Irreducibility of Moduli Spaces of Vector Bundles on Birationally Ruled Surfaces

Charles H. Walter∗
Laboratoire de Mathématiques
Université de Nice
F-06108 Nice Cedex 02 FRANCE

Abstract

Let $S$ be a birationally ruled surface. We show that the moduli schemes $M_{S}(r, c_1, c_2)$ of semistable sheaves on $S$ of rank $r$ and Chern classes $c_1$ and $c_2$ are irreducible for all $(r, c_1, c_2)$ provided the polarization of $S$ used satisfies a simple numerical condition. This is accomplished by proving that the stacks of prioritary sheaves on $S$ of fixed rank and Chern classes are smooth and irreducible.

One important recent result in the theory of vector bundles on algebraic surfaces is the theorem of Gieseker and Li that for any smooth projective surface $S$ and any ample divisor $H$ on $S$, the moduli scheme $M_{S,H}(2, c_1, c_2)$ of $H$-semistable torsion-free sheaves of rank 2, determinant $c_1 \in \text{Pic}(S)$, and second Chern class $c_2$ is irreducible if $c_2 \gg 0$. If $S$ is a surface of general type, the condition $c_2 \gg 0$ is necessary because of an example of Gieseker with small $c_2$ where the moduli space is reducible. In contrast it has been known for quite some time that the moduli schemes $M_{\mathbb{P}_2,H}(r, c_1, c_2)$ is irreducible for all $(r, c_1, c_2)$ for which there exist semistable sheaves on the projective plane, and the same result is also known for $\mathbb{P}_1 \times \mathbb{P}_1$.

In this paper we extend this strong irreducibility result from $\mathbb{P}_2$ and $\mathbb{P}_1 \times \mathbb{P}_1$ to all smooth projective surfaces of negative Kodaira dimension. To simplify our exposition, we will omit $\mathbb{P}_2$ although it can be handled by the same method. (Indeed our method is based on a method of Ellingsrud and Strømme which was developed for $\mathbb{P}_2$.) So our surface $S$ possesses a morphism $\pi: S \rightarrow C$ onto a smooth curve with connected fibers and with general fiber isomorphic to $\mathbb{P}_1$. We fix such a $\pi$. (Such a $\pi$ is unique if $g(S) = g(C) > 0$ or if $S = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(e))$ with $e > 0$, but there can be many, even infinitely many, possible $\pi$ for certain rational surfaces.) For $p \in C$, let $f_p = \pi^{-1}(p)$. These $f_p$ are all numerically equivalent, and we write $f \in \text{NS}(X)$ for the numerical class of these $f_p$. We prove

Theorem 1. Let $\pi: S \rightarrow C$ be a birationally ruled surface and $f \in \text{NS}(S)$ the numerical class of a fiber of $\pi$. Let $H$ be an ample divisor on $S$ such that $H \cdot (K_S + f) < 0$. Suppose $r \geq 2$, $c_1 \in \text{NS}(S)$, and $c_2 \in \mathbb{Z}$ are given. If the moduli scheme $M_{S,H}(r, c_1, c_2)$ of $S$-equivalence

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classes of $H$-semistable torsion-free sheaves of rank $r$ and Chern classes $c_1$ and $c_2$ is non-empty, then it is irreducible and normal. In addition, the open subscheme $M_{S,H}^c(r, c_1, c_2)$ parametrizing stable sheaves is smooth.

Our methods also show that the general $H$-semistable torsion-free sheaf in any of the $M_{S,H}(r, c_1, c_2)$ is locally free and that there is a dominant, generically finite map from an open subscheme of $\text{Jac}(C) \times \text{Jac}(C) \times \mathbb{P}^m$ to $M_{S,H}(r, c_1, c_2)$ with $m = 2rc_2 - (r - 1)c_1 + (r^2 - 2)g - r^2 + 1$ where $g$ is the genus of $C$. So if $S$ is a rational surface, then $M_{S,H}^c(r, c_1, c_2)$ is unirational.

Ample divisors $H$ satisfying the hypothesis $H \cdot (K_S + f) < 0$ exist on any birationally ruled surface because of Lemma 5 below. On certain surfaces there may exist a divisor $D$ of degree 1 on $C$ such that the divisor class $-K_S - \pi^*(D)$ is effective. In that case all ample divisors $H$ satisfy $H \cdot (K_S + f) < 0$, and the theorem holds for all possible polarizations. Examples of such surfaces include Del Pezzo surfaces, rational ruled surfaces, and ruled surfaces of the form $\mathbb{P}(O_C \oplus O_C(D_0))$ with $D_0$ a divisor of degree at least $2g - 1$ on the smooth projective curve $C$ of genus $g$.

Our method of proof begins by adapting a definition from [11]. If $\pi : S \rightarrow C$ is a birationally ruled surface, then we will say that a coherent sheaf $\mathcal{E}$ on $S$ is prioritary (with respect to $\pi$) if it is torsion-free and if $\text{Ext}^2(\mathcal{E}, \mathcal{E}(-f_p)) = 0$ for all $p \in C$. By the semicontinuity theorem the prioritary sheaves in any locally noetherian flat family of coherent sheaves on $S$ form an open subfamily. Hence the prioritary sheaves on $S$ are parametrized by an open substack of the stack of coherent sheaves on $S$.

For a given $r \geq 1$, $c_1 \in \text{NS}(S)$, and $c_2 \in \mathbb{Z}$, we will write $\text{Coh}_S(r, c_1, c_2)$ for the stack of coherent sheaves of rank $r$ and Chern classes $c_1$ and $c_2$ (modulo numerical equivalence), and $\text{TF}_S(r, c_1, c_2)$ and $\text{Prior}_S(r, c_1, c_2)$ for the open substacks of, respectively, torsion-free and prioritary sheaves. We will derive Theorem 4 from:

**Proposition 2.** Let $\pi : S \rightarrow C$ be a birationally ruled surface. Suppose $r \geq 2$, $c_1 \in \text{NS}(S)$, and $c_2 \in \mathbb{Z}$ are given. Then the stack $\text{Prior}_S(r, c_1, c_2)$ of prioritary sheaves on $S$ of rank $r$ and Chern classes $c_1$ and $c_2$ is smooth and irreducible.

The Proposition is proven in two steps. First we prove it for geometrically ruled surfaces. Then we prove that for $S \rightarrow S_1$ the blowup of a point, if the proposition holds for $S_1$ then it holds for $S$. Our method is based on the version of the method of Ellingsrud and Strømme ([2], [53], [59]) as presented in [5] §9.

The reader unfamiliar with algebraic stacks may wish to consult [LM]. We use stacks because in that context there exist natural universal families of coherent (or torsion-free or prioritary) sheaves. Alternative universal families which stay within the category of schemes would be certain standard open subschemes of $\text{Quot}$ schemes. But these depend on the choice of a polarization $\mathcal{O}_S(1)$, of the Hilbert polynomial $P$, of twists $m \gg 0$ and of a vector space $H_m^r$ of dimension $P(m)$. One then deals with the scheme $\text{Quot}_S^0(\mathcal{O}_S(1))(P, m)$ parametrizing all quotients $\gamma : H_m \otimes \mathcal{O}_S(-m) \rightarrow \mathcal{F}$ such that $H^i(\mathcal{F}(m)) = 0$ for all $i \geq 1$ and such that the induced map $H_m \rightarrow H^0(\mathcal{F}(m))$ is an isomorphism. One would prefer not to work in a context where one constantly has to refer to all these choices, particularly since no single set of choices will work when one deals with unlimited families. But strictly speaking, these $\text{Quot}$
schemes are not that far from our point of view because the verification in [LM] (4.14.2) that the coherent sheaves on $S$ are parametrized by an algebraic stack $\text{Coh}_S$ is done essentially by gluing together all the $\text{Quot}_{S \times \text{Spec}(P),m}^0(\mathcal{O}_S(1))$ in the smooth Grothendieck topology.

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### 1 Proof of the Theorem

We begin with two lemmas about coherent sheaves on $\mathbb{P}^1$ and the restriction of torsion-free sheaves on surfaces to curves in the surface. These lemmas are well known although they have usually been stated in terms of complete families or of versal deformation spaces instead of stacks. We state them without proof.

**Lemma 3.** Let $r \geq 2$ and $0 \leq d < r$ be integers. Let $\text{Coh}_{\mathbb{P}^1}(r, -d)$ be the stack of coherent sheaves of rank $r$ and degree $-d$ on $\mathbb{P}^1$.

(i) If $d > 0$, then sheaves not isomorphic to $\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ form a closed substack of $\text{Coh}_{\mathbb{P}^1}(r, -d)$ of codimension at least 2.

(ii) If $d = 0$, then sheaves not isomorphic to $\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ form a closed substack of $\text{Coh}_{\mathbb{P}^1}(r, 0)$ of codimension 1. Sheaves isomorphic neither to $\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ nor to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ form a closed substack of $\text{Coh}_{\mathbb{P}^1}(r, 0)$ of codimension at least 2.

**Lemma 4.** Let $D$ be an effective Cartier divisor on a projective surface $S$. If $E$ is a torsion-free sheaf on $S$ such that $\text{Ext}^2(E, E(D)) = 0$, then the restriction map $\text{TF}_S(r, c_1, c_2) \to \text{Coh}_D(r, c_1 \cdot D)$ is smooth (and therefore open) in a neighborhood of $[E]$.

We also need two lemmas for reduction steps in the proof of Proposition 2.

**Lemma 5.** Let $\pi: S \to C$ be a geometrically ruled surface with a section $\sigma \subset S$. If $E$ is a coherent sheaf on $S$ such that $\pi_*(E(-\sigma)) = R^1\pi_*(E) = 0$, then there is an exact sequence

$$0 \to \pi^*(\pi_*(E)) \to E \to \pi^*(R^1\pi_*(E(-\sigma))) \otimes \Omega_{S/C}(\sigma) \to 0.$$ 

**Proof.** This is a special case of a relative version of Beilinson’s spectral sequence, but for lack of a precise reference we give the proof in full. Let $Y := S \times_C S$. Then the diagonal $\Delta$ of $Y$ has Beilinson’s resolution (cf. [3])

$$0 \to \Omega_{S/C}(\sigma) \otimes \mathcal{O}_S(-\sigma) \to \mathcal{O}_Y \to \mathcal{O}_\Delta \to 0.$$ 

Applying $R^1\pi_*(- \otimes \pi^*(E))$ to this exact sequence gives a long exact sequence which is equivalent to the one asserted by the lemma because one always has $R^i\pi_*(\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes \pi^*(R^i\pi_*(\mathcal{G}))$ if $\mathcal{F}$ is locally free and $\mathcal{G}$ coherent on $S$ because of the projection formula and [H], Chapter III, Proposition 9.3. □
Lemma 6. Let $S_1$ be a smooth surface $\alpha: S \to S_1$ the blowup of a point $x$ of $S_1$. Let $E$ be the exceptional divisor in $S$. Suppose that $\mathcal{E}$ is a coherent sheaf of rank $r$ on $S$ such that $\mathcal{E}|_E \cong \mathcal{O}_E^d \oplus \mathcal{O}_E(-1)^d$ for some $d$. Then $\alpha_*(\mathcal{E})$ is locally free in a neighborhood of $x$, and there are exact sequences

$$0 \to \alpha^*(\alpha_*(\mathcal{E})) \to \mathcal{E} \to \mathcal{O}_E(-1)^d \to 0,$$

$$0 \to \mathcal{E}(\mathcal{E}) \to \alpha^*(\alpha_*(\mathcal{E})) \to \mathcal{O}_E^d \to 0.$$

Moreover, for any divisor $D$ on $S_1$ we have $\text{Ext}^2(\mathcal{E}, \mathcal{E}(\alpha^*(D))) \cong \text{Ext}^2(\alpha_*(\mathcal{E}), \alpha_*(\mathcal{E})(D))$.

Proof. Let $\mathcal{F}$ be the kernel of the composition $\mathcal{E} \to \mathcal{E}|_E \to \mathcal{O}_E(-1)^d$. By general properties of elementary transforms the exact sequence $0 \to \mathcal{O}_E^d \to \mathcal{E}|_E \to \mathcal{O}_E(-1)^d \to 0$ transforms into $0 \to \mathcal{O}_E^d \to \mathcal{F}|_E \to \mathcal{O}_E^r \to 0$. So $\mathcal{F}$ is trivial along $E$, and $\mathcal{F} \cong \alpha^*(\alpha_*(\mathcal{F}))$. Applying $\alpha_*$ to the exact sequence $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}_E(-1)^d \to 0$, we see that $\alpha_*(\mathcal{F}) \cong \alpha_*(\mathcal{E})$. Hence $\mathcal{F} \cong \alpha^*(\alpha_*(\mathcal{E}))$. The exact sequences asserted by the lemma are now the standard exact sequences of an elementary transform.

By adjunction and the formula $K_S = \alpha^*(K_{S_1}) + E$ we see that

$$\text{Hom}(\alpha_*(\mathcal{E}), \alpha_*(\mathcal{E})(-D + K_{S_1})) \cong \text{Hom}(\mathcal{F}, \mathcal{E}(-\alpha^*(D) + K_S - E))$$

or by Serre duality that $\text{Ext}^2(\alpha_*(\mathcal{E}), \alpha_*(\mathcal{E})(D)) \cong \text{Ext}^2(\mathcal{E}, \mathcal{F}(\alpha^*(D) + E))$. If we now apply the functor $\text{Ext}^2(\mathcal{E}(\alpha^*(D)), -)$ to the exact sequence $0 \to \mathcal{E} \to \mathcal{F}(\mathcal{E}) \to \mathcal{O}_E(-1)^r \to 0$ and note that

$$\text{Ext}^i(\mathcal{E}(\alpha^*(D)), \mathcal{O}_E(-1)) \cong H^i(E, (\mathcal{E}|_E)^\vee(-1)) = 0$$

for $i = 1, 2$, then we see that $\text{Ext}^2(\mathcal{E}, \mathcal{F}(\alpha^*(D) + E)) \cong \text{Ext}^2(\mathcal{E}, \mathcal{E}(\alpha^*(D)))$. This completes the proof of the lemma. \hfill $\square$

We now begin the proof of Proposition 2. The smoothness of $\text{Prior}_S(r, c_1, c_2)$ follows from $\text{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$ because this is the obstruction space for deformations of $\mathcal{E}$. So we concentrate on irreducibility. We begin with the special case of geometrically ruled surfaces.

Proof of Proposition 2 when $\pi: S \to C$ is a geometrically ruled surface. We follow the method of Ellingsrud and Stromme as presented in [L] §9.

We fix a section $\sigma \subset S$. Replacing $\mathcal{E}$ by an appropriate twist $\mathcal{E}(n\sigma)$ if necessary we may assume that $d := -c_1 \cdot f$ satisfies $0 \leq d < r$. The proof now divides briefly into two cases $d > 0$ and $d = 0$.

If $d > 0$, then by Lemmas 3 and 4, those $\mathcal{E}$ such that $\mathcal{E}|_{f_p} \not\cong \mathcal{O}_p^{r-d} \oplus \mathcal{O}_p(-1)^d$ for some $p \in C$ are parametrized by a closed substack of $\text{Prior}_S(r, c_1, c_2)$ of codimension at least 1. So we may restrict ourselves to the dense open substack $\text{Prior}_0^0$ where $\mathcal{E}|_{f_p} \cong \mathcal{O}_p^{r-d} \oplus \mathcal{O}_p(-1)^d$ for all $p \in C$.

If $d = 0$, then by an analogous argument, we may restrict ourselves to a dense open substack $\text{Prior}_0^0$ where $\mathcal{E}|_{f_p} \cong \mathcal{O}_p^r$ for all $p \in C$ except for a finite number of $p$ where $\mathcal{E}|_{f_p} \cong \mathcal{O}_p^1 \oplus \mathcal{O}_p^{-2} \oplus \mathcal{O}_p(-1)$.

In either case, we set $K := \pi_*^{\mathcal{E}}$. Since $R^1\pi_*(\mathcal{E}) = 0$, the Leray-Serre spectral sequence implies that $\chi(\mathcal{E}) = \chi(K)$. We may now calculate by Riemann-Roch that $K$ is a vector bundle on $C$ of rank $r - d$ and degree $k := \chi(\mathcal{E}) + (r - d)(g - 1)$ where $g$ is the genus of $C$. 


Let $\mathcal{L} = R^1\pi_*(\mathcal{E}(-\sigma))$. Then $\mathcal{L}$ is a sheaf on $C$ of rank $d$ and degree $l := -\chi(\mathcal{E}(-\sigma)) + d(g-1) = -\chi(\mathcal{E}) + (c_1 \cdot \sigma) - (r-d)(g-1)$. The sheaf $\mathcal{L}$ is locally free if $d > 0$.

By Lemma \ref{lem:existence}, there is an exact sequence

$$0 \to \pi^*(\mathcal{K}) \to \mathcal{E} \to \pi^*(\mathcal{L}) \otimes \Omega_{S/C}(\sigma) \to 0.$$ 

Now using the notations $\text{Ext}_+$ and $\text{Ext}_-$ of [DL], p. 200, we have

$$\text{Ext}_i^+(\mathcal{E}, \mathcal{E}) = H^i(\pi^*(\mathcal{K}) \otimes \mathcal{L}) \otimes \Omega_{S/C}(\sigma) = 0$$

for all $i$. Hence $\text{Ext}_i(\mathcal{E}, \mathcal{E}) \cong \text{Ext}_i^+(\mathcal{E}, \mathcal{E})$ for all $i$. Thus the infinitesimal deformations of $\mathcal{E}$ are the same as the infinitesimal deformations of the filtered sheaf $0 \subset \pi^*(\mathcal{K}) \subset \mathcal{E}$. Furthermore, since $\text{Ext}^2(\pi^*(\mathcal{L}) \otimes \Omega_{S/C}(\sigma), \pi^*(\mathcal{K})) = 0$, we have a surjection

$$\text{Ext}_1^+(\mathcal{E}, \mathcal{E}) \to \text{Ext}_1^+(\mathcal{K}, \mathcal{K}) \oplus \text{Ext}_1^+(\mathcal{L}, \mathcal{L}) \to 0.$$ 

Hence a general infinitesimal deformation of $\mathcal{E}$ induces general infinitesimal deformations of $\mathcal{K}$ and $\mathcal{L}$. Since none of these deformations are obstructed, the morphism $\text{Prior}^0 \to \text{Coh}_C(r-d, k) \times \text{Coh}_C(d, l)$ defined by $[\mathcal{E}] \mapsto ([\mathcal{K}], [\mathcal{L}])$ is dominant. The fibers of this morphism are irreducible since they are stack quotients of an open subscheme of the affine space $\text{Ext}_1^+(\pi^*(\mathcal{L}) \otimes \Omega_{S/C}(\sigma), \pi^*(\mathcal{K}))$. The target of the morphism is irreducible since stacks of coherent sheaves of a fixed rank and degree on a smooth connected curve $C$ are irreducible.

So $\text{Prior}^0$ and hence also $\text{Prior}_S(r, c_1, c_2)$ are irreducible. This completes the proof of Proposition \ref{prop:irreducibility} when $\pi: S \to C$ is a geometrically ruled surface. \hfill $\square$

**Proof of Proposition \ref{prop:irreducibility} in general.** We go by induction on the Picard number $\rho(S) := \text{rk}_\mathbb{Z}(\text{NS}(S))$. The initial value is $\rho(S) = 2$ which is the case of geometrically ruled surfaces which we just proved. So we may assume that $\rho(S) \geq 3$. Then $\pi: S \to C$ is birationally but not geometrically ruled. So some fiber of $\pi$ contains an irreducible component $E \cong \mathbb{P}^1$ such that $E^2 = -1$. We let $\alpha: S \to S_1$ be the contraction of $E$, and we let $\beta: S_1 \to C$ be the morphism such that $\pi$ factors as $\pi = \beta \circ \alpha$. Since $\rho(S_1) = \rho(S) - 1$, we may assume by induction that Proposition \ref{prop:irreducibility} holds for $\beta: S_1 \to C$.

Let $d = -c_1 \cdot E$. By replacing $\mathcal{E}$ with an appropriate twist $\mathcal{E}(nE)$ we may assume that $0 \leq d < r$. Because $f_{\pi(E)} - E$ is effective, the condition $\text{Ext}^2(\mathcal{E}, \mathcal{E}(-f_{\pi(E)})) = 0$ implies $\text{Ext}^2(\mathcal{E}, \mathcal{E}(-E)) = 0$. So by Lemmas \ref{lem:existence} and \ref{lem:dimension} the substack $\text{Prior}^1 \subset \text{Prior}_S(r, c_1, c_2)$ parametrizing priority sheaves $\mathcal{E}$ such that $\mathcal{E}|_E \cong \mathcal{O}_E^{-d} \oplus \mathcal{O}_E(-1)^d$ is open and dense. By Lemma \ref{lem:dimension}, the application $[\mathcal{E}] \mapsto [\alpha_*(\mathcal{E})]$ defines a morphism $\text{Prior}^1 \to \text{Prior}_S(r, c_1, c_2)$ parametrizing priority sheaves which are locally free at the center of the blowup. Since $\text{Prior}_S(r, c_1, c_2)$ is irreducible by the inductive hypothesis, we see that $\text{Prior}^1$ and hence also $\text{Prior}_S(r, c_1, c_2)$ are irreducible. This completes the proof of Proposition \ref{prop:irreducibility}. \hfill $\square$

**Proof of Theorem \ref{thm:irreducibility}**. First we show that if $H \cdot (K_S + f) < 0$, then any $H$-semistable sheaf $\mathcal{E}$ is priority. But if $\mathcal{E}$ is $H$-semistable, then any nonzero torsion-free quotient $Q$ of $\mathcal{E}$ would have $H$-slope satisfying $\mu_H(Q) \geq \mu_H(\mathcal{E})$, while any nonzero subsheaf $S$ of $\mathcal{E}$ would have
contradicting Proposition 2. So if there exist
Ext by an open substack $H$ classes, then
the contraction of an ample divisor
generated by global sections. Let
tients $\gamma$ such that $\text{Ext}^2(H, \mathcal{E}(m)) = 0$ for $i \geq 1$ and have $\mathcal{E}(m)$
generated by global sections. Let $H_m$ be a vector space of dimension $h^0(\mathcal{E}(m))$ and let $Q^{ss} := \text{Quot}_{ss}(H_m; r, c_1, c_2)$ denote the Hilbert-Grothendieck scheme parametrizing all quotients $\gamma: H_m \otimes \mathcal{O}_S(-m) \to F$ such that $F$ is $H$-semistable of rank $r$ and Chern classes $c_1$ and $c_2$ with $H^1(F(m)) = 0$ for all $i \geq 1$ and such that the induced map $H_m \to H^0(F(m))$ is an isomorphism. Then according to the construction of [H] (4.14.2), $H$–SS may be identified
with the quotient stack $[Q^{ss}/\text{GL}(H_m)]$. Hence $Q^{ss}$ is a smooth and irreducible scheme. But $M_{S,H}(r, c_1, c_2)$ is the GIT quotient scheme $Q^{ss}/(\text{GL}(H_m), \mathcal{L}_N)$ where $\mathcal{L}_N$ is Simpson’s polarization of $Q^{ss}$ defined by $\mathcal{L}_N := \det(pr_1(U \otimes pr_2^*(\mathcal{O}_S(N))))$, where $U$ is the universal sheaf on $Q^{ss} \times S$, the $pr_i$ are the two projections, and $N \gg m$. Thus $M_{S,H}(r, c_1, c_2)$ is the GIT quotient of a smooth and irreducible variety. But such quotients, when nonempty, are always normal and irreducible varieties, and the points of the quotient corresponding to stable points are smooth. 

2 Existence of Ample Divisors

We show that on any birationally ruled surface there exists an ample divisor satisfying the hypothesis of Theorem 4.

Lemma 7. Let $S$ be a birationally ruled surface, $\pi: S \to C$ a birational ruling, and $f \in \text{NS}(X)$ the class of a fiber of $\pi$. Then there exists an ample divisor $H$ on $S$ such that $H \cdot (K_S + f) < 0$.

Proof. First we consider the case of $S$ a geometrically ruled surface. Let $\sigma$ be a section of minimal self-intersection $-e$. According to [H] Chapter V, Corollary 2.11, Propositions 2.20 and 2.21 and Exercise 2.14, we see that $K_S \equiv -2\sigma + (2g - 2 - e)f$ and that an $H = \sigma + bf$ is ample if $b$ is sufficiently large. Since $H \cdot (K_S + f) = 2g - 1 + e - 2b$, by picking $b$ large enough we get an ample $H$ such that $H \cdot (K_S + f) < 0$.

Now we consider the case where $\pi: S \to C$ is birationally but not geometrically ruled. Then some fiber of $\pi$ contains an exceptional divisor of the first kind $E$. Let $\alpha: S \to S_1$ be the contraction of $E$. By induction on the Picard number we may assume there exists an ample divisor $H_1$ on $S_1$ such that $H_1 \cdot (K_{S_1} + f) < 0$. From [H] Chapter V, Proposition 3.3
and Exercise 3.3, we see that $K_S = \alpha^*(K_{S_1}) + E$ and that $H := 2\alpha^*(H_1) - E$ is ample. Then $H \cdot (K_S + f) = 2(H_1 \cdot (K_{S_1} + f)) - E^2 < 0$.

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