CAT(0) SPACES ON WHICH A CERTAIN TYPE OF SINGULARITY IS BOUNDED

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Abstract. In this paper, we will consider a family $\mathcal{Y}$ of complete CAT(0) spaces such that the tangent cone $TC_p^\infty Y$ at each point $p \in Y$ of each $Y \in \mathcal{Y}$ is isometric to a (finite or infinite) product of the Euclidean cones $\text{Cone}(X_\alpha)$ over elements $X_\alpha$ of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of CAT(1) spaces. Each element of such $\mathcal{Y}$ is a space presented by Gromov [4] as an example of a “CAT(0) space with “bounded” singularities”. We will show that the Izeki-Nayatani invariants of spaces in such a family are uniformly bounded from above by a constant strictly less than 1.

1. Introduction

In [4], Gromov introduced the term “CAT(0) space with ‘bounded’ singularities”, and remarked that there exist infinite groups which admit no uniform embeddings into such a space. He used this terminology without providing its precise definition, but as examples of such spaces, he presented CAT(0) spaces $Y$ such that the tangent cone $TC_p^\infty Y$ at each point $p \in Y$ is isometric to a (finite or infinite) product of Euclidean cones $\text{Cone}(X_\alpha)$ over elements $X_\alpha$ of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of CAT(1) spaces.

On the other hand, Izeki and Nayatani [5] defined an invariant $\delta(Y) \in [0,1]$ of a complete CAT(0) space $Y$. And some general results for CAT(0) spaces whose Izeki-Nayatani invariants are bounded from above were proved by Izeki, Kondo, and Nayatani ([5], [6], [7], [8], [9]). Group $\Gamma$ is said to have the fixed-point property for a metric space $Y$, if for any group homomorphism $\rho: \Gamma \to \text{Isom}(Y)$ there exists a point $p \in Y$ such that $\rho(\gamma)p = p$ for all $\gamma \in \Gamma$. Izeki, Kondo and Nayatani [7] proved that a random group of Gromov’s graph model has the fixed-point property for all elements $Y$ of a family $\mathcal{Y}$ of CAT(0) spaces whose Izeki-Nayatani invariants are uniformly bounded from above by a constant strictly less than 1:

$$\sup\{\delta(Y) \mid Y \in \mathcal{Y}\} < 1.$$ 

Moreover, it is straightforward to see that an expander admits no uniform embedding into a complete CAT(0) space $Y$ with $\delta(Y) < 1$ (see [9]). Combining this with Gromov’s argument in [4], the existence of infinite groups which admit no uniform embeddings into a space $Y$ with $\delta(Y) < 1$ follows. This seems to suggest that the

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Izeki-Nayatani invariant measures a certain type of “singularity” similar to Gromov’s notion.

Although these general results were proved, the computation of the Izeki-Nayatani invariant is difficult. It is still unclear what kind of CAT(0) spaces $Y$ or families $\mathcal{Y}$ of CAT(0) spaces have the boundedness property as above. It had been even unknown whether there exists a complete CAT(0) space $Y$ with $\delta(Y) = 1$ or not, until Kondo [9] showed the existence of CAT(0) spaces with $\delta = 1$ fairly recently.

In this paper, we prove the following theorem.

**Theorem 1.1.** Let $\mathcal{Y}$ be a family of complete CAT(0) spaces such that the tangent cone $TC_p Y$ at each point $p \in Y$ on each $Y \in \mathcal{Y}$ is isometric to a (finite or infinite) product of the Euclidean cones Cone($X_\alpha$) over elements $X_\alpha$ of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of complete CAT(1) spaces. Then we have

$$\sup_{Y \in \mathcal{Y}} \delta(Y) < 1.$$ 

Here, we use the word *product* of Euclidean cones $T_1, T_2, \ldots$ in the sense of $\ell^2$-product of the pointed metric spaces $(T_1, O_1), (T_2, O_2), \ldots$, where each $O_n$ is the cone point of $T_n$. That is, the product $T$ of the cones $T_1, T_2, \ldots$ consists of all sequences $(x_n)_n$ such that $x_n \in T_n$ and $\sum_n d_n(O_n, x_n)^2 < \infty$, and $T$ is equipped with the metric function $d$ defined by

$$d(x, y)^2 = \sum_{n=1}^{\infty} d_n(x_n, y_n)^2$$

for any $x = (x_1, x_2, \ldots) \in T$ and any $y = (y_1, y_2, \ldots) \in T$, where $d_n$ is the metric function on $T_n$ for each $n$. Then, $T$ also has a cone structure with the cone point $O = (O_1, O_2, \ldots)$. And completeness and CAT(0) condition are preserved by this construction.

Combining Theorem 1.1 with the general results mentioned above, we have the following corollary.

**Corollary 1.2.** (i) If $Y$ is a complete CAT(0) space such that the tangent cone at each point $y \in Y$ is isometric to a (finite or infinite) product of Euclidean cones Cone($X_\alpha$) over elements $X_\alpha$ of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of CAT(1) spaces, then there exists infinite groups which admit no uniform embeddings into $Y$. (ii) There exist infinite groups which has the fixed-point property for all elements $Y$ in such a family $\mathcal{Y}$ as in Theorem 1.1.

Here, (i) has already been remarked in [4]. And (ii) follows from the general result in [7]. (ii) can be stated in terms of random groups(see [7]).

In the end of this paper, we claim that by the same technique used in the proof of Theorem 1.1, we can prove a more general statement, which includes Theorem 1.1 as a special case (Proposition 5.4).
2. Preliminaries on CAT(0) spaces

In this section we recall some basic definitions and facts concerning CAT(0) spaces. For a detailed exposition, we refer the reader to [1], [2] or [11].

For $\kappa > 0$ let $M^2_\kappa$ denote the simply connected, complete 2-dimensional Riemannian manifold of constant Gaussian curvature $\kappa$, and let $d_\kappa$ be its distance function. Let $D_\kappa \in (0, \infty)$ be the diameter of $M^2_\kappa$.

Let $(Y, d_Y)$ be a metric space. A geodesic in $Y$ is an isometric embedding $\gamma$ of a closed interval $[a, b]$ into $Y$. A geodesic triangle in $Y$ is a triple $\triangle = (\gamma_1, \gamma_2, \gamma_3)$ of geodesics $\gamma_i : [a_i, b_i] \to Y$ such that

$$\gamma_1(b_1) = \gamma_2(a_2), \quad \gamma_2(b_2) = \gamma_3(a_3), \quad \gamma_3(b_3) = \gamma_1(a_1).$$

If $\triangle$ has a perimeter less than $2D_\kappa$: $\sum_{i=1}^{3} |b_i - a_i| < 2D_\kappa$, then there is a geodesic triangle

$$\triangle^\kappa = (\gamma_1^\kappa, \gamma_2^\kappa, \gamma_3^\kappa), \quad \gamma_i : [a_i, b_i] \to M^2_\kappa$$

in $M^2_\kappa$, which has the same side lengths as $\triangle$. This triangle $\triangle^\kappa$ is unique up to isometry of $M^2_\kappa$, and we call it the comparison triangle of $\triangle$ in $M^2_\kappa$. Then $\triangle$ is said to be $\kappa$-thin if

$$d_Y(\gamma_i(s), \gamma_j(t)) \leq d_\kappa(\gamma_i^\kappa(s), \gamma_j^\kappa(t))$$

whenever $i, j \in \{1, 2, 3\}$ and $s \in [a_i, b_i]$, and $t \in [a_j, b_j]$.

**Definition 2.1.** A metric space $(Y, d)$ is called a $\text{CAT}(\kappa)$ space, if for any pair of points $p, q \in Y$ with $d(p, q) < D_\kappa$ there exists a geodesic from $p$ to $q$, and any geodesic triangle in $Y$ with perimeter $< 2D_\kappa$ is $\kappa$-thin.

Next, we recall the definition of the Euclidean cone. Let $(X, d_X)$ be a metric space. The cone $\text{Cone}(X)$ over $X$ is the quotient of the product $X \times [0, \infty)$ obtained by identifying all points in $X \times \{0\} \subset X \times [0, \infty)$. The point represented by $(x, 0)$ is called the cone point of $\text{Cone}(X)$ and we will denote this point by $O_{\text{Cone}(X)}$ in this paper. The cone distance $d_{\text{Cone}(X)}(v, w)$ between two points $v, w \in \text{Cone}(X)$ represented by $(x, t), (y, s) \in X \times [0, \infty)$ respectively, is defined by

$$d_{\text{Cone}(X)}(v, w) = \sqrt{t^2 + s^2 - 2ts \cos(\min\{\pi, d_X(x, y)\})}.$$

Then $(\text{Cone}(X), d_{\text{Cone}(X)})$ is a metric space, and we call it the Euclidean cone over $(X, d_X)$. It is known that a metric space $(X, d_X)$ is a CAT(1) space if and only if $(\text{Cone}(X), d_{\text{Cone}(X)})$ is a CAT(0) space.

Suppose that $Y$ is a CAT(0) space. Then by the definition of CAT(0) space, there is a unique geodesic joining any pair of points in $Y$. So, for any triple of points $(p, q, r)$ in $Y$, it makes sense to denote by $\triangle(p, q, r)$ the geodesic triangle consisting of three geodesics joining each pair of the three points.

Let $\gamma : [a, b] \to Y, \gamma' : [a', b'] \to Y$ be two geodesics in a CAT(0) space $Y$ such that

$$\gamma(a) = \gamma'(a') = p \in Y.$$

We define the angle $\angle_p(\gamma, \gamma')$ between $\gamma$, $\gamma'$ as

$$\angle_p(\gamma, \gamma') = \lim_{t \to a, t' \to a'} \angle_p^0(\gamma(t), \gamma(t')).$$
where \( \angle_p^2(\gamma(t), \gamma(t')) \) is the corresponding angle of the comparison triangle of \( \triangle(p, \gamma(t), \gamma'(t')) \) in \( M_0^2 = \mathbb{R}^2 \). The existence of the limit follows from the definition of CAT(0) space.

**Definition 2.2.** Let \((Y, d_Y)\) be a complete CAT(0) space, and let \( p \in Y \). We denote by \((S_pY)^\circ\) the set of all geodesics \( \gamma : [a, b] \to Y \) such that \( \gamma(a) = p \). Then the angle \( \angle_p \) defines a pseudometric on \((S_pY)^\circ\). The space of directions \( S_pY \) at \( p \) is the metric completion of the quotient space of \((S_pY)^\circ\) where we identify any \( x, y \in S_pY \) with \( \angle_p(x, y) = 0 \). We define the tangent cone \( TC_pY \) of \( Y \) at \( p \) to be the Euclidean cone \( \text{Cone}(S_pY) \) over the space of directions at \( p \).

If \((Y, d_Y)\) is a complete CAT(0) space and if \( p \in Y \), then it can be proved that the space of directions \( S_pY \) at \( p \) is a complete CAT(1) space. Hence, the tangent cone \( TC_pY \) at \( p \) is a complete CAT(0) space.

Finally, we recall some basic notions and facts about probability measures on a metric space \((Y, d_Y)\). In this paper, we will treat only finitely supported measures. Measure \( \nu \) on \( Y \) is finitely supported if there exists a finite subset \( S \subset Y \) such that \( \nu(Y \setminus S) = 0 \). We call the minimal subset \( S \) with such a property the support of \( \nu \), and denote it by \( \text{supp}(\nu) \). We denote by \( \mathcal{P}(Y) \) the set of all finitely supported probability measures on \( Y \). If \( \text{supp}(\nu) = \{p_1, \ldots, p_n\} \), then \( \nu \) can be represented as

\[
\nu = \sum_{i=1}^{n} t_i \text{Dirac}_{p_i}
\]

by nonnegative real numbers \( t_1, \ldots, t_n \) with \( \sum_{i=1}^{n} t_i = 1 \), where \( \text{Dirac}_{p_i} \) stands for the Dirac measure at \( p_i \in Y \). We will also use the notation \( \mathcal{P}'(Y) \) to denote the subset of \( \mathcal{P}(Y) \) consisting of all measures whose supports contain at least two points. Let \( Z \) be a set and let \( \phi : Y \to X \) be a map. Then for any \( \nu \in \mathcal{P}(Y) \), we define the pushforward measure \( \phi_*\mu \) on \( X \) as

\[
\phi_*\nu(A) = \mu(\phi^{-1}(A)) \quad A \subset X
\]

If we write \( \nu \) as in the form (2.1), we can write \( \phi_*\nu \) as

\[
\phi_*\nu = \sum_{i=1}^{n} t_i \text{Dirac}_{\phi(p_i)}
\]

If \((Y, d_Y)\) is a complete CAT(0) space, and if \( \nu \in \mathcal{P}(Y) \), there exists a unique point \( \text{bar}(\nu) \in Y \) which minimizes the function

\[
y \mapsto \int_Y d(y, z)^2 \nu(dz)
\]

defined on \( Y \). This point is called the barycenter of \( \nu \). We refer the reader to [11] for the existence and uniqueness of barycenter.

### 3. Hilbert sphere valued maps and an invariant of a CAT(1) space

In this section, we define a certain invariant of complete CAT(1) spaces. First we set up some notations for Hilbert sphere valued maps on CAT(1) spaces. Let \( \mathcal{H} \)
be a real Hilbert space, and let $\phi : X \to \mathcal{H}$ be a map whose image is contained in the unit sphere in $\mathcal{H}$. Thus $\|\phi(x)\| = 1$ for all $x \in X$. Let $\mu \in \mathcal{P}(X)$ be a finitely supported probability measure on $X$. We define the vector $E_\mu[\phi] \in \mathcal{H}$ as

$$E_\mu[\phi] = \int_X \phi(x) \mu(dx).$$

And if the vector $E_\mu[\phi]$ is not the zero vector, we denote by $\tilde{E}_\mu[\phi]$ the unit vector parallel to $E_\mu[\phi]$: $\tilde{E}_\mu[\phi] = \frac{1}{\|E_\mu[\phi]\|} E_\mu[\phi]$.

Then the value $\|E_\mu[\phi]\| \in [0, 1]$ amounts to a sort of concentration of the pushforward measure $\phi_*\mu$ around $\tilde{E}_\mu[\phi]$ on the unit sphere. By simple calculation, we have

$$\|E_\mu[\phi]\| = \int_X \langle \tilde{E}_\mu[\phi], \phi(x) \rangle \mu(dx)$$

whenever $\|E_\mu[\phi]\| \neq 0$.

Now we define an invariant of a complete CAT(1) space by using the notations introduced above. This invariant is designed for estimating the Izeki-Nayatani invariant of a CAT(0) space, whose definition will be recalled in the next section.

**Definition 3.1.** Let $(X, d_X)$ be a metric space, and let $\mu \in \mathcal{P}(X)$. We define $\tilde{\delta}(\mu) \in [0, 1]$ to be

$$\tilde{\delta}(\mu) = \inf_{\phi} \|E_\mu[\phi]\|^2,$$

where the infimum is taken over all maps $\phi : X \to \mathcal{H}$ to some Hilbert space $\mathcal{H}$ such that

$$\|\phi(x)\| = 1, \quad \angle (\phi(x), \phi(y)) \leq d_X(x, y)$$

for any $x, y \in X$. Here and henceforth, we denote the angle between two vectors $v, w$ in any Hilbert space by $\angle(v, w)$.

Suppose $(X, d_X)$ is a complete CAT(1) space and $\iota : X \to \text{Cone}(X)$ is the canonical inclusion of $X$ into its Euclidean cone. Then, we define $\tilde{\delta}(X)$ to be

$$\tilde{\delta}(X) = \sup\{\delta(\mu) \mid \mu \in \mathcal{P}(X), \text{bar}(\iota_*\mu) = O_{\text{Cone}(X)}\}.$$

When there is no measure satisfying such a condition, we define $\tilde{\delta}(X) = -\infty$.

To estimate this invariant in the proceeding sections, we will use the following fact:

**Lemma 3.2.** Let $(X, d_X)$ be a complete CAT(1) space. For $v, w \in \text{Cone}(X)$ represented by $(x, t), (y, s) \in X \times \mathbb{R}$ respectively, we set

$$\langle v, w \rangle = ts \cos(\min\{\pi, d_X(x, y)\}).$$

Then for any $\nu \in \mathcal{P}(\text{Cone}(X))$ the following two conditions are equivalent:

(i): $\text{bar}(\nu) = O_{\text{Cone}(X)}$.

(ii): $\int_{\text{Cone}(X)} \langle E_x, v \rangle \nu(dv) \leq 0$, whenever $x \in X$ and $E_x$ is an element of $\text{Cone}(X)$ represented by $(x, 1)$. 
Proof. For \( w \in \text{Cone}(X) \) represented by \( w = (y, s) \in X \times \mathbb{R} \), we write \( \|w\| = s \). Fix \( x \in X \) and let \( v_t \) be an element of \( \text{Cone}(X) \) represented by \( (x, t) \in X \times \mathbb{R} \). Suppose that \( \overline{\nu} = O_{\text{Cone}(X)} \). Then the function
\[
F_x(t) = \int_{\text{Cone}(X)} d_{\text{Cone}(X)}(v_t, w)^2 \nu(dw)
= \int_{\text{Cone}(X)} \{t^2 + \|w\|^2 - 2t\langle E_x, w \rangle \} \nu(dw),
\]
defined on \([0, \infty)\) must attain its minimum at \( t = 0 \). This happens if and only if
\[
F'_x(t) = 2 \left( t - \int_{\text{Cone}(X)} \langle E_x, w \rangle \nu(dw) \right) \geq 0.
\]
for all \( t \in \mathbb{R} \). So (ii) follows.

Conversely, if (ii) holds, then the function \( F_x \) on \([0, \infty)\) as (3.3) attains its minimum at \( t = 0 \) for each \( x \in X \). And it is easily seen that \( \overline{\nu} = O_{\text{Cone}(X)} \). \( \square \)

In the final section, we will use this lemma in the following form.

Corollary 3.3. Let \((X, d_X)\) be a complete CAT(1) space, and let \( \iota : X \to \text{Cone}(X) \) be the canonical inclusion. If \( \mu \in \mathcal{P}(X) \) satisfies \( \overline{\iota^*\mu} = O_{\text{Cone}(X)} \), then we have
\[
\mu \left( \left\{ y \in X \mid d_X(x, y) \leq \theta \right\} \right) \leq \frac{1}{1 + \cos \theta}
\]
for any \( x \in X \) and any \( 0 \leq \theta < \frac{\pi}{2} \). In particular, we have
\[
\mu \left( \left\{ y \in X \mid d_X(x, y) \leq \frac{\pi}{3} \right\} \right) \leq \frac{2}{3}
\]
for all \( x \in X \).

Proof. Suppose there is \( x_0 \in X \) such that
\[
\mu \left( \left\{ y \in X \mid d_X(x_0, y) \leq \theta \right\} \right) > \frac{1}{1 + \cos \theta}.
\]
Then we would have
\[
\int_X \cos \left( \min \{ \pi, d_X(x_0, x) \} \right) \mu(dx) = \int_{\{x \in X \mid d_X(x, x_0) \leq \theta \}} \cos \left( \min \{ \pi, d_X(x_0, x) \} \right) \mu(dx)
+ \int_{X \setminus \{x \in X \mid d_X(x, x_0) \leq \theta \}} \cos \left( \min \{ \pi, d_X(x_0, x) \} \right) \mu(dx)
> \cos \theta \times \frac{1}{1 + \cos \theta} + (-1) \times \left( 1 - \frac{1}{1 + \cos \theta} \right)
= 0.
\]
This implies \( \overline{\iota^*\mu} \neq O_{\text{Cone}(X)} \) by Lemma 3.2, which is a contradiction. \( \square \)
4. IZEKI-NAYATANI INVARIANT

In this section, we recall the definition of the invariant \( \delta \) of a complete CAT(0) space introduced by Izeki and Nayatani [5]. We will then derive a relation between \( \delta \) and the invariant \( \tilde{\delta} \) of a complete CAT(1) space defined in the previous section. More information about the Izeki-Nayatani invariant \( \delta \) can be found in [5], [6], [7], [8] and [10].

**Definition 4.1** ([5]). Let \((Y, d_Y)\) be a complete CAT(0) space. Recall that \( P'(Y) \) is the subset of \( P(Y) \) consisting of all measures whose supports contain at least two points. For any \( \nu \in P'(Y) \), we define \( \delta(\nu) \) to be

\[
\delta(\nu) = \inf_{\phi} \frac{\int_Y \| \phi(p) \nu(dp) \|^2}{\int_Y \| \phi(p) \|^2 \nu(dp)},
\]

where the infimum is taken over all maps \( \phi: \text{supp}(\nu) \to H \) from the support of \( \nu \) to some Hilbert space \( H \) such that

\[
\| \phi(p) \| = d(\text{bar}(\nu), p),
\]

\[
\| \phi(p) - \phi(q) \| \leq d(p, q)
\]

for all \( p, q \in \text{supp}(\nu) \). Then the Izeki-Nayatani invariant \( \delta(Y) \) of \( Y \) is defined by

\[
\delta(Y) = \sup \{ \delta(\nu) \mid \nu \in P'(Y) \}.
\]

By definition, we have \( 0 \leq \delta(\nu) \leq 1 \) and \( 0 \leq \delta(Y) \leq 1 \). When \( Y \) is a Euclidean cone, we define \( \delta(Y, O_Y) \in [0, 1] \) to be

\[
\delta(Y, O_Y) = \sup \{ \delta(\nu) \mid \nu \in P'(Y), \text{bar}(\nu) = O_Y \},
\]

where \( O_Y \) is the cone point of \( Y \). When there is no measure satisfying such a condition, we define \( \delta(Y, O_Y) = -\infty \). The following lemma is shown in [5].

**Lemma 4.2** ([5]). Suppose that \( Y \) is a complete CAT(0) space, and \( \nu \in P'(Y) \). Then we have

\[
\delta(\nu) \leq \delta(TC_{\text{bar}(\nu)}Y, O_{TC_{\text{bar}(\nu)}Y}).
\]

In particular, we have

\[
\delta(Y) \leq \sup \{ \delta(TC_pY, O_{TC_pY}) \mid p \in Y \}.
\]

The following lemma is a slight generalization of Proposition 6.5 in [5].

**Lemma 4.3.** Let \((T_1, d_1), (T_2, d_2), (T_3, d_3), \ldots \) be complete CAT(0) spaces which are isometric to Euclidean cones, and let \( O_1, O_2, \ldots \) be their cone points respectively. Let \( T \) be the cone obtained as the product of \( T_1, T_2, \ldots \) with the cone point \( O = (O_1, O_2, \ldots) \). Then we have

\[
\delta(T, O) = \sup_n \delta(T_n, O_n).
\]
Proof. The following proof is almost the same argument as in the proof of Proposition 6.5 in [5]. We however include it for the sake of completeness.

First, the inequality $\delta(T, O) \geq \sup_n \delta(T_n, O_n)$ is obvious. Because we have the canonical isometric embedding $I_n : T_n \to T$ for each $n$, and for each $\mu \in \mathcal{P}'(T_n)$ with $\text{bar}(\mu) = O_n$, it is easy to see that $\text{bar}(I_n \ast \mu) = O$ and $\delta(\mu) = \delta(I_n \ast \mu)$.

Let 

$$
\mu = \sum_{i=1}^{m} t_i \text{Dirac}_{v_i} \in \mathcal{P}'(T)
$$

be an arbitrary measure in $\mathcal{P}'(T)$ with $\text{bar}(\mu) = O$, where $v_1, \ldots, v_m \in T$ and $t_1, \ldots, t_m > 0$ with $\sum_{i=1}^{m} t_i = 1$. Write $v_i = (v_i^{(1)}, v_i^{(2)}, \ldots)$ and let

$$
\mu_n = \sum_{i=1}^{m} t_i \text{Dirac}_{v_i^{(n)}} \in \mathcal{P}'(T_n), \quad n = 1, 2, \ldots.
$$

Then $\text{bar}(\mu_n) = O_n$ for each $n$. Because if we have $\text{bar}(\mu_n) \neq O_n$ for some $n$, it is easy to show that

$$
\int_T d(w, B)^2 \mu(dw) < \int_T d(w, O)^2 \mu(dw),
$$

where $B \in T$ is a point in $T$ such that all of its components are the cone points but $\text{bar}(\mu_n)$ for the $n$-th component, and it contradicts the assumption that $\text{bar}(\mu) = O$.

Let $\varepsilon > 0$ be an arbitrary positive number. By the definition of $\delta(T_n, O_n)$, there exists a map $\phi_n : \text{supp}(\mu_n) \to \mathcal{H}_n$ from the support of $\mu_n$ to some Hilbert space $\mathcal{H}_n$ with the properties (4.1) and (4.2) with respect to $\mu_n$, satisfying

$$
\frac{\| \int_{T_n} \phi_n(v) \mu_n(dv) \|^2}{\int_{T_n} \| \phi_n(v) \|^2 \mu_n(dv)} \leq \delta(T_n, O_n) + \varepsilon.
$$

We define a map $\phi : \text{supp}(\mu) \to \mathcal{H}$ from the support of $\mu$ to the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$ to be

$$
\phi(v_i) = (\phi_1(v_i^{(1)}), \phi_2(v_i^{(2)}), \ldots), \quad i = 1, \ldots, m.
$$

Then it is straightforward to see that $\phi$ satisfies the properties (4.1) and (4.2) with respect to $\mu$. And we have

$$
\delta(\mu) \leq \frac{\| \int_T \phi(v) \mu(dv) \|^2}{\int_T \| \phi(v) \|^2 \mu(dv)} = \frac{\sum_{n=1}^{\infty} \| \sum_{i=1}^{m} t_i \phi_n(v_i^{(n)}) \|^2}{\sum_{n=1}^{\infty} \sum_{i=1}^{m} t_i \| \phi_n(v_i^{(n)}) \|^2} \leq \sup_n \frac{\| \sum_{i=1}^{m} t_i \phi_n(v_i^{(n)}) \|^2}{\sum_{n=1}^{\infty} \sum_{i=1}^{m} t_i \| \phi_n(v_i^{(n)}) \|^2} \leq \sup_n (\delta(T_n, O_n) + \varepsilon).
$$

Since this holds for an arbitrary $\varepsilon > 0$ and an arbitrary $\mu \in \mathcal{P}'(T)$ with $\text{bar}(\mu) = O$, we have $\delta(T, O) \leq \sup_n \delta(T_n, O_n)$. \qed

For a CAT(1) space $X$, we prove the following relation between $\delta(\text{Cone}(X), O_{\text{Cone}(X)})$ and $\delta(X)$.
Proposition 4.4. Let \((X, d_X)\) be a complete \(\text{CAT}(1)\) space. Then we have
\[
\delta(\text{Cone}(X)) = 0 \quad \text{if and only if} \quad \delta(O_{\text{Cone}(X)}) \leq \tilde{\delta}(X).
\]

Before proving Proposition 4.4, we establish the following two lemmas.

Lemma 4.5. Let \((X, d_X)\) be a complete \(\text{CAT}(1)\) space. Let
\[
\nu = \sum_{i=1}^{m} t_i \text{Dirac}_{v_i} \in \mathcal{P}'(\text{Cone}(X)),
\]
where \(v_i \in \text{Cone}(X)\) for \(i = 1, \ldots, m\) and \(t_1, \ldots, t_m > 0\) with \(\sum_{i=1}^{m} t_i = 1\). Suppose that \(\text{bar}(\nu) = O_{\text{Cone}(X)}\). If \(v_1 = O_{\text{Cone}(X)}\) and if
\[
\nu' = \sum_{i=2}^{m} \frac{t_i}{1-t_1} \text{Dirac}_{v_i},
\]
then \(\text{bar}(\nu') = O_{\text{Cone}(X)}\) and \(\delta(\nu) \leq \tilde{\delta}(\nu')\).

Proof. The former assertion follows immediately from Lemma 3.2. Let \(\phi' : \text{supp}(\nu') \rightarrow \mathcal{H}\) be a map from the support of \(\nu'\) to some Hilbert space \(\mathcal{H}\) satisfying (4.1) and (4.2) with respect to \(\nu'\). Define \(\phi : \text{supp}(\nu) \rightarrow \mathcal{H}\) by
\[
\phi(v_1) = 0, \\
\phi(v_i) = \phi'(v_i), \quad i = 2, \ldots, m.
\]
Then \(\phi\) satisfies (4.1) and (4.2) with respect to \(\nu\). Moreover, an easy computation shows that
\[
\frac{\| \int_{\text{Cone}(X)} \phi(v)\nu(dv) \|^2}{\int_{\text{Cone}(X)} \|\phi(v)\|^2 \nu(dv)} \leq \frac{\| \int_{\text{Cone}(X)} \phi'(v)\nu'(dv) \|^2}{\int_{\text{Cone}(X)} \|\phi'(v)\|^2 \nu'(dv)}.
\]
Hence, by the definition of \(\delta\), the latter assertion follows. \(\Box\)

Lemma 4.6. Let \((X, d_X)\) be a complete \(\text{CAT}(1)\) space and let
\[
\nu = \sum_{i=1}^{m} t_i \text{Dirac}_{x_i, r_i} \in \mathcal{P}'(\text{Cone}(X)),
\]
where \([x_i, r_i]\) is the point on \(\text{Cone}(X)\) represented by \((x_i, r_i) \in X \times [0, \infty)\). Suppose that \(\alpha > 0\), \(l \in \{1, 2, \ldots, m-1\}\), and
\[
\nu' = \frac{1}{\sum_{i=1}^{l} \frac{t_i}{\alpha} + \sum_{i=l+1}^{m} t_i} \left( \sum_{i=1}^{l} \frac{t_i}{\alpha} \text{Dirac}_{x_i, \alpha r_i} + \sum_{i=l+1}^{m} t_i \text{Dirac}_{x_i, r_i} \right).
\]
Then \(\text{bar}(\nu') = O_{\text{Cone}(X)}\) if and only if \(\text{bar}(\nu) = O_{\text{Cone}(X)}\). Moreover, if \(\text{bar}(\nu) = \text{bar}(\nu') = O_{\text{Cone}(X)}\) and if \(\alpha > 1\) (resp. \(0 < \alpha < 1\)), then the inequality \(\delta(\nu) \leq \tilde{\delta}(\nu')\) holds if and only if
\[
\alpha \sum_{i=1}^{l} t_i r_i^2 \sum_{i=l+1}^{m} t_i r_i^2 \leq \sum_{i=1}^{l} t_i \left( \sum_{i=l+1}^{m} t_i r_i^2 \right) \quad (\text{resp.} \quad \alpha \sum_{i=1}^{l} t_i r_i^2 \sum_{i=l+1}^{m} t_i r_i^2 \geq \sum_{i=1}^{l} t_i \left( \sum_{i=l+1}^{m} t_i r_i^2 \right)).
\]
Proof. The equivalence between $\overline{\text{bar}}(\nu) = O_{\text{Cone}(X)}$ and $\overline{\text{bar}}(\nu') = O_{\text{Cone}(X)}$ is an immediate consequence of Lemma 3.2. Assume that $\overline{\text{bar}}(\nu) = \overline{\text{bar}}(\nu') = O_{\text{Cone}(X)}$, and fix some real Hilbert space $\mathcal{H}$ of dimension $\geq m$. Then there is a natural bijection $\phi \mapsto \phi'$ between the set of all maps from $\text{supp}(\nu)$ to $\mathcal{H}$ satisfying (4.1) and (4.2) with respect to $\nu$, and the set of all maps from $\text{supp}(\nu')$ to $\mathcal{H}$ satisfying (4.1) and (4.2) with respect to $\nu'$: it is given by

$$\phi'[x_i, \alpha r_i] = \alpha \phi[x_i, r_i], \quad i = 1, \ldots, l,$$

$$\phi'[x_i, r_i] = \phi[x_i, r_i], \quad i = l + 1, \ldots, m.$$  

Let $\phi : \text{supp}(\nu) \to \mathcal{H}$ and $\phi' : \text{supp}(\nu') \to \mathcal{H}$ be the maps satisfying (4.1) and (4.2) with respect to $\nu$ and $\nu'$ respectively, and corresponding to each other under this bijection. Let

$$T = \frac{1}{\alpha} \sum_{i=1}^{l} t_i + \sum_{i=l+1}^{m} t_i.$$  

Then we have

$$\frac{\| \int_{\text{Cone}(X)} \phi'(p) \nu'(dp) \|^2}{\int_{\text{Cone}(X)} \| \phi'(p) \|^2 \nu'(dp)} = T \frac{\| \sum_{i=1}^{l} t_i \phi[x_i, r_i] \|^2}{\alpha \sum_{i=1}^{l} t_i \| \phi[x_i, r_i] \|^2 + \sum_{i=l+1}^{m} t_i \| \phi[x_i, r_i] \|^2}.$$  

Hence,

$$\frac{\| \int_{\text{Cone}(X)} \phi'(p) \nu'(dp) \|^2}{\int_{\text{Cone}(X)} \| \phi'(p) \|^2 \nu'(dp)} = \frac{\| \int_{\text{Cone}(X)} \phi(p) \nu(dp) \|^2}{\int_{\text{Cone}(X)} \| \phi(p) \|^2 \nu(dp)} \leq T \frac{\| \sum_{i=1}^{l} t_i \phi[x_i, r_i] \|^2}{\alpha \sum_{i=1}^{l} t_i \| \phi[x_i, r_i] \|^2 + \sum_{i=l+1}^{m} t_i \| \phi[x_i, r_i] \|^2}.$$  

We also have

$$T \sum_{i=1}^{m} t_i r_i^2 - \alpha \sum_{i=1}^{l} t_i \| \phi[x_i, r_i] \|^2 - \sum_{i=l+1}^{m} t_i \| \phi[x_i, r_i] \|^2$$

$$= \frac{1 - \alpha}{1 - \alpha} \left( \sum_{i=1}^{l} t_i \right) + \alpha \left\{ \alpha \left( \sum_{i=1}^{m} t_i \right) \left( \sum_{i=1}^{l} t_i r_i^2 \right) - \left( \sum_{i=1}^{l} t_i \right) \left( \sum_{i=l+1}^{m} t_i r_i^2 \right) \right\}.$$  

By (4.4) and (4.5), the inequality

$$\frac{\| \int_{\text{Cone}(X)} \phi'(p) \nu'(dp) \|^2}{\int_{\text{Cone}(X)} \| \phi'(p) \|^2 \nu'(dp)} \geq \frac{\| \int_{\text{Cone}(X)} \phi(p) \nu(dp) \|^2}{\int_{\text{Cone}(X)} \| \phi(p) \|^2 \nu(dp)}$$  

holds if and only if

$$\alpha \geq 1, \quad \alpha \left( \sum_{i=1}^{l} t_i \right) \left( \sum_{i=1}^{l} t_i r_i^2 \right) - \left( \sum_{i=1}^{l} t_i \right) \left( \sum_{i=l+1}^{m} t_i r_i^2 \right) \leq 0.$$
or
\[
0 < \alpha \leq 1, \quad \alpha \left( \sum_{i=l+1}^{m} t_i \right) \left( \sum_{i=1}^{l} t_i r_i^2 \right) - \left( \sum_{i=1}^{l} t_i \right) \left( \sum_{i=l+1}^{m} t_i r_i^2 \right) \geq 0.
\]

The lemma follows easily from this equivalence and the bijectivity of the correspondence \( \phi \leftrightarrow \phi' \).

**Proof of Proposition 4.4.** First suppose that \( \mu \in \mathcal{P}(\text{Cone}(X)) \), \( \text{bar}(\mu) = O_{\text{Cone}(X)} \), and \( \text{supp}(\mu) \subset \iota(X) \). Let \( \iota : X \to \text{Cone}(X) \) be the canonical inclusion, and let \( \iota^{-1} : \iota(X) \to X \) be the inverse map. Let \( \tilde{\phi} : X \to \mathcal{H} \) be a map from \( X \) to some Hilbert space \( \mathcal{H} \) satisfying (3.2). Then the restriction \( \phi = [\tilde{\phi} \circ \iota^{-1}]|_{\text{supp}(\mu)} \) of \( \tilde{\phi} \circ \iota^{-1} : \iota(X) \to \mathcal{H} \) to \( \text{supp}(\mu) \) satisfies (4.1) and (4.2). Moreover we have
\[
\| E_{\iota^{-1} \mu}[\tilde{\phi}] \|^2 = \frac{\| \int_{\text{Cone}(X)} \phi(v) \mu(dv) \|^2}{\int_{\text{Cone}(X)} \| \phi(v) \|^2 \mu(dv)}.
\]

Hence by the definitions of \( \delta(\iota^{-1} \mu) \) and \( \delta(\mu) \), we have
\[
\delta(\mu) \leq \delta(\iota^{-1} \mu).
\]

Thus, if we prove the existence of \( \nu' \in \mathcal{P}(\text{Cone}(X)) \) such that
\[
\delta(\nu) \leq \delta(\nu'), \quad \text{supp}(\nu') \subset \iota(X)
\]
for any
\[
\nu = \sum_{i=1}^{m} t_i \text{Dirac}_{[x_i, r_i]} \in \mathcal{P}'(\text{Cone}(X))
\]
with \( \text{bar}(\nu) = O_{\text{Cone}(X)} \), then the desired assertion follows. Here, we can assume \( r_i > 0 \) for all \( i \in \{1, \ldots, m\} \) by Lemma 4.5. And, if \( r_1 = r_2 = \cdots = r_m \), we can take
\[
\nu' = \sum_{i=1}^{m} t_i \text{Dirac}_{[x_i, 1]},
\]
and \( \nu' \) satisfies (4.6) because it is straightforward that \( \delta(\nu) = \delta(\nu') \). So we can assume \( r_1 = \cdots = r_{l} < r_{l+1} \leq \cdots \leq r_m \) without loss of generality. Then we have
\[
\left( \frac{\sum_{i=1}^{l} t_i}{\sum_{i=l+1}^{m} t_i} \right) \left( \frac{\sum_{i=1}^{l} t_i r_i^2}{\sum_{i=l+1}^{m} t_i r_i^2} \right) \geq \frac{r_{l+1}^2}{r_1^2} \geq \frac{r_{l+1}}{r_1}.
\]
Hence, if we set
\[
\nu_0 = \frac{1}{r_{l+1} \sum_{i=1}^{l} t_i + \sum_{i=l+1}^{m} t_i} \left( \sum_{i=1}^{l} r_i t_i \text{Dirac}_{[x_i, r_{l+1}]} + \sum_{i=l+1}^{m} t_i \text{Dirac}_{[x_i, r_i]} \right),
\]
then we have
\[
\delta(\nu_0) \geq \delta(\nu)
\]
by Lemma 4.6. Repeating this procedure, we finally get
\[
\nu_1 = \sum_{i=1}^{m} s_i \text{Dirac}_{[x_i, r_m]},
\]
which satisfies $\delta(\nu_1) \geq \delta(\nu)$. If we set $\nu' = \sum_{i=1}^m s_i \text{Dirac}_{[x_i,1]}$, it is easily seen that $\delta(\nu') = \delta(\nu_1)$, and the assertion follows. \qed

5. Proof of the theorem

Recall that the Gromov-Hausdorff precompactness is known to be equivalent to the uniformly total boundedness. We call the family $\mathcal{X}$ of metric spaces uniformly totally bounded if the following two conditions are satisfied:

- There is a constant $D$ such that $\text{diam}(X) \leq D$ for all $X \in \mathcal{X}$.
- For any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that each $X \in \mathcal{X}$ contains a subset $S_{X,\varepsilon}$ with the following property: the cardinality of $S_{X,\varepsilon}$ is no greater than $N(\varepsilon)$ and $X$ is covered by the union of all $\varepsilon$-balls whose centers are in $S_{X,\varepsilon}$.

By Lemma 4.2, Lemma 4.3 and Proposition 4.4 to prove Theorem 1.1 it suffices to prove the following proposition.

**Proposition 5.1.** Let $(X, d_X)$ be a complete CAT(1) space. Assume that there exist $N \in \mathbb{N}$ and a subset $S = \{x_i\}_{i=1}^N \subset X$ such that $X$ is covered by the union of all $\frac{\sqrt{2}}{12}$-balls whose centers are in $S$. Then there exists a constant $C(N) < 1$, depending only on $N$, such that

$$\tilde{\delta}(X) < C(N).$$

**Remark 5.2.** It follows from the argument in the proof of Proposition 5.1, we can take

$$C(N) = \left(\frac{2}{3} + \frac{1}{3} \sqrt{\frac{e^{-\frac{N^2}{2N} + 1}}{2}}\right)^2,$$

as a constant $C(N)$ in the proposition.

Before proving Proposition 5.1, we will recall a well-known construction of a map from a Hilbert space to the unit sphere in another Hilbert space, and derive some necessary estimates for them. We follow Dadarlat and Guentner [3] to explain this construction. Let $\mathcal{H}$ be a Hilbert space. Let $\text{Exp}(\mathcal{H}) = \mathbb{R} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots,$ and define $\text{Exp} : \mathcal{H} \to \text{Exp}(\mathcal{H})$ by

$$\text{Exp}(\zeta) = 1 \oplus \zeta \oplus \left(\frac{1}{\sqrt{2!}} \zeta \otimes \zeta\right) \oplus \left(\frac{1}{\sqrt{3!}} \zeta \otimes \zeta \otimes \zeta\right) \oplus \cdots.$$

For $t > 0$, define a map $G_t$ from $\mathcal{H}$ to $\text{Exp}(\mathcal{H})$ to be

$$G_t(\zeta) = e^{-t\|\zeta\|^2} \text{Exp}(\sqrt{2t}\zeta).$$

Then simple computation shows that

$$\cos \angle(G_t(\zeta), G_t(\zeta')) = \langle G_t(\zeta), G_t(\zeta') \rangle = e^{-t\|\zeta - \zeta'\|^2}$$

for all $\zeta, \zeta' \in \mathcal{H}$. In particular, $\|G_t(\zeta)\| = 1$ for all $\zeta \in \mathcal{H}$. Hence we can regard $G_t$ as a map from $\mathcal{H}$ to the unit sphere in $\text{Exp}(\mathcal{H})$.

We need the following estimate to prove Proposition 5.1.
Lemma 5.3. Let \((X, d_X)\) be a metric space, and let \(F : X \to H\) be an \(L\)-Lipschitz map \((L > 0)\) to some Hilbert space. Suppose that \(0 < tL^2 \leq \frac{1}{2}\). Then the map \(\phi = G_t \circ F : X \to \text{Exp}(H)\) satisfies
\[
\angle(\phi(x), \phi(y)) \leq \min\{\pi, d_X(x, y)\}
\]
for all \(x, y \in X\).

Proof. By (5.1) and \(L\)-Lipschitz continuity of \(F\), it is sufficient to show that
\[
e^{-tL^2d_X(x,y)^2} \geq \cos(\min\{\pi, d_X(x, y)\})
\]
for all \(x, y \in X\) and all \(t \in (0, \frac{1}{2L^2})\). When \(d_X(x, y) \geq \frac{\pi}{2}\), (5.2) is obvious. So, if we put \(a = tL^2\) and \(d = d_X(x, y)\), then what we have to show is that
\[
a \leq -\frac{\log(\cos d)}{d^2}
\]
holds for any \(a \in (0, \frac{1}{2}]\) and any \(d \in [0, \frac{\pi}{2}]\). But this is obvious because the right-hand side of (5.3) is non-decreasing with respect to \(d\). \(\square\)

Now we are ready to prove Proposition 5.1.

Proof of Proposition 5.1. First we define a map \(F_S\) from \(X\) to \(\mathbb{R}^N\) by
\[
F_S(x) = (d_X(x, x_1), d_X(x, x_2), \ldots, d_X(x, x_N))
\]
for \(x \in X\). Then \(F_S\) is \(\sqrt{N}\)-Lipschitz since
\[
\|F_S(x) - F_S(y)\| = \left\{ \sum_{i=1}^{N} (d_X(x, x_i) - d_X(y, x_i))^2 \right\}^{\frac{1}{2}} \leq \sqrt{N} \cdot d_X(x, y).
\]

On the other hand, by the definition of the subset \(S\), for any \(x, y \in X\) with \(d_X(x, y) \geq \frac{\pi}{3}\), there exist \(i_0, i_1 \in \{1, \ldots, N\}\) such that
\[
d_X(x_{i_0}, x) \geq \frac{\pi}{4}, \quad d_X(x_{i_0}, y) \leq \frac{\pi}{12},
\]
\[
d_X(x_{i_1}, y) \geq \frac{\pi}{4}, \quad d_X(x_{i_1}, x) \leq \frac{\pi}{12}
\]
Hence
\[
\|F_S(x) - F_S(y)\| \geq \sqrt{(d_X(x_{i_0}, x) - d(x_{i_0}, y))^2 + (d_X(x_{i_1}, x) - d(x_{i_1}, y))^2} \geq \frac{\pi}{3\sqrt{2}}
\]
for any \(x, y \in X\) with \(d_X(x, y) \geq \frac{\pi}{3}\).

We now set \(\phi = G_{\frac{1}{3\sqrt{N}}} \circ F_S : X \to \text{Exp}(\mathbb{R}^N)\). Then the all values of \(\phi\) are contained in the unit sphere of \(\text{Exp}(\mathbb{R}^N)\), and \(\phi\) satisfies
\[
\angle(\phi(x), \phi(y)) \leq \min\{\pi, d_X(x, y)\}
\]
for all \(x, y \in X\) by Lemma 5.3. Moreover (5.1) and (5.4) imply that
\[
\angle(\phi(x), \phi(y)) \geq \arccos(e^{-\frac{\pi^2}{36N}})
\]
for any \( x, y \in X \) with \( d_X(x, y) \geq \frac{\pi}{3} \).

Set \( \eta = \arccos(e^{-\frac{\pi^2}{36N}}} \), and let \( \mu \) be an arbitrary measure in \( \mathcal{P}(X) \) with \( \text{bar}(\iota_*\mu) = O_{\text{Cone}(X)} \), where \( \iota : X \to \text{Cone}(X) \) is the canonical inclusion and \( O_{\text{Cone}(X)} \) is the cone point of \( \text{Cone}(X) \). Then we have

\[
\phi_*\mu \left( B \left( v, \frac{\eta}{2} \right) \right) \leq \frac{2}{3}
\]

for any point \( v \) on the unit sphere in \( \text{Exp}(\mathbb{R}^N) \), where

\[
B \left( v, \frac{\eta}{2} \right) = \left\{ u \in \text{Exp}(\mathbb{R}^N) \mid \| u \| = 1, \angle(v, u) < \frac{\eta}{2} \right\}.
\]

This is because if there exists some vector \( \phi(x_0) \) contained in \( B \left( v, \frac{\eta}{2} \right) \cap \phi(X) \), then by (5.5) and Corollary 3.3 we have

\[
\phi_*\mu \left( B \left( v, \frac{\eta}{2} \right) \right) \leq \phi_*\mu \left( B \left( \phi(x_0), \eta \right) \right) = \mu \left( \phi^{-1} \left( B \left( \phi(x_0), \eta \right) \right) \right) \leq \mu \left( B \left( x_0, \frac{\pi}{3} \right) \right) \leq \frac{2}{3},
\]

where \( B \left( x_0, \frac{\pi}{3} \right) \) is the open ball in \( X \) centered at \( x_0 \) with radius \( \frac{\pi}{3} \). In the case \( B \left( v, \frac{\eta}{2} \right) \cap \phi(X) = \phi \), (5.6) obviously holds.

By (5.6), we have

\[
\int_X \langle v, \phi(x) \rangle \mu(dx) = \int_S \langle v, u \rangle \phi_*\mu(du)
\]

\[
= \int_{B(v, \frac{\eta}{2})} \langle v, u \rangle \phi_*\mu(du) + \int_{S \setminus B(v, \frac{\eta}{2})} \langle v, u \rangle \phi_*\mu(du)
\]

\[
\leq 1 \times \phi_*\mu \left( B \left( v, \frac{\eta}{2} \right) \right) + \cos \frac{\eta}{2} \times \left\{ 1 - \phi_*\mu \left( B \left( v, \frac{\eta}{2} \right) \right) \right\}
\]

\[
\leq 1 \times \frac{2}{3} + \left( \cos \frac{\eta}{2} \right) \times \frac{1}{3},
\]

where \( S \) is the unit sphere in \( \text{Exp}(\mathbb{R}^N) \). Setting \( \nu = \mathbb{E}_\mu[\phi] \) in the above inequality and using (3.1), we have

\[
\| \mathbb{E}_\mu[\phi] \| = \left\| \int_X \langle \mathbb{E}_\mu[\phi], \phi(x) \rangle \mu(dx) \right\| \leq c_N,
\]

where

\[
c_N = 1 \times \frac{2}{3} + \left( \cos \frac{\eta}{2} \right) \times \frac{1}{3} = \frac{2}{3} + \frac{1}{3} \sqrt{e^{-\frac{\pi^2}{36N}}} + 1
\]

Thus, by the definition of \( \tilde{\delta}(X) \),

\[
\tilde{\delta}(X) \leq c_N^2 < 1
\]

which proves the proposition.
Finally, we remark that the proof of Proposition 5.1 works for the following more general statement.

**Proposition 5.4.** Let $0 < \theta < \frac{\pi}{2}$, $0 < \alpha < 1$ and $\varepsilon > 0$. Let $(X, d_X)$ be a complete CAT(1) space. Assume that there exists a finite subset $S \subset X$ such that

$$\# \{ s \in S \mid \|d_X(x, s) - d_X(y, s)\| \geq \varepsilon \} \geq \alpha \# S$$

whenever $x, y \in X$ and $d(x, y) \geq \theta$. Here, $\# S$ stands for the cardinality of $S$. Then there exists a constant $C = C(\theta, \alpha, \varepsilon) < 1$ such that

$$\bar{\delta}(X) \leq C.$$

**Proof.** We denote the cardinality of $S$ by $N$. Let $F_S$ be the map from $X$ to $\mathbb{R}^N$ as in the proof of Proposition 5.1 with respect to our set $S$. Then $F_S$ is $\sqrt{N}$-Lipschitz and we have

$$\|F_S(x) - F_S(y)\| \geq \sqrt{\alpha N \varepsilon}$$

for any $x, y \in X$ with $d_X(x, y) \geq \theta$. If we set $\phi = G_{\frac{1}{\sqrt{N}}} \circ F_S : X \to \text{Exp}(\mathbb{R}^N)$, then all the values of $\phi$ are contained in the unit sphere of $\text{Exp}(\mathbb{R}^N)$, and $\phi$ satisfies

$$\angle(\phi(x), \phi(y)) \leq \min\{\pi, d_X(x, y)\}$$

for all $x, y \in X$ by Lemma 5.3. Moreover (5.1) and (5.7) imply that

$$\angle(\phi(x), \phi(y)) \geq \arccos(e^{-\frac{\alpha \varepsilon^2}{2}})$$

for any $x, y \in X$ with $d_X(x, y) \geq \theta$.

Now the rest of the proof is done exactly in the same manner as in the proof of Proposition 5.1, and we have

$$\bar{\delta}(X) \leq (c_{\theta, \alpha, \varepsilon})^2,$$

where

$$c_{\theta, \alpha, \varepsilon} = 1 \times \frac{1}{1 + \cos \theta} + \left(\cos \frac{\arccos(e^{-\frac{\alpha \varepsilon^2}{2}})}{2}\right) \times \left(1 - \frac{1}{1 + \cos \theta}\right)$$

$$= \frac{1}{1 + \cos \theta} + \sqrt{\frac{e^{-\frac{\alpha \varepsilon^2}{2}} + 1}{2}} \times \frac{\cos \theta}{1 + \cos \theta} < 1.$$

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