HYPERgeometric functions DIfferential REduction (HYPERDIRE): MATHEMATICA based packages for differential reduction of generalized hypergeometric functions: $F_D$ and $F_S$ Horn-type hypergeometric functions of three variables

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**ABSTRACT**

HYPERDIRE is a project devoted to the creation of a set of Mathematica based programs for the differential reduction of hypergeometric functions. The current version includes two parts: the first one, $F_D$Function, for manipulations with Appell hypergeometric functions $F_D$ of $r$ variables; and the second one, $F_S$Function, for manipulations with Lauricella–Saran hypergeometric functions $F_S$ of three variables. Both functions are related with one-loop Feynman diagrams.

**Program summary**

Program title: HYPERDIRE
Catalogue identifier: AEPP_v3_0
Program summary URL: http://cpc.cs.qub.ac.uk/summaries/AEPP_v3_0.html
Program obtainable from: CPC Program Library, Queen’s University, Belfast, N. Ireland
Licensing provisions: GNU General Public License
No. of lines in distributed program, including test data, etc.: 310
No. of bytes in distributed program, including test data, etc.: 7666
Distribution format: tar.gz
Programming language: Mathematica
Computer: All computers running Mathematica
Operating system: All operating systems running Mathematica
Classification: 4.4.
Catalogue identifier of previous version: AEPP_v1_0
Journal reference of previous version: Comput. Phys. Comm. 184 (2013) 2332
Does the new version supersede the previous version?: No. It is an extension to the previous program, which performs reductions of hypergeometric functions $F_D$ and $F_S$ to the set of basis functions $F_0$, $F_1$, $F_2$, $F_3$ and $F_4$
Nature of problem: Reduction of hypergeometric functions $F_D$ and $F_S$
Solution method: Differential reduction

Reasons for new version: New algorithms for the reduction of multivariable Lauricella functions, Horn functions, (hypergeometric functions $F_D$ and $F_S$)

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This paper and its associated computer program are available via the Computer Physics Communication homepage on ScienceDirect (http://www.sciencedirect.com/science/journal/00104655).

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Summary of revisions: HYPERDIRE is a project devoted to the creation of a set of Mathematica based programs for the differential reduction of hypergeometric functions. The current version includes two parts: the first one, FdFunction, for manipulations with Appell hypergeometric functions $F_2$ of $r$ variables; and the second one, FsFunction, for manipulations with Lauricella–Saran hypergeometric functions $F_S$ of three variables. Both functions are related with one-loop Feynman diagrams.

Running time: Depends on the complexity of problem

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1. Introduction

The study of solutions of linear partial differential equations (PDEs) of a few variables in terms of multiple series, i.e., a multivariable generalization of Gauss hypergeometric function [1], was started a long time ago [2]. Following the Horn definition, a multiple series is called Horn-type hypergeometric function [4], if around some point $\tilde{z} = \tilde{z}_0$, there are series representations

$$H(\tilde{z}) = \sum_{\tilde{m}} C(\tilde{m}) (\tilde{z} - \tilde{z}_0)^{\tilde{m}},$$

where $\tilde{m}$ is a set of integers and the ratio of two coefficients can be represented as a ratio of two polynomials:

$$\frac{C(\tilde{m} + \tilde{e}_j)}{C(\tilde{m})} = \frac{P_j(\tilde{m})}{Q_j(\tilde{m})},$$

(1)

where $\tilde{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, is the $j$th unit vector. The coefficients $C(\tilde{m})$ of such a series are expressible as product/ratio of Gamma-functions (up to some factors irrelevant for our consideration) [5]:

$$C(\tilde{m}) = \frac{\prod_{j=1}^{r} \Gamma \left( \sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j \right)}{\prod_{k=1}^{p} \Gamma \left( \sum_{b=1}^{p} v_{kb} m_b + \sigma_k \right)},$$

(2)

where $\mu_{ja}$, $v_{kb}$, $\gamma_j$, $\sigma_k \in \mathbb{Z}$ and $m_a$ are elements of $\tilde{m}$.

The Horn-type hypergeometric function, Eq. (1), satisfies the following system of differential equations:

$$0 = D_j(\tilde{z}) H(\tilde{z}) = \left[ Q_j \left( \sum_{k=1}^{r} z_k \frac{\partial}{\partial z_k} \right) - P_j \left( \sum_{k=1}^{r} z_k \frac{\partial}{\partial z_k} \right) \right] H(\tilde{z}),$$

(3)

where $j = 1, \ldots, r$. The degree of polynomials $P_j$ and $Q_j$ is $p_j$ and $q_j$, respectively. The largest of these numbers, $r = \max\{p_j, q_j\}$, is called the order of the hypergeometric series.

Any Horn-type hypergeometric function is a function of two kind of variables, continuous variables, $z_1, z_2, \ldots, z_r$, and discrete variables: $\{J_b : = \{y_b, \sigma_b\}\}$, where the latter can change by integer numbers and are often referred to as the parameters of the hypergeometric function. For any Horn-hypergeometric function there are linear differential operators changing the value of the discrete variables by one unit:

$$R_k(\tilde{z}) \frac{\partial^{K}}{\partial \tilde{z}^{K}} H(\tilde{J}; \tilde{z}) = H(\tilde{J} \pm \epsilon_k; \tilde{z}),$$

(4)

where $R_k(\tilde{z})$ are polynomial (rational) functions. In Refs. [6,7] it was shown that there is an algorithmic solution for the construction of inverse linear differential operators:

$$B_l(\tilde{z}) \frac{\partial^{l}}{\partial \tilde{z}^{l}} \left( R_k(\tilde{z}) \frac{\partial^{K}}{\partial \tilde{z}^{K}} H(\tilde{J}; \tilde{z}) \equiv B_l(\tilde{z}) \frac{\partial^{l}}{\partial \tilde{z}^{l}} H(\tilde{J} \pm \epsilon_k; \tilde{z}) = H(\tilde{J}; \tilde{z}). \right)$$

(5)

Applying the direct or inverse differential operators to the hypergeometric function, the value of parameters can be changed by an arbitrary integer number:

$$S(\tilde{z}) H(\tilde{J} + \tilde{m}; \tilde{z}) = \sum_{l=0}^{r} S_l(\tilde{z}) \frac{\partial^{l}}{\partial \tilde{z}^{l}} H(\tilde{J}; \tilde{z}),$$

(6)

where $\tilde{m}$ is a set of integers, $S$ and $S_l$ are polynomials and $r$ is the holonomic rank (the number of linearly independent solutions) of the system of differential equations, Eq. (3). Additionally, the construction of inverse differential operators defined by Eq. (5) allows to

(i) find a set of exceptional parameters for any hypergeometric function, and this set coincides with the condition of reducibility of the monodromy group of the corresponding hypergeometric functions (see discussion in [8] and Section 3.4 for $F_2$ and $F_S$ functions);
(ii) convert the system of linear PDEs, Eq. (3), into Pfaff form for any hypergeometric functions, including functions with Puiseux monomials as one of the solution, see details in [9].

The interest of physicists in hypergeometric functions is related with

(i) the necessity of an analytical evaluation of multiple series generated by multiple residues of Mellin–Barnes integrals [10];
(ii) the restricted set of values of parameters of hypergeometric functions or multiple series, where the algorithms [11–14] are applicable;
(iii) the complicated analytical structure of one-loop massive Feynman diagrams, where, nevertheless, a simple hypergeometric representation exists [15–17].

It was pointed out in [18] that the differential reduction algorithm, defined as a full system of differential operators, Eqs. (3)–(5), can be applied to the construction of analytical coefficients of the so-called $\varepsilon$-expansions of hypergeometric functions about any rational values of parameters via the direct solution of the linear systems of differential equations.

This is the motivation for creating a package for the manipulation of the parameters of Horn-type hypergeometric functions of several variables.

In the previous publications the algebraic reduction of $F_2$ functions has been considered [19], the program pfq, for the manipulation of hypergeometric functions, $F_{p_1}F_{p_2}$ $(p \geq 1)$ [8], the program AppellIF4, for the manipulation of Appell hypergeometric functions, $F_1$, $F_2$, $F_3$, and $F_4$ [20], the program Horn, for the manipulation of Horn-hypergeometric functions of two variables [20 hypergeometric functions in addition to four Appell functions] [9].

The aim of this paper is to present a further extension of the Mathematica [21] based package HYPERDIRE for the differential reduction of the Horn-type hypergeometric function with arbitrary values of parameters to a set of basis functions. The current version consists of two parts; one, FdFunction, for the manipulation of Lauricella hypergeometric functions, $F_2$, of $r$ variables, and the second one, FsFunction, for the manipulation of Lauricella–Saran hypergeometric functions $F_3$ with three variables.

2. The structure of hypergeometric functions related with one-loop off-shell Feynman diagrams

A generic scalar one-loop $N$-point function is defined by the following integral in $d$ space–time dimensions

$$
I_{N,a_1,\ldots,a_N}^{(d)} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{((l-p_1)^2 + m_{12}^2)^{a_1} \cdots ((l-(p_1+\cdots+p_{N-1}))^2 + m_{N-1,N}^2)^{a_N} (l^2 + m_0^2)^{a_0}},
$$

where $l$ is the loop momentum to be integrated, $p_i$ are the external momenta and $m_{ij}^2$ the masses of the internal propagators, $i, j = 1, \ldots, N$. Energy–momentum conservation enforces $\sum_i p_i = 0$.

2.1. Massive case

In accordance with the algorithm described in [22], one-loop $N$-point diagrams with all powers of propagators equal to unity, i.e., all $a_i = 1$ in Eq. (7), satisfy to the following difference equation

$$
I_{N}^{(d)} = b_N(d) + \sum_{k=1}^{N} \left( \frac{\partial}{\partial a_k} \Delta_N \right) \sum_{r=0}^{\infty} \left( \frac{d - N + 1}{2} \right)^r \left( \frac{G_{N-1}}{\Delta_N} \right) \kappa^{-1} I_{N-1}^{(d+2r)},
$$

where $I_{N}^{(d)} \equiv I_{N,a_1,\ldots,a_N}^{(d)}$ $(a_k)$ is a Pochhammer symbol, $(a)_k = (\Gamma(a + k))/\Gamma(a)$, the operator $\kappa^{-1}$ shifts the value of index $a_k \rightarrow a_k - 1$ (in our case, $a_k \equiv 1 \Rightarrow \kappa^{-1} = 0$), $d$ is the dimension of space–time, and $b_N(d)$ is an arbitrary periodic function depending on the kinematic invariants. Moreover, we are working in Euclidean space–time, which is the source of the sign “$+$” instead of “$-$” as it was defined in [22], $G_N$ is a Gram determinant, $\Delta_N$ is a Cayley determinant for the $N$-point diagram and $\partial/\partial a_k = \frac{\partial}{\partial a_k} \Delta_N$. For details we refer to [22,15]. Eq. (8) can be solved iteratively [22,15] and the result for the one-loop $N$-point diagram in an arbitrary dimension $d$ can be written as linear combinations of the following hypergeometric functions:

$$
I_{N \geq 2}^{(d)} \sim \sum_{j=2}^{N} \prod_{k=2}^{j} \left( \frac{\partial}{\partial a_k} \Delta_j \right) \times \sum_{r_1+r_2+\ldots+r_{N-1}=0} \sum_{r_1+r_2+\ldots+r_{N-1}=0} \cdots \left( \frac{G_{N-1}}{\Delta_N} \right)^{r_1} \left( \frac{G_{N-1}}{\Delta_N} \right)^{r_2} \cdots \left( \frac{G_{N-1}}{\Delta_N} \right)^{r_{N-1}}
$$

$$
\times \left( \frac{d - N + 1}{2} \right)^{r_1} \cdots \left( \frac{d - N + 1}{2} \right)^{r_{N-1}} \left( \frac{d^2}{2} + r_1 + r_2 + \cdots + r_{N-1} \right) \cdots \left( \frac{d^2}{2} + r_1 + r_2 + \cdots + r_{N-1} \right)
$$

$$
\times \left( \frac{d^2}{2} + r_1 + r_2 + \cdots + r_{N-1} \right) \cdots \left( \frac{d^2}{2} + r_1 + r_2 + \cdots + r_{N-1} \right) \cdots \left( \frac{d^2}{2} + r_1 + r_2 + \cdots + r_{N-1} \right)
$$

$$
+ \sum_{j=1}^{N} \frac{b_j(d)c_j(d)}{j},
$$

where $m_j^2$ are some masses, cf., Eq. (7). In accordance with Proposition 1 of [23],2 the holonomic rank of the product $b_j(d)c_j(d)$, is expected to be equal to the holonomic rank of the hypergeometric function, defined by Eq. (10). Dropping all irrelevant factors, the hypergeometric

2 For completeness, we recall it here: A multiple Mellin–Barnes integrals can be presented as a linear combination of Horn-type hypergeometric functions about some point. Therefore, the holonomic rank of the corresponding system of linear differential equations related with the Mellin–Barnes integral is equal to the holonomic rank of any hypergeometric function in the corresponding hypergeometric representation. This statement has been proven only for one-fold Mellin–Barnes integrals.
In accordance with Eq. (3), the order of differential equations of hypergeometric functions Eq. (10), increases with the number of external integer.

There recursive application of the linear-fractional transformation, Eq. (15), to Eq. (10) gives rise to the following hypergeometric function:

\[
H_{N \geq 2}^{(d)} = \sum_{r_1, r_2, \ldots, r_{N-1} = 0}^{\infty} \frac{\Gamma \left( \frac{d-1}{2} + r_1 + r_2 \cdots + r_{N-1} \right)}{\Gamma \left( \frac{d}{2} + r_1 + r_2 \cdots + r_{N-1} \right)}
\times \frac{\Gamma \left( \frac{d-1}{2} + r_2 \cdots + r_{N-1} \right)}{\Gamma \left( \frac{d}{2} + r_2 \cdots + r_{N-1} \right)} \frac{\Gamma \left( \frac{d-3}{2} + r_3 \cdots + r_{N-1} \right)}{\Gamma \left( \frac{d-1}{2} + r_3 \cdots + r_{N-1} \right)} \cdots \frac{\Gamma \left( \frac{d-N+1}{2} + r_{N-1} \right)}{\Gamma \left( \frac{d-N+2}{2} + r_{N-1} \right)} z_1^r z_2^s \ldots z_{N-1}^t. \tag{10}
\]

For the lowest values of \( N = 2, 3, 4, 5 \), Eq. (10) has the following form:

\[
H_2^{(d)} = \sum_{r_1} \frac{\left( \frac{d-1}{2} \right)_r}{\left( \frac{d}{2} \right)_r} z_1^r,
\tag{11}
\]

\[
H_3^{(d)} = \sum_{r_1, r_2} \frac{\left( \frac{d-1}{2} \right)_{r_1+r_2}}{\left( \frac{d}{2} \right)_{r_1+r_2}} \frac{\left( \frac{d-2}{2} \right)_{r_1}}{\left( \frac{d}{2} \right)_{r_1}} z_1^{r_1} z_2^{r_2},
\tag{12}
\]

\[
H_4^{(d)} = \sum_{r_1, r_2, r_3} \frac{\left( \frac{d-1}{2} \right)_{r_1+r_2+r_3}}{\left( \frac{d}{2} \right)_{r_1+r_2+r_3}} \frac{\left( \frac{d-2}{2} \right)_{r_2+r_3}}{\left( \frac{d}{2} \right)_{r_2+r_3}} \frac{\left( \frac{d-1}{2} \right)_{r_1}}{\left( \frac{d}{2} \right)_{r_1}} z_1^{r_1} z_2^{r_2} z_3^{r_3},
\tag{13}
\]

\[
H_5^{(d)} = \sum_{r_1, r_2, r_3, r_4} \frac{\left( \frac{d-1}{2} \right)_{r_1+r_2+r_3+r_4}}{\left( \frac{d}{2} \right)_{r_1+r_2+r_3+r_4}} \frac{\left( \frac{d-2}{2} \right)_{r_2+r_3+r_4}}{\left( \frac{d}{2} \right)_{r_2+r_3+r_4}} \frac{\left( \frac{d-1}{2} \right)_{r_1}}{\left( \frac{d}{2} \right)_{r_1}} z_1^{r_1} z_2^{r_2} z_3^{r_3} z_4^{r_4}.
\tag{14}
\]

In accordance with Eq. (3), the order of differential equations of hypergeometric functions Eq. (10), increases with the number of external legs:

\[
\frac{P_{ij}}{Q_{ij}} = j,
\]

where the index \( j \) is the same as the summation index \( r_1 \) and \( j+1 \) is equal to the number of external legs of the Feynman diagrams, cf. Eq. (7).

To reduce the order of differential equations of the hypergeometric function Eq. (10), we apply recursively the following transformation:

\[
\sum_{r=0}^{\infty} \frac{\Gamma \left( A + r \right)}{\Gamma \left( B + r \right)} z^r = \frac{\Gamma \left( A \right)}{\Gamma \left( B \right)} F_1 \left( 1, A \left| B \right| z \right) = \frac{1}{1 - z} \frac{\Gamma \left( A \right)}{\Gamma \left( B \right)} F_1 \left( 1, B - A \left| B \right| \frac{z}{z - 1} \right)
\]

\[
= \frac{1}{1 - z} \frac{\Gamma \left( A \right)}{\Gamma \left( B - A \right)} \sum_{r=0}^{\infty} \left( \frac{z}{z - 1} \right)^r \frac{\Gamma \left( B - A + r \right)}{\Gamma \left( B + r \right)}.
\tag{15}
\]

Let us introduce new variables:

\[
x_1 = -\frac{z}{1 - z}, \quad z_1 = -\frac{x_1}{1 - x_1}, \quad 1 - z_1 = \frac{1}{1 - x_1}.
\tag{16}
\]

The recursive application of the linear-fractional transformation, Eq. (15), to Eq. (10) gives rise to the following hypergeometric function:

\[
\prod_{k=1}^{N-1} \left( 1 - z_k \right) H_{N \geq 2}^{(d)} = \sum_{r_1, r_2, \ldots, r_{N-1} = 0}^{\infty} \frac{\Gamma \left( \frac{d}{2} \right)_{r_1 + r_2 \cdots + r_{N-1}}}{\Gamma \left( \frac{d}{2} + r_1 + \cdots + r_{N-1} \right)} \prod_{i=1}^{N-1} x_i^{r_i} \left[ \frac{\Gamma \left( \frac{d}{2} + r_i + \cdots + r_{N-1} \right)}{\Gamma \left( \frac{d}{2} + r_i + \cdots + r_{N-1} \right)} \right].
\tag{17}
\]

For completeness, we present explicitly the hypergeometric terms defined by Eq. (17) for the first few values of \( N = 2, 3, 4, 5 \):

\[
H_2^{(d)} = \sum_{r_1} \frac{\left( \frac{d}{2} \right)_{r_1}}{\left( \frac{d}{2} \right)_{r_1}} x_1^{r_1},
\tag{18}
\]

\[
H_3^{(d)} = \sum_{r_1, r_2} \frac{\left( \frac{d}{2} \right)_{r_1}}{\left( \frac{d}{2} \right)_{r_1+r_2}} x_1^{r_1} x_2^{r_2},
\tag{19}
\]

\[
= \sum_{r_1, r_2} \frac{\left( \frac{d}{2} \right)_{r_1}}{\left( \frac{d}{2} \right)_{r_1+r_2}} \left( \frac{1}{2} \right)_{r_1+r_2} x_1^{r_1} x_2^{r_2}.
\tag{20}
\]

\(^3\) For our discussion we drop all irrelevant factors, like \((1 - z_i)^{-1}\) and assume, wherever it does not cause any problems, that \(\Gamma \left( \frac{d}{2} \pm k \right) = \left( \frac{d}{2} \pm k \right)_k\) with \(k\) being integer.
The three-point diagram (vertex) is not algebraically reducible to a simpler diagram. The hypergeometric functions).

\[ H^{(d)}_4 = \sum_{r_1, r_2, r_3} \left( \frac{1}{2} \right) \frac{1}{2, r_1 + r_2} \frac{1}{2, r_1 + r_2 + r_3} \frac{1}{2, r_1 + r_2 + r_3 + r_4} x_1^r_1 x_2^r_2 x_3^r_3 \]

where the \( x_1 \) are defined in Eq. (16). As follows from Eqs. (19) and (20), the vertex diagrams are described by the Appell hypergeometric functions \( F_2 \) or \( F_1 [25]. \) For the pentagon \( (N = 5), \) the hypergeometric function has the following form:

\[ H^{(d)}_5 = \sum_{r_1, r_2, r_3, r_4} \left( \frac{1}{2} \right) \frac{1}{2, r_1 + r_2 + r_3} \frac{1}{2, r_1 + r_2 + r_3 + r_4} \frac{1}{2, r_1 + r_2 + r_3 + r_4 + r_5} x_1^r_1 x_2^r_2 x_3^r_3 x_4^r_4 \]

After the linear-fractional transformation, Eq. (15), the order of the differential equation for the hypergeometric function related to the box diagram is reduced from three to two, see Eqs. (21) and (22), [15]. The pentagon, Eq. (25), corresponds to a hypergeometric function satisfying a differential equation of order two. As follows from Eqs. (10) and (17), the massive hexagon is expressible in terms of hypergeometric functions of five variables satisfying a differential equations of order three. However, since the difference between parameters of hypergeometric functions, Eqs. (10) or (17), are integer or half-integer, these functions possess extended symmetries with respect to non-linear transformations of their arguments [26] (multivariable generalizations of quadratic transformations related to Gauss hypergeometric functions). It is still an open question, whether or not it is possible, to reduce the order of differential equations with the help of non-linear transformations.

### 2.2. Off-shell massless case

Let us consider an off-shell massless one-loop \( N \)-point diagram, where for some \( i, \) we have \( \langle p^2 \rangle \neq 0, \) cf., Eq. (7). In these kinematics, the three-point diagram (vertex) is not algebraically reducible to a simpler diagram. The \( I_2^{(d)} \) integral can be written (up to some irrelevant normalization) as

\[ I_2^{(d)} = \frac{1}{\Gamma \left( \frac{d-1}{2} \right)} \]

and the iterative solution of Eq. (8) is

\[ H^{(d)}_{N \geq 3} \bigg|_{\langle p^2 \rangle = 0} = \sum_{r_1, r_2, \ldots, r_{N-2}} \frac{\Gamma \left( \frac{d-2}{2} + r_1 + r_2 \cdots + r_{N-2} \right)}{\Gamma \left( \frac{d-1}{2} + r_1 + r_2 \cdots + r_{N-2} \right)} \]

\[ \times \frac{\Gamma \left( \frac{d-3}{2} + r_2 + r_3 \cdots + r_{N-2} \right)}{\Gamma \left( \frac{d-2}{2} + r_2 + r_3 \cdots + r_{N-2} \right)} \frac{\Gamma \left( \frac{d-4}{2} + r_3 + r_4 \cdots + r_{N-2} \right)}{\Gamma \left( \frac{d-3}{2} + r_3 + r_4 \cdots + r_{N-2} \right)} \]

\[ \times \frac{\Gamma \left( \frac{d-5}{2} + r_4 \cdots + r_{N-2} \right)}{\Gamma \left( \frac{d-4}{2} + r_4 \cdots + r_{N-2} \right)} \frac{\Gamma \left( \frac{d-6}{2} + r_4 \cdots + r_{N-2} \right)}{\Gamma \left( \frac{d-5}{2} + r_4 \cdots + r_{N-2} \right)} \]

\[ \frac{\Gamma \left( \frac{d-N+2}{2} + r_{N-2} \right)}{\Gamma \left( \frac{d-N+1}{2} + r_{N-2} \right)} \frac{\Gamma \left( \frac{d-N+1}{2} + r_{N-2} \right)}{\Gamma \left( \frac{d-N+2}{2} + r_{N-2} \right)} \frac{\Gamma \left( \frac{d-N+2}{2} + r_{N-2} \right)}{\Gamma \left( \frac{d-N+1}{2} + r_{N-2} \right)} \]

From Eqs. (10) and (29) we see, that the structure of hypergeometric functions related with off-shell massive and off-shell massless integrals is related as follows [17]:

\[ H^{(d)}_{N+1} \bigg|_{\langle p^2 \rangle = 0} \approx H^{(d-1)}_N \]

---

4 All hypergeometric functions, defined by Eqs. (11)-(27), belong to the class of multiple Gauss hypergeometric functions [4]; the series representation can be written as infinite sum(s) with respect to the index of summation over the parameters of \( F_1 \) hypergeometric functions: in Eqs. (20), (23), (27) the Gauss hypergeometric functions enter via the last index of summation; in Eq. (21) via summation over \( r_1; \) in Eq. (22) via summation over \( r_2 \) or \( r_3; \) in Eq. (24) via summation over \( r_1 \) or \( r_2; \) in Eq. (25) via summation over \( r_1 \) or \( r_3; \) in Eq. (26) via summation over \( r_1. \) Exploring the transformation properties of Gauss hypergeometric functions, see Eq. (15) for an example, the transformation of hypergeometric functions, Eqs. (11)-(27), can be performed.

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where $d$ is the dimension of space–time and $N$ denotes the number of external legs. The symbol $\sim$ in Eq. (30) indicates that this relation is valid for hypergeometric functions related with the corresponding Feynman diagram. Eq. (30) is also valid for hypergeometric functions, Eq. (17), after application of the linear-fractional transformation, Eq. (15).

3. Differential reduction of Horn-type hypergeometric functions of three variables

3.1. System of differential equations

Let us consider the system of linear differential operators of second order $L_i$ for the hypergeometric functions $\omega(\vec{z})$:

$$ L_i \omega(\vec{z}) : \quad \theta_i^2 \omega(\vec{z}) = \left[ \theta_i \sum_{j=1, j \neq i}^3 P_j \theta_j + \sum_{m=1}^3 R_m \theta_m + S_i \right] \omega(\vec{z}), \quad i = 1, 2, 3, \tag{31} $$

where $\vec{z} = (z_1, z_2, z_3)$ with $z_1$, $z_2$, $z_3$ being variables, $\{P_j, R_m, S_j\}$ are rational functions, $\theta_j = z_j \theta_j$ for $j = 1, 2, 3$, and $\theta_1 \cdots \theta_k = \theta_1 \cdots \theta_k$. Taking the derivative, $\theta_k L_i \omega(\vec{z})$, we finally obtain from Eq. (31):

$$ \theta_k L_i \omega(\vec{z}) : \quad \left[ (1 - P_k P_k) \theta_k \theta_i^2 - \sum_{j=1, j \neq k}^3 (P_j + P_k P_j) \theta_j \theta_i \theta_k \right] \omega(\vec{z}) $$

$$ = \left[ P_k R_k P_k + R_k \theta_k R_k + P_k R_k + (\theta_k P_k) \theta_k + \sum_{j=1, j \neq k}^3 \left[ P_k R_k P_j + P_k R_j P_k + (\theta_k P_k) \right] \theta_j \theta_k \theta_i \right] $$

$$ + \sum_{j=1, j \neq k}^3 \left[ R_k P_j + P_k \theta_k \right] \theta_j \theta_i + \sum_{m=1}^3 \left[ P_k R_k R_m + R_k \theta_m + (\theta_k R_m) \right] \theta_m $$

$$ + P_k S_k \theta_i \theta_k + S_k \theta_k + P_k R_k S_k + R_k S_k + (\theta_k S_k) \omega(\vec{z}), \quad i, k = 1, 2, 3. \tag{32} $$

For a function of three variables, the sum $\sum_{j=1, j \neq k}^3 \omega(\vec{z})$ can be replaced by the index $j$, where $j \neq i \neq k$. The conditions of complete integrability are defined via the relations:

$$ \theta_i \left[ \theta_j L_k \right] \omega(\vec{z}) = \theta_j \left[ \theta_k L_i \right] \omega(\vec{z}), \quad i, j, k = 1, 2, 3. \tag{33} $$

The number of independent solutions of the system of differential equations, Eq. (31), of three variables is defined by coefficients on the l.h.s. of Eq. (32) and the validity of Eq. (33). When the coefficients

$$ (1 - P_k P_k), \quad i, k = 1, 2, 3, \tag{34} $$

and

$$ \sum_{j=1, j \neq k}^3 (P_j + P_k P_j), \quad i, k = 1, 2, 3, \tag{35} $$

are not equal to zero for all $i, j, k$, Eqs. (31) and (32) can be reduced to the Pfaff system of eight independent differential equations:

$$ d\vec{G} = \vec{f}, \tag{36} $$

where $f = (\omega(\vec{z}), \theta_1 \omega(\vec{z}), \theta_2 \omega(\vec{z}), \theta_3 \omega(\vec{z}), \theta_{12} \omega(\vec{z}), \theta_{13} \omega(\vec{z}), \theta_{23} \omega(\vec{z}), \theta_{123} \omega(\vec{z})).$ When some of the coefficients in Eq. (34) are zero, the coefficients in front of the terms $\theta_{123} \omega(\vec{z})$, defined by Eq. (35), start to play a role. For non-zero values of Eq. (35), the terms $\theta_{123} \omega(\vec{z})$ can be excluded, and the rank of differential system is reduced to seven independent functions. When for some values of $i$ and $k$ both coefficients, defined by Eqs. (34) and (35) are zero, a further simplification can be performed, so that the rank of system is reduced to six or to an even smaller number.

The locus of singularities $L_j$ of the linear system of differential equations of second order of three variables defined by Eq. (31) follows from singularities of higher rank differential operators on the l.h.s. of Eqs. (31) and (32):

$$ L_j = \cup_{k=1}^3 \{ \lambda_k = 0 \} \cup_{i,k=1}^3 \{ (1 - P_k P_k)^{-1} = 0 \} \cup_{i,k=1}^3 \{ (P_j + P_k P_j)^{-1} = 0 \}. \tag{37} $$

This result is valid only for the full (non-degenerate) case of Pfaff system including eight elements.

3.2. Lauricella hypergeometric function $F_D$

3.2.1. General consideration

Let us consider the $F_D^{(r)}$ functions of $r$ variables, defined around $z_i = 0$ as

$$ F_D^{(r)}\left( a; b_1, \ldots, b_k; c; z_1, \ldots, z_r \right) = \sum_{m_1, \ldots, m_r=0}^{\infty} \frac{(a)_m}{(c)_m} \prod_{j=1}^r \left( b_j \right)_{m_j} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_r^{m_r}}{m_r!}, \tag{38} $$
For $r = 1$ this function coincides with the Gauss hypergeometric function, for $r = 2$, it coincides with Appell function $F_1$ [27]. As follows from the definition, Eq. (38), this function is symmetric with respect to the transformation

$$b_i \leftrightarrow b_j, \quad z_i \leftrightarrow z_j.$$  

Generally, $F_0$ functions and their properties have been analyzed in detail in many references [6,28]. The differential operators for $F_0$ hypergeometric function, Eq. (3), are given by

$$D_{i} F_{0}^{(i)} : \quad \partial_{i} \left( c - 1 + \sum_{j=1}^{r} \theta_{j} \right) F_{0}^{(i)} = \left( a + \sum_{j=1}^{r} \theta_{j} \right) (b_i + \theta_i) F_{0}^{(i)}, \quad i = 1, \ldots, r.$$  

(39)

where

$$F_0 = F_0^{(i)} (a; b_1, \ldots, b_r; c; z_1, \ldots, z_r).$$  

(40)

They can be written in canonical form, cf. Eq. (31):

$$L_i F_0 : \quad \theta^2 F_0 = \left[ -\theta_i + \sum_{j \neq i} \theta_j + \frac{(a + b_i) z_i - (c - 1)}{1 - z_i} \theta_i + \frac{b_i z_i}{1 - z_i} \sum_{j \neq i} \theta_j + \frac{ab_i z_i}{1 - z_i} \right] F_0, \quad i = 1, \ldots, r.$$  

(41)

From these equations we have:

$$P_i = P_j = -1, \quad S_i = \frac{ab_i z_i}{1 - z_i} \equiv aP_i,$$

$$R_{ai} = \frac{(a + b_i) z_i - (c - 1)}{1 - z_i} \equiv R_i, \quad R_{am} = \frac{b_i z_i}{1 - z_i} \equiv P_i, \quad m \neq i.$$  

(42)

Upon substitution of these values for $P_j$, $R_{ab}$, $S_i$ into Eq. (32), we obtain

$$\left[ (P_k - P_i + R_i - R_k) \partial_i \partial_k [R_{ai} R_{am} - R_{ak} R_{am}] \sum_{m=1}^{r} \theta_m - S_i \partial_i + S_j \partial_k - P_k S_i + P_k S_k \right] F_0 = 0.$$  

(43)

Eq. (43) can be simplified with the help of Eq. (42) by taking into account that the sum of the last two terms in Eq. (43), $P_i S_k - P_k S_i$, is equal to zero, and by splitting the sum over $m$ into $i, k$ and $j$, where $j \neq i \neq k$. In this way, we get

$$\left( R_k - P_k - R_i + P_i \right) \partial_i \partial_k F_0 = \left( P_k \left[ P_i - R_i - a \right] \partial_i + P_k \left| P_k - R_k - a \right| \partial_k \right) F_0,$$  

(44)

where

$$R_i - P_i \equiv R_{ai} - R_{ak} = \frac{az_i - (c - 1)}{1 - z_i}.$$  

(45)

Eq. (44) can be rewritten in a more familiar form, see [28]:

$$\left[ (z_i - z_j) \partial_i \partial_j - b_j z_i \theta_i + b_i z_j \theta_j \right] F_0 = 0.$$  

(46)

After factorization of $z_i$, $z_j$, Eq. (46) can be expressed as follows,

$$\left[ (z_i - z_j) \partial_i + b_j \partial_i - b_i \partial_j \right] F_0 = 0.$$  

(47)

In this way, all second derivatives of an $F_0$ function are expressible in terms of the corresponding first derivatives and function, see Eqs. (41) and (46). As a consequence, there are only $r + 1$ linearly independent solutions of linear differential equations, Eq. (39). The locus of the singularities $L_0$ of an $F_0$ function is defined from the singularities of the differential equations, Eqs. (41) and (46):

$$L_0 = \bigcup_{i=1}^{r} \left[ z_i = 0 \right] \cup \bigcup_{i=1}^{r} \left[ z_i = 1 \right] \cup \bigcup_{i=1}^{r} \left[ z_i = z_j \right].$$  

(48)

The Pfaff system for an $F_0$ hypergeometric function has the following form:

$$d \omega(\vec{z}) = \left( \sum_{i<j} A_{ij} d \log(z_i - z_j) \right) \omega(\vec{z}),$$  

where $\omega(\vec{z}) = \{ F_0, \theta_j F_0 \}$ and the matrices $A_{ij}$ have been constructed explicitly in [28].
3.2.2. Differential reduction of $F_D$

The direct differential operators are the following:

$$a F_D^{(r)}(a + 1; b_1, \ldots, b_r; c; z_1, \ldots, z_r) = \left( a + \prod_{i=1}^r \theta_i \right) F_D,$$

$$b_r F_D^{(r)}(a; b_1, \ldots, b_r; c; z_1, \ldots, z_r) = (b_r + \theta_r) F_D,$$

$$(c - 1) F_D^{(r)}(a; b_1, \ldots, b_r; c - 1; z_1, \ldots, z_r) = \left( c - 1 + \prod_{i=1}^r \theta_i \right) F_D,$$  \hspace{1cm} (49)

and $F_D$ is defined by Eq. (40). The inverse differential operators have been constructed in [6]:

$$(c - a) F_D^{(r)}(a - 1; b_1, \ldots, b_r; c; z_1, \ldots, z_r) = \left[ \sum_{j=1}^{r} (1 - z_j) \theta_j - \sum_{j=1}^{r} b_j z_j + c - a \right] F_D,$$  \hspace{1cm} (50)

$$\left( c - \sum_{j=1}^{r} b_j \right) F_D^{(r)}(a; b_1, \ldots, b_r - 1, \ldots, b_r; c; z_1, \ldots, z_r) = \left[ z_r \sum_{j=1}^{r} (1 - z_j) \delta_j - a z_r + c - \sum_{j=1}^{r} b_j \right] F_D,$$  \hspace{1cm} (51)

$$(c - a) \left( c - \sum_{j=1}^{r} b_j \right) F_D^{(r)}(a; b_1, \ldots, b_r; c + 1; z_1, \ldots, z_r) = c \left[ \sum_{j=1}^{r} (1 - z_j) \delta_j + c - a - \sum_{j=1}^{r} b_j \right] F_D,$$  \hspace{1cm} (52)

where $F_D$ is defined by Eq. (40). In this case, the results of the differential reduction, Eq. (6), have the following form

$$S(\varepsilon) F_D((a; \varepsilon; c) + \tilde{m}; \tilde{z}) = S_0(\varepsilon) F_D(a; \varepsilon; c; \tilde{z}) + \sum_{i=1}^{r} S_i(\varepsilon) \frac{\partial}{\partial z_i} F_D^{(r)}(a; \varepsilon; c; \tilde{z}),$$

where $\tilde{m}$ is a set of integers and $S, S_i$ are polynomials.

3.3. Hypergeometric function $F_3$

3.3.1. General consideration

The Lauricella–Saran hypergeometric function of three variables $F_3$ \cite{29} ($F_7$ in the notation of [2]) is defined around the point $z_1 = z_2 = z_3 = 0$ as follows

$$F_3(a_1; a_2; b_1, b_2, b_3; c; z_1, z_2, z_3) = \sum_{m_1, m_2, m_3 = 0}^{\infty} \frac{(a_1)_{m_1} (a_2)_{m_2 + m_3}}{(c)_{m_1 + m_2 + m_3}} \sum_{j=1}^{3} (b_j)^{m_1} z_1^{m_1} z_2^{m_2} z_3^{m_3},$$

\hspace{1cm} (54)

It is one of the 14 functions of three variables of order two,\footnote{The complete set of programs for the differential reduction for other functions from the Lauricella–Srivastava list \cite{4} will be presented in a separate publication.} introduced by Lauricella [2]. In this case, the differential operators, Eq. (3), are

$$D_1 F_3 : \ \partial_1 \left( c - 1 + \sum_{j=1}^{3} \theta_j \right) F_3 = (a_1 + \theta_1) (b_1 + \theta_1) F_3,$$  \hspace{1cm} (55)

$$D_i F_3 : \ \partial_i \left( c - 1 + \sum_{j=1}^{3} \theta_j \right) F_3 = (a_i + \theta_2 + \theta_3) (b_i + \theta_i) F_3, \ \ i = 2, 3,$$  \hspace{1cm} (56)

where

$$F_3 = F_3(a_1, a_2; b_1, b_2, b_3; c; z_1, z_2, z_3).$$

The canonical form of these differential equations are the following:

$$L_1 F_3 : \ \partial_1^2 F_3 = \left[ -\frac{1}{1 - z_1} \theta_1 (\theta_2 + \theta_3) + \frac{(a_1 - b_1 - z_1) (c - 1)}{1 - z_1} \theta_1 + \frac{a_1 b_1 z_1}{1 - z_1} \right] F_3,$$  \hspace{1cm} (58)

$$L_2 F_3 : \ \partial_2^2 F_3 = \left[ -\theta_2 \theta_3 - \frac{1}{1 - z_2} \theta_2 \theta_1 + \frac{(a_2 + b_2) z_2 (c - 1)}{1 - z_2} \theta_2 + \frac{b_2 z_2}{1 - z_2} \theta_3 + \frac{a_2 b_2 z_2}{1 - z_2} \right] F_3,$$  \hspace{1cm} (59)

$$L_3 F_3 : \ \partial_3^2 F_3 = \left[ -\theta_3 \theta_2 - \frac{1}{1 - z_3} \theta_3 \theta_1 + \frac{(a_3 + b_3) z_3 (c - 1)}{1 - z_3} \theta_3 + \frac{b_3 z_3}{1 - z_3} \theta_2 + \frac{a_3 b_3 z_3}{1 - z_3} \right] F_3,$$  \hspace{1cm} (60)
These equations define the values of functions $P_{ij}$, $R_{ij}$, $S_i$ entering in Eq. (31):
$$
R_{12} = R_{13} = R_{21} = R_{31} = 0, \quad P_{23} = P_{32} = -1, \\
P_{13} = P_{12} = \frac{1}{1 - z_1}, \quad P_{21} = \frac{1}{1 - z_2}, \quad P_{31} = \frac{1}{1 - z_3}, \\
R_{11} = \frac{(a_1 + b_1)z_1 - (c - 1)}{1 - z_1}, \quad R_{ii} = \frac{(a_i + b_i)z_i - (c - 1)}{1 - z_i}, \quad i = 2, 3, \\
R_{23} = \frac{b_2z_2}{1 - z_2}, \quad R_{32} = \frac{b_3z_3}{1 - z_3}, \quad S_1 = \frac{a_1b_1z_1}{1 - z_1}, \quad S_i = \frac{a_i b_i z_i}{1 - z_i}, \quad i = 2, 3. \quad (61)
$$

With the substitution of these values of $P_{ij}$ into Eq. (32) and, since $1 - P_{23}P_{32} = 0$, we can express the third mixing derivatives of the hypergeometric function, $\theta_{123\theta}(z)$, via second derivatives of the hypergeometric function.

The series representation of the hypergeometric function $F_3$ can be rewritten in the following form:
$$
F_3(a_1, a_2, a_2; b_1, b_2, b_3; c; z_1, z_2, z_3) = \sum_{m=0}^{\infty} \frac{(a_1)_m (b_1)_m z_1^m}{(c)_m m!} F_1(a_2; b_2, b_3; c + m_1; z_2, z_3), \quad (62)
$$

where $F_1(a; b_1, b_2; c; z_1, z_2)$ is the Appell function of two variables: $F_1 \equiv F_1^{(2)}$. From this representation it is easy to get the following relation:
$$
[(z_2 - z_3)\theta_{13}] F_3 = (b_3 z_3 \theta_2 - b_2 z_2 \theta_3) F_5. \quad (63)
$$

Eq. (32) allows us to express all higher derivatives of hypergeometric functions $F_5$ in terms of second derivatives only. In particular,
$$
\theta_2 L_1 : \left(1 - \frac{1}{1 - z_1(1 - z_2)}\right) \theta_{12} F_5 = \left\{ (P_{12} R_{22} + R_{11}) \theta_{12} + P_{12} R_{23} \theta_{13} + P_{12} S_2 \theta_1 + S_1 \theta_2 \right\} F_5 \\
= \left\{ \frac{(a_1 + b_1)z_1 - (c - 1)}{1 - z_1} - \frac{(a_2 + b_2)z_2 - (c - 1)}{1 - z_1(1 - z_2)} \right\} \theta_{12} \\
- \frac{b_2 z_2}{(1 - z_1)(1 - z_2)} \theta_{13} - \frac{a_2 b_2 z_2}{(1 - z_1)(1 - z_2)} \theta_1 + \frac{a_1 b_1 z_1 \theta_2}{1 - z_1} \right\} F_5, \quad (64)
$$

$$
\theta_3 L_1 : \left(1 - \frac{1}{1 - z_1(1 - z_3)}\right) \theta_{13} F_5 = \left\{ (P_{13} R_{33} + R_{11}) \theta_{13} + P_{13} R_{32} \theta_{12} + P_{13} S_3 \theta_1 + S_1 \theta_2 \right\} F_5 \\
= \left\{ \frac{(a_1 + b_1)z_1 - (c - 1)}{1 - z_1} - \frac{(a_2 + b_3)z_3 - (c - 1)}{1 - z_1(1 - z_3)} \right\} \theta_{13} \\
- \frac{b_3 z_3}{(1 - z_1)(1 - z_3)} \theta_{12} - \frac{a_2 b_3 z_3}{(1 - z_1)(1 - z_3)} \theta_1 + \frac{a_1 b_1 z_1 \theta_3}{1 - z_1} \right\} F_5, \quad (65)
$$

$$
\theta_3 L_2 = -\theta_2 L_3 : \left(\frac{z_3 - z_2}{(1 - z_2)(1 - z_3)}\right) \theta_{123} F_5 = \left\{ \frac{P_{23} R_{23} \theta_{12} - P_{31} R_{33} \theta_{13} - (R_{22} - R_{23} + R_{32} - R_{33})}{1 - z_3} \left( b_3 z_3 \theta_2 - b_2 z_2 \theta_3 \right) \right\} F_5 \\
- \frac{R_{23} (a_2 + R_{33} - R_{32}) \theta_3 + R_{32} (a_2 + R_{22} - R_{32}) \theta_2}{1 - z_1(1 - z_3)} \right\} F_5 \\
= \left\{ \frac{b_3 z_3}{(1 - z_2)(1 - z_3)} \theta_{13} - \frac{b_2 z_2}{(1 - z_2)(1 - z_3)} \theta_{12} \right\} F_5. \quad (66)
$$

The last equation, Eq. (66), coincides with the derivative of the Eq. (63) with respect to $z_1$. The remaining two differential equations we can write in the following form:
$$
\theta_1 L_2 : \left(1 - \frac{1}{1 - z_1(1 - z_2)}\right) \theta_{122} F_5 = -\left(1 - \frac{1}{1 - z_1(1 - z_2)}\right) \theta_{123} F_5 \\
+ \left\{ (P_{12} R_{11} + R_{22}) \theta_{12} + R_{23} \theta_{13} + P_{2} S_1 \theta_2 + S_2 \theta_1 \right\} F_5 \\
= -\left(1 - \frac{1}{1 - z_1(1 - z_2)}\right) \theta_{123} F_5 \\
+ \left\{ \frac{(a_2 + b_2)z_2 - (c - 1)}{1 - z_2} - \frac{(a_1 + b_1)z_1 - (c - 1)}{1 - z_1(1 - z_2)} \right\} \theta_{12} \\
+ \frac{b_2 z_2}{(1 - z_2)} \theta_{13} - \frac{a_1 b_1 z_1}{(1 - z_1)(1 - z_2)} \theta_2 + \frac{a_2 b_2 z_2}{1 - z_2} \theta_1 \right\} F_5, \quad (67)
$$
\[ \theta_1 L_3 : \left( 1 - \frac{1}{(1 - z_1)(1 - z_3)} \right) \theta_{13} F_5 = - \left( 1 - \frac{1}{(1 - z_1)(1 - z_3)} \right) \theta_{123} F_5 \]

\[ + \left\{ (P_1 R_{11} + R_{13}) \theta_{13} + (R_{12} R_{12} + P_3 S_i \theta_3 + S_3 \theta_3) F_5 \right\} \]

\[ = - \left( 1 - \frac{1}{(1 - z_1)(1 - z_3)} \right) \theta_{123} F_5 \]

\[ + \left\{ \left( a_2 + b_3 \right) z_3 - (c - 1) \right\} \left( a_1 + b_1 \right) \frac{z_1}{1 - z_3} \theta_{13} \]

\[ + \frac{b_3 z_3}{(1 - z_3)} \theta_{12} - \frac{a_1 b_1 z_1}{(1 - z_1)(1 - z_3)} \theta_{1} + \frac{a_2 b_3 z_3}{1 - z_3} \theta_{1} \right\} F_5, \]

where the mixed derivative \( \theta_{123} F_5 \) is defined by Eq. (66).

In this way, we have proven:

**Theorem 1.** The Lauricella–Saran hypergeometric function \( F_5 \) of three variables, Eq. (54), has six linearly independent solutions around the points \( z_1 = z_2 = z_3 = 0 \).

The locus of singularities \( L_9 \) of the hypergeometric function \( F_5 \) follows from the singularities of the differential operators, Eqs. (58)–(60), (66)–(68):

\[ L_9 = \bigcup_{i=1}^{3} \{ z_i = 0 \} \bigcup_{i=1}^{3} \{ z_i = 1 \} \bigcup_{i=1}^{3} \{ z_1 + z_2 = z_1 z_2 \}. \]

**3.3.2. Differential reduction of \( F_5 \)**

The direct differential operators are the following:

\[ a_i F_5(a_1 + 1, a_2; \bar{b}; c; \bar{x}) = (a_1 + \theta_1) F_5, \]

\[ a_2 F_5(a_1, a_2 + 1; b; c; \bar{x}) = (a_2 + \theta_2 + \theta_3) F_5, \]

\[ b_i F_5(a_1, a_2; \cdots, b_i + 1, \ldots, c; \bar{x}) = (b_i + \theta_i) F_5, \]

\[ (c - 1) F_5(\bar{a}; \bar{b}; c - 1; \bar{x}) = \left( c - 1 + \sum_{j=1}^{3} \theta_j \right) F_5. \]

and \( F_5 \) is defined by Eq. (57). The corresponding inverse differential operators we define for the parameters \( X \in \{ a_1, a_2, b_1, b_2, b_3 \} \) as follows:

\[ F_5(X; c; \bar{z}) = \left[ A_{X, F_5} + B_{X, F_5} \theta_1 + C_{X, F_5} \theta_2 + D_{X, F_5} \theta_3 + E_{X, F_5} \theta_{12} + F_{X, F_5} \theta_{13} \right] F_5(X; 1; c; \bar{z}), \]

and for the parameter \( c \):

\[ F_5(a_1, a_2; b_1, b_2, b_3; c; \bar{z}) = \left[ A_{c, F_5} + B_{c, F_5} \theta_1 + C_{c, F_5} \theta_2 + D_{c, F_5} \theta_3 + E_{c, F_5} \theta_{12} + F_{c, F_5} \theta_{13} \right] F_5(a_1, a_2; b_1, b_2, b_3; c - 1; \bar{z}). \]

The full list of inverse differential operators are the following:

\[ A_{a_1, F_5} = \frac{a_1^2 + a_1 (b_1 z_1 + D_1 - D_3 - 2 b_1) + a_2 (b_1 z_1 + D_2 - a_1) + (b_1 z_1 + c - 1)(D_2 - a_1)}{D_0 D_2}, \]

\[ B_{a_1, F_5} = \frac{(z_1 - 1)(a_2 + D_2)}{D_0 D_2}, \]

\[ C_{a_1, F_5} = \frac{b_1 z_1 (z_2 - 1)}{z_2 D_0 D_2}, \]

\[ D_{a_1, F_5} = \frac{b_1 z_1 (z_3 - 1)}{z_3 D_0 D_2}, \]

\[ E_{a_1, F_5} = \frac{z_1 + z_2 - z_1 z_2}{z_2 D_0 D_2}, \]

\[ F_{a_1, F_5} = \frac{z_1 + z_3 - z_1 z_3}{z_3 D_0 D_2}, \]

\[ A_{a_2, F_5} = \frac{(b_2 z_2 + b_2 z_3 + D_1)(a_1 + D_1) - b_1 D_1}{D_0 D_1}, \]

\[ B_{a_2, F_5} = \frac{(z_1 - 1)(b_2 z_2 + b_2 z_3)}{z_1 D_0 D_1}, \]

\[ C_{a_2, F_5} = \frac{(z_2 - 1)(b_1 + D_0)}{D_0 D_1}, \]

\[ D_{a_2, F_5} = \frac{(z_3 - 1)(b_1 + D_0)}{D_0 D_1}, \]

\[ E_{a_2, F_5} = \frac{z_1 + z_2 - z_1 z_2}{z_1 D_0 D_1}, \]

\[ F_{a_2, F_5} = \frac{z_1 + z_3 - z_1 z_3}{z_1 D_0 D_1}, \]

\[ A_{c, F_5} = \frac{(c - 1) D_0 D_1 D_2 D_3}{D_0 D_1 D_2 D_3}, \]

\[ B_{c, F_5} = \frac{(c - 1)(z_1 - 1)(a_2 (D_1 + D_2) + D_1 D_3)}{z_1 D_0 D_1 D_2 D_3}, \]

\[ C_{c, F_5} = \frac{(c - 1)(1 - z_2)(a_1 (D_1 + D_2) + D_1 D_3)}{z_2 D_0 D_1 D_2 D_3}. \]
where

\[ F = \frac{(c - 1)(1 - z_3)}{z_3D_0D_1D_2} \]

\[ E = \frac{(c - 1)(z_1 + z_2 - z_1z_2)(D_1 + D_2)}{z_1z_2D_0D_1D_2D_3} \]

\[ A_{b_1} = \frac{a_2(a_1z_1 + D_3) + (a_1z_1 + D_1 - a_2)D_3}{D_1D_3} \]

\[ B_{b_1} = \frac{(z_1 - 1)(a_2 + D_3)}{D_1D_3} \]

\[ C_{b_1} = \frac{a_2z_1(z_2 - 1)}{z_2D_1D_3} \]

\[ E_{b_1} = \frac{- z_1 + z_2 - z_1z_2}{z_2D_1D_3} \]

\[ A_{b_2} = \frac{a_1(a_2z_2 + D_3) + D_3(a_2z_2 + D_3 - b_1)}{D_2D_3} \]

\[ B_{b_2} = \frac{a_2(z_1 - 1)z_2}{z_1D_2D_3} \]

\[ C_{b_2} = \frac{(z_2 - 1)(a_1 + D_3)}{D_2D_3} \]

\[ E_{b_2} = \frac{- z_2(z_1 + z_2 - z_1z_2)}{z_2D_1D_3} \]

\[ A_{b_3} = \frac{a_1(a_2z_3 + D_3) + D_3(a_2z_3 + D_3 - b_1)}{D_2D_3} \]

\[ B_{b_3} = \frac{a_2(z_1 - 1)z_3}{z_1D_2D_3} \]

\[ C_{b_3} = \frac{(z_2 - 1)(a_1 + D_3)}{D_2D_3} \]

\[ E_{b_3} = \frac{- z_2(z_1 + z_2 - z_1z_2)}{z_1D_2D_3} \]

\[ F_{b_3} = \frac{- z_2(z_1 + z_2 - z_1z_2)}{z_1D_2D_3} \]

where

\[ D_0 = a_1 + a_2 - (c - 1) \]

\[ D_1 = a_2 + b_1 - (c - 1) \]

\[ D_2 = a_1 + b_2 + b_3 - (c - 1) \]

\[ D_3 = b_1 + b_2 + b_3 - (c - 1) \]

and

\[ D_1 + D_2 = D_0 + D_3 \]

The results of the differential reduction, Eq. (6), have the following form in this case:

\[ S(\tilde{z}) = S_0(\tilde{z})F_3(\tilde{a}; \tilde{b}; \tilde{c}; \tilde{z}) + \sum_{j=1}^{3} S_j(\tilde{z}) \frac{\partial}{\partial z_j} F_3(\tilde{a}; \tilde{b}; \tilde{c}; \tilde{z}) + \sum_{j=2}^{3} S_j(\tilde{z}) \frac{\partial^2}{\partial z_j \partial \tilde{z}_j} F_3(\tilde{a}; \tilde{b}; \tilde{c}; \tilde{z}) \]

where \( \tilde{m} \) is a set of integers, \( S_i \) and \( S_j \) are polynomials.

3.4. Exceptional values of parameters: \( F_0 \) and \( F_3 \)

It was pointed out in [8], that the subset of parameters for which the results of the differential reduction, Eqs. (53) and (84), have simpler forms, can be defined from the conditions

(i) that the hypergeometric function entering the l.h.s. of Eqs. (50)–(52), (73)–(78), is expressible in terms of simpler hypergeometric functions (\( F_0^{(s)} \) for \( F_0 \) and \( F_1 \) or \( F_3 \) for \( F_3 \) hypergeometric function);

(ii) that some of the coefficients entering the inverse differential relations are equal to zero (infinity).

For the hypergeometric functions \( F_0(\equiv F_0^{(s)} \) and \( F_3 \), the exceptional sets of parameters are listed in Table 1.
4. Mathematica based program for the differential reduction of $F_D$ and $F_3$ hypergeometric functions

In this section, we will present the Mathematica based programs $FdFunction$ and $FsFunction$ for the differential reduction of Horn-type hypergeometric functions $F_D$ of $r$ variables and $F_3$ of three variables. In particular, in the application to Lauricella functions $F_D$, the reduction algorithm, Eq. (6), has the following form:

$$R(x, y) F_D^{(r)} (a + m_c; b + m_b; c + m_c; \vec{z}) = \left[ P_0(\vec{z}) + P_1(\vec{z}) \partial_{\theta_1} + \cdots + P_r(\vec{z}) \partial_{\theta_r} \right] F_D^{(r)} (a; b; c; \vec{z}),$$  \hspace{1cm} (85)

where $m_b, m_a, m_c$ are sets of integers and $b, a, c$ denote the set of parameters. $R, P_i$ are some polynomial and $\theta_i = z_i \partial_{z_i}$. The differential reduction algorithm in application to the Lauricella–Saran function $F_3$ is:

$$R(\vec{z}) F_3 (\vec{a} + \vec{m}_a; \vec{b} + \vec{m}_b; c + \vec{m}_c; \vec{z}) = \left[ P_0(\vec{z}) + P_1(\vec{z}) \partial_{\theta_1} + P_2(\vec{z}) \partial_{\theta_2} + P_3(\vec{z}) \partial_{\theta_3} + P_{12}(\vec{z}) \partial_{\theta_1 \theta_2} + P_{13}(\vec{z}) \partial_{\theta_1 \theta_3} \right] F_3 (\vec{a}; \vec{b}; c; \vec{z}),$$  \hspace{1cm} (86)

where, again, $m_a, \vec{m}_b, \vec{m}_c, m_c$ denote sets of integers, $\vec{a}, \vec{b}, c$ sets of parameters, and $R, [P_i]$ some polynomials.

The program is freely available from [30] subject to the license conditions specified. The current version, 1.0, deals with non-exceptional values of a parameters only.

4.1. Package $FdFunction$

The package can be loaded in the standard way:

```mathematica
<< "FdFunction.m"
```

and it includes the following basic routines:

```
FdIndexChange[changingVector, parameterVector],
```

(87)

and

```
FdDiffSeries[\cdot \cdot \cdot],  
FsSeries[\cdot \cdot \cdot]  
```

(88)

The list "changingVector" in Eq. (87) provides the set of integers by which the values of parameters of the Lauricella function $F_D$ are to be changed, i.e., the vector $m_a$, $m_b$, $m_c$ in Eq. (85). The set of initial parameters of $F_D$ function are defined in the list "parameterVector" corresponding to the vector $a + m_a; b + m_b, c + m_c$ and arguments $\vec{z}$ in the l.h.s. of Eq. (85).

The structure of the output of $FdIndexChange[]$ is the following:

```mathematica
{[A_1, A_2, \ldots, A_{r+1}], \{parameterVectorNew\},
```

(89)

where

(i) "parameterVectorNew" is the set of new parameters of $F_D^{(r)}$ hypergeometric function;
(ii) $A_1, A_2, \ldots, A_{r+1}$ are the rational functions corresponding to the ratios of $P_0/R, P_1/R, P_2/R \ldots$ of functions entering in Eq. (85).

The functions $FdDiffSeries[]$ and $FsSeries[]$ are designed for the numerical evaluation of $F_D$ hypergeometric functions. They return the Taylor series of $F_D$ in its derivates, respectively:

```
FdDiffSeries[numberOfvariable, vectorInit, numbSer],
```

(90)

```
FsSeries[vectorInit, numbSer].
```

(91)

where

(i) "numberOfvariable" is the list of variable numbers for differentiation;
(ii) "vectorInit" is the set of Fd parameters;
(iii) "numbSer" is the number of terms in Taylor expansion.

Let us present a number of examples for the usage of $FdFunction$.

**Example 1.** Differential reduction of the hypergeometric function $F_D^{(2)}$ of two variables.

```
FdIndexChange[{-1, {1, 0}, 1}, {a, {b_1, b_2}, c, {z_1, z_2}}]
```

```
\left\{ \\
\begin{align*}
& c(-1 + z_2) - b_2 z_2 + z_1 (-1 + a + b_1 (-1 + z_2) + z_2 - a z_2 + b_2 z_2), \\
& 1 - z_2 - b_2 z_2 + z_1 (-1 + b_2 (-1 + z_2) + z_2 + b_2 z_2) + a(-1 + z_1 + z_2 - z_1 z_2), \\
& 1 - a(1 + z_1 (-2 + z_2)) - b_2 z_2 + c z_2 + z_1 (-2 - c + b_1 (-1 + z_2) + z_2 + b_2 z_2), \\
& -1 - a, \{b_1 + b_2, 1 + c, \{z_1, z_2\}\}
\end{align*}
\right\}
```

(92)

---

7 The programs have been tested for Mathematica version 8.0.
8 All functions in the package HYPERDIRR generate output without additional simplification. This is done for the maximum efficiency of the algorithm. To bring the output into a simpler form, we recommend to use in addition the command Simplify. All examples considered here have been treated with the command Simplify[]. Subsequent to the call of HYPERDIRR.
9 When $r = 2$, the Lauricella function $F_D$ coincides with the Appell function $F_1$ and the package AppellF1F4 [8] can be used for the differential reduction.
In an explicit form:

\[
F_D^{(2)}(a; b_1, b_2, c; z_1, z_2) = \frac{c(-1 + z_2) - b_2 z_2 + z_1(-1 + a + b_1(-1 + z_2) + z_2 - a z_2 + b_2 z_2)}{c(-1 + z_2)} \\
+ 1 - z_2 - b_2 z_2 + z_1(-1 + b_1(-1 + z_2) + z_2 + b_2 z_2) + a(-1 + z_1 + z_2 - z_1 z_2) \frac{\theta_1}{(-1 + a)c(-1 + z_2)} \\
+ 1 - a(1 + z_1(-2 + z_2)) - b_2 z_2 + c z_2 + z_1(-2 - c + b_1(-1 + z_2) + z_2 + b_2 z_2) \frac{\theta_2}{(-1 + a)c(-1 + z_2)} \\
\times F_D^{(2)}(a - 1; b_1 + 1, b_2, c + 1; z_1, z_2).
\]  

(93)

**Example 2.** Reduction of the hypergeometric function \(F_D^{(3)}\) of three variables.

\[
\texttt{FIndexChange} \left[ \left[-1, \{1, -1, 0\}, 0, \{a, \{b_1, b_2, b_3\}, c, \{z_1, z_2, z_3\}\}\right] \\
\left\{ \left\{ z_1 - 1 \right\} \frac{z_1 - 1}{z_2 - 1} \theta_1 + \frac{z_1(a - c + (b_2 - 1) z_2) - (a - c + b_2 - 1) z_2}{(a - 1)(b_2 - 1) (z_2 - 1) z_2} \theta_2 \right\} \\
\left\{ \frac{z_1 - 1}{(a - 1)(z_2 - 1)} \theta_3 \right\} \times F_D^{(3)}(a - 1; b_1 + 1, b_2, b_3; c, z_1, z_2, z_3).
\]

(94)

This has the explicit form:

\[
F_D^{(3)}(a; b_1, b_2, b_3; c; z_1, z_2, z_3) \\
= \left\{ \frac{z_1 - 1}{z_2 - 1} + \frac{z_1 - 1}{(a - 1)(z_2 - 1)} \theta_1 + \frac{z_1(a - c + (b_2 - 1) z_2) - (a - c + b_2 - 1) z_2}{(a - 1)(b_2 - 1) (z_2 - 1) z_2} \theta_2 \right\} \\
\times F_D^{(3)}(a - 1; b_1 + 1, b_2 - 1, b_3; c, z_1, z_2, z_3).
\]

**Example 3.** Reduction of the hypergeometric function \(F_D^{(5)}\) of five variables.

\[
\texttt{FIndexChange} \left[ \left[-1, \{0, 1, 0, 0, -1\}, 0, \{a, \{b_1, b_2, b_3, b_4, b_5\}, c, \{z_1, z_2, z_3, z_4, z_5\}\}\right] \\
\left\{ \left\{ z_1 - 1 \right\} \frac{z_2 - 1}{z_5 - 1} \theta_1 + \frac{z_2 - 1}{(a - 1)(z_5 - 1)} \theta_2 + \frac{z_2 - 1}{(a - 1)(z_5 - 1)} \theta_3 \right\} \\
\left\{ \frac{z_2 (a + (b_5 - 1) z_5 - c) - z_5 (a + b_5 - c - 1)}{(a - 1)(b_5 - 1) (z_5 - 1) z_5} \right\} \times \left\{ \frac{z_1 - 1}{(a - 1)(z_5 - 1)} \theta_4 \right\} \times F_D^{(5)}(a - 1; b_1 + 1, b_2 - 1, b_3, b_4, b_5, c, z_1, z_2, z_3, z_4, z_5).
\]

This has the explicit form:

\[
F_D^{(5)}(a; b_1, b_2, b_3, b_4, b_5; c; z_1, z_2, z_3, z_4, z_5) \\
= \left\{ \frac{z_2 - 1}{z_5 - 1} + \frac{z_2 - 1}{(a - 1)(z_5 - 1)} \theta_1 + \frac{z_2 - 1}{(a - 1)(z_5 - 1)} \theta_2 + \frac{z_2 - 1}{(a - 1)(z_5 - 1)} \theta_3 \right\} \\
\times F_D^{(5)}(a - 1; b_1 + 1, b_2 - 1, b_3, b_4, b_5 - 1, c, z_1, z_2, z_3, z_4, z_5).
\]

The hypergeometric function \(F_D\) is not built into the current version of Mathematica. The series representation of the hypergeometric function \(F_D^{(3)}\), Eq. (38), is implemented in our package. The functions \texttt{FdDiffSeries[]} and \texttt{FdSeries[]} allow to make numerical cross-checks of the results of the differential reduction. The corresponding examples for using these functions are all collected in the file \texttt{example-FdFunction.m}, which is available in [30].

4.2. Package \texttt{FsFunction}

Again, the program can be loaded in a standard way:

\[
\ll "FsFunction.m"
\]

and its structure and output are similar to the \texttt{FdFunction} package. The package \texttt{FsFunction} includes the following basic routines:

\[
\texttt{FsIndexChange}[\text{changingVector}, \text{parameterVector}].
\]  

(95)
Here, again, “changingVector” is the list of integers by which the values of the parameters of the function \( F_5 \) are to be changed, i.e., the vectors \( \vec{m}_a, \vec{m}_b, m_c \) in Eq. (86), while the set of initial parameters of the function \( F_5 \) is defined in the list “parameterVector” corresponding to the vector \( \vec{a} + \vec{m}_a; \vec{b} + \vec{m}_b, c + m_c \) and the arguments \( \vec{z} \) in the l.h.s. of Eq. (86).

The structure of the output of \texttt{FsIndexChange} is the following:

\[
\{A, B, C, D, E, F\}, \{\text{parameterVectorNew}\},
\]

where

(i) “parameterVectorNew” is the set of new parameters of the function \( F_5 \);

(ii) \( A, B, C, D, E, F \) are the rational functions corresponding to the ratios of \( P_0/R, P_1/R, P_2/R, P_3/R, P_{12}/R \) and \( P_{13}/R \) entering Eq. (86).

**Example 4. Reduction of \( F_5 \).**

\[
\text{FsIndexChange } \left\{ \{1, -1, 0, 0, 0, 0\}, \{a_1, a_2, b_1, b_2, b_3, c, z_1, z_2, z_3\} \right\}
\]

\[
\begin{align*}
\left\{ & a_1 + b_1 z_1 + b_2 + b_3 - c + 1, \frac{1 - z_1}{a_1 + b_2 + b_3 - c + 1}, \frac{z_2 (-a_1 - b_2 - b_3 + c - 1) - b_1 z_1 (z_2 - 1)}{(a_2 - 1) z_2 (a_1 + b_2 + b_3 - c + 1)}, \\
& z_3 (z_2 - z_1 (z_3 - 1)) (a_1 + b_2 + b_3 - c + 1) - z_3 (z_3 (z_3 - 1)) (a_2 - 1) z_3 (a_1 + b_2 + b_3 - c + 1) \right\}
\end{align*}
\]

This has the explicit form:

\[
F_5(a_1, a_2; b_1, b_2, b_3; c; z_1, z_2, z_3) = \left[ \frac{a_1 + b_1 z_1 + b_2 + b_3 - c + 1}{a_1 + b_2 + b_3 - c + 1} - \frac{1 - z_1}{a_1 + b_2 + b_3 - c + 1} \right] \theta_1
\]

\[
- \frac{z_2 (-a_1 - b_2 - b_3 + c - 1) - b_1 z_1 (z_2 - 1)}{(a_2 - 1) z_2 (a_1 + b_2 + b_3 - c + 1)} \theta_2
\]

\[
- \frac{z_3 (z_2 - z_1 (z_3 - 1)) (a_1 + b_2 + b_3 - c + 1) - z_3 (z_3 (z_3 - 1)) (a_2 - 1) z_3 (a_1 + b_2 + b_3 - c + 1)}{(a_2 - 1) z_3 (a_1 + b_2 + b_3 - c + 1)} \theta_3
\]

\[
\times F_5(a_1 + 1, a_2 - 1; b_1, b_2, b_3; c; z_1, z_2, z_3).
\]

**Example 5. Reduction of \( F_5 \).**

\[
\text{FsIndexChange } \left\{ \{1, 0, 0, 0, 1, 2\}, \{a_1, a_2, b_1, b_2, b_3, c, z_1, z_2, z_3\} \right\}
\]

\[
\left\{ \frac{b_1 z_1 (z_2 - 1) (-a_2 z_3 - a_1 + c) - z_2 (a_2 (z_3 (b_1 - c) + b_2 (z_3 - 1)) + c (c + 1)) + a_2 b_3 z_3 - a_2 c z_3 + c^2 + c}{c (c + 1) (z_2 - 1)} \right\}
\]

This has the explicit form:

\[
F_5(a_1, a_2; b_1, b_2, b_3; c; z_1, z_2, z_3)
\]

\[
= \left[ \frac{-b_1 z_1 (z_2 - 1) (-a_2 z_3 - a_1 + c) - z_2 (a_2 (z_3 (b_1 - c) + b_2 (z_3 - 1)) + c (c + 1))}{c (c + 1) (z_2 - 1)} \right] + \frac{a_2 b_3 z_3 - a_2 c z_3 + c^2 + c}{c (c + 1) (z_2 - 1)} + \frac{z_1 (a_2 z_3 + a_1 - c) - a_2 z_3 + c}{c (c + 1) (z_2 - 1)} \theta_1
\]

\[
- \frac{z_2 (a_2 - b_2 z_1 - b_3 z_1 + b_2 + c z_3 - 2 c - 1) - a_2 z_1 - b_1 z_1 (z_2 - 1) (z_3 - 1) + b_3 z_3 + c + z_3}{c (c + 1) (z_2 - 1)} \theta_2
\]

\[
- \frac{z_3 (z_3 (b_3 - c) + b_2 (z_3 - 1) + c) + b_1 z_1 (z_2 - 1) (z_3 - 1) - b_3 z_3 + c z_3 - c}{c (c + 1) (z_2 - 1)} \theta_3
\]

\[
= \frac{z_1 (z_2 (z_2 - 1) - z_2 (z_3 - 1)) (z_2 - 1) - z_2 (z_3 - 1)}{c (c + 1) (z_2 - 1)} \theta_1 \theta_2 - \frac{z_3 (z_3 (z_3 - 1)) (z_3 - 1)}{c (c + 1) (z_2 - 1)} \theta_1 \theta_3
\]

\[
\times F_5(a_1 + 1, a_2 - 1; b_1, b_2, b_3; c; z_1, z_2, z_3).
\]
where For the hypergeometric functions, where algorithms [11,12] are applicable, the result of the differential reduction and the corresponding examples are gathered in the file example-FsFunction.m available from [30].

5. On the construction of coefficients of the ε-expansion of Horn-type hypergeometric functions

For physical applications, the construction of analytical coefficients of the Laurent expansion of hypergeometric functions around particular values of parameters (integer, half-integer, rational) is necessary [11,12,31]. Many efforts have been made in the past in attempts to write results of the Laurent expansion in terms of multiple polylogarithms [32]. There are a number of different though entirely equivalent ways to describe the hypergeometric functions [3]:

(i) as a multiple series;
(ii) as a solution of a system of differential equations;
(iii) as an integral of the Euler type.

Each of these approaches can be used for the construction of the Laurent expansion of hypergeometric functions and each of them has some technical advantage or disadvantage in comparison with the other ones.

The most universal technique which does not depend on the order of differential equation is based on the algebra of multiple sums [11,12]. For the hypergeometric functions, where algorithms [11,12] are applicable, the result of the ε-expansion is automatically written in terms of multiple polylogarithms. The algorithms [11] have been implemented in a few packages [33]. However there are some technical problems with the extension of this approach to rational values of parameters and its application specific classes of hypergeometric functions.

The differential equation approach [18] allows to analyze arbitrary sets of parameters simultaneously and to construct the solution in terms of iterated integrals, but for any hypergeometric function the Pfaff system of differential equations should be constructed.

Purely numerical approaches [34] can be applied to arbitrary values of the parameters. However this technique typically does not produce stable numerical result around the region of singularities of hypergeometric function.

The integral representation is well suited for the construction of the ε-expansion for a limited number of types of hypergeometric functions [35].

For hypergeometric functions considered in this paper, the physically interesting cases correspond to the construction of the ε-expansion for the hypergeometric function $F_3$ around integer values of parameters [36], and for $F_5$ around half-integer values of parameters [15], respectively. For the Appell functions $F_{0j}$, the following theorem about structure of coefficients of all-order ε-expansion is valid:

**Theorem 2.** The ε-expansions of the Appell hypergeometric function $F_{0j}$ of r-variables and its first derivative have the following structure:

\[
F_{0j}(ae; \{b_j\}; 1 + ce; \bar{z}) = 1 + O \sum_{k=0}^{\infty} \epsilon_k \Phi_k \left(1, \bar{z}, \frac{z_1}{z_j}\right) e^k,
\]

\[
z_1 \frac{d}{dz_1} F_{0j}(ae; \{b_j\}; 1 + ce; \bar{z}) = \sum_{k=0}^{\infty} d_k \epsilon_k \left(1, \bar{z}, \frac{z_1}{z_j}\right) e^k,
\]

where $a$, $\{b_j\}$, $c$ are arbitrary numbers, $\epsilon$ is an arbitrary small parameter, $r_k$ and $d_k$ are the polynomial in $a$, $\{b_j\}$, $c$ and $\Phi_k$ and $\rho_k$ include only multiple polylogarithms of weight $k$.

The proof of this theorem is based on the explicit construction of the iterative solution of the system of differential equations. Up to functions of weight 3 the explicit value of the coefficients are the following

\[
F_{0j}^{(3)}(ae; \{b_j\}; 1 + ce; \bar{z}) = 1 + \sum_{k=0}^{\infty} \omega_0^{(3)}(\bar{z}) e^k,
\]

\[
z_1 \frac{d}{dz_1} F_{0j}^{(3)}(ae; \{b_j\}; 1 + ce; \bar{z}) \equiv z_1 \frac{\partial_b e^2}{1 + ce} F_{0j}^{(3)}(1 + ae; \{b_j\}; 2 + ce; \bar{z}) \equiv \sum_{k=0}^{\infty} \omega_1^{(3)}(\bar{z}) e^k,
\]

where

\[
\omega_0^{(3)}(\bar{z}) = \sum_{j=1}^{r} a_j L_{1,2} (z_j)
\]

\[
\omega_0^{(3)}(\bar{z}) = \sum_{j=1}^{r} (a - c + b_j) b_j S_{1,2} (z_j) - c \sum_{j=1}^{r} b_j L_{1,2} (z_j) + \sum_{i < j < l} b_i b_j \left[ L_{1,2} \left( \frac{z_i}{z_l}, z_j \right) + L_{1,2} \left( \frac{z_j}{z_l}, z_i \right) \right],
\]
and

\[ \frac{a_{j}^{(2)}(\bar{z})}{a} = -b_{j} \ln(1 - z_{j}), \]  
[104]

\[ \frac{a_{j}^{(3)}(\bar{z})}{a} = b_{j} \left[ \frac{1}{2} (a + b_{j} - c) \ln^{2}(1 - z_{j}) - c \text{Li}_{2}(z_{j}) + \sum_{k=1, k \neq j}^{r} b_{k} \text{Li}_{1,1}(\frac{z_{k}}{z_{j}}, z_{j}) \right], \]  
[105]

where \( j = 1,\ldots,r \)

\[ \text{Li}_{k_{1},k_{2},\ldots,k_{n}}(x_{1}, x_{2},\ldots,x_{n}) = \sum_{0 \leq m_{1} < m_{2} < \cdots < m_{n} < m_{0}} \frac{x_{1}^{m_{1}} x_{2}^{m_{2}} \cdots x_{n}^{m_{n}}}{m_{1}^{k_{1}} m_{2}^{k_{2}} \cdots m_{n}^{k_{n}}} \]  
[106]

denotes a multiple polylogarithm [32]. \( \text{Li}_{k}(x) \) is a classical and \( S_{1,2} \) a Nielsen polylogarithm, respectively.

Starting from \( a_{j}^{(4)}(\bar{z}) \), the result of the \( \varepsilon \)-expansion has more a complicated form. Since the \( F_{0}^{\varepsilon} \) hypergeometric function is symmetric with respect to the exchange of the variables \((b_{i}, z_{i}) \leftrightarrow (b_{j}, z_{j})\) we present here result only for \( a_{j}^{(4)}(\bar{z}) \). For the case of two variables, we have:

\[ \frac{a_{j}^{(4)}(x,y)}{ab_{1}} = \Delta_{0}^{(4)} \text{Li}_{1,1,1,1} (1, 1, x) + \left[ a b_{1} - c \Delta_{0} \right] \text{Li}_{2,1,1} (1, 1, x) - c \Delta_{0} \text{Li}_{1,2,1} (1, x) + c^{2} \text{Li}_{3} (x) \]

\[ + b_{2} \Delta_{0} \text{Li}_{1,1,1} (\frac{y}{x}, 1, 1) + b_{2} \Delta_{0} \text{Li}_{2,1,1,1} (1, \frac{y}{x}, 1, x) + b_{1} b_{2} \text{Li}_{1,1,1,1} (\frac{y}{x}, \frac{y}{x}, \frac{x}{y}, x) \]

\[ - c b_{2} \text{Li}_{2,1,1} (\frac{y}{x}, x) + (a - c) b_{2} \text{Li}_{2,1} (\frac{y}{x}, x), \]  
[107]

where

\[ \Delta_{0} = a - c + b. \]

Eq. (107) coincides with results of [33].

For \( r \)-variable case \((r \geq 3)\) the iterative solution is:

\[ \frac{1}{ab_{1}b_{r}} \left[ a_{j}^{(4)}(z_{1},\ldots,z_{r}) = a_{j}^{(4)}(z_{1},\ldots,z_{r-1},z_{r}) = 0 \right] \]

\[ = (a - c) \left[ \text{Li}_{1}(\frac{z_{r}}{z_{1}}) \text{Li}_{1,1} (1, z_{1}) - \text{Li}_{1,1,1} (1, z_{1}, \frac{z_{r}}{z_{1}}) - \text{Li}_{1,2,1} (1, z_{1}) \right] \]

\[ + b_{1} \text{Li}_{1,1,1} (1, \frac{z_{r}}{z_{1}}, z_{1}) - c \text{Li}_{1,2} (\frac{z_{r}}{z_{1}}, z_{1}) \]

\[ + b_{1} \left[ \text{Li}_{1,1,1} (\frac{z_{1}}{z_{r}}, \frac{z_{r}}{z_{1}}, z_{1}) + \text{Li}_{1,1,1} (\frac{z_{r}}{z_{1}}, 1, z_{1}) \right] \]

\[ + \sum_{j=1,j \neq 1}^{r} b_{j} \left[ \text{Li}_{1,1,1} (\frac{z_{j}}{z_{r}}, \frac{z_{r}}{z_{1}}, z_{1}) + \text{Li}_{1,1,1} (\frac{z_{r}}{z_{j}}, \frac{z_{j}}{z_{1}}, z_{1}) \right], \]  
[108]

and further simplifications can be done by the help of the stuffle relation:

\[ \text{Li}_{1}(\frac{z_{r}}{z_{1}}) \text{Li}_{1,1,1} (1, z_{1}) = \text{Li}_{1,1,1} (1, z_{1}, \frac{z_{r}}{z_{1}}) + \text{Li}_{1,1,1} (1, \frac{z_{r}}{z_{1}}, z_{1}) + \text{Li}_{1,1,1} (\frac{z_{r}}{z_{1}}, 1, z_{1}) \]

\[ + \text{Li}_{1,2} (1, z_{1}) + \text{Li}_{1,1} (\frac{z_{r}}{z_{1}}, z_{1}) \]  
[109]

As an example for the application of these coefficients, let us evaluate the function \( F_{0}^{(r)} (1; 1,\ldots,1; 2; z_{1},\ldots,z_{r}) \). Using the differential reduction algorithm, we get the following expression:

\[ F_{0}^{(r)} (1 + a; \{1 + b_{1}\}; 1 + c; z_{1},\ldots,z_{r}) = \frac{c}{a} \sum_{i=1}^{r} z_{i}^{r-2} \frac{\partial_{i}^{(r)}}{\Pi_{i=1,j \neq 1}^{r} (z_{i} - z_{j})} b_{i} F_{0}^{(r)} (a; \{b_{i}\}; c; z_{1},\ldots,z_{r}). \]  
[110]

After replacing, \( a \rightarrow a \varepsilon, b_{i} \rightarrow b_{i} \varepsilon, c \rightarrow 1 + c \varepsilon \) and taking limit \( \varepsilon \rightarrow 0 \), we get (see [37]):

\[ F_{0}^{(r)} (1; \{1\}; 2; z_{1},\ldots,z_{r}) = \sum_{i=1}^{r} \Pi_{i=1,j \neq 1}^{r} (z_{i} - z_{j}) \frac{a_{j}^{(2)}(\bar{z})}{ab_{i}} \]

\[ = - \sum_{i=1}^{r} \Pi_{i=1,j \neq 1}^{r} (z_{i} - z_{j}) \ln(1 - z_{i}). \]  
[111]
In the end, we also present the theorem\(^\text{10}\) about structure of the \(\varepsilon\)-expansion for two other Appell hypergeometric functions of two variables:

**Theorem 3.** The \(\varepsilon\)-expansions of the Appell hypergeometric functions \(F_2\) and \(F_3\) of 2-variables and their derivatives have the following structure:

\[
F_2(\{a, b; c, d\}; z_1, z_2) = 1 + \sum_{k=2}^\infty r_k \Phi_k \left( \left( \begin{array}{c} 1, \tilde{z}_1 \tilde{z}_2 \\ z_1 z_2 \end{array} \right) \right) \varepsilon^k ,
\]

(112a)

\[
z_1 \frac{d}{d\tilde{z}_1} F_2(\{a, b; c, d\}; z_1, z_2) = \sum_{k=2}^\infty d_k \rho_{k-1} \left( \left( \begin{array}{c} 1, \tilde{z}_1 \tilde{z}_2 \\ z_1 z_2 \end{array} \right) \right) \varepsilon^k ,
\]

(112b)

\[
z_1 z_2 \frac{d}{d\tilde{z}_1} \frac{d}{d\tilde{z}_2} F_2(\{a, b; c, d\}; z_1, z_2) = \sum_{k=2}^\infty q_k \sigma_{k-2} \left( \left( \begin{array}{c} 1, \tilde{z}_1 \tilde{z}_2 \\ z_1 z_2 \end{array} \right) \right) \varepsilon^k ,
\]

(112c)

\[
F_3(\{a_1, a_2, a_3; b_1, b_2, b_3\}; c, z_1, z_2) = 1 + \sum_{k=2}^\infty \tilde{r}_k \Phi_k \left( \left( \begin{array}{c} 1, \tilde{z}_1 \tilde{z}_2 \\ z_1 z_2 \end{array} \right) \right) \varepsilon^k ,
\]

(113a)

\[
z_1 \frac{d}{d\tilde{z}_1} F_3(\{a_1, a_2, a_3; b_1, b_2, b_3\}; c, z_1, z_2) = \sum_{k=2}^\infty \tilde{d}_k \tilde{\rho}_{k-1} \left( \left( \begin{array}{c} 1, \tilde{z}_1 \tilde{z}_2 \\ z_1 z_2 \end{array} \right) \right) \varepsilon^k ,
\]

(113b)

\[
z_1 z_2 \frac{d}{d\tilde{z}_1} \frac{d}{d\tilde{z}_2} F_3(\{a_1, a_2, a_3; b_1, b_2, b_3\}; c, z_1, z_2) = \sum_{k=2}^\infty \tilde{q}_k \tilde{\sigma}_{k-2} \left( \left( \begin{array}{c} 1, \tilde{z}_1 \tilde{z}_2 \\ z_1 z_2 \end{array} \right) \right) \varepsilon^k ,
\]

(113c)

where \(a, b, c\) are arbitrary numbers, \(\varepsilon\) is an arbitrary small parameter, \(r_k, d_k, q_k\) and \(\tilde{r}_k, \tilde{d}_k, \tilde{q}_k\) are polynomials in \(a, b, c, \Phi_k, \rho_k, \sigma_k\) and \(\tilde{\Phi}_k, \tilde{\rho}_k, \tilde{\sigma}_k\) include only multiple polylogarithms of weight \(k\).

**Example.**

\[
F_3(\{a_1, a_2, a_3; b_1, b_2, b_3\}; c, z_1, z_2) = 1 + \varepsilon^2 \sum_{j=1}^{3} a_j b_j \text{Li}_2(z_j) + \varepsilon^3 \sum_{j=1}^{3} a_j b_j \{(a_j + b_j - c)S_{1,2}(z_j) - c\text{Li}_3(z_j)\} + O(\varepsilon^4) .
\]

(114)

The \(\varepsilon\)-expansion for \(F_3\) around half-integer values of parameters will be considered in another paper.

6. Conclusion

The differential-reduction algorithm for Horn-type hypergeometric functions allows one to compare different functions in this class whose values for the parameters differ by integers. This proceeds in an entirely algorithmic manner suitable for automation in a computer algebra system. In this paper, we have presented the Mathematica-based programs \(\text{FdFunction}\) and \(\text{FsFunction}\) for the differential reduction of the generalized hypergeometric function \(F_0\) of \(r\) variables and the Lauricella–Saran hypergeometric function \(F_3\) of three variables. Both functions are related with one-loop massive Feynman diagrams and both belong to the class of Horn-type hypergeometric function of order two.

For hypergeometric function \(F_3\), Eq. (54), we have presented a detailed analysis of the dimension of the solution space (see **Theorem 1** in Section 3.3). Regarding the expansion of the hypergeometric functions in a small coefficient \(\varepsilon\) around specific values of their parameters, we have proven two theorems (**Theorems 2** and **3**). These concern the structure of the all-order \(\varepsilon\)-expansion of the hypergeometric functions, \(F_0^{(r)}\). \(F_2, F_3\) around integer values of parameters. As illustration, the first three coefficients (up to functions of weight 3) have been constructed explicitly.

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**Appendix**

For completeness, we present also the integral representation of functions \(F_0\) and \(F_3\):

\[
\frac{\Gamma(a)\Gamma(c - a)}{\Gamma(a)} F_0(a; b_1, \ldots, b_r; c_1, \ldots, c_q; z_1, \ldots, z_k) = \int_0^1 du u^{a-1}(1 - u)^{c-a-1}(1 - uz_1)^{-b_1} \cdots (1 - uz_n)^{-b_r}.
\]

(115)

\(^{10}\) The proof is straightforward and similar to the technique described in [18,38].
\[
\Gamma(a_1) \Gamma(a_2) \Gamma(c - a_1 - a_2) \frac{\Gamma(F_c)}{\Gamma(c)} = \int_{0 \leq u, v \leq 1} du dv u^{a_1 - 1} v^{a_2 - 1} (1 - u - v)^{c - a_1 - a_2 - 1} (1 - uz_1)^{-b_1} (1 - vz_2)^{-b_2} (1 - uz_3)^{-b_3}.
\]  
(116)

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