Quantum mechanics with coordinate dependent noncommutativity

V.G. Kupriyanov*
CMCC, Universidade Federal do ABC, Santo André, SP, Brazil

May 10, 2014

Abstract

Noncommutative quantum mechanics can be considered as a first step in the construction of quantum field theory on noncommutative spaces of generic form, when the commutator between coordinates is a function of these coordinates. In this paper we discuss the mathematical framework of such a theory. The noncommutativity is treated as an external antisymmetric field satisfying the Jacobi identity. First, we propose a symplectic realization of a given Poisson manifold and construct the Darboux coordinates on the obtained symplectic manifold. Then we define the star product on a Poisson manifold and obtain the expression for the trace functional. The above ingredients are used to formulate a nonrelativistic quantum mechanics on noncommutative spaces of general form. All considered constructions are obtained as a formal series in the parameter of noncommutativity. In particular, the complete algebra of commutation relations between coordinates and conjugated momenta is a deformation of the standard Heisenberg algebra. As examples we consider a free particle and an isotropic harmonic oscillator on the rotational invariant noncommutative space.

*e-mail: vladislav.kupriyanov@gmail.com
1 Introduction

Quantum field theory on noncommutative spaces has been studied extensively during the last decades [1]. The main attention was given to the case of flat noncommutative space-time, realized by the coordinate operators $\hat{x}^i$, $i = 1, ..., N$, satisfying the algebra $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$, with a constant $\theta^{ij}$ being the parameters of noncommutativity. Different phenomenological consequences of the presence of this type of noncommutativity in the theory were studied. The comparison of the theoretical predictions with the experimental data gives rise to the bounds of noncommutativity in this model, see e.g. [2] and references therein. However, the restriction to flat noncommutative spaces does not seem to be very natural. The physical motivation for noncommutativity comes from a combination of the general quantum mechanical arguments with Einstein relativity [3], and space-time in any dynamical theory of gravity cannot be flat.

The presence of a more general type of noncommutativity, i.e., when the parameters of noncommutativity depend on coordinates may lead to absolutely different phenomenological consequences. For example, in [4] it was shown that the Lagrangian of the Grosse-Wulkenhaar (renormalizable) model [5] can be written as a Lagrangian of a scalar field propagating in a curved noncommutative space, defined by the truncated Heisenberg algebra. The problem is to construct a consistent quantum field theory on noncommutative spaces of general form.

Our point of view is the following: one may construct noncommutative field theory of generic form considering relativistic version of quantum mechanics (QM) with coordinate operators satisfying the commutation relations

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} (\hat{x})^q, \quad i, j = 1, ..., N,$$

(1)

where $\hat{\omega}^{ij} (\hat{x})$ is an operator defined from physical considerations with a specified ordering which describes the noncommutativity of the space. In this paper we discuss the mathematical framework of nonrelativistic QM with coordinate operators satisfying commutation relations of the type (1). A two-dimensional model of position-dependent noncommutativity in QM was proposed in [4]. The particular example was considered in [7], where also was observed that canonical operators in these models are in general non Hermitian with respect to standard inner products [8]. For an example of QM and field theory on kappa-Minkowski space see e.g. [9] and references therein. Quantum mechanical models on fuzzy spaces were discussed in [10].

To formulate a consistent QM on noncommutative spaces [11] first we need to introduce momenta $\hat{p}_i$, conjugated to $\hat{x}^i$, i.e., to obtain the complete algebra of commutation
relations, obeying the Jacobi identities, as a deformation in $\theta$ of the Heisenberg algebra

$$
[x^i, x^j] = i\theta \omega^{ij}_q (\hat{x}) ,
$$

$$
[x^i, \hat{p}_j] = i\delta^{ij}_q (\hat{x}, \hat{p}) = \delta^{ij} + \theta \delta_1^{ij} (\hat{x}, \hat{p}) + O (\theta^2) ,
$$

$$
[\hat{p}_i, \hat{p}_j] = i\bar{\omega}^{ij}_q (\hat{x}, \hat{p}) = \theta \bar{\omega}_1^{ij} (\hat{x}, \hat{p}) + O (\theta^2) ,
$$

where $\delta^{ij}$ is the Kronecker delta. However, $\delta_1^{ij} (\hat{x}, \hat{p})$ and $\bar{\omega}_1^{ij} (\hat{x}, \hat{p})$ are already an operator-valued function of $\hat{x}$ and $\hat{p}$. Then one should construct a representation of this algebra. And finally, it is necessary to define a Hilbert space and to introduce an internal product $\langle \phi | \psi \rangle$ and $\langle \psi | \phi \rangle$ for any two states $|\phi\rangle$ and $|\psi\rangle$ from the Hilbert space.

In the Sec. 2 we show that the algebra (2) can be obtained from a quantization of $2N$ dimensional symplectic manifold with coordinates $\xi^\mu = (x^i, p_i)$, $\mu = 1, ..., 2N$, and a symplectic structure

$$
\Omega_{\mu\nu} (\xi) = \Omega^0_{\mu\nu} + O (\theta) ,
$$

such that $\Omega_{ij} = \theta \omega^{ij} (x)$, where $\omega^{ij} (x)$ is a Poisson bi-vector, corresponding to the operator $\hat{\omega}_q^{ij} (\hat{x})$ and $\Omega^0_{\mu\nu}$ is a canonical symplectic matrix. That is, to construct the complete algebra of commutation relations (2) one should start with a finding of a symplectic realization of the corresponding Poisson manifold. In the Sec. 3 we give a recursive solution to this problem in a form of power series in $\theta$.

Then, in the Sec. 4 we construct the Darboux coordinates $\eta^\mu = (y^i, \pi_i)$ on the obtained symplectic manifold in a form of perturbative series $\eta^\mu (\xi) = \xi^\mu + \theta \eta^\mu_1 (\xi) + O (\theta^2)$, providing an explicit formulas $\eta^\mu_1 (\xi)$ of each order in $\theta$ and giving a complete description of the arbitrariness in our construction. The particular case $y^i = x^i + \theta / 2 \omega^{ij} (x) p_j + O (\theta^2)$ and $\pi_i = p_i$ is considered in the Sec. 5, where we also present a direct method of finding of Darboux coordinates starting from a Poisson bi-vector $\omega^{ij} (x)$.

In fact, it is shown that the problem of the construction of the symplectic structure $\Omega$ in each order in $\theta$ is reduced to the solution of algebraic equations. We give the explicit formulae for the solution of these equations in each order in the deformation parameter. In the Sec. 6 we discuss a general form of Darboux coordinates obtained from the direct method:

$$
x^i = y^i - \frac{\theta}{2} \omega^{ij} (y) \pi_i + O (\theta^2) , \quad p_i = \pi_i - \theta j_i (y, \pi, \theta),
$$

where $j_i (y, \pi, \theta)$ is an arbitrary vector.

Finally, we consider the canonical quantization of the obtained symplectic manifold. In Sec. 7 we use the expressions for Darboux coordinates $\eta^\mu (\xi)$ to construct the polydifferential representation of the algebra (2) and to define the star product on the corresponding Poisson manifold. The explicit form of the star product is given up to the third order.
in the deformation parameter $\theta$. To complete our construction we also need the expression for the trace functional on the algebra of star product. In Sec. 8 we describe the perturbative procedure of the construction of the trace.

Using the above ingredients in Sec. 9 we formulate the quantum mechanics with noncommutative coordinates satisfying the algebra (1), describing the Hilbert space, the internal product on it and the action of the canonical operators on the states from the Hilbert space. As an example we solve the eigenvalue problem for the free particle on coordinate dependent NC space. Also we discuss the three-dimensional isotropic harmonic oscillator on rotational invariant NC space. This example shows that the noncommutativity can be introduced in a minimal way in the theory, i.e., one may obtain nonlocality without violating physical observables like the energy spectrum, etc.

2 Necessary and sufficient conditions

Consider the noncommutative space defined by the commutation relations (1). We choose the symmetric Weyl ordering for the operator $\hat{\omega}^{ij}_{q}(\hat{x})$. Let $\omega^{ij}_{q}(x)$ be a symbol of the operator $\hat{\omega}^{ij}_{q}(\hat{x})$, i.e.,

$$\hat{\omega}^{ij}_{q}(\hat{x}) = \int \frac{d^{N}p}{(2\pi)^{N}} \tilde{\omega}^{ij}_{q}(p) e^{-ip_{m}\hat{x}^{m}},$$

where $\tilde{\omega}^{ij}_{q}(p)$ is a Fourier transform of $\omega^{ij}_{q}(x)$. The consistency condition for the algebra (1) is a consequence of the Jacobi identity (JI) and reads:

$$[\hat{x}^{i}, \hat{\omega}^{jk}_{q}] + [\hat{x}^{k}, \hat{\omega}^{ij}_{q}] + [\hat{x}^{j}, \hat{\omega}^{ki}_{q}] = 0. \tag{4}$$

It implies, see e.g. [17], that

$$\omega^{ij}_{q}(x) = \omega^{ij}(x) + \omega^{ij}_{co}(x), \tag{5}$$

where $\omega^{ij}(x)$ should be a Poisson bi-vector, i.e., satisfy the equation

$$\omega^{il}_{q} \partial_{l} \omega^{jk} + \omega^{kl}_{q} \partial_{l} \omega^{ij} + \omega^{jl}_{q} \partial_{l} \omega^{ki} = 0, \tag{6}$$

and the term $\omega^{ij}_{co}(x)$ stands for non-Poisson corrections to $\omega^{ij}(x)$ of higher order in $\theta$, expressed in terms of $\omega^{ij}(x)$ and its derivatives, which depend on specific ordering of the operator $\hat{\omega}^{ij}_{q}(\hat{x})$. So, telling that the operator $\hat{\omega}^{ij}_{q}(\hat{x})$ is defined from physical considerations we mean that the Poisson bi-vector $\omega^{ij}(x)$ is given and the ordering is specified. In what follows we treat $\omega^{ij}(x)$ as an external antisymmetric field obeying the eq. (6).
It is convenient to introduce the following notations:

\[ \hat{\xi}^\mu = (\hat{x}^i, \hat{p}_i), \quad \mu = 1, \ldots, 2N, \]

\[ \hat{\Omega}^q_{\mu\nu} = \begin{pmatrix} \theta \hat{\omega}^i_j q & \delta^i_j q \\ -\delta^i_j q & \hat{\omega}^i_j q \end{pmatrix}. \]

In these notations the algebra of commutation relations (2) is written as

\[ [\hat{\xi}^\mu, \hat{\xi}^\nu] = i\hat{\Omega}^q_{\mu\nu}. \tag{7} \]

The Jacobi identity for (7) implies that

\[ [\hat{\xi}^\mu, \hat{\Omega}^q_{\alpha\nu}] + [\hat{\xi}^\alpha, \hat{\Omega}^q_{\mu\nu}] + [\hat{\xi}^\nu, \hat{\Omega}^q_{\alpha\mu}] = 0. \tag{8} \]

This equation we call the consistency condition for the algebra (2). Let \( \Omega^q_{\mu\nu}(\xi) \) be a symbol of the operator \( \hat{\Omega}^q_{\mu\nu} \). The eq. (8) means that

\[ \Omega^q_{\mu\nu}(\xi) = \Omega_{\mu\nu}(\xi) + \Omega^{co}_{\mu\nu}(\xi), \tag{9} \]

where \( \Omega_{\mu\nu}(\xi) \) should be a Poisson bi-vector, i.e.,

\[ \Omega_{\mu\sigma} \partial_\sigma \Omega_{\nu\alpha} + \Omega_{\nu\alpha} \partial_\sigma \Omega_{\mu\nu} + \Omega_{\nu\sigma} \partial_\sigma \Omega_{\alpha\mu} = 0, \tag{10} \]

with \( \partial_\sigma = \partial / \partial \xi^\sigma \), and \( \Omega^{co}_{\mu\nu}(\xi) \) is a non-Poisson correction due to ordering. Comparing (9) with (5) we conclude that

\[ \Omega_{\mu\nu}(\xi) = \begin{pmatrix} \theta \omega^{ij}(x) & \delta^{ij}(x, p) \\ -\delta^{ij}(x, p) & \omega^{ij}(x, p) \end{pmatrix}, \tag{11} \]

where \( \omega^{ij}(x) \) is a given Poisson bi-vector and the functions \( \delta^{ij}(x, p) \) and \( \omega^{ij}(x, p) \) to be determined from the equation (10). Writing this equation in components, e.g., \( \mu = i, \nu = j, \alpha = N + k \), one obtains partial differential equations for the functions \( \delta^{ij}(x, p) \) and \( \omega^{ij}(x, p) \) in terms of given \( \omega^{ij}(x) \).

Thus, in order to obtain the expression for the operator \( \hat{\Omega}^q_{\mu\nu} \), satisfying the consistency condition (8), first we need to find the Poisson bi-vector \( \Omega_{\mu\nu}(\xi) \), such that \( \Omega_{ij}(\xi) = \theta \omega^{ij}(x) \), which is a necessary condition, and then determine the corrections due to ordering \( \Omega^{co}_{\mu\nu}(\xi) \), which is a sufficient condition.
3 Symplectic realization of a Poisson manifold

This section is devoted to the solution of the necessary condition, i.e., to the construction of the Poisson bi-vector $\Omega_{\mu \nu} (\xi)$. From a mathematical point of view the problem of finding a symplectic realization of a Poisson manifold is the problem of the construction of a local symplectic groupoid. The existence of a local symplectic groupoid for any Poisson structure was shown in [11] and independently in [12]. In the present paper we are interested in explicit formulae for the symplectic structure $\Omega$ in a form (124), which is motivated by the condition that the complete algebra of commutation relations (2) should be a deformation of the Heisenberg algebra.

Note that the condition (124) implies that the matrix $\Omega_{\mu \nu}$ can be represented as a perturbative series

$$\Omega_{\mu \nu} (\xi) = \sum_{n=0}^{\infty} \theta^n \Omega_{\mu \nu}^n (\xi), \quad \Omega^0 = \begin{pmatrix} 0 & 1_{N \times N} \\ -1_{N \times N} & 0 \end{pmatrix}. \quad (12)$$

In particular,

$$\delta^{ij} (x, p) = \delta^{ij} + \theta \delta^{ij}_1 (x, p) + O (\theta^2), \quad (13)$$

$$\omega^{ij} (x, p) = \theta \omega^{ij}_1 (x, p) + O (\theta^2).$$

So, at least perturbatively, $\Omega_{\mu \nu}$ is non-degenerate, $\det \Omega \neq 0$. Let us denote its inverse matrix by $\bar{\Omega}_{\mu \nu}$.

From the identity $\bar{\Omega}_{\mu \sigma} \Omega_{\sigma \nu} = \delta_{\mu \nu}$ one obtains

$$\partial_\sigma \Omega_{\mu \nu} = -\Omega_{\mu \beta} \partial_\sigma \bar{\Omega}_{\beta \gamma} \Omega_{\gamma \nu}. \quad (14)$$

Which means that non-degenerate matrix $\Omega_{\mu \nu}$ obeys the JI (10) if and only if its inverse obeys the equation

$$\partial_\sigma \bar{\Omega}_{\beta \gamma} + \partial_\gamma \bar{\Omega}_{\sigma \beta} + \partial_\beta \bar{\Omega}_{\gamma \sigma} = 0. \quad (15)$$

If the matrix $\bar{\Omega}_{\mu \nu}$ has a form

$$\bar{\Omega}_{\mu \nu} = \partial_\mu J_\nu - \partial_\nu J_\mu, \quad (16)$$

where vector $J_\mu (\xi)$ is called symplectic potential, then (15) is automatically satisfied. We conclude that if the matrix $\Omega_{\mu \nu}$ obey the equation

$$\partial_\mu J_\nu - \partial_\nu J_\mu = \Omega^{-1}_{\mu \nu}, \quad (17)$$

then it obeys the JI (10).

Our idea is to study the equation (17), where $\Omega_{ij} = \theta \omega^{ij} (x)$ is given and satisfy the JI (6), while $\delta^{ij}$, $\omega^{ij}$ and $J_\mu$ are unknown, instead of looking for solution of eq. (10). The eq. (10) is an integrability condition for (17).
3.1 Perturbative solution

We are interested in a perturbative solution of (17). To this end let us first represent (17) as a perturbative series and obtain equation in each order in $\theta$. Since $\Omega_{\mu\nu}$ has a perturbative form (12), its inverse matrix also can be written as

$$\bar{\Omega}_{\mu\nu}(\xi) = \sum_{n=0}^{\infty} \theta^n \bar{\Omega}^n_{\mu\nu}(\xi),$$

where $\bar{\Omega}^0 = -\Omega^0$ and

$$\bar{\Omega}^n_{\mu\nu} = \begin{pmatrix} \bar{\omega}^n_{ij}(x, p) & \delta^n_{ij}(x, p) \\ -\delta^n_{ji}(x, p) & \bar{\omega}^n_{ij}(x, p) \end{pmatrix}.$$  \hspace{1cm} (19)

By the definition

$$\bar{\omega}^n_{ij} = \partial_i J^n_j - \partial_j J^n_i,$$  \hspace{1cm} (20)

$$\delta^n_{ij} = \partial_i J^n_{N+j} - \partial_j J^n_{N+i},$$

$$\bar{\omega}^n_{ij} = \partial_i J^n_{N+j} - \partial_j J^n_{N+i},$$

where $\partial_i = \partial/\partial x^i$ and $\partial^i = \partial/\partial p^i$.

Substituting (12) and (18) in the identity $\bar{\Omega}_{\mu\sigma} \Omega_{\sigma\nu} = \delta_{\mu\nu}$ one has

$$\left(\sum_{n=0}^{\infty} \theta^n \bar{\Omega}^n\right) \left(\sum_{m=0}^{\infty} \theta^m \Omega^m\right) = \sum_{n=0}^{\infty} \theta^n \sum_{m=0}^{n} \Omega^{n-m} \Omega^m = 1.$$  \hspace{1cm} (21)

Equating the powers in $\theta$ in the right and in the left hand sides of (21) we obtain

$$-\bar{\Omega}^n \Omega^0 = \bar{\Omega}^0 \Omega^n + \sum_{m=1}^{n-1} \Omega^{n-m} \Omega^0 \Omega^m, \hspace{1cm} n \geq 1,$$  \hspace{1cm} (22)

which can be rewritten as

$$\Omega^0 \bar{\Omega}^n \Omega^0 = \Omega^n + \sum_{m=1}^{n-1} \Omega^0 \bar{\Omega}^{n-m} \Omega^0 \Omega^m, \hspace{1cm} n \geq 1.$$  \hspace{1cm} (23)

Then, iterating (23) we come to the following equation

$$-\bar{\Omega}^0 \bar{\Omega}^n \Omega^0 + \Omega^n + \sum_{m_1=1}^{n-1} \Omega^{n-m_1} \Omega^0 \Omega^{m_1} + \ldots +$$

$$\sum_{m_1=1}^{n-r} \ldots \sum_{m_r=1}^{n-m_1-\ldots-m_{r-1}-1} \Omega^{n-m_1-\ldots-m_r} \Omega^0 \Omega^{m_r} \Omega^0 \Omega^{m_1} + \ldots + \Omega^1 \Omega^0 \Omega^1)^{n-1} = \Omega^0 \bar{\Omega}^n \Omega^0 + \Omega^n + \sum_{r=1}^{n-1} \left\{ \sum_{m_1=1}^{n-r} \ldots \sum_{m_r=1}^{n-m_1-\ldots-m_{r-1}-1} \Omega^{n-m_1-\ldots-m_r} \Omega^0 \Omega^{m_r} \Omega^0 \Omega^{m_1} \right\} = 0.$$  \hspace{1cm} (24)
In particular, for \( n = 1 \) \((24)\) means
\[
- \Omega^0 \bar{\Omega}^1 \Omega^0 + \Omega^1 = 0, \tag{25}
\]
for \( n = 2 \):
\[
- \Omega^0 \bar{\Omega}^2 \Omega^0 + \Omega^2 + \Omega^1 \Omega^0 \Omega^1 = 0, \tag{26}
\]
etc.

Eq. \((25)\) in components reads
\[
\partial^p J^1_{N+j} - \partial^p J^1_{N+i} = -\omega^{ij} (x), \tag{27}
\]
\[
\delta^1_{ij} = \partial_i J^1_{N+j} - \partial_j J^1_{N+i} = \delta^1_{ij} (x, p), \tag{28}
\]
\[
\partial_t J^1_i - \partial_t J^1_j = -\omega^{ij} (x, p), \tag{29}
\]
where \( \omega^{ij} (x) \) is given and \( \delta^1_{ij} \) and \( \omega^{ij}_1 \) should be found. From the eq. \((27)\) one defines, \( J^1_{N+i} = \frac{1}{2} \omega^{ij} (x) p_j + \partial^p f^1 (x, p) \) in terms of \( \omega^{ij} (x) \) and arbitrary function \( f^1 (x, p) \). \( J^1_i \) remains arbitrary and eqs. \((28)\) and \((29)\) define \( \delta^1_{ij} \) and \( \omega^{ij}_1 \) in terms of \( \omega^{ij} \), \( f^1 \) and \( J^1_i \).

Eq. \((26)\) in components reads
\[
\partial^p J^2_{N+j} - \partial^p J^2_{N+i} - \omega^{il} \delta^1_{lj} - \delta^1_{il} \omega^{lj} = 0, \tag{30}
\]
\[
\delta^2_{ij} = \partial_i J^2_{N+j} - \partial_j J^2_{N+i} + \delta^1_{il} \delta^1_{lj} - \omega^{il} \omega^{lj}_1, \tag{31}
\]
\[
\omega^{ij}_2 = \partial_j J^2_i - \partial_i J^2_j + \delta^1_{in} \omega^{ij}_1 + \omega^{il}_1 \delta^1_{lj}. \tag{32}
\]
Again, eq. \((30)\) is a differential equation on \( J^2_{N+i} \) which defines it up to the gradient of an arbitrary function \( f^2 (x, p) \), while eqs. \((31)\) and \((32)\) are the definition of the functions \( \delta^2_{ij} \) and \( \omega^{ij}_2 \) in terms of \( J^2_{N+i} \), arbitrary \( J^2_i \) and first order functions.

The integrability condition for the eq. \((30)\) is
\[
\partial^p \left( \omega^{il} \delta^1_{lj} + \delta^1_{il} \omega^{lj} \right) + \text{cycl.}(ijk) = 0. \tag{33}
\]
We rewrite it as
\[
\omega^{jl} \left( \partial^p \delta^1_{kl} - \partial^p \delta^1_{ki} \right) + \text{cycl.}(ijk) = 0. \tag{34}
\]
Then, using \((27)\) and \((28)\) one finds
\[
\partial^p \delta^1_{kl} - \partial^p \delta^1_{ki} = \partial_t \omega^{ki}. \tag{35}
\]
That is, \((33)\) is exactly the JI \((6)\) for the matrix \( \omega^{ij} (x) \).
Now let us write the eq. (24) in components in the \( n \)-th order in \( \theta \), for \( n > 1 \):

\[
\partial_i^p J_{N+i} - \partial_j^p J_{N+i} = \\
- \sum_{r=1}^{n-1} \left\{ \sum_{m_1=1}^{n-r} \cdots \sum_{m_r=1}^{n-m_1-\ldots-m_{r-1}-1} \left[ \Omega^{n-m_1-\ldots-m_r} \Omega^0 \Omega^0 \Omega^0 \Omega^0 \Omega^0 \Omega^{m_1} \right]_{ij} \right\},
\]

(36)

\[
\delta_{nj} = - \bar{\delta}_{nj}^i, \\
\bar{\omega}_{nj} = - \bar{\omega}_{nj}^i
\]

(37)

\[
- \sum_{r=1}^{n-1} \left\{ \sum_{m_1=1}^{n-r} \cdots \sum_{m_r=1}^{n-m_1-\ldots-m_{r-1}-1} \left[ \Omega^{n-m_1-\ldots-m_r} \Omega^0 \Omega^0 \Omega^0 \Omega^0 \Omega^0 \Omega^{m_1} \right]_{N+iN+j} \right\},
\]

(38)

where functions \( \bar{\delta}_{nj}^i \) and \( \bar{\omega}_{nj}^i \) are determined in (20). The logic is the same as in the first two orders, from the eq. (36) one finds \( J_{N+i}^n \) up to the gradient of an arbitrary function \( f_n(x,p) \), and then uses the eqs. (37) and (38) to define functions \( \delta_{nj}^i \) and \( \bar{\omega}_{nj}^i \) in terms of \( J_{N+i}^n \), arbitrary \( J_i^p \) and functions \( \delta_{mj}^i \) and \( \bar{\omega}_{mj}^i \) of lower orders, \( m < n \).

### 3.2 Integrability condition in the \( n \)-th order

Let us study the integrability condition for the eq. (36). Since \( \partial_i^p = \partial_{N+i} = \Omega_{\sigma}^0 \partial_{\sigma} \), it can be written as

\[
C = \Omega_{k\sigma}^0 \partial_{\sigma} \sum_{r=1}^{n-1} \left\{ \sum_{m_1=1}^{n-r} \cdots \sum_{m_r=1}^{n-m_1-\ldots-m_{r-1}-1} \left[ \Omega^{n-m_1-\ldots-m_r} \Omega^0 \Omega^0 \Omega^0 \Omega^0 \Omega^0 \Omega^{m_1} \right]_{ij} \right\}
\]

(39)

In the first order the integrability condition was satisfied automatically, since right hand side of the eq. (27) does not depend on \( p \). In the second order the integrability condition was satisfied as a consequence of the JI (10) for the matrix \( \omega^{ij}(x) \). To prove the existence of the symplectic potential \( J_{\mu}(\xi) \) we should prove that the integrability condition (39) is satisfied in all orders.

We will do it by the induction. Suppose that the eq. (36) is solvable up to the \((n-1)\)-th order. It means that both the symplectic potential \( J_{\mu}(\xi) \) and symplectic structure \( \Omega_{\mu\nu} \) can be defined from the eq. (24) up to the \((n-1)\)-th order. That is, JI (10) is satisfied
up to the \((n - 1)\)-th order,

\[
\Omega^0_{\mu\sigma} \partial_{\sigma} \Omega^r_{\nu\alpha} + \sum_{s=1}^{r-1} \Omega^{r-s}_{\mu\sigma} \partial_{\sigma} \Omega^s_{\nu\alpha} + \text{cycl.}(\mu\nu\alpha) = 0, \quad r = 2, \ldots, n - 1, \tag{40}
\]

\[
\Omega^0_{\mu\sigma} \partial_{\sigma} \Omega^1_{\nu\alpha} + \text{cycl.}(\mu\nu\alpha) = 0.
\]

Let us show that (33) is satisfied. To begin with we prove the following Lemma.

**Lemma 1** If \(J_\mu\) exists up to the \((n - 3)\)-th order, then

\[
D = \sum_{m_1=1}^{n-1} \sum_{m_2=1}^{n-r} \sum_{m_3=1}^{n-m_1-1} \Omega^{m_2+1}_0 \frac{\partial_{\sigma}}{\partial_{\alpha}} \frac{\partial_{\sigma}}{\partial_{\beta}} \Omega^{m_3}_{\alpha\beta} \Omega^{n-m_2-1}_{\alpha\beta} \Omega^{0}_{\alpha\beta} \Omega^{1}_{\beta} + \text{cycl.}(ijk) = 0.
\]

**Proof.** Using the eq. (24) the first term of (41) can be written as

\[
\sum_{m_1=1}^{n-1} \sum_{m_2=1}^{n-r} \sum_{m_3=1}^{n-m_1-1} \Omega^{m_2+1}_0 \frac{\partial_{\sigma}}{\partial_{\alpha}} \frac{\partial_{\sigma}}{\partial_{\beta}} \Omega^{m_3}_{\alpha\beta} \Omega^{n-m_2-1}_{\alpha\beta} \Omega^{0}_{\alpha\beta} \Omega^{1}_{\beta} + \text{cycl.}(ijk) = 0.
\]

\[
\sum_{m_1=1}^{n-1} \sum_{m_2=1}^{n-r} \sum_{m_3=1}^{n-m_1-1} \Omega^{m_2+1}_0 \frac{\partial_{\sigma}}{\partial_{\alpha}} \frac{\partial_{\sigma}}{\partial_{\beta}} \Omega^{m_3}_{\alpha\beta} \Omega^{n-m_2-1}_{\alpha\beta} \Omega^{0}_{\alpha\beta} \Omega^{1}_{\beta} + \text{cycl.}(ijk) = 0.
\]
Now, if to substitute (42) back to (41) one finds that

\[
D = \sum_{m_1=1}^{n-3} \sum_{m_2=1}^{n-m_1-1} \sum_{m_3=1}^{n-m_1-m_2-1} \Omega_{j\alpha}^{m_1} \Omega_{j\beta}^{m_2} \partial_{\alpha} \tilde{\Omega}_{\alpha\beta}^{n-m_1-m_2-m_3} + \text{cycl.}(ijk) = \\
- \sum_{m_1=1}^{n-3} \sum_{m_2=1}^{n-m_1-1} \sum_{m_3=1}^{n-m_1-m_2-1} \Omega_{j\alpha}^{m_1} \Omega_{j\beta}^{m_2} \partial_{\alpha} \tilde{\Omega}_{\alpha\beta}^{n-m_1-m_2-m_3} + \text{cycl.}(\sigma\alpha\beta) = 0,
\]

which complete the proof. ■

The sum \( C + D \) can be represented as

\[
C + D = \sum_{r=1}^{n-1} T_r,
\]  

(43)

where

\[
T_1 = \sum_{m_1=1}^{n-1} \Omega_{k\sigma}^{0} \partial_{\sigma} \left[ \Omega^{n-m_1} \Omega^{0} \Omega^{m_1} \right]_{ij} + \text{cycl.}(ijk),
\]  

(44)

\[
T_2 = \sum_{m_1=1}^{n-2} \sum_{m_2=1}^{n-m_1-1} \Omega_{i\alpha}^{0} \partial_{\sigma} \left[ \Omega^{n-m_1-m_2} \Omega^{0} \Omega^{m_2} \Omega^{0} \Omega^{m_1} \right]_{ij} + \text{cycl.}(ijk),
\]  

(45)

\[
T_r = \sum_{m_1=1}^{n-r-1} \sum_{m_r=1}^{n-m_1-\ldots-m_{r-1}-1} \left\{ \Omega_{k\sigma}^{0} \partial_{\sigma} \left[ \Omega^{n-m_1-\ldots-m_r} \Omega^{0} \Omega^{m_r} \Omega^{0} \ldots \Omega^{0} \Omega^{m_1} \right]_{ij} + \Omega_{j\alpha}^{m_1} \Omega_{j\beta}^{m_r} \partial_{\alpha} \tilde{\Omega}_{\alpha\beta}^{0} \Omega^{m_1} \Omega^{m_r} + \text{cycl.}(ijk) \right\},
\]  

(46)

\[
3 \leq r \leq n - 1.
\]
Let us introduce the following notations:

\[ I(n, 1) = \sum_{m_1=1}^{n-1} \Omega_{k\sigma}^m \partial_\sigma \Omega_{ij}^{n-m_1} + \text{cycl.}(ijk) \] (47)

\[ I(n, r) = \sum_{m_1=1}^{n-r} \sum_{m_r=1}^{n-m_r-1-1} \left\{ \left( \Omega_{i\sigma}^m \partial_\sigma \Omega_{ja}^{n-m_1} + \text{cycl.}(i\alpha\beta) \right) \left[ \Omega^0 \Omega^{m_r-1} ... \Omega^0 \Omega^1 \right]_{ak} + ... + \left[ \Omega^{m_r-1} \Omega^0 ... \Omega^0 \Omega^{m_r-s} \Omega^0 \right]_{ja} \left( \Omega_{j\sigma}^m \partial_\sigma \Omega_{ia}^{n-m_1} + \text{cycl.}(i\alpha\beta) \right) \left[ \Omega^0 \Omega^{m_s-1} \Omega^0 ... \Omega^0 \Omega^1 \right]_{bk} + ... + \left[ \Omega^{m_r-1} \Omega^0 ... \Omega^0 \Omega^{m_r-s} \Omega^0 \right]_{ja} \left( \Omega_{j\sigma}^m \partial_\sigma \Omega_{ia}^{n-m_1} + \text{cycl.}(i\alpha\beta) \right) \left[ \Omega^0 \Omega^1 \right]_{bk} \right\} + \text{cycl.}(ijk), \quad 2 \leq r \leq n-1, \]

\[ I(n, n) = \left( \Omega_{i\sigma}^0 \partial_\sigma \Omega_{j1}^1 + \text{cycl.}(ij\alpha) \right) \left[ \left( \Omega^0 \Omega^1 \right)^{n-1} \right]_{ak} + \sum_{s=1}^{n-3} \left[ \left( \Omega^0 \Omega^0 \right)^{s} \right]_{ja} \left( \Omega_{j\sigma}^0 \partial_\sigma \Omega_{ia}^1 + \text{cycl.}(i\alpha\beta) \right) \left[ \left( \Omega^0 \Omega^1 \right)^{n-s-1} \right]_{bk} + \text{cycl.}(ijk). \] (49)

In particular,

\[ I(n, 2) = \sum_{m_1=1}^{n-2} \sum_{m_2=1}^{n-m_1-1} \left( \Omega_{i\sigma}^m \partial_\sigma \Omega_{ja}^{n-m_1-m_2} + \text{cycl.}(ij\alpha) \right) \left[ \Omega^0 \Omega^1 \right]_{ak} + \text{cycl.}(ijk), \]

etc.

Note that since \( \Omega_{ij}^m = 0 \) for \( m > 1 \) and \( \Omega_{ij}^1 = \omega_{ij}(x) \) does not depend on \( p \), one has

\[ I(n, 1) = \sum_{m_1=1}^{n-1} \Omega_{k\sigma}^m \partial_\sigma \Omega_{ij}^{n-m_1} + \text{cycl.}(ijk) = \omega^{kl} \partial_k \omega_{ij} + \text{cycl.}(ijk) = 0. \] (50)

Also, due to Jacobi identity \([10]\)

\[ I(n, n) = 0. \] (51)

**Lemma 2** If symplectic potential \( J_\mu \) exists up to the \( n-1 \) order, then

\[ T_r = I(n, r) - I(n, r+1), \quad 1 \leq r \leq n-1. \] (52)

**Proof.** First we write \( T_1 \) as

\[ T_1 = \sum_{m_1=1}^{n-1} \left( \Omega_{i\sigma}^0 \partial_\sigma \Omega_{ja}^{n-m_1} + \Omega_{j\sigma}^0 \partial_\sigma \Omega_{ia}^{n-m_1} \right) \left[ \Omega^0 \Omega^1 \right]_{ak} + \text{cycl.}(ijk). \] (53)
Then using (40) we represent it in the form

\[ T_1 = -\sum_{m_1=1}^{n-1} \Omega^0_{\alpha\sigma} \partial_\sigma \Omega_{ij}^{n-m_1} \left[ \Omega^0 \Omega_{m_1} \right]_{ak} \]

\[ - \sum_{m_1=1}^{n-2} \sum_{m_2=1}^{n-m_1-1} \left( \Omega_{\alpha\sigma}^{m_2} \partial_\sigma \Omega_{ja}^{n-m_1-m_2} + \text{cycl.}(ij\alpha) \right) \left[ \Omega^0 \Omega_{m_1} \right]_{ak} + \text{cycl.}(ijk). \]

The sum in the second line is exactly \( I(n, 2) \). Since,

\[ \Omega^0_{\alpha\sigma} \Omega^0_{-\alpha\beta} = \delta_{\sigma\beta}, \]

(54)

the sum in the first line can be rewritten as

\[ \sum_{m_1=1}^{n-1} \Omega_{k\sigma}^{m_1} \partial_\sigma \Omega_{ij}^{n-m_1} + \text{cycl.}(ijk). \]

That is,

\[ T_1 = I(n, 1) - I(n, 2). \]

(55)

For the second term we write

\[ T_2 = \sum_{m_1=1}^{n-2} \sum_{m_2=1}^{n-m_1-1} \left\{ \left( \Omega^0_{\alpha\sigma} \partial_\sigma \Omega_{ja}^{n-m_1-m_2} + \Omega^0_{\beta\sigma} \partial_\sigma \Omega_{\alpha i}^{n-m_1-m_2} \right) \left[ \Omega^0 \Omega_{m_2} \Omega^0 \Omega_{m_1} \right]_{ak} \right\} \]

(56)

\[ + \left[ \Omega^0 \Omega_{m_2} \right]_{ja} \Omega^0_{\alpha\sigma} \partial_\sigma \Omega_{\alpha i}^{n-m_1-...-m_k} \left[ \Omega^0 \Omega_{m_1} \right]_{jk} + \text{cycl.}(ijk) \]

Applying the eq. (40) in (56) one transforms it as

\[ \sum_{m_1=1}^{n-2} \sum_{m_2=1}^{n-m_1-1} \left\{ -\Omega^0_{\alpha\sigma} \partial_\sigma \Omega_{ij}^{n-m_1-m_2} \left[ \Omega^0 \Omega_{m_2} \Omega^0 \Omega_{m_1} \right]_{ak} \right\} \]

\[ - \sum_{m_1=1}^{n-3} \sum_{m_2=1}^{n-m_1-1} \sum_{m_3=1}^{n-m_1-m_2-1} \left\{ \left( \Omega^0_{\alpha\sigma} \partial_\sigma \Omega_{ja}^{n-m_1-m_2-1} + \text{cycl.}(i\alpha\beta) \right) \left[ \Omega^0 \Omega_{m_2} \Omega^0 \Omega_{m_1} \right]_{ak} \right\} \]

\[ + \left[ \Omega^0 \Omega_{m_2} \right]_{ja} \left( \Omega^0_{\alpha\sigma} \partial_\sigma \Omega_{\alpha i}^{n-m_1-m_2-1} + \text{cycl.}(i\alpha\beta) \right) \left[ \Omega^0 \Omega_{m_1} \right]_{jk} + \text{cycl.}(ijk). \]

The last sum here is \( I(n, 3) \). Using the identity (54) we simplify the first sum and see that

\[ T_2 = I(n, 2) - I(n, 3). \]

(57)
For the terms $T_r$ with $3 \leq r \leq n - 1$ the logic is the same as in the first two cases, but we should also take into account the contribution from the sum (41). First using Jacobi identity (40) in each order we represent it as

\[ T_r = \sum_{m_1=1}^{n-r} \sum_{m_r=1}^{n-m_1-\ldots-m_{r-1}-1} \{ -\Omega^0_{\alpha\sigma} \partial_{\sigma} \Omega^{n-m_1-\ldots-m_r}_{ij} \left[ \Omega^0 \Omega^{m_r} \Omega^0 \ldots \Omega^0 \Omega^1 \right]_{ak} 
- \ldots - \left[ \Omega^{m_r} \Omega^0 \ldots \Omega^0 \Omega^{m_r-s} \Omega^0 \right]_{j\alpha} \left( \Omega^0_{\beta\sigma} \partial_{\sigma} \Omega^{n-m_1-\ldots-m_r}_{ia} \right)_{\beta k} 
+ \left[ \Omega^{m_r-1} \Omega^0 \ldots \Omega^0 \Omega^{m_r-s-1} \Omega^0 \ldots \Omega^0 \Omega^1 \right]_{\beta k} \} + \text{cycl.}(ijk) 
- I(n, r + 1). \]

Then simplifying the first sum we end up with

\[ T_r = I(n, r) - I(n, r + 1), \quad 3 \leq r \leq n - 1. \]

Using this Lemma and eqs. (50) and (51) we represent the sum (43) as

\[ C + D = \sum_{r=1}^{n-1} \left[ I(n, r) - I(n, r + 1) \right] = I(n, 1) - I(n, n) = 0. \quad (58) \]

Since $D = 0$ due to Lemma 1 we conclude that $C = 0$. That is, by the induction, the integrability condition (39) holds true in all orders.

### 3.3 Concluding remarks

Once (39) is satisfied in all orders, from the eqs. (36)-(38) we can define $J^0_{N+1}, \delta^i_j$ and $\varpi^i_j$ for any $n$ and construct the perturbative solution for the eq. (17). The arbitrariness in the construction of functions $\delta^i_j(x, p)$ and $\varpi^i_j(x, p)$ is described by an arbitrary vector $J_i(x, p, \theta)$ and arbitrary function $f(x, p, \theta)$.

Note that since we have a symplectic structure $\Omega_{\mu\nu}$ obtained from a symplectic potential $J_\mu$, one can always write a first-order action leading to this symplectic structure

\[ S = \int dt \left[ J_\mu(\xi) \dot{\xi}^\mu - H(\xi) \right], \quad (59) \]
where \( H(\xi) \) is a Hamiltonian of a system. If an external field \( \omega^{ij}(x) \) transforms as a tensor and \( J_i(x,p,\theta) \) transforms as a vector with respect to the Lorentz group, one can also write a relativistic generalization of the action (59), like it was done, e.g., in [18] for the case of spin noncommutativity and in [19] for canonical noncommutativity.

4 Darboux coordinates

Since \( \Omega_{\mu\nu}(\xi) \) is a Poisson bi-vector, we may define the Poisson bracket (PB) as

\[
\{\xi^\mu, \xi^\nu\} = \Omega_{\mu\nu}(\xi). \tag{60}
\]

The construction of a quantum algebra (2) is, in fact, the quantization of a classical system with a Poisson Bracket (60).

One of the methods of quantization of classical systems with noncanonical PB is a quantization in Darboux coordinates. That is, first one should change the phase space coordinates: \( \xi_\mu = (x^i, p_i) \to \eta_\mu = (y^i, \pi_i): \xi_\mu = \xi_\mu(\eta) \), where new variables have canonical PB:

\[
\{\eta_\mu, \eta_\nu\} = \Omega^0_{\mu\nu}, \tag{61}
\]

and are called Darboux coordinates. And then construct quantization in these new phase space coordinates. Since the existence of a Darboux coordinates is guarantied by the Darboux theorem [13], here we are interested in an explicit expression for them.

We will search for a Darboux coordinates in a perturbative form:

\[
\xi_\mu(\eta) = \eta_\mu + \sum_{n=1}^{\infty} \theta^n \xi^n_{\mu}(\eta). \tag{62}
\]

The inverse transformation reads

\[
\eta_\mu(\xi) = \xi_\mu + \sum_{n=1}^{\infty} \theta^n \eta^n_{\mu}(\xi). \tag{63}
\]

Note that once (63) is known, the inverse transformation (62) may be easily calculated from the algebraic equation \( \eta_\mu(\eta(\xi)) = \eta_\mu \). In particular,

\[
\xi_\mu(\eta) = \eta_\mu - \theta \eta^1_\mu(\eta) + \theta^2 \left( \eta^1_\nu(\eta) \partial_\nu \eta^1_\mu(\eta) - \eta^2_\mu(\eta) \right) + O(\theta^3).
\]

So, we will look for (63) and then construct (62). To obtain differential equation on \( \eta_\mu(\xi) \) let us write (61) as

\[
\{\eta_\mu(\xi), \eta_\nu(\xi)\} = \Omega^0_{\mu\nu},
\]

15
and take into account (60), so
\[ \partial_\sigma \eta_\mu \Omega_{\rho \nu} \partial_\rho \eta_\nu = \Omega^0_{\mu \nu}. \] (64)

This equation is just a consequence of the definition of a Darboux coordinates. Substituting in (64) perturbative expressions (12) and (63) one has
\[ \left( \delta_{\sigma \mu} + \sum_{n=1}^{\infty} \theta^n \partial_\sigma \eta^n_\mu \right) \left( \Omega^0_{\sigma \rho} + \sum_{n=1}^{\infty} \theta^n \Omega^n_{\sigma \rho} \right) \left( \delta_{\rho \nu} + \sum_{n=1}^{\infty} \theta^n \partial_\rho \eta^n_\nu \right) = \Omega^0_{\mu \nu}. \] (65)

From this equation, equating the powers in \( \theta \) in the right and in the left hand sides, we obtain the differential equations on the functions \( \eta^n_\mu \) in each order \( n \). Thus, in the first order one has
\[ \partial_\sigma \eta^1_\mu \Omega^0_{\sigma \nu} + \Omega^0_{\mu \sigma} \partial_\sigma \eta^1_\nu + \Omega^1_{\mu \nu} = 0. \] (66)

In the second order we have
\[ \partial_\sigma \eta^2_\mu \Omega^0_{\sigma \nu} + \Omega^0_{\mu \sigma} \partial_\sigma \eta^2_\nu + \partial_\sigma \eta^1_\mu \Omega^0_{\sigma \nu} \partial_\rho \eta^1_\nu + \partial_\sigma \eta^1_\mu \Omega^1_{\sigma \nu} + \Omega^1_{\mu \sigma} \partial_\sigma \eta^1_\nu = 0. \] (67)

The equation in the \( n \)-th order reads
\[ \partial_\sigma \eta^n_\mu \Omega^0_{\sigma \nu} + \Omega^0_{\mu \sigma} \partial_\sigma \eta^n_\nu + F^n_{\mu \nu} = 0, \] (68)

where
\[ F^1_{\mu \nu} = \Omega^1_{\mu \nu}, \]
\[ F^2_{\mu \nu} = \Omega^2_{\mu \nu} + \partial_\sigma \eta^1_\mu \Omega^0_{\sigma \nu} \partial_\rho \eta^1_\nu + \partial_\sigma \eta^1_\mu \Omega^1_{\sigma \nu} + \Omega^1_{\mu \sigma} \partial_\sigma \eta^1_\nu, \]
\[ F^n_{\mu \nu} = \Omega^n_{\mu \nu} + \sum_{m=1}^{n-1} \left[ \partial_\sigma \eta^{n-m}_\mu \Omega^0_{\sigma \nu} \partial_\rho \eta^m_\nu + \partial_\sigma \eta^{n-m}_\mu \Omega^m_{\sigma \nu} + \Omega^n_{\mu \sigma} \partial_\sigma \eta^m_\nu \right] + \sum_{m=1}^{n-2} \sum_{k=1}^{n-m-1} \Omega^{n-m-k}_{\sigma \rho} \partial_\sigma \eta^k_\mu \partial_\rho \eta^m_\nu = 0, \quad n \geq 3. \]

**Lemma 3** The integrability condition for the equation (68) is
\[ \Omega^0_{\alpha \beta} \partial_\beta F^n_{\mu \nu} + \text{cycl.}(\alpha \mu \nu) = 0. \] (70)

**Proof.** One can easily verify that
\[ \Omega^0_{\alpha \beta} \partial_\beta \left( \partial_\sigma \eta^n_\mu \Omega^0_{\sigma \nu} + \Omega^0_{\mu \sigma} \partial_\sigma \eta^n_\nu \right) + \text{cycl.}(\alpha \mu \nu) = 0, \] (71)
so (70) is a necessary condition for a solution of eq. (68). To prove, that (70) is a sufficient condition let us write (68) in components. Note that \( \Omega^0_{\sigma} \partial_\sigma = \partial_\rho^\rho \) and \( \Omega^0_{N+\sigma} \partial_\sigma = -\partial_i \). For \( \mu = i \) and \( \nu = j \), (68) has a form

\[
\partial_\rho^\rho y_n^l - \partial_\rho^\rho y_n^i = F^n_{ij}.
\]

(72)

Its integrability condition is

\[
\partial_i F^n_{N+iN+k} + \text{cycl.}(ijk) = 0,
\]

(73)

which is exactly (70) for \( \alpha = i, \mu = j \) and \( \nu = k \). For \( \mu = N+i \) and \( \nu = N+j \), (68) reads

\[
\partial_j \pi^n_i - \partial_i \pi^n_j = F^n_{N+iN+j},
\]

(74)

with the integrability condition

\[
\partial_i F^n_{N+jN+k} + \text{cycl.}(ijk) = 0,
\]

(75)

which is eq. (70) for \( \alpha = N+i, \mu = N+j \) and \( \nu = N+k \). Finally, for \( \mu = N+i \) and \( \nu = j \), eq. (68) is

\[
- \partial_i y_n^l - \partial_\rho^\rho \pi^n_i = F^n_{N+i},
\]

(76)

If treat this equation as an equation for \( y_n^l \):

\[
\partial_i y_n^l = -\partial_\rho^\rho \pi^n_i - F^n_{N+i},
\]

then its integrability condition is

\[
\partial_k \left( \partial_j \pi^n_i + F^n_{N+i} \right) - \partial_i \left( \partial_\rho^\rho \pi^n_k + F^n_{N+i+k} \right) = \partial_i F^n_{N+i+k} + \partial_\rho^\rho F^n_{N+i+k} - \partial_k F^n_{N+i} = 0.
\]

(77)

If treat (76) as an equation for \( \pi^n_i \):

\[
\partial_j \pi^n_i = -\partial_i y_n^l - F^n_{N+i},
\]

its integrability condition is

\[
\partial_k \left( \partial_i y_n^l + F^n_{N+i} \right) - \partial_j \left( \partial_i y_n^k + F^n_{N+i+k} \right) = \partial_i F^n_{N+i} + \partial_\rho^\rho F^n_{N+i} + \partial_k F^n_{N+i} = 0.
\]

(78)

Equations (77) and (78) are (70) for \( \alpha = N+i, \mu = N+k, \nu = j \) and for \( \alpha = N+i, \mu = k, \nu = j \) correspondingly. So, the eq. (68) written in components gives eqs. (72), (74) and (76) with integrability conditions (73), (75), (77) and (78), which are exactly eq. (70) written in components. 

\[\square\]
For the eq. (66) in the first order the condition (70) is
\[ \Omega_0^{\alpha\beta} \partial_\beta \Omega_1^{\mu\nu} + \text{cycl.}(\alpha\mu\nu) = 0. \quad (79) \]
This equation is a Jacobi identity (10) in the first order. In the second order in \( \theta \) (70) reads
\[ \Omega_0^{\alpha\beta} \partial_\beta \left( \Omega_2^{\mu\nu} + \partial_\sigma \eta_\mu^{1} \Omega_0^{\sigma\rho} \partial_\rho \eta_\nu^{1} + \partial_\sigma \eta_\mu^{1} \Omega_1^{\sigma\nu} + \Omega_1^{1} \partial_\sigma \eta_\nu^{1} \right) + \text{cycl.}(\alpha\mu\nu) = 0. \quad (80) \]
First we calculate
\[ \Omega_0^{\alpha\beta} \partial_\beta \left( \partial_\sigma \eta_\mu^{1} \Omega_2^{\sigma\rho} \partial_\rho \eta_\nu^{1} + \partial_\sigma \eta_\mu^{1} \Omega_1^{\sigma\nu} + \Omega_1^{1} \partial_\sigma \eta_\nu^{1} \right) + \text{cycl.}(\alpha\mu\nu) = \]
\[ \partial_\sigma \eta_\alpha^{1} \Omega_0^{\rho\sigma} \partial_\rho \eta_\beta^{1} \Omega_0^{\beta\nu} + \partial_\sigma \eta_\mu^{1} \Omega_2^{\sigma\rho} \partial_\rho \eta_\nu^{1} + \partial_\sigma \eta_\mu^{1} \Omega_1^{1} \partial_\sigma \eta_\nu^{1} + \Omega_0^{1} \partial_\sigma \Omega_1^{\sigma\nu} + \Omega_0^{1} \partial_\sigma \Omega_1^{\sigma\nu} + \text{cycl.}(\alpha\mu\nu). \quad (81) \]
Then, taking into account (66) and (79), we represent (81) as
\[ \Omega_1^{1} \partial_\sigma \partial_\rho \Omega_2^{\mu\nu} + \text{cycl.}(\alpha\mu\nu). \]
Finally, for (80) one gets
\[ \Omega_0^{\alpha\beta} \partial_\beta \Omega_2^{\mu\nu} + \Omega_1^{1} \partial_\beta \Omega_1^{\mu\nu} + \text{cycl.}(\alpha\mu\nu) = 0, \quad (82) \]
which is a Jacobi identity (10) in the second order.

In the appendix A we will prove by the induction that the integrability condition (70) in the \( n \)-th order is satisfied as a consequence of a Jacobi identity (10), like in the first two orders.

The solutions of the eqs. (72) and (74) are given by
\[ y^n_j = \int_{1}^{0} p_j F^n_{nij} (x, sp) sds + \partial^p_i f^n (x, p), \quad (83) \]
\[ \pi^n_i = \int_{1}^{0} x^j F^n_{ni} N + (sx, p) sds + \partial^p_i g^n (x, p), \quad (84) \]
correspondingly, where \( f^n (x, p) \) and \( g^n (x, p) \) are arbitrary functions. Then, from (70) we may define the relation between \( f^n \) and \( g^n \) in terms of \( F^n_{ij}, F^n_{N+iN+j} \) and \( F^n_{N+iN+j} \). So, the arbitrariness of the solution of the eq. (68) is described by only one function
\[ g (x, p, \theta) = \sum_{n=0}^{\infty} \theta^n g^n (x, p). \quad (85) \]
Since the addition of this function in (68) does not change the canonical PB (61), it can be interpreted as a generating function of the canonical transformation.
5 Particular case, \( J_i = p_i + \partial_i r(x, p, \theta) \).

Let us consider the particular case of our construction, choosing the vector \( J_i = p_i + \partial_i r(x, p, \theta) \). In this case, by the definition (20), \( \bar{\omega}^{ij} = 0 \), and the matrix

\[
\Omega = \begin{pmatrix}
(\bar{\delta}^{-1})^T \bar{\omega} \bar{\delta}^{-1} & (\bar{\delta}^{-1})^T \\
\bar{\delta}^{-1} & 0
\end{pmatrix}.
\]

(86)

That is,

\[
\{p_i, p_j\} = \bar{\omega}^{ij}(x, p) = 0.
\]

(87)

In section 3 we proved that such \( \Omega \) exists and gave an iterative procedure of its construction. In particular, from eqs. (27) and (30) one finds

\[
J_{N+i} = \frac{\theta}{2} \omega^{il} p_l + \frac{\theta^2}{6} \omega^{lk} \partial_k \omega^{mi} p_l p_m + O(\theta^3),
\]

(88)

where the function \( f(x, p, \theta) \) which describe the arbitrariness in the construction of \( J_{N+i} \) was chosen to cancel the contribution from \( r(x, p, \theta) \), i.e., \( f = r \). And then we calculate

\[
\delta^{ij}(x, p) = \delta^{ij} + \frac{\theta}{2} \partial_j \omega^{il} p_l + \theta^2 \left( \frac{1}{12} \partial_j \omega^{kl} \partial_k \omega^{im} + \frac{1}{6} \omega^{lk} \partial_j \partial_k \omega^{im} \right) p_l p_m + O(\theta^3).
\]

(89)

From the eqs. (86)-(88) and (86) we may see that the structure of \( J_{N+i}^n \) and \( \delta^{ij}_n \) for \( n > 2 \) will be the same as in first two orders: they will be polynomials in \( p \) of order \( n \).

The next step is to construct the Darboux coordinates. From (68) and (94) we may see that choosing \( g(x, p, \theta) = 0 \), one obtains that \( \pi_i^n = 0, \ n \geq 1, \ i.e., \)

\[
\pi_i = p_i.
\]

(90)

For the coordinates \( y^i \) one has

\[
y^i = x^i + \frac{\theta}{2} \omega^{ij}(x) p_j + \frac{\theta^2}{12} \left( \omega^{lk} \partial_l \omega^{ij} + \omega^{lj} \partial_j \omega^{lk} \right) p_j p_k + O(\theta^3).
\]

(91)

Note that since \( \delta^{ij}_n \) are polynomials in \( p \) of order \( n \), from (72) and (76) we conclude that each \( y^i_n \) with \( n \geq 1 \) also will be a polynomial in \( p \) of order \( n \). Making the inverse transformation we find that the expression for \( x^i \) in terms of \( y^i \) and \( \pi_i \) has a form

\[
x^i = y^i + \sum_{n=1}^{\infty} \Gamma^{i(n)}(y)(\theta \pi)^n,
\]

(92)

19
where \( \Gamma^{i(n)} (y) = \Gamma^{ij_1...j_n} (y) \). That is, each \( x^i_n \) is a polynomial in \( \pi \) of order \( n \). In particular, up to the second order one has:

\[
x^i = y^i + \theta \Gamma^{ij} (y) \pi_j + \theta^2 \Gamma^{ijk} (y) \pi_j \pi_k + O (\theta^3) .
\]  

(93)

The coefficients \( \Gamma^{ij_1...j_n} (y) \) can be found using the standard procedure described in two previous sections, using the explicit form of \( \delta^{ij} \) and formulas (83) and (76) to construct \( y^i_n \), and then constructing the inverse transformation.

However, in this case there is a much simpler procedure, based on the form (92) of coordinates \( x^i \), Poisson brackets (60) with \( \mu = i \) and \( \nu = j \), i.e., \( \{ x^i , x^j \} = \theta \omega^{ij} (x) \), and the definition of the Darboux coordinates (61). First we write

\[
\left\{ y^i + \sum_{n=1}^{\infty} \Gamma^{i(n)} (y) (\theta \pi)^n , y^j + \sum_{n=1}^{\infty} \Gamma^{j(n)} (y) (\theta \pi)^n \right\} =
\theta \omega^{ij} \left( y^i + \sum_{n=1}^{\infty} \Gamma^{i(n)} (y) (\theta \pi)^n \right).
\]

(94)

Then, equating coefficients in the left and in the right-hand sides of (94) in each order in \( \theta \), one obtains algebraic equations on the coefficients \( \Gamma^{i(n)} (y) \) in terms of \( \omega^{ij} \) and lower order coefficients \( \Gamma^{i(m)} (y) , m < n \). The existence of the solution of these equations is a consequence of the combination of three facts: the existence of a symplectic structure \( \Omega_{\mu \nu} (\xi) = \Omega_{\mu \nu}^0 + O (\theta) \), such that \( \Omega^{ij} (\xi) = \theta \omega^{ij} (x) \); the existence of a perturbative form (62) of a Darboux coordinates and the fact that coordinates \( x^i \) have a polynomial form (92) in terms of \( \pi_j \). The procedure is analogous to the one proposed in [17] for the construction of polydifferential representation of the algebra \( [\hat{x}^i , \hat{x}^k] = 2 \alpha \hat{\omega}^{jk}(\hat{x}) \), where also was given a solution of these algebraic equations.

In the first order in \( \theta \) we have from (94):

\[
\Gamma^{ji} - \Gamma^{ij} = \omega^{ij},
\]

with a solution \( \Gamma^{ij} = -\omega^{ij} / 2 + s^{ij} \), where \( s^{ij} \) is an arbitrary symmetric matrix. Choosing \( s^{ij} = 0 \), which correspond to the choice \( f = r \), we end up with \( \Gamma^{ij} = -\omega^{ij} / 2 \). The equation in the second order is:

\[
8 (\Gamma^{ijk} - \Gamma^{jik}) = -\omega^{mk} \partial_m \omega^{ij}.
\]  

(95)

With a solution:

\[
\Gamma^{ijk} = \frac{1}{24} \omega^{km} \partial_m \omega^{ij} + \frac{1}{24} \omega^{jm} \partial_m \omega^{ik}.
\]  

(96)

That is, up to the second order the expression for \( x^i \) reads

\[
x^i = y^i - \frac{\theta}{2} \omega^{ij} \pi_j + \frac{\theta^2}{24} (\omega^{km} \partial_m \omega^{ij} + \omega^{jm} \partial_m \omega^{ik}) \pi_j \pi_k + O (\theta^3) .
\]  

(97)
One can verify that the inverse transformation of (91) gives exactly the eq. (97).

In the $n$-th order the algebraic equation for $\Gamma^{(n)}_i(y)$ reads

$$n\Gamma^{ij_{1}...i_{n}}(y) = G^{ij_{1}...i_{n}}(y), \quad n > 1,$$

where

$$G^{ij_{1}...i_{n}}\pi_{i_{1}}...\pi_{i_{n}} = \omega^{ij}_{n-1} - \sum_{m=1}^{n-1} \{ \Gamma^{i(n-m)}(\pi)^{n-m}, \Gamma^{(m)}(\pi) \} ,$$

$$\omega^{ij}_{n-1} = \frac{d^{n-1}}{d\theta^{n-1}} \omega^{ij} \left( y^i + \sum_{n=1}^{\infty} \Gamma^{i(n)}(\theta \pi)^n \right) \bigg|_{\theta = 0} .$$

The solution of the eq. (98) is given by

$$\Gamma^{ji_{1}...i_{n}} = \frac{1}{n(n+1)} \left( G^{ji_{1}...i_{n}} + G^{ji_{2}...i_{n}} + \ldots + G^{ji_{n+1}...i_{n+1}} \right) .$$

Once we know the explicit expression for the Darboux coordinates: $\eta_\mu = \eta_\mu(\xi)$, we can invert it: $\xi_\mu = \xi_\mu(\eta)$ and find the explicit form for the symplectic structure

$$\Omega_{\mu\nu}(\xi) = \{ \xi_\mu(\eta), \xi_\nu(\eta) \} \big|_{\eta_\mu = \eta_\mu(\xi)}.$$ 

By the construction (101) obey the Jacobi identity (10).

6 General solution

Now suppose that $J_i(x,p,\theta)$ is any arbitrary function. It means that the functions $\varpi^{ij}(x,p)$ and $\delta^{ij}(x,p)$ will have expressions different from (87) and (89):

$$\delta^{ij}(x,p) = \delta^{ij} + \theta(\partial^{i}J_{j}^{1} - \frac{1}{2}\partial_{j}\omega^{i}p_{l}) + O(\theta^{2}) ,$$

$$\varpi^{ij}(x,p) = \theta(\partial_{i}J_{j}^{1} - \partial_{j}J_{i}^{1}) + O(\theta^{2}) .$$

Consequently, the expressions for $y^i$ and $\pi_i$ in terms of $x^i$ and $p_i$ will also change. However, the expression (92) for $x^i$ in terms of $y^i$ and $\pi_i$ will remain the same, since $\Omega_{ij} = \theta \omega^{ij}(x)$ does not change, and the (92) continue obeying the eq.

$$\{ x^i(y,\pi), x^j(y,\pi) \} = \theta \omega^{ij}(x(y,\pi)) .$$

What will change now is the expression for $p_i$. In the general case we write

$$p_i = \pi_i - j_i(y,\pi,\theta), \quad j_i(y,\pi,\theta) = \partial_{i}f(y) + \theta j^{1}_{i}(y,\pi) + O(\theta^{2}) .$$
Which together with (97) implies that
\[
\pi_i = p_i + \theta j^1_i (x, p) + \theta^2 \left( \frac{1}{2} \partial_i j^1_i \omega^{ik} p_k + \partial_i^p j^1_i j^1_i \right) + O (\theta^3) .
\] (104)

The Poisson brackets between \( x^i (y, \pi) \) and \( p_i (y, \pi) \) are
\[
\{ x^i (y, \pi), p_j (y, \pi) \} = \delta^{ij} + \theta (\partial^p_j j^1_i (y, \pi) - \frac{1}{2} \partial_j \omega^{it} (y) p_t) -
\]
\[
\frac{\theta^2}{2} \left( \partial_i \omega^{ik} (y) \epsilon_k j^1_j (y, \pi) - \omega^{it} (y) \partial_j j^1_j (y, \pi) \right) + O (\theta^3) ,
\] (105)
\[
\{ p_i (y, \pi), p_j (y, \pi) \} = \theta (\partial_j j^1_i (y, \pi) - \partial_i j^1_j (y, \pi)) +
\]
\[
\theta^2 \left( \partial_j j^2_i (y, \pi) - \partial_i j^2_j (y, \pi) + \partial_j^p j^1_j (y, \pi) - \partial_i^p j^1_j (y, \pi) \partial_j j^1_j (y, \pi) \right) + O (\theta^3) .
\] (106)

Taking into account \(97\) and \(104\) we find that
\[
\{ x^i, p_j \} = \delta^{ij} (x, p) ,
\]
\[
\{ p_i, p_j \} = \omega^{ij} (x, p) ,
\] (107)

where
\[
\delta^{ij} (x, p) = \delta^{ij} + \theta (\partial^p_j j^1_i - \frac{1}{2} \partial_j \omega^{it} p_t)
\]
\[
+ \frac{\theta^2}{2} \left( \partial_i \omega^{it} j^1_t - \frac{1}{2} \partial_m \partial_i \omega^{it} \omega^{nk} p_k p_l + \frac{1}{2} \partial_j \Gamma^{ikl}_0 p_k p_l + \omega^{it} \partial_i j^1_j - \partial_i \omega^{ik} \partial^p_l j^1_j p_k \right) + O (\theta^3) ,
\]
\[
\omega^{ij} (x, p) = \theta (\partial_j j^1_i - \partial_i j^1_j) + \frac{\theta^2}{2} \left( \partial_i (\partial_j j^1_i - \partial_i j^1_j) \omega^{jk} p_k - 2 \partial_i^p (\partial_j j^1_i - \partial_i j^1_j) j^1_l \right) +
\]
\[
\theta^2 \left( \partial_i j^2_j - \partial_j j^2_i + \partial_i j^1_j - \partial_j j^1_i \partial_j j^1_j \right) + O (\theta^3) .
\] (108)

By the construction the Poisson brackets \(107\) together with \( \{ x^i, x^j \} = \theta \omega^{ij} (x) \), obey the Jacobi identity and equations \(92\) and \(103\) express phase space variables \( x^i \) and \( p_i \) in terms of Darboux coordinates \( y^i \) and \( \pi_i \). In fact, this process of construction of functions \( \delta^{ij} (x, p) \) and \( \omega^{ij} (x, p) \) and corresponding Darboux coordinates is much simpler than one described in sections 3 and 4. However, the existence of this perturbative procedure is a consequence of the existence of the procedures described in previous sections.

As far as the description of the arbitrariness in solution is concerned, we see comparing \(102\) and \(108\) that \( j^1_i = J^1_i \) and \( j^2_i \) can be expressed in terms of \( J^2_i \) and combinations of \( J^1_i \), \( \omega^{ik} \) and its derivatives, etc. That is, the vector \( J_i \) describing the arbitrariness in the construction of the symplectic structure \( \Omega_{\mu \nu} (\xi) \) can be expressed in terms of vector \( j_i \) and vice versa. So, in this direct method of the construction of the Darboux coordinates and the symplectic structure the arbitrariness is described by the vector \( j_i (y, \pi, \theta) \).
7 Canonical quantization and the star product

In this section we will discuss the quantization of the obtained in symplectic manifold. From a mathematical point of view the quantization of a symplectic realization of a Poisson manifold is a formal deformation of a local symplectic groupoid. This problem was considered in [14], where a universal generating function for a formal symplectic groupoid was provided in terms of Kontsevich trees and the Kontsevich weights [15]. However, it should be noted that there is no systematic way to compute the Kontsevich weights beyond the second order in the deformation, see e.g. [16].

We will construct the quantization in the Darboux coordinates, using the expressions (97) and (103) of original momenta \( p_i \) and coordinates \( x^i \), choosing the normal ordering, operators of momenta \( \hat{\pi}_i \) stand on the right of the operators of coordinates \( \hat{y}_i \).

Since variables \( \eta_{\mu} \) have the canonical PB (61), the corresponding operators \( \hat{\eta}_\mu = (\hat{y}_i, \hat{\pi}_i) \) obey the standard Heisenberg algebra:

\[
[\hat{y}_i, \hat{y}_j] = [\hat{\pi}_i, \hat{\pi}_j] = 0, \quad [\hat{y}_i, \hat{\pi}_j] = i\delta^i_j.
\] (109)

We choose the coordinate representation for this algebra, the operators of momenta are the derivatives, \( \hat{\pi}_i = -i\partial_i \), and the operators of coordinates are the operators of multiplication, \( \hat{y}_i = x^i \). Due to the normal ordering, we get for the operators \( \hat{x}^i \) and \( \hat{p}_i \) the following expression:

\[
\hat{p}^{Db}_i = -i\partial_i + j_i (x, -i\partial, \theta), \quad \hat{x}^{Db}_i = x^i + \frac{i\theta}{2} \omega^{dl} \partial_l - \theta^2 \Gamma^nlm \partial_l \partial_m - i\theta^3 \Gamma^nlmn \partial_l \partial_m \partial_n + O (\theta^4),
\] (110)

where the differential operator \( j_i (x, -i\partial, \theta) \) correspond to the vector \( j_i (y, \pi, \theta) \) describing an arbitrariness in our construction. As we will see this arbitrariness may be fixed if to require that the operators \( \hat{p}_i \) should be self-adjoint.

Let us calculate the commutator

\[
[\hat{x}^{Db}_i, \hat{x}^{Db}_j] = i\theta \omega^{ij} (x) - \frac{\theta^2}{2} \omega^{kl} \partial_k \omega^{ij} \partial_l - \frac{i\theta^3}{8} \omega^{nk} \omega^{ml} \partial_n \partial_m \omega^{ij} \partial_k \partial_l.
\] (111)

Using the eqs. (110) and supposing the normal ordering between coordinate and momentum operators, i.e., all \( \hat{x}^{Db}_i \) should stay on the left from \( \hat{p}^{Db}_i \), one can see that right hand side of the first commutator can be represented as

\[
[\hat{x}^{Db}_i, \hat{x}^{Db}_j] = i\theta \omega^{ij} (\hat{x}^{Db}_i) = i\theta \omega^{ij} (\hat{x}^{Db}_i) + i\theta \omega^{ij} (\hat{x}^{Db}_i, \hat{p}^{Db}_i),
\] (112)

where

\[
\omega^{ij} (\hat{x}^{Db}_i, \hat{p}^{Db}_i) = \frac{i\theta^2}{3} \omega^{nk} \partial_k \omega^{ml} \partial_l \partial_m \omega^{ij} (\hat{x}^{Db}_i) \hat{p}^{Db}_i + O (\theta^3).
\] (113)

23
stands for quantum corrections to $\hat{\omega}^{ij}(\hat{x}_{Db})$, that are needed to make the algebra (2) consistent i.e., obeying the sufficient condition. Note that due to dependence of $\hat{\omega}^{ij}_{Db}$ on operators of momenta $\hat{p}^{Db}$, the operators of coordinates $\hat{x}^{i}_{Db}$ do not form the subalgebra anymore, like it was stated in the beginning (1).

This problem can be solved, introducing the quantum corrections to the coordinate operators

$$\hat{x}^{i} = \hat{x}^{i}_{Db} + \hat{x}^{i}_{co},$$

(114)

where operator $\hat{x}^{i}_{co}$ should be chosen in the way to cancel the dependence of the commutator between coordinates (114) on momenta, i.e.,

$$\left[\hat{x}^{i}_{Db} + \hat{x}^{i}_{co}, \hat{x}^{j}_{Db} + \hat{x}^{j}_{co}\right] = i\theta \hat{\omega}^{ij}_{q} (\hat{x}_{Db} + \hat{x}_{co}).$$

(115)

Since corrections (113) starts from the second order in $\theta$ and are of first order in derivatives $\hat{p}^{Db} = -i\partial_{l}$ in this order, the corrections $\hat{x}^{i}_{co}$ to the operators of coordinates $\hat{x}^{i}_{Db}$ should start from the third order in $\theta$ and be quadratic in derivatives to cancel contribution from $\hat{\omega}^{ij}_{co} (\hat{x}_{Db}, \hat{p}^{Db})$, i.e.,

$$\hat{x}^{i}_{co} = -i\theta^{3} \Gamma^{ilm}_{1} (x) \partial_{l}\partial_{m} + O (\theta^{4}).$$

Therefore we write for the coordinate operators

$$\hat{x}^{i} = x^{i} + \frac{i\theta}{2} \omega^{il} \partial_{l} - \theta^{2} \Gamma^{ilm}_{1} \partial_{l} \partial_{m} - i\theta^{3} \left( \Gamma^{ilmn}_{1} \partial_{l} \partial_{m} \partial_{n} + \Gamma^{ilm}_{1} \partial_{l} \partial_{m} \right) + O (\theta^{4}).$$

(116)

The commutator between these operators reads

$$\left[\hat{x}^{i}, \hat{x}^{j}\right] = i\theta \hat{\omega}^{ij} (\hat{x}) + \frac{i\theta^{3}}{3} \omega^{nk} \partial_{k} \omega^{ml} \partial_{n} \partial_{m} \omega^{ij} \partial_{l} + i\theta^{3} \Gamma^{[ij]}^{[l]} \partial_{l} + O (\theta^{4}).$$

(117)

That is, if to choose $\Gamma^{ilm}_{1}$ obeying the equation

$$\Gamma^{[ij]}_{1} = -\frac{1}{3} \omega^{nk} \partial_{k} \omega^{ml} \partial_{n} \partial_{m} \omega^{ij},$$

(118)

with a solution

$$\Gamma^{[ij]}_{1} = \frac{1}{6} \omega^{nl} \partial_{l} \omega^{mk} \partial_{n} \partial_{m} \omega^{ij} + \frac{1}{6} \omega^{nl} \partial_{l} \omega^{mj} \partial_{n} \partial_{m} \omega^{ik},$$

(119)

the eq. (117) takes the form

$$\left[\hat{x}^{i}, \hat{x}^{j}\right] = i\theta \hat{\omega}^{ij} (\hat{x}) + O (\theta^{4}).$$

(120)

We see that new operators of coordinates $\hat{x}^{i}$ with coefficients $\Gamma^{ijk}_{1}$ defined in (119) form the subalgebra (120). It should be noted that the construction of the corrections $\hat{x}^{i}_{co}$ in the next order in $\theta$ will also require the corrections $\hat{\omega}^{ij}_{co} (\hat{x})$ to the operator $\hat{\omega}^{ij} (\hat{x})$. However,
these corrections will depend only on \( \hat{x}^i \). In this case, the expressions for both \( \hat{x}^i \) and \( \hat{\omega}_{ij}^q (\hat{x}) = \hat{\omega}_{ij}^q (\hat{x}) + \hat{\omega}_{ij}^q (\hat{x}) \) coincide with corresponding expressions from [17]. This fact admits us to define a star product on the algebra (1) by the standard rule,

\[
f (\hat{x}) g (\hat{x}) = (f \star g) (\hat{x}),
\]

where

\[
(f \star g)(x) = f (\hat{x}) g(x) = f g + \frac{i \theta}{2} \partial_i f \omega^{ij} \partial_j g
- \frac{\theta^2}{4} \left[ \frac{1}{2} \omega^{ij} \omega^{kl} \partial_i \partial_k f \partial_j \partial_l g - \frac{1}{3} \omega^{ij} \partial_j \omega^{kl} (\partial_i \partial_k f \partial_l g - \partial_k \partial_l f \partial_i g) \right]
- \frac{i \theta^3}{8} \left[ \frac{1}{3} \omega^{nl} \partial_i \omega^{mk} \partial_n \partial_m \omega^{ij} (\partial_i \partial_j f \partial_k g - \partial_i g \partial_j \partial_k f) + \frac{1}{6} \omega^{nk} \partial_n \omega^{jm} \partial_m \partial_j \partial_i f \partial_k g - \frac{1}{3} \omega^{nl} \partial_i \partial_j \partial_k g \partial_n \partial_m f
+ \frac{1}{3} \omega^{ln} \partial_i \omega^{jm} \omega^{ik} (\partial_i \partial_j f \partial_k \partial_n \partial_m g - \partial_i g \partial_j \partial_k \partial_n \partial_m f) + \frac{1}{6} \omega^{il} \omega^{im} \omega^{nk} \partial_i \partial_j \partial_k \partial_l \partial_n \partial_m g
+ \frac{1}{6} \omega^{mk} \omega^{nl} \partial_n \partial_m \omega^{ij} (\partial_i f \partial_j \partial_k \partial_l g - \partial_i g \partial_j \partial_k \partial_l f) \right] + O (\theta^4).
\]

For more details of this construction see [17].

8 Trace functional

To formulate the QM on noncommutative spaces we also need an expression for the trace functional on the algebra with a star product, i.e., a functional \( Tr (f) = \int \Omega (x) f (x) \), where \( \Omega (x) \) is an integration measure, satisfying

\[
Tr (f \star g) - Tr (fg) = 0.
\]

The condition (123) implies the cyclic property of trace. The existence of a trace functional for the Kontsevich star-product related to a Poisson bi-vector \( \omega^{ij} \) was proven in [20]. The recursive procedure for the construction of a trace was proposed in [21], which consists in the following steps. First to find a function \( \mu (x) \) such that

\[
\partial_i (\mu \omega^{ij}) = 0,
\]

25
and to define the measure as \( \Omega(x) = d^N x \mu(x) \), that is,

\[
Tr(f) = \int d^N x \mu(x) f(x).
\]

(125)

Note that if \( \det \omega^{ij} \neq 0 \), a natural integration measure is \( \Omega(x) = dx^N / \sqrt{\det \omega^{ij}(x)} \). Then, using the gauge freedom in the definition of the star product [15] to construct a new star product

\[
f \ast' g = D^{-1} (Df \ast Dg),
\]

(126)

choosing a gauge operator \( D \) in such a way that the condition (123) holds true for this new product. Because one may verify that (123) is not satisfied for the star-product (122), defined in the previous section. In particular, substituting (122) in (123) for two arbitrary functions \( f \) and \( g \) vanishing on the infinity, in the second order in \( \theta \) in the left hand side of (123) one gets

\[
- \frac{\theta^2}{4} \int d^N x \mu(x) \left[ \frac{1}{2} \omega^{ij} \omega^{kl} \partial_i \partial_k f \partial_j \partial_l g - \frac{1}{3} \omega^{ij} \partial_j \omega^{kl} (\partial_i \partial_k f \partial_l g - \partial_k \partial_i \partial_l g) \right]
\]

(127)

Integrating this expression by parts on \( f \) and \( g \) and using (124) we rewrite it as

\[
\frac{\theta^2}{12} \int d^N x \partial_i f \partial_l (\mu \omega^{ij} \partial_j \omega^{lk}) \partial_k g,
\]

(128)

where the matrix \( \partial_l (\mu \omega^{ij} \partial_j \omega^{lk}) \) is symmetric, i.e.,

\[
\partial_l (\mu \omega^{ij} \partial_j \omega^{lk}) = \partial_l (\mu \omega^{kj} \partial_j \omega^{il}),
\]

(129)

due to the JI and (124). The expression (128) is different from zero.

Due to (128) and (129) we search the gauge operator \( D \) in the form

\[
D = 1 + \theta^2 b^{ik} \partial_i \partial_k + O(\theta^3).
\]

(130)

In this case the new star product reads

\[
f \ast' g = f \ast g - 2 \theta^2 b^{ik} \partial_i f \partial_k g + O(\theta^3).
\]

(131)

The condition (123) for the star product (131) in the second order implies that

\[
\frac{\theta^2}{12} \partial_i f \partial_l (\mu \omega^{ij} \partial_j \omega^{lk}) \partial_k g - 2 \theta^2 \mu b^{ik} \partial_i f \partial_k g = 0.
\]

(132)

That is,

\[
b^{ik} = \frac{1}{24 \mu} \partial_l (\mu \omega^{ij} \partial_j \omega^{lk}).
\]

(133)

We conclude that given a Poisson bi-vector \( \omega^{ij} \) and a function \( \mu(x) \) obeying (124), the modified star product (131) admits the trace (125).
9 Quantization scheme and examples

Now we have all necessary tools to define the consistent noncommutative quantum mechanics.

The Hilbert space is determined as a space of complex-valued functions which are square-integrable with a measure \( \Omega(x) \).

The internal product between two states \( \varphi(x) \) and \( \psi(x) \) from the Hilbert space is defined as

\[
\langle \varphi | \psi \rangle = \text{Tr} (\varphi^* \star \psi).
\]

(134)

The action of the coordinate operators \( \hat{x}^i \) on functions \( \psi(x) \) from the Hilbert space is defined through the modified star product (131), for any function \( V(x) \) one has

\[
V(\hat{x}) \psi(x) = V(x) \star' \psi(x).
\]

(135)

The definitions (134) and (135) means that the coordinate operators are self-adjoint with respect to the introduced scalar product:

\[
\langle \hat{x}^i | \varphi \rangle = \text{Tr} \left( (x^i \star' \varphi)^* \star' \psi \right) = \text{Tr} \left( \varphi^* \star' \left( x^i \star' \psi \right) \right) = \langle \varphi | \hat{x}^i \psi \rangle.
\]

(136)

The momentum operators \( \hat{p}_i \) are fixed from the condition that they also should be self-adjoint with respect to (134). One of the possibilities is to choose it in the form

\[
\hat{p}_i = -i \partial_i - \frac{i}{2} \partial_i \ln \mu(x).
\]

(137)

One can easily verify that in this case \( \langle \hat{p}_i | \varphi \rangle = \langle \varphi | \hat{p}_i \psi \rangle \).

The choice (137) for the representation of the momentum operators imply that they commute: \( [\hat{p}_i, \hat{p}_j] = 0 \). The commutator between \( \hat{x}^i \) and \( \hat{p}_j \) is

\[
[\hat{x}^i, \hat{p}_j] = i \delta^i_j - \frac{i \theta}{2} \left( \partial_j \omega^d (\hat{x}) \hat{p}_l + i \partial_j \left( \omega^d \partial_l \ln \mu (\hat{x}) \right) \right) + O (\theta^2).
\]

(138)

That is, as it was stated in the beginning the complete algebra of commutation relations involving \( \hat{x}^i \) and \( \hat{p}_j \) is a deformation in \( \theta \) of a standard Heisenberg algebra.

9.1 Free particle

As an example we consider the eigenvalue problem for the Hamiltonian

\[
\hat{H} = \frac{1}{2} \hat{p}_i \hat{p}^i,
\]

(139)
describing a free particle. In this case (139) takes the form
\[
\hat{H} = -\frac{1}{2} \left( \partial_i + \frac{1}{2} \partial_i \ln \mu(x) \right)^2 ,
\]
(140)
The eigenstates of this Hamiltonian are
\[
\psi = e^{-ik_i x^i} \mu(x)^{-\frac{1}{2}},
\]
(141)
with eigenvalues
\[
E = \frac{1}{2} k_i k^i ,
\]
(142)
where \( k_i \) are the eigenvalue of momenta \( \hat{p}_i \). We see that the spectrum of energy \( E \) is non-negative and continuous like in undeformed case, however the eigenstates differ from the plane wave on the commutative space by the factor \( \mu(x)^{-\frac{1}{2}} \), which may lead to different phenomenological consequences studying the processes of scattering of plane waves on curved noncommutative spaces.

9.2 Three-dimensional isotropic harmonic oscillator

It should be noted that the quantum mechanical scale of energies is rather different from the Planck scale, therefore from the physical point of view it is useless to look for the effects caused by the noncommutativity in QM. However some important properties like preservation of symmetries and corresponding consequences can be studied already in QM. In particular, it is well known fact that the canonical noncommutativity, \( \left[ \hat{x}^i , \hat{x}^j \right] = i\theta^{ij} \), breaks the rotational symmetry of a particle in a central potential, which removes the degeneracy of the energy levels [22] over the magnetic quantum number \( m \). This fact leads to the bounds of noncommutativity. The same logic remains in the field theory [2]. We will show here that the noncommutativity can be introduced in a way to preserve the symmetries and the corresponding degeneracy.

Let us consider three-dimensional isotropic harmonic oscillator described by the Hamiltonian
\[
\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2} r^2 ,
\]
(143)
where \( r^2 = x^2 + y^2 + z^2 \). And let us choose the external antisymmetric field in a way to preserve the rotational symmetry of the system, \( \omega^{ij}(x) = \varepsilon^{ijk} x^k \). The corresponding algebra of noncommutative coordinates is the algebra of fuzzy sphere [23],
\[
\left[ \hat{x}^i , \hat{x}^j \right] = i\theta \varepsilon^{ijk} \hat{x}^k .
\]
(144)
Note that the rotationally invariant NC space can be obtained as a foliation of fuzzy spheres [24].
One may see that any function μ(r^2) obeys the equation (124):

\[ \partial_i (\mu(r^2)\varepsilon^{ijk}x^kf(r^2)) = 0, \]

and can be chosen as a measure to define a trace functional. For simplicity we set \( \mu(x) = 1 \). So, the momentum operators are just derivatives \( \hat{p}_i = -i\partial_i \). The specific choice of the external field implies that the coordinate operators transform as a vectors under rotations:

\[ [L_i, x^j] = i\varepsilon^{ijk}x^k, \quad (145) \]

where \( L^i = -i\varepsilon^{ijk}x_j\partial_k \) is the angular momentum operator. That is, the rotational symmetry \((143)\) will be conserved, preserving the degeneracy of the energy spectrum over the magnetic quantum number \( m \). The Hamiltonian can be written as

\[ \hat{H} = -\frac{1}{2}\Delta + \frac{\omega^2}{2}r^2 + \frac{\theta^2\omega^2}{24}L^2 + O(\theta^4), \quad (146) \]

where \( L^2 = L_x^2 + L_y^2 + L_z^2 \) is the the orbital momentum. We use the usual perturbation theory to calculate the leading corrections to the energy levels,

\[ \Delta E_n^{NC} = \left\langle \psi^0 \left| \frac{\theta^2\omega^2}{24}L^2 \right| \psi^0 \right\rangle = \frac{\theta^2\omega^2l(l+1)}{24}, \quad (147) \]

where \( |\psi^0\rangle = R_{nl}(r)Y_l^m(\vartheta, \varphi) \) is the unperturbed wave function, corresponding to the energy \( E_n = \omega(n + \frac{3}{2}) \), here \( n \) is a principal quantum number and \( l \) is the azimuthal quantum number. The corresponding nonlocality is given by

\[ \Delta x\Delta y \geq \frac{\theta^2}{4} |m|, \quad (148) \]

where \( m = -l, ..., l \) and \( l = 0, 1, ..., n \). That is, the more the energy of the system the more the nonlocality.

10 Conclusions and perspectives

In conclusion we would like to overview some perspectives of the present activity. As it was stated in the introduction our aim is to construct the consistent quantum field theory on noncommutative spaces of general form. In this connection the next step is to construct a relativistic generalization of the proposed nonrelativistic quantum mechanics. If the external field \( \omega^{\rho\sigma}(x) \) transforms as a two tensor with respect to a Lorentz group, e.g., \( \omega^{\rho\sigma}(x) = \varepsilon^{\rho\sigma\lambda}x_\lambda f(x^2) \) in \((2 + 1)\) dimensions, the operators \( \hat{x}^\rho = x^\rho + i\theta/2\omega^{\rho\sigma}\partial_\sigma + O(\theta^2) \) and \( \hat{p}_\rho = -i\partial_\rho - i\partial_\rho\mu(x) \) will transform as vectors. This fact may be used to construct
relativistic wave equations on coordinate dependent NC space-time. In particular, the free noncommutative Klein-Gordon equation can be written as
\[
\left[ \left( \partial_\rho - \frac{1}{2} \partial_\rho \ln \mu \right)^2 - m^2 \right] \Phi = 0. \tag{149}
\]
This equation is covariant under Lorentz transformation. The action leading to the equation (149) is
\[
S_{\text{free}} = \int d^N x \mu(x) \left[ \frac{1}{2} \left( \hat{\partial}_\rho \Phi \right)^* \star \left( \hat{\partial}_\rho \Phi \right) + \frac{m^2}{2} \Phi^* \star \chi \Phi \right]. \tag{150}
\]
One may also add the interaction term to this action, e.g.,
\[
S_{\text{int}} = \frac{\lambda}{4!} \int d^N x \mu(x) \Phi^* \chi \Phi \chi \Phi^* \chi \Phi^* \chi \Phi, \tag{151}
\]
and to study the corresponding quantum theory. Different questions may be addressed here like unitarity and renormalizability. An important problem here is to describe the physical mechanism which define the external field \( \omega^{\rho\sigma}(x) \).

**Acknowledgements**

I am grateful to Marcelo Gomes and Pedro Gomes for fruitful discussions and to Dima Vassilevich for useful comments.

**A Integrability condition for the Darboux coordinates**

Let us consider the eq. (68). In section 4 it was shown that integrability condition for this equation in the first two orders in \( \theta \) is satisfied as a consequence of the Jacobi identity (10) for the symplectic structure \( \Omega_{\mu\nu} \). Suppose that the solution of (68) was found up to the \((n-1)\)-th order. Here we will prove that the integrability condition (70) in the \( n \)-th order is exactly Jacobi identity (10) in the \( n \)-th order. Let us write
\[
L = \sum_{m=1}^{n-1} \Omega_{\alpha\beta}^0 \partial_\beta \left[ \partial_\sigma \eta^{n-m}_\mu \Omega_{\sigma\rho}^0 \partial_\rho \eta^m_\nu + \partial_\sigma \eta^{n-m}_\mu \Omega^m_{\sigma\nu} + \Omega_{\rho\sigma}^0 \partial_\lambda \eta^{m}_\nu \right] \tag{152}
\]
\[
+ \sum_{m=1}^{n-2} \sum_{k=1}^{n-m-1} \Omega_{\alpha\beta}^0 \partial_\beta \left( \Omega_{\sigma\rho}^{n-m-k} \partial_\sigma \eta^{k}_\mu \partial_\rho \eta^{m}_\nu \right) + \text{cycl.}(\alpha\mu\nu).
\]

30
Using JI \((40)\) in the first line of \((152)\) one rewrites \(L\) as

\[
L = \sum_{m=1}^{n-1} \left[ \partial_\sigma \eta_\mu^m \Omega_\rho_\sigma \partial_\rho \left( \partial_\beta \eta_\nu^{n-m} \Omega_\beta_\alpha + \Omega_\nu_\beta \partial_\beta \eta_\alpha^{n-m} + \Omega_\nu_\alpha \right) - \Omega_{\mu_\sigma} \partial_\sigma \left( \partial_\beta \eta_\nu^{n-m} \Omega_\beta_\alpha + \Omega_\nu_\beta \partial_\beta \eta_\alpha^{n-m} \right) \right]
\]

\[
+ \sum_{m=1}^{n-2} \sum_{n-m-1}^{n-1} \left[ \Omega_\alpha_\beta \partial_\beta \Omega_\alpha^{n-m-k} \partial_\sigma \eta_\mu^l \partial_\rho \eta_\nu^m + \partial_\sigma \eta_\mu^m \Omega_\sigma_\alpha \partial_\sigma \partial_\rho \eta_\nu^{n-m-k} \partial_\rho \left( \Omega_\nu_\beta \partial_\beta \eta_\alpha^{k} + \Omega_\nu_\alpha \partial_\beta \eta_\alpha^{k} \right) \right]
\]

\[
+ \partial_\sigma \eta_\mu^m \left( \Omega_{\alpha_\beta}^{n-m-k} \partial_\beta \Omega_\sigma_\nu^{k} \right) \right] + \text{cycl.}(\alpha_\sigma \nu) \right] \right] + \text{cycl.}(\alpha \mu \nu).
\]

Then, taking into account the eq. \((68)\) in the first line of \((153)\) we represent it in the form

\[
L = \sum_{m=1}^{n-1} \Omega_{\mu_\sigma} \partial_\sigma \Omega_\nu_\alpha^{n-m} + \sum_{m=1}^{n-2} \left[ \partial_\sigma \eta_\mu^m \Omega_\rho_\sigma \partial_\rho \left( \partial_\beta \eta_\nu^{n-m} \Omega_\beta_\alpha + \Omega_\nu_\beta \partial_\beta \eta_\alpha^{n-m} + \Omega_\nu_\alpha \right) \right]
\]

\[
+ \sum_{k=1}^{n-3} \sum_{m=1}^{n-2} \sum_{l=1}^{n-m-1} \Omega_\delta_\sigma \partial_\sigma \left( \Omega_\alpha_\beta^{n-m-k-l} \partial_\beta \eta_\nu^l \partial_\beta \eta_\alpha^{k} \right) + \text{cycl.}(\alpha \mu \nu).
\]

Using the JI in the second term of the second line and also the eq. \((68)\) for \(n = 1\), we have

\[
L = \sum_{m=1}^{n-1} \Omega_{\mu_\sigma} \partial_\sigma \Omega_\nu_\alpha^{n-m} + \sum_{m=1}^{n-2} \left[ \partial_\sigma \eta_\mu^m \Omega_\rho_\sigma \partial_\rho \left( \partial_\beta \eta_\nu^{n-m} \Omega_\beta_\alpha + \Omega_\nu_\beta \partial_\beta \eta_\alpha^{n-m} + \Omega_\nu_\alpha \right) \right]
\]

\[
+ \sum_{k=1}^{n-3} \sum_{m=1}^{n-2} \sum_{l=1}^{n-m-1} \left[ \Omega_\alpha_\beta^{n-m-k-l} \partial_\beta \eta_\nu^l \partial_\beta \eta_\alpha^{k} \right] + \text{cycl.}(\alpha \mu \nu).
\]

We will need the following Lemma:
Lemma 4  The identities

\[ M = \sum_{m=1}^{n-3} \sum_{k=1}^{n-m-2} \sum_{l=1}^{n-m-k-1} \left[ \partial_\sigma \eta_\mu^m \Omega^0_{\rho\sigma} \partial_\rho \left( \partial_\beta \eta_\nu^l \Omega^{n-m-k-l}_{\beta\gamma} \partial_\gamma \eta_\alpha^k \right) \right. \]
\[ + \partial_\sigma \eta_\mu^m \Omega^0_{\rho\sigma} \partial_\rho \left( \partial_\beta \eta_\nu^l \Omega^0_{\beta\gamma} \partial_\gamma \eta_\alpha^k \right) \]
\[ + \partial_\sigma \eta_\mu^m \Omega^0_{\rho\sigma} \partial_\rho \left( \partial_\beta \eta_\nu^l \Omega^{n-m-k-l-1}_{\beta\gamma} \partial_\gamma \eta_\alpha^k \right) \]
\[ + \text{cycl.}(\alpha\mu\nu) = 0, \]

and

\[ N = \sum_{m=1}^{n-2} \sum_{k=1}^{n-m-1} \partial_\sigma \eta_\mu^m \Omega^0_{\rho\sigma} \partial_\rho \left( \partial_\beta \eta_\nu^l \Omega^{n-m-k}_{\beta\gamma} \partial_\gamma \eta_\alpha^k \right) + \text{cycl.}(\alpha\mu\nu) = 0. \]

hold true.

Proof. One can verify that

\[ \sum_{m=1}^{n-3} \sum_{k=1}^{n-m-2} \sum_{l=1}^{n-m-k-1} \partial_\sigma \eta_\mu^m \Omega^0_{\rho\sigma} \partial_\rho \left( \partial_\beta \eta_\nu^l \Omega^{n-m-k-l}_{\beta\gamma} \partial_\gamma \eta_\alpha^k \right) + \text{cycl.}(\alpha\mu\nu) = 0. \]

Also using (150) one can see that

\[ \sum_{m=1}^{n-3} \sum_{k=1}^{n-m-2} \sum_{l=1}^{n-m-k-1} \partial_\sigma \eta_\mu^m \Omega^0_{\rho\sigma} \partial_\rho \left( \partial_\beta \eta_\nu^l \Omega^{n-m-k-l-1}_{\beta\gamma} \partial_\gamma \eta_\alpha^k \right) + \text{cycl.}(\alpha\mu\nu) = 0. \]

Substituting the right-hand side of the eq. (157) in the right hand side of the eq. (156) one finds that (154) holds true. The proof of (155) is straightforward. \[ \square \]
The sum $L + M + N$ using the eq. (68) for $n = 2$ can be written as

$$L + M + N = \sum_{m=1}^{n-1} \Omega_{\mu\alpha}^m \partial_{\nu} \Omega_{\nu\alpha}^{n-m} + \sum_{m=1}^{n-3} \partial_{\sigma} \eta_{\mu}^m \Omega_{\nu\sigma}^0 \partial_{\rho} \left( \Omega_{\nu\beta}^0 \partial_{\beta} \eta_{\alpha}^{n-m} + \Omega_{\beta\alpha}^0 \partial_{\beta} \eta_{\nu}^{n-m} + \Omega_{\nu\alpha}^{n-m} \right)$$

$$+ \sum_{k=1}^{n-m-1} \left( \Omega_{\mu\beta}^{n-m-k} \partial_{\beta} \eta_{\nu}^{k} + \Omega_{\beta\nu}^{n-m-k} \partial_{\beta} \eta_{\mu}^{k} + \partial_{\beta} \eta_{\nu}^{n-m-k} \Omega_{\beta\gamma}^0 \partial_{\gamma} \eta_{\alpha}^{k} \right)$$

$$+ \sum_{k=1}^{n-m-2} \sum_{l=1}^{n-m-1} \partial_{\beta} \eta_{\nu}^l \Omega_{\beta\gamma}^{n-m-k-l} \partial_{\gamma} \eta_{\alpha}^{k} \right)$$

$$+ \sum_{m=1}^{n-4} \sum_{k=1}^{n-m-3} \partial_{\sigma} \eta_{\mu}^m \Omega_{\nu\sigma}^0 \partial_{\rho} \left( \Omega_{\nu\beta}^0 \partial_{\beta} \eta_{\alpha}^{n-m-k} + \Omega_{\beta\alpha}^0 \partial_{\beta} \eta_{\nu}^{n-m-k} + \Omega_{\nu\alpha}^{n-m-k} \right)$$

$$+ \sum_{l=1}^{n-m-k-1} \left( \Omega_{\mu\beta}^{n-m-k-l} \partial_{\beta} \eta_{\nu}^{l} + \Omega_{\beta\nu}^{n-m-k-l} \partial_{\beta} \eta_{\mu}^{l} + \partial_{\beta} \eta_{\nu}^{n-m-k-l} \Omega_{\beta\gamma}^0 \partial_{\gamma} \eta_{\alpha}^{l} \right)$$

$$+ \sum_{l=1}^{n-m-k-2} \sum_{p=1}^{n-m-k-l-1} \partial_{\beta} \eta_{\nu}^l \Omega_{\beta\gamma}^{n-m-k-l-p} \partial_{\gamma} \eta_{\alpha}^{k} \right) + \text{cycl.}(\alpha\mu\nu).$$

Again using eq. (68) we end up with

$$L + M + N = \sum_{m=1}^{n-1} \Omega_{\mu\sigma}^m \partial_{\nu} \Omega_{\nu\alpha}^{n-m}.$$ 

Since $M = N = 0$ due to Lemma 4, the integrability condition (70) has the form

$$\Omega_{\mu\sigma}^0 \partial_{\nu} \Omega_{\nu\alpha}^{m} + \sum_{m=1}^{n-1} \Omega_{\mu\sigma}^m \partial_{\nu} \Omega_{\nu\alpha}^{n-m} + \text{cycl.}(\mu\nu\alpha) = 0,$$

which is exactly Jacobi Identity (10) in the $n$-th order.

References

[1] M. Douglas, N. Nekrasov, Rev.Mod.Phys.73 (2001) 977-1029, R. Szabo, Phys.Rept.378 (2003) 207-299.

[2] T.C. Adorno, D.M. Gitman, A.E. Shabad, D.V. Vassilevich, Phys.Rev.D84 (2011) 085031.
[3] S. Doplicher, K. Fredenhagen and J. Roberts, Commun.Math.Phys. **172** (1995) 187.

[4] M. Buric, M. Wohlgenannt, JHEP **1003** (2010) 053.

[5] H. Grosse, R. Wulkenhaar, JHEP **12** (2003) 019

[6] M. Gomes, V.G. Kupriyanov, Phys.Rev.**D79** (2009) 125011.

[7] A. Fring, L. Gouba, F.G. Scholtz, J.Phys. **A43** (2010) 345401.

[8] B. Bagchi, A. Fring, Phys. Lett. A373 (2009) 4307.

[9] E. Harikumar, T. Juric, S. Meljanac, Phys.Rev.**D84** (2011) 085020; Phys.Rev.**D86** (2012) 045002.

[10] V. Galikova, P. Presnajder, J. Phys: Conf. Ser. **343** (2012) 012096.

[11] M.V. Karasev, *Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets*. (Russian) Izv.Akad.Nauk SSSR Ser. Mat. **50** (3) 638 (1986).

[12] A. Coste, P. Dazord, A. Weinstein, *Groupoides symplectiques*. (French) Publ. Dép. Math. Nouvelle Sér. A, 87-2, Lyon: Univ. Claude-Berbard, 1987, pp. 1-62.

[13] V.I. Arnold, *Mathematical methods of Classical Mechanics*, Springer, New York (1978).

[14] A.S. Cattaneo, B. Dherin, G. Felder, Commun.Math.Phys. **253** (2005) 645.

[15] M. Kontsevich, Lett. Math. Phys. **66** (2003) 157.

[16] G. Dito, D. Sternheimer, *Deformation quantization: Genesis, developments and metamorphoses*. 9–54, IRMA Lect. Math. Theor. Phys., 1, de Gruyter, Berlin, 2002. [arXiv:math/0201168](http://arxiv.org/abs/math/0201168).

[17] V.G. Kupriyanov, D.V. Vassilevich, Eur.Phys.J.C. **58** (2008) 627-637.

[18] M. Gomes, V.G. Kupriyanov, A.J. da Silva, Phys.Rev.**D81** (2010) 085024.

[19] D.M. Gitman and V.G. Kupriyanov, Eur.Phys.J. C **54** (2008) 325.

[20] G. Felder, B. Shoikhet, Lett. Math. Phys. **53** (2000) 75-86.

[21] V.G. Kupriyanov, *Hydrogen atom on curved noncommutative space*, [arXiv:1209.6105](http://arxiv.org/abs/1209.6105).

[22] M. Chaichian, M.M. Sheikh-Jabbari and A. Tureanu, Phys.Rev.Lett. **86** (2001) 2716.
[23] J. Madore, Class.Quant.Grav. 9 (1992) 69; G. Alexanian, A. Piznul, A. Stern, Nucl.Phys. B600 (2001) 531; H. Grosse, J. Madore, H. Steinacker, Int. J. Mod. Phys. A 17 (2002) 2095.

[24] A. B. Hammou, M. Lagraa and M. M. Sheikh-Jabbari, Phys. Rev. D 66 (2002) 025025; E. Moreno, Phys. Rev. D 72 (2005) 045001.