Two-grid algorithm for the solution of singularly perturbed two-parameter problem on Shishkin mesh

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Abstract. A boundary value problem for a second-order semilinear singularly perturbed ordinary differential equation with two small parameters affecting the convection and diffusion terms is considered. We use Newton and Picard iterations for a linearization. To solve the problem at each iteration we apply the second order difference scheme on the Shishkin mesh which converges uniformly with respect to both singular perturbation parameters. To decrease the required number of arithmetical operations for resolving the difference scheme, a cascadic two-grid method is proposed. To increase the accuracy of difference scheme, we investigate the possibility to apply Richardson extrapolation using known solutions of the difference scheme on both meshes. The results of some numerical experiments are discussed.

1. Introduction

It is well known that the application of classical difference schemes for a singularly perturbed problem leads to large errors for small values of perturbation parameters. The uniform convergence of a difference scheme for such problem can be provided by fitting the scheme to a boundary layer component [1, 2] or by using a mesh which is dense in a boundary layer [3, 4]. We consider a singular perturbation boundary value problem for a second-order nonlinear ordinary differential equation with two small parameters affecting the convection and diffusion terms on a piecewise-uniform mesh. Difference schemes of higher accuracy for nonlinear singular perturbation boundary value problems are very important, see [5–9] and the references therein. In [8], a scheme with an accuracy of \( O(\ln^2 N/N^2) \) based on asymptotic constructions is proposed for a nonlinear elliptic equation at small values of the parameter \( \varepsilon \). A nonlinear equation of the convection-diffusion type is investigated as a model one.

The two-grid method were investigated in [9–18] and in other works. According to the conception of the two-grid algorithm, if the difference scheme is resolved based on iterations then at first the problem is solved on a coarse mesh. Secondly, the new mesh solution is interpolated to nodes of the fine mesh and is used as the initial guess for following iterations. It leads to reduction of the number of iterations on the fine mesh and also reduces the number of arithmetical operations. A two-grid method to solve a nonlinear singular perturbation boundary value problem was investigated in [9, 12–18]. To increase \( \varepsilon \)-uniform accuracy of the difference scheme on Shishkin mesh without additional calculations, we use Richardson extrapolation formula [6, 19, 20] in two-grid method.
In [21] a second order monotone numerical method for a second order linear singularly perturbed ordinary differential equation with two small parameters affecting the convection and diffusion terms based on the difference scheme of a second order of accuracy on a second order semilinear singularly perturbed equation with two small parameters affecting the uniformly with respect to both singular perturbation parameters.

The aim of this work is to investigate a two-grid method using Richardson extrapolation for a second order semilinear singularly perturbed equation with two small parameters affecting the convection and diffusion terms based on the difference scheme of a second order of accuracy on the Shishkin mesh.

**Notation:** Let \( ||f|| = \max_{x \in \Omega} |f(x)| \) be the norm of a continuous argument function, and \( ||f^N||_N = \max_{0 \leq i \leq N} |f^N_i| \) be the norm of the mesh function. Let \([u]_\Omega\) be the projection of a function \(u(x)\) on a mesh \(\Omega\). Here \(C\), sometimes subscripted, denotes a generic positive constant that is independent of the perturbation parameters \(\varepsilon\), \(\mu\) and the step size of the mesh.

### 2. Preliminaries

Consider the boundary value problem:

\[
Lu(x) = \varepsilon u''(x) + \mu a(x)u'(x) = f(x, u(x)), \quad x \in \Omega = (0, 1),
\]

where the functions \(a\) and \(f\) are sufficiently smooth and

\[
0 < \varepsilon \leq 1, \quad 0 \leq \mu \leq 1, \quad a(x) \geq \alpha > 0, \quad f_u'(x, u) \geq \beta > 0, \quad (x, u) \in \Omega \times R.
\]

The solution \(u(x)\) generally has two boundary layers near \(x = 0\) and \(x = 1\).

If conditions (2) are satisfied, the solution problem (1) is uniformly bounded with respect to \(\varepsilon\) and \(\mu\):

\[
||u|| \leq C_0 = \beta^{-1}\|f(x, 0)\| + \max\{|A|, |B|\}
\]

and in accordance with [21, 22] in the case of \(\mu^2 = \varepsilon\) the following estimate of the derivative is valid:

\[
|u'(x)| \leq C \left(1 + \varepsilon^{-1/2} \left(\varepsilon^{-0.5} + \varepsilon^{-0.5} \right)\right), \quad 0 \leq x \leq 1,
\]

\[
\mu_1 = \frac{1}{2} \left(\sqrt{\alpha^2(0) + 4\beta + a(0)}\right), \quad \mu_2 = \frac{1}{2} \left(\sqrt{\alpha^2(1) + 4\beta + a(1)}\right).
\]

Assume that, in addition to (2), the following conditions are satisfied:

\[
\beta^* \geq f_u'(x, u) \geq \beta > 0, \quad (x, u) \in \Omega \times R,
\]

and \(\gamma = \min_{x \in \Omega} \{f_u'(x, u)/a(x)\}\).

According to [4, 21], we specify a mesh:

\[
\Omega_N = \{x_i : x_i = x_{i-1} + h_i, \quad x_0 = 0, \quad x_N = 1, \quad i = 1, 2, \ldots, N\},
\]

where

\[
h_i = \frac{4\sigma_1}{N}, \quad 1 \leq i \leq N/4, \quad h_i = \frac{2(1 - \sigma_1 - \sigma_2)}{N}, \quad N/4 < i \leq 3N/4, \quad h_i = \frac{4\sigma_2}{N}, \quad 3N/4 < i \leq N,
\]

\[
\sigma_1 = \left\{\min \left\{\frac{1}{4}, \frac{\varepsilon}{\mu a \ln N}\right\}, \quad \mu^2 \leq \frac{\varepsilon}{\alpha}, \quad \min \left\{\frac{1}{4}, \frac{\varepsilon}{\mu a \ln N}\right\}, \quad \mu^2 \geq \frac{\varepsilon}{\alpha}\right\},
\]

\[
\sigma_2 = \left\{\min \left\{\frac{1}{2}, \frac{\varepsilon}{\mu a \ln N}\right\}, \quad \mu^2 \leq \frac{\varepsilon}{\alpha}, \quad \min \left\{\frac{1}{2}, \frac{\varepsilon}{\mu a \ln N}\right\}, \quad \mu^2 \geq \frac{\varepsilon}{\alpha}\right\}.
\]
Consider the finite difference scheme that uses the central difference, upwind and mid-point schemes on the Shishkin mesh (4):

\[
L^N u_j^N = r_j^- u_{j-1}^N + r_j^+ u_{j+1}^N + Q^N(f(x_j, u_j^N)), \quad 0 < j < N,
\]

\[
L^N u_0^N = A, \quad u_N^N = B,
\]

where

\[
L^N = \begin{cases} L_{cd}, & \text{if } 1 \leq j < \frac{N}{4}, \\ L_{mp}, & \text{if } j = \frac{N}{4}, \\ L_{up}, & \text{otherwise}. \end{cases}
\]

\[
L^N = \begin{cases} L_{cd}, & \text{if } j = \frac{N}{4}, \\ L_{mp}, & \text{if } N < j < \frac{3N}{4}, \\ L_{up}, & \text{otherwise}. \end{cases}
\]

\[
L^N = \begin{cases} L_{cd}, & \text{if } j = \frac{3N}{4}, \\ L_{mp}, & \text{if } \frac{3N}{4} < j < N, \end{cases}
\]

Here we use the notation from [21]:

\[
L_{cd}^N u_j^N \equiv \varepsilon \delta^2 u_j^N + \mu a_j D^0 u_j^N = f(x_j, u_j^N),
\]

\[
L_{up}^N u_j^N \equiv \varepsilon \delta^2 u_j^N + \mu a_j D^+ u_j^N = f(x_j, u_j^N),
\]

\[
L_{mp}^N u_j^N \equiv \varepsilon \delta^2 u_j^N + \mu a_j D^- u_j^N = \tilde{f}(x_j, u_j^N),
\]

where \( a_j = a(x_j) \), \( z_j = (z_j + z_{j+1})/2 \) and

\[
\delta^2 u_j^N = \frac{1}{h_j} \left( \frac{u_{j+1}^N - u_j^N}{h_{j+1}} - \frac{u_j^N - u_{j-1}^N}{h_j} \right), \quad D^0 u_j^N = \frac{u_{j+1}^N - u_{j-1}^N}{h_{j+1} + h_j}, \quad D^+ u_j^N = \frac{u_{j+1}^N - u_j^N}{h_{j+1}}.
\]

According to [21], assume that

\[
N (\ln N)^{-1} > 8 \max \{ \|a\|/\alpha, \beta^*/(\alpha \gamma) \}.
\]

The solution of the difference scheme (5) can be resolved based on iterations. Consider the Newton linearization:

\[
L^{m,N} u_{j+1}^{m+1,N} = r_j^- u_{j-1}^{m+1,N} + \left( r_j^c - f'_{u_j}^c(x_j, u_j^{m,N}) \right) u_j^{m+1,N} + r_j^+ u_{j+1}^{m+1,N} = Q^N(f(x_j, u_j^{m,N}) - f_{u_j}^c(x_j, u_j^{m,N})u_j^{m,N}), \quad 0 < j < N,
\]

\[
u_0^{m+1,N} = A, \quad u_N^{m+1,N} = B, \quad m \geq 0.
\]
Then analogically consider the Picard linearization.

\[
    r_j^{-1} u_{j-1}^{m+1,N} + (r_j^c - \beta^*) \ u_j^{m+1,N} + r_j^{+} u_{j+1}^{m+1,N} = Q_N^{N} (f(x_j, u_j^{m,N}) - \beta^* u_j^{m,N}), \quad 0 < j < N, \]

\[
    u_0^{m+1,N} = A, \quad u_N^{m+1,N} = B, \quad m \geq 0.
\]

Consider the case of the linear problem (1):

\[
    f(x, u) = b(x)u + g(x), \quad b(x) \geq \beta > 0.
\]

According to [21], the following theorem is valid.

**Theorem 1** Let \( u(x) \) be the solution to linear problem (1) with sufficiently smooth coefficients and right-hand side, and let \( u^N \) be the solution to the difference scheme (5). Then on the Shishkin mesh (4) for a constant \( C \) we have:

\[
    \| [u]_{\Omega_N} - u^N \|_N \leq C \Delta_N = \begin{cases} 
    C \ln^3 N/N^2, & \gamma \varepsilon < \alpha \mu^2 \\
    C \ln^2 N/N^2, & \gamma \varepsilon \geq \alpha \mu^2.
\end{cases}
\]

uniformly with respect to \( \varepsilon \) and \( \mu \).

### 3. Two-grid method using Richardson extrapolation

To decrease the required number of arithmetical operations for resolving the difference scheme, a two-grid method is proposed. According to the idea of the two-grid algorithm [12], at first, the initial problem (1) is solved on a coarse mesh. Secondly, the found mesh solution is interpolated to nodes of the fine mesh and is used as the initial guess for following iterations. It leads to reduction of the number of iterations on the fine mesh and also reduces the number of arithmetical operations. Note that the interpolation formula must be uniform with respect to the singular perturbation parameters else the accuracy of the found mesh solution may be lost.

Thus, let \( \Omega_n \) be the Shishkin mesh corresponding to (4) and containing \( n \) mesh intervals, where we take \( n \ll N \). At first, problem (1) is preliminarily solved on the mesh \( \Omega_n \). The Picard or Newton iterations on the mesh \( \Omega_n \) are performed until the inequality

\[
    \| u^{m,n} - u^n \|_n \leq \Delta_n.
\]

is satisfied. Then the mesh solution \( u^{m,n} \) found on the mesh \( \Omega_n \) is interpolated to the nodes of the initial mesh \( \Omega_N \) using an appropriate interpolation which is uniform with respect to the parameters \( \varepsilon \) and \( \mu \) and for some constant \( C \)

\[
    \| \text{Int}([u]_{\Omega_n}, x) - u(x) \| \leq C \Delta_n.
\]

Therefore, we have for some constant \( C \)

\[
    \|[u]_{\Omega_n} - u^{m,n} \|_n \leq C \Delta_n.
\]

Now we specify an initial guess for the iterations on the initial mesh \( \Omega_N \) and we have for some constant \( C \)

\[
    \| u^{0,N} - u \|_N \leq C \Delta_n, \quad u^{0,N} = \text{Int}(u^{m,n}, x)|_{\Omega_N}.
\]

Thus, using iterations on the coarse mesh and an appropriate interpolation, an initial guess \( u^{0,N} \) for iterations on the mesh \( \Omega_N \) with an accuracy of \( O(\Delta_n) \) is constructed. It remains to perform iterations on the mesh \( \Omega_N \) until an accuracy of \( O(\Delta_N) \) is achieved.
Let $M_N$ be the necessary number of arithmetical operations required for a one-grid method and $M_{Nn}$ be the necessary number of arithmetical operations required for a two-grid method. We assume that it is necessary to fulfill $dN$ arithmetical operations to perform one iteration. Then according to [13, 14], the decrease in the number of arithmetic operations in the two-grid Picard method is estimated as

$$M_N - M_{Nn} \approx d(N - n) \frac{\ln(\Delta_n/\delta)}{\ln(1 - \beta/\beta^*)} - J_n,$$

where $\delta = \|u^N - u^0\|$ and $\beta$, $\beta^*$ corresponds to (3). For the two-grid Newton method we have:

$$M_N - M_{Nn} \approx d(N - n) \log_2 \frac{\ln(\alpha^{-1}\theta\Delta_n)}{\ln(\alpha^{-1}\theta\delta)} - J_n,$$

where $\theta = \max_{x \in \Omega} \{ | \partial^n u(x, \xi) | \}$ and $J_n$ is the necessary number of arithmetical operations required for the interpolation method.

To increase the accuracy of the difference scheme, the Richardson extrapolation in two-grid method is applied [6, 13–15, 19, 20]. For this reason the mesh $\Omega_n$ must have the same value of the parameters $\sigma_1$ and $\sigma_2$ as the mesh $\Omega_N$. Thus these meshes are nested that is

$$\Omega_n = \{ x_j \} \subset \Omega_N = \{ x_i \}.$$

Let us $N = kn$, where $k$ is some integer. Then obviously that the initial mesh $\Omega_N$ can be obtained from the coarse mesh $\Omega_n$ by dividing each intervals into $k$ equal parts.

Let $u^n$ be the solution of the difference scheme on the mesh $\Omega_n$. According to Richardson method we introduce $u^{Nn}$ on the mesh $\Omega_N$. At first, let us define $u^{Nn}$ on the mesh $\Omega_n$ as

$$u^{Nn}(X_j) = k_n u^n(X_j) + k_N u^N(X_j), \quad X_j \in \Omega_n,$$

where $k_n = -n^2/(N^2 - n^2) = -1/(k^2 - 1)$, $k_N = N^2/(N^2 - n^2) = k^2/(k^2 - 1)$.

At the nodes of the initial mesh don’t coinciding with nodes of the coarse mesh, we define $u^{Nn}(x_i)$ for any $x_i \in \Omega_N$ by using an appropriate interpolation. Note that the interpolation formula must be uniform with respect to the singular perturbation parameters $\varepsilon$ and $\mu$ else the accuracy of the found mesh solution may be lost [13–18, 23].

So, we constructed the mesh solution $u^{Nn}$ on the mesh $\Omega_N$ using Richardson extrapolation.

4. Results of numerical experiments

Consider the following boundary value problem:

$$\varepsilon u'' + 2 \mu u' = 4u - 1 + u e^u + g(x), \quad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) = 1,$$

where $g(x)$ corresponds to the exact solution:

$$u(x) = 0.25 + \frac{3 + e^{-m_2/\varepsilon}}{4(e^{-m_1/\varepsilon} - e^{-m_2/\varepsilon})} e^{-m_1 x/\varepsilon} - \frac{3 + e^{-m_1/\varepsilon}}{4(e^{-m_1/\varepsilon} - e^{-m_2/\varepsilon})} e^{-m_2 x/\varepsilon},$$

where $m_1 = \mu - \sqrt{\mu^2 + 4\varepsilon}$, $m_2 = \mu + \sqrt{\mu^2 + 4\varepsilon}$. Note that $0 \leq u(x) \leq 1$ and $\alpha = 2$, $\beta = 5$, $\|a\| = 2$, $\beta^* = 4 + 2\varepsilon$, $\gamma = 2.5$. 

Taking into account the error of the scheme (6) we stop iterations on the mesh $\Omega_N$ when the following condition is satisfied

$$\|L_N^m u_{m,N} - f_N\|_N \leq \beta \Delta_N,$$

where $\beta$ corresponds to (2). Then from the estimate

$$\|u_{m,N} - u^N\|_N \leq \beta^{-1} \|L_N^m u_{m,N} - f_N\|_N$$

follows:

$$\|u_{m,N} - u^N\|_N \leq \Delta_N.$$

Table 1 contains the number of Picard iterations for a two-grid method for $\mu = 2^{-3}$ (left table) and for $\mu = 2^{-5}$ (right table) on the mesh $\Omega_N$ in the case of $n = N/2$ for various values of $N$ and $\varepsilon$ in upper line. The number of Picard iterations on the mesh $\Omega_n$ is given in brackets. The number of iterations for a one-grid method depending on $N$ is given in the bottom line of the table.

| $\varepsilon$ | 64  | 256 | 1024 | 8192 |
|--------------|-----|-----|------|------|
| 1            | 1(4)| 1(5)| 1(6) | 1(8) |
| 4            | 5   | 7   | 9    |
| $2^{-4}$     | 2(4)| 1(6)| 1(8) | 1(10)|
| 5            | 7   | 9   | 11   |
| $2^{-7}$     | 2(5)| 2(7)| 2(8) | 3(11)|
| 6            | 7   | 9   | 12   |
| $2^{-9}$     | 3(4)| 3(7)| 3(8) | 3(11)|
| 6            | 7   | 9   | 11   |
| $2^{-11}$    | 3(4)| 3(6)| 3(8) | 3(10)|
| 5            | 7   | 9   | 11   |
| $2^{-14}$    | 3(4)| 3(6)| 3(8) | 3(10)|
| 5            | 7   | 9   | 11   |

Table 2 contains the number of Newton iterations for a two-grid method for $\mu = 2^{-3}$ (left table) and for $\mu = 2^{-5}$ (right table) on the mesh $\Omega_N$ in the case of $n = N/2$ for various values of $N$ and $\varepsilon$ in upper line. The number of Newton iterations on the mesh $\Omega_n$ is given in brackets. The number of iterations for a one-grid method depending on $N$ is given in the bottom line of the table.

Table 3 contains the error norm for a one-grid method (left table) and for a two-grid method with Richardson extrapolation (right table) in the case of $n = N/2$ for various values of $N$ and $\varepsilon$ for $\mu = 1$.

Table 4 contains the error norm for a one-grid method (left table) and for a two-grid method with Richardson extrapolation (right table) in the case of $n = N/2$ for various values of $N$ and $\varepsilon$ for $\mu = 2^{-3}$. 

Table 1. The number of Picard iterations for one-grid and two-grid methods for $\mu = 2^{-3}$ (left) and for $\mu = 2^{-5}$ (right).

| $\varepsilon$ | 64  | 256 | 1024 | 8192 |
|--------------|-----|-----|------|------|
| 1            | 1(4)| 1(5)| 1(6) | 1(8) |
| 4            | 5   | 7   | 9    |
| $2^{-4}$     | 2(4)| 1(6)| 1(8) | 1(10)|
| 5            | 7   | 9   | 11   |
| $2^{-7}$     | 2(5)| 2(7)| 2(8) | 3(11)|
| 6            | 7   | 9   | 12   |
| $2^{-9}$     | 3(4)| 3(7)| 3(8) | 3(11)|
| 6            | 7   | 9   | 11   |
| $2^{-11}$    | 3(4)| 3(6)| 3(8) | 3(10)|
| 5            | 7   | 9   | 11   |
| $2^{-14}$    | 3(4)| 3(6)| 3(8) | 3(10)|
| 5            | 7   | 9   | 11   |
The error norm for a one-grid method (left) and for a two-grid method (right) for \( \mu = 2^{-5} \) (right).

| Table 2 |
| --- |
| **\( \varepsilon \)** | 64 | 256 | 1024 | 8192 |
| --- | --- | --- | --- | --- |
| 1 | 1(2) | 1(3) | 1(3) | 1(3) |
| 2^{-4} | 1(3) | 1(3) | 1(3) | 1(4) |
| 2^{-7} | 1(3) | 1(3) | 1(3) | 1(4) |
| 2^{-9} | 2(3) | 1(3) | 1(3) | 1(4) |
| 2^{-11} | 2(3) | 1(3) | 1(3) | 1(4) |
| 2^{-14} | 2(3) | 1(3) | 1(3) | 1(4) |
| 3 | 3 | 4 | 4 |

The error norm for a one-grid method (left) and for a two-grid method (right) for \( \mu = 2^{-3} \) (left) and for \( \mu = 2^{-3} \) (right).

| Table 3 |
| --- |
| **\( \varepsilon \)** | 64 | 256 | 1024 | 8192 |
| --- | --- | --- | --- | --- |
| 2^{-4} | 2.59e-3 | 1.64e-4 | 1.01e-5 | 1.58e-7 |
| 2^{-7} | 1.31e-2 | 1.30e-3 | 1.26e-4 | 3.31e-6 |
| 2^{-9} | 1.32e-2 | 1.31e-3 | 1.26e-4 | 3.33e-6 |
| 2^{-11} | 1.32e-2 | 1.31e-3 | 1.26e-4 | 3.33e-6 |
| 2^{-14} | 1.32e-2 | 1.31e-3 | 1.26e-4 | 3.34e-6 |

| **\( \varepsilon \)** | 64 | 256 | 1024 | 8192 |
| --- | --- | --- | --- | --- |
| 2^{-4} | 3.71e-4 | 2.01e-6 | 3.57e-7 | 6.89e-9 |
| 2^{-7} | 5.08e-3 | 6.92e-5 | 5.63e-7 | 2.64e-9 |
| 2^{-9} | 5.16e-3 | 7.25e-5 | 5.68e-7 | 2.10e-9 |
| 2^{-11} | 5.18e-3 | 7.24e-5 | 6.24e-7 | 1.98e-9 |
| 2^{-14} | 5.19e-3 | 7.23e-5 | 5.99e-7 | 1.44e-9 |

| Table 4 |
| --- |
| **\( \varepsilon \)** | 64 | 256 | 1024 | 8192 |
| --- | --- | --- | --- | --- |
| 2^{-4} | 2.16e-4 | 1.14e-5 | 6.38e-7 | 1.17e-8 |
| 2^{-7} | 1.17e-3 | 7.48e-5 | 4.67e-6 | 7.30e-8 |
| 2^{-9} | 7.93e-3 | 7.88e-4 | 7.72e-5 | 1.60e-6 |
| 2^{-11} | 8.97e-3 | 8.89e-4 | 8.60e-5 | 2.27e-6 |
| 2^{-14} | 9.35e-3 | 9.26e-4 | 8.95e-5 | 2.36e-6 |

| **\( \varepsilon \)** | 64 | 256 | 1024 | 8192 |
| --- | --- | --- | --- | --- |
| 2^{-4} | 1.42e-4 | 2.40e-6 | 1.42e-7 | 4.27e-9 |
| 2^{-7} | 1.43e-4 | 7.70e-6 | 1.03e-6 | 6.14e-9 |
| 2^{-9} | 2.57e-3 | 4.81e-5 | 4.22e-7 | 6.75e-9 |
| 2^{-11} | 3.29e-3 | 1.06e-4 | 4.63e-7 | 4.47e-9 |
| 2^{-14} | 3.58e-3 | 5.52e-5 | 3.76e-6 | 4.93e-9 |

Table 5 contains the error norm for a one-grid method (left table) and for a two-grid method with Richardson extrapolation (right table) in the case of \( n = N/2 \) for various values of \( N \) and \( \varepsilon \) for \( \mu = 2^{-5} \).

It follows from Tables 3-5 that the application Richardson extrapolation in a two-grid method increases the accuracy of the difference scheme to \( O(\ln^3 N/N^3) \) uniformly with respect to both singular perturbation parameters.
Table 5. The error norm for a one-grid method (left) and for a two-grid method (right) for $\mu = 2^{-5}$

| $\varepsilon$ | $N$ | $\varepsilon$ | $N$ |
|---------------|-----|---------------|-----|
| 64            | 1024 | 8192          |     |
| $2^{-4}$      | 1.85e-4 | 1.01e-5 | 3.62e-7 | 3.37e-9 |
| $2^{-7}$      | 1.20e-3 | 7.38e-5 | 4.14e-6 | 7.24e-8 |
| $2^{-9}$      | 4.12e-3 | 2.66e-4 | 1.66e-5 | 2.62e-7 |
| $2^{-11}$     | 7.26e-3 | 7.51e-4 | 5.60e-5 | 1.17e-6 |
| $2^{-14}$     | 1.62e-2 | 2.03e-3 | 1.04e-4 | 2.18e-6 |
| $2^{-4}$      | 9.30e-5 | 2.04e-6 | 3.11e-7 | 2.57e-9 |
| $2^{-7}$      | 1.30e-4 | 2.44e-6 | 2.23e-8 | 2.55e-9 |
| $2^{-9}$      | 6.75e-4 | 2.47e-6 | 2.63e-7 | 1.56e-9 |
| $2^{-11}$     | 2.96e-2 | 3.84e-5 | 1.01e-6 | 2.79e-9 |
| $2^{-14}$     | 5.43e-3 | 7.22e-4 | 4.08e-5 | 7.15e-9 |

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