GROMOV-WITTEN THEORY AND NOETHER-LEFSCHETZ THEORY

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Abstract. Noether-Lefschetz divisors in the moduli of $K3$ surfaces are the loci corresponding to Picard rank at least 2. We relate the degrees of the Noether-Lefschetz divisors in 1-parameter families of $K3$ surfaces to the Gromov-Witten theory of the 3-fold total space. The reduced $K3$ theory and the Yau-Zaslow formula play an important role. We use results of Borcherds and Kudla-Millson for $O(2,19)$ lattices to determine the Noether-Lefschetz degrees in classical families of $K3$ surfaces of degrees 2, 4, 6 and 8. For the quartic $K3$ surfaces, the Noether-Lefschetz degrees are proven to be the Fourier coefficients of an explicitly computed modular form of weight 21/2 and level 8. The interplay with mirror symmetry is discussed. We close with a conjecture on the Picard ranks of moduli spaces of $K3$ surfaces.

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0. Introduction

0.1. K3 families. Let $C$ be a nonsingular complete curve, and let

$$\pi : X \to C$$
be a 1-parameter family of nonsingular quasi-polarized $K3$ surfaces. Let $L \in \text{Pic}(X)$ denote the quasi-polarization of degree

$$\int_{K3} L^2 = l \in 2\mathbb{Z}^>0.$$  

The family $\pi$ yields a morphism,

$$\iota_\pi : C \to M_l,$$

to the 19 dimensional moduli space of quasi-polarized $K3$ surfaces of degree $l$. A review of the definitions can be found in Section 1.

0.2. Noether-Lefschetz numbers. Noether-Lefschetz numbers are defined by the intersection of $\iota_\pi(C)$ with Noether-Lefschetz divisors in $M_l$. Noether-Lefschetz divisors can be described via Picard lattices or Picard classes. We briefly review the two approaches.

Let $(L, v)$ be a rank 2 integral lattice with an even symmetric bilinear form

$$\langle , \rangle : L \times L \to \mathbb{Z}$$

and a distinguished primitive vector $v \in L$ satisfying

$$\langle v, v \rangle = l.$$  

The invariants of $(L, v)$ are the discriminant $\Delta \in \mathbb{Z}$ and the coset

$$\delta \in \left(\frac{\mathbb{Z}}{l\mathbb{Z}}\right) / \pm.$$  

If the data are presented as

$$L_{h, d} = \begin{pmatrix} l & d \\ d & 2h - 2 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then the discriminant is

$$\Delta_l(h, d) = - \det \begin{vmatrix} l & d \\ d & 2h - 2 \end{vmatrix} = d^2 - 2lh + 2l$$

and the coset is

$$\delta = d \text{ mod } l \in \left(\frac{\mathbb{Z}}{l\mathbb{Z}}\right) / \pm.$$  

Two lattices $(L_{h, d}, v)$ and $(L_{h', d'}, v')$ are equivalent if and only if

$$\Delta_l(h, d) = \Delta_l(h', d') \quad \text{and} \quad \delta_{h, d} = \delta_{h', d'}.$$  

However, not all pairs $(\Delta, \delta)$ are realized.

The first type of Noether-Lefschetz divisor is defined by specifying a Picard lattice. Let

$$P_{\Delta, \delta} \subset M_l$$
be the closure of the locus of quasi-polarized $K3$ surfaces $(S, L)$ of degree $l$ for which $(\text{Pic}(S), L)$ is of rank 2 with discriminant $\Delta$ and coset $\delta$. By the Hodge index theorem, $P_{\Delta, \delta}$ is empty unless $\Delta > 0$.

The second type of Noether-Lefschetz divisor is defined by specifying a Picard class. In case $\Delta_l(h, d) > 0$, let

$$D_{h,d} \subset \mathcal{M}_l$$

have support on the locus of quasi-polarized $K3$ surfaces $(S, L)$ for which there exists a class $\beta \in \text{Pic}(S)$ satisfying

$$\int_S \beta^2 = 2h - 2 \quad \text{and} \quad \int_S \beta \cdot L = d.$$

More precisely, $D_{h,d}$ is the weighted sum

$$D_{h,d} = \sum_{\Delta, \delta} \mu(h, d | \Delta, \delta) \cdot [P_{\Delta, \delta}]$$

where the multiplicity

$$\mu(h, d | \Delta, \delta) \in \{0, 1, 2\}$$

is defined to be the number of elements $\beta$ of the lattice $(\mathbb{L}, v)$ associated to $(\Delta, \delta)$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \quad \text{and} \quad \langle \beta, v \rangle = d.$$ 

If no lattice corresponds to $(\Delta, \delta)$, the multiplicity $\mu(h, d | \Delta, \delta)$ vanishes and $P_{\Delta, \delta}$ is empty. If the multiplicity is nonzero, then

$\Delta | \Delta_l (h, d).$

Hence, the sum on the right of (1) has only finitely many terms.

As relation (1) is easily seen to be triangular, the divisors $P_{\Delta, \delta}$ and $D_{h,d}$ are essentially equivalent. However, the divisors $D_{h,d}$ will be seen to have better formal properties.

A natural approach to studying the divisors $D_{h,d}$ is via intersections with test curves. In case $\Delta_l(h, d) > 0$, the Noether-Lefschetz number $NL_{h,d}^\pi$ is the classical intersection product

$$NL_{h,d}^\pi = \int_C t_\pi^* [D_{h,d}].$$

If $\Delta_l(h, d) < 0$, the divisor $D_{h,d}$ vanishes by the Hodge index theorem.

A definition of $NL_{h,d}^\pi$ for all values $\Delta_l(h, d) \geq 0$ is given by classical intersection in the period domain for $K3$ surfaces in Section \(\square\).
The divisibility of a nonzero element $\beta$ of a lattice is the maximal positive integer $m$ dividing $\beta$. Refined divisors $D_{m,h,d}$ are defined by

$$D_{m,h,d} = \sum_{\Delta, \delta} \mu(m, h, d | \Delta, \delta) \cdot [P_{\Delta, \delta}]$$

where the multiplicity

$$\mu(m, h, d | \Delta, \delta) \in \{0, 1, 2\}$$

is the number of elements $\beta$ of divisibility $m$ of the lattice $(\mathbb{L}, v)$ associated to $(\Delta, \delta)$ satisfying (2). Refined Noether-Lefschetz number are defined by

$$NL_{m,h,d}^{\pi} = \int_C \iota^*[D_{m,h,d}].$$

0.3. Invariants. We will study three types of invariants associated to a 1-parameter family $\pi$ of quasi-polarized $K3$ surfaces in case the total space $X$ is nonsingular:

(i) the Noether-Lefschetz numbers of $\pi$,
(ii) the Gromov-Witten invariants of $X$,
(iii) the reduced Gromov-Witten invariants of the $K3$ fibers.

The Noether-Lefschetz numbers (i) are classical intersection products while the Gromov-Witten invariants (ii)-(iii) are quantum in origin.

The Gromov-Witten invariants (ii) of the 3-fold $X$ and the reduced Gromov-Witten invariants (iii) of a $K3$ surface are defined via integration against virtual classes of moduli spaces of stable maps. We view both of these Gromov-Witten theories in terms of the associated BPS state counts defined by Gopakumar and Vafa [17, 18].

Let $n^{X}_{g,d}$ denote the Gopakumar-Vafa invariant of $X$ of genus $g$ for $\pi$-vertical curve classes of degree $d$ with respect to $L$. Let $r_{g,m,h}$ denote the Gopakumar-Vafa reduced $K3$ invariant of genus $g$ and curve class $\beta \in H_2(K3, \mathbb{Z})$ of divisibility $m$ satisfying

$$\int_{K3} \beta^2 = 2h - 2.$$ 

A review of these quantum invariants is presented in Section 2.

A geometric result intertwining the invariants (i)-(iii) is derived in Section 3 by a comparison of the reduced and usual deformation theories of maps of curves to the $K3$ fibers of $\pi$.

**Theorem 1.** For $d > 0$,

$$n^{X}_{g,d} = \sum_{h} \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL_{m,h,d}^{\pi}.$$
Theorem 1 is the main geometric result of the paper. The proof is given in Section 3.

0.4. Applications. Since Theorem 1 relates three distinct geometric invariants, the result can be effectively used in several directions.

An application for studying reduced invariants of $K3$ surfaces is given in [25]. A central conjecture discussed in Section 2.3 is the independence of $r_{g,m,h}$ on $m$. In genus 0, the independence is the non-primitive Yau-Zaslow conjecture proven in [25] as a consequence of Theorem 1.

The approach taken there is the following. For a specific 1-parameter family of $K3$ surfaces, known in the physics literature as the STU model, the BPS states $n_{0,d}^{STU}$ are known by proven mirror transformations and the Noether-Lefschetz numbers $NL_{m,h,d}^{STU}$ can by exactly determined. Theorem 1 is then used in [25] to solve for $r_{0,m,h}$:

$$r_{0,m,h} = r_{0,1,h}, \quad \sum_{h \geq 0} r_{0,1,h} = \prod_{n \geq 1} \frac{1}{(1 - q^n)^24}.$$

The genus 1 results

$$r_{1,m,h} = r_{1,1,h} = \frac{h}{12} r_{0,1,h}$$

are an easy consequence, see Section 2.3. We write $r_{g,m,h} = r_{g,h}$ independent of $m$ for $g = 0, 1$.

Using [25], the genus 0 and 1 specialization takes a much simpler form.

**Corollary 1.** For $g \leq 1$ and $d > 0$,

$$n_{g,d}^X = \sum_{h=g}^{\infty} r_{g,h} \cdot NL_{h,d}^\pi.$$

By Corollary 1, the Gromov-Witten invariants $n_{g,d}^X$ are completely determined by the Noether-Lefschetz numbers of $\pi$ for any 1-parameter family of quasi-polarized $K3$ surfaces. The result may be viewed as giving a fully classical interpretation of the Gromov-Witten invariants of $X$ in $\pi$-vertical classes.

Theorem 1 can also be used to constrain the Noether-Lefschetz degrees themselves. An important approach to the Noether-Lefschetz numbers (already used in the STU calculation) is via results of Borcherds [6] and Kudla-Millson [27]. The Noether-Lefschetz numbers of $\pi$ are

\footnote{If $m^2$ does not divide $2h - 2$, then $r_{g,m,h} = 0$. The independence is conjectured only when $m^2$ divides $2h - 2$. When we write $r_{g,m,h}$, the divisibility condition is understood to hold.}
proven to be the Fourier coefficients of a vector-valued modular form\footnote{While the paper \cite{6,27} have considerable overlap, we will follow the point of view of Borcherds.}. For several classical families of $K3$ surfaces, Corollary 1 in genus 0 provides an alternative method of calculating the Noether-Lefschetz numbers via the invariants $n_{0,d}^X$. Together, we obtain a remarkable sequence of identities intertwining hypergeometric series from mirror transformations (calculating $n_{0,d}^X$) and modular forms. The Harvey-Moore identity \cite{20} for the STU model is a special case.

As a basic example, we provide a complete calculation of the Noether-Lefschetz numbers for the family of $K3$ surfaces determined by a Lefschetz pencil of quartics in $\mathbb{P}^3$. The required mirror symmetry calculations (iii) for the quartic pencil have long been established rigorously \cite{15,16}. We give the derivation of the Noether-Lefschetz numbers via Gromov-Witten calculations in Section 5. The resulting hypergeometric-modular identity follows immediately in Section 5.5. A second approach to calculating Noether-Lefschetz numbers directly via more sophisticated modular form techniques is explained for quartics and several other classical families in Section 6.

Once the Noether-Lefschetz numbers are calculated for the 1-parameter family $\pi$, Corollary 1 yields the genus 1 Gromov-Witten invariants of $X$ in $\pi$-vertical classes. There are very few methods for the exact calculation of genus 1 invariants in Calabi-Yau geometries\footnote{See \cite{49} for a different mathematical approach to genus 1 invariants for complete intersections.}. Corollary 1 provides a new class of complete solutions.

0.5. Heterotic duality. In rather different terms, approach (i)-(iii) was pursued in the string theoretic work of Klemm, Kreuzer, Riegler, and Scheidegger \cite{24} with the goal of calculating the BPS counts $n_{g,d}^X$ from the genus 0 values $n_{0,d}^X$. Heterotic duality was used in \cite{24} for (i) since the connection to the intersection theory of the Noether-Lefschetz divisors

$$D_{h,d} \subset \mathcal{M}_l$$

and the work of Borcherds was not made. The perspective of \cite{24} can be turned upside down by using Gromov-Witten theory to calculate the Noether-Lefschetz numbers. On the other hand, modularity allows the calculations of \cite{24} to be pursued in much greater generality.

In fact, the back and forth here between heterotic duality and mathematical results is older. Borcherds’ paper on automorphic functions \cite{5} which underlies \cite{6} was motivated in part by the work of Harvey...
and Moore [20, 21] on heterotic duality. The first higher genus results for K3 fibrations were by Mariño and Moore [36].

Finally, we mention the circle ideas here can be considered for interesting isotrivial families of K3 surfaces with double Enriques fibers [26, 37]. While heterotic duality arguments apply there, Borcherds’ result does not directly apply.

0.6. Modular forms. Let $A$ and $B$ be modular forms of weight $1/2$ and level 8,

$$A = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad B = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}.$$ 

Let $\Theta$ be the modular form of weight $21/2$ and level 8 defined by

$$2^{22} \Theta = 3A^{21} - 81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 - 20007A^{16}B^5 - 169092A^{15}B^6 - 120636A^{14}B^7 - 621558A^{13}B^8 - 292796A^{12}B^9 - 1038366A^{11}B^{10} - 346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} - 361908A^7B^{14} - 56364A^6B^{15} - 60021A^5B^{16} - 4812A^4B^{17} - 1881A^3B^{18} - 27A^2B^{19} + B^{21}.$$ 

We can expand $\Theta$ as a series in $q^{1/8}$,

$$\Theta = -1 + 108q + 320q^{9/8} + 5016q^{16/8} + \ldots.$$ 

The modular form $\Theta$ was first found in calculations of [24].

Let $\pi$ be the family of quasi-polarized K3 surfaces determined by a Lefschetz pencil of quartics in $\mathbb{P}^4$. Let $\Theta[m]$ denote the coefficient of $q^m$ in $\Theta$.

**Theorem 2.** The Noether-Lefschetz numbers of the quartic pencil $\pi$ are coefficients of $\Theta$,

$$NL^\pi_{h,d} = \Theta \left[ \frac{\Delta_4(h,d)}{8} \right].$$ 

0.7. Classical quartic geometry. Let $V$ be a 4-dimensional $\mathbb{C}$-vector space. A quartic hypersurface in $\mathbb{P}(V)$ is determined by an element of $\mathbb{P}(\text{Sym}^4 V^*)$. Let

$$U \subset \mathbb{P}(\text{Sym}^4 V^*)$$

be the Zariski open set of nonsingular quartic hypersurfaces. Since $[S] \in U$ corresponds to a polarized K3 surface of degree 4, we obtain a canonical morphism

$$\phi : U \to \mathcal{M}_4.$$
If $\triangle_4(h, d) > 0$, the pull-back
\[ \mathcal{D}_{h,d} = \phi^{-1}(D_{h,d}) \subset U \]
is a closed subvariety of pure codimension 1. As a Corollary of Theorem 2, we obtain a complete calculation of the degrees of the hypersurfaces
\[ \overline{D}_{h,d} \subset \mathbb{P}(\text{Sym}^4 V^*) . \]

**Corollary 2.** If $\triangle_4(h, d) > 0$, the degree of $\overline{D}_{h,d}$ is
\[ \deg(\overline{D}_{h,d}) = \Theta \left[ \frac{\triangle_4(h, d)}{8} \right] - \Psi \left[ \frac{\triangle_4(h, d)}{8} \right] \]
where the correction term is
\[ \Psi = 108 \sum_{n>0} q^{n^2} . \]

The correction term, obtained from the contribution of the nodal quartics, is explained in Section 5.6. Formulas for the degrees of
\[ \overline{\phi^{-1}(P_{\Delta,\delta})} \subset \mathbb{P}(\text{Sym}^4 V^*) \]
are easily obtained from (1) and a parallel nodal analysis. While Corollary 2 answers a classical question about the Hodge theory of quartic $K3$ surfaces, the method of proof is modern.

**0.8. Outline.** In Section 1, we give a precise definition of Noether-Lefschetz numbers and establish several elementary properties. The definitions of BPS invariants for 3-folds and reduced Gromov-Witten invariants of $K3$ surfaces are recalled in Section 2. Two central conjectures about the reduced theory of $K3$ surfaces are stated in Section 2.3. The proof of Theorem 1 is presented in Section 3.

We review of the work of Borcherds on Heegner divisors and explain the application to families of $K3$ surfaces in Section 4. The results are applied with Theorem 1 to prove Theorem 2 via mirror symmetry calculations in Section 5. A direct approach to Noether-Lefschetz degrees for classical families of $K3$ surfaces of degrees 2, 4, 6, and 8 is given in Section 6 via a deeper study of vector-valued modular forms. Finally, in Section 7, we state a conjecture regarding Picard ranks of moduli spaces of $K3$ surfaces of degree $l$.

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1. NOETHER-LEFSCHETZ NUMBERS

1.1. Picard lattice. Let $S$ be a K3 surface. The second cohomology of $S$ is a rank 22 lattice with intersection form

$$H^2(S, \mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is the (negative) Cartan matrix. The intersection form (4) is even.

The divisibility of $\beta \in H^2(S, \mathbb{Z})$ is the maximal positive integer dividing $\beta$. If the divisibility is 1, $\beta$ is primitive. Elements with equal divisibility and norm are equivalent up to orthogonal transformation of $H^2(S, \mathbb{Z})$.

The Hodge decomposition of the second cohomology of $S$ has dimensions $(1, 20, 1)$,

$$H^2(S, \mathbb{Z}) \otimes \mathbb{C} = H^{2,0}(S, \mathbb{C}) \oplus H^{1,1}(S, \mathbb{C}) \oplus H^{0,2}(S, \mathbb{C}).$$

The Picard lattice of $S$ is

$$\text{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S, \mathbb{C}).$$

1.2. Quasi-polarization. A quasi-polarization on $S$ is a line bundle $L$ with primitive Chern class $c_1(L) \in H^2(S, \mathbb{Z})$ satisfying

$$\int_S L^2 > 0 \quad \text{and} \quad \int_S L \cdot [C] \geq 0$$
for every curve $C \subset S$. A sufficiently high tensor power $L^n$ of a quasi-polarization is base point free and determines a birational morphism

$$S \to \tilde{S}$$

contracting A-D-E configurations of $(-2)$-curves on $S$. Hence, every quasi-polarized $K3$ surface $(S, L)$ is algebraic.

Let $X$ be a compact 3-dimensional complex manifold equipped with a holomorphic line bundle $L$ and a holomorphic map

$$\pi : X \to C$$

to a nonsingular complete curve. The triple $(X, L, \pi)$ is a family of quasi-polarized $K3$ surfaces of degree $l$ if the fibers $(X_\xi, L_\xi)$ are quasi-polarized $K3$ surfaces satisfying

$$\int_{X_\xi} L_\xi^2 = l$$

for every $\xi \in C$. The family $(X, L, \pi)$ yields a morphism,

$$\iota : C \to M_l,$$

to the moduli space of quasi-polarized $K3$ surfaces of degree $l$.

We will often refer to the triple $(X, L, \pi)$ just by $\pi$. Associated to $\pi$ is the projective variety $\tilde{X}$ obtained from the relative quasi-polarization,

$$X \to \tilde{X} \subset \mathbb{P}(R^0\pi_*(L^n)^*) \to C,$$

for sufficiently large $n$. The complex manifold $X$ may be a non-projective small resolution of the singular projective variety $\tilde{X}$.

1.3. Period domain. Let $V$ be a rank 22 integer lattice with intersection form $\langle , \rangle$ obtained from the second homology of a $K3$ surface,

$$V \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

A 1-dimensional subspace $\mathbb{C} \cdot \omega \in V \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying

$$\langle \omega, \omega \rangle = 0 \text{ and } \langle \omega, \overline{\omega} \rangle > 0$$

(5)

determines a Hodge structure of type $(1, 20, 1)$ on $V$,

$$V \otimes_{\mathbb{Z}} \mathbb{C} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2} = \mathbb{C} \cdot \omega \oplus (\mathbb{C} \cdot \omega \oplus \mathbb{C} \cdot \overline{\omega})^\perp \oplus \mathbb{C} \cdot \overline{\omega}.$$ 

Conversely, a Hodge structure of type $(1, 20, 1)$ determines a 1-dimensional subspace $\mathbb{C} \cdot \omega$ satisfying (5).

The moduli space $M^V$ of Hodge structures of type $(1, 20, 1)$ on $V$ is therefore an analytic open set of the 20-dimensional nonsingular isotropic quadric $Q$,

$$M^V \subset Q \subset \mathbb{P}(V \otimes_{\mathbb{Z}} \mathbb{C}).$$
The moduli space $M^V$ is the period domain.

For nonzero $\beta \in V$, let $D^V_\beta \subset M^V$ denote the locus of Hodge structures for which $\beta \in V^{1,1}$. Certainly,

$$D^V_\beta = M^V \cap \beta^\perp \subset \mathbb{P}(V \otimes \mathbb{C})$$

where $\beta^\perp$ is the linear space orthogonal to $\beta$. Hence, $D^V_\beta$ is simply a 19-dimensional hyperplane section of $M^V$.

1.4. Local systems. Let $(X, L, \pi)$ be a quasi-polarized family of $K3$ surfaces over a nonsingular curve $C$. Let

$$\mathcal{V} = R^2\pi_*(\mathbb{Z}) \to C$$

denote the rank 22 local system determined by the middle cohomology of the fibration

$$\pi : X \to C.$$ 

The local system $\mathcal{V}$ is equipped with the fiberwise intersection form $\langle \cdot, \cdot \rangle$.

Let $\mathcal{M}^V$ be the $\pi$-relative moduli space of Hodge structures

$$\mu : \mathcal{M}^V \to C$$

with fiber

$$\mu^{-1}(\xi) = M^V_\xi.$$

The moduli space $\mathcal{M}^V$ is a complex manifold, and $\mu$ is a locally trivial fibration in the analytic topology.

Duality and homological push-forward yield a canonical map

$$\epsilon : \mathcal{V} \to H_2(X, \mathbb{Z})$$

where the right side can be viewed as a trivial local system. Let $H_2(X, \mathbb{Z})^\pi$ denote the kernel of the projective map

$$\pi_* : H_2(X, \mathbb{Z}) \to H_2(C, \mathbb{Z}).$$

For $h \in \mathbb{Z}$ and $\gamma \in H_2(X, \mathbb{Z})^\pi$, we will define a Noether-Lefschetz number $NL^\pi_{h, \gamma}$ for the $K3$ fibration $\pi$.

Informally, $NL^\pi_{h, \gamma}$ counts the number of point $\xi \in C$ for which there exists an integral class $\beta \in V_\xi$ of type $(1,1)$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \quad \text{and} \quad \epsilon(\beta) = \gamma.$$ 

The formal definition is given in Section 1.5.
1.5. Classical intersection. Define the relative divisor
\[ \mathcal{D}_{h,\gamma}^V \subset \mathcal{M}^V \]
by the set of Hodge structures which contain a class \( \beta \in \mathcal{V}_\xi \) of type \((1, 1)\) satisfying
\[ \langle \beta, \beta \rangle = 2h - 2 \text{ and } \epsilon(\beta) = \gamma. \]
When \( \mathcal{M}^V \) is trivialized over a Euclidean open set \( U \subset C \),
\[ \mathcal{M}^V_U = \mathcal{M}^V \times U, \]
the subset \( \mathcal{D}_{h,\gamma}^V \) restricts to
\[ \mathcal{D}_{h,\gamma}^V_U = \bigcup_\beta \mathcal{D}_\beta^V \times U \]
where the union is over all \( \beta \in \mathcal{V} \) satisfying
\[ \langle \beta, \beta \rangle = 2h - 2 \text{ and } \epsilon(\beta) = \gamma. \]
Hence, \( \mathcal{D}_{h,\gamma}^V \subset \mathcal{M}^V \) is a countable union of divisors.

The Noether-Lefschetz number is defined by a tautological intersection product. The family \( \pi \) determines a canonical section
\[ \sigma : C \to \mathcal{M}^V. \]
where
\[ \sigma(\xi) = [H^{2,0}(X_\xi, \mathbb{C})] \in \mathcal{M}^V_\xi \]
is the Hodge structure determined by the K3 surface \( X_\xi \). Let
\[ (6) \quad NL_{h,\gamma}^\pi = \int_C \sigma^*|\mathcal{D}_{h,\gamma}^V|. \]
The divisor \( \mathcal{D}_{h,\gamma}^V \) may have infinitely many components. However, by the finiteness result of Proposition 1, \( NL_{h,\gamma}^\pi \) is well-defined.

While \( NL_{h,\gamma}^\pi \) is a classical intersection number, an excess calculation is required in case \( \sigma(C) \subset \mathcal{D}_{h,\gamma}^V \). The informal counting interpretation is not always well-defined.

**Proposition 1.** \( NL_{h,\gamma}^\pi \) is finite.

**Proof.** Let \( L \) be the quasi-polarization on \( X \). If there exists a point \( \xi \in C \) for which \( L_\xi \) is ample, then \( L \) is \( \pi \)-relatively ample over an open set of \( C \). If \( L_\xi \) is never ample, then the morphism
\[ X \to \tilde{X} \subset \mathbb{P}(R^0\pi_*(L^n)) \]

\footnote{We take trivializations obtained from trivializing \( R^2\pi_*(\mathbb{Z}) \) compatibly with \( \epsilon \).}
for sufficiently large \( n \) contracts divisors on \( X \) which intersect the generic fiber \( X_\xi \) in (-2)-curves. After modification of \( L \) by these contracted divisors, a new quasi-polarization \( L' \) of \( X \) may be obtained which is \( \pi \)-relatively ample over a nonempty open set of \( C \).

We assume now (after possible modification) the quasi-polarization \( L \) is \( \pi \)-relatively ample over a nonempty open set \( U \subset C \). Let

\[ d = \int_\gamma L \]

be the degree of \( \gamma \). Let

\[ l = \int_{X_\xi} L^2_\xi > 0 \]

be the degree of the \( K3 \) fibers of \( \pi \).

Let \( \beta \in \mathcal{V}_\xi \) of type (1, 1) satisfy

\[ \langle \beta, \beta \rangle = 2h - 2 \quad \text{and} \quad \epsilon(\beta) = \gamma. \]

We will prove

\[ \sigma(C) \subset \mathcal{M}^V \]

intersects only finitely many components of \( \mathcal{D}^V_{h, \gamma} \).

Let \( k \) be an integer satisfying

\[ d +lk > 0 \quad \text{and} \quad lk^2 + 2dk + 2h - 2 > -4. \]

The first step is to show

\[ \tilde{\beta} = \beta + kc_1(L_\xi) \]

is an effective curve class on \( X_\xi \) by Riemann-Roch.

Let \( L_\tilde{\beta} \) denote the unique line bundle on \( X_\xi \) with

\[ c_1(L_\tilde{\beta}) = \tilde{\beta}. \]

By Serre duality,

\[ H^2(X_\xi, L_\tilde{\beta}) = H^0(X_\xi, L^*_\tilde{\beta})^* \]

Since

\[ \langle c_1(L^*_\tilde{\beta}), L_\xi \rangle \leq -d - lk < 0, \]

\[ \text{A base change of } \pi : X \to C \text{ is not required since the modification can be averaged over the symmetries of the (-2)-curve configuration.} \]
$h^0(X_\xi, L_{\tilde{\beta}}^*)$ vanishes. Then, by Riemann-Roch,
\[
\begin{align*}
h^0(X_\xi, L_{\tilde{\beta}}) &\geq \chi(X_\xi, L_{\tilde{\beta}}) - h^2(X_\xi, L_{\tilde{\beta}}) \\
&= \chi(X_\xi, L_{\tilde{\beta}}) \\
&= \frac{1}{2}\langle \tilde{\beta}, \tilde{\beta} \rangle + 2 \\
&> 0.
\end{align*}
\]
Hence, $\tilde{\beta}$ is an effective curve class on $X_\xi$.

Consider first the open set $U \subset C$ over which $L$ is $\pi$-relatively ample. Let
\[
\mathcal{H} \to U
\]
be the $\pi$-relative Hilbert scheme parameterizing of curves in $X_{\xi \in U}$ of degree
\[
\langle \tilde{\beta}, c_1(L_\xi) \rangle = d + lk
\]
and Euler characteristic
\[
\chi(X_\xi, \mathcal{O}_{X_\xi}) - \chi(X_\xi, L_{\tilde{\beta}}^*) = -\frac{1}{2}\langle \tilde{\beta}, \tilde{\beta} \rangle = -\frac{1}{2}(lk^2 + 2dk + 2h - 2).
\]
The scheme $\mathcal{H}$ is projective over $U$ and of finite type.

An irreducible component $\mathcal{H}_{irr} \subset \mathcal{H}$ either dominates $U$ or maps to a point $\xi \in U$. In the former case, the classes of curves represented by $\mathcal{H}_{irr}$ yield a finite monodromy invariant subset of $V$. In the latter case, the curves represented by $\mathcal{H}_{irr}$ yield a single element of $V_\xi$.

After shifting the finiteness statements back by $kc_1(L_\xi)$, we obtain the finiteness of the intersection geometry
\[
(7) \quad \sigma(C) \cap D_{h,\gamma}^V
\]
over $U \subset C$. Indeed, the dominant components $H_{irr}$ correspond to finitely many excess intersections and the non-dominant components correspond to finitely many true intersections.

Finally consider the complement $U^c \subset C$. The complement is a finite set. For each $\xi^c \in U^c$, let $L_{\xi^c}$ be an ample line bundle. The above argument using the ample bundles $L_{\xi^c}$ for the fibers $X_{\xi^c}$ shows there are finitely many intersections in (7) over $U^c \subset C$ as well.

We conclude the intersection geometry is finite over all of $C$ and the product
\[
NL_{h,\gamma}^\pi = \int_C \sigma^* [D_{h,\gamma}^V]
\]
is well-defined.

Let $\gamma_L$ denote the push-forward of the ample class on the fibers,
\[
\gamma_L = c_1(L) \cap [X_\xi] \in H_2(X, \mathbb{Z})^\pi.
\]
By an elementary comparison,
\[ \sigma^*[D^V_{h,\gamma}] = \sigma^*[D^V_{h+d+\frac{1}{2},\gamma+\gamma_L}]. \]

We obtain the following result.

**Proposition 2.** \( NL_{h,\gamma}^\pi = NL_{h+d+\frac{1}{2},\gamma+\gamma_L}^\pi \).

The proof of Proposition 1 show the vanishing of the Noether-Lefschetz number for high \( h \).

**Proposition 3.** For fixed \( \gamma \), the numbers \( NL_{h,\gamma}^\pi \) vanish for sufficiently high \( h \).

The Noether-Lefschetz numbers \( NL_{h,\gamma}^\pi(\pi) \) have non-trivial dependence on \( \gamma \) despite the linear equivalence
\[ D^V_\beta \cong D^V_\beta' \]
on \( M^V \). The Noether-Lefschetz numbers involve also the twisting of the local system \( V \) over \( C \).

1.6. **Refinements.** The Noether-Lefschetz numbers \( NL_{h,d}^\pi \) defined in Section 0.3 are obtained from the relation
\[ (8) \quad NL_{h,d}^\pi = \sum_{\gamma L = d} NL_{h,\gamma}^\pi. \]
The finiteness of the sum on the right is a consequence of the negative definiteness of the intersection matrix of divisors in \( X_\xi \) contracted by \( L_\xi \). The invariants \( NL_{h,\gamma}^\pi \) may be viewed as a refinement of \( NL_{h,d}^\pi \) with the nonvanishing discriminant hypothesis lifted.

Further refined Noether-Lefschetz numbers may be defined with respect to any additional monodromy invariant data. For example, the divisibility \( m \) of an element \( \beta \in V_\xi \) is a monodromy invariant. Let
\[ D^V_{m,h,\gamma} \subset M^V \]
be the divisor of Hodge structures which contain a class \( \beta \in V_\xi \) of type \((1,1)\) of divisibility \( m \) satisfying
\[ \langle \beta, \beta \rangle = 2h - 2 \quad \text{and} \quad \epsilon(\beta) = \gamma. \]

We define
\[ NL_{m,h,\gamma}^\pi = \int_C \sigma^*[D_{m,h,\gamma}]. \]
The relation
\[ (9) \quad NL_{h,\gamma}^\pi = \sum_{m \geq 1} NL_{m,h,\gamma}^\pi \]
certainly holds.
1.7. Intersection theory of \( \mathcal{M}_l \). Let \( v \in V \) be a vector of norm \( l \), and let
\[
\mathcal{M}_v^V = v^\perp \cap \mathcal{M}^V.
\]
Let \( \Gamma \) denote the group of orthogonal transformations of the lattice \( V \), and let
\[
\Gamma_v \subset \Gamma
\]
be the subgroup fixing \( v \). The moduli space of quasi-polarized \( K3 \) surfaces of degree \( l \) is the quotient
\[
\mathcal{M}_l = \mathcal{M}_v^V / \Gamma_v.
\]
The moduli space is a nonsingular orbifold. We refer the reader to \[12\] for a more detailed discussion.

In case \( \Delta_l(h, d) \neq 0 \), the above construction of \( \mathcal{M}_l \) shows the definitions of the Noether-Lefschetz number by (3) and (8) agree.

2. Gromov-Witten theory

2.1. BPS states for 3-folds. Let \((X, L, \pi)\) be a quasi-polarized family of \( K3 \) surfaces. While \( X \) may not be a projective variety, \( X \) carries a \((1,1)\)-form \( \omega_K \) which is Kähler on the \( K3 \) fibers of \( \pi \). The existence of a fiberwise Kähler form is sufficient to define Gromov-Witten theory for vertical classes
\[
0 \neq \gamma \in H_2(X, \mathbb{Z})^\pi.
\]
The fiberwise Kähler form \( \omega_K \) is obtained by a small perturbation of the quasi-Kähler form obtained from the quasi-polarization. The associated Gromov-Witten theory is independent of the perturbation used.

Let \( \mathcal{M}_g(X, \gamma) \) be the moduli space of stable maps from connected genus \( g \) curves to \( X \). Gromov-Witten theory is defined by integration against the virtual class,
\[
N_{g, \gamma}^X = \int_{[\mathcal{M}_g(X, \gamma)]^{vir}} 1.
\]
The expected dimension of the moduli space is 0.

The Gromov-Witten potential \( F^X(\lambda, v) \) for nonzero vertical classes is the series
\[
F^X = \sum_{g \geq 0} \sum_{0 \neq \gamma \in H_2(X, \mathbb{Z})^\pi} N_{g, \gamma}^X \lambda^{2g-2} v^\gamma.
\]
\[6\]See \[28, 34\] for treatments of Gromov-Witten invariants for fiberwise Kähler geometry.
where \( \lambda \) and \( v \) are the genus and curve class variables. The BPS counts \( n_{g,\gamma}^X \) of Gopakumar and Vafa are uniquely defined by the following equation:

\[
F^X = \sum_{g \geq 0} \sum_{\gamma \in H_2(X,\mathbb{Z})^\pi} n_{g,\gamma}^X \lambda^{2g-2} \sum_{d>0} \frac{1}{d} \left( \frac{\sin(d\lambda/2)}{\lambda/2} \right)^{2g-2} v^{d\gamma}.
\]

Conjecturally, the invariants \( n_{g,\gamma}^X \) are integral and obtained from the cohomology of an as yet unspecified moduli space of sheaves on \( X \).

2.2. Reduced theory. Let \( C \) be a connected, nodal, genus \( g \) curve. Let \( S \) be a \( K3 \) surface, and let \( \beta \in \text{Pic}(S) \) be a nonzero class. The moduli space \( M_C(S, \beta) \) parameterizes maps from \( C \) to \( S \) of class \( \beta \). Let \( \nu : C \times M_C(S, \beta) \to M_C(S, \beta) \) denote the projection, and let \( f : C \times M_C(S, \beta) \to S \) denote the universal map. The canonical morphism

\[
R^\bullet \nu_*(f^*S)^\vee \to L^\bullet_{MC}
\]

determines a perfect obstruction theory on \( M_C(S, \beta) \), see \([2, 3, 32]\). Here, \( L^\bullet_{MC} \) denotes the cotangent complex of \( M_C(S, \beta) \).

Let \( \Omega_S \) denote the cotangent bundle of \( S \). Let \( \Omega_\nu \) and \( \omega_\nu \) denote respectively the sheaf of relative differentials of \( \nu \) and the relative dualizing sheaf of \( \nu \). There are canonical maps

\[
f^*(\Omega_S) \to \Omega_\nu \to \omega_\nu
\]

The sections of the canonical bundle \( H^0(S, K_S) \) determine a 1-dimensional space of holomorphic symplectic forms. Hence, there is a canonical isomorphism

\[
T_S \otimes H^0(S, K_S) \cong \Omega_S
\]

where \( T_S \) is the tangent bundle. We obtain a map

\[
f^*(T_S) \to \omega_\nu \otimes (H^0(S, K_S))^\vee
\]

and a map

\[
R^\bullet \nu_*(\omega_\nu)^\vee \otimes H^0(S, K_S) \to R^\bullet \nu_*(f^*T_S)^\vee.
\]

From \((\ref{eq:R^null})\), we obtain the cut-off map

\[
\iota : \tau_{-1} R^\bullet \nu_*(\omega_\nu)^\vee \otimes H^0(S, K_S) \to R^\bullet \nu_*(f^*T_S)^\vee.
\]

The complex \( \tau_{-1} R^\bullet \nu_*(\omega_\nu)^\vee \otimes H^0(S, K_S) \) is represented by a trivial bundle of rank 1 tensored with \( H^0(S, K_S) \) in degree \(-1\). Consider the mapping cone \( C(\iota) \) of \( \iota \). Certainly \( R^\bullet \pi_*(f^*T_S)^\vee \) is represented by a
two term complex. An elementary argument using nonvanishing $\beta \neq 0$ shows the complex $C(\iota)$ is also two term.

By Ran’s results\footnote{The required deformation theory can also be found in a recent paper by M. Manetti \cite{35}.} on deformation theory and the semiregularity map, there is a canonical map

\begin{equation}
C(\iota) \to L^*_{M_C}
\end{equation}

induced by (11), see \cite{14}. Ran proves the obstructions to deforming maps from $C$ to a holomorphic symplectic manifold lie in the kernel of the semiregularity map. After dualizing, Ran’s result precisely shows (11) factors through the cone $C(\iota)$.

The map (14) defines a new perfect obstruction theory on $M_C(S, \beta)$. The conditions of cohomology isomorphism in degree 0 and the cohomology surjectivity in degree $-1$ are both induced from the perfect obstruction theory (11). We view (11) as the standard obstruction theory and (14) as the reduced obstruction theory.

Following \cite{2, 3}, the morphism (14) is an obstruction theory of maps to $S$ relative to the Artin stack $\mathcal{M}_g$ of genus $g$ curves. A reduced absolute obstruction theory

\begin{equation}
E^* \to L^*_{\mathcal{M}_g(S, \beta)}
\end{equation}

is obtained via a distinguished triangle in the usual way, see \cite{2, 3, 32}. The obstruction theory (15) yields a reduced virtual class

$$
[M_g(S, \beta)]_{\text{red}}^{\text{rel}} \in A_g(M_g(S, \beta), \mathbb{Q})
$$

of dimension $g$.

2.3. **BPS for K3 surfaces.** Let $(S, \omega_K)$ be a K3 surface with a Kähler form $\omega_K$. Let $\beta \in \text{Pic}(S)$ be a nonzero class of positive degree

$$
\int_{\beta} \omega_K > 0.
$$

We are interested in the following reduced Gromov-Witten integrals,

\begin{equation}
R_{g, \beta} = \int_{[M_g(S, \beta)]_{\text{red}}} (-1)^g \lambda_g.
\end{equation}

Here, the integrand $\lambda_g$ is the top Chern class of the Hodge bundle

$$
E_g \to \mathcal{M}_g(S, \beta)
$$

with fiber $H^0(C, \omega_C)$ over moduli point

$$
[f : C \to S] \in \mathcal{M}_g(S, \beta).
$$
See [13, 19] for a discussion of Hodge classes in Gromov-Witten theory. The definition of the BPS counts associated to the Hodge integrals (16) is straightforward. Let $\alpha \in \text{Pic}(S)$ be a primitive class of positive degree with respect to $\omega_K$. The Gromov-Witten potential $F_\alpha(\lambda, v)$ for classes proportional to $\alpha$ is

$$F_\alpha = \sum_{g \geq 0} \sum_{m > 0} R_{g,ma} \lambda^{2g-2} v^{ma}.$$ 

The BPS counts $r_{g,ma}$ are uniquely defined by the following equation:

$$F_\alpha = \sum_{g \geq 0} \sum_{m > 0} r_{g,ma} \lambda^{2g-2} \sum_{d > 0} \frac{1}{d} \left( \frac{\sin(d\lambda/2)}{\lambda/2} \right)^{2g-2} v^{dma}.$$ 

We have defined BPS counts for both primitive and divisible classes. The string theoretic calculations of Katz, Klemm and Vafa [22] via heterotic duality yield two conjectures.

**Conjecture 1.** The BPS count $r_{g,\beta}$ depends upon $\beta$ only through the square $\int_S \beta^2$.

Assuming the validity of Conjecture 1, let $r_{g,h}$ denote the BPS count associated to a class $\beta$ satisfying $\int_S \beta^2 = 2h - 2$.

Conjecture 1 is rather surprising from the point of view of Gromov-Witten theory. By deformation arguments, the invariants $R_{g,\beta}$ depend upon both the divisibility $m$ of $\beta$ and $\int_S \beta^2$. Hence, BPS counts $r_{g,m,h}$ depending upon both the divisibility and the norm are well-defined unconditionally.

**Conjecture 2.** The BPS counts $r_{g,h}$ are uniquely determined by the following equation:

$$\sum_{g \geq 0} \sum_{h \geq 0} (-1)^g r_{g,h}(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{2g} q^h = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{20}(1 - yq^n)^2(1 - y^{-1}q^n)^2}.$$ 

As a consequence of Conjecture 2, $r_{g,h}$ vanishes if $g > h$ and

$$r_{g,g} = (-1)^g (g + 1).$$ 

The first values are tabulated below:
The right side Conjecture 2 is related to the generating series of Hodge numbers of the Hilbert schemes of points $\text{Hilb}(S, n)$. The genus 0 specialization of Conjecture 2 recovers the Yau-Zaslow formula

$$\sum_{h \geq 0} r_{0,h} q^h = \prod_{n \geq 1} \frac{1}{(1-q^n)^{24}}$$

related to the Euler characteristics of $\text{Hilb}(S, n)$.

The Conjectures are proven in very few cases. A mathematical approach to the genus 0 Yau-Zaslow formula following [47] can be found in [4, 11, 14]. The Yau-Zaslow formula is proven for primitive classes $\beta$ by Bryan and Leung [9]. If $\beta$ has divisibility 2, the Yau-Zaslow formula is proven by Lee and Leung in [29]. Using Theorem 1, a complete proof of the Yau-Zaslow formula for all divisibilities is given in [25]. Since

$$R_{1,\beta} = \int_{[M_1(S, \beta)]^{red}} -\lambda_1 = -\frac{\langle \beta, \beta \rangle}{24} R_{0,\beta},$$

we obtain

$$r_{1,h} = -\frac{h}{12} r_{0,h}$$

and Conjectures 1 and 2 from the genus 0 results.

Conjecture 2 for primitive classes $\beta$ is connected to Euler characteristics of the moduli spaces of stable pairs on $K3$ by the correspondence of [42, 43]. A proof of Conjecture 2 for primitive classes is given in [38].

3. Theorem 1

3.1. Result. Consider a quasi-polarized family of $K3$ surfaces of degree $l$ as in Section 1.2

$$\pi : X \to C.$$

We restate Theorem 1 in terms of $\gamma \in H_2(X, \mathbb{Z})^\pi$ following the notation of Section 1.4.

Theorem 1. For $\gamma \neq 0$,

$$n_{g,\gamma} = \sum_{h} \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL_{m,h,\gamma}^\pi.$$
3.2. Proof. Since the formulas relating the BPS counts to Gromov-Witten invariants are the same for $X$ and the $K3$ surface, Theorem 1 is equivalent to the analogous Gromov-Witten statement:

$$N^{X}_{g,\gamma} = \sum_{h} \sum_{m=1}^{\infty} R_{g,m,h} \cdot NL_{m,h,\gamma}^\pi$$

for $\gamma \neq 0$.

Following the notation of Section 1.5, let $\sigma$ denote the section

$$\sigma : C \to M^\nu$$

determined by the Hodge structure of the $K3$ fibers

$$\sigma(\xi) = [H^0(X, K_{X_{\xi}})] \in M^\nu.$$

For each $\xi \in C$, let

$$V_\xi(m, h, \gamma) \subset V$$

be the set of classes with divisibility $m$, square $2h-2$, and push-forward $\gamma$. Let

$$B_\xi(m, h, \gamma) = \{ \beta \in V_\xi(m, h, \gamma) \mid \sigma(\xi) \in \beta^\perp \}.$$

By Proposition 1, the set $B_\xi(m, h, \gamma)$ is finite.

Equation (17) is proven by showing the contributions of the classes $B_\xi(m, h, \gamma)$ to both sides are the same. The set

$$B(m, h, \gamma) = \bigcup B_\xi(m, h, \gamma) \subset V$$

can be divided into two disjoint subsets

$$B(m, h, \gamma) = B_{iso}(m, h, \gamma) \cup B_{\infty}(m, h, \gamma).$$

The elements of $B_{iso}(m, h, \gamma)$ are isolated while the the elements of $B_{\infty}(m, h, \gamma)$ form a finite local system over $C$,

$$\epsilon : B_{\infty}(m, h, \gamma) \to C.$$

We address the contributions of the isolated issues and the local system separately.

Consider first the local system (18). The contribution of $\epsilon$ to the Gromov-Witten invariant $N^{X}_{g,\gamma}$ is the integral

$$N^{X}_{g,\epsilon} = \int_{[\overline{\mathcal{M}}_g(X,\epsilon)]^{vir}} 1$$
where $\overline{M}_g(X, \epsilon) \subset \overline{M}_g(X, \gamma)$ is the connected component of the moduli space of stable maps which represent curve classes in $\epsilon$. Alternatively, 

\begin{equation}
N^X_{g, \epsilon} = \int_{[\overline{M}_g(\pi, \epsilon)]^{vir}} c_g(\mathbb{E}_g^* \otimes T_C)
\end{equation}

where $\overline{M}_g(\pi, \epsilon) \subset \overline{M}_g(\pi, \gamma)$ is a connected component of the relative moduli space of maps. By standard arguments, the difference in the absolute and relative obstruction theories yields the Hodge integrand in (19).

The family $\pi$ determines a canonical line bundle $K \to C$

with fiber $H^0(X_\xi, K_{X_\xi})$ over $\xi \in C$. By the construction of the reduced class in Section 2.2,

\[ [\overline{M}_g(\pi, \epsilon)]^{vir} = c_1(K^*) \cap [\overline{M}_g(\pi, \epsilon)]^{red} \]

where, on the right side, the reduced virtual class for the relative moduli space of maps appears. Expanding (19) yields

\[ N^X_{g, \epsilon} = \int_{[\overline{M}_g(\pi, \epsilon)]^{red}} c_g(\mathbb{E}_g^* \otimes T_C) \cdot c_1(K^*) \]

\[ = \int_{[\overline{M}_g(K^3, m \alpha)]^{red}} (-1)^g \lambda_g \cdot \int_{B_\infty(m, h, \gamma)} c_1(K^*) \]

\[ = R_{g, m, h} \cdot \int_{B_\infty(m, h, \gamma)} c_1(K^*). \]

In the second equality, $\alpha$ is primitive and satisfies

\[ \langle m \alpha, m \alpha \rangle = 2h - 2. \]

The contribution of the local system $\epsilon$ to the Noether-Lefschetz number $NL^\pi_{m, h, \gamma}$ is much easier to calculate. The local system represents an excess intersection contribution

\[ \int_{B_\infty(m, h, \gamma)} c_1(\text{Norm}) \]

where $\text{Norm}$ is the line bundle with fiber

\[ \text{Hom}(H^0(X_\xi, K_{X_\xi}), \mathbb{C} \cdot \beta) \]

\(\text{By connected component, we mean both open and closed. Formally, the condition is usually stated as a union of connected components.}\)
at $\beta \in B_\infty(m, h, \gamma)$ lying over $\xi \in C$. Over $B_\infty(m, h, \gamma)$, the fibration $C \cdot \beta$ is a trivial line bundle. Hence, the excess contribution of $B_\infty(m, h, \gamma)$ to $NL^\pi_{m,h,\gamma}$ is
\[ \int_{B_\infty(m,h,\gamma)} c_1(K^\ast). \]

We conclude the contributions of $B_\infty(m, h, \gamma)$ to the left and right sides of equation (17) exactly match.

We consider now the contributions of the isolated classes $B_{iso}(m, h, \gamma)$ to the two sides of (17). Let
\[ \beta \in B_{iso}(m, h, \gamma) \]
be an isolated class lying over $\xi \in C$. We trivialize $\mathcal{M}^V$ over a Euclidean open set $U \subset C$ as in Section 1.5. The local intersection of the section $\sigma$ with the divisor $D_{\mathcal{V}}^\beta \times U \subset M^V \times U$ has an isolated point corresponding to $(\beta, \xi)$. The local intersection multiplicity may not be 1. However, by deformation equivalence of the Gromov-Witten contributions on the left side of (17) and the intersection products on the right side of (17), we may assume the local intersection multiplicity is 1 after local holomorphic perturbation of the section $\sigma$. Then, the contribution of the isolated class $\beta$ to $NL^\pi_{m,h,\gamma}$ is certainly 1.

The final step is to show the contribution of the isolated class $\beta$ with intersection multiplicity 1 to $N^X_{g,\gamma}$ is simply $R_{g,m,h}$. The result is obtained by a comparison of obstruction theories.

By the multiplicity 1 hypothesis, a connected component of the moduli space of stable maps to $X$ coincides with the moduli stable of stable maps to fiber $X_\xi$,
\[ (20) \quad \overline{\mathcal{M}}_g(X_\xi, \beta) \subset \overline{\mathcal{M}}_g(X, \gamma). \]

At the level of points, the assertion is obvious. The multiplicity 1 conditions prohibits any infinitesimal deformations of maps away from the fiber $X_\xi$ and implies the scheme theoretic assertion.

From the fibration $\pi$, we obtain an exact sequence
\[ (21) \quad 0 \to T_{X_\xi} \to T_X|_{X_\xi} \to T_{C_\xi} \to 0, \]
and an induced map
\[ \overline{\nu} : R^\ast \nu_*(f^*T_{X_\xi})^\vee \to T^*_{C_\xi}. \]
where the second complex is a trivial bundle in degree \(-1\). Following the notation of Section 2.2 we have a canonical map
\[ \iota: H^0(X_\xi, K_{X_\xi}) \to R^*_\nu(f^*T_{X_\xi})^\vee \]
where the first complex is a trivial bundle with fiber \(H^0(X_\xi, K_{X_\xi})\) in degree \(-1\). By Lemma 1 below, the composition
\[ \tilde{\iota} \circ \iota: H^0(X_\xi, K_{X_\xi}) \to T^*_{C,\xi} \]
is an isomorphism. Hence, by sequence (21), the obstruction theories \(R^*_\nu(f^*T_X)^\vee\) and \(C(\iota)\) differ by only the Hodge bundle \(E_g \otimes T^*_{C,\xi}\).

We conclude
\[ [\overline{M}_g(X_\xi, \beta)]^{vir}_X = (-1)^g \lambda_g \cap [\overline{M}_g(X_\xi, \beta)]^{red} \]
where the virtual class on the left is obtained from the obstruction theory of maps to \(X\) via (20). The contribution of the isolated class \(\beta\) to \(N^X_{g,\gamma}\) is thus \(R_{g,h,m}\).

Since the contributions of \(B_{iso}(m, h, \gamma)\) to the left and right sides of equation (17) also match, the proof of Theorem 1 is complete. \(\square\)

**Lemma 1.** The composition
\[ \tilde{\iota} \circ \iota: H^0(X_\xi, K_{X_\xi}) \to T^*_{C,\xi} \]
is an isomorphism.

**Proof.** Consider the differential of the period map at \(\xi\),
\[ T_{C,\xi} \to H^1(T_{X_\xi}) \to \text{Hom}(H^0(K_{X_\xi}), H^1(\Omega_{X_\xi})). \]
The multiplicity 1 condition implies that the image of this map is not contained in the tangent space to the hyperplane \(\beta^\perp = 0\). More explicitly, if we apply the cup-product pairing of \(H^1(\Omega_{X_\xi})\) with the class \(\beta \in H^2(X_\xi, \mathbb{Z})\), the composition
\[ T_{C,\xi} \to H^0(K_{X_\xi})^* \otimes H^1(\Omega_{X_\xi}) \xrightarrow{\beta \cup} H^0(K_{X_\xi})^* \otimes \mathbb{C} \]
is nonzero. This sequence can be included in the diagram
\[
\begin{array}{ccccccc}
T_{C,\xi} & \xrightarrow{\iota} & H^1(T_{X_\xi}) & \xrightarrow{\beta \cup} & H^0(K_{X_\xi})^* \otimes H^1(\Omega_{X_\xi}) & \xrightarrow{\beta \cup} & H^0(K_{X_\xi})^* \\
| & & | & & | & & |
T_{C,\xi} & \xrightarrow{\iota} & R^*_\nu(f^*T_{X_\xi}) & \xrightarrow{\beta \cup} & H^0(K_{X_\xi})^* \otimes R^*_\nu(f^*\Omega_{X_\xi}) & \xrightarrow{\beta \cup} & H^0(K_{X_\xi})^* \\
\end{array}
\]
where the vertical maps are given by base-change morphisms and the bottom row is the map \((\tilde{\iota} \circ \iota)^*\). Standard comparison results imply that this diagram commutes. Since the top row is nonvanishing, so is the bottom row. \(\square\)
3.3. Conjectures 1 and 2 revisited. The proof of Conjectures 1 and 2 in the following case allows us to bound from below the $h$ summation in Theorem 1.

**Lemma 2.** If $\int_{K^3} \beta^2 < 0$, then $r_{g,\beta} = 1$ if

$$g = 0 \text{ and } \int_{K^3} \beta^2 = -2$$

and $r_{g,\beta} = 0$ otherwise.

**Proof.** Let $S$ be a $K3$ surface, and let $\beta \in \text{Pic}(S)$ be primitive with

$$\int_{S} \beta^2 = -2.$$  

We may assume $\beta$ is represented by an isolated $-2$ curve $P \subset S$. Let

$$\pi : X \to \Delta_0$$

be a 1-parameter deformation of $S$ over the the disk $\Delta_0$ for which $\beta$ fails (even infinitesimally) to remain algebraic. By the proof of Theorem 1, the reduced invariants $r_{g,m,\beta}$ are obtained from the contribution of $P$ to the BPS state counts of $X$. Since $P$ is a rigid $(-1,-1)$ curve, $P$ contributes a single BPS state $[13]$. We conclude

$$r_{g,m,\beta} = 1$$

if $(g,m) = (0,1)$ and $r_{g,m,\beta} = 0$ otherwise.

If $\beta \in \text{Pic}(S)$ is primitive with square $2h - 2$ strictly less than $-2$, then all reduced invariants $r_{g,m,\beta}$ vanish. The proof is obtained by considering elliptically fibered $K3$ surfaces $S \to \mathbb{P}^1$. Let

$$[s], [f] \in \text{Pic}(S)$$

be the classes of a section and a fiber respectively. Then,

$$[s] + h[f], -[s] - h[f] \in \text{Pic}(S)$$

are both primitive with square $2h - 2$. Since the moduli spaces

$$\overline{M}_g (S, m([s] + h[f])), \overline{M}_g (S, m([-s] - h[f]))$$

are easily seen to be empty, all reduced invariants $r_{g,m,\beta}$ vanish. □

By Lemma 2, the integrals $r_{g,m,h<0}$ all vanish. Hence, Theorem 1 may be written as

$$n^X_{g,\gamma} = \sum_{h \geq 0} \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL^{\pi}_{m,h,\gamma}.$$  

9The local NL intersection number here is 1.
If Conjecture 1 and the vanishing \( r_{g,h} \) for \( g > h \) of Conjecture 2 hold, then

\[ r_{g,h} = r_{g,m,h} \]

and Theorem 1 implies the following result by relation (9).

**Theorem 1**. For \( \gamma \neq 0 \),

\[ n_{g,\gamma}^X = \sum_{h \geq g} r_{g,h} \cdot NL_{h,\gamma}^\pi. \]

The asterisk here indicates the dependence of Theorem 1* upon Conjectures 1 and 2.

3.4. **Invertibility.** Theorem 1* and Conjecture 2 imply the BPS states \( n_{g,\gamma}^X \) of the total space contain exactly the same information as the Noether-Lefschetz numbers \( NL_{h,\gamma}^\pi \).

**Proposition 4**. For \( \gamma \in H_2(X,\mathbb{Z})^\pi \) of positive degree, the invariants \( \{n_{g,\gamma}(\pi)\}_{g \geq 0} \) determine the Noether-Lefschetz numbers \( \{NL_{h,\gamma}(\pi)\}_{h \geq 0} \) in terms of the invariants \( \{r_{g,h}\}_{g,h \geq 0} \).

**Proof.** Fix \( \gamma \in H_2(X,\mathbb{Z})^\pi \). By Proposition 2, the numbers \( NL_{h,\gamma}(\pi) \) vanish for \( h > h_{\text{top}} \). So we need only determine

\[ NL_{0,\gamma}, \ldots, NL_{h_{\text{top}},\gamma}. \]

The equations

\[ n_{g,\gamma}(\pi) = \sum_{h = g}^{h_{\text{top}}} r_{g,h} \cdot NL_{h,\gamma}(\pi) \]

for \( g = 0, \ldots, h_{\text{top}} \) of Theorem 1* are triangular and invertible by Conjecture 2.

\( \square \)

4. **Modular forms**

4.1. **Overview.** We explain here Borcherds’ work [6] relating Noether-Lefschetz numbers to Fourier coefficients of modular forms [14]. His results apply in great generality to arithmetic quotients of symmetric spaces associated to the orthogonal group \( O(2, n) \) for any \( n \). While we are mainly interested in the case of \( O(2, 19) \), we will first explain the statement in full generality. Other values of \( n \) play a role, for example,

\(^{10}\)Borcherds’ original result is modular only up to a Gal(\( \overline{\mathbb{Q}} / \mathbb{Q} \))-action. The strengthening of [6] by the more recent rationality result of [39] removes the Gal(\( \overline{\mathbb{Q}} / \mathbb{Q} \)) issue.
in studying 1-parameter families of $K3$ surfaces with generic Picard rank at least 2.

4.2. Vector-valued modular forms of half-integral weight. We first summarize standard facts and notation regarding modular forms of half-integral weight. In order to make sense of the modular transformation law with half-integer exponents, a double cover of the standard modular group $SL_2(\mathbb{Z})$ is required.

The metaplectic group $Mp_2(\mathbb{R})$ is the unique connected double cover $SL_2(\mathbb{R})$. The elements of $Mp_2(\mathbb{R})$ can be written in the form

$$\left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \phi(\tau) = \pm \sqrt{c\tau + d} \right)$$

where $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{R})$ and $\phi(\tau)$ is a choice of square root of the function $c\tau + d$ on the upper-half plane $\mathcal{H}$. The group structure is defined by the product

$$(A_1, \phi_1(\tau)) \cdot (A_2, \phi_2(\tau)) = (A_1 A_2, \phi_1(A_2 \tau) \phi_2(\tau)).$$

Here, we write $A\tau$ for the usual action of $SL_2(\mathbb{R})$ on $\tau \in \mathcal{H}$.

The group $Mp_2(\mathbb{Z})$ is the preimage of $SL_2(\mathbb{Z})$ under the projection map

$$\pi : Mp_2(\mathbb{R}) \to SL_2(\mathbb{R}).$$

It is generated by the two elements

$$T = \left( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), 1 \right), S = \left( \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \sqrt{\tau} \right),$$

where $\sqrt{\tau}$ denotes the choice of square root with positive real part.

Suppose we are given a representation $\rho$ of $Mp_2(\mathbb{Z})$ on a finite-dimensional complex vector space $V$ with the property that $\rho$ factors through a finite quotient. Given $k \in \frac{1}{2} \mathbb{Z}$, we define a modular form of weight $k$ and type $\rho$ to be a holomorphic function

$$f : \mathcal{H} \to V$$

such that, for all $g = (A, \phi(\tau)) \in Mp_2(\mathbb{Z})$, we have

$$f(A\tau) = \phi(\tau)^{2k} : \rho(g)(f(\tau)).$$

For $k \in \mathbb{Z}$ and $\rho$ trivial, this reduces to the usual transformation rule.

If we fix an eigenbasis $\{v_\gamma\}$ for $V$ with respect to $T$, we can take the Fourier expansion of each component of $f$ at the cusp at infinity. That is, we write

$$f(\tau) = \sum_{\gamma} \sum_{k \in \mathbb{Z}} c_{k,\gamma} q^{k/R} v_\gamma \in V$$
where
\[ q = e^{2\pi i \tau} \]
and \( R \) is the smallest positive integer for which \( T^R \in \text{Ker}(\rho) \). The function \( f \) is holomorphic at infinity if \( c_{k,r} = 0 \) for \( k < 0 \). The space \( \text{Mod}(Mp_2(\mathbb{Z}), k, \rho) \) of holomorphic modular forms of weight \( k \) and type \( \rho \) is finite-dimensional.

Given an integral lattice \( M \) with an even bilinear form \( \langle , \rangle \) with signature \( (2, n) \), we associate to \( M \) the following unitary representation of \( Mp_2(\mathbb{Z}) \). Let
\[ M^\vee \subset M \otimes \mathbb{Q} \]
denote the dual lattice and \( M^\vee / M \) the finite quotient. The pairing \( \langle , \rangle \) extends linearly to a \( \mathbb{Q} \)-valued pairing on \( M^\vee \). The functions \( \frac{1}{2} \langle \gamma, \gamma \rangle \) and \( \langle \gamma, \delta \rangle \) descend to \( \mathbb{Q}/\mathbb{Z} \)-valued functions on \( M^\vee / M \).

We construct a representation \( \rho_M \) of \( Mp_2(\mathbb{Z}) \) on the group algebra \( \mathbb{C}[M^\vee / M] \). It suffices to define \( \rho_M \) in terms of the action of the generators \( T \) and \( S \) with respect to the standard basis \( v_\gamma \) for \( \gamma \in M^\vee / M \),
\[
\rho_M(T)v_\gamma = e^{2\pi i \langle \gamma, \gamma \rangle / 2} v_\gamma ,
\]
\[
\rho_M(S)v_\gamma = \frac{\sqrt{i}^{n-2}}{\sqrt{|M^\vee / M|}} \sum_\delta e^{-2\pi i \langle \gamma, \delta \rangle} v_\delta .
\]

Let \( N \) denote the smallest integer for which \( N\langle \gamma, \gamma \rangle / 2 \in \mathbb{Z} \) for all \( \gamma \in M^\vee \). The representation factors through the finite index subgroup
\[ \tilde{\Gamma}(N) \subset Mp_2(\mathbb{Z}) \]
consisting of elements \( (A, \phi) \) for which
\[ A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N. \]
We will be primarily interested in the dual representation \( \rho_M^* \) of \( Mp_2(\mathbb{Z}) \) on \( \mathbb{C}[M^\vee / M] \). We have given the action of \( \rho_M \) to match Borcherds’ notation.

4.3. Heegner divisors. Given the lattice \( M \) of type \( (2, n) \) as before, consider the Hermitian symmetric domain
\[ \mathcal{D} = \{ \omega \in \mathbb{P}(M \otimes_{\mathbb{Z}} \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0 \} \]
naturally associated to \( M \). We will study the quotient
\[ (22) \quad \mathcal{X}_M = \mathcal{D} / \Gamma_M \]
of \( \mathcal{D} \) by the arithmetic subgroup of \( O(2, n) \)
\[ \Gamma_M = \{ g \in \text{Aut}(M) \mid g \text{ acts trivially on } M^\vee / M \} . \]
The quotient (22) is a quasi-projective algebraic variety.

For every $n \in \mathbb{Q}^{<0}$ and $\gamma \in M^\vee/M$, we associate a divisor class $y_{n,\gamma} \in \text{Pic}(\mathcal{X}_M)$ as follows. Given an element $v \in M^\vee$, there is an associated hyperplane

$$v^\perp = \{ \omega \in \mathcal{D} \mid \langle \omega, v \rangle = 0 \}.$$ 

Both $\langle v, v \rangle$ and the residue class $v \mod M$ are invariant under the action of $\Gamma_M$. Therefore, if we fix $n \in \mathbb{Q}$ and $\gamma \in M^\vee/M$, the set of $v \in M^\vee$ with

$$\langle v, v \rangle = n, \; v \equiv \gamma \mod M$$

is also $\Gamma_M$-invariant. The union over the set of the associated hyperplanes

$$\sum_{\langle v, v \rangle = n, \; v \equiv \gamma \mod M} v^\perp$$

is $\Gamma_M$-invariant and descends to an algebraic divisor

$$y_{n,\gamma} = \left( \sum_{\langle v, v \rangle = n, \; v \equiv \gamma \mod M} v^\perp \right) / \Gamma_M.$$

The $y_{n,\gamma}$ are the *Heegner divisors* of $\mathcal{X}_M$. Because of the symmetry $v^\perp = (-v)^\perp$, there is a redundancy

$$y_{n,\gamma} = y_{n,-\gamma}$$

in our notation, and $y_{n,\gamma}$ is multiplicity 2 everywhere if $2\gamma \equiv 0 \mod M$.

In the degenerate case where $n = 0$, we have the following prescription. The line bundle $\mathcal{O}(-1)$ on $\mathcal{D} \subset \mathbb{P}(M \otimes \mathbb{Z})$ admits a natural $\Gamma_M$ action and therefore descends to a line bundle $K$ on $\mathcal{X}_M$. If $n = 0$ and $\gamma = 0$, we set

$$y_{0,0} = K^*.$$ 

If $n = 0$ and $\gamma \neq 0$, we set $y_{n,\gamma} = 0$.

We place the Heegner divisors in a formal power series $\Phi_M(q)$ with coefficients in $\text{Pic}(\mathcal{X}_M) \otimes \mathbb{C}[M^\vee/M]$. More precisely, we consider the generating function

$$\Phi(q) = \sum_{n \in \mathbb{Q}^{\geq 0}} \sum_{\gamma \in M^\vee/M} y_{-n,\gamma} q^n v_\gamma \in \text{Pic}(\mathcal{X}_M)[[q^{1/N}]] \otimes \mathbb{C}[M^\vee/M].$$

The main result of [6] together with the refinement of [39] yield the following Theorem.
Theorem ([6],[39]) Let $M$ have signature $(2,n)$. The generating function $\Phi(q)$ is an element of

$$\text{Pic}(\mathcal{X}_M) \otimes \mathbb{Z} \text{Mod}(M\mathcal{P}_2(\mathbb{Z}), 1 + \frac{n}{2}, \rho^*_M).$$

As a consequence, given any linear functional

$$\lambda : \text{Pic}(\mathcal{X}_M) \otimes \mathbb{C} \to \mathbb{C},$$

the contraction $\lambda(\Phi_M(q))$ is the Fourier expansion of a vector-valued modular form of weight $1 + \frac{n}{2}$ and type $\rho^*_M$.

Borcherds’ proof uses the singular theta lift of [5] to construct automorphic forms on $\mathcal{X}_M$ starting from vector-valued meromorphic modular forms on the upper half-plane. The zeroes and poles of these automorphic forms lie precisely along the Heegner divisors with multiplicity determined by the singular part of the initial modular form. Each such lifting gives a relation in $\text{Pic}(\mathcal{X}_M)$. The total collection of relations arising in this way are encoded in the modularity statement.

In [5], Borcherds only shows that $\Phi_M(q)$ lies in a certain Galois closure of the space of modular forms. For the representations $\rho$ arising in [5], MacGraw proves in [39] that $\text{Mod}(M\mathcal{P}_2(\mathbb{Z}), k, \rho)$ admits a basis with rational coefficients. Therefore, the Galois closure does not enlarge the space.

4.4. Application to K3 surfaces. Let $V$ be the rank 22 lattice obtained from the second cohomology of a $K3$ surface with fixed polarization $L$ of norm $l$. In order to apply Borcherds’ results to the moduli spaces $\mathcal{M}_l$, we consider the lattice of signature $(2,19)$

$$M = L^\perp = \{v \in V \mid \langle L, v \rangle = 0\}.$$

A direct check yields

$$M \cong \mathbb{Z}w \oplus U^2 \oplus E_8(-1)^2$$

where $\langle w, w \rangle = -l$. Therefore

$$M^\vee / M = \mathbb{Z}/l\mathbb{Z}$$

and is generated by $\frac{1}{l}w$. Here, we will write $\rho_l$ for the representation $\rho_M$.

From the definitions, we find $\text{Aut}(V, L) = \Gamma_M$, so we have the identification

$$\mathcal{M}_l = \mathcal{X}_M.$$

We claim the Heegner divisors correspond precisely to our Noether-Lefschetz divisors.
Lemma 3. We have $D_{h,d} = y_{n,\gamma}$, where
\[
n = -\frac{\Delta_l(h,d)}{2l} \quad \text{and} \quad \gamma \equiv d\left(\frac{1}{l}w\right) \mod M.
\]

Proof. The Noether-Lefschetz divisor $D_{h,d}$ is the quotient by $\Gamma_M$ of the union of hyperplanes
\[
\sum \beta^\perp.
\]
\[
\langle \beta, \beta \rangle = 2h - 2
\]
\[
\langle L, \beta \rangle = d
\]
It therefore suffices to establish a bijection between the two sets of hyperplanes. Given an element $\beta \in V$ satisfying
\[
\langle \beta, \beta \rangle = 2h - 2, \quad \langle \beta, L \rangle = d,
\]
let $v = \beta - \frac{d}{l}L \in M \otimes \mathbb{Q}$ be the projection of $\beta$ to $M = L^\perp$. A direct calculation shows
\[
\frac{1}{2} \langle v, v \rangle = h - 1 - \frac{d^2}{2l} = \frac{\Delta_l(h,d)}{2l},
\]
\[
v \equiv d \cdot \left(\frac{1}{l}w\right) \mod M.
\]

Conversely, given $v \in M^\vee$ satisfying the above conditions,
\[
\beta = v + \frac{d}{l}L
\]
gives the inverse construction. Since $\beta^\perp = v^\perp$, we obtain the result. \qed

It is important for our applications that the constant term $y_{0,0}$ of $\Phi_M(q)$ matches with the line bundle $K^*$ from our excess calculation in the proof of Theorem 1. This occurs because automorphic forms can be viewed as sections of powers of $K^*$ on $\mathcal{M}_l$.

Let $\pi$ be a 1-parameter family of quasi-polarized $K3$ surfaces of degree $l$, and let $\iota$ be the associated morphism to moduli space:
\[
\pi : X \to C,
\]
\[
\iota : C \to \mathcal{M}_l.
\]
We can apply Borcherds’ theorem to the functional on $\text{Pic}(\mathcal{M}_l)$ given by
\[
D \mapsto \int_C \iota^* D.
\]
Corollary 3. There is a vector-valued modular form of weight $21/2$ and type $\rho^*_4$, 

$$\Phi^\pi(q) = \sum_{r=0}^{l-1} \Phi^\pi_r(q)v_r \in \mathbb{C}[[q^{1/2l}]] \otimes \mathbb{C}[\mathbb{Z}/l\mathbb{Z}],$$

with nonzero coefficients determined by the equality 

$$NL^\pi_{h,d} = \Phi^\pi_r \left[ \frac{\Delta_4(h, d)}{2l} \right]$$

where $r \equiv d \mod l$.

4.5. Quartic K3 surfaces. We now apply Borcherds’ modularity to the study of K3 surfaces of degree 4. If $l = 4$, the isomorphism class of a rank two lattice $(\mathbb{L}, v)$ with primitive polarization $\langle v, v \rangle = l$ is determined only by the discriminant $\Delta$.

Given a 1-parameter family $\pi : X \to C$ of quasi-polarized K3 surfaces of degree 4, we have the generating function 

$$\Phi^\pi(q) = \Phi^\pi_0(q)v_0 + \Phi^\pi_1(q)v_1 + \Phi^\pi_2(q)v_2 + \Phi^\pi_3(q)v_3$$

which is a modular form of weight $21/2$ and type $\rho^*_4$ by Corollary 3.

Consider the scalar-valued power series 

$$\phi^\pi(q) = \Phi^\pi_0(q) + \frac{1}{2}\Phi^\pi_1(q) + \Phi^\pi_2(q) + \frac{1}{2}\Phi^\pi_3(q).$$

By chasing definitions, we see $\phi^\pi(q)$ has the following property:

$$(23) \quad NL^\pi_{h,d} = \phi^\pi \left[ \frac{\Delta_4(h, d)}{8} \right].$$

The factor of $1/2$ is included to correct for the redundancy 

$$\Phi^\pi_1(q) = \Phi^\pi_3(q).$$

Proposition 5. The function $\phi^\pi(q)$ is a homogeneous polynomial of degree 21 in 

$$A = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{8}} \quad \text{and} \quad B = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{8}}.$$

Proof. While the vector $\Phi^\pi(q)$ is modular with respect to the full metaplectic group, $\phi^\pi(q)$ is a priori only modular with respect to the subgroup $\tilde{\Gamma}(8) = \text{Ker}(\rho^*_4)$. However, we can write $\phi^\pi(q)$ as a sum 

$$\phi^\pi(q) = \frac{3}{4}\phi_+(q) + \frac{1}{4}\phi_-(q).$$
where
\[
\phi_+(q) = \Phi_0^\pi(q) + \Phi_1^\pi(q) + \Phi_2^\pi(q) + \Phi_3^\pi(q),
\]
\[
\phi_-(q) = \Phi_0^\pi(q) - \Phi_1^\pi(q) + \Phi_2^\pi(q) - \Phi_3^\pi(q).
\]

Consider the congruence subgroup of $SL_2(\mathbb{Z})$
\[
\Gamma^0(8) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \mid b \equiv 0 \mod 8 \right\}.
\]

A direct calculation of the representation $\rho^*_4$ shows that $\phi_+(q)$ and $\phi_-(q)$ are modular forms of weight $21/2$ with respect to
\[
\tilde{\Gamma}^0(8) = \{(A, \phi) \in Mp_2(\mathbb{Z}) \mid A \in \Gamma^0(8)\}
\]
and distinct characters
\[
\chi_+, \chi_- : \tilde{\Gamma}^0(8) \to \mathbb{C}^*.
\]

Moreover, $A$ and $B$ are modular forms of weight $1/2$ with respect to $\tilde{\Gamma}^0(8)$ and the same characters $\chi_+$ and $\chi_-$ respectively.

We will not describe $\chi_\pm$ explicitly. While they are distinct, their squares are equal and $\chi = \chi_+^2 = \chi_-^2$ descends to a character
\[
\chi : \Gamma^0(8) \to \mathbb{C}^*.
\]

The character $\chi$ is specified completely by the following evaluations:
\[
\chi(\Gamma^1(8)) = 1, \quad \chi \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) = -1, \quad \chi \left( \begin{array}{cc} 3 & 8 \\ 1 & 3 \end{array} \right) = -1
\]
where
\[
\Gamma^1(8) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \mid b \equiv 0 \mod 8, a \equiv d \equiv 1 \mod 8 \right\}.
\]

Consider the space $\text{Mod}(\Gamma^0(8), 11, \chi)$ of holomorphic modular forms of weight $11$ and type $\chi$. The space $\text{Mod}(\Gamma^0(8), 11, \chi)$ is $12$-dimensional space with basis
\[
A^{22}, A^{20}B^2, \ldots, A^2B^{20}, B^{22}.
\]

Both $\phi_+(q) \cdot A$ and $\phi_-(q) \cdot B$ lie in $\text{Mod}(\Gamma^0(8), 11, \chi)$. Since $A^{22}/B$ and $B^{22}/A$ are not holomorphic at the boundary, we conclude $\phi_\pm(q)$ are each homogeneous polynomials of degree $21$ in $A$ and $B$ and therefore so is $\phi^\pi(q)$. 

\[\square\]
5. LEFSCHETZ PENCIL OF QUARTICS

5.1. Quartics. A general Lefschetz pencil of quartics can be viewed as a hypersurface of type \((4,1)\),
\[
\pi : X_{4,1} \subset \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^1
\]
where the last projection is onto the second factor. Unfortunately, \(\pi\) contains 108 nodal fibers, so the family (24) does not fit the specifications of Section 1.2.

A family of quasi-polarized \(K3\) surfaces of degree 4 can be obtained from the Lefschetz pencil \(\pi\) by the following construction. Let
\[
\epsilon : C_{53} \to \mathbb{P}^1
\]
be the genus 53 hyperelliptic curve branched over the 108 points of \(\mathbb{P}^1\) corresponding to the nodal fibers of \(\pi\). The family
\[
\epsilon^*(X_{4,1}) \to C_{53}
\]
has 3-fold double point singularities over the 108 nodes of the fibers of the original family \(\pi\). Let
\[
\tilde{\pi} : \tilde{X} \to C_{53}
\]
be obtained from a small resolution
\[
\tilde{X} \to \epsilon^*(X_{4,1}).
\]
Then, \(\tilde{\pi}\) is easily seen to be a family of quasi-polarized \(K3\) surfaces of degree 4. The quasi-polarization is the pull-back of \(\mathcal{O}_{\mathbb{P}^3}(1)\).

5.2. Invariants. The Noether-Lefschetz numbers are defined in Section 1 only for the family \(\tilde{\pi}\). However, for convenience, we define
\[
NL^{\pi}_{g,d} = \frac{1}{2}NL^{\tilde{\pi}}_{g,d}.
\]
Instead of a curve class \(\gamma\), the degree \(d\) against the polarization is taken as the second subscript.

The family \(\tilde{\pi}\) may be viewed as twice the Lefschetz pencil of quartics. Let
\[
\pi_{4,2} : X_{4,2} \subset \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^1
\]
be the family obtained from a nonsingular Calabi-Yau hypersurface. The family \(\pi_{4,2}\) may also be viewed as twice the Lefschetz pencil.

**Lemma 4.** \(n_{g,d}^{\tilde{X}} = n_{g,d}^{X_{4,2}}\).
Proof. It suffices to prove the analogous statement for Gromov-Witten invariants. Consider the degeneration of $X_{4,2}$ to the union

$$X_{4,1} \cup_{K3} X_{4,1}$$

of two $(4,1)$ hypersurfaces along a smooth $K3$ surface. The degeneration formula of \cite{30,31} implies

$$N_{g,d}^{X_{4,2}} = 2N_{g,d}^{X_{4,1}/K3}$$

where the latter term denotes the Gromov-Witten theory of $X_{4,1}$ relative to the $K3$ fiber. Since the Gromov-Witten theory of $K3 \times \mathbb{P}^1$ vanishes, the trivial degeneration

$$X_{4,1} \cup_{K3} (K3 \times \mathbb{P}^1)$$

yields the equality of relative and absolute invariants

$$N_{g,d}^{X_{4,1}} = N_{g,d}^{X_{4,1}/K3}.$$

To study the small resolution $\tilde{\pi}$, consider the family of double covers

$$\epsilon_t : C_t \mapsto \mathbb{P}^1$$

ramified at 108 generic points which specializes to our particular double cover \cite{25} as $t \to 0$. The behavior of Gromov-Witten theory in the conifold transition from

$$X_t = \epsilon_t^*(X_{4,1})$$

to $\tilde{X}$ has been calculated by Li and Ruan \cite{30}:

$$N_{g,d}^{\tilde{X}} = N_{g,d}^{X_t}.$$ 

By degenerating the base $C_t$ to two copies of $\mathbb{P}^1$, we have a degeneration of $X_t$ to two copies of $X_{4,1}$ attached at 54 smooth $K3$ fibers. As before, we apply the degeneration formula and the identification of relative and absolute invariants to obtain the equality

$$N_{g,d}^{\tilde{X}} = N_{g,d}^{X_t} = 2N_{g,d}^{X_{4,1}} = N_{g,d}^{X_{4,2}}.$$

Instead of studying the Gromov-Witten invariants of $\tilde{X}$, we may study the Gromov-Witten invariants of $X_{4,2}$.

5.3. Mirror symmetry.

5.3.1. Overview. The genus 0 invariants of $X_{4,2}$ are determined from hypergeometric series by the mirror transformation. The mirror formulas of Candelas, de la Ossa, Green, and Parkes \cite{10} have been proven mathematically in many settings \cite{15,16,33}. In particular, the case of $X_{4,2}$ is understood rigorously. We follow the notation of \cite{41}. 

5.3.2. Potential. Let the variables $T_1, T_2$ correspond to the hyperplane classes

$$H_1 \subset \mathbb{P}^3, \quad H_2 \subset \mathbb{P}^1$$

respectively. The genus 0 potential of $X_{4,2}$ for classes restricted from $\mathbb{P}^3 \times \mathbb{P}^1$ is

$$\mathcal{F}(T_1, T_2) = \frac{1}{3} T_1^3 + 2 T_1^2 T_2 + \sum_{d_1, d_2 \geq 0, (d_1, d_2) \neq (0,0)} N_{X_{4,2}}^{4,1} e^{d_1 T_1} e^{d_2 T_2}$$

where we follow the Gromov-Witten notation of Section 2. The curve class $(d_1, d_2)$ is not a fiber class for $\pi_{4,2}$ if $d_2 > 0$.

5.3.3. Hypergeometric series. Let $t_1, t_2$ be new variables. Define the hypergeometric series $I_{i,j}(t_1, t_2)$ by

$$\sum_{i=0}^{3} \sum_{j=0}^{1} I_{i,j}(t_1, t_2) H_1^i H_2^j =$$

$$\sum_{d_1, d_2 \geq 0} e^{(H_1+d_1)t_1} e^{(H_2+d_2)t_2} \frac{\prod_{r=0}^{d_1+2d_2} (4H_1 + 2H_2 + r)}{\prod_{r=1}^{d_1} (H_1 + r)^4 \prod_{r=1}^{d_2} (H_2 + r)^2}.$$

The right side, taken mod $H_1^4$ and $H_2^2$, is valued in $H^*(\mathbb{P}^3 \times \mathbb{P}^1, \mathbb{Q})$. Formally,

$$I_{i,j}(t_1, t_2) \in \mathbb{Q}[\![t_1, e^{t_1}, t_2, e^{t_2}]\!]$$

The functions $I_{i,j}(t)$ form a solution of the Picard-Fuchs differential equation associated to the mirror geometry.

5.3.4. Mirror transformation. The mirror transformation is defined using two auxiliary functions. Let

$$F(e^{t_1}, e^{t_2}) = \sum_{d_1=0}^{\infty} \sum_{d_2=0}^{\infty} e^{d_1 t_1} e^{d_2 t_2} \frac{(4d_1 + 2d_2)!}{(d_1!)^4 (d_2!)^2},$$

and let

$$G_{a,b}(e^{t_1}, e^{t_2}) = \sum_{d_1=0}^{\infty} \sum_{d_2=0}^{\infty} e^{d_1 t_1} e^{d_2 t_2} \frac{(4d_1 + 2d_2)!}{(d_1!)^4 (d_2!)^2} \left( \sum_{r=1}^{ad_1+bd_2} \frac{1}{r} \right)$$

for $a, b \geq 0$.

The mirror transformation relating the variables $T_i$ and $t_i$ is determined by the following equations:

$$T_1 = t_1 + \frac{4(G_{4,2}(e^{t_1}, e^{t_2}) - G_{1,0}(e^{t_1}, e^{t_2}))}{F(e^{t_1}, e^{t_2})},$$
\[ T_2 = t_2 + \frac{2(G_{4,2}(e^{t_1}, e^{t_2}) - G_{0,1}(e^{t_1}, e^{t_2}))}{F(e^{t_1}, e^{t_2})}. \]

Exponentiation yields
\[ e^{T_1} = e^{t_1} \exp \left( \frac{4(G_{4,2}(e^{t_1}, e^{t_2}) - G_{1,0}(e^{t_1}, e^{t_2}))}{F(e^{t_1}, e^{t_2})} \right), \]
\[ e^{T_2} = e^{t_2} \exp \left( \frac{2(G_{4,2}(e^{t_1}, e^{t_2}) - G_{0,1}(e^{t_1}, e^{t_2}))}{F(e^{t_1}, e^{t_2})} \right). \]

Together, the above four equations define a change of variables from formal series in \( T_1, e^{T_1}, T_2, e^{T_2} \) to formal series in \( t_1, e^{t_1}, t_2, e^{t_2} \). The mirror transformation is easily seen to be invertible.

5.3.5. Genus 0 invariants. The genus 0 potential \( F \) is determined by mirror symmetry,
\[ F(T_1(t_1, t_2), T_2(t_1, t_2)) = \left( \frac{2I_{1,1} - I_{2,0}}{I_{1,0}} \right) \left( \frac{I_{3,0}}{I_{1,0}} \right) + 2 \left( \frac{I_{2,0}}{I_{1,0}} \right) \left( \frac{I_{2,1}}{I_{1,0}} \right) - 2 \left( \frac{I_{3,1}}{I_{1,0}} \right). \]

The arguments of the functions on the right side are understood to be \( t_1 \) and \( t_2 \). The genus 0 BPS states \( n_{X_{4,2}} \) are determined by \( F \).

5.4. Proof of Theorem 2. Consider twice the Lefschetz pencil of quartics \( \tilde{\pi} : \tilde{X} \to C_{53} \).

Corollary 1 in genus 0 is
\[ n_{0,d}^{\tilde{X}} = \sum_{h=0}^{\infty} r_{0,h} \cdot NL_{h,d}^{\tilde{\pi}}. \]

We now solve for the Noether-Lefschetz numbers of \( \tilde{\pi} \). By (23),
\[ NL_{h,d}^{\tilde{\pi}} = \phi^{\tilde{\pi}} \left[ \frac{\Delta_4(h, d)}{8} \right] \]

where \( \phi^{\tilde{\pi}}(q) \) is a homogeneous polynomial of degree 21 in \( A \) and \( B \). We need only 22 equations to determine \( \phi^{\tilde{\pi}}(q) \). Using the mirror symmetry calculation of \( n_{0,d}^{\tilde{X}} \), equation (26) provides infinitely many relations. In particular, \( \phi^{\tilde{\pi}}(q) \) is easily determined by linear algebra.

The precise formula for \( \phi^{\tilde{\pi}} \) is \( 2\Theta \) where \( \Theta \) is given in Section 0.6 since \( \tilde{\pi} \) is twice the Lefschetz pencil of quartics. The modular form \( \Theta \) was first computed in [23].
5.5. Modular identity. Equation (26) may be viewed as a rather intricate relation between hypergeometric functions (after mirror transformation) on the left and modular forms on the right. Let

\[ G(q) = -\frac{2}{q} + 168 + \sum_{d \geq 1} n_{0,d}^X q^d \]

be the generating function determined by the property

\[ \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} n_{0,d}^X \frac{1}{k^3} e^{d k T_1} = \left( F(T_1, T_2) - \frac{1}{3} T_1^3 - 2 T_1^2 T_2 \right) |_{e^{T_2} = 0} \]

where \( F \) is determined as above.

**Corollary 4.** We have the equality

\[ G(q) = 2 \frac{\Theta(q)}{\Delta(q)} \]

where \( \Theta(q) \) is given in Section 0.6 and

\[ \Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \].

Such relations are produced by Theorem 1 for many classical examples. For any 1-parameter family of K3 surfaces obtained via a toric complete intersection, there is an associated identity of special functions. The relation obtained from the STU model studied in [25] is the Harvey-Moore identity. In fact, the Harvey-Moore identity is the only one for which a direct proof (avoiding Theorem 1) is known. The proof is due to Zagier and can be found in [25].

5.6. Proof of Corollary 2. Let \( \pi \) be the Lefschetz pencil of quartic K3 surfaces. The difference between \( NL_{h,d}^\pi \) and the degree of \( NL_{h,d}^\pi \) is the contribution of the nodal quartics. The nodal quartics contribute to \( NL_{h,d}^\pi \) but not the hypersurface \( \overline{D}_{h,d} \).

Using the relation \( NL_{h,d}^\pi = \frac{1}{2} NL_{h,d}^\pi \), we can study instead the doubled family. The Picard lattice of each of the 108 fibers of \( \pi \) corresponding to the original nodal fibers of \( \pi \) is

\[ \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \]

(27)

We use here the genericity of the Lefschetz pencil \( \pi \).
The equation $\langle \beta, L \rangle = d$ is solvable in the lattice (27) if and only if $d$ is divisible by 4. Then, $\langle \beta, \beta \rangle = 2h - 2$ is solvable if and only if

$$4\left(\frac{d}{4}\right)^2 - 2n^2 = 2h - 2$$

in which case there are two solutions. In the solvable cases,

$$\triangle_4(h, d) = 8n^2.$$

Hence, the contribution of the nodal fiber to the Noether-Lefschetz numbers of $\tilde{\pi}$ is

$$\Psi(q) = 108 \cdot 2 \sum_{n>0} q^{n^2}. $$

The Corollary follows by halving. □

6. Direct Noether-Lefschetz calculations

6.1. Overview. We apply Corollary 3 to directly study $K3$ surfaces of low degree via a more sophisticated approach to modular forms. The key idea is to construct a basis of the space of vector-valued modular forms of Corollary 3 instead of working with the much larger space of scalar-valued modular forms as in Section 4.5. For many classical families, the dimensions of the associated spaces of vector-valued modular forms are very small. The Noether-Lefschetz numbers can often be specified by a few classical calculations. In particular, we see another derivation of Theorem 2.

6.2. Rankin-Cohen brackets. Since each component of a vector-valued modular form is a half-weight modular form of level $2l$, we can use a basis of the latter to construct all vector-valued modular forms. In practice, however, the method is tedious since the dimensions of the spaces of scalar-valued modular forms are much larger. We will instead apply the following shortcut for low degree $K3$ surfaces.

Let $f(q)$ and $g(q)$ be scalar-valued level $N$ modular forms on the upper-half plane $\mathcal{H}$ of weights $k_1$ and $k_2$ respectively. For each integer $n \geq 0$, the $n$-th Rankin-Cohen bracket is a bilinear differential operator defined by the expression

$$[f(q), g(q)]_n = \sum_{r=0}^{n} (-1)^r \binom{n+k-1}{n-r} \binom{n+l-1}{r} f^{(r)}(q) \cdot g^{(n-r)}(q),$$

where $f^{(r)}$ denote $r$ applications of the differential operator

$$\frac{d}{d\tau} = q \frac{d}{dq}.$$

For $n = 0$, the 0-th bracket is just multiplication.
The key feature of Rankin-Cohen brackets is the preservation of modularity. Suppose we are given a representation \( \rho \) of \( \text{Mp}_2(\mathbb{Z}) \) on \( V \), a modular form \( f \in \text{Mod}(\text{Mp}_2(\mathbb{Z}), k_1, \rho) \) of weight \( k_1 \) and type \( \rho \), and a scalar-valued modular form \( g \in \text{Mod}(\text{SL}_2(\mathbb{Z}), k_2) \) of weight \( k_2 \) and level 1. Let

\[ f(q) = \sum_{\gamma} f_\gamma(q)v_\gamma \in V \]

denote the decomposition of \( f \) into components with respect to some basis of \( V \). For each integer \( n \geq 0 \), the Rankin-Cohen bracket is a holomorphic function on \( \mathcal{H} \) with values in \( V \) defined by

\[ [f, g]_n(q) = \sum_{\gamma} [f_\gamma(q), g(q)]_nv_\gamma. \]

We then have the following result.

**Lemma 5.** \( [f, g]_n(q) \in \text{Mod}(\text{Mp}_2(\mathbb{Z}), k_1 + k_2 + 2n, \rho) \).

**Proof.** For scalar-valued modular forms, a proof is given in [48]. Since \( g \) is scalar-valued and level 1, the same argument translates to the vector-valued context without change. \( \Box \)

### 6.3. Bases of modular forms

Following the notation of Corollary 3, we now look for modular forms of weight \( 21/2 \) and type \( \rho^* \) for even \( l = 2, 4, 6, 8 \).

From the dimension formula given in Section 7 below,

\[ \dim(\text{Mod}(\text{Mp}_2(\mathbb{Z}), 21/2, \rho^*)) = 2, 3, 4, 5 \]

for \( l = 2, 4, 6, 8 \) respectively. We are only interested\(^{11}\) in the subspace

\[ \text{Mod}_0(\text{Mp}_2(\mathbb{Z}), 21/2, \rho^*) \]

of forms \( \sum f_i(q)v_i \) where \( f_r(q) \) is a cusp form for \( r \neq 0 \). In the \( l = 8 \) case, we have a 4-dimensional subspace.

We can use Rankin-Cohen brackets to construct explicit bases. Indeed, for each \( l \), there is a canonical weight \( 1/2 \) modular form given by the Siegel theta function (see [3], Section 4),

\[ \theta^{(l)}(q) = \sum_{i=0}^{l} \sum_{s} q^{\frac{(l+s)^2}{2}}v_i \in \text{Mod}(\text{Mp}_2(\mathbb{Z}), 1/2, \rho^*). \]

Therefore, for \( n = 0, 1, 2, 3 \), Lemma 5 gives us a modular form,

\[ F_n^l(q) = [\theta^{(l)}(q), E_{10-2n}(q)]_n \in \text{Mod}(\text{Mp}_2(\mathbb{Z}), 21/2, \rho^*). \]

\(^{11}\)The cusp condition is obtained from Borcherds’ results and was omitted in the statement of Corollary 3 for simplicity.
of weight $21/2$ where $E_{2k}(q)$ denotes Eisenstein series of weight $2k$.

Using the explicit formula for Rankin-Cohen brackets and the dimension formula, the following Lemma is obtained by calculating the initial Taylor coefficients.

**Lemma 6.** For $l = 2, 4, 6$, the modular forms

$$F^l_n(q) = \left[ \theta^{(l)}(q), E_{10-2n}(q) \right]_n, n = 0, \ldots, l/2$$

form a basis of $\text{Mod}(M_p(\mathbb{Z}), 21/2, \rho^*_l)$. For $l = 8$, the modular forms for $n = 0, \ldots, 3$ form a basis of the subspace $\text{Mod}_0(M_p(\mathbb{Z}), 21/2, \rho^*_l)$.

### 6.4. Classical families of $K3$ surfaces

A general $K3$ surface of degree $l = 2, 4, 6, 8$ is either a branched cover of $\mathbb{P}^2$ (for $l = 2$) or a complete intersection in projective space. We obtain 1-parameter families of quasi-polarized $K3$ surfaces of degree $l$ by taking a generic Lefschetz pencil of these constructions (and resolving singularities as discussed in Section 5.1). Because the space of vector-valued forms is of low dimension, we only need a few classical constraints to completely determine the associated modular form. In fact, we will use only the following constraints:

(i) the degree of the Hodge bundle $R^2\pi_*\mathcal{O}$ (the coefficient of $q^0v_0$),
(ii) the number of nodal fibers (the coefficient of $q^1v_0$),
(iii) vanishing obtained from Castelnuovo’s bound in Lemma 7 below.

The following result is a special case of Castelnuovo’s bound for projective curves [1].

**Lemma 7.** Given a $K3$ surface with very ample bundle $L$ and an primitive curve class $\beta$, we have the inequality

$$\langle \beta, \beta \rangle \leq 2 \left( \frac{L \cdot \beta - 1}{2} \right) - 2.$$

We now apply these constraints for 1-parameter families of $K3$ given by Lefschetz pencils for $l = 2, 4, 5, 6$.

- **Degree 2 $K3$ surfaces**

A generic degree $K3$ surface of degree 2 is a double cover of $\mathbb{P}^2$ branched along a nonsingular sextic plane curve. Consider a family

$$R \subset \mathbb{P}^1 \times \mathbb{P}^2$$
of sextics defined by a generic hypersurface of type $(2, 6)$. Let $X$ be the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified over $R$. Since all the singular fibers of $R \to \mathbb{P}^1$ are irreducible and nodal, the associated family of $\pi : X \to \mathbb{P}^1$ of $K3$ surfaces has only 3-fold double-point singularities (which admit small resolutions).

The degree of the Hodge bundle is $-1$ by a Riemann-Roch calculation. The number of nodal fibers of $\pi$ is 150, twice the degree of the discriminant locus of sextics. Since we have a 2-dimensional space of forms, the generating series of Noether-Lefschetz numbers is the vector-valued modular form

$$\Theta(q) = -F^{(2)}_0(q) - \frac{1}{2} F^{(2)}_1(q).$$

In the case of $l = 2$, the discriminant $\Delta$ of a rank 2 lattice with degree 2 polarization determines the coset class $\delta$ by $\delta = \Delta \mod 2$. So there is no loss of information if we replace $\Theta(q)$ by the sum of the components $\Theta(q) = \Theta_0 + \Theta_1$.

If we consider the theta functions

$$U = \sum_{n \in \mathbb{Z}} q^{n^2/4}, \quad V = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/4},$$

we can express $\Theta$ as a polynomial function of $A$ and $B$:

$$\Theta(q) = \frac{1}{1024} (U^{21} - 12U^{17}V^4 - 402U^{13}V^8 - 572U^9V^{12} - 39U^5V^{16}$$

$$= -1 + 150q + 1248q^{5/4} + 108600q^2 + 332800q^{9/4} + 5113200q^3 \cdots .$$

To see equivalence of the two expressions, we observe both are modular forms of weight $21/2$ with respect to $\Gamma(4)$ and check the agreement of sufficiently many coefficients.

- **Degree 4 $K3$ surfaces**

A generic $K3$ surface of degree 4 is a quartic hypersurface in $\mathbb{P}^3$. If we take a generic Lefschetz pencil of such quartics, the degree of the Hodge bundle is $-1$. Using Lemma 7, the Noether-Lefschetz degrees associated to the lattices

$$\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$$
both vanish. Indeed, by choosing a generic pencil, we can assume all fibers containing these Picard lattices have very ample quasi-polarization. The coefficients of $q^0 v_0, q^{1/8} v_1$, and $q^{1/2} v_2$ determine

$$
\Theta(q) = - F_0^{(4)}(q) - \frac{5}{4} F_1^{(4)}(q) - \frac{16}{21} F_2^{(4)}(q).
$$

Again, as in the degree 2 case, we can recover all Noether-Lefschetz degrees from

$$
\Theta(q) = \Theta_0(q) + \Theta_1(q) + \Theta_2(q).
$$

In terms of

$$
A = \sum_{n \in \mathbb{Z}} q^{n^2/8}, \quad B = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/8},
$$

we recover the expression for $\Theta(q)$ given in Section 0.6 since both are modular forms of weight $21/2$ and level 8 which agree on initial terms.

- Degree 6 K3 surfaces

A generic K3 surface of degree 6 is the intersection of a quadric and cubic hypersurface in $\mathbb{P}^4$. We have two basic families. We can fix a quadric and take a Lefschetz pencil of cubics or vice versa. In each case, we have vanishings associated to the lattices

$$
\begin{pmatrix}
6 & 1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
6 & 2 \\
2 & 0
\end{pmatrix}
$$

from the Castelnuovo bound. Along with the Hodge bundle degree and the number of nodal fibers, we completely determine the Noether-Lefschetz series.

For the first family, the Hodge and nodal degrees are $-1$ and 98 respectively. We obtain the series

$$
\Theta(q) = - F_0^{(6)}(q) - \frac{49}{24} F_1^{(6)}(q) - \frac{8}{3} F_2^{(6)}(q) - \frac{12}{5} F_3^{(6)}(q).
$$

For the second family, the Hodge and nodal degrees are $-1$ and 7. We obtain the series

$$
\Theta(q) = - F_0^{(6)}(q) - \frac{17}{8} F_1^{(6)}(q) - \frac{22}{7} F_2^{(6)}(q) - \frac{18}{5} F_3^{(6)}(q).
$$

One can read off other classical calculations from our results. For example, the number of surfaces containing elliptic plane curves or
containing lines are the Noether-Lefschetz degrees associated to the lattices
\[
\begin{pmatrix}
6 & 3 \\
3 & 0
\end{pmatrix},
\begin{pmatrix}
6 & 1 \\
1 & -2
\end{pmatrix}
\]
respectively. In the first family, the degrees are 0 and 168 respectively. In the second family, the degrees are 10 and 198. In both cases, the numbers agree with earlier enumerative calculations.

- **Degree 8 K3 surfaces**

A generic $K3$ surface of degree 8 is the intersection of three quadric hypersurfaces in $\mathbb{P}^5$. The basic family comes from fixing two quadrics and allowing the third to vary in a Lefschetz pencil. Again, the series is determined by the Hodge term, the nodal term, and the two Castelnuovo vanishings from Lemma 7. The Hodge term is given by $-1$, and the number of nodal fibers is 80. We find

\[
\Theta(q) = -F_0^{(8)}(q) - \frac{49}{18}F_1^{(8)}(q) - \frac{128}{27}F_2^{(8)}(q) - \frac{256}{45}F_3^{(8)}(q).
\]

Again, we can read off that the number of fibers containing a line is 128, agreeing with the classical calculation.

For all the classical examples discussed above, the mirror symmetry calculation of the genus 0 Gromov-Witten invariants is solvable in terms of hypergeometric functions. In each case, Theorem 1 yields a remarkable identity with hypergeometric functions (after mirror transformation) on the left and modular forms on the right, as in Section 5.5.

7. **Picard rank of $\mathcal{M}_t$**

The Picard ranks of the moduli spaces of quasi-polarized $K3$ surfaces $\mathcal{M}_t$ are unknown. By an argument of O’Grady, the ranks can grow arbitrarily large [40]. Let

\[
\text{Pic}(\mathcal{M}_t)^{NL} \otimes \mathbb{Q} \subset \text{Pic}(\mathcal{M}_t) \otimes \mathbb{Q}
\]
denote the span of the Noether-Lefschetz divisors $D_{h,d}$. We make the following conjecture.

**Conjecture 3.** The inclusion is an isomorphism,

\[
\text{Pic}(\mathcal{M}_t)^{NL} \otimes \mathbb{Q} \cong \text{Pic}(\mathcal{M}_t) \otimes \mathbb{Q}.
\]
Bruinier has calculated the dimension of the space $\text{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q}$ in [7]. If Conjecture 3 holds, we obtain a formula for the Picard rank of $\mathcal{M}_l$.

We now recount Bruinier’s formula for the span of the Noether-Lefschetz divisors. By Borcherds’ work, we have a map

$$\text{Mod}(Mp_2(\mathbb{Z}), \rho^*_l, 21/2)^* \to \text{Pic}(\mathcal{M}_l) \otimes \mathbb{C}. \quad (29)$$

Let $\text{Cusp}(Mp_2(\mathbb{Z}), \rho^*_l, 21/2)$ denote the subspace of cusp forms — modular forms for which the Fourier coefficients $c_{0,\gamma}$ vanish for all $\gamma$. The map (29) induces a map

$$\text{Cusp}(Mp_2(\mathbb{Z}), \rho^*_l, 21/2)^* \to (\text{Pic}(\mathcal{M}_l) \otimes \mathbb{C})/\mathbb{C}K, \quad (30)$$

where $K$ is the Hodge bundle on $\mathcal{M}_l$. Bruinier shows the map (30) is injective [7]. Specifically, if $L$ is a $(2, n)$ lattice containing two copies of $U$ as direct summands, Bruinier shows that every relation among Heegner divisors is obtained from Borcherds’ theta lifting. Therefore,

$$\dim \text{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} = 1 + \dim \text{Cusp}(Mp_2(\mathbb{Z}), \rho^*_l, 21/2).$$

A direct calculation of the dimension of the space of cusp forms via Riemann-Roch yields the following evaluation [7]:

$$\dim \text{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} = \frac{38}{24} + \frac{31}{24} l - \frac{1}{8\sqrt{l}} \Re(G(2, 2l)) - \frac{1}{6\sqrt{3}l} \Re(e^{-2\pi i \frac{l}{12}}(G(1, 2l) + G(-3, 2l))) - \sum_{k=0}^{1/2} \left\{ \frac{k^2}{2l} \right\} - C,$$

where $G(a, b)$ denotes the quadratic Gauss sum

$$G(a, b) = \sum_{k=0}^{b-1} e^{2\pi i \frac{ak^2}{b}},$$

the braces $\{,\}$ denote fractional part, and $C$ is the cardinality of the set

$$\left\{ k \mid 0 \leq k \leq \frac{l}{2}, \frac{k^2}{2l} \in \mathbb{Z} \right\}.$$

For $l = 2, 4, 6$, the formula yields

$$\dim \text{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} = 2, 3, 4$$

respectively. For $l = 2$ and 4, we have agreement with the Picard ranks of $\mathcal{M}_l$ calculated in [23, 45, 46]. Hence, the inclusion (28) is an isomorphism in at least the first two cases.
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