The Langlands functoriality conjecture in the bisemialgebra framework

C. Pierre

Institut de Mathématique pure et appliquée
Université de Louvain
Chemin du Cyclotron, 2
B-1348 Louvain-la-Neuve, Belgium
pierre@math.ucl.ac.be

Mathematics subject classification (2000): 11G18, 11R34, 11R37, 11R39.

Abstract
The Langlands functoriality conjecture envisaged in the bisemistructure framework is proved to correspond to the non-orthogonal completely reducible cuspidal representation of bilinear algebraic semigroups.
1 Historical frame of the Langlands functoriality

The Langlands program originated from Artin’s reciprocity law of abelian class field theory. The simplest version of Artin’s reciprocity law states that if \(\sigma : \text{Gal}(E/Q) \to \mathbb{C}^*\) is the homomorphism from the Galois group of a finite extension \(E\) of \(Q\) into \(\mathbb{C}^*\), then there exists a Dirichlet character \(\chi_\sigma(\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*\) such that \(\sigma(\text{Frob}_p) = \chi_\sigma(p)\) for all primes \(p\) (unramified) in \(E\).

The non abelian equivalent of this reciprocity law concerns the \(n\)-dimensional representations of the Galois group \(\text{Gal}(E/Q)\) throughout homomorphisms:

\[
\sigma : \text{Gal}(E/Q) \longrightarrow \text{GL}_n(\mathbb{C})
\]

which led Artin to introduce \(L\)-functions:

\[
L(s, \sigma) = \prod_p (\det[I_n - \sigma(\text{Frob}_p) p^{-s}]^{-1}
\]

where \(\det[I_n - \sigma(\text{Frob}_p) p^{-s}]\) are characteristic polynomials related to conjugacy classes \(A_p\) of \(\text{Gal}(E/Q)\) described by \(\sigma(\text{Frob}_p)\) [Gel], [Rog]. But, he did not find the \(n\)-dimensional analogues of Dirichlet characters and \(L\)-functions.

It was Langlands [Lan1] who generalizes the concept of Dirichlet characters by introducing:

1) a (unitary) cuspidal automorphic representation \(\pi\) of \(\text{GL}_n(\mathbb{A}_F)\), where \(\mathbb{A}_F\) is the ring of adeles of the global number field \(F\) of characteristic zero.

2) the \(L\)-function associated with \(\pi\).

These considerations where then formulated in the Langlands (global) reciprocity conjecture which asserts that:

For any irreducible representation \(\sigma\) of \(\text{Gal}(\overline{F}/F)\) in \(\text{GL}_n(\mathbb{C})\), there exists a cuspidal automorphic representation \(\pi\) of \(\text{GL}_n(\mathbb{A}_F)\) in such a way that the Artin \(L\)-function of \(\sigma\) agrees with the Langlands \(L\)-function of \(\pi\) at almost every place where \(\pi\) is unramified [Kna].

In the local case, i.e. when the envisaged field is a finite extension \(K\) of \(\mathbb{Q}_p\), Langlands conjectured the existence of bijections between the set \(WD\text{Rep}_n(W_K)\) of equivalence classes of \(n\)-dimensional Frobenius semisimple Weil-Deligne representations of the Weil group \(W_K\) and the set \(\text{Irr cusp}(\text{GL}_n(K))\) of equivalence classes of (super)cuspidal irreducible representations of \(\text{GL}_n(K)\).

1
The global correspondences of Langlands over function fields on a smooth curve over \( \mathbb{F}_q \) were extensively studied by L. Lafforgue [Laf] while the local correspondences of Langlands over finite number fields were worked out by G. Henniart [Hen] and by M. Harris and R. Taylor [H-T], [Cara].

One of the major innovations introduced by Langlands in this context is the existence of \( L \)-groups \( L^G \) consisting in the semidirect product \( \widehat{G} \rtimes \text{Gal}(\overline{F}/F) \) of the complex reductive group \( \widehat{G} \) by the Galois groups \( \text{Gal}(\overline{F}/F) \) [Lan2].

This allows to tie the cuspidal automorphic representation \( \pi \) of a reductive group \( G(\mathbb{A}_F) \), as well as the associated cuspidal representation \( \Pi \) of \( \text{GL}_n(\mathbb{A}_F) \), to the \( n \)-dimensional holomorphic representation \( \rho \) of the corresponding \( L \)-group \( L^G \).

According to [Lan3], if \( A_v(\pi) \) denotes the \( v \)-th cuspidal conjugacy class of \( G(\mathbb{A}_F) \) and \( A_v(\Pi) \) the \( v \)-th cuspidal conjugacy class of \( \text{GL}_n(\mathbb{A}_F) \), the following equalities can be stated:

\[
\begin{align*}
\text{a)} & \quad \{ A_v(\Pi) \}_v = \{ \rho( A_v(\pi) ) \}_v, \quad \forall \ v \in \mathbb{N}, \text{ prime}, \\
\text{b)} & \quad L(s, \pi, \rho) = L(s, \Pi),
\end{align*}
\]

relating the \( L \)-function \( L(s, \Pi) \) on \( \text{GL}_n(\mathbb{A}_F) \) to the \( L \)-function \( L(s, \pi, \rho) \) on \( G(\mathbb{A}_F) \) and \( L^G \).

This shows [Lau] that the \( L \)-group \( L^G \) can be interpreted as a dual group of \( G(\mathbb{A}_F) \) in such a way that:

\[
\begin{align*}
\text{a)} & \quad \text{the cuspidal representation } \rho \text{ of } L^G \text{ be the contragradient representation } \widehat{\pi} \text{ with respect to the cuspidal representation } \pi \text{ of } G(\mathbb{A}_F) \quad [\text{Cara}] ; \\
\text{b)} & \quad \text{the coroots of } \widehat{\pi} \text{ correspond to the roots of } \pi \quad [\text{J-L}] .
\end{align*}
\]

To proceed further into the program of Langlands, the principle of functoriality must be envisaged. In its general form, it can be stated as follows:

Let \( G \) and \( G' \) be two reductive groups and let \( L^G \) and \( L^{G'} \) be the corresponding \( L \)-groups. The two homomorphisms:

\[
\begin{align*}
\phi_G : & \quad G \longrightarrow G' \\
\text{and} & \quad \phi_{L^G} : \quad L^G \longrightarrow L^{G'}
\end{align*}
\]

then imply the corresponding homomorphisms:

\[
\begin{align*}
\pi_{\phi_G} : & \quad \pi = \bigotimes \pi_v \longrightarrow \pi' = \bigotimes \pi'_v \\
\text{and} & \quad \pi_{\phi_{L^G}} : \quad \widehat{\pi} = \bigotimes \widehat{\pi}_v \longrightarrow \widehat{\pi'} = \bigotimes \widehat{\pi'}_v
\end{align*}
\]

2
of their respective cuspidal representations in such a way that:

\[ \pi_v' = \phi_{G|v}(\pi_v) , \quad \tilde{\pi}_v' = \phi_{L|v}(\tilde{\pi}_v) . \]

More concretely, let \( \text{sym}^m : \text{GL}_2(\mathbb{A}_F) \to \text{GL}_{m+1}(\mathbb{A}_F) \) be the \( m \)-th symmetric power representation of \( \text{GL}_2(\mathbb{A}_F) \). If \( \pi = \otimes \pi_v \) is the cuspidal representation of \( \text{GL}_2(\mathbb{A}_F) \), then:

\[ \text{sym}^m(\pi) = \otimes \text{sym}^m(\pi_v) \]

will be the searched cuspidal representation of \( \text{GL}_{m+1}(\mathbb{A}_F) \) in such a way that \( \text{sym}^m(\pi_v) \) be the \( v \)-th cuspidal conjugacy class of \( \text{GL}_{m+1}(F_v) \). This constitutes the Langlands functoriality conjecture extensively studied now.

It was especially proved by H. Kim and F. Shahidi [C-K-P-S], [K-S] by \( L \)-function methods for \( m = 3, 4 \) when \( F \) is a number field.

In order to try a new breakthrough in this problem of functoriality, the bisemisturcture framework introduced in [Pie1] will be considered in this paper.

Indeed, it was shown in [Pie2] that, to each mathematical structure, it can be generally associated a triple (left semistructure, right semistructure, bisemistructure) where the left and right semistructures, referring (or localised in) respectively to the upper and lower half spaces, operate on each other by means of their product giving rise to a bisemistructure in such a way that its bielements be either diagonal bielements [Lang] or cross bielements.

Taking into account bisemistructures allows:

- to generate richer mathematical structures by the consideration of cross (bisemi)-substructures;

- to have a better visibility of the endomorphisms of these (bisemi)structures of which objects are in some cases de facto bilinear, as for instance (square) matrices \( (n \times n) \).

Before taking up functoriality in this new bilinear mathematical frame, I shall introduce or recall the structure of the considered algebraic groups and of their cuspidal representations in this new context as well as the connection of these with their classical correspondents.

This will constitute the contents of the next two sections.
2 Structure of the algebraic bilinear semigroups

- In the framework of bisemistructures, a group $G$ must be viewed as a triple $(G_L, G_R, G_{R \times L})$ where:
  - $G_L$ (resp. $G_R$) is a left (resp. right) semigroup under the addition of its left (resp. right) elements $g_L$ (resp. $g_R$) referring to the upper (resp. lower) half space;
  - $G_{R \times L}$ is a bisemigroup, or bilinear semigroup, such that its bielements $(g_R \times g_L)$ be submitted to the cross binary operation $\times$ defined by:

$$G_{R \times L} \times G_{R \times L} \rightarrow G_{R \times L}$$

$$(g_R \times g_L) \times (g_R \times g_L) \rightarrow (g_R + g_R) \times (g_L + g_L).$$

By this way, pairs of right and left elements are sent either in diagonal bielements $(g_R \times g_L)$ and $(g_R \times g_L)$ or in cross bielements $(g_R \times g_L)$ and $(g_R \times g_L)$.

- Thus, an algebraic group $GL_n(F)$ of $(n \times n)$ square invertible matrices with entries in a field $F$ can be viewed as a bilinear algebraic semigroup $GL_n(F_R \times F_L)$ with entries in the product $F_R \times F_L$ of the right and left finite algebraic symmetric extensions $F_R$ and $F_L$ of a global number field $k$ of characteristic zero.

This bilinear algebraic semigroup $GL_n(F_R \times F_L)$ can then be decomposed into the product of the right semigroup $T_n(F_R)$ of lower triangular matrices with entries in the semifield $F_R$ by the left semigroup $T_n(F_L)$ of upper triangular matrices with entries in the semifield $F_L$ according to:

$$GL_n(F_R \times F_L) = T_n(F_R) \times T_n(F_L).$$

The left (resp. right) algebraic semigroup $T_n(F_L)$ (resp. $T_n(F_R)$) can be viewed as an operator:

$$T_n(F_L) : F_L \rightarrow T^{(n)}(F_L) \equiv V_L$$

(resp. $T_n(F_R) : F_R \rightarrow T^{(n)}(F_R) \equiv V_R$),

sending the left (resp. right) semifield $F_L$ (resp. $F_R$) into the left (resp. right) $T_n(F_L)$-semimodule $T^{(n)}(F_L)$ (resp. $T_n(F_R)$-semimodule $T^{(n)}(F_R)$) which is a left (resp. right) affine semispace $V_L$ (resp. $V_R$) of dimension $n$ localized in the upper (resp. lower) half space.

- The left semigroup $T_n(F_L)$ then corresponds to the parabolic subgroup $P_n(F)$ of Borel upper triangular matrices over the number field $F$ such that the homomorphism:

$$\text{Ind}_{T \rightarrow G} : T_n(F_L) \rightarrow GL_n(F_R \times F_L)$$
produces an induction \(\text{[Rod]}\) from the parabolic subsemigroup \(T_n(F_L)\) to the algebraic bilinear semigroup \(\text{GL}_n(F_R \times F_L)\).

- Therefore, it appears that if \(T_n(F_L)\) is assumed to be a semisimple (reductive) (semi)group \(G\), then the associated (semi)group \(T_n^l(F_R)\), on which a contragradient (cuspidal) representation can be defined, is the dual (semi)group of \(G\) and would correspond to the Langlands \(L\)-group \(L^G\) of \(G\) if \(L^G\) is interpreted as the dual group of \(G\) as suggested in section 1.

- Proposition 2.1. The representation (bisemi)space of the algebraic bilinear semigroup of matrices \(\text{GL}_n(F_R \times F_L)\) is given by the \(\text{GL}_n(F_R \times F_L)\)-bisemimodule \(G^{(n)}(F_R \times F_L)\) which is a \(n^2\)-dimensional affine bisemispace \((V_R \otimes_{F_R \times F_L} V_L)\) belonging to a neutral Tannakian tensor category \(\mathcal{C}_{R \times L}\) equivalent to the category of finite dimensional representations of affine bisemigroup schemes.

Proof. 1. It was proved in [Pie1] and in [Pie2] that the \(n^2\)-dimensional affine bisemispace \((V_R \otimes_{F_R \times F_L} V_L)\) is a \(\text{GL}_n(F_R \times F_L)\)-bisemimodule \(G^{(n)}(F_R \times F_L)\) under the action right by left of \(\text{GL}_n(F_R \times F_L)\) on an irreducible (unitary) \(n^2\)-dimensional affine bisemispace in such a way that \(G^{(n)}(F_R \times F_L) = T^{(n)}(F_R) \otimes T^{(n)}(F_L)\) be the tensor product \((V_R \otimes_{F_R \times F_L} V_L)\) of the right and left affine semispaces \(V_R\) and \(V_L\) over the bisemifield \(F_R \times F_L\).

2. A left (resp. right) algebraic semigroup \(T_n(F_L)\) (resp. \(T_n^l(F_R)\)) is in fact a left (resp. right) semigroup scheme over \(k\), i.e. a representable functor from the quotient \(k\)-algebra \(Q_L\) (resp. \(Q_R\)) of the polynomial ring \(k[x]\) modulo the ideal \(I_L\) (resp. \(I_R\)) to the left (resp. right) affine semispace \(V_L\) (resp. \(V_R\)) [Pie1].

And, \(\text{GL}_n(F_R \times F_L)\) is an affine bisemigroup scheme, i.e. a representable (bi)functor from \(Q_R \otimes Q_L\) to the affine bisemispace \((V_R \otimes_{F_R \times F_L} V_L)\).

Then, the category of finite dimensional representations of affine bisemigroup schemes is equivalent in the perspective of [D-M] to the neutral Tannakian tensor category \(\mathcal{C}_{R \times L}\) for which there exists a \(F_R \times F_L\)-(bi)linear tensor functor \(\omega_{R \times L} : \mathcal{C}_{R \times L} \to \mathcal{C}^{(n)}(F_R \times F_L)\) where \(\mathcal{C}^{(n)}(F_R \times F_L)\) denotes the category of \(n^2\)-dimensional bisemimodules \(G^{(n)}(F_R \times F_L)\) over \((F_R \times F_L)\).

\[
\square
\]

- Considering the Artin’s reciprocity law recalled in section 1 and the congruence subgroups used in the theories of cuspidal forms, (pseudo-ramified) complex completions of symmetric finite (closed) algebraic extensions of the number field \(k\) of characteristic zero were defined [Pie1] at infinite complex places by their degree given by the integers modulo \(N\). They correspond to \(\text{Hom}_k(F_L, \mathbb{C})\) and \(\text{Hom}_k(F_R, \mathbb{C})\) where \(k\) may be \(\mathbb{Q}\) [Kna].
By this way, we get a left (resp. right) tower
\[ F_ω = \{ F_{ω_{1}}, \ldots, F_{ω_{j,m_j}}, \ldots, F_{ω_{r,m_r}} \} \]
(resp. \( F_ω = \{ F_{ω_{1}}, \ldots, F_{ω_{j,m_j}}, \ldots, F_{ω_{r,m_r}} \} \))
of packets of equivalent complex completions localized in the upper (resp. lower) half space and characterized by their degrees (or ranks):
\[ [F_{ω_j} : k] = * + m^{(j)} \cdot j \cdot N , \quad j \in \mathbb{N} , \]
(resp. \[ [F_{ω_j} : k] = * + m^{(j)} \cdot j \cdot N ) \]
\[ 1 \leq j \leq r \leq \infty , \]
at every complex place \( ω_j \) (resp. \( \overline{ω_j} \))
where:
- the closure \( \overline{ω_j(F)} \) of \( ω_j(F) \) is a compact field;
- \( * \) denotes an integer inferior to \( N \);
- \( m^{(j)} \) is the number (or multiplicity) of the real completions covering \( F_{ω_j} \) (resp. \( F_{ω_j} \));
- \( j \) is a global residue degree \( f_{ω_j} \) (resp. \( f_{ω_j} \)).

When the global Weil groups are considered, the degrees of the finite algebraic extensions \( \bar{F}_{ω_j} \) (resp. \( \bar{F}_{ω_j} \)) associated with the completions \( F_{ω_j} \) (resp. \( F_{ω_j} \)) are restricted to [Pie1]:
\[ [\bar{F}_{ω_j} : k] \equiv [\bar{F}_{ω_j} : k] = 0 \mod N \]
leading to
\[ [\bar{F}_{ω_j} : k] \equiv [\bar{F}_{ω_j} : k] = j \cdot m^{(j)} \cdot N . \]
Each complex (resp. conjugate complex) completion \( F_{ω_p} \) (resp. \( F_{ω_p} \)) at the primary complex infinite place \( ω_p \) (resp. \( \overline{ω_p} \)), associated with the corresponding finite extension of \( k \), is then characterized by a degree:
\[ [F_{ω_p} : k] \equiv [F_{ω_p} : k] = m^{(p)} \cdot p \cdot N \]
and a number of elements:
\[ n_e(p) = m^{(p)} \cdot p \cdot N \cdot (n_{nu}) \]
where \( (n_{nu}) \) is the number of nonunits.

If the global residue degree is given by
\[ f_{ω_{p+i}} = k \ p + i' = p + i , \quad k \in \mathbb{N} , \quad 0 \leq i' \leq p - 1 , \quad 1 \leq i \leq \infty , \]
then the number of elements \( n_e(p + r) \) of a complex completion \( F_{ω_{p+i}} \) (resp. \( F_{ω_{p+i}} \)) at the infinite place \( ω_{p+i} \) (resp. \( \overline{ω_{p+i}} \)) is equal to:
\[ n_e(p + i) = m^{(p+i)} \cdot (p + i) \cdot N \cdot (n_{nu}) . \]
The connection between the sets $\{\omega_1, \ldots, \omega_r\}$ and $\overline{\omega} = \{\overline{\omega}_1, \ldots, \overline{\omega}_r\}$ of infinite places of $F$ and a set of finite places corresponding to finite extensions $K_p$ of $k_p$, for every prime $p$, can be obtained as follows:

The field $K_p$, which is a finite extension of $k_p$ that may be $Q_p$, is a $p$-adic field. Let $\Theta_{K_p}$ denote its ring of integers, $\wp_{K_p}$ the unique maximal ideal of $\Theta_{K_p}$, $k(v_{K_p}) = k(\wp_{K_p}) = \Theta_{K_p}/\wp_{K_p}$ its residue field, $\overline{\wp}_{K_p}$ a uniformizer in $\Theta_{K_p}$ and $v_{K_p}: K_p^* \to \mathbb{Z}$ be the unique valuation.

The number of elements in $k(\wp_{K_p})$ is $q_p = p^{f_{K_p}}$ where $f_{K_p} = [k(v_{K_p}) : \mathbb{F}_p]$ is the residue degree over $k_p$.

On the other hand, let $E_p$ be an imaginary quadratic field in which $p$ splits. Let $F^+ = F_R^+ \cup F_L^+$ be the real field covering the complex field $F = F_R \cup F_L$ as described below. If $F$ verifies $F = E_p \cdot F^+$, then $F$ is a CM field.

A finite extension $K_p$ of $k_p$ can be identified with the completion of the real field $F^+$ in a place $v_{p+1} = v_{p1}'$ above $p$. As, $p$ is decomposed in $E_p$ into two places $\wp$ and $\overline{\wp}$, the places of $F$ dividing $p$ are divided into a set of left infinite complex places $\wp_p' = \{\ldots, \wp_{p1}', \ldots, \wp_{ps}'\}$ above $\wp$ and into a set of right infinite complex places $\overline{\wp}_p' = \{\ldots, \overline{\wp}_{p1}', \ldots, \overline{\wp}_{ps}'\}$ above $\overline{\wp}$ in such a way that:

$$
\#\wp_{pi}' = \#\wp_{pi} + p + i, \quad 1 \leq i \leq \infty,
$$

(resp. $\#\overline{\wp}_{pi}' = \#\overline{\wp}_{pi} + p + i$),

where $(p + i)$ is in general given by $k_p + i'$.

By this way, finite extensions $K_p$ can correspond to etale coverings of completions of $F$ in $\wp_p'$ and $\overline{\wp}_p'$.

Let $G^{(n)}(F_\wp \times F_\omega)$ be the bilinear algebraic semigroup with entries in the product, right by left, $F_\wp \times F_\omega$ of towers of packets of equivalent complex completions at the set $\wp \times \omega = \{\wp_1, \ldots, \wp_r\} \times \{\omega_1, \ldots, \omega_r\}$ of bilaces.

$G^{(n)}(F_\wp \times F_\omega)$ is then composed of conjugacy class representatives $G^{(n)}(F_{\wp_j,m_j} \times F_{\omega_j,m_j})$, $1 \leq j \leq r \leq \infty$, having multiplicities $m^{(j)}$, $1 \leq m_j \leq m^{(j)}$, and corresponding to the $r$ bilaces $(\wp \times \omega)$.

Each conjugacy class representative $G^{(n)}(F_{\wp_j,m_j} \times F_{\omega_j,m_j})$ is a $\text{GL}_n(F_{\wp_j,m_j} \times F_{\omega_j,m_j})$-subbisemimodule $\subset G^{(n)}(F_\wp \times F_\omega)$.

The decomposition of $G^{(n)}(F_\wp \times F_\omega)$ into conjugacy classes can be realized by considering the cutting of the bilattice $\Lambda_\wp \times \Lambda_\omega$, referring to it, into subbilattices in $G^{(n)}(F_\wp \times F_\omega)$ under the action of the product $T_R(n;r) \otimes T_L(n;r)$ of Hecke operators having as representation $\text{GL}_n((\mathbb{Z}/N \mathbb{Z})^2)$ [Pie1].
• The algebraic representation of the bilinear algebraic semigroup of matrices $GL_n(F_\omega \times F_\omega)$ into the $GL_n(F_\omega \times F_\omega)$-bisemimodule $G^{(n)}(F_\omega \times F_\omega)$ corresponds to the algebraic morphism from $GL_n(F_\omega \times F_\omega)$ into $GL(G^{(n)}(F_\omega \times F_\omega))$ which is the group of automorphisms of $G^{(n)}(F_\omega \times F_\omega)$.

• Let $F_\omega \oplus = \bigoplus_{j,m_j} F_{\omega,j,m_j}$ and $F_\omega \oplus = \bigoplus_{j,m_j} F_{\omega,j,m_j}$ denote the sums of completions.

Then, $G^{(n)}(F_\omega \times F_\omega) = \bigoplus_{j,m_j} G^{(n)}(F_{\omega,j,m_j} \times F_{\omega,j,m_j})$ will correspond to the sums of conjugacy class representatives of the bilinear algebraic semigroup $G^{(n)}(F_\omega \times F_\omega)$.

In this respect, $GL(G^{(n)}(F_\omega \times F_\omega))$ constitutes the $n$-dimensional equivalent of the product, right by left, $W_{ab}^{ab} F_R \times W_{ab}^{ab} F_L$ of the sums of the equivalence classes of the global Weil groups $W_{ab}^{ab} F_R$ and $W_{ab}^{ab} F_L$ [Pie1] and $G^{(n)}(F_\omega \oplus \times F_\omega \oplus)$ becomes naturally its $n$-dimensional (irreducible) representation space according to:

$$\text{Irr Rep}_{W_{ab}^{ab} F_R \times W_{ab}^{ab} F_L}^{(n)}(W_{ab}^{ab} F_R \times W_{ab}^{ab} F_L) = G^{(n)}(F_\omega \oplus \times F_\omega \oplus)$$

if we have the injective morphism:

$$W_{ab}^{ab} F_R \times W_{ab}^{ab} F_L \rightarrow \text{GL}(G^{(n)}(F_\omega \oplus \times F_\omega \oplus)) .$$
3 Cuspidal representations of the bilinear algebraic semigroups

- The next step then consists in providing a (super)cuspidal representation of the bilinear algebraic semigroup $G^{(n)}(F_T \times F_{\omega})$ or of $G^{(n)}(F_{\omega} \times F_{\omega}^\sigma)$. The procedure can be summarized as follows:

  a) finding the cuspidal subrepresentation of each conjugacy class representative $G^{(n)}(F_{\omega,j,m_j} \times F_{\omega,m_j})$ of $G^{(n)}(F_{\omega} \times F_{\omega})$.

  b) showing that the sum of these cuspidal subrepresentations correspond to the searched cuspidal representation $\text{Irr cusp}(G^{(n)}(F_{\omega} \times F_{\omega}))$ of the algebraic bilinear semigroup $G^{(n)}(F_{\omega} \times F_{\omega})$.

- Let $
\gamma_{F_{\omega,j,m_j}}^T : F_{\omega,j,m_j} \rightarrow F_{\omega,j,m_j}^T, \quad 1 \leq j \leq r \leq \infty \ ,
$
(resp. $
\gamma_{F_{\omega,j,m_j}}^T : F_{\omega,j,m_j} \rightarrow F_{\omega,j,m_j}^T, \quad
$
be the toroidal isomorphism mapping each left (resp. right) complex completion $F_{\omega,j,m_j}$ (resp. $F_{\omega,j,m_j}$ ) into its toroidal equivalent $F_{T_{\omega,j,m_j}}^T$ (resp. $F_{T_{\omega,j,m_j}}^T$ ) which is a complex one-dimensional semitorus $T_L^n[j,m_j]$ (resp. $T_R^n[j,m_j]$ ) localized in the upper (resp. lower) half space.

Then, the morphisms:

$$T_n(F_{\omega,j,m_j}^T) : F_{\omega,j,m_j}^T \rightarrow T^{(n)}(F_{\omega,j,m_j}^T) = T_L^n[j,m_j],$$

(resp. $T_n(F_{\omega,j,m_j}^T) : F_{\omega,j,m_j}^T \rightarrow T^{(n)}(F_{\omega,j,m_j}^T) = T_R^n[j,m_j]$),

introduced in section 2, sends the left (resp. right) toroidal complex completion $F_{\omega,j,m_j}^T$ (resp. $F_{\omega,j,m_j}^T$ ) into the upper (resp. lower) semispace $T^{(n)}(F_{\omega,j,m_j})$ (resp. $T^{(n)}(F_{\omega,j,m_j}^T)$ ) which is a $n$-dimensional complex semitorus $T_L^n[j,m_j]$ (resp. $T_R^n[j,m_j]$ ), corresponding to the conjugacy class representative $G^{(n)}(F_{\omega,j,m_j}^T)$ (resp. $G^{(n)}(F_{\omega,j,m_j}^T)$ ).

So, the composition of bimorphisms

$$((T_n \circ T_n) \circ (\gamma_{F_{\omega,j,m_j}}^T \times \gamma_{F_{\omega,j,m_j}}^T) : F_{\omega,j,m_j} \times F_{\omega,j,m_j} \rightarrow G^{(n)}(F_{\omega,j,m_j}^T \times F_{\omega,j,m_j}^T).$$

a) is responsible for the generation of conjugacy class representatives $G^{(n)}(F_{\omega,j,m_j}^T \times F_{\omega,j,m_j}^T)$, $\forall j,m_j$ of the bilinear semigroup $G^{(n)}(F_{\omega}^T \times F_{\omega}^T)$ from the products, right by left, $F_{\omega,j,m_j} \times F_{\omega,j,m_j}$ of complex completions;
b) allows to envisage the bimorphisms:

\[ \gamma^{T}_{G^{(\alpha)}(F_{j,m})} \times \gamma^{T}_{G^{(\alpha)}(F_{j,m})} : \quad G^{(\alpha)}(F_{j,m} \times F_{\omega,j,m}) \]

\[ \longrightarrow G^{(\alpha)}(F_{j,m}^{T} \times F_{\omega,j,m}^{T}) \]

which send each conjugacy class representative \( G^{(\alpha)}(F_{j,m} \times F_{\omega,j,m}) \) of \( G^{(\alpha)}(F_{\omega} \times F_{\omega}) \) into its toroidal equivalent \( G^{(\alpha)}(F_{j,m}^{T} \times F_{\omega,j,m}^{T}) \).

- Next, we consider the left (resp. right) (semi)algebra \( \hat{G}^{(\alpha)}(F_{\omega}^{T}) \) (resp. \( \hat{G}^{(\alpha)}(F_{\omega}^{T}) \)) of continuous complex valued measurable functions \( \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{l}}) \) (resp. \( \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{r}}) \)) on \( G^{(\alpha)}(F_{\omega}^{T}) \) (resp. \( G^{(\alpha)}(F_{\omega}^{T}) \)) satisfying:

\[ \int_{G^{(\alpha)}(F_{\omega}^{T})} \left| \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{l}}) \right| \, dx_{g_{l}} < \infty \]

(esp.

\[ \int_{G^{(\alpha)}(F_{\omega}^{T})} \left| \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{r}}) \right| \, dx_{g_{r}} < \infty \),

with respect to a unique Haar measure on \( G^{(\alpha)}(F_{\omega}^{T}) \) (resp. \( G^{(\alpha)}(F_{\omega}^{T}) \)): it is also noted \( L_{1}^{1}(G^{(\alpha)}(F_{\omega}^{T})) \) (resp. \( L_{1}^{1}(G^{(\alpha)}(F_{\omega}^{T})) \).

The bisemialgebra \( \hat{G}^{(\alpha)}(F_{\omega}^{T} \times F_{\omega}^{T}) \) of continuous complex valued measurable bifunctions \( \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{r}}) \otimes \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{l}}) \) on \( G^{(\alpha)}(F_{\omega}^{T} \times F_{\omega}^{T}) \) satisfying

\[ \int_{G^{(\alpha)}(F_{\omega}^{T} \times F_{\omega}^{T})} \left| \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{r}}) \otimes \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{l}}) \right| \, dx_{g_{r}} \, dx_{g_{l}} < \infty , \]

is noted \( L_{1}^{0}(G^{(\alpha)}(F_{\omega}^{T} \times F_{\omega}^{T})) \).

If the right functions \( \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{r}}) \) are projected on the left functions \( \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{l}}) \), they become cofunctions and \( L_{1}^{0}(G^{(\alpha)}(F_{\omega}^{T} \times F_{\omega}^{T})) \) is transformed under this involution into the (bisei)algebra \( L_{1}^{0}(G^{(\alpha)}(F_{\omega}^{T} \times F_{\omega}^{T})) \) of square integrable functions.

- Each left (resp. right) function \( \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{l}}) \) (resp. \( \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{r}}) \)) of \( L_{1}^{1}(G^{(\alpha)}(F_{\omega}^{T})) \) (resp. \( L_{1}^{1}(G^{(\alpha)}(F_{\omega}^{T})) \)) is defined on a conjugacy class representative \( G^{(\alpha)}(F_{\omega,j,m}^{T}) \) (resp. \( G^{(\alpha)}(F_{\omega,j,m}^{T}) \)) which is a \( n \)-dimensional complex left (resp. right) semitorus.

Thus, \( \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{l}}) \) (resp. \( \phi_{G^{(\alpha)}G^{(\alpha)}}(x_{g_{r}}) \)) is a left (resp. right) function \( \phi_{L}(T_{L}^{n}[j,m]) \) (resp. \( \phi_{R}(T_{R}^{n}[j,m]) \)) on the semitorus

\[ G^{(\alpha)}(F_{\omega,j,m}^{T}) \equiv T_{L}^{n}[j,m] \quad \text{(resp. \( G^{(\alpha)}(F_{\omega,j,m}^{T}) \equiv T_{R}^{n}[j,m] \))} \]
Proposition 3.1. Each left (resp. right) function \( \phi^{(n)}_{G_{jL}}(x_{G_{jL}}) \) (resp. \( \phi^{(n)}_{G_{jR}}(x_{G_{jR}}) \)) on the conjugacy class representative \( G^{(n)}(F^{T}_{\omega_{j},m_{j}}) \) (resp. \( G^{(n)}(F^{T}_{\omega_{j},m_{j}}) \)) is a function on the left (resp. right) \( n \)-dimensional complex semitorus \( T^{n}_{L}[j,m_{j}] \) (resp. \( T^{n}_{R}[j,m_{j}] \)) having the analytic development:

\[
\phi_{L}(T^{n}_{L}[j,m_{j}]) = \lambda_{j}^{\frac{1}{2}}(n,j,m_{j}) e^{2\pi ijz}
\]

(resp. \( \phi_{R}(T^{n}_{R}[j,m_{j}]) = \lambda_{j}^{\frac{1}{2}}(n,j,m_{j}) e^{-2\pi ijz} \)),

where:

- \( \vec{z} = \sum_{d=1}^{n} z_{d} \vec{e}_{d} \) is a point of \( G^{(n)}(F^{T}_{\omega_{j},m_{j}}) \);

- \( \lambda_{j}(n,j,m_{j}) \simeq j^{n} N^{n} (m^{(j)})^{n} \) can be considered as a Hecke character.

Proof. 1. As we are concerned with the \( j \)-th infinite complex place \( \omega_{j} \) (resp. \( \overline{\omega}_{j} \)), we have to take into account the global Frobenius substitution given by the mapping:

\[
e^{2\pi iz} \rightarrow e^{2\pi ijz} \quad \text{(resp. } e^{-2\pi iz} \rightarrow e^{-2\pi ijz} \text{)},
\]

in such a way that

\[
e^{e\pi ijz} \simeq e^{2\pi ijz_{1}} \times \ldots \times e^{2\pi ijz_{n}}
\]

(resp. \( e^{-2\pi ijz} \simeq e^{-2\pi ijz_{1}} \times \ldots \times e^{-2\pi ijz_{n}} \))

because the \( n \)-dimensional complex semitorus \( T^{n}_{L}[j,m_{j}] \) (resp. \( T^{n}_{R}[j,m_{j}] \)) is diffeomorphic to the \( n \)-fold product:

\[
T^{n}_{L}[j,m_{j}] \simeq T^{1}_{L}[j,m_{j}] \times \ldots \times T^{1}_{L}[j,m_{j}]
\]

(resp. \( T^{n}_{R}[j,m_{j}] \simeq T^{1}_{R}[j,m_{j}] \times \ldots \times T^{1}_{R}[j,m_{j}] \));

2. The scalar \( \lambda(n,j,m_{j}) \) corresponds to the \( (j,m_{j}) \)-th coset representative \( U_{j,m_{j}L} \times U_{j,m_{j}R} \) of the product \( T_{R}(n;r) \otimes T_{L}(n;r) \) of Hecke operators being represented by \( GL_{n}((\mathbb{Z}/N \mathbb{Z})^{2}) \).

More precisely, let \( \{\lambda_{d}(n,j,m_{j})\}_{d=1}^{2n} \) be the set of eigen(bi)values of \( U_{j,m_{j}L} \times U_{j,m_{j}R} \) and let \( \lambda(n,j,m_{j}) = \prod_{d=1}^{2n} \lambda_{d}(n,j,m_{j}) \) be the product of these eigenvalues.

According to [Pie1], we have that:

\[
\lambda(n,j,m_{j}) = \prod_{d=1}^{2n} \lambda_{d}(n,j,m_{j}) = \det(\alpha_{n^{2},j^{2}} \times D_{j^{2},m_{j}^{2}}) \simeq j^{2n} \cdot N^{2n} \cdot (m^{(j)})^{2n}
\]

where:

- \( D_{j^{2},m_{j}^{2}} \) is the decomposition group of the \( j \)-th bisublattice with representative \( m_{j} \);
Proposition 3.2. As every left (resp. right) function $j, m$ of the $j, m$-th split Cartan subgroup.

It then appears that the square root $\lambda^2(n, j, m_j)$ of $\lambda(n, j, m_j)$ can be considered as a Hecke character having an inflation action on $e^{2\pi ijz}$.

3. Referring to the composition of $T_{m}^{n}[j, m_j]$ into a $n$-fold product of $T_{m}^{1}[j, m_j]$ and to the analytic development $\phi_L(T_{m}^{n}[j, m_j]) = \lambda^2(n, j, m_j) e^{2\pi ijz}$ of the function $\phi_L$ on $T_{m}^{n}[j, m_j]$, it is clear that $\phi_L(T_{m}^{1}[j, m_j])$ is a function on the 1-dimensional complex semitorus $T_{m}^{1}[j, m_j]$.

Indeed, we have that:

$$\phi_L(T_{m}^{1}[j, m_j]) = \lambda_{T_{m}^{1}} e^{2\pi ijz_d}$$

for $z_d \in \mathbb{C}$, where:

- $\lambda_{d_1}(n, j, m_j) \cdot \lambda_{d_2}(n, j, m_j)$ is the product of the eigenvalues of $U_{j, m_j} \subset T_{m}^{1}(1, r) \times T_{m}^{1}(1, r)$;
- $r_{d_1}^{S_1}$ and $r_{d_2}^{S_1}$ are radii of the circles $S_{d_1}^{1}[j, m_j]$ and $S_{d_2}^{1}[j, m_j]$.

Thus, the left 1D-complex semitorus $T_{m}^{1}[j, m_j]$ is diffeomorphic to the product $S_{d_1}^{1}[j, m_j] \times S_{d_2}^{1}[j, m_j]$ of two circles localized in perpendicular planes with $cos(2\pi ijy_{d_2})$ and $sin(2\pi ijy_{d_2})$ of $e^{2\pi ij(y_{d_2})}$ defined over $i\mathbb{R}$.

- **Proposition 3.2.** As every left (resp. right) function $\phi_L(T_{m}^{n}[j, m_j])$ (resp. $\phi_R(T_{m}^{n}[j, m_j])$) constitutes the cuspidal representation $\Pi_{j, m_j}(GL_{n}(F_{\omega, j, m_j}))$ (resp. $\Pi_{j, m_j}(GL_{n}(F_{\omega'} j, m_j)))$ of the $(j, m_j)$-th conjugacy class representative of the algebraic semigroup $GL_{n}(F_{\omega})$ (resp. $GL_{n}(F_{\omega'})$), the sum $\oplus \phi_{R}(T_{m}^{n}[j, m_j]) \otimes \phi_{L}(T_{m}^{n}[j, m_j])$ of the cuspidal subrepresentations of all conjugacy class representatives of the bilinear algebraic semigroup $GL_{n}(F_{\omega} \times F_{\omega'})$ is the searched cuspidal representation $\Pi(GL_{n}(F_{\omega'} \times F_{\omega'}))$ according to:

$$\Pi(GL_{n}(F_{\omega'} \times F_{\omega'})) = \oplus \Pi_{j, m_j}(GL_{n}(F_{\omega, j, m_j} \times F_{\omega, j, m_j}))$$

$$= \oplus \phi_{R}(T_{m}^{n}[j, m_j] \otimes \phi_{L}(T_{m}^{n}[j, m_j])) .$$

Proof. If the sum $\oplus \phi_{R}(T_{m}^{n}[j, m_j])$ tends to infinity, i.e. $r \to \infty$, then $\oplus \phi_{L}(T_{m}^{n}[j, m_j])$ represents the Fourier development of a left (resp. right) cuspidal form over $\mathbb{C}^n$.

And thus, $\Pi(GL_{n}(F_{\omega'} \times F_{\omega'}))$ constitutes clearly the cuspidal representation of the bilinear algebraic semigroup $GL_{n}(F_{\omega} \times F_{\omega})$, for $1 \leq j \leq r \leq \infty$. 

12
Proposition 3.3 (Langlands global correspondence). Let \( \sigma_{j,m}(W_{F_{\omega_{j,m}}}) = G^{(n)}(F_{\omega_{j,m}} \times F_{\omega_{j,m}}) \) denote the \( n \)-dimensional representation subspace of the product, right by left, \( W_{F_{\omega_{j,m}}} \times W_{F_{\omega_{j,m}}} \) of the Weil subgroups restricted to \( F_{\omega_{j,m}} \) and \( F_{\omega_{j,m}} \) and given by

\[
\sigma_{j,m}(W_{F_{\omega_{j,m}}} \times W_{F_{\omega_{j,m}}}) = \text{Irr Rep}^n(W_{F_{\omega_{j,m}}} \times W_{F_{\omega_{j,m}}})
\]
as described in section 2.

Let \( \Pi_{j,m}(\text{GL}_n(F_{\omega_{j,m}} \times F_{\omega_{j,m}})) = \Pi^\vee_{j,m}(\text{GL}_n(F_{\omega_{j,m}})) \) be its cuspidal (sub)representation in such a way that \( \Pi^\vee_{j,m}(\text{GL}_n(F_{\omega_{j,m}})) \) be the contra-gradient cuspidal subrepresentation restricted to the Weil subgroup \( W_{F_{\omega_{j,m}}} \).

Then, there exists bijective morphisms:

\[
T_{j,m} : \sigma_{j,m}(W_{F_{\omega_{j,m}}} \times W_{F_{\omega_{j,m}}}) \rightarrow \Pi_{j,m}(\text{GL}_n(F_{\omega_{j,m}} \times F_{\omega_{j,m}})), \quad 1 \leq j \leq r \leq \infty,
\]
between the \( n \)-dimensional conjugacy class representatives of the products, right by left, of the Weil subgroups and the corresponding \( n \)-dimensional cuspidal class representatives, leading to the bijective morphism:

\[
T : (\sigma(W_{F_{\omega}}^{ab} \times W_{F_{\omega}}^{ab})) \rightarrow \Pi(\text{GL}_n(F_{\omega} \times F_{\omega}))
\]
between the sum \( \sigma(W_{F_{\omega}}^{ab} \times W_{F_{\omega}}^{ab}) \) of the \( n \)-dimensional conjugacy class representatives of the Weil subgroups given by the algebraic bilinear semigroup \( G^{(n)}(F_{\omega} \times F_{\omega}) \) and its cuspidal representation given by \( \Pi(\text{GL}_n(F_{\omega} \times F_{\omega})) \).

Proof. 1. The \( n \)-dimensional conjugacy class representative of the product, right by left, of the Weil subgroups \( W_{F_{\omega_{j,m}}} \times W_{F_{\omega_{j,m}}} \) is given by:

\[
G^{(n)}(F_{\omega_{j,m}} \times F_{\omega_{j,m}}) = \sigma_{j,m}(W_{F_{\omega_{j,m}}} \times W_{F_{\omega_{j,m}}}).
\]

The toroidal compactification

\[
T_{j,m}(G^{(n)}(F_{\omega_{j,m}} \times F_{\omega_{j,m}})) \simeq \Pi_{j,m}(\text{GL}_n(F_{\omega_{j,m}} \times F_{\omega_{j,m}}))
\]

\[
= \lambda^{\frac{1}{2}}(n,j,m_j) e^{-2\pi i j z} \times \lambda^{\frac{1}{2}}(n,j,m_j) e^{2\pi i j z}
\]

of the conjugacy class representative \( \text{GL}_n(F_{\omega_{j,m}} \times F_{\omega_{j,m}}) \) of the bilinear algebraic semigroup \( \text{GL}_n(F_{\omega} \times F_{\omega}) \) is in bijection with the corresponding cuspidal conjugacy class representative \( \Pi_{j,m}(\text{GL}_n(F_{\omega_{j,m}} \times F_{\omega_{j,m}})) \) given by \( \phi_R(T_R^{n}[j,m_j]) \otimes \phi_L(T_L^{n}[j,m_j]) \) as developed in proposition 3.1.
2. Then, the sum $\sigma(W_{FR}^{ab} \times W_{FL}^{ab})$ of the $n$-dimensional conjugacy class representatives of the Weil subgroups given by

$$G^{(n)}(F_{\varpi,\omega} \times F_{\omega,\varpi}) = \bigoplus_{j,m} G^{(n)}(F_{\varpi,j,m} \times F_{\omega,j,m})$$

is in one-to-one correspondence with the searched cuspidal representation $\Pi(GL_n(F_{\varpi,j,m} \times F_{\omega,j,m}))$ according to:

$$T(\sigma(W_{FR}^{ab} \times W_{FL}^{ab})) \simeq \bigoplus_{j,m} \Pi_{j,m}(GL_n(F_{\varpi,j,m} \times F_{\omega,j,m})) = \Pi(GL_n(F_{\varpi} \times F_{\omega})).$$

\[\blacksquare\]
4 The Langlands functoriality in this new bilinear mathematical framework

- The Langlands global correspondence(s) having been stated in the irreducible complex case, it is now time to ask in what extent the cuspidal representation $\Pi^{(2n)}(GL_{2n}(F_\omega \times F_\omega))$ of the algebraic bilinear semigroup $GL_{2n}(F_\omega \times F_\omega)$ can be reached from the knowledge of the cuspidal representation $\Pi^{(2)}(GL_{2}(F_\omega \times F_\omega))$ of the algebraic bilinear semigroup $GL_{2}(F_\omega \times F_\omega)$.

As recalled in section 1, the Langlands functoriality conjecture consists in proving that:

$$\Pi^{(2n)} = \text{sym}^{(n)}(\Pi^{(2n)}) = \bigotimes_v \text{sym}^{(n)}(\Pi^{(2)}_v)$$

where $\Pi^{(2)}_v$ denotes the cuspidal subrepresentation of the considered algebraic group at the primary place $v_v$.

This case of functoriality, transposed in the considered bilinear framework, will be considered in the following, but also a case of functoriality associated with the cuspidal reducible representation of $GL_{2n}(F_\omega \times F_\omega)$, which constitutes the content of our main proposition 4.2.

- Proposition 4.1 (Functoriality sym$^{(n)}$ in a bilinear framework). Let

$$\text{sym}^{(n)}: \quad GL_2(F_\omega \times F_\omega) \longrightarrow GL_{2n}(F_\omega \times F_\omega)$$

or equivalently:

$$\text{sym}^{(n)}: \quad G^{(2)}(F_\omega \times F_\omega) \longrightarrow G^{(2n)}(F_\omega \times F_\omega)$$

be the $n$-th symmetric power representation of the algebraic bilinear semigroup $GL_{2}(F_\omega \times F_\omega)$.

Then, the cuspidal representation $\Pi^{(2n)}(GL_{2n}(F_\omega \times F_\omega))$ of the bilinear algebraic semigroup $GL_{2n}(F_\omega \times F_\omega)$ can be reached functorially from the cuspidal representation $\Pi^{(2)}(GL_{2}(F_\omega \times F_\omega))$ of the algebraic bilinear semigroup $GL_{2}(F_\omega \times F_\omega)$ throughout the injective morphism:

$$\Pi^{cusp}_{2\rightarrow 2n}: \quad \Pi^{(2)}(GL_{2}(F_\omega \times F_\omega)) \longrightarrow \Pi^{(2n)}(GL_{2n}(F_\omega \times F_\omega))$$

which corresponds to the morphism $\text{sym}^{(n)}$ on the cuspidal representation $\Pi^{(2)}(GL_{2}(F_\omega \times F_\omega))$ of $GL_{2}(F_\omega \times F_\omega)$.

Proof. 1. Referring to proposition 3.3, we have that the cuspidal representation $\Pi^{(2)}(GL_{2}(F_\omega \times F_\omega))$ of the bilinear algebraic semigroup $GL_{2}(F_\omega \times F_\omega)$ can be de-
developed according to:

\[
\Pi^{(2)}(\text{GL}_2(F_\omega \times F_\omega)) = \bigoplus_{j,m_j} \Pi^{(2)}_{j,m_j}(\text{GL}_2(F_{\omega,j,m_j} \times F_{\omega,j,m_j}))
\]

\[
= \bigoplus_{j,m_j} (\phi_R(T^2_R[j,m_j]) \otimes \phi_L(T^2_L[j,m_j]))
\]

\[
= \bigoplus_{j,m_j} (\lambda^{\frac{1}{2}}(2,j,m) e^{-2\pi ijy^2} \otimes \lambda^{\frac{1}{2}}(2,j,m) e^{2\pi ijy^2},
\]

with respect to the cuspidal subrepresentations of the conjugacy class representatives \(\text{GL}_2(F_{\omega,j,m_j} \times F_{\omega,j,m_j})\) of \(\text{GL}_2(F_\omega \times F_\omega)\).

Remark that the complex dimensions are here envisaged in real notations.

2. Then, every cuspidal conjugacy class representative \(\Pi^{(2n)}_{j,m_j}(\text{GL}_{2n}(F_{\omega,j,m_j} \times F_{\omega,j,m_j}))\) of the bilinear algebraic semigroup \(\text{GL}_{2n}(F_\omega \times F_\omega)\) can be obtained from the corresponding cuspidal conjugacy class representative \(\Pi^{(2)}_{j,m_j}(\text{GL}_2(F_{\omega,j,m_j} \times F_{\omega,j,m_j}))\) of \(\text{GL}_2(F_\omega \times F_\omega)\) by means of the injective morphism:

\[
\Pi^{\text{cusp}}_{2 \rightarrow 2n}(j,m) : \lambda^{\frac{1}{2}}(2,j,m) e^{-2\pi ijy^2} \otimes \lambda^{\frac{1}{2}}(2,j,m) e^{2\pi ijy^2}
\]

\[
\longrightarrow \lambda^{\frac{1}{2}}(2n,j,m) e^{-2\pi ijy} \otimes \lambda^{\frac{1}{2}}(2n,j,m) e^{2\pi ijy},
\]

sending:

- \(\lambda^{\frac{1}{2}}(2,j,m)\) into \(\lambda^{\frac{1}{2}}(2n,j,m)\)
- \(y^2\) into \(y\).

This is possible if proposition 3.1 is taken into account.

3. And, the searched cuspidal representation \(\Pi^{(2n)}(\text{GL}_{2n}(F_\omega \times F_\omega))\) results from the sum of the injective morphisms:

\[
\bigoplus_{j,m_j} \left[ \Pi^{\text{cusp}}_{2 \rightarrow 2n}(j,m) : \Pi^{(2)}_{j,m_j}(\text{GL}_2(F_{\omega,j,m_j} \times F_{\omega,j,m_j})) \longrightarrow \Pi^{(2n)}_{j,m_j}(\text{GL}_{2n}(F_{\omega,j,m_j} \times F_{\omega,j,m_j})) \right]
\]

in such a way that

\[
\Pi^{(2n)}(\text{GL}_{2n}(F_\omega \times F_\omega)) = \bigoplus_{j,m_j} \Pi^{(2n)}_{j,m_j}(\text{GL}_{2n}(F_{\omega,j,m_j} \times F_{\omega,j,m_j})).
\]

• This treatment of functoriality is rather trivial in the considered bilinear framework.

A more interesting way of envisaging functoriality is to take into account the fact that the algebraic bilinear semigroup \(\text{GL}_2(F_\omega \times F_\omega)\) is a bisemigroup submitted to the cross binary operation \(\times\) which allows to reduce the problem of Langlands functoriality to the reducibility of representations of groups.
Let $2n = 2_1 + 2_2 + \cdots + 2_\ell + \cdots + 2_n$ be a partition of the integer $2n$ and let

$$\text{GL}_2(F_{\overline{\sigma}} \times F_\omega) \times \text{GL}_2(F_{\overline{\sigma}} \times F_\omega) \times \cdots \times \text{GL}_2(F_{\overline{\sigma}} \times F_\omega) \times \cdots \times \text{GL}_2_n(F_{\overline{\sigma}} \times F_\omega)$$

be the $n$-th symmetric power of $\text{GL}_2(F_{\overline{\sigma}} \times F_\omega)$ according to this partition.

Referring to section 2 and to [Pie2], it appears that the product “×” between two bilinear algebraic semigroups is the cross binary operation “×” which enables to develop this product according to:

$$\text{GL}_2(F_{\overline{\sigma}} \times F_\omega) \times \cdots \times \text{GL}_2(F_{\overline{\sigma}} \times F_\omega) \times \cdots \times \text{GL}_2_n(F_{\overline{\sigma}} \times F_\omega)$$

$$= (\text{GL}_2(F_{\overline{\sigma}}) \oplus \cdots \oplus \text{GL}_2(F_{\overline{\sigma}}) \oplus \cdots \oplus \text{GL}_2_n(F_{\overline{\sigma}}))$$

$$\times (\text{GL}_2(F_\omega) \oplus \cdots \oplus \text{GL}_2(F_\omega) \oplus \cdots \oplus \text{GL}_2_n(F_\omega))$$

and to state the main proposition.

**Main Proposition 4.2.** The cuspidal representation $\Pi^{(2n)}(\text{GL}_{2n}(F_{\overline{\sigma}} \times F_\omega))$ of the bilinear algebraic semigroup $\text{GL}_{2n}(F_{\overline{\sigma}} \times F_\omega)$ is (non orthogonally) completely reducible if it decomposes diagonally according to the direct sum $\bigoplus_{\ell=1}^{n} \Pi^{(2\ell)}(\text{GL}_{2\ell}(F_{\overline{\sigma}} \times F_\omega))$ of irreducible cuspidal representations of the algebraic bilinear semigroups $\text{GL}_{2\ell}(F_{\overline{\sigma}} \times F_\omega)$ and off-diagonally according to the direct sum $\bigoplus_{k \neq \ell} \Pi^{(2k)}(\text{GL}_{2k}(F_{\overline{\sigma}})) \otimes \Pi^{(2\ell)}(\text{GL}_{2\ell}(F_\omega))$ of the (tensor) products of irreducible cuspidal representations of cross algebraic linear semigroups $\text{GL}_{2k}(F_{\overline{\sigma}}) \times \text{GL}_{2\ell}(F_\omega) \equiv T_{2^k}^2(F_{\overline{\sigma}}) \times T_{2^\ell}^2(F_\omega), \ \forall \ k \neq \ell, \ 1 \leq k, \ell \leq n$.

This reducible cuspidal representation

$$\Pi^{(2n)}(\text{GL}_{2n}(F_{\overline{\sigma}} \times F_\omega)) = \bigoplus_{\ell=1}^{n} \Pi^{(2\ell)}(\text{GL}_{2\ell}(F_{\overline{\sigma}} \times F_\omega)) \bigoplus_{k \neq \ell} \Pi^{(2k)}(\text{GL}_{2k}(F_{\overline{\sigma}})) \otimes \Pi^{(2\ell)}(\text{GL}_{2\ell}(F_\omega))$$

of $\text{GL}_{2n}(F_{\overline{\sigma}} \times F_\omega)$ then corresponds to the Langlands functoriality:

$$\Pi^{\text{Cusp}}_{2 \to 2n} : \Pi^{(2)}(\text{GL}_2(F_{\overline{\sigma}} \times F_\omega)) \longrightarrow \Pi^{(2n)}(\text{GL}_{2n}(F_{\overline{\sigma}} \times F_\omega))$$

**Proof.** 1. As it was noticed above, the cross binary operation “×” allows to develop the $n$-th symmetric power of $\text{GL}_2(F_{\overline{\sigma}} \times F_\omega)$ according to:

$$\text{GL}_2(F_{\overline{\sigma}} \times F_\omega) \times \cdots \times \text{GL}_2(F_{\overline{\sigma}} \times F_\omega) \times \cdots \times \text{GL}_2_n(F_{\overline{\sigma}} \times F_\omega)$$

$$= \bigoplus_{\ell=1}^{n} \text{GL}_2(F_{\overline{\sigma}}) \times \bigoplus_{\ell=1}^{n} \text{GL}_2(F_\omega)$$

$$= \bigoplus_{\ell=1}^{n} \text{GL}_2(F_{\overline{\sigma}}) \times \bigoplus_{k \neq \ell} \bigoplus_{\ell=1}^{n} \text{GL}_2(F_\omega) \times (T_{2^k}^2(F_{\overline{\sigma}}) \times T_{2^\ell}^2(F_\omega)) \ .$$
The sum of the cuspidal representations of these algebraic bilinear semigroups \( GL_2(\mathbb{F}_\omega \times \mathbb{F}_\omega) \) and \( (T_{2k}(\mathbb{F}_\omega) \times T_{2\ell}(\mathbb{F}_\omega)) \) is the reducible cuspidal representation of the algebraic bilinear semigroup \( GL_{2n}(\mathbb{F}_\omega \times \mathbb{F}_\omega) \) according to:

\[
\Pi^{(2n)}(GL_{2n}(\mathbb{F}_\omega \times \mathbb{F}_\omega)) = \bigoplus_{\ell=1}^{n} \Pi^{(2\ell)}(GL_2(\mathbb{F}_\omega)) \bigoplus_{k \neq \ell=1}^{n} \left( \Pi^{(2k)}(GL_2(\mathbb{F}_\omega)) \times \Pi^{(2\ell)}(GL_2(\mathbb{F}_\omega)) \right).
\]

2. This decomposition of the cuspidal representation of \( GL_{2n}(\mathbb{F}_\omega \times \mathbb{F}_\omega) \) then corresponds clearly to the Langlands functoriality statement:

\[
\Pi_{2 \rightarrow 2n}^{\text{cusp}} : \Pi^{(2)}(GL_2(\mathbb{F}_\omega \times \mathbb{F}_\omega)) \longrightarrow \Pi^{(2n)}(GL_{2n}(\mathbb{F}_\omega \times \mathbb{F}_\omega))
\]

**Corollary 4.3.** The cuspidal representation \( \Pi^{(2n)}(GL_{2n}(\mathbb{F}_\omega \times \mathbb{F}_\omega)) \) of the bilinear algebraic semigroup \( GL_{2n}(\mathbb{F}_\omega \times \mathbb{F}_\omega) \) is orthogonally completely reducible if it is decomposed only diagonally according to the direct sum of the irreducible cuspidal representations of the algebraic bilinear semigroups \( GL_{2\ell}(\mathbb{F}_\omega \times \mathbb{F}_\omega), \ 1 \leq \ell \leq n \), as follows:

\[
\Pi^{(2n)}(GL_{2n}(\mathbb{F}_\omega \times \mathbb{F}_\omega)) = \bigoplus_{\ell=1}^{n} \Pi^{(2\ell)}(GL_2(\mathbb{F}_\omega)).
\]

*Proof.* This cuspidal representation of \( GL_{2n}(\mathbb{F}_\omega \times \mathbb{F}_\omega) \) is orthogonally completely reducible with respect to the non orthogonally completely reducible cuspidal representation considered in proposition 4.2 in the sense that the off-diagonal cuspidal representations \( \Pi^{(2k)}(GL_2(\mathbb{F}_\omega)) \otimes \Pi^{(2\ell)}(GL_2(\mathbb{F}_\omega)) \) are not taken into account. \[\blacksquare\]
References

[Cara] H. CARAYOL, Preuve de la conjecture de Langlands locale pour $GL_n$ : travaux de Harris-Taylor et Henniart, *Sém. Bourbaki*, **857** (1999), 1–52.

[Cart] P. CARTIER, Les représentations des groupes réductifs $p$-adiques et leurs caractères, *Sém. Bourbaki*, **471** (1975–76).

[C-K-P-S] J.W. COGDELL, H. KIM, I. PIATETSKI-SHAPIRO, F. SHAHIDI, Functoriality for the classical groups, *Publ. Math. IHES*, **99** (2004), 163–233.

[D-M] P. DELIGNE, J.S. MILNE, Tannakian categories, *Lect. Notes Math.*, **900** (1982), 101–279.

[H-T] M. HARRIS, R. TAYLOR, On the geometry and cohomology of some simple Shimura varieties, *Annals of Math. Stud.*, **151** (2002), Princeton Univ. Press.

[Hen] G. HENNIART, Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps $p$-adique, *Invent. Math.*, **139** (2000), 439–455.

[J-L] H. JACQUET, R.P. LANGLANDS, Automorphic forms on $GL(2)$, *Lect. Notes Math.*, **114** Springer (1970).

[Gel] S. GELBART, An elementary introduction fo the Langlands program, *Bull. Amer. Math. Soc.*, **10** (1984), 177–219.

[K-S] H. KIM, F. SHAHIDI, Functorial products for $GL_2 \times GL_3$ and the symmetric cube for $GL_2$, *Annals of Math.*, **155** (2002), 837–893.

[Kna] A. KNAPP, Introduction to the Langlands program, *Proceed. Symp. Pure Math.*, **63** (1997), 245–302.

[Laf] L. LAFFORGUE, Chôticas de Drinfeld et correspondance de Langlands, *Invent Math.*, **147** (2002), 1–242.

[Lang] S. LANG, Algebra, Addison-Wesley.

[Lan1] R.P. LANGLANDS, Problems in the theory of automorphic forms, *Lect. Mod. Analysis and Appl. III; Lect. Notes Math.*, **170** (1970), 18–61.

[Lan2] R.P. LANGLANDS, Base change for $GL(2)$, *Annals Math. Stud.*, **96** (1980), Princeton University Press.

[Lan3] R.P. LANGLANDS, Where stands functoriality today?, *Proceed. Symp. Pure Math.*, **61** (1997), 457–471.

[Lau] G. LAUMON, The work of Laurent Lafforgue, *I.C.M.*, Vol. **1** (2002), 91–97.
[Pie1] C. Pierre, $n$-dimensional global correspondences of Langlands, Preprint Arxiv: Math-RT/0510348 (2005).

[Pie2] C. Pierre, Introducing bisemistructures, Preprint (2006) archiv.org.math. GM/0607624.

[Rod] F. Rodier, Représentations de $\text{GL}(n, k)$ où $k$ est un corps $p$-adique, Sém. Bourbaki, 587 (1981–82).

[Rog] J. Rogawski, Functoriality and the Artin conjecture, Proc. Symp. Pure Math., 61 (1997), 331–353.