Grassmann Tensors and their Applications in Multiview Geometry

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Abstract

In this paper, we introduce the Grassmann tensor by tensor product of vectors and some basic terminology in tensor theory. Some basic properties of the Grassmann tensors are investigated and the tensor language is used to rewrite some relations and correspondences in the multiview geometry. Finally we show that a polytope in the Euclidean space $\mathbb{R}^n$ can also be concisely expressed as the Grassmann tensor generated by its vertices.

Keywords: Anti-symmetric tensor; Grassmann tensor; multiview geometry; polytope; tensor product.

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1. Introduction

Tensors play a very important role in deep learning and machine learning\cite{32}. They are at the heart of algorithms such as convolutional neural networks (CNNs) and recurrent neural networks (RNNs) which are widely used in many fields e.g. image recognition\cite{7}, natural language processing (NLP)\cite{8}, and time series analysis (TSA)\cite{29}.

A tensor is a multi-dimensional array of numerical values used to describe high dimensional data such as the physical state or properties of a material in physics and mechanics\cite{20, 21, 18}. In image analysis, we usually use a matrix to capture the intensity of light along with the spatial coordinates of the image to illustrate a grayscale image, and a third-order tensor to represent a batch of grayscale images with two spatial modes and one mode indexing different images in the batch. A color image can be algebraically described by a 3-order tensor identical to a triple of matrices $(R, G, B)$, each representing the color image in...
one channel. In this way, a color video can be represented by a fourth-order tensor $A = (A_{ijkl}) \in \mathbb{R}^{m \times n \times 3 \times p}$ where the discretized video flow is sampled into a sequence of $p$ frames with each a color image of size $m \times n$. In general, the more indices a tensor has, the more degrees of freedom it represents. Thus tensors can compactly represent very high-dimensional data. Tensors also make many operations easier to perform than matrices. For example, they are easily manipulated using automatic differentiation software like TensorFlow and ByTorch, which can automatically compute the gradient of a tensor with respect to any other tensor.

Deep learning models are composed of layers each of which takes some input data and produces the output after some transformations. There are many types of layers, including convolutional layers, pooling layers and recurrent layers, which are fully connected and parameterized by tensors. Tensors are also used to represent the weights of neural networks. A weight tensor is simply a tensor that is used as a parameter in a layer. When we train a neural network, we need to optimize the values of these weight tensors to minimize some loss functions.

Tensors have already found tremendous applications in image analysis and signal processing [3, 6, 24, 33], facial recognitions [11] and computer vision [14, 16, 17, 6], including object and motion recognition and video understanding since 1997 [26]. Quan and Kanade [28] and Faugeras et al. [10] and Quan [27] studied projections from $P^2$ to $P^1$, solved the reconstruction problem by trifocal tensors and presented two possible reconstructions in the case. Faugeras et al. [10] investigated the self-calibration of a camera undergoing motion in a plane. Hartley and Vidal [14] use tensors to compute non-rigid structure and motion under perspective projection. The bifocal, trifocal and quadrifocal tensor have been established to reconstruct a 3D scene from its projection respectively in two, three or four images ([15, 12, 16, 13, 11]).

There are some already known research work to reconstruct the scene points by tensors [17], including the trifocal tensors [1, 2, 4, 3] and the quadrifocal tensors [30]. Wolf and Shashua [31] investigated the projections from $P^n$ to $P^2$ to analyze several different problems in dynamic scene, but no general way of defining such tensors. Hartley and Schaffalitzky [13] initialized the Grassmann tensors for the unification of different form of the view tensors and the algorithms for estimating the projection matrices. They show that the projection matrices can be determined uniquely by a Grassmann tensor up to projective equivalence. However, the computations involved to extract the projections and the reconstruction are still limited to matrix techniques.

In this paper, we introduce the Grassmann (Plücker) tensor by the antisymmetric operation on the tensor product of vectors, also introduced are some basic terminology and notations of tensor theory. We investigate properties of Grassmann tensor and then use Grassmann tensors to simplify some relations and correspondences in multiview geometry. We show in the end of the paper that a polytope in Euclidean space $\mathbb{R}^n$ can be concisely expressed as the Grassmann tensor generated by its vertices.

Recall that a tensor is a multi-way array which is also called a hypermatrix.
A matrix is a 2-order tensor with two modes: row and column, and an m-order tensor has m modes with entries addressed by m indices. A tensor can also be regarded as a multilinear mapping. For more detail in tensor theory we refer the reader to [23].

We denote by $[m]$ the set $\{1, 2, \ldots, m\}$ for any positive integer $m$ and by $\mathcal{P}_m$ the set of permutations on $[m]$. Throughout the paper, we use the tensorial notation, i.e., tensors of order 0 (scalars) are denoted by means of italic type letters $a, b, x, y$ and some Greek letters $\lambda, \mu$ etc., tensors of order 1 (vectors) by means of boldface italic letters $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and Greek letters $\alpha, \beta, \gamma$, tensors of order two (matrices) by capital boldface letters $A, B, X, Y, M$, and tensors of higher orders by curlicue letters $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}, \ldots$. An m-order tensor $\mathcal{A}$ is of size $I_m := I_1 \times \ldots \times I_m$ if the dimension of the $k$-mode of $\mathcal{A}$ is $I_k$ for $k \in [m]$. Denote by $\mathcal{T}_m$ for the set of all real tensors of size $I_m$. An m-order tensor $\mathcal{A}$ is called an $m$th order $n$-dimensional real tensor if $I_1 = I_2 = \ldots = I_m := n$ for some positive integer $m$. The set of all $m$th order $n$-dimensional real tensors is denoted by $\mathcal{T}_{m,n}$. For our convenience, we denote

$$S(m, n) = \{(i_1, i_2, \ldots, i_m) : i_k \in [n], \forall k \in [m]\}$$

which usually serves as the index set of a tensor in $\mathcal{T}_{m,n}$. An $m$th order $n$-dimensional real tensor $\mathcal{A} \in \mathcal{T}_{m,n}$ is an $m$-array whose entries are indexed by $\sigma := (i_1, i_2, \ldots, i_m) \in S(m, n)$. We sometimes denote $A_{i_1i_2\ldots i_m}$ by $A_\sigma$ for $\sigma = (i_1, \ldots, i_m)$. A tensor $\mathcal{A} \in \mathcal{T}_{m,n}$ is called symmetric if each of its entries $A_\sigma$ is invariant under any permutation of its indices, i.e.,

$$A_\sigma = A_{\tau(\sigma)} \quad \forall \tau \in \mathcal{P}_m, \sigma \in S(m, n).$$

A symmetric tensor $\mathcal{A} \in \mathcal{T}_{m,n}$ is uniquely associated with an $m$-order $n$-variate homogeneous polynomial

$$f_\mathcal{A}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \sum_{i_1, i_2, \ldots, i_m} A_{i_1i_2\ldots i_m} x_{i_1}x_{i_2}\ldots x_{i_m} \quad (1.1)$$

$\mathcal{A}$ is called positive semidefinite (positive definite) if $f_\mathcal{A}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m > 0 (\geq 0)$ for all nonzero vector $\mathbf{x} \in \mathbb{R}^n$.

We need some preparations on the multiplications of tensors and the related terminology and notations before we introduce the Grassmann tensor. There are mainly three kind of multiplications of tensors: the outer product (also called tensor product), contractive (mode) product, and the t-product defined on 3-order tensors. We will introduce the first two in this paper and generalize the outer-product of two tensors. For the t-product of two 3-order tensors, we refer the reader to [22, 23, 34].

Let $p, q$ be positive integers and $\mathcal{A} \in \mathcal{T}_{p,n}$, $\mathcal{B} \in \mathcal{T}_{q,n}$. Let $\theta := \{\theta_1, \theta_2, \ldots, \theta_p\}$ be a nonempty subset of $[m]$ with complement $\theta^c := \{\phi_1, \phi_2, \ldots, \phi_q\}$, both ordered increasingly. The outer product of tensors $\mathcal{A}, \mathcal{B}$ along $(\theta, \theta^c)$, which is denoted by $\mathcal{A} \times_{(\theta, \theta^c)} \mathcal{B}$ or simply $\mathcal{A} \times_{\theta} \mathcal{B}$, is defined as

$$C = \mathcal{A} \times_{\theta} \mathcal{B} = (C_\theta), \text{with } C_\theta = A_{\theta_1\theta_2\ldots \theta_p}B_{\phi_1,\phi_2\ldots\phi_q} \quad (1.2)$$
where $m := p + q, \sigma = (i_1, i_2, \ldots, i_m)$. For $\theta := [p]$, we denote $A \times_{\theta} B$ simply by $A \times B$.

The outer product of tensors satisfies the law of the association, i.e.,

$$ (A \times B) \times C = A \times (B \times C) \quad (1.3) $$

for any tensors $A, B$ and $C$, and thus applies to any number of tensors. A special case is when all tensors involved are vectors, i.e.,

$$ \theta \subset S \quad (1.5) $$

$A$ is a rank-one $m$th order $n$-dimensional real tensor and is denoted $x^m$ when all $x^{(k)}$ are identical (to $x \in \mathbb{C}^n$). It is shown [9] that every $m$th order $n$-dimensional real tensor can be written as the sum of some rank-one tensors.

Given a set $S$ with cardinality $|S| = n$ and a positive number $q \in [n]$, we denote by $S[q]$ the set of all $q$-sets of $S$, i.e., $S[q] = \{ T \subset S : |T| = q \}$. For example, if $S = [4] = \{1, 2, 3, 4\}$ and $q = 2$, we have

$$ [4][2] = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

**Example 1.1.** The outer product of two matrices is another special case of (1.2). Let $A \in \mathbb{C}^{m_1 \times n_1}, B \in \mathbb{C}^{m_2 \times n_2}$, and let $\theta = \{s, t\} \subset [4], s < t, \{p, q\} = \theta^c, p < q$. Then $A \times_{\theta} B$ is a $4$-order tensor satisfying

$$ (A \times_{\theta} B)_{i_1 i_2 i_3 i_4} = A_{s i_1} B_{p i_4} \quad (1.5) $$

There are six outer-products corresponding resp. to six elements in $[4][2]$, i.e., $A \times_{\theta} B$ with $\theta \in \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. When $A \in \mathbb{R}^{n \times m}$, we have $A \times_{\theta} A = A \times_{\theta^c} A$.

For any matrix $A \in \mathbb{R}^{p \times q}$, we can also generate different kind of $2k$-order tensor $A^{[k]} := A \times A \times \cdots \times A$. For example, we can define $A = (A_{i_1 \ldots i_k : j_1 \ldots j_k})$ as

$$ A_{i_1 i_2 \ldots i_k : j_1 j_2 \ldots j_k} = \prod_{s=1}^k A_{i_s j_s} \quad (1.6) $$

which is a paired symmetric tensor (e.g. [11]) since for any permutation $\alpha \in \mathbb{P}_3$ we have

$$ A_{i_1 i_2 \ldots i_k : j_1 j_2 \ldots j_k} = A_{\alpha i_1 \ldots i_k : \alpha j_1 \ldots j_k} \quad (1.7) $$

Given any $2k$-order tensor $A = (A_{i_1 \ldots i_k : j_1 \ldots j_k})$ and an $k$-order tensor $B$. The contractive product of $A$ with $B$, denoted $AB$, is defined as

$$ (AB)_{i_1 i_2 \ldots i_k} = \sum_{j_1, j_2, \ldots, j_k} A_{i_1 i_2 \ldots i_k : j_1 j_2 \ldots j_k} B_{j_1 j_2 \ldots j_k} \quad (1.8) $$

Given a tensor $A \in T_{m \times n}$ and matrix $B \in \mathbb{R}^{n \times p}$. The contractive product of $A$ by $B$ along the $k$-mode, denoted $A \times_{(k)} B$ ($\forall k \in [m]$), is defined by

$$ (A \times_{(k)} B)_{i_1 \ldots i_k \ldots i_{k+1} \ldots i_n} = \sum_{j=1}^{n} A_{i_1 \ldots i_{k-1} j_{k+1} \ldots i_n} B_{j i_k} \quad (1.9) $$
We write $AB$ for $A \times_{(m)} B$. Note that it reduces to a matrix product for $m = 2$, i.e., $A \times_{(2)} B = AB$, $A \times_{(1)} B = A^\top B$.

The contractive product along one mode can be generalized to two modes resulting in an $(m - 2)$th order tensor, and it can preserve or compress (other than expand as a sequence of outer product) the tensors and is useful in many aspects. For example, the homogeneous polynomial $f(x) := Ax^m$ defined by (1.1) can be regarded as the contractive product of $A$ with the rank-one tensor $x^m$, and $A(x^m)$ is employed to define various eigen-pairs (see e.g. [25]).

Now we let $v_1, v_2, \ldots, v_m \in \mathbb{C}^n$ with each $v_j \neq 0$, and define the linear map

$$L := \sum_{\sigma \in P_m} (-1)^{\tau(\sigma)} \sigma$$

on $\mathcal{T}_{m,n}$ as

$$v_1 \wedge \cdots \wedge v_m = L(v_1 \times v_2 \times \cdots \times v_m) \quad (1.10)$$

We call the tensor $v_1 \wedge \cdots \wedge v_m$ defined by (1.10) the Grassmann (or Plücker) tensor associated with \{ $v_k$ \} \_k=1^m and denote it by $\mathcal{P}[v_1, \ldots, v_m]$. Note that $\mathcal{L}^2 = \mathcal{L}$ and $\mathcal{P}[v_1, \ldots, v_m] \neq 0$ if and only if $v_1, \ldots, v_m$ are linearly independent [30].

For $m = 2$, $\mathcal{P}[v_1, v_2] \in \mathbb{R}^{n \times n}$ ($n \geq 2$) is an anti-symmetric matrix of rank 2 when $v_1, v_2 \in \mathbb{R}^n$ are linearly independent.

It is known [30] that a Grassmann tensor is an anti-symmetric tensor. Hartley and Schaffalitzky [13] first introduced the Grassmann tensors in the context of multiview geometry and used Grassmann tensors to extend the fundamental matrices (bifocal tensors) to higher order multi-focal tensors e.g. trifocus tensor and quadrifocus tensor to establish the relationships between sets of corresponding subspaces in various views.

2. Grassmann tensors and their properties

Let $A = [a^1, a^2, \ldots, a^m] \in \mathbb{R}^{n \times m}$ be a matrix with $a^1, \ldots, a^m \in \mathbb{R}^n$ being its columns and $m \leq n$. Denote $\mathcal{P} := a^1 \wedge \cdots \wedge a^m$. Then $\mathcal{P} \in \mathcal{T}_{m,n}$ is a $m$th order $n$-dimensional real tensor. We have

**Lemma 2.1.** $\mathcal{P} x^m = 0$ for all $x \in \mathbb{R}^n$.

**Proof.** By definition, we have

$$\mathcal{P} x^m = \left( \sum_{\sigma} \text{sgn}(\sigma) a_i^1 \wedge a_i^2 \wedge \cdots \wedge a_i^m \right) x^m$$

$$= \left[ \sum_{\sigma} \text{sgn}(\sigma) \left( x^\top a_i^1 \right) \wedge \cdots \wedge \left( x^\top a_i^m \right) \right]$$

$$= \prod_{k=1}^m (x_i^\top a_i^k) \left( \sum_{\sigma} \text{sgn}(\sigma) \right)$$

$$= 0.$$
Here \( \sigma = (i_1, i_2, \ldots, i_m) \in \mathcal{P}_m \) is any possible permutation on \([m]\).

Lemma 2.1 shows that each anti-symmetric tensor corresponds to a zero polynomial, which is obvious in the matrix case.

**Theorem 2.2.** Let \( A = [a^1, a^2, \ldots, a^m] \in \mathbb{R}^{n \times m}, A = a^1 \wedge a^2 \wedge \ldots \wedge a^m \). Then \( A = (A_{i_1 \ldots i_m}) \in \mathcal{T}_{m,n} \) with

\[
A_{i_1 \ldots i_m} = \det A[i_1, \ldots, i_m:].
\]

where \( A[i_1, \ldots, i_m:] \) denotes the \( m \times m \) submatrix of \( A \) consisting of the \( (i_1, \ldots, i_m) \)-th rows of \( A \).

**Proof.** For any \( \eta := (i_1, i_2, \ldots, i_m) \in S(m, n) \), we denote \( r = |\eta| \) for the number of distinct elements in set \( \{i_1, i_2, \ldots, i_m\} \). Then we have

\[
A_{i_1i_2\ldots i_m} = A_\eta = (a^1 \wedge a^2 \wedge \ldots \wedge a^m)_\eta \\
= [\mathcal{L}(a^1 \times a^2 \times \ldots \times a^m)]_\eta \\
= \sum_\sigma \text{sgn}(\sigma) a_{i_1\sigma(1)} a_{i_2\sigma(2)} \ldots a_{i_m\sigma(m)} \\
= \det A[i_1, \ldots, i_m:].
\]

Given an \( m \)-th order \( n \)-dimensional real tensor \( A \in \mathcal{T}_{m,n} \) and \( \kappa := (P_1, \ldots, P_m) \) where \( P_k \) is a nonempty subset of \([n]\) for each \( k \). A subtensor of \( A \) determined by \( \kappa \), denoted \( A[\kappa] \), is an \( m \)-order tensor whose elements are indexed within \( P_1 \times \cdots \times P_m \). If \( P_1 = \ldots = P_m = S \) with \( |S| = r \), we denote \( A[\kappa] \) by \( A[S] \) and call it an \( r \)-dimensional principal subtensor, i.e., \( A[S] \in \mathcal{T}_{m,r} \). For \( S = [r] \), \( A[S] \) is called a leading principal subtensor (abbrev. LPS).

Now we let \( A = a^1 \wedge a^2 \wedge \ldots \wedge a^m \). By Theorem 2.2, we see that \( A = 0 \) if and only if vectors \( a^1, a^2, \ldots, a^m \) are linearly dependent. When \( a^1, a^2, \ldots, a^m \) are linearly independent, we have

**Corollary 2.3.** Let \( A = [a^1, a^2, \ldots, a^m] \in \mathbb{R}^{n \times m} \) with \( \text{rank}(A) = m \leq n \) and \( A = a^1 \wedge a^2 \wedge \ldots \wedge a^m \). Then for any \( \sigma := (i_1, i_2, \ldots, i_m) \in S(m, n), A[\sigma] \neq 0 \) if and only if \( |\sigma| = m \) and \( \det A[\sigma:] \neq 0 \).

**Corollary 2.4.** Let \( a^1, a^2, \ldots, a^m \in \mathbb{R}^n \) be linearly independent and \( A = a^1 \wedge a^2 \wedge \ldots \wedge a^m \). Then \( A \) has at most \( \frac{n^m}{(n-m)!} \) nonzero elements.

**Proof.** By Corollary 2.3 there are at most \( \binom{n}{m} \) nonzero LPS subtensors \( A[\alpha] \in \mathcal{T}_{m,n} \) where \( |\alpha| = m \) and \( \alpha \in S(m, n) \) is an \( m \)-tuple chosen from \([n]\) with no repetitions and there are \( m \) nonzero elements in \( A[\alpha] \) if \( \det(A[\alpha:] \neq 0 \).

It follows from Corollary 2.3 that
Corollary 2.5.\[
A = a^1 \wedge a^2 \wedge \ldots \wedge a^n = \det(A)\mathcal{H}
\] (2.2)
where \(\mathcal{H} = (H_\sigma) \in T_{n;n}\) with \(H_\sigma = sgn(\sigma)\) being the generalized sign function defined as
\[
sgn(\sigma) = \begin{cases} 
1, & \text{if } \sigma \in E_n, \\
-1, & \text{if } \sigma \in O_n, \\
0, & \text{otherwise.}
\end{cases}
\]
Here \(E_n\) and \(O_n\) denote respectively the set of even and odd permutations on \([n]\).

As a special case of Corollary 2.3, we have

Corollary 2.6. Let \(\alpha, \beta \in \mathbb{R}^2\). Then
\[
\mathcal{P}[\alpha, \beta] = \det[\alpha, \beta]E_2
\] (2.3)
where \(E_2\) is the elementary antisymmetric matrix defined as
\[
E_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\] (2.4)

Given a tensor \(A \in T_{m;n}\) and a vector \(b \in \mathbb{R}^n\). The wedge product of \(A\) with \(b\), denoted \(A \wedge b\), is defined by
\[
A \wedge b = \sum_{k=1}^{m+1} (-1)^{k-1} A \times_k b
\] (2.5)
where \(A \times_k b \in T_{m+1;n}\) is the outer-product of \(A\) and \(b\) with the \(k\)-mode assigned to \(b\). We will show in the following that \(A \wedge c = a \wedge (b \wedge c)\) if \(A = a \wedge b\) for some \(a, b \in \mathbb{R}^n\).

Example 2.7. Let \(a \in \mathbb{R}^n\) be a nonzero vector with \(a_i\) as its \(i\)th coordinate, and \(A = (A_{ijk}) = I_n \wedge a\). Then \(A \in T_{3;n}\) with elements
\[
\begin{align*}
A_{iii} &= a_i, \\
A_{ijj} &= A_{jii} = -A_{iji} = a_j, & \text{if } i \neq j, \\
A_{ijk} &= 0, & \text{if } i, j, k \text{ are distinct.}
\end{align*}
\]
Thus we have \(Ax^3 = \langle x, x \rangle \langle a, x \rangle = \sum_{i=1}^{n} a_i x_i^3\).

The following proposition offers the connection of the matrix-vector wedge and the vector wedges as well as the association law on wedge products of vectors.

Proposition 2.8. Let \(a, b, c \in \mathbb{R}^n\) be any vectors with \(n \geq 3\). Then
\[
(a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b \wedge c
\] (2.6)
Proof. We need only to show
\[(a \land b) \land c = a \land b \land c \quad (2.7)\]
Denote \(A = (a_{ij}) = a \land b\). By (2.5), the left hand side of (2.7) can be rewritten equivalently as
\[A \land c = (a \times b - b \times a) \times_1 c - (a \times b - b \times a) \times_3 c + (a \times b - b \times a) \times_3 c\]
\[= c \times a \times b - c \times b \times a - a \times c \times b + b \times c \times a + a \times b \times c - b \times a \times c\]
\[= a \times b \times c + b \times c \times a + c \times a \times b - a \times c \times b - c \times b \times a - b \times a \times c\]
\[= a \land b \land c\]
The equality \(a \land (b \land c) = a \land b \land c\) can also be proved by similar arguments. \(\Box\)

To generalize (2.5) to any pair of tensors \((A, B)\) with \(A \in T_{p,m}, B \in T_{q,n}\), we let \(m := p+q, \theta \subseteq [m], |\theta| = p\) and \(\theta^c\) be the complement of \(\theta\) in \([p+q]\) \((|\theta^c| = q)\).

We abuse the notation \(\theta (\theta^c)\) both for the subset and the corresponding sequence ordered increasingly. Then \(\theta \cup \theta^c = [m]\) is a permutation of \([m]\). Now we define
\[A \land B = \sum_{\theta \in Q_{p,m}} \text{sgn}(\theta)A \times_\theta B \quad (2.8)\]
where \(Q_{p,m}\) is the set of \(p\)-sequences whose entries are chosen from \([m]\) with increasing order. For \(p = q = 2\) and \(A, B \in \mathbb{R}^{n \times n}\), (2.8) implies
\[A \land B = A \times_{(1,2)} B - A \times_{(1,3)} B + A \times_{(1,4)} B - A \times_{(2,4)} B + A \times_{(2,3)} B - A \times_{(3,4)} B \quad (2.9)\]
It is out of our expectation when we find two more plus items than the minus ones in (2.9), other than a balance between the number of positive and that of negative items in the expression of the vector wedges. Furthermore, If we take \(A = B\), then
\[A \land A = 2(A \times A - A \times_{(1,3)} A + A \times_{(1,4)} A) \quad (2.10)\]

**Corollary 2.9.** Let \(A \in \mathbb{R}^{n \times n}\) be either a symmetric matrix or a rank-one matrix. Then \(A \land A = 2(A \times A)\).

**Proof.** First we assume that \(\text{rank}(A) = 1\). Then there exist some vectors \(\alpha, \beta \in \mathbb{R}^n\) such that \(A = \alpha \times \beta\). Note that
\[(\alpha \times \beta) \times_{(1,3)} (\alpha \times \beta) = (\alpha \times \beta) \times_{(1,4)} (\alpha \times \beta) = \alpha \times \alpha \times \beta \times \beta,\]
by (2.9), we get
\[A \land A = (\alpha \times \beta) \land (\alpha \times \beta) = 2[(\alpha \times \beta) \times (\alpha \times \beta) - (\alpha \times \beta) \times_{(1,3)} (\alpha \times \beta) + (\alpha \times \beta) \times_{(1,4)} (\alpha \times \beta)] = 2(\alpha \times \beta \times \alpha \times \beta) = 2A \times A.\]
If $A$ is symmetric, then $A$ can be written as $A = \sum_{i=1}^{r} \alpha_i \times \alpha_i$ for some $\alpha_i \in \mathbb{R}^n$.

Thus we have by

$$A \wedge A = \sum_{i,j} (\alpha_i \times \alpha_i) \wedge (\alpha_j \times \alpha_j) = 2 \sum_{i,j} \alpha_i \times \alpha_i \times \alpha_j \times \alpha_j = 2(A \times A).$$

The last equation comes from the expansion of $A \times A$. \hfill \Box

**Corollary 2.10.** For any matrix $X \in \mathbb{R}^{n \times n}$, we have

$$(I_n \wedge I_n)X = 2\text{Tr}(X)I_n \quad (2.11)$$

where the tensor-matrix product is contractive on the last two modes.

**Proof.** We take $A$ to be the identity matrix $I_n$ in (2.10) and denote $J = I_n \wedge I_n$, then $J = 2I_n \times I_n \in T_{4,n}$ by Corollary 2.9 and so

$$(JX)_{ij} = 2[(I_n \times I_n)X]_{ij} = 2\delta_{ij} \sum_{k_1,k_2} \delta_{k_1,k_2}X_{k_1,k_2} = 2\text{Tr}(X)\delta_{ij},$$

for all $i, j$. Thus (2.11) holds. \hfill \Box

We remark that tensor $K_n := I_n \times (1,3) I_n \in T_{4,n}$ acts as an identity map since $K_nX = X = XK_n$ for all $X \in \mathbb{R}^{n \times n}$ where

$$(AX)_{ij} = \sum_{i',j'} A_{i,j',i,j'}X_{i',j'}, (XA)_{ij} = \sum_{i',j'} X_{i,j',i,j'}A_{i,j'}. \quad (2.8)$$

$K_n$ is also called the commutation tensor in statistics [35].

The following lemma, which is also of interest in itself and will be used to prove the main result in the next section, is to be generalized to Lemma 3.7.

**Lemma 2.11.** Let $\alpha, \beta, \gamma \in \mathbb{R}^n$ be linearly independent and $n \geq 3$. Then $\alpha \wedge \beta, \beta \wedge \gamma, \gamma \wedge \alpha$ are also linearly independent in the Grassmann algebra.

**Proof.** If $\alpha \wedge \beta, \beta \wedge \gamma, \gamma \wedge \alpha$ are linearly dependent, then there exist some scalar $\lambda, \mu \in \mathbb{R}$ such that

$$\alpha \wedge \beta = \lambda(\beta \wedge \gamma) + \mu(\gamma \wedge \alpha)$$

which implies that $\alpha \wedge \beta = (\lambda\beta - \mu\alpha) \wedge \gamma$ and thus $\alpha \wedge \beta \wedge \gamma = 0$ (e.g. [36]). It turns out by [36] that $\alpha, \beta, \gamma$ must be linearly dependent, a contradiction to the hypothesis. \hfill \Box
3. Applications of Grassmann tensors in multiview geometry and geometry

Tensors can be employed to express the correspondences in computer vision. They can also be used to estimate the fundamental matrix, which is crucial in 3D reconstruction from two-views in the vision. A fundamental matrix $F$ can be described as the homography transforming a point $x$ in an image plane to a line $l' =Fx$ in another plane, so

$$(x')^TFx = 0 \quad (3.1)$$

holds for every corresponding point pair $(x,x')$. (3.1) can alternatively be written in tensor form as

$$F \times_1 x' \times_2 x = 0 \quad (3.2)$$

where $F$ is viewed as a second-order tensor. Note that here all points and planes are expressed in homogeneous coordinate system, i.e., $x,x' \in \mathbb{R}^3$. Another expression for $F$ is through the two camera matrices.

The Grassmann tensor can be used to simplify some expressions in the geometry. For example, a plane determined by three points at general positions can be usually expressed by a determinant equation as in the following.

**Proposition 3.1.** Let $X_1, X_2, X_3 \in \mathbb{P}^3$ ($X_k$ is 4-dimensional) be located in general positions, i.e., they are non-colinear. Then a point $X \in \mathbb{R}^4$ lies on the plane determined by $X_1, X_2, X_3$ if and only if

$$\det[X, X_1, X_2, X_3] = 0 \quad (3.3)$$

By the Grassmann tensor, however, we can put it in a more concise form, as stated in the following theorem.

**Theorem 3.2.** Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^4$ be vectors representing respectively planes $\pi_1, \pi_2$ and $\pi_3$ located at general positions. Then these planes meet at a unique point $x \in \mathbb{R}^4$ if and only if $Ax = 0$ where $A = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \in T_{3,4}$.

**Proof.** We let $x \in \mathbb{R}^4$ be the intersection point of the three planes $\pi_1, \pi_2$ and $\pi_3$. By definition, we have

$$\alpha_i^\top x = 0, \quad i = 1, 2, 3. \quad (3.3)$$

Then by definition we have

$$Ax = (\alpha_1 \wedge \alpha_2 \wedge \alpha_3)x$$

$$= (\alpha_1 \times \alpha_2 \times \alpha_3 + \alpha_2 \times \alpha_3 \times \alpha_1 + \alpha_3 \times \alpha_1 \times \alpha_2 - \alpha_2 \times \alpha_1 \times \alpha_3 - \alpha_3 \times \alpha_2 \times \alpha_1 - \alpha_1 \times \alpha_3 \times \alpha_2)x$$

$$= (\alpha_3^\top x)\alpha_1 \wedge \alpha_2 + (\alpha_1^\top x)\alpha_2 \wedge \alpha_3 + (\alpha_2^\top x)\alpha_3 \wedge \alpha_1$$

So we have

$$Ax = (\alpha_3^\top x)\alpha_1 \wedge \alpha_2 + (\alpha_1^\top x)\alpha_2 \wedge \alpha_3 + (\alpha_2^\top x)\alpha_3 \wedge \alpha_1 \quad (3.4)$$

Since $\alpha_1, \alpha_2, \alpha_3$ are linearly independent, we know that $\alpha_1 \wedge \alpha_2, \alpha_2 \wedge \alpha_3, \alpha_3 \wedge \alpha_1$ are also linearly independent by Lemma 2.11. It follows from (3.4) that $Ax = 0$ is equivalent to (3.3). \[\square\]
By the symmetry, we can deduce the following result which is analog to Theorem 3.2.

**Corollary 3.3.** Let \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^4 \) be vectors representing three points in \( \mathbb{R}^3 \) at general positions. Then they uniquely determine a plane \( \pi \in \mathbb{R}^4 \). Furthermore, a point \( x \in \mathbb{R}^4 \) lies on \( \pi \) if and only if \( x \) satisfies condition \( Ax = 0 \) where \( A = \alpha_1 \land \alpha_2 \land \alpha_3 \in T_{3;4} \).

By Corollary 3.3, we call \( A = \alpha_1 \land \alpha_2 \land \alpha_3 \) the Grassmann tensor or Plücker tensor associated with plane \( \pi \) which is determined by points \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^4 \). This is actually the extension of the Plücker matrix of a line, as explained in the follows.

Let \( L(X, Y) \) denote the line determined by two distinct points \( X, Y \in \mathbb{R}^4 \) which are homogeneous represented. The Plücker matrix associated with \( X, Y \), denoted by \( P := P[X, Y] \), is defined as

\[
P := X \land Y = X \times Y - Y \times X = XY^\top - YX^\top
\]

\( P[X, Y] \in \mathbb{R}^{4 \times 4} \) is a rank-2 anti-symmetric matrix. Different pairs \( (X, Y) \) may give the same Plücker matrix \( P \). But we do have

**Theorem 3.4.** Let \( (X, Y) \) and \( (X', Y') \) be two point pairs with \( X, Y, X', Y' \in \mathbb{R}^4 \). Then \( P[X, Y] = \lambda P[X', Y'] \) for some nonzero scalar \( \lambda \) if and only if there is a nonsingular matrix \( Q = (q_{ij}) \in \mathbb{R}^{2 \times 2} \) such that

\[
\begin{align*}
X' &= q_{11} X + q_{12} Y, \\
Y' &= q_{21} X + q_{22} Y
\end{align*}
\]

**Proof.** We let \( M = [X, Y, X', Y'] \). Then \( M \in \mathbb{R}^{4 \times 4} \). We want to prove that \( \text{rank}(M) = 2 \). We write \( M_1 = [X, Y] \) and \( M_2 = [X', Y'] \), then we have \( M_k \in \mathbb{R}^{4 \times 2} \). If \( X, Y \) are linearly dependent, then \( P[X, Y] = 0 \), thus we may assume that \( X, Y \) (and \( X', Y' \)) are linearly independent. So \( \text{rank}(M_k) = 2 \) for \( k = 1, 2 \).

Denote by \( N(A) \) the null space of a given matrix \( A \). We now show that

\[
N(P[X, Y]) = N(M_1^\top)
\]

(3.7)

\[
N(P[X', Y']) = N(M_2^\top)
\]

(3.8)

To prove (3.7), we let \( Z \in N(P[X, Y]) \). Then

\[
0 = P[X, Y]Z = (XY^\top - YX^\top)Z = (Y^\top Z)X - (X^\top Z)Y,
\]

which implies \( X'Z = 0 \) and \( Y'Z = 0 \), and thus \( Z \in N(M^\top) \). It follows that \( N(P[X, Y]) \subseteq N(M_1^\top) \). Conversely, we can show \( N(M_1^\top) \subseteq N(P[X, Y]) \) by reversing the argument. Thus (3.7) holds. Similarly we can show (3.8).

To show the sufficiency of (3.6), we first note that

\[
P[X, Y] = M_1DM_1^\top, P[X', Y'] = M_2DM_2^\top
\]

(3.9)
where $D = E_2$ is defined by (3.4). If (3.6) holds, i.e., $M_2 = M_1 Q$ with $Q \in \mathbb{R}^{2 \times 2}$ nonsingular, then we have

$$
P[X', Y'] = M_2 DM_2^T = M_1 QDQ^T M_1^T = M_1 P[q_1, q_2]M_1^T
$$

(3.10)

where $Q = [q_1, q_2]$ with $q_k \in \mathbb{R}^2 (k = 1, 2)$. We note that $q_1, q_2$ are linearly independent due to the nonsingularity of $Q$, and thus $P[q_1, q_2]$ is not zero. Furthermore, we have $P[q_1, q_2] = \det(Q) D$ by simple computations. Consequently we get by (3.10)

$$
P[X', Y'] = \det(Q)M_1 DM_1^T = \det(Q)P[X, Y].
$$

To prove the necessity, we assume w.l.o.g. that $\lambda = 1$ and $P[X, Y] = P[X', Y']$. By (3.7) and (3.8), we have $N(M_1^T) = N(M_1^T)$. Thus

$$
N(M^T) = N(M_1^T) \cap N(M_2^T) = N(M_1^T),
$$

which implies $\text{rank}(M) = \text{rank}(M^T) = \text{rank}(M_1^T) = 2$. Since $X, Y$ are linearly independent, there exists a matrix $Q = (q_{ij}) \in \mathbb{R}^{2 \times 2}$ such that $M_2 = M_1 Q$. Similarly we have $M_1 = M_2 Q'$ for some $Q' = (q'_{ij}) \in \mathbb{R}^{2 \times 2}$ by swapping $(X, Y)$ with $(X', Y')$. It follows that $Q$ (and also $Q'$) is invertible (nonsingular). The proof is completed.

Theorem 3.5 allows us to choose an invertible matrix $Q \in \mathbb{R}^{2 \times 2}$ such that $\{X', Y'\}$ is orthonormal, i.e.,

$$
\langle X', X' \rangle = \langle Y', Y' \rangle = 1, \quad \langle X', Y' \rangle = 0
$$

(3.11)

with $P[X, Y] = P[X', Y']$. In fact, we can use the Schmidt orthognormal process to obtain $X', Y'$:

$$
X' = \lambda_1 X, \quad \lambda_1 := \frac{1}{\|X\|},
$$

$$
Y = Y - \langle X', Y' \rangle X', \quad Y' = \lambda_2 Y, \quad \lambda_2 := \frac{1}{\|Y\|}
$$

We now show the following theorem.

**Theorem 3.5.** Let $X, Y \in \mathbb{R}^4$ be orthonormal. Then the Plücker matrix $P = P[X, Y]$ is the reflection with respect to $Y$.

**Proof.** We first assume that $X, Y \in \mathbb{R}^n$ are orthonormal vectors, i.e.,

$$
\|X\| = \|Y\| = 1, \quad \langle X, Y \rangle = X^TY = 0,
$$

where the norm is 2-norm. Then we have

$$
PX = (XY^T - YX^T)X = -Y, \quad PY = (XY^T - YX^T)Y = X,
$$

Denote $\alpha = X + tY, \beta = X - tY$ where $t := \sqrt{-1}$. Then $\alpha, \beta$ are orthonormal (thus also linearly independent). Furthermore, we can check easily that

$$
P[\alpha, \beta] = [\alpha, \beta]K,
$$

where $K = \text{diag}(t, -t)$, i.e., $P$ maps $\alpha$ to $\alpha' = i \alpha$ and $\beta$ to $\beta' = -i\beta$. 

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The Plücker matrix $P = ab^\top - ba^\top$ can be used to represent the intersection line $l$ of two planes (see e.g. [13]).

**Proposition 3.6.** Let $a, b \in \mathbb{R}^4$ be linearly independent vectors representing two distinct planes $\pi_1$ and $\pi_2$ respectively, and $l$ be their intersection line, i.e.,

$$l : \begin{cases} \pi_1 : a^\top x = 0, \\ \pi_2 : b^\top x = 0. \end{cases}$$

Then $P = ab^\top - ba^\top$ is the Plücker matrix of $l$, i.e., $x \in l$ if and only if $Px = 0$. Furthermore, If $Q \in \mathbb{R}^{4 \times 4}$ is also a Plücker matrix of $l$, then $Q = \lambda P$ for some nonzero scalar $\lambda \in \mathbb{R}$.

In order to generalize Proposition 3.6, we first need the following lemma which is an extension of Lemma 2.11.

**Lemma 3.7.** Let vectors $a^1, a^2, \ldots, a^m \in \mathbb{R}^n$ be linearly independent with $1 < m \in [n]$. Then the $(m - 1)$-vectors in set

$$\Gamma_{m-1} := \{ a^i \wedge a^j \wedge \cdots \wedge a^{i_{m-1}} : \{i_1, i_2, \ldots, i_{m-1}\} \subset [m]. \} \quad (3.12)$$

are linearly independent.

**Proof.** The result is obvious for $m = 1, 2,$ and it is also true for $m = 3$ by Lemma 2.11. Now we assume that $m > 3$ ($m \leq n$). Denote

$$\alpha^{(k)} = \bigwedge_{j \neq k, j = 1}^m a^j, \quad k = 1, 2, \ldots, m. \quad (3.13)$$

Then $\Gamma_{m-1} = \{ \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)} \}$. Suppose there are some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, not all zeros, such that

$$\lambda_1 \alpha^{(1)} + \lambda_2 \alpha^{(2)} + \cdots + \lambda_m \alpha^{(m)} = 0. \quad (3.14)$$

We may assume that the last nonzero scalar in $\{\lambda_k\}$ is $\lambda_{m'}$ with $1 < m' \leq m$, i.e., $0 \neq \lambda_{m'}$, and $\lambda_{m'+1} = \lambda_{m'+2} = \cdots = \lambda_m = 0$ (if $m' = m$, then $\lambda_m \neq 0$). For simplicity, we may assume that $m' = m$. Then (3.14) is equivalent to

$$\alpha^{(m)} = \sum_{k=1}^{m-1} \lambda'_k \alpha^{(k)}, \quad \lambda'_k = \frac{\lambda_k}{\lambda_m}, k \in [m-1]. \quad (3.15)$$

By (3.15), we have

$$\alpha^{(m)} \wedge a^m = \sum_{k=1}^{m-1} \lambda'_k (\alpha^k \wedge a^m) = 0. \quad (3.16)$$

On the other hand, we have

$$\alpha^{(m)} \wedge a^m = a^1 \wedge a^2 \wedge \cdots \wedge a^m$$

which is nonzero due to the linear independency of $\{a^k : k \in [m]\}$ (see e.g. [36]), a contradiction to (3.16). Thus we have $\lambda_1 = \cdots = \lambda_m = 0$, and so result holds. \qed
To state our last result, we recall that a polytope in \( \mathbb{R}^d \) generated by a set of points \( X := \{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}^d \) is defined as the set of the affine combinations of the points in \( X \), i.e.,

\[
W = \text{conv}(X) := \left\{ \sum_{j=1}^m \lambda_j x_j : \sum_{j=1}^m \lambda_j = 1, \lambda_j \geq 0 \forall j \in [m] \right\}
\]

\( W \) can also be represented as the bounded solution set of a finite system of linear inequalities, i.e.,

\[
W = W(A, b) := \{ x \in \mathbb{R}^d : A^\top x \leq b \} \quad (3.17)
\]

Here \( A = [a_1, a_2, \ldots, a_m] \in \mathbb{R}^{d \times m}, b \in \mathbb{R}^m \).

We end the paper by the following theorem in which the Grassmann tensor is used to describe a polytope in geometry.

**Theorem 3.8.** Let \( \hat{W} \) be the surface (bounder) of the polytope \( W \) defined by \((3.17)\) where \( A = [a_1, a_2, \ldots, a_m] \in \mathbb{R}^{d \times m}, b \in \mathbb{R}^m \) with \( \text{rank}(A) = r > 1 \). Then \( x \in \hat{W} \) if and only if

\[
(a_1 \wedge a_2 \wedge \cdots \wedge a_r)x = 0 \quad (3.18)
\]

where \( \{a_1, a_2, \ldots, a_r\} \) is a basis of the set of vectors \( X = \{a_1, a_2, \ldots, a_m\} \).

**Proof.** We may assume without loss of generality that \( r = m \) since otherwise we can replace \( A \) by its submatrix \( A_1 \in \mathbb{R}^{d \times r} \) whose column vectors form a basis of \( X := \{a_1, a_2, \ldots, a_m\} \). Denote \( P := a_1 \wedge a_2 \wedge \cdots \wedge a^r \) and \( \alpha^k \) be defined by \((3.13)\). Then

\[
0 = Px = (a_1 \wedge a_2 \wedge \cdots \wedge a^m)x = \sum_{k=1}^m (-1)^{m-k}(x^\top a^k)\alpha^k
\]

Since \( X \) is a set of linearly independent vectors, \( \alpha^1, \alpha^2, \ldots, \alpha^m \) are also linearly independent by Lemma [3.7]. Thus we have \( x^\top a^k = (a_k^\top) x = 0 \) for all \( k \in [m] \), which is equivalent to \( A^\top x = 0 \), that is, \( x \) is on the surface of the polytope \( W \). The converse can be proved by reversing the arguments.

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