PT symmetric lattices with a local degree of freedom

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Recently, open systems with balanced, spatially separated loss and gain have been realized and studied using non-Hermitian Hamiltonians that are invariant under the combined parity and time-reversal (PT) operations. Here, we model and investigate the effects of a local, two-state, quantum degree of freedom, called a pseudospin, on a one-dimensional tight-binding lattice with position-dependent tunneling amplitudes and a single pair of non-Hermitian, PT-symmetric impurities. We show that if the resulting Hamiltonian is invariant under exchange of two pseudospin labels, the system can be decomposed into two uncoupled systems with tunable threshold for PT symmetry breaking. We discuss implications of our results to systems with specific tunneling profiles, and open or periodic boundary conditions.

Introduction: A non-Hermitian Hamiltonian $H \neq H^\dagger$ that is invariant under the combined parity and time-reversal (PT) operations is called PT symmetric. Since the groundbreaking discovery of such Hamiltonians in continuum models fifteen years ago [1], significant research has been carried out to identify and characterize the properties of PT-symmetric Hamiltonians that, typically, are decomposed into a Hermitian kinetic term and a non-Hermitian, PT-symmetric potential term, $V(x) = V^\ast(-x) \neq V^\dagger(x)$ [2, 3]. The spectrum $\epsilon_\lambda$ of such a non-Hermitian Hamiltonian is purely real over a region of the parameter space, called the PT-symmetric phase; in this region, its (non-orthonormal) eigenvectors $|\phi_\lambda\rangle$ are simultaneous eigenfunctions of the PT operation. For parameters outside the PT-symmetric phase, the eigenvalues of the Hamiltonian occur in complex conjugate pairs, and due to the anti-linear nature of the time-reversal operator $T$, the corresponding eigenfunctions are not simultaneous eigenfunctions of the PT operation. This emergence of complex eigenvalues is called PT-symmetry breaking. PT-symmetric Hamiltonians are ideally suited to model non-equilibrium phenomena that transition from a quasi steady-state behavior (PT-symmetric phase) to loss of reciprocity (broken PT-symmetry) [4, 5].

In the past three years, experiments on coupled optical waveguides [6, 9], coupled electrical circuits [10], and coupled pendulums [11] have shown that instead of being a mathematical curiosity, PT-symmetric Hamiltonians represent open (quantum) systems with spatially separated, balanced, loss and gain. The discrete nature of these systems has also sparked new interest in the properties of PT-symmetric tight-binding lattice models with different topologies [12, 13]; such lattice models are most readily realized in evanescently coupled optical waveguides [14, 15]. Recent theoretical work has led to the identification of robust and fragile PT-symmetric phases in a lattice with open boundary conditions [16, 19], tunable PT-symmetric threshold in a lattice with periodic boundary conditions [20], and substantially strengthened PT-symmetric phase in finite lattices with position-dependent tunneling profile [21, 22]. All of this work is, however, restricted to systems in one spatial dimension where the parity operation is defined as $P : x \rightarrow -x$ in the continuum case (with a suitably defined origin) and $P : k \rightarrow k = N+1-k$ in a lattice with $N$ sites. In particular, the properties of PT-symmetric Hamiltonians in two (or higher) dimensions have been barely explored [23].

In this paper, we investigate PT-symmetric lattices with a local, two-state, quantum degree of freedom labeled by a pseudospin $\sigma = \pm 1$. We present a class of models that can be mapped onto one-dimensional lattice models that have been investigated in the past, and thus are solvable in a straightforward manner. Such degree of freedom can represent, for example, two orthogonal polarizations of a mode in a single elliptical waveguide [24] in an array of coupled elliptical waveguides. Thus, although we use the term “pseudospin” to denote this degree of freedom, we emphasize that its properties are modeled after the so-called pseudospin labels $\sigma \leftrightarrow -\sigma$. For a pair of balanced loss or gain impurities at mirror-symmetric locations, the potential is...
given by

\[ V = a_m^\dagger i \Gamma a_m - a_m^\dagger i \Gamma a_m \]

where \( \tilde{m} = N + 1 - m \), \( i [\Gamma] = i \gamma_s + i \gamma_d \tau_x \) denotes the non-Hermitian gain matrix at site \( m \), and \( 0 \leq \gamma_d \leq \gamma_s \) denote the gain amplitudes for mode preserving and mode exchanging processes. The potential \( V \) is also invariant under the exchange of pseudospin labels, and is \( \mathcal{PT} \) symmetric irrespective of the time-reversal properties of the pseudospin.

The eigenvalue difference equation obeyed by a two-component eigenfunction \( \Psi(k) = (f_k, g_k)^T \) with energy \( \epsilon \) is given by

\[ -T(k-1)\Psi(k-1) - T(k)\Psi(k+1) + (\delta_{k,m} - \delta_{k,\tilde{m}})i\epsilon \Psi(k) = \epsilon \Psi(k) \]

where \( k = 1, \ldots, N \). We note that open boundary conditions are implemented by assigning \( T(0) = 0 = T(N) \) whereas periodic boundary conditions imply \( T(0) = T(N) \neq 0 \). Using the symmetric and antisymmetric basis that diagonalizes the tunneling matrix \( T(k) \) at every site, it is straightforward to obtain the following coupled equations,

\[ -(t_{k+1}^S f_k^S + t_{k-1}^S f_{k-1}^S) + i\gamma_s f_k^S (\delta_{k,m} - \delta_{k,\tilde{m}}) = \epsilon f_k^S \]
\[ -(t_k^A f_{k+1}^A + t_{k-1}^A f_{k-1}^A) + i\gamma_s f_k^A (\delta_{k,m} - \delta_{k,\tilde{m}}) = \epsilon f_k^A \]

Here \( t_{k}^{(S,A)} = [t_s(k) \pm t_d(k)] \) are the symmetric and antisymmetric combinations of the tunneling rates, \( \gamma^{(S,A)} = (\gamma_s \pm \gamma_d) \), and \( f_k^{(S,A)} = (f_k \pm g_k) \) are the eigenfunction components in the symmetric-antisymmetric basis.

Eqs. (4,5) show that the \( \mathcal{PT} \)-symmetric Hamiltonian is a direct sum of Hamiltonians for two lattices with no internal structure: \( H = H_0 + V = H_S \oplus H_A \) where \( H_S \) is the \( \mathcal{PT} \)-symmetric Hamiltonian with tunneling profile \( t_k^S \) and a pair of non-Hermitian impurities at mirror-symmetric locations \( (m, \tilde{m}) \) with strength \( \gamma^S \), and \( H_A \) is obtained correspondingly. We emphasize that this decomposition into uncoupled problems is valid for arbitrary, position-dependent tunneling profiles \( t_s(k) \), mode-mixing amplitudes \( t_d(k) \), open or periodic boundary conditions, and arbitrary loss or gain strengths, as long as the underlying Hamiltonian is invariant under the exchange pseudospin indices.

**Specific Cases and Numerical Results:** When there is no mixing between the two pseudospin states, \( t_d = 0 = \gamma_d \), the problem is trivial. In general, the \( \mathcal{PT} \)-symmetric threshold for \( \lambda_0 \) is equal to the smaller of the corresponding thresholds for \( \lambda_S \) and \( \lambda_A \).

When \( \gamma_d = 0 \), the loss (or gain) potential couples maximally to the pseudospin eigenmodes \( \sigma = \pm 1 \), and not to a linear combination of them. In this case, if the tunneling is constant, the \( \mathcal{PT} \)-symmetric phase diagram \( \gamma_{\mathcal{PT}}(m) \) is given by a U-shaped curve, obtained in Ref. [18], with the maximum value \( \gamma_{\mathcal{PT}} = (t_s - t_d) \). For parity-symmetric, non-constant tunneling profiles, the appreciably strong \( \mathcal{PT} \)-symmetric threshold, obtained in Ref. [22], is now selectively suppressed by increasing the mode-mixing tunneling amplitude \( t_d(k) \). For a lattice with periodic boundary conditions, we consider the model with tunneling matrices \( T_0 = t_{0A}^{-1} + t_{0d}\tau_x \) and \( T_b = t_{0b}^{-1} + t_{bd}\tau_x \) that are constant along each of the two paths that connect the gain site to the loss site (Fig. 2). It then follows that the \( \mathcal{PT} \)-symmetric threshold is independent of the distance between the loss and gain sites, as discussed in Ref. [20], and is given by the smaller of the two combinations, \( (t_0^S - t_b^S) \) and \( (t_0^A - t_b^A) \). Thus, a \( \mathcal{PT} \)-symmetric ring with a local degree of freedom offers significant threshold tunability independent of the distance between the loss and gain impurities.
When $\gamma_d \neq 0$, the analysis carried out here predicts bounds on the gain matrix, given by $(\gamma_s + \gamma_d) \leq (t_s + t_d)$ and $(\gamma_s - \gamma_d) \leq (t_s - t_d)$; however, these bounds do not determine the individual thresholds for $\gamma_s$ and $\gamma_d$. In the extreme case of $\gamma_s = \gamma_d$ (meaning the gain potential only couples to the symmetric combination), we find that $\gamma^A = 0$, $H_A$ is a purely Hermitian Hamiltonian and therefore, the $PT$-symmetric threshold is solely determined by the Hamiltonian $H_S$. Note that, in general, a direct-sum decomposition of the Hamiltonian $H$ is possible if and only if the tunneling matrix $T(k)$ at every site $k$ and the non-Hermitian potential matrix $i\Gamma$ can be simultaneously diagonalized.

Finally we consider the case where the full Hamiltonian $H$ cannot be decomposed into two non-interacting pieces. Generically, for an open lattice or a ring with constant tunneling matrix $T$ and a single pair of gain/loss matrix $i\Gamma$, wave function matching approach [17] [18] leads to a characteristic equation for eigenvalues of $H$ that results from the determinant of a $6 \times 6$ matrix. It is, thus, of little analytical value to calculate the $PT$-symmetric threshold $\gamma_{PT}(m)$ and instead, we obtain the $PT$-symmetric phase diagram numerically. We restrict ourselves to the simplest case of a constant-tunneling Hamiltonian $H_0$ and an impurity potential matrix $i\Gamma = \gamma_s \tau_z$ where $\tau_z$ is the $z$-Pauli matrix. In contrast to the previous cases, where the losses or gains for both modes occurred in the same waveguide, this non-Hermitian potential represents gain for one mode, $\sigma = +1$, and loss for the other, $\sigma = -1$, at site $m$.

Figure [3] shows the numerically obtained results for the threshold $\gamma_{PT}(\mu)/t_s$ as a function of the fractional location $\mu = m/N$ of the first impurity for three different values of mode-mixing tunneling $t_d/t_s = \{0, 0.4, 0.7\}$. The left-hand panel shows the results for an even lattice with $N = 40$. When $t_d = 0$ (solid blue circles), the two degrees of freedom are uncoupled and $PT$-symmetric phase diagram is identical to that for an open lattice with no internal degree of freedom [18]. As $t_d$ increases (solid red squares and black stars), generically, we find that the critical $\gamma_{PT}(\mu)$ is non-monotonically suppressed for different values of impurity locations $\mu$. The right-hand panel shows corresponding results for an odd lattice with $N = 41$. When $t_d = 0$ (solid blue circles), the threshold impurity strength is given by $\gamma_{PT}/t_s = \sqrt{1+1/N} \approx 1.012$ when $m = 1$ [17], and therefore, does not appear in the figure. Once again, when $t_d$ increases, the $PT$-symmetric phase is (mostly) suppressed in a non-monotonic way. These results suggest that $PT$-symmetry breaking in such systems shows a rich behavior that cannot be described with a simple analytical model.

Discussion: In this paper, we have introduced $PT$-symmetric lattices with a local, two-state, quantum degree of freedom. By imposing invariance requirements on the Hermitian tunneling term, and $PT$-symmetric potential term that represents spatially separated gain and loss impurities, we have shown that a broad class of such lattice systems can be expressed as the direct sum of two, uncoupled, $PT$-symmetric systems. In such cases, we have predicted that $PT$-symmetric threshold can be tuned by mode-mixing tunneling amplitude. Since
the mapping is exact, all signatures of $\mathcal{PT}$-symmetry breaking, such as the ubiquitous, maximal chirality at $\mathcal{PT}$-symmetry breaking threshold \[20\], the even-odd effect \[22\], tunable amplification \[25\], etc. will be applicable in these cases as well.

Since we have used the mode polarization as an example of the local degree of freedom, a microscopic calculation of the mode structure and the overlap between modes in adjacent waveguides is necessary to obtain typical tunneling matrix elements. Similarly a detailed study of the selection rules for different polarizations will be necessary to characterize the relative strengths of elements of the gain matrix $\gamma_s$ and $\gamma_d$.

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