Shell-model versus clustering effects in heavy nuclei

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Abstract. We analyze the interplay between the shell-model and \( \alpha \)-clustering in heavy nuclei. In particular, we show that the standard shell-model approach underestimates \( B(E2) \) values and \( \alpha \)-decay widths in \( \alpha \)-like nuclei. By using surface Gaussian-like components in the single particle orbitals, mocking four body correlations, it becomes possible to enhance both electromagnetic and \( \alpha \) transition probabilities up to the experimental values in the case of \(^{212}\)Po.

1. Introduction
Nuclei are typical finite size mesoscopic systems with a specific interplay between volume and surface effects. One important property concerns the nuclear stability with respect to the \( \alpha \)-particle and electromagnetic emission. In the simplest approach of the \( \alpha \)-decay process the preformed \( \alpha \)-particle was considered a cluster moving around the core since the time when Gamow proposed the penetrability theory [1]. Later on, the cluster was built microscopically from two proton and two neutron single particle configurations [2, 3]. Anyway, the decay widths were by several orders of magnitude smaller than the experimental values. In the late 1970’s it was shown that the inclusion of many configurations increases the value of the absolute decay width by more than four orders of magnitude [4]. Yet, the calculated widths were two orders of magnitude too small. Only a theory where a combined shell- and cluster-model configurations were considered could reproduce the absolute decay width [5, 6]. This may justify the so-called extreme cluster models, where the \( \alpha \)-particle moves around the core [7, 8, 9].

In this presentation we will show that a mixed shell-model and cluster representation is able to describe not only \( \alpha \) decay widths, but also electromagnetic transitions.

2. Shell-model description of decay processes
In the standard description of low-lying states the nuclear many-body Hamiltonian can be approximated by a mean-field plus a residual two-body part

\[
H \approx H_{mf} + H_{res} .
\]

(1)

The mean field can be estimated within the self-consistent Hartree-Fock-Bogoliubov scheme, but a good approximation is given by the Woods-Saxon ansatz [10]. The spin of the nucleon is strongly coupled to the orbital motion in the mean field. Therefore the mean field part generates a single particle shell structure of the following form

\[
\psi_{eljm}^{(SM)}(r,s) \rightarrow a_{eljm}^{\dagger}|0\rangle ,
\]

(2)
characterized by the eigenvalue (\(e\)), orbital angular momentum (\(l\)), total spin (\(j=l+s\)) and its projection (\(m\)). The single particle levels in the mean field potential are clustered in major shells characterized by the principal harmonic oscillator quantum number. Shell structure is a volume feature which is characteristic for mesoscopic finite size systems.

The most important residual interaction describing ground state is given by the monopole pairing force

\[
H_{\text{res}} \approx -G \sum_j \left( \sum_{m=-j}^j a_{jm}^\dagger a_{j-m}^\dagger \right) \sum_{j'=-j'}^{j'} \left( \sum_{m'=-j'}^{j'} a_{j'-m'} a_{j'=-m'} \right),
\]

where we denoted by \(G\) the strength of the pairing interaction and introduced the short-hand index notation \((e,l,j) \rightarrow j\). It turns out that the residual interaction is peaked on the nuclear surface, as can be shown in figure 1, where we plotted radial center of mass two-particle wave functions for protons (solid line) and neutrons (dashed line) in \(^{220}\text{Ra}\). This justify the use of some schematic interactions in describing spectroscopic properies, like the surface delta interaction. Surface effects are also characteristic for mesoscopic systems.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Center of mass two particle wave function for protons (solid line) and neutrons (dashed line) in \(^{220}\text{Ra}\).}
\end{figure}

Let us consider an \(\alpha\)-decay process

\[
P \rightarrow D + \alpha.
\]

The amplitude to find an \(\alpha\)-particle in a shell-model wave function of the parent nucleus is given by the following overlap

\[
F(R_\alpha) = \langle \Psi_D \psi_\alpha | \Psi_P \rangle,
\]
depending on the $\alpha$-daugher distance $R_\alpha$. Here, we denoted by $\Psi_{D/P}$ the many-body wave function of the daughter/parent nucleus and by $\psi_\alpha$ the intrinsic wave function of the $\alpha$-particle. Its radial part is a product of two-proton and two-neutron Gaussians, while the spin component is a singlet state [2]. By using as a residual interaction the pairing force, the formation amplitude becomes also peaked on the nuclear surface, as can be seen from figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{$\alpha$-particle formation probability in $^{220}$Ra versus the $\alpha$-daughter radius.}
\end{figure}

For spherical emitters the decay width is defined in terms of the monopole scattering amplitude $N$ [11] as follows

$$\Gamma = \hbar v |N|^2 = \hbar v \left| \frac{R_\alpha F(R_\alpha)}{G_0(R_\alpha)} \right|^2, \quad (6)$$

where $v$ is the velocity of the $\alpha$-daugher system at infinity and $G_0(R_\alpha)$ is the monopole irregular Coulomb function.

Of course, the above ratio does not depend on the radius, because in principle both internal and external wave functions should satisfy the same Schrödinger equation. Actually the formation amplitude $F(R_\alpha)$ is estimated independently from the external wave function $G_0(R_\alpha)/R_\alpha$ and their ratio depends on the radius, but this dependence is rather week around the geometrical touching point, as can be seen from figure 3.

Traditionally the decay width is rewritten

$$\Gamma = 2P_0 \gamma^2, \quad (7)$$
in terms of the penetrability and reduced width squared

$$P_0 = \frac{\kappa R_\alpha}{|G_0(R_\alpha)|^2}$$
$$\gamma^2 = \frac{\hbar^2}{2\mu R_\alpha} |F(R_\alpha)|^2, \quad (8)$$
where $\kappa$ is the momentum and $\mu$ the reduced $\alpha$-daughter mass.

It turns out that the residual pairing interaction is not able to reproduce the absolute value of the decay width. Indeed, in figure 3 we give by solid line the logarithm of the ratio between the computed decay width according to Eq. (6) and the experimental value as a function of the radius, for a typical decay process $^{220}\text{Ra} \rightarrow ^{216}\text{Rn} + \alpha$. One sees that the result underestimates the experiment by four orders of magnitude.

![Figure 3](image)

**Figure 3.** Logarithm of the ratio between the theoretical and experimental $\alpha$-decay width for $^{220}\text{Ra} \rightarrow ^{216}\text{Rn} + \alpha$ as a function of the $\alpha$-daughter radius (solid line). By dashed and dot-dashed lines are given the two terms in Eq. (6).

3. **Cluster description of decay processes**

The binding energy per nucleon for an $\alpha$-particle is much larger with respect to their neighbours. Due to a smaller nuclear density at surface, an $\alpha$-cluster structure is energetically more favourable in this region. Therefore, clustering is a surface effect and it can be simulated by assuming that the $\alpha$-particle moves in a pocket-like potential centered at some radius $R_0$ on the nuclear surface. By considering the Q-value as the first eigenvalue corresponding to a shifted harmonic oscillator wave function

$$|F(R_0)|^2 = A^2 e^{-\beta(R_0 - R_0)^2},$$

one easily obtains an universal law for the reduced width squared (8) [12]

$$\log_{10} \gamma^2(R_B) = \frac{\log_{10} e^2}{\hbar \omega} V_{\text{frag}}(R_B) + \log_{10} \frac{\hbar^2 A^2}{2 e \mu R_B},$$

depending on the fragmentation potential, defined as the difference between the Coulomb barrier and Q-value, i.e.

$$V_{\text{frag}}(R_B) = V_C(R_B) - Q,$$
where $R_B$ is the radius of the Coulomb barrier. This law does not depend upon the pocket shift radius $R_0$ and it is satisfied with a good accuracy not only by all $\alpha$-decays between ground states, but also by proton and heavy cluster reduced widths [12]. This law is a direct evidence that a cluster should exist on the nuclear surface with the amplitude $A$.

![Figure 4](image)

**Figure 4.** (a) Gaussian distribution centered on the nuclear surface (solid line) and various expansion terms (dot-dashed lines).
(b) Expanssion coefficients of a Gaussian centered on the nuclear surface given by Eq. (14).

Thus, in order to properly describe clustering properties at the single particle level one has to consider two components of the single particle wave function, i.e.

$$\psi_l(r) = N_l^{(SM)} \psi^{(SM)}_l(r) + N_l^{(clus)} \psi^{(clus)}_l(r) \; .$$

(12)

The shell model part has a standard spherical harmonic oscillator representation

$$\psi_l^{(SM)}(r) = \sum_n b_n R_{nl}^{(\beta)}(r) \; ,$$

(13)

where $\beta = M_N \omega / \hbar$ is the harmonic oscillator parameter and the principal quantum number has the values $N = 2n + l \leq 6$.

For the cluster part one uses a wave function of the shifted oscillator

$$\psi_l^{(clus)}(r) = e^{-\beta_c (r-R_0)^2/2} = \sum_n c_n R_{nl}^{(\beta)}(r) \; ,$$

(14)

where $N$ is spread around a larger value $N \sim 8 - 10$, as can be seen from figure 4. Notice that in Ref. [6] the harmonic oscillator parameter of the cluster part has a smaller value, in order to reduce the dimension of the single particle basis (12). Here all coefficients are found by a diagonalization procedure of the Woods-Saxon mean field.
Figure 5. (a) $\alpha$-particle formation probability for $^{212}\text{Po} \rightarrow ^{208}\text{Pb} + \alpha$. Dashed line: shell model, Solid line: shell model + $\alpha$-cluster.
(b) Logarithm of the ratio between the decay width and the experimental value as a function of the daughter-$\alpha$-particle radius

The typical example of a heavy nucleus where the $\alpha$-clustering effects are very important is $^{212}\text{Po}$. Here, the structure of low-lying states can be explained in terms of two proton and two neutron orbitals above the double magic inert core $^{208}\text{Pb}$ [13]. Anyway, by using the standard components of the two protons and two neutrons the decay width is underestimated by two orders of magnitude. It is necessary to consider the second cluster component in Eq. (12), with $N_l^{(clus)} = 0.3$, in order to describe the absolute value of the $\alpha$-decay width, as can be seen in figure 5 (b) (solid line). In spite of the fact that in figure 5 (a) the difference between the formation probability within the shell model approach (dashed line) and mixed approach (solid line) is rather small, the enhancement of the decay width is about two orders of magnitude. Notice that the same value was obtained in Ref. [5], by using a diagonalisation procedure of the residual interaction.

This picture is also supported by analyzing electromagnetic transitions. The electric transition probability of the multipolarity $\lambda$ is proportional to the reduced matrix element of the transition operator squared, i.e.

$$B(E\lambda : J \rightarrow J') = \frac{1}{2J+1} |\langle J' | \hat{T}_\lambda | J \rangle|^2,$$

(15)

where $|J\rangle$ and $|J'\rangle$ denote initial and final states and the transition operator is proportional to spherical harmonics

$$\hat{T}_{\lambda \mu} = r^\lambda Y_{\lambda \mu}.$$

(16)

The radial part of the quadrupole transition matrix element is proportional to the principal
Figure 6. $B(E2 : J \rightarrow J - 2)$ values (W.u.). Shell-model values in $^{210}$Po and shell-model + cluster values in $^{212}$Po (solid lines). By dashes are given experimental values.

quantum number

$$\langle n_1 l_1 | r^2 | n_2 l_2 \rangle \sim N = 2n + l .$$

(17)

As it is shown in figure 4, the clustering components are centered around the region $N \sim 8 - 10$ and therefore they should enhance transition probability.

It is known that $B(E2)$ values are satisfactory described within the shell model for $^{210}$Pb, but they are by one order of magnitude less than the experimental values in $^{212}$Po. They are of the same order of magnitude as those in $^{210}$Pb, given by the lower solid line in figure 6 [14]. In order to explain this discrepancy we used the mixed representation (12) with the same $\alpha$-cluster amplitude. In figure 6 one indeed obtains a very good agreement of the computed values (upper solid line) with respect to experimental data (upper dashed line).

This is a very convincing evidence in favour of the $\alpha$-clustering structure of this nucleus.

4. Conclusions
Electromagnetic and $\alpha$-transitions in heavy nuclei cannot be described within the standard shell-model approach. The analysis of experimental $\alpha$-decay reduced widths have shown that an $\alpha$-cluster component centered on the nuclear surface is necessary in order to describe decay processes. In heavy nuclei this cluster component cannot be build from usual single particle orbitals because protons and neutron lie in different major shells. Thus, an additional gaussian centered on the nuclear surface is necessary to be included in the single particle basis. We described both $\alpha$-decay width and $B(E2)$ value in $^{212}$Po by using a mixed single particle basis, containing shell-model and $\alpha$-cluster components.
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