Decentralized Proximal Gradient Algorithms
with Linear Convergence Rates

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Abstract

This work studies a class of non-smooth decentralized multi-agent optimization problems where the agents aim at minimizing a sum of local strongly-convex smooth components plus a common non-smooth term. We propose a general algorithmic framework that captures many existing state-of-the-art algorithms including the adapt-then-combine gradient-tracking methods for smooth costs. We then prove linear convergence of the proposed method in the presence of the non-smooth term. Moreover, for the more general class of problems with agent specific non-smooth terms, we show that linear convergence cannot be achieved (in the worst case) for the class of algorithms that uses the gradients and the proximal mappings of the smooth and non-smooth parts, respectively. We further provide a numerical counterexample that shows some state-of-the-art algorithms fail to converge linearly for strongly-convex objectives and different local non-smooth terms.

Index Terms

Decentralized optimization, proximal gradient algorithms, linear convergence, adapt-then-combine (ATC).

I. INTRODUCTION

In this work, we consider a static and undirected network of $K$ agents connected over some graph where each agent $k$ owns a private cost function $J_k(w) : \mathbb{R}^M \rightarrow \mathbb{R}$. Through only local interactions (i.e.,

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with agents only communicating with their immediate neighbors), each node is interested in finding a solution to the following problem:

$$w^* = \arg \min_{w \in \mathbb{R}^M} \frac{1}{K} \sum_{k=1}^{K} J_k(w) + R(w)$$  \hspace{1cm} (1)

where $R(w) : \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ is a convex function (not necessarily differentiable). We adopt the following assumption throughout this work.

**Assumption 1. (Cost function):** We assume that a solution exists to problem (1) and each cost function $J_k(w)$ is first-order differentiable and $\nu$-strongly-convex:

$$(w^o - w^\star)^T(\nabla J_k(w^o) - \nabla J_k(w^\star)) \geq \nu ||w^o - w^\star||^2$$  \hspace{1cm} (2)

with $\delta$-Lipschitz continuous gradients:

$$||\nabla J_k(w^o) - \nabla J_k(w^\star)|| \leq \delta ||w^o - w^\star||$$  \hspace{1cm} (3)

for any $w^o$ and $w^\star$. Constants $\nu$ and $\delta$ are strictly positive and satisfy $0 < \nu \leq \delta$. We also assume $R(w)$ to be a proper and lower-semicontinuous convex function.

Note that from the strong-convexity condition (2), we know the objective function in (1) is also strongly convex and, thus, the global solution $w^*$ is unique.

A. Related Works

Various algorithms have been proposed to solve decentralized optimization problems of the form (1) – see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. Only few works have attempted to unify some of these various algorithms [12], [13], [14]. For example, the work [12] proposed a general method that includes EXTRA [7] and DIGing [6] (for static and undirected network) as special cases. However, the method in [12] does not include the adapt-then-combine (ATC) gradient-tracking algorithms [1], [2], [3]. The work [13] proposed a canonical form that captures decentralized algorithms that require a single round of communication and gradient computation per iteration, which does not include the Aug-DGM (ATC-DIGing) [1], [2]. Reference [13] only focused on the canonical form without focusing on the analysis of this form. Later, the work [14] studied a special class of the algorithms in [13] and provided worst case linear convergence rates through numerical solution of semidefinite programs. This

1The function $f(.)$ is proper if $-\infty < f(x)$ for all $x$ in its domain and $f(x) < \infty$ for at least one $x$.

2The Adapt-then-Combine (ATC) structure appeared in [15] to distinguish between different implementations of diffusion learning strategies – see also [16, Ch. 7].
special class does not include algorithms that require communicating two different vectors per iteration such as gradient tracking methods [1], [2], [3], [4], [5], [6].

Different from [12], [13], [14] we propose an adapt-then-combine (ATC) primal-dual framework that captures more existing algorithms including EXTRA [7], Exact diffusion [8], NIDS [9], and different implementations of the gradient tracking methods [1], [2], [3], [4], [5] including Aug-DGM [1]. Our framework shows that the ATC gradient-tracking methods can be represented as primal-dual recursions. Moreover, building on [17] we extend this framework to handle the non-smooth term $R(w)$ and prove linear convergence of the proposed method in the presence of the non-smooth term.

In order to establish global linear convergence, this work considers the non-smooth term to be common across all agents [17]. One might wonder whether it is possible for a decentralized proximal gradient algorithm to achieve global linear convergence in the presence of different local non-smooth $R_k(w)$ terms. As far as we know, this question has not been explicitly answered in the literature. Many decentralized optimization problems where each agent $k$ has a local non-smooth term $R_k(w)$ possibly different from other agents [9], [18], [19], [20], [21] have been proposed. None of these methods have been shown to achieve global linear convergence in the presence of general non-smooth terms. By adjusting the results from [22] to the decentralized optimization set-up with agent-specific non-smooth terms $\{R_k(w)\}$, it can be shown that it is impossible for any proximal gradient based algorithm to achieve linear convergence in the worst case – see Section VI. Note that the works [23], [24] showed that global linear convergence is not possible for non-smooth strongly-convex functions in the worst case for the class of algorithms limited to one communication round but unlimited in the amount of computation and access to the functions per iteration. In contrast, we consider algorithms unlimited in the number of communications rounds but limited to one gradient and proximal computations per iteration.

B. Contribution

Given the above, this paper has three contributions. First, when $R(w) = 0$ we present an adapt-then-combine (ATC) unifying primal-dual framework that covers many existing state-of-the-art algorithms [1], [2], [3], [4], [5], [6], [7], [8], [9], which to our knowledge, is the first primal-dual interpretation of the ATC gradient-tracking methods [1], [2], [3], [4]. Second, we extend this framework to handle a common non-smooth regularization term and provide a unifying linear convergence analysis under proper conditions. Our step-size and convergence rate upper bounds shed light on the stability and performance of these various methods. Third, by tailoring a result from [22], we show that if each agent owns a non-smooth term, then linear convergence cannot be achieved in the worst case for the class of decentralized algorithms where each agent can compute one gradient and one proximal mapping per iteration for the smooth and
non-smooth part, respectively. We further provide a numerical counter example where PG-EXTRA \cite{18} and proximal linearized ADMM \cite{19,20} fail to achieve global linear convergence for strongly-convex objectives.

C. Notation

For a vector \( x \in \mathbb{R}^M \) and a positive semi-definite matrix \( C \succeq 0 \), we let \( \|x\|_C^2 = x^T C x \). Moreover, for any symmetric matrices \( A \) and \( B \) with the same dimension, we let \( A \succeq B \) if \( A - B \) is positive semi-definite (positive definite). The \( N \times N \) identity matrix is denoted by \( I_N \). We let \( 1_N \) be a vector of size \( N \) with all entries equal to one. The Kronecker product is denoted by \( \otimes \). We let \( \text{col}\{x_n\}_{n=1}^N \) denote a column vector (matrix) that stacks the vector (matrices) \( x_n \) of appropriate dimensions on top of each other. The subdifferential \( \partial f(x) \) of a function \( f : \mathbb{R}^M \to \mathbb{R} \) at some \( x \in \mathbb{R}^M \) is the set of all subgradients \( \partial f(x) = \{ g | g^T (y - x) \leq f(y) - f(x), \forall y \in \mathbb{R}^M \} \). The proximal operator with parameter \( \mu > 0 \) of a function \( f : \mathbb{R}^M \to \mathbb{R} \) is

\[
\text{prox}_{\mu f}(x) = \arg \min_z \ f(z) + \frac{1}{2\mu} \|z - x\|^2
\]

II. ADAPT-THEN-COMBINE FRAMEWORK

In this section, we present an adapt-then-combine (ATC) algorithmic framework that covers various state-of-the-art algorithms as special cases. To this end, we will first focus on the smooth case \( (R(w) = 0) \) in this section, which will then be extended to handle the non-smooth component \( R(w) \) in the following section.

A. General Algorithm

For algorithm derivation and motivation purposes, we will rewrite problem (1) in an equivalent manner. To do that, we let \( w_k \in \mathbb{R}^M \) denote a local copy of \( w \) available at agent \( k \) and introduce the network quantities:

\[
\mathcal{W} \triangleq \text{col}\{w_1, \ldots, w_K\} \in \mathbb{R}^{KM}, \quad \mathcal{J}(w) \triangleq \frac{1}{K} \sum_{k=1}^K J_k(w_k) \quad (5)
\]

Further, we introduce two general symmetric matrices \( B \in \mathbb{R}^{MK \times MK} \) and \( C \in \mathbb{R}^{MK \times MK} \) that satisfy the following conditions:

\[
\begin{cases}
B \mathcal{W} = 0 \iff w_1 = \cdots = w_K \\
C = 0 \quad \text{or} \quad C \mathcal{W} = 0 \iff B \mathcal{W} = 0
\end{cases}
\]

(6a)

(6b)

For algorithm derivation, the matrices \{\( B, C \)\} can be any general consensus matrices \cite{25}. Later, we will see how to choose these matrices specifically to get different decentralized implementations – see Section
II-C With these quantities, it is easy to see that problem (1) with \( R(w) = 0 \) is equivalent to the following problem:

\[
\min_{w \in \mathbb{R}^{KM}} \mathcal{J}(w) + \frac{1}{2\mu} \|w\|_C^2, \quad \text{s.t. } Bw = 0
\]  

(7)

where \( \mu > 0 \) and the matrix \( C \in \mathbb{R}^{MK \times MK} \) is a positive semi-definite consensus penalty matrix satisfying (6b). To solve problem (7), we consider the saddle-point formulation:

\[
\min_w \max_y \mathcal{L}(w, y) \triangleq \mathcal{J}(w) + \frac{1}{\mu} y^T Bw + \frac{1}{2 \mu} \|w\|_C^2
\]  

(8)

where \( y \in \mathbb{R}^{MK} \) is the dual variable. To solve (8), we propose the following algorithm: let \( y_{-1} = 0 \) and \( w_{-1} \) take any arbitrary value. Repeat for \( i = 0, 1, \ldots \)

\[
\begin{align*}
z_i &= (I - C)w_{i-1} - \mu \nabla \mathcal{J}(w_{i-1}) - B y_{i-1} && \text{(primal-descent)} \\
y_i &= y_{i-1} + B z_i && \text{(dual-ascent)} \\
w_i &= \bar{A} z_i && \text{(Combine)}
\end{align*}
\]  

(9a) (9b) (9c)

In the above algorithm, \( \bar{A} = \bar{A} \otimes I_M \) where \( \bar{A} \) is a symmetric and doubly-stochastic combination matrix. In the above algorithm, step (9a) is a gradient descent followed by a gradient ascent step in (9b), both applied to the saddle-point problem (8) with step-size \( \mu \). The last step (9c) is a combination step that enforces further agreement. Next we show that by proper choices of \( \bar{A}, B, \) and \( C \) we can recover many state of the art algorithms. To do that, we need to introduce the combination matrix associated with the network.

B. Network Combination Matrix

Thus, we introduce the combination matrices

\[
A = [a_{sk}] \in \mathbb{R}^{K \times K}, \quad \bar{A} = \bar{A} \otimes I_M
\]  

(10)

where the entry \( a_{sk} = 0 \) if there is no edge connecting agents \( k \) and \( s \). The matrix \( A \) is assumed to be symmetric and doubly stochastic matrix (different from \( \bar{A} \)). We further assume the matrix to be primitive, i.e., there exists an integer \( j > 0 \) such that all entries of \( A^j \) are positive. Under these conditions it holds that \( (I_{MK} - A)w = 0 \) if, and only, if \( w_k = w_s \) for all \( k, s \) — see [7], [8].
C. Specific Instances

We start by rewriting recursion (9) in an equivalent manner by eliminating the dual variable \( y_i \). Thus, from (9a) it holds that

\[
Z_i - Z_{i-1} = (I - C)(w_{i-1} - w_{i-2}) - B(y_{i-1} - y_{i-2}) - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))
\]

(9b)

Rearranging the previous equation we get:

\[
Z_i = (I - B^2)Z_{i-1} + (I - C)(w_{i-1} - w_{i-2}) - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))
\]

(11)

Utilizing this property, we will now choose specific matrices \( \{\bar{A}, B, C\} \) and show that we can recover many state of the art algorithms:

1) Exact diffusion [8]: If we choose \( \bar{A} = 0.5(I + A) \), \( C = 0 \) and \( B^2 = 0.5(I - A) \) in (11), we get:

\[
z_i = \bar{A}z_{i-1} + w_{i-1} - w_{i-2} - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))
\]

(12)

Multiplying the previous equation by \( \bar{A} \) and noting from (9c) that \( w_i = \bar{A}z_i \), we get:

\[
w_i = \bar{A}
\left(2w_{i-1} - w_{i-2} - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))\right)
\]

(13)

The above recursion is the exact diffusion recursion first proposed in [8]. We also note that if we choose \( C = 0 \), \( B^2 = c(I - A) \) (\( c \in \mathbb{R} \)), and \( \bar{A} = I - B^2 \) then we recover the smooth case of the NIDS algorithm from [9]. As highlighted in [9], NIDS is identical to exact diffusion for the smooth case when \( c = 0.5 \).

2) Aug-DGM [1]: Let \( C = 0 \), \( \bar{A} = A^2 \), and \( B = I - A \). Substituting into (11):

\[
z_i = (2A - A^2)z_{i-1} + w_{i-1} - w_{i-2} - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))
\]

(14)

By multiplying the previous equation by \( \bar{A} = A^2 \) and noting from (9c) that \( w_i = A^2z_i \), we get the recursion:

\[
w_i = A
\left(2w_{i-1} - Aw_{i-2} - \mu A(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))\right)
\]

(15)

The above recursion is equivalent to the Aug-DGM [1] (also known as ATC-DIGing [2]) algorithm:

\[
w_i = A(w_{i-1} - \mu x_{i-1})
\]

(16a)

\[
x_i = A(x_{i-1} + \nabla J(w_i) - \nabla J(w_{i-1}))
\]

(16b)

By eliminating the gradient tracking variable \( x_i \), we can rewrite the previous recursion as (15) – see Appendix [B]
3) ATC tracking method \([4]\): Let \(C = I - A\) and \(B = I - A\). Substituting into (11):

\[
z_i = (2A - A^2)z_{i-1} + Aw_{i-1} - Aw_{i-2} - \mu\left(\nabla J(w_{i-1}) - \nabla J(w_{i-2})\right)
\]

By multiplying the previous equation by \(\bar{A} = A\) and noting from (9c) that \(w_i = Az_i\), we get the recursion:

\[
w_i = A\left(2w_{i-1} - Aw_{i-2} - \mu\left(\nabla J(w_{i-1}) - \nabla J(w_{i-2})\right)\right)
\]

The above recursion is equivalent to the following variant of the ATC tracking method \([4], [3]\):

\[
\begin{align*}
 w_i &= A(w_{i-1} - \mu x_{i-1}) \quad \text{(19a)} \\
 x_i &= Ax_{i-1} + \nabla J(w_i) - \nabla J(w_{i-1}) \quad \text{(19b)}
\end{align*}
\]

By eliminating the gradient tracking variable \(x_i\), we can show that the previous recursion is exactly (18) – see Appendix B.

4) EXTRA and DIGing: We note that EXTRA \([7]\) and DIGing \([6]\) can also be represented as in (9) with \(\bar{A} = I\) and proper choices of \(B\) and \(C\). Since \(\bar{A} = I\), these algorithms are not of the ATC form and we do not focus on them in this work. We refer the readers to \([17]\) for details.

**Remark 1** (COMMUNICATION COST). Notice that exact diffusion \([13]\) communicates one vector per iteration. On the other hand, the gradient tracking method \([19]\) requires communicating the two vectors \(w_{i-1} - \mu x_{i-1}\) and \(x_{i-1}\) at each iteration \(i\). Similarly, the Aug-DGM (ATC-DIGing) method \([16]\) also requires sharing two variables, \(w_{i-1} - \mu x_{i-1}\) and \(x_{i-1} + \nabla J(w_i) - \nabla J(w_{i-1})\); moreover, it requires communicating these two variables sequentially (at different communication rounds).

### III. Proximal ATC Algorithms

In this section, we extend (9) to handle the non-differentiable component \(R(w)\). Let us introduce the network quantity

\[
R(w) \triangleq \frac{1}{K} \sum_{k=1}^{K} R(w_k)
\]

With this definition, we propose the following recursion: let \(y_{-1} = 0\) and \(w_{-1}\) take any arbitrary value. Repeat for \(i = 0, 1, \ldots\)

\[
\begin{align*}
 z_i &= (I - C)w_{i-1} - \mu\nabla J(w_{i-1}) - By_{i-1} \\
 y_i &= y_{i-1} + Bz_i \\
 w_i &= \text{prox}_{\mu R}(\bar{A}z_i)
\end{align*}
\]
We refer the reader to Appendix C for specific instances of the above algorithm and how to implement them in a decentralized manner. In the following, we will show that \( w_i \) in the above recursion converges to \( 1_K \otimes w^* \) where \( w^* \) is the desired solution of (1). We first prove the existence and optimality of the fixed points of recursion (21).

**Lemma 1 (Optimality Point).** Under Assumption 1 and condition (6), a fixed point \((w^*, y^*, z^*)\) exists for recursions (21a)–(21c), i.e., it holds that

\[
\begin{align*}
    z^* &= w^* - \mu \nabla J(w^*) - B y^* \\
    0 &= B z^* \\
    w^* &= \text{prox}_{\mu R}(\bar{A} z^*)
\end{align*}
\]

Moreover, \( w^* \) and \( z^* \) are unique with \( w^* = 1_K \otimes w^* \) where \( w^* \) is the solution of problem (1).

**Proof.** See Appendix A.

IV. LINEAR CONVERGENCE

Note that there exists a particular fixed point \((w^*, y^*_b, z^*)\) where \( y^*_b \) is a unique vector that belongs to the range space of \( B \) – see [17, Remark 2]. In the following we will show that the iterates \((w_i, y_i, z_i)\) converge linearly to this particular fixed point \((w^*, y^*_b, z^*)\). To this end, we introduce the error quantities:

\[
\begin{align*}
    \tilde{w}_i &\triangleq w_i - w^*, \quad \tilde{y}_i \triangleq y_i - y^*_b, \quad \tilde{z}_i = z_i - z^*
\end{align*}
\]

Note that from condition (6) we have \( Cw^* = 0 \). Therefore, from (21a)–(21c) and (22a)–(22c) we can reach the following error recursions:

\[
\begin{align*}
    \tilde{z}_i &= (I - C) \tilde{w}_{i-1} - \mu (\nabla J(w_{i-1}) - \nabla J(w^*)) - B \tilde{y}_{i-1} \\
    \tilde{y}_i &= \tilde{y}_{i-1} + B \tilde{z}_i \\
    \tilde{w}_i &= \text{prox}_{\mu R}(\bar{A} \tilde{z}_i) - \text{prox}_{\mu R}(\bar{A} z^*)
\end{align*}
\]

For our convergence result, we need the following technical conditions.

**Assumption 2 (Combination Matrices).** It is assumed that both condition (6) and the following condition hold:

\[
\begin{align*}
    0 &< I - B^2 \\
    \bar{A}^2 &\leq I - B^2 \text{ and } 0 \leq C < 2I
\end{align*}
\]

**Remark 2 (Convergence Conditions).** Note that the above conditions are satisfied for exact diffusion [26] and NIDS [9]. For the ATC tracking methods (16) and (19), the conditions translate to the requirement

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that the eigenvalues of $A$ must be in $(0, 1]$, rather than the typical $(-1, 1]$. Although this condition is not necessary, it can be easily satisfied by redefining $A \leftarrow 0.5(I + A)$. We also impose it to unify the analysis of these methods through a short proof. Note that most works that analyze decentralized methods under more relaxed conditions on the network topology impose restrictive step-size conditions that depend on the network and on the order of $O(\nu^{\theta_1}/\delta^{\theta_2})$ where $0 < \theta_1 \leq 1$ and $\theta_2 > 1$ – see [2], [5], [10], [12]. On the other hand, we require step sizes of order $O(1/\delta)$. Moreover, we will show that any algorithm that fits into our setup with $C = 0$ can use a step-size as large as the centralized gradient descent – see discussion after Theorem 1.

To state our result, we let $\sigma(B^2)$ denote the minimum non-zero singular value of the matrix $B^2$. Since $B^2$ is symmetric, its singular values are equal to its eigenvalues and condition (25a) implies $0 < \sigma(B^2) < 1$. We also let $\sigma_{\max}(C) < 2$ denote the maximum singular value of $C$.

**Theorem 1 (LINEAR CONVERGENCE).** Under Assumptions [1][2] if $\gamma_0 = 0$ and the step-size satisfies $\mu < \frac{2-\sigma_{\max}(C)}{\delta}$, it holds that

\[
\|\tilde{v}_i\|^2 + \|\tilde{v}_{i-1}\|^2 \leq \gamma^i \left(\|\tilde{v}_{i-1}\|^2 + \|\tilde{v}_{i-2}\|^2\right)
\]

where $\gamma = \max \{1 - \mu(2 - \sigma_{\max}(C) - \mu\delta), 1 - \sigma(B^2)\} < 1$.

**Proof.** Squaring both sides of (24a) and (24b) we get

\[
\|\tilde{z}_i\|^2 = \|(I - C)\tilde{w}_{i-1} - \mu (\nabla J(w_{i-1}) - \nabla J(w^*))\|^2 + \|B\tilde{y}_{i-1}\|^2
- 2\tilde{y}_{i-1}^T B \left((I - C)\tilde{w}_{i-1} - \mu (\nabla J(w_{i-1}) - \nabla J(w^*))\right)
\]

and

\[
\|\tilde{v}_i\|^2 = \|\tilde{v}_{i-1} + B\tilde{z}_i\|^2 = \|\tilde{v}_{i-1}\|^2 + \|B\tilde{z}_i\|^2 + 2\tilde{v}_{i-1}^T B\tilde{z}_i
\]

\[
\geq \|\tilde{v}_{i-1}\|^2 + \|\tilde{z}_i\|^2 - 2\|B\tilde{y}_{i-1}\|^2
+ 2\tilde{v}_{i-1}^T B \left((I - C)\tilde{w}_{i-1} - \mu (\nabla J(w_{i-1}) - \nabla J(w^*))\right)
\]

Adding equation (28) to (27) and rearranging, we get

\[
\|\tilde{z}_i\|^2 + \|\tilde{v}_i\|^2 = \|(I - C)\tilde{w}_{i-1} - \mu (\nabla J(w_{i-1}) - \nabla J(w^*))\|^2 + \|\tilde{y}_{i-1}\|^2 - \|B\tilde{y}_{i-1}\|^2
\]

where $Q = I - B^2$ is positive definite from (25a). Since $\gamma_0 = 0$ and $\gamma_i = \gamma_{i-1} + B\gamma_{i-1}$, we know $\gamma_i \in \text{range}(B)$ for any $i$. Thus, both $\gamma_i$ and $\gamma^*_i$ lie in the range space of $B$, and it holds that $\|B\tilde{y}_{i-1}\|^2 \geq \sigma(B^2)\|\tilde{y}_{i-1}\|^2$. Therefore, we can bound (29) by

\[
\|\tilde{z}_i\|^2 + \|\tilde{v}_i\|^2 \leq \|\tilde{w}_{i-1} - \mu (\nabla J(w_{i-1}) - \nabla J(w^*)) + \frac{1}{\mu} C\tilde{w}_{i-1}\|^2 + (1 - \gamma(B^2))\|\tilde{y}_{i-1}\|^2
\]
Also, since \( J(v) + \frac{1}{2\mu} \| v \|_C^2 \) is \( \delta \mu = \delta + \frac{1}{\mu} \sigma_{\max}(C) \)-smooth, it holds that [27, Theorem 2.1.5]:
\[
\| (\nabla J(w_{i-1}) - \nabla J(w^*) + \frac{1}{\mu} C \bar{w}_{i-1}) \|_C^2 \leq \delta \mu \bar{w}_{i-1}^T (\nabla J(w_{i-1}) - \nabla J(w^*) + \frac{1}{\mu} C \bar{w}_{i-1})
\]
(31)

Using this bound, it can be easily verified that:
\[
\| \bar{w}_{i-1} - \mu (\nabla J(w_{i-1}) - \nabla J(w^*) + \frac{1}{\mu} C \bar{w}_{i-1}) \|_C^2
\]
\[
\leq \| \bar{w}_{i-1} \|_C^2 - \mu (2 - \mu \delta \mu) \bar{w}_{i-1}^T (\nabla J(w_{i-1}) - \nabla J(w^*) + \frac{1}{\mu} C \bar{w}_{i-1})
\]
\[
\leq (1 - \mu \nu (2 - \mu \delta \mu)) \| \bar{w}_{i-1} \|_C^2 = (1 - \mu \nu (2 - \sigma_{\max}(C) - \mu \delta)) \| \bar{w}_{i-1} \|_C^2
\]
(32)

where in the last step we used the fact that \( 2 - \mu \delta \mu > 0 \), which follows from the condition \( \mu < (2 - \sigma_{\max}(C)) / \delta \), and the fact that \( J(v) + \frac{1}{2\mu} \| v \|_C^2 \) is \( \nu \)-strongly convex. Thus, we can substitute the previous inequality in (30) and get
\[
\| \bar{z}_i \|_Q^2 + \| \bar{y}_i \|_Q^2 \leq (1 - \mu \nu (2 - \sigma_{\max}(C) - \mu \delta)) \| \bar{w}_{i-1} \|_C^2 + (1 - \sigma(B^2)) \| \bar{y}_{i-1} \|_C^2
\]
(33)

From (24c) and the nonexpansive property of the proximal operator, we have
\[
\| \bar{w}_i \|_C^2 = \| \text{prox}_{\mu R}(\bar{A} \bar{z}_i) - \text{prox}_{\mu R}(\bar{A} \bar{z}^*) \|_C^2 \leq \| \bar{A} \bar{z}_i \|_C^2 \leq \| \bar{z}_i \|_Q^2
\]
(34)

where the last step holds because of condition (25b) so that \( \| \bar{A} \bar{z}_i \|_C^2 = \| \bar{z}_i \|_Q^2 \leq \| \bar{z}_i \|_Q^2 \). Substituting (34) into (33) we reach our result. Finally we note that:
\[
(1 - \mu \nu (2 - \sigma_{\max}(C) - \mu \delta)) < 1 \iff \mu < \frac{2 - \sigma_{\max}(C)}{\delta}
\]
(35)

The convergence rate \( \gamma \) in Theorem 1 clearly shows how the matrices \( B \) and \( C \) affect the convergence rate. Notice that when \( C = 0 \), which is the case for exact diffusion (13) and Aug-DGM (ATC-DIGing) (16), the step size bound becomes \( \mu < \frac{2}{\delta} \). This bound is as large as the centralized gradient descent. Moreover, for \( C = 0 \) the convergence rate becomes \( \gamma \geq \max\{1 - \mu \nu (2 - \mu \delta), 1 - \sigma(B^2)\} < 1 \), which separates the network effect from the cost function. A similar conclusion appears for NIDS [9], which is subsumed in our framework.

V. SIMULATIONS ON REAL DATA

In this section we test the performance of three different instantiations of the proposed method (21) against some state-of-the-art algorithms. We consider the following sparse logistic regression problem:

\[
\min_{w \in \mathbb{R}^M} \frac{1}{K} \sum_{k=1}^{K} J_k(w) + \rho \| w \|_1 \quad \text{where} \quad J_k(w) = \frac{1}{L} \sum_{\ell=1}^{L} \ln(1 + \exp(-y_{k,\ell} x_{k,\ell}^T w)) + \frac{\lambda}{2} \| w \|_2^2
\]
where \( \{x_{k,\ell}, y_{k,\ell}\}_{\ell=1}^{L} \) are local data kept by agent \( k \) and \( L \) is the size of the local dataset. We consider three real datasets: Covtype.binary, MNIST, and CIFAR10. The last two datasets have been transformed into binary classification problems by considering data with two labels, digits two and four (‘2’ and ‘4’) classes for MNIST, and cat and dog classes for CIFAR-10. In Covtype.binary we use 50,000 samples as training data and each data has dimension 54. In MNIST we use 10,000 samples as training data and each data has dimension 784. In CIFAR-10 we use 10,000 training data and each data has dimension 3072. All features have been preprocessed and normalized to the unit vector with sklearn’s normalize 3.

\[ \text{Fig. 1: The network topology used in the simulation.} \]

For the network, we generated a randomly connected network with \( K = 20 \) nodes, which is shown in Fig. 1. The associated combination matrix \( A \) is generated according to the Metropolis rule [16]. For all simulations, we assign data evenly to each agent. We set \( \lambda = 10^{-4} \) and \( \rho = 2 \times 10^{-3} \) for Covtype, \( \lambda = 10^{-2} \) and \( \rho = 5 \times 10^{-4} \) for CIFAR-10, and \( \lambda = 10^{-4} \) and \( \rho = 2 \times 10^{-3} \) for MNIST. The simulation results are shown in Figure 2. The decentralized implementations of Prox-ED, Prox-ATC I, and prox-ATC II are given in Appendix C. For each algorithm, we tune the step-sizes manually to achieve the best possible convergence rate. We notice that the performance of each algorithm differs in each data set. Prox-ED always performs the best in our simulation setup. Prox-ATC I and prox-ATC II have comparable performance to Prox-ED in the MNIST data set but performs worse in the other two data-sets.

VI. SEPARATE NON-SMOOTH TERMS: SUBLINEAR RATE

In this section, we will show that global linear convergence cannot be attained (in the worst case) if there exists more than one non-smooth term. Consider the more general problem with agent specific regularizers:

\[ \min_{w \in \mathbb{R}^M} \frac{1}{K} \sum_{k=1}^{K} J_k(w) + R_k(w), \quad (36) \]
where \( J_k(w) \) is a strongly convex smooth function and \( R_k(w) \) is non-smooth convex with closed form proximal mappings (each \( J_k(w) \) and \( R_k(w) \) are further assumed to be closed and proper functions). Although many algorithms (centralized and decentralized) exist that solve (36), none have been shown to achieve linear convergence in the presence of general non-smooth proximal terms \( R_k(w) \). In the following, by tailoring the results from [22], we show that this is not possible when having access to the proximal mapping of each individual non-smooth term \( R_k(w) \) separately.

A. Sublinear lowerbound

Let \( \mathcal{H} \) be a deterministic algorithm that queries

\[
\{ J_k(\cdot), R_k(\cdot), \nabla J_k(\cdot), \text{prox}_{\mu_{i,k} R_k(\cdot)} | \mu_{i,k} > 0, k = 1, \ldots, K \}
\]

once for each iteration \( i = 0, 1, \ldots \). To clarify, the scalar parameter \( \mu_{i,k} > 0 \) can differ for \( i = 0, 1, \ldots \) and \( k = 1, \ldots, K \) or they can be constants (e.g. \( \mu_{i,k} = \mu > 0 \)). Note that \( \mathcal{H} \) has the option to combine the queried values in any possible combination (it can only use certain information from certain communications). Thus, \( \mathcal{H} \) includes decentralized algorithms in which communication is restricted to edges on a graph.

Consider the specific instance of (36)

\[
\min_{w \in \mathbb{R}^m} F_\nu(w) = \frac{\nu}{2} \|w\|^2 + \frac{1}{K} \sum_{k=1}^{K} R_k(w)
\]  

(37)

where \( \nu > 0 \) and \( J_k(w) = \frac{\nu}{2K} \|w\|^2 \). Assume \( R_k(w) < \infty \) if and only if \( \|w\| \leq B \) and \( |R_k(w_1) - R_k(w_2)| \leq G \|w_1 - w_2\| \) for all \( w_1, w_2 \) (where \( B \) and \( G \) are some positive constants) such that \( \|w_1\| \leq B \) and \( \|w_2\| \leq B \). To prove that linear convergence is not possible, we will reduce our setup to \( \min_{w \in \mathbb{R}^m} F_0(w) \), which has a known lower bound [22]. Let \( \mathcal{H}_0 \) be a deterministic algorithm that queries

\[
\{ R_k(\cdot), \text{prox}_{\mu_{i,k} R_k(\cdot)} | \mu_{i,k} > 0, k = 1, \ldots, K \}
\]

Fig. 2: Simulation results. The y-axis indicates the relative squared error \( \sum_{k=1}^{K} \|w_{k,i} - w^*\|^2/\|w^*\|^2 \). Prox-ED refers to (21) with \( A = 0.5(I + A), B^2 = 0.5(I - A), \) and \( C = 0 \). Prox-ATC I refers to (21) with \( A = A^2, B = I - A, \) and \( C = 0 \). Prox-ATC II refers to (21) with \( A = A, B = I - A, \) and \( C = I - A \). DL-ADMM [19], PG-EXTRA [18], NIDS [9].
once for each iteration \( i = 0, 1, \ldots \) and communicates through a fully connected network. The following result is a special case of the more general result [22, Theorem 1].

**Theorem 2.** Let \( 0 < B, 0 < G, 2 \leq K, \) and \( 0 < \varepsilon < GB/12. \) For a large enough problem dimension \( M = \mathcal{O}(KGB/\varepsilon), \) the algorithm \( \mathcal{H}_0 \) (in the worst case) requires \( \mathcal{O}(LB/\varepsilon) \) or more iterations to find a \( \hat{w} \) such that \( F_0(\hat{w}) - \inf_w F_0(w) < \varepsilon. \)

We argue that algorithm \( \mathcal{H} \) cannot be too efficient at solving \( \min_w F_{\nu}(w) \) with \( \nu > 0 \) as otherwise it can be used to efficiently solve \( \min_w F_0(w) \) and contradict Theorem 2.

**Theorem 3.** Let \( 0 < \nu, 0 < B, 0 < G, 2 \leq K, \) and \( 0 < \varepsilon < G^2/(288\nu). \) For a large enough problem dimension \( M = \mathcal{O}(KG/\sqrt{\nu\varepsilon}), \) the algorithm \( \mathcal{H} \) (in the worst case) requires \( \mathcal{O}(G/\sqrt{\nu\varepsilon}) \) or more iterations to find a \( \hat{w} \) such that \( F_\nu(\hat{w}) - \inf_w F_\nu(w) < \varepsilon. \)

**Proof.** This argument modifies the proof of [22, Theorem 2], which makes a similar but slightly different claim. Let \( \nu = \varepsilon/B^2 \) and \( w^*_\nu \) denotes the minimizer of \( F_\nu. \) Assume for contradiction that \( \mathcal{H} \) can find a \( \hat{w} \) such that

\[
F_\nu(\hat{w}) - F_\nu(w^*_\nu) < \frac{\varepsilon}{2}
\]  

in \( o(G/\sqrt{\nu\varepsilon}) \) iterations. Note that for all \( w \) such that \( \|w\| \leq B, \) it holds from [37]:

\[
F_\nu(w) \leq F_0(w) + \frac{\nu B^2}{2} = F_0(w) + \frac{\varepsilon}{2}.
\]  

Putting these together, we get

\[
F_0(\hat{w}) - F_0(w^*_\nu) - \frac{\varepsilon}{2} \leq F_0(\hat{w}) - F_\nu(w^*_\nu) \overset{(a)}{\leq} F_\nu(\hat{w}) - F_\nu(w^*_\nu) < \frac{\varepsilon}{2},
\]  

where in step (a) we used \( F_0(w) \leq F_\nu(w) \) and \( F_\nu(w^*_\nu) \leq F_\nu(w^*_\nu). \) We conclude that \( F_0(\hat{w}) - F_0(w^*_0) < \varepsilon. \)

Since \( \nabla F_\nu(\cdot) = \nu I \) is just a scaled identity, querying \( \nabla J_k(\cdot) \) does not provide a new direction that \( \mathcal{H}_0 \) could otherwise not use. Thus, algorithm \( \mathcal{H} \) applied to minimizing \( F_\nu \) is an instance of algorithm \( \mathcal{H}_0. \)

This means that we have an algorithm for minimizing \( F_0 \) in \( o(G/\sqrt{\nu\varepsilon}) = o(GB/\varepsilon) \) iterations, which contradicts Theorem 2. Note that \( 0 < \varepsilon < GB/12 \) from Theorem 2 and by using \( \nu = \varepsilon/B^2 \) we require \( 0 < \varepsilon < G^2/(288\nu) \) (an extra factor of 2 appears because of (38)).

**Corollary 1.** For the problem setup of [36] with strongly convex \( J_k(\cdot) \) for all \( k = 1, 2, \ldots, K, \) any algorithm that accesses the functions through evaluations of \( J_k(\cdot) \) and \( R_k(\cdot), \) the gradients of \( J_k(\cdot), \) and proximal operators of \( R_k(\cdot) \) is not globally linearly convergent (in the worst case).
Remark 3. The lower bound of Theorem 3 is dimension independent in the same way other Nesterov-type lower bounds are [22], [27]. The result implies that it is not possible to establish linear convergence of $\mathcal{H}$ with a rate depending on $\nu$ and $G$, but not on the problem dimension $K$. That said, a dimension dependent linear convergence may be established. For example, asymptotic linear convergence has been established in [28] when the functions $\{J_k(\cdot), R_k(\cdot)\}$ are piecewise linear quadratic. This result does not contradict our result as the linear rate and the number of iterations needed to observe the linear rate are dependent on the problem dimension. Our linear convergence result of Theorem 1 is dimension independent as it holds for any dimension $M$.

B. Numerical counter example

In this section, we numerically show that linear convergence is not possible. We consider an instance of (36) with $K = 2$, $M$ is a very large even number, and quadratic smooth terms $J_k(w) = \eta/2\|w\|^2$ for some $\eta > 0$. We let the non-smooth terms be

$$R_1(w) = |\sqrt{2}w(1) - 1| + |w(2) - w(3)| + |w(4) - w(5)| + \cdots + |w(M-2) - w(M-1)|$$

(41a)

$$R_2(w) = |w(1) - w(2)| + |w(3) - w(4)| + \cdots + |w(M-1) - w(M)|$$

(41b)

Both $\text{prox}_{R_1}$ and $\text{prox}_{R_2}$ have closed forms — see Appendix D for details. The above construction is related to the one in [23], which was used to derive lower bounds for a different class of algorithms as explained in the introduction.

In the numerical experiment, we test the performance of two well known decentralized proximal methods, PG-EXTRA [18] and DL-ADMM [19], [20]. We set $M = 2000$ and $\eta = 1$. The step-sizes for both PG-EXTRA and DL-ADMM are set to 0.005. The combination matrix is set as $A = \frac{1}{2}I_2 \frac{I_2}{2}^T$. The numerical results in Fig. 3 show that both PG-EXTRA and DL-ADMM perform almost the same, and they converge sublinearly to the solution. No global linear convergence is observed in the simulation for sufficiently large dimension $M$ and algorithms independent of $M$, which is consistent with our discussion in Remark 3.

VII. CONCLUDING REMARKS

In this work, we proposed a proximal primal-dual algorithmic framework, which subsumes many existing algorithms in the smooth case, and established its linear convergence under strongly-convex

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4 A sequence $\{x_i\}_{i=0}^\infty$ has asymptotic linear convergence to $x^*$ if there exists a sufficiently large $i_o$ such that $\|x_i - x^*\| \leq \gamma^iC$ for some $C > 0$ and all $i \geq i_o$.
Fig. 3: Both PG-EXTRA [18] and DL-ADMM [19, 20] converge sublinearly to the solution of the proposed numerical counter example.

objectives. Our analysis provides wider step-size conditions than many existing works, which provides insightful indications on the performance of each algorithm. That said, these step-size bound comes at the expense of stronger assumption on the combination matrices – see Remark 2. It is therefore of interest to study the interrelation between the step-sizes and combination matrices for linear convergence.

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APPENDIX A

PROOF OF LEMMA 1

To show the existence we will construct a point \((w^*, y^*, z^*)\) that satisfies equations (22a)–(22c).
Since each \(J_k(w)\) is strongly convex, there exists a unique solution \(w^*\) for problem (1), i.e., \(0 \in \frac{1}{K} \sum_{k=1}^{K} \nabla J_k(w^*) + \partial R(w^*)\). This also indicates that there must exist a subgradient \(r^* \in \partial R(w^*)\) such that
\[
\frac{1}{K} \sum_{k=1}^{K} \nabla J_k(w^*) + r^* = 0 \tag{42}
\]
Now we define \(z^* = \frac{\Delta}{\mu} r^* + w^*\), it holds that \(r^*/K + (w^* - z^*)/\mu = 0\), i.e., \(0 \in (1/K)\partial R(w^*) + (1/\mu)(w^* - z^*)\). This implies that
\[
w^* = \arg \min_w \left\{ \frac{1}{K} R(w) + \frac{1}{2\mu} \|w - z^*\|^2 \right\} \tag{43}
\]
We next define \(W^* = \frac{1}{K} \otimes w^*\) and \(Z^* = \frac{1}{K} \otimes z^*\). Since \(Z^* = \frac{1}{K} \otimes z^*\), it belongs to the null space of \(B\) so that \(Bz^* = 0\) and, moreover, \(\bar{A}z^* = z^*\) since \(\bar{A} = \bar{A} \otimes I_M\) where \(\bar{A}\) is doubly stochastic. Therefore, relation (43) implies that equation (22c) holds. It remains to construct \(y^*\) that satisfies equation (22a).
Note that
\[
(\mathbb{1}_N \otimes I_M)^T (w^* - z^* - \mu \nabla \mathcal{J}(w^*)) = -\mu r^* - \frac{\mu}{K} \sum_{k=1}^{K} \nabla J_k(w^*) = 0, \tag{44}
\]
where the last equality holds because \(w^*\) is the optimal solution of problem (1). Equation (44) implies
\[
\frac{1}{\mu} (w^* - z^* - \mu \nabla \mathcal{J}(w^*)) \in \text{Null}(\mathbb{1}_N \otimes I_M) = \text{Null}(B) = \text{Range}(B). \tag{45}
\]
where \(\perp\) denotes the orthogonal complement. Therefore, there exist a vector \(y^*\) satisfying equation (22a).

We now show that any fixed point is of the form \(w^* = \mathbb{1}_K \otimes w^*\) and \(w^*\) is the solution to problem (1). From (22b) and (6), it holds that the block elements of \(z^*\) are equal to each other, i.e. \(z^*_1 = \cdots = z^*_K\), and we denote each block element by \(z^*\). Thus, \(\bar{A}z^* = z^* = \mathbb{1}_K \otimes z^*\) because \(\bar{A} = \bar{A} \otimes I_M\) where \(\bar{A}\) is doubly stochastic. Therefore, from (22c) and the definition of the proximal operator it holds that
\[
w_k^* = \arg \min_{w_k} \left\{ R(w_k)/K + \|w_k - z^*\|^2/2\mu \right\} \tag{46}
\]
where we used \(z_k^* = z^*\) for each \(k\). Thus, we must have \(w_1^* = \cdots = w_K^* \vdash w^*\). It is easy to verify that (46) implies
\[
0 \in \partial R(w^*)/K + (w^* - z^*)/\mu. \tag{47}
\]
Multiplying \((1_K \otimes I_M)^T\) from the left to both sides of equation (22a), we get

\[
K z^* = Kw^* - \mu \frac{1}{K} \sum_{k=1}^{K} \nabla J_k(w^*)
\]  

(48)

Combining (47) and (48), we get \(0 \in \frac{1}{K} \sum_{k=1}^{K} \nabla J_k(w^*) + \partial R(w^*).\) Thus, \(w^*\) is the unique solution to problem (1). Due to the uniqueness of \(w^*\), we see from (48) that \(z^*\) is unique. Consequently, \(w^* = 1_K \otimes w^*\) and \(z^* = 1_K \otimes z^*\) must be unique.

**APPENDIX B**

**EQUIVALENT REPRESENTATION**

**A. Aug-DGM (ATC-DIGing)**

Here we show that (16) is equivalent to (15). From (16a) we have

\[
w_i - Aw_{i-1} = A \left( w_{i-1} - Aw_{i-2} - \mu (x_{i-1} - Ax_{i-2}) \right)
\]

(16b)

\[
A \left( w_{i-1} - Aw_{i-2} - \mu A \left( \nabla J(w_{i-1}) - \nabla J(w_{i-2}) \right) \right)
\]

Rearranging the previous equation we get:

\[
w_i = A \left( 2w_{i-1} - Aw_{i-2} - \mu A \left( \nabla J(w_{i-1}) - \nabla J(w_{i-2}) \right) \right)
\]

which is recursion (15).

**B. ATC-Tracking**

In a similar manner we can show that (19) is equivalent to (18). From (19a) we have

\[
w_i - Aw_{i-1} = A \left( w_{i-1} - Aw_{i-2} - \mu (x_{i-1} - Ax_{i-2}) \right)
\]

(19b)

\[
A \left( w_{i-1} - Aw_{i-2} - \mu \left( \nabla J(w_{i-1}) - \nabla J(w_{i-2}) \right) \right)
\]

Rearranging the previous equation we get:

\[
w_i = A \left( 2w_{i-1} - Aw_{i-2} - \mu \left( \nabla J(w_{i-1}) - \nabla J(w_{i-2}) \right) \right)
\]

which is recursion (18).
APPENDIX C
IMPLEMENTATION OF (21)

A. Prox-ED: $\bar{A} = 0.5(I + A)$, $B^2 = 0.5(I - A)$, and $C = 0$

Recursion (21) with $\bar{A} = 0.5(I + A)$, $B^2 = 0.5(I - A)$, and $C = 0$ is equivalent to the proximal exact diffusion (Prox-ED) recursion listed in (53a)-(53d). To see this, note that for $i = 0$, it is straightforward to check that each block in (21c) is the same as $w_{k,0}$ in (53). Now we will show the equivalence for $i \geq 1$. From (21a), we know that:

$$Z_{i} - Z_{i-1} = W_{i-1} - W_{i-2} - \mu(\nabla J(W_{i-1}) - \nabla J(W_{i-2})) - \mathcal{B}(\gamma_{i-1} - \gamma_{i-2})$$

$$= W_{i-1} - W_{i-2} - \mu(\nabla J(W_{i-1}) - \nabla J(W_{i-2})) - B^2 Z_{i-1}$$

(49)

where we used (21b) in the last step. Rearranging and noting that $B^2 = 0.5(I - A)$ we get

$$z_{i} = \bar{A} z_{i-1} + w_{i-1} - w_{i-2} - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))$$

(50)

By multiplying $\bar{A}$ to both sides of the previous equation and introducing $x_{i} = \bar{A} z_{i}$ we get

$$x_{i} = \bar{A} \left( x_{i-1} + w_{i-1} - w_{i-2} - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2})) \right)$$

(51)

Thus from (21c) we get

$$x_{i} = \bar{A} \left( x_{i-1} + w_{i-1} - w_{i-2} - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2})) \right)$$

(52a)

$$w_{i} = \text{prox}_{\mu R}(x_{i})$$

(52b)

The above recursion is equivalent to (53a)-(53d). This can be easily seen by substituting (53a)-(53b) into (53c).

Algorithm (Prox-ED)
Setting: Let $A = [\bar{a}_{sk}] = (I_K + A)/2$. Initialize $x_{k,-1} = \psi_{k,-1}$ and $w_{k,-1}$ arbitrary. For every agent $k$, repeat for $i = 0, 1, 2, ...$

$$\psi_{k,i} = w_{k,i-1} - \mu \nabla J_{k}(w_{k,i-1})$$

(53a)

$$z_{k,i} = x_{k,i-1} + \psi_{k,i} - \psi_{k,i-1}$$

(53b)

$$x_{k,i} = \sum_{s \in N_{k}} \bar{a}_{sk} z_{s,i} \quad \text{(Communication step)}$$

(53c)

$$w_{k,i} = \text{prox}_{\mu R}(x_{k,i})$$

(53d)
B. Prox-ATC I: \( \bar{A} = A^2, B^2 = (I - A)^2, \) and \( C = 0 \)

For the choice \( \bar{A} = A^2, B^2 = (I - A)^2, \) and \( C = 0, \) we can represent (21) as listed in (56). This can be seen by following the same approach as the previous subsection. To see this, note that with \( B^2 = (I - A)^2 \) to get

\[
Z_i = (2A - A^2)Z_{i-1} + w_{i-1} - w_{i-2} - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))
\] (54)

By multiplying \( A^2 \) to both sides of the previous equation and introducing \( X_i \) to get

\[
x_i = A\left((2I - A)x_{i-1} + Aw_{i-1} - Aw_{i-2} - \mu(A(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))\right)
\] (55)

Thus from (21c) we have \( w_i = \text{prox}_{\mu R}(x_i). \)

Algorithm: Prox-ATC I

Setting: Initialize \( x_{k,-1} = \psi_{k,-1} = 0 \) and \( w_{k,-1} \) arbitrary. For every agent \( k, \) repeat for \( i = 0, 1, 2, \ldots \)

\[
\psi_{k,i} = w_{k,i-1} - \mu\nabla J_k(w_{k,i-1})
\] (56a)

\[
z_{k,i} = 2x_{k,i-1} - \sum_{s \in N_k} a_{sk}(x_{s,i-1} - \psi_{s,i} + \psi_{s,i-1}) \quad \text{(Communication step)}
\] (56b)

\[
x_{k,i} = \sum_{s \in N_k} a_{sk}z_{s,i} \quad \text{(Communication step)}
\] (56c)

\[
w_{k,i} = \text{prox}_{\mu R}(x_{k,i})
\] (56d)

C. Prox-ATC II: \( \bar{A} = A, B = I - A, \) and \( C = I - A \)

For the choice \( \bar{A} = A, B = I - A, \) and \( C = I - A \), we can represent (21) as listed in (59). This can be seen by following the same approach as the previous subsection. To see this, note that with \( B^2 = (I - A)^2 \) to get

\[
z_i = (2A - A^2)z_{i-1} + Aw_{i-1} - Aw_{i-2} - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))
\] (57)

By multiplying \( A \) to both sides of the previous equation and introducing \( x_i \) to get

\[
x_i = A\left((2I - A)x_{i-1} + Aw_{i-1} - Aw_{i-2} - \mu(\nabla J(w_{i-1}) - \nabla J(w_{i-2}))\right)
\] (58)

Thus from (21c) we have \( w_i = \text{prox}_{\mu R}(x_i). \)
Algorithm: Prox-ATC II

Setting: Initialize $x_{k,-1} = \psi_{k,-1} = 0$ and $w_{k,-1}$ arbitrary. For every agent $k$, repeat for $i = 0, 1, 2, \ldots$

$$\psi_{k,i} = 2x_{k,i-1} - \mu \left( \nabla J_k(w_{k,i-1}) - \nabla J_k(w_{k,i-2}) \right)$$  \hspace{1cm} (59a)

$$z_{k,i} = \psi_{k,i} - \sum_{s \in \mathcal{N}_k} a_{sk}(x_{s,i-1} - w_{s,i-1} + w_{s,i-2}) \quad \text{(Communication step)}$$  \hspace{1cm} (59b)

$$x_{k,i} = \sum_{s \in \mathcal{N}_k} a_{sk} z_{s,i} \quad \text{(Communication step)}$$  \hspace{1cm} (59c)

$$w_{k,i} = \text{prox}_{\mu R}(x_{k,i})$$  \hspace{1cm} (59d)

\[\text{APPENDIX D}\]

Proximal Mapping of (41)

To rewrite the non-smooth terms (41) more compactly, we introduce

$$D_1 \triangleq \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{M \times M}$$  \hspace{1cm} (60)

$$D_2 \triangleq \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{M \times M}$$  \hspace{1cm} (61)

and $b_1 \triangleq e_1$ where $e_1$ is the first column of the identity matrix $I_{M/2}$. With $D_1$, $D_2$ and $b_1$, we can rewrite $R_1(w)$ and $R_2(w)$ in (41) as

$$R_1(w) = \|D_1w - b_1\|_1, \quad R_2(w) = \|D_2w\|_1.$$

(62)

Let us introduce $g(w) = \|w\|_1$ so that $R_1(w) = g(D_1w - b_1)$ and $R_2(w) = g(D_2w)$. It can be verified that $D_1D_1^T = 2I$ and $D_2D_2^T = 2I$. Thus, from [29] Theorem 6.15 it holds that

$$\text{prox}_{\mu R_1}(w) = w + \frac{1}{2\mu} D_1^T[\text{prox}_{2\mu g}(\mu D_1w - \mu b_1) - \mu D_1w + \mu b_1],$$

(63a)

$$\text{prox}_{\mu R_2}(w) = w + \frac{1}{2\mu} D_2^T[\text{prox}_{2\mu g}(\mu D_2w) - \mu D_2w].$$

(63b)
In other words, both $\text{prox}_{\mu R_1} (w)$ and $\text{prox}_{\mu R_2} (w)$ have closed forms which are easy to calculate since

$$\text{prox}_{\kappa g} (w) = \text{col} \left\{ \text{sgn}(w[j]) \max\{|w[j]| - \kappa, 0\} \right\}_{j=1}^{M} \in \mathbb{R}^{M}.$$