Polar decomposition of regularly varying time series in star-shaped metric spaces

Johan Segers · Yuwei Zhao · Thomas Meinguet

Abstract There exist two ways of defining regular variation of a time series in a star-shaped metric space: either by the distributions of finite stretches of the series or by viewing the whole series as a single random element in a sequence space. The two definitions are shown to be equivalent. The introduction of a norm-like function, called modulus, yields a polar decomposition similar to the one in Euclidean spaces. The angular component of the time series, called angular or spectral tail process, captures all aspects of extremal dependence. The stationarity of the underlying series induces a transformation formula of the spectral tail process under time shifts.

Keywords extremal index · extremogram · regular variation · spectral tail process · stationarity · tail dependence · time series

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1 Introduction

The concept of regular variation plays an important role in the study of heavy-tailed phenomena, which appear in diverse contexts such as financial risk management, telecommunications, and meteorology, to name a few. Traditionally, regular variation has been defined and studied for univariate functions and random variables in $\mathbb{R}$, see for instance Bingham et al (1987) and Resnick (1987) and the references therein. Later on, it has been extended to random vectors and stochastic processes (Resnick, 1986; Hult and Lindskog, 2005; Resnick, 2007). Basrak and Segers (2009) study the polar decomposition of a regularly varying time series and Hult and Lindskog (2006), by introducing the $M_0$-convergence, build a framework to define regular variation for measures on metric spaces endowed with scalar multiplication. Combining results and methods in these two papers, Meinguet and Segers (2010) provide a detailed study of regularly varying time series in Banach spaces. Our aim is to extend and generalize results in the latter concerning two aspects: regular variation of the time series when seen as a single random element in a sequence space and the polar decomposition in star-shaped metric spaces.

Let $X = (X_t)_{t \in \mathbb{Z}}$ be a discrete-time stochastic process taking values in a star-shaped metric space $S$, i.e., a complete, separable metric space equipped with a scalar multiplication (see Section 2). Regular variation of random elements in such spaces has been introduced in Hult and Lindskog (2006), generalizing theory in Kuelbs and Mandrekar (1974) and Mandrekar and Zinn (1980) for regular variation in Hilbert and Banach spaces, respectively; see also Giné et al (1990) and de Haan and Lin (2001) for regular variation of random continuous functions. Regular variation of a time series $X$ can be defined via its finite-dimensional distributions, that is, $(X_{-m}, \ldots, X_m)$ is regularly varying as a random element in $S^{m+1}$ for each $m \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Alternatively, $X$ can be required to be regularly varying as a random element in the sequence space $S^\infty$. In Samorodnitsky and Owada (2013), it is shown that, under mild conditions, these two ways of defining regular variation of a real-valued stochastic process are equivalent. As one of the paper’s aims, the equivalence is shown for $X$ taking values in a general metric space.

The polar decomposition of stationary regularly varying time series in Euclidean spaces is introduced by Basrak and Segers (2009) and generalized to Banach spaces by Meinguet and Segers (2010). Let $\mathcal{B}$ be a Banach space equipped with a norm $\| \cdot \|$. Regular variation of a $\mathcal{B}$-valued stationary time series $X$ is equivalent to the existence of the limit in distribution of

\[
\left( \frac{\|X_0\|}{u}, (X_t/\|X_0\|)_{t \in \mathbb{Z}} \right) \text{ conditionally on } \|X_0\| > u \text{ as } u \to \infty,
\]

where the limit of $\|X_0\|/u$ given $\|X_0\| > u$ is assumed to be non-degenerate. This leads to a natural decomposition of the limit process into independent modular and angular components. The modular component, the limit in distribution of $\|X_0\|/u$ given $\|X_0\| > u$ as $u \to \infty$, is fully determined by the index of regular variation, $\alpha$, of the random variable $\|X_0\|$, while the angular component, the limit in distribution of $(X_t/\|X_0\|)_{t \in \mathbb{Z}}$ given $\|X_0\| > u$, captures all aspects of extremal dependence. The angular component is called spectral tail process. Stationarity of $X$ induces a
transformation formula for the spectral tail process under time shifts. The spectral
tail process provides a single apparatus to describe a large variety of objects de-
scribing extremal dependence: the extremal index (Leadbetter, 1983), the cluster in-
dex (Mikosch and Wintenberger, 2014), the extremogram (Davis and Mikosch, 2009;
Davis et al, 2013), limits of sums or maxima (Basrak et al, 2012; Meinguet, 2012),
and Markov tail chains (Janssen and Segers, 2014; Drees et al, 2015).

A general metric space may not possess a norm. However, an alternative function
possessing some key properties of a norm, named modulus, might still exist. If this is
the case, then the above polar decomposition still goes through, and the time-change
formula for the spectral tail process is shown to be preserved.

The structure of the paper is as follows. The conditions on the metric space and the
definition of a modulus function are introduced in Section 2. In Section 3, the polar
decomposition of a regularly varying random element in a metric space is studied.
Regular variation of a time series seen as a random element in a sequence space
is investigated in Section 4. Results on the spectral tail process and on the time-
change formula are given in Sections 5 and 6, respectively. Section 7 provides some
brief discussion in connection to hidden regular variation and Appendix A contains
auxiliary results on convergence of measures.

2 Star-shaped metric spaces

Let \((S,d)\) be a complete, separable metric space and let \(0_S \in S\) be a point in \(S\) called
‘origin’. (To avoid trivialities, assume that \(S\) is not equal to \(\{0_S\}\).) To define regular
variation of measures on the metric space \(S\), Hult and Lindskog (2006) assume that
\(S\) is equipped with a scalar multiplication. The following is a formal definition of such a
multiplication. In the cited paper, conditions (i) and (ii) are not mentioned explicitly.

Definition 2.1 A scalar multiplication on \(S\) is a map \([0, \infty) \times S \to S : (\lambda, x) \to \lambda x\)
satisfying the following properties:

(i) \(\lambda_1 (\lambda_2 x) = (\lambda_1 \lambda_2) x\) for all \(\lambda_1, \lambda_2 \in [0, \infty)\) and \(x \in S\);
(ii) \(1 x = x\) for \(x \in S\);
(iii) the map is continuous with respect to the product topology;
(iv) if \(x \in S_0 = S \setminus \{0_S\}\) and if \(0 \leq \lambda_1 < \lambda_2\), then \(d(\lambda_1 x, 0_S) < d(\lambda_2 x, 0_S)\).

Let \(x \in S_0\). For any \(\lambda \in [0, \infty)\), we have \(\lambda (0x) = (\lambda 0)x = 0x\) by (i) in Definition 2.1. It follows that \(d(\lambda_1 (0x), 0_S) = d(0x, 0_S) = d(\lambda_2 (0x), 0_S)\) for all \(\lambda_1, \lambda_2 \in [0, \infty)\). By (iv), it can therefore not be true that \(0x \in S_0\). We find that \(0x = 0_S\) for all \(x \in S\). In addition, we necessarily have \(\lambda 0_S = 0_S\) for all \(\lambda \in [0, \infty)\); indeed, by the
property just established, we have \(\lambda 0_S = \lambda (00_S) = (\lambda 0)0_S = 00_S = 0_S\).

We think of \(S\) as ‘star-shaped’ with rays emanating from the origin. Alternatively,
think of \(S\) as a kind of cone. We will sometimes write \(x/\lambda := \lambda^{-1} x\) for \(\lambda > 0\) and
\(x \in S\).

The distance function \(x \mapsto d(x, 0_S)\) need not be homogeneous. This will be important in Section 4
where we will consider metrics on sequence spaces inducing the
product topology. To decompose a point in $S_0$ in a ‘modular’ component and an ‘angular’ component, a modulus function needs to be present. The following definition captures the properties needed on such a function.

**Definition 2.2** A function $\rho : S \to [0, \infty)$ is a modulus if it satisfies the following properties:

(i) $\rho$ is continuous;
(ii) $\rho$ is homogeneous: $\rho(\lambda x) = \lambda \rho(x)$ for $\lambda \in [0, \infty)$ and $x \in S$;
(iii) for every $\varepsilon > 0$, we have $\inf\{\rho(x) : d(x,0_\mathcal{S}) > \varepsilon\} > 0$.

Since $0_\mathcal{S} = 0_\delta$ for all $\delta \in [0, \infty)$, homogeneity implies $\rho(0_\delta) = 0$. The third condition on the modulus $\rho$ will be needed to ensure that every subset of $S_0$ which is bounded away from the origin is included in a set of the form $\{x : \rho(x) \geq \delta\}$ for some $\delta > 0$. In particular, $\rho(x) > 0$ for $x \neq 0_\mathcal{S}$. Therefore, $x = 0_\mathcal{S}$ if and only if $\rho(x) = 0$. Furthermore, the third condition implies that there exist positive scalars $(\varepsilon_r)_{r > 0}$ such that $\lim_{r \to 0} \varepsilon_r = 0$ and $\{x : \rho(x) < r\} \subset \{x : d(x,0_\mathcal{S}) < \varepsilon_r\}$ for every $r > 0$. Since $\rho(0_\mathcal{S}) = 0$ and since $\rho$ is continuous, the collection of sets $\{x : \rho(x) < r\}$, for $r > 0$, therefore forms an open neighbourhood base for $0_\mathcal{S} \in S$.

We think of $\rho(x)$ as the ‘modulus’ of $x$. We further define the ‘angle’ of $x \in S_0$ as $\theta(x) = (\rho(x))^{-1} x$. Note that $\rho(\theta(x)) = 1$, that is, $\theta(x) \in \{\theta : \rho(\theta) = 1\} =: \mathbb{R}$, the ‘unit sphere’ of $S$. Clearly, $x = \rho(x) \theta(x)$ for $x \in S_0$. The map $T : S_0 \to (0, \infty) \times \mathbb{R} : x \mapsto T(x) = (\rho(x), \theta(x))$ is the polar decomposition.

**Example 2.1** In case the function $x \mapsto d(x,0_\mathcal{S})$ is itself homogeneous, it is a modulus as in Definition 2.2. This is the case, for instance, if $S$ is a Banach space and the distance is the one induced by the norm, which brings us back to the set-up in Meinguet and Segers (2010). Another example is the Skorohod space $D = D([0, 1], \mathbb{R}^d)$ of càdlàg functions $[0, 1] \to \mathbb{R}^d$ equipped with the $J_1$-metric: in that case, the zero element $0_D$ is the zero function, and the Skorohod distance of $x \in D$ to $0_D$ is given by $d(x, 0_D) = \sup_{t \in [0, 1]} ||x(t)||$. Regular variation of $D$-valued random elements was considered in Hult and Lindskog (2005).

**Example 2.2** Assume that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\{x : d(\delta^{-1} x, 0_\mathcal{S}) \leq 1\} \subset \{x : d(x,0_\mathcal{S}) \leq \varepsilon\}$. Then it can be shown that the map $\rho : S \to [0, \infty)$ defined by

$$\rho(x) = \begin{cases} \inf\{\lambda \in (0, \infty) : d(\lambda^{-1} x, 0_\mathcal{S}) \leq 1\} & \text{if } x \neq 0_\mathcal{S}, \\ 0 & \text{if } x = 0_\mathcal{S}. \end{cases}$$

is a modulus as in Definition 2.2. Intuitively, the condition on the metric $d$ is that scalar multiplication increases distances to the origin in a uniform way.

**Example 2.3** Let $\mathbb{D}$ be a nonempty compact subset of some Euclidean space and let $S = \text{USC}_+(\mathbb{D})$ be the space of upper semicontinuous functions $x : \mathbb{D} \to [0, \infty)$. Each such function $x$ is identified with its hypograph, i.e., the set $\text{hypo} x = \{(\alpha, s) \in \mathbb{R} \times \mathbb{D} : \alpha \leq x(s)\}$, a closed subset of $\mathbb{R} \times \mathbb{D}$. The hypo-topology on $\text{USC}_+(\mathbb{D})$ is
the one induced by the Fell hit-and-miss topology on the space of closed subsets of $\mathbb{R} \times \mathcal{D}$; see [Molchanov 2005, Section 5.3] for the dual theory of epi-convergence of lower semicontinuous functions. The map $\rho(x) = \sup_{x \in \mathcal{B}} x(s)$, for $x \in \text{USC}_+(\mathcal{D})$, then defines a modulus on $\text{USC}_+(\mathcal{D})$.

If $S$ is locally compact, then condition (iii) in Definition 2.2 may be relaxed to the seemingly weaker assumption that $\rho(x) > 0$ for all $x \neq 0_S$. In general, however, the latter condition does not imply (iii); see Example 2.4. See also Section 7 for a discussion on condition (iii).

Example 2.4 Let $\mathbb{H}$ be a separable, infinite-dimensional Hilbert space, the metric being the one induced by the scalar product. Let $e_1, e_2, \ldots$ be an orthonormal basis in $\mathbb{H}$, and define $\rho(x) = \left(\sum_{i \geq 1} \lambda_i |\langle x, e_i \rangle|^2 \right)^{1/2}$, where $(\lambda_i)_{i \geq 1}$ is a positive scalar sequence such that $\lambda_i \to 0$ as $i \to \infty$. Then $\rho$ satisfies conditions (i) and (ii) in Definition 2.2 and $\rho(x) > 0$ as soon as $x \neq 0_\mathbb{H}$. Still, condition (iii) in Definition 2.2 is not satisfied, since $\rho(e_i) \to 0$ as $i \to \infty$ while $d(e_i, 0_\mathbb{H}) = 1$ for every $i \geq 1$.

3 Regular variation and the polar decomposition

Let $(S, d)$ be a complete, separable metric space equipped with an origin $0_S \in S$ and a scalar multiplication (Definition 2.1). Let $\mathcal{B}(S)$ denote the Borel $\sigma$-field on $S$ and let $M_0(S)$ be the space of Borel measures on $S_0 = S \setminus \{0_S\}$ that are bounded on complements of neighbourhoods of the origin. Let $C_0$ denote the collection of bounded and continuous functions $f : S_0 \to \mathbb{R}$ for which there exists $r > 0$ such that $f$ vanishes on $B_{0,r} = \{x \in S : d(x, 0_S) < r\}$. The convergence of measures $\mu_n \to \mu$ in $M_0(S)$ holds as said in [Hult and Lindskog 2006] if and only if $\int f d\mu_n \to \int f d\mu$ for all $f \in C_0$. Versions of the Portmanteau and continuous mapping theorems for $M_0$-convergence are stated as Theorems 2.4 and 2.5, respectively, in [Hult and Lindskog 2006].

For $\tau \in \mathbb{R}$, let $\mathcal{R}_\tau$ denote the class of regularly varying functions at infinity with index $\tau$, i.e., positive, measurable functions $g$ defined in a neighbourhood of infinity such that $\lim_{u \to \infty} g(\lambda u) / g(u) = \lambda^\tau$ for every $\lambda \in (0, \infty)$.

Definition 3.1 ([Hult and Lindskog 2006]) A random element $X$ in $S$ is regularly varying with index $\alpha \in (0, \infty)$ if and only if there exists a function $V \in \mathcal{R}_{-\alpha}$ and a nonzero measure $\mu \in M_0(S)$ such that

$$
\frac{1}{\mu(V)} \Pr[u^{-1}X \in \cdot] \to \mu(\cdot), \quad u \to \infty.
$$

The measure $\mu$ must be homogeneous: $\mu(\lambda \cdot) = \lambda^{-\alpha} \mu(\cdot)$ for every $\lambda \in (0, \infty)$ [Hult and Lindskog 2006, Theorem 3.1].

Let $\rho$ be a modulus on $S$ (Definition 2.2). Our aim is now to extend the present set-up the familiar decomposition of a regularly varying random vector into a regularly varying ‘modulus’ and an asymptotically independent ‘angle’. First, we need a preliminary result linking the auxiliary function $V$ to the tail function $u \mapsto \Pr[\rho(X) > u]$. 

Lemma 3.1 Let $X$ be a regularly varying random element in $S$ with index $\alpha$ and limit measure $\mu$. If $\rho$ is a modulus on $S$, then $\mu(\{x \in S : \rho(x) = \lambda\}) = 0$ for every $\lambda \in (0, \infty)$ and

$$\lim_{u \to \infty} \frac{1}{V(u)} \Pr[\rho(X) > u] = \mu(\{x \in S : \rho(x) > 1\}) \in (0, \infty).$$

Proof We have $\{x : \rho(x) = \lambda\} = \{\lambda x : \rho(x) = 1\}$ for $\lambda \in (0, \infty)$. The set $\{x : \rho(x) = \lambda\}$ is closed due to the continuity of $\rho$ and does not contain the origin. Hence $\mu(\{x : \rho(x) = \lambda\}) = \lambda^{-\alpha} \mu(\{x : \rho(x) = 1\})$ must be finite. Since the sets $\{x : \rho(x) = \lambda\}$ are disjoint for different $\lambda$, we conclude that $\mu(\{x : \rho(x) = \lambda\}) = 0$ for all $\lambda \in (0, \infty)$.

The set $\{x : \rho(x) = 1\}$ is the boundary of the open set of $\{x : \rho(x) > 1\}$. The latter is thus a $\mu$-continuity set, and its closure, $\{x : \rho(x) \geq 1\}$, does not contain the origin, so that $\mu(\{x : \rho(x) \geq 1\}) < \infty$. We obtain

$$\frac{1}{V(u)} \Pr[\rho(X) > u] = \frac{1}{V(u)} \Pr[\rho(u^{-1}X) > 1] \to \mu(\{x : \rho(x) > 1\}) = \mu(\{x : \rho(x) \geq 1\}), \quad u \to \infty.$$ 

The latter quantity must be nonzero: indeed, $\mu$ is nonzero and we have $S_0 = \bigcup_{k \geq 1} \{x : \rho(x) > k^{-1}\}$ and $\mu(\{x : \rho(x) > k^{-1}\}) = \mu(k^{-1} \{x : \rho(x) > 1\}) = k^{\alpha} \mu(\{x : \rho(x) > 1\})$.

Let the arrow $\Rightarrow$ denote convergence in distribution, and let $\mathcal{L}(Y \mid A)$ denote the law of a random object $Y$ conditionally on an event $A$. For $\alpha > 0$, let $\text{Pareto}(\alpha)$ denote the probability distribution of a random variable $Y$ such that $\Pr(Y > y) = y^{-\alpha}$ for $y \in [1, \infty)$. Recall $T(x) = (\rho(x), \theta(x))$ with $\theta(x) = \rho(x)^{-1}x$ for $x \in S_0$ and recall $\mathcal{R} = \{x \in S : \rho(x) = 1\}$. Let $\otimes$ signify product measure and let $1_B$ denote the indicator function of a set $B$.

Proposition 3.1 Let $X$ be a random element in $S$ and let $\alpha \in (0, \infty)$. Assume that a modulus $\rho : S \to [0, \infty)$ exists. The following properties are equivalent:

(i) $X$ is regularly varying with index $\alpha > 0$.

(ii) The function $u \mapsto \Pr[\rho(X) > u]$ is in $\mathcal{R}_{-\alpha}$ and there exists a probability measure $H$ on $\mathcal{R} = \{x \in S : \rho(x) = 1\}$ such that

$$\mathcal{L}(\Theta(X) \mid \rho(X) > u) \Rightarrow H, \quad u \to \infty, \quad (1)$$

(iii) There exists a probability measure $H$ on $\mathcal{R}$ such that

$$\mathcal{L}(\rho(X)/u, \Theta(X) \mid \rho(X) > u) \Rightarrow \text{Pareto}(\alpha) \otimes H, \quad u \to \infty. \quad (2)$$

In that case, we have

$$\frac{1}{\Pr[\rho(X) > u]} \Pr[u^{-1}X \in \cdot] \to \mu, \quad u \to \infty, \quad (3)$$

where $\mu$ is determined by

$$\mu \circ T^{-1}(dr, d\theta) = \alpha r^{-\alpha - 1} dr H(d\theta), \quad (r, \theta) \in (0, \infty) \times \mathcal{R}. \quad (4)$$
In terms of integrals, equation (4) means that, for \( \mu \)-integrable functions \( f : S_0 \to \mathbb{R} \), we have

\[
\int_{S_0} f(x) d\mu(x) = \int_0^\infty \int_{\theta \in \mathbb{R}} f(r\theta) dH(\theta) \alpha r^{-\alpha - 1} dr.
\]

Proposition 3.1 is to be compared with Corollary 4.4 in Lindskog et al (2014). In the latter paper, the set excluded from the space \( S \) is not necessarily just a single point but is allowed to be a closed cone. In contrast, the metric in Lindskog et al (2014) is supposed to be homogeneous, as in our Example 2.1 an assumption that we avoid here.

**Proof (Proof of Proposition 3.1)** We break down the equivalence claim into a number of implications.

(i) implies (ii) and (iii). Let \( \tilde{V} \) and \( \tilde{\mu} \) be the auxiliary function and the limit measure in the definition of regular variation of \( X \). By Lemma 3.1, we have \( \Pr[\rho(X) > u]/\tilde{V}(u) \to \tilde{\mu}((x : \rho(x) > 1)) \) as \( u \to \infty \), the limit being finite and nonzero. Hence, the function \( V(u) := \Pr[\rho(X) > u] \) is a valid auxiliary function for \( X \) too. With this choice, the limit measure is then just a rescaled version of the old one: \( \mu(\cdot) = \tilde{\mu}(\cdot)/\tilde{\mu}((x : \rho(x) > 1)) \). In particular, \( \mu((x : \rho(x) > 1)) = 1 \).

Define a Borel measure \( H \) on \( \mathbb{R} \) by

\[
H(\cdot) = \mu((x : \rho(x) > 1, \theta(x) \in \cdot)).
\]

By construction, \( H(\mathbb{R}) = 1 \), i.e., \( H \) is a probability measure.

For \( r \in (0, \infty) \) and Borel sets \( B \subset \mathbb{R} \), we have

\[
\mu \circ T^{-1}((r, \infty) \times B) = \mu(B - r, \theta(x) \in B)) \]
\[
= \mu(r \{x : \rho(x) > 1, \theta(x) \in B\}) \]
\[
= r^{-\alpha} \mu(B - r, \theta(x) \in B)) \]
\[
= r^{-\alpha} H(B).
\]

Since the collection of sets of the form \( \{(r, \infty) \times B : r \in (0, \infty), B \in \mathcal{B}(\mathbb{R})\} \) is a \( \pi \)-system generating the Borel \( \sigma \)-field on \( (0, \infty) \times \mathbb{R} \), we find (4).

We prove (1). For \( G \subset \mathbb{R} \) open, we have

\[
\liminf_{u \to \infty} \Pr[\theta(X) \in G \mid \rho(X) > u] = \liminf_{u \to \infty} \frac{\Pr[u^{-1}X \in T^{-1}((1, \infty) \times G)]}{\Pr[\rho(X) > u]} \]
\[
\geq \mu \circ T^{-1}((1, \infty) \times G) = H(G).
\]

The inequality on the second line follows from the Portmanteau theorem for \( M_0 \) convergence (Hult and Lindskog 2006, Theorem 2.4); indeed, the set \( T^{-1}((1, \infty) \times G) \) is open in \( S \) and its closure does not contain the origin. The fact that the above display implies (1) follows from the Portmanteau theorem for weak convergence of probability measures on metric spaces (Billingsley 1999, Theorem 2.1).

Further, for \( \lambda \in [1, \infty) \), we have

\[
\Pr[\rho(X)/u > \lambda \mid \rho(X) > u] = \frac{V(\lambda u)}{V(u)} \to \lambda^{-\alpha}, \quad u \to \infty. \tag{5}
\]
It follows that \( \mathcal{L}(\rho(X)/u \mid \rho(X) > u) \rightarrow \text{Pareto}(\alpha) \) as \( u \rightarrow \infty \).

By (1) and (5), it follows that the distributions \( \mathcal{L}(\rho(X)/u, \theta(X) \mid \rho(X) > u) \) are asymptotically tight as \( u \rightarrow \infty \). It remains to show that there is only a single accumulation point.

Let \( B \in \mathcal{B}(\mathbb{R}) \) be a \( H \)-continuity set and let \( I \) be the open interval \((\lambda_1, \lambda_2)\) with \( 1 \leq \lambda_1 < \lambda_2 < \infty \). The set \( T^{-1}(I \times B) \subset S_0 \) is bounded away from the origin and is a continuity set with respect to \( \mu \). It follows that

\[
Pr[\rho(X)/u \in I, \theta(X) \in B \mid \rho(X) > u] = \frac{1}{V(u)} Pr[u^{-1}X \in T^{-1}(I \times B)]
\rightarrow \mu \circ T^{-1}(I \times B) = (\lambda_1^{-\alpha} - \lambda_2^{-\alpha}) H(B) = (\text{Pareto}(\alpha) \otimes H)(I \times B)
\]
as \( u \rightarrow \infty \). This fixes the value of \( L(I \times B) \) for any law \( L \) that can arise as the limit in distribution of a sequence \([\rho(X)/u_n, \theta(X) \mid \rho(X) > u_n]\) where \( u_n \rightarrow \infty \) as \( n \rightarrow \infty \). The collection of such sets \( I \times B \) forms a \( \pi \)-system generating \( \mathcal{B}((1, \infty) \times \mathbb{R}) \). (Use the Lindelöf property to write every open subset of the separable metric space \((1, \infty) \times \mathbb{R}\) as a countable union of sets of the form \( I \times B \), with \( B \) an open ball in \( \mathbb{R} \) whose boundary is an \( H \)-null set.) It follows that all sequences \([\rho(X)/u_n, \theta(X) \mid \rho(X) > u_n]\) converge in distribution to the same limit.

(iii) implies (ii). Convergence in distribution (1) is a consequence of convergence in distribution (2) and the continuous mapping theorem. Moreover, for \( \lambda \geq 1 \), we have

\[
Pr[\rho(X) > \lambda u \mid \rho(X) > u] = Pr[\rho(X)/u > \lambda \mid \rho(X) > u] 
\rightarrow \lambda^{-\alpha} \text{ as } u \rightarrow \infty.
\]
It follows that the function \( u \mapsto Pr[\rho(X) > u] \) belongs to \( \mathcal{B}_{-\alpha} \).

(ii) implies (i). Define a measure \( \mu \) on \( S_0 \) by

\[
\mu(B) = \int_{r=0}^{\infty} \int_{\theta \in \mathbb{R}} 1_B(r\theta) dH(\theta) \alpha r^{-\alpha-1} \, dr,
\]
where \( \alpha \) is the push-forward of the product measure \( \alpha r^{-\alpha-1} dr dH(\theta) \) on \((0, \infty) \times \mathbb{R}\) induced by the map \((0, \infty) \times \mathbb{R} \rightarrow S_0 : (r, \theta) \mapsto r\theta\).

The measure \( \mu \) is finite on complements of neighbourhoods of the origin. Indeed, let \( \epsilon > 0 \). By assumption, there exists \( \delta > 0 \) such that \( d(x, 0) > \epsilon \) implies that \( \rho(x) > \delta \). Therefore, \( \{x : d(x, 0) > \epsilon\} \subset \{x : \rho(x) > \delta\} \). The \( \mu \)-measure of the latter set is equal to \( \delta^{-\alpha} \), and thus finite.

We show that (3) holds. Let \( B \in \mathcal{B}(\mathbb{R}) \) be a \( H \)-continuity set, i.e., \( H(\partial B) = 0 \), where \( \partial B \) denotes the topological boundary of \( B \) in \( \mathbb{R} \). Let \( 0 < \lambda < \infty \). Put \( V(u) = Pr[\rho(X) > u] \). By the Portmanteau theorem for weak convergence of probability measures,

\[
\frac{1}{V(u)} Pr[u^{-1}X \in \{x : \rho(x) > \lambda, \theta(x) \in B\}]
= \frac{V(\lambda u)}{V(u)} Pr[\rho(X)^{-1}X \in B \mid \rho(X) > \lambda u]
\rightarrow \lambda^{-\alpha} H(B) = \mu(\{x : \rho(x) > \lambda, \theta(x) \in B\}), \quad u \rightarrow \infty.
\]
Since the limit is continuous in \( \lambda \) and since \( \{x : \rho(x) > \lambda\} \subset \{x : \rho(x) > \lambda\} \subset \{x : \rho(x) > (1 - \epsilon)\lambda\} \) for every \( \epsilon \in (0, 1) \), we find that the above display remains
valid if we replace \( \rho(x) > \lambda \) by \( \rho(x) \geq \lambda \). It follows that, for every open interval \( I = (\lambda_1, \lambda_2) \) with \( 0 < \lambda_1 < \lambda_2 < \infty \) and for each \( H \)-continuity set \( B \in \mathcal{B}(\mathbb{R}) \), we have

\[
\frac{1}{V(u)} \Pr[u^{-1}X \in \{x : \rho(x) \in I, \theta(x) \in B\}] 
\rightarrow \mu(\{x : \rho(x) \in I, \theta(x) \in B\}), \quad u \rightarrow \infty. \tag{6}
\]

Let \( G \subset S \) be open and such that \( 0_S \notin G^- \). The set \( \{(r, \theta) \in (0, \infty) \times \mathbb{R} : r \theta \in G\} \) is open by continuity of the scalar multiplication map. For every \( x \in G \), there exists \( 0 < \varepsilon < \rho(x) \) such that the set \( (\rho(x) - \varepsilon, \rho(x) + \varepsilon) \times \{\theta \in \mathbb{R} : d(\theta, \theta(x)) < \varepsilon\} \) is a subset of \( \{(r, \theta) : r \theta \in G\} \). By decreasing \( \varepsilon \) if needed, we can ensure that the ball \( \{\theta \in \mathbb{R} : d(\theta, \theta(x)) < \varepsilon\} \) is a \( H \)-discontinuity set. Let \( \varepsilon(x) \) denote the value of \( \varepsilon \) thus obtained, depending on \( x \in G \).

The sets \( A(x) = \{y \in S : |\rho(y) - \rho(x)| < \varepsilon(x), d(\theta(y), \theta(x)) < \varepsilon(x)\} \), for \( x \in G \), are open subsets of \( G \) and they cover \( G \) as \( x \) ranges over \( G \). By the Lindelöf property, there exists a countable subcover of \( G \) by sets \( A(x_i) \), say. Finite intersections of the sets \( A(x_i) \) are of the form \( \{y : \rho(y) \in I, \theta(x) \in B\} \), where \( I \) is an open interval of \((0, \infty)\), bounded away from 0 and \( \infty \), and \( B \) is an \( H \)-continuity subset of \( \mathbb{R} \).

Fix \( \eta > 0 \). Since \( \mu(G) < \infty \), we can find a finite number \( k \) such that \( \mu(G) \leq \mu(\bigcup_{i=1}^k A(x_i)) + \eta \). Write \( \mu_G(\cdot) = V(u)^{-1} \Pr[u^{-1}X \in \cdot] \). By (6) and the inclusion-exclusion formula, \( \mu_G(\bigcup_{i=1}^k A(x_i)) \rightarrow \mu(\bigcup_{i=1}^k A(x_i)) \) as \( u \rightarrow \infty \). But then we have \( \liminf_{u \rightarrow \infty} \mu_G(G) \geq \mu(G) - \eta \). Since \( \eta > 0 \) was arbitrary, we can conclude that \( \liminf_{u \rightarrow \infty} \mu_G(G) \geq \mu(G) \).

Finally, let \( F \subset S \) be closed and such that \( 0_S \notin F \). Since the complement of \( F \) is open, there exists \( \varepsilon > 0 \) such that \( d(x, 0_S) \leq \varepsilon \) implies \( x \notin F \). Further, there exists \( \delta > 0 \) such that \( \rho(x) \leq \delta \) implies \( d(x, 0) \leq \varepsilon \). As a consequence, \( F \subset \{x : \rho(x) > \delta\} \). Define \( G = \{x : \rho(x) > \delta\} \setminus F \). Then \( G \) is open and \( 0_S \notin G^- \). From the previous paragraph, recall \( \liminf_{u \rightarrow \infty} \mu_G(G) \geq \mu(G) \). Further, \( \mu_G(G) = V(\delta u)^{-1}(V(u) - \mu_G(F)) \) and \( \mu(G) = \delta^{-1} - \mu(F) \). It follows that \( \limsup_{u \rightarrow \infty} \mu_G(G) \leq \mu(F) \). Conclude by the Portmanteau Theorem 2.4 in [Hult and Lindskog (2006)].

4 Regularly varying time series

Recall that \((S, d)\) is a complete, separable metric space equipped with an origin and a scalar multiplication. In this section, no modulus will be needed. For simplicity, let from now on the origin of \( S \) be denoted simply by \( 0 \) rather than by \( 0_S \).

Let \( S^d \) be the space of all sequences \((x_t)_{t \in \mathbb{Z}}\) with elements in \( S \). For nonnegative integer \( m \), identify the set \( S^{[-m, m]} \) with the set \( S^{2m+1} \), so that we can write \((x_{-m}, \ldots, x_m) \in S^{2m+1} \). The sets \( S^d \) and \( S^{2m+1} \) are endowed with the respective product topologies, and these topologies can be metrized by the metrics \( d_m \) and \( d_m \), respectively, where

\[
d_m(x, y) = \sum_{t \in \mathbb{Z}} 2^{-|t|} \frac{d(x_t, y_t)}{1 + d(x_t, y_t)}, \quad x, y \in S^d, \\
d_m(x, y) = \sum_{t = -m}^m 2^{-|t|} \frac{d(x_t, y_t)}{1 + d(x_t, y_t)}, \quad x, y \in S^{2m+1}.
\]
The metric spaces \((S^Z, d_u)\) and \((S^{2m+1}, d_m)\) are complete and separable too. The 
precise choice of the metrics is not essential, and we could have chosen equivalent ones, 
replacing \(d(x_t, y_t)/(1 + d(x_t, y_t))\) by \(\min\{d(x_t, y_t), 1\}\), for instance. 

Let \(0 \in S^Z\) be the zero sequence and let \(0^{(m)} = (0, \ldots, 0) \in S^{2m+1}\). These are 
the origins of the spaces \(S^Z\) and \(S^{2m+1}\), respectively. Scalar multiplication on these 
spaces is defined componentwise.

Let \(X = (X_t)_{t \in \mathbb{Z}}\) be a discrete-time stochastic process, not necessarily stationary, 
taking values in \(S\). There are two ways of defining regular variation of \(X\): either 
by viewing \(X\) as a random element of \(S^Z\) or via its finite-dimensional distributions.

According to the following theorem, these two definitions are essentially equivalent. 
For integer \(m \geq 0\), write \(X^{(m)} = (X_{-m}, \ldots, X_m)\), a random element in \(S^{2m+1}\).

**Theorem 4.1** Let \((S, d)\) be a complete, separable metric space equipped with an 
origin \(0 \in S\) and a scalar multiplication. Let \(X = (X_t)_{t \in \mathbb{Z}}\) be a stochastic process in 
\(S\). Let \(\alpha \in (0, \infty)\) and \(V \in \mathbb{R}_{-\alpha}\). The following two statements are equivalent:

(a) There exists \(\mu^{(\infty)} \in M_0(S^\infty)\) such that \(\mu^{(\infty)}(\{x : x_0 \neq 0\}) > 0\) and such that, as \(u \to \infty\),

\[
\frac{1}{V(u)} \Pr[u^{-1} X \in \cdot] \to \mu^{(\infty)}(\cdot) \quad \text{in } M_0(S^\infty).
\] (7)

(b) For each nonnegative integer \(m\), there exists a non-zero \(\mu^{(m)} \in M_0^{(m)}\) such that, as \(u \to \infty\),

\[
\frac{1}{V(u)} \Pr[u^{-1} X^{(m)} \in \cdot] \to \mu^{(m)}(\cdot) \quad \text{in } M_0^{(m)}(S^{2m+1}).
\] (8)

If \((X_t)_{t \in \mathbb{Z}}\) is strictly stationary, the condition \(\mu^{(\infty)}(\{x : x_0 \neq 0\}) > 0\) in (a) is equiv-
alent to the condition that \(\mu^{(\infty)}\) is non-zero.

In case the two equivalent conditions of Theorem 4.1 hold, we say that the stochastic 
process \((X_t)_{t \in \mathbb{Z}}\) is regularly varying.

**Proof** (Proof of Theorem 4.1) Regarding the last statement: if \((X_t)_{t \in \mathbb{Z}}\) is strictly sta-
tionary, then the value of \(\mu^{(\infty)}(\{x : x_t \neq 0\})\) does not depend on \(t \in \mathbb{Z}\), and since 
\(S^Z \setminus \{0\} = \bigcup_{t \in \mathbb{Z}}\{x : x_t \neq 0\}\), we find that \(\mu^{(\infty)}(\{x : x_0 \neq 0\}) > 0\) if and only if \(\mu^{(\infty)}\)
is non-zero.

For integer \(n \geq m \geq 0\), define the projections \(Q_m : S^Z \to S^{2m+1}\) and \(Q_{n,m} : S^{2n+1} \to S^{2m+1}\) by

\[
Q_m(x) = (x_{-m}, \ldots, x_m), \quad x \in S^Z, \\
Q_{n,m}(x_{-n}, \ldots, x_n) = (x_{-m}, \ldots, x_m), \quad (x_{-n}, \ldots, x_n) \in S^{2n+1}.
\]

Let \(Q_m^{-1}\) and \(Q_{n,m}^{-1}\) denote the usual inverse images, inducing maps from the Borel 
\(\sigma\)-field \(\mathcal{B}(S^{2m+1})\) to the Borel \(\sigma\)-fields \(\mathcal{B}(S^Z)\) and \(\mathcal{B}(S^{2n+1})\), respectively. The pro-
jections are continuous and we have \(Q_m(0) = Q_{n,m}(0^{(m)}) = 0^{(m)}\).
Further, for scalar \( u > 0 \) and integer \( m \geq 0 \), define the measures

\[
\begin{align*}
\nu_u^{(m)}(\cdot) &= \frac{1}{V(u)} \Pr[\mu^{-1} X^{(m)} \in \cdot], \\
\nu_u^{(\infty)}(\cdot) &= \frac{1}{V(u)} \Pr[\mu^{-1} X \in \cdot].
\end{align*}
\]

For integer \( n \geq m \geq 0 \), we have \( Q_{n,m}(X^{(n)}) = Q_m(X) = X^{(m)} \). We obtain

\[
\nu_u^{(n)} \circ Q_{n,m}^{-1} = \nu_u^{(\infty)} \circ Q_m^{-1} = \nu_u^{(m)}.
\]

(a) implies (b). By assumption, \( \nu_u^{(\infty)} \to \mu^{(\infty)} \) as \( u \to \infty \) in \( M_0(S^\infty) \). Theorem 2.5 in [Hult and Lindskog (2006)] yields

\[
\nu_u^{(m)} = \nu_u^{(\infty)} \circ Q_m^{-1} \to \mu^{(\infty)} \circ Q_m^{-1} = \mu^{(m)}, \quad \text{as } u \to \infty
\]

in \( M_{Q_m}(S^{2m+1}) \). The measure \( \mu^{(m)} \) is non-zero, since \( \mu^{(m)}(\{ x : x_0 \neq 0 \}) = \mu^{(n)}(\{ x : x_0 = 0 \}) > 0 \).

(b) implies (a). Since \( \nu_u^{(n)} \circ Q_{n,m}^{-1} = \nu_u^{(m)} \) for integer \( n \geq m \geq 0 \), we have, letting \( u \to \infty \) and using again [Hult and Lindskog (2006) Theorem 2.5],

\[
\mu^{(n)} \circ Q_{n,m}^{-1} = \mu^{(m)}.
\]

This self-consistency property of the measures \( \mu^{(m)} \) suggests the use of the Daniell–Kolmogorov extension theorem to construct a Borel measure \( \mu^{(\infty)} \) on \( S^\infty \) such that \( \mu^{(n)} \circ Q_m^{-1} = \mu^{(m)} \). Care is needed, however, since the measures \( \mu^{(m)} \) are finite only on complements of neighbourhoods of \( \{0\} \) in \( S^{2m+1} \). Moreover, the spaces \( S^{2m+1} \setminus \{0(m)\} \) are not product spaces. A more delicate construction is therefore needed to obtain \( \mu^{(\infty)} \), starting from a decreasing sequence of neighborhoods of the zero sequence \( 0 \) in \( S^\infty \). Convergence to the limit measure \( \mu^{(\infty)} \) will then be shown using Theorem A.1

Let \( \mathcal{B}_f(S^\infty) \) be the class of cylinders of \( S^\infty \), that is,

\[
\mathcal{B}_f(S^\infty) = \bigcup_{m=0}^{\infty} \{ Q_m^{-1}(A) : A \in \mathcal{B}(S^{2m+1}) \}.
\]

For integer \( m \geq 0 \) and for real \( r > 0 \), define

\[
N_{m,r}(x) = \{ y \in S^{2m+1} : d_m(x, y) < r \}, \quad x \in S^{2m+1}.
\]

For \( x, y \in S^\infty \) we have \( d_m(x, y) \leq d_m(Q_m(x), Q_m(y)) + 2^{-m} \). We obtain

\[
Q_m^{-1}(N_{m,r}(Q_m(x))) \subset \{ y \in S^\infty : d_m(x, y) < r + 2^{-m} \}, \quad x \in S^\infty.
\]

For every \( \varepsilon > 0 \) we can find \( r > 0 \) and integer \( m \geq 1 \) such that \( r + 2^{-m} \leq \varepsilon \). Therefore, we can write any open subset of \( S^\infty \) as a countable union of open cylinders; apply the Lindelöf property, using the separability of the metric space \( (S^\infty, d_m) \). The \( \sigma \)-field generated by \( \mathcal{B}_f(S^\infty) \) is thus equal to \( \mathcal{B}(S^\infty) \). Clearly, \( \mathcal{B}_f(S^\infty) \) is a \( \pi \)-system.
Fix integer $m \geq 0$ and let
\[ \mathcal{B}_f(S^{2m+1}) = \{ B \in \mathcal{B}(S^{2m+1}) : \mu^{(m)}(\partial B) = 0 \}, \] (11)
i.e., the collection of $\mu^{(m)}$-smooth Borel sets of $S^{2m+1}$. Since $\partial(A \cap B) \subset \partial A \cup \partial B$ for all subsets $A$ and $B$ of a topological space, $\mathcal{B}_f(S^{2m+1})$ is a $\pi$-system. Moreover, finiteness of $\mu^{(m)}$ on complements of neighbourhoods of $0^{(m)}$ together with separability of the metric space $(S^{2m+1}, d_m)$ implies, via the Lindelöf property, that every open subset of $S^{2m+1}$ can be covered by a countable collection of $\mu^{(m)}$-smooth open balls. In particular, the $\sigma$-field generated by $\mathcal{B}_f(S^{2m+1})$ is equal to $\mathcal{B}(S^{2m+1})$.

For integer $m \geq k \geq 1$, define the subset $A_k^{(m)}$ of $S^{2m+1}$ by
\[ A_k^{(m)} = \left\{ x \in S^{2m+1} : \max_{-k \leq j \leq k} d(x, 0) \leq 1/k \right\}. \]
By homogeneity of $\mu^{(m)}$, we have $\mu^{(m)}(\{ x \in S^{2m+1} : d(x, 0) = c \}) = 0$ for all integer $m \geq 0$, $j \in \{-m, \ldots, m\}$, and real $c > 0$. Therefore, $\mu^{(m)}(\partial A_k^{(m)}) = 0$ for all integer $m \geq k \geq 1$. The set $S^{2m+1} \setminus A_k^{(m)}$ is $\mu^{(m)}$-smooth and open. By construction, $d_m(x, 0^{(m)}) \geq 2^{-k}/(1 + k)$ for $x \in S^{2m+1} \setminus A_k^{(m)}$ while $d_m(x, 0^{(m)}) \leq 3/(1 + k)$ for $x \in A_k^{(m)}$. As a consequence of the former inequality, $\mu^{(m)}(S^{2m+1} \setminus A_k^{(m)}) < \infty$.

For integer $m \geq k \geq 1$, write
\[ \mu_k^{(m)} = \mu^{(m)}(\cdot \setminus A_k^{(m)}). \] (12)
If additionally $n \geq m$, we have, since $Q_{n,m}^{-1}(A_k^{(m)}) = A_k^{(n)}$, by (9),
\[ \mu_k^{(n)} \circ Q_{n,m}^{-1} = \mu_k^{(m)}, \quad n \geq m \geq k \geq 1. \] (13)

Let $R_k = \mu^{(k)}(S^{2k+1} \setminus A_k^{(k)})$ be the common value of the total mass of the measures $\mu_k^{(m)}$ for $m \geq k$. Then $0 < R_k < \infty$; positivity follows from the fact that $\mu^{(k)}$ is nonzero and homogenous; finiteness follows because $A_k^{(k)}$ is a neighbourhood of $0^{(k)}$ in $S^{2k+1}$.

Fix integer $k \geq 1$. For integer $m \geq k$, consider the probability measures $P_k^{(m)} = R_k^{-1} \mu_k^{(m)}$ on $S^{2m+1}$. By (13), we have
\[ P_k^{(n)} \circ Q_{n,m}^{-1} = P_k^{(m)}, \quad n \geq m \geq k. \] (14)
By (14), the family $(P_k^{(m)})_{m \geq k}$ is consistent in the sense of Pollard (2002, Chapter 4, Section 8). Since the metric space $(S^{2m+1}, d_m)$ is separable and complete for all $m \geq k$, every probability measure $P_k^{(m)}$ is tight (Billingsley, 1999, Theorem 1.3). According to the Daniell–Kolmogorov extension theorem (Pollard, 2002, Theorem 53), there exists a tight probability measure $P_k^{(\infty)}$ on $S^2$ such that
\[ P_k^{(\infty)} \circ Q_{n,m}^{-1} = P_k^{(m)}, \quad m \geq k. \] (15)
Define $\mu_k^{(m)} = R_k P_k^{(m)}$. Then (15) implies

$$\mu_k^{(m)} \circ Q_m^{-1} = \mu_k^{(m)}, \quad m \geq k. \quad (16)$$

The sets

$$A_k^{(\infty)} = \left\{ x \in S^\infty : \max_{k \leq j \leq k} d(x_j, 0) \leq k^{-1} \right\}, \quad k \geq 1,$$

form a decreasing sequence of closed neighbourhoods of 0 in $S^\infty$. Clearly, $A_k^{(\infty)} \subset \{ x : d_m(x, 0) \leq 3/(1 + k) \}$. For $B \subset S^\infty$ such that $0 \notin B^-$, there exists $k_0 \geq 1$ such that $A_k^{(\infty)} \cap B = \emptyset$ for all $k \geq k_0$.

By (12) and (16), the measure $\mu_k^{(m)}$ is finite and vanishes on $A_k^{(\infty)} = Q_k^{-1}(A_k^{(k)})$:

$$\mu_k^{(m)}(A_k^{(m)}) = \mu_k^{(k)}(A_k^{(k)} \setminus A_k^{(k)}) = 0, \quad \mu_k^{(\infty)}(S^\infty \setminus A_k^{(\infty)}) = \mu_k^{(k)}(S_0 \setminus A_k^{(k)}) < \infty. \quad (17) \quad (18)$$

Moreover, we have

$$\mu_k^{(\infty)}(\cdot \setminus A_k^{(\infty)}) = \mu_k^{(m)}(\cdot \setminus A_k^{(m)}) = \mu_k^{(\infty)}(\cdot), \quad \ell \geq k \geq 1. \quad (19)$$

Indeed, for $m \geq \ell$ and $B \in \mathcal{B}(S^{2m+1})$, we have, by (16),

$$\mu_k^{(\infty)}(Q_m^{-1}(B) \setminus A_k^{(\infty)}) = \mu_k^{(\infty)}(Q_m^{-1}(B \setminus A_k^{(m)})) = \mu_k^{(\infty)}(B \setminus A_k^{(m)}) = \mu_k^{(m)}(B) = \mu_k^{(\infty)}(Q_m^{-1}(B)),$$

and the cylinders of $S^\infty$ form a $\pi$-system generating $\mathcal{B}(S^\infty)$; apply Theorem 3.3 in Billingsley (1995) to arrive at (19).

According to (19), the measures $\mu_k^{(\infty)}$ are successive extensions of one another, each measure being supported on $S^\infty \setminus A_k^{(\infty)}$, a sequence of subsets of $S^\infty$ which is growing to $S^\infty \setminus \{0\}$ as $k \to \infty$. These properties can be used to define a measure $\mu^{(\infty)}$ concentrated on $S^\infty \setminus \{0\}$ by

$$\mu^{(\infty)}(B) = \mu_1^{(\infty)}(B \setminus A_1^{(\infty)}) + \sum_{k=2}^{\infty} \mu_k^{(\infty)}((B \cap A_{k-1}^{(\infty)}) \setminus A_k^{(\infty)}), \quad B \in \mathcal{B}(S^\infty).$$

By properties (17), (13) and (19), we have

$$\mu^{(\infty)}(\cdot \setminus A_k^{(\infty)}) = \mu_k^{(\infty)}(\cdot), \quad k \geq 1. \quad (20)$$

The measures $\mu^{(m)}$ and $\mu^{(\infty)}$ are connected through the formula

$$\mu^{(\infty)} \circ Q_m^{-1} = \mu^{(m)}, \quad m \geq 0. \quad (21)$$

Indeed, let $B \in \mathcal{B}(S^{2m+1})$ be such that $0^{(m)} \notin B^-$. Then we can find $\ell \geq \max(m, 1)$ such that $\max_{j=-m,\ldots,m} d(x_j, 0) > 1/\ell$ for all $x \in B$ and thus $A_\ell^{(\infty)} \cap Q_m^{-1}(B) = \emptyset$ and
\[ A_i^\ell \cap Q_{\ell,m}^{-1}(B) = \emptyset. \] Since \( Q_{\ell,m}^{-1}(B) = Q_{\ell,m}^{-1}(Q_{\ell,m}^{-1}(B)) \), we find, applying successively equations (20), (16), (12) and (9).

\[ \mu^{(m)}(Q_{m}^{-1}(B)) = \mu^{(m)}(Q_{m}^{-1}(B)) = \mu^{(m)}(Q_{\ell,m}^{-1}(B)) = \mu^{(m)}(Q_{\ell,m}^{-1}(B)) = \mu^{(m)}(B), \]

as required. In particular, \( \mu^{(m)}(\{x : x_0 \neq 0\}) = \mu(\emptyset) = 0 \).

To prove \( M_0 \)-convergence in Theorem 4.1(a), we apply Theorem A.1. Recall the \( \mu^{(m)} \)-smooth Borel sets \( \mathcal{B}(S^{2m+1}) \) in (11). Define the collection \( \mathcal{A} \subset \mathcal{B}(S^2) \) by

\[ \mathcal{A} = \bigcup_{m=0}^{\infty} \{ Q_{m}^{-1}(B) : B \in \mathcal{B}_s(S^{2m+1}), 0^{(m)} \notin B^- \}. \]

We show that \( \mathcal{A} \) satisfies conditions (C1) and (C2) of Theorem A.1.

(C1) Put \( N_i = A_i^{\infty} \) for \( i \in \mathbb{N} \). We have \( d_m(x, \emptyset) < 3/(1+i) + 2^{-i} \) for \( x \in N_i \). Further, let \( i \in \mathbb{N} \) and let \( A = Q_{m}^{-1}(B) \) with \( B \in \mathcal{B}_s(S^{2m+1}) \) and \( 0^{(m)} \notin B^- \), so \( A \in \mathcal{A} \).

We need to show that \( A \setminus N_i \in \mathcal{A} \) too. Put \( \ell = \max(m, i) \) and note that \( A = Q_{\ell,m}^{-1}(Q_{\ell,m}^{-1}(B)) \) and \( N_i = Q_{\ell,i}^{-1}(Q_{\ell,i}^{-1}(A_i^{(i)})) \), whence

\[ A \setminus N_i = Q_{\ell,m}^{-1}(Q_{\ell,m}^{-1}(B) \setminus Q_{\ell,i}^{-1}(A_i^{(i)})). \]

The set on the right-hand side is of the desired form \( Q_{\ell,m}^{-1}(C) \) for some set \( C \in \mathcal{B}_s(S^{2(\ell+1)}) \) such that \( 0^{(\ell)} \notin C^- \); indeed, we have for instance \( \mu^{(\ell)}(\partial Q_{\ell,m}^{-1}(B)) = \mu^{(\ell)}(Q_{\ell,m}^{-1}(\partial B)) = \mu^{(\ell)}(\partial B) = 0 \) by (15) and the fact that \( 0^{(m)} \notin B^- \). As a consequence, \( A \setminus N_i \in \mathcal{A} \) for \( A \in \mathcal{A} \).

(C2) Recall from the paragraph containing (10) that every open subset of \( S^2 \) can be written as the union of a countable collection of open cylinders. Moreover, recall from the paragraph containing (11) that every open subset of \( S^{2m+1} \) can be written as a countable collection of \( \mu^{(m)} \)-smooth open balls, and this for arbitrary integer \( m \geq 0 \). As a consequence, every open subset \( G \) of \( S^2 \) such that \( \emptyset \notin G^- \) can be written as a countable union of \( \mathcal{A} \)-sets.

Finally, by Theorem 4.1(b) and the Portmanteau theorem (Hult and Lindskog 2006, Theorem 2.4), we have, for every \( A = Q_{m}^{-1}(B) \in \mathcal{A} \) with \( B \in \mathcal{B}_s(S^{2m+1}) \) and \( 0^{(m)} \notin B^- \),

\[ \frac{1}{V(u)} \Pr[u^{-1} X \in A] = \frac{1}{V(u)} \Pr[u^{-1} X^{(m)} \in B] \rightarrow \mu^{(m)}(B) = \mu^{(\infty)}(A), \quad u \rightarrow \infty, \]

the final identity following from (27). Apply Theorem A.1 to conclude that the \( M_0 \)-convergence stated in Theorem 4.1(a) holds.
5 Angular or spectral tail processes

Let $X = (X_t)_{t \in \mathbb{Z}}$ be a strictly stationary, regularly varying discrete-time stochastic process taking values in $S$. According to Theorem 4.1, the random variable $X_0$ is a regularly varying random element in the space $(S, \nu)$. With the assumption that a modulus $\rho : S \to [0, \infty)$ exists, Proposition 3.1 describes the joint limit of the rescaled modulus $\rho(X_t)/u$ and the angle $\theta(X_0) = X_0/\rho(X_0)$ given that $\rho(X_0) > u$ as $u \to \infty$. In the following theorem, we will extend this by considering the entire self-normalized process $X_t/\rho(X_t)$, $t \in \mathbb{Z}$. The theorem generalizes Theorem 2.1 in Basrak and Segers (2009) and Theorem 3.1 in Meinguet and Segers (2010). Let $\mathcal{V}_0(S^k)$ be the space of functions $S^k \setminus \{(0, \ldots, 0)\} \to \mathbb{R}$ which are bounded and continuous and vanish on the complement of a neighbourhood of the origin, $(0, \ldots, 0)$, in $S^k$. Recall that the arrow $\rightsquigarrow$ signifies weak convergence.

**Theorem 5.1** Let $X = (X_t)_{t \in \mathbb{Z}}$ be a strictly stationary time series taking values in a complete, separable metric space $S$, endowed with an origin, a scalar multiplication, and a modulus $\rho$. The following properties are equivalent:

(i) $X$ is regularly varying with index $\alpha \in (0, \infty)$.

(ii) The function $u \mapsto \Pr[\rho(X_0) > u]$ belongs to $\mathcal{R}_{-\alpha}$ and there exists a random element $(\Theta_t)_{t \in \mathbb{Z}} \in S^\mathbb{Z}$ such that, as $u \to \infty$,

$$
\mathcal{L}((X_t/\rho(X_0))_{t \in \mathbb{Z}} \mid \rho(X_0) > u) \rightsquigarrow (\Theta_t)_{t \in \mathbb{Z}}.
$$

(iii) There exist a Pareto($\alpha$) random variable $Y$ and a random element $(\Theta_t)_{t \in \mathbb{Z}} \in S^\mathbb{Z}$, independent of each other, such that, as $u \to \infty$,

$$
\mathcal{L}(\rho(X_0)/u, (X_t/\rho(X_0))_{t \in \mathbb{Z}} \mid \rho(X_0) > u) \rightsquigarrow (Y, (\Theta_t)_{t \in \mathbb{Z}}).
$$

In this case, the law of $(\Theta_t)_{t \in \mathbb{Z}}$ is the same across (ii) and (iii), and for every integer $t$ and every positive integer $k$,

$$
\mathbb{E}[\rho(\Theta_t)^k] = \lim_{r \uparrow 0} \lim_{u \to \infty} \Pr[\rho(X_t) > ru \mid \rho(X_0) > u] \leq 1,
$$

$$
\frac{1}{\Pr[\rho(X_0) > u]} \Pr[(X_1/u, \ldots, X_k/u) \in \cdot] \to \nu_k(\cdot), \quad u \to \infty,
$$

in $M_0(S^k)$, where \(\int f \, d\nu_k\) for $f \in \mathcal{V}_0(S^k)$ is given by

$$
\sum_{i=1}^k \int_0^\infty \mathbb{E} \left[ f(0, \ldots, 0, z\Theta_0, \ldots, z\Theta_{t-i}) \mathbb{I} \left( \max_{1 \leq j \leq k} \rho(\Theta_j) = 0 \right) \right] d((-z)^{-\alpha}).
$$

**Proof** We prove the implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i) & (24) & (25).

(i) implies (ii). Let $V$, $\hat{\mu}^{(m)}$ and $\hat{\mu}^{(m)}$ be the auxiliary function and the limit measures, respectively, in (7) and (3). Proposition 3.1 and the equation (8) imply that when $m = 0$, we have

$$
\frac{1}{V(u)} \Pr[\rho(X_0) > u] \to \hat{\mu}^{(0)}(\{x \in S : \rho(x) > 1\}), \quad u \to \infty,
$$

where $\mathcal{L}(\rho(X_0)/u, (X_t/\rho(X_0))_{t \in \mathbb{Z}} \mid \rho(X_0) > u) \rightsquigarrow (\Theta_t)_{t \in \mathbb{Z}}$ is the same across (ii) and (iii), and for every integer $t$ and every positive integer $k$,
the limit being finite and nonzero. Hence, the function \( V(u) := \Pr(\rho(X_0) > u) \) is a valid auxiliary function for \( X \). With this choice, the limit measures are rescaled versions of the old ones:

\[
\mu^{(\omega)}(\cdot) = \tilde{\mu}^{(\omega)}(\cdot)/\tilde{\mu}^{(0)}(\{x : \rho(x) > 1\}), \\
\mu^{(m)}(\cdot) = \tilde{\mu}^{(m)}(\cdot)/\tilde{\mu}^{(0)}(\{x : \rho(x) > 1\}).
\]

For every \( \lambda > 0 \), the homogeneity of the measure \( \mu^{(m)} \) and the modulus \( \rho \) implies that \( \mu^{(\omega)}(\{x \in S^2 : \rho(x_0) = \lambda\}) = \lambda^{-a} \mu^{(\omega)}(\{x \in S^2 : \rho(x_0) = 1\}) \). The set \( \{x \in S^2 : \rho(x_0) = 1\} \) is the boundary of the open set \( \{x \in S^2 : \rho(x_0) > 1\} \). The latter is thus a \( \mu^{(\omega)} \)-continuity set, and its closure, \( \{x \in S^2 : \rho(x_0) \geq 1\} \), does not contain the origin 0. Similarly, we have that, for every nonnegative integer \( m \), the set \( \{x^{(m)} = (x_{-m}, \ldots, x_m) \in S^{2m + 1} : \rho(x_0) > \lambda\} \) is a \( \mu^{(m)} \)-continuity set, whose closure does not contain the origin \( 0^{(m)} \).

Let \( m \) be a nonnegative integer and write \( k = 2m + 1 \). Put

\[
\mathcal{K}_m = \{(\theta_{-m}, \ldots, \theta_m) \in S^k : \rho(\theta_0) = 1\} \tag{27}
\]

and define a probability measure on \( \mathcal{K}_m \) by

\[
H_m(B) = \mu^{(m)}(\{x^{(m)} = (x_{-m}, \ldots, x_m) \in S^k : \rho(x_0) > 1, x^{(m)}/\rho(x_0) \in B\}),
\]

for Borel sets \( B \subset \mathcal{K}_m \). Let \( g : \mathcal{K}_m \to \mathbb{R} \) be bounded and continuous and define \( f : S^k \to \mathbb{R} \) by

\[
f(x_{-m}, \ldots, x_m) = g(x_{-m}/\rho(x_0), \ldots, x_m/\rho(x_0)) 1_{\{\rho(x_0) > 1\}},
\]

to be interpreted as 0 if \( x_0 = 0 \). The function \( f \) is bounded and vanishes on the set \( \{x^{(m)} \in S^{2m+1} : \rho(x_0) \leq 1\} \), which is a closed neighbourhood of the origin \( 0^{(m)} \) in \( S^{2m+1} \). Moreover, it is continuous everywhere except perhaps on \( \mathcal{K}_m \), which is a \( \mu^{(m)} \)-null set. By Lemma [A.1]

\[
E[g(X_{-m}/\rho(X_0), \ldots, X_m/\rho(X_0)) | \rho(X_0) > u]
\]

\[
= \frac{1}{\Pr[\rho(X_0) > u]} E[f(X_{-m}/\rho(X_0), \ldots, X_m/\rho(X_0))] \\
\to \int_{S^k} f \, d\mu^{(m)} = \int_{\mathcal{K}_m} g \, dH_m, \quad u \to \infty.
\]

If \( (\Theta_{-m}, \ldots, \Theta_m) \) is a random element of \( \mathcal{K}_m \) with distribution \( H_m \), then

\[
\mathcal{L}\left((X_{-m}/\rho(X_0), \ldots, X_m/\rho(X_0)) | \rho(X_0) > u\right) \sim (\Theta_{-m}, \ldots, \Theta_m),
\]
as \( u \to \infty \). The Daniell–Kolmogorov extension theorem [Pollard, 2002, Theorem 53] yields that there exists a random element \( (\Theta_0) \in S^2 \) such that, for every nonnegative integer \( m \), the distribution of \( (\Theta_{-m}, \ldots, \Theta_m) \) is \( H_m \). Weak convergence of finite stretches characterizes weak convergence in the product space \( S^2 \) [van der Vaart and Wellner, 1996, Theorem 1.4.8], and statement (ii) follows.
(ii) implies (iii). Fix a nonnegative integer \( m \). Let \( y \geq 1 \) and let \( g : \mathbb{R}_m \to \mathbb{R} \) be continuous and bounded, with \( \mathbb{R}_m \) as in (27). We have
\[
E\{ (\rho(X_0)/u) > y \} g(X_{-m}/\rho(X_0), \ldots, X_m/\rho(X_0)) | \rho(X_0) > u \}
= \frac{\Pr[\rho(X_0) > uy]}{\Pr[\rho(X_0) > u]} E[g(X_{-m}/\rho(X_0), \ldots, X_m/\rho(X_0)) | \rho(X_0) > uy]
\to y^{-\alpha} E[g(\Theta_{-m}, \ldots, \Theta_m)], \quad u \to \infty.
\]
In view of Lemma A.2 as \( u \to \infty \),
\[
\mathcal{L} \left( (\rho(X_0)/u, X_{-m}/\rho(X_0), \ldots, X_m/\rho(X_0)) | \rho(X_0) > u \right) \sim (Y, \Theta_{-m}, \ldots, \Theta_m),
\]
where \( Y \) is a Pareto(\( \alpha \)) random variable independent of \( (\Theta_{-m}, \ldots, \Theta_m) \). Statement (iii) follows.

(iii) implies (i), (24) and (25). To prove (i), we will show that (3) holds with \( V(u) = \Pr[\rho(X_0) > u] \). The latter function belongs to \( \mathcal{R}_\alpha \) because (iii) implies that \( V(au)/V(u) = \Pr[\rho(X_0)/u > y | \rho(X_0) > u] \to y^{-\alpha} \) as \( u \to \infty \), for all \( y \geq 1 \). By Hult and Lindskog (2006, Theorem 2.1), equation (3) is equivalent to the condition that for every \( m \geq 0 \) and every \( f \in C_0(\mathbb{R}^{2m+1}) \), we have
\[
\lim_{u \to \infty} \frac{1}{V(u)} E[f(X_{-m}/u, \ldots, X_m/u)] = \int_{\mathbb{R}^{2m+1}} f(x_{-m}, \ldots, x_m) \, d\mu^{(m)}.
\]
By stationarity, this limit relation is a consequence of (25): just replace \( (X_{-m}, \ldots, X_m) \) by \( (X_1, \ldots, X_k) \) with \( k = 2m + 1 \).

We start with proving (23). Fix integer \( t \) and real \( r > 0 \). Put \( V(u) = \Pr[\rho(X_0) > u] \).
Statement (iii) implies the independence between \( Y \) and \( (\Theta_{1})_{\geq 0} \). Writing
\[
\frac{X_i}{u} = \rho(X_0) \frac{X_i}{\rho(X_0)} \quad \text{for} \quad i \in \mathbb{N},
\]
we have, by stationarity and Fubini’s theorem,
\[
\lim_{u \to \infty} \Pr[\rho(X_{-m}) > ru | \rho(X_0) > u]
= \lim_{u \to \infty} \frac{V(ru)}{V(u)} \Pr[\rho(X_0) > u | \rho(X_0) > ru]
= r^{-\alpha} \Pr[r \rho(\Theta_1) > 1]
= E \left[ \int_1^\infty I\{ r \rho(\Theta_1) > 1 \} \, d(-y^{-\alpha}) \right]
= E \left[ \int_r^\infty I\{ r \rho(\Theta_1) > 1 \} \, d(-z^{-\alpha}) \right]
\]
By monotone convergence, we have \( E[\min\{\rho(\Theta_1), r^{-1}\}^{\alpha}] \to E[\rho(\Theta_1)^\alpha] \) as \( r \downarrow 0 \), whence (23).

Fix \( f \in C_0(\mathbb{R}^k) \). There exists \( r_0 > 0 \) such that \( f \) vanishes on the set \( \{ x \in \mathbb{R}^k : \max_{1 \leq i \leq k} \rho(x_i) \leq r_0 \} \). Indeed, \( f \) vanishes on a neighbourhood of the origin \( (0, \ldots, 0) \) in \( \mathbb{R}^k \) and sets of the stated form constitute a neighbourhood basis of this origin, by Definition 2.2(iii) and by definition of the product topology.
Fix $r \in (0, r_0)$. Form a partition of $\{ \max_{1 \leq i \leq k} \rho(X_i) > ur \}$ according to the smallest index $i$ such that $\rho(X_i) > ur$. By stationarity, we find

\[
\frac{1}{V(u)} E[f(X_1/u, \ldots, X_k/u)] \\
= \frac{1}{V(u)} \sum_{i=1}^k \int_r^\infty E \left[ f(x_i, \ldots, X_k/u) 1 \left( \max_{1 \leq i \leq k} \rho(x_i) > r \right) \right] d(-y^{-\alpha}) \\
= \frac{1}{V(u)} \sum_{i=1}^k \int_r^\infty E \left[ f(x_i, \ldots, X_k/u) 1 \left( \max_{1 \leq i \leq k} \rho(x_i) \leq ur \right) \right] d(-y^{-\alpha}),
\]

where we substituted $z = ry$. The final expression involves an arbitrary scalar $r \in (0, r_0)$ but, in view of the left-hand side of (28), it does not depend on the exact value of $r$. We show that we can take the limit as $r \downarrow 0$, obtaining (26). To that end, we apply dominated convergence to each term $i \in \{1, \ldots, k\}$ separately. For fixed $z \in (0, \infty)$, we have, since $f$ is bounded,

\[
\lim_{r \downarrow 0} E \left[ f(z\Theta_1, \ldots, z\Theta_k) 1 \left( \max_{1 \leq i \leq k} \rho(z\Theta_i) < r \right) \right] \\
= E \left[ f(0, \ldots, 0, 0, \ldots, 0) 1 \left( \max_{1 \leq i \leq k} \rho(\Theta_i) = 0 \right) \right].
\]

Next, we need to show that we can integrate this limit over $z \in (0, \infty)$ according to the measure $d(-z^{-\alpha})$. Since $f$ is bounded and vanishes on the set $\{x \in S^k : \max_{1 \leq i \leq k} \rho(x_i) \leq r_0 \}$, there exists $c > 0$ such that

\[
|f(x)| \leq c 1 \left( \max_{1 \leq i \leq k} \rho(x_i) \geq r_0 \right), \quad x \in S^k \setminus \{(0, \ldots, 0)\}.
\]
It follows that, for all \( z \in (0, \infty) \) and all \( r \in (0, r_0) \),

\[
\left| \mathbb{E} \left[ f(\Theta_{-i}, \ldots, \Theta_{k-i}) I \left( \max_{1 \leq j \leq k-i} \rho(\Theta_j) < r \right) \right] \right| 
\leq c \Pr \left[ \max_{1 \leq j \leq k} \rho(\Theta_j) \geq r_0 \right] \leq c \sum_{j=1}^{k} \Pr[\rho(\Theta_{j-i}) \geq z^{-1}r_0].
\]

For any integer \( r \), we have, by Fubini’s theorem,

\[
\int_0^\infty \Pr[\rho(\Theta_s) \geq z^{-1}r_0] d(-z^{-\alpha}) = r_0^{-\alpha} \mathbb{E}[\rho(\Theta_s)^{\alpha}] \leq r_0^{-\alpha},
\]

the inequality following from (24). This justifies the use of the dominated convergence theorem when passing to the limit \( r \downarrow 0 \) on the right-hand side of (23). We arrive at (25) with limit measure \( \nu_k \) as given in (26). This completes the proof of Theorem 5.1.

6 The time-change formula

In general, the spectral process \( \Theta \) of a stationary regularly varying time series \( (X_i)_{i \in \mathbb{Z}} \) is itself nonstationary. Still, the fact that \( (X_i)_{i \in \mathbb{Z}} \) is stationary induces a peculiar structure on the distribution of the spectral process. In particular, the distribution of \( \Theta \) is determined by the distribution of its restriction to the nonnegative time axis, that is, of the forward spectral process \( \Theta \) with \( \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \). The same is true for the backward spectral process \( \Theta \) with \( \mathbb{Z}^- = \{0, -1, -2, \ldots\} \).

**Theorem 6.1** Statements (ii) and (iii) in Theorem 5.1 are equivalent to the statements with \( \mathbb{Z} \) replaced by \( \mathbb{Z}^+ \) or \( \mathbb{Z}^- \). In that case,

\[
\mathbb{E}[f(\Theta_{-s}, \ldots, \Theta_t)] = \mathbb{E} \left[ f \left( \frac{\Theta_0}{\rho(\Theta_s)}, \ldots, \frac{\Theta_{t+s}}{\rho(\Theta_s)} \right) \rho(\Theta_t)^{\alpha} \right] \quad (29)
\]

for all nonnegative integers \( s \) and \( t \) and for all integrable functions \( f : S^{t+s+1} \to \mathbb{R} \) with the property that \( f(\theta_{-s}, \ldots, \Theta_t) = 0 \) whenever \( \theta_{-s} = 0 \).

By ‘integrable functions’ is meant real-valued, Borel-measurable functions such that one of the expectations, and hence the other one, exists. In (29) and in later formulas in which expressions like \( \rho(\Theta_s) \) appear both in the denominator and as a term in a product, the integrand is to be interpreted as zero when \( \rho(\Theta_s) \) is zero. A time-change formula for general integrable functions, without the zero-property, is given in (40) inside the proof of Theorem 6.1.

By considering the time-reversed process \( \check{X}_i = X_{-i} \), equation (29) can be reversed in the obvious way. A simple case occurs when \( f \) only depends on its first component, that is, \( f(\theta_{-s}, \ldots, \Theta_t) = f(\theta_{-s}) \) and \( f(0) = 0 \); equation (29) then reduces to

\[
\mathbb{E}[f(\Theta_{-s})] = \mathbb{E}[f(\theta_{-s}) \rho(\Theta_s)] \rho(\Theta_t)^{\alpha} \quad s \in \mathbb{Z}. \quad \quad (30)
\]
This yields an expression of the distribution of $\Theta_s$ in terms of the joint law of $\Theta_0$ and $\Theta_s$. In particular, we find
\[
\Pr[\Theta_s \neq 0] = E[\rho(\Theta_s)^\alpha], \quad s \in \mathbb{Z}.
\]

If the common value in the preceding display is equal to unity, then (30) is valid for arbitrary integrable $f$, that is, without the restriction that $f(0) = 0$.

**Proof (Proof of Theorem 5.1)** By symmetry, we only need to consider the forward case, $\mathbb{Z}_+ = \{0, 1, 2, \ldots \}$. Consider the statements (ii) and (iii) in Theorem 5.1 with $\mathbb{Z}_+$ replaced by $\mathbb{Z}^+$.

(ii+) The function $u \mapsto \Pr[\rho(X_0) > u]$ belongs to $\mathcal{R}_\alpha$, and in $\mathbb{S}^+$,
\[
\mathcal{L} \left( (X_t / \rho(X_0))_{t \in \mathbb{Z}_+} \mid \rho(X_0) > u \right) \sim (\Theta_t)_{t \in \mathbb{Z}_+} \quad (u \to \infty).
\]

(iii+) In $(0, \infty) \times \mathbb{S}^+$, as $u \to \infty$,
\[
\mathcal{L} \left( \rho(X_0) / u, (X_t / \rho(X_0))_{t \in \mathbb{Z}_+} \mid \rho(X_0) > u \right) \sim (Y, (\Theta_t)_{t \in \mathbb{Z}_+}),
\]

where $Y$ is a Pareto($\alpha$) random variable independent from $(\Theta_t)_{t \in \mathbb{Z}_+}$.

We have to show that the statements (i)--(iii) in Theorem 5.1 are equivalent with each of (ii+) and (iii+). We already know that (i) implies (ii). Trivially, (ii) implies (ii+). To show that (ii+) implies (iii+), just set $s = 0$ in the part of the proof of Theorem 5.1 that (ii) implies (iii). Since (iii) implies (i) by Theorem 5.1, all that remains to be shown is that (iii+) implies (iii).

The proof of (24) in Theorem 5.1 ensures that if (iii+), then for every $t \in \mathbb{Z}_+$, (24) holds.

**Lemma 6.1** If (iii+), then for every $t \in \mathbb{Z}_+$,
\[
\mathcal{L} \left( X_t / \rho(X_0) \mid \rho(X_0) > u \right) \sim \nu_t, \quad u \to \infty,
\]
where $\nu_t$ is a probability measure on $\mathbb{S}$ given for $\nu_t$-integrable $g : \mathbb{S} \to \mathbb{R}$ by
\[
\int g \, d\nu_t = g(0) \{1 - E[\rho(X_0)^\alpha] \} + E[g(\Theta_0 / \rho(\Theta_0)) \rho(\Theta_0)^\alpha].
\]

**Proof (Proof of Lemma 6.1)** Let $g : \mathbb{S} \to \mathbb{R}$ be continuous and bounded. Fix $r > 0$. We have
\[
E[g(X_t / \rho(X_0)) \mid \rho(X_0) > u]
= g(0) \Pr[\rho(X_t) \leq ru \mid \rho(X_0) > u]
+ E[\{g(X_t / \rho(X_0)) - g(0)\} \mathbb{I}(\rho(X_t) \leq ru) \mid \rho(X_0) > u]
+ E[g(X_t / \rho(X_0)) \mathbb{I}(\rho(X_t) > ru) \mid \rho(X_0) > u]
= Q_1 + Q_2 + Q_3.
\]
The first term $Q_1$ on the right-hand side has been treated in (24). If $\rho(X_0) > u$ and $\rho(X_{-t}) \leq ru$, then $\rho(X_{-t}/\rho(X_0)) = \rho(X_{-t})/\rho(X_0) < r$. Recall that there exist positive scalars $(z_t)_{t \geq 0}$ such that $\{ x : \rho(x) < r \} \subset \{ x : d(x, 0) < z_t \}$ and $\lim_{t \to 0} z_t = 0$. Since $g$ is continuous,

$$\lim_{r \to 0} \limsup_{u \to \infty} |Q_2| \leq \lim_{r \to 0} \sup_{x \in \rho(x) < r} |g(x) - g(0)| = 0.$$ 

For $Q_3$, writing $V(u) = \Pr[\rho(X_0) > u]$, we have, by stationarity of $X$,

$$Q_3 = \frac{V(ru)}{V(u)} E[g(X_0/\rho(X_0)) 1(\rho(X_0) > u) \mid \rho(X_0) > ru]$$

$$= \frac{V(ru)}{V(u)} E \left[ g \left( \frac{X_0/\rho(X_0)}{\rho(X_0)/\rho(X_0)} \right) 1 \left( r \frac{\rho(X_0)}{\rho(X_0)} > 1 \right) \mid \rho(X_0) > ru \right]$$

$$\to r^{-\alpha} E[g(\Theta_0/\rho(\Theta_0)) 1(r \rho(\Theta_0) > 1)], \quad u \to \infty.$$ 

The last step is justified by (iii.), which implies the continuity of the law of $Y$ and the independence between $Y$ and $\Theta$. Moreover, this limit relation holds for every $r > 0$ in a neighborhood of zero. The limit is equal to $E[g(\Theta_0/\rho(\Theta_0)) \min\{ \rho(\Theta_0), r^{-1}\alpha \}]$, which, by dominated convergence, tends to $E[g(\Theta_0/\rho(\Theta_0)) \rho(\Theta_0)^{\alpha}]$ as $r \downarrow 0$. Therefore, Lemma 6.1 is established.

Fix nonnegative integer $s$ and $t$. If (iii.), then in view of Lemma 6.1, the converse half of Prohorov’s theorem (Billingsley, 1999, Theorem 5.2) and Tychonoff’s theorem, there exists $u_0 > 0$ such that the collection of probability measures

$$\mathcal{L}(X_{-s}/\rho(X_0), \ldots, X_t/\rho(X_0) \mid \rho(X_0) > u), \quad u > u_0,$$

(31)
is tight, that is, for every $\epsilon > 0$ there exists a compact subset $K_\epsilon$ of $S^{s+t+1}$ so that the probability mass of $K_\epsilon$ under each of the laws above is at least $1 - \epsilon$. By the direct half of Prohorov’s theorem (Billingsley, 1999, Theorem 5.1), the collection of probability measures above is relatively compact: for every sequence $u_n \to \infty$ there exists a subsequence $u_{n_0} \to \infty$ for which the laws have a limit in distribution. To prove convergence in distribution of (31) as $u \to \infty$, it is then sufficient to show uniqueness of the possible sequential limits. As probability distributions are determined by their integrals of bounded, Lipschitz continuous functions (Billingsley, 1999, proof of Theorem 1.2), it is sufficient to show the following lemma.

**Lemma 6.2** If (iii.), then for every nonnegative integer $s$ and $t$ and for every bounded, Lipschitz continuous function $f : S^{s+t+1} \to \mathbb{R}$, the following limit exists:

$$\lim_{u \to \infty} E[f(X_{-s}/\rho(X_0), \ldots, X_t/\rho(X_0)) \mid \rho(X_0) > u].$$

If moreover $f(\Theta_{-s}, \ldots, \Theta_t) = 0$ as soon as $\Theta_{-s} = 0$, then the limit is equal to

$$E \left[ f \left( \frac{\Theta_0}{\rho(\Theta_0)} \ldots, \frac{\Theta_t}{\rho(\Theta_t)} \right) \rho(\Theta_t)^{\alpha} \right].$$

(33)
Proof (Proof of Lemma 6.2) We fix an integer \( t \geq 0 \) and proceed by induction on the integer \( s \geq 0 \). The case \( s = 0 \) is already included in (ii+) or (iii+). Note that \( \rho(\Theta_0) = 1 \) with probability one.

Let \( s \geq 1 \) be an integer and assume the stated convergence holds for \( s \) replaced by \( s - 1 \), and all bounded, Lipschitz continuous functions from \( S^{+r-1+1} \) into \( \mathbb{R} \). Let \( f : S^{+r+1} \to \mathbb{R} \) be bounded and Lipschitz continuous with Lipschitz constant \( L > 0 \). Define \( f_0 : S^{+r+1} \to \mathbb{R} \) by

\[
f_0(\theta_s, \ldots, \theta_t) = f(\theta_s, \ldots, \theta_t) - f(0, \theta_{s-1}, \ldots, \theta_t).
\]

We have

\[
E[f(X_{-s}/\rho(X_0), \ldots, X_t/\rho(X_0)) \mid \rho(X_0) > u] = E[f(0, X_{-s+1}/\rho(X_0), \ldots, X_t/\rho(X_0)) \mid \rho(X_0) > u] + E[f_0(X_{-s}/\rho(X_0), \ldots, X_t/\rho(X_0)) \mid \rho(X_0) > u].
\]

By the induction hypothesis, the following limit already exists:

\[
\lim_{u \to \infty} E[f(0, X_{-s+1}/\rho(X_0), \ldots, X_t/\rho(X_0)) \mid \rho(X_0) > u].
\]

We will show that

\[
\lim_{u \to \infty} E\left[f_0(\frac{X_{-s}}{\rho(X_0)}, \ldots, \frac{X_t}{\rho(X_0)}) \mid \rho(X_0) > u \right] = E\left[f_0\left(\frac{\Theta_0}{\rho(\Theta_s)}, \ldots, \frac{\Theta_{t+s}}{\rho(\Theta_s)}\right) \rho(\Theta_s)^{\alpha}\right].
\]

Fix \( r > 0 \) and split the integrand on the left-hand side into two parts, according to whether \( \rho(X_{-s}) \leq ru \) or \( \rho(X_{-s}) > ru \). By the triangle inequality, equation (35) will be the consequence of the following three limits:

\[
\lim_{r \uparrow 0} \lim_{u \to \infty} E\left[f_0\left(\frac{X_{-s}}{\rho(X_0)}, \ldots, \frac{X_t}{\rho(X_0)}\right) \mid \rho(X_0) > u \right] \mathbb{I}\{\rho(X_{-s}) \leq ru\} = 0,
\]

\[
\lim_{u \to \infty} E\left[f_0\left(\frac{X_{-s}}{\rho(X_0)}, \ldots, \frac{X_t}{\rho(X_0)}\right) \mid \rho(X_0) > u \right] \mathbb{I}\{\rho(X_{-s}) > ru\} \rho(X_0)^{-1} \min\{\rho(\Theta_s), r^{-1}\}^{\alpha} = E\left[f_0\left(\frac{\Theta_0}{\rho(\Theta_s)}, \ldots, \frac{\Theta_{t+s}}{\rho(\Theta_s)}\right) \min\{\rho(\Theta_s), r^{-1}\}^{\alpha}\right],
\]

\[
\lim_{r \downarrow 0} E\left[f_0\left(\frac{\Theta_0}{\rho(\Theta_s)}, \ldots, \frac{\Theta_{t+s}}{\rho(\Theta_s)}\right) \min\{\rho(\Theta_s), r^{-1}\}^{\alpha}\right] = E\left[f_0\left(\frac{\Theta_0}{\rho(\Theta_s)}, \ldots, \frac{\Theta_{t+s}}{\rho(\Theta_s)}\right) \rho(\Theta_s)^{\alpha}\right].
\]

We will show equations (36), (37), and (38).
First we show (36). Recall that there exist positive scalars \((z_r)_{r>0}\) such that \(\{ x : \rho(x) < r \} \subset \{ x : d(x,0) < z_r \}\) for every \(r > 0\) \(\) and \(\lim_{r \downarrow 0} z_r = 0\). By definition of \(f_0\) and the fact that \(\rho\) is Lipschitz continuous with some constant \(L > 0\), we find that the expectation on the left-hand side in (36) is bounded by \(Lz_r\). This converges to zero as \(r \downarrow 0\).

Next we show (37). Let \(V(u) = \Pr[\rho(X_0) > u]\). By stationarity of \((X_t)_{t \in \mathbb{Z}}\), regular variation of \(V\), and (iii\(\_+\)), we have

\[
E[f_0(X_{-\alpha}/\rho(X_0), \ldots, X_{\alpha}/\rho(X_0)) \mathbb{I}(\rho(X_{-\alpha}) > ru) | \rho(X_0) > u] = V(\frac{ru}{u})E[f_0(X_0/\rho(X_0), \ldots, X_{\alpha}/\rho(X_0)) \mathbb{I}(\rho(X_0) > u) | \rho(X_0) > ru] \rightarrow r^{-\alpha}E[f_0(\frac{\Theta_0}{\rho(\Theta_0)}, \ldots, \frac{\Theta_{\alpha}}{\rho(\Theta_0)}) \mathbb{I}(r\rho(\Theta_0) > 1)], \quad u \rightarrow \infty.
\]

The passage to the limit is justified by (iii\(\_+\)), the continuity of \(Y\), and the independence of \(Y\) and \((\Theta_t)_{t \in \mathbb{Z}}\). By Fubini’s theorem, the expression on the right-hand side is equal to

\[
E[f_0(\Theta_0, \ldots, \Theta_{\alpha}) \mathbb{I}(\rho(\Theta_0) > 1)] d(\rho(\Theta_0)) - \int_r^\infty E[f_0(\Theta_0, \ldots, \Theta_{\alpha}) \mathbb{I}(\rho(\Theta_0) > 1)] d(\rho(\Theta_0)) = \int_r^\infty E[f_0(\Theta_0, \ldots, \Theta_{\alpha}) \mathbb{I}(\rho(\Theta_0) > 1)] d(\rho(\Theta_0)) - \int_r^\infty E[f_0(\Theta_0, \ldots, \Theta_{\alpha}) \mathbb{I}(\rho(\Theta_0) > 1)] d(\rho(\Theta_0)) = \int_r^\infty E[f_0(\Theta_0, \ldots, \Theta_{\alpha}) \mathbb{I}(\rho(\Theta_0) > 1)] d(\rho(\Theta_0)).
\]

We arrive at (37).

Finally, the proof of (38) is immediate in view of the dominated convergence theorem, the boundedness of \(f\), and the integrability of \(\rho(\Theta_0)^\alpha\), see (24).

We have now proven (36), (37) and (38) and thus (35). If the function \(f\) is such that \(f(\theta_{-s}, \ldots, \theta_t) = 0\) as soon as \(\theta_{-s} = 0\), then \(f = f_0\) and (33) follows. This finishes the proof of Lemma 6.2.

By Lemma 6.2 and the tightness argument preceding it, condition (iii\(\_+\)) implies that the limit in distribution

\[
\mathcal{L}(X_{-\alpha}/\rho(X_0), \ldots, X_{\alpha}/\rho(X_0) | \rho(X_0) > u), \quad u \rightarrow \infty,
\]

exists for all nonnegative integer \(s\) and \(t\). By the Daniell–Kolmogorov extension theorem (Pollard 2002, Chapter 4, Theorem 53), these limits in distributions are the ‘finite-dimensional’ distributions of a random element \((\Theta_t)_{t \in \mathbb{Z}}\) in the product space \(S^\mathbb{Z}\). Statement (iii) concerning weak convergence in \(S^\mathbb{Z}\) then follows from the convergence in the previous display for all \(s\) and \(t\) together with van der Vaart and Wellner (1996, Theorem 1.4.8).
It remains to show equation (29). The weak convergence established in the previous paragraph together with Lemma 6.2 imply that for bounded, Lipschitz continuous functions \( f : \mathbb{S}^{t+s+1} \to \mathbb{R} \) vanishing on \( \{ (\theta_s, \ldots, \theta_t) \in \mathbb{S}^{t+s+1} : \theta_s = 0 \} \), we have

\[
\mathbb{E}[f(\Theta_{t-s}, \ldots, \Theta_t)] = \lim_{u \to \infty} \mathbb{E} \left[ f \left( \frac{X_{-1}}{\rho(\Theta_0)}, \ldots, \frac{X_t}{\rho(\Theta_0)} \right) \bigg| \rho(X_0) > u \right] = \mathbb{E} \left[ f \left( \frac{\Theta_0}{\rho(\Theta_0)}, \ldots, \frac{\Theta_{t+1}}{\rho(\Theta_0)} \right) \rho(\Theta_0)^\alpha \right]. \tag{39}
\]

Let \( g : \mathbb{S}^{t+s+1} \to \mathbb{R} \) be bounded and Lipschitz continuous. Write \( g(\Theta_{t-s}, \ldots, \Theta_t) \) as a telescoping sum of \( s+1 \) terms:

\[
g(\Theta_{t-s}, \ldots, \Theta_t) = g(\Theta_{t-s}, \ldots, \Theta_t) - g(0, \Theta_{s-1}, \ldots, \Theta_t) + g(0, \Theta_{s-1}, \ldots, \Theta_t) - g(0, 0, \Theta_{s-2}, \ldots, \Theta_t) + \cdots + g(0, \ldots, 0, \Theta_{s-1}, \ldots, \Theta_t) - g(0, \ldots, 0, \Theta_0, \ldots, \Theta_t) + g(0, \ldots, 0, \Theta_0, \ldots, \Theta_t).
\]

Take expectations on both sides and apply (39) to the first \( s \) lines of the right-hand side of the previous display at \( s \) replaced by \( s, s-1, \ldots, 1 \), respectively, to obtain

\[
\mathbb{E}[g(\Theta_{t-s}, \ldots, \Theta_t)] = \mathbb{E} \left[ g \left( \frac{\Theta_0}{\rho(\Theta_0)}, \ldots, \frac{\Theta_{s-1}}{\rho(\Theta_0)} \right) \right] - \mathbb{E} \left[ g \left( 0, \frac{\Theta_0}{\rho(\Theta_t)}, \ldots, \frac{\Theta_{s-2}}{\rho(\Theta_t)} \right) \right] + \mathbb{E} \left[ g \left( 0, \frac{\Theta_0}{\rho(\Theta_t)}, \ldots, \frac{\Theta_{s-2}}{\rho(\Theta_t)} \right) \right] - \mathbb{E} \left[ g \left( 0, 0, \frac{\Theta_{s-3}}{\rho(\Theta_t)}, \ldots, \frac{\Theta_{s-2}}{\rho(\Theta_t)} \right) \right] + \cdots + \mathbb{E} \left[ g \left( 0, \ldots, 0, \frac{\Theta_{s-2}}{\rho(\Theta_t)}, \ldots, \frac{\Theta_0}{\rho(\Theta_t)} \right) \right] - \mathbb{E} \left[ g \left( 0, \ldots, 0, \frac{\Theta_{s-2}}{\rho(\Theta_t)}, \ldots, \frac{\Theta_0}{\rho(\Theta_t)} \right) \right] + \mathbb{E} \left[ g \left( 0, \ldots, 0, \frac{\Theta_{s-2}}{\rho(\Theta_t)}, \ldots, \frac{\Theta_0}{\rho(\Theta_t)} \right) \right]. \tag{40}
\]

The equality in the preceding display being true for all bounded and Lipschitz continuous functions \( g : \mathbb{S}^{t+s+1} \to \mathbb{R} \), it must hold whenever \( g \) is the indicator function of a closed set (Billingsley, 1999, proof of Theorem 1.2) and then, by a standard argument, also for all measurable functions \( \mathbb{S}^{t+s+1} \to \mathbb{R} \) that are integrable with respect to the law of \( (\Theta_{t-s}, \ldots, \Theta_t) \). For such functions that vanish whenever their first argument is equal to zero, the formula in the preceding display simplifies to (29) again.

This concludes the proof of Theorem 6.1.

7 Discussion

On \( S = [0, \infty)^2 \), the function \( \rho(\alpha, \beta) = \min(\alpha, \beta) \) is not a modulus, since condition (iii) in Definition 2.2 is not satisfied. Similarly, Dombry and Ribatet (2015) consider ‘cost
functionals’ that satisfy conditions (i) and (ii) but not necessarily (iii) in Definition 3.2. Without the latter condition, however, regular variation in $S$ can no longer be characterized via a polar decomposition as in Proposition 5.1 since sets of the form \( \{ x : \rho(x) < r \} \) do no longer form a neighbourhood base of the origin.

Relative to such ‘pseudo-moduli’, hidden regular variation (Resnick, 2007) on subcones may still occur. The notion of $M_0$-convergence then needs to be replaced by a more general one, involving sets that are bounded away from some ‘forbidden set’ which may be larger than a singleton. In $S = [0, \infty)^2$, one could for instance exclude the union of the two coordinate axes. Such a concept of regular variation is relevant for stochastic volatility models, for example, which exhibit asymptotic independence and therefore trivial spectral tail processes in the sense of this paper (Davis et al., 2013; Janssen and Drees, 2016). A complication, however, is that the index of hidden regular variation may depend on the time lag (Kulik and Soulier, 2013). A general treatment for such time series in metric spaces is an interesting research problem.

### A Convergence of measures

We consider a complete, separable metric space $(S, d)$ and some point $0 \in S$. For $A \subset S$, let $A^+$ and $A^-$ denote the interior and closure of $A$, respectively, and let $\partial A = A^- \setminus A^+$ be the boundary of $A$. Recall $B_u = \{ x \in S : d(x, 0) < u \}$ for $u > 0$ as well as the space $M_0(S)$ from Section 3. Let $M_0(\mathcal{X})$ denote the set of finite Borel measures on some metric space $\mathcal{X}$ and define convergence of measures in $M_0(\mathcal{X})$ by the usual notion of weak convergence, i.e., convergence of integrals of bounded, continuous functions from $\mathcal{X}$ into $\mathbb{R}$. We begin with a variation on Theorem 2.2 in (Hult and Lindskog, 2006).

**Lemma A.1**

(i) Assume $\mu_n \rightarrow \mu$ in $M_0(S)$ as $n \rightarrow \infty$ and let $f : S_0 \rightarrow \mathbb{R}$ be bounded, measurable, and vanish on $B_u$, for some $u > 0$. Let $D$ be the discontinuity set of $f$. If $\mu(D) = 0$, then $\int f \, d\mu_n \rightarrow \int f \, d\mu$ as $n \rightarrow \infty$.

(ii) If there exists a decreasing sequence of positive scalars $(r_i)_{i \in \mathbb{N}}$ with $r_i \rightarrow 0$ as $i \rightarrow \infty$ such that for each $i$, there exists a neighbourhood of the origin $0$, say $N_i$, such that $N_i \subset B_{r_i}$, and $\mu_n(\cdot \setminus N_i) ightarrow \mu(\cdot \setminus N_i)$ in $M_0(S \setminus N_i)$, then $\mu_n \rightarrow \mu$ in $M_0(S)$ as $n \rightarrow \infty$.

**Proof** (i) Let $r \in (0, u)$ be such that $\mu(\partial B_{r_0}) = 0$. Let $\mu_n^{(r)}$ and $\mu_n^{(i)}$ denote the restrictions of $\mu_n$ and $\mu$ to $S \setminus B_{r_0}$, respectively. By the (proof of) Theorem 2.2 in (Hult and Lindskog, 2006), we have weak convergence $\mu_n^{(r)} \rightarrow \mu^{(r)}$ as $n \rightarrow \infty$ in the space $M_0(S \setminus B_{r_0})$. By the continuous mapping theorem for weak convergence of finite measures, $\int_{S_0} f \, d\mu_n = \int_{S \setminus B_{r_0}} f \, d\mu_n \rightarrow \int_{S \setminus B_{r_0}} f \, d\mu^{(r)} = \int_{S_0} f \, d\mu$ as $n \rightarrow \infty$.

(ii) For any $f \in \mathcal{C}(S)$, there exists $i \in \mathbb{N}$ such that $f$ vanishes on $B_{r_i}$, and consequently on $N_i$. Since $\mu_n(\cdot \setminus N_i) \rightarrow \mu(\cdot \setminus N_i)$ in $M_0(S \setminus N_i)$, we have $\int_{S \setminus N_i} f \, d\mu_n = \int_{S \setminus N_i} f \, d\mu \rightarrow \int_{S \setminus N_i} f \, d\mu = \int_{S_0} f \, d\mu$. Therefore, $\mu_n \rightarrow \mu$ in $M_0(S)$ as $n \rightarrow \infty$.

The following lemma is useful for proving convergence in distribution.

**Lemma A.2** Let $(S, d)$ be a separable metric space. Let $(X_n, Y_n)$ and $(X, Y)$ be random elements in $\mathbb{R} \times S$. Then $(X_n, Y_n) \Rightarrow (X, Y)$ if and only if

$$E[\mathbb{I}[X_n \leq x]g(Y_n)] \rightarrow E[\mathbb{I}[X \leq x]g(Y)] \quad (n \rightarrow \infty)$$

for every continuity point $x \in \mathbb{R}$ of $X$ and every bounded and continuous function $g : S \rightarrow \mathbb{R}$.

**Proof** The ‘only if’ part is a special case of the continuous mapping theorem. So assume (41) holds. Taking $g \equiv 1$ yields $X_n \Rightarrow X$. Taking $x$ arbitrarily large so that $P(X > x)$ is arbitrarily small yields $Y_n \Rightarrow Y$. As a consequence, the sequence $(X_n, Y_n)$ is tight. It remains to show that the joint distribution of $(X, Y)$ is determined by expectations as in the right-hand side (41). By Lemma 1.4.2 in van der Vaart and Wellner (1996), the joint distribution of $(X, Y)$ is determined by expectations of the form $E[\mathbb{I}(f(X)g(Y))]$ with $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : S \rightarrow \mathbb{R}$ nonnegative, Lipschitz continuous, and bounded. It then suffices to write $f$ as the limit of an increasing sequence of step functions whose jump locations are continuity points of $X$. 

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The following theorem is similar to Theorems 2.2 and 2.3 in [Billingsley (1999)] and provides a criterion for convergence in $M_0(S)$.

**Theorem A.1** Suppose that $\mathcal{A}$ is a $\pi$-system on $S$ satisfying the following two conditions:

(C1) There exists a decreasing sequence $(r_i)_{i \in \mathbb{N}}$ of positive scalars with $r_i \to 0$ as $i \to \infty$ such that for each $i$, there exists a neighbourhood of the point 0, say $N_i$, such that $N_i \subset B_{r_i}$, and $A \setminus N_i \in \mathcal{A}$ for all $A \in \mathcal{A}$.

(C2) Each open subset $G$ of $S$ with $0 \notin G$ is a countable union of $\mathcal{A}$-sets.

If $\mu_n(A) \to \mu(A)$ as $n \to \infty$, for all $A$ in $\mathcal{A}$, then $\mu_n \Rightarrow \mu$ in $M_0(S)$ as $n \to \infty$.

**Proof** Let $i \in \mathbb{N}$; by Lemma A.1 it is sufficient to show that $\mu_n(\cdot \cap N_i) \to \mu(\cdot \cap N_i)$ in $M_0(S,N_i)$ as $n \to \infty$.

To do so, we apply the Portmanteau theorem for weak convergence of finite measures [Billingsley (1999), Theorem 2.1]. Any open subset of $(S,i)$ can be written as $G \setminus N_i$ where $G \subset S$ is open and $0 \notin G$; we need to show that $\liminf_{n \to \infty} \mu_n(G \setminus N_i) \geq \mu(G \setminus N_i)$. Let $A_1,A_2,\ldots$ be a sequence in $\mathcal{A}$ such that $G = \bigcup_{j=1}^{\infty} A_j$.

Write $A_j = A_j \setminus N_i \in \mathcal{A}$. Since $\mathcal{A}$ is a $\pi$-system and by the condition that $\lim_{n \to \infty} \mu_n(A) = \mu(A)$ for every $A \in \mathcal{A}$, we find, in view of the inclusion-exclusion formula, $\lim_{n \to \infty} \mu_n(\bigcup_{j=1}^{k} A_j) = \mu(\bigcup_{j=1}^{k} A_j)$ for every integer $k \geq 1$. But $\mu(\bigcup_{j=1}^{k} A_j) = \lim_{n \to \infty} \mu_n(\bigcup_{j=1}^{k} A_j)$ is bounded by $\liminf_{n \to \infty} \mu_n(G \setminus N_i)$, as required.

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