Analytical study and numerical simulation of diffusive prey predator model with Holling type II functional response

A Satria¹, A R Putri¹*, M Syafwan¹

¹Department of Mathematics, Andalas University, Padang, Indonesia

*Corresponding author: arrivalputri@gmail.com

Abstract. A prey predator model which consists of two distinct population is discussed. The model used Holling response function of type II without limiting on prey population growth. Equilibrium points of the model was determined and stability of the model was analyzed by phase plane analysis. Furthermore, the model is reformulated by adding a diffusive terms to understand the spatial effect of the dynamical system behaviour. Solutions of the diffusive model were numerically illustrated with Neumann boundary conditions. Numerical simulations are presented to confirm the analytical results.

1. Introduction

The prey predator interaction is fundamental and important process in population dynamics. There are two population known as prey and predator in this interaction. The interaction affects the growth of both population. It shows that prey grows at a certain rate and also die at a certain rate because of predator, otherwise predator dies at a certain rate and it also grows by eating prey [1,2]. Alfred Lotka and Vito Volterra first introduced the prey predator model in the 1920s [4].

Interaction between prey and predator need the specific form of the response function that shows predator population increasing or prey population decreasing. Holling and Tanner has been adding different response functions to prey predator model. Those models are known as the Holling-Tanner model [7]. One of those models is Holling type II. The model describes the rate of predator consumption increases as increasing prey population although it will tend to a certain level and consumption will remain constant in the end.

Spatial effect is a big problem in ecology causing appearance of spatial case. System reaction diffusion is used to explain that phenomena. The diffusion prey predator model has been studied [3,5,6]. We begin by analyzing the prey predator model with the functional response of Holling type II without limiting on prey population growth, then determine equilibrium points of the model and analyzed stability of the model by phase plane analysis. Furthermore, the model is reformulated by adding a diffusive term to understand the spatial effect of the dynamical system behavior.
The paper is organized as follows. The prey predator model with Holling type II is given in section 2. The diffusive model is also given in section 2. In section 3 we have presented stability analysis of the system. In section 4, we have provided numerical solution to confirm the analytical solutions.

2. Mathematical Model of Prey Predator Holling Type II

Prey predator model with holling type II functional response lead to the following system of equations

\[
\begin{align*}
\dot{u}(t) &= au - \frac{buw}{1+mu}, \\
\dot{w}(t) &= -dw + \frac{f_uw}{1+mu},
\end{align*}
\]  

(1)

where \(u\) is prey population, \(w\) is predator population, \(a\) is rate of prey growth, \(d\) is rate of predator death, \(b\) and \(f\) are rate of prey-predator interaction, and \(m\) is rate of predator saturation for hunting prey. Predator cannot continue to grow linearly, but experience saturation at certain levels \(l/m\) [3,5,6]. Thus, prey will grow exponentially over time without predation and predators will decrease exponentially without prey.

Furthermore, the system (1) is reformulated by adding a diffusive terms. The model (1) becomes

\[
\begin{align*}
\dot{u}(t) &= D_1 \frac{\partial^2 u}{\partial x^2} + au - \frac{buw}{1+mu}, \\
\dot{w}(t) &= D_2 \frac{\partial^2 w}{\partial x^2} - dw + \frac{f_uw}{1+mu},
\end{align*}
\]  

(2)

where \(D_1\) and \(D_2\) are positive diffusive constants. Let \(x\) be the one dimensional coordinate variable, \(u = u(x,t)\) and \(w = w(x,t)\) denote the population of prey and predator at position \(x\) and time \(t\), respectively. Solutions of the system (2) are defined in the plane \(\{(x,t)|0 \leq x \leq L, t > 0\}\). For \(t > 0\) and \(0 \leq x \leq L\), the system of equations (2) is bounded with homogeneous Neumann boundary conditions

\[
\frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} = 0,
\]

and initial conditions

\[
\begin{align*}
u(x,0) &= u(0), \\
w(x,0) &= w(0).
\end{align*}
\]

3. Stability Analysis

There are two equilibrium points of the system (1), one is \(E_1 = (0,0)\) and the other one is \(E_2 = (\frac{d}{f - dm}, \frac{af}{bf - bdm})\). Jacobian matrix of the system (1) is

\[
J = \begin{bmatrix}
a - \frac{bw}{(1+mu)^2} & -bu \\
\frac{fw}{(1+mu)^2} & \frac{fu}{1+mu} - d
\end{bmatrix}.
\]  

(3)
Jacobian matrix (3) at equilibrium point at $E_1 = (0,0)$ is

$$J_{E_1} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix},$$

and characteristic equation of matrix $J_{E_1}$ is given by $\det(J_{E_1} - \lambda I) = 0$,

$$\begin{vmatrix} a - \lambda & 0 \\ 0 & -d - \lambda \end{vmatrix} = 0.$$

There are two distinct real eigenvalues, one is positive $\lambda_1 = a$, and the other one is negative $\lambda_2 = -d$. It means that the equilibrium point $E_1$ is a saddle point and unstable. For the other equilibrium point, $E_2 = \left( \frac{d}{f-dm}, \frac{af}{bf-bdm} \right)$, Jacobian matrix (3) is

$$J_{E_2} = \begin{bmatrix} \frac{adm}{f} & -\frac{bd}{f} \\ -\frac{a(dm-f)}{b} & 0 \end{bmatrix}.$$

The characteristic equation for matrix $J_{E_2}$ is given by $\det(J_{E_2} - \lambda I) = 0$

$$\begin{vmatrix} \frac{adm}{f} - \lambda & -\frac{bd}{f} \\ -\frac{a(dm-f)}{b} & -\lambda \end{vmatrix} = 0.$$

There are two distinct eigenvalues, one is $\lambda_1 = \frac{1}{2} adm + \sqrt{ad(adm^2 + 4dfm - 4f^2)}$, and the other one is $\lambda_2 = \frac{1}{2} adm - \sqrt{ad(adm^2 + 4dfm - 4f^2)}$. If $ad(adm^2 + 4dfm - 4f^2) > 0$ then the equilibrium point $E_2$ is an unstable node, and if $ad(adm^2 + 4dfm - 4f^2) < 0$ then the equilibrium point $E_2$ is an unstable focus.

4. Numerical Solutions

Stability analysis of the prey predator model was confirmed numerically by plot solutions and portrait phase in phase plane. Let $a = 1.3$, $b = 0.5$, $d = 0.7$, $f = 1.6$ [2], with initial conditions $u(0) = 0.9$ and $w(0) = 1.8$. Figure 1 and 3 show solutions of the system (1) and (2), respectively. Figure 2 show phase portrait of the system (1) that both the critical points $E_1 = (0,0)$ and $E_2 = \left( \frac{d}{f-dm}, \frac{af}{bf-bdm} \right)$ are unstable for $m = 0.01$ and $m = 0.1$. Figure 4 shows profile of prey and predator in spatial coordinate, respectively.
Figure 1. Solutions of the system (1).

Figure 2. Phase plane the system system (1).
Figure 3. Positive Solutions of the system system (2) for $m = 0.1$.

5. Conclusion
We have discussed prey predator model by considering diffusive terms. Stability of diffusive prey predator model with Holling response function of type II has been analyzed. Stability of the prey predator system depends on parameter of eigenvalues including the value $m$, rate of predator saturation for hunting prey. The equilibrium points are unstable for value of $m = 0.01$ and $m = 0.1$. In the further research, we will analyze traveling wave solutions and the numerical simulations technique of prey predator interactions in one dimension.

Figure 4. Profile of the prey and predator system (2) for $m = 0.1$. 
Acknowledgments
Author would like acknowledge to the Directorate General of Higher Education, Republic of Indonesia for the funding of this research.

References
[1] Kim B 2004 Computing Traveling-Wave Front Solution in a Diffusive Predator-Prey Model. J. of Department of mathematics 23 1-19
[2] Mukhopadhyay B and Bhattachryya R 2006 Modelling the Role of Diffusion Coefficients on Turing Instability in a Reaction-Diffusion Prey-Predator System Bulletin Of Mathematical Biology 68 293-313
[3] Freedman HI 1980 Deterministic Mathematical Models in Population Ecology (New York: Dakker)
[4] Murray JD 1989 Mathematical Biology (New York: Springer-Verlag)
[5] Huang J, Lu G and Ruan S 2003 Existence of Traveling Wave Solution in a Diffusive Predator-Prey Model J. of Math. Bio. 46 132-152
[6] May R 1974 Stability and Complexity in Model Ecosystem (Princeton: Princeton University Press)
[7] Dawes JHP and Souza MO 2013 A derivation of Holling Type I, II, III Functional Response In Predator Prey J.Theo. Bio. 327 11-12