Defective Colouring of Hypergraphs

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Abstract

We prove that the vertices of every \((r + 1)\)-uniform hypergraph with maximum degree \(\Delta\) may be coloured with \(c\left(\frac{\Delta}{d+1}\right)^{1/r}\) colours such that each vertex is in at most \(d\) monochromatic edges. This result, which is best possible up to the value of the constant \(c\), generalises the classical result of Erdős and Lovász who proved the \(d = 0\) case.

1 Introduction

Hypergraph colouring is a widely studied field with numerous deep results [4, 8–10, 15–17, 22–24, 28]. In a seminal contribution, Erdős and Lovász [13] proved that every \((r + 1)\)-uniform hypergraph with maximum degree \(\Delta\) has a vertex-colouring with at most \(c\Delta^{1/r}\) colours and with no monochromatic edge, where \(c\) is an absolute constant. The proof is a simple application of what is now called the Lovász local lemma, introduced in the same paper. Indeed, hypergraph colouring was the motivation for the development of the Lovász local lemma, which has become a staple of probabilistic combinatorics.

A vertex-colouring of a (hyper)graph is \(d\)-defective if each vertex is in at most \(d\) monochromatic edges (equivalently, the maximum degree of each monochromatic component is at most \(d\)). Defective colouring of graphs has been widely studied; the comprehensive survey [32] has over one hundred references to papers dedicated to defective colouring. One of the early results in the area, due to Lovász [25], is that every graph with maximum degree \(\Delta\) has a \(d\)-defective colouring with \(\left\lfloor \frac{\Delta}{d+1} \right\rfloor + 1\) colours. An example of one of the more recent highlights is that the defective analogue of Hadwiger’s conjecture holds. In particular, Edwards et al. [11] showed that every \(K_t\)-minor-free graph has a \(d(t)\)-defective \((t − 1)\)-colouring, for some function \(d(t)\). Here \(t − 1\) colours is best possible regardless of \(d\). The best defect bound known [30] is \(d(t) = \mathcal{O}(t)\). Very little is known about defective colouring of hypergraphs.

This paper proves the common generalisation of the results of Lovász [25] and Erdős and Lovász [13] mentioned above.

**Theorem 1.** For all integers \(r \geq 1\) and \(d \geq 0\) and \(\Delta \geq \max\{d + 1, 50^{100^r}\}\), every \((r + 1)\)-uniform hypergraph \(G\) with maximum degree at most \(\Delta\) has a \(d\)-defective \(k\)-colouring, where

\[
k \leq 100 \left(\frac{\Delta}{d+1}\right)^{1/r}.
\]

Several notes on Theorem 1 are in order.

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• The bound on the number of colours in the theorem of Erdős and Lovász [13] and in Theorem 1 is best possible (up to the multiplicative constant) because of complete hypergraphs. Indeed, let $G$ be the $(r + 1)$-uniform complete hypergraph on $n$ vertices, which has maximum degree $\Delta = \binom{n}{r}/\binom{n-1}{r-1} \leq (\frac{n}{r})^r$. In any $d$-defective $k$-colouring of $G$, at least $\frac{k}{d}$ vertices are monochromatic, implying $d \geq \left( \frac{n}{2rk} \right)^r > \frac{\Delta}{(2rk)^r}$. Thus $k \geq \frac{1}{2r} \left( \frac{\Delta}{d} \right)^{1/r}$, which is within a constant factor of the upper bound in Theorem 1. It remains tight even for $(r + 1)$-uniform hypergraphs with no complete $(r + 2)$-vertex subhypergraph. For example, a hypergraph construction by Cooper and Mubayi [10, § 3.2.2] has this property.

• The correct multiplicative constant is not known even for $d = 0$ and $r = 2$ (that is, even for proper colouring of 3-uniform hypergraphs). In this case, the best upper bound known [31] is $[2\Delta^{1/2}]$ while the lower bound given by complete 3-uniform hypergraphs is $(1/\sqrt{2} + o(1))\Delta^{1/2}$.

• The assumption $\Delta \geq d + 1$ in Theorem 1 is reasonable, since if $\Delta \leq d$ then one colour suffices. The assumption that $\Delta \geq 50100^d$ enables the uniform constant 100 in the bound on $k$. Of course, one could drop the assumption and replace 100 by some constant $c_r$ depending on $r$.

• If $G$ is a linear hypergraph (that is, any two edges intersect in at most one vertex), then Theorem 1 may be proved directly with the Lovász local lemma. Non-linear hypergraphs are hard because the number of neighbours of a vertex $v$ is not precisely determined by the degree of $v$. See the start of Section 2 for details.

• Theorem 1 can be rephrased as saying that for any $k$, $G$ has a $k$-colouring with maximum monochromatic degree $O\left( \frac{n}{k^2} \right)$ for fixed $r$. This is similar to a result of Bollobás and Scott [6] who showed that for any $k$ every $(r + 1)$-uniform hypergraph with $m$ edges has a $k$-colouring with $O\left( \frac{m}{k} \right)$ monochromatic edges of each colour. In this light, Theorem 1 is a variant on so-called judicious partitions [1, 5–7, 19, 20, 29, 33, 34].

1.1 Notation

Let $G$ be a hypergraph, which consists of a finite vertex-set $V(G)$ and an edge-set $E(G) \subseteq 2^{V(G)}$. Let $e(G) := |E(G)|$. $G$ is $r$-uniform if every edge has size $r$. The link hypergraph of a vertex $v$ in $G$, denoted $G_v$, is the hypergraph with vertex-set $V(G) \setminus \{v\}$ and edge-set $\{e \subseteq V(G) \setminus \{v\} : e \cup \{v\} \in E(G)\}$. If $G$ is $(r + 1)$-uniform, then $G_v$ is $r$-uniform. The degree of a set of vertices $S \subseteq V(G)$, denoted $\deg(S)$, is the number of edges in $G$ that contain $S$. We often omit set parentheses, so $\deg(x)$ and $\deg(u, v)$ denote the number of edges containing $x$ and the number of edges containing both $u$ and $v$, respectively. Let $\Delta(G) := \max\{\deg(v) : v \in V(G)\}$.

1.2 Probabilistic Tools

We use the following standard probabilistic tools.

Lemma 2 (Lovász local lemma [13]). Let $A$ be a set of events in a probability space such that each event in $A$ occurs with probability at most $p$ and for each event $A \in A$ there is a collection $A'$ of at most

\footnote{Let $e_i$ denote the $r$-dimensional vector with 1 in the $i^{th}$ coordinate and 0 elsewhere. Let $G$ be the $(r + 1)$-uniform hypergraph with vertex set $\{1, \ldots, n\}^r$ and whose edges are $\{v, v_1, \ldots, v_r\}$ where, for each $i$, $v_i - v$ is a positive multiple of $e_i$. Any $r + 2$ vertices induce at most two edges, so $G$ has contains no $(r + 2)$-clique. $G$ has maximum degree $\Delta = (n - 1)^r < n^r$. Suppose that $V(G)$ is coloured with $k \leq (\Delta/(d + 1))^{1/r} < n/(r(d + 1)^{1/r})$ colours. Then there is a monochromatic set $S \subseteq V(G)$ of size at least $(d + 1)^{1/r}r^{-1}$. Apply the following iterative deletion procedure to $S$: if, for some coordinate $j$ and integers $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_r \in \{1, \ldots, n\}$, there are less than $(d + 1)^{1/r}$ vertices in $S$ whose $i^{th}$ coordinate is $a_i$ for all $i \neq j$, then delete all these vertices. Let $S'$ be the set remaining after applying all such deletions. Each step deletes less than $(d + 1)^{1/r}$ vertices and at most $r^{-1}r^{-1}$ steps occur so $S'$ is non-empty. Let $v \in S'$ have the smallest coordinate sum. By definition of $S'$, for each $i$, there are at least $(d + 1)^{1/r}$ vertices $v_i \in S'$ with $v_i - v$ being a positive multiple of $e_i$. Hence, $v$ has degree at least $d + 1$ in $S'$. Therefore, every $d$-defective colouring of $G$ uses more than $(\Delta/(d + 1))^{1/r}/r$ colours.}
We will need a version of Chernoff for negatively correlated random variables, for example, vertex of the edges in $F$ changes the value of $X$ by at most $c^2$. Proof

Finally we need McDiarmid’s bounded differences inequality [26].

Lemma 4 (Chernoff bound). Let $X \sim \text{Bin}(n, p)$. For any $\epsilon \in [0, 1]$,

$$
\mathbb{P}(X \geq (1 + \epsilon)\mathbb{E}(X)) \leq \exp(-\epsilon^2 np / 3),
$$

$$
\mathbb{P}(X \leq (1 - \epsilon)\mathbb{E}(X)) \leq \exp(-\epsilon^2 np / 2).
$$

We will need a version of Chernoff for negatively correlated random variables, for example, see [21, Thm. 1]. Boolean random variables $X_1, \ldots, X_n$ are negatively correlated if, for all $S \subseteq \{1, \ldots, n\}$,

$$
\mathbb{P}(X_i = 1 \text{ for all } i \in S) \leq \prod_{i \in S} \mathbb{P}(X_i = 1).
$$

Lemma 5 (Chernoff for negatively correlated variables). Suppose $X_1, \ldots, X_n$ are negatively correlated Boolean random variables with $\mathbb{P}(X_i = 1) \leq p$ for all $i$. Then, for any $t \geq 0$,

$$
\mathbb{P}(\sum X_i \geq pn + t) \leq \exp(-2t^2 / n).
$$

Finally we need McDiarmid’s bounded differences inequality [26].

Lemma 6 (McDiarmid’s inequality). Let $T_1, \ldots, T_n$ be $n$ independent random variables. Let $X$ be a random variable determined by $T_1, \ldots, T_n$, such that changing the value of $T_j$ (while fixing the other $T_i$) changes the value of $X$ by at most $c_j$. Then, for any $t \geq 0$,

$$
\mathbb{P}(X \geq \mathbb{E}(X) + t) \leq \exp\left(-\frac{2t^2}{\sum c_j^2}\right).
$$

2 Proof

For motivation we first consider a na"ıve application of the Lovász local lemma. Suppose $G$ is a linear $(r + 1)$-uniform hypergraph. Colour $G$ with $k := \lceil 100(\frac{A}{d+1})^{1/r} \rceil > 99(\frac{A}{d+1})^{1/r}$ colours uniformly at random. For each set $F$ of $d + 1$ edges all containing a common vertex, let $B_F$ be the event that the vertex set of $F$ is monochromatic. Then, since $G$ is linear, $p := \mathbb{P}(B_F) = k^{-r(d+1)}$. For a fixed $F$, the number of $F'$ sharing a vertex with $F$ is at most $D := (r(d + 1) + 1)A(r + 1)(\frac{1}{A})$; here we have specified the vertex shared with $F$, the edge containing that vertex, the common vertex of the edges in $F'$, and the remaining $d$ edges of $F'$. Now $D \leq 3r^2d\Delta(e\Delta/d)^d$ and so

$$
4pD \leq 4 \cdot 99^{-r(d+1)} (\frac{d+1}{A})^{d+1} \cdot 3r^2e^d d^{-d+1} \Delta^{d+1}
$$

$$
= 12r^2e^d \cdot 99^{-r(d+1)} \cdot d(d + 1)(\frac{d+1}{A})^d
$$

$$
\leq 24r^2e^d d^d \cdot 99^{-r(d+1)} \leq 1.
$$

Hence, by the Lovász local lemma, there is a colouring in which no $B_F$ occurs; that is, there is a $d$-defective $k$-colouring of $G$. It was crucial in this argument that $G$ was linear so that the powers of $\Delta$ in $D$ and $p$ cancelled out exactly. For non-linear $G$, the number of neighbours of a vertex $v$ is not determined by the degree of $v$ and so $p$ may be larger without a corresponding decrease in $D$. A more involved argument is required.
2.1 First Steps

Here we outline our colouring strategy before diving into the details. We are given an \((r+1)\)-uniform hypergraph \(G\) with maximum degree \(\Delta\) and wish to colour its vertices so that every vertex is in at most \(d\) monochromatic edges. For a fixed colouring \(\phi\), the monochromatic degree of a vertex \(v\), denoted \(\deg_{\phi}(v)\), is the number of monochromatic edges containing \(v\) (which must have colour \(\phi(v)\)).

First we colour the vertices of \(G\) uniformly at random with \(k\) colours where \(k = \lfloor 49\left(\frac{\Delta}{d} + 1\right)^{1/r} \rfloor\). Since \(\Delta \geq d + 1\), we have \(k > 48\left(\frac{\Delta}{d} + 1\right)^{1/r}\). Say a vertex is bad if its monochromatic degree is greater than \(d\) and good otherwise. We are aiming for a colouring in which every vertex is good. The expected monochromatic degree of a vertex \(v\) in such a colouring is \(k^{-r}\deg(v) \leq k^{-r}\Delta < 48^{-r}(d + 1)\). In particular, each individual vertex has small (certainly, by Markov’s inequality, less than \(48^{-r}\)) probability of being bad. However, the goodness of a vertex \(v\) depends on the colours assigned to vertices in the neighbourhood of \(v\) and so \(48^{-r}\) is not a sufficiently small probability to conclude (by, say, the Lovász local lemma) that there is a particular colouring for which all vertices are good.

Instead of colouring all of \(G\) with a single random colouring, we do so over many rounds. After a round (where we coloured a hypergraph \(G\)), any good vertices will keep their colours and be discarded (they have been coloured appropriately). Let \(G'\) be the subhypergraph of \(G\) induced by the bad vertices. In the next round we uniformly and randomly colour the vertices of \(G'\) with a new palette of colours completely disjoint from those used in previous rounds. Using new colours ensures that monochromatic edges can only be produced within individual rounds. If the palettes all have the same size and the process runs for too many rounds, then we will end up using too many colours. However, if \(\Delta(G') \leq 2^{-r}\Delta(G)\), then we can use half the number of colours in the next round and so use \(O\left(\left(\frac{\Delta}{d} + 1\right)^{1/r}\right)\) colours across all the rounds. Thus, our aim is to prove the following nibble-style lemma from which Theorem 1 easily follows.

**Lemma 7.** Fix non-negative integers \(r, \Delta, d\) with \(r \geq 1\) and \(\Delta \geq \max\{d + 1, 50^{50r^3}\}\). Then every \((r+1)\)-uniform hypergraph \(G\) with maximum degree at most \(\Delta\) has a partial colouring with at most \(49\left(\frac{\Delta}{d} + 1\right)^{1/r}\) colours such that every coloured vertex has monochromatic degree at most \(d\) and the subhypergraph \(G'\) of \(G\) induced by the uncoloured vertices satisfies \(\Delta(G') \leq 2^{-r}\Delta\).

**Proof of Theorem 1 assuming Lemma 7.** We start with a \((r+1)\)-uniform hypergraph \(G\) with maximum degree at most \(\Delta = \Delta_0\) for some \(\Delta_0 \geq \max\{d + 1, 50^{100r^4}\}\). Apply Lemma 7 to get a partial colouring of \(G\) where:

- every vertex has monochromatic degree at most \(d\),
- at most \(49\left(\frac{\Delta_0}{d+1}\right)^{1/r}\) colours are used, and
- the subhypergraph \(G_1\) of \(G\) induced by uncoloured vertices has \(\Delta(G_1) \leq \Delta_1 = 2^{-r}\Delta_0\).

Iterate this procedure (using a palette of new colours each round) to obtain, for \(i = 0, 1, \ldots\), an induced subhypergraph \(G_i\) of \(G\) with \(\Delta(G_i) \leq \Delta_i = 2^{-r}\Delta\) such that \(G[V(G) - V(G_i)]\) has been coloured with at most

\[
49\left(\frac{\Delta_0}{d+1}\right)^{1/r} + 49\left(\frac{\Delta_1}{d+1}\right)^{1/r} + \cdots + 49\left(\frac{\Delta_{i-1}}{d+1}\right)^{1/r} = 49\left(\frac{\Delta}{d+1}\right)^{1/r}(1 + 2^{-1} + \cdots + 2^{-(i-1)}) \leq 98\left(\frac{\Delta}{d+1}\right)^{1/r}
\]

colours and every monochromatic degree is at most \(d\). Continue carrying out rounds of colouring until \(\Delta_i < d + 1\) or \(\Delta_i < 50^{50r^3}\).

First suppose that \(\Delta_i < d + 1\) and so \(\Delta(G_i) < d\). Use a single new colour on the entirety of \(G_i\) to give a \(d\)-defective colouring of \(G\). Now suppose that \(d + 1 \leq \Delta_i < 50^{50r^3}\). Properly colour \(G_i\)
with \( \Delta(G_i) + 1 \leq 50^{50|\mathcal{V}|} \) colours. This gives a \( d \)-defective colouring of \( G \) with at most

\[
98\left( \frac{\Delta}{d+1} \right)^{1/r} + 50^{50|\mathcal{V}|} \leq 100\left( \frac{\Delta}{d+1} \right)^{1/r}
\]

colours. The final inequality uses the fact that \( \Delta \geq 50^{100|\mathcal{V}|} \) and \( d + 1 < 50^{50|\mathcal{V}|} \).

Recall that a vertex is bad for a colouring \( \phi \) if it has monochromatic degree at least \( d + 1 \). Say that an edge \( e \) is bad for a colouring \( \phi \) if every vertex in \( e \) is bad (note that a bad edge is not necessarily monochromatic). Furthermore, say that a vertex is terrible for a colouring \( \phi \) if it is incident to more than \( 2^{-r}\Delta \) bad edges. Lemma 7 says that there is some colouring for which no vertex is terrible. The key to the proof of Lemma 7 is to show that a vertex is terrible with low probability.

In the remainder of the paper, we use the definitions of good, bad, and terrible given above and also set \( k := \lceil 49\left( \frac{\Delta}{d+1} \right)^{1/r} \rceil \).

**Lemma 8.** Let \( \Delta \geq \max\{ d + 1, 50^{50|\mathcal{V}|} \} \). Let \( G \) be an \((r + 1)\)-uniform hypergraph with maximum degree at most \( \Delta \). In a uniformly random \( k \)-colouring of \( V(G) \), each vertex \( v \) of \( G \) is terrible with probability at most \( \Delta^{-5} \).

**Proof of Lemma 7 assuming Lemma 8.** Randomly and independently assign each vertex of \( G \) one of \( k \) colours. For each vertex \( v \), let \( A_v \) be the event that \( v \) is terrible. By Lemma 8, \( \mathbb{P}(A_v) \leq \Delta^{-5} \). The event \( A_v \) depends solely on the colours assigned to vertices in the closed second neighbourhood of \( v \). Thus if two vertices \( v \) and \( w \) are at distance at least 5 in \( G \), then \( A_v \) and \( A_w \) are independent. Thus each event \( A_v \) is mutually independent of all but at most \( 2(r\Delta)^4 \) other events \( A_w \). Since \( 4\Delta^{-5} \cdot 2(r\Delta)^4 = 8r^4/\Delta \leq 1 \), by the Lovász local lemma, with positive probability, no event \( A_v \) occurs. Thus, there exists a \( k \)-colouring \( \phi \) of \( G \) such that no vertex is terrible. Let \( G' \) be the subgraph of \( G \) induced by the bad vertices. Since no vertex is terrible, \( \Delta(G') \leq 2^{-r}\Delta \). Uncolour all the bad vertices: every coloured vertex is good and so has monochromatic degree at most \( d \).

It remains to prove Lemma 8, which we do in Section 2.4. We have now reduced the question to a local property of a random \( k \)-colouring.

A vertex \( v \) is terrible if it is bad and at least \( 2^{-r}\Delta \) edges in its link graph, \( G_v \), are bad. Analysing the dependence between the badness of different edges in \( G_v \) is difficult. We sidestep this issue by using a sunflower decomposition. A **sunflower with \( p \) petals** is a collection \( A_1, \ldots, A_p \) of sets for which \( A_i \setminus K \) are pairwise disjoint where \( K := A_1 \cap \cdots \cap A_p \) (that is, \( A_i \cap A_j = K \) for all distinct \( i, j \)). \( K \) is the **kernel** of the sunflower and \( A_1 \setminus K, \ldots, A_p \setminus K \) are its **petals**.

If \( A_1, \ldots, A_p \) are distinct edges of a uniform hypergraph that form a sunflower, then the petals are pairwise disjoint, non-empty and have the same size. The kernel may be empty in which case the sunflower is a matching of size \( p \). In a random colouring, the colourings on different petals of a sunflower are independent. Hence, it will be useful to partition the edges of hypergraphs into sunflowers with many petals together with a few edges left over.

**Lemma 9** (Sunflower decomposition). Let \( H \) be an \( r \)-uniform hypergraph and \( a \) be a positive integer. There are edge-disjoint subhypergraphs \( H_1, \ldots, H_s \) of \( H \) such that:

- Each \( H_i \) is a sunflower with exactly \( a \) petals.
- \( H' = H - (E(H_1) \cup \cdots \cup E(H_s)) \) has fewer than \( (ra)^r \) edges.
Applying Lemma 5 to vertex-set is established in Lemma 11. By symmetry of the colours, this is just the Harris inequality. Fix \( \ell > V \) increasing in all other \( C \).

Fix distinct colours \( c_1, \ldots, c_\ell \) and let \( V_j \) be the set of vertices given colour \( c_j \). Conditioned on the event \( C_j = \{ v_j \text{ is coloured } c_j \} \), \( B_j \) is increasing in \( V_j \), while \( D \) is non-increasing in \( V_j \). Hence, by the Harris inequality [18], \( \mathbb{P}(B_j \cap D \mid C_j) \leq \mathbb{P}(B_j \mid C_j) \mathbb{P}(D \mid C_j) \). Using this and the symmetry of the colours gives

\[
\mathbb{P}(B_j \mid D) = \mathbb{P}(B_j \mid D \cap C_j) = \frac{\mathbb{P}(B_j \cap D \mid C_j)}{\mathbb{P}(D \mid C_j)} \leq \mathbb{P}(B_j \mid C_j) = \mathbb{P}(B_j).
\]

But \( \mathbb{P}(B_j) \leq 48^{-r} \), so \( \mathbb{E}(X \mid D) = \sum_j \mathbb{P}(B_j \mid D) \leq \ell \cdot 48^{-r} \).

Let \( C \) be the event \( \{ \text{each } v_i \text{ is coloured } c_i \} \). Conditioned on \( C \), \( B_j \) is increasing in \( V_j \) and non-increasing in all other \( V_i \). We claim the \( B_i \) are negatively correlated on the event \( C \). For \( \ell = 2 \) this is just the Harris inequality. Fix \( \ell > 2 \) and let \( S \) be a set of indices: we need to show

\[
\mathbb{P}((\cap_{i \in S} B_i) \mid C) \leq \prod_{i \in S} \mathbb{P}(B_i \mid C).
\]

If \( |S| \leq 1 \), then there is equality. Otherwise let \( i_1, i_2 \in S \). Now \( B_{i_1} \cap B_{i_2} \) is increasing in \( V_{i_1} \cup V_{i_2} \) and non-increasing in all other \( V_i \). By induction,

\[
\mathbb{P}((\cap_{i \in S} B_i) \mid C) \leq \mathbb{P}(B_{i_1} \cap B_{i_2} \mid C) \cdot \prod_{i \in S \setminus \{i_1, i_2\}} \mathbb{P}(B_i \mid C) \leq \prod_{i \in S} \mathbb{P}(B_i \mid C).
\]

By symmetry of the colours, \( \mathbb{P}(B_i \mid C) = \mathbb{P}(B_i \mid D) \) for all \( i \) and also \( \mathbb{P}((\cap_{i \in S} B_i) \mid C) = \mathbb{P}((\cap_{i \in S} B_i) \mid D) \) for any set of indices \( S \). In particular, the \( B_i \) are negatively correlated on the event \( D \).

Applying Lemma 5 to \( X_1, \ldots, X_\ell \) gives the result. \( \square \)
**Lemma 11.** Let \( S \) be a set of vertices of \( G \) with \( 10^{6r} \leq |S| \leq k^{1/2} \). In a uniformly random \( k \)-colouring of \( V(G) \), with failure probability at most \( 2(e|S|^{-1/2})^{|S|^{1/2}} \), fewer than \( 3^{-r} |S| \) vertices of \( S \) are bad.

**Proof.** Let \( A \) be the event that the number of distinct colours on \( S \) is at most \( |S| - |S|^{1/2} \). We first give an upper bound for \( \mathbb{P}(A) \). The probability that a fixed vertex does not have a unique colour is at most \( |S|/k \). If \( A \) does occur, then at least \( |S|^{1/2} \) vertices of \( S \) do not have a unique colour. Hence,

\[
\mathbb{P}(A) \leq \left( \frac{|S|}{|S|^{1/2}} \right) \left( \frac{|S|}{k} \right)^{|S|^{1/2}} \leq \left( \frac{|S|}{|S|^{1/2}} \right) |S|^{-|S|^{1/2}}.
\]

If \( A \) does not occur, then there is an subset \( S' \subseteq S \) of size \( |S| - |S|^{1/2} \) where the vertices are all given different colours. Fix such an \( S' \) and let \( X \) be the number of bad vertices in \( S \) and \( X' \) be the number of bad vertices in \( S' \). Note that if \( X' < 4^{-r} |S'| \), then \( X < 4^{-r} |S'| + |S|^{1/2} \leq 4^{-r} |S| + |S|^{1/2} \leq 3^{-r} |S| \).

Let \( D \) be the event that all vertices of \( S' \) get different colours. By Lemma 10 and the previous paragraph,

\[
\mathbb{P}(X \geq |S| \cdot 3^{-r} \mid D) \leq \mathbb{P}(X' \geq |S'| \cdot 4^{-r} \mid D) \\
\leq \mathbb{P}(X' \geq |S'| \cdot 48^{-r} + |S'| \cdot 4^{-r} / \sqrt{2} \mid D) \leq \exp(-|S'| \cdot 4^{-2r}).
\]

Let \( \overline{A} \) be the complement of \( A \). Taking a union bound over all \( S' \),

\[
\mathbb{P}(\{X \geq |S| \cdot 3^{-r}\} \cap \overline{A}) \leq \left( \frac{|S|}{|S|^{1/2}} \right) \exp(-|S'| \cdot 4^{-2r}).
\]

Finally,

\[
\mathbb{P}(X \geq |S| \cdot 3^{-r}) \leq \left( \frac{|S|}{|S|^{1/2}} \right) (\exp(-|S'| \cdot 4^{-2r}) + |S|^{-|S|^{1/2}}) \\
\leq (e|S|^{1/2}) |S|^{1/2} \cdot 2 |S|^{-|S|^{1/2}} = 2(e|S|^{-1/2})|S|^{1/2}.
\]

\( \square \)

### 2.3 When \( k \) is small: \( k \leq \Delta^{1/(6r^2)} \)

Recall that \( 48(\frac{\Delta}{2^{r+1}})^{1/r} < k \leq 49(\frac{\Delta}{2^{r+1}})^{1/r} \) throughout. We need a simple max cut lemma.

**Lemma 12 (Max cut).** Let \( G \) be a hypergraph whose edges have size at most \( r + 1 \) and let \( \ell \) be a positive integer. There is a partition \( V_1 \cup \cdots \cup V_\ell \) of \( V(G) \) such that, for every vertex \( x \in V_i \), the number of edges containing \( x \) and at least one more vertex from \( V_i \) is at most \( r \deg(x)/\ell \).

**Proof.** Throughout the proof, vertices \( u, v, x \) are distinct. Choose a partition \( V_1 \cup \cdots \cup V_\ell \) of \( V(G) \) into \( \ell \) parts that minimises

\[
\sum_i \sum_{u,v \in V_i} \deg(u, v).
\]

(1)

Fix a vertex \( x \) and suppose it is in some part \( V_a \). By minimality, for all \( i \),

\[
\sum_{u \in V_a} \deg(u, x) \leq \sum_{u \in V_i} \deg(u, x),
\]

or else we could increase (1) by moving \( x \) to \( V_i \). But

\[
\sum_i \sum_{u \in V_i} \deg(u, x) = \sum_{u \in V(G)} \deg(u, x) \leq r \deg(x),
\]

and so \( \sum_{u \in V_a} \deg(u, x) \leq r \deg(x)/\ell \). This last sum is at least the number of edges containing \( x \) and at least one more vertex from \( V_a \). \( \square \)
Given a large vertex-set $S$ we aim to show that, with high probability, a small proportion of its vertices are bad. We use Lemma 12 to split $S$ into parts so that very few edges have two vertices in the same part. Consider an arbitrary part $P$. We will show that, with high probability, a small proportion of the vertices in $P$ are bad. We do this by first revealing the random $k$-colouring on $V(G) - P$. Since $k$ is small, we get strong concentration on the distribution of colours on $V(G) - P$. We then reveal the colouring on $P$ and use this concentration to show that it is unlikely that $P$ has a high proportion of bad vertices.

**Lemma 13.** Suppose $\Delta \geq 50r^{3/2}$, $k \leq \Delta^{1/2}$ and let $S$ be a set of at least $(3k)^3\Delta^{1/(6r)}$ vertices of $G$. With failure probability at most $\Delta^{-6}$, in a uniformly random $k$-colouring of $V(G)$, fewer than $3^{-r}|S|$ vertices of $S$ are bad.

**Proof.** It will be helpful to partition $S$ into multiple parts such that not too many edges meet one part in more than one vertex. We therefore apply the max cut lemma, Lemma 12, to $G$ with $\ell = rk^r$, and restrict the resulting partition to $S$. We obtain a partition $P$ of $S$ into $rk^r$ parts such that, for every vertex $x \in S$, the number of edges containing $x$ and at least one more vertex from $x$’s part is at most $\deg(x)/k^r$. We say a part $P \in \mathcal{P}$ is big if $|P| \geq |S|/((50r(3k)^r))$ and is small otherwise.

Since there are $rk^r$ parts in $P$ and small parts have less than $|S|/(50r(3k)^r)$, the number of vertices of $S$ in small parts is less than $|S|/(50r(3k)^r) \cdot rk^r = 0.02 \cdot 3^{-r}|S|$. Hence, if $3^{-r}|S|$ vertices of $S$ are bad, then at least $0.98 \cdot 3^{-r}$ proportion of the vertices in big parts are bad, so some big part $P$ has at least $0.98 \cdot 3^{-r}|P|$ bad vertices. We now focus on a big part $P \in \mathcal{P}$ and show that, with failure probability at most $\Delta^{-6}$, at most $0.98 \cdot 3^{-r}|P|$ vertices of $P$ are bad.

For each vertex $x \in P$, let $G'_x$ be the $r$-uniform graph on $V(G) - P$, whose edges are those $e$ with $e \cup \{x\} \in E(G)$ (that is, $G'_x$ is the link graph of $x$ restricted to $V(G) - P$). Define the $r$-uniform auxiliary (multi)hypergraph $H_P$ to have vertex set $V(G) - P$ and edge set

$$E(H_P) = \bigcup_{x \in P} E(G'_x),$$

where edges are counted with multiplicity. Let $\phi$ be a uniformly random $k$-colouring of $V(G)$ and $\phi'$ be the restriction of $\phi$ to $V(G) - P$. Reveal $\phi'$ and let $X$ be the number of monochromatic edges of $H_P$, again counted with multiplicity.

We now apply McDiarmid’s inequality to show that $X$ concentrates. First note that $e(H_P) \leq |P| \cdot \Delta$ and $\mathbb{E}(X) = e(H_P)k^{(r-1)} \leq |P| \cdot \Delta k^{(r-1)}$. For a vertex $v \in V(H_P)$, changing $\phi'(v)$ changes the value of $X$ by at most $\deg_{H_P}(v)$. Now,

$$\sum_v \deg_{H_P}(v)^2 \leq \Delta \sum_v \deg_{H_P}(v) = r\Delta e(H_P) \leq r\Delta^2|P|.$$

By McDiarmid’s inequality (Lemma 6),

$$\mathbb{P}\left(X \geq \frac{1.1 \cdot \Delta|P|}{k^{r-1}}\right) \leq \mathbb{P}\left(X \geq \mathbb{E}(X) + \frac{0.1 \cdot \Delta|P|}{k^{r-1}}\right) \leq \exp\left(-\frac{|P|}{50r^2k^{2(r-1)}}\right) \leq \exp\left(-\frac{|S|}{2500r^2 \cdot 3^r \cdot k^{6r-2}}\right) \leq \exp\left(-k^2\Delta^{1/(6r)} / (2500r^2)\right) \leq \Delta^{-8}/2.\$$

For a vertex $x \in P$, say a colour is $x$-unhelpful if there are more than $(48^r - 1)/\Delta/k^r$ monochromatic edges of $G'_x$ of that colour. Say $x$ is unhelpful if there are more than $0.45 \cdot 3^{-r}k$ $x$-unhelpful colours. Note that if $x$ is unhelpful, then the number of monochromatic edges in $G'_x$ is greater than $0.45(48^r - 1) \cdot \Delta \cdot 3^{-r}/k^{r-1}$. Hence, if more than $0.45 \cdot 3^{-r} \cdot |P|$ vertices of $P$ are unhelpful, then
the number of monochromatic edges in $H_P$ is greater than $1.1 \cdot \Delta|P|/k'^{-1}$. We have just shown this occurs with probability less than $\Delta^{-8}/2$. Hence, with failure probability at most $\Delta^{-8}/2$, at least $(1 - 0.48 \cdot 3^{-i})|P|$ vertices of $P$ are helpful.

Suppose that at least $(1 - 0.48 \cdot 3^{-i})|P|$ vertices of $P$ are helpful; call the set of helpful vertices $P'$. Now reveal $\phi$ on $P$. For each vertex $x \in P'$, the probability that $x$ gets given an $x$-unhelpful colour is at most $0.45 \cdot 3^{-i}$. Let $Y$ be the number of $x \in P'$ coloured with an $x$-unhelpful colour. For different $x \in P'$, these events are independent (we have already revealed $\phi$ on $V(G) - P$) and so we may couple $Y$ with a random variable $Z \sim \text{Bin}(|P'|, 0.45 \cdot 3^{-i})$ so that $Y \leq Z$. Hence, by the Chernoff bound (Lemma 4),

$$\mathbb{P}(Y \geq 0.5 \cdot 3^{-i}|P'|) \leq \mathbb{P}(Z \geq 0.5 \cdot 3^{-i}|P'|) \leq \mathbb{P}(Z \geq 1.1 \cdot \mathbb{E}(Z)) \leq \exp(-0.45 \cdot 3^{-i}|P'|/300) \leq \exp(-k^2\Delta^{1/6r}/(6000r)) \leq \Delta^{-8}/2.$$

Hence, with failure probability at most $\Delta^{-8}/2 + \Delta^{-8}/2 = \Delta^{-8}$, at least $(1 - 0.5 \cdot 3^{-i})|P'| \geq (1 - 0.98 \cdot 3^{-i})|P|$ vertices of $P$ are coloured with an $x$-helpful colour.

We now show that if a vertex $x$ is given an $x$-helpful colour, then $x$ will be a good vertex (for $\phi$). There are at most $\deg(x)/k' \leq \Delta/k'$ edges of $G$ containing $x$ that have at least one more vertex in $P$, and as $x$ is given an $x$-helpful colour, there are at most $(48' - 1)\Delta/k'$ other monochromatic edges containing $x$. In particular, if $x$ is given an $x$-helpful colour, then at most $48'\Delta/k' < d + 1$ monochromatic edges contain $x$ and so $x$ is good. Hence, with failure probability at most $\Delta^{-8}$, at least $(1 - 0.98 \cdot 3^{-i})|P|$ vertices of $P$ are good, that is, at most $0.98 \cdot 3^{-i}|P|$ vertices of $P$ are bad.

Finally, taking a union bound over the big parts shows that the probability some big part $P$ has at least $0.98 \cdot 3^{-i}|P|$ bad vertices is at most $rk^2\Delta^{-8} \leq r\Delta^{-8+1/r} \leq \Delta^{-6}$, as required. $\square$

### 2.4 Proof of Lemma 8

To prove Lemma 8 we use the sunflower decompositions given by Lemma 9 to show that if a vertex is terrible, then some reasonably large set of vertices $S$ must have at least $3^{-r}$ proportion of its vertices being bad. Lemmas 11 and 13 show that this is unlikely.

**Proof of Lemma 8.** Recall that $\Delta \geq 50^{50r^3}$. Fix a vertex $v$ of $G$ and consider the link graph $G_v$, which is an $r$-uniform hypergraph. Recall that an edge of $G_v$ is bad if all its vertices are bad and is good otherwise. If $v$ is terrible, then at least $2^{-r}\Delta$ edges of $G_v$ are bad.

First suppose that $k \geq \Delta^{1/(6r^2)}$. By Lemma 9, there are edge-disjoint subgraphs $G_1, \ldots, G_6$ of $G_v$ each of which is a sunflower with exactly $\lfloor \Delta^{1/(12r^2)} \rfloor$ petals and such that $e(G_v - E(G_1 \cup \cdots \cup G_6)) < r^6 \cdot \Delta^{1/(12r)} \leq 6^{-6}\Delta$. Let $G' = G_1 \cup \cdots \cup G_6$. For each $G_i$, choose a vertex from each petal to form a vertex-set $S_i$. If $v$ is terrible, then the number of bad edges in $G'$ is at least

$$(2^{-r} - 6^{-r})\Delta \geq 3^{-r}\Delta \geq 3^{-r}e(G').$$

Hence, if $v$ is terrible, then there is some $i$ for which at least $3^{-r}e(G_i)$ edges of $G_i$ are bad. But, since $S_i$ contains exactly one vertex from each petal of $G_i$, at least $3^{-r}|S_i|$ vertices of $S_i$ are bad. Also, each $S_i$ has size $\lfloor \Delta^{1/(12r^2)} \rfloor \geq \Delta^{2/(25r^2)} \geq 50^{4r} \geq 10^{6r}$ and $\lfloor \Delta^{1/(12r^2)} \rfloor \leq k^{1/2}$. Hence, by Lemma 11, at least $3^{-r}|S_i|$ vertices of $S_i$ are bad with probability at most

$$2(e|S|^{-1/2})|S|^{1/2} \leq 2(e\Delta^{-1/(25r^2)})^{1/2} \leq 2(\Delta^{-1/(50r^2)})^{50^{4r}} \leq 2(\Delta^{-1/(50r^2)})^{400^2} = 2\Delta^{-8}.$$ 

Taking a union bound over $i$ shows that $v$ is terrible with probability at most $2s\Delta^{-8} \leq \Delta^{-5}$. 

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Now suppose that \( k \leq \Delta^{1/(6r^2)} \). By Lemma 9, there are edge-disjoint subgraphs \( G_1, \ldots, G_s \) of \( G_0 \) each of which is a sunflower with at least \( \Delta^{1/r}/(6r) \) petals and such that \( e(G_0) - E(G_1 \cup \ldots \cup G_s) < 6^{-r}\Delta \). Let \( G' = G_1 \cup \ldots \cup G_s \). For each \( G_i \), choose a vertex from each petal to form a vertex-set \( S_i \). If \( v \) is terrible, then the number of bad edges in \( G' \) is at least
\[
(2^{-r} - 6^{-r}) \Delta \geq 3^{-r} \Delta \geq 3^{-r} e(G').
\]
Hence, if \( v \) is terrible, then there is some \( i \) for which at least \( 3^{-r} e(G_i) \) edges of \( G_i \) are bad and so at least \( 3^{-r} |S_i| \) vertices of \( S_i \) are bad. Now, \( (3k)^3 \Delta^{1/(6r)} \leq 3^r \Delta^{1/(2r)} \Delta^{1/(6r)} \leq \Delta^{1/r}/(6r) \leq |S_i| \). Hence, by Lemma 13, at least \( 3^{-r} |S_i| \) vertices of \( S_i \) are bad with probability at most \( \Delta^{-6} \). Taking a union bound over \( i \) shows that \( v \) is terrible with probability at most \( s \Delta^{-6} \leq \Delta^{-5} \). \( \square \)

### 3 Open problems

As noted in the introduction, Erdős and Lovász proved that every \((r+1)\)-uniform hypergraph with maximum degree at most \( \Delta \) has chromatic number \( \chi(G) = O(\Delta^{1/r}) \). Frieze and Mubayi [17] improved this to \( O((\Delta / \log \Delta)^{1/r}) \) when \( G \) is a linear hypergraph and there have been similar improvements [8, 9, 24] when \( G \) satisfies other sparsity conditions (such as being triangle-free\(^2\)).

It would be interesting to know whether logarithmic improvements occur for defective colourings of sparse hypergraphs. Frieze and Mubayi [16] showed that there exist \((r+1)\)-uniform linear hypergraphs \( G \) with maximum degree \( \Delta \) and \( \chi(G) = \Omega((\Delta / \log \Delta)^{1/r}) \). Consider a \( d \)-defective \( k \)-colouring of \( G \) (where \( d \geq 2 \)). Each colour class induces a linear \((r+1)\)-uniform hypergraph with maximum degree \( d \) and so is \( O((d / \log d)^{1/r}) \)-colourable. In particular,
\[
k = \Omega\left(\left(\frac{\Delta}{\log \Delta} \cdot \frac{\log d}{d}\right)^{1/r}\right).
\]
We conjecture this is tight.

**Conjecture 14.** Every \((r+1)\)-uniform linear hypergraph is \( k \)-colourable with defect \( d \geq 2 \), where
\[
k = O\left(\left(\frac{\Delta}{\log \Delta} \cdot \frac{\log d}{d}\right)^{1/r}\right).
\]

Finally, it would be interesting to extend Theorem 1 to the list colouring setting.

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\(^2\)A triangle in a hypergraph consists of edges \( e, f, g \) and vertices \( u, v, w \) such that \( u, v \in e \) and \( v, w \in f \) and \( w, u \in g \) and \( \{u, v, w\} \cap e \cap f \cap g = \emptyset \).
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