(Conformally) semisymmetric spaces and special semisymmetric Weyl tensors

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Abstract. Semisymmetric spaces are a natural generalisation of symmetric spaces. For semisymmetric spaces in four dimensions with Lorentz signature, the Weyl tensor is easily seen (via spinors) to have a particularly simple quadratic property, which we call a special semisymmetric Weyl tensor. Using dimensionally dependent tensor identities, all (conformally) semisymmetric spaces are confirmed to have special semisymmetric Weyl tensors for all signatures in four dimensions. Furthermore, all Ricci-semisymmetric spaces with special semisymmetric Weyl tensors are shown to be semisymmetric for all signatures in four dimensions. Counterexamples demonstrate that these two properties have no direct generalisations in higher dimensions.

1. Introduction

It is known, for spaces in dimensions:

(i) $n \geq 5$ for all signatures [3, 9]
(ii) $n = 4$ for Lorentz signature (using spinors) [8]

that semi-symmetry [2, 14, 15] is equivalent to conformal semi-symmetry when the Weyl conformal tensor $C_{abcd}$ is non-zero:

$$\nabla_\alpha \nabla_\beta R_{\alpha\beta\gamma\delta} = 0 \iff \nabla_\alpha \nabla_\beta C_{\gamma\delta\alpha\beta} = 0 \quad (if \ C \neq 0).$$

Here, $R_{abcd}$ is the Riemann tensor. The Lorentzian result was also implicit in [10, 11]. For dimensions $n = 4$ a single simple counterexample is known [5], which has proper Riemannian signature. Using four-dimensional tensor identities [7], it is possible to show precisely when the result fails for proper Riemannian signature, and for neutral signature [6].

2. The spinor inspiration

Using the spinor formalism with the standard notation [12, 13] for the curvature spinors, the semi-symmetric conditions $\nabla_\alpha \nabla_\beta R_{\alpha\beta\gamma\delta} = 0$ can be rewritten as:

$$0 = \Box_{AB} \Psi_{CDEFG} = X_{AB}(C^G \Psi_{DEF})^G$$

$$0 = \Box_{A'B'} \Psi_{CDEFG} = \Phi_{A'B'}(C^G \Psi_{DEF})^G$$

$$0 = \Box_{AB} \Phi_{CDEFG} = 2X_{AB}(C^E \Phi_{DE'})^E + 2\Phi_{AB}(C^E \Phi_{CD'E'})^E$$
where $X_{ABCD} = \Psi_{ABCD} + R \epsilon_A(BC)\epsilon_D/6$ and $R$ is the scalar curvature. One must observe that (1) and (2) correspond to $\nabla_a \nabla_b C_{cdef} = 0$, while (3) corresponds to $\nabla_{[a} \nabla_{b]} R_{cdf} = 0$ (this is the so-called “Ricci semi-symmetry” condition).

One can prove that [(1) and (2)] $\implies$ (3) ($C \neq 0$) [8]. Similarly [(1) and (3)] $\implies$ (2) [6].

There are some special cases having a zero Weyl tensor which have been fully discussed in [1], see also Aman’s contribution to this volume.

It springs to mind the idea to concentrate solely on the expression (1), which is equivalent to

$$24 \Psi_{AB(C}^{\epsilon} \Psi_{DEF)G} = -R (\epsilon_A(C\Psi_{DEF})B + \epsilon_B(C\Psi_{DEF})A).$$

Its tensor expression reads

$$C_{k[a}^{\epsilon}C^k_{b]cd} + C_{k[e}^{\epsilon}C^k_{d]ab} = \frac{R}{6} \left( \delta_{[b}^{[e} C_{a]}^{f]}_{cd} + \delta_{[a}^{[e} C_{c]}^{f]}_{ab} \right).$$

This can be straightforwardly generalised to all dimensions $n \geq 4$ as follows [6]:

**Definition 1 (Edgar, 2010)** A Weyl tensor is said to be special semi-symmetric if it satisfies:

$$C_{k[a}^{\epsilon}C^k_{b]cd} + C_{k[e}^{\epsilon}C^k_{d]ab} = \frac{2R}{n(n-1)} \left( \delta_{[b}^{[e} C_{a]}^{f]}_{cd} + \delta_{[a}^{[e} C_{c]}^{f]}_{ab} \right).$$

### 3. The main result

Once the concept of special semi-symmetric Weyl tensor has been introduced, one can prove the following result.

**Theorem 1 (Edgar, 2010)** All (conformally) semi-symmetric spaces have a special semi-symmetric Weyl tensor in four dimensions. This is not the case in dimensions higher than four.

**Remark:** Observe that the result holds for arbitrary signature of the metric.

**Proof.** The conformal semisymmetric condition $\nabla_a \nabla_b C_{cdef} = 0$ reads

$$C_{k[a}^{\epsilon}C^k_{b]cd} + C_{k[e}^{\epsilon}C^k_{d]ab} = \frac{2}{n(n-1)} R \left( \delta_{[b}^{[e} C_{a]}^{f]}_{cd} + \delta_{[a}^{[e} C_{c]}^{f]}_{ab} \right) = \frac{2}{n-2} \left( \tilde{R}_{i[a}^{[e} \delta_{b]}^{f]} C_{i]}^{c]}_{cd} + \tilde{R}_{i[a}^{[e} \delta_{b]}^{f]} C_{i]}^{c]}_{ab} - \tilde{R}_{[a}^{i[a} \delta_{b]}^{f]} C_{i]}^{c]}_{cd} - \tilde{R}_{[a}^{i[a} \delta_{b]}^{f]} C_{i]}^{c]}_{ab} \right),$$

where $R_{ab}$ is the Ricci tensor and $\tilde{R}_{ab} \equiv R_{ab} - \frac{R}{n} g_{ab}$ its trace-free part. This relation has a non-trivial trace

$$0 = R_{ij}^{\epsilon} i[a C^i_{b]}^{\epsilon} + R_{ij}^{\epsilon} i[c C^i_{j]}^{\epsilon}$$

which can be split into two parts

$$0 = (n-1) C_{a[b}^{(c} \tilde{R}^{d)}_{i]} - 2 C_{a[b}^{(c} \tilde{R}^{d)}_{i]} - 2 \delta_{a[b}^{(c} \tilde{R}^{d)}_{i]} R_{ij},$$

$$C_{abij}^{cdeij} + 4 C_{a[i} \delta_{b]j}^{cd} - \frac{2R}{n} C_{ab}^{cd}$$

$$= \frac{1}{2(n-2)} \left( (n-3) \tilde{R}_{i[a}^{[c} C_{i]}^{d]}_{ab} + \tilde{R}_{i[a}^{i[c} C_{b]}^{d]}_{ab} + 2 \tilde{R}_{ij} \delta_{a[b}^{(c} \delta_{i]}^{d)} \right).$$

A basic “fundamental dimensionally dependent identity” (feldi) [7] in 4 dimensions is

$$C_{ab}^{(c} \delta_{i]}^{d)} = 0$$
from where one immediately deduces

\[ 0 = 9 \tilde{R}^i_j C_{[a}^{[i \delta_j]} = 2 \left( \tilde{R}^{[i} C_{|a|}^{d]_i} + \tilde{R}^i_{|a|} C^{cd}_{|a|} + 2 \tilde{R}_{ij} \delta^{[i}_{[a} C^{d]}_{b]j} \right). \]

Thus, (5) becomes, for \( n = 4 \),

\[ C_{abij} C^{cdij} + 4 C^{[a}_{ij} C^{b]}_{ij} = 0. \] (7)

Surprisingly, this implies by itself that the Weyl tensor is (for \( n = 4 \)) special semi-symmetric: use again the fddi (6) to build

\[ C^{i}_j e f C_{[a}^{[i \delta_j]} + C^{i}_j a b C_{[e f}^{[cd} \delta_j] = 0 \]

which expands into

\[ C^{i}_e f [a C_{d]}^{i a b} + C^{i}_{j a b} C^{e f} [c d i j] = 0. \]

Managing this (and its trace) in a judicious manner \([6]\), and using (7), one arrives at

\[ C^{ef}_{k[a} C^{k}_{b]d} = C^{ef}_{k[a} C^{k}_{d]}_{ab} = R^{R}_{6} \left( \delta^{[e}_{[a} C^{f]}_{cd} + \delta^{[e}_{[d} C^{f]}_{ca} \right) \]

in the case \( n = 4 \). This is precisely the condition given in Definition 1 for the Weyl tensor to be special semi-symmetric four dimensions. This proves the \( n = 4 \) part of the theorem.

The result in higher dimensions follows because one can easily construct explicit counterexamples, showing that this property is exclusive to \( n = 4 \). One particular simple counterexample is to consider a semi-riemannian manifold decomposable into two submanifolds, each of which has constant curvature. It is then very simple to compute the Weyl tensor and check that the condition in Definition 1 fails to hold for all \( n > 4 \) \([6]\).

\[ \begin{align*}
4. \textbf{Final remarks: extension to pseudo-symmetric spaces} \\
	ext{Actually, all the previous results hold for the more general classes of (conformally) pseudo-symmetric spaces. Pseudo-symmetric spaces are defined by (e.g. \([4]\) and references therein)}
\end{align*} \]

\[ R^{ef}_{k[a} R^{k}_{b]d} + R^{ef}_{k[c} R^{k}_{d]}_{ab} = L \left( \delta^{[e}_{[a} R^{f]}_{cd} + \delta^{[e}_{[d} R^{f]}_{ca} \right) \]

while conformally pseudo-symmetric spaces satisfy

\[ R^{ef}_{k[a} C^{k}_{b]d} + R^{ef}_{k[c} C^{k}_{d]}_{ab} = \tilde{L} \left( \delta^{[e}_{[a} C^{f]}_{cd} + \delta^{[e}_{[d} C^{f]}_{ca} \right) \]

for some scalars \( L \) and \( \tilde{L} \). Observe that they correspond respectively to

\[ \nabla^{e} \nabla^{\prime} R_{abcd} = -L \left( \delta^{[e}_{[a} R^{f]}_{cd} + \delta^{[e}_{[d} R^{f]}_{ca} \right), \]

\[ \nabla^{e} \nabla^{\prime} C_{abcd} = -\tilde{L} \left( \delta^{[e}_{[a} C^{f]}_{cd} + \delta^{[e}_{[d} C^{f]}_{ca} \right) \]

so that (8) implies (9) —for appropriate \( \tilde{L} \). Then, analogously to the semi-symmetric case, one can split (9) by decomposing the Riemann tensor into Weyl plus Ricci terms and then check (for instance via spinors) that the Weyl tensor satisfies a particular quadratic relation for \( n = 4 \). Generalizing it to arbitrary dimension, one can introduce the following definition...
Definition 2 (Edgar, 2010) A Weyl tensor is said to be special pseudo-symmetric if it satisfies:

\[ C^e f_{k[a} C^k_{b]cd} + C^e f_{k[c} C^k_{d]ab} = M \left( \delta_b^{[e} C_{a]}_{c} f_{]cd} + \delta_d^{[e} C_{c]}_{a} f_{]ab} \right) \]

for some scalar function \( M \).

Clearly, the semi-symmetric Weyl tensor condition of Definition 1 is a special case with \( n(n - 1)M = 2R \).

Using exactly the same type of arguments —based again on the fddi (6) for \( n = 4\) —as in the proof of Theorem 1, one can then prove [6]

Theorem 2 (Edgar, 2010) All (conformally) pseudo-symmetric spaces have a special pseudo-symmetric Weyl tensor in four dimensions. This is not the case in dimensions higher than four.

Again, the result holds for arbitrary signature.

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