A Stable Implementation of a Well-posed 2-D Curvilinear Shallow Water Equations with No-Penetration Wall and Far-Field Boundary conditions

Reindorf N. Borkor\textsuperscript{a,b}, Magnus Svärd\textsuperscript{c}, Adu Sakyi\textsuperscript{a,b,*}, Peter Amoako-Yirenkyi\textsuperscript{a,b}

\textsuperscript{a}Center for Scientific and Technical Computing, National Institute for Mathematical Sciences (NIMS), Ghana
\textsuperscript{b}Department of Mathematics, Kwame Nkrumah University of Science and Technology (KNUST), Ghana
\textsuperscript{c}Department of Mathematics, University of Bergen, Bergen, Norway

Abstract

This paper presents a more stable implementation and a highly accurate numerical tool for predicting flooding in urban areas. We started with the (linearised) well-posedness analysis by [1], where far-field boundary conditions were proposed but extended their analysis to include wall boundaries. Specifically, high-order Summation-by-parts (SBP) finite-difference operators were employed to construct a scheme for the Shallow Water Equations. Subsequently, a stable SBP scheme with Simultaneous Approximation Terms that imposes both far-field and wall boundaries was developed. Finally, we extended the schemes and their stability proofs to non-cartesian domains.

To demonstrate the strength of the schemes, computations for problems with exact solutions were performed and a theoretical design-order with second-, third- and fourth (2,3,4) convergence rates obtained. Finally, we apply the 4th-order scheme to steady river channel, canal (or flood control channel simulations), and dam-break problems. The results show that the imposition of the boundary conditions are stable, and that they cause no visible reflections at the boundaries. The analysis adequately becomes nec-

*Corresponding author
Email addresses: reinbork@knust.edu.gh (Reindorf N. Borkor), Magnus.Svard@uib.no (Magnus Svärd), asakyi@knust.edu.gh (Adu Sakyi), amoakoyirenkyi@knust.edu.gh (Peter Amoako-Yirenkyi)
ecessary in flood risk management decisions since they can confirm stable and accurate future predictions of floodplain variables.

**Keywords:** Hydrodynamic, Summation-by-Parts (SBP), Simultaneous Approximation Terms (SAT), high-order finite-difference methods, Stability, Well-posedness

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1. Introduction

Over the years, tragic events due to flooding have been recorded annually in most cities in Ghana. Mostly, these involve loss of both lives and properties. Over a hundred and fifty (150) persons died as a result of a combination of floods and fire outbreaks in June 2015. (See the report [2].) Aside natural factors that cause flooding such as intense rainfall stimulating internal erosion and surface run-offs, other contributing factors such as urbanization, land use and poor channels have aggravated the impact of floods in cities [3, 4, 5]. The problem of flooding is not only limited to Ghana, it is a general problem which is encountered in many parts of the world today. Due to this fact, a simulation tool which accurately predict flood events is needed.

Propagation of flood can, in principle, be modelled by the Navier-Stokes Equations (NSE) but these equations are too computationally expensive to solve [6, 7]. Instead, we consider the Shallow Water equations (SWE) that are obtained by depth-integrating the NSE. The SWE describe the evolution of a shallow layer of fluid and are useful in the simulation of open-channel flows, large-scale hydraulics, to mention but a few examples. (See [6, 7, 8, 9, 10, 11] for more details). Mathematically, SWE is a system of non-linear conservation laws that often requires computational methods to approximate solutions. Solutions can contain shocks but herein we focus on smooth solutions. For smooth solutions, high-order accurate numerical schemes are more efficient [12, 13, 14, 15, 16, 17, 18].

The primary focus of this paper is to derive well-posed boundary conditions of the Shallow Water Equations and stable high-order schemes for the resulting initial-boundary value problems. The sets of well-posed far-field boundary conditions for the Shallow Water Equations have been derived for the one-dimensional (1-D) case in [19] and 2-D in [1]. Here, we re-derive the set of far-field boundary conditions in [1] and prove stability for a Summation-by-Parts Simultaneous Approximation Term (SBP-SAT) implementation. In
addition, we demonstrate that the no-penetration wall boundary conditions are well-posed and derive a stable SBP-SAT implementation. Furthermore, since realistic flows are usually limited to Cartesian domains, we generalize the schemes and the stability proofs to a curvilinear frame for the complete initial-boundary value problem. Furthermore, we verify the scheme using the method of manufactured solutions (see [20, 21]) and then apply it to a set of flood propagation problems. To further enhance the capability of the scheme, stable schemes for multi-block grids will be derived in a future article.

The paper is outlined as follows; Section 2 introduces the definitions and concepts underpinning the theory of linear Initial Boundary Value problem (IBVP). The 2-D SWE are presented in section 3. Section 4 establishes the linear well-posedness analysis with continuous energy estimates of the 2-D SWE including boundary conditions. Further in Section 5, we construct high-order SBP-SAT finite-difference schemes that satisfy discrete energy estimates that mimic the corresponding estimates. Also, in Section 6 we propose a general case of the stability proofs by deriving a Curvilinear Shallow Water Equations. Finally, section 7 provides numerical experiments that verify the analysis carried out in section 4, 5 and 6.

2. Concepts and Definitions

Let \( \Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \) denote the spatial domain in 2-D and \( \partial \Omega \) denote its boundary. Consider the following IBVP:

\[
\begin{align*}
    u_t + P u &= F(x, y, t), \quad (x, y) \in \Omega, \quad t > 0, \\
    Bu &= g(x, y, t), \quad (x, y) \in \partial \Omega, \quad t \geq 0, \\
    u &= f(x, y), \quad (x, y) \in \Omega, \quad t = 0,
\end{align*}
\]

(1)

where \( u(x, y, t) \) represents the solution vector. \( P \) and \( B \), respectively denote the spatial linear differential and boundary operators. Furthermore, \( F \) is a forcing function. Finally, \( g \) and \( f \) are, respectively, smooth boundary and initial data.

The scalar product and norm are defined as,

\[
(w, v)_\Omega = \int_\Omega w^T v dx dy, \quad \|w(\cdot, t)\|\Omega^2 = (w, w)_\Omega,
\]

(2)

for real-valued vector functions \( w \) and \( v \). We need the Definitions 1-2 below, that are found in [14, 22].
Definition 1. The IBVP \([1]\) with \(F = g = 0\) is well posed if for every \(f \in C^\infty\) that vanishes in a neighborhood of \(\partial \Omega\), a unique smooth solution exists that satisfies the estimate
\[
\|u(\cdot, t)\|^2_\Omega \leq K^c_1 e^{\alpha_c t} \|f\|^2_\Omega, \quad 0 < t < T,
\]
where \(K^c_1\) and \(\alpha_c\) are constants value which are also bounded independently of \(f\).

Sometimes the solution can be bounded with \(F \neq 0, g \neq 0\). Then we say that the problem is strongly well-posed.

Next, we turn to the discretization of (1). Let \(x_i = i h_1, i = 0, 1, \ldots, M\) and \(y_j = j h_2, j = 0, 1, \ldots, N\) where respectively, \(h_1 = 1/M\) and \(h_2 = 1/N\) are the grid spacings. Now, the following sets of indices are, respectively, defined for the interior and boundaries—\(\mathcal{I}_1 = \{(i, j) : i = 1; \ldots, M - 1, j = 1, \ldots, N - 1\}\), \(\mathcal{I}_2 = \{(i, j) : i = 0, M, \forall j\}\), and \(\mathcal{I}_3 = \{(i, j) : \forall i; j = 0, N\}\).

We introduce the semi-discrete approximation of (1),
\[
(v_{ij})_t + P_h v_{ij} = F_{ij}, \quad (i, j) \in \mathcal{I}_1, \quad t \geq 0,
\]
\[
B_h v_{ij} = g_{ij}, \quad (i, j) \in \mathcal{I}_2, \mathcal{I}_3, \quad t \geq 0,
\]
\[
v_{ij} = f_{ij}, \quad t = 0,
\]
(3)

where \(v_{ij}(t)\) is the approximate solution at grid point \((x_i, y_j)\). \(P_h\) is the difference operator approximating \(P\) and \(B_h\) is a discrete boundary operator approximating \(B\). \(F_{ij} = F(x_i, y_j, t), g_{ij} = g(x_i, y_j, t)\) and \(f_{ij} = f(x_i, y_j)\) are the known compatible data of (1). (See [14] for an explanation of compatibility.)

The definition of the discrete scalar product and norm are given as
\[
(w, v)_h = w^T P v \quad \|w\|^2_h = (w, w)_h,
\]
(4)
for real grid vector functions \(w\) and \(v\). Here, \(P\) is a positive definite symmetric matrix to be defined later.

Definition 2. Let \([3]\) be a semi-discrete approximation of the well-posed \([1]\) with \(F = g = 0\). Then the semi-discrete approximation \([3]\) is stable if the solution \(v\) satisfies the estimate
\[
\|v(t)\|^2_h \leq K^d_1 e^{\alpha_d t} \|f\|^2_h,
\]
where \(K^d_1\) and \(\alpha_d\) are constants which are bounded and independent of \(f\).

Analogously to the continuous problem, if the numerical solution can be bounded with \(F \neq 0\) and \(g \neq 0\), we say that the problem is strongly stable.
3. The Continuous Problem

The 2-D nonlinear Shallow Water Equations are

\[ W_t + [F(W)]_x + [G(W)]_y = 0, \quad (x, y) \in \Omega, \quad t > 0, \quad (5) \]

where

\[ W = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \quad F(W) = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \\ hv \end{pmatrix}, \quad G(W) = \begin{pmatrix} hv \\ huv \\ huv + gh^2/2 \end{pmatrix}. \]

\( W \) is the vector consisting of the conserved variables. The convective flux-vector functions are \( F(W) \) and \( G(W) \), respectively, in the \( x- \) and \( y- \) direction. \( h(t,x,y) \) is the depth of water (or height) and \( hu \) and \( hv \) are, respectively, the momentum in the \( x- \) and \( y- \) direction. Furthermore, \( g \) is the gravitational constant and \( u(t,x,y) \) and \( v(t,x,y) \) are the velocity components in \( x- \) and \( y- \) direction, respectively (see [6]).

The 1-D form of (5) is given as

\[ W_t + [F(W)]_x = 0, \quad x \in \Omega, \quad t > 0, \quad (6) \]

where \( W = \begin{pmatrix} h \\ hu \end{pmatrix}, \quad F(W) = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \end{pmatrix}. \)

4. Well-posedness analysis

In this section, we demonstrate well-posedness of (5) using the energy method. (See [22, 23] for further information on the energy method.) To this end, we linearize, symmetrize (5) (see Appendix A) and freeze the coefficients, and arrive at

\[ w_t + Aw_x + Bw_y = 0, \quad (x, y) \in \Omega, \quad t > 0, \quad (7) \]

where

\[ w = \begin{pmatrix} g\tilde{h}/c \\ \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad A = \begin{pmatrix} u_0 & c & 0 \\ c & u_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix}, \quad B = \begin{pmatrix} v_0 & 0 & c \\ 0 & v_0 & 0 \\ c & 0 & v_0 \end{pmatrix}. \]

The variables \( \tilde{h}, \tilde{u}, \tilde{v} \), respectively, represent perturbations of the height and velocities. Also, \( u_0, v_0 \) and \( h_0 \) are, respectively, the constant mean velocities and the height. Furthermore, \( c = \sqrt{gh_0} \) is the gravity wave speed.
Applying the energy method on (7), leads to
\[ \int_\Omega w^T w_t \, dx \, dy + \int_\Omega (w^T A w)_x \, dx \, dy + \int_\Omega (w^T B w)_y \, dx \, dy = 0. \] (8)

Using the Green-Gauss theorem on (8) yields
\[ \|w\|_t^2 + \oint_{\partial \Omega} w^T [(A, B) \hat{n}] w \, ds = 0, \] (9)

where \( \hat{n} = (\hat{n}_1, \hat{n}_2) \) denotes the outward unit normal vector of \( \partial \Omega \); \((A, B) \hat{n} = \hat{n}_1 A + \hat{n}_2 B \) and \( s \) is the coordinate along \( \partial \Omega \).

First, we consider the computational domain \( \Omega = [0, L_1] \times [0, L_2] \). At \( x = \{0, L_1\} \), \( \hat{n} = \pm e_1 \) and at \( y = \{0, L_2\} \), \( \hat{n} = \pm e_2 \). We recast (9) as
\[ \|w(t, x, y)\|_t^2 = \int_{x=0} \, w^T A w \, dy - \int_{x=L_1} \, w^T A w \, dy + \int_{y=0} \, w^T B w \, dx - \int_{y=L_2} \, w^T B w \, dx. \] (10)

To obtain a bound on the solution \( w \), the right-hand side of (10) must be bounded with appropriate boundary conditions. However, different flow conditions require different boundary conditions.

4.0.1. Far-field (open) boundary conditions
Far-field (open) boundary conditions were derived in [1] and for the reader’s convenience, we repeat their results. To reduce notation, we focus on the boundary at \( x = 0 \),
\[ \|w(t, x, y)\|_t^2 = \int_{x=0} \, w^T A w \, dy, \] (11)

and ignore the other boundaries. (They can be treated in a similar way.)

Since \( A \) is symmetric we can diagonalize it as \( R_1^T A R_1 = \Lambda_1 \), where
\[ \Lambda_1 = \begin{pmatrix} u_0 & 0 & 0 \\ 0 & u_0 + c & 0 \\ 0 & 0 & u_0 - c \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix}. \]

Next, \( A \) is decomposed as
\[ A = R_1 \Lambda_1^+ R_1^T + R_1 \Lambda_1^- R_1^T = A^+ + A^-, \] (12)
where, $\Lambda^+_{1,-\infty}$ are diagonal matrices with the positive and negative eigenvalues, respectively. The continuous energy (11) then becomes

$$\|w(t, x, y)\|^2_t = \int_{x=0} w^T A^+ w dy + \int_{x=0} w^T A^- w dy. \quad (13)$$

Boundary conditions must ensure that the growth of $\|w\|^2_t$ is bounded. At $x = 0$, $A^-$ do not contribute to the growth of $\|w\|^2_t$. To bound $A^+$, we need as many boundary conditions as positive eigenvalues. The sign of the eigenvalues change with the sign of the normal velocity $u_n$ and the Froude number, which is given as

$$\left|u_n\right| \leq \frac{c}{\sqrt{\text{Froude number}}} = \begin{cases} < 1, & \text{Subcritical inflow/outflow} \\ > 1, & \text{Supercritical inflow/outflow} \end{cases} \quad (14)$$

The four possibilities and the necessary (and sufficient) number of boundary conditions are shown in Table 1.

Table 1: Different flow conditions based on (14) and their sign of eigenvalues. The last column list the number of boundary conditions in 2-D.

| Condition                  | $u_0$ | $u_0 + c$ | $u_0 - c$ | 2-D |
|----------------------------|-------|-----------|-----------|-----|
| Subcritical (outflow)      | -     | +         | -         | 1   |
| Subcritical (inflow)       | +     | +         | -         | 2   |
| Supercritical (outflow)    | -     | -         | -         | 0   |
| Supercritical (inflow)     | +     | +         | +         | 3   |

Following [1], we use the characteristic far-field conditions,

$$A^+ w = A^+ g, \quad (15)$$

and recast (13) as,

$$\|w(t, x, y)\|^2_t \leq \int_{x=0} w^T A^+ w dy - 2 \int_{x=0} w^T (w - g) dy = 0, \quad (16)$$

or,

$$\|w(t, x, y)\|^2_t \leq - \int_{x=0} (w - g)^T A^+ (w - g) dy + \int_{x=0} g^T A^+ g dy. \quad (17)$$

The results are then summarized in the following proposition:
Proposition 1. Let $R_T^T AR_1 = \Lambda_1$ and $R_T^T BR_2 = \Lambda_2$ where $\Lambda_i = \text{diag}(\omega_i, \omega_i + c, \omega_i - c), i = 1, 2,$ with $\omega_1 = u_0$ and $\omega_2 = v_0$, respectively. Then the far-field boundary conditions

\begin{align*}
A^+ w &= A^+ g_1, & x &= 0, \\
B^+ w &= B^+ g_3, & y &= 0,
\end{align*}

lead to (strong) well-posedness of (7), where $g_l, l = 1, ..., 4$ are bounded boundary data.

4.0.2. No-penetration wall boundary conditions

In the case of a wall at $x = 0$, the physically relevant boundary condition is the no-penetration boundary condition $u_0 = 0$. (That is, the normal velocity is zero.) We insert $u_0 = 0$ in $A$, and obtain

$$A = \begin{pmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ (18)

The no-penetration condition also implies that perturbation $\tilde{u} = 0$. Using (18) and $\tilde{u} = 0$ result in $w^T A w = 0$, and (11) becomes

$$\|w(t, x, y)\|_l^2 = 0.$$ (19)

We have proven the following proposition:

Proposition 2. The symmetric constant-coefficient SWE (7) with wall boundary conditions $\tilde{u} = u_0 = 0$ at $x = \{0, L_1\}$ and $\tilde{v} = v_0 = 0$ at $y = \{0, L_2\}$ is well-posed.

5. Semi-Discrete Problem

We start by introducing a 1-D computational grid, $x_i = i h, i \in 0, 1, 2, ..., N$, where $h$ is the grid spacing, and a scalar grid function $u(t) = (u_0(t), \cdots, u_N(t))^T$. In addition, we introduce the vectors $e_0 = (1, 0, 0, \cdots, 0)^T, e_1 = (0, 0, 0, \cdots, 1)^T$, and the matrices $E_0 = e_0 e_0^T$ and $E_1 = e_1 e_1^T$. Also, $D = P^{-1} Q$ is the spatial first-derivative finite-difference summation-by-parts (SBP) operator, where $P$ is a positive definite symmetric matrix and $Q + Q^T = E_1 - E_0 = \text{diag}(-1, 0, \cdots, 0, 1)$. Herein, the matrix $P$ is diagonal and with entries scaled.
by the stepsize \( h \). \( P \) defines a discrete \( L_2 \)-equivalent norm, \( \| u \|^2 = u^T P u \) (See [24, 25] for a review of SBP schemes.)

Furthermore, we say that \( D \) is a first-derivative SBP operator that is \((s,p)\)-accurate, if its truncation error is
\[
T = (O(h^s), \ldots, O(h^s), O(h^p), \ldots, O(h^p), \ldots, O(h^s))^T,
\]
where \( s < p \).

(See [26, 27].) For diagonal \( P \), \( s = p/2 \) at most. For first-order energy-stable hyperbolic problems (like (7)), this leads to a convergence rate of \( s + 1 \). (See [28].)

With the help of Kronecker products, denoted as \( \otimes \), we can extend the (scalar and 1-D) finite-difference operator to systems of equations and higher dimensional problems. (See [29, 30] for definition and properties.) To define the schemes for (5), we follow the approach in [31, 32] for Navier-Stokes equations. We introduce a 2-D grid with \((M + 1)\) points along the x-axis and \((N + 1)\) points along the y-axis and denote the numerical solution as
\[
V(t) = [V_{001}, V_{002}, V_{003}, V_{101}, V_{102}, V_{103}, \ldots, V_{MN1}, V_{MN2}, V_{MN3}]^T,
\]
where \( V_{ijk} = V(t)_{ijk} \) \((i \in (0, \ldots, M), j \in (0, \ldots, N) \) and \( k \in (1, 2, 3)\)) denotes the numerical approximation of \( W_k(t,x_i,y_j) \), that is \( k \)th component of \( W \) at \( x_i, y_j \).

We can now define SBP operators for a system with three variables in 2-D as
\[
D_x = (I_y \otimes D_x \otimes I_3), \quad D_y = (D_y \otimes I_x \otimes I_3),
\]
where \( D_{x,y} = P_{x,y}^{-1} Q_{x,y} \), respectively, denote the operators in the \( \{x,y\}\)-directions. In all the Kronecker products, matrices in first place have size \((N + 1) \times (N + 1)\), the second place are of size \((M + 1) \times (M + 1)\) and the third place is a \( 3 \times 3 \) identity matrix. In the same way, we define
\[
P_x = (I_y \otimes P_x \otimes I_3), \quad P_y = (P_y \otimes I_x \otimes I_3), \quad P = P_x P_y = (P_y \otimes P_x \otimes I_3),
\]
and
\[
E_{0x} = (I_y \otimes E_0 \otimes I_3), \quad E_{1x} = (I_y \otimes E_1 \otimes I_3), \quad E_{0y} = (E_0 \otimes I_x \otimes I_3), \quad E_{1y} = (E_1 \otimes I_x \otimes I_3).
\]

Lastly, we define the norm
\[
\| V \|^2 = V^T P V.
\]
Analogously to \([20]\), we define the discrete flux vectors, \(\bar{\mathbf{F}}, \bar{\mathbf{G}}\) with components \(\bar{F}_{kij}, \bar{G}_{kij}\). Furthermore, \(\bar{g}\) is the discrete boundary data which has the same structure as the discrete fluxes. The form of \(\bar{g}\) with (far-field/wall) boundary data vector projected at \(x = 0\) have entries given by

\[
\bar{g}_{ij}(t) = \begin{cases} 
g(t, x_i, y_j) & \text{for } i = 0, \forall j, \\
0 & \text{otherwise.} \end{cases}
\]

(23)

Here, \(g(t, x_i, y_j)\) is the boundary vector function (data) evaluated at \((i, j)\)th discrete point. Note, we will use \(f\) or \(w\) as superscript on \(\bar{g}\) to represent far-field or wall boundary data when the need be.

5.1. Energy stability of the 2D Shallow Water Equations

A semi-discrete approximation of (5) is

\[
\mathcal{V}_t + D_x \bar{\mathbf{F}} + D_y \bar{\mathbf{G}} = \mathcal{S}(\mathcal{V}, \bar{g}),
\]

(24)

where \(\mathcal{S}(\mathcal{V}, \bar{g})\) denotes Simultaneous Approximation Term (SAT) that enforces the boundary conditions.

In analogy to continuous case, we linearize, symmetrize and freeze the coefficients to arrive at the semi-discrete counterpart of (7)

\[
v_t + D_x (A v) + D_y (B v) = \mathcal{S}(v, \bar{g}).
\]

(25)

In (25), \(v\) is the numerical solution vector of \(w\), \(A = (I_y \otimes I_x \otimes A)\) and \(B = (I_y \otimes I_x \otimes B)\). We now apply the energy analysis to (25), and use the SBP properties to arrive at

\[
\|v\|^2_t - v^T P_y E_{0x}(Av) + v^T P_y E_{1x}(Av) - v^T P_x E_{0y}(Bv) + v^T P_x E_{1y}(Bv) = 2v^T PS(v, \bar{g}).
\]

(26)

The above expression corresponds to (10). To reduce notation, we focus on the boundary \(x = x_0\) and ignore all other boundary terms by assuming that they are stable. Hence, we consider

\[
\|v\|^2_t - v^T P_y E_{0x}(Av) = 2v^T PS_{x_0}(v, \bar{g}),
\]

(27)

where \(S_{x_0}\) is the \(x_0\)-part of \(S\).
5.1.1. Discrete far-field boundary conditions

To enforce (15), we propose the following Sx0 in (27):

\[ S_{x_0}(v, \bar{g}^f) = -P_x^{-1}E_{0x}A^+(v - \bar{g}^f). \quad (28) \]

We introduce \( A = A^+ + A^- \) where \( A^+/− = (I_y \otimes I_x \otimes A^+/−) \) and \( A^+/− \) are given in (12), and substitute (28) into (27) to obtain

\[ \|v\|_t^2 - v^T P_y E_{0x} (A^+ - A^-) v = -2v^T E_{0x} P_y A^+ (v - \bar{g}^f), \quad (29) \]

In analogy with (17), we recast (29) to obtain the estimate,

\[ \|v\|_t^2 \leq -(v - \bar{g}^f)^T P_y E_{0x} A^+ (v - \bar{g}^f) + \bar{g}^f P_y E_{0x} P_y A^+ \bar{g}^f. \]

The other boundaries can be handled similarly, and we summarize the results in the following proposition:

**Proposition 3.** The following penalty (SAT) terms lead to stability of the scheme (25):

\[ -P_x^{-1}E_{0x} A^+ (v - \bar{g}^f_1), \quad x = x_0, \quad P_x^{-1}E_{1x} A^- (v - \bar{g}^f_2), \quad x = x_M, \]
\[ -P_y^{-1}E_{0y} B^+ (v - \bar{g}^f_3), \quad y = y_0, \quad P_y^{-1}E_{1y} B^- (v - \bar{g}^f_4), \quad y = y_N, \]

where \( \bar{g}^f_l, l = 1, \ldots, 4 \) is the boundary data defined as in (23).

5.1.2. Discrete no-penetration wall boundary condition

To enforce the no-penetration condition \( u_0 = \bar{u} = 0 \) discretely at \( x = x_0 \), we use (28) and replace \( \bar{g}^f \) with \( \bar{g}^w = (\bar{g}h/c, -\bar{u}, \bar{\nu}) \), which is obtained by substituting \( \bar{u} \to -\bar{\nu} \) into the solution \( w = (\bar{g}h/c, \bar{u}, \bar{\nu})^T \) at \( x_0 \). (Note that, \( \bar{g}^w - w = 0 \), is equivalent to \( \bar{u} = 0 \).)

Furthermore, the no-penetration boundary condition implies that \( u_0 = 0 \) in \( A \).

Now, for stability of the semi-discrete problem, we require that

\[ v^T P_y E_{0x} (Av - 2A^+ (v - \bar{g}^w)) \leq 0. \quad (30) \]

In (30), \( A = (I_y \otimes I_x \otimes A) \) where \( A \) is the symmetric matrix defined in (7). Let \( v_{0j} = (v_{01j}, v_{20j}, v_{30j})^T \) be the solution vector at the \((x_0, y_j)^{th}\) point at the boundary. Hence, we have the solution vector \( v_{0j} \) as \((\bar{g}h_{0j}, \bar{u}_{0j}, \bar{v}_{0j})^T \). We rewrite (30) in component form

\[ \sum_j v_{0j}^T P_{yj} (Av_{0j} - 2A^+ (v_{0j} - \bar{g}^w_{0j})) \leq 0, \quad (31) \]
where $P_{ij} > 0$ is the $j$th diagonal element of $P$. Using $u_0 = 0$ in $A$, we have

$$A = \begin{pmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad A^+ = \begin{pmatrix} c/2 & c/2 & 0 \\ c/2 & c/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ (dropping the indices), we obtain

$$v^T A v - 2v^T A^+(v - \bar{g}) = 2gh\bar{u} - 2gh\bar{v} - 2c^2 \leq 0.$$ 

We summarize the results for all other boundaries in the following proposition.

**Proposition 4.** The following penalty (SAT) terms lead to stability of the scheme (25):

$$-P^{-1}_x E_0 A^+(v - \bar{g}_w), \quad x = x_0,$$

$$-P^{-1}_y E_0 B^+(v - \bar{g}_w), \quad y = y_0,$$

$$-P^{-1}_y E_1 B^-(v - \bar{g}_w), \quad y = y_N,$$

where $\bar{g}_w, l = 1, ..., 4$ is boundary data projected at a boundary.

6. Curvilinear Shallow Water Equations

We begin by briefly summarizing the mapping of the SWE from a Cartesian to a curvilinear coordinate frame. Let

$$x = x(\xi, \eta), \quad \text{and} \quad y = y(\xi, \eta),$$

be the mapping from $\Omega_\xi(\xi, \eta) : [0, 1] \times [0, 1]$ to $\Omega$. We will need the following metric relations:

$$J_{\xi x} = y_\eta, \quad J_{\eta x} = -y_\xi,$$

$$J_{\xi y} = -x_\eta, \quad J_{\eta y} = x_\xi,$$

and $J = x_\xi y_\eta - x_\eta y_\xi$ which denotes the determinant of the metric Jacobian.

The curvilinear form of the SWE are

$$(JW)_t + F_\xi + G_\eta = 0,$$

where $F = (J_{\xi x} F + J_{\xi y} G)$ and $G = (J_{\eta x} F + J_{\eta y} G)$ expressed respectively as:

$$J \begin{pmatrix} \xi_x hu + \xi_y hv \\ \xi_x hu^2 + \xi_x gh^2/2 + \xi_y huv \\ \xi_x huv + \xi_y hv^2 + \xi_y gh^2/2 \end{pmatrix} \quad \text{and} \quad J \begin{pmatrix} \eta_x hu + \eta_y hv \\ \eta_x hu^2 + \eta_x gh^2/2 + \eta_y huv \\ \eta_x huv + \eta_y hv^2 + \eta_y gh^2/2 \end{pmatrix}.$$
We term (35) as Curvilinear Shallow Water Equations (CSWE). (See e.g. [33, 34] for derivations of some transformed equations.) The linear and symmetric form of (35) then becomes,
\[
(\mathcal{J}w)_t + (\hat{A}w)_\xi + (\hat{B}w)_\eta = 0,
\]
where the new jacobian matrices are \(\hat{A} = A\mathcal{J}\xi_x + B\mathcal{J}\xi_y\) and \(\hat{B} = A\mathcal{J}\eta_x + B\mathcal{J}\eta_y\).

6.1. Well-posedness

Since the computational domain \(\Omega_{\xi}\) is a square, we can substitute the matrices \(A\) and \(B\) in Section 4 with \(\hat{A}\) and \(\hat{B}\) for the well-posedness analysis. The derivatives with regard to \(x\) and \(y\) turn to be \(\xi\) and \(\eta\), respectively, in the curvilinear frame. We obtain
\[
\|\sqrt{\mathcal{J}}w\|^2_t = \int_{\xi=0}^{\xi=1} w^T \hat{A}wd\eta + \int_{\eta=0}^{\eta=1} w^T \hat{B}wd\xi - \int_{\eta=0}^{\eta=1} w^T \hat{B}wd\xi. \tag{37}
\]
As before, we focus on one boundary term, \(\xi = 0\). This reduces (37) to
\[
\|\sqrt{\mathcal{J}}w\|^2_t = \int_{\xi=0}^{\xi=1} w^T \hat{A}wd\eta, \quad \text{where} \quad \hat{A} = A\mathcal{J}\xi_x + B\mathcal{J}\xi_y. \tag{38}
\]
Let \(\mathbf{n} = (n_1, n_2) = \mathcal{J}(\xi_x, \xi_y)\) (at \(\xi = 0\)), which in fact is a normal to the boundary (not necessarily of unit length but that is of no importance here).

Then \(\hat{A} = An_1 + Bn_2\), or
\[
\hat{A} = \begin{pmatrix}
  u_0n_1 + v_0n_2 & cn_1 & cn_2 \\
  cn_1 & u_0n_1 + v_0n_2 & 0 \\
  cn_2 & 0 & u_0n_1 + v_0n_2
\end{pmatrix}. \tag{39}
\]
Keeping in mind that, all other non-unit normal at the other boundaries are given as
\[
(n_1, n_2) = \mathcal{J}(\xi_x, \xi_y) \quad \text{(at \(\xi = 1\)) and} \quad (n_1, n_2) = \mathcal{J}(\eta_x, \eta_y) \quad \text{(at \(\eta = 0, \eta = 1\)).} \tag{40}
\]

6.1.1. Boundary Conditions

From the above derivation it is clear that the curvilinear far-field conditions are completely analogous to the Cartesian in Prop. 1. Simply exchange \(A, B\) with \(\hat{A}, \hat{B}\).

We summarize the far-field boundary conditions for (36) in the following proposition:
Proposition 5. Let \( R_1^T \hat{A} R_1 = \Lambda_1 \) and \( R_2^T \hat{B} R_2 = \Lambda_2 \) where \( \Lambda_{1,2} = \text{diag}( (u_0, v_0) \cdot n + c, (u_0, v_0) \cdot n - c ) \), then the far-field boundary conditions
\[
\hat{A}^+ w = \hat{A}^+ g^I_1, \quad \xi = 0, \quad \hat{A}^- w = \hat{A}^- g^I_2, \quad \xi = 1, \\
\hat{B}^+ w = \hat{B}^+ g^I_3, \quad \eta = 0, \quad \hat{B}^- w = \hat{B}^- g^I_4, \quad \eta = 1,
\]
lead to well-posedness of (36), where \( g^I_i, i = 1, ..., 4 \), are known bounded functions.

Similarly, by inserting the no-penetration boundary condition, \( u_0 n_1 + v_0 n_2 = 0 \) at the boundary \( \xi = 0 \), we obtain \( \hat{A} = \begin{pmatrix} 0 & c n_1 & c n_2 \\ c n_1 & 0 & 0 \\ c n_2 & 0 & 0 \end{pmatrix} \). From
\[
w^T \hat{A} w = 2 g (n_1 \tilde{u} + n_2 \tilde{v}) \tilde{h} \text{ where } w = (\tilde{g} \tilde{h}/c, \tilde{u}, \tilde{v})^T \text{ and setting } n_1 \tilde{u} + n_2 \tilde{v} = 0, \text{ we obtain } \| \sqrt{\mathcal{J}} w \|_t^2 = 0. \text{ The results are summarize in a proposition as follows:}

Proposition 6. The curvilinear Shallow Water Equations (36) is well-posed at \( \xi = \{0, 1\} \) and \( \eta = \{0, 1\} \), respectively, with wall boundary conditions \( (u_0 n_1 + v_0 n_2) = 0 \), where the normals are given in (40).

6.2. Stability of the Curvilinear Shallow Water Equations

All vectors and matrices are defined as in the Cartesian case but the indices \( i, j \) now refer to \( \xi_i, \eta_j \). A semi-discrete approximation of (35) is
\[
(\mathcal{J} \nu)_t + D_\xi \bar{F} + D_\eta \bar{G} = S(\nu, \bar{g}).
\]
(41)

After linearizing and symmetrizing (41), leads to
\[
(\sqrt{\mathcal{J}} \nu)_t + D_\xi (\hat{A} \nu) + D_\eta (\hat{B} \nu) = S(\nu, \bar{g}).
\]
(42)

We perform the energy analysis and apply the SBP properties to obtain
\[
\| \sqrt{\mathcal{J}} \nu \|_t^2 - v^T P_\eta E_{0\xi}(\hat{A} \nu) + v^T P_\eta E_{1\xi}(\hat{A} \nu) - v^T P_\xi E_{0\eta}(\hat{B} \nu) + v^T P_\xi E_{1\eta}(\hat{B} \nu) = 2 v^T P S(\nu, \bar{g}),
\]
(43)
just as section (5.1). Here, Equation (43) corresponds to (37).
6.2.1. Discrete far-field boundary conditions

With focus on $\xi = \xi_0$, we propose the penalty term
\[ S_{\xi_0}(v, \bar{g}^f) = -P_{\xi}^{-1}E_{0\xi} \hat{A}^+(v - \bar{g}^f). \] (44)
Substituting (44) into (43), we arrive at
\[ \|\sqrt{\mathcal{J}} v\|^2_t - v^T P_{\eta} E_{0\eta} (\hat{A} v) = -2v^T P_{\eta} E_{0\eta} \hat{A}^+(v - \bar{g}^f). \]
\[ \text{As before, we decompose the matrix } \hat{A} \text{ into positive and negative parts, and drop the terms that do not contribute to the energy growth rate,} \]
\[ \|\sqrt{\mathcal{J}} v\|^2_t \leq \left( v^T P_{\eta} E_{0\eta} (\hat{A}^+) v - 2v^T P_{\eta} E_{0\eta} \hat{A}^+(v - \bar{g}^f) \right), \] (45)
and recast it as,
\[ \|\sqrt{\mathcal{J}} v\|^2_t \leq \left( -v^T P_{\eta} E_{0\eta} (\hat{A}^+) v + \bar{g}^f P_{\eta} E_{0\eta} \hat{A}^+ \bar{g}^f \right). \] (46)
The RHS of (46) is bounded with the last term representing the energy sent across the boundary $\xi = \xi_0$ into the domain. The results are summarized in the proposition below:

**Proposition 7.** Given the discrete version of the preamble in proposition 5, the following penalty (SAT) terms lead to stability of the scheme (42):
\[ \begin{align*}
-\mu_{\xi}^{-1} E_{0\xi} \hat{A}^+(v - \bar{g}^f), & \quad \xi = \xi_0, \\
-\mu_{\eta}^{-1} E_{0\eta} \hat{B}^+(v - \bar{g}^f), & \quad \eta = \eta_0,
\end{align*} \]
\[ \begin{align*}
-\mu_{\xi}^{-1} E_{0\xi} \hat{A}^- (v - \bar{g}^f), & \quad \xi = \xi_M, \\
-\mu_{\eta}^{-1} E_{0\eta} \hat{B}^- (v - \bar{g}^f), & \quad \eta = \eta_N,
\end{align*} \]
where $\bar{g}^f_l, l = 1, \ldots, 4$ is boundary data projected at a boundary and depending on its position takes entries defined similar to (23).

6.2.2. Discrete wall boundary conditions

Similar to previous section, we propose a penalty term (SAT)
\[ S_{\xi_0}(v, \bar{g}^w) = -2\mu_{\xi}^{-1} E_{0\xi} \hat{A}^+(v - \bar{g}^w), \] (47)
at the wall boundary. Substituting (47) into (43) with focus on $\xi = \xi_0$ yields
\[ \|\sqrt{\mathcal{J}} v\|^2_t - v^T P_{\eta} E_{0\eta} (\hat{A} v - 4\hat{A}^+(v - \bar{g}^w)) = 0.\]
Next, we focus on the construction of $\bar{g}^w$ at a boundary point $(\xi_0, \eta_j)$. No flow through the boundary implies that both $(u \cdot n)n = 0$ and $(\tilde{u} \cdot n)n = 0$. The first condition goes into the matrix $\hat{A}$ and the second is used to construct $\bar{g}$ at $(\xi_0, \eta_j)^{th}$ boundary point. Now in complete analogy with the Cartesian case, the continuous boundary data function is $g^w = (\bar{g}h/c, \tilde{u} - (\tilde{u} \cdot n)n_1, \tilde{v} - (\tilde{u} \cdot n)n_2)^T$.

For stability of the semi-discrete problem, we require to show that

$$v^T P_\eta E_\eta (\hat{A}v - 4\hat{A}^+(v - \bar{g}^w)) \leq 0.$$  

(48)

Just as the cartesian case, we rewrite (48) in component form

$$\sum_j v^T_0 P_{\eta j} (\hat{A}v_0 - 4\hat{A}^+(v_0 - \bar{g}^w_0)) \leq 0.$$  

(49)

We insert the wall boundary condition $u_0n_1 + v_0n_2 = 0$ into $\hat{A}$ and $\hat{A}^+$, and consider one term of (49)

$$v^T \hat{A}v - 4v^T \hat{A}^+(v - \bar{g}^w) = 2gn_1\tilde{h}\tilde{u} + 2gn_2\tilde{h}\tilde{v} - 2g_n_1\tilde{h}\tilde{u} - 2g_n_2\tilde{h}\tilde{v} - 2n_1^2c\tilde{u}^2 - 4n_1n_2\tilde{c}\tilde{u}\tilde{v} - 2n_2^2c\tilde{v}^2$$  

(50)

$$= -2c(n_1\tilde{u} + n_2\tilde{v})^2 \leq 0.$$ 

Hence, the stability proof of bounded solution when wall boundary condition is imposed.

We summarize the results for all other boundaries in the following proposition.

**Proposition 8.** Let $\bar{g}^w_l, l = 1, \ldots, 4$ be the boundary data projected at the boundary. Then the following penalty (SAT) terms,

$$-2P_{\xi}^{-1} E_\xi \hat{A}^+(v - \bar{g}^w_1), \quad \xi = \xi_0, \quad 2P_{\xi}^{-1} E_\xi \hat{A}^-(v - \bar{g}^w_2), \quad \xi = \xi_M,$$

$$-2P_{\eta}^{-1} E_\eta \hat{B}^+(v - \bar{g}^w_3), \quad \eta = \eta_0, \quad 2P_{\eta}^{-1} E_\eta \hat{B}^-(v - \bar{g}^w_4), \quad \eta = \eta_N,$$

lead to stability of the scheme (42).

7. Numerical experiments

We have implemented the numerical scheme (41) with (44) and (47). We use SBP operators of order $2s$ in the interior and order $s$ close to the spatial...
boundaries (denoted SBP(s, 2s)), where $s \in \{1, 2, 3\}$. We march in time with standard 4th-order Runge Kutta scheme and use a CFL number of 0.50 in all the simulations.

Whenever an exact solution is available, one can compute the errors using the $L^2$ norm defined in (22)

$$
\|e\|_P = \sqrt{e^T Pe},
$$

where $e$ is the vector of the pointwise difference between the exact and the approximate solution.

The rate of convergence $r$ is given by

$$
r = \frac{\log(\|e_1\|_{P_1}/\|e_2\|_{P_2})}{\log((M + 1)/(N + 1))},
$$

where the subscripts $\{1, 2\}$ represent two different mesh sizes with $M + 1$ and $N + 1$ grid points respectively. $e_1$ and $e_2$ are the errors with grid spacing $h_1$ and $h_2$, respectively. (See [35, 27, 28, 36] for more details on order of accuracy of initial boundary value problem.)

### 7.1. Convergence rate for 1-D Swallow Water Equations with far-field boundaries

Here, we use the scheme given in (24) (reduced to 1-D) on the domain $[0, 1]$ with far-field boundary conditions (see Proposition 3) on both ends of the domain. The function

$$
\mathbf{W} = (h, hu) = [2 + \sin(5x - 10t), 2 + \sin(5x - 10t)]^T,
$$

solves (5) if the additional source term

$$
T(\mathbf{W}) = [-\cos(5x-10t), (10g-5)\cos(5x-10t)+5g\sin(5x-10t)\cos(5x-10t)]^T,
$$

is added to the right-hand side. (This does not affect the stability proof. It is known as the "method of manufactured solutions". See [20, 21].) From (51), we can deduce initial condition ($t = 0$) and time-dependent boundary data $g$ (at $x = 0$ and $x = 1$).

We compute the solution on grids with $N = 50, 100, 200, 400$ spatial nodes. The errors are computed at $t = 1.0$. The rates of convergence for height and momentum are shown in Tables 2 and 3 respectively. We use the
Table 2: The rates of convergence and errors at $t = 1.0$ for height under a mesh refinement sequence.

| Grid points | SBP(1,2) $L^2$ error | SBP(2,4) $L^2$ error | SBP(3,6) $L^2$ error |
|-------------|------------------------|------------------------|------------------------|
| 50          | -                      | 0.0049                 | -                      | 4.2269×10$^{-4}$ | 1.6532×10$^{-4}$ |
| 100         | 2.0299                 | 0.0012                 | 3.0458                 | 5.1185×10$^{-5}$ | 4.0167 | 1.0214×10$^{-5}$ |
| 200         | 2.0111                 | 2.9621×10$^{-4}$      | 3.0203                 | 6.3087×10$^{-6}$ | 3.9607 | 6.5596×10$^{-7}$ |
| 400         | 2.0033                 | 7.3881×10$^{-5}$      | 3.0060                 | 7.8533×10$^{-7}$ | 3.9319 | 4.2978×10$^{-8}$ |

Table 3: The rates of convergence and errors at $t = 1.0$ for momentum under a mesh refinement sequence.

| Grid points | SBP(1,2) $L^2$ error | SBP(2,4) $L^2$ error | SBP(3,6) $L^2$ error |
|-------------|------------------------|------------------------|------------------------|
| 50          | -                      | 0.0037                 | -                      | 3.1666×10$^{-4}$ | 1.9787×10$^{-4}$ |
| 100         | 2.0258                 | 9.0348×10$^{-4}$      | 3.0370                 | 3.8581×10$^{-5}$ | 4.0036 | 1.2336×10$^{-5}$ |
| 200         | 2.0106                 | 2.2422×10$^{-4}$      | 3.0117                 | 4.7837×10$^{-6}$ | 3.9816 | 7.8091×10$^{-7}$ |
| 400         | 2.0036                 | 5.5915×10$^{-5}$      | 2.9984                 | 5.9865×10$^{-7}$ | 3.9624 | 5.0097×10$^{-8}$ |

SBP schemes of order (1, 2), (2, 4) and (3, 6) with design order 2, 3, and 4, respectively. As expected, we obtain design order for the numerical simulations of 1-D SWE in all three cases in agreement with our analysis.

We also consider simulations of the 1-D model with manufactured solution (51) for a longer time. The errors until $t = 40$ with 100 grid points are presented in Fig. 1. In Fig. 1 we observe that there is a bounded error growth with respect to time in agreement with [37].

Fig. 1. Long time simulation of 1-D SWE. The $L^2$ error for the height and momentum solution components versus time using 100 grid points.

(a) SBP(1,2) Scheme  
(b) SBP(2,4) Scheme  
(c) SBP(3,6) Scheme
7.1.1. Convergence rates for Curvilinear Shallow Water Equations with far-field boundaries

Next, we turn to the Curvilinear Shallow Water Equations where we examine the scheme given in (41) with far-field boundary conditions at all boundaries (see Proposition 7). We consider a circular bend domain (see Fig. 2). In polar coordinate, the circular bend is given as \( \Omega(r, \theta) = [0.5, 1] \times [0.2, 2] \). We use the manufactured solution

\[
W = [2 + \sin(5x + 5y - 10t), 2 + \sin(5x + 5y - 10t), 2 + \sin(5x + 5y - 10t)]^T
\]

and the source

\[
T = J[0, 10g \sin(5x + 5y - 10t) + 5g \sin(5x + 5y - 10t) \cos(5x + 5y - 10t), 10g \sin(5x + 5y - 10t) + 5g \sin(5x + 5y - 10t) \cos(5x + 5y - 10t)]^T.
\]

We compute the solution on equidistant \( N \times N \) grids in \((\xi, \eta)\), where \( N = 25, 50, 75, 100 \). Errors and rates of convergence for the conserved variables at time \( t = 1.0 \) are computed for the circular domain and displayed in Table 4-6. We show in Fig. 3, the solution at \( t = 1.0 \) on both domains obtained with SBP(3,6). The numerical experiments on these grids show the robustness of the CSWE scheme for any domain.

Fig. 2. A schematic of a physical curvilinear domain in a Cartesian coordinate system.
Table 4: The rates of convergence and errors at $t = 1.0$ for height $h$ under a mesh refinement sequence.

| Grid points | SBP(1,2) $L^2$ error | SBP(2,4) $L^2$ error | SBP(3,6) $L^2$ error |
|-------------|-----------------------|-----------------------|-----------------------|
| 25          | 0.0084                | 0.0015                | 0.0021                |
| 50          | 1.9736                | 2.9898                | 4.0045                |
| 75          | 1.9361                | 2.9709                | 3.9988                |
| 100         | 1.9872                | 2.9559                | 4.0670                |

Table 5: The rates of convergence and errors at $t = 1.0$ for momentum $hu$ under a mesh refinement sequence.

| Grid points | SBP(1,2) $L^2$ error | SBP(2,4) $L^2$ error | SBP(3,6) $L^2$ error |
|-------------|-----------------------|-----------------------|-----------------------|
| 25          | -                     | -                     | -                     |
| 50          | 2.0945                | 3.0561                | 3.9650                |
| 75          | 2.0097                | 3.0979                | 3.9713                |
| 100         | 1.9841                | 3.0685                | 4.0112                |

Table 6: The rates of convergence and errors at $t = 1.0$ for momentum $hv$ under a mesh refinement sequence.

| Grid points | SBP(1,2) $L^2$ error | SBP(2,4) $L^2$ error | SBP(3,6) $L^2$ error |
|-------------|-----------------------|-----------------------|-----------------------|
| 25          | 0.0205                | -                     | -                     |
| 50          | 1.9578                | 3.0618                | 4.0341                |
| 75          | 2.0131                | 3.0513                | 3.9907                |
| 100         | 2.0023                | 3.0881                | 4.0458                |

Fig. 3. Height solution at $t = 1.0$ in Curvilinear domains
7.1.2. Convergence rates for Curvilinear Shallow Water Equations with wall boundaries

Here, we demonstrate the imposition of wall boundary conditions (see proposition 8) on our circular bend by using the manufactured solution given by

\[ W = [2 + \sin(5x + 5y - 10t), 0, 0]^T, \quad (53) \]

and the source

\[ T = \mathcal{J} \cos(5x + 5y - 10t)[-10, 5g(2 + \sin(5x + 5y - 10t)), 5g(2 + \sin(5x + 5y - 10t))]^T. \]

Errors and rates of convergence are given in Table 7-9. Furthermore, we show in Fig. 4 the momentum field at time \( t = 1.0 \) obtained with SBP(3,6). Note how the trajectories are aligned with the wall in agreement with the no-penetration boundary condition. This plot is zoomed in at various corners of the domain for clear view of the behaviour of the solution.

Table 7: The rates of convergence and errors at \( t = 1.0 \) for height \( h \) under a mesh refinement sequence.

| Grid points | SBP(1,2) | SBP(2,4) | SBP(3,6) |
|------------|----------|----------|----------|
|            | \( r \)  | \( L^2 \) error | \( r \)  | \( L^2 \) error | \( r \)  | \( L^2 \) error |
| 25         | -        | 0.0081   | -        | 0.0041   | -        | 0.0052   |
| 50         | 1.9733   | 0.0021   | 3.0706   | 4.8802\times10^{-4} | 3.9161   | 3.4446\times10^{-4} |
| 75         | 2.0490   | 9.1498\times10^{-4} | 3.0176   | 5.9675\times10^{-5} | 4.0139   | 2.1914\times10^{-5} |
| 100        | 2.0289   | 5.1041\times10^{-4} | 3.0176   | 5.9675\times10^{-5} | 4.0139   | 2.1914\times10^{-5} |

Table 8: The rates of convergence and errors at \( t = 1.0 \) for momentum \( hu \) under a mesh refinement sequence.

| Grid points | SBP(1,2) | SBP(2,4) | SBP(3,6) |
|------------|----------|----------|----------|
|            | \( r \)  | \( L^2 \) error | \( r \)  | \( L^2 \) error | \( r \)  | \( L^2 \) error |
| 25         | -        | 0.0146   | -        | 0.0054   | -        | 0.0094   |
| 50         | 1.9890   | 0.0037   | 2.9810   | 6.8395\times10^{-4} | 3.9741   | 5.9814\times10^{-4} |
| 75         | 1.9486   | 0.0017   | 3.0148   | 2.0144\times10^{-4} | 4.0173   | 1.1733\times10^{-4} |
| 100        | 1.9667   | 9.6545\times10^{-4} | 2.9086   | 8.7247\times10^{-5} | 4.0238   | 3.6817\times10^{-5} |

7.2. Applications

Testing the schemes on a more realistic problems, we consider the propagation of a flood wave in a stream, river or any open channel. Being able to predict such flows is of great practical utility, as it can be used in early flood warning systems. Here, we consider three examples of such flows: a 1-D and a 2-D river flow in a channel and a dam-break simulation.
Table 9: The rates of convergence and errors at \( t = 1.0 \) for momentum \( hv \) under a mesh refinement sequence.

| Grid points | SBP(1,2) | SBP(2,4) | SBP(3,6) |
|-------------|---------|---------|---------|
|             | \( r \) | \( L^2 \) error | \( r \) | \( L^2 \) error | \( r \) | \( L^2 \) error |
| 25          | -       | 0.0097  | -       | 0.0068  | -       | 0.0096  |
| 50          | 1.9191  | 0.0026  | 2.9690  | 8.6846\( \times 10^{-4} \) | 3.9498  | 6.2125\( \times 10^{-4} \) |
| 75          | 2.0197  | 0.0011  | 2.9118  | 2.6669\( \times 10^{-4} \) | 3.9657  | 1.2442\( \times 10^{-4} \) |
| 100         | 2.0303  | 9.4795\( \times 10^{-4} \) | 3.0301  | 1.1026\( \times 10^{-4} \) | 4.0246  | 3.9090\( \times 10^{-5} \) |

(a) zoom in of the momentum field at the north-east boundary
(b) zoom in of the momentum field at the north-west boundary
(c) zoom in of the momentum field at the west boundary

Fig. 4. Momentum field at time \( t = 1.0 \)

7.2.1. Steady River channel

Next, we consider the domain \([0, 1]\) as a 1-D model of a river. We initiate the simulation by a hump of water. The hump is given as a Gaussian curve centered at \( x = 0.5 \) and overlaid on a river at rest (initial velocity \( u_0 = 0 \)) with height \( h_0 = 1 \). Hence, the initial height \( h \) is given by

\[
h(x, 0) = h_0 + 0.5e^{(-250(x-0.5)^2)}, \quad 0 \leq x \leq 1.
\] (54)

The initial velocity \( u \) is set to \( u_0 \). The gravitational constant is taken as \( g = 1 \). Fig. 5 shows solution to the height and momentum observed at different time instances. In Fig. 6, \((x, t)\)-space plots are shown. We observe two flood waves (or wavefronts) traveling in opposite directions with different wave speeds (\( \pm \sqrt{gh_0} \)). The characteristic far-field boundary conditions are implemented and the flood waves leave the domain freely at the boundary after time 0.60s without causing any visible reflections at the boundaries.
This demonstrates the efficacy of the characteristic boundary conditions and the stability of the SAT implementation.

Fig. 5. Evolution of height and momentum concentrated at the center. The small disturbance generates two waves propagating in both directions. The left column shows height($h$) and right shows momentum ($hu$). Each row represents the height and momentum at different time $t = 0, 0.20, 0.40, 0.60$. 
Fig. 6. $(x,t)$-space plot of Fig. 5 respectively.

(a) water depth (or height)  
(b) momentum
7.2.2. Flood control channel simulation

Next, we consider flood waves in a channel. Flood control channels are mostly situated in cities for conveying water and/or serving as a way to reduce flooding. They are usually built with erected wall structures along the sides (see Fig. 7). We model a portion of a Cartesian and circular channel by imposing wall boundary conditions along the wall structures and far-field boundary conditions at the open ends. Our aim is to show the robustness of the wall and far-field boundary conditions. To this end, we consider a flow with initial height given by

\[ h(x, y, 0) = 1 + 0.5e^{-25((x-0.5)^2+(y-0.5)^2)}, \]  

(55)

and initial velocity \((u, v) = (0, 0)\). In Figs. 8, 9 and 10 we show the height and momentum in Cartesian and circular domains at an instance in time \(t = 1\). The flood waves have propagated to affect the flow in a large part of both domains. We also observe in the momentum field plots that, as flood waves move in a direction towards the wall structures, they deviate to move along the wall towards the open boundaries. The wall boundary conditions are accurately and stably enforced by the SATs.
7.2.3. Dam break problem

Next, we examined another important case which involves an area bounded by solid walls and a dam that separate a reservoir and a floodplain. Here, we demonstrate a 1-D domain bounded by walls at \( x = \{0, 1\} \). The dam is broken instantly at time \( t = 0 \), which gives rise to the following discontinuous initial data

\[
\begin{align*}
    h(x, 0) &= \begin{cases} 
4 & \text{if } x < 0.5 \\
1 & \text{if } x > 0.5
\end{cases} \\
    u(x, 0) &= 0.
\end{align*}
\]

(56)

The domain \([0, 1]\) is discretized with 100 grid points. The exact solution consists of a right-going shock and a left-going rarefaction. It is shown in
Fig. 9. Plots of water depth at time $t = 1.0$

Fig. 11 that the numerical solution behaves in a similar way. However, since our scheme lacks shock capturing capabilities, we have large oscillations in the solutions of Fig. 11_a-b caused by the discontinuity in the data. In Fig. 11_c-d, we have added a little diffusion which yields a non-oscillatory solution. Details of the treatment is beyond the scope of this work, and left to a future article. (Note that in the previous smooth test cases there are no oscillations in the solution despite there being no artificial diffusion added to the schemes.) Furthermore, all outputs in figure 11 are stable with the boundary conditions.
8. Conclusions

The study carried out a continuous and semi-discrete energy estimate analysis to prove well-posedness and stability of the Shallow Water Equations. Stability proofs and analysis for the Shallow Water Equations (SWE) having both wall (no-penetration) and far-field (open) boundary conditions have been provided. Particularly, we derived a well-posed wall and far-field boundary conditions for SWE in both Cartesian and curvilinear coordinates using the energy method. With a weak boundary procedure, implementa-
tion of these boundary conditions were done to obtain continuous energy estimates. Furthermore, a high-order summation-by-parts finite-difference schemes was constructed for the SWE in both coordinate systems that includes the derived boundary conditions on a single domain. This led to stability proofs of energy stable numerical scheme for the SWE in Cartesian coordinates and the curvilinear coordinates, including the wall (no-penetration) and open (far-field) boundary conditions.

Using the method of manufactured solution, numerical computations were performed creditably well with verifiable and correct convergence rates toward the exact solution. Furthermore, simulations were conducted to show the robustness of the boundary conditions for the SWE. Finally we applied the stable schemes to the propagation of flood in river canals, and a dam break. Except for the dam-break which had an oscillating solution and therefore damped by the addition of artificial diffusion, it was observed that the simulated solution of these applications obeys the boundary conditions and confirms our analysis.
Appendix A. Linearization and symmetrization of 2-D Shallow Water Equations

We briefly present the process in obtaining a linear and symmetric Shallow Water Equations. First, we rewrite the system (5) in quasi-linear form,

\[ W_t + \bar{A} W_x + \bar{B} W_y = 0, \quad \text{(A.1)} \]

where the Jacobian matrices \( \bar{A} \) and \( \bar{B} \) are

\[
\bar{A} = \begin{pmatrix}
0 & 1 & 0 \\
-u^2 + gh & 2u & 0 \\
-uv & v & u \\
\end{pmatrix}, \quad \bar{B} = \begin{pmatrix}
0 & 0 & 1 \\
-uv & v & u \\
-v^2 + gh & 0 & 2v \\
\end{pmatrix}.
\]

Next, we shift the variables from the conservative \( W = (h, hu, hv) \) to the primitive variables \( U = (h, u, v) \) using the transformation matrix

\[
M = \frac{\partial W}{\partial U} = \begin{pmatrix}
1 & 0 & 0 \\
u & h & 0 \\
v & 0 & h \\
\end{pmatrix}.
\quad \text{(A.2)}
\]

We rewrite Equation (A.1) as

\[ W_t + M(M^{-1} \bar{A} M) M^{-1} W_x + M(M^{-1} \bar{B} M) M^{-1} W_y = 0, \]

and introduce the following

\[
\mathcal{A} = M^{-1} \bar{A} M = \begin{pmatrix}
u & h & 0 \\
g & u & 0 \\
0 & 0 & u \\
\end{pmatrix}, \quad \mathcal{B} = M^{-1} \bar{B} M = \begin{pmatrix}
v & 0 & h \\
0 & v & 0 \\
g & 0 & v \\
\end{pmatrix},
\]

\[
U_t = M^{-1} W_t = \begin{pmatrix}h \\
u \\
v \end{pmatrix}, \quad U_x = M^{-1} W_x = \begin{pmatrix}h \\
u \\
v \end{pmatrix}, \quad U_y = M^{-1} W_y = \begin{pmatrix}h \\
u \\
v \end{pmatrix},
\]

to obtain

\[ U_t + \mathcal{A} U_x + \mathcal{B} U_y = 0. \quad \text{(A.3)} \]

Equation (A.3) is the non-linear non-conservation form of SWE (5) with primitive variables.
We linearize (A.3) by perturbing a smooth solution \((\bar{h}, \bar{u}, \bar{v}), t = h(\cdot, t) = \bar{h} + \tilde{h}, u(\cdot, t) = \bar{u} + \tilde{u} \) and \(v(\cdot, t) = \bar{v} + \tilde{v}\). Next, we freeze the coefficients and arrive at the constant-coefficient linear problem

\[
U_t + A_0 U_x + B_0 U_y = 0,
\]  
(A.4)

where

\[
U = \begin{pmatrix} \tilde{h} \\ \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad A_0 = \begin{pmatrix} u_0 & h_0 & 0 \\ g & u_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} v_0 & 0 & h_0 \\ 0 & v_0 & 0 \\ g & 0 & v_0 \end{pmatrix}.
\]

Additionally, \(u_0, v_0\) and \(h_0\) represent the constant mean fluid velocities and height, respectively. (See [22, 23] for more information on linearizing and localizing of equations.)

The system (A.4) is symmetrized by

\[
S = \begin{pmatrix} g/c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where \(c = \sqrt{gh_0}\) is the gravity wave speed. (See [1, 33] for further information on symmetrization.) We multiply (A.4) by \(S\) to obtain

\[
w_t + Aw_x + Bw_y = 0,
\]  
(A.5)

where \(A = SA_0S^{-1}\) as well as \(B = SB_0S^{-1}\) are symmetrized matrices given by

\[
w = SU = \begin{pmatrix} g\bar{h}/c \\ \bar{u} \\ \bar{v} \end{pmatrix}, \quad A = \begin{pmatrix} u_0 & c & 0 \\ c & u_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix}, \quad B = \begin{pmatrix} v_0 & 0 & c \\ 0 & v_0 & 0 \\ c & 0 & v_0 \end{pmatrix}.
\]

Equation (A.5) is the linearized and symmetrized form of the SWE in a Cartesian domain.

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