New mixed solutions generated by velocity resonance in the (2 + 1)-dimensional Sawada–Kotera equation

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Abstract Based on the N-soliton solutions of the (2 + 1)-dimensional Sawada–Kotera equation, the collisions among lump waves, line waves, and breather waves are studied in this paper. By introducing new constraints, the lump wave does not collide with other waves forever, or stays in collision forever. Under the condition of velocity resonance, the soliton molecules consisting of a lump wave, a line wave, and any number of breather waves are derived for the first time. In particular, the interaction of a line wave and a breather wave will generate two breathers under certain conditions, which is worth exploring, and the method can also be extended to other (2 + 1)-dimensional integrable equations.

Keyword Velocity resonance · Mixed solutions · Soliton molecules · Lump wave

1 Introduction

The problem of soliton molecules is a very hot topic and has been studied extensively in recent years [1–9]. Soliton molecules are the stable bound state of solitons generated by velocity resonance [7], which breaks the previous understanding of traveling wave motion. Soliton molecules can appear in optical systems, fluid systems, and can be used in many fields, including ocean surface waves, electromagnetic waves in discrete transmission lines and plasma waves, etc. It is currently known that multiple soliton solutions can be derived by the Hirota bilinear method [3,4] and the Darboux transformation method [5,8,12,13], and then, soliton molecules can be obtained by constraints on parameters. Soliton molecule is a special kind of solution of nonlinear partial differential equation, and there are many kinds of solitons and methods to solve nonlinear partial differential equation, among which numerical method is popular to solve nonlinear time-dependent partial differential Eq. [14–21].

It is known that the collision between two solitary waves is elastic in the soliton theory, and the shape and velocity of the soliton do not change with the collision, only the phase changes. In fact, soliton molecules can be regarded as several waves with the same velocity. In (2 + 1)-dimensional integrable system, soliton molecules include line wave molecules [3–6,8,9], breather molecules [4,22,23], lump-line-breather molecules [10]. The definition of a lump-line molecule is a hybrid solution consisting of line and lump waves with the same velocity. Similarly, the lump-breather molecule is a hybrid solution consisting of lump wave and breather wave with the same velocity. However, lump-line molecules and lump-breather molecules have rarely been studied, which deserve further study.

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The (2 + 1)-dimensional Sawada–Kotera equation was introduced by Konopelchenko and Dubrovsky firstly [24], and it is a significant equation in integrable systems, this equation can be further regarded as an integrable extension of the (1 + 1)-dimensional Sawada–Kotera equation, and it may appear in various physical fields such as shallow water, ion acoustic waves in plasma and others, its research is beneficial to nonlinear science. The solutions of (2+1)-dimensional Sawada–Kotera equation have been extensively studied by scholars [4,28–38], such as soliton molecules [4], Y-type resonance soliton [29,34], mixed solutions [30–32,35–38], and so on. It is of great physical significance to study (2 + 1)-dimensional Sawada–Kotera equation, and the solutions under velocity resonance mechanism are worth further exploration. Therefore, the paper mainly uses bilinear method to study the new mixed solutions consisting of a lump wave and other waves or always collide or always collide with line waves , a lump wave never collide or always collide with breathers. The (2 + 1)-dimensional Sawada–Kotera equation studied in this paper is given as follows:

\[ u_t + u_{5x} + 15uu_xu_{xx} + 15uu_3x + 45u^2u_x + 5u_{xx}y + 15uu_y + 15u_x \int u_sdx - 5 \int u_ydyx = 0. \]  

(1)

Let us expand \( f \) formally in powers of an arbitrary parameter \( \epsilon \) as

\[ f(x, y, t) = 1 + f^{(1)}\epsilon + f^{(2)}\epsilon^2 + f^{(3)}\epsilon^3 + \cdots, \]  

(3)

where \( f^{(1)} = f^{(1)}(x, y, t) \), \( f^{(2)} = f^{(2)}(x, y, t) \), \( f^{(3)} = f^{(3)}(x, y, t) \), \( \cdots \).

Substituting Eq. (3) into Eq. (2) and comparing the coefficient of the same power of \( \epsilon \), we can get

\[
\begin{align*}
&f^{(1)}_{tx} - 5f^{(1)}_{xy} + f^{(1)}_{6x} + 5f^{(1)}_{3xy} = 0, \\
&2f^{(2)}_{tx} - 5f^{(2)}_{xy} + f^{(2)}_{6x} + 5f^{(2)}_{3xy} = - (D_xD_t + \eta^6(x, y) + 5D_xD_y - 5D_y^2)(f^{(1)}D_x + f^{(1)}), \\
&f^{(3)}_{tx} - 5f^{(3)}_{xy} + f^{(3)}_{6x} + 5f^{(3)}_{3xy} = - (D_xD_t + \eta^6(x, y) + 5D_xD_y - 5D_y^2)(f^{(1)}D_x + f^{(2)}), \\
&\cdots \cdots
\end{align*}
\]

(4)

By solving Eq. (4), we find that \( f^{(1)} \) has a solution in exponential form \( f^{(1)} = e^{\xi_1} \), and \( \xi_1 = \frac{k_1x - \frac{k_1}{5k_jp_j - 5p_{j}^2} \phi_1 - 5\phi_1}{k_1} \) is one soliton solution of Eq. (1) [39]. The \( N \)-soliton solutions of the equation can be obtained as follows

\[ f_N = \sum_{\mu=0,1}^N \sum_{j<s} \sum_{j=1}^N \mu_j A_{js} + \sum_{j=1}^N N \mu_j \xi_j, \]  

(7)

with

\[ A_{js} = - \frac{(k_j - k_s)(w_j - w_s) + (k_j - k_s)^6 + 5(k_j - k_s)^3 (p_j - p_s) - 5(p_j - p_s)^2}{(k_j + k_s)(w_j + w_s) + (k_j + k_s)^6 + 5(k_j + k_s)^3 (p_j + p_s) - 5(p_j + p_s)^2}. \]  

(8)

Here, \( u = u(x, y, t) \) is a potential function, \( x \) and \( y \) are spatial variables, and \( t \) is a time variable, the subscript in the formula means partial differentiation.

Substituting the dependent variable transformation \( u = 2\ln(f) \) into Eq. (1), it is reduced to the following bilinear form

\[ \left(D_xD_t + \eta^6 + 5D_x^2D_y - 5D_y^2\right) f \cdot f = 0, \]  

(2)

where \( D \) is the famous Hirota bilinear derivative operator defined by

\[ D^mD^nD^h f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^h f(x, y, t) \cdot g(x', y', t')|_{x'=x, y'=y, t'=t}. \]

\[ \xi_j = k_jx + w_jt + p_jy + \phi_j, \]  

(9)

and

\[ k_j^6 + 5k_j^3 p_j - 5p_j^2 + w_jk_j = 0, \]  

where \( k_j, p_j, w_j, \phi_j \) are parameters, \( x, t, y \) are the independent variables, and the sum of \( \mu \) is going to be all possible combinations of \( \mu_j = 0,1 (j = 1, 2, ..., N) \) in Eq. (7).

Based on the \( N \)-soliton solutions, a new type of mixed solutions is derived by using the partial long wave limit method and the new constraints mentioned in this paper, in which the lump wave will never collide with other waves or always collide with other waves. The discovery of this new type of nonlinear superposition will enrich the interaction solutions in fluid systems. In particular, the trajectories of lump wave before
and after its collision with other waves have been studied in some papers [25–28], it is of great significance to study the dynamic properties of soliton solutions.

2 The velocities of line waves, breather waves, lump waves

It is known that the orthogonal decomposition method can be used to study the uniform motion of objects, so the motion properties of solitons can also be studied by this method. The soliton moves at a uniform velocity in space with the change of time, so it can be regarded as moving in two mutually perpendicular directions, which represent the velocity in the \( x \)-direction and the \( y \)-direction, respectively.

Line wave is the most common kind of solitary waves, and its motion properties are also the most easily studied. In fact, Eq. (7) describes an interaction among \( N \) line waves and velocities of line waves are jointly determined by parameters \( k_s, p_s, (s = 1, 2, \cdots, N) \). The velocity formula for line waves is given by

\[
V_{\text{line}} = \left[ V_{\text{line}, x}, V_{\text{line}, y} \right]^T = \left[ \frac{-w_5 k_s}{k_s^2 + p_s^2}, \frac{-w_5 p_s}{k_s^2 + p_s^2} \right]^T,
\]  

(10)

where \( V_{\text{line}} \) represents the velocity of line wave, and \( V_{\text{line}, x} \) represents the velocity in the \( x \)-direction, \( V_{\text{line}, y} \) represents the velocity in the \( y \)-direction. Note that \( k_s, p_s, w_s \) are real parameters.

It is well known that a breather wave can be derived from two line waves via module resonance condition. When \( N = 2 \) in Eq. (7) with the parameters \( k_1 = k_2^s, p_1 = p_2^s, \phi_1 = \phi_2^s \), a breather wave will be derived. According to previous research [10], the velocity formulas of breathers wave can be given by

\[
V_{\text{breather}} = \left[ V_{\text{breather}, x}, V_{\text{breather}, y} \right]^T = \left[ \frac{-\Re(w_s)\Re(k_s)}{\Re(k_s)^2 + \Re(p_s)^2}, \frac{-\Re(w_s)\Re(p_s)}{\Re(k_s)^2 + \Re(p_s)^2} \right]^T,
\]  

(11)

where \( V_{\text{breather}} \) represents the velocity of a breather wave, and \( V_{\text{breather}, x} \) represents the velocity in the \( x \)-direction, \( V_{\text{breather}, y} \) represents the velocity in the \( y \)-direction. Note that \( k_s, p_s, w_s, (s = 1, 2, \cdots) \) are complex parameters, and \( \Re(w_s) \) means the real part of \( w_s \).

Lump wave can be derived from two line waves by the long wave limit method and module resonance [10, 11, 25–27]. Based on Eq. (7) with parameters \( N = 2, k_1 = K_1 \epsilon, k_2 = K_2 \epsilon, p_1 = P_1 \epsilon, p_2 = P_2 \epsilon, \phi_1 = \phi_2 = \pi i, K_1 = K_2^s, P_1 = P_2^s \), and considering \( \epsilon \rightarrow 0 \), a lump wave will be obtained. The velocity formula of a lump wave is given by

\[
V_{\text{lump}} = \left[ V_{\text{lump}, x}, V_{\text{lump}, y} \right]^T = \left[ \frac{5P_1 P_2}{K_1 K_2}, \frac{-5(K_1 P_2 + K_2 P_1)}{K_1 K_2} \right]^T,
\]  

(12)

where \( V_{\text{lump}} \) represents the velocity of a lump wave, and \( V_{\text{lump}, x} \) represents the velocity in the \( x \)-direction, \( V_{\text{lump}, y} \) represents the velocity in the \( y \)-direction.

3 The new nonlinear superposition of a lump wave and line waves

In this section, some new mixed solutions of a lump wave and line waves are derived by putting forward new constraints in \( N \)-soliton solutions Eq. (7). The lump wave is localized, while line wave is infinite in space, so can we regard the movement of a lump wave as the movement of a point, and the movement of a line wave as the movement of a straight line.

If a lump wave and a line wave never collide, there are two possibilities: The first is that the movement trajectory of lump wave is parallel to the line wave, and we put forward Proposition 1 to find this nonlinear phenomenon; The second is that the velocity of a lump wave and a line wave are completely the same, which is called a lump-line molecule.

Proposition 1 On the basis of the \( N \)-soliton solutions Eq. (7), if it satisfies the partial long wave limit constraints, the new nonlinear superposition of a lump wave and \( M \) line waves can be derived through the following constraints:

\[
N = 2 + M, k_i = K_i \epsilon, p_i = P_i \epsilon, \epsilon^{\phi_i} = -1, K_1 = K_2^s, P_1 = P_2^s, \epsilon \rightarrow 0, \quad (i = 1, 2)
\]  

(13)

and it is also restricted by the new constraint among parameters

\[
\lambda_s = 0, \quad (s = 3, 4, \cdots, N)
\]

with

\[
\lambda_s = -\frac{k_s^6 + 5k_s^3 p_s - 5p_s^2}{k_s} + \frac{5P_1 P_2 k_s}{K_1 K_2} - \frac{5(K_1 P_2 + K_2 P_1) p_s}{K_1 K_2},
\]
here, \( k_s \), \( p_s \) are real parameters, the movement trajectory of lump wave can be parallel to the line wave or lump wave moves along the line wave, and the parameters satisfy Eq. (8) and Eq. (9).

In the mixed solutions described by Proposition 1, when \( N = 3 \), the new nonlinear superposition can be expressed as

\[
u = 2(\ln f)_{xx},
\]

with

\[
f = \theta_1 \theta_2 + a_{12} + e^{\xi_1}(a_{13}a_{23} + \theta_1 a_{23}) + e^{\xi_2}(a_{14}a_{24} + \theta_1 a_{24}),
\]

\[
\theta_s = k_s x + P_s y + \frac{5p_s^2}{K_s}, \quad (s = 1, 2)
\]

and

\[
a_{js} = \begin{cases} 
- \frac{6K_j^2k_s^2(K_j P_s + P_j K_s)}{(K_j P_s - P_j K_s)^2}, & j < s < 3, \\
- \frac{6K_j^2k_s^2(K_j k_s^3 + K_j p_s + P_j k_s)}{K_j^2k_s^6 + 2K_j^2k_s^3 p_s + K_j P_j k_s + k_s^4 + K_j^2 p_s^2 - 2K_j P_j k_s p_s + k_s^2 p_j^2}, & j = 1, 2, s \geq 3.
\end{cases}
\]

Proving that the movement trajectory of a lump wave is parallel to that of the line wave, the lump wave moves uniformly along a straight line parallel to

\[
x = \frac{5P_1 P_2 t}{K_1 K_2} + l_1, \quad y = \frac{-5(K_1 P_2 + K_2 P_1)t}{K_1 K_2} + l_2,
\]

where \(|t|\) is large enough and \(l_1, l_2\) are real parameters, and Eq. (7) can be expressed as

\[
f_3 = \eta_1 \eta_2 + \eta_3 a_{12} + e^{\beta_3}(a_{13}a_{23} + \eta_1 a_{23}) + \eta_2 a_{13} + \eta_1 \eta_2 + a_{12},
\]

\[
\eta_s = k_s l_1 + P_s l_2, \quad \beta_3 = k_s l_1 + p_3 l_2 + \phi_3, \quad (s = 1, 2)
\]

here the parameters \(a_{js}, k_s, P_s\) satisfying Eqs. (13) and (14).

Lump wave can never collide with the line wave, but if the phase of the line wave is changed, let \(l_1 = l_2 = 0\) in Eq. (15), we found an interesting phenomenon that the motion trajectory of a lump wave is coincident with the line wave as shown in Fig. 1. As a result of the collision, a lump wave changes from a single-peaked soliton to double-peaked one, and Fig. 2a vividly shows the three-dimensional image of the double-peaked soliton, Fig. 2b shows the two-dimensional image of the double-peaked soliton at \(y = 0\). In addition, a lump wave and two line waves also produce the never-colliding situation and it is shown in Fig. 3.

When \( N = 4 \) in Proposition 1, the new nonlinear superposition consisting of a lump wave and two line waves are derived, it can be express as

\[
u = 2(\ln f)_{xx},
\]

with

\[
f = \theta_1 \theta_2 + a_{12} + e^{\xi_1}(a_{13}a_{23} + \theta_1 a_{23}) + \theta_1 a_{23} + e^{\xi_2}(a_{14}a_{24} + \theta_1 a_{24}) + \theta_1 a_{24} + \theta_2 a_{13} + a_{14}a_{24} + \theta_1 a_{24} + \theta_2 a_{14} + \theta_1 \theta_2 + a_{12},
\]

here \(\xi_s, \theta_s, A_{js}, a_{js}\) are give by Eqs. (8), (9), and (14).

At exactly the same velocity, the lump wave and the line wave can be seen as a whole moving in a certain direction together, and this phenomenon can be observed in Fig. 4. In particular, it has been verified that the velocity resonance among a lump and two line waves does not exist, so only a lump-line molecule consisting of a lump wave and two line waves are derived, it can be express as

\[
N = 3, \quad k_i = K_i \epsilon, \quad p_i = P_i \epsilon, \quad e^{\phi_i} = -1, \quad K_1 = K_s,
\]

\[
P_i = P_s, \quad \epsilon \to 0, \quad (i = 1, 2)
\]

with

\[
\left[5P_1 P_2, \frac{5(K_1 P_2 + K_2 P_1)}{K_1 K_2}, \frac{-5(K_1 P_2 + K_2 P_1)}{K_1 K_2}ight]^T = \left[-\frac{w_3 k_3}{k_3^2 + p_3}, \frac{w_3 p_3}{k_3^2 + p_3}ight]^T,
\]

and the parameters are given by Eqs. (8) and (9), \(k_3, p_3, w_3\) are real parameters.

4 The new nonlinear superposition of a lump wave and breathers

It is widely known that soliton molecules and soliton-breather molecules [4, 22, 23] have been extensively studied in many papers, but the lump-breather molecules have rarely been studied. The new nonlinear superposition consisting of a lump wave and breathers are derived
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Fig. 1 (Color online) A lump wave and a line wave always collide described by Proposition 1 with the parameters \( N = 3, K_1 = 1 + i, K_2 = 1 - i, P_1 = 2, P_2 = 2, k_3 = 1, p_3 = \frac{3}{2} - \frac{3\sqrt{5}}{10}, \phi_1 = i\pi, \phi_2 = i\pi, \phi_3 = -3 \)

Fig. 2 (Color online) The double-peaked soliton produced by Proposition 1 with the parameters \( N = 3, K_1 = 1 + i, K_2 = 1 - i, P_1 = 2, P_2 = 2, k_3 = 1, p_3 = \frac{3}{2} - \frac{3\sqrt{5}}{10}, \phi_1 = i\pi, \phi_2 = i\pi, \phi_3 = -3 \)

Fig. 3 (Color online) A lump and two line waves never collide described by Proposition 1 with the parameters \( N = 4, K_1 = 1 + i, K_2 = 1 - i, P_1 = 2, P_2 = 2, k_3 = 1, k_4 = -1, p_3 = \frac{1}{2} + \frac{3\sqrt{5}}{10}, p_4 = -\frac{1}{2} + \frac{3\sqrt{5}}{10}, \phi_1 = i\pi, \phi_2 = i\pi, \phi_3 = 20, \phi_4 = -20 \)
by proposing new constraints in this section. There are two types of solutions where a lump wave never collide with breather waves, the constraints are in Eqs. (18) and (20), respectively.

**Proposition 2** On the basis of the N-soliton solution Eq. (7), if it satisfies the partial long wave limit and module resonance constraints, the new nonlinear superposition of a lump wave and M breather waves can be derived through the following constraints

\[ N = 2 + 2M, k_i = K_i \epsilon, p_i = P_i \epsilon, e^{\phi_i} = -1, \]

\[ K_1 = K_2^s, P_1 = P_2^s, \epsilon \rightarrow 0, \quad (i = 1, 2) \]

\[ k_{1+2j} = k^u_{2+2j}, p_{1+2j} = p^u_{2+2j}, \]

\[ \phi_{1+2j} = \phi^u_{2+2j}, \quad (j = 1, 2, ..., M) \]

and it is also restricted by the new constraint between parameters

\[ \Re(\lambda_s) = 0, \quad (s = 3, 4, ..., N) \]

with

\[ \Im(\lambda_s) = \Im(-\frac{k_s^6 + 5k_s^3p_s - 5p_s^2}{k_s} + \frac{5P_1P_2k_s}{K_1K_2} - \frac{5(K_1P_2 + K_2P_1)p_s}{K_1K_2}), \]

here, \( k_s, p_s (s = 3, 4, ..., N) \) are complex parameters, the movement trajectory of lump wave is parallel to the breather wave, the parameters satisfy Eqs. (8) and (9).

When \( N = 4 \) in Proposition 2, a lump wave can never collide with a breather wave, and Fig. 5 vividly shows the interaction between a lump wave and a breather wave. Similarly, a lump wave can never collide with \( M \) breather waves.

In lump-breather molecules, a lump wave also does not collide with \( M \) breather waves under the velocity resonance mechanism. The parameters constraints are given by

\[ N = 2 + M, k_i = K_i \epsilon, p_i = P_i \epsilon, e^{\phi_i} = -1, \]

\[ K_1 = K_2^s, P_1 = P_2^s, \epsilon \rightarrow 0, \quad (i = 1, 2) \]

with

\[ \begin{bmatrix} 5P_1P_2 & -5(K_1P_2 + K_2P_1) \\ K_1K_2 & K_1K_2 \end{bmatrix}^T \]

\[ = \begin{bmatrix} -\Re(w_j)\Re(k_j) - \Re(w_j)\Re(p_j) & -\Re(w_j)\Re(p_j) \\ \Re(k_j)^2 + \Re(p_j)^2 & \Re(k_j)^2 + \Re(p_j)^2 \end{bmatrix}^T, \]

\( (s = 3, 4, ..., N) \)

and

\[ p_{2j+1} = p^u_{2j+2}, k_{2j+1} = k^u_{2j+2}, \quad (j = 1...M) \]

here, the parameters are based on Eq. (7), and \( k_s, p_s, w_s \) are complex parameters.

The breather-line molecules can be generated by line waves and breather waves. We also find a very interesting exact solution where a line wave and a breather wave will generate two breathers. And this amazing phenomenon is vividly shown in Fig. 7. As a line wave and a breather wave gradually approaching, two breathers are produced by controlling the phase of the line wave, the reasons for this phenomenon deserve in-depth study. On the basis of the 3-soliton solutions, the

Fig. 4 (Color online) The lump-line molecule with the parameters \([N = 3, k_1 = \frac{9\sqrt{19}}{19} + i, k_2 = \frac{9\sqrt{19}}{19} - i, k_3 = -\frac{\sqrt{2\sqrt{53} + 10}}{2}, p_1 = -\frac{\sqrt{19}}{19} + i, p_2 = -\frac{\sqrt{19}}{19} - i, p_3 = \frac{\sqrt{2\sqrt{53} + 10}}{2}, \phi_1 = i\pi, \phi_2 = i\pi, \phi_3 = 50]\)
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condition for the formation of a breather-line molecule is given by

\[
\begin{bmatrix}
-\frac{w_1 k_1}{k_1^2 + p_3^2}, & -\frac{w_3 p_3}{k_1^2 + p_3^2} \\
-\frac{\Re(w_1)\Re(k_1)}{\Re(k_1)^2 + \Re(p_3)^2}, & -\frac{\Re(w_1)\Re(p_3)}{\Re(k_1)^2 + \Re(p_3)^2}
\end{bmatrix}^T, \quad (s = 1, 2)
\]

with

\( k_1 = k_2^s, \ p_1 = p_2^s, \ \phi_1 = \phi_2^s, \)

here, \( k_3, \ p_3, \ w_3 \) are real parameters and satisfy Eq. (9).

We can also derived lump-line-breather molecules by comprehensively considering Propositions 1 and 2. Based on the \( N \)-soliton solutions, using partial long-wave limit and module resonance, and satisfying the following conditions Eq. (21), the molecule consisting of a lump, a line waves and \( Q \) breather waves can be derived. Once the initial state is determined, the position among them will not be changes over time, a lump-line-breather molecule consisting of a lump wave, a line wave and a breather wave can be seen in Fig. 8. By extension, the new nonlinear superposition of a lump wave and a line wave and \( Q \) breather waves can be derived with the conditions follows

\[ V_{\text{lump}} = V_{\text{line}} = V_{\text{breather}}. \]
Fig. 7 (Color online) A line wave gradually approaches the breather to form two breathers with the parameters \( N = 3, k_1 = 1 + i, k_2 = 1 - i, k_3 = \frac{\sqrt{10 + 2 \sqrt{37}}}{2}, p_1 = -1 + i \left(1 + \frac{\sqrt{70}}{5}\right), p_2 = -1 - i \left(1 + \frac{\sqrt{70}}{5}\right), p_3 = -\frac{1}{\sqrt{10 + 2 \sqrt{37}}}, \phi_1 = 0, \phi_2 = 0, \phi_3 = 3 \) at \( t = 0 \).

Fig. 8 (Color online) The lump-line-breather molecule with the parameters \( N = 5, K_1 = 3 i, K_2 = -3 i, k_3 = \frac{1}{3} + \frac{1}{2} i, k_4 = \frac{1}{3} - \frac{1}{2} i, k_5 = \frac{\sqrt{10 + 2 \sqrt{85}}}{2}, p_1 = \frac{3}{2} \sqrt{3} + \frac{3}{2} i, p_2 = \frac{3}{2} \sqrt{3} - \frac{3}{2} i, p_3 = -\frac{1}{3} + i \left(\frac{23}{72} + \frac{\sqrt{64285}}{270}\right), p_4 = -\frac{1}{3} - i \left(\frac{23}{72} + \frac{\sqrt{64285}}{270}\right), p_5 = -\frac{1}{\sqrt{10 + 2 \sqrt{85}}}, \phi_1 = \pi, \phi_2 = \pi, \phi_3 = -5, \phi_4 = 5, \phi_5 = 0 \) with

\[
\begin{bmatrix}
5P_1P_2 - 5(K_1P_2 + K_2P_1) \\ K_1K_2
\end{bmatrix}^T = \begin{bmatrix}
-\frac{w_1k_1k_2}{k_1^2 + p_3^2} - \frac{w_3p_3}{k_3^2 + p_3^2} \\
-\frac{\Re(w_1)\Re(k_1) - \Re(w_1)\Re(p_1)}{\Re(k_1)^2 + \Re(p_1)^2} - \frac{\Re(w_3)\Re(p_3)}{\Re(k_3)^2 + \Re(p_3)^2}
\end{bmatrix}^T,
\]

\( (s = 4, 5, \ldots, N) \)

and

\( N = 3 + 2Q, k_i = K_i\epsilon, p_i = P_i\epsilon, e^{\phi_i} = -1, K_1 = K_2^*, P_1 = P_2^*, \epsilon \to 0, \) (\( i = 1, 2 \))

\( k_{2j+2} = k_{2j+3}^*, p_{2j+2} = p_{2j+3}^*, \phi_{2j+2} = \phi_{2j+3}, \) (\( j = 1, 2, \ldots, Q \))

where the parameters based on Eq. (7), and \( k_3, p_3, \phi_3 \) are real parameters, \( k_5, p_5, (s = 4, 5, \ldots, N) \) are complex parameters.

There are still many questions about the velocity resonance mechanism of solitons that can be studied, and it is very meaningful in nonlinear dynamical system.

5 Conclusion

According to the velocity formulas of various solitons, this paper studies some new mixed solutions of \((2 + 1)-\)
dimensional Sawada–Kotera equation by some special constraints among parameters. This paper explores two new mixed solutions of lump waves never collide or always collide with other waves. The first hybrid solution is a nonlinear superposition between a single lump wave and other waves, in which the motion trajectory of the lump wave parallel to other waves. And these waves can collide with each other forever by means of adjustment of phase parameters. The graph of these relevant solutions is shown in Figs. 1, 3 and 5. The second hybrid solution is a soliton molecule consisting of a lump wave and other waves, where the lump wave and other waves remain relatively stationary. These interesting solutions are shown in Figs. 4, 6, and 8 vividly. In addition, there is a very interesting phenomenon that in a breather-line molecule, two breather waves will be obtained as the line wave and breather wave approach each other gradually, it is shown in Fig. 7. These studies enrich the type of hybrid solutions in the integrable system.

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Data availability All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests.

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