Density and path-connectedness in $St(n, H)$

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July 27, 2021

Abstract

If $H$ is a Hilbert space, the Stiefel manifold $St(n, H)$ is formed by all the independent $n$-tuples in $H$. In this article, we contribute to the topological study of Stiefel manifolds by proving density and path-connectedness-related results. Regarding the density aspect, we generalize the fact that $St(n, H)$ is dense in $H^n$ and prove that $St(n, H) \cap S$ is dense in $S$ whenever $S \subseteq H^n$ is connected by polynomial paths of finite degree to some $\Theta \in St(n, H) \cap S$. We provide special examples of such sets $S$ in the context of finite-dimensional continuous frames (we set $H := L^2(X, \mu; F)$ and we identify $St(n, H)$ with $\mathcal{F}(X, \mu), n)$ which are constructed from the inverse image of singletons by some familiar linear and pseudo-quadratic functions. In the second part devoted to path-connectedness, we prove that the intersection of translates of $St(n, H)$ is path-connected under a condition on the codimension of the span of the components of the translating $n$-tuples. These results are also a contribution to the topological theory of Hilbert space frames which is presently an active area of research.

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2020 Mathematics Subject Classification. 57N20; 42C15; 54D99; 54D05.
Key words and phrases. Stiefel manifold, continuous frame, dense subset, path-connected space.
1 Introduction

Duffin and Shaeffer introduced in 1952 [13] the notion of a Hilbert space frame to study some deep problems in nonharmonic Fourier series. However, the general idea of signal decomposition in terms of elementary signals was known to Gabor [16] in 1946. The landmark paper of Daubechies, Grossmann, and Meyer [12] (1986) accelerated the development of the theory of frames which then became more widely known to the mathematical community. Nowadays, frames have a wide range of applications in both engineering science and mathematics: they have found applications in signal processing, image processing, data compression, and sampling theory. They are also used in Banach space theory. Intuitively, a frame in a Hilbert space $K$ is an overcomplete basis allowing non-unique linear expansions, though technically, it must satisfy a double inequality called the frame inequality. There are many generalizations of frames in the literature, for instance frames in Hilbert C*-modules [15]. A general introduction to frame theory can be found in ([8],[10]).

The space $\mathcal{F}_{(X,\mu),n}$ of continuous frames indexed by $(X,\mu)$ and with values in $\mathbb{F}^n$ is isometric to the Stiefel manifold $St(n,L^2(X,\mu;\mathbb{F}))$. If $H$ is a Hilbert space, the Stiefel manifold $St(n,H)$ is the set of independent $n$-frames in $H$ in the geometric sense. The term $n$-frame is used in geometry to simply denote an independent $n$-tuple. Stiefel manifolds are studied in differential topology and are one of the fundamental examples in this area. The theory of finite dimensional Stiefel manifolds is summarized in [23] and also covered in ([20],[22]). The theory of infinite dimensional Stiefel manifolds is less studied and some recent results can be found in ([5],[19]).

There have been also many studies directly devoted to the geometry of frames and their subsets. Connectivity properties of some important subsets of the frame space $\mathcal{F}_{k,n}$ were studied in ([7],[25]). Differential and algebro-geometric properties of these subsets were studied in ([14],[27],[28],[29]) and (chapter 4 of [9]) respectively. A fiber bundle structure with respect to the $L^1$ and $L^\infty$ norms was established for continuous frames in ([1],[2]). A notion of density for general frames analogous to Beurling density was introduced and studied in [3]. Finally, connectivity and density properties were studied for Gabor ([4],[11],[21],[24]) and wavelet ([6],[17],[18],[26]) frames.

In this article, we contribute to the topological study of Stiefel manifolds by proving density and path-connectedness-related results. Regarding the density
aspect, we generalize the fact that \( St(n, H) \) is dense in \( H^n \) and prove that \( St(n, H) \cap S \) is dense in \( S \) when \( S \subseteq H^n \) is connected by polynomial paths of finite degree to some \( \Theta \in St(n, H) \cap S \). We provide special examples of such sets \( S \) in the context of finite-dimensional continuous frames (we set \( H = L^2(X, \mu; \mathbb{F}) \) and we identify \( St(n, H) \) with \( F_n(X, \mu) \)), which are constructed from the inverse image of singletons by some familiar linear and pseudo-quadratic functions. The chosen linear function is the integration of \( \Phi \in L^2(X, \mu; \mathbb{F}) \) against a function in \( L^2(X, \mu; \mathbb{F}) \) as well as a generalized version of this function. The pseudo-quadratic function is the integration of \( (\langle b, \varphi_x \rangle \varphi_x)_{x \in X} \) against a function in \( L^\infty(X, \mu; \mathbb{F}) \) where \( b \in \mathbb{F}^n \). In the second part devoted to path-connectedness, we prove that the intersection of translates of \( St(n, H) \) is path-connected under a condition on the codimension of the span of the components of the translating \( n \)-tuples.

**Plan of the article.** This article is organized as follows. In section 2, we set some notations, introduce the definition of continuous Bessel and frame families and their basic properties in \( \mathbb{F}^n \), and present Stiefel manifolds with a special emphasis on their topological aspects. In section 3, we prove a result saying that \( St(n, H) \cap S \) is dense in \( S \) when \( S \subseteq H^n \) is connected by polynomial paths of finite degree to some \( \Theta \in St(n, H) \cap S \), and we provide some special examples. In the final section 4, we prove that the intersection of translates of \( St(n, H) \) is path-connected under a condition on the codimension of the span of the components of the translating \( n \)-tuples.

# 2 Preliminaries

## 2.1 Notation

The following notations are used throughout this article.

- \( \mathbb{N} \) denotes the set of natural numbers including 0 and \( \mathbb{N}^\ast = \mathbb{N} \setminus \{0\} \).
- We denote by \( n \) an element of \( \mathbb{N}^\ast \) and by \( \mathbb{F} \) one of the fields \( \mathbb{R} \) or \( \mathbb{C} \).
- If \( K \) is a Hilbert space, we denote by \( L(K) \) and \( B(K) \) respectively the set of linear and bounded operators in \( K \). \( \text{Id}_K \) is the identity operator of \( K \).
- If \( K \) is a Hilbert space, \( m \in \mathbb{N}^\ast \), and \( \theta_1, \ldots, \theta_m \in H \), the Gram matrix of \( (\theta_1, \ldots, \theta_m) \) is the matrix \( \text{Gram}(\theta_1, \ldots, \theta_m) \) whose \( k, l \)-coefficient is \( \text{Gram}(\theta_1, \ldots, \theta_m)_{k, l} = \langle \theta_k, \theta_l \rangle \).
- If \( \sigma, \tau \in \mathbb{N}^\ast \), we denote by \( M_{\sigma, \tau}(\mathbb{F}) \) the algebra of matrices of size \( \sigma \times \tau \) over the field \( \mathbb{F} \). When \( \sigma = \tau \), we denote this algebra \( M_{\sigma}(\mathbb{F}) \).
- An element \( x \in \mathbb{F}^n \) is a \( n \)-tuple \( (x^1, \ldots, x^n) \) with \( x^k \in \mathbb{F} \) for all \( k \in [1, n] \).
- If \( S \in L(\mathbb{F}^n) \), we denote by \( [S] \in M_n(\mathbb{F}) \) the matrix of \( S \) in the standard basis of \( \mathbb{F}^n \), and we write \( I_n \) as a shorthand for \( [\text{Id}_{\mathbb{F}^n}] \).
If $U = (u_x)_{x \in X}$ is a family in $\mathbb{F}^n$ indexed by $X$, then for each $k \in [1, n]$, we denote by $U^k$ the family $(u_x^k)_{x \in X}$.

### 2.2 Continuous frames in $\mathbb{F}^n$

Let $K$ be a Hilbert space and $(X, \Sigma, \mu)$ a measure space.

**Definition 2.1.** [10] We say that a family $\Phi = (\varphi_x)_{x \in X}$ with $\varphi_x \in K$ for all $x \in X$ is a continuous frame in $K$ if

$$\exists 0 < A \leq B : \forall v \in K : A\|v\|^2 \leq \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq B\|v\|^2$$

A frame is tight if we can choose $A = B$ as frame bounds. A tight frame with bound $A = B = 1$ is called a Parseval frame. A Bessel family is a family satisfying only the upper inequality. A frame is discrete if $\Sigma$ is the discrete $\sigma$-algebra and $\mu$ is the counting measure. We denote by $\mathcal{F}(X, \mu, K)$ and $\mathcal{F}^n(X, \mu, K)$ respectively the set of continuous frames with values in $K$ and the set of continuous frames with values in $\mathbb{F}^n$.

If $U = (u_x)_{x \in X}$ with $u_x \in K$ for all $x \in K$ is a continuous Bessel family in $K$, we define its analysis operator $T_U : K \to L^2(X, \mu; \mathbb{F})$ by

$$\forall v \in K : T_U(v) := (\langle v, u_x \rangle)_{x \in X}.$$

The adjoint of $T_U$ is an operator $T_U^* : L^2(X, \mu; \mathbb{F}) \to K$ given by

$$\forall c \in L^2(X, \mu; \mathbb{F}) : T_U^*(c) = \int_X c(x) u_x d\mu(x).$$

The composition $S_U = T_U^* T_U : K \to K$ is given by

$$\forall v \in K : S_U(v) = \int_X \langle v, u_x \rangle u_x d\mu(x)$$

and called the frame operator of $U$. Since $U$ is a Bessel family, $T_U$, $T_U^*$, and $S_U$ are all well defined and continuous. If $U$ is a frame in $K$, then $S_U$ is a positive self-adjoint operator satisfying $0 < A \leq S_U \leq B$ and thus, it is invertible.

We now recall a proposition preventing that a frame belongs to $L^2(X, \mu; K)$ when $\dim(K) = \infty$. Here the set $L^2(X, \mu; K)$ refers to Bochner square integrable (classes) of functions in $\mathcal{M}(X; K)$, where the latter refers to the set of measurable functions from $X$ to $K$. It explains why we only study the $L^2$ topology of frame subspaces in the finite dimensional case. However, notice that even if $\dim(K) = \infty$, one can find a subset $S$ of $\mathcal{M}(X; K)$ and a continuous frame $\Phi$ with values in $K$ such that $L^2(X, \mu; K) \cap (\Phi + S) = \{ U \in L^2(X, \mu; K) : U - \Phi \in S \}$ is non-empty. This set inherits the topology of $L^2(X, \mu; K)$ and can be investigated topologically.
Proposition 2.1. Let $K$ be a Hilbert space with $\dim K = \infty$. Then $\mathcal{F}_{(X, \mu), K} \cap L^2(X, \mu; K) = \emptyset$.

Proof. Let $\Phi = (\varphi_x)_{x \in X} \in \mathcal{F}_{(X, \mu), K} \cap L^2(X, \mu; K)$. Let $\{e_m\}_{m \in M}$ be an orthonormal basis of $K$. We have

$$\text{Tr}(S\Phi) = \text{Tr}(T\Phi T^{*}\Phi) = \sum_{m \in M} \|T(e_m)\|^2$$

$$= \sum_{m \in M} \int_X |(e_m, \varphi_x)|^2 d\mu(x)$$

$$= \int_X \left( \sum_{m \in M} |(e_m, \varphi_x)|^2 \right) d\mu(x)$$

$$= \int_X \|\varphi_x\|^2.$$

Since $\Phi \in \mathcal{F}_{(X, \mu), K}$, there exists a constant $A > 0$ such that $S\Phi \geq A \cdot I_d$, so

$$\int_X \|\varphi_x\|^2 = Tr(S\Phi) = +\infty$$

since $\dim(K) = \infty$. Hence $\Phi \notin L^2(X, \mu; K)$. \qed

From now on, we consider $K = \mathbb{F}^n$. In what follows, we will recall some elementary facts about Bessel sequences and frames in this setting.

Proposition 2.2. A family $U = (u_x)_{x \in X}$ with $u_x \in \mathbb{F}^n$ for all $x \in X$ is a continuous Bessel family if and only if it belongs to $L^2(X, \mu, \mathbb{F}^n)$.

Proof. ($\Rightarrow$) Suppose that $U = (u_x)_{x \in X}$ is a continuous Bessel family. For each $k \in [1, n]$, denote by $e_k$ the $k$-th vector of the standard basis of $\mathbb{F}^n$.

Applying the definition to the vector $e_k$, we have for each $k \in \{1, \cdots, n\}$:

$$\|U\|_{L^2(X, \mu, \mathbb{F}^n)}^2 = \sum_{k=1}^{n} \|U_k\|_{L^2(X, \mu, \mathbb{F}^n)}^2 < \infty,$$

which implies $U \in L^2(X, \mu; \mathbb{F}^n)$.

($\Leftarrow$) Suppose that $U = (u_x)_{x \in I} \in L^2(X, \mu; \mathbb{F}^n)$. We have

$$\forall v \in \mathbb{F}^n : \int_{x \in X} |(v, u_x)|^2 d\mu(x) \leq \|U\|_{L^2(X, \mu, \mathbb{F}^n)}^2 \|v\|^2 < \infty$$

by the Cauchy-Schwarz inequality, which implies that $U = (u_x)_{x \in X}$ is a continuous Bessel family. \qed
Lemma 2.1. If \( U \in L^2(X, \mu; \mathbb{F}^n) \), then \([S_U] = \text{Gram}(U^1, \ldots, U^n)\).

Proof. Let \((e_k)_{k \in [1,n]}\) the standard basis of \(\mathbb{F}^n\). Let \(i, j \in [1,n]\). Then

\[
[S_U]_{i,j} = \langle Se_j, e_i \rangle = \int_X \langle e_j, u_x \rangle \langle u_x, e_i \rangle d\mu(x) = \int_X \overline{u_x} u_x^i d\mu(x) = \langle U^i, U^j \rangle.
\]

\(\square\)

Proposition 2.3. \([10]\) Suppose \(\Phi = (\varphi_x)_{x \in X}\) is a family in \(\mathbb{F}^n\). Then

\(\Phi\) is a continuous frame \(\iff\) \(\Phi \in L^2(X, \mu; \mathbb{F}^n)\) and \(S_\Phi\) is invertible

\(\iff\) \(\Phi \in L^2(X, \mu; \mathbb{F}^n)\) and \(\det(\text{Gram}(\Phi^1, \ldots, \Phi^n)) > 0\)

\(\iff\) \(\Phi \in L^2(X, \mu; \mathbb{F}^n)\) and \(\{\Phi^1, \ldots, \Phi^n\}\) is free.

Proposition 2.4. \([10]\) Suppose \(\Phi = \{\varphi_x\}_{x \in X}\) is a family in \(\mathbb{F}^n\) and let \(a > 0\). Then

\(\Phi\) is a measurable \(a\)-tight frame \(\iff\) \(\Phi \in L^2(X, \mu; \mathbb{F}^n)\) and \(S_\Phi = aI_n\)

\(\iff\) \(\Phi \in L^2(X, \mu; \mathbb{F}^n)\) and \(\text{Gram}(\Phi^1, \ldots, \Phi^n) = aI_n\)

\(\iff\) \(\Phi \in L^2(X, \mu; \mathbb{F}^n)\) and \(\{\Phi^1, \ldots, \Phi^n\}\) is an orthogonal family of \(L^2(X, \mu; \mathbb{F})\) and \((\forall i \in [1,n]) : \|\Phi^i\| = \sqrt{a}\).

Example 2.1. Define \(\varphi^1_m = \frac{1}{m} e^{2\pi ima}\) and \(\varphi^2_m = \frac{1}{m} e^{2\pi imb}\) with \(a, b\) two real numbers such that \(a - b\) is not an integer. Then \(\Phi^1 = (\varphi^1_m)_{m \in \mathbb{N}}\) and \(\Phi^2 = (\varphi^2_m)_{m \in \mathbb{N}}\) are square summable with sum \(\frac{x^2}{b}\). Since the sequences \(\Phi^1\) and \(\Phi^2\) are not proportional due to the constraint on \(a\) and \(b\), it follows by 2.3 that \(\Phi\) is a discrete frame in \(\mathbb{C}^2\). It is not however a tight frame since \(\Phi^1\) and \(\Phi^2\) are not orthogonal.

2.3 Basic topological properties of \(St(n, H)\) and \(St_o(n, H)\)

In this subsection, we introduce \(St(n, H)\) and \(St_o(n, H)\) as well as some of their basic topological properties. We recall that \(n\) is a fixed element of \(\mathbb{N}^+\). If \(H\) is a Hilbert space, then \(St(n, H)\) is non-empty exactly when \(\dim(H) \geq n\). In the following, we will always suppose this condition.

Definition 2.2. The Stiefel manifold of independent \(n\)-frames in \(H\) is defined by \(St(n, H) := \{h = (h_1, \ldots, h_n) \in H^n : \{h_1, \ldots, h_n\}\) is free\}. The Stiefel manifold of orthonormal \(n\)-frames in \(H\) is defined by \(St_o(n, H) := \{h = (h_1, \ldots, h_n) \in H^n : \{h_1, \ldots, h_n\}\) is an orthonormal system\}.

Proposition 2.5. We have

1. \(St(n, L^2(X, \mu; \mathbb{F}))\) is isometric to \(\mathcal{F}_{(X, \mu), n}^\mathbb{F}\).
2. \( St_o(n, L^2(X, \mu; F)) \) is isometric to the set of continuous \((X, \mu)\)-Parseval frames with values in \( \mathbb{F}^n \).

**Proof.** Define

\[
\text{Transpose} : \begin{cases} 
L^2(X, \mu; \mathbb{F}^n) & \to L^2(X, \mu; \mathbb{F})^n \\
F = (f_x)_{x \in X} & \mapsto ((f^1_x)_{x \in X}, \ldots, (f^n_x)_{x \in X})
\end{cases}
\]

Then **Transpose** is clearly an isometry, and it sends \( F^{\mathbb{F}^n}_{(X, \mu), n} \) to \( St(n, L^2(X, \mu; \mathbb{F})) \) and \( St_o(n, L^2(X, \mu; \mathbb{F})) \) to the set of continuous \((X, \mu)\)-Parseval frames with values in \( \mathbb{F}^n \) by propositions 2.3 and 2.4 respectively.

**Remark 2.1.** Because of proposition 2.5, the reader should keep in mind that the following topological properties and the new results of this article are also shared, for any measure space \((X, \Sigma, \mu)\), by \( F_{(X, \mu), n} \) or the set of continuous \((X, \mu)\)-Parseval frames with values in \( \mathbb{F}^n \), depending on the context.

**Proposition 2.6.** We have

1. \( St(n, H) \) is open in \( H^n \).
2. \( St_o(n, H) \) is closed in \( H^n \).

**Proof.**

1. \( St(n, H) \) is open because \( St(n, H) = (\det \circ \text{Gram})^{-1}((0, \infty)) \).

2. \( St_o(n, H) \) is closed because \( St_o(n, H) = \text{Gram}^{-1}(I_n) \). \( \square \)

**Proposition 2.7.** \( St(n, H) \) is dense in \( H^n \).

**Proof.** Consider \( h = (h_1, \ldots, h_n) \in H^n \). Pick some \( \theta = (\theta_1, \ldots, \theta_n) \in St(n, H) \). Let \( \gamma \) be the straight path connecting \( \theta \) to \( h \), i.e. for each \( t \in [0, 1] : \gamma(t) = th + (1 - t)\theta \in H^n \). Let \( \Gamma(t) = \det(\text{Gram}((\gamma(t)_1, \ldots, \gamma(t)_n))) \). Clearly, \( \Gamma(t) \) is a polynomial function in \( t \) which satisfies \( \Gamma(0) \neq 0 \) since \( \theta \in St(n, H) \). Therefore

\[
\Gamma(t) \neq 0 \text{ except for a finite number of } t \text{'s.}
\]

Moreover,

\[
||\gamma(t) - u||_{H^n}^2 = \sum_{k=1}^{n} ||\gamma(t)_k - h_k||_{H}^2 = \sum_{i=1}^{n} |1 - t|^2 ||\theta_k - h_k||_{H}^2 \to 0 \text{ when } t \to 1
\]

Hence, there exists \( t \in [0, 1] \) such that \( \gamma(t) \) is close to \( h \) and \( \Gamma(t) \neq 0 \), and so \( \gamma(t) \in St(n, H) \). \( \square \)
By joining continuously each element of $St(n, H)$ to its corresponding Gram-Schmidt orthonormalized system in $St_o(n, H)$, we can prove

**Proposition 2.8.** $St_o(n, H)$ is a deformation retract of $St(n, H)$.

**Definition 2.3.** Let $X$ be a topological space and $m \in \mathbb{N}$. Then $X$ is said to be $m$-connected if its homotopy groups $\pi_i(X)$ are trivial for all $i \in [0, m]$.

**Proposition 2.9.** (see pp. 382-383 of [20] and [31]) We have

1. $St(n, \mathbb{R}^k)$ is $(k - n - 1)$-connected.
2. $St(n, \mathbb{C}^k)$ is $(2k - 2n)$-connected.
3. If $H$ is infinite dimensional, then $St(n, H)$ is contractible.

We also include the following proposition on the differential structure of the Stiefel manifolds.

**Proposition 2.10.** [23] We have

1. $St(n, \mathbb{R}^k)$ is a real manifold of dimension $nk$.
2. $St_o(n, \mathbb{R}^k)$ is a real manifold of dimension $nk - \frac{n(n+1)}{2}$.
3. $St(n, \mathbb{C}^k)$ is a real manifold of dimension $2nk$.
4. $St_o(n, \mathbb{C}^k)$ is a real manifold of dimension $2nk - n^2$.
5. If $\dim(H) = \infty$, then $St(n, H)$ and $St_o(n, H)$ are Hilbert manifolds of infinite dimension.

We ask the reader to keep in mind remark 2.1 when reading the remaining parts of this article.

### 3 Density

In the next proposition, we generalize the fact that $St(n, H)$ is dense in $H^n$. Before that, we need a definition.

**Definition 3.1.** Let $V$ be a $\mathbb{F}$-vector space, $q \in \mathbb{N}$, and $v, v' \in V$. We say that $\gamma : [0, 1] \to V$ is a polynomial path up to reparametrization of degree less than or equal to $q$ joining $v$ and $v'$ if there exist a measure space $(Z, \nu)$, a measurable family $(P_z)_{z \in Z}$ of polynomials with $P_z \in \mathbb{F}_q[X]$ for all $z \in Z$, a measurable family $(V_z)_{z \in Z}$ with $V_z \in H^n$ for all $z \in Z$ and a homeomorphism $\phi : [0, 1] \to [a, b] \subseteq \mathbb{R}$.
such that \( \forall t \in [0, 1] : \forall k \in [0, q] : \int_Z c_k(P_z)V^zdv(z) < \infty \) (\( c_k(P_z) \) denotes the k-th coefficient of \( P_z \)), \( \forall t \in [0, 1] : \gamma(t) = \int_Z P_z(\phi(t))V^zdv(z) \), \( \gamma(0) = v \) and \( \gamma(1) = v' \).

If \( V \) is equipped with a topology, then we say that \( \gamma \) is a continuous polynomial path when it is continuous as a map from \([0, 1] \) to \( V \).

**Lemma 3.1.** Let \( E \) be a normed vector space, \( q \in \mathbb{N} \) and \( v, v' \in E \). Then each polynomial path up to reparametrization \( \gamma : [0, 1] \to E \) of degree less than or equal to \( q \) joining \( v \) and \( v' \) is continuous.

**Proof.** Let \( \gamma : [0, 1] \to E \) be a polynomial path up to reparametrization of degree less than or equal to \( q \) joining \( v \) and \( v' \) up to reparametrization. Hence we can write \( \forall t \in [0, 1] : \gamma(t) = \int_Z P_z(\phi(t))V^zdv(z) \). The continuity of \( \gamma \) follows from

\[
\|\gamma(t) - \gamma(t')\| \leq \sum_{k=0}^{q} \left| \int_Z c_k(P_z)V^zdv(z) \right| |\phi(t)^k - \phi(t')^k| 
\]

which goes to 0 when \( t \) goes to \( t' \) by continuity of \( \phi^k \) for all \( k \in [0, q] \). \( \square \)

**Proposition 3.1.** Let \( q \in \mathbb{N} \), \( S \subseteq St(n, H) \), and \( \Theta \in St(n, H) \cap S \) such that for all \( U \in S \) there exists a polynomial path up to reparametrization of degree less than or equal to \( q \) joining \( \Theta \) and \( U \) and contained in \( S \). Then \( St(n, H) \cap S \) is dense in \( S \).

**Proof.** Consider \( U \in S \). Pick a polynomial path up to reparametrization \( \gamma \) of degree less than or equal to \( q \) in \( S \) connecting \( \Theta \) to \( U \) and contained in \( S \). Composing \( \gamma \) with \( \phi^{-1} \) if necessary, we can w.l.o.g. assume that \( \gamma \) is defined on \([a, b], \forall t : [a, b] : \gamma(t) = \int_Z P_z(t)V^zdv(z), \gamma(a) = \Theta \) and \( \gamma(b) = U \). Let \( \Gamma(t) = \det(Gram((\gamma(t)_1, \cdots, \gamma(t)_n))) \). Since for all \( i, j \in [1, n] \) and \( t \in [a, b] \)

\[
\langle \gamma(t)_i, \gamma(t)_j \rangle = \left\langle \int_Z P_z(t)v^i_zdv(z), \int_Z P_z(t)v^j_zdv(z) \right\rangle
= \sum_{k, k'=0}^q \left\langle \int_Z c_k(P_z)v^i_zdv(z), \int_Z c_{k'}(P_z)v^j_zdv(z) \right\rangle t^{k+k'}
\]

is a polynomial function in \( t \in [a, b] \), and the determinant of a matrix in \( M_{n,n}(\mathbb{F}) \) is a polynomial function in its coefficients, \( \Gamma(t) \) is a polynomial function in \( t \in [a, b] \) which satisfies \( \Gamma(a) \neq 0 \) since \( \Theta \in St(n, H) \). Therefore

\[
\Gamma(t) \neq 0 \text{ except for a finite number of } t's.
\]

Hence, using continuity of \( \gamma \) (see lemma 3.1) at \( b \), there exists \( t \in [a, b] \) such that \( \Phi := \gamma(t) \in S \) is close to \( U \) and \( \Phi \in St(n, H) \). \( \square \)
This result shows the abundance of independent \( n \)-frames not only in \( H^n \) but in all polynomially curved subsets containing at least one independent \( n \)-frame. For instance, the previous proposition applies when \( S \) is a star domain of \( H^n \) with respect to some \( \Theta \in St(n, H) \cap S \), and in particular if it is a convex subset of \( H^n \) (such as an affine subspace) and contains some \( \Theta \in Y \cap S \).

The following propositions provide examples of subsets of \( L^2(X, \mu; F^n) \) to which the previous proposition applies. These examples are special because they are constructed from the inverse image of singletons by some familiar linear and pseudo-quadratic functions. In general, there are a multitude of examples of polynomial path-connected subsets \( S \subseteq St(n, H) \) with respect to some \( \Theta \in St(n, H) \cap S \).

**Proposition 3.2.** Let \( h \in L^2(X, \mu; F) \). Define the linear operator

\[
T : \begin{cases} L^2(X, \mu; F^n) & \rightarrow F^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) f_x d\mu(x). \end{cases}
\]

If there exists a measurable subset \( Y \subseteq X \) such that \( \dim(L^2(Y, \mu; F^n)) \geq n \) and \( \mu((X \setminus Y) \cap h^{-1}(\{0\})) > 0 \), then for all \( d \in F^n \), \( T^{-1}(\{d\}) \) contains a continuous frame \( \Phi = (\varphi_x)_{x \in X} \).

**Proof.** Since \( \dim(L^2(Y, \mu; F^n)) \geq n \), there exists a continuous frame \( (\varphi_y)_{y \in Y} \in F_{(Y, \mu), n} \). We extend \( (\varphi_y)_{y \in Y} \) by setting

\[
\varphi_x := \frac{h(x)}{\|h\|_{L^2(X \setminus Y, \mu; F)}^2} \left( d - \int_Y h(y) \varphi_y d\mu(y) \right) \quad \text{for all } x \in X \setminus Y.
\]

Let \( \Phi = (\varphi_x)_{x \in X} \). We have

\[
T(\Phi) = \int_X h(x) \varphi_x d\mu(x) = \int_Y h(y) \varphi_y d\mu(y) + \left( \int_{X \setminus Y} h(x) \frac{h(x)}{\|h\|_{L^2(X \setminus Y, \mu; F)}^2} d\mu(x) \right) \left( d - \int_Y h(y) \varphi_y d\mu(y) \right) = d.
\]

Moreover, \( \Phi \in F_{(X, \mu), n} \) since we have only completed \( (\varphi_y)_{y \in Y} \) by a function in \( L^2(X \setminus Y, \mu; F^n) \).

**Corollary 3.1.** Let \( (X, \Sigma, \mu) \) be a measure space, \( d \in \mathbb{C}^n \), and \( h \in L^2(X, \mu; F) \) such that \( T^{-1}(\{d\}) \) contains a continuous frame \( \Phi = (\varphi_x)_{x \in X} \) (see for instance proposition 3.2), where

\[
T : \begin{cases} L^2(X, \mu; F^n) & \rightarrow F^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) f_x d\mu(x). \end{cases}
\]

Then by proposition 3.1, \( F_{(X, \mu), n} \cap T^{-1}(\{d\}) \) is dense in \( T^{-1}(\{d\}) \).
Proposition 3.3. Let $(X, \Sigma, \mu)$ be a measure space, $l \in \mathbb{N}^*$, $(X_j)_{j \in [1,l]}$ a partition of $X$ by measurable subsets, and $h \in L^2(X, \mu; \mathbb{F})$ such that there exist a family $(Y_j)_{j \in [1,l]}$ with $Y_j$ a measurable subset of $X_j$ for all $j \in [1,l]$, $\mu((X_j \setminus Y_j) \cap h^{-1}(\mathbb{F}^*)) > 0$ for all $j \in [1,l]$, and $\sum_{j=1}^l \dim(L^2(Y_j, \mu; \mathbb{F})) \geq n$.

Define the operator
\[
W : \begin{cases} 
L^2(X, \mu; \mathbb{F}^n) \to \prod_{j \in [1,l]} \mathbb{F}^n \\
F = (f_x)_{x \in X} \mapsto (f_{X_j} h(x) f_x d\mu(x))_{j \in [1,l]}.
\end{cases}
\]

Then for all $D = (d_j)_{j \in [1,l]} \in \prod_{j \in [1,l]} \mathbb{F}^n$, $W^{-1}(\{D\})$ contains at least one continuous frame $\Phi \in \mathcal{F}^F_{(X,\mu),n}$.

Remark 3.1. Proposition 3.2 results from proposition 3.3 by taking $l = 1$.

Remark 3.2. Proposition 3.3 can be generalized to $l = +\infty$ or to partitions indexed by a general index set $J$ if we restrict to $D = 0$ (due to convergence issues).

Proof. For each $i \in [1,n]$, let $e_i$ be the $i$-th vector of the standard basis of $\mathbb{F}^n$.

Since $\sum_{j=1}^l \dim(L^2(Y_j, \mu; \mathbb{F})) \geq n$, we can find distinct $j_1, \cdots, j_r \in [1,l]$ such that for each $u \in [1,r]$, $\dim(L^2(Y_{j_u}, \mu; \mathbb{F})) \geq 1$ and $\sum_{u=1}^r \dim(L^2(Y_{j_u}, \mu; \mathbb{F})) \geq n$. Take a partition $P_1, \cdots, P_r$ of $\{e_1, \cdots, e_n\}$ with $|P_u| \leq \dim(L^2(Y_{j_u}, \mu; \mathbb{F}))$ for all $u \in [1,r]$. For all $u \in [1,r]$, let $(g_{p,}^u)_{p \in P_u}$ be an orthonormal family in $L^2(Y_{j_u}, \mu; \mathbb{F})$ and define $\varphi_{x \in X_{j_u}}$ by
\[
\varphi_y = \sum_{p \in P_u} g_{p,y}^u \quad \text{for all } y \in Y_{j_u}
\]
and
\[
\varphi_x := \frac{h(x)}{\|h\|_{L^2(X_{j_u} \setminus Y_{j_u}, \mu; \mathbb{F})}^2} \left( d_{j_u} - \int_{Y_{j_u}} \varphi_y d\mu(y) \right) \quad \text{for all } x \in X_{j_u} \setminus Y_{j_u}.
\]

For all $j \notin \{j_u : u \in [1,r]\}$, define $\varphi_{x \in X_j}$ by
\[
\varphi_y = 0 \quad \text{for all } y \in Y_j
\]
and
\[
\varphi_x := \frac{h(x)}{\|h\|_{L^2(X_j \setminus Y_j, \mu; \mathbb{F})}^2} \quad \text{for all } x \in X_j \setminus Y_j.
\]
Let $\Phi = (\varphi_x)_{x \in X}$. We have

$$W(\Phi) = \left( \int_{X_j} h(x) \varphi_x d\mu(x) \right)_{j \in [1,l]}$$

$$= \left( \int_{Y_j} h(y) \varphi_y d\mu(y) + \left( \int_{X_j \setminus Y_j} h(x) \frac{h(x)}{\|h\|_{L^2(X_j \setminus Y_j, \mu; \mathbb{F})}} d\mu(x) \right) (d_j - \int_{Y_j} h(y) \varphi_y d\mu(y)) \right)_{j \in [1,l]}$$

$$= (d_j)_{j \in [1,l]} = D.$$ 

Moreover, $\Phi \in \mathcal{F}^F_{(X,\mu),n}$ since

$$\forall v \in \mathbb{F}^n : \|v\|^2 = \sum_{u=1}^r \int_{Y_{j_u}} |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x)$$

and

$$\int_X |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq \|v\|^2 + \left( \sum_{u=1}^r \left\| d_{j_u} - \int_{Y_{j_u}} h(y) \varphi_y d\mu(y) \right\|_2 \right) \|v\|^2$$

$$+ \left( \sum_{j \notin \{j_u : u \in [1,r]\}} \left\| d_j \right\|_{L^2(X_j, \mu; \mathbb{F})} \right) \|v\|^2$$

\[ \square \]

**Corollary 3.2.** Let $(X, \Sigma, \mu)$ be a measure space, $l \in \mathbb{N}^*$, $(X_j)_{j \in [1,l]}$ a partition of $X$ by measurable subsets, $h \in L^2(X, \mu; \mathbb{F})$, and $D \in \mathbb{F}^n$ such that $W^{-1}(\{D\})$ contains a continuous frame $\Phi = (\varphi_x)_{x \in X}$ (see for instance proposition 3.3), where

$$W : \left\{ \begin{array}{c} L^2(X, \mu; \mathbb{F}^n) \\ F = (f_x)_{x \in X} \end{array} \right\} \rightarrow \prod_{j \in [1,l]} \mathbb{F}^n$$

Then $\mathcal{F}^F_{(X,\mu),n} \cap W^{-1}(\{D\})$ is dense in $W^{-1}(\{D\})$.

**Proposition 3.4.** Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space, $b \in \mathbb{C}^n \setminus \{0\}$, $d \in \mathbb{C}^n$, $\epsilon > 0$ and $h \in L^\infty(X, \mu; \mathbb{C})$ such that

- if $d \neq 0$, then there exists a measurable subset $Y \subseteq X$, two measurable subsets $B_1 \subseteq \{ z \in \mathbb{C} : \text{Re}(\langle b, d \rangle z) > \epsilon \text{ and } \text{Im}(\langle b, d \rangle z) < -\epsilon \}$ and $B_2 \subseteq \{ z \in \mathbb{C} : \text{Re}(\langle b, d \rangle z) > \epsilon \text{ and } \text{Im}(\langle b, d \rangle z) > \epsilon \}$ such that $\dim(L^2(Y, \mu; \mathbb{C})) \geq n$ and $\mu((X \setminus Y) \cap h^{-1}(B_1)), \mu((X \setminus Y) \cap h^{-1}(B_2)) > 0$.

- if $d = 0$, then there exist a measurable subset $Y \subseteq X$ such that $\dim(L^2(Y, \mu; \mathbb{C})) \geq n$ and $h(x) < 0$ $\mu$-almost everywhere on $Y$, and \( |\text{two} \)
measurable subsets $B_1 \subseteq \{ z \in \mathbb{C} : \text{Re}(z) > 0 \text{ and } \text{Im}(z) < 0 \}$ and $B_2 \subseteq \{ z \in \mathbb{C} : \text{Re}(z) > 0 \text{ and } \text{Im}(z) > 0 \}$ such that $\mu((X \setminus Y) \cap h^{-1}(B_1)), \mu((X \setminus Y) \cap h^{-1}(B_2)) > 0$ or (a measurable subset $B_3 \subseteq \{ z \in \mathbb{C} : \text{Re}(z) > 0 \text{ and } \text{Im}(z) = 0 \}$ such that $\mu((X \setminus Y) \cap h^{-1}(B_3)) > 0$).

Then there exists a continuous frame $\Phi = (\varphi_x)_{x \in X} \in q^{-1}\{\{d\}\}$, where

$$q : \begin{cases} L^2(X, \mu; \mathbb{C}^n) & \rightarrow \mathbb{C}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) \langle b, f_x \rangle f_x d\mu(x) \end{cases}.$$  

Proof.  

• Suppose $d \neq 0$. Let $\widetilde{B}_1 = (X \setminus Y) \cap h^{-1}(B_1)$ and $\widetilde{B}_2 = (X \setminus Y) \cap h^{-1}(B_2)$. There is no loss in generality in assuming that $\mu(\widetilde{B}_1)$ and $\mu(\widetilde{B}_2)$ are finite since $\mu$ is $\sigma$-finite. Let $a < \frac{\epsilon}{\|h\|_{L^2(Y,\mu;\mathbb{C})}^2}$. Since $\dim(L^2(Y,\mu;\mathbb{C})) \geq n$, we can pick an $a$-tight frame $(\varphi_y)_{y \in Y} \in F_{(Y,\mu),n}$. Let $\widetilde{h}(x) = (\langle b, d \rangle - \int_Y h(y) \langle b, \varphi_y \rangle^2 d\mu(y)) h(x)$ for all $x \in X$. Notice that we have

$$\text{Re}(\widetilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_1,$$

$$\text{Im}(\widetilde{h}(x)) < 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_1,$$

$$\text{Re}(\widetilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_2,$$

and

$$\text{Im}(\widetilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_2.$$

Let

$$A = \frac{1}{\frac{\langle -\text{Im}(h), \text{Re}(h) \rangle_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}}{\| \text{Im}(h) \|_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}^2} + \frac{\langle \text{Im}(h), \text{Re}(h) \rangle_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}}{\| \text{Im}(h) \|_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}^2}} > 0$$

and

$$g(x) = \sqrt{A} \frac{\sqrt{-\text{Im}(\widetilde{h}(x))}}{\| \text{Im}(\widetilde{h}) \|_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}} 1_{\widetilde{B}_1}(x) + \sqrt{A} \frac{\sqrt{\text{Im}(\widetilde{h}(x))}}{\| \text{Im}(\widetilde{h}) \|_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}} 1_{\widetilde{B}_2}(x) \text{ for all } x \in X \setminus Y$$

Then it is easily seen that

$$\left( \int_{X \setminus Y} h(x) |g(x)|^2 d\mu(x) \right) \left( -\langle b, d \rangle + \int_Y h(y) \langle b, \varphi_y \rangle^2 d\mu(y) \right) = -1. \quad (1)$$

Consider $(\varphi_x)_{x \in X \setminus Y}$ defined by $\varphi_x = g(x) (-d + \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y))$ for all $x \in X \setminus Y$. Then $\Phi = (\varphi_x)_{x \in X} \in F_{(X,\mu),n}$ since we have only completed
\((\varphi_y)_{y \in Y}\) by a function in \(L^2(X \setminus Y, \mu; \mathbb{C}^n)\). Moreover

\[
q(\Phi) = \int_X h(x) \langle b, \varphi_x \rangle \varphi_x d\mu(x)
\]
\[
= \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y)
\]
\[
+ \int_{X \setminus Y} h(x) \langle b, g(x) \rangle \left( -d + \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \right) g(x) \left( -d + \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \right)
\]
\[
= \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) - \left[ \int_{X \setminus Y} h(x) |g(x)|^2 \left( -\langle b, d \rangle + \int_Y h(y) \langle b, \varphi_y \rangle |\varphi_y|^2 d\mu(y) \right) d\mu(x) \right] d
\]
\[
+ \left[ \int_{X \setminus Y} h(x) |g(x)|^2 \left( -\langle b, d \rangle + \int_Y h(y) \langle b, \varphi_y \rangle |\varphi_y|^2 d\mu(y) \right) d\mu(x) \right] \left( \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \right)
\]
\[
= d
\]
using equality 1.

- Suppose \(d = 0\). Let \(\widetilde{B}_1 = (X \setminus Y) \cap h^{-1}(B_1)\) and \(\widetilde{B}_2 = (X \setminus Y) \cap h^{-1}(B_2)\). Suppose first that \(\mu(\widetilde{B}_1), \mu(\widetilde{B}_2) > 0\). There is no loss in generality in assuming that \(\mu(\widetilde{B}_1)\) and \(\mu(\widetilde{B}_2)\) are finite since \(\mu\) is \(\sigma\)-finite. Since \(\text{dim}(L^2(Y, \mu; \mathbb{C})) \geq n\), we can pick a frame \((\varphi_y)_{y \in Y} \in \mathcal{F}_{(Y, \mu), n}\). Let

\[
\tilde{h}(x) = -\left( \int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y) \right) h(x)
\]

for all \(x \in X\). Notice that we have

\[
\text{Re}(\tilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_1,
\]
\[
\text{Im}(\tilde{h}(x)) < 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_1,
\]
\[
\text{Re}(\tilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_2,
\]

and

\[
\text{Im}(\tilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_2.
\]

Let

\[
A = \frac{1}{\text{Re}(\tilde{h}(x))_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}^2 + \text{Im}(\tilde{h}(x))_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}^2} > 0
\]

and

\[
g(x) = \sqrt{A} \frac{-\text{Im}(\tilde{h}(x))}{\text{Re}(\tilde{h}(x))_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}} 1_{\widetilde{B}_1}(x) + \sqrt{A} \frac{\text{Im}(\tilde{h}(x))}{\text{Re}(\tilde{h}(x))_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}} 1_{\widetilde{B}_2}(x)
\]
for all \(x \in X \setminus Y\).
Then it is easily seen that
\[
\left( \int_{X \setminus Y} h(x)|g(x)|^2d\mu(x) \right) \left( \int_Y h(y)|\langle b, \varphi_y \rangle|^2d\mu(y) \right) = -1. \tag{2}
\]
Consider \((\varphi_x)_{x \in X \setminus Y}\) defined by \(\varphi_x = g(x)\int_Y h(y)\langle b, \varphi_y \rangle \varphi_y d\mu(y)\) for all \(x \in X \setminus Y\). Then \(\Phi = (\varphi_x)_{x \in X} \in \mathcal{F}_{(X, \mu), n}\) since we have only completed \((\varphi_y)_{y \in Y}\) by a function in \(L^2(X \setminus Y, \mu; \mathbb{C}^n)\). Moreover
\[
q(\Phi) = \int_X h(x)\langle b, \varphi_x \rangle \varphi_x d\mu(x)
= \int_Y h(y)\langle b, \varphi_y \rangle \varphi_y d\mu(y)
+ \int_{X \setminus Y} h(x)\langle b, g(x) \int_Y h(y)\langle b, \varphi_y \rangle \varphi_y d\mu(y) \rangle \varphi_x d\mu(x)
= \int_Y h(y)\langle b, \varphi_y \rangle \varphi_y d\mu(y)
+ \int_{X \setminus Y} h(x)|g(x)|^2 \left( \int_Y h(y)|\langle b, \varphi_y \rangle|^2d\mu(y) \right) d\mu(x)
\]
\[
= 0
\]
using equality 2.
Now let \(\tilde{B}_3 = (X \setminus Y) \cap h^{-1}(B_3)\) and suppose instead that \(\mu(\tilde{B}_3) > 0\). There is no loss in generality in assuming that \(\mu(\tilde{B}_3)\) is finite since \(\mu\) is \(\sigma\)-finite. Since \(\dim(L^2(Y, \mu; \mathbb{C})) \geq n\), we can pick a frame \((\varphi_y)_{y \in Y} \in \mathcal{F}_{(Y, \mu), n}\). Let \(\tilde{h}(x) = -\left( \int_Y h(y)|\langle b, \varphi_y \rangle|^2d\mu(y) \right) h(x)\) for all \(x \in X\). Notice that we have
\[
\tilde{h}(x) > 0 \quad \mu - \text{almost everywhere on } \tilde{B}_3.
\]
Let
\[
g(x) = \frac{\sqrt{\tilde{h}(x)}}{\|\tilde{h}\|_{L^2(\tilde{B}_3, \mu; \mathbb{C})}} 1_{\tilde{B}_3}(x) \quad \text{for all } x \in X \setminus Y
\]
Then it is easily seen that
\[
\left( \int_{X \setminus Y} h(x)|g(x)|^2d\mu(x) \right) \left( \int_Y h(y)|\langle b, \varphi_y \rangle|^2d\mu(y) \right) = -1. \tag{3}
\]
Consider \((\varphi_x)_{x \in X \setminus Y}\) defined by \(\varphi_x = g(x)\int_Y h(y)\langle b, \varphi_y \rangle \varphi_y d\mu(y)\) for all \(x \in X \setminus Y\). Then \(\Phi = (\varphi_x)_{x \in X} \in \mathcal{F}_{(X, \mu), n}\) since we have only completed \((\varphi_y)_{y \in Y}\) by a function in \(L^2(X \setminus Y, \mu; \mathbb{C}^n)\). Moreover we can prove that \(q(\Phi) = 0\) as before using equality 3.
\[
\square
\]
Remark 3.3. Consider the function \( q \) of the previous proposition. If \( \Phi = (\varphi_x)_{x \in X} \in q^{-1}(\{0\}) \), then we also have \( \Phi \in q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi) \) since \( q(2\Phi) = 0 \).

Proposition 3.5. Consider the function \( q \) of the previous proposition. Let \( \Phi = (\varphi_x)_{x \in X} \in q^{-1}(\{0\}) \) and \( U = (u_x)_{x \in X} \in q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi) \). Then for all \( \lambda, \mu \in \mathbb{R} \), \( \lambda \Phi + \mu U \in q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi) \). In particular, there exists a polynomial path of degree 1 joining \( U \) and \( \Phi \) and contained in \( q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi) \).

Proof. Let \( \lambda, \mu \in \mathbb{R} \). Let \( s \) be the sesquilinear form

\[
\begin{align*}
L^2(X, \mu; \mathbb{C}^n) &\to \mathbb{C}^n \\
(F, G) &\mapsto f_X(b, g_x)f_xd\mu(x).
\end{align*}
\]

We have

\[
q(\lambda \Phi + \mu U) = \lambda^2 q(\Phi) + \lambda \mu(s(\Phi, U) + s(U, \Phi)) + \mu^2 q(U)
\]

\[
= \lambda \mu(q(\Phi + U) - q(\Phi) - q(U))
\]

\[
= 0.
\]

We can show similarly that \( q((\lambda + 1)\Phi + \mu U) = 0 \).

Corollary 3.3. Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite measure space, \(b \in \mathbb{C}^n \setminus \{0\}\), and \(h \in L^\infty(X, \mu; \mathbb{C})\) such that \(q^{-1}(\{0\})\) contains a continuous frame \(\Phi = (\varphi_x)_{x \in X}\) (see for instance proposition 3.4), where

\[
q : \begin{cases}
L^2(X, \mu; \mathbb{C}^n) &\to \mathbb{C}^n \\
F = (f_x)_{x \in X} &\mapsto f_X h(x)\langle b, f_x \rangle f_x d\mu(x).
\end{cases}
\]

Then by propositions 3.1 and 3.5, \( \mathcal{F}_{(X, \mu), n}^c \cap q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi) \) is dense in \( q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi) \).

4 Path-connectedness

Definition 4.1. Let \( E \) be a topological vector space and \( \gamma : [0,1] \to E \) be a continuous path. We say that \( \gamma \) is a polygonal path if there exists \( q \in \mathbb{N}^* \), \((e_k)_{k \in [1,q]}\) and \((f_k)_{k \in [1,q]}\) two finite sequences with \( e_k, f_k \in E \) for all \( k \in [1,q] \), and \((\gamma_k)_{k \in [1,q]}\) a finite sequence of (continuous) straight paths with \( \gamma_k = \begin{cases} [t^q_{k-1}, t^q_k] &\to E \\
&\mapsto q(t - \frac{k-1}{q})f_k + q\frac{k}{q} - t e_k
\end{cases} \) such that \( \gamma = \gamma_1 \cdots \gamma_q \), where \( \ast \) is the path composition operation. We say that a subset \( S \subseteq E \) is polygonally connected if every two points of \( S \) are connected by a polygonal path.
In the following, when we say that \( S \subseteq E \) is polygonally connected, we mean that each two points of \( S \) are connected by a polygonal path of the type \( \gamma_1 * \gamma_2 \) where \( \gamma_1 \) and \( \gamma_2 \) are two straight paths.

Before we prove the main proposition of this section, let’s prove a useful lemma.

**Lemma 4.1.** 1. Suppose we have a family \( (a(j))_{j \in J} \) indexed by \( J \) where each \( a(j) \) belongs to \( St(n,H) \). Then if \( \bigcup_{j \in J} \bigcup_{k \in [1,n]} \{a(j)k\} \) is free, we have

\[
\text{span}\{\{a(j)\}_{j \in J}\} \setminus \{0\} \subseteq St(n,H)
\]

2. Suppose we have a family indexed by \( J \) where each \( a(j) \) belongs to \( St_o(n,H) \) for all \( j \in J \). Then if \( \bigcup_{j \in J} \bigcup_{k \in [1,n]} \{a(j)k\} \) is an orthonormal system, we have

\[
\{x \in \text{span}\{\{a(j)\}_{j \in J}\} : \|x\| = \sqrt{n}\} \subseteq St_o(n,H)
\]

**Proof.** 1. Let \( h = (h_1, \ldots, h_n) \in \text{span}\{\{a(j)\}_{j \in J}\} \setminus \{0\} \). We can write

\[
h = \sum_{u=1}^{r} \lambda_u a(j_u)
\]

with \( \lambda_u \in \mathbb{F} \) for all \( u \in [1,r] \) and the \( \lambda_u \)’s are not all zeros. We need to show that \((h_1, \ldots, h_n)\) is an independent system. Suppose otherwise \( \sum_{k=1}^{n} c_k h_k = 0 \). This means that \( \sum_{k=1}^{n} c_k (\sum_{u=1}^{r} \lambda_u a(j_u)k) = 0 \), and so \( \sum_{k=1}^{r} \sum_{u=1}^{n} \lambda_u c_k a(j_u)k = 0 \). Since \( \bigcup_{j \in J} \bigcup_{k \in [1,n]} \{a(j)k\} \) is free, we deduce that \( \lambda_u c_k = 0 \) for all \( u \in [1,r] \) and \( k \in [1,n] \), which implies that \( c_k = 0 \) for all \( k \in [1,n] \) since the \( \lambda_u \)’s are not all zeros. Therefore, \( h \in St(n,H) \).

2. Let \( h = (h_1, \ldots, h_n) \in \text{span}\{\{a(j)\}_{j \in J}\} \) such that \( \|h\| = \sqrt{n} \). We can write \( h = \sum_{u=1}^{r} \lambda_u a(j_u) \) with \( \lambda_u \in \mathbb{F} \) for all \( u \in [1,r] \). We need to show that \( \langle h_k, h_l \rangle = \delta_{k,l} \) for all \( k,l \in [1,n] \). For \( k \neq l \), we have:

\[
\langle h_k, h_l \rangle = \langle \sum_{u=1}^{r} \lambda_u a(j_u), \sum_{v=1}^{n} \lambda_v a(j_v) \rangle = \sum_{u=1}^{r} \sum_{v=1}^{n} \lambda_u \lambda_v \langle a(j_u), a(j_v) \rangle = 0
\]

since \( \bigcup_{j \in J} \bigcup_{k \in [1,n]} \{a(j)k\} \) is an orthogonal system. Moreover, \( \|h_k\|^2 = \sum_{u=1}^{r} \sum_{v=1}^{n} \lambda_u \lambda_v \|a(j_u)\|^2 = \sum_{u=1}^{r} |\lambda_u|^2 |a(j_u)\|^2 = \sum_{u=1}^{r} |\lambda_u|^2 |a(j_u)|^2 = \sum_{u=1}^{r} |\lambda_u|^2 = \sum_{u=1}^{n} |\lambda_u|^2 = n \sum_{u=1}^{n} |\lambda_u|^2 = n \sum_{u=1}^{n} |\lambda_u|^2 = 1 \) for all \( k \in [1,n] \) as desired.

\( \square \)

**Proposition 4.1.** Let \( H \) be a Hilbert space with \( \dim(H) \geq n \), \( l \in \mathbb{N}^* \cup \{+\infty\} \) and \((U(j))_{j \in [1,l]} \) a family with \( U(j) \in H^n \) for all \( j \in [1,l] \). If \( \text{codim}_H(\text{span}(\bigcup_{j \in [1,l]} \bigcup_{k \in [1,n]} \{u(j)k\})) \geq 3n \), then \( \bigcap_{j \in [1,l]} (U(j) + St(n,H)) \) is polygonally-connected.

**Remark 4.1.** Tangentially, we can observe that since the translation maps are homeomorphisms and \( St(n,H) \) is open and dense in \( H^n \) (propositions 2.6 and 2.7), \( \bigcap_{j=1}^{l} (U(j) + St(n,H)) \) is open and dense in \( H^n \) by Baire’s theorem.
Proof. Let $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$ in $\cap_{j=1}^l (U(j) + St(n, H))$. Let $(z_1, \cdots, z_n)$ be an independent family in $H$ such that
\[
\text{span} \left( \cup_{k \in [1,n]} \{z_k\} \right) \cap \text{span} \left( (\cup_{k \in [1,n]} \{x_k\}) \cup (\cup_{k \in [1,n]} \{y_k\}) \cup (\cup_{j \in [1,l]} \cup_{k \in [1,n]} \{u(j)_k\}) \right) = \{0\}
\]
This is possible since $\text{codim}_H (\text{span}(\cup_{j \in [1,l]} \cup_{k \in [1,n]} \{u(j)_k\})) \geq 3n$.
This ensures that we have for all $j \in [1,l]$,
\[
\text{span} \left( \cup_{k \in [1,n]} \{-u(j)_k + z_k\} \right) \cap \text{span} \left( \cup_{k \in [1,n]} \{-u(j)_k + x_k\} \right) = \{0\},
\]
\[
\text{span} \left( \cup_{k \in [1,n]} \{-u(j)_k + z_k\} \right) \cap \text{span} \left( \cup_{k \in [1,n]} \{-u(j)_k + y_k\} \right) = \{0\}
\]
and
\[
\{-u(j)_k + z_k\}_{k \in [1,n]} \text{ is independent.}
\]
We define the straight paths
\[
\begin{cases}
\gamma_1 : [0, 1] \to H^n \\
\gamma_2 : [0, 1] \to H^n 
\end{cases}
\]
by $\gamma_1(t) = tZ + (1-t)X$ and $\gamma_2(t) = tY + (1-t)Z$ respectively. We have $\gamma_1(0) = X$, $\gamma_1(1) = \gamma_2(0) = Z$, and $\gamma_2(1) = Y$.
Since for all $j \in [1,l]$
\[
\text{span} \left( \{-u(j)_1 + z_1, \cdots, -u(j)_n + z_n\} \cap \text{span} \left( \{-u(j)_1 + x_1, \cdots, -u(j)_n + x_n\} \right) = \{0\},
\]
we have for all $t \in [0, 1]$ and $j \in [1,l]$
\[
-U(j) + tZ + (1-t)X = t(-U(j) + Z) + (1-t)(-U(j) + X) \in St(n, H)
\]
by lemma 4.1, and so $\gamma_1(t) \in \cap_{j \in [1,l]} (U(j) + St(n, H))$.
Similarly, $\gamma_2(t) \in \cap_{j=1}^l (U(j) + St(n, H))$ for all $t \in [0, 1]$.
Composing $\gamma_1$ with $\gamma_2$, we see that $\cap_{j=1}^l (U(j) + St(n, H))$ is polygonally connected. 

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