Pincer-based vs. Same-direction Search Strategies
After Smart Evaders by Swarms of Agents

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Abstract—Suppose in a given planar region, there are smart mobile evaders and we want to detect them using sweeping agents. We assume that the agents have line sensors of equal length. We propose procedures for designing cooperative sweeping processes that ensure successful completion of the task, thereby deriving conditions on the sweeping velocity of the agents and their paths. Successful completion of the task means that evaders with a known limit on their velocity cannot escape the sweeping agents. A simpler task for the sweeping swarm is the confinement of the evaders to their initial domain. The feasibility of completing these tasks depends on geometric and dynamic constraints that impose a lower bound on the velocity the sweeping agent must have. This critical velocity is derived to ensure the satisfaction of the confinement task. Increasing the velocity above the lower bound enables the agents to complete the search task as well. We present a quantitative and qualitative comparison analysis between the total search time of same-direction sweep processes and pincer- movement search strategies. We evaluate the different strategies by using two metrics, total search time and the minimal critical velocity required for a successful search. We compare two types of pincer-movement search processes, circular and spiral, with their same-direction counterparts, for any even number of sweeping agents. We prove that pincer based strategies provide superior results for all practical scenarios and that the spiral pincer sweep process allows detection of all evaders while sweeping at nearly theoretically optimal velocities.

Index Terms—Multiple Mobile Robot Systems, Motion Planning, Aerial Robots, Search and Rescue Robotics, Robot Surveillance and Security, Cooperating Robots, Swarms

I. INTRODUCTION

The aim of this work is to provide an efficient "must-win" search policy for a swarm of \( n \) sweeping agents that must guarantee detection of an unknown number of smart evaders initially residing inside a given circular region of radius \( R_0 \) while minimizing the search time. The evaders move and try to escape the initial region at a maximal velocity of \( V_T \), known to the sweepers. All sweepers move at a velocity \( V_s > V_T \) and detect the evaders using linear sensors of length \( 2r \). Each "must-win" policy requires a minimal velocity that depends on the trajectory of the sweepers. Finding an efficient algorithm requires that, throughout the sweep, the footprint of the sweepers’ sensors maximally overlaps the evader region (the region where evaders may possibly be). This work develops two "must-win" same-direction search strategies, circular and spiral, for a swarm consisting of an even number of searchers that sweep the evader region until all evaders are detected. Afterwards, a comparison between the developed same-direction and pincer-based search strategies developed in [1] is performed.

II. OVERVIEW OF RELATED RESEARCH

An interesting challenge for multi-agent systems is the design of searching or sweeping algorithms for static or mobile targets in a region, which can either be fully mapped in advance or unknown, see e.g. [2]–[3]. Often the aim is to continuously patrol a domain in order to detect intruders or to systematically search for mobile targets known to be located within a given area [6]. Search for static targets involves complete covering of the area where they are located, but a much more interesting and realistic scenario is the question of how to efficiently search for targets that are dynamic and smart. A smart target is one that detects and responds to the motions of searchers by performing optimal evasive maneuvers, to avoid interception.

Several such problems originated in the second world war due to the need to design patrol strategies for aircraft aiming to detect ships or submarines in the English channel, see [7]. The problem of patrolling a corridor using multi agent sweeping systems in order to ensure the detection and interception of smart targets was also investigated by Vincent et al. in [8] and provably optimal strategies were provided by Altshuler et al. in [9]. A somewhat related, discrete version of the problem, was also investigated by Wagner et al. and later by Altshuler et al. in [10], [11] and [12]. It focuses on a dynamic variant of the cooperative cleaners problem, a problem that requires several simple agents to a clean a connected region on the grid with contaminated pixels. This contamination is assumed to spread to neighbors at a given rate.

In [6], Tang et al. propose a non-escape search procedure for evaders. Evaders are originally located in a convex region of the plane and may move out of it. Tang et al. propose a cooperative progressing spiral-in algorithm performed by several agents with disk shaped sensors in a leader-follower formation. The authors establish a sufficient condition for the number of searching agents required to guarantee that no evader can escape the region undetected. This lower bound is based on the sensor radius, searcher and evader velocities and the initial perimeter of the region. In [13], McGee et al. also investigate a search problem for smart targets that do not have any maneuverability restrictions except for an upper limit on their velocity. The sensor that the agents are equipped with detects targets within a disk shaped area around the searcher location. Search patterns consisting of spiral and linear sections are considered. In [14], Hew proposes searching for smart evaders using concentric arc trajectories with agents sensors similar to [13]. Such a search is aimed at detecting submarines in a channel or in a half plane. The paper focuses on determining the size of a region that can be successfully
patrolled by a single searcher, where the searcher and evader velocities are known. The search problem is formulated as an optimization problem. The search progress per arc or linear iteration is maximized while guaranteeing that the evader cannot slip past the searcher undetected.

Another set of related problems are pursuit-evasion games, where the pursuers’ objective is to detect evaders and the evaders objective is to avoid the pursuers. In this context several works considered the problem of defending a region from the entrance of intruders under the name of perimeter defense games by Shishika et al. in [15]–[17], with a focus on utilizing cooperation between pursuers to improve the defense strategy.

III. SAME-DIRECTION SWEEPS

This paper considers a scenario in which a multi-agent swarm of identical agents search for mobile targets or evaders that are to be detected. The information the agents perceive only comes from their own sensors, and all evaders that intersect a sweeper’s field of view are detected. We assume that all agents have a linear sensor of length $2r$. The evaders are initially located in a disk shaped region of radius $R_0$. There can be many evaders, and we consider the domain to be continuous, meaning that evaders can be located at any point in the interior of the circular region at the beginning of the search process. The sweeping protocols proposed are predetermined and deterministic, hence the sweepers can perform them using a minimal amount of memory and computations. All sweepers move with a speed of $V_e$ (measured at the center of the linear sensor). By assumption the evaders move at a maximal speed of $V_T$, without any maneuverability restrictions. The sweeper swarm’s objective is to “clean” or to detect all evaders that can move freely in all directions from their initial locations in the circular region of radius $R_0$.

Search time clearly depends on the type of sweeping movement the swarm employs. Detection of evaders is based on deterministic and preprogrammed search protocols. We consider two types of search patterns, circular and spiral patterns. The desired result is that after each sweep around the region, the radius of the circle that bounds the evader region (for the circular sweep), or the actual radius of the evader region (for the spiral sweep), decreases by a value that is strictly positive. This guarantees complete cleaning of the evader region (for the circular sweep), or the actual radius of the evader region (for the spiral sweep), decreases by a value that is strictly positive. This guarantees complete cleaning of the evader region, by shrinking in finite time the possible area in which evaders can reside to zero. At the beginning of the circular search process we assume that only half the length of the agents’ sensors is inside the evader region, i.e. a footprint of length $r$, while the other half is outside the region in order to catch evaders that may move outside the region while the search progresses. At the beginning of the spiral search process we assume that the entire length of the agents’ sensors is inside the evader region, i.e. a footprint of length $2r$.

In the single agent search problem described in [18], we observed that there can be escape from point $P = (0, R_0)$ (shown in Fig. 1 (a)), when basing the searcher’s velocity only on a single traversal around the evader region. Therefore we had to increase the agent’s critical velocity to deal with this possible escape. If we were to distribute a multi-agent swarm say, equally along the boundary of the initial evader region, we would have the same problem of possible escape from the points adjacent to the starting locations of the sweepers.

![Fig. 1. (a) - Initial placement of 2 agents employing the same-direction circular sweep process. (b) - Initial placement of 2 agents employing the same-direction spiral sweep process. The sweepers sensors are shown in green. The angle $\phi$ is the angle between the tip of a sweeper’s sensor and the normal of the evader region. $\phi$ is an angle that depends on the ratio between the sweeper and evader velocities.](image)

IV. PINCER-BASED SEARCH

Since we wish the sweepers to have the lowest possible critical velocity, in [1] we propose a different idea than same-direction sweeps for the search process. The idea is to have pairs of sweeping agents move out in opposite directions along the boundary of the evader region and sweep in a pincer movement rather than having a convoy of sweepers all moving in the same-direction along the boundary.

Our method is readily applicable for any even number of sweepers. The sweepers are initially positioned in pairs back to back. One sweeper in the pair moves counterclockwise while the other sweeper in the pair moves clockwise. In case the search is planar, once the sweepers meet, i.e. their sensors are again superimposed at a meeting point, they switch the directions in which they move. For example, if the search carried out by two sweepers, after the first sweep this switching point is located at $(0, -R_0)$. This changing of directions occurs every time a sweeper bumps into another. Each sweeper is responsible for an angular sector of the evader region that is proportional to the number of participating agents in the search. The search process can be viewed as a 2 dimensional search in which the actual agents travel on a plane or as a 3 dimensional search where the sweepers are drone like agents which fly over the evader area. In case the search is 3 dimensional, where sweepers fly at different heights above the evader region, every time a sweeper is directly above
another, they exchange the angular section they are responsible to sweep between them, and continue the search. The analysis of the two cases is exactly the same.

Sweepers that employ a pincer movement solve the problem of evader region’s spread from the “most dangerous points”, points located at the tips of their sensors closest to the evader region’s center. These points have the maximum time to spread during sweeper movement and therefore if evaders that try to escape from these points are detected, evaders trying to escape from other points are detected as well. When a sweeper returns to a location, the evader region has a smaller or equal radius than it had 2 cycles previously. If all sweepers were to rotate in the same-direction after being deployed equally around the circle, the evader region’s points that need to be considered for limiting the region’s spread are points that are adjacent to the center of the sensor (for a circular sweep) and points that are adjacent to the sensors’ tips that are furthest from the center of the evader region (for a spiral sweep). This consideration would lead to higher critical velocities for sweepers that employ same-direction sweeps. Higher critical velocities also imply that, for a given sweeper velocity above the critical velocity, that is sufficient for both same-direction and pincer based sweep processes, sweep time is reduced when sweepers perform pincer movement sweep. In [18], the analysis of a single agent circular sweep process indicates that the critical velocity for agents employing same-direction sweeps is indeed higher compared to the pincer based critical velocities developed in [1]. In this paper it is proven that for swarms with multiple sweepers same-direction critical velocities are indeed higher than their pincer-based counterparts.

We analyze the proposed sweep processes’ performance in terms of the total time to complete the search, defined as the time at which all potential evaders that resided in the initial evader region were detected. Expressions for the complete cleaning times of the evader region as a function of the search parameters, \( R_0, r, V_T \) and the number of agents, \( n \), in the swarm are derived, evaluated and discussed for each developed sweep process.

Illustrative simulations that demonstrate the evolution of the search processes were generated using NetLogo software [19] and are presented in Fig. 3. and Fig. 4. Green areas are locations that are free from evaders and red areas indicate locations where potential evaders may still be located. Fig. 3. shows the cleaning progress of the evader region when 6 agents employ the circular pincer sweep process. Fig. 4. shows the cleaning progress of the evader region when 4 agents employ the spiral pincer sweep process.

![Fig. 3. Swept areas and evader region status for different times in a scenario where 6 agents employ the circular pincer sweep process. (a) - Beginning of first cycle. (b) - Midway of the first cycle. (c) - toward the completion of the first cycle. (d) - Beginning of the second cycle. (e) - toward the end of the third cycle. (f) - Beginning of the one before last cycle. Green areas are locations that are free from evaders and red areas indicate locations where potential evaders may still be located.](image)

Note that in the considered problems, we consider the exact locations of evaders and even their numbers is a priori unknown. The only information the sweepers have about the evaders locations is that the evaders are located somewhere inside a given circular region at the beginning of the search process, and that the evaders may try to move and slip undetected out of this region as the search progresses, to avoid interception. Since the sweepers do not have any additional knowledge about the evaders whereabouts, or even if all evaders were found at some intermediate point of time during the search, the search is continued until the whole region is searched, thus reducing the uncertainty region where potential evaders might be located to have an area of 0.

The contributions of the paper are as follows. A complete theoretical analysis of trajectories, critical velocities and search times for a swarm of \( n \) cooperative agents whose mission is to guarantee detection of all smart evaders that are initially
to compare between different search methods. We denote the resulting cleaning rate to the optimal derived bound in order is independent of the particular search pattern employed. For a line shaped sensor of length $2r$ this happens when the entire length of the sensor fully overlaps the evader region and it moves perpendicular to its orientation. The rate of sweeping when this happens has to be higher than the minimal expansion rate of the evader region (given its total area) otherwise no sweeping process can ensure detection of all evaders.

We analyze the search process when the sweeper swarm is comprised of $n$ identical agents. The smallest searcher velocity satisfying this requirement is defined as the critical velocity and denoted by $V_{LB}$, we have:

**Theorem 1.** No sweeping process is able to successfully complete the confinement task if its velocity, $V_s$, is less than,

$$V_{LB} = \frac{\pi R_0 V_T}{nT}$$  \hspace{1cm} (1)

For proof see [1]. Hopefully, after the first sweep the evader region is within a circle with a smaller radius than the initial evader region’s radius. Since the sweepers travel along the perimeter of the evader region and this perimeter decreases after the first sweep, ensuring a sufficient sweeper velocity that guarantees that no evader escapes during the initial sweep guarantees also that the sweeper velocity is sufficient to prevent escape in subsequent sweeps as well.

**VI. Pincer Sweep Process**

A. Multiple Agents with Linear Sensors: the Circular Pincer Sweep Process

We analyze the case that the multi-agent swarm consists of $n$ agents, where $n$ is an even number, and each sweeper has a sensor length of $2r$. At the beginning of the search process the footprint of each sweeper’s sensor that is over the evader region is equal to $r$. When employing this type of search pattern the symmetry between the two agent trajectories prevents the escape from point $P = (0, R_0)$ that is the most dangerous point an evader can escape from as proved in [18]. Therefore each sweeper’s critical velocity is based only upon the time it takes to traverse the angular section it is required to scan, i.e. an angle of $\frac{\pi}{n}$. For example, if the sweeper swarm consists of only two sweepers, each sweeper is required to scan an angle of $\frac{\pi}{2}$. If the sweepers’ velocities are above the critical velocity of the scenario the agents can advance inward toward the center of the evader region after completing a cycle. The notion of a cycle or an iteration corresponds to an agent’s traversal of the angular section it is required to scan, i.e. an angle of $\frac{\pi}{n}$. Therefore, its definition varies with the number of sweepers. Once the agents finish scanning the angular section they are responsible for, and if their velocities allow it, they advance inward together. In case the search is performed in the $2D$ plane, the sweepers change their scanning direction only after the completion of an inward advancement maneuver. In case the search is 3 dimensional, i.e. the sweepers fly over the evader region, the sweepers first advance inward together, and only then exchange between them the angular section they located in given circular region from which they may move out of in order to escape the pursuing sweeping agents. The theoretical analysis is provided by designing search algorithms for any even number of sweeping agents the employ same-direction circular and spiral search processes and qualitatively and quantitatively comparing between them to their pincer-based counterparts. The theoretical analysis is complemented by simulation experiments that verify the theoretical results and illustrate them graphically.

For both circular and spiral sweep processes, it is quantitatively proven that sweeping with pairs of sweepers that employ pincer movements between themselves and between adjacent sweeper pairs yields a lower critical velocity than the case where sweepers are distributed evenly around the region and sweep in the same-direction. Additionally, the derived expressions for the complete sweep time of the evader region, show that the spiral pincer sweep process allows sweepers to successfully detect all evaders in the region while sweeping with velocities that are nearly theoretically optimal and also enables the sweepers to detect all evaders in the region at a significantly shorter amount of time compared to its same-direction spiral counterpart and to the circular pincer sweep process.

**V. A Universal Bound On Cleaning Rate**

In this section we present an optimal bound on the cleaning rate of a searcher with a linear shaped sensor. This bound is independent of the particular search pattern employed. For each of the proposed search methods we then compare the resulting cleaning rate to the optimal derived bound in order to compare between different search methods. We denote the searcher’s velocity as $V_s$, the sensor length as $2r$, the evader region’s initial radius as $R_0$ and the maximal velocity of an evading agent as $V_T$. The maximal cleaning rate occurs when the footprint of the sensor over the evader region is maximal. For a line shaped sensor of length $2r$ this happens when the entire length of the sensor fully overlaps the evader region and it moves perpendicular to its orientation. The rate of sweeping when this happens has to be higher than the minimal expansion rate of the evader region (given its total area) otherwise no sweeping process can ensure detection of all evaders.

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are responsible to sweep. Afterwards the sweepers continue to scan a section with a smaller radius. Contrary to the works in [15–17] that devise methods for intercepting as many intruders as possible with agents that can intercept only a single invader, the circular pincer sweep process guarantees evaders are detected regardless of their number and the number of pursuing sweepers. Since each sweeper has a sensor length of \( r \) outside the evader region, in order to guarantee that no evader escapes the sweepers, we must demand that the spread of the evader region, from any potential location where an evader might be located, is confined to a radius of no more than \( r \) from its origin point at the beginning of the cycle. During an angular traversal of \( \frac{2\pi}{n} \) around the evader region radius of \( R_0 \), this yields the circular pincer sweep process’s critical velocity,

\[
V_c = \frac{2\pi R_0 V_T}{nr}
\]

Therefore, we obtain that the circular critical velocity equals twice the optimal minimal critical velocity,

\[
V_c = 2V_{LB}
\]

**Theorem 2.** For an \( n \) agent swarm for which \( n \) is even, that performs the circular sweep process, where the sweeper distribution is as described, the number of iterations it takes the swarm to reduce the evader region to be bounded by a circle with a radius that is less than or equal to \( r \) is given by,

\[
N_n = \left[ \ln \left( \frac{2\pi V_T - nr V_c}{2\pi R_0 V_T - nr V_c} \right) \right] \ln \left( 1 + \frac{2\pi V_T}{n(V_s + V_T)} \right)
\]

After \( N_n \) sweeps the sweeper swarm performs an additional circular sweep and cleans the entire evader region. We denote by \( T_{in} \) the sum of all inward advancement times and by \( T_{circular} \) the sum of all the circular traversal times. Therefore it takes the swarm to clean the entire evader region is given by,

\[
T(n) = T_{in}(n) + T_{circular}(n)
\]

Where \( T_{in}(n) \) is given by,

\[
T_{in}(n) = R_0 V_s + \left( \frac{2\pi R_0 V_T - nr V_c}{nV_s(V_s + V_T)} \right) \left( 1 + \frac{2\pi V_T}{n(V_s + V_T)} \right)^{N_n-1}
\]

And \( T_{circular}(n) \) is given by,

\[
T_{circular}(n) = -R_0(V_s + V_T) + \frac{nr(V_s + V_T) + 2\pi V_T}{2\pi V_T^2} + \left( 1 + \frac{2\pi V_T}{nV_s + V_T} \right) \frac{N_n}{2\pi V_T^2} \left( V_s + V_T \right) - V_T^2 \left( \frac{nr V_c}{2\pi V_T} \right)
\]

For proof see [1].

### B. Multiple Agents with Linear Sensors: the Spiral Pincer Sweep Process

Since at the start of every circular sweep process half of the sweepers’ sensors are outside of the evader region, we would like the sweepers to employ a more efficient motion throughout the cleaning process. This means that throughout the motion of the searcher the footprint of its sensor that is above the evader region should maximally overlap the evader region. This can be achieved with a spiral scan, where the agent’s sensor tracks the expanding evader region wavefront, while preserving its shape to be as close as possible to a circle.

We analyze the case that the multi-agent swarm consists of \( n \) agents, where \( n \) is an even number, and each sweeper has a line sensor length of \( 2r \). Choosing all sweepers to have sensors of equal lengths implies that when the sweepers’ sensors reach the same point and are tangent to each other, escape will not be possible from the gap between the sensors. This is a necessary requirement since the process that is described below relies on the symmetry between the agents sensors that are over the evader region.

At the beginning of the search process the footprint of each sweeper’s sensor that is over the evader region is equal to \( 2r \). When employing this type of search pattern the symmetry between the two agent trajectories prevents the escape from point \( P = (0, R_0) \) that is the most dangerous point an evader can escape from. Therefore, each sweeper’s critical velocity can be based only upon the time it takes it to traverse the angular section it responsible for, namely \( \frac{2\pi}{2} \). Similarly to the circular sweep process, when sweepers move at a velocity greater than their critical velocity and if the search is carried out on the 2D plane, sweepers change their direction of scanning after performing an inward advancement. If the search is 3 dimensional, the sweepers advance inward together, exchanging afterwards between them the angular section they are responsible to sweep. After this motion the sweepers start scanning a section with a smaller radius.

Each searcher begins its spiral traversal with the tip of its sensor tangent to the edge of the evader region. In order to keep its sensor tangent to the evader region throughout the sweep, the searcher must travel at angle \( \phi \) to the normal of the evader region. \( \phi \) is an angle that depends on the ratio between the sweeper and evader velocities. The incentive of a sweeper to travel in a constant angle \( \phi \) to the normal of the evader region is to preserve the evader region’s circular shape and keep the entire length of its sensor inside the evader region at all times. Fig 2 (b) shows an initial placement of 2 agents that employ the pincer movement spiral sweep process. \( \phi \) is calculated by,

\[
\sin \phi = \frac{V_T}{V_s}
\]

Thus we have,

\[
\phi = \arcsin \left( \frac{V_T}{V_s} \right)
\]

The isoperimetric inequality states that for a given area, the shape of the curve that bounds the area which has the smallest perimeter is circular. Since the agent travels along the perimeter of the evader region while keeping the evader region’s shape close to circular, the isoperimetric inequality implies that the time it takes to complete a sweep around the region is minimal. The agent’s angular velocity or rate of change of its angle with respect to the center of the evader region, \( \theta_s \), can be described as a function of \( \phi \) as,

\[
\frac{d\theta_s}{dt} = \frac{V_s \cos \phi}{R_0(t)} = \frac{\sqrt{V_s^2 - V_T^2}}{R_0(t)}
\]
And the instantaneous growth rate of the searcher radius will be given by,
\[
\frac{dR_s(t)}{dt} = V_s \sin \phi = V_T
\]  
\[(11)\]
Integrating equation \[(10)\] between the initial and final sweep times of the angular section yields that,
\[
\theta(t_\theta) = \frac{\sqrt{V_s^2 - V_T^2}}{V_T} \ln \left( \frac{V_Tt_\theta + R_0 - r}{R_0 - r} \right)
\]  
\[(12)\]
Applying the exponent function to both sides of the equation results in,
\[
(R_0 - r) e^{\sqrt{V_s^2 - V_T^2}} = V_Tt_\theta + R_0 - r = R_s(t_\theta)
\]  
\[(13)\]
Each sweeper begins its spiral traversal with the tip of its sensor tangent to the edge of the evader region. The time it takes to complete a spiral traversal around the evader region is responsible to scan it takes a sweeper to complete a spiral traversal around the sensor tangent to the edge of the evader region. The time it takes for the evader to be cleaned is given by,
\[
\text{number of iterations} = \frac{360}{\theta(t_\theta)}
\]
\[(14)\]
For an evader distribution is as described, the number of iterations it takes the swarm to clean the entire evader region is given by,
\[
N_n = N_n + \eta + 1 = \left( \frac{\ln \left( \frac{V_T + V_s e^{n \sqrt{V_s^2 - V_T^2}}}{V_T} \right)}{\ln \left( \frac{V_T + V_s e^{n \sqrt{V_s^2 - V_T^2}}}{V_T} \right) + e^{2\sqrt{V_s^2 - V_T^2}}} \right)^n + \eta + 1
\]  
\[(16)\]
where \(\eta = 0, or \eta = 1\).

**Theorem 3.** For an \(n\) agent swarm for which \(n\) is even, that performs the spiral sweep process, where the sweepers distribution is as described, the number of iterations it takes the swarm to clean the entire evader region is given by,
\[
N_n = N_n + \eta + 1 = \left( \frac{\ln \left( \frac{V_T + V_s e^{n \sqrt{V_s^2 - V_T^2}}}{V_T} \right)}{\ln \left( \frac{V_T + V_s e^{n \sqrt{V_s^2 - V_T^2}}}{V_T} \right) + e^{2\sqrt{V_s^2 - V_T^2}}} \right)^n + \eta + 1
\]  
\[(17)\]
Therefore, the time it takes the swarm to clean the entire evader region is given by,
\[
T(n) = T_{in}(n) + T_{spiral}(n)
\]  
\[(18)\]
Where \(T_{in}(n)\) is given by,
\[
T_{in}(n) = T_{in}(n) + T_{in,last}(n) + \eta T_{in}(n)
\]  
\[(19)\]
Where \(T_{in}(n)\) is given by,
\[
T_{in}(n) = \frac{2r}{V_s + V_T} + \frac{R_0 - r}{V_s} + \frac{2r}{V_s + V_T} 
\]  
\[(20)\]
\[ T_{\text{in, last}}(n) \text{ is given by } T_{\text{in, last}}(n) = \frac{R_V}{V_s} \text{ and } T_{\text{in, f}}(n) \text{ is given by } T_{\text{in, f}}(n) = \frac{T_V V_T}{V_s}. \] Therefore,

\[ T_{\text{in}}(n) = \overline{T}_{\text{in}}(n) + \frac{R_N}{V_s} + \frac{\eta}{V_s} e^{\frac{2 \pi V_T}{v_s} - 1} \tag{20} \]

\[ T_{\text{s, spiral}}(n) \text{ is given by,} \]

\[ T_{\text{s, spiral}}(n) = \overline{T}_{\text{s, spiral}}(n) + T_{\text{last}}(n) + \eta T_{f}(n) \tag{21} \]

Where \( \overline{T}_{\text{s, spiral}}(n) \) is given by,

\[ \overline{T}_{\text{s, spiral}}(n) = \left( \frac{e - R_0}{V_s} + \frac{2 \pi r (N_n - 1)}{V_T} \right) - \frac{2 \pi r (N_n - 1)}{V_T} \left( V_T + V_s e^{\frac{2 \pi V_T}{v_s} - 1} \right) \]

\[ + \left( V_T + V_s e^{\frac{2 \pi V_T}{v_s} - 1} \right) \left( \frac{N_n}{V_s} - \frac{\eta}{V_s} e^{\frac{2 \pi V_T}{v_s} - 1} + 1 \right) \]

\[ T_{\text{last}}(n) \text{ is given by } T_{\text{last}}(n) = \frac{2 \pi r}{V_T} \text{ and } T_{f}(n) \text{ is given by} \]

\[ T_{f}(n) = \left( \frac{e - R_0}{V_s} + \frac{2 \pi r (N_n - 1)}{V_T} \right). \text{ Hence, } T_{\text{s, spiral}}(n) \text{ is given by,} \]

\[ T_{\text{s, spiral}}(n) = \overline{T}_{\text{s, spiral}}(n) + \frac{2 \pi r}{V_T} + \eta - \frac{e^{\frac{2 \pi V_T}{V_s} - 1} - 1}{V_T}. \tag{22} \]

For a derivation of the sweep times for the spiral pincer sweep process see [1].

\[ \text{VII. SAME-DIRECTION CIRCULAR SWEEP} \]

Previously, we tried to find the tightest lower bound of a searcher’s velocity by constructing a function of 2 variables \( f(t, V_s) \), by demanding that the furthest possible spread of the evader region is cleaned by the furthest tip of the sweeper’s line sensor. A lesser requirement is to demand that by the time the most problematic point in the evader region, point \( P \), spreads to a possible circle of radius \( r \) around point \( P \), the sweeping swarm completes in addition to a sweep of \( \frac{2 \pi}{n} \) around the evader region an additional angular traversal that is proportional to traversing an arc of length \( r \). This means that the agent travels an angle of \( \frac{2 \pi}{n} + \beta_0 \) where \( \beta_0 \) is marked in Fig. 6. We denote the time it takes the most problematic points to spread a distance of \( r \) as \( T_s \). These points are adjacent to the starting locations of the sweeper, and 2 such points \( P_1 \) and \( P_2 \) exist in case the search is performed with 2 sweeper, as shown in Fig. 6. We have that \( T_s = \frac{r}{V_T}. \) We can see from Fig. 6. that \( \sin \beta_0 = \frac{r}{R_0} \) therefore \( \beta_0 = \arcsin \frac{r}{R_0} \). The time it takes the sweeper to travel an angle of \( \frac{2\pi}{n} + \beta_0 \) is therefore given by \( T_s = \frac{\frac{2\pi}{n} + \arcsin \left( \frac{r}{R_0} \right)}{V_s} \). In order to guarantee no escape, we demand that \( T_s \leq T_e \). Therefore, rearranging terms in the previous equation and plugging \( T_e \) instead of \( T_s \) we get that,

\[ V_e \geq \left( \frac{2 \pi}{n} + \arcsin \left( \frac{r}{R_0} \right) \right) \frac{R_0 V_T}{r} \tag{24} \]

The lower bound on a sweeper velocity that ensures confinement is obtained when we have equality in (24), i.e.

\[ V_{\text{LB, } s} = \left( \frac{2 \pi}{n} + \arcsin \left( \frac{r}{R_0} \right) \right) \frac{R_0 V_T}{r} \]. \]

In future derivations we use the first order Taylor approximation for the arcsine function in (24), in order to enable the construction of analytical results for the sweep times of the evader region. Such an approximation is valid since in all practical scenarios the ratio between \( \frac{r}{R_0} \) is sufficiently small. Applying this approximation to (24) allows us to define \( V_e \), the chosen critical velocity, given by,

\[ V_{\text{LB, } s} = \frac{2 \pi R_0 V_T}{r} + V_T \tag{25} \]

In order for the sweeper swarm to advance inward toward the center of the evader region it must travel in a velocity that is greater than the critical velocity. We denote by \( \Delta V > 0 \) the increment in the sweeping agents’ velocity that is above the critical velocity. Each agent’s velocity \( V_s \) is therefore given by the sum of the critical velocity and \( \Delta V \), namely \( V_s = V_e + \Delta V \). The total sweep times it takes the sweeper swarm to reduce the evader region to a region bounded by a circle with a radius that is smaller or equal to \( r \) is given by the sum of the circular motions and inward advancements that are performed after the completion of each circular sweep. The time it takes the sweeper to perform the circular sweeps is given by,

\[ T_{\text{circular}} = - \frac{R_0 (V_s + V_T)}{V_T} + \frac{r (V_s + V_T) (n (V_s + V_T) + 2 \pi V_T n)}{2 \pi V_T V_s} + \frac{r (V_s + V_T)}{2 \pi V_T V_s} + \frac{\Delta V}{2 \pi V_T} \] \tag{26}
The time it takes the sweepers to perform the inward advancement is given by,

\[
T_{in} = \frac{R_0}{V_s} + \left(1 + \frac{2\pi V_T}{V_s + V_T} \right)^{N-1} \left( \frac{2\pi R_0 V_T - r(V_s - V_T)}{V_s (V_s + V_T)} \right) \tag{27}
\]

The full analytical development is provided in Appendix A.

### A. same-direction Circular Sweep End-Game

In order to entirely clean the evader region the sweepers need to change the scanning method when the evader region is bounded by a circle of radius \( r \). This is due to the fact that a smart evader that is very close to the center of the evader region can travel at a very high angular velocity compared to the angular velocity of the pursuing agents. This constraint is described by the following two equations, \( \omega_s = \frac{V_s}{r} \), \( \omega_T = \frac{V_T}{r} \). The first describes the searcher’s angular velocity and the second the evader’s angular velocity. Since \( \varepsilon \) can be arbitrarily small the evader can move just behind a sweeper’s sensor and never be detected. Thus a slight modification to the sweep process needs to be applied in order to clean the entire evader region with the sweeper swarm that employs a circular scan.

After completing sweep number \( N_n - 1 \) the sweepers move toward the center of the evader region until the tip of the sweeper’s sensors closest to the center of the evader region are placed at the center of the evader region. Following this motion the sweepers perform a circular sweep of radius \( r \) around the center of the evader region. The time this last circular sweep is bounded by a circle of radius \( R_{last} \), given by,

\[
R_{last} = T_{last} V_T = \frac{2\pi r V_T}{n V_s} \tag{28}
\]

In order to overcome the challenges in the circular search that were described we propose that after scan number \( N_n \) the sweepers will travel to the right until cleaning the waveform that propagates from the right portion of the remaining evader region and then travel to the left until cleaning the remaining evader region. A depiction of the scenario at the beginning of the end-game is presented in Fig. 7.

Theorem 4 states the conditions for this demand to hold.

**Theorem 4.** When defining \( \alpha = \frac{R_0}{r} \), if \( \Delta V \) satisfies that,

\[
\Delta V \geq -4\pi V_T \alpha + \pi V_T + V_T \sqrt{\pi^2 + 8\pi n} \frac{2n}{2n} \tag{29}
\]

then the evader region will be completely cleaned by \( n \) sweepers that employ the linear scan after \( N_n + 1 \) iterations.

**Proof.** During the previously mentioned movement the margin between the tip of the sensor in each direction to the evader region boundaries must satisfy,

\[
\frac{2r - R_{last}}{V_T} > T_{linear} \tag{30}
\]

in order to guarantee no escape. \( T_{linear} \) denotes the time it takes the sweepers to clean the right section of the remaining evader region in addition to the time it takes them to scan from the rightmost point they got to until the leftmost point of the expansion. These times are respectively denoted as \( t \) and \( \hat{t} \). Therefore, \( T_{linear} \) is given by \( T_{linear} = \hat{t} + t \).

The evader region’s rightmost point of expansion starts from the point \( (R_{last}, 0) \) and spreads at a velocity of \( V_T \). Therefore, if the constraint in (30) is satisfied we can view the rightward and leftward linear sweeps as a one dimensional scan. This geometric constraint can be observed in Fig. 7. Therefore, the time \( t \) it takes the sweepers to clean the spread of potential evaders from the right section of the region can be calculated from, \( V_s t = R_{last} + V_T \hat{t} \). Therefore, \( t \) is given by, \( t = \frac{R_{last} - V_T}{V_s - V_T} \hat{t} \). \( \hat{t} \) is computed by calculating the time it takes the sweepers located at point \( (tV_s, 0) \) to change their scanning direction and perform a leftward scan to a point that spread at a velocity of \( V_T \) from the leftmost point in the evader region at the origin of the search, the point \( (R_{last}, 0) \), for a time given by \( \hat{t} + t \).

We have that, \( -R_{last} - V_T (\hat{t} + t) = 6\pi V_T V_s - 2\pi V_T^2 \frac{2n}{n V_s (V_s - V_T)^2} \). Plugging in the value of \( t \) yields \( T_{linear} = \frac{2\pi V_T R_{last}}{V_s - V_T} \). Therefore given by,

\[
T_{linear} = \hat{t} + t = \frac{2\pi V_T R_{last}}{V_s - V_T} - 2\pi V_T^2 \frac{2n}{n V_s (V_s - V_T)^2} \tag{31}
\]

**Fig. 7.** Depiction of the end-game steps for the same-direction circular sweep performed by 2 sweepers. The sweepers sensors’ are shown in green and red areas indicate locations where potential evaders may still be located. In order to overcome the challenges in the circular search that were described, we propose that after scan number \( N_n \) the sweepers will travel to the right until cleaning the waveform that propagates from the right portion of the remaining evader region and then travel to the left until cleaning the remaining evader region. (a) - Evader region status and sweepers’ locations prior to the last inward advancement. (b) - Evader region status and sweepers’ locations prior to the linear sweep.

**Fig. 8.** shows a plot of \( T_{linear} \) as a function of \( n \). Therefore, the total scan time until a complete cleaning of the evader region is given by \( T_{total} = T_{circular} + T_{in} + T_{linear} \). For the one dimensional scan to be valid and ensure a non escape search and complete cleaning of the evader region must be satisfied. This demand implies that,

\[
\frac{2r - R_{last}}{V_T} > \frac{R_{last} (3V_s - V_T)}{(V_s - V_T)^2} \tag{32}
\]

By rearranging terms, \( -R_{last} - V_T (\hat{t} + t) = V_s t - V_T \hat{t} \) can be written as,

\[
2r(V_s - V_T)^2 > R_{last} V_s (V_s + V_T) \tag{33}
\]

By substitution of \( R_0 \) with \( \alpha r \) where \( \alpha > 1 \) and by substituting the terms for \( V_s \) and \( R_{last} \), (33) resolves to a quadratic
equation in $\Delta V$ that has only one positive root. This root is a monotonically decreasing function in $\alpha$, given by

$$\Delta V \geq \frac{-4\pi V_T \alpha + \pi V_T + V_T \sqrt{\pi^2 + 8\pi n}}{2n}$$

(34)

Therefore, for a given $\alpha$, the designer of the sweep process can infer which $\Delta V$ needs to be chosen in order to completely clean the evader region using the final linear sweeping motion.

For a complete derivation see Appendix B.

**Theorem 5.** For a valid circular search process the total search time until a complete cleaning of the evader region is given by, $T = T_{\text{circular}} + T_{\text{in}} + T_{\text{linear}}$, or as,

$$T = \frac{R_0}{V_T} + \frac{\tau (V_v - V_T) (n(V_v + V_T) + 2\pi V_T N_n)}{2\pi V_T^2 V_v} + \left(\frac{1}{n(V_v + V_T)}\right)^{N_n - 1} \frac{2\pi R_0 V_v - \tau n (V_v - V_T)}{2\pi V_T^2 V_v} + \left(\frac{1}{n(V_v + V_T)}\right)^{N_n - 1} \frac{V_v}{2\pi V_T} + \frac{1}{nV_v} \left(\frac{1}{nV_v} + \frac{1}{nV_T}\right) + \frac{2\pi}{nV_v}$$

(35)

Fig. 8. End-game search times until complete cleaning of the evader region for the same-direction circular sweep processes. We simulated sweep processes with an even number of agents, ranging from 2 to 32 agents, that employ the multi-agent same-direction circular sweep processes. We show results obtained for different values of velocities above the circular critical velocity. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$.

VIII. **SAME-DIRECTION SPIRAL SWEEP**

Since our aim is to provide a sweep process that improves the same-direction circular sweep process we would like the sweepers to employ a more efficient motion throughout the sweep process. Therefore, we wish that throughout the sweep process the sweepers must travel at angle $\phi$ normal to the evader region. $\phi$ is calculated in equation (9) in section VI. This method of traveling at angle $\phi$ preserves the evader region’s circular shape.

$\theta$ ($t_\theta$) describes the angle a sweeper travelled around the region and is calculated in section VI, equation (12). $t_\theta$ denotes the sweep time after a sweeper completes a traversal of angle $\theta$ around the center of the evader region. The time it takes a sweeper to travel an angle of $\theta$ around the center of the evader region, $t_\theta$ is obtained from (12).

Contrary to the pincer-based strategy where each sweeper travels only an angle of $\frac{2\pi}{n}$ at each sweep iteration, in same-direction sweeps, each sweeper travels a larger angle than $\frac{2\pi}{n}$ at each iteration around the evader region in order to detect all escaping smart evaders. The additional angle, denoted by $\beta$, needs to be traversed in order to detect all evaders that may have spread from the "most dangerous points" at the beginning of each sweep. Such points are adjacent to the starting locations of every sweeper. The angle $\beta$ depends on the radius of the circle that bounds the evader region. After a sweeper traverses the additional angle $\beta$, the evader region’s boundary is due to spread from points that resided at the lower tips of the sensors. When the tips of the sensors leave these points, evaders may spread from them in all directions at a velocity of $V_T$.

There exists a geometric relation between $\beta$, which is measured with respect to the center of the evader region, and $\varphi$. $\varphi$ is an angle that measures the angle between the center of the sweeper after it completes a sweep of $\frac{2\pi}{n}$ with respect to the center of the spread from the most dangerous points at each iteration. $\varphi$ is measured only for the additional sweeping after the sweeper completes a sweep of $\frac{2\pi}{n}$ and is shown in Fig. 10.
We have that the coordinates of the outer tip of the sweeper that starts to sweep in the counter-clockwise direction from point \( p_2 = (0, -R_0) \) after it completes a traversal of \( \frac{2\pi}{n} \) as a function of \( \phi \) are given by,

\[
x = -V_T t_\phi \sin \varphi_i, \quad y = \left( R_i + V_T t_\frac{2\pi}{n} - V_T t_\phi \right) \cos \varphi_i \tag{37}
\]

During the time that the sweeper executes the outward spiral for time \( t_\frac{2\pi}{n} \) and the inward spiral for time \( t_\phi \) the evader region spreads from the "most dangerous point" \( q_1 = (0, R_0 - 2r) \) at a velocity of \( V_T \) in all directions. The equation of the points \((x, y)\) on the wavefront that originates from \( q_1 \) are given by,

\[
x^2 + (y - R_0 + 2r)^2 = V_T^2 \left( t_\frac{2\pi}{n} + t_\phi \right)^2 \tag{38}
\]

The sweeper will travel in an inward spiral until the upper tip of its sensor satisfies \((38)\). \( \beta_0 \) is given by,

\[
\tan \beta_0 = \frac{x}{y} \tag{39}
\]

The subscript 0 in \( \beta_0 \) denotes the iteration or cycle number, indicating that the value of \( \beta \) changes as the sweep process progresses. The same reasoning and notations hold for \( \varphi \). Substituting the expressions for \( x \) and \( y \) from \((37)\) in \((39)\) yields,

\[
\tan \beta_i = \tan \beta_0 = \frac{-V_T t_\phi \sin \varphi}{(R_0 + V_T t_\frac{2\pi}{n} - V_T t_\phi) \cos \varphi} \tag{40}
\]

Therefore,

\[
\beta_i = \arctan \left( -\frac{V_T t_\phi \tan \varphi_i}{R_0 + V_T (t_\frac{2\pi}{n} - t_\varphi)} \right) \tag{41}
\]

When the sweeper completes a traversal of \( \frac{2\pi}{n} \), its center is located at a distance of \( R_0 - r + V_T t_\frac{2\pi}{n} \) from the center of the evader region. It then starts sweeping in an inward spiral for a time given by,

\[
t_\varphi = \frac{(R_0 - r + V_T t_\frac{2\pi}{n}) \left( 1 - e^{-\frac{2\pi V_T}{R_0 - r + V_T t_\frac{2\pi}{n}}} \right)}{V_T} \tag{42}
\]

The angle \( \varphi \) is calculated from the equation,

\[
\int_0^{t_\varphi} \varphi(\zeta) d\zeta = \int_0^{t_\varphi} \frac{\sqrt{V_s^2 - V_T^2}}{R_0 - r + V_T t_\frac{2\pi}{n} - V_T \zeta} d\zeta \tag{43}
\]

Hence \( \varphi \) is given by,

\[
\varphi_0(t_\varphi) = -\sqrt{V_s^2 - V_T^2} \ln \left( \frac{R_0 - r + V_T t_\frac{2\pi}{n} - V_T t_\varphi}{R_0 - r + V_T t_\frac{2\pi}{n}} \right) \tag{44}
\]

In order to solve for the critical velocity that satisfies the confinement task for the same-direction spiral sweep process, we need to determine both \( \varphi_0 \) and the critical velocity. The critical velocity and \( \varphi_0 \) are calculated from the demand that the expansion of the evader region after the first sweep at time \( t_\frac{2\pi}{n} + t_\varphi \), has to satisfy that,

\[
V_T \left( t_\frac{2\pi}{n} + t_\varphi \right) \leq 2r \tag{45}
\]
Substitution of terms yields,

\[
(R_0 - r) \left( 2 e^{n \sqrt{V_s^2 - V_T^2}} - e^{n \sqrt{V_s^2 - V_T^2}} - 1 \right) = \frac{t_{2n} + t_{\phi_i}}{V_T}
\]  

(46)

In order to solve for the critical velocity of the same direction spiral sweep process, denoted by \( V_{c,s,p,i} \), and \( \varphi_0 \) we write (44) as,

\[
H(V_s, \varphi_i) = (R_0 - r) \left( 2 e^{n \sqrt{V_s^2 - V_T^2}} - e^{n \sqrt{V_s^2 - V_T^2}} - 1 \right) - 2r
\]

(47)

The critical velocity and \( \varphi_0 \) can be computed numerically using the Newton method. Since \( \varphi_0 \leq \beta_0 \) we choose the initial estimate for \( \varphi_0 \) to be,

\[
\varphi_0 = \arcsin \left( \frac{2r}{R_0 - 2r} \right)
\]

(48)

We choose the initial estimate for the critical velocity as the lower bound on the sweeper velocity given by,

\[
V_s = \frac{\pi R_0 V_T}{r} = V_{LB}
\]

(49)

From (45), we find \( V_{c,s,p,i} \) and \( \varphi_0 \) using the Newton iterative method for the velocity whose equation is given by,

\[
V_{s_{n+1}} = V_{s_n} - H(V_{s_n}, \varphi_i) \frac{\partial H(V_{s_n}, \varphi_i)}{\partial V_{s_n}}
\]

(50)

Using,

\[
\frac{\partial H(V_s, \varphi_i)}{\partial V_s} = \frac{(R_0 - r) V_T V_s}{n(V_s^2 - V_T^2)^2} \left( -4 \pi e^{n \sqrt{V_s^2 - V_T^2}} + (2 \pi - n \varphi_i) e^{n \sqrt{V_s^2 - V_T^2}} \right)
\]

(51)

And by using the Newton iterative method for \( \varphi_0 \) whose equation is given by,

\[
\varphi_{i_{n+1}} = \varphi_{i_n} - H(V_{s_n}, \varphi_{i_n}) \frac{\partial H(V_{s_n}, \varphi_{i_n})}{\partial \varphi_{i_n}}
\]

(52)

Using,

\[
\frac{\partial H(V_s, \varphi_i)}{\partial \varphi_i} = \frac{V_T (R_0 - r) e^{n \sqrt{V_s^2 - V_T^2}}}{\sqrt{V_s^2 - V_T^2}}
\]

(53)

With an initial estimate for \( \varphi_0 \) given in (48), an iterative convergence to a solution for \( V_s \) satisfying (45) with equality, is achieved by substituting the solution for \( V_s \) in equation (50), solving for an updated \( \varphi_0 \) using (52), and repeating this process until converging to a solution with a desired tolerance. This alternating root finding approach yields solution for \( \varphi_0 \) and \( V_{c,s,p,i} \). After the sweepers traverse an angle of \( \frac{2\pi}{T} + \beta_0 \) around the evader region the evader region’s boundary is due to evaders that originated from points \( q_1 \) and \( q_2 \) (in case the sweep process is performed with 2 sweepers), and not from points \( p_1 \) and \( p_2 \) as can be seen in Fig. 10.

Let us denote by \( \Delta V > 0 \) the addition to the sweeper’s velocity above the critical velocity. The velocity is therefore given by,

\[
V_s = V_{c,s,p,i} + \Delta V.
\]

If a sweeper moves with a velocity greater than the critical velocity, after the completion of the inwards spiral sweep the evader region is bounded in a circle with a radius that is less than \( R_0 \). After the sweepers complete an inwards spiral sweep the new radius of the evader region is given by,

\[
R_{i+1} = R_i - V_T \left( t_{2n} + t_{\varphi_0} \right)
\]

(54)

Therefore after a set of outward and inward spirals as described, the sweepers sweep around an evader region that is bounded by a circle with a smaller radius. After each sweep \( \varphi_i \) is calculated using the Newton method with respect to the new radius of the circle that bounds the evader region, allowing for the calculation of \( t_{\varphi_i} \). This search methodology continues until the evader region is bounded by a circle with a radius that is less than or equal to \( 2r \).

**IX. SAME-DIRECTION SPIRAL SWEEP END-GAME**

**A. The end-game**

In order to entirely clean the evader region, the sweepers need to change the scanning method when the evader region is bounded by a circle of radius \( 2r \), due to the same consideration that are described in the end-game of the same-direction circular sweep process. The depiction of the scenario at the beginning of the end-game is shown in Fig. 11(a). In the last inward advancement, the sweepers place the tips of their
sensors at the center of the evader region. The time it takes the sweepers to complete this movement is given by,
\[
T_e = \frac{R_N}{V_s}
\] (55)

Following the last inward advancement, the sweepers perform the last spiral sweep when the tips of their sensors are placed at the center of the evader region as can be seen in Fig. 11(b). In order to apply a linear sweeping movement the last spiral sweep has to reduce the evader region to be bounded by a circle of radius less than \(2r\). Following the last inward advancement, the sweepers perform an additional spiral sweep when the center of each sweeper’s sensor is at a distance of \(r\) from the center of the evader region. The time it takes to complete this sweep is denoted by \(T_l\) and is given by,
\[
T_l = r \left( \frac{e^{\frac{2\pi V_T}{V_s r}} - 1}{V_T} \right)
\] (56)

The depiction after the sweepers complete this last spiral movement can be seen in Fig. 11(c). During the last spiral sweep, the evader region spreads from its center point to a circle with a radius of,
\[
R_{last} = T_l V_T = r \left( \frac{e^{\frac{2\pi V_T}{V_s r}} - 1}{V_T} \right)
\] (57)

In order for a linear scan to be applicable \(R_{last}\) has to be smaller than \(2r\). This leads to a requirement on the sweeper’s velocity,
\[
r \left( \frac{e^{\frac{2\pi V_T}{V_s r}} - 1}{V_T} \right) < 2r
\] (58)

Which resolves to,
\[
\frac{e^{\frac{2\pi V_T}{V_s r}} - 1}{V_T} < 2
\] (59)

Yielding the requirement on the velocity,
\[
V_s > V_T \sqrt{\frac{4\pi^2}{(n \ln 2)^2} + 1}
\] (60)

Since we previously observed that the spiral critical velocity (for the considered spiral process) is close to the lower bound on the critical velocity, \(V_{LB}\), we can check whether the condition in (58) is automatically satisfied. If the requirement on \(V_s\) in (60) is less than \(V_{LB}\), then since the sweepers move with a velocity above it, then (58) is always satisfied. Therefore if,
\[
V_T \sqrt{\frac{4\pi^2}{(n \ln 2)^2} + 1} < \frac{\pi R_0 V_T}{nr} = V_{LB}
\] (61)

Or if the ratio \(\frac{R_0}{r}\) satisfies that,
\[
\sqrt{\frac{4}{(n \ln 2)^2} + \frac{1}{\pi^2}} < \frac{R_0}{r}
\] (62)

Then \(R_{last}\) is smaller than \(2r\). Following this last spiral sweep, two sweepers place the tips of their sensors at the center of the evader region, advancing a distance of,
\[
R_{in} = 2r - R_{last}
\] (63)

The time it takes the sweepers to complete this motion is given by,
\[
T_f = \frac{2r - R_{last}}{V_s}
\] (64)

During this time the evaders spread to a circle of radius \(R_f\) around the center of the region given by,
\[
R_f = T_f V_T + R_{last}
\] (65)

The depiction of the scenario at this time instance is shown in Fig. 11(d). Following this movement, the sweepers perform a linear motion and complete the search process. During the previously mentioned movement the margin between the edge of the sensors in each direction to the evader region boundaries must satisfy,
\[
\frac{2r - R_f}{V_T} > T_{linear}
\] (66)

In order to guarantee no evader escapes undetected. \(T_{linear}\) denotes the time it takes the sweepers to clean the right section of the remaining evader region in addition to the time it takes it to scan from the rightmost point it got to until the leftmost point of the expansion. These times are respectively denoted as \(t\) and \(\tilde{t}\). Therefore, \(T_{linear}\) is given by \(T_{linear} = t + \tilde{t}\). The evader region’s rightmost point of expansion starts from the point \((R_f, 0)\) and spreads at a velocity of \(V_T\). Therefore, if the constraint in (66) is satisfied, we can view the rightward and leftward linear sweeps as a one dimensional scan. Therefore, the time \(t\) it takes the sweepers to clean the spread of potential evaders from the right section of the region can be calculated from, \(V_s t = R_f + V_T \tilde{t}\). Therefore, \(t\) is given by,
\[
t = \frac{R_f}{V_s - V_T}
\] (67)

\(\tilde{t}\) is computed by calculating the time \(t\) it takes to see the sweepers located at point \((tV_s, 0)\) to change their scanning direction and perform a leftward scan to a point that spread at a velocity of \(V_T\) from the leftmost point in the evader region at the origin of the search, the point \((-R_f, 0)\), for a time given by \(\tilde{t} + t\). We have that,
\[
-R_f - V_T (\tilde{t} + t) = tV_s - V_s \tilde{t}
\] (68)

Plugging the value of \(t\) yields,
\[
\tilde{t} = \frac{2V_s R_f}{(V_s - V_T)^2}
\] (69)

\(T_{linear}\) is therefore given by,
\[
T_{linear} = t + \tilde{t}
\] (70)

Therefore, the total scan time until a complete cleaning of the evader region is given by,
\[
T_{total} = T_{spiral} + T_{in} + T_e + T_t + T_f + T_{linear}
\] (71)

For the one dimensional scan to be valid and ensure a non escape search and complete cleaning of the evader region, (66) must be satisfied. This demand implies that,
\[
\frac{2r - R_f}{V_T} > \frac{R_f (3V_s - V_T)}{(V_s - V_T)^2}
\] (72)
By rearranging terms, \( \frac{72}{4} \) can be written as,
\[
2r(V_s - V_T)^2 > R_f V_s (V_s + V_T)
\]  
(73)

By substitution of \( R_0 \) with \( \alpha r \) where \( \alpha > 1 \) and by substituting the terms for \( V_s \) and \( R_f \), (73) resolves to a quadratic equation in \( \Delta V \) that has only one positive root. This root is a monotonically decreasing function in \( \alpha \), given by
\[
V_s^2 (2r - R_f) - V_s V_T (4r + R_f) + 2rV_T^2 > 0
\]  
(74)

The quadratic equation in (74) has 2 positive roots. Therefore, in order for the one dimensional linear scan to be valid \( V_s \) has to be greater than the largest positive root. Imposing that,
\[
V_s \geq \frac{2rV_T + V_T R_f}{2r - R_f}
\]  
(75)

Fig. 12. End-game search times until complete cleaning of the evader region for the same-direction spiral sweep processes. We simulated sweep processes with an even number of agents, ranging from 2 to 32 agents, that employ the multi-agent same-direction spiral sweep processes. We show results obtained for different values of velocities above the spiral critical velocity. The chosen values of the parameters are \( r = 10 \), \( V_T = 1 \) and \( R_0 = 100 \).

Fig. 13. Total search times until complete cleaning of the evader region for the same-direction spiral sweep processes. We simulated sweep processes with an even number of agents, ranging from 2 to 32 agents, that employ the multi-agent same-direction spiral sweep processes. We show results obtained for different values of velocities above the spiral critical velocity. The chosen values of the parameters are \( r = 10 \), \( V_T = 1 \) and \( R_0 = 100 \).

**X. COMPARATIVE ANALYSIS OF PINCER MOVEMENT SEARCH STRATEGIES AND SAME-DIRECTION SWEEPS**

The purpose of this section is to compare between the obtained results for the circular and spiral same-direction sweep processes that were developed in the previous sections and compare them to pincer sweep processes developed in [1]. In order to make a fair comparison between the total sweep times of sweeper swarms that can perform both the circular and spiral sweep processes the number of sweepers and sweeper velocity must be the same in each of the tested spiral and circular swarms. The critical velocity that is required for sweepers that perform the same-direction circular or spiral sweep processes is higher than the minimal critical velocity of their pincer sweep counterparts. Additionally, since pincer sweep processes require a smaller critical velocity compared to same-direction sweep processes, in order to make a fair comparison all sweepers in the swarm move at velocities above the critical velocity of 2 sweepers that perform the same-direction circular sweep.

In Fig. 14 we show the ratio between the time it takes swarms that employ same-direction circular sweeps to the time it takes swarms that perform their circular pincer sweep processes counterparts to detect all evaders in the region. We see that for all choices of velocities above the same-direction circular critical velocity, the ratio is greater than 1, implying that same-direction circular sweeps require more time in order to clean the entire evader region. The sweep times are obtained for different values of velocities above the same-direction circular critical velocity of 2 sweepers, since this velocity is greater than the critical velocity of search processes performed with more sweepers.

In Fig. 15 we show the ratio between the time it takes swarms that employ same-direction spiral sweeps to the time it takes swarms that perform their spiral pincer sweep processes counterparts to detect all evaders in the region. We see that when 4 or more sweepers perform the search process, for all choices of velocities above the same-direction circular critical velocity, the ratio is greater than 1, implying that same-direction spiral sweeps require more time in order to clean the entire evader region. The only cases where same-direction spiral sweeps are better than their pincer based counterparts are when sweepers move at velocities that are close to the critical velocity and where the sweep process is performed with more than 2 sweeping agents. We observe that as the number of sweepers increases, the gain in utilizing the cooperation between the sweeping pairs in the pincer-based sweep processes decreases the sweeping time dramatically. When the sweepers perform the same-direction spiral sweeps they have to sweep large sections at each iteration in order to ensure no evader escapes, while in pincer-based spiral search strategies, sweeping these additional sections in unnecessary to the complementary and symmetric trajectories of the sweepers. The sweep times in Fig 15 are obtained for different values of velocities above the same-direction spiral critical velocity of 2 sweepers, since this velocity is greater than the critical velocity of search processes performed with more sweepers. This means that the values of \( \Delta V \) that are mentioned in Fig.
14 correspond to sweeper velocities that are almost twice the same-direction spiral critical velocities presented in Fig. 15. Requiring a higher critical velocity implies that there are entire regions of operation where an evader region with a given radius could be cleaned by a sweeper swarm that performs the same-direction spiral sweep process but cannot be cleaned by a sweeper swarm that performs the same-direction circular sweep process. This also implies that swarms that perform pincer movement search strategies can sweep larger regions than their same-direction sweeps counterparts.

Fig. 14. Ratio between total search times until complete cleaning of the evader region for circular same-direction and pincer sweep processes. We simulated sweep processes with an even number of agents, ranging from 2 to 32 agents, that employ the multi-agent same-direction and pincer circular sweep processes. We show results obtained for different values of velocities above the same-direction circular critical velocity. The chosen values of the parameters are \( r = 10 \), \( V_T = 1 \) and \( R_0 = 100 \).

Fig. 15. Ratio between total search times until complete cleaning of the evader region for spiral same-direction and pincer sweep processes. We simulated sweep processes with an even number of agents, ranging from 2 to 32 agents, that employ the multi-agent same-direction and pincer spiral sweep processes. We show results obtained for different values of velocities above the same-direction spiral critical velocity. The chosen values of the parameters are \( r = 10 \), \( V_T = 1 \) and \( R_0 = 100 \).

XI. CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

This research considers a scenario in which a multi-agent swarm of agents performs a search of an area containing smart mobile evaders that are to be detected. There can be many evaders in this area, and potential evaders can be located at any point in an initial circular region of radius \( R_0 \). The sweepers objective is to “clean” or to detect all evaders that can move freely in all directions from their initial locations. The search time depends on the type of sweeping movement the sweeper swarm employs. This work compares same-direction sweeps and pincer movement sweep protocols demonstrating and proving the superiority of the latter. We perform a quantitative comparison between pincer based sweep protocols and same-direction sweep protocols for any number of even sweepers where the sensing capabilities and velocities of the swarms are equivalent. We prove that critical velocities for pincer based search methods are lower than their same-direction counterparts and therefore allow to sweep successfully larger regions. Afterwards, we provide a comparison between the different search methods in terms of completion times of the sweep processes.

APPENDIX A

The inward advancement time at iteration \( i \) is denoted by \( T_{in,i} \). It is given by,

\[
T_{in,i} = \frac{\delta_{eff}(\Delta V)}{V_s} = \frac{rn(V_s - V_T) - 2\pi r_i V_T}{nV_s(V_s + V_T)}
\]  

(76)

The total advancement time until the evader region is bounded by a circle of with a radius that is less than or equal to \( r \) is denoted as \( \bar{T}_{in} \). It is given by,

\[
\bar{T}_{in} = \sum_{i=0}^{N-2} T_{in,i} = \frac{(N_n - 1) rn(V_s - V_T)}{nV_s(V_s + V_T)} - \frac{2\pi V_T N_n}{nV_s(V_s + V_T)}
\]  

(77)

We have that,

\[
R_{N_n-2} = \frac{c_1}{1 - c_2} + c_2 N_n - 2 \left( \frac{R_0 - c_1}{1 - c_2} \right)
\]  

(78)

The sum of the radii is given by,

\[
\sum_{i=0}^{N_n-2} R_i = \frac{R_0 - c_2 R_{N_n-2} + (N_n - 2)c_1}{1 - c_2}
\]  

(79)

Where the coefficients \( c_1 \) and \( c_2 \) are given by,

\[
c_1 = \frac{r(V_s - V_T)}{V_s + V_T}, \quad c_2 = 1 + \frac{2\pi V_T}{n(V_s + V_T)}
\]  

(80)

Substitution of terms for the expression of \( R_{N_n-2} \) in \( 78 \) yields,

\[
R_{N_n-2} = \frac{r(V_s - V_T)}{2\pi V_T} + \left( 1 + \frac{2\pi V_T}{V_s + V_T} \right)^{N_n-2} \left( \frac{2\pi R_0 V_T - r(V_s - V_T)}{2\pi V_T} \right)
\]  

(81)

Substitution of terms in \( 79 \) yields,

\[
\sum_{i=0}^{N_n-2} R_i = -\frac{R_0 n(V_s + V_T)}{2\pi V_T} + \frac{rn^2(V_s - V_T)(V_s + V_T)}{(2\pi V_T)^2} + \frac{r n^2(V_s - V_T)}{2\pi V_T} + \left( 1 + \frac{2\pi V_T}{n(V_s + V_T)} \right) N_n - 1 \left( \frac{2\pi R_0 V_T - r n(V_s - V_T)}{2\pi V_T} \right)
\]  

(82)
We therefore obtain that,
\[
\tilde{T}_{in} = \frac{R_0}{V_s} - \frac{r n(V_s - V_T)}{2\pi V_T n - 1} N_{n-1}^{-1} \left( \frac{2\pi R_0 V_T - r n(V_s - V_T)}{2\pi V_T V_s} \right)
\]  
(83)
The last inward advancement is given by,
\[
T_{in,last} = \frac{R_{N_n}}{V_s}
\]  
(84)
We have that,
\[
R_{N_n} = \frac{c_1}{1 - c_2} + c_2 N_n \left( R_0 - \frac{c_1}{1 - c_2} \right)
\]  
(85)
Therefore,
\[
R_{N_n} = \frac{r n(V_s - V_T)}{2\pi V_T} + \left( 1 + \frac{2\pi V_T}{n(V_s + V_T)} \right) N_n \left( \frac{2\pi R_0 V_T - r n(V_s - V_T)}{2\pi V_T V_s} \right)
\]  
(86)
Substitution of terms yields,
\[
T_{in,last} = R_{N_n} + \left( 1 + \frac{2\pi V_T}{n(V_s + V_T)} \right) N_n \left( \frac{2\pi R_0 V_T - r n(V_s - V_T)}{2\pi V_T V_s} \right)
\]  
(87)
The total inward advancement times is therefore given by,
\[
T_{in} = \frac{R_0}{V_s} + \left( 1 + \frac{2\pi V_T}{V_s + V_T} \right)^{N-1} \left( \frac{2\pi R_0 V_T - r (V_s - V_T)}{V_s (V_s + V_T)} \right)
\]  
(88)
The time it takes the sweepers to perform the circular sweeps before the evader region is bounded by a circle with a radius that is smaller or equal to \( r \) is given by,
\[
\tilde{T}_{circular} = T_0 - c_2 T_{N_n - 1} + (N_n - 1) c_3
\]  
(89)
Where the coefficient \( c_3 \) is given by,
\[
c_3 = -\frac{2\pi r (V_s - V_T)}{n V_s (V_s + V_T)}
\]  
(90)
The time it takes the sweepers to perform the first sweep is given by,
\[
T_0 = \frac{2\pi R_0}{n V_s}
\]  
(91)
The time it takes the sweepers to perform the last circular sweep is given by,
\[
T_{N_n - 1} = r \left( V_s - V_T \right) + \left( 1 + \frac{2\pi V_T}{n(V_s + V_T)} \right)^{N_n - 1} \left( \frac{2\pi R_0 V_T}{n V_s V_T} - r n(V_s - V_T) \right)
\]  
(92)
Therefore \( \tilde{T}_{circular} \) is given by,
\[
\tilde{T}_{circular} = R_{N_n} \left( V_s + V_T \right) + \frac{r V_s (V_s - V_T) n (V_s + V_T) + 2\pi V_T N_n}{2\pi V_T V_s} + \left( 1 + \frac{2\pi V_T}{n(V_s + V_T)} \right)^N \left( V_s + V_T \right) \left( \frac{2\pi R_0 V_T - r n(V_s - V_T)}{2\pi V_T V_s} \right)
\]  
(93)
The last circular sweep occurs when the lowest tips of the sweepers’ sensors are located at the center of the evader region. It is given by,
\[
T_{last} = \frac{2\pi r}{n V_s}
\]  
(94)
Therefore the total circular traversal times are given by,
\[
T_{circular} = -\frac{R_0(V_s + V_T)}{V_s V_T} + \frac{r (V_s - V_T) n (V_s + V_T) + 2\pi V_T N_n}{2\pi V_T V_s} + \left( 1 + \frac{2\pi V_T}{n(V_s + V_T)} \right)^N \left( V_s + V_T \right) \left( \frac{2\pi R_0 V_T - r n(V_s - V_T)}{2\pi V_T V_s} \right) + \frac{2\pi r}{n V_s}
\]  
(95)
\[\text{APPENDIX B}\]

We have that,
\[
2r(V_s - V_T)^2 > R_{last} V_s (V_s + V_T)
\]  
(96)
And,
\[
V_s = \frac{2\pi R_0 V_T}{r n} + V_T + \Delta V
\]  
(97)
Denoting \( \alpha = \frac{R_0}{r} \) and substituting the following terms in yields,
\[
2r \left( \frac{2\pi V_T}{n} + \Delta V \right)^2 > \frac{2\pi r V_T}{n} \left( \frac{2\pi \alpha V_T}{n} + 2V_T + \Delta V \right)
\]  
(98)
Rearranging terms yields a quadratic equation in \( \Delta V \),
\[
\Delta V^2 + \Delta V \left( \frac{4\pi V_T \alpha - \pi V_T}{n} \right) + \frac{4\pi^2 V_T \alpha^2 - 2\pi \alpha V_T - 2\pi n V_T^2}{n^2} > 0
\]  
(99)
The equation (99) has a positive and a negative root. Since \( \Delta V \) is non-negative we are interested only in the positive root. Therefore, in order to completely clean the evader region \( \Delta V \) has to satisfy
\[
\Delta V \geq \frac{-4\pi V_T \alpha + \pi V_T + V_T \sqrt{\pi^2 + 8\pi n}}{2n}
\]  
(100)
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