Sub-signature operators and the Kastler-Kalau-Walze type theorem for manifolds with boundary

Tong Wu¹, Sining Wei², Yong Wang¹∗

¹School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, China
²School of Data Science and Artificial Intelligence, Dongbei University of Finance and Economics, Dalian 116025, P.R.China

Abstract

In this paper, we obtain two Lichnerowicz type formulas for sub-signature operators. And we give the proof of Kastler-Kalau-Walze type theorems for sub-signature operators on 4-dimensional and 6-dimensional compact manifolds with (resp.without) boundary.

Keywords: Sub-signature operators; Lichnerowicz type formulas; Noncommutative residue; Kastler-Kalau-Walze type theorems.

1. Introduction

Until now, many geometers have studied noncommutative residues. In [8, 23], authors found noncommutative residues are of great importance to the study of noncommutative geometry. In [3], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Connes showed us that the noncommutative residue on a compact manifold \( M \) coincided with the Dixmier’s trace on pseudodifferential operators of order \(-\text{dim } M\) in [4]. And Connes claimed the noncommutative residue of the square of the inverse of the Dirac operator was proportioned to the Einstein-Hilbert action. Kastler [11] gave a brute-force proof of this theorem. Kalau and Walze proved this theorem in the normal coordinates system simultaneously in [10]. Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac operator \( \text{Wres}(D^2) \) in turn is essentially the second coefficient of the heat kernel expansion of \( D^2 \) in [1].

On the other hand, Wang generalized the Connes’ results to the case of manifolds with boundary in [19, 20], and proved the Kastler-Kalau-Walze type theorem for the Dirac operator and the signature operator on lower-dimensional manifolds with boundary [21]. In [21, 22], Wang computed \( \text{Wres}(\pi^{\dagger}D^{-1} \circ \pi^{\dagger}D^{-1}) \) and \( \text{Wres}(\pi^{\dagger}D^{-2} \circ \pi^{\dagger}D^{-2}) \), where the two operators are symmetric, in these cases the boundary term vanished. But for \( \text{Wres}(\pi^{\dagger}D^{-1} \circ \pi^{\dagger}D^{-3}) \), Wang got a nonvanishing boundary term [16], and give a theoretical explanation for gravitational action on boundary. In others words, Wang provides a kind of method to study the Kastler-Kalau-Walze type theorem for manifolds with boundary. In [25] and [26], Zhang introduced the sub-signature operators and proved a local index formula for these operators. In [5] and [12], by computing the adiabatic limit of \( \eta \)-invariants associated to the so-called sub-signature operators, a new proof of the Riemann-Roch-Grothendieck type formula of Bismut-Lott was given. In [2], Bao, Wang and Wang proved a local equivariant index theorem for sub-signature operators which generalized the Zhang’s index theorem for sub-signature operators.

∗Corresponding author.

Email addresses: wut977@nenu.edu.cn (Tong Wu¹), weisn835@nenu.edu.cn (Sining Wei²), wangy581@nenu.edu.cn (Yong Wang¹)

Preprint submitted to Elsevier January 25, 2022
The motivation of this paper is to prove the Kastler-Kalau-Walze type theorem for the sub-signature operators on 4-dimensional and 6-dimensional compact manifolds.

Actually, for 4-dimensional manifolds, generally $\text{Wres}[\pi^+ D^{-1} \circ \pi^+ D^{-1}]$ has a vanishing boundary term. In order to get the non-vanishing boundary term, for $D$ which is not self-adjoint, we usually calculate $\text{Wres}[\pi^+ (D^*)^{-1} \circ \pi^+ D^{-1}]$. Because $D^* \neq D$, it’s not symmetry, we might get the boundary term. See [17, 18].

In this paper, the operator $D_t$ for $t \in \mathbb{R}$ is self-adjoint. At the moment, the boundary term of $\text{Wres}[\pi^+ D_t^{-1} \circ \pi^+ D_t^{-1}]$ disappears. We want to get the non-vanishing boundary term, so we consider the operator $D_t$ for $t \in \mathbb{C}$, where $D_t$ is not self-adjoint. Similarly to [17, 18], we want to get the boundary term from $\text{Wres}[\pi^+ (D_t^*)^{-1} \circ \pi^+ D_t^{-1}]$, that’s our motivation of thinking about $D_t$ for $t \in \mathbb{C}$. After taking trace and some computations, we find that we don’t get the non-vanishing boundary term. Let $\{e_1 \cdots e_n\}$ is fixed orthonormal frame, $c(e_i)$ is the Clifford action as (2.3), $S$ is the tensor defined by (2.7) and $f_\alpha$ is defined by (2.8). Our main theorems are as follows.

**Theorem 1.1.** Let $M$ be a 4-dimensional oriented compact manifold with boundary $\partial M$ and the metric $g^T$ be defined as (3.1), $D_t$ and $D_t^*$ be sub-signature operators on $M$ ($M$ is a collar neighborhood of $M$) as in (2.3), (2.4), then the following identities hold:

$$\text{Wres}[\pi^+ D_t^{-1} \circ \pi^+ (D_t^*)^{-1}] = 32\pi^2 \int_M \left( -\frac{4}{3}K - \frac{(7-t)^2}{2} \sum_{i=1}^4 \sum_{\alpha=1}^k |S(e_i)f_\alpha|^2 \right) d\text{Vol}_M, \quad (1.1)$$

$$\text{Wres}[\pi^+ t^{-1} \circ \pi^+ D_t^{-1}] = 32\pi^2 \int_M \left( -\frac{4}{3}K \right) d\text{Vol}_M, \quad (1.2)$$

where $K$ is the scalar curvature. In particular, the boundary term vanishes.

In general, for $\text{Wres}[\pi^+ D_t^{-1} \circ \pi^+ D_t^{-1}]$ in the 4-dimensional case and for $\text{Wres}[\pi^+ t^{-2} \circ \pi^+ D_t^{-2}]$ in the 6-dimensional situations, we get the vanishing boundary terms. In order to get the boundary term that doesn’t disappear, we usually consider the case of asymmetry, that is, the calculations of $\text{Wres}[\pi^+ D_t^{-1} \circ \pi^+ D_t^{-3}]$ and $\text{Wres}[\pi^+ t^{-1} \circ \pi^+ (D_t^* D_t D_t^*)^{-1}]$ in 6-dimensional situation. As in the following Theorem 1.2 we get the non-vanishing boundary terms. Our main motivation is to obtain the non-vanishing boundary term in the 6-dimensional case.

**Theorem 1.2.** Let $M$ be a 6-dimensional oriented compact manifold with boundary $\partial M$ and the metric $g^T$ be defined as (3.1), $D_t$ and $D_t^*$ be sub-signature operators on $M$ ($M$ is a collar neighborhood of $M$) as in (2.3), (2.4), then the following identities hold:

$$\text{Wres}[\pi^+ D_t^{-1} \circ \pi^+ (D_t^* D_t D_t^*)^{-1}] = 128\pi^3 \int_M \left( -\frac{16}{3}K - (7-t)^2 \sum_{i=1}^6 \sum_{\alpha=1}^k |S(e_i)f_\alpha|^2 \right) d\text{Vol}_M + \int_{\partial M} \left( \left( \frac{65}{8} - \frac{41}{8}i\pi h'(0) \right) \Omega_4 \right) d\text{Vol}_M, \quad (1.3)$$

$$\text{Wres}[\pi^+ t^{-1} \circ \pi^+ (D_t^{-3})] = 128\pi^3 \int_M \left( -\frac{16}{3}K \right) d\text{Vol}_M + \int_{\partial M} \left( \left( \frac{65}{8} - \frac{41}{8}i\pi h'(0) \right) \Omega_4 \right) d\text{Vol}_M, \quad (1.4)$$

where $K$ is the scalar curvature, $h$ is defined by (3.1) and $\Omega_4$ is the canonical volume of $S^4$.

The paper is organized in the following way. In Section 2 by using the definition of sub-signature operators, we compute the Lichnerowicz formulas for sub-signature operators. In Section 3 and in Section 4 we prove the Kastler-Kalau-Walze type theorem for 4-dimensional and 6-dimensional manifolds with boundary for sub-signature operators respectively.
2. Sub-signature operators and their Lichnerowicz formulas

Firstly we introduce some notations about sub-signature operators. Let $M$ be an $n$-dimensional ($n \geq 3$) oriented compact Riemannian manifold with a Riemannian metric $g^{TM}$. And let $F$ be a subbundle of $TM$, $F^\perp$ be the subbundle of $TM$ orthogonal to $F$. Then we have the following orthogonal decomposition:

$$TM = F \bigoplus F^\perp,$$

$$g^{TM} = g^F \bigoplus g^{F^\perp},$$

(2.1)

where $g^F$ and $g^{F^\perp}$ are the induced metric on $F$ and $F^\perp$.

Let $\nabla^L$ be the Levi-Civita connection about $g^{TM}$. In the fixed orthonormal frame $\{e_1, \cdots, e_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(e_1, \cdots, e_n) = (e_1, \cdots, e_n)(\omega_{s,t}).$$

(2.2)

Let $\epsilon(e_j)$, $\iota(e_j)$ be the exterior and interior multiplications respectively, where $e_j = g^{TM}(e_j, \cdot)$. Write

$$\hat{\epsilon}(e_j) = \epsilon(e_j) + \iota(e_j); \quad c(e_j) = \epsilon(e_j) - \iota(e_j),$$

(2.3)

which satisfies

$$\hat{\epsilon}(e_i)\hat{\epsilon}(e_j) + \hat{\epsilon}(e_j)\hat{\epsilon}(e_i) = 2g^{TM}(e_i, e_j);$$

$$c(e_i)c(e_j) + c(e_j)c(e_i) = -2g^{TM}(e_i, e_j);$$

$$c(e_i)\hat{\epsilon}(e_j) + \hat{\epsilon}(e_j)c(e_i) = 0.$$

(2.4)

By [24], we have

$$D = d + \delta = \sum_{i=1}^{n} c(e_i) \left[ e_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)[\hat{\epsilon}(e_s)\hat{\epsilon}(e_t) - c(e_s)c(e_t)] \right].$$

(2.5)

Let $\pi^F$ (resp. $\pi^{F^\perp}$) be the orthogonal projection from $TM$ to $F$ (resp. $F^\perp$). Set

$$\nabla^F = \pi^F \nabla^L \pi^F,$$

$$\nabla^{F^\perp} = \pi^{F^\perp} \nabla^L \pi^{F^\perp},$$

(2.6)

then $\nabla^F$ (resp. $\nabla^{F^\perp}$) is a Euclidean connection on $F$ (resp. $F^\perp$), let $S$ be the tensor defined by

$$\nabla^L = \nabla^F + \nabla^{F^\perp} + S.$$

(2.7)

Let $e_1, e_2, \cdots, e_n$ be the orthonormal basis of $TM$ and $f_1, \cdots, f_k$ be the orthonormal basis of $F^\perp$. The sub-signature operators $D_t$ and $D_t^*$ acting on $\bigwedge^* TM \otimes \mathbb{C}$ are defined by

$$D_t = d + \delta + t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i)\hat{\epsilon}(S(e_i)f_{\alpha})\hat{\epsilon}(f_{\alpha})$$

$$= \sum_{i=1}^{n} c(e_i) \left[ e_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)[\hat{\epsilon}(e_s)\hat{\epsilon}(e_t) - c(e_s)c(e_t)] \right] + t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i)\hat{\epsilon}(S(e_i)f_{\alpha})\hat{\epsilon}(f_{\alpha});$$

3
Theorem 2.1. Then we have the following theorem, by Lichnerowicz formulas,

\[ D_t^* = d + \delta + \sqrt{-1} \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_\alpha) \tilde{c}(f_\alpha) \]

\[ = \sum_{i=1}^{n} c(e_i) \left[ e_i + \frac{1}{\sqrt{-1}} \sum_{s,t} \omega_{s,t}(e_i)[\tilde{c}(e_s) \tilde{c}(e_t) - c(e_s)c(e_t)] \right] + \sqrt{-1} \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_\alpha) \tilde{c}(f_\alpha), \]

where \( t \) is a complex number.

Then when \( t = -\frac{1}{2} \),

\[ D_t = (\sqrt{-1})^{-\frac{k(k+1)}{2}} (-1)^{\frac{k(k+1)}{2}} \tilde{c}(f_1) \cdots \tilde{c}(f_k) \tilde{D}_F, \]

where \( \tilde{D}_F \) is the sub-signature operator defined in Proposition 2.1 in [2], so we call that \( D_t \) is the sub-signature operator.

Set \( A = \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_\alpha) \tilde{c}(f_\alpha) \). Set

\[ \nabla^1_{e_i} := \nabla^{A^*T^*M} - \frac{1}{2} (t c(e_i) A + \tilde{t} A c(e_i)), \quad \nabla^2_{e_i} := \nabla^{A^*T^*M} - \frac{1}{2} (t (c(e_i) A + A c(e_i)). \]

Let \( \Delta^j \) be the Laplacian with respect to \( \nabla^j \) for \( j = 1, 2 \):

\[ \Delta^j := -\nabla^j_{e_i} \nabla^j_{e_i} + \nabla^2_{\nabla^2_{e_i} e_i}. \]

Then we have the following theorem.

**Theorem 2.1.** The following equalities hold:

\[ D_t^* D_t = \Delta^1 - \frac{1}{8} \sum_{ijkl} R_{ijkl} c(e_i) \tilde{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} K + \frac{1}{4} \sum_j (t c(e_j) A + \tilde{t} A c(e_j))^2 \]

\[ + \frac{1}{2} (t c(e_j)(\nabla^{A^*T^*M}_{e_j} A) - \tilde{t} (\nabla^{A^*T^*M}_{e_j} A) c(e_j)) + \tilde{t} A^2, \]

(2.10)

\[ D_t^2 = \Delta^2 - \frac{1}{8} \sum_{ijkl} R_{ijkl} c(e_i) \tilde{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} K - \frac{1}{4} \sum_j \bar{t}^2 (c(e_j) A + A c(e_j))^2 \]

\[ + \frac{1}{2} (c(e_j)(\nabla^{A^*T^*M}_{e_j} A) - (\nabla^{A^*T^*M}_{e_j} A) c(e_j)) + \bar{t}^2 A^2, \]

(2.11)

where \( K \) is the scalar curvature.

**Proof.** From (2.8) and (2.9),

\[ D_t^* D_t = (d + \delta)^2 + t (d + \delta) \circ A + \tilde{t} A \circ (d + \delta) + \tilde{t} A^2. \]

(2.12)

By Lichnerowicz formulas,

\[ (d + \delta)^2 = \Delta^{A^*T^*M} - \frac{1}{8} \sum_{ijkl} R_{ijkl} c(e_i) \tilde{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} K. \]

4
Using normal coordinates, $\Delta^{\Lambda^* T^*M} = -\nabla^{\Lambda^* T^*M}_{\ell_i} \nabla^{\Lambda^* T^*M}_{\ell_i}$. Since $d + \delta = c(e_i)\nabla^{\Lambda^* T^*M}_{\ell_i}$, we have

$$(d + \delta) \circ A = c(e_i) A \nabla^{\Lambda^* T^*M}_{\ell_i} + c(e_i)(\nabla^{\Lambda^* T^*M}_{\ell_i} A), \quad A \circ (d + \delta) = A \circ c(e_i) \nabla^{\Lambda^* T^*M}_{\ell_i}.$$

(2.14)

Notice that

$$\Delta^1 = -\nabla^1_{\ell_i} \nabla^1_{\ell_i} = -(\nabla^1_{\ell_i} \nabla^1_{\ell_i} - \frac{1}{2}(tc(e_i)A + TAc(e_i))^2) = -\nabla^{\Lambda^* T^*M}_{\ell_i} \nabla^{\Lambda^* T^*M}_{\ell_i} + (tc(e_i)A + TAc(e_i))\nabla^{\Lambda^* T^*M}_{\ell_i} + \frac{1}{2}(\nabla^{\Lambda^* T^*M}_{\ell_i}(tc(e_i)A + TAc(e_i)) - \frac{1}{4}(tc(e_i)A + TAc(e_i))^2).$$

(2.15)

Thus we obtain (2.10). The proof of (2.11) is similar.

By Theorem 2.1 we can define that

$$E_{D_1^* D_1}(x_0) = \frac{1}{8} \sum_{ijkl} R_{ijkl} \tilde{c}(e_i) \tilde{c}(e_j) c(e_k) c(e_l) - \frac{1}{4} K - \frac{1}{2} A^2 - \frac{1}{4} \sum_j [c(e_j) tA + TAc(e_j)]^2 - \frac{1}{2} tc(e_j)(\nabla^*_{\ell_i} T^*M A) - \frac{1}{4} (\nabla^*_{\ell_i} T^*M A)c(e_j)].$$

(2.16)

$$E_{D_1^* D_1^2}(x_0) = \frac{1}{8} \sum_{ijkl} R_{ijkl} \tilde{c}(e_i) \tilde{c}(e_j) c(e_k) c(e_l) - \frac{1}{4} K - t^2 A^2 - \frac{1}{4} \sum_j [c(e_j) tA + TAc(e_j)]^2 + \frac{1}{2} [t(\nabla^{\Lambda^* T^*M}_{\ell_i} A)c(e_j) - tc(e_j)(\nabla^{\Lambda^* T^*M}_{\ell_i} A)].$$

(2.17)

From [1], we know that the noncommutative residue of operator of Laplace type $\Delta$ is expressed as

$$(n - 2) \Phi_2(\Delta) = (4\pi)^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \text{Wres}(\Delta^{\frac{n}{2} + 1}),$$

(2.18)

where $\Phi_2(\Delta)$ denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of $\Delta$ and Wres denotes the noncommutative residue.

Now let $\Delta = D_1^* D_1$. Since $D_1^* D_1$ is operator of Laplace type, and $D_1^* D_1 = \Delta^1 - E_{D_1^* D_1}$, then, we have (see [1])

$$\text{Wres}(D_1^* D_1) = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M \text{tr}(\frac{1}{6} K + E_{D_1^* D_1})d\text{Vol}_M.$$

(2.19)

$$\text{Wres}(D_1^* D_1) = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M \text{tr}(\frac{1}{6} K + E_{D_1^* D_1})d\text{Vol}_M.$$

(2.20)

Next, we need to compute $\text{tr}(E_{D_1^* D_1})$ and $\text{tr}(E_{D_1^* D_1})$. Obviously, we have

$$\text{tr}\left(-\frac{1}{4} K\right) = -\frac{1}{4} K\text{tr}[\text{id}].$$

5
and

$$\sum_{ijkl} \text{tr}[R_{ijkl}\hat{c}(e_i)\hat{c}(e_j)c(e_k)c(e_l)] = 0. \tag{2.21}$$

Note that

$$\text{tr}\left[\sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha)^2 \right] = \sum_{i,j,\alpha,\beta} \text{tr}[c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha)c(e_j)\hat{c}(S(e_j)f_\beta)\hat{c}(f_\beta)]. \tag{2.22}$$

**case(a)** When $i \neq j$.

By $c(e_i)c(e_j) = -c(e_j)c(e_i)$, $c(e_i)\hat{c}(S(e_i)f_\alpha) = -\hat{c}(S(e_i)f_\alpha)c(e_i)$, $c(e_i)\hat{c}(f_\alpha) = -\hat{c}(f_\alpha)c(e_i)$ and by $\text{tr}ab = \text{tr}ba$, we have

$$\sum_{i,j,\alpha,\beta, i \neq j} \text{tr}[c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha)c(e_j)\hat{c}(S(e_j)f_\beta)\hat{c}(f_\beta)] = 0. \tag{2.23}$$

**case(b)** When $i = j, \alpha \neq \beta$.

By $c(e_i)^2 = -1$, $\hat{c}(f_\alpha)\hat{c}(f_\beta) = -\hat{c}(f_\beta)\hat{c}(f_\alpha)$, $\hat{c}(f_\alpha)\hat{c}(S(e_i)f_\alpha) = -\hat{c}(S(e_i)f_\alpha)c(f_\alpha)$ and by $\text{tr}ab = \text{tr}ba$, we have

$$\sum_{i=j, \alpha \neq \beta} \text{tr}[c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha)c(e_j)\hat{c}(S(e_j)f_\beta)\hat{c}(f_\beta)] = 0. \tag{2.24}$$

**case(c)** When $i = j, \alpha = \beta$.

By $\hat{c}(f_\alpha)^2 = 1$, we have

$$\sum_{i=j, \alpha = \beta} \text{tr}[c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha)c(e_j)\hat{c}(S(e_j)f_\beta)\hat{c}(f_\beta)] = \sum_{i=1}^{n} \sum_{\alpha=1}^{k} |S(e_i)f_\alpha|^2 \text{tr}[^1\text{id}], \tag{2.25}$$

therefore

$$\text{Tr}A^2 = \sum_{i=1}^{n} \sum_{\alpha=1}^{k} |S(e_i)f_\alpha|^2 \text{tr}[^1\text{id}]. \tag{2.26}$$

Note that

$$\text{tr} \left[ \sum_{j} c(e_j) \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha) \right]^2 = \sum_{i,j,\alpha,\beta} \text{tr}[c(e_j)c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha)c(e_j)c(e_i)\hat{c}(S(e_i)f_\beta)\hat{c}(f_\beta)]. \tag{2.27}$$

**case(a)** When $i = j = l$.

$$\sum_{i=j=l, \alpha, \beta} \text{tr}[c(e_j)c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha)c(e_j)c(e_i)\hat{c}(S(e_i)f_\beta)\hat{c}(f_\beta)] = -\sum_{i=1}^{n} \sum_{\alpha=1}^{k} \text{tr}[\hat{c}(S(e_i)f_\alpha)\hat{c}(S(e_i)f_\alpha)] \tag{2.28}$$
\[ = - \sum_{i=1}^{n} \sum_{\alpha=1}^{k} |S(e_i)f_\alpha|^2 \text{tr}[\mathbf{d}]. \]

(2.29)

case(b) When \( i \neq j \).

By \( \hat{c}(f_a)\hat{c}(f_\beta) = -\hat{c}(f_\beta)\hat{c}(f_a) \), \( \hat{c}(f_a)\hat{c}(S(e_i)f_\alpha) = -\hat{c}(S(e_i)f_\alpha)\hat{c}(f_a) \) and by \text{trab} = \text{trba}, we have

\[
\sum_{i=j \neq \alpha, \beta} \text{tr}[c(e_j)c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_a)c(e_j)c(e_i)\hat{c}(S(e_i)f_\beta)\hat{c}(f_\beta)]
\]

\[
= - \sum_{i \neq j, \alpha, \beta} \text{tr}[\hat{c}(S(e_i)f_\alpha)\hat{c}(f_a)c(e_i)c(e_j)\hat{c}(S(e_i)f_\beta)\hat{c}(f_\beta)]
\]

\[= 0. \]

(2.30)

case(c) When \( i \neq j, i = l \).

\[
\sum_{i=\neq j, \alpha, \beta} \text{tr}[c(e_j)c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_a)c(e_j)c(e_i)\hat{c}(S(e_i)f_\beta)\hat{c}(f_\beta)]
\]

\[= (n - 1) \sum_{i=\neq j, \alpha, \beta} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{k} |S(e_i)f_\alpha|^2 \text{tr}[\mathbf{d}]. \]

(2.31)

case(d) When \( i \neq j, i \neq l, j \neq l \) and \( i \neq j = l \).

By \( \hat{c}(f_a)\hat{c}(f_\beta) = -\hat{c}(f_\beta)\hat{c}(f_a), \hat{c}(f_a)\hat{c}(S(e_i)f_\alpha) = -\hat{c}(S(e_i)f_\alpha)\hat{c}(f_a) \) and by \text{trab} = \text{trba}, we have

\[
\sum_{i=\neq j, \alpha, \beta} \text{tr}[c(e_j)c(e_i)\hat{c}(S(e_i)f_\alpha)\hat{c}(f_a)c(e_j)c(e_i)\hat{c}(S(e_i)f_\beta)\hat{c}(f_\beta)] = 0,
\]

(2.32)

therefore

\[
\text{Tr} \sum_j |c(e_j)A|^2 = (n - 2) \sum_{i=1}^{n} \sum_{\alpha=1}^{k} |S(e_i)f_\alpha|^2 \text{tr}[\mathbf{d}].
\]

(2.33)

Note that

\[
\text{tr} \left( \sum_j c(e_j) \nabla_{e_j}^{\Lambda^* T^* M} \left( \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \hat{c}(S(e_i)f_\alpha)\hat{c}(f_a) \right) \right)
= \sum_{i=1}^{n} \sum_{\alpha=1}^{k} \text{tr}[c(e_j)\nabla_{e_j}^{\Lambda^* T^* M}(c(e_i))\hat{c}(S(e_i)f_\alpha)\hat{c}(f_a)]
\]

\[+ \sum_{j=1}^{n} \sum_{\alpha=1}^{k} \text{tr}[c(e_j)c(e_i)\nabla_{e_j}^{\Lambda^* T^* M}(\hat{c}(S(e_i)f_\alpha))\hat{c}(f_a)]
\]

\[+ \sum_{j=1}^{n} \sum_{\alpha=1}^{k} \text{tr}[c(e_j)c(e_i)\hat{c}(S(e_i)f_\alpha)\nabla_{e_j}^{\Lambda^* T^* M}(\hat{c}(f_a))],
\]

(2.34)
by
\[
\nabla^{\wedge}_{e_i} T^M (c(e_i)) = c(\nabla^L_{e_i} e_i), \quad \nabla^{\wedge}_{e_i} T^M (\tilde{c}(S(e_i) f_\alpha)) = \tilde{c}(\nabla^L_{e_i} (S(e_i) f_\alpha)),
\]
\[
\nabla^{\wedge}_{e_i} T^M (\tilde{c}(f_\alpha)) = \tilde{c}(\nabla^L_{e_i} f_\alpha), \quad c(e_j) c(\nabla^L_{e_i} e_i) + c(\nabla^L_{e_i} e_i) c(e_j) = -2 g^M (e_j, \nabla^L_{e_i} e_i),
\]
(2.35)

and \(\text{tr}_a b = \text{tr}_b a\) and \(\tilde{c}(S(e_i) f_\alpha) \tilde{c}(f_\alpha) + \tilde{c}(f_\alpha) \tilde{c}(S(e_i) f_\alpha) = 2 g^M (f_\alpha, S(e_i) f_\alpha) = 0\), we have
\[
\sum_{j = 1}^{n} \sum_{i = 1}^{n} \sum_{\alpha = 1}^{k} \text{tr}[c(e_j) \nabla^{\wedge}_{e_i} T^M (c(e_i)) \tilde{c}(S(e_i) f_\alpha) \tilde{c}(f_\alpha)] = 0.
\]
(2.36)

Similarly,
\[
\sum_{j = 1}^{n} \sum_{i = 1}^{n} \sum_{\alpha = 1}^{k} \text{tr}[c(e_j) c(e_i) \nabla^{\wedge}_{e_i} T^M (\tilde{c}(S(e_i) f_\alpha)) \tilde{c}(f_\alpha)] = - \sum_{i = 1}^{n} \sum_{\alpha = 1}^{k} g^M (f_\alpha, \nabla^L_{e_i} (S(e_i) f_\alpha)) \text{tr}[\text{id}],
\]
(2.37)
\[
\sum_{j = 1}^{n} \sum_{i = 1}^{n} \sum_{\alpha = 1}^{k} \text{tr}[c(e_j) c(e_i) \tilde{c}(S(e_i) f_\alpha) \tilde{c}(\nabla^L_{e_i} f_\alpha)] = - \sum_{i = 1}^{n} \sum_{\alpha = 1}^{k} g^M (\nabla^L_{e_i} f_\alpha, S(e_i) f_\alpha) \text{tr}[\text{id}],
\]
(2.38)

therefore,
\[
\text{Tr}[\sum_j c(e_j) \nabla^{\wedge}_{e_i} T^M A] = - \sum_{j = 1}^{n} \sum_{i = 1}^{n} \sum_{\alpha = 1}^{k} [g^M (f_\alpha, \nabla^L_{e_i} (S(e_i) f_\alpha)) + g^M (\nabla^L_{e_i} f_\alpha, S(e_i) f_\alpha)] \text{tr}[\text{id}] = 0.
\]
(2.39)

Then, by (2.21)-(2.39), we get
\[
\text{tr}(E_{D_t^* D_t}) = \left(- \frac{K}{4} - \frac{1}{4} (t^2 + \tilde{t}^2) (n - 2) - 2 n \tilde{t} + 4 \tilde{t} \right) \sum_{i = 1}^{n} \sum_{\alpha = 1}^{k} |S(e_i) f_\alpha|^2 \text{tr}[\text{id}],
\]
(2.40)
\[
\text{tr}(E_{D_t^2}) = \left(- \frac{K}{4} \right) \text{tr}[\text{id}],
\]
(2.41)

Then, by (2.19) and (2.20), we have the following theorem,

**Theorem 2.2.** If \(M\) is a \(n\)-dimensional compact oriented manifolds without boundary, and \(n\) is even, then we get the following equalities:
\[
\text{Wres}(D_t^* D_t)^{-\frac{n-2}{2}} = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M 2^n \left(- \frac{1}{12} K - \frac{1}{4} [(t^2 + \tilde{t}^2) (n - 2) - 2 n \tilde{t} + 4 \tilde{t}] \sum_{i = 1}^{n} \sum_{\alpha = 1}^{k} |S(e_i) f_\alpha|^2 \right) d\text{Vol}_M,
\]
(2.42)
\[
\text{Wres}(D_t^2)^{-\frac{n-2}{2}} = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M 2^n \left(- \frac{1}{12} K \right) d\text{Vol}_M,
\]
(2.43)

where \(K\) is the scalar curvature.
3. A Kastler-Kalau-Walze type theorem for 4-dimensional manifolds with boundary

In this section, we prove the Kastler-Kalau-Walze type theorem for 4-dimensional oriented compact manifolds with boundary. We firstly recall some basic facts and formulas about Boutet de Monvel’s calculus and the definition of the noncommutative residue for manifolds with boundary which will be used in the following. For more details, see Section 2 in [21].

Let \( M \) be a 4-dimensional compact oriented manifold with boundary \( \partial M \). We assume that the metric \( g^{TM} \) on \( M \) has the following form near the boundary,

\[
g^{TM} = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2,
\]

(3.1)

where \( g^{\partial M} \) is the metric on \( \partial M \) and \( h(x_n) \in C^\infty([0,1]) := \{ \tilde{h}|_{[0,1]} \} \) for some \( \varepsilon > 0 \) and \( h(x_n) \) satisfies \( h(x_n) > 0, h(0) = 1 \) where \( x_n \) denotes the normal directional coordinate. Let \( U \subset M \) be a collar neighborhood of \( \partial M \) which is diffeomorphic with \( \partial M \times [0,1) \). By the definition of \( h(x_n) \in C^\infty([0,1]) \) and \( h(x_n) > 0 \), there exists \( \tilde{h} \in C^\infty((\varepsilon, 1)) \) such that \( h|_{[0,1]} = \tilde{h} \) and \( \tilde{h} > 0 \) for sufficiently small \( \varepsilon > 0 \). Then there exists a metric \( g' \) on \( \tilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon,0] \) which has the form on \( U \cup_{\partial M} \partial M \times (-\varepsilon,0] \)

\[
g' = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2,
\]

(3.2)

such that \( g'|_{\partial M} = g \). We fix a metric \( g' \) on the \( \tilde{M} \) such that \( g'|_{\partial M} = g \).

Let the Fourier transformation \( F' \) be

\[
F' : L^2(\mathbb{R}_+^4) \to L^2(\mathbb{R}_+^4); \quad F'(u)(v) = \int_{\mathbb{R}} e^{-ivt} u(t) dt
\]

and let

\[
r^+ : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}_+^4); \quad f \to f|_{\mathbb{R}_+^4}; \quad \mathbb{R}_+^4 = \{ x \geq 0; x \in \mathbb{R} \}.
\]

We define \( H^+ = F'(\Phi(\mathbb{R}_+^4)); \quad H^- = F'(\Phi(\mathbb{R}_-^4)) \) which satisfies \( H^+ \uplus H^- = H^0 \), where \( \Phi(\mathbb{R}_+^4) = r^+ \Phi(\mathbb{R}) \), \( \Phi(\mathbb{R}_-^4) = r^- \Phi(\mathbb{R}) \) and \( \Phi(\mathbb{R}) \) denotes the Schwartz space. We have the following property: \( h \in H^+ \) (resp. \( H^- \)) if and only if \( h \in C^\infty(\mathbb{R}) \) which has an analytic extension to the lower (resp. upper) complex half-plane \( \{ \text{Im} \xi < 0 \} \) (resp. \( \{ \text{Im} \xi > 0 \} \) such that for all nonnegative integer \( l \),

\[
\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l c_k}{\xi^l},
\]

as \( |\xi| \to +\infty \), \( \text{Im} \xi \leq 0 \) (resp. \( \text{Im} \xi \geq 0 \)) and where \( c_k \in \mathbb{C} \) are some constants.

Let \( H' \) be the space of all polynomials and \( H^- = H_0^- \uplus H^+ \); \( H = H^+ \uplus H^- \). Denote by \( \pi^+ \) (resp. \( \pi^- \)) the projection on \( H^+ \) (resp. \( H^- \)). Let \( H = \{ \text{rational functions having no poles on the real axis} \} \). Then on \( H \),

\[
\pi^+ h(\xi_0) = \lim_{u \to 0^+} \frac{1}{2\pi i} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi,
\]

(3.3)

where \( \Gamma^+ \) is a Jordan closed curve included \( \text{Im}(\xi) > 0 \) surrounding all the singularities of \( h \) in the upper half-plane and \( \xi_0 \in \mathbb{R} \). In our computations, we only compute \( \pi^+ h \) for \( h \in \tilde{H} \). Similarly, define \( \pi^- \) on \( \tilde{H} \),

\[
\pi^- h = \frac{1}{2\pi i} \int_{\Gamma^-} h(\xi) d\xi.
\]

(3.4)

So \( \pi'(H^-) = 0 \). For \( h \in H \cap L^1(\mathbb{R}) \), \( \pi^+ h = \frac{1}{2\pi} \int_{\mathbb{R}} h(v) dv \) and for \( h \in H^+ \cap L^1(\mathbb{R}) \), \( \pi^- h = 0 \).

An operator of order \( m \in \mathbb{Z} \) and type \( d \) is a matrix

\[
\tilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} : \begin{array}{c} C^\infty(M, E_1) \\ \oplus \\ C^\infty(\partial M, F_1) \end{array} \to \begin{array}{c} C^\infty(M, E_2) \\ \oplus \\ C^\infty(\partial M, F_2) \end{array},
\]
where $M$ is a manifold with boundary $\partial M$ and $E_1, E_2$ (resp. $F_1, F_2$) are vector bundles over $M$ (resp. $\partial M$). Here, $P : C^\infty_0(\Omega, E_1) \to C^\infty(\Omega, E_2)$ is a classical pseudodifferential operator of order $m$ on $\Omega$, where $\Omega$ is a collar neighborhood of $M$ and $E_i|M = E_i$ $(i = 1, 2)$. $P$ has an extension: $E'(\Omega, E_1) \to D'(\Omega, E_2)$, where $E'(\Omega, E_1)$ $(D'(\Omega, E_2))$ is the dual space of $C^\infty(\Omega, E_1)$ $(C^\infty_0(\Omega, E_2))$. Let $e^+ : C^\infty(M, E_1) \to E'(\Omega, E_1)$ denote extension by zero from $M$ to $\Omega$ and $r^+ : D'(\Omega, E_2) \to D'(\Omega, E_2)$ denote the restriction from $\Omega$ to $X$, then define

$$
\pi^+ P = r^+ P e^+ : C^\infty(M, E_1) \to D'(\Omega, E_2).
$$

In addition, $P$ is supposed to have the transmission property; this means that, for all $j,k,\alpha$, the homogeneous component $p_j$ of order $j$ in the asymptotic expansion of the symbol of $P$ in local coordinates near the boundary satisfies:

$$
\partial_x^k \partial_x^\alpha p_j(x', 0, 0, +1) = (-1)^{j-|\alpha|} \partial_x^k \partial_x^\alpha p_j(x', 0, 0, -1),
$$

then $\pi^+ P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ by Theorem 4 in [14] page 139. Let $G,T$ be respectively the singular Green operator and the trace operator of order $m$ and $d$. Let $K$ be a potential operator and $S$ be a classical pseudodifferential operator of order $m$ along the boundary. Denote by $B^{m,d}$ the collection of all operators of order $m$ and type $d$, and $B$ is the union over all $m$ and $d$.

Recall that $B^{m,d}$ is a Fréchet space. The composition of the above operator matrices yields a continuous map: $B^{m,d} \times B^{m,d'} \to B^{m+m',\max(m+d,d')}$. Write

$$
\tilde{\pi}^+ = \left( \begin{array}{c} \pi^+ P + G \\ T \\ K \\ S \end{array} \right) \in B^{m,d}, \tilde{\pi}'^+ = \left( \begin{array}{c} \pi'^+ P' + G' \\ T' \\ K' \\ S' \end{array} \right) \in B^{m',d'}.
$$

The composition $\tilde{\pi}^+ \tilde{\pi}'^+$ is obtained by multiplication of the matrices (For more details see [12]). For example $\pi^+ P \circ G'$ and $G \circ G'$ are singular Green operators of type $d'$ and

$$
\pi^+ P \circ \pi'^+ P' = \pi^+(PP') + L(P, P'),
$$

Here $PP'$ is the usual composition of pseudodifferential operators and $L(P, P')$ called leftover term is a singular Green operator of type $m' + d$. For our case, $P, P'$ are classical pseudo differential operators, in other words $\pi^+ P \in B^\infty$ and $\pi'^+ P' \in B^\infty$.

Let $M$ be a $n$-dimensional compact oriented manifold with boundary $\partial M$. Denote by $B$ the Boutet de Monvel’s algebra. We recall that the main theorem in [6, 21].

**Theorem 3.1.** [Fedosov-Golse-Leichtnam-Schrohe] Let $M$ and $\partial M$ be connected, $\dim M = n \geq 3$, and let $\tilde{S}$ (resp. $\tilde{S}'$) be the unit sphere about $\xi$ (resp. $\xi'$) and $\sigma(\xi)$ (resp. $\sigma(\xi')$) be the corresponding canonical $n-1$ (resp. $(n-2)$) volume form. Set $\tilde{\pi} = \left( \begin{array}{c} \pi^+ P + G \\ T \\ K \\ S \end{array} \right) \in B$, and denote by $p, b$ and $s$ the local symbols of $P, G$ and $S$ respectively. Define:

$$
\widetilde{\text{Wres}}(\tilde{\pi}) = \int_X \int_{\tilde{S}} \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx
$$

$$
+ 2\pi \int_{\partial X} \int_{\tilde{S}'} \left[ \text{tr}_E [p_{-n}(x', \xi')] + \text{tr}_F [s_{-n}(x', \xi')] \right] \sigma(\xi') dx',
$$

(3.5)

where $\widetilde{\text{Wres}}$ denotes the noncommutative residue of an operator in the Boutet de Monvel’s algebra. Then a) $\text{Wres}(\tilde{\pi}, B) = 0$, for any $\tilde{\pi}, B \in B$; b) It is the unique continuous trace on $B/B^\infty$.

**Definition 3.2.** [21] Lower dimensional volumes of spin manifolds with boundary are defined by

$$
\text{Vol}_n^{p_1,p_2} M := \widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}].
$$

(3.6)
By \[21\], we get
\[
\text{Wres}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}] = \int_M \int_{|\xi'| = 1} \text{trace}\lambda T^M \otimes c[\sigma^{-n}(D^{-p_1-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi, \quad (3.7)
\]
and
\[
\Phi = \int_{|\xi'| = 1} \int_{-\infty}^{\infty} \sum_{j,k=0}^{\infty} \left( \frac{(-1)^{i+j+k+1}}{\alpha(i+j+k+1)} \right) \times \text{trace}_{\lambda T^M \otimes c} \left[ \partial^j_{\xi_n} \partial^k_{\xi_n} \sigma_r(D^{-p_1})(x', 0, \xi', \xi_n) \right] \times \partial^j_{\xi_n} \partial^k_{\xi_n} \sigma_0(D^{-p_2})(x', 0, \xi', \xi_n) dx' \quad (3.8)
\]
where the sum is taken over \( r + l - k - \alpha - j - 1 = -n, \quad r \leq -p_1, l \leq -p_2 \).

Since \([\sigma^{-n}(D^{-p_1-p_2})]\)_{|M} has the same expression as \(\sigma^{-n}(D^{-p_1-p_2})\) in the case of manifolds without boundary, so locally we can compute the first term by \([10, 11, 13, 21\).

For any fixed point \(x_0 \in \partial M\), we choose the normal coordinates \(U\) of \(x_0\) in \(\partial M\) (not in \(M\)) and compute \(\Phi(x_0)\) in the coordinates \(\tilde{U} = U \times (0, 1) \subset M\) and with the metric \(\frac{1}{\lambda x_0}g^{\partial M} + dx_n^2\). The dual metric of \(g^T_M\) on \(\tilde{U}\) is \(h(x_0)g^{\partial M} + dx_n^2\). Write \(g_{ij}^{TM} = g^{TM}(\partial_{x_i}, \partial_{x_j}); \quad g_{ij}^TM = g^TM(dx_i, dx_j), \) then
\[
[g_{ij}^{TM}] = \begin{bmatrix}
\frac{1}{\lambda x_0}g_{ij}^{\partial M} & 0 \\
0 & 1
\end{bmatrix}; \quad [g_{ij}^{TM}] = \begin{bmatrix}
h(x_0)g_{ij}^{\partial M} & 0 \\
0 & 1
\end{bmatrix}, \quad (3.9)
\]
and
\[
\partial_{x_i} g^{\partial M}_{ij}(x_0) = 0, 1 \leq i, j \leq n - 1; \quad g^{TM}_{ij}(x_0) = \delta_{ij}. \quad (3.10)
\]

From \([21\), we can get three lemmas.

**Lemma 3.3.** \([21\) With the metric \(g^{TM}\) on \(M\) near the boundary
\[
\partial_{x_j}(\lambda'_{x^2})^i_j(x_0) = \begin{cases}
0, & \text{if } j < n, \\
h'(0)|\xi'| g^{\partial M}, & \text{if } j = n,
\end{cases} \quad (3.11)
\]
\[
\partial_{x_j}[c(\xi)](x_0) = \begin{cases}
0, & \text{if } j < n, \\
\partial_{x_n}[c(\xi')](x_0), & \text{if } j = n,
\end{cases} \quad (3.12)
\]
where \(\xi = \xi' + \xi_n dx_n\).

**Lemma 3.4.** \([21\) With the metric \(g^{TM}\) on \(M\) near the boundary
\[
\omega_{s,t}(e_i)(x_0) = \begin{cases}
\omega_{s,t}(e_i)(x_0) = \frac{1}{2} h'(0), & \text{if } s = n, t = i, \quad i < n, \\
\omega_{s,t}(e_i)(x_0) = -\frac{1}{2} h'(0), & \text{if } s = i, t = n, \quad i < n, \\
\omega_{s,t}(e_i)(x_0) = 0, & \text{otherwise},
\end{cases} \quad (3.13)
\]
where \((\omega_{s,t})\) denotes the connection matrix of Levi-Civita connection \(\nabla^L\).

**Lemma 3.5.** \([21\) When \(i < n\), then
\[
\Gamma_{st}^k(x_0) = \begin{cases}
\Gamma_{st}^k(x_0) = \frac{1}{2} h'(0), & \text{if } s = t, k = n, \\
\Gamma_{nt}^k(x_0) = -\frac{1}{2} h'(0), & \text{if } s = n, t = i, k = i, \\
\Gamma_{ni}^k(x_0) = -\frac{1}{2} h'(0), & \text{if } s = i, t = n, k = i,
\end{cases} \quad (3.14)
\]
in other cases, \(\Gamma_{st}^k(x_0) = 0\).

By \([37\) and \([58\), we firstly compute
\[
\text{Wres}[\pi^+ D_{t^{-1}} \circ \pi^+(D_t^{-1})^{-1}] = \int_M \int_{|\xi'| = 1} \text{trace}\lambda T^M \otimes c[\sigma^{-4}((D_t^{-1})^*)] \sigma(\xi) dx + \int_{\partial M} \Phi. \quad (3.15)
\]
Lemma 3.6. The following identities hold:

\[
\sigma_1(D_t) = \sigma_1(D_t^*) = ic(\xi);
\]

\[
\sigma_0(D_t) = \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_i)\tilde{c}(e_s)\tilde{c}(e_t) - \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_i)\tilde{c}(e_s)c(e_t) + tA;
\]

\[
\sigma_0(D_t^*) = \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_i)\tilde{c}(e_s)\tilde{c}(e_t) - \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_i)c(e_s)\tilde{c}(e_t) + \tilde{t}A.
\]

Write

\[
D^\alpha_t = (-i)^{|\alpha|} \partial^\alpha_t; \quad \sigma(D_t) = p_1 + p_0; \quad (\sigma(D_t)^{-1}) = \sum_{j=1}^{\infty} q_j.
\]

By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma(D_t \circ D_t^{-1}) = \sum_{\alpha} \frac{1}{|\alpha|!} \partial^\alpha_t [\sigma(D_t)]D^\alpha_t [\sigma(D_t^{-1})]
\]

\[
= (p_1 + p_0)(q_1 + q_2 + q_3 + \cdots)
\]

\[
+ \sum_{j} (\partial_{\xi_j}p_1 + \partial_{\xi_j}p_0)(D_{x_j}q_1 + D_{x_j}q_2 + D_{x_j}q_3 + \cdots)
\]

\[
= p_1q_1 + (p_1q_2 + p_0q_1 + \sum_{j} \partial_{\xi_j}p_1 D_{x_j}q_1 + \cdots)
\]

so

\[
q_1 = p_1^{-1}; \quad q_2 = -p_1^{-1} [p_0 p_1^{-1} + \sum_{j} \partial_{\xi_j} p_1 D_{x_j}(p_1^{-1})].
\]
Lemma 3.7. The following identities hold:

\[ \sigma_{-1}(D_t^{-1}) = \sigma_{-1}((D_t^*)^{-1}) = \frac{ic(\xi)}{||\xi||^2}; \]
\[ \sigma_{-2}(D_t^{-1}) = \frac{c(\xi)\sigma_0(D_t)c(\xi)}{||\xi||^4} + \frac{c(\xi)}{||\xi||^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi))||\xi||^2 - c(\xi)\partial_{x_j}(||\xi||^2) \right]; \]
\[ \sigma_{-2}((D_t^*)^{-1}) = \frac{c(\xi)\sigma_0(D_t^*)c(\xi)}{||\xi||^4} + \frac{c(\xi)}{||\xi||^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi))||\xi||^2 - c(\xi)\partial_{x_j}(||\xi||^2) \right]. \] (3.22)

Theorem 3.8. Let \( M \) be a 4-dimensional oriented compact manifold with boundary \( \partial M \) and the metric \( g^{TM} \) be defined as \( 3\mathbb{R}^4 \). \( D_t \) and \( D_t^* \) be sub-signature operators on \( TM \) (\( M \) is a collar neighborhood of \( M \)) as in (3.4), (3.7), then

\[ \text{Wres}[\pi^+D_t^{-1} \circ \pi^+(D_t^*)^{-1}] = 32\pi^2 \int_M \left( -\frac{4}{3}K - \frac{(\bar{T} - t)^2}{2} \sum_{i=1}^k \sum_{\alpha=1}^{|\alpha|} |S(e_i)f_{\alpha}|^2 \right) d\text{Vol}_M, \] (3.23)

where \( K \) is the scalar curvature. In particular, the boundary term vanishes.

Proof. When \( n = 4 \), then \( \text{tr}_{\Lambda T^*M}[\text{id}] = \text{dim}(\Lambda^*(\mathbb{R}^4)) = 16 \), the sum is taken over \( r + l - k - j - |\alpha| = -3 \), \( r \leq l, l \leq -1 \), then we have the following five cases:

case a) I) \( r = -1, l = -1, k = j = 0, |\alpha| = 1 \).

By (3.16), we get

\[ \Phi_1 = -\int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \sum_{|\alpha| = 1} \text{tr}[\partial_{x_i}^2 \pi^+_\xi \sigma_{-1}(D_t^{-1}) \times \partial_{x_i}^2 \sigma_{-1}((D_t^*)^{-1})](x_0) d\xi_0 \sigma(\xi') dx'. \] (3.24)

By Lemma 3.3, for \( i < n \), then

\[ \partial_{x_i} \left( \frac{ic(\xi)}{||\xi||^2} \right)(x_0) = \frac{i\partial_{x_i} c(\xi)(x_0)}{||\xi||^2} - \frac{ic(\xi)\partial_{x_i}(||\xi||^2)(x_0)}{||\xi||^4} = 0, \] (3.25)

so \( \Phi_1 = 0 \).

case a) II) \( r = -1, l = -1, k = |\alpha| = 0, j = 1 \).

By (3.16), we get

\[ \Phi_2 = -\frac{1}{2} \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_i}^2 \pi^+_\xi \sigma_{-1}(D_t^{-1}) \times \partial_{x_i}^2 \sigma_{-1}((D_t^*)^{-1})](x_0) d\xi_0 \sigma(\xi') dx'. \] (3.26)

By Lemma 3.7, we have

\[ \partial_{x_n}^2 \sigma_{-1}((D_t^*)^{-1})(x_0) = i \left( \frac{6\xi_n c(dx_n) + 2c(\xi')}{||\xi||^4} + \frac{8\xi_n^2 c(\xi)}{||\xi||^6} \right); \] (3.27)
\[ \partial_{x_n} \sigma_{-1}(D_t^{-1})(x_0) = \frac{i\partial_{x_n} c(\xi')(x_0)}{||\xi||^2} - \frac{ic(\xi)\xi_n^2 h(0)}{||\xi||^4}. \] (3.28)

By (3.3), (3.4) and the Cauchy integral formula we have

\[ \pi^+_\xi \left( \frac{c(\xi)}{||\xi||^4} \right)(x_0)|_{|\xi'| = 1} = \pi^+_\xi \left[ \frac{c(\xi') + \xi_n c(dx_n)}{(1 + \xi_n^2)^2} \right]. \]
Similarly, we have,

$$\pi_\xi^+ \left[ \frac{i\partial_\sigma e(\xi')}{|\xi|^2} \right] (x_0)|_{|\xi'|=1} = \frac{\partial_\xi e(\xi')(x_0)}{2|\xi'|} + i\hbar'(0) \left[ \frac{(i\xi_n + 2)e(\xi') + ic(dx_n)}{4(|\xi_n - i|)^2} \right].$$

By (3.29), then

$$\pi_\xi^+ \partial_\sigma e^{-1}(D_t^{-1})|_{|\xi'|=1} = \frac{\partial_\xi [e(\xi')](x_0)}{2|\xi'|} + i\hbar'(0) \left[ \frac{(i\xi_n + 2)e(\xi') + ic(dx_n)}{4(|\xi_n - i|)^2} \right].$$

By the relation of the Clifford action and trab, we have the equalities:

$$\text{tr}[e(\xi')c(dx_n)] = 0; \quad \text{tr}[e(dx_n)^2] = -16; \quad \text{tr}[e(\xi')^2] = -16;$$

$$\text{tr}[\partial_\xi e(\xi')c(dx_n)] = 0; \quad \text{tr}[\partial_\xi e(\xi')c(\xi')](x_0)|_{|\xi'|=1} = -8h'(0);$$

$$\text{tr}[e(\xi)\xi(\xi)c(\xi)c(\xi)] = 0(i \neq j).$$

By (3.29), we have

$$\hbar'(0) \text{tr} \left[ \frac{(i\xi_n + 2)e(\xi') + ic(dx_n)}{4(|\xi_n - i|)^2} \times \left( \frac{8\xi_n e(dx_n) + 2c(\xi')}{1 + \xi_n^2} - \frac{8\xi_n c(e(\xi') + \xi_n c(dx_n))}{1 + \xi_n^2} \right) \right] (x_0)|_{|\xi'|=1}$$

$$= -16\hbar'(0) \frac{-2\xi_n^2 - \xi_n + i}{(\xi_n - i)^2(\xi_n + i)^2}.$$

Similarly, we have

$$-i \text{tr} \left[ \frac{\partial_\xi [e(\xi')](x_0)}{2(\xi_n - i)} \times \left( \frac{6\xi_n e(dx_n) + 2c(\xi')}{1 + \xi_n^2} - \frac{8\xi_n c(e(\xi') + \xi_n c(dx_n))}{1 + \xi_n^2} \right) \right] (x_0)|_{|\xi'|=1}$$

$$= -8i\hbar'(0) \frac{3\xi_n^2 - 1}{(\xi_n - i)^2(\xi_n + i)^2}.$$

Then

$$\Phi_2 = -\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{4ih'(0)(\xi_n - i)^2}{(\xi_n - i)^2(\xi_n + i)^3} \xi_n \sigma(\xi') \sigma(\xi') dx'$$

$$= -4i\hbar'(0)\Omega_3 \int_{\Gamma^+} \frac{1}{(\xi_n - i)^2(\xi_n + i)^3} \xi_n dx'$$

$$= -4i\hbar'(0)\Omega_3 2\pi i \left( \frac{1}{(\xi_n + i)^3} \right)_{\xi_n = i} dx'$$

$$= ^{-\frac{3}{2}} \pi h'(0)\Omega_3 dx',$$

(3.35)

where $\Omega_3$ is the canonical volume of $S^3$.

**case a) III** $r = -1, \ l = -1, \ j = |\alpha| = 0, \ k = 1.$

By (3.16), we get

$$\Phi_3 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_\sigma e_{\xi}^+ \xi \sigma(\sigma^{-1}(D_t^{-1}) \times \partial_\xi \sigma, \sigma^{-1}(D_t^{-1})) (x_0) \right] dx_\xi \sigma(\xi') dx'.$$  

(3.36)
By Lemma 3.7, we have
\[ \partial_{\xi_n} \partial_{x_n} \sigma^{-1}_-(D_t^{-1})(x_0)|_{|\xi'|=1} = -ih'(0) \left[ c(dx_n) \frac{c(\xi')}{|\xi|^4} - 4\xi_n c(\xi') \frac{\xi_n c(dx_n)}{|\xi|^6} \right] - \frac{2\xi_n i \partial_{x_n} c(\xi')(x_0)}{|\xi|^4}; \]  
(3.37)

\[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma^{-1}_-(D_t^{-1})(x_0)|_{|\xi'|=1} = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}. \]  
(3.38)

Similar to case a) ii), we have
\[ \text{tr} \left\{ \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} \times \frac{\partial(\xi')}{\partial x}, c(\xi')(x_0) \right\} = \frac{8h'(0)}{(\xi_n - i)^4(\xi_n + i)^2}; \]  
(3.39)

and
\[ \text{tr} \left\{ \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} \times \frac{2\xi_n i \partial_{x_n} c(\xi')(x_0)}{|\xi|^4} \right\} = -\frac{8ih'(0)\xi_n}{(\xi_n - i)^2(\xi_n + i)^2} \]  
(3.40)

So we have
\[ \Phi_3 = -\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{hi(4i - 3\xi_0)}{(\xi_n - i)^4(\xi_n + i)^2} d\xi_n \sigma(\xi')dx' - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{hi(4i - 3\xi_0)}{(\xi_n - i)^4(\xi_n + i)^2} d\xi_n \sigma(\xi')dx' \]  
(3.41)

Case b) \( r = -2, l = -1, k = j = |\alpha| = 0. \)

By 3.10, we get
\[ \Phi_1 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} [\pi^+_{\xi_n} \sigma^{-2}_-(D_t^{-1}) \times \partial_{\xi_n} \sigma^{-1}_-(D_t^{-1})][(x_0) d\xi_n \sigma(\xi')dx']. \]  
(3.42)

By Lemma 3.7, we have
\[ \sigma^{-2}_-(D_t^{-1})(x_0) = \frac{c(\xi)\sigma_0(D_t)(x_0) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) [\partial_{x_n} c(\xi')](x_0) |\xi|^2 - c(\xi)h'(0) |\xi|^2 a_M], \]  
(3.43)

where
\[ \sigma_0(D_t)(x_0) = \frac{1}{4} \sum_{s.t,i} \omega_{s,t}(e_i)(x_0) c(e_i) c(e_i) c(e_i) - \frac{1}{4} \sum_{s.t,i} \omega_{s,t}(e_i)(x_0) c(e_i) c(e_i) c(e_i) + tA. \]  
(3.44)

We denote
\[ Q^1_0(x_0) = \frac{1}{4} \sum_{s.t,i} \omega_{s,t}(e_i)(x_0) c(e_i) c(e_i) c(e_i); \]  
(3.45)
\[ Q^2_0(x_0) = -\frac{1}{4} \sum_{s.t,i} \omega_{s,t}(e_i)(x_0) c(e_i) c(e_i) c(e_i). \]  
(3.45)
Then
\[
\pi_{\xi_n}^+ \sigma_{-2}(D_{t^{-1}}(x_0))|_{\xi'|=1} = \pi_{\xi_n}^+ \left[ \frac{c(\xi)Q_0^1(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[ \frac{c(\xi)tA(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[ \frac{c(\xi)Q_0^2(x_0)c(\xi')}{(1 + \xi_n^2)^2} \right] - h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^3)^3} \right]. \tag{3.46}
\]

By computations, we have
\[
\pi_{\xi_n}^+ \left[ \frac{c(\xi)Q_0^1(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] = \pi_{\xi_n}^+ \left[ \frac{c(\xi')Q_0^1(x_0)c(\xi')}{(1 + \xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[ \frac{\xi_n c(dx_n)Q_0^1(x_0)c(\xi')}{(1 + \xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[ \frac{\xi_n^2 c(dx_n)Q_0^1(x_0)c(dx_n)}{(1 + \xi_n^2)^2} \right] = - \frac{c(\xi')Q_0^1(x_0)c(\xi')(2 + i\xi_n)}{4(\xi_n - i)^2} + \frac{ic(\xi')Q_0^1(x_0)c(dx_n)}{4(\xi_n - i)^2} + \frac{-i\xi_n c(dx_n)Q_0^1(x_0)c(dx_n)}{4(\xi_n - i)^2}. \tag{3.47}
\]

Since
\[
c(dx_n)Q_0^1(x_0) = - \frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(e_i)\tilde{c}(e_i)c(e_n)\tilde{c}(e_n), \tag{3.48}
\]
then by the relation of the Clifford action and trab, we have the equalities:
\[
\tr[c(e_i)\tilde{c}(e_i)c(e_n)\tilde{c}(e_n)] = 0 \quad (i < n); \quad \tr[Q_0^1c(dx_n)] = 0;
\]
\[
\tr\left[ \sum_{i=1}^{n-k} \sum_{a=1}^{k} c(e_i)\tilde{c}(s(e_i)f_a)\tilde{c}(f_a)c(dx_n) \right] = 0; \quad \tr[\tilde{c}(\xi')\tilde{c}(dx_n)] = 0. \tag{3.49}
\]

Since
\[
\partial_{\xi_n}^{-1}((D_{t^*})^{-1}) = \partial_{\xi_n}^{-1}((D_{t^*})^{-1})(x_0)|_{\xi'|=1} = i \left[ \frac{c(dx_n)}{1 + \xi_n^2} - \frac{2\xi_n c(\xi') + 2\xi_n^2 c(dx_n)}{(1 + \xi_n^2)^2} \right]. \tag{3.50}
\]

By (3.47) and (3.50), we have
\[
\tr[\pi_{\xi_n}^+ \left[ \frac{c(\xi)Q_0^1(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] \times \partial_{\xi_n}^{-1}((D_{t^*})^{-1})(x_0)|_{\xi'|=1} = \frac{1}{2(1 + \xi_n^2)^2} \tr[c(\xi')Q_0^1(x_0)] + \frac{i}{2(1 + \xi_n^2)^2} \tr[c(dx_n)Q_0^1(x_0)] = \frac{1}{2(1 + \xi_n^2)^2} \tr[c(\xi')Q_0^1(x_0)]. \tag{3.51}
\]

We note that $i < n$, $\int_{\xi'|=\{\xi_1, \xi_2, \ldots, \xi_{i2+1}\}} \sigma(\xi') = 0$, so $\tr[c(\xi')Q_0^1(x_0)]$ has no contribution for computing case b).

By computations, we have
\[
\pi_{\xi_n}^+ \left[ \frac{c(\xi)Q_0^1(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{\xi_n} c(\xi')(x_0) + h'(0)\pi_{\xi_n}^+ \left[ \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^2} \right] := C_1 - C_2, \tag{3.52}
\]
where
\[
C_1 = - \frac{1}{4(\xi_n - i)^2} (2 + i\xi_n) c(\xi') Q_0^1(x_0) c(\xi') + i\xi_n c(dx_n) Q_0^1(x_0) c(dx_n)
\]
+ (2 + iξ_n)c(ξ')c(dx_n)\partial_{x_n}c(\xi') + ic(dx_n)Q_0^2(x_0)c(\xi') + ic(\xi')Q_0^2(x_0)c(dx_n) - i\partial_{x_n}c(\xi')] \tag{3.53}

and

\[ C_2 = \frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^2} [ic(\xi') - c(dx_n)] \right]. \tag{3.54} \]

By (3.50) and (3.53), we have

\[ \text{tr}[C_2 \times \partial_{\xi_n} \sigma_{-1}((D_t^*)^{-1})]|_{|\xi'| = 1} = \frac{i}{2}h'(0) \frac{\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)(\xi_n + i)^2} \text{tr}[\text{id}] \tag{3.55} \]

By (3.50) and (3.53), we have

\[ \text{tr}[C_1 \times \partial_{\xi_n} \sigma_{-1}((D_t^*)^{-1})]|_{|\xi'| = 1} = \frac{-8ic_0}{(1 + \xi_n^2)^2} + 2h'(0) \frac{\xi_n^2 - i\xi_n - 2}{(\xi_n - i)(1 + \xi_n^2)^2}, \tag{3.56} \]

where \( Q_0^2 = c_0 c(dx_n) \) and \( c_0 = -\frac{1}{4}h'(0). \)

By (3.54) and (3.50), we have

\[ -i \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[(C_1 - C_2) \times \partial_{\xi_n} \sigma_{-1}((D_t^*)^{-1})]|_{x_0} d\xi_n \sigma(\xi') dx' \]

\[ = -\Omega_3 \int_{\Gamma_+} \frac{8c_0(\xi_n - i) + ih'(0)}{(\xi_n - i)^3(\xi_n + i)^2} d\xi_n dx' \]

\[ = \frac{9}{2} \pi h'(0) \Omega_3 dx'. \tag{3.57} \]

Similar to (3.51), we have

\[ \text{tr}[\pi_{\xi_n}^+ \left[ \frac{c(\xi)tA(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] \times \partial_{\xi_n} \sigma_{-1}((D_t^*)^{-1})]|_{|\xi'| = 1} = \frac{i}{2(1 + \xi_n^2)^2} \text{tr}[c(dx_n)tA(x_0)]. \tag{3.58} \]

By (3.58), we have

\[ -i \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \left[ \frac{c(\xi)tA(\xi)}{(1 + \xi_n^2)^2} \right] \times \partial_{\xi_n} \sigma_{-1}((D_t^*)^{-1})]|_{x_0} d\xi_n \sigma(\xi') dx' \]

\[ = \pi \text{tr}[c(dx_n)tA] \Omega_3 dx' \]

\[ = 0. \tag{3.59} \]

Then, we have

\[ \Phi_4 = \frac{9}{2} \pi h'(0) \Omega_3 dx'. \tag{3.60} \]

case c) \( r = -1, \ l = -2, \ k = j = |\alpha| = 0. \)

By (3.56), we get

\[ \Phi_5 = -i \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}(D_t^{-1}) \times \partial_{\xi_n} \sigma_{-2}((D_t^*)^{-1})]|_{x_0} d\xi_n \sigma(\xi') dx'. \tag{3.61} \]
By (3.3) and (3.4), Lemma 3.7 we have

\[
\pi_{\xi_n}^+ \sigma_{-1}(D_t^{-1})|_{\xi'|=1} = \frac{e(\xi') + ice(dx_n)}{2(\xi_n - i)}.
\]  

(3.62)

Since

\[
\sigma_{-2}((D_t^*)^{-1})(x_0) = \frac{c(\xi)\sigma_0(D_t^*)c(\xi)}{\xi^4} + \frac{c(\xi)}{\xi^6}c(dx_n)\left[\partial_{x_n}[c(\xi')](x_0)\right] - c(\xi)h'(0)\xi_n^2
\]

(3.63)

where

\[
\sigma_0(D_t^*)c(x_0) = \frac{1}{4}\sum_{s,t,i} \sigma_s^t e_i(x_0)e_i c(e_i)c(e_i) - \frac{1}{4}\sum_{s,t,i} \sigma_s^t e_i(x_0)e_i c(e_i)c(e_i) + \overline{TA}(x_0)
\]

(3.64)

then

\[
\frac{\partial_{\xi_n}\sigma_{-2}((D_t^*)^{-1})(x_0)|_{\xi'|=1}}{\xi^4} = \frac{\partial_{\xi_n} \frac{c(\xi)Q_0^1(x_0)c(\xi)}{\xi^4} + c(\xi)\partial_{x_n}[c(\xi')](x_0)\frac{\xi^2}{\xi^6} - c(\xi)h'(0)\xi_n^2}{\xi^4}
\]

(3.65)

By computations, we have

\[
\frac{\partial_{\xi_n} \frac{c(\xi)Q_0^1(x_0)c(\xi)}{\xi^4}}{\xi^4} = \frac{c(dx_n)Q_0^1(x_0)c(\xi)}{\xi^4} + c(\xi)\frac{Q_0^1(x_0)c(dx_n)}{\xi^6} - \frac{4\xi_n c(\xi)Q_0^1(x_0)c(\xi)}{\xi^6}.
\]  

(3.66)

\[
\frac{\partial_{\xi_n} \frac{c(\xi)\overline{TA}(x_0)c(\xi)}{\xi^4}}{\xi^4} = \frac{c(dx_n)\overline{TA}(x_0)c(\xi)}{\xi^4} + c(\xi)\overline{TA}(x_0)c(dx_n) - \frac{4\xi_n c(\xi)\overline{TA}(x_0)c(\xi)}{\xi^4}.
\]  

(3.67)

We denote

\[
q_{-2}^1 = \frac{c(\xi)Q_0^1(x_0)c(\xi)}{\xi^4} + \frac{c(\xi)}{\xi^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)]\frac{\xi^2}{\xi^6} - c(\xi)h'(0)
\]

then

\[
\frac{\partial_{\xi_n} q_{-2}^1}{\xi^4} = \frac{1}{(1 + \xi_n^2)^4}\left[(2\xi_n - 2\xi_n^2)c(dx_n)Q_0^1c(dx_n) + (1 - 3\xi_n^2)c(dx_n)Q_0^2c(\xi') + (3\xi_n^2 - 1)\partial_{x_n}c(\xi')
\]

\[
+ (1 - 3\xi_n^2)c(\xi')Q_0^2c(dx_n) - 4\xi_n c(\xi')Q_0^2c(\xi') + 2h'(0)\xi_n c(dx_n) - 4\xi_n c(\xi')\partial_{x_n}c(\xi') + 2h'(0)\xi_n c(dx_n)
\]

\[
+ 6\xi_n h'(0)\frac{c(\xi)\partial_{x_n}c(\xi)}{\xi^4}.
\]  

(3.68)

By (3.62) and (3.66), we have

\[
\text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_t^{-1})] = \frac{c(\xi)Q_0^1c(\xi)}{\xi^4}(x_0)|_{\xi'|=1}
\]

18
Then, by (3.16)-(3.18), we obtain Theorem 3.8.

We note that $i < n$, \( \int_{|\xi'|=1} \{ \xi_1, \xi_2 \cdots \xi_{2d+1} \} \sigma(\xi') = 0 \), so \( \text{tr}[c(\xi')Q_0^1(x_0)] \) has no contribution for computing case c). By (3.62) and (3.68), we have

\[
\text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_{t^{-1}}^{-1}) \times \partial_{\xi_n} c(\xi')Q_0^1(x_0)]|_{|\xi'|=1} = \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^3}.
\]  

(3.71)

Then

\[
-i\Omega_3 \int_{\Gamma^n} \left[ \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^3} \right] d\xi_n dx' = \frac{9}{2} \pi h'(0)\Omega_3 dx' .
\]  

(3.72)

By (3.62) and (3.68), we have

\[
\text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_{t^{-1}}^{-1}) \times \partial_{\xi_n} c(\xi')\hat{T}A(x_0)]|_{|\xi'|=1} = \frac{1}{(\xi - i)^3(\xi + i)^3} \text{tr}[c(\xi')\hat{T}A(x_0)] + \frac{i}{(\xi - i)^3(\xi + i)^3} \text{tr}[c(dx_n)\hat{T}A(x_0)].
\]  

(3.73)

By \( \int_{|\xi'|=1} \{ \xi_1, \xi_2 \cdots \xi_{2d+1} \} \sigma(\xi') = 0 \) and (3.49), we have

\[
-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_{t^{-1}}^{-1}) \times \partial_{\xi_n} c(\xi')\hat{T}A(x_0)]d\xi_n \sigma(\xi')dx' = \frac{1}{4} \text{tr}[c(dx_n)\hat{T}A] \Omega_3 dx' = 0 .
\]  

(3.74)

Then,

\[
\Phi_5 = \frac{9}{2} \pi h'(0)\Omega_3 dx'.
\]  

(3.75)

So \( \Phi = \sum_{i=1}^{5} \Phi_i = 0 \).

Then, by (3.10)- (3.13), we obtain Theorem 3.8.

Next, we also prove the Kastler-Kalan-Walze type theorem for 4-dimensional manifolds with boundary associated to \( D_{t^{-1}}^2 \). By (3.7) and (3.8), we will compute

\[
\mathcal{W}\text{Res}[\pi^+D_{t^{-1}} \circ \pi^+D_{t^{-1}}] = \int_M \int_{|\xi'|=1} \text{trace}_{\lambda^* T^* M \otimes C[\sigma_{-1}(D_{t^{-1}}^2)]} \sigma(\xi) dx + \int_{\partial M} \overline{\Phi} ,
\]  

(3.76)

where

\[
\overline{\Phi} = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^\infty \sum_{a=0}^\infty \frac{(-i)^{a+j+k+1}}{a!(j+k+1)!} \times \text{trace}_{\lambda^* T^* M \otimes C[\partial_{\xi_n}^a \partial_{\xi_n}^j \partial_{\xi_n}^k \sigma^+]}(D_{t^{-1}})(x', 0, \xi', \xi_n)
\]  

19
\[ \times \partial_{\xi}^{k} \partial_{\xi'}^{k+1} \sigma_{1}(D_{t}^{-1})(x',0,\xi',\xi)\] 
and the sum is taken over \( r + l - k - j - |\alpha| = -3, \ r \leq -1, l \leq -1. \)

By Theorem 3.9, we compute the interior of \( \operatorname{Wres}[\pi^{+} D_{t}^{-1} \circ \pi^{+} D_{t}^{-1}] \), then

\[ \int_{M} \int_{|\xi'|=1} \operatorname{trace}_{\Lambda^{*} T^{*} M} \otimes \mathbb{C}[\sigma_{-4}(D_{t}^{-2})] \sigma(\xi)\] 

(3.77)

Theorem 3.9. Let \( M \) be a 4-dimensional oriented compact manifold with boundary \( \partial M \) and the metric \( g^{TM} \) be defined as (3.7), \( D_{t} \) be sub-signature operator on \( M \) (\( M \) is a collar neighborhood of \( M \)) be defined as in (3.9), (3.10), then

\[ \overline{\operatorname{Wres}[\pi^{+} D_{t}^{-1} \circ \pi^{+} D_{t}^{-1}]} = 32 \pi^{2} \int_{M} \left(- \frac{4}{3} K\right) d\operatorname{Vol}_{M}. \] 

(3.78)

where \( K \) is the scalar curvature. In particular, the boundary term vanishes.

Proof. When \( n = 4 \), then \( \operatorname{tr}_{\Lambda^{*} T^{*} M}[\operatorname{id}] = \dim(\Lambda^{*}(\mathbb{R}^{4})) = 16 \), the sum is taken over \( r + l - k - j - |\alpha| = -3, \ r \leq -1, l \leq -1 \), then we have the following five cases:

case a) I) \( r = -1, \ l = -1, \ k = j = 0, \ |\alpha| = 1. \)

By (3.77), we get

\[ \overline{\Phi}_{1} = - \int_{|\xi'|=1} \int_{-\infty}^{\infty} \sum_{|\alpha|=1} \operatorname{tr}[\partial_{\xi}^{k} \pi_{\xi}^{+} \sigma_{-1}(D_{t}^{-1}) \times \partial_{\xi}^{k} \partial_{\xi} \sigma_{-1}(D_{t}^{-1})](x_{0}) d\xi d\sigma(\xi') \] 

(3.80)

case a) II) \( r = -1, \ l = -1, \ k = |\alpha| = 0, \ j = 1. \)

By (3.77), we get

\[ \overline{\Phi}_{2} = - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{\infty} \operatorname{trace}[\partial_{\xi}^{k} \pi_{\xi}^{+} \sigma_{-1}(D_{t}^{-1}) \times \partial_{\xi}^{k} \sigma_{-1}(D_{t}^{-1})](x_{0}) d\xi d\sigma(\xi') \] 

(3.81)

case a) III) \( r = -1, \ l = -1, \ j = |\alpha| = 0, \ k = 1. \)

By (3.77), we get

\[ \overline{\Phi}_{3} = - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{\infty} \operatorname{trace}[\partial_{\xi}^{k} \pi_{\xi}^{+} \sigma_{-1}(D_{t}^{-1}) \times \partial_{\xi} \sigma_{-1}(D_{t}^{-1})](x_{0}) d\xi d\sigma(\xi') \] 

(3.82)

By Lemma 3.7 we have \( \sigma_{-1}(D_{t}^{-1}) = \sigma_{-1}(D_{t}^{-1}) \). Similarly, \( \sum_{i=1}^{3} \overline{\Phi}_{i} = 0. \)

case b) \( r = -2, \ l = -1, \ k = j = |\alpha| = 0. \)

By (3.77), we get

\[ \overline{\Phi}_{4} = - i \int_{|\xi'|=1} \int_{-\infty}^{\infty} \operatorname{trace}[\pi_{\xi}^{+} \sigma_{-2}(D_{t}^{-1}) \times \partial_{\xi} \sigma_{-1}(D_{t}^{-1})](x_{0}) d\xi d\sigma(\xi') \] 

(3.83)

By Lemma 3.7 we have \( \sigma_{-1}(D_{t}^{-1}) = \sigma_{-1}(D_{t}^{-1}) \). By (3.42), (3.66), we have

\[ \overline{\Phi}_{4} = \frac{9}{2} \pi h'(0) \Omega_{3} d\xi', \]

(3.84)
where $\Omega_3$ is the canonical volume of $S^3$.

**case c)** $r = -1$, $l = -2$, $k = j = |\alpha| = 0$.

By (3.85), we get
\[
\overline{\Theta}_N = -i \int_{|\xi'|=1} \int_{-\infty}^{\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \times \partial_{\xi_n} \sigma_{-2}(D_t^{-1})](x_0) d\xi_n \sigma(\xi') dx'.
\] (3.85)

By (3.3) and (3.4), Lemma 3.7, we have
\[
\pi_{\xi_n}^+ \sigma_{-1}(D_t^{-1})|_{|\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}.
\] (3.86)

Since
\[
\sigma_{-2}(D_t^{-1})(x_0) = \frac{c(\xi) \sigma_0(D_t)(x_0)e(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n)[\partial_{\xi_n}[c(\xi')](x_0)]|\xi|^2 - c(\xi)h'(0)|\xi|^2_{\partial_M},
\] (3.87)

where
\[
\sigma_0(D_t)(x_0) = \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(e_i)(x_0)e(e_i)e(e_i) + \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(e_i)(x_0)e(e_i)e(e_i) + tA(x_0)
\]
\[
= Q_0^3(x_0) + Q_0^3(x_0) + tA(x_0),
\] (3.88)

then
\[
\partial_{\xi_n} \sigma_{-2}(D_t^{-1})(x_0)|_{|\xi'|=1} = \partial_{\xi_n} \left\{ \frac{c(\xi)Q_0^3(x_0) + Q_0^3(x_0) + tA(x_0)}{|\xi|^4} \right\}
\]
\[
+ \frac{c(\xi)}{|\xi|^6} c(dx_n)[\partial_{\xi_n}[c(\xi')](x_0)]|\xi|^2 - c(\xi)h'(0)
\] (3.89)

By computations, we have
\[
\partial_{\xi_n} \frac{c(\xi)Q_0^3(x_0)e(\xi)}{|\xi|^4} = \frac{c(dx_n)Q_0^3(x_0)e(\xi)}{|\xi|^4} + \frac{c(\xi)Q_0^3(x_0)e(dx_n)}{|\xi|^4} - \frac{4\xi_n c(\xi)Q_0^3(x_0)e(\xi)}{|\xi|^6},
\] (3.90)

\[
\partial_{\xi_n} \frac{c(\xi)tA(x_0)e(\xi)}{|\xi|^4} = \frac{c(dx_n)tA(x_0)e(\xi)}{|\xi|^4} + \frac{c(\xi)tA(x_0)e(dx_n)}{|\xi|^4} - \frac{4\xi_n c(\xi)tA(x_0)e(\xi)}{|\xi|^4}.
\] (3.91)

We denote
\[
q_{-2} = \frac{c(\xi)Q_0^3(x_0)e(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n)[\partial_{\xi_n}[c(\xi')](x_0)]|\xi|^2 - c(\xi)h'(0),
\]
then
\[
\partial_{\xi_n} (q_{-2}^2) = \frac{1}{1 + q_{-2}^2} \left[ (2\xi_n - 2q_{-2}^2)c(dx_n)Q_0^3(x_0)e(dx_n) + (1 - 3q_{-2}^2)c(dx_n)Q_0^3(x_0) \right]
\]
\[
+ (1 - 3q_{-2}^2)c(\xi')Q_0^3(x_0) - 4\xi_n c(\xi')Q_0^3(x_0) + (3q_{-2}^2 - 1)\partial_{\xi_n} c(\xi')
\]
\[
- 4\xi_n c(\xi')c(dx_n)\partial_{\xi_n} c(\xi') + 2h'(0)c(\xi') + 2h'(0)\xi_n c(dx_n) \right]
\]
21
By (3.86) and (3.90), we have
\[
\text{tr}[\pi_{\xi_n}^\sigma \sigma_{-1}(D_{t}^{-1}) \times \partial_{\xi_n} \frac{c(\xi)Q_0^1 c(\xi)}{|\xi|^4}](x_0)|_{|\xi'|=1} = \frac{-1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi')Q_0^1](x_0)] + \frac{i}{(\xi - i)(\xi + i)^3} \text{tr}[c(dx_n)Q_0^1](x_0)].
\] (3.93)

By (3.49), we have
\[
\text{tr}[\pi_{\xi_n}^\sigma \sigma_{-1}(D_{t}^{-1}) \times \partial_{\xi_n} \frac{c(\xi)Q_0^1 c(\xi)}{|\xi|^4}](x_0)|_{|\xi'|=1} = \frac{-1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi')Q_0^1](x_0)].
\] (3.94)

We note that \(i < n\), \(\int_{|\xi'|=1} \{\xi_1, \xi_2, \cdots, \xi_{i+1}\} \sigma(\xi') = 0\), so \(\text{tr}[c(\xi')Q_0^1](x_0)]\) has no contribution for computing case c). By (3.86) and (3.92), we have
\[
\text{tr}[\pi_{\xi_n}^\sigma \sigma_{-1}(D_{t}^{-1}) \times \partial_{\xi_n} \frac{c(\xi)Q_0^1 c(\xi)}{|\xi|^4}](x_0)|_{|\xi'|=1} = \frac{12h'(0)(i\xi_0^2 + \xi_0 - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^3},
\] (3.95)

then
\[
-i\Omega_3 \int_{\Gamma_+} \left[ \frac{12h'(0)(i\xi_0^2 + \xi_0 - 2i)}{(\xi_0 - i)^3(\xi_0 + i)^3} + \frac{48h'(0)i\xi_n}{(\xi_0 - i)^3(\xi_0 + i)^3} \right] d\xi_n d\xi' = -\frac{9}{2} \pi h'(0)\Omega_3 d\xi'.
\] (3.96)

By (3.86) and (3.91), we have
\[
\text{tr}[\pi_{\xi_n}^\sigma \sigma_{-1}(D_{t}^{-1}) \times \partial_{\xi_n} \frac{c(\xi)A c(\xi)}{|\xi|^4}](x_0)|_{|\xi'|=1} = \frac{-1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi')A](x_0)] + \frac{i}{(\xi - i)(\xi + i)^3} \text{tr}[c(dx_n)A](x_0)].
\] (3.97)

By \(\int_{|\xi'|=1} \{\xi_1, \xi_2, \cdots, \xi_{i+1}\} \sigma(\xi') = 0\) and (3.49), we have
\[
-i \int_{|\xi'|=1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{tr}[\pi_{\xi_n}^\sigma \sigma_{-1}(D_{t}^{-1}) \times \partial_{\xi_n} \frac{c(\xi)A c(\xi)}{|\xi|^4}] d\xi_n d\xi'\Omega_3 d\xi' = -\frac{\pi}{4} \text{tr}[c(dx_n)A]\Omega_3 d\xi'.
\] (3.98)

Then,
\[
\mathcal{F}_5 = -\frac{9}{2} \pi h'(0)\Omega_3 d\xi'.
\] (3.99)

So \(\mathcal{F} = \sum_{i=1}^{5} \mathcal{F}_i = 0\). By (3.70)-(3.78), we obtain Theorem 3.9.
4. A Kastler-Kalau-Walze type theorem for 6-dimensional manifolds with boundary

Firstly, we prove the Kastler-Kalau-Walze type theorems for 6-dimensional manifolds with boundary. From [11], we know that
\[
\text{Wres}[\pi^+ D_t^{-1} \circ \pi^+ (D_t^* D_t)^{-1}] = \int_M \int_{[\xi'] = 1} \text{trace}_{\Lambda^* T^* M \otimes \mathbb{C}}[\sigma_{-4}((D_t^* D_t)^{-2})] \sigma(\xi) dx + \int_{\partial M} \Psi, \quad (4.1)
\]
where
\[
\Psi = \int_{[\xi'] = 1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \left\{ \frac{1}{\alpha!(j+k+1)} \times \text{trace}_{\Lambda^* T^* M \otimes \mathbb{C}}[\partial_i^j \partial_\xi^k \partial_\xi^k \sigma_+ (D_t^{-1})(x', 0, \xi', \xi_n)]
\]
\[
\times \partial_\xi^k \partial_\xi^k \partial_\xi^k \sigma_+ (D_t^* D_t)^{-1}(x', 0, \xi', \xi_n)) d\xi_n \sigma(\xi') dx', \quad (4.2)
\]
and the sum is taken over \( r + \ell - k - j - |\alpha| - 1 = -6, \ r \leq -1, \ell \leq -3. \)

By Theorem 2.2, we compute the interior term of (4.1), then
\[
\int_M \int_{[\xi'] = 1} \text{trace}_{\Lambda^* T^* M \otimes \mathbb{C}}[\sigma_{-4}((D_t^* D_t)^{-2})] \sigma(\xi) dx' = 128\pi^3 \int_M \left( -\frac{16}{3} t - t^2 \right) \sum_{i=1}^{6} \sum_{\alpha=1}^{k} |S(e_{\alpha}) f_\alpha|^2 ) d\text{Vol}_M, \quad (4.3)
\]
(4.3) holds by the similar reason for (3.17).

Next, we compute \( \int_{\partial M} \Psi \). Let \( \xi = \sum_j \xi_j dx_j \) and \( \nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k \), we denote that
\[
\sigma_i = -\frac{1}{4} \sum_{x,k} \omega_{x,i}(e_{x}) c(e_x) c(e_i); \quad a_i = \frac{1}{4} \sum_{x,k} \omega_{x,i}(e_{x}) \hat{c}(e_x) c(e_i); \quad \xi^j = g^{ij} \xi_i; \quad \Gamma^k = g^{ij} \Gamma_{ij}^k; \quad \sigma^j = g^{ij} \sigma_i; \quad a^j = g^{ij} a_i, \quad (4.4)
\]
Since \( E \) is globally defined on \( M \), taking normal coordinates at \( x_0 \), we have \( \sigma^i(x_0) = 0, \ a^i(x_0) = 0, \partial^i[c(\partial_j)](x_0) = 0, \Gamma^k(x_0) = 0, \ g^{ij}(x_0) = \delta^i_j, \) then by computations, we get
\[
D_t^* D_t^* = \sum_{i=1}^{n} c(e_i)(e_i, dx_i)(-g^{ij} \partial_i \partial_j) + \sum_{i=1}^{n} c(e_i)(e_i, dx_i) \left\{ -(\partial_i g^{ij}) \partial_i \partial_j - g^{ij} \left( 4(\sigma_i + a_i) \partial_j - 2\Gamma_{ij}^k \partial_k \right) \right\} \right)
\]
\[
+ \sum_{i=1}^{n} \left[ \partial_j \left( t \sum_{i=1}^{n} \sum_{j=1}^{k} c(e_i) \hat{c}(S(e_{i}) f_{\alpha}) \hat{c}(f_{\alpha}) c(e_j) - c(e_j) \hat{t} \sum_{i=1}^{n} \sum_{j=1}^{k} c(e_i) \hat{c}(S(e_{i}) f_{\alpha}) \hat{c}(f_{\alpha}) \right) \right] \partial_j \hat{t}
\]
\[
+ \sum_{j,k} \Gamma_{ij}^k \partial_k \left( \partial_j \left( \sum_{i=1}^{n} \sum_{j=1}^{k} c(e_i) \hat{c}(S(e_{i}) f_{\alpha}) \hat{c}(f_{\alpha}) c(e_j) - c(e_j) \hat{t} \sum_{i=1}^{n} \sum_{j=1}^{k} c(e_i) \hat{c}(S(e_{i}) f_{\alpha}) \hat{c}(f_{\alpha}) \right) \right] \partial_j \hat{t}
\]
\[
+ \sum_{i=1}^{n} \left[ g^{ij} \left( \partial_i \partial_j \right) - g^{ij} \left( \partial_i a_j + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k \right) \right] \partial_j \hat{t}
\]
\[
+ \sum_{i,j} g^{ij} \left[ t \sum_{i=1}^{n} \sum_{j=1}^{k} c(e_i) \hat{c}(S(e_{i}) f_{\alpha}) \hat{c}(f_{\alpha}) c(\partial_i) \sigma_i + t \sum_{i=1}^{n} \sum_{j=1}^{k} c(e_i) \hat{c}(S(e_{i}) f_{\alpha}) \hat{c}(f_{\alpha}) c(\partial) a_i \right]
\]
\[
+ \sum_{i,j} g^{ij} \left[ \partial_j \hat{t} \sum_{i=1}^{n} \sum_{j=1}^{k} c(e_i) \hat{c}(S(e_{i}) f_{\alpha}) \hat{c}(f_{\alpha}) c(\partial_i) a_i + c(\partial_i) \partial_j \hat{t} \sum_{i=1}^{n} \sum_{j=1}^{k} c(e_i) \hat{c}(S(e_{i}) f_{\alpha}) \hat{c}(f_{\alpha}) \right] \partial_i \right\} \right)
\]
\[
23
\]
By the composition formula of pseudodifferential operators, we have

\[ \sum_{\alpha=1}^{n} \sum_{i=1}^{n} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \] + \frac{1}{4} K - \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k) c(e_l)

\[ + \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\} \left\{ \left( \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right) \right\}

\[ + \sum_{ij} g^{ij} \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\} \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\}

\[ + \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\} \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\}

\[ + \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\} \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\}

\[ + \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\} \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\}

\[ - \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\} \left\{ \sum_{i=1}^{n} \sum_{l} \sum_{k} \sum_{c} c(e_i) \hat{c}(S(e_i)f_0) \hat{c}(f_0) \right\}

(4.5)

Then, we obtain

**Lemma 4.1.** The following identities hold:

\[ \sigma_2(D_t^* D_t D_t^*) = \sum_{i,j,l} c(dx_i) \hat{d}(g^{ij}) \xi_j + c(\xi)(4A^k + 4A^k - 2\Gamma^k \xi_k + 2[\xi] T A - \xi T A) \xi^2 \]

\[ + \frac{1}{4} \xi^2 \sum_{s,t,l} \omega_{s,t} c(e_i) \hat{c}(e_s) c(e_l) - c(e_l) c(e_s) c(e_i) + |\xi|^2 |TA|^2;\]

\[ \sigma_3(D_t^* D_t D_t^*) = ic(\xi)|\xi|^2. \]

(4.6)

Write

\[ \sigma(D_t^* D_t D_t^*) = p_3 + p_2 + p_1 + p_0; \quad \sigma((D_t^* D_t D_t^*)^{-1}) = \sum_{j=3}^{\infty} q_{-j}. \]

(4.7)

By the composition formula of pseudodifferential operators, we have

\[ 1 = \sigma((D_t^* D_t D_t^*) \circ (D_t^* D_t D_t^*)^{-1}) = \sum_{\alpha=1}^{n} \frac{1}{\alpha!} \partial^\alpha _{\alpha !} [\sigma(D_t^* D_t D_t^*)] D_0^n [(D_t^* D_t D_t^*)^{-1}] \]

24
By (4.2), we get
\[(D_x q_3 + D_x q_4 + D_x q_5 + \cdots) = p_3 q_3 + (p_3 q_4 + p_2 q_4 + \sum_j \partial_{\xi_i} p_3 D_x (p_3^{-1})) + \cdots, \tag{4.8}\]
by (4.3), we have
\[q_3 = p_3^{-1}; q_4 = -p_3^{-1} \partial_{\xi_i} p_3 D_x (p_3^{-1})]. \tag{4.9}\]

**Lemma 4.2.** The following identities hold:
\[
\begin{align*}
\sigma_{-3}((D_t^* D_t D_t^*)^{-1}) &= \frac{i c(\xi)}{|\xi|^4} \\
\sigma_{-4}((D_t^* D_t D_t^*)^{-1}) &= \frac{c(\xi) c_2((D_t^* D_t D_t^*)-1) c(\xi)}{|\xi|^8} + \frac{i c(\xi)}{|\xi|^6} \left( |\xi|^4 c(dx_n) \partial_{\xi_n} c(\xi') - 2 h'(0) c(dx_n) c(\xi) \right) \\
&+ 2 \xi_n c(\xi) \partial_{\xi_n} c(\xi') + 4 \xi_n h'(0).
\end{align*}
\tag{4.10}\]

**Theorem 4.3.** Let \( M \) be a 6-dimensional compact oriented manifold with boundary \( \partial M \) and the metric \( g_T^{TM} \) be defined as (3.7). Let \( D_t \) and \( D_t^* \) be sub-signature operators on \( M \) (\( M \) is a collar neighborhood of \( M \)) as in (2.9), (2.10), then
\[
\begin{align*}
\widehat{\text{Res}}[\pi^+ D_t^{-1} \circ \pi^+ (D_t^* D_t D_t^*)^{-1}] = 128 \pi^3 \int_M \left( -\frac{16}{3} K - (t - l)^2 \sum_{i=1}^{6} \sum_{\alpha=1}^{k} |S(e_i) f_{\alpha}|^2 \right) d\text{Vol}_M + \int_{\partial M} \left( \frac{65}{8} - \frac{41}{8} i h'(0) \right) \Omega_4 d\text{Vol}_M, \tag{4.11}\end{align*}
\]
where \( K \) is the scalar curvature.

**Proof.** When \( n = 6 \), then \( \text{tr}_\pi \cdot \pi^+ [\Omega d] = 64 \). Since the sum is taken over \( r + \ell - k - j - |\alpha| - 1 = -6 \), \( r \leq -1, \ell \leq -3 \), then we have \( \int_{\partial M} \Psi \) is the sum of the following five cases:

**case (a) (I)** \( r = -1, l = -3, j = k = 0, |\alpha| = 1 \).

By (4.2), we get
\[
\Psi_1 = -\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \left[ \partial_{\xi_n} \pi^+_\xi \sigma_{-3} ((D_t^* D_t D_t^*)^{-1}) \times \partial_{\xi_n} \sigma_{-3} ((D_t^* D_t D_t^*)^{-1}) \right] (x_0) d\xi_n c(\xi') dx'. \tag{4.12}\]
By Lemma (4.2) for \( i < n \), we have
\[
\partial_{\xi_n} \sigma_{-3} ((D_t^* D_t D_t^*)^{-1}) (x_0) = \partial_{\xi_n} \left[ \frac{i c(\xi)}{|\xi|^4} (x_0) = i \partial_{\xi_n} [c(\xi)] |\xi|^{-4} (x_0) - 2 i c(\xi) \partial_{\xi_n} |\xi|^2 |\xi|^{-6} (x_0) = 0, \tag{4.13}\right.
so \( \Psi_1 = 0 \).

**case (a) (II)** \( r = -1, l = -3, |\alpha| = k = 0, j = 1 \).

By (4.2), we have
\[
\Psi_2 = \left[ \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left[ \partial_{\xi_n} \pi^+_\xi \sigma_{-3} ((D_t^* D_t D_t^*)^{-1}) \times \partial_{\xi_n} \sigma_{-3} ((D_t^* D_t D_t^*)^{-1}) \right] (x_0) d\xi_n c(\xi') dx'. \tag{4.14}\right]
By computations, we have
\[
\partial_{\xi_n}^2 \sigma_3((D_t^* D_t D_t^*)^{-1}) = i \left[ \frac{(20\xi_n^2 - 4)\xi' + 12(\xi_n^2 - \xi_n)c(dx_n)}{(1 + \xi_n^2)^4} \right].
\] (4.15)

Since \( n = 6, \) \( \text{tr}[-id] = -64. \) By the relation of the Clifford action and \( \text{trab} = \text{trba}, \) then
\[
\text{tr}[\xi'(dx_n')] = 0; \text{tr}[\xi(dx_n')] = -64; \text{tr}[\partial_{\xi_n}[\xi'(dx_n)] = 0; \text{tr}[\partial_{\xi_n}[\xi'(dx_n)]|_{\xi'|=1} = -32h'(0).
\] (4.16)

By (4.15) and (4.16), we get
\[
\text{trace} \left[ \partial_{\xi_n} \pi_0^+ \sigma_\alpha^{-1}(D_t^{-1}) \times \partial_{\xi_n} \partial_{\xi_n} \sigma_\alpha^{-1}(D_t^* D_t D_t^*)^{-1} \right](x_0) = 64h'(0) \frac{-1 - 3\xi_n + 5\xi_n^2 + 3i\xi_n^3}{(\xi_n - i)^6(\xi_n + i)^4}
\] (4.17)

Then we obtain
\[
\Psi_2 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} h'(0) \text{dim} F \frac{-8 - 24\xi_n i + 40\xi_n^2 + 24i\xi_n^3}{(\xi_n - i)^6(\xi_n + i)^4} d\xi_n \sigma(\xi') dx'
\]
\[
= 8h'(0)\Omega_4 \int_{\Gamma^+} \frac{4 + 122\xi_n i - 20\xi_n^2}{(\xi_n - i)^6(\xi_n + i)^4} d\xi_n dx'
\]
\[
= h'(0)\Omega_4 \frac{\pi i}{32} \left[ \frac{8 + 24\xi_n i - 40\xi_n^2 - 24i\xi_n^3}{(\xi_n + i)^4} \right]|_{\xi_n = i} dx'
\]
\[
= -\frac{15}{2} \pi h'(0)\Omega_4 dx'.
\] (4.18)

where \( \Omega_4 \) is the canonical volume of \( S^4. \)

**Case (a) (III)** \( r = -1, l = -3, |\alpha| = j = 0, k = 1. \)

By (4.12), we have
\[
\Psi_3 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_0^+ \sigma_\alpha^{-1}(D_t^{-1}) \times \partial_{\xi_n} \partial_{\xi_n} \sigma_\alpha^{-1}(D_t^* D_t D_t^*)^{-1} \right](x_0) d\xi_n \sigma(\xi') dx'.
\] (4.19)

By computations, we have
\[
\partial_{\xi_n} \partial_{\xi_n} \sigma_\alpha^{-1}(D_t^* D_t D_t^*)^{-1} = -\frac{4i\xi_n \partial_{\xi_n} \xi'(dx_n)(x_0)}{(1 + \xi_n^2)^3} + i 12h'(0)\xi_n e(\xi') = i \frac{2 - 10\xi_n^2}{(1 + \xi_n^2)^4}.
\] (4.20)

Combining (3.35) and (4.20), we have
\[
\text{trace} \left[ \partial_{\xi_n} \pi_0^+ \sigma_\alpha^{-1}(D_t^{-1}) \times \partial_{\xi_n} \partial_{\xi_n} \sigma_\alpha^{-1}(D_t^* D_t D_t^*)^{-1} \right](x_0)|_{\xi'|=1} = 8h'(0) \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^6(\xi_n + i)^4}.
\] (4.21)

Then
\[
\Psi_3 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} 8h'(0) \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^6(\xi_n + i)^4} d\xi_n \sigma(\xi') dx'
\]
\[
= -\frac{1}{2} h'(0)\Omega_4 \int_{\Gamma^+} \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^6(\xi_n + i)^4} d\xi_n dx'
\]
\[
= -8h'(0)\Omega_4 \frac{\pi i}{4!} \left[ \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n + i)^4} \right]|_{\xi_n = i} dx'
\]
\[
= \frac{25}{2} \pi h'(0)\Omega_4 dx'.
\] (4.22)
case (b) $r = -1, l = -4, |\alpha| = j = k = 0$.

By (4.22), we have

\[
\Psi_4 = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_n^+ \sigma_{-1}(D_t^{-1}) \times \partial_{\xi_n} \sigma_{-4}(D_t^* D_t D_t^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
= i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_n^+ \sigma_{-1}(D_t^{-1}) \times \sigma_{-4}(D_t^* D_t D_t^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
\]

(4.23)

In the normal coordinate, $g^{ij}(x_0) = \delta^i_j$ and $\partial x_j (g^{\alpha\beta})(x_0) = 0$, if $j < n$; $\partial x_j (g^{\alpha\beta})(x_0) = h'(0)\delta^\beta_\alpha$, if $j = n$. So by [21], when $k < n$, we have $\Gamma^\alpha(x_0) = \frac{2}{i} h'(0)$, $\Gamma^k(x_0) = 0$, $\delta^\alpha(x_0) = 0$ and $\delta^k = \frac{1}{i} h'(0)c(e_\alpha)c(e_n)$. Then, we obtain

\[
\sigma_{-4}(D_t^* D_t D_t^{-1}) (x_0) |_{|\xi'|=1} = \frac{c(\xi)\sigma_2(D_t^* D_t D_t^{-1})(x_0) |_{|\xi'|=1} c(\xi)}{|\xi|^2} - \frac{c(\xi)}{|\xi|^2} \sum_{j \neq k} \partial_{\xi_j} (c(\xi)|\xi|^2) D_x_j \left( \frac{i c(\xi)}{|\xi|^2} \right) \\
= \frac{1}{|\xi|^2} \left( \frac{1}{2} h'(0) c(\xi) \sum_{k < n} \xi_k c(e_k)c(e_n) \right) - \frac{1}{2} h'(0) c(\xi) \sum_{k < n} \xi_k \tilde{c}(e_k) \tilde{c}(e_n) \\
- \frac{5}{2} h'(0) \xi_n c(\xi) - \frac{1}{4} |\xi|^2 c(dx_n) + 2[|\xi|^2 \gamma A - c(\xi) \gamma A c(\xi)] + |\xi|^2 \gamma A c(\xi) \\
+ \frac{i c(\xi)}{|\xi|^2} \left( |\xi|^4 c(dx_n) \partial_{\xi_n} c(\xi') - 2h'(0)c(dx_n)c(\xi) + 2\xi_n c(\xi) \partial_{\xi_n} c(\xi') + 4\xi_n h'(0) \right).
\]

(4.24)

By [3.36] and (4.23), we have

\[
\text{tr}[\partial_{\xi_n} \pi_n^+ \sigma_{-1}(D_t^{-1}) \times \sigma_{-4}(D_t^* D_t D_t^{-1}) (x_0) |_{|\xi'|=1}] = \\
- \frac{1}{2(\xi_n - i)^2(1 + \xi_n^2)^2} \left( \frac{3 i}{4} + \frac{1}{2} \left[ 3 i \xi_n + (-6 + 2i) \xi_n^2 + 3 \xi_n^3 + \frac{9i}{4} \xi_n^4 \right] h'(0) \text{tr}[id] \right) \\
+ \frac{1}{2(\xi_n - i)^2(1 + \xi_n^2)^2} \left( -1 - 3 i \xi_n - 2 \xi_n^2 - 4 i \xi_n^3 - \xi_n^4 - i \xi_n^5 \right) c(\xi') \tilde{c}(e_k) c(\xi') \]

\[
- \frac{1}{2(\xi_n - i)^2(1 + \xi_n^2)^2} \left( 3 + \frac{1}{2} \sum_{k < n} \frac{\xi_n + 1}{2} \xi_n^2 + \frac{3}{2} \xi_n^3 + \frac{1}{2} \xi_n^4 \right) c(\xi') \tilde{c}(e_k) c(dx_n) \tilde{c}(dx_n) \\
+ \frac{-\xi_n i + 3}{2(\xi_n - i)^4(1 + \xi_n^2)^3} \text{tr}[\gamma A c(\xi')] - \frac{3 \xi_n + i}{2(\xi_n - i)^4(1 + \xi_n^2)^3} \text{tr}[\gamma A c(\xi')].
\]

(4.25)

By the relation of the Clifford action and trab = trba, we have equalities:

\[
\text{tr}[\gamma A (x_0) c(dx_n)]] = 0; \quad \text{tr}[\gamma A (x_0) c(\xi')] = 0; \quad \text{tr}[c(e_i) \tilde{c}(e_i) c(e_n) \tilde{c}(e_n)] = 0 \quad (i < n).
\]

(4.26)

Then

\[
\text{tr}[c(\xi') \tilde{c}(\xi') c(dx_n) \tilde{c}(dx_n)] = \sum_{i,j < n} \text{tr}[\xi_i \xi_j c(e_i) \tilde{c}(e_j) c(dx_n) \tilde{c}(dx_n)] = 0.
\]

(4.27)

So, we have

\[
\Psi_4 = i h'(0) \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} 64 \times \frac{3 i + 2 + (3 + 4i) \xi_n + (-6 + 2i) \xi_n^2 + 3 \xi_n^3 + \frac{9i}{4} \xi_n^4}{2(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n c(\xi') dx' \\
+ i h'(0) \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} 32 \times \frac{1 + 3 i \xi_n + 2 \xi_n^2 + 4 i \xi_n^3 + \xi_n^4 + \frac{9i}{4} \xi_n^5}{2(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n c(\xi') dx'.
\]

27
By (4.30), (3.3) and (3.4), we have
\[ + i \int_{\xi' = 1}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_n - i - 2\xi_n i + 1}{2(\xi_n - i)^3(i + \xi_n)^3} \text{tr}[tAc(dx_n)]d\xi_n\sigma(\xi')dx' \]
\[ - i \int_{\xi' = 1}^{\infty} \int_{-\infty}^{\infty} \frac{3\xi_n + i}{2(\xi_n - i)^3(i + \xi_n)^3} \text{tr}[tAc(\xi')]d\xi_n\sigma(\xi')dx' \]
\[ = (-\frac{19}{4} - 15)i\pi h'(0)\Omega_d dx' + (-\frac{3}{8} - \frac{75}{8})i\pi h'(0)\Omega_d dx' \]
\[ = (-\frac{41}{8}i - 195)\pi h'(0)\Omega_d dx'. \] (4.28)

**Case (c)** \( r = -2, l = -3, |\alpha| = j = k = 0. \)

By \( 4.2 \), we have
\[ \Psi_5 = -i \int_{|\xi'| = 1}^{\infty} \int_{-\infty}^{\infty} \text{trace}\left[ \pi_{\xi_n}^+ \sigma_{-2}(D_t^{-1}) \times \partial_{\xi_n} \sigma_{-3}(D_t^* D_t D_t^*)^{-1} \right] (x_0) d\xi_n\sigma(\xi')dx'. \] (4.29)

By Lemma 4.1 and Lemma 4.2, we have
\[ \sigma_{-2}(D_t^{-1})(x_0) = \frac{c(\xi)\sigma_0(D_t)c(\xi)(x_0) + c(\xi)}{|\xi|^4} \sum_j c(dx_j) \left[ \partial_{x_j} c(\xi) \right] |\xi|^2 - c(\xi) \partial_{x_j} (|\xi|^2) \right] (x_0), \] (4.30)

where
\[ \sigma_0(D_t) = \frac{1}{4} \sum_{i,n,t} \omega_s_t(e_i)c(e_i)c(e_i) + \frac{1}{4} \sum_{i,n,t} \omega_s_t(e_i)c(e_i)c(e_i)c(e_i) + tA. \] (4.31)

On the other hand,
\[ \partial_{\xi_n} \sigma_{-3}(D_t^* D_t D_t^*)^{-1} = -\frac{4i\xi_n c(\xi')}{(1 + \xi_n^2)^3} + \frac{i(1 - 3\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3}. \] (4.32)

By \( 4.30, 3.3 \) and \( 3.4 \), we have
\[ \pi_{\xi_n}^+ \left[ \sigma_{-2}(D_t^{-1}) \right] (x_0)_{|\xi'| = 1} = \pi_{\xi_n}^+ \left[ \frac{c(\xi)\sigma_0(D_t)(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{\xi_n} c(\xi')}{|\xi|^2} \right] (x_0) \]
\[ - h'(0)\pi_{\xi_n}^+ \left[ \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3} \right]. \] (4.33)

We denote
\[ \sigma_0(D_t)(x_0)_{|\xi'| = 1} = Q_0(x_0) = Q_0^0(x_0) + Q_0^2(x_0) + tA. \] (4.34)

Then, we obtain
\[ \pi_{\xi_n}^+ \left[ \sigma_{-2}(D_t^{-1}) \right] (x_0)_{|\xi'| = 1} = \pi_{\xi_n}^+ \left[ \frac{c(\xi)Q_0^2(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{\xi_n} c(\xi')}{(1 + \xi_n^2)^2} \right] - h'(0)\pi_{\xi_n}^+ \left[ \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3} \right] \]
\[ + \pi_{\xi_n}^+ \left[ \frac{c(\xi)c(dx_0)c(\xi)(x_0)}{(1 + \xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[ \frac{c(\xi)tAc(\xi)(x_0)}{(1 + \xi_n^2)^2} \right]. \] (4.35)

Furthermore,
\[ \pi_{\xi_n}^+ \left[ \frac{c(\xi)tAc(\xi)(x_0)}{(1 + \xi_n^2)^2} \right] = \pi_{\xi_n}^+ \left[ \frac{c(\xi)tAc(\xi)(x_0)}{(1 + \xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[ \frac{\xi_n c(\xi')tAc(x_0)c(dx_n)}{(1 + \xi_n^2)^2} \right]. \]
By computations, we have

\[ \begin{align*}
&+ \pi_n^+ \left[ \frac{\xi_n c(dx_n) t A(x_n) c(\xi')}{(1 + \xi_n^2)^2} \right] \\
&= - \frac{\epsilon(\xi') t A(x_n) c(\xi')(2 + i \xi_n)}{4(\xi_n - i)^2} + \frac{\iota c(\xi') t A(x_n) c(dx_n)}{4(\xi_n - i)^2} + \frac{ic(dx_n) t A(x_n) c(\xi')}{4(\xi_n - i)^2} \\
&+ \frac{-i\xi_n c(dx_n) t A(x_n) c(dx_n)}{4(\xi_n - i)^2},
\end{align*} \]

\[ (4.36) \]

By the relation of the Clifford action and trab = trba, we have equalities:

\[ \text{tr}[\hat{Q}_0^2 c(dx_n)] = 0; \quad \text{tr}[\hat{c}(\xi')\hat{c}(dx_n)] = 0. \]

Then we have

\[ \text{tr} \left[ \pi_n^+ \left\{ \frac{c(\xi) Q_0^1(x_0) c(\xi)}{(1 + \xi_n^2)^2} \right\} \times \partial_{\xi_n} \sigma_3 ((D_t^* D_t D_t^*)^{-1})(x_0) \right] \bigg|_{\xi' = 1} = \frac{2 - 8i \xi_n - 6 \xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[\hat{Q}_0^1(x_0) c(\xi')]. \]

By computations, we have

\[ \pi_n^+ \left[ \frac{c(\xi) Q_0^2(x_0) c(\xi) + c(\xi) c(dx_n) \partial_{\xi_n} \sigma_3 (\xi') (x_0)}{(1 + \xi_n^2)^2} \right] - h'(0) \pi_n^+ \left[ \frac{c(\xi) c(dx_n) c(\xi)}{(1 + \xi_n^2)^2} \right] = C_1 - C_2, \]

\[ (4.40) \]

where

\[ C_1 = \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i \xi_n)c(\xi') Q_0^2 c(\xi') + i \xi_n c(dx_n) Q_0^2 c(dx_n) \right. \]

\[ + (2 + i \xi_n)c(\xi') c(dx_n) \partial_{\xi_n} c(\xi') + ic(dx_n) Q_0^2 c(\xi') + ic(\xi') Q_0^2 c(dx_n) - i \partial_{\xi_n} c(\xi') \]

\[ = \frac{1}{4(\xi_n - i)^2} \left[ \frac{h'(0) c(dx_n)}{2} - \frac{5 i}{2} h'(0) c(\xi') - (2 + i \xi_n)c(\xi') c(dx_n) \partial_{\xi_n} c(\xi') + i \partial_{\xi_n} c(\xi') \right]; \]

\[ (4.41) \]

\[ C_2 = \frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3 \xi_n - 7i}{8(\xi_n - i)^3} (ic(\xi') - c(dx_n)) \right]. \]

\[ (4.42) \]

By (13.32) and (14.42), we have

\[ \text{tr}[C_2 \times \partial_{\xi_n} \sigma_3 ((D_t^* D_t D_t^*)^{-1})(x_0)] \bigg|_{\xi' = 1} \]

\[ = \text{tr} \left\{ \frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3 \xi_n - 7i}{8(\xi_n - i)^3} (ic(\xi') - c(dx_n)) \right] \right\}. \]
\[
\begin{align*}
&\times -4i\xi_n(\xi') + (i - 3i\xi_n^2)c(dx_n) \bigg) \\
&= 8h'(0) \frac{4i - 11\xi_n - 6i\xi_n^2 + 3\xi_n^3}{(\xi_n - i)^4(\xi_n + i)^3}.
\end{align*}
\]

Similarly, we have
\[
\text{tr}[C_1 \times \partial_{\xi_n} \sigma_{-3}(D_t^* D_t D_t^*)^{-1})]_{|\xi'|=1} = \text{tr}\left\{ \frac{1}{4(\xi_n - i)^2} \left[ \frac{5}{2} h'(0)c(dx_n) - \frac{5i}{2} h'(0)c(\xi') - (2 + i\xi_n)c(\xi')c(dx_n)\partial_{\xi_n} c(\xi) + i\partial_{\xi_n} c(\xi') \right] \right\} \\
&= 8h'(0) \frac{3 + 12\xi_n + 3\xi_n^2}{4(\xi_n - i)^2(\xi_n + i)^3}.
\]

By \(\int_{|\xi'|=1} \{\xi_1, \xi_2, \ldots, \xi_{d+1}\} \sigma(\xi') = 0\), we have
\[
\Psi_5 = -ih'(0) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} 8 \times \frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n - i)^3(\xi_n + i)^3} \text{d}\xi_n \sigma(\xi') \text{d}x' \\
= -8ih'(0) \times \frac{2\pi i}{4!} \left[ \frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n + i)^3} \right]_{|\xi_n=0} \Omega_4 \text{d}x' \\
= \frac{55}{2} \pi h'(0) \Omega_4 \text{d}x'.
\]

Now \(\Psi\) is the sum of the cases (a), (b) and (c), then
\[
\Psi = \left( \frac{65}{8} - \frac{41}{8}i \right) \pi h'(0) \Omega_4 \text{d}x'.
\]

By (4.1)-(4.3), we obtain Theorem 4.3 \(\square\)

Next, we prove the Kastler-Kalau-Walze type theorem for 6-dimensional manifold with boundary associated to \(D_t^3\). From [14], we know that
\[
\text{Wres}[\pi^+ D_t^{-1} \circ \pi^+ D_t^{-3}] = \int_M \int_{|\xi'|=1} \text{trace}_{\lambda^* T^* M} \otimes C[\sigma_{-4}(D_t^{-4})] \sigma(\xi) \text{d}x + \int_{\partial M} \Psi,
\]
where \(\text{Wres}\) denote noncommutative residue on minifolds with boundary,
\[
\Psi = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^{j+k+1}}{\alpha! (j + k + 1)!} \times \text{trace}_{\lambda^* T^* M} \otimes C[\partial_{\xi_n}^j \partial_{\xi_n}^k \sigma_t(D_t^{-3})(x', 0, \xi', \xi_n)] \text{d}\xi_n \sigma(\xi') dx',
\]

\[(4.49)\]
and the sum is taken over \( r + \ell - k - j - |\alpha| = -6 \), \( r \leq -1 \), \( \ell \leq -3 \).

By Theorem 2.2, we compute the interior term of (4.49), then

\[
\int_M \int_{|\xi|=1} \text{trace}_{\gamma^* T^* M} c(\sigma \Sigma(D_{t}^{-4})|\sigma(\xi)) dx = 128\pi^3 \int_M \left(-\frac{16}{3} K\right) d\text{Vol}_M.
\]

(4.50)

So we only need to compute \( \int_M \nabla \). Let us now turn to compute the specification of \( D_{t}^3 \).

\[
D_{t}^3 = \sum_{i=1}^{n} c(e_i) \langle e_i, dx_i \rangle \left(-g^{ij} \partial_i \partial_j \right) + \sum_{i=1}^{n} c(e_i) \langle e_i, dx_i \rangle \left\{ \begin{array}{c}
-(\partial_t g^{ij}) \partial_i \partial_j - g^{ij} \left(4(\sigma_i + |\alpha|) \partial_j - 2\Gamma_{ij}^k \partial_k\right)
\end{array} \right\}
\]

\[
+ \sum_{i=1}^{n} c(e_i) \langle e_i, dx_i \rangle \left\{ -2(\partial_t g^{ij})(\sigma_i + |\alpha|) \partial_j + g^{ij}(\partial_t \Gamma_{ij}^k) \partial_k - 2g^{ij} \left[(\partial_t \sigma_i) + (\partial_a \sigma_i) + (\partial_t \Gamma_{ij}^k) \right] \Gamma_{ij}^k \partial_k
\end{array} \right\}
\]

\[
+ \sum_{i=1}^{n} c(e_i) \langle e_i, dx_i \rangle \left\{ -g^{ij} \left[(\partial_t \sigma_j) + (\partial_t a_j) + \sigma_t + \sigma_j a_j + a_i a_j + a_i a_j - \Gamma_{ij}^k a_k \right]
\end{array} \right\}
\]

\[
+ \sum_{i,j} g^{ij} \left[ \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha}) c(\partial_t) \sigma_i + t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha}) c(\partial_t) a_i
\end{array} \right]
\]

\[
+ c(\partial_t) \sigma_i t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha})] + \frac{1}{4} K - \frac{1}{8} \sum_{i,j,k} R_{ij,k} \tilde{c}(e_i) c(e_k) c(e_i)
\]

\[
- \left\{ \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha})^2 \right\} + \left\{ (\sigma_i + |\alpha|) + \left( t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha}) \right) \right\}
\]

\[
+ \left\{ \sum_{i,j=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \langle e_i, dx_i \rangle \left\{ \begin{array}{c}
2(\partial_t g^{ij})(\sigma_i + |\alpha|) \\
+ (t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha}) \right) \left\{ -\sum_{i,j} g^{ij} \left[2\sigma_t \partial_j + 2a_t \partial_j - \Gamma_{ij}^k \partial_k
\end{array} \right\}
\end{array} \right\}
\]

\[
+ (\partial_t \sigma_j) + (\partial_t a_j) + \sigma_t + a_i a_j + a_i a_j - \Gamma_{ij}^k a_k \right\} + \sum_{i,j} g^{ij} \left[ c(\partial_t) \sigma_i t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha}) c(\partial_t) a_i
\end{array} \right]
\]

\[
+ c(\partial_t) \sigma_i t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha})] + \frac{1}{4} K - \frac{1}{8} \sum_{i,j,k} R_{ij,k} \tilde{c}(e_i) c(e_k) c(e_i)
\]

\[
+ \left\{ \begin{array}{c}
(\sigma_i + |\alpha|) + (t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha}) \right) \right\}
\]

\[
+ \left\{ \sum_{i,j=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \langle e_i, dx_i \rangle \left\{ \begin{array}{c}
2(\partial_t g^{ij})(\sigma_i + |\alpha|) + (t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha}) \right) \left\{ -\sum_{i,j} g^{ij} \left[2\sigma_t \partial_j + 2a_t \partial_j - \Gamma_{ij}^k \partial_k
\end{array} \right\}
\end{array} \right\}
\]

\[
+ (\partial_t \sigma_j) + (\partial_t a_j) + \sigma_t + a_i a_j + a_i a_j - \Gamma_{ij}^k a_k \right\} + \sum_{i,j} g^{ij} \left[ c(\partial_t) \sigma_i t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha}) c(\partial_t) a_i
\end{array} \right]
\]

\[
+ c(\partial_t) \sigma_i t \sum_{i=1}^{n} \sum_{\alpha=1}^{k} c(e_i) \tilde{c}(S(e_i) f_{\alpha}) \tilde{c}(f_{\alpha})] + \frac{1}{4} K - \frac{1}{8} \sum_{i,j,k} R_{ij,k} \tilde{c}(e_i) c(e_k) c(e_i)
\]

31
By (4.51)-(4.55), we have some symbols of operators. By (4.54), we have

The following identities hold:

Lemma 4.4. The following identities hold:

\[
\sigma_2(D_t^3) = \sum_{i,j,l} c(dx_i) \partial_i (g^{ij}) \xi_j + c(\xi)(4\sigma^k + 4a^k - 2\Gamma^k)\xi_k - 2[c(\xi)tAc(\xi) - |\xi|^2 tA]
\]

\[
+ \frac{1}{4} |\xi|^2 \sum_{s,t,i,j} \omega_{s,t} c(e_i) c(e_s) c(e_t) - c(e_i) c(e_s) c(e_t)\]

\[
+ |\xi|^2 (e_i) c(e_i) c(e_s) c(e_t));
\]

\[
\sigma_3(D_t^{-3}) = ic(\xi)|\xi|^2.
\]

Write

\[
\sigma(D_t^3) = p_3 + p_2 + p_1 + p_0; \quad \sigma(D_t^{-3}) = \sum_{j=3}^{\infty} q_{-j}.
\]

By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma(D_t^3 \circ D_t^{-3}) = \sum_{\alpha} \frac{1}{\alpha !} \partial_\xi^\alpha [\sigma(D_t^3)] D_\xi^\alpha[\sigma(D_t^{-3})]
\]

\[
= (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \cdots)
\]

\[
+ \sum_j (\partial_{\xi_j} p_3 + \partial_{\xi_j} p_3 + \partial_{\xi_j} p_1 + \partial_{\xi_j} p_0)(D_{x_j} q_{-3} + D_{x_j} q_{-4} + D_{x_j} q_{-5} + \cdots)
\]

\[
= p_3 q_{-3} + (p_3 q_{-4} + p_2 q_{-3} + \sum_j \partial_{\xi_j} p_3 D_{x_j} q_{-3}) + \cdots.
\]

By (4.51), we have

\[
q_{-3} = p_3^{-1}; \quad q_{-4} = -p_3^{-1}[p_2 p_3^{-1} + \sum_j \partial_{\xi_j} p_3 D_{x_j} (p_3^{-1})].
\]

By (4.51)-(4.55), we have some symbols of operators.

Lemma 4.5. The following identities hold:

\[
\sigma_{-3}(D_t^{-3}) = \frac{ic(\xi)}{|\xi|^4};
\]

\[
\sigma_{-4}(D_t^{-3}) = \frac{c(\xi)\sigma_2(D_t^3)c(\xi)}{|\xi|^8} + \frac{ic(\xi)}{|\xi|^4}
\]

\[
\left(|\xi|^4 c(dx_n) \partial_{x_n} c(\xi') - 2h'(0)c(dx_n) c(\xi)
\]

\[
+ 2\xi_n c(\xi) \partial_{x_n} c(\xi') + 4\xi_n h'(0)\right).
\]

(4.56)
Theorem 4.6. Let $M$ be a 6-dimensional compact oriented manifold with boundary $\partial M$ and the metric $g^{TM}$ be defined as (7.1), $D_t$ be a sub-signature operator on $M$ ($\bar{M}$ is a collar neighborhood of $M$) as in [2,3], [2.3], then

$$
\overline{\text{Wres}}[\pi^+ D_t^{-1} \circ \pi^+ (D_t^{-3})]
= 128\pi^3 \int_M \left(-\frac{16}{3} K\right) d\text{Vol}_M + \int_{\partial M} \left(\frac{65}{8} - \frac{41}{8-i}\pi h'(0)\right) \Omega_4 d\text{Vol}_M.
$$

(4.57)

where $K$ is the scalar curvature.

Proof. When $n = 6$, then $\text{tr}_{\mathcal{T} \cdot T \cdot M[\text{id}]} = 64$. Since the sum is taken over $r + \ell - k - j - |\alpha| = -6$, $r \leq -1$, $\ell \leq -3$, then we have the $\int_{\partial M} \Psi$ is the sum of the following five cases:

**case (a) (I)** $r = -1, l = -3, j = k = 0, |\alpha| = 1$.

By (4.49), we get

$$
\Psi_1 = -\int_{|\xi'| = 1} \int_{|\xi| = 1}^{+\infty} \sum_{|\alpha| = 1} \text{trace} \left[ \partial^\alpha \xi \xi' \sigma_{-1}(D_t^{-1}) \times \partial^\alpha \xi \xi' \sigma_{-3}(D_t^{-3}) \right](x_0) d\xi_n \sigma(\xi') dx'.
$$

(4.58)

**case (a) (II)** $r = -1, l = -3, |\alpha| = k = 0, j = 1$.

By (4.49), we have

$$
\Psi_2 = -\frac{1}{2} \int_{|\xi'| = 1} \int_{|\xi| = 1}^{+\infty} \text{trace} \left[ \partial^\alpha \xi \xi' \sigma_{-1}(D_t^{-1}) \times \partial^\alpha \xi \xi' \sigma_{-3}(D_t^{-3}) \right](x_0) d\xi_n \sigma(\xi') dx'.
$$

(4.59)

**case (a) (III)** $r = -1, l = -3, |\alpha| = j = 0, k = 1$.

By (4.49), we have

$$
\Psi_3 = -\frac{1}{2} \int_{|\xi'| = 1} \int_{|\xi| = 1}^{+\infty} \text{trace} \left[ \partial^\alpha \xi \xi' \sigma_{-1}(D_t^{-1}) \times \partial^\alpha \xi \xi' \sigma_{-3}(D_t^{-3}) \right](x_0) d\xi_n \sigma(\xi') dx'.
$$

(4.60)

By Lemma 4.2 and Lemma 4.3, we have $\sigma_{-3}(D_t^* D_t D_t^*)^{-1} = \sigma_{-3}(D_t^{-3})$, by (4.58) - (4.60), we obtain

$$
\sum_{i=1}^{3} \Psi_i = 5\pi h'(0)\Omega_4 dx',
$$

where $\Omega_4$ is the canonical volume of $S^4$.

**case (b)** $r = -1, l = -4, |\alpha| = j = k = 0$.

By (4.49), we have

$$
\Psi_4 = -\int_{|\xi'| = 1} \int_{|\xi| = 1}^{+\infty} \text{trace} \left( \pi^+ \xi \xi' \sigma_{-1}(D_t^{-1}) \times \partial^\alpha \xi \sigma_{-4}(D_t^{-3}) \right)(x_0) d\xi_n \sigma(\xi') dx'
$$

$$
= i \int_{|\xi'| = 1} \int_{|\xi| = 1}^{+\infty} \text{trace} \left[ \partial^\alpha \xi \xi' \sigma_{-1}(D_t^{-1}) \times \sigma_{-4}(D_t^{-3}) \right](x_0) d\xi_n \sigma(\xi') dx'.
$$

(4.61)

Then, we obtain

$$
\sigma_{-4}(D_t^{-3})(x_0)|_{|\xi'| = 1} = \frac{c(\xi)\sigma_2(D_t^3)(x_0)|_{|\xi'| = 1} c(\xi)}{|\xi|^{3}} - \frac{c(\xi)}{|\xi|^{3}} \sum_{j} \partial_{\xi_j} c(\xi)|\xi|^{2} D_j \left(\frac{ic(\xi)}{|\xi|^{3}}\right)
$$

$$
= \frac{1}{|\xi|^{3}}(\xi) \left(\frac{1}{2} h'(0) c(\xi) \sum_{k<n} \xi_k c(e_k) c(e_n) - \frac{1}{2} h'(0) c(\xi) \sum_{k<n} \xi_k \bar{c}(e_k) \bar{c}(e_n)\right)
$$

33
By (3.38) and (4.62), we have

\[ \Psi = \psi(0) + \psi(\xi) + \int_{\mathbb{R}^n} \psi(x) \, dx. \]

By Lemma 4.1 and Lemma 4.5, we have case (c)

\[ \psi(\xi) = \left( 1 + 3i \xi_\alpha + 2i \xi_\beta + 4i \xi_\gamma + \frac{9i}{4} \xi_\delta \right) \psi(0) + \int_{\mathbb{R}^n} \psi(x) \, dx. \]

By (4.60) and (4.63), we have

\[ \Psi_4 = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} 64 \times \frac{3i + 2 + (3 + 4i) \xi_\alpha + (-6 + 2i) \xi_\beta + 3 \xi_\gamma + 3i \xi_\delta}{2(\xi - i)^2(1 + \xi_\delta)} \, d\xi_\delta \, d\xi \]

\[ + i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} 32 \times \frac{1 + 3i \xi_\alpha + 2i \xi_\beta + 4i \xi_\gamma + \xi_\delta}{2(\xi - i)^2(1 + \xi_\delta)} \, d\xi_\delta \, d\xi \]

\[ = (-\frac{41}{8} - \frac{195}{8}) \pi h'(0) \Omega_4 \eta. \]

**Case (c)** \( r = -2, l = -3, |\alpha| = j = k = 0. \)

By (4.49), we have

\[ \Psi_5 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[ (D_t^{-1}) \times \partial_{\xi_\delta} (D_t^{-3}) \right] (x_0) \, d\xi_\delta \, d\xi. \]

By Lemma 4.3 and Lemma 4.5, we have \( \sigma_{-3}(D_t^* D_t D_t^{-1}) = \sigma_{-3}(D_t^{-3}) \), by (4.28)-(4.31), we obtain

\[ \Psi_5 = \frac{55}{2} \pi h'(0) \Omega_4 \eta. \]

Now \( \Psi \) is the sum of the cases (a), (b) and (c), then

\[ \Psi = \left( \frac{65}{8} - \frac{41}{8} i \right) \pi h'(0) \Omega_4 \eta. \]

By (4.48)-(4.50), we obtain Theorem 4.6
Acknowledgements

This work was supported by NSFC. 11771070. The authors thank the referee for his (or her) careful reading and helpful comments.

References

[1] Ackermann T.: A note on the Wodzicki residue. J. Geom. Phys. 20, 404-406, (1996).
[2] Bao K, Wang J, Wang Y.: A local equivariant index theorem for sub-signature operators. arXiv:1312.3721, 2013.
[3] Connes A.: Quantized calculus and applications. 11th International Congress of Mathematical Physics (Paris, 1994), Internat Press, Cambridge, MA, 15-36, (1995).
[4] Connes A.: The action functional in Noncommutative geometry. Comm. Math. Phys. 117, 673-683, (1998).
[5] Dai X, Zhang W.: Adiabatic limit, Bismut-Freed connection, and the real analytic torsion form. J. reine angew. Math. 647, 87-113, (2010).
[6] Fedosov B V, Golse F, Leichtnam E, Schrohe E.: The noncommutative residue for manifolds with boundary. J. Funct. Anal. 142, 1-31, (1996).
[7] Gilkey P B.: Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem. Vol 11 of mathematics Lecture Series, 1984.
[8] Guillemin V W.: A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. Adv. Math. 55, no. 2, 131-160, (1985).
[9] Iochum B, Levy C.: Tadpoles and commutative spectral triples. J. Noncommut. Geom. 299-329, (2011).
[10] Kalau W, Walze M.: Gravity, Noncommutative geometry and the Wodzicki residue. J. Geom. Physics. 16, 327-344, (1995).
[11] Kastler D.: The Dirac Operator and Gravitation. Comm. Math. Phys. 166, 633-645, (1995).
[12] Ma X, Zhang W.: η-invariants, torsion forms and flat vector bundles. Math. Ann. 340: 569-624, (2008).
[13] Ponge R.: Noncommutative geometry and lower dimensional volumes in Riemannian geometry. Lett. Math. Phys. 83, no. 1, 19-32, (2008).
[14] Rempel S, Schulze B W.: Index theory of elliptic boundary problems. Akademie-Verlag, Berlin, 1982, 393 pp.
[15] Schrohe, E.: Noncommutative residue, Dixmier's trace, and heat trace expansions on manifolds with boundary. Contemp. Math. 242, 161-186, (1999).
[16] Wang J, Wang Y.: The Kastler-Kalau-Walze type theorem for six-dimensional manifolds with boundary. J. Math. Phys. 56, 052501, (2015).
[17] Wang J, Wang Y.: Twisted Dirac Operators and the Noncommutative Residue for Manifolds with Boundary. J pseudo-Differ. Oper. Appl. 10, (2016), 7, (no. 2) 181-211.
[18] Wang J, Wang Y, Yang C L.: Dirac Operators with Torsion and the Noncommutative Residue for Manifolds with Boundary. Journal of Geometry and Physics, 2014, 81, 92-111.
[19] Wang Y.: Differential forms and the Wodzicki residue for Manifolds with Boundary. J. Geom. Physics. 56, 731-753, (2006).
[20] Wang Y.: Differential forms the Noncommutative Residue for Manifolds with Boundary in the non-product Case. Lett. Math. Phys. 77, 41-51, (2006).
[21] Wang Y.: Gravity and the Noncommutative Residue for Manifolds with Boundary. Lett. Math. Phys. 80, 37-56, (2007).
[22] Wang Y.: Lower-Dimensional Volumes and Kastler-Kalau-Walze Type Theorem for Manifolds with Boundary. Commun. Theor. Phys. 54, 38-42, (2010).
[23] Wodzicki M.: Local invariants of spectral asymmetry. Invent. Math. 75(1), 143-178, (1995).
[24] Yu Y.: The Index Theorem and The Heat Equation Method, Nankai Tracts in Mathematics Vol 2, World Scientific Publishing, (2001).
[25] Zhang W.: Sub-signature operator and its local index theorem. Chinese Sci. Bull. 41, 294-295, (1996). (in Chinese)
[26] Zhang W.: Sub-signature operators, η-invariants and a Riemann-Roch theorem for flat vector bundles. Chin. Ann. Math. 25B, 7-36, (2004).