A poor man’s improvement on Zhang’s result: there are infinitely many prime gaps less than 60 million

T. S. Trudgian∗
Mathematical Sciences Institute
The Australian National University
timothy.trudgian@anu.edu.au
June 5, 2013

Consider a set $H = \{h_1, h_2, \ldots, h_{k_0}\}$, composed of distinct, non-negative integers, in which $k_0$ denotes a natural number. Call the set $H$ admissible if $\nu_p(H) < p$ for every prime $p$, where $\nu_p(H)$ is the number of distinct residue classes modulo $p$ that are covered by the elements $h_i$.

In [3, Thm. 1] Zhang shows that if $k_0 = 3.5 \times 10^6$ and $H$ is admissible, then there are infinitely many positive integers $n$ for which \(\{n + h_1, \ldots, n + h_{k_0}\}\) contains at least two primes. He follows this with three lines of deduction to prove the remarkable result

$$\lim \inf_{n \to \infty} (p_{n+1} - p_n) < 7 \times 10^7.$$ 

It is with this three-line deduction — the infinitely easier portion of Zhang’s paper — that this article is concerned. Given a value of $k_0$, how can one find an admissible set $H$ for which the length of $H$, defined to be $h_{k_0} - h_1$, is small?

**An upper bound for the length of $H$**

Consider a set of $k_0$ primes

$H = \{p_{m+1}, p_{m+2}, \ldots, p_{m+k_0}\},$

where $m$ is a non-negative integer to be determined momentarily; Zhang takes $m = k_0$. When is $H$ an admissible set?

Consider primes $p \leq p_m$. Since $p_{m+i} \not\equiv 0 \pmod{p}$ for all $1 \leq i \leq k_0$, it follows that $\nu_p(H) \leq p - 1 < p$ for all $p \leq p_m$.

Now consider primes $p \geq p_{m+1}$. We should like to show that there is not enough room in the set to fill all of the residue classes modulo $p$. Therefore for

∗Supported by ARC Grant DE120100173.
primes $p \in \mathcal{H}$ we wish to show that $k_0 - 1 < p - 1$. It is sufficient to show that $p_{m+1} > k_0$. A quick computation shows that one may choose $m = 250, 150$. Therefore the maximal gap between primes in \{ $n + p_{m+1}, \ldots, n + p_{m+k_0}$ \} is

\[ p_{m+k_0} - p_{m+1} = 59,874,594. \]

In 2013, between 30th May and 3rd June, a considerable amount of work was undertaken by Morrison, Tao, et al. \[2\] which not only improved on the method of exhibiting small gaps, but also improved on the value of $k_0$. To date, the smallest permissible value of $k_0$ is 341,640, which leads to a prime gap not exceeding 4,180,222.

**An lower bound for the length of $\mathcal{H}$**

The set $\{1, 2 \ldots, k_0\}$ cannot be admissible since, *inter alia*, both even and odd numbers are present. Therefore, at the very least, we must impose that our set be of the form $\{r_1, r_1 + 2, \ldots, r_1 + 2(k_0 - 1)\}$, where $r_1$ is either 1 or 0 modulo 2. Such a set has $k_0$ elements, and length $2(k_0 - 1)$. It may be that this set is not admissible; the point to note is that the minimal length of an admissible set must be bounded below by $2(k_0 - 1)$.

We may generalise this approach by noting that we can fill, at the most, $p_i - 1$ residue classes modulo $p_i$, for each $p_i$. At best, we may include $R_m = (p_1 - 1) \cdots (p_m - 1)$ integers modulo $M_m = p_1 \cdots p_m$, where $m$ is an integer that we shall determine momentarily. For $1 \leq i \leq j \leq R_m$ let $r_i \leq r_j$ run through the $R_m$ residues modulo $M_m$. Consider the sets

\[ T = \{r_1, r_2, \ldots, r_{R_m}, \ldots, r_1 + (a-1)M_m, r_2 + (a-1)M_m, \ldots, r_{R_m} + (a-1)M_m\} \]

and

\[ T' = \{r_1 + aM_m, \ldots, r_n + aM_m\}, \]

in which $1 \leq n \leq R_m$ and in which $a$ is chosen such that $|T| < k_0$ and $|T \cup T'| \geq k_0$. It follows that $R_m a < k_0 \leq R_m (a+1)$, whence the length of $T \cup T'$ bounded below by

\[ aM_m \geq M_m \left( \frac{k_0}{R_m} - 1 \right). \]

Given a value of $k_0$ one may choose the value of $m$ maximising the right-side of (1). When $k_0 = 341,640$, one should choose $m = 6$, which gives a gap at least as large as 1,751,112.

**Comparison of bounds**

These bounds appear to be wasteful. Consider, for example, the data in \[3\] p. 832. There, conditional values of $k_0$ are given along with the corresponding minimal length of the $k_0$-tuple. Table 1 compares the results in \[3\], obtained by an exhaustive computational search, with the upper and lower bounds obtained here.

While the upper bound gives the correct answer for small values of $k_0$, it becomes increasingly profligate as $k_0$ increases; the lower bound appears to be ubiquitously impotent.
Table 1: Comparison of bounds for gaps between successive primes

| $k_0$ | Upper bound | Lower bound | Length in [4] p. 829, 832 |
|-------|-------------|-------------|---------------------------|
| 6     | 16          | 12          | 16                        |
| 10    | 32          | 24          | 32                        |
| 12    | 46          | 30          | 42                        |
| 65    | 364         | 189         | 336                       |
| 193   | 1292        | 694         | 1204                      |
| 1000  | 8424        | 4165        |                            |
| 10000 | 109152      | 45815       |                            |
| 341,640 | 5,005,362*  | 1,751,112   |                            |

*Note that, as mentioned on page 2, this has been improved to 4,802,222.

The method of searching by brute force, potentially another poor man’s improvement, appears to be next to hopeless. For, given $k_0$ distinct non-negative integers of size at most $N$, there are $\binom{N}{k_0}$ possible $k_0$-tuples. Even with the modest value of $N = 7 \times 10^6$ one faces the daunting prospect of searching for an admissible $(341,640)$-tuple amongst more than $10^{2.5} \times 10^5$ possible candidates.

I am grateful for Scott Morrison’s providing me with data for $k_0 = 341,640$, and for the interesting work undertaken by him, Terry Tao, and others in [2].

References

[1] D.A. Goldston, J. Pintz, and C.Y. Yildirim. Primes in tuples I. *Ann. Math.*, 170(2):819–862, 2009.

[2] S. Morrison. [http://sbseminar.wordpress.com/2013/05/30](http://sbseminar.wordpress.com/2013/05/30) Website, 2013.

[3] Y. Zhang. Bounded gaps between primes. *Ann. Math.*, to appear.