DARBOUX TRANSFORMATION AND SOLITON SOLUTIONS
OF THE SEMI-DISCRETE MASSIVE THIRRING MODEL

TAO XU AND DMITRY E. PELINOVSKY

ABSTRACT. A one-fold Darboux transformation between solutions of the semi-discrete massive Thirring model is derived using the Lax pair and dressing methods. This transformation is used to find the exact expressions for soliton solutions on zero and nonzero backgrounds. It is shown that the discrete solitons have the same properties as solitons of the continuous massive Thirring model.

1. INTRODUCTION

The massive Thirring model (MTM) in laboratory coordinates is an example of the nonlinear Dirac equation arising in two-dimensional quantum field theory [23], optical Bragg gratings [7], and diatomic chains with periodic couplings [1]. This model received much of attention because of its integrability [17] which was used to study the inverse scattering [13–16, 21, 27, 28], soliton solutions [2–4, 20], spectral and orbital stability of solitons [6, 12, 22], and construction of rogue waves [8].

Several integrable semi-discretizations of the MTM in characteristic coordinates were proposed in the literature [18, 19, 24–26] by discretizing one of the two characteristic coordinates. These semi-discretizations are not relevant for the time-evolution problem related to the MTM in laboratory coordinates. It was only recently [11] when the integrable semi-discretization of the MTM in laboratory coordinates was derived. The corresponding semi-discrete MTM is written as the following system of three coupled equations:

\[
\begin{align*}
\frac{4i}{h} \frac{dU_n}{dt} + Q_{n+1} + Q_n + \frac{2i}{h} (R_{n+1} - R_n) + U_n^2 (\bar{R}_n + \bar{R}_{n+1}) & = 0, \\
-2i \frac{dQ_n}{dx} + U_n - |U_n|^2 Q_n & = 0, \\
R_{n+1} + R_n - 2U_n + \frac{i}{2} U_n^2 (R_{n+1} - R_n) & = 0,
\end{align*}
\]

where \( h \) is the lattice spacing of the spatial discretization and \( n \) is the discrete lattice variable. \( \bar{R}_n \) and \( \bar{Q}_n \) denote the complex conjugate of \( R_n \) and \( Q_n \) respectively. Only the first equation of the system (1) represents the time evolution problem, whereas the other two equations represent the constraints which define components of \( \{R_n\}_{n \in \mathbb{Z}} \) and \( \{Q_n\}_{n \in \mathbb{Z}} \) in terms of \( \{U_n\}_{n \in \mathbb{Z}} \) instantaneously in time \( t \).

In the continuum limit \( h \to 0 \), the slowly varying solutions to the system (1) can be represented by

\[
U_n(t) = U(x = hn, t), \quad R_n(t) = R(x = hn, t), \quad Q_n(t) = Q(x = nh, t),
\]

where the continuous variables satisfy the following three equations:

\[
\begin{align*}
2i \frac{\partial U}{\partial t} + i \frac{\partial R}{\partial x} + Q + U^2 \bar{R} - U(|Q|^2 + |R|^2) & = 0, \\
-2i \frac{\partial Q}{\partial x} + U - |U|^2 Q & = 0, \\
R - U & = 0.
\end{align*}
\]
The system (2) in variables $U(x, t) = u(x, t - x)$ and $Q(x, t) = v(x, t - x)$ yields the continuous MTM system in the form:

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{du}{dt} + \frac{\partial u}{\partial x} + v = |v|^2 u, \\
\frac{dv}{dt} - \frac{\partial v}{\partial x} + u = |u|^2 v.
\end{array} \right.
\end{align*}
$$

(3)

It is shown in [11] that the semi-discrete MTM system (1) is the compatibility condition

$$
\frac{d}{dt} N_n(\lambda) = P_{n+1}(\lambda)N_n(\lambda) - N_n(\lambda)P_n(\lambda),
$$

(4)
of the following Lax pair of two linear equations:

(5a) \quad \Phi_{n+1}(\lambda) = N_n(\lambda)\Phi_n(\lambda), \quad N_n(\lambda) = \left( \begin{array}{cc} \lambda + \frac{2i}{h\lambda} & \frac{1 + \frac{h}{2}|U_n|^2}{1 - \frac{h}{2}|U_n|^2} \\
\frac{2U_n}{1 - \frac{h}{2}|U_n|^2} & \lambda - \frac{1 + \frac{h}{2}|U_n|^2}{1 - \frac{h}{2}|U_n|^2} \end{array} \right),

(5b) \quad \frac{d}{dt} \Phi_n(\lambda) = P_n(\lambda)\Phi_n(\lambda), \quad P_n(\lambda) = \frac{i}{2} \left( \begin{array}{cc} \lambda^2 - |R_n|^2 & \lambda R_n - Q_n\lambda^{-1} \\
\lambda R_n - Q_n\lambda^{-1} |Q_n|^2 - \lambda^{-2} \end{array} \right),

where $\Phi_n(\lambda) \in \mathbb{C}^2$ is defined for $n \in \mathbb{Z}$ and $\lambda$ is a spectral parameter.

Because the passage from the discrete system (1) to the continuum limit (3) involves the change of the coordinates $U(x, t) = u(x, t - x)$ and $Q(x, t) = v(x, t - x)$, the initial-value problem for the semi-discrete MTM system (1) does not represent the initial-value problem for the continuous MTM system (3) in time variable $t$. In addition, numerical explorations of the semi-discrete system (1) are challenging because the last two constraints in the system (1) may lead to appearance of bounded but non-decaying sequences $\{R_n\}_{n \in \mathbb{Z}}$ and $\{Q_n\}_{n \in \mathbb{Z}}$ in response to the bounded and decaying sequence $\{U_n\}_{n \in \mathbb{Z}}$. On the other hand, since the semi-discrete MTM system (1) has the Lax pair of linear equations (5), it is integrable by the inverse scattering transform method which implies existence of infinitely many conserved quantities, exact solutions, transformations between different solutions, and reductions to other integrable equations [10]. These properties of integrable systems were not explored for the semi-discrete MTM system (1) in the previous work [11].

The purpose of this work is to derive the one-fold Darboux transformation between solutions of the semi-discrete MTM system (1). We employ the Darboux transformation in order to generate one-soliton and two-soliton solutions on zero background in the exact analytical form. By looking at the continuum limit $h \to 0$, we show that the discrete solitons share many properties with their continuous counterparts. We also construct one-soliton solutions on a nonzero constant background. Further properties of the model, e.g. conserved quantities and solvability of the initial-value problem, are left for further studies.

The following theorem represents the main result of this work.

**Theorem 1.** Let $\Phi_n(\lambda_1) = (f_n, g_n)^T$ be a nonzero solution of the Lax pair (5) with $\lambda = \lambda_1$ and $(U_n, R_n, Q_n)$ be a solution of the semi-discrete MTM system (1). Another solution of the semi-discrete MTM system (1) is given by

$$
\begin{align*}
U_n^{[1]} &= -\frac{2i(\bar{\lambda}_1|f_n|^2 + \lambda_1|g_n|^2)U_n - h|\lambda_1|^2(\lambda_1|f_n|^2 + \bar{\lambda}_1|g_n|^2)U_n + 2i(\lambda_1^2 - \bar{\lambda}_1^2)|f_n\bar{g}_n|}{2i(\lambda_1|f_n|^2 + \lambda_1|g_n|^2) - h|\lambda_1|^2(\lambda_1|f_n|^2 + \bar{\lambda}_1|g_n|^2) + h(\lambda_1^2 - \bar{\lambda}_1^2)|f_n\bar{g}_n|U_n}, \\
R_n^{[1]} &= -\frac{-\lambda_1|f_n|^2 + \lambda_1|g_n|^2}{\lambda_1|f_n|^2 + \lambda_1|g_n|^2}R_n + \lambda_1(\lambda_1^2 - \bar{\lambda}_1^2)|f_n\bar{g}_n|, \\
Q_n^{[1]} &= -\frac{|\lambda_1|^2(\lambda_1|f_n|^2 + \lambda_1|g_n|^2)Q_n + (\lambda_1^2 - \bar{\lambda}_1^2)|f_n\bar{g}_n|}{|\lambda_1|^2(\lambda_1|f_n|^2 + \lambda_1|g_n|^2)}.
\end{align*}
$$

(6a)\quad (6b)\quad (6c)
Theorem 1 is proven in Section 2 using the Lax pair (5) and the dressing methods. One-soliton and two-soliton solutions on zero background are obtained in Section 3. One-soliton solutions on a nonzero constant background are constructed in Section 4. Both zero and nonzero constant backgrounds are modulationally stable in the evolution of the semi-discrete MTM system (1). A summary and further directions are discussed in Section 5.

2. PROOF OF THE ONE-FOLD DARBOUX TRANSFORMATION

The one-fold Darboux transformation takes an abstract form (see, e.g., [9]):

\[ \Phi^{[1]}(\lambda) = T(\lambda)\Phi(\lambda), \]

where \( T(\lambda) \) is the Darboux matrix, \( \Phi(\lambda) \) is a solution to the system (5), whereas \( \Phi^{[1]}(\lambda) \) is a solution of the transformed system

\[ \Phi^{[1]}_{n+1}(\lambda) = N^{[1]}_n(\lambda)\Phi^{[1]}_n(\lambda), \]

\[ \frac{d}{dt}\Phi^{[1]}_n(\lambda) = P^{[1]}_n(\lambda)\Phi^{[1]}_n(\lambda), \]

with \( N^{[1]}_n(\lambda) \) and \( P^{[1]}_n(\lambda) \) having the same form as \( N_n(\lambda) \) and \( P_n(\lambda) \) except that the potentials \( (U_n, Q_n, R_n) \) are replaced by \( (U^{[1]}_n, Q^{[1]}_n, R^{[1]}_n) \). By substituting (7) into the linear equations (8) and using the linear equations (5), we obtain the following system of equations for the Darboux matrix \( T(\lambda) \):

\[ T_{n+1}(\lambda)N_n(\lambda) = N^{[1]}_n(\lambda)T_n(\lambda), \]

\[ \frac{d}{dt}T_n(\lambda) + T_n(\lambda)P_n(\lambda) = P^{[1]}_n(\lambda)T_n(\lambda). \]

Since \( N_n(\lambda) \) and \( P_n(\lambda) \) in (5) contain both the positive and negative powers of \( \lambda \), we take the one-fold Darboux matrix \( T(\lambda) \) in the following form (used in [9] in the context of the semi-discrete nonlocal nonlinear Schrödinger equation):

\[ T_n(\lambda; t) = \begin{pmatrix}
\sum_{l=-1}^{1} a_{l,n}(t)\lambda^l & \sum_{l=-1}^{1} b_{l,n}(t)\lambda^l \\
\sum_{l=-1}^{1} c_{l,n}(t)\lambda^l & \sum_{l=-1}^{1} d_{l,n}(t)\lambda^l
\end{pmatrix},
\]

where the coefficients are to be determined. Before further work, we shall simplify the Darboux matrix in (10) by using some constraints following from the system (9). Expanding Eq. (9b) in powers of \( \lambda \) and equating the coefficients of \( \lambda^3 \) and \( \lambda^{-3} \) to 0, we verify that

\[ b_{1,n} = c_{1,n} = b_{-1,n} = c_{-1,n} = 0. \]

Collecting coefficients of other powers of \( \lambda \) yields the following system of equations:

\[ \lambda^2 : a_{1,n}R_n - b_{0,n} - d_{1,n}R_n^{[1]} = 0, \]
\[ \lambda^2 : a_{1,n}R_n^{[1]} - c_{0,n} - d_{1,n}R_n = 0, \]
\[ \lambda^{-2} : a_{-1,n}Q_n + b_{0,n} - d_{-1,n}Q_n^{[1]} = 0, \]
\[ \lambda^{-2} : a_{-1,n}Q_n^{[1]} + c_{0,n} - d_{-1,n}Q_n = 0, \]
\[ \lambda^1 : a_{0,n}R_n - d_{0,n}R_n^{[1]} = 0, \]
\[ \lambda^1 : -a_{0,n}R_n^{[1]} + d_{0,n}R_n = 0, \]
\[ \lambda^1 : (|R_n^{[1]}|^2 - |R_n|^2)a_{1,n} + R_n b_{0,n} - R_n^{[1]} c_{0,n} - 2\frac{da_{1,n}}{dt} = 0. \]
allows us to find simultaneously both the coefficients of $T$

In order to determine difficult to compute relations between the new and old potentials from these four equations. Therefore, we will

Let Lemma 2.

It follows from Eqs. (12e), (12f), (12i) and (12j) that if $(|Q[^1]|, |R[^1]|) \neq (|Q|, |R|)$, then $a_{0,n} = d_{0,n} = 0$, after which Eqs. (12m) and (12n) are identically satisfied. Solving Eqs. (12a), (12b), (12c), and (12d) yields

Plugging (13) into Eqs. (12g) and (12l) gives

It follows from Eqs. (13a) and (13b) that $b_{0,n} = a_{1,n}R_n - d_{1,n}R_n[^1] = d_{-1,n}Q_n[^1] - a_{-1,n}Q_n$.

Expanding Eq. (9a) in powers of $\lambda$ and equating the coefficients of $\lambda^2$ and $\lambda^{-2}$ to 0, we verify that

Combining Eqs. (14) and (15), we conclude that $a_{1,n}(t)$ and $d_{-1,n}(t)$ are constants both in $t$ and $n$. For normalization purposes, we set $a_{1,n}(t) = 1$ and $d_{-1,n}(t) = |\lambda|^2$. We also re-enumerate the remaining coefficients as follows: $a_{-1,n}(t) = a_n(t)|\lambda|^2$, $b_{0,n}(t) = b_n(t)$, $c_{0,n}(t) = c_n(t)$, and $d_{1,n}(t) = d_n(t)$. The Darboux matrix $T_n[^1]$ given previously by (10) is now rewritten in the simplified form:

In order to determine $a_n(t)$, $b_n(t)$, $c_n(t)$, and $d_n(t)$, we use the symmetry properties of the Lax pair [5]. This allows us to find simultaneously both the coefficients of $T(\lambda)$ and the transformations between the potentials $(U, Q, R)$ and $(U[^1], Q[^1], R[^1])$.

**Lemma 2.** Let $\Phi(\lambda_1) = (f, g)^T$ be a nonzero solution of the Lax pair [5] at $\lambda = \lambda_1$. Then,

$$|\Phi(\lambda_1)|_n = \Omega_n \left(-\frac{g_n}{f_n}\right), \quad |\Phi(-\lambda_1)|_n = (-1)^n \left(-\frac{f_n}{g_n}\right), \quad |\Phi(-\lambda_1)|_n = (-1)^n \Omega_n \left(\frac{\bar{g}_n}{f_n}\right),$$

where

$$\Omega_n = \frac{\prod_{j=0}^{n-1} f_{-j}}{\prod_{j=0}^{n-1} g_{-j}},$$

$$f_n = \frac{\partial a_n}{\partial t} + \frac{\partial a_{n-1}}{\partial t} + b_{n-1} a_n,$$

$$g_n = \frac{\partial a_n}{\partial t} + \frac{\partial a_{n-1}}{\partial t} + b_{n-1} a_n.$$
are solutions of the Lax pair (8) at \( \lambda = \bar{\lambda}_1 \), \( \lambda = -\lambda_1 \), and \( \lambda = -\bar{\lambda}_1 \) respectively, where \( \Omega_n(t) \) satisfies:

\[ \begin{align*}
(18a) & \quad \Omega_{n+1} = -S_n \Omega_n, \quad S_n := \frac{1 + \frac{i}{2} h |U_n|^2}{1 - \frac{i}{2} h |U_n|^2}, \\
(18b) & \quad \frac{d\Omega_n}{dt} = M_n \Omega_n, \quad M_n := \frac{i}{2} \left( \bar{\lambda}_1^2 - \lambda_1^2 + |Q_n|^2 - |R_n|^2 \right).
\end{align*} \]

**Proof.** It follows from (5a) that components of \( \Phi(\lambda_1) \) satisfy the system of difference equations:

\[ \begin{align*}
(19) & \quad \begin{cases}
    f_{n+1} = \left( \lambda_1 + \frac{2i}{n \lambda_1} S_n \right) f_n + \frac{2U_n}{1 - \frac{i}{2} h |U_n|^2} g_n, \\
    g_{n+1} = -\frac{2U_n}{1 - \frac{i}{2} h |U_n|^2} f_n + \left( \frac{2i}{n \lambda_1} - \lambda_1 S_n \right) g_n,
\end{cases}
\end{align*} \]

whereas components of \( \Phi(\bar{\lambda}_1) \) satisfy the system of difference equations:

\[ \begin{align*}
(20) & \quad \begin{cases}
    \Omega_{n+1} g_{n+1} = \left( \bar{\lambda}_1 + \frac{2i}{n \lambda_1} S_n \right) \Omega_n g_n - \frac{2U_n}{1 - \frac{i}{2} h |U_n|^2} \Omega_n \tilde{f}_n, \\
    \Omega_{n+1} \tilde{f}_{n+1} = -\frac{2U_n}{1 - \frac{i}{2} h |U_n|^2} \Omega_n \tilde{g}_n + \left( \frac{2i}{n \lambda_1} - \bar{\lambda}_1 S_n \right) \Omega_n \tilde{f}_n.
\end{cases}
\]

Dividing (20) by \( \Omega_{n+1} \) and taking the complex conjugation yields (19) if and only if \( \Omega \) satisfies the difference equation (18a). Similarly, it follows from (5b) that components of \( \Phi(\lambda_1) \) satisfy the time evolution equations:

\[ \begin{align*}
(21) & \quad \begin{cases}
    \frac{df_n}{dt} = \frac{i}{2} \left( \lambda_1^2 - |R_n|^2 \right) f_n + \left( \lambda_1 R_n - \lambda_1^{-1} Q_n \right) g_n, \\
    \frac{dg_n}{dt} = \frac{i}{2} \left( \lambda_1 R_n - \lambda_1^{-1} Q_n \right) f_n + (-\lambda_1^2 + |Q_n|^2) g_n,
\end{cases}
\end{align*} \]

whereas components of \( \Phi(\bar{\lambda}_1) \) satisfy the time evolution equations:

\[ \begin{align*}
(22) & \quad \begin{cases}
    \frac{d\Omega_n}{dt} \tilde{g}_n + \Omega_n \frac{dg_n}{dt} = \frac{i}{2} \left( \bar{\lambda}_1^2 - |R_n|^2 \right) \Omega_n \tilde{g}_n - (\bar{\lambda}_1 R_n - \bar{\lambda}_1^{-1} Q_n) \Omega_n \tilde{f}_n, \\
    \frac{d\Omega_n}{dt} \tilde{f}_n + \Omega_n \frac{df_n}{dt} = \frac{i}{2} \left( -\bar{\lambda}_1 R_n + \bar{\lambda}_1^{-1} Q_n \right) \Omega_n \tilde{g}_n + (-\bar{\lambda}_1^2 + |Q_n|^2) \Omega_n \tilde{f}_n.
\end{cases}
\end{align*} \]

Taking the complex conjugation of (22) yields (21) if and only if \( \Omega \) satisfies the time evolution equation (18b). The other two solutions in (17) are obtained by the symmetry of the system (5) with respect to the reflection \( \lambda \to -\lambda \).

**Lemma 3.** Let \( \Phi(\lambda_1) = (f, g)^T \) be in the kernel of the Darboux matrix \( T(\lambda_1) \) and \( \Phi(\bar{\lambda}_1) = \Omega(-\tilde{g}, \tilde{f})^T \) be in the kernel of \( T(-\bar{\lambda}_1) \). Then, the coefficients of \( T(\lambda) \) in (16) are given by

\[ \begin{align*}
(23) & \quad a_n = -\frac{\Delta_n}{\Delta_n}, \quad b_n = -\frac{\left( \lambda_1^2 - \bar{\lambda}_1^2 \right) f_n g_n}{\Delta_n}, \quad c_n = \frac{(\lambda_1^2 - \bar{\lambda}_1^2) \tilde{f}_n \tilde{g}_n}{\Delta_n}, \quad d_n = -\frac{\Delta_n}{\Delta_n},
\end{align*} \]

where \( \Delta_n := |f_n|^2 + \lambda_1 |g_n|^2 \). Furthermore, \( \Phi(-\lambda_1) \) and \( \Phi(-\bar{\lambda}_1) \) in (17) are in the kernel of \( T(-\lambda_1) \) and \( T(-\bar{\lambda}_1) \) respectively.

**Proof.** We rewrite the linear equations for \( T(\lambda_1)\Phi(\lambda_1) = 0 \) and \( T(-\bar{\lambda}_1)\Phi(-\bar{\lambda}_1) = 0 \) in the following explicit form:

\[ \begin{align*}
(24) & \quad \begin{cases}
    (\lambda_1 + a_n \bar{\lambda}_1) f_n + b_n g_n = 0, \\
    c_n f_n + (d_n \lambda_1 + \bar{\lambda}_1) g_n = 0, \\
    -(\bar{\lambda}_1 + a_n \lambda_1) \tilde{g}_n + b_n \tilde{f}_n = 0, \\
    -c_n \tilde{g}_n + (d_n \lambda_1 + \bar{\lambda}_1) \tilde{f}_n = 0,
\end{cases}
\end{align*} \]

where the scalar factor \( \Omega \) has been canceled out. Solving the linear system (24) with Cramer’s rule yields (23). Then, it follows from (16) and (23) that \( T_n(\lambda) \) can be written in the form:

\[ (25) T_n(\lambda) = \frac{(\lambda_1^2 - \bar{\lambda}_1^2) (\lambda^2 - \bar{\lambda}_1^2)}{2 \lambda \Delta_n} T_n(\lambda), \]
where
\[
\hat{T}_n(\lambda) = \frac{1}{\lambda - \lambda_1} \left( \frac{\bar{g}_n}{\bar{f}_n} \right) \left( \begin{array}{cc} f_n & -f_n \\ g_n & -g_n \end{array} \right) + \frac{1}{\lambda + \lambda_1} \left( \begin{array}{cc} -\bar{g}_n \frac{f_n}{\bar{f}_n} & 0 \\ 0 & -\bar{g}_n \frac{f_n}{\bar{f}_n} \end{array} \right) ( g_n f_n ) \\
+ \frac{1}{\lambda - \lambda_1} \left( \begin{array}{cc} f_n & -g_n \\ -\bar{g}_n & \bar{f}_n \end{array} \right) + \frac{1}{\lambda + \lambda_1} \left( \begin{array}{cc} f_n & -\bar{g}_n \\ -\bar{g}_n & \bar{f}_n \end{array} \right) ( g_n f_n ).
\]

It follows from (25) that
\[
T_n(\lambda_1) \left( \begin{array}{c} f_n \\ g_n \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \quad T_n(-\lambda_1) \left( \begin{array}{c} -f_n \\ g_n \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \\
T_n(\bar{\lambda}_1) \left( \begin{array}{c} -\bar{g}_n \\ \bar{f}_n \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \quad T_n(-\bar{\lambda}_1) \left( \begin{array}{c} \bar{g}_n \\ \bar{f}_n \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right),
\]
hence \(T(\pm \lambda_1)\Phi(\pm \lambda_1) = 0\) and \(T(\pm \bar{\lambda}_1)\Phi(\pm \bar{\lambda}_1) = 0\). \(\Box\)

**Lemma 4.** Let the Darboux matrix \(T(\lambda)\) be in the form (16) with the coefficients given by Eqs. (23). Then, the determinant of \(T(\lambda)\) is given by
\[
\det T_n(\lambda) = \frac{\Delta_n}{\Delta_n} \left( \lambda^2 - \lambda_1^2 \right) \left( \lambda^2 - \bar{\lambda}_1^2 \right).
\]

**Proof:** Expanding \(\det T_n(\lambda)\) given by (16) yields
\[
\det T_n(\lambda) = d_n \lambda^2 + a_n d_n |\lambda_1|^2 - b_n c_n + |\lambda_1|^2 + a_n |\lambda_1|^4 \lambda^{-2}.
\]
Since \(\pm \lambda_1\) and \(\pm \bar{\lambda}_1\) are the roots of \(\det T(\lambda)\), we obtain (26). Alternatively, substituting (23) into (27) yields (26). \(\Box\)

For \(\lambda \neq \pm \lambda_1\) and \(\lambda \neq \pm \bar{\lambda}_1\), we define
\[
adT_n(\lambda) = \det T_n(\lambda)[T_n(\lambda)]^{-1} = \left( \begin{array}{cc} d_n \lambda + \frac{|\lambda_1|^2}{\lambda} & -b_n \\
-c_n & \lambda + a_n \frac{|\lambda_1|^2}{\lambda} \end{array} \right),
\]
and obtain \(adT_n(\lambda)\) from (16) and (23) in the form:
\[
adT_n(\lambda) = \frac{(\lambda^2 - \lambda_1^2)(\lambda^2 - \bar{\lambda}_1^2)}{2\lambda \Delta_n} \hat{T}_n(\lambda),
\]
where
\[
ad\hat{T}_n(\lambda) = \frac{1}{\lambda - \lambda_1} \left( \begin{array}{cc} f_n & -g_n \\ \bar{g}_n & -\bar{f}_n \end{array} \right) + \frac{1}{\lambda + \lambda_1} \left( \begin{array}{cc} -\bar{g}_n \frac{f_n}{\bar{f}_n} & 0 \\ 0 & -\bar{g}_n \frac{f_n}{\bar{f}_n} \end{array} \right) ( g_n f_n ) \\
+ \frac{1}{\lambda - \lambda_1} \left( \begin{array}{cc} f_n & \bar{g}_n \\ \bar{g}_n & -\bar{f}_n \end{array} \right) + \frac{1}{\lambda + \lambda_1} \left( \begin{array}{cc} f_n & -\bar{g}_n \\ -\bar{g}_n & \bar{f}_n \end{array} \right) ( g_n f_n ).
\]

New potentials \(N_{n1}^{[1]}(\lambda)\) and \(P_{n1}^{[1]}(\lambda)\) are derived from Eqs. (9) by using the Darboux matrix \(T(\lambda)\). Assuming \(\lambda \neq \pm \lambda_1\) and \(\lambda \neq \pm \bar{\lambda}_1\), we obtain from (9) and (28) that
\[
N_{n1}^{[1]}(\lambda) = \frac{1}{\det T_n(\lambda)} T_{n+1}(\lambda) N_n(\lambda) adT_n(\lambda)
\]
(30) \[
= -\frac{\lambda}{2\Delta_n} T_{n+1}(\lambda) N_n(\lambda) ad\hat{T}_n(\lambda)
\]
and
\[
P_{n1}^{[1]}(\lambda) = \frac{1}{\det T_n(\lambda)} \left[ \frac{d}{dt} T_n(\lambda) + T_n(\lambda) P_n(\lambda) \right] adT_n(\lambda)
\]
Substituting this expression into (30), we finally obtain

\[ N_n(\lambda) \left( \frac{f_n}{g_n} \right) = \left( \frac{f_{n+1}}{g_{n+1}} \right) + (\lambda - \lambda_1) \left( \begin{array}{cc} 1 - \frac{2i}{\hbar \lambda \lambda_1} S_n & 0 \\ \frac{2i}{\hbar \lambda \lambda_1} - S_n & 0 \end{array} \right) \left( \begin{array}{c} f_n \\ g_n \end{array} \right), \]

where the expressions (26) and (29) have been used.

First, we compute the products in the right-hand side of Eq. (30). By Lemma 4 and direct computations, we obtain

\begin{align*}
(32a) \quad & N_n(\lambda) \left( \frac{f_n}{g_n} \right) = \left( \frac{f_{n+1}}{g_{n+1}} \right) + (\lambda - \lambda_1) \left( \begin{array}{cc} 1 - \frac{2i}{\hbar \lambda \lambda_1} S_n & 0 \\ \frac{2i}{\hbar \lambda \lambda_1} - S_n & 0 \end{array} \right) \left( \begin{array}{c} f_n \\ g_n \end{array} \right), \\
(32b) \quad & N_n(\lambda) \left( \frac{f_n}{-g_n} \right) = \left( \frac{-f_{n+1}}{g_{n+1}} \right) + (\lambda + \lambda_1) \left( \begin{array}{cc} 1 + \frac{2i}{\hbar \lambda \lambda_1} S_n & 0 \\ \frac{2i}{\hbar \lambda \lambda_1} - S_n & 0 \end{array} \right) \left( \begin{array}{c} f_n \\ -g_n \end{array} \right), \\
(32c) \quad & N_n(\lambda) \left( \frac{-\bar{g}_n}{f_n} \right) = -S_n \left( \frac{-\bar{g}_{n+1}}{-f_{n+1}} \right) + (\lambda - \bar{\lambda}_1) \left( \begin{array}{cc} 1 - \frac{2i}{\hbar \lambda \lambda_1} S_n & 0 \\ \frac{2i}{\hbar \lambda \lambda_1} - S_n & 0 \end{array} \right) \left( \begin{array}{c} -\bar{g}_n \\ f_n \end{array} \right), \\
(32d) \quad & N_n(\lambda) \left( \frac{-\bar{g}_n}{f_n} \right) = S_n \left( \frac{-\bar{g}_{n+1}}{-f_{n+1}} \right) + (\lambda + \bar{\lambda}_1) \left( \begin{array}{cc} 1 + \frac{2i}{\hbar \lambda \lambda_1} S_n & 0 \\ \frac{2i}{\hbar \lambda \lambda_1} - S_n & 0 \end{array} \right) \left( \begin{array}{c} -\bar{g}_n \\ f_n \end{array} \right),
\end{align*}

where \( S_n \) is defined in Eq. (18a). By using this table, we compute the first product in (30):

\[
N_n(\lambda) \text{ad} \tilde{T}_n(\lambda) = \frac{1}{\lambda - \lambda_1} \left( \frac{f_{n+1}}{g_{n+1}} \right) \left( -\bar{f}_n \right) \left( \bar{g}_n \right) + \frac{1}{\lambda + \lambda_1} \left( \frac{-f_{n+1}}{g_{n+1}} \right) \left( \bar{f}_n \right) \left( \bar{g}_n \right)
\]
\[
+ \frac{1}{\lambda - \lambda_1} S_n \left( \frac{g_{n+1}}{-f_{n+1}} \right) \left( g_n \right) \left( f_n \right) + \frac{1}{\lambda + \lambda_1} S_n \left( \frac{g_{n+1}}{f_{n+1}} \right) \left( g_n \right) \left( -f_n \right)
\]
\[
+ \frac{4i}{\hbar \lambda |\lambda_1|^2} \left( S_n \Delta_n \right) \left( 0 \right) \left( -\Delta_n \right).
\]

By Lemma 3 and direct computations, we obtain

\begin{align*}
(33a) \quad & T_n(\lambda) \left( \frac{f_n}{g_n} \right) = (\lambda - \lambda_1) \left( \begin{array}{cc} 1 - \frac{\lambda}{\lambda_1} & 0 \\ \frac{\lambda}{\lambda_1} - \frac{\lambda}{\lambda_1} & 0 \end{array} \right) \left( \begin{array}{c} f_n \\ g_n \end{array} \right), \\
(33b) \quad & T_n(\lambda) \left( \frac{f_n}{-g_n} \right) = (\lambda + \lambda_1) \left( \begin{array}{cc} 1 + \frac{\lambda}{\lambda_1} & 0 \\ \frac{\lambda}{\lambda_1} + \frac{\lambda}{\lambda_1} & 0 \end{array} \right) \left( \begin{array}{c} f_n \\ -g_n \end{array} \right), \\
(33c) \quad & T_n(\lambda) \left( \frac{-\bar{g}_n}{f_n} \right) = (\lambda - \bar{\lambda}_1) \left( \begin{array}{cc} 1 - \frac{\lambda}{\lambda_1} & 0 \\ \frac{\lambda}{\lambda_1} - \frac{\lambda}{\lambda_1} & 0 \end{array} \right) \left( \begin{array}{c} -\bar{g}_n \\ f_n \end{array} \right), \\
(33d) \quad & T_n(\lambda) \left( \frac{-\bar{g}_n}{f_n} \right) = (\lambda + \bar{\lambda}_1) \left( \begin{array}{cc} 1 + \frac{\lambda}{\lambda_1} & 0 \\ \frac{\lambda}{\lambda_1} + \frac{\lambda}{\lambda_1} & 0 \end{array} \right) \left( \begin{array}{c} -\bar{g}_n \\ f_n \end{array} \right).
\end{align*}

By using this table, we compute the second product in (30):

\[
T_{n+1}(\lambda) N_n(\lambda) \text{ad} \tilde{T}_n(\lambda) = 2 \left( \frac{-(f_{n+1} f_n - S_n \bar{g}_{n+1} g_n)}{\lambda (\lambda_1 g_n f_n + S_n \lambda_1 \bar{f}_{n+1} g_n)} - \frac{a_{n+1}}{\lambda} \left( \bar{\lambda}_1 f_{n+1} \bar{g}_n + S_n \lambda_1 \bar{g}_{n+1} f_n \right) \right)
\]
\[
+ \frac{4i}{\hbar \lambda |\lambda_1|^2} \left( \lambda + \frac{a_{n+1}}{\lambda} \frac{|\lambda_1|^2}{\lambda_1} \right) \left( S_n \Delta_n \right) \left( 0 \right) \left( -\Delta_n \right).
\]

Substituting this expression into (30), we finally obtain

\[
N_n^{[1]}(\lambda) = \left( \begin{array}{ccc} \delta_0 & \frac{2i}{\hbar \lambda} & \frac{\delta_2}{\lambda} \\ \delta_1 & \frac{\delta_3}{\lambda} & \frac{2i}{\hbar \lambda} - \delta_4 \lambda \end{array} \right),
\]
where
\[ \delta_0 = \frac{\bar{f}_n f_{n+1} - S_n g_n \bar{g}_{n+1}}{\Delta_n} - \frac{2i \lambda \Delta_n}{h |\lambda|^2 \Delta_n}, \]
\[ \delta_1 = -\frac{a_{n+1} S_n \Delta_n}{\Delta_n}, \]
\[ \delta_2 = a_{n+1} \frac{\bar{\lambda}_1 f_{n+1} g_n + S_n \lambda_1 g_{n+1} f_n}{\Delta_n} + \frac{2i b_{n+1}}{h |\lambda_1|^2}, \]
\[ \delta_3 = -\frac{\bar{\lambda}_1 g_n f_{n+1} + S_n \lambda_1 \bar{f}_{n+1} g_n}{\Delta_n} - \frac{2i c_{n+1} S_n \Delta_n}{h |\lambda|^2 \Delta_n}, \]
\[ \delta_4 = -\frac{2i d_{n+1}}{h |\lambda_1|^2} + d_{n+1} \frac{g_{n+1} \bar{g}_n - S_n \bar{f}_{n+1} f_n}{\Delta_n}. \]

It follows from substitution of (19) and (20) for \( f_{n+1}, g_{n+1}, \bar{f}_{n+1} \) and \( \bar{g}_{n+1} \) that
\[ \bar{f}_n f_{n+1} - S_n g_n \bar{g}_{n+1} = \bar{\Delta}_n + \frac{2i S_n \Delta_n}{h |\lambda|^2} \]
and
\[ g_{n+1} \bar{g}_n - S_n \bar{f}_{n+1} f_n = -S_n \Delta_n + \frac{2i \Delta_n}{h |\lambda|^2}. \]

As a result, we verify that \( \delta_0 = 1 \) and \( \delta_1 = \delta_4 \). We represent \( N_n^{(1)}(\lambda) \) in (34) in the same form as \( N_n(\lambda) \) in (5a), therefore, we write
\[ \delta_1 = \frac{1 + \frac{i}{2} h W_n}{1 - \frac{i}{2} h W_n}, \quad \delta_2 = \frac{2i Y_n}{1 - \frac{i}{2} h W_n}, \quad \delta_3 = \frac{2i Z_n}{1 - \frac{i}{2} h W_n} \]
for some \( Y_n, Z_n, \) and \( W_n \). Using Eqs. (23) for \( a_{n+1}, b_{n+1}, \) and \( c_{n+1} \) and solving Eq. (35) for \( W_n, Y_n, \) and \( Z_n \) yield
\[ W_n = \frac{2i (\bar{\Delta}_n \Delta_{n+1} - S_n \bar{\Delta}_{n+1} \Delta_n)}{h (\Delta_n \Delta_{n+1} + S_n \bar{\Delta}_{n+1} \Delta_n)}, \]
\[ Y_n = -\frac{h |\lambda|^2 \bar{\Delta}_{n+1} (\lambda_1 S_n f_n \bar{g}_{n+1} + \bar{\lambda}_1 f_{n+1} \bar{g}_n) + 2i (\lambda_1^2 - \bar{\lambda}_1^2) \bar{\Delta}_n f_{n+1} \bar{g}_{n+1}}{h |\lambda|^2 (\Delta_n \Delta_{n+1} + S_n \bar{\Delta}_{n+1} \Delta_n)}, \]
\[ Z_n = -\frac{h |\lambda|^2 \bar{\Delta}_{n+1} (\lambda_1 S_n \bar{f}_{n+1} g_n + \bar{\lambda}_1 f_{n+1} g_n) + 2i (\lambda_1^2 - \bar{\lambda}_1^2) S_n \Delta_n \bar{f}_{n+1} \bar{g}_{n+1}}{h |\lambda|^2 (\Delta_n \Delta_{n+1} + S_n \bar{\Delta}_{n+1} \Delta_n)}. \]

Substituting Eqs. (19) and (20) into Eqs. (36b)--(36c) simplifies \( Y_n \) and \( Z_n \) to the form:
\[ Y_n = \frac{h |\lambda|^2 \bar{\Delta}_n U_n - 2i (\lambda_1^2 - \bar{\lambda}_1^2) f_n \bar{g}_n - 2i \Delta_n U_n}{h (\lambda_1^2 - \bar{\lambda}_1^2) f_n \bar{g}_n U_n - h |\lambda|^2 \Delta_n + 2i \Delta_n}, \]
\[ Z_n = \frac{h |\lambda|^2 \bar{\Delta}_n \bar{U}_n - 2i (\bar{\lambda}_1^2 - \lambda_1^2) g_n f_n + 2i \Delta_n \bar{U}_n}{h (\bar{\lambda}_1^2 - \lambda_1^2) f_n \bar{g}_n U_n - h |\lambda|^2 \Delta_n - 2i \Delta_n}. \]

It follows from Eqs. (37) that \( Y_n = \bar{Z}_n \). We have checked with the aid of Wolfram’s MATHEMATICA from Eq. (36a) that \( W_n = Y_n Z_n \) is satisfied. As a result, we conclude that \( N_n^{(1)}(\lambda) \) in (34) is the same as that of \( N_n(\lambda) \) in (5a) with the correspondence: \( U_n^{[1]} = Y_n, \bar{U}_n^{[1]} = \bar{Z}_n = \bar{Y}_n, \) and \( |U_n^{[1]}|^2 = W_n = |Y_n|^2 \). Thus, Eq. (6a) follows from the transformation formula (37a).
Next, we prove Eq. (31) and derive the transformations for $R_n$ and $Q_n$ in Eqs. (6b) and (6c). Again, using Lemma 2 and direct computations, we obtain

\begin{align}
(38a) & \quad P_n(\lambda) \left( \begin{array}{c} f_n \\ g_n \end{array} \right) = \left( \begin{array}{c} f_{n,t} \\ g_{n,t} \end{array} \right) + (\lambda - \lambda_1) H_1(\lambda) \left( \begin{array}{c} f_n \\ g_n \end{array} \right), \\
(38b) & \quad P_n(\lambda) \left( \begin{array}{c} f_n \\ -g_n \end{array} \right) = \left( \begin{array}{c} f_{n,t} \\ -g_{n,t} \end{array} \right) + (\lambda + \lambda_1) H_2(\lambda) \left( \begin{array}{c} f_n \\ -g_n \end{array} \right), \\
(38c) & \quad P_n(\lambda) \left( \begin{array}{c} \bar{g}_n \\ -\bar{f}_n \end{array} \right) = \left( \begin{array}{c} g_{n,t} \\ -f_{n,t} \end{array} \right) + M_n \left( \begin{array}{c} g_n \\ -f_n \end{array} \right) + (\lambda - \bar{\lambda}_1) H_3(\lambda) \left( \begin{array}{c} \bar{g}_n \\ -\bar{f}_n \end{array} \right), \\
(38d) & \quad P_n(\lambda) \left( \begin{array}{c} \bar{g}_n \\ \bar{f}_n \end{array} \right) = \left( \begin{array}{c} \bar{g}_{n,t} \\ \bar{f}_{n,t} \end{array} \right) + M_n \left( \begin{array}{c} \bar{g}_n \\ \bar{f}_n \end{array} \right) + (\lambda + \bar{\lambda}_1) H_4(\lambda) \left( \begin{array}{c} \bar{g}_n \\ \bar{f}_n \end{array} \right),
\end{align}

where $M_n$ is defined in Eq. (18b) and matrices $H_{1,2,3,4}(\lambda)$ are given by

\begin{align}
H_1(\lambda) &= \frac{i}{2} \left( \begin{array}{cc} \lambda + \lambda_1 & R_n + \frac{1}{\lambda_1} Q_n \\ \bar{R}_n + \frac{1}{\lambda_1} \bar{Q}_n & \frac{\lambda + \lambda_1}{\lambda_1^2} \end{array} \right), \\
H_2(\lambda) &= \frac{i}{2} \left( \begin{array}{cc} \lambda - \lambda_1 & R_n - \frac{1}{\lambda_1} Q_n \\ \bar{R}_n - \frac{1}{\lambda_1} \bar{Q}_n & \frac{\lambda - \lambda_1}{\lambda_1^2} \end{array} \right), \\
H_3(\lambda) &= \frac{i}{2} \left( \begin{array}{cc} \lambda + \bar{\lambda}_1 & R_n + \frac{1}{\bar{\lambda}_1} \bar{Q}_n \\ \bar{R}_n + \frac{1}{\bar{\lambda}_1} \bar{Q}_n & \frac{\lambda + \bar{\lambda}_1}{\bar{\lambda}_1^2} \end{array} \right), \\
H_4(\lambda) &= \frac{i}{2} \left( \begin{array}{cc} \lambda - \bar{\lambda}_1 & R_n - \frac{1}{\bar{\lambda}_1} \bar{Q}_n \\ \bar{R}_n - \frac{1}{\bar{\lambda}_1} \bar{Q}_n & \frac{\lambda - \bar{\lambda}_1}{\bar{\lambda}_1^2} \end{array} \right).
\end{align}

Based on the results in Eq. (38), the product in the right-hand side of Eq. (31) can be obtained as

\begin{align}
\left[ \frac{d}{dt} T_n(\lambda) + T_n(\lambda) P_n(\lambda) \right] \text{ad} \tilde{T}_n(\lambda) \\
= \frac{1}{\lambda - \lambda_1} \left[ T_n(\lambda) \left( \begin{array}{c} f_n \\ g_n \end{array} \right) \right]_t (-\tilde{f}_n \; \bar{g}_n) + \frac{1}{\lambda + \lambda_1} \left[ T_n(\lambda) \left( \begin{array}{c} f_n \\ -g_n \end{array} \right) \right]_t (\bar{f}_n \; g_n) \\
+ \frac{1}{\lambda - \lambda_1} \left[ T_n(\lambda) \left( \begin{array}{c} \bar{g}_n \\ -\bar{f}_n \end{array} \right) \right]_t (-g_n \; -f_n) + \frac{1}{\lambda + \lambda_1} \left[ T_n(\lambda) \left( \begin{array}{c} \bar{g}_n \\ \bar{f}_n \end{array} \right) \right]_t (g_n \; -f_n) \\
+ T_n(\lambda) H_1(\lambda) \left( \begin{array}{c} f_n \\ g_n \end{array} \right) (-\tilde{f}_n \; \bar{g}_n) + T_n(\lambda) H_2(\lambda) \left( \begin{array}{c} f_n \\ -g_n \end{array} \right) (\bar{f}_n \; g_n) \\
+ T_n(\lambda) H_3(\lambda) \left( \begin{array}{c} \bar{g}_n \\ -\bar{f}_n \end{array} \right) (-g_n \; -f_n) + T_n(\lambda) H_4(\lambda) \left( \begin{array}{c} \bar{g}_n \\ \bar{f}_n \end{array} \right) (g_n \; -f_n) \\
+ M_n \left[ \frac{1}{\lambda - \lambda_1} T_n(\lambda) \left( \begin{array}{c} \bar{g}_n \\ -\bar{f}_n \end{array} \right) (-g_n \; -f_n) + \frac{1}{\lambda + \lambda_1} T_n(\lambda) \left( \begin{array}{c} \bar{g}_n \\ \bar{f}_n \end{array} \right) (g_n \; -f_n) \right].
\end{align}

Expanding the above equation and substituting it into (31) gives

\begin{align}
P_n^{(1)}(\lambda) = \frac{1}{\Delta_n} \left( -\bar{\lambda}_1 \tilde{f}_n (a_n f_n)_t - \lambda_1 g_n (a_n \bar{g}_n)_t \lambda (f_n \bar{g}_n, t - f_n, t \bar{g}_n) \right)
\end{align}
where we have used Eq. (23) in obtaining the last term. Thus, \( P_n^{[1]} \) can be formally written in the form

\[
P_n^{[1]}(\lambda) = \frac{i}{2} \begin{pmatrix}
\lambda_1^2 - \pi_1 \pi_3 & \pi_1 \lambda - \pi_2 \lambda^{-1} \\
\pi_3 \lambda - \pi_4 \lambda^{-1} & \pi_2 \pi_4 - \lambda^{-2}
\end{pmatrix},
\]

Comparing Eqs. (39) and (40) and using Eqs. (23) together with (21), we can express \( \pi_i \)'s (1 \( \leq \) i \( \leq \) 4) as

\[
\begin{align*}
\pi_1 &= -\frac{\Delta_n R_n + (\lambda_1^2 - \lambda_1^n) f_n g_n}{\Delta_n}, \\
\pi_2 &= -\frac{|\lambda_1|^2 \Delta_n Q_n + (\lambda_1^2 - \lambda_1^n) f_n g_n}{|\lambda_1|^2 \Delta_n}, \\
\pi_3 &= -\frac{\Delta_n R_n + (\lambda_1^2 - \lambda_1^n) f_n g_n}{\Delta_n}, \\
\pi_4 &= -\frac{|\lambda_1|^2 \Delta_n Q_n + (\lambda_1^2 - \lambda_1^n) f_n g_n}{|\lambda_1|^2 \Delta_n},
\end{align*}
\]

where Wolfram's MATHEMATICA has been used for simplification. It is obvious from (41) that \( \bar{\pi}_1 = \pi_3 \) and \( \bar{\pi}_2 = \pi_4 \). As a result, we conclude that \( P_n^{[1]}(\lambda) \) in (39) is the same as that of \( P_n(\lambda) \) in (5b) with the correspondence: \( R_n^{[1]} = \pi_1 \) and \( Q_n^{[1]} = \pi_2 \). Thus, Eqs. (6b)–(6c) follow from the transformation formulas (41a)–(41b). Theorem [1] is proven with the algorithmic computations.

3. Soliton solutions on zero background

Here we use the one-fold Darboux transformation of Theorem [1] and construct soliton solutions on zero background. Hence we take zero potentials \((U, R, Q) = (0, 0, 0)\) in the transformation formula (6) and obtain

\[
\begin{align*}
U_n^{[1]} &= -\frac{2i(\lambda_1^2 - \lambda_1^n) f_n g_n}{2i(\lambda_1 |f_n|^2 + \lambda_1 |g_n|^2) - \hbar |\lambda_1|^2 (\lambda_1 |f_n|^2 + \lambda_1 |g_n|^2)}, \\
R_n^{[1]} &= -\frac{(\lambda_1^2 - \lambda_1^n) f_n g_n}{\lambda_1 |f_n|^2 + \lambda_1 |g_n|^2}, \\
Q_n^{[1]} &= -\frac{(\lambda_1^2 - \lambda_1^n) f_n g_n}{|\lambda_1|^2 (\lambda_1 |f_n|^2 + \lambda_1 |g_n|^2)},
\end{align*}
\]

where \( \Phi_n(\lambda_1) = (f_n, g_n)^T \) is a nonzero solution of the Lax pair (5) with \( \lambda = \lambda_1 \) at the zero background. First, we prove that the zero background is linearly stable in the semi-discrete MTM system (1). Next, we construct Jost solutions of the Lax pair (5) at the zero background. At last, we obtain and study the exact expressions for one-soliton and two-soliton solutions.
3.1. **Stability of zero background.** Linearization of the semi-discrete MTM system \((1)\) at the zero background is written as the linear system

\[
\begin{align*}
4i \frac{du_n}{dt} + q_{n+1} + q_n + \frac{2i}{h} (r_{n+1} - r_n) &= 0, \\
q_{n+1} - q_n + ihu_n &= 0, \\
r_{n+1} + r_n - 2u_n &= 0.
\end{align*}
\]

(43)

Thanks to the linear superposition principle, we use the discrete Fourier transform on the lattice,

\[
u_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{u}(\theta) e^{i n \theta} d\theta, \quad n \in \mathbb{Z},
\]

(44)

invert the second and third equations of the differential-difference system \((43)\), and obtain the following differential equation with parameter \(\theta \in (-\pi, \pi) \setminus \{0\} ):

\[
\frac{h}{4} \frac{d\hat{u}}{dt} = \left( \frac{h^2 e^{i\theta} + 1}{4} - \frac{e^{-i\theta}}{4} \right) \hat{u}.
\]

(45)

Separating variables in \(\hat{u} = \hat{u}_0(\theta) e^{-i\omega(\theta)}\) yields the dispersion relation for the Fourier mode \(\hat{u}_0(\theta)\):

\[
\omega(\theta) = \frac{1}{h \sin \theta} \left[ \left( \frac{h^2}{4} + 1 \right) + \left( \frac{h^2}{4} - 1 \right) \cos \theta \right], \quad \theta \in (-\pi, \pi) \setminus \{0\}.
\]

(46)

Since \(\omega(\theta) \in \mathbb{R}\) for every \(\theta \in (-\pi, \pi) \setminus \{0\} \), the zero background is linearly stable. Note however that \(|\omega(\theta)\| \to \infty\) as \(\theta \to 0\) and \(\theta \to \pm \pi\). Divergences of the dispersion relation in \((46)\) as \(\theta \to 0\) and \(\theta \to \pm \pi\) are related to inversion of the second and third difference equations in the linear system \((43)\).

3.2. **Solutions of the Lax pair \((5)\) at zero background.** Lax pair \((5)\) at the zero background is decoupled into two systems which admit the following two linearly independent solutions:

\[
[\Phi_+(\lambda)]_n(t) = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_+^n e^{\frac{i\lambda^2 t}{2}}, \quad [\Phi_-(\lambda)]_n(t) = \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mu_-^n e^{-\frac{i\lambda^2 t}{2}},
\]

(47)

where \(\alpha, \beta \in \mathbb{C} \setminus \{0\}\) are parameters and

\[
\mu_{\pm}(\lambda) := \frac{2i}{h\lambda} \pm \lambda.
\]

(48)

We say that \(\Phi(\lambda)\) is the Jost function if \(\lambda \in \mathbb{C}\) yields either \(|\mu_+(\lambda)| = 1\) or \(|\mu_-(\lambda)| = 1\), in which case one of the two fundamental solutions in \((47)\) is bounded in the limit \(|n| \to \infty\). Constraints \(|\mu_{\pm}(\lambda)| = 1\) for \(\lambda = |\lambda| e^{i\theta}/2\) in the polar form are equivalent to the following equation:

\[
|\lambda|^2 \pm \frac{4}{h} \sin(\theta) + \frac{4}{h^2 |\lambda|^2} = 1.
\]

(48)

Roots of Eq. \((48)\) in the complex plane for \(\lambda \in \mathbb{C}\) are shown on Fig. (1) for \(h < 4\) (left) and \(h > 4\) (right). For every \(\lambda\) on each curve of the Lax spectrum, there exists one Jost function in \((47)\) which remains bounded in the limit \(|n| \to \infty\). On the other hand, thanks to the time dependence in \((47)\), Jost functions remain bounded also in the limit \(|t| \to \infty\) if and only if \(\lambda^2 \in \mathbb{R}\). No such Jost functions exist for \(h < 4\) as is seen from the left panel of Fig. (1). In other words, all Jost functions diverge exponentially either as \(t \to -\infty\) or as \(t \to +\infty\) if \(h < 4\).
3.3. One-soliton solutions. Fix $\lambda_1 \in \mathbb{C}$ such that $\mu_{\pm}(\lambda_1) \neq 0$ and $\lambda_1^2 \notin \mathbb{R}$. Taking a general solution for \( \Phi(\lambda_1) = (f, g)^T \), we write \( f \) and \( g \) in the form:

\[
(49) \quad f_n(t) = \alpha_1 e^{\xi_{1,n}(t)}, \quad g_n(t) = \beta_1 e^{\eta_{1,n}(t)},
\]

where

\[
(50) \quad \xi_{1,n}(t) = \eta \log \left( \lambda_1 + \frac{2i}{h \lambda_1} \right) + \frac{i}{2} \lambda_1 t, \quad \eta_{1,n}(t) = \eta \log \left( -\lambda_1 + \frac{2i}{h \lambda_1} \right) - \frac{it}{2 \lambda_1^2},
\]

and $\alpha_1, \beta_1 \in \mathbb{C}\setminus\{0\}$ are parameters. Without loss of generality, we set $\lambda_1 = \delta_1 e^{i \theta_1/2}$ with some $\delta_1 > 0$ and $\theta_1 \in (0, \pi)$. Substituting Eq. (49) into Eqs. (42) yields the exact one-soliton solution in the form:

\[
U_{n}^{[1]} = -\frac{4i \delta_1 \alpha_1 \beta_1 \sin \theta_1 e^{i \theta_1/2}}{|\beta_1|^2 (2 + i h \delta_1^2 e^{i \theta_1}) e^{\eta_{1,n} - \xi_{1,n}} + |\alpha_1|^2 (2 e^{i \theta_1} + i h \delta_1^2 e^{-\eta_{1,n} + \xi_{1,n}})},
\]

\[
R_{n}^{[1]} = -\frac{2i \delta_1 \alpha_1 \beta_1 \sin \theta_1 e^{i \theta_1/2}}{|\beta_1|^2 e^{\eta_{1,n} - \xi_{1,n}} + |\alpha_1|^2 e^{-\eta_{1,n} + \xi_{1,n} + i \theta_1}},
\]

\[
Q_{n}^{[1]} = -\frac{2i \alpha_1 \beta_1 \sin \theta_1 e^{i \theta_1/2}}{\delta_1 (|\beta_1|^2 e^{\eta_{1,n} - \xi_{1,n} + i \theta_1} + |\alpha_1|^2 e^{-\eta_{1,n} + \xi_{1,n}})},
\]

where

\[
\xi_{1,n}(t) - \eta_{1,n}(t) = n \log \left( \frac{2 - i h \delta_1^2 e^{i \theta_1}}{2 + i h \delta_1^2 e^{i \theta_1}} \right) + \frac{i}{2} \left( \delta_1^2 + \frac{1}{\delta_1^2} \right) \cos \theta_1 t - \frac{1}{2} \left( \delta_1^2 - \frac{1}{\delta_1^2} \right) \sin \theta_1 t.
\]

Fig. 2(a), 2(c) presents the one-soliton solutions (51) for $\alpha_1 = 1$, $\beta_1 = 1 + i$, $\lambda_1 = 2 e^{\frac{\pi}{4}}$, and $h = 1$.

Let us check that the discrete solitons (51) recover solitons of the continuous MTM system (2). In order to simplify the computations, we set $\delta_1 = 1$, which corresponds to the case of stationary solitons [6, 22]. By defining $x_n = h n$, $n \in \mathbb{Z}$ and taking the limit $h \to 0$, we obtain for $\delta_1 = 1$:

\[
U_{n}^{[1]} \to U(x, t) = \frac{2i \alpha_1 \beta_1 \sin \theta_1 e^{i \cos \theta_1 (t-x)}}{|\alpha_1|^2 e^{\sin \theta_1 x + i \theta_1/2} + |\beta_1|^2 e^{-\sin \theta_1 x - i \theta_1/2}},
\]

\[
R_{n}^{[1]} \to R(x, t) = \frac{2i \alpha_1 \beta_1 \sin \theta_1 e^{i \cos \theta_1 (t-x)}}{|\alpha_1|^2 e^{\sin \theta_1 x + i \theta_1/2} + |\beta_1|^2 e^{-\sin \theta_1 x - i \theta_1/2}},
\]

\[
Q_{n}^{[1]} \to Q(x, t) = \frac{2i \alpha_1 \beta_1 \sin \theta_1 e^{i \cos \theta_1 (t-x)}}{|\alpha_1|^2 e^{\sin \theta_1 x + i \theta_1/2} + |\beta_1|^2 e^{-\sin \theta_1 x - i \theta_1/2}}.
\]
This expression yields in the limit expanding to the first order in $\epsilon$ discrete algebraic soliton for the case (Darboux transformation (6) with $x, \xi$ where $\lambda$ and $\xi$ terms$^{2}$).

Fix (obtain the one-soliton solutions $T^{[1]}$). Two-soliton solutions.

Similarly, one can prove that the discrete soliton (53), degenerates to the zero solution in the limit (55)

$$U(x, t) \rightarrow U_{a}(x, t) = -\frac{e^{-i(t-x)}}{x-\frac{1}{2}}.$$ 

Discrete solitons (51) enjoy the same properties as the continuous solitons. In particular, let us recover the discrete algebraic soliton for the case $\alpha_1 = \beta_1$ and $\delta_1 = 1$ in the limit $\theta_1 \rightarrow \pi$. By setting $\theta_1 = \pi - \epsilon$ and expanding to the first order in $\epsilon$, we obtain from (51)

$$U^{[1]}_n = \frac{4(\epsilon + O(\epsilon^2))e^{-it}}{(2 - ih - e\imath h + O(\epsilon^2))(\frac{2-ih+ir(2+ih)/2+O(\epsilon^2)}{2+ih+ir(2-ih)/2+O(\epsilon^2)})^n - (2 - ih - 2\epsilon i + O(\epsilon^2))(\frac{2-ih+ir(2+ih)/2+O(\epsilon^2)}{2+ih+ir(2-ih)/2+O(\epsilon^2)})^n}.$$ 

This expression yields in the limit $\epsilon \rightarrow 0$ the discrete algebraic soliton

$$(54) \quad U^{[1]}_n \rightarrow [U_a]_n = -\frac{4e^{-it}}{8nh(2-ih)}\frac{(2 + ih)^n}{2 - ih}.$$ 

If $x_n = hn, n \in \mathbb{Z}$, the discrete algebraic soliton (54) reduces in the limit $h \rightarrow 0$ to the continuous algebraic soliton (53). Similarly, one can prove that the discrete soliton (51) degenerates to the zero solution in the limit $\theta_1 \rightarrow 0$.

3.4. Two-soliton solutions. In order to construct two-soliton solutions, one needs to use the one-fold Darboux transformation (6) twice. Fix $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ such that $\mu_{\pm}(\lambda_{1,2}) \neq 0, \lambda_1^2 \notin \mathbb{R}, \lambda_2 \neq \pm 1$, and $\lambda_2 \neq \pm \lambda_1$. A general solution of the Lax pair (5) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ at zero background is written in the form

$$(55) \quad [\Phi(\lambda_1)]_n(t) = \left(\frac{\alpha_1 e^{\xi_n(t)}}{\beta_1 e^{\eta_n(t)}}\right), \quad [\Phi(\lambda_2)]_n(t) = \left(\frac{\alpha_2 e^{\xi_n(t)}}{\beta_2 e^{\eta_n(t)}}\right),$$

where $\xi_{j,n}$ and $\eta_{j,n}$ with $j = 1, 2$ are given by (50) for $\lambda_1, \lambda_2$, and $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C} \setminus \{0\}$ are parameters.

By using the one-fold Darboux transformation (6) with zero potentials, $\lambda = \lambda_1$, and $\Phi = \Phi(\lambda_1)$, we obtain the one-soliton solutions $(U_{1}^{[1]}, R_{1}^{[1]}, Q_{1}^{[1]})$ in the form (51). The transformed eigenfunction $\Phi^{[1]}(\lambda_2) = T^{[1]}(\lambda_2)\Phi(\lambda_2)$ satisfies the Lax pair (5) with the potentials $(U_{1}^{[1]}, R_{1}^{[1]}, Q_{1}^{[1]})$ and $\lambda = \lambda_2$. By using the one-fold Darboux transformation (6) with $(U_n, R_n, Q_n)$ replaced by $(U_{1}^{[1]}, R_{1}^{[1]}, Q_{1}^{[1]})$, $\lambda_1$ replaced by $\lambda_2$, and $\Phi(\lambda_1)$
replaced by $\Phi^{[1]}(\lambda_2)$, we obtain the two-soliton solutions $(U_n^{[2]}, R_n^{[2]}, Q_n^{[2]})$ in the explicit form (which is not given here).

Fig. 3(a)–3(c) shows the two-soliton solutions for $\alpha_1 = 1$, $\beta_1 = 1 + i$, $\alpha_2 = 1$, $\beta_2 = 1$, $\lambda_1 = \sqrt{3}e^{\pi i / 6}$, $\lambda_2 = \sqrt{5}e^{\pi i \arctan \frac{2}{2}}$, and $h = 1$. The two-soliton solutions feature elastic collisions of two individual solitons with preservation of their shapes. Such collisions are very common in integrable equations including the continuous MTM system (3).

4. SOLITON SOLUTIONS ON NONZERO CONSTANT BACKGROUND

Here we use the one-fold Darboux transformation of Theorem 1 and construct soliton solutions on nonzero constant background $(U, R, Q) = (\rho, \rho, \rho - 1)$, where $\rho > 0$ is a real parameter. Similarly to Section 3, we prove that the nonzero constant background is linearly stable in the semi-discrete MTM system (11) for every $\rho > 0$, construct Jost solutions of the Lax pair (5) at nonzero constant background, and then finally obtain the exact expressions for one-soliton solutions.

4.1. Stability of nonzero constant background. Linearization of the semi-discrete MTM system (11) at the nonzero constant background $(U, R, Q) = (\rho, \rho, \rho - 1)$ with $\rho > 0$ yields the linear system of equations:

$$\begin{align*}
4i\frac{d u_n}{dt} + 2\left(\rho^2 - \frac{1}{\rho^2}\right) u_n + \left(1 + \frac{i\rho^2}{2}\right) \left(\frac{2i}{h} r_{n+1} - \bar{q}_{n+1}\right) - \left(1 - \frac{i\rho^2}{2}\right) \left(\frac{2i}{h} r_n + \bar{q}_n\right) &= 0, \quad (56) \\
\left(1 + \frac{i\rho^2}{2}\right) \bar{q}_{n+1} + \left(1 - \frac{i\rho^2}{2}\right) \bar{q}_n + ihu_n &= 0, \\
\left(1 + \frac{i\rho^2}{2}\right) r_{n+1} + \left(1 - \frac{i\rho^2}{2}\right) r_n - 2u_n &= 0.
\end{align*}$$

By using the discrete Fourier transform on the lattice (44), we close the linear system (56) at the following differential equation with parameter $\theta \in (-\pi, \pi)$:

$$\begin{align*}
\frac{d \hat{u}}{dt} + \frac{h}{2} \left(\rho^2 - \frac{1}{\rho^2}\right) \hat{u} + \left(h^2 \cos \frac{\theta}{2} - \frac{h\rho^2}{2} \sin \frac{\theta}{2} - \sin \frac{\theta}{2} + \frac{h\rho^2}{2} \cos \frac{\theta}{2}\right) \frac{h}{2} \cos \frac{\theta}{2} &= 0. \\
\frac{d \hat{\bar{q}}}{dt} + \frac{h}{2} \left(\rho^2 - \frac{1}{\rho^2}\right) \hat{\bar{q}} + \left(h^2 \cos \frac{\theta}{2} - \frac{h\rho^2}{2} \sin \frac{\theta}{2} - \sin \frac{\theta}{2} + \frac{h\rho^2}{2} \cos \frac{\theta}{2}\right) \frac{h}{2} \cos \frac{\theta}{2} &= 0.
\end{align*}$$

The dispersion relation following from linear equation (57) is purely real, which implies that the nonzero constant background is linearly stable for every $\rho > 0$. Note that the linear equation (57) does not reduce to equation (45) in the limit $\rho \to 0$ because the nonzero constant background $(U, R, Q) = (\rho, \rho, \rho^{-1})$ is singular in this limit, hence the variable $q$ in the linearized system (43) is replaced by $\bar{q}$ in the system (56).
Note that \((u, v) = (\rho, \rho^{-1})\) is also the nonzero constant solution of the continuous MTM system \((5)\). However, computations similar to those in \((56)\) and \((57)\) show that the nonzero constant background for any \(\rho > 0\) is modulationally unstable. This is different from the conclusion on the nonzero constant background in the semi-discrete MTM system \((1)\).

### 4.2. Solutions of the Lax pair \((5)\) at nonzero constant background.

Solving Lax pair \((5)\) with the potentials \((U, R, Q) = (\rho, \rho, \rho^{-1})\), we have two independent solutions:

\[
[\Phi_+(\lambda)]_n(t) = \alpha \left( \frac{\rho}{\lambda} \right) \mu^n e^{\frac{1}{2}(\frac{\lambda^2}{\rho^2} - \lambda^2) t}, \quad [\Phi_-^*(\lambda)]_n(t) = \beta \left( \frac{\lambda}{\rho} \right) \mu^n e^{\frac{1}{2}(\lambda^2 - \frac{1}{\rho^2}) t},
\]

where \(\alpha, \beta \in \mathbb{C} \setminus \{0\}\) are parameters and

\[
\mu_+(\lambda) := \frac{2i}{h \lambda} - \lambda, \quad \mu_-(\lambda) := \frac{2i}{h \lambda} + \lambda.
\]

Similarly to the case of zero potentials, we say that \(\Phi(\lambda)\) is a Jost function if \(\lambda \in \mathbb{C}\) yields either \(|\mu_+(\lambda)| = 1\) or \(|\mu_-(\lambda)| = 1\). Interestingly, the constraints \(|\mu_+(\lambda)| = 1\) with \(\lambda = |\lambda|e^{i\theta/2}\) yield the same equation \((48)\). Hence, any point on each curve of the Lax spectrum shown on Fig. \((4)\) gives one Jost function in \((58)\) which remains bounded in the limit \(|n| \to \infty\). The function of \(\Phi_+(\lambda)\) is always bounded in the limit \(|t| \to \infty\) since \(\rho > 0\). On the other hand, \(\Phi_-^*(\lambda)\) is bounded as \(|t| \to \infty\) if and only if \(\lambda^2 \in \mathbb{R}\), and no such Jost functions exist for \(\Phi_-(\lambda)\) if \(h < 4\).

### 4.3. One-breather solutions.

Fix \(\lambda_1 \in \mathbb{C}\) such that \(\mu_+^1(\lambda_1) \neq 0\) and \(\lambda_1^2 \notin \mathbb{R}\). Let \(\Phi(\lambda_1) = (f, g)^T\) be the general solution of Lax pair \((5)\) with \((U, R, Q) = (\rho, \rho, \rho^{-1})\) and \(\lambda = \lambda_1\). We write \(f\) and \(g\) in the form

\[
f_{1,n} = \alpha_1 \rho e^{\mu_{1,n}(t)} + \beta_1 \lambda_1 e^{\nu_{1,n}(t)}, \quad g_{1,n} = -\alpha_1 \lambda_1 e^{\mu_{1,n}(t)} + \beta_1 \rho e^{\nu_{1,n}(t)},
\]

with

\[
\mu_{1,n}(t) = n \log \left[ \frac{2i}{h \lambda_1} - \lambda_1 \right] + \frac{i}{2} \left( 1 - \rho^2 - \lambda_1^2 \right) t,
\]

\[
\nu_{1,n}(t) = n \log \left[ \frac{2i}{h \lambda_1} + \lambda_1 \right] + \frac{i}{2} \left( \lambda_1^2 - 1 \right) t,
\]

where \(\alpha_1, \beta_1 \in \mathbb{C} \setminus \{0\}\) are parameters. Substituting Eq. \((59)\) into Eqs. \((6)\), we obtain the one-breather solutions at nonzero constant background as follows:

\[
U_{1,n}^1 = -\frac{|\alpha_1|^2 \rho \lambda_1 \lambda_{11} e^{\Theta_{1,n}} + |\beta_1|^2 \rho \lambda_1 \lambda_{11} e^{-\Theta_{1,n}} + \alpha_1 \beta_1 |\lambda_1|^2 h \rho (\lambda_1^2 - \lambda_{11}^2) e^{-i \Xi_{1,n}}}{|\alpha_1|^2 \rho \lambda_1 \lambda_{11} e^{\Theta_{1,n}} + |\beta_1|^2 \rho \lambda_1 \lambda_{11} e^{-\Theta_{1,n}} - \alpha_1 \beta_1 |\lambda_1|^2 h \rho (\lambda_1^2 - \lambda_{11}^2) e^{i \Xi_{1,n}}}.
\]

\[
R_{1,n}^1 = -\frac{|\alpha_1|^2 \rho \lambda_1 \lambda_{11} e^{\Theta_{1,n}} + |\beta_1|^2 \rho \lambda_1 \lambda_{11} e^{-\Theta_{1,n}} - \alpha_1 \beta_1 |\lambda_1|^2 h \rho (\lambda_1^2 - \lambda_{11}^2) e^{-i \Xi_{1,n}}}{|\alpha_1|^2 \rho \lambda_1 \lambda_{11} e^{\Theta_{1,n}} + |\beta_1|^2 \rho \lambda_1 \lambda_{11} e^{-\Theta_{1,n}} + \alpha_1 \beta_1 |\lambda_1|^2 h \rho (\lambda_1^2 - \lambda_{11}^2) e^{i \Xi_{1,n}}}.
\]

\[
Q_{1,n}^1 = -\frac{|\alpha_1|^2 \rho \lambda_1 \lambda_{11} e^{\Theta_{1,n}} + |\beta_1|^2 \rho \lambda_1 \lambda_{11} e^{-\Theta_{1,n}} + \alpha_1 \beta_1 |\lambda_1|^2 h \rho (\lambda_1^2 - \lambda_{11}^2) e^{i \Xi_{1,n}}}{|\alpha_1|^2 \rho \lambda_1 \lambda_{11} e^{\Theta_{1,n}} + |\beta_1|^2 \rho \lambda_1 \lambda_{11} e^{-\Theta_{1,n}} - \alpha_1 \beta_1 |\lambda_1|^2 h \rho (\lambda_1^2 - \lambda_{11}^2) e^{-i \Xi_{1,n}}}.
\]

where

\[
\Theta_{1,n} = \Re(\mu_{1,n} - \nu_{1,n}), \quad \Xi_{1,n} = \Im(\mu_{1,n} - \nu_{1,n}),
\]

\[
\lambda_{11} = \rho^2 + \lambda_1^2, \quad \bar{\lambda}_{11} = \rho^2 + \bar{\lambda}_1^2,
\]

\[
h_{\lambda_1} = -2i + h \lambda_1^2, \quad h_{\bar{\lambda}_1} = -2i + h \bar{\lambda}_1^2, \quad h_\rho = 2i + h \rho^2.
\]
Due to the presence of the oscillatory terms $e^{i\Xi_{1,n}}$ and $e^{-i\Xi_{1,n}}$, solutions \((60)\), in general, exhibit the localized breathers which oscillate periodically both in \(n\) and \(t\). Fig. 4(a)–4(c) illustrates the one-breather solutions \((60)\) at the constant background for \(\alpha_1 = 1, \beta_1 = 1 + i, \rho = 1, \lambda_1 = 2e^{i\pi/8}, \) and \(h = 3/4\).

**Figure 4.** An example of the one-breather solutions \((60)\) at the nonzero background.

No periodic oscillations occur in the one-breather solutions \((60)\) if and only if \(\Xi_{1,n} = 0\). In this case, solutions \((60)\) describe one-solitons illustrated on Fig. 5(a)–5(c) for \(\alpha_1 = 1, \beta_1 = 1 + i, \rho = 2^{3/4}/\sqrt{7}, \lambda_1 = \sqrt{2}e^{i\pi/4}, \) and \(h = \sqrt{3}\).

**Figure 5.** An example of the one-breather solutions \((60)\) without periodic oscillations.

We show that the one-breather solutions \((60)\) feature no periodic oscillations if the modulus and argument of \(\lambda_1\) are given by

\[
|\lambda_1| = \frac{1}{\rho}\sqrt{\frac{2}{h}}, \quad \arg(\lambda_1) = \frac{1}{2}\arccos \left( \frac{2h - 1 - \rho^4}{4 - h^2\rho^4} \right) \tag{61}
\]

in the two regions described by

\[
\text{either } h > \frac{2}{\rho^4}, \rho < 1, \text{ or } 0 < h < \frac{2}{\rho^4}, \rho > 1. \tag{62}
\]

Note that the two regions intersect at \(\rho = 1, h = 2\), for which \(|\lambda_1| = 1\) whereas \(\arg(\lambda_1)\) is not determined. In fact, we show that \(\arg(\lambda_1) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)\). The existence region for non-oscillating one-soliton solutions \((60)\) on the \((h, \rho)\) plane is displayed in Fig. 6.
In order to verify (61), we note that the condition \( \Xi_{1,n} = 0 \) is equivalent to the system of two equations

\[
\begin{align*}
\frac{2}{\rho^2} - 2\rho^2 - \frac{\lambda_1^2}{h} + \frac{1}{\lambda_1} + \frac{1}{\lambda_1} &= 0, \\
\frac{4\rho^2}{h^2|\lambda_1|^2} - h|\lambda_1|^2\rho^2 + \frac{2}{h} \left( \frac{\lambda_1}{\lambda_1} + \frac{\lambda_1}{\lambda_1} \right) \left( 1 - \frac{h^2\rho^4}{4} \right) &= 0,
\end{align*}
\]

subject to the constraint

\[
\left( \frac{4}{h^2|\lambda_1|^2} - |\lambda_1|^2 \right) \left( 1 - \frac{h^2\rho^4}{4} \right) - 2\rho^2 \left( \frac{\lambda_1}{\lambda_1} + \frac{\lambda_1}{\lambda_1} \right) > 0.
\]

By using the polar form \( \lambda_1 = \delta_1 e^{i\theta_1/2} \) with \( \delta_1 > 0 \) and \( \theta_1 \in (0, \pi) \), we rewrite the constraints (63)–(64) in the form:

\[
\begin{align*}
\frac{1}{\rho^2} - \rho^2 + \left( \frac{1}{\delta_1^2} - \delta_1^2 \right) \cos \theta_1 &= 0, \\
\frac{\delta_1^4 h^2 \rho^2}{4} + \delta_1^2 \left( h^2 \rho^4 - 4 \right) \cos \theta_1 - 4\rho^2 &= 0,
\end{align*}
\]

subject to the constraint

\[
\frac{\delta_1^4 h^2 - 4}{4\delta_1^2 h^2} - 4\rho^2 \cos \theta_1 > 0.
\]

Let us first assume that \( \delta_1 \neq 1 \), in which case the first equation in (65) gives a unique solution for \( \theta_1 \):

\[
\cos \theta_1 = \frac{\rho^2 - \rho^{-2}}{\delta_1^2 - \delta_1^2}.
\]

Substituting (67) into the second equation in (65) yields the following equation

\[
\delta_1^8 h^2 \rho^4 - \delta_1^4 (h^2 \rho^4 + 4) + 4\rho^4 = 0
\]

with two roots \( \delta_1^4 = \rho^4 \) and \( \delta_1^4 h^2 \rho^4 = 4 \). Since \( \delta_1 = \rho \) implies \( \cos \theta_1 = -1 \) in (67), which is not admissible, we only have one positive root for \( \delta_1 \) given by

\[
\delta_1 = \frac{\sqrt{2}}{\rho \sqrt{h}},
\]

which implies

\[
\cos \theta_1 = 2h \frac{1 - \rho^4}{4 - h^2 \rho^4}.
\]
thanks to (67). Solutions (68) and (69) are equivalent to (61). The constraint (66) with the solutions (68–69) is rewritten in the form

\[
\frac{(1 - \rho^4)(h^2\rho^4 + 4)^2}{2h^2(h^2\rho^4 - 4)} > 0,
\]

from which the two regions in (62) follow. In the exceptional case, \( \delta_1 = 1 \), we have from the first equation in (65) that \( \rho = 1 \) whereas \( \cos \theta_1 \) is not determined. Then, the second equation in (65) implies that \( h = 2 \) since \( \cos \theta_1 = -1 \) is not admissible. The constraint (66) yields \( \cos \theta_1 < 0 \) so that \( \theta_1 \in \left( \frac{\pi}{2}, \pi \right) \).

5. Conclusion

We have derived the one-fold Darboux transformation between solutions of the semi-discrete MTM system using the Lax pair and the dressing methods. When one solution of the semi-discrete MTM system is either zero or nonzero constant, the one-fold Darboux transformation generates one-soliton solution on the zero or nonzero constant background respectively. When the one-fold Darboux transformation is used recursively, it also allows us to construct two-soliton solutions and generally multi-soliton solutions. We have showed that properties of the discrete solitons in the semi-discrete MTM system are very similar to properties of the continuous MTM solitons.

Among further problems related to the semi-discrete MTM system, we mention construction of conserved quantities which may clarify orbital stability of the discrete MTM solitons, similar to the work [22]. Another direction is to develop the inverse scattering transform for solutions of the Cauchy problem associated with the semi-discrete MTM system, similar to the work [21]. Since numerical simulations of the semi-discrete MTM system (1) present serious challenges, it may be interesting to look for another version of the integrable semi-discretization of the continuous MTM system [3].

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(Tao Xu) State Key Laboratory of Heavy Oil Processing, China University of Petroleum, Beijing 102249, China and College of Science, China University of Petroleum, Beijing 102249, China

(Dmitry E. Pelinovsky) Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1 and Department of Applied Mathematics, Nizhny Novgorod State Technical University, 24 Minin street, 603950 Nizhny Novgorod, Russia