GLOBAL REGULARITIES OF TWO-DIMENSIONAL DENSITY PATCH FOR INHOMOGENEOUS INCOMPRESSIBLE VISCOUS FLOW WITH GENERAL DENSITY

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Abstract. Toward the open question proposed by P.-L. Lions in [22] concerning the propagation of regularities of density patch for viscous inhomogeneous flow, we first establish the global in time well-posedness of two-dimensional inhomogeneous incompressible Navier-Stokes system with initial density being of the form: \( \eta_1 \Omega_0 + \eta_2 \Omega_0 \), for any pair of positive constants \((\eta_1, \eta_2)\), and for any bounded, simply connected \( W^{k+2, p}(\mathbb{R}^2) \) domain \( \Omega_0 \). We then prove that the time evolved domain \( \Omega(t) \) also belongs to the class of \( W^{k+2, p} \) for any \( t > 0 \). Thus in some sense, we have solved the aforementioned Lions’ question in the two-dimensional case. Compared with our previous paper [21], here we remove the smallness condition on the jump, \(|\eta_1 - \eta_2|\), moreover, the techniques used in the present paper are completely different from those in [21].

Keywords: Inhomogeneous incompressible Navier-Stokes equations, density patch, striated distributions, Littlewood-Paley theory.

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1. Introduction

We consider the following two-dimensional inhomogeneous incompressible Navier-Stokes equations:

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho v) &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\partial_t (\rho v) + \text{div} (\rho v \otimes v) - \Delta v + \nabla \pi &= 0, \\
\text{div } v &= 0, \\
(\rho, v)|_{t=0} &= (\rho_0, v_0).
\end{align*}
\]

Here the unknown \( \rho \) is a positive function\(^1\) which represents the density of the fluid at time \( t \) and point \( x \), \( v = (v_1, v_2) \) stands for the velocity field of the fluid, and \( \pi \) designates the pressure at time \( t \) and point \( x \), which ensures the incompressibility of the fluid.

This system (1.1) can be used as a model to describe a viscous fluid that is incompressible but has nonconstant density. Basic examples are mixtures of incompressible and non-reactant components, flows with complex structure (e.g. blood flow or model of rivers), fluids containing a melted substance, etc.

Let us notice that in the case where \( \rho_0 \equiv 1 \), the system (1.1) turns out to be the classical incompressible Navier-Stokes system \( (NS) \). We have to keep in mind that the system (1.1) is much more complex than \( (NS) \).

This system (1.1) has three major basic features, and let us state them in general space dimension \( d \geq 2 \). Firstly, the incompressibility condition on the convection velocity field in

\(^1\)We do want to avoid vacuum.
the density transport equation ensures that
\begin{equation}
\|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty} \quad \text{and} \quad \text{meas}\{ x \in \mathbb{R}^d \mid \alpha \leq \rho(t,x) \leq \beta \} \text{ is independent of } t \geq 0,
\end{equation}
for any pair of non-negative real numbers \((\alpha, \beta)\). Secondly the kinetic energy is formally conserved
\begin{equation}
\frac{1}{2} \int_{\mathbb{R}^d} \rho |v(t)|^2 dx + \int_0^t \| \nabla v(t') \|_{L^2(\mathbb{R}^d)}^2 dt' = \frac{1}{2} \int_{\mathbb{R}^d} \rho_0(x) |v_0(x)|^2 dx.
\end{equation}

The third basic feature is the scaling invariance property. Indeed, if \((\rho, v, \pi)\) is a solution of (1.1) on \([0, T] \times \mathbb{R}^d\), then the rescaled triplet \((\rho, v, \pi)_\lambda\) defined by
\begin{equation}
(\rho, v, \pi)_\lambda(t,x) \overset{\text{def}}{=} (\rho(\lambda^2 t, \lambda x), \lambda v(\lambda^2 t, \lambda x), \lambda^2 \pi(\lambda^2 t, \lambda x)), \quad \lambda \in \mathbb{R}
\end{equation}
is also a solution of (1.1) on \([0, T/\lambda^2] \times \mathbb{R}^d\). This leads to the notion of critical regularity.

Based on the energy estimate (1.3), J. Simon constructed in [26] the global weak solutions of (1.1) with finite energy. More generally, P.-L. Lions proved the global existence of weak solutions to the inhomogeneous incompressible Navier-Stokes system with variable viscosity in the book [22].

In the case of smooth initial data with no vacuum, the existence result of strong unique solutions goes back to the work of O.-A. Ladyzhenskaya and V.-A. Solonnikov [20]. Motivated by (1.4), R. Danchin [12] established the well-posedness of (1.1) in the whole space \(\mathbb{R}^d\) in the so-called critical functional framework for small perturbations of some positive constant density. The basic idea in [12] is to use functional spaces (or norms) that have the same scaling invariance as (1.4). This result was extended to more general Besov spaces in [1, 2, 3, 4, 23].

Given that in all those aforementioned works, the density has to be at least in the Besov space \(\dot{B}^d_{p,\infty}(\mathbb{R}^d)\), one cannot capture discontinuities across some hypersurface. In fact, the Besov regularity of the characteristic function of a smooth domain is only \(\dot{B}^d_{p,\infty}(\mathbb{R}^d)\). Therefore, those results do not apply to a mixture flow composed of two separate fluids with different densities.

The first breakthrough along this line was made by R. Danchin and P.-B. Mucha in [13], where they basically proved the global well-posedness of (1.1) with initial density allowing discontinuity across a \(C^1\) interface with a sufficiently small jump. Later on, even \(\rho_0\) is only bounded and with small fluctuation to some positive constant, J. Huang, M. Paicu, and the second author in [19] could show the global existence of solutions to (1.1) in a critical functional framework, and the uniqueness was obtained provided assuming slightly more regularities on the initial velocity field. R. Danchin and the second author extended this result to the half-space setting in [15].

In the general case where \(\rho_0 \in L^\infty(\mathbb{R}^d)\) with a positive lower bound and \(v_0 \in H^2(\mathbb{R}^d)\), R. Danchin and P.-B. Mucha [14] proved that the system (1.1) has a unique local in time solution. Furthermore, with the initial density fluctuation being sufficiently small, for any initial velocity \(v_0 \in B^{1,2}_{1,2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)\), or \(v_0 \in B^{2/3}_{4,5}(\mathbb{R}^d)\) with small size for \(1 < p < \infty, d < q < \infty\) and \(2 - \frac{2}{p} \neq \frac{1}{q}\), they also proved the global well-posedness of (1.1). M. Paicu, Z. Zhang and the second author [24] improved the well-posedness results in [14] with less regularity assumptions on the initial velocity.

A natural question to ask is whether it is possible to propagate the boundary regularities of the interface of the fluids. In particular, P.-L. Lions proposed the following open question in [22]: suppose the initial density \(\rho_0 = 1_D\) for some smooth domain \(D\), Theorem 2.1 of [22]
ensures at least one global weak solution \((\rho, v)\) of (1.1) such that for all \(t \geq 0\), \(\rho(t) = 1_{D(t)}\) for some set \(D(t)\) with \(\text{vol}(D(t)) = \text{vol}(D)\). Then whether or not the regularity of \(D\) is preserved by time evolution? Since this problem is very sophisticated due to the possible appearance of vacuum, as a first step toward this question, here we aim at establishing the global well-posedness of (1.1) with the initial density \(\rho_0(x) = \eta_1 1_{\Omega_0} + \eta_2 1_{\Omega_0^c}\) for any pair of positive constants \((\eta_1, \eta_2)\), and for any bounded simply connected \(W^{k+2,p}(\mathbb{R}^2)\) domain \(\Omega_0\).

More precisely, let \(\Omega_0\) be a simply connected \(W^{k+2,p}(\mathbb{R}^2), k \geq 1, p \in [2, 4]\), bounded domain. Let \(f_0 \in W^{k+2,p}(\mathbb{R}^2)\) such that \(\partial \Omega_0 = f_0^{-1}(\{0\})\) and \(\nabla f_0\) does not vanish on \(\partial \Omega_0\). Then we can parametrize \(\partial \Omega_0\) as

\[
(1.5) \quad \gamma_0 : S^1 \rightarrow \partial \Omega_0 \text{ via } s \mapsto \gamma_0(s) \text{ with } \partial_s \gamma_0(s) = \nabla \perp f_0(\gamma_0(s)).
\]

For any \(\eta_1, \eta_2 > 0\), we take the initial density \(\rho_0\) and the initial velocity \(v_0\) of (1.1) as follows

\[
(1.6) \quad \rho_0(x) = \eta_1 1_{\Omega_0}(x) + \eta_2 1_{\Omega_0^c}(x), \quad v_0 \in L^2 \cap \dot{B}^0_{2,1} \quad \text{and} \quad \partial_{X_0} v_0 \in L^2 \cap \dot{B}^\epsilon_{2,1} \quad \text{for } \epsilon = 1, \ldots, k, \quad \text{and} \quad X_0 \text{ def } = \nabla \perp f_0, \quad \partial_{X_0} v_0 \text{ def } = X_0 \cdot \nabla v_0,
\]

where \(s_0 \in ]0, 1[, \quad s_\ell \text{ def } = s_0 - \ell \epsilon/k \text{ for some fixed } \epsilon \in ]0, s_0[, \text{ and } p \in ]2, 2/(1 - s_k)[].

The main ideas for us to solve the above problem come from J.-Y. Chemin [7, 8, 9, 10]. Indeed, by using the idea of conormal distributions or striated distributions, J.-Y. Chemin [8, 9] (see also [6]) proved the global regularities of the two-dimensional vortex patch for ideal incompressible flow. One may also check [11, 16, 17, 18, 25] for some extensions. We emphasize that the initial velocity field \(v_0\) satisfying (1.6) belongs to some striated distribution space. To avoid technicalities, here we prefer not to present details of such spaces. Interested readers may check the book [10] and the references therein.

The main result of this paper states as follows:

**Theorem 1.1.** Let the initial data \((\rho_0, v_0)\) be given by (1.5) and (1.6), for any pair of positive constants \((\eta_1, \eta_2)\). Then (1.1) has a unique global solution \((\rho, v)\) such that \(\rho(t, x) = \eta_1 1_{\Omega(t)}(x) + \eta_2 1_{\Omega(t)^c}(x),\) with \(\Omega(t)\) being a bounded, simply connected \(W^{k+2,p}(\mathbb{R}^2)\) domain for any \(t > 0\).

**Remark 1.1.** For \(\Omega_0\) given by (1.5), it follows from [21] that the initial velocity \(v_0\) with vorticity \(\omega_0 = 1_{\Omega_0}(x)\) satisfies the assumptions of (1.6). Hence in particular, Theorem 1.1 ensures the global well-posedness of (1.1) with initial density \(\rho_0 = \eta_1 1_{\Omega_0}(x) + \eta_2 1_{\Omega_0^c}(x)\) and initial vorticity \(\omega_0 = 1_{\Omega_0}(x)\).

**Remark 1.2.**

1. Besides the difficulties encountered in [21], here we remove the smallness assumption on the jump, \(|\eta_1 - \eta_2|\), which is crucial in [21] to exploit the maximal regularity estimate for the time evolutionary Stokes operator to handle the \(W^{2,p}\) estimate of the velocity field.

2. The method of time-weighted energy estimate in [24] will play an important role here. However, in order to propagate the \(W^{2,p}\) regularity of the tangential vector field \(X_0\) of \(\partial \Omega_0\), we need to deal with the energy estimate of \(\nabla \partial_t v\), which we can not go through under the mere bounded density assumption. We overcome this difficulty by working with the energy estimate of \(\nabla D_t v\), where \(D_t = \partial_t + v \cdot \nabla\) denotes the material derivative.

3. With initial density being only in the bounded function space, we can not apply the classical ideas in [12] to propagate the Besov regularity of the velocity field as in the previous papers [15, 19, 24]. Namely, given the initial velocity field \(v_0 \in \dot{B}^\epsilon_{p,r}\), they did not prove the solution \(v\) belonging to \(C([0, \infty]; \dot{B}_{p,r}^\epsilon)\). In this paper, motivated
by the characterisation of Besov spaces with positive regularity index, we succeed in establishing the propagation of the Besov regularity of the velocity field. One may check Theorem 2.1 below for details.

2. Structure and main ideas of the proof

Before proceeding, let us first recall the definitions of Besov norms from [5] for instance.

**Definition 2.1.** Let us consider a smooth radial function $\varphi$ on $\mathbb{R}$, the support of which is included in $[3/4, 8/3]$ such that

$$\forall \tau > 0, \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \tau) = 1 \quad \text{and} \quad \chi(\tau) \overset{\text{def}}{=} 1 - \sum_{j \geq 0} \varphi(2^{-j} \tau) \in \mathcal{D}([0, 4/3]).$$

Let us define

$$\Delta_j a \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|) \hat{a}), \quad \text{and} \quad S_j a \overset{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j}|\xi|) \hat{a}).$$

Let $(p, r, \kappa)$ be in $[1, +\infty]^3$ and $s$ in $\mathbb{R}$. We define the Besov norms by

$$\|a\|_{\dot{B}^{s}_{p,r}} \overset{\text{def}}{=} \left\| (2^{js}\|\Delta_j a\|_{L^p})_j \right\|_{L^r} \quad \text{and} \quad \|a\|_{\dot{B}^{s}_{p,r}(\dot{B}^{s}_{p,r})} \overset{\text{def}}{=} \left\| (2^{js}\|\Delta_j a\|_{L^p})_j \right\|_{L^r}.$$  

We remark that in the particular case when $p = r = 2$, the Besov spaces $\dot{B}^{s}_{p,r}$ coincide with the classical homogeneous Sobolev spaces $\dot{H}^s$.

For notational simplicity, we always denote $\dot{B}^s \overset{\text{def}}{=} \dot{B}^s_{2,1}$.

As a matter of fact, we shall prove a much more general version of Theorem 1.1. More precisely, we assume the initial data satisfying

$$0 < \rho_* \leq \rho_0 \leq \rho^*, \quad v_0 \in L^2 \cap \mathcal{B}^{s_0} \text{ for some } s_0 \in [0, 1[,$$

along with the following striated regularity assumptions for $\ell = 1, \cdots, k$,

$$\partial_{X_0}^{\ell-1} X_0 \in W^{2,p}, \quad \partial_{X_0}^{\ell} \rho_0 \in L^\infty, \quad \partial_{X_0}^{\ell} v_0 \in L^2 \cap \mathcal{B}^{s_{\ell}}, \quad \text{with}$$

$$s_{\ell} \overset{\text{def}}{=} s_0 - \ell/k, \quad \text{for some } \epsilon \in [0, s_0[, \quad \text{and } \quad p \in ]2, 2/(1-s_k)[,$$

for some vector field $X_0 = (X_0^1, X_0^2)$ and $\partial_{X_0} f \overset{\text{def}}{=} X_0 \cdot \nabla f$. Moreover, to avoid cumbersome calculations, without loss of generality, we assume the divergence free condition on the initial vector field $X_0$:

$$\text{div } X_0 = 0.$$

Given the convection velocity field $v$, we define $X(t) = (X_1(t), X_2(t))$ via

$$\begin{cases} 
\partial_t X + v \cdot \nabla X = X \cdot \nabla v, \\
X(0, x) = X_0(x).
\end{cases}$$

**Conventions.** In this whole context, we shall use the following conventions:

$$\partial_X \overset{\text{def}}{=} X \cdot \nabla, \quad D_t \overset{\text{def}}{=} \partial_t + v \cdot \nabla, \quad f_t \overset{\text{def}}{=} \partial_t f, \quad \sigma(t) \overset{\text{def}}{=} \min \{1, t\}, \quad \text{and} \quad \theta_0 \overset{\text{def}}{=} \epsilon/k,$$

$$\rho^\ell \overset{\text{def}}{=} \partial_{X}^{\ell} \rho, \quad v^\ell \overset{\text{def}}{=} \partial_{X}^{\ell} v, \quad \pi^\ell \overset{\text{def}}{=} \partial_{X}^{\ell} \pi, \quad X^\ell \overset{\text{def}}{=} \partial_{X}^{\ell} X, \quad \text{and} \quad \partial_{X}^{\ell-1} f \overset{\text{def}}{=} 0,$$

$$C(v_0, u_0, s) \overset{\text{def}}{=} \|u_0\|_{\mathcal{B}^s}, \exp(\mathcal{A}_0 \exp(\mathcal{A}_0 \|u_0\|_{L^2}^{4/2})) \quad \text{and} \quad C(v_0, s) \overset{\text{def}}{=} C(v_0, v_0, s),$$

$$\mathcal{H}_t(t) \overset{\text{def}}{=} \exp \exp \exp \cdots \exp (A_t(t^2)) \quad \text{with} \quad \langle t \rangle \overset{\text{def}}{=} (1 + t^2)^{1/2}.$$
Thus by virtue of Theorem 2.1, the coupled system (1.1) with (2.4) has a unique global solution
\[ A_0(t) = C(v_0, s_0) \]
where
\[ A_0(t) \leq \mathcal{H}_k(t) \]
\[ A_\ell(t) \leq \mathcal{H}_k(t) \]
and for \( k \in \{1, \ldots, k\} \)

\[ \mathcal{H}_k(t) = C(v_0, s_0) \]
Here we used the fact that $2 < p < 2/(1-s_k)$ so that $1-1/p-s_0/2 < 1/2$.

Let $\Omega(t) \overset{\text{def}}{=} \psi(t, \Omega_0)$. Due to (1.6), we deduce from the transport equation of (1.1) that

\begin{equation}
\rho(t, x) = \eta_1 1_{\Omega(t)}(x) + \eta_2 1_{\Omega(t)'}(x).
\end{equation}

Next we are going to prove that $\Omega(t)$ is of class $W^{3,p}$. Indeed let $\partial \Omega(t)$ be the level surface of $f(t, \cdot)$. Then $f$ solves

\[
\begin{cases}
\partial_t f + v \cdot \nabla f = 0, \\
f(0, x) = f_0(x),
\end{cases}
\]

which implies that $X(t, x) \overset{\text{def}}{=} \nabla^\bot f(t, x)$ solves (2.4). We thus deduce from (2.4) and (2.10) that the tangential vector field $X(t, x) = \nabla^\bot f(t, x)$ satisfies

\begin{equation}
X(t, \psi(t, x)) = X_0(x) \cdot \nabla \psi(t, x).
\end{equation}

Moreover, in view of (2.8), we have

\begin{equation}
X^{\ell-1} \in L^\infty_{\text{loc}}(\mathbb{R}^+; W^{2,p}), \quad \text{for } \ell = 1, \cdots, k.
\end{equation}

Then it follows then from (2.11), (2.12) and (2.15) that

\[
\partial_i \partial_j (X(t, \psi(t, x))) = \partial_i \partial_j \psi(t, x) \cdot \nabla X(t, \psi(t, x)) + (\partial_i \psi \otimes \partial_j \psi)(t, x) : \nabla^2 X(t, \psi(t, x)) \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^p), \quad \forall 1 \leq i, j \leq 2.
\]

Therefore we deduce from (1.5) and (2.14) that

\begin{equation}
\partial_t (\psi(t, \gamma_0(s))) = X_0(\gamma_0(s)) \cdot \nabla \psi(t, \gamma_0(s)) = X(t, \psi(t, \gamma_0(s))) \in L^\infty_{\text{loc}}(\mathbb{R}^+; W^{2,p}),
\end{equation}

that is, $\Omega(t)$ belongs to the class of $W^{3,p}$ for any $t > 0$.

Along the same line to the derivation of (2.16), we write

\[
\partial^\ell_t (\psi(t, \gamma_0(s))) = \partial^\ell_{\gamma_0}(\psi)(t, \gamma_0(s))) = \cdots = (\partial^\ell_{X_0} \psi)(t, \gamma_0(s)), \quad \forall \ell \geq 1.
\]

Thus in order to prove that $\partial \Omega(t) = \psi(t, \gamma_0(\partial \Omega_0))$ belongs to the class of $W^{k+2,p}$ for any $t > 0$, it suffices to show that $\partial^\ell_{X_0} \psi \in L^\infty_{\text{loc}}(\mathbb{R}^+; W^{2,p})$ for any $\ell \in \{1, 2, \cdots, k\}$.

To this end, we notice that (2.14) implies that

\[
X(t, x) = (\partial_{X_0} \psi)(t, \psi^{-1}(t, x)),
\]

so that for any smooth function $f$, we write

\[
\partial_X (f(\psi^{-1}(t, x))) = \sum_{i=1}^{2} \sum_{\alpha=1}^{2} X^i(t, x) \partial_\alpha f(\psi^{-1}(t, x)) \frac{\partial (\psi^{-1})^\alpha(t, x)}{\partial x_i} \\
= \sum_{i=1}^{2} \sum_{\alpha=1}^{2} \sum_{j=1}^{2} (X_0^i \partial_j \psi^j)(t, \psi^{-1}(t, x)) \frac{\partial (\psi^{-1})^\alpha(t, x)}{\partial x_i} \partial_\alpha f(\psi^{-1}(t, x)) \\
= (\partial_{X_0} \psi)(\psi^{-1}(t, x)).
\]

Then an inductive argument leads to

\[
X^{\ell-1}(t, x) = \partial^{\ell-1}_X ((\partial_{X_0} \psi)(t, \psi^{-1}(t, x))) = (\partial^\ell_{X_0} \psi)(t, \psi^{-1}(t, x)), \quad \forall \ell \geq 1,
\]

which together with (2.11), (2.12) and (2.15) implies $\partial^\ell_{X_0} \psi \in L^\infty_{\text{loc}}(\mathbb{R}^+; W^{2,p})$ for any $\ell \in \{1, 2, \cdots, k\}$. This completes the proof of Theorem 1.1. \qed
The rest of this section is devoted to the outline of the proof to Theorem 2.1. Let us begin with the notations we are going to use in the whole context.

**Notations:** We denote \((a|b) \overset{\text{def}}{=} \int_{\mathbb{R}^2} a|b\,dx\) to be the \(L^2(\mathbb{R}^2)\) inner product of \(a\) and \(b\), and \([A;B] = AB - BA\) to be the commutator of the operators \(A\) and \(B\). For \(a \lesssim b\), we mean that there is a uniform constant \(C\), which may be different on different lines, such that \(a \leq Cb\). Finally we denote \((d_q)_{q\in \mathbb{Z}}\) to be a generic element of \(\ell^1(\mathbb{Z})\) so that \(\sum_{q\in \mathbb{Z}} d_q = 1\).

We shall first prove in Section 3 that there holds (2.6) under the assumption (2.1).

**Proposition 2.1.** Let \((\rho, v, \nabla \pi)\) be a smooth enough solution of (1.1) and \(A_{01}(t)\) be the functional given by (2.7). Then under the hypothesis of (2.1), the following inequality is valid for any \(t \geq 0\)

\[
A_{01}(t) \leq C(v_0, s_0).
\]

Let us remark that the essence of the proof of Proposition 2.1 is to derive the \textit{a priori} estimates for \(\|v\|_{\tilde{L}^{\infty}_t(L^{s_0}_q)} \) and \(\|\sigma^{\frac{1-s_0}{2}} v_l\|_{L^2_t(L^2)}\), which is completely new compared with the previous references [15, 19, 24]. Due to the fact that the density function only belongs to the bounded function space, we can not use the classical ideas in [12], namely, we can not apply the operator \(\Delta_j\) to the momentum equation of (1.1) and then perform energy estimate for \(\Delta_j v\).

Here the main idea of the proof to the propagation of the Besov regularities of the velocity field is motivated by the characterization of Besov norms with positive regularity index. More precisely, we write

\[
(2.17) \quad v = \sum_{q \in \mathbb{Z}} v_q \quad \text{and} \quad \nabla \pi = \sum_{q \in \mathbb{Z}} \nabla \pi_q,
\]

with \((v_q, \nabla \pi_q)\) solving

\[
(2.18) \quad \begin{cases}
\rho \partial_t v_q + \rho v \cdot \nabla v_q - \Delta v_q + \nabla \pi_q = 0, \\
\text{div} v_q = 0, \\
v_q|_{t=0} = \Delta_q v_0.
\end{cases}
\]

Then by performing \(H^1\) energy estimate to (2.18), one has

\[
(2.19) \quad \|v_q\|_{L^\infty_t(L^2)} + \|\sigma^{\frac{1}{2}} \partial_t v_q\|_{L^2_t(L^2)} + 2^{-q}(\|\nabla v_q\|_{L^\infty_t(L^2)} + \|\partial_t v_q\|_{L^2_t(L^2)}) \lesssim d_q 2^{-q s_0} \|v_0\|_{B^{s_0}_2}.
\]

We thus deduce from Bernstein-type lemma (see Lemma 2.1 of [5] for instance) that

\[
\|\Delta_j v\|_{L^\infty_t(L^2)} \lesssim \sum_{q > j} \|\Delta_j v_q\|_{L^\infty_t(L^2)} + 2^{-j} \sum_{q \leq j} \|\Delta_j \nabla v_q\|_{L^\infty_t(L^2)}
\]

\[
\lesssim \sum_{q > j} \|v_q\|_{L^\infty_t(L^2)} + 2^{-j} \sum_{q \leq j} \|\nabla v_q\|_{L^\infty_t(L^2)} \lesssim d_j 2^{-j s_0} \|v_0\|_{B^{s_0}_2},
\]

which together with Definition 2.1 ensures that \(v \in \tilde{L}^{\infty}_t(B^{s_0}_2)\). Indeed the same procedure would imply that \(v \in \tilde{L}^{\infty}_t(\tilde{B}^{s_0}_{2,r})\) for any \(r \in [1, \infty]\) provided that \(v_0 \in \tilde{B}^{s_0}_{2,r}\). However in this case, (2.19) would become

\[
\|\sigma^{\frac{1}{2}} \partial_t v_q\|_{L^2_t(L^2)} + 2^{-q} \|\partial_t v_q\|_{L^2_t(L^2)} \lesssim c_{q,r} 2^{-q s_0} \|v_0\|_{\tilde{B}^{s_0}_{2,r}},
\]

where \((c_{q,r})_{q \in \mathbb{Z}}\) is a generic element of \(\ell^r(\mathbb{Z})\) so that \(\|(c_{q,r})_{q \in \mathbb{Z}}\|_{\ell^r(\mathbb{Z})} = 1\). As a result, we infer

\[
\|\sigma^{\frac{1-s_0}{2}} \partial_t v\|_{L^2_t(L^2)} \leq \sum_{q \in \mathbb{Z}} \|\partial_t v_q\|_{L^2_t(L^2)}^{s_0} \|\sigma^{\frac{1}{2}} \partial_t v_q\|_{L^2_t(L^2)}^{1-s_0} \lesssim \|v_0\|_{\tilde{B}^{s_0}_{2,r}} \sum_{q \in \mathbb{Z}} c_{q,r}.
\]
In order to guarantee the series, \( \sum_{q \in \mathbb{Z}} c_{q,r} \), to be convergent, the only choice is \( r = 1 \). That is the reason why we choose to work with the initial velocity in the Besov space, \( \mathcal{B}^{s_0} \), instead of the classical Sobolev space \( \dot{H}^{s_0} \).

With Proposition 2.1, we shall exploit the 2-D interpolation inequality

\[
\|a\|_{L^r} \leq C\|a\|_{L^2}^{\frac{2}{r}}\|\nabla a\|_{L^2}^{1-\frac{2}{r}} \quad \forall \ r \in [2, \infty[,
\]

to prove that

**Corollary 2.1.** Let \( r \in [2, \infty[. \) Then under the same assumptions of Proposition 2.1, we have for any \( t \geq 0 \),

\[
\begin{align*}
\|\sigma^{1-s_0} v\|_{L^\infty_t(L^r)} &+ \|\sigma^{1-s_0}(1-\frac{3}{2})v\|_{L^\infty_t(L^r)} + \|\sigma^{1-s_0}\nabla v\|_{\dot{L}^{r_0}_t(L^r)} \\
&+ \|\sigma^{(1-\frac{3}{2})} \nabla v\|_{L^\infty_t(L^r)} + \|\sigma^{(1-\frac{3}{2})} (v_t, \nabla^2 v, \nabla \pi)\|_{L^2_t(L^r)} \\
&+ \|\sigma^{(1-s_0)} (v_t, \nabla^2 v, \nabla \pi)\|_{L^{2r_0}_t(L^r)} + \|\sigma^{1-s_0}\nabla v\|_{L^\infty_t(L^r)} \leq C(v_0, s_0).
\end{align*}
\]

In order to derive the *a priori* estimate of \( \|\nabla X\|_{L^\infty_t(W^{1,p})} \), not only we need to perform the energy estimate for \( v_t \) but also the energy estimate of \( \nabla v_t \). To this end, we get, by applying \( \partial_t \) to the momentum equation of (1.1), that

\[
\rho \partial_t v_t + \rho v \cdot \nabla v_t - \Delta v_t + \nabla \pi_t = -\rho_t (v_t + v \cdot \nabla v) - \rho v_t \cdot \nabla v.
\]

To perform the energy estimate for \( \nabla v_t \), we need to deal with such terms as

\[
\int_{\mathbb{R}^2} \rho_t D_t v|v_{tt}| \, dx = -\int_{\mathbb{R}^2} v \cdot \nabla \rho D_t v|v_{tt}| \, dx,
\]

which is impossible to go through with non-Lipschitz density function \( \rho \). This is nevertheless the case here.

The idea to overcome this difficulty is to apply the material derivative \( D_t \) instead of \( \partial_t \) to the momentum equation of (1.1), which gives

\[
\rho D_t^2 v - \Delta D_t v + \nabla D_t \pi = F_D(v, \pi) \quad \text{with}
\]

\[
F_D(u, \Pi) \overset{\text{def}}{=} -2\nabla v_\alpha \cdot \partial_\alpha \nabla u - \Delta v \cdot \nabla u + \nabla v_\alpha \partial_\alpha \Pi.
\]

Here and in what follows, repeated indices of \( \alpha \) means summation of \( \alpha \) from 1 to 2. We remark that the advantage of exploiting the operator \( D_t \) is that \( D_t \rho = 0 \), so that difficult terms mentioned before do not appear anymore.

Note that due to \( \text{div} v = 0 \), one has

\[
\begin{align*}
\text{div} D_t v &= \text{div} (v \cdot \nabla v) = \partial_\alpha v \cdot \nabla v_\alpha \quad \text{and} \\
\text{div} D_t^2 v &= \text{div} b_0 \quad \text{with} \quad b_0 \overset{\text{def}}{=} v \cdot (\nabla v_t + D_t \nabla v) + D_t v \cdot \nabla v.
\end{align*}
\]

In Subsection 3.3, we shall use the following lemma to perform \( \dot{H}^1 \) energy estimate for \( D_t v \):

**Lemma 2.1.** Let \( (w, \nabla q) \) be a smooth enough solution of the following system:

\[
\begin{align*}
\left\{\begin{array}{l}
\rho D_t^2 w - \Delta D_t w + \nabla q = F, \\
\text{div} D_t w = \text{div} a \quad \text{and} \quad \text{div} D_t^2 w = \text{div} b.
\end{array}\right.
\end{align*}
\]
Lemma 2.2. Let (2.33) be satisfied. Then under the assumptions of Proposition 2.1, the following estimate is valid for any \( t \geq 0 \)

\[
A_{02}(t) \leq C(v_0, s_0).
\]

Then for any \( s \in [0, 1] \), we have

\[
\| \sigma^{s-\frac{\alpha}{2}} D_t w \|^2_{L^\infty(L^2)} + \| \sigma^s \nabla D_t w \|^2_{L^\infty(L^2)} + \| \sigma^{s-\frac{\alpha}{2}} (D_t^2 w, \nabla^2 D_t w, \nabla q) \|^2_{L^2(L^2)}
\]

(2.25)

\[
\leq C \exp \left( C \| v_0 \|^2_{L^2} \right) \left( \| \sigma^{s-\frac{\alpha}{2}} (D_t w, a) \|^2_{L^2(L^2)} + \| \sigma^{s-\frac{\alpha}{2}} \nabla a \|^2_{L^2(L^2)} + \| \sigma^{s-\frac{\alpha}{2}} (\nabla \text{div} a, b, F) \|^2_{L^2(L^2)} \right).
\]

The main result of Subsection 3.3 states as follows:

Proposition 2.2. Let \( A_{02}(t) \) be the functional given by (2.7). Then under the assumptions of Proposition 2.1, the following estimate is valid for any \( t \geq 0 \)

\[
A_{02}(t) \leq C(v_0, s_0).
\]

This verifies the Estimate (2.6). Next let us turn to the proof of (2.8) for \( \ell = 1 \). To this end, let us first derive the equations satisfied by \((\rho^1, v^1, \nabla \pi^1)\). It is easy to observe from (2.4) and (2.5) that

(2.26)

\[[\partial_X; D_t] = 0, \text{ that is, } \partial_X D_t f = D_t \partial_X f.
\]

Then we get, by applying \( \partial_X^\ell \) to the transport equation of (1.1), that

\[
D_t \rho^\ell = \partial_t \rho^\ell + v \cdot \nabla \rho^\ell = 0, \quad \ell = 1, \ldots, k,
\]

which implies

(2.27)

\[
\| \rho^\ell \|_{L^\infty(\mathbb{R}^+; L^\infty)} \leq \| \partial_X^\ell \rho_0 \|_{L^\infty}, \quad \ell = 1, \ldots, k.
\]

While due to \( \text{div} \ X_0 = 0 = \text{div} v \), we deduce from (2.4) that

(2.28)

\[
\partial_t (\text{div} X) + v \cdot \nabla (\text{div} X) = 0 \quad \text{with} \quad \text{div} X_0 = 0.
\]

Hence \( \text{div} v = \text{div} X = 0 \), a straightforward calculation shows that

(2.29)

\[
\text{div} v^1 = \text{div} (\partial_X v) = \partial_X v \cdot \nabla \alpha = \text{div} (v \cdot \nabla X).
\]

It is also easy to observe that

(2.30)

\[
[\partial_X; \partial_i] f = -\partial_i X \cdot \nabla f \quad \text{and} \quad [\partial_X; \partial_i^2] f = -\partial_i^2 X \cdot \nabla f - 2 \partial_i X \cdot \nabla \partial_i f, \quad i = 1, 2.
\]

In view of (2.26), (2.29) and (2.30), by taking \( \partial_X \) to the momentum equation of (1.1), we find that \((v^1, \nabla \pi^1)\) solves

(2.31)

\[
\begin{aligned}
\rho \partial_t v^1 + \rho v \cdot \nabla v^1 - \Delta v^1 + \nabla \pi^1 &= F_1(v, \pi), \\
\text{div} v^1 &= \text{div} (v \cdot \nabla X), \\
v^1|_{t=0} &= \partial_X v_0,
\end{aligned}
\]

where the source term \( F_1(v, \pi) \) is given by

(2.32)

\[
F_1(v, \pi) \overset{\text{def}}{=} -\rho^1 D_t v - (\Delta X \cdot \nabla v + 2 \partial_X X \cdot \nabla \partial_t v) + \nabla X^\alpha \partial_t \pi.
\]

In Section 4, we shall use the following lemma to work with the \( H^1 \) energy estimate of \( v^1 \):

Lemma 2.2. Let \((\rho, v, \nabla \pi)\) be a smooth enough solution of (1.1) and \((u, \nabla \Pi)\) be determined by

(2.33)

\[
\begin{aligned}
\rho \partial_t u + \rho v \cdot \nabla u - \Delta u + \nabla \Pi &= f, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\text{div} u &= \text{div} g,
\end{aligned}
\]
with initial data \( u_0 = 0 \). Then for any \( s \in [0, 1] \) and \( \delta \in [s, 1] \), one has
\[
\left\| \sigma^{1-s} \nabla u \right\|_{L^2(L^2)}^2 + \left\| \sigma^{1-s} (u_t, \nabla^2 u, \nabla \Pi) \right\|_{L^2(L^2)}^2 \\
\leq A_0 \left( \frac{(t)}{s} \left\| g(0) \right\|_{L^2}^2 + \left\| \sigma^{-1-(s-n)} \nabla g \right\|_{L^2(L^2)}^2 \\
+ \left\| \sigma^{-\frac{s}{2}} (f, g_t) \right\|_{L^2(L^2)}^2 + \left\| \sigma^{-\frac{s}{2}} \nabla g \right\|_{L^2(L^2)}^2 + \left\| \sigma^{1-s} (g_t, \nabla \text{div} g, f) \right\|_{L^2(L^2)}^2 \right).
\]

The main result concerning the \( H^1 \) energy estimate of \( v^1 \) lists as follows:

**Proposition 2.3.** Suppose that the initial data, \((\rho_0, v_0, X_0)\), satisfies the assumption (2.1), (2.3) and \( \partial X_0 \rho_0 \in L^\infty, \partial X_0 v_0 \in L^2 \cap B^{\alpha_1} \). Let \((\rho, v, \nabla \pi, X)\) be a smooth enough solution of the coupled system (1.1) and (2.4). Then one has
\[
A_{11}^2(t) \leq \exp(A_0(t^2)) \left( A_1 + \int_0^t \left( \mathcal{B}_0(t') + \sigma(t')^{-1 + \frac{\theta_0}{2}} \right) \left\| \nabla X(t') \right\|_{W^{1, \rho}(t')}^2 \, dt' \right),
\]
where \( \theta_0 \) is defined in (2.5), the functional \( A_{11}(t) \) is determined by (2.9) and
\[
\mathcal{B}_0(t) \overset{\text{def}}{=} \left\| \nabla v(t) \right\|_{L^2}^2 + \left\| \sigma^{1-\frac{\alpha_2}{2}} (v_t, \nabla^2 v, \nabla \pi, D_t v, v \otimes \nabla v)(t) \right\|_{L^2}^2 \\
+ \left\| \sigma^{1-\frac{\alpha_2}{2}} \nabla v(t) \right\|_{L^2}^2 + \left\| \sigma^{1-\frac{\alpha_2}{2}} \nabla v(t) \right\|_{L^\infty}^2 + \left\| \sigma^{1-\frac{\alpha_2}{2}} (D_t v, v \otimes \nabla v, v_t)(t) \right\|_{L^2}^2 \\
+ \left\| \sigma^{\frac{\alpha_2}{2}} (D_t^2 v, \nabla^2 D_t v, D_t^2 \nabla v, D_t^2 \nabla \pi, D_t \nabla \pi)(t) \right\|_{L^2}^2 \\
+ \left\| \sigma^{\frac{\alpha_2}{2}} ( \nabla D_t v, D_t \nabla v, \nabla (v \otimes \nabla v), \nabla v_t)(t) \right\|_{L^2}^2 + \left\| \sigma^{1-\frac{\alpha_2}{2}} b_0(t) \right\|_{L^2}^2,
\]
for any \( r \in [2, \infty] \), and where \( b_0 \) is given by (2.23).

To derive the equation of \((D_t v^1, \nabla D_t \pi^1)\), we get, by applying the operator \( D_t \) to the \( v^1 \) equation of (2.31) and using \( D_t \rho = 0 \), that
\[
\rho D_t^2 v^1 - \Delta D_t v^1 + \nabla D_t \pi^1 = F_D(v^1, \pi^1) + D_t F_1(v, \pi) \overset{\text{def}}{=} F_{1D},
\]
where \( F_D \) and \( F_1 \) are given by (2.22) and (2.32) respectively, and we thus obtain
\[
F_{1D} = -2\partial_\alpha v \cdot \nabla \partial_\alpha v^1 - \Delta v \cdot \nabla v^1 + \nabla v_0 \partial_\alpha \pi^1 \\
- \rho^1 D_t^2 v - D_t(\Delta X \cdot \nabla v + 2\partial_\alpha X \cdot \nabla \partial_\alpha v) + D_t(\nabla X^\alpha \partial_\alpha \pi).
\]

In Section 5, we shall apply Lemma 2.1 to deal with the time-weighted \( H^1 \) energy estimate of \( D_t v^1 \). This together with Proposition 2.3 leads to the energy estimate of \( D_t v^1 \) and \( \nabla D_t v^1 \), namely

**Proposition 2.4.** Let \((\rho, v, \nabla \pi, X)\) be a smooth solution of the coupled system (1.1) and (2.4). Then under the hypothesis of (2.1), (2.2) for \( \ell = 1 \) and (2.3), the Estimate (2.8) is valid for \( \ell = 1 \).

To handle the estimate (2.8) for \( \ell \) varying from 2 to \( k \) when \( k \geq 2 \), we need to derive the equations satisfied by \((v^\ell, \nabla \pi^\ell)\) and \((D_t v^\ell, \nabla D_t \pi^\ell)\) respectively. Indeed by taking \( \partial_{X}^{\ell-1} \) with \( \ell \geq 2 \) to (2.31), we write
\[
\rho \partial_t v^\ell + \rho v \cdot \nabla v^\ell - \Delta v^\ell + \nabla \pi^\ell = F_\ell(v, \pi),
\]
where the source term \( F_\ell(v, \pi) \) is determined inductively by
\[
F_\ell(v, \pi) = F_1(v^{\ell-1}, \pi^{\ell-1}) + \partial_X F_{\ell-1}(v, \pi), \quad \ell \geq 2,
\]
with the function $F_1(\cdot, \cdot)$ being given by (2.32). We thus get by induction that

$$F_\ell(v, \pi) = \sum_{i=0}^{\ell-1} \partial_X^i F_1(v^{\ell-1-i}, \pi^{\ell-1-i})$$

$$= \sum_{i=0}^{\ell-1} \partial_X^i \left( -\rho^1 D_t v^{\ell-1-i} - 2\partial_\alpha X \cdot \nabla \partial_\alpha v^{\ell-1-i} - \Delta X \cdot \nabla v^{\ell-1-i} + \nabla X \cdot \nabla \pi^{\ell-1-i} \right).$$

Taking into account of the fact (2.26) that $[\partial_X; D_t] = 0$, we obtain

$$F_\ell(v, \pi) = \sum_{i=0}^{\ell-1} \sum_{j=0}^{i} C^i_j \left( -\rho^{j+1} D_t v^{\ell-1-j} - 2\partial_X^j \partial_\alpha X \cdot \partial_X^{i-j} \nabla \partial_\alpha v^{\ell-1-i} - \partial_X^i \Delta X \cdot \partial_X^{i-j} \nabla v^{\ell-1-i} + \partial_X^i \nabla X \cdot \partial_X^{i-j} \nabla \pi^{\ell-1-i} \right).$$

(2.40)

On the other hand, by applying the operator $D_t$ to (2.39) and using $D_t \rho = 0$ once again, we find

$$\rho D_t^2 v^\ell - \Delta D_t v^\ell + \nabla D_t \pi^\ell = D_t F_\ell(v, \pi) + F_D(v^\ell, \pi^\ell) \overset{\text{def}}{=} F_{LD},$$

(2.41)

where $F_\ell(\cdot, \cdot)$ and $F_D(\cdot, \cdot)$ are given respectively by (2.40) and (2.22). Again thanks to $[\partial_X; D_t] = 0$, we have

$$F_{LD} = -2\partial_\alpha v \cdot \nabla \partial_\alpha v^\ell - \Delta v \cdot \nabla v^\ell + \nabla v \cdot \nabla \pi^\ell + \sum_{i=0}^{\ell-1} \sum_{j=0}^{i} C^i_j \left( -\rho^{j+1} D_t^2 v^{\ell-1-j} \right.$$

$$\left. - D_t \left( 2\partial_X^j \partial_\alpha X \cdot \partial_X^{i-j} \nabla \partial_\alpha v^{\ell-1-i} + \partial_X^i \Delta X \cdot \partial_X^{i-j} \nabla v^{\ell-1-i} - \partial_X^i \nabla X \cdot \partial_X^{i-j} \nabla \pi^{\ell-1-i} \right) \right).$$

(2.42)

With (2.39) and (2.41), by repeating the proof of Proposition 2.4 and through an inductive argument, we shall prove in Sections 6 and 7 that

**Proposition 2.5.** Let $(\rho, v, \nabla \pi, X)$ be the smooth solution to the coupled system (1.1) and (2.4). Then under the assumptions of (2.1), (2.2) and (2.3) with $k \geq 2$, the Estimate (2.8) is valid for $\ell = 2, \ldots, k$.

Now we are in the position to complete the proof of Theorem 2.1.

**Proof of Theorem 2.1.** By mollifying the initial data satisfying the assumptions (2.1), (2.2) and (2.3), we first construct the global approximate solutions to (1.1). Then along the same lines to the proof of the above propositions, we obtain similar uniform estimates as (2.6) and (2.8) for such approximate solutions. Finally a standard compactness argument as that used in the proof of Theorem 1.1 of [19] completes the existence part of Theorem 2.1. The uniqueness of such solutions has been proved in [24]. We skip the details here. \[\square\]

3. The Propagation of Besov Regularities of the Velocity Field

The goal of this section is to prove the Estimate (2.6). Toward this, let us first present some preliminary estimates for the linearized system (2.33).
3.1. Some preliminary energy estimates. Let \((\rho, v, \nabla \pi)\) be a smooth enough solution of (1.1). We first deduce from (1.1) and (2.1) that
\[
\rho_s \leq \rho(t, x) \leq \rho^*, \quad \text{and} \quad \frac{1}{2} \|\sqrt{\rho} v(t)\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = \frac{1}{2} \|\sqrt{\rho_0} v_0\|_{L^2}^2, \quad \forall \ t \geq 0,
\]
so that we deduce from (2.20) that
\[
\|v\|_{L^2_t(L^2)} \leq C \|v\|_{L^\infty_t(L^2)} \|\nabla v\|_{L^2_t(L^2)} \leq C \|v_0\|_{L^2}, \quad \forall \ t \geq 0.
\]

Lemma 3.1. Let \((u, \nabla \Pi)\) be a smooth enough solution of (2.33). Then we have
\[
\frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\nabla u, \nabla^2 u, \nabla \Pi\|_{L^2}^2 \leq C \left(\|v\|_{L^4}^4 \|\nabla u\|_{L^2}^2 + \|(g_t, \nabla \text{div} g, f)\|_{L^2}^2\right).
\]

Proof. By taking \(L^2(\mathbb{R}^2)\) inner product of the \(u\) equation of (2.33) with \(u_t\), we obtain
\[
\int_{\mathbb{R}^2} \rho |u_t|^2 \, dx + \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 = \int_{\mathbb{R}^2} \left( f - \nabla \Pi - \rho \nabla \cdot u \right) |u_t| \, dx.
\]

It follows from the 2-D interpolation inequality (2.20) that
\[
\left| \int_{\mathbb{R}^2} \rho \nabla \cdot u |u_t| \, dx \right| \leq C \|v\|_{L^4} \|\nabla u\|_{L^4} \|\sqrt{\rho u_t}\|_{L^2}
\leq C \|v\|_{L^4} \|\nabla u\|_{L^2}^4 \|
\leq \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} \|\sqrt{\rho u_t}\|_{L^2}.
\]

On the other hand, we get, by taking the space divergence operator to the \(u\) equation of (2.33), that
\[
\Delta \Pi = \text{div} \left( f - \rho u_t - \rho \nabla \cdot u \right) + \Delta \text{div} u,
\]
which together with the condition \(\text{div} u = \text{div} g\) yields
\[
\|\nabla \Pi\|_{L^2} \leq C \left( \|\nabla \text{div} g\|_{L^2} + \|\sqrt{\rho u_t}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla u\|_{L^2} + \|f\|_{L^2} \right),
\]
from which, and the \(u\) equation of (2.33), we deduce that \(\|\nabla^2 u\|_{L^2}\) satisfies the same estimate. This together with (2.20) ensures
\[
\|\nabla^2 u\|_{L^2} + \|\nabla \Pi\|_{L^2} \leq C \left( \|\nabla \text{div} g\|_{L^2} + \|\sqrt{\rho u_t}\|_{L^2} + \|\nabla u\|_{L^2} + \|f\|_{L^2} \right).
\]

Hence we get, by using Young’s inequality, that
\[
\left| \int_{\mathbb{R}^2} \rho \nabla \cdot u |u_t| \, dx \right| \leq C \left( \|v\|_{L^4}^4 \|\nabla u\|_{L^2}^2 + \|\nabla \text{div} g\|_{L^2}^2 + \|f\|_{L^2}^2 \right) + \frac{1}{6} \|\sqrt{\rho u_t}\|_{L^2}^2.
\]

While we get, by using integration by parts and (3.4), that
\[
\left| \int_{\mathbb{R}^2} \nabla \Pi |u_t| \, dx \right| = \left| \int_{\mathbb{R}^2} \nabla \Pi g_t \, dx \right| \leq \|\nabla \Pi\|_{L^2} \|g_t\|_{L^2}
\leq C \left( \|v\|_{L^4}^4 \|\nabla u\|_{L^2}^2 + \|(g_t, \nabla \text{div} g, f)\|_{L^2}^2 \right) + \frac{1}{6} \|\sqrt{\rho u_t}\|_{L^2}^2.
\]

As a result, it comes out
\[
\frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\sqrt{\rho u_t}\|_{L^2}^2 \leq C \left( \|v\|_{L^4}^4 \|\nabla u\|_{L^2}^2 + \|(g_t, \nabla \text{div} g, f)\|_{L^2}^2 \right).
\]

This together with (3.4) leads to (3.3). \(\Box\)
Proposition 3.1. Let \((u, \nabla \Pi)\) be a smooth enough solution of (2.33) with \(g = 0\). Then there hold for any \(t \geq 0\),
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho |u(t)|^2 \, dx + \|\nabla u\|_{L^2}^2 &\leq \int_{\mathbb{R}^2} f |u\|_{L^2} \, dx \\
\|\nabla u\|_{L^\infty}^2 + \|(u_t, \nabla^2 u, \nabla \Pi)\|_{L^2}^2 &\leq C\left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|f\|_{L^2}^2 \right), \\
\|\sigma \frac{1}{2} \nabla u\|_{L^\infty(L^2)}^2 + \|\sigma \frac{1}{2} (u_t, \nabla^2 u, \nabla \Pi)\|_{L^2(L^2)}^2 &\leq C\left( \|u_0\|_{L^2}^2 + \|f\|_{L^2(L^2)}^2 \right) \exp (C\|v_0\|_{L^2}^4),
\end{align*}
\]
and
\[
(3.6)
\]
\[
\frac{1}{2} \|\sqrt{\rho} u\|_{L_t^\infty(L^2)}^2 + \|\nabla u\|_{L_t^2(L^2)}^2 \leq \frac{1}{2} \|\sqrt{\rho} u_0\|_{L^2}^2 + C\|f\|_{L_t^1(L^2)}^2 + \frac{1}{4} \|\sqrt{\rho} u\|_{L_t^\infty(L^2)}^2,
\]
which leads to the first inequality of (3.5).

In view of (3.2), the second inequality of (3.5) can be achieved by applying Gronwall’s inequality to (3.3).

While we get, by multiplying (3.3) by \(\sigma(t)\) and then integrating the resulting inequality over \([0, t]\), that
\[
\begin{align*}
\sigma(t) \|\nabla u(t)\|_{L^2}^2 + \int_0^t \sigma(t') \|u_t, \nabla^2 u, \nabla \Pi\|_{L^2} \, dt' &\leq \int_0^t \|\nabla u(t')\|_{L^2}^2 \, dt' + C\int_0^t \|v\|_{L^2}^4 \sigma(t') \|\nabla u\|_{L^2}^2 + \sigma(t') \|f(t')\|_{L^2}^2 \, dt'.
\end{align*}
\]
Then in view of (3.2) and the first inequality of (3.5), we achieve (3.6) by applying Gronwall’s inequality. \(\square\)

In particular, taking \(u = v\) and \(f = g = 0\) in (3.6) gives rise to

Corollary 3.1. Let \((\rho, v, \nabla \pi)\) be a smooth enough solution of (1.1). Then there holds
\[
(3.9)
\|\sigma \frac{1}{2} \nabla v\|_{L_t^\infty(L^2)}^2 + \|\sigma \frac{1}{2} (v_t, \nabla^2 v, \nabla \pi)\|_{L_t^2(L^2)}^2 \leq C\|v_0\|_{L^2}^2 \exp (C\|v_0\|_{L^2}^4).
\]

Lemma 3.2. Let \((u, \nabla \Pi)\) be a smooth enough solution of (2.33) with \(g = 0\) and \(\tau(t)\) be a non-negative Lipschitz function. Then we have
\[
\begin{align*}
\|\tau \frac{1}{2} u_t\|_{L_t^\infty(L^2)}^2 + \|\tau \frac{1}{2} \nabla u_t\|_{L_t^2(L^2)}^2 &\leq \exp (C\|v_0\|_{L^2}^4) \left( \|\tau \frac{1}{2} f_t\|_{L_t^2(L^2)}^2 \right) \\
&+ \|\tau \frac{1}{2} \sigma \frac{1}{2} f\|_{L_t^2(L^2)}^2 + \|\tau \frac{1}{2} \sigma \frac{1}{2} \nabla u\|_{L_t^\infty(L^2)}^2 + \int_0^t \left( \frac{\tau^2}{2} + \tau \sigma^{-1} \right) \|u_t\|_{L^2}^2 \, dt'.
\end{align*}
\]

Proof. We first get, by taking \(\partial_t\) to the \(u\) equation of (2.33), that
\[
\rho \partial_t^2 u + pv \cdot \nabla u_t - \Delta u_t + \nabla \Pi_t = -\rho_t(u_t + v \cdot \nabla u) - pv_t \cdot \nabla u + f_t.
\]
Taking \(L^2(\mathbb{R}^2)\) inner product of the above equation with \(u_t\) gives
\[
(3.11)
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} u_t(t)\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 = -\int_{\mathbb{R}^2} (\rho_t(u_t + v \cdot \nabla u) + pv_t \cdot \nabla u) u_t \, dx + \int_{\mathbb{R}^2} f_t u_t \, dx.
\]
By using the transport equation of (1.1) and integrating by parts, one has
\[
\left| \int_{\mathbb{R}^2} \rho v u_t \, dx \right| = 2 \int_{\mathbb{R}^2} \rho v \cdot \nabla u_t \, dx \leq C \|v\|_{L^4} \|\nabla u_t\|_{L^2} \|u_t\|_{L^4},
\]
so that by applying the 2-D inequality (2.20) and Young’s inequality, we obtain
\[
(3.12) \quad \left| \int_{\mathbb{R}^2} \rho v u_t \, dx \right| \leq \frac{1}{6} \|\nabla u_t\|_{L^2}^2 + C \|v\|_{L^4}^4 \|u_t\|_{L^2}^2.
\]
Similarly, we get, by using integration by parts, that
\[
\int_{\mathbb{R}^2} \rho v \cdot \nabla u |u_t| \, dx = \int_{\mathbb{R}^2} (v \cdot \nabla) \rho v \cdot \nabla u |u_t| \, dx
+ \int_{\mathbb{R}^2} \rho (v \otimes \nabla) : \nabla^2 u |u_t| \, dx + \int_{\mathbb{R}^2} v \cdot \nabla u \rho v \cdot \nabla u |u_t| \, dx.
\]
Applying 2-D interpolation inequality (2.20) and the Estimate (3.4) gives
\[
\left| \int_{\mathbb{R}^2} \rho (v \cdot \nabla) v \cdot \nabla u |u_t| \, dx \right| \lesssim \|v\|_{L^4} \|\nabla v\|_{L^4} \|\nabla u\|_{L^2} \|u_t\|_{L^4}
\leq \frac{1}{6} \|\nabla u_t\|_{L^2}^2 + C \left( \|v\|_{L^4}^4 + \|\nabla v\|_{L^2}^2 \right) \|u_t\|_{L^2}^2
\times \left( \|u_t\|_{L^2}^\frac{1}{2} + \|v\|_{L^4} \|\nabla u\|_{L^2}^\frac{1}{2} + \|f\|_{L^2}^\frac{1}{2} \right) \|u_t\|_{L^2}^\frac{1}{2} \|\nabla u_t\|_{L^2}^\frac{1}{2},
\]
from which and Young’s inequality, we infer
\[
\left| \int_{\mathbb{R}^2} \rho (v \cdot \nabla) v \cdot \nabla u |u_t| \, dx \right| \leq \frac{1}{18} \|\nabla u_t\|_{L^2}^2 + C \left( \|v\|_{L^4}^4 + \|\nabla v\|_{L^2}^2 \right) \|u_t\|_{L^2}^2
+ \|\nabla v\|_{L^2}^2 \|f\|_{L^2}^2 + C \left( 1 + \|v\|_{L^2}^2 \right) \|\nabla v\|_{L^2}^2 \|u_t\|_{L^2}^2.
\]
Exactly along the same line, one has
\[
\left| \int_{\mathbb{R}^2} \rho (v \otimes \nabla) : \nabla^2 u |u_t| \, dx \right| \leq C \|v\|_{L^2}^2 \|\nabla^2 u\|_{L^2} \|u_t\|_{L^4}
\leq C \|v\|_{L^2}^\frac{1}{2} \|\nabla v\|_{L^2}^\frac{3}{2} \left( \|u_t\|_{L^2} + \|v\|_{L^4} \|\nabla u\|_{L^2} + \|f\|_{L^2} \right) \|u_t\|_{L^2}^\frac{1}{2} \|\nabla u_t\|_{L^2}^\frac{1}{2}
\leq \frac{1}{18} \|\nabla u_t\|_{L^2}^2 + C \left( \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^4 \|\nabla u\|_{L^2}^2 \right.
+ \|\nabla v\|_{L^2}^2 \|f\|_{L^2}^2 + C \left( 1 + \|v\|_{L^2}^2 \right) \|\nabla v\|_{L^2}^2 \|u_t\|_{L^2}^2,
\]
and
\[
\left| \int_{\mathbb{R}^2} v \cdot \nabla u \rho v \cdot \nabla u_t \, dx \right| \leq C \|v\|_{L^2}^2 \|\nabla u\|_{L^4} \|\nabla u_t\|_{L^2}
\leq C \|v\|_{L^2}^\frac{1}{2} \|\nabla v\|_{L^2}^\frac{3}{2} \|\nabla u\|_{L^2}^\frac{1}{2} \left( \|\rho u_t\|_{L^2}^\frac{1}{2} + \|v\|_{L^4} \|\nabla u\|_{L^2}^\frac{1}{2} + \|f\|_{L^2}^\frac{1}{2} \right) \|\nabla u_t\|_{L^2}
\leq \frac{1}{18} \|\nabla u_t\|_{L^2}^2 + C \left( \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \left( \|u_t\|_{L^2}^2 + \|f\|_{L^2}^2 \right) \right).
\]
Hence it comes out
\[
(3.13) \quad \left| \int_{\mathbb{R}^2} \rho v \cdot \nabla u |u_t| \, dx \right| \leq \frac{1}{6} \|\nabla u_t\|_{L^2}^2 + C \left( 1 + \|v\|_{L^2}^2 \right) \|\nabla v\|_{L^2}^2 \|u_t\|_{L^2}^2
+ C \left( \|v\|_{L^2}^2 + \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \right) \|\nabla u\|_{L^2}^2 + C \|\nabla v\|_{L^2}^2 \|f\|_{L^2}^2.
\]
Finally it is easy to observe that
\[
\left| \int_{\mathbb{R}^2} \rho v_t \cdot \nabla u | u_t \, dx \right| \leq C \|v_t\|_{L^2} \|\nabla u\|_{L^4} \|u_t\|_{L^4} \\
\leq C \|v_t\|_{L^2} \|\nabla u\|_{L^2}^2 \left( \|u_t\|_{L^2}^2 + \|v\|_{L^4}^2 \|\nabla u\|_{L^2}^2 + \|f\|_{L^2}^2 \right) \|u_t\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2.
\]

Applying Young’s inequality gives rise to
\[
\int_{\mathbb{R}^2} \rho v_t \cdot \nabla u | u_t \, dx \leq \frac{1}{6} \|\nabla u_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \\
+ \sigma^{-1}(t) (\|u_t\|_{L^2}^2 + \|f\|_{L^2}^2) + C (\|v\|_{L^4}^4 + \sigma(t) \|v_t\|_{L^2}^2) \|u_t\|_{L^2}^2.
\]

Inserting the Estimates (3.12), (3.13) and (3.14) into (3.11) gives rise to
\[
\frac{d}{dt} \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \leq C \left( (1 + \|v\|_{L^2}^2) \|\nabla v\|_{L^2}^2 + \sigma(t) \|v_t\|_{L^2}^2 \right) \|u_t\|_{L^2}^2 \\
+ C (\|v_t\|_{L^2}^2 + \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2) \|\nabla u_t\|_{L^2}^2 \\
+ \sigma^{-1}(t) \|u_t\|_{L^2}^2 + C (\|\nabla v\|_{L^2}^2 + 1) \sigma^{-1}(t) \|f\|_{L^2}^2 + \|f\|_{L^2} \|u_t\|_{L^2}^2.
\]

Multiplying the above inequality by \(\tau(t)\) and then applying Gronwall’s inequality and Young’s inequality to the resulting inequality leads to
\[
\|\sigma^{\frac{1}{2}} u_t\|_{L_t^\infty(L^2)}^2 + \|\sigma^{\frac{1}{2}} \nabla u_t\|_{L_t^2(L^2)}^2 \leq C \left( \|\tau^{\frac{1}{2}} f_t\|_{L_t^2(L^2)}^2 + (\|\nabla v\|_{L_t^\infty(L^2)}^2 + 1) \|\tau^{\frac{1}{2}} \sigma^{-\frac{1}{2}} f\|_{L_t^2(L^2)}^2 \\
+ (\|\nabla v\|_{L_t^2(L^2)}^2 + \|\nabla v\|_{L_t^\infty(L^2)}^2) \|\nabla v\|_{L_t^2(L^2)}^2 \|\nabla u_t\|_{L_t^2(L^2)}^2 \\
+ \int_0^t (\tau' + \sigma^{-1}) \|u_t\|_{L^2}^2 \, dt' \right) \times \exp \left( C \int_0^t (1 + \|v\|_{L^2}^2) \|\nabla v\|_{L^2}^2 + \sigma \|u_t\|_{L^2}^2 \, dt' \right),
\]

from which, (3.1) and (3.9), we conclude the proof of (3.10).

**Proposition 3.2.** Let \((u, \nabla \Pi)\) be a smooth enough solution of (2.33) with \(f = g = 0\). Then one has
\[
\|\sigma^{\frac{1}{2}} (u_t, \nabla^2 u, \nabla \Pi)\|_{L_t^\infty(L^2)}^2 + \|\sigma^{\frac{1}{2}} \nabla u_t\|_{L_t^2(L^2)}^2 \leq \|\nabla u_0\|_{L^2}^2 \exp \left( C \exp(C\|v_0\|_{L^2}^4) \right),
\]
and
\[
\|\sigma (u_t, \nabla^2 u, \nabla \Pi)\|_{L_t^\infty(L^2)}^2 + \|\sigma \nabla u_t\|_{L_t^2(L^2)}^2 \leq \|u_0\|_{L^2}^2 \exp \left( C \exp(C\|v_0\|_{L^2}^4) \right).
\]

**Proof.** Taking \(f = 0\) and \(\tau(t) = \sigma(t)\) in (3.10) and using (3.5) gives
\[
\|\sigma^{\frac{1}{2}} u_t\|_{L_t^\infty(L^2)}^2 + \int_0^t \sigma(t') \|\nabla u_t\|_{L^2}^2 \, dt' \leq \|\nabla u_0\|_{L^2}^2 \exp \left(C \exp(C\|v_0\|_{L^2}^4)\right).
\]

Resuming the Estimate (3.17) into (3.4) and using (3.1), (3.5) and (3.9) yields
\[
\|\sigma^{\frac{1}{2}} (\nabla^2 u, \nabla \Pi)\|_{L_t^\infty(L^2)}^2 \leq C \left( \|\sigma^{\frac{1}{2}} u_t\|_{L_t^\infty(L^2)}^2 + \|v\|_{L_t^\infty(L^2)}^2 \|\nabla v\|_{L_t^2(L^2)}^2 \|\nabla u\|_{L_t^\infty(L^2)}^2 \right) \\
\leq C \|\nabla u_0\|_{L^2}^2 \exp \left( C \exp(C\|v_0\|_{L^2}^4) \right).
\]

This achieves (3.15).

While by taking \(f = 0\) and \(\tau(t) = \sigma^2(t)\) in (3.10) and using (3.6) leads to
\[
\|\sigma u_t\|_{L_t^\infty(L^2)}^2 + \int_0^t \sigma^2 \|\nabla u_t\|_{L^2}^2 \, dt' \leq \|u_0\|_{L^2}^2 \exp \left(C \exp(C\|v_0\|_{L^2}^4)\right).
\]
And hence thanks to (3.4) and (3.1), (3.6), (3.9), we obtain
\[
\|\sigma(\nabla^2 u, \nabla \Pi)\|_{L^\infty_t(\mathcal{L}^2)}^2 \leq C \left( \|\sigma u_t\|_{L^\infty_t(\mathcal{L}^2)}^2 + \|v\|_{L^\infty_t(\mathcal{L}^2)}^2 \|\sigma \frac{1}{2} \nabla v\|_{L^\infty_t(\mathcal{L}^2)}^2 \|\sigma \frac{1}{2} \nabla u\|_{L^\infty_t(\mathcal{L}^2)}^2 \right)
\leq \|u_0\|_{L^2}^2 \exp \left( C \exp (C \|v_0\|_{L^2}^4) \right).
\]
This yields (3.16), and we complete the proof of the proposition. \qed

In particular, taking \(u = v\) in (3.16) gives rise to

**Corollary 3.2.** The following estimate is valid for smooth enough solution \((\rho, v, \nabla \pi)\) of (1.1)
\[
\|\sigma(v_t, \nabla^2 v, \nabla \pi)\|_{L^\infty_t(\mathcal{L}^2)}^2 + \|\sigma \nabla v_t\|_{L^2_t(\mathcal{L}^2)}^2 \leq \exp \left( C \exp (C \|v_0\|_{L^2}^4) \right).
\]

### 3.2. Time-weighted \(B^s\) estimate of \(u\).

**Proposition 3.3.** Let \((\rho, v, \nabla \pi)\) be a smooth enough solution of (1.1) and \((u, \nabla \Pi)\) solve (2.33) with \(f = g = 0\) and with initial data \(u_0 \in B^s\) for some \(s \in [0, 1]\). Then there holds
\[
\|u\|_{L^\infty_t(B^s)} + \|\nabla u\|_{L^2_t(B^s)} \leq C \|u_0\|_{B^s} \exp (C \|v_0\|_{L^2}^4),
\]
\[
\|\sigma \frac{1}{2} \nabla u\|_{L^\infty_t(B^s)} + \|\sigma \frac{1}{2} u_t\|_{L^2_t(B^s)} \leq C(v_0, u_0, s),
\]
\[
\|\sigma \frac{1}{2} \nabla u\|_{L^\infty_t(B^s)} + \|\sigma \frac{1}{2} (u_t, \nabla^2 u, \nabla \Pi)\|_{L^2_t(\mathcal{L}^2)} \leq C(v_0, u_0, s),
\]
where \(C(v_0, u_0, s)\) is given by (2.5).

**Proof.** Let \((u_q, \nabla \Pi_q)\) be determined by
\[
\begin{cases}
\rho \partial_t u_q + \rho v \cdot \nabla u_q - \Delta u_q + \nabla \Pi_q = 0, \\
\text{div } u_q = 0, \\
|u_q|_{t=0} = \Delta u_0.
\end{cases}
\]
Then by the uniqueness of solution to (2.33) with \(f = g = 0\), we have
\[u = \sum_{q \in \mathbb{Z}} u_q \text{ and } \nabla \Pi = \sum_{q \in \mathbb{Z}} \nabla \Pi_q.\]

While it follows from Definition 2.1, Bernstein type inequality (see [5]) and (3.5) that
\[
\|u_q\|_{L^\infty_t(\mathcal{L}^2)}^2 + \|\nabla u_q\|_{L^2_t(\mathcal{L}^2)}^2 \leq C \|\Delta_4 u_0\|_{L^2}^2 \leq d_q^2 2^{-2qs} \|u_0\|_{B^s}^2,
\]
and
\[
\|\nabla u_q\|_{L^\infty_t(\mathcal{L}^2)}^2 + \| (\partial_t u_q, \nabla^2 u_q, \nabla \Pi_q)\|_{L^2_t(\mathcal{L}^2)}^2 \leq C \|\Delta_4 u_0\|_{H^1}^2 \exp (C \|v_0\|_{L^2})
\leq d_q^2 2^{2q(1-s)} \|u_0\|_{B^s} \exp (C \|v_0\|_{L^2}).
\]
Exploiting the idea used in the proof of the characterization of Besov space with positive index (see Lemma 2.88 of [5] for instance) gives for any $j \in \mathbb{Z}$ that

$$
\|\Delta_j u\|_{L^\infty_t(L^2)} + \|\Delta_j \nabla u\|_{L^2_t(L^2)} \leq \sum_{q>j} \left( \|\Delta_j u_q\|_{L^\infty_t(L^2)} + \|\Delta_j \nabla u_q\|_{L^2_t(L^2)} \right) + 2^{-j} \sum_{q \leq j} \left( \|\Delta_j \nabla u_q\|_{L^\infty_t(L^2)} + \|\Delta_j \nabla^2 u_q\|_{L^2_t(L^2)} \right)
$$

which together with Definition 2.1 ensures the Inequality (3.19).

Similarly, we deduce from (3.6) that

$$
\|\sigma^{\frac{1}{2}} \nabla u_q\|_{L^\infty_t(L^2)} + \|\sigma^{\frac{1}{2}} (\partial_t u_q, \nabla^2 u_q, \nabla u_q)\|_{L^2_t(L^2)} \leq C \|\Delta_j u_0\|_{L^2} \exp(C \|v_0\|_{L^2}^4) \lesssim d_j 2^{-j s} \|u_0\|_{B^s} \exp(C \|v_0\|_{L^2}^4).
$$

And it follows from (3.15) that

$$
\|\sigma^{\frac{1}{2}} \nabla^2 u_q\|_{L^\infty_t(L^2)} + \|\sigma^{\frac{1}{2}} \partial_t u_q\|_{L^2_t(L^2)} \leq \|\nabla \Delta_q u_0\|_{L^2} \exp(C \exp(C \|v_0\|_{L^2}^4)) \lesssim d_j 2^{-2q(1-s)} C^2(v_0, u_0, s).
$$

Then exactly along the same line of the proof to the Inequality (3.19), we achieve the Inequality (3.20).

Finally we deduce from the following interpolation inequality in Besov spaces:

$$
\|\sigma^{\frac{1-s}{2}} u\|_{L^\infty_t(B^s_v)} \lesssim \|u\|_{L^\infty_t(B^s_v)} \|\sigma^{\frac{1}{2}} \nabla u\|_{L^\infty_t(B^s_v)}^{1-s},
$$

and (3.19) and (3.20), that

$$
\|\sigma^{\frac{1-s}{2}} u\|_{L^\infty_t(B^1_v)} \leq C(v_0, u_0, s).
$$

Whereas in view of (3.24) and (3.25), we have

$$
\int_{\mathbb{R}^+} \sigma(t)^{-s} \|\partial_t u_q(t)\|^2_{L^2} \, dt \leq \left( \int_{\mathbb{R}^+} \sigma(t)^s \|\partial_t u_q(t)\|^2_{L^2} \, dt \right)^{1-s} \left( \int_{\mathbb{R}^+} \|\partial_t u_q(t)\|^2_{L^2} \, dt \right)^s \lesssim d_q^2 C^2(v_0, u_0, s),
$$

which implies

$$
\left( \int_{\mathbb{R}^+} \sigma(t)^{-s} \|u_t\|^2_{L^2} \, dt \right)^{\frac{1}{2}} \leq \sum_{q \in \mathbb{Z}} \left( \int_{\mathbb{R}^+} \sigma(t)^{-1-s} \|\partial_t u_q\|^2_{L^2} \, dt \right)^{\frac{1}{2}} \lesssim C(v_0, u_0, s).
$$

The same estimate holds for $\|\sigma^{\frac{1-s}{2}} (\nabla^2 u, \nabla u)\|_{L^2_t(L^2)}$. This together with (3.27) ensures (3.21). We thus complete the proof of the proposition.

Let us remark that the estimate of $\|\sigma^{\frac{1-s}{2}} u_t\|_{L^2_t(L^2)}$ presented in Proposition 3.3 will be crucial for us to deal with the energy estimate of $u_t$. 

\[\square\]
Corollary 3.3. Under the same assumptions of Proposition 3.3, one has

\begin{equation}
\|\sigma^{1-\frac{s}{2}}(u_t, \nabla^2 u, \nabla \Pi)\|_{L^\infty_t(L^2)} + \|\sigma^{1-\frac{s}{2}} \nabla u_t\|_{L^2_t(L^2)} \leq C(v_0, u_0, s).
\end{equation}

Proof. Taking \( f = 0 \) and \( \tau(t) = \sigma(t)^{2-s} \) in Lemma 3.2 gives rise to

\[ \|\sigma^{1-\frac{s}{2}} u_t\|_{L^\infty_t(L^2)}^2 + \|\sigma^{1-\frac{s}{2}} \nabla u_t\|_{L^2_t(L^2)}^2 \leq \exp(\exp(C\|v_0\|_{L^2})) \left( \|\sigma^{\frac{1-s}{2}} \nabla u\|_{L^\infty_t(L^2)}^2 + \|\sigma^{1-s} u_t\|_{L^2_t(L^2)}^2 \right), \]

which together with Proposition 3.3 ensures that

\begin{equation}
\|\sigma^{1-\frac{s}{2}} u_t\|_{L^\infty_t(L^2)} + \|\sigma^{1-\frac{s}{2}} \nabla u_t\|_{L^2_t(L^2)} \leq C(v_0, u_0, s).
\end{equation}

As a result, by virtue of (3.4) and Corollary 3.1, Proposition 3.3, we obtain

\[ \|\sigma^{1-\frac{s}{2}} (\nabla^2 u, \nabla \Pi)\|_{L^\infty_t(L^2)} \lesssim \|\sigma^{1-\frac{s}{2}} u_t\|_{L^\infty_t(L^2)} \]

\[ + \|v\|_{L^\infty_t(L^2)} \|\sigma^{\frac{1-s}{2}} \nabla v\|_{L^\infty_t(L^2)} \|\sigma^{1-s} \nabla u\|_{L^\infty_t(L^2)} \lesssim C(v_0, u_0, s). \]

By summing up (3.29) and the above inequality, we achieve (3.28).

Thanks to (3.1), Proposition 3.3 and Corollary 3.3, we conclude the proof of Proposition 2.1 by taking \( u = v \) and \( f = g = 0 \) in (2.33). Next let us present the proof of Corollary 2.1.

Proof of Corollary 2.1. We first deduce from the 2-D interpolation inequality (2.20) that for any \( r \in [2, +\infty] \),

\[ \|\sigma^{\frac{1-\mu}{2r}} (1-\frac{2}{2r}) v\|_{L^r_t(L^r)} \lesssim \|v\|_{L^2_t(L^2)} \|\sigma^{\frac{1-\mu}{2r}} \nabla v\|_{L^\infty_t(L^2)} \]

\[ \|\sigma^{\frac{1-s}{2}} \nabla v\|_{L^2_t(L^2)} \lesssim \|\sigma^{\frac{1-\mu}{2r}} (1-\frac{2}{2r}) v_t\|_{L^\infty_t(L^2)} \|\sigma^{\frac{1-s}{2}} \nabla v\|_{L^\infty_t(L^2)} \]

\[ \|\sigma^{(1-\frac{1}{2r} - \frac{2}{2r})} \nabla v\|_{L^2_t(L^2)} \lesssim \|v_t\|_{L^\infty_t(L^2)} \|\sigma^{(1-\frac{1}{2r} - \frac{2}{2r})} \nabla v_t\|_{L^\infty_t(L^2)} \]

\[ \|\sigma^{1-\frac{2}{2r}} v_t\|_{L^2_t(L^2)} \lesssim \|v_t\|_{L^\infty_t(L^2)} \|\sigma^{1-\frac{2}{2r}} \nabla v_t\|_{L^\infty_t(L^2)} \]

from which and Proposition 2.1, we infer

\begin{equation}
\|\sigma^{\frac{1-\mu}{2r}} v\|_{L^\infty_t(L^r)} + \|\sigma^{\frac{1-\mu}{2r}} (1-\frac{2}{2r}) v\|_{L^\infty_t(L^r)} + \|\sigma^{\frac{1-\mu}{2r}} \nabla v\|_{L^2_t(L^2)} \]

\[ + \|\sigma^{(1-\frac{1}{2r} - \frac{2}{2r})} \nabla v\|_{L^r_t(L^r)} + \|\sigma^{(1-\frac{1}{2r} - \frac{2}{2r})} v_t\|_{L^2_t(L^2)} + \|\sigma^{1-\frac{2}{2r}} v_t\|_{L^2_t(L^2)} \lesssim C(v_0, s_0). \]

While it follows from the classical estimate of Stokes operator, (2.20) and the 2-D interpolation inequality \( \|a\|_{L^{2r}} \lesssim \|a\|_{L^r}^{\frac{1}{2r}} \|\nabla a\|_r^{\frac{1}{2r}} \), that

\[ \|(\nabla^2 v, \nabla \Pi)\|_{L^r} \lesssim \|v_t\|_{L^r} + \|v\|_{L^{2r}} \|\nabla v\|_{L^r} \]

\[ \lesssim \|v_t\|_{L^r} + \|v\|_{L^2} \|\nabla v\|_{L^r} \|\nabla v\|_{L^r}, \]

which implies

\[ \|(\nabla^2 v, \nabla \Pi)\|_{L^r} \lesssim \|v_t\|_{L^r} + \|v\|_{L^2} \|\nabla v\|_{L^2} \|\nabla v\|_{L^r}. \]
We then deduce from (3.30) that
\[
\left\| \sigma^{1-\frac{4\omega}{2}} (\nabla^2 v, \nabla \pi) \right\|_{L^2_t(L^r)} \lesssim \left\| \sigma^{1-\frac{4\omega}{2}} v_t \right\|_{L^2_t(L^r)} \\
+ \left\| v \right\|_{L^\infty_t(L^2)} \left\| \nabla v \right\|_{L^2_t(L^2)} \left\| \sigma^{1-\frac{4\omega}{2}} \nabla v \right\|_{L^\infty_t(L^r)} \lesssim C(v_0, s_0).
\] (3.31)

Along the same line, it follows from Corollary 3.1 and (3.30) that
\[
\left\| \sigma^{1-\frac{4\omega}{2}} (\nabla^2 v, \nabla \pi) \right\|_{L^2_t(L^r)} \lesssim \left\| \sigma^{1-\frac{4\omega}{2}} v_t \right\|_{L^2_t(L^r)} \\
+ \left\| v \right\|_{L^\infty_t(L^2)} \left\| \frac{1}{\sigma} \nabla v \right\|_{L^2_t(L^2)} \left\| \sigma^{1-\frac{8\omega}{2}} \nabla v \right\|_{L^\infty_t(L^r)} \lesssim C(v_0, s_0).
\]

Finally taking \( r = \frac{2}{1-\frac{4\omega}{3}} \) in (3.31) and using the 2-D interpolation inequality
\[
\left\| a \right\|_{L^\infty} \lesssim \left\| a \right\|_{L^\frac{7}{2}} \left\| \nabla a \right\|_{L^\frac{7}{2}} \lesssim \left\| a \right\|_{H^s} \left\| \nabla a \right\|_{L^\frac{7}{2}},
\]
and Proposition 2.1, we infer
\[
\left\| \sigma^{1-\frac{4\omega}{2}} \nabla v \right\|_{L^2_t(L^\infty)} \lesssim \left\| \nabla v \right\|_{L^\infty_t(H^s)} \left\| \sigma^{\frac{9}{2}} \nabla^2 v \right\|_{L^\infty_t(L^{\frac{1}{2}})} \lesssim C(v_0, s_0).
\]

This ends the proof of Corollary 2.1. \( \square \)

3.3. Time-weighted \( H^1 \) estimate of \( D_t v \). The main result of this subsection is to perform the time-weighted \( \dot{H}^1 \) estimate of \( D_t v \). To the end, let us first present the following lemma:

Lemma 3.3. Under the assumptions of Proposition 2.1, for any \( r \in [2, \infty[ \), we have
\[
\left\| \sigma^{1-\frac{4\omega}{2}} (D_t v, v \otimes \nabla v) \right\|_{L^\infty_t(L^2)} + \left\| \sigma^{\frac{3-4\omega}{2}} (\nabla v \otimes \nabla v, v \otimes \nabla^2 v) \right\|_{L^\infty_t(L^2)} \\
+ \left\| \sigma^{1-\frac{4\omega}{2}} (D_t v_t, v \otimes \nabla v, v \otimes \nabla^2 v) \right\|_{L^\infty_t(L^2)} \\
+ \left\| \sigma^{1-\frac{4\omega}{2}} (D_t v, v \otimes \nabla v, v \otimes \nabla^2 v) \right\|_{L^\infty_t(L^2)} \\
+ \left\| \sigma^{1-\frac{4\omega}{2}} (D_t v, v \otimes \nabla v, D_t v_t, v \otimes \nabla v) \right\|_{L^\infty_t(L^2)} \\
+ \left\| \sigma^{1-\frac{4\omega}{2}} (D_t v, v \otimes \nabla v, v \otimes \nabla^2 v) \right\|_{L^\infty_t(L^2)} \leq C(v_0, s_0).
\] (3.32)

Proof. By virtue of (2.20) and
\[
\left\| a \right\|_{L^\infty} \leq C \left\| a \right\|_{L^\frac{7}{2}} \left\| \nabla^2 a \right\|_{L^2},
\]
we deduce from (3.2), (3.9), (3.18) that for any \( t \in \mathbb{R}^+ \) and \( r \in [2, \infty[ \)
\[
\left\| v \right\|_{L^\infty_t(L^r)} + \left\| \sigma^{\frac{2}{7}} v \right\|_{L^\infty_t(L^r)} + \left\| \sigma^{\frac{3}{7}} v \right\|_{L^\infty_t(L^r)} + \left\| \sigma^{\frac{4}{7}} v \right\|_{L^\infty_t(L^r)} \\
+ \left\| \sigma^{\frac{5}{7}} v \right\|_{L^\infty_t(L^r)} + \left\| \sigma^{\frac{6}{7}} v \right\|_{L^\infty_t(L^r)} \leq \exp \left( C \exp \left( C \left\| v_0 \right\|_{L^2}^4 \right) \right).
\] (3.33)

It is easy to observe that
\[
\left\| \sigma^{1-\frac{4\omega}{2}} (D_t v, v \otimes \nabla v) \right\|_{L^\infty_t(L^2)} \leq \left\| \sigma^{1-\frac{4\omega}{2}} v_t \right\|_{L^\infty_t(L^2)} + \left\| \sigma^{\frac{3}{2}} v \right\|_{L^\infty_t(L^4)} \left\| \sigma^{\frac{3-4\omega}{2}} \nabla v \right\|_{L^\infty_t(L^4)},
\]
and
\[
\left\| \sigma^{\frac{3-4\omega}{2}} (\nabla v \otimes \nabla v, v \otimes \nabla^2 v) \right\|_{L^\infty_t(L^2)} \leq \left\| \sigma^{\frac{3}{2}} \nabla v \right\|_{L^\infty_t(L^4)} \left\| \sigma^{\frac{3-4\omega}{2}} \nabla v \right\|_{L^\infty_t(L^4)} \\
+ \left\| \sigma^{\frac{3}{2}} v \right\|_{L^\infty_t(L^4)} \left\| \sigma^{1-\frac{4\omega}{2}} \nabla^2 v \right\|_{L^\infty_t(L^4)}.
\]

Hence we deduce from Proposition 2.1, Corollary 2.1 and (3.33) that
\[
\left\| \sigma^{1-\frac{4\omega}{2}} (D_t v, v \otimes \nabla v) \right\|_{L^\infty_t(L^2)} + \left\| \sigma^{\frac{3-4\omega}{2}} (\nabla v \otimes \nabla v, v \otimes \nabla^2 v) \right\|_{L^\infty_t(L^2)} \leq C(v_0, s_0).
\]
Exactly along the same line, we have
\[ \| \sigma^{1-s_0} (D_t v, v \otimes \nabla v) \|_{L^2_t(L^2)} \leq \| \sigma^{1-s_0} v_t \|_{L^2_t(L^2)} + \| v \|_{L^4_t(L^4)} \| \sigma^{1-s_0} \nabla v \|_{L^8_t(L^8)} \leq C(v_0, s_0), \]
and
\[ \| \sigma^{-1-2s_0} \left( \nabla D_t v, D_t \nabla v, \nabla v \otimes \nabla v, v \otimes \nabla^2 v \right) \|_{L^2_t(L^2)} \leq \| \sigma^{-1-2s_0} \nabla v_t \|_{L^2_t(L^2)} + \| \sigma^{-1-2s_0} \nabla v \|_{L^8_t(L^8)} \| \sigma^{-1-2s_0} \nabla^2 v \|_{L^2_t(L^2)} \leq C(v_0, s_0), \]
from which and (2.20), we infer for any \( r \in [2, \infty] \),
\[ \| \sigma^{(1-\frac{1}{r}-\frac{s_0}{2})} (D_t v, v \otimes \nabla v) \|_{L^2_t(L^r)} \leq C \| \sigma^{1-s_0} (D_t v, v \otimes \nabla v) \|_{L^2_t(L^2)} \]
\[ \times \| \sigma^{1-\frac{s_0}{2}} (\nabla D_t v, \nabla (v \otimes \nabla v)) \|^{1-\frac{1}{2}}_{L^2_t(L^2)} \leq C(v_0, s_0), \]
and
\[ \| \sigma^{3-2s_0} (v \otimes (\nabla v_t, \nabla D_t v)) \|_{L^2_t(L^2)} \leq \| \sigma^{3} v \|_{L^\infty_t(L^\infty)} \| \sigma^{1-2s_0} (v \otimes D_t v) \|_{L^2_t(L^2)} \leq C(v_0, s_0). \]
Moreover, we deduce from Proposition 3.3 and Corollary 3.1 that
\[ \| \sigma^{3-2s_0} (\nabla v \otimes \nabla^2 v, \nabla v \otimes \nabla \pi) \|_{L^2_t(L^2)} \leq \| \sigma^{1} \nabla v \|_{L^4_t(L^4)} \| \sigma^{1} (\nabla^2 v, \nabla \pi) \|_{L^1_t(L^1)} \leq C(v_0, s_0), \]
and
\[ \| \sigma^{3-2s_0} (\nabla v \otimes \nabla v) \|_{L^2_t(L^2)} \leq \| \sigma \nabla v \|_{L^\infty_t(L^\infty)} \| \sigma^{1} (\nabla^2 v, \nabla \pi) \|_{L^1_t(L^1)} \leq C(v_0, s_0), \]
This completes the proof of the Lemma. \( \square \)

Before proceeding the time-weighted \( \dot{H}^1 \) energy estimate, let us first present the proof of Lemma 2.1.

**Proof of Lemma 2.1.** By taking \( L^2(\mathbb{R}^2) \) inner product of (2.24) with \( D_t w \), one has
\[ \frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} D_t w(t) \|_{L^2}^2 + \| \nabla D_t w \|_{L^2}^2 = \int_{\mathbb{R}^2} (F - \nabla q) |D_t w| \ dx. \]

Multiplying the above equality by \( \sigma(t)^{2-s} \) and then integrating the resulting equality over \([0, t]\) leads to
\[ \frac{1}{2} \| \sigma^{1-\frac{s}{2}} \sqrt{\rho} D_t w(t) \|_{L^2}^2 + \| \sigma^{1-\frac{s}{2}} \nabla D_t w \|_{L^2_t(L^2)}^2 \]
\[ = \frac{2 - s}{2} \| \sigma^{1-\frac{s}{2}} \sqrt{\rho} D_t w \|_{L^2_t(L^2)}^2 + \int_{0}^{t} \int_{\mathbb{R}^2} \sigma^{2-s} (F - \nabla q) |D_t w| \ dx \ dt'. \]

It is easy to observe that
\[ \left| \int_{0}^{t} \int_{\mathbb{R}^2} \sigma^{2-s} F |D_t w| \ dx \ dt' \right| \leq \| \sigma^{\frac{1-s}{2}} F \|_{L^2_t(L^2)}^2 + \| \sigma^{\frac{1-s}{2}} D_t w \|_{L^2_t(L^2)}^2. \]

While due to \( \text{div} D_t w = \text{div} a \), by virtue of (2.24), we write
\[ - \int_{0}^{t} \int_{\mathbb{R}^2} \sigma^{2-s} \nabla q |D_t w| \ dx \ dt' = \int_{0}^{t} \int_{\mathbb{R}^2} \sigma^{2-s} (\rho D_t^2 w - \Delta D_t w - F) a \ dx \ dt'. \]
As a result, for any $\varepsilon > 0$, we obtain
\[
\left| \int_0^t \int_{\mathbb{R}^2} \sigma^{2-s} \nabla q |D_t w| \, dx \, dt' \right| \leq \frac{1}{2} \left( \|\sigma^{1-s} \nabla v \|_{L^2_2(H^3)}^2 + \varepsilon \|\sigma^{3-s} \sqrt{\rho} D_t^2 w \|_{L^2_2}^2 \right) + C_\varepsilon \left( \|\sigma^{1-s} a \|_{L^2_2}^2 + \|\sigma^{1-s} \nabla a \|_{L^2_2}^2 + \|\sigma^{3-s} F \|_{L^2_2}^2 \right).
\]
Inserting the above inequalities into (3.34) yields
\[
\begin{aligned}
\|\sigma^{1-s} \sigma^{3-s} \sqrt{\rho} D_t w(t) \|^2_{L^2_2} + \|\sigma^{1-s} \nabla D_t w \|^2_{L^2_2} & \leq C \|\sigma^{1-s} \sigma^{3-s} \sqrt{\rho} D_t w \|^2_{L^2_2} \\
& + \varepsilon \|\sigma^{1-s} \sigma^{3-s} \sqrt{\rho} D_t^2 w \|^2_{L^2_2} + C_\varepsilon \left( \|\sigma^{1-s} a \|_{L^2_2}^2 + \|\sigma^{1-s} \nabla a \|_{L^2_2}^2 + \|\sigma^{3-s} F \|_{L^2_2}^2 \right).
\end{aligned}
\tag{3.35}
\]
On the other hand, due to $\text{div} v = 0$, one has
\[
\int_{\mathbb{R}^2} - \Delta D_t w |D_t^2 w| \, dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla D_t w|^2 \, dx + \int_{\mathbb{R}^2} \nabla D_t w : (\nabla v_t \partial_a D_t w) \, dx,
\]
and
\[
\left| \int_{\mathbb{R}^2} \nabla D_t w : (\nabla v_t \partial_a D_t w) \, dx \right| \leq \|\nabla v_t \|_{L^2} \|\nabla D_t w \|^2_{L^4} \leq C \|\nabla v_t \|_{L^2} \|\nabla D_t w \|_{L^2} \|\nabla^2 D_t w \|_{L^2},
\]
so that we get, by taking the $L^2(\mathbb{R}^2)$ inner product of (2.24) with $D_t^2 w$, that
\[
\left| \|\sqrt{\rho} D_t^2 w \|^2_{L^2} + \frac{1}{2} \frac{d}{dt} \|\nabla D_t w \|^2_{L^2} \right| = \int_{\mathbb{R}^2} (F - \nabla q) |D_t^2 w| \, dx + C \|\nabla v_t \|_{L^2} \|\nabla D_t w \|_{L^2} \|\nabla^2 D_t w \|_{L^2}.
\]
For any $\eta > 0$, multiplying the above equality by $\sigma(t)^{3-s}$ and then integrating the resulting equality over $[0, t]$ leads to
\[
\begin{aligned}
\frac{1}{2} \|\sigma^{3-s} \nabla D_t w(t) \|^2_{L^2} + \|\sigma^{3-s} \sqrt{\rho} D_t^2 w \|^2_{L^2} & \leq \frac{3-s}{2} \|\sigma^{1-s} \nabla D_t w \|^2_{L^2} \\
& + \frac{1}{4} \|\sigma^{3-s} \sqrt{\rho} D_t^2 w \|_{L^2} + C \|\sigma^{3-s} F \|^2_{L^2} + \eta \|\sigma^{3-s} \nabla D_t w \|^2_{L^2} \\
& + C_\eta \int_0^t \|\nabla v_t \|^2_{L^2} \|\sigma^{3-s} \nabla D_t w \|^2_{L^2} \, dt' - \int_0^t \int_{\mathbb{R}^2} \sigma^{3-s} \nabla q |D_t^2 w| \, dx \, dt'.
\end{aligned}
\tag{3.36}
\]
By using integration by parts and the condition: $\text{div} D_t^2 w = \text{div} b$, we get that
\[
\begin{aligned}
\left| \int_0^t \int_{\mathbb{R}^2} \sigma^{3-s} \nabla q |D_t^2 w| \, dx \, dt' \right| = \left| \int_0^t \int_{\mathbb{R}^2} \sigma^{3-s} \nabla q |b| \, dx \, dt' \right| \leq \eta \|\sigma^{3-s} \nabla q \|^2_{L^2} + C_\eta \|\sigma^{3-s} b \|^2_{L^2}.
\end{aligned}
\tag{3.37}
\]
To deal with the estimate of $\nabla^2 D_t w$ and $\nabla q$, in view of (2.24), we write
\[
- \Delta (D_t w - \nabla \Delta^{-1} \text{div} D_t w) + \nabla q = -\rho D_t^2 w + \nabla \text{div} D_t w + F.
\]
Since $\text{div}(D_t w - \nabla \Delta^{-1} \text{div} D_t w) = 0$, we deduce from the classical estimate on Stokes’ operator that
\[
\|\nabla^2 D_t w, \nabla q \|_{L^2} \lesssim \|\rho D_t^2 w \|_{L^2} + \|\nabla \text{div} D_t w \|_{L^2} + \|F \|_{L^2},
\]
which implies
\[
\|\sigma^{3-s} (\nabla^2 D_t w, \nabla q) \|_{L^2} \leq C \|\sigma^{3-s} (\sqrt{\rho} D_t^2 w, \nabla \text{div} a, F) \|_{L^2}.
\tag{3.38}
\]
Inserting (3.38) into (3.37) and then substituting the resulting inequality into (3.36), we get, by taking \( \eta \) so small that \( C^2 \eta \leq \frac{1}{4} \), that
\[
\left\| \sigma^{\frac{3-s}{2}} \nabla D_t w \right\|_{L_t^\infty (L^2)}^2 + \left\| \sigma^{\frac{3-s}{2}} \sqrt{\rho} D_t^2 v \right\|_{L_t^2 (L^2)}^2 \leq C \left\| \sigma^{\frac{3-s}{2}} (\nabla \text{div} a, b, F) \right\|_{L_t^2 (L^2)}^2
\]
\[
+ 3 \left\| \sigma^{1-s} \nabla D_t w \right\|_{L_t^2 (L^2)}^2 + 2 \int_0^t \left\| \nabla v \right\|_{L^2}^2 \left\| \sigma^{\frac{3-s}{2}} \nabla D_t w \right\|_{L^2}^2 \, dt'.
\]

Taking \( \varepsilon \leq \frac{1}{8} \), we deduce (2.25) by summing up the above inequality with \( 4 \times (3.35) \) and then applying Gronwall’s inequality and using (3.38). This completes the proof of Lemma 2.1. \( \square \)

Let us now turn to the proof of Proposition 2.2.

Proof of Proposition 2.2. Taking into account of (2.23), we get, by applying Lemma 2.1 to (2.22), that
\[
\left\| \sigma^{\frac{3-s}{2}} (D_t v, v \cdot \nabla v) \right\|_{L_t^2 (L^2)}^2 + \left\| \sigma^{1-s} (\nabla (v \cdot v)) \right\|_{L_t^2 (L^2)}^2 \leq C_0 (v_0, s_0).
\]

In view of (2.22), (2.23), we deduce from Lemma 3.3 that
\[
\left\| \sigma^{\frac{3-s}{2}} F_D (v, \pi) \right\|_{L_t^2 (L^2)}^2 \leq \left\| \sigma^{\frac{3-s}{2}} (\nabla v \otimes \nabla v, \nabla v \otimes \nabla \pi) \right\|_{L_t^2 (L^2)}^2 \leq C (v_0, s_0),
\]
and
\[
\left\| \sigma^{\frac{3-s}{2}} b_0 \right\|_{L_t^2 (L^2)} \leq \left\| \sigma^{\frac{3-s}{2}} (v \otimes (\nabla v_t, D_t \nabla v), D_t v \otimes \nabla v) \right\|_{L_t^2 (L^2)} \leq C (v_0, s_0).
\]

Substituting the above inequalities into (3.39) gives rise to
\[
\left\| \sigma^{\frac{3-s}{2}} \nabla D_t v \right\|_{L_t^\infty (L^2)}^2 + \left\| \sigma^{\frac{3-s}{2}} (D_t^2 v, \nabla^2 D_t v, \nabla \nabla \pi) \right\|_{L_t^2 (L^2)}^2 \leq C (v_0, s_0),
\]
from which and Lemma 3.3 again, we infer
\[
\left\| \sigma^{\frac{3-s}{2}} (D_t \nabla v, \nabla v_t) \right\|_{L_t^\infty (L^2)} \leq \left\| \sigma^{\frac{3-s}{2}} \nabla D_t v \right\|_{L_t^\infty (L^2)}
\]
\[
+ \left\| \sigma^{\frac{3-s}{2}} (\nabla v \otimes \nabla v, v \otimes \nabla^2 v) \right\|_{L_t^\infty (L^2)} \leq C (v_0, s_0).
\]

Along with (3.41), we complete the proof of Proposition 2.2. \( \square \)

We conclude this section with the following corollary concerning the estimates of such terms as \( \left\| \sigma^{1-s} \nabla v \right\|_{L_t^\infty (L^\infty)}, \left\| b_0 \right\|_{L_t^2} \), which will play an important role in the following context.

Corollary 3.4. Let \((\rho, v, \nabla \pi)\) be a smooth enough solution of (1.1). Let \( r \in [2, \infty[ \) and \( \mathfrak{B}_0 (t) \) be determined by (2.36). Then one has
\[
\left\| \sigma^{\frac{3-s}{2}} (D_t v, v_t, v \otimes \nabla v, \nabla^2 v, \nabla \pi) \right\|_{L_t^\infty (L^r)}^2
\]
\[
+ \left\| \sigma^{1-s} \nabla v \right\|_{L_t^\infty (L^\infty)}^2 + \int_0^t \mathfrak{B}_0 (t') \, dt' \leq C^2 (v_0, s_0).
\]
Proof. It follows from (2.20) that for any \(r \in [2, \infty]\),

\[
\| \sigma \left( \frac{3}{2} - \frac{1}{2} - \frac{2m}{2} \right) D_t v \|_{L^2_t(L^r)} \leq C \| \sigma^{1-\frac{m}{2}} D_t v \|_{L^\infty_t(L^2)} \| \sigma^{\frac{3+2m}{2}} \nabla D_t v \|_{L^2_t(L^r)}^{\frac{1}{2}} \| \sigma^{\frac{3-2m}{2}} \nabla D_t v \|_{L^2_t(L^r)}^{\frac{1}{2}},
\]

\[
\| \sigma \left( \frac{3}{2} - \frac{1}{2} - \frac{2m}{2} \right) \nabla D_t v \|_{L^2_t(L^r)} \leq C \| \sigma^{1-\frac{m}{2}} \nabla D_t v \|_{L^\infty_t(L^2)} \| \sigma^{\frac{3+2m}{2}} \nabla^2 D_t v \|_{L^2_t(L^r)}^{\frac{1}{2}} \| \sigma^{\frac{3-2m}{2}} \nabla^2 D_t v \|_{L^2_t(L^r)}^{\frac{1}{2}},
\]

which together with (3.32), (3.41) and classical estimates on Stokes operator \(-\Delta v + \nabla \pi = -\rho D_t v\), ensures that for any \(r \in [2, \infty]\),

\[
(3.43) \quad \| \sigma \left( \frac{3}{2} - \frac{1}{2} - \frac{2m}{2} \right) (D_t v, \nabla^2 v, \nabla \pi) \|_{L^2_t(L^r)} + \| \sigma \left( \frac{3}{2} - \frac{1}{2} - \frac{2m}{2} \right) \nabla D_t v \|_{L^2_t(L^r)} \leq C(v_0, s_0).
\]

Hence thanks to

\[
\| \sigma^{1-\frac{m}{2}} \nabla v \|_{L^\infty_t(L^\infty)} \leq C \| \sigma^{1-\frac{m}{2}} \nabla v \|_{L^\infty_t(L^2)} \| \sigma^{\frac{3+2m}{2}} \nabla^2 v \|_{L^2_t(L^\infty)}^{\frac{1}{2}} \| \sigma^{\frac{3-2m}{2}} \nabla^2 D_t v \|_{L^2_t(L^\infty)}^{\frac{1}{2}},
\]

\[
\| \sigma^{1-\frac{m}{2}} D_t v \|_{L^2_t(L^\infty)} \leq C \| \sigma^{1-\frac{m}{2}} D_t v \|_{L^\infty_t(L^2)} \| \sigma^{\frac{3+2m}{2}} \nabla^2 D_t v \|_{L^2_t(L^\infty)}^{\frac{1}{2}} \| \sigma^{\frac{3-2m}{2}} \nabla^2 D_t v \|_{L^2_t(L^\infty)}^{\frac{1}{2}},
\]

and Proposition 2.1, (3.32), (3.41), (3.43), we deduce that

\[
(3.44) \quad \| \sigma^{1-\frac{m}{2}} \nabla v \|_{L^\infty_t(L^\infty)} + \| \sigma^{1-\frac{m}{2}} D_t v \|_{L^2_t(L^\infty)} \leq C(v_0, s_0).
\]

While it follows from Corollary 2.1 and (3.33) that for any \(r \in [2, \infty]\),

\[
\| \sigma \left( \frac{3}{2} - \frac{1}{2} - \frac{2m}{2} \right) (v \otimes \nabla v) \|_{L^2_t(L^r)} \leq C \| \sigma^{\frac{1}{2}} v \|_{L^\infty_t(L^\infty)} \| \sigma^{1-\frac{1}{2}} \nabla v \|_{L^\infty_t(L^r)} \leq C(v_0, s_0),
\]

from which and \(v_t = D_t v - v \cdot \nabla v\), we infer from (3.43) that for \(r \in [2, +\infty]\)

\[
\| \sigma \left( \frac{3}{2} - \frac{1}{2} - \frac{2m}{2} \right) (v \otimes \nabla v, v_t) \|_{L^2_t(L^r)} \leq C(v_0, s_0).
\]

Finally it is easy to observe that

\[
\| \sigma^{1-\frac{m}{2}} (v \otimes \nabla v) \|_{L^2_t(L^\infty)} \leq C \| \sigma^{\frac{1}{2}} v \|_{L^\infty_t(L^\infty)} \| \sigma^{1-\frac{m}{2}} \nabla v \|_{L^2_t(L^\infty)},
\]

and

\[
\| \sigma^{\frac{3+2m}{2}} (D_t \nabla^2 v, \nabla D_t v, D_t \nabla \pi) \|_{L^2_t(L^2)} \leq \| \sigma^{\frac{3+2m}{2}} (\nabla^2 D_t v, \nabla D_t \pi) \|_{L^2_t(L^2)} + C \| \sigma^{\frac{3+2m}{2}} (\nabla v \otimes \nabla^2 v, \nabla v \otimes \nabla \pi) \|_{L^2_t(L^2)},
\]

and

\[
\| \sigma^{\frac{3}{2} - \frac{1}{2} - \frac{2m}{2}} (\nabla v \otimes \nabla v, v \otimes \nabla^2 v) \|_{L^2_t(L^r)} \leq \| \sigma^{\frac{1}{2}} \nabla v \|_{L^\infty_t(L^r)} \| \sigma^{1-\frac{1}{2}} \nabla v \|_{L^2_t(L^\infty)} + \| \sigma^{\frac{1}{2}} v \|_{L^\infty_t(L^\infty)} \| \sigma^{1-\frac{1}{2}} \nabla v \|_{L^2_t(L^\infty)}.
\]

As a consequence, we deduce from (3.32), (3.33), (3.41) and Corollary 2.1 that for \(r \in [2, +\infty]\)

\[
\| \sigma^{1-\frac{m}{2}} (v \otimes \nabla v) \|_{L^2_t(L^\infty)} + \| \sigma^{\frac{3+2m}{2}} (D_t \nabla^2 v, \nabla D_t v, D_t \nabla \pi) \|_{L^2_t(L^2)} + \| \sigma^{\left( \frac{3}{2} - \frac{1}{2} - \frac{2m}{2} \right)} (\nabla v \otimes \nabla v, v \otimes \nabla^2 v) \|_{L^2_t(L^r)} \leq C(v_0, s_0).
\]

This together with

\[
D_t \nabla v = \nabla D_t v - \nabla v \cdot \nabla v, \quad v_t = D_t v - v \cdot \nabla v
\]

and (3.43), (3.44) ensures that for \(r \in [2, +\infty]\)

\[
\| \sigma^{1-\frac{m}{2}} v_t \|_{L^2_t(L^\infty)} + \| \sigma^{\left( \frac{3}{2} - \frac{1}{2} - \frac{2m}{2} \right)} (\nabla D_t v, D_t \nabla v, \nabla v_t) \|_{L^2_t(L^r)} \leq C(v_0, s_0).
\]
This together with Propositions 2.1 and 2.2, Corollary 2.1, Lemma 3.3 and (3.40) ensures that
\[ \int_0^t \mathcal{B}_0(t') \, dt' \leq C^2(v_0, s_0). \]
This completes the proof of (3.42). \qed

4. Propagation of the time-weighted $H^1$ regularity of $v^1$

The goal of this section is to study the propagation of time-weighted $H^1$ regularity of $v^1$. To this end, in view of (2.31), we split $(v^1, \nabla \pi^1)$ as
\begin{equation}
(4.1) \quad v^1 = v_{11} + v_{12} \quad \text{and} \quad \nabla \pi^1 = \nabla p_{11} + \nabla p_{12},
\end{equation}
with $(v_{11}, \nabla p_{11})$ and $(v_{12}, \nabla p_{12})$ solving the following systems respectively
\begin{equation}
(4.2) \quad \begin{cases}
\rho \partial_t v_{11} + \rho v \cdot \nabla v_{11} - \Delta v_{11} + \nabla p_{11} = 0, \\
\operatorname{div} v_{11} = 0, \\
v_{11}|_{t=0} = \partial_{X_0} v_0,
\end{cases}
\end{equation}
and
\begin{equation}
(4.3) \quad \begin{cases}
\rho \partial_t v_{12} + \rho v \cdot \nabla v_{12} - \Delta v_{12} + \nabla p_{12} = F_1(v, \pi), \\
\operatorname{div} v_{12} = \operatorname{div} (v \cdot \nabla X), \\
v_{12}|_{t=0} = 0,
\end{cases}
\end{equation}
where the source term $F_1(v, \pi)$ is given by (2.32).

Then we deduce from Proposition 3.3 that
\begin{equation}
(4.4) \quad \|v_{11}\|_{L^\infty_t(B^{s+1})} + \|\nabla v_{11}\|_{L^2_t(B^{s+1})} \leq C \|\partial_{X_0} v_0\|_{B^s} \exp(C \|v_0\|_{L^2}^4),
\end{equation}
and
\begin{equation}
(4.5) \quad \|\sigma^{-s} v_{11}\|_{L^\infty_t(B^s)} + \|\sigma^{-s} (\partial_t v_{11}, \nabla^2 v_{11}, \nabla p_{11})\|_{L^2_t(L^2)} \leq C(v_0, \partial_{X_0} v_0, s). \tag{4.5}
\end{equation}

4.1. Time-weighted $H^1$ estimate for $v_{12}$. Let us first present the proof of Lemma 2.2.

**Proof of Lemma 2.2.** We first get, by multiplying (3.3) by $\sigma^{-s}$, that
\[ \frac{d}{dt} \|\sigma^{-s} \nabla u(t)\|_{L^2}^2 + \|\sigma^{-s} (u_t, \nabla^2 u, \nabla \Pi)\|_{L^2}^2 \leq (1-s) \sigma^{-s} \|\nabla u\|_{L^2}^2 \\
+ C \sigma^{1-s} (\|v\|_{L^4}\|\nabla u\|_{L^2}^2 + \|(g_t, \nabla \operatorname{div} g, f)\|_{L^2}^2), \]
so that applying Gronwall’s inequality gives rise to
\begin{equation}
(4.6) \quad \|\sigma^{-s} \nabla u\|_{L^\infty_t(L^2)}^2 + \|\sigma^{-s} (u_t, \nabla^2 u, \nabla \Pi)\|_{L^2_t(L^2)}^2 \leq C \exp(C \|v\|_{L^4_t(L^4)}^4) \\
\times \int_0^t (\sigma^{-s} \|\nabla u\|_{L^2}^2 + \sigma^{1-s} (\|g_t, \nabla \operatorname{div} g, f\|_{L^2}^2) \, dt').
\end{equation}
To handle the term $\int_0^t \sigma^{-s} \|\nabla u\|_{L^2}^2 \, dt'$, let us denote
\begin{equation}
(4.7) \quad \eta \overset{\text{def}}{=} u - g, \quad \eta = \eta_1 + \eta_2 \quad \text{and} \quad \nabla \Pi = \nabla \Pi_1 + \nabla \Pi_2,
\end{equation}
with $(\eta_1, \nabla \Pi_1)$ and $(\eta_2, \nabla \Pi_2)$ solving respectively
\begin{equation}
(4.8) \quad \begin{cases}
\rho \partial_t \eta_1 + \rho v \cdot \nabla \eta_1 - \Delta \eta_1 + \nabla \Pi_1 = 0, \\
\operatorname{div} \eta_1 = 0, \\
\eta_1|_{t=0} = -g(0),
\end{cases}
\end{equation}
and
\begin{equation}
\begin{aligned}
\rho \partial_t \eta_2 + \rho v \cdot \nabla \eta_2 - \Delta \eta_2 + \nabla \Pi_2 &= f - (\rho \partial_t g + \rho v \cdot \nabla g - \Delta g), \\
\text{div } \eta_2 &= 0, \\
\eta_2|_{t=0} &= 0.
\end{aligned}
\end{equation}

For any $\delta \in [0, 1]$, it follows from Proposition 3.3 that
\begin{equation}
\| \sigma^{1-\delta} \nabla \eta_1 \|_{L^\infty_t(L^2)} \leq A_0 |g(0)|_{B^s},
\end{equation}
which in particular implies for $s \in [0, \delta]$ that
\begin{equation}
\| \sigma^{-\frac{s}{2}} \nabla \eta_1 \|_{L^2_t(L^2)}^2 \leq \| \sigma^{1-\frac{s}{2}} \nabla \eta_1 \|_{L^\infty_t(L^2)}^2 \| \sigma^{-1+\delta-s} \|_{L^1_t} \leq \frac{A_0(t)}{\delta - s} |g(0)|_{B^s}^2.
\end{equation}

Whereas for any fixed $t_0 \in [0, 1]$, we denote $\sigma_{t_0}(t) = \begin{cases} 
\sigma(t) & \text{if } t \geq t_0, \\
t_0 & \text{if } t \leq t_0.
\end{cases}$ Then we get, by first taking the $L^2$ inner product of (4.9) with $\eta_2$, then multiplying the equality by $\sigma_{t_0}^{-s}(t)$ and finally integrating the resulting inequality over $[0, t]$, that
\begin{align*}
\| \sigma_{t_0}^{-\frac{s}{2}} \sqrt{\rho} \eta_2(t) \|_{L^2_t(L^2)}^2 + \int_0^t \sigma_{t_0}^{-s} \| \nabla \eta_2 \|_{L^2_t}^2 dt' \\
&= -s \int_0^t \sigma_{t_0}^{-1-s} \sigma_{t_0} \| \sqrt{\rho} \eta_2 \|_{L^2_t}^2 dt' + \int_0^t \int_{\mathbb{R}^2} \sigma_{t_0}^{-s} (f - \rho \partial_t g - \rho v \cdot \nabla g + \Delta g) |\nabla \eta_2| dx dt' \\
&\leq C \| \sigma_{t_0}^{-\frac{s}{2}} (f, g_t) \|_{L^1_t(L^2)}^2 + C \| \sigma^{-1+s} (f, g_t) \|_{L^\infty_t(L^\infty)}^2 \| \sigma^{-1-(s-\frac{1}{2})} \nabla g \|_{L^1_t(L^2)}^2 \\
&\quad + \frac{1}{2} \| \sigma_{t_0}^{-\frac{s}{2}} \sqrt{\rho} \eta_2 \|_{L^\infty_t(L^2)}^2 - \int_0^t \int_{\mathbb{R}^2} \sigma_{t_0}^{-s} \nabla g |\nabla \eta_2| dx dt'.
\end{align*}

Noting that
\begin{align*}
\int_0^t \int_{\mathbb{R}^2} \sigma_{t_0}^{-s} \nabla g |\nabla \eta_2| dx dt' \leq \frac{1}{2} \| \sigma_{t_0}^{-\frac{s}{2}} \nabla g \|_{L^2_t(L^2)}^2 + \frac{1}{2} \| \sigma_{t_0}^{-\frac{s}{2}} \nabla \eta_2 \|_{L^2_t(L^2)}^2.
\end{align*}

This together with Corollary 2.1 ensures that
\begin{align*}
\int_0^t \sigma_{t_0}^{-s} \| \nabla \eta_2 \|_{L^2_t}^2 dt' \leq A_0 \left( \| \sigma^{-\frac{s}{2}} (f, g_t) \|_{L^1_t(L^2)}^2 + \| \sigma^{-1+(s-\frac{1}{2})} \nabla g \|_{L^1_t(L^2)}^2 + \| \sigma^{-\frac{s}{2}} \nabla g \|_{L^2_t(L^2)}^2 \right),
\end{align*}
from which, (4.7) and (4.10), we infer
\begin{align*}
\int_0^t \sigma_{t_0}^{-s} \| \nabla u \|_{L^2_t}^2 dt' \leq A_0 \left( \frac{t}{\delta - s} |g(0)|_{B^s}^2 + \| \sigma^{-\frac{s}{2}} (f, g_t) \|_{L^1_t(L^2)}^2 \\
&\quad + \| \sigma^{-1+(s-\frac{1}{2})} \nabla g \|_{L^1_t(L^2)}^2 + \| \sigma^{-\frac{s}{2}} \nabla g \|_{L^2_t(L^2)}^2 \right).
\end{align*}

Applying Fatou’s Lemma, we deduce that as $t_0 \to 0_+$ in the above inequality, there holds
\begin{align*}
\int_0^t \sigma^{-s} \| \nabla u \|_{L^2_t}^2 dt' \leq A_0 \left( \frac{t}{\delta - s} |g(0)|_{B^s}^2 + \| \sigma^{-\frac{s}{2}} (f, g_t) \|_{L^1_t(L^2)}^2 \\
&\quad + \| \sigma^{-1+(s-\frac{1}{2})} \nabla g \|_{L^1_t(L^2)}^2 + \| \sigma^{-\frac{s}{2}} \nabla g \|_{L^2_t(L^2)}^2 \right).
\end{align*}

Inserting the above estimate into (4.6) leads to (2.34). \qed

Applying Lemma 2.2 to the system (4.3) yields the time-weighted $\dot{H}^1$ estimate of $v_{12}$. 
Proposition 4.1. Let \((v_{12}, \nabla p_{12})\) be a smooth enough solution of (4.3). Then there holds
\[
\begin{align*}
\left\|\sigma^{\frac{s-1}{2}} \nabla v_{12}\right\|_{L^2_{t} (L^2)}^2 + \left\|\sigma^{\frac{s-1}{2}} (\partial_t v_{12}, \nabla^2 v_{12}, \nabla p_{12})\right\|_{L^2_{t} (L^2)}^2 \leq \mathfrak{B}_{1,X}(t) \quad \text{with} \\
\mathfrak{B}_{1,X}(t) \overset{\text{def}}{=} \exp (A_0(t)^2) \left(A_1 + \int_0^t (\mathfrak{B}_0(t') + \sigma(t')^{-1 + \theta_0}) \left\|\nabla X(t')\right\|_{W^{1,p}}^2 dt'\right),
\end{align*}
\]
where \(\mathfrak{B}_0(t)\) is given by (2.36).

Proof. Noticing that for \(\theta_0 = s_0 - s_1\),
\[
\begin{align*}
\left\|\sigma^{\frac{s-1}{2}} \nabla g\right\|_{L^2_{t} (L^2)}^2 &\leq \left\|\sigma^{\frac{s-1}{2}} \nabla g\right\|_{L^2_{t} (L^2)}^2 \left\|\sigma^{s_0 - s_0}\right\|_{L^1} \leq C(t) \left\|\sigma^{\frac{s-1}{2}} \nabla g\right\|_{L^2_{t} (L^2)}^2, \\
\text{and} \\
\left\|\sigma^{\frac{s-1}{2}} (f, g_t)\right\|_{L^2_{t} (L^2)}^2 &\leq \left\|\sigma^{s_0 - s_0}\right\|_{L^1} \left\|\sigma^{\frac{s-1}{2}} (f, g_t)\right\|_{L^2_{t} (L^2)}^2 \leq C(t) \left\|\sigma^{\frac{s-1}{2}} (f, g_t)\right\|_{L^2_{t} (L^2)}^2
\end{align*}
\]
we get, by applying Lemma 2.2 (with \(s = s_1\) and \(\delta = s_0\)) to the system (4.3), that
\[
\begin{align*}
\left\|\sigma^{\frac{s-1}{2}} \nabla v_{12}\right\|_{L^2_{t} (L^2)}^2 &\leq \left\|\sigma^{\frac{s-1}{2}} (\partial_t v_{12}, \nabla^2 v_{12}, \nabla p_{12})\right\|_{L^2_{t} (L^2)}^2 \overset{A_0(t)}{\leq} \left\|v_0 \cdot \nabla X_0\right\|_{B^{s_0}_p}^2 \\
&+ \left\|\sigma^{\frac{s-1}{2}} (v \cdot \nabla X)\right\|_{L^2_{t} (L^2)}^2 + \left\|\sigma^{s_0 - s_0}(\partial_t (v \cdot \nabla X), \nabla \text{div}(v \cdot \nabla X), F_1(v, \pi))\right\|_{L^2_{t} (L^2)}^2
\end{align*}
\]
It follows from the law of product in Besov spaces (see [5] for instance) that
\[
\left\|v_0 \cdot \nabla X_0\right\|_{B^{s_0}_p} \lesssim \left\|v_0\right\|_{L^2 \cap B^{s_0}_p} \left\|\nabla X_0\right\|_{W^{1,p}} \lesssim \left\|v_0\right\|_{L^2 \cap B^{s_0}_p} \left\|\nabla X_0\right\|_{W^{1,p}} \lesssim A_1.
\]
It follows from Corollary 2.1 that
\[
\begin{align*}
\left\|\sigma^{-\frac{s}{2}} \nabla (v \cdot \nabla X)\right\|_{L^2_{t} (L^2)}^2 &\leq \int_0^t \sigma^{-1 + \theta_0} \left\|\sigma^{\frac{s-1}{2}} (v)\right\|_{L^2_{t} (L^2)}^2 \left\|\sigma^{\frac{s-1}{2}} (v)\right\|_{L^2_{t} (L^2)}^2 \left\|\nabla X\right\|_{W^{1,p}}^2 dt' \\
&\leq A_0 \int_0^t \sigma(t')^{-1 + \theta_0} \left\|\nabla X(t')\right\|_{W^{1,p}}^2 dt'.
\end{align*}
\]
While in view of (2.4), we have
\[
\partial_t (v \cdot \nabla X) = v_t \cdot \nabla X - (v \cdot \nabla) v \cdot \nabla X - v_a v \cdot \nabla \partial_a X + v \cdot \nabla v^1,
\]
and thus it comes out
\[
\begin{align*}
\left\|\sigma^{\frac{s-1}{2}} \partial_t (v \cdot \nabla X)\right\|_{L^2_{t} (L^2)}^2 &\leq \int_0^t \left\|\sigma^{\frac{s-1}{2}} v_t\right\|_{L^2}^2 + \sigma^{-1 + s_0} \left\|\sigma^{\frac{s-1}{2}} (v)\right\|_{L^\infty_{t} (L^{\frac{2p}{p-2}})}^4 \\
&\quad + \left\|\sigma^{\frac{s-1}{2}} v\right\|_{L^2_{t} (L^\infty)} \left\|\sigma^{\frac{s-1}{2}} \nabla v\right\|_{L^2_{t} (L^2)}^2 \left\|\nabla X\right\|_{W^{1,p}}^2 dt' \\
&\quad + \int_0^t \sigma^{-1 + s_1} \left\|\sigma^{\frac{s-1}{2}} v\right\|_{L^2_{t} (L^\infty)} \left\|\sigma^{\frac{s-1}{2}} \nabla v_{12}\right\|_{L^2_{t} (L^2)}^2 + \left\|\sigma^{\frac{s-1}{2}} \nabla p_{12}\right\|_{L^2_{t} (L^2)}^2 dt',
\end{align*}
\]
from which, Corollary 2.1 and (4.5), we infer that
\[
\begin{align*}
\left\|\sigma^{\frac{s-1}{2}} \partial_t (v \cdot \nabla X)\right\|_{L^2_{t} (L^2)}^2 &\leq A_1 (t) + A_0 \int_0^t \left((\mathfrak{B}_0(t') + \sigma(t')^{-1 + s_0}) \left\|\nabla X(t')\right\|_{W^{1,p}}^2 \right. \\
&\left. + \sigma(t')^{-1 + s_1} \left\|\sigma^{\frac{s-1}{2}} \nabla p_{12}(t')\right\|_{L^2_{t} (L^2)}^2 \right) dt'.
\end{align*}
\]
Notice that due to \(\text{div} X = 0\), we write
\[
\begin{align*}
\left\|\nabla \text{div} (v \cdot \nabla X)\right\|_{L^2} = \left\|\nabla (\partial_a v \cdot \nabla X^a)\right\|_{L^2} &\leq C \left(\left\|\nabla^2 v\right\|_{L^2} + \left\|\nabla v\right\|_{L^\frac{2p}{p-2}}\right) \left\|\nabla X\right\|_{W^{1,p}},
\end{align*}
\]
Proof. We get, by using standard energy estimate to (4.3), that
\[ \| \sigma^{\frac{1-\alpha}{2}} \nabla (v \cdot \nabla X) \|_{L_t^2(L^2)}^2 \leq \int_0^t \left( \| \sigma^{\frac{1-\alpha}{2}} \nabla^2 v \|_{L^2}^2 + \| \sigma^{\frac{1-\alpha}{2}} \nabla v \|_{L_t^2(L^2)}^2 \right) \| \nabla X \|_{W^{1,p}}^2 \, dt' \]
\[ \quad \leq \int_0^t \mathcal{B}_0(t') \| \nabla X(t') \|_{W^{1,p}}^2 \, dt'. \]

Similarly, we deduce from (2.27) and (2.32) that
\[ \| \sigma^{\frac{1-\alpha}{2}} F_1(v, \pi) \|_{L_t^2(L^2)}^2 \lesssim \| \sigma^{\frac{1-\alpha}{2}} D_t v \|_{L_t^2(L^2)}^2 \]
\[ + \int_0^t \left( \| \sigma^{\frac{1-\alpha}{2}} \nabla v \|_{L_t^2(L^2)}^2 + \| \sigma^{\frac{1-\alpha}{2}} (\nabla^2 v, \nabla \pi) \|_{L_t^2(L^2)}^2 \right) \| \nabla X \|_{W^{1,p}}^2 \, dt', \]
which together with Lemma 3.3 and Corollary 3.4 ensures that
\[ \| \sigma^{\frac{1-\alpha}{2}} F_1(v, \pi) \|_{L_t^2(L^2)}^2 \leq A_0 + \int_0^t \mathcal{B}_0(t') \| \nabla X(t') \|_{W^{1,p}}^2 \, dt'. \]

Inserting the above estimates into (4.12) gives rise to
\[ \| \sigma^{\frac{1}{2}} \nabla v_{12} \|_{L_t^\infty(L^2)}^2 + \| \sigma^{\frac{1}{2}} (\partial_t v_{12}, \nabla^2 v_{12}, \nabla p_{12}) \|_{L_t^2(L^2)}^2 \leq A_1(t)^2 \]
\[ + A_0(t) \int_0^t \left( (\mathcal{B}_0(t') + \sigma(t')^{-1+\theta_0}) \| \nabla X(t') \|_{W^{1,p}}^2 + \sigma(t')^{-1-s_1} \| \sigma^{\frac{1-\alpha}{2}} \nabla v_{12}(t') \|_{L_t^2(L^2)}^2 \right) \, dt'. \]

Applying Gronwall’s inequality leads to (4.11). This completes the proof of the proposition. \( \square \)

Combining (4.5) with Proposition 4.1, we obtain

**Corollary 4.1.** Let \((\rho, v, \nabla \pi, X)\) be a smooth enough solution of the coupled system (1.1) and (2.4). Then one has
\[ \| \sigma^{\frac{1}{2}} \nabla v \|_{L_t^\infty(L^2)}^2 + \int_0^t \sigma(t')^{1-s_1} \| (\partial_t v^1, \nabla^2 v^1, \nabla \pi^1) \|_{L_t^2(L^2)}^2 \, dt' \leq \mathcal{G}_{1,X}(t), \]
with \(\mathcal{G}_{1,X}(t)\) being given by (4.11).

4.2. The proof of Proposition 2.3. We first present the \(L^2\) energy estimate of \(v_{12}\).

**Lemma 4.1.** Let \((v_{12}, \nabla p_{12})\) be a smooth enough solution of (4.3). Then there holds
\[ \| v_{12} \|_{L_t^\infty(L^2)}^2 + \| \nabla v_{12} \|_{L_t^2(L^2)}^2 \leq \mathcal{G}_{1,X}(t), \]
for \(\mathcal{G}_{1,X}(t)\) given by (4.11).

**Proof.** We get, by using standard energy estimate to (4.3), that
\[ \frac{1}{2} \| \sqrt{\rho} v_{12} \|_{L_t^\infty(L^2)}^2 + \| \nabla v_{12} \|_{L_t^2(L^2)}^2 = \int_0^t \int_{\mathbb{R}^2} \left( - \nabla p_{12} + F_1(v, \pi) \right) v_{12} \, dx \, dt'. \]

By using \(\text{div} v_{12} = \text{div}(v \cdot \nabla X)\), one has
\[ \int_{\mathbb{R}^2} \nabla p_{12} v_{12} \, dx = \int_{\mathbb{R}^2} \nabla p_{12} (v \cdot \nabla X) \, dx, \]
from which and the momentum equation of (4.3), we infer
\[
\frac{1}{2} \| \sqrt{\rho v_{t2}} \|_{L_t^\infty(L^2)}^2 + \| \nabla v_{t2} \|_{L_t^2(L^2)}^2 = \int_0^t \int_{\mathbb{R}^2} F_1(v, \pi)(v_{t2} - v \cdot \nabla X) \, dx \, dt' + \int_0^t \int_{\mathbb{R}^2} (\rho \partial_t v_{t2} + \rho v \cdot \nabla v_{t2} - \Delta v_{t2}) \, v \cdot \nabla X \, dx \, dt'.
\] (4.18)

It is easy to observe from (2.27) and (2.32) that
\[
| \int_0^t \int_{\mathbb{R}^2} F_1(v, \pi)(v_{t2} - v \cdot \nabla X) \, dx \, dt' | \leq \frac{1}{4} \| \sqrt{\rho v_{t2}} \|_{L_t^\infty(L^2)}^2 
\]
\[
+ C \left( \int_0^t \sigma^{-\frac{1-s_0}{2}} \times \sigma^{-\frac{1-s_0}{2}} (\| \nabla v \|_{L_p^\infty}^2 + \| (D_t v, \nabla^2 v, \nabla \pi) \|_{L_t^2}) (1 + \| \nabla X \|_{W_{1,p}}) \, dt' \right)^2 
\]
\[
+ C \int_0^t \sigma^{-\frac{1-s_0}{2}} \times \sigma^{-\frac{1-s_0}{2}} (\| \nabla v \|_{L_p^\infty}^2 + \| (D_t v, \nabla^2 v, \nabla \pi) \|_{L_t^2}) \| v \|_{L_t^2} (1 + \| \nabla X \|_{W_{1,p}}) \, dt'.
\]

Hence by virtue of Corollary 2.1 and (2.36), we deduce
\[
| \int_0^t \int_{\mathbb{R}^2} F_1(v, \pi)(v_{t2} - v \cdot \nabla X) \, dx \, dt' | \leq \frac{1}{4} \| \sqrt{\rho v_{t2}} \|_{L_t^\infty(L^2)}^2 + \mathfrak{G}_{1,X}(t)
\] (4.19)
for \( \mathfrak{G}_{1,X}(t) \) given by (4.11).

Let us now turn to the estimate of the second line of (4.18). Once again, we get, by applying Hölder’s inequality, that
\[
\int_0^t \int_{\mathbb{R}^2} \rho (\partial_t v_{t2} - \Delta v_{t2}) \, v \cdot \nabla X \, dx \, dt' \leq C \int_0^t \| (\partial_t v_{t2}, \nabla^2 v_{t2}) \|_{L_t^2} \| v \|_{L_t^2} \| \nabla X \|_{L_t^\infty} \, dt' 
\]
\[
\leq C \| \sigma^{-\frac{1-s_0}{2}} (\partial_t v_{t2}, \nabla^2 v_{t2}) \|_{L_t^2}^2 + C \int_0^t \sigma^{-1-\sigma}(1) \| v \|_{L_t^\infty}^2 \| \nabla X \|_{W_{1,p}}^2 \, dt',
\]

and
\[
\int_0^t \int_{\mathbb{R}^2} \rho v \cdot \nabla v_{t2} \, v \cdot \nabla X \, dx \, dt' \leq \int_0^t \| v \|_{L_t^\infty} \| \nabla v_{t2} \|_{L_t^2} \| v \|_{L_t^2} \| \nabla X \|_{L_t^\infty} \, dt' 
\]
\[
\leq \frac{1}{2} \| \nabla v_{t2} \|_{L_t^2}^2 + \int_0^t \sigma^{-1-\sigma}(1) \| \sigma^{-\frac{1-s_0}{2}} v \|_{L_t^\infty}^2 \| v \|_{L_t^\infty}^2 \| \nabla X \|_{W_{1,p}}^2 \, dt'.
\]

Together with Corollary 2.1, the above inequalities ensure that
\[
| \int_0^t \int_{\mathbb{R}^2} (\rho \partial_t v_{t2} + \rho v \cdot \nabla v_{t2} - \Delta v_{t2}) \, v \cdot \nabla X \, dx \, dt' | \leq \frac{1}{2} \| \nabla v_{t2} \|_{L_t^2(L^2)}^2 
\] (4.20)
\[
+ C \| \sigma^{-\frac{1-s_0}{2}} (\partial_t v_{t2}, \nabla^2 v_{t2}) \|_{L_t^2(L^2)}^2 + A_0 \int_0^t \sigma(t')^{-1+s_1} \| \nabla X \,(t') \|_{W_{1,p}}^2 \, dt'.
\]

Inserting the Inequalities (4.19) and (4.20) into (4.18) leads to
\[
\| \sqrt{\rho v_{t2}} \|_{L_t^\infty(L^2)}^2 + \| \nabla v_{t2} \|_{L_t^2(L^2)}^2 \leq C \| \sigma^{-\frac{1-s_0}{2}} (\partial_t v_{t2}, \nabla^2 v_{t2}) \|_{L_t^2(L^2)}^2 + \mathfrak{G}_{1,X}(t)
\]
for \( \mathfrak{G}_{1,X}(t) \) given by (4.11). This together with Proposition 4.1 leads to (4.17). We finish the proof of the lemma. \( \square \)

Thanks to (4.1), we conclude the proof of Proposition 2.3 by combining the Estimates (4.4) and (4.5) with Proposition 4.1 and Lemma 4.1. Moreover, we have the following corollary, which will be used in Section 5:
Corollary 4.2. Let $\mathcal{G}_{1, X}(t)$ be given by (4.11). Then under the assumptions of Proposition 2.3, we have

$$
\left\| \sigma^{1-s_1} (D_tv^1,v^1 \otimes \nabla v, v^1 \otimes \nabla^2 v, \Delta D_t X, D_t \Delta X) \right\|_{L_t^2(L^2)}^2 + \left\| \sigma^{1-s_1} \nabla v^1 \right\|_{L_t^2(L^2)}^2 + \left\| \sigma^{1-s_1} (D_tv^1, D_t \nabla v_t) \right\|_{L_t^2(L^2)}^2 
+ \left\| \sigma^{3-s_1} \nabla v \otimes (\nabla^2 v, \nabla \pi, D_tv^1, D_t \Delta X) \right\|_{L_t^2(L^2)}^2 
+ \left\| \sigma^{3-s_1} (\nabla v^1, D_t \nabla X) \otimes (\nabla^2 v, \nabla \pi, D_tv^1, v^1 \otimes \nabla v) \right\|_{L_t^2(L^2)}^2 \leq \mathcal{G}_{1, X}(t).
$$

(4.21)

**Proof.** We first deduce from (2.20) and Proposition 2.3 that

$$
\left\| v^1 \right\|_{L_t^4(L^4)}^2 + \left\| \sigma^{1-s_1} \nabla v^1 \right\|_{L_t^4(L^4)}^2 \leq \mathcal{G}_{1, X}(t).
$$

(4.22)

Then due to

$$
\left\| \sigma^{1-s_1} (v \otimes \nabla v^1, v^1 \otimes \nabla v) \right\|_{L_t^2(L^2)} \leq \left\| v \right\|_{L_t^4(L^4)} \left\| \sigma^{1-s_1} \nabla v^1 \right\|_{L_t^4(L^4)} + \left\| \sigma^{1-s_1} \nabla v \right\|_{L_t^4(L^4)} \left\| v^1 \right\|_{L_t^4(L^4)},
$$

and

$$
\left\| \sigma^{1-s_1} D_tv^1 \right\|_{L_t^2(L^2)} \leq \left\| \sigma^{1-s_1} \partial_t v^1 \right\|_{L_t^2(L^2)} + \left\| \sigma^{1-s_1} \partial_t v \cdot \nabla v^1 \right\|_{L_t^2(L^2)},
$$

we deduce from Corollary 2.1 and Proposition 2.3 that

$$
\left\| \sigma^{1-s_1} (v \otimes \nabla v^1, v^1 \otimes \nabla v) \right\|_{L_t^2(L^2)}^2 + \left\| \sigma^{1-s_1} D_tv^1 \right\|_{L_t^2(L^2)}^2 \leq \mathcal{G}_{1, X}(t).
$$

(4.23)

Note that $D_t X = v^1$, we have

$$
D_t \Delta X = \Delta D_t X - \Delta v \cdot \nabla X - 2\nabla v_a \cdot \partial_a \nabla X,
$$

from which and Proposition 2.3, we infer

$$
\left\| \sigma^{1-s_1} (\Delta D_t X, D_t \Delta X) \right\|_{L_t^2(L^2)}^2 \leq \left\| \sigma^{1-s_1} \Delta v^1 \right\|_{L_t^2(L^2)}^2 
+ \int_0^t \left( \left\| \sigma^{1-s_1} \Delta v \right\|_{L^2}^2 + \left\| \sigma^{1-s_1} \nabla v \right\|_{L^2 (L^2)}^2 \right) \left\| \nabla X \right\|_{L^2}^2 \mathrm{d}t' \leq \mathcal{G}_{1, X}(t).
$$

(4.24)

On the other hand, it is easy to observe from Corollary 2.1, Proposition 2.3 and (4.22) that

$$
\left\| \sigma^{1-s_1} (\nabla v \otimes \nabla v^1, v \otimes \nabla^2 v^1, v^1 \otimes \nabla^2 v) \right\|_{L_t^2(L^2)} \leq \left\| \sigma^{1-s_1} \nabla v \right\|_{L_t^4(L^4)} \left\| \sigma^{1-s_1} \nabla v^1 \right\|_{L_t^4(L^4)} 
+ \left\| \sigma^{1-s_1} \nabla v \right\|_{L_t^2(L^2)} \left\| \sigma^{1-s_1} \nabla^2 v \right\|_{L_t^2(L^2)} + \left\| \sigma^{1-s_1} \nabla^2 v \right\|_{L_t^2(L^2)} \left\| \sigma^{1-s_1} \nabla^3 v \right\|_{L_t^2(L^2)} \left\| v^1 \right\|_{L_t^2(L^2)} \leq \mathcal{G}_{1, X}(t).
$$

(4.25)

Similarly we deduce from Corollaries 2.1 and 3.4, Proposition 2.3 and (4.24) that

$$
\left\| \sigma^{1-s_1} (v^1 \otimes \nabla v, v \otimes \nabla v^1) \right\|_{L_t^2(L^2)} \leq \left\| \sigma^{1-s_1} \nabla v \right\|_{L_t^4(L^4)} \left\| \sigma^{1-s_1} \nabla v^1 \right\|_{L_t^4(L^4)} 
+ \left\| \sigma^{1-s_1} \nabla v \right\|_{L_t^2(L^2)} \left\| \sigma^{1-s_1} \nabla^2 v \right\|_{L_t^2(L^2)} \left\| v^1 \right\|_{L_t^2(L^2)} \leq \mathcal{G}_{1, X}(t),
$$

(4.25)
and
\[ \left\| \frac{3-s}{2} \nabla v \otimes (\nabla^2 v, \nabla \pi, D_t v, D_t \Delta X) \right\|_{L_t^2(L^2)} \leq \left\| \sigma \nabla v \right\|_{L_t^\infty(L^\infty)} \left\| \frac{1+s}{2} \left( \nabla^2 v, \nabla \pi, D_t v, D_t \Delta X \right) \right\|_{L_t^2(L^2)} \leq \mathcal{G}_{1,X}(t), \]
\[ \left\| \frac{3-s}{2} \nabla v_1 \otimes (\nabla^2 v, \nabla \pi, D_t v_1, D_t \Delta X) \right\|_{L_t^2(L^2)} \leq \left\| \sigma \nabla v \right\|_{L_t^\infty(L^\infty)} \left\| \frac{1+s}{2} \left( \nabla^2 v, \nabla \pi, D_t v_1, D_t \Delta X \right) \right\|_{L_t^2(L^2)} \leq \mathcal{G}_{1,X}(t). \]

As a result, we infer from Corollary 2.1 and Corollary 3.4 that
\[ \left\| \frac{3-s}{2} D_t \nabla X \otimes (\nabla^2 v, \nabla \pi, D_t v_1, D_t \Delta X) \right\|_{L_t^2(L^2)} \leq \left\| \sigma \nabla v \right\|_{L_t^\infty(L^\infty)} \left\| \frac{1+s}{2} \left( \nabla^2 v, \nabla \pi, D_t v_1, D_t \Delta X \right) \right\|_{L_t^2(L^2)} \leq \mathcal{G}_{1,X}(t). \]

Finally it follows from Corollary 3.4 and Proposition 2.3 that for any \( r \in ]2, +\infty[ \),
\[ \left\| \frac{3-s}{2} v_1 \otimes D_t \nabla v \right\|_{L_t^2(L^2)} \leq \left\| \sigma \frac{1-s}{2} v_1 \right\|_{L_t^\infty(L^\infty)} \left\| \sigma \left( \frac{1}{2} - \frac{1}{2} \cdot \frac{3}{2} \right) D_t \nabla v \right\|_{L_t^2(L^r)} \leq C \left\| v_1 \right\|_{L_t^\infty(L^\infty)} \left\| \frac{1+s}{2} \nabla v \right\|_{L_t^\infty(L^\infty)} \left\| \sigma \left( \frac{1}{2} - \frac{1}{2} \cdot \frac{3}{2} \right) D_t \nabla v \right\|_{L_t^2(L^2)} \leq \mathcal{G}_{1,X}(t), \]

which together with the above estimate, (4.25) and Corollary 2.1 ensures
\[ \left\| \frac{3-s}{2} v_1 \otimes \nabla v \right\|_{L_t^2(L^2)} \leq \left\| \sigma \frac{1-s}{2} v_1 \otimes D_t \nabla v \right\|_{L_t^2(L^2)} + \left\| \sigma \frac{3-s}{2} v_1 \otimes (v \cdot \nabla^2 v) \right\|_{L_t^2(L^2)} \leq \mathcal{G}_{1,X}(t). \]

This completes the proof of the corollary. \( \square \)

5. Propagation of the first-order striated regularity (continued)

The aim of this section is to prove Proposition 2.4. In order to do it, we shall first present the time-weighted \( H^1 \)-energy estimate for \( D_t v^1 \), and then we derive the global \textit{a priori} estimate of \( \parallel \nabla X \parallel_{L_t^\infty(W^{1,p})} \). Finally we study the propagation of \( B^{s_1} \) regularity of \( v^1 \).

5.1. \( H^1 \) energy estimates for \( D_t v^1 \). In view of of Lemma 2.1, to close the time-weighted \( H^1 \) estimate of \( D_t v^1 \), it remains to deal with the estimates of \( \text{div} \, D_t v^1 \) and \( \text{div} \, D_t^2 v^1 \), which are the lemmas as follows:

**Lemma 5.1.** Let \( a_1 \defeq 2v^1 \cdot \nabla v + v_1 \cdot \nabla X + v_1 \partial \alpha \nabla \alpha - (v \cdot \nabla X) \cdot \nabla v \). Then we have \( \text{div} \, D_t v^1 = \text{div} \, a_1 \), and for \( \mathcal{G}_{1,X}(t) \) given by (4.11), there holds
\[ \frac{1-s}{2} a_1 \]
and
\[
\text{div} (\partial_X (v \cdot \nabla v)) = \text{div} (v^1 \cdot \nabla v + v \cdot \nabla v^1 - (v \cdot \nabla X) \cdot \nabla v)
\]
\[
= \text{div} (2v^1 \cdot \nabla v + v \text{div} v^1 - (v \cdot \nabla X) \cdot \nabla v)
\]
\[
= \text{div} (2v^1 \cdot \nabla v + v \partial_\alpha X \cdot \nabla v_\alpha - (v \cdot \nabla X) \cdot \nabla v).
\]

This shows that \( \text{div} D_t v^1 = a_1 \). Moreover, it is easy to observe that
\[
\| \sigma^{-\frac{3}{2}} \nabla a_1 \|^2_{L^2_t(L^2)} \leq \int_0^t \| \sigma^{-\frac{3}{2}} (\nabla v^1 \cdot \nabla v, v \cdot \nabla v^1) \|^2_{L^2_t(L^2)} \, dt'
\]
\[
+ \int_0^t \| \sigma^{-\frac{3}{2}} (v, v \cdot \nabla v) \|^2_{L^2_t(L^2)} \| \nabla^2 X \|_{L^p_t} \, dt'
\]
\[
\leq \mathcal{G}_{1,X}(t).
\]

While we deduce from (2.36) and Corollary 4.2 that
\[
\| \sigma^{-\frac{3}{2}} \nabla a_1 \|^2_{L^2_t(L^2)} \leq \int_0^t \| \sigma^{-\frac{3}{2}} (\nabla v^1 \cdot \nabla v, v \cdot \nabla v^1) \|^2_{L^2_t(L^2)} \, dt'
\]
\[
+ \int_0^t \| \sigma^{-\frac{3}{2}} (v, v \cdot \nabla v) \|^2_{L^2_t(L^2)} \| \nabla^2 X \|_{L^p_t} \, dt'
\]
\[
\leq \mathcal{G}_{1,X}(t).
\]

On the other hand, it follows from (5.2) that
\[
\text{div} D_t v^1 = \text{div} \partial_X D_t v = \partial_\alpha X \cdot \nabla D_t v^\alpha + 2t \text{tr}(\nabla v^1 \nabla v) - 2t \text{tr}((\nabla X^\alpha \partial_\alpha v) \nabla v),
\]
from which, Corollaries 2.1, 3.4 and 4.2, we infer
\[
\| \sigma^{-\frac{3}{2}} \nabla a_1 \|^2_{L^2_t(L^2)} \leq \int_0^t \| \sigma^{-\frac{3}{2}} (\nabla v^1 \cdot \nabla v, v \cdot \nabla v^1) \|^2_{L^2_t(L^2)} \, dt'
\]
\[
+ \int_0^t \| \sigma^{-\frac{3}{2}} (v, v \cdot \nabla v) \|^2_{L^2_t(L^2)} \| \nabla^2 X \|_{L^p_t} \, dt'
\]
\[
\leq \mathcal{G}_{1,X}(t).
\]

This completes the proof of (5.1). \( \square \)

**Lemma 5.2.** Let \( b_0(t) \) and \( \mathcal{G}_{1,X}(t) \) be given by (2.23) and (4.11) respectively. Let \( b_1 \overset{def}{=} D_t^2 v \cdot \nabla X - b_0 \cdot \nabla X + \partial_X b_0 \). Then we have
\[
(5.3) \quad \text{div} D_t^2 v^1 = b_1,
\]
and for any \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon > 0 \) so that
\[
(5.4) \quad \| \sigma^{-\frac{3}{2}} b_1 \|^2_{L^2_t(L^2)} \leq \varepsilon \| \sigma^{-\frac{3}{2}} \sqrt{\text{div} D_t^2 v^1} \|^2_{L^2_t(L^2)} + C_\varepsilon \mathcal{G}_{1,X}(t).
\]

**Proof.** It follows from (2.23) and \( [\partial_X; D_t] = 0 \) that
\[
\text{div} D_t^2 v^1 = \text{div} (\partial_X D_t^2 v) = \text{div} (D_t^2 v \cdot \nabla X) + (X \text{div} (D_t^2 v))
\]
\[
= \text{div} (D_t^2 v \cdot \nabla X) + (X \text{div} b_0).
\]

This together with \( \text{div} X = 0 \) ensures (5.3).
Observing from (2.23) that
\[ \partial_X b_0 = v^1 \cdot (\nabla v_t + D_t \nabla v) + D_t v^1 \cdot \nabla v + D_t v \cdot \nabla v^1 - (D_t v \cdot \nabla X) \cdot \nabla v 
+ v \cdot (D_t \nabla v^1 + D_t \nabla v - \nabla \sigma \partial \alpha v_t - \nabla (X_t \cdot \nabla v) - D_t (\nabla \sigma \partial \alpha v)), \]
we have
\[
\begin{align*}
\| \sigma^{\frac{3-\alpha}{2}} \partial_X b_0 \|_{L_t^2(L^2)}^2 & \leq \| \sigma^{\frac{3-\alpha}{2}} v^1 \otimes (\nabla v_t + D_t \nabla v) \|_{L_t^2(L^2)}^2 \\
& \quad + \| \sigma^{\frac{3-\alpha}{2}} (D_t v^1 \otimes \nabla v, D_t v \otimes \nabla v^1) \|_{L_t^2(L^2)}^2 \\
& \quad + \| \sigma^{\frac{3-\alpha}{2}} \nabla v \|_{L_t^\infty(L^2)}^2 \int_0^t \| \sigma^{1-\frac{\alpha}{2} - \frac{\alpha}{2}} D_t v(t') \|_{L^\infty}^2 \| \nabla X(t') \|_{W^{1,p}}^2 \, dt' \\
& \quad + \| \sigma^{\frac{3-\alpha}{2}} \|_{L_t^\infty(L^\infty)}^2 \| \sigma (\nabla \partial_t v^1 + D_t \nabla v^1 \otimes \nabla (X_t \cdot \nabla v), D_t (\nabla \nabla X \cdot \nabla v)) \|_{L_t^2(L^2)}^2.
\end{align*}
\]
(5.5)
To handle the last line of (5.5), we first deduce from (2.37) and (3.35) that
\[
\begin{align*}
\| \sigma^{1-\frac{\alpha}{2}} \nabla D_t v^1 \|_{L_t^2(L^2)}^2 & \leq C \| \sigma^{\frac{3-\alpha}{2}} \sqrt{D_t} v^1 \|_{L_t^2(L^2)}^2 + \| \sigma^{\frac{3-\alpha}{2}} \nabla \sqrt{D_t} v^1 \|_{L_t^2(L^2)}^2 \\
& \quad + C_{\varepsilon} \left( \| \sigma^{\frac{3-\alpha}{2} + 2} a_1 \|_{L_t^2(L^2)}^2 + \| \sigma^{1-\frac{\alpha}{2}} \nabla a_1 \|_{L_t^2(L^2)}^2 + \| \sigma^{\frac{3-\alpha}{2}} F_1 \|_{L_t^2(L^2)}^2 \right).
\end{align*}
\]
(6.6)
While in view of (2.27) and (2.38), we have
\[
\begin{align*}
\| \sigma^{\frac{3-\alpha}{2}} F_1 D \|_{L_t^2(L^2)}^2 & \leq \| \sigma^{\frac{3-\alpha}{2}} \nabla v \otimes (\nabla^2 v^1, \nabla v^1, D_t \Delta X) \|_{L_t^2(L^2)}^2 \\
& \quad + \| \sigma^{\frac{3-\alpha}{2}} (\nabla v^1, D_t \nabla v) \otimes (\nabla v^1, \nabla v) \|_{L_t^2(L^2)}^2 + \| \sigma^{\frac{3-\alpha}{2}} \nabla^2 v \|_{L_t^2(L^2)}^2 \\
& \quad + \int_0^t \left( \| \sigma^{\frac{3-\alpha}{2}} D_t v \|_{L^2(L^2)}^2 + \| \sigma^{\frac{3-\alpha}{2}} (D_t v, D_t \nabla v) \|_{L^2(L^2)}^2 \right) \| \nabla X \|_{W^{1,p}}^2 \, dt',
\end{align*}
\]
from which and Proposition 2.2 and Corollary 4.2, we infer
\[
\| \sigma^{\frac{3-\alpha}{2}} F_1 \|_{L_t^2(L^2)}^2 \leq \mathfrak{G}_{1,X}(t).
\]
(5.7)
Inserting the Inequality (5.7) into (6.6) and using Corollary 4.2 and Lemma 5.1, we achieve
\[
\| \sigma^{1-\frac{\alpha}{2}} \nabla D_t v^1 \|_{L_t^2(L^2)}^2 \leq \varepsilon \| \sigma^{\frac{3-\alpha}{2}} \sqrt{D_t} v^1 \|_{L_t^2(L^2)}^2 + C_{\varepsilon} \mathfrak{G}_{1,X}(t),
\]
which together with Corollary 4.2 implies
\[
\begin{align*}
\| \sigma (\nabla \partial_t v^1 + D_t \nabla v^1) \|_{L_t^2(L^2)}^2 & \leq \| \sigma^{1-\frac{\alpha}{2}} \nabla D_t v^1 \|_{L_t^2(L^2)}^2 + \| \sigma^{1-\frac{\alpha}{2}} (v \otimes \nabla^2 v^1, \nabla v \otimes \nabla v^1) \|_{L_t^2(L^2)}^2 \\
& \leq \varepsilon \| \sigma^{\frac{3-\alpha}{2}} \sqrt{D_t} v^1 \|_{L_t^2(L^2)}^2 + C_{\varepsilon} \mathfrak{G}_{1,X}(t).
\end{align*}
\]
Whereas due to $D_t X = v^1$, we have
\[
\begin{align*}
\| \sigma \nabla (X_t \cdot \nabla v) \|_{L_t^2(L^2)}^2 & \leq \| \sigma^{1-\frac{\alpha}{2}} (\nabla v \otimes \nabla v^1, v^1 \otimes \nabla^2 v) \|_{L_t^2(L^2)}^2 \\
& \quad + \int_0^t \left( \| \sigma^{\frac{3-\alpha}{2}} v \|_{L_t^\infty(L^2)}^2 + \| \sigma^{\frac{3-\alpha}{2}} \nabla v \|_{L_t^\infty}^2 \right) \| \nabla X \|_{W^{1,p}}^2 \, dt',
\end{align*}
\]
Proposition 5.1. Under the assumptions of Proposition 2.4, we have
\[ \| \sigma(\nabla X \cdot \nabla \nu) \|_{L_t^2(L^2)}^2 \leq \| \sigma X \nabla \nu \|_{L_t^2(L^2)}^2 + \int_0^t \| \sigma \nabla \nu \|_{L_t^2(L^2)}^2 \| \nabla X \|_{W^{1,p}}^2 \, dt', \]
we thus deduce from Corollaries 2.1 and 4.2 that
\[ \| \sigma(\nabla(X_t \cdot \nabla \nu), D_t(\nabla X \cdot \nabla \nu)) \|_{L_t^2(L^2)}^2 \leq \mathcal{G}_{1,X}(t). \]
Inserting the above inequalities into (5.5) and using Corollaries 2.1 and 4.2, we find
\[ \| \sigma \nabla X \|_{L_t^2(L^2)}^2 \leq \varepsilon \| \sigma \nabla \nu \|_{L_t^2(L^2)}^2 + \mathcal{G}_{1,X}(t), \]
which together with the fact that
\[ \| \sigma \frac{3-s_1}{2} \|_{L_t^2(L^2)}^2 \leq \| \sigma \frac{3-s_1}{2} \|_{L_t^2(L^2)}^2 + \int_0^t \left( \| \sigma \frac{3-s_1}{2} \|_{L_t^2(L^2)}^2 + \| \sigma \frac{3-s_1}{2} \|_{L_t^2(L^2)}^2 \right) \| \nabla X \|_{L_t^2(L^2)}^2 \, dt', \]
ensures (5.4).

With the above preparations, we can close the time-weighted \( H^1 \) energy estimate of \( D_t \nu \).

**Proposition 5.1.** Under the assumptions of Proposition 2.4, we have
\[ A_{11}(t) + A_{12}(t) + \| X \|_{L_t^\infty(W^{2,p})} \leq \mathcal{H}_{1}(t), \]
where \( A_{11}(t), A_{12}(t) \) are given by (2.9) for \( \ell = 1 \).

**Proof.** We get, by applying Lemma 2.1 to (2.37), that
\[ \| \sigma^{1-\frac{s_1}{2}} D_t \nu \|_{L_t^2(L^2)}^2 + \| \sigma^{\frac{3-s_1}{2}} \nabla D_t \nu \|_{L_t^2(L^2)}^2 + \| \sigma^{1-\frac{s_1}{2}} (D_t^2 \nu, \nabla D_t \nu, D_t \pi) \|_{L_t^2(L^2)}^2 \]
\[ \leq \mathcal{A}_0(\| \sigma^{1-\frac{s_1}{2}} \|_{L_t^2(L^2)}^2 + \| \sigma^{\frac{3-s_1}{2}} \|_{L_t^2(L^2)}^2 + \| \sigma^{1-\frac{s_1}{2}} \|_{L_t^2(L^2)}^2 + \| \sigma^{\frac{3-s_1}{2}} \|_{L_t^2(L^2)}^2). \]
Taking into account (4.21), by substituting the Estimates (5.1), (5.4) and (5.7) into the above inequality and then choosing \( \varepsilon \) sufficiently small, we achieve
\[ \| \sigma^{1-\frac{s_1}{2}} D_t \nu \|_{L_t^2(L^2)}^2 + \| \sigma^{\frac{3-s_1}{2}} \nabla D_t \nu \|_{L_t^2(L^2)}^2 \leq \mathcal{G}_{1,X}(t). \]
Moreover, it follows from the 2-D interpolation inequality (2.20) that for any \( r \in [2, \infty] \)
\[ \| \sigma^{1-\frac{s_1}{2}} D_t \nu \|_{L_t^2(L^r)} \leq C \| \sigma^{1-\frac{s_1}{2}} D_t \nu \|_{L_t^2(L^2)}^{2} \| \sigma^{1-\frac{s_1}{2}} \|_{L_t^2(L^2)}^{2} \| \sigma^{1-\frac{s_1}{2}} \|_{L_t^2(L^2)} \leq \mathcal{G}_{1,X}(t). \]
Along the same line, we deduce from Proposition 2.2 that
\[ \| \sigma^{1-\frac{s_1}{2}} D_t \nu \|_{L_t^2(L^r)} \leq C(t_0, s_0) \ \forall \ r \in [2, \infty]. \]
Let us now turn to the estimate of \( \| X(t) \|_{W^{2,p}} \). We first get, by using \( L^p \) energy estimate to (2.4) and (4.23), that
\[ \frac{d}{dt} \| X \|_{W^{2,p}} \leq C \left( \| \nabla X \|_{L^\infty} \| X \|_{W^{2,p}} + \| \Delta v \|_{L^p} \| \nabla X \|_{L^\infty} \right) + \| \Delta v \|_{L^p}. \]
To deal with the estimate of $\|\Delta v^1\|_{L^p}$, we rewrite (2.31) as
$$-\Delta (v^1 - \nabla \Delta^{-1} (\partial_\alpha v \cdot \nabla X^\alpha)) + \nabla \pi = -\rho Dtv^1 + \nabla (\partial_\alpha v \cdot \nabla X^\alpha) + F_1(v, \pi),$$
for $F_1(v, \pi)$ given by (2.32). Then due to $\text{div } v^1 = \partial_\alpha v \cdot \nabla X^\alpha$, we deduce from classical estimate for the Stokes operator and (2.32) that for any $r \in [2, p]$
$$\| (\nabla^2 v^1, \nabla \pi) \|_{L^r} \leq C (\| \nabla (\nabla v \otimes \nabla X) \|_{L^r} + \| \rho Dtv^1 \|_{L^r} + \| F_1(v, \pi) \|_{L^r})$$
$$\leq C \left( \| (Dtv^1, Dtv) \|_{L^r} + \left( \| \nabla v \|_{L^\frac{r}{p}} + \| (\nabla^2 v, \nabla \pi) \|_{L^r} \right) \| \nabla X \|_{W^{1,p}} \right).$$
(5.14)

Inserting (5.14) for $r = p$ into (5.13), and integrating the resulting inequality over $[0, t]$, we obtain
$$\| X(t) \|_{W^{2,p}} \leq \| X_0 \|_{W^{2,p}} + C \| (Dtv^1, Dtv) \|_{L^1_t(L^p)}$$
$$+ C \int_0^t \left( \| \nabla v \|_{L^\infty} + \| (\nabla^2 v, \nabla \pi) \|_{L^p} \right) \| X \|_{W^{2,p}} dt'.$$
(5.15)

Note from the hypothesis that $p \in [2, 2/(1 - s_1)]$, which implies
$$\frac{2p}{p + 2} (1 - \frac{s_1}{2}) < 1.$$ This together with (5.11) and (5.12) ensures that
$$\| (Dtv^1, Dtv) \|_{L^1_t(L^p)} \leq \| \sigma^{1 - \frac{4}{p}} (Dtv^1, Dtv) \|^2_{L^\frac{2p}{p+2}} \| \sigma^{-(1 - \frac{4}{p})} \|^2_{L^{\frac{2p}{p+2}}([0, t])} \leq \mathcal{G}_1, X(t).$$

And it is obvious to observe from $2/p + s_1 > 1$ that
$$\left( \int_0^t \left( \| \nabla v \|_{L^\infty} + \| (\nabla^2 v, \nabla \pi) \|_{L^p} \right) \| X \|_{W^{2,p}} dt' \right)^2 \leq \left( \int_0^t \| \nabla v \|_{L^\infty}^2 \| X \|_{W^{2,p}}^2 dt' \right)^2$$
$$+ \int_0^t \| \sigma^{1 - \frac{2}{p} + \frac{4}{p}} (\nabla^2 v, \nabla \pi) \|_{L^p}^2 \| X \|_{W^{2,p}}^2 dt' \int_0^t \sigma^{-2 + \frac{4}{p} + s_1} dt' \leq \mathcal{G}_1, X(t).$$

Substituting the above estimates into (5.15) and using the definition of $\mathcal{G}_1, X(t)$ given by (4.11) leads to
$$\| X(t) \|^2_{W^{2,p}} \leq \| X_0 \|^2_{W^{2,p}} + \exp (A_0(t)^2) \left( A_1 + \int_0^t \left( \mathcal{B}_0(t') + \sigma(t')^{-1 + \frac{2p}{p}} \right) \| X(t') \|^2_{W^{2,p}} dt' \right).$$

Then by virtue of (3.42), we get, by applying Gronwall’s inequality, that
$$\| X \|_{L^\infty_t(W^{2,p})} \leq A_1 (1 + \| X_0 \|^2_{W^{2,p}}) \exp \left( \exp (A_0(t)^2) \right) \leq \mathcal{H}_1(t),$$
which together with Proposition 2.3 and (3.42) implies that
$$A_{11}(t) \leq \mathcal{H}_1(t).$$
(5.17)

Similarly, by inserting (5.16) into (5.10), we arrive at
$$\| \sigma^{1 - \frac{4}{p}} Dtv^1 \|^2_{L^\infty_t(L^2)} + \| \sigma^{\frac{4}{p - 1}} \nabla v \|^2_{L^\infty_t(L^2)}$$
$$+ \| \sigma^{1 - \frac{4}{p}} \nabla Dtv^1 \|^2_{L^\infty_t(L^2)} + \| \sigma^{\frac{4}{p - 1}} (Dtv^1, \nabla Dtv^1, \nabla Dtv^1) \|^2_{L^\infty_t(L^2)} \leq \mathcal{H}_1(t).$$
(5.18)

On the other side, it follows from (5.14) that for $r = 2$
$$\| \sigma^{1 - \frac{4}{p}} (\nabla^2 v^1, \nabla \pi^1) \|_{L^\infty_t(L^2)} \leq C \| \sigma^{1 - \frac{4}{p}} (Dtv^1, Dtv) \|_{L^\infty_t(L^2)}$$
$$+ C \left( \| \sigma^{1 - \frac{4}{p}} \nabla v \|_{L^\frac{2p}{p+2}_t(L^p)} + \| \sigma^{1 - \frac{4}{p}} (\nabla^2 v, \nabla \pi) \|_{L^\infty_t(L^2)} \right) \| X \|_{L^\infty_t(W^{2,p})}.$$
from which, Proposition 2.1, (5.12), and (5.18), we infer
\[\|\sigma^{1-\frac{3}{2}}(\nabla^2 v^1, \nabla \pi^1)\|_{L_t^\infty(L^2)} \leq \mathcal{H}_1(t),\]
which together with (5.18) implies
\[A_{12}(t) \leq \mathcal{H}_1(t).\]
Along with (5.16), (5.17), we conclude the proof of the proposition. \qed

5.2. Some implications. In this subsection, we shall derive some useful estimates implied by Proposition 5.1.

**Corollary 5.1.** Under the assumptions of of Proposition 2.4, for any \(\varepsilon_0 \in [0, 1]\) and \(r \in \{2, p\}\), one has

\[\mathcal{A}_1(t) \overset{\text{def}}{=} \|v^1\|_{L_t^\infty(L^2)} + \|\sigma^{\frac{1}{2}} v^1\|_{L_t^\infty(L^2)} + \|\sigma^{1+\frac{\varepsilon_0-1}{2}}(v^1, X_t)\|_{L_t^\infty(L^2)} \leq \mathcal{H}_1(t),\]

\[\|\sigma^{1-\frac{3}{2}}(\partial_X v^1, \nabla v^1, \nabla X_t)\|_{L_t^\infty(L^2)} + \|\sigma^{(1-\frac{3}{2})}\nabla X v^1, \sigma v^1, D_t \nabla X)\|_{L_t^\infty(L^2)} \leq \mathcal{H}_1(t),\]

\[\|\sigma^{1-\frac{3}{2}}(\partial_X v^1, \nabla v^1, D_t \nabla X)\|_{L_t^\infty(L^2)} + \|\sigma^{1-\frac{3}{2}}(\partial_X v^1, \nabla v^1, D_t \nabla X)\|_{L_t^\infty(L^2)} \leq \mathcal{H}_1(t).\]

**Proof.** In view of Proposition 5.1, we have for any \(r \in [2, +\infty[\)

\[\|\sigma^{1-\frac{3}{2}}(1-\frac{3}{2})v^1\|_{L_t^\infty(L^r)} \leq C\|v^1\|_{L_t^\infty(L^2)} \|\sigma^{1-\frac{3}{2}} \nabla v^1\|_{L_t^\infty(L^2)} \leq \mathcal{H}_1(t),\]

\[\|\sigma^{1-\frac{3}{2}} \nabla v^1\|_{L_t^\infty(L^2)} \leq C\|\sigma^{1-\frac{3}{2}} \nabla v^1\|_{L_t^\infty(L^2)} \|\sigma^{1-\frac{3}{2}} \nabla v^1\|_{L_t^\infty(L^2)} \leq \mathcal{H}_1(t),\]

which together with 2-D the interpolation inequality, \(\|a\|_{L^\infty} \lesssim \|a\|_{L^\infty}^{\frac{3}{2}} \|\nabla a\|_{L^\infty}^{\frac{1}{2}}\), ensures that for any \(\varepsilon_0 \in [0, 1]\)

\[\|\sigma^{1+\frac{\varepsilon_0-1}{2}} v^1\|_{L_t^\infty(L^\infty)} \leq C\|v^1\|_{L_t^\infty(L^2)} \|\sigma^{1+\frac{\varepsilon_0-1}{2}} \nabla v^1\|_{L_t^\infty(L^2)}^{\frac{1}{1+\varepsilon_0}} \|\nabla v^1\|_{L_t^\infty(L^2)}^{\frac{1}{1+\varepsilon_0}} \leq \mathcal{H}_1(t).\]

Furthermore, in view of (2.4) and Corollary 2.1, for any \(\varepsilon_0 \in [0, 1]\), we infer

\[\|\sigma^{1+\frac{\varepsilon_0-1}{2}} X_t\|_{L_t^\infty(L^\infty)} + \|\sigma^{1+\frac{\varepsilon_0-1}{2}} X_t\|_{L_t^\infty(L^\infty)} \leq \mathcal{H}_1(t) + C\|\sigma^{1+\frac{\varepsilon_0-1}{2}} v\|_{L_t^\infty(L^2 \cap L^\infty)} \|\nabla X\|_{L_t^\infty(L^\infty)} \leq \mathcal{H}_1(t),\]

and

\[\|\sigma^{1+\frac{\varepsilon_0-1}{2}} (\partial_X v^1, \nabla v^1, \nabla X_t)\|_{L_t^\infty(L^2)} \leq \|\sigma^{1+\frac{\varepsilon_0-1}{2}} (\nabla v^1, \nabla X v^1, \nabla X v^1, \nabla v^1)\|_{L_t^\infty(L^2)} \leq \left(\|\sigma^{1+\frac{\varepsilon_0-1}{2}} \nabla v^1\|_{L_t^\infty(L^2)} + \|\sigma^{1+\frac{\varepsilon_0-1}{2}} v\|_{L_t^\infty(L^2 \cap L^\infty)} \right) \leq \mathcal{H}_1(t).\]

As a result, we obtain

\[\|\sigma^{1+\frac{\varepsilon_0-1}{2}} (v^1, X_t)\|_{L_t^\infty(L^\infty)} + \|\sigma^{1+\frac{\varepsilon_0-1}{2}} (v^1, X_t)\|_{L_t^\infty(L^\infty)} \leq \mathcal{H}_1(t),\]

\[\sum_{i+j=1} \|\sigma^{1+\frac{\varepsilon_0-1}{2}} (\partial_X v^j, \nabla X_t)\|_{L_t^\infty(L^2)} \leq \mathcal{H}_1(t).\]
Whereas it follows from Proposition 5.1 and the 2-D interpolation inequality (2.20) that for any 
\( r \in [2, +\infty] \),
\[
\|\sigma(\frac{3}{2} - \frac{4}{3}) D_t v^1\|_{L_t^\infty(L^r)} \leq \|\sigma^{1 - \frac{4}{3}} D_t v^1\|_{L_t^\infty(L^2)}^{\frac{2}{3}} \|\sigma^{\frac{3}{2} - \frac{4}{3}} \nabla D_t v^1\|_{L_t^\infty(L^2)}^{\frac{1 - \frac{4}{3}}{3}} \leq H_1(t),
\]
which together with (5.14), Corollaries 2.1 and 3.4 ensures that for \( r \in \{2, p\} \)
\[
\|\sigma(\frac{3}{2} - \frac{4}{3} - \frac{r}{3}) (D_t v^1, \nabla^2 v^1, \nabla^4)\|_{L_t^\infty(L^r)} \leq C(1 + \|X\|_{L_t^\infty(W_2, p)})
\times \left(\|\sigma(\frac{3}{2} - \frac{4}{3}) (D_t v^1, D_t v, \nabla^2 v, \nabla^4)\|_{L_t^\infty(L^r)} + \|\sigma^{1 - \frac{4}{3}} \nabla v\|_{L_t^\infty(L^\infty \cap L_t^{2 p})}\right) \leq H_1(t).
\]
Then we deduce from 2-D interpolation inequality that
\[
\|\sigma(1 - \frac{4}{3}) \nabla v^1\|_{L_\infty^\infty} \leq C\|\sigma^{\frac{1 - \frac{4}{3}}{2}} \nabla v^1\|_{L_2^{\frac{2}{1 - \frac{4}{3} p}}}^{\frac{1/2}{1 - \frac{4}{3} p}} \|\sigma^{\frac{3}{2} - \frac{4}{3} - \frac{4}{3}} \nabla^2 v^1\|_{L_p^{2 p}}^{\frac{1/2}{1 - \frac{4}{3} p}} \leq H_1(t).
\]
Observing that
\[
\partial_X \nabla v = \nabla v^1 - \nabla X^\alpha \partial_\alpha v \quad \text{and} \quad D_t \nabla X = \nabla v^1 - \nabla v_\alpha \partial_\alpha X.
\]
Then in view of Corollaries 2.1, 3.4 and (5.19), (5.22), we infer
\[
\|\sigma(\frac{3}{2} - \frac{4}{3} - \frac{r}{3}) (D_t \nabla v, \partial_X \nabla v, D_t \nabla X)\|_{L_t^\infty(L^r)} \leq C(1 + \|X\|_{L_t^\infty(W_2, p)})
\times \left(\|\sigma(\frac{3}{2} - \frac{4}{3}) (D_t \nabla v, D_t v, \nabla^2 v, \nabla^4)\|_{L_t^\infty(L^r)} + \|\sigma^{1 - \frac{4}{3}} \nabla v\|_{L_t^\infty(L^\infty \cap L_t^{2 p})}\right) \leq H_1(t).
\]
Finally it is easy to observe that
\[
\sum_{i+j+l=1} \|\sigma(\frac{3}{2} - \frac{4}{3} - \frac{r}{3}) (\partial_X^i \nabla \partial_X^j \nabla v^1, \partial_X^i \partial_\alpha v^1, D_t v^1, \partial_X^i \nabla^4, D_t \nabla^2 X)\|_{L_t^\infty(L^r)}
\leq \|\sigma(\frac{3}{2} - \frac{4}{3}) (\partial_X^i \nabla \partial_X^j \nabla v^1, \partial_X^i \partial_\alpha v^1, D_t v^1, \partial_X^i \nabla^4, D_t \nabla^2 X)\|_{L_t^\infty(L^r)}
\leq C(1 + \|X\|_{L_t^\infty(W_2, p)})
\times \left(\|\sigma^{\frac{1 - \frac{4}{3}}{2}} \nabla v\|_{L_t^\infty(L^\infty \cap L_t^{2 p})} + \|\sigma^{\frac{3}{2} - \frac{4}{3} - \frac{4}{3}} \nabla^2 v\|_{L_t^\infty(L^\infty \cap L_t^{2 p})}\right) \leq H_1(t).
\]
While it follows from Corollary 2.1 and (5.19) that for any \( r \in [2, \infty] \)
\[
\|\sigma(\frac{3}{2} - \frac{4}{3} - \frac{r}{3}) (v^1 \cdot \nabla v, v \cdot \nabla^2 v)\|_{L_t^\infty(L^r)} \leq \|\sigma^{\frac{1 - \frac{4}{3}}{2}} (v, v^1)\|_{L_t^\infty(L^\infty \cap L_t^{2 p})} \|\sigma^{\frac{3}{2} - \frac{4}{3} - \frac{4}{3}} (\nabla v, \nabla^2 v)\|_{L_t^\infty(L^r)} \leq H_1(t),
\]
which together with (2.4) and (3.42) ensures that for \( r \in \{2, p\} \),
\[
\|\sigma(\frac{3}{2} - \frac{4}{3} - \frac{r}{3}) (\nabla^2 X \otimes \nabla v, \nabla X \otimes \nabla^2 v, \nabla X \otimes \nabla^4, X_t \otimes \nabla v, v \otimes \nabla v^1)\|_{L_t^\infty(L^r)} \leq C(1 + \|X\|_{L_t^\infty(W_2, p)})
\times \left(\|\sigma^{\frac{1 - \frac{4}{3}}{2}} \nabla v\|_{L_t^\infty(L^\infty \cap L_t^{2 p})} + \|\sigma^{\frac{3}{2} - \frac{4}{3} - \frac{4}{3}} (\nabla^2 v, \nabla^4, v \otimes \nabla v, v \otimes \nabla v^1)\|_{L_t^\infty(L^r)}\right) \leq H_1(t),
\]
from which and (5.21), we infer that for \( r \in \{2, p\} \),
\[
\sum_{i+j+l=1} \|\sigma(\frac{3}{2} - \frac{4}{3} - \frac{r}{3}) (\partial_X^i \nabla \partial_X^j \nabla v^1, \partial_X^i \partial_\alpha v^1, D_t v^1, \partial_X^i \nabla^4, D_t \nabla^2 X)\|_{L_t^\infty(L^r)} \leq H_1(t).
\]
This together with (5.20) and (5.23) completes the proof of the corollary \( \square \)

**Corollary 5.2.** Under the assumptions Corollary 5.1, one has

\[
(5.24) \quad \int_0^t \mathfrak{B}_1(t') \, dt' \leq \mathcal{H}_1(t),
\]

where

\[
\mathfrak{B}_1(t) \coloneqq \sum_{i+j=1} \left( \| \sigma^{\left( \frac{1}{2} + \frac{1}{p} - \frac{2}{p'} \right)} \right) \left( 2 \cdot t \right) \| \nabla \mathfrak{D}_i \mathfrak{D}_j \|_{L^2_{\infty}(L^p_{2p})}^2 + \| \sigma^{\left( 1 - \frac{2}{2} \right)}(\partial^i_X \mathfrak{D}_l \nabla \mathfrak{V}^j, \partial^j_X \nabla \mathfrak{D}_l \mathfrak{V}^i) \|_{L^2_{\infty}}^2 + \| \sigma^{\left( 1 - \frac{2}{2} \right)}(\partial^i_X \mathfrak{D}_l \nabla \mathfrak{V}^j, \partial^j_X \nabla \mathfrak{D}_l \mathfrak{V}^i) \|_{L^2_{\infty}}^2.
\]

**Proof.** It follows from Corollary 4.2 and Proposition 5.1 that

\[
\| \sigma^{\left( \frac{1}{2} + \frac{1}{p} - \frac{2}{p'} \right)} \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \leq C \| \sigma^{\left( 1 - \frac{2}{2} \right)} \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \| \sigma^{\left( 1 - \frac{2}{2} \right)} \nabla \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \leq \mathcal{H}_1(t),
\]

and for any \( r \in [2, +\infty[ \)

\[
\| \sigma^{\left( \frac{1}{2} + \frac{1}{r} - \frac{2}{p} \right)} \nabla \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \leq \| \sigma^{\left( 1 - \frac{2}{2} \right)} \nabla \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \| \sigma^{\left( 1 - \frac{2}{2} \right)} \nabla ^2 \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \leq \mathcal{H}_1(t).
\]

As a result, for \( r = 2, \frac{2n}{p-2} \)

\[
\sum_{i+j=1} \| \sigma^{\left( \frac{1}{2} + \frac{1}{r} - \frac{2}{p} \right)} \partial^i_X \nabla \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \leq \| \sigma^{\left( 1 - \frac{2}{2} \right)} \nabla \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \| \sigma^{\left( 1 - \frac{2}{2} \right)} \nabla ^2 \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \| \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})}(1 + \| \nabla \mathfrak{X} \|_{L^\infty_{\infty}(L^\infty_{\infty})}) \leq \mathcal{H}_1(t).
\]

Moreover, for \( r = 2 \) or \( r = \frac{2n}{p-2} \), by virtue of Corollary 3.4 and Corollary 5.1, we infer

\[
\sum_{i+j=1} \| \sigma^{\left( \frac{1}{2} + \frac{1}{r} - \frac{2}{p} \right)} \partial^i_X \mathfrak{D}_l \nabla \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \leq \| \sigma^{\left( 1 - \frac{2}{2} \right)} \mathfrak{D}_l \nabla \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \| \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \| \mathfrak{X} \|_{L^\infty_{\infty}(L^\infty_{\infty})} \| \| \mathfrak{D}_l \nabla \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \| \| \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \| \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \| \mathfrak{D}_l \mathfrak{V}^i \|_{L^2_{\infty}(L^p_{2p})} \leq \mathcal{H}_1(t).
\]

It is easy to observe that

\[
[D_t; \nabla \mathfrak{V}]^1 = -\nabla \mathfrak{V} \partial_\alpha \mathfrak{V} - 2 \mathfrak{V} \partial_\alpha \mathfrak{V}, \quad [D_t; \nabla \pi]^1 = -\nabla \mathfrak{V} \partial_\alpha \pi,
\]

\[
D_t [\partial_X; \nabla]^2 = -D_t(\nabla^2 \mathfrak{V} \partial_\alpha \mathfrak{V} + 2 \mathfrak{V} \partial_\alpha \mathfrak{V}) \nabla \mathfrak{V}, \quad D_t [\partial_X; \nabla] \pi = -D_t(\nabla \mathfrak{X} \partial_\alpha \pi),
\]

\[
\nabla [\partial_X; \nabla] D_t \mathfrak{V} = -\nabla (\nabla \mathfrak{X} \partial_\alpha D_t \mathfrak{V}),
\]
we deduce from Corollaries 3.4, 4.2 and 5.1, Proposition 5.1 that
\[
\sum_{i+j=1} \| \sigma^{\frac{3+s}{2}} (\Delta_i^2 v^1, \partial_i^j X_i^j D_{t_i}^2 v^1, \nabla \partial_i^j X_i^j D_{t_i}^2 v^1, \partial_i^j X_i^j D_{t_i}^2 \nabla \pi^j) \|_{L_t^2(L^2)}(L^2) \\
\lesssim \| \sigma^{\frac{3-s}{2}} (\Delta^2 v^1, \nabla^2 D_{t_i} v^1, \nabla D_{t_i} \pi^1, \nabla^2 v \otimes \nabla v^1, \nabla v \otimes \nabla \pi^1, \nabla v \otimes \nabla v^1) \|_{L_t^2(L^2)}(L^2)
\]
\[
+ \| \sigma^{\frac{3-s}{2}} (\Delta^2 X \otimes \nabla v, \nabla X \otimes (\nabla^2 v, \nabla \pi)) \|_{L_t^2(L^2)}(L^2) + \| \sigma^{\frac{3-s}{2}} \nabla (\nabla X \otimes \nabla D_{t_i} v) \|_{L_t^2(L^2)}(L^2) \\
\lesssim \mathcal{H}_1(t) + \| \sigma^{\frac{3}{2}} (\nabla v, D_{t_i} \nabla X) \|_{L_t^2(L^\infty)}(L^\infty) \| \sigma^{1-s} (\Delta^2 v, \nabla^2 v \pi) \|_{L_t^\infty(L^2)}(L^2)
\]
\[
+ \| X \|_{L_t^\infty(W^{2,2})} (\| \sigma^{\frac{3-s}{2}} (\Delta^2 v, \nabla D_{t_i} v) \|_{L_t^2(L^\frac{2p}{p-2})} + \| \sigma^{\frac{3-s}{2}} (\Delta^2 v, \nabla^2 D_{t_i} v, D_{t_i} \nabla \pi) \|_{L_t^2(L^2)}) \leq \mathcal{H}_1(t).
\]
Summing up the above estimates leads to (5.24). \(\square\)

5.3. Propagation of the \(B^{s_1}\)-regularity for \(v^1\). We shall use the same strategy as that used in the proof of Proposition 3.3 to study the Besov regularity of \((v^1, \nabla \pi^1)\). Indeed let \((v_q, \nabla \pi_q)\) be the unique solution of (2.18), we shall first investigate the following system for the unknown \((u_q, \nabla p_q)\):
\[
\begin{cases}
\rho \partial_t u_q + \nu \cdot \nabla u_q - \Delta u_q + \nabla p_q = F_{1,q} & \text{with} \\
F_{1,q} \overset{\text{def}}{=} -\nu \Delta v_q - (\Delta X \cdot \nabla v_q + 2 \partial_\alpha X \cdot \nabla \partial_\alpha v_q) + \nabla X \partial_\alpha \pi_q, \\
\text{div} u_q = \text{div}(v_q \cdot \nabla X), \\
u_q|_{t=0} = 0.
\end{cases}
\]
Let \((v_{12}, \nabla p_{12})\) be a smooth enough solution of (4.3). Then by virtue of (2.31), (4.3) and (5.25), we have
\[
v_{12} = \sum_{q \in \mathbb{Z}} u_q \quad \text{and} \quad \nabla p_{12} = \sum_{q \in \mathbb{Z}} \nabla p_q.
\]

5.3.1. The \(L^2\) energy estimate. The following lemma concerning the norms of solution to (2.18) will be very useful in the estimates that follows.

**Lemma 5.3.** Let \((v_q, \nabla \pi_q)\) be the unique solution determined by the System (2.18). Let \(\theta_0\) and \(C(v_0, s_0)\) be given by (2.5), then the following inequality is valid
\[
\|v_q\|_{L_t^\infty(L^\frac{2p}{p-2})} + \|\sigma^{\frac{1-s}{2}} v_q\|_{L_t^\infty(L^\infty)} + \|\nabla v_q\|_{L_t^2(L^\frac{2p}{p-2})} + \|\sigma^{\frac{1-s}{2}} \nabla v_q\|_{L_t^\infty(L^2)}(L^2)
\]
\[
+ \|\sigma^{\frac{1-s}{2}} \nabla v_q\|_{L_t^2(L^\infty)}(L^\infty) + \|\sigma^{\frac{1-s}{2}} (D_t v_q, \partial_t v_q, \nabla^2 v_q, \nabla \pi_q)\|_{L_t^2(L^2)}(L^2)
\]
\[
+ \|\sigma^{\frac{1-s}{2}} (D_t v_q, \partial_t v_q, \nabla^2 v_q, \nabla \pi_q)\|_{L_t^2(L^\frac{2p}{p-2})} \leq C(v_0, s_0) (t)^\frac{1}{2} d_q 2^{-q s_1}.
\]

**Proof.** It follows from (2.20), (2.18) and (3.23), (3.24), (3.25) that for any \(\delta \in [0, 1]\)
\[
\|v_q\|_{L_t^\infty(L^\frac{2p}{p-2})} \leq \|v_q\|_{L_t^2(L^2)}^{\frac{1-s}{2}} \|v_q\|_{L_t^2(L^\infty)}^{\frac{s_0}{2}} \leq C(v_0, s_0) d_q 2^{-q s_1},
\]
\[
\|\nabla v_q\|_{L_t^2(L^\frac{2p}{p-2})} \leq \|\nabla v_q\|_{L_t^2(L^2)}^{\frac{1-s}{2}} \|\nabla^2 v_q\|_{L_t^2(L^\infty)}^{\frac{s_0}{2}} \leq C(v_0, s_0) d_q 2^{-q s_1},
\]
\[
\|\sigma^{\frac{1-s}{2}} v_q\|_{L_t^\infty(L^\frac{2p}{p-2})} \leq \|v_q\|_{L_t^2(L^2)}^{\delta} \|\nabla v_q\|_{L_t^2(L^\infty)}^{1-\delta} \leq C(v_0, s_0) d_q 2^{-q s_0},
\]
which together with the 2-D interpolation inequality that
\[ \|a\|_{L^2} \lesssim \|a\|_{L^{1+rac{1}{\delta}}} \|\nabla^2 a\|_{L^{\frac{1}{1+\delta}}}^{\frac{1}{1+\delta}}, \]
for \( \delta = \frac{\theta_0}{1-\theta_0} \) and (3.26) ensures that
\[ \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} v_q\|_{L^\infty_t(L^\infty_x)} \lesssim \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} v_q\|_{L^\infty_t(L^{\frac{1}{2} - \frac{\theta_0}{2}})} \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \nabla^2 v_q\|_{L^\infty_t(L^2)} \leq C(v_0, s_0) d_q 2^{-q s_1}. \]
It is easy to observe that
\[ \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \nabla v_q\|_{L^\infty_t(L^2)} \leq \|\nabla v_q\|_{L^\infty_t(L^2)} \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \nabla^2 v_q\|_{L^\infty_t(L^2)}, \]
\[ \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \partial_t v_q\|_{L^2_t(L^2)} \leq \|\partial_t v_q\|_{L^2_t(L^2)} \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \partial_t v_q\|_{L^2_t(L^2)}, \]
which together with (3.24) and (3.25) implies that
\[ \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \nabla v_q\|_{L^\infty_t(L^2)} + \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \partial_t v_q\|_{L^2_t(L^2)} \leq C(v_0, s_0) d_q 2^{-q s_1}. \]
While note from (2.20), (3.25) and (3.26) that for \( \delta \in [0, 1] \), there holds
\[ \|\sigma^{\frac{1}{2} \delta} \nabla v_q\|_{L^\infty_t(L^{\frac{1}{1+\delta}})} \leq C \|\sigma^{\frac{1}{2} \delta} \nabla v_q\|_{L^\infty_t(L^2)} \|\sigma^{\frac{1}{2} \delta} \nabla^2 v_q\|_{L^\infty_t(L^2)} \leq C(v_0, s_0) d_q 2^{-q(s_0 - \delta)}, \]
(5.28)
\[ \|\sigma^{\frac{1}{2} \delta} \partial_t v_q\|_{L^2_t(L^{\frac{1}{1+\delta}})} \leq C \|\sigma^{\frac{1}{2} \delta} \partial_t v_q\|_{L^2_t(L^2)} \|\sigma^{\frac{1}{2} \delta} \nabla \partial_t v_q\|_{L^2_t(L^2)} \leq C(v_0, s_0) d_q 2^{-q(s_0 - \delta)}. \]
Then it follows from the classical estimate on Stokes operator and Corollary 2.1, (2.18) that
\[ \|\sigma^{\frac{1}{2}} (\nabla^2 v_q, \nabla \pi_q)\|_{L^2_t(L^{\frac{1}{1+\delta}})} \lesssim \|\sigma^{\frac{1}{2}} \partial_t v_q\|_{L^2_t(L^{\frac{1}{1+\delta}})} \]
(5.29)
\[ + \langle t \rangle^{\frac{\delta}{2}} \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} v_q\|_{L^\infty_t(L^{\frac{1}{2} - \frac{\theta_0}{2}})} \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \nabla v_q\|_{L^\infty_t(L^{\frac{1}{2} - \frac{\theta_0}{2}})} \leq C(v_0, s_0) \langle t \rangle^{\frac{\delta}{2}} d_q 2^{-q(s_0 - \delta)}. \]
Taking \( \delta = \frac{\theta_0}{2 - \theta_0} \) in the second inequality of (5.28) and using (3.24) gives rise to
\[ \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \partial_t v_q\|_{L^2_t(L^{\frac{1}{2} - \frac{\theta_0}{2}})} \leq \|\partial_t v_q\|_{L^2_t(L^{\frac{1}{2} - \frac{\theta_0}{2}})} \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \partial_t v_q\|_{L^2_t(L^{\frac{1}{2} - \frac{\theta_0}{2}})} \leq C(v_0, s_0) d_q 2^{-q s_1}, \]
which together with a similar derivation of (5.29) ensures that
\[ \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} (\nabla^2 v_q, \nabla \pi_q)\|_{L^2_t(L^{\frac{1}{2} - \frac{\theta_0}{2}})} \]
\[ + \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} (\nabla^2 v_q, \nabla \pi_q)\|_{L^2_t(L^{\frac{1}{2} - \frac{\theta_0}{2}})} \leq C(v_0, s_0) \langle t \rangle^{\frac{\delta}{2}} d_q 2^{-q s_1}. \]
Furthermore, thanks to the following interpolation inequality
\[ \|a\|_{L^\infty} \lesssim \|a\|_{L^2} \|\nabla a\|_{L^{\frac{1}{1+\delta}}}^{\frac{1}{1+\delta}}, \]
and (5.29), we infer
\[ \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \nabla v_q\|_{L^2_t(L^{\frac{1}{1+\delta}})} \lesssim \|\nabla v_q\|_{L^2_t(L^2)} \|\sigma^{\frac{1}{2} - \frac{\theta_0}{2}} \nabla^2 v_q\|_{L^2_t(L^{\frac{1}{2} - \frac{\theta_0}{2}})} \leq C(v_0, s_0) \langle t \rangle^{\frac{\delta}{2}} d_q 2^{-q s_1}. \]
By summing up the above estimates and making use of the fact that \( \rho D_t v_q = \Delta v_q - \nabla \pi_q \), we complete the proof of (5.27).

Let us turn to the \( L^2 \) energy estimate of \( u_q \).

**Lemma 5.4.** Let \((u_q, \nabla p_q)\) be the unique solution of (5.25). Then there holds
(5.30)
\[ \|u_q\|_{L^\infty_t(L^2)} + \|\nabla u_q\|_{L^2_t(L^2)} \leq H_1(t) d_q 2^{-q s_1}. \]
Proof. We first get, by using $L^2$ energy estimate to the $u_q$ equation of (5.25), that

\begin{equation}
\frac{1}{2} \| \sqrt{\rho} u_q(t) \|_{L^2}^2 + \| \nabla u_q \|_{L^2_t(L^2)}^2 = - \int_0^t \int_{\mathbb{R}^2} \nabla p_q \cdot u_q \, dx \, dt' + \int_0^t \int_{\mathbb{R}^2} F_{1,q} |u_q| \, dx \, dt'.
\end{equation}

(5.31)

It follows from the definition of $F_{1,q}$ and (2.27) that

\begin{align*}
\| \sigma^{\gamma_0} F_{1,q} \|_{L^2_t(L^2)} & \leq \| \sigma^{\gamma_0} D_t v_q \|_{L^2_t(L^2)} + \| \Delta X \|_{L^\infty_t(L^p)} \| \sigma^{\gamma_0} \nabla v_q \|_{L^2_t(L^\frac{2m}{m-2})} \\
& \quad + \| \nabla X \|_{L^\infty_t(L^\frac{4}{m})} \| \sigma^{\gamma_0} (\nabla^2 v_q, \nabla \pi_q) \|_{L^2_t(L^\frac{4}{m-2})},
\end{align*}

from which, Proposition 5.1 and Lemma 5.3, we infer for $\beta = \frac{-p-2}{(1-\gamma_0)p} \in [0, 1]$

\begin{equation}
\| \sigma^{\gamma_0} F_{1,q} \|_{L^2_t(L^2)} \leq \mathcal{H}_1(t) d_q 2^{-qs_1} + \left( \| \nabla v_q \|_{L^2_t(L^\frac{2m}{m-2})} \| \sigma^{\gamma_0} \nabla v_q \|_{L^2_t(L^{1-\beta})} \right) \| \nabla X \|_{L^\infty_t(W^{1,p})} \leq \mathcal{H}_1(t) d_q 2^{-qs_1}.
\end{equation}

(5.32)

We thus obtain

\begin{equation}
\left| \int_0^t \int_{\mathbb{R}^2} F_{1,q} |u_q| \, dx \, dt' \right| \leq \int_0^t \sigma^{-1+\gamma_0} \| u_q \|_{L^2}^2 \, dt' + \mathcal{H}_1(t) d_q 2^{-2qs_1}.
\end{equation}

(5.33)

While we get, by using integration by parts and (5.25), that

\begin{equation}
\int_{\mathbb{R}^2} \nabla p_q \cdot u_q \, dx = \int_{\mathbb{R}^2} \nabla p_q \cdot \nabla X \, dx \\
= \int_{\mathbb{R}^2} (F_{1,q} - \partial_t (\rho u_q) - \text{div} (\rho v \otimes u_q) + \Delta u_q) \cdot \nabla X \, dx.
\end{equation}

(5.34)

In view of Proposition 5.1 and Lemma 5.3, one has

\begin{equation}
\| v_q \cdot \nabla X \|_{L^\infty_t(L^2)} \leq \| v_q \|_{L^\infty_t(L^\frac{2m}{m-2})} \| \nabla X \|_{L^\infty_t(L^\frac{2}{m-2})} \leq \mathcal{H}_1(t) d_q 2^{-qs_1},
\end{equation}

(5.35)

so that a similar derivation as (5.33) ensures

\begin{equation}
\left| \int_0^t \int_{\mathbb{R}^2} F_{1,q} |v_q \cdot \nabla X| \, dx \, dt' \right| \leq \mathcal{H}_1(t) d_q 2^{-2qs_1}.
\end{equation}

Again by integration by parts, we write

\begin{equation}
\int_{\mathbb{R}^2} \partial_t (\rho u_q) \cdot v_q \cdot \nabla X \, dx = \frac{d}{dt} \int_{\mathbb{R}^2} \rho u_q |v_q \cdot \nabla X| \, dx - \int_{\mathbb{R}^2} \rho u_q \partial_t (v_q \cdot \nabla X) \, dx.
\end{equation}

Yet it follows from Proposition 5.1, Corollary 5.1 and Lemma 5.3 that

\begin{align*}
\| \sigma^{\gamma_0} \partial_t (v_q \cdot \nabla X) \|_{L^2_t(L^2)} \leq \| \nabla X \|_{L^\infty_t(W^{1,p})} \| \sigma^{\gamma_0} \partial_t v_q \|_{L^2_t(L^\frac{2}{m-2})} \\
+ C\langle t \rangle \| \sigma^{\gamma_0} v_q \|_{L^\infty_t(L^\infty)} \| \sigma^{\gamma_0} \nabla X \|_{L^\infty_t(L^2)} \leq \mathcal{H}_1(t) d_q 2^{-qs_1},
\end{align*}

(5.37)
which together with (5.35) implies
\[
\int_0^t \int_{\mathbb{R}^2} \partial_t (\rho u_q) |v_q \cdot \nabla X| \, dx \, dt' \leq \frac{1}{4} \| \sqrt{\rho} u_q(t) \|_{L^2}^2 + C \| v_q \cdot \nabla X \|_{L^\infty_t(L^2)}^2 \\
+ C \int_0^t \sigma^{-1+\frac{6q}{n}} \| u_q(t') \|_{L^2}^2 \, dt' + C \| \sigma^{-1+\frac{6q}{n}} \partial_t (v_q \cdot \nabla X) \|_{L^2_t(L^2)}^2 \\
\leq \frac{1}{4} \| \sqrt{\rho} u_q \|_{L^2_t(L^2)}^2 + C \int_0^t \sigma^{-1+\frac{6q}{n}} \| u_q(t') \|_{L^2}^2 \, dt' + \mathcal{H}_1(t) d_q^2 2^{-2q_1},
\]
Along the same line to the derivation of (5.32), one has
\[
\| \nabla (v_q \cdot \nabla X) \|_{L^2_t(L^2)} \leq \| (\nabla v_q \|_{L^2_t(L^{2(1-s_0)})}^2 + \| v_q \|_{L^\infty_t(L^{2(1-s_0)})}^2 \| \nabla X \|_{L^\infty_t(W^{1,p})} \\
\leq \mathcal{H}_1(t) d_q 2^{-qs_1},
\]
which ensures that
\[
\int_0^t \int_{\mathbb{R}^2} \text{div}(\rho v \otimes u_q) |v_q \cdot \nabla X| \, dx \, dt' \leq C \int_0^t \| v \|_{L^4} \| u_q \|_{L^4} \| \nabla (v_q \cdot \nabla X) \|_{L^2} \, dt' \\
\leq \frac{1}{6} \| \nabla u_q \|_{L^2_t(L^2)}^2 + \int_0^t \| v \|_{L^4}^2 \| u_q \|_{L^2}^2 \, dt' + \mathcal{H}_1(t) d_q^2 2^{-2qs_1},
\]
and
\[
\int_0^t \int_{\mathbb{R}^2} \Delta u_q |v_q \cdot \nabla X| \, dx \, dt' \leq \frac{1}{2} \| \nabla u_q \|_{L^2_t(L^2)}^2 + \mathcal{H}_1(t) d_q^2 2^{-2qs_1}.
\]
Inserting the above estimates into (5.34) gives rise to
\[
\int_0^t \int_{\mathbb{R}^2} \nabla \rho_q |u_q| \, dx \, dt' \leq \frac{1}{3} \| \sqrt{\rho} u_q(t) \|_{L^2}^2 + \| \nabla u_q \|_{L^2_t(L^2)}^2 \\
+ C \int_0^t \left( \| v \|_{L^4}^4 + \sigma^{-1+\frac{6q}{n}} \right) \| u_q(t') \|_{L^2}^2 \, dt' + \mathcal{H}_1(t) d_q^2 2^{-2qs_1}.
\]
Substituting the Inequalities (5.33) and (5.36) into (5.31), we arrive at
\[
\| \sqrt{\rho} u_q(t) \|_{L^2_t}^2 + \| \nabla u_q \|_{L^2_t(L^2)}^2 \leq C \int_0^t \left( \| v \|_{L^4}^4 + \sigma^{-1+\frac{6q}{n}} \right) \| u_q(t') \|_{L^2}^2 \, dt' + \mathcal{H}_1(t) d_q^2 2^{-2qs_1}.
\]
Then applying Gronwall’s inequality leads to (5.30). This completes the proof of the lemma.
\[
\square
\]
5.3.2. The $H^1$ energy estimate.

Lemma 5.5. Under the same assumptions of Lemma 5.4, one has
\[
\| \nabla u_q \|_{L^\infty_t(L^2)} + \|(\partial_t u_q, \nabla^2 u_q, \nabla p_q)\|_{L^2_t(L^2)} \leq \mathcal{H}_1(t) d_q 2^{q(1-s_1)}.
\]

Proof. In view of (5.25), we get, by applying Lemma 3.1, that
\[
\frac{d}{dt} \| \nabla u_q(t) \|_{L^2}^2 + \| \partial_t (v_q \cdot \nabla X) \|_{L^2}^2 \\
\leq C \left( \| v \|_{L^4}^4 \| \nabla u_q \|_{L^2}^2 + \| \partial_t (v_q \cdot \nabla X) \|_{L^2}^2 + \| (F_1, \nabla \text{div}(v_q \cdot \nabla X)) \|_{L^2}^2 \right).
\]
Applying Gronwall’s inequality yields
\[
\left\| \nabla u_q \right\|_{L^\infty_t(L^2)}^2 + \left\| (\partial_t u_q, \nabla^2 u_q, \nabla p_q) \right\|_{L^2_t(L^2)}^2 \leq \exp(C\left\| v_0 \right\|_{L^2}^2) \left( \left\| \partial_t(v_q \cdot \nabla X) \right\|_{L^2_t(L^2)}^2 + \left\| (F_{1q}, \nabla \text{div}(v_q \cdot \nabla X)) \right\|_{L^2_t(L^2)}^2 \right).
\]
(5.38)

Note that since \( v_0 \in L^2 \cap B^{s_0} \), we observe that the Inequalities (3.23) to (3.26) hold for any \( s \in [0, s_0] \). In particular, it follows from (3.24), (3.26) and \( \rho D_t v_q = \Delta v_q - \nabla \pi_q \) that
\[
\left\| (\nabla v_q, \sigma^{\frac{1}{2}} \nabla^2 v_q) \right\|_{L^\infty_t(L^2)} + \left\| (D_t v_q, \nabla^2 v_q, \partial_t v_q, \nabla \pi_q) \right\|_{L^2_t(L^2)} \lesssim C(v_0, s_1) d_q 2^q(1-s_1),
\]
which together with (2.20) ensures that for any \( \delta \in [0, 1] \),
\[
\left\| \nabla v_q \right\|_{L^{\frac{2}{1-\delta}}_t(L^\infty_x)} \lesssim \left\| \nabla v_q \right\|_{L^\infty_t(L^2)}^{\delta} \left\| \nabla^2 v_q \right\|_{L^\infty_t(L^2)}^{1-\delta} \lesssim C(v_0, s_1) d_q 2^q(1-s_1).
\]
Similarly for any \( \delta \in [0, 1] \), we have
\[
\left\| v_q \right\|_{L^{\frac{2}{1-\delta}}_t(L^\infty_x)} \lesssim \left\| v_q \right\|_{L^\infty_t(L^2)}^{\delta} \left\| \nabla^2 v_q \right\|_{L^\infty_t(L^2)}^{1-\delta} \lesssim C(v_0, s_1 - \delta) d_q 2^q(1-s_1),
\]
and
\[
\left\| \sigma^{\frac{\delta}{2(1-\delta)}} v_q \right\|_{L^\infty_t(L^\infty_x)} \lesssim \left\| v_q \right\|_{L^\infty_t(L^2)}^{\frac{1}{1-\delta}} \left\| \sigma^{\frac{\delta}{2}} \nabla^2 v_q \right\|_{L^\infty_t(L^2)}^{\frac{\delta}{1+\delta}} \lesssim A_0 d_q 2^q(1-s_1).
\]
(5.41)

Taking \( \delta \in [0, s_1/(1-s_1)] \), we deduce from (5.39), (5.41) and Corollary 5.1 that
\[
\left\| \partial_t v_q \cdot \nabla X \right\|_{L^2_t(L^2)} \leq \left\| \partial_t v_q \right\|_{L^2_t(L^2)} \left\| \nabla X \right\|_{L^\infty_t(L^\infty)} + \left\| \sigma^{\frac{\delta}{2(1+\delta)}} v_q \right\|_{L^\infty_t(L^\infty_x)} \left\| \sigma^{\frac{1-s_1}{2}} \nabla X_t \right\|_{L^\infty_t(L^2)} \left\| \sigma^{-\frac{1}{2}} (1+s_1)^{-1} \right\|_{L^2_t} \leq \mathcal{H}_1(t) d_q 2^q(1-s_1).
\]
(5.42)

While due to \( \text{div} X = 0 \), we have
\[
\nabla \text{div}(v_q \cdot \nabla X) = \nabla \partial_\alpha v_q \cdot \nabla X^\alpha + \partial_\alpha v_q \cdot \nabla \nabla X^\alpha,
\]
from which, (2.27) and (5.25), we infer
\[
\left\| (F_{1q}, \nabla \text{div}(v_q \cdot \nabla X)) \right\|_{L^2_t(L^2)} \lesssim \left( 1 + \left\| \nabla X \right\|_{L^\infty_t(W^{1,p})} \right) 
\times \left( \left\| (D_t v_q, \nabla^2 v_q, \nabla \pi_q) \right\|_{L^2_t(L^2)} + t^{\left( \frac{1}{2} - \frac{1}{p} \right)} \left\| \nabla v_q \right\|_{L^\infty_t(L^{\frac{2p}{p-2}})} \right).
\]

As a result, by virtue of (5.39) and (5.40) (for \( \delta = (p-2)/p \), it comes out
\[
\left\| (F_{1q}, \nabla \text{div}(v_q \cdot \nabla X)) \right\|_{L^2_t(L^2)} \leq \mathcal{H}_1(t) d_q 2^q(1-s_1).
\]
(5.43)

Substituting the Inequalities (5.42) and (5.43) into (5.38) gives rise to (5.37). This finishes the proof of Lemma 5.5.

The main result of this subsection is as follows

**Proposition 5.2.** Under the assumptions of Proposition 2.4, we have
\[
\left\| \nabla^1 \right\|_{L^\infty_t(B^{s_1})} + \left\| \nabla \nabla^1 \right\|_{L^2_t(B^{s_1})} \leq \mathcal{H}_1(t).
\]
(5.44)

**Proof.** Indeed in view of (5.26), Lemmas 5.4 and 5.5, we deduce by a similar proof of Proposition 3.3 that
\[
\left\| v_{12} \right\|_{L^\infty_t(B^{s_1})} + \left\| \nabla v_{12} \right\|_{L^2_t(B^{s_1})} \leq \mathcal{H}_1(t),
\]
which together with (4.1) and (4.4) leads to (5.44). □
Combing Proposition 5.1 with Proposition 5.2, we achieve Proposition 2.4.

6. Propagation of higher order striated regularity

The goal of this section is to present the proof of Proposition 2.5. To this end, for the functional $A_\ell(t)$ given by (2.8), we inductively assume that

\begin{equation}
A_\ell(t) \leq \mathcal{H}_\ell(t), \quad \text{for any } l \leq \ell - 1 \quad \text{and} \quad \ell \leq k.
\end{equation}

We shall always assume (6.1) throughout this section. We aim at establishing the Estimate (6.1) for $l = \ell$.

6.1. Deductive estimates from (6.1). In this subsection we shall derive some estimates from the inductive assumption (6.1), which will be used constantly in the following context. For $\ell \geq 1$, let

\begin{equation}
R_\ell(t) \overset{\text{def}}{=} \sum_{m+n+\kappa \leq \ell - 1} \left( \| \partial_X^{\kappa} \nabla X^n \|_{L_t^\infty(W^{1,p})} + \| \partial_X^{m} \nabla \partial_X^{\kappa} \|_{L_t^\infty(L^p(L^p))} \right).
\end{equation}

Let

\begin{equation}
\begin{aligned}
r_1 &\in \{2, 2p/(p - 2), +\infty\}, \quad r \in \{2, p\}, \\
r_2 &\in \{2p/(p - 2), +\infty\}, \quad \text{and} \quad r_3 \in \{2, 2p/(p - 2)\},
\end{aligned}
\end{equation}

and

\begin{equation}
0 < \varepsilon_0 < \min \left( s_k/2, (p/2 - 1)(1 - s_0) \right),
\end{equation}

we denote

\begin{align*}
\hat{A}_\ell(t) &\overset{\text{def}}{=} \sum_{i+j+l=\ell} \left( \| v^i \|_{L_t^\infty(L^2)} + \| \sigma^{1-s_0} v^i \|_{L_t^\infty(L^{2p/(2p + 2)})} + \| \sigma^{1+s_0} (v^i, \partial_X^{m} \partial_t X) \|_{L_t^\infty(L^{p})} \right) \\
&\quad + \| \sigma^{1-s_0} \partial_X^{m} \nabla X_t \|_{L_t^\infty(L^2)} + \| \sigma^{s_0} (\partial_X^i \partial^j, \partial_X^{m} D_t \nabla^2 X^n) \|_{L_t^\infty(L^2)} \\
&\quad + \| \sigma^{(1-s_0)2} \partial_X^i \partial^j v^i \|_{L_t^\infty(L^{p/2})} + \| \sigma^{(s_0-1)2} \partial_X^i \partial^j D_t v^j \|_{L_t^\infty(L^{p/2})} \\
&\quad + \| \sigma^{s_0} (\partial_X^i \nabla D_t^j \nabla \partial_X \partial_t v^j) \|_{L_t^\infty(L^p)}
\end{align*}

and

\begin{align*}
\hat{B}_\ell(t) &\overset{\text{def}}{=} \sum_{i+j=\ell} \left( \| \sigma^{3-s_0} (\partial_X^i \nabla D_t^j, \partial_X^{m} D_t \nabla v^j) \|_{L_t^{2p/(2p + 2)}} \right) \\
&\quad + \| \sigma^{3-s_0} (D_t^2 v^i, \partial_X D_t \nabla^2 v^j, \nabla \partial_X \nabla D_t v^j, \partial_X D_t \nabla \pi^j) \|_{L_t^2}
\end{align*}

and

\begin{equation}
\mathcal{A}_\ell(t) \overset{\text{def}}{=} \sum_{l \leq \ell} \hat{A}_l(t) \quad \text{and} \quad \mathcal{B}_\ell(t) \overset{\text{def}}{=} \mathcal{B}_0(t) + \sum_{l \leq \ell} \hat{B}_l(t),
\end{equation}

where $\mathcal{B}_0(t)$ is given by (2.36).

Under the inductive assumption (6.1), the estimates of $R_{\ell-1}(t), \mathcal{A}_{\ell-1}(t), \mathcal{B}_{\ell-1}(t)$ relies on the following lemma concerning the commutative estimates, the proof of which will be postponed in the appendix.
Lemma 6.1. Let \( \ell \in \{1, \cdots, k\} \) and \((i, j)\) be any pair of nonnegative integers with \(i + j \leq \ell\). Then for \( r_1, r, r_3 \) satisfying (6.3), there exists a positive constant \( C \) such that

\[
\| \partial_X^{\ell-i} \nabla X^{\ell-i} - \nabla X^{\ell} \|_{L^\infty(W^{1,p})} + \| \partial_X \partial_X^{\ell-i} \nabla X^{\ell-i-j} - \nabla^2 \nabla^{\ell} \|_{L^\infty(L^p)} \leq CR^2(t),
\]

and

\[
\| \sigma(t) \left( \frac{1}{2} - \frac{2\eta}{3} \right) (\partial_X^{\ell} \nabla v^{\ell-i} - \nabla v^\ell, \partial_X^{\ell} \nabla v^{\ell-i} - \nabla v^\ell) \|_{L^\infty(L^r)}^2 + \| \sigma(t) \left( \frac{1}{2} - \frac{2\eta}{3} \right) (\partial_X^{\ell} \nabla v^{\ell-i} - \nabla v^\ell, \partial_X^{\ell} \nabla v^{\ell-i} - \nabla v^\ell) \|_{L^\infty(L^r)}^2
\]

\[
+ \| \sigma(t) \left( \frac{1}{2} - \frac{2\eta}{3} \right) (\partial_X^{\ell} \nabla v^{\ell-i} - \nabla v^\ell, \partial_X^{\ell} \nabla v^{\ell-i} - \nabla v^\ell) \|_{L^\infty(L^r)}^2 \leq CR^2(t) \mathcal{A}_{\ell-1}(t),
\]

and

\[
(6.8)
\]

\[
\|
\]

\[
(6.9)
\]

\[
(6.10)
\]

Lemma 6.2. Under the assumptions of Proposition 2.5 and the inductive assumption (6.1), one has

\[
R_l(t) + \mathcal{A}_l(t) + \int_0^t \mathcal{B}_l(t') dt' \leq \mathcal{H}_l(t), \quad 1 \leq l \leq \ell - 1.
\]

Proof. We first deduce from Corollaries 2.1 and 3.4, Proposition 2.4, Corollaries 5.1 and 5.2 that (6.11) holds for \( l = 1 \). Inductively, we assume that

\[
R_\kappa(t) + \mathcal{A}_\kappa(t) + \int_0^t \mathcal{B}_\kappa(t') dt' \leq \mathcal{H}_\kappa(t), \quad 2 \leq \kappa \leq \ell - 1.
\]

We intend to show the Estimate (6.12) for \( \kappa + 1 \). Indeed it follows from (6.1) that

\[
\|
\]

\[
(6.12)
\]

which together with the commutator estimate (6.8) for \( \ell = \kappa \) and (6.12) ensures that

\[
(6.13)
\]

while we deduce from (2.20) and (6.1) that for any \( \beta \leq \ell \) and \( r \in [2, \infty] \),

\[
(6.14)
\]

from which, the second inequality of (6.30) below and (6.13), we infer

\[
(6.15)
\]

Furthermore, by Corollary 2.1 and (6.14) one has

\[
(6.16)
\]

from which, (6.14) and (2.20) we deduce

\[
(6.17)
\]
On the other side, in view of (6.14) and (6.15), we get, by applying (2.20) and

\[ \| \sigma^{(1-\frac{1}{p})} a \|_{L^\infty} \leq C \| \sigma^{\frac{1-s_\beta}{p}} a \|_{L^2}^{\frac{1-2/p}{2(1-1/p)}} \| \sigma^{(1-\frac{1}{s})} \nabla a \|_{L^p}^{\frac{1}{2(1-1/p)}}, \]

that

\[ \| \sigma^{(1-\frac{1}{p})} a \|_{L^\infty} \leq C \| \sigma^{\frac{1-s_\beta}{p}} a \|_{L^2}^{\frac{1-2/p}{2(1-1/p)}} \| \sigma^{(1-\frac{1}{s})} \nabla a \|_{L^p}^{\frac{1}{2(1-1/p)}}, \]

(6.17)

\[ \| \sigma^{(1-\frac{1}{p})} \nabla v^{\kappa+1} \|_{L^\infty(L^\infty)} \leq \mathcal{H}_{\kappa+1}(t), \quad \forall \tau_1 \in [2, +\infty]. \]

Noticing that for any \( q \in [2, \infty[ \),

\[ \| \sigma^{(1-\frac{1}{q})} a \|_{L^\infty} \leq C \| a \|_{L^2}^{\frac{1-2/q}{2(1-1/q)}} \| \sigma^{(1-\frac{1}{s})} \nabla a \|_{L^p}^{\frac{1}{2(1-1/q)}}, \]

and (2.20)

\[ \| \sigma^{1-s_\beta} \|_{L^2}^{\frac{2p}{2p-2}} \leq C \| a \|_{L^2}^{\frac{1}{p}} \| \sigma^{1-s_\beta} \nabla a \|_{L^2}^{\frac{2}{p}}, \]

we deduce from (6.14) and (6.17) (by taking \( q = \frac{2(s_\kappa+1-s_\beta)}{s_\kappa+1-2\varepsilon_0} \)) that

\[ \| \sigma^{(1-s_\beta)} v^{\kappa+1} \|_{L^\infty(L^\infty)} + \| \sigma^{1-s_\beta} v^{\kappa+1} \|_{L^\infty(L^{\frac{2p}{2p-2}})} \leq \mathcal{H}_{\kappa+1}(t). \]

To deal with the estimates pertaining to \( X \) in \( \hat{\mathcal{A}}_{\kappa+1}(t) \), we first get from (2.4) that

\[ \partial^\kappa_X \partial_t X = \partial^\kappa_X (v^1 - v \cdot \nabla X) = v^{\kappa+1} - \sum_{n=0}^{\kappa} C^{(n)} \partial^{\kappa-n}_X \nabla X, \]

and

\[ \partial^\kappa_X \partial_t X = \partial^\kappa_X (v^1 - v \cdot \nabla X) - \sum_{n=0}^{\kappa} C^{(n)} \partial^{\kappa-n}_X \nabla X + v^n \cdot \partial^{\kappa-n}_X \partial^2 X. \]

We then apply the operator \( \partial^{-1}_X \) to (2.4) to get

\[ D_{t} X^{\kappa-1} = \partial_t X^{\kappa-1} + v \cdot \nabla X^{\kappa-1} = v^\kappa, \]

which together with (2.30) ensures that for any nonnegative integer \( m \leq \kappa \),

\[ \partial^m_X D_{t} \nabla X^{\kappa-m} = \partial_X^m \nabla D_{t} X^{\kappa-m} - \partial_X^m (\nabla v \cdot \nabla X^{\kappa-m}) \]

\[ = \partial_X^m \nabla v^{\kappa+1-m} - \sum_{n=0}^{m} C^{(n)} \partial_X^m \nabla v_a \partial_X^{\kappa-n} \partial_a X^{\kappa-m}, \]

and

\[ \partial_X^m D_{t} \nabla^2 X^{\kappa-m} = \partial_X^m \nabla^2 D_{t} X^{\kappa-m} - \partial_X^m (\nabla^2 v_a \partial_a X^{\kappa-m} + 2 \nabla v_a \partial_a \nabla X^{\kappa-m}) \]

\[ = \partial_X^m \nabla^2 v^{\kappa+1-m} - \sum_{n=0}^{m} C^{(n)} \partial_X^m \nabla^2 v_a \partial_X^{\kappa-n} \partial_a X^{\kappa-m} + 2 \partial_X^m \nabla v_a \partial_X^{\kappa-n} \partial_a \nabla X^{\kappa-m}. \]

By virtue of (6.2), we deduce from (6.19)-(6.20) that

\[ \| \sigma^{(1-s_\beta)} \partial_X^\kappa X_t \|_{L^\infty(L^\infty)} + \| \sigma^{(1-s_\beta)} \partial_X \nabla X_t \|_{L^\infty(L^2)} \leq C (1 + R_{\kappa+1}(t)) \]

\[ \times \sum_{i+j \leq \kappa+1} \left( \| \sigma^{(1-s_\beta)} v^i \|_{L^\infty(L^\infty)} + \| \sigma^{(1-s_\beta)} \partial_X \nabla^j v \|_{L^\infty(L^2)} \right), \]
and it follows from (6.22)-(6.23) that for \( r_2 = \frac{2p}{p-2} \) or \( r_2 = \infty \) and for any \( m \leq \kappa \),

\[
\| \sigma^{1-\frac{1}{r_2}} D_t \sigma^{\frac{1}{2} - \frac{\kappa + 1}{r_2}} \partial_X D_t \nabla X^{\kappa - m} \|_{L_p^\infty(L^2)} + \| \sigma^{1-\frac{1}{r_2}} D_t \sigma^{\frac{1}{2} - \frac{\kappa + 1}{r_2}} \partial_X^m D_t \nabla^2 X^{\kappa - m} \|_{L_p^\infty(L^2)} \\
\leq C(1 + R_{\kappa+1}(t)) \sum_{i+j \leq \kappa+1} \left( \| \sigma^{1-\frac{1}{r_2}} D_t \sigma^{\frac{1}{2} - \frac{\kappa + 1}{r_2}} \partial_X^i \nabla^j \|_{L_p^\infty(L^2)} \right) \\
+ \| \sigma^{1-\frac{1}{r_2}} D_t \sigma^{\frac{1}{2} - \frac{\kappa + 1}{r_2}} \partial_X^i \nabla^j \|_{L_p^\infty(L^\frac{2p}{p-2})},
\]

The Estimates (6.13) to (6.18), along with (6.1), (6.9), (6.12), the above two inequalities and the following fact for any nonnegative integer \( i \leq \kappa + 1 \),

\[
\| \sigma^{1-\frac{1}{r_2}} \partial_X^i \partial_t v^{\kappa+1-i} \|_{L_p^\infty(L^2)} = \| \sigma^{1-\frac{1}{r_2}} (\partial_X^i D_t v^{\kappa+1-i} - \partial_X^i (v \cdot \nabla v^{\kappa+1-i})) \|_{L_p^\infty(L^2)} \\
\leq \| \sigma^{1-\frac{1}{r_2}} D_t \nabla^{\kappa+1} \|_{L_p^\infty(L^2)} + C \sum_{n=0}^i \| \sigma^{\frac{1}{2}} \nabla^n \|_{L_p^\infty(L^\infty)} \| \sigma^{1-\frac{1}{r_2}} \partial_X^{\kappa-n} \nabla v^{\kappa+1-i} \|_{L_p^\infty(L^2)},
\]

implies that

\[
\dot{H}_{\kappa+1}(t) \leq \mathcal{H}_{\kappa+1}(t).
\]

Finally under the assumption (6.1), we deduce from the commutative estimates (6.10) and (6.12), (6.13), (6.14), (6.16), (6.24), that

\[
\int_0^t \dot{\mathcal{H}}_{\kappa+1}(t') dt' \leq \mathcal{H}_{\kappa+1}(t).
\]

Whence in view of (6.7), by summing up (6.12), (6.13), (6.24) and (6.25), we conclude that (6.12) is valid for \( \kappa + 1 \). Hence Lemma 6.2 follows. \( \square \)

6.2. Some preliminary estimates. According to Lemma 2.2 and (2.39), the time-weighted \( H^1 \) energy estimates of \( v^\ell \) relies on the estimates of \( F_\ell(v, \pi, \nabla v) \), \( \partial_X v^\ell \) and \( \nabla \partial_X v^\ell \), which is the goal of this subsection.

Let us first calculate \( \partial_X v^\ell \). Notice that \( \partial_X v^1 = \partial_X (v \cdot \nabla X) \). Suppose inductively that \( \partial_X v^{\ell-1} = \partial_X g_{\ell-1} \), then due to \( \partial_X X = 0 \) we have

\[
\partial_X v^\ell = \partial_X (v^\ell \cdot \nabla X) = \partial_X \left( v^{\ell-1} \cdot \nabla X + \partial_X g_{\ell-1} \right) \\
= \partial_X \left( v^{\ell-1} \cdot \nabla X + \partial_X g_{\ell-1} - g_{\ell-1} \cdot \nabla X \right) \overset{\text{def}}{=} \partial_X g_{\ell}.
\]

We denote

\[
E_X f \overset{\text{def}}{=} \partial_X f - f \cdot \nabla X.
\]

It is easy to observe that

\[
\partial_X E_X f = \partial_X \partial_X f.
\]

We thus get, by using (6.26) and an inductive argument, that

\[
g_{\ell} = \sum_{i=0}^{\ell-1} E_X^i (v^{\ell-1-i} \cdot \nabla X),
\]
Lemma 6.3. For \( \ell = 2, \cdots, k, \) and \( r \in \{2, p\}, \) there hold

\[
\|\sigma\left(\frac{3}{2} - \frac{1}{r} - \frac{s_r - 1}{2}\right)(F_\ell(v, \pi), \nabla \div v^\ell)(t)\|_{L^r} \leq \mathcal{H}_{\ell-1}(t)(1 + \|\nabla X^{\ell-1}\|_{W^{1,p}}) \quad \text{and}
\]

\[
\|\sigma\left(\frac{3}{2} - \frac{1}{r} - \frac{s_r - 1}{2}\right)(\nabla^2 v^\ell, \nabla \pi^\ell)(t)\|_{L^r} \leq C \|\sigma\left(\frac{3}{2} - \frac{1}{r} - \frac{s_r - 1}{2}\right)D_t v^\ell(t)\|_{L^r} + \mathcal{H}_{\ell-1}(t)(1 + \|\nabla X^{\ell-1}(t)\|_{W^{1,p}}).
\]

Proof. For \( r = 2 \) or \( p, \) we deduce from (2.40) that

\[
\|F_\ell(v, \pi)\|_{L^r} \leq C\left(R_{\ell-1}(t) + \sum_{j=1}^{\ell} \|\rho^j\|_{L^\infty} + \|\partial_{X}^{\ell-1}\nabla X\|_{L^\infty} + \|\partial_{X}^{\ell-1}\Delta X\|_{L^p}\right)
\times \sum_{i+j \leq \ell-1} \left(\|D_t v^j, \partial_X v^j, \partial_X^2 \nabla v^{j}, \partial_X^2 \nabla \pi^{j}\|_{L^r} + \|\partial_X^i \nabla v^j\|_{L^{\frac{mr}{m-r}}}\right),
\]

which together with (6.5) and (6.8), ensures that for \( r = 2 \) or \( r = p \)

\[
\|F_\ell(v, \pi)\|_{L^r} \leq C\left(R_{\ell-1}(t) + R_{\ell-1}^2(t) + \sum_{j=1}^{\ell} \|\rho^j\|_{L^\infty} + \|\nabla X^{\ell-1}\|_{W^{1,p}}\right)\sigma\left(\frac{3}{2} - \frac{1}{r} - \frac{s_r - 1}{2}\right)A_{\ell-1}(t).
\]

As a result, we deduce from Lemma 6.2 and (2.27) that for \( r = 2 \) or \( r = p \)

\[
\|F_\ell(v, \pi)\|_{L^r} \leq \mathcal{H}_{\ell-1}(t)\sigma(t)^{-\left(\frac{3}{2} - \frac{1}{r} - \frac{s_r - 1}{2}\right)}(1 + \|\nabla X^{\ell-1}\|_{W^{1,p}}).
\]

While it follows from (6.29) that

\[
\nabla \div v^\ell = \sum_{i=0}^{\ell-1} \sum_{j=0}^{i} C^{ij}_i \nabla \left(\partial_X^j \partial_{X}^{\ell-1-i-j} \nabla X^\alpha\right),
\]

from which, we infer for \( r = 2 \) or \( r = p \)

\[
\|\nabla \div v^\ell\|_{L^r} \leq C\left(R_{\ell-1}(t) + \|\partial_{X}^{\ell-1}\nabla X\|_{L^\infty} + \|\nabla \partial_{X}^{\ell-1}\nabla X\|_{L^p}\right)
\times \sum_{i+j \leq \ell-1} \left(\|\nabla \partial_X^i \nabla v^j\|_{L^r} + \|\partial_X^i \nabla v^j\|_{L^{\frac{mr}{m-r}}}\right).
\]

Then we deduce from (6.5) and the commutative estimate (6.8) that for \( r = 2, p, \)

\[
\|\nabla \div v_t\|_{L^r} \leq C\left(R_{\ell-1}(t) + R_{\ell-1}^2(t) + \|\nabla X^{\ell-1}\|_{W^{1,p}}\right)\sigma\left(\frac{3}{2} - \frac{1}{r} - \frac{s_r - 1}{2}\right)A_{\ell-1}(t).
\]

By virtue of Lemma 6.2 and (6.31), we obtain the first inequality of (6.30).

On the other hand, in view of (2.39), we write

\[
- \Delta (v^\ell - \nabla \Delta^{-1} \div v^\ell) + \nabla \pi^\ell = \nabla \div v^\ell - \rho D_t v^\ell + F_\ell(v, \pi),
\]

so that for any \( r \in [1, \infty[, \) it follows from classical estimates for Stokes operator that

\[
\|\left(\nabla^2 v^\ell, \nabla \pi^\ell\right)\|_{L^r} \leq C\left(\|\nabla \div v^\ell\|_{L^r} + \|\rho D_t v^\ell\|_{L^r} + \|F_\ell(v, \pi)\|_{L^r}\right),
\]

and

\[
\div v^\ell = \sum_{i=0}^{\ell-1} \div E_X^i (v^{\ell-1-i} \cdot \nabla X) = \sum_{i=0}^{\ell-1} \partial_X^i \div (v^{\ell-1-i} \cdot \nabla X)
\]

(6.29)
which together with the first inequality of (6.30) gives rise to the second one of (6.30). This completes the proof of the lemma.

\textbf{Lemma 6.4.} Let $g_\ell$ be given by (6.28). Then one has

\begin{equation}
\|g_\ell(t)\|_{L^2} + \sigma(t) \frac{1}{\frac{1}{2}} \|\nabla g_\ell(t)\|_{L^2} \leq \mathcal{H}_{\ell-1}(t) \left( 1 + \|X^{\ell-1}(t)\|_{W^{1,p}} \right),
\end{equation}

\begin{equation}
\|\partial_t g_\ell(t)\|_{L^2} \leq \|v\|_{L^\infty} \|\nabla v^f\|_{L^2} + \mathcal{H}_{\ell-1}(t) \sigma(t) \frac{1}{\frac{1}{2}} \left( 1 + \|X^{\ell-1}(t)\|_{W^{1,p}} \right).
\end{equation}

\textbf{Proof.} It follows from (6.28) that

\begin{equation}
g_\ell = \sum_{i=0}^{\ell-1} (\partial_X - (\nabla X)^T \ell_i) (v^{\ell-1-i} \cdot \nabla X)
\end{equation}

\begin{equation}
= \sum_{i=0}^{\ell-1} \sum_{i_1 + \cdots + i_r = i} \partial_X^{i_1} (v^{\ell-1-i} \cdot \nabla X) \cdot \partial_X^{i_2} (-\nabla X)^{i_3} \cdots \partial_X^{i_{r-1}} (-\nabla X)^{i_r}.
\end{equation}

It is obvious to observe from (6.34), Lemmas 6.1 and 6.2 that

\begin{equation}
\|g_\ell\|_{L^2} \leq \|v\| \cdot \|X^{\ell-1} \nabla X\|_{L^2} + \sum_{i \leq \ell-2, i \leq \ell-1} \|v^i\|_{L^2} \|\partial_X^{i-1} \nabla X\|_{L^\infty}
\end{equation}

\begin{equation}
\leq \left( \sum_{i \leq \ell-1} \|v^i\|_{L^2} \right) \left( \|\partial_X^{\ell-1} \nabla X\|_{L^\infty} + R_{\ell-1}^{\ell-1}(t) \right)
\end{equation}

\begin{equation}
\leq \mathcal{H}_{\ell-1}(t) \left( 1 + \|X^{\ell-1}\|_{W^{1,p}} \right).
\end{equation}

While by taking $\partial_\kappa$ with $\kappa = 0, 1, 2$ (here $\partial_0 \overset{\text{def}}{=} \partial_t$) to (6.34) and using (2.30), we write

\begin{equation}
\partial_\kappa g_\ell = g_1^{\kappa, \ell} + g_2^{\kappa, \ell},
\end{equation}

where

\begin{equation}
g_1^{\kappa, \ell} \overset{\text{def}}{=} \sum_{i=1}^{\ell-1} \sum_{i_1 + \cdots + i_r = i} \left( \partial_X^{i_1} (\partial_\kappa X \cdot \nabla \partial_X^{i_2} (v^{\ell-1-i} \cdot \nabla X)) \cdots \partial_X^{i_{r-1}} (-\nabla X)^{i_r} \right.
\end{equation}

\begin{equation}
+ \partial_X^{i_1} (v^{\ell-1-i} \cdot \nabla X) \cdots \partial_X^{i_{j-1}} (\partial_\kappa X \cdot \nabla \partial_X^{i_{j+1}} (-\nabla X)^{i_{j+1}}) \cdots \partial_X^{i_{r-1}} (-\nabla X)^{i_r}
\end{equation}

\begin{equation}
+ \partial_X^{i_1} (v^{\ell-1-i} \cdot \nabla X) \cdot \partial_X^{i_2} (-\nabla X)^{i_3} \cdots \partial_X^{i_{\ell-1}} (-\nabla X)^{i_r},
\end{equation}

and

\begin{equation}
g_2^{\kappa, \ell} \overset{\text{def}}{=} \sum_{i=0}^{\ell-1} \sum_{i_1 + \cdots + i_r = i} \partial_X^{i_1} \partial_\kappa (v^{\ell-1-i} \cdot \nabla X) \cdot \partial_X^{i_2} (-\nabla X)^{i_3} \cdots \partial_X^{i_{\ell-1}} (-\nabla X)^{i_r}.
\end{equation}

Notice that the indices $i_1, i_2, \ldots, i_r$ satisfy $i_1 + \cdots + i_r \leq \ell - 2$ in $g_1^{\kappa, \ell}$. We thus get, by applying Lemma 6.2, that for $\kappa = 1, 2$,

\begin{equation}
\|g_1^{\kappa, \ell}\|_{L^2} \leq C \sum_{i+j \leq \ell-2} \mathcal{H}_{\ell-1}(t) \left( \|\partial_X^{i} \nabla v^j\|_{L^2} + \|v^i\|_{L^{2\kappa}}^{\frac{2\kappa}{\kappa-2}} \right) \leq C \mathcal{H}_{\ell-1}(t) \sigma \frac{1}{\frac{1}{2}} \mathfrak{A}_{\ell-1}(t).
\end{equation}

To handle $g_2^{\kappa, \ell}$, we separate the case when $i_1 = i = \ell - 1$ from others to get

\begin{equation}
\|g_2^{\kappa, \ell}\|_{L^2} \leq \|\nabla v\|_{L^2} \|X^{\ell-1} \nabla X\|_{L^\infty} + \|v\|_{L^{2\kappa}} \|X^{\ell-1} \nabla^2 X\|_{L^p}
\end{equation}

\begin{equation}
+ C \sum_{i+j \leq \ell-1} \mathcal{H}_{\ell-1}(t) \left( \|\partial_X^{i} \nabla v^j\|_{L^2} + \|v^i\|_{L^{2\kappa}}^{\frac{2\kappa}{\kappa-2}} \right),
\end{equation}
from which and the commutative estimate (6.8), we infer
\[ \|g_{\ell,\kappa}^2\|_{L^2} \leq C(\mathcal{H}_{\ell-1}(t) + \|\nabla X^{\ell-1}\|_{W^{1,p}})\sigma^{-\frac{1-s\ell-1}{2}}\mathcal{A}_{\ell-1}(t). \]

Combining the above estimates of \(g_{\ell,\kappa}^1\) and \(g_{\ell,\kappa}^2\) with (6.35) and Lemma 6.2, we obtain the first estimate of (6.33).

Along the same line, we deduce from (6.36) that
\[ \|g_{\ell,0}^1\|_{L^2} \leq \mathcal{H}_{\ell-1}(t)\sigma^{-\frac{1-s\ell-1}{2}}\mathcal{A}_{\ell-1}(t) \leq \mathcal{H}_{\ell-1}(t)\sigma^{-\frac{1-s\ell-1}{2}}. \]

While separating the case when \(i_1 = i = \ell - 1\) from others and taking into account of the fact that: \(X_t = -v \cdot \nabla X + v^1\) one has
\[ \|g_{\ell,0}^2\|_{L^2} \lesssim \|v_1 \cdot \partial^{\ell-1}_{X} \nabla X\|_{L^2} + \|v \otimes \partial^{\ell-1}_{X} \nabla (v \cdot \nabla X)\|_{L^2} + \|v \cdot \partial^{\ell-1}_{X} \nabla v^1\|_{L^2} + C \sum_{i+j \leq \ell - 1}_{\ell - m \leq \ell - 1, m \leq 2} \mathcal{H}_{\ell-1}(t)(\|\partial^{m}_{X} \nabla X_t\|_{L^2}), \]

which yields
\[ \|g_{\ell,0}^2\|_{L^2} \leq \left( \|v_1 \cdot \nabla v\|_{L^2} + \|v \otimes \nabla v\|_{L^{2\ell-2}} \right) \left( \|\partial^{\ell-1}_{X} \nabla X\|_{L^\infty} + \|\partial^{\ell-1}_{X} \nabla X\|_{L^p} \right) + \|v \cdot \nabla v^1\|_{L^2} + \sum_{i+j \leq \ell - 1} \left( \|v \otimes \partial^{i}_{X} \nabla v^1\|_{L^2} + \|v \otimes \nabla v^1\|_{L^{2\ell-2}} \right) \mathcal{H}_{\ell-1}(t) + C \mathcal{H}_{\ell-1}(t)\sigma^{-\frac{1-s\ell-1}{2}}(\mathcal{A}_{\ell-1}(t) + \mathcal{A}_{\ell-1}(t)). \]

However, due to
\[ \partial^{\ell-1}_{X} \nabla v^1 - \nabla v^1 = \sum_{i=0}^{\ell-2} \partial^{\ell-1}_{X} \left( \partial_X \nabla v^{i-1} \right) = -\sum_{i=0}^{\ell-2} \partial^{\ell-1}_{X} \left( \partial_X \nabla v^{i-1} \right) = -\sum_{i=0}^{\ell-2} \sum_{m=0}^i C^m_i \partial^{m}_{X} \nabla X \cdot \partial^{-m}_{X} \nabla v^{i-1}, \]
we deduce from the commutative estimate (6.8) and Lemma 6.2 that
\[ \|g_{\ell,0}^2\|_{L^2} \leq \|v\|_{L^\infty} \|\nabla v^1\|_{L^2} + \mathcal{H}_{\ell-1}(t)\sigma^{-\frac{1-s\ell-1}{2}} + C(\mathcal{H}_{\ell-1}(t) + \|\nabla X^{\ell-1}\|_{W^{1,p}}) \times \sum_{i+j \leq \ell - 1} \left( \|v_1\|_{L^2} + \|v\|_{L^\infty} (\|\partial^{i}_{X} \nabla v^1\|_{L^2} + \|v^1\|_{L^{2\ell-2}}) \right) \]
\[ \leq \|v\|_{L^\infty} \|\nabla v^1\|_{L^2} + \mathcal{H}_{\ell-1}(t)\sigma^{-\frac{1-s\ell-1}{2}}(1 + \|\nabla X^{\ell-1}\|_{W^{1,p}}), \]
which together with the Estimate (6.37) ensures the second estimate of (6.33). This finishes the proof of Lemma 6.4. \(\square\)
6.3. Time-weighted $H^1$ energy estimate of $v^\ell$. In this subsection, we follow the same lines as that in Section 4 to derive the time-weighted $H^1$ estimate of $v^\ell$. Similar to the beginning of Section 4, due to (2.39), we first decompose $(v^\ell, \nabla v^\ell)$ as

$$v^\ell = v_{\ell 1} + v_{\ell 2}, \quad \nabla v^\ell = \nabla p_{\ell 1} + \nabla p_{\ell 2},$$

with $(v_{\ell 1}, \nabla p_{\ell 1})$ and $(v_{\ell 2}, \nabla p_{\ell 2})$ solving the following systems respectively

\[
\begin{aligned}
\rho \partial_t v_{\ell 1} + \rho v \cdot \nabla v_{\ell 1} - \Delta v_{\ell 1} + \nabla p_{\ell 1} &= 0, \\
\text{div } v_{\ell 1} &= 0, \\
v_{\ell 1}|_{t=0} &= \partial_X v_0,
\end{aligned}
\]

and

\[
\begin{aligned}
\rho \partial_t v_{\ell 2} + \rho v \cdot \nabla v_{\ell 2} - \Delta v_{\ell 2} + \nabla p_{\ell 2} &= F_\ell(v, \pi), \\
\text{div } v_{\ell 2} &= \text{div } g_\ell, \\
v_{\ell 2}|_{t=0} &= 0,
\end{aligned}
\]

where $F_\ell(v, \pi)$ and $g_\ell$ are given by (2.40) and (6.28) respectively.

It follows from Proposition 3.3 that

$$\|v_{\ell 1}\|_{L^\infty_t(B^{s_\ell})} + \|\nabla v_{\ell 1}\|_{L^2_t(B^{s_\ell})} \leq C \|\partial_X v_0\|_{B^{s_\ell}} \exp(C \|v_0\|_{L^2}^4),$$

and

$$\|\sigma^{1-s_\ell} \nabla v_{\ell 1}\|_{L^\infty_t(L^2)} + \|\sigma^{1-s_\ell} (\partial_t v_{\ell 2}, \nabla^2 v_{\ell 2}, \nabla p_{\ell 2})\|_{L^2_t(L^2)} \leq C (v_0, \partial_X v_0, s_\ell).$$

**Proposition 6.1.** Let $A_\ell(t)$ be given by (2.9). Then under the assumptions of Proposition 2.5, we have

$$A_\ell^2(t) \leq A_\ell \exp \left(A_0(t)^2 + \mathcal{H}_{t-1}(1 + \int_0^t \sigma^{-(1-s_\ell/2)} \|\nabla X^{t-1}(t')\|_{W^{1,p}}^2 dt' \right) \overset{\text{def}}{=} \tilde{\Theta}_{t, X}(t).$$

**Proof.** We first deduce from Lemma 2.2 and (6.40) that

$$\|\sigma^{1-s_\ell} \nabla v_{\ell 2}\|_{L^\infty_t(L^2)} + \|\sigma^{1-s_\ell} (\partial_t v_{\ell 2}, \nabla^2 v_{\ell 2}, \nabla p_{\ell 2})\|_{L^2_t(L^2)} \leq A_0(t) \left( \|g_\ell(0)\|_{B^{s_\ell-1}}^2 + \|\sigma^{1-s_\ell} (\partial_t \text{div } g_\ell, F_\ell(v, \pi))\|_{L^1_t(L^2)}^2 + \|\sigma^{1-s_\ell} \nabla g_\ell\|_{L^1_t(L^2)}^2 \right).$$

In view of (3.34), we get, by applying the law of product in Besov spaces (see [5] for instance), that

$$\|g_\ell(0)\|_{B^{s_\ell-1}} \leq C \sum_{i,j \leq t-1} \|v^i(0)\|_{B^{s_\ell-1}} \|\partial_X^j \nabla X_0 v_0\|_{B_{p,1}^{s_\ell}} \leq C \sum_{i,j \leq t-1} \|v^i(0)\|_{B^{s_\ell-1}} \|\partial_X^j \nabla X_0 v_0\|_{W^{1,p}} \leq C (\|v\|_{L^2} + \cdots, \|\partial_X^{t-1} v_0\|_{B^{s_\ell-1}}, \|X_0\|_{W^{2,p}} + \cdots, \|\partial_X^{t-1} X_0\|_{W^{2,p}}) \leq A_\ell.$$
And Lemma 6.4 ensures that
\[
\|\sigma^{-\frac{s}{2}}\partial_t g\|_{L_t^1(L^2)}^2 + \|\sigma^{-\frac{s}{2}}(|\mathbf{v}|^2)\nabla g\|_{L_t^1(L^2)}^2 + \|\sigma^{-\frac{s}{2}}\nabla g\|_{L_t^1(L^2)}^2 + \|\sigma^{-\frac{s}{2}}\partial_t g\|_{L_t^1(L^2)}^2
\]
\[
\lesssim (t)\|\sigma^{-\frac{s}{2}}\nabla g\|_{L_t^\infty(L^\infty)}^2 \int_0^t \sigma^{-\frac{s}{2}}(\|\nabla v\|_{L^2}^2) dt'
\]
\[
+ \mathcal{H}_{t-1}^2(t)\int_0^t \sigma^{-\frac{s}{2}}(1 + \|\nabla X^{-1}\|_{W^{1,p}}^2) dt'.
\]
Yet it follows from (6.38) that
\[
\int_0^t \sigma^{-\frac{s}{2}}(\|\nabla v\|_{L^2}^2) dt' \lesssim (t)\|\sigma^{-\frac{s}{2}}\nabla v\|_{L_t^\infty(L^2)}^2 + \int_0^t \sigma^{-\frac{s}{2}}(\|\nabla v\|_{L^2}^2) dt'.
\]
This together with (6.42) implies
\[
\|\sigma^{-\frac{s}{2}}\partial_t g\|_{L_t^1(L^2)}^2 + \|\sigma^{-\frac{s}{2}}(|\mathbf{v}|^2)\nabla g\|_{L_t^1(L^2)}^2 + \|\sigma^{-\frac{s}{2}}\nabla g\|_{L_t^1(L^2)}^2 + \|\sigma^{-\frac{s}{2}}\partial_t g\|_{L_t^1(L^2)}^2
\]
\[
\lesssim A(t)^2 + A_0(t) \int_0^t \sigma^{-\frac{s}{2}}(\|\nabla v\|_{L^2}^2) dt'
\]
\[
+ \mathcal{H}_{t-1}^2(t)\int_0^t \sigma^{-\frac{s}{2}}(1 + \|\nabla X^{-1}\|_{W^{1,p}}^2) dt'.
\]
Inserting the above estimates into (6.44) and applying Gronwall’s inequality gives rise to
\[
(6.45) \quad \|\sigma^{-\frac{s}{2}}\nabla v\|_{L_t^\infty(L^2)}^2 + \|\sigma^{-\frac{s}{2}}(\|\partial_t v\|_2^2 + \|\nabla v\|_2^2)\|_{L_t^1(L^2)}^2 \leq \mathcal{G}_t, X(t),
\]
for $\mathcal{G}_t, X(t)$ given by (6.43).

Let us now turn to the $L^2$ energy estimate of $v^\ell$. Similar to the derivation of (4.18), we get, by taking $L^2(\mathbb{R}^2)$ inner product of (6.40) with $v_{t_2}$ and making use of the fact: $\text{div} v_{t_2} = \text{div} g_t$, that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla v_{t_2}\|_{L^2}^2 + \|\nabla v_{t_2}\|_{L^2}^2 = \int_{\mathbb{R}^2} F_\ell(v, \pi)(v_{t_2} - g_t) dx
\]
\[
+ \int_{\mathbb{R}^2} (\rho |v| \nabla v_{t_2} - \nabla v_{t_2}) g_t dx.
\]
By virtue of Lemmas 6.3 and 6.4, we find
\[
\left| \int_{\mathbb{R}^2} F_\ell(v, \pi)(v_{t_2} - g_t) dx \right| \leq \sigma^{-\frac{s}{2}} \|F_\ell(v, \pi)\|_{L^2}^2 + \sigma^{-\frac{s}{2}} \|v_{t_2}\|_{L^2}^2 + \sigma^{-\frac{s}{2}} \|g_t\|_{L^2}^2
\]
\[
\leq \sigma^{-\frac{s}{2}} \|v_{t_2}\|_{L^2}^2 + \mathcal{H}_{t-1}(t) \sigma^{-\frac{s}{2}}(1 + \|\nabla X^{-1}\|_{W^{1,p}}^2),
\]
and
\[
\left| \int_{\mathbb{R}^2} (\rho |v| \nabla v_{t_2} - \nabla v_{t_2}) g_t dx \right| \lesssim \|v\|_{L^\infty} \sigma^{-\frac{s}{2}} \|\nabla v_{t_2}\|_{L^2}^2 + \mathcal{H}_{t-1}(t) \sigma^{-\frac{s}{2}}(1 + \|\nabla X^{-1}\|_{W^{1,p}}^2).
\]
Inserting the above inequalities into (6.46) and then applying Gronwall’s inequality to the resulting inequality, we achieve
\[
\|v_{t_2}\|_{L_t^\infty(L^2)}^2 + \|\nabla v_{t_2}\|_{L_t^1(L^2)}^2 \leq C \exp ((t)) \left( \|\sigma^{-\frac{s}{2}}(\|\partial_t v\|_2^2 + \|\nabla v\|_2^2)\|_{L_t^1(L^2)}^2 + \int_0^t \sigma^{-\frac{s}{2}}(1 + \|\nabla X^{-1}\|_{W^{1,p}}^2) dt' \right),
\]
which together with Corollary 2.1 and (6.45) ensures that

\[(6.47) \quad \|v_{\ell}\|_{L^2_t(L^2)}^2 + \|\nabla v_{\ell}\|_{L^2_t(L^2)}^2 \leq \bar{\Theta}_{\ell,X}(t).\]

By summing up (6.41), (6.42), (6.45) and (6.47), we conclude the proof of (6.43). □

**Corollary 6.1.** Under the assumptions of Proposition 6.1, one has

\[(6.48) \quad \|\sigma^{1-s}_L v^\ell\|_{L^\infty_t(L^2)}^2 + \|\sigma^{1-s}_L (D_t v^\ell, \nabla \partial_X^\ell \nabla v, D_t \partial_X^\ell \Delta X)\|_{L^2_t(L^2)}^2
\]

\[+ \|\sigma^{1-s}_L (\nabla v^\ell, \partial_X^\ell \nabla v, D_t \partial_X^\ell - 1 \nabla X)\|_{L^2_t(L^2 \rightarrow L^2)}^2 \leq \bar{\Theta}_{\ell,X}(t).\]

**Proof.** We first deduce from Proposition 6.1 and the 2-D interpolation inequality (2.20) that

\[\|\sigma^{1-s}_L v^\ell\|_{L^\infty_t(L^2)}^2 \leq C \left( \|v^\ell\|_{L^\infty_t(L^2)}^2 \|\sigma^{1-s}_L \nabla v^\ell\|_{L^\infty_t(L^2)}^2 \right)^2 \leq \bar{\Theta}_{\ell,X}(t).\]

It follows from Corollary 2.1 and Proposition 6.1 that

\[\|\sigma^{1-s}_L D_t v^\ell\|_{L^2_t(L^2)}^2 \leq \|\sigma^{1-s}_L \partial_t v^\ell\|_{L^2_t(L^2)}^2 + \|v\|_{L^2_t(L^\infty)}^2 \|\sigma^{1-s}_L \nabla v^\ell\|_{L^\infty_t(L^2)}^2 \leq \bar{\Theta}_{\ell,X}(t).\]

While it is easy to observe from (2.30) that

\[\nabla \partial_X^\ell \nabla v = \nabla \sum_{i=0}^{\ell-1} \partial_X^i [\partial_X; \nabla] v^{\ell-1-i} + \nabla^2 v^\ell = -\sum_{i=0}^{\ell-1} \nabla \partial_X^i (\nabla X \cdot \nabla v^{\ell-1-i}) + \nabla^2 v^\ell,\]

from which, we infer

\[\|\sigma^{1-s}_L (\nabla \partial_X^\ell \nabla v - \nabla^2 v^\ell)\|_{L^2_t(L^2)}^2 \leq \left( \|\sigma^{1-s}_L \nabla v^\ell\|_{L^2_t(L^2)}^2 + \|\sigma^{1-s}_L \nabla^2 v^\ell\|_{L^2_t(L^2)}^2 \right)^2 \times \int_0^\ell \sigma^{-(1-\theta_0)} \left( \|\nabla \partial_X^{\ell-1} \nabla X\|_{L^2_t(L^2)}^2 + \|\partial_X^{\ell-1} \nabla X\|_{L^\infty_t(L^2)}^2 \right) dt'
\]

\[+ (t) \sum_{l \leq \ell-2, i+j \leq \ell-1} \left( \|\nabla \partial_X^i \nabla X\|_{L^2_t(L^2)}^2 \|\sigma^{1-s}_L \partial_X^j \nabla v^\ell\|_{L^\infty_t(L^2)}^2 \right) + \|\partial_X^i \nabla X\|_{L^\infty_t(L^2)}^2 \|\sigma^{1-s}_L \partial_X^j \nabla v^\ell\|_{L^\infty_t(L^2)}^2 \right).\]

As a result, we deduce from Lemmas 6.1 and 6.2, and Proposition 6.1 that

\[(6.49) \quad \|\sigma^{1-s}_L \nabla \partial_X^\ell \nabla v\|_{L^2_t(L^2)}^2 \leq \bar{\Theta}_{\ell,X}(t).\]

Next due to \(D_t X^{\ell-1} = v^\ell\), we have

\[D_t \partial_X^{\ell-1} \Delta X = D_t \sum_{i=0}^{\ell-2} \partial_X^i [\partial_X; \Delta] X^{\ell-2-i} + D_t \Delta X^{\ell-1}\]

\[= -\sum_{i=0}^{\ell-2} D_t \partial_X^i (\Delta X \cdot \nabla X^{\ell-2-i} + 2 \partial_X^i X \cdot \nabla \partial_X X^{\ell-2-i}) \]

\[+ \Delta v^\ell - (\Delta v \cdot \nabla X^{\ell-1} + 2 \partial_X v \cdot \nabla \partial_X X^{\ell-1}),\]
and it thus comes out
\[
\| \sigma^{1-\frac{s}{p}} (D_t^{\ell-1} \Delta X - \Delta v^\ell) \|_{L_t^2 L_x^p}^2 
\leq \langle t \rangle \sum_{m+n+k+l \leq \ell-2} \left( \| \sigma^{1-\frac{s}{p} + \frac{1}{2} - \frac{l}{p} - \frac{m}{2} - \frac{n}{2} - \frac{k}{2} - \frac{l}{2} \|_{L_t^2 L_x^p} \right) \]
\]+ \left( \| \sigma^{1-\frac{s}{p}} \Delta v^\ell \|_{L_t^2 L_x^p} \right) \int_0^t \sigma^{-(1-\theta_0)} \| \nabla X^\ell-1 \|_{W^{1,p}}^2 \ dt',
\]

from which and Lemma 6.2, Proposition 6.1, we infer that
\[
\| \sigma^{1-\frac{s}{p}} D_t^{\ell-1} \Delta X \|_{L_t^2 L_x^p}^2 \leq \overline{\Theta}_{\ell,X}(t).
\]

On the other hand, by virtue of (6.43), we get from (2.20) that
\[
(6.50) \quad \| \sigma^{1-\frac{s}{p}} \nabla^2 v^\ell \|_{L_t^2 L_x^{2p}}^2 \leq C \left( \| \nabla v^\ell \|_{L_t^2 L_x^p} \right) \| \sigma^{1-\frac{s}{p}} \nabla^2 v^\ell \|_{L_t^2 L_x^p}^2 \leq \overline{\Theta}_{\ell,X}(t).
\]

Then, due to
\[
\partial_X^\ell \nabla v = \sum_{i=0}^{\ell-1} \partial_X^i [\partial_X; \nabla] v^{\ell-1-i} + \nabla v^\ell = -\sum_{i=0}^{\ell-1} \partial_X^i (\nabla X \cdot \nabla v^{\ell-1-i}) + \nabla v^\ell,
\]

we have (noticing \( p > 2 \))
\[
\| \sigma^{1-\frac{s}{p}} (\partial_X^\ell \nabla v - \nabla v^\ell) \|_{L_t^2 L_x^{2p}}^2 \leq \int_0^t \left( \| \sigma^{1+s_0 - \frac{s}{p}} \|_{L_t^2 L_x^{2p}} \right) \| \partial_X^{\ell-1} \nabla X \|_{L_x^\infty} \| \sigma^{1+\frac{1}{p} - \frac{2s}{p}} \nabla v \|_{L_t^2 L_x^p}^2 \dt' \]
\]+ \left( \| \sigma^{1-\frac{s}{p}} \nabla^2 v^\ell \|_{L_t^2 L_x^p} \right) \int_0^t \sigma^{-(1-\theta_0)} \| \nabla X^\ell-1 \|_{W^{1,p}}^2 \ dt',
\]

This together with Lemma 6.2 and (6.50) ensures that (6.50) also holds for \( \partial_X^\ell \nabla v \).

Finally observing from (2.4) and (2.30) that
\[
D_t^{\ell-1} \nabla X = D_t \sum_{i=0}^{\ell-2} \partial_X^i [\partial_X; \nabla] X^{\ell-2-i} + D_t \nabla X^{\ell-1}
\]
\[= -\sum_{i=0}^{\ell-2} D_t \partial_X^i (\nabla X \cdot \nabla X^{\ell-2-i}) + \nabla v^\ell - \nabla v \cdot \nabla X^{\ell-1},
\]

we infer from Lemma 6.2 and (6.50)
\[
\| \sigma^{1-\frac{s}{p}} (D_t \partial_X^{\ell-1} \nabla X - \Delta v^\ell) \|_{L_t^2 L_x^{2p}}^2 
\leq \int_0^t \left( \| \sigma^{1+s_0 - \frac{s}{p}} \|_{L_t^2 L_x^{2p}} \right) \| \nabla X^{\ell-1} \|_{L_x^\infty} \| \sigma^{1+\frac{1}{p} - \frac{2s}{p}} \nabla v \|_{L_t^2 L_x^p}^2 \dt' \]
\]+ \left( \| \sigma^{1-\frac{s}{p}} \nabla^2 v^\ell \|_{L_t^2 L_x^p} \right) \int_0^t \sigma^{-(1-\theta_0)} \| \nabla X^\ell-1 \|_{W^{1,p}}^2 \ dt',
\]

This completes the proof of (6.48). \( \square \)
7. Energy estimate of $D_t v^\ell$

The goal of this section is to derive the time-weighted $H^1$ energy estimate for $D_t v^\ell$. To this end, for $F_{\ell D}$ given by (2.42), we denote

$$
\widetilde{F}_{\ell D} \overset{\text{def}}{=} F_{\ell D} + 2 \partial_t v \cdot \nabla \partial_t v^\ell + \Delta v \cdot \nabla v^\ell - \nabla v \cdot \nabla v^\ell + \rho^2 D_t^2 v
+ 2D_t(\partial_{2-l}^\ell \partial_\alpha X \cdot \nabla \partial_\alpha v) + D_t(\partial_{2-l}^\ell \Delta X \cdot \nabla v) - D_t(\partial_{2-l}^\ell \Delta X \cdot \nabla v).
$$

Then by (2.27) and (2.42), we find

$$
|\widetilde{F}_{\ell D}| \lesssim \sum_{i+j \leq \ell-1, l \leq \ell-2} \left( |D_{\ell}^2 v^i| + |\partial_\chi D_t \nabla X||(|\partial_\chi \nabla^2 v^j| + |\partial_\chi \nabla v^j|)
+ |\partial_\chi \nabla X||(|\partial_\chi D_t \nabla v^2 v^j| + |\partial_\chi D_t \nabla v^j|)
+ |\partial_\chi D_t \Delta X||\partial_\chi \nabla v^j| + |\partial_\chi \Delta X||\partial_\chi D_t \nabla v^j|\right),
$$

which together with (6.5), (6.6) and Lemma 6.2 implies

$$
\|\widetilde{F}_{\ell D}\|_{L^2}^2 \leq \mathcal{H}_{\ell-1}(t) \sum_{i+j \leq \ell-1} \left( \|D_{\ell}^2 v^i, \partial_\chi D_t \nabla v^j, \partial_\chi D_t \nabla v^j\|_{L^2}^2 + \|\partial_\chi D_t \nabla v^j\|_{L^{2(2l-2)}}^2 \right)
+ C \sum_{i+j \leq \ell-1, l \leq \ell-2} \left( \|\partial_\chi D_t \nabla v^j, \partial_\chi \nabla v^j\|_{L^\infty}^2 \|\partial_\chi \nabla^2 v^j, \partial_\chi \nabla^2 v^j, \partial_\chi D_t \Delta X\|_{L^2}^2 \right)
\leq \mathcal{H}_{\ell-1}(t) \left( \sigma(t)^{-(3-6\ell-1)} \mathcal{B}_{\ell-1}(t) + \sigma(t)^{-(4-2s\ell-1)} \right).
$$

As a result, we deduce from (7.1) and Lemma 6.2 that

$$
\|\sigma^{3-6\ell} \widetilde{F}_{\ell D}\|_{L^2(L^2)}^2 \leq \mathcal{H}_{\ell-1}(t) \left( C \|\sigma \nabla v\|_{L^\infty(L^\infty)}^2 \|\sigma^{1-2\ell} (\nabla^2 v^\ell, \nabla v^\ell, D_t \partial_\chi^{\ell-1} \Delta X)\|_{L^2(L^2)}^2
+ C \|\sigma^{3-6\ell} (\nabla^2 v^\ell, \nabla v^\ell, D_t \partial_\chi^{\ell-1} \nabla v^\ell)\|_{L^2(L^2)}^2 \right)
\leq \mathcal{H}_{\ell-1}(t) \left( C \int_0^t \left( \|\sigma^{\frac{3-6\ell}{2}} (D_t^2 v, D_t \nabla v^\ell, D_t \nabla v^\ell)\|_{L^2}^2 + \|\sigma^{3-6\ell} D_t \nabla v^\ell\|_{L^{2(2l-2)}}^2 \right) \, dt'.
$$

By virtue of (2.27), Corollary 3.4, Proposition 6.1 and Corollary 6.1, we arrive at

**Lemma 7.1.** For $\ell = 2, \cdots, k$, there holds

$$
\|\sigma^{3-6\ell} F_{\ell D}\|_{L^2(L^2)}^2 \leq \mathcal{H}_{\ell-1}(t) \left( \mathcal{A}_t + \int_0^t \left( \mathcal{B}_0(t') + \sigma(t')^{-\left(1-\frac{6\ell}{2}\right)} \|\nabla X^{\ell-1}(t')\|_{W^{1,p}}^2 \right) \, dt' \right),
$$

for $\mathcal{B}_0(t)$ given by (2.36).

Let the operator $\mathcal{E}_X$ be given by (6.27). Then in view of (5.2), we have

$$
\mathcal{A} \overset{\text{def}}{=} D_t v \cdot \nabla X + \mathcal{E}_X (v \cdot \nabla v) \quad \text{and} \quad \text{div} D_t v^1 = \text{div} \mathcal{A}.
$$

Inductively, let us assume that $\text{div} D_t v^{\ell-1} = \text{div} \mathcal{A}_{\ell-1}$ for $\ell \geq 2$. We then deduce that

$$
\text{div} D_t v^\ell = \text{div} (\partial_\chi D_t v^{\ell-1}) = \text{div} (D_t v^{\ell-1} \cdot \nabla X + X \text{div} D_t v^{\ell-1})
= \text{div} (D_t v^{\ell-1} \cdot \nabla X + \mathcal{E}_X \mathcal{A}_{\ell-1}).
$$

This gives

$$
\mathcal{A}_\ell \overset{\text{def}}{=} D_t v^{\ell-1} \cdot \nabla X + \mathcal{E}_X \mathcal{A}_{\ell-1} \quad \text{and} \quad \text{div} D_t v^\ell = \text{div} \mathcal{A}_\ell.
$$
We thus get by induction that for \( \ell \geq 2 \),

\[
a_\ell = \sum_{i=0}^{\ell-1} \mathcal{E}_X^i(D_t v^{\ell-1-i} \cdot \nabla X) + \mathcal{E}_X^\ell (v \cdot \nabla v)
\]

(7.4)

\[
= \sum_{i=0}^{\ell-1} \sum_{i_1 + \cdots + i_r = i} \partial_X^{i_1} (D_t v^{\ell-1-i} \cdot \nabla X) \cdot \partial_X^{i_2} (-\nabla X)^{i_3} \cdots \partial_X^{i_r} (-\nabla X)^r + \partial_X^{\ell} (v \cdot \nabla v).
\]

The main result concerning the estimate of \( a_\ell \) is as follows:

**Lemma 7.2.** For \( \ell = 2, \cdots, k \), let \( a_\ell \) be given by (7.3). Then there holds

\[
\| \sigma^{1-s_\ell} a_\ell \|_{L^2_t(L^2)} + \| \sigma^{1-s_\ell} \nabla a_\ell \|_{L^2_t(L^2)} + \| \sigma^{1-s_\ell} \text{div} a_\ell \|_{L^2_t(L^2)} \leq \mathcal{G}_{t,X}(t).
\]

Here and in all that follows, for \( \mathcal{B}_t(t) \) determined by (6.7), we always designate

(7.6)

\[
\mathcal{G}_{t,X}(t) \equiv \mathcal{H}_{t-1}^2(\ell) \left( \mathcal{A}_\ell + \int_0^t \left( \mathcal{B}_{t-1}(t') + \sigma(t')^{-1} \| \nabla X_{t-1}^\ell(t') \|_{W^{1,p}}^2 \right) dt' \right).
\]

**Proof.** Let

(7.7)

\[
\mathcal{S}_\ell \equiv a_\ell - v \cdot \nabla v - v \cdot \partial_X \nabla v.
\]

Then we deduce from Lemma 6.2 and (7.4) that

\[
\| \mathcal{S}_\ell(t) \|_{L^2} \leq \left( \mathcal{H}_{t-1}(t) + \| \partial_X^{\ell-1} \nabla X \|_{L^\infty} \sigma(t)^{-1} \right) \times \sum_{i+j \leq \ell-1} \left( \| \sigma^{1-s_{i+j}} D_t v^i \|_{L^2} + \| \sigma^{1-s_{i+j}} \nabla v^i \|_{L^\infty} \| \sigma^{1-s_{i+j}} \partial_X \nabla v \|_{L^2} \right),
\]

which together with Lemma 6.2 implies

(7.8)

\[
\| \sigma^{1-s_{\ell}} \mathcal{S}_\ell(t) \|_{L^2}^2 \leq \mathcal{H}_{t-1}^2(\ell) \sigma(t)^{-1} \left( 1 + \| \nabla X_{t-1}^\ell \|_{W^{1,p}}^2 \right).
\]

To handle the estimate of \( \nabla \mathcal{S}_\ell \), by virtue of (7.4) and (7.7), we write

\[
\nabla \mathcal{S}_\ell = 3\ell_1 + 3\ell_2 + 3\ell_3 + \sum_{i+j=\ell \atop i \neq \ell, j \neq \ell} \nabla (v^i \cdot \partial_X \nabla v),
\]

where

\[
3\ell_1 \equiv \sum_{i=1}^{\ell-1} \sum_{i_1 + \cdots + i_r = i} \left( \partial_X^{i_1} (\nabla X \cdot \nabla \partial_X^{i_2} (D_t v^{\ell-1-i} \cdot \nabla X)) \cdots \partial_X^{i_r} (-\nabla X)^r, \right.
\]

\[
+ \partial_X^{i_1} (D_t v^{\ell-1-i} \cdot \nabla X) \cdots \partial_X^{i_{r-1}} (\nabla X \cdot \nabla \partial_X^{i_r} (-\nabla X)^{i_{r+1}}) \cdots \partial_X^{i_r} (-\nabla X)^r)
\]

\[
+ \partial_X^{i_1} (D_t v^{\ell-1-i} \cdot \nabla X) \cdot \partial_X^{i_2} (-\nabla X)^{i_3} \cdots \partial_X^{i_r} \nabla (-\nabla X) \cdots \partial_X^{i_r} (-\nabla X)^r \biggr),
\]
and
\[
3\ell_2 \overset{\text{def}}{=} \sum_{j_1 + \cdots + j_\ell = \ell - 1} \left( \partial_{X}^{j_1} (\nabla X \cdot \nabla \partial_{\theta}^{j_\ell}(v \cdot \nabla v)) \cdots \partial_{\theta}^{j_{\ell-1}} (-\nabla X)^{j_1} 
+ \partial_{X}^{j_1}(v \cdot \nabla v) \cdots \partial_{\theta}^{j_{\ell-1}}(-\nabla X)^{j_\ell} \right),
\]
and
\[
3\ell_3 \overset{\text{def}}{=} \sum_{\ell - 1}^{\ell - 1} \sum_{i=0}^{j_1 + \cdots + j_\ell = \ell} \partial_{X}^{j_i} \nabla (D_t v^{\ell - i} \cdot \nabla X) \cdots \partial_{\theta}^{j_{\ell-1}}(-\nabla X)^{j_i}
+ \sum_{j_1 + \cdots + j_\ell = \ell, j_\ell \neq \ell} \partial_{X}^{j_i} (v \cdot \nabla v) \cdots \partial_{\theta}^{j_{\ell-1}}(-\nabla X)^{j_i}.
\]
This gives rise to
\[
\|\nabla_3(t)\|^2_{L^2} \leq \mathcal{H}_{\ell-1}(t)(1 + \|\nabla X^{\ell-1}\|^2_{W^{1,p}}) \sum_{i+j_i \leq \ell-1} \left( \|\partial_{X}^{j_1} \nabla D_t v^i\|^2_{L^p} + \|D_t v^i\|^2_{L^p} \right)
+ \sigma(t)^{3-\frac{s}{2}} \|\nabla^{j_1} \nabla v^2\|^2_{L^p} \left( \|\nabla X^{\ell-1}\|^2_{L^p} \right)
\]
from which and Lemma 6.2, we infer that
\[
(7.9) \quad \|\sigma^{1-\frac{s}{2}} \nabla_3(t)\|^2_{L^2} \leq \mathcal{H}_{\ell-1}(t)(1 + \|\nabla X^{\ell-1}\|^2_{W^{1,p}})(\mathcal{B}_{\ell-1}(t) + \sigma(t)^{-(1-\theta_0)}).
\]
Taking into account of (7.7), we deduce from (7.8) and (7.9) that
\[
\|\sigma^{1-\frac{s}{2}} a^2\|^2_{L^2(L^2)} \leq \|\sigma^{\frac{1}{2}} \nabla v^2\|^2_{L^2(L^2)} \|\sigma^{\frac{1}{2}} \nabla v^2\|^2_{L^2(L^2)}
+ \|v^2\|^2_{L^2(L^2)} \|\sigma^{1-\frac{s}{2}} \nabla v^2\|^2_{L^2(L^2)} + \mathcal{H}_{\ell-1}(t) \left( 1 + \int_{0}^{t} \sigma(t')^{-(1-\theta_0)} \|\nabla X^{\ell-1}(t')\|^2_{W^{1,p}} \mathrm{d}t' \right),
\]
and
\[
\|\sigma^{1-\frac{s}{2}} \nabla a\|^2_{L^2(L^2)} \leq \|\sigma^{\frac{1}{2}} \nabla v^2\|^2_{L^2(L^2)} \|\sigma^{1-\frac{s}{2}} \nabla v^2\|^2_{L^2(L^2)}
+ \|v^2\|^2_{L^2(L^2)} \|\sigma^{1-\frac{s}{2}} \nabla v^2\|^2_{L^2(L^2)} + \mathcal{H}_{\ell-1}(t) \left( 1 + \int_{0}^{t} \mathcal{B}_{\ell-1}(t') + \sigma(t')^{-(1-\theta_0)} \|\nabla X^{\ell-1}(t')\|^2_{W^{1,p}} \mathrm{d}t' \right).
\]
Together with Corollary 2.1, Proposition 6.1 and Corollary 6.1, we conclude
\[
(7.10) \quad \|\sigma^{1-\frac{s}{2}} a^2\|^2_{L^2(L^2)} + \|\sigma^{1-\frac{s}{2}} \nabla a\|^2_{L^2(L^2)}
\leq \mathcal{H}_{\ell-1}(t) \left( A + \int_{0}^{t} (\mathcal{B}_{\ell-1}(t') + \sigma(t')^{-(1-\theta_0)}) \|\nabla X^{\ell-1}(t')\|^2_{W^{1,p}} \mathrm{d}t' \right).
\]
Finally let us turn to the estimate of $\nabla \text{div} \, \mathbf{a}_\ell$. Indeed due to $\text{div} \, E_X f = \partial_X \text{div} \, f$ and (7.4), we write
\[
\text{div} \, \mathbf{a}_\ell = \sum_{i=0}^{\ell-1} \partial^i_X \text{div} \, (D_t v^{\ell-1-i} \cdot \nabla X) + \partial^\ell_X \text{div} \, (v \cdot \nabla v)
\]
so that it comes out
\[
\| \sigma^{3-s/4} \nabla \text{div} \, \mathbf{a}_\ell \|_{L_t^2(L^2)} \leq \left( \int_0^t \left( \| \sigma^{3-s/4} \nabla^2 \partial^i_X \text{div} \, (D_t v^{\ell-1-i} \cdot \nabla X) \|_{L_t^2(L^2)} + \| \sigma^{3-s/4} \nabla \partial^i_X \text{div} \, (v \cdot \nabla v) \|_{L_t^2(L^2)} \right) \, dt \right)^{\frac{1}{2}} + C R_{\ell-1}(t) \sum_{i+j \leq \ell-1} \left( \| \sigma^{3-s/4} \nabla \partial^i_X \nabla D_t v^j \|_{L_t^2(L^2)} + \| \sigma^{3-s/4} \partial^i_X \nabla D_t v^j \|_{L_t^2(L^2)} \right)
\]
Then by virtue of (7.10) and Corollary 3.4, Lemma 6.2, (6.8), Proposition 6.1 and Corollary 6.1, we complete the proof of (7.5).

In view of Lemma 2.1, to close the time-weighted $H^1$ energy estimate of $D_t v^\ell$, we also need the estimate of $\text{div} \, D_t^2 v^\ell$. Indeed taking into account of (5.3), we inductively assume that $\text{div} \, D_t^2 v^{\ell-1} = \text{div} \, \mathbf{b}_{\ell-1}$, then one has
\[
(7.11) \quad \text{div} \, D_t^2 v^\ell = \text{div} \, (D_t^2 v^{\ell-1} \cdot \nabla X + X \text{div} \, D_t^2 v^{\ell-1})
\]
\[
= \text{div} \, (D_t^2 v^{\ell-1} \cdot \nabla X + E_X \mathbf{b}_{\ell-1}) \equiv \text{div} \, \mathbf{b}_\ell.
\]

**Lemma 7.3.** Let $\mathbf{b}_\ell$ be given by (7.11). Then for $\ell = 2, \ldots, k$, one has
\[
(7.12) \quad \| \sigma^{3-s/4} \mathbf{b}_\ell \|_{L_t^2(L^2)} \leq C \varepsilon \| \sigma^{3-s/4} \nabla \text{div} \, D_t^2 v^\ell \|_{L_t^2(L^2)} + C \varepsilon \mathcal{G}_{\ell,X}(t),
\]
with $\mathcal{G}_{\ell,X}(t)$ given by (7.6).

**Proof.** For $\mathbf{b}_0$ given by (2.23), we thus get, by using induction to (7.11), that
\[
\mathbf{b}_\ell = \sum_{i=0}^{\ell-1} \mathcal{E}_X^i(D_t^2 v^{\ell-1-i} \cdot \nabla X) + \mathcal{G}_X^\ell \mathbf{b}_0
\]
\[
= \sum_{i=0}^{\ell-1} \sum_{i_1 + \cdots + i_r = i} \partial_X^{i_1}(D_t^2 v^{\ell-1-i} \cdot \nabla X) \cdots \partial_X^{i_r}(-\nabla X)^{i_r} + \sum_{j_1 + \cdots + j_l = \ell} \partial_X^{j_1} \mathbf{b}_0 \cdots \partial_X^{j_l}(-\nabla X)^{j_l}.
\]
It is easy to observe from (2.23) that
\[
\mathbf{b}_0 = v \cdot (\nabla v_t + D_t \nabla v) + D_t v \cdot \nabla v = 2v \cdot D_t \nabla v - v \cdot \nabla (v \cdot \nabla v) + D_t v \cdot \nabla v,
\]
we obtain
\[
\|\sigma^\frac{3-\varepsilon}{2} b_\ell\|^2_{L^2_t(L^2)} \leq \int_0^t \|\sigma^\frac{3-\varepsilon}{2} (D_t^2 v, b_0)\|^2_{L^2} \|\nabla X^\ell-1\|^2_{L^\infty} \, dt' \\
+ 2\|\sigma^\frac{1}{2} v\|^2_{L^\infty_t(L^\infty)} \|\sigma^\frac{1-\varepsilon}{2} \partial_X^\ell D_t v\|^2_{L^2_t(L^2)} + \|\sigma v\|^2_{L^\infty_t(L^\infty)} \|\sigma^\frac{1-\varepsilon}{2} D_t v\|^2_{L^2_t(L^2)} \\
+ C\|\frac{\partial}{\partial y} v\|^2 \|\sigma^\frac{3-\varepsilon}{2} \frac{2\partial_X^\ell}{\partial_y} (D_t v, \nabla (v \otimes \nabla v))\|^2_{L^2_t(L^p)} \\
+ \|\sigma \partial_X^\ell \nabla v\|^2_{L^2_t(L^{2p/2})} \|\sigma^\frac{3-\varepsilon}{2} \frac{1}{p} \partial_X^\ell \nabla (v \otimes \nabla v, D_t v)\|^2_{L^2_t(L^p)} \\
+ C(1 + R_{t\ell-1}(t))^2 \sum_{i,j \leq \ell - 1} \left( \langle t \rangle \|\sigma^\frac{3-\varepsilon}{2} D_t v\|^2_{L^\infty_t(L^\infty)} \|\sigma \partial_X^i \nabla (v \cdot \nabla v)\|^2_{L^\infty_t(L^\infty)} \\
+ \|\sigma^\frac{3-\varepsilon}{2} D_t^2 v^j\|^2_{L^2_t(L^2)} + \|\sigma v\|^2_{L^\infty_t(L^\infty)} \|\sigma^\frac{1-\varepsilon}{2} (\partial_X^i \nabla (v \cdot \nabla v), D_t \partial_X^j \nabla v)\|^2_{L^2_t(L^2)} \right).
\]

Noticing that
\[
\partial_X^\ell D_t \nabla v = \partial_X^\ell [D_t; \nabla] v + \sum_{i=0}^{\ell - 1} \partial_X^i \partial_X^{\ell - 1 - i} v D_t v^{\ell - 1 - i} + \nabla D_t v^\ell,
\]
we obtain
\[
\|\sigma^\frac{3-\varepsilon}{2} \partial_X^\ell D_t v\|^2_{L^2_t(L^2)} \leq \|\sigma^\frac{3-\varepsilon}{2} \nabla D_t v\|^2_{L^2_t(L^2)} + 2\|\sigma^\frac{1}{2} \partial_X^\ell \nabla v\|^2_{L^2_t(L^{2p/2})} \|\sigma^\frac{3-\varepsilon}{2} \frac{2\partial_X^\ell}{\partial_y} \nabla v\|^2_{L^\infty_t(L^p)} \\
+ \left( \int_0^t \|\sigma^\frac{3-\varepsilon}{2} D_t v\|^2_{L^2} \|\nabla X^\ell-1\|^2_{L^\infty} \, dt' \right)^\frac{1}{2} \\
+ \sum_{i,j \leq \ell - 1} \|\sigma \partial_X^i \nabla v\|^2_{L^\infty_t(L^\infty)} \|\sigma^\frac{1-\varepsilon}{2} \partial_X^\ell \nabla v\|^2_{L^2_t(L^2)} + \|\sigma^\frac{3-\varepsilon}{2} \partial_X^\ell \nabla D_t v^j\|^2_{L^2_t(L^2)},
\]
from which, Corollary 2.1, Lemma 6.2 and Corollary 6.1, we infer
\[
\|\sigma^\frac{3-\varepsilon}{2} \partial_X^\ell D_t v\|^2_{L^2_t(L^2)} \leq \|\sigma^\frac{3-\varepsilon}{2} \nabla D_t v\|^2_{L^2_t(L^2)} \\
+ \tilde{g}_{\ell, X}(t) + \int_0^t \|\sigma^\frac{3-\varepsilon}{2} \nabla D_t v\|^2_{L^2} \|\nabla X^\ell-1\|^2_{L^\infty} \, dt'.
\]

Yet for any \( \varepsilon > 0 \), we get, by applying (3.35) to the equation (2.41), that
\[
\|\sigma^\frac{3-\varepsilon}{2} \nabla D_t v\|^2_{L^2_t(L^2)} \leq C\|\sigma^\frac{1-\varepsilon}{2} D_t v\|^2_{L^2_t(L^2)} + \varepsilon\|\sigma^\frac{3-\varepsilon}{2} D_t^2 v^\ell\|^2_{L^2_t(L^2)} \\
+ C\varepsilon \left( \|\sigma^\frac{3-\varepsilon}{2} a_\ell\|^2_{L^2_t(L^2)} + \|\sigma^\frac{3-\varepsilon}{2} \nabla a_\ell\|^2_{L^2_t(L^2)} + \|\sigma^\frac{3-\varepsilon}{2} F_{\ell D}\|^2_{L^2_t(L^2)} \right).
\]
Inserting the above estimate into (7.14) and using Corollary 6.1, Lemmas 7.1 and 7.2, we achieve
\[
\|\sigma^\frac{3-\varepsilon}{2} \partial_X^\ell D_t v\|^2_{L^2_t(L^2)} \leq \varepsilon\|\sigma^\frac{3-\varepsilon}{2} D_t^2 v^\ell\|^2_{L^2_t(L^2)} + C\varepsilon \tilde{g}_{\ell, X}(t).
\]
Finally, we notice that
\[
\sum_{j \leq \ell - 1} \left\| \sigma^{1 - \frac{s}{2}} \partial_X^j \nabla (v \cdot \nabla v) \right\|_{L^2_t(L^2)} \leq C(t) \frac{1}{2} \sum_{i + \ell \leq \ell - 1} \left( \left\| \sigma^{\frac{1}{2}} v_i \right\|_{L^\infty_t(L^\infty)} \left\| \sigma^{1 - \frac{s}{2}} \partial_X^i \nabla v \right\|_{L^\infty_t(L^2)} + \left\| \sigma^{1 - \frac{s}{2}} \partial_X^i \nabla v \right\|_{L^\infty_t(L^2)} \right).
\]
Substituting the above inequality and (7.15) into (7.13) and applying Gronwall’s inequality to
\[
A_{t \ell}^2(t) \leq A_0 \left( \left\| \sigma^{\frac{s}{2}} D_t v \right\|^2_{L^2_t(L^2)} + \left\| \sigma^{1 - \frac{s}{2}} \nabla v \right\|^2_{L^2_t(L^2)} + \left\| \sigma^{\frac{s}{2}} \nabla \div (a_{t \ell}, b_{t \ell}, F_{t \ell}) \right\|^2_{L^2_t(L^2)} \right),
\]
from which, we infer
\[
A_{t \ell}^2(t) + A_{t \ell}^2(t) \leq C_\varepsilon \left\| \sigma^{\frac{s}{2}} D_t v \right\|^2_{L^2_t(L^2)} + C_\varepsilon \mathcal{G}_{t, X}(t),
\]
for \(A_{t \ell}(t), A_{t \ell}^2(t)\) and \(\mathcal{G}_{t, X}(t)\) given by (2.9) and (7.12) respectively. Taking \(\varepsilon\) so small that
\[
C_\varepsilon \leq \frac{1}{2}
\]
ensures
\[
A_{0 \ell}(t) + A_{0 \ell}(t) \leq \mathcal{G}_{t, X}(t).
\]
By virtue of (7.17), we get, by applying (2.20), that for any \(r \in [2, \infty]\)
\[
\left\| \sigma^{1 - \frac{s}{2}} D_t v \right\|_{L^r_t(L^r)} \leq \mathcal{G}_{t, X}(t),
\]
which together with the fact: \(p < 2/(1 - s_\ell)\), ensures that
\[
\left\| D_t v \right\|_{L^p_t(L^p)} = \left\| \sigma^{1 - \frac{s}{2}} D_t v \right\|_{L^2_t(L^p)} \left\| \sigma^{1 - \frac{s}{2}} \right\|_{L^p_t(\mathbb{R}^2)} \leq \mathcal{G}_{t, X}(t).
\]
On the other hand, taking into account of (6.21) and \(v^\ell = X \cdot \nabla v^\ell - 1\), we get, by using the \(L^p\) type energy estimate and Lemma 6.2, that
\[
\|X^\ell(t)\|_{L^p} \leq \|\partial_X^\ell 0\|_{L^p} + \|v^\ell\|_{L^1_t(L^p)} \leq \|\partial_X^\ell 0\|_{L^p} + \|X\|_{L^\infty_t(L^\infty)} \left\| \sigma^{1 - \frac{s}{2}} \nabla v^\ell - 1 \right\|_{L^\infty_t(L^p)} \leq A_\ell + \mathcal{H}_{t - 1}(t).
\]
Applying \(\Delta\) to (6.21) gives
\[
\partial_\ell \Delta X^\ell - v \cdot \nabla \Delta X^\ell = -\Delta v \cdot \nabla X^\ell - 2\partial_\alpha v \cdot \nabla \partial_\alpha X^\ell + \Delta v^\ell,
\]
from which, we infer
\[
\|\Delta X^\ell(t)\|_{L^p} \leq \|\Delta \partial_X^\ell 0\|_{L^p} + \int_0^t \left( \|\Delta v\|_{L^p} + \|\nabla v\|_{L^\infty} \right) \|\nabla X^\ell(t)\|_{W^{1,p}} dt + \|\Delta v^\ell\|_{L^1_t(L^p)}.
\]
Summing up the above inequality with (7.19), and then applying Gronwall’s inequality to the resulting inequality and using Corollary 2.1, we achieve
\[
\|X^\ell\|_{L^\infty_t(W^{2,p})} \leq \exp(A_0) \left( A_\ell + \mathcal{H}_{t - 1}(t) + \|\Delta v^\ell\|_{L^1_t(L^p)} \right),
\]
which together with the second inequality of (6.30) ensures that
\[
\|X^{\ell-1}\|_{L^\infty_p(W^{2,p})} \leq H_{\ell-1}(t)\left(\mathcal{A}_\ell + \|D_t v^\ell\|_{L^1_p(L^p)} + \int_0^t \sigma^{-\left(\frac{3}{2} - \frac{5}{p} - \frac{\nu}{2}\right)} \|\nabla X^{\ell-1}\|_{W^{1,p}} \, dt\right).
\]

Inserting (7.18) into the above inequality and using the fact that (noticing also \(\frac{1}{p} > \frac{1-\nu}{2}\))
\[
\int_0^t \sigma^{-\left(\frac{3}{2} - \frac{5}{p} - \frac{\nu}{2}\right)} \|\nabla X^{\ell-1}\|_{W^{1,p}} \, dt \leq \langle t \rangle^{\frac{3}{2}}\left(\int_0^t \sigma^{-\left(1 - \frac{\nu}{2}\right)} \|\nabla X^{\ell-1}\|_{W^{1,p}}^2 \, dt\right)^{\frac{1}{2}},
\]
we infer
\[
\|X^{\ell-1}\|_{L^\infty_p(W^{2,p})} \leq \Theta_{\ell,X}(t).
\]
Taking into account of the definition of \(\Theta_{\ell,X}(t)\) given by (7.6), we get, by applying Gronwall’s inequality and Lemma 6.2, that
\[
\|X^{\ell-1}\|_{L^\infty_p(W^{2,p})} \leq H(t).
\]
Substituting (7.20) into (7.17) leads to (7.16). This completes the proof of Proposition 7.1. \(\square\)

With Proposition 7.1, to complete the proof of Proposition 2.5, it remains to prove that
\[
\|v^\ell\|_{L^\infty_p(B^{\ell})} + \|\nabla v^\ell\|_{L^2_p(B^{\ell})} \leq H(t).
\]
The proof of (7.21) follows exactly the same argument as those in proofs of Propositions 3.3 and 5.2. It is quite involved but does not contain new ideas. We skip the details here.

**APPENDIX A. PROOF OF LEMMA 6.1**

In this appendix we present the proof of Lemma 6.1, which basically follows the same arguments as the proof of Lemma 4.1 in [21].

Let us first present some identities which will be used in what follows. Let \((m, n)\) be nonnegative integer pair such that \(m + n \leq \ell - 1\), \(f\) be a smooth enough function. Then it follows from (2.30) that
\[
\partial_X^{m+1} \nabla \partial_X^{\ell-m-1} f - \nabla \partial_X^\ell f = \sum_{\kappa=0}^m \partial_X^\kappa [\partial_X; \nabla] \partial_X^{\ell-\kappa-1} f
\]
(A.1)
\[
= - \sum_{\kappa=0}^m \sum_{l=0}^m C^l_{\kappa} \partial_X^l \nabla X_\alpha \partial_X^{\kappa-l} \partial_X^{\ell-\kappa-1} f;
\]
and
\[
\partial_X^{m+1} \nabla^2 \partial_X^{\ell-m-1} f - \nabla^2 \partial_X^\ell f = \sum_{\kappa=0}^m \partial_X^\kappa [\partial_X; \nabla^2] \partial_X^{\ell-\kappa-1} f
\]
(A.2)
\[
= - \sum_{\kappa=0}^m \sum_{l=0}^m C^l_{\kappa} \left(\partial_X^l \nabla^2 X_\alpha \partial_X^{\kappa-l-1} \partial_X^{\ell-\kappa-1} f + 2 \partial_X^l \nabla X_\alpha \partial_X^{\kappa-l} \nabla \partial_X^{\ell-\kappa-1} f\right);
and
\[
\partial_X^m \nabla X^{n+1} \nabla X^{\ell-m-n-1} - \partial_X^m \nabla^2 X^{\ell-m} f = \sum_{k=0}^n \partial_X^m \nabla X^k [\partial X; \nabla] \partial_X^{\ell-m-k-1} f
\]
\[= - \sum_{k=0}^n \sum_{\kappa=0}^k C^\kappa_X \nabla \partial_X^k \nabla (\partial_X^k \nabla X_\alpha \partial_X^{\kappa-l-m-q} \partial_\alpha \partial_X^{\ell-m-k-1} f)
\]
(A.3)
\[= - \sum_{q=0}^m \sum_{\kappa=0}^q \sum_{\ell=0}^\infty C_m \partial_X^q \nabla \partial_X^q \nabla X_\alpha \partial_X^{\kappa-l-m-q} \partial_\alpha \partial_X^{\ell-m-k-1} f
\]
\[+ \partial_X^{q+1} \nabla X_\alpha \partial_X^{\kappa-l-m-q} \partial_\alpha \partial_X^{\ell-m-k-1} f\);  
and
\[
\partial_X^{m+1} \nabla X^{n+1} \nabla X^{\ell-m-n-1} f - \nabla \partial_X^{m+n+1} \nabla X^{\ell-m-n-1} f
\]
\[= \sum_{k=0}^m \partial_X^k [\partial X; \nabla] \partial_X^{m-k+n} \nabla \partial_X^{\ell-m-n-1} f
\]
(A.4)
\[= \sum_{k=0}^m \sum_{\kappa=0}^k C^\kappa_X \partial_X^k \nabla X_\alpha \partial_X^{k-l} \partial_\alpha \partial_X^{m-k+n} \nabla \partial_X^{\ell-m-n-1} f.
\]

A.1. Proof of (6.8). We shall first present the estimate to the second term in (6.8). It is easy to observe that (6.8) holds trivially when \(i + j = 0\). Suppose by induction that (6.8) is valid for any pair of nonnegative integers \((i, j)\) with \(i + j \leq \ell - 1\), and it suffices to show (6.8) in the following two cases:

- Case \(i \leq \ell - 1, j \leq \ell\) such that \(i + j \leq \ell\) \quad or \quad \(i \leq \ell, j \leq \ell - 1\) such that \(i + j \leq \ell\).

- We first deduce from (A.3) and (6.2) that for nonnegative integers \(m, n \leq \ell - 1\) with \(m+n \leq \ell-1\)

\[
\| \partial_X^m \nabla X^{n+1} \nabla X^{\ell-m-n-1} - \partial_X^m \nabla^2 X^{\ell-m} \|_{L^\infty(L^p)}
\]
\[\leq C \sum_{q=0}^m \sum_{\kappa=0}^q \sum_{\ell=0}^\infty (\| \partial_X^q \nabla \partial_X^q \nabla X \|_{L^\infty(L^p)} \| \partial_X^{\kappa-l-m-q} \nabla X^{\ell-m-k-1} \|_{L^\infty(L^\infty)})
\]
\[+ \| \partial_X^{q+1} \nabla X \|_{L^\infty(L^\infty)} \| \partial_X^{m-q} \nabla X^{k-l-m-q} \nabla X^{\ell-m-k-1} \|_{L^\infty(L^p)})
\]
\[\leq C \sum_{\ell \leq \ell-1} R_{\ell+1}(t) R_{\ell-1}(t) \leq CR_{\ell}^2(t).
\]

This in turn shows that for any pair of nonnegative integers \((i, j)\) with \(i \leq \ell - 1, j \leq \ell, i + j \leq \ell\),

\[
\| \partial_X^i \nabla X^j \nabla X^{\ell-i-j} - \partial_X^i \nabla^2 X^{\ell-i} \|_{L^\infty(L^p)} \leq CR_{\ell}^2(t).
\]
(A.5)

Taking \(i = 0\) in (A.5) leads to

\[
\| \nabla \partial_X^j \nabla X^{\ell-j} - \nabla^2 X^{\ell} \|_{L^\infty(L^p)} \leq CR_{\ell}^2(t), \quad \forall j \leq \ell.
\]
(A.6)

- Case \(i \leq \ell, j \leq \ell - 1\) with \(i + j \leq \ell\).
It follows from (A.4) that for nonnegative integer pair \((m, n)\) satisfying \(m + n \leq \ell - 1\),
\[
\| \partial_X^{m+1} \nabla \partial_X^n \nabla X^{\ell-m-n-1} - \nabla \partial_X^{m+1} \nabla X^{\ell-m-n-1} \|_{L^p} \\
\leq \sum_{\kappa=0}^m \sum_{l=0}^\kappa C_l \| \partial_X^l \nabla X \|_{L^\infty} \| \partial_X^{\kappa-l} \nabla \partial_X^{m+1-n} \nabla X^{\ell-m-n-1} \|_{L^p} \leq CR_2^2(t).
\]
This together with (A.6) ensures that for \((i, j)\) satisfying \(i \leq \ell, j \leq \ell - 1\), and \(i + j \leq \ell\),
\[
\| \partial_X^i \nabla \partial_X^j \nabla X^{\ell-i-j} - \nabla^2 X^\ell \|_{L^\infty(L^p)} \leq \| \partial_X^i \nabla \partial_X^j \nabla X^{\ell-i-j} - \nabla \partial_X^{i+j} \nabla X^{\ell-i-j} \|_{L^\infty(L^p)} \\
+ \| \nabla \partial_X^{i+j} \nabla X^{\ell-i-j} - \nabla^2 X^\ell \|_{L^\infty(L^p)} \leq CR_2^2(t) .
\]
(A.7)
Taking \(j = 0\) in (A.7) gives rise to
\[
\| \partial_X^i \nabla^2 X^\ell - \nabla^2 X^\ell \|_{L^\infty(L^p)} \leq CR_2^2(t), \quad \forall i \leq \ell .
\]
(A.8)
Combining (A.5) with (A.8), we obtain for \((i, j)\) satisfying \(i \leq \ell - 1, j \leq \ell\), and \(i + j \leq \ell\) that
\[
\| \partial_X^i \nabla \partial_X^j \nabla X^{\ell-i-j} - \nabla^2 X^\ell \|_{L^\infty(L^p)} \leq \| \partial_X^i \nabla \partial_X^j \nabla X^{\ell-i-j} - \partial_X^i \nabla^2 X^{\ell-i} \|_{L^\infty(L^p)} \\
+ \| \partial_X^i \nabla^2 X^{\ell-i} - \nabla^2 X^\ell \|_{L^\infty(L^p)} \leq CR_2^2(t),
\]
which together with (A.7) ensures that (A.7) holds for all nonnegative integer pair \((i, j)\) satisfying \(i + j \leq \ell\).
Along the same line, it follows from (A.1) that for any integer \(0 \leq m \leq \ell - 1\),
\[
\| \partial_X^{m+1} \nabla X^{\ell-m-1} - \nabla X^\ell \|_{L^\infty(L^p)} \leq C \sum_{\kappa=0}^m \sum_{l=0}^\kappa \| \partial_X^l \nabla X \|_{L^\infty(L^\infty)} \| \partial_X^{\kappa-l} \nabla X^{\ell-k-1} \|_{L^\infty(L^p)} ,
\]
which together with the definition of \(R_2(t)\) given by (6.2) implies
\[
\| \partial_X^i \nabla X^{\ell-i} - \nabla X^\ell \|_{L^\infty(L^p)} \leq CR_2^2(t), \quad \forall 0 \leq i \leq \ell .
\]
This together with (A.7) shows (6.8).
A.2. Proof of (6.9).
A.2.1. The estimates of \((\partial_X^i \nabla v^{\ell-i} - \nabla v^\ell)\) and \((\partial_X^i \nabla \pi^{\ell-i} - \nabla \pi^\ell)\). It is easy to observe from (A.1) that for any nonnegative integer \(m \leq \ell - 1\) and for any \(r \in [1, +\infty]\) that
\[
\| \partial_X^{m+1} \nabla v^{\ell-m-1} - \nabla v^\ell \|_{L^r} \leq C \sum_{\kappa=0}^m \sum_{l=0}^\kappa \| \partial_X^l \nabla X \|_{L^\infty(L^\infty)} \| \partial_X^{\kappa-l} \nabla v^{\ell-k-1} \|_{L^r} , \quad \forall m \leq \ell - 1 .
\]
Hence we get for \(r_1 \in \{2, \frac{2p}{p-2}, +\infty\}\) and for \(1 \leq i \leq \ell\),
\[
\| \sigma^{(1 - \frac{1}{r} - \frac{\ell-1}{2})} (\partial_X^i \nabla v^{\ell-i} - \nabla v^\ell) \|_{L^\infty(L^{r_1})} \\
\leq C \sum_{\kappa=0}^i \sum_{l=0}^\kappa \| \partial_X^l \nabla X \|_{L^\infty(L^\infty)} \| \sigma^{(1 - \frac{1}{r_1} - \frac{\ell-1}{2})} \partial_X^{\kappa-l} \nabla v^{\ell-k-1} \|_{L^\infty(L^{r_1})} \leq CR_2(t) \mathfrak{A}_{\ell-1}(t) .
\]
(A.9)
Along the same line, we obtain for \(r \in \{2, p\}\),
\[
\| \sigma^{\left(\frac{1}{2} - \frac{1}{r} - \frac{\ell-1}{2}\right)} (\partial_X^i \nabla \pi^{\ell-i} - \nabla \pi^\ell) \|_{L^\infty(L^r)} \leq CR_2(t) \mathfrak{A}_{\ell-1}(t) .
\]
(A.10)
A.2.2. The estimate of \( (\partial^m_X \nabla \partial^i_X \nabla^\ell-i-j - \nabla^2 v^\ell) \). It follows from (A.3) that for the nonnegative integer pair \((m, n)\) with \(m + n \leq \ell - 1\) and for \(r \in \{2, p\}\)

\[
\| \sigma \left( \frac{3}{2} - \frac{\ell - 1}{2} \right) \left( \partial^m_X \nabla \partial^{m+1}_X \nabla^\ell-m-n-1 - \partial^m_X \nabla^2 v^\ell-m \right) \|_{L^r} \\
\leq C \sum_{q=0}^m \sum_{\kappa=0}^n \sum_{l=0}^\kappa \left\| \partial^q_X \nabla \partial^l_X \nabla X \right\|_{L^r} \| \sigma \left( \frac{3}{2} - \frac{\ell - 1}{2} \right) \partial^{\kappa-l+m-q}_X \nabla^\ell-m-k-1 \|_{L^{\frac{2p}{r}}} \|

+ \left\| \partial^{l+q}_X \nabla X \right\|_{L^\infty} \| \sigma \left( \frac{3}{2} - \frac{\ell - 1}{2} \right) \partial^{m+q}_X \nabla \partial^{\kappa-l}_X \nabla^\ell-m-k-1 \|_{L^r},
\]

which together with (6.2) and (6.5) implies that for \(r \in \{2, p\}\) and any nonnegative integer pair \((i, j)\) satisfying \(i \leq \ell - 1, j \leq \ell\) with \(i + j \leq \ell\)

\[
(A.11) \quad \| \sigma \left( \frac{3}{2} - \frac{\ell - 1}{2} \right) \left( \partial^i_X \nabla \partial^j_X \nabla^\ell-i-j - \partial^i_X \nabla^2 v^\ell-i \right) \|_{L^\infty(L^r)} \leq CR_\ell(t) \mathcal{A}_{\ell-1}(t).
\]

Similarly for non-negative integer pair \((m, n)\) satisfying \(m + n \leq \ell - 1\), we deduce from (A.4) that for \(r \in \{2, p\}\)

\[
\| \partial^{m+1}_X \nabla \partial^m_X \nabla^\ell-m-n-1 - \nabla \partial^{m+n+1}_X \nabla^\ell-m-n-1 \|_{L^r} \\
\leq C \sum_{\kappa=0}^m \sum_{l=0}^\kappa \left\| \partial^l_X \nabla X \right\|_{L^\infty} \| \partial^{\kappa-l}_X \nabla \partial^{m+n}_X \nabla^\ell-m-n-1 \|_{L^r},
\]

which ensures that for \(r \in \{2, p\}\) and any nonnegative integer pair \((i, j)\) satisfying \(i \leq \ell, j \leq \ell - 1\) with \(i + j \leq \ell\)

\[
(A.12) \quad \| \sigma \left( \frac{3}{2} - \frac{\ell - 1}{2} \right) \left( \partial^i_X \nabla \partial^j_X \nabla^\ell-i-j - \nabla \partial^i_X \nabla^2 v^\ell-i-j \right) \|_{L^\infty(L^r)} \leq CR_\ell(t) \mathcal{A}_{\ell-1}(t).
\]

With (A.11) and (A.12), we can repeat the proof of (6.8) to conclude that for any integer pair \((i, j)\) satisfying \(i + j \leq \ell\) and \(r \in \{2, p\}\)

\[
(A.13) \quad \| \sigma \left( \frac{3}{2} - \frac{\ell - 1}{2} \right) \partial^i_X \nabla \partial^j_X \nabla^\ell-i-j - \nabla^2 v^\ell \|_{L^\infty(L^r)} \leq CR_\ell(t) \mathcal{A}_{\ell-1}(t).
\]

Along with (A.9) and (A.10), we complete the proof of (6.9).

A.3. Proof of (6.10).

A.3.1. The estimates of \( (\partial^m_X \nabla D_t v^\ell-i - \nabla D_t v^\ell, \partial^m_X D_t \nabla^\ell-i - \nabla D_t v^\ell, \partial^m_X D_t \nabla^\ell-i - \nabla D_t v^\ell) \).

It is easy to observe from (A.1) and \([D_t; \partial^m_X] = 0\) that for nonnegative integer \(m \leq \ell - 1\),

\[
\partial^{m+1}_X \nabla D_t v^\ell-m-1 - \nabla D_t v^\ell = - \sum_{\kappa=0}^m \sum_{l=0}^\kappa C^l_\kappa \partial^l_X \nabla X \alpha \partial^{\kappa-l}_X \partial^m \alpha D_t v^\ell-k-1,
\]

and

\[
\partial^{m+1}_X D_t \nabla v^\ell-m-1 - D_t \nabla v^\ell = D_t (\partial^{m+1}_X \nabla^\ell-m-1 - \nabla v^\ell) \\
= - \sum_{\kappa=0}^m \sum_{l=0}^\kappa C^l_\kappa \left( \partial^l_X D_t \nabla X \alpha \partial^{\kappa-l}_X \partial^m \alpha v^\ell-k-1 + \partial^l_X \nabla X \partial^{\kappa-l}_X D_t \partial^m \alpha v^\ell-k-1 \right),
\]
so that we write
\[
\left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{m+1} \nabla D_t v^{\ell-m-1} - \nabla D_t v^{\ell}, \partial_X^{m+1} D_t \nabla v^{\ell-m-1} - D_t \nabla v^{\ell} \right) \right\|_{L^3}
\]
\[
\leq C \sum_{\kappa=0}^{m} \sum_{l=0}^{\kappa} \left( \left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} \nabla D_t v^{\ell-\kappa-1}, \partial_X^{\kappa+1} D_t \nabla v^{\ell-\kappa-1} \right) \right\|_{L^3} \right)
\]
\[
+ \sigma(t)^{-\frac{1-s}{2}} \left\| \sigma^{1-\frac{s}{\rho}} \partial_X^{\kappa+1} D_t \nabla X \right\|_{L^3} \left( \left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} \nabla D_t v^{\ell-\kappa-1} \right) \right\|_{L^3} \right)
\]
which together with the fact
\[
\left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) (D_t \nabla v^{\ell} - \nabla D_t v^{\ell}) \right\|_{L^3} = \left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \nabla v \cdot \nabla v^{\ell} \right\|_{L^3}
\]
\[
\leq \sigma^{-\frac{1-s}{2}} \left\| \nabla \cdot \nabla \right\|_{L^3} \left( \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \nabla v \cdot \nabla v^{\ell} \right)_{L^3},
\]
ensures that for nonnegative integer \( i \leq \ell \) and \( r_3 = 2 \) or \( \frac{2p}{p-2} \)
(A.14)
\[
\left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} \nabla D_t v^{\ell-i}, \partial_X^{\kappa+1} D_t \nabla v^{\ell-i} \right) \right\|_{L^3}
\]
\[
\leq CR^2(t) \mathcal{B}_{\ell - 1}(t) + C \mathcal{A}_{\ell}(t) \sigma^{-(1-s_i)}.
\]
The same argument yields for \( m \leq \ell - 1 \) that
\[
\left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} D_t \nabla \nabla^{\ell-m-1} - D_t \nabla \nabla^{\ell} \right) \right\|_{L^2}
\]
\[
\leq C \sum_{\kappa=0}^{m} \sum_{l=0}^{\kappa} \left( \left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} D_t \nabla v^{\ell-\kappa-1} \right) \right\|_{L^2} \right)
\]
\[
+ \sigma(t)^{-\frac{1-s}{2}} \left\| \sigma^{1-\frac{s}{\rho}} \partial_X^{\kappa+1} D_t \nabla v^{\ell-\kappa-1} \right\|_{L^2},
\]
which together with the fact
\[
\left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) (D_t \nabla v^{\ell} - \nabla D_t v^{\ell}) \right\|_{L^2} = \left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \nabla v \cdot \nabla v^{\ell} \right\|_{L^2}
\]
\[
\leq \sigma^{-\frac{1-s}{2}} \left\| \nabla \cdot \nabla \right\|_{L^2} \left( \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \nabla v \cdot \nabla v^{\ell} \right)_{L^2},
\]
implies that
(A.15)
\[
\left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} D_t \nabla \nabla^{\ell-i} - \nabla D_t v^{\ell} \right)(t) \right\|_{L^2}
\]
\[
\leq CR^2(t) \mathcal{B}_{\ell - 1}(t) + C \mathcal{A}_{\ell}(t) \sigma^{-(1-s_i)}.
\]
A.3.2. The estimate of \( (\partial_X^{\kappa+1} D_t \nabla \nabla^{\ell-i} - \nabla D_t v^{\ell}) \). We first deduce from (A.2) that for nonnegative integer \( m \leq \ell - 1 \)
\[
\left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) (\partial_X^{\kappa+1} D_t \nabla v^{\ell-m} - D_t \nabla v^{\ell}) \right\|_{L^2}
\]
\[
\leq C \sum_{\kappa=0}^{m} \sum_{l=0}^{\kappa} \left( \left\| \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} D_t \nabla v^{\ell-\kappa-1} \right) \right\|_{L^2} \right)
\]
\[
+ \left\| \partial_X^{\kappa+1} \left( \partial_X^{\ell-m} - \partial_X^{\ell} \right) \right\|_{L^2} \left( \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} D_t \nabla v^{\ell-\kappa-1} \right) \right)_{L^2}
\]
\[
+ \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left\| \partial_X^{\kappa+1} D_t \nabla v^{\ell-\kappa-1} \right\|_{L^2} \left( \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} D_t \nabla v^{\ell-\kappa-1} \right) \right)_{L^2}
\]
\[
+ \left\| \partial_X^{\kappa+1} \nabla X \right\|_{L^\infty} \left( \sigma \left( \frac{3}{2} - \frac{s}{\rho} - \frac{s - 1}{2} \right) \left( \partial_X^{\kappa+1} D_t \nabla v^{\ell-\kappa-1} \right) \right)_{L^2}.\]
which implies that for any nonnegative integer \( i \leq \ell \),
\[
\left\| \sigma \frac{3 - \ell - 1}{2} (\partial_X^m \partial^2 \nu_{\ell-i} - \partial^2 \nu_{\ell-i}) \right\|_{L^2} \leq CR^2(\ell) \mathfrak{B}_{\ell-1}(t) + C \mathfrak{A}^4_{\ell}(t) \sigma^{-1-s_\ell}.
\]
We notice then
\[
\left\| \sigma \frac{3 - \ell - 1}{2} (\partial_X^m \partial^2 \nu_{\ell-i} - \partial^2 \nu_{\ell-i}) \right\|_{L^2} \leq \sigma(\ell) \frac{1-s_\ell}{2} \left( \left\| \sigma \frac{1-s_\ell}{2} \partial^2 \nu_{\ell-i} \right\|_{L^2} \right) \leq C \mathfrak{A}^4_{\ell}(t) \sigma(\ell)^{-1-s_\ell},
\]
from which and the above inequality, we infer that for any nonnegative integer \( i \leq \ell \),
\[
(A.16) \quad \left\| \sigma \frac{3 - \ell - 1}{2} (\partial_X^m \partial^2 \nu_{\ell-i} - \partial^2 \nu_{\ell-i}) \right\|_{L^2} \leq CR^2(\ell) \mathfrak{B}_{\ell-1}(t) + C \mathfrak{A}^4_{\ell}(t) \sigma^{-1-s_\ell}.
\]

A.3.3. The estimate of \( (\nabla \partial^i_X \nabla \partial^2 \nu_{\ell-i} - \nabla^2 \nu^i_{\ell-i}) \). Again it is easy to observe from (A.1) that for nonnegative integer \( m \leq \ell - 1 \)
\[
\left\| \nabla \partial_X^{m+1} \nabla \partial^2 \nu_{\ell-m-1} - \nabla^2 \nu^i_{\ell-m-1} \right\|_{L^2} \leq \left\| \nabla \partial_X^{m+1} \nabla \partial^i_X \nu_{\ell-m} - \nabla \partial^i_X \nu_{\ell-m} \right\|_{L^2} \leq C \sum_{\kappa=0}^m \sum_{\ell=0}^\infty \left( \left\| \nabla \partial_X^{\kappa} \nabla X \right\|_{L^p} \left\| \sigma \frac{3 - \ell - 1}{2} \partial^i_X \nabla \partial^2 \nu_{\ell-m-1} \right\|_{L^2} \right)^\frac{2}{p},
\]
which implies for any integer \( 0 \leq i \leq \ell \) that
\[
(A.17) \quad \left\| \sigma \frac{3 - \ell - 1}{2} (\partial_X^m \partial^2 \nu_{\ell-i} - \partial^2 \nu_{\ell-i}) \right\|_{L^2} \leq CR^2(\ell) \mathfrak{B}_{\ell-1}(t).
\]
By summing up the Estimates (A.14), (A.15), (A.16) and (A.17), we achieve (6.10). This completes the proof of the lemma.

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References

1. H. Abidi, Équations de Navier-Stokes avec densité et viscosité variables dans l’espace critique, Rev. Mat. Iberoam., 23 (2007), 537–586.
2. H. Abidi and M. Paicu, Existence globale pour un fluide inhomogène, Ann. Inst. Fourier, 57 (2007), 883–917.
3. H. Abidi, G. Gui and P. Zhang, On the wellposedness of 3–D inhomogeneous Navier-Stokes equations in the critical spaces, Arch. Ration. Mech. Anal., 204 (2012), 189–230.
4. H. Abidi, G. Gui and P. Zhang, Wellposedness of 3–D inhomogeneous Navier-Stokes equations with highly oscillating initial velocity field, J. Math. Pures Appl., 100 (2013), 166–203.
5. H. Bahouri, J.-Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der mathematischen Wissenschaften, Springer, 2010.
6. A.-L. Bertozzi and P. Constantin, Global regularity for vortex patches, Comm. Math. Phys., 152 (1993), 19–28.
7. J.-Y. Chemin, Calcul paracadifférentiel précisé et applications à des équations aux dérivées partielles non semilinéaires, Duke Math. J., 56 (1988), 431–469.
8. J.-Y. Chemin, Sur le mouvement des particules d’un fluide parfait incompressible bidimensionnel, Invent. Math., 103 (1991), 599–629.
[9] J.-Y. Chemin, Persistance de structures géométriques dans les fluides incompressibles bidimensionnels, *Ann. Sci. École Norm. Sup.*, **26** (1993), 517-542.

[10] J.-Y. Chemin, *Perfect incompressible fluids*, Oxford Lecture Series in Mathematics and its Applications, **14**, The Clarendon Press, Oxford University Press, New York, 1998.

[11] R. Danchin, Poches de tourbillon visqueuses, *J. Math. Pures Appl. (9)*, **76** (1997), 609-647.

[12] R. Danchin, Density-dependent incompressible viscous fluids in critical spaces, *Proc. Roy. Soc. Edinburgh Sect. A*, **133** (2003), 1311-1334.

[13] R. Danchin and P.-B. Mucha, A Lagrangian approach for the incompressible Navier-Stokes equations with variable density, *Comm. Pure Appl. Math.*, **65** (2012), 1458-1480.

[14] R. Danchin and P.-B. Mucha, Incompressible flows with piecewise constant density, *Arch. Ration. Mech. Anal.*, **207** (2013), 991–1023.

[15] R. Danchin and P. Zhang, Inhomogeneous Navier-Stokes equations in the half-space, with only bounded density, *J. Funct. Anal.*, **267** (2014), 2371-2436.

[16] T. Hmidi, Régularité Hölderienne des poches de tourbillon visqueuses, *J. Math. Pures Appl. (9)*, **84** (2005), 1455-1495.

[17] T. Hmidi, Poches de tourbillon singulières dans un fluide faiblement visqueux, *Rev. Mat. Iberoam.*, **22** (2006), 489-543.

[18] J.-Y. Chemin, Persistance de structures géométriques dans les fluides incompressibles bidimensionnels, *Ann. Sci. École Norm. Sup.*, **26** (1993), 517-542.

[19] J. Huang, M. Paicu and P. Zhang, Global well-posedness of incompressible inhomogeneous fluid systems with bounded density or non-Lipschitz velocity, *Arch. Ration. Mech. Anal.*, **209** (2013), 631–682.

[20] O.-A. Ladyzhenskaja and V.-A. Solonnikov, The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids. (Russian) Boundary value problems of mathematical physics, and related questions of the theory of functions, 8, *Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **52** (1975), 52-109, 218-219.

[21] X. Liao and P. Zhang, On the global regularity of two-dimensional density patch for inhomogeneous incompressible viscous flow, *Arch. Ration. Mech. Anal.*, **220** (2016), 937-981.

[22] P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models*, Oxford Lecture Series in Mathematics and its Applications, **3**, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[23] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier-Stokes system, *J. Funct. Anal.*, **262** (2012), 3556–3584.

[24] M. Paicu, P. Zhang and Z. Zhang, Global unique solvability of inhomogeneous Navier-Stokes equations with bounded density, *Comm. Partial Differential Equations*, **38** (2013), 1208-1234.

[25] P. Zhang and Q. Qiu, Propagation of higher-order regularities of the boundaries of 3-D vortex patches, *Chinese Ann. Math. Ser. A*, **18** (1997), 381–390.

[26] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, *SIAM J. Math. Anal.*, **21** (1990), 1093–1117.

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