A Note On Gorenstein Projective Conjecture II

Xiaojin Zhang

School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, P. R. China

Abstract In this paper, we prove that Gorenstein projective conjecture is left and right symmetric and the co-homology vanishing condition can not be reduced in general. Moreover, Gorenstein projective conjecture is proved to be true for CM-finite algebras.

Keywords Gorenstein projective, CM-finite algebras, Gorenstein projective conjecture

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1 Introduction

For the representation theory of Artinian algebras, the Auslander-Reiten conjecture (ARC) which is related to generalized Nakayama conjecture (GNC) is everything. It was proposed by Auslander and Reiten, which says that $M$ is projective if $\text{Ext}^i_\Lambda(M \oplus \Lambda, M \oplus \Lambda) = 0$ for any $i \geq 1$(See [2,3]). Achievements for special cases have been got by K. R. Fuller, B. Zimmermann-Huisgen, A. Marótí and G. Wilson...(See [10,15,16]). In general it is still open now. As a special case of Auslander-Reiten conjecture, Luo and Huang proposed the following Gorenstein projective conjecture (GPC) in 2008:

Let $\Lambda$ be an Artinian algebra and let $M$ be a Gorenstein projective module. Then $M$ is projective if and only if $\text{Ext}^i_\Lambda(M, M) = 0$ for any $i \geq 1$(See [14]).

It is still unknown whether the Auslander-Reiten conjecture is left and right symmetric. But as we stated Gorenstein projective conjecture is a special case. So what about the left and right symmetric property of Gorenstein projective conjecture? In this paper, we will give a positive answer to this question.

By the definition of Gorenstein projective conjecture, for an algebra $\Lambda$ the truth of Auslander-Reiten conjecture implies the truth of Gorenstein projective conjecture. So we can get a large class of algebras satisfying Gorenstein projective conjecture. It is interesting to ask: Is there an algebra satisfying Gorenstein projective conjecture while for which the Auslander-Reiten conjecture is unknown?

*E-mail address: xjzhang@nuist.edu.cn
Recall that an algebra is called CM-finite (of finite Cohen-Macaulay type) if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective modules. CM-finite algebras are studied by several authors recently (see [5,6,7,12,13]). Although the Auslander-Reiten conjecture for this class of algebras is unknown, we will give a positive answer to the second question above.

The paper is organized as follows:

In Section 2, based on some facts of Gorenstein projective modules, we will show the symmetric property of Gorenstein projective conjecture. Moreover, an example is given to show that the condition \( \text{Ext}_i^A(M, M) = 0 \) for any \( i \geq 1 \) in Gorenstein projective conjecture can not be reduced to \( \text{Ext}_t^A(M, M) = 0 \) for some positive integer \( t \) and any \( 1 \leq i \leq t \).

In Section 3, we will prove that CM-finite algebras satisfy the Gorenstein projective conjecture by showing the Gorenstein projective conjecture holds for algebras with finite self-orthogonal indecomposable Gorenstein projective modules (up to isomorphisms).

Throughout the paper, \( A \) is an Artinian algebra and all modules are finitely generated left \( A \)-modules.

2 Symmetric property of Gorenstein projective conjecture

In this section we will show the symmetric property of Gorenstein projective conjecture. First, we need to recall some notions and lemmas. The following definition is due to Auslander, Briger, Enochs and Jenda (see [1,8,9]).

**Definition 2.1** A module \( M \) is called **Gorenstein projective** if for any \( i \geq 1 \)

1. \( \text{Ext}_i^A(M, A) = 0 \)
2. \( \text{Ext}_i^A(\text{Tr} M, A) = 0 \)

Where \( \text{Tr} M \) denotes the Auslander transpose of \( M \).

Let \( \cdots \to P_2(M) \to P_1(M) \to M \to 0 \) be a minimal projective resolution of \( M \). Denoted by \( \Omega^i M \) the \( i \)-th syzygy of \( M \). Dually, one can define \( \Omega^{-i} M \). We remark that for any \( i \geq 0 \) \( \Omega^i M \) is a Gorenstein projective if so is \( M \). Let \( C \) be the subcategory of mod \( A \) consisting of modules \( M \) such that \( \text{Ext}_j^A(M, A) = 0 \) for any \( j \geq 1 \) and \( D \) a subcategory of mod \( A \) consisting of Gorenstein projective modules. We use \( C \) and \( D \) to denote the stable subcategory of \( C \) and \( D \) modulo projectives, respectively. We recall the following proposition from [1].

**Proposition 2.2**

1. \( \Omega : C \to C \) is a fully faithful functor.
2. \( \Omega : D \to D \) is an equivalence.
3. \( (\cdot)^* = \text{Hom}(\cdot, A) : D \to D^\circ \) is a duality, where \( D^\circ \) denotes the subcategory of Gorenstein projective right \( A \)-modules.
Proof. (1) is a result of Auslander and Bridger. One can get (2) by (1) and the remark above. (3) is well-known.

Recall that a module $M$ is called self-orthogonal if $\text{Ext}_\Lambda^j(M, M) = 0$ for any $j \geq 1$. The following self-orthogonal property is essential to the main result in this section.

**Lemma 2.3** Let $\Lambda$ be an algebra. Then for any $M \in \mathcal{D}$ and $i \geq 1$, $M$ is self-orthogonal if and only if $M^\ast$ is self-orthogonal.

**Proof.** Since $(-)^\ast$ is a duality between $\mathcal{D}$ and $\mathcal{D}^\circ$, it is enough to show that $\text{Ext}_\Lambda^i(M, M) = 0$ implies $\text{Ext}_\Lambda^i(M^\ast, M^\ast) = 0$.

One can take the following minimal projective resolution of $M$:

$$
\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \quad (1)
$$

Applying the functor $\text{Hom}(-, M)$ to sequence (1) above, since $\text{Ext}_\Lambda^i(M, M) = 0$ we get the following exact sequence

$$
0 \rightarrow \text{Hom}(M, M) \rightarrow \text{Hom}(P_0, M) \rightarrow \text{Hom}(P_1, M) \rightarrow \cdots \quad (2)
$$

On the other hand, applying the functor $(-)^\ast$ to the sequence (1), since $M \in \mathcal{D} \subseteq \mathcal{C}$ one can show the following exact sequence

$$
0 \rightarrow M^\ast \rightarrow P_0^\ast \rightarrow P_1^\ast \rightarrow \cdots \quad (3)
$$

Then by using the functor $\text{Hom}(M^\ast, -)$ on the exact sequences (3), one has the following exact sequence

$$
0 \rightarrow \text{Hom}(M^\ast, M^\ast) \rightarrow \text{Hom}(M^\ast, P_0^\ast) \rightarrow \text{Hom}(M^\ast, P_1^\ast) \rightarrow \cdots \quad (4)
$$

Using Proposition 2.2(3), we get $\text{Ext}_\Lambda^i(M^\ast, M^\ast) \simeq \text{Ext}_\Lambda^i(M, M) = 0$ by comparing sequences (2) with (4). □

Although the symmetric property of Auslander-Reiten conjecture is still unknown, we are able to show the symmetric properties of Gorenstein projective conjecture.

**Theorem 2.4** Let $\Lambda$ be an algebra and let $\Lambda^\circ$ be the opposite ring of $\Lambda$. Then $\Lambda$ satisfies the Gorenstein projective conjecture if and only if $\Lambda^\circ$ satisfies the Gorenstein projective conjecture.

**Proof.** $\Rightarrow$ Assume that $N \in \mathcal{D}^\circ$ and $\text{Ext}_\Lambda^i(N, N) = 0$ for any $i \geq 1$. By Proposition 2.2, there is a $M \in \mathcal{D}$ such that $M^\ast \simeq N$. By Lemma 2.3 one gets that $\text{Ext}_\Lambda^i(M, M) = 0$. Note that $\Lambda$ satisfies the Gorenstein projective conjecture, we have $M$ is projective, and hence $N \simeq M^\ast$ is projective. Conversely, one can formula the proof above. □
Notice that Gorenstein projective conjecture is a special case of Auslander-Reiten conjecture. It is natural to consider whether the assumption of Gorenstein projective conjecture can be reduced. In particular, whether can the condition \( \text{Ext}_i^\Lambda (M, M) = 0 \) for any \( i \geq 1 \) in GPC be reduced to \( \text{Ext}_i^\Lambda (M, M) = 0 \) for some positive integer \( t \) and any \( 1 \leq i \leq t \)? At the end of this section, we construct an example to give a negative answer to the question.

**Example 2.5** Let \( n > t + 1 \) be a positive integer and let \( \Lambda \) be the algebra generated by the following quiver

\[ \begin{array}{ccccccc}
& & & n & & & \\
& a_n & & a_1 & & a_2 & \\
n-1 & & & n & & & \\
a_{n-1} & & & a_2 & & & \\
n-2 & & & 2 & & & \\
& & a_n & & & & \\
& & & 3 & & & \\
& & & 5 & & & \\
& & & n-2 & & & \\
& & & 4 & & & \\
& & & 1 & & & \\
\end{array} \]

modulo the ideal \( \{ a_n a_1 = 0, a_i a_{i+1} = 0 \mid 1 \leq i \leq n - 1 \} \). Denoted by \( S(j) \) the simple module according to the dot \( j \). Then

1. \( \Lambda \) is a Nakayama self-injective algebra.
2. \( S(j) \) is Gorenstein projective such that \( \text{Ext}_i^\Lambda (S(j), S(j)) = 0 \) for \( t \geq i \geq 1 \) and \( 1 \leq j \leq n \), but it is not projective.

3 **Gorenstein projective conjecture for CM-finite algebras**

In this section we try to find a class of algebras which satisfy Gorenstein projective conjecture and for which the Auslander-Reiten conjecture is unknown. They are CM-finite algebras. We begin with the following definition due to Beligiannis

**Definition 3.1** An algebra is called CM-finite (of finite Cohen-Macaulay type) if there are only finite number of indecomposable Gorenstein projective modules (up to isomorphisms).

**Remark 3.2** (1) Algebras of finite representation type or finite global dimension are CM-finite.

(2) There does exist a CM-finite algebra \( \Lambda \) such that \( \Lambda \) is of infinite type and the global dimension of \( \Lambda \) is infinite [13].

(3) There does exist a CM-finite algebra which is not Gorenstein [5].

(4) An algebra with a trivial maximal n-orthogonal subcategory for some positive integer
is CM-finite [11].

Let \( \mathcal{C}, \mathcal{D} \) and \( \mathcal{D}^o \) be as in Section 2. The following lemma partly from [1] plays an important role in the proof of the main results.

**Lemma 3.3** For any \( M \in \mathcal{C} \) and \( N \in \text{mod} \Lambda \), then \( \text{Ext}_\Lambda^1(M, N) \cong \text{Hom}_\Lambda(\Omega^1M, N) \) and hence \( \text{Ext}_\Lambda^i(M, N) \cong \text{Hom}_\Lambda(\Omega^iM, N) \) for any \( i \geq 1 \).

**Proof.** The first assertion is a result of Auslander and Bridger. For the second one, the case \( i = 1 \) is clear. We only need to show the case \( i \geq 2 \). Taking a minimal projective resolution of \( M \), one gets \( \text{Ext}_\Lambda^i(M, N) \cong \text{Ext}_\Lambda^1(\Omega^{i-1}M, N) \) for any \( i \geq 2 \). Notice that \( M \in \mathcal{C} \), by Proposition 2.2 one can show \( \Omega^{i-1}M \in \mathcal{C} \). Using the first assertion, we are done. \( \square \).

The following proposition gives a connection between the self-orthogonal property of \( M \) and that of \( \Omega^iM \) for any \( i \geq 0 \).

**Proposition 3.4** Let \( M \in \mathcal{C} (\mathcal{D}) \). Then

(1) \( \Omega^iM \) is self-orthogonal in \( \mathcal{C} (\mathcal{D}) \) for any \( i \geq 0 \) if \( M \) is self-orthogonal.

(2) If \( M \in \mathcal{D} \) is self-orthogonal, then \( \text{Tr}M \) is self-orthogonal in \( \mathcal{D}^o \).

**Proof.** (1) For the case \( i = 0 \), there is nothing to prove. By Proposition 2.2, we only need to prove the case \( i = 1 \), that is, \( \text{Ext}_\Lambda^1(\Omega M, \Omega M) = 0 \) for any \( j \geq 1 \). One gets \( \text{Ext}_\Lambda^j(\Omega M, \Omega M) \cong \text{Hom}_\Lambda(\Omega^{j+1}M, \Omega M) \cong \text{Hom}_\Lambda(\Omega^j M, M) \) by Proposition 2.2 and Lemma 3.3. Using the second equation of Lemma 3.3, one can show \( \text{Hom}_\Lambda(\Omega^j M, M) \cong \text{Ext}_\Lambda^j(M, M) = 0 \) since \( M \) is self-orthogonal.

(2) Taking a minimal projective resolution of \( M \), it is not difficult to show that \( \text{Tr}M \cong (\Omega^2 M)^* \) since \( M \in \mathcal{D} \). By Propositions 2.2 and 3.4(1), \( \Omega^2 M \) is also self-orthogonal in \( \mathcal{D} \). Then by Lemma 2.3 and Proposition 2.2 one gets the assertion. \( \square \)

The following proposition is crucial to the main results.

**Proposition 3.5** Let \( \Lambda \) be an algebra with only finite (up to isomorphism) self-orthogonal indecomposable modules in \( \mathcal{D} (\mathcal{C}) \) and let \( M \) be a self-orthogonal indecomposable module in \( \mathcal{D} (\mathcal{C}) \). Then \( M \) is projective.

**Proof.** Denoted by \( \{M_1, M_2, ..., M_t\} \) the complete set of non-isomorphic self-orthogonal indecomposable modules in \( \mathcal{D} (\mathcal{C}) \). Then \( M \cong M_i \) for some \( 1 \leq i \leq t \).

Suppose that \( M \) is not projective. Then by Proposition 2.2 and Proposition 3.4, we have the following set of self-orthogonal indecomposable modules \( \mathcal{S} = \{\Omega^i M | 1 \leq i \} \) in \( \mathcal{D} (\mathcal{C}) \).

We claim that there are two modules \( \Omega^i M, \Omega^j M \) in \( \mathcal{S} \) such that \( \Omega^i M \cong \Omega^j M \) for some \( i < j \). Otherwise, one gets infinite number of non-isomorphic self-orthogonal indecomposable modules in \( \mathcal{D} (\mathcal{C}) \), a contradiction. Again by Proposition 2.2, one gets \( M \cong \Omega^{i-1} M \).
Considering the following exact sequence $0 \to \Omega^{j-i}M \to P \to \Omega^{j-i-1}M \to 0$, we will show $\text{Ext}^1_\Lambda(\Omega^{j-i-1}M, \Omega^{j-i}M) = 0$. Since $\Omega^{j-i-1}M \in \mathcal{D}$ $(\mathcal{C})$ and $M \simeq \Omega^{j-i}M$, we get $\text{Ext}^1_\Lambda(\Omega^{j-i-1}M, \Omega^{j-i}M) \simeq \text{Hom}_\Lambda(\Omega^{j-i}M, M) \simeq \text{Ext}^{j-i}_\Lambda(M, M) = 0$ by Proposition 2.2 and Lemma 3.3, and hence $M$ is projective, a contradiction. The assertion holds true. □

Now we are in the position to show the main result of this section.

**Theorem 3.6** Let $\Lambda$ be CM-finite. Then $\Lambda$ satisfies Gorenstein projective conjecture.

**Proof.** Since $\Lambda$ is CM-finite, then there are only finite (up to isomorphisms) indecomposable modules in $\mathcal{D}$. One can show the result by Proposition 3.5. □

Although the Auslander-Reiten conjecture for CM-finite algebras is unknown now, we have the following

**Proposition 3.7** Let $\Lambda$ be a CM-finite algebra and let $M$ be a $\Lambda$-module satisfying $\text{Ext}^i_\Lambda(M, M \bigoplus \Lambda) = 0$ for any $i \geq 1$. Then the following are equivalent.

1. $M$ is projective.
2. $M$ is Gorenstein projective.

**Proof.** (1) $\Rightarrow$ (2) is trivial. The converse follows from Theorem 3.6. □

We end this section with two open questions related to this paper.

**Question 1** Does the Gorenstein projective conjecture hold for virtually Gorenstein algebras (see [5])?

**Question 2** Does the Auslander-Reiten conjecture hold for CM-finite algebras?

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**References**

[1] Auslander M and Bridger M. *Stable module theory*, Memoirs Amer. Math. Soc. 94, Amer. Math. Soc., Providence, RI, 1969.

[2] Auslander M and Reiten I. *On a generalized version of Nakayama conjecture*, Proc. Amer. Math. Soc. 52(1975), 69–74.
[3] Auslander M and Reiten I. Applications of contravariantly finite subcategories. Adv. Math. 86(1991), 111–152.

[4] Bass H. Injective dimension in Noetherian Ring, Trans. Amer. Math. Soc. 102 (1962), 18-29.

[5] Beligiannis A. On Algebras of Finite Cohen-Macaulay Type. Adv. Math 226, (2011), 1973–2019.

[6] Chen X W. An Auslander-type result for gorenstein-projective modules. Adv. Math. 218 (2008), 2043–2050.

[7] Christensen L W, Piepmeyer G, Striuli J and Takahashi R. Finite Gorenstein representation type implies simple singularity. Adv. Math. 218 (2008) 1012–1026.

[8] Enochs E E and Jenda O M G. Gorenstein injective and projective modules. Math. Zeit. 220 (1995), 611–633.

[9] Enochs E E and Jenda O M G. Relative Homological Algebra. de Gruyter Exp. Math., vol. 30, Walter de Gruyter Co., 2000.

[10] Fuller K R and Zimmermann-Huisgen B. On the generalized Nakayama conjecture and the Cartan determinant problem. Tran. Amer. Math. Soc 294(2) (1986), 679–691.

[11] Huang Z Y and Zhang X J. The existence of maximal n-orthogonal subcategories. J. Algebra 321(10) (2009), 2829–2842.

[12] Li Z W and Zhang P. Gorenstein algebras of finite Cohen-Macaulay type. Adv. Math. 218 (2010), 728–734.

[13] Li Z W and Zhang P. A construction of Gorenstein-projective modules. J. Algebra 323 (2010) 1802–1812.

[14] Luo R and Huang Z Y. When are torsionless modules projective? J. Algebra, 320(5)(2008), 2156–2164.

[15] Maróti A. A proof of a generalized Nakayama conjecture. Bull. Lond. Math. Soc 38(5) (2006), 777–785.

[16] Wilson G. The Cartan map on categories of graded modules. J. Algebra 85 (1983), 390–398.