Volterra-type operators on the minimal Möbius-invariant space

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Abstract. In this note, we mainly study operator-theoretic properties on the Besov space $B_1$ on the unit disk. This space is the minimal Möbius-invariant space. First, we consider the boundedness of Volterra-type operators. Second, we prove that Volterra-type operators belong to the Deddens algebra of a composition operator. Third, we obtain estimates for the essential norm of Volterra-type operators. Finally, we give a complete characterization of the spectrum of Volterra-type operators.

1 Introduction and preparation

In this paper, $\mathbb{D}$ denotes the open unit disk and $\mathbb{T}$ be the unit circle. Let $H(D)$ be the space of all analytic functions on $\mathbb{D}$. For $0 < p < \infty$, the Hardy space $H^p$ consists of analytic functions $f \in \mathbb{D}$ such that

$$\|f\|_{H^p}^p = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$ 

If $p = \infty$, then $H^\infty$ is the space of bounded analytic functions $f$ on $H(\mathbb{D})$ with

$$\|f\|_\infty = \sup \{|f(z)| : z \in \mathbb{D}\}.$$ 

For $0 < p < \infty$, the Bergman space $A^p$ consists of all functions $f$ analytic on $\mathbb{D}$ such that

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue area measure on $\mathbb{D}$. It is clear that $H^p \subset A^p$. Moreover, $H^p \subset A^{2p}$ and $\|f\|_{A^{2p}} \leq \|f\|_{H^p}$ for $0 < p < \infty$. See [44] for example.

The Dirichlet-type space $D^p$ is the set of all functions $f \in H(\mathbb{D})$ with

$$\|f\|_{D^p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \, dA(z) < \infty.$$ 

The space of all conformal automorphisms of $\mathbb{D}$ forms a group, called the Möbius group, and is denoted by $\text{Aut}(\mathbb{D})$. It is well known that $\varphi$ belongs to $\text{Aut}(\mathbb{D})$ if and only if there exists a real number $\theta$ and a point $a \in \mathbb{D}$ such that

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\[ \varphi(z) = e^{i\theta} \sigma_a(z) \text{ and } \sigma_a(z) = \frac{a - z}{1 - az}, \ z \in \mathbb{D}. \]

Let \( X \) be a Banach space of analytic functions on \( \mathbb{D} \). Then \( X \) is said to be Möbius-invariant whenever \( f \circ \varphi \in X \) for all \( f \in X \) and \( \varphi \in \text{Aut}(\mathbb{D}) \) and \( \| f \circ \varphi \|_X = \| f \|_X \).

For \( 1 < p < \infty \), the Besov space \( B_p \) consists of analytic functions \( f \) on \( \mathbb{D} \) such that
\[
\int_{\mathbb{D}} (1 - |z|^2)^{p-2} |f'(z)|^p dA(z) < \infty.
\]
The norm of \( B_p \) is defined as
\[
\| f \|_{B_p} = |f(0)| + \left( \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |f'(z)|^p dA(z) \right)^{\frac{1}{p}}.
\]
When \( p = \infty \), \( B_\infty =: \mathcal{B} \) is called the classical Bloch space. We define a norm on \( \mathcal{B} \) as
\[
\| f \|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.
\]
When \( p = 2 \), \( B_2 =: \mathcal{D} \) is the classical Dirichlet space. When \( p = 1 \), we get the analytic Besov space \( B_1 \), which is the minimal Möbius-invariant space consisting of all functions \( f \in H(\mathbb{D}) \) with
\[
f(z) = \sum_{k=1}^{\infty} c_k \sigma_{a_k}(z),
\]
where the sequences \( \{ c_k \}_{k \geq 1} \in \ell^1 \) and \( \{ a_k \}_{k \geq 1} \in \mathbb{D} \). An equivalent norm of \( B_1 \) is defined as
\[
\| f \|_{B_1} = |f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(z)| \, dA(z).
\]
Arazy, Fisher, and Peetre [5] first studied minimal Möbius-invariant space systematically. More results related to minimal Möbius-invariant space may be seen from [6, 7, 9, 32, 33, 45].

Now, we define several operators on \( B_1 \). For \( g \in H(\mathbb{D}) \), the multiplication operator \( M_g \) on \( B_1 \) is defined as
\[
(M_g) f(z) = f(z) g(z), \ f \in B_1, \ z \in \mathbb{D}.
\]
The differentiation operator is given by \( Df = f' \) for each \( f \in H(\mathbb{D}) \). Given \( g \in H(\mathbb{D}) \), the Volterra-type operator \( T_g \) is defined as
\[
(T_g f)(z) = \int_0^z f(w) g'(w) \, dw, \text{ for all } f \in B_1.
\]
When \( g(z) = z \), the operator \( T_z f(z) = \int_0^z f(w) \, dw \) becomes the simplest Volterra operator. An integral operator related to \( T_g \), denoted by \( I_g \), is defined as
\[
(I_g f)(z) = \int_0^z f'(w) g(w) \, dw, \text{ for all } f \in B_1.
\]

The Volterra-type operator \( T_g \) was originally studied by Pommerenke [38]. Later, a series of articles appeared on the study of Volterra-type integration operators on classical spaces of analytic functions, such as Hardy spaces, Bergman spaces, and
Volterra-type operators on the minimal Möbius-invariant space

Dirichlet-type spaces. For more details, please refer to [1, 3, 4, 16, 18]. In [13], ˇCuˇckovi´c and Paudyal describe the lattice of the closed invariant subspaces of Volterra-type operators. Lin, Liu, and Wu [22] generalized some of the works of [13] to the general case when \( 1 \leq p < \infty \), and obtained the boundedness of the Volterra-type operators \( T_g \) and \( I_g \) on the derivative Hardy space \( S_p(\mathbb{D}) \). And then, they also considered strict singularity of Volterra-type operators on Hardy spaces in [23]. Meanwhile, Lin [21] characterized the boundedness and compactness of the Volterra-type operators between Bloch-type spaces and weighted Banach spaces. In [31], Miihkinen et al. completely characterize the boundedness of the Volterra-type operators acting from the weighted Bergman spaces to the Hardy spaces of the unit ball.

In this paper, we mainly study the operator-theoretic properties in minimal Möbius-invariant space \( B_1 \). The structure of this article is as follows. In Section 2, we discuss the boundedness of the Volterra operator on \( B_1 \). In Section 3, it is shown whether the integral operator belongs to Deddens algebras. In Section 4, we will be concerned with the essential norms of integral operators on \( B_1 \). Section 5 is devoted to the study of the spectrum of integral operators on \( B_1 \).

Throughout this paper, we use the following convention. For two nonnegative functions \( F \) and \( G \) defined on some function space \( X \), we write \( F \lesssim G \) if \( F(f) \leq C \cdot G(f) \) for all \( f \in X \) and for some positive constant \( C \) which is independent of \( F \) and \( G \). Denote by \( F \approx G \) whenever \( F \lesssim G \lesssim F \).

2 Volterra-type operators on \( B_1 \)

First, we need the following Hardy inequality.

**Lemma 1** [15] If \( f \in H^1 \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then

\[
\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \pi \| f \|_{H^1}.
\]

The following lemma is a classic exercise in mathematical analysis, but it might be worth to give a brief details of the proof for completeness.

**Lemma 2** Suppose that \( f(x) \) is a continuously increasing function on \( [a, b] \). Then,

\[
\int_a^b x f(x) \, dx \geq \frac{a+b}{2} \int_a^b f(x) \, dx.
\]

**Proof** As \( f(x) \) is monotonically increasing on \( [a, b] \), the “integral mean value theorem” shows that there exists a \( \xi \in [a, b] \) such that

\[
\int_a^b \left( x - \frac{a+b}{2} \right) f(x) \, dx = f(a) \int_a^\xi \left( x - \frac{a+b}{2} \right) \, dx + f(b) \int_{a}^b \left( x - \frac{a+b}{2} \right) \, dx
\]

\[
= \frac{1}{2} (f(b) - f(a)) (a \xi + b \xi - \xi^2 - ab)
\]

\[
= \frac{1}{2} (f(b) - f(a)) (b - \xi)(\xi - a) \geq 0.
\]

This completes the proof of the lemma. ■
Lemma 3  If \( f \in D^1 \), then \( \| f \|_{H^1} \leq \| f \|_{D^1} \).

Proof  Let \( f \in D^1 \). Then, we can see that
\[
|f(e^{i\theta})| - |f(0)| \leq |f(e^{i\theta}) - f(0)| \leq \left| \int_0^1 f'(re^{i\theta}) \, dr \right| \leq \int_0^1 |f'(re^{i\theta})| \, dr.
\]

Thus,
\[
\| f \|_{H^1} = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| \, d\theta \\
\leq \frac{1}{2\pi} \int_0^{2\pi} \left[ |f(0)| + \int_0^1 |f'(re^{i\theta})| \, dr \right] d\theta \\
\leq |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f'(re^{i\theta})| \, dr \, d\theta.
\]

On the other hand,
\[
\| f \|_{D^1} = |f(0)| + \int_D |f'(z)| \, dA(z) = |f(0)| + \frac{1}{\pi} \int_0^{2\pi} \int_0^1 rf'(re^{i\theta}) \, dr \, d\theta.
\]

By Hardy’s convexity theorem (see [28, 46]), we find that
\[
F(r) = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})| \, d\theta, \quad 0 < r < 1,
\]
is a nondecreasing function of \( r \). It follows from Lemma 2 that
\[
\int_0^1 F(r) \, dr \leq 2 \int_0^1 rF(r) \, dr.
\]

This shows that \( \| f \|_{H^1} \leq \| f \|_{D^1} \). \qed

Remark 1  In Lemma 3, set \( f(z) = z \). Then, we see that \( 1 = \| z \|_{H^1} \leq \| z \|_{D^1} = 1 \), showing that the norm estimate is sharp. This improves the previous conclusion, namely, \( \| f \|_{H^1} \leq 2 \| f \|_{D^1} \) from the works of Girela and Merchán [17].

Lemma 4  If \( f \in B_1 \), then \( \| f \|_{\infty} \leq \pi \| f \|_{B_1} \) and \( B_1 \subset H^\infty \).

Proof  Assume that \( f \in B_1 \) and write \( f(z) = \sum_{k=0}^\infty a_k z^k \). By Lemmas 1 and 3, we have
\[
|f(z)| = \left| \sum_{k=0}^\infty a_k z^k \right| \leq \sum_{k=0}^\infty |a_k| z^k \leq \sum_{k=0}^\infty |a_k| \leq \pi \| f' \|_{H^1} + |f(0)| \leq \pi \| f' \|_{D^1} + |f(0)| \leq \pi \| f \|_{B_1},
\]
for all \( z \in \mathbb{D} \). Hence, we obtain that \( \| f \|_{\infty} \leq \pi \| f \|_{B_1} \). \qed

Remark 2  In [22, Theorem 1], they obtained that \( \| f \|_{\infty} \leq \pi \| f \|_{S^1} \) for each \( f \in S^1 \), where the space \( S^1 \) is defined as \( S^1 = \{ f \in H^1 : f' \in H^1 \} \). The norm on \( S^1 \) is given by
\[
\| f \|_{S^1} = |f(0)| + \| f' \|_{H^1}.
\]

Moreover, we obtain the following norm estimate:
\[
\| f \|_{\infty} \leq \pi \| f \|_{S^1} \leq \pi \| f \|_{B_1}, \quad \text{for all} \ f \in B_1.
\]
Volterra-type operators on the minimal Möbius-invariant space

In the following, we discuss the boundedness of \( T_g \) and \( I_g \) on minimal Möbius-invariant space.

**Theorem 1** The operator \( T_g \) is bounded on \( B_1 \) if and only if \( g \in B_1 \). Moreover,
\[
\| g - g(0) \| \leq \| T_g \| \leq (1 + \pi) \| g - g(0) \|_{B_1}.
\]

**Proof** Let \( f \in B_1 \). By Hölder's inequality and Lemma 3, we have
\[
\int_{\mathbb{D}} |f'(z)g'(z)| dA(z) \leq \left( \int_{\mathbb{D}} |f'(z)|^{2} dA(z) \right)^{\frac{1}{2}} \left( \int_{\mathbb{D}} |g'(z)|^{2} dA(z) \right)^{\frac{1}{2}} = \| f' \|_{A^2} \| g' \|_{A^2}.
\]
Hence, we get
\[
\| T_g f \|_{B_1} = \| f g' \|_{A^1(D)} = |f(0)g'(0)| + \int_{\mathbb{D}} |f'(z)g'(z)| dA(z) + \int_{\mathbb{D}} |f(z)g''(z)| dA(z) \leq \| f \|_{B_1} \| g - g(0) \|_{B_1} + \| f \|_{\infty} \| g - g(0) \|_{B_1} \leq (1 + \pi) \| f \|_{B_1} \| g - g(0) \|_{B_1},
\]
showing that \( T_g \) is a bounded operator on \( B_1 \).

Conversely, assume that \( T_g \) is a bounded operator on \( B_1 \) and let \( f = 1 \). Then, we obtain
\[
\| T_g \| \geq \| T_g 1 \|_{B_1} \geq \| g - g(0) \|_{B_1},
\]
which gives that \( g \in B_1 \). Thus,
\[
\| g - g(0) \| \leq \| T_g \| \leq (1 + \pi) \| g - g(0) \|_{B_1}.
\]

For \( 0 < p < \infty, -2 < q < \infty, \) and \( 0 \leq s < \infty \), we define the general family of function spaces \( F(p, q, s) \) as the set of all analytic functions \( f \) on \( \mathbb{D} \) such that
\[
\| f \|_{p,q,s} = |f(0)| + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^{p} |1 - |z|^{2}|^{q} g^{s}(z, a) dA(z) < \infty,
\]
where \( g(z, a) = \log \frac{1}{|\varphi_{a}(z)|} \). These spaces were introduced by Zhao in [47]. In 2003, Rättyä provided the following \( n \)th derivation characterization of functions in spaces \( F(p, q, s) \).

**Lemma 5** [39, Theorem 3.2] Let \( f \) be an analytic function on \( \mathbb{D} \), and let \( 0 < p < \infty, \) \( -2 < q < \infty, \) and \( 0 \leq s < \infty \). Let \( n \in \mathbb{N} \) and \( q + s > -1; \) or \( n = 0 \) and \( q + s - p > -1 \). Then, \( f \in F(p, q, s) \) if and only if
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^{p} |1 - |z|^{2}|^{q} \left( 1 - |\varphi_{a}(z)|^{2} \right)^{s} dA(z) < \infty.
\]
For $p > 0$, the space $Z_p$ consists of all analytic functions $f$ on $\mathbb{D}$ such that

$$
\|f\|_{Z_p} = |f(0)| + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \sigma_a(z))'/(1 - |z|^2)|^{p-1} dA(z) < \infty.
$$

It is clear that $Z_1 = F(1, -1, 1)$. For more results about $Z_p$ space, see [25, 48].

**Theorem 2** The operator $I_g$ is bounded on $B_1$ if and only if $g \in Z_1 \cap H^\infty$.

**Proof** Assume that $g \in Z_1 \cap H^\infty$. Using Lemma 6 of [25], we obtain that

$$
\int_{\mathbb{D}} |f'(w)| \cdot |g'(w)| dA(w) \lesssim \|g\|_{Z_1} \left( |f'(0)| + \int_{\mathbb{D}} |f''(w)| dA(w) \right).
$$

Thus, we have

$$
\| (I_g f)(z) \|_{B_1} = \| f' g \|_{D^1} 
\leq |f'(0)g(0)| + \int_{\mathbb{D}} |f''(w)| \cdot |g(w)| dA(w) + \int_{\mathbb{D}} |f'(w)| \cdot |g'(w)| dA(w)
\lesssim |f'(0)g(0)| + \|g\|_{\infty} \int_{\mathbb{D}} |f''(w)| dA(w) + \|g\|_{Z_1} \left( |f'(0)| + \int_{\mathbb{D}} |f''(w)| dA(w) \right)
\lesssim (\|g\|_{\infty} + \|g\|_{Z_1}) \|f\|_{B_1}
$$

for all $f \in B_1$. This implies that $I_g$ is a bounded operator on $B_1$.

Conversely, suppose that $I_g$ is a bounded operator on $B_1$. For each $a \in \mathbb{D}$, let $f(z) = \sigma_a(z)$. Then, $\|f\|_{B_1} \leq 1$ and

$$
\| I_g \| \geq \|I_g f\|_{B_1} = \|g(w) f'(w)\|_{D^1}
\geq \|g(w) f'(w)\|_{H^1} = \int_0^{2\pi} |g(e^{i\theta})| \cdot |\sigma_a'(e^{i\theta})| d\theta = \int_0^{2\pi} |g(\sigma_a(e^{i\theta}))| d\theta
\geq |g(\sigma_a(0))| = |g(a)|,
$$

which shows that $\|I_g\| \geq \|g\|_{\infty}$. Hence, $g \in H^\infty$. Moreover,

$$
\|I_g\| \geq \|I_g \sigma_a\|_{B_1}
\geq \int_{\mathbb{D}} |\sigma_a'(w)| \cdot |g'(w)| dA(w) - \int_{\mathbb{D}} |\sigma_a''(w)| \cdot |g(w)| dA(w)
\geq \int_{\mathbb{D}} |\sigma_a'(w)| \cdot |g'(w)| dA(w) - \|g\|_{\infty} \|\sigma_a\|_{B_1},
$$

from which it follows that

$$
\|I_g\| + \|g\|_{\infty} \|\sigma_a\|_{B_1} \geq \int_{\mathbb{D}} |\sigma_a'(w)| \cdot |g'(w)| dA(w).
$$

This implies that $g \in Z_1$. Therefore, $g \in Z_1 \cap H^\infty$. This completes the proof of the theorem.
Remark 3 Note that $B_1 \subset Z_1 \cap H^\infty$. By Lemma 4, we know that $f \in H^\infty$ whenever $f \in B_1$. Using the second derivation characterization of functions on $Z_1$, we have

$$|f|_{Z_1} \approx |f(0)| + \sup_{a \in D} \int_D |f'''(z)|(1 - |\sigma_a(z)|^2)\,dA(z) \leq |f(0)| + \int_D |f'''(z)|\,dA(z) \leq \|f\|_{B_1}.$$ 

Then $f \in Z_1$, and thus $B_1 \subset Z_1 \cap H^\infty$.

Now, we define the space $B^0_1$ as

$$B^0_1 = \{ f \in B_1 : f(0) = 0 \}.$$

The following theorem gives the connection between the Volterra operator $T_z$ on $\mathcal{D}^1$ and the multiplication operator $M_z$ on $B^0_1$.

**Theorem 3** The Volterra-type operator $T_z : \mathcal{D}^1 \rightarrow B^0_1$ is bounded and invertible with $T_z^{-1} = D$.

**Proof** First, we show that $T_z(\mathcal{D}^1) = B^0_1$. For $f \in \mathcal{D}^1$, we consider

$$F(z) := (T_zf)(z) = \int_0^z f(w)\,dw.$$

Clearly, $F' = f \in \mathcal{D}^1$ so that $F \in B^0_1$ and $T_z(\mathcal{D}^1) \subset B^0_1$.

Conversely, for each $F \in B^0_1$, we have $F' \in \mathcal{D}^1$. Then,

$$(T_z(F'(z))) = \int_0^z F'(w)\,dw = F(z) - F(0) = F(z).$$

Then, $B^0_1 \subset T_z(\mathcal{D}^1)$. This implies that $T_z(\mathcal{D}^1) = B^0_1$.

Second, we show that $T_z : \mathcal{D}^1 \rightarrow B^0_1$ is a bounded isomorphism, and its inverse $T_z^{-1} = D$. Recall from the above discussion that $(T_z(F'(z))) = F(z)$ for $F \in B^0_1$. Then, for each $f \in \mathcal{D}^1$, we have that $(D(T_zf))(z) = f(z)$. This implies that $T_z$ is a bijective operator from $\mathcal{D}^1$ onto $B^0_1$, since $T_z$ is linear and $T_z$ is an isomorphism from $\mathcal{D}^1$ onto $B^0_1$.

Finally, we need to prove that $T_z$ is a bounded operator on $\mathcal{D}^1$. For $f \in \mathcal{D}^1$, we have

$$\|T_zf\|_{B^0_1} \leq \|((T_zf)(0)) + \|f\|_{\mathcal{D}^1}\| \leq \|f\|_{\mathcal{D}^1}.$$ 

Therefore, $T_z$ is a bounded isomorphism from $\mathcal{D}^1$ onto $B^0_1$.

Let us introduce an addition operator $P$ defined as

$$(Pf)(z) = (M_zf)(z) + (T_zf)(z) \quad \text{for } f \in H(\mathbb{D}) \text{ and } z \in \mathbb{D}. $$

**Theorem 4** Let $T_z : \mathcal{D}^1 \rightarrow B^0_1$ and $M_z : B^0_1 \rightarrow \mathcal{D}^1$ be the Volterra-type operator and the multiplication operator, respectively. Then, $P$ is an operator on $\mathcal{D}^1$ with $P = T_z^{-1}M_zT_z$.

**Proof** Let $f \in \mathcal{D}^1$ and $F = T_zf$. Then, we get

$$(Pf)(z) = (M_zf)(z) + (T_zf)(z) = zf(z) + F(z) = D(zF(z)) = (T_z^{-1}M_zT_zf)(z),$$

which shows that $P = T_z^{-1}M_zT_z$.

**Theorem 5** If $f, g \in B_1$, then $\|fg\|_{B_1} \leq (2\pi + 2)\|f\|_{B_1}\|g\|_{B_1}$. 
Proof. For \( f, g \in B_1 \), we get
\[
\|fg\|_{B_1} = |f(0)g(0)| + |(fg)'(0)| + \int_D |f''(w)g(w) + 2f'(w)g'(w) + f(w)g''(w)| \, dA(w)
\]
\[
\leq (|f(0)| + |f'(0)|)(|g(0)| + |g'(0)|) + \int_D |f''(w)g(w)| \, dA(w)
\]
\[
+ 2\int_D |f'(w)g'(w)| \, dA(w) + \int_D |f(w)g''(w)| \, dA(w)
\]
\[
\leq \|g\|_{B_1}(|f(0)| + |f'(0)|) + \|g\|_{\infty}(\|f\|_{B_1} - |f(0)| - |f'(0)|)
\]
\[
+ 2\|f\|_{B_1}\|g\|_{B_1} + \|f\|_{\infty} \int_D |g''(w)| \, dA(w)
\]
\[
\leq \|g\|_{B_1}(|f(0)| + |f'(0)|) + \pi\|g\|_{B_1}(\|f\|_{B_1} - |f(0)| - |f'(0)|)
\]
\[
+ 2\|f\|_{B_1}\|g\|_{B_1} + \pi|f|_{B_1}\|g\|_{B_1}
\]
\[
\leq (2\pi + 2)\|f\|_{B_1}\|g\|_{B_1},
\]
and the proof is complete. \( \blacksquare \)

Remark 4. In [5, Theorem 10], Arazy et al. obtained \( \|fg\| \leq 7\|f\|\|g\| \) for \( f, g \in B_1 \), in which they defined the norm of \( f \in B_1 \) as
\[
\|f\| = \inf \left\{ \sum_{k=1}^{\infty} c_k : f(z) = \sum_{k=1}^{\infty} c_k a_k(z) \right\}.
\]

Inspired by their work, we derived Theorem 5 and we are not sure whether the constant \( 2\pi + 2 \) in Theorem 5 is optimal or not.

3 Deddens algebras

Let \( \mathcal{L}(X) \) denote the algebra of all bounded linear operators on a complex Banach space \( X \). A nontrivial invariant subspace of an operator \( A \in \mathcal{L}(X) \) is, by definition, a closed subspace \( M \) of \( X \) such that \( M \neq \{ 0 \}, M + X, \) and \( Ax \in M \) for every \( x \in M \); or, briefly, \( A(M) \subset M \).

Let \( A \in \mathcal{L}(X) \). The operator \( T \) is said to belong to the Deddens algebra \( \mathcal{D}_A \) if there exists \( M = M(T) > 0 \) such that
\[
\|A^nTf\| \leq M\|A^n f\|
\]
for each \( n \in \mathbb{N} \) and \( f \in X \).

The study of the Deddens algebra was originally introduced by Deddens [14], where he assumed that \( A \) is an invertible operator and \( \sup_{n \in \mathbb{N}} \|A^nTA^{-n}\| < \infty \). Later, it received the attention of many scholars (see [14, 19, 20, 34–37, 42, 43]). Recently, Petrovic and Sievewright [37] studied the Deddens algebra associated with compact composition operators \( C_p \) on Hardy spaces, where \( A \) is not necessarily invertible, and they have demonstrated that the operators \( M_g \) and \( T_z \) belong to the Deddens algebra \( \mathcal{D}_{C_A} \). It is worth to point out that compact operators on \( H^2 \) are not invertible.

Let us begin to present the boundedness of composition operators and multiplication operators on \( B_1 \).
Lemma 6 [45] Let $\varphi$ be an analytic self-map of $\mathbb{D}$. Then, the composition operator $C_{\varphi}$ is bounded on $B_1$ if and only if
\[
\sup_{a \in \mathbb{D}} \| C_{\varphi}a \|_{B_1} < \infty.
\]

In particular, we give another description of sufficiency condition for the boundedness of the composition operator in the following theorem.

Theorem 6 Let $\varphi$ be an analytic self-map of $\mathbb{D}$ such that $\varphi(0) = 0$. Then, the composition operator $C_{\varphi}$ is bounded on $B_1$ whenever $\varphi' \in Z_1 \cap H^\infty$.

Proof Suppose that $\varphi' \in Z_1 \cap H^\infty$. Then, we get
\[
\| C_{\varphi}f \|_{B_1} = \| f(\varphi(z)) \|_{B_1}
\]
\[
= |f(\varphi(0))| + |f'(\varphi(0))\varphi'(0)| + \int_{\mathbb{D}} |f''(\varphi(w)) \cdot (\varphi'(w))^2
\]
\[
+ f'(\varphi(w)) \cdot \varphi''(w)|\, dA(w)
\]
\[
\leq |f(\varphi(0))| + |f'(\varphi(0))\varphi'(0)| + \int_{\mathbb{D}} |f''(\varphi(w)) \cdot (\varphi'(w))^2|\, dA(w)
\]
\[
+ \int_{\mathbb{D}} |f'(\varphi(w)) \cdot \varphi''(w)|\, dA(w)
\]
\[
\leq \| f \|_\infty + \| \varphi' \|_\infty \| f \|_{B_1} + \| \varphi \|_\infty^2 \| f \|_{B_1} + \| \varphi \|_{Z_1} \| \varphi' \|_\infty \| f \|_{B_1}
\]
\[
\leq (\| \varphi' \|_\infty^2 + \| \varphi \|_{Z_1} \| \varphi' \|_\infty + \| \varphi' \|_\infty + 1) \| f \|_{B_1},
\]
which implies that $C_{\varphi}$ is bounded on $B_1$ if $\varphi' \in Z_1 \cap H^\infty$. 

Theorem 7 Suppose that $M_g$ is a multiplication operator on $B_1$. Then, $M_g$ is bounded if and only if $g \in B_1$.

Proof For any $f \in B_1$, if $g \in B_1$, we have $M_g$ is bounded on $B_1$, by Theorem 5.

Conversely, let $M_g$ be a bounded operator on $B_1$. Then, with $f = 1$, we get $\| M_g \|_1 \|_{B_1} = \| g \|_{B_1}$, which implies that $g \in B_1$. 

In the following theorem, we will consider the algebra $D_{C_{\varphi}}$, in which the operator $C_{\varphi}$ is a bounded composition operator. For $n \in \mathbb{N}$, it clear that $C_{\varphi}^n f = f \circ \varphi \circ \cdots \circ \varphi$. For simplicity of the notation, we write $\varphi_n$ instead of $\varphi \circ \cdots \circ \varphi$.

Theorem 8 Let $g \in B_1$, and let $\varphi$ be an analytic self-map of $\mathbb{D}$ with $\varphi(0) = 0$ such that $C_{\varphi}$ is bounded on $B_1$. Then, the operators $M_g$, $T_g$, and $I_g$ belong to the Deddens algebra $D_{C_{\varphi}}$.

Proof For each $n \in \mathbb{N}$, we see that
\[
C_{\varphi}^n M_g f = C_{\varphi}^n (g f) = (g \circ \varphi_n) (f \circ \varphi_n) = M_{g \circ \varphi_n} C_{\varphi}^n f.
\]
Since $\varphi_n(\mathbb{D}) \subset \mathbb{D}$, it follows that
\[
\| M_{g \circ \varphi_n} f \|_{B_1} = \| (g \circ \varphi_n) f \|_{B_1} \leq \| g \|_{B_1} \| f \|_{B_1}.
\]
and therefore
\[ \| C^n M g f \|_{B_1} = \| M_{g \circ \varphi_n} C^n f \|_{B_1} \leq \| g \|_{B_1} \| C^n f \|_{B_1}, \]
where \( f \in B_1(\mathbb{D}) \). This implies that \( M_g \in \mathcal{D}_{C^n} \).

Next, we have
\[ C^n T g f(z) = C^n \left( \int_0^z f(w) g'(w) dw \right) = \int_0^\varphi_n(z) f(w) g'(w) dw. \]
Since \( \varphi \) is an analytic self-map of \( \mathbb{D} \) satisfying \( \varphi(0) = 0 \), we have \( \varphi_n(0) = 0 \), and therefore
\[ T g C^n f(z) = T g \varphi_n \left( (f \circ \varphi_n)(z) \right) = \int_0^z (f \circ \varphi_n)(w) g'(w) \varphi'_n(w) dw = \int_0^\varphi_n(z) f(w) g'(w) dw, \]
where \( f \in B_1 \). It shows that \( C^n T g = T g \varphi_n \varphi_n \). By Theorem 1, we have
\[ \| C^n T g f(z) \|_{B_1} = \| T g \varphi_n C^n f(z) \|_{B_1} \leq \| \varphi_n \|_{B_1} \| C^n f(z) \|_{B_1} \leq \| g \|_{B_1} \| C^n f(z) \|_{B_1}, \]
which gives \( T g \in \mathcal{D}_{C^n} \).

Finally, we have
\[ C^n I g f(z) = C^n \left( \int_0^z f'(w) g(w) dw \right) = \int_0^\varphi_n(z) f'(w) g(w) dw \]
and
\[ I g C^n f(z) = I g \varphi_n \left( (f \circ \varphi_n)(z) \right) = \int_0^z \varphi_n(w) f'(w) g(\varphi_n(w)) \varphi'_n(w) dw = \int_0^\varphi_n(z) f'(w) g(w) dw, \]
where \( f \in B_1 \). Therefore, \( C^n I g = I g \varphi_n C^n \). By Theorem 2 and Remark 3, we find that
\[ \| C^n I g f(z) \|_{B_1} = \| I g \varphi_n C^n f(z) \|_{B_1} \leq \| \varphi_n \|_{B_1} \| C^n f(z) \|_{B_1} \leq \| g \|_{B_1} \| C^n f(z) \|_{B_1} \]
for all \( f \in B_1 \). We thus deduce that \( I g \in \mathcal{D}_{C^n} \).

**4 Essential norms of Volterra-type operators on \( B_1 \)**

Suppose that \( X \) is a Banach space and \( T \) is a bounded linear operator on \( X \). The essential norm of \( T \) is defined to be
\[ \| T \|_e = \inf \{ \| T - K \| : K \text{ is a compact operator on } B_1 \}. \]
Obviously, the essential norm of \( T \) is 0 if and only if \( T \) is compact. For more results, we invite the reader to refer to [26, 41]. In this section, we characterize the essential norm of linear operator on \( B_1 \), which generalizes the conclusion of Liu et al. [24]
Theorem 9  Every bounded operator $T_g$ on $B_1$ is compact.

Proof  By definition, we know $\| T_g \|_e \geq 0$.

Next, we show that $\| T_g \|_e \leq 0$. To do this, we define the following operators:

$$T_g, f := \int_0^z rf(w)g'(w) \, dw,$$

where $g_r(z) = g(rz)$ and $r \in (0, 1)$. It is easy to see that $T_g, r$ is a compact operator on $B_1$ for $g \in B_1$. In fact, if $T_g$ is a bounded operator on $B_1$, then $g \in B_1$. Suppose that $\{f_n\}_{n=1}^\infty \subset B_1$ with $\| f_n \|_{B_1} \leq 1$, and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then,

$$\| T_g, f_n \|_{B_1} = \| f_n(raz)g'(raz) \|_{B_1} \leq \| f_n \|_{B_1} \| g \|_{B_1},$$

where $\delta_1 < 1$, and $\mathbb{D}_{\delta_1} = \{ z : |z| < \delta_1 \}$ are compact subsets of $\mathbb{D}$.

Note that

$$\int_{\mathbb{D}} |f_n'(raz)g'(raz)| \, dA(z) \leq \| f_n \|_{B_1} \| g \|_{B_1} \quad \text{and} \quad \int_{\mathbb{D}} |f_n(raz)g''(raz)| \, dA(z) \leq \| f_n \|_{B_1} \| g \|_{B_1}.$$  

Using the theorem of absolute continuity of Lebesgue measure, we conclude that

$$\int_{\mathbb{D}_{\delta_1}} |f_n'(raz)g'(raz)| \, dA(z) < \varepsilon \quad \text{and} \quad \int_{\mathbb{D}_{\delta_1}} |f_n(raz)g''(raz)| \, dA(z) < \varepsilon.$$  

On the other hand, using a basic result of complex analysis (see page 151 of [12]), we know that if $f_n \to 0$, then $f_n' \to 0$ uniformly on compact subsets of $\mathbb{D}$. Consequently,

$$\int_{\mathbb{D}_{\delta_1}} |f_n'(raz)g'(raz)| \, dA(z) \quad \text{and} \quad \int_{\mathbb{D}_{\delta_1}} |f_n(raz)g''(raz)| \, dA(z)$$

converge to 0 when $n \to \infty$. This implies that $T_{g, r}$ is compact.

Meanwhile, we have

$$\| T_g \|_e \leq \| T_g - T_{g, r} \| = \| T_{g - g_r} \| \leq \| g - g_r \|_{B_1},$$

for $r \in (0, 1)$. Similarly, with the above computation, we have

$$\lim_{r \to 1^-} \| g - g_r \|_{B_1} = 0,$$

so that $\| T_g \|_e \leq 0$. Hence, we deduce that $\| T_g \|_e = 0$, which completes the proof. 

Theorem 10  If $I_g$ is bounded operator on $B_1$, then $\| g \|_\infty \leq \| I_g \|_e \leq ( \| g \|_\infty + \| g \|_{Z_1})$.

Proof  From the proof of Theorem 2, we have

$$\| I_g \|_e = \inf \| I_g - K \| \leq \| I_g \| \leq C(\| g \|_\infty + \| g \|_{Z_1}),$$

where $C$ is a positive constant.
We next show that $\|I_g\|_e \geq \|g\|_\infty$. Choose $a_n \in \mathbb{D}$ such that $|a_n| \to 1$ as $n \to \infty$. Let $f_n(z) = \sigma_{a_n}(z) - a_n$. It is obvious that $\|f_n\|_{B_1} = 1$. Since $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, for every compact operator $K$ on $B_1$, we obtain $\|Kf_n\|_{B_1} \to 0$ as $n \to \infty$. Therefore,

$$\|I_g - K\| \geq \lim_{n \to \infty} \sup \| (I_g - K)f_n \|_{B_1}$$

$$\geq \lim_{n \to \infty} \sup (\| I_g f_n \|_{B_1} - \| Kf_n \|_{B_1})$$

$$= \lim_{n \to \infty} \sup \| I_g f_n \|_{B_1}.$$ 

Similarly to the proof of Theorem 2, we get

$$\|I_g f_n\|_{B_1} = \|g(z) f'_n(z)\|_{\mathbb{D}^1} \geq |g(\sigma_{a_n}(0))| = |g(a_n)|.$$ 

As the choice of the sequence $\{a_n\} \subset \mathbb{D}$ is arbitrary, we have $\|I_g\|_e \geq \|g\|_\infty$, which completes the proof.

**Theorem 11** Every bounded operator $M_g$ on $B_1$ is compact.

**Proof** The proof is similar to Theorem 9, so we omit its details.

## 5 Spectrum of Volterra-type operators on $B_1(\mathbb{D})$

The spectrum of integral operators on different spaces has attracted the attention of many scholars. The spectra of integral operators on weighted Bergman spaces are characterized by Aleman and Constantin [2]. Later, Constantin [10] obtained the spectrum of Volterra-type operators on Fock spaces. Mengestie [29] studied the spectrum of Volterra-type operators on Fock–Sobolev spaces. Mengestie [30] also obtained the spectrum of $T_g$ in terms of a closed disk of radius twice the coefficient of the highest degree term in a polynomial expansion of $g$. For more results, see [8, 27]. Recently, Lin et al. described the spectra of the multiplication operator and the Volterra-type operator $I_g$ in [23], respectively. Inspired by the above results, it is natural to study the spectra of the multiplication operator and the Volterra-type operators on $B_1$.

**Theorem 12** Suppose that $M_g$ is bounded on $B_1$. Then, we have $\sigma(M_g) = \overline{g(\mathbb{D})}$.

**Proof** Suppose that $\lambda \notin \sigma(M_g)$. Then, $M_g - \lambda I$ is invertible. As $1 \in B_1$, there exists an $f \in B_1$ such that $(g(z) - \lambda)f(z) = 1$ for all $z \in \mathbb{D}$, which implies that $\lambda \notin g(\mathbb{D})$. Thus, $g(\mathbb{D}) \subset \sigma(M_g)$.

For the other way inclusion, we let $\lambda \notin \overline{g(\mathbb{D})}$. Then, we can choose a $t > 0$ such that $|g(z) - \lambda| > t$ for all $z \in \mathbb{D}$. This shows that $h = (g - \lambda)^{-1}$ is a bounded analytic function on $\mathbb{D}$. For all $g \in B_1$, we get

$$\|h\|_{B_1} \leq t|\lambda(0)| + \|h'\|_{\mathbb{D}^1}$$

$$\leq \left( \frac{1}{|g(0) - \lambda|} + \frac{\|g'\|_{\mathbb{D}^1}}{(g - \lambda)^2} \right).$$
Volterra-type operators on the minimal Möbius-invariant space

\[
\left| \frac{1}{g(0) - \lambda} \right| + \left| \frac{g'(0)}{(g(0) - \lambda)^2} \right| + \int_D \left| \frac{g''(z)}{(g(z) - \lambda)^2} - \frac{2(g')^2}{(g(z) - \lambda)^3} \right| \, dA(z)
\]

\[\leq \|g\|_{B_1} + \|g\|^3_{B_1}.
\]

Hence, \( h \in B_1 \). Then, \( M_h \) is bounded on \( B_1 \), by Theorem 7. Since \( M_h = M_{(g-\lambda)^{-1}} = M_{g^{-1}-\lambda} \), we see that \( M_{g-\lambda} \) is invertible, and thus \( \lambda \notin \sigma(M_g) \). Therefore, \( \sigma(M_g) \subset g(\mathbb{D}) \).

Since the spectrum set is closed, we conclude that \( \sigma(M_g) = g(\mathbb{D}) \). \( \blacksquare \)

**Lemma 7** \([40]\) Let \( T \) be a bounded linear operator on a Banach space \( X \), and let \( T \) be compact. If \( \dim X = \infty \), then \( \sigma(T) = \{0\} \cup \{ \text{eigenvalues of } T \} \).

**Theorem 13** Suppose that \( T_g \) is a bounded operator on \( B_1 \). Then, \( \sigma(T_g) = \{0\} \).

**Proof** Let \( T_g \) be a bounded operator on \( B_1 \). Then, \( T_g \) is compact, by Theorem 9. By Lemma 7, we obtain \( 0 \in \sigma(T) \).

Next, we prove that \( T_g \) has no nonzero eigenvalue. Assume that \( T_g \) has an eigenvalue \( \lambda \neq 0 \) with eigenvector \( f \). Then,

\[
T_g f(z) = \int_0^z f(w) g'(w) \, dw = \lambda f(z).
\]

Differentiating equation (1) with respect to \( z \), we get

\[
f(z) g'(z) = \lambda f'(z).
\]

All nonzero solutions of this equation are of the form \( f(z) = ce^{\frac{\lambda z}{1-\lambda}} \) for some \( c \neq 0 \). Setting \( z = 0 \) in (1) shows that \( 0 = \lambda f(0) \), which contradicts the last relation about \( f \). Therefore, there is no nonzero eigenvalue for \( T_g \). From this, we deduce that \( \sigma(T_g) = \{0\} \). \( \blacksquare \)

**Theorem 14** If \( I_g \) is a bounded operator on \( B_1 \), then

\[
\sigma(I_g) = \{0\} \cup g(\mathbb{D}^c).
\]

**Proof** For any constant function \( a \), we have

\[
(I_g a)(z) = \int_0^z a'(w) g(w) \, dw = 0,
\]

which gives \( 0 \in \sigma(I_g) \).

Suppose that \( \lambda \in \mathbb{C} \setminus \{0\} \). Note that the equation

\[
f - \frac{1}{\lambda} I_g f = h, \quad \text{for } h \in B_1,
\]

has a unique analytic solution

\[
f(z) = R_{\lambda,g} h(z) = \int_0^z \frac{h'(\xi)}{1 - \frac{\lambda}{g(\xi)}} \, d\xi + h(0) = I_{(1-\frac{1}{\lambda}g)^{-1}} h(z) + h(0)
\]
(see [11] for more details). Hence, the resolvent set $\rho(I_g)$ of the bounded operator $I_g$ consists precisely of all points $\lambda \in \mathbb{C}$ for which $R_{\lambda,g}$ is a bounded operator on $B_1$.

If $\lambda \in \mathbb{C}\setminus\{0\} \cup \{g(z)\}$, then it is clear that $1 - \frac{1}{\lambda}g(z)$ is bounded away from 0, which implies that $\frac{1}{1 - \frac{1}{\lambda}g(z)} \in H^\infty$. If $I_g$ is a bounded operator on $B_1$, then $g \in H^\infty \cap Z_1$ by Theorem 2. Moreover, it is easy to show that $\frac{1}{1 - \frac{1}{\lambda}g(z)} \in Z_1$. This implies that the operator $R_{\lambda,g}$ is a bounded operator on $B_1$. It follows that $\mathbb{C}\setminus\{0\} \cup \{g(D)\} \subset \rho(I_g)$. Thus, $\sigma(I_g) \subset \{0\} \cup \{g(D)\}$.

On the other hand, if $\lambda \in g(D)$ and $\lambda \neq 0$, then $\frac{1}{1 - \frac{1}{\lambda}g(z)}$ is not bounded, which shows that the operator $R_{\lambda,g}$ is not bounded on $B_1$. Therefore, we obtain that $g(D)\setminus\{0\} \subset \sigma(I_g)$. This together with the fact that $0 \in \sigma(I_g)$ shows that $g(D) \cup \{0\} \subset \sigma(I_g) \subset g(D) \cup \{0\}$.

Since the spectrum $\sigma(I_g)$ is closed, we deduce that $\sigma(I_g) = g(D) \cup \{0\}$.

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