Optimal Reconstruction of Inviscid Vortices

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Abstract

This study addresses the question whether, given some measurements of the velocity field induced by a vortex, one can stably determine the structure of the vortex. Assuming that the flow is incompressible, inviscid and stationary in the frame of reference moving with the vortex, the “structure” of the vortex is uniquely characterized by the functional relation between the streamfunction and vorticity. It is demonstrated how the inverse problem of reconstructing this functional relation from data can be framed as an optimization problem which can be efficiently solved using variational techniques. To focus attention, we consider 3D axisymmetric vortex rings and use measurements of the tangential velocity on the boundary of the vortex bubble. In contrast to earlier studies, the vorticity function defining the streamfunction-vorticity relation is reconstructed in the continuous setting subject to a minimum number of assumptions. To validate our approach, two test cases are presented, involving Hill’s and Norbury-Fraenkel’s vortices, in which good reconstructions are obtained. A key result of this study is the application of our approach to obtain an optimal inviscid vortex model in an actual viscous flow problem based on DNS data.
Keywords: vortex dynamics; vortex rings; inverse problems; variational methods

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1 Introduction

In this study we are concerned with an inverse problem in vortex dynamics where, given some information about the induced velocity field, we seek to recreate the structure of the vortex. As an example, we will consider incompressible axisymmetric flows with vortex rings which is motivated by their particular importance in applications ranging from biological propulsion [1, 2] to the fuel injection in internal combustion engines.
For convenience, in the following we use the cylindrical coordinates \((z, r, \theta)\) with \(z\) the longitudinal (propagation) direction of the flow. We denote the velocity field by \(\mathbf{v} = (v_z, v_r, 0)^T\) and by \(\omega = \nabla \times \mathbf{v} = (0, 0, \omega_\theta)^T\) the corresponding vorticity. Thus, given our assumption that the flow is axisymmetric, the only nonzero vorticity component is the azimuthal one, denoted by \(\omega_\theta := \omega \theta\).

A vortex ring is then defined as the axisymmetric region \(\Omega_b\) of \(\mathbb{R}^3\) such that \(\omega \neq 0\) in \(\Omega_b\) and \(\omega = 0\) elsewhere (see Figure 1). The domain \(\Omega_b\), also called vortex bubble, is delimited by the streamline corresponding to \(\psi = 0\), where \(\psi(r, z)\) is the Stokes streamfunction in the frame of reference moving with the vortex ring.

![Figure 1: Direct numerical simulation of the incompressible Navier-Stokes equations representing a physical vortex ring with axial symmetry [5]. Velocity vectors (a) and corresponding streamlines (b) in the frame of reference traveling with the vortex ring.](image)

Classical vortex ring models are stationary solutions of Euler equations. The key feature of such models is that the vorticity transport equation reduces (e.g. [6, 7]) to

\[
\frac{\omega}{r} = \begin{cases} 
  f(\psi) & \text{in } \Omega_b, \\
  0 & \text{elsewhere,}
\end{cases}
\]

with \(f : \mathbb{R} \to \mathbb{R}\) called vorticity function. In other words, \(\omega\) propagates with a time-invariant profile \(f(\psi)\) along streamlines in the frame of reference moving with the vortex. As described in detail in the next section, the associated mathematical problem consists in solving an elliptic partial differential equation (PDE) for \(\psi\) with a right-hand-side term depending on the solution itself. The main difficulty in solving this PDE comes from the fact that the boundary \(\partial \Omega_b\) of the vortex bubble is not known in advance, which makes it a free boundary problem.
The only known analytical solution of this problem considers \( f(\psi) = \text{const} \) within \( \Omega_b \) which has the shape of a sphere. This solution is known as Hill’s spherical vortex \([8]\) (see also \([6, 7]\)). The mathematical theory of inviscid axisymmetric vortex rings was developed in the ’70s and in the early ’80s \([9]\) around Hill’s vortex, by considering the following particular form of the vorticity function

\[
f(\psi) = \text{const}, \quad \forall \psi > k, \quad \text{and} \quad f(\psi) = 0, \quad \forall \psi \leq k, \tag{2}
\]

with \( k > 0 \) defining the vortex-ring core as \( \Omega_c = \{(z, r, \theta) \in \mathbb{R}^3, \psi(r, z) > k\} \subset \Omega_b \) (see Figure 1 b). Existence and uniqueness results for the inviscid vortex ring problem are presented in \([9, 10, 11]\) for the general case and in \([12, 13]\) for vortex rings bifurcating from Hill’s vortex. Numerical solutions of the vortex-ring problem, using (2) as the vorticity function, where presented by Norbury \([14]\) in a tabulated form allowing to reconstruct numerically the boundaries \( \partial \Omega_c \) and \( \partial \Omega_b \) of, respectively, the vortex core and vortex bubble. The obtained Norbury-Fraenkel family of vortex rings \([15, 14]\) (hereafter referred to as NF vortices) became quite popular in the physics literature, because vortex rings could be identified based on a single geometric parameter. These models were also extended to allow for swirling vortex rings (with nonzero azimuthal velocity); analytical closed-form solutions for swirling Hill’s spherical vortex were obtained in \([16]\) and numerical solutions for vorticity functions generalizing (2) for swirling flows were presented in \([17]\).

From the practical point of view, vortex ring models are useful as approximations to actual vortex structures observed in experiments or generated by the Direct Numerical Simulation (DNS) of the Navier-Stokes equations. For the purposes of fitting such models to DNS data, the NF inviscid vortex-ring model \([15, 14]\) was widely adopted and proved very useful in estimating integral quantities and global properties of actual vortex rings \([18, 19, 20]\). This is quite remarkable, since the vorticity function (2) gives a linear vorticity distribution in the vortex core, i.e. proportional to the distance from the axis of symmetry, which is in fact quite different from the Gaussian vorticity distribution typically observed in experiments (e.g. \([21, 22]\)).

The main feature of the NF vortex ring model is that the vorticity function \( f(\psi) \) is prescribed by (2) as a hypothesis of the model. While experimental studies \([23, 24]\) reported some scatter in the plots of \( \omega/r \) versus \( \psi \), this data was rather well fitted by an empirical formula for the vorticity function in the exponential form \( f(\psi) = a \exp(b\psi) \) with \( a \) and \( b \) representing two constants adjusted during the fitting procedure \([24]\). This supports the idea that steady inviscid models could be used as good approximations of unsteady viscous vortex rings arising in real flows if the vorticity function \( f(\psi) \) is accurately determined.

In the present contribution we develop a computational algorithm that provides an optimal numerical approximation of the vorticity function \( f(\psi) \) of the vortex ring, start-
ing from some incomplete and possibly noisy measurements of the induced velocity field (obtained via experiments or DNS). We consider here measurements of the velocity field on \( \partial \Omega_b \), the boundary of the vortex bubble, which opens the possibility for using the present approach in practical applications. In other words, we will formulate and solve an inverse problem for identifying the structure of the vorticity distribution in the vortex bubble. This will be done by framing this inverse problem as a suitable optimization problem. The novelty of the present study is that the vorticity function \( f(\psi) \) is reconstructed in a very general form with no assumptions other than smoothness and the behaviour at the endpoints of its domain of definition. This is fundamentally different from classical approaches (e. g. [25]) reducing the reconstruction problem to fitting a small number of variables parameterizing the vorticity distribution of a given vortex-ring model (for instance, the NF model with (2)). From the computational point of view, the key enabler of our approach is a technique for the reconstruction of constitutive relations developed recently in [26, 27].

The structure of the paper is as follows: in the next two sections we introduce the equations satisfied by the steady inviscid vortex rings and formulate the reconstruction problem in terms of an optimization approach. In Section 4 we propose a gradient-based solution method and derive the gradient formula. The computational algorithm is described in Section 5, together with the tests validating the method used for the computation of the gradients. The proposed method is first validated against known analytical (Hill's vortex) or numerical (Norbury-Fraenkel) vortex ring solutions in Section 6. The approach is then applied in Section 7 to a challenging problem consisting in the reconstruction of the vorticity function from realistic DNS data. Discussion and final conclusion are deferred to Section 8.

2 Physical Problem and Governing Equations

In this section we present the equations satisfied by our vortex model. We consider incompressible axisymmetric vortex rings without swirl. If a stationary solution is sought, it is more convenient to describe the flow in the frame of reference moving with the translation velocity \( W e_z \) (assumed constant) of the vortex ring (see Figure 1). A divergence-free velocity field is constructed by defining the Stokes streamfunction \( \psi \) [6, 7] such that:

\[
v_z = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad v_\theta = 0.
\]

(3)

The azimuthal component of the vorticity vector is then given by

\[
\omega = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}.
\]

(4)
Combining (3) and (4) results in an elliptic partial differential equation for the stream-function

\[ \mathcal{L} \psi = -\omega \quad \text{in} \quad \Pi = \{ (r, z) \in \mathbb{R}^2, r > 0 \}, \]  

(5)

where \( \mathcal{L} \) is a self-adjoint elliptic operator

\[ \mathcal{L} := \frac{\partial}{\partial z} \left( \frac{1}{r} \frac{\partial}{\partial z} \right) + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) = \nabla \cdot \left( \frac{1}{r} \nabla \right), \quad \text{where} \quad \nabla := \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial r} \right)^T. \]  

(6)

The boundary condition required for equation (5) accounts for an external flow around the vortex which is uniform at infinity with velocity \(-W \mathbf{e}_z\) (see Figure 1)

\[ \Psi := \psi + \frac{1}{2} W r^2 \rightarrow 0 \quad \text{as} \quad |x| := \sqrt{z^2 + r^2} \rightarrow \infty. \]  

(7)

We note that \( \Psi \) is the Stokes streamfunction in the laboratory frame of reference; \( \Psi \) also satisfies the PDE (5), since \( \mathcal{L}(k + \frac{1}{2} W r^2) = 0 \) for any constants \( W \) and \( k \).

We recall that for inviscid and steady flows in the frame of reference moving with the vortex ring, the transport equation for the vorticity reduces to (1). Problem (5) can be reduced to a semi-linear elliptic system by considering a particular form of the vorticity function as given, for example, in (2) [9]. A different reformulation of the problem, namely, as a semi-linear Dirichlet boundary-value problem for the Laplacian operator in cylindrical coordinates in \( \mathbb{R}^3 \), was introduced in [10]. This made possible the use of variational techniques to prove existence results [10, 28], symmetry [29] and asymptotic behaviour [30] of solutions.

![Figure 2: Domain \( \Omega \) (in the meridian plane) for the vortex ring problem. Contours represent the level sets of the streamfunction \( \psi \) in the frame of reference moving with the vortex ring.](image-url)
In the present study, we formulate the vortex-ring problem in the domain \( \Omega \subset \mathbb{R}^2 \) defined as the cross-section of the vortex bubble \( \Omega_b \) in the meridian half plane \( r > 0 \) (see Figure 2). The domain \( \Omega \) is then bounded by the dividing streamline \((\psi = 0)\) together with the front \((A)\) and rear \((B)\) stagnation points. On the axis of symmetry \((r = 0)\) the radial velocity \(v_r\) vanishes and consequently \(\psi = 0\) there. These two parts of the boundary of the vortex bubble will be denoted \(\gamma_b\) and \(\gamma_z\), respectively. Thus, the governing system for our vortex rings takes the final form

\[
\mathcal{L}\psi = -rf(\psi) \quad \text{in } \Omega, \\
\psi = 0 \quad \text{on } \gamma := \gamma_z \cup \gamma_b. \tag{8a-8b}
\]

The point \(C\) inside the vortex (Figure 2) corresponds to the maximum of the vorticity and is also a stagnation point. Although the fore-and-aft symmetry is not enforced in solving system (8), most (albeit not all) solutions of this system obtained in our study will have this property. Hereafter we will use as diagnostic quantities the following integral characteristics of the vortex ring: circulation \(\Gamma\), impulse (in the horizontal direction \(z\)) \(I\), and energy \(E\). Using the vorticity function \(f(\psi)\), they can be expressed \([7]\) in terms of the following integrals over domain \(\Omega\) (figure 2)

\[
\Gamma := \int_{\Omega} rf(\psi(r, z)) \, drdz, \tag{9a}
\]

\[
I := \pi \int_{\Omega} r^3 f(\psi(r, z)) \, drdz, \tag{9b}
\]

\[
E := \pi \int_{\Omega} rf(\psi(r, z)) \psi(r, z) \, drdz. \tag{9c}
\]

3 Formulation of the Reconstruction Problem

In this section we formulate the reconstruction problem as an inverse problem of source identification amenable to solution using variational optimization techniques. Before we can precisely state this formulation, we need to characterize the admissible vorticity functions \(f\). Their domain of definition will be restricted to the interval \(\mathcal{I} := [0, \psi_{\max}]\), where \(\psi_{\max} > \max_{x \in \Omega} \psi(x)\) is chosen arbitrarily, so that \(f : \mathcal{I} \to \mathbb{R}\). We will refer to \(\mathcal{I}\) as the “identifiability interval” \([26]\). Next, we note that in order to guarantee the existence of nontrivial solutions to nonlinear elliptic boundary-value problems of the type (8), the vorticity function \(f(\psi)\) must be positive (we refer the reader to the monographs \([31, 32]\) for a more detailed discussion of this issue). Regarding the regularity of the vorticity function, \(f\) may be a discontinuous (piecewise-continuous) function as was the case in the NF model, cf. equation (2). However, in practical applications motivating
this study, one is typically interested in smoother vorticity functions. Therefore, we will assume that $f$ belongs to the Sobolev space $H^1(I)$ of continuous functions defined on $I$ with square-integrable gradients. The inner product defined in this space is

$$\forall z_1, z_2 \in H^1(I), \quad \langle z_1, z_2 \rangle_{H^1(I)} = \int_0^{\psi_{\max}} \left( z_1 z_2 + \ell \frac{\partial z_1}{\partial s} \frac{\partial z_2}{\partial s} \right) ds,$$

where $\ell \in \mathbb{R}^+$ is a parameter with the meaning of a “length scale” raised to the power 2. We can now state the reconstruction problem as follows

**Problem 1** Given the measurements $m : \gamma_b \cup \gamma_z \rightarrow \mathbb{R}$ of the velocity component tangential to the boundaries $\gamma_b$ and $\gamma_z$, find a vorticity function $\hat{f} \in H^1(I)$, such that the corresponding solution of (8) matches data $m$ as well as possible in the least-squares sense.

For the purpose of the numerical solution, we will recast Problem 1 as a variational optimization problem which can be solved using a suitable gradient-based method. Since the tangential velocity at the boundary is $v \cdot n^\perp = \frac{1}{r} \frac{\partial \psi}{\partial n}$, we define a cost functional $J : H^1(I) \rightarrow \mathbb{R}$ as

$$J(f) := \frac{\alpha_b}{2} \int_{\gamma_b} \left( \left. \frac{1}{r} \frac{\partial \psi}{\partial n} \right|_{\gamma_b} - m \right)^2 d\sigma + \frac{\alpha_z}{2} \int_{\gamma_z} \left( \left. \frac{1}{r} \frac{\partial \psi}{\partial n} \right|_{\gamma_z} - m \right)^2 d\sigma,$$

where $\alpha_b$ and $\alpha_z$ assume the values $\{0, 1\}$ depending on which part(s) of the domain boundary the measurements are available on. The optimal reconstruction $\hat{f}$ will thus be obtained via solution of the following minimization problem

$$\hat{f} := \arg\min_{f \in H^1(I)} J(f).$$

In some situations it will be necessary to enforce the nonnegativity $f(\psi) \geq 0$ of the vorticity function (functions obtained by imposing this property will be denoted $f_+$). Rather than including an inequality constraint in optimization problem (12), this can be achieved in a straightforward manner by expressing $f_+ = (1/2)g^2$, where $g$ is a real-valued function defined on $I$, and then recasting problem (12) in terms of the new function $g$ as the control variable.

Problem 1 is an example of an inverse problem of source identification. However, in contrast to the most common problems of this type [33], in which the source function depends on the independent variables (e.g., on $x$), in Problem 1 the source $f$ is sought as a function of the state variable $\psi$. As will be shown in the following section, to address this aspect of the problem, a specialized version of the adjoint-based gradient approach will be developed.
4 Gradient-Based Solution Approach

In this section we first describe the general optimization formulation which is followed by the derivation of a convenient expression for the cost functional gradient. Finally, we discuss the calculation of smoothed Sobolev gradients.

4.1 Minimization Algorithm

For simplicity, the solution approach to Problem 1 we present below will not address the positivity constraint which is deferred to the end of the discussion. Solutions to problem (12) are characterized by the following first-order optimality condition

\[ \forall f' \in H^1(I) \quad J'(\hat{f}; f') = 0, \]  

(13)

where the Gâteaux differential \( J'(f; f') := \lim_{\epsilon \to 0} \epsilon^{-1} [J(f + \epsilon f') - J(f)] \) of functional (11) is

\[ J'(f; f') = \alpha_b \int_{\gamma_b} \left( \frac{1}{r} \frac{\partial \psi}{\partial n} \bigg|_{\gamma_b} - m \right) \frac{1}{r} \frac{\partial \psi'}{\partial n} \bigg|_{\gamma_b} d\sigma + \alpha_z \int_{\gamma_z} \left( \frac{1}{r} \frac{\partial \psi}{\partial n} \bigg|_{\gamma_z} - m \right) \frac{1}{r} \frac{\partial \psi'}{\partial n} \bigg|_{\gamma_z} d\sigma, \]  

(14)

in which the variable \( \psi' \) satisfies the linear perturbation equation

\[ \nabla \cdot \left( \frac{1}{r} \nabla \psi' \right) + rf^2(\psi) \psi' = -rf^2 \quad \text{in } \Omega, \]  

(15a)

\[ \psi' = 0 \quad \text{on } \gamma, \]  

(15b)

where \( f_\psi := \frac{df}{d\psi} \) and \( f' \) is the “direction” in which the differential is computed in (14).

The optimal reconstruction can be obtained as \( \hat{f} = \lim_{k \to \infty} f^{(k)} \), where the approximations \( f^{(k)} \) can be computed with the following gradient descent algorithm

\[ f^{(k+1)} = f^{(k)} - \tau_k \nabla J(f^{(k)}), \quad k = 1, 2, \ldots \]  

(16)

\[ f^{(1)} = f_0, \]

in which \( f_0 \) is the initial guess and \( \tau_k \) represents the length of the step along the descent direction at the \( k \)-th iteration. For the sake of simplicity, formulation (16) corresponds to the steepest-descent algorithm, however, in actual computations we shall prefer more advanced minimization techniques, such as the conjugate gradient method [34] (see Section 5).
4.2 Derivation of the Gradient Expression

A key element of descent algorithm (16) is the cost functional gradient $\nabla J(f)$. Assuming that Gâteaux differential (14) is a bounded linear functional defined on a Hilbert space $X$ (e.g., $X = L^2(\mathcal{I})$ or $X = H^1(\mathcal{I})$), i.e., $J'(f; \cdot) : X \to \mathbb{R}$, an expression for the gradient $\nabla_X J(f)$ can be obtained from (14) employing the Riesz representation theorem [35]

$$J'(f; f') = \left\langle \nabla_X J(f), f' \right\rangle_X,$$  \hspace{1cm} (17)

with $\langle \cdot, \cdot \rangle_X$ denoting the inner product in space $X$. We note that representation (14) is not yet consistent with (17), since the perturbation $f'$ is not explicitly present in it, but is instead hidden in the source term of perturbation equation (15a). In order to identify an expression for the gradient consistent with (17), we introduce the adjoint variable $\psi^* : \Omega \to \mathbb{R}$. Integrating (15a) against $\psi^*$ over $\Omega$ and then integrating by parts twice we obtain

$$0 = \int_\Omega \psi^* \left[ \nabla \cdot \left( \frac{1}{r} \nabla \psi' \right) + rf_\psi(\psi) \psi^* \right] \, d\Omega + \int_\Omega \psi^* r f' \, d\Omega$$

$$= \int_\Omega \psi' \left[ \nabla \cdot \left( \frac{1}{r} \nabla \psi^* \right) + rf_\psi(\psi) \psi^* \right] \, d\Omega + \int_\Omega \psi^* r f' \, d\Omega$$

$$+ \int_{\gamma_b \cup \gamma_z} \frac{1}{r} \left( \psi^* \frac{\partial \psi'}{\partial n} - \psi' \frac{\partial \psi^*}{\partial n} \right) \, d\sigma.$$  \hspace{1cm} (18)

Using boundary condition (15b) and defining the adjoint system as follows

$$\nabla \cdot \left( \frac{1}{r} \nabla \psi^* \right) + rf_\psi(\psi) \psi^* = 0$$ \hspace{1cm} in $\Omega,$  \hspace{1cm} (19a)

$$\psi^* = \alpha_b \left( \frac{1}{r} \frac{\partial \psi}{\partial n} \bigg|_{\gamma_b} - m \right) \text{ on } \gamma_b,$$  \hspace{1cm} (19b)

$$\psi^* = \alpha_z \left( \frac{1}{r} \frac{\partial \psi}{\partial n} \bigg|_{\gamma_z} - m \right) \text{ on } \gamma_z,$$  \hspace{1cm} (19c)

identity (18) simplifies to

$$J'(f; f') = -\int_\Omega \psi^* r f' \, d\Omega.$$  \hspace{1cm} (20)

Although perturbation $f'$ appears explicitly in (20), this expression still is not in a form consistent with Riesz representation (17), because the latter requires an inner product with $s$ (equivalently, $\psi$) as the integration variable. We address this issue by expressing $f'(\psi)$ in terms of the following integral transform

$$f'(\psi(x)) = \int_0^{\psi_{\max}} \delta(\psi(x) - s)f'(s) \, ds, \quad x \in \Omega,$$  \hspace{1cm} (21)
where \( \delta(\cdot) \) is the Dirac delta distribution. Plugging (21) into (20) and then using Fubini’s theorem to reverse the order of integration, we obtain

\[
\mathcal{J}'(f; f') = - \int_\Omega \psi^* r \left[ \int_0^{\psi_{\text{max}}} \delta(\psi(x) - s) f'(s) \, ds \right] d\Omega
\]

\[
= - \int_0^{\psi_{\text{max}}} f'(s) \left[ \int_\Omega \psi^* r \delta(\psi(x) - s) \, d\Omega \right] \, ds
\]

(22)

which is already consistent with Riesz representation (17). Although this is not the gradient used in our actual calculations, we first identify the \( L^2 \) gradient of \( \mathcal{J} \) and then obtain from it the required Sobolev gradient \( \nabla_{H^1} \mathcal{J} \) as shown in Section 4.3. Thus, setting \( X = L^2(I) \), relation (17) becomes

\[
\mathcal{J}'(f; f') = \int_I \nabla \mathcal{J}(s) f'(s) \, ds
\]

which, together with (22), yields

\[
\nabla_{L^2} \mathcal{J}(s) = - \int_\Omega \psi^* r \delta(\psi(x) - s) \, d\Omega = - \int_{\gamma_s} \psi^* r \left( \frac{\partial \psi}{\partial n} \right)^{-1} \, d\sigma, \quad s \in [0, \psi_{\text{max}}].
\]

(23)

The expression on the right-hand side (RHS) of (23) shows that, for a given \( s \in I \), the gradient \( \nabla_{L^2} \mathcal{J}(s) \) can be evaluated as a contour integral on the level set

\[
\gamma_s := \{ x \in \Omega : \psi(x) = s \}.
\]

(24)

Positivity of the reconstructed vorticity function can be enforced in a straightforward manner by redefining cost functional (11) in terms of a new variable \( g : I \to \mathbb{R} \) as \( \mathcal{J}_g(g) := \mathcal{J}((1/2)g^2) \), where \( (1/2)g^2 = f_+ \). The corresponding optimizer \( \hat{g} \) can then be found with a gradient descent method analogous to (16), i.e., \( g^{(n+1)} = g^{(n)} - \tau_n \nabla_g \mathcal{J}_g(g^{(n)}) \), where \( \nabla_g \mathcal{J}_g \) is the gradient of the cost functional with respect to the new variable \( g \). It can be obtained from (23) through the following transformation. Noting that \( f' = gg' \), Riesz representation identity (17) yields

\[
\mathcal{J}'(f_+; f') = \left< \nabla_{L^2} \mathcal{J}(f_+), f' \right>_{L^2} = \mathcal{J}'(f_+(g); gg')
\]

\[
= \left< g \nabla_{L^2} \mathcal{J}(f_+(g)), g' \right>_{L^2} = \left< \nabla_{L^2} \mathcal{J}_g(g), g' \right>_{L^2},
\]

(25)

so that

\[
\nabla_{L^2} \mathcal{J}_g = \sqrt{2f} \nabla_{L^2} \mathcal{J}.
\]

(26)

Since the gradients of the cost functional with respect to \( f \) and \( g \) are so closely related, our description of the computational algorithm and its validation in Section 5 will focus on \( \nabla \mathcal{J} \) with extensions to \( \nabla_g \mathcal{J} \) being obvious.
4.3 Sobolev Gradients

We now proceed to discuss how Sobolev gradients $\nabla J = \nabla^{H^1} J$ employed in gradient-descent approach (16) can be derived from the $L_2$ gradients obtained in (23). This will be done using inner product (10) in Riesz identity (17). In addition to enforcing smoothness of the reconstructed vorticity functions, this formulation also allows us to impose the desired behavior at the endpoints of interval $\mathcal{I}$ via suitable boundary conditions (we refer the reader to [26] for a more in-depth discussion of these issues). As regards the behavior of the gradients $\nabla^{H^1} J$ at the endpoints of interval $\mathcal{I}$, we can require the vanishing of either the gradient itself or its derivative with respect to $s$. In the present study we prescribe the homogeneous Neumann boundary condition at the right endpoint of the identifiability interval $\mathcal{I}$

$$\frac{d}{ds}(\nabla^{H^1} J) = 0 \quad \text{at} \quad s = \psi_{\text{max}}$$

which implies that, with respect to the initial guess $f_0$, at $s = \psi_{\text{max}}$ iterations (16) can modify the values, but not the slope, of the reconstructed functions $f^{(k)}$. As regards the behavior of the Sobolev gradients at the left endpoint, we will consider either Dirichlet or Neumann boundary conditions

$$\nabla^{H^1} J = 0 \quad \text{at} \quad s = 0,$$

$$\frac{d}{ds}(\nabla^{H^1} J) = 0 \quad \text{at} \quad s = 0$$

which will preserve, respectively, the value or the slope of the initial guess $f_0$ at $s = 0$. We refer the reader to [36] for a discussion of other possible choices of boundary conditions imposed on the Sobolev gradients in an identification problem with a similar structure.

Identifying expression (17) in which $\mathcal{X} = H^1(\mathcal{I})$ with the inner product given in (10), integrating by parts and using boundary conditions (27)–(28) we obtain the following elliptic boundary-value problem on $\mathcal{I}$ defining the Sobolev gradient $\nabla^{H^1} J$

$$\left(1 - \ell \frac{d^2}{ds^2}\right) \nabla^{H^1} J = \nabla^{L^2} J \quad \text{in} \quad \mathcal{I},$$

$$\nabla^{H^1} J = 0 \quad \text{at} \quad s = 0,$$

$$\frac{d}{ds}\nabla^{H^1} J = 0 \quad \text{at} \quad s = \psi_{\text{max}},$$

where the expression for $\nabla^{L^2} J$ is given in (23). The same approach can be employed in the positivity-enforcing formulation with $g$ as the control variable, cf. (26). A slightly different way of obtaining Sobolev gradients in identification problems with analogous structure is discussed in [26].
It is well known [37] that extraction of cost functional gradients in the space $H^1$ with the inner product defined as in (10) can be regarded as low-pass filtering of $L_2$ gradients with the cut-off wavenumber given by $\ell^{-1/2}$. While the choice of the parameter $\ell$ is also known to significantly affect the performance of gradient algorithms, in general, there is no systematic way of finding an optimal value of this parameter which is typically determined by trial and error. One approach which has been found to work particularly well [37] is to start with a relatively large value of $\ell$, which gives smooth gradients suitable for reconstructing large-scale features of the solution, and then progressively decrease it with iterations, which allows one to zoom in on smaller features of the solution. This is the approach we adopt here by setting $\ell^{(k)}$, the value of the length-scale used at the $k$-th iteration, as

$$\ell^{(k)} = \zeta^k \ell^{(0)}, \quad k > 0,$$

where $\ell^{(0)}$ is some initial value and $0 < \zeta < 1$ the decrement factor.

5 Computational Algorithm

As is evident from Section 3, the reconstruction algorithm requires of the solution of several linear and nonlinear elliptic boundary-value problems in one or two spatial dimensions, namely, the governing system (8), the adjoint system (19) and the preconditioning system for the Sobolev gradients (29). In addition, evaluation for the $L_2$ gradients is somewhat involved, because the integrals in (23) are evaluated on the level sets $\gamma_s$ which have to be identified, cf. definition (24). All of these technical issues were easily handled using the freely available finite-element software FreeFem++ [38, 39]. This generic PDE solver offers the possibility of using a large variety of triangular finite elements, with an integrated grid generator in two or three dimensions. FreeFem++ is equipped with its own high-level programming language with syntax close to mathematical formulations. It was recently used to solve different types of partial differential equations, e.g. Schrödinger and Gross-Pitaevskii equations [40, 41], incompressible Navier-Stokes equations [27], Laplace equations with nonlinear source terms [42] and Navier-Stokes-Boussinesq equations [43]. The main advantage of using FreeFem++ for the present problem is the simplicity in using different finite-element meshes for each sub-problem making the interpolation or computation of integrals very easy and accurate. Below we briefly describe the implementation of key elements of the computational algorithm which are then validated in the following section.

5.1 Main Computational Modules

The computational algorithm consists of the following main modules:
• Definition of the mesh and the associated finite-element spaces] We define here the boundaries $\gamma_b$ and $\gamma_z$ and build a triangular mesh covering the vortex domain $\Omega$ (see Figure 3). The mesh density is characterized by $N_x$ representing the number of segments per unit length in the discretization of the domain boundaries. The finite element space $V_h$ is defined such that all variables are represented using piecewise quadratic $P^2$ finite elements. Cost function (11) is computed with a 6-th order Gauss quadrature formula.

• One-dimensional interpolation] The vorticity function $f(s)$ is tabulated at $N_f$ discrete values $s_i \in [0, \psi_{\text{max}}]$, $i = 1, \ldots, N_f$. The value $\psi_{\text{max}}$ (cf. Section 3) is set depending on a particular reconstruction case. To obtain values for $f$ and its derivative $f_\psi$ for non-tabulated values of $\psi$, we use cubic spline interpolation.

• Solution of direct problem (8)] Given the nonlinearity of this problem, we use Newton’s method with $p$-th iteration consisting in computing the solution $q := (\psi^p - \psi^{p+1})$ of the following variational problem

$$
\int_{\Omega} \frac{1}{r} \nabla q \cdot \nabla v \, d\Omega - \int_{\Omega} f_\psi(\psi^p) q v \, d\Omega = \int_{\Omega} \frac{1}{r} \nabla \psi^p \cdot \nabla v \, d\Omega - \int_{\Omega} f(\psi^p) v \, d\Omega, \quad \forall v \in V_h. \tag{31}
$$

This problem is solved efficiently in FreeFem++ by building the corresponding matrices (spline interpolation is used to evaluate $f(\psi(x))$ and $f_\psi(\psi(x))$). Newton’s iterations are stopped when $||q||_2 \leq \varepsilon_N$, with $\varepsilon_N = 10^{-6}$.

• Solution of adjoint problem (19)] Given the linearity of this problem, this consists in solving the weak formulation

$$
\int_{\Omega} -\frac{1}{r} \nabla \psi^* \nabla v \, d\Omega + \int_{\Omega} f_\psi(\psi) \psi^* v \, d\Omega = 0, \quad \forall v \in V_h, \tag{32}
$$

with Dirichlet boundary conditions (19b)-(19c), which takes two lines of code in FreeFem++.

• Computation of $L_2$ gradient] To use formula (23) for the $L_2$ gradient $\nabla L^2 J(s)$, for each value $s_i$, $i = 1, \ldots, N_f$, in the table defining the discretized vorticity function $f(s_i)$, we construct the corresponding level set $\gamma_{s_i}$ and mesh its interior (see Figure 3). The values of $\psi^*$ and $\psi$ are $P^2$ interpolated on the new mesh and the integral in (23) is then computed with a 6-th order Gauss quadrature formula.

• Computation of $H^1$ gradient] To obtain the $H^1$ gradient from the $L_2$ gradient we solve the one-dimensional boundary-value problem (29) with either (29b) or (29c) as the boundary condition. This is a standard problem which can be solved in a straightforward manner using $P^1$ piecewise linear finite elements or second-order accurate centered finite differences.

• Minimization algorithm] With the cost functional gradient evaluated as described above, we approximate the optimal vorticity function $\hat{f}$ using the Polak-Ribiere variant of the conjugate gradients algorithm [34] which is an improved version of descent algorithm (16). The length of the step $\tau_k$ at every iteration $k$ is determined by solving a line
minimization problem

$$\tau_k = \arg\min_{\tau > 0} J(f^{(k)} - \tau \nabla J(f^{(k)}))$$  \hspace{1cm} (33)

using Brent’s method [44].

Clearly, accurate evaluation of the cost functional gradients $\nabla J(f)$ is a key element of the proposed reconstruction approach and these calculations are thoroughly validated in the following section.

5.2 Validation of Cost Functional Gradients

In this section we analyze the consistency of the gradient $\nabla J$ evaluated based on formula (23) with respect to refinement of the two key numerical parameters in the problem, namely, $N_x$ and $N_f$ (see the previous section for definitions). A standard test [45] consists in computing the Gâteaux differential $J'(f; f')$ in some arbitrary direction $f'$ using relations (22)–(23) and comparing it to the result obtained with a forward finite-difference formula. Thus, deviation of the quantity

$$\kappa(\epsilon) := \frac{\epsilon^{-1} [J(\psi_b + \epsilon \psi'_b) - J(\psi_b)]}{\int_{\psi_{\text{max}}}^{\psi_{\text{min}}} f'(s) \nabla J(s) \, ds}$$  \hspace{1cm} (34)

from the unity is a measure of the error in computing $J'(f; f')$ (we note that, in the light of identity (17), expression in the denominator of (34) may be based on the $L_2$ gradients).

The dependence of the quantity $\log |\kappa(\epsilon) - 1|$, which captures the number of significant digits of accuracy achieved in the evaluation of (34), on $\epsilon$ is shown in Figures 4a and
Figure 4: Dependence of the diagnostic quantity $\kappa(\epsilon)$ defined in (34) on $\epsilon$ for (a) different discretizations of the identifiability interval $I$ given by $N_{f} = 100, 150, 200$ with $N_{x} = 150$ fixed, and (b) different discretizations of the domain $\Omega$ given by $N_{x} = 75, 150, 300$ with $N_{f} = 200$ fixed.

These results were obtained in a configuration representing Hill’s vortex in which $C = 1/2$ and $\Omega$ is a half-circle of radius $a = 2$ (see Section 6.1 for a precise definition of this test problem), and some generic forms of the reference vorticity function $f$ and its perturbation $f'$ were used. As is evident from Figures 4a and 4b, the values of $\kappa(\epsilon)$ approach the unity for $\epsilon$ ranging over approximately 7 orders of magnitude as the discretization is refined (i.e., as $N_{f}$ and $N_{x}$ increase). We emphasize that, since we are using the “differentiate–then–discretize” rather than “discretize–then–differentiate” approach, the gradient should not be expected to be accurate up to the machine precision [46]. The deviation of $\kappa(\epsilon)$ from the unity for very small values of $\epsilon$ is due to the arithmetic round–off errors, whereas for the large values of $\epsilon$ it is due to the truncation errors, both of which are well known effects [45]. These results thus demonstrate high accuracy of the computed gradients.

The computational results presented in next two sections were obtained with the numerical resolution $N_{x} = 75$, corresponding to $N_{e} = O(10^{4} - 10^{5})$ finite elements discretizing domain $\Omega$ (the exact number varied depending on the specific test problem), and $N_{f} = 100$. 

4b, respectively, for increasing $N_{f}$ and $N_{x}$ while keeping the other parameter fixed.
6 Reconstruction of Inviscid Vortex Rings — Problems with Known Solutions

In this section we employ algorithm (16) to reconstruct the vorticity function $f$ in two test cases: one involving Hill’s spherical vortex introduced in Section 6.1 and the second one involving Norbury-Fraenkel’s vortex discussed in Section 6.2. In fact, Hill’s vortex is a special case of Norbury-Fraenkel’s family of vortices; however, for reasons which will become apparent below, we treat the two cases separately here. In both cases, the vorticity function $f$ has a known form specific to each problem. Then, in Section 7, we will use our approach to reconstruct the vorticity function $f(\psi)$ in a steady Euler flow assumed to model an actual high-Reynolds number flow with concentrated vortex rings. Data for this reconstruction will be obtained from a DNS of such a flow.

6.1 Hill’s Spherical Vortex

6.1.1 Analytical solution for Hill’s Spherical Vortex

Hill’s spherical vortex is a well known [6, 7] closed-form solution for which the vortex bubble $\Omega$ is a sphere of radius $a$ and the vorticity function is constant everywhere in $\Omega$, i.e.,

$$f(\psi) = C, \quad C > 0, \quad \forall \psi(x), \quad x \in \Omega.$$  \hspace{1cm} (35)

The flow outside the bubble approaches the uniform flow $We_z$ as $|x| \to \infty$. By matching the solution inside the bubble with the external solution, the continuity of $\psi$ and $\nabla \psi$ on $\gamma$ gives the compatibility relationship

$$W = \frac{2}{15}Ca^2.$$ \hspace{1cm} (36)

The complete solution in the frame of reference traveling with velocity $W$ is [47, 6, 7]

$$\psi(r, z) = \left\{ \begin{array}{ll}
\frac{C}{10} r^2(a^2 - r^2 - z^2), & \text{if } r^2 + z^2 \leq a^2, \\
\frac{C r a^2}{15} \left[ \frac{a^3}{(r^2 + z^2)^{3/2}} - 1 \right], & \text{if } r^2 + z^2 > a^2.
\end{array} \right.$$  \hspace{1cm} (37)
The corresponding velocity components, which will be needed later, are given by

\begin{align}
  v_r(r, z) &= \begin{cases} 
    \frac{C}{2}rz, & \text{if } r^2 + z^2 \leq a^2, \\
    \frac{Ca^5}{5} \frac{r}{(r^2 + z^2)^{5/2}}, & \text{if } r^2 + z^2 > a^2,
  \end{cases} \\
  v_z(r, z) &= \begin{cases} 
    \frac{-C}{5}(2r^2 + z^2 - a^2), & \text{if } r^2 + z^2 \leq a^2, \\
    \frac{-Ca^5}{15} \left[ \frac{r^2 - 2z^2}{(r^2 + z^2)^{5/2}} + \frac{2}{a^2} \right], & \text{if } r^2 + z^2 > a^2.
  \end{cases}
\end{align}

(38a) (38b)

The circulation, impulse and energy then take the following values, cf. (9),

\begin{align}
  \Gamma_{\text{Hill}} &= \frac{2}{3} C a^3, \\
  I_{\text{Hill}} &= \frac{4}{15} C a^5 \pi, \\
  E_{\text{Hill}} &= \frac{4}{525} C^2 a^7 \pi.
\end{align}

(39)

It is interesting to note that Hill’s vortex is not only an Euler solution, but also satisfies the Navier-Stokes equation (in this sense, it is related to the “controllable flows” introduced by Truesdell [48]). Indeed, if an additional pressure $-2C \mu z$ is included inside the bubble to balance the viscous term $\mu \Delta v = -2C \mu e_z$, the Navier-Stokes equation is satisfied both inside and outside the vortex (see [7]). However, at the boundary of the vortex ring, only the continuity of the velocity is satisfied. The normal and tangential stresses are not continuous across the boundary, therefore (37) is not an exact solution of the complete Navier-Stokes system.

6.1.2 Reconstruction of Vorticity Function in Hill’s Vortex

In the test problem analyzed here we consider Hill’s vortex discussed in Section 6.1.1 in which without the loss of generality we set $a = 2$ and $C = 1/2$. We assume that the measurements $m$ of the tangential velocity component, cf. (38), are available on the entire separatrix streamline with $\psi = 0$, i.e., $\gamma_b \cup \gamma_z = \gamma_0$, cf. (24), in cost functional (11). Since contour $\gamma_0$ is closed, by Stokes’ theorem, measurements $m$ of the tangential velocity determine the total circulation $\Gamma$ contained in the region $\Omega$. Thus, the reconstruction problem formulated in this way is quite complete.

In order to assess the effect of the initial guess $f_0$ on the convergence of gradient algorithm (16), we analyze iterations staring from two distinct initial guesses $f_0$, one underestimating and one overestimating the exact vorticity function (35) (since these initial guesses are representative of a broad range of functions with similar structure, the exact formulas are not important). The Sobolev gradients are computed using Neumann boundary condition (28b) at the left endpoint ($s = 0$) of the identifiability interval $I$. Using homogeneous Dirichlet boundary condition (28a) together with $f_0(0) = C = 1/2$ would make the reconstruction problem too easy, whereas imposing $f_0(0) \neq$
Figure 5: [Hill’s vortex] Decrease of cost functional $J(f^{(k)})$ with $k$ for iterations starting with initial guess $f_0$ (a) underestimating and (b) overestimating the exact vorticity function (35). Different lines correspond to the values of $\ell(0)$ and $\zeta$, cf. equation (30), indicated in the figure legends.

1/2 would be inconsistent with measurements $m$. In the present problem uniformly positive reconstructions $\hat{f}$ were obtained for the vorticity function without any positivity enforcement.

The histories of cost functional $J(f^{(k)})$ with iterations corresponding to the two initial guesses are shown in Figures 5a and 5b, where we consider cases with different $\ell(0)$ and $\zeta$ (see Section 4.3, in particular, formula (30)). We note that in most cases the cost functional decreases by about five orders of magnitude over a few iterations. Reducing the length-scale $\ell(k)$ with iterations has an effect on the rate of convergence when one initial guess is used, but appears to play little role when the other initial guess is employed. Hence, in this problem, we will adopt the values $\ell(0) = 10^{-1/2}$ and $\zeta = 10^{-2/10}$. This choice of $\zeta$ ensures that $\ell(k)$ decreases by an order of magnitude every five iterations, cf. (30).

In Figures 6a and 6b we show the optimal reconstructions $\hat{f}$ obtained in the two cases together with the corresponding initial guesses. We observe that in both cases the reconstructed vorticity function $\hat{f}$ is very close to the exact solution $C = 1/2$ on the interval $[0, \psi_{\text{max}}]$, where $\psi_{\text{max}} = 0.2$. The convergence of circulation $\Gamma$, impulse $I$ and energy $E$ with iterations $k$ to the values characterizing the exact solution is shown in
Figure 6: [Hill’s vortex] Reconstructed source functions $\hat{f}$ (blue solid lines) and the corresponding initial guesses (red dashed lines) when $f_0$ (a) underestimates and (b) overestimates the exact vorticity function (35). The black horizontal dotted line represents exact solution (35) with $a = 2$ and $C = 1/2$, whereas the vertical dotted lines mark the maximum values achieved by the streamfunction in exact solution (37) (i.e., $\psi_{\max} = 0.2$).

Figures 7a and 7b for the two cases. In these figures we plot the relative error

$$\varepsilon^{(k)}(\Gamma) := \left| \frac{\Gamma(f^{(k)})}{\Gamma_{\text{Hill}}} - 1 \right|$$

for the vortex circulation and analogous expressions for the impulse and energy using logarithmic scale to determine the number of significant digits captured in the reconstruction. In the figures we note a fast, though nonmonotonous, convergence of the three diagnostic quantities to the corresponding exact values. We obtain approximately two digits of accuracy for the energy and three or more for the circulation and impulse.

6.2 Norbury-Fraenkel’s Family of Vortex Rings

6.2.1 Numerical solution of Norbury-Fraenkel Family of Vortex Rings

This model relies on the existence of a vortex core $\Omega_c$, defined by the streamline $\psi = k$, with $k$ a positive constant (see figures 1 and 2). Physically, the quantity $2\pi k$ represents the flow rate between the axis $0z$ and the boundary $\partial\Omega_c$. Norbury [14] numerically solved the general problem (8) with a discontinuous vorticity function (2) using a dimensionless
Figure 7: [Hill’s vortex] Evolution of the relative error $\varepsilon^{(k)}(\Gamma)$ for the circulation, cf. (40), and of analogously defined expressions for the impulse $I$ and energy $E$ showing convergence to the corresponding values $\Gamma_{\text{Hill}}, I_{\text{Hill}}$ and $E_{\text{Hill}}$ characterizing the exact solution (37). Iterations starting with initial guess $f_0$ underestimating and overestimating the exact vorticity function (35) are shown in panels (a) and (b), respectively.

formulation and a Fourier series representation for the coordinates of the core and bubble boundaries, respectively, $\partial \Omega_c$ and $\gamma$. The variables are scaled as follows

$$\bar{z} = \frac{z}{L}, \quad \bar{r} = \frac{r}{L}, \quad \bar{v} = \frac{v}{U_0}, \quad \bar{\psi} = \frac{\psi}{U_0 L^2},$$

where $L$ and $U_0$ are, respectively, the length and velocity scales. Equation (8a) is invariant with respect to this change of variables provided the vorticity function on the RHS is replaced with

$$\bar{f}(\bar{\psi}) = \frac{f(\psi) L^2}{U_0}.$$  \hspace{1cm} (42)

The Norbury-Fraenkel family of vortex rings is thus parameterized by a single geometric quantity $\alpha$ defined as a non-dimensional mean core radius

$$\alpha = \frac{\sqrt{|\Omega_c|/\pi}}{R_c},$$

where $R_c = |OC|$ (see figure 2). Norbury also chose a particular velocity scale related to the vorticity through the following relationship

$$C = \frac{1}{\alpha^2} \frac{U_0}{L^2} \quad \implies \quad \bar{f}(\bar{\psi}) = \begin{cases} 0, & \bar{\psi} \leq k, \\ \frac{1}{\alpha^2}, & \bar{\psi} > k. \end{cases}$$

$$21$$
Hereafter in this test problem we shall use the nondimensionalized variables (with the bars dropped to simplify the notation).

For $\alpha \to 0$ the classical thin-core vortex [7] is recovered, while Hill’s spherical vortex is obtained as the terminal member of the Norbury-Fraenkel family corresponding to $\alpha = \sqrt{2}$. Norbury calculated the shape of the vortex core $\Omega_c$ and the vortex bubble $\Omega$ for different values of the parameter $0 < \alpha < \sqrt{2}$, and presented the results in tabular form [14]. This data is used in the next section to set up test cases based on the Fraenkel-Norbury vortex rings.

6.2.2 Reconstruction of Vorticity Function in Norbury-Fraenkel’s Vortex Rings

We now go on to discuss the results obtained in the second test problem which is based on the Norbury-Fraenkel vortices introduced in the previous section. To fix attention, we will focus on the case with $\alpha = 0.6$. The constant in (44) will then assume the value $C = 1/\alpha^2 = 2.777$ and the diagnostic quantities (9) are equal to

$$\Gamma_{\text{NF}} = 3.066, \quad I_{\text{NF}} = 11.185, \quad E_{\text{NF}} = 2.140.$$  \hspace{1cm} (45)

We emphasize that, from the point of view of our reconstruction problem, the key difference between the source functions (35) and (44) corresponding to Hill’s and Norbury-Fraenkel’s vortices is that the latter is discontinuous which, as will be discussed below, makes the reconstruction problem more challenging.

In order to obtain a consistent reconstruction of the vorticity function for the Norbury-Fraenkel vortex, we need to guarantee that $\hat{f}(0) = 0$, cf. (44). This is will be ensured by using the initial guesses with the property $f_0(0) = 0$ and Sobolev gradients (29) subject to the Dirichlet boundary condition (28a) at the left endpoint of the identifiability interval. As we did in the reconstruction of Hill’s vortex, here as well we analyze iterations staring from two distinct initial guesses $f_0$, one underestimating and one overestimating the exact vorticity function (44). In the calculations we use $\ell_0 = 0.2$ and $\zeta = 10^{-2/10}$ as the parameters defining the Sobolev gradients, cf. (30). In the present case the reconstruction problem is solved with and without explicit enforcement of the positivity constraint $\hat{f} > 0$.

The history of cost functional $\mathcal{J}(f^{(k)})$ with iterations $k$ for the two initial guesses is shown in Figures 8a and 8b. We see that a significant decrease of more than three orders of magnitude is achieved in just over 10 iterations and faster convergence is obtained for the algorithm without the enforcement of positivity. This can be understood by recognizing that, subject to the positivity constraint, the reconstruction space is “smaller”. Interestingly, in all cases this is achieved in two phases, at the beginning and at the end of the reconstruction, with the intermediate iterations producing little
Figure 8: [Norbury-Fraenkel’s vortex] Decrease of cost functional $\mathcal{J}(f^{(k)})$ with $k$ for iterations starting with initial guess $f_0$ (a) underestimating and (b) overestimating the exact vorticity function (44) with and without the enforcement of positivity of $\hat{f}$.

decrease of the cost functional. The optimal reconstructions $\hat{f}$ and $\hat{f}_+$ are shown in Figures 9a and 9b together with the corresponding initial guesses and the exact source function (44). We see that the reconstructed source functions are quite similar for the two different initial guesses and interpolate smoothly the step function (44). The optimal reconstructions without positivity enforcement $\hat{f}$ exhibit significant overshoots and undershoots. This behavior is in fact to be expected given how the reconstruction problem is formulated with the optimizers sought in the Sobolev space $H^1(I)$, cf. (12), whose element must be continuous, but are otherwise arbitrary. It is reminiscent of the Gibbs phenomenon known in numerical analysis to occur when smooth interpolants are used to interpolate discontinuous functions [49]. On the other hand, optimizers $\hat{f}_+$, which are nonnegative by construction, exhibit no undershoots towards negative values. Finally, the convergence of the diagnostic quantities (9) with iterations to the corresponding values (45) characterizing the exact solution is shown in Figures 10a–d for the two initial guesses with and without the enforcement of positivity. We see that, similarly to the case of Hill’s vortex discussed in Section 6.1.2, in all cases the circulation $\Gamma$ and impulse $I$ are captured with the accuracy of roughly three significant digits, and two digits in the case of energy $E$, although the convergence of these quantities is not monotonous in these cases as well.
Figure 9: [Norbury-Fraenkel’s vortex] Reconstructed vorticity functions \( \hat{f} \) (blue solid lines), \( \hat{f}_+ \) (green dash-dotted lines) and the corresponding initial guesses (red dashed lines) when \( f_0 \) (a) underestimates and (b) overestimates the exact vorticity function (44). The black dotted line represents the exact solution (44) with \( C = 2.777 \), whereas the vertical dotted lines mark the maximum value \( \psi_{\text{max}} = 0.345 \) achieved by the stream-function.
Figure 10: [Norbury-Fraenkel’s vortex] Evolution of the relative error $\varepsilon^{(k)}(\Gamma)$ for the circulation, cf. (40), and of analogously defined expressions for the impulse $I$ and energy $E$ showing convergence to the corresponding values $\Gamma_{NF}$, $I_{NF}$ and $E_{NF}$ characterizing the NF numerical solution. Iterations starting with initial guess $f_0$ underestimating and overestimating the exact vorticity function (44) are shown in panels (a1–a2–a3) and (b1–b2–b3), respectively. The results obtained with and without positivity enforcement are marked with empty and solid symbols, respectively.
7 Reconstruction of Vorticity from DNS Data

In this section we describe a more challenging task of reconstructing the vorticity function characterizing a realistic vortex ring. In the following we use a high-resolution DNS of the axisymmetric Navier-Stokes equations to generate a realistic evolution of a viscous vortex ring (see [5] for details of the numerical simulation). The vortex ring is generated by prescribing an appropriate axial velocity profile at the inlet section of the computational domain. We used the specified discharge velocity (SDV) model proposed in [50] to mimic the flow generated by a piston/cylinder mechanism pushing a column of fluid through a long pipe of diameter $D$. In the following, all presented quantities will be normalized using $D$ as the length scale and the maximum (piston) velocity $U_0$ at the entry of the pipe as the velocity scale. The corresponding reference time is thus $D/U_0$. The main physical parameter of the flow is the Reynolds number based on the characteristic velocity: $Re_D = U_0 D/\nu = 3400$, with $\nu$ the viscosity of the fluid. The injection is characterized by the stroke length ($L_p$) of the piston. We prescribed the piston velocity program used in actual experiments with $L_p/D = 1.28$ [51].

For the reconstruction problem, we consider the vortex ring data obtained from the DNS at the nondimensional time $t = 20$. This time instant corresponds to the post-formation phase, since the injection stopped at $t_{off} = 2.26$. The DNS streamfunction $\psi_{DNS}(r,z)$ in the frame of reference moving with the vortex is computed by solving the general equation (5) within the rectangular domain used for the DNS together with the corresponding boundary conditions. We then use the level set $\psi_{DNS} = 0$ to define the reconstruction domain $\Omega$ (see Figure 2) and from this data extract the measurements $m(r,z)$ on $\gamma_b$ and $\gamma_z$, which serve as the target data in optimization problem (11)–(12).

The source function $f$ on the RHS in (8a) is reconstructed so that the inviscid vortex ring model provides the most accurate representation of the DNS data. As a starting point, the empirical relation $\{\omega(r_p,z_p)/r_p, \psi(r_p,z_p)\}_p$ between the vorticity and the streamfunction, cf. (1), at the points $(r_p,z_p)$ discretizing the flow domain $\Omega$ is shown as a scatter plot in Figure 11. The deviation of this set of points from a continuous curve is a manifestation of the fact that the original Navier-Stokes flow is viscous and not strictly steady in the chosen frame of reference. An approximation to this data obtained with a least-squares fit of a power function yields (cf. Figure 11)

$$f_{DNS}(\psi) = 18.837 \psi^{1.6364}.$$  \hspace{1cm} (46)

This may be, arguably, a natural candidate for the initial guess $f_0$ in gradient-descent algorithm (16). Surprisingly however, it turns out that $f_0 = f_{DNS}$ is already a local minimizer of optimization problem (11)–(12), i.e., $\nabla J(f_{DNS}) \approx 0$ and algorithm (16) is not able to perform even a single iteration starting from $f_{DNS}$. The corresponding values of cost functional (11) and the diagnostic quantities (9) are presented in Table 1. On
the other hand, using an initial guess $f_0$ which does not approximate the empirical data \{\omega(r_p, z_p)/r_p, \psi(r_p, z_p)\} \_p very well results in iterations (16) converging to a minimum characterized by a value of cost functional $\mathcal{J}(\hat{f}) \geq 33\%$ smaller than $\mathcal{J}(f_{DNS})$, see Table 1. In the present problem as well uniformly positive reconstructions $\hat{f}$ of the vorticity function were obtained without any positivity enforcement. The initial guess $f_0$ and the corresponding optimal reconstruction $\hat{f}$ of the source function are shown in Figure 11. We remark that convergence to this minimizer was in fact achieved with a broad range of quite different initial guesses and the particular form of $f_0$ shown in Figure 11 is used only for illustration.

This somewhat surprising finding is analyzed in Figures 12a and 12b where the tangential velocity $\partial \psi / \partial n$ corresponding to the actual DNS and the vortex ring model (8) with $f_{DNS}$ and $\hat{f}$ is shown on the boundaries $\gamma_z$ and $\gamma_b$ as functions of the arc-length coordinate $s$. We see that, while the quality of the reconstruction of the tangential velocity on $\gamma_b$ is comparable in both cases, the iterations starting with a seemingly poor initial guess ultimately lead to a better reconstruction on $\gamma_z$ than the iterations starting with the empirical fit (46). The corresponding values of the diagnostic quantities (9) are shown in Table 1, where we observe that the optimal reconstruction $\hat{f}$ also captures
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
 & $\mathcal{J}(f)$ & $\Gamma$ & $\varepsilon(\Gamma)$ & $I$ & $\varepsilon(I)$ & $E$ & $\varepsilon(E)$ \\
\hline
DNS & & 6.8724 & & 18.211 & & 14.520 & \\
$\hat{f}_\text{DNS}$ & 0.4225 & 5.8544 & 0.148 & 19.988 & 0.097 & 10.7347 & 0.260 \\
$\hat{f}$ & 0.282103 & 6.6815 & 0.027 & 21.8954 & 0.202 & 10.6063 & 0.269 \\
\hline
\end{tabular}
\caption{Values of the cost functional (11), the diagnostic quantities (9) and the corresponding relative errors (40) obtained for the fitted vorticity function $f_{\text{DNS}}$, cf. (46), and the optimally reconstructed vorticity function $\hat{f}$. The relative errors are computed with respect to the diagnostic quantities corresponding to the original DNS data.}
\end{table}

the circulation $\Gamma$ of the actual flow with a better accuracy than does the fit (46). The impulse $I$ and energy $E$ of the original flow are captured with a comparable accuracy by both $\hat{f}$ and $f_{\text{DNS}}$.

8 Discussion, Conclusions and Outlook

In this study we have formulated and validated a novel solution approach to an inverse problem in vortex dynamics concerning the reconstruction of the vorticity function in 3D axisymmetric Euler flows. More generally, this is an example of the reconstruction of a nonlinear source term in an elliptic PDE and as such has many applications in fluid mechanics and beyond (more on this below). It also has some similarities to the “Calderon problem” which is one of the classical inverse problems studied in the context of elliptic PDEs. In particular, many questions concerning the uniqueness of the reconstructions remain open and we in fact encountered such behavior in Section 7. In contrast to a number of earlier approaches which relied on finite-, and usually low-dimensional, parameterizations of the reconstructed vorticity function (e.g., [25]), the method proposed here is non-parametric and allows us to reconstruct the vorticity function in a very general form in which only the smoothness and boundary behavior are prescribed. A key element of the computational approach is a suitable reformulation of the adjoint-based optimization which was developed for the reconstruction of constitutive relations in [26, 27] and was already applied to other estimation problems in fluid mechanics in [36, 52].

In addition to standard tests on the accuracy of the adjoint gradients presented in Section 5.2, our approach was validated by reconstructing the vorticity functions in two classical problems, namely, the vortex rings of Hill and Norbury-Fraenkel. Given that we required our reconstructions to be continuous ($H^1$) functions, the second problem involving a discontinuous vorticity function (44) turned out to be particularly challeng-
Figure 12: [DNS] Tangential velocity on the boundary segment (a) $\gamma_b$ and (b) $\gamma_z$ as a function of the arc-length coordinate $s$ along the boundary. Curves correspond to the original DNS data (blue dotted line) and the vortex ring model (8) with the vorticity function $f_{\text{DNS}}$ fitted to the DNS data, cf. (46) (green dashed line) and with the vorticity function $\hat{f}$ obtained as a solution of the reconstruction problem (red solid line).

As can be anticipated from the analogy with the Gibbs phenomenon in numerical interpolation [49], the optimal reconstructions $\hat{f}$ obtained with different initial guesses $f_0$ exhibit “wiggles” around the discontinuity of $f$. This issue was partially remedied by enforcing the positivity of the reconstructed vorticity function $\hat{f}_+$. Thus, in spite of the “pathology” of this test problem, our approach behaves as expected. Since our main goal was to validate the method, the reconstruction problem was chosen to be “well-determined” in the sense that the velocity measurements $m$ were available on the entire boundary $\gamma$ of the domain (which, in particular, determined the vortex circulation $\Gamma$ through Stokes’ theorem). The accuracy of the reconstruction as measured by the corresponding values of the cost functional was indeed very good (see Figures 5 and 8). It may be, however, expected that in the presence of incomplete (“gappy”) measurements the aforementioned question of the uniqueness of reconstructed vorticity functions will become more pronounced.

The results obtained for the case with the actual DNS data in Section 7 are interesting. First, they indicate that the empirical fit (46) to the data $\{\omega(r_p,z_p)/r_p, \psi(r_p,z_p)\}_p$ is already a local minimizer of the cost functional (11). Surprisingly, however, a different locally optimal reconstruction with a rather nonintuitive form was also obtained leading in fact to a more accurate representation of the velocity field. While such multiplicity of minimizers can be (at least partially) mitigated by including constraints (for exam-
example, on the circulation, impulse or energy) in optimization problem (12), this example demonstrates the utility of our method for finding solutions with nonintuitive structure. In addition, it also shows that construction of an inviscid model by simply interpolating the empirical data for $\omega/r$ versus $\psi$ may lead to suboptimal results.

On the more practical side, in regard to the case based on the DNS data, it would be interesting to see how the form and accuracy of the optimal reconstructions depend on the Reynolds number characterizing the data. This could provide valuable insights about the applicability of inviscid system (8) as a model for actual viscous flows. Another future research direction concerns applications of the method to datasets obtained experimentally with techniques such as the Particle Image Velocimetry (PIV). As discussed in Introduction, this is relevant for a number of both classical and emerging application problems. In such situations, due to a number of practical issues, the measurements $m$ are likely to be incomplete (in the sense of being available on a part of the boundary $\gamma$ only). A well-determined problem can then be recovered by providing additional information about the circulation, impulse and/or energy of the vortex which can be incorporated into optimization problem (12) as equality constraints. These extensions will be pursued in the future work. In this context, another interesting and relevant question is the comparison of the inviscid vortex-ring models constructed as proposed here with different viscous models derived based on linearized equations [53, 54] or using perturbation techniques [55]. In order for the conclusions to be generally valid, such comparisons will need to be made for flows with vortex rings under different conditions (such as Reynolds number, etc.). Investigations aiming to address these questions are already underway and results will be reported in the near future.

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