A proof of the Biswas–Mitra–Bhattacharyya conjecture for the ideal quantum gas trapped under the generic power law potential $U = \sum_{i=1}^{d} C_i x_i^{a_i} n_i$ in $d$ dimensions

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Abstract. The well known relation for the ideal classical gas $\Delta \epsilon^2 = kT^2 C_V$, which does not remain valid for quantum systems, is revisited. A new connection is established between the energy fluctuation and specific heat for quantum gases, which is valid in both the classical limit and the degenerate quantum regime. Most importantly, the proposed Biswas–Mitra–Bhattacharyya conjecture (Biswas et al 2015 J. Stat. Mech. P03013) relating the hump in energy fluctuation and the discontinuity of specific heat is essentially proved and made precise in this manuscript.

Keywords: Bose–Einstein condensation (theory), classical phase transitions (theory), quantum fluids, quantum gases
1. Introduction

Increasing attraction to the subject of quantum gases has been observed since it was made possible to create Bose–Einstein condensates (BEC) in magnetically trapped alkali gases [2–4] and experimentally confirm Fermi degeneracy [5]. Different theoretical and experimental studies analysing the effects of the temperature dependence of energy and the specific heat of ultracold Fermi gases [5–7], the momentum distribution for harmonically trapped quantum gas [8, 9], the temperature dependence of the chemical potential [10], the critical number of particles for the collapse of attractively interacting Bose gas [11], the Casimir effect [12, 13], the equivalence of Bose and Fermi systems [14, 15] and q deformed systems [16, 17] have already been reported. Although a lot of theoretical studies [18–25] have been done on quantum gases trapped under the generic power law potential, none of these contained a detailed discussion on the energy fluctuation of trapped quantum gases, until the recent paper of Biswas et al [1]. In the case of the ideal classical gas, the specific heat $C_V$ is regarded as the energy fluctuation $\Delta \varepsilon^2$, as $\Delta \varepsilon^2$ is related to $C_V$, as $\Delta \varepsilon^2 = kT^2C_V$. But this connection becomes invalid for both types of quantum gases (Bose and Fermi) in the quantum degenerate regime [1]. Biswas et al [1] have conjectured (the Biswas–Mitra–Bhattacharyya (BMB) conjecture) a connection between the discontinuity of $C_V$ and energy fluctuation. According to the BMB conjecture, the appearance of a hump in $\frac{\Delta \varepsilon^2}{kT^2}$ over its classical limit may indicate a discontinuity in $C_V$. They have shown this to be true for free and harmonically trapped Bose gases [1], without proving that the inverse of this statement may not always be true. However, this conjecture is yet to be proven for any quantum system trapped under the generic power law potential in arbitrary dimensions, and is an open problem. In this manuscript, we have essentially proved and made precise the BMB conjecture for ideal quantum gases trapped under the generic power law potential, $U = \sum_{i=1}^{d} c_i \left| \frac{x_i}{a_i} \right|^ {n_i}$ in $d$ dimensions. Thus, in principle one can reconstruct the results of Biswas et al by choosing all $n_i = 2$ for harmonically trapped systems and all $n_i = \infty$ for free systems.
Besides this, a connection is established between $C_V$ and $\Delta \epsilon^2$, which is not only valid in the classical limit, but in the quantum limit as well.

The manuscript is organized in the following way: in section 2, we have determined the grand potential in a unified way for both types of quantum gases, from which we are able to calculate quantities such as $C_V$ and $\Delta \epsilon^2$. In the next section, we elaborate on two theorems which eventually guide us to prove the conjecture. The results are discussed in section 4 and the concluding remarks are presented in section 5.

2. Grand potential, specific heat and energy fluctuations

For a quantum gas, the average number of particles occupying the $i$th single particle energy eigenstate and the grand potential (in units of $kT$) is given by

$$\bar{n}_i = \frac{1}{z^{-1}e^{\beta\epsilon_i} - a}$$

and,

$$q = \frac{1}{a} \sum_\epsilon \ln(1 - aze^{-\beta\epsilon})$$

where $a = 1(-1)$ stands for Bose systems (Fermi systems), $z$ is the fugacity and $\beta = \frac{1}{kT}$ is the Boltzmann constant. Let us consider an ideal quantum system trapped in a generic power law potential in $d$ dimensional space with a single particle Hamiltonian of the form,

$$\epsilon(p, x_i) = bp^l + \sum_{i=1}^d c_i \left| \frac{x_i}{a_i} \right|^{n_i}$$

where $b$, $l$, $a_i$, $c_i$ and $n_i$ are all positive constants, $p$ is the momentum and $x_i$ is the $i$th component of the coordinates of a particle. Here, $c_i$, $a_i$ and $n_i$ determine the depth and confinement power of the potential, $l$ is the kinematic parameter, and $x_i < a_i$. As $\left| \frac{x_i}{a_i} \right| < 1$, the potential term goes to zero, as all $n_i \rightarrow \infty$. Using $l = 2$ and $b = \frac{1}{2m}$ one can get the energy spectrum of the Hamiltonian used in the literature [19–22]. If one uses $l = 1$ and $b = c$, one finds the Hamiltonian of massless systems, such as photons [20].

The density of states for such systems is [18, 25],

$$\rho(\epsilon) = C(b, V'_d) e^{\lambda - 1}$$

where $C(b, V'_d)$ is a constant depending on the effective volume $V'_d$ [14]. For a detailed derivation of the density of states see [25]. Now, replacing the sum by an integral we obtain the grand potential (in units of $kT$),

$$q = q_0 + \frac{V'_d}{\lambda^d} Li_{\lambda+1}(\sigma).$$

Note that $Li_{\lambda}(m)$ is known as the polylog function, defined as [15]
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\[ L_i(q)(m) = \frac{1}{\Gamma(q)} \int_0^m \left[ \ln \left( \frac{m}{\eta} \right) \right]^{q-1} \frac{d\eta}{1-\eta} \]  

and \( q_0 \) is,

\[ q_0 = \frac{1}{a} \ln(1 - az). \]  

\( L_i(q)(m) \) is a real-valued function if \( m \in \mathbb{R} \) and \(-\infty < m < 1\). Also, the effective volume \( V'_d \), the effective thermal de Broglie wavelength \( \lambda'_d \) along with \( \chi \) and \( \sigma \) are defined as

\[ V'_d = V_d \prod_{i=1}^{d} \left( \frac{kT}{c_i} \right)^{1/n_i} \Gamma \left( \frac{1}{n_i} + 1 \right) \]  

\[ \lambda' \equiv \frac{\hbar b^{1/2}}{\pi^{1/2} (kT)^{1/2} d^{1/2} (d+1)^{1/2}} \]  

\[ \chi = \frac{d}{l} + \sum_{i=1}^{d} \frac{1}{n_i} \]  

\[ \sigma = \begin{cases} -z, & \text{Fermi system} \\ z, & \text{Bose system} \end{cases} \]

A detailed idea of the effective thermal de Broglie wavelength and the effective volume for trapped quantum gases can be found in [23–26]. Note that when \( l = 2 \) and \( b = \frac{1}{2m} \), we get from equation (9) \( \lambda' = \lambda_0 = \frac{\hbar}{(2\pi m kT)^{1/2}} \), which is the thermal de Broglie wavelength of nonrelativistic massive fermions as well as massive bosons. However, it should also be noted that when \( l \neq 2 \), \( \lambda' \) depends on the dimension. In \( d = 3 \) and \( d = 2 \), the thermal de Broglie wavelength of massless particles such as photons is \([19]\) \( \frac{\hbar c}{2\pi^{1/2}kT} \) and \( \frac{\hbar c}{(2\pi^{1/2}kT)^{3/2}} \), respectively, which can be obtained from equation (9) by choosing \( b = c \), \( c \) being the velocity of light. However, one needs to consider the effects of antiparticles in order to calculate the thermodynamic quantities of ultrarelativistic quantum gas [27]. So, one can reproduce the thermal de Broglie wavelength of both massive and massless particles from the definition of the effective thermal de Broglie wavelength with a more general energy spectrum. It is also seen that the effective volume \( V'_d \) is a salient feature of the trapped system, as this allows us to treat the trapped quantum gas [14, 23–25] as a free one. The difference between \( V'_d \) and \( V_d \) is that the former depends on the temperature and the power law exponent while the latter does not. But as all \( n_i \to \infty \), \( V'_d \) approaches \( V_d \). The great benefit of evaluating \( V'_d \) and \( \lambda' \) is that it enables us to write all the thermodynamic functions of the trapped quantum system in compact

\[ \text{For a detailed conceptual overview of the effective thermal de Broglie wavelength, see [26].} \]

\[ \text{The effective volume is also referred to as the pseudo-volume. For a detailed conceptual overview on how the equation of state of trapped quantum systems can be obtained, see [24].} \]

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forms similar to those of a free quantum gas [14, 23]. It is well known that the Bose and Fermi functions represent the thermodynamics of Bose and Fermi systems, respectively, and can be written in terms of polylogarithmic functions,

$$L_i(z) = g_i(z) = \frac{1}{\Gamma(l)} \int_0^\infty dx \frac{x^{l-1}}{z^{-1}e^x - 1}$$  \hspace{1cm} (12)

$$-L_i(-z) = f_i(z) = \frac{1}{\Gamma(l)} \int_0^\infty dx \frac{x^{l-1}}{z^{-1}e^x + 1}.$$  \hspace{1cm} (13)

An important point to note is that as $z \to 1$, the Bose function $g_\chi(z)$ approaches $\zeta(\chi)$, for $\chi > 1$ [19]. From the grand potential we can now determine the energy $E$ and the specific heat $C_V$,

$$E = -\left(\frac{\partial g}{\partial \beta}\right)_{z, V_d} = NkT \chi \left(\frac{L_{\chi+1}(\sigma)}{L_\chi(\sigma)} \frac{L_{\chi}(\chi + 1)}{L_{\chi}(\sigma)}\right)$$  \hspace{1cm} (14)

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N, V_d} = Nk \left[\chi(\chi + 1) \frac{L_{\chi+1}(\sigma)}{L_\chi(\sigma)} - \chi^2 \frac{L_{\chi}(\sigma)}{L_{\chi-1}(\sigma)} \right]$$  \hspace{1cm} (15)

where $N$ is the number of particles. In the classical limit of quantum gases, $C_V$ equals $Nk\chi$. Now, the energy fluctuation can be expressed as

$$\Delta^2 = \bar{\epsilon}^2 - \bar{\epsilon}^2 = \sum_i \bar{\tilde{\epsilon}}_i^2 - \left(\sum_i \bar{\tilde{\epsilon}}_i\right)^2 = \int d\epsilon \rho(\epsilon)\epsilon^2 n(\epsilon) - \left(\int d\epsilon \rho(\epsilon)\epsilon n(\epsilon)\right)^2$$

$$= (kT)^2 \left[\chi(\chi + 1) \frac{L_{\chi+2}(\sigma)}{L_\chi(\sigma)} - \chi^2 \frac{L_{\chi+1}(\sigma)}{L_{\chi}(\sigma)} \right].$$  \hspace{1cm} (16)

So, it is clear from equation (15) and (16) that $\Delta^2 \approx kT^2C_V$. However, it is valid in the classical limit as $z \to 0$ equation (15) and (16) indicates

$$\Delta^2 \longrightarrow kT^2C_V = N\chi(kT)^2.$$  \hspace{1cm} (17)

Still, we can establish that such a relation is valid within the whole temperature range. We can write the energy fluctuation as [20],

$$\Delta^2 = \bar{\epsilon}^2 - \bar{\epsilon}^2 = -\left(\frac{\partial E}{\partial \beta}\right)_{z, V_d} = kT^2 \left(\frac{\partial E}{\partial T}\right)_{z, V_d} = kT^2 \left(\frac{\partial E}{\partial T}\right)_{N, V_d} + kT^2 \left(\frac{\partial E}{\partial N}\right)_{T, V_d} \left(\frac{\partial N}{\partial T}\right)_{z, V_d}$$

$$= kT^2 C_V + kT \left(\frac{\partial E}{\partial N}\right)_{T, V_d} \left(\frac{\partial E}{\partial \mu}\right)_{T, V_d}.$$  \hspace{1cm} (18)

where in the last line we have used $\frac{1}{T} \left(\frac{\partial E}{\partial \mu}\right)_{T, V_d} = \left(\frac{\partial N}{\partial T}\right)_{z, V_d}$ [19]. In the high temperature limit, the second term of equation (18) becomes zero and equation (18) coincides with equation (17). It can easily be justified that equation (18) is valid, not only in the classical limit but also in the quantum degenerate regime.
An important point to note is that the expressions $C_V$ and $\Delta \epsilon^2$ in equations (14) and (15) represent both Bose and Fermi systems in a unified manner. In the case of Fermi systems,

$$C_V = N k \left[ \chi (\chi + 1) \frac{f_{\chi + 1}(z)}{f_{\chi}(z)} - \chi^2 \frac{f_{\chi}(z)}{f_{\chi - 1}(z)} \right]$$

$$\Delta \epsilon^2 = (kT)^2 \left[ \chi (\chi + 1) \frac{f_{\chi + 2}(z)}{f_{\chi}(z)} - \chi^2 \frac{f_{\chi + 1}(z)}{f_{\chi}(z)} \right].$$

The above equations coincide exactly with the results of [1] for appropriate choices of $n_i$ and $d$. Writing the expressions for the Bose system (per particle) we get

$$C_V = \begin{cases} k \left[ \chi (\chi + 1) \frac{\nu'}{\nu D} g_{\chi + 1}(z) - \chi^2 \frac{g_{\chi}(z)}{g_{\chi - 1}(z)} \right], & T > T_c \\ k\chi (\chi + 1) \frac{\nu'}{\nu D} \zeta (\chi + 1), & T \leq T_c \end{cases}$$

$$\Delta \epsilon^2 = \begin{cases} (kT)^2 \left[ \chi (\chi + 1) \frac{g_{\chi + 2}(z)}{g_{\chi}(z)} - \chi^2 \frac{g_{\chi + 1}(z)}{g_{\chi}(z)} \right], & T > T_c \\ (kT)^2 \left[ A_1 \left( \frac{T}{T_c} \right)^\chi - A_2 \left( \frac{T}{T_c} \right)^{2\chi} \right], & T \leq T_c \end{cases}$$

Here, $T_c$ denotes the critical temperature and $A_1$ and $A_2$ are defined as

$$A_1 = \chi (\chi + 1) \frac{\zeta (\chi + 2)}{\zeta (\chi)}$$

$$A_2 = \left[ \chi \frac{\zeta (\chi + 1)}{\zeta (\chi)} \right]^2.$$}

Equation (22) agrees with the expressions for free and harmonically trapped Bose systems in three-dimensional space [1]. It is noteworthy that for both types of trapped quantum gases, $C_V$ approaches its classical value $\chi N k$ [28].

### 3. Theorems regarding $C_V$ and $\Delta \epsilon^2$ of Bose gas

In this section, we present the necessary theorems to prove the conjecture. It was shown by Biswas et al that a hump does exist in $\Delta \epsilon^2/kT^2 C_V^{cl}$ in the condensed phase for harmonically trapped Bose gas. We need to find a general criteria to locate a hump in $\Delta \epsilon^2/kT^2 C_V^{cl}$ over its classical limit in arbitrary dimensions with any trapping potential.

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As there is no condensed phase in an ideal Fermi gas trapped under a potential \[29\], this theorem bears no significance for them.

**Theorem 4.1.** Let us consider an ideal Bose gas in an external potential \[U = \sum_{i=1}^{d} c_i \left| \frac{\mathbf{z}}{a_i} \right|^{n_i}\]. In the condensed phase, a hump will exist in \[\Delta \epsilon^2 / kT^2 C_V \] over the classical limit if

\[
\chi < \frac{A_1^2}{4A_2} = \gamma(\chi) \\
\Rightarrow \chi \geq 2.3 \pm \epsilon
\]

where \(\epsilon \ll 1\).

**Proof.** Re-writing equation \((21)\) in the condensed phase we get

\[
\frac{\Delta \epsilon^2}{(kT)^2} = f(\tau) = A_1 \tau^\chi - A_2 \tau^{2\chi} \\
\Rightarrow f'(\tau) = \frac{\partial f}{\partial \tau} = A_1 \chi \tau^{\chi-1} - 2A_2 \chi \tau^{2\chi-2}.
\]

The condition for the maximum is,

\[
f'(\tau)|_{\tau=0} = 0 \\
\Rightarrow \tau_0^\chi = \frac{A_1}{2A_2},
\]

(26)

The hump will be over its classical limit if

\[
f(\tau)|_{\tau=0} > \chi \\
\Rightarrow A_1 \tau_0^\chi - A_2 \tau_0^{2\chi} > \chi \\
\Rightarrow \chi < \gamma(\chi) = \frac{A_1^2}{4A_2},
\]

(27)

From table 1, one can conclude that the relation \((24)\) will be maintained (a hump will exist in the condensed phase over the classical limit) when

\[
\chi \geq 2.3 \pm \epsilon
\]

(28)

where, \(\epsilon \ll 1\). Note that it is possible to go beyond one decimal digit for the value of \(\chi\) and test the criterion \(\chi < \gamma(\chi)\), but no significant change in result was observed in this case. This is why we have used the constant \(\epsilon\) in equation \((28)\) and essentially proved theorem 4.1. □

**Theorem 4.2:** Consider an ideal Bose gas in an external potential, \(U = \sum_{i=1}^{d} c_i \left| \frac{\mathbf{z}}{a_i} \right|^{n_i}\),

(i) \(C_V\) will be discontinuous at \(T = T_c\) if

\[
\chi = \frac{d}{l} + \sum_{i=1}^{d} \frac{1}{n_i} > 2
\]

(ii) and the difference between the heat capacities at constant volume at \(T = T_c\) is

\[
\Delta C_V|_{T=T_c} = C_V|_{T=T_c} - C_V|_{T=0} = Nk\chi^2 \frac{\zeta(\chi)}{\zeta(\chi - 1)}
\]
Proof. As $T \to T_c$, $z \to 1$ and $\eta \to 0$, where $\eta = -\ln z$. For $T \to T_c^+$,
\begin{equation}
C_V(T_c^+) = Nk \left[ \chi(\chi + 1) \frac{\nu'}{\lambda'} g_{\chi+1}(z) |_{z \to 1} - \chi^2 \frac{g_\chi(z)}{g_{\chi-1}(z)} |_{z \to 1} \right]
\end{equation}

\begin{equation}
= Nk \left[ \chi(\chi + 1) \frac{\nu'}{\lambda'} \zeta(\chi + 1) - \chi^2 \frac{\zeta(\chi)}{g_{\chi-1}(z)} |_{z \to 1} \right] \tag{29}
\end{equation}

Since the denominator of the second term on the right-hand side contains $g_{\chi-1}(z)$, we cannot simply substitute it with the zeta function as $z \to 1$. So, using the representation of the Bose function by Robinson [30],
\begin{equation}
g_{\chi}(e^{-\eta}) = \frac{\Gamma(1 - \chi)}{\eta^{1-\chi}} + \sum_{i=0}^{\infty} (-1)^i \frac{\zeta(\chi - i)\eta^i}{i!} \tag{30}
\end{equation}

we get from the above,
\begin{equation}
C_V(T_c^+) = Nk \left[ \chi(\chi + 1) \frac{\nu'}{\lambda'} \zeta(\chi + 1) - \chi^2 \frac{\zeta(\chi)}{\Gamma(2 - \chi)\eta^{2-\chi} |_{\eta \to 0}} \right]. \tag{31}
\end{equation}

On the other hand,
\begin{equation}
C_V(T_c^-) = Nk \left[ \chi(\chi + 1) \frac{\nu'}{\lambda'} \zeta(\chi + 1) \right]. \tag{32}
\end{equation}

Taking the difference between $C_V(T_c^+)$ and $C_V(T_c^-)$, we get
\begin{equation}
\Delta C_V|_{T=T_c} = \chi^2 \frac{\zeta(\chi)}{\Gamma(2 - \chi)\eta^{2-\chi} |_{\eta \to 0}}. \tag{33}
\end{equation}
This dictates that $\Delta C_V|_{T=T_c}$ will be non-zero for $\chi > 2$. So, $C_V$ will be discontinuous when $\chi > 2$, thus completing the first part of the theorem.

Since at $T = T_c$, $\chi$ should be greater than two for $\Delta C_V$ to be non-zero, we can re-write equation (21), by substituting $g_{\chi-1}(z)$ by the zeta function.

$$C_V(T_c) = Nk \left[ \chi(\chi + 1) \frac{\nu'\zeta(\chi + 1) - \chi^2 \zeta(\chi - 1)}{\chi B} \right].$$  \hspace{1cm} (34)

From equations (32) and (34) we can write,

$$\Delta C_V|_{T=T_c} = C_V|_{T=T} - C_V|_{T=T_c} = Nk\chi^2 \frac{\zeta(\chi)}{\zeta(\chi - 1)}. \hspace{1cm} (35)$$

Note that the same result has also been derived by Yan et al for a Bose gas trapped in a symmetric power law potential [25].

Now, based on the above two theorems, we can come to a conclusion. We have seen that a hump will exist in $\Delta \epsilon^2/kT^2 C_V^cl$ over the classical limit when $\chi \geq 2.3$ and $C_V$ will be discontinuous while $\chi > 2$. Therefore, the existence of a hump in $\Delta \epsilon^2/kT^2 C_V^cl$ over the classical limit automatically ensures a discontinuity in $C_V$, but a discontinuity in $C_V$ does not imply the presence of a hump in $\Delta \epsilon^2/kT^2 C_V^cl$ over the classical limit when the value of $\chi$ is in between $2 < \chi < 2.3$, thus proving the conjecture.

\[\square\]

4. Results and discussion

In this section, we discuss the energy fluctuation and specific heat in detail, and check the prediction of the above theorems. We have illustrated in this paper how specific heat can differ from the energy fluctuation for the entire range of temperature. It is important to note that the difference between the probabilities of the classical and the quantum gases essentially arises from the non-zero fugacity of the quantum gas.

In the case of trapped quantum gases, all thermodynamic quantities are expressed by polylogarithmic functions depending on the fugacity and the value of $\chi$. Thus, apart from the fugacity, the value of $\chi$ bears the signature of the difference between different quantum systems. Also, as seen from the theorems, the value of $\chi$ dictates whether there will be a hump over the classical limit as well as the discontinuity in $C_V$. In figure 1 we have described the influence of different power law potentials on the energy fluctuation of Bose systems. It is clearly seen that $\Delta \epsilon^2/kT^2 C_V^cl$ has a hump way over its classical limit when $\chi > 2.3$. At $\chi = 2.3$, the hump is just over its classical limit. There is no hump over the classical limit when $\chi < 2.3$. All of these are in accordance with theorem 4.1. It has also been noticed that the results in Biswas et al [1] are in agreement with the theorem. In their manuscript, they found a hump over the classical limit in three-dimensional harmonically trapped Bose systems, where $\chi = 3 > 2.3$. Although they did find a hump in two-dimensional harmonically trapped Bose systems, this hump was below the classical limit. In this case, $\chi = 2 > 2.3$; i.e. there is
Figure 1. The energy fluctuation of an ideal trapped Bose gas as a function of \( \tau = \frac{T}{T_c} \) with different power law potentials.

Figure 2. The specific heat of an ideal trapped Bose gas as a function of \( \tau = \frac{T}{T_c} \) with different power law potentials.

no hump over the classical limit. Therefore, it can be said that theorem 4.1 can perfectly determine whether the humps will be below or above the classical limit.

Figure 2 illustrates the \( C_V \) of Bose systems with different trapping potentials. It is seen from the figure that \( C_V \) is continuous when \( \chi \leq 2 \), and it becomes discontinuous when \( \chi > 2 \), in agreement with theorem 4.2. Now, as \( \chi \geq 2.3 \) denotes a hump in
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Table 2. Status of energy fluctuation and specific heat of a Bose system trapped under a generic power law potential.

| Range of $\chi$ | Hump over classical limit in $\Delta \varepsilon^2 / kT^2 C_V^{\text{cl}}$ | Discontinuity in $C_V$ |
|-----------------|--------------------------------------------------|---------------------|
| $0 < \chi \leq 2$ | No hump over classical limit | Continuous |
| $2 < \chi < 2.3$ | No hump over classical limit | Discontinuous |
| $\chi \geq 2.3$ | Hump over classical limit | Discontinuous |

$\Delta \varepsilon^2 / kT^2 C_V^{\text{cl}}$ over its classical limit, this automatically depicts a discontinuity in $C_V$. Thus, we can conclude that the appearance of a hump in $\Delta \varepsilon^2 / kT^2 C_V^{\text{cl}}$ over its classical limit does indicate a discontinuity in $C_V$. However, a discontinuity in $C_V$ does not ensure the appearance of a hump in $\Delta \varepsilon^2 / kT^2 C_V^{\text{cl}}$ over its classical limit, because a discontinuity in $C_V$ may arise even if $2 < \chi < 2.3$, but no hump in $\Delta \varepsilon^2$ will exist in this interval of $\chi$. On the other hand, $\chi \geq 2.3$ will denote a discontinuity in $C_V$ as well as the appearance of a hump in $\Delta \varepsilon^2 / kT^2 C_V^{\text{cl}}$ over its classical limit (see table 2).

5. Conclusion

We proposed two theorems regarding the conditions of the hump in energy fluctuation over the classical limit and discontinuity in the specific heat of an ideal Bose gas trapped under the generic power law potential in arbitrary dimensions. Our general result can express why we do not get any hump over the classical limit in $\Delta \varepsilon^2 / kT^2$ in three-dimensional ideal free Bose gas ($\chi = 1.5$), but do get a hump for harmonically trapped Bose gas in $d = 3$ ($\chi = 3$). However, even if the system is trapped with a harmonic potential, one cannot have the hump in energy fluctuation when $d = 2$ and $d = 1$. Using our theorems, one can check out the cases in which it is possible to find the hump in energy fluctuation or discontinuity in specific heat for an ideal Bose gas trapped with any power law potential in an arbitrary dimension. The present theorems not only reproduce all the known results, but also extend the calculation for generic potential, leading us to prove the BMB conjecture for ideal Bose gas. The whole calculation is done considering the thermodynamic limit, and calculations are yet to be performed beyond this limit. We are currently undertaking this using Yukalov’s [31] modified semiclassical approximation. It will also be highly intriguing to generalize the theorems for relativistic quantum gases by taking into account the effect of antiparticles.

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References

[1] Biswas S, Mitra J and Bhattacharyya S 2015 J. Stat. Mech. P03013
[2] Bradley C C, Sackett C A, Tollett J J and Hulet R G 1995 Phys. Rev. Lett. 75 1687
[3] Anderson M H, Escher J R, Mathews M R, Wieman C E and Cornell E A 1995 Science 269 195
[4] Davis K B, Mewes M O, Andrew M R, Van N J, Durfee D S, Kurn D M and Ketterle W 1995 Phys. Rev. Lett. 75 3969
[5] DeMarco B and Jin D S 1999 Science 285 1703
[6] Kinast J, Turlapov A, Thomas J E, Chen Q, Stajic J and Levin K 2005 Science 307 1296
[7] Biswas S, Manna R K and Jana D 2012 Eur. Phys. J. D 66 217
[8] Truscott A G, Strecker K E, McAlexander W I, Partridge G B and Hulet R G 2001 Science 291 2570
[9] Luo L and Thomas J E 2008 J. Low Temp. Phys. 154 1
[10] Lee M H 1989 J. Math. Phys. 30 1837
[11] Biswas S 2009 Eur. Phys. J. D 55 653
[12] Casimir H B G and Polder D 1948 Phys. Rev. 73 360
[13] Napiorkowski M, Jakubczyk P and Nowak K 2013 J. Stat. Mech. P06015
[14] Faruk M M 2015 J. Stat. Mech. 161 679–87
[15] Lee M H 1997 Phys. Rev. E 55 1518
[16] Cai S, Su G and Chen J 2007 Phys. A: Math. Theor. 40 11245
[17] Narayana Swamy P 2006 Eur. Phys. J. B 50 291–4
[18] Faruk M M 2015 Eur. J. Phys. 36 058003
[19] Pathria R K 2004 Statistical Mechanics (Amsterdam: Elsevier)
[20] Huang K 1991 Statistical Mechanics (New York: Wiley)
[21] Ziff R M, Uhlenbeck G E and Kac M 1977 Phys. Rep. 32 169
[22] Salasnich L 2000 J. Math. Phys. 41 8016
[23] Yan Z 1999 Phys. Rev. A 59 4657
[24] Yan Z 2000 Phys. Rev. A 61 063607
[25] Yan Z, Li M, Chen L, Chen C and Chen J 1999 J. Phys. A: Math. Gen. 32 4069–78
[26] Yan Z 2000 Eur. J. Phys. 21 625
[27] Haber H E and Weldon H A 1981 Phys. Rev. Lett. 46 1497
[28] Faruk M M and Bhuiyan G M 2015 Acta Phys. Pol. 46 12
[29] Faruk M M 2015 Acta Phys. Pol. 46 12
[30] Robinson J E 1951 Phys. Rev. E 83 678
[31] Yukalov V I 2005 Phys. Rev. A 72 033608

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