ASYMPTOTIC IMPROVEMENT OF THE SUNFLOWER BOUND

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Abstract. A sunflower with a core $Y$ is a family $B$ of sets such that $U \cap U' = Y$ for each two different elements $U$ and $U'$ in $B$. The well-known sunflower lemma states that a given family $F$ of sets, each of cardinality at most $s$, includes a sunflower of cardinality $k$ if $|F| > (k-1)^s s!$. Since Erdős and Rado proved it in 1960, it has not been known for more than half a century whether the sunflower bound $(k-1)^s s!$ can be improved asymptotically for any $k$ and $s$. It is conjectured that it can be reduced to $c^s k$ for some real number $c_k > 0$ depending only on $k$, which is called the sunflower conjecture. This paper shows that the general sunflower bound can be indeed reduced by an exponential factor: We prove that $F$ includes a sunflower of cardinality $k$ if $|F| \geq \left(\sqrt{10} - 2\right)^2 \left[ k \cdot \min\left(\frac{1}{\sqrt{10} - 2}, \frac{c}{\log \min(k, s)}\right)\right]^s s!$, for a constant $c > 0$, and any $k \geq 2$ and $s \geq 2$. For instance, whenever $k \geq s^\epsilon$ for a given constant $\epsilon \in (0, 1)$, the sunflower bound is reduced from $(k-1)^s s!$ to $(k-1)^s \cdot \left[ O\left(\frac{1}{\log s}\right)\right]^s$, achieving the reduction ratio of $\left[ O\left(\frac{1}{\log s}\right)\right]^s$. Also any $F$ of cardinality at least $\left(\sqrt{10} - 2\right)^2 \left(\frac{1}{\sqrt{10} - 2}\right)^s s!$ includes a sunflower of cardinality $k$, where $\frac{1}{\sqrt{10} - 2} = 0.866025404...$. Our result demonstrates that the sunflower bound can be improved by a factor of less than a small constant to the power $s$, giving hope for further update.

1. Introduction

A set means a subset of a given universal set $X$. Denote by $F$ a family of sets, and by $B$ its sub-family. For a set $Y \subset X$, a sunflower with a core $Y$ is a family $B$ of sets such that $U \cap U' = Y$ for each two different elements $U$ and $U'$ in $B$. Equivalently, $B$ is a sunflower if $U \cap U' = \bigcap_{V \in B} V$ for any $U, U' \in B$ such that $U \neq U'$. A sunflower of cardinality $k$ is called $k$-sunflower for short. A constant is a fixed positive real number depending on no variable.

The sunflower lemma shown by Erdős and Rado \cite{erdos} states that:

Lemma 1.1. A family $F$ of sets, each of cardinality at most $s$, includes a $k$-sunflower if $|F| > (k-1)^s s!$.

Since its proof was given in 1960, it has not been known whether the sunflower bound $(k-1)^s s!$ can be asymptotically improved for any $k$ and $s$, despite its usefulness in combinatorics and various applications \cite{erdos} \cite{rado}. It is conjectured that the bound can be reduced to $c^s k$ for a real number $c_k > 0$ only depending on $k$, which is called the sunflower conjecture. The results known so far related to this topic include:

2010 Mathematics Subject Classification. 05D05:Extremal Set Theory (Primary).
Key words and phrases. Sunflower Lemma, Sunflower Conjecture, $\Delta$-System.
- Kostochka [8] showed that the sunflower bound for \( k = 3 \) is reduced from \( 2^s s! \) to \( c s! \left( \frac{\log \log \log s}{\log \log s} \right)^s \) for a constant \( c \). The case \( k = 3 \) of the sunflower conjecture is especially emphasized by Erdős [4], which other researchers also believe includes some critical difficulty.

- It has also been shown [9] that \( F \) of cardinality greater than \( k^s \left( 1 + c_s k^{-2^{-s}} \right) \) includes a \( k \)-sunflower for some \( c_s \in \mathbb{R}^+ \) depending only on \( s \).

- With the sunflower bound \((k-1)^s s!\), Razborov proved an exponential lower bound on the monotone circuit complexity of the clique problem [10]. Alon and Boppana strengthened the bound [1] by relaxing the condition to be a sunflower from \( U \cap U' = \bigcap_{V \in B} V \) to \( U \cap U' \supset \bigcap_{V \in B} V \) for all \( U, U' \in F, U \neq U' \).

- [2] discusses the sunflower conjecture and its variants in relation to fast matrix multiplication algorithms. Especially, it is shown in the paper that if the sunflower conjecture is true, the Coppersmith-Winograd conjecture implying a faster matrix multiplication algorithm [3] does not hold.

In this paper we show that the general sunflower bound can be indeed improved by an exponential factor. We prove the following theorem.

**Theorem 1.2.** A family \( F \) of sets, each of cardinality at most \( s \), includes a \( k \)-sunflower if

\[
|F| \geq \left( \sqrt{10} - 2 \right)^2 \left[ k \cdot \min \left( \frac{1}{\sqrt{10} - 2}, \frac{c}{\log \min(k, s)} \right) \right]^s s!,
\]

for a constant \( c \) and any integers \( k \geq 2 \) and \( s \geq 2 \).

This improves the sunflower bound by the factor of \( \left[ O \left( \frac{1}{\log s} \right) \right]^s \) whenever \( k \) exceeds \( s^2 \) for a given constant \( \epsilon \in (0, 1) \). Also any \( F \) of cardinality at least \( \left( \sqrt{10} - 2 \right)^2 \left( \frac{k}{\sqrt{10} - 2} \right)^s s! \) includes a \( k \)-sunflower.

We split its proof in two steps. We will show:

**Statement I:** \( F \) includes a \( k \)-sunflower if \( |F| \geq \left( \sqrt{10} - 2 \right)^2 \left( \frac{k}{\sqrt{10} - 2} \right)^s s! \) for any positive integers \( s \) and \( k \).

**Statement II:** \( F \) includes a \( k \)-sunflower if \( |F| \geq \left[ \frac{c k}{\log \min(s, k)} \right]^s s! \) for some constant \( c \) and any integers \( s \geq 2 \) and \( k \geq 2 \).

It is clear that the two statements mean Theorem 1.2. The rest of the paper is dedicated to the description of their proofs.

### 2. Terminology and Related Facts

Denote an arbitrary set by \( S \) that is a subset of \( X \). Given a family \( F \) of sets of cardinality at most \( s \), define

\[
\mathcal{F}(S) \overset{\text{def}}{=} \{ U : U \in F \text{ and } U \cap S \neq \emptyset \} ,
\]

\[
\mathcal{F}_j(S) \overset{\text{def}}{=} \{ U : U \in F \text{ and } |U \cap S| = j \} \text{ for positive integer } j,
\]

\[
\mathcal{F}_{\text{sup}}(S) \overset{\text{def}}{=} \{ U : U \in F \text{ and } U \supset S \} , \text{ and}
\]

\[
\mathcal{P}(S) \overset{\text{def}}{=} \{ (v, U) : v \in X, U \in F \text{ and } v \in U \cap S \} .
\]
Let \( \epsilon \in (0, 1/8) \) be a constant and \( k \in \mathbb{Z}^+ \). Given such numbers, we use the following two functions as lower bounds on \(|\mathcal{F}|\):

\[
\Phi_1(s) \overset{def}{=} \left( \frac{\sqrt{10} - 2}{e} \right)^s s!, \quad \text{and} \\
\Phi_2(s) \overset{def}{=} \frac{k^s s!}{p_1 p_2 \cdots p_s},
\]

where \( p_j \overset{def}{=} \left\{ \begin{array}{ll}
\epsilon \ln \min (j, k) & \text{if } j \geq 2, \\
\epsilon & \text{if } j = 1.
\end{array} \right. \)

Here \( \ln \cdot \) denotes the natural logarithm of a positive real number. We regard \( \Phi_1(j) = 0 \) if \( j \not\in \mathbb{Z}^+ \) for each \( i = 1, 2 \).

The following lemma shows that Statement II is proved if \(|\mathcal{F}| \geq \Phi_2(s)\) means a \( k \)-sunflower in \( \mathcal{F} \).

**Lemma 2.1.** There exists a constant \( c \) such that \( \Phi_2(s) \leq \left( \frac{e k}{\ln \min (k, s)} \right)^s s! \) for any \( s \geq 2 \) and \( k \geq 2 \).

It is shown by \( p_1 p_2 \cdots p_s \geq (c' \ln \min (k, s))^s \) for another constant \( c' \). Its exact proof is found in Appendix.

We also have

\[
\Phi_2(s) = \frac{k^s s!}{p_1 p_2 \cdots p_s} \Phi_2(s - 1) = \frac{k^s k(s - 1)}{p_s p_{s-1}} \Phi_2(s - 2) = \cdots
\]

\[
= \frac{k^j s(s - 1) \cdots (s - j + 1)}{p_s p_{s-1} \cdots p_{s-j+1}} \Phi_2(s - j) \\
= \frac{k^j}{p_s p_{s-1} \cdots p_{s-j+1}} \frac{s!}{(s-j)!} \Phi_2(s - j) \\
\geq \left( \frac{k}{p_s} \right)^j \frac{s!}{(s-j)!} \Phi_2(s - j),
\]

for each positive integer \( j < s \). The last inequality is due to \( p_2 \leq p_3 \leq \cdots \leq p_s \).

To derive another inequality from (2.1), we use Stirling’s approximation

\[
\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} \left( \frac{n}{e} \right)^n} = 1 \quad \text{where } e = 2.71828... \text{ denotes the natural logarithm base. In}
\]

a form of double inequality, it is known as

\[
\frac{n!}{\sqrt{2\pi n} \cdot n^n e^{-n + \frac{1}{12n}}} < n! < \sqrt{2\pi n} \cdot n^n e^{-n + \frac{1}{12n}},
\]

for \( n \in \mathbb{Z}^+ \). This means

\[
(2.2) \quad \sqrt{2\pi n} \cdot n^n e^{-n} < n! \leq \sqrt{n} \cdot n^n e^{-n+1}.
\]

Thus,

\[
(2.3) \quad \left( \frac{n}{m} \right) < \frac{n^m}{m!} < \exp \left( m \ln \frac{n}{m} + m - \ln \sqrt{2\pi m} \right) < \exp \left( m \ln \frac{n}{m} + m \right),
\]

for positive integers \( n \) and \( m \) such that \( n \geq m \). We substitute \( s! > \sqrt{2\pi s} \cdot s^s e^{-s} \) and \((s-j)! < \sqrt{s} \cdot s^{s-j} e^{-s+j+1}\) from (2.2) into (2.1) to see:

**Lemma 2.2.** \( \Phi_2(s) > \Phi_2(s-j) \exp \left( j \ln \frac{k^s}{p_s} - \frac{j^2}{s} - 1 \right) \) for a positive integer \( j < s \).

A precise proof is also given in Appendix.
3. The Improvement Method

We will show Statements I and II by improving the original proof of the sunflower lemma in [5]. We review it in a way to introduce our proof method easily. The original proof shows by induction on $s$ that $\mathcal{F}$ includes a $k$-sunflower if $|\mathcal{F}| > \Phi_0(s)$, where

$$\Phi_0(s) \overset{\text{def}}{=} (k - 1)^s!.$$  

The claim is clearly true in the induction basis $s = 1$; the family $\mathcal{F}$ consists of more than $k - 1$ different sets of cardinality at most 1 including a desired sunflower.

To show the induction step, let

$$B = \{B_1, B_2, \ldots, B_r\}$$

be a sub-family of $\mathcal{F}$ consisting of pairwise disjoint sets $B_i$, whose cardinality $r$ is maximum. We say that such $B$ is a maximal coreless sunflower in $\mathcal{F}$ for notational convenience in this paper. Also write

$$B \overset{\text{def}}{=} B_1 \cup B_2 \cup \cdots \cup B_r.$$  

We show $r \geq k$ to complete the induction step. We have

$$|\mathcal{F}_{\sup}(S)| \leq \Phi_0(s - |S|)$$  

for $\mathcal{F}_{\sup}(S)$, the sub-family of $\mathcal{F}$ consisting of the sets including $S$ as defined in Section 2. Otherwise a $k$-sunflower exists in $\mathcal{F}$ by induction hypothesis. Then the sub-family $\mathcal{F}(B)$ consisting of the sets intersecting with $B$ meets $|\mathcal{F}(B)| \leq |B|\Phi_0(s - 1)$; because for every element $v$ in $B$, the cardinality of $\mathcal{F}(\{v\}) = \mathcal{F}_{\sup}(\{v\})$ is bounded by $\Phi_0(s - 1)$. Also $\Phi_0(s - 1) = \frac{\Phi_0(s)}{ks}$. Thus,

$$|\mathcal{F}(B)| \leq |B|\Phi_0(s - 1) \leq rs\Phi_0(s - 1)$$

$$= \frac{r\Phi_0(s)}{ks} = \frac{r}{k}\Phi_0(s) < \frac{|\mathcal{F}|}{k}.$$  

Now $|\mathcal{F}(B)|$ is less than $|\mathcal{F}|$ unless $r \geq k$. In other words, if $r < k$, then $\mathcal{F}$ would have a set disjoint with any $B_i \in \mathcal{B}$, contradicting the maximality of $r = |\mathcal{B}|$. Hence $r \geq k$, meaning $\mathcal{B}$ includes a $k$-sunflower with an empty core. This proves the induction step.

To improve this argument, we note that the proof works even if $\mathcal{F}_{\sup}(\{v\})$ for all $v \in B$ are disjoint. If so, each $\mathcal{F}_{\sup}(\{v\})$ includes elements $U \in \mathcal{F}$ only intersecting with $\{v\}$, and disjoint with $B - \{v\}$. Let $v \in B_i \in \mathcal{B}$. If we replace $B_i$ by any set $U$ in $\mathcal{F}_{\sup}(\{v\})$ such that $U \cap (B - \{v\}) = \emptyset$, then $\mathcal{B}$ is still a maximal coreless sunflower in $\mathcal{F}$.

On the other hand, if there are sufficiently many $U \in \mathcal{F}_{\sup}(\{v\})$ disjoint with $B - \{v\}$, we can find $U$ among them such that $|U \cap B_i|$ is much smaller than $s$. (Here we assume both $k$ are $s$ are large enough.) This gives us the following contradiction: Due to the maximality of $r = |\mathcal{B}|$, the family $\mathcal{F}(U)$ must contain $\mathcal{F}(B_i) - \mathcal{F}(B - B_i)$. So $\mathcal{F}(U) \cap \mathcal{F}(B_i)$ includes most sets in $\mathcal{F}_{\sup}(\{v\})$ being a not
too small family. But by (3.1), $|\mathcal{F}(U) \cap \mathcal{F}(B_i)|$ is upper-bounded by
\[
s^2\Phi_0(s-2) + |U \cap B_i|\Phi_0(s-1) = s^2 \cdot \frac{\Phi_0(s)}{k^2 s(s-1)} + |U \cap B_i| \frac{\Phi_0(s)}{ks} < \frac{1}{k^2 (1 - \frac{1}{s})} + \frac{|U \cap B_i|}{ks}.
\]
As $|U \cap B_i|$ is much smaller than $s$, the cardinality $|\mathcal{F}(U) \cap \mathcal{F}(B_i)|$ is also small, i.e. bounded by $|\mathcal{F}|$ times $O \left( \frac{1}{k^2} + \frac{|U \cap B_i|}{ks} \right)$. This contradiction on $|\mathcal{F}(U) \cap \mathcal{F}(B_i)|$ essentially means that if $r$ is around $k$, we can construct a larger coreless sunflower in $\mathcal{F}$. Hence $r$ must be more than $k$ with the cardinality lower bound $\Phi_0(s)$.

Our proof of Theorem 1.2 in the next section generalizes the above observation. By finding $B_i \in \mathcal{B}$ with sufficiently large $|\mathcal{F}(B_i) - \mathcal{B}(B - B_i)|$, we will show Statement I that $\mathcal{F}$ such that $|\mathcal{F}| \geq (\sqrt{10} - 2)^2 \left( \frac{k}{\sqrt{10} - 2} \right)^s s!$ includes a $k$-sunflower.

We will further extend this argument to show Statement II. Instead of finding just one such $B_i \in \mathcal{B}$, we will find $\mathcal{B}' \subset \mathcal{B}$ such that a sub-family $\mathcal{H}$ of
\[
\mathcal{F} \left( \bigcup_{S \in \mathcal{B}'} S \right) - \mathcal{F} \left( \bigcup_{S' \in \mathcal{B} - \mathcal{B}'} S' \right)
\]
is sufficiently large. Then we show a maximal coreless sunflower in $\mathcal{H}$ whose cardinality is larger than $|\mathcal{B}'|$. This again contradicts the maximality of $r = |\mathcal{B}|$ to prove the second statement.

4. PROOF OF THEOREM 1.2

4.1. Statement I. We prove Statement I in this section. Put
\[
\delta = \sqrt{10} - 3 = 0.16227 \ldots, \quad x = \frac{k}{1 + \delta} = \frac{k}{\sqrt{10} - 2},
\]
then
\[
\Phi_1(s) = \left( \sqrt{10} - 2 \right)^2 \left( \frac{k}{\sqrt{10} - 2} \right)^s s! = (1 + \delta)^2 x^s s!,
\]
as defined in Section 2. We show that $|\mathcal{F}| \geq \Phi_1(s)$ means a $(1 + \delta)x$-sunflower included in $\mathcal{F}$.

We prove it by induction on $s$. Its basis occurs when $s \leq 2$. The claim is true by the sunflower lemma, since $\Phi_1(s) = (1 + \delta)^2 x^s s! \geq (\alpha(s + 1) + \delta) x^s s! \geq k^s s! > (k - 1)^2 s!$ if $s \leq 2$. Assume true for $1, 2, \ldots, s - 1$ and prove true for $s \geq 3$. As in Section 3, let $\mathcal{B} = \{ B_1, B_2, \ldots, B_r \}$ be a maximal coreless sunflower of cardinality $r$ in $\mathcal{F}$, and $B = B_1 \cup B_2 \cup \cdots \cup B_r$. Contrarily to the claim, let us assume
\[
r < (1 + \delta)x.
\]
We will find a contradiction caused by (4.1).

Observe the following facts.

- For any nonempty set $S \subset X$ such that $|S| < s$, if $|\mathcal{F}_{\text{sup}}(S)| \geq \Phi_1(s - |S|)$, the family $\mathcal{F}_{\text{sup}}(S)$ contains a $(1 + \delta)x$-sunflower by induction hypothesis. Thus we assume
\[
|\mathcal{F}_{\text{sup}}(S)| < \Phi_1(s - |S|) \quad \text{for } S \subset X \text{ such that } 1 \leq |S| < s.
\]
• We also have \( k \geq 3 \), because \( \Phi_1(s) > 0 \) for \( k = 1 \), and
  \[
  \Phi_2(s) = \left( \frac{\sqrt{10} - 2}{\sqrt{10} - 2} \right)^s > s! = (k - 1)s! \text{ if } k = 2.
  \]
• Since \( r \) has the maximum value,
  \[
  (4.3) \quad F = F(B),
  \]
i.e., every set in \( F \) intersects with \( B \).
• \( \mathcal{P}(B) \), defined in Section 2 as the family of pairs \((v, U)\) such that \( U \in F \) and \( v \in U \cap B \), has a cardinality bounded by
  \[
  (4.4) \quad |\mathcal{P}(B)| \leq |B| \Phi_1(s - 1) \leq rs \Phi_1(s - 1) < (1 + \delta) x s \cdot \frac{\Phi_1(s)}{x s} \leq (1 + \delta)|F|,
  \]
due to (4.1) and (4.2). Here \( |\mathcal{P}(B)| \leq |B| \Phi_1(s - 1) \) because for each \( v \in B \), there are at most \( \Phi_1(s - 1) \) pairs \((v, U) \in \mathcal{P}(B)\).

We first see that many \( U \in F \) intersect with \( B \) by cardinality 1, i.e., \( |F_1(B)| \) is sufficiently large. Observe two lemmas.

**Lemma 4.1.** \( |F_1(B)| > (1 - \delta)|F| \).

*Proof.* Let \( |F_1(B)| = (1 - \delta')|F| \) for some \( \delta' \in [0, 1] \). By (4.3), there are \( \delta'|F| \) elements \( U \in F \) such that \( |U \cap B| \geq 2 \), each of which creates two or more pairs in \( \mathcal{P}(B) \). If \( \delta' \geq \delta \),

\[
|\mathcal{P}(B)| \geq (1 - \delta')|F| + 2 \delta'|F| = (1 + \delta')|F| \geq (1 + \delta)|F|,
\]
contradicting (4.3). Thus \( \delta' < \delta \) proving the lemma. \( \square \)

**Lemma 4.2.** There exists \( B_i \in \{B_1, B_2, \ldots, B_r\} \) such that \( |F_1(B_i) - F(B - B_i)| \geq \frac{1 - \delta}{1 + \delta} \cdot \frac{\Phi_1(s)}{x} \).

*Proof.* By Lemma 4.1 there exists \( B_i \in \{B_1, B_2, \ldots, B_r\} \) such that the number of \( U \in F \) intersecting with \( B_i \) by cardinality 1, and disjoint with \( B - B_i \), is at least

\[
\frac{(1 - \delta)|F|}{r} > \frac{1 - \delta}{1 + \delta} \cdot \frac{\Phi_1(s)}{x}.
\]

The family of such \( U \) is exactly \( F_1(B_i) - F(B - B_i) \), so its cardinality is no less than \( \frac{1 - \delta}{1 + \delta} \cdot \frac{\Phi_1(s)}{x} \). \( \square \)

Assume such \( B_i \) is \( B_1 \) without loss of generality. Then

\[
|F_1(B_1) - F(B - B_1)| \geq |F_1(B_1) - F(B - B_1)| \geq \frac{1 - \delta}{1 + \delta} \cdot \frac{\Phi_1(s)}{x} > 0.
\]

We choose any element \( B'_1 \in F_1(B_1) - F(B - B_1) \) that is not \( B_1 \). Switch \( B_1 \) with \( B'_1 \) in \( B \). Since \( B'_1 \) is disjoint with any of \( B_2, B_3, \ldots, B_r \), the obtained family \( \{B'_1, B_2, \ldots, B_r\} \) is another maximal coreless sunflower in \( F \). We see the following inequality.
Lemma 4.3.

\[ |\mathcal{F}(B_1) \cap \mathcal{F}(B'_1)| < \frac{\Phi_1(s)}{x} \left( \frac{1}{s} + \frac{1}{x} \right) \]

Proof. A set \( U \in \mathcal{F}(B_1) \cap \mathcal{F}(B'_1) \) intersects with \( B_1 \cap B'_1 \) of cardinality 1, or both \( B_1 - B'_1 \) and \( B'_1 - B_1 \) of cardinality \( s - 1 \). By (4.12), there are at most

\[ \Phi_1(s - 1) + (s - 1)^2 \Phi_1(s - 2) = \frac{\Phi_1(s)}{sx} + (s - 1)^2 \frac{\Phi_1(s)}{s(s - 1)x^2} \]

such \( U \in \mathcal{F}(B_1) \cap \mathcal{F}(B'_1) \). The lemma follows. \( \square \)

\( \mathcal{F} = \mathcal{F}(B'_1 \cup B_2 \cup B_3 \cup \cdots \cup B_r) \) would be true if the cardinality \( r \) of the new coreless sunflower \( \{B'_1, B_2, B_3, \ldots, B_r\} \) were maximum. However, it means that every element in \( \mathcal{F}(B_1) - \mathcal{F}(B - B_1) \) is included in \( \mathcal{F}(B'_1) \). Thus, \( \mathcal{F}(B_1) \cap \mathcal{F}(B'_1) \supset \mathcal{F}(B_1) - \mathcal{F}(B - B_1) \), leading to

\[ |\mathcal{F}(B_1) \cap \mathcal{F}(B'_1)| \geq |\mathcal{F}(B_1) - \mathcal{F}(B - B_1)| \geq \frac{1 - \delta}{1 + \delta} \frac{\Phi_1(s)}{x} \geq \frac{\Phi_1(s)}{x} \left( \frac{1}{s} + \frac{1}{x} \right). \]

Its last inequality is confirmed with \( s \geq 3, k \geq 3, x = \frac{k}{1 + s} \), and \( \frac{1 - \delta}{1 + s} = \frac{1}{3} + \frac{1 + \delta}{3} \) as \( \delta = \sqrt{10} - 3 \). This contradicts Lemma 4.3, completing the proof of Statement I.

4.2. Statement II. We prove the second statement by further developing the above method. We show that \( \mathcal{F} \) includes a \( k \)-sunflower if \( |\mathcal{F}| > \Phi_2(s) \) for a choice of sufficiently small constant \( \epsilon \in (0, 1/8) \). This suffices to prove Statement II thanks to Lemma 2.1. The proof is by induction on \( s \). Its basis occurs when \( \min(k, s) \) is smaller than a sufficiently large constant \( c_1 \), i.e., when \( \min(k, s) \) is smaller than a lower bound \( c_1 \) on \( \min(k, s) \) required by the proof below. To meet this case, we choose \( \epsilon \in (0, 1/8) \) to be smaller than \( 1/\ln c_1 \). Then \( \Phi_2(s) \geq k^s! > (k - 1)^s! \) by the definition of \( \Phi_2 \) in Section 2. The family \( \mathcal{F} \) thus includes a \( k \)-sunflower-by the sunflower lemma in the basis.

Assume true for \( 1, 2, \ldots, s - 1 \) and prove true for \( s \). The two integers \( k \) and \( s \) satisfy

\[ \min(k, s) \geq c_1, \]

i.e., they are sufficiently large. Put

\[ p \overset{\text{def}}{=} \frac{1}{8} \ln \min(k, s), \quad \text{and} \quad x \overset{\text{def}}{=} \frac{k}{p}, \]

\( p \) is sufficiently large since both \( k \) and \( s \) are.

Note. The lower bound \( c_1 \) on \( \min(s, k) \) is required in order to satisfy (4.8), (4.10), (4.11), and (4.12) below, which are inequalities with fixed coefficients and no \( \epsilon \). We choose \( c_1 \) as the minimum positive integer such that \( \min(k, s) \geq c_1 \) satisfies the inequalities, and also \( \epsilon \) as \( \min \left( \frac{1}{2m c_1}, \frac{1}{9} \right) \).

As \( p \overset{\text{def}}{=} \epsilon \ln \min(k, s) < p \) in (2.1),

\[ \Phi_2(s - 1) = \frac{p}{k} \Phi_2(s) < \frac{p}{k} \Phi_2(s) = \frac{\Phi_2(s)}{x s}. \]
Lemma 4.5. There exists a nonempty sub-family $\mathcal{B}$ of $\mathcal{B}$ such that $|\mathcal{F}| < \Phi_2(s - |S|)$ for any $S \subset X$ with $1 \leq |S| < s$, as induction hypothesis. We also keep denoting a maximal coreless sunflower in $\mathcal{F}$ by $\mathcal{B} = \{B_1, B_2, \ldots, B_r\}$, and $B_1 \cup B_2 \cup \cdots \cup B_r$ by $\mathcal{B}$. Also (4.3) holds due to the maximality of $r = |\mathcal{B}|$.

We prove $r \geq k$ for the induction step. Suppose contrarily that
\begin{equation}
(4.7)\quad x \leq r < k,
\end{equation}
and we will find a contradiction. Here $r \geq x$ is confirmed similarly to (5.2) in Section 5, i.e., by
\[|\mathcal{F}(B)| \leq |B|\Phi_2(s - 1) \leq rs \Phi_2(s - 1) \leq \frac{r}{x} |\mathcal{F}|,
\]
with (4.4), so $r < x$ would contradict the maximality of $r = |\mathcal{B}|$.

We start our proof by showing a claim seen similarly to Lemma 4.1.

Lemma 4.4. There exists a positive integer $j \leq 2p$ such that $|\mathcal{F}_j(B)| \geq \frac{2p(s)}{4p}$.

Proof. We first show $\left|\sum_{0 \leq j \leq 2p} \mathcal{F}_j(B)\right| \geq \frac{1}{2} |\mathcal{F}|$. If not, there would be at least $\frac{1}{2} |\mathcal{F}|$ sets $U \in \mathcal{F}$ such that $|U \cap B| > 2p$ by the definition of $\mathcal{F}_j$ given in Section 2. Each such $U$ creates at least $[2p]$ pairs $(v, U) \in \mathcal{P}(B)$, so
\[|\mathcal{P}(B)| \geq 2p \cdot \frac{|\mathcal{F}|}{2} = p|\mathcal{F}|.
\]
However, similarly to (4.4),
\[|\mathcal{P}(B)| \leq |B|\Phi_2(s - 1) \leq rs \Phi_2(s - 1) \leq ks \cdot \frac{p \Phi_2(s)}{ks} \leq p|\mathcal{F}|,
\]
by induction hypothesis and (4.6). By the contradictory two inequalities, $\left|\sum_{0 \leq j \leq 2p} \mathcal{F}_j(B)\right| < \frac{1}{2} |\mathcal{F}|$ is false. Thus $\left|\sum_{0 \leq j \leq 2p} \mathcal{F}_j(B)\right| \geq \frac{1}{2} |\mathcal{F}|$.

Let $j$ be an integer in $[1, 2p]$ with the maximum cardinality of $\mathcal{F}_j(B)$. By the above claim and (4.3), $|\mathcal{F}_j(B)| \geq \frac{|\mathcal{F}|}{2p} \cdot \frac{1}{4p} \geq \frac{2p(s)}{4p}$, proving the lemma. \qed

Fix this integer $j \in [1, 2p]$. Next we find a small sub-family $\mathcal{B}'$ of $\mathcal{B}$ such that (3.3) is large enough. For each non-empty $\mathcal{B}' \subset \mathcal{B}$, define
\[\mathcal{G}(\mathcal{B}') \overset{\text{def}}{=} \{U : U \in \mathcal{F}_j(B), \forall S \in \mathcal{B}', U \cap S \neq \emptyset \text{ and } \forall S' \in \mathcal{B} - \mathcal{B}', U \cap S' = \emptyset\}.
\]
Observe a lemma regarding $\mathcal{G}(\mathcal{B}')$.

Lemma 4.5. There exists a nonempty sub-family $\mathcal{B}' \subset \mathcal{B}$ of cardinality at most $j$ such that $|\mathcal{G}(\mathcal{B}')| \geq \frac{2p(s)}{8p(j)}$.

Proof. By definition, the cardinality of $\mathcal{B}'$ such that $\mathcal{G}(\mathcal{B}') \neq \emptyset$ does not exceed $j$. Thus there are at most
\[\binom{r}{j} + \binom{r}{j - 1} + \binom{r}{j - 2} + \cdots + \binom{r}{1}\]
\[< \binom{r}{j} \left(1 + \frac{j}{r - j + 1} + \left(\frac{j}{r - j + 1}\right)^2 + \left(\frac{j}{r - j + 1}\right)^3 + \cdots\right) \leq 2 \binom{r}{j},\]
such possible $\mathcal{B}' \subset \mathcal{B}$. Here its truth is confirmed by the following arguments.
\[ \binom{n}{m} = \frac{m}{n-m+1} \binom{n}{m} \] for any \( n, m \in \mathbb{Z}^+ \) such that \( m \leq n \). So \( \binom{j}{r} = \frac{1}{r-j+1} \binom{r}{j} \), \( \binom{j-2}{r-j+2} \binom{r}{j-1} < \left( \frac{j}{r-j+1} \right)^2 \binom{r}{j} \), \( \binom{j-3}{r-j+3} \binom{r}{j-2} < \left( \frac{j}{r-j+1} \right)^3 \binom{r}{j} \), \ldots.

- The last inequality is due to \( r \geq x = \frac{k}{p} = \frac{8k}{\ln \min(k, s)} > k \ln k \) by (4.7), and \( j \leq 2p < \ln k \).

Thus
\[ \frac{j}{r-j+1} < \frac{\ln k}{\ln k - \ln k + 1} < \frac{1}{2}, \]
by (4.8) where \( k \) is sufficiently large. So the last inequality holds in the above.

Then by Lemma 4.5, there exists at least one nonempty \( B' \subset B \) such that
\[ |G(B')| > \frac{|F_j(B)|}{2^j} \geq \frac{\Phi_2(s)}{2^j} \cdot 4p = \frac{\Phi_2(s)}{8p^j}. \]
The lemma follows.

We now construct a sub-family \( H \) of (3.3) in which we will find a larger maximal coreless sunflower. Fix a sub-family \( B' \subset B \) decided by Lemma 4.5. Put
\[ B' \overset{\text{def}}{=} \bigcup_{S \in B'} S, \]
\[ r' \overset{\text{def}}{=} |B'| \leq j \leq 2p, \]
and
\[ H \overset{\text{def}}{=} F_j(B') - F(B - B') = F_j \left( \bigcup_{S \in B'} S \right) - F \left( \bigcup_{S' \in B - B'} S' \right). \]
The family \( H \) includes \( G(B') \) by definition, so
\[ |H| \geq \frac{\Phi_2(s)}{8p^j}. \]
by Lemma 4.5. If we find a maximal coreless sunflower in \( H \) whose cardinality is larger than \( r' = |B'| \), it means the existence of a coreless sunflower in \( F \) with cardinality larger than \( r \), since any \( U \in H \) is disjoint with \( B - B' \).

Extending the notation \( F(S) \), write
\[ H(S) \overset{\text{def}}{=} \{ U : U \in H \text{ and } U \cap S \neq \emptyset \}, \]
for a nonempty set \( S \subset X \). Then \( H(\{v\}) \) for an element \( v \in X \) is the family of \( U \in H \subset F \) containing \( v \).

Let us show two lemmas on \( H \) and \( H(\{v\}) \). By them we will see that the latter is sufficiently smaller than the former.

**Lemma 4.6.** \(|H| > s^j \Phi_2(s - j) \exp(-4p)\).

**Proof.** We have two facts on (4.9).

- The natural logarithm of the denominator \( 8p^j \) is upper-bounded by
\[ \ln 8p^j < j \ln \frac{r}{j} + j + \ln 8p < j \ln \frac{k}{p} + j + \ln 8p = j \ln \frac{k}{p} + j + \ln 8p, \]
due to (2.3), (4.7) and \( x = \frac{k}{p} \).
Let $d = \ln \Phi_2(s) - \ln \Phi_2(s - j)$, or $\Phi_2(s) = \Phi_2(s - j) \exp(d)$. It satisfies
\[
d > j \ln \frac{ks}{p} - \frac{j^2}{s} - 1 > j \ln \frac{ks}{p} - \frac{j^2}{s} - 1 = j \ln s - \frac{j^2}{s} - 1 > j \ln s - j - 1,
\]
by Lemma 2.2, $p_s < p$, and $j \leq s$.

Then,
\[
|\mathcal{H}| \geq \frac{\Phi_2(s)}{8p(s)} > \Phi_2(s - j) \exp \left( j \ln s - j - 1 - \left( j \ln \frac{2p}{j} + j + \ln 8p \right) \right)
\]
\[
= s^j \Phi_2(s - j) \exp \left( -j \ln \frac{2p}{j} - 2j - \ln 8p - 1 \right).
\]

Find $\max_{1 \leq j \leq 2p} \left( j \ln \frac{2p}{j} + 2j \right)$ regarding $j$ as a real parameter. The maximum value $(4 - 2 \ln 2)p$ is achieved when $j = 2p$. Since $p$ is sufficiently large by (4.5),
\[
2p \ln 2 > 1 + \ln 8p.
\]

Hence,
\[
|\mathcal{H}| > s^j \exp \left( -j \ln \frac{2p}{j} - 2j - \ln 8p - 1 \right) \geq s^j \exp \left( -(4 - 2 \ln 2)p - \ln 8p - 1 \right)
\]
\[
= s^j \Phi_2(s - j) \exp \left( -4p \right),
\]
completing the proof. \hfill \Box

**Lemma 4.7.** The following two statements hold true.

i) $|\mathcal{H}(\{v\})| \leq \frac{1}{s} \cdot e^{7p} \cdot |\mathcal{H}|$ for every $v \in B'$.

ii) $|\mathcal{H}(\{v\})| \leq \frac{1}{(s-j)p} \cdot e^{7p} \cdot |\mathcal{H}|$ for every $v \in X - B'$.

**Proof.** i): Let $U$ be any set in $\mathcal{H}(\{v\})$ for the given $v \in B'$, and write $U' = U \cap B'$. Since $U'$ has cardinality $j$ containing $v$, there are no more than
\[
\binom{|B'| - 1}{j - 1} \leq \binom{r's - 1}{j - 1} = \frac{j}{r's} \binom{r's}{j} \leq \frac{j}{s} \binom{r's}{j}
\]
choices of $U'$. Here the identity $\binom{r's}{j} = \frac{j}{s} \binom{r's}{j}$ is used. By the induction hypothesis on $s$, the number of $U \in \mathcal{H}(\{v\})$ is upper-bounded by
\[
|\mathcal{H}(\{v\})| \leq \frac{j}{s} \binom{r's}{j} \cdot \Phi_2(s - j) \leq \frac{\Phi_2(s - j)}{s} \exp \left( j \ln \frac{r's}{j} + j \right)
\]
\[
\leq s^j \Phi_2(s - j) \exp \left( \frac{j \ln \frac{r's}{j} + j + \ln j}{s} \right) \leq s^j \Phi_2(s - j) \exp \left( \frac{j + \ln j}{s} \right)
\]
\[
\leq s^j \Phi_2(s - j) \exp \left( \frac{2p + \ln 2p}{s} \right) < s^j \Phi_2(s - j) \exp \left( \frac{3p}{s} \right),
\]
where $\binom{r's}{j} \leq \exp \left( j \ln \frac{r's}{j} \right)$ is due to (4.8), and
\[
2p + \ln 2p < 3p,
\]
due to (4.5).

With the previous lemma, we see that the ratio $|\mathcal{H}(\{v\})|/|\mathcal{H}|$ does not exceed
\[
\frac{s^j \Phi_2(s - j) \exp \left( \frac{3p}{s} \right) / s}{s^j \Phi_2(s - j) \exp \left( -4p \right)} = \frac{e^{7p}}{s},
\]
proving i).

ii): As \( v \in X - B' \), the number of choices of \( U' = U \cap B' \) is now \( \binom{|B'|}{r'} \leq \binom{s - j}{r'} \).

For each such \( U' \), the number of choices of \( U' \cap B' \) is at most

\[
\Phi_2(s - j - 1) = \frac{p_{s-j}}{k(s-j)} \Phi_2(s - j) < \frac{p}{k(s-j)} \Phi_2(s - j),
\]

by induction hypothesis and \( p_{s-j} \leq p_s < p \). Hence \( |\mathcal{H}(\{v\})| \) is upper-bounded by \( \Phi_2(s - j) \binom{s}{r'} \). Then argue similarly to i). \( \square \)

Lemma 4.7 means that \( \mathcal{H} \) includes a coreless sunflower of cardinality more than \( r' = |B'| \). Let us formally prove it with the following lemma.

**Lemma 4.8.** \( |B'| \cdot |\mathcal{H}(V)| \leq \frac{1}{2} |\mathcal{H}| \) for every \( V \in \mathcal{H} \).

**Proof.** Let

\[
y \overset{\text{def}}{=} \min(s, k), \text{ so that } p = \frac{1}{8} \ln y, y = e^{8p}, s \geq y, \text{ and } x = k \frac{p}{y} = \frac{y}{p}.
\]

We have

\[
(4.12) \quad p \geq 1 \text{ and } 8p^3 e^{-p} < \frac{1}{2},
\]

due to (4.5).

Fix each \( V \in \mathcal{H} \). By Lemma 4.7, the family \( \mathcal{H}(V) \) has a cardinality bounded by

\[
|\mathcal{H}(V)| = \bigcup_{v \in V} |\mathcal{H}(\{v\})| \leq \left( j \cdot \frac{e^{7p}}{s} + (s-j) \frac{e^{7p}}{(s-j)x} \right) |\mathcal{H}|
\]

\[
\leq \frac{je^{7p}}{s} \left( \frac{1}{s} + \frac{1}{x} \right) |\mathcal{H}| \leq \frac{je^{7p}}{y} \left( \frac{1}{y} + \frac{p}{y} \right) |\mathcal{H}| \leq j e^{7p} \cdot \frac{2p}{y} \cdot |\mathcal{H}|
\]

\[
\leq 4p^2 e^{7p} \cdot \frac{|\mathcal{H}|}{y} = 4p^2 e^{7p} \cdot \frac{|\mathcal{H}|}{e^{8p}} = 4p^2 e^{-p} \cdot |\mathcal{H}|.
\]

Therefore, by \( |B'| = r' \leq j \leq 2p \),

\[
|B'| \cdot |\mathcal{H}(V)| \leq 2p |\mathcal{H}(V)| \leq 8p^3 e^{-p} \cdot |\mathcal{H}| < \frac{1}{2} |\mathcal{H}|.
\]

The lemma follows. \( \square \)

Hence each \( V \in \mathcal{H} \) intersects with at most \( \frac{|\mathcal{H}|}{2|B'|} \) sets in \( \mathcal{H} \). There exist more than \( |B'| \) pairwise disjoint sets in \( \mathcal{H} \). By definition, every element in \( \mathcal{H} \) is disjoint with a set in \( B - B' \). The cardinality \( r \) of the coreless sunflower \( B \) in \( \mathcal{F} \) is therefore not maximum. This contradiction proves Statement II, completing the proof of Theorem 1.2.

**Appendix: Proofs of Lemmas 2.1 and 2.2**

**Lemma 2.1.** There exists a constant \( c > 0 \) such that \( \Phi_2(s) \leq \left( \frac{ck}{\ln \min(k, s)} \right)^s s! \) for any \( s \geq 2 \) and \( k \geq 2 \).

**Proof.** We first show the lemma when \( s \leq k \). By the definition of \( \Phi_2(s) \) in Section 2,

\[
\Phi_2(s) = \frac{1}{\ln 2 \cdot \ln 3 \cdots \ln s} \left( \frac{k}{s} \right)^s s!.
\]
We assume $s \geq 3$ since the claim is trivially true for $s = 2$. It suffices to show
\[
\frac{1}{\ln 2 \ln 3 \cdots \ln s} \left( \frac{k}{e} \right)^s s! \leq \left( \frac{\ln s}{e^2} \right)^s (\ln \frac{k}{e})^s s!,
\]
or
\[
\ln 2 \cdot \ln 3 \cdots \ln s \geq \left( \frac{\ln s}{e^2} \right)^s
\]
\[
\Leftrightarrow \ln 2 + \ln 3 + \cdots + \ln \ln s \geq s \ln \ln s - 2s.
\]

\[
\ln \ln x
\]
is a smooth, monotonically increasing function of $x \in \mathbb{R}^+$, so
\[
\ln \ln 3 + \ln \ln 4 + \cdots + \ln \ln s > \int_{2}^{s} \ln \ln x \, dx
\]
\[
= (s \ln s - li(s)) - (2 \ln 2 - li(2)).
\]
Here $li(s)$ is the logarithmic integral $\int_{0}^{s} \frac{dx}{\ln x}$. As $\ln 2 < 0$,
\[
\ln 2 + \ln 3 + \cdots + \ln \ln s > s \ln s - li(s) + li(2),
\]
where $li(s) - li(2) = \int_{2}^{s} \frac{dx}{\ln x}$ is upper-bounded by $(s - 2) / \ln 2 < 2s$. Hence, $\ln 2 + \ln 3 + \cdots + \ln \ln s \geq s \ln s - 2s$, proving the lemma when $s \leq k$.

If $s > k$, the lemma is also proved by (4.13):
\[
\Phi_2(s) = \frac{1}{\ln 2 \ln 3 \cdots \ln k \cdot (\ln k)^{s-k}} \left( \frac{k}{e} \right)^s s! \leq \frac{1}{(\ln k)^k \cdot (\ln k)^{s-k}} \left( \frac{k}{e} \right)^s s! < \left( \frac{e^2 k}{\epsilon \ln k} \right)^s s!.
\]

This completes the proof. \hfill \Box

**Lemma 2.2.** \( \Phi_2(s) > \Phi_2(s - j) \exp \left( j \ln \frac{k}{p_r} - \frac{j^2}{s} - 1 \right) \) for a positive integer $j < s$.

**Proof.** It suffices to show \( \frac{\left( \frac{k}{p_r} \right)^j s!}{(s-j)!} \geq \exp \left( j \ln \frac{k}{p_r} - \frac{j^2}{s} - 1 \right) \) due to (2.1), or
\[
\frac{s!}{(s-j)!} > \exp \left( j \ln s - \frac{j^2}{s} - 1 \right).
\]
By (2.2), \( \ln \frac{s!}{(s-j)!} \) is at least
\[
\ln \left( s \ln s + \ln \sqrt{2\pi s} \right) - \left( (s-j) \ln (s-j) - (s-j) + \ln \sqrt{s-j} + 1 \right)
\]
\[
= s \ln s - (s-j) \ln (s-j) - j + \ln \frac{\sqrt{2\pi s}}{\sqrt{s-j} - 1}
\]
\[
> s \ln s - (s-j) \left( \ln s + \ln \left( 1 - \frac{j}{s} \right) \right) - j - 1
\]
\[
= j \ln s - (s-j) \ln \left( 1 - \frac{j}{s} \right) - j - 1.
\]
By the Taylor series of natural logarithm, $-\ln \left( 1 - \frac{j}{s} \right) = \sum_{i=1}^{\infty} \frac{1}{i} \left( \frac{j}{s} \right)^i > \frac{j}{s}$. From these,
\[
\ln \frac{s!}{(s-j)!} > j \ln s + (s-j) \frac{j}{s} - j - 1 \geq j \ln s - \frac{j^2}{s} - 1,
\]
which is equivalent to the desired inequality to prove the lemma. \hfill \Box
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