A Microscopic Model of Edge States of Fractional Quantum Hall Liquid: From Composite Fermions to Calogero-Sutherland Model

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I. INTRODUCTION

The basic characteristic of the quantum Hall states is the incompressibility of the two-dimensional (2-d) electron system in a strong perpendicular magnetic field [1]. While there is a finite energy gap for particle-hole excitations in the bulk, the low-lying gapless excitations are located at the edge of the quantum Hall liquid [2]. The theoretical picture of the edge states of fractional quantum Hall (FQH) effect is beyond the Fermi liquid framework and known as the chiral Luttinger liquid (CLL) [3,4]. Recently, the edge excitations of FQHE effect (FQHE) were studied by several groups numerically [5] or in accordance with the Calogero-Sutherland model (CSM) [6] as well as the composite fermion (CF) picture in the Hartree approximation [7].

In previous works [8], we have given a microscopic model of the CF for the edge excitations at the filling factor \( \nu = 1/m \), \( m \) is odd integer. It was seen that at \( \nu = 1/m \) the CF system [8] reduces to the original CSM [9]. And the low-lying excitations of the edge, then, are governed by a CLL. We further applied this microscopic theory to analyze the tunneling experiment [8] and got a better fit to the features of the measured current-temperature curve [8]. However, the CLL is applied not only to the one channel edge excitation, i.e., \( \nu = 1/m \) but also to the multi-channel case, for example, at \( \nu = \frac{\nu^*}{\phi^*+1} \) for integer \( \nu^* \) and even number \( \phi > 0 \) [8]. Furthermore, a systematic experimental study of the current-voltage (I-V) characteristic for the electron tunneling between a metal and the edge of a 2-d electron gas at a fractional filling factor \( \nu \) shows a continuous non-Ohmic exponents \( \alpha = 1/\nu \), i.e., \( I \sim V^\alpha \) [10]. This is contradict with the prediction of the CLL theory in which, say, \( I \sim V^3 \) for the primary filling factor being 1/3. The experimentalists even found that such an I-V characteristic is observed at the filling factor \( \nu = 1/2 \) for which the bulk states are compressible. Several authors have made their efforts in explaining these phenomena [11,12]. Lee and Wen recently proposed a two-boson model for FQHE regime in which the spin velocity is much slower than the charge’s and then the long time behavior shows the exponent \( \alpha = 1/\nu \) while the short time behavior complies with the Fermi statistics of the electrons [13]. They also use the theory of the half-filling Landau level proposed by Lee [14] to explain the I-V curve observed in the experiment.

In this work, we would like to generalize our microscopic derivation to the CLL at \( \nu = 1/m \) to that at \( \nu = \frac{\nu^*}{\phi^*+1} \). (For the bulk state, Cappelli et al have discussed this stable hierarchical quantum Hall liquids from W(1 + \infty) minimal model [13].) Tracing the clue of the previous works [8], it is seen that the edge theory at \( \nu = \frac{\nu^*}{\phi^*+1} \) is basically described by the SU(\( \nu^* \)) CSM, i.e., the \( \nu^*-\)branch theory whose low-lying excitations behaves a charge-‘spin separation form where the \( \nu^* - 1 \) spin branches run in the same direction as the charge branch’s. However, it is not the case for that at \( \nu = \frac{|\nu^*|}{\phi^*+1} \) (\( \nu^* < 0 \)). The edge theory at these filling factors does not correspond to an exact soluble model. Fortunately, we could still find a good approximation ground state wave function and then the \( |\nu^*| \) branch low-lying excitations where the charge branch runs in the opposite direction to the spin branches. We show that the exclusion statistics between the branches is described by the K matrices [13]. Thus, using the bosonization procedure developed in [13] the system with the exclusion statistics matrix K can be taken as the fixed point of the multi-channel Luttinger liquid. From the radial wave equation of the system, one can show that the residual magnetic field provides a gap between right- and left-moving modes in a single branch of these low-lying excitations. This verifies the CLL at the \( T \rightarrow 0 \) limit. From this microscopic picture, the spin and charge velocities can be estimated and one finds that \( v_s^* \ll v_p^* \). This supports the two-boson theory of Lee and Wen.

This paper is organized as follows: In section II, we present our frame of work and explain the approximation we used. In section III, we discuss the solution of the edge Hamiltonian and get the SU(\( \nu^* \)) Calogero-Sutherland model. In section IV, we discuss the robustness of the exponents of the CLL. In section V, we prove the chirality of the edge states. In section VI, we discuss...
the edge states with filling factors \( \nu = 2/3, 3/5, \ldots \). The section VII gives our conclusion.

II. COMPOSITE FERMIONS AT EDGE

A. General Formalism

The two-dimensional interacting electrons which are polarized by a high magnetic field are governed by the following Hamiltonian

\[
H_{el} = \sum_{\alpha=1}^{N} \frac{1}{2m_b} [\vec{p}_\alpha - e \vec{A}(\vec{r}_\alpha)]^2 + \sum_{\alpha<\beta} V(\vec{r}_\alpha - \vec{r}_\beta) + \sum_\alpha U(\vec{r}_\alpha),
\]

where \( V(\vec{r}) \) is the interaction between electrons. \( m_b \) is the band mass of the electron and \( U(\vec{r}) \) is the external potential. The composite particle transformation will bring us to a good starting point to involve in the FQHE physics as many successful investigations told us \([20]\). We begin with the CF transformation which reads

\[
\Phi(z_1, \ldots, z_N) = \prod_{\alpha<\beta} \left[ \frac{z_\alpha - z_\beta}{|z_\alpha - z_\beta|} \right]^\delta \Psi(z_1, \ldots, z_N),
\]

where \( \Phi \) is the electron wave function. The CF consists of an electron attached by \( \phi \) flux quanta. By using the CF theory, the bulk behavior of the FQHE has been well-understood \([3,22]\). We, now, would like to study the microscopic theory of the CF edge excitations. The partition function of the system is given by

\[
Z = \sum_{N^e} C_N^{N^e} \int_\partial d^2 z_1 \ldots d^2 z_{N^e} \int_B d^2 z_{N^e+1} \ldots d^2 z_N \times \left( \prod_\delta |\Psi_\delta|^2 e^{-\beta(E_\delta + E_g)} + \prod_\gamma |\Psi_\gamma|^2 e^{-\beta(E_\gamma + E_g)} \right),
\]

where we have divided the sample into the edge \( \partial \) and the bulk \( B \). \( E_g \) is the ground state energy and \( E_\delta \) are the low-lying gapless excitation energies with \( \delta \) being the excitation branch index. \( E_\gamma \) are the gapful excitation energies. At \( \nu = 1/\hat{\phi} \), the low-lying excitations are everywhere in the sample and we do not consider this case here. We are interested in the case \( \nu = \nu_{\varphi^*} \pm 1 \), where the bulk states are gapful. The low-lying excitations are confined in the edge of the sample. For convenience, we consider a disc geometry sample here. The advantage of the CF picture is we have a manifestation that the FQHE of the electrons in the external field \( B \) could be understood as the IQHE of the CFs in the effective field \( B^* \) defined by \( B^* \nu^* = B \nu \). The energy gap in the bulk is of the order \( h\omega_c^* \) with the effective cyclotron frequency \( \omega_c^* = e B^* / m^* \). Hereafter, we use the unit \( \hbar = e/c = 2m^* = 1 \) except the explicit expressions. By the construction of the CF, the FQHE of the electrons can be described by the IQHE of the CFs \([9]\) while the electrons in the field with the filling factor \( \nu = 1/\hat{\phi} \) could be thought as the CFs in a zero effective field. Thus, a Fermi-liquid like theory could be used \([24]\) and we have a set of CF-type quasiparticles. Applying the single particle picture, which Halperin used to analyze the edge excitations of the IQHE of the electrons, to the edge excitations of the CFs, one could have a microscopic theory of the quasiparticles at the edge. In the low-temperature limit, the domination states contributing to the partition function are those states that the lowest Landau level of the CF-type excitations is fully filled in the bulk but only allow the edge CF-type excitations to be gapless because the gap is shrinked in the edge due to the sharp edge potential. The other states with their energy \( E_g + E_\delta \) open a gap at least in the order of \( h\omega_c^* \) to the ground state. In the low-temperature limit, \( k_B T \ll h\omega_c^* \), the effective partition function is

\[
Z \simeq \sum_{\delta, N^e} C_N^{N^e} \int_\partial d^2 z_1 \ldots d^2 z_{N^e} |\Psi_{e,\delta}|^2 e^{-\beta(E_\delta(N^e) + E_g,\delta)} \times \sum_{N^e} C_N^{N^e} \text{Tr}_{(edge)} e^{-\beta(H_g + E_g,\delta)},
\]

where the trace runs over the low-lying set of the quantum state space for a fixed \( N_e \) and, according to the single particle picture, \( \Psi_{e,\delta} \) are the edge many-quasiparticles wave functions. \( E_\delta(N^e) \) is the eigen energy of the edge quasiparticle excitations and \( E_{g,\delta} \) is the bulk state contribution to the ground state energy. For the disc sample, the edge quasiparticles are restricted in a circular strip near the boundary with its width \( \delta R(\vec{r}) \ll R \) while the radius of the disc is \( R \). The edge Hamiltonian of CFs reads

\[
H_e = \sum_{i=1}^{N^e} [\hat{p}_i - \hat{A}(\vec{r}_i)] + \sum_{i<j} V(\vec{r}_i - \vec{r}_j) + \sum_i U_{eff}(\vec{r}_i),
\]

where the external potential \( U_{eff} \) is the effective potential including the interaction between the edge and bulk particles. The band mass \( m_b \) has been phenomenologically replaced by the CF effective mass. We suppose the potential is an infinity wall for \( r \geq R \). The statistics gauge field \( \vec{a} \) is given by

\[
\hat{a}_e(\vec{r}_i) = \frac{\hat{\phi}}{2\pi} \sum_{j \neq i} \frac{\hat{z} \times (\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2}, \quad \hat{a}_b(\vec{r}_i) = \frac{\hat{\phi}}{2\pi} \sum_a \frac{\hat{z} \times (\vec{r}_i - \vec{r}_a)}{|\vec{r}_i - \vec{r}_a|^2},
\]

where \( a \) is the index of the bulk electrons. Taking the polar coordinate \( x_i = r_i \cos \varphi_i, y_i = r_i \sin \varphi_i \), the vector
potential $A_x(r_i) = \frac{B_i}{2} r_i$ and $A_z(r_i) = 0$. In the mean-field approximation, $a_{i,b}(r_i) = 0$ and $a_{z,b}(r_i) = B_z r_i/2$. Substituting the polar variations and the vector potential to $H_e$ while using the mean-field value of $\bar{a}$, one has

$$H_e = \sum_i \left[ \frac{\partial^2}{\partial r_i^2} - \frac{i}{r_i} \partial_{r_i} \frac{B_z^*}{2} r_i + \frac{m(N_e - 1)}{2r_i} \right]$$

where the residual magnetic field $B^* = \frac{\omega_c}{2} B$. The manifestation of CF picture is that the $\nu^*$ denotes the highest Landau level index of CF in the residual magnetic field. Our central focus is to solve the many-body problem $H_e\Psi_e(z_1\sigma_1, ..., z_N\sigma_N) = EW_e(z_1\sigma_1, ..., z_N\sigma_N)$ where $\sigma_1 = 1, ..., \nu^*$ is the Landau level index which we call spin hereafter. And the many-body wave function $\Psi_e$ has to be consistent with the bulk state.

B. Reduce to Calogero-Sutherland Model

In the previous works [8], we have presented an example to solve the problem at $\nu = 1/m$, i.e., $\nu^* = 1$, and see that the edge ground states can be directly related to Laughlin wave function in the bulk. However, it is not applied to a general FQH state with $\nu = \frac{\nu^*}{\nu^* + 1}$.

It seems that the FQH states with $\nu^* > 0$ can be more easily handled than the states with $\nu^* < 0$ as seen below. We start from the easier one. To the zero order of $V$, we first switch off this interaction. Without loss of the generality, one takes the trial wave function is of the form

$$\Psi_e(z_1\sigma_1, ..., z_N\sigma_N) = \exp \left[ \sum_{i<j} \frac{t_{\sigma_1,\sigma_j}}{R} \cot \frac{\phi_{ij}}{2} - \frac{A}{2} \frac{r_i - r_j}{R} (J_{\sigma_1} - J_{\sigma_j}) \right]$$

where $t_{\sigma_1,\sigma_j}$ is a parameter matrix to be determined and so is $A$. $J_{\sigma_1}$ is the spin quantum number. The radial wave function $f$ is symmetric and the azimuthal wave function $\Psi_e$ is anti-symmetric in the particle exchange. To be consistent with the bulk wave function, the azimuthal wave function takes its form,

$$\Psi_e(\varphi_1\sigma_1, ..., \varphi_N\sigma_N) = \prod_{i<j} \phi_{ij} \cdot \prod_k \xi_k^{l_{\varphi}}$$

where $\phi_{ij} = |\xi_i - \xi_j| \hat{\phi}(\xi_i - \xi_j)^{\delta_{i,j}} \exp(i\pi \frac{n}{2} \text{sgn}(\sigma_i - \sigma_j))$, (10)

where $\xi_i = e^{i\varphi_i}$. In the one-dimensional (1-d) limit, taking $\delta R/R \to 0$ in [8] after acting on $\Psi_e$, the Hamiltonian on $\Psi_e$ yields $H_{cs}$ on $\Psi_e$ with

$$H_{cs} = \sum_i (i \frac{\partial}{\partial r_i} + B_z^* R)^2 + \frac{\omega_c}{2} \sum_{i<j} \phi(\phi + P_{\delta\sigma\sigma})$$

where $x_{ij} = x_i - x_j$, $\varphi_i = \frac{2\pi x_i}{L}$ and $L = 2\pi R$ is the size of the boundary. $P_{\sigma\delta\sigma'}$ is the spin exchange operator. To arrive at (11), the matrix $t_{\sigma_1,\sigma_j}$ is taken as $\frac{1}{2} \delta_{\sigma_1,\sigma_j} - i$ and $A = 1/N_e$. The Hamiltonian (11) is just the SU(\nu*) CSM Hamiltonian with a constant shift to the momentum operator [9,10] and the ground state wave function is given by taking CFs in each branch have the same number $M$ and $J_1 = ... = J_{\nu^*} = (M - 1)/2$ in $\Psi_e$. In principle, $H_e$’s Hilbert space in the 1-d limit can be larger than $H_{cs}$’s. However, we believe that all interesting azimuthal physics have been included in $H_{cs}$.

Moreover, from the zeros of the ground state wave function, one can read out the exclusion statistics matrix $K$ [9]

$$K_{\sigma\delta} = \hat{\phi} + \delta_{\sigma\delta}$$

Here we see that the mutual exclusion statistics can be different from the mutual exchange statistics by a Klein factor. It is easy to see that $\nu = \frac{\nu^-}{\nu^* + 1} = \sum_{\sigma\delta}(K^{-1})_{\sigma\delta}$, which shows the consistence between the edge and the bulk states. Furthermore, the asymptotic Bethe ansatz (ABA) equation which determines the pseudomomentum $n_{\sigma\delta}$, according to the $K$-matrix (12). The pseudomomentum $n_{\sigma\delta}$ relates to $J_{\sigma\delta}$ in a complicated way and we do not show it explicitly. Because we have used an SU(\nu*) symmetric form to construct the azimuthal wave function, the ABA equations are symmetric for the spin indices, which coincides with the symmetric $K$ matrix.

III. NON-RENORMALIZATION OF THE EXPOENTS

In a work characterizing the Luttinger liquid in terms of the ideal exclusion gas [10], we have bosonized the single component CSM and arrived at the single branch Luttinger liquid. This procedure can be generalized to bosonize the SU(\nu*) CSM. The generalization is somewhat trivial but tedious. We do not present the details for the many branch model here because the result is just as expected–a $\nu^*$ branch Luttinger liquid with the commutation relations between the neutral edge excitations $\rho_{mn}$ [8]

$$[\rho^a_{\sigma\delta\sigma'}, \rho^b_{\sigma'\rho_{\sigma'd}}] = \delta_{ab}(K^{-1})_{\sigma\delta\sigma'} \frac{n}{2\pi R} \delta_{n+n'}$$

where $a,b = L, R$ are the indices of the left- and right-movings. The SU(\nu*) symmetry leads to all excitations have the same velocity. However, we have to face two problems: i) The edge excitations are chiral which has not shown in the previous discussion. ii) The robustness of the CLL exponents to the perturbations. In this section, we focus on the latter by taking the one-component CCL as an example.
It is well-known that the CSM is an example of one dimensional ideal excusion gas (IEG) with the statistical parameter $m$. And the IEG is proved to describe the fixed point of the Luttinger liquid. The bosonization of CSM shows that the low-lying excitations are governed by a $c = 1$ CFT with the compactified radius $1/\sqrt{m}$ . In this section, we would like to show that at least some kinds of short-range interactions between the CFs do not renormalize the topological exponent $g = m$ under the condition that the scatterings with large momentum transfer (including backward scattering and umklapp scattering) are absent because of the chirality.

To deal with the CSM with interactions, we begin with the asymptotic Bethe ansatz (ABA) equation, 

$$k_i L = 2\pi I_i + \sum_j t \theta(k_i - k_j),$$

where $k_i$ is the pseudomomentum of particle $i$, $L$ is the size of the one dimensional system concerned, and $I_i$ gives the corresponding quantum number, which is an integer or half-odd. $\theta(k)$ represents the phase shift of a particle after a single collision with a pseudomomentum transfer of $k$. It has been proved that ABA equations give exact solutions to the energy spectrum of the CSM. We assume this approach could be generalized to the situations of CSM plus some other kind of interaction with the asymptotic Bethe ansatz (ABA) equation [10],

$$\frac{d^2 \psi(x)}{dx^2} + \left( E - V \frac{l(l+1)}{x^2} \right) \psi(x) = 0.$$  

The asymptotic solution of (16) for $x \gg 0$ is given by

$$\psi(x) \approx 2 \sin(kx - \frac{1}{2}(\pi + 2\delta_l)), \quad \text{for} \quad x \gg 0, \quad \delta_l = \frac{\pi}{2} \frac{l(l+1)}{m}.$$

where $\delta_l$ is the three-dimensional phase shift corresponding to the scattering potential $V$. In the sense of 1-d scattering,

$$\theta(k) = \pi(m-1) \text{sgn}(k) - 2\delta_l(k).$$

We see that the contribution of $V$ to the phase shift is

$$\theta_{\text{eig}}(k) = -2\delta_l(k),$$

which is continuous and vanishing at $k = 0$ if $V$ is short-ranged (shorter than $1/r^2$).

Now, let’s make the relation to the macroscopic theory. In terms of the partition function (??), there is a most probable edge CF number $\bar{N}_c$ which is given by $\delta Z/\delta N_c = 0$. $\bar{N}_c = \int dx \rho(x)$ with the edge density $\rho(x) = h(x) \rho_e$, [3]. Here $h(x)$ is the edge deformation and $\rho_e$ is the average density of the bulk electrons. We do not distinguish $\bar{N}_c$ and $N_c$ hereafter if there is no ambiguity. The low energy properties of the CSM can be obtained from the ABA equations,

$$\rho(k) = \rho_0(k) - \int_{-k_F}^{k_F} g(k-q) \rho(q) dq$$

$$\epsilon(k) = \epsilon_0(k) - \int_{-k_F}^{k_F} g(k-q) \epsilon(q) dq$$

changes like a phase transition. This assumption is reasonable, for the low energy limit of the CSM is the fixed point of Luttinger liquid which is robust against perturbations. Therefore, what follows from ABA, as we believe, is credible. As a matter of fact, for a large class of short-range interactions, we can expect the ABA works in describing the low-lying excitations of the system. Indeed, there are several kinds of short-range interactions whose low-lying excitations are governed by the ABA. An example of them is the $e^{(|l|^{1/2})}$-function interaction with $(l)$ representing the l-th derivative of the $\delta$-function and $l$ being restricted to $l < m$. The pseudopotentials used by Haldane [27] are other examples because of the vanishing of the expectation value of the pseudopotentials in the ground state.

To calculate the phase shift, we note the analog of the Schrodinger equation of the two-body CSM with an additional short-range interaction in the limit $L \rightarrow \infty$ to the radial equation of a three-dimensional scattering problem of a centrally symmetric potential. The topological exponent $m$ corresponds to the total angular momentum $l$, i. e. $m = l + 1$. The Schrodinger equation reads

$$\frac{d^2 \psi(x)}{dx^2} + \left( E - V \frac{l(l+1)}{x^2} \right) \psi(x) = 0.$$
where \( k_F = \pi m N_0^2 / L \),

\[
g(k) = \frac{1}{2\pi} \frac{d\theta(k)}{dk}, \quad (22)
\]

\( \epsilon_0(k) = k^2 - k_F^2 \) and \( \rho_0(k) = \frac{1}{2\pi} \).

If we consider only the CSM without other interactions, then we have

\[
\theta_{cs}(k) = \pi (m - 1) \text{sgn}(k). \quad (23)
\]

Substituting (23) into (21) and after the linearization, we get

\[
\epsilon_{cs\pm}(k) = \begin{cases} 
\pm v_+(k \mp k_F), & \text{if } |k| > k_F \\
\pm v_-(k \mp k_F), & \text{if } |k| < k_F,
\end{cases} \quad (24)
\]

where

\[
v_+ = \frac{de(k)}{dk} \Bigg|_{k=k_F + 0^+} = v_F \quad (25)
\]

\[
v_- = \frac{de(k)}{dk} \Bigg|_{k=k_F - 0^+} = \frac{v_F}{m}
\]

with \( v_F = 2k_F \) and

\[
\rho_+ = \rho(k_F + 0^+) = \frac{L}{2\pi} \quad (26)
\]

\[
\rho_- = \rho(k_F - 0^+) = \frac{L}{2\pi m}
\]

We rewrite the important equations essential to the bosonization for CSM as follows

\[
v_+ = mv_- \quad (27)
\]

\[
\rho_+ = m\rho_-. \quad (28)
\]

A successful bosonization of the theory with the refraction dispersion (24) has been done by the authors of [19] and one shows that the low-lying excitations of the CSM are controlled by the \( c = 1 \) CFT with its compactified radius \( R = 1 / \sqrt{m} \) [28]. This implies that the low-lying states of the CSM have the Luttinger liquid behaviors with the exponent \( g = m \). We will be back to this issue later after we supplies the chiral constraint and then show that the edge states of FQHE have the CLL behaviors.

Now, let us see the effects of the interactions. Following our discussion that leads to the \( c = 1 \) CFT with the compactified radius \( R = 1 / \sqrt{m} \), the relations (27) are essential. We would like to check if they are renormalized by the interactions between CFs. Here, we limit our discussion to the case \( m \neq 1 \). We assume the ABA works to describe the low-lying excitations of the system with an additional short range interaction, which is consistent with the chirality of the edge excitations. Differentiating the phase shift (18) with respect to \( k \), one has

\[
g(k) = (m - 1)\delta(k) + g_{\text{reg}}(k). \quad (31)
\]

The continuity of \( \theta_{\text{reg}} \) implies that \( g_{\text{reg}} \) is no more singular than the \( \delta \)-function at \( k \to 0 \). Therefore, we can prove that the relations (29) still hold even after we have introduced a short-range interaction. After differentiating the dressed energy equation (21) that is assumed holding for the short-range interaction we are using and in the dilute gas approximation, with respect to \( k \), we obtain

\[
v_\pm = v_0 + \int_{-k_F}^{k_F} \frac{d}{dq} \epsilon(q) (k_F \pm 0^+ - q) dq \quad (32)
\]

\[
= v_0 - \int_{-k_F}^{k_F} \frac{d}{dq} \epsilon(q) (k_F \pm 0^+ - q) dq
\]

\[
+ \epsilon(k_F) g(k_F \pm 0^+ - k_F) - \epsilon(-k_F) g(k_F \pm 0^+ + k_F).
\]

The definition of \( k_F \), i.e., \( \epsilon(\pm k_F) = 0 \), leads to

\[
v_+ - v_- = \int_{-k_F}^{k_F} \frac{d}{dq} \epsilon(q)(m - 1)\delta(q - k) \quad (33)
\]

Hence

\[
v_+ = m v_- \quad (34)
\]

The value of \( v_\pm \) can be modified by the interactions but the above relation does not change. Note that if \( \epsilon(k_F) = 0 \) and \( \epsilon(-k_F) g(k_F \pm 0^+ + k_F) \) is continuous at \( k_F \), the above conclusion still holds, which will be the case in the CLL derivation of Sec IV. By performing a similar procedure to \( \rho(k) \), we can show (31) for \( \rho_\pm \) as well. Therefore one can see that the bosonization process of the CSM is still applicable in the presence of perturbative interactions, and the topological exponent \( g = m \) is not renormalized by the short-range interaction. As a result, the compactified radius of the \( c = 1 \) CFT which governs the low-lying excitations of the theory does not change.

Let us give more comments on the conclusion drawn above. This result seems remarkable at the first sight, when compared with the standard Luttinger liquid theory, in which we will have the characteristic exponent renormalized once a short-range perturbative interaction is switched on. In fact, no inconsistencies exist here. In the bosonization of the general Luttinger liquid, only short range interactions are considered, whose Fourier transformation \( V(k) \) at \( k = 0 \) possesses no singularity.
Even if the divergence of $V(k)$ as $k$ approaches zero does show up, it is suppressed by introducing something like a short-range cutoff or a long-range cutoff which makes the problem concerned more subtle. The exponent so obtained may be cutoff-dependent. So we can not naively apply it here. In contrast to the standard approach, the bosonization of the CSM is based on the especially simple form of the phase shift function of the $1/r^2$ interaction that is essential to the solution of ABA. The singularity here manifests itself as a step discontinuity which can be handled easily (no cutoff is needed). Because of the critical property of the $1/r^2$ interaction, no other interactions with shorter ranges can alter this discontinuity, which guarantees the robustness of the bosonization process. In short, the bosonization of CSM is not so general as the standard one, but it surely makes a step forward in understanding the low energy physics of nontrivial interactions.

We emphasize once again that both $1/r^2$ interaction and the chirality contribute to the robustness of $g = 1/m$ when $m > 1$. In general, the critical exponent will be changed by the introduction of other short-range interactions if the chirality is not present and backward scattering is allowed. In contrast, in case of $m = 1$, where we are actually dealing with a Fermi liquid, the discontinuity of the phase shift $\theta(k)$ is absent. So the above argument of robustness fails. An simple example is to consider a $\delta$-function interaction. For $m > 1$, the short-range divergence of the $1/r^2$ potential requires that the wave function vanishes when two particles approach each other. Hence the $\delta$-function contribution to the phase shift is completely suppressed in case of $m > 1$, while it does show up for $m = 1$ [?]. On the occasion of $m = 1$, however, the chirality alone serves as the determinant factor to ensure the non-renormalizability of $g = 1$, by prohibiting the left-right scattering part of perturbative interactions from modifying $g$. Therefore, one can see that the different microscopic mechanisms for $m > 1$ and $m = 1$ give the same macroscopic result.

From the above arguments, we see that the topological exponent is invariant to the perturbations introduced by additional interactions between particles, if their interaction range is shorter than that of $1/r^2$. However, the long range nature of Coulomb interaction allows it to dominate the $1/r^2$ interaction which gives $g = \nu$. Considering its especially singular behavior at $k = 0$, we believe that the so called topological index can no longer survive, if an unscreened Coulomb interaction without any cut-off really exists. Fortunately, we have several possibilities that will lead to partial screening of the Coulomb interaction. In real experiments, the edge electrons actually are not isolated to a wire-like structure. There are bulk electrons adjacent to them, which can provide mirror charges and reduce the original Coulomb interaction to a shorter range interaction. What is more, metal electrodes commonly used in experiments to supply a confinement potential can also serve as a mirror charges provider. So we only have to concern ourselves with partly screened Coulomb interaction instead of the bare one. The effect of short-range interactions has been discussed in this section.

IV. CHIRAL Luttinger Liquid: The Microscopic Point of View

A. Microscopic Derivation of CLL from the Radial Equation

In the previous sections, we freeze the radial degree of freedom of the edge particles and see that the azimuthal dynamics is described by the CSM. However, there are two branches of gapless excitations in the CSM and the chirality of the edge excitations are not shown. To arrive at the conclusion of chirality, we take the radial degree of freedom into account. Let us first make some simplifications before going into details. The interactions between CFs are assumed to be independent of the radial degree of freedom because of the small width of the edge. Moreover, we can think of the interaction between the CFs as consisting of only the $1/x^2$-type as we have demonstrated that short-range interactions do not renormalize the topological exponent $g = m$.

In Sec. II, we take the approximation $r_i \simeq R$ and arrive at the CSM. Restoring the radial variable, one has the radial eigen equation, which reads

$$\sum_i \left[ -\frac{\partial^2}{\partial r_i^2} + \left( \frac{n}{r_i} - \frac{|B|^2 r_i^2}{2} \right) g(r_1, \ldots, r_N) \right]$$

$$+ [U_{eff} + O(\delta r_i/R)] g(r_1, \ldots, r_N) = E g(r_1, \ldots, r_N),$$

where the terms $\frac{1}{R} \frac{\partial}{\partial r_i}$ have been absorbed into $g$ by a simple transformation like the multiplication of $e^{-\sum r_i/R}$. One can see that the radial eigenstate equation can be treated in the single particle picture except that the pseudomomenta $k = nR$ are related to one another by the ABA equations [14]. It is reasonable to arrive at such a result because the interactions between CFs are the functions of $r_i - r_j$ and the radius-dependent part of the interactions is of order $\delta r/R$. Now we employ the harmonic approximation used by Halperin in the case of IQHE edge states [3]. Let us first turn off the applied electric field. The radial single particle wave equation in the stripe approximation reads

$$- \frac{d^2}{dy^2} + B^2 y^2 g = \varepsilon_+ g,$$

for $n > 0$ and

$$- \frac{d^2}{dy^2} + B^2 y^2 g + |nB|^2 g = \varepsilon_- g,$$

for $n < 0$. Here $y = r - R_n$ and

$$R_n = \sqrt{\frac{|n|}{|B|^2}}.$$
Comparing (36) with (37), we see that the magnetic field separates the \( n < 0 \) sector from the \( n > 0 \) sector by an energy gap \(|n|\hbar \omega^*_c\). Therefore only the \( n > 0 \) (or equivalently, \( k > -K_F \)) sector needs to be considered for the low-lying excitations. This is the first sign of chirality. The harmonic equation (36) has its eigenstate energy

\[
\varepsilon_{+,*} = \hbar \omega^*_c (|\nu^*| - 1 + \frac{1}{2})
\]  

if the center of the harmonic potential \( R_n \ll R \). This is consistent with the mean-field approximation to the bulk state because \( R_n \ll R \) actually corresponds to the bulk state of the theory if we recognize that the width of the harmonic oscillator wave function is about several times the cyclotron motion radius \( R^*_c \). Since \( R_n \) is the function of \( n \), \( p_n = m^*\omega^* R_n \) can be regarded as a momentum-like quantity. The harmonic oscillator energy for \( R - R_n \ll R^*_c \) implies that there is no left-side Fermi point. This provides a necessary condition of the chirality. To justify the CLL, one should show the existence of gapless excitations on the right side. It is known that the eigenstate energy at \( R_n = R \) is raised to

\[
\varepsilon_{R,\nu^*} = \hbar \omega^*_c \left( 2(|\nu^*| - 1) + \frac{3}{2} \right),
\]

because of the vanishing of the wave function at \( r = R \). One asks that what happens if \( R_n \) is slightly away from \( R \). To see it clearly, we rewrite (36) as

\[
\begin{align*}
-\frac{d^2\hat{q}}{dt^2} + B^2 \gamma^2 g + g^2 (R - R_n)\hat{y}g \\
+ B^2 (R_n - R)^2 g = \varepsilon_{+,*} g,
\end{align*}
\]

where \( \hat{y} = r - R \). If \( R_n \) is very close to \( R \), i.e., \(|R - R_n| \leq R^*_c\), one can take the third term as perturbation and a first order perturbative calculation shows

\[
\delta \varepsilon_{0,\nu^*} = \varepsilon_{+,*} - \varepsilon_{R,\nu^*} = v^*_c (p_n - p_R) + (p_n - p_R)^2,
\]

where \( v^*_c = 2\pi^{-1/2} B^* \omega^*_c \) is of the order of the cyclotron velocity of the CF corresponding to \( B^* \) and then of the order of \( v_F \). So, if we take \( v^*_c \approx v_F \) and note that \( p_n - p_R \approx k - K_0 \), the dispersion (42) can be simply rewritten as

\[
\delta \varepsilon_0 (k) \approx (k - K_0 + k_F^2) - k_F^2,
\]

where \( k \) is given by the ABA equations (14). Near the Fermi point \( K_0 \), the dispersion can be linearized as

\[
\delta \varepsilon_0 (k) = v_F (k - K_0),
\]

which implies that there is a right-moving sound wave excitation along the edge with the sound velocity \( v_F \). There is another Fermi point \( k = K_0 - 2k_F \), which corresponds to \( R_n \approx R - 2R^*_c \) and is outside of the region we are considering and in fact belongs to the bulk state. To show that the above chiral theory has a Luttinger liquid behavior, we extend continuously \( \delta \varepsilon_0 (k) \) to all possible pseudomomenta which obey the ABA (14). Then, the equation (34) and (14) mean that the system is an IEG (19). The problem can be solved by using a bosonization procedure developed in ref. (19). The edge excitations can be obtained by considering only the properties of such an IEG system near \( k \sim K_0 \). Consequently, the low-lying excitations of the theory are controlled by the \( c = 1 \) CFT with its compactified radius \( R = 1/\sqrt{m} \) as we point out in the discussion of the CSM in Sec. III. However, the relevant excitations of the edge states include only the right-moving branch. In other words, the edge states are chiral and the sound wave excitations correspond to the non-zero modes of the right-moving sector of the \( c = 1 \) CFT. There are two other kinds of edge excitations which correspond to the particle additions to the ground state and the current excitations along the edge respectively. The velocity relations of these excitations are given by

\[
v_M = mv_F, v_J = v_F/m, v_F = \sqrt{v_M v_J}.
\]  

The relations resemble those of Haldane’s Luttinger liquid if one identifies \( m \) with the characteristic parameter \( e^{-2\rho} \) in the Luttinger liquid theory (28). These observations are crucial to the conclusion that the edge states are controlled by the \( c = 1 \) CFT with its compactified radius \( R = 1/\sqrt{m} \).

To arrive at the effective theory of CLL, let us perform the following bosonization procedure.

According to (13) and (14), the edge excitations with the pseudomomentum \( k \) have their dressed energy

\[
\varepsilon (k) = \begin{cases} \\
(k^2 - k_F^2)/m, & |k| < k_F, \\
(2k^2 - k_F^2)/m, & |k| > k_F.
\end{cases}
\]  

Here we have made a translation \( k \to k + K_0 - k_F \). The linearization approximation of the dressed energy near \( k \sim \pm k_F \) is given by (24). In terms of the linearized dressed energy, we obtain a free fermion-like representation of the theory and then can easily bosonize it (9). The Fourier transformation of the right-moving density operator is given by

\[
\rho^{(+)}_q = \sum_{k > k_F} c_{k-q}^\dagger c_k^\dagger + \sum_{k < k_F - m} c_{k+m}^\dagger c_k^\dagger,
\]

\[
\rho^{(-)}_q = \sum_{k < k_F - m} c_{k+m}^\dagger c_{k+q}^\dagger,
\]

for \( q > 0 \) is the sound wave vector. Here \( c_k \) is a fermion annihilation operator. And a similar \( \rho^{(-)}_q \) can be defined near \( k \sim -k_F \). The bosonized Hamiltonian is given by

\[
H_B = v_F \left\{ \sum_{q > 0} q (b_q^\dagger b_q + \bar{b}_q^\dagger \bar{b}_q) + \frac{1}{2} \frac{\pi}{L} n M^2 + \frac{m}{1} J^2 \right\}.
\]  

Thus, we have a current algebra like
\[ [\rho_q^{(\pm)}, \rho_q^{(\pm)^*}] = \frac{L}{2\pi} \delta_{q,q'}, \quad [H_B, \rho_q^{(\pm)}] = \pm v_F q \rho_q^{(\pm)}. \] (50)

In the coordinate-space formulation, the normalized density field \( \rho(x) \) is given by \( \rho(x) = \rho_R(x) + \rho_L(x) \):

\[ \rho_L(x) = \frac{M}{2L} + \sum_{q>0} \sqrt{q/2\pi} L m(e^{iqx} b_q + e^{-iqx} b_q^\dagger), \] (51)

and \( \rho_R(x) \) is similarly constructed from \( \tilde{b}_q \) and \( \tilde{b}_q^\dagger \). Here \( b_q = \sqrt{2\pi/qL} \rho_q^{(\pm)^*} \) and so on. The boson field \( \phi(x) \), which is conjugated to \( \rho(x) \) and satisfies \([\phi(x), \rho(x')] = i\delta(x-x')\), is \( \phi(x) = \phi_R(x) + \phi_L(x) \) with

\[ \phi_L(x) = \frac{\phi_0 L}{2L} + \frac{\pi J x}{2L} + i \sum_{q>0} \sqrt{\pi m/2qL} (e^{iqx} b_q - e^{-iqx} b_q^\dagger), \]

and a similar \( \phi_L(x) \). Here \( M \) and \( J \) are operators with integer eigenvalues, and \( \phi_0 = \phi_0 + \phi_{\gamma_0} \) is an angular variable conjugated to \( M: [\phi_0, M] = i \). The Hamiltonian becomes

\[ H_B = \frac{v_F}{2\pi} \int_0^L dx \left( \Pi(x)^2 + (\partial_x X(x))^2 \right), \] (52)

where \( \Pi(x) = m^2/\rho(x) \) and \( X(x) = m^{-1/2} \phi(x) \). With \( X(x,t) = e^{iHt} X(x)e^{-iHt} \), the Lagrangian density reads

\[ \mathcal{L} = \frac{v_F}{2\pi} \partial_x X(x,t) \partial^\alpha X(x,t). \] (53)

We recognize that \( \mathcal{L} \) is the Lagrangian of a \( c = 1 \) CFT. Since \( \phi_0 \) is an angular variable, there is a hidden invariance in the theory under \( \phi \to \phi + 2\pi \). The field \( X \) is thus said to be “compartmented” on a circle, with a radius that is determined by the exclusion statistics:

\[ X \sim X + 2\pi R_c, \quad R_c^2 = 1/m. \] (54)

States \( V[X]|0\rangle \) or operators \( V[X] \) are allowed only if they respect this invariance, so quantum numbers of quasiparticles are strongly constrained.

In the present case, only the right-moving sector is relevant. So, we have an ‘almost’ chiral edge state theory whose sound wave excitation is chiral but there are charge leakages between the bulk and the edge. The leakages are reflected in the zero-mode particle number and current excitations.\(^{[29]}\) In this almost chiral theory, the charge-one fermion operator is defined by

\[ \Psi_R^\dagger(x) = \sum_{l=-\infty}^{\infty} \exp(i2(l + 1/2m)\theta_R(x)) \exp(i\phi_R(x)), \] (55)

where

\[ \theta_R(x) = \pi \int_{-\infty}^x \rho_R(x') dx'. \] (56)

The correlation function, then, can be calculated \(^{[3]}\)

\[ <\Psi_R(x,t)\Psi_R(0,0)> = \sum_{l=-\infty}^{\infty} C_l \left( \frac{1}{x - v_F t} \right)^{(l+m)^2/m} \exp(i2\pi(l + 1/2)/x/L)). \] (57)

The \( l = 0 \) sector recovers Wens result \(^{[3]}\). In other words, the present theory justifies microscopically Wens suggestion of CCL of the FQHE edge states.

V. MANY BRANCH THEORY

To see the chirality in many branch theory, we can apply the discussion in the previous section to the \( \nu' \)-branch. The eigenenergy of the single wave function \( g_+(y) = g(r_n - R) \) with \( n > 0 \) is \( |n|\hbar \omega_{nu'}^* \) lower than that of \( g_-(y) = g(r_n - R) \) with \( n < 0 \). Therefore, the states with the negative \( n \) are ranged out of the low-lying state sector. This implies that only the left-moving mode in the azimuthal dynamics belongs to the low-lying excitation sector. The width of the wave function \( g_{\nu'} \) is several times of \( R_{\nu'}^* \), the cyclotron radius of the CF in the effective field. For the highest spin \( \nu' \), one can see that the eigenenergy of \( g_{\nu'}(0) \) (\( R_{\nu'}^* = R \) and \( \omega_{nu'}^* \)) since the wave function vanishes at \( r = R \) while \( \epsilon_{nu'} = (|\nu' - 1| + 1/2)\hbar \omega_{nu'}^* \) for \( R - r_n \gg R_{\nu'}^* \), a harmonic oscillator energy and coinciding with the mean field theory applied to the bulk states. The gapless excitations appear when \( |R - r_n| \sim R_{\nu'}^* \). Using a perturbative calculation, one has the bare excitation energy is \( \epsilon_0 = \epsilon_{nu'} - \epsilon_{nu''} \sim v_F \pi \hbar \rho_{\nu'} \left( r_n - \bar{R} \right) \) with the Fermi velocity \( v_F = \pi \hbar \rho_{\nu'} \left( \phi + \nu' \right)^{[22]} \). So, we see that the Fermi velocity then the CF effective mass \( m^* \) are determined by the slope of the edge spectrum.

The \( SU(\nu') \) symmetry forces all other branches of the edge excitations have the same velocity both in their magnitude and direction as the \( \nu' \)-branch. This is contradict to the recent edge tunneling experiment \(^{[14]}\). Lee and Wen \(^{[15]}\) argued that this inconsistency could be dispelled if the spin mode velocities are much smaller than the charge’s. ( Charge mode is given by \( \rho_{\sigma} \propto \sum \rho_{\sigma} \) and so on.) In our model, there are two factors to change the sound velocity \( v_c \), which equals to \( v_F \) before considering those factors. First, we have taken the effective potential as an infinite wall to simplify our model. In real samples, this potential is also smooth. This implies that the real edge spectrum is much flatter than that in the infinite wall potential case and then the real sound velocity \( \hat{v}_c \) is much smaller than \( v_F \). Equivalently, the CF effective mass gains due to the smoothness of the edge potential. The other factor to affect the sound velocity is the interaction \( V \) which violates the \( SU(\nu) \) symmetry. The interaction is of the form \( \rho \nu' V_{\nu'\nu'} \rho_{\nu'} \). In the bosonic form, it can be rewritten as \( \rho_{\nu'} V_{\nu'\nu'} \rho_{\nu'} \) where \( \rho_{\nu'} \) is the charge density wave and \( \rho_{\sigma} \) are the \( \nu' - 1 \) branches of spin density wave.
All the interactions stem from the electron-electron interaction, which yields $V_{ss'}, V_{sc} \ll V_{\rho}$. We can assume $V_{ss'} = V_{sc} = 0$. Therefore, only the charge density wave velocity is renormalized by the interaction $V_{\rho}$. Finally, we have the renormalized velocities

$$v^* = v_c + V_\rho, \quad v^*_c = v_c.$$  \hspace{1cm} (58)

We see that the smooth edge potential suppresses the spin wave velocity whereas the interaction $V_{\rho}$ is not changed. Here we consider the interaction to be short range. In reality, it may be an unscreened Coulomb interaction. It is reasonable to take $V_{\rho} \approx \rho^{\phi}$, by the inverse of the bulk CF effective mass. However, $\rho^{\phi}$ is proportional to the inverse of the edge CF effective mass. It is reasonable to take $V_{\rho} \propto \rho^{\phi}$, which is almost the inverse of the bulk CF effective mass. As the edge potential is flatted, the edge effective mass increases a lot such that $\sqrt{\rho^{\phi}} v^* \ll v^*_c$. Therefore, $v^* \approx V_{\rho} >> v^*_c$ which is just what Lee and Wen predicted.

VI. $\nu = \frac{|\nu^*|}{|\nu^*| + 1}$ EDGE STATE

Turn to much subtle problem at $\nu^* < 0$, i.e., $\nu = \frac{|\nu^*|}{|\nu^*| + 1}$.$\nu^*$ To solve the problem, we make an anyon transformation for the edge CF with the statistical parameter $b\delta_{s,s'}$, where $b$ is a real number which is given by a solution of equation $(\delta - 1)b^2 + 2\delta + 2 = 0$. Although the 1-d limit model is still not soluble even with this transformation, we may attract the low-lying excitation sector by using a trial wave function. A trial wave function enlightened by the bulk wave function may be taken of the form

$$\Psi_s(z_1, \ldots, z_{N_s}, \sigma_{\nu}) = \exp \left[ \frac{1}{2} \sum_{i<j} s_{\sigma,\sigma'} r_i - r_j \right] \times f(r_1, \ldots, r_{N_s})\Psi_s(\phi_{\nu^*}, \ldots, \phi_{\nu^*}, \sigma_{\nu}),$$  \hspace{1cm} (59)

where $s_{\sigma,\sigma'} = \frac{1}{2} + c\delta_{\sigma,\sigma'}$ with $c = (\frac{1}{4}\rho^{\phi} - \frac{1}{2})b - \frac{1}{2}$. The azimuthal wave function $\Psi_s$ can be set according to the $K$ matrix of the bulk state and reads

$$\Psi_s(\varphi_{\nu^*}, \ldots, \varphi_{\nu^*}, \sigma_{\nu}) = \prod_{i>j} \phi_{ij} \cdot \prod_k \xi_k^{lK_{ij}}$$

$$\phi_{ij} = [\xi_i - \xi_j]|_{\sigma_j}^{\delta_{\sigma,\sigma}} \exp \frac{1}{2} \left[ \text{sgn}(\sigma_i - \sigma_j) + \text{sgn}(i-j)\delta_{\sigma,\sigma'} \right].$$  \hspace{1cm} (60)

This wave function is indeed an eigen wave function of the anyon-transformed Hamiltonian if $\delta R/R \to 0$. Taking a suitable set of the quantum numbers as that for $\nu^* > 0$, we have the ground state wave function. The important matter is that the exclusion statistics of the azimuthal wave function is given by the expected bulk $K$ matrix

$$K_{\sigma,\sigma'} = \delta - \delta_{\sigma,\sigma'}.$$  \hspace{1cm} (61)

According to this exclusion statistics matrix and our bosonization approach, we can finally arrive the CLL theory in which the charge mode travels in the opposite direction than the $\nu^* - 1$ spin-modes, which in the clean edge, leads to the absence of the edge equilibration. However, the effective potential $U_{eff}$ includes all possible external potential. Of them, a most relevant one is the random impurity potential. Kane et al have shown that this random potential drives the edge to a stable fixed point and restores the edge equilibration.[1]

VII. CONCLUSION

In conclusion, we have proposed a microscopic model of edge excitations for FQHE at $\nu = \frac{|\nu^*|}{|\nu^*| + 1}$. The SU($\nu^*$) CSM is a good candidate for the edge states for $\nu^*$ while there is no exact soluble counterpart for $\nu^* < 0$. The low-lying excitations for both $\nu^* > 0$ and $\nu^* < 0$, however, are proven to be described by the CLL. We also argued that the two-boson theory of Lee-Wen is valid to explain the experiments by Grayson at el in FQH regime while we did not discuss the explanation to the tunneling experimental result for the other filling factors such as $\nu = 1/2$ and in the non-FQH regime.

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