Schrödinger’s Interpolating Dynamics and Burgers’ Flows

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Abstract

We discuss a connection (and a proper place in this framework) of the unforced and deterministically forced Burgers equation for local velocity fields of certain flows, with probabilistic solutions of the so-called Schrödinger interpolation problem. The latter allows to reconstruct the microscopic dynamics of the system from the available probability density data, or the input-output statistics in the phenomenological situations. An issue of deducing the most likely dynamics (and matter transport) scenario from the given initial and terminal probability density data, appropriate e.g. for studying chaos in terms of densities, is here exemplified in conjunction with Born’s statistical interpretation postulate in quantum theory, that yields stochastic processes which are compatible with the Schrödinger picture free quantum evolution.

1 The Schrödinger reconstruction problem: most likely microscopic dynamics from the input-output statistics data

Probability measures, both invariant and nontrivially time-dependent, often on different levels of abstraction, are ubiquitous in diverse areas of physics. According

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to pedestrian intuitions, one normally expects that any kind of time development (dynamics, be it deterministic or random), which is analyzable in terms of probability, under suitable mathematical restrictions may give rise to a well defined stochastic process. Non-Markovian implementations are regarded to be close to reality, but the corresponding Markovian approximations (when appropriate) are easier to handle analytically.

Given a dynamical law of motion (for a particle as example), in many cases one can associate with it (compute or approximate the observed frequency data) a probability distribution and various mean values. In fact, it is well known that inequivalent finite difference random motion problems may give rise to the same continuous approximant (like e.g. in case of the diffusion equation representation of discrete processes). As well, in the study of nonlinear dynamical systems, specifically those exhibiting the so-called deterministic chaos, given almost any (basically one-dimensional in the cited references) probability density, it is possible to construct an infinite number of deterministic finite difference equations, whose iterates are chaotic and which give rise to this a priori prescribed density.

Studying dynamics in terms of densities of probability measures instead of individual paths (trajectories) of a physical system is a respectable tool, even if we know exactly the pertinent microscopic dynamics.

Under general circumstances, the main task of a physicist is to fit a concrete dynamical model (through a clever guess or else) to available phenomenological data. Then, the distinction between the chaotic (nonlinear, deterministic) and purely stochastic implementations may not be sharp enough to allow for a clean discrimination between those options: the intrinsic interplay between the stochastic and deterministic modelling of physical phenomena, blurs the access to reality and certainly precludes a definitive choice of one type of modelling against the other.

An inverse operation of deducing the detailed (possibly individual, microscopic) dynamics, which is either compatible with a given probability measure (we shall be mostly interested in those admitting densities) or induces its own time evolution, cannot have a unique solution. However, the level of ambiguities can be substantially reduced, if we invoke the so-called Schrödinger problem of reconstructing the microscopic dynamics from the given input-output statistics data and/or from the a priori known time development of a given probability density. The problem is known to give rise to a particular class of solutions (most likely interpolations), in terms of Markov diffusion processes.

In its original formulation, due to Schrödinger, one seeks the answer to the following question:

given two strictly positive (usually on an open space-interval) boundary probability densities $\rho_0(\vec{x}), \rho_T(\vec{x})$ for a process with the time of duration $T \geq 0$. Can we uniquely
identify the stochastic process interpolating between them?

Another version of the same problem, \cite{6}, departs from a given (Fokker-Planck-type) probability density evolution and investigates the circumstances allowing to deduce a unique random process from this dynamics. We shall pay some attention to this issue in Section 3.

The answer to the above Schrödinger’s question is known to be affirmative, if we assume the interpolating process to be Markovian. In particular, we can get here a unique Markovian diffusion process which is specified by the joint probability distribution

$$m_T(A, B) = \int_A d^3x \int_B d^3y \, m_T(\vec{x}, \vec{y})$$

where

$$m_T(\vec{x}, \vec{y}) = u_0(\vec{x}) \, k(x, 0, y, T) \, v_T(\vec{y})$$

and the two unknown functions $u_0(\vec{x}), v_T(\vec{y})$ come out as solutions of the same sign of the integral identities (1). Provided, we have at our disposal a continuous bounded strictly positive (ways to relax this assumption were discussed in Ref. \cite{12}) integral kernel $k(\vec{x}, s, \vec{y}, t), 0 \leq s < t \leq T$.

We shall confine further attention to cases governed by the familiar Feynman-Kac kernels. Then, the solution of the Schrödinger boundary-data problem in terms of the interpolating Markovian diffusion process is found to be completely specified by the adjoint pair of parabolic equations. In case of gradient forward drift fields, the pertinent process can be determined by checking (this imposes limitations on the admissible potential) whether the Feynman-Kac kernel

$$k(\vec{y}, s, \vec{x}, t) = \int \exp[- \int_s^t c(\vec{\omega}(\tau), \tau)d\tau]d\mu_{(\vec{y}, s)}(\omega)$$

is positive and continuous in the open space-time area of interest (then, additional limitations on the path measure need to be introduced, \cite{11}), and whether it gives rise to positive solutions of the adjoint pair of generalised heat equations:

$$\partial_t u(\vec{x}, t) = \nu \Delta u(\vec{x}, t) - c(\vec{x}, t)u(\vec{x}, t)$$

$$\partial_t v(\vec{x}, t) = -\nu \Delta v(\vec{x}, t) + c(\vec{x}, t)v(\vec{x}, t) .$$

Here, a function $c(\vec{x}, t)$ is restricted only by the positivity and continuity demand for the kernel (3), see e.g. \cite{11}. In the above, $d\mu_{(\vec{y}, s)}(\omega)$ is the conditional Wiener measure over sample paths of the standard Brownian motion.
Solutions of (4), upon suitable normalisation give rise to the Markovian diffusion process with the factorised probability density \( \rho(\vec{x}, t) = u(\vec{x}, t)v(\vec{x}, t) \) which, while evolving in time, interpolates between the boundary density data \( \rho(\vec{x}, 0) \) and \( \rho(\vec{x}, T) \). The interpolation admits a realisation in terms of Markovian diffusion processes with the respective forward and backward drifts defined as follows:

\[
\vec{b}(\vec{x}, t) = 2\nu \frac{\nabla v(\vec{x}, t)}{v(\vec{x}, t)}
\]

\[
\vec{b}_*(\vec{x}, t) = -2\nu \frac{\nabla u(\vec{x}, t)}{u(\vec{x}, t)}
\]

in the prescribed time interval \([0, T]\).

The related transport equations for the densities easily follow. For the forward interpolation, the familiar Fokker-Planck equation holds true:

\[
\partial_t \rho(\vec{x}, t) = \nu \Delta \rho(\vec{x}, t) - \nabla [\vec{b}(\vec{x}, t) \rho(\vec{x}, t)]
\]

while for the backward interpolation we have:

\[
\partial_t \rho(\vec{x}, t) = -\nu \Delta \rho(\vec{x}, t) - \nabla [\vec{b}_*(\vec{x}, t) \rho(\vec{x}, t)] .
\]

We have assumed that drifts are gradient fields, \( \text{curl} \vec{b} = 0 \). As a consequence, those that are allowed by the prescribed choice of \( c(\vec{x}, t) \) must fulfill the compatibility condition

\[
c(\vec{x}, t) = \partial_t \Phi + \frac{1}{2} \left( b^2 - 2\nu \nabla b \right)
\]

which establishes the Girsanov-type connection of the forward drift \( \vec{b}(\vec{x}, t) = 2\nu \nabla \Phi(\vec{x}, t) \) with the Feynman-Kac, cf. \([11, 10]\), potential \( c(\vec{x}, t) \). In the considered Schrödinger’s interpolation framework, the forward and backward drift fields are connected by the identity \( \vec{b}_* = \vec{b} - 2\nu \nabla \ln \rho \).

One of the distinctive features of Markovian diffusion processes with the positive density \( \rho(\vec{x}, t) \) is that, given the transition probability density of the (forward) process, the notion of the backward transition probability density \( p_*(\vec{y}, s, \vec{x}, t) \) can be consistently introduced on each finite time interval, say \( 0 \leq s < t \leq T \):

\[
\rho(\vec{x}, t) p_*(\vec{y}, s, \vec{x}, t) = p(\vec{y}, s, \vec{x}, t) \rho(\vec{y}, s)
\]

so that \( \int \rho(\vec{y}, s) p(\vec{y}, s, \vec{x}, t) d^3y = \rho(\vec{x}, t) \) and \( \rho(\vec{y}, s) = \int p_*(\vec{y}, s, \vec{x}, t) \rho(\vec{x}, t) d^3x \).

The transport (density evolution) equations (6) and (7) refer to processes running in opposite directions in a fixed, common for both, time-duration period.
forward one, (6), executes an interpolation from the Borel set $A$ to $B$, while the backward one, (7), executes an interpolation from $B$ to $A$, compare e.g. the defining identities (1).

The knowledge of the Feynman-Kac kernel (3) implies that the transition probability density of the forward process reads:

$$p(\vec{y}, s, \vec{x}, t) = k(\vec{y}, s, \vec{x}, t) \frac{v(\vec{x}, t)}{v(\vec{y}, s)} \cdot$$

while the corresponding (derivable from (10), since $\rho(\vec{x}, t)$ is given) transition probability density of the backward process has the form:

$$p_{\ast}(\vec{y}, s, \vec{x}, t) = k(\vec{y}, s, \vec{x}, t) \frac{u(\vec{y}, s)}{u(\vec{x}, t)} \cdot$$

Obviously, [10, 8], in the time interval $0 \leq s < t \leq T$ there holds:

$$u(\vec{x}, t) = \int u_{0}(\vec{y})k(\vec{y}, s, \vec{x}, t)d^{3}y$$

$$v(\vec{y}, s) = \int k(\vec{y}, s, \vec{x}, T)v_{T}(\vec{x})d^{3}x \cdot$$

Consequently, the system (4) fully determines the underlying random motions, forward and backward, respectively.

2 The Burgers equation in Schrödinger’s interpolation

The prototype nonlinear field equation named the Burgers or "nonlinear diffusion" equation (typically without, [14, 15], the forcing term $\vec{F}(\vec{x}, t))$:

$$\partial_{t}\vec{v}_{B} + (\vec{v}_{B}\nabla)\vec{v}_{B} = \nu \Delta \vec{v}_{B} + \vec{F}(\vec{x}, t)$$

recently has acquired a considerable popularity in the variety of physical contexts, [13].

By dropping the force term in (13), we are left with a commonly used form of the "nonlinear diffusion equation" whose solutions are known exactly, in view of the Hopf-Cole linearising transformation mapping (13) into the heat equation. Here, $\partial_{t}\vec{v}_{B} + (\vec{v}_{B}\nabla)\vec{v}_{B} = \nu \Delta \vec{v}_{B}$ is mapped into $\partial_{t}\theta = \nu \Delta \theta$, by means of the substitution $\vec{v}_{B} = -2\nu \nabla ln\theta$. This linearisation of the Burgers equation is normally regarded to be devoid of any deeper physical meaning, and specifically the link with stochastic
processes determined by the heat equation has not received a proper attention. Our previous analysis shows that the intrinsic interplay between the deterministic and random evolution, appropriate for a large class of classically chaotic systems, extends to much wider framework.

Burgers velocity fields can be analysed on their own with different (including random) choices of the initial data and/or force fields. However, we are interested in the possible diffusive matter transport that is locally governed by Burgers flows, cf. [13]. In this particular connection, let us point out a conspicuous hesitation that could have been observed in attempts to establish the most appropriate matter transport rule, if any diffusion-type microscopic dynamics assumption is adopted to underly the "nonlinear diffusion" (13).

Depending on the particular phenomenological departure point, one either adopts the standard continuity equation, [16, 17], that is certainly valid to a high degree of accuracy in the so-called low viscosity limit $\nu \downarrow 0$, but incorrect on mathematical grounds if there is a genuine Markovian diffusion process involved and simultaneously a solution of (13) stands for the respective current velocity of the flow:

$$\partial_t \rho(\vec{x}, t) = -\nabla[\vec{v}(\vec{x}, t)\rho(\vec{x}, t)] .$$

Alternatively, following the white noise calculus tradition telling that the stochastic integral $\vec{X}(t) = \int_0^t \vec{v}_B(\vec{X}(s), s)ds + \int_0^t \vec{\eta}(s)ds$ necessarily implies the Fokker-Planck equation, one is tempted to adopt:

$$\partial_t \rho(\vec{x}, t) = \nu \Delta \rho(\vec{x}, t) - \nabla[\vec{v}_B(\vec{x}, t)\rho(\vec{x}, t)]$$

which is clearly problematic in view of the classic McKean’s discussion of the propagation of chaos for the Burgers equation, [18, 19, 20] and the derivation of the stochastic "Burgers process" in this context: "the fun begins in trying to describe this Burgers motion as the path of a tagged molecule in an infinite bath of like molecules", [18].

To put things on the solid ground, let us consider a Markovian diffusion process, which is characterised by the transition probability density (generally inhomogeneous in space and time law of random displacements) $p(\vec{y}, s, \vec{x}, t)$, $0 \leq s < t \leq T$, and the probability density $\rho(\vec{x}, t)$ of its random variable $\vec{X}(t)$, $0 \leq t \leq T$. The process is completely determined by these data. For clarity of discussion, we do not impose any spatial boundary restrictions, nor fix any concrete limiting value of $T$ which, in principle, can be moved to infinity.

Let us confine attention to processes defined by the standard backward diffusion equation. Under suitable restrictions (boundedness of involved functions, their continuous differentiability) the function:

$$g(\vec{x}, s) = \int p(\vec{x}, s, \vec{y}, T)g(\vec{y}, T)d^3y \quad (14)$$

satisfies the equation

$$-\partial_s g(\vec{x}, s) = \nu \Delta g(\vec{x}, s) + [\vec{b}(\vec{x}, s)\nabla]g(\vec{x}, s) . \quad (15)$$
Let us point out that the validity of (14) is known to be a necessary condition for the existence of a Markov diffusion process, whose probability density $\rho(\vec{x}, t)$ is to obey the Fokker-Planck equation (the forward drift $\vec{b}(\vec{x}, t)$ replaces the previously utilized Burgers velocity $\vec{v}_B(\vec{x}, t)$).

The case of particular interest, in the traditional nonequilibrium statistical physics literature, appears when $p(\vec{y}, s, \vec{x}, t)$ is a fundamental solution of (15) with respect to variables $\vec{y}, s$, [21, 22, 23], see however [10] for an analysis of alternative situations. Then, the transition probability density satisfies also the second Kolmogorov (e.g. the Fokker-Planck) equation in the remaining $\vec{x}, t$ pair of variables. Let us emphasize that these two equations form an adjoint pair of partial differential equations, referring to the slightly counterintuitive for physicists, though transparent for mathematicians, [24, 25, 26, 27, 8], issue of time reversal of diffusions.

We can consistently introduce the random variable of the process in the form $\vec{X}(t) = \int_0^t \vec{b}(\vec{X}(s), s) ds + \sqrt{2\nu} \vec{W}(t)$. Then, in view of the standard rules of the Itô stochastic calculus, [28, 27, 8], we realise that for any smooth function $f(\vec{x})$ of the random variable $\vec{X}(t)$ the conditional expectation value:

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[ \int p(\vec{x}, t, \vec{y}, t + \Delta t)f(\vec{y}, t + \Delta t)d^3y - f(\vec{x}, t) \right] = (D_+ f)(\vec{X}(t), t) = (16)$$

$$= (\partial_t + \vec{b} \nabla + \nu \Delta)f(\vec{x}, t) ,$$

where $\vec{X}(t) = \vec{x}$, determines the forward drift $\vec{b}(\vec{x}, t)$ of the process (if we set components of $\vec{X}$ instead of $f$) and, moreover, allows to introduce the local field of (forward) accelerations associated with the diffusion process, which we constrain by demanding (see e.g. Refs. [27, 28, 8] for prototypes of such dynamical constraints):

$$(D_+^2 \vec{X})(t) = (D_+ \vec{b})(\vec{X}(t), t) = (\partial_\mu \vec{b}+(\vec{b} \nabla)\vec{b}+\nu \Delta \vec{b})(\vec{x}, t) = \vec{F}(\vec{x}, t) (17)$$

where $\vec{X}(t) = \vec{x}$ and, at the moment arbitrary, function $\vec{F}(\vec{x}, t)$ may be interpreted as an external forcing applied to the diffusing system, [11].

By invoking (9), we can also define the backward derivative of the process in the conditional mean (cf. [11, 29, 30] for a discussion of these concepts in case of the most traditional Brownian motion and Smoluchowski-type diffusion processes)

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} [\vec{x} - \int p_*(\vec{y}, t - \Delta t, \vec{x}, t)\vec{y}d^3y] = (D_- \vec{X})(t) = \vec{b}_*(\vec{X}(t), t) (18)$$

$$(D_- f)(\vec{X}(t), t) = (\partial_t + \vec{b}_* \nabla - \nu \Delta)f(\vec{X}(t), t)$$

Accordingly, the backward version of the acceleration field reads

$$(D_-^2 \vec{X})(t) = (D_+^2 \vec{X})(t) = \vec{F}(\vec{X}(t), t) (19)$$
where in view of $\tilde{b}_* = \tilde{b} - 2\nu \nabla \ln \rho$ we have explicitly fulfilled the forced Burgers equation:

$$\partial_t \tilde{b}_* + (\tilde{b}_* \nabla)\tilde{b}_* - \nu \triangle \tilde{b}_* = \tilde{F}$$

(20)

and, \[72, 8, 11\], under the gradient-drift field assumption, $\text{curl} \tilde{b}_* = 0$, we deal with $\tilde{F}(\vec{x}, t) = 2\nu \nabla c(\vec{x}, t)$ where the Feynman-Kac potential (3) is explicitly involved.

Let us notice that the familiar (linearization of the nonlinear problem) Hopf-Cole transformation, \[15, 31\], of the Burgers equation into the generalised diffusion equation (yielding explicit solutions in the unforced case) has been explicitly used before (the second formula (4)) in the framework of the Schrödinger interpolation problem. In fact, by defining $\Phi_* = \log u$, we immediately recover the traditional form of the Hopf-Cole transformation for Burgers velocity fields: $\tilde{b}_* = -2\nu \nabla \Phi_*$. In the standard considerations that allows to map a nonlinear (unforced Burgers) equation into a linear, heat, equation. In the special case of the standard free Brownian motion, there holds $\tilde{b}(\vec{x}, t) = 0$ while $\tilde{b}_*(\vec{x}, t) = -2\nu \nabla \log \rho(\vec{x}, t)$.

Let us point out that the equation (7) is in fact the only transport equation where the Burgers velocity field is allowed to be undisputably present, under the diffusive scenario assumption, \[13\]. The standard continuity equation is certainly inappropriate for nonzero values of the diffusion constant $\nu$.

3 Reconstruction of the microscopic dynamics from the probability density data: obstacles exemplified

We have mentioned before, that another version of the Schrödinger boundary data problem, \[6\], departs directly from a given (Fokker-Planck-type) probability density evolution and investigates the circumstances allowing to deduce a unique random process from this dynamics. Surely, solutions of the Fokker-Plack equation itself do not yet determine the underlying stochastic process. Additional assumptions are always necessary and a number of traps to be avoided.

As a particular guide to those obstacles, we shall refer to the familiar free quantum evolution that is regarded as the time adjoint parabolic problem, exactly in the spirit of our previous discussion.

In our previous paper, \[30\], the major conclusion was that in order to give a definitive probabilistic description of the quantum dynamics as a unique diffusion process solving Schrödinger’s interpolation problem, a suitable Feynman-Kac semigroup must be singled out. Let us point out that the measure preserving dynamics,
permitted in the presence of conservative force fields, was investigated in \[11\], see also \[?, 32\].

The present analysis was performed quite generally and extends to the dynamics affected by time dependent external potentials, with no clear-cut discrimination between the nonequilibrium statistical physics and essentially quantum evolutions. The formalism of Section 1 encompasses both groups of problems. Nevertheless, it is quite illuminating to see directly how sensitive, even in simplest cases, the formalism is with respect to any attempt of relaxing our previous assumptions and the Schrödinger interpolation problem rules-of-the-game. Specifically in the quantum domain, where the seemingly trivial case of the free evolution, which is nonstationary, needs the general parabolic system (4) to be considered. Even worse, then the system (4) displays a nontrivial nonlinearity: the parabolic equations are coupled by the effective, solution dependent potential. At the first glance, this feature might seem to exclude the existence of any conceivable Feynman-Kac (dynamical semigroup) kernel, and in consequence any common-sense law of random displacements (i.e. the transition probability density) governing the pertinent stochastic evolution. Certainly, the existence of fundamental solutions in this case is far from being obvious.

At this point, let us emphasize that our principal goal is to take seriously the Schrödinger picture quantum dynamics under the premises of the Born statistical postulate. Hence, once we select as appropriate a concrete quantal interpolation between the prescribed (phenomenologically supported in particular) input-output statistics data \(\rho_0(x)\) and \(\rho_T(x)\) in terms of \(\rho(x,t) = \psi(x,t)\psi^*(x,t)\), \(t \in [0,T]\), where \(\psi(x,t)\) solves the Schrödinger equation then, on exactly the same footing, we are entitled to look for an alternative probabilistic explanation (or appropriate description) of the very same interpolation, in terms of a well defined Markov stochastic (eventually diffusion) process.

We shall proceed in the spirit of Section 1, while restricting our discussion to the free Schrödinger dynamics. Following Ref. \[30\] we shall discuss the rescaled problem so as to eliminate all dimensional constants.

The free Schrödinger evolution \(i\partial_t \psi = -\Delta \psi\) implies the following propagation of a specific Gaussian wave packet:

\[
\psi(x, 0) = (2\pi)^{-1/4}exp \left(-\frac{x^2}{4}\right) \rightarrow \psi(x, t) = \frac{2}{\pi}^{1/4} (2 + 2it)^{-1/2}exp\left[-\frac{x^2}{4(1 + it)}\right]
\]

So that

\[
\rho_0(x) = |\psi(x, 0)|^2 = (2\pi)^{-1/2} exp\left[-\frac{x^2}{2}\right] \rightarrow
\]

\[
\rho_0(x) = \frac{2}{\pi}^{1/4} (2 + 2it)^{-1/2}exp\left[-\frac{x^2}{4(1 + it)}\right] \rightarrow
\]

\[ \rho(x, t) = |\psi(x, t)|^2 = [2\pi(1 + t^2)]^{-1/2} \exp[-\frac{x^2}{2(1 + t^2)}] \]

and the Fokker-Planck equation (easily derivable from the standard continuity equation \( \partial_t \rho = -\nabla(v\rho) \), \( v(x, t) = xt/(1 + t^2) \)) holds true:

\[
\partial_t \rho = \Delta \rho - \nabla(b\rho), \quad b(x, t) = -\frac{1 - t}{1 + t^2} x
\] (23)

The Madelung factorization \( \psi = \exp(R + iS) \) implies (notice that \( v = 2\nabla S \) and \( b = 2\nabla(R + S) \)) that the related real functions \( \theta(x, t) = \exp(R + iS) \) and \( \theta^*(x, t) = \exp(R - iS) \) read:

\[
\theta(x, t) = [2\pi(1 + t^2)]^{-1/4} \exp(-\frac{x^2}{4} \frac{1 - t}{1 + t^2} - \frac{1}{2} \arctan t)
\]

\[
\theta^*(x, t) = [2\pi(1 + t^2)]^{-1/4} \exp(-\frac{x^2}{4} \frac{1 + t}{1 + t^2} + \frac{1}{2} \arctan t)
\] (24)

They solve a suitable version of the general parabolic equations (4), namely:

\[
\partial_t \theta = -\Delta \theta + \frac{1}{2} \Omega \theta
\] (25)

\[
\partial_t \theta^* = \Delta \theta^* - \frac{1}{2} \Omega \theta^*
\]

with

\[
\frac{1}{2} \Omega(x, t) = \frac{x^2}{2(1 + t^2)^2} - \frac{1}{1 + t^2} = 2\frac{\Delta \rho^{1/2}}{\rho^{1/2}} = Q(x, t)
\] (26)

By setting \( t = T \) we associate with the above dynamics the terminal density \( \rho_T(x) \), and then the concrete Schrödinger boundary data problem for the stochastic interpolation \( \rho_0(x) \rightarrow \rho_T(x) \), (1).

To capture the spirit of our previous discussion, we shall replace equations (25) by the more general equations (4), where only the potential \( c(x, t) \) will be identified with the above \( \frac{1}{2} \Omega(x, t) \). Then, we shall look for solutions \( u(x, t) \), \( v(x, t) \) of these parabolic equations, and in particular we shall identify the quantally implemented functions \( \theta(x, t), \theta^*(x, t), (24) \), among them. Effectively, it amounts to the previously mentioned linearisation of the nonlinear parabolic system.

In view of the relatively simple form of the probability density \( \rho(x, t) \), (22) one might be tempted to guess (more or less fortunately) the transition probability density, consistent with the propagation (22). However, it is well known that there are many stochastic processes implying (22) for all \( t \in [0, T] \), which not necessarily
have much in common with the original wave function dynamics (21). In general they are incompatible with the corresponding parabolic system (cf. (4) and (25)). If it happens otherwise, the reason for this proliferation of would-be consistent stochastic processes is rooted in exploiting the particular functional form of solutions, instead of relying on the form-independent arguments, e.g. (4).

Let us consider simple examples which, albeit coming under very special circumstances (free dynamics with a specific initial wave packet choice, and no zeros admitted in the course of the propagation), clearly indicate how important is the proper choice of the Feynman-Kac kernel. The virtue of a parabolic system (4) is that its form is universal for the Schrödinger dynamics, and thus does not depend on a particular functional form of solutions nor this of external potentials. It appears that the system (4) sets a very rigid framework for the probabilistic manifestations (e.g. stochastic processes) of the quantum Schrödinger dynamics.

**Example 1**: We shall demonstrate that an improper (not through (4) or (25)), but fortunate, choice of the kernel might lead to an alternative stochastic representation of the quantum dynamics (22).

Let us begin from directly introducing the transition probability density

\[ p(y, s, x, t) = [2\pi(t^2 - s^2)]^{-1/2} \exp\left[-\frac{(x - y)^2}{2(t^2 - s^2)}\right] \]

which for all intermediate times \(0 \leq s < t \leq T\) executes a desired propagation

\[ \rho(x, t) = \int p(y, s, x, t) \rho(y, s)dy, \] (22)

Clearly, the Chapman-Kolmogorov identity

\[ \int p(y, s, z, \tau) p(z, \tau, x, t) d\tau = p(y, s, x, t) \]

holds true, and the properties (the first one, for all \(\epsilon > 0\)):

\[ \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \int_{|x-y|>\epsilon} p(y, t, x, t + \Delta t) dx = 0 \]

\[ \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{+\infty} (x - y) p(y, t, x, t + \Delta t) dx = 0 \] (28)

\[ \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{+\infty} (x - y)^2 p(y, t, x, t + \Delta t) dx = 2t \]

tell us that the law of random displacements \(p(y, s, x, t), (27)\), can be attributed to a Markov diffusion process associated with the parabolic (Fokker-Planck) equation

\[ \partial_t \rho = t \Delta_x \rho \] (29)

In fact, our \(p(y, s, x, t)\) is a fundamental solution of this equation with respect to \(x, t\) variables, while obeying the time adjoint parabolic equation in the remaining (e.g. \(y, s\)) pair of variables

\[ \partial_s p(y, s, x, t) = -s \Delta_y p(y, s, x, t) \] (30)
This diffusion has a vanishing forward drift and the quadratic in time variance (the diffusion coefficient equals $t^2$), hence its local characteristics are completely divorced from those of the Nelson process [30] derivable from the solution (21) of the Schrödinger equation.

Interestingly, since $p(y, s, x, t)$ itself is a perfect, strictly positive and continuous in all variables (Markov) semigroup kernel, nothing prevents us from performing the Schrödinger problem analysis (1) with the boundary densities $\rho_0(x)$ and $\rho_T(x)$ defined by the above free evolution problem. However, we shall proceed otherwise and having given explicit solutions of the parabolic system (25) we introduce another strictly positive and continuous in all variables function:

$$k_1(y, s, x, t) = p(y, s, x, t) \frac{\theta(y, s)}{\theta(x, t)} = \left[2\pi(t^2 - s^2)\right]^{-1/2} \left(\frac{1 + t^2}{1 + s^2}\right)^{1/4} \exp\left[-\frac{(x - y)^2}{2(t^2 - s^2)}\right] \exp\left[-\frac{y^2 - s}{4(1 + s^2)} + \frac{x^2 - t}{4(1 + t^2)}\right] \exp\left[\frac{1}{2}\frac{1}{2}\arctan t - \arctan s\right]$$

(31)

and observe that the Schrödinger system (1) in the present situation is involved as well, since trivially there holds:

$$\rho_0(x) = \theta_*(x, 0) \int k_1(x, 0, y, T) \theta(y, T) dy$$

(32)

$$\rho_T(x) = \theta(x, T) \int k_1(y, 0, x, T) \theta_*(y, 0) dy$$

Disregarding the derivation which has led us to (22), we can simply consider (22) as the Schrödinger system of equations with a fixed kernel and boundary density data. Then, we immediately infer that by Jamison’s theorem, [4], its unique (up to a coefficient) solution is constituted by the pair $\theta_*(x, 0), \theta(x, T)$ of functions, already determined by (24). Moreover, $k_1(y, s, x, t)$ obeys the Chapman-Kolmogorov composition rule:

$$\int k_1(y, s, z, \tau)k_1(z, \tau, x, t)d\tau =$$

(33)

$$\int p(y, s, z, \tau) \frac{\theta(y, s)}{\theta(z, \tau)}p(z, \tau, x, t) \frac{\theta(z, u)}{\theta(x, t)} d\tau = p(y, s, x, t) \theta(y, s) \theta(x, t) = k_1(y, s, x, t)$$

In view of $\int p(y, s, x, t) dx = 1$ for all $s < t$, we have

$$\int k_1(x, s, y, t) \theta(y, t) dy = \theta(x, s)$$

(34)

and, since $\theta \theta_* = \rho$, we get

$$\int k_1(y, s, x, t) \theta_*(y, s) dy = \int \theta_*(y, s) p(y, s, x, t) \frac{\theta(y, s)}{\theta(x, t)} dy =$$

(35)
Thus, undoubtedly we have in hands a complete solution of the Schrödinger boundary data problem (1): for the once chosen kernel $k_1$, this solution is unique, and compatible with the dynamics of the corresponding Schrödinger wave function. But, the constructed stochastic process is completely incongruent with the standard wisdom about Nelson’s diffusion processes [8, 27, 28, 30]. The reason is clear: our analysis was performed for a particular solution, whose functional form allows for an alternative stochastic representation. But, let us stress the point, if we look for the functional-form-independent construction, it is the parabolic system (4) from which one should depart.

Anyway, even the inappropriate choice of the integral kernel $k_1$, does allow to derive the quantum mechanically implemented dynamics (22) from respectively $\theta(x, T)$ and $\theta_*(x, 0)$, by means of the propagation formulas (4). The probability density evolves in time correctly, but the vanishing drift and the linear in time diffusion coefficient situate this stochastic process outside the scope set by (25) and (4).

**Example 2**: We shall demonstrate, that another choice of the kernel, still with no reference to the system (4), will allow to reproduce the stochastic propagation with the probability density, drifts and diffusion coefficient of Nelson’s stochastic mechanics, which however is not Nelson’s process for the quantum evolution (22).

We are inspired by our previous paper [30], where an interesting stochastic propagation, compatible with (22), was introduced by means of the transition probability density:

$$p_{y,s}(x,t) = \left[4\pi(t-s)\right]^{-1/2} \exp\left[-\frac{(x - c_{t,s}y)^2}{4(t-s)}\right]$$

$$p_{y,s}(x,s) = \delta(x-y), \quad 0 \leq s < t \leq T$$

$$c_{t,s} = \left[\frac{(1-t)^2 + 2s}{1 + s^2}\right]^{1/2}$$

(36)

Here, the density $\rho(y, s)$, (22), is propagated into the corresponding $\rho(x, t)$ according to the rule $\rho(x, t) = \int p_{y,s}(x,t)\rho(y, s)dy$, for all intermediate times $0 \leq s < t \leq T$.

As noticed in [30], this propagation is somewhat pathological since it does not obey the Chapman-Kolmogorov composition rule: $\int p_{y,s}(z, \tau)p_{z,t}(x, t)d\tau \neq p_{y,s}(x, t)$ and thus $p$ cannot be interpreted as a transition density of the Markov process.

However, if we would naively proceed like in the Example 1 and define the strictly positive continuous function

$$k_2(y, s, x, t) = p_{y,s}(x,t)\frac{\theta(y, s)}{\theta(x, t)}$$

(37)
where $0 \leq s < t \leq T$ and $\theta(x,t)$ is given by (24), then the Schrödinger system (32), with $k_2$ replacing $k_1$, trivially appears. Indeed, because of $\int p_{y,s}(x,t) dx = 1$ for $s < t$, there holds:

$$\theta_*(x,0) \int k_2(x,0,y,T)\theta(y,T)dy = \theta_*(x,0) \int p_{x,0}(y,T)\theta(x,0)dy = \theta_*(x,0)\theta(x,0) = \rho_0(x)$$

$$\theta(x,T) \int k_2(y,0,x,T)\theta_*(y,0)dy = \int p_{y,0}(x,T)\theta(y,0)\theta_*(y,0)dy = \int p_{y,0}(x,T)\rho(y,0)dy = \rho_T(x)$$

As a consequence, if we analyze the above Schrödinger system with the boundary data $\rho_0(x)$ and $\rho_T(x)$ fixed by (22) (as before), but with the new kernel $k_2$ then, somewhat unexpectedly, the same as before pair $\theta(x,0), \theta_*(x,T)$ necessarily comes out as a solution. Let us emphasize that the solution is unique for the chosen kernel $k_2$, albeit it coincides with the unique (as well) solution previously associated with the kernel $k_1$ (cf. Example 1).

The meaning of the uniqueness of solution of the Schrödinger system [7] becomes clear: if we have prescribed the boundary density data the solution is unique for a chosen kernel, but there are many kernels which may give rise to the very same solution.

The pathology (non-Markovian density) of $p_{y,s}(x,t)$ extends to $k_2(y,s,x,t)$ and the semigroup composition rule is invalid in this case. Nevertheless, we can blindly repeat the step (32), with $k_2$ instead of $k_1$, so reproducing the evolution (22). Moreover, in the present case, [30], we can exploit the standard recipe to evaluate the forward drift of a conventional diffusion:

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \int y p_{x,t}(y, t + \Delta t)dy - x] = b(x, t) = \frac{-1}{1 + t^2} x$$

Clearly, it is the forward drift of the Nelson diffusion [27, 30] associated with (24), and it consistently appears in the corresponding Fokker-Planck equation (6).

Let us observe that $p_{y,s}(x,t)$ solves the first Kolmogorov equation with respect to $x, t$:

$$\partial_t p_{y,s}(x,t) = \Delta_x p_{y,s}(x,t) - b_{y,s}(t)\nabla_x p_{y,s}(x,t)$$

$$b_{y,s}(t) = y \frac{\partial c_{t,s}}{\partial t}$$
As such, it can be exploited to construct a genuine Markov process, albeit disconnected from the quantal dynamics (22). Namely, we can define another solution of the equation (40), in variables $x_1, t_1$:

$$p_{y,s}(x_1, t_1, x_2, t_2) = [4\pi(t_2 - t_1)]^{-1/2} e^{-\left(\frac{(x_2 - x_1 - cy)^2}{4(t_2 - t_1)}\right)}$$

(41)

$c = c_{t_2,s} - c_{t_1,s}$, $0 \leq s < t_1 < t_2 \leq T$

with $c_{t,s}$ given by (36). It is easy to verify that the transition density (41) actually is a fundamental solution, and as such satisfies the second Kolmogorov equation with respect to $x_2, t_2$, for each fixed $y, s$ label, $0 \leq s < t_1 < t_2 \leq T$. Consequently, we have in hands the $(y, s)$-family of well defined Markovian transition probability densities $p_{y,s}$ for random propagation scenarios. Indeed, to this end one needs to check the (apparent) compatibility conditions: (a) $p_{y,s}(x_1, t_1, x_2, t_2) = \delta(x_2 - x_1)$, (b) $\int p_{y,s}(x_1, t_1, x_2, t_2) p_{y,s}(x_1, t_1) dx_1 = p_{y,s}(x_2, t_2)$ and in addition (c) $\int p_{y,s}(x_1, t_1, x_2, t_2) p_{y,s}(x_2, t_2, x_3, t_3) dx_2 = p_{y,s}(x_1, t_1, x_3, t_3)$, where $0 \leq s < t_1 < t_2 < t_3 \leq T$ and $p_{y,s}(x, t)$ plays the role of the density of the Markov process. The identity (c) in the above is the Chapman-Kolmogorov formula.

To avoid the above obstacles, the only how-to-proceed procedure is provided by the route outlined before, e.g. that leading from the Feynman-Kac kernel to the associated Markov diffusion process via Schrödinger’s boundary-data problem. The complete solution to this particular issue, in the quantum dynamics context, has been given elsewhere, [30].

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References

[1] M. Kac, J. Logan, in: Fluctuation phenomena, eds. E. W. Montroll, J. L. Lebowitz, North-Holland, Amsterdam, 1976

[2] M. C. Mackey, L. Glass, From clocks to chaos: rhythms of life, Princeton University Press, Princeton, 1988

[3] A. Lasota, M. C. Mackey, ”Chaos, Fractals, and Noise”, Springer-Verlag, Berlin, 1994

[4] C. Beck, pp. 3, in: ”Chaos-The Interplay Between Stochastic and Deterministic Behaviour”, eds. P. Garbaczewski, M. Wolf, A. Weron, LNP vol 457, Springer-Verlag, Berlin, 1995

15
[5] E. Schrödinger, Ann. Inst. Henri Poincare, 2, (1932), 269
[6] T. Mikami, Commun. Math. Phys., 135, (1990), 19
[7] B. Jamison, Z. Wahrsch. verw. Geb. 30, (1974), 65
[8] J. C. Zambrini, J. Math. Phys. 27, (1986), 3207
[9] P. Garbaczewski, J. R. Klauder, R. Olkiewicz, Phys. Rev. E 51, (1995), 4114
[10] P. Garbaczewski, R. Olkiewicz, J. Math. Phys. 37, (1996), 732
[11] Ph. Blanchard, P. Garbaczewski, Phys. Rev. E 49, (1994), 3815
[12] P. Garbaczewski, Acta Phys. Polon. B 27, (1996), 617
[13] P. Garbaczewski, G. Kondrat, Phys. Rev. Lett. 77, (1996), 2608
[14] J. M. Burgers, ”The Nonlinear Diffusion Equation”, Reidel, Dordrecht, 1974
[15] E. Hopf, Commun. Pure Appl. Math., 3, (1950), 201
[16] S. F. Shandarin, B. Z. Zeldovich, Rev. Mod. Phys. 61, (1989), 185
[17] S. Albeverio, A. A. Molchanov, D. Surgailis, Prob. Theory Relat. Fields, 100, (1994), 457
[18] H. P. McKean, pp. 177, in: ”Lecture Series in Differential Equations”, vol. II, ed. A. K. Aziz, Van Nostrand, Amsterdam, 1969
[19] P. Calderoni, M. Pulvirenti, Ann. Inst. Henri Poincaré, 39, (1983), 85
[20] H. Osada, S. Kotani, J. Math. Soc. Japan, 37, (1985), 275
[21] M. Krzyżanński, A. Szybiak, Lincei-Rend. Sc. fis. mat. e nat. 28, (1959), 26
[22] A. Friedman, ”Partial Differential Equations of Parabolic type”, Prentice-Hall, Englewood, NJ, 1964
[23] W. Horsthemke, R. Lefever, ”Noise-Induced Transitions”, Springer-Verlag, Berlin, 1984
[24] U. G. Haussmann, E. Pardoux, Ann. Prob. 14, (1986), 1188
[25] H. Föllmer, pp. 119, in: ”Stochastic Processes-Mathematics and Physics”, ed. S. Albeverio, Ph. Blanchard, L. Streit, LNP vol. 1158, Springer-Verlag, Berlin, 1985

[26] H. Hasegawa, Progr. Theor. Phys. 55, (1976), 90

[27] E. Nelson, ”Quantum Fluctuations”, Princeton University Press, Princeton, 1985

[28] E. Nelson, ”Dynamical Theories of the Brownian Motion”, Princeton University Press, Princeton, 1967

[29] P. Garbaczewski, J. P. Vigier, Phys. Rev. A46, (1992), 4634

[30] P. Garbaczewski, R. Olkiewicz, Phys. Rev. A 51, (1995), 3445

[31] W. H. Fleming, H. M. Soner, ”Controlled Markov Processes and Viscosity Solutions”, Springer-Verlag, Berlin, 1993

[32] M. Freidlin, ”Functional Integration and Partial Differential Equations”, Princeton University Press, Princeton, 1985