RESTRICTION OF EIGENFUNCTIONS TO TOTALLY GEODESIC SUBMANIFOLDS

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Abstract. We prove a number of results on the Fourier coefficients \( \langle \gamma_H \varphi_j, e_k \rangle_{L^2(H)} \) of restrictions \( \gamma_H \varphi_j \) of Laplace eigenfunctions \( \varphi_j \) of eigenvalue \(-\lambda_j^2\) of a compact Riemannian manifold \((M, g)\) of dimension \(n\) relative to the eigenfunctions \(\{e_k\}\) of eigenvalues \(-\mu_k^2\) of a totally geodesic submanifold \(H\) of dimension \(d\). The results pertain to the ‘edge case’ \(c = 1\) where \(|\mu_k - \lambda_j| \leq \epsilon\) for some \(\epsilon > 0\) of Kuznecov-Weyl sums

\[
N_{\epsilon, H}^1(\lambda) = \sum_{j, \lambda_j \leq \lambda k: |\mu_k - \lambda_j| \leq \epsilon} \left| \int_H \varphi_j \overline{e_k} dV_H \right|^2.
\]

We prove a universal asymptotic formula \(N_{\epsilon, H}^1(\lambda) \sim C_{n, d} a_{j, H}^0(\lambda) \lambda^{n+4} \), together with universal estimates on the remainder and on jumps in \(N_{\epsilon, H}^1(\lambda)\). The growth of the Kuznecov-Weyl sums depends on \(d = \text{dim} \ H\), in contrast to the “bulk cases” where \(|\mu_k - c \lambda_j| \leq \epsilon, 0 < c < 1\), where the order of growth is \(\lambda^{n-1}\) for submanifolds of any dimension (as shown by Y. Xi, E. Wyman and the author).

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1. Introduction

Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold without boundary, let $\Delta_M = \Delta_g$ denote its Laplacian, and let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis of its eigenfunctions,

$$(\Delta_M + \lambda_j^2)\phi_j = 0, \quad \int_M \phi_j \overline{\varphi_k} dV_M = \delta_{jk},$$

where $dV_M$ is the volume form of $g$, and where the eigenvalues are enumerated in increasing order, $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \cdots \uparrow \infty$. Let $H \subset M$ be a $d$-dimensional embedded totally geodesic submanifold with induced metric $g|_H$ with volume form $dV_H$, let $\Delta_H$ denote the Laplacian of $(H, g|_H)$, and let $\{\psi_k\}_{k=1}^{\infty}$ be an orthonormal basis of its eigenfunctions on $H$,

$$(\Delta_H + \mu_k^2)\psi_k = 0, \quad \int_H \psi_k \overline{\varphi_j} dV_H = \delta_{jk}.$$

The purpose of this article is to study the Fourier coefficients in the eigenfunction expansion of the restriction $\gamma_H \varphi_j = \varphi_j|_H$ of $\varphi_j$ to $H$,

$$\gamma_H \varphi_j(y) = \sum_{k=1}^{\infty} \langle \gamma_H \varphi_j, \psi_k \rangle_H \psi_k(y), \quad \langle \gamma_H \varphi_j, \psi_k \rangle_{L^2(H)} := \int_H \varphi_j(y) \overline{\psi_k(y)} dV_H, \quad (1.1)$$

in the ‘edge case’ where $\mu_k \simeq \lambda_j$ in the sense made precise below.

Ideally, we would like to obtain asymptotics or estimates on the Fourier coefficients of individual eigenfunctions. In special cases this is possible, e.g. for closed geodesics on certain surfaces. But in general the Fourier coefficients may vary erratically as $\lambda_j$ varies or
as $\mu_k$ varies, and to obtain asymptotics of Fourier coefficients it is usually necessary to study averages of squares of Fourier coefficients in both the $\mu_k$ and $\lambda_j$ spectral parameters. We therefore study thin window or “ladder Kuznecov sums” in the sense of [WXZ21],

$$
\begin{align*}
N^c_{\psi,H}(\lambda) &:= \sum_{j,\lambda_j \leq \lambda} \sum_{k=0}^{\infty} \psi(c\lambda_j - \mu_k) \left| \int_H \varphi_j c_k dV_H \right|^2, \\
N^c_{\epsilon,H}(\lambda) &:= \sum_{j,\lambda_j \leq \lambda} \sum_{k=1=\lambda} \sum_{|\epsilon\lambda_j - \mu_k| \geq \epsilon} \left| \int_H \varphi_j c_k dV_H \right|^2,
\end{align*}
$$

(1.2)

where the test function $\psi \in \mathcal{S}(\mathbb{R})$ (Schwartz class). It is shown in [WXZ21] that the sums decay rapidly if $c > 1$. The asymptotics for $c = 0$ were first determined in [Zel92] and those are those for $0 \leq c < 1$ are determined in [WXZ21] for any submanifold. In this article, and its sequel [Z22], the asymptotics are studied for the ‘edge case’ $c = 1$ in the case where $H$ is a totally geodesic submanifold. It turns out that even the order of growth of the sums (1.2) are quite different from the case $c < 1$ and depend on $d = \dim H$. This is because the edge Fourier coefficients often have enhanced magnitudes compared to ‘bulk’ Fourier coefficients with $c < 1$. The enhancement depends on the second fundamental form of $H$, and is largest when $H$ is totally geodesic. The order of growth of the sums is smaller when $H$ has non-degenerate second fundamental form, and that case is not considered here. We refer to Section 1.2 and Section 4.6 for a comparison of the cases $c < 1$ and $c = 1$.

Remark 1.1. The ‘ordering’ $\lambda_j - \mu_k$ in (1.2) and (1.7) is important, and is adhered to throughout, because it will imply that asymptotically the argument of $\psi$ is positive.

The Kuznecov sums (1.2) involve two types of localization: (i) localization of $\varphi_j$ along $H$; and (ii) Fourier localization of $\varphi_j|_H$ to a thin window of Fourier modes on $H$ of frequencies $\mu_j$ close to $\lambda_j$. In [WXZ21] the first sum in (1.2) is called ‘smooth-sharp’ since it involves the smoothed inner sum with $\psi$; the second sum is called ‘sharp-sharp’ since it involves indicator functions in both $\lambda_j$ and in $\mu_k$. The smoothed sum $N^1_{\psi,H}(\lambda)$ is technically simpler to work with and, perhaps surprisingly, often has better applications. Indeed, the inner sum of the sharp-sharp average can jump at certain $c$, as discussed in [WXZ21].

To state first result we need some notation. By the injectivity radius $\text{inj}(M,g)$ we mean the largest $R > 0$ so that $\exp_x : B_x(R) \to M$ is a diffeomorphism to its image for all $x \in M$; $B_x(R) \subset T_x M$ denotes the ball of radius $r$ in the tangent space. We say that the geodesic flow $G^t_x$ of any Riemannian manifold $(X,g)$ is ‘aperiodic’ if the set of closed geodesics of $(X,g)$ has Liouville measure zero in $S^n X$. We denote the volume form in geodesic coordinates $y = \exp_x \xi$ based at $x \in M$ by $\Theta(x,y)dy$. If we change to geodesic polar coordinates $(r,\omega)$, we get $dV_g = J(r,\omega)drd\omega$ where

$$
J(r,\omega) = r^{n-1} \Theta(r,\omega, y) = \|V_1(r) \wedge \cdots \wedge V_{n-1}(r) \wedge \frac{\partial}{\partial r}\|
$$

(1.3)

where $\Theta(r,\omega) = \Theta(x, y)$ when $x = (r, \omega)$. Here, $V_j$ is a basis of vertical Jacobi fields along the geodesic with direction $\omega$ and initial point $x$, i.e. Jacobi fields satisfying $V_j(0) = 0$ and $\frac{DV_j}{dt}(0)$ is an orthonormal basis of the normal space to the geodesic. Also, $\frac{\partial}{\partial r}$ is the unit tangent vector to the geodesic. For example, on the standard sphere $S^n$, $J(r,\omega) = \sin^{n-1} r$.

We assume throughout that $d = \dim H \geq 1$. If $d = 0$ and $H = \{x_0\}$ is a single point, then the eigenfunctions of $H$ are constants, the only Fourier coefficient (1.1) is the pointwise square $|\varphi_j(x_0)|^2$, and the Kuznecov-Weyl asymptotics of (1.2) reduce to pointwise Weyl law results (see [DG73, HoIV, SV] for background). The main result is a generalization of such pointwise Weyl laws to higher dimensional submanifolds.
Theorem 1.2. Let \( \dim M = n \) and let \( \dim H = d \geq 1 \). Assume that \( H \) is totally geodesic. Let \( \hat{\psi} \in C_c^\infty(\mathbb{R}) \) be a real, positive, even test function supported in the set \( (-r_0, r_0) \) where \( r_0 < \text{inj}(M, g) \). Then, there exist universal constants \( C_{n,d} \) such that for any \( \epsilon > 0 \),

\[
N_{\hat{\psi},H}^1(\lambda) = C_{n,d} a_1^0(H, \psi) \frac{n}{d} + R_{\hat{\psi},H}^1(\lambda), \quad \text{where } R_{\hat{\psi},H}^1(\lambda) = O(\lambda^{n/d-1}).
\]

The leading coefficient is given by,

\[
a_1^0(H, \psi) := \int_\mathbb{R} \int_H \int_{S^*H} \hat{\psi}(s) (s + i0)^{-\frac{n}{d}} \Theta_{M}^\frac{1}{2}(q, \exp_q s\omega) \Theta_{H}^\frac{1}{2}(q, \exp_q s\omega) dsdV_H(q)dS(\omega).
\]

If the geodesic flow \( G_t^H \) of \( H \) is aperiodic, then

\[
R_{\hat{\psi},H}^1(\lambda) = o(\lambda^{n/d-1}). \quad (1.5)
\]

In the sequel [Z22], we study two term asymptotics of (1.2), which give necessary conditions to obtain maximal growth of individual terms (see Section 1.8 for further remarks). The last statement for aperiodic flows is a two-term asymptotic in which the second term of order \( \lambda^{\frac{n}{d}-1} \) vanishes. The proof that it vanishes is the same as in [WXZ21] Section 5.3 but will be reviewed in Section 7.6. Before stating further results, or discussing the proof of Theorem 1.2 we address some issues in both the assumptions and conclusions of the theorem which explain the organization and the length of this article.

Note that when \( d = n - 1 \), the density \( (s + i0)^{-\frac{n}{d}} \) is integrable and there is no need for regularization. When \( d < n - 1 \), the asymptotics of Theorem 1.2 are somewhat subtle because the contribution of very short (wave-length scale) distances in \( s \) is larger than the asymptotics in Theorem 1.2 i.e. substantial cancellation is required in the integral over \( \text{supp} \hat{\psi} \) to produce the relatively small order of growth \( \lambda^{\frac{n}{d}} \). The unusual leading coefficient (1.4) signals the existence of a ‘blow-down’ singularity in the oscillatory integrals used in the proof of Theorem 1.2. The geometric origin of the blow-down singularity is simple to understand: it is due to the collapse of distance spheres on \( H \) when \( s = 0 \) (see Section 1.3 and Section 8.1). This singularity does not occur for \( c < 1 \) in [WXZ21]. It indicates that the asymptotics in Theorem 1.2 cannot be deduced purely from the Fourier integral operator theory under clean compositions, or equivalently, by a stationary phase analysis. To prove the formula for the leading term, we use stationary phase outside of a small interval around \( s = 0 \) and then reduce to a universal model integral for the short distance part (Section 4). It is proved in Section 1.3 that the procedure used in the model case extends to give the result for the general case, completing the proof of the leading term and remainder in Theorem 1.2 but without calculating the full amplitude; that is done in Section 5. It is simple to obtain the coefficient under the (rather artificial) assumption that \( \hat{\psi} = 0 \) in some interval \( [-\epsilon, \epsilon] \) around \( s = 0 \), but not so simple to determine the regularization or to show that there does not exist another term supported at \( s = 0 \), as for \( c < 1 \) in [WXZ21]. The evaluation of the coefficient (1.4) is given in Section 1.3 and is indirect: Once the existence of the asymptotic expansion is proved, it is fairly obvious that \( \psi \to a_1^0(H, \psi) \) is a positive distribution, hence a positive measure. Due to the ordering \( \lambda_\beta - \mu_k \geq 0 \) it is supported in \( \mathbb{R}_+ \). As a result, the Fourier transform of the distribution (1.4) has a holomorphic extension to the upper half plane, as indicated in (1.4).

The most computable examples corroborating the leading order term and proving sharpness of the remainder are that of totally geodesic spheres \( S^d \subset S^n \) of standard spheres,
subspaces of $\mathbb{R}^n$ (Section 4.5) and non-degenerate closed geodesics of any Riemannian manifold $(M, g)$. In [22], the asymptotics are derived by Gaussian beam techniques.

A second aspect of the asymptotics is due to the dependence on the test function $\psi$. In special cases, again such as $\mathbb{S}^d \subset \mathbb{S}^n$, the eigenvalue differences $\lambda_j - \mu_k$ are essentially differences $N - M$ of positive integers. This raises the general question (for any $c \in (0, 1]$) of when the ordered difference spectrum,

$$
\Sigma_{M,H}(c) = \{c\lambda_j - \mu_k\}
$$

is dense in $\mathbb{R}$. If instead of assuming that $\hat{\psi} \in C_0^\infty(\mathbb{R})$ we assumed that $\psi \in C_0^\infty(\mathbb{R})$, we could take $\psi$ supported in a short interval $[-\frac{1}{2}, \frac{1}{2}]$ which is a gap in the difference spectrum $N - M$, and for such $\psi$ (1.2) would equal zero. Thus, we see that there are different types of asymptotics problems, some involving $\hat{\psi}$ with small support around 0, including wavelength support, and some involving the opposite regime where $\psi$ has small support, and that different techniques are needed for the different regimes. The question of when (1.6) is dense is reminiscent of the Helton clustering theorem; it appears that the remainder term $R_{\psi,H}^1$ is maximal only when the difference spectrum (1.6) fails to be dense. We leave these questions to future work.

In this article we assume that $\hat{\psi} \in C_0^\infty$ and $\text{Supp} \hat{\psi} \subset (-r_0, r_0)$. The assumption that $\hat{\psi} \in C_0^\infty$ often arises in a Fourier integral analysis of spectral problems to limit the number of singularities of the dual sum,

$$
S(t, \psi) : = \sum_{j,k} e^{it\lambda_j} \psi(\lambda_j - \mu_k) |\int_H \varphi_j e_k dV_H|^2.
$$

By studying the wave equations of $(M, g)$ and of $(H, g|_H)$, one finds that for $\hat{\psi} \in C_0^\infty(\mathbb{R})$ the leading order term (1.4) involves the times $t$ of periodic orbits of the geodesic flow $G^t_M$ of $(M, g)$ and all times $s$, including the times $s$ of periodic orbits, of the geodesic flow of $G^s_H$ of $(H, g|_H)$ (see Section 2.2). It is evident from the factor $\Theta_{M}^{-\frac{1}{2}}(q, \exp_q s\omega)\Theta_{H}^{-\frac{1}{2}}(q, \exp_q s\omega)$ in (1.4) that the distribution $a_1^0(H, \psi)$ for general $\hat{\psi} \in C_0^\infty$ depends on the structure of periodic orbits and conjugate points. The formula (1.4) uses the assumption that $\hat{\psi}$ is supported in the interval $(-r_0, r_0)$, since the volume densities $\Theta_M(q, q')$, resp. $\Theta_H(q, q')$ become singular when $q'$ is a conjugate point of $q$. The formula suggests that the leading coefficient is valid for any $\hat{\psi} \in C_0^\infty$ if one suitably regularizes the integrand at conjugate points. For instance, in the case of a totally geodesic subsphere $\mathbb{S}^d \subset \mathbb{S}^n$, one has the global in $s$ formula,

$$
a_1^0(\mathbb{S}^d, \psi) = \int_{\mathbb{R}} \hat{\psi}(s)(\sin(s + i0))^{-\frac{n-d}{2}} ds.
$$

The additional assumption that $\text{Supp} \hat{\psi} \subset (-r_0, r_0)$ ensures that there is only one singularity of (1.7), namely at $t = 0$, and it does not involve any periods or conjugate points of the geodesic flow on $H$. This is already sufficient to prove one term asymptotics with a sharp remainder. If we allow $\psi \in C_0^\infty(\mathbb{R})$, the Fourier integral operator techniques of this article would require a sum over all periods of the geodesic flows and the test function $\hat{\psi}(s)$ would have to have a compensating decay. This is far from saying that the case of $\psi \in C_0^\infty(\mathbb{R})$ is uninteresting. Such high localization in the difference spectrum is often of the highest interest in applications. But it explains why we do not consider that regime in this article. For further discussion of the singularity of (1.4) we refer to Section 4.4.

Two final notation remarks. First, the constant $C_{n,d}$ is computed from the Hadamard parametrix method in [Be] (see Section 5). The formula for (1.4) requires determining the
precise regularization of the distribution \( s^{-\frac{n-d}{2}} \) at \( s = 0 \). We often make use of the relations [GS, Page 93 and Page 172],

\[
\left\{ \begin{array}{l}
(s \pm i0)^{-\frac{n-d}{2}} = s_{\mp}^{-\frac{n-d}{2}} e^{\mp i\pi \frac{n-d}{2}} = s_{\mp}^{-\frac{n-d}{2}} + (\mp i)^{n-d} s_{\mp}^{-\frac{n-d}{2}}, \\
\int_0^\infty e^{it\lambda} dt = i e^{i\lambda \pi/2} \Gamma(\lambda + 1)(\sigma + i0)^{-\lambda-1}
\end{array} \right.
\]

(1.9)

which explain the origins of the regularization and of the normalizing constants \( C_{n,d} \). Second, we use the convention that the propagator of an operator \( P \) by \( U(t) = e^{itP} \) (rather than \( e^{-itP} \)) and always integrate it against \( e^{-it\lambda} \).

1.1. **Jumps in the Kuznecov-Weyl sums.** Theorem 1.2 concerns asymptotics of a double average (1.2), an outer average over the eigenvalues \( \lambda_j \) of \( \sqrt{-\Delta_M} \) and an inner average over the eigenvalues \( \mu_k \) of \( \sqrt{-\Delta_H} \). One may obtain results on eigenfunctions of individual eigenvalues to some extent by studying the jumps of the Kuznecov-Weyl sums (1.2) at the eigenvalues \( \lambda_j \), weighted by either smooth test functions \( \psi \) or sharp indicator functions:

\[
\left\{ \begin{array}{l}
J^1_{\psi,H}(\lambda_j) := \sum_{\ell: \lambda_\ell = \lambda_j} \sum_k \psi(\lambda_j - \mu_k) |\int_H \varphi_k e^{i\ell k} dV_H|^2, \\
J^1_{\ell,H}(\lambda_j) := \sum_{\ell: \lambda_\ell = \lambda_j} \sum_{k: |\lambda_j - \mu_k| \leq \epsilon} |\int_H \varphi_k e^{i\ell k} dV_H|^2.
\end{array} \right.
\]

(1.10)

The sum over \( \ell \) is a sum over an orthonormal basis for the eigenspace \( H(\lambda_j) \) of \( \Delta_M \) of eigenvalue \( \lambda_j \). If the \( M \)-eigenvalues have multiplicity one, then there is a single term in the \( \lambda_j \) sum. On the other hand, unless the \( H \)-eigenvalues come in clusters with high multiplicities and are separated by gaps, the \( \mu_k \)-sum runs over roughly \( O(\lambda_j^{d-1}) \) eigenvalues. We refer to the sums of (1.10) as the ‘inner sums’.

Theorem 1.2 implies the following universal bounds on remainders and jumps.

**Corollary 1.3.** With the same assumptions and notations as in Theorem 1.2, for any positive test function \( \psi \) with \( \hat{\psi} \in C_0^\infty(\mathbb{R}) \) having small support, there exists a constant \( C_\psi > 0 \) such that

\[
(i) \quad J^1_{\psi,H}(\lambda) \leq C_\psi \lambda^{\frac{n-d}{2} - 1}.
\]

Moreover, for any \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) so that

\[
(ii) \quad J^1_{\ell,H}(\lambda_j) \leq C_\epsilon \lambda^{\frac{n-d}{2} - 1}.
\]

In the aperiodic case,

\[
J^1_{\psi,H}(\lambda) = o(\lambda^{\frac{n-d}{2} - 1}).
\]

(1.11)

The first statement follows from the standard observation that,

\[
J^1_{\psi,H}(\lambda_j) = N^1_{\psi,H}(\lambda_j) - N^1_{\psi,H}(\lambda_j - 0)
\]

(1.12)

since the leading order term in Theorem 1.2 is continuous. The remainder bound (i) thus follows from Theorem 1.2. To prove (ii) we use (i) and an appropriate choice of \( \psi \) (see Section 7.5).

To obtain sharper results, it is necessary to study the long time behavior of geodesic flows and wave groups of \( (M,g) \) and of \( (H,g|_H) \). This is carried out in the sequel [Z22]. The proof of Theorem 1.2 and of the other results of this article mainly use the ‘small time’ behavior of
the geodesic flows $G^t_M$ of $(M,g)$, resp. $G^t_H$ of $(H,g|_H)$. An exception is the statement \([1.5]\), which follows from the wave front set analysis in Section \([2.2]\) analogous to the wave front analysis in \([WXZ21]\). In the short time analysis we employ both the Hörmander parametrix (Section \([3.1]\)) and the Hadamard parametrix method (Section \([3]\)).

1.2. Comparison to $c < 1$. Let us compare Theorem \([1.2]\) to the analogous result \([WXZ21]\) Theorem 1.1] in the case $c < 1$. In that case,

$$N_{\psi,H}^c(\lambda) = C_{n,d} a^0_c(H,\psi)\lambda^{n-1} + O(\lambda^{n-2}), \quad (1.13)$$

where the leading coefficient is given by the temperate distribution,

$$a^0_c(H,\psi) := \hat{\psi}(0)^{d-1}(1-c^2)^{\frac{n-d-2}{2}} H^d(H)$$

if $\hat{\psi}$ is supported in a sufficiently small interval around $s = 0$. Note that the power of $\lambda$ in the $c < 1$ case is independent of $d = \dim H$. To bridge the two results, note that if $c = 1 - \frac{1}{d-1}$ then

$$(1 - c)^{\frac{n-d-2}{2}} = \left(\frac{d-2}{d-1}\right)^{\frac{n-d-2}{2}} r(\epsilon)\lambda^{-\frac{n-d-2}{2}}, \quad \text{and} \quad (1 - c)^{\frac{n-d-2}{2}}\lambda^{n-1} = r(\epsilon)\lambda^{-\frac{n-d-2}{2}}\lambda^{n-1} = r(\epsilon)\lambda^{\frac{n-d-2}{2}}.$$

If $\hat{\psi} = 0$ near $s = 0$, then the order of magnitude of $N_{\psi,H}^c(\lambda)$ drops by 1 when $c < 1$, but not when $c = 1$. A further result, \([WXZ21]\) Theorem 1.18\([WXZ21]\) in the case $c < 1$ shows that the leading coefficient increases as the support of $\hat{\psi}$ increases (assuming $\hat{\psi} \geq 0$), with jumps at special values $s_j$. In contrast, Theorem \([1.2]\) shows that when $c = 1$, the leading coefficient is continuous and has increases with the support of $\hat{\psi}$ (again, assuming $\hat{\psi} \geq 0$). Further comparisons are given in Section \([1.8]\) and in Section \([4.6]\).

We also see that the order of growth $\lambda^{\frac{n-d}{2}}$ in the case $c = 1$ is greater than the order of growth $\lambda^{n-1}$ for $c < 1$ if and only if $d = n - 1$ (the hypersurface case), that the two orders are equal when $d = n - 2$ and that the asymptotics for $c < 1$ are of higher order than for $c = 1$ if $d \leq n - 3$.

The very different powers in Theorem \([1.2]\) and for \([1.13]\) is due to the different type of contributions of conormal directions to $H$. It is explained below in Section \([1.3]\) that the powers of $\lambda$ are controlled by the dimension of the set $S_H^* M$ of geodesic bi-angles. The parameter $c$ is defined by $c = \frac{\|\xi^T\|}{|\xi|}$ where $\pi_H : T_q^* M \to T^*_q H$ is the orthogonal projection at $q \in H$. We denote by $S^*_H M$ the covectors this equality with footpoint on $H$, and we denote the decomposition into tangent and normal components by $\xi = \xi_T + \xi^\perp$. When $c < 1$, $|\xi^\perp|^2 = \sqrt{1 - |\xi_T|^2} = \sqrt{1 - c^2}$ and there the set of conormal parts of unit vectors has dimension $n - d - 2$. When $c = 1$ the co-normal part $\xi^\perp$ must vanish, the dimension drops by $n - d - 2$, and the power drops by $\frac{n-d-2}{2}$. As mentioned above, the singularity is of the “blow-down” type, exhibited by integrals of Bessel functions (cf. Section \([1.5]\)). We refer to Lemma \([2.8]\) for the modified order calculation following the methods of \([WXZ21]\), and to Section \([8.1]\) for discussion of the blow-down singularity.

1.3. Geodesic geometry. Before sketching the proof of Theorem \([1.2]\) and before stating additional results, we discuss the geodesic geometry that is responsible for the singularity at $s = 0$ of the coefficient \([1.13]\), and the geodesic geometry which is responsible for maximal jumps in Corollary \([1.3]\). We will prove the statements on maximal jumps in \([ZZ22]\).

In the case $c < 1$ of \([WXZ21]\), the relevant dynamics was given by geodesic bi-angle geometry. For any $c$, the singularities of \([1.7]\) are governed by the dynamical equation,

$$G^t_M \pi_H G^{-s} (q,\xi) = \pi_H(q,\xi), \quad q \in H, \xi \in S^c H. \quad (1.14)$$
where $G^t_M$ (resp. $G^*_H$) is the geodesic flow of $(M, g)$ resp. $(H, g|_{TH})$, where $\pi_H : S^*_g M \rightarrow S^*_g H$ is the orthogonal projection, where $c = \frac{\pi_H(\xi)}{\xi}$, and where $S^*_H M$ is the set of covectors $\xi \in T^*_H M$ satisfying this constraint. To be precise, (1.14) is the equation that holds on the critical set (or wave front relation) of the governing Fourier integral operator. We think of the condition $\xi \in S^*_H M$ as a ‘constraint’ on the shape of the bi-angles. For $c < 1$, the solutions $(t, s, q, \xi)$ correspond to geodesic bi-angles starting at $q$ and ending at $\exp_q s \pi_H \xi$, consisting of one leg given by an $M$-geodesic of length $s + t$, one leg consisting of an $H$-geodesic of length $s$, and with compatible initial and terminal velocity vectors. When $c = 1$, and $H$ is totally geodesic, $\xi \in S^*_H$ and $G^*_H(q, \xi) = G^*_M(q, \xi)$, so the equation (1.14) simplifies to

$$G^*_H(q, \xi) = (q, \xi), \quad q \in H, \xi \in S^*_H. \quad (1.15)$$

The equation is independent of $s$, explaining why (1.1) is an integral over all $s$. It also indicates that an important geodesic condition is periodicity of the geodesic flow is that of $H$. In general, we say that the geodesic flow of a Riemannian manifold $(X, g)$ is periodic if there exists $T \neq 0$ such that $G^T_X = Id$.

The simplest example where maximal jumps occur is the case of totally geodesic subspheres $S^d \subset S^n$ of spheres, where both $G^*_H$ and $G^t_M$ are periodic. Examples where maximal jumps do not occur are Riemannian products $S^d \times S^{n-d}$, despite periodicity of $G^*_H$ (see Section 2.1). This raises the question whether periodicity of $G^*_M$ is also necessary for maximal jumps. In fact, it is not necessary. An example is where $M$ is a convex surface of revolution and $H$ is its unique rotationally invariant geodesic. The conditions for maximal jumps to occur will be given in [Z22]. Some indications of what is involved are given in Section 1.7.

To tie the discussion of the equation (1.14) together with that in [WXZ21], we recall the following standard definition from [DG75].

**Definition 1.4.** We say that the set $G^*_c$, resp. $G^*_c$ of solutions of (1.14), resp. (1.15), is clean if $G^*_c$, resp. $G^*_c$, is a submanifold of $\mathbb{R} \times \mathbb{R} \times S^*_H M$, resp. $\mathbb{R} \times S^*_H M$, and if its tangent space at each point is the subspace fixed by $D_\xi G^*_H \circ \pi_H \circ G^*_M + t$ (resp. the same with $s = 0$), where $\xi$ denotes a point of $S^*_H M$.

The equation (1.15) is simply the equation for periodic geodesics of $G^*_H$ and cleanliness is the condition that for each period $T$, the set of closed geodesics of period $T$ is a submanifold of $S^*_M$ whose tangent space is the fixed point set of $DG^*_H$ (see [DG75] for this case). Even when this occurs, the Fourier integral operator compositions of Section 2.2 fail to be clean when $c = 1$ and $s = 0$ essentially due to the collapse of the fibers of the co-normal sphere bundle, as discussed in Section 1.2 (see also Section 4.6). Since this failure is universal, it does not depend on the metrics or geodesic flows.

**1.4. Outline of the proof of Theorems 1.2 and further results.** To prove Theorem 1.2 we first study a doubly-smoothed version,

$$N^1_{\psi, \rho, H}(\lambda) = \sum_{j,k=0}^{\infty} \rho(\lambda - \lambda_j) \psi(\lambda_j - \mu_k) \left| \int_H \varphi_j e^{ik} dV_H \right|^2, \quad (1.16)$$

with a second test function $\rho \in S(\mathbb{R})$ with $\hat{\rho} \in C^\infty_0$. To obtain the asymptotics of (1.16), we study the singularities of the Fourier transform,

$$N^1_{\psi, \rho, H}(\lambda) = \int_{\mathbb{R}} \hat{\rho}(t) e^{it\lambda} S(t, \psi) dt,$$
where \( S(t, \psi) \) is defined in (1.7). In terms of wave kernels, we define

\[
S(s, t) = \int_H \int_H \gamma_H U_M \gamma_H^*(t + s, q, q') U_H(-s, q, q') dV_H(q) dV_H(q'),
\]
and then,

\[
S(t, \psi) = \int_{\mathbb{R}} \hat{\psi}(s) S(s, t) ds.
\]

In Theorem 1.2 we assumed that the only period of \( G_H^s \) in the support of \( \hat{\psi} \) is \( s = 0 \). In studying (1.16), we further need assumptions of the periods of \( G_M^s \) in the support of \( \hat{\rho} \). We study the short time singularities of \( S(t, \psi) \) using a reduction to a model phase, unlike the methods employed in [WXZ21]. The model phase is most visible if one uses the Hörmander type parametrix, whose phase is linear in \( t \) (Section 3.1). In Section 5 we use a Hadamard parametrix to calculate the amplitude.

To prove the ‘aperiodicity’ result of Theorem 1.2 we also need some input from long time singularities from Section 2.2. To this end, we repeat two definitions from [WXZ21] but point out that they simplify when \( c = 1 \) and when \( H \) is totally geodesic. The following is the \( c = 1 \) analogue of [WXZ21, Definition 1.15].

**Definition 1.5.** Let

\[ \Sigma^1 := \{ t : \text{Fix}(G_H^1) \neq \emptyset \} \]

be the set of periods of closed geodesics of \( H \).

In [WXZ21] Definition 1.15, the analogous set \( \Sigma^c(\psi) \) for \( c < 1 \) was defined by as the set of singular points \( t \) of \( S(t, \psi) \); it consists of \( t \) for which there exist solutions of \((1.14)\) with \( s \in \text{supp} \hat{\psi} \). When \( c = 1 \) and \( H \) is totally geodesic, it follows from (1.15) that \( \Sigma^1(\psi) \) is independent of \( \psi \). The order of the singularity at \( t \) corresponds to the dimension of the solution set of (1.15).

**Definition 1.6.** We say that the singularity at \( t = 0 \) is dominant if the dimension of the solution set \((1.14)\) at \( t = 0 \) is strictly greater than any other \( t \in \Sigma^1 \).

By the remarks above, the singularity at \( t = 0 \) is dominant except when \( G_H^t \) has a positive measure of periodic orbits. In the clean case (cf. Definition 1.4), this would mean that \( G_H^t \) is periodic, i.e. that \( (H, g_H) \) is a Zoll manifold.

Although \( \Sigma^1 \) is independent of \( \psi \), the periods \( s \in \Sigma^1 \) in the support of \( \hat{\psi} \) play an important role in the coefficient (1.4) and indeed cause the difficulties which led to the assumption in Theorem 1.2 that \( \text{Supp} \psi \subset (-r_0, r_0) \). These periods do not affect the order of the singularities at \( t \in \Sigma^1 \) (including \( t = 0 \)) but they do affect the coefficient generalizing (1.4).

The following theorem is analogous to Theorem 1.2 but employs a smooth test function \( \rho \) instead of the sharp interval sum. Its assumptions qualify it as a ‘short-time’ theorem.

**Theorem 1.7.** Let \( \dim M = n, \dim H = d \), and assume \( H \) is totally geodesic. Let \( \psi, \rho \in S(\mathbb{R}) \) with \( \hat{\psi}, \hat{\rho} \in C_0^\infty(\mathbb{R}) \). Assume that \( \text{supp} \hat{\rho} \cap \Sigma^1 = \{0\} \) and \( \hat{\rho}(0) = 1 \). Let \( \hat{\psi} \in C_0^\infty(\mathbb{R}) \) be a real, even test function supported in the set \((−r_0, r_0)\) where \( r_0 < \text{inj}(M, g) \). Then, there exists a complete asymptotic expansion of \( N_{\rho, \psi, H}^1(\lambda) \) of the form,

\[
N_{\rho, \psi, H}^1(\lambda) \sim \lambda^{\frac{n+d}{2}-1} \sum_{j=0}^{\infty} \beta_j \lambda^{-j},
\]

with \( \beta_0 = a_0^0(H, \psi) (1.4) \), and where \( C_{n,d} \) is a universal dimensional constant.
This theorem seems similar to [DG75, Proposition 2.1] and many similar asymptotics results that are proved under clean composition hypotheses. Equivalently, they are proved by the stationary phase method. As mentioned above, the compositions relevant to Theorem \ref{thm:main} are not clean due to a blow-down singularity at \( s = 0 \) and cannot be proved solely by stationary phase methods. Rather we take a hybrid approach in Section 4 that combines stationary phase in certain variables and a direct integration in others. The order of the asymptotics is consistent with a formal application of stationary phase, but it is not clear apriori that this formal order is correct because the blow-down singularity could dominate the asymptotics. We emphasize this point with the Euclidean case (Bessel integrals) in Section 4.5.

In Proposition 6.1 we extend the proof of Theorems 1.2–Theorem 1.7 to test functions \( \psi \) with arbitrary support but retain the assumption that \( \text{Supp} \hat{\psi} \subset (-r_0, r_0) \). The extension then does not involve any new inputs beyond the wave front analysis in Section 2.2.

Once Theorem 1.7 is proved, the ‘sharp’ Weyl asymptotics of Theorem 1.2 (except for the last statement about aperiodic flows) follow from Theorem 1.7 by a standard cosine Tauberian theorem (Section 7.1). We discuss the proof of the aperiodic flow in the next subsection.

To prove Theorem 1.7, we construct parametrices for (1.16) when \( \text{Supp} \hat{\rho} \) lies in a sufficiently short interval around \( t = 0 \) so that no other singularities of (1.7) lie in the interval. We use the parametrix to calculate the leading order contribution in Theorem 1.2. Due to the fact that the clean composition calculus breaks down at small distances, we rely on a small time parametrix when studying small distances. When the geodesic flow of \((M, g)\) is periodic, the small distance problem will recur for larger times as well; but the main term of the asymptotics is due to the \( t = 0 \) singularity and that can be analysed using the small time parametrix.

1.5. The remainder estimate in the case where \( G_t \) is aperiodic. The last statement of Theorem 1.2 says that the remainder is ‘small oh” of the universal remainder when \( G_t \) is aperiodic. This requires inputs from the long time behavior of the wave group and geodesic flow. But the inputs are less than those necessary to generalize Theorem 1.2 to general \( \hat{\rho}, \hat{\psi} \in C_0^\infty \). We retain the assumptions on \( \hat{\psi} \) but relax the assumption on \( \hat{\rho} \) and let its support be arbitrarily large, i.e. we consider all singularities of \( S(t, \psi) \) where \( \text{Supp} \hat{\psi} \subset (-r_0, r_0) \).

1.6. Dependence of \( N_{c,H}^1 \) and of the jumps \( J^1_{c,H}(\lambda_j) \) on \( \epsilon \). It may seem more interesting to analyse the sharp-sharp sums \( N_{c,H}^1 \) of (1.2). However, they have an erratic dependence on \( \epsilon \), just as \( N_{\hat{\psi},H}^1(\lambda) \) has a complicated dependence on \( \psi \) if we drop the support assumptions in Theorem 1.2. The situation is similar to that for \( c < 1 \), discussed in detail in [WXZ21, Section 1.10] (see [WXZ21, Theorem 1.24]. The asymptotics of the sharp-sharp sum \( N_{c,H}^c(\lambda) \) for \( 0 < c < 1 \) are proved to have the form,

\[
N_{c,H}^c(\lambda) = a_c^0(H, \epsilon)\lambda^{n-1} + o(\lambda^{n-1}),
\]

where the leading coefficient \( a_c^0(H, \epsilon) \) is linear in \( \epsilon \). The surprisingly large remainder estimate is sharp on spheres, due to jumps in the jump \( J_{c,H}^c \) at special values of \( \epsilon \). Note that \( \hat{\psi}(0) = \int \psi = 2\epsilon \) when \( \psi = 1_{[-\epsilon, \epsilon]} \). This case is similar to (but more irregular than) the case where \( \psi \in C_0^\infty \).
1.7. **Background, related results and problems.** Fourier coefficients of restrictions of eigenfunctions are central to the theory of automorphic forms. They were studied classically by Hermite and Jacobi for Fourier coefficients of modular forms around closed horocycles for the modular surface $\mathbb{H}^2/SL(2,\mathbb{Z})$, and later by H. Petersson [P32] around closed geodesics. The study of $C^\infty$ eigenfunctions of the Laplacian on a hyperbolic surface was developed by H. Masss and was studied systematically by N.V. Kuznecov [K80] using the Kuznecov sum formula. Since then, the study of Fourier coefficients around various submanifolds has formed an important part of automorphic forms on very general arithmetic locally symmetric manifolds. By far the sharpest estimates on individual Fourier coefficients are those of [M16] for geodesic Fourier coefficients of Hecke eigenfunctions.

Averages of Fourier coefficients have been studied in many articles since Kuznecov's article [K80]. In the automorphic forms literature, the weights or test functions used in Kuznecov's formula are adapted to the setting of hyperbolic quotients or other locally symmetric manifolds. In this article, as in [Zel92, WXZ20, WXZ21], we use the wave equation and associated test functions. As often happens, wave equation methods give sharper remainder terms than other methods, and this is true in the present applications.

A distant goal is to obtain asymptotics of “empirical measure of the Fourier coefficients” (1.1) of an individual eigenfunction $\varphi_j$ as $\mu_k$ varies, i.e. the measure whose mass at $\mu_k$ is the modulus square $|\int_H \varphi_j e_k|^2$ of the Fourier coefficient. The results of this article and of [WXZ21] concern the integrals of the empirical measure over short intervals.

We have already discussed the long-time refinement of the results of Theorem 1.2 above; they are studied in [Z22]. In addition, we point out the following closely related problems.

1.7.1. **Simple generalizations.** With no additional effort, all results of this article extend to the more general Kuznecov-Weyl sums, for any $c \in [0,1]$, and for any $f \in C^\infty(H)$,

$$N_{\psi,H,f}^c(\lambda) := \sum_{j,\lambda_j \leq \lambda} \sum_{k=0}^{\infty} \psi(\mu_k - c\lambda_j) \left| \int_H f \varphi_j e_k dV_H \right|^2. \quad (1.19)$$

In the leading coefficient (1.4), for $c = 1$, $\mathcal{H}^d(H)$ is replaced by $(\int_H f dV_H)$. More generally, we could replace $f$ by a semi-classical pseudo-differential operator $O_{PH}(a)$ along $H$ and replace the inner products by $\langle O_{PH}(a)\gamma_{Hj},e_k \rangle_{L^2(H)}$ and then the coefficient is $\int_{S^*H} a_0 d\mu_H$ where $a_0$ is the principal symbol and $d\mu_H$ is the Liouville measure on $S^*H$. We will use the generalization to $O_{PH}(a)$ in Section 7.2.

1.7.2. **More refined remainder estimates.** The Hadamard parametrix of Section 5 can be used on manifolds without conjugate points to obtain a global-in-time parametrix for the half-wave group [Be]. Such manifolds always have aperiodic geodesic flows. By controlling the exponential growth rate of Jacobi fields and of the number of periodic orbits, one can probably obtain logarithmic improvements to the remainder estimates of Theorem 1.2; see [SXZh17, CG21] among many papers on logarithmic improvements on related problems.

1.8. **Two term asymptotics.** We briefly indicate the results of the long time analysis in [Z22]. There, Theorem 1.2 is strengthened to a two term asymptotics roughly of the form,

$$C_{n,d}a^0_0(H,\psi)\lambda^{\frac{d}{2d-1}} + Q_H(\lambda)\lambda^{\frac{d}{2d-1}-1} + o(\lambda^{\frac{d}{2d-1}-1}),$$

where $Q_H(\lambda)$ is a bounded, oscillatory function. To be more precise, we prove somewhat weaker asymptotic inequalities of the type proved for the pointwise Weyl law by Yu. Safarov.
In statement (1.5) of Theorem 1.2 the second \( Q_H \) term vanishes due to aperiodicity of \( G^*_H \). In general, \( Q_H(\lambda) \) may be continuous or discontinuous, depending on the long time dynamics of the geodesic flows. When it is discontinuous, the two-term asymptotics imply that the jump estimates above are sharp. Recently, E. L. Wyman and Y. Xi have proved a two-term asymptotics for the \( c = 0 \) Kuznecov formula (i.e. integrals of a fixed function \( f \) against restricted eigenfunctions) [WX22].

We give two examples to illustrate the significance of the continuity of \( Q_H(\lambda) \) when it is non-zero. A model case is supplied by totally geodesic subspheres \( S^d \subset S^n \) of spheres (see [Z22]). When \( n = 2 \) and \( H = \gamma \) is the equator, the standard basis \( \{Y^m_1\} \) of spherical harmonics has the property that \( Y^m_1|_{\gamma} \) has a single non-zero Fourier coefficient. Its size depends on the ratio \( \frac{m}{N} \), illustrating the purpose of the constraint \( |cN - m| < \epsilon \). The edge case \( m = N \) corresponds to highest weight spherical harmonics, which are special cases of Gaussian beams (see Section 1.8.1 for background). The \( N \)th Fourier coefficient of the restriction to a stable elliptic closed geodesic \( \gamma \) of a general Gaussian beam \( \{\varphi^j_N\} \) of frequency \( \sim N \) has the size of its \( L^2 \) norm, \( ||\varphi^j_N||_{L^2(\gamma)} \). As this shows, universal bounds on individual Fourier coefficients (i.e. without any assumptions on \( (M, g, H) \)) are the same as universal bounds on the restricted \( L^2 \) norm (see [BGT] for the relevant results). On the other hand, restrictions of other spherical harmonics \( Y^m_1|_{\gamma} \) have again just one non-zero Fourier coefficient but in general it is \( O(1) \) when \( \frac{m}{N} = c < 1 \). More drastically, if we restrict \( \varphi^j_N \) to another geodesic \( \gamma' \neq \gamma \), the restricted Fourier coefficients and \( L^2 \) norms are exponentially decaying in \( N \).

A more general example when \( \dim M = 2 \) consists of general closed geodesics. The remainder estimates are sharp when one can construct a Gaussian beam along the closed geodesic. Such a Gaussian beam exists along the unique rotationally invariant closed geodesic of a convex surface of revolution, but does not exist along a hyperbolic closed geodesic for an hourglass of revolution. In the latter case, \( Q_H(\lambda) \) is continuous and does not contribute to the jumps. The results of [Z22] prove such heuristic statements.

1.8.1. The role of Gaussian beams. Maximal Fourier coefficients for \( c = 1, d = 1 \) arise when there exist Gaussian beams along elliptic closed geodesics. The classical examples are highest weight spherical harmonics, which are roughly of the form, \( \varphi_N^{n,\gamma}(s,y) = C(n, 1, N)e^{\frac{2\pi i N s}{L}}e^{-N|y|^2/2} \) of eigenvalue \(-\lambda^2 \sim N^2\); they oscillate along a stable elliptic closed geodesic \( s \in \gamma \) of length \( L \), and have Gaussian decay in the normal directions \( y \). Gaussian beams are the most highly localized eigenfunctions, both in phase space \( S^* M \) and in configuration space \( M \); they concentrate in tubes of radius \( \lambda^{-\frac{1}{2}} \sim N^{-\frac{1}{2}} \) around the phase space geodesic. Their Fourier coefficients concentrate at the ‘edge’ \( \mu_k = N \), so they are a \( c = 1 \) phenomenon. Restrictions of Gaussian beams do not contribute strongly to the asymptotics of \( J^{e_H}_c(\lambda) \) for \( c < 1 \), since the Fourier coefficients of \( \varphi^2_N|_{\gamma} \) concentrate at the edge. They are studied in detail in [Z22].

1.8.2. Quantum Birkhoff normal form analysis when \( H \) is a closed geodesic. For \( d = 1 \), the eigenvalues of \( |\frac{d^2}{ds^2}|^{\frac{1}{4}} \) form the arithmetic progression \( 2\pi Z \) and have multiplicity at most 2. When \( H \) is a closed geodesic, it is possible to fix \( n \) and study the sums \( \sum_j \rho(\lambda - \lambda_j)|\int_H \varphi_j e_n|^2 \). In higher dimensions, the eigenvalues are generically distributed uniformly modulo one [DG75] and can have various types of multiplicities.

In the case of a closed geodesic, one could use quantum Birkhoff normal form techniques to study Fourier coefficients; such techniques do not seem available for higher dimensional totally geodesic submanifolds.
One could also let $H$ be an arc of a non-closed geodesic, but there never exist maximal jumps for restrictions to such geodesic arcs and in applications they do not seem important.

### 1.8.3. Submanifolds with non-degenerate second fundamental form.

In this article, we only study totally geodesic submanifolds $H \subset M$ in the case $c = 1$. These are the extremal cases for Kuznecov-Weyl asymptotics, and have been the focus of eigenfunction restriction problems (e.g. [G79, BGT, CG21, M16, T09]). But the case of manifolds with non-degenerate second fundamental form and with $c = 1$ are also important, and in addition are generic. For instance, horocycles and distance spheres are non-degenerate and are fundamental in the theory of automorphic forms. The proof of Proposition 3.1 breaks down at the steps in Section 3.1.1 where $\exp_M^{-1}$ and $\exp_H^{-1}$ are equated on $TH$ and for the same reason (1.14) is more complicated than (1.15). In the case of a hypersurface $H$ with non-degenerate second fundamental form phase function in Proposition 3.1 has a degeneracy of fold type when $c = 1$ rather than the collapse of $N^*H$ along $\text{Diag}(H \times H)$. The fold singularity when $(n, d) = (2, 1)$ and $(M, g)$ is a finite area hyperbolic surface with cusps is responsible for the analytic results in [Wo04]. It would be interesting to generalize the results of this article (and the case $c < 1$ in [WXZ21], in which $H$ is a general submanifold) to the case $c = 1$ and $H$ has non-degenerate second fundamental form. Explicit examples where one can expect relative extremals for Kuznecov-Weyl asymptotics are non-equatorial latitude spheres $S^d \subset S^n$.

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### 2. Geodesic geometry

In this section we discuss the geodesic geometry underlying the Kuznecov-Weyl asymptotics, in particular the role of (1.14). We begin with a short list of examples of $(M, g, H)$ where $H \subset M$ is totally geodesic. A study of Kuznecov-Weyl asymptotics in these examples illuminates the basic question of when the jumps in Corollary 1.3 are of maximal size.

#### 2.1. Model examples of totally geodesic $H \subset M$.

Closed geodesics of Riemannian manifolds are always totally geodesic submanifolds of dimension $d = 1$ and many articles are devoted to norms and Fourier coefficients of restrictions. See for instance [P32, G83, GRS17, M16].

A generic Riemannian manifold does not possess even the germ of a totally geodesic submanifold of dimension $d > 1$. Hence we provide a number of well-known models which do have such submanifolds.

- Totally geodesic subspheres $S^d \subset S^n$ of spheres. The jump estimates are shown to be sharp for certain cases of $(n, d)$ in [Z22]. Spheres are special in that they are compact rank one symmetric spaces, and the eigenspaces of $\Delta$ are spanned in an appropriate sense by Gaussian beams along closed geodesics (see Section 1.8.1). The leading term and second term of the Kuznecov-Weyl sums reflect the different types of restriction of Gaussian beams of $S^n$ to $S^d$, in particular of Gaussian beams of $S^n$ along closed geodesics of $S^d$ versus Gaussian beams along closed geodesics transverse to $S^d$. Spheres also illustrate the need for the sum $\sum_{\ell}$ over repeated eigenvalues in an eigenspace in (1.10). In some cases of $(n, d)$, there exist classical
results on asymptotics of Legendre functions which also prove the sharpness of the jump estimates.

- Totally geodesics submanifolds of other compact rank one symmetric spaces. For instance, the results for subspheres $\mathbb{S}^d \subset \mathbb{S}^n$ of spheres generalize to sub-projective spaces $\mathbb{CP}^d \subset \mathbb{CP}^n$. These are examples of Zoll manifolds, all of whose geodesics are closed, and their eigenspaces are spanned by Gaussian beams. One would not expect maximal jumps to occur on general Zoll manifolds, where the Gaussian beams are only quasi-modes and not actual eigenfunctions.

- Maximal flats of higher rank compact symmetric spaces or of compact locally symmetric quotients are totally geodesic. The asymptotics of eigenfunctions of higher rank compact symmetric spaces are studied in [G79], although not the Fourier coefficients of restrictions. $L^p$ norms of eigenfunctions on higher rank locally symmetric quotients are studied in [M15].

- QCI (Quantum Completely integrable) systems [TZ03, T09]. Quantum integrability means that $\Delta$ commutes with $n = \dim M$ independent first order pseudo-differential operators. Compact symmetric spaces are QCI and compact locally symmetric quotients of rank $r$ are partially QCI ($\Delta$ commutes with $n - r$ additional operators). An open question is whether there exist examples other than compact rank one symmetric spaces where jumps achieve their maximal growth, since joint eigenfunctions concentrate on level sets of the moment map and as in [TZ03, T09], the eigenfunctions which concentrate on singular levels have large restricted $L^p$ norms.

Examples not already discussed include ellipsoids $\mathcal{E}_n \subset \mathbb{R}^{n+1}$ of the form $\sum_{j=1}^d x_j^2 + \sum_{j=d+1}^{n+1} \frac{x_j^2}{a_j}$ where $\{1, a_j\}_{j=d+1}^{n+1}$ are independent over $\mathbb{Q}$. The subsphere $\mathbb{S}^{d-1} \subset \mathcal{E}_n$ is totally geodesic and $G^\mu_{\mathbb{S}^{d-1}}$ is periodic but $G^\mu_{\mathcal{E}_n}$ is aperiodic.

- Riemannian products $M = H \times K$; $H \times \{k\}$ or $\{h\} \times K$ is totally geodesic for any $h \in H, k \in K$. For instance if $M = \mathbb{S}^1 \times \mathbb{S}^d$, the geodesic flow is periodic on $H$ but not on $M$. Flat tori are often products of lower dimensional tori. Maximal jumps are never achieved on product manifolds (see [222]).

- Warped product metrics $M = K \times_w H$. Given metrics $h$ on $H$ and $k$ on $K$, a warped product has the form $k \oplus wh$ where $w : K \to \mathbb{R}_+$ is a positive smooth function. One often views $M$ as a bundle over $K$ with fiber $H$. A submanifold $S \times K \subset M$ is totally geodesic in $M$ if and only if $S$ is totally geodesic in $H$ and in particular each submanifold $H \times \{k\}$ is totally geodesic. More generally, Riemannian submersions with totally geodesic fibers are examples with totally geodesic submanifolds.

2.2. Wave front calculations. In this section, we generalize the results of [WXZ21, Sections 3-5] on the compositions of canonical relations relevant to smoothed Kuznecov-Weyl sums [L10] to the case $c = 1$. Much of the analysis in the case $c = 1$ is almost identical to that for $c < 1$, and the proofs are omitted when they are essentially the same for $c < 1$ in [WXZ21] and $c = 1$. However, it is necessary to repeat some of the calculations because the order of $S(t, \psi)$ [1.7] at $t = 0$ in Lemma 2.8 is very different from the order in the case $c < 1$, and we need to track down the change in order.
A second major change is the existence of a blow-down singularity at \(s = 0\), leading to the singularity in the leading coefficient \((1.3)\), viewed as a distribution on the test function \(\psi\). The calculations in this section are used to explain the relevance of the equation \((1.14)\) to Kuznecov-Weyl asymptotics and are also used to prove \((1.5)\) of Theorem 1.2

The analysis in \([WXZ21]\) for the case \(c < 1\) was based on ladder theory for Fourier integral operators, in the sense of \([GU89]\). The geodesic geometry arises in the analysis of the compositions of the Fourier integral operators in Section 2.2.

The smoothed Kuznecov-Weyl sums \((1.16)\) are Fourier dual to traces \((1.7)\) arising from the following operators on \(C^\infty(M \times H)\),

\[
\begin{align*}
P := P_M := \sqrt{-\Delta_M} \otimes I, \\
Q_1 := \sqrt{-\Delta_M} \otimes I - I \otimes \sqrt{-\Delta_H} = P_M - P_H.
\end{align*}
\]

As discussed at length in \([WXZ21]\), the system \((P, Q_1)\) is elliptic; \(Q_1\) is a non-elliptic first order pseudo-differential operator of real principal type with characteristic variety,

\[
\text{Char}(Q_1) := \{(x, \xi, q, \eta) \in T^*M \times T^*H : |\xi|_g - |\eta|_{gh} = 0\}.
\]

Here, \(g_H\) denotes the restriction of \(g\) to \(TH\).

Given \(\psi \in S(\mathbb{R})\) with \(\hat{\psi} \in C^\infty(\mathbb{R})\), we define the ‘fuzzy ladder projection’,

\[
\psi(Q_1) : L^2(M \times H) \to L^2(M \times H), \quad \psi(Q_1) = \int_\mathbb{R} \hat{\psi}(s) e^{itQ_1} ds
\]

Then, the trace \((1.7)\) is given by,

\[
S^1(t, \psi) = \Pi_s(\Delta_H \times \Delta_H)^* (\gamma_H \otimes I) e^{itP} \psi(Q_1)(\gamma_H \otimes I)^* = \sum_{j,k} e^{it\lambda_j} \psi(\lambda_j - \mu_k) \left| \int_H \varphi_{j,k}(x, x) dV_H(x) \right|^2.
\]

They are Fourier dual to the \((1.16)\) in the sense that,

\[
S^1(t, \psi) = \mathcal{F}_{\lambda \to t} dN^1_{\psi, H}(t)
\]

The composition theory of \((2.3)\) is discussed in Section 2.2 below.

The following Lemma is analogous to \([WXZ21\, Lemma 3.1]\) and the proof is the same.

**Lemma 2.1.** \(\psi(Q_1)\) of \((2.3)\) is a Fourier integral operator in the class \(I^{-\frac{1}{8}}((M \times H) \times (M \times H))\), \(\mathcal{I}_{\psi}^c\) with canonical relation

\[
\mathcal{I}_{\psi}^c := \{(x, \xi, q, \eta; x', \xi', q', \eta') \in \text{Char}(Q_1) \times \text{Char}(Q_1) : \exists s \in \text{supp} (\hat{\psi}) \text{ such that } G_M^s \times G_H^{-s}(x, \xi, q, \eta) = (x', \xi', q', \eta')\}
\]

The symbol of \(\psi(Q_1)\) is the transport of \((2\pi)^{-\frac{1}{2}} \hat{\psi}(s)|ds|^\frac{1}{2} \otimes |d\mu_L|^\frac{1}{2} \) via the implied parametrization \((s, \zeta) \mapsto (\zeta, G_M^s \times G_H^{-s}(\zeta))\), where \(\mu_L\) is Liouville surface measure on \(\text{Char}(Q_1)\).

We introduce a second smooth cutoff \(\rho \in S(\mathbb{R})\), with \(\hat{\rho} \in C^\infty_0\) and define

\[
\rho(P - \lambda) = \frac{1}{2\pi} \int_\mathbb{R} \hat{\rho}(t) e^{-it\lambda} e^{itP} dt.
\]
By Fourier inversion,
\[
\rho(P - \lambda)\psi(Q_1) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\rho}(t)e^{-it\lambda}e^{itP}\psi(Q_1)\,dt
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\rho}(t)\hat{\psi}(s)e^{-it\lambda}e^{-isQ_1}\,ds\,dt. \tag{2.6}
\]

As is well-known [DG75], \(e^{itP} \in I^{-\frac{1}{4}}(\mathbb{R} \times M \times M, \text{Graph}(g'))\) where \(
\text{Graph}(g') = \{(t, \tau, g^t(\zeta), \zeta) : \tau + \sigma_P(\zeta) = 0\}\) is the space-time graph of the flow. For simplicity of notation we denote \(\zeta = (\zeta_1, \zeta_2) \in T^*(M \times H)\). Since the canonical relation of \(e^{itP}\) is the graph of the bicharacteristic flow of the symbol \(\sigma_P\) of \(P\) on \(T^*(M \times H)\), the composition theorem for Fourier integral operators gives,

The following Lemma is analogous to [WXZ21] Lemma 3.4.

**Lemma 2.2.** \(e^{itP}\psi(Q_1) : L^2(M \times H) \to L^2(\mathbb{R} \times M \times H)\) is a Fourier integral operator in the class \(I^{-\frac{3}{4}}((\mathbb{R} \times M \times H) \times (M \times H), \mathcal{C}_\psi^1)\), with canonical relation

\[
\mathcal{C}_\psi^1 := \{(t, \tau, G_M^{s+t} \times G_H^{-s}(\zeta), \zeta) \in T^*\mathbb{R} \times \text{Char}(Q_1) \times \text{Char}(\hat{\psi}) : s \in \text{supp } \hat{\psi}, \, \tau + |\zeta_M|_g = 0\}
\]

In the natural parametrization of \(\mathcal{C}_\psi^1\) by \((s, t, \zeta) \in \text{supp } \hat{\psi} \times \mathbb{R} \times \text{Char}(Q_1)\) given by

\[
(t, -|\zeta_M|_g, G_M^{s+t} \times G_H^{-s}(\zeta), \zeta),
\]

the symbol of \(e^{itP}\psi(Q_1)\) is \((2\pi)^{-\frac{3}{2}}\hat{\psi}(s)|ds|^\frac{1}{2} \otimes |dt|^\frac{1}{2} \otimes |d\mu_L|^\frac{1}{2}\), where \(\mu_L\) is Liouville surface measure on \(\text{Char}(Q_1)\).

**Remark 2.3.** Geometrically, this wave front relation consists (initial, terminal) data of pairs of geodesic arcs, an \(H\) arc of length \(s\) and an \(M\)-arc of length \(t + s\), the only constraint on the initial and terminal covectors being that the \(H\) initial (resp. terminal) covectors have the same length as the \(M\) initial (resp. terminal) covectors.

As in [WXZ21] for \(c < 1\), \(e^{itP} \circ \psi(Q_1)\) is a transversal composition, and therefore its order is the sum of the order \(-\frac{1}{4}\) of \(e^{itP}\) [DG75] and the order \(-\frac{3}{4}\) of \(\psi(Q_1)\) (Lemma 2.1).

To reduce to \(H\), we introduce the restriction operator, \(\gamma_H \otimes I : C(M \times H) \to C(H \times H)\), and define

\[
\gamma_{\mathbb{R} \times H \times H} \circ (B_\epsilon(x, D) \otimes I) e^{itP} \psi(Q_1) \circ \gamma_H^* \tag{2.7}
\]

Here, \(B_\epsilon(x, D)\) is a cutoff operator away from normal directions. We refer to [WXZ21] for a discussion of such cutoff operators, but note that in the totally geodesic case the cutoff away from tangential directions is un-necessary. For fixed \(\epsilon > 0\), we define the cutoff operator \(\chi^{(n)}(x, D) = Op(\chi^{(n)}_\epsilon) \in Op(S^0_d(T^*M))\) has its homogeneous symbol \(\chi^{(n)}_\epsilon(x, \xi)\) supported in an \(\epsilon\)-conic neighbourhood of \(N^*H\) with \(\chi^{(n)}_\epsilon \equiv 1\) in an \(\frac{\epsilon}{2}\) subcone. We put, \(B_\epsilon(x, D) = I - \chi^{(n)}_\epsilon(x, D)\).

The following is the analogue of [WXZ21] Lemma Proposition 4.7 when \(c = 1\).
Proposition 2.4. The wave front relation of (2.7) is given by,
\[
\Gamma^1_{\psi,\epsilon} := (\pi_{\mathbb{R} \times H \times H})_! (T^* \mathbb{R} \times T^*_H M \times T^* H \times T^*_H M \times T^* H)
\]
\[
= \{(t, \tau, \pi_{\mathbb{R} \times H \xi}, \pi_{\mathbb{R} \times H}(G^{s+t}_M \times G^{s-t}_H)(\zeta)) : |\zeta_M| + \tau = 0,
\zeta \in \text{Char} Q_1 \cap T^*_H M \times H, G^{s+t}_M(\zeta_M) \in T^*_H M
\]
\[
(1 - \chi_\epsilon)(G^{s+t}_M(\zeta_M)) \neq 0, s \in \text{supp} \psi \}
\subset T^* \mathbb{R} \times (T^* H \times T^* H \times T^* H \times T^* H).
\]

Moreover, on the support of the cutoff operator $B_c(x, D)$ away from $N^* H$, $\Gamma^1_{\psi,\epsilon}$ is a Lagrangian submanifold and the ‘reduced’ Fourier integral operator
\[
\gamma_{\mathbb{R} \times H \times H} \circ (B_c(x, D) \otimes I) e^{itP} \psi(Q_1) \circ \gamma^*_H \times H
\]
belongs to the class
\[
I^{\rho(m,d)}(\mathbb{R} \times (H \times H) \times (H \times H), \Gamma^1_{\psi,\epsilon}),
\]
with
\[
\rho(m, d) = \text{ord} e^{itP} \psi(Q_1) + \frac{1}{2}(n - d) + 2d + \frac{1}{2} - \frac{1}{2}(4d + 1) = \text{ord} e^{itP} \psi(Q_1) + \frac{1}{2}(n - d).
\]

The statement and proof are essentially the same as in the case $c < 1$ of Lemma Proposition 4.7] and we therefore omit the proof and refer there for the details.

Next we pullback under the diagonal embedding to obtain the following is analogue of Proposition 5.2. We use the notation,
\[
\zeta_H = (q, \eta), \zeta^*_H = (q', \eta'), G^s(q, \eta) = (q', \eta')
\]
\[
\zeta = (\zeta_M, \zeta^*_H) = (x, \xi, \eta, \eta') \in \text{Char}(Q_1), (x, \xi) \in T^*_H M, G^{s+t}_M(x, \xi) \in T^*_H M
\]

Proposition 2.5. The wave front relation of $(\Delta_H \times \Delta_H)^* \Gamma^1_{\psi,\epsilon} \subset T^* \mathbb{R} \times T^* H \times T^* H$ is given by,
\[
(\Delta_H \times \Delta_H)^* \Gamma^1_{\psi,\epsilon} = \{(t, \tau, (q, \eta - \pi_H \xi); (q', \eta' - \pi_H \xi')) \in T^* \mathbb{R} \times T^* H \times T^* H : \xi \in T_q M, \xi' \in T_{q'} M, \exists (s, \sigma) : G^s(q, \eta) = (q', \eta'), G^{t+s}(q, \xi) = (q', \xi'), \tau = -|\eta| = -|\xi|, |\eta'| = |\xi'|\}.
\]

Remark 2.6. This wave front set consists of analogues for bi-angles of geodesic loops. A geodesic loop of length $T$ at a point $x \in M$ is given by a geodesic arc $\exp_x T \xi = x$. Unlike a closed geodesic, the initial and term directions do not have to be the same. A bi-angle is the analogue of a closed geodesic but the ‘bi-angle-loop’ consists of two geodesic arcs, an $M$-arc and an $H$-arc from $q$ to $q'$, with no constraint that the projection of the initial or terminal directions of the $M$ arc agree with those of the $H$ arc.

Proof. The calculation is similar to that of [DG75 (1.20)] for the pullback to the ‘single diagonal’ in $M \times M$. The pullback to the ‘double-diagonal’ $\Delta_H \times H \subset H \times H \times H$ subtracts the two covectors at the same base points in the double-diagonal.
In terms of the above notation,
\[(\Delta_H \times \Delta_H)^* \Gamma^1_{\psi,\epsilon} = \{(t, \tau, (q, \eta - \pi_H \xi); (q', \eta' - \pi_H \xi')) \in T^* \mathbb{R} \times T^* H \times T^* H : \exists s (t, \tau, (q, \pi_H \xi), (G^s(q, \eta), \pi_H G^{t+s}(q, \xi))) \in \Gamma^1_{\psi,\epsilon}\} \]

\[= \{(t, \tau, (q, \eta - \pi_H \xi) \exists (s, \sigma, \pi_{H \times H}(x, \xi, y, \eta''), \pi_{H \times H}(G^{s+t}_M \times G^{-s}_H)((x, \xi, y, \eta''))) \in \Gamma^1_{\psi,\epsilon}, (q, q', q') = (x, y, \pi G^{s+t}_M(x, \xi, \pi G^{-s}_H(y, \eta'')), (\zeta_H, \zeta'_H) = (\Delta_H \times \Delta_H)^* (\pi_{H \times H} \zeta, \pi_{H \times H}(G^{s+t}_M \times G^{-s}_H)(\zeta))\}\]

Indeed, on the base $M \times H$, the pullback restricts to the double diagonal,
\[\Delta_H \times \Delta_H(q, q') = (q, q, q') = \pi(\pi_{H \times H}(q, \pi_{H \times H}(G^{s+t}_M \times G^{-s}_H)(\zeta))) \iff q = x = y, q' = \pi G^{s+t}_M(x, \xi) = \pi G^{-s}_H(y, \eta'').\]

Moreover, on the (co-)vector level, $\eta'' = \eta$ and,
\[(\zeta_H, \zeta'_H) = (\Delta_H \times \Delta_H)^* \pi_{H \times H}(\zeta, (G^{s+t}_M \times G^{-s}_H)^*(\zeta)), \iff \pi_H(q, \xi) = (q, \eta), \pi_H G^{t+s}(q, \xi) = (q', \eta') = G^{-s}_H(q, \eta).\]

These conditions imply that $(q, \xi) \in S^* H$, since $\pi_H \xi = \eta$ and $|\xi| = |\eta|$. \hfill \Box

The next step is to pushforward under $\Pi : \mathbb{R} \times H \times H \to \mathbb{R}$, which results in ‘closing’ the bi-angle-loop wave front set to the set of ‘closed bi-angles’.

**Proposition 2.7.** The pushforward wave front set is given by,
\[\Lambda^1_\psi := \Pi_* (\Delta_H \times \Delta_H)^* \Gamma^1_{\psi,\epsilon} \subset T^* \mathbb{R} = \{(t, \tau) \in T^* \mathbb{R} : \exists (q, \eta) : G^t(q, \eta) = (q, \eta)\}\]  
(2.9)

**Proof.** As in [DG75] (1.21]), the pushforward operation erases points of
\[(\Delta_H \times \Delta_H)^* \Gamma^1_{\psi,\epsilon} = \{(t, \tau, (q, \eta - \pi_H \xi); (q', \eta' - \pi_H \xi'))\} \]

unless $\eta - \pi_H \xi = \eta' - \pi_H \xi' = 0$ and the output for such vectors is $(t, \tau)$. Since $H$ is totally geodesic, and $\pi_H \xi = \eta$, one has $G^{t+s}(q, \eta) = G^s(q, \eta)$. Cancelling the $s$ factors gives the result. \hfill \Box

2.3. **Cleanliness issues.** We recall that except for the last statement of Theorem 1.2 and Theorem 1.7, it is assumed that $\text{supp} \hat{\rho}, \text{supp} \hat{\rho} \subset (-r_0, r_0)$. The wave front calculations above explain the difficult cleanliness issues that arise if we drop the assumption that $\text{supp} \hat{\rho} \subset (-r_0, r_0)$. In the intermediate wave front set calculations, one would need that the set of arcs from $q$ to $q'$ of arbitrary length form clean submanifolds. This is obviously true for $H$ arcs or $M$-arcs of lengths in $(-r_0, r_0)$, where the arc from $q$ to $q'$ is unique but is rarely true for long arcs. In particular, there are problems if $q, q'$ are conjugate points along an $H$ arc.

It is possible that the wave front relations are not Lagrangian submanifolds until the last step, where the set of arcs consists of closed geodesics. In that case, we would need that the fixed point sets $\text{Fix} G^i_H$ are all clean in the sense of [DG75]; this is substantially simpler than cleanliness of the arc-sets in the intermediate steps, but is still non-generic.
2.4. The order of \((1.7)\) at \(t = 0\). We now calculate the order of \((1.7)\) after further composing with the diagonal pullback to \((H \times H) \times (H \times H)\) and then the pushforward to \(\mathbb{R}\). The trace is the composition defined by the fiber product diagram of \(\Gamma_{\psi, \epsilon}^{1}\) with the conormal bundle of the diagonal of \(H \times H\) (see [WXZ11 (8.6)]). Note that the order at \(t = 0\) is quite different from the case \(c < 1\) in [WXZ11], and justifies including the material in the previous section.

In a neighborhood of \(t = 0\), \(\hat{\psi}\) is a Lagrangian distribution and we may calculate its order using the calculus of Fourier integral operators despite the blow-down singularity at \(s = 0\).

**Lemma 2.8.** As in [WXZ11 (5.10)], if \(\hat{\psi} = 0\) in an interval around \(s = 0\), then the order of \(S(t, \psi)\) is \(-3 + \frac{1}{2}(n - d) + \frac{1}{2} \dim \mathcal{G}^T_1\), where \(\mathcal{G}^T_1 = \text{Fix}(\mathcal{G}^1_1)\). From the fact that \(\dim \mathcal{G}^0_1 = \dim S^*H = 2d - 1\),

\[
\text{ord}S(t, \psi)|_{t=0} = -\frac{3}{4} + \frac{1}{2}(n - d) + \frac{1}{2} - \frac{5}{4} + \frac{n + d}{2} = -\frac{1}{4}.
\]

Recall that \(I_{\pi - \frac{1}{2}}^n\) has symbols of order \(s^{\frac{n - 1}{2}}\).

**Proof.** The composition is clean away from the singularity at \(s = 0\). The order and composition are therefore computed precisely as in [DG75] and [WXZ11 (5.7)-(5.8)]. However the excess \(e(0)\) is different and results in the different order. \(\square\)

3. **Proof of Theorem \((1.2)\) and Theorem \((1.7)\) using Hörmander parametrices**

In this section, we use the Hörmander parametrices of the wave proof to prove Theorem \((1.7)\) for \(\psi\) such that \(\hat{\psi} = 0\) in some interval around \(0\). Although this does not clarify the regularization at \(s = 0\), its phase is simple and exhibits the blow-down singularity responsible for the singular coefficient in Theorem \((1.2)\). In fact, it is precisely the phase of a certain Bessel-type integral that we refer to as a “double-Bessel function” \((4.6)\). In Section 4 we build on this parametrix approach to construct model oscillatory integrals. We then reduce the actual problem to the model case. The Hörmander parametrix method is also used in [WXZ11] and makes it easy to compare with the case \(c < 1\) (see Section 4.6).

3.1. **Hörmander small time parametrix.** In this section we review the Hörmander parametrix of the half-wave kernel and use it to derive the following.

**Proposition 3.1.** Let \((M, g)\) be any compact Riemannian manifold of dimension \(n\), and let \(H \subset M\) be a totally geodesic submanifold of dimension \(d \leq n - 1\). Then, there exists a semi-classical amplitude \(\tilde{A}(\langle y, \omega \rangle, \langle y, \bar{\omega} \rangle, q, y, \omega, \bar{\omega})\) of order zero such that,

\[
N^{1}_{\psi, \rho, \theta, H}(\lambda) = \lambda^{n + d - 2} \int_{\infty}^{0} \int_{H} \int_{\mathbb{T}_{q}H} \int_{\mathbb{T}_{q}M} \hat{\psi}(\langle y, \bar{\omega} \rangle) \hat{\rho}(\langle y, \omega \rangle) e^{i\lambda \langle y, \omega + \bar{\omega} \rangle} dV_{H}(y)dV_{H}(q)dS_{q}(\omega)dS_{q}(\bar{\omega}),
\]

(3.1)
When \(d = 1\), \(\mathbb{S}_{q}H = \{\pm e_{1}\}\) and the integral is a sum over \(\pm\).

**Proof.** On any Riemannian manifold, we may construct small time parametrix for \(U(t) = e^{it\sqrt{-\Delta}}\) an oscillatory integral of the form,

\[
U(t, x, y) = \int_{T_{q}M} e^{i\langle \exp_{x}^{-1}(y), \xi \rangle} e^{it|\xi|} A(t, y, \xi) d\xi,
\]

(3.2)
where $A(t, x, y, \xi)$ is a homogeneous amplitude of order 0 and supported in the set $r(x, y) \leq t + \delta$ for any small $\delta > 0$. Here, $exp_x : B_x(\epsilon) \subset T_xM \to M$ is the exponential map and $B_x(\epsilon)$ is a sufficiently small ball so that $exp_x$ is a diffeomorphism to its image. The amplitude is independent of $x$ and for $|t| < \text{inj}(M, g)$ satisfies, for all $(y, \xi) \in T^*M$,

$$
\begin{cases}
A(0, y, \xi) = 1, \\
A(t, y, \xi) - 1 \in S^{-1}.
\end{cases}
$$

(3.3)

Then, if $\hat{\rho}$ and $\hat{\psi}$ have sufficiently small support so that the parametrix (3.2) is valid for both $U_M(t, U_H(s)$,

$$
N^1_{H, \psi, \rho}(\lambda) = \int_{\mathbb{R}} \int_{H \times H} \hat{\rho}(t)e^{it\lambda} \Pi_{H \times H} \Delta^*_H \gamma_H \hat{\psi}(Q_t) e^{itP} \gamma_H dt dv_H(q) dv_H(q')
$$

$$= \int_{\mathbb{R}} \int_{H \times H} \hat{\psi}(s) \rho(t)e^{it\lambda} U_H(-s, q, q') U_M(t + s, q, q') dv_H(q) dv_H(q') ds dt
$$

has the parametrix,

$$
\int_{H \times H} \int_{T_qH} \int_{\mathbb{R}} \int_{T_qM} \int_{T_{q'H}} \hat{\psi}(s) \rho(t) e^{it\lambda} \rho \tilde{A}_1(t, s, q, q', \xi, \eta) dv_H(q) dv_H(q') ds dt ds d\xi d\eta.
$$

(3.4)

where

$$
\Psi_1 = \langle (\exp^M_q)^{-1}(q'), \xi \rangle + (\exp^H_q)^{-1}(q', \eta) + (t + s)|\xi| - s|\eta|,
$$

and where

$$
\tilde{A}_1(t, s, q, q', \xi, \eta) = A_M(t + s, q, q', \xi)A_H(-cs, q, q', \eta).
$$

We change variables $\xi \to \lambda \xi$, $\eta \to \lambda \eta$ and obtain a semi-classical oscillatory integral,

$$
\lambda^{n+d} \int_{H \times H} \int_{\mathbb{R}} \int_{T_qH} \int_{T_qM} \int_{T_{q'H}} \hat{\psi}(s) \rho(t) e^{it\lambda} e^{i\lambda \Psi_1} \tilde{A}(t, s, q, q', \xi, \eta) dv_H(q) dv_H(q') ds dt ds d\xi d\eta.
$$

(3.5)

To simplify the integral, we set $\xi = \rho \omega, \eta = \sigma \tilde{\omega}$ and change variables to get the phase,

$$
\Psi_2(t, s, q, q', \rho, \omega, \sigma, \bar{\omega}) := t(1 - \rho) + s(\rho - \sigma) + \rho(\langle (\exp^M_q)^{-1}(q'), \omega \rangle + \sigma(\langle (\exp^H_q)^{-1}(q'), \bar{\omega} \rangle.
$$

We eliminate the pair of variables $dsdt$ by stationary phase, which introduces a new factor $\lambda^{-1}$ and sets $\rho = 1, t = \langle (\exp^M_q)^{-1}(q'), \omega \rangle$. The Hessian determinant equals 1, reducing the integral to,

$$
\lambda^{n+d-1} \int_{H \times H} \int_{\mathbb{R}} \int_{T_qH} \int_{T_qM} \int_{T_{q'H}} \hat{\psi}(s) \rho(\langle (\exp^M_q)^{-1}(q'), \omega \rangle) e^{i\lambda \Psi_2}
$$

$$
\tilde{A}_2(\langle (\exp^M_q)^{-1}(q'), \omega \rangle, sq, q', \xi, \eta) dv_H(q) dv_H(q') ds dS(\omega) \sigma^{d-1} d\sigma dS(\bar{\omega}),
$$

where

$$
\Psi_2 = s(1 - \sigma) + \langle (\exp^M_q)^{-1}(q'), \omega \rangle + \sigma(\langle (\exp^H_q)^{-1}(q'), \bar{\omega} \rangle,
$$

and where the new amplitude $\tilde{A}_2$ still satisfies (3.3).

Next we eliminate $d\sigma ds$ by stationary phase to obtain $\sigma = 1$ and $s = \langle (\exp^H_q)^{-1}(q'), \bar{\omega} \rangle$. The Hessian again equals 1 and the oscillatory integral reduces to and reduce to,
\[ \lambda^{n+d-2} \int_{H \times H} \int_{\hat{S}_q^* M} \int_{\hat{S}_{q'}^* H} \hat{\psi}((\exp_q^H)^{-1}(q'), \hat{\omega})) \hat{\rho}((\exp_{q'}^M)^{-1}(q'), \hat{\omega})) e^{i\lambda \Psi_3} \]

\[ \tilde{A}_3((\exp_q^M)^{-1}(q'), \omega), ((\exp_{q'}^H)^{-1}(q'), \hat{\omega})q, q', \omega, \hat{\omega}) dV_H(q) dV_H(q') dS(\omega) dS(\hat{\omega}), \]

where \[
\Psi_3(q, q', \omega, \hat{\omega}) := ((\exp_q^M)^{-1}(q'), \omega) + ((\exp_{q'}^H)^{-1}(q'), \hat{\omega}),
\]

and where the new phase satisfies \[
\tilde{A}_3((\exp_q^M)^{-1}(q'), \omega), ((\exp_{q'}^H)^{-1}(q'), \hat{\omega})q, q', \omega, \hat{\omega}) = 1, \quad \text{(when } q = q').
\]

3.1.1. \textbf{H totally geodesic.} We now use that \( H \) is totally geodesic for the first time. Since \( q, q' \in H, (\exp_q^M)^{-1}(q') = (\exp_{q'}^H)^{-1}(q') \), and we may combine terms to simplify the phase to, \[
\Psi(q, q', \omega, \hat{\omega}) := ((\exp_q^H)^{-1}(q'), \omega + \hat{\omega}).
\]

For simplicity of notation, we henceforth denote \[
y = (\exp_q^H)^{-1}(q') \in T_q H, r = r_H(q, q'), y = rv
\]

and obtain the universal phase, \[
\Psi(y, \omega; \hat{\omega}) := \langle y, \omega + \hat{\omega} \rangle, \quad (y \in T_q^* H, \omega \in S^*_q M, \hat{\omega} \in S^*_q H).
\]

Henceforth, we regard \((q, \omega)\) as parameters and consider the oscillatory integral, \[
N_{H, \psi, \rho}^1(\lambda, q, \omega) = \lambda^{n+d-2} \int_{\hat{S}_q^* M} \int_{\hat{S}_{q'}^* H} \hat{\psi}((\exp_q^H)^{-1}(q') \omega) \hat{\rho}((\exp_q^M)^{-1}(q') \omega)) e^{i\lambda (y, \omega + \hat{\omega})} \]

\[ \tilde{A}(\langle y, \omega \rangle, \langle y, \hat{\omega} \rangle, y, \omega, \hat{\omega}) dy dS(\omega), \]

where \( \tilde{A} = 1 \) when \( y = 0 \).

This completes the proof of Proposition 3.1.

\[ \square \]

3.2. \textbf{Analysis of the phase.} For purposes of comparison to model phases in Section 4 we extract from (3.1) the sub-integrals, \[
J_{H, M}^1(\lambda, q, rv, \chi, \psi, \rho) := \int_{\hat{S}_q^* H} \int_{\hat{S}_{q'}^* M} \hat{\psi}(r(v, \omega)) \hat{\rho}(r(v, \omega)) e^{i\lambda (v, \omega + \hat{\omega})} \]

\[ \tilde{A}(\langle v, \omega \rangle, \langle v, \hat{\omega} \rangle, q, rv, \omega, c\hat{\omega}) dS_q(\omega) dS_q(\hat{\omega}). \]

As will be seen in Section 4 they are closely related to Bessel (or, rather, double-Bessel) integrals (4.6) and exhibit the same blow-down singularities. Their relation to (3.1) is given by, \[
N_{\psi, \rho, H}^1(\lambda) = \lambda^{n+d-2} \int_0^\infty \int_H \int_{\hat{S}_q^* H} \int_{\hat{S}_{q'}^* M} J_{H, M}^1(\lambda, q, rv, \chi, \psi, \rho) r^{-d-1} dr dV_H(q) dS_q(v).
\]

As before, we regard the variables \((q, \omega) \in S^*_q H\) as parameters and the the variables \((q', \omega) \in H \times S^{n-1}\) as the “phase variables” of the oscillatory integral. We change variables as in that section, to write \[
q' = \exp_q y = \exp_q(rv), \quad y = rv \in T_q H, v \in S_q H, r > 0, \quad (y, \omega) \in T_q^* H \times S^*_q M.
\]
This is possible since the short-time parametrix is used only for \((q, q')\) near the diagonal \(\Delta_H \times H \times H\). It should be kept in mind that \(y = 0\) or \(r = 0\) corresponds to the diagonal in \(H \times H\). As usual, we identify \(T_q^*H = T_qH\) using the metric \(g\) without further comment.

We therefore study the inner integral of Proposition 3.1

\[
u(\lambda, q, \tilde{\omega}) := \int_{T_qH} \int_{S^1} \psi((y, \tilde{\omega})) \rho((y, \omega)) e^{i\lambda(y, \omega + \tilde{\omega})} \tilde{A} \, dydS_q(\omega),
\]

where \(\tilde{A} = \langle (y, \omega), \langle y, \tilde{\omega} \rangle, q, y, \omega, \tilde{\omega} \rangle\). Recall that this parametrix expression is only valid for \(|t|\) smaller than the injectivity radius of \(H\).

The phase function of (3.11) or (3.14) in the coordinates (3.13) is,

\[
\Phi(\tilde{\omega}; y, \omega) := \langle y, \omega - \tilde{\omega} \rangle : S^*_qM \times T_qH \times S^*_qH \to \mathbb{R}.
\]

Here, \(q \in H\) is fixed and is not indicated further in the notation for (3.14), henceforth, we view the phase (3.15) as defined on \(S^d_{\omega} \times \mathbb{R}^d \times S^{n-1}\).

In this section, we analyze the Lagrangian submanifold generated by the phase (3.15) and, in particular, the singularity of its projection at \(y = 0\) (i.e., on the diagonal). In Section 4, we show that the cubic Taylor approximation to (3.15) is the normal form phase function generating a Lagrangian submanifold with a blow-down singularity. We follow [DG75] and [HoIV] for background on clean phase functions of homogeneous oscillatory integrals, [D74] for background on semi-classical (non-homogeneous) oscillatory integrals with a large parameter, and [G89] for background on blow-down maps.

We recall (cf. [DG75, Page 71]) that a phase function \(\varphi(x, \theta)\) on \(X \times \mathbb{R}^N\) is clean if

\[
C_\varphi := \{(x, \theta) \in X \times \mathbb{R}^N : \nabla_\theta \varphi(x, \theta) = 0\}
\]

is a submanifold of \(X \times \mathbb{R}^N\) and at each point of \(C_\varphi\) the tangent space is the space of vectors annihilated by \(d(\varphi)_{\theta_1}, \ldots, d(\varphi)_{\theta_N}\). The excess \(e\) of the clean phase is defined so that \(N - e\) is the dimension of the space spanned by these differentials. When the phase is clean, the map

\[
\iota_\varphi : C_\varphi \to \Lambda_\varphi \subset T^*X, \quad \iota_\varphi(x, \theta) = (x, \nabla_x \varphi)
\]

is a fiber mapping of fiber dimension \(e\) over its image \(\Lambda_\varphi\). The Leray measure \(dC_\varphi\) on \(C_\varphi\) is the pullback \(\delta_0(\nabla_\theta \varphi)\) of \(\delta_0\) under the map \(\nabla_\theta \varphi : X \times \mathbb{R}^N \to \mathbb{R}^N\). It is well-defined if \(\nabla_\theta \varphi\) is a submersion.

For the phase (3.15) with \(q\) fixed, the role of \(X\) is played by \(\tilde{\omega} \in S^{d-1}\) and the role of the phase variable \(\theta \in \mathbb{R}^N\) is played by \(\theta = (y_1, \ldots, y_d, \omega \in S^{n-1}\), with \(N = d + n - 1\). Then,

\[
\nabla_{y, \omega} \Psi = (\omega - \tilde{\omega}, y - \langle y, \tilde{\omega} \rangle, \omega) \in S^{n-1} \times H \simeq S^{n-1} \times \mathbb{R}^d
\]

and the critical set of the phase (3.15) is given by,

\[
C_\Psi = \{(\tilde{\omega}; y, \omega) : \omega = \tilde{\omega}, \quad y = \langle y, \tilde{\omega} \rangle \omega \} \subset S^*_qM \times T_qH \times S^*_qM.
\]

As the next Lemma shows, the phase (3.15) fails to be clean (much less, non-degenerate).

**Lemma 3.2.** We have,

- (i) The phase (3.15) is a clean phase of excess \(e = 1\) on the complement of the diagonal \(q = q'\), i.e., for \(y \neq 0\). That is, the kernel of \(d\Psi_{y, \omega}'\) on \(T_{\tilde{\omega}, y, \omega}(S^{d-1} \times \mathbb{R}^d \times S^{n-1})\) equals \(T_{\tilde{\omega}, y, \omega} C_\Psi\) when \(y \neq 0\). If \(y = rv\), all components of \(d_{\tilde{\omega}, v, \omega} \Psi\) are independent on \(C_\Psi\).
• (ii) The phase is not clean along the set where \( \{ y = 0 \} \) (the diagonal). The kernel of \( d\Psi(y, \omega) \), \( T^*_\omega \) \( (S^{d-1} \times \mathbb{R}^d \times S^{n-1}) \) jumps at \( y = 0 \) to include \( SN^*\Delta_{\text{H} \times H} \), i.e. \( \omega \in S^*_q \mathcal{M} : \pi_H \omega = 0 \).

• (iii) The map \( \iota_\Psi : C_\Psi \setminus \{ y = 0 \} \rightarrow \Lambda_\Psi \) is an \( \mathbb{R}^* \) bundle over the zero section \( \Lambda_\Psi = 0_{T^*S^{d-1}} \subset T^*S^{d-1} \). It has a blow-down singularity over \( \{ y = 0 \} \) (see Section 8.7).

Proof. Proof of (i)

For fixed \( \tilde{\omega} \), the equation \( y = (y, \tilde{\omega}) \tilde{\omega} \) determines \( v = \frac{\partial}{\partial y} = \tilde{\omega} \) and therefore the slice \( C_\Psi(\tilde{\omega}) \) with fixed \( \tilde{\omega} \) can be identified with \( r \in (0, \infty) \) when \( r > 0 \). \( C_\Psi(\tilde{\omega}) \) also contains the point \( y = 0 \) for every \( \tilde{\omega} \). It follows that \( C_\Psi \simeq S^*_q H \times S^+_+ \) is a manifold with boundary \( S^*_q \times \{ 0 \} \). Henceforth, we put

\[
\Psi_\pm(y, \omega) = \Psi(\tilde{\omega}, y, \omega).
\]

To calculate the covectors \( d(\Psi_y \Psi) = d\nabla_{y, \omega} \Psi \), we use that the phase is symmetric under the diagonal action of \( SO(d) \) on \( S^{d-1} \times \mathbb{R}^d \times S^{n-1} \), where it acts on the third factor by \( g \cdot \omega = (g \pi_H \omega, \omega^\perp) \). We fix \( q \in H \) and define linear coordinates \( (y_1, \ldots, y_d) \) on \( T_qH \simeq \mathbb{R}^d \). Without loss of generality we may assume that \( \tilde{\omega} = e_d \). We also fix linear coordinates \( (x_1, \ldots, x_n) \) on \( T^*_q \mathcal{M} \simeq \mathbb{R}^n \) and endow \( S^*_q \mathcal{M} \simeq S^{n-1} \) with the coordinates \( x' := (x_1, \ldots, x_{d-1}, x_{d+1}, \ldots, x_n) \), with \( x_d = \pm \sqrt{1 - |x'|^2} \). Then \( \omega = (x_1, \ldots, x_{d-1}, \pm \sqrt{1 - |x'|^2}, x_{d+1}, \ldots, x_n) \).

In these coordinates, \( e_d = (0, 1, 0) \) corresponds to \( x' = 0 \) in \( B_1(\mathbb{R}^{n-1}) \). Then,

\[
\omega - \tilde{\omega} = (x_1, \ldots, x_{d-1}, \pm \sqrt{1 - |x'|^2} - 1, x_{d+1}, \ldots, x_n), \quad (\tilde{\omega} = e_d)
\]

so that for \( |x'| < 1 \) and on the \pm hemisphere \( S^{d-1}_+ \) where \( x_d = \pm \sqrt{1 - |x'|^2} \), (3.15) reduces to

\[
\Psi_\pm := \sum_{j=1}^{d-1} y_j x_j \pm y_d(\sqrt{1 - |x'|^2} - 1) = \sum_{j=1}^{d-1} y_j x_j + y_d(x_d - 1). \tag{3.17}
\]

We note that the choice of \( \tilde{\omega} = e_d \) creates the sign asymmetry between \( \pm \sqrt{1 - |x'|^2} - 1 \).

Next, we calculate the covectors \( d(\Psi_y \Psi) = d\nabla_{y, \omega} \Psi \) in these coordinates. Let \( \partial_{x_j} = \frac{\partial}{\partial x_j} \).

Since \( \partial_{x_j} \sqrt{1 - |x'|^2} = \frac{x_j}{\sqrt{1 - |x'|^2}} \), \( j \neq d \), the critical point equations of the phase (3.17) on the complement of \( x_d = 0 \) are,

\[
\begin{align*}
(i) & \quad \partial_{y_j} \Psi^{e_d} = x_j = 0 \quad (j = 1, \ldots, d - 1); \\
(ii) & \quad \partial_{y_d} \Psi^{e_d} = x_d - 1 = 0 \iff x_d = 1 \iff \pm \sqrt{1 - |x'|^2} = 1 \iff +, x' = 0; \\
(iii) & \quad \partial_{x_j} \Psi^{e_d} = y_j + y_d \frac{x_j}{\sqrt{1 - |x'|^2}} = 0 \iff y_j = 0, j = 1, \ldots, d - 1; \\
(iv) & \quad \partial_{x_j} \Psi^{e_d} = y_d \frac{x_j}{\sqrt{1 - |x'|^2}} = 0 \iff y_d = 0 \text{ or } x_j = 0, \quad j = d + 1, \ldots, n; \tag{3.18}
\end{align*}
\]

There is a sign asymmetry in \( \pm \) due to the choice of \( \tilde{\omega} = e_d \) in (ii), since there are no solutions for the \( - \) sign. In (iii) we use (ii) to simplify the equation. By (i), \( x_j = 0 \), for \( j = 0, \ldots, d - 1 \), hence on the critical set, \( x' = (0, 0, \ldots, 0, x_d, x_{d+1}, \ldots, x_n) \). The critical point equation in \( y_d \)
implies $x_d = \pm 1$ (hence $1$ as in (ii) since we have chosen $e_d$ as the base point). It follows that $x_{d+1}, \ldots, x_n = 0$ at a critical point. Combining the first critical point equation with the second critical point equation, one finds that $y_j = 0$ for $j = 1, \ldots, d - 1$. When $y_d = 0$, equation (iv) does not force $x_{d+1}, \ldots, x_n = 0$ but it is forced by (ii).

To check the condition that $TC_{\Psi}$ is the nullspace of the differentials, we form the Hessian matrix of $\Psi$, 

$$
(D\nabla \Psi^\omega)|_{C_{\Psi}} = \begin{pmatrix}
y & \omega \\
y D_2^y \Psi^\omega & D_2^2 \Psi^\omega \\
\omega & D_\omega D_\omega^\omega \Psi^\omega
\end{pmatrix} = \begin{pmatrix}
A & B \\
B^T & D
\end{pmatrix}.
$$

Here, $A$ is $d \times d$, $B$ is $(n - 1) \times d$ and $D$ is $(n - 1) \times (n - 1)$. Intrinsically, the Hessian acts on $T_qH \oplus T(S^*_qM)$.

For future reference, we recall [L88] that the signature of an invertible symmetric matrix $M$ as above, with inverse 

$$
(B')^T D' \begin{pmatrix}
A' \\
\vdots \\
B'
\end{pmatrix}
$$

is given by 

$$
\text{sgn } M = \text{sgn } A + \text{sgn } D'.
$$

(3.19) 

**Proof of (ii)** The following Lemma implies (i)-(ii).

**Lemma 3.3.** Let $r = \text{Rank}(D\nabla \Psi^\omega)|_{C_{\Psi}}$ and let $V = \text{Ker}(D\nabla \Psi^\omega)|_{C_{\Psi}}$.

1. When $y \neq 0$, $r = n + d - 2$ and $V = \text{span} \frac{\partial}{\partial n}$, the radial vector in $T_qH$.

2. When $y = 0$, $r = 2d - 2$ and $V = \text{span} \frac{\partial}{\partial n} \oplus T(S^*_qM)$.

**Proof.** Since $A = 0$, the rank of the Hessian is $\text{rank } B + \text{rank } \begin{pmatrix} B \\ D \end{pmatrix}$. $D_y \Psi^\omega = \pi_{\Psi}^\omega = \tilde{\omega}$ on the critical set and its derivative there always has rank $d - 1$. Since $\Psi^\omega = H_{\omega\omega}(y) = -|y|I_{n-1}$ is the second fundamental form of $S^{n-1}$ at point where $y$ is normal, it equals $-|y|$ times the identity matrix of rank $n - 1$. When $y = 0$ then $D = 0$ and the rank equals $2 \text{ rank } B = 2d - 2$. Thus, there is a drop in rank by $n - d$ when $y = 0$ and the phase fails to be clean when $y = 0$. But it is of constant rank and is clean for $y \neq 0$.

**Proof of (iii)**

The associated Lagrange map is 

$$
\iota_{\Psi} : C_{\Psi} \rightarrow T^*S^{d-1}, \quad \iota_{\Psi}(\tilde{\omega}, y, \omega) = (\tilde{\omega}, d_\omega \Psi(\tilde{\omega}, y, \omega)) = (\tilde{\omega}, y - \langle y, \tilde{\omega} \rangle \tilde{\omega}) = (\tilde{\omega}, 0) \in 0_{T^*S^{d-1}}.
$$

The fiber of this map (with $q$ fixed and suppressed) is given by 

$$
\iota_{\Psi}^{-1}(\tilde{\omega}, 0) = \{(\tilde{\omega}, y, \omega) : \omega = \tilde{\omega}, y = \langle y, \tilde{\omega} \rangle \tilde{\omega}\} \simeq \mathbb{R}^*.
$$

(3.20) since the equation $y = \langle y, \tilde{\omega} \rangle \tilde{\omega}$ is homogeneous under multiplication of $y$ by $x \in \mathbb{R}$. If we denote $v = \frac{\omega}{|y|}$ then $v = \pm \tilde{\omega}$.

It will be explained in Section 8.1 that $\iota_{\Psi}$ has a blow down singularity over $\{y = 0\}$. 

3.3. **Stationary phase.** For the sake of completeness, we recall the stationary phase method (cf. [HoIV, Volume I]).

**Theorem 1.** Let \( K \subset \mathbb{R}^n \) be compact, let \( U \) be an open neighborhood of \( K \), and let \( k \in \mathbb{N} \). Let \( a \in C^\infty_0(K) \), \( S \in C^\infty(U) \) with \( \text{Im} \ S = 0 \). Assume \( S'(x_0) = 0, \det \ S''(x_0) \neq 0, \ S' \neq 0 \) in \( K \setminus \{x_0\} \). Then:

\[
\int_{\mathbb{R}^n} a(x)e^{i\lambda S(x)} \, dx = e^{i\lambda S(x_0)} \sqrt{\det(\lambda S''(x_0))/2\pi i} \sum_{j<k} \lambda^{-j} L_j a(x_0)
\]

+ \( O(\lambda^{-k} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u(x)|) \).

Here, if \( g_{x_0}(x) = S(x) - S(x_0) - \langle S''(x_0) (x - x_0), (x - x_0) \rangle / 2 \) then

\[
L_j a = \sum_{\nu-\mu=j} \sum_{2\mu \geq 2j} \frac{i^{-j/2-\nu}}{\mu!\nu!} \langle S''(x_0)^{-1} D, D \rangle^\nu (g_{x_0}^\mu a).
\]

3.4. **Stationary phase analysis away when \( \hat{\psi}(s) \) vanishes near \( s = 0 \).**

**Proof.** By Lemma 3.2 stationary phase applies to the subintegral of (3.11) defined by,

\[
I_2(q, \bar{\omega}, \lambda) \simeq \lambda^{n+d-2} \int_{T_H} \int_{S_M} \hat{\psi}(y, \bar{\omega}) \hat{\rho}(y, \omega) e^{i\lambda \Psi} \tilde{\Lambda} dy dS(\omega),
\]

in the ‘phase variables’ \( (\omega, y) \), with \( \Psi \) defined in (3.15), and with amplitude

\[
\tilde{\Lambda} = \tilde{A}(r \langle v, \omega \rangle, r \langle \bar{\omega}, \bar{\omega} \rangle, q, r v, \omega, \bar{\omega}),
\]

where \( y = rv \), and where \( (q, \bar{\omega}) \) are regarded as parameters. By (3.16), the phase is stationary if and only if \( \omega = \bar{\omega} = \pm v = \pm \frac{y}{|y|} \) with \( |\det \text{Hess} \Psi|^{-\frac{1}{2}} = r^{-(n-d)} \). The critical set for fixed \( (q, \bar{\omega}) \) may be identified with \( \mathbb{R}_x \); the phase equals zero on the critical set. When \( y \neq 0 \) we can use the Schur determinant formula to obtain,

\[
\det \text{Hess} = \det D \det(B^T D^{-1} B) = |y|^{n-1} |y|^{-(d-1)} = |y|^{-(n-d)} = r^{-(n-d)}.
\]

The signature of the Hessian at the critical point is given by,

\[
\text{sgn Hess} = 0.
\]

Indeed, it is recalled in (3.19) that the signature of the Hessian is the signature of \( D' \) where the inverse of the Hessian has lower right block \( D' \); here we use that the upper left block of the Hessian equals zero. By the Schur complement inverse formula, when \( A = 0 \) the Schur complement is \( M/D = -BD^{-1}B^T \) and

\[
M^{-1} = \begin{pmatrix}
(M/D)^{-1} & -(M/D)^{-1} BD^{-1} \\
-D^{-1}B^T(M/D)^{-1} & D^{-1} + D^{-1}B^T(M/D)^{-1}BD^{-1}
\end{pmatrix},
\]

so (again assuming \( A = 0 \))

\[
\text{sgn} M = \text{sgn} \left( D^{-1} + D^{-1}B^T(M/D)^{-1}BD^{-1} \right) = \text{sgn} \left( D^{-1} + D^{-1}B^T(-BD^{-1}B^T)^{-1}BD^{-1} \right)
\]

\[
= -\text{sgn}(I_{n-1} - B^T(BB^T)^{-1}B),
\]

since by Lemma 3.3 \( D = -|y|I_{n-1} \), while \( B = D \omega \pi_H \omega \). Now \( BB^T = I_{n-1} \), concluding the proof.
Since
\[ \langle y, \omega \rangle = r \langle v, \omega \rangle = \pm r = x \]
on the critical set, modulo dimensional constants \( C_{n,d} \),
\[
I_2(q, \omega, \lambda) \approx \lambda^{n+d-2} \int_{\omega \in S^{n-1}} \int_{\mathbb{R}^d} \tilde{\psi}(\langle y, \omega \rangle) \hat{\rho}(\langle y, \omega \rangle) \chi_2(y) e^{i\lambda(\langle y, \omega \rangle)} \tilde{A} dy dS(\omega)
\]
\[
\approx \lambda^{n+d-2} \lambda^{-\frac{n-d}{2}} \int_{-\infty}^{\infty} \hat{\rho}(x) \tilde{\psi}(x) A_0(q, \omega, x) |x|^{-\frac{n-d}{2}+d-1} dx,
\]
where \( A_0(q, \omega, x) := \tilde{A}_0(x, x, q, x\omega, \omega, \omega) \).

Assuming that \( \hat{\rho} = 1 \) on \( \text{Supp} \tilde{\psi} \) and \( \text{Supp} \hat{\psi} \subset (0, \infty) \), the integral has the same singularity (to leading order) as,
\[
C_{n,d} \lambda^{\frac{n+d}{2}-1} \int_{0}^{\infty} \tilde{\psi}(s) s^{-\frac{n-d}{2}} ds.
\]

This is a preliminary result since we have assumed the \( \hat{\psi} \) vanishes near 0. The stationary phase expansion is not valid all the way down to \( s = 0 \), as one can easily verify in model cases such as for Bessel functions (Section 4.5). In the next sections we will prove the complete formula for the leading coefficient. We also calculate the amplitude explicitly by using a Hadamard parametrix in Section 5. In the next section, we use a model integral with a sufficiently accurate Taylor approximation to the phase, which clarifies the distributional coefficient (1.4) when \( [0, 1] \subset \text{Supp} \tilde{\psi} \).

4. **Proof of Theorem 1.2 and 1.7 using a model phase**

In this section, we complete the proof of Theorem 1.2 and Theorem 1.7 by approximating the phase \( \langle y, \omega - \omega \rangle \) by its Taylor expansion up to order 4. This allows one to distinguish one variable that causes degeneracy of the stationary phase method. It is possible to apply the stationary phase method in the remaining variables and then to integrate the result in the distinguished variable.

4.1. **Model phase.** In determining properties of the Hessian, and in calculating asymptotics of integrals, it is convenient to Taylor expand (3.17) around its critical point \( x' = 0 \). We define the ‘model’ phase,
\[
\Psi_{\text{model}}(\bar{y}, \bar{x}) = \sum_{j=1}^{d-1} y_j x_j - \frac{1}{2} y_d |x'|^2 : \mathbb{R}^d \times B_1(\mathbb{R}^{n-1}) \rightarrow \mathbb{R}
\]
\[
= \sum_{j=1}^{d-1} y_j x_j - \frac{1}{2} y_d (x_1^2 + \cdots + x_{d-1}^2 + x_{d+1}^2 + \cdots + x_n^2).
\]
We view the variables \( (y_d, x_d, \ldots, x_n) \) as parameters and consider the phase (3.17) as a function \( \Psi_{\text{model, d-1}} \) of \( (y_1, \ldots, y_{d-1}, x_1, \ldots, x_{d-1}) \). The critical point analysis in Lemma 3.2 applies to the model phase as much as to (3.17) since they agree modulo terms of order 4 by the next Lemma. Since \( \sqrt{1 - |x'|^2} = 1 - \frac{1}{2} |x'|^2 + O(|x'|^4) \), we have

**Lemma 4.1.** In the above coordinates, when \( y \neq 0 \) the universal phase (3.17) has a unique critical point \( \bar{x} = e_d \) and \( y = y_d e_d \), and satisfies,
\[
\langle y, \omega - \omega \rangle = \Psi_{\text{model}}(\bar{y}, \bar{x}) + O(y_d |x'|^4).
\]
The Hessian of (3.17) equals that of (4.1) at the critical points.
Lemma 4.2. For any fixed \((y_d, x_d, x_{d+1}, \ldots, x_n)\), the Hessian of (3.17) (or equivalently, \(\Psi_{\text{model}, d}\)) at \(y' = 0 = x'\) is non-degenerate in the variables \((y', x') = (y_1, \ldots, y_{d-1}, x_1, \ldots, x_{d-1})\).

Indeed,

\[
(D^2\Psi_{\text{model}})|_{x'=0, y=y_d} = \begin{pmatrix}
\bar{y} & \bar{x} \\
\bar{y} & 0_{d-1,d-1} & I_{d-1,d-1} \\
\bar{x} & I_{d-1,d-1} & -y_d I_{d-1,d-1}
\end{pmatrix}.
\]

Moreover, \(\det(D^2\Psi_{\text{model}})|_{x'=0, y=y_d} = 1\), and its inverse is given by,

\[
(D^2\Psi_{\text{model}})^{-1}|_{x'=0, y=y_d} = \begin{pmatrix}
y_d I_{d-1,d-1} & I_{d-1,d-1} \\
I_{d-1,d-1} & 0_{d-1,d-1}
\end{pmatrix}.
\]

In particular, we note that the determinant and inverse of this Hessian are uniformly bounded, i.e. do not blow up when \(y_d \to 0\). However, when \(y_d = 0\) the critical point equations do not imply that \(x_{d+1} = \cdots = x_n = 0\). In invariant terms, \(y_d = 0\) corresponds to \(\Delta_H \times H\) and the coordinates \((x_{d+1}, \ldots, x_n)\) run over the fiber of \(N^*\Delta_H \times H\).

Proof. The Hessian has the form

\[
(D^2\Psi_{\text{model}})|_{x'=0, y=y_d} = \begin{pmatrix}
y & x' \\
y & y_d^2 \Psi_{\text{model}} & D^2_{y,x'} \Psi_{\text{model}} \\
x' & D^2_{x,y} \Psi_{\text{model}} & D^2_{x,x'} \Psi_{\text{model}}
\end{pmatrix} |_{C_y}
\]

with \(A = 0\) and with,

\[
B_{kj} = \frac{D^2_{y_k x_j}}{y_k y_j} \left(\sum_{j=1}^{d-1} y_j x_j - \frac{1}{2} y_d |x'|^2\right) \delta_{jk} = I_{d-1,d-1} \quad k = 1, \ldots, d-1, j = 1, \ldots, d-1,
\]

\[
D_{jk} := \frac{D^2_{x_j x_k}}{y_d} \left(\sum_{j=1}^{d-1} y_j x_j - \frac{1}{2} y_d |x'|^2\right) = -y_d \delta_{jk}, \quad j, k = 1, \ldots, d-1,
\]

proving that the Hessian has the stated form. Since we can multiply the top row block by \(y_d\) and add it to the bottom row block without changing the rank, the matrix has full rank.

We calculate the determinant Schur formula by interchanging the two columns and using the Schur formula \(\det M = \det D \det(A - BD^{-1}C)\).

\[
\square
\]

4.2. Asymptotics of the model integral. In this section, we drop the factors \(\bar{A} e^{i\lambda R_4}\) from the amplitude. (3.10) and (3.14), and study the model oscillatory integral,

\[
I_{\text{model}}(\lambda) := \int_{B_3(\mathbb{R}^{n-1})} \int_{\mathbb{R}^d} \chi(y) \hat{\psi}(y_d) e^{i\lambda \left(\sum_{j=1}^{d-1} y_j x_j - \frac{1}{2} y_d (x_1^2 + \cdots + x_{d-1}^2 + x_{d+1}^2 + \cdots + x_n^2)\right)} dy d\bar{x}. \tag{4.2}
\]

In the next section, we restore the factors and explain their role in the final answer.

By Lemma 4.1 we can remove the variables \((y_1, \ldots, y_{d-1}, x_1, \ldots, x_{d-1})\) by applying stationary phase to the sub-integral,

\[
I(\lambda, y_d, x_{d+1}, \ldots, x_n) := \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi(y', y_d) e^{i\lambda \left(\sum_{j=1}^{d-1} y_j x_j - \frac{1}{2} y_d (x_1^2 + \cdots + x_{d-1}^2 + x_{d+1}^2 + \cdots + x_n^2)\right)} dy' d\bar{x}'.
\]
As in Section 3.3 there exists a complete asymptotic expansion with leading term,

$$I(\lambda, y_d, x_{d+1}, \ldots, x_n) \simeq \lambda^{-\frac{d-1+d-1}{2}} \chi(0, y_d) e^{-\frac{1}{2}i\lambda y_d(x_{d+1}^2 + \cdots + x_n^2)}.$$

The higher order terms in $\lambda^{-1}$ involve the inverse Hessian derivatives of the amplitude multiplied by $e^{i\lambda R_3}$ where $R_3$ is the third and higher order terms of the phase. As noted below Lemma 4.2, the inverse Hessian operators have smooth coefficients and so the remainders are as stated in Section 3.3.

After applying stationary phase, the integral is reduced to a series of which the leading term is,

$$\lambda^{-\frac{d-1+d-1}{2}} \int_{\mathbb{R}} \int_{B_1(\mathbb{R}^{n-d})} \chi(0, y_d) \hat{\psi}(y_d) e^{-\frac{1}{2}i\lambda y_d(x_{d+1}^2 + \cdots + x_n^2)} dy_d dx_{d+1} \cdots dx_n.$$

We put the integral in polar coordinates with radial variable $R = (x_{d+1}^2 + \cdots + x_n^2)$ to get,

$$I(\lambda, y_d, x_{d+1}, \ldots, x_n) \simeq \lambda^{-\frac{d-1+d-1}{2}} \int_{\mathbb{R}} \int_{0}^{1} \chi(0, y_d) \hat{\psi}(y_d) e^{-\frac{1}{2}i\lambda y_d R} dy_d R^{n-d-1} dR,$$

where we obtain the new amplitude $\tilde{A}_1$ from the stationary phase expansion and integration over the unit sphere in $\mathbb{R}^{n-d}$. Let us write $\rho = \frac{1}{2} R^2$ to get

$$I(\lambda, y_d, x_{d+1}, \ldots, x_n) \simeq \lambda^{-\frac{d-1+d-1}{2}} \int_{\mathbb{R}} \int_{0}^{\frac{\lambda}{2}} \chi(0, y_d) \hat{\psi}(y_d) e^{-\frac{1}{2}i\lambda y_d \rho} dy_d \rho^{\frac{n-d-1}{2}} \rho^{-\frac{1}{2}} d\rho.$$

Note that the integrand is in $L^1$ for any $d \leq n - 1$. Thus, the model oscillatory integral reduces to

$$\int_{\mathbb{R}} \int_{0}^{\frac{\lambda}{2}} \hat{\psi}(y_d) e^{i\lambda y_d R^2} R^{n-d-1} dR dy_d = \int_{0}^{\frac{\lambda}{2}} \psi(\lambda R^2) R^{n-d-1} dR$$

$$= \lambda^{-\frac{n-d-1}{2}} \int_{0}^{\lambda} \psi(\rho) \rho^{\frac{n-d-2}{2}} d\rho$$

$$= \lambda^{-\frac{n-d}{2}} \int_{0}^{\infty} \psi(\rho) \rho^{\frac{n-d-2}{2}} d\rho + O(\lambda^{-\infty}).$$

We then rewrite the answer in terms of the Fourier transform (1.9),

$$\int_{\mathbb{R}} \psi(\rho) \rho^{\frac{n-d-2}{2}} d\rho = \int_{\mathbb{R}} \hat{\psi}(s) \mathcal{F}_{\rho^x} \rho^{\frac{n-d-2}{2}} s = i e^{i\pi/2} \Gamma(\frac{n-d-2}{2} + 1) \int_{\mathbb{R}} \hat{\psi}(s) (s + i0)^{-\frac{n-d}{2}} ds. (4.3)$$

Multiplying by the factor $\lambda^{n+d-2} \lambda^{-d-1}$ from the prior calculations, the model integral becomes

$$I_{\text{model}}(\lambda) \sim C_{n,d} \lambda^{n+d-2} \lambda^{-d-1} \lambda^{-\frac{n-d-2}{2}} \int_{\mathbb{R}} \hat{\psi}(s) (s + i0)^{-\frac{n-d}{2}} ds$$

$$= C_{n,d} \lambda^{n+d-2} \int_{\mathbb{R}} \hat{\psi}(s) (s + i0)^{-\frac{n-d}{2}} ds. (4.4)$$

4.3. Completion of the proof of Theorem 1.7. The purpose of the above calculation was to exhibit a simple model which gives the same type of leading coefficient. We now complete the proof of the first (short-time) statement of Theorem 1.2 and Theorem 1.7 by including the additional amplitude factors $e^{i\lambda R_4} \tilde{A}$ of Lemma 4.1 in the integrand.
Proof. We repeat the analysis in Section 4.2 but replacing the amplitude $\chi(y)\hat{\psi}_d$ in (4.2) by the full amplitude

$$
\chi(y)\hat{\psi}(y_d)\tilde{A}(x_1, \ldots, x_{d-1}, y_1, \ldots, y_d, y_d, x_{d+1}, \ldots, x_n) e^{i\lambda R_4(x_1, \ldots, x_{d-1}, y_1, \ldots, y_d, y_d, x_{d+1}, \ldots, x_n)}.
$$

As in the proof of the stationary phase method in [HoIV, Volume I], the factor $e^{i\lambda R_4}$$\tilde{A}$ can be absorbed into the amplitude and then produces the expansion reviewed in Section 3.3. The stationary phase procedure applies with this additional factor as in the model case since the phase is the same as the model case and since, by Lemma 4.2, the inverse Hessian derivatives are smooth.

Stationary phase in the variables $(x_1, \ldots, x_{d-1}, y_1, \ldots, y_d)$ localizes the integrand to $(x_1, \ldots, x_{d-1}, y_1, \ldots, y_d) = 0$. The new part of the integrand is,

$$
\tilde{A}(0, 0, y_d, x_{d+1}, \ldots, x_n) e^{i\lambda R_4(0, 0, y_d, x_{d+1}, \ldots, x_n)}.
$$

We then use polar coordinates $(x_{d+1}, \ldots, x_n) = R\omega'$ and again use that the model phase is $e^{-\frac{1}{2}i\lambda y d R^2}$.

The other factor in the amplitude is

$$
\tilde{A}(0, 0, y_d, x_{d}, \ldots, x_n)
$$

We set

$$
A(y_d) = \int_{\omega'} \tilde{A}(0, 0, y_d, x_{d}, \ldots, x_n) d\mu.
$$

The resulting integrals have the form,

$$
\lambda^{-\frac{d+1}{2}} \int_{\mathbb{R}} \int_{B_1(\mathbb{R}^{n-d})} \chi(0, y_d)\hat{\psi}(y_d)A(y_d)e^{-\frac{1}{2}i\lambda y_d\sqrt{1-R^2}} R^{n-d-1} dy_d dR.
$$

Next we integrate in $y_d$ to get

$$
\mathcal{F}_{y_d \to \eta}(0, y_d)\hat{\psi}(y_d)A(y_d)|_{\eta = (\frac{1}{2}\lambda \sqrt{1-R^2})}.
$$

We then get the explicit integral above, and inverse Fourier transform to get

$$
\int_{\mathbb{R}} \chi(0, y_d)\hat{\psi}(y_d)A(y_d) (y_d + i0)^{-\frac{n-d}{2}} dy_d.
$$

Modulo calculating $A(y_d)$, this gives all the details of the leading order term in Theorem 1.2 and Theorem 1.7. The calculation of the density is given in Section 5. \qed

4.4. Apriori properties of $a_1^0(H, \psi)$. In view of the complications in computing the leading coefficient using a hybrid stationary phase and Fourier inversion method, we use an indirect argument to prove that the regularization $(s + i0)^{-\frac{n-d}{2}}$ in (1.4) is the correct regularization.

Lemma 4.3. The functional $\psi \to a_1^0(H, \psi)$ in (1.4) is a positive measure on $\mathbb{R}$ which is supported on $\mathbb{R}^+$. 

Proof. Since it is now proved that (1.2) has an asymptotic expansion of order $\lambda^{-\frac{n+d}{2}}$, the leading coefficient is given by,

$$
a_1^0(H, \psi) := \lim_{\lambda \to \infty} \lambda^{-\frac{n+d}{2}} N_{\psi, H}^1(\lambda).
$$

Since $N_{\psi, H}^1(\lambda) \geq 0$ when $\psi \geq 0$, also $a_1^0(H, \psi) \geq 0$ if $\psi \geq 0$. To prove that it is supported on $\mathbb{R}^+$ we note that by the convention in (1.2), the differences of eigenvalues are ordered as...
\( \lambda_j - \mu_k \), and from the fact that the Fourier coefficients (1.11) are negligible for \( \mu_k \geq \lambda_j + \epsilon \) we see that the positive measure \( d\rho(y) \) is supported on the positive reals.

In (4.4) we get the inverse Fourier transform of this measure. It is determined on \((0, \infty)\) by the stationary phase analysis above, which shows that it agrees with the formula (1.14) for \( y \) such that \( \psi = 0 \) in some interval around 0. This leaves two points unclear: (i) how the Fourier transform is regularized at \( s = 0 \); (ii) whether there exists a component of the Fourier transform supported at \( s = 0 \). Regarding point (i), there is apriori only one regularization \( s - \frac{n+1}{2} \) for \( s > 0 \) whose Fourier transform is a temperate positive measure supported on \( \mathbb{R}_+ \). Indeed, all regularizations must agree on \((0, \infty)\) and their differences must be supported at \( s = 0 \). Regarding point (ii) the analysis of the model phase shows that there is no component supported at \( s = 0 \). \( \square \)

4.5. **Relation to Bessel integrals.** The singularity at \( s = 0 \) is therefore universal. The above model integral with \( \tilde{A} = 1 \) is essentially the ‘double Bessel integral’. To explain this, we recall the well-known formula,

\[
|\lambda \xi|^{-\frac{n+2}{2}} J_{\frac{n-2}{2}}(2\pi \lambda |\xi|) = \int_{S^{n-1}} e^{2\pi i \lambda \xi} dS(\xi), \quad (\xi \in \mathbb{R}^n).
\]

Stationary phase asymptotics apply when \( |\lambda \xi| \to \infty \) but do not apply when \( |\lambda \xi| \leq M \) for some \( M > 0 \). The rapid decay of the integral (as \( \lambda \to \infty \))

\[
\int_{\mathbb{R}^n} \hat{\psi}(y) \left( \int_{S^{n-1}} e^{i\lambda \langle y, \omega \rangle} d\omega \right) dy = \int_{S^{n-1}} \psi(\lambda \omega) d\omega = \psi(\lambda), \quad \text{if} \ \psi(y) = \psi(|y|),
\]

for a radial function \( \psi \) on \( \mathbb{R}^n \) is obvious from the fact that there are no critical points of \( y \to \langle y, \omega \rangle \), but becomes opaque if one tries to first apply stationary phase in \( \omega \), or to express it in terms of Bessel functions,

\[
\int_{\mathbb{R}^n} \hat{\psi}(y) \left( \int_{S^{n-1}} e^{i\lambda \langle y, \omega \rangle} d\omega \right) dy = \int_0^\infty \frac{\hat{\psi}(r)}{r} \left( J_{\frac{n-2}{2}}(\lambda r) \right) r^{n-1} dr.
\]

The well-known (stationary phase) asymptotics of \( (\lambda r)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda r) \) only apply when \( \lambda r \to \infty \).

The singularity at \( s = 0 \) is essentially of this type. To be more exact, it arises in the Euclidean case as the “double-Bessel” function,

\[
J^1_{\mathbb{R}^d, \mathbb{R}^n}(\lambda, y, \psi) := \int_{S^{d-1}} \int_{S^{n-1}} \frac{\hat{\psi}(\omega, \omega') e^{i\lambda \langle y, \omega \rangle + \omega' \cdot (y - \omega)}}{\sigma_d(\omega) \sigma_d(\omega')} dS_n(\omega) dS_d(\omega'), \quad y \in \mathbb{R}^d,
\]

\[
= (r \lambda)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda r) (r \lambda)^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(\lambda r), \quad \text{if} \ \hat{\psi} = 1, \ r = |y|.
\]

The model phase (1.11) approximates the phase of this integral. The parameter \( s \) is \( |y| \). The asymptotics of (4.6) for \( \lambda R \leq C \) are obtained by Taylor expansion and are clearly of larger order in \( \lambda \) than the stationary phase asymptotics for \( \lambda r \to \infty \).

4.6. **Comparison to the case** \( c < 1 \). The asymptotics above involve test functions \( \rho \) such that the support of \( \hat{\rho} \) is an interval \([-\epsilon, \epsilon]\) which contains no periods \( T \neq 0 \) of the \( G''_M \). We now compare the results for such test functions in the case \( c < 1 \) and \( c = 1 \). Later, we will compare results for general test functions \( \hat{\rho} \). In the case \( c < 1 \) of [WXZ21] the phase \( \langle y, \omega + \tilde{w} \rangle \) of Proposition 3.1 and of (3.15) gets replaced by \( \hat{\Psi}_c(\tilde{w}; y, \omega) = \langle y, \omega - c\tilde{w} \rangle \). Its critical set is given by \( C_{\hat{\Psi}_c} = \{ \pi_H \omega = c\tilde{w}, y = \langle y, \omega \rangle \pi_H \omega \} \), rather than the set (3.16). The main difference is that the critical point equations do not constrain \( \pi_H \omega \in N^*H \) when \( c < 1 \) except in its norm \( |\pi_H \omega| = \sqrt{1 - c^2} \); when \( c = 1 \), the normal component
vanishes. Hence, \( \dim C_{\Psi_c} = \dim C_{\Psi} + (n - d - 1) \), and \( \dim C_{\Psi_c} = d - 1 + (n - d - 1) = n - 2 \). When \( H \) is a hypersurface, \( \dim N_q^c H = 0 \) and so \( \dim C_{\Psi} = \dim C_{\Psi_c} + 1 \). When \( \dim H = n - 2 \) then \( \dim N_q^c H = 1 \) and \( \dim C_{\Psi} = \dim C_{\Psi_c} \). Otherwise, \( \dim C_{\Psi} > \dim C_{\Psi_c} \). This difference in dimensions is responsible for the change in order of growth of \( N_{\Psi,H}(\lambda) \) from \( \lambda^{n-1} \) when \( c < 1 \) [WXZ21] Theorem 1.1] to \( \lambda^{n+d} \) in Theorem 1.2. Moreover, the degeneracy of the \( (3.15) \) when \( c = 0 \) does not occur when \( c < 1 \) since in the coordinates above, \( \omega = (x_1, \ldots, x_d, x_{d+1}, \ldots, x_n) \) with \( x_1^2 + \cdots + x_d^2 = c^2, x_{d+1}^2 + \cdots + x_n^2 = 1 - c^2 \) and \( \langle y, \omega - c \tilde{\omega} \rangle = \sum_{j=1}^{d-1} y_j x_j + y_d(x_d - c) \), and the analogue of the local model \( (4.1) \) for \( c = 1 \) is \( \Psi_{c,\text{model}}(\vec{y}, \vec{x}) = \sum_{j=1}^{d-1} y_j x_j - y_d\sqrt{c^2 - (x_1^2 + \cdots + x_{d-1}^2) - c} \). The critical points are given by, \( x_j = y_j = 0, (j = 1, \ldots, d-1), x_d = c \) but \( y_d \) and \( x_{d+1}, \ldots, x_n \) are unconstrained by the critical point equation except that the norm of \( (x_{d+1}, \ldots, x_n) \) is \( \sqrt{1 - c^2} \). The analogue of the equation \( (1.14) \) when \( c < 1 \) is \( G_{H}^{-s} \pi_H G_{H} G^{t+es}(q, \xi) = \pi_H(q, \xi) \), and there are solutions along the diagonal only when \( s = t = 0 \), since otherwise the bi-angle must make a non-zero angle with \( H \). Since \( (x_{d+1}, \ldots, x_n) \) lie in the critical set, the normal Hessian has no \( (x_{d+1}, \ldots, x_n) \)-block as in Section 3.3 and therefore does not acquire the factor \( |y|^{-(n-d)} \).

5. Calculation of the amplitude in \( (1.4) \) using the Hadamard parametrix

In this section, we use Hadamard parametrices for \( U_M(t, x, y) \) resp. \( U_H(s, x, y) \) to give a formula \( (5.7) \) for \( N_{\Psi,\rho,H} \). The Hadamard parametrices are simple and explicit enough to identify the geometric invariants in \( (1.4) \). On the other hand, the calculations in polar coordinates become singular at \( r = 0 \) and the Hadamard parametrix does not seem to give a simple approach to the singularity at \( s = 0 \).

The Hadamard parametrix uses the phase \( \sigma(t^2 - r^2(x, y)) \) where \( \sigma > 0 \) and where \( r(x, y) \) is the Riemannian distance between \( x, y \). The distance squared \( r^2(x, y) \) is smooth in a neighborhood of the diagonal \( x = y \) but is not smooth when \( y \) is a cut point to \( x \). Hence we need to cutoff the parametrix using a cutoff \( \chi(x, y) \) sufficiently near the diagonal so that \( r^2 \) is smooth. We absorb the cutoff into the amplitude and suppress it from the notation. The neighborhood is the union of the same balls \( B_j(\epsilon) \) in the definition of the Hörmander parametrix. We then denote the volume density in geodesic coordinates centered at \( x \) by \( dV_q = \Theta(x, y) dy \) (see Section 5 and [Be] for background on \( \Theta(x, y) \)).

The Hadamard parametrices for the half-wave group \( U_M(t, x, y) \) of \( M \), resp. \( U_H(t, x, y) \) of \( H \) are given (at least for \( t \geq 0 \)) by

\[
\begin{align*}
U_M(t, x, y) &= \int_0^{\infty} e^{i\sigma(t^2 - r_M^2)} A_M(t, x, y, \sigma) \sigma_+^{\frac{n+1}{2}} d\sigma, \\
U_H(t, x, y) &= \int_0^{\infty} e^{i\sigma(t^2 - r_H^2)} A_H(t, x, y, \sigma) \sigma_+^{\frac{n+1}{2}} d\sigma,
\end{align*}
\]

where \( r_M \) resp. \( r_H \) are the distance functions of \( M \) resp. \( H \). Since \( H \) is totally geodesic, \( r_H = r_M \) on \( H \times H \). The amplitudes \( A_M \), resp. \( A_H \) are divisible by \( t \) have the asymptotic symbol expansions as \( \sigma \to \infty \),

\[
\begin{align*}
A_M(t, x, y, \sigma) &\sim t \sum_{j=0}^{\infty} U_M^j(x, y) \sigma^{-j}, \\
A_H(t, x, y, \sigma) &\sim t \sum_{j=0}^{\infty} U_H^j(x, y) \sigma^{-j}.
\end{align*}
\]

The original Hadamard-Riesz parametrix constructions did not express these wave kernels as oscillatory integrals. Rather, they expressed them as infinite series in what are now called
Riesz kernels. For instance,

\[
\cos t \sqrt{\Delta}(x, y) = C_0 |t| \sum_{j=0}^{\infty} (-1)^j U_j(x, y) \left( \frac{t^2 - i^j}{4\tau^{j+1}} \right) \mod C^\infty. \tag{5.1}
\]

The oscillatory integral formula in [Be] is obtained by using the Fourier transform formula (1.9).

For small \( t \), and for any Riemannian manifold \((X, g)\) of dimension \( n \), \( U_X(t, x, y) \) is to leading order similar to the Euclidean case of \( \mathbb{R}^n \), whose half-wave kernel is given by

\[
U_{\mathbb{R}^n}(t, x, y) = C'_n \left( \frac{t}{((t+i0)^2 - r^2)^{\frac{n+1}{2}}} \right) = C_n \lim_{\tau \to 0} \left( \frac{it}{((t+i\tau)^2 - r(x, y)^2)^{\frac{n+1}{2}}} \right). \tag{5.2}
\]

for constants \( C'_n, C_n \) depending only on the dimension. A good way to understand the relevant regularizations is that \( U(t) = e^{it\sqrt{-\Delta}} \) has a holomorphic extension to \( U(t+i\tau) \) for \( \tau > 0 \) because \( \sqrt{-\Delta} \) is a positive operator. Note that (5.2) is Poisson kernel at imaginary time and that \( U(t, x, y) \) does not have finite propagation speed, hence does not satisfy Huyghen’s principle and is therefore different from the cosine kernel (5.1).

We briefly review the construction of the amplitude by a series of transport equations. Our main references are [Be, Zel12]. The following complications should be kept in mind.

- As is carefully explained in [Be], the parametrices for \( \cos t \sqrt{-\Delta} \) and for \( \frac{\sin t \sqrt{-\Delta}}{\sqrt{-\Delta}} \) are first derived for \( t > 0 \) and then extended to all \( t \) using the even/odd property of cosine/sine. The coefficient \( |t| \) in the formula (5.1) seems singular but of course the kernel is analytic in \( t \). This only indicates that the parametrix (but not the wave kernel) is singular at \( t = 0 \) and \( t = 0 \) due to polar coordinate singularities; this is discussed further below.

- The regularization procedure of Riesz (by analytic continuation of Riesz kernels) introduces many constants. One arises from the Fourier transform in (1.9). More are introduced by requiring that \( U(0, x, y) = \delta_x(y) \). In [Be] (11)], the coefficients \( U_k \) are converted to coefficients \( u_k \) on a manifold of dimension \( d \) by the formula \( U_k(x, y) = C_0 e^{-i\frac{d-k}{2}\pi/2} \frac{4^{-k}}{1} u_k(x, y) \). The product of these constants and others arising in stationary phase constitute the constant \( C_{n,d} \) in Theorem 1.2 and are ultimately responsible for the shape of (1.4). To avoid a lengthy (and futile) chasing of constants, we note that the final coefficients are universal and can be calculated on \( \mathbb{R}^n \) or \( S^n \) (see Section 5.1 for the exact formulae on spheres).

We now recall some of the details of the Hadamard parametrix construction for \( U_M(t) = \exp \sqrt{\sqrt{-\Delta}} \) as an oscillatory integral of the form,

\[
U_M(t, x, y) = C_{n,d} t \int_0^{\infty} e^{i\sigma \left( t^2 - 1 \right)} \sum_{j=0}^{\infty} W_j(x, y) \sigma^{-j} d\sigma \mod C^\infty \tag{5.3}
\]

where \( \sigma^+ \) is the regularization of the distribution \( \sigma^+ \) for \( s \) with negative real part (see [Be]).

Hadamard himself did not treat \( U_M(t) \) but rather the cosine propagator \( \cos t \sqrt{-\Delta} \) and the sine propagator \( (-\Delta^{-\frac{1}{2}}) \sin t \sqrt{-\Delta} \), both of which are formally Taylor series in \( \Delta \). To obtain the Hadamard parametrix for \( U_M(t) \) one may apply \( \sqrt{-\Delta} \) to \( (-\Delta^{-\frac{1}{2}}) \sin t \sqrt{-\Delta} \) to obtain its imaginary part and add it to \( \cos t \sqrt{-\Delta} \).
The Hadamard-Riesz coefficients $W_j$ for $(-\Delta^{-\frac{1}{2}})\sin t\sqrt{-\Delta}$ are determined inductively by the transport equations,

$$ \frac{\Theta'}{2\Theta} W_0 + \frac{\partial W_0}{\partial r} = 0 $$

(5.4)

$$ 4ir(x, y)\left\{(\frac{k+1}{r(x,y)}) + \frac{\Theta'}{2\Theta} W_{k+1} + \frac{\partial W_{k+1}}{\partial r}\right\} = \Delta_y W_k. $$

The solutions are given by:

$$ W_0(x, y) = \Theta^{-\frac{1}{2}}(x, y) $$

(5.5)

$$ W_{j+1}(x, y) = \Theta^{-\frac{1}{2}}(x, y) \int_0^1 s^k \Theta(x, x_s) \frac{1}{2} \Delta_2 W_j(x, x_s) ds $$

where $x_s$ is the geodesic from $x$ to $y$ parametrized proportionately to arc-length and where $\Delta_2$ operates in the second variable. As above, $\Theta(x, y)$ is the volume density in geodesic normal coordinates based at $x$, $dV_\Theta = \Theta(x,y) dy$. If we change to geodesic polar coordinates $(r, \omega)$, we get $dV_\Theta = \Theta(r, \omega) dr d\omega$. We can see that the leading order amplitude remains $\mathcal{J}$, except for an overall factor of $\mathcal{J}$. In particular, we see that apart from an overall factor of $t$, the amplitudes $A_M$, resp. $A_H$, above are independent of $t$. The parametrix for $\cos t\sqrt{-\Delta}$ is obtained by differentiating that for $(-\Delta^{-\frac{1}{2}})\sin t\sqrt{-\Delta}$ in $t$ and the parametrix for $U_M(t)$ is of course obtained by applying $\sqrt{-\Delta}$ to the latter and adding the real part. It is straightforward to see that the leading order amplitude remains $\mathcal{J}$. The details are given in [Zel12] and will not be repeated here.

Since $U_H(-s) = U_H(s)^*$, the Hadamard parametrix for $U_H(-s)$ has the form,

$$ U_H(-s, x, y) = \overline{U_H(s, y, x)} = \int_0^\infty e^{-it\sigma(s^2-r_H^2)} A_H(s, y, x, \sigma) d\sigma. $$

(5.6)

If we use the Hadamard parametrices and recall (5.6), (1.16) becomes,

$$ N_{\psi, \rho, H}(\lambda) = \lambda^{\frac{n+1}{2}+1} \int_{\mathbb{R} \times M} \int_0^\infty \hat{\psi}(s) \hat{\rho}(t) e^{-it\lambda e^{i\lambda(\sigma_1(t+s)^2 - \sigma_2 s^2 - (\sigma_1 - \sigma_2)^2)}(q, q')} dsdtd\sigma_1d\sigma_2 dV_H(q) dV_H(q'). $$

(5.7)

Again the amplitudes $A_H, A_M$ are defined above and incorporate the notational conventions for the constants. This is the starting point for the stationary phase analysis in the next section [5.2]

We remark that the Hadamard parametrix is singularity at $t = 0$ on the diagonal, which motivated the use of the Hörmander parametrix method in Section 3.1. The singularity arises because the phase of the Hadamard parametrix is expressed in geodesic polar coordinates which become singular on the diagonal. As a result, the canonical relation,

$$ C = \{(t, 2t\sigma, x, -\sigma d_x r^2, y, \sigma d_y r^2) : t^2 = r^2\} \subset T^*(\mathbb{R} \times M \times M), $$

generated by the phase $\sigma(t^2 - r^2)$ of the half-wave kernel has an apparent singularity (a 0 in the wave front relation) when $r = t = 0$, whereas in fact when $t = 0$ it is the graph of the identity map. I.e. the co-normals to the distance spheres collapse to the unit cotangent space at the origin.
5.1. **Exact calculations for** $M = S^n$. Since $U_M(t)$ is more complicated than $(−Δ^{−}\frac{1}{2}) \sin t\sqrt{−Δ}$ or $\cos t\sqrt{−Δ}$, we illustrate the result in the case of the standard sphere. We define the wave kernel $U_M(t) = \exp iA$ in terms of the degree operator $A = \sqrt{−Δ + \frac{(n−1)^2}{4} − \frac{n−1}{2}}$, which has eigenvalue $N$ in the space of $N$th degree spherical harmonics. Then as calculated in [1ay],

$$U_M(t, x, y) = \frac{2i \sin t}{|S^{n−1}|} \lim_{\epsilon \to 0} (2 \cos (t + i\epsilon) − 2 \cos r(x, y))^\frac{n+1}{2}.$$

As above, $U_M(t, x, y)$ has a holomorphic extension in $t$ to the upper half-plane and on the real $t$ axis is the boundary value of this holomorphic function.

5.2. **Determination of the amplitude in Theorem 1.2 by the Hadamard parametrix method.** We employ the Hadamard parametrix to give a simple determination of the amplitude in the leading coefficient of Theorem 1.2. As mentioned above, we denote any dimensional constant by $C_{n,d}$; it is understood that the constant may change in each usage.

In what follows, we restrict the stationary phase analysis to the regimes $t + s > 0$ for $M$ and $s > 0$ for $H$ or $t + s < 0$ for $M$ and $s < 0$ for $H$ and show that there are no points in the canonical relation $C$ in the complementary cases. We therefore break up the proof into the cases $t + s > 0, s > 0$ and $t + s < 0, s < 0$ and explain (in more detail than above) why the complementary cases $t + s < 0, s > 0$ and $t + s > 0, s < 0$ do not contribute to the asymptotics. The universal constants arising in the two cases may be different and we denote them by $C_{n,d}^+$ and $C_{n,d}^−$.

5.3. **Critical point analysis for** $t + s > 0, s > 0$.

**Proof.** We rewrite the integral (5.7) in geodesic polar coordinates centered at $q \in H$.

$$N^1_{\rho, H}(\lambda) = \lambda^{\frac{n+1}{2}+1} \int_{S^*H} \hat{\psi}(s)I_\rho(q, s, \omega, \lambda)dV_H(q)dS(\omega),$$

with phase,

$$\Psi = −t + (\sigma_1(t + s)^2 − \sigma_2 s^2 − (\sigma_1 − \sigma_2)r_H^2(q, q'))$$

$$= −t + \sigma_1[(t + s)^2 − r^2] − \sigma_2[s^2 − r^2],$$

where

$$I_\rho(q, s, \omega, \lambda) := \int_0^\infty \int_0^\infty \int_0^\infty \hat{\rho}(t)e^{i\lambda \Psi} \tilde{A}(s, t, r, \lambda \sigma_1, \lambda \sigma_2) r^{d−1} \sigma_1^{\frac{n+1}{2}−}\sigma_2^\frac{d−1}{2} dt dr d\sigma_1 d\sigma_2,$$

(5.10)

where $\sigma_j^z$ is regularized by $(\sigma_j)^z_+$ as described in the previous section, and where

$$\tilde{A}(s, t, r, \lambda \sigma_1, \lambda \sigma_2) := (s + t)s \int_{S^*H} A_{M \times H}(r, \omega, \lambda \sigma_1, \lambda \sigma_2)dV_H(q)dS(\omega).$$

Here,

$$A_{M \times H}(q, q', \lambda \sigma_1, \lambda \sigma_2) := A_M(q, q', \lambda \sigma_1)A_H(q, q', \lambda \sigma_2),$$

is a semi-classical symbol with principal term as $\lambda \to \infty$,

$$A^0_{M \times H} = \Theta_M^{-\frac{1}{2}}(q, q')\Theta_H^{-\frac{1}{2}}(q, q') = \Theta_H^{-1}(q, q').$$

(5.11)

We treat $q$ as a parameter and write $q' = \exp_q \iota \omega$ with $\omega \in T_q H$ (identified with $S^{d−1}$) and $r = r_H(q, q')$. As above, we note that although $\tilde{A}(s, t, r, \sigma_1, \sigma_2)$ apriori depends on $s, t$, in fact it is independent of $s, t$. We now impose the restriction that $t + s > 0, s > 0$ and therefore write (5.10) as $I^+$. x
Lemma 5.1. The phase has non-degenerate critical points
\[ t = 0, r = s, \sigma_2 = \sigma_1 = \frac{1}{2s}, \]
for \( s \neq 0 \) in the variables \( t, r, \sigma_1, \sigma_2 \). The determinant and signature of the Hessian are given by,
\begin{itemize}
  \item \( \det \tilde{H}_2 = (2s)^4 \), hence, \( \det^{-\frac{1}{2}} = (2s)^{-2} \).
  \item \( \text{sgn} \tilde{H}_2 = 0 \)
\end{itemize}

The calculations are straightforward; since our only purpose is to calculate the amplitude, in the interest of brevity, we leave the calculations in Lemma 5.1 to the reader.

We then apply stationary phase to \((5.12)\) in the variables \((t, \sigma)\) to obtain the complete asymptotic expansion claimed in the Proposition. Using Lemma 5.1, and cancelling the determinant factor of \(s^{-2}\) with the Hadamard parametrix factors \((s + t)s|_{t=0}\), we obtain for \( s \neq 0 \), and for a dimensional constant \( C_{n,d} \), the leading term in the asymptotic expansion has the form, for \( s > 0 \),
\[ I^+_\rho(q, s, \omega, \lambda) \simeq C_{n,d}^+ \lambda^{-2} \hat{\rho}(0) s^{d-1} s^{-\frac{n+1}{2}} s^{-\frac{d+2}{2}} \tilde{A}(s, \frac{\lambda}{2s}, \frac{\lambda}{2s}). \tag{5.12} \]
We further integrate over \( S^*H \) to obtain the final result. Taking into account the regularization of \( \sigma_+^* \), the leading term with \( s + t \geq 0, s \geq 0 \) takes the form,
\[ N^+_{\psi, \rho, H}(\lambda) \simeq C_{n,d}^+ \lambda^{\frac{n+d-1}{2}-1} \hat{\rho}(0) \int_{\mathbb{R}} \hat{\psi}(s) s^{d-1} s^{-\frac{n+1}{2}} s^{-\frac{d+2}{2}} \left( \int_{S^*H} \tilde{A}(s, \frac{\lambda}{2s}, \frac{\lambda}{2s}) dV_H(q) dS(\omega) \right) ds, \tag{5.13} \]
or more precisely, replacing \( \tilde{A} \) by its principal term \((5.11)\),
\[ \lambda^{\frac{n+d-1}{2}} C_{n,d}^+ \hat{\rho}(0) \int_{\mathbb{R}} \hat{\psi}(s) s^{-\frac{n+1}{2}} \left( \int_{S^*H} \Theta_M^{-\frac{1}{2}}(q, \exp_q s\omega) \Theta_H^{-\frac{1}{2}}(q, \exp_q s\omega) dV_H(q) dS(\omega) \right) ds. \]

The stationary phase expansion for \( t + s < 0, s < 0 \) is the complex conjugate of that for \( t + s > 0, s > 0 \), as one sees by changing variables \( s = -S, t = -T \) with \( S, T > 0 \). Hence the amplitude is the same.

There are no critical points when \( t + s > 0, s < 0 \), resp. \( t + s < 0, s > 0 \). This is because, by Proposition 2.5, the only \((s, t)\) for which there exist points in the canonical relation contributing to the singularities of \( S(t, \psi) \) are those for which there exist solutions of \((1.14)\), i.e. for which there exist \( \xi \in T^*H \) such that \( G_{H}^{-s} \pi_H G_{M}^{t-s}(q, \xi) = \pi_H(q, \xi) \). This forces \( t = 0 \) and then \( s > 0 \) and \( s < 0 \) are incompatible.

Thus, we have proved that the amplitude in \((1.14)\) is as claimed in Theorem 1.2

6. Singularities of \( S(t, \psi) \) for Long Times

To prove the last statement of Theorem 1.2 we will need a generalization of Theorem 1.7 on the asymptotics of \( N^1_{\psi, \rho, H}(\lambda) \) to the case where \( \text{supp} \hat{\rho} \) is an arbitrarily long interval. We assume as before that the solution set of \((1.14)\) is clean. The statement and proof are analogous to \([WXZ21, \text{Proposition 1.20}]\), and only involve the wave front analysis in Section 2.2. We only sketch the main points and refer to the discussion of the \( c < 1 \) case in \([WXZ21]\) for further details.
Proposition 6.1. Let \( \rho \in \mathcal{S}(\mathbb{R}) \) with \( \hat{\rho} \in C_0^\infty \) and with \( 0 \notin \text{supp}\hat{\rho} \). Assume that the fixed point set of \( G^S_H \) at a period \( S \in \Sigma^1 \) is clean, and denote by \( d_j \) the dimension of a component \( Z_j(T) \) of the fixed point set. Then, there exists \( \beta_j \in \mathbb{R} \) and a complete asymptotic expansion, 
\[
N_{\rho,\psi,H}^c(\lambda) \sim \lambda^{-1+\frac{1}{2}(n-d)} \sum_{T \in \Sigma^1} \sum_{\ell=0}^\infty \beta_\ell(t - T) \lambda^{\frac{d_j(T)}{2} - \ell},
\]

The asymptotics corresponding to \( T \) are of lower order than the principal term of Theorem 1.7 unless \( G^T_H = \text{id} \).

Proof. We follow [DG75, WXZ21]. The singularities of \( S(t,\psi) \) are isolated and Lagrangian and we treat them one at a time. For \( t \) sufficiently close to \( T \),
\[
S(t,\psi) = \sum_j \beta_j(t - T),
\]
where \( \beta_j \) is a homogeneous Lagrangian distribution given by,
\[
\beta_j(t) = \int_{\mathbb{R}} \alpha_j(s)e^{-ist}ds, \quad \text{with} \quad \alpha_j(s) \sim \left(\frac{s}{2\pi}\right)^{-1+\frac{1}{2}(n-d)+\frac{d_j(T)}{2}} i^{-\sigma_j} \sum_{k=0}^\infty \alpha_{j,k}s^{-k},
\]
where \( d_j(T) \) is the dimension of the component \( Z_j(T) \).

7. Tauberian theorems and proofs of Theorem 1.2 and Corollary 1.3

In the next section, we use the following Tauberian theorems. Let \( \rho \) be a nonnegative Schwartz-class function on \( \mathbb{R} \) with compact Fourier support let \( N \) be a tempered, monotone non-decreasing function with \( N(\lambda) = 0 \) for \( \lambda < 0 \), and \( N' \) its distributional derivative as a nonnegative measure on \( \mathbb{R} \).

Proposition 7.1 (Corollary B.2.2 in [SV]). Let \( \rho \in \mathcal{S}(\mathbb{R}) \) be a positive, even test function with \( \hat{\rho}(0) = 1 \) and \( \hat{\rho} \in C_0^\infty(\mathbb{R}) \). Let \( N(\lambda) \) be a monotone non-decreasing temperate function. Fix \( \nu \geq 0 \). If \( N' \ast \rho(\lambda) = O(\lambda^\nu) \), then
\[
N(\lambda) = (N \ast \rho)(\lambda) + O(\lambda^\nu).
\]
This estimate holds uniformly for a set of such \( N \) provided \( N' \ast \rho(\lambda) = O(\lambda^\nu) \) holds uniformly.

The next one is [SV Theorem B.5.1] with \( \nu = \frac{n+d}{2} - 1 \).

Proposition 7.2. Let \( \rho \in \mathcal{S}(\mathbb{R}) \) be a positive, even test function with \( \hat{\rho}(0) = 1 \) and \( \hat{\rho} \in C_0^\infty(\mathbb{R}) \). Let \( N(\lambda) \) be a monotone non-decreasing temperate function. If \( N' \ast \rho(\lambda) = O(\lambda^{\frac{n+d}{2} - 1}) \) and additionally
\[
N' \ast \chi(\lambda) = o(\lambda^{\frac{n+d}{2} - 1})
\]
for every Schwartz-class \( \chi \) on \( \mathbb{R} \) whose Fourier support is contained in a compact subset of \( (0, \infty) \). Then,
\[
N(\lambda) = N \ast \rho(\lambda) + o(\lambda^{\frac{n+d}{2} - 1}).
\]

Last, we recall [HoIV Theorem 29.1.5-Corollary 29.1.6] in a form stated by Ivrii [IS0].
Proposition 7.3. Let \( \beta \in C_0^\infty(\mathbb{R}) \), \( \beta \equiv 1 \) in \((-\frac{1}{2}, \frac{1}{2})\), \( \beta = 0 \) for \( |t| \geq 1 \). Let \( \beta_T(t) = \beta(t/T) \). Let \( N(\lambda) \) be a non-decreasing function such that \( N(\lambda) \leq C\lambda^d \). Suppose that
\[
\int_0^\infty \hat{\beta}_T(\lambda - \mu) dN(\mu) = a_0 d\lambda^{d-1} + a_1 (d-1)\lambda^{d-2} + o(\lambda^{d-2}).
\]
Then,
\[
|N(\lambda) - a_0 \lambda^d + a_1 \lambda^{d-1}| \leq C \frac{a_0}{T} \lambda^{d-1} + o(\lambda^{d-1}).
\]

7.1. Completion of the proof of Theorem 1.2. Except for the last statement, Theorem 1.2 follows from Theorem 1.7 and a standard cosine Tauberian theorem: Except for the last statement on aperiodic manifolds, the remainder of the proof of Theorem 1.2 is similar to the end of the proof of Theorems 1.16 and 1.20 of [WXZ21]. We therefore sketch the overlapping proofs and refer to the earlier paper for complete details.

Proof. Theorem 1.2 pertains to the Weyl function \( N^1_{\psi,H}(\lambda) \) of (1.2), which for convenience we repeat here,
\[
N^1_{\psi,H}(\lambda) := \sum_{j,k: \lambda_k \leq \lambda} \psi(\lambda_j - \mu_k) \left| \int_H \varphi_j e^\lambda dV_H \right|^2.
\]
We assume with no essential loss of generality that \( \psi \geq 0 \). Then, \( N^c_{\psi,H}(\lambda) \) is monotone non-decreasing and has Fourier transform \( S(t, \psi) \) (2.4), (1.7).

For \( \psi \geq 0 \), we apply Proposition 7.1 with \( \hat{\rho} \cap \text{singsupp } S(t, \psi) = \{0\} \) and to \( dN^1_{\psi,H}(\lambda) \).

By Theorem 1.7, \( \rho \ast dN^1_{\psi,H}(\lambda) = \beta_0 \lambda^{\frac{n+d}{2} - 1} + O(\lambda^{\frac{n+d}{2} - 2}) \), and therefore,
\[
N^1_{\psi,H}(\lambda) = \rho \ast N^1_{\psi,H}(\lambda) + O(\lambda^{\frac{n+d}{2} - 2})
\]
\[
= \beta_0 \lambda^{\frac{n+d}{2}} + O(\lambda^{\frac{n+d}{2} - 1}),
\]
where \( \beta_0 \) is the principal coefficient, concluding the proof of Theorem 1.2. \( \square \)

7.2. Aperiodic case: Proof of the last statement of Theorem 1.2. It remains to prove the last statement of Theorem 1.2, that if the geodesic flow \( G^S_H \) of \( H \) is aperiodic, then
\[
N^1_{\psi,H}(\lambda) = C_{n,d} a^0_1(H, \psi) \lambda^{\frac{n+d}{2}} + R^1_{\psi,H}(\lambda), \quad \text{where } R^1_{\psi,H}(\lambda) = o(\lambda^{\frac{n+d}{2} - 1}).
\]

In the case where the fixed point sets of \( G^S_H \) at a period \( S \in \Sigma^1 \) are all clean, the last statement follows immediately from Proposition 6.1 and Proposition 7.2. Hence, the problem is to prove the same estimates without assuming cleanliness. We only assume that the closed geodesics at all non-zero periods forms a set of Liouville measure zero.

Remark: In principle, there could exist a second term of order \( \lambda^{\frac{n+d}{2} - 1} \). It requires a calculation to prove that it vanishes in the Kuznetsov \( c = 1 \) case (Section 7.6), as it did in the case \( c < 1 \) [WXZ21].

Proof. In Theorem 1.2, Theorem 1.7, we have already proved an asymptotic result when \( \hat{\rho} \) only contains \( \{0\} \) among the singularities. To obtain the two-term Weyl law for longer times, we use a pseudo-differential cutoff argument generalizing the one for pointwise Weyl asymptotics in [Hol14, Theorem 29.1.5-Corollary 29.1.6].

Let \( \hat{B}_T, \hat{b}_T := I - \hat{b}_T \in \Psi^0(H) \) be zeroth order pseudo-differential operators on \( H \) so that the support of the principal symbol \( b_T \) of \( \hat{b}_T \) contains the union of all closed geodesics of \( H \).
of period \( \leq T \). Let us briefly review the construction of \( B_T, b_T \) from [HolV]. First define the microlocal period function of \( H \),

\[
L^*_H(q, \eta) = \inf \{ t > 0 : G'_{\eta}(t, \eta) = (q, \eta) \},
\]

where \( L^* \) is defined to be \(+\infty\) if no such \( t \) exists. It is homogeneous of degree zero and lower semicontinuous. Henceforth, we restrict it to \( S^* H \). The set of periodic points of \( G'_{\eta} \) is the closed set defined by

\[
P_H = \{(q, \eta) \in S^* H : 1/L_H^*(q, \eta) \neq 0\}.
\]

If \( T > 1 \) is a large parameter, then we can find a function \( b_T \in C^\infty(S^* H, [0, 1]) \) so that

\[
\int_{S^* H} b_T(q, \eta) d\mu_L(q, \eta) \leq 1/T^2,
\]

and so that

\[
1/L_H^*(q, \eta) \leq 1/T, \quad \text{on supp } B_T (= \text{supp}(1 - b_T)).
\]

We then define \( \hat{B}_T = \text{Op}_H(B_T) \) for a fixed choice of quantization; similarly for \( \hat{b}_T \) and use the partition of unity \( I = \hat{B}_T + b_T \) to introduce pseudo-differential cutoffs on \( L^2(H) \) to decompose the trace [1.18]. There are several ways to introduce cutoffs in the composition \( \gamma_H U_M^* \gamma_H^* (t+s, q, q') U_H(-s, q, q') \): (i) to introduce \( I = \hat{B}_T + b_T \) only on the left and right sides of \( U_H(-s) \) (with adjoint on the right side) or only on the left and right sides of \( \gamma_H U_M^* (t+s) \gamma_H^* \); (ii) to introduce the partition of unity on both sides of both factors. It turns out that (ii) is a convenient choice in apply Proposition 7.2. In terms of eigenfunction expansions, it corresponds to

\[
N_{\psi, \rho, H}^1(\lambda) = \sum_{j, k=0}^{\infty} \rho(\lambda - \lambda_j) \psi(\lambda_j - \mu_k) \left| \langle (B_T + b_T)\gamma_H \varphi_j, (\hat{B}_T + b_T)\xi_k \rangle_H \right|^2,
\]

where \( \langle f, g \rangle_H = \int_H f \bar{g} dV_H \). The goal is to use Proposition 7.2 to show that (7.2) is \( o(\lambda^{-\frac{n+d}{2}}) \).

A crude but effective approach is to multiply out the inner product and the modulus-square and estimate each resulting term separately. Multiplying out the inner product \( \langle (B + b)\gamma_H \varphi_j, (B + b)\xi_k \rangle_H \) we obtain four terms. We call the ones with \( (B, B) \), resp. \( (b, b) \) ‘diagonal terms’ \( A_1 \), resp. \( A_2 \). Then multiplying out \( |D_1 + D_2 + A_1 + A_2|^2 \) gives the ‘pure’ products \( |D_1|^2 + |D_2|^2 + |A_1|^2 + |A_2|^2 \) plus the ‘mixed’ products, of which one is \( \text{Re } D_1 \bar{D}_2 \), four are of the form \( \text{Re } D_k \bar{D}_j \) and one is \( \text{Re } A_1 \bar{A}_2 \). Using \( |\text{Re } ab| \leq \frac{1}{2}(|a|^2 + |b|^2) \) we can bound all of the mixed products by a universal constant times the pure products. It therefore suffices to show that the analogue of (7.4) with summand \( |D_1|^2 + |D_2|^2 + |A_1|^2 + |A_2|^2 \) is \( o(\lambda^{-\frac{n+d}{2}}) \). We denote the corresponding sums by,

\[
N_{\text{pure}, \psi, \rho, H}^1(\lambda) = N_{|D_1|^2, \psi, \rho, H}^1(\lambda) + N_{|D_2|^2, \psi, \rho, H}^1(\lambda) + N_{|A_1|^2, \psi, \rho, H}^1(\lambda) + N_{|A_2|^2, \psi, \rho, H}^1(\lambda).
\]

Each term is a monotone non-decreasing temperate function in the sense of hypotheses of Proposition 7.2.

As mentioned in Section 1.7.1, the asymptotics of the four terms of (7.5) can be determined by the same method as for (1.16). We express each term as a semi-classical Fourier transform \( \mathcal{F}_{\lambda} \) of the corresponding part (e.g. \( S_{|D_1|^2}(t, \psi) \)) of of the Kuznecov trace, which we express in terms of the wave kernels composed with the designated pseudo-differential operators. The wave front analysis of these four operator traces is the same as in Section 2.2 the only change being in the formulae for the amplitudes and symbols.
7.2.1. The \((B, B)\) term. By the assumption on \(B\) and the wave front analysis in Section 2.2, the kernel
\[
K_{BB}(s, t, q, q') = B_T \gamma_H U(t + s) \gamma_H B_T^* \circ B_T U_H(-s) B_T^*(q, q')
\]
is smooth for \(0 < |t| < T\). Indeed, there do not exist any solutions of \(1.14\) for \(t \neq 0\) in the support of \(B_T\) and therefore the only solutions are those of \(1.15\). It follows that, as long as \(\text{Supp} \hat{\chi} \subset (-T, T), \chi \ast dN^1_{|D_1|^2, \psi, \rho, H}(\lambda)\) has a complete asymptotic expansion as in Theorem 1.7 and Proposition 6.1. To employ Theorem 7.2, we start with a given \(\chi \in \mathcal{S}(\mathbb{R})\) with \(\hat{\chi}\) vanishing near 0 and supported in \([-T, T]\) and then decompose \(1 = B_T + b_T\). Then \(S_{BB}(t, \psi)\) is smooth for \(t \in \text{Supp} \hat{\chi}\). Hence, \(\chi \ast dN^1_{|D_1|^2, \psi, \rho, H}(\lambda) = O(\lambda^{-\infty})\).

7.2.2. The \((b, b)\) term. With \(\chi, T\) fixed as above, we next consider \(\chi \ast dN^1_{|D_2|^2, \psi, \rho, H}(\lambda)\). To prove that this monotone function is \(O(\lambda^{\frac{n+d}{2} - 2})\) we use Proposition ?? to prove that for any \(\epsilon > 0\),
\[
\chi \ast dN^1_{\psi, H}(\lambda) \leq C \epsilon \lambda^{\frac{n+d}{2} - 1}.
\]
Indeed, this estimate holds if \(\chi\) is replaced by \(\rho\) with \(\hat{\rho}\) supported in \((-r_0, r_0)\). As mentioned above, the proofs of the asymptotic expansion of Theorem 1.7 and of the Kuznetsov-Weyl law Theorem 1.2 extend with only minor modifications if we compose with \(b(x, D)\). The modification is that the integrand of the principal term \(a^0_1(H, \psi)\) acquires the additional factor of the principal symbol \(b_0\) of \(b(x, D)\) and is therefore of order \(\frac{1}{T} < \epsilon\). By Proposition ?? the same estimate holds general \(\chi\).

7.3. \((b, B)\) terms. For such ‘off-diagonal terms, we move both cutoffs onto the \(e_k\) factor of the inner products, as in
\[
\left| \langle \gamma_H \varphi_j, (b_T)^* (B_T e_k)_H \rangle \right|^2.
\]
As in the case of \((B, B)\) terms, \((b_T)^* (B_T) U_H(s)\) has a smooth kernel for \(s \in (0, T)\). Hence, the contributions of these term is the same as their cutoff to a small interval around \(s = 0\) multiplied by \(\frac{1}{T}\).

7.4. Conclusion. It follows that if the expansion of \(N^1_{\psi, \rho, H}(\lambda)\) is the same as the expansion in the case where \(\text{Supp} \hat{\rho} \subset (-r_0, r_0)\) plus \(\epsilon \lambda^{\frac{n+d}{2} - 1}\) plus \(\frac{1}{T} \lambda^{\frac{n+d}{2} - 1}\). Hence, assuming that the second term in the expansion at \(t = 0\) vanishes,
\[
N^1_{\psi, \rho, H}(\lambda) = a^0_1(H, \psi) \lambda^{\frac{n+d}{2} - 1} + \epsilon \lambda^{\frac{n+d}{2} - 1}
\]
for any \(\epsilon > 0\), proving the last statement of Theorem 1.2.

7.5. Proof of Corollary 1.3.

Proof. Theorem 1.3 is a consequence of the remainder estimate of Theorem 1.2. To prove Theorem 1.3 it suffices to prove that, for any \(\epsilon > 0\) there exists a test function \(\psi \geq 0, \hat{\psi} \in C_0^\infty(\mathbb{R}), \hat{\psi}(0) = 1\) with \(\text{Supp} \hat{\psi} \subset (-r_0, r_0)\) and \(\psi \geq 1_{[-\epsilon, \epsilon]}\). Then there exists a universal constant \(C(\epsilon)\) depending only on \((\epsilon, \delta)\) so that for all \(\lambda_j\),
\[
J^1_{\psi, H}(\lambda_j) \geq C(\epsilon) J^1_{\epsilon, H}(\lambda_j).
\]
(7.7)
Then,
\[ \sum_{k:|\mu_k - c\lambda_j| \leq \varepsilon} \left| \int_H \varphi_j e^{\xi_k} dV_H \right|^2 \leq \sum_{k} \psi(\lambda_j - \mu_k) \left| \int_H \varphi_j e^{\xi_k} dV_H \right|^2, \]
and the upper bound for \( J^1_{\psi,H}(\lambda_j) \) given in Corollary 1.3 provides the upper bound for \( J^1_{\psi,H}(\lambda_j) \).

The construction of \( \psi = \psi_\varepsilon \) is elementary and we follow the discussion in [DG75, Lemma 2.3]. Let \( \psi \in \mathcal{S}(\mathbb{R}) \) with \( \hat{\psi} \in C^\infty_0 \). Replacing \( \psi \) by \( \psi \cdot \hat{\psi} \) one has \( \psi \geq 0, \psi(0) > 0 \) and \( \psi \in C^\infty_0 \). Replacing \( \hat{\psi} \) by \( \hat{\psi}(\frac{x}{\delta}) \) and \( \psi \) by \( \delta \psi(x,\delta) \), and taking \( \delta \) sufficiently small one can assume that \( \text{supp} \hat{\psi} \subset (-r_0, r_0) \) and by multiplying by a positive scalar we have, \( \psi > 0 \) on \([-K, K]\) for any \( K > 0 \).

\[ \square \]

In the aperiodic case, the same argument gives (1.11).

7.6. Subprincipal term. The vanishing of the subprincipal term is a result pertaining to the smooth expansions in Theorem 6.1 and is independent of the Tauberian argument. As in [WXZ21], it follows from the fact that the subprincipal symbol of \( \sqrt[4]{\Delta_X} \) vanishes for any Riemannian manifold \( X \), together with some parity arguments from [DG75]. We assume that \( \hat{\psi} \) has small support and both \( \hat{\psi} \) and \( \hat{\rho} \) are even, and that \( \hat{\rho} \equiv 1 \) near 0.

We claim that the subprincipal symbol is odd, so that its integrals over cospheres vanishes. We first note that the subprincipal symbols of \( \sqrt{-\Delta_M} \otimes I \) and of \( Q_c \) both vanish. The homogeneous part of degree \( k \) in \( \sigma_P(x, \xi) \) is even, resp. odd if \( k \) is even, resp. odd. By induction with respect to \( r \) it follows that \( (\frac{\partial}{\partial \mu})^r a_{-j} \) is an even, resp. odd. if \( r - j \) is even, resp. odd. The amplitude of \( e^{itP} e^{isQ_c} \) is obtained by integrating \( e^{it(\xi-P_M \otimes \xi + \bar{P}_H)} \). The parities of the terms in the amplitude agree with those of [DG75], for \( s = t = 0 \). The restriction of the \( M \)-amplitudes to \( H \) have the same parity. The further restriction to the diagonals in \( H \times H \) seems to multiply the amplitudes, but the subprincipal term can only be obtained as the product of the principal symbol and the subprincipal symbol. Hence it is odd.

8. Appendix

8.1. Blow-down singularity. In this section, we review the definition of a blowdown map \( f : \mathbb{R}^N \to \mathbb{R}^N \), following [G89, Page 111]. \( f : X^N \to Y^N \) is a blow-down map with singularity along a submaifold \( S \) if

- \( S = \{ x : \det Df(x) = 0 \} \) is the critical set of \( f \). One assumes that \( d(\det Df(x)) \neq 0 \) on \( S \), so that \( S \) is a smooth submanifold.
- \( \ker d_{s} f \subset T_{s} S \) for all \( s \in S \);
- \( f \) is of constant rank along \( S \) and \( f^{*} dV_x \) vanishes to order \( n - k \) along \( S \).

Roughly, \( f|_{S} : S \to W \) is a fibration over a submanifold \( W \subset Y \) of codimension \( n - k + 1 \). Under these assumptions, there exist coordinates \( x_1, \ldots, x_n \) around each \( s \in S \) and \( y_1, \ldots, y_n \) around \( f(s) \) in \( Y \) so that \( f^{*} y_{j} = x_j \) for \( j = 1, \ldots, k \) and \( f^{*} y_{i} = x_{i} x_1 \) for \( i = k + 1, \ldots, n \). In this case, \( S = \{ x_1 = 0 \} \) and \( W = \{ \bar{y} : y_1 = y_{k+1} = \cdots y_n = 0 \} \).
We claim that the Lagrange map $\iota_\Psi$ of Lemma 3.3 and of the model phase (4.1) is a blow-down map. The critical point set of (4.1) is given by,
\[
\nabla_{y_1,\ldots,y_{d-1}} \Psi_{\text{model}} = \vec{x} \in \mathbb{R}^{d-1}, \quad \nabla_{y_d} \Psi_{\text{model}} = -\frac{1}{2} (x_1^2 + \cdots + x_{d-1}^2 + x_{d+1}^2 + \cdots + x_d^2) = 0,
\]
\[
\nabla_{(x_1,\ldots,x_{d-1})} \Psi_{\text{model}} = \vec{y} \in \mathbb{R}^{d-1}, \quad \nabla_{(x_{d+1},\ldots,x_n)} \Psi_{\text{model}} = y_d \vec{x} \in \mathbb{R}^{n-d}.
\]
The phase variables are $(y_1,\ldots,y_{d-1},x_1,\ldots,x_{d-1})$. The second equation forces $x_d = 1$.

In the definition of $f$, we let
\[
X = \{(y_d, x_{d+1}, \ldots, x_n)\} \simeq \mathbb{R}^{n-d}, \quad Y = \{(y_1,\ldots,y_{d-1},x_1,\ldots,x_{d-1})\} \simeq \mathbb{R}^{2(d-1)},
\]
and define the critical set of the phase by,
\[
C_{\Psi_{\text{model}}} = \{(y_1,\ldots,y_d,x_1,\ldots,x_n) : (x_1,\ldots,x_{d-1}) = 0 = (y_1,\ldots,y_{d-1})\} \subset X \times \mathbb{R}^{2(d-1)}.
\]
Then the associated Lagrange map $\iota_{\Psi_{\text{model}}}: C_{\Psi_{\text{model}}} \to T^* \mathbb{R}^{n-d}$ is given by,
\[
\iota_{\Psi_{\text{model}}}((\vec{0}, y_d, \vec{0}, x_{d+1} \ldots, x_n)) = (\vec{0}, y_d, d_y \Psi_{\text{model}}, \vec{0}, (x_{d+1}, \ldots, x_n), d_x^* \Psi_{\text{model}})
\]
\[
= (\vec{0}, y_d, 0, \vec{0}, (x_{d+1}, \ldots, x_n), y_d (x_{d+1}, \ldots, x_n)).
\]
The image is a (non-homogeneous) Lagrangian submanifold of $T^* \mathbb{R}^{n-d}$, in which the fiber $(x_{d+1}, \ldots, x_n) \in \mathbb{R}^{n-d-1}$ gets blown down to a point when $y_d = 0$. In the original model over $T^* H$ and with $\vec{\omega} = \vec{c}_d$, the set $y_d$ corresponds to the diagonal, and $S^* H$ gets blown down to a point when $s = 0$.

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