On y-closed Rickart Modules

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Abstract
In a previous work, Ali and Ghawi studied closed Rickart modules. The main purpose of this paper is to define and study the properties of y-closed Rickart modules. We prove that, Let M and N be two R-modules such that N is singular. Then M is N-y-closed Rickart module if and only if Hom(M, N) = 0. Also, we study the direct sum of y-closed Rickart modules.

Keywords: y-closed submodule, y-closed simple, y-closed Rickart modules.

1. INTRODUCTION
A module M is called closed Rickart if for any \( f \in \text{End}(M) \), \( \text{ann}_M(f) = \text{Ker}f \) is closed submodule of M [1]. Recall that a submodule A of an R-module M is called a y-closed submodule of M if \( \frac{M}{A} \) is nonsingular [2]. It is known that every y-closed submodule is closed.

In this paper, we give some results on the y-closed Rickart modules.

In §2, we give the definition of the y-closed Rickart modules with some examples and basic properties. For example, we prove that for two R-modules M and N such that N is nonsingular module, then M is N-y-closed Rickart module, see proposition (2.3).

In section 3, we study the direct sum of y-closed Rickart module. For example, we prove that for two R-modules M and N such that \( M = A \oplus B \), where A and B are submodules of M. If M is N-y-closed Rickart module, then A is N-y-closed Rickart module, see Theorem (3.1).

Throughout this article, R is a ring with identity and M is a unitary left R-module. \( S = \text{End}_R(M) \) will denote the endomorphism ring of M.

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§2: Y-Closed Rickart Modules

In this section, we introduce the definition of y-closed Rickart module. Also we give some basic properties of this concept.

**Definition 2.1:** Let M and N be two R-modules. We say that M is N-y-closed Rickart module if for each \( f \in \text{End}(M, N) \), \( \text{ann}_M(f) = \text{Ker}f \) is a y-closed submodule of M.

For a module M, if M is M-y-closed Rickart module, then we say that M is y-closed Rickart module.

**Examples 2.2:**
1- Consider the modules Z and Q as Z-modules. Then Z is Q-y-closed Rickart module. To show that, let \( f: Z \to Q \) be an R-homomorphism, by the first isomorphism theorem \( \frac{Z}{\text{Ker}f} \cong \text{Im}f \). Since Q is nonsingular, then \( \text{Im}f \) is nonsingular. Therefore \( \text{Ker}f \) is a y-closed submodule of Z. Thus Z is Q-y-closed Rickart module.

2- Consider the modules \( Z_4 \) and \( Z_2 \) as Z-modules and let \( f: Z_4 \to Z_2 \) be a map defined by \( f(x) = 3x, \forall x \in Z_4 \). Hence Kerf = \( \{ x \in Z_4, f(x) = 0 \} = \{0, 2\} \). But \( \frac{Z_4}{\{0, 2\}} \cong Z_2 \) and \( Z_2 \) singular as Z-module. Thus \( Z_4 \) is not \( Z_2 \)-y-closed Rickart module.

**Note:** A Rickart (closed Rickart) module needs not to be a y-closed Rickart module. For example, the module \( Z_6 \) as Z-module is a Rickart (closed Rickart) module, where \( Z_6 \) is semisimple. We claim that \( Z_6 \) is not y-closed Rickart module. To verify this, let \( f: Z_6 \to Z_6 \) be a map defined by \( f(x) = 3x, \forall x \in Z_6 \). Clearly, f is an R-homomorphism and Kerf = \( \{ x \in Z_4, f(x) = 0 \} = \{0, 2, 4\} \). By the first isomorphism theorem, \( \frac{Z_6}{\{0, 2, 4\}} \cong Z_2 \) and \( Z_2 \) singular as Z-module. Thus \( Z_6 \) is not y-closed Rickart module.

**Proposition 2.3:** Let M and N be two R-modules such that N is nonsingular module. Then M is N-y-closed Rickart module.

**Proof:** Let \( f: M \to N \) be an R-homomorphism. Since N is nonsingular and \( \text{Im}f \) is a submodule of N, then \( \text{Im}f \) is nonsingular module. By the first isomorphism theorem, \( \frac{M}{\text{Ker}f} \cong \text{Im}f \). Therefore \( \frac{M}{\text{Ker}f} \) is nonsingular. Hence Kerf is a y-closed of M. Thus M is N a y-closed Rickart module.

**Corollary 2.4:** Let R be an integral domain and let M be torsion free R-module. Then M is a y-closed Rickart module.

No, we give the following characterization.

**Propositions 2.5:** Let M and N be two R-modules. Then M is N-y-closed Rickart module if and only if, for every R-homomorphism \( f: M \to N \), \( \text{Im}f \) is a nonsingular module.

**Proof:** Let \( f: M \to N \) be an R-homomorphism. Since \( \text{Im}f \) is nonsingular module and \( \frac{M}{\text{Ker}f} \cong \text{Im}f \). Therefore \( \frac{M}{\text{Ker}f} \) is nonsingular. Hence Kerf is a y-closed submodule of M. Thus M is N-y-closed Rickart module.

Conversely, let \( f: M \to N \) be an R-homomorphism. Since \( \text{Im}f \) is nonsingular and \( \frac{M}{\text{Ker}f} \cong \text{Im}f \), then \( \frac{M}{\text{Ker}f} \) is nonsingular. Therefore Kerf is a y-closed submodule of M. Thus M is N-y-closed Rickart module.

Recall that a module M is said to be K-nonsingular if for every homomorphism \( f: M \to M \) such that kerf is essential in M, implies \( f = 0 \) [1].

**Proposition 2.6:** Every y-closed Rickart module is K-nonsingular.

**Proof:** Suppose that M is a y-closed Rickart module and let \( f: M \to M \) be an R-homomorphism such that kerf is essential in M. Then \( \frac{M}{\text{Ker}f} \) is singular, by [2]. But M is a y-closed Rickart module, therefore kerf is a y-closed submodule of M, which implies that kerf = M and so \( f = 0 \). Thus M is K-nonsingular.

**Propositions 2.7:** Let M and N be two R-modules such that N is singular. Then M is N-y-closed Rickart module if and only if \( \text{Hom}(M, N) = 0 \).
Proof: Assume that M is N-y-closed Rickart module and let f: M → N be an R-homomorphism. Then Kerf is a y-closed submodule of M and hence \( \frac{M}{Kerf} \) is nonsingular. So Imf is nonsingular. But N is singular, therefore Imf = 0. Thus Hom(M, N) = 0.

The converse is clear.

Corollary 2.8: Let A be a proper essential submodule of a module M. Then M is not \( \frac{M}{A} \) -y-closed Rickart module.

Proof. Since A is an essential submodule of M, then by [2], \( \frac{M}{A} \) is a singular module. Let \( \pi: M \rightarrow \frac{M}{A} \) be the natural epimorphism. It is clear that 0 ≠ π \( \in \) Hom(\( \frac{M}{A} \)). Thus by Proposition (2.7) M is not \( \frac{M}{A} \) -y-closed Rickart module.

§3 DIRECT SUM OF Y-CLOSED RICKART MODULES

In this section, we study the direct sum of the y-closed Rickart modules. We begin with the following theorem.

Theorem 3.1: Let M and N be two R-modules such that M = A\( \oplus \)B, where A and B are submodules of M. If M is N-y-closed Rickart module, then A is N-y-closed Rickart module.

Proof. Let \( \psi: A \rightarrow N \) be an R-homomorphism and let p: M \rightarrow A be the projection map. Consider the map \( \psi \circ p: M \rightarrow N \). Since M is N-y-closed Rickart module, then Ker(\( \psi \circ p \)) is a y-closed submodule of M. But

\[
\text{ker}(\psi \circ p) = \{x \in M, \ \psi \circ p(x) = 0\} = \{a + b \in A \oplus B, \ \psi(p(a + b)) = 0, \ a \in A, b \in B\} = \{a + b \in A \oplus B, \ \psi(a) = 0, \ a \in A, b \in B\} = \text{ker}\psi \circ \text{B}
\]

Therefore \( \frac{M}{\text{ker}\psi \circ \text{B}} = \frac{A \oplus \text{B}}{\text{ker}\psi \circ \text{B}} \cong \frac{A}{\text{ker}\psi} \) is nonsingular. So Ker\psi is a y-closed submodule of A. Thus A is N-y-closed Rickart module.

Propositions 3.2: Let M = \( \bigoplus_{i \in I} M_i \) and N = \( \bigoplus_{i \in I} N_i \) be two R-modules, such that for every f \( \in \) Hom(M, N), f(M_i) \( \subseteq \) N, \( \forall i \in I \). If M_i is N_i -y-closed Rickart module, \( \forall i \in I \), then M is N-y-closed Rickart module.

Proof. Assume that M_i is N_i -y-closed Rickart module, \( \forall i \in I \), and let f: M \rightarrow N be an R-homomorphism. We want to show that Kerf is a y-closed submodule of M. By our assumption,

\( f \big|_{M_i}: M_i \rightarrow N_i, \ \forall i \in I \). It is clear that kerf \( \big|_{M_i} = \text{kerf} \cap M_i \), for each i \( \in I \). We claim that kerf = \( \bigoplus_{i \in I} \text{ker}(f \big|_{M_i}) \). To show that, let x \( \in \) Kerf. Then x = \( \sum_{i \in I} x_i \), where x_i \( \in \) Mi, for each i \( \in I \) and x_i ≠ 0 for at most a finite number of i \( \in I \) and f(x) = 0. Then f(x) = f(\( \sum_{i \in I} x_i \)) = \( \sum_{i \in I} f(x_i) = 0 \), where f(x_i) \( \in \) N_i. But N = \( \bigoplus_{i \in I} N_i \). Therefore f(x_i) = 0, \( \forall i \in I \). So x_i \( \in \) Kerf \( \cap M_i \), \( \forall i \in I \) and hence x = \( \sum_{i \in I} x_i \in \bigoplus_{i \in I} \text{Ker}(f \big|_{M_i}) \). Thus Kerf = \( \bigoplus_{i \in I} \text{Ker}(f \big|_{M_i}) \). Since M_i is N_i-y-closed Rickart module for each i \( \in I \), then Ker(\( f \big|_{M_i} \)) is a y-closed submodule of M_i. Therefore Kerf = \( \bigoplus_{i \in I} \text{Ker}(f \big|_{M_i}) \) is a y-closed submodule of M, by [3]. Thus M is N-y-closed Rickart module.

Let M be an R-module, then M is called a y-closed simple if M and 0 are the only y-closed submodules of M.

Example 3.3:
1- The module Z as Z-module is a y-closed simple module, where \( \frac{Z}{nZ} \cong Z_n, \ \forall n \geq 2 \) and Z_n is singular as Z-module. Thus nZ is not y-closed submodule of Z, \( \forall n \geq 2 \).
2- The module Z_6 as Z-module is not y-closed simple module, where \( \frac{Z}{(6)} \cong Z_6 \) and Z_6 as Z-module is singular. Hence the submodule \( \{0\} \) of Z_6 is not y-closed submodule.
Propositions 3.4: Let M be a y-closed simple R-module and let N be an R-module. If M is N-y-closed Rickart, then either
(1) \( \text{Hom}(M,N)=0 \)
or
(2) Every nonzero R-homomorphism from M to N is a monomorphism.
Proof. Assume that \( \text{Hom}(M,N) \neq 0 \) and let \( f:M \rightarrow N \) be a non-zero R-homomorphism. Since M is N-y-closed Rickart, then kerf is y-closed submodule of M. But M is y-closed simple, therefore kerf = \( \{0\} \) and f is a monomorphism.
Recall that an R-module M is called a Quasi-Dedekind R-module if every nonzero endomorphism of M is a monomorphism [4, Th(1.5), CH2].

Corollary 3.5: Let M be a y-closed simple R-module and let N be any R-module such that \( \text{Hom}(M,N) \neq 0 \). If M is N-y-closed Rickart module, then M is Quasi-Dedekind. In particular, if M is y-closed Rickart, then M is Quasi-Dedekind.

Proof. By Proposition (3.4), there is a monomorphism \( f:M \rightarrow N \). Assume that M is not Quasi-Dedekind R-module. So there exists a homomorphism \( g:M \rightarrow M \) such that \( \text{Ker}(g) \neq 0 \). Since f is a monomorphism, then \( \text{Ker}(f \circ g) = \text{Ker}(g) \neq 0 \). But M is N-y-closed Rickart module, therefore \( \text{Ker}(f \circ g) = \text{Ker}(f) \) is a y-closed submodule of M. So \( \text{Ker}(f) \), where \( \text{Ker}(f) \) is a y-closed simple. Thus \( g = 0 \), which is a contradiction. Thus M is a Quasi-Dedekind R-module.

Proposition 3.6: Let M be an R-module. If R is M-y-closed Rickart module, then every cyclic submodule of M is projective. In particular, if R is y-closed Rickart ring, then every principal ideal is projective, i.e., R is a principal projective ring.

Proof. Let M be an R-module such that R is M-y-closed Rickart module and let \( m \in M \). Now consider the following short exact sequence

\[
0 \rightarrow \text{ker}f \rightarrow R \rightarrow M \rightarrow 0
\]

where \( i \) is the inclusion homomorphism and g is a map defined by \( f(r) = rm, \forall r \in R \). It is clear that \( g \) is an epimorphism. Let \( i_2:R \rightarrow M \) be the inclusion map. Since R is M-y-closed Rickart module and \( i_2 \circ f:R \rightarrow M \), then \( \text{Ker}(i_2 \circ f) \) is a y-closed ideal of R. But \( i_2 \) is a monomorphism, therefore \( \text{Ker}(i_2 \circ f) = \text{ker}f \) is a y-closed ideal of R. Hence \( \frac{R}{\text{ker}f} \) is nonsingular. By the first isomorphism theorem, \( \frac{R}{\text{ker}f} \cong \text{Rm} \). So \( \text{Rm} \) is nonsingular, by [2,corollary(1.25),p35]. Thus \( \text{Rm} \) is projective.

Recall that an R-module M is called dualizable if \( \text{Hom}(M,R) \neq 0 \) [5].

Corollary 3.7: Let M be a y-closed simple dualizable R-module. If M is R-y-closed Rickart module, then M is isomorphic to an ideal of R. Hence, if R has nonzero nilpotent elements, then \( \text{End}(M) \) is commutative.

Proof. Since \( \text{Hom}(M,R) \neq 0 \), then by Proposition (3.4), M is isomorphic to an ideal I of R and hence \( \text{End}(M) \cong \text{End}(I) \). For the second part, since R has no nonzero elements and I is an ideal in R, then \( \text{End}(I) \) is commutative [6, proposition(2.1),CH1]. Thus \( \text{End}(M) \) is commutative.

Recall that an R-module M is called a multiplication module if for each submodule N of M there exists an ideal I of R such that \( N = IM \) [6].

Corollary 3.8: Let M be a y-closed simple projective R-module and R has no nonzero nilpotent element. If M is R-y-closed Rickart module and \( \text{Hom}(M,R) \neq 0 \), then M is a multiplication module.

Proof. By the same argument of the proof of Corollary (3.7), \( \text{End}(M) \) is a commutative and hence M is a multiplication [7].

Proposition 3.9: Let M be an R-module with the property that the intersection of any two y-closed submodules of M is a y-closed submodule of M. Then the following statements are equivalent:
(a) M is a y-closed Rickart module,
(b) The left annihilator in M of every left finitely generated ideal I = \( \langle f_1, \ldots, f_n \rangle \) of \( \text{End}_R(M) \) is a y-closed submodule of M.

Proof. (a) \( \Rightarrow \) (b) Let I = \( \langle f_1, \ldots, f_n \rangle \) be a left finitely generated ideal of the \( \text{End}_R(M) \). Since M is a y-closed Rickart module, then \( \text{ann}_M(f_i) \) is a y-closed submodule of M, \( \forall 1 \leq i \leq n \). Hence
\( \bigcap_{i=1}^{n} \mathrm{ann}_M(f_i) \) is a \( y \)-closed submodule of \( M \), by [3]. But \( \mathrm{ann}_M(1) = \mathrm{ann}_M(S_f + \cdots + S_{f_n}) = \bigcap_{i=1}^{n} \mathrm{ann}_M(S_f) \). Therefore \( \mathrm{ann}_M(1) \) is a \( y \)-closed submodule of \( M \).

(b) \( \Rightarrow \) (a) Clear.

Now, we give the following characterization.

**Theorem 3.10.** Let \( M_1 \) and \( M_2 \) be two \( R \)-modules. Then the following statements are equivalent.

1. \( M_1 \) is \( M_2 \)-\( y \)-closed Rickart module;
2. For every submodule \( N \) of \( M_2 \), every direct summand \( K \) of \( M_1 \) is \( N \)-\( y \)-closed Rickart;
3. For every direct summand \( K \) of \( M_1 \), every \( y \)-closed submodule \( L \) of \( M_2 \) and every \( f \in \mathrm{Hom}_R(M, L) \). The kernel of the restricted map \( f|_K \) is a \( y \)-closed submodule of \( K \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( N \) be submodule of \( M_2 \). Let \( K \) be a direct summand of \( M_1 \) and let \( f : K \rightarrow N \) be an \( R \)-homomorphism. Then \( M_1 = K \oplus K_1 \), for some submodule \( K_1 \) of \( M_1 \). Let \( g : M_1 \rightarrow M_2 \) be a map defined by \( g(x) = \begin{cases} f(x), & \text{if } x \in K \\ 0, & \text{if } x \in K_1 \end{cases} \).

It is clear that \( g \) is an \( R \)-homomorphism. Since \( M_1 \) is \( M_2 \)-\( y \)-closed Rickart module, then \( \ker g \) is a \( y \)-closed submodule of \( M_1 \). But \( \ker g = \{ a + b \in M_1, \ g(a + b) = 0 , \ a \in K, b \in K_1 \} = \{ a + b \in M_1, \ f(a) = 0 , \ a \in K, b \in K_1 \} = \ker f \oplus K_1 \)

Therefore \( \ker f \oplus K_1 \) is a \( y \)-closed submodule of \( M_1 \) and hence \( \frac{M_1}{\ker f \oplus K_1} \) is nonsingular. But \( \frac{M_1}{\ker f \oplus K_1} \cong \frac{M}{K} \), so \( \ker f \) is a \( y \)-closed submodule of \( K \). Thus \( K \) is \( N \)-\( y \)-closed Rickart module.

(2) \( \Rightarrow \) (3) Let \( K \) be a direct summand of \( M_1 \) and \( L \) be a submodule of \( M_2 \). Let \( f : M_1 \rightarrow L \) be an \( R \)-homomorphism. Consider the map \( f|_K : K \rightarrow L \). Since \( K \) is \( L \)-\( y \)-closed Rickart module, then \( \ker f|_K \) is a \( y \)-closed submodule of \( K \).

(3) \( \Rightarrow \) (1) Let \( f : M_1 \rightarrow M_2 \) be an \( R \)-homomorphism. Take \( L = M_2 \) and \( K = M_1 \). Since \( f|_K : K \rightarrow L \) and \( K \) is \( L \)-\( y \)-closed Rickart module, therefore \( \ker f \) is a \( y \)-closed submodule of \( M_2 \). Thus \( M_1 \) is \( M_2 \)-\( y \)-closed Rickart module.

**Remark 3.11.** Let \( M \) and \( N \) be two \( R \)-modules and \( f : M \rightarrow N \) be an \( R \)-homomorphism. Let \( A_M = M \oplus 0 \), \( A_N = N \oplus 0 \), \( f : A_M \rightarrow A_N \) be a map defined by \( f(m,0) = (0,f(m)) \), for every \( m \in M \) and

\[ T_f = \{ x + f(x), x \in A_M \}. \]

1. \( M \oplus N = A_M \oplus A_N \)
2. \( T_f \) is a \( y \)-closed submodule
3. \( \ker f = \ker f \oplus 0 \)
4. \( T_f \) is a submodule of \( M \oplus N \)
5. \( A_M + T_f = A_M \oplus \text{Im} f \)

In the following theorem by \( A_M, B_M, T_f, T_f \), we mean the same concepts in the previous above Remark.

Now, we give another characterization for the relative \( y \)-closed Rickart module.

**Theorem 3.12.** Let \( M \) and \( N \) be two \( R \)-modules. Then \( M \) is \( N \)-\( y \)-closed Rickart module if and only if for every homomorphism \( f : M \rightarrow N \), \( A_M \cap T_f \) is \( y \)-closed submodule of \( A_M \).

**Proof.** Let \( f : M \rightarrow N \) be an \( R \)-homomorphism. Since \( M \) is \( N \)-\( y \)-closed Rickart module, then \( \ker f \) is a \( y \)-closed submodule of \( M \) and hence \( \frac{M}{\ker f} \) is nonsingular. Then \( \frac{A_M}{\ker f} = \frac{M \oplus 0}{\ker f \oplus 0} \cong \frac{M}{\ker f} \) is nonsingular. So \( \ker f \) is a \( y \)-closed submodule of \( A_M \).

By the same argument of the proof of the [8,Theorem(2.2)], \( \ker f = A_M \cap T_f \).

For the converse, let \( f : M \rightarrow N \) be an \( R \)-homomorphism. Then by our assumption, \( A_M \cap T_f \) is a \( y \)-closed submodule of \( A_M \) since \( A_M \cap T_f = A_M \cap T_f \), \( \ker f \) is a \( y \)-closed submodule of \( A_M \) and hence \( \frac{A_M}{\ker f} \) is nonsingular. Therefore \( \frac{M \oplus 0}{\ker f \oplus 0} \cong \frac{M}{\ker f} \) is nonsingular. So \( \ker f \) is a \( y \)-closed submodule of \( M \). Thus \( M \) is \( N \)-\( y \)-closed Rickart module.
But, we have the following.

**Theorem 3.13:** Let $M$ and $N$ be two $R$-modules and let $f : M \to N$ be an $R$-homomorphism. Then $M$ is $N$-$y$-closed Rickart module if and only if $T_f$ is $y$-closed submodule of $A_M + T_f$.

**Proof.** Let $f : M \to N$ be an $R$-homomorphism. Now consider the following short exact sequences:

$$
\begin{align*}
0 & \longrightarrow A_M \cap T_f & i_1 & \longrightarrow A_M & \pi_1 & \longrightarrow A_M & \longrightarrow 0 \\
0 & \longrightarrow T_f & i_2 & \longrightarrow A_M + T_f & \pi_2 & \longrightarrow T_f & \longrightarrow 0
\end{align*}
$$

where $i_1, i_2$ are the inclusion homomorphisms and $\pi_1, \pi_2$ are the natural epimorphisms. Since $M$ is $N$-$y$-closed Rickart, then $\ker f$ is $y$-closed submodule of $M$ and hence $M/\ker f$ is nonsingular. So $A_M/\ker f = M \oplus 0 \cong M/\ker f$ is nonsingular. Thus $\ker f = A_M \cap T_f$ is a $y$-closed submodule of $A_M$. Hence $A_M/\ker f = A_M \cap T_f$ is nonsingular. By the second isomorphism theorem, $A_M/\ker f \cong A_M + T_f/\ker f$. Since $T_f$ is $y$-closed submodule of $A_M + T_f$, then $T_f$ is a $y$-closed submodule of $A_M + T_f$.

For the converse, let $f : M \to N$ be an $R$-homomorphism. Consider the following short exact sequences:

$$
\begin{align*}
0 & \longrightarrow A_M \cap T_f & i_1 & \longrightarrow A_M & \pi_1 & \longrightarrow A_M/\ker f & \longrightarrow 0 \\
0 & \longrightarrow T_f & i_2 & \longrightarrow A_M + T_f & \pi_2 & \longrightarrow T_f & \longrightarrow 0
\end{align*}
$$

where $i_1, i_2$ are the inclusion homomorphisms and $\pi_1, \pi_2$ are the natural epimorphisms. By the second isomorphism theorem, $A_M/\ker f \cong A_M + T_f/\ker f$. Since $T_f$ is $y$-closed submodule of $A_M + T_f$, then $A_M/\ker f \cong A_M + T_f/\ker f$ is nonsingular, therefore $A_M/\ker f$ is nonsingular. Hence $A_M \cap T_f$ is a $y$-closed submodule of $A_M$. So $\ker f = \ker f \oplus 0$ is a $y$-closed submodule of $A_M = M \oplus 0$. Thus $\ker f$ is $y$-closed submodule of $M$.

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