Tonal harmony, the topology of dynamical score networks and the Chinese postman problem

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Abstract. We introduce the concept of dynamical score networks for the representation and analysis of tonal compositions: a score can always be interpreted as a dynamical network where every chord is a node and each progression links successive chords. We demonstrate that in a tonal harmony context this network displays scale-free properties, and optimal (most economical) chord progressions can be found by solving a path optimization like the Chinese postman problem. Moreover, the dynamical network can be viewed as a time series of a non-stationary signal and as such can be partitioned for the automatic identification of key regions using well-established techniques for time series analysis and change point detection. Based on this interpretation we introduce a key-finding algorithm that does not rely on comparisons with pre-determined reference sets, as in the Krumhansl-Schmuckler model, or extensive corpora, as in machine-learning approaches. Finally, we show how the principles discussed in this work can be used to design a generative model of tonal compositional design. We illustrate the algorithms and discuss their accuracy with an analysis of L. van Beethoven’s string quartet Op. 127, n. 12 and J.S. Bach’s chorale BWV 267.

1. Introduction.

[1] In the article Topology of Networks in Generalized Musical Spaces, recently published on the Leonardo Music Journal, (Buongiorno Nardelli, Topology of Networks in Generalized Musical Spaces 2020) the author introduces the concept of harmony as a network representation of the musical structures built out of all possible combinatorial pitch class sets in any arbitrary temperament. Inspired by a long tradition of network representations of musical structures such as the circle of fifths (Heinichen 1969), the Tonnentz (Euler 1739), and recent works on the spiral array model of pitch space, (Chew 2014) the geometry of musical chords (D. Tymoczko, The Geometry of Musical Chords 2006) and generalized voice-leading spaces (Callender, Quinn and Tymoczko 2008) (D. Tymoczko, The Generalized Tonnetz 2012), the author interprets the harmonic structure of a composition as a large-scale complex network whose topological properties uncover its underlying organizational principles and demonstrates how classifications or rule-based frameworks (such as common-practice harmony, for instance) can be interpreted as emerging phenomena in a complex network system. Since the conclusions of that study serve as foundations for the present paper, let us review some of its principal results.

[2] Network analysis methods exploit the use of graphs or networks as convenient tools for modeling relations in large data sets. If the elements of a data set are thought of as “nodes”, then the emergence of pairwise relations between them, “edges”, yields a network representation of the underlying set. Similarly to social networks, biological networks and other well-known real-world complex networks, entire dataset of musical structures can be treated as networks, where each individual musical entity (pitch class set (pcs), chord, rhythmic progression, etc.) is represented by a node, and a pair of nodes is connected by a link if the respective two objects exhibit a certain
level of similarity according to a specified quantitative metric. Pairwise similarity relations between nodes are thus defined through the introduction of a measure of “distance” in the network: a “metric” (Albert and Barabási 2002). As in more well-known social or biological networks, individual nodes are connected if they share a certain property or characteristic (i.e., in a social network people are connected according to their acquaintances, collaborations, common interests, etc.) Clearly, different properties of interest can determine whether a pair of nodes is connected; therefore, different networks connecting the same set of nodes can be generated.

[3] In this paper we construct networks where nodes are the individual chords that can be extracted from the score, and edges are built between successive chords in the progression: nodes are connected if they appear as neighbors in the sequence. Naturally, nodes are visited numerous times, and the score evolution implies a directionality of the links. The networks are thus “directed”, and each edge will have a weight (strength) proportional to the times the link is visited.

[4] Given a network we can perform many statistical operations that shed light on the internal structure of the data. In this work we will consider only two of such measures, degree centrality and modularity class. (Barabasi and Posfai 2016) The degree of a node is measured by the number of edges that depart from it. It is a local measure of the relative “importance” of a node in the network. Modularity is a measure of the strength of division of a network into communities: high modularity (above 0.6 in a scale from 0 to 1) corresponds to networks that have a clearly visible community structure. (Zinoviev 2018). Isolating communities through modularity measures provides a way to operate within regions of higher similarity, and thus, as we will demonstrate in this paper, of closer harmonic content. Of particular interest in the context of this paper is the observation that tonal score networks exhibit scale-free properties, that is, their distribution of node degrees follows some form of power law. This implies that the network is characterized by a few nodes with many connections (hubs) while the remainder exhibits relatively few nodes. (Barabasi and Posfai 2016) As we will see in the following, this is a property that relates directly to some of the salient characteristics of tonal harmony.

[5] As summarized very effectively by the authors of (Moss, et al. 2019), tonal harmony compositions share four essential features: centricity, referentiality, directedness, and hierarchy. We argue here that all of these properties are completely captured by the network representation outlined above: centricity, the observation that tonal harmony is governed by a few central chords, (Neuwirth, et al. 2018) is captured by the degree distribution and the free-scale character of the score network, as discussed in (Buongiorno Nardelli, Topology of Networks in Generalized Musical Spaces 2020) and in this paper. Chords do not occur in random order, but they are introduced following syntactical rules: referentiality introduces the idea that chords occur within a hierarchical structure at both global (among tonal regions) and local level (between individual chords); (Schoenberg 1969) (Lerdahl and Jackendoff 1983) (Rohrmeier 2011) (D. Tymoczko, Root Motion, Function, Scale-degree: A Grammar for Elementary Tonal Harmony 2003) directedness, is the preference for asymmetric chord progressions in tonal music: (Hedges and Rohrmeier 2011) it has direct implications in the formation of listening expectations and the directionality of chord progressions towards a build-up on expectation and release; finally, hierarchy involves the construction of hierarchical relationships at every level of the composition, from the locality of chords to the global organization of tonal regions and keys within a single piece. (Schenker 1954)

[6] Let’s illustrate these key concepts and how they relate to the network representation of a given score, with an example. In Example 1, we show the score network of J.S. Bach’s chorale from the
Cantata “An Wasserflüssen Babylon”, BWV 267. The edges here are directional, to indicate the pcs progression in the piece (from now on, the term “pcs” and “chord” will be equally used to denote any vertical arrangement of pitch classes). A simple statistical analysis shows that the network has an average degree of 1.9 per node, and a modularity index of 0.5: the distribution of edges is relatively sparse, and many nodes are visited numerous times. One of the main result of such analysis is that the classification of nodes based on the modularity index clearly individuates tonal regions visited in the short piece (color-coded in Example 1): G Major/C Major (purple), A minor (cyan), D minor (green), E minor (orange). Although we will come back for a more thorough analysis of the chorale in Section 5, we can still make some important observations: first, a statistical measure of modularity (a well-defined mathematical measure), individuates broadly the tonal regions of the piece without any a priori knowledge of the harmonic structure (referentiality and hierarchy); second, that the distribution of nodes’ degrees gives us a way of identifying the individual tonal regions: the most important functional chords (tonic and dominant) are the ones that are visited more often (centricity); finally, the network is clearly directional: some progressions are more frequent than others, or directedness, a manifestation of the asymmetry of chord progressions in tonal harmony (Moss, et al. 2019).

Example 1. Score network of J.S. Bach’s chorale from the Cantata “An Wasserflüssen Babylon”, BWV 267. Nodes’ radius is proportional to their degree and colors individuate different modularity classes. Edges are directional (arrows) to reflect the harmonic progression of the composition, and their thickness is proportional to the number of times that edge is visited. In this figure and in the following, network graphics and basic statistics are obtained using the Gephi app at www.gephi.org.
[7] In the discussion above, we are taking a static perspective on the score network: we build its graph representation by collecting all the nodes (chords) and progressions (directional edges) from the whole piece. However, this is a procedure that ignores completely the fact that a piece of music is a temporal system, that is, each occurrence of any pcs is aligned according to a time scale, from the first to the last note. This observation leads to the interpretation of the temporal network as a time series allowing the use of well-established techniques for time series analysis and change point detection in the evaluation of the harmonic content of the composition. In particular, we will demonstrate how we can automatically identify key regions in tonal compositions without relying on comparisons with pre-determined perceptual reference sets, such as in the Krumhansl-Schmuckler key-finding algorithm, (Temperley 1999) or machine-learning of comprehensive corpora (Micchi, Gotham and Giraud 2020).

2. Methodology and data preparation.

[8] We will illustrate all the ideas and models using a particular example: the score of the string quartet in Eb Major, Op. 127 n. 12 by L. van Beethoven. However, all the algorithms and procedure developed here are generally applicable to any composition, even beyond those written in the tonal harmony idiom. We will make use of two principal software libraries for computational music analysis, both written in the Python language: MUSICNTWRK (at www.musicntwrk.com) and music21 (at https://web.mit.edu/music21). MUSICNTWRK is an open source python library for pitch class set and rhythmic sequences classification and manipulation, the generation of networks in generalized music and sound spaces, deep learning algorithms for timbre recognition, and the sonification of arbitrary data (Buongiorno Nardelli, MUSICNTWRK: data tools for music theory, analysis and composition, 2019). music21, developed at MIT (Cuthbert 2010), is an object-oriented toolkit for analyzing, searching, and transforming music in symbolic (score-based) forms of great versatility, whose modularity allows a seamless integration with MUSICNTWRK and other applications. All the results are compiled in a set of IPython Jupyter notebooks available for download as supplemental information to this paper.

[9] Scores are read in musicxml format by the readSCORE function of MUSICNTWRK, where their harmonic content (and other relevant information, like in which measure the chord is found) is extracted using the music21 parser and converter (using the “chordify” method). With this we obtain easily the full sequence of pcs, chord by chord, where each change to a new pitch results in a new chord. Upon “chordification”, each pcs is reduced to its normal form. While such “quantization” of pcs is quite adequate for the analysis of pieces with a harmonic movement where each vertical pcs plays some functional values (for instance in the corpus of J.S. Bach’s chorales), for compositions where there is more contrapuntal development, the number of individual pcs in the sequence can become very large, without providing necessarily more detailed information, since many such chords are only modifications via passing notes or vagrant harmonies. In order to circumvent this problem and make the analysis more manageable without losing any functional value, we have devised a “filtering” algorithm based on the cumulative measure of how many times an individual pcs appears in the sequence. All pcs with a frequency lower than a threshold are eliminated.

[10] To illustrate this initial data preparation process we will consider the first movement of the Beethoven string quartet Op. 127 and then extend the analysis to the full piece. We use the score from the Annotated Beethoven Corpus (ABC-1.0) (Neuwirth, et al. 2018) exported to musicxml
using the MuseScore scorewriter app (at www.musescore.com). The first movement of the quartet is comprised of 282 measures, for a total of 1298 individual pcs in the sequence. Of these 1298, 246 are the unique pcs combinations that are used throughout. If we number individually each unique pcs, we can build a map of the occurrences of the various chords and construct a time-series as the one displayed in Example 2.

Example 2. Time series of the full sequence of all the normal-ordered pcs extracted from the score of the first movement of the string quartet Op. 127 n. 12.

From this plot we can draw already some observations:

a. Repetition: since the onset of the composition, we observe a number of pcs that are repeated with higher frequency
b. Clustering: there are clear temporal clusters that appear during the time evolution
c. Incrementality: new pcs are continuously added for throughout the duration of the piece: new harmonic material is continuously added until the end.

These observations have a musical counterpart that will become more apparent in the following. Let’s suffice for now to say that repeating pcs are probably a manifestation of the concept of centrality, while clustering and incrementality are likely a manifestation of the referentiality and hierarchy properties discussed above.

[11] We exploit the above observations to curate the data by filtering out all the pcs that occur a number of times that lies below a certain threshold. From the analysis of the histogram of occurrences in Example 3, we observe that there is a clear separation between a few pcs that have high occurrences, and the rest of them with lower occurrences, many of which appear only once in the course of the piece. Note that here we do not make any assumption on which pcs is more “important” although, as we will demonstrate below, this is a clear quantitative manifestation of centrality in tonal music. With this observation in mind, we can now decide a filtering threshold and generate the set of curated data for further analysis. We choose to fix the threshold for filtering pcs to 10%, that is we ignore all pcs that have occurrences less that 10% of the max (here the reference is pcs n. 0 with 109 occurrences and that happens to be the Eb major triad [7,10,2], the key of the composition). Applying this procedure, we generate the “curated” sequence of pcs displayed in Example 4. Now, the number of time stamps (the successive occurrences of pcs) is
reduced to 752, with only 33 unique chords. By inspection of Example 4 we can confirm even more clearly the three observations made above: repetition, clustering and incrementality. We are now ready to use these data for a more complete analysis of the score network.

Example 3. Histogram of occurrences of each individual pcs. We plot here the number of times a given pcs appears in the composition. Again, we see a predominance of a few chords with high occurrence, another evidence of the centrality property.

Example 4. Time series of the filtered sequence of the normal-ordered pcs from the first movement of the string quartet Op. 127 n. 12.

3. The static score network.

[12] With these curated data we can now build the score network as outlined in the Introduction: each of the 33 unique pcs are the nodes of a directional network. The full score network is displayed in Example 5. Nodes are labelled with the chord name; their radii are proportional to
their degree (the number of connections each node has with the rest of the network) and are color-coded according to their modularity class. Similar to the case of Bach’s chorale discussed above, also here belonging to a particular modularity class implies an underlying tonal region: we can broadly identify the orange with the principal tonality, Eb major (the Eb major triad is central in this partition), C minor (green) and G minor (purple). However, a better understanding of the tonal dynamics of the piece can be gleaned only from the analysis of the changing topology during the time evolution of the network (see next Section). Here, we would like to extend our discussion with a quantitative analysis of the network topology and its scale-free properties.

**Example 5.** Score network of the first movement of the Beethoven’s string quartet Op. 127 n. 12.

[13] In Example 6 we show, in a loglog scale, the degree distribution of the score network. As it is clear from the figure, there are few nodes with very high degree (centrality) that act as hubs, while the vast majority of them follows a power law distribution: a characteristic of scale-free
networks. For a thorough discussion of the properties of scale-free networks and the implications for practical applications we refer the reader to (Barabasi and Posfai 2016), here we summarize some of the most important consequences of scale-freedom. In the context of tonal harmony, the most important consequence of the appearance of hub chords in a score network is that their presence decreases the number of hops from one node to another: distances in a scale-free network are smaller than the distances observed in networks where the node degrees follow a random (flat) distribution. Hub chords act as “pivots” within progressions and between them, allowing for seamless transitions between tonal regions (they satisfy both the referentiality and hierarchy properties). After all, this is a manifestation of the elementary fact well known to any harmony student: the same chord coexists in different keys and with different functional properties, I in C major is also V in F major and so on. Our results align perfectly with the conclusions of a previous study of the statistical characteristics of tonal harmony in Beethoven string quartets, where the power law behavior was inferred from the chord lexicon analyzed as an n-gram model as in Natural Language Processing (Moss, et al. 2019). However, while Moss et al. relied on a manually annotated corpus (Neuwirth, et al. 2018), our results stem from a purely statistical analysis of the raw pcs material.

Example 6. Loglog plot of the degree distribution in the score network. Dashed line is the best fit to a power law distribution, readily demonstrating that indeed the degree distribution in the network displays the scale-free property.

4. The dynamic score networks.

[14] One of the main goals of this paper is to introduce a novel approach to automatic identification of key regions in tonal compositions. The analysis conducted so far, however, is only able to give us a broad perspective on the characteristics of the piece, without elucidating the hierarchy of the different regions in the overall compositional design. In order to arrive to this, we need to look at the score not as a static network, but as a dynamic graph: a time evolution where the system explores different regimes. A practical way to achieve this is to look at the pcs data in Example 4 as a time series and exploit the analytical techniques that have been developed to tackle this kind of data systems. The observation of repetition, clustering and incrementality in [10], suggests that
time-series data can indeed provide the information on the different regimes, the tonal regions. In this perspective, we analyze the pcs sequence of Example 4 (the filtered set) as a non-stationary signal and apply various algorithms of change point detection and segmentation (Truong, Oudre and Vayatis 2020). Among those, we chose a binary change point detection algorithm designed to perform fast signal segmentation. It is a sequential approach: first, one change point is detected in the complete input signal, then the series is split around this change point, and the operation is repeated on the two resulting sub-signals (Fryzlewicz 2014). The results of this analysis on the filtered chord sequence is shown in Example 7a.

Example 7. a) Result of the segmentation algorithm on the score time-series. Alternating blue and pink areas show detection of change points. We used a binary segmentation algorithm with a kernelized mean change cost function and the ruptures Python package (at https://github.com/deepcharles/ruptures). b) Time series of the filtered sequence of the normal-ordered pcs after the segmentation. Vertical lines correspond to the breaking points in the time series. We identify 11 different regions (0 to 10).
Example 8. Panel 0 to 10, score network for the different sections as identified by the segmentation algorithm. Last bottom right panel, heat map of the similarity measure between the sub-networks.
Each segment (in alternating colors) corresponds to a chord range identified by the breaking points at the following measures: 32, 58, 63, 101, 120, 150, 166, 181, 187, 207. Each segment is sufficiently different in the pcs, and thus harmonic, content to be warrant a separate section, as seen in Example 7b, where we show the same data as in Example 4, but now we separate the different regions with vertical lines. Just from a visual inspection, we can observe that each of the segments display characteristics that set it apart from the contiguous ones. With these results in hand, we can now proceed to the analysis of each individual region as a network to capture the topology of pcs interactions in a window of time. The result of this process is summarized in Example 8, where we display in each panel (0-10) the network representation of the score in the range of measures indicated by the segmentation algorithm. In these figures we fix the node configuration as in the full static network of Example 5, and we update only the node size to reflect the change in degree. It is clear that each section displays strikingly different link topologies, where different groups of nodes are connected at different times.

Example 9. Loglog plot of the degree distribution in the score sub-networks compared with the full one from Example 6. The tails of the individual degree distributions are still largely following a power law behavior.

We can now apply the same analytical techniques that we considered for the full score network to these sub-networks. The subdivisions of the score network now acquire a more definite musical meaning, reflecting the harmonic evolution of the compositional design. Before me move to the quantification of the harmonic content is important to make some observations that are at the same time structural and stylistic. Given the finite nature of the score, some subnetworks are comprised of very few nodes (see for instance panel 2 and 8), an indication of regions of the composition where there is less harmonic movement in preparation of more dramatic transitions. Of the more articulated sections, the ones that last longer and thus are comprised of more nodes and edges, still display a structure centered around a few main nodes (hubs with larger degree) and their degree distribution still largely follows a power law distribution that reflects their scale-free properties. In Example 9 we show the degree distribution of sections 0, 1, 4, 7, 9 and 10, where this behavior is clearly evident. This is a quantitative evidence of the referentiality character of tonal harmony and of its hierarchical structure: we can say that these properties manifest in the self-similarity, or quasi-fractal, characteristics of the tonal progressions.
The determination of the score sections and their representation as sub-networks allows us to quantify the relationships between the different regions using well established algorithms for graph similarity measurements. Here we use a graph distance metric based on the maximal common subgraph, as originally proposed in (Bunke and Shearer 1998) and implemented in the GMatch4py library at https://github.com/Jacobe2169/GMatch4py. The results of the comparisons between the networks of the sections are shown as a heat map in the bottom right panel of Example 8. Before we analyze these results more closely, let’s relate the individual degree distribution of the subnetworks to the attribution of a particular tonal character (key) of the individual regions. Given the predominance of the tonic and dominant in tonal music, we assume that the most frequently visited pcs in each subnetwork are either major or minor triads or dominant seventh chords (see also (Pardo and Birmingham 2002)). With this proviso, we obtain the assignment of the tonal regions shown in Example 10. The partitions identified by our algorithm agree well with the expert analysis of the score from the ABC-1.0 data (Neuwirth, et al. 2018), as it will be reviewed more in depth in the discussion of the full score analysis in Section 5 below.

| Section | measures | prevalent_chord | region |
|---------|----------|-----------------|--------|
| 0       | 1-32     | [Eb, G, Bb]     | I      |
| 1       | 32-58    | [G, Bb, D]      | iii    |
| 2       | 58-63    | [G, Bb, D]      | iii    |
| 3       | 63-101   | [G, B, D]       | III    |
| 4       | 101-120  | [C, Eb, G]      | vi     |
| 5       | 120-150  | [C, E, G]       | VI     |
| 6       | 150-166  | [Eb, G]         | I      |
| 7       | 166-181  | [Eb, G, Bb]     | I      |
| 8       | 181-187  | [Eb, G, Bb]     | I      |
| 9       | 187-207  | [G, Bb, Db, Eb] | IV     |
| 10      | 207-282  | [Eb, G, Bb]     | I      |

Example 10. Assignment of relative tonal regions in the first movement of the string quartet Op. 127 n. 12.

A comparison between the key region assignments and the similarity map for the sub-networks shows a greater variability than one might naively expect: even regions that have the same tonal character might show a relatively low similarity score. Notable exceptions are regions 0, 8 and 10 (the regions in the predominant key of Eb major) that have a similarity score of about 0.8 (they show an 80% overlap). Of course, stylistic considerations play a central role here, since the observed variability serves as an expressive tool in the development of the composition. Moreover, also the definition of “prevalent chord” will have to be adapted to the particular style or historical period. Finally, the determination of the tonal regions allows for the straightforward application of computational analysis tools, such as the ones in music21, for the identification of the chord types and their roman numeral representation (Tymoczko, et al. 2019).
5. Score analysis, calibration and uncertainty quantification.

[19] So far, we have discussed the general principles behind our dynamical score network approach for harmonic analysis using the first movement of Beethoven’s string quartet Op. 127, n. 12 without optimizing the two most important free parameters that enter into the algorithm. Let’s summarize them here: first, in order to speed up the computational analysis we have the filtering of the chord sequence discussed in [11] and Example 3. This procedure allows for a reduction of the number of pcs in the time series, and in general does not influence greatly the outcome of the analysis if the threshold is kept lower than 10%. Second, and most important, are the parameters that enter into the segmentation algorithm discussed in [14] and Example 7. In the situation in which the number of change points is unknown, the binary change point detection algorithm uses a penalty parameter (>0) that increases (smaller penalty) or reduces (larger penalty) the number of breaking points. Controlling the penalty value allows us to zoom from the harmonic macro- to micro-structure the of the composition: if the cost of breaking the series is high (large penalty) we eventually obtain a single section that corresponds to the static network of [12] and Example 5. From there we can basically extract the overall key of the composition (Eb major in the mov. 1). Reducing the penalty, the number of breakpoints increases and so does the harmonic partitioning of the score. In all the examples so far, a penalty=3 was chosen.

### mov. 1

| measures | region (ABC) | region (this work) |
|----------|--------------|--------------------|
| 0-34     | Eb:b         | Eb:b              |
| 34-36    | i            | iii                |
| 36-39    | vi           | iii                |
| 39-63    | iii          | iii                |
| 63-75    | iii          | III               |
| 75-101   | III          | III               |
| 101-107  | III          | vi                |
| 107-117  | vi           | vi                |
| 117-124  | i            | vi                |
| 124-129  | i            | ii                |
| 129-133  | ii           | ii                |
| 133-135  | ii           | VI                |
| 135-147  | VI           | VI                |
| 147-150  | i            | VI                |
| 150-162  | i            | i                 |
| 162-166  | ii           | VI                |
| 166-167  | ii           | i                 |
| 167-187  | i            | i                 |
| 187-199  | IV           | IV                |
| 199-203  | ii           | IV                |
| 203-207  | i            | IV                |
| 207-282  | i            | i                 |

### mov. 2

| measures | region (ABC) | region (this work) |
|----------|--------------|--------------------|
| 1-4      | Ab:b         | Ab:b              |
| 4-27     | i            | i                 |
| 27-29    | i            | vi                |
| 29-32    | i            | V                 |
| 32-37    | i            | i                 |
| 37-38    | IV           | i                 |
| 38-48    | i            | i                 |
| 48-50    | V            | i                 |
| 50-61    | i            | i                 |
| 62-62    | i            | #V                |
| 62-81    | #V           | #V                |
| 81-87    | i            | i                 |
| 87-88    | V            | i                 |
| 88-99    | i            | i                 |
| 99-104   | IV           | IV                |
| 104-110  | #iii         | #iii              |
| 110-111  | #i           | #iii              |
| 111-130  | i            | i                 |
### Example 11.
Harmonic region assignments in the ABC compared to the automatic segmentation of the dynamical score network (this work).

[20] In order to explore quantitatively the accuracy of the partitioning of the dynamical score network, we have compared the result of our analysis with the expert annotations of the Annotated
Beethoven Corpus (ABC) (Neuwirth, et al. 2018) for all the movement of the string quartet. The results of this comparison are summarized in Example 11. The notation is the same as in the ABC. In order to obtain a fair comparison between the manual annotations and the automatic partitioning, we have adjusted the penalty parameter in order to obtain a similar number of partitions in the two sets. The values chosen are thus: 1.8 for mov. 1, 2.8 for mov. 2, 2.6 for mov. 3, and 2.6 for mov. 4. In Example 11 we show, for each movement, the range of measures that corresponds to different harmonic regions. On average, the agreement between the ABC and this work is greater than 80%, a remarkable result, considering a certain degree of arbitrariness in the assignment of roman numerals to chords in any composition: Mov. 1 matches to 81%, mov. 2 to 89%, mov. 3 to 86% and mov. 4 to 77%. Moreover, we did not refine the definition of prevalent chord introduced in [17] for the automatic key identification, so further refinements can indeed improve upon the present numbers. These results show an overall accuracy that is comparable or better than a key assignment with the classical perception-based algorithm of Krumhansl-Schmuckler (Temperley 1999).

[21] To conclude this section, let’s go back to the chorale BWV 267 of Example 1, and run the same algorithm. In general chorales are very short pieces with a small number of chords (BWV 267 contains only 17 measures and 123 chords of which 52 unique pcs), a characteristic that makes them relatively poor candidates for analysis based on statistical inference. However, even in this case, our algorithms identify tonal regions with an accuracy of 85% relative to expert annotations by Jones, Tymoczko and Robb, as provided in the Bach chorale corpus of the music21 package (Cuthbert 2010) (Tymoczko, et al. 2019) proving once more the accuracy and flexibility of the method.

6. Bach, Euler and the Chinese Postman Problem

[22] The identification and classification of the dynamical score network as a convincing representation of a composition suggests avenues for the design of autonomous systems for the generation of harmonic progressions. Let’s reexamine the score network from Bach’s chorale BWV 267 and let’s ask ourselves the following question: given this graph, how many links do I have to visit in order to complete a full circuit and travel all the chord progressions at least once? In other words, how “economical” is Bach’s solution to use the harmonic material at his disposal? This is the well-known Chinese postman optimization problem: to find a shortest closed path or circuit that visits every edge of a graph. When the graph has a Eulerian circuit (a closed walk that covers every edge once), that circuit is an optimal solution. Otherwise, the optimization problem is to find the smallest number of graph edges to duplicate (or the subset of edges with the minimum possible total weight) so that the resulting multigraph does have a Eulerian circuit (Edmonds and Johnson 1973).1 If we analyze the network of Example 1 in this context, Bach completes his “Eulerian circuit” in 117 steps where the naked network has 52 nodes and 93 undirected links (101 directed). This means that in Bach’s circuit 24 edges have been duplicated. We can run the Chinese postman optimization algorithm both on the naked undirected network, and on the directed version that takes into account the “direction” of the individual progressions (i.e. V to I or I to V). The results are striking in the case of the directed network, the minimum

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1 It is interesting to note here that both Bach and Euler, the Swiss mathematician who is credited with the foundation of modern topology and network theory, were at the Prussian court of Frederik II in the mid 1740s (Shavin 2000). A coincidence that I find rather remarkable in the context of this study.
circuit length is 110, only 7 links shy of the Bach’s solution! The undirected case gives a shorter circuit, 101 links, but still very close to the original. A close inspection of the paths on the score network can be done in Example 12. In the directed case the solution is built imposing \textit{a priori} the \textit{directedness} of the underlying tonal harmony, and the minimal path overlaps greatly with the Bach’s version. It is important to note that most loops in the graphs can be visited clockwise or counterclockwise without changing the total path length, thus generating multiple “minimal length paths”. This allows to retrieve the analogy with the Bach’s original straightforwardly. In the undirected network the algorithm is not constrained with \textit{a priori} directionality; nonetheless the resulting circuit is still very close to the previous ones.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{network_example.png}
\caption{Example 12. Score network of BWV 267 with the three circuits: the original Bach’s chord progression, a Eulerian path on the directed network and a Eulerian path on the undirected network. The paths start always from the G major triad and return to it.}
\end{figure}

7. \textit{Towards a generative model of tonal compositional design.}

[23] To summarize, the main results of this whole study are: a score can always be interpreted as a dynamical network; in a tonal harmony context this network (and its sub-networks) displays scale-free properties; and optimal (most economical) chord progressions can be found by solving a path optimization problem on said network. In the following we will expand on how these three principles lay the foundation for a generative model of tonal compositional design.

[24] Looking at the discussion in [22] from a different perspective, the very geometrical characteristics of the optimal path point to the origin of the \textit{directedness} and \textit{centrality} in tonal compositions as constraints that emerge naturally from the network topology. In particular, the presence of chord “hubs” forces the path to return to these nodes in order to complete the circuit, thus naturally enforcing the existence of harmonic centers. From here, one can envision a compositional design process comprised of the following steps: the generation of a scale-free networks with the desired topological characteristics, the assignment of a chord distribution to the
nodes, and the determination of the optimal path. Of these steps, the last one is well defined once the network has been generated, so we will now focus on the first two.

[25] There are many generators for scale-free networks (Hagberg, Schult and Swart 2008). In this study we use the original model proposed by Barabasi and Albert (Barabasi and Albert 1999). It incorporates two important general concepts, growth and preferential attachment: growth means that the number of nodes in the network increases over time; preferential attachment means that the more connected a node is, the more likely it is to receive new links. Nodes with a higher degree have a stronger ability to grab links added to the network. Both characteristics ensure that the generated network displays scale-free properties and fall into the general taxonomy of tonal harmony, where centrality is a representation of preferential attachment and growth is intrinsic to the dynamical score network concept. It is then straightforward to generate random scale-free networks that can be interpreted as the skeleton of a dynamical score networks. In Example 13 we show one such network generated on 52 nodes (same number of nodes as in BWV 267).

Example 13. Scale-free network generated with the Barabasi-Albert algorithm including the chord-node assignments and one of the directional Eulerian paths.

[26] We now get to the most complex step: the assignment of specific chords to each of the nodes. This is definitely a very arbitrary task that can be undertaken in a myriad of ways depending on the choices of the composing agent (being it human or machine). All the characteristics of tonality come into play here: the most obvious, centrality, coincides with the existence of chord hubs, so that highest degree nodes can be assigned to the most prevalent chords (tonic, dominant, etc.) according to one of the numerous fitness models (see for instance (Pardo and Birmingham 2002), (Navarro-Caceres, Caetano and Bernardes 2020)), functionality scores (Cuthbert 2010) in the preferential attachment, or according to degree distributions in existing corpora (Moss, et al. 2019). As illustration, in the network of Example 13 we have assigned to the nodes the same degree distribution of BWV 267, where we match the lists after sorting from the highest to the lowest degree. This gives the following for the six highest degree nodes: GBD (degree 13 in BWV and
13 in the Barabasi-Albert (BA), DF#A (11 and 13, respectively), CEG (10 and 11), ACE (9 and 11), EGB (8 and 8), and DFA (8 and 6). Finally, we run the Chinese postman algorithm to generate a dynamical score network, also depicted in Example 13. A realization of the first cycle of 44 chords in the Eulerian path is shown in Example 14. It is important to note that besides the highest degree nodes, all other chords are distributed randomly, without taking into account any directionality or hierarchy in the pcs assignment (two thirds of the nodes have degree 2, so the assignment depends on the random distribution of these nodes in the BA network). Nonetheless, the result is still strongly tonal, reinforcing once more the dominance of centrality in the perception of tonality.

![Barabasi-Albert dynamical score network (pcs 0-44)](image)

**Example 14 and Audio example 1.** A realization of the first cycle of 44 chords in the Eulerian path built on the Barabasi-Albert network of Example 13 including a MIDI realization as mp3. None of the pcs in the sequence have been altered in any way.

8. **Conclusions and acknowledgements**

[27] In conclusion, with this study we have built on the concept of network representation of musical spaces and introduced the idea of a composition as a dynamical score network, we have developed an efficient key-finding algorithm that relies only on the topology of the network, we characterized the full chord progression of the composition as a route optimization problem (the Chinese postman) and shown how these principles can be used to design a generative model of tonal compositional design.

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Works Cited
Albert, R., and A.L. Barabási. 2002. "Statistical mechanics of complex networks." Reviews of Modern Physics 74 (1): 47–97.
Barabasi, A.L., and R. Albert. 1999. "Emergence of scaling in random networks." Science 286: 509-512.
Barabasi, A.L., and M. Posfai. 2016. Network Science. Cambridge: Cambridge University Press.
Bunke, H., and K. Shearer. 1998. "A graph distance metric based on the maximal common subgraph." Pattern Recognition Letters 19 (3-4): 255-259.
Buongiorno Nardelli, M.. 2019. "MUSICNTWRK: data tools for music theory, analysis and composition.," Computer Music Multidisciplinary Research. Marseille: Springer Lecture Notes in Computer Science.
Buongiorno Nardelli, M.. 2020. "Topology of Networks in Generalized Musical Spaces." Leonardo Music Journal Just Accepted: 1-9.
Callender, C., I. Quinn, and D. Tymoczko. 2008. "Generalized Voice-Leading Spaces." Science 320: 346.
Chew, E. 2014. Mathematical and Computational Modeling of Tonality. Springer US.
Cuthbert, M.S. and Ariza, C. 2010. "Music21: A Toolkit for Computer-Aided Musicology and Symbolic Music Data." International Society for Music Information Retrieval. 637-642.
Edmonds, J., and E.L. Johnson. 1973. "Matching, Euler tours and the Chinese postman." Mathematical Programming 5 (1): 111-114.
Euler, L. 1739. Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae. Saint Petersburg Academy.
Fryzlewicz, P. 2014. "Wild binary segmentation for multiple change-point detection." The Annals of Statistics 42 (6): 2243–2281.
Hagberg, E.A., D.A. Schult, and P.J. Swart. 2008. "Exploring network structure, dynamics, and function using NetworkX." Edited by Travis Vaught and Jarrod Millman. Proceedings of the 7th Python in Science Conference (SciPy2008). Pasadena, CA.
Hedges, T., and M. Rohrmeier. 2011. "Exploring Rameau and Beyond: A Corpus Study of Root Progression Theories." In Mathematics and Computation in Music. Lecture Notes in Artificial Intelligence (6726), by C. Agon, M. Andreatta, G. Assayag, J. Bresson and J. Manderau, 334-337. Berlin: Springer.
Heinichen, J. D. 1969. "Der General-Bass in der Composition." (G. Olms).
Lerdahl, F, and R. S. Jackendoff. 1983. A Generative Theory of Tonal Music. Cambridge: MIT Press.
Micchi, G., M. Gotham, and M. Giraud. 2020. "Not All Roads Lead to Rome: Pitch Representation and Model Architecture for Automatic Harmonic Analysis." Transactions of the International Society for Music Information Retrieval 3 (1): 42-54.
Moss, F.C., M. Neuwirth, D. Harasim, and M. Rohrmeier. 2019. "Statistical characteristics of tonal harmony: A corpus study of Beethoven’s string quartets." PLOS One 1-16.
Navarro-Caceres, M., M. Caetano, and G. Bernardes. 2020. "Objective Evaluation of Tonal Fitness for Chord Progressions Using the Tonal Interval Space." In Artificial Intelligence in Music, Sound, Art and Design, by Juan Romero, Ekart Aniko and Joao Correia, 150-164. Springer International Publishing.
Neuwirth, M., D. Harasim, F. C. Moss, and M. Rohrmeier. 2018. "The Annotated Beethoven Corpus (ABC): A Dataset of Harmonic Analyses of All Beethoven String Quartets." Frontiers Dig. Human. 5 (16).
Pardo, B., and W.P. Birmingham. 2002. "Algorithms for chordal analysis." *Computer Music Journal* 26 (2): 27-49.
Rorhmeir, M. 2011. "Towards a Generative Syntax of Tonal Harmony." *J. Mathematics & Music* 5 (1): 35-53.
Schenker, H. 1954. *Harmony*. Cambridge: MIT Press.
Schoenberg, A. 1969. *Structural Functions of Harmony*. W. W. Norton & Company.
Shavin, D. 2000. "Thinking Through Singing." *Fidelio* 9 (4): 60-84.
Temperley, D. 1999. "What's Key for Key? The Krumhansl-Schmuckler Key-Finding Algorithm Reconsidered," *Music Perception* 17 (1): 65-100.
Truong, C., L. Oudre, and N. Vayatis. 2020. "Selective review of offline change point detection methods." *Signal Processing* 167: 107299.
Tymoczko, D. 2003. "Root Motion, Function, Scale-degree: A Grammar for Elementary Tonal Harmony." *Musurgia* X (304): 35-64.
Tymoczko, D. 2012. "The Generalized Tonnetz." *Journal of Music Theory* 56 (1): 1-52.
Tymoczko, D. 2006. "The Geometry of Musical Chords." *Science* 313: 72-75.
Tymoczko, D., M. Gotham, M.S. Cuthbert, and C. Aritza. 2019. "The Romantext Format: A flexible and standard method for representing Roman numeral analyses." *International Society for Music Information Retrieval Conference (ISMIR 2019).* Delft.
Zinoviev, D. 2018. *Complex Network Analysis in Python: Recognize - Construct - Visualize - Analyze – Interpret.* Pragmatic Bookshelf.