NEW RANDOMIZED RESPONSE TECHNIQUE FOR ESTIMATING THE POPULATION TOTAL OF A QUANTITATIVE VARIABLE

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Abstract. In this paper, a new randomized response technique aimed at protecting respondents’ privacy is proposed. It is designed for estimating the population total, or the population mean, of a quantitative characteristic. It provides a high degree of protection to the interviewed individuals, hence it may be favorably perceived by them and increase their willingness to cooperate. Instead of revealing the true value of the characteristic under investigation, a respondent only states whether the value is greater (or smaller) than a number which is selected by him/her at random, and is unknown to the interviewer. For each respondent this number, a sort of individual threshold, is generated as a pseudorandom number from the uniform distribution. Further, two modifications of the proposed technique are presented. The first modification assumes that the interviewer also knows the generated random number. The second modification deals with the issue that, for certain variables, such as income, it may be embarrassing for the respondents to report either high or low values. Thus, depending on the value of the pseudorandom lower bound, the respondent is asked different questions to avoid being embarrassed. The suggested approach is applied in detail to the simple random sampling without replacement, but it can also be applied to many currently used sampling schemes, including cluster sampling, two-stage sampling, etc. Results of simulations illustrate the behavior of the proposed procedure.

1. Introduction

A steady decline in response rates has been reported for many surveys in most countries around the world; see, e.g., Stoop (2005), Steeh et al. (2001) or Synodinos and Yamada (2000). This decline is observed regardless of the mode of the survey, e.g., face-to-face survey, paper/electronic questionnaire, Internet survey or telephone interviewing. Furthermore, this trend has continued despite additional procedures aimed at reducing refusal and increasing contact rates (Brick (2013)). For some time we have observed that people are getting more and more suspicious with respect to any kind of sampling surveys, a priori assuming that the other side cheats (or can cheat). It is especially due to the overall spread of the Internet, where we communicate with anonymous computer robots, leaving us no chance to check their trustworthiness.

The growing concern about “invasion of privacy” thus also represents an important challenge for statisticians. Quite naturally, a respondent may be hesitant or even evasive in providing any information which may indicate a deviation from a social or legal norm, and/or which he/she feels might be used against him/her some time later. Therefore, if we ask sensitive or pertinent questions in a survey, conscious reporting of false values would often occur (Särndal et al. (1992), pp 547). Unfortunately, standard techniques such as reweighting or model-based imputation cannot usually be applied; for a thorough discussion, see Särndal et al. (1992) or Särndal and Lundström (2005). On the other
hand, this issue can, at least partially, be resolved using randomized response techniques (RRT).

For all of the reasons mentioned above, different RRTs have been developed with the goal to obtain unbiased estimates and to reduce the non-response rate. These techniques started with a seminal paper by Warner (1965), who aimed at estimating the proportion of people in a given population with sensitive characteristics, such as substance abuse, unacceptable behavior, criminal past, controversial opinions, etc. Eriksson (1973) and Chaudhuri (1987) modified Warner’s method to estimate the population total of a quantitative variable. However, in our opinion based on personal practical experience, these standard RRTs aimed at estimating the population total are rather complicated and demanding on both respondents and survey statisticians for various real life applications, see also the discussion in Chaudhuri (2017). They require “non-trivial arithmetic operations” from respondents within Chaudhuri’s approach, while the survey statistician must expend a lot of effort connected with the design of a suitable deck of cards, or other randomization mechanisms to be used for masking the sensitive variables (such as income, personal wealth) and, at the same time, providing accurate enough estimates.

In this paper, we propose a method which is simpler in comparison with those proposed previously and is practically applicable. The respondent is only asked whether the value of a sensitive variable attains at least a certain random lower bound. This technique, and its modifications, are developed in detail, applied to the simple random sampling without replacement, and illustrated using simulations.

The main advantages of the suggested method include the ease of implementation, simpler use by the respondent, and practically acceptable preciseness. Moreover, respondents’ privacy is well protected because they never report the true value of the sensitive variable. Unlike in Chaudhuri’s or Eriksson’s approach, there is no issue with the cards design. From a certain point of view, a small disadvantage may be a lower degree of confidence in anonymity, due to the extrinsic device/technique used for generating random numbers.

This paper is organized as follows. In sec. 2, selected randomized response techniques for estimation of the population total, or population mean, are concisely described. In sec. 3, a new randomized response technique and its two modifications are proposed, and their properties studied. Sec. 4 illustrates the suggested ideas with the aid of a simulation study. Finally, sec. 5 provides the main conclusions of the paper.

2. Selected randomized response techniques for estimating the population total and their properties

Let us consider a finite population \( U = \{1, \ldots, N\} \) of \( N \) identifiable units, where each unit can unambiguously be identified by its label. Let \( Y \) be a sensitive quantitative variable; the goal of the survey is to estimate the population total \( t_Y = \sum_{i \in U} Y_i \) or, alternatively, the population mean \( \bar{t}_Y = t_Y / N \), of the surveyed variable. To that end, we use a random sample \( s \) selected with probability \( p(s) \), described by a sampling plan with a fixed sample size \( n \). Let us denote by \( \pi_i \) the probability of inclusion of the \( i^{th} \) element in the sample, i.e., \( \pi_i = \sum_{s \ni i} p(s) \), and by \( \xi_i \) the indicator of inclusion of the \( i^{th} \) element in the sample \( s \), i.e., \( \xi_i = 1 \) if \( s \ni i \) and \( \xi_i = 0 \) otherwise. To keep the length of the paper acceptable, we do not introduce all notions from scratch and refer the reader to Tillé (2006) if needed.

As argued above, in practice it is often impossible to obtain the values of the surveyed variable \( Y \) in sufficient quality because of its sensitivity. Therefore, statisticians try to
obtain from each respondent at least a randomized response \( R \) that is correlated to \( Y \). Randomization of the responses is carried out independently for each population unit in the sample.

Note that, in such a case, the survey has two phases. First, a sample \( s \) is selected from \( U \) and then, given \( s \), responses \( R_i \) are realized using the selected RRT. We denote the corresponding probability distributions by \( p(s) \) and \( q(r \mid s) \). In this setting, the notions of the expected values, unbiasedness and variances are tied to a twofold averaging process:

- Over all possible samples \( s \) that can be drawn using the selected sampling plan \( p(s) \).
- Over all possible response sets \( r \) that can be realized given \( s \) under the response distribution \( q(r \mid s) \).

Below we follow the literature and, where appropriate, denote the expectation operators with respect to these two distributions by \( E_p \) and \( E_q \), respectively.

In a direct survey, the population total \( t_Y \) is usually estimated from the observed values \( Y_i \) using a linear estimator \( t(s, Y) = \sum_{i\in s} b_{si} Y_i \), where the weights \( b_{si} \) follow the unbiasedness constraint \( \sum_{s\ni i} p(s)b_{si} = 1, \ i = 1, \ldots, N \). If \( \pi_i > 0 \ \forall i \in U \), then Horvitz-Thompson’s estimator

\[
t_{HT}(s, Y) = \sum_{i\in s} \frac{Y_i}{\pi_i}
\]

is a linear unbiased estimator with the weights \( b_{si} = 1/\pi_i \), and \( E_p(t_{HT}(s, Y)) = t_Y \), see Horvitz and Thompson (1952), or sec. 2.8 in Tillé (2006) for details.

If the survey is conducted by means of RRT, the true values of \( Y_i \) for the sample \( s \) are unknown and, instead of them, values of random variables \( R_i \) correlated to \( Y_i \) are collected. The population total is then usually estimated using a Horvitz-Thompson’s type estimator

\[
t_{HT}(s, R) = \sum_{i\in s} \frac{R_i}{\pi_i}.
\]

Suppose now that we have an estimator (a formula, or a computational procedure) for estimating the population total \( t_Y \) or population mean \( \bar{Y}_Y \); we denote it by \( \hat{Y}_R \) and \( \hat{\bar{Y}}_R \), respectively. The subscript \( R \) emphasizes that the estimator is based on the values of \( R_i \) in the sample, i.e., on randomized responses. Moreover, we assume that the randomized responses \( R_i \) follow a model for which it holds \( E(R_i) = Y_i \), \( \var(R_i) = \phi_i \ \forall i \in U \), and \( \text{cov}(R_i, R_j) = 0 \ \forall i \neq j, \ i, j \in U \). Note that \( \phi_i \) is a function of \( Y_i \).

Recall that the estimator \( \hat{Y}_R \) of the population total \( t_Y \) is conditionally unbiased if the conditional expectation of \( \hat{Y}_R \) given the sample \( s \) is equal to the current estimator \( \hat{Y}_s \) that would be obtained if no randomization took place, i.e., if \( E_q(\hat{Y}_R \mid s) = \hat{Y}_s \). The subscript \( s \) indicates that the “usual” estimator based on the non-randomized sample, e.g., the Horvitz-Thompson’s one, is used, and \( E_q(\hat{Y}_R \mid s) \) stands for the conditional expectation of \( \hat{Y}_R \) given the sample \( s \) with respect to the distribution induced by the randomization of responses. For the estimator \( \hat{\bar{Y}}_R \) of the population mean, we proceed analogously.

If \( \hat{Y}_R \) is conditionally unbiased and \( \hat{Y}_s \) is unbiased, then \( \hat{Y}_R \) is unbiased as well, since it holds \( E(\hat{Y}_R) = E_p(E_q(\hat{Y}_R \mid s)) = E_p(\hat{Y}_s) = t_Y \). Analogously it holds \( E(\hat{\bar{Y}}_R) = \bar{Y}_Y \).
By a standard formula of probability theory, we get the variance of \( \hat{Y}_R \) in the form

\[
\text{var}(\hat{Y}_R) = E_p\left(\text{var}_q(\hat{Y}_R | s)\right) + \text{var}_p\left(E_q(\hat{Y}_R | s)\right)
\]

\[
= E_p\left(\text{var}_q(\hat{Y}_R | s)\right) + \text{var}_p(\hat{Y}_s).
\]

(3)

The second term on the right-hand side of (3) is, obviously, the variance of the estimator that would apply if no randomization of responses was deemed necessary, while the first term represents the increase of the variance produced by the randomization. In other words, the two terms on the right-hand side of (3) represent, respectively, “contribution by randomized response technique used” and “contribution by sampling variation” to the total variance of \( \hat{Y}_R \). When treating \( \hat{Y}_R \), we proceed analogously.

Because the variances of \( \hat{Y}_s \) are well known for many currently used sampling procedures, it remains to find the contribution by randomization and to suggest methods for its estimation.

For the estimator \( t_{HT}(s, R) \) given by (2), we have

\[
\text{var}\left(t_{HT}(s, R)\right) = E_p\left(\text{var}_q(t_{HT}(s, R) | s)\right) + \text{var}_p\left(E_q(t_{HT}(s, R) | s)\right)
\]

\[
= E_p\left(\sum_{i \in U} \frac{\phi_i \xi_i}{\pi_i^2}\right) + \text{var}_p(t_{HT}(s, Y))
\]

\[
= \sum_{i \in U} \frac{\phi_i}{\pi_i} + \text{var}\left(t_{HT}(s, Y)\right).
\]

(4)

When any RRT is used instead of direct surveying, the variance of the population total estimator is always higher. This increase in variability of \( t_{HT}(s, R) \) is described by the first term in (4), which represents additional variability caused by using a randomized response \( R \) instead of the directly surveyed variable \( Y \).

Let us take a look at two RRT proposals that are recommended in the literature and used in practice. Note that the subscript \( E, C \) (respectively) emphasizes Eriksson’s, Chaudhuri’s (respectively) approach; each of them is concisely revisited below.

Eriksson (1973) proposed a technique in which the respondent randomly draws a card from a deck. The deck contains \( 100 C\% \), \( 0 < C < 1 \), cards with the text “True value”, while the remaining cards have values \( x_1, \ldots, x_T \) with relative frequencies \( q_1, \ldots, q_T \), \( \sum_{t=1}^{T} q_t = 1 - C \). The values of cards \( x_1, \ldots, x_T \) are chosen to mask the true values of the surveyed variable \( Y \). Each respondent randomly draws one card from a deck. If a card with the text “True value” is selected, then the true value of \( Y \) is reported, otherwise the value \( x_t \) shown on the card is given. The respondent then returns the selected card to the deck, and the interviewer does not know which card it was. The answer from the \( i^{th} \) respondent is thus a random variable

\[
Z_{i,E} = \begin{cases} Y_i, & \text{with probability } C, \\ x_t, & \text{with probability } q_t, \ t = 1, \ldots, T. \end{cases}
\]

The answer \( Z_{i,E} \) from the \( i^{th} \) respondent is then transformed to \( R_{i,E} = \frac{Z_{i,E} - \sum_{t=1}^{T} q_t x_t}{C} \). It follows from the definition of \( Z_{i,E} \) that the transformed randomized responses \( R_{i,E} \) have the expectation and variance values
\[ E(R_{i,E}) = Y_i, \]
\[ \text{var}(R_{i,E}) = \frac{C(1-C)Y_i^2 + \sum_{t=1}^{T} q_t x_i^2 - \left( \sum_{t=1}^{T} q_t x_i \right)^2}{C^2} - 2CY_i \sum_{t=1}^{T} x_i q_t, \]

so that the corresponding Horvitz-Thompson's type estimator is unbiased. Unfortunately, if the value reported by a respondent differs from any of \( x_t \), the interviewer can deduce the true value of the sensitive variable; this fact may decrease the credibility and the willingness of some respondents to cooperate.

Later on, Chaudhuri (1987) suggested that two decks of cards should be used. The first deck contains cards with values \( a_1, \ldots, a_K \), and the second deck values \( b_1, \ldots, b_L \). Both decks of cards should mask the behavior of the studied variable \( Y \). Moreover, the following relationships must hold:

\[ \mu_a = \frac{1}{K} \sum_{k=1}^{K} a_k \neq 0, \quad \sigma_a^2 = \frac{1}{K} \sum_{k=1}^{K} (a_k - \mu_a)^2 > 0, \]
\[ \mu_b = \frac{1}{L} \sum_{l=1}^{L} b_l \neq 0, \quad \sigma_b^2 = \frac{1}{L} \sum_{l=1}^{L} (b_l - \mu_b)^2 > 0. \]

The respondent randomly draws one card from each deck, say \( a_k \) and \( b_l \), whereas the interviewer does not know the values on the drawn cards. Then the respondent returns both cards, and instead of the true value \( Y_i \) the value of \( Z_{i,C} = a_k Y_i + b_l \) is reported. This response is then transformed to the randomized response \( R_{i,C} = Z_{i,C} - \mu_b \mu_a \). It follows from the definition of \( Z_{i,C} \) that the randomized response \( R_{i,C} \) has the expectation and variance

\[ E(R_{i,C}) = Y_i, \quad \text{and} \quad \text{var}(R_{i,C}) = Y_i^2 \frac{\sigma_a^2}{\mu_a^2} + \frac{\sigma_b^2}{\mu_a^2}, \]

so that corresponding Horvitz-Thompson’s type estimator is also unbiased.

Both Eriksson’s and Chaudhuri’s techniques have been further developed and improved by other researchers, see, e.g., an interesting papers by Arnab (1995, 1998), Gjestvanga and Singh (2009) or Bose and Dihidar (2018). The ideas and a representative review of further research are presented in a monograph by Chaudhuri (2017). Other types of randomization techniques were suggested in a series of papers by Dalenius and his colleagues, e.g., Bourke and Dalenius (1976) or Dalenius and Vitale (1979). From among the recent papers about dealing with sensitive questions in population surveys, we would like to mention, for example, papers by Trappmann et al. (2014) and Kirchner (2015). In both of them, long lists of relevant references can be found. Finally, recall that probably the most comprehensive account of recent developments in sample survey theory and practice can be found in Handbook of Statistics 29 A, B, edited by Pfeffermann and Rao (2009).

3. New randomized response technique

In this section, we suggest a completely different approach. Assume that the studied sensitive variable \( Y \) is non-negative and bounded from above, i.e., \( 0 \leq Y \leq M \). Parameter \( M \) should be chosen taking into account both bias and privacy. For that purpose, knowledge of the empirical quantiles of the studied population, or at least reasonably
guessing them, is vital. Each respondent carries out, independently of the others, a random experiment generating a pseudorandom number $\Upsilon$ from the uniform distribution on interval $(0, M)$, whereas the interviewer does not know this value. The respondent can generate the pseudorandom number $\Upsilon$ using, for example, a laptop online/offline application; for some other possibilities see sec. 3.4. The respondent then answers a simple question: “Is the value of $\Upsilon$ at least $\Upsilon$?” (e.g., “Is your monthly income at least $\Upsilon$?”). Note that the subscript $\text{AV}$ used below indicates that the estimator, as well as random variables used for its construction, are based on the new idea of randomization suggested in this Section.

Answer of the $i^{\text{th}}$ respondent follows the alternative distribution with the parameter $Y_i/M$, i.e.,

$$
E(Z_{i,\text{AV}}(0,M)) = \begin{cases} 
1 & \text{with probability} \frac{Y_i}{M}, \text{ if } \Upsilon_i \leq Y_i, \\
0 & \text{with probability} \ 1 - \frac{Y_i}{M}, \text{ otherwise}
\end{cases}
$$

Evidently, $E(R_{i,\text{AV}}(0,M)) = Y_i$ and $\text{var}(R_{i,\text{AV}}(0,M)) = Y_i(M - Y_i)$.

For certain sensitive variables, such as the total amount of alcohol consumed within a certain period, it is better to use a question: “Is the value of $\Upsilon$ lower than $\Upsilon$?” In such a case we recode the answer $Z_{i,\text{AV}}(0,M)$ to $Z_{i,\text{AV}}^*(0,M) = 1 - Z_{i,\text{AV}}(0,M)$, and apply the suggested RRT to $Z_{i,\text{AV}}^*(0,M)$.

3.1. Application to the simple random sampling. Consider now the situation in which the sampling plan $p(s)$ is a simple random sampling without replacement with a fixed sample size $n$. Denote by $\overline{Y} = \frac{1}{N} \sum_{i \in U} Y_i$ the population mean, by $S_Y^2 = \frac{1}{N-1} \sum_{i \in U} (Y_i - \overline{Y})^2$ the population variance, and by $f = n/N$ the corresponding sampling fraction. In this case, the inclusion probabilities are constant, i.e., $\pi_i = P(\xi_i = 1) = n/N \ \forall i \in U$.

Let the population total $t_Y$ be estimated using the Horvitz-Thompson’s type estimator

$$
t_{\text{HT,AV}}(0,M) = \frac{N}{n} \sum_{i \in s} R_{i,\text{AV}}(0,M).
$$

This estimator is evidently unbiased, and we calculate its variance. First, taking into account the independence of outcomes of the randomization experiments performed by the respondents, we have

$$
\text{var}_{q}(t_{\text{HT,AV}}(0,M) | s) = \frac{N^2}{n^2} \sum_{i \in s} Y_i (M - Y_i).
$$

Using the well-known identity $\sum_{i \in U} (Y_i - \overline{Y})^2 = \sum_{i \in U} Y_i^2 - N\overline{Y}^2$, we can calculate the contribution of the suggested RRT to the variance as
\[
E_p\left(\text{var}_q(t_{HT,AV,(0,M)} \mid s)\right) = E_p\left(\frac{N^2}{n^2} \sum_{i \in s} Y_i (M - Y_i)\right) = E_p\left(\frac{N^2}{n^2} \sum_{i \in U} Y_i (M - Y_i)\xi_i\right) = \frac{N}{n} \sum_{i \in U} Y_i (M - Y_i) = \frac{N^2}{n} \left(\overline{Y}(M - \overline{Y}) - \frac{N - 1}{N} S_Y^2\right).
\]

Finally, taking into account the variance of the simple random sampling without replacement, see sec. 4.4 in Tillé (2006) for details, we get

\[
(8) \quad \text{var}(t_{HT,AV, (0,M)}) = \frac{N^2}{n} \left(\overline{Y}(M - \overline{Y}) - \frac{n - 1}{N} S_Y^2\right).
\]

To characterize the variance of the suggested estimators more profoundly, and to get a more transparent insight into the variance of the suggested \( RRT \), we introduce two auxiliary “measures of concentration”. More precisely, let us denote

\[
(9) \quad \Gamma_{Y,M} = \frac{1}{N} \sum_{i \in U} \frac{Y_i}{M} (1 - \frac{Y_i}{M}) = \frac{1}{MN} \sum_{i \in U} Y_i - \frac{1}{MN^2} \sum_{i \in U} \frac{Y_i^2}{M} = \overline{Y} - \frac{\overline{Y}^2}{M^2}
\]

and

\[
(10) \quad \Gamma_{\overline{Y},M} = \frac{\overline{Y}}{M} (M - \overline{Y}) = \frac{\overline{Y}}{M} - \frac{\overline{Y}^2}{M^2}.
\]

We call \( \Gamma_{Y,M} \) the mean relative concentration measure, and \( \Gamma_{\overline{Y},M} \) the proximity measure of the population mean \( \overline{Y} \) to \( \frac{M}{2} \).

If \( Y_i \) are i.i.d. random variables with a finite variance \( \sigma^2 \) and an expectation \( \mu \), then, by the law of large numbers, both \( \Gamma_{Y,M} \) and \( \Gamma_{\overline{Y},M} \) converge, as \( N \to \infty \), with probability 1 to

\[
(11) \quad \Gamma_{Y,M,as} = \frac{\mu}{M} (1 - \frac{\mu}{M}) - \frac{\sigma^2}{M^2} \quad \text{and} \quad \Gamma_{\overline{Y},M,as} = \frac{\mu}{M} (1 - \frac{\mu}{M}).
\]

We call \( \Gamma_{Y,M,as} \) the asymptotic mean relative concentration measure, and \( \Gamma_{\overline{Y},M,as} \) the asymptotic proximity measure of the population mean \( \overline{Y} \) to \( \frac{M}{2} \). Note that both \( \Gamma_{Y,M,as} \) and \( \Gamma_{\overline{Y},M,as} \) exist if \( 0 \leq Y_i \leq M \quad \forall i \in U \).

Let us focus on these measures in more detail. First, note that in our setting both these measures are population characteristics, not random variables. Second, both \( \Gamma_{Y,M} \) and \( \Gamma_{\overline{Y},M} \) take on their values in the interval \([0, \frac{1}{4}]\), and are equal to zero only in the pathological cases when either \( Y_i = 0 \quad \forall i \in U \) or \( Y_i = M \quad \forall i \in U \). The higher these measures, the higher the variance of \( t_{HT,AV, (0,M)} \). The mean relative concentration measure \( \Gamma_{Y,M} \) attains its maximum 1/4 when all values lie at the center of the interval \((0, M)\), i.e., if \( Y_i = M/2 \quad \forall i \in U \). The higher the mean relative concentration measure \( \Gamma_{\overline{Y},M} \) of the population mean’s proximity to the center of the interval \((0, M)\) attains its maximum 1/4 only if the population mean is at the interval center, i.e., \( \overline{Y} = M/2 \). This case occurs, e.g., when random variable \( Y \) is symmetric around the center of interval \( M/2 \); this feature is certainly true for the uniform distribution on \((0, M)\).
For a fixed value of the upper bound $M$, population size $N$ and sample size $n$, the contribution of the suggested RRT to the variance of $t_{HT,AV,(0,M)}^R$ depends, up to a multiplicative constant, on $\Gamma_{Y,M}$, because it holds

\begin{equation}
E_p \left( \text{var}_q \left( t_{HT,AV,(0,M)}^R \mid s \right) \right) = \frac{M^2N^2}{n} \frac{1}{N} \sum_{i \in U} \frac{Y_i (M - Y_i)}{M^{2}} \Gamma_{Y,M} = \frac{M^2N^2}{n} \Gamma_{Y,M}.
\end{equation}

Analogously, this contribution can also be expressed, up to multiplicative constants, by $\Gamma_{Y,M}$ and $S_Y^2$, because it holds

\begin{equation}
E_p \left( \text{var} \left( t_{HT,AV,(0,M)}^R \mid s \right) \right) = \frac{M^2N^2}{n} \Gamma_{Y,M} - \frac{N(N-1)}{n} S_Y^2.
\end{equation}

Both $\Gamma_{Y,M}$ and $\Gamma_{Y,M}$ thus help us explain how the suggested RRT increases the variance of the estimator of the population total $\tau_Y$ for distributions symmetrical around $M/2$; for distributions concentrated closely to the center of $(0, M)$, symmetrical around $M/2$; or uniformly distributed. Moreover, they show that the suggested approach is especially suitable for skewed distributions provided they are concentrated around their mean values. Let us sum up: both these measures help us not only describe the variance of the estimator used, as well as compare (12) and (13), but also interpret it better.

**Remark 1.** Notice that, if the values of $Y$ are bounded both from below and above, i.e., $0 < m \leq Y \leq M$, then variance of $t_{HT,AV,(0,M)}^R$ can be significantly reduced by generating pseudorandom numbers $Y_i$ from the uniform distribution on the interval $(m, M)$ instead on $(0, M)$. Indeed; if this is the case, we replace $Z_{i,AV,(0,M)}$, described by (5), with

\[
Z_{i,AV,(m,M)} = \begin{cases} 
1 \text{ with probability } \frac{Y_i - m}{M - m}, & m \leq Y_i \leq Y, \\
0 \text{ with probability } 1 - \frac{Y_i - m}{M - m}, & \text{otherwise},
\end{cases}
\]

transform these variables to $R_{i,AV,(m,M)} = m + (M - m)Z_{i,AV,(m,M)}$, and estimate population total $\tau_Y$ analogously to (7), i.e., using the Horvitz-Thompson’s type estimator

\begin{equation}
t_{HT,AV,(m,M)}^R \equiv t_{AV,(m,M)}^R = \frac{N}{n} \sum_{i \in s} R_{i,AV,(m,M)}.
\end{equation}

It is easy to show that the variance of $t_{HT,AV,(m,M)}^R$ is smaller than that of $t_{HT,AV,(0,M)}^R$, namely, by the value $\frac{N^2m}{n}(M - Y)$.

When choosing parameters $m$ and $M$, both bias and privacy should be taken into account. While the lower bound $m$ affects mostly bias and is not crucial for respondents’ privacy, the choice of $M$ affects both bias and privacy. Thus, the knowledge of empirical quantiles for the studied characteristic, or at least a reasonable guess about them, is vital for setting the values of $m$ and $M$ properly.

An immediate question arises of what happens if the interval $[m, M]$ has not been set correctly. Evidently, if some values of $Y_i$ lie outside of the interval $[m, M]$, then with probability 1 it holds $Z_{i,AV,(m,M)} = 0$ if $Y_i < m$ and $Z_{i,AV,(m,M)} = 1$ if $Y_i > M$. The bias of the suggested estimator then equals

\begin{equation}
\sum_{i \in U \mid Y_i < m} (Y_i - m) + \sum_{i \in U \mid Y_i > M} (Y_i - M)
\end{equation}
Let us discuss some advantages of the suggested approach in comparison with other currently used RRTs, including Eriksson’s and Chaudhuri’s:

- It is simple; this fact increases respondents’ confidence and cooperation, and thus reduces the estimation error.
- Respondents’ privacy is well protected, because they never report the true value of the sensitive variable.
- One can avoid a demanding task of designing the deck of cards to mask the studied variable.
- It enables us to estimate the population total at an acceptable level of accuracy, see sec. 4 for details.

Due to the device/technique used for generating random numbers, some respondents may feel a lower degree of confidence in preserving their anonymity.

A natural question arises whether we could improve the accuracy of the suggested method. We discuss two modifications of the RRTs suggested above and their properties in the subsections below. The heuristics behind this approach are based on the following observations. All the techniques presented up to now have assumed that the interviewer does not know the outcome of the random mechanism leading to the randomized response, such as the card drawn, the value of the pseudorandom number, etc. It is plausible to ask what would happen if we also knew the outcome of that random experiment on the one hand, while protecting respondents’ privacy on the other hand. More precisely: can statisticians improve the accuracy of the proposed estimator, i.e., to decrease its variance, if they also know the values of the generated pseudorandom numbers? We surmise it is feasible, and suggest one possible way of reaching this goal. Let us point out, however, that the success of the suggested approach, to a considerable extent, depends on the statistician’s insight into the problem. It may be embarrassing to report either high or low values of the variables in question, say, the personal income. Depending on the value of the pseudorandom number $\Upsilon$, a different question is then asked with the aim to reduce the respondent’s potential embarrassment.

### 3.2. Estimators using knowledge of $\Upsilon$.

Assume again that the studied sensitive variable $Y$ is non-negative and bounded from above, i.e., $0 \leq Y \leq M$. Each respondent carries out, independently of the others, a random experiment generating a pseudorandom number $\Upsilon$ from the uniform distribution on interval $(0, M)$, and informs the interviewer of both its value and whether $\Upsilon \leq Y$ or not. For example, the response is that the simulated number has been $xxx$ and the respondent earns more/less. Assume further that the corresponding random response is now described not by (5), but using a dichotomous random variable

$$Z_{i,AV,\alpha} = \begin{cases} 1 - \alpha + 2\alpha \frac{\Upsilon_i}{M}, & \text{if } \Upsilon_i \leq Y_i, \\ -\alpha + 2\alpha \frac{\Upsilon_i}{M}, & \text{otherwise}, \\ 0 \leq \alpha < 1. \end{cases}$$

For random responses $Z_{i,AV,\alpha}$ it holds

$$\mathbb{E}(Z_{i,AV,\alpha}) = \mathbb{P}(\Upsilon_i \leq Y_i) = \frac{1}{M} \int_0^{Y_i} \left(1 - \alpha + 2\alpha \frac{u}{M}\right) du + \frac{1}{M} \int_{Y_i}^{M} \left(-\alpha + 2\alpha \frac{u}{M}\right) du = \frac{Y_i}{M},$$

$$\text{var}(Z_{i,AV,\alpha}) = \frac{1 - 2\alpha}{M^2} Y_i (M - Y_i) + \frac{\alpha^2}{3}.$$
The random responses $Z_{i,AV,\alpha}$ are transformed to $R_{i,AV,\alpha} = MZ_{i,AV,\alpha}$, and the desired estimator of the population total $t_Y$ can be constructed analogously to (7) and (14). More precisely, we suggest using again the Horvitz-Thompson’s type of estimator in the form

$$t_{HT,AV,\alpha}^R = t_{AV,\alpha}^R = \frac{N}{n} \sum_{i \in s} R_{i,AV,\alpha}.$$  

Because $E(R_{i,AV,\alpha}) = Y_i$, estimator (17) is unbiased, and the contribution of the randomization to its variance is

$$E_p\left( \text{var}_q(t_{HT,AV,\alpha}^R | s) \right) = \frac{M^2N^2}{n} \sum_{i \in U} \left( \frac{1}{N} (1 - 2\alpha) \frac{Y_i}{M} \left(1 - \frac{Y_i}{M}\right) + \frac{\alpha^2}{3N} \right).$$

An easy calculation shows that (18) takes on its global minimum at $\alpha_{opt} = 3\Gamma_{Y,M} \in [0,3/4]$. Substituting $\alpha_{opt}$ back to (18), we get

$$E_p\left( \text{var}_q(t_{HT,AV,\alpha_{opt}}^R | s) \right) = \frac{M^2N^2}{n} \sum_{i \in U} \left( \left(1 - 6\Gamma_{Y,M}\right) \frac{1}{N} \frac{Y_i}{M} \left(1 - \frac{Y_i}{M}\right) + \frac{3\Gamma_{Y,M}^2}{N} \right)$$

$$= \frac{M^2N^2}{n} \Gamma_{Y,M} \left(1 - 3\Gamma_{Y,M}\right).$$

We would like to point out that the knowledge of pseudorandom numbers $Y_i$ and the use of $\alpha_{opt}$ can considerably decrease variability depending on the suggested RRT – compare (19) with (12). Note also that our simulations summarized in sec. 4 confirm these findings.

The value of the parameter $\alpha$, which is a priori set by the interviewer, is fixed and unknown to the respondent. For $\alpha = 0$ we have the original method described in sec. 3.1. The response to $Z_{i,AV,\alpha}$ is transformed not by the respondent, but by the interviewer off-line.

Parameter $\alpha$ should be set to its optimal value $\alpha_{opt} = 3\Gamma_{Y,M}$, where the mean relative concentration measure $\Gamma_{Y,M}$ is introduced in sec. 3, formula (9). If the interviewer has some prior information about the mean $\mu$ and variance $\sigma^2$ values for the theoretical distribution of the surveyed variable $Y$, he/she should rather apply asymptotic concentration measure (11), which can be estimated using a plug-in moment estimator. More precisely, the population mean $\overline{Y}$ should be replaced with $\mu$, and the population variance $S_Y^2$ with $\sigma^2$. Since the population second moment $\overline{Y^2}$ can be expressed as $\frac{N-1}{N}S_Y^2 + \overline{Y^2}$, it is sufficient to substitute $\mu$ and $\sigma^2$ into this expression. Recall that the prior information is often available for regular surveys in official statistics, such as EU-SILC, because in such a case we can either use results from previous years updated by inflation, or we can rely on the expert opinion. If no prior information is available, we recommend choosing small values of $\alpha$, such as 0.5, to decrease the negative values of $R_{i,AV,\alpha}$. Note that in such a case the resulting estimator may attain unacceptably low or even negative values for estimates of non-negative variables. However this issue can, to some extent, be resolved by properly tuning parameter $\alpha$ and increasing the sample size.

Notice that if a non-negative surveyed random variable $Y$ is bounded not only from above, but also from below, i.e., $0 < m \leq Y \leq M$, we generate $Y_i$ from the uniform distribution on the interval $(m, M)$, modifying $Z_{i,AV,\alpha}$ given by (16) to
\[ Z_{i,AV,\alpha,(m,M)} = \begin{cases} 1 - \alpha + 2\alpha \frac{Y_i}{M-m}, & \text{if } Y_i \leq Y_i, \\ -\alpha + 2\alpha \frac{Y_i}{M-m}, & \text{otherwise,} \end{cases} \]

transforming \( Z_{i,AV,\alpha,(m,M)} \) to \( R_{i,AV,\alpha,(m,M)} = (M-m)Z_{i,AV,\alpha,(m,M)} + m(1-2\alpha) \), and forming

an estimator of the population total \( t_Y \) of the Horvitz-Thompson’s type analogously to (17), i.e.,

\[ t^R_{HT,AV,\alpha,(m,M)} = t^R_{AV,\alpha,(m,M)} = \frac{N}{n} \sum_{i \in s} R_{i,AV,\alpha,(m,M)}. \]

Because \( E(R_{i,AV,\alpha,(m,M)}) = Y_i \), the estimate (20) is again unbiased.

We must firmly emphasize here that the information about neither the value of pseudo-random number \( \Upsilon \) nor of the value \( \alpha \) enables us to guess the exact value of the sensitive variable \( Y \), except for the case \( Y = M \). In other words, knowing them does not intrude on the respondent’s privacy.

The heuristics behind the proposed modification are the following:

- If the answer is YES, then a high value of the pseudorandom number \( \Upsilon \) implies a high value of the studied variable \( Y \), because \( Y \geq \Upsilon \), and these observations “considerably” increase the value of the estimator.
- On the other hand, if the answer is NO, then a low value of the pseudorandom number \( \Upsilon \) implies a low value of \( Y \), because \( Y < \Upsilon \), and these observations “considerably” decrease the value of the estimator.

Unfortunately, in both of these situations, i.e., when the value of the response is either (too) low or (too) high, the respondent may be more prone to fabricate his/her answer.

3.3. Estimators using switching questions. Let us emphasize that for some characteristics, such as monthly income of a household, it may be sensitive for respondents to report either high or low values. This led us to modifying the suggested RRT approach in the following way.

First, we set a proper fixed threshold \( T, 0 < T < M \), unknown to the respondent. Depending on whether the pseudorandom number \( \Upsilon \), which is distributed according to the uniform distribution on \((0, M)\), does or does not exceed the fixed threshold \( T \), we ask one of the following questions:

(i) If \( \Upsilon \leq T \): “Is the value of \( Y \) at least \( \Upsilon \)?”
(ii) If \( \Upsilon > T \): “Is the value of \( Y \) smaller than \( \Upsilon \)?”

Second, we form random variables

\[ Z_{i,AV,T} = \begin{cases} 1, & \text{if } \Upsilon_i \leq T, \ U_i \leq Y_i, \\ 0, & \text{if } \Upsilon_i \leq T, \ U_i > Y_i, \text{ or if } \ U_i > T, \ U_i \leq Y_i, \\ -1, & \text{if } \Upsilon_i > T, \ U_i > Y_i. \end{cases} \]

If we know only the answer concerning the value of \( Y \) but not the question asked, i.e., whether \( \Upsilon_i \leq T \) or not, \( Z_{i,AV,T} \) has the expectation \( E(Z_{i,AV,T}) = 1 - \left| \frac{T}{M} - \frac{Y_i}{M} \right| \).

Unfortunately, in such a case it is impossible to construct either an estimator of the population total \( t_Y \) or of the population mean \( \bar{Y} \).

On the other hand, if we know both the answer concerning the value of \( Y \) and the question asked, i.e., whether \( \Upsilon_i \leq T \) or not, then \( E(Z_{i,AV,T}) = \frac{Y_i}{M} + \frac{T}{M} - 1 \). In this case, the transformation of \( Z_{i,AV,T} \) to \( R_{i,AV,T} = MZ_{i,AV,T} + M - T \) enables us to construct
an unbiased estimator of the population total \( t_Y \) of the Horvitz-Thompson’s type, which has the form

\[
(22) \quad t_{HT,AV,T}^{R} \equiv t_{AV,T}^{R} = \frac{N}{n} \sum_{i \in s} R_{i,AV,T}.
\]

An unbiased estimator of the population mean \( \bar{t}_Y \) can be constructed analogously.

As regards the variance of \( R_{i,AV,T} \), we must distinguish between \( Y_i > T \) and the complementary inequality. It holds

\[
(23) \quad \text{var}(R_{i,AV,T}) = \begin{cases} 
Y_i(M - Y_i) + (M - T)(2Y_i + T), & Y_i \leq T, \\
Y_i(M - Y_i) + T(3M - 2Y_i - T), & Y_i > T.
\end{cases}
\]

If we compare (23) with (6), we can see that the variance of \( R_{i,AV,T} \) is always higher than that of \( R_{i,AV,(0,M)} \). Because negative values of \( Z_{i,AV,T} \) may occur, this may occasionally lead to negative values of \( R_{i,AV,T} \). More precisely, note that \( R_{i,AV,T} \) attains only three values, i.e., positive (equal to \( 2M - T \)), zero, and negative (equal to \( -T \)), being the source of its poor performance. Recall that \( t_{HT,AV,T}^{R} \) is intended to estimate non-negative variable \( Y \). Unfortunately, looking at the results of our simulations we observe that \( t_{HT,AV,T}^{R} \) quite often returns inadmissibly low, or even negative values; this is a big drawback.

A simple, but somewhat tedious, analysis of (23) shows that we cannot find the optimal value of the threshold from an open interval \( 0 < T < M \) minimizing \( \text{var}(R_{i,AV,T}) \). Moreover, numerical experiments show that the variance of \( \text{var}(R_{i,AV,T}) \) is acceptable only for very low, or very high, values of the threshold \( T \), like \( T = 0.1 \) or \( T = 0.9M \). For example, the variance contribution of the modified \( R \) for \( T = 0.9M \) is

\[
E_p\left( \text{var}_d(t_{HT,AV,T=0.9M,s}^{R}) \right) = \frac{N}{n} \left( \sum_{i \in U} Y_i(M - Y_i) + \sum_{i \in U | Y_i \leq T} (0.20Y_i + 0.09M) \right. \\
\left. + \sum_{i \in U | Y_i > T} (-1.80Y_i + 1.89M) \right).
\]

Notice that if a non-negative surveyed random variable \( Y \) is bounded not only from above, but also from below, i.e., \( 0 < m \leq Y \leq M \), we generate \( Y \) from the uniform distribution on \((m,M)\), and, analogously to (21), form random variables

\[
Z_{i,AV,T,(m,M)} = \begin{cases} 
1, & \text{if } Y_i \leq T, Y_i \leq Y_i, \\
0, & \text{if } Y_i \leq T, Y_i > Y_i, \text{ or if } Y_i > T, Y_i \leq Y_i, \\
-1, & \text{if } Y_i > T, Y_i > Y_i.
\end{cases}
\]

It is easy to show that \( E\left(Z_{i,AV,T,(m,M)}\right) = (T + Y_i - m - M)/(M - m) \), so that if we transform \( Z_{i,AV,T,(m,M)} \) to \( R_{i,AV,T,(m,M)} = (M - m)Z_{i,AV,T,(m,M)} + m + M - T \), then \( E\left(R_{i,AV,T,(m,M)}\right) = Y_i \). Now we can form unbiased estimator of the population total \( t_Y \) of the Horvitz-Thompson’s type, analogously to (17), of the form

\[
(24) \quad t_{HT,AV,T,(m,M)}^{R} \equiv t_{AV,T,(m,M)}^{R} = \frac{N}{n} \sum_{i \in s} R_{i,AV,T,(m,M)}.
\]
Let us point out that modifications described in this Section are interesting especially from the theoretical point of view. Despite them offering a seemingly nice idea, they cannot be recommended for practical use. We can also compare the results of the simulations.

3.4. Random number generation. In all RRTs we are aware of, the preparation of the random mechanism is probably the trickiest point. For example, it is not clear how to design an acceptably large deck of cards that would sufficiently mask the true values (interviewer cannot guess very close to the true values using respondents’ answers and the knowledge of cards from this deck) and provide sufficient accuracy. Assume now direct face-to-face interviewing and describe several possibilities for generating random numbers.

(1) We allow the respondent to select the random number according to the European ISO 28640:2010(en) Standard, which provides not only the methods suitable for generation, but also tables of random numbers and random digits. Recall that equivalents of this Standard, as well as of the tables of random numbers, exist all over the world. We are convinced that existence of an international standard can increase credibility of the survey and willingness of respondents to respond truthfully. The selected random number is then used according to the RRT used.

(2) To those who feel they are “experts in the field of randomness”, the reviewer can offer that they select a random number from the uniform distribution using his/her own method. The remaining procedure is the same as described above.

(3) Another possibility is, e.g., using a huge deck of cards, for example cards with 100-CZK value steps in our case, but it would require additional calculations to find the bias of such an approach.

On the other hand, we would like to point out that the question of credibility is not only a matter for statisticians, but more and more a task for psychologists. While statisticians must suggest procedures which are “sufficiently random” in their eyes, psychologists must find and offer ways to convince the respondents that they are not cheated. Unfortunately, a detailed discussion of this topic would go beyond the scope of this paper.

4. Simulation Study

In many countries, income is recognized as a private and (highly) sensitive item of information. The respondents often refuse to respond at all or provide strongly biased answers. This in particular happens if their income is (very) high or (very) low. That leads us to assessing the performance of the proposed RRT by a simulation study using Czech wage data from the Average Earnings Information System (IPSV) of the Ministry of Labor and Social Affairs of the Czech Republic.

Based on the extensive analysis of monthly wage data provided by IPSV from the years 2004–2014, Vrabec and Marek (2016) recommended a model of wages in the Czech Republic as a three-parameter log-logistic distribution with the density

\[
 f(y; \tau, \sigma, \delta) = \begin{cases} 
 \frac{\tau}{\sigma} \left(\frac{y - \delta}{\sigma}\right)^{\tau-1} \left(1 + \left(\frac{y - \delta}{\sigma}\right)^{-\tau}\right)^{-2}, & y \geq \delta > 0, \ \tau > 0, \ \sigma > 0, \\
 0, & \text{otherwise},
\end{cases}
\]

where \( \tau > 0 \) is a shape parameter, \( \sigma > 0 \) is a scale parameter, and \( \delta \) is a location parameter.
We estimate parameters of (25) using the data from 2nd quarter 2014, and receive (26) \( \hat{\tau} = 4.0379, \hat{\sigma} = 21,687 \) and \( \hat{\delta} = 250. \) The corresponding estimated average monthly income is 24,290 CZK (approximately 950 EUR). Note that the estimates (26) are based on roughly \( 2.1 \times 10^6 \) observations, covering practically half of the overall relevant population.

Histograms of the data with the bin width 500 (CZK), and density of the log-logistic distribution (25) with the unknown parameters replaced by their estimates (26), are presented in fig. 1. Moreover, the corresponding sample quantile function of the observed wages is presented in fig. 2. It is interesting to take a look at both lower and upper sample quantiles of the data used. While 8,000 CZK corresponds to the 0.01 sample quantile, 40,000 CZK corresponds to the 0.91 sample quantile, 60,000 CZK to the 0.97 sample quantile and, finally, 80,000 CZK to the 0.98 sample quantile, compare visually fig. 2.

Figure 1: Probability histogram of monthly wages in the Czech Republic in the 2nd quarter of 2014, and the density (in red) of approximating model (25) with the parameters estimated by (26).

Figure 2: The sample quantile function of monthly wages in the Czech Republic in the 2nd quarter of 2014.
It is evident from fig. 1 that the original data is highly skewed. Therefore, it is not surprising that the mean relative concentration measure $\Gamma_{Y,M} = 0.198$ is close to its attainable maximum. In such a case, as follows from sec. 3, we can expect higher variance of the estimators using the suggested RRT than for the Horvitz-Thompson’s estimator based on non-randomized data. Moreover, the estimator $t_{HT,AV,\alpha}^R$ based on the knowledge of $\Upsilon_i$’s and “almost-optimal” choice of the parameter $\alpha \approx 3\Gamma_{Y,M}$, should have smaller variance than $t_{HT,AV,(m,M)}^R$ (corresponding to $\alpha = 0$). That conjecture is confirmed by our simulations.

From the “theoretical” wage distribution corresponding to model (25), in which unknown parameters have been replaced with their estimates (26), 1,000 replications of populations sized $N = 200$, or $N = 400$ are simulated. The simulations are carried out with the aid of statistical freeware R, version 3.5.1; for details, see R Core Team (2018). Data from the log-logistic distribution is generated using the package flexsurv.

From each replication of the population, we draw, without replacement, 1,000 random samples of the size $n = 20$, or $n = 50$. Such population and sample sizes are standard for separate strata in the business sampling surveys, and also resemble the usual social statistical surveys, such as the EU Statistics of Income Living Condition. In such a survey for a medium sized country like the Czech Republic with the population of 10,000,000 inhabitants and approximately 4,300,000 households, the samples approximately include 9,500 households surveyed in a two-dimensional stratification (region and size of municipality), giving $78 \times 4 = 312$ strata. The average sample size is then about 30 per stratum. In EU-SILC, detailed results are presented for eight income groups, leading on average to the population size of approximately $N = 1,250,000$ inhabitants per one income group. For a more detailed description of the stratification, strata, sample sizes and sampling design see EU-SILC 2016.

For each sample, both $t_Y$ and $\Upsilon_Y$ are estimated using the techniques described in sec. 3. Estimates of the total mean values, instead of the population totals, are presented to enable easier comparison between the results obtained for populations with different sizes $N$ and different sample sizes $n$.

In the simulations, we are especially interested in the impact of “tuning parameters” $m, M, T, \alpha$ and $\alpha_{opt}$ on the estimates. Taking into account the type and nature of the data we are simulating, we set the parameters as described in tab. 1. The values of $\alpha_{opt}$ were set using the formulae for the optimal variance described in sec. 3. Other parameters were chosen with regard to our experience, in particular, which monthly salary can be perceived to be high. Because practically all the available data is larger than 7,000 CZK, we set the lower bound of the interval for generating pseudorandom numbers $\Upsilon_i$ to $m = 7,000$. 

Table 1: Choice of tuning parameters for the simulations.

| $m$     | $M$     | $T$     | $\alpha$ | $\alpha_{opt}$ |
|---------|---------|---------|----------|----------------|
| 7,000   | 40,000  | 30,000  | 0.75     | 0.72           |
| 7,000   | 60,000  | 45,000  | 0.75     | 0.59           |
| 7,000   | 80,000  | 45,000  | 0.75     | 0.52           |

Note that the simulation results virtually do not change after 100 replications of the population; differences begin at the third significant digit.
The results are summarized in tab. 2–4 and in fig. 3–5. They show that for large populations the accuracy of the suggested estimators is acceptable even for the method of the switching questions. The reason for the lower standard deviation of $\hat{t}_{AV,\alpha}$, and especially $\hat{t}_{AV,\alpha,\text{opt}}$, in comparison with $\hat{t}_{AV,T}$ and $\hat{t}_{AV,(m,M)}$ is that this estimator efficiently uses the information on the generated numbers of $\Upsilon$. Note that we have used the moment plug-in estimate for the optimal value of $\alpha$.

As expected, the variance values of our new estimator and its modifications are higher than those of Horvitz-Thompson’s estimator based on the non-randomized data. The precision of our basic proposal is practically acceptable, because, according to the simulations, the corresponding sample standard deviation of the estimates has gone up by a mere 60% in comparison with the Horvitz-Thompson estimate for $M = 60000$; this result is quite reasonable, taking into account that $Y$ is a very sensitive variable. Notice, however, that the modification using the knowledge of the values of $\Upsilon_i$ leads to a substantial reduction in variance. Thus, while mildly relaxing respondents’ privacy on the one hand but still keeping secret the true response because the true value of the sensitive variable is never reported, this modification provides estimates whose precision is comparable with directly surveying under zero non-response. On the other hand, the high variability of the estimates, even the presence of negative estimates for the mean wages, shows that the modification using the switching questions described in sec. 3.3 is only a theoretical exercise and cannot be recommended for practical use. Its improvement remains an open question.

Comparing contents of all tables, we can see that the mean has practically not changed; however, the expected decrease occurs in the variability of the estimates, of about 9%, which shows that it pays “to tune up” the procedure and its parameters according to the given problem and potential data.

Both results of sec. 3.1 and simulations show that variance of estimators can be greatly reduced by choice of bounds $m$ and $M$. We see that for low value of the upper bound $M = 40,000$ the proposed estimators are competitive even with Horvitz-Thompson estimator. It follows from the bias formula 15 that approximately unbiased estimators with low variance can be constructed if we use prior information on population quantiles for choice of bounds $m$ and $M$. Optimal choice of bounds with respect to the minimization of the mean square error is field of further research.

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2In tab. 2–4 both the sample averages (mean) and sample standard deviations (sd) of the simulated values are presented. For simplicity, we omit “HT” in the descriptions of the analyzed estimators in all figures and tables because all the estimators we compare here are of the Horvitz–Thompson’s type.
Table 2: Numerical results of simulations. The mean estimated salaries (in $10^3$ CZK) and the corresponding sample standard deviations (in $10^3$ CZK) for different population sizes $N$ and sample sizes $n$. Random numbers $\Upsilon_i$ are generated from the uniform distribution on the interval $[m, M] = [7,000; 40,000]$, $T = 30,000$, $\alpha = 0.75$, $\alpha_{opt} = 0.72$, 1,000 simulated populations, 1,000 replications of each.

| Estimator          | $N = 200$ | $N = 400$ |
|--------------------|-----------|-----------|
|                   | $n = 20$  | $n = 50$  | $n = 20$  | $n = 50$  |
| $\bar{y}_{HT}$    | mean      | 24.270    | 24.272    | 24.287    | 24.288    |
|                   | sd        | 2.782     | 1.757     | 2.773     | 1.758     |
| $\bar{y}^R_{AV,(m,M)}$ | mean     | 23.189    | 23.192    | 23.203    | 23.205    |
|                   | sd        | 3.687     | 2.333     | 3.690     | 2.336     |
| $\bar{y}^R_{AV,\alpha}$ | mean    | 23.192    | 23.194    | 23.206    | 23.207    |
|                   | sd        | 3.000     | 1.897     | 3.001     | 1.902     |
| $\bar{y}^R_{AV,\alpha_{opt}}$ | mean | 23.192    | 23.194    | 23.206    | 23.207    |
|                   | sd        | 2.965     | 1.875     | 2.966     | 1.880     |
| $\bar{y}^R_{AV,T}$ | mean      | 23.185    | 23.189    | 23.199    | 23.202    |
|                   | sd        | 6.066     | 3.836     | 6.068     | 3.837     |

Table 3: Numerical results of simulations. The mean estimated salaries (in $10^3$ CZK) and the corresponding standard deviations (in $10^3$ CZK) for different population sizes $N$ and sample sizes $n$. Random numbers $\Upsilon_i$ are generated from the uniform distribution on the interval $[m, M] = [7,000; 60,000]$, $T = 45,000$, $\alpha = 0.75$, $\alpha_{opt} = 0.59$, 1,000 simulated populations, 1,000 replications of each.

| Estimator          | $N = 200$ | $N = 400$ |
|--------------------|-----------|-----------|
|                   | $n = 20$  | $n = 50$  | $n = 20$  | $n = 50$  |
| $\bar{y}_{HT}$    | mean      | 24.297    | 24.301    | 24.288    | 24.290    |
|                   | sd        | 2.773     | 1.758     | 2.813     | 1.779     |
| $\bar{y}^R_{AV,(m,M)}$ | mean     | 23.983    | 23.984    | 23.965    | 23.974    |
|                   | sd        | 5.530     | 3.501     | 5.529     | 3.495     |
| $\bar{y}^R_{AV,\alpha}$ | mean    | 23.974    | 23.976    | 23.956    | 23.965    |
|                   | sd        | 4.401     | 2.786     | 4.398     | 2.780     |
| $\bar{y}^R_{AV,\alpha_{opt}}$ | mean | 23.976    | 23.977    | 23.958    | 23.967    |
|                   | sd        | 4.164     | 2.637     | 4.161     | 2.631     |
| $\bar{y}^R_{AV,T}$ | mean      | 23.991    | 23.992    | 23.973    | 23.982    |
|                   | sd        | 9.066     | 5.729     | 9.067     | 5.726     |
Table 4: Numerical results of simulations. The mean estimated salaries (in $10^3$ CZK) and the corresponding standard deviations (in $10^3$ CZK) for different population sizes $N$ and sample sizes $n$. Random numbers $\Upsilon_i$ are generated from the uniform distribution on the interval $[m, M] = [7,000; 80,000]$, $T = 45,000$, $\alpha = 0.75$, $\alpha_{opt} = 0.53$, 1,000 simulated populations, 1,000 replications of each.

| Estimator         | $N = 200$  |     | $N = 400$  |     |
|-------------------|------------|-----|------------|-----|
|                   | $n = 20$   | $n = 50$ | $n = 20$   | $n = 50$ |
| $\bar{\tau}_{HT}$| mean       | 24.275 | 24.273     | 24.299 | 24.299 |
|                   | sd         | 2.765  | 1.739      | 2.753  | 1.737  |
| $\bar{\tau}^R_{AV,(m,M)}$ | mean       | 24.138 | 24.140     | 24.158 | 24.168 |
|                   | sd         | 6.911  | 4.372      | 6.921  | 4.378  |
| $\bar{\tau}^R_{AV,\alpha}$ | mean      | 24.145 | 24.146     | 24.165 | 24.174 |
|                   | sd         | 5.962  | 3.770      | 5.950  | 3.767  |
| $\bar{\tau}^R_{AV,\alpha_{opt}}$ | mean     | 24.143 | 24.145     | 24.163 | 24.173 |
|                   | sd         | 5.404  | 3.417      | 5.398  | 3.417  |
| $\bar{\tau}^R_{AV,T}$ | mean      | 24.136 | 24.137     | 24.156 | 24.165 |
|                   | sd         | 13.018 | 8.236      | 13.036 | 8.244  |
Figure 3: Behavior of considered estimators applied to different sizes of the population $N$ and sample sizes $n$; $(m, M) = (7,000; 40,000)$, $T = 30,000$, $\alpha = 0.75$ and $\alpha_{opt} = 0.72$. 

- N=200, n=20, m=7000, M=40000
- N=200, n=50, m=7000, M=40000
- N=400, n=20, m=7000, M=40000
- N=400, n=50, m=7000, M=40000

$\bar{t}_{HT}$, $\bar{t}_{AV,(m,M)}^R$, $\bar{t}_{AV,\alpha}^R$, $\bar{t}_{AV,\alpha_{opt}}^R$, $\bar{t}_{AV,T}^R$

$\alpha = 0.75$, $\alpha_{opt} = 0.72$, $T = 30000$
Figure 4: Behavior of considered estimators applied to different sizes of the population $N$ and sample sizes $n$: $(m, M) = (7,000; 6,000), T = 45,000$, $\alpha = 0.75$ and $\alpha_{opt} = 0.59$. 

\[ \bar{t}_{HT}, \bar{t}_{AV,(m,M)}, \bar{t}_{AV,\alpha}, \bar{t}_{AV,\alpha_{opt}}, \bar{t}_{AV,T} \]

$\alpha = 0.75$, $\alpha_{opt} = 0.59$, $T = 45000$.
Figure 5: Behavior of considered estimators applied to different sizes of the population $N$ and sample sizes $n$: $(m, M) = (7,000; 80,000)$, $T = 45,000$, $\alpha = 0.75$ and $\alpha_{opt} = 0.53$.

5. Conclusions

The purpose of this paper is to present a new randomized response technique possessing two attractive properties, namely:

- It is simple to use.
- It provides a high level of anonymity to the respondent.

Though a quantitative estimate is the final end, the respondent is only asked for a qualitative response. Two modifications are discussed as well. The suggested estimators are based on the values of pseudorandom numbers generated by the respondents, which are used for masking sensitive information.

A small disadvantage of the suggested method may, for some respondents, be a feeling of infringement on their privacy due to an extrinsic device/technique being used for generating the random numbers. This problem is of mainly a psychological nature and
can, at least partially, be resolved by a proper explanation of the approach by the interviewer. Unfortunately, all currently used RRT procedures suffer, to a certain extent, from the same problem – see, for example, the thorough discussion in Chaudhuri (2017) and Chaudhuri and Christofides (2013).

The first modification assumes that not only the respondent, but also the interviewer knows the generated random number that masks the true value of the response. The second modification makes use of switching questions with the aim to make the survey less embarrassing for respondents in certain specific situations. For all suggested RRT procedures, we show their unbiasedness, and derive the corresponding variance for the Horvitz-Thompson’s type estimator under the simple random sampling without replacement. The optimal values of the tuning parameters enabling us to minimize the variance of the suggested procedures are also discussed. The first modification seems to be especially promising because we have shown that knowing the random number and properly setting the tuning parameters can sufficiently increase the precision of the estimator. For the second modification, it is good to know that it would not work in practice. On the other hand, we admit that, for some readers, the suggested modifications may be of interest, even if only from the theoretical point of view.

As a technical tool, two auxiliary measures are proposed, called the mean relative concentration measure of the values of \( Y \) around the center of interval \((0, M)\), and the proximity measure of the population mean to the center of interval \((0, M)\). With the aid of these measures we can explain why, and especially how, the suggested RRTs increase the variance of the estimators of \( t_Y \) and \( \bar{t}_Y \) for symmetrical distributions; distributions closely concentrated around their centers; or uniform distributions.

We would like to summarize the merits of the method proposed in this paper. In our opinion, we are bringing progress in this field. The first advantage is that our method is easy to implement because there exist many more or less easily available online/offline generators of random numbers from the uniform distribution. If the main goal of a survey is to estimate a continuous random variable with a large span, like income or personal wealth then, for the “classical RRT methods” described in sec. 2, we need to design a very large deck of cards to mask the true values of the surveyed variable. For example, if we assume an income range from 7,000 CZK to 60,000 CZK, as is reasonable in our example, the number of cards needed for Eriksson’s RRTs, provided the income values are rounded to 1,000 CZK, is 54. If the rounding step is 500 CZK, then 107 cards are needed. Finally, if the rounding step is 100 CZK, then 531 cards are needed. Manipulations with such a large deck of cards can be cumbersome for both the respondent and the interviewer. Even if the span of the surveyed variable is not very large, it is not easy to find precise instructions, or algorithms, concerning how to design the corresponding deck of cards. Difficulties with this design may pose a problem for the field survey statisticians, discouraging them from the use of such RRTs.

Our technique also shares the ease of use with Eriksson’s technique. Unlike within Chaudhuri’s approach, which requires quite demanding arithmetic operations from the respondent, each respondent only states whether his/her true income is higher than a certain number. Let us point out that the respondent never reports the true value of the variable. In our original proposal, described in sec. 3, the interviewer moreover does not know the value of \( \bar{Y} \). Thus, the privacy of respondent is protected better than in Eriksson’s approach, which intrudes on the privacy of the respondents to a certain extent. Indeed, if the value reported by a respondent differs from any of \( x_t \), the interviewer learns about the true value of the sensitive variable.
Finally, note that we find rather problematic any comparison of our approach with the methods employed by Eriksson or Chaudhuri, because their performance strongly depends on the choice of the cards used. In our opinion, it is tricky to design a deck of cards for a continuous variable with a high range, such as the income in the Czech Republic, and a reliable estimator of this type with an acceptably small variance value would need an excessively large deck of cards.

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