Limits of small functors *

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Abstract

For a small category \( \mathcal{K} \) enriched over a suitable monoidal category \( \mathcal{V} \), the free completion of \( \mathcal{K} \) under colimits is the presheaf category \( [\mathcal{K}^{\text{op}}, \mathcal{V}] \). If \( \mathcal{K} \) is large, its free completion under colimits is the \( \mathcal{V} \)-category \( \mathcal{P}\mathcal{K} \) of small presheaves on \( \mathcal{K} \), where a presheaf is small if it is a left Kan extension of some presheaf with small domain. We study the existence of limits and of monoidal closed structures on \( \mathcal{P}\mathcal{K} \).

A fundamental construction in category theory is the category of presheaves \( [\mathcal{K}^{\text{op}}, \text{Set}] \) on a small category \( \mathcal{K} \). Among many other important properties, it is the free completion of \( \mathcal{K} \) under colimits. If the category \( \mathcal{K} \) is large, then the full presheaf category \( [\mathcal{K}^{\text{op}}, \text{Set}] \) is not the free completion of \( \mathcal{K} \) under colimits; indeed it is not even a legitimate category, insofar as its hom-sets are not in general small.

In some contexts it is more appropriate to consider not all the presheaves on \( \mathcal{K} \), but only the small ones: a presheaf \( F : \mathcal{K}^{\text{op}} \to \text{Set} \) is said to be small if it is the left Kan extension of some presheaf whose domain is small. This is equivalent to \( F \) being the left Kan extension of its restriction to some small full subcategory of its domain, or equally to its being a small colimit of representables. The natural transformations between two small presheaves on \( \mathcal{K} \) do form a small set, and so the totality of small presheaves on \( \mathcal{K} \) forms a genuine category \( \mathcal{P}\mathcal{K} \) with small hom-sets. Furthermore, \( \mathcal{P}\mathcal{K} \) is in fact the free completion of \( \mathcal{K} \) under colimits. Of course if \( \mathcal{K} \) is small, then every presheaf on \( \mathcal{K} \) is small, and so \( \mathcal{P}\mathcal{K} \) is just \( [\mathcal{K}^{\text{op}}, \text{Set}] \), but in general this is not the case.

Although \( \mathcal{P}\mathcal{K} \) is the free completion of \( \mathcal{K} \) under colimits, it does not have all the good properties of \( [\mathcal{K}^{\text{op}}, \text{Set}] \) for small \( \mathcal{K} \). For example it is not necessarily complete or cartesian closed. In this paper we study, among other things, when \( \mathcal{P}\mathcal{K} \) does have such good properties.

In fact we work not just with ordinary categories, but with categories enriched over a suitable monoidal category \( \mathcal{V} \). Once again, if \( \mathcal{K} \) is small then \( [\mathcal{K}^{\text{op}}, \mathcal{V}] \) is the free completion of \( \mathcal{K} \) under colimits, but for large \( \mathcal{K} \) this is no longer the case; the illegitimacy of \( [\mathcal{K}^{\text{op}}, \mathcal{V}] \) in that case is more drastic: it is not even a \( \mathcal{V} \)-category. The free completion of \( \mathcal{K} \) under colimits is the \( \mathcal{V} \)-category \( \mathcal{P}\mathcal{K} \) of small presheaves on \( \mathcal{K} \), where once again a presheaf is small if it is the left Kan extension of some presheaf with small domain; and once again the two reformulations of this notion can be made.

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The case $\mathcal{V} = \textbf{Set}$ is closely related to work by various authors. Freyd [7] introduced two smallness notions for presheaves on large categories. He called a functor $F : \mathcal{K}^{\text{op}} \to \textbf{Set}$ petty if there is a small family $(C_\lambda \in \mathcal{K})_{\lambda \in \Lambda}$ with an epimorphism

$$\sum_\lambda \mathcal{K}(-, C_\lambda) \to F;$$

and lucid if it is petty and for any representable $\mathcal{K}(-, A)$ and any pair of maps $u, v : \mathcal{K}(-, A) \to F$, their equalizer is petty. Freyd studied when the category of petty presheaves on $\mathcal{K}$ is complete, and when the category of lucid presheaves on $\mathcal{K}$ is complete, obtaining results similar to our Theorem 3.8 below. Rosický [15] showed that if $\mathcal{K}$ is complete, then a presheaf $F$ on $\mathcal{K}$ is lucid if and only if it is small; one can then deduce our Corollary 3.9 from the results of Freyd. Rosický also characterized, in the case $\mathcal{V} = \textbf{Set}$, when $\mathcal{P}\mathcal{K}$ is cartesian closed; see Example 7.4 below. In a slightly different direction, the existence of limits in free completions under some class of colimits was studied in [9].

In the enriched case, the fact, mentioned above, that $\mathcal{P}\mathcal{K}$ is the free completion of $\mathcal{K}$ under colimits, is due to Lindner [14]. The existence of limits or monoidal closed structures on $\mathcal{P}\mathcal{K}$ seems not to have been considered in the enriched setting.

Some of our results have been used in abstract homotopy theory; for example Corollary 3.9 was used in [5]. The idea is that one wants to have a complete and cocomplete category of diagrams of some particular type, where the indexing category is large. In this context one is particularly interested in the case $\mathcal{V} = \textbf{SSet}$, the category of simplicial sets.

In Section 1 we review the required background from enriched category theory, and in Section 2 the notion of small functor. Then in Section 3 we prove the fundamental result that $\mathcal{P}\mathcal{K}$ is complete if and only if it has limits of representables; thus in particular $\mathcal{P}\mathcal{K}$ is complete if $\mathcal{K}$ is so. In Section 4 we refine the results of the previous section to deal not with arbitrary (small) limits, but with limits of some particular type, such as finite limits or finite products. In Section 5 we deduce from the earlier results various known results about the case $\mathcal{V}_0 = \textbf{Set}$ of ordinary categories, before extending them to the case where $\mathcal{V}_0$ is a presheaf category. Section 6 concerns not the existence of limits in $\mathcal{P}\mathcal{K}$ but the preservation of limits by functors $\mathcal{P}\mathcal{F} : \mathcal{P}\mathcal{K} \to \mathcal{P}\mathcal{L}$ given by left Kan extension along $F^{\text{op}} : \mathcal{K}^{\text{op}} \to \mathcal{L}^{\text{op}}$. In Section 7 we study monoidal closed structures on $\mathcal{P}\mathcal{K}$ using the notion of promonoidal category. In Section 8 we consider limits of small functors with codomain a locally presentable category $\mathcal{M}$, generalizing the earlier case of $\mathcal{M} = \mathcal{V}$. Finally in Section 9 we briefly discuss Isbell conjugacy for large categories.

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1 Review of relevant enriched category theory

We shall work over a symmetric monoidal closed category $\mathcal{V}$. The tensor product is denoted $\otimes$, the unit object $I$, and the internal hom $[\cdot, \cdot]$. Where necessary the underlying ordinary category is denoted $\mathcal{V}_0$. 
We suppose that this underlying ordinary category is locally presentable \([3,2]\): thus for some regular cardinal \(\alpha\) and some small category \(\mathcal{C}\) with \(\alpha\)-small limits \(\mathcal{V}_0\) is equivalent to the category of \(\alpha\)-continuous functors from \(\mathcal{C}\) to \(\text{Set}\). It follows that \(\mathcal{V}_0\) is complete and cocomplete, and it turns out that \(\mathcal{C}\) is equivalent to the opposite of the full subcategory \((\mathcal{V}_0)_\alpha\) of \(\mathcal{V}_0\) consisting of the \(\alpha\)-presentable objects: these are the \(X \in \mathcal{V}_0\) for which \(\mathcal{V}_0(X,-) : \mathcal{V}_0 \to \text{Set}\) preserves \(\alpha\)-filtered colimits. By \([13]\), after possibly changing \(\alpha\), we may suppose that \((\mathcal{V}_0)_\alpha\) is closed in \(\mathcal{V}_0\) under the monoidal structure, so that \(\mathcal{V}\) is \textit{locally \(\alpha\)-presentable as a closed category}, in the sense of \([10]\).

We shall work throughout the paper over such a locally presentable closed category. This includes many important examples, such as the categories \(\text{Set}, \text{Ab}, \text{R-Mod}, \text{Cat}, \text{Gpd}\), and \(\text{SSet}\), of sets, abelian groups, \(R\)-modules (over a commutative ring \(R\)), categories, groupoids, and simplicial sets, as well as the two-element lattice \(\mathfrak{2}\). All these examples are locally \textit{finitely} presentable (that is, locally \(\aleph_0\)-presentable) but there are further examples which require a higher cardinal than \(\aleph_0\): for example any Grothendieck topos, the category \(\text{Ban}\) of Banach spaces and linear contractions, Lawvere’s category \([0, \infty]\) of extended non-negative real numbers, or the first-named author’s \(*\)-autonomous category \([−\infty, \infty]\) of extended real numbers. All categorical notions are understood to be enriched over \(\mathcal{V}\), even if this is not explicitly stated. (Thus category means \(\mathcal{V}\)-category, functor means \(\mathcal{V}\)-functor, and so on.) We fix a regular cardinal \(\alpha_0\) for which \(\mathcal{V}_0\) is locally \(\alpha_0\)-presentable and \((\mathcal{V}_0)_{\alpha_0}\) is closed under the monoidal structure. Henceforth “\(\alpha\) is a regular cardinal” will mean “\(\alpha\) is a regular cardinal and \(\alpha \geq \alpha_0\)”.

For such a \(\mathcal{V}\), it was shown in \([10]\) that there is a good notion of locally \(\alpha\)-presentable \(\mathcal{V}\)-category, for any regular cardinal \(\alpha \geq \alpha_0\). A locally \(\alpha\)-presentable \(\mathcal{V}\)-category \(\mathcal{K}\) is complete and cocomplete, and is equivalent to the \(\mathcal{V}\)-category of \(\alpha\)-continuous \(\mathcal{V}\)-functors from \(\mathcal{C}\) to \(\mathcal{V}\) for some small \(\mathcal{V}\)-category \(\mathcal{C}\) with \(\alpha\)-small limits. This \(\mathcal{C}\) can be identified with the opposite of the category of \(\alpha\)-presentable objects in \(\mathcal{K}\).

A \textit{weight} is a presheaf \(F : \mathcal{C}^{\text{op}} \to \mathcal{V}\), usually, although not always with small domain. The \textit{colimit} of a functor \(S : \mathcal{C} \to \mathcal{K}\) is denoted by \(F * S\), while the \textit{limit} of a functor \(S : \mathcal{C}^{\text{op}} \to \mathcal{K}\) is denoted by \(\{F,S\}\). When \(\mathcal{C}^{\text{op}}\) is the unit \(\mathcal{V}\)-category \(\mathcal{J}\), we may identify \(F\) with an object of \(\mathcal{V}\) and \(S\) with an object of \(\mathcal{C}\); we sometimes write \(F \cdot S\) for \(F * S\) and call it a tensor, and we sometimes write \(F \bowtie S\) for \(\{F,S\}\) and call it a cotensor.

## 2 Small functors

A functor \(F : \mathcal{K} \to \mathcal{V}\) is said to be \textit{small} if it is the left Kan extension of its restriction to some small full subcategory of \(\mathcal{K}\). This will clearly be the case if \(F\) is a small colimit of representables, for then we may take as the subcategory precisely those objects corresponding to the representables in the colimit. On the other hand, if \(F : \mathcal{K} \to \mathcal{V}\) is the left Kan extension of \(FJ\) along the inclusion \(J : \mathcal{C} \to \mathcal{K}\) of some small full subcategory, then \(F = (FJ) * \mathcal{K}(J,1)\), and so \(F\) is a small colimit of representables. Thus the small functors are precisely the small colimits of representables.

Of course if \(\mathcal{K}\) is itself small, then every functor from \(\mathcal{K}\) to \(\mathcal{V}\) is small. If on the other hand \(\mathcal{K}\) is locally presentable, then a functor \(F : \mathcal{K} \to \mathcal{V}\) is small if and only if it is accessible: that is, if and only if it preserves \(\alpha\)-filtered colimits for some regular cardinal \(\alpha\). For if \(F\) is accessible, then we may choose \(\alpha\) so that \(\mathcal{K}\) is locally \(\alpha\)-presentable and \(F\) preserves \(\alpha\)-filtered colimits; then \(F\) is the left Kan extension of its restriction to the full subcategory of \(\mathcal{K}\) consisting of the \(\alpha\)-presentable objects. Conversely, if \(F\) is the left Kan extension of its restriction to a small full subcategory \(\mathcal{C}\) of \(\mathcal{K}\), then we may choose a regular cardinal \(\alpha\) in such a way that \(\mathcal{K}\) is locally \(\alpha\)-presentable and
every object in $\mathcal{C}$ is $\alpha$-presentable in $\mathcal{K}$, and then $F$ preserves $\alpha$-filtered colimits.

**Remark 2.1** There is a corresponding result for the case where $\mathcal{K}$ is accessible, but we have not taken the trouble to formulate it here, since as usual there is a greater sensitivity to the choice of regular cardinal in the accessible case than in the locally presentable one.

The totality of small functors from $\mathcal{K}^{\text{op}}$ to $\mathcal{V}$ forms a $\mathcal{V}$-category $\mathcal{P}\mathcal{K}$ which is cocomplete and is in fact the free cocompletion of $\mathcal{K}$ via the Yoneda embedding $Y : \mathcal{K} \to \mathcal{P}\mathcal{K}$. In the case where $\mathcal{K}$ is small, $\mathcal{P}\mathcal{K}$ is simply the presheaf category $[\mathcal{K}^{\text{op}}, \mathcal{V}]$, but in general not every presheaf is small.

**Example 2.2** Let $\mathcal{V}$ be $\text{Set}$, and let $\mathcal{K}$ be any large set $X$, seen as a discrete category. Then a presheaf on $\mathcal{K}$ can be seen as an $X$-indexed set $A \to X$, and it is small if and only if $A$ is so.

The construction $\mathcal{P}\mathcal{K}$ is pseudofunctorial in $\mathcal{K}$, and forms part of a pseudomonad $\mathcal{P}$ on $\mathcal{V}$-$\text{Cat}$. We shall also consider free completions under certain types of colimit. Let $\Phi$ be a class of weights with small domain. For a $\mathcal{V}$-category $\mathcal{K}$ write $\Phi(\mathcal{K})$ for the closure of $\mathcal{K}$ in $\mathcal{P}\mathcal{K}$ under $\Phi$-colimits. The Yoneda embedding $Y : \mathcal{K} \to \Phi(\mathcal{K})$ exhibits $\Phi(\mathcal{K})$ as the free completion of $\mathcal{K}$ under $\Phi$-colimits. The class $\Phi$ is said to be saturated if, whenever $\mathcal{K}$ is small, $\Phi(\mathcal{K})$ consists exactly of the presheaves on $\mathcal{K}$ lying in $\Phi$. (This idea goes back to [3], where the word “closed” was used rather than “saturated”.') Once again the construction $\Phi(\mathcal{K})$ is pseudofunctorial in $\mathcal{K}$ and forms part of a pseudomonad $\Phi^*$ on $\mathcal{V}$-$\text{Cat}$. The union $\Phi^*$ of all the $\Phi(\mathcal{C})$ with $\mathcal{C}$ small is a new class of weights called the saturation of $\Phi$.

Thus far we have spoken only of smallness of presheaves, but we shall also have cause to consider smallness of more general functors. Once again, we say that a $\mathcal{V}$-functor $S : \mathcal{K} \to \mathcal{M}$ is small if it is the left Kan extension of some $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{M}$ with small domain, or equivalently, if it is the left Kan extension of its restriction to some small full subcategory of $\mathcal{K}$. This definition works best when $\mathcal{M}$ is cocomplete, so that one can form the relevant left Kan extensions, and we shall only use it in this context. An important case is where $\mathcal{M} = [\mathcal{C}, \mathcal{V}]$ for some small $\mathcal{C}$. We say that $S : \mathcal{K} \to [\mathcal{C}, \mathcal{V}]$ is pointwise small if the composite of $S$ with each evaluation functor $\text{ev}_C : [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ is small.

**Lemma 2.3** A functor $S : \mathcal{K} \to [\mathcal{C}, \mathcal{V}]$ is small if and only if it is pointwise small.

**Proof:** Since the evaluation functors preserve Kan extensions the “only if” part is immediate. Conversely, if $S$ is pointwise small, then for each $C$ there is a small full subcategory $\mathcal{B}_C$ of $\mathcal{K}$ with the property that $\text{ev}_C S$ is the left Kan extension of its restriction to $\mathcal{B}_C$. Since $\mathcal{C}$ is small, the union $\mathcal{B}$ of the $\mathcal{B}_C$ is small, and now each $\text{ev}_C S$ is the left Kan extension of its restriction to $\mathcal{B}$, hence the same is true of $S$. □

In Section 3 we shall also consider the case where $\mathcal{M}$ is locally presentable.

### 3 Limits of small functors

As observed above, if $\mathcal{K}$ is small then $\mathcal{P}\mathcal{K}$ is the full presheaf category $[\mathcal{K}^{\text{op}}, \text{Set}]$ which is of course not just cocomplete but also complete. In general, however, a category of the form $\mathcal{P}\mathcal{K}$ need not be complete, as the following example, based on Example 2.2 shows:
Example 3.1 If \( \mathcal{V} \) is \( \text{Set} \) and \( \mathcal{K} \) is a large discrete category then \( \mathcal{PK} \) has no terminal object.

We investigate which categories \( \mathcal{K} \) have the property that \( \mathcal{PK} \) is complete. First observe that since \( \mathcal{PK} \) contains the representables, any limit in \( \mathcal{PK} \) must be formed pointwise. Thus the question "is \( \mathcal{PK} \) complete?" may be rephrased as "are limits of small presheaves on \( \mathcal{K} \) small?" This may appear to involve consideration of the illegitimate \( [\mathcal{K}^{\text{op}}, \mathcal{V}] \), but in fact this is unnecessary. Given a weight \( \varphi: \mathcal{C} \to \mathcal{V} \), where \( \mathcal{C} \) is small, and a diagram \( S: \mathcal{C} \to \mathcal{PK} \), we may regard \( S \) as a functor \( \tilde{S}: \mathcal{K}^{\text{op}} \to [\mathcal{C}, \mathcal{V}] \), and compose \( \tilde{S} \) with \( \{\varphi, -\}: [\mathcal{C}, \mathcal{V}] \to \mathcal{V} \), and ask whether the composite \( \{\varphi, S-\}: \mathcal{K}^{\text{op}} \to \mathcal{V} \) is small.

An arbitrary \( \tilde{S}: \mathcal{K}^{\text{op}} \to [\mathcal{C}, \mathcal{V}] \) arises in this way from some \( S: \mathcal{C} \to \mathcal{PK} \) if and only if \( \tilde{S} \) is pointwise small; recall from the previous section that this means that each \( \text{ev}_C\tilde{S}: \mathcal{K}^{\text{op}} \to \mathcal{V} \) is small, but that it is equivalent to \( \tilde{S} \) itself being small.

**Proposition 3.2** The limit of \( S: \mathcal{C} \to \mathcal{PK} \) weighted by \( \varphi: \mathcal{C} \to \mathcal{V} \) exists if and only if \( \{\varphi, S-\} \) is small; \( \mathcal{PK} \) has all \( \varphi \)-limits if and only if \( \{\varphi, R-\} \) is small for every small \( R: \mathcal{K}^{\text{op}} \to [\mathcal{C}, \mathcal{V}] \).

Related to the existence of limits in \( \mathcal{K} \) is the existence of a right adjoint to \( \mathcal{PF}: \mathcal{PK} \to \mathcal{PL} \) for a functor \( F: \mathcal{K} \to \mathcal{L} \). Here \( \mathcal{PF} \) is given by left Kan extensions along \( F \), so if \( \mathcal{K} \) were small then \( \mathcal{PF} \) would have a right adjoint given by restriction along \( F \). In general, however, the restriction \( GF \) of a small \( G: \mathcal{L}^{\text{op}} \to \mathcal{V} \) need not be small; indeed the restriction \( \mathcal{L}(F, L): \mathcal{K}^{\text{op}} \to \mathcal{V} \) of a representable \( \mathcal{L}(-, L) \) need not be small. But if each \( \mathcal{L}(F, L) \) is small, we have the right adjoint:

**Proposition 3.3** For an arbitrary functor \( F: \mathcal{K} \to \mathcal{L} \), there is a right adjoint to \( \mathcal{PF}: \mathcal{PK} \to \mathcal{PL} \) if and only if \( \mathcal{L}(F, L) : \mathcal{K}^{\text{op}} \to \mathcal{V} \) is small for every object \( L \) of \( \mathcal{L} \), and then the right adjont is given by restriction along \( F \).

**Proof:** If \( \mathcal{PF} \) has a right adjoint \( R \), then

\[
RGA \cong \mathcal{PK}(YA, RG) \cong \mathcal{PL}(\mathcal{PF}YA,G) \cong \mathcal{PL}(YFA,G) \cong GFA
\]

for any \( G \) in \( \mathcal{PL} \), and so \( R \) must be given by restriction along \( F \). Thus \( RYL = \mathcal{L}(F, L) \), which must therefore be small.

Suppose conversely that each \( \mathcal{L}(F, L) \) is small. Each \( G \) in \( \mathcal{PL} \) is a small colimit of representables. Since restricting along \( F \) preserves colimits, \( GF \) is a small colimit of functors of the form \( \mathcal{L}(F, L) \), but these are small by assumption, so \( GF \) is small. \( \square \)

Our first example of a large category \( \mathcal{K} \) with \( \mathcal{PK} \) complete is the opposite of a locally presentable category.

**Proposition 3.4** \( \mathcal{PK} \) is complete if \( \mathcal{K}^{\text{op}} \) is locally presentable.

**Proof:** If \( \mathcal{K}^{\text{op}} \) is locally presentable and \( R: \mathcal{K}^{\text{op}} \to [\mathcal{C}, \mathcal{V}] \) is small, then for each object \( C \) of \( \mathcal{C} \) there is a regular cardinal \( \alpha_C \) for which \( \text{ev}_C R \) is \( \alpha_C \)-accessible. Since \( \mathcal{C} \) is small, we may choose a regular cardinal \( \alpha \) for which \( \mathcal{K}^{\text{op}} \) is an \( \alpha \)-accessible category, \( R \) is an \( \alpha \)-accessible functor, and \( \varphi \) is \( \alpha \)-presentable in \( [\mathcal{C}, \mathcal{V}] \). Then \( R \) and \( \{\varphi, -\} \) preserve \( \alpha \)-filtered colimits, hence so does \( \{\varphi, R-\} \). \( \square \)
Remark 3.5 The proposition remains true if $\mathcal{K}^{\text{op}}$ is accessible; the comments made in Remark 2.1 still apply.

Corollary 3.6 $\mathcal{P}\mathcal{K}$ is complete if $\mathcal{K}$ is $[\mathcal{A}, \mathcal{V}]^{\text{op}}$ for a small category $\mathcal{A}$.

In other words, $\mathcal{P}\mathcal{K}$ is complete if $\mathcal{K} = \mathcal{P}(\mathcal{A}^{\text{op}})^{\text{op}}$ for a small $\mathcal{A}$. We shall now show how to remove the hypothesis that $\mathcal{A}$ is small. First observe $\mathcal{P}\mathcal{J} : \mathcal{P}\mathcal{K} \to \mathcal{P}\mathcal{L}$ is given by left Kan extension along $J$, so is fully faithful if $J$ is so.

Proposition 3.7 $\mathcal{P}\mathcal{K}$ is complete if $\mathcal{K} = \mathcal{P}(\mathcal{L}^{\text{op}})^{\text{op}}$.

Proof: Let $\mathcal{C}$ be a small category and let $R : \mathcal{K}^{\text{op}} \to [\mathcal{C}, \mathcal{V}]$ be small; we must show that $\{\varphi, R-\}$ is small. Now $R$ is the left Kan extension of its restriction to a small full subcategory $\mathcal{D}$ of $\mathcal{P}(\mathcal{L}^{\text{op}})$. Each $D \in \mathcal{D}$ is a small functor $\mathcal{L} \to \mathcal{V}$, so is the left Kan extension of its restriction to some small $B_D$. The union $\mathcal{B}$ of the $B_D$ is small, and now the full inclusion $J : \mathcal{B}^{\text{op}} \to \mathcal{L}^{\text{op}}$ induces a full inclusion $\mathcal{P}J : \mathcal{P}(\mathcal{B}^{\text{op}}) \to \mathcal{P}(\mathcal{L}^{\text{op}})$ containing $\mathcal{D}$.

Now $\mathcal{B}$ is small, so $\mathcal{P}J$ has a right adjoint $J^*$ given by restriction along $J$, and thus $\text{Lan}_{\mathcal{P}J}$ is itself given by restriction along $J^*$. Since $R$ is the left Kan extension of its restriction $S$ along $\mathcal{P}J$, we have

$$\{\varphi, R-\} = \{\varphi, -\}R \cong \{\varphi, -\}\text{Lan}_{\mathcal{P}J}S \cong \{\varphi, -\}SJ^* \cong \text{Lan}_{\mathcal{P}J}\{\varphi, -\}S = \text{Lan}_{\mathcal{P}J}\{\varphi, S-\}$$

and so $\{\varphi, R-\}$ will be small if $\{\varphi, S-\}$ is so. Now $S : \mathcal{P}(\mathcal{B}^{\text{op}}) \to \mathcal{V}$ is the left Kan extension of its restriction to $\mathcal{D}$, hence small, and $\mathcal{B}$ is small, so by Corollary 3.6 we conclude that $\{\varphi, S-\}$ is small.

We are now ready to prove the main result of this section:

Theorem 3.8 $\mathcal{P}\mathcal{K}$ is complete if and only if it has limits of representables.

Proof: The “only if” part is trivial, so suppose that $\mathcal{P}\mathcal{K}$ has limits of representables. Let $\mathcal{L} = \mathcal{P}(\mathcal{K}^{\text{op}})^{\text{op}}$, and let $Z : \mathcal{K} \to \mathcal{L}$ be the Yoneda embedding. By Proposition 3.3 the fully faithful $\mathcal{P}Z : \mathcal{P}\mathcal{K} \to \mathcal{P}\mathcal{L}$ has a right adjoint if $\mathcal{L}(Z, L)$ is small for each $L$. But $\mathcal{L}(Z, L) = \mathcal{P}(\mathcal{K}^{\text{op}})(L, Y)$, where $L : \mathcal{K} \to \mathcal{V}$ is a small functor. Then $L$ is the left Kan extension of its restriction to some small full subcategory $J : \mathcal{B} \to \mathcal{K}$, and now $\mathcal{P}(\mathcal{K}^{\text{op}})(\text{Lan}_{J}(LJ), Y) = \mathcal{P}(\mathcal{B}^{\text{op}})(LJ, YJ)$ which is the $LJ$-weighted limit of a diagram of representables, thus small by assumption. This proves that $\mathcal{P}\mathcal{K}$ is a full coreflective subcategory of $\mathcal{P}\mathcal{L}$; since $\mathcal{P}\mathcal{L}$ is complete by Proposition 3.7 it follows that $\mathcal{P}\mathcal{K}$ is so.

Corollary 3.9 $\mathcal{P}\mathcal{K}$ is complete if $\mathcal{K}$ is so.

4 Particular types of limit

This section gives a more refined result, dealing with particular classes of limits. It also provides an alternative proof for the main results of the previous section. It is based on the ideas of [1].

Let $\Phi$ be a class of weights. For a $\mathcal{V}$-category $\mathcal{C}$, we write $\Phi\mathcal{C}$ for the closure of the representables in $\mathcal{P}\mathcal{C}$ under $\Phi$-weighted colimits. We suppose that the class $\Phi$ satisfies the following conditions:
(a) (smallness) If $C$ is small then so is $\Phi C$;
(b) (soundness) If $D$ is small and $\Phi$-complete, and $\psi : D \to V$ is $\Phi$-continuous, then $\psi \ast - : [D^{\text{op}}, V] \to V$ is $\Phi$-continuous.

**Example 4.1**

1. If $V$ is $\text{Set}$, then any sound doctrine in the sense of [1] provides an example. Thus one could take $\Phi$ to be the (class of weights corresponding to the) finite limits, or the $\alpha$-small limits for some regular cardinal $\alpha$, or the finite products, or the finite connected limits.

2. For any locally $\alpha$-presentable $V$, by the results of [10, (6.11),(7.4)] one can take $\Phi$ to be the class $P_{\alpha}$ of $\alpha$-small limits.

3. If $V$ is cartesian closed, then by the results of [4] (see also [12]) one can take $\Phi$ to be the class of finite products. In fact by the results of [4] this is still the case if $V$ is the algebras of any commutative finitary theory over a cartesian closed category.

**Lemma 4.2** If $K$ is $\Phi$-cocomplete and $J : C \to K$ is a small full subcategory, then the closure $\bar{C}$ of $C$ in $K$ under $\Phi$-colimits is small.

**Proof:** By the smallness assumption on $\Phi$, the free $\Phi$-cocompletion $\Phi C$ of $C$ is small. Then $\bar{C}$ is given, up to equivalence, by the full image of the $\Phi$-continuous extension $\bar{J} : \Phi C \to K$ of $J$; thus $\bar{C}$ is small since $\Phi C$ is so. □

**Proposition 4.3** If $K$ is $\Phi$-complete then so is $\mathcal{P}K$.

**Proof:** Let $\varphi : C \to V$ be in $\Phi$, with $C$ small, and let $S : \mathcal{K}^{\text{op}} \to [C, V]$ be small. Then $S$ is the left Kan extension of its restriction to some small full subcategory $J^{\text{op}} : B^{\text{op}} \to \mathcal{K}^{\text{op}}$. By the lemma, $S = \text{Lan}_{J^{\text{op}}} R$, where $B$ is small and $\Phi$-complete, $J : B \to \mathcal{K}$ is $\Phi$-continuous, and $R : B^{\text{op}} \to [C, V]$. Now $\mathcal{K}(K, J \ast -) : B \to V$ is $\Phi$-continuous for all $K \in \mathcal{K}$, so $\mathcal{K}(K, J \ast -) : B^{\text{op}} \to V$ is $\Phi$-continuous, so $\{\varphi, -\} : [C, V] \to V$ preserves the left Kan extension $S = \text{Lan}_{J^{\text{op}}} R$. In other words

$$\{\varphi, S\} = \{\varphi, \text{Lan}_{J^{\text{op}}} R\} = \text{Lan}_{J^{\text{op}}} \{\varphi, R\}$$

and so $\{\varphi, S\}$ is small. □

**Proposition 4.4** $\mathcal{P}K$ has all $\Phi$-limits if and only if it has $\Phi^*$-limits of representables.

**Proof:** Recall from Section 2 that $\Phi^*$ is the “saturation” of $\Phi$, so that a $V$-category has $\Phi$-limits if and only if it has $\Phi^*$-limits, and in particular if $\mathcal{P}K$ has $\Phi$-limits then it certainly has $\Phi^*$-limits of representables.

Let $Z : \mathcal{K} \to \mathcal{L}$ be the free $\Phi$-completion of $\mathcal{K}$ under $\Phi$-limits; explicitly, $\mathcal{L} = \Phi^*(\mathcal{K}^{\text{op}})^{\text{op}}$, and $Z$ is the restricted Yoneda embedding. Then $\mathcal{P}L$ is $\Phi$-complete by the previous proposition. Since $\mathcal{P}Z : \mathcal{P}K \to \mathcal{P}L$ is fully faithful, $\mathcal{P}K$ will be $\Phi$-complete provided that $\mathcal{P}Z$ has a right adjoint. But this will happen if and only if $\mathcal{L}(Z-, F) : \mathcal{K}^{\text{op}} \to V$ is small for all $F \in \mathcal{L}$. Now

$$\mathcal{L}(Z-, F) = \mathcal{P}(\mathcal{K}^{\text{op}})(F, Y-)$$

and the latter is an $F$-weighted limit of representables, with $F \in \Phi^*$. □
Corollary 4.5 \( \mathcal{P}\mathcal{K} \) is complete if \( \mathcal{K} \) is.

Proof: If \( \mathcal{K} \) is complete, then it is \( \mathcal{P}\alpha \) complete for any regular cardinal \( \alpha \). Thus \( \mathcal{P}\mathcal{K} \) is \( \mathcal{P}\alpha \) complete for any regular cardinal \( \alpha \), and so is complete. \( \square \)

5 The case where \( \mathcal{V}_0 \) is a presheaf category

For the first part of this section we suppose that \( \mathcal{V} = \text{Set} \), leading to Theorem 5.1. The latter should be attributed to Freyd, although it may not have been written down by him in exactly this form; it is a special case of [9, Theorem 4.8]. We include it as a warm-up for the more general case where the underlying category \( \mathcal{V}_0 \) of \( \mathcal{V} \) is a presheaf category. This includes the case of the cartesian closed categories of directed graphs, or of simplicial sets, as well as such non-cartesian cases as the category of \( G \)-graded sets, for a group \( G \), or the category of \( M \)-sets, for a commutative monoid \( M \).

Suppose then that \( \mathcal{V} = \text{Set} \). First observe that the statement \( \mathcal{P}\mathcal{K} \) has limits if and only if it has limits of representables remains true if by limit we mean conical limit. To say that \( \mathcal{P}\mathcal{K} \) has conical limits of representables is to say that for any \( S : \mathcal{C} \to \mathcal{K} \) with \( \mathcal{C} \) small, the limit of \( YS \) is small.

For the first part of this section we suppose that \( \mathcal{V} = \text{Set} \) with \( \mathcal{V}_0 \) as in Corollary 4.5. With \( \mathcal{V} = \text{Set} \), it is a special case of [9, Theorem 4.8]. We include it as a warm-up for the more general case where the underlying category \( \mathcal{V}_0 \) of \( \mathcal{V} \) is a presheaf category. This includes the case of the cartesian closed categories of directed graphs, or of simplicial sets, as well as such non-cartesian cases as the category of \( G \)-graded sets, for a group \( G \), or the category of \( M \)-sets, for a commutative monoid \( M \).

The existence of a small full subcategory \( \mathcal{B} \) satisfying (i) is clearly equivalent to the existence of a small set of cones through which every cone factorizes: this is the solution set condition. In fact, however, if this solution set condition holds for any \( S : \mathcal{D} \to \mathcal{K} \) with \( \mathcal{D} \) small then \( \mathcal{P}\mathcal{K} \) is complete, for we shall show below that if \( \mathcal{B} \) satisfies (i), then we may enlarge \( \mathcal{B} \) to a new small full subcategory \( \overline{\mathcal{B}} \) which satisfies (i) and (ii). This is done as follows. Let \( \mathcal{B}_0 = \mathcal{B} \). We construct inductively small full subcategories \( \mathcal{B}_n \) for each natural number \( n \), and then define \( \overline{\mathcal{B}} \) to be the union of the \( \mathcal{B}_n \).

Let \( \mathcal{D} \) be the category obtained from \( \mathcal{D} \) by freely adjoining two cones, with vertices 0 and 1, say. Let \( \mathcal{B}_n \) be a small full subcategory, and consider all functors \( S' : \mathcal{D}' \to \mathcal{K} \) extending \( \mathcal{D} \), and sending the vertices 0 and 1 to objects of \( \mathcal{B}_n \). For each such \( S' \), we may by hypothesis choose a small full subcategory \( \mathcal{B}_{S'} \) of \( \mathcal{K} \) which is a “solution set” for \( S' \). Take \( \mathcal{B}_{n+1} \) to be the union of \( \mathcal{B}_n \) and all the \( \mathcal{B}_{S'} \). This is a small union of small full subcategories, so is itself a small full subcategory. Once again, \( \overline{\mathcal{B}} \) is a small union of the small full subcategories \( \mathcal{B}_n \), and so is small. Clearly it satisfies (i); we check that it satisfies (ii) as well. Suppose then that \( \beta : \Delta B \to S \) and \( \beta' : \Delta B' \to S \) are cones over \( S \) with \( B, B' \in \overline{\mathcal{B}} \), and that \( f : A \to B \) and \( f' : A \to B' \) are arrows with \( \beta \Delta f = \beta' \Delta f' \). Then \( B, B', \beta \), and \( \beta' \) together define a functor \( S' : \mathcal{D}' \to \mathcal{K} \) extending \( S \); while to give \( A, f \), and \( f' \) is precisely to give a cone under \( S' \). Since \( B, B' \in \overline{\mathcal{B}} \), there is some \( n \in \mathbb{N} \) for which \( B, B' \in \mathcal{B}_n \), so there is a cone \( S' \) with vertex \( C \in \mathcal{B}_{S'} \) through which the cone \( (A, f, f') \) factorizes. But this \( C \) is in \( \mathcal{B}_{n+1} \), and so in \( \overline{\mathcal{B}} \). This proves:

**Theorem 5.1** \( \mathcal{P}\mathcal{K} \) is complete if and only if for every diagram \( S : \mathcal{D} \to \mathcal{K} \) with \( \mathcal{D} \) small, there is a small set of cones under \( S \) through which every cone factorizes.
We now extend this argument to the case where \( \mathcal{V}_0 \) is a presheaf category \( [\mathcal{G}^{\text{op}}, \text{Set}] \). To extend the argument, we need to assume that the \( \mathcal{V} \)-category \( \mathcal{K} \) admits tensors and cotensors by the representables; this means that for all \( A, B \in \mathcal{K} \) and \( G \in \mathcal{G} \), there are natural isomorphisms

\[
\mathcal{K}(A, G \sqcup B) \cong [\mathcal{G}^{\text{op}}, \text{Set}](\mathcal{G}(\_, G), \mathcal{K}(A, B)) \cong \mathcal{K}(G \cdot A, B)
\]

for objects \( G \cdot A \) and \( G \sqcup B \) of \( \mathcal{K} \); the first operation is called a tensor by \( G \) and the second a cotensor by \( G \). When these exist, we say that \( \mathcal{K} \) is \( \mathcal{G} \)-tensored and \( \mathcal{G} \)-cotensored. (Of course in the case \( \mathcal{V} = \text{Set} \) we have \( \mathcal{G} = \{ I \} \) and so this is automatic.)

**Proposition 5.2** If \( \mathcal{K} \) is \( \mathcal{G} \)-cotensored, then \( \mathcal{PK} \) is complete if and only if its underlying ordinary category \( (\mathcal{PK})_0 \) has conical limits of representables.

**Proof:** Recall [11] 3.10 that every weighted limit has a canonical expression as a conical limit of cotensors. On the other hand, since every object of \( \mathcal{V}_0 = [\mathcal{G}^{\text{op}}, \text{Set}] \) is (canonically) a conical colimit of representables, and we have \( (\colim_i G_i) \sqcup A \cong \lim_i (G_i \sqcup A) \), it follows that every cotensor is canonically a conical limit of \( \mathcal{G} \)-cotensors, and so finally that every weighted limit is canonically a conical limit of \( \mathcal{G} \)-cotensors. Suppose now that we have a diagram of representables \( YS : \mathcal{D} \to \mathcal{PK} \) and a weight \( \varphi \). We can therefore express this as a conical limit of \( \mathcal{G} \)-cotensors of representables. But \( \mathcal{K} \) was assumed to be \( \mathcal{G} \)-cotensored, so a \( \mathcal{G} \)-cotensor of representables in \( \mathcal{PK} \) exists, and is representable. Thus \( \mathcal{PK} \) will have weighted limits of representables provided that it has conical limits of representables. Finally, \( \mathcal{PK} \) is cocomplete, so is certainly tensored, thus conical limits in \( \mathcal{PK} \) exist provided that they exist in the underlying ordinary category \( (\mathcal{PK})_0 \) of \( \mathcal{PK} \), consisting of small \( \mathcal{V} \)-functors \( \mathcal{K}^{\text{op}} \to \mathcal{V} \) and \( \mathcal{V} \)-natural transformations between them. \( \square \)

We now adapt the argument from the \( \mathcal{V} = \text{Set} \) case to the \( V = [\mathcal{G}^{\text{op}}, \text{Set}] \) case to prove:

**Proposition 5.3** If \( \mathcal{K} \) is \( \mathcal{G} \)-tensored, then \( (\mathcal{PK})_0 \) has conical limits of representables if and only if, for every diagram \( S : \mathcal{C} \to \mathcal{K}_0 \), there is a small set of cones through which every cone factorizes.

**Proof:** Suppose that \( (\mathcal{PK})_0 \) has conical limits of representables, and let \( S : \mathcal{C} \to \mathcal{K}_0 \) be given. Then \( YS : \mathcal{C} \to (\mathcal{PK})_0 \) has a limit \( L \) with cone \( \eta_C : L \to \mathcal{K}(\_, SC) \). Also \( L \) is a small colimit of representables \( \colim_i \mathcal{K}(\_, B_i) \). For each \( i \), there is an induced cone \( \mathcal{K}(\_, B_i) \to \mathcal{K}(\_, SC) \), or equivalently \( \beta_i C : B_i \to SC \) under \( S \). We claim that any cone \( \alpha C : A \to SC \) factorizes through one of these. Now \( \mathcal{K}(\_, \alpha C) : \mathcal{K}(\_, A) \to \mathcal{K}(\_, SC) \) must factorize through \( L \), but \( \mathcal{K}(\_, A) \) is representable, so homming out of it preserves colimits, and so we get a factorization \( \mathcal{K}(\_, A) \to \mathcal{K}(\_, B_i) \) for some \( i \), and so the desired \( A \to B_i \).

For the harder part, suppose that for each \( S : \mathcal{C} \to \mathcal{K}_0 \), there is a small set of cones \( \beta_i C : B_i \to SC \) through which each cone factorizes. Then there is a small full subcategory \( \mathcal{B}_S \) of \( \mathcal{K} \) such that each cone under \( S \) factorizes through one whose vertex is in \( \mathcal{B}_S \). Exactly as before, we construct the (possibly larger but still) small full subcategory \( J : \mathcal{B} \to \mathcal{K} \) with the property that if two cones with vertices in \( \mathcal{B} \) are connected, then they are connected using cones with vertices in \( \mathcal{B} \). This implies that for all \( A \in \mathcal{K} \), we have \( \lim_{B \in \mathcal{B}} \mathcal{K}_0(A, SC) \cong \colim_{B \in \mathcal{B}} \mathcal{K}_0(A, B) \). For any \( G \in \mathcal{G} \),
we have

\[ V_0(G, \lim \mathcal{K}(A, SC)) \cong \lim_C V_0(G, \mathcal{K}(A, SC)) \]

\[ \cong \lim_C K_0(G \cdot A, SC) \]

\[ \cong \colim_{B \in \mathcal{B}} K_0(G \cdot A, B) \]

\[ \cong \colim_{B \in \mathcal{B}} V_0(G, \mathcal{K}(A, B)) \]

\[ \cong V_0(G, \colim_{B \in \mathcal{B}} \mathcal{K}(A, B)) \]

where the last step uses the fact that \( G \) is representable, so \( V_0(G, -) \) preserves colimits. Now \( \mathcal{G} \) is dense in \( V_0 \), so we have

\[ \lim_C \mathcal{K}(A, SC) \cong \colim_{B \in \mathcal{B}} \mathcal{K}(A, B) \]

\[ \lim_C \mathcal{K}(-, SC) \cong \colim_{B \in \mathcal{B}} \mathcal{K}(-, B) \]

but the left hand side is the presheaf \( \mathcal{K}^{\text{op}} \to \mathcal{V} \) which is the pointwise limit of \( YS \), and which we are to prove small, while the right hand side is a small colimit of representables, since \( \mathcal{B} \) is small. □

Combining the last two results, we have:

**Theorem 5.4** Suppose the underlying category \( V_0 \) of \( V \) is a presheaf category \([\mathcal{G}^{\text{op}}, \text{Set}]\) and that \( \mathcal{K} \) is a \( \mathcal{G} \)-tensored and \( \mathcal{G} \)-cotensored \( \mathcal{V} \)-category. Then \( \mathcal{P} \mathcal{K} \) is complete if and only if the following condition is satisfied. For every small ordinary category \( C \) and every functor \( S : C \to \mathcal{K} \), there is a small set of cones \( \lambda C : B \to SC \) through which every such cone factorizes.

6 Preservation of limits

Having studied the categories \( \mathcal{K} \) for which \( \mathcal{P} \mathcal{K} \) is complete, we now turn to the functors \( F : \mathcal{K} \to \mathcal{L} \) for which \( \mathcal{P} F \) is continuous.

We saw \( \mathcal{P} \mathcal{K} \) is always complete if \( \mathcal{K} \) is small; the situation for functors is totally different:

**Example 6.1** Let \( \mathcal{V} = \text{Set} \), let \( \mathcal{K} \) be the terminal category 1, let \( \mathcal{L} \) be the discrete category 2, and let \( F : \mathcal{K} \to \mathcal{L} \) be the first injection. Then \( \mathcal{P} \mathcal{K} = \text{Set} \) and \( \mathcal{P} \mathcal{L} = \text{Set}^2 \), which are of course complete; but \( \mathcal{P} F : \text{Set} \to \text{Set}^2 \) is the functor sending a set \( X \) to \((X, 0)\), which clearly fails to preserve the terminal object.

Consider a functor \( F : \mathcal{K} \to \mathcal{L} \), where \( \mathcal{P} \mathcal{K} \) and \( \mathcal{P} \mathcal{L} \) are complete, a weight \( \phi : \mathcal{C} \to \mathcal{V} \) and a diagram \( S : \mathcal{C} \to \mathcal{P} \mathcal{K} \). Let \( R : \mathcal{K}^{\text{op}} \to [\mathcal{C}, \mathcal{V}] \) be the corresponding pointwise small functor. To say that \( \mathcal{P} F \) preserves the limit \( \{\phi, S\} \) is to say that \( \{\phi, -\} : [\mathcal{C}, \mathcal{V}] \to \mathcal{V} \) preserves the left Kan extension \( \text{Lan}_F R \); that is, the colimit \( \mathcal{L}(L, F-)*R \) for each object \( L \) of \( \mathcal{L} \).

**Proposition 6.2** If \( F : \mathcal{K} \to \mathcal{L} \) is a right adjoint then \( \mathcal{P} F \) is continuous.

**Proof:** If \( F \) has a left adjoint \( G \), then

\[ \{\phi, \mathcal{L}(L, F-) * R\} \cong \{\phi, \mathcal{K}(GL, -) * R\} \cong \{\phi, RGL\} \]
while
\[ \mathcal{L}(L, F^{-}) \star \{ \varphi, R \} \cong \mathcal{L}(GL, -) \star \{ \varphi, R \} \cong \{ \varphi, RGL \}. \]

Along the same lines, observe that \( \mathcal{P}F.Y \cong YF \), so that if \( \mathcal{P}F \) is continuous then \( F \) must preserve any limits which exist.

Suppose that \( F : \mathcal{K} \to \mathcal{L} \) is given, with \( \mathcal{P}\mathcal{K} \) and \( \mathcal{P}\mathcal{L} \) complete. Then \( \mathcal{P}F \) is continuous if and only if each \( ev_L.\mathcal{P}F \) is so; but \( ev_L.\mathcal{P}F \) is just \( \mathcal{L}(L, F) \star - \). If \( \mathcal{K} \) is small, then \( \mathcal{L}(L, F) \star - \) is continuous if and only if it is \( \alpha \)-continuous for every regular cardinal \( \alpha \); in other words, if \( \mathcal{L}(L, F) \) is \( \alpha \)-flat for every \( \alpha \).

More generally, if \( \mathcal{P}\mathcal{K} \) is \( \alpha \)-complete, we say that a functor \( G : \mathcal{K} \to \mathcal{V} \) is \( \alpha \)-flat if \( G \star - : \mathcal{P}\mathcal{K} \to \mathcal{V} \) is \( \alpha \)-continuous, and \( \alpha \)-flat if \( G \star - \) is continuous; that is, if \( G \) is \( \alpha \)-flat for every \( \alpha \). Thus \( \mathcal{P}F \) will be continuous if and only if each \( \mathcal{L}(L, F) \) is \( \alpha \)-flat. Similarly, if \( \Phi \) is a class of weights satisfying the conditions in Section 4 and \( \mathcal{P}\mathcal{K} \) is \( \Phi \)-complete, we say that \( G : \mathcal{K} \to \mathcal{V} \) is \( \Phi \)-flat if \( G \star - \) is \( \Phi \)-continuous.

**Lemma 6.3** If \( \mathcal{K} \) is complete and \( G : \mathcal{K} \to \mathcal{V} \) continuous then \( \text{Lan}_Y G : \mathcal{P}\mathcal{K} \to \mathcal{V} \) is continuous.

**Proof:** First observe that if \( \mathcal{K} \) is complete then \( \mathcal{P}\mathcal{K} \) is so. Let \( \varphi : \mathcal{D} \to \mathcal{V} \) be a weight, and let \( S : \mathcal{D} \to \mathcal{P}\mathcal{K} \) correspond to the pointwise small functor \( R : \mathcal{K}^{\text{op}} \to [\mathcal{D}, \mathcal{V}] \). For \( X \in \mathcal{P}\mathcal{K} \), we have the formula \( (\text{Lan}_Y G)X = X \star G \), thus to say that \( \text{Lan}_Y G \) preserves the limit \( \{ \varphi, S \} \) is to say that \( \{ \varphi, - \} \) preserves the colimit \( G \star R \). Since \( R \) is pointwise small, it is the left Kan extension of its restriction to some full subcategory \( \mathcal{K}^{\text{op}} \) of \( \mathcal{K}^{\text{op}} \). Let \( \alpha \) be a regular cardinal for which \( \varphi \) is \( \alpha \)-small. We may choose \( \mathcal{B} \) to be closed in \( \mathcal{K} \) under \( \alpha \)-limits, then the inclusion \( J : \mathcal{B} \to \mathcal{K} \) preserves \( \alpha \)-limits. Then \( G \star R \cong G \star (\text{Lan}_J(RJ)) \cong GJ \star RJ \) by \([11\ 4.1]\), and \( GJ : \mathcal{B} \to \mathcal{V} \) preserves \( \alpha \)-limits, hence so does \( GJ \star - \) by \([11\ 6.11, 7.4]\), and now
\[ \{ \varphi, G \star R \} \cong \{ \varphi, GJ \star RJ \} \cong GJ \star \{ \varphi, RJ \} \cong G \star \text{Lan}_J \{ \varphi, RJ \}. \]

On the other hand \( \text{Lan}_J \) preserves \( \alpha \)-limits since \( J \) does so, thus
\[ G \star \text{Lan}_J \{ \varphi, RJ \} \cong G \star \{ \varphi, \text{Lan}_J(RJ) \} \cong G \star \{ \varphi, R \}. \]
This proves that \( G \star - \) preserves the limit \( \{ \varphi, R \} \), and so that \( \text{Lan}_Y G \) preserves \( \{ \varphi, S \} \).

**Theorem 6.4** Let \( \mathcal{K} \) and \( \mathcal{L} \) be complete. Then \( F : \mathcal{K} \to \mathcal{L} \) is continuous if and only if \( \mathcal{P}F : \mathcal{P}\mathcal{K} \to \mathcal{P}\mathcal{L} \) is so.

**Proof:** The “if part” was observed above. Suppose then that \( F \) is continuous. Then each \( \mathcal{L}(L, F) \) is continuous, so \( \text{Lan}_Y \mathcal{L}(L, F) \) is continuous, but \( \text{Lan}_Y \mathcal{L}(L, F) \cong ev_L.\mathcal{P}F \), and so \( \mathcal{P}F \) is continuous, since limits in \( \mathcal{P}\mathcal{L} \) are constructed pointwise.

**Remark 6.5** The Yoneda embedding \( Y : \mathcal{K} \to \mathcal{P}\mathcal{K} \) preserves any existing limits, and is continuous if \( \mathcal{K} \) is complete. The pseudomonad \( \mathcal{P} \) is of the Kock-Zöberlein type, and so the multiplication \( \mathcal{P}\mathcal{K} \to \mathcal{P}\mathcal{K} \) has both adjoints so also preserves any existing limits (or colimits). Thus the pseudomonad \( \mathcal{P} \) lifts from \( \mathcal{V}\text{-Cat} \) to the 2-category of complete \( \mathcal{V} \)-categories, continuous \( \mathcal{V} \)-functors, and \( \mathcal{V} \)-natural transformations.
Remark 6.6 Suppose once again that $\Phi$ is a class of weights satisfying the conditions of Section 4. Suppose that $K$ and $L$ are $\Phi$-complete and $F : K \to L$ is $\Phi$-continuous. Then each $\mathcal{L}(L, F)$ is $\Phi$-continuous, so each $ev_L \mathcal{P}(F)$ is $\Phi$-continuous, and so finally $\mathcal{P}F : \mathcal{P}K \to \mathcal{P}L$ is $\Phi$-continuous. Thus the pseudomonad $\Phi^*$ lifts from $\mathcal{V}$-$\mathcal{C}$at to the 2-category of $\Phi$-complete $\mathcal{V}$-categories, $\Phi$-continuous $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations.

7 Monoidal structure on $\mathcal{P}K$

In this section we suppose that $\mathcal{K}$ is a $\mathcal{V}$-category for which $\mathcal{P}K$ is complete. If $K$ is small, so that $\mathcal{P}K$ is $[K^{op}, \mathcal{V}]$, monoidal closed structures on $\mathcal{P}K$ correspond to promonoidal structures on $\mathcal{K}^{op}$ [6]. These consist of $\mathcal{V}$-functors $P : \mathcal{K}^{op} \otimes \mathcal{K} \otimes \mathcal{K} \to \mathcal{V}$ and $J : \mathcal{K}^{op} \to \mathcal{V}$ equipped with coherent associativity and unit isomorphisms.

If $K$ is large, we shall insist that $P(-; A, B) : \mathcal{K}^{op} \to \mathcal{V}$ and $J : \mathcal{K}^{op} \to \mathcal{V}$ be small, and we write $P : \mathcal{K} \otimes \mathcal{K} \to \mathcal{P}K : (A, B) \mapsto P(-; A, B)$ and $J \in \mathcal{P}K$. If $F, G \in \mathcal{P}K$ are given, we define $F \otimes G$ using the usual convolution formula:

$$F \otimes G = \int^{A, B} P(-; A, B) \otimes FA \otimes GB.$$ 

This is small, since each $P(-; A, B)$ is small by assumption, so $\int^A P(-; A, B) \otimes FA$ is a small ($F$-weighted) colimit of small presheaves for each $B$, and so $\int^{A, B} P(-; A, B) \otimes FA \otimes GB$ is itself a small colimit of small presheaves, hence small.

In the usual case, where $\mathcal{K}$ is small, this monoidal structure is closed, with (right) internal hom given by

$$[G, H] \simeq \int_{B, C} [P(C; -, B) \otimes GB, HC]$$

$$\simeq \int_{B, C} [GB, [P(C; -, B), HC]]$$

If $\mathcal{K}$ is large, this need not lie in $\mathcal{P}K$, but if it does so, then it will still provide the internal hom. Now $G$ is small, and the expression above for $[G, H]$ is precisely the $G$-weighted limit of the functor sending $B$ to $\int_C [P(C; -, B), HC]$. Since $\mathcal{P}K$ is complete this limit will exist provided that this functor actually lands in $\mathcal{P}K$; that is, provided that

$$\int_C [P(C; -, B), HC] : \mathcal{K}^{op} \to \mathcal{V}$$

is small for all $B \in \mathcal{K}$.

The case of the other internal hom is similar, and we have:

**Proposition 7.1** The convolution monoidal category $\mathcal{P}K$ is closed if and only if the presheaves $\int_C [P(C; -, B), HC]$ and $\int_C [P(C; B, -), HC]$ are small for all $B \in \mathcal{K}$.

An important special case is where the promonoidal structure $P$ is a filtered colimit $P = \text{colim}_i P_i$ of promonoidal structures $P_i$ which are in fact monoidal, as in

$$P_i(C; A, B) = \mathcal{K}(C, A \otimes_i B).$$
We call such a promonoidal structure $P$ \textit{approximately monoidal}; of course every monoidal structure is approximately monoidal. (We are using the fact that the colimit is filtered to obtain the associativity and unit isomorphisms; a general colimit of promonoidal structures need not be promonoidal.)

In the approximately monoidal case a simplification is possible, since

$$\int_C [P(C; -, B), HC] \cong \int_C [\colim_i P_i(C; -, B), HC]$$

$$\cong \lim_i \int_C [P_i(C; -, B), HC]$$

$$\cong \int C [\mathcal{K}(C, - \otimes_i B), HC]$$

$$\cong \lim_i H(- \otimes_i B)$$

which is small provided each $H(- \otimes_i B)$ is so. But $H$ is small, so has the form $\text{Lan}_J(HJ)$ for some $J : \mathcal{D} \to \mathcal{K}^{\text{op}}$ with $\mathcal{D}$ small. Then

$$H(- \otimes_i B) \cong \int^D \mathcal{K}^{\text{op}}(D, - \otimes_i B) \cdot HJD$$

$$\cong \int^D \mathcal{K}(- \otimes_i B, D) \cdot HJD$$

which is a small ($HJ$-weighted) colimit of presheaves $\mathcal{K}(- \otimes_i B, D)$ with $D \in \mathcal{D}$, so will be small provided that the $\mathcal{K}(- \otimes_i B, D)$ are so. Once again the case of the other internal hom is similar, and we have:

\textbf{Proposition 7.2} The convolution monoidal category $\mathcal{P}\mathcal{K}$ arising from an approximately monoidal structure on $\mathcal{K}$ is closed if and only if the presheaves $\mathcal{K}(- \otimes_i B, D)$ and $\mathcal{K}(B \otimes_i -, D)$ are small for all $B$ and $D$ in $\mathcal{K}$, and for each monoidal structure $\otimes_i$.

In particular we have:

\textbf{Proposition 7.3} The convolution monoidal category $\mathcal{P}\mathcal{K}$ arising from a monoidal structure on $\mathcal{K}$ is closed if and only if the presheaves $\mathcal{K}(- \otimes B, D)$ and $\mathcal{K}(B \otimes -, D)$ are small for all $B$ and $D$ in $\mathcal{K}$.

\textbf{Example 7.4}

1. The special case where $\mathcal{V} = \text{Set}$ and the monoidal structure is cartesian was proved in \textbf{[15]}.

2. If $\mathcal{K}$ is not just monoidal but closed then the $\mathcal{K}(- \otimes B, D)$ and $\mathcal{K}(B \otimes -, D)$ are not just small but representable, and so $\mathcal{P}\mathcal{K}$ is monoidal closed.

3. If $\mathcal{V}$ is cartesian monoidal (so that $\otimes = \times$), and $\mathcal{K} = \mathcal{E}^{\text{op}}$ where $\mathcal{E}$ is also cartesian monoidal, then $\mathcal{K}(- \otimes B, D) = \mathcal{E}(D, \times B) = \mathcal{E}(D, B) \times \mathcal{E}(D, -)$ which is given by tensoring the representable $\mathcal{E}(D, -)$ by the $\mathcal{V}$-object $\mathcal{E}(D, B)$, and so is small. Thus once again $\mathcal{P}\mathcal{K}$ is monoidal closed.
8 Functors with codomain other than $V$

In this section we consider small functors $K^{op} \to M$ where $M$ is cocomplete, building on our earlier work on the case $M = V$ and $M = [C, V]$.

In that earlier work, we considered when, for a small functor $S : K^{op} \to [C, V]$, each $\{\varphi, S\}$ was small. But $\{\varphi, -\}$ is just the representable functor $[C, V](\varphi, -)$, which motivates the following definition: a functor $S : K \to M$ is representably small if each $M(M, S) : K \to V$ is small. Thus Corollary 3.9 asserts that if $K$ is complete then every small functor $K^{op} \to [C, V]$ is representably small.

In this section we investigate the relationship between smallness and representable smallness for more general $M$. We have already seen that smallness does not in general imply representable smallness. For an explicit counterexample in the case $M = V$ we have:

Example 8.1 As in Example 3.1 let $V$ be $Set$, and let $K$ be any large set $X$, seen as a discrete category. Then a presheaf on $K$ is an $X$-indexed set $A \to X$, and it is small if and only if $A$ is so. Certainly $x : 1 \to X$ is small, for any $x \in X$; this corresponds to the representable presheaf $X(-, x) : X \to Set$ sending $x$ to 1, and all other elements to 0. Now $Set(0, X(-, x))$ is the terminal presheaf, which as we have seen is not small. Thus $X(-, x)$ is small but not representably small.

To see that a representably small functor need not be small, we have:

Example 8.2 If $K$ is a large $V$-category for which $P K$ is complete (for example if $K$ is complete), then the Yoneda embedding $Y_K^{op} : K^{op} \to P(K^{op})$ is representably small. For if $F \in P(K^{op})$, the composite $P(K^{op})(F, Y) : K^{op} \to V$ is the $F$-weighted limit of $Y : K \to P K$, so is small since $F$ is small and $P K$ is complete. But $Y_K^{op} : K^{op} \to P(K^{op})$ is not small unless $K$ is so. For if $Y_K^{op}$ were small, $K^{op}$ would have a small full subcategory $J : C^{op} \to K^{op}$ for which $Y = \text{Lan}_J(Y J)$, so

$$K(-, A) = \int_{C \in C} K(J C, A) \cdot K(-, J C)$$

for all $A$, and in particular

$$K(A, A) = \int_{C} K(J C, A) \cdot K(A, J C)$$

and so the identity $1 : A \to A$ must factorize through some $J C$; in other words, each $A \in K$ is a retract of some object in $C$. But this clearly implies that $K$ is small.

As a first positive result we have:

Proposition 8.3 If $K$ is a $V$-category for which $P K$ admits cotensors, a presheaf $F : K^{op} \to V$ is small if and only if it is representably small. In particular this will be the case if $P K$ is complete.

Proof: Representably small presheaves are always small, since $V(I, F)$ is just $F$, for any presheaf $F$. It remains to show that any small presheaf $F : K^{op} \to V$ is representably small. Suppose then that $X \in V$. Then $V(X, F)$ is the cotensor $X \otimes F$ of $F$ by $X$, which is small by assumption. \[\square\]
For the remainder of the section we suppose that $\mathcal{K}$ is a $\mathcal{V}$-category for which $\mathcal{P}\mathcal{K}$ is complete, and that $\mathcal{M}$ is a locally presentable $\mathcal{V}$-category. If $\beta$ is a regular cardinal for which $\mathcal{M}$ is locally $\beta$-presentable, write $\mathcal{M}_{\beta}$ for the full subcategory of $\mathcal{M}$ consisting of the $\beta$-presentable objects, and $W : \mathcal{M} \to [\mathcal{M}_{\beta}^{\mathcal{V}^{\text{op}}}, \mathcal{V}]$ for the canonical (fully faithful) inclusion.

Lemma 8.4 For a $\mathcal{V}$-functor $S : \mathcal{K}^{\mathcal{V}^{\text{op}}} \to \mathcal{M}$, the following are equivalent:

(a) $S$ is representably small;

(b) $WS$ is small;

(c) $S$ is small.

Proof: (a) $\Rightarrow$ (b). To say that $S$ is representably small is to say that $\mathcal{M}(M,S)$ is small for all $M \in \mathcal{M}$; to say that $WS$ is small is to say that this is so for all $M \in \mathcal{M}_{\beta}$, so this is immediate.

(b) $\Rightarrow$ (c). For each $M \in \mathcal{M}_{\beta}$ we have $\mathcal{M}(M,S)$ small, so it is the left Kan extension of its restriction to some full subcategory $\mathcal{D}_M$ of $\mathcal{M}^{\mathcal{V}^{\text{op}}}$. Since $\mathcal{M}_{\beta}$ is small, the union $\mathcal{D}$ of the $\mathcal{D}_M$ is small, and each $\mathcal{M}(M,S)$ is the left Kan extension of its restriction to $\mathcal{D}$. Thus $WS$ is the left Kan extension of its restriction to $\mathcal{D}$. But $W$ is fully faithful, and so reflects Kan extensions; thus also $S$ is the left Kan extension of its restriction to $\mathcal{D}$.

(c) $\Rightarrow$ (a). This is by far the hardest implication; we prove it in several steps, analogous to the main steps used in preparation for the proof of Theorem 3.8. Suppose then that $S$ is small and $M \in \mathcal{M}$; we must show that $\mathcal{M}(M,S)$ is small.

Case 1: $\mathcal{K}^{\mathcal{V}^{\text{op}}}$ is locally presentable. Since $S$ is small, it is the left Kan extension $\text{Lan}_{\mathcal{K}^{\mathcal{V}^{\text{op}}}}R$ along $J^{\mathcal{V}^{\text{op}}} : \mathcal{K}^{\mathcal{V}^{\text{op}}} \to \mathcal{K}^{\mathcal{V}^{\text{op}}}$ of some $R : \mathcal{C}^{\mathcal{V}^{\text{op}}} \to \mathcal{M}$ with $\mathcal{C}$ small. Since $\mathcal{C}$ is small and $\mathcal{K}^{\mathcal{V}^{\text{op}}}$ and $\mathcal{M}$ are locally presentable, there exists a regular cardinal $\gamma \geq \beta$ for which each $JC$ is $\gamma$-presentable in $\mathcal{K}^{\mathcal{V}^{\text{op}}}$ and $M$ is $\gamma$-presentable in $\mathcal{M}$. Now (a) $\mathcal{K}^{\mathcal{V}^{\text{op}}}$ is the free completion under $\gamma$-filtered colimits of the full subcategory $(\mathcal{K}^{\mathcal{V}^{\text{op}}})_{\gamma}$ of $\mathcal{K}^{\mathcal{V}^{\text{op}}}$ consisting of the $\gamma$-presentable objects, (b) $S$ preserves $\gamma$-filtered colimits, and (c) $\mathcal{M}(M,-)$ preserves $\gamma$-filtered colimits. Thus $\mathcal{M}(M,S)$ preserves $\gamma$-filtered colimits, so is the left Kan extension of its restriction to $(\mathcal{K}^{\mathcal{V}^{\text{op}}})_{\gamma}$. This proves that $\mathcal{M}(M,S)$ is small, and so that $S$ is representably small.

Case 2: $\mathcal{K}^{\mathcal{V}^{\text{op}}} = \mathcal{P}(\mathcal{L}^{\mathcal{V}^{\text{op}}})$. Then $S$ is the left Kan extension of its restriction to some small full subcategory $\mathcal{D}$ of $\mathcal{P}(\mathcal{L}^{\mathcal{V}^{\text{op}}})$. Each $D \in \mathcal{D}$ is a small functor $\mathcal{L} \to \mathcal{V}$, so is the left Kan extension of its restriction to some small $\mathcal{B}_D$. The union $\mathcal{B}$ of the $\mathcal{B}_D$ is small, and now the full inclusion $J : \mathcal{B}^{\mathcal{V}^{\text{op}}} \to \mathcal{L}^{\mathcal{V}^{\text{op}}}$ induces a full inclusion $\mathcal{P}J : \mathcal{P}(\mathcal{B}^{\mathcal{V}^{\text{op}}}) \to \mathcal{P}(\mathcal{L}^{\mathcal{V}^{\text{op}}})$ whose image contains $\mathcal{D}$.

Now $\mathcal{B}$ is small, so $\mathcal{P}J$ has a right adjoint $J^\ast$ given by restriction along $J$, and thus $\text{Lan}_{\mathcal{P}J}$ is itself given by restriction along $J^\ast$. Since $S$ is the left Kan extension of its restriction $Q$ along $\mathcal{P}J$, we have

$$\mathcal{M}(M,S) = \mathcal{M}(M,\text{Lan}_{\mathcal{P}J}Q) = \mathcal{M}(M,QJ^\ast) = \mathcal{M}(M,Q)J^\ast = \text{Lan}_{\mathcal{P}J}\mathcal{M}(M,Q)$$

and so $\mathcal{M}(M,S)$ will be small if $\mathcal{M}(M,Q)$ is so. Now $Q$ is the left Kan extension of its restriction to $\mathcal{D}$, hence small, so $\mathcal{M}(M,Q)$ is small by Case 1. This proves that $\mathcal{M}(M,S)$ is small, and so that $S$ is representably small.

Case 3: $\mathcal{P}\mathcal{K}$ is complete. The left Kan extension $\text{Lan}_Y(S) : \mathcal{P}(\mathcal{K}^{\mathcal{V}^{\text{op}}}) \to \mathcal{M}$ of $S$ along the Yoneda embedding is small, so by Case 2 is representably small. Thus each $\mathcal{M}(M,\text{Lan}_Y(S)) : \mathcal{P}(\mathcal{K}^{\mathcal{V}^{\text{op}}})^{\mathcal{V}^{\text{op}}} \to \mathcal{V}$ is small; that is, a small colimit of representables. Now restriction along the
Yoneda embedding preserves colimits, so it will send small presheaves to small presheaves provided that it sends representables to small presheaves; but the latter is equivalent to completeness of \( \mathcal{KH} \). Thus each \( \mathcal{M}(M,S) \) is small, and \( S \) is representably small. \( \square \)

Write \([\mathcal{K}^{\text{op}}, \mathcal{M}]_s\) for the \( \mathcal{V} \)-category of all small \( \mathcal{V} \)-functors from \( \mathcal{K}^{\text{op}} \) to \( \mathcal{M} \).

**Theorem 8.5** Let \( \mathcal{M} \) be a locally presentable \( \mathcal{V} \)-category, and \( \mathcal{K} \) a \( \mathcal{V} \)-category for which \( \mathcal{K} \) is complete. Then \([\mathcal{K}^{\text{op}}, \mathcal{M}]_s\) is complete.

**Proof:** Let \( \varphi : \mathcal{D} \to \mathcal{V} \) and \( S : \mathcal{D} \to [\mathcal{K}^{\text{op}}, \mathcal{M}]_s \) be given, where \( \mathcal{D} \) is small. Since \( \mathcal{D} \) is small, the functor \( S : \mathcal{K}^{\text{op}} \to [\mathcal{D}, \mathcal{M}] \) corresponding to \( S \) is small. The “pointwise limit” is the composite

\[
\begin{array}{ccc}
\mathcal{K}^{\text{op}} & \xrightarrow{S} & [\mathcal{D}, \mathcal{M}] \xrightarrow{\{\varphi,-\}} \mathcal{M}
\end{array}
\]

and provided that this is small, and so lies in \([\mathcal{K}^{\text{op}}, \mathcal{M}]_s\), it will be the limit. Since \( \mathcal{M} \) is locally presentable, by the lemma it will suffice to show that each composite with \( \mathcal{M}(M,-) \) is small. But for any \( X : \mathcal{D} \to \mathcal{M} \) we have

\[
\mathcal{M}(M,\{\varphi,X\}) \cong \{\varphi,\mathcal{M}(M,X)\} \\
\cong \int_D [\varphi D, \mathcal{M}(M,XD)] \\
\cong \int_D \mathcal{M}(\varphi D \cdot M, XD) \\
\cong [\mathcal{D}, \mathcal{M}](\varphi_M, X)
\]

where \( \varphi_M : \mathcal{D} \to \mathcal{M} \) is the functor sending \( D \) to \( \varphi D \cdot M \), so now \( \mathcal{M}(M,\{\varphi,-\}) \) is representable as \([\mathcal{D}, \mathcal{M}](\varphi_M, -) : [\mathcal{D}, \mathcal{M}] \to \mathcal{V} \).

Now \([\mathcal{D}, \mathcal{M}] \) is locally presentable, so by the lemma once again the small \( \tilde{S} \) is representably small, and so \([\mathcal{D}, \mathcal{M}](\varphi_M, \tilde{S}) : \mathcal{K}^{\text{op}} \to \mathcal{V} \) is small; but we have just seen that this is the composite of \( \tilde{S} \) with \( \mathcal{M}(M,\{\varphi,-\}) \). This now proves that \( \{\varphi,-\} \circ \tilde{S} \) is representably small, and so small, and it therefore provides the desired limit \( \{\varphi, S\} \). \( \square \)

9 \textbf{Isbell conjugacy}

If \( \mathcal{C} \) is a small category then as well as the Yoneda embedding \( Y : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathcal{V}] \) there is also the “dual” Yoneda embedding \( Z : \mathcal{C} \to [\mathcal{C}, \mathcal{V}]^{\text{op}} \), and this induces an adjunction between \([\mathcal{C}^{\text{op}}, \mathcal{V}]\) and \([\mathcal{C}, \mathcal{V}]^{\text{op}} \) called “Isbell conjugacy”. The left adjoint \( L : [\mathcal{C}^{\text{op}}, \mathcal{V}] \to [\mathcal{C}, \mathcal{V}]^{\text{op}} \) is given by \( \text{Lan}_Y Z \).

What happens if we replace \( \mathcal{C} \) be an arbitrary category \( \mathcal{K} \)? Then we have \( Y : \mathcal{K} \to \mathcal{P} \mathcal{K} \) and \( Z : \mathcal{K} \to (\mathcal{P} \mathcal{K})^{\text{op}} \), but do we still have the adjunction between them? A sufficient condition for the left adjoint \( L : \mathcal{P} \mathcal{K} \to (\mathcal{P} \mathcal{K})^{\text{op}} \) to exist is that \( \mathcal{P} \mathcal{K}^{\text{op}} \) be cocomplete, or equivalently \( \mathcal{P} \mathcal{K}^{\text{op}} \) complete, but in fact this is also necessary. For if \( \text{Lan}_Y Z \) does exist, then for each small \( F : \mathcal{C}^{\text{op}} \to \mathcal{V} \) the colimit \( F \ast Z \) in \( \mathcal{P} \mathcal{K}^{\text{op}} \) exists. But then for any \( \varphi : \mathcal{C}^{\text{op}} \to \mathcal{V} \) and \( S : \mathcal{C} \to \mathcal{K} \), we have \( \text{Lan}_S \varphi \) small, and \( (\text{Lan}_S \varphi) \ast Z = \varphi Z S \), and so \( \mathcal{P} \mathcal{K}^{\text{op}} \) has arbitrary colimits of representables, \( \mathcal{P} \mathcal{K}^{\text{op}} \) has arbitrary limits of representables, and so \( \mathcal{P} \mathcal{K}^{\text{op}} \) is in fact complete.
Thus \( \mathcal{KH} \to \mathcal{P}(\mathcal{X}^{\text{op}})^{\text{op}} \) exists if and only if \( \mathcal{P}(\mathcal{X}^{\text{op}}) \) is complete, and dually the putative right adjoint \( \mathcal{P}(\mathcal{X}^{\text{op}})^{\text{op}} \to \mathcal{KH} \) exists if and only if \( \mathcal{KH} \) is complete.

In particular, both will exist if \( \mathcal{X} \) is complete and cocomplete.

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