QUANTUM SOLVABLE MODELS WITH NONLOCAL
ONE POINT INTERACTIONS

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Abstract. Within the framework of quantum mechanics working with one-dimensional, manifestly non-Hermitian Hamiltonians $H = -\frac{d^2}{dx^2} + V$ the traditional class of the exactly solvable models with local point interactions $V = V(x)$ is generalized and studied. The consequences of the use of the nonlocal point interactions such that $(Vf)(x) = \int K(x,s)f(s)ds$ are discussed using the suitably adapted formalism of boundary triplets.

1. Introduction

The authors of introductory textbooks on Quantum Mechanics have to combine a persuasive survey of its heuristics (involving, e.g., the explanation of the so called principle of correspondence) and applicability (say, to hydrogen atom) with a maximally compact presentation of the underlying mathematics. This means that a more advanced understanding of the theory proceeds, typically, either beyond the naive forms of the classical-quantum correspondence, or beyond the oversimplified usage of the underlying language of functional analysis.

Both of these tendencies appeared re-unified after the mind-boggling discovery [1] - [3] of the existence of certain rather anomalous one-dimensional Schrödinger operators

$$H = -\frac{d^2}{dx^2} + V(x)$$

in Hilbert space $L_2(\mathbb{R})$ which appeared to possess real spectra and to support stable bound states in spite of being manifestly non-self-adjoint.

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The existence of such an apparent puzzle encouraged an intensification of the study of similar non-self-adjoint operators which led, recently, to its more or less satisfactory clarification (cf., e.g., the mathematically oriented collection of reviews [1] of the current situation in the field). *A priori*, it is not too surprising that the reliable physical interpretation of the manifestly non-self-adjoint bound-state models (1.1) may prove mathematically deeply nontrivial [5].

In the context of the non-self-adjoint-operator phenomenology serious difficulties emerged in the scattering dynamical regime [6]. In this regime the (naturally, highly desirable!) unitarity of the evolution can only be guaranteed *after* a replacement of the local forces in (1.1) by their suitable *non-local-interaction* generalizations [7].

In such a situation one is exposed to the necessity of a *simultaneous*, *viz.*, non-self-adjoint and nonlocal generalization of interactions.

In the present paper we study non-self-adjoint Schrödinger operators with nonlocal one point interactions. Such kind of new solvable models with point interactions has recently been proposed and studied (for self-adjoint case) by S. Albeverio and L. Nizhnik [8] (see also [9] - [13]). Our interest to the non-self-adjoint case was inspired in part by an intensive development of Pseudo-Hermitian (PT-Symmetric) Quantum Mechanics PHQM (PTQM) during last decades [14]–[16].

Non-self-adjoint point-interaction solvable models (see, e.g., [17]–[19]) require more detailed analysis in comparison with their self-adjoint counterparts. In contrast to the self-adjoint case [20], one should illustrate a typical PHQM/PTQM evolution of spectral properties which can be obtained by changing parameters of the model: complex eigenvalues → spectral singularities / exceptional points → similarity to a self-adjoint operator. One of the simplest examples of such kind is the well-studied δ-interaction model \(-d^2/dx^2 + a < \delta \cdot > \delta(x)\) with complex parameter \(a \in \mathbb{C}\) (see [21], [22] or section 6 below). However, this model seems to be sufficiently trivial due to the very simple structure of the singular potential that leads to ‘poor’ spectral properties of the corresponding operator-realizations \(H_a\) (for instance, \(H_a\) have no exceptional points and bound states on continuous spectrum).

One of possible ‘reasonable complication’ of the model consists in the addition of the nonlocal interaction term \(\int_{-\infty}^{\infty} K(x, s)f(s)ds\). Trying to keep the solvability of the model and its intimate relationship with
δ-interaction, we assume that
\[ K(x, s) = q(x)\delta(s) + \delta(x)q^*(s), \]
where \( q \in L_2(\mathbb{R}) \) is a given piecewise continuous function. The corresponding nonlocal δ-interaction
\[ -\frac{d^2}{dx^2} + a < \delta, \cdot > \delta(x) + < \delta, \cdot > q(x) + (q, \cdot)\delta(x), \quad a \in \mathbb{C}, \quad (1.2) \]
where \((\cdot, \cdot)\) is the inner product in \( L_2(\mathbb{R}) \) linear in the second argument, is studied in Section 5 with the use of boundary triplet technique (see the Appendix). Namely, the formal expression (1.2) gives rise to the family of operators \( \{H_a\} \):
\[ H_a f = -\frac{d^2 f}{dx^2} + f(0)q(x), \quad a \in \mathbb{C}, \quad q \in L_2(\mathbb{R}) \]
with domains of definition (5.3) which are determined by the singular part of perturbation \( a < \delta, \cdot > \delta(x) + (q, \cdot)\delta(x) \) in (1.2). Our investigation of \( \{H_a\} \) is based on the fact that each operator \( H_a \) is the proper extension of the symmetric operator \( \tilde{S}_{\min} \) (5.5), i.e., \( \tilde{S}_{\min} \subset H_a \subset \tilde{S}_{\max} \), where \( \tilde{S}_{\max} = \tilde{S}_{\min}^\dagger \) is the adjoint of \( \tilde{S}_{\min} \), see section 5.1.

We show that spectral properties of \( H_a \) are completely characterized by the pair \( \{a, \tilde{W}_\lambda\} \), where \( a \in \mathbb{C} \) distinguishes \( H_a \) among all proper extensions of \( \tilde{S}_{\min} \), while the Weyl-Titchmarsh function \( \tilde{W}_\lambda \) (5.10) characterizes the symmetric operator \( \tilde{S}_{\min} \) which is ‘the common part’ of all \( H_a \); see Theorems 5.1, 5.4, and 5.6.

One of interesting features of the model is fact that \( a \in \mathbb{C} \) determines the measure of non-self-adjointness of the operators \( H_a \), while the choice of \( q \) defines the symmetric operator \( \tilde{S}_{\min} \) and, therefore, the structure of the holomorphic function \( \tilde{W}_\lambda \). Such ‘a separation of responsibility’ of parameters of the model allows one to preserve its solvability and illustrate the possible appearance of exceptional points and eigenvalues on continuous spectrum, see Example 5.3 and subsec. 6.

The proposed approach to the construction of non-self-adjoint nonlocal point interaction models is not restricted to the case of δ-interactions only and it can be applied to the wider class of ordinary point interaction models. We illustrate this point in sections 2–4 which are devoted to general case of one point interactions including combinations of δ- and δ'-interactions.

Throughout the paper, \( \mathcal{D}(H) \), \( \mathcal{R}(H) \), and \( \ker H \) denote the domain, the range, and the null-space of a linear operator \( H \), respectively, while \( H \upharpoonright_D \) stands for the restriction of \( H \) to the set \( D \). The adjoint of \( H \)
with respect to the natural inner product $(\cdot, \cdot)$ (linear in the second argument) in $L_2(\mathbb{R})$ is denoted by $H^\dagger$.

2. One point interactions

2.1. Ordinary one point interactions. A one-dimensional Schrödinger operator with interactions supported at the point $x = 0$ can be defined by the formal expression

$$\frac{d^2}{dx^2} + a < \delta, \cdot > \delta(x) + b < \delta', \cdot > \delta'(x) + c < \delta, \cdot > \delta'(x) + d < \delta', \cdot > \delta'(x),$$

(2.1)

where $\delta$ and $\delta'$ are, respectively, the Dirac $\delta$-function and its derivative, the parameters $a, b, c, d$ are complex numbers, and

$$< \delta, f > = f(0), \quad < \delta', f > = -f'(0), \quad \forall f \in W_2^2(\mathbb{R}).$$

Denote $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then (2.1) can be rewritten in more compact form

$$-\frac{d^2}{dx^2} + [\delta, \delta']T \begin{bmatrix} < \delta, \cdot > \\ < \delta', \cdot > \end{bmatrix}.$$

(2.2)

The expression (2.2) determines the symmetric (non-self-adjoint) operator

$$S = -\frac{d^2}{dx^2}, \quad \mathcal{D}(S) = \{ f \in W_2^2(\mathbb{R}) : f(0) = f'(0) = 0 \},$$

in $L_2(\mathbb{R})$, which does not depend on the choice of $a, b, c, d$. In order to take into account the impact of these parameters, we should extend the action of $\delta$ and $\delta'$ onto $W_2^2(\mathbb{R}\setminus\{0\})$. The most natural way is

$$< \delta, f > := f_r(0) = \frac{f(0^+) + f(0^-)}{2}, \quad < \delta', f > := -f'_r(0) = -\frac{f'(0^+) + f'(0^-)}{2}.$$

Furthermore, we assume that the second derivative in (2.2) acts on $W_2^2(\mathbb{R}\setminus\{0\})$ in the distributional sense, that is

$$-f'' = -\{ f''(x) \}_{x \neq 0} - f_s(0)\delta'(x) - f'_s(0)\delta(x), \quad f \in W_2^2(\mathbb{R}\setminus\{0\}),$$

where

$$f_s(0) = f(0^+) - f(0^-), \quad f'_s(0) = f'(0^+) - f'(0^-).$$

Then, the action of (2.2) on functions $f \in W_2^2(\mathbb{R}\setminus\{0\})$ can be represented as follows:

$$-\{ f''(x) \}_{x \neq 0} + [\delta, \delta'][t \Gamma_0 f - \Gamma_1 f],$$

(2.3)
where

\[
\Gamma_0 f = \begin{bmatrix} \langle \delta, f \rangle \\ \langle \delta', f \rangle \end{bmatrix} = \begin{bmatrix} f_r(0) \\ -f'_r(0) \end{bmatrix}, \quad \Gamma_1 f = \begin{bmatrix} f'_s(0) \\ f_s(0) \end{bmatrix}.
\]

Obviously, (2.3) determines a function from \(L^2(\mathbb{R})\) if and only if \(T \Gamma_0 f = \Gamma_1 f\). Therefore, the expression (2.1) gives rise to the operator \(-d^2/dx^2\) in \(L^2(\mathbb{R})\) with the domain of definition \(\{f \in W^2_2(\mathbb{R}\{0\}) : T \Gamma_0 f - \Gamma_1 f = 0\}\).

2.2. **Nonlocal one point interactions.** Let us generalize the one point interactions potential considered in (2.1) by adding a nonlocal point interaction part

\[
\langle \delta, \cdot \rangle q_1(x) + (q_1, \cdot)\delta(x) + (q_2, \cdot)\delta'(x) + \langle \delta', \cdot \rangle q_2(x),
\]

where functions \(q_j \in L^2(\mathbb{R})\) are assumed to be piecewise continuous and \((\cdot, \cdot)\) is the standard inner product (linear in the second argument) of \(L^2(\mathbb{R})\). Then the generalization of (2.2) takes the form

\[
-\frac{d^2}{dx^2} + [\delta, \delta']\left(T\begin{bmatrix} \langle \delta, \cdot \rangle \\ \langle \delta', \cdot \rangle \end{bmatrix} + \begin{bmatrix} (q_1, \cdot) \\ (q_2, \cdot) \end{bmatrix}\right) + [q_1, q_2]\begin{bmatrix} \langle \delta, \cdot \rangle \\ \langle \delta', \cdot \rangle \end{bmatrix}.
\]

(2.4)

Extending, by analogy with (2.2), the action of (2.4) onto \(W^2_2(\mathbb{R}\{0\})\) we obtain

\[
-\{f''(x)\}_{x \neq 0} + [\delta, \delta'][T \Gamma_0 f - \Gamma_1 f] + [q_1, q_2]\Gamma_0 f,
\]

(2.5)

where

\[
\Gamma_0 f = \begin{bmatrix} \langle \delta, f \rangle \\ \langle \delta', f \rangle \end{bmatrix} = \begin{bmatrix} f_r(0) \\ -f'_r(0) \end{bmatrix}, \quad \Gamma_1 f = \begin{bmatrix} f'_s(0) - (q_1, f) \\ f_s(0) - (q_2, f) \end{bmatrix}.
\]

(2.6)

The expression (2.5) has sense as a function from \(L^2(\mathbb{R})\) if and only if the second term of (2.5) is vanished, i.e., if \(T \Gamma_0 f - \Gamma_1 f = 0\). This means that the formula (2.4) determines the following operator in \(L^2(\mathbb{R})\):

\[
H_T f = -\frac{d^2 f}{dx^2} + [q_1, q_2]\Gamma_0 f = -\{f''(x)\}_{x \neq 0} + f_r(0)q_1(x) - f'_r(0)q_2(x)
\]

(2.7)

with the domain of definition

\[
\mathcal{D}(H_T) = \{f \in W^2_2(\mathbb{R}\{0\}) : (T \Gamma_0 - \Gamma_1)f = 0\}.
\]

(2.8)

The maximal operator in the Hilbert space \(L^2(\mathbb{R})\) that can be determined by (2.4) coincides with

\[
S_{max} f = -\frac{d^2 f}{dx^2} + [q_1, q_2]\Gamma_0 f, \quad f \in \mathcal{D}(S_{max}) = W^2_2(\mathbb{R}\{0\}).
\]

(2.9)
Taking (2.6) into account, we obtain
\[ S_{\text{max}} f = -\{f''(x)\}_{x \neq 0} + f_r(0)q_1 - f'_r(0)q_2. \]

The operator \( S_{\text{max}} \) satisfies the Green’s identity
\[ (S_{\text{max}} f, g) - (f, S_{\text{max}} g) = (\Gamma_1 f) \cdot \Gamma_0 g - (\Gamma_0 f) \cdot \Gamma_1 g, \tag{2.10} \]
where the dot “." in the right hand side means the standard inner product in \( \mathbb{C}^2 \). Moreover, according to [8, Lemma 1], for any vectors \( h_0, h_1 \in \mathbb{C}^2 \), there exists \( f \in \mathcal{D}(S_{\text{max}}) \) such that \( \Gamma_0 f = h_0 \) and \( \Gamma_1 f = h_1 \).

The next operator plays an important role in what follows:
\[ H_{\infty} = S_{\text{max}} \mid_{\mathcal{D}(H_{\infty})}, \quad \mathcal{D}(H_{\infty}) = \{ f \in \mathcal{D}(S_{\text{max}}) : \Gamma_0 f = 0 \}. \tag{2.11} \]

In view of (2.6) and (2.9), \( H_{\infty} = -\frac{d^2 f}{dx^2} \), \( f \in \mathcal{D}(H_{\infty}) = \{ f \in W_2^2(\mathbb{R}\setminus\{0\}) : f_r(0) = f'_r(0) = 0 \} \).

It is easy to check that \( H_{\infty} \) is a positive self-adjoint operator in \( L_2(\mathbb{R}) \).

Due to [23, Corollary 2.5], the self-adjointness of \( H_{\infty} \), the Green identity (2.10), and the surjectivity of the mapping \( (\Gamma_0, \Gamma_1) : \mathcal{D}(S_{\text{max}}) \to \mathbb{C}^2 \oplus \mathbb{C}^2 \) lead to the conclusion that the operator \( S_{\text{min}} = S_{\text{max}} \mid_{\mathcal{D}(S_{\text{min}})} \) with the domain of definition \( \mathcal{D}(S_{\text{min}}) = \{ f \in \mathcal{D}(S_{\text{max}}) : \Gamma_0 f = \Gamma_1 f = 0 \} \) is a closed symmetric operator in \( L_2(\mathbb{R}) \). Precisely, \( S_{\text{min}} f = -\frac{d^2 f}{dx^2} \) with the domain
\[ \mathcal{D}(S_{\text{min}}) = \left\{ f \in W_2^2(\mathbb{R}\setminus\{0\}) : \begin{array}{ll} f_r(0) = 0 & f_s(0) = (q_2, f) \\ f'_r(0) = 0 & f'_s(0) = (q_1, f) \end{array} \right\}. \tag{2.12} \]

Moreover, the relation \( S_{\text{min}}^* = S_{\text{max}} \) holds and the collection \( (\mathbb{C}^2, \Gamma_0, \Gamma_1) \) is a boundary triplet\(^2\) of \( S_{\text{max}} \). The latter property is especially important because operators \( H_T \), are intermediate extensions between \( S_{\text{min}} \) and \( S_{\text{max}} \) and their domains of definition are determined in terms of boundary operators \( \Gamma_j \), see (2.8). Therefore, the well developed methods of boundary triplet theory [24] can be applied for the investigation of \( H_T \).

3. Special cases of nonlocal one point interactions

3.1. Self-adjoint nonlocal one point interactions.

**Lemma 3.1.** If the entries of \( T \) satisfy the conditions \( a, d \in \mathbb{R}, b = c^* \), then the corresponding operator \( H_T \) defined by (2.7) is self-adjoint in \( L_2(\mathbb{R}) \) for any choice of \( q_j \in L_2(\mathbb{R}) \).

\(^1\)since \( (H_{\infty} f, f) = \int_{\mathbb{R}} |f'(x)|^2 dx > 0 \) for nonzero \( f \in \mathcal{D}(H_{\infty}) \)
\(^2\)see the Appendix
Proof. It follows from the theory of boundary triplets (see the Appendix) that $H_T^\dagger = H_T^*$, where $T^\dagger = (T^*)^t$. Therefore, $H_T$ is a self-adjoint operator if and only if the matrix $T$ is Hermitian. The latter is equivalent to the conditions $a, d \in \mathbb{R}$, $b = c^*$.

3.2. $\mathcal{PT}$-symmetric nonlocal one point interactions. As usual [14] we consider the space parity operator $P f(x) = f(-x)$ and the conjugation operator $\mathcal{T} f = f^\ast$. An operator $H$ acting in $L_2(\mathbb{R})$ is called $\mathcal{PT}$-symmetric if $\mathcal{PT} H = H \mathcal{PT}$.

Lemma 3.2. If the entries of $T$ and the functions $q_j$ satisfy the conditions

$$a, d \in \mathbb{R}, \quad b, c \in i\mathbb{R}, \quad \mathcal{PT} q_1 = q_1, \quad \mathcal{PT} q_2 = -q_2, \quad (3.1)$$

then the corresponding operator $H_T$ defined by (2.7) is $\mathcal{PT}$-symmetric.

Proof. It is easy to check that, for any $f \in W^2_2(\mathbb{R}\setminus \{0\})$,

$$(\mathcal{P} f)_r(0) = f_r(0), \quad (\mathcal{P} f)_s(0) = -f_s(0), \quad (\mathcal{P} f)'_r(0) = -f'_r(0), \quad (\mathcal{P} f)'_s(0) = f'_s(0).$$

These relations, the definition (2.6) of $\Gamma_j$, and (3.1) lead to the conclusion that

$$\Gamma_j \mathcal{PT} f = \sigma_3 \mathcal{T} \Gamma_j f, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad j = 0, 1. \quad (3.2)$$

Therefore, if (3.1) holds, then the operator $S_{\text{max}}$ defined by (2.9) is $\mathcal{PT}$-symmetric

$$\mathcal{PT} S_{\text{max}} f = -\frac{d^2}{dx^2} \mathcal{PT} f + [q_1, q_2] \sigma_3 \mathcal{T} \Gamma_0 f = S_{\text{max}} \mathcal{PT} f.$$

Since $H_T$ is the restriction of $S_{\text{max}}$ onto $\mathcal{D}(H_T)$, the invariance of $\mathcal{D}(H_T)$ with respect to $\mathcal{PT}$ will guarantee the $\mathcal{PT}$-symmetry of $H_T$.

Let us prove that $\mathcal{PT} : \mathcal{D}(H_T) \to \mathcal{D}(H_T)$. To do that, we consider an arbitrary $f \in \mathcal{D}(H_T)$. Then, according to (2.8), $\Gamma_0 f = \Gamma_1 f$ and the inclusion $\mathcal{PT} f \in \mathcal{D}(H_T)$ is equivalent to the condition $\mathcal{T} \Gamma_0 \mathcal{PT} f = \Gamma_1 \mathcal{PT} f$. By virtue of (3.2), $\Gamma_0 \mathcal{PT} f = \mathcal{T} \sigma_3 \mathcal{T} \Gamma_0 f$ and

$$\Gamma_1 \mathcal{PT} f = \sigma_3 \mathcal{T} \Gamma_1 f = \sigma_3 \mathcal{T} \mathcal{T} \Gamma_0 f = \sigma_3 \mathcal{T}^* \mathcal{T} \Gamma_0 f.$$

This means that the required identity $\mathcal{T} \Gamma_0 \mathcal{PT} f = \Gamma_1 \mathcal{PT} f$ is true if and only if $\mathcal{T} \sigma_3 = \sigma_3 \mathcal{T}^*$. The latter matrix relation holds if the entries of $T$ satisfy (3.1). ■

\footnote{The same symbol $\mathcal{T}$ are used for the conjugation operators in $L_2(\mathbb{R})$ and in $\mathbb{C}^2$.}
3.3. \( P \)-self-adjoint nonlocal one point interactions. An operator \( H_T \) defined by (2.7) is called \( P \)-self-adjoint if \( PH_T = H_T^\dagger P \).

**Lemma 3.3.** If the entries of \( T \) and the functions \( q_j \) satisfy the conditions

\[
a, d \in \mathbb{R}, \quad b = -c^*, \quad Pq_1 = q_1, \quad Pq_2 = -q_2, \quad (3.3)
\]

then the operator \( H_T \) is \( P \)-self-adjoint.

**Proof.** Similarly to the proof of Lemma 3.2 we check that \( \Gamma_j Pf = \sigma_3 \Gamma_j f \) and show that the conditions (3.3) ensure the commutation relation \( S_{max}P = PS_{max} \).

The operators \( H_T \) and \( H_T^\dagger \) are restrictions of \( S_{max} \). Therefore, the condition \( P : \mathcal{D}(H_T) \to \mathcal{D}(H_T^\dagger) \) means the identity \( PH_T = H_T^\dagger P \).

Let us verify that \( P : \mathcal{D}(H_T) \to \mathcal{D}(H_T^\dagger) \). Since \( H_T^\dagger = H_T^* \), the domains of definition \( \mathcal{D}(H_T) \) and \( \mathcal{D}(H_T^\dagger) \) are determined by (2.8) with the matrices \( T \) and \( T^* \), respectively. Let \( f \in \mathcal{D}(H_T) \). Then \( T\Gamma_0 f = \Gamma_1 f \) and the inclusion \( Pf \in \mathcal{D}(H_T^\dagger) \) is equivalent to the condition \( T^*\Gamma_0 Pf = \Gamma_1 Pf \).

Taking into account that \( \Gamma_j Pf = \sigma_3 \Gamma_j f \), we obtain \( T^*\Gamma_0 Pf = T^*\sigma_3 \Gamma_1 f \) and \( \Gamma_1 Pf = \sigma_3 \Gamma_1 f = \sigma_3 T\Gamma_0 f \). Hence, \( T^*\Gamma_0 Pf = \Gamma_1 Pf \) holds if and only if \( T^*\sigma_3 = \sigma_3 T \). This matrix relation holds if the entries \( a, b, c, d \) of \( T \) satisfy (3.3).

4. Spectral Analysis of \( H_T \)

The relations (2.7), (2.8) lead to the conclusion that operators \( H_T \) are finite rank perturbations of the self-adjoint operator \( H_\infty \) defined by (2.11). The spectrum of \( H_\infty \) is purely continuous and it coincides with \([0, \infty)\). This means that the continuous spectrum of each \( H_T \) coincides with \([0, \infty)\) and only eigenvalues of \( H_T \) may appear in \( \mathbb{C} \setminus [0, \infty) \).

An eigenfunction of \( H_T \) should be the eigenfunction of \( S_{max} \) corresponding to the same eigenvalue (since \( S_{max} \) is an extension of \( H_T \)).

The kernel subspace \( \ker(S_{max} - \lambda I) \) has the dimension 2 for any choice of \( \lambda \in \mathbb{C} \setminus [0, \infty) \). Let \( u_\lambda, v_\lambda \) be a basis of \( \ker(S_{max} - \lambda I) \). Then, any \( f \in \ker(S_{max} - \lambda I) \) has the form \( f = c_1 u_\lambda + c_2 v_\lambda \) and \( f \) turns out to be the eigenfunction of \( H_T \) corresponding to the eigenvalue \( \lambda \) if and only if \( f \) belongs to the domain \( \mathcal{D}(H_T) \) determined by (2.8), i.e., if \( c_1, c_2 \) are nonzero solutions of the linear system

\[
c_1(T\Gamma_0 - \Gamma_1)u_\lambda + c_2(T\Gamma_0 - \Gamma_1)v_\lambda = 0.
\]
Therefore, the eigenvalues \( \lambda \in \mathbb{C} \setminus [0, \infty) \) of \( H_T \) coincide with the roots of the characteristic equation
\[
\det((T \Gamma_0 - \Gamma_1)u_\lambda, (T \Gamma_0 - \Gamma_1)v_\lambda) = 0. \tag{4.1}
\]

Let us assume, without loss of generality, that the eigenfunctions \( u_\lambda, v_\lambda \) are chosen in such a way that \( \Gamma_0 u_\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \Gamma_0 v_\lambda = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

Then the characteristic equation (4.1) for the determination of eigenvalues of \( H_T \) takes the form
\[
\det(T - W_\lambda) = 0, \tag{4.2}
\]
where \( 2 \times 2 \)-matrix \( W_\lambda = [\Gamma_1 u_\lambda, \Gamma_1 v_\lambda] \) is called the Weyl-Titchmarsh function associated to the boundary triplet \((\mathbb{C}^2, \Gamma_0, \Gamma_1)\). The Weyl-Titchmarsh function \( W_\lambda \) is holomorphic on \( \mathbb{C} \setminus [0, \infty) \) and it satisfies the relation \( (W_\lambda^*)^T = W_\lambda \) (see the Appendix).

4.1. Eigenfunctions of \( S_{\text{max}} \). Let us write any \( \lambda \in \mathbb{C} \setminus [0, \infty) \) as \( \lambda = k^2 \), where \( k \in \mathbb{C}_+ = \{k \in \mathbb{C} : \text{Im } k > 0\} \) and consider the function
\[
G(x) = \frac{i}{2k} e^{i k |x|}.
\]
Obviously, \( G(\cdot) \) belongs to \( W^2_2(\mathbb{R} \setminus \{0\}) \) and
\[
-G'' - k^2 G = 0, \quad -(G')'' - k^2 G' = 0, \quad x \neq 0.
\]

Moreover,
\[
G_r(0) = \frac{i}{2k}, \quad G'_r(0) = 0, \quad G''_r(0) = -\frac{i k}{2}, \quad G_s(0) = 0, \quad G'_s(0) = -1, \quad G''_s(0) = 0.
\]

The convolution
\[
f = (G \ast q)(x) = \int_{-\infty}^{\infty} G(x - s)q(s) \, ds
\]
\( (q \in L_2(\mathbb{R}) \) is a piecewise continuous function) is the solution of the differential equation \(-f'' - k^2 f = q \) in \( L_2(\mathbb{R}) \).

**Lemma 4.1.** The functions
\[
u(x) = -(G \ast q_2)(x) - 2ik[1 + (G \ast q_1)(0)]G(x) + \frac{2i}{k}(G' \ast q_1)(0)G'(x)
\]
\[
v(x) = -(G \ast q_2)(x) - 2ik(G \ast q_2)(0)G(x) - \frac{2i}{k}[1 - (G' \ast q_2)(0)]G'(x)
\]
form the basis of the eigenfunction subspace \( \ker(S_{\text{max}} - k^2 I) \).
Proof. An elementary analysis shows that the functions \( u, v \) belong to \( W^2_2(\mathbb{R}\setminus\{0\}) \) and

\[
\begin{align*}
u_r(0) &= 1, \quad u_s(0) = -\frac{2i}{k}(G' * q_1)(0), \quad v_r(0) = 0, \quad v_s(0) = \frac{2i}{k}[1 - (G' * q_2)(0)] \\
u'_r(0) &= 0, \quad u'_s(0) = 2ik[1 + (G' * q_1)(0)], \quad v'_r(0) = -1, \quad v'_s(0) = 2ik(G' * q_2)(0)
\end{align*}
\] (4.3)

The first and the third columns in (4.3) mean that \( u, v \) are linearly independent and \( \Gamma_0 u = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \), \( \Gamma_0 v = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \). Furthermore, taking into account (2.9) and (4.3) we obtain for almost all \( x \in \mathbb{R} \)

\[
(S_{\text{max}} - k^2 I)u = -u'' - k^2 u + q_1 = -q_1 + q_1 = 0.
\]
Similarly, \( (S_{\text{max}} - k^2 I)v = -v'' - k^2 v + q_2 = -q_2 + q_2 = 0 \). Hence, the functions \( u, v \) belong to \( \text{ker}(S_{\text{max}} - k^2 I) \) and they form a basis of this subspace. \( \blacksquare \)

4.2. **The Weyl-Titchmarsh function associated to** \( (\mathbb{C}^2, \Gamma_0, \Gamma_1) \).

Since \( \Gamma_0 u = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \) and \( \Gamma_0 v = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \), the Weyl-Titchmarsh function associated to \( (\mathbb{C}^2, \Gamma_0, \Gamma_1) \) has the form \( W_\lambda = [\Gamma_1 u, \Gamma_1 v] \), where, in view of (2.6) and (4.3),

\[
\Gamma_1 u = \left[ \begin{array}{c} 2ik[1 + (G' * q_1)(0)] - (q_1, u) \\ -\frac{2i}{k}(G' * q_1)(0) - (q_2, u) \end{array} \right], \quad \Gamma_1 v = \left[ \begin{array}{c} 2ik(G' * q_2)(0) - (q_1, v) \\ \frac{2i}{k}[1 - (G' * q_2)(0)] - (q_2, v) \end{array} \right].
\]

Making some additional rudimentary calculations (mainly related to the calculation of scalar products \( (q, u), (q, v) \) for functions \( u, v \) from Lemma 4.1), we obtain

\[
W_\lambda = \left[ \begin{array}{cc} (q_1, G' * q_1) & (q_1, G' * q_2) \\ (q_2, G' * q_1) & (q_2, G' * q_2) \end{array} \right] + \left[ \begin{array}{cc} r_{11} & r_{12} \\ r_{21} & r_{22} \end{array} \right],
\]

(4.4)

where

\[
\begin{align*}
r_{11} &= 2i k[1 + (G' * q_1)(0)][1 + (G' * q_1^*)(0)] + \frac{2i}{k}(G' * q_1)(0)(G' * q_1^*)(0), \\
r_{22} &= \frac{2i}{k}[1 - (G' * q_2)(0)][1 - (G' * q_2^*)(0)] + 2ik(G' * q_2)(0)(G' * q_2^*)(0), \\
r_{12} &= 2i k(G' * q_2)(0)[1 + (G' * q_1^*)(0)] - \frac{2i}{k}(G' * q_1^*)(0)[1 - (G' * q_2)(0)], \\
r_{21} &= 2i k(G' * q_2^*)(0)[1 + (G' * q_1)(0)] - \frac{2i}{k}(G' * q_1)(0)[1 - (G' * q_2^*)(0)].
\end{align*}
\]

Denote

\[
B_{q_1, q_2} = \left[ \begin{array}{cc} 1 + (G' * q_1)(0) & (G' * q_2)(0) \\ -(G' * q_1^*)(0) & 1 - (G' * q_2^*)(0) \end{array} \right].
\]
Then (4.4) can be rewritten as follows:

$$W_\lambda = \begin{pmatrix} (q_1, G \ast q_1) & (q_1, G \ast q_2) \\ (q_2, G \ast q_1) & (q_2, G \ast q_2) \end{pmatrix} + B_{q_1, q_2} \begin{bmatrix} 2ik & 0 \\ 0 & 2i/k \end{bmatrix} B_{q_1, q_2}. \quad (4.5)$$

Substituting (4.5) into (4.2) we obtain the characteristic equation for eigenvalues $\lambda \in \mathbb{C}\setminus[0, \infty)$ of $H_T$. In particular, if $q_1 = q_2 = 0$, the Weyl function $W_\lambda$ coincides with $\begin{bmatrix} 2ik & 0 \\ 0 & 2i/k \end{bmatrix}$ and the equation (4.2) is transformed to the polynomial

$$2dk^2 + ik(\det T - 4) + 2a = 0, \quad (4.6)$$

which determines spectra of ordinary point interactions considered in subsection 2.1.

5. Nonlocal $\delta$-interaction

5.1. Definition and description of eigenvalues. The classical one point $\delta$-interaction is given by the formal expression

$$-\frac{d^2}{dx^2} + a < \delta, \cdot > \delta(x), \quad a \in \mathbb{C} \quad (5.1)$$

It is natural to suppose that the generalization of (5.1) to the nonlocal case consists in the addition of the nonlocal part $< \delta, \cdot > q(x) + (q, \cdot)\delta(x)$ of $\delta$-interaction. For this reason, a nonlocal one-point $\delta$-interaction can be defined via the formal expression

$$-\frac{d^2}{dx^2} + a < \delta, \cdot > \delta(x) + < \delta, \cdot > q(x) + (q, \cdot)\delta(x), \quad a \in \mathbb{C}, \ q \in L^2(\mathbb{R}),$$

which is a particular case of (2.4) with $T = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \ q_1 = q, \ q_2 = 0$. This means that the corresponding operator $H_T \equiv H_a$ defined by (2.7) and (2.8) acts as

$$H_a f = -\frac{d^2 f}{dx^2} + f_r(0)q(x), \quad (5.2)$$

on the domain of definition

$$\mathcal{D}(H_a) = \left\{ f \in W^2_2(\mathbb{R}\setminus\{0\}) : \begin{array}{l} f_r(0) = 0 \\ f''_r(0) = af_r(0) + (q, f) \end{array} \right\} \quad (5.3)$$

In view of Lemma 3.2, the operator $H_a$ is $\mathcal{PT}$-symmetric if $a \in \mathbb{R}$ and $\mathcal{PT} q = q$. In this case, due to Lemma 3.1 the operator $H_a$ should be self-adjoint. Therefore, $\mathcal{PT}$-symmetric nonlocal $\delta$-interactions are
realized via self-adjoint operators. The same result is true for the case of \( \mathcal{P} \)-self-adjoint operators \( H_a \) (see Lemma 3.3).

**Theorem 5.1.** The operator \( H_a \) defined by (5.2) has an eigenvalue \( \lambda = k^2 \in \mathbb{C} \setminus [0, \infty) \) if and only the following relation holds:

\[
a = (q, G \ast q) + 2ik[1 + (G \ast q)(0)][1 + (G \ast q^*)(0)], \quad k \in \mathbb{C}_+.
\]

**Proof.** If \( q = q_1 \) and \( q_2 = 0 \), then the Weyl-Titchmarsh function (4.5) has the form

\[
W_{\lambda} = \begin{pmatrix}
(q, G \ast q) + r_{11} & -\frac{2i}{k}(G' \ast q^*)(0) \\
-\frac{2i}{k}(G' \ast q)(0) & \frac{2i}{k}
\end{pmatrix},
\]

where \( r_{11} = 2ik[1 + (G \ast q)(0)][1 + (G \ast q^*)(0)] + \frac{2i}{k}(G' \ast q)(0)(G' \ast q^*)(0) \).

By virtue of (4.2), \( \lambda \in \sigma_p(H_a) \) if and only if \( \det(T - W_{\lambda}) = 0 \), where

\[
T = \begin{pmatrix}
a & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

The direct calculation of \( \det(T - W_{\lambda}) \) in the latter equation gives (5.4).\( \blacksquare \)

Each operator \( H_a \) satisfies the relation \( S_{\text{min}} \subset H_a \subset S_{\text{max}} \) because \( H_a = H_T \) with the matrix \( T \) determined above. This important general relation (which holds for any \( H_T \)) can be made more precise for the particular case of operators \( H_a \). Indeed, it follows from (5.3) that \( H_a \) are extensions of the following operator:

\[
\tilde{S}_{\text{min}} f = -\frac{d^2 f}{dx^2}, \quad \mathcal{D}(\tilde{S}_{\text{min}}) = \left\{ f \in W^2_2(\mathbb{R}\setminus\{0\}) : f_s(0) = f_r(0) = 0 \right\}.
\]

(5.5)

It is easy to see (comparing \( \mathcal{D}(\tilde{S}_{\text{min}}) \) with the domain \( \mathcal{D}(S_{\text{min}}) \) determined by (2.12)) that \( \tilde{S}_{\text{min}} \) is an extension of \( S_{\text{min}} \), i.e., \( S_{\text{min}} \subset \tilde{S}_{\text{min}} \). Moreover, the operator \( \tilde{S}_{\text{min}} \) is symmetric. This fact follows from the Green identity (4.2) because \( \Gamma_1 f = 0 \) for all \( f \in \mathcal{D}(\tilde{S}_{\text{min}}) \).

Denote \( \tilde{S}_{\text{max}} = \tilde{S}_{\text{min}}^* \). The calculation of the adjoint operator gives

\[
\tilde{S}_{\text{max}} f = -\frac{d^2 f}{dx^2} + f_s(0)q(x), \quad \mathcal{D}(\tilde{S}_{\text{max}}) = \left\{ f \in W^2_2(\mathbb{R}\setminus\{0\}) : f_s(0) = 0 \right\}.
\]

(5.6)

It is easy to check that \( S_{\text{min}} \subset \tilde{S}_{\text{min}} \subset H_a \subset \tilde{S}_{\text{max}} \subset S_{\text{max}} \). Thus, \( H_a \) is a proper extension of the symmetric operator \( \tilde{S}_{\text{min}} \). Furthermore, an elementary analysis shows that:

(i) the kernel subspace \( \ker(\tilde{S}_{\text{max}} - \lambda I) \) is one-dimensional and it is generated by the function (cf. Lemma 4.1)

\[
u_\lambda(x) = -(G \ast q)(x) - 2ik[1 + (G \ast q)(0)]G(x);
\]

(5.6)
(ii) the triple \((\mathbb{C}, \tilde{\Gamma}_0, \tilde{\Gamma}_1)\), where
\[
\tilde{\Gamma}_0 f = f_r(0), \quad \tilde{\Gamma}_1 f = f_s'(0) - (q, f), \quad f \in \mathcal{D}(\tilde{S}_{\text{max}}) \tag{5.7}
\]
is the boundary triplet of \(\tilde{S}_{\text{max}}\) and
\[
\tilde{\Gamma}_0 u_\lambda = 1, \quad \tilde{\Gamma}_1 u_\lambda = (q, G \ast q) + 2ik[1 + (G \ast q)(0)][1 + (G \ast q^*)(0)], \tag{5.8}
\]
where \(u_\lambda\) is determined by \([5.6]\);
(iii) the operators \(H_a\) initially defined by \([5.2]\) and \([5.3]\) can be rewritten in terms of the boundary triplet \((\mathbb{C}, \tilde{\Gamma}_0, \tilde{\Gamma}_1)\) (cf. \([2.8]\)):
\[
H_a = \tilde{S}_{\text{max}} \upharpoonright \mathcal{D}(H_a), \quad \mathcal{D}(H_a) = \{ f \in \mathcal{D}(\tilde{S}_{\text{max}}) : (a\tilde{\Gamma}_0 - \tilde{\Gamma}_1)f = 0 \}; \tag{5.9}
\]
(iv) the operator (cf. \([2.11]\))
\[
\tilde{H}_\infty = \tilde{S}_{\text{max}} \upharpoonright \mathcal{D}(\tilde{H}_\infty), \quad \mathcal{D}(\tilde{H}_\infty) = \{ f \in \mathcal{D}(\tilde{S}_{\text{max}}) : \tilde{\Gamma}_0 f = 0 \}
\]
is positive self-adjoint and its spectrum coincides with \([0, \infty)\).

The items \((i) - (iv)\) allow one to simplify the investigation of \(H_a\).

First of all we note that the Weyl-Titchmarsh function \(\tilde{W}_\lambda\) associated to the boundary triplet \((\mathbb{C}, \tilde{\Gamma}_0, \tilde{\Gamma}_1)\) is a holomorphic function on \(\rho(\tilde{H}_\infty) = \mathbb{C} \setminus [0, \infty)\) and, due to \([5.8]\), it has the form
\[
\tilde{W}_\lambda = \tilde{\Gamma}_1 u_\lambda = (q, G \ast q) + 2ik[1 + (G \ast q)(0)][1 + (G \ast q^*)(0)]. \tag{5.10}
\]

The obtained formula immediately justifies \([5.4]\) because \(\lambda \in \mathbb{C} \setminus [0, \infty)\) is an eigenvalue of \(H_a\) if and only if \(\det(a - \tilde{W}_\lambda) = 0\) or, that is equivalent, if \(a = \tilde{W}_\lambda\). The latter identity shows that at least one of subspaces \(\mathbb{C}_+\) belongs to \(\rho(H_a)\). Indeed, if \(a \in \mathbb{R}\), then \(\rho(H_a) \supset \mathbb{C}_+\).

If \(a \in \mathbb{C} \setminus \mathbb{R}\), then only non-real eigenvalues of \(H_a\) might be in \(\mathbb{C}_\pm\). Let us assume that \(\lambda_\pm \in \sigma_p(H_a)\) with \(\text{Im} \lambda_+ > 0\) and \(\text{Im} \lambda_- < 0\). Then, simultaneously, \(\text{Im} a > 0\) and \(\text{Im} a < 0\) (since \(\text{Im} \tilde{W}_\lambda = a\) and \(\text{Im} \lambda)(\text{Im} \tilde{W}_\lambda) > 0\) for \(\text{Im} \lambda \neq 0\), see the Appendix) that is impossible. Therefore, at least one of \(\mathbb{C}_\pm\) does not belong to \(\sigma(H_a)\). This result is not true for the general case of one point interactions considered in section \([2]\). For instance, if \(q_1 = q_2 = 0\) and \(a = d = 0\), \(bc = 4\), then the characteristic equation \([4.6]\) is vanished and the eigenvalues of \(H_{\mathbf{T}}\) fill the whole domain \(\mathbb{C} \setminus [0, \infty)\).

**Corollary 5.2.** The existence of a real eigenvalue of \(H_a\) means that \(H_a\) is a self-adjoint operator in \(L_2(\mathbb{R})\).

**Proof.** Let \(u_\lambda \in L_2(\mathbb{R})\) be an eigenfunction of \(H_a\) corresponding to a real eigenvalue \(\lambda\). It follows from the definition of \(\tilde{S}_{\text{min}}\) that \(\ker(\tilde{S}_{\text{min}} -}
\( \lambda I \) = \{0\}. Therefore, the domain of \( H_a \) can be represented as
\[
\mathcal{D}(H_a) = \{ f = v + cu_\lambda : v \in \mathcal{D}(\tilde{S}_{\text{min}}), \ c \in \mathbb{C} \}
\]
(since the symmetric operator \( \tilde{S}_{\text{min}} \) has the defect index 1) and
\[
H_a f = H_a(v + cu_\lambda) = \tilde{S}_{\text{min}} v + \lambda cu_\lambda.
\]
Using the last expression we check that \( \text{Im} (H_a f, f) = 0 \) for all \( f = v + cu_\lambda \) from the domain of \( H_a \). Therefore, \( H_a \) is a self-adjoint operator.

In contrast to the case of ordinary one point interactions considered in subsec. 2.1, the operators \( H_a \) may have real eigenvalues embedded into continuous spectrum \([0, \infty)\). To see this we rewrite the function \( u_\lambda \) in (5.6) as follows:
\[
u_\lambda(x) = \begin{cases} A_k(x)e^{ikx} + B_k(x)e^{-ikx}, & x > 0 \\ C_k(x)e^{ikx} + D_k(x)e^{-ikx}, & x < 0 \end{cases}, \quad \lambda = k^2, \quad (5.11)
\]
where
\[
A_k(x) = 1 + \frac{i}{2k} \int_0^\infty e^{iks}q(s)ds - \frac{i}{2k} \int_0^x e^{-iks}q(s)ds,
\]
\[
D_k(x) = 1 + \frac{i}{2k} \int_{-\infty}^0 e^{-iks}q(s)ds - \frac{i}{2k} \int_x^0 e^{iks}q(s)ds,
\]
\[
B_k(x) = -\frac{i}{2k} \int_x^\infty e^{iks}q(s)ds, \quad C_k(x) = -\frac{i}{2k} \int_{-\infty}^x e^{-iks}q(s)ds.
\]

If \( \lambda = k^2 \) with \( k \in \mathbb{C}_+ \), then the function \( u_\lambda \) belongs to \( L^2(\mathbb{R}) \) and it solves the differential equation \( -f''(x) + f(0)q(x) = \lambda f(x) \) for \( x \neq 0 \). According to (5.8) and (5.10), \( u_\lambda \) belongs to the domain of definition (5.3) of the operator \( H_a \) with \( a = \tilde{W}_\lambda \). In other words, \( u_\lambda \) is the eigenfunction of \( H_a \).

If \( \lambda = k^2 \) with \( k \in \mathbb{R} \setminus \{0\} \), then the function \( u_\lambda \) defined by (5.11) turns out to be generalized eigenfunction of \( H_a \). This means that \( u_\lambda \) preserves all properties above except the property of being in \( L^2(\mathbb{R}) \). It should be noted that \( u_\lambda \) may belong to \( L^2(\mathbb{R}) \). In this case the generalized eigenfunction coincides with the ordinary eigenfunction and the corresponding operator \( H_a \) will have a positive eigenvalue \( \lambda = k^2 \) located on continuous spectrum \([0, \infty)\). In view of Corollary 5.2 this phenomenon is possible only for self-adjoint operators \( H_a \).

**Example 5.3.** The case of an even function with finite support.
Let \( q \) be an even function with support in \([-\rho, \rho]\). The elementary
calculation in (5.11) gives that for all $|x| > \rho$

$$u_\lambda(x) = \beta_k e^{ik|x|}, \quad \beta_k = 1 - \frac{1}{k} \int_0^\rho \sin ks q(s) ds.$$  

It is easy to see that $u_\lambda$ will be in $L_2(\mathbb{R})$ if and only if $\beta_k = 0$. If $k \in \mathbb{R} \setminus \{0\}$ is a solution of the last equation, then $u_\lambda$ turns out to be an eigenfunction of the self-adjoint operator $H_a$, where $a = \tilde{W}_\lambda$ and $\tilde{W}_\lambda$ is formally defined by (5.10) with $\lambda = k^2 \in (0, \infty)$.

It should be noted that the case of odd functions with finite support is completely different. Indeed, if $q$ is odd with the support in $[-\rho, \rho]$, then

$$u_\lambda(x) = \begin{cases} 
(1 - \frac{1}{k} \int_0^\rho \sin ks q(s) ds) e^{ikx}, & x > \rho \\
(1 + \frac{1}{k} \int_0^\rho \sin ks q(s) ds) e^{-ikx}, & x < -\rho 
\end{cases}$$

Obviously, such a function $u_\lambda$ does not belong to $L_2(\mathbb{R})$ and it cannot be an eigenfunction of $H_a$. Therefore, in the case of odd function $q$ with finite support, the corresponding operators $H_a$ $(a \in \mathbb{C})$ have no positive eigenvalues.

Let us consider the simplest example of even function

$$q(x) = Z \chi_{[-\rho, \rho]}(x) = \begin{cases} 
Z, & x \in [-\rho, \rho] \\
0, & x \in \mathbb{R} \setminus [-\rho, \rho] 
\end{cases} \quad Z \in \mathbb{R}, \quad \rho > 0.$$  

The characteristic equation $\beta_k = 0$ takes the form $Z(1 - \cos k\rho) = k^2$. Let $k_0 \in \mathbb{R} \setminus \{0\}$ be the solution of this equation. Then the function

$$u_\lambda(x) = \frac{Z(1 - \cos k_0(\rho - |x|))}{k_0^2} \chi_{[-\rho, \rho]}(x) \quad \lambda = k_0^2,$$

belongs to the domain of definition

$$\mathcal{D}(H_a) = \left\{ f \in W^2_2(\mathbb{R}\setminus\{0\}) : f(0-) = f(0+) \equiv f(0) \right\}$$

of the self-adjoint operator $H_a f = -\frac{d^2 f}{dx^2} + Z f(0) \chi_{[-\rho, \rho]}(x)$, where

$$a = [u_\lambda]'s(0) - Z \int_{-\rho}^\rho u_\lambda(x) dx = \frac{Z^2}{k_0^2} \left( \frac{\sin 2k_0\rho}{k_0} - 2\rho \right).$$

The function $u_\lambda$ is an eigenfunction of $H_a$ corresponding to the positive eigenvalue $\lambda = k_0^2$. 
5.2. **Exceptional points.** The geometric multiplicity of any \( \lambda \in \sigma_p(H_a) \) is 1 due to \((i)\) and the fact that \( \ker(\tilde{S}_{\min} - \lambda I) = \{0\} \). The algebraical multiplicity can be calculated with the use of general formula \((8.4)\).

An eigenvalue of \( H_a \) is called *exceptional point* if its geometrical multiplicity does not coincide with the algebraic multiplicity. The presence of an exceptional point means that \( H_a \) cannot be self-adjoint for any choice of inner product in \( L_2(\mathbb{R}) \). By virtue of Corollary 5.2, the operators \( H_a \) may only have non-real exceptional points.

**Theorem 5.4.** A non-real eigenvalue \( \lambda_0 \) of \( H_a \) is an exceptional point if and only if \( \tilde{W}'_{\lambda_0} = 0 \), where \( \tilde{W}'_{\lambda} = \frac{d}{d\lambda} \tilde{W}_{\lambda} \).

**Proof.** The resolvent \((\tilde{H}_\infty - \lambda I)^{-1}\) of a self-adjoint operator \( \tilde{H}_\infty \) is a holomorphic operator-valued function on \( \rho(\tilde{H}_\infty) = \mathbb{C} \setminus [0, \infty) \). On the other hand, the resolvent \((H_a - \lambda I)^{-1}\) may be a meromorphic function on \( \mathbb{C} \setminus [0, \infty) \) and its poles are eigenvalues of \( H_a \).

Let \( \lambda_0 \in \mathbb{C} \setminus \mathbb{R} \) be a pole of \((H_a - \lambda I)^{-1}\). Then its order coincides with the maximal length of Jordan vectors associated with \( \lambda_0 \) (see, e.g., [25, Chapt. 2]). Therefore, the existence of an exceptional point \( \lambda_0 \) of \( H_a \) is equivalent to the existence of pole \( \lambda_0 \) of order greater than one for the meromorphic operator-valued function

\[
\Xi(\lambda) = (H_a - \lambda I)^{-1} - (\tilde{H}_\infty - \lambda I)^{-1}.
\]

In other words, \( \lambda_0 \) turns out to be an exceptional point of \( H_a \) if and only if there exists \( v \in L_2(\mathbb{R}) \) such that

\[
\lim_{\lambda \to \lambda_0} \| (\lambda - \lambda_0) \Xi(\lambda) v \| = \infty.
\]

It is sufficient to suppose in \((5.14)\) that \( v = u_{\lambda^*} \in \ker(\tilde{S}_{\max} - \lambda^* I) \) (since \( H_a \) and \( \tilde{H}_\infty \) are extensions of \( \tilde{S}_{\min} \) and, hence, \( \Xi(\lambda) \downharpoonright_{\mathcal{R}(\tilde{S}_{\min} - \lambda I)} = 0 \)).

It follows from the Krein-Naimark resolvent formula \((8.3)\) that

\[
\| (\lambda - \lambda_0) \Xi(\lambda) u_{\lambda^*} \| = \left| \frac{\lambda - \lambda_0}{\lambda - \lambda} \right| \| \gamma(\lambda) \gamma(\lambda^*)^{\dagger} u_{\lambda^*} \|.
\]

Let us evaluate the part \( \| \gamma(\lambda) \gamma(\lambda^*)^{\dagger} u_{\lambda^*} \| \) in \((5.15)\). In view of \((8.2)\),

\[
\gamma(\lambda^*)^{\dagger} u_{\lambda^*} = \tilde{\Gamma}_1(\tilde{H}_\infty - \lambda I)^{-1} u_{\lambda^*}.
\]

The operator \( \tilde{H}_\infty \) is defined in \((iv)\) and it acts as \( \tilde{H}_\infty f = -\frac{d^2 f}{dx^2} \) for all functions \( f \in \mathcal{D}(\tilde{H}_\infty) = \{ f \in W_2^2(\mathbb{R} \setminus \{0\}) : f(0-) = f(0+) = 0 \} \). The resolvent of \( \tilde{H}_\infty \) is well known and it takes especially simple form
for \( f = u_{\lambda^*} \)

\[
(H_\infty - \lambda I)^{-1}u_{\lambda^*} = \frac{1}{2i(Im \lambda)}(u_{\lambda} - u_{\lambda^*}).
\]

The definition of the Weyl-Titchmarsh function \( \widetilde{W}_\lambda \) associated to the boundary triplet \((\mathbb{C}, \Gamma_0, \Gamma_1)\) and the relation \( \Gamma_0 u_\lambda = 1 \) in (5.8) imply that \( \Gamma_1 u_\lambda = \widetilde{W}_\lambda \) for all \( \lambda \in \mathbb{C} \setminus [0, \infty) \). Therefore,

\[
\gamma(\lambda^*)^\dagger u_{\lambda^*} = \Gamma_1(\widetilde{H}_\infty - \lambda I)^{-1}u_{\lambda^*} = \frac{\Gamma_1(u_\lambda - u_{\lambda^*})}{2i(Im \lambda)} = \frac{\widetilde{W}_\lambda - \widetilde{W}_{\lambda^*}}{2i(Im \lambda)} = \frac{Im \widetilde{W}_\lambda}{Im \lambda}.
\]

Further, it follows from the definition of \( \gamma \)-field \( \gamma(\cdot) \) associated with \((\mathbb{C}, \Gamma_0, \Gamma_1)\) (the Appendix) and (5.8) that \( \gamma(\lambda)c = cu_\lambda \) for all \( c \in \mathbb{C} \). Hence, \( \gamma(\lambda)\gamma(\lambda^*)^\dagger u_{\lambda^*} = \frac{Im \widetilde{W}_\lambda}{Im \lambda} u_\lambda \). Setting \( f_\lambda = u_\lambda \) in (8.1) we decide that

\[
||u_\lambda||^2 = \frac{Im \widetilde{W}_\lambda}{Im \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\] (5.16)

Therefore,

\[
\alpha(\lambda) = ||\gamma(\lambda)\gamma(\lambda^*)^\dagger u_{\lambda^*}|| = \left( \frac{Im \widetilde{W}_\lambda}{Im \lambda} \right)^{3/2}.
\]

The function \( \alpha(\lambda) \) is continuous in a neighborhood of the non-real point \( \lambda_0 \) and \( \alpha(\lambda_0) \neq 0 \). Therefore, taking (5.15) into account, we decide that (5.14) is equivalent to the condition

\[
\lim_{\lambda \to \lambda_0} \frac{a - \widetilde{W}_\lambda}{\lambda - \lambda_0} = 0.
\]

Remembering that \( a = \widetilde{W}_{\lambda_0} \) (since \( \lambda_0 \) is an eigenvalue of \( H_a \)) we complete the proof. ■

**Corollary 5.5.** If \( H_a \) has an exceptional point \( \lambda_0 \), then \( \lambda_0^* \) is an exceptional point for \( H_{a^*} \).

The proof follows from Theorem 5.4 and the relation \( \widetilde{W}_\lambda^* = \widetilde{W}_{\lambda^*} \).

5.3. **Spectral singularities.** Let \( H_a \) be a non-self-adjoint operator with real spectrum. The operator \( H_a \) cannot have real eigenvalues due to Corollary 5.2. Therefore, the spectrum of \( H_a \) is continuous and it coincides with \([0, \infty)\).

If \( H_a \) turns out to be self-adjoint with respect to an appropriative choice of inner product of \( L_2(\mathbb{R}) \) (i.e, if \( H_a \) is similar to a self-adjoint
operator in $L_2(\mathbb{R})$, then its resolvent $(H_a - \lambda I)^{-1}$ should satisfy the standard evaluation

$$
\| (H_a - \lambda I)^{-1} f \| \leq \frac{C}{|\text{Im} \lambda|} \| f \|, \tag{5.17}
$$

where $C > 0$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $f \in L_2(\mathbb{R})$.

The case where $H_a$ is not similar to a self-adjoint operator in $L_2(\mathbb{R})$ deals with the existence of special spectral points of $H_a$ which are impossible for the spectra of self-adjoint operators. Traditionally, these spectral points are called spectral singularities if they are located at the continuous spectrum of $H_a$. Particular role pertaining to the spectral singularities was discovered for the first time by Naimark [26]. Nowadays, various aspects of spectral singularities including the physical meaning and possible practical applications has been analyzed with a wealth of technical tools (see, e.g., [27], [28]).

It is natural to suppose that a spectral singularity $\lambda_0 \in (0, \infty)$ of $H_a$ is characterized by an untypical behaviour of the resolvent $(H_a - \lambda I)^{-1}$ in a neighborhood of $\lambda_0$. This assumption leads to the following definition: a positive number $\lambda_0$ is called spectral singularity of $H_a$ if there exists $f \in L_2(\mathbb{R})$ such that the evaluation (5.17) does not hold when non real $\lambda$ tends to $\lambda_0$.

**Theorem 5.6.** Let $\lambda_0 \in (0, \infty)$ and let there exist a sequence of non-real $\lambda_n$ such that $\lambda_n \to \lambda_0$ and $\lim_{n \to \infty} \tilde{W}_{\lambda_n} = a \in \mathbb{C} \setminus \mathbb{R}$. Then $\lambda_0$ is a spectral singularity of non-self-adjoint operators $H_a$ and $H_{a^*}$.

**Proof.** The inequality (5.17) is equivalent to the inequality

$$
\| \Xi(\lambda)f \| \leq \frac{C}{|\text{Im} \lambda|} \| f \|, \tag{5.18}
$$

where $\Xi(\lambda)$ is defined by (5.13). Moreover, it follows from the proof of Theorem 5.4 that it is sufficient to verify (5.18) for $f = u_{\lambda^*}$ only. By virtue of (5.15) and the proof of Theorem 5.4,

$$
\| \Xi(\lambda)u_{\lambda^*} \| = \frac{\| \gamma(\lambda)\gamma^*(\lambda^*)u_{\lambda^*} \|}{|a - W_{\lambda}|} = \frac{\text{Im} \tilde{W}_\lambda}{\text{Im} \lambda} \frac{\| u_\lambda \|}{|a - W_{\lambda}|}. \tag{5.19}
$$

It follows from (5.16) that $\| u_\lambda \| = \| u_{\lambda^*} \|$. Replacing $\| u_\lambda \|$ by $\| u_{\lambda^*} \|$ in (5.19) we rewrite (5.18) in the following equivalent form

$$
\frac{|\text{Im} \tilde{W}_\lambda|}{|a - W_{\lambda}|} \leq C, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{5.20}
$$
If the condition of Theorem 5.6 is satisfied, then the inequality (5.20) cannot be true in neighborhood of $\lambda_0$. Therefore, $\lambda_0$ should be a spectral singularity of $H_a$. The same result holds for $H_{a^*}$ if we consider the sequences $\lambda_n^* \to \lambda_0$, $W_{\lambda_n}^* = W_{\lambda_n} \to a^*$ and take into account that $H_{a^*}^\dagger = H_a$. 

If $\lambda = k^2$ with $k \in \mathbb{R} \setminus \{0\}$, then the formula (5.11) allows one to define two functions $u_{\lambda}^\pm$ corresponding to positive/negative values of $k$, respectively. In this case, the formula 

$$\tilde{W}_\lambda^\pm = [u_{\lambda}^\pm]_0(0) - (q, u_{\lambda}^\pm) = 2ik \left( 1 + \frac{i}{k} \int_0^\infty e^{iks} q^{ev}(s) ds \right) - (q, u_{\lambda}^\pm)$$

($q^{ev}$ is the even part of $q$) gives two values of the Weyl-Titchmarsh function $\tilde{W}_\lambda$ on $(0, \infty)$.

The conditions imposed on $q$ guaranties that $\tilde{W}_\lambda^\pm$ are well-posed (i.e. $\tilde{W}_\lambda^\pm \neq \infty$). Moreover, the functions $\tilde{W}_\lambda^\pm$ can be interpreted as limits on $(0, \infty)$ of the holomorphic functions $\tilde{W}_\lambda$ considered on $\mathbb{C}_\pm$, respectively. Taking the relation $\tilde{W}_\lambda^* = \tilde{W}_\lambda$, $\lambda \in \mathbb{C} \setminus [0, \infty)$ into account, we deduce that $(\tilde{W}_\lambda^* )^* = \tilde{W}_\lambda^-$ for $\lambda > 0$. This relation and the definition of $\tilde{W}_\lambda^\pm$ imply that $u_{\lambda}^+$ and $u_{\lambda}^-$ are generalized eigenfunctions of the operators $H_a$ and $H_{a^*}$, respectively with $a = \tilde{W}_\lambda^+$. 

If $a = \tilde{W}_\lambda^+$ is non-real, then, due to Theorem 5.6 $\lambda$ is a spectral singularity of the non-self-adjoint operators $H_a$ and $H_{a^*}$. The corresponding generalized eigenfunctions coincide with $u_{\lambda}^+$ and $u_{\lambda}^-$. If $a = \tilde{W}_\lambda^+$ is real, then the evaluation (5.17) holds (since $H_a$ is self-adjoint) and $\lambda$ cannot be a spectral singularity of $H_a$.

### 6. Examples

#### 6.1. Ordinary $\delta$-interaction.

This simplest case corresponds to $q = 0$. The operators $H_a = -\frac{d^2}{dx^2}$ have the domains:

$$D(H_a) = \left\{ f \in W_2^2(\mathbb{R} \setminus \{0\}) : \begin{array}{l} f(0-) = f(0+) \equiv f(0) \\ f''(0+) - f''(0-) = af(0) \end{array} \right\}. $$

The Weyl-Titchmarsh function has the form $\tilde{W}_\lambda = 2ik = 2i\sqrt{\lambda}$. There are no exceptional points for operators $H_a$ because $\tilde{W}_\lambda' = i/\sqrt{\lambda}$ does not vanish on $\mathbb{C} \setminus [0, \infty)$.

The limit functions $\tilde{W}_\lambda^\pm = 2ik$, $k > 0/k < 0$ takes non-real values. Hence, the operators $H_{\tilde{W}_\lambda^+}$ and $H_{\tilde{W}_\lambda^-}$ have the spectral singularity $\lambda = k^2$. 

The ordinary $\delta$-interaction are well-studied [21], [22] and the evolution of spectral properties of $H_a$ when $a$ runs $\mathbb{C}$ can be illustrated as follows:

\begin{align*}
\text{Re}(a) & \quad \text{Im}(a) \\
\text{\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\text\tex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The limit functions \( \tilde{W}_\lambda^\pm \) are determined by (6.2) for \( k > 0 \) and \( k < 0 \), respectively. It is easy to check that the imaginary part of \( \tilde{W}_\lambda^\pm \):

\[
\text{Im} \tilde{W}_\lambda^\pm = 2k + \frac{|Z|^2}{k^3} (2 \cos^2 k \rho - 4 \cos k \rho + 2)
\]
do not vanish when \( k \) runs \( \mathbb{R} \setminus \{0\} \). Hence, any positive \( \lambda \) turns out to be a spectral singularity for some operators \( H_a \). Namely, the operators \( H_a \) and \( H_a^* \) with \( a = \tilde{W}_\lambda^+ \) will have the spectral singularity \( \lambda \).

6.3. The case of even function \( q = ce^{-\mu|x|} \) (\( \mu > 0 \)). The corresponding operators \( H_a f = -\frac{d^2 f}{dx^2} + f(0)ce^{-\mu|x|} \) have the domains

\[
D(H_a) = \left\{ f \in W^2_2(\mathbb{R} \setminus \{0\}) : \begin{array}{l}
f(0-) = f(0+) \equiv f(0) \\
f'(0+) - f'(0-) = af(0) + c^* \int_\mathbb{R} e^{-\mu|x|} f(x) dx
\end{array} \right\}.
\]

The eigenfunctions \( u_\lambda \) (see (5.11)) are given by the expression

\[
u_\lambda = \left( 1 - \frac{c}{\mu^2 + \lambda} \right) e^{ik|x|} + \frac{q(x)}{\mu^2 + \lambda}, \quad \lambda = k^2. \tag{6.3}\]

The Weyl-Titchmarsh function

\[
\tilde{W}_\lambda = 2ik - \langle q, u_\lambda \rangle = 2ik - \frac{4\Re c}{\mu - ik} + \frac{||q||^2}{(\mu - ik)^2} \tag{6.4}
\]
is defined on \( \mathbb{C} \setminus [0, \infty) \) and its limit functions \( \tilde{W}_\lambda^\pm \) are determined by (6.1) with \( k > 0 \) and \( k < 0 \), respectively.

Each \( \lambda \in \mathbb{C} \setminus [0, \infty) \) is an eigenvalue of the operator \( H_a \) with \( a = \tilde{W}_\lambda \) and the corresponding eigenfunction is given by (6.3).

It follows from (6.3) that a positive eigenvalue \( \lambda \) exists for some operator \( H_a \) if and only if \( c \geq \mu^2 \). In this case, \( \lambda = c - \mu^2 \), the corresponding eigenfunction \( u_\lambda \) coincides with \( \frac{q(x)}{\mu^2 + \lambda} = e^{-\mu|x|} \) and \( u_\lambda \) an eigenfunction of a self-adjoint operator \( H_a \) with \( a = \tilde{W}_\lambda^\pm = -3\mu - \frac{\lambda}{\mu} \).

Let us assume for the simplicity that \( c \in i\mathbb{R} \) and \( ||q||^2 = \frac{|c|^2}{\mu} = 1. \)

Then

\[
\tilde{W}_\lambda = 2ik + \frac{1}{(\mu - ik)^2} = 2i\sqrt{\lambda} + \frac{1}{(\mu - i\sqrt{\lambda})^2}. \tag{6.5}
\]

If \( k \) is real in (6.5), then the imaginary part of \( \tilde{W}_\lambda^\pm \):

\[
\text{Im} \tilde{W}_\lambda^\pm = 2k + \frac{2k\mu}{|\mu - ik|^2}
\]
does not vanish when \( \lambda = k^2 \in (0, \infty) \). Hence, any positive \( \lambda \) is a spectral singularity of operators \( H_a \) and \( H_a^* \) with \( a = \tilde{W}_\lambda^+ \).
It follows from (6.5) that
\[ \tilde{W}'_{\lambda} = \frac{i}{k} \left[ 1 + \frac{1}{(\mu - ik)^3} \right] = \frac{i}{\sqrt{\lambda}} \left[ 1 + \frac{1}{(\mu - i\sqrt{\lambda})^3} \right]. \]
Therefore, \( \tilde{W}'_{\lambda} = 0 \) for certain \( \lambda \in \mathbb{C} \setminus [0, \infty) \) if and only if \((\mu - ik)^3 = -1\) for \( k \in \mathbb{C}_+ \). The latter equation has two required solutions
\[ k_0 = \frac{\sqrt{3}}{2} + i\left(\frac{1}{2} - \mu\right), \quad k_1 = -k_0^* \]
when \( 0 < \mu < \frac{1}{2} \). By virtue of Theorem 5.4, \( \lambda_0 = k_0^2 \) is an exceptional point of the operator \( H_a \) with
\[ a = \tilde{W}_{\lambda_1} = 2ik_0 + \frac{1}{(\mu - ik_0)^2} = 2ik_0 + \frac{\mu - ik_0}{(\mu - ik_0)^3} = 3ik_0 - \mu, \]
while \( \lambda_1 = k_1^2 = \lambda_0^* \) will be an exceptional point of its adjoint \( H_a^* = H_a^d \), cf. Corollary 5.5.

The obtained result shows that the existence of exceptional points for some operators from the collection \( \{H_a\}_{a \in \mathbb{C}} \) depends on the behaviour of the function \( q(x) = ce^{-\mu|x|} \). If \( q(x) \) decrease (relatively) slowly on \( \infty \) (the case \( 0 < \mu < \frac{1}{2} \)) then exist two operators \( H_a \) and \( H_a^d \) with exceptional points \( \lambda_0 \) and \( \lambda_0^* \), respectively.

7. Summary

Although the knowledge of the merits of the pseudo-Hermitian representation of observables (and, in particular, of Hamiltonians) in quantum theory dates back to the middle of the last century, its applicability still remains restricted, mainly due to the presence and emergence of multiple technical obstacles [29]. In the present paper we paid attention to the possibilities of circumventing the obstacles via introduction of interactions which combined the exact solvability feature of the traditional point interactions with the necessity of extension of the latter class of local potentials to some maximally friendly nonlocal generalizations.

For the sake of a reasonable length of our paper we only considered a subset of the eligible candidates for the interaction and we also did not pay any explicit attention to the possible connection of our models with physics and with the possible experimental realizations of the systems. This enabled us to pay more attention to the usually neglected mathematical features of the models and to the explicit description of the qualitative differences between the self-adjoint and non-selfadjoint choices and/or between the local and nonlocal versions and special cases of the Hamiltonians.
We would like to emphasize the importance of our present successful transition from the traditional study of finite-matrix models (i.e., of the simplified, difference Schrödinger equations as sampled, e.g., in [30]) to the full-fledged differential operators (albeit with the mere ultralocal-distribution interactions). Obviously, such a step still remains to be followed by several future resolutions of challenges incorporating, first of all, the construction of the physical inner products, etc.

In a way inspired by the older developments in self-adjoint context [8] we found a key to the technical new results in the use of the language of the formalism of boundary triplets. We managed to demonstrate that even after a restriction of our attention to the first nontrivial class of one point nonlocal interactions the wealth of the spectral properties of the models remains satisfactorily rich involving not only the usual regularities/anomalies in the discrete spectra but, equally well, also the advanced (and, in the finite-dimensional models, inaccessible) features of the presence of the exceptional points and of the spectral singularities.

Naturally, we expect that the set of the present results will be complemented, in some not too remote future, not only by the similar rigorous coverage of the more general nonlocal interactions (and of the related enhanced flexibility, say, in the quantum spectral design) but also by the development of some parallels to the success of transfer of the applicability of the manifestly non-selfadjoint models in the scattering dynamical regime, with a particular emphasis upon the possible restoration of the unitarity of the S matrix (in this direction our future plans will be inspired by the encouraging success of Ref. [31] in the analysis of certain local point-interaction predecessors of our present models).

8. Appendix: Boundary triplets

Let $S_{\text{min}}$ be a closed symmetric (densely defined) operator in a Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$. Denote $S_{\text{max}} = S_{\text{min}}^\dagger$. Obviously, $S_{\text{min}} \subset S_{\text{max}}$.

A triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1$ are linear mappings of $\mathcal{D}(S_{\text{max}})$ into $\mathcal{H}$, is called a boundary triplet of $S_{\text{max}}$ if the Green identity
\[
(S_{\text{max}} f, g) - (f, S_{\text{max}} g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathcal{D}(S_{\text{max}})
\]
is satisfied and the map $(\Gamma_0, \Gamma_1) : \mathcal{D}(S_{\text{max}}) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

The symmetric operator $S_{\text{min}}$ is the restriction of $S_{\text{max}}$ onto $\mathcal{D}(S_{\text{min}}) = \{ f \in \mathcal{D}(S_{\text{max}}) : \Gamma_0 f = \Gamma_1 f = 0 \}$. The defect indices of $S_{\text{min}}$ coincides with the dimension of $\mathcal{H}$. Boundary triplets of $S_{\text{max}}$ are not determined
uniquely and they exist only in the case where the symmetric operator $S_{\min}$ has self-adjoint extensions\footnote{see [32] for various generalization of boundary triplets}.

Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of $S_{\max}$. Then the operator
$$H_\infty = S_{\max} \upharpoonright \mathcal{D}(H_\infty), \quad \mathcal{D}(H_\infty) = \{ f \in \mathcal{D}(S_{\max}) : \Gamma_0 f = 0 \}$$
is a self-adjoint extension of $S_{\min}$.

The Weyl-Titchmarsh function $W_\lambda$ associated to the boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is defined for all $\lambda \in \rho(H_\infty)$ \footnote{\cite{33}}:
$$W_\lambda \Gamma_0 f_\lambda = \Gamma_1 f_\lambda, \quad \forall f_\lambda \in \ker(S_{\max} - \lambda I).$$

Let $f_\lambda \in \ker(S_{\max} - \lambda I)$. It follows from the Green identity that
$$(\text{Im } \lambda) \| f_\lambda \|^2 = (\Gamma_0 f_\lambda, (\text{Im } W_\lambda) \Gamma_0 f_\lambda), \quad \text{where } \text{Im } W_\lambda = \frac{W_\lambda - W_\lambda^\dagger}{2i}. \tag{8.1}$$

Therefore, $(\text{Im } \lambda) (\text{Im } W_\lambda) > 0$ for non-real $\lambda$. The latter means that $W_\lambda$ is a Herglotz (Nevanlinna) function \footnote{\cite{34}}.

Let $T$ be a bounded operator in the auxiliary Hilbert space $\mathcal{H}$. The operator
$$H_T = S_{\max} \upharpoonright \mathcal{D}(H_T), \quad \mathcal{D}(H_T) = \{ f \in \mathcal{D}(S_{\max}) : (T \Gamma_0 - \Gamma_1) f = 0 \}$$
is a proper extension of $S_{\min}$ (i.e., $S_{\min} \subset H_T \subset S_{\max}$). Moreover, the adjoint operator $H_T^\dagger$ is also a proper extension and $H_T^\dagger = H_T^\dagger$, where $T^\dagger$ is the adjoint operator of $T$ in the auxiliary space $\mathcal{H}$. Hence, the self-adjointness of unbounded operator $H_T$ in $\mathcal{H}$ is equivalent to the self-adjointness of bounded operator $T$ in the auxiliary space $\mathcal{H}$.

The spectrum of $H_T$ is described in terms of $T$ and $W_\lambda$. Namely \footnote{\cite{33}}, $\lambda \in \rho(H_\infty)$ belongs to the point $\sigma_p(H_T)$, to the residual $\sigma_r(H_T)$, and to the continuous $\sigma_c(H_T)$ parts of the spectrum of $H_\infty$ if and only if $0$ belongs to the same parts of spectrum of $T - W_\lambda$, i.e., if $0 \in \sigma_\alpha(T - W_\lambda)$, $\alpha \in \{p, r, c\}$.

For each $\lambda \in \rho(H_\infty)$, the operator $\Gamma_0$ is a bijective mapping of the subspace $\ker(S_{\max} - \lambda I)$ onto $\mathcal{H}$. Its bounded inverse
$$\gamma(\lambda) = (\Gamma_0 \upharpoonright \ker(S_{\max} - \lambda I))^{-1} : \mathcal{H} \to \ker(S_{\max} - \lambda I)$$
is called the $\gamma$-field associated with $(\mathcal{H}, \Gamma_0, \Gamma_1)$.

The $\gamma$-field $\gamma(\cdot)$ is a holomorphic operator-valued function on $\rho(H_\infty)$ and \footnote{\cite{24}, Prop. 14.14, 14.15}
$$d W_\lambda = \gamma(\lambda^*)^\dagger \gamma(\lambda) \tag{8.2}$$
where the adjoint operator $\gamma(\lambda^*)^\dagger$ maps $\ker(S_{\max} - \lambda^* I)$ into $\mathcal{H}$.

For any $\lambda \in \rho(H_\infty) \cap \rho(H_T)$, the Krein-Naimark resolvent formula

$$(H_T - \lambda I)^{-1} - (H_\infty - \lambda I)^{-1} = \gamma(\lambda)(T - W_\lambda)^{-1}\gamma(\lambda^*)^\dagger$$  \hspace{1cm} (8.3)

holds [24, Theorem 14.18].

Let us assume for simplicity that the auxiliary space $\mathcal{H}$ is finite-dimensional, i.e., $\dim \mathcal{H} = m$ and the spectrum of $H_\infty$ is purely continuous. Then, the continuous spectrum of each $H_T$ coincides with $\sigma(H_\infty)$ and only eigenvalues of $H_T$ may appear in $\mathbb{C} \setminus \sigma(H_\infty)$ (since $H_T$ are finite rank perturbations of the self-adjoint operator $H_\infty$). Without loss of generality, we may assume that $H = \mathbb{C}^m$. In this case, the operator $T$ and the Weyl-Titchmarsh function $W_\lambda$ can be replaced by $m \times m$-matrices and $\lambda \in \mathbb{C} \setminus \sigma(H_\infty)$ is an eigenvalue of $H_T$ if and only if $\det(T - W_\lambda) = 0$.

The geometric multiplicity of an eigenvalue $\lambda$ coincides with $m - \text{rank}(T - W_\lambda)$.

In our presentation we assume that $\sigma(H_T) \neq \mathbb{C}$. Then, the presence of an eigenvalue $\lambda_0 \in \mathbb{C} \setminus \sigma(H_\infty)$ of $H_T$ can be characterized as follows: $\lambda_0$ should be a zero of finite-type [35, Definition 3.1] of the matrix-valued holomorphic function $T - W_\lambda$.

According to the Fredholm theorem [36, Thm. VI.14], $(T - W_\lambda)^{-1}$ is holomorphic on the punctured disk $D(\lambda_0; \epsilon_0) = \{\lambda \in \mathbb{C} \mid 0 < |\lambda - \lambda_0| < \epsilon_0\}$ for some $0 < \epsilon_0$ sufficiently small. In this case, we may define the index of $T - W_\lambda$ with respect to the counterclockwise oriented circle $C(\lambda_0; \epsilon) = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| = \epsilon\}$:

$$\text{ind}_{C(\lambda_0; \epsilon)}(T - W_\lambda) = \text{tr}_{\mathbb{C}^m} \oint_{C(\lambda_0; \epsilon)} W'_\xi(W_\xi - T)^{-1}d\xi, \hspace{1cm} 0 < \epsilon < \epsilon_0. \hspace{1cm} (8.4)$$

By virtue of [35, Theorem 6.4] the algebraic multiplicity of the eigenvalue $\lambda_0$ of $H_T$ coincides with $\text{ind}_{C(\lambda_0; \epsilon)}(T - W_\lambda)$. The latter quantity is also the algebraic multiplicity of the zero of $T - W_\lambda$ at $\lambda_0$.

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