The integer homology threshold in $Y_d(n, p)$

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Abstract

We prove that in the $d$-dimensional Linial–Meshulam stochastic process the $(d-1)$st homology group with integer coefficients vanishes exactly when the final isolated $(d-1)$-dimensional face is covered by a top-dimensional face. This generalizes the $d = 2$ case proved recently by Luczak and Peled and establishes that $p = d \log n \over n$ is the sharp threshold for homology with integer coefficients to vanish in $Y_d(n, p)$, answering a 2003 question of Linial and Meshulam.

1 Statement of the Result

Here we consider the stochastic process version of the Linial–Meshulam random simplicial complex model. Recall that the Linial–Meshulam random simplicial complex model (introduced in [9]), denoted $Y_d(n, p)$ for $d$ a fixed dimension, $n \in \mathbb{N}$, and $p = p(n) \in [0, 1]$, is the probability space on $d$-dimensional simplicial complexes with complete $(d-1)$-skeleton generated by including each possible $d$-dimensional face independently with probability $p$. Accordingly the (discrete-time) stochastic process version of $Y_d(n, p)$, which we denote here as $Y_d(n)$, following [10], is a Markov process $Y_d(n, 0) \subseteq Y_d(n, 1) \subseteq \cdots \subseteq Y_d(n, (n\over d+1))$ where $Y_d(n, 0)$ is the complete $(d-1)$-complex on $n$ vertices and $Y_d(n, k)$ is generated by adding a $d$-dimensional face to $Y_d(n, k-1)$ chosen uniformly at random from among all $d$-dimensional faces not included in $Y_d(n, k-1)$.

For a topological property $P$ and a single instance of $Y_d(n)$, the hitting time for property $P$ is defined to be the minimal $m$ so that $Y_d(n, m)$ satisfies property $P$. A statement $S$ about $Y_d(n)$ is said to hold with high probability if the probability $S$ holds tends to 1 as $n \to \infty$.

Our main new result on $Y_d(n)$ is the following theorem: following tradition, we call a $(d-1)$-dimensional face isolated if it is not covered by any $d$-dimensional face.

**Theorem 1.** Fix $d \geq 2$. With high probability the $(d-1)$st homology group of $Y = Y_d(n)$ with integer coefficients vanishes exactly when the last isolated $(d-1)$-dimensional face is covered by a $d$-dimensional face. That is the hitting time for the property that no $(d-1)$-dimensional face of $Y$ is isolated exactly coincides with the hitting time for the property that $H_{d-1}(Y; \mathbb{Z}) = 0$.

In fact we prove a slightly stronger result in Corollary which shows that slightly before the final isolated $(d-1)$-dimensional face is covered, the $(d-1)$st homology group is a free abelian group with rank given by the number of isolated faces.

The $d = 2$ case of Theorem was previously established by Luczak and Peled [10]. Moreover, the $d = 1$ is the classic result of Bollobas and Thomason [1] that the stochastic random graph becomes connected at the exact moment its last isolated vertex is covered by an edge.

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2 Background

The Linial–Meshulam model is a higher-dimensional generalization of the Erdős–Rényi random graph, and one of the most fundamental results in random graph theory is the following theorem of Erdős and Rényi which establishes the connectivity threshold for $G(n, p)$.

**Theorem ([3]).** For $c < 1$ and $p = \frac{c \log n}{n}$, with high probability $G \sim G(n, p)$ is not connected, and for $c > 1$ and $p = \frac{c \log n}{n}$, with high probability $G \sim G(n, p)$ is connected.

This statement can be given a homological reformulation by observing that a graph is connected if and only if its zeroth reduced homology group is trivial. This motivates the general definition of homological connectivity over abelian group $R$ higher dimensional simplicial complexes. A $d$-dimensional simplicial complex $X$ is said to be homologically connected over $R$ provided that $	ilde{H}_i(X; R) = 0$ for all $i \leq d - 1$. In $Y_d(n, p)$, all complexes have complete $(d - 1)$-skeleton so homological connectivity over $R$ of $Y \sim Y_d(n, p)$ is equivalent to $\tilde{H}_{d-1}(Y; R) = 0$.

In the case that $R = \mathbb{Z}$, we will simply say the complex is homologically connected. Indeed, by the universal coefficient theorem, a complex $X$ is homologically connected if and only if it is homologically connected over all abelian groups $R$.

Generalizing the connectivity result of Erdős and Rényi, Linial and Meshulam prove the following.

**Theorem ([9]).** For $c < 2$ and $p = \frac{c \log n}{n}$, with high probability $Y \sim Y_2(n, p)$ satisfies $H_1(Y; \mathbb{Z}/2\mathbb{Z}) \neq 0$, and for $c > 2$ and $p = \frac{c \log n}{n}$, with high probability $Y \sim Y_2(n, p)$ satisfies $H_1(Y; \mathbb{Z}/2\mathbb{Z}) = 0$.

For any $d$-complex $Y$, $H_{d-1}(Y; \mathbb{Z}/2\mathbb{Z}) = 0$ implies that $H_{d-1}(Y; \mathbb{Q}) = 0$. By the universal coefficient theorem, therefore, the above result of Linial and Meshulam implies that $H_1(Y; \mathbb{Z})$ is finite for $Y \sim Y_2(n, \frac{c \log n}{n})$ and $c > 2$. However, the result does not imply that $H_1(Y, \mathbb{Z}) = 0$ in this case; a priori, it may be some other finite group. Thus, unlike the case of the Erdős–Rényi random graph, it is not sufficient to consider only $\mathbb{Z}/2\mathbb{Z}$ coefficients to show that $H_{d-1}(Y)$ is trivial. Generalizing the Linial–Meshulam result to higher dimensions and to other coefficients rings, Meshulam and Wallach prove the following result.

**Theorem ([11]).** Fix $d \geq 1$, and let $R$ be a fixed finite abelian group. For $c < d$ and $p = \frac{c \log n}{n}$, with high probability $Y \sim Y_d(n, p)$ satisfies $H_{d-1}(Y; R) \neq 0$, and for $c > d$ and $p = \frac{c \log n}{n}$, with high probability $Y \sim Y_d(n, p)$ satisfies $H_{d-1}(Y; R) = 0$.

If $Y$ is a $d$-dimensional simplicial complex with $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0$ for all primes $q$ then $H_{d-1}(Y; \mathbb{Z}) = 0$. However, the theorem of Meshulam and Wallach does not rule out the possibility that $H_{d-1}(Y; \mathbb{Z})$ has $q$-torsion for a sequence of primes $q$ which grow with $n$. Indeed until now the question of the homological connectivity threshold (with integer coefficients) has been open for all $d \geq 3$. Previously, the best result for $d \geq 3$ about the homological connectivity threshold was the following result of Hoffman, Kahle and Paquette:

**Theorem ([3]).** For $d \geq 2$ and $p \geq \frac{8d \log n}{n}$, with high probability $Y \sim Y_d(n, p)$ satisfies $H_{d-1}(Y) = 0$.

For $d = 2$, the main hitting-time result of [10] establishes that $\frac{2 \log n}{n}$ is the sharp threshold for the first homology group with integer coefficients to vanish in $Y_2(n, p)$, and our result here generalizes this hitting-time result to higher dimensions.

We should also mention that over the field $\mathbb{Z}/2\mathbb{Z}$, the hitting-time result has been established. If one considers homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients the hitting-time result was previously known in the $d = 2$ case due to Kahle and Pittel [7], and more recently the $\mathbb{Z}/2\mathbb{Z}$ version of the hitting-time for homological connectivity was proved by Cooley et al. [2] in all dimensions. In addition a hitting-time result that for $\mathbb{Q}$-coefficients is proved in [3].
3 Cocycle counting

While the $d = 2$ case of Theorem 1 has already been established in [10], we develop a new approach based on the methods of Meshulam and Wallach [11]. They develop the technique of cocycle counting to show that for any fixed finite abelian group $R$, $H_{d-1}(Y_d(n, p); R) = 0$ when $p = \frac{d \log n + \omega(1)}{n}$. That is, rather than considering homology, they consider cohomology and bound the probability that $Y \sim Y_d(n, p)$ has a nontrivial cocycle. When $R$ is fixed and finite this may be accomplished by showing that the expected cardinality $E|H^{d-1}(Y; R)|$ tends to 0.

We will adapt this technique to work over many fields simultaneously. Following [11], we start by defining some useful notation.

**Definition 1.** For a $(d-1)$-cochain $\phi$ of the simplex on $n$ vertices with coefficients in any field $R$, the **weight** of $\phi$, denoted $w(\phi)$, is defined to be the minimum of the support of $\phi'$ for all $\phi'$ with $\phi - \phi'$ a coboundary, and $b(\phi)$ is defined to be the number of $d$-dimensional faces $\sigma$ in the simplex on $n$ vertices so that $\partial^*_\sigma(\phi)(\sigma) \neq 0$.

With this notation in hand, Meshulam and Wallach prove the following coisoperimetric inequality [11, Proposition 3.1].

**Lemma 1** (Coisoperimetric inequality). For any abelian group $R$ and any $(d-1)$-chain of the simplex on $n$ vertices,

$$b(\phi) \geq \frac{nw(\phi)}{d+1}.$$

Note that although $b(\phi)$ and $w(\phi)$ depend on the underlying ring $R$, the coisoperimetric inequality is uniform over all abelian groups $R$.

We will now sketch the basic cocycle counting method. Observe that if $\phi$ is a $(d-1)$-cochain then the probability that it is a cocycle over $Y \sim Y_d(n, p)$ is $(1 - p)^{b(\phi)}$. Consider each equivalence class of cochains modulo coboundaries and choose a minimal-support element from each equivalence class. If $\phi$ is a cochain with support size $k$ and weight equal to $k$, then by Lemma 1 the probability that $\phi$ is a cocycle is at most $(1 - p)^{nk/(d+1)}$. Thus if $R$ is a fixed finite field of size $r$, we have that for $Y \sim Y_d(n, p)$,

$$\Pr(H^{d-1}(Y; R) \neq 0) \leq \sum_{k=1}^{(n)\choose{k}} \left( \binom{n}{d} \right) (r - 1)^k (1 - p)^{nk/(d+1)}.$$

Indeed there are $\binom{n}{k}$ choices for the support of a cochain of weight $k$, and from there at $(r - 1)$ choices for the coefficient associated to each facet in the support. It follows that if $p = \frac{e \log n}{n}$ for $e > d(d + 1)$ then $H^{d-1}(Y; R) = 0$ for $Y \sim Y_d(n, p)$. In order to improve on this, Meshulam and Wallach find a better bound than $\binom{n}{k}(r - 1)^k$ for the number of nontrivial cocycles and use the coisoperimetric inequality in a more subtle way.

In the current situation we want to show that homology with integer coefficients vanishes. Our approach is based on the elementary observation that for any simplicial complex $X$, if $H_{d-1}(X; \mathbb{Z}/q\mathbb{Z}) = 0$ for all primes $q$ then $H_{d-1}(X; \mathbb{Z}) = 0$. So, we adapt the cocycle counting method of Meshulam and Wallach to work over $\mathbb{Z}/q\mathbb{Z}$ for all primes $q$ simultaneously.

It is worth pointing out that a direct first-moment argument alone cannot work if $q$ is very large. If we sample $Y$ from $Y_d(n, \frac{e \log n}{n})$ then the probability that $Y$ has no $d$-dimensional faces is $\exp(-\Theta(n^d \log n))$. In this case, the dimension of $H_{d-1}(Y; R)$ for any field $R$ is $\binom{n}{d}$, so if $R = \mathbb{Z}/q\mathbb{Z}$, for $q$ a prime larger than $\exp(n^d)$, then the expected number of cocycles over $R$ is at least $\exp(\Theta(n^{2d}) - \Theta(n^d \log n)) \to \infty$. Thus is critical that we eliminate the $(r - 1)^k$ term from the cocycle counting method.
To do so, rather than consider \((d - 1)\)-cochains, we consider \((d - 1)\)-dimensional complexes and bound the probability that any such complexes support a cocycle over any prime-order finite field. We will make use of the coisoperimetric inequality (Lemma \(\text{I}\)), but now in a more geometric way. We first define the following geometric analogue to \(b(\phi)\).

**Definition 2.** For a fixed field \(R\) and any \((d - 1)\)-dimensional subcomplex \(X\) of the simplex on \(n\) vertices we define \(b(X, R)\) as:

\[
b(X, R) := \inf \{ b(\phi) : \phi \text{ is supported exactly on } X \text{ with coefficients in } R \text{ with } w(\phi) = |X| \}.
\]

Note that \(b(X, R)\) can be infinity but only in the situation where there are no cochains of minimum weight supported on \(X\). We also define \(b(X)\) to be the infimum of \(b(X, R)\) over \(R = \mathbb{Z}/q\mathbb{Z}\) for all primes \(q\) and \(R = \mathbb{Q}\).

Now \(b(X)\) is closely related to \(b(\phi)\) where \(X\) is a \((d - 1)\)-complex and \(\phi\) is a cochain supported on \(X\), but it removes everything about an underlying coefficient ring. Meshulam and Wallach also define a geometric quantity \(\beta(X)\), closely related to \(b(\phi)\) for \(\phi\) supported on \(X\), as follows.

**Definition 3.** For a \((d - 1)\)-dimensional subcomplex \(X\) of the simplex on \(n\) vertices we define \(\beta(X)\) to be the number of \(d\)-dimensional faces which contain exactly one \((d - 1)\)-dimensional face of \(X\).

We make use of this definition too in Section \(\text{V}\). In outlining their proof in \(\text{III}\), Meshulam and Wallach point out that the coisoperimetric inequality does not hold if \(b(\phi)\) is replaced with \(\beta(X)\) and that this is a major obstacle to applying their technique to prove that integer homology vanishes. Nonetheless, it is a useful quantity because for any cochain \(\phi\) with coefficients in \(R\), minimally-supported on \(X\), one has \(\beta(X) \leq b(X) \leq b(\phi)\).

So while \(\beta(X)\) does not satisfy the coisoperimetric inequality, the geometric quantity \(b(X)\) does satisfy it. Indeed, due to the uniformity in \(R\) in the coisoperimetric inequality (Lemma \(\text{I}\)),

\[
b(X) \geq \frac{n|X|}{d + 1}.
\]

There is one potential disadvantage to using \(b(X)\) in place of \(b(\phi)\). If \(R\) is a coefficient ring and if \(\phi\) is a cochain over \(R\) which is minimally supported on \(X\), then it is clear from the definition of \(b(\phi)\) that the probability that \(\phi\) is a cocycle of \(Y_d(n, p)\) is \((1 - p)^{b(\phi)}\). However, if instead we wish to bound the probability that \(X\) is the support of a cocycle over any prime-order finite field, we no longer have the simple bound \((1 - p)^{b(X)}\). Nonetheless, as we will show in Lemma \(\text{III}\) this bound is true up to a lower–order correction.

To frame Lemma \(\text{III}\) we begin by introducing the following notation.

**Definition 4.** For a fixed field \(R\) and any \((d - 1)\)-dimensional subcomplex \(X\) of the simplex on \(n\) vertices, we let \(z(X, R)\) denote the event that there exists a cocycle \(\phi\) over \(R\) with \(w(\phi) = |X|\) and \(\text{supp}(\phi) = X\). We let \(z(X)\) denote the event that there exists \(R\) in \(\{\mathbb{Z}/q\mathbb{Z} : q\text{ is prime and at most } \sqrt{d + 1}^{|X|}\} \cup \{\mathbb{Q}\}\) so that \(z(X, R)\) holds.

The choice of \(\sqrt{d + 1}^{|X|}\) in the definition of \(Z(X)\) comes from a bound on the size of the torsion group of the cokernel of an integral matrix, given as Claim \(\text{II}\). This claim is essentially Proposition 3 of \(\text{I}\) who credits it to Gabber. However, we do have a sharper exponent in our bound than in \(\text{I}\) \((t^{\text{rank}(M)}\) compared to \(t^{\text{min}(n, m)}\)). This sharper exponent is not necessary to our application here, but in the interest of keeping the proof self-contained we give a proof of Claim \(\text{II}\) and no additional work is require to obtain the sharper exponent.

**Claim 2.** If \(M\) is a matrix with integer entries so that the norm of every column of \(M\) is at most \(t\), then the torsion part of the cokernel of \(M\), denoted \(\text{coker}(M)_T\), has size at most \(t^{\text{rank}(M)}\).
Proof. Let $M$ be a matrix which satisfies our assumptions. First, define $N$ to be a restriction of $M$ to a maximal set of $\mathbb{Q}$-linearly independent columns of $M$. We have that $\text{coker}(M)_T \leq \text{coker}(N)_T$. Indeed, this immediately as clearly $\text{Im}_{\mathbb{Z}}(N) \leq \text{Im}_{\mathbb{Z}}(M)$.

Now we want to construct a square matrix from $N$ in a canonical way. Beginning with $N$ let $i_1$ be the smallest index in $\{1, \ldots, m\}$ so that the standard basis vector $e_{i_1}$ is not in the $\mathbb{Q}$-span of $N$; add $e_{i_1}$ to $N$. Now let $i_2$ be the smallest index in $\{1, \ldots, m\}$ so that $e_{i_2}$ is not in the $\mathbb{Q}$-span of $N$ and $e_{i_1}$, add $e_{i_2}$ to the matrix. Continue in this way to arrive at a (necessarily square) matrix $N'$. We check that $\text{coker}(N)_T \leq \text{coker}(N')_T$.

We check the subgroup inclusion inductively. Suppose $v$ is a torsion element of the span of the columns of $N$ together with standard basis vector $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$, but $v$ is not a torsion element of the cokernel after adding the column $e_{i_{k+1}}$. Then $v$ can be written as an integral linear combination of columns of $N$ and standard basis vectors $e_{i_1}, e_{i_2}, \ldots, e_{i_k}, e_{i_{k+1}}$, with nonzero coefficient $\alpha$ on $e_{i_{k+1}}$. However since $v$ is a torsion element of the cokernel before adding $e_{i_{k+1}}$, we have that there exists an nonzero integer $s$ so that $sv$ is in the integer span of the columns $N$ and $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$. This gives us two ways to write $sv$ as a linear combination of columns of $N$ and $e_{i_1}, e_{i_2}, \ldots, e_{i_k}, e_{i_{k+1}}$, one with coefficient $s\alpha$ on $e_{i_{k+1}}$ and one with coefficient $0$ on $e_{i_{k+1}}$. By linear independence of the columns of $N'$ we have that $s\alpha = 0$, a contradiction. Thus $\text{coker}(N)_T \leq \text{coker}(N')_T$ and we complete the proof.

Similar bounds appear in [5][8][10]. In [8], it is shown that if $X$ is a $d$-dimensional simplicial complex on $n$ vertices then $|H_{d-1}(X)_T| \leq \sqrt{d+1}^{\binom{n}{d-2}}$. Claim 2 essentially gives a local version of this result.

Lemma 3. For any $c > (d-1)/2$, there is an $n_0$ sufficiently large so that for all $n \geq n_0$, for all $(d-1)$-dimensional complexes $X$ with $b(X) = (1-\theta)nk \geq nk/(d+1)$, where $k := |X|$, and for $Y \sim Y_d(n,\lceil c\log n \over n \rceil)$, the probability that $z(X)$ holds is at most $n^{-c/(1-\theta)(d-1/2)}$.

Proof. Fix a $(d-1)$-dimensional complex with $b(X) = (1-\theta)nk$ where $k := |X|$, and fix a field $R$ to be either $\mathbb{Z}/q\mathbb{Z}$ for $q \leq \sqrt{d+1}^k$ or to be $\mathbb{Q}$. We will bound the probability of $z(X,R)$ for $R$ fixed and then take a union bound over all at-most $\sqrt{d+1}^k$ necessary fields to bound $z(X)$.

We will use the Linial–Meshulam stochastic process $Y_d(n) = \{Y_d(n,i)\}_{i=0}^{\lceil c\log n \over n \rceil}$ to sample from $Y_d(n,m)$ where $m := \lceil c\log n \over n \rceil$. For each $i$, let $z(X,R,i)$ be the event that $X$ is the support of a cocycle over $R$ of weight $k$ in $Y_d(n,i)$. At each step $i$, we let $N(i)$ denote the dimension of the kernel of the coboundary matrix of $Y_d(n,i)$ restricted to the columns associated to $X$. Clearly if $N(m) = 0$, then $X$ is not the support of a cocycle over $R$.

For each $i$ let $p_i$ denote the probability that $N(i+1) < N(i)$. Now if $X$ is the support of a cocycle of weight $k$ over $R$ in $Y_d(n,i)$ then

$$p_i \geq {b(X) \over \binom{n}{d+1}} \geq (1-\theta)nk \over \binom{n}{d+1}.$$ 

We want to bound the following probability

$$\Pr((N(m) > 0) \cap z(X,R,m)).$$

Note that $z(X,R,m)$ implies $N(m) > 0$ so the probability above is actually equal to
Pr(z(X, R)). Clearly,

\[
\Pr(X(m) > 0 \cap z(X, R, m)) \leq \Pr \left( \bigvee_{i \leq m} \left( X(m) > 0 \cap z(X, R, m) \mid p_i \geq \frac{(1 - \theta)nk}{(d+1)} \right) \right) \\
+ \Pr \left( \bigvee_{i \leq m} \left( X(m) > 0 \cap z(X, R, m) \mid p_i < \frac{(1 - \theta)nk}{(d+1)} \right) \right).
\]

Now the second summand is zero. Indeed while \( z(X, R, i) \) holds \( p_i \geq \frac{(1 - \theta)nk}{(d+1)} \) and if \( z(X, R, i) \) fails for some \( i \leq m \), then so does \( z(X, R, m) \). The goal is to bound the first summand. This will be accomplished by comparison to a binomial random variable. Let \( B \) be a binomial random variable with \( m \) trials and success probability \( \psi = \frac{(1 - \theta)nk}{(d+1)} \). Since \( X(0) = k \), it follows that

\[
\Pr \left( \bigvee_{i \leq m} \left( X(m) > 0 \mid p_i \geq \frac{(1 - \theta)nk}{(d+1)} \right) \right) \leq \Pr(B < k).
\]

We bound the probability that \( B \) is less than \( k \), which since \( k \) is less than \( \mathbb{E}(B) = m\psi = \Omega(k \log n) \), allows the following version of Chernoff’s inequality to apply:

\[
Pr(B < k) \leq \exp(mH_\psi(k/m)),
\]

where \( H_\psi(x) = x \log(\psi/x) + (1 - x) \log((1 - \psi)/(1 - x)). \)

Observe that \( k \) is no more than \( n^d \), and that \( n^d/m = O(1/\log n) \). Hence uniformly in \( k \leq n^d \), we have that

\[
mH_\psi(k/m) = k \log(m\psi/k) + (m - k) \log(1 - \psi) - (m - k) \log(1 - k/m)
\]

\[
\leq k \log(m\psi/k) - m\psi + k(1 + \psi) + O(k^2/m)
\]

\[
\leq -m\psi + k \log \log n + k(1 + O(1/\log n)).
\]

Thus there is a constant \( C > 0 \) so that for all \( n \) large enough and for all \( k \leq n^d \)

\[
\Pr(B < k) \leq n^{-(1-\theta)ck} (C \log n)^k.
\]

Now we sum over all fields \( \mathbb{Z}/q\mathbb{Z} \) with \( q \leq \sqrt{d+1}^k \) and the field \( \mathbb{Q} \) to get that the probability that \( X \) is the support of a cocycle over any such field is at most:

\[
\Pr(z(X)) \leq n^{-(1-\theta)ck} (C \sqrt{d+1} \cdot \log n)^k.
\]  

(1)

This gives the desired bound for all \( n \) sufficiently large.

\[
\square
\]

**Remark 1.** We should point out here that there is nothing particularly meaningful about the choice of \((d-1/2)\) in the lemma. Later, this turns out to be a convenient value to have, and so we use it here to simplify some notation.

Also, recalling the definition of \( b(X) \), there is some cochain \( \phi \) over some field \( R \) such that \( \phi \) is supported on \( X \) and so that \( b(\phi) = b(X) \). For this particular cochain, the probability it is a cocycle is \((1 - p)^{b(X)} = n^{-(1-\theta)ck} \). Hence we have

\[
n^{-(1-\theta)ck} \leq \Pr(z(X)) \leq n^{-(1-\theta)ck} (C \sqrt{d+1} \cdot \log n)^k,
\]

showing that the above bound is accurate up to subleading factors.
4 Overview of the proof

Now that we have defined $z(X)$, we can give an outline of the proof of Theorem\textsuperscript{1}. Essentially the idea of the proof will be to prove that with high probability $Y \sim Y_d(n, \frac{c \log n}{n})$ satisfies three particular conditions for $c > d - 1/2$ and then to show that these three conditions deterministically imply Theorem\textsuperscript{2}. The majority of the work of this paper is to prove the former, which we state below as Lemma\textsuperscript{3}. To simplify notation, for a $d$-dimensional simplicial complex, we use “face”, “facet”, and “ridge” to refer to $d$-dimensional faces, $(d-1)$-dimensional faces, and $(d-2)$-dimensional faces respectively. We also say that a cocycle $\phi$ is inclusion-minimal over a field $R$ if there is no cocycle over $R$ supported on any proper subset of the support of $\phi$.

Lemma 4. Fix $d \geq 2$ and $c > d - 1/2$, then with high probability $Y \sim Y_d(n, \frac{c \log n}{n})$ satisfies the following three conditions:

1. $z(X)$ fails to hold for all $(d-1)$-subcomplexes $X$ with $|X| \geq n/(3d)$
2. $Y$ has no inclusion minimal $(d-1)$- cocycles of support size $k$ over any field for $2 \leq k \leq n/(3d)$.
3. $Y$ has no isolated facets that meet at a ridge.

These three conditions, in turn, imply the desired homology vanishing on $Y$, as a consequence of the following deterministic lemma.

Lemma 5. Suppose that $Y$ is a $d$-dimensional simplicial complex with complete $(d-1)$-skeleton so that conditions 1 and 2 from Lemma\textsuperscript{3} hold, then $H_{d-1}(Y)$ is a free abelian group of rank equal to the number of isolated facets of $Y$. Moreover if all three conditions hold then the stochastic process of adding $d$-dimensional faces uniformly at random to $Y$ will result in a complex $Y' \supseteq Y$ which has $H_{d-1}(Y') = 0$ exactly at the moment the final isolated facet of $Y$ is covered.

In the proof of this lemma we encounter the term strongly-connected which we define here and will use again in Section\textsuperscript{6}.

Definition 5. For a $d$-dimensional simplicial complex $X$ we define the dual graph $G(X)$ to be the graph whose vertex set is the set $d$-dimensional faces of $X$ with an edge between $\sigma$ and $\tau$ provided that $\sigma$ and $\tau$ intersect at a $(d-1)$-dimensional face. We say that $X$ is strongly-connected if its dual graph is connected. Equivalently, $X$ is strongly connected if for every $\sigma$ and $\tau$ there is a path $\sigma = \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_k = \tau$, so that for every $1 \leq i \leq k-1$, $|\sigma_i \cap \sigma_{i+1}| = d-1$.

Proof of Lemma\textsuperscript{6}. First suppose that $Y$ is such that conditions 1 and 2 hold. Then we have that over $\mathbb{Q}$ every inclusion-minimal cocycle of $Y$ is an isolated facet. Thus $H^{d-1}(Y; \mathbb{Q})$ is generated by isolated facets of $Y$. It follows that the same holds for $H^{d-1}(Y; \mathbb{Z})$ (recall that torsion subgroups “shift up” one dimension when we change from homology to cohomology, so we don’t immediately have that $H_{d-1}(Y; \mathbb{Z})$ is free). We claim that $\beta^{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = \beta^{d-1}(Y; \mathbb{Q})$ for every prime $q$. This will imply that there is no torsion in homology and so $H^{d-1}(Y; \mathbb{Z})$ will be isomorphic to $H_{d-1}(Y; \mathbb{Z})$ proving the first part of the claim. Suppose there is a prime $q$ so that $\beta^{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) > \beta^{d-1}(Y; \mathbb{Q})$. Then $Y$ has a nontrivial cocycle $\phi$ with coefficients in $\mathbb{Z}/q\mathbb{Z}$ that is not the image of an integral cocycle modulo $q$. Let $X$ be the support of a minimal-weight representative of $\phi$. We may assume that $X$ has no isolated facets (otherwise we could subtract from $\phi$ an appropriate multiple of a cocycle supported on exactly an isolated facet of $X$ to arrive at a new cocycle over $\mathbb{Z}/q\mathbb{Z}$ which is not the image of a cocycle over $\mathbb{Z}$ and has smaller support).

Now over $\mathbb{Z}$ we have $\partial_{d}^{*}|_{X}(\phi) = q\psi$ for some integer vector $\psi$. Moreover by conditions 1 and 2, $q > \sqrt{d+1} \cdot |X|$ (and $|X| \geq n/(3d)$). But this $q$ is too large relative to $|X|$ for the cokernel of $\partial_{d}^{*}|_{X}$ to have $q$-torsion by Claim\textsuperscript{6} thus $q\psi$ in the image of $\partial_{d}^{*}|_{X}$ over $\mathbb{Z}$ implies that $\psi$ is also
in the image over $\mathbb{Z}$. Thus there exists an integral vector $\phi'$ so that $\phi - q\phi'$ is supported on $X$ and is a cocycle over $\mathbb{Z}$. However since $X$ has no isolated facets and $H^{d-1}(X, \mathbb{Z})$ is generated by isolated facets we have that $\phi - q\phi'$ is a coboundary over $\mathbb{Z}$, but then modding out by $q$ gives us that $\phi$ is a coboundary over $\mathbb{Z}/q\mathbb{Z}$ contradicting our assumption on $\phi$. This proves the first part of the claim.

Now suppose that $Y$ satisfies conditions 1, 2, and 3 and let $Y'$ be the complex at the moment in the stochastic process where the last isolated facet of $Y$ is covered. We wish to show that $H_{d-1}(Y') = 0$. We will prove this by induction on the number of isolated facets of $Y$.

By conditions 1 and 2, $H_{d-1}(Y)$ is a free abelian group generated by the isolated facets of $Y$. Thus if $Y$ has no isolated facets then $H_{d-1}(Y) = 0$ and $Y' = Y$ so we have the result. For the inductive step, we prove that if $Y$ satisfies conditions 1, 2, and 3 then for any face $\sigma$ which could be added to $Y$, we have that $Y \cup \{\sigma\}$ still satisfies 1, 2, and 3. This will prove that 1, 2, and 3 hold at every step and eventually we cover some isolated facet of the complex and then can apply induction.

Conditions 1 and 3 are clearly monotone, so we only have to show that condition 2 is monotone under our other assumptions. Suppose not. Let $\sigma$ be a face so that $Y$ satisfies conditions 1, 2, and 3 but $Y \cup \{\sigma\}$ does not satisfy condition 2. Let $\phi$ be an inclusion-minimal cocycle, with weight and support size at least two, over some field $R$ for $Y \cup \{\sigma\}$. Since $\phi$ is a cocycle of $Y \cup \{\sigma\}$ it is a cocycle for $Y$. But because $Y$ satisfies 1 and 2, we have that $H^{d-1}(Y; R)$ is generated by isolated facets. Therefore the support of $\phi$ is a union of isolated facets. By inclusion-minimality (after adding $\sigma$) the support of $\phi$ is strongly connected too. However, by 3 we have that the support of $\phi$ must be a single isolated facet, so $\phi$ has support size one, contradicting our assumption. This shows that $Y \cup \{\sigma\}$ satisfies condition 2 and we finish the proof by induction.

Before we get to the proof of Theorem 1 we state the following corollary which characterizes the structure of $H_{d-1}(Y)$ for $Y \sim Y_d(n, \frac{c \log n}{n})$ and $c > d - 1/2$. The proof is immediate from Lemmas 4 and 5.

**Corollary 6.** If $c > d - 1/2$, then with high probability $H_{d-1}(Y)$ for $Y \sim Y_d(n, \frac{c \log n}{n})$ is a free abelian group of rank equal to the number of isolated facets of $Y$.

Together with a first moment argument showing that $Y \sim Y_d(n, \frac{c \log n}{n})$ has no isolated facets with high probability for $c > d$, this corollary establishes the sharp threshold for integral homology to vanish in $Y_d(n, p)$. Now we give the proof of the main result.

**Proof of Theorem 1.** Consider an instance of $Y_d(n)$, and let $m_0$ be the hitting time for the event that the final isolated facet of $Y_d(n)$ is covered. Clearly $Y_d(n, i)$ has nontrivial $(d-1)$st homology group for $i < m_0$. That is the hitting time for $(d-1)$st homology to vanish is not earlier than the hitting time for the final isolated facet to be covered. It therefore suffices to show that with high probability $H_{d-1}(Y_d(n, m_0)) = 0$.

First generate $Y \sim Y_d\left(n, \frac{(d-1/4) \log n}{n}\right)$. If $Y$ has isolated facets, then run the stochastic Linial–Meshulam process starting at $Y$ and continuing until the moment the last isolated facets is covered. In the case that $Y$ has isolated facets, this generates a complex $Y_d(n, m_0)$ in $Y_d(n)$. By Lemma 4 $Y$ satisfies the three stated conditions with high probability and so by Lemma 5 the probability that $Y_d(n, m_0)$ has nontrivial $(d-1)$st homology group given that $Y$ has isolated facets is $o(1)$. Thus it suffices to check that the probability that $Y$ has no isolated facets is also $o(1)$, but this follows from a straightforward second moment argument. □
The rest of the paper will be devoted to proving Lemma 4. Condition 1 will be referred to as the “large cocycle” condition and will be proved in Section 5. Condition 2 will be referred to as the “small cocycle” condition and will be proved in Section 6. Condition 3 is much easier and we prove it now.

**Lemma 7.** If $Y \sim Y_d(n, c \log n/n)$ and $c > (d + 1)/2$ then with high probability $Y$ does not contain two isolated facets that meet at a ridge.

**Proof.** We will use the first moment method. Two isolated facets that meet a ridge is a subcomplex with two $(d - 1)$-dimensional faces, which are both isolated, and $d + 1$ vertices. The number of such complexes is at most \( \binom{n}{d+1}^2 \). The probability that both facets are isolated is at most \((1 - p)^{2(n-d)-1}\). Thus the expected number of pairs of isolated facets that meet at a ridge is at most

\[
\binom{n}{d+1}^2 (1-p)^{2(n-d)-1} \leq n^{d+1}(d+1)^2 \exp \left( -\frac{c \log n}{n} (2(n-d) - 1) \right) \leq n^{d+1}(d+1)^2 n^{-2c(1-o(1))}
\]

This is $o(1)$ since $2c > (d + 1)$.

\[\square\]

## 5 Large cocycles

The goal of this section is to prove the following lemma about the large cocycle condition. This will be accomplished by using Lemma 3 together with an enumeration result from [11] to bound the probability that $Y \sim Y_d(n, c \log n/n)$ contains a $(d-1)$-dimensional subcomplex $X$, with $|X| \geq n/(2d)$, for which $z(X)$ holds.

**Lemma 8.** If $Y \sim Y_d(n, c \log n/n)$ and $c > d - \frac{1}{2}$ then with high probability $z(X)$ fails to hold for all $(d-1)$-dimensional complexes on $n$ vertices of size at least $n/(3d)$.

Similar to the approach in [11], but now avoiding having to deal with coefficients, we want to count the number of $(d-1)$-complexes on $n$ vertices with $b(X) = (1 - \theta)|X|$. To do so we recall that $\beta(X) \leq b(X)$, and we count the number of complexes with $\beta(X) \leq (1 - \theta)n|X|$. We make use of the following lemma from Meshulam–Wallach.

**Lemma 9** (Claim 4.2 from [11]). Let $0 < \epsilon \leq 1/2$ and then for $n$ large enough and $X$ so that $\beta(X) \leq (1 - \theta)|X|/(n-d)$ for some $0 < \theta \leq 1$, there exists a subfamily $S \subseteq X$ of size less than $C^{|X|} + 2 \log \frac{1}{\epsilon} n$ such that $\Gamma(S) := \{ \tau \in X : |\tau \cap \sigma| = d - 1 \text{ for some } \sigma \in S \}$ has size at least $(1 - \epsilon)\theta|X|$, where $C$ is a constant depending only on $\epsilon$ and $d$.

We are now ready to use Lemma 9 to count the number of complexes $X$ with $\beta(X) \leq (1 - \theta)|X|/n$. This will upper bound the number of complexes $X$ with $b(X) = (1 - \theta)|X|/n$.

**Lemma 10** (Modification to Proposition 4.1 from [11]). For $n$ large enough, $k \geq n/(3d)$, and $\theta \geq 1/(2d)$ there exists a constant $c = c(d)$ so that the number of $(d-1)$-complexes $X$ with $|X| = k$ and $\beta(X) \leq (1 - \theta)kn$ is at most

\[
\left( c n^{(d-1)(1-\theta(1-\frac{1}{2d}))} \right)^k
\]

We give the proof here, essentially as it appears in [11], though we omit any consideration of an underlying coefficient ring. The proof follows directly from Lemma 9.
Proof. For \( n, k, \) and \( \theta \), let \( \mathcal{F}_n(k, \theta) \) denote the collection of \((d - 1)\)-dimensional subcomplexes \( X \) of the simplex on \( n \) vertices with \( k \) facets and \( \beta(X) \leq (1 - \theta)kn \). If \( X \in \mathcal{F}_n(k, \theta) \) then

\[
\beta(X) \leq \left(1 - \frac{\theta n - d}{n - d}\right)k(n - d)
\]

Suppose that \( \theta \geq 1/(2d) \) and let \( \theta' = \frac{\theta n - d}{n - d} \) and \( \epsilon = 1/(2d^2) \), by Lemma \( \ref{lem:bound} \) when \( n \) is large enough we obtain for every \( X \in \mathcal{F}_n(k, \theta') \) where \( k \geq n/(3d) \) a set \( S \) of size at most \( C \frac{k}{n} \), for some constant \( C \) depending on \( \epsilon \) so that \( \Gamma(S) \) has size at least \( (1 - 1/(2d^2))\theta'k \). Thus we get a map taking \( X \) in \( \mathcal{F}_n(k, \theta) \) to \((S, \Gamma(S), X - \Gamma(S))\). Since the latter two coordinates of this 3-tuple give a partition of \( X \), this map is injective, so the cardinality of \( \mathcal{F}_n(k, \theta) \) is at most the number of such tuples. Therefore it is at most

\[
\left(\sum_{i=0}^{Ck/n}(\frac{n}{i})\right)(2^{\Omega(d)dn})(\sum_{j=0}^{k-\theta'k(1-1/(2d^2))}(\frac{n}{j})^{n/d})
\]

Now the first two factors in the product above are at most \( c_1^k \) and \( c_2^k \) respectively for some constants \( c_1 \) and \( c_2 \) depending on \( d \). Thus for \( n \) large enough and \( k \leq \frac{(n)}{2} \), we have

\[
|\mathcal{F}_n(k, \theta)| \leq (c_1c_2)^kk(1-\theta'(1-1/(2d^2)))(k(1-\theta'(1-1/(2d^2)))^n/d)
\]

Therefore there exists a constant \( c \) so that for \( n/(3d) \leq k \leq \frac{(n)}{2} \),

\[
|\mathcal{F}_n(k, \theta)| \leq c^{kn(d-1)(1-\theta(1-1/2d))}
\]

This finishes the proof of the lemma in the case that \( k \leq \frac{(n)}{2} \). In the case that \( k \) is larger than \( \frac{(n)}{2} \), we may use the trivial bound of \( 2^{(n)} \) on \( |\mathcal{F}_n(k, \theta)| \) and so there is nothing to prove. \( \square \)

We are now ready to combine Lemma \( \ref{lem:base} \) with Lemma \( \ref{lem:z} \) to show that with high probability \( z(X) \) fails to hold for every \((d - 1)\)-dimensional complex \( X \) on \( n \) vertices with at least \( n/(3d) \) facets.

**Lemma 11.** If \( Y \sim Y_d(n, \frac{\log n}{n}) \) for \( c > d - 1/2 \) then with high probability \( z(X) \) fails to hold for all \((d - 1)\)-dimensional complexes on \( n \) vertices of size at least \( n/(3d) \).

**Proof.** For any \((d - 1)\)-dimensional subcomplex \( X \), we have that the probability of \( z(X) \) is at most \( n^{-(1-\theta)(d-1/2)} \) where \( b(X) = (1-\theta)nk \) by Lemma \( \ref{lem:z} \). Now if we define \( f_n(k, \theta) \) to be the number of \( X \) with \( |X| = k \) and \( b(X) = (1-\theta)nk \), then we wish to show that

\[
\sum_{k \geq n/(3d)} \sum_{\theta \in \Upsilon} f_n(k, \theta)n^{-(1-\theta)(d-1/2)} = o(1),
\]

where we set \( \Upsilon = \{ \theta : (1-\theta)nk \in \mathbb{Z}, (1-\theta)nk \leq n^{d+1}, \theta \leq d/(d+1) \} \).

---

\(1\)This is where we use the fact that \( k \geq n/(3d) \) and \( \theta \geq 1/(2d) \). Indeed the \( C \) from Lemma \( \ref{lem:bound} \) gives us a bound of \( C\frac{k}{n} + 2\log\frac{1}{\theta'} \leq C\frac{k}{n} + 2\log(4d^3) \leq C\frac{k}{n} + 6d\log(4d^3)\frac{k}{n} \). So the \( C \) in this proof should be the \( C \) in Lemma \( \ref{lem:bound} \) plus \( 6d\log(4d^3) \). Of course we could set any \( \delta, \theta_0 > 0 \) and assume that \( k \geq \delta n \) and \( \theta \geq \theta_0 \), but the choices of \( 1/(3d) \) and \( 1/(2d) \) respectively are convenient in other parts of our paper.
For $\theta \in T_1 := \{\theta \in \Upsilon : \theta < 1/(2d)\}$, we cannot apply Lemma 10 but the trivial bound on $\sum f(k, \theta) \leq \binom{n}{k}$ works instead. Indeed we have,

$$\sum_{k \geq n/(3d)} \sum_{\theta \in T_1} f(k, \theta)n^{-(1-\theta)(d-1/2)k} \leq \sum_{k \geq n/(3d)} n^{(d-1)k}n^{-(1-1/(2d))(d-1/2)k}$$

$$\leq \sum_{k \geq n/(3d)} n^{-k/(4d)} = o(1).$$

For $\theta \in T_2 := \{\theta \in \Upsilon : \theta \geq 1/(2d)\}$ we will apply Lemma 10 as $b(X) = (1-\theta)nk$ implies that $\beta(X) \leq (1-\theta)nk$ so $f_n(k, \theta) \leq |F_n(k, \theta)|$. We have

$$\sum_{k \geq n/(3d)} \sum_{\theta \in T_2} f(k, \theta)n^{-(1-\theta)(d-1/2)k} \leq \sum_{k = n/(3d)}^{n^d} \sum_{\theta \in T_2} \left(cn^{(d-1)(1-\theta(1-\frac{1}{2d}))} \right)^k n^{-(1-\theta)(d-1/2)k}$$

$$\leq \sum_{k = n/(3d)}^{n^d} \sum_{\theta \in T_2} \left(cn^{-\frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4} - \frac{1}{2d}\right)\theta} \right)^k$$

$$\leq \sum_{k = n/(3d)}^{n^d} \sum_{\theta \in T_2} \left(cn^{-\frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4} - \frac{1}{2d}\right)\frac{1}{2d}} \right)^k$$

$$\leq \sum_{k = n/(3d)}^{n^d} \left(cn^{-\frac{1}{2d(4d+1)}} \right)^k$$

$$\leq (n^d)^{n^d} \left(cn^{-\frac{1}{2d(4d+1)}} \right)^k \leq n^{2d+1}n^{-\Theta(n)} = o(1).$$

Finally, we use a simple coupling argument to finish the proof of Lemma 8.

**Proof of Lemma 8.** Fix $c' \in (d - 1/2, c)$. Let $Y_1 \sim Y_d(n, \frac{c'\log n}{n} \binom{n}{d+1})$. With high probability $Y_1 \subseteq Y$. Indeed, by a routine application of Chernoff's bound, if $Y'$ is distributed as $Y_d(n, \frac{c\log n}{n})$ then the probability that $Y$ has fewer than $\frac{c\log n}{n} \binom{n}{d+1}$ faces is at most

$$\exp \left( -\frac{\log n \binom{n}{d+1} (c - c')^2}{2c} \right) = o(1).$$

And by Lemma 11 with high probability $z(X)$ fails to hold for all $X$ on $n$ vertices of size at least $n/(2d)$ in $Y_1$, and hence the same holds in $Y$ since the kernel of the $d$th coboundary map of $Y$ is contained in the kernel of the $d$th coboundary map of $Y_1$.

## 6 Small cocycles

To show that the small cocycle condition holds for $Y \sim Y_d(n, \frac{c\log n}{n})$ with high probability for $c > d-1/2$ holds we rely on the fact that the support of an inclusion minimal cocycle is strongly-connected. Strongly connected $(d-1)$-complexes with $k$ facets on $n$ vertices are relatively few, in comparison to those that are not strongly connected. Specifically, we have the following:
Lemma 12. The number of strongly-connected \((d-1)\)-complexes with \(k\) facets on \(n\) vertices is at most
\[
n^{d+k-1}(2d)^k.
\]

Proof. A complex \(X\) is strongly connected if and only if the dual graph of \(G(X)\) is a connected subgraph of the dual graph \(H\) of the \((d-1)\)-skeleton of the full simplex on \(n\) vertices. As each \(G(X)\) has a spanning tree, we estimate the number of \(X\) above by the number of rooted subtrees of \(H\) with \(k\) vertices. There are at most
\[
\binom{n}{d} \cdot 2^{k-1} \cdot (dn)^{k-1}
\]
such rooted subtrees, enumerated in breadth–first–search order. In this enumeration, the \(\binom{n}{d}\) counts the number of choices for the root, the \(2^{k-1}\) counts the ways to partition the remaining \(k-1\) into the sizes of the neighborhoods of each vertex in the breadth–first–search, and the \((dn)^{k-1}\) overestimates the number of ways to pick the neighborhoods.

Lemma 13. If \(c > (d+1)/2\) then with high probability \(Y \sim Y_d(n, c \log n/n)\) has no inclusion-minimal \((d-1)\)-cocycles of support size \(k\) over any field for \(2 \leq k \leq \log n\).

Proof. We use the first moment method. If \(X\) is the support of an inclusion-minimal cocycle of \(Y\) over some field then \(X\) is a strongly-connected \((d-1)\) complex. Moreover, if \(|X| = k \leq \log n\) then at least \(n - d \log n\) vertices do not belong to \(X\). Now if \(X\) is to be the support of a cocycle over any field, then any face \(\tau\) obtained as the union of a \((d-1)\)-dimensional face of \(X\) and a vertex outside of \(X\) must be excluded from \(Y\). Thus the probability that a fixed \(X\) of size \(k \leq \log n\) is the support of a cocycle over any field is at most \((1 - p)^{k(n-d \log n)}\).

Since \(X\) must be strongly connected, the number of choices for \(X\) with size \(k\) is at most \((2d)^k n^{d-1+k}\) by Lemma [12] Applying the union bound over \(k \in \{2, 3, ..., \log n\}\) for the probability that there exists an inclusion-minimal cocycle of support size \(k\) we have:

\[
\sum_{k=2}^{\log n} (2d)^k n^{d-1+k}(1-p)^{kn(1-o(1))} \leq n^{d-1} \sum_{k=2}^{\log n} (2d)^k n^{k-ck(1-o(1))}
\leq n^{d-1} (\log n)n^{2-2c+o(1)}
= (\log n)n^{d-1-2(c-1)+o(1)} = o(1).
\]

Lemma 14. If \(c > 3/2\), then with high probability \(Y \sim Y_d(n, c \log n/n)\) has no inclusion minimal \((d-1)\)-cocycles of support size \(k\) over any field for \(\log n \leq k \leq n/(3d)\).

Proof. If \(X\) is the support of such a cocycle then there are at least \(n - n/3 = 2n/3\) vertices of \(Y\) outside of \(X\). Taking this consideration and the same argument as the proof of \(k \leq \log n\), we bound the following to prove the lemma:

\[
\sum_{k=\log n}^{n/(3d)} n^{d-1+k}(1-p)^{2kn/3} \leq n^{d-1} \sum_{k=\log n}^{n/(3d)} n^{k-2ck/3}
\leq n^{d} (n^{2c/3-1})^{-\log n}
= n^{-\Theta(\log n)}.
\]

This finishes the proof of Lemma [4] and hence the proof of our main result.
7 Conclusion

Our result finally establishes \( p = \frac{d \log n}{n} \) as the sharp threshold for homological connectivity of \( Y_d(n, p) \). Moreover, Corollary 6 tells us about the structure of the \((d - 1)\)st homology group immediately before it vanishes. However, the following two questions are closely related to our main result and remain open.

- What is the homological connectivity threshold for the random hypergraph model? This model is similar to the Linial–Meshulam model except that one does not start with the complete \((d - 1)\)-skeleton. Rather the \(d\)-dimensional faces are included independently and the complex is obtained by taking the downward closure of the top-dimensional faces. In [2], Cooley et al. show the hitting-time result for homological connectivity with \( \mathbb{Z}/2\mathbb{Z} \) coefficients in the random hypergraph model. Their result establishes that the sharp threshold for homological connectivity with \( \mathbb{Z}/2\mathbb{Z} \)-coefficients for the random hypergraph model is at \( \frac{d \log n}{2n} \). Can our methods be adapted to obtain the corresponding result with integer coefficients?

- The question about torsion in homology of \( Y_d(n, p) \) is raised in [6, 10]. Namely, experimental evidence strongly suggests that shortly before the first nontrivial cycle appears in the top homology group of \( Y_d(n) \), there is an exceptionally large (on the order of \( \exp(\Theta(n^d)) \)) torsion group which appears in the \((d - 1)\)st homology group. Outside of this however, it is believed that \( Y_d(n) \) has no torsion in homology; [10] formulates this conjecture precisely. Our paper in fact grew out of an attempt to prove the stronger result that for \( c \) a sufficiently large constant and \( p = c/n \), one has that with high probability the \((d - 1)\)st homology group of \( Y_d(n, p) \) is torsion free. However this problem remains open.

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