THE REAL FREED-HOPKINS-TELEMAN THEOREM

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ABSTRACT. Let $G$ be a compact, connected, and simply-connected Lie group viewed as a $G$-space via the conjugation action. The Freed-Hopkins-Teleman Theorem (FHT) asserts a canonical link between the equivariant twisted $K$-homology of $G$ and its Verlinde algebra. In this paper we give a generalization of FHT in the presence of a Real structure of $G$. Along the way we develop preliminary materials necessary for this generalization, which are of independent interest in their own right. These include the definitions of Real Dixmier-Douady bundles, the Real third cohomology group which is shown to classify the former, and Real Spin$^c$ structures.

2010 Mathematics Subject Classification: 19L50; 55N91

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Date: March 2, 2015.
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References

1. Introduction

Let $G$ be a compact connected Lie group. The Freed-Hopkins-Teleman Theorem (FHT) is a recent deep result which asserts a canonical isomorphism between the twisted equivariant $K$-homology of $G$ with its Verlinde algebra, the representation group of positive energy representations of the loop group $LG$ equipped with the intricately defined ring structure called fusion product (cf. [Fr], [FHT1], [FHT2], [FHT3], Theorem 5.1). Verlinde algebra is an object of great interest in mathematical physics and algebraic geometry. One of the remarkable aspects of Freed-Hopkins-Teleman Theorem is that it provides an algebro-topological approach to interpreting the fusion product, which is usually defined using conformal blocks or moduli spaces of $G$-bundles on Riemann surfaces (cf. [Be], [BL] and [V]). Moreover, Freed-Hopkins-Teleman also provides the framework for a formulation of geometric quantization of q-Hamiltonian spaces (cf. [M2] and [M3]).

Motivated by the index theory of real elliptic operators, Atiyah introduced $KR$-theory in [At]. It is a version of topological $K$-theory for the category of Real spaces, i.e. topological spaces with involutions (see Appendix for definition and basic properties), and used by Atiyah to derive the 8-periodicity of $KO$-theory from the 2-periodicity of complex $K$-theory. $KR$-theory can be regarded as a unifying thread of $KO$-theory, complex $K$-theory and $KSC$-theory (cf. [At]). In recent years there is a rekindled interest in $KR$-theory; in particular it has found applications in string theory, as it classifies the D-brane charges in orientifold string theory (cf. [DMR]). Seeing the possible applications in string theory as well as moduli spaces of nonorientable surfaces, we find it is of interest to obtain a generalization of FHT in the context of $KR$-theory. For simplicity, we will from now on assume in addition that $G$ is simple and simply-connected.

Our main result shows that, by incorporating a group anti-involution of $G$ (i.e. composition of group inversion and an automorphic involution), the corresponding equivariant twisted $KR$-homology of $G$ is essentially a module over the equivariant $KR$-homology coefficient ring, generated by the irreducible positive energy representations of real, complex and quaternionic types (cf. Corollary 5.5). Moreover, the ring structure of the equivariant twisted $KR$-theory induced by the Pontryagin product, when restricted to those generators of positive energy representations, is precisely the fusion product. This is the content of Theorem 5.6. In short, the group anti-involution as the additional Real structure in the equivariant twisted $KR$-homology respects the algebra structure of the Verlinde algebra.

The organization of this paper is as follows. In Section 3 we review equivariant $K$-homology and incorporate such additional structures as involutions and twists (i.e. local coefficients), which are necessary for the generalization of FHT in the Real context. We define Real Dixmier-Douady bundles which are used to realize the twists of $KR$-theory (homology) and the Real third equivariant cohomology group which is shown to classify the twists. We also develop the notion of Real Spin$^c$ structures, which are the $KR$-theoretic orientation, and which we utilize to conclude that $KH$-theory (Quaternionic $K$-theory, see Definition A.5) can be viewed as twisted $KR$-theory.
Section 4 concerns the computation of the group of the equivariant Real Dixmier-Douady classes of $G$ equipped with an anti-involution, and the construction of an equivariant Real fundamental Dixmier-Douady bundle (cf. Definition 4.3) following the idea in [M1]. We find that equipping $G$ with an anti-involution rather than an automorphic involution enables it to support an equivariant Real fundamental Dixmier-Douady bundle, making it the appropriate candidate for formulating the Real FHT. In Section 5 we put together the preliminary material in previous sections and prove the main results Corollary 5.5 and Theorem 5.6. The Appendix reviews the definitions and basic properties of $KR$-theory.

Acknowledgment. The author would like to thank Reyer Sjamaar for suggesting the problem of generalizing FHT in the Real context and his patient guidance and support in the course of writing this paper.

2. Notations, definitions and conventions

Throughout this paper, we let $G$ be a compact, simple and simply-connected Lie group. We use $\Gamma = \{1, \gamma\}$ to denote the group of order 2, and $\sigma_X$ to denote the involutive homeomorphism induced by $\gamma$ on the Real space $X$. If $X$ is a group $G$, $\sigma_G$ always means an involutive automorphism, while $a_G$ means the corresponding anti-involution $\sigma_G \circ \text{inv}$, where $\text{inv}$ means group inversion. We sometimes suppress the notations for involution if there is no danger of confusion about the involutive homeomorphism. For example, we simply use $G$ with the understanding that it is equipped with $\sigma_G$, while we use $G^-$ to mean the group $G$ equipped with $a_G$. By abuse of notation, we denote by $G^-$ the group $G$ viewed as a $G \times \Gamma$-space, where the first factor acts by conjugation while the second acts by the anti-involution $a_G$. Following the convention in [At], we use the word ‘Real’ (with a capital R) in all contexts involving involutions, so as to avoid confusion with the word ‘real’ with the usual meaning. For example, ‘Real $K$-theory’ is used interchangeably with $KR$-theory, whereas ‘real $K$-theory’ means $KO$-theory. Moreover, we do not discriminate between the meanings of the terms ‘Real structure’ and ‘involution’.

We let $T$ be a fixed choice of maximal torus, and $W$ the corresponding Weyl group. We let $\Lambda \in t$ be the coroot lattice and $\Lambda^* \subset t^*$ the weight lattice. We fix a choice of simple roots $\{\alpha_0, \cdots, \alpha_l\}$ (we adopt the convention that $\alpha_0 = -\alpha_{\text{max}}$, where $\alpha_{\text{max}}$ is the highest root) and the positive Weyl chamber $t^* \subset \Lambda^*$. We let $B$ be the basic inner product on $g^*$, which is the bi-invariant inner product such that $B(\alpha_{\text{max}}, \alpha_{\text{max}}) = 2$. $B$ is used to identify $t$ and $t^*$. The dual Coxeter number, $h^\vee$, of $G$ is then defined to be $1 + B(\rho, \alpha_{\text{max}})$, where $\rho$ is the half sum of positive roots. We let $\Delta^k$ be the level $k$ closed Weyl alcove of $t^*$, defined by the inequalities

$$\lambda(\alpha_i^\vee) + k\delta_{i,0} \geq 0 \text{ for } i = 0, \cdots, l$$

with vertices labelled by $\{0, \cdots, l\}$, so that the origin is labelled 0. We use $\Delta$ to denote the ordinary closed Weyl alcove $\Delta^1$. Let $\Lambda^*_k$ be $\Delta^k \cap \Lambda^*$, the level $k$ weights.

By abuse of notation, we also use $\Delta$ to denote $B^* (\Delta) \subset t$. Let $I \subseteq \{0, \cdots, l\}$. Let $\Delta_I$ be the closed subsimplex of $\Delta$ spanned by vertices with labels from the index set $I$. Let $W_I$ be the subgroup of $W$ fixing $\Delta_I$. We also let $G_I$ be the stabilizer subgroup of $\Delta_I$ and $\Lambda_I$, the coroot lattice of $G_I$. 


If we view $R(G)$ as the ring of characters of $G$, then the level $k$ Verlinde ideal $I_k$ can be defined as the vanishing ideal of

$$\left\{ \exp_T \left( B^2 \left( \frac{\lambda + \rho}{k + h^\vee} \right) \right) \mid \lambda \in \Lambda_k^\ast \right\}$$

and the level $k$ Verlinde algebra can be alternatively defined as $R_k(G) := R(G)/I_k$ (cf. [Be]).

3. Twisted $KR$-homology

This Section is devoted to background material on $K$-homology, its Real and twisted version, the classification of (Real) twists (which are realized by Dixmier-Douady bundles in this paper), and the notion of Real Spin$^c$ structures. We refer the reader to Appendix or [At], [AS] and [F] for the definitions and basic properties of $KR$-theory.

3.1. (Real) $K$-homology. $K$-homology is a homology theory dual to $K$-theory through the $K$-theory version of Poincaré duality, where a manifold is oriented in $K$-theory if it has a Spin$^c$ structure. Inspired by the Atiyah-Singer index theorem, which he used to realize the $K$-theoretic Poincaré duality, Kasparov gave the first definition of $K$-homology (cf. [Kas]). In this Section and the next, we follow Kasparov’s definition, give a quick review of $K$-homology and introduce its Real and twisted version. Most of the materials in this Section are directly culled from [BHS], [M1] and [HR], to the latter of which we refer the reader for more details on the subject.

**Definition 3.1.** Let $A$ be a separable $\mathbb{Z}_2$-graded $G-C^*$-algebra. A (zero-graded) Fredholm module over $A$ is a triple $(\rho, \mathcal{H}, F)$ where

1. $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is a separable $\mathbb{Z}_2$-graded $G$-Hilbert space,
2. $\rho : A \to B(\mathcal{H})$ is a representation of $A$ by bounded linear operators on $\mathcal{H}$ which preserves the grading and $G$-equivariant, and
3. $F$ is an odd graded $G$-equivariant operator on $\mathcal{H}$ such that
   $$(F^2 - 1)\rho(a) \sim 0, (F - F^*)\rho(a) \sim 0, [F, \rho(a)] \sim 0$$
   for all $a \in A$, where $\sim$ means equality modulo compact operators.

A $q$-graded Fredholm module is one equipped with odd-graded skew adjoint operators $\varepsilon_1, \cdots, \varepsilon_q$ on $\mathcal{H}$, called the grading operators, such that

$$\varepsilon_i^2 = -\text{Id}, \quad \varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i \quad \text{for } i \neq j$$

and that they commute with $F$ and $\rho(a)$ for all $a \in A$.

**Definition 3.2.** (1) We say the two Fredholm modules $(\rho, \mathcal{H}, F)$ and $(\rho', \mathcal{H}', F')$ are unitarily equivalent if there is a degree zero unitary isomorphism $U : \mathcal{H}' \to \mathcal{H}$ such that

$$(\rho', \mathcal{H}', F') = (U^* \rho U, \mathcal{H}', U^* FU)$$

Two $q$-graded Fredholm modules are unitarily equivalent if there exists a unitary isomorphism as above which in addition intertwines with the grading operators $\varepsilon_i$ and $\varepsilon_i'$. 
(2) Two Fredholm modules \((\rho, H, F_0)\) and \((\rho, H, F_1)\) are operator homotopic if there exists a norm continuous function \(t \mapsto F_t\) for \(t \in [0, 1]\).

**Definition 3.3** (Kasparov's \(K\)-homology). The equivariant \(K\)-homology group of a \(\mathbb{Z}_2\)-graded \(G - C^*\)-algebra \(A\), \(K^0_G(A)\), is the abelian group with one generator \([x]\) for each unitary equivalent class of Fredholm modules over \(A\) with the following relations

1. if \(x_0\) and \(x_1\) are operator homotopic then \([x_0] = [x_1]\) in \(K^0_G(A)\), and
2. \([x_0 \oplus x_1] = [x_0] + [x_1]\)

We define \(K^{-q}_G(A)\) similarly using \(q\)-graded Fredholm modules. If \(A\) is an ungraded \(G - C^*\)-algebra, its \(K\)-homology is defined to be the one for \(A \oplus A\) with the obvious \(\mathbb{Z}_2\)-grading.

**Remark 3.4.** One may, by regarding the grading operators as the canonical generators of the (complex) Clifford algebra, equivalently define \(K^{-q}_G(A)\) to be \(K^0_G(A \otimes \text{Cl}_q)\).

Like \(K\)-theory, \(K\)-homology is also 2-periodic (cf. [HR Proposition 8.2.13]). The use of superscript to denote the grading of \(K\)-homology of \(G - C^*\)-algebra is due to the fact that the \(K\)-homology functor on the category of \(G - C^*\)-algebras is contravariant. For if \((\rho, \mathcal{H}, F)\) is a Fredholm module over \(A\), and \(\alpha : A' \to A\) is a \(G - C^*\)-algebra homomorphism, then \((\rho \circ \alpha, \mathcal{H}, F)\) is a Fredholm module over \(A'\).

**Definition 3.5.** The equivariant \(K\)-homology of a locally compact \(G\)-space \(X\) is defined to be the equivariant \(K\)-homology of the \(G - C^*\)-algebra of space of continuous functions on \(X\) vanishing at infinity.

\[ K^G_q(X) := K^{-q}_G(C_0(X)) \]

The \(K\)-homology functor on the category of topological spaces is covariant, as opposed to the contravariance of its counterpart on the category of \(G - C^*\)-algebras, hence the use of subscript to denote the grading.

**Example 3.6.** Let \(X = \text{pt}\) equipped with the trivial \(G\)-action. Then \(C(X)\) is isomorphic to \(\mathbb{R}\) with trivial \(G\)-action. There is a unique representation \(\rho\) of \(\mathbb{R}\) by bounded linear operators on the \(G\)-Hilbert space \(\mathcal{H}\), namely the scalar multiplication. It follows that the odd-graded operator \(F\) in any Fredholm module over \(C(X)\) is a \(G\)-Fredholm operator on \(\mathcal{H}\) by definition. The \(K\)-homology class of any given Fredholm module is then determined by the \(G\)-Fredholm index of \(F\), which is a (finite-dimensional) representation of \(G\). Hence \(K^G_0(\text{pt}) \cong R(G)\). It can also be shown easily that \(K^G_1(\text{pt}) = 0\).

**Definition 3.7.** Recall that \(\text{Spin}^c(n) = \text{Spin}(n) \times_{\mathbb{Z}_2} U(1)\), where \(\mathbb{Z}_2\) acts on the central circle \(U(1)\) by multiplication by \(-1\). Let \(V\) be a rank \(n\) orientable Euclidean \(G\)-vector bundle over \(M\). It is \(G\)-\(\text{Spin}^c\) if there exists, over \(M\), an equivariant \(G\)-principal \(\text{Spin}^c(n)\)-bundle \(\tilde{P}\) whose structure group lifts that of the oriented frame bundle \(P\) \(G\)-equivariantly. Equivalently, \(V\) is \(G\)-\(\text{Spin}^c\) if \(\bigwedge^* V \otimes_{\mathbb{R}} \mathbb{C}\) is of the form \(\tilde{P} \times_{\text{Spin}^c(n)} \mathbb{C}_n\).

**Example 3.8.** Let \(M\) be an \(n\)-dimensional oriented Riemannian \(G\)-manifold. It is \(G\)-\(\text{Spin}^c\) if \(TM\) is. Let \(S := \bigwedge^* TM \otimes_{\mathbb{R}} \mathbb{C} = \tilde{P} \times_{\text{Spin}^c(n)} \mathbb{C}_n\) be the spinor bundle, which comes equipped with the Clifford multiplication of \(\mathbb{C}(TM)\) on the left and the Clifford multiplication of \(\mathbb{C}_n\) on the right. The fundamental class \([M] \in K^G_0(M)\) corresponding to a given \(G\)-\(\text{Spin}^c\) structure of \(M\) is given by the \(n\)-graded Fredholm module \((\rho, \mathcal{H}, F)\), where
(1) $\mathcal{H} = L^2(M, S)$, graded by the even and odd degree parts of $S$, and the $n$ grading operators are right multiplication by the $n$ generators by $\mathbb{C}l_n$,

(2) $\rho$ is the representation of $C_0(M)$ on $\mathcal{H}$ by left multiplication, and

(3) $F$ is the Dirac operator $D$ suitably normalized so that it becomes a bounded operator.

The Poincaré duality pairing between $K$-homology and $K$-theory is realized by the index pairing, which produces the Fredholm index of a certain operator on a Hilbert space constructed out of the Fredholm module and the vector bundle representing the relevant $K$-homology and $K$-theory classes respectively. For instance, for an oriented $G$-Spin$^c$ Riemannian manifold $M$ of dimension $n$, the index pairing of $[M]$ and the $K$-class $[E] \in K^0_G(M)$ of a $G$-vector bundle $E$ gives the (analytic) $G$-index of the twisted Dirac operator $D_E$ on $S \otimes E$ (cf. [HR, Lemma 11.4.1]). This pairing with the fundamental class is tantamount to the wrong way map in $K$-theory induced by the map collapsing $M$ to a point. The $K$-theoretic Poincaré duality asserts that

$$K^*_G(M) \to K^{*+n}_G(M)$$

$$[E] \mapsto [(\rho, L^2(M, S \otimes E), D_E)]$$

is an isomorphism.

As one would expect, the equivariant $KR$-homology can be defined by adding natural Real structures throughout the definition of equivariant $K$-homology (cf. [HR, Appendix B] for a brief discussion on the set-up of $KR$-homology). For the convenience of the reader, we shall spell out the definitions of $KR$-homology.

**Definition 3.9.** $A$ is a separable $\mathbb{Z}_2$-graded Real $G$–$C^*$-algebra if it is a graded $G$–$C^*$-algebra with an anti-linear, zero graded involution $\sigma_A$ such that

(1)

$$\sigma_A(g \cdot a) = \sigma_G(g) \cdot \sigma_A(a)$$

where $\sigma_G$ is a group involution on $G$. A Real Fredholm module over $A$ is one where $\mathcal{H}$ is equipped with an anti-linear zero-graded involution $\sigma_\mathcal{H}$ satisfying a compatibility relation with the $G$-action similar to Equation (1), $\rho$ intertwines with $\sigma_A$ and $\sigma_\mathcal{H}$, and $F$ commutes with $\sigma_\mathcal{H}$. A $(p, q)$-graded Real Fredholm module satisfies an additional condition that the $p + q$ grading operators commute with $\sigma_\mathcal{H}$ and satisfy

$$\varepsilon_i^2 = \begin{cases} -\text{Id} & \text{if } 1 \leq i \leq q \\ \text{Id} & \text{if } q + 1 \leq i \leq p + q \end{cases}$$

and $\varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i$ for $i \neq j$.

We simply call a $(0, q)$-graded Real Fredholm module $q$-graded Real Fredholm module. The notions of unitary equivalence and operator homotopy of Real Fredholm modules, and hence equivariant $KR$-homology $KR^{p,q}_G(A)$ (resp. $KR^{q}_G(A)$), can be defined straightforwardly by requiring that $\sigma_\mathcal{H}$ respect various structures in Definition 3.3 and using $(p, q)$-graded Real Fredholm modules (resp. $q$-graded Real Fredholm modules). Define, for a Real locally compact $G$-space $X$, its equivariant $KR$-homology $KR^{p,q}_G(X)$ (resp. $KR^{q}_G(X)$) to be $KR^{p,q}_G(C_0(X))$ (resp. $KR^{q}_G(C_0(X))$).

We shall point out notable changes brought by the addition of Real structures. First, the $KR$-homology is $(1, 1)$-periodic if double grading is used, and 8-periodic if single grading.
is used. Repeated application of \((1,1)\)-periodicity yields \(KR_G^{p,q}(\mathbb{A}) \cong KR_G^{p-q}(\mathbb{A})\). Second, in defining the Poincaré duality pairing between \(KR\)-homology and \(KR\)-theory, parallel to that between \(K\)-homology and \(K\)-theory, we shall need the notion of Real Spin\(^c\) structure as the \(KR\)-theoretic orientation. As we are unable to find a precise definition of Real Spin\(^c\) structure in the literature, we shall hereby give one.

Let \(V \to X\) be a \(\Gamma\)-equivariant, orientable real vector bundle of rank \(n\) (caveat: this should be distinguished from Real vector bundles). Equip each fiber with an inner product such that \(\sigma_V\) is orthogonal. Let \(P\) be the oriented orthonormal frame bundle of \(V\). \(P\) has an \(SO(n)\)-action defined by

\[
g \cdot (v_1, \cdots, v_{2n}) = \left( \sum_{j=1}^{2n} g_1 j v_j, \cdots, \sum_{j=1}^{2n} g_{2n} j v_j \right)
\]

making it a principal \(SO(n)\)-bundle. We make the structure group \(SO(n)\) a Real Lie group by assigning the involutive automorphism \(g \mapsto (I_q - I_p) g (I_q - I_p)\) where \(p + q = n\). We also assign \(Spin^c(n)\) with the involutive automorphism which descends to the one on \(SO(n)\) defined above and restricts to complex conjugation on the central circle. We call \(P\) a Real principal \(SO(n)\)-bundle of type \((p, q)\) if

1. the Real structure on it is defined by
   \[
   \sigma_P(v_1, v_2, \cdots, v_n) = (\sigma_V(v_1), \sigma_V(v_2), \cdots, \sigma_V(v_q), -\sigma_V(v_{q+1}), \cdots, -\sigma_V(v_n))
   \]
2. \(P\) is preserved by \(\sigma_P\).

Note that another way of saying condition (2) is that the (unoriented) frame bundle of \(V\) with Real structure given by condition (1) is the trivial double cover of \(P\) in the Real sense (a trivial double cover is trivial in the Real sense if the involution preserves the two connected components). Recall that an ordinary real vector bundle is orientable if its (unoriented) frame bundle is the trivial double cover of the oriented frame bundle. This prompts the following

**Definition 3.10.** Suppose \(V\) is orientable. \(V\) is Real \((p, q)\)-orientable if the oriented frame bundle \(P\) is preserved by \(\sigma_P\) defined above.

\(V\) is Real \((p, q)\)-orientable if and only if either \(\sigma_V\) is orientation-preserving and \(p\) is even, or \(\sigma_V\) is orientation-reversing and \(p\) is odd.

**Example 3.11.** Consider the real vector bundle \(\mathbb{R}^{r,s} \to \text{pt}\). It is Real \((p, q)\)-orientable if and only if \(q\) and \(s\) are of the same parity or, equivalently, \(p - q - (r - s)\) is divisible by 4.

**Definition 3.12.** Let \(V\) be a \(\Gamma\)-equivariant, orientable real vector bundle over a Real space \(X\). Suppose further that \(V\) is Real \((p, q)\)-orientable. We say \(V\) is Real \((p, q)\)-\(Spin^c\) if there exists, over \(X\), a Real principal \(Spin^c(n)\)-bundle \(\tilde{P}\) of type \((p, q)\) whose structure group lifts that of \(P\) as the Real \(U(1)\)-central extension as defined above. Equivalently, \(V\) is Real

\[1\]We cannot resist to point out that we thought of using the phrase 'Really trivial double cover' but quickly realized that it was really a bad pun.
$(p, q)$-$\text{Spin}^c$ if \( \bigwedge^* V \otimes_\mathbb{R} \mathbb{C} \) is of the form of the Real spinor bundle \( S = \tilde{P} \times_{\text{Spin}^c(n)} \mathbb{C}l_{p,q} \), where \( \mathbb{C}l_{p,q} \) is the complex Clifford algebra \( \mathbb{C}l_n \) equipped with an antilinear algebra involution \( \sigma \) with \( \sigma(\varepsilon_i) = \varepsilon_i \) for \( 1 \leq i \leq q \) and \( \sigma(\varepsilon_j) = -\varepsilon_j \) for \( q + 1 \leq j \leq n \). The equivariant version of \( \text{Real} \ (p,q) \)-Spin\(^c \) structure can be defined by incorporating the \( G \)-action compatibly throughout the above definition.

With Definition 3.12, we can define, similar to Example 3.8, the fundamental class \([M] \in KR_{G,T}^c(M)\) of the \( \text{Real} \ (p,q)-\text{Spin}^c \) Riemannian \( G \)-manifold \( M \), and formulate the Poincaré duality for \( KR \)-theory:

\[
KR_{G,T}^c(M) \xrightarrow{\cong} KR_{G,T-r,p-s}^c(M)
\]

There is one technical point to note, however. When defining the Real Fredholm module \((p, L^2(M, S), D)\) of type \((p,q)\) representing \([M]\), the grading operators are given by the right multiplication by \( \varepsilon_1, \cdots, \varepsilon_q, i\varepsilon_{q+1}, \cdots, i\varepsilon_n \in \mathbb{C}l_{p,q} \) on \( S \).

**Example 3.13.** Consider again the \( \Gamma \)-equivariant real vector bundle \( \mathbb{R}^{r,s} \rightarrow pt \), which we assume is \( \text{Real} \ (p,q)\)-orientable. The Real principal \( SO(n) \)-bundle \( P \) of type \((p,q)\) is isomorphic, through the map \((v_1, \cdots, v_n) \mapsto \left( \begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} \right)\), to \( SO(n) \rightarrow pt \) with involution being (up to a rearrangement of the orthonormal vectors \( v_1, \cdots, v_n \))

\[
g \mapsto \left( \begin{array}{cc} -I_{|p-r|} & I_{n-|p-r|} \\ \end{array} \right) g
\]

Though obviously \( \tilde{P} = (\text{Spin}^c(n) = \text{Spin}(n) \times_{\mathbb{Z}_2} U(1) \rightarrow pt \) is the principal \( \text{Spin}^c(n) \)-bundle which lifts the structure group of \( P \), making \( \mathbb{R}^n \rightarrow pt \) a \( \text{Spin}^c \) vector bundle, it is not true that one can always equip \( \tilde{P} \) with a compatible Real structure which descends to the one on \( P \) so that \( \mathbb{R}^{r,s} \) is \( \text{Real} \ \text{Spin}^c \). We would like to determine when it is \( \text{Real} \ (p,q)-\text{Spin}^c \). Note that \( \text{Real} \ (p,q)\)-orientability of \( \mathbb{R}^{r,s} \) implies that \( p - r \) is even. Let \( |p - r| = 2k \). The two elements in \( \text{Spin}(n) \) that lift \( \left( \begin{array}{cc} -I_{2k} \\ I_{n-2k} \end{array} \right) \) are \( \pm \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2k} \). So \( \mathbb{R}^{r,s} \) is \( \text{Real} \ (p,q)-\text{Spin}^c \) if and only if the bundle automorphism on \( \tilde{P} \) given by \( [(g, z)] \mapsto [(\pm \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2k} g, z)] \) is an involution, which is the case exactly if and only if \( k \) is even. If \( k \) is odd, the above bundle automorphism only acts as a ‘quarter turn’ and may be deem a ‘Quaternionic structure’, as we will see in Section 3.3.

**Proposition 3.14.** Let \( p + q = r + s = n \).

1. \( \mathbb{R}^{r,s} \rightarrow pt \) is not \( \text{Real} \ (p,q)\)-orientable if and only if \( \frac{p-q-(r-s)}{2} \) is odd.
2. \( \mathbb{R}^{r,s} \rightarrow pt \) is \( \text{Real} \ (p,q)\)-orientable but not \( \text{Real} \ (p,q)-\text{Spin}^c \) if and only if \( \frac{p-q-(r-s)}{4} \) is odd.
3. \( \mathbb{R}^{r,s} \rightarrow pt \) is \( \text{Real} \ (p,q)-\text{Spin}^c \) if and only if \( p - q - (r - s) \) is divisible by 8.

### 3.2. Equivariant Real Dixmier-Douady bundles and their classification.

The study of local coefficient systems for \( K \)-theory was pioneered in [DK]. Since then various models for the local coefficient systems of \( K \)-theory and \( K \)-homology have been proposed, namely,
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Definition 3.15. Let $X$ be a locally compact Real $G$-space. A $G$-equivariant Real DD bundle $\mathcal{A}$ is a $G$-equivariant, locally trivial $\mathcal{K}(\mathcal{H})$-bundle with structure group $PU(\mathcal{H})$, equipped with an involution $\sigma_\mathcal{A}$ such that

1. $(\mathcal{A}, \sigma_\mathcal{A})$ is a Real $G$-space,
2. $\sigma_\mathcal{A}$ descends to $\sigma_X$ on $X$, and
3. $\sigma_\mathcal{A}$ maps fiber to fiber anti-linearly.

Definition 3.16. (1) An equivariant Real DD bundle $\mathcal{A}$ is Morita trivial if there exists an equivariant Real Hilbert space bundle $\mathcal{E}$ such that $\mathcal{A}$ is isomorphic to $\mathcal{K}(\mathcal{E})$. We say $\mathcal{E}$ Morita trivializes $\mathcal{A}$ if $\mathcal{A} \cong \mathcal{K}(\mathcal{E})$.

(2) The opposite DD bundle of $\mathcal{A}$, denoted by $\mathcal{A}^{opp}$, is the $\mathcal{K}(\mathcal{H})$-bundle with the same underlying space as that of $\mathcal{A}$ except that it is modeled on the opposite Hilbert space $\mathcal{H}^{opp}$ with the conjugate complex structure.

(3) The (completed) tensor product of two equivariant Real DD bundles $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the equivariant Real DD bundle modeled on $\mathcal{K}(\mathcal{H}_1) \otimes \mathcal{K}(\mathcal{H}_2) \cong \mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

(4) Two equivariant Real DD bundles $\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita isomorphic if $\mathcal{A}_1 \otimes \mathcal{A}_2^{opp}$ is isomorphic to a Morita trivial equivariant Real DD bundle. We say $\mathcal{E}$ witnesses the Morita isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$ if $\mathcal{E}$ Morita trivializes $\mathcal{A}_1 \otimes \mathcal{A}_2^{opp}$.

(5) Suppose $(X_i, \mathcal{A}_i), i = 1, 2$ are two equivariant Real DD bundles modeled on $\mathcal{K}(\mathcal{H}_i)$. A Morita morphism

$$f, \mathcal{E} : (X_1, \mathcal{A}_1) \rightarrow (X_2, \mathcal{A}_2)$$

consists of an equivariant Real proper map $f : X_1 \rightarrow X_2$ and an equivariant Real Hilbert space bundle $\mathcal{E}$ on $X_1$ which is a $(f^* \mathcal{A}_2, \mathcal{A}_1)$-bimodule, locally modeled on the $(\mathcal{K}(\mathcal{H}_2), \mathcal{K}(\mathcal{H}_1))$-bimodule $\mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, the space of compact operators mapping from $\mathcal{H}_1$ to $\mathcal{H}_2$.

If $(f, \mathcal{E}_i), i = 1, 2$, are two Morita morphisms from $(X_1, \mathcal{A}_1)$ to $(X_2, \mathcal{A}_2)$, then the $(f^* \mathcal{A}_2, \mathcal{A}_1)$-bimodules $\mathcal{E}_1$ and $\mathcal{E}_2$ differ by an equivariant Real line bundle $L$. More precisely,

$$\mathcal{E}_2 = \mathcal{E}_1 \otimes L, \text{ with } L = \text{Hom}_{f^* \mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}_1, \mathcal{E}_2)$$

Definition 3.17. We say the two Morita morphisms $(f, \mathcal{E}_i), i = 1, 2$, are 2-isomorphic if the equivariant Real line bundle $L$ is trivial.

It follows that, if there is a Morita morphism between $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ covering the equivariant Real map $f : X_1 \rightarrow X_2$, the set of 2-isomorphism classes of Morita morphisms covering $f$ is a principal homogeneous space acted upon by the equivariant Real Picard group of $X_1$. 

The category of Real DD bundles over a Real space is therefore endowed with a group structure where multiplication is given by the tensor product, and group inversion the opposite bundle construction. It is well-known that, analogous to complex line bundles being classified by the second integral cohomology group up to isomorphism, DD bundles are classified, up to Morita isomorphism, by the third integral cohomology group (cf. [DiDo]). In what follows we will prove an analogous result for equivariant Real DD bundle by adding one additional structure (Real and equivariant structures) at a time to ordinary DD bundles. We first consider the classification of Real DD bundles.

For a Real space \(X\), let \(U = \{U_\alpha\}_{\alpha \in J}\) be a \(\Gamma\)-cover of \(X\) where the \(\Gamma\)-action on the index set \(J\) is free. For a Real DD bundle \(\mathcal{A}\), there are transition functions \(r_{\alpha \beta} : U_\alpha \to PU(H)\) satisfying \(r_{\alpha \beta}(x) = r_{\Gamma(\alpha)\Gamma(\beta)}(\sigma_X(x))\). If \(U\) is fine enough, \(r_{\alpha \beta}\) can be lifted to a \(U(H)\)-valued function \(\tilde{r}_{\alpha \beta}\). Let \(s_{\alpha \beta \gamma} = \tilde{r}_{\alpha \beta \gamma} \tilde{r}_{\gamma \alpha}\). \(\{s_{\alpha \beta \gamma}\}\) defines a \(\Gamma\)-equivariant \(U(1)\)-valued 2-cocycle in the Cech cohomology group \(H^2(\tilde{C}(U, U(1)_\Gamma))^\Gamma)\), where \(U(1)_\Gamma\) is the \(\Gamma\)-sheaf of continuous \(U(1)\)-valued functions with \(\Gamma\) acting on \(U(1)\) by complex conjugation. The short exact sequence

\[
1 \longrightarrow \mathbb{Z}_\Gamma \longrightarrow \mathbb{R}_\Gamma \longrightarrow U(1)_\Gamma \longrightarrow 1
\]

where \(\Gamma\) acts on \(\mathbb{Z}\) and \(\mathbb{R}\) by negation, and the fact that \(\mathbb{R}_\Gamma\) is a fine \(\Gamma\)-sheaf (i.e. it admits a \(\Gamma\)-equivariant partition of unity), lead to the isomorphism \(H^2(\tilde{C}(U, U(1)_\Gamma)^\Gamma) \cong H^3(\tilde{C}(U, \mathbb{Z}_\Gamma)^\Gamma)\) induced by the coboundary map in the long exact sequence of Cech cohomology groups. By the Corollary in Section 5.5 of [Gr], \(\lim U(1)_\Gamma\) is isomorphic to the sheaf cohomology \(H^3(X; \mathbb{Z}_\Gamma)\) defined as the third right derived functor of the \(\Gamma\)-invariant global section functor. According to [St], because \(\Gamma\) is a discrete group, \(H^3(X; \mathbb{Z}_\Gamma)\) is isomorphic to \(H^3_\Gamma(X, \mathbb{Z}_\Gamma)\), which is Borel’s equivariant cohomology defined more generally as follows.

**Definition 3.18.** Let \(G\) be a topological group with a \(\Gamma\)-action. Define the Real cohomology

\[H^3_\Gamma(X, G_\Gamma) := H^n(X \times \Gamma E\Gamma, X \times E\Gamma \times G/\Gamma)\]

where \(X \times E\Gamma \times G/\Gamma\) is the local coefficient system with fiber \(G\) over \(X \times \Gamma E\Gamma\).

**Definition 3.19.** The Real DD-class of \(\mathcal{A}\), denoted by \(DD_R(\mathcal{A})\), is defined to be the image of the 2-cocycle \(\{s_{\alpha \beta \gamma}\}\) in \(H^3_\Gamma(X, \mathbb{Z}_\Gamma)\) under the various isomorphisms of cohomology groups discussed above, namely \(H^3(\tilde{C}(U, \mathbb{Z}_\Gamma)^\Gamma) \cong H^3(X; \mathbb{Z}_\Gamma) \cong H^3_\Gamma(X, \mathbb{Z}_\Gamma)\).

**Remark 3.20.**

1. The introduction of the Real cohomology \(H^3_\Gamma(X, \mathbb{Z}_\Gamma)\) as the home where the Real DD-classes live is inspired by [Kah], where Kahn introduced Real Chern classes and use \(H^2(X, \mathcal{A}_2)\) to classify Real line bundles over \(X\).

2. The map \(DD_R : \text{category of Real DD bundles over } X \to H^3_\Gamma(X, \mathbb{Z}_\Gamma)\) is a group homomorphism. If \((f, \mathcal{E}) : (X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2)\) is a Morita morphism, then \(f^*DD_R(\mathcal{A}_2) = DD_R(\mathcal{A}_1)\).

By definition, one can see that \(\mathcal{A}\) is Morita trivial if and only if \(DD_R(\mathcal{A}) = 0 \in H^3_\Gamma(X, \mathbb{Z}_\Gamma)\). To show that \(H^3_\Gamma(X, \mathbb{Z}_\Gamma)\) classifies the Morita isomorphism classes of Real DD bundles over \(X\), we need the following
Proposition 3.21. For any class $\alpha \in H^3_F(X, \mathbb{Z}_\Gamma)$, there exists a Real DD bundle $A$ such that $DD_\mathbb{R}(A) = \alpha$.

Proof. The coboundary maps of the long exact sequences cohomology groups induced by the short exact sequences give rise to the following string of isomorphisms

$$H^3_F(X, \mathbb{Z}_\Gamma) \cong H^1_F(X, U(1)_\Gamma) \cong H^1_F(X, PU(\mathcal{H})_\Gamma) \cong H^0_F(X, BPU(\mathcal{H})_\Gamma) \cong \Gamma \cdot [X, BPU(\mathcal{H})]_\mathbb{R}$$

where $[X, Y]_\mathbb{R}$ is the set of $\Gamma$-equivariant homotopy equivalence classes of Real maps from $X$ to $Y$. Therefore $\alpha$ corresponds to a Real map $f : X \to BPU(\mathcal{H})$. The Real DD bundle $f^*(EPU(\mathcal{H}) \times_{PU(\mathcal{H})} \mathcal{K}(\mathcal{H}))$ can be easily checked to have $\alpha$ as the Real DD-class. □

Remark 3.22. The proof above also shows that the Real DD bundle $EPU(\mathcal{H}) \times_{PU(\mathcal{H})} \mathcal{K}(\mathcal{H})$ is a universal Real DD bundle.

Let $G \rtimes \Gamma$ be the semi-direct product such that conjugation by $\gamma$ on $G$ amounts to the automorphic involution $\sigma_G$. One can define the equivariant version of the Real cohomology similarly by Borel’s construction.

Definition 3.23. Let $X$ be a Real $G$-space. Define the equivariant Real cohomology $H^n_F(X, \mathbb{Z})_\Gamma$ to be $H^n((X \times_G EG) \times_\Gamma E\Gamma, (X \times_G EG) \times E\Gamma \times \mathbb{Z}/\Gamma)$.

Atiyah and Segal showed in [AS2] that the equivariant cohomology $H^3_G(X, \mathbb{Z})_\Gamma$ classifies the equivariant DD bundles up to Morita isomorphism. Putting together the equivariant and Real structure, one can, with little effort, show that

Proposition 3.24. The Morita isomorphism classes of equivariant Real DD bundles on the Real $G$-space $X$ is classified by $H^3_G(X, \mathbb{Z})_\Gamma$.

The following simple observation is helpful in computing the free part of $H^*_G(X, \mathbb{Z})_\Gamma$.

Proposition 3.25. $H^*_G(X, \mathbb{R})_\Gamma$ is isomorphic to $H^*_G(X, \mathbb{R})_\Gamma$, the $\Gamma$-invariant subgroup of $H^*_G(X, \mathbb{R})$, where the $\Gamma$-action on the cohomology group is defined by $\gamma \cdot \alpha = -\gamma^\ast \alpha$.

Proof. $H^*_G(X, \mathbb{R})_\Gamma$ is isomorphic to the sheaf cohomology $H^*_G(X \times_G EG; \Gamma, \mathbb{R})_\Gamma$ (cf. [St]). $\mathbb{R}_\Gamma$ has the following resolution by fine $\Gamma$-sheaves

$$\mathbb{R}_\Gamma \xrightarrow{d} \Omega^0(X \times_G EG) \xrightarrow{d} \Omega^1(X \times_G EG) \xrightarrow{d} \cdots$$

Note that $\Gamma$ acts on the global sections of $\Omega^n(X \times_G EG)$ by $-\gamma^\ast$. We have that $H^n(X; \Gamma, \mathbb{R})_\Gamma$ is the $n$-th cohomology of the cochain

$$\Omega^0(X \times_G EG)^_\Gamma \xrightarrow{d} \Omega^1(X \times_G EG)^_\Gamma \xrightarrow{d} \cdots$$

An easy averaging argument shows that taking $\Gamma$-invariants and taking cohomology commute. Hence $H^n(X \times_G EG; \Gamma, \mathbb{R})_\Gamma \cong H^n(X \times_G EG, \mathbb{R})^_\Gamma \cong H^*_G(X, \mathbb{R})^_\Gamma$. □
Example 3.26. We shall show that $H^3_{\Gamma}(pt, \mathbb{Z}_\Gamma) \cong \mathbb{Z}_2$ and exhibit the DD bundle whose DD class is the nontrivial 2-torsion. Consider the $S^0$-bundle $E\Gamma \to B\Gamma$. Being non-orientable, this bundle has Euler class $e$ which lives in the first cohomology group $H^1(B\Gamma, E\Gamma \times_\Gamma \mathbb{Z})$ with twisted coefficient system. Applying the Gysin sequence

$$\cdots \to H^2(E\Gamma, \mathbb{Z}) \to H^2(B\Gamma, \mathbb{Z}) \to H^3(B\Gamma, E\Gamma \times_\Gamma \mathbb{Z}) \xrightarrow{\pi^*} H^3(E\Gamma, \mathbb{Z}) \to \cdots$$

and by its exactness we have that $H^3(B\Gamma, E\Gamma \times_\Gamma \mathbb{Z}) \cong H^2(B\Gamma, \mathbb{Z}) \cong \mathbb{Z}_2$. Thus, there is at least a copy of $\mathbb{Z}_2$ as a summand in $H^3_{\Gamma}(X, \mathbb{Z}_\Gamma)$ which is the image of the split-injective pullback map $H^3_{\Gamma}(pt, \mathbb{Z}_\Gamma) \to H^3_{\Gamma}(X, \mathbb{Z}_\Gamma)$. The nonzero 2-torsion of $H^3_{\Gamma}(pt, \mathbb{Z}_\Gamma)$ is the Real DD class of the $\mathcal{K}(\mathcal{H})$-bundle over a point where $\mathcal{H} = \mathcal{L}(\mathbb{N})$ and the involution on $\mathcal{K}(\mathcal{H})$ is induced by the ‘quaternionic quarter turn’ on $\mathcal{H}$, i.e. $(z_1, z_2, z_3, z_4, \cdots) \mapsto (\overline{z_2}, \overline{z_1}, \overline{z_4}, \overline{z_3}, \cdots)$. In Section 3.3 we give another interpretation of this non-zero 2-torsion class as the obstruction for the $\Gamma$-equivariant real vector bundle $\mathbb{R}^{r, s} \to pt$ to possess a Real $(p, q)$-Spin$^c$ structure or, equivalently, the $KR$-theory orientation, when $\frac{p-q}{4}$ is odd.

3.3. Twisted $K$-theory (homology). Twisted $K$-theory, or $K$-theory with local coefficient systems, was first studied in [DK], where the case of local coefficient systems with torsion DD-classes was explored. The general case was taken up in [R], where Rosenberg defined twisted $K$-theory as homotopy equivalence classes of sections of a twisted bundle of Fredholm operators, with twisting data given by the local coefficient system. In recent years there have been extensive works done on twisted $K$-theory and its various models (cf. [AS2], [BCMMS], [CW], [Kar] and the references therein) due to its connection with string theory. It has been conjectured that $D$-branes and Ramond-Ramond field strength are classified by twisted $K$-theory, where the twist is defined by a $B$-field. An impetus to the whole enterprise of studying this mathematical physical connection is a deep result by Freed-Hopkins-Teleman, which is explained in more details in Section 5.

In what follows, we shall use the following definition of twisted $KR$-theory (homology) obtained by incorporating the Real structures into the analytic definition of twisted $K$-theory and homology, as in [M1].

**Definition 3.27.** For an equivariant Real $G$-space $X$ with an equivariant Real DD bundle $\mathcal{A}$, we define the **twisted $KR$-homology**

$$KR^G_q(X, \mathcal{A}) := KR^-C^G_q(S_0(\mathcal{A}))$$

where $S_0(\mathcal{A})$ is the Real $G-C^*$-algebra of space of sections of $\mathcal{A}$ vanishing at infinity. Similarly, we define the twisted $KR$-theory

$$KR^G_q(X, \mathcal{A}) := KR^G_q(S_0(\mathcal{A}))$$

We list some useful features of twisted equivariant (Real) $K$-homology, adapted from [M1].

1. A Morita morphism $(f, \mathcal{E}) : (X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2)$ induces a pushforward map

$$f_* : KR^G_q(X_1, \mathcal{A}_1) \to KR^G_q(X_2, \mathcal{A}_2)$$
which depends only on the 2-isomorphism class of \((f, \mathcal{E})\). In particular, the push forward map \((\text{Id}, (X \times \mathcal{H}) \otimes L)_*\) induces an automorphism on \(KR_*^G(X, \mathcal{A})\), which depends on the isomorphism class of the equivariant Real line bundle \(L\). In other words, the equivariant Real Picard group of \(X\) acts on the automorphism group of \(KR_*^G(X, \mathcal{A})\). If the equivariant Real Picard group of \(X\), which is \(H^2_{G \times \Gamma} (X, \mathbb{Z} \Gamma)\) by straightforwardly adapting the arguments in Section 3.2, is trivial, then there is only one canonical pushforward map independent of \(\mathcal{E}\). From now on, we will, for brevity, sometimes use \(f_*\) to denote the pushforward map if it is independent of \(\mathcal{E}\).

2. One can define the crossed product, which is a special case of Kasparov product (cf. [HR])

\[ KR_*^G(X_1, A_1) \otimes KR_*^G(X_2, A_2) \to KR_*^G(X_1 \times X_2, A_1 \otimes A_2) \]

where \(A_1 \otimes A_2\) is the external tensor product.

3. Recall that one of the motivations for introducing local coefficient systems to generalize cohomology theory is to formulate Thom isomorphism and Poincaré duality for spaces which are non-orientable in the sense of the relevant cohomology theory. For an even rank (resp. odd rank) \(G\)-equivariant real vector bundle \(V \to X\), which is orientable in the usual sense but not necessarily \(G\)-Spin\(^c\) (i.e. equivariant \(K\)-theoretic orientable), the \(K\)-theoretic local coefficient system reflecting the obstruction for \(V\) to be \(G\)-Spin\(^c\) can be realized by the Clifford bundle \(\text{Cl}(V)\) (resp. \(\text{Cl}(V \otimes \mathbb{R})\)), which is a matrix algebra bundle (and hence a DD bundle) with its DD-class being a 2-torsion. This is because \(V\) is \(G\)-Spin\(^c\) if and only if its Clifford bundle is Morita trivial (and trivialized by the reduced spinor bundle). We will call \(\text{Cl}(V)\) (resp. \(\text{Cl}(V \otimes \mathbb{R})\)) and any other Morita isomorphic DD bundles the orientation twist of \(V\) and denote it by \(o_V\). The Thom isomorphism now can be formulated as

\[ K^*_G(X, A \otimes o_V) \cong K^{*+n}_G(V, \pi^* A) \]

whereas the Poincaré duality is

\[ K^{n-*}_G(X, A) \cong K^*_G(X, A^{\text{opp}} \otimes o_{TX}) \]

The corresponding statement of Thom isomorphism and Poincaré duality in the Real case are completely analogous, just that we need to treat the degree shift carefully. Let \(V\) and \(TX\) be Real \((p, q)\)-orientable Euclidean \(G\)-vector bundles. Then the Thom isomorphism should be

\[ KR_{G,r,p-s}^{r+p,s}(X, A \otimes o_V) \cong KR_{G}^{r,s}(V, \pi^* A) \]

while the Poincaré duality is

\[ KR_{G,r,s}^{r,s}(X, A) \cong KR_{G,q-r,p-s}^{q-r,s}(X, A^{\text{opp}} \otimes o_{TX}) \]

**Example 3.28.** By Proposition 3.13 \(\mathbb{R}^{r,s} \to \text{pt}\) is Real \((p, q)\)-Spin\(^c\) if \(p - q - (r - s)\) is divisible by 8. The Thom isomorphism together with \((1, 1)\)-periodicity gives

\[ KR_{G}^{i+p-j,q}(X) \cong KR_{G}^{i+p-j,q}(X) \cong KR_{G}^{i+j}(X \times \mathbb{R}^{r,s}) \cong KR_{G}^{i+r,j+s}(X) \cong KR_{G}^{i+r-j-s}(X) \]

which is in consistence with the 8-periodicity of \(KR\)-theory. On the other hand, if \(p - q - (r - s)\) is only divisible by 4 but not 8, then \(\mathbb{R}^{r,s} \to \text{pt}\) is only Real \((p, q)\)-orientable but not Real \((p, q)\)-Spin\(^c\). The DD class of \(o_{\mathbb{R}^{r,s}}\) is exactly the non-zero two torsion of the
image of the split-injective map $H^3_{\Gamma}(pt, \mathbb{Z}_G) \to H^3_{G, \Gamma}(pt, \mathbb{Z}_G)$. In particular, if $r = 0, s = 4$, $p = q = 2$, the twisted version of Thom isomorphism yields

**Proposition 3.29.** $KR^*(G, \pi^*(o_{\mathbb{R}^0.4})) \cong KR^*(G \times \mathbb{R}^{0.4})$, which in turn is isomorphic to the Quaternionic $K$-theory $KH^4_r(X)$ by Proposition A.6

So Quaternionic $K$-theory is Real $K$-theory twisted by $o_{\mathbb{R}^0.4}$.

### 4. The equivariant fundamental DD bundle over $G^-$

4.1. **The group of Morita isomorphism classes of equivariant Real DD bundles over $G^-$.** In this section, we shall compute $H^3_{G, \Gamma}(G^-, \mathbb{Z}_G)$, the group of Morita isomorphism classes of $G$-equivariant Real DD-bundles over $G^-$. It is well-known that $H^3_{G}(G, \mathbb{R}) \cong \mathbb{R}$ and the integral generator is represented by the equivariant differential form

$$\eta_G(\xi) = \frac{1}{12}B(\theta^L, [\theta^L, \theta^L]) - \frac{1}{2}B(\theta^L + \theta^R, \xi)$$

where $\theta^L$ and $\theta^R$ are the left and right Maurer-Cartan forms respectively, and $\xi \in \mathfrak{g}$ (cf. [M2] Section 3).

**Lemma 4.1.** $H^3_{G}(G, \mathbb{Z})$ is invariant under $\Gamma$.

**Proof.** It suffices to show that $\eta_G$ is $\Gamma$-invariant. Note that

$$-(\Gamma^* \eta_G)(\xi) = -\frac{1}{12}B((\sigma_G \circ \text{inv})^* \eta_G^L, [(\sigma_G \circ \text{inv})^* \theta^L, (\sigma_G \circ \text{inv})^* \theta^L]) + \frac{1}{2}B((\sigma_G \circ \text{inv})^*(\theta^L + \theta^R), \sigma_G(\xi))$$

$$= -\frac{1}{12}B(-\sigma_G^* \theta^R, \sigma_G^*[\theta^R, \theta^R]) + \frac{1}{2}B(-\sigma_G^*(\theta^L + \theta^R), \sigma_G(\xi))$$

$$= \frac{1}{12}B(\theta^R, [\theta^R, \theta^R]) - \frac{1}{2}B(\theta^L + \theta^R, \xi)$$

$$= \eta_G(\xi)$$

The result follows.\[\square\]

By Lemma 3.25, $H^3_{G, \Gamma}(G^-, \mathbb{R}_G) \cong H^3_{G}(G^-, \mathbb{R}_G)^\Gamma \cong \mathbb{R}$. It follows that $H^3_{G, \Gamma}(G^-, \mathbb{Z}_G)$ is of rank 1 and its free part is generated by $[\eta_G]$. On the other hand, there must be a $\mathbb{Z}_2$ summand in $H^3_{G, \Gamma}(G^-, \mathbb{Z}_G)$ by the discussion in Example 3.26. The non-zero 2-torsion is $\text{DD}_{\mathbb{R}}(\pi^*o_{\mathbb{R}^0.4})$. Thus we have that $H^3_{G, \Gamma}(G^-, \mathbb{Z}_G)$ contains the subgroup generated by $[\eta_G]$ and $\text{DD}_{\mathbb{R}}(\pi^*o_{\mathbb{R}^0.4})$. In fact, it is all that this equivariant Real cohomology group contains.

**Proposition 4.2.** $H^3_{G, \Gamma}(G^-, \mathbb{Z}_G) \cong \mathbb{Z}[\eta_G] \oplus \mathbb{Z}_2 \text{DD}_{\mathbb{R}}(\pi^*o_{\mathbb{R}^0.4})$.

**Proof.** By definition,

$$H^3_{G, \Gamma}(G^-, \mathbb{Z}_G) = H^3(((G^- \times EG)/G \times E\Gamma)/\Gamma, ((G^- \times EG)/G \times E\Gamma \times \mathbb{Z})/\Gamma)$$

Applying Serre spectral sequence to the fiber bundle $(G^- \times EG)/G \hookrightarrow ((G^- \times EG)/G \times E\Gamma)/\Gamma \to B\Gamma$, we have that the $E_2$-page is

$$E^{p,q}_2 = H^p(B\Gamma, H^q_G(G^-, \mathbb{Z}_G) \times \Gamma E\Gamma)$$
Note that
\[ E_2^{0,3} = H^0(B\Gamma, Z) \cong Z \quad (H^3_G(Z, Z) \cong Z) \text{ is invariant under the } \Gamma\text{-action by Lemma 1.1} \]
\[ E_2^{1,2} = E_2^{2,1} = 0 \text{ as } H^i_G(Z, Z) = 0 \text{ for } i = 1, 2 \]
\[ E_2^{3,0} = H^3_G(\text{pt}, Z_G) \cong Z_2 \quad (\text{Note that } \Gamma\text{ acts on } H^0_G(Z, Z) \cong Z\text{ by negation}) \]

The convergence of the spectral sequence implies that \( H^3_G(\Gamma(Z, Z)) \) is a certain extension of subquotients of \( Z_2 \) by \( Z \). But from the discussion preceding this Proposition we have that \( H^3_G(\Gamma(Z, Z)) \) contains \( Z \oplus Z_2 \) as a subgroup. We conclude that indeed \( H^3_G(\Gamma(Z, Z)) \cong Z \oplus Z_2 \). \( \square \)

**Definition 4.3.** Any equivariant Real DD bundles over \( G^- \) whose equivariant Real DD class is \([ -\eta_G ]\) is called an equivariant Real fundamental DD bundle.

**Remark 4.4.** Using the ideas in this Section, one can show that \( H^3_G(Z, Z) \cong Z_2 \text{DD}_2(\pi^*o_{20,4}) \) if \( \Gamma\) acts on \( G \) by an involutive automorphism. In this case the equivariant twisted KR-homology of \( G, KR^{G'}(G, \pi^*o_{20,4}) \) is isomorphic to \( KR^{G'}_{-4}(G) \cong KH^*(G) \) by Proposition 3.29. We refer the reader to [4] for a complete description of the algebra structure of \( KR^{G'}(G) \) (and hence \( KR^{G'}(G) \) by Poincaré duality and \( KH^*(G) \) by shifting degree by \(-4\)).

### 4.2. A distinguished maximal torus with respect to \( \sigma_G \)

In constructing the equivariant Real fundamental DD-bundle over \( G \), which are done in Sections 4.3 we need to work with a particular kind of maximal torus associated with \( \sigma_G \). In this Section we record the results about this maximal torus, directly taken from [4,5].

Let \( g = k \oplus q \), where \( k \) and \( q \) are the \( \pm 1 \) eigenspaces of \( \sigma_g \) respectively. Let \( a \) be the maximal abelian subspace of \( q \), \( t \) a choice of maximal abelian subalgebra of \( g \) containing \( a \). Let \( t' \) be the centralizer of \( a \) in \( t \), and \( t = t \cap t' \). It is known that \( t = t' \oplus a \) (c.f. [4,5, Appendix B]) and \( \sigma_G \) respects this decomposition. Let \( K' = \exp_G t' \), \( T' = \exp_G t \) and \( T = \exp_G t \). \( T \) is the maximal torus we will use from now on. Note that \( T' \) is a maximal torus of \( K' \). Let \( W' \) be the Weyl group of \( K' \) with respect to \( T' \). Let \( w_0' \) be the longest element in \( W' \). Define \( \sigma_+ : t^* \to t^* \) by

\[ \sigma_+(\lambda) = -\sigma_g(w_0'\lambda) \]

and \( \sigma_+ : t \to t \) similarly. Let \( k_0 \in N_{K'} \) be any representative of \( w_0' \). Let \( R \) be the root system of \( (g, t) \). We define a positive root system \( R_+ \) as follows. Let \( R' = \{ \alpha \in R | \sigma_+(\alpha) = \alpha \} \), \( R'^{\alpha} = \{ \alpha |_a \ | \alpha \notin R' \} \)

\( R' \) is the root system of \( (t', t') \), thus a root subsystem of \( R \). \( R'^{\alpha} \) is the system of restricted roots of the symmetric pair \( (g, t) \). Define

\[ R_+ := R'_+ \cup \{ \alpha \in R \ | \ \alpha|_a \in R_+^{\alpha} \} \]

Now we can fix a choice of the closed Weyl alcove \( \Delta \) with respect to \( R_+ \).

**Proposition 4.5.** \( (1) \) ([4,5, Lemma 4.7(i)] \( \sigma_+ \) is an involution and \( \sigma_+(R_+) = R_+ \). \( \quad \) Hence \( \sigma_+ \) preserves \( t^*_+ \) and \( \Delta \).
(2) ([OS], Addendum 4.11) We have $\sigma^\text{G}G_{\lambda}^{\circ} \cong V_{\sigma^+}(\lambda)$, where $V_{\lambda}$ is the irreducible complex $G$-representation with highest weight $\lambda$. $V_{\lambda}$ is in $\text{RR}(G, \mathbb{R})$ (resp. $\text{RH}(G, \mathbb{R})$) iff $\sigma^+ (\lambda) = \lambda$ and $(k_0^2)^{\lambda} = 1$ (resp. $(k_0^2)^{\lambda} = -1$). $V_{\lambda} \oplus V_{\sigma^+ (\lambda)}$ is in $\text{RR}(G, \mathbb{C})$ iff $\sigma^+ (\lambda) \neq \lambda$.

(3) ([OS], Lemma 4.7(ii)) $\sigma^+ (\lambda) = \lambda$ iff either $\lambda \in a^*$, or $\lambda \in (t')^*$ and $\lambda = -w_0^* \lambda$.

(4) ([OS], Lemma 4.7(iii)) If $\lambda \in a^*$, then $(k_0^2)^{\lambda} = 1$. If $\lambda \in (t')^*$ and $\lambda = -w_0^* \lambda$, then $(k_0^2)^{\lambda} = \pm 1$.

(5) Let

$$\sigma_{G/T} : G/T \to G/T$$

$$gT \mapsto \sigma_G(g)k_0^{-1}T$$

Then the Weyl covering map

$$(G/T, \sigma_{G/T}) \times (T, \text{Id}) \to G^-$$

is a Real map.

**Proof.** We only show the last part. Let $\xi = \xi_1 + \xi_2$ with $\xi_1 \in t'$ and $\xi_2 \in a$. Then

$$\sigma_G(g)k_0^{-1}\exp(\sigma^+ (\xi_1 + \xi_2))k_0\sigma_G(g)^{-1}$$

$$= \sigma_G(g)\exp(w_0^{-1}(-\sigma_g(w_0^* (\xi_1 + \xi_2))))\sigma_G(g)^{-1}$$

$$= \sigma_g(g)\exp(-\xi_1 + \xi_2)\sigma_G(g)^{-1}$$

$$= \sigma_G(g)\sigma_G(\exp(\xi_1 + \xi_2))^{-1}\sigma_G(g)^{-1}$$

□

### 4.3. Construction of the equivariant Real fundamental DD bundle.

In this Section, we shall first review the construction, as in [M1], of the equivariant DD-bundle $A$ over $G$ whose equivariant DD-class is $[-\eta_G] \in H^3_G(G, \mathbb{Z})$. Then we point out how to equip $A$ with a suitable Real structure so that it becomes an equivariant Real DD-bundle over $G^-$ with equivariant Real DD-class $[-\eta_G] \in H^3_{G_\mathbb{R}}(G^-, \mathbb{Z})$.

Consider the restriction of $A$ to a maximal torus $T$, viewed as a $T$-equivariant DD-bundle. The central extension

$$1 \to U(1) \to U(H) \to \text{Aut}(K(H)) \to 1$$

is pulled back to a central extension $\hat{T}$ of $T$ via the $T$-action on $\text{Aut}(K(H))$. In this way $A|_T$ gives rise to a family of central extensions $\bigsqcup_{t \in T} \hat{T}_t$. Now for any $t_1, t_2 \in T$, $\hat{T}_{t_1} \cong \hat{T}_{t_2} \cong T \times U(1)$ as central extensions of $T$ up to $H^2_T(pt, \mathbb{Z}) \cong \Lambda^*$. Since the latter group is discrete, any path from $t_1$ to $t_2$ defines an isomorphism $\hat{T}_{t_1} \to \hat{T}_{t_2}$ up to an element in $H^2_T(pt, \mathbb{Z})$ which only depends on the homotopy class of the path. We therefore can define a holonomy homomorphism

$$\pi_1(T) \cong H_1(T, \mathbb{Z}) \cong \Lambda \to H^2_T(pt, \mathbb{Z}) \cong \Lambda^*$$

which is an element in $H^1(T, \mathbb{Z}) \otimes H^2_R(pt, \mathbb{Z}) \cong \Lambda^* \otimes \Lambda^*$. 


On the other hand, the restriction map
\[ i^*_T : H^3_G(G, \mathbb{Z}) \to H^3_G(pt, \mathbb{Z}) \cong H^2_T(\mathbb{Z}) \oplus H^1(T, \mathbb{Z}) \oplus H^0(T) \]
is injective, and sends the generator \( \eta_G \) to minus the basic inner product \( -B|_t \in \Lambda^* \otimes \Lambda^* \cong H^2_T(\mathbb{Z}) \oplus H^1(T, \mathbb{Z}) \) (cf. [M2, Proposition 3.1]). Hence \( \coprod_{t \in T} \hat{T}_t \) is the \( \hat{T} \)-bundle \( t \times \Lambda \hat{T} \) where \( \Lambda \) acts on \( \hat{T} \cong T \times U(1) \) by
\[ \lambda \cdot (t, z) = (t, t^{-B^T(\lambda)} z) \]
and the Weyl group \( W \) acts on \( t \times \Lambda \hat{T} \) by
\[ w \cdot [\xi, t, z] = [w \xi, wt, z] \]
The next step is to construct a suitable family of central extensions \( \coprod_{g \in G} \hat{G}_g \) of the stabilizer subgroup \( G_g \) which extends \( t \times \Lambda \hat{T} \) in such a way that the induced action by conjugation on this family extends that of \( W \) on \( t \times \Lambda \hat{T} \). This is done in [M1] by way of simplicial techniques, which we shall recall here.

Note that \( \Delta \) parametrizes the orbit spaces of \( W \)-action on \( T \), and
\[ T = \coprod_I W/W_I \times \Delta_I / \sim \]
where, for \( J \subset I \),
\[ (\varphi_I^J(w), x) \sim (w, t^I_J(x)), \]
where \( \varphi_I^J : W/W_I \to W/W_J \) is the natural projection and \( t^I_J : \Delta_J \to \Delta_I \) the inclusion of simplices.

**Definition 4.6.** Let \( \lambda_I : W_I \to \Lambda \) be defined by the equation \( \Delta_I = w \cdot \Delta_I + \lambda_I(w) \) for \( w \in W_I \).

Note that \( \lambda_I \) is a group cocycle and \( \lambda_I|_{W_J} = \lambda_J \) for \( J \subset I \).

**Proposition 4.7.** The family of central extensions \( t \times \Lambda \hat{T} \) is isomorphic to \( \coprod_I (W \times W_I \hat{T}) \times \Delta_I / \sim \) \( W \)-equivariantly, where \( W_I \) acts on \( \hat{T} \) by
\[ w \cdot (t, z) = (wt, h^{B_T(\Lambda_I)}(w^{-1}))) \]

**Proof.** We first check that the map \( \hat{T} \times \Delta_I \to t \times \Lambda \hat{T} \) defined by \( (t, z, \xi) \mapsto [(\xi, t, z)] \) is \( W_I \)-equivariant. Indeed, for \( \xi \in \Delta_I \),
\[ w \cdot [(\xi, t, z)] = [(w \xi, wh, z)] = [(\xi - \lambda_I(w); wh, z)] = [(\xi; wh, (wh)^{B_T(\Lambda_I)}(w))] = [(\xi; wh, h^{-B_T(\Lambda_I)}(w^{-1}) z)] \]
So it extends to the map \( \coprod_I (W \times W_I \hat{T}) \times \Delta_I / \sim \to t \times \Lambda \hat{T} \), which is \( W \)-equivariant. □
Since $G$ admits the following similar simplicial description

$$G = \coprod_I G/G_I \times \Delta_I / \sim$$

the desired family of central extension $\coprod_{g \in G} \hat{G}_g$ should be of the form

$$\coprod_I (G \times_{G_I} \hat{G}_I) \times \Delta_I / \sim$$

where $G_I$ acts by conjugation on $\hat{G}_I$, which is a central extension of $G_I$ such that

1. $\hat{G}_I$ contains $\hat{T}$ as the common maximal torus,
2. there are lifted inclusions $\hat{i}_J : \hat{G}_I \hookrightarrow \hat{G}_J$ satisfying $\hat{i}_K = \hat{i}_J \circ \hat{i}_K$ for $K \subset J \subset I$, and
3. the Weyl group element $w \in W_I \cong N_{G_I}(\hat{T})/\hat{T} \cong N_{G_I}(T)/T$ acts on $\hat{T}$ by the action given in Proposition 4.7.

**Proposition 4.8 ([M1]).** The central extension

$$\hat{G}_{I,t} := \hat{G}_I \times_{\pi_1(G_I)} U(1)$$

where $\tilde{G}_I$ is the universal cover of $G_I$, $t = \exp(\xi)$ for $\xi \in \text{int}(\Delta_I)$ and $\pi_1(G_I) = \Lambda/\Lambda_I$ acts on $U(1)$ by

$$\lambda \cdot t = t^{-B^0(\lambda)} = \exp(-2\pi \sqrt{-1}B(\lambda, \xi))$$

satisfies the properties stated above.

**Proof.** The first two properties are easy. For the last one, we only need to note that if we choose $\hat{T}$ to be the image of the map

$$i_I : T \times U(1) \rightarrow \hat{G}_{I,t}$$

$$(\exp_T(\zeta), z) \mapsto [(\exp_{\hat{G}_I}(\zeta), \exp(-2\pi \sqrt{-1}B(\zeta, \xi))z)]$$

then the conjugation action by a representative of $w \in W_I$ restricted to $\hat{T}$ is indeed the one given in Proposition 4.7. \qed

The isomorphism classes of the central extensions $\hat{G}_{I,t}$ are independent of $t$ because $\exp(\text{int}(\Delta_I))$ is contractible. We shall simply fix a choice of $t$ and drop $t$ from the subscript in the notation for the central extension of $G_I$ from now on.

**Lemma 4.9. ([M2] Lemma 3.5)** There exists a Hilbert space $\mathcal{H}$ equipped with unitary representations of the central extensions $\tilde{G}_I$ such that the central circle acts with weight $-1$ and, for $J \subset I$, the action of $\tilde{G}_J$ restricts to that of $\tilde{G}_I$.

Putting $\mathcal{A}_I = G \times_{G_I} \mathcal{K}(\mathcal{H})$ and

$$\mathcal{A} = \coprod_I (\mathcal{A}_I \times \Delta_I) / \sim$$
where the relation \( \sim \) is similar to the one used in Equation \( 2 \) we have that the family of central extensions induced by \( \mathcal{A} \) is the opposite of \( \coprod_{g \in G} \widehat{G}_g = \coprod_I (G \times_{G_I} \widehat{G}_I) \times \Delta_I / \sim \). The equivariant DD-class of \( \mathcal{A} \) is therefore \([-\eta_G]\).

We end this section by pointing out how to endow \( \mathcal{A} \) with a suitable Real structure which descends to the anti-involution on \( G \), as promised at the beginning of this section. We choose \( T \) to be the distinguished maximal torus with respect to \( \sigma_G \) as in Section \( 4.2 \). Let \( \sigma_{G/G_I} : G/G_I \to G/G_I \) be defined by \( gG_I \to \sigma_G(g)k_0^{-1}G_I \). By \( 3 \) of Proposition \( 4.5 \) the simplicial piece \( G/G_I \times \Delta_I \) used in the simplicial description of \( G \) induces the anti-involution on \( G \). Setting \( \mathcal{H}' := \mathcal{H} \oplus \sigma_G \mathcal{H} \) with the Real structure being swapping the two summands, we obtain a Real representation of \( \widehat{G}_I \) with the central circle acting by \(-1\), and \( \mathcal{A}_I = G \times_{G_I} \mathcal{K}(\mathcal{H}') \). We find that the simplicial piece \( \mathcal{A}_I \times \Delta_I \) with the involution

\[
([g, F], \xi) \mapsto ([\sigma_G(g)k_0^{-1}, \overline{F}], \sigma_+(\xi))
\]

induces the desired Real structure on \( \mathcal{A} \).

**Remark 4.10.** The other generator of \( H^3_{G\times I}(G^{-}, \mathbb{Z}_I) \) of infinite order, \([ -\eta_G ] + \text{DD}_{\mathbb{R}}(\pi^*\mathfrak{o}_{\mathbb{R}^2, 4}) \), is the DD class of the same DD bundle \( \mathcal{A} \) except that the involution \( \sigma_{\mathcal{A}} \) is induced by the involution

\[
([g, F], \xi) \mapsto ([\sigma_G(g)k_0^{-1}, J\overline{F}, J^{-1}], \sigma_+(\xi))
\]

on the simplicial pieces \( \mathcal{A}_I \times \Delta_I \), where \( J \) is a ‘quaternionic quarter turn’ on \( \mathcal{H} \).

5. **Generalization of the Freed-Hopkins-Teleman in the Real context**

5.1. **Freed-Hopkins-Teleman Theorem.** Let \( \mathcal{A} \) be an equivariant DD bundle whose DD class is the generator of \( H^3_G(G, \mathbb{Z}) \cong \mathbb{Z} \). The equivariant twisted \( \mathcal{K} \)-homology \( K^\mathcal{G}_G(G, \mathcal{A}^p) \) has a multiplicative structure induced by the crossed product (see \( 2 \) in Section \( 3.3 \)),

\[
K^\mathcal{G}_*(G, \mathcal{A}^p) \otimes K^\mathcal{G}_*(G, \mathcal{A}^p) \to K^\mathcal{G}_*(G \times G, \pi_1^*\mathcal{A}^p \otimes \pi_2^*\mathcal{A}^p)
\]

followed by the pushforward map induced by the group multiplication

\[
m_* : K^\mathcal{G}_*(G \times G, \pi_1^*\mathcal{A}^p \otimes \pi_2^*\mathcal{A}^p) \to K^\mathcal{G}_*(G, \mathcal{A}^p)
\]

Note that there is a Morita isomorphism \( m_*\mathcal{A}^p \cong \pi_1^*\mathcal{A}^p \otimes \pi_2^*\mathcal{A}^p \) because \( m_*[\eta_G] = \pi_1^*[\eta_G] + \pi_2^*[\eta_G] \). \( m_* \) is independent of the equivariant Hilbert space bundle \( \mathcal{E} \) on \( G \times G \) which witnesses this Morita isomorphism, and hence canonically defined, since \( H^2_G(G \times G, \mathbb{Z}) = 0 \) (cf. (1) of Section \( 3.3 \)). Freed-Hopkins-Teleman Theorem asserts that

**Theorem 5.1** (Freed-Hopkins-Teleman Theorem, \[ FHT1, FHT2, FHT3 \]). Let \( h' \) be the dual Coxeter number of \( G \) (for definition see Section \( 2 \)). The equivariant twisted \( \mathcal{K} \)-homology \( K^\mathcal{G}_*(G, \mathcal{A}^{k+h'h'}) \) is isomorphic to the level \( k \) Verlinde algebra \( R_k(G) \) (to be explained below), for \( k \geq 0 \). More precisely, the pushforward map

\[
t_* : R(G) \cong K^\mathcal{G}_*(pt) \to K^\mathcal{G}_*(G, \mathcal{A}^{k+h'h'})
\]

is onto with kernel being \( I_k \), the level \( k \) Verlinde ideal (for definition see Section \( 2 \)).
The above Theorem merits some remarks. On one end of the isomorphism is Verlinde algebra, which is an object of great interest in mathematical physics. It is the Grothendieck group of the positive energy representations of the level $k$ central extension of the (free) loop group $LG$, equipped with an intricately defined ring structure called the fusion product. It is known that $R_k(G)$, as an abelian group, is generated freely by the isomorphism classes $V_{\lambda}$ with highest weight $\lambda$ in $\Lambda_k^*$, the set of level $k$ weights (see Section 2 for definition). The fusion product rule can be stipulated by defining its structural constants $c_{\gamma\lambda\mu}$ with respect to those generators satisfying

$$[V_{\lambda}] \cdot [V_{\mu}] = \sum_{\gamma \in \Lambda_k^*} c_{\gamma\lambda\mu}[V_{\gamma}]$$

to be the dimension of a certain vector space associated with the Riemann surface of genus 0, with three punctures labelled by $\lambda$, $\mu$ and $\gamma$ $: = -w_0 \gamma$. This vector space has its root in Conformal Field Theory (see [V]) and can be interpreted as the space of conformal block (for one of its models see [Be]), which was shown to be canonically isomorphic to the space of generalized theta functions of the moduli space of $G$-bundles (cf. [BL] and the references therein). Thus one of the novelties of Freed-Hopkins-Teleman Theorem is that it provides an algebro-topological perspective of the fusion product, in addition to the conformal field theory and algebro-geometric approach. An alternative definition of Verlinde algebra as a quotient ring of $R(G)$ is given in Section 2.

Let $A$ be an equivariant Real fundamental DD bundle on $G^-$, whose construction is given in Section 4.3. We will consider $KR^*_G(G^-, A^{k+h^\vee})$ to formulate the Real version of FHT and give its proof in this Section. Note that in this Real situation the pushforward map $m_*$ induced by the group multiplication is a canonical one because the equivariant Real cohomology $H^2_G \times TL(G^- \times G^-, Z_T) = 0$ (cf. (1) in Section 3.3), which can be proved along the lines of thought in Section 4.1.

5.2. A structure theorem. The following structure theorem is a generalization of [Se, Theorem 4.2].

**Theorem 5.2.** Let $X$ be a Real $G$-space and $A$ an equivariant Real DD bundle on $X$. Suppose $K^*_G(X, A)$ is a free abelian group which is decomposed by the involution $\sigma_G \circ \overline{\sigma}_X$ into the following summands

$$K^*_G(X, A) = M_+ \oplus M_- \oplus T \oplus \sigma_G \circ \overline{\sigma}_X T$$

where $\sigma_G \circ \overline{\sigma}_X$ is identity on $M_+$ and negation on $M_-$. Suppose further that there exist $x_1, \cdots, x_n \in KR^*_G(X, A)$ such that their images in $K^*_G(X, A)$ under the forgetful map (forgetting the Real structure) form a basis of $M_+ \oplus M_-$. Let $F$ be the free $KR^*(pt)$-module generated by $x_1, \cdots, x_n$, and $K^*(+) \text{ the complex } K$-theory of a point extended to a $\mathbb{Z}_8$-graded module over $K^*(pt) \cong \mathbb{Z}$. Then the following map

$$f : F \oplus r(K^*(+) \otimes T) \to KR^*_G(X, A)$$

is an isomorphism of $KR^*(pt)$-modules, where $r(x) = x + \sigma_G \circ \overline{\sigma}_X x \in KR^*_G(X, A)$. 
Theorem 5.2 still holds if Corollary 5.3.

equivariant KR out its statement.

H-twisted equivariant setting. Consider the following The proof will be a straightforward adaptation of that of \cite[Theorem 4.2]{Se} in the So r we use \(\pi\) to denote various projection maps by abuse of notations, and \(G\) acts on the spheres trivially. A generalization of the Gysin sequence (cf. \cite[Proposition 3.2]{At})

\[
\cdots \to KR^{p-q}_G(X, A) \xrightarrow{(-q)^p} KR^{q}_G(X, A) \xrightarrow{\pi^*} KR^{q}_G(X \times S^{p, 0}, \pi^* A) \xrightarrow{\delta} \cdots
\]

implies the following short exact sequence for \(p \geq 3\)

\[
0 \to KR^{q}_G(X, A) \xrightarrow{\pi^*} KR^{q}_G(X \times S^{p, 0}, \pi^* A) \xrightarrow{\delta} KR^{p+1-q}_G(X, A) \to 0
\]

If we take inverse limits as \(p\) tends to infinity, then \(\lim_{\to p} KR^{p+1-q}_G(X, A) = 0\) and hence

\[
KR^{p}_G(X, A) = \lim_{\to p} KR^{p+1-q}_G(X, A)
\]

This shows that \(HR^*(p, q)(X, A)\) does converge to \(KR^*(X, A)\). That the system \(H^*_G(p, q)(X, A)\) converges to \(K^*_G(X, A)\) follows from the proof of \cite[Lemma 4.1]{Sc}. Now we define a map

\[
f(p, q) : HR^*(p, q)(pt) \otimes_{KR^*(pt)} F \oplus r(H^*(p, q)(pt) \otimes T) \to HR^{p}_G(p, q)(X, A)
\]

\[
\rho_1 \otimes a_1 \oplus r(\rho_2 \otimes a_2) \mapsto \rho_1a_1 + r(\rho_2a_2)
\]

As \(F\) is a free \(KR^*(pt)\)-module and \(T\) a free abelian group, and tensoring free objects and taking cohomology commute, \(f\) is the abutment of \(f(p, q)\). On the \(E_1^{p,q}\)-page, \(f(p, q)\) becomes

\[
f^{p,q}_1 : KR^{-q}(S^{1,0}) \otimes_{KR^*(pt)} F \oplus r(K^{-q}(S^{1,0}) \otimes T) \to KR^{q}_G(X \times S^{1,0}, \pi^* A)
\]

Note that \(KR^{-q}(S^{1,0}) \cong K^{-q}(pt)\), \(K^{-q}(S^{1,0}) \cong K^{-q}(pt) \oplus K^{-q}(pt)\) and \(KR^{q}_G(X \times S^{1,0}, \pi^* A) \cong K^{q}_G(X, A)\). With the above identifications,

\[
r : K^{-q}(S^{1,0}) \otimes T \to KR^{q}_G(X \times S^{1,0}, \pi^* A) \cong K^{q}_G(X, A)
\]

\[
(u_1, 0) \otimes t_1 + (0, u_2) \otimes t_2 \mapsto u_1t_1 + \overline{u_2}^*\sigma^*_G \circ \sigma^*_X t_2
\]

So \(r(K^*(S^{1,0}) \otimes T) = T \oplus \sigma^*_G \circ \sigma^*_X T\). Together with the identification \(KR^*(S^{1,0}) \otimes_{KR^*(pt)} F \cong M_+ \oplus M_-\), we have that \(f^{p,q}_1\) is an isomorphism, and so is its abutment \(f\).

Corollary 5.3. Theorem \cite{Sc} still holds if \(X\) is assumed to be a Real \(G\)-manifold, twisted equivariant \(KR\)-theory is replaced by twisted equivariant \(KR\)-homology, \(\sigma^*_X\) by \(\sigma^*_X\) throughout its statement.
Proof. Let $X$ be Real $(p, q)$-orientable and $pd : KR^G_*(X, A) \to KR^G_{-p-*(X, A^{opp} \otimes o_{TX})}$ be the Poincaré duality map. We shall show that the following square commutes. The Corollary then follows from Poincaré duality. That $\sigma^*_G$ commutes with $pd$ being obvious, it remains to show that $pd \circ \sigma^*_X = \sigma^*_{X*} \circ pd$. On the one hand, the pushforward map $\sigma_{X*}$ on $KR^G_*(X, A)$ satisfies $pd \circ \sigma_{X*} = \sigma_{X*} \circ pd$ by definition. On the other hand, the push-pull formula
\[
\sigma_{X*}(\sigma^*_X(x) \cdot y) = x \cdot \sigma_{X*}(y)
\]
for $x \in KR^G_*(X, A)$ and $y \in KR^G_*(X, A)$, and $\sigma_{X*}(1) = 1$ imply that $\sigma_{X*} = (\sigma^*_X)^{-1} = \sigma_X^*$, concluding the proof. \hfill \Box

5.3. The Real Freed-Hopkins-Teleman.

**Proposition 5.4.** Let $P \subset \Lambda^*_k$ be a set of representatives of the 2-element orbits of the action of $\sigma_+$ on $\Lambda^*_k$. The involution $\sigma^*_G \sigma_{G^{-*}}$ decomposes $K^*_G(G^-, A^{k+h'}$) $\cong R_k(G)$ into the following summands
\[
\bigoplus_{\{\lambda \in \Lambda^*_k|\sigma_+(\lambda) = \lambda\}} \mathbb{Z}[V_\lambda] \oplus \bigoplus_{\mu \in P} \mathbb{Z}[V_{\mu}] \oplus \bigoplus_{\mu \in P} \mathbb{Z}[V_{\sigma_+(\mu)}]
\]

Proof. It suffices to show that $\sigma^*_G \sigma_{G^{-*}}(V_\lambda) = V_{\sigma_+(\lambda)}$. The pushforward map $\iota_* : K^G_*(pt) \to K^G_*(G, A^{k+h'})$ can be easily seen to satisfy $\sigma^*_G \sigma_{G^{-*}} \circ \iota_* = \iota_* \circ \sigma_{G}$. Note that for $\lambda \in \Lambda^*_k$, $W_\lambda \in K^G_*(pt)$, the irreducible representation of $G$ with highest weight $\lambda$, is sent by $\iota_*$ to $V_\lambda \in K^G_*(G, A^{k+h'})$, the irreducible positive energy representation of $LG$ with highest weight $\lambda$. It follows that
\[
\sigma^*_G \sigma_{G^{-*}}(V_\lambda) = \iota_* \circ \sigma_{G}^*(W_\lambda) = \iota_*(W_{\sigma_+(\lambda)}) = V_{\sigma_+(\lambda)}
\]

\hfill \Box

**Corollary 5.5.** Let $A$ be an equivariant Real fundamental DD bundle over $G^-$ constructed in Section 4.3. Using the same notations as in the previous Proposition, we have the following $KR_*(pt)$-module isomorphism
\[
KR^G_*(G^-, A^{k+h'}) \cong KR_*(pt) \otimes \bigoplus_{\{\lambda \in \Lambda^*_k|\sigma_+(\lambda) = \lambda\}} \mathbb{Z}[V_\lambda] \oplus r(K_*(+) \otimes \bigoplus_{\mu \in P} \mathbb{Z}[V_{\mu}])
\]

Here $V'_\lambda$ is the element in $KR^G_*(G^-, A^{k+h'})$ which is mapped by the forgetful map to $V_\lambda \in K^G_*(G, A^{k+h'})$, and is assigned with degree 0 or $-4$ according as whether $k_\lambda^0 = 1$ or $-1$.
Proof. We shall show that those $V'_\lambda \in KR^G_\ast(G^-, \mathcal{A}^{k+h^\vee})$ which are sent by the forgetful map to $V_\lambda \in K^G_\ast(G, \mathcal{A}^{k+h^\vee})$ do exist. Consider the commutative diagram

\[
\begin{array}{ccc}
KR^G_\ast(pt) & \xrightarrow{\iota^G_\ast} & KR^G_\ast(G^-, \mathcal{A}^{k+h^\vee}) \\
\downarrow{c} & & \downarrow{c^{\text{twist}}} \\
K^G_\ast(pt) & \xrightarrow{\iota^G_\ast} & KR^G_\ast(G, \mathcal{A}^{k+h^\vee})
\end{array}
\]

where the vertical maps are forgetful maps. Let $W'_\lambda \in KR^G_\ast(pt)$ be the Real or Quaternionic irreducible representation (depending on whether $k_0^\lambda = 1$ or $-1$, respectively) satisfying $c(W'_\lambda) = W_\lambda$. Let $V'_\lambda = i^G_\ast(W'_\lambda)$. By the commutativity of the above diagram, $c^{\text{twist}}(V'_\lambda)$ is indeed $V_\lambda$. Now we can apply Corollary 5.3 and Proposition 5.4 to obtain the desired isomorphism. □

**Theorem 5.6.** Let $A$ be the Real fundamental DD bundle over $G^-$ constructed in Section 4.3. Let the level $k$ Verlinde ideal $I_k$ be generated by $r_1, \cdots, r_m \in R(G)$, and $RI_k$ be the ideal in $KR^G_\ast(pt)$ with generators obtained from $r_1, \cdots, r_m$ by the following.

1. Assigning each irreducible component of $r_1$ which is not in $R(G, \mathbb{C})$ with degree 0 (resp. $-4$) according as whether it can be made a Real representation (resp. Quaternionic representation), and
2. replacing each irreducible component $s$ of $r_i$ which is in $R(G, \mathbb{C})$ with the double $s + \sigma^G_i s$, which is assigned with degree 0.

Then the pushforward map $i^G_\ast : KR^G_\ast(pt) \to KR^G_\ast(G^-, \mathcal{A}^{k+h^\vee})$ is onto with kernel $RI_k$.

Proof. Note that $V'_\lambda$ and $r(V_\mu)$, which are generators of $KR^G_\ast(G^-, \mathcal{A}^{k+h^\vee})$ as a $KR^G_\ast(pt)$-module by Corollary 5.3, are images of $W'_\lambda$ and $r(W_\mu)$ under $i^G_\ast$, where $W'_\lambda \in KR^G_\ast(pt)$ is the Real or Quaternionic irreducible representation with highest weight $\lambda$, and $W_\mu \in K^G_\ast(pt)$ the complex irreducible representation with highest weight $\mu$. Thus $i^G_\ast$ is onto. Comparing the descriptions of the coefficient ring $KR^G_\ast(pt)$ and $KR^G_\ast(G^-, \mathcal{A}^{k+h^\vee})$ in Proposition A.9 and Corollary 5.3, respectively, and observing how the map $i^G_\ast$ works yield that $RI_k$ indeed is the kernel of $i^G_\ast$. □

**Remark 5.7.**

1. In [Dou2], Douglas gave an explicit description of Verlinde algebra of $G$. In particular, he gave a list of generators of $I_k$ for each type of simple, connected, simply-connected and compact Lie groups in such a way that the number of generators is independent of the level $k$. Together with Theorem 5.6 and the description of the ring structure of the coefficient ring $KR^G_\ast(pt)$ given in Proposition A.9 (and which can also be deduced from Corollary 5.3), one can obtain the ring structure of $KR^G_\ast(G^-, \mathcal{A}^{k+h^\vee})$ explicitly.
2. By Theorem 5.6, the degree 0 part $KR^G_\ast(G^-, \mathcal{A}^{k+h^\vee}) \cong RR(G)/(RI_k \cap RR(G))$ gives the Real Verlinde algebra, the Grothendieck group of the isomorphism classes
of Real positive energy representations of the Real loop group $LG$, where the Real structure of $LG$ is given by

$$\sigma_{LG} : LG \to LG$$

$$\ell \mapsto \sigma_G \circ \ell \circ c$$

with $c$ meaning reflection on the unit circle.

(3) In [M1], FHT was proved using Segal’s spectral sequence (cf. [S1] for its definition), which was shown to collapse on the $E_2$-page, which in turn amounts to a free resolution of the Verlinde algebra. In fact Segal’s spectral sequence is a generalized form of Mayer-Vietoris sequence. The $G$-invariant open cover used in FHT consists of certain open subsets in $G$ which deformation retract to conjugacy classes $G$-equivariantly. In [F2], the twisted equivariant $KR$-homology of $G$ is computed in the case where $R(G, \mathbb{C}) = 0$, following the spectral sequence arguments in [M1]. The condition that there is no complex type representation is put in place just to ensure that the open cover is also invariant under the anti-involution on $G$, so that the Segal’s spectral sequence can be adapted easily in this Real context. It was found that the Real $E_2$-page is the tensor product of the free resolution of the Verlinde algebra in the proof of [M1] and the coefficient ring $KR_*(pt)$, by virtue of the equivariant Real formality. Due to the nullhomotopy of the free resolution of the Verlinde algebra, the Real spectral sequence also collapses on the Real $E_2$-page, whose homology is the tensor product of the Verlinde algebra and the coefficient ring. Thus in this case $KR_*(G^{-, A^{k+h'}}) \cong R_k(G) \otimes KR_*(pt)$, in agreement with Theorem 5.6.

**Appendix A.**

This Appendix is devoted to the background material for $KR$-theory and Real representation rings, which is culled directly from [F].

**A.1. $KR$-theory.**

**Definition A.1.**

1. A **Real space** is a pair $(X, \sigma_X)$ where $X$ is a topological space equipped with an involutive homeomorphism $\sigma_X$, i.e. $\sigma_X^2 = \text{Id}_X$. A **Real pair** is a pair $(X, Y)$ where $Y$ is a closed subspace of $X$ invariant under $\sigma_X$.

2. Let $\mathbb{R}^{p,q}$ be the Euclidean space $\mathbb{R}^{p+q}$ equipped with the involution which is identity on the first $q$ coordinates and negation on the last $p$-coordinates. Let $B^{p,q}$ and $S^{p,q}$ be the unit ball and sphere in $\mathbb{R}^{p,q}$ with the inherited involution.

3. A **Real vector bundle** (to be distinguished from the usual real vector bundle) over $X$ is a complex vector bundle $E$ over $X$ which itself is also a Real space with involutive homeomorphism $\sigma_E$ satisfying
   
   (a) $\sigma_X \circ p = p \circ \sigma_E$, where $p : E \to X$ is the projection map,
   
   (b) $\sigma_E$ maps $E_x$ to $E_{\sigma_X(x)}$ anti-linearly.

A **Quaternionic vector bundle** (to be distinguished from the usual quaternionic vector bundle) over $X$ is a complex vector bundle $E$ over $X$ equipped with an anti-linear lift $\sigma_E$ of $\sigma_X$ such that $\sigma_E^2 = -\text{Id}_E$. 
(4) A complex of Real vector bundles over the Real pair \((X, Y)\) is a complex of Real vector bundles over \(X\)

\[
0 \to E_1 \to E_2 \to \cdots \to E_n \to 0
\]

which is exact on \(Y\).

(5) Let \(X\) be a Real space. The ring \(KR(X)\) is the Grothendieck group of the isomorphism classes of Real vector bundles over \(X\), equipped with the usual product structure induced by tensor product of vector bundles over \(\mathbb{C}\). The relative \(KR\)-theory for a Real pair \(KR(X, Y)\) can be similarly defined using complexes of Real vector bundles over \((X, Y)\), modulo homotopy equivalence and addition of acyclic complexes (cf. \([S2]\)). In general, the graded \(KR\)-theory ring of the Real pair \((X, Y)\) is given by

\[
KR^*(X, Y) := \bigoplus_{q=0}^{7} KR^{-q}(X, Y),
\]

where

\[
KR^{-q}(X, Y) := KR(X \times B^{0,q}, X \times S^{0,q} \cup Y \times B^{0,q}).
\]

The ring structure of \(KR^*\) is extended from that of \(KR\), in a way analogous to the case of complex \(K\)-theory. The number of graded pieces, which is 8, is a result of Bott periodicity for \(KR\)-theory (cf. \([AO]\)).

Note that when \(\sigma_X = \text{Id}_X\), then \(KR(X) \cong KO(X)\). On the other hand, if \(X \times \mathbb{Z}_2\) is given the involution which swaps the two copies of \(X\), then \(KR(X \times \mathbb{Z}_2) \cong K(X)\). Also, if \(X\) is equipped with the trivial involution, then \(KR(X \times S^2, 0) \cong KSC(X)\), the Grothendieck group of homotopy classes of self-conjugate bundles over \(X\) (cf. \([AI]\)). In this way, it is natural to view \(KR\)-theory as a unifying thread of \(KO\)-theory, \(K\)-theory and \(KSC\)-theory.

On top of the Real structure, we may further add compatible group actions and define equivariant \(KR\)-theory.

**Definition A.2.**

1. A **Real G-space** \(X\) is a quadruple \((X, G, \sigma_X, \sigma_G)\) where a group \(G\) acts on \(X\) and \(\sigma_G\) is an involutive automorphism of \(G\) such that

\[\sigma_X(g \cdot x) = \sigma_G(g) \cdot \sigma_X(x).\]

2. A **Real G vector bundle** \(E\) over a Real G-space \(X\) is a Real vector bundle and a \(G\)-bundle over \(X\), and it is also a Real G-space.

3. In a similar spirit, one can define equivariant \(KR\)-theory \(KR^*_G(X, Y)\). Notice that the \(G\)-actions on \(B^{0,q}\) and \(S^{0,q}\) in the definition of \(KR^{-q}_G(X, Y)\) (cf. Equation (5)) are trivial.

**Definition A.3.**

1. Let \(K^*(+)\) be the complex \(K\)-theory of a point extended to a \(\mathbb{Z}_8\)-graded algebra over \(K^0(\text{pt}) \cong \mathbb{Z}\), i.e. \(K^*(+) \cong \mathbb{Z}[\beta] / \beta^4 - 1\). Here \(\beta \in K^{-2}(+)\), called the **Bott class**, is the class of the reduced canonical line bundle on \(\mathbb{C}\mathbb{P}^1 \cong S^2\).

2. Let \(\sigma_X^E\) be the map defined on (equivariant) vector bundles on \(X\) by \(\sigma_X^E := \sigma_X^* E\).

The involution induced by \(\sigma_X^*\) on \(K^*_G(X)\) is also denoted by \(\sigma_X^E\) for simplicity.
In the following proposition, we collect, for reader’s convenience, some basic results of KR-theory (cf. [Sc Section 2]), some of which are stated in the more general context of equivariant KR-theory.

**Proposition A.4.** (1) We have
\[KR^*(pt) \cong \mathbb{Z}[[\eta, \mu]]/(2\eta, \eta^3, \mu\eta, \mu^2 - 4),\]
where \(\eta \in KR^{-1}(pt), \mu \in KR^{-4}(pt)\) represents the reduced Hopf bundles of \(\mathbb{R}P^1\) and \(\mathbb{H}P^1\) respectively.

(2) Let \(c : KR^*_G(X) \to K^*_G(X)\) be the homomorphism which forgets the Real structure of Real vector bundles, and \(r : K^*_G(X) \to KR^*_G(X)\) be the realification map defined by \([E] \mapsto [E \oplus \sigma^*_G \sigma^*_X E]\), where \(\sigma^*_G\) means twisting the original G-action on E by \(\sigma_G\). Then we have the following relations
(a) \(c(1) = 1, c(\eta) = 0, c(\mu) = 2\beta^2\), where \(\beta \in K^{-2}(pt)\) is the Bott class,
(b) \(r(1) = 2, r(\beta) = \eta^2, r(\beta^2) = \mu, r(\beta^3) = 0\),
(c) \(r(xc(y)) = r(x)y, rc(x) = x + \sigma^*_G \sigma^*_X x\) and \(rc(y) = 2y\) for \(x \in K^*_G(X)\) and \(y \in KR^*_G(X)\), where \(K^*_G(X)\) is extended to a \(\mathbb{Z}_R\)-graded algebra by Bott periodicity.

**Proof.** (1) is given in [Sc Section 2]. The proof of (2) is the same as in the nonequivariant case, which is given in [AI]. \(\Box\)

**Definition A.5.** A Quaternionic G-vector bundle over a Real space \(X\) is a complex vector bundle \(E\) equipped with an anti-linear vector bundle endomorphism \(J\) on \(E\) such that \(J^2 = -\text{Id}_E\) and \(J(g \cdot v) = \sigma_G(g) \cdot J(v)\). Let \(KH^*_G(X)\) be the corresponding K-theory constructed using Quaternionic G-bundles over \(X\).

By generalizing the discussion preceding in [Sc Lemma 5.2] to the equivariant and graded setting, we define a natural transformation
\[t : KH^*_G(X) \to KR^*_G(X)\]
which sends
\[0 \to E_1 \xrightarrow{f} E_2 \to 0\]
to
\[0 \to \pi^*(\mathbb{H} \otimes \mathbb{C} E_1) \xrightarrow{g} \pi^*(\mathbb{H} \otimes \mathbb{C} E_2) \to 0.\]

Here
(1) \(E_i, i = 1, 2\) are equivariant Quaternionic vector bundles on \(X \times \mathbb{R}^{0,q}\) equipped with the Quaternionic structures \(J_{E_i}\),
(2) \(f\) is an equivariant Quaternionic vector bundle homomorphism which is an isomorphism outside \(X \times \{0\}\),
(3) \(\pi : X \times \mathbb{R}^{0,q} \to X \times \mathbb{R}^{0,q}\) is the projection map,
(4) \(\mathbb{H} \otimes \mathbb{C} E_i\) is the equivariant Real vector bundles equipped with the Real structure \(J \otimes J_{E_i}\),
(5) \(g\) is an equivariant Real vector bundle homomorphism defined by \(g(v, w \otimes e) = (v, vw \otimes f(e))\).

The discussion in the last section of [AS] can be extended to the equivariant setting and yields
Proposition A.6. $t$ is an isomorphism.

A.2. The Real representation ring $RR(G)$. In this Section, we review the basics of Real representations elaborated in [E].

Definition A.7. (1) A Real representation $V$ of $G$ is a finite-dimensional complex representation of $G$ equipped with an anti-linear involution $\sigma_V$ such that $\sigma_V(g \cdot v) = \sigma_G(g) \cdot \sigma_V(v)$. Similarly a Quaternionic representation is one equipped with an anti-linear endomorphism $J_V$ such that $J_V^2 = -\Id_V$ and $J_V(g \cdot v) = \sigma_G(g) \cdot J_V(v)$. For $F = \mathbb{R}$ or $\mathbb{H}$, a morphism between $V$ and $W \in \mathcal{R}ep_F(G)$ is a linear transformation from $V$ to $W$ which commutes with $G$ and respects both $\sigma_V$ and $\sigma_W$. Let $\mathcal{R}ep_F(G)$ (resp. $\mathcal{R}ep_\mathbb{H}(G)$) be the category of Real (resp. Quaternionic) representations of $G$. The Real (resp. Quaternionic) representation group of $G$, denoted by $RR(G)$ (resp. $RH(G)$) is the Grothendieck group of $\mathcal{R}ep_F(G)$ (resp. $\mathcal{R}ep_\mathbb{H}(G)$).

(2) Let $V$ be an irreducible Real (resp. Quaternionic) representation of $G$. Its commuting field is defined to be $\text{Hom}_G(V, V)^{\sigma_V}$, which is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $RR(G, F)$ (resp. $RH(G, F)$) be the abelian group generated by the isomorphism classes of irreducible Real (resp. Quaternionic) representations with $F$ as the commuting field.

(3) Let $R(G, \mathbb{C})$ be the abelian group generated by the isomorphism classes of those irreducible complex representations $V$ satisfying $V \ncong \sigma_G^* V$.

(4) Let $V$ be a $G$-representation. We use $\sigma_G^* V$ to denote the $G$-representation with the same underlying vector space where the $G$-action is twisted by $\sigma_G$, i.e. $\rho_{\sigma_G^* V}(g)v = \rho_V(\sigma_G(g))v$. We will use $\sigma_G^{\sigma_V}$ to denote the map on $R(G)$ defined by $[V] \mapsto [\sigma_G^* V]$.

The following Proposition gives characterizations of Real and Quaternionic representations of various types. See Proposition [43][2] for alternative characterizations for $RR(G, \mathbb{R})$ and $RH(G, \mathbb{R})$.

Proposition A.8. [E] Propositions 2.16 and 2.18

1. Let $V$ be an irreducible Real representation of $G$.
   - (a) The commuting field of $V$ is isomorphic to $\mathbb{R}$ iff $V$ is an irreducible complex representation and there exists $f \in \text{Hom}_G(V, \sigma_G^* V)$ such that $f^2 = \Id_V$.
   - (b) The commuting field of $V$ is isomorphic to $\mathbb{C}$ iff $V \cong W \oplus \sigma_G^* W$ as complex $G$-representations, where $W$ is an irreducible complex $G$-representation and $W \ncong \sigma_G^* W$, and $\sigma_V(w_1, w_2) = (w_2, w_1)$.
   - (c) The commuting field of $V$ is isomorphic to $\mathbb{H}$ iff $V \cong W \oplus \sigma_G^* W$ as complex $G$-representations, where $W$ is an irreducible complex $G$-representation and there exists $f \in \text{Hom}_G(W, \sigma_G^* W)$ such that $f^2 = \Id_W$, and $\sigma_V(w_1, w_2) = (w_2, w_1)$.

2. Let $V$ be an irreducible Quaternionic representation of $G$.
   - (a) The commuting field of $V$ is isomorphic to $\mathbb{H}$ iff $V \cong W \oplus \sigma_G^* W$ as complex $G$-representations, where $W$ is an irreducible complex $G$-representation and there exists $f \in \text{Hom}_G(W, \sigma_G^* W)$ such that $f^2 = \Id_W$ with $J(w_1, w_2) = (-w_2, w_1)$.
   - (b) The commuting field of $V$ is isomorphic to $\mathbb{C}$ iff $V \cong W \oplus \sigma_G^* W$ as complex $G$-representations, where $V$ is an irreducible complex $G$-representation and $W \ncong \sigma_G^* W$ with $J(w_1, w_2) = (-w_2, w_1)$. 


(c) The commuting field of $V$ is isomorphic to $\mathbb{R}$ iff $V$ is an irreducible complex $G$-representation and there exists $f \in \text{Hom}_G(V, \sigma_G^*V)$ such that $f^2 = -\text{Id}_V$.

**Proposition A.9.** [F] Proposition 3.3] The map

$$f : (RR(G, \mathbb{R}) \oplus RH(G, \mathbb{R})) \otimes KR^*(pt) \oplus r(R(G, \mathbb{C}) \otimes K^*(+)) \rightarrow KR_G^*(pt),$$

$$\rho_1 \otimes x_1 \oplus \rho_2 \otimes x_2 \oplus r(\rho_3 \otimes \beta^i) \mapsto \rho_1 \cdot x_1 + \rho_2 \cdot x_2 + \rho_3 \cdot \beta^i + \sigma_G \rho_3 \cdot (-1)^i \beta^i$$

is an isomorphism of graded rings. Here elements in $RR(G, \mathbb{R})$ and $RH(G, \mathbb{R})$ are assigned with degree 0 and $-4$ respectively.

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