\section{Introduction}

Paracontact metric structures were introduced in \cite{KaneyukiW} by S. Kaneyuki and F. L. Williams. The importance of paracontact geometry, and in particular of para-Sasakian geometry, has been indicated in the recent years by many authors. Its role in pseudo-Riemannian geometry as well as in mathematical physics was emphasized in several papers (see e.g. \cite{Besse}, \cite{Gray}, \cite{Libermann}). Recently (\cite{Erken}) I. Küpeli Erken studied normal almost paracontact metric manifolds provided they satisfy some additional projective flatness conditions.

Relations between affine differential geometry and paracomplex geometry can be found in \cite{Ghosh} and \cite{Szancer} for example. Moreover, affine immersions with an almost product structures are also studied (see e.g. \cite{Ziemian}).

In \cite{Szancer} the author studied affine hypersurfaces with an arbitrary $J$-tangent transversal vector field, where $J$ was the canonical complex structure on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. It was proved that if the induced almost contact structure is metric relative to the second fundamental form then it is a Sasakian structure and the hypersurface itself is a piece of hyperquadric. In this paper we study affine hypersurfaces $f: M \to \mathbb{R}^{2n+2}$ with an arbitrary $\tilde{J}$-tangent transversal vector field, where $\tilde{J}$ is the canonical paracomplex structure on $\mathbb{R}^{2n+2}$. Such a vector field...
induces in a natural way an almost paracontact structure \((\varphi, \xi, \eta)\) as well as the second fundamental form \(h\). We prove that if \((\varphi, \xi, \eta, h)\) is an almost paracontact metric structure then it is a para \(\alpha\)-Sasakian structure with \(\alpha = -1\). Moreover, the hypersurface is a piece of a hyperquadric.

In Section 2, we briefly recall the basic formulas of affine differential geometry. We introduce the notion of a \(\tilde{J}\)-tangent transversal vector field and a \(\tilde{J}\)-invariant distribution \(D\).

In Section 3 we recall the definitions of an almost paracontact metric structure, para \(\alpha\)-Sasakian structure and para \(\alpha\)-contact structure. We introduce the notion of an induced almost paracontact structure and prove some results related to this structure.

Section 4 contains main results of this paper. We prove that if \((\varphi, \xi, \eta, h)\) is an almost paracontact metric structure then the hypersurface is equiaffine and the shape operator \(S = -\text{id}\). In consequence, the structure is para \((-1)\)-Sasakian. We also prove that the hypersurface is a piece of a hyperquadric and give an explicit formula for it.

2. Preliminaries

We briefly recall the basic formulas for affine differential geometry. For more details, we refer to [7]. Let \(f: M \rightarrow \mathbb{R}^{n+1}\) be an orientable connected differentiable \(n\)-dimensional hypersurface immersed in affine space \(\mathbb{R}^{n+1}\) equipped with its usual flat connection \(D\). Then for any transversal vector field \(C\) we have

\[
D_X f_* Y = f_*(\nabla_X Y) + h(X, Y)C
\]

and

\[
D_X C = -f_*(SX) + \tau(X)C,
\]

where \(X, Y\) are tangent vector fields. For any transversal vector field \(\nabla\) is a torsion-free connection, \(h\) is a symmetric bilinear form on \(M\), called the second fundamental form, \(S\) is a tensor of type \((1, 1)\), called the shape operator and \(\tau\) is a 1-form.

In this paper we assume that \(h\) is nondegenerate so that \(h\) defines a pseudo-Riemannian metric on \(M\). If \(h\) is nondegenerate, then we say that the hypersurface or the hypersurface immersion is \textit{nondegenerate}. We have the following
Theorem 2.1 ([7], Fundamental equations). For an arbitrary transversal vector field $C$ the induced connection $\nabla$, the second fundamental form $h$, the shape operator $S$, and the 1-form $\tau$ satisfy the following equations:

\begin{align*}
R(X,Y)Z &= h(Y,Z)SX - h(X,Z)SY, \\
(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) &= (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \\
(\nabla_X S)(Y) - \tau(X)SY &= (\nabla_Y S)(X) - \tau(Y)SX, \\
h(X,SY) - h(SX,Y) &= 2d\tau(X,Y).
\end{align*}

The equations (2.1), (2.2), (2.3), and (2.4) are called the equation of Gauss, Codazzi for $h$, Codazzi for $S$ and Ricci, respectively.

For an affine hypersurface the cubic form $Q$ is defined by the formula

\begin{equation}
Q(X,Y,Z) = (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z).
\end{equation}

It follows from the equation of Codazzi (2.2) that $Q$ is symmetric in all three variables.

For a hypersurface immersion $f: M \to \mathbb{R}^{n+1}$ a transversal vector field $C$ is said to be equiaffine (resp. locally equiaffine) if $\tau = 0$ (resp. $d\tau = 0$).

Let $\dim M = 2n + 1$ and $f: M \to \mathbb{R}^{2n+2}$ be a nondegenerate (relative to the second fundamental form) affine hypersurface. We always assume that $\mathbb{R}^{2n+2}$ is endowed with the standard paracomplex structure $\tilde{J}$

\begin{equation}
\tilde{J}(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) = (y_1, \ldots, y_{n+1}, x_1, \ldots, x_{n+1}).
\end{equation}

Let $C$ be a transversal vector field on $M$. We say that $C$ is $\tilde{J}$-tangent if $\tilde{J}C_x \in f_* (T_x M)$ for every $x \in M$. We also define a distribution $D$ on $M$ as the biggest $\tilde{J}$-invariant distribution on $M$, that is

\begin{equation}
D_x = f^{-1}_x (f_*(T_x M) \cap \tilde{J}(f_*(T_x M)))
\end{equation}

for every $x \in M$. We have that $\dim D_x \geq 2n$. If for some $x$ the $\dim D_x = 2n + 1$ then $D_x = T_x M$ and it is not possible to find $\tilde{J}$-tangent transversal vector field in a neighbourhood of $x$. In this paper we study $f$ with a $\tilde{J}$-tangent transversal vector field $C$, so in particular $\dim D = 2n$. The distribution $D$ is smooth as the intersection of two smooth distributions and because $\dim D$ is constant. A vector field $X$ is called a $D$-field if $X_x \in D_x$ for every $x \in M$. We use the notation
$X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields. We say that the distribution $\mathcal{D}$ is nondegenerate if $h$ is nondegenerate on $\mathcal{D}$.

Additionally, we define a 1-dimensional distribution $\mathcal{D}_h$ as follows

$$\mathcal{D}_{h,x} := \{X \in T_x M : h(X, Y) = 0 \forall Y \in \mathcal{D}_x\},$$

where $h$ is the second fundamental form on $M$ relative to any transversal vector field.

To simplify the writing, we will be omitting $f_* \mathcal{X}$ in front of vector fields in most cases.

3. ALMOST PARACONTACT STRUCTURES

A $(2n + 1)$-dimensional manifold $M$ is said to have an almost paracontact structure if there exist on $M$ a tensor field $\varphi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ which satisfy

$$\varphi^2(X) = X - \eta(X)\xi, \quad (3.1)$$

$$\eta(\xi) = 1 \quad (3.2)$$

for every $X \in TM$ and the tensor field $\varphi$ induces an almost paracomplex structure on the distribution $\mathcal{D} = \ker \eta$. That is the eigendistributions $\mathcal{D}^+, \mathcal{D}^-$ corresponding to the eigenvalues $1, -1$ of $\varphi$ have equal dimension $n$. If additionally there is a pseudo-Riemannian metric $g$ on $M$ of signature $(n + 1, n)$ such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y) \quad (3.3)$$

for every $X, Y \in TM$ then $(\varphi, \xi, \eta, g)$ is called an almost paracontact metric structure. In particular, for an almost paracontact metric structure we have

$$\eta(X) = g(X, \xi) \quad (3.4)$$

for all $X \in TM$. Hence $\xi$ is $g$-orthogonal to $\mathcal{D}$.

An almost paracontact metric structure is called para $\alpha$-Sasakian if

$$(\hat{\nabla}_X \varphi)(Y) = \alpha(-g(X, Y)\xi + \eta(Y)X), \quad (3.5)$$

where $\hat{\nabla}$ is the Levi-Civita connection for $g$ and $\alpha$ is some smooth function on $M$. In particular, when $\alpha = 1$ we get the standard para-Sasakian structure. An
almost paracontact metric manifold is called \textit{para} $\alpha$-\textit{contact} if

\[ d\eta(X, Y) = \alpha g(X, \varphi Y) \]  

(3.6)

for a certain non-zero function $\alpha$ and for every $X, Y \in TM$. When $\alpha = 1$ an almost paracontact metric structure $(\varphi, \xi, \eta, g)$ satisfying (3.6) is called a \textit{paracontact metric structure}.

We say that an almost paracontact structure $(\varphi, \xi, \eta)$ is \textit{normal} if

\[ [\varphi, \varphi] - 2d\eta \otimes \xi = 0, \]  

(3.7)

where $[\varphi, \varphi]$ is the Nijenhuis tensor for $\varphi$. We have the following theorem

\textbf{Theorem 3.1} \cite{9}. An almost paracontact metric manifold is para $\alpha$-Sasakian if and only if it is normal and para $\alpha$-contact.

Let $f: M \to \mathbb{R}^{2n+2}$ be a nondegenerate affine hypersurface with a $\tilde{J}$-tangent transversal vector field $C$. Then we can define a vector field $\xi$, a 1-form $\eta$ and a tensor field $\varphi$ of type (1,1) as follows:

\[ \xi := \tilde{J}C; \]  

(3.8)

\[ \eta|_D = 0 \text{ and } \eta(\xi) = 1; \]  

(3.9)

\[ \varphi|_D = \tilde{J}|_D \text{ and } \varphi(\xi) = 0. \]  

(3.10)

It is easy to see that $(\varphi, \xi, \eta)$ is an almost paracontact structure on $M$. This structure will be called the \textit{induced} almost paracontact structure.

We shall now prove

\textbf{Theorem 3.2}. Let $f: M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a $\tilde{J}$-tangent transversal vector field $C$. If $(\varphi, \xi, \eta)$ is an induced almost paracontact structure
on $M$ then the following equations hold:

\begin{align}
\eta(\nabla_X Y) &= h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \quad (3.11) \\
\varphi(\nabla_X Y) &= \nabla_X \varphi Y - \eta(Y) SX - h(X, Y)\xi, \quad (3.12) \\
\eta([X, Y]) &= h(X, \varphi Y) - h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) \\
&\quad + \eta(Y)\tau(X) - \eta(X)\tau(Y), \quad (3.13) \\
\varphi([X, Y]) &= \nabla_X \varphi Y - \nabla_Y \varphi X + h(X) SY - \eta(Y)SX, \quad (3.14) \\
\eta(\nabla_X \xi) &= \tau(X), \quad (3.15) \\
\eta(SX) &= -h(X, \xi) \quad (3.16)
\end{align}

for every $X, Y \in \mathcal{X}(M)$.

**Proof.** For every $X \in TM$ we have

$$\tilde{J}X = \varphi X + \eta(X)C.$$ 

We also have

$$\tilde{J}(D_X Y) = \tilde{J}(\nabla_X Y + h(X, Y)C) = \tilde{J}(\nabla_X Y) + h(X, Y)\tilde{J}C$$

$$= \varphi(\nabla_X Y) + \eta(\nabla_X Y)C + h(X, Y)\xi$$

and

$$D_X \tilde{J}Y = D_X(\varphi Y + \eta(Y)C) = D_X \varphi Y + X(\eta(Y))C + \eta(Y)D_X C$$

$$= \nabla_X \varphi Y + h(X, \varphi Y)C + X(\eta(Y))C + \eta(Y)(-SX + \tau(X)C)$$

$$= \nabla_X \varphi Y - \eta(Y) SX + (h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X))C.$$ 

Since $D_X \tilde{J}Y = \tilde{J}(D_X Y)$, comparing transversal and tangent parts, we obtain (3.11) and (3.12), respectively. Equations (3.13)–(3.16) follow directly from (3.11) and (3.12).

From the above theorem we immediately get
Corollary 3.3. For every $Z, W \in \mathcal{D}$ we have

\begin{align*}
\eta(\nabla_Z W) &= h(Z, \varphi W), \quad (3.17) \\
\eta(\nabla_\xi Z) &= h(\xi, \varphi Z), \quad (3.18) \\
\varphi(\nabla_Z W) &= \nabla_Z \varphi W - h(Z, W)\xi, \quad (3.19) \\
\eta([Z, W]) &= h(Z, \varphi W) - h(W, \varphi Z), \quad (3.20) \\
\eta([Z, \xi]) &= -h(\xi, \varphi Z) + \tau(Z). \quad (3.21)
\end{align*}

Almost paracontact normal structures can be characterized as follows

Proposition 3.4. The induced almost paracontact structure $(\varphi, \xi, \eta)$ is normal if and only if

$$S\varphi Z - \varphi SZ + \tau(Z)\xi = 0 \quad \text{for every } Z \in \mathcal{D}.$$ 

Proof. It is an immediate consequence of (3.7), the identity

$$d\eta(X, Y) = \frac{1}{2} \left( X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \right)$$

and the formulas (3.13) and (3.14). \qed

4. Main results

In this section we always assume that $(\varphi, \xi, \eta)$ is an induced almost paracontact structure. In order to prove the main theorem of this section we need the following two lemmas:

Lemma 4.1. If $(\varphi, \xi, \eta, h)$ is an almost paracontact metric structure then

\begin{align*}
\eta(X) &= h(X, \xi), \text{ for every } X \in TM, \quad (4.1) \\
S(\mathcal{D}) &\subset \mathcal{D}, \quad (4.2) \\
S\xi &= -\xi + Z_0, \text{ where } Z_0 \in \mathcal{D} \quad (4.3) \\
\tau(Z) &= -h(Z, \varphi Z_0) \text{ for every } Z \in \mathcal{D}. \quad (4.4)
\end{align*}

Proof. Properties (4.1), (4.2) and (4.3) are immediate consequence of (3.3) and (3.16). The Codazzi equation for $S$ implies that

$$\nabla_X S\xi - S(\nabla_X \xi) - \tau(X)S\xi = \nabla_\xi SX - S(\nabla_\xi X) - \tau(\xi)SX.$$
Formula (3.18) and the fact that \((\varphi, \xi, \eta, h)\) is metric structure imply that \(\nabla_{\xi} Z \in D\) for every \(Z \in D\). By (4.2) and (4.3) we obtain
\[
\tau(Z) = -\eta(\nabla_{\xi} Z_0) + \eta(\nabla_Z \xi) + \eta(S(\nabla_Z \xi)) \quad (4.5)
\]
for every \(Z \in D\). Now, using (3.11), (3.16), (4.1) and the fact that \((\varphi, \xi, \eta, h)\) is metric structure we get
\[
\eta(\nabla_Z Z_0) = h(Z, \varphi Z_0), \quad \eta(S(\nabla_Z \xi)) = -\eta(\nabla_Z \xi)
\]
for every \(Z \in D\). Hence, equation (4.5) can be rewritten as
\[
\tau(Z) = -h(Z, \varphi Z_0),
\]
which proves (4.4). □

**Lemma 4.2.** If \((\varphi, \xi, \eta, h)\) is an almost paracontact metric structure then
\[
Q(X, W, Z) = -Q(X, \varphi W, \varphi Z), \quad (4.6)
\]
\[
Q(W_1, W_2, W_3) = 0, \quad (4.7)
\]
\[
Q(\xi, W, W) = -h(SW, \varphi W) = h(S\varphi W, W) \quad (4.8)
\]
for every \(X \in \mathcal{X}(M)\) and \(W, W_1, W_2, W_3, Z \in D\).

**Proof.** Let \(X \in \mathcal{X}(M)\) and \(W, Z \in D\). Then by (2.5) and (3.3) we have
\[
Q(X, \varphi W, \varphi Z) = X(h(\varphi W, \varphi Z)) - h(\nabla_X \varphi W, \varphi Z) - h(\varphi W, \nabla_X \varphi Z)
\]
\[
+ \tau(X) h(\varphi W, \varphi Z)
\]
\[
= -X(h(W, Z)) - h(\nabla_X \varphi W, \varphi Z) - h(\varphi W, \nabla_X \varphi Z)
\]
\[
- \tau(X) h(W, Z).
\]
By Theorem 3.2 we see that
\[
\nabla_X \varphi W = \varphi(\nabla_X W) + h(X, W) \xi
\]
and
\[
\nabla_X \varphi Z = \varphi(\nabla_X Z) + h(X, Z) \xi.
\]
Thus using the above and (3.3) we get
\[ Q(X, \varphi W, \varphi Z) = -X(h(W, Z)) - h(\nabla_X W, \varphi Z) - h(X, W) h(\xi, \varphi Z) - h(\varphi W, \varphi (\nabla_X Z)) - h(X, Z) h(\varphi W, \xi) - \tau(X) h(W, Z) \]
\[ = -X(h(W, Z)) + h(\nabla_X W, Z) + h(W, \nabla_X Z) - \tau(X) h(W, Z) \]
\[ = -Q(X, W, Z), \]
which proves (4.6). To prove (4.7) observe that from (4.6) we have
\[ Q(W, W, W) = -Q(W, \varphi W, \varphi W) = -Q(\varphi W, \varphi W, W) = 0 \]
for every \( W \in D \), because \( h(\varphi W, W) = 0 \) and \( \nabla_{\varphi W} W = \varphi(\nabla_{\varphi W} W) - h(W, W) \xi \).
Since \( Q \) is symmetric in all three variables, the last equation implies that \( Q(W_1, W_2, W_3) = 0 \) for every \( W_1, W_2, W_3 \in D \). To prove (4.8) note first that
\[ Q(\xi, W, W) = Q(W, \xi, W) = -h(\nabla_W \xi, W) - h(\xi, \nabla_W W), \]
since \( h(\xi, W) = 0 \). Formula (3.12) imply that
\[ \varphi(\nabla_W \xi) = -SW. \]
From (3.3) and (3.17) we get
\[ \nabla_W W \in D. \]
Now we have
\[ Q(\xi, W, W) = -h(\nabla_W \xi, W) = h(\varphi(SW), W) = -h(SW, \varphi W) \]
for every \( W \in D \). From (4.6) we obtain
\[ Q(\xi, W, W) = -Q(\xi, \varphi W, \varphi W) \]
and in consequence
\[ -h(SW, \varphi W) = h(S\varphi W, W), \]
which completes the proof of (4.8).

We shall now prove

**Theorem 4.3.** Let \( f: M \to \mathbb{R}^{2n+2} \) be a nondegenerate hypersurface with a \( \tilde{J} \)-tangent transversal vector field and let \( (\varphi, \xi, \eta) \) be the induced almost paracontact structure on \( M \). If \( (\varphi, \xi, \eta, h) \) is the almost paracontact metric structure then
\[ S = -\text{id} \quad \text{and} \quad \tau = 0. \]
Proof. Let \( W, Z \in \mathcal{D} \). Formulas (2.3), (4.2) and (3.9) imply that

\[
\eta(\nabla_W S Z) - \eta(S(\nabla_W Z)) = \eta(\nabla_Z S W) - \eta(S(\nabla_Z W)).
\]

Thus, by (3.16) and (3.3),

\[
\eta(\nabla_W S Z) - \eta(\nabla_Z S W) = \eta(S([W, Z])) = -\eta([W, Z]) = \eta([Z, W]).
\]

By Corollary 3.3 (the formulas (3.17), (3.20)) we get

\[
h(W, \varphi S Z) - h(Z, \varphi S W) = -2h(W, Z) \quad \text{for every } W, Z \in \mathcal{D}. \tag{4.9}
\]

Using the Gauss equation we get

\[
(R(W, \varphi W) \cdot h)(\varphi W, \varphi W) = -2h(R(W, \varphi W)\varphi W, \varphi W) \tag{4.10}
\]

\[
= 2h(W, W)h(SW, \varphi W)
\]

for every \( W \in \mathcal{D} \). On the other hand

\[
(R(W, \varphi W) \cdot h)(\varphi W, \varphi W) = (\nabla_W \nabla_\varphi W h)(\varphi W, \varphi W)
\]

\[-(\nabla_\varphi W \nabla_W h)(\varphi W, \varphi W) - (\nabla_{[\varphi W]} h)(\varphi W, \varphi W).
\]

The following formulas are obvious:

\[
(\nabla_W \nabla_\varphi W h)(\varphi W, \varphi W) = W((\nabla_\varphi W h)(\varphi W, \varphi W)) - 2(\nabla_\varphi W h)(\nabla_W \varphi W, \varphi W),
\]

\[
(\nabla_\varphi W \nabla_W h)(\varphi W, \varphi W) = \varphi W((\nabla_W h)(\varphi W, \varphi W)) - 2(\nabla_W h)(\nabla_\varphi W \varphi W, \varphi W).
\]

We have

\[
(\nabla_X h)(Y, Z) = Q(X, Y, Z) - \tau(X)h(Y, Z)
\]
for every $X, Y, Z \in \mathcal{X}(M)$. Thus Lemma 4.2 and the above formulas imply

\[
(\nabla_W \nabla_{\varphi W} h)(\varphi W, \varphi W) = W \left( Q(W, \varphi W, \varphi W) - \tau(\varphi W) h(\varphi W, \varphi W) \right)
\]

\[
- 2Q(W, \nabla_{\varphi W} \varphi W, \varphi W) + 2\tau(\varphi W) h(\nabla_{\varphi W} \varphi W, \varphi W)
\]

\[
= -W(\tau(\varphi W)) h(\varphi W, \varphi W) - \tau(\varphi W) W(h(\varphi W, \varphi W))
\]

\[
+ 2Q(\nabla_W \varphi W, W, W) + 2\tau(\varphi W) h(\nabla_W \varphi W, \varphi W)
\]

\[
= W(\tau(\varphi W)) h(W, W) - \tau(\varphi W) W(h(W, W))
\]

\[
+ 2h(W, W)Q(\xi, W, W)
\]

where, in the last equality, we used (3.17). In a similar way we obtain

\[
(\nabla_{\varphi W} \nabla_W h)(\varphi W, \varphi W) = \varphi W(\tau(W) h(W, W))
\]

\[
- 2\varphi W(\tau(W) h(W, W)) - \tau(W) \varphi W(h(W, W))
\]

\[
+ 2\tau(W) h(\nabla_{\varphi W} \varphi W, \varphi W)
\]

\[
= \varphi W(\tau(W) h(W, W)) - \tau(W)(\varphi W) h(W, W)
\]

\[
= \varphi W(\tau(W)) h(W, W) - \tau(W) \tau(\varphi W) h(W, W).
\]

From (3.20) and Lemma 4.2 we also have

\[
(\nabla_{[W, \varphi W]} h)(\varphi W, \varphi W) = Q([W, \varphi W], \varphi W, \varphi W) - \tau([W, \varphi W]) h(\varphi W, \varphi W)
\]

\[
= -\eta([W, \varphi W]) Q(\xi, W, W) + \tau([W, \varphi W]) h(W, W)
\]

\[
= -2h(W, W)Q(\xi, W, W) + \tau([W, \varphi W]) h(W, W).
\]

Using (1.8) and the Ricci equation (2.4), we get

\[
2Q(\xi, W, W) = -h(SW, \varphi W) + h(W, S\varphi W) = 2d\tau(W, \varphi W).
\]

From (4.11) and the preceding formulas, we obtain

\[
(R(W, \varphi W) \cdot h)(\varphi W, \varphi W) = 6d\tau(W, \varphi W) h(W, W)
\]

and so, by (4.11) and (1.8),

\[
(R(W, \varphi W) \cdot h)(\varphi W, \varphi W) = 6Q(\xi, W, W) h(W, W) = -6h(W, W) h(SW, \varphi W),
\]
which, combined with (4.10), yields
\[ h(W, W) \cdot h(SW, \varphi W) = 0 \]
for every \( W \in D \). Using the fact that \( h \) is nondegenerate on \( D \) we get
\[ h(SW, \varphi W) = 0 \quad (4.12) \]
for every \( W \in D \).

Now (4.12) implies
\[ 0 = h(S(W + 2\varphi Z), \varphi W + 2Z) = 2h(SW, Z) + 2h(S\varphi Z, \varphi W). \]
Therefore
\[ h(S\varphi Z, \varphi W) = -h(SW, Z). \]

By (4.9) we also have
\[ h(S\varphi Z, \varphi W) = 2h(W, Z) + h(SW, Z). \]
The above formulas imply that
\[ h(SW, Z) = -h(W, Z) \]
for every \( Z \in D \). Now, since \( D \) is nondegenerate and by (4.2) it follows that
\[ SW = -W \]
for every \( W \in D \). From Lemma 4.1 we get
\[ SX = -X + \eta(X)Z_0 \quad (4.13) \]
for every \( X \in X(M) \). We shall show that \( Z_0 = 0 \). Assume \( Z_0 \neq 0 \), then using the Codazzi equation for \( S \) we have
\[ \nabla_W S Z_0 - S(\nabla_W Z_0) - \tau(W)S Z_0 = \nabla_{Z_0} SW - S(\nabla_{Z_0} W) - \tau(Z_0)SW. \]
Since \( \tau(Z_0) = 0 \) (Lemma 4.1, using (4.13) we can rewrite the above equality in the form
\[ -\eta(\nabla_W Z_0)Z_0 + \tau(W)Z_0 = -\eta(\nabla_{Z_0} W)Z_0, \]
Now, by (3.20) and (4.4) we have
\[ -\tau(W)Z_0 = \eta([Z_0, W])Z_0 = -2h(W, \varphi Z_0)Z_0 = 2\tau(W)Z_0. \]
The last equality implies that \( \tau|_D = 0 \). Now, formula (4.4) implies \( Z_0 = 0 \) which contradicts our assumption.
The property $\tau = 0$ easily follows from the fact that $S = -\text{id}$ and the equation of Codazzi for $S$. The proof is completed. □

The following theorem gives equivalent conditions for being the induced almost paracontact metric structure.

**Theorem 4.4.** Let $f : M \to \mathbb{R}^{2n+2}$ be a nondegenerate hypersurface with a $\tilde{J}$-tangent transversal vector field and let $(\varphi, \xi, \eta)$ be the induced almost paracontact structure on $M$. The following conditions are equivalent:

1. $(\varphi, \xi, \eta, h)$ is an almost paracontact metric structure, \hspace{1cm} (4.14)
2. $(\varphi, \xi, \eta, h)$ is a para $\alpha$-contact metric structure, where $\alpha = -1$, \hspace{1cm} (4.15)
3. $(\varphi, \xi, \eta, h)$ is a para $\alpha$-Sasakian structure, where $\alpha = -1$. \hspace{1cm} (4.16)

**Proof.** If $(\varphi, \xi, \eta, h)$ is an almost paracontact metric structure then by Theorem 4.3, we obtain $\tau = 0$. Theorem 3.2 (eq. (3.13)) implies that $(\varphi, \xi, \eta, h)$ is a para $(-1)$-contact metric structure. Again by Theorem 4.3, we get $S = -\text{id}$. Hence $(\varphi, \xi, \eta)$ is normal (Prop. 3.4). Now Theorem 3.1 completes the proof. □

Using Pick-Berwald theorem we get

**Theorem 4.5.** Let $f : M \to \mathbb{R}^{2n+2}$ be a nondegenerate hypersurface with a $\tilde{J}$-tangent transversal vector field and let $(\varphi, \xi, \eta)$ be the induced almost paracontact structure on $M$. If $(\varphi, \xi, \eta, h)$ is the almost paracontact metric structure, then $f(M)$ is a piece of a hyperquadric.

**Proof.** It is enough to show that $Q \equiv 0$. By Lemma 4.2, we have

$$Q(W_1, W_2, W_3) = 0 \quad \text{for every } W_1, W_2, W_3 \in \mathcal{D}$$

and

$$Q(\xi, W_1, W_2) = 0 \quad \text{for every } W_1, W_2 \in \mathcal{D}.$$

Since $\tau = 0$ by Theorem 4.3, using (3.11) and (4.1) we obtain

$$Q(X, \xi, \xi) = -2h(\nabla_X \xi, \xi) = -2\eta(\nabla_X \xi) = 0$$

for every $X \in \mathcal{X}(M)$. The above equalities imply that

$$Q(X_1, X_2, X_3) = 0$$
for every $X_1, X_2, X_3 \in \mathcal{X}(M)$. □

Finally, we can find an explicit formula for such hyperquadrics. We have the following theorem

**Theorem 4.6.** The nondegenerate hyperquadric of center $0$ such that the induced almost paracontact structure $(\varphi, \xi, \eta)$ is metric relative to the second fundamental form $h$ can be expressed in the form

$$H = \{x \in \mathbb{R}^{2n+2} : x^T Ax = 1\},$$

where $\det A \neq 0$ and

$$A = \begin{bmatrix} P & R \\ -R & -P \end{bmatrix},$$

$P^T = P$, $R^T = -R$, $P, R \in M(n+1, n+1, \mathbb{R})$.

Moreover, the induced almost paracontact structure for hyperquadrics of the above form is metric relative to the second fundamental form.

**Proof.** Every nondegenerate hyperquadric of center $0$ has a form

$$H = \{x \in \mathbb{R}^{2n+2} : x^T Ax = 1\},$$

where $\det A \neq 0$, $A^T = A$, $A \in M(2n+2, 2n+2, \mathbb{R})$. Since $(\varphi, \xi, \eta)$ is the induced metric structure, Theorem 4.3 implies that the shape operator $S = -\text{id}$ and $\tau = 0$. So $C := x$ is a $\tilde{J}$-tangent transversal vector field and $\xi = \tilde{J}x$ is tangent. Since $Ax$ is orthogonal (relative to the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n+2}$) to $H$ we have

$$0 = \langle \tilde{J}x, Ax \rangle = x^T \tilde{J}Ax$$

for every $x \in H$. We also have

$$x^T A \tilde{J}x = 0,$$

so in consequence

$$x^T (\tilde{J}A + A\tilde{J})x = 0 \quad (4.17)$$

for every $x \in H$. Since $\tilde{J}A + A\tilde{J}$ is symmetric, formula (4.17) implies that

$$\tilde{J}A = -A\tilde{J}.$$ 

The last formula implies that

$$A = \begin{bmatrix} P & R \\ -R & -P \end{bmatrix},$$
$P^T = P$, $R^T = -R$, $P, R \in M(n + 1, n + 1, \mathbb{R})$.

Now, we shall show that the induced almost paracontact structure is metric. Since $\langle Ax, C \rangle = 2$, it is enough to prove that

$$\langle D_{\tilde{J}Z} \tilde{J}W, Ax \rangle = -\langle D_Z W, Ax \rangle$$

for every $Z, W \in \mathcal{D}$. For $Z, W \in \mathcal{D}$ we have

\begin{align*}
\langle Z, Ax \rangle &= 0, \quad (4.18) \\
\langle \tilde{J}Z, Ax \rangle &= 0, \quad (4.19) \\
\langle W, Ax \rangle &= 0, \quad (4.20) \\
\langle \tilde{J}W, Ax \rangle &= 0. \quad (4.21)
\end{align*}

We also have

$$\langle \tilde{J}X, Y \rangle = \langle X, \tilde{J}Y \rangle$$

(4.22)

for every $X, Y$ tangent to hyperquadric. Using the fact that $DA = 0$ we obtain

$$D_Z Ax = AZ \quad (4.23)$$

and

$$D_{\tilde{J}Z} Ax = A \tilde{J}Z. \quad (4.24)$$

Since $D\langle \cdot, \cdot \rangle = 0$ we also have

$$\langle D_{\tilde{J}Z} \tilde{J}W, Ax \rangle = \tilde{J}W(\langle \tilde{J}Z, Ax \rangle) - \langle \tilde{J}W, D_{\tilde{J}Z} Ax \rangle.$$  

Using (1.19), (4.24) and (4.22) we obtain

$$\langle D_{\tilde{J}Z} \tilde{J}W, Ax \rangle = -\langle \tilde{J}W, A \tilde{J}Z \rangle = \langle \tilde{J}W, \tilde{J}AZ \rangle = \langle W, AZ \rangle$$

On the other hand (4.23), (4.20) and $D\langle \cdot, \cdot \rangle = 0$ imply

$$\langle W, AZ \rangle = \langle W, D_Z Ax \rangle = -\langle D_Z W, Ax \rangle,$$

what completes the proof. \qed

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