3$ as a competitor to the experimental dimension $d = 4$, but if we assume that we only have access to low energies and thus only shall see mass-protected particles we could on the that ground exclude the odd space-time dimensions anyway.
sponding to a set of left handed quarks this field would have anomalies in
the gauge symmetry. I.e. the gauge symmetry could not be upheld at the
quantum level unless further particles were added to the model to compen-
sate the anomaly. But then, of course, the added field might be in danger of
lacking stability again.

Hence, a claim that there should be enough gauge fields to realize the
“form stability” proposed seems not to be fully realized in Nature - although
Nature comes close - but it is also not realizable with mass protected fermions:
One simply cannot cancel the gauge anomalies and at the same time have
all the fermions in the defining representations unless we choose a trivial
cancellation which will spoil the mass-protection. With so many defining
representations as are used in the Standard Model one might then think that
the Standard Model does its best possible to be maximally “stable”.

4.1 Could our observation be an accident?

Which “signals” in phenomenology carry much information and which “sig-
als” carry only small amounts of information?
The purpose of reading out “signals” from phenomenology is of course that
this is the way we may really learn from Nature rather than only from our own
speculations. It is, however, only to be trusted when the number explained
is sufficiently complicated that it is not too easy to find an explanation for it.
There must be a fair amount information (which have previously not been
understood) that finds an explanation, otherwise the explanation will not be
convincing.

Moreover, there can only be one explanation of a given phenomenon -
such as the dimensionality of space-time considered here. If there were two
fundamental (independent) explanations of a given interesting number (such
as \(4 = 3 + 1\) dimensions) then it would be a strange mathematical accident
that these two explanations would give the same result.

In that sense fundamentally different attempts of arriving at the number
4 (for instance, the alternative attempts [11] - [21] mentioned in the list of
references) cannot be true - provided, for instance, that our observation is
not an “accident”.

\[19\]In fact, what favours the stability is that the representation is so “small” as possible
compared to the “size” of the gauge group or rather the part that is not represented
trivially. So maximal stability of the Weyl equation would roughly correspond to smallest
possible representations of the fermions. Now it happens that S.Chadha and one of us
pointed out that the Standard Model representations are characterized as the “smallest”
possible ones with mass-protection and no gauge-anomalies (we thank J. Sidenius for
finding that a single alternative solution is excluded by the requirement of no mixed
anomaly). So it is from that work really suggested that the Standard Model is as stable
as it would have been possible to construct it without loosing mass-protection and/or the
anomaly cancellations.

\[20\]That our observation is not an accident would be strongly supported if we could
obtain from the Weyl equation - in a related way - not only the dimension 4 but also its
splitting into 3+1. The Weyl equation supplemented with the - rather mild and reasonable
- assumption that by appropriate multiplication with a matrix the Weyl-equation operator
Note, that the dimensionality of space-time is known with several digits precision,

\[ d = 4 = 4.00. \]

But almost all these digits are “0”s, and therefore they do not carry so much information - modulo that most underlying theories, by construction, have an integer number of dimensions.

It would be easier to get reliable information from the electromagnetic finestructure constant, say,

\[ \alpha = \frac{e^2}{\hbar c} = \frac{1}{(137.0359895 \pm 0.0000061)} \]

which is such an “irregular” number (measured to a precision of 8 digits or so) that it probably carries a huge amount of information to support or destroy a theory which aims at predicting this constant (we remark that no theory exist today which claims to arrive at \( \alpha \) with 8 digits precision\(^\text{21}\)).

Compare, also, e.g. with the Standard Model group \( S(U_2 \times U_3) \). If we want to predict this group, relative to other possible groups with up to 12 generators (gauge bosons), we have of the order of 240 groups \(^\text{22}\). We conclude that to predict the true gauge group is comparable to predicting a number with two significant digits.

If we, by accident, find several fundamentally different explanations for the observed dimensionality \( d = 4 \) of our space-time then it would presumably indicate that the number “4” is too small and too simple (there is not encoded information enough in this number) for reading out interesting structural information from it. In the light of that several explanations - which seems reasonable and fundamental, e.g. the stability of (classical or quantum) systems governed by the Coulomb or Newtons law - coexist, we fear that the number “4” is too simple to be readable in this sense.

\( \sigma^\mu D_\mu \) becomes Hermitean can actually only be realized provided the splitting is into an odd number of times (and an odd number of space-dimensions; we think of the even \( d \) case). A relatively easy series of algebraic manipulations shows this fact. For the troubles (as far as the Dirac or Weyl equations are concerned) caused in “euklidianization” due to the difference between “4” and “3+1”, see e.g. S. Coleman \(^\text{23}\).

\(^{21}\) Cf. e.g., “Review of Particle Properties”, Phys. Lett. B 204 (April 1988).

\(^{22}\) Actually it happens that one of us would claim to have quite an impressive “explanation” of all the three finestucture constants in the Standard Model by relating them to the (or a) “multi-critical point” (where several phases of a - lattice - gauge theory meet) But these fits have rather only one significant digit, although that seems to us already suggesting that there is some truth behind. Cf. D.L. Bennett & H.B. Nielsen, in preparation.
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