Rayleigh-Taylor instability in binary condensates

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(Dated: October 4, 2009)

We propose a scheme to initiate and examine Rayleigh-Taylor instability in the two species Bose-Einstein condensates. We identify $^{85}\text{Rb}-^{87}\text{Rb}$ mixture as an excellent candidate to observe it experimentally. The instability is initiated by tuning the $^{85}\text{Rb}-^{85}\text{Rb}$ interaction through magnetic Feshbach resonance. We show that the observable signature of the instability is the damping of the radial oscillation. This would perhaps be one of the best controlled experiments on Rayleigh-Taylor instability. We also propose a semi analytic scheme to determinate stationary state of binary condensates with the Thomas-Fermi approximation for the axis symmetric traps.

PACS numbers: 03.75.Mn, 03.75.Kk.

Introduction.—Rayleigh-Taylor instability (RTI) sets in when lighter fluid supports a heavier one. It is present across a wide spectrum of phenomena related to interface of two fluids. The turbulent mixing in astrophysics, inertial confinement fusion and geophysics originate from RTI. In superfluids, RTI sets up crystallization waves at the superfluid-solid $^4\text{He}$ interface [1]. Despite the ubiquitous nature and importance, controlled experiments with RTI are difficult and rare. However, we show that the two species Bose-Einstein condensates (TBECs) or binary condensates are ideal systems for a controlled study of RTI in superfluids. The remarkable feature of TBECs, absent in single component BECs, is the phenomenon of phase separation. The TBECs, first realized in a mixture of two hyperfine states of $^{87}\text{Rb}$ [2], are rich systems to explore nonlinear phenomena. Several theoretical works have examined various aspects of TBECs. These include stationary states [3, 4, 5, 6], modulational instability [7, 8, 9], collective excitations [10, 11, 12, 13] and domain walls solitons[14]. Another instability related to RTI, which has attracted growing interest, is the Kelvin Helmholtz instability (KHI). The prerequisites of KHI are, phase separation and relative tangential velocities at the interface. Quantum KHI have been observed in experiments with $^3\text{He}$ [15] and recently studied theoretically for TBEC [16].

To initiate RTI we start with the phase separated state. Then, increase the scattering length of the species at the core. At a certain value it creates a quantum analogue of RTI in fluid dynamics. As a case study we choose the TBEC of $^{85}\text{Rb}-^{87}\text{Rb}$ mixture. In this system, the $^{85}\text{Rb}$ intra species interaction is tunable through a Feshbach resonance [18] and was recently used to study the miscibility [19]. More recently, the dynamical pattern formation during the growth of this TBEC was theoretically investigated [8]. The other feature is, the inter species $^{85}\text{Rb}-^{87}\text{Rb}$ interaction is also tunable and well studied [20]. Considering the parameters of the experimental realization, we choose the axis symmetric (cigar shaped) trap geometry.

Phase separated cigar shaped TBECs.—In the mean field approximation, the TBEC is described by a set of coupled Gross-Pitaevskii equations

$$\left[ -\frac{\hbar^2}{2m_i} \nabla^2 + V_i(\rho,z) + \sum_{j=1}^{2} U_{ij} |\psi_j|^2 \right] \psi_i(\rho,z) = \mu_i \psi_i(\rho,z),$$

where $i=1,2$ is the species index, $U_{ii} = 4\pi \hbar^2 a_i/m_i$ with $m_i$ as mass and $a_i$ as s-wave scattering length, is the intra-species interaction; $U_{ij} = 2\pi \hbar^2 a_{ij}/m_{ij}$ with $m_{ij} = m_im_j$ as reduced mass and $a_{ij}$ as inter-species scattering length, is inter-species interaction and $\mu_i$ is the chemical potential of the $i^{th}$ species.

To study the RTI, we consider the phase separated state $(U_{12} > \sqrt{U_{11}U_{22}})$ in axis symmetric trapping potentials $V_i(\rho,z) = m_i \omega^2(\alpha_i^2 \rho^2 + \lambda_i^2 z^2)/2$. In the present work we consider cigar shaped potentials, that is the anisotropy parameters $\alpha_i > \lambda_i$ and $U_{ij}$ are all positive. Neglecting the inter species overlap, the Thomas-Fermi (TF) solutions are $|\psi_i(\rho,z)|^2 = [\mu_i - V_i(\rho,z)]/U_{ii}$. The chemical potentials $\mu_i$ are fixed through the normalization conditions. When $\alpha_i > \lambda_i$, the interface of the phase separated state is planar and species having larger scattering length sandwiches the other one [17].

For simplicity of analysis consider trapping potentials with coincident centers. Then, let $z = \pm L_1$ be the planes separating the two components and $\pm L_2$, the spatial extent of the outer species along z-axis. The density distributions $n_1$ and $n_2$ of the TBEC are

$$n_1(\rho,z) = \frac{\mu_1 - V_1(\rho,z)}{U_{11}}, \quad -L_1 < z < L_1,$$

$$n_2(\rho,z) = \frac{\mu_2 - V_2(\rho,z)}{U_{22}}, \quad L_1 < |z| < L_2.$$  

This assumes no overlap between the two species. Then the problem to determine the stationary state is equivalent to calculating $L_1$. Theoretically, $L_1$ can be determined by minimising the total energy of the TBEC with fixed number of particles of each species. If $N_i$ and $\rho_i$ are the number of atoms and radial size of $i^{th}$ species
respectively, then

$$N_i = 2\pi \int_0^\rho d\rho \int_{-L_i}^{L_i} dz |\psi_i(\rho, z)|^2. \quad (4)$$

From the TF approximation

$$N_1 = \frac{\pi L_1 (3\omega^2 L_1^4 m_1 \lambda_1^2 - 20 L_1^2 \lambda_1^2 \mu_1 - 60(\omega^2 m_1 - 2)\mu_1^2)}{30U_1 \alpha_1^2}$$

$$N_2 = \frac{2\pi}{3\lambda_2 U_{22}} \left[ \frac{L_1^2 \lambda_2^2}{20a_0^2} (5\omega^2 \lambda_2^2 \mu_2^2 - 8\omega^2 m_2 (L_1^2 \lambda_2^2)^{3/2}} - 60\lambda_2 L_1(\omega^2 m_2 - 1)\mu_2 + 40(\omega^2 m_2 - 1)L_1 \lambda_2 \mu_2 \right]
- \frac{\mu_2}{3a_0^2} (-5\omega^2 \lambda_2^3 \mu_2^3 - 15\lambda_2 L_1(\omega^2 m_2 - 2)\mu_2
+ 4\sqrt{2}(3\omega^2 m_2 - 5)\mu_2^3/2). \quad (6)$$

The total energy of the binary condensate is

$$E = \int dV \left[ V_1(\rho, z)|\psi_1(\rho, z)|^2 + V_2(\rho, z)|\psi_2(\rho, z)|^2 + \frac{1}{2} U_{11}|\psi_1(\rho, z)|^4 + \frac{1}{2} U_{22}|\psi_2(\rho, z)|^4 \right]. \quad (7)$$

We minimize $E$ numerically, with Eq. (5) and (6) as constraints, to obtain the required value of $L_1$. Substituting the value of $L_1$ back into Eqs. (5) and (6), one can determine $\mu_1$ and $\mu_2$. Thus Eqs. (5) and (6) uniquely define the stationary state of the TBEC.

As mentioned earlier, we consider the parameters of the recent experiment [12] with $^{85}$Rb and $^{87}$Rb as the first and second atomic species. The radial trapping frequencies are identical ($\alpha_i = 1$) and for the axial trapping frequencies $\lambda_1 = 0.022$ and $\lambda_2 = 0.020$. The scattering lengths are $a_{11} = 51a_0$, $a_{22} = 99a_0$ and $a_{12} = a_{21} = 214a_0$, and we take $N_i = 50,000$. Then, Fig. 1 shows the variation in $E$ as a function of $L_1$. The value of $L_1$ where minimum of $E$ occurs is $32.5a_{\text{osc}}$. Here the unit of length $a_{\text{osc}} = \sqrt{\hbar/m\omega}$ with $\omega = 130$Hz, is the radial trapping frequency. This is in agreement with the numerical result $33.8a_{\text{osc}}$ calculated using split-step Crank-Nicholson method (imaginary time propagation) [21]. We refer to this state as phase I, where $^{85}$Rb and $^{87}$Rb are at the center and flanks respectively. We have also calculated the equations of interface planes for trapping potentials whose minima do not coincide. The expressions are much more complicated, however the numerical and semi-analytic results are in agreement.

**Binary condensate evolution.**—In the fluid dynamics parlance, the gradient of the trapping potentials are the equivalent of gravity. If $s$ is the oscillation frequency of the interface between the two condensates, one placed over the other. Then from Bernoulli’s principle along with proper boundary conditions [22, 23], we find from linear stability analysis

$$s = \pm \left[ \sqrt{\frac{k_x^2 + k_y^2 m\omega^2 \lambda^2 L(n_1 - n_2)}{n_1 + n_2}} \right]^{1/2} \quad (8)$$

Here $k_x$ and $k_y$ are wave numbers along $x$ and $y$ coordinates. The densities $n_1$ and $n_2$ are at a point $(\rho, L)$ on the interface. For the sake of simplicity, we consider $m_1 = m_2 = m$ and $\lambda_1 = \lambda_2 = \lambda$ while deriving the above relation. There is an instability, referred to as Rayleigh-Taylor instability, at the interface when $n_1 < n_2$. From the TF approximation this condition is equivalent to $a_{11} > a_{22}(\mu_1 - V)/(\mu_2 - V)$. Here $V$ is the trapping potential of the two species at the interface. Normal fluids with RTI, any perturbation at the interface however small grows exponentially. Then the lighter fluid rises to the top as bubbles and heavier fluid sinks as finger like extensions till the entire bulk of the lighter fluid is on top of the denser one. On the other hand, binary condensates in a similar situation evolve in a very different way.

To examine the dynamical evolution of the binary condensate with RTI, we take phase I ($a_{11} < a_{22}$) as the initial state. In this phase, the $^{87}$Rb BEC at the flanks is considered as resting over the $^{85}$Rb BEC at the core. Then through the $^{85}$Rb–$^{87}$Rb magnetic Feshbach resonance [18] increase $a_{11}$ till $a_{11} > a_{22}(\mu_1 - V)/(\mu_2 - V)$ to set up RTI. However, maintain $U_{12} > \sqrt{U_{11}/U_{22}}$ so that the TBEC is still immiscible. Let us call this as phase Ia and it is an unstable state. The stationary state of the new parameters is phase separated and similar in structure to the initial state. But with the species interchanged. Let us call the stationary state of the new parameters as phase II. The binary condensate should dynamically evolve from phase Ia to II. However, unlike in normal fluids with RTI, there are no bulk flows of either $^{85}$Rb or $^{87}$Rb atoms, to the periphery of the trap. Instead the condensates tunnel with modulations. This occurs due to the coherence in the quantum liquids. To examine the evolution, we solve the pair of

![FIG. 1: The variation in energy $E$ with $L_1$ in phase separated regime. The upward arrow indicates the position of minimum $E$, which is at $L_1 = 32.5a_{\text{osc}}$. Inset shows the same plot along with the variation of $\mu_1$ and $\mu_2$ with respect to $L_1$, the blue and black curves correspond to $\mu_1$ and $\mu_2$ respectively.](image-url)
time-dependent GP equations

\[ i\hbar \frac{\partial \psi_i(\rho, z)}{\partial t} = \left[ -\frac{\hbar^2}{2m_i} \nabla^2 + V_i(\rho, z) + \sum_{j=1}^{2} U_{ij} |\psi_j|^2 \right] \psi_i(\rho, z), \tag{9} \]

which describe the TBEC. During the evolution, the density profiles is approximated as \( n_i(\rho, z) = n_i^{eq}(\rho, z) + \delta n_i(\rho, z) \). Here \( n_i^{eq}(\rho, z) \) and \( \delta n_i(\rho, z) \) are the equilibrium density and fluctuation arising from the increase in \( a_{11} \). Following the hydrodynamic approximations, the \( \delta n_i(\rho, z) \) or collective modes follow the equations

\[ m_i \frac{\partial^2 \delta n_i}{\partial t^2} = \nabla n_i \cdot \nabla \sum_{j=1}^{2} U_{ij} \delta n_j + n_i \nabla^2 \sum_{j=1}^{2} U_{ij} \delta n_i. \tag{10} \]

Consider \( \delta n_i(\rho, z, t) = a_i(t) \rho^l \exp(\pm i\phi) \) as the form of the solution, where \( a_i(t) \) subsumes the time dependent part of the solution including temporal variation of the amplitude and \( l \) is an integer. Then as \( \nabla^2 \delta n_i = 0 \) and for the miscible phase, considered for simplicity of the boundary conditions, we get

\[ \delta n_i = \frac{m^2}{U_{ij}} (U_{ii} a_{11} + U_{ij} a_{22}). \tag{11} \]

We can also get a similar set of coupled equations for the other form of the collective modes \( \delta n_i(\rho, z, t) = a_i(t) \rho^{l-1} \exp(\pm i(l-1)\phi) \). In this case the prefactor is \((l-1 + \lambda^2)\) instead of \( l \). In either of the cases, the equations are similar to two coupled oscillators. For the phase separated state, the form of the TF solutions are significantly different from the miscible one. However, when RTI sets in, the collective modes like in miscible case, are damped and coupled as the condensates interpenetrate each other.

**TBEC evolution with RTI.**—To examine the evolution of TBEC with RTI, as mentioned earlier, we choose the phase I as the initial state. Then change \( a_{11} \) to 80\( a_0 \), 102\( a_0 \), 200\( a_0 \), 306\( a_0 \), 408\( a_0 \) and 780\( a_0 \), the last value is in the miscible parameter region. The dynamical variables which are coarse grained representative of the dynamical evolution are \( \rho_{rms} \) and \( z_{rms} \), the \( \rho_{rms} \) radial and axial sizes.

When \( a_{11} \) is increased to 80\( a_0 \), the \( ^{85}\text{Rb} \) condensate oscillates radially to accommodate excess repulsion energy. This is the only available degree of freedom as tight confinement, arising from \( ^{87}\text{Rb} \) at the flanks, along \( z \)-axis restricts axial oscillations. In TF approximation the effective potential \( V_{eff} = V + (\mu_2 - V)U_{12}/U_{22} \). The angular frequency of the oscillation is \( \approx 0.32 \omega \). This is close to one of the eigen modes of the Bogoliubov equations. The temporal variation of \( \rho_{rms} \) is shown in Fig.2 (inset plot). The plots show that, the oscillation of the \( ^{87}\text{Rb} \) is sympathetically initiated. This is due to the coupling between the two condensate species. The oscillations are more prominent with less number of atoms.

There is a change in the nature of oscillations when \( a_{11} > a_{22}(\mu_1 - V)/(\mu_2 - V) \). The corresponding station-ary state has \( ^{87}\text{Rb} \) and \( ^{85}\text{Rb} \) at the core and flank respectively. The \( \rho_{rms} \) oscillation frequency is the same as in \( a_{11} < a_{22} \) case. But there is a temporal decay of the amplitude till it equillibrates. The decay is due to the expansion of \( ^{85}\text{Rb} \) along \( z \)-axis and is an unambiguous signature of RTI. The expansion is clearly discernible in the density profile as shown in Fig.2 and the rate of decay increases with \( \Delta a_{11} \). The main plot in Fig.2 shows temporal variation of \( \rho_{rms} \) for \( a_{11} = 408 a_0 \), close to the miscible domain. There is a strong correlation between the decay rate and nature of oscillation. For \( a_{11} \) marginally larger than \( a_{22} \), the \( ^{85}\text{Rb} \) condensate tunnels through the \( ^{87}\text{Rb} \) condensate. Where as at larger values the \( ^{85}\text{Rb} \) expands and spreads into the \( ^{87}\text{Rb} \).

A dramatic change of the coupled oscillations occurs when \( U_{12} < \sqrt{U_{11}U_{22}} \), the TBEC is then miscible. The \( ^{85}\text{Rb} \) expands through the \( ^{87}\text{Rb} \) cloud and the two species undergo radial oscillations which has a beat pattern. The Fig.3 shows the \( \rho_{rms} \) when \( a_{11} = 780 a_0 \). Besides the radial oscillations, as to be expected when \( a_{11} > a_{22}(\mu_1 - V)/(\mu_2 - V) \), \( z_{rms} \) increases steadily. This accommodates the excess repulsion energy along the axial direction. Along with the oscillations there are higher frequency density fluctuations reminiscent of modulational...
instability. It is to be mentioned that, in earlier works [7, 8] modulational instability in the miscibility domain was analysed in depth. For the present case the detailed analysis of modulational instability shall be the subject of a future publication.

FIG. 4: The variation in $r_{\text{rms}}$ (in units of $a_{\text{osc}}$) for $^{85}\text{Rb}$ and $^{87}\text{Rb}$ with time (in units of $\omega^{-1}$) when $a_1$ is changed from $51a_0$ to $780a_0$. The blue and red curves correspond to $^{85}\text{Rb}$ and $^{87}\text{Rb}$ respectively.

Summary and outlook.—We have examined the onset of Rayleigh-Taylor instability in TBEC and identified the observable signature in the dynamics. We have specifically chosen the experimentally well studied $^{85}\text{Rb}$-$^{87}\text{Rb}$ mixture as case study and propose observing RTI with the $^{85}\text{Rb}-^{85}\text{Rb}$ Feshbach resonance. Starting from $a_{11} < a_{22}$, RTI sets in when the TBEC is tuned to $a_{11} > a_{22}(\mu_1 - V)/(\mu_2 - V)$ in the TF approximation. Then damping of $r_{\text{rms}}$ of $^{85}\text{Rb}$, species at the core, oscillations marks the onset of RTI. To analyse the stationary states we have proposed a semi analytic scheme, applicable when $\lambda \ll 1$, to minimize the energy functional with TF approximation. The results of which are in excellent agreement with the numerical results. The $\lambda \ll 1$ is also the case when the interface is planar and RTI is more prominent.

Acknowledgements.—We thank S. A. Silotri, B. K. Mani and S. Chattoppadhyay for very useful discussions. We acknowledge the help of P. Muruganandam while doing the numerical calculations.

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