Strichartz estimates
for the Wave and Schrödinger Equations
with the Inverse-Square Potential

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Abstract

We prove spacetime weighted-$L^2$ estimates for the Schrödinger and wave equation with an inverse-square potential. We then deduce Strichartz estimates for these equations.

1 Introduction

Consider the following linear equations

\begin{align}
  i\partial_t u + \Delta u - \frac{a}{|x|^2} u &= 0, & u(0, x) &= f(x) \\
  -\partial^2_t u + \Delta u - \frac{a}{|x|^2} u &= 0, & u(0, x) &= f(x), \partial_t u(0, x) &= g(x)
\end{align}

where $\Delta$ is the $n$ dimensional Laplacian and $a$ is a real number. The Schrödinger equation (1) is of interest in quantum mechanics (see [3, 12, 2] and references therein), while the wave equation (2) arises in the study of wave propagation on conic manifolds [4]. We also note that the heat flow for the elliptic operator $-\Delta + a|x|^{-2}$ has been studied in the theory of combustion (see [28] and references therein).

The mathematical interest in these equations however comes mainly from the fact that the potential term is homogeneous of degree $-2$ and therefore scales exactly the same as the Laplacian. This in particular implies that perturbation methods cannot be used in studying the effect of this potential. Indeed, the $|x|^{-2}$ decay is in some sense the borderline case for the existence of global-in-time estimates for wave or Schrödinger equations with a potential [23, 6]. In particular, it is known that a negative potential $V$ decaying slower than inverse-square results in the spectrum of $-\Delta + V$ being unbounded from below [21 §XIII, pp.87–88].

On the other hand, the scale-covariance present in the elliptic operator $-\Delta + \frac{a}{|x|^2}$ appearing in the above equations is a key feature of many problems in physics and in geometry, where such scaling behavior manifests itself if not everywhere, at least in a certain region
of space, for example near a singular point, or near infinity, or both. Consider for example the Dirac equation with a Coulomb potential, (which can be recast in the form of a Klein-Gordon equation with an inverse-square potential, plus other terms which exhibit the same scaling as the Laplacian) [3]. Another family of examples is given by linearized perturbations of spacetime metrics that are well-known solutions of the Einstein equations of general relativity, such as the Schwarzschild solution [22, 30] or the Reissner-Nordström solution [31, 16].

There are also nonlinear problems, of a geometric nature, where such critical behavior potentials make an appearance, for example the perturbation of equivariant stationary solutions of a 2+1-dimensional wave map from the Minkowski space into a 2-sphere gives rise to a system of two wave equations, with a potential that behaves like $|x|^{-2}$ both near zero and near infinity, as well as another term with the same scaling, i.e. $|x|^{-2}x \cdot \nabla$. The occurrence of this phenomenon in a nonlinear setting is significant, since it is clear that to study a nonlinear wave equation one must have estimates for the linear inhomogeneous wave equation that bound various (perhaps fractional) number of derivatives of the solution in terms of the correct number of derivatives of the source and the data. Such estimates are by-and-large unknown for problems involving a potential, except for those that are of much faster decay than $|x|^{-2}$ ([29] and references therein).

Equations such as (1) and (2) with the inverse-square potential thus represents the simplest case, where the scaling holds exactly and everywhere. These are to be thought of as model problems, used to develop and test new tools and methods that we hope are capable of being generalized to the more complicated situations that are of actual physical and geometric interest, such as those named above.

In [19] we showed for the wave equation (2) that in the radial case, i.e. when the data – and thus the solution – are radially symmetric, the solution to (2) satisfies generalized spacetime Strichartz estimates as long as

$$a > -\frac{(n-2)^2}{4}. \tag{3}$$

The corresponding Strichartz estimates would hold for the Schrödinger equation (1) as well, since our proof was based solely on estimates for the elliptic operator

$$P_a := -\Delta + \frac{a}{|x|^2}.$$ 

**Remark 1** $P_a$ is in fact the *self-adjoint extension* of $-\Delta + a|x|^{-2}$. It is known that in the range $-(n-2)^2/4 < a < 1 - (n-2)^2/4$ the extension is not unique [27, 12]. In this case, when we do make a choice among the possible extensions, such as in [19], it corresponds to the Friedrichs extension (see [19, 12] for details).

In this work we intend to remove the assumption on the data being radially symmetric. As explained in [19, 20], one cannot hope to get any kind of dispersive (be it at fixed $t$ or spacetime) estimate if (3) is not satisfied. We also note that when $a < 0$, the classical $L^\infty - L^1$ estimate for the wave equation does not hold [20], and thus one cannot obtain Strichartz estimates by interpolation between this dispersive estimate and the energy estimate (see also the remark at the end of Subsection 3.1).
This paper is divided into four sections. In Section 2, we obtain weighted-$L^2$ estimates for (1) and (2). Such estimates are known for the free Schrödinger equation, and are often referred to as local smoothing estimates ([1] and references therein). In Section 3 we deduce Strichartz estimates for solutions of the Schrödinger equation (1) through Duhamel’s formula, combining the smoothing estimate (6) for (1) with Strichartz estimates for the free Schrödinger equation. We then do the same for the wave equation. We note that for the Schrödinger equation such a strategy was successfully applied in [11] for rapidly decaying potentials, to obtain dispersive estimates (from which Strichartz estimates are then obtained by the usual duality argument). This result was recently extended to potentials that decay strictly faster than $|x|^{-2}$ [23]. While our potential obviously does not satisfy this condition, we are able to take advantage of its special form to extend the approach in [23] to our setting. Finally, in Section 4 we obtain the frequency-localized version of the above estimates and use them to obtain the generalized Strichartz estimates (with derivatives) for these equations, which we then apply to obtain an optimal global well-posedness result for a nonlinear wave equation.

1.1 Notations

In this paper we will be using the following notations. For integer $n \geq 2$ let

$$\lambda(n) := \frac{n - 2}{2}.$$ (4)

For integer $d \geq 0$ and real number $a \geq -\lambda^2(n)$ let

$$\nu_d(n, a) = \sqrt{(\lambda(n) + d)^2 + a}.$$ (5)

We will suppress the arguments of the above functions whenever doing so does not cause confusion.

We also define multiplication operators $\Omega^s$ by

$$(\Omega^s \phi)(x) = |x|^s \phi(x).$$

Abusing notation we use the same symbol for the operators which are pointwise equal to these for all times,

$$(\Omega^s \phi)(t, x) = |x|^s \phi(t, x).$$

For $|s| < n/2$ and integer $d \geq 0$ let $\dot{H}^s_{\leq d}$ denote the subspace of the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ consisting of functions that are orthogonal to all spherical harmonics of degree less than $d$, and let $\dot{H}^s_{\geq d}$ denote the orthogonal complement of this space. Finally, let

$$d_0(n) := \begin{cases} 1 & n = 2 \\ 0 & n \geq 3. \end{cases}$$
2 Weighted-$L^2$ estimates for the Schrödinger and wave equations

2.1 Local smoothing for the Schrödinger equation

Except for the case $n = 2$ the following theorem is well-known when $a = 0$. See [24] for the sharp constants and the references therein for the history. In this paper we need only the case $\alpha = 1/4$.

**THEOREM 1** Let $n \geq 2$, $d \geq d_0(n)$, $0 < \alpha < \frac{1}{4} + \frac{1}{2} \nu d$ and let $u$ be the unique solution of (1). There exists a constant $C > 0$, depending on $n$, $a$, $d$ and $\alpha$, such that for all $f \in L^2_{\geq d}(\mathbb{R}^n)$,

$$\|\Omega^{-1/2 - 2\alpha} P_a^{1/4 - \alpha} u\|_{L^2(\mathbb{R}^{n+1})} \leq C \|f\|_{L^2(\mathbb{R}^n)}$$

(6)

**Proof of Theorem 1** We begin by noting that, by rotational symmetry and the $L^2$ orthogonality of the various spherical harmonic spaces, it suffices to prove the estimate (6) for $f$ belonging to the $l$’th harmonic subspace, where $l \geq 0$ if $\lambda^2 + a > 0$ and $l > 0$ if $\lambda^2 + a = 0$. This, of course, requires the constants to be uniformly bounded in $l$, but this will be clear from the explicit form of the constants given below. On the $l$’th spherical harmonic subspace

$$P_a = A_\nu,$$

(7)

where

$$A_\nu = -\partial^2_r - (n-1)r^{-1}\partial_r + [(l+2\lambda) + a]r^{-2}$$

$$= -\partial^2_r - (n-1)r^{-1}\partial_r + [\nu^2 - \lambda^2]r^{-2},$$

where

$$\nu = \nu_l(n, a) = \sqrt{(\lambda + l)^2 + a}.$$  

(8)

Our assumptions imply that

$$\nu > 0.$$  

(9)

The above considerations allow us to restate the problem as follows. We are to prove the estimate

$$\|\Omega^{-1/2 - 2\alpha} A_\nu^{1/4 - \alpha} S_\nu f\|_{L^2(\mathbb{R}^{n+1})} \leq C \|f\|_{L^2(\mathbb{R}^n)}$$

where $S_\nu f$ is the unique solution of the initial value problem

$$i\partial_t u - A_\nu u = 0, \quad u(0, x) = f(x).$$

In this we are allowed to assume that $f$ belongs to the $l$’th spherical harmonic subspace, but in fact we have no further use for this assumption.

We define the Hankel transform of order $\nu$ in the usual way:

$$(\mathcal{H}_\nu \phi)(\xi) = \int_0^{\infty} (r|\xi|)^{-\lambda} J_\nu(r|\xi|)\phi(r|\xi|)r^{n-1} dr,$$
where $J_\nu$ is the Bessel function of the first kind of order $\nu$. By abuse of notation we use the same symbol $H_\nu$ to denote the operator on functions on $\mathbb{R}^{n+1}$ which is just the Hankel transform of order $\nu$ pointwise for all times:

$$(H_\nu \phi)(t, \xi) = \int_0^\infty (r|\xi|)^{-\lambda} J_\nu(r|\xi|) \phi(t, r\xi/|\xi|) r^{n-1} dr.$$ 

The Hankel transform has the following properties:

(i) $H_\nu^2 = 1$,

(ii) $H_\nu$ is self-adjoint,

(iii) $H_\nu$ is an $L^2$ isometry, and

(iv) $H_\nu A_\nu = \Omega^2 H_\nu$.

The first of these is an immediate consequence of properties of the Fourier-Bessel integral defining $H_\nu$, but a proof may be found in [19] along with a proof of the fourth. The second is obvious from the definition, and the third follows from the first and second.

We define fractional powers of $A_\nu$ using the fourth property above:

$$A_\nu^{\sigma/2} = H_\nu \Omega^\sigma H_\nu.$$ 

An integral kernel for $A_\nu^{\sigma/2}$ is given in [19],

$$(A_\nu^{\sigma/2} \phi)(r, \theta) = \int_0^\infty k_{\nu,\nu}^{\sigma}(r, s) \phi(s, \theta) s^{n-1} ds$$  \hspace{1cm} (10)$$

where

$$k_{\nu,\nu}^{\sigma}(r, s) = \begin{cases} \frac{2^{\sigma+1} \Gamma(\nu + \frac{\sigma}{2} + 1)}{\Gamma(\frac{\sigma}{2} + 1)} \frac{s^{\sigma - \lambda}}{(s^2 + r^2)^{\nu + \frac{\sigma}{2} + 1}} F(\nu + \frac{\sigma}{2} + 1, \frac{\sigma}{2} + 1; \nu + 1; \frac{r^2}{s^2}) & \text{if } s < r, \\ \frac{2^{\sigma+1} \Gamma(\nu + \frac{\sigma}{2} + 1)}{\Gamma(\frac{\sigma}{2} + 1)} \frac{(s^2 + r^2)^{\frac{\sigma}{2} - \lambda}}{s^{\nu + \frac{\sigma}{2} + 1}} F(\nu + \frac{\sigma}{2} + 1, \frac{\sigma}{2} + 1; \nu + 1; \frac{s^2}{r^2}) & \text{if } s > r. \end{cases}$$

Here $F$ is the hypergeometric function

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} z + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{1 \cdot 2 \cdot \gamma(\gamma + 1)} z^2 + \ldots$$  \hspace{1cm} (11)$$

The integral in (10) may be interpreted in the usual Lebesgue sense for $\sigma < 0$. See [19] for the correct interpretation when $\sigma \geq 0$.

Hankel transforming both sides of the estimate we are trying to prove, we see that we are reduced to proving

$$\| A_\nu^{-1/4 - \alpha}\Omega^{1/2 - 2\alpha} H_\nu S_\nu f \|_{L^2(\mathbb{R}^{n+1})} \leq C \| H_\nu f \|_{L^2(\mathbb{R}^n)}$$

where $H_\nu S_\nu f$ solves

$$i \partial_t H_\nu S_\nu f - \Omega^2 H_\nu S_\nu f = 0, \quad (H_\nu S_\nu f)(0, \xi) = (H_\nu f)(\xi).$$
But the solution to this initial value problem is just

$$(\mathcal{H}_\nu S_\nu f)(t, \xi) = \exp(-it|\xi|^2)(\mathcal{H}_\nu f)(\xi).$$

Let $\mathcal{F}_t$ denote the Fourier transform in the $t$ variable,

$$\mathcal{F}_t f(\tau, x) = \frac{1}{\sqrt{2\pi}} \int e^{-it\tau} f(t, x) dt.$$

It is an isometry of $L^2(\mathbb{R}^{n+1})$ and it commutes with both $A_{\nu}^{-1/4-\alpha}$ and $\Omega^{1/2-2\alpha}$, since these are defined pointwise in $t$. Thus

$$\|A_{\nu}^{-1/4-\alpha} \Omega^{1/2-2\alpha} \mathcal{H}_\nu S_\nu f\|_{L^2(\mathbb{R}^{n+1})} = \|A_{\nu}^{-1/4-\alpha} \Omega^{1/2-2\alpha} \mathcal{F}_t \mathcal{H}_\nu S_\nu f\|_{L^2(\mathbb{R}^{n+1})}.$$

From the calculation of the last paragraph we see that

$$(\mathcal{F}_t \mathcal{H}_\nu S_\nu f)(\tau, \xi) = (\mathcal{H}_\nu f)(\xi) \delta(\tau - |\xi|^2)$$

and hence

$$(A_{\nu}^{-1/4-\alpha} \Omega^{1/2-2\alpha} \mathcal{F}_t \mathcal{H}_\nu S_\nu f)(\tau, \xi)$$

$$= \int k_{\nu,\nu}^{-1/2-2\alpha}(|\xi|, s) s^{1/2-2\alpha} \delta(\tau - s^2) \mathcal{H}_\nu f(s\xi/|\xi|) s^{n-1} ds$$

$$= \frac{1}{2} \tau^{\lambda-\alpha+1/4} k_{\nu,\nu}^{-1/2-2\alpha}(|\xi|, \sqrt{\tau})(\mathcal{H}_\nu f)(\sqrt{\tau} \xi/|\xi|).$$

We now compute the square of the $L^2(\mathbb{R}^{n+1})$ norm of this quantity. We square the absolute value and integrate over $\mathbb{R}^{n+1}$, replacing the Cartesian coordinates $\xi$ with spherical coordinates $\rho$, $\theta$ to obtain

$$\frac{1}{4} \int_0^\infty \int_0^\infty \int_{S^{n-1}} \tau^{2\lambda-2\alpha+1/2} (k_{\nu,\nu}^{-1/2-2\alpha}(\rho, \sqrt{\tau}))^2 |(\mathcal{H}_\nu f)(\sqrt{\tau} \theta)|^2 d\theta \rho^{n-1} d\rho d\tau$$

or, making the change of variable $\omega = \sqrt{\tau},$

$$\frac{1}{2} \int_0^\infty \int_0^\infty \int_{S^{n-1}} \omega^{4\lambda-4\alpha+2} (k_{\nu,\nu}^{-1/2-2\alpha}(\rho, \omega))^2 |(\mathcal{H}_\nu f)(\omega \theta)|^2 d\theta \rho^{n-1} d\rho d\omega.$$

Since $A_{\nu}^{-1/2-2\alpha} = A_{\nu}^{-1/4-\alpha} A_{\nu}^{-1/4-\alpha}$ we have

$$k_{\nu,\nu}^{-1-4\alpha}(r, t) = \int_0^\infty k_{\nu,\nu}^{-1/2-2\alpha}(r, s) k_{\nu,\nu}^{-1/2-2\alpha}(s, t) s^{n-1} ds.$$

We apply this with $r = t = \omega$ and $s = \rho$ to evaluate the integral over $\rho$, since $k(r, s) = k(s, r)$, obtaining,

$$\frac{1}{2} \int_0^\infty \int_{S^{n-1}} \omega^{4\lambda-4\alpha+2} k_{\nu,\nu}^{-1-4\alpha}(\omega, \omega) |(\mathcal{H}_\nu f)(\omega \theta)|^2 d\theta d\omega$$
or, using the explicit formula for $k_{\nu,\sigma}$ given above and Gauss’s formula for the value of the hypergeometric function at $z = 1$,

$$\frac{2^{1-4\alpha}\pi}{\sqrt{\Gamma(\nu-2\alpha+\frac{1}{2})\Gamma(4\alpha)}} \int_0^\infty \int_{S^{n-1}} \omega^{n-1}|(\mathcal{H}_\nu f)(\omega\theta)|^2d\theta d\omega.$$  

The double integral is just the square of the $L^2(\mathbb{R}^n)$ norm of $\mathcal{H}_\nu f$ and therefore of $f$. Thus we see that

$$\|\Omega^{-1/2-2\alpha} A^{1/4-\alpha}_\nu S\nu f\|_{L^2(\mathbb{R}^{n+1})} = C_{\nu,\alpha} \|f\|_{L^2(\mathbb{R}^n)}$$

where

$$C_{\nu,\alpha} = 2^{1/2-2\alpha} \pi^{1/2} \frac{\Gamma(\nu-2\alpha+\frac{1}{2})\Gamma(4\alpha)}{\Gamma(\nu+2\alpha+\frac{1}{2})\Gamma(2\alpha+\frac{1}{2})^2}.$$  

(12)

As promised, one can see immediately from this formula that $C_{\nu,\alpha}$ is finite if $0 < \alpha < \frac{1}{2} + \frac{n}{d}$, and that it is a decreasing function of $\nu$ and hence of $l$. Thus to finish the proof of the Theorem, we expand $f$ in spherical harmonics, $f = \sum_{l=0}^\infty f_l$, use the triangle inequality on the left and the $L^2$-orthogonality of spherical harmonic subspaces on the right, to obtain (6), with the constant $C = \min_{l \geq d} C_{\nu,l,\alpha} = C_{\nu,l,\alpha}$.

\[ \square \]

2.2 Generalized Morawetz estimate for the wave equation

In this section we will obtain a weighted-$L^2$ estimate for the wave equation (2) which is analogous to the one obtained above for the Schrödinger equation. The estimate for the free case $a = 0$ was proved in [8]. The Morawetz estimate, which is the estimate for the free wave equation obtainable from the Morawetz radial identity [17] is a special case of our estimate below (see Corollary 1). We note that the simple proof of (6) from [1] applies as well to the free wave equation.

THEOREM 2 Let $n \geq 2$ and $d \geq d_0(n)$ be integers, let $0 < \alpha < \frac{1}{4} + \frac{1}{2}d$ and let $u$ be the solution to (2). There exists a constant $C > 0$ depending on $n, a, d, \alpha$ such that for all $f \in H^{1/2}(\mathbb{R}^n)$ and $g \in H^{-1/2}(\mathbb{R}^n)$,

$$\|\Omega^{-1/2-2\alpha} F^{1/4-\alpha}_a u\|_{L^2(\mathbb{R}^{n+1})} \leq C (\|f\|_{H^{1/2}(\mathbb{R}^n)} + \|g\|_{H^{-1/2}(\mathbb{R}^n)}).$$  

(13)

Proof: Once again, we can work one spherical harmonic at a time, and thus we are solving

$$\partial_t^2 u + A_{\nu} u = 0, \quad u_t(0, x) = f_l(x), \quad \partial_l u_t(0, x) = g_l(x)$$

with $\nu > 0$ as before. Applying the Hankel transform (and suppressing the $l$ subscripts) we obtain the solution as

$$\mathcal{H}_\nu u(t, \xi) = \cos(t|\xi|) \mathcal{H}_\nu f(\xi) + \frac{\sin(t|\xi|)}{|\xi|} \mathcal{H}_\nu g(\xi),$$

and Fourier-transforming in time we have

$$\mathcal{F}_t \mathcal{H}_\nu u(\tau, \xi) = \frac{1}{|\xi|} (\delta(\tau + |\xi|) h_+(\xi) + \delta(\tau - |\xi|) h_-(\xi))$$

and

$$\mathcal{F}_t \mathcal{H}_\nu u(\tau, \xi) = \frac{1}{|\xi|} (\delta(\tau + |\xi|) h_+(\xi) + \delta(\tau - |\xi|) h_-(\xi)).$$
where
\[ h_\pm (\xi) = \frac{1}{2} (\sqrt{\xi} |\mathcal{H}_\nu f(\xi)| \pm \frac{1}{i\sqrt{|\xi|}} \mathcal{H}_\nu g(\xi)) \]
so that
\[ \|h_\pm\|_{L^2(\mathbb{R}^n)} \leq C (\|A_\nu^{1/4} f\|_{L^2(\mathbb{R}^n)} + \|A_\nu^{-1/4} g\|_{L^2(\mathbb{R}^n)}) . \] (14)
Thus for \( \tau > 0, \)
\[ (A_\nu^{-1/4-\alpha}\Omega^{1/2-2\alpha} \mathcal{F}_l \mathcal{H}_\nu u)(\tau, \xi) = \tau^{n-1-2\alpha} k_{\nu,\nu'}^{-1/2-2\alpha}(\xi, \tau) h_-(\tau \xi/|\xi|) \]
and for \( \tau < 0 \) we change \( \tau \) to \(-\tau\) and \( h_- \) to \( h_+ \) in the right hand side above. By the same calculation as in the Schrödinger case performed in the previous section we then have
\[ \|A_\nu^{-1/4-\alpha}\Omega^{1/2-2\alpha} \mathcal{F}_l \mathcal{H}_\nu u\|_{L^2(\mathbb{R}^n+1)} = C_{\nu,\alpha} \|h_\pm\|_{L^2(\mathbb{R}^n)} \]
with \( C_{\nu,\alpha} \) as in (12).

In light of (14) we have the desired estimate provided we can show that the Sobolev norms based on (small) powers of the operator \( P_\alpha \) are equivalent to standard Sobolev norms based on the Laplacian. This can be accomplished in several ways. Here we will give a simple proof of a such a result based on Hardy’s inequality. (The range of \( s \) here is not optimal, one could improve on it by using the more sophisticated machinery developed in Section 4).

**Proposition 1** Let \( n \geq 3, \ a + \lambda^2 > 0 \) and \(-1 \leq s \leq 1. \) There exists constants \( C_1, C_2 > 0 \) depending on \( n, a, s \) such that
\[ C_1 \|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|P_\alpha^{s/2} f\|_{L^2(\mathbb{R}^n)} \leq C_2 \|f\|_{\dot{H}^s(\mathbb{R}^n)} \] (15)
for all \( f \in \dot{H}^s(\mathbb{R}^n). \) For \( n = 2 \) the same result holds for functions \( f \in \dot{H}_\square^s. \)

**Proof:** We use the following version of Hardy’s inequality: For \( n \geq 3, \)
\[ \|\Omega^{-1} f\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{\lambda} \|f\|_{\dot{H}^1(\mathbb{R}^n)} \] (16)
A similar inequality is easy to obtain for \( n = 2 \) and \( f \) orthogonal to radial functions: Let \( f(r, \theta) = \sum_{n \neq 0} f_n(r) e^{in\theta} \) be the Fourier series expansion of such an \( f. \) Then
\[ \int_0^{2\pi} |f(r, \theta)|^2 d\theta = \sum_{n \neq 0} |f_n|^2 \leq \sum_{n \neq 0} n^2 |f_n|^2 = \int_0^{2\pi} |\partial_\theta f|^2 d\theta \]
and thus
\[ \|\Omega^{-1} f\|_{L^2}^2 \leq \int_0^\infty \int_0^{2\pi} \frac{1}{r^2} |\partial_\theta f(r, \theta)|^2 d\theta dr \leq \|\nabla f\|_{L^2}^2 \]
which establishes the Hardy inequality in 2 dimensions, with constant = 1. Denoting now the \( L^2(\mathbb{R}^n) \) inner product by \( \langle , \rangle, \) we have
\[ \|P_\alpha^{1/2} f\|_{L^2}^2 = \langle f, -\Delta f \rangle + a \|\Omega^{-1} f\|_{L^2}^2 \]
Thus for $a > 0$ by (16),
\[ \| \nabla f \|_2^2 \leq \| P_a^{1/2} f \|_2^2 \leq (1 + \frac{a}{\lambda^2}) \| \nabla f \|_2^2 \]
while for $a < 0$ we have the same as above with the two inequality signs reversed. Thus we obtain (15) for $n \geq 3$ and $s = 1$ with $C_1 = \min \{ \nu_0 / \lambda, 1 \}$ and $C_2 = \max \{ \nu_0 / \lambda, 1 \}$. For $n = 2$ we have $C_1 = 1$ and $C_2 = 1 + a$. By duality, we obtain (15) for $s = -1$ as well, and interpolating between the two endpoints establishes the claim. 

3 Strichartz estimates

3.1 The Schrödinger equation

Here we prove that the solution to (1) satisfies the same set of estimates as that of the free Schrödinger equation, for $n \geq 2$:

THEOREM 3 Let $f \in L^2$, $p \geq 2, q$ such that
\[ \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad (n, p) \neq (2, 2). \] (17)

Let $u$ be the unique solution of (1). Then, provided $a + \lambda^2 > 0$, there exists a constant $C > 0$ depending on $n, p, a$, such that
\[ \| u \|_{L^p_t(L^q_x)} \leq C \| f \|_{L^2}. \] (18)

We will follow the strategy from [23] (notice however that using the end-point allows to shorten the argument for $n \geq 3$, as well as to recover said end-point). We consider the potential term as a source term,
\[ i\partial_t u + \Delta u = \frac{a}{|x|^2} u, \] (19)

and integrate using $S(t) = \exp(it\Delta)$, the free evolution, to get
\[ u(t) = S(t)f - ia \int_0^t S(t - s)\Omega^{-2}u(s)ds. \] (20)

The first term can be ignored, and we focus on the Duhamel term. We postpone the $n = 2$ case, and assume $n \geq 3$. Then, for the free evolution, one has Strichartz estimates up to the end-point, namely the pair $(p, q) = (2, \frac{2n}{n-2})$. We recall that these Strichartz estimates hold in a slightly relaxed setting ([14]),
\[ \| \int_0^t S(t - s)F(x, s)ds \|_{L^2_t(L^{\frac{2n}{n-2}}_x)} \leq C \| F \|_{L^2_t(L^{\frac{2n}{n-2}}_x)}, \] (21)
where $L^{\alpha, \beta}$ are Lorentz spaces. We also note that the estimate (18) for general $(p, q)$ satisfying (17) follows by interpolating between the above endpoint estimate and the well-known
estimate \( p = \infty, q = 2 \) corresponding to the conservation of charge for the Schrödinger equation.

Hence to prove our estimate, all we need to check is \( F = \Omega^{-2}u \in L^2(\frac{2n}{n+2}, \Omega) \). We have from Theorem \([\ref{thm:1}]\) with \( d = 0, \alpha = 1/4 \) that

\[
\|F\|_{L^2_t L^{\frac{2n}{n+2}}} = \|\Omega^{-1}\Omega^{-1}u\|_{L^2_t L^{\frac{2n}{n+2}}} \leq \frac{1}{|x|} \|\Omega^{-1}\Omega^{-1}u\|_{L^\infty_x} \leq C \|f\|_{L^2},
\]

where we have made use of the generalized Hölder inequality \([\ref{lem:2}]\). This ends the proof for large dimensions.

In the 2D case, one needs to follow \([\ref{lem:3}]\) more closely, and resort to the following lemma, proved in \([\ref{lem:4}]\).

**Lemma 3.1** Let \( X, Y \) be two Banach spaces and let \( T \) be a bounded linear operator from \( L^\beta(\mathbb{R}^+; X) \) to \( L^\gamma(\mathbb{R}^+; Y) \), such that \( T f(t) = \int_0^\infty K(t, s) f(s) ds \). Then the operator \( \tilde{T} f(t) = \int_0^\infty K(t, s) f(s) ds \) is bounded from \( L^\beta(\mathbb{R}^+; X) \) to \( L^\gamma(\mathbb{R}^+; Y) \) when \( \beta < \gamma \), and \( \|T\| \leq c_{\beta, \gamma} \|T\| \) with \( c_{\beta, \gamma} = (1 - 2^{1/\gamma - 1/\beta})^{-1} \).

Using this lemma, one may forget about the \( \int_0^t \) in the Duhamel formula \([\ref{lem:5}]\), and replace it with an integral over all times. Thus we define, for \( h \in L^2(\mathbb{R}^2) \),

\[
Th(t) = S(t) \int S(-s)\Omega^{-1} h(s) ds.
\]

Consider now the estimate dual to \([\ref{lem:6}]\), with \( d = 1, a = 0 \) and \( \alpha = 1/4 \). It reads

\[
\| \int S(s) F(s) ds \|_{L^2} \leq C \| OF \|_{L^2_x}
\]

Combining this with the Strichartz estimate for the free Schrödinger group \( S(t) \), we see that \( T : L^2(\mathbb{R}, L^2) \rightarrow L^p(\mathbb{R}, L^q) \) with \( p, q \) as in the statement of the Theorem. Since \( p > 2 \), by the above Lemma \([\ref{lem:7}]\) the operator \( \tilde{T} \) is bounded on the same spaces. On the other hand, from \([\ref{lem:8}]\) we have that, for \( u \) the solution to \([\ref{thm:1}]\),

\[
u(t) = S(t) f - ia\tilde{T}(\Omega^{-1} u).
\]

We now use \([\ref{lem:9}]\) again with \( \alpha = 1/4 \), to conclude

\[
\|u\|_{L^p_t L^q_x} \leq C(1 + |a|) \|f\|_{L^2}.
\]

Since we are in two space dimensions, we need to assume that the data \( f \) is orthogonal to radial functions, in order for \([\ref{thm:1}]\) to hold in the case \( a = 0, \alpha = 1/4 \). On the other hand, Strichartz estimates for \([\ref{thm:1}]\) in the case of radial data are obtainable from the case \( a = 0 \) using the conjugation procedure presented in \([\ref{lem:10}]\). For future reference, we state a more general result here:

**Theorem 4** Let \( f \in \dot{H}^s_{d, a}(\mathbb{R}^n) \). Then there is a constant \( C > 0 \) (depending on \( n, d, p, s \)) such that \( u \), the solution to \([\ref{thm:1}]\) satisfies

\[
\|(-\Delta)^{s/2} u\|_{L^p_t L^q_x} \leq C \|f\|_{\dot{H}^s}.
\]
for all \((p, q)\) as in (14) and all \(s\) such that

\[
-\min\{1 + \sqrt{a + \lambda^2} \cdot \frac{n}{2} + \frac{2}{p}\} < s < \min\{1 + \sqrt{a + \lambda^2} - \frac{2}{p} \cdot \frac{n}{2}\}
\]

**Proof:** Let \(f = \sum_{l=0}^{d-1} f_l\) be the spherical harmonics decomposition of \(f\), and let \(u = \sum_{l=0}^{d-1} u_l\) be the corresponding decomposition of the solution \(u\). On the \(l\)'th harmonic subspace, 

\[-\Delta = A_\mu \quad \text{and} \quad -\Delta + a|x|^{-2} = A_\nu, \quad \text{with} \quad \mu = \lambda + l, \nu = \nu_l(n, a).\]

We thus have

\[i\partial_t u_l - A_\nu u_l = 0, \quad u_l(0) = f_l, \quad l = 0, \ldots, d - 1\]

In [19] it was shown that the operator \(K_{\mu, \nu}^0 := \mathcal{H}_\mu \mathcal{H}_\nu\) is a conjugation operator between \(A_\mu\) and \(A_\nu\), i.e. \(A_\mu K_{\mu, \nu}^0 = K_{\mu, \nu}^0 A_\nu\). Thus, to obtain the estimate (22) for each \(u_l\) by conjugation from the corresponding estimate in the case \(a = 0\), all we need to know are the continuity properties of \(K_{\mu, \nu}^0\) on appropriate Sobolev spaces. In particular, for the above estimate (22) we need continuity of \(K_{\mu, \nu}^0\) on \(\dot{H}^s\) for the right-hand side and continuity of \(K_{\nu, \mu}^0\) on \(\dot{H}^q\) for the left. It was shown in [19] that \(K_{\mu, \nu}^0\) is continuous on \(\dot{H}^s\) provided that

\[-\min\{\lambda, \mu, \nu, \mu - s\} < n/r - \lambda < 2 + \min\{\lambda, \mu, \nu, \nu + s\}\]

Applying this in the two cases we need here gives the restriction on \(s\) in the statement of the theorem. We thus obtain the desired Strichartz estimate on each spherical harmonic subspace. These need to be added up, which can be done since there is only a finite number of them, and finally \(\|f\|_{\dot{H}^s} \leq C_{l,s}\|f\|_{\dot{H}^s}\), where \(C_{l,s}\) is the norm of the projection operator onto the \(l\)'th spherical harmonic subspace. \(\square\)

To conclude the proof of Theorem 3 we use Theorem 4 with \(n = 2\), \(s = 0\) and \(d = 1\) to obtain the estimate for the radial part of the data. \(\square\)

**Remark 2** We end this section by some comments on dispersive estimates and their relationship to Strichartz estimates. The main goal in [11] was to obtain the \(L^\infty - L^1\) dispersive estimate, and they deduced Strichartz estimates as a corollary, applying the usual duality argument from the free case. However, it required considerably more work to obtain such a dispersive estimate in the presence of a potential, and it imposed the assumption of rapid decay on this potential. The arguments from [23] bypass the dispersive estimate to prove Strichartz directly. In our setting, this type of approach is required since the \(L^\infty - L^1\) dispersive estimate is known to fail (20), at least for negative \(a\). However, to perform the duality argument referred to in the above, one only needs an \(L^p' - L^p\) dispersive estimate, for \(p \sim 2n/(n - 2)\). Hence, one may wonder whether such a restricted dispersive estimate holds true. Estimates of this type have been obtained in [6] for the wave equation with a potential decaying strictly faster than \(|x|^{-2}\). In the remaining part of this section we will indicate a simple way to get a dispersive estimate with a epsilon loss. The same argument would apply to the wave equation.

We will prove the following result
PROPOSITION 2 Let \( n \geq 5 \). Let \( u \) be the unique solution of (1). There exists a constant \( C_{\varepsilon} \) (depending also on \( n, a \)), such that

\[
\|\Delta_j u(t, \cdot)\|_{L^2} \leq C_{\varepsilon}^{2j_{\varepsilon}} \|\Delta_j f\|_{L^2},
\]

where \( \Delta_j \) is the usual frequency localization at \(|\xi| \sim 2^j\).

Let us assume momentarily that we have obtained a weighted \( L^2 \) dispersive estimate of Kato-Jensen type (9),

\[
\int |u|^2 \frac{dx}{|x|^{2\alpha}} \lesssim \frac{1}{t^{2\alpha}} \int |f|^2 |x|^{2\alpha} dx,
\]

for some range of \( \alpha \) (which will depend on the dimension). This estimate will play the role assigned to (6) in our proof of Strichartz.

We proceed with a nice observation we learned from J. Ginibre (7). Recall (20),

\[
u(t) = S(t) f - ia \int_0^t S(t-s)\Omega^{-2} u(s) ds,
\]

and write also the reversed Duhamel formula, where \( S_a(t) = \exp(-itP_a) \) plays the role of the free group. Replace \( u(s) \) in the first Duhamel formula by its expression coming from the second, we get

\[
u(t) = S(t) f - ia \int_0^t S(t-s)\Omega^{-2} S(s) f ds
\]

+ \( a^2 \int_0^t \int_0^s S(t-s)\Omega^{-2} S_a(s-\tau) \Omega^{-2} S(\tau) f d\tau ds.
\]

Recall the dispersive estimate (with Lorentz spaces) for the free group, which we state for a frequency localized data: \( 1 < p' \leq 2 \),

\[
\|\Delta_j S(t) f\|_{L^{p',2}} \leq C \min(2^{nj_{\frac{1}{p'}-\frac{1}{p}}}, t^{-n\left(\frac{1}{2} - \frac{1}{p}\right)}) \|\Delta_j f\|_{L^{p',2}}.
\]

For the second term in (25) we use \( p = \frac{2n}{n-2} \), and compute explicitly the integral after using the free dispersion, to get \( \log(1 + 2^{2j_{\varepsilon}}t)^{-1} \). So we focus on the third term. Freezing the time variables, we have, with \( p = \frac{2n}{n-2-2\mu} \), (\( \mu \) should be thought as small)

\[
\|\Delta_j S(t-s)\Omega^{-2} S_a(s-\tau) \Omega^{-2} S(\tau) \Delta_j f\|_{L^{p',2}} \leq \min(2^{2j_{\left(1+\mu\right)}}), \frac{1}{(t-s)^{1+\mu}}, \frac{1}{(s-\tau)^{1-\mu}}, \frac{1}{\tau^{1+\mu}} \|\Delta_j f\|_{L^{p',2}}
\]

where we have successively used free dispersion, generalized Hölder, weighted \( L^2 \) dispersion (21) with \( \alpha = 1 - \mu \), generalized Hölder again, and finally free dispersion. Evaluating the double time integral yields the desired decay \( 2^{4j_{\varepsilon}}t^{-1+\mu} \), and interpolation with the \( L^2 \) bound gives the result (said \( L^2 \) bound resulting from Strichartz which hold for both \( S(t) \) and \( S_a(t) \)).

We now prove (24). In (20) we obtained some dispersive estimates in the radial case, but the argument would equally apply to any spherical harmonic with uniform constants. Thus,
one could deduce (24) from these estimates, using generalized H"older in the radial variable.
However, we choose to give a simple proof here, based on the following observation, which
was pointed out to us by I. Rodnianski: introduce $L = x/2 + it \nabla$ the pseudo-conformal
vectorfield. $L$ commutes with the free evolution, but not with the potential term. Introduce
$C = L^2 + at^2 \Omega^{-2}$. Then a simple sequence of computations gives
\[ [i \partial_t - P_a, C] u = it(2a \Omega^{-2} + ax \cdot \nabla \Omega^{-2}) u. \]
Due to the special form of the potential, $2\Omega^{-2} + x \cdot \nabla \Omega^{-2} = 0$, and $C$ commutes with
the equation. Therefore, $\|Cu\|_2^2$ is conserved. Now, another simple computation gives
$\Delta Mf = \frac{1}{t^2}ML^2f$, where $Mf = e^{it\frac{2}{t^2}}f$. Adding the potential, $P_aMf = -\frac{1}{t^2}MCf$, since
obviously $M$ and $\Omega^{-2}$ commute. Provided $a > -(n-2)^2/4$, we can apply Hardy to the
left-hand side (this will require $n \geq 5$), set $f = u$, to get
\[
\| \frac{M u}{|x|^2} \|_2 \leq \frac{C}{t^2} \|MCu\|_2,
\]
and the $M$ is now irrelevant, so that
\[
\| \frac{u}{|x|^2} \|_2 \leq \frac{C}{t^2} \|Cu_0\|_2 = \frac{C}{t^2} \| |x|^2 u_0 \|_2.
\]
Interpolating this with $L^2$ conservation for $u$ gives the desired estimate. \hfill \square

Remark 3 We note that such a weighted $L^2$ dispersive estimate (with a decay greater
than $t^{-1}$) is all which is required to run the perturbative argument from [23], with the
free evolution replaced by $S_a(t)$. In fact, all the other estimates which are required are
contained in Theorem [11]. Hence, one could in principle obtain Strichartz estimates for (non
radial) perturbations of the inverse square potential which decay $\varepsilon$ faster.

3.2 The wave equation
In this section we will obtain Strichartz estimates for the operator $\Box_a = \partial^2_t + P_a$ from
the generalized Morawetz estimate (13) as we did for the Schr"odinger equation. We present
two different results: the first one is in some sense the true equivalent of Theorem [11]; it
uses Lemma 3.1 to bypass issues related to the varying degrees of smoothness in Strichartz
estimates for the wave equation. The second result shows one can in fact recover the
end-point Strichartz estimate if needed. These two results will merge into a theorem on
generalized Strichartz estimates with derivatives, in the last section.

THEOREM 5 Let $u$ be the solution to (2) with Cauchy data $(f, g) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$. Let
$p \geq 2$, and $q$ be such that $\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}$ ($p > 2$ if $n = 3$ and $p > 4$ if $n = 2$). Then,
provided $a + \lambda^2 > 0$,
\[
\| (-\Delta)^{\sigma/2} u \|_{L^p_t(L^q_x)} \leq C(\|f\|_{\dot{H}^{\frac{1}{2}}} + \|g\|_{\dot{H}^{-\frac{1}{2}}}),
\]
where $\sigma = \frac{1}{p} + \frac{n}{q} - \frac{n-1}{2}$ (gap condition).
One could of course replace the norm on the left with the appropriate Besov norm.

Proof: We write the solution to the wave equation with a potential \(2\) as the sum of the linear solution to the free wave equation

\[
W(t)f + W(t)g = \mathcal{F}^{-1}(\cos(t|\xi|)\mathcal{F}f + \frac{\sin(t|\xi|)}{|\xi|} \mathcal{F}g),
\]

where \(\mathcal{F}\) is the Fourier transform in the space variable \(x\), plus a Duhamel term

\[
a \int_0^t W(t-s)\Omega^{-2}u(s)ds.
\]  

(27)

Since \(W(t-s) = -\dot{W}(t)W(s) + W(t)\dot{W}(s)\), we obtain two terms in the above. We will deal with the first one, the treatment of the second term being similar. We are going to use Lemma 3.1, thus we set

\[
T h(t) := \dot{W}(t) \int W(s)\Omega^{-1}h(s)ds.
\]

Once again, using the Strichartz estimate for the free wave equation, combined with the dual to estimate (13) (for \(n \geq 2, d = d_0(n), a = 0\) and \(\alpha = 1/4\)),

\[
\|T h\|_{L^p H^\sigma_q} \leq C \|\int W(s)\Omega^{-1}h(s)ds\|_{\dot{H}^{1/2}({\mathbb R}^n)} \leq C\|h\|_{L^2({\mathbb R}^{n+1})}
\]

with \(p, q\) and \(\sigma\) as in the statement of the Theorem. By Lemma 3.1 the corresponding operator \(\tilde{T}\) satisfies the same estimate as \(T\) (with a different constant). On the other hand, the solution to (2) is

\[
u(t) = \dot{W}(t)f + W(t)g + a\tilde{T}(\Omega^{-1}u)
\]

thus using (13) one more time we obtain the desired result. For \(n = 2\) we need to assume that the data \(f, g\) are orthogonal to radial functions, for (13) to hold in the \(a = 0, \alpha = 1/4\) case. On the other hand, Strichartz estimates for the wave equation (2) in case of radial data were proven in [19] using the conjugation method. For future reference we quote a more general result here. We note that the restriction on \(\gamma\) and hence on \(\sigma\), which is the number of derivatives that can be taken, comes from the requirement of the continuity of the conjugation operator \(K_{\mu, \nu}^0\) on the spaces involved (see [19] for details).

THEOREM 6 Let \(n \geq 2, 2 \leq q < \infty\) and let \(p, \gamma, \sigma\) be such that

\[
1 \leq \frac{1}{p} \leq \min\left\{\frac{1}{2}, \frac{n - 1}{2}(\frac{1}{2} - \frac{1}{q})\right\}, \quad \sigma = \gamma + \frac{1}{p} - n(\frac{1}{2} - \frac{1}{q}).
\]

(28)

For integer \(d \geq 1\) let \(f \in \dot{H}_d^\gamma\) and \(g \in \dot{H}_d^{-1}\). Then there exists a constant \(C > 0\) depending on \(n, q, p, \gamma, d\) such that the solution to (2) satisfies

\[
\|u\|_{L^p H^\sigma_q} \leq C(\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{-1}}).
\]

(29)

provided

\[-1 - \nu_0 < \gamma < 1 + \nu_0 - \frac{1}{p}\]

(29)
To complete the proof of Theorem 5, we use the above estimate with \( n = 2, d = 1, \) and \( \gamma = 1/2 \) for the radial part of the data.

We now turn to an endpoint estimate:

**THEOREM 7** Let \( n \geq 4, \) let

\[
(p, q, \gamma, \sigma) = \left(2, \frac{2(n - 1)}{n - 3}, \frac{n - 3}{2(n - 1)}, \frac{-2}{n - 1}\right),
\]

and let \( u \) be the solution to (2). Then, provided \( a + \lambda^2 > 0, \)

\[
\|(-\Delta)^{-\sigma/2}u\|_{L^p_t(L^q_x)} \leq C(\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^\gamma-1}).
\]

(31)

The exponents here are chosen so as to postpone until Section 4 the unpleasant issues related to commuting the free Laplacian with its counterpart with potential.

**Proof of Theorem 7** We start with a simple corollary of Theorem 2. Using the fact that \( P_a \) commutes with the solution, and the equivalence (15), we have

**COROLLARY 1** Let \( n, d, \alpha, u \) be as in Theorem 2. Then

\[
\|\Omega^{-1/2-2\alpha}u\|_{L^2_t(L^2_x)} \leq C(\|f\|_{\dot{H}^{2\alpha}} + \|g\|_{\dot{H}^{2\alpha-1}}).
\]

(32)

We note that the case \( a = 0, \alpha = \frac{1}{2} \) is the Morawetz estimate [17], which states that for the solution \( u \) of the linear wave equation \( \Box u = 0 \) in four or more dimensions, \( \int \int u^2/|x|^3 \text{d}x\text{d}t \) is bounded by the energy of the initial data.

Using the above estimate (32), one can proceed as we did for Schrödinger on the Duhamel term (27). Take \( 2\alpha = \frac{n-3}{2(n-1)} \), then

\[
\Omega^{-\frac{1}{n-1}}u \in L^2_t(L^2_x).
\]

From this, writing

\[
\Omega^{-2}u = \Omega^{-(1+\frac{n-1}{n})} \Omega^{-\frac{1}{n-1}}u,
\]

and using \(|x|^{-\frac{1}{n-1}} \in L^{n-1,\infty}_x\), the generalized Hölder inequality for Lorentz spaces yields

\[
\Omega^{-2}u \in L^2_t(L^{\frac{2(n-1)}{n+1}}_x).
\]

We can then finish the proof by applying the usual end-point to end-point Strichartz estimate,

\[
\||\nabla|^{-\frac{2}{n-1}}\Box^{-1}F\|_{L^2_t(L^{\frac{2(n-1)}{n+1}}_x)} \leq C\|F\|_{L^2_t(L^{\frac{2(n-1)}{n+1}}_x)}.
\]

(33)
4 Estimates with derivatives

When dealing with nonlinear applications for either the Schrödinger or the wave equation, the $f \in L^2$ or $\dot{H}^{1/2}$ estimates obtained above are often not enough, and one needs to consider estimates involving derivatives of the solution and the data. Moreover, the correct number of derivatives needed could be fractional. For the wave equation this is the case even at the linear level, as we already saw when trying to prove Strichartz estimates, where spaces such as $\dot{H}^{1/2}$ and $\dot{H}^{-1/2}$ naturally appear. In the free case, frequency localizations commute with the flow, hence one may immediately deduce estimates for $f \in \dot{H}^s$ from an $L^2$ or $\dot{H}^{1/2}$ estimate, through Littlewood-Paley. In our setting, this is no longer true however, and one needs to replace frequency localizations $\phi(\sqrt{-\Delta})$ by those based on the operator $P_a$. At the same time we need the final estimate to be phrased in terms of standard Sobolev spaces, based on powers of $-\Delta$, otherwise the estimate would be useless in nonlinear applications (unless one studies carefully the multiplication properties of spaces based on the operator $P_a$). At issue is therefore the lack of commutation between the two localizations, which we are now going to address. Here we will obtain various estimates in weighted spaces for the products of the two projections, which we will then use to deduce generalized Strichartz estimates with derivatives from the same weighted-$L^2$ estimates obtained above.

We will end this Section with a nonlinear application that illustrates the need for derivative estimates.

Let $\Delta_j$ be the usual dyadic frequency localization at $|\xi| \sim 2^j$, and let $\Pi_k$ be the localization with respect to $\sqrt{P_a}$. More precisely, let $\beta_0 \in C_0^\infty(\mathbb{R})$ denote the standard bump function supported in $[1/2, 2]$, with the property that $\sum_j (\beta_0(2^{-j}x))^2 = 1$ for all $x \in \mathbb{R}$. and let $\beta_j(\xi) := \beta_0(2^{-j}|\xi|)$. Let $\mathcal{H}_\nu$ denote the Hankel transform of order $\nu$. Let $\Delta_l^j$ and $\Pi_l^k$ denote the restrictions to $L^2_l$, the $l$-th spherical harmonic subspace of $L^2$, of the above projections $\Delta_j$ and $\Pi_k$ respectively. It was shown in [20] that

$$\Delta_l^j = \mathcal{H}_\mu \beta_j \mathcal{H}_\mu, \quad \Pi_l^k = \mathcal{H}_\nu \beta_k \mathcal{H}_\nu$$

where $\mu = \lambda + l$, $\lambda = \frac{a-2}{2}$, and $\nu = \sqrt{\mu^2 + a}$. Let us define the following operators:

$$J_{jk} := \Delta_j \Omega^{-2} \Delta_k, \quad M_{kl} := \Delta_k \Pi_l, \quad N_{lm} := \Pi_l \Delta_m.$$

We are going to need estimates on the operators $J_{jk}, M_{kl}, N_{lm}$, in weighted-$L^2$ spaces. These will be provided in the next three Lemmas. The estimate needed for $N_{lm}$ is the almost orthogonality lemma for the projectors, the radial version of which was given in [20]:

**Lemma 4.1** Let $n \geq 2$, $d \geq 0$. For all positive $\epsilon_1 < \min\{\lambda + d, \nu d\} + 1$,

there exists a constants $C > 0$ such that for all $j, k \in \mathbb{Z}$ and $f \in L^2_{2d}(\mathbb{R}^n),

$$\|M_{jk} f\|_{L^2}, \|N_{jk} f\|_{L^2} \leq C 2^{-\epsilon_1 |j-k|} \|f\|_{L^2}. \quad (34)$$

**Proof:** Recalling the definition of $K^0_{\mu, \nu}$,

$$\Delta_l^j \Pi_l^k = \mathcal{H}_\mu \beta_j K^0_{\mu, \nu} \beta_k \mathcal{H}_\nu.$$
Since the Hankel transforms appearing at the extremes are $L^2$-isometries, the problem reduces to showing that the operator

$$L_{jk} := \beta_j K_{\mu,\nu}^0 \beta_k$$

is bounded on $L^2_l$, with a norm that is bounded independent of $l$. For $j$ close to $k$ this is obviously true by the boundedness of each factor. We note that here as well as in the Proposition that follows, only the support properties of $\beta_j$ are used, and not that their squares form a partition of unity.

Now for $j \neq k$,

$$L_{jk} f = \beta_j(r) \int_0^\infty k_{\mu,\nu}^0(r,s) \beta_k(s) f(s) s^{-1} ds \quad (35)$$

We recall the formula for the integral kernel $k_{\mu,\nu}^0$ obtained in [19]:

$$k_{\mu,\nu}^0(r,s) = \frac{2\Gamma\left(\frac{\mu+\nu}{2} + 1\right)}{\Gamma\left(\frac{\nu-\mu}{2}\right)\Gamma(\nu + 1)} \frac{s^{\nu-\lambda}}{r^{\lambda+\nu+2}} F\left(\mu + \nu + 1, \frac{\nu - \mu}{2} + 1; \frac{\nu + 1}{2}; \frac{s^2}{r}\right) \quad (36)$$

Here $F$ is the hypergeometric function defined by (11). The above formula for $k_{\mu,\nu}^0$ is valid for $s < r$. For $s > r$ one needs to switch $s$ and $r$, and switch $\mu$ and $\nu$ in the formula. Turning now to $L_{jk}$ for $|j-k| \geq 3$, we see that in (35) we have $s \sim 2^j$ and $r \sim 2^k$, thus we either have $s \leq r/2$ or $s \geq 2r$. Therefore the last argument of the hypergeometric function in (36) will always be in $[0, \frac{1}{2}]$. It is then easy to see from (11) that $|F| \leq C$ independent of $l$. Moreover, $\nu = \mu + O\left(\frac{1}{l}\right)$ for large $l$, and from Stirling’s formula,

$$\frac{2\Gamma\left(\frac{\mu+\nu}{2} + 1\right)}{\Gamma\left(\frac{\nu-\mu}{2}\right)\Gamma(\nu + 1)} = O(l^{-1+\epsilon/l}) < C.$$

By the procedure outlined in [19] §3.1], the resulting pointwise bound for $k_{\mu,\nu}^0$ gives the desired $L^2$ bound, namely,

$$\|L_{jk} f\|_{L^2_0} \leq C 2^{-\delta|j-k|} \|f\|_{L^2_0},$$

for any $\delta < \delta_l := \min\{\mu, \nu\} + 1$. Summing over $l$ and using orthogonality of spherical harmonics in $L^2$, we obtain (34), with

$$\epsilon_1 < \min_{l \geq d} \delta_l = \min\{\lambda + d, \nu_d\} + 1.$$  

Next we obtain a weighted-$L^2$ estimate for $M_{kl}$. First we need the following general result:

**PROPOSITION 3** Let $n \geq 2$ and $d \geq d_0(n)$ be fixed integers. There exists a constant $C > 0$ (depending on $n$ and $d$) such that for all $f \in L^2_\geq d(\mathbb{R}^n)$,

$$\|\beta_0(-\Delta)^{-1} \beta_m f\|_{L^2} \leq C 2^{-|m|\langle\lambda+d\rangle^+ + |m|\|f\|_{L^2}}$$

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Proof: Let us first assume $|m| \geq 3$. Let $K(x)$ denote the Newtonian potential, and let $T_d(x; y)$ be the Taylor polynomial of degree $d - 1$ in $y$ for $K(x - y)$. For example, for $n \geq 3$, we have that up to a constant factor,

$$T_d(x; y) = |x|^{-n+2} + (n-2)|x|^{-n}x \cdot y + \frac{n-2}{2} |x|^{-n-2} \{n(x \cdot y)^2 - |x|^2|y|^2\}.$$  

Let $K_d(x, y) := \max\{T_d(x; y), T_d(y; x)\}$ It is easy to see that $K_d \in L^2_{<d}$ and thus for $f \in L^2_{\geq d}$ we may write

$$(-\Delta)^{-1} \beta_m f(x) = \int K(x - y)\beta_m(y)f(y)dy = \int (K(x - y) - K_d(x, y))\beta_m(y)f(y)dy.$$  

We have $|x| \sim 1$ and $|y| \sim 2^m$. Since $|m| \geq 3$ we have that either $|y| \leq \frac{1}{2}|x|$ or $|y| \geq 2|x|$. We can then check easily that

$$|K(x - y) - K_d(x, y)| \leq C \frac{(\min\{|x|, |y|\})^d}{(\max\{|x|, |y|\})^{n+d-2}} \leq C \left\{ \begin{array}{ll} 2^{-m(n-2)} & m \geq 3 \\ 2^m & m \leq -3 \end{array} \right.$$ 

We then have

$$\|\beta_0(x)(-\Delta)^{-1} \beta_m(y)f(y)\|_{L^2} \leq \sup |K(x - y) - K_d(x, y)|\|f\beta_m\|_{L^1} \leq C\|f\|_{L^2} \left\{ \begin{array}{ll} 2^{-m(\lambda + d - 1)} & m \geq 3 \\ 2^{m(\lambda + d + 1)} & m \leq -3 \end{array} \right.$$ 

which establishes the claim, for $|m| \geq 3$. For $|m| \leq 2$ we simply observe that if we let $\tilde{K}(x, y) := \beta_0(x)K(x-y)\beta_m(y)$ then $\|\tilde{K}\|_{L^1} \leq \|\tilde{K}\|_{L^\infty} < C$ and thus the operator corresponding to $\tilde{K}$ maps any $L^p$ into itself.

**Lemma 4.2** Let $n \geq 2$, $d \geq d_0(n) + 1$, and $0 \leq \eta \leq 2$. There exist a constant $C > 0$ such that for all $j, k \in \mathbb{Z}$ and for all $f \in (L^2_{<d}(\mathbb{R}^n))$,

$$\|\Omega^{-\eta}M_{jk}\Omega^{\eta} f\|_{L^2} \leq C2^{-\epsilon_2 |j-k|}\|f\|_{L^2} \quad (37)$$

for any $\epsilon_2 < \min\{\lambda + d, \nu_d\} + 1 - \eta$.

Proof: We can obtain this estimate by interpolating between (34) and the following estimate

$$\|\Omega^{-2} M_{jk} \Omega^{2} f\|_{L^2} \leq C2^{-\gamma |j-k|}\|f\|_{L^2} \quad (38)$$

for some appropriate $\gamma > 0$. To prove this estimate, or equivalently, its dual estimate, we note that on $L^2_{\tilde{f}}$,

$$\Omega^2 \Pi_j \Delta_k \Omega^{-2} = \Omega^2 \mathcal{H}_\nu \beta_j \mathcal{H}_\nu \mathcal{H}_\mu \beta_k \mathcal{H}_\mu \Omega^{-2} = \mathcal{H}_\nu \mathcal{A}_\nu \beta_j \mathcal{K}_{\nu, \mu} \beta_k \mathcal{A}_{\mu} \mathcal{A}_{\mu}^{-1} \mathcal{H}_\mu.$$
where the $A$’s are as in (8). Once again, the Hankel transforms at the two extremes being $L^2$-isometries, the above reduces to proving an estimate for the operator $L_{jk}$ defined above, namely,

$$\| A_\nu L_{jk} A_\mu^{-1} f_l \|_{L^2} \leq C 2^{-\gamma |j-k|} \| f_l \|_{L^2}$$

We have $A_\nu = A_\mu + \varphi$ and $A_\nu K^0_{\nu, \mu} = K^0_{\nu, \mu} A_\mu$. Thus

$$A_\nu L_{jk} A_\mu^{-1} f_l = (A_\mu \beta_j) K^0_{\nu, \mu} \beta_k A_\mu^{-1} f_l + \beta_j K^0_{\nu, \mu} (A_\mu \beta_k) A_\mu^{-1} f_l + \beta_j K^0_{\nu, \mu} \partial_k \beta_k A_\mu^{-1} f_l + \partial_j \beta_j (\partial_k K^0_{\nu, \mu}) \beta_k A_\mu^{-1} f_l + L_{jk} f_l$$

$$:= I + II + III + IV + V.$$ 

We have estimated $V$ already. For the other four, first note that since what we want to prove, namely (8), is scale-invariant, in estimating any of the pieces $I$-$IV$ we can set either $j$ or $k$ equal to zero. Thus for $I$ it is enough to show

$$\sum_l \| (A_\mu \beta_j) K^0_{\nu, \mu} \beta_0 A_\mu^{-1} f_l \|_{L^2} \leq C 2^{-\gamma |j|} \| f_l \|_{L^2}.$$  \hspace{1cm} (39)

Now $A_\mu \beta_j(x) = 2^{-2j} \tilde{\beta}_j(x)$ where $\tilde{\beta}_j(x) = (A_\mu \beta_0)(2^{-j} x)$ is bounded independently of $j$ and has the same support as $\beta_j$. Using that $\sum_k \beta_k^2 = 1$ and that $\beta_j \beta_k = 0$ for $|j-k| > 2$, we have

$$\tilde{\beta}_j K^0_{\nu, \mu} \beta_0 A_\mu^{-1} f_l = \tilde{\beta}_j K^0_{\nu, \mu} \beta_0 \sum_p \beta_p^2 A_\mu^{-1} \sum_m \beta_m^2 f_l$$

$$= \sum_{p=-1}^1 \sum_{m \in \mathbb{Z}} \tilde{\beta}_j K^0_{\nu, \mu} \beta_0 \beta_p^2 A_\mu^{-1} \beta_m^2 f_l.$$

It is enough to estimate the term with $p = 0$, the other two being similar. We then have

$$\| \tilde{\beta}_j K^0_{\nu, \mu} \beta_0 \beta_0^2 A_\mu^{-1} \beta_m^2 f_l \|_{L^2} \leq C 2^{-|j|} \| \beta_0^2 A_\mu^{-1} \beta_m^2 f_l \|_{L^2}$$

by the previous Lemma, for any $\delta < \delta_l := \min \{\mu, \nu\} + 1$. On the other hand, we have, by Proposition 3 that

$$\sum_l \sum_m \| \beta_0^2 A_\mu^{-1} \beta_m^2 f_l \|_{L^2} = \sum_m \| \beta_0^2 \Delta^{-1} \beta_m^2 f_l \|_{L^2} \leq C \| f \|_{L^2} \sum_m 2^{-|m| (\lambda + d) + m} \leq C \| f \|_{L^2}$$

since by our assumptions $\lambda + d > 1$, and thus

$$\sum_{l=1}^\infty \| (A_\mu \beta_j) K^0_{\nu, \mu} \beta_0 A_\mu^{-1} f_l \|_{L^2} \leq C 2^{-2j} 2^{-|j|} \| f \|_{L^2} \leq C 2^{-\gamma j} \| f \|_{L^2}$$
for
\[ \gamma < \min_{l \geq d} \delta_l - 2 = \min\{\lambda + d, \nu_d\} - 1. \]

Estimating II and III is entirely analogous to the above, and for IV we need to use the explicit form of the kernel (36). Using the fact that \( r \) and \( s \) are well-separated, the series (11) can be differentiated term-by-term, thus obtaining corresponding decay rates for the derivative kernel, which in turn give the desired estimate for IV by the same procedure as above.

We thus have the estimate (38), with \( \gamma \) as in the above. Now, interpolating between (34) and (38) gives the desired estimate (37). ✷

Finally, we need to estimate \( J_{jk} \) on weighted-\( L^2 \) as well:

**Lemma 4.3** Let \( n \geq 2, d \geq d_0(n) \) and \( 0 \leq \zeta \leq 2 \). There exist a constant \( C > 0 \) such that for all \( f \in L^2 \geq d(R^n) \) and \( j, k \in \mathbb{Z} \),
\[ \| \hat{\Omega}^{\zeta} J_{jk} \hat{\Omega}^{2-\zeta} f \|_{L^2} \leq C 2^{-\epsilon_3|j-k|} \| f \|_{L^2} \]

for all \( \epsilon_3 < \lambda + d - |1 - \zeta| \).

**Proof:** After taking the Fourier transform, the estimate to prove is
\[ \| \Delta^{\zeta/2} \beta_j \Delta^{\lambda-1} \beta_k \Delta^{1-\zeta/2} \hat{f} \|_{L^2} \leq C 2^{-\epsilon_3|j-k|} \| \hat{f} \|_{L^2} \]
where \(-\Delta_\xi = \sum_i \partial^2/\partial \xi_i^2 \) is the Laplacian in the Fourier variable. We will prove this by interpolation: Let \( T_{jk} := \beta_j \Delta^{\lambda-1} \beta_k \). We then need to show that \( T_{jk} \) maps \( L^2 \) into the homogeneous Sobolev space \( \tilde{H}^2 \), and that it maps \( \tilde{H}^{-2} \) into \( L^2 \). Also note that \( T_{jk} = T_{kj}^* \). Thus it is enough to show that
\[ \| T_{jk} g \|_{\tilde{H}^2} \leq C_{jk} \| g \|_{L^2} \]
for some constant \( C_{jk} \), in order to obtain via interpolation that
\[ \| T_{jk} g \|_{\tilde{H}^{\zeta-2}} \leq C_{jk}^{\zeta/2} C_{kj}^{1-\zeta/2} \| g \|_{\tilde{H}^{\zeta-2}} \]
The desired estimate would then follow by setting \( g = \Delta^{1-\zeta/2} \hat{f} \).

To prove (41) we again note that by scaling we can set \( j = 0 \), and estimate
\[ \| \Delta_\xi \beta_0 \Delta^{\lambda-1} \beta_k g \|_{L^2} \]
We have
\[ \Delta_\xi \beta_0 \Delta^{\lambda-1} \beta_k g = \beta_0 \beta_k g + 2 \nabla \beta_0 \cdot \nabla \Delta_\xi^{\lambda-1} \beta_k g + (\Delta_\xi \beta_0)(\Delta^{\lambda-1} \beta_k g) \]
The first term on the right is easy to estimate since \( \beta_0 \beta_k \equiv 0 \) if \( |k| \geq 2 \). For the third term, we use Proposition 3 which gives
\[ \|(\Delta_\xi \beta_0)(\Delta^{\lambda-1} \beta_k g)\|_{L^2} \leq C 2^{-|k|(|\lambda+d|+k)} \]
Similar argument applies to the middle term in (42), and we thus obtain (41) with
\[ C_{jk} := C 2^{-|j-k|(|\lambda+d|+j-k)} \]
Hence,
\[ C^{\zeta/2}_{jk}C^{1-\zeta/2}_{kj} = C'2^{-|j-k|((\lambda+d)-(j-k)(1-\zeta)} \leq C2^{-|j-k|((\lambda+d-[1-\zeta])}, \]
and we have the desired estimate for \( |j-k| \geq 3 \). For \( j \) close to \( k \), on the other hand, we can estimate the two factors that make up \( J_{jk} \) separately, i.e. it is enough to show that \( \Delta^2_\zeta \beta_j \Delta^{-1}_\zeta \) is bounded on \( L^2 \) independent of \( j \). By interpolation, this further reduces to proving that \( \Delta^2_\zeta \beta_j \Delta^{-1}_\zeta \) is bounded on \( L^2 \), or equivalently, that multiplication by \( \beta_j \) is bounded on \( \dot{H}^2 \), which is easily seen to be the case.

4.1 The Schrödinger equation

To obtain a frequency-localized version of (18), we recall that the solution to (1) can be expressed as
\[ u(t) = S(t)f - i\alpha \int_0^t S(t-s)\Omega^{-2}S_\alpha(s)f \, ds \]
where \( S_\alpha(s) = e^{-isP_\alpha} \) is the Schrödinger group associated to \( P_\alpha \). We begin by applying \( \Delta_j \) to both sides of the above, using that it commutes with \( S(t) \). We then insert resolutions of identity based on the \( \Pi \)'s and the \( \Delta \)'s before and after the \( S_\alpha(s) \) factor, to obtain
\[ \Delta_ju = \Delta_jS(t)u_0 - i\alpha \sum_{k,l,m} \int_0^t S(t-s)\Delta_j\Omega^{-2}\Delta_k\Pi_lS_\alpha(s)\Pi_l\Delta_m\Delta_mf, \] (43)

By the endpoint Strichartz estimate for the free group \( S(t) \), and generalized Hölder inequality we then have, for \( n \geq 3 \),
\[
\| \int_0^t S(t-s)J_{jk}M_{kl}S_\alpha(s)N_{lm}\Delta_mf \, ds \|_{L^2_tL^2_x} \leq C\| J_{jk}M_{kl}S_\alpha(s)N_{lm}\Delta_mf \|_{L^2_tL^2_x} \\
\leq C\| |x|^{-1}\|_{L^\infty_x} \| \Omega J_{jk}\Omega^{-1}M_{kl}\Omega^{-1}s_\alpha N_{lm}\Delta_{m}f \|_{L^2_tL^2_x}
\]
while for \( n = 2 \) we again proceed as before, utilizing Lemma 3.1 Strichartz estimate for the free Schrödinger equation with \( (p,q) \) as in (17), and the dual to the weighted-\( L^2 \) estimate (with \( a = 0, d = 1, \alpha = 1/4 \) to obtain
\[
\| S(t) \int S(-s)J_{jk}M_{kl}S_\alpha(s)N_{lm}\Delta_mf ds \|_{L^2_tL^2_x} \leq C\| \Omega J_{jk}\Omega^{-1}M_{kl}\Omega^{-1}s_\alpha N_{lm}\Delta_{m}f \|_{L^2_tL^2_x}
\]
Thus in either case we need to apply Lemma 1.3 with \( \zeta = 1 \), Lemma 1.2 with \( \eta = 1 \), Lemma 4.1 (all three Lemmas with \( d = d_0(n) + 1 \)), and estimate 6 with \( \alpha = 1/4 \) to conclude that
\[
\| \Delta_ju \|_{L^2_tL^2_x} \leq C \sum_{k,l,m} 2^{-\epsilon_1|j-k|+\epsilon_2|k-l|+\epsilon_3|l-m|} \| \Delta_mf \|_{L^2_x} \\
\leq C \sum_m 2^{-\epsilon|j-m|} \| \Delta_mf \|_{L^2_x},
\] (44)
for $\epsilon = \min_{i=1}^{3} \epsilon_i < \min\{n/2 + d_0(n), \nu_{d_0(n)+1}\}$, which is the desired frequency-localized version of the endpoint Strichartz estimate, valid for data $f \in L^2_{\geq d_0(n)+1}(\mathbb{R}^n)$. Here $(p, q) = (2, 2n/(n - 2))$ if $n \geq 3$ and $(p, q)$ are as in (17) if $n = 2$.

The above estimate can now be used to obtain generalized Strichartz estimates in Sobolev, or more generally, Besov spaces, as follows: Let $(p, q)$ be as in the above and $f \in L^2_{\geq 1+d_0(n)}$. We then have

$$
\sum_j 2^{2sj}\|\Delta_j u\|_{L^p_t(L^{q,2}_x)}^2 \leq C \sum_m 2^{2sm}\|\Delta_m f\|_{L^2_tL^2_x}^2 \sum_j 2^{2s(j-m) - 2s|j-m|} \\
\leq C\|f\|^2_{H^s}
$$
as long as $|s| < \epsilon$.

In other words, combining this with Theorem 3 for $d = d_0(n) + 1$ we have proved

**THEOREM 8** Let $s$ be such that

- $-\min\{1 + \nu_0, \delta, \nu_1\} < s < \min\{1 - \frac{2}{p} + \nu_0, \delta, \nu_1\}$ if $n \geq 3$
- $-\min\{1 + \sqrt{a}, 1 + \frac{2}{p}\} < s < \min\{1 + \sqrt{a} - \frac{2}{p}, 1\}$ if $n = 2$

where $(p, q)$ are as in (17). Then there exists a constant $C > 0$ depending on $n, p, s, \lambda$ such that the unique solution $u$ of (1) satisfies

$$
\|(-\Delta)^{s/2}u\|_{L^p_t(L^2_x)} \leq C\|u\|_{L^p_t(B^s_{q,2} \cap \dot{B}^s_{\infty,2})} \leq C\|f\|_{H^s}.
$$

**Proof:** When $n \geq 3$ and $p = 2$ this estimate is a simple consequence of the definition of Besov spaces and (14). Interpolating with the energy estimate gives the full range of $p$. $\square$

### 4.2 The wave equation

We will proceed exactly as in the previous subsection. We write

$$
\Delta_j u = \dot{W}(t)\Delta_j f + W(t)\Delta_j g \\
+ a \sum_{k,l,m} \int_0^t W(t-s)\Delta_j \Omega^{-2}\Delta_k \Delta_l \Pi_l [W_a(s)\Pi_l \Delta_m \Delta_m g] \\
+ \dot{W}_a(s)\Pi_l \Delta_m \Delta_m f \quad ds,
$$

where $W_a(s)$ is the propagator corresponding to $\partial^2_t + P_a$, i.e. on the $l$-th spherical harmonic subspace $W_a(s) = H_{\nu} \frac{\sin(s[\xi])}{s[\xi]} H_{\nu}$.

For simplicity, let us assume $f \equiv 0$. Again, by the endpoint Strichartz estimate 33 we then have, for $n \geq 4$,

$$
2^{n-2}\|\int_0^t W(t-s)J_{jk}M_{kl}W_a(s)N_{lm}\Delta_m g \; ds\|_{L^2_tL^{n-2}_x} \leq C\|J_{jk}M_{kl}W_a(s)N_{lm}\Delta_m g\|_{L^2_tL^{n-2}_x} \leq C\|x^{-\frac{n}{n-2}}\|_{L^{n-1,\infty}_x} \|\Omega^{-\frac{n}{n-2}}J_{jk}\Omega^{-\frac{n}{n-2}} \Omega^{-\frac{n-2}{n-2}}M_{kl}\Omega^{-\frac{n-2}{n-2}} \Omega^{-\frac{n-2}{n-2}}W_a N_{lm}\Delta_m g\|_{L^2_tL^2_x},
$$

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Now, we use Lemma 3.3 with \( \zeta = \frac{n}{n-1} \), Lemma 4.2 with \( \eta = \frac{n-1}{n} \), Lemma 4.1 (all three with \( d = d_0(n) + 1 \)), and Corollary 4 with \( \alpha = \frac{n-3}{4(n-1)} \) to conclude

\[
2^{-2\gamma j} \| \Delta_j u \|_{L^2_t L^2_x} \leq C \sum_{j,l,m} 2^{-\gamma j - m} \| \Delta_m g \|_{L^2_t L^2_x} \leq C \sum_{m} 2^{-\gamma j - m} \| \Delta_m g \|_{L^2_t L^2_x},
\]

for \( \epsilon = \min_{i=1}^{3} \epsilon_i < \min \{ n/2, \kappa_1 \} - \frac{1}{n-1}, \) which is the desired frequency-localized version of the endpoint Strichartz estimate.

The above can be used to obtain an endpoint Strichartz estimate with derivatives, as follows: Multiply both sides of (47) by \( 2^{j(\gamma - \frac{n-3}{2(n-1)})} \), then square both sides, sum over \( j \), and change the order of summation on the right, to get

\[
\sum_j 2^{2\gamma j} \| \Delta_j u \|_{L^2_t L^2_x}^2 \leq C \sum_{m} 2^{2(\gamma - 1)m} \| \Delta_m g \|_{L^2_t L^2_x}^2,
\]

where \( \sigma = \gamma - \frac{n+1}{2(n-1)} \). This holds provided \( |\gamma - \frac{n-3}{2(n-1)}| < \epsilon \).

For the cases \( n = 2, 3 \), once again appealing to Lemma 3.1 we use Strichartz estimate for the free wave equation to obtain

\[
2^{\sigma j} ||\hat{W}(t)\int W(-s)J_{jk}M_{kl}W_a(s)N_{lm}\Delta_m g ds||_{L^p L^q} \leq C ||\int W(-s)G(s)ds||_{H^{1/2}},
\]

where \( G := J_{jk}M_{kl}W_a(s)N_{lm}\Delta_m g \) and \( p, q, \sigma \) are as in (48) with \( \gamma = 1/2 \). Once again, we use the dual to (32) with \( \alpha = 1/4 \) and \( a = 0 \) to obtain

\[
||\int W(-s)G(s)ds||_{H^{1/2}} \leq C ||G||_{L^2(\mathbb{R}^{n+1})}
\]

Next we use Lemma 3.3 with \( \zeta = 1 \), Lemma 4.2 with \( \eta = 1 \), and (32) with \( \alpha = 1/4 \), all three with \( d = d_0(n) + 1 \) to arrive at

\[
||G||_{L^2(\mathbb{R}^{n+1})} \leq C 2^{-(\epsilon j - m)} \| P_{\alpha}^{-1/4} N_{lm} \Delta_m g \|_{L^2}
\]

and from here the proof proceeds as in the case \( n \geq 4 \) and we obtain

\[
2^{\sigma j} \| \Delta_j u \|_{L^p L^q} \leq C \sum_{m} 2^{-\gamma j - m} \| \Delta_m g \|_{L^2_t L^2_x}
\]

Multiplying by \( 2^{j(\gamma - 1/2)} \) and carrying on as above we get

\[
\sum_j 2^{2\sigma' j} \| u \|_{L^p L^q}^2 \leq C \sum_m 2^{2m(\gamma - 1)} \| g \|_{L^2}^2
\]

where \( \sigma' = \gamma + \frac{1}{p} - n(\frac{1}{2} - \frac{1}{q}) \). This holds as long as \( |\gamma - \frac{1}{2}| < \epsilon \).

As before, combining the above with Theorem 6 we can deduce generalized Strichartz estimates for the wave equation (2).
THEOREM 9 For $n \geq 2$ let $(p,q,\gamma,\sigma)$ be as in \cite{26}. There exists a constant $C > 0$ depending on $n, a, p, q, \gamma$ such that the solution $u$ of \eqref{2} satisfies
\begin{equation}
\|(-\Delta)^{\sigma/2} u\|_{L^p L^q} \leq C(\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}})
\end{equation}
provided
\[-\min\left\{ \frac{n-1}{2}, \nu_1 - \frac{1}{2}, 1 + \nu_0 \right\} < \gamma < \min\left\{ \frac{n+1}{2}, \nu_1 + \frac{1}{2}, 1 + \nu_0 - \frac{1}{p} \right\}\]
if $n = 2, 3$ and
\[-\min\left\{ \frac{n}{2} - \frac{n + 3}{2(n-1)}, \nu_1 - \frac{n + 3}{2(n-1)}, 1 + \nu_0 \right\} < \gamma < \min\left\{ \frac{n+1}{2}, \nu_1 + \frac{1}{2}, 1 + \nu_0 - \frac{1}{p} \right\}\]
if $n \geq 4$.

We end this section by giving a nonlinear application for the above estimates. In \cite{26} the authors study the following equation,
\begin{equation}
\Box u + V(x) u = g(x, u),
\end{equation}
where $V(x) = C/|x|^{2-\delta}$ and $g$ behaves like $|u|^\kappa$ for some $\kappa > 1$. When $C = 0$ (no potential term) the behavior of global solutions of small amplitude depend on whether $\kappa$ is larger than a critical value $\kappa_c$ (\cite{10}). In the range $\kappa > \kappa_c$ no blow-up occurs. However, adding a potential term can affect the solution, leading to blow-up in finite time. Essentially in \cite{26} blow-up is proved when $\delta > 0$ for $\kappa > 1$ while global existence of smooth solutions is proved when $\delta < 0$ for $\kappa > \kappa_c$ and $|C|$ small.

The inverse-square potential corresponds to the borderline case $\delta = 0$. Consider now the following Cauchy problem
\begin{equation}
\begin{cases}
\partial_t^2 u + P_a u = \pm |u|^\kappa, \\
u(0, x) = u_0(x) \\
\partial_t u(0, x) = u_1(x)
\end{cases}
\end{equation}
for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, where $P_a = -\Delta + a|x|^2$ with $a > -\lambda(n)^2$ (See \cite{19} for the precise meaning of this operator when $-\lambda^2 < a < 1 - \lambda^2$). Set $\varsigma := \frac{n}{\kappa - 1}$. A simple computation shows the equation \eqref{50} to be invariant under the scaling $u_{\theta} = \theta^\varsigma u(\theta x, \theta t)$. This suggests that the equation should be well-posed at the critical level $\dot{H}^{s_c}$ where $s_c = \frac{n}{2} - \varsigma$, as the $\dot{H}^{s_c}$ norm of $u$ is left unchanged by this rescaling. Indeed, for the usual wave equation with no potential, well-posedness holds for all $s \geq s_c$ when $s_c \geq \frac{1}{2}$, or equivalently $\kappa \geq \frac{n+3}{n-1}$ (see \cite{25}, \cite{13} and \cite{15}). In \cite{19} a similar result was shown to hold for \eqref{50} under the additional assumption that the initial data are radial. This assumption was made only because the linear estimates on which the proof was based were available only in the radial case. The proof of global wellposedness itself was based on standard iteration arguments and did not use the assumption of spherical symmetry in any way. Using the generalized Strichartz estimate \cite{18} that are now available to us we can remove the assumption of spherical symmetry from this wellposedness result.
THEOREM 10 Let \( n \geq 2, \kappa \geq \frac{n+3}{n-1}, s_c := \frac{n}{2} - \frac{2}{\kappa-1} \). Suppose \( a \in \mathbb{R} \) is such that
\[
\sqrt{a + \lambda(n)^2} > \lambda(n) - \frac{2}{\kappa-1} + \max \left\{ \frac{1}{2\kappa}, \frac{2}{(n+1)(\kappa-1)} \right\}.
\]
Let \((u_0, u_1) \in (\dot{H}^{s_c}, \dot{H}^{s_c-1})\) be functions with small norms. Then there exists a unique global solution to (50) such that
\[
u(x, t) \in C_t(\dot{H}^{s_c}) \cap L^\sigma_t(\dot{H}^{\alpha q}_q), \quad \partial_t u(x, t) \in C_t(\dot{H}^{s_c-1}) \cap L^\sigma_t(\dot{H}^{\alpha-1}_q), \quad (51)
\]
where \( q, \sigma, \alpha \) are as follows:

1. For \( n \leq 3 \) or \( \frac{n+3}{n-1} \leq \kappa \leq \frac{n+1}{n-3} \)
\[
\frac{1}{q} = \frac{1}{2} - \frac{4}{(n^2-1)(\kappa-1)}, \quad \frac{1}{\sigma} = \frac{2}{(n+1)(\kappa-1)}, \quad \alpha = s_c - \frac{2}{(n-1)(\kappa-1)}.
\]

2. For \( n \geq 4 \) and \( \kappa > \frac{n+1}{n-3} \)
\[
\frac{1}{q} = \frac{1}{2} - \frac{1}{(n-1)\kappa}, \quad \frac{1}{\sigma} = \frac{1}{2\kappa}, \quad \alpha = s_c - \frac{n+1}{2(n-1)\kappa}.
\]

Proof: This result is proved by a contraction mapping argument. A sequence of Picard iterates is constructed in the function space
\[
\mathcal{E} = C_t(\dot{H}^{s_c}) \cap L^\sigma_t(\dot{H}^{\alpha q}_q).
\]
The nonlinearity in (50) maps \( \mathcal{E} \) into another suitably chosen space \( \mathcal{F} \), and the particular choice of the parameters ensures, via the Strichartz estimate (48), that \((-\partial_t^2 + P_0)^{-1}\) maps \( \mathcal{F} \) back into \( \mathcal{E} \). It is then easy to show that for data of sufficiently small Sobolev norm, this will be a contraction mapping. See [19] for details.

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