Conformally Flat Metric, Position-Dependent Mass and Cold Dark Matter

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Abstract

The maximal acceleration (MA) problem associated with the position-dependent rest mass concept is considered. New arguments in favor of the mass-dependent maximal acceleration (MDMA) are put forward. The hypothesis that there exists a maximal force with the numerical value equal to the inverse Einstein’s gravitation constant is advanced. The Lagrangian and Hamiltonian classical dynamics of a point-like particle with the coordinate-dependent mass is given. The effective Lagrangian for the pure gravitational interaction of a test particle is proposed. Within the scope of this model the typical spiral galaxy rotation is described. It is shown that by this model the peculiar form of the corresponding rotation curve is as a whole reproduced without recourse to the dark matter concept. Also, it is demonstrated that the canonical quantization of this model leads directly to the Dirac oscillator model for a particle with Plank’s mass.

1 Introduction

The maximal acceleration hypothesis was first conjectured by Caianiello [1]. Different aspects, formulation and inferences concerning possible ex-

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istence of the limiting value for the proper acceleration of a particle have been considered in various works on the classical and quantum bases (see, for example [2] and the references therein). Despite the fact that there are many arguments supporting the existence of MA, its actual status is still open to dispute. Specifically, it is not clear whether the numerical value of MA should be considered as a universal constant, similar to the speed of light, or as a parameter depending on the individual mass due to the action of force (Mass-Dependent Maximal Acceleration - MDMA). It was shown that the effective conformally flat metric can arise directly from the existence of maximal acceleration [3]. This metric reveals interesting confinement aspects. Namely, the Lorentz-scalar potential and damping normalization factor introduced in particular relativistic phenomenological quarkonium models to provide quark confinement occur (in this case) purely geometrically as a consequence of the existence of the effective conformally flat static metric [1][8]. As it was first shown by A.K.Gorbatshevich and L.M.Tomilchik [1] (see also [5]-[7]), and somewhat later independently by M.Gasperini [8], such a metric leads to the appearance of the coordinate-dependent rest mass. The position-dependent mass was introduced by several authors in nonrelativistic quantum mechanical models developed to study the electronic properties of condensed media (see, for example [9]), including rather interesting attempts of pure geometrical interpretation of such a dependence via the constant curvature space [10].

The apparent efficiency of the quantum-mechanical models based on the ad hoc hypothesis of a coordinate-dependent rest mass suggests that this concept has universal nature. Therefore, it seems natural to extend it to the classical (nonquantum) objects as well as to find the general principles that could naturally lead to such a dependence. It is likely that a search for kinematic restrictions involving the maximal-acceleration hypothesis has considerable promise.

In any case, it is natural to expect that the observable effects possible due to the existence of MA should appear already at a level of the point-like particle classical dynamics.

In the present paper new arguments in favor of MDMA are put forward. The Lagrangian and Hamiltonian classical dynamics of a point-like particle with the coordinate-dependent mass is considered. Under the special choice of such dependence, the effective Lagrangian for the pure gravitational interaction is proposed. It is shown that within this model the peculiar form of the corresponding rotation curve is as a whole reproduced.
without the use of the cold dark matter concept. It is demonstrated that
the canonical quantization of this model leads directly to the Dirac oscil-
lator model for a particle with Plank’s mass.

2 Some Heuristic Considerations

It is well known that the existence of a maximal transmission velocity for
the signal synchronizing clocks separated spatially in each fixed inertial
reference frame leads to the appearance of four-dimensional space-time
with the pseudo-Euclidean structure.

But according to Einstein and Poincare, the procedure of clock syn-
chronization suggests the transmission of an instantaneous signal, i.e. a
signal whose duration can be made as small as is wished. On the other
hand, a synchronizing signal can be transmitted and accepted only due to
the exchange of any finite portion of energy $\delta E$ (and hence momentum $\delta p$).
If $E$ and $p$ are certain time-dependent functions, we have evident relations

$$\delta E = \frac{dE}{dt} \delta t, \delta p = \frac{dp}{dt} \delta t$$

relating the formation duration of a synchronizing signal ($\delta t$) to the val-
ues of the energy and momentum carried by the transmitted signal. In
principle, the quantities $\delta E$ and $\delta p$ should be finite, whereas the inter-
val $\delta t$ should always tend to zero, implicitly suggesting fulfillment of the
conditions

$$\lim_{\delta t \to 0} \frac{dE}{dt} = \lim_{\delta t \to 0} \frac{dp}{dt} = \infty.$$  \hspace{1cm} (1)

However, if the quantities $(\frac{dE}{dt})_{\text{lim}}$ and $(\frac{dp}{dt})_{\text{lim}}$, respectively representing
maximal power and maximal force, are assumed to be limiting, a nonzero
extra time retardation occurs:

$$\delta t_{\text{extra}} = (\frac{dE}{dt})_{\text{lim}}^{-1} \delta E, \text{ or } (\frac{dp}{dt})_{\text{lim}}^{-1} \delta p$$

that should be taken into account in clock synchronization. It is clear that
the smaller the distance between the synchronized clocks the more evident
the corrections (recall that the Einstein clock is point-like by definition).
Basing on this reasoning, it is suggested that there exists a constant $\kappa_0$ re-
representing the upper limit for possible values of the proper energy changing
with time \([11]\). In this paper, it has been proposed to combine the infinitesimal intervals of the Minkowski space \(ds\) and momentum space \(dp\) in accordance with the Born Reciprocity Principle \([12]\):

\[
dS^2 = ds^2 + \frac{1}{\kappa_0^2} dp^2 = dx^\mu dx_\mu + \frac{1}{\kappa_0^2} dp^\mu dp_\mu. \tag{2}
\]

Using the customary definition \(ds = cd\tau\) (\(\tau\) is the proper time), we obtain

\[
dS^2 = ds^2 \left\{1 + \frac{1}{F_0^2} \frac{dp^\mu dp_\mu}{d\tau d\tau}\right\} \tag{3}
\]

where the parameter \(F_0 = \kappa_0 c\) is a constant with the dimension of force.

The model proposed by Caianiello and his coworkers (see \([1]\), \([2]\), \([3]\)) to include the effects of maximal acceleration into the particle dynamics consisted in extension of the space-time manifold to the eight-dimensional space-time tangent bundle.

The fundamental infinitesimal interval for a particle is represented by the following eight-dimensional line element:

\[
dS^2 = dx^\mu dx_\mu + \frac{c^2}{A^2} d\dot{x}^\mu d\dot{x}_\mu \tag{4}
\]

where \(\dot{x}_\mu = \frac{dx_\mu}{d\tau}\). \(A\) is a parameter with the dimension of acceleration. Assuming the Minkowski background metric, i.e. \(g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\), and taking into account that \(d\dot{x}^\mu = d\ddot{x}^\mu d\tau\), we obtain from (4) that

\[
dS^2 = \left(1 + \frac{\ddot{x}^\mu \ddot{x}_\mu}{A^2}\right) ds^2. \tag{5}
\]

Here \(\ddot{x}^\mu = \frac{d^2x^\mu}{d\tau^2}\), \(ds^2 = dx^\mu dx_\mu = c^2 d\tau^2\), and \(\ddot{x}^\mu\) is the space-like vector (i.e. \(\ddot{x}^\mu \ddot{x}_\mu < 0\)). The explicit form of \(\ddot{x}^\mu \ddot{x}_\mu\) in the noncovariant notation is

\[
\ddot{x}^\mu \ddot{x}_\mu = -\gamma^3 \left\{W^2 - \frac{1}{c^2} (W \cdot V)^2\right\} \tag{6}
\]

where \(\beta = \frac{dr}{c dt} = \frac{\dot{r}}{c}, \ W = \frac{d^2r}{dt^2} = \ddot{r}, \ \gamma = (1 - \beta^2)^{-\frac{1}{2}}\). In case of \(\dot{r} = 0\) we obtain from (5), (6) the formula

\[
dS^2 = c^2 d\tau^2 \left(1 - \frac{W^2}{A^2}\right) \tag{7}
\]
demonstrating the limiting role of $A$. Taking the interval (3) for a material point, i.e. assuming $\frac{dE}{dt} = \frac{1}{c} \left( \vec{V} \frac{dp}{dt} \right)$, we obtain the expression

$$dS^2 = ds^2 \left\{ 1 - \left( 1 - \frac{\dot{r}^2}{c^2} \right)^{-1} \left( \frac{1}{F_o^2} \left( \frac{dp}{dt} \right)^2 \left( 1 - \frac{\dot{r}^2}{c^2} \cos^2 \phi \right) \right) \right\}$$

(8)

where $\phi$ is an angle between $\dot{r}$ and $\dot{p} = \frac{dp}{dt}$. From this formula it follows that using the same assumptions as in the derivation of relation (7) and with $\frac{dp}{dt} = f$ we obtain the expression

$$dS = c dt \left( 1 - \frac{f^2}{F_o^2} \right)^{1/2}$$

(9)

from whence the constant $F_o$ plays the limiting role. It is natural to identify this constant as maximal force (MF). We assume that this constant has purely classical nature, i.e. it is not related to a minimum value of the action $a_{min} = \hbar$ and represents the inverse of the Einstein gravitation constant, being numerically determined as

$$F_o = \frac{c^4}{G}.$$ 

(10)

Postulating this universal constant as $\kappa_o (F_o = c \kappa_o)$, we come to the representation of the mass-dependent maximal acceleration (MDMA)

$$\left( \frac{dV}{dt} \right)_{max} = W_{max} = A = \frac{F_o}{m} = \frac{c^4}{mG} = \frac{2c^2}{r_g(m)}$$

(11)

where $r_g(m) = \frac{2mG}{c^2}$ is the gravitation radius corresponding to the (point-like!) mass $m$.

It is seen from (10) and (11) that MDMA has an obvious classical, Newtonian meaning of the centripetal acceleration of a test point-like particle rotating uniformly in a circle whose radius is equal to the ”radius” of the Schwarzschild sphere.

Needless to say that this descriptive pattern should not be considered literally. Nevertheless, the idea that the Schwarzschild parameter $r_g$ is related to the MDMA scheme holds much promise. A model for hyperbolic motion of the point-like mass in the Special Relativity also indicates the existence of this relation. Here, as it is known, the corresponding world
line is given by the equation \( x^2 - ct^2 = r_0^2 \), where \( r_0 \) is a fixed parameter having the dimension of length. If the initial velocity is equal to zero, for the absolute value of the three-dimensional acceleration \( W \) we obtain the expression \( W = \frac{c^2}{r_0} \left( 1 + \left( \frac{ct}{r_0} \right)^2 \right)^{-\frac{3}{2}} \), from which it is seen that the quantity \( W_0 = \frac{c^2}{r_0} \) represents acceleration at the initial instant of time, i.e. it is the MDMA for the given mass \( m \). If \( W_0 \) is determined by formula (11), the parameter \( r_0 \) is described as \( r_0 = \frac{mG}{c^2} = \frac{1}{2} r_g \).

It may be inferred intuitively that, because of the existence of maximal acceleration, deviations of the mechanical motion of a point-like particle from the standard dynamics should manifest themselves under conditions when its acceleration tends to \( W_0 \) (or is comparable to it). It is clear that in case when \( A \) is determined by expression (11), the ”kinematic” manifestations of MDMA will be more evident for the material points with greater mass. Let us estimate the quantity \( \frac{W_0}{A} \) for a case of electromagnetic interactions. To this end, we use the expression for the force of the static Coulomb interaction of two charges \( e \) and \( Ze \) (i.e. \( mW = \frac{e^2Z}{r^2} \)). In this case the expression is as follows:

\[
\frac{W}{A} = \frac{f}{F_o} = \frac{e^2 Z}{r^2 F_o} \sim \frac{e^2 G Z}{c^4} r^2 = r_o^2
\]

where \( r_o^2 = \frac{e^2 G Z}{c^4} \) and hence the corrections to unity in (9) will be equal to \( \sim \left( \frac{r_o}{r} \right)^4 \).

As it is seen, \( r_o^2 \sim \frac{1}{2} r_c r_g \), where \( r_c = \frac{e^2}{mc^2} \), \( r_g = \frac{2mG}{c^2} \) are the classical and gravitation radii of the point-like charged mass for the interaction of two equal charges.

For all the elementary particles, the quantity \( r_o \) vanishes, being equal to \( r_o \sim (10^{-68})^{1/2} \sim 10^{-34} \text{ cm} \).

Needless to say that such reasoning gives only a rough estimate, since the use of the classical parameters of electromagnetic interaction is inadequate in real situation (it is well known that for such interactions the quantum effects become significant at distances on the order of the atomic dimensions). Nevertheless, a fixed value of the parameter \( r_0 \) (\( r_0 \) is of the order of Plank’s length) points to the fact that the effects related to MDMA
can influence the known elementary particles only at distances comparable to Plank’s length (i.e. at energies of $\sim 10^{19}$ GeV) associated with the Plank energy scales. The same may be valid for strong interactions too.

A distinct situation is observed in case of gravitation interactions.

First, note that the concepts of force and acceleration are used in modern physics only in the classical context. This being so, it is not imperative, in our opinion, to relate the numerical value of maximal acceleration to the constant $\hbar$, as this is made commonly (see [2]). The existence of the maximal force interpreted purely classically, however, should exclude a fall at the center for the pattern of mutual attraction of two point-like particles, or should lead to the appearance of some effective repulsion. From this standpoint, the situation is qualitatively similar to the well-known effects of quantum dynamics caused by the Plank’s quantum of action and hence noncommutativity of the canonically conjugate coordinates and momenta.

To illustrate the "repulsion" effect caused by the classical maximal force, we use the elementary concepts based on the Newtonian law of gravitation. If there exists the above-postulated maximal force, there should exist a minimum distance $r_0$ between two attracting point-like masses. This distance may be determined from the condition

$$\frac{mMG}{r_0^2} = \frac{c^4}{G}$$

whence the expression for parameter $r_0$ may be found

$$r_0^2 = \frac{1}{4}r_gR_g$$

where $r_g, R_g$ are the corresponding Schwarzschild radii.

This result correlates well with the conclusion that the Schwarzschild sphere is principally impenetrable for the test classical particle obtained in [13] on the basis of the solution of the motion problem in the Schwarzschild field with regard to maximal acceleration.

It is obvious that the absence of fall at the center in such a two-particle problem means that there exists some nonzero angular momentum. Actually, from the standpoint of a "naive" model, a minimum distance, where the centers of two small balls with masses $m$ and $M$ and hence radii $r_g = \frac{2mG}{c^2}, R_g = \frac{2MG}{c^2}$ can come of each other, is equal to $r_g + R_g$. We easily calculate that the moment of inertia of such a system with respect to its center of mass (disregarding the proper rotation of the balls) is equal
For such a system there should exist a maximal frequency of rotation around the center of mass. An approximate estimation of the quantity $\omega_{\text{max}}$ within the scope of the model considered gives the expression

$$\omega_{\text{max}} = \frac{c}{R_g}.$$  

(15)

Proceeding from this expression, for a minimal value of the angular momentum we obtain

$$L_{\text{min}} = I_0 \omega_{\text{max}}.$$  

(16)

Actually, this nonzero angular momentum cannot be attributed to either of the two particles but belongs to the system as a whole. The situation is similar to Extra Spin in the electric charge - magnetic monopole system.

Now consider the applicability of the model under study to classical systems. Apart from such a fairly perfect classical theory of gravitation as the Standard General Relativity, there is quite a number of purely gravitational, comparatively isolated systems, the observed mechanical properties of which may be described in reality on the basis of the models correlating with the ordinary Newton approximation. First, consider rotation of the most abundant, typical spiral galaxies, in a sense, belonging to the simplest astrophysical objects. There is a reason to believe that such systems may be considered within the framework of the Newton approximation: the observed intragalaxy velocities do not exceed several thousandth of the speed of light, and the corresponding Schwarzschild radius measures fraction of one parsec (this value is by a factor of $10^4 - 10^5$ smaller than the characteristic dimensions of a galaxy, even having a mass on the order of $10^{12}$ of the Sun mass). Therefore, modeling of the observed rotation of stars around the galactic center by the nonrelativistic movement of a material point in a centrally-symmetric field, representing a combination of the Newton attraction and gravitational attraction potential linearly increasing with the distance, seems to be wholly warranted.

At the same time, the observed asymptotic behavior of the rotational curves for the typical spiral galaxies is in drastic contradiction with such a theoretical representation. The most popular idea that should eliminate this contradiction is currently associated with the cold-dark-matter concept. Alternative explanation schemes based on the attempts to modify
the Newton model of gravitational interactions (see [14] and the literature therein) are also available. In our opinion, using the coordinate dependence of the rest mass offers great promise here. As it will be shown below, the use of a simple classical model enables one to reproduce the general shape of the rotational curve for the typical spiral galaxy practically over the whole range of distances from its center.

When using the conception of phase space and Hamiltonian dynamics, it is convenient to introduce, \textit{a la} M.Born [12], the parameters \( q_0, p_0 \) having the dimensions of length and momentum, respectively. For each specific physical system with a finite action these parameters are assumed to be determined by the relations \( q_0 p_0 = a, p_0/q_0 = \kappa_0 \), where \( \kappa_0 \) is a universal constant, \( \kappa_0 = \frac{c^3}{G} \), and \( a \) is the parameter with the dimension of action determined for each classical or quantum system. Its minimum value is equal to the Planck universal constant \( \hbar \). Thus, the following definitions are true for the parameters: \( q_0 = (a \kappa_0^{-1})^{\frac{1}{2}}, p_0 = (a \kappa_0)^{\frac{1}{2}} \).

It is evident that for \( a_{\min} = \hbar \) the parameters \( q_0, p_0 \) are equal to the corresponding Planck's quantities \( l_p, p_p \). And the ratio \( \frac{p_0}{q_0} \) is independent of \( a \), having the same value both for classical and quantum systems \(^1\).

Let us consider the relation between maximal acceleration and conformal transformation.

It is well known that attempts to interpret the special conformal transformation (SCT) in the Minkowski Space were made even by L.Page and N.I.Adams (see [15] and papers cited there). This transformation may be written in the following form:

\[
x'^\mu = \sigma(x, b)(x^\mu + b^\mu x^2)
\]

where

\[
\sigma(x, b) = (1 + 2b x + b^2 x^2)^{-1},
\]

\[
b x = b^\mu x_\mu = b_\mu x^\mu,
\]

\( b^\mu \) is the four-vector parameter with the dimension of \( \text{(length)}^{-1} \). The parameter \( b^\mu \) is traditionally related to the constant relative four-dimensional acceleration of the reference frame.

\(^1\)Note that from the geometrical standpoint, the constant \( \kappa_0 = p_0/q_0 \) determines maximal deformation ("Prokrust strain") of a given phase area (including an elementary phase cell).
Using this interpretation, we write \( b^\mu = c^{-2} A^\mu \), where \( A^\mu \) denotes the constant relative four-dimensional acceleration. Besides, it is required that the obtained expression be coincident, in the limit, with the path formula for the uniformly accelerated movement along the \( x \) axis. The vector \( b^\mu \) should be space-like.

Let us write \( b^\mu \) in the following form:

\[
 b^\mu = \{0, c^{-2} W, 0, 0\} \tag{18}
\]

where \( W \) is the \( x \)-component of the three-dimensional acceleration.

For world lines of the point-like particles only the interior of the light cone is available. Specifically, it is assumed that \( x^\mu = \{ct, 0, 0, 0\} \). Then we obtain

\[
x^2 = c^2 t^2, \quad bx = 0, \quad b^2 = -\frac{w^2}{4c^4}, \quad \sigma(x, b) = 1 - \left(\frac{Wt}{2c}\right)^2.
\]

Consequently, transformations (17) take the following form:

\[
x^\mu = \frac{Wt^2}{2} \left(1 - \left(\frac{Wt}{2c}\right)^2\right)^{-1}, \quad t' = t \left(1 - \left(\frac{Wt}{2c}\right)^2\right)^{-1}. \tag{19}
\]

If the term \( \left(\frac{Wt}{2c}\right)^2 \) in the denominator is ignored, as it might be expected, we obtain nonrelativistic expressions \( x' = \frac{Wt^2}{2}, \quad t' = t \).

It is seen that higher values of \( \frac{Wt}{2c} \) are limited by \( c \).

Evidently, the condition \( \frac{Wt}{2c} < 1 \) suggests two variants of maximal values for the acceleration and corresponding time interval

\[
(a) \quad W_{\text{min}} \Delta t_{\text{max}} = 2c,
\]

\[
(b) \quad W_{\text{max}} \Delta t_{\text{min}} = 2c.
\]

It may be conclusively advocated that the existence of maximal acceleration requires the existence of a minimal time interval, and conversely if there is a maximal time interval, there should exist a certain minimal acceleration.

It seems probable that, in reality, both these opportunities should be taken into account. In principle, simultaneous existence of any large and small time intervals is a necessary condition for any model realization of a
physical clock per se. To illustrate, consider the circle of a clock dial and the primes on it as well as, in the general case, large and small periods and the possibility of comparing them to a set of integers.

For a uniform and isotropic model of the Universe as a whole there are obvious candidates for $\Delta t_{\text{min}}$ and $\Delta t_{\text{max}}$: Planck’s time $\tau_p$ and the Universe age (inverse of the Hubble constant).

It is interesting that the existence of $\tau_{\text{max}}$ (in combination with the assumption that there exists a maximal force $F_0 = \frac{c^4}{G}$) should lead to the upper limit of the total action in the metagalaxy (assuming $\tau_{\text{max}} = H_o^{-1}$)

$$S_{\text{max}} = F_0 c \tau_{\text{max}}^2 = \frac{c^5}{G H_o^2}$$

that at a given experimental value of $H_o^{-1} \sim 10^{18}s$ results in $a_{\text{max}} = 10^{95}\text{erg} \cdot \text{s}$.

Since the quantum of action $a_{\text{min}} = h \sim 10^{-27}\text{erg} \cdot \text{s}$, the total number of quanta is equal to

$$N \sim 10^{122} \sim e^{280}.$$  

3 Classical Dynamics of a Particle with the Position-Dependent Mass

As it has been shown in [4] (see also [3]-[7]), a geodesic equation in the conformally flat metric

$$g_{\mu\nu} = U^2(x)\eta_{\mu\nu}, \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$  

(20)

for the static case with $U(x) = U(r), \frac{\partial(U^2)}{\partial t} = 0$ may be written in the following form:

$$\frac{dp}{dt} + \frac{p_0^2}{2m}\text{grad}(U^2) = 0, \quad \frac{dm}{dt} = 0.$$  

(21)

Here $p = m \dot{r}, m = c^{-1}p_0 U(r)(1 - (\frac{\dot{r}}{c})^2)^{-\frac{1}{2}}$ and $p_0$ is a parameter with the dimension of momentum. This equation is formally coincident with a nonrelativistic equation of motion for a "particle" of "mass" $m$ in a "potential field"

$$f = -\frac{p_0^2}{2m}\text{grad}(U^2).$$  

(22)
The solution of equation (21) gives an essentially new result: the existence of the peculiar parametric dependence of "mass" \( m \) on the initial conditions due to its appearance in (21) as an integral of motion rather than as a numerical constant. The momentum \( p \) is related to the energy \( E \) by the standard relation

\[
E^2 - c^2p^2 = c^2p_0^2U^2(r),
\]

(23)
demonstrating that the quantity \( c^{-1}p_0U(r) \) plays a role of the coordinate-dependent rest mass.

The same result can be obtained using the standard variational procedure with an action determined by the linear element

\[
ds = (g_{\mu\nu}dx^\mu dx^\nu)^{1/2}
\]

(24)
that is defined by the metric (20). The associated Lagrangian is of the form

\[
L = -cp_0U(r) \left(1 - \frac{r^2}{c^2}\right)^{1/2}.
\]

(25)
The form of the corresponding Hamiltonian is as follows:

\[
H = c(p^2 + p_0^2U(r)^2)^{1/2}.
\]

(26)

In centrally symmetric case (apart from the energy \( E \)) there exists a vector integral of motion (angular momentum)

\[
\mathbf{M} = \mathbf{r} \times \mathbf{p} = Ec^{-2}\mathbf{L}, \quad \mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}.
\]

(27)
In this case, equation of motion (21) takes the form

\[
\ddot{r} + \frac{c^2}{2} \epsilon^{-2}r^{-1} \frac{dU^2}{dr} \dot{r} = 0
\]

(28)
where \( \epsilon = (cp_0)^{-1}E = U(r)(1 - \left(\frac{1}{c}\dot{r}\right)^2)^{-\frac{1}{2}} \) is the reduced energy. A motion will be finite when \( U(r) \) is an increasing function of \( r \). It is assumed that this function has no singularities throughout the domain of its definition. Since equation (28) is formally coincident with the nonrelativistic dynamics equation for a point-like particle in a centrally symmetric potential field, the motion is investigated in a standard way. In this case the radial velocity \( \dot{r} \) is determined by the following expression:

\[
\dot{r}^2 = c^2(1 - \frac{U^2(r)}{c^2}) - \frac{L^2}{r^2}
\]

(29)
where ϵ and $L$ are the integrals of motion defined above. The turning points are obtained from the equation $\dot{r} = 0$. The motion will be finite if the function $U(r)$ is such that this equation has two real positive roots and, in exceptional cases, the trajectories are closed.

Consider a particular model. To this end, we choose the expression for the function $U(r)$ of the form

$$U(r) = \left(1 + \frac{r^2}{R_0^2}\right)^{1/2}$$  \hspace{1cm} (30)

where $R_0$ is a parameter having the dimension of length. The reasoning in favor of this choice is heuristic in character:

. the problem is exactly solvable;
. at small $r(\ll R_0)$ the form of the potential corresponding to the field of a gravitating mass, continuously distributed with a constant density, is reproduced by $U^2(r)$;
. the required asymptotic behavior of the velocity is provided at large $r(\gg R_0)$;
. the model is reciprocally symmetric in the sense of M. Born.

Furthermore, we assume that the Lagrangian contains the constant $\overline{C}$ representing the asymptotic limit of the velocity of mechanical motion, for a test particle in the closed system considered, rather than the speed of light. In other words, we deal with a model defined by the effective Lagrangian of the form

$$L_{eff} = -m_0\overline{C}^2 \left(1 + \frac{r^2}{R_0^2}\right)^{1/2} \left(1 - \frac{r^2}{\overline{C}^2}\right)^{1/2}$$  \hspace{1cm} (31)

where $\overline{C}$ and $R_0$ are experimentally determined parameters. $m_0$ is the mass of the test particle that, as will be shown later, disappears in the final result. In this case, the condition $\dot{r} = 0$ leads to a biquadratic equation

$$r^4 - R_0^2(\epsilon^2 - 1)r^2 + R_0^2(\overline{C})^{-2}\epsilon^2L^2 = 0$$  \hspace{1cm} (32)

whose solution determines the semiaxes of the elliptic trajectories as follows:

$$r_{\pm} = \frac{R_0}{\sqrt{2}} \left(\epsilon^2 - 1\right)^{1/2} \left\{ \pm \sqrt{1 + 4L^2\epsilon^2 R_0^{-2}(\overline{C})^{-2}(\epsilon^2 - 1))^{-2}} \right\}^{1/2}. \hspace{1cm} (33)$$
The circular orbits correspond to the equality $r_+ = r_-$, namely the condition $R_0^2C^2(\epsilon^2 - 1)^2 = 4\epsilon^2L^2$ that, considering $L^2 = r^2 \times \dot{r}^2$, leads to an explicit expression relating the velocity $v = (\dot{r}^2)^{1/2}$ of the circular orbital motion of a point-like particle to the orbital radius

$$v(r) = C \left(2 + \frac{R_0^2}{r^2}\right)^{-1/2}.$$  \hfill (34)

As can be seen, the function (34) reproduces remarkably well the general shape of the rotation curve for a spiral galaxy. Figure 1 shows experimental data characteristic for the rotation curve of the NGC 3198 galaxy (this figure has been taken from [14]).

![Figure 1](image1.png)

**Figure 1**: Experimental curve of $v(km/s)$ as a function of $r(kpc)$ for NGC 3198 galaxy

Figure 2 demonstrates the rotation curve $v(r)$ calculated by formula (34) using the following parameters:

$$C = 212.132 km/s, \quad R_0 = 3.182 kpc.$$  \hfill (35)

Note that such a numerical value of the parameter $R_0$ seems to be reasonable as regards the MDMA concept proposed by us. The problem considered is, to a certain extent, a classical analog of the quantum oscillator problem. As it is known, here the characteristic parameter with the dimension of length is determined as $x_0^2 = \frac{\hbar}{m\omega}$, where $m$ is the mass of an oscillating particle and $\omega$ is the eigenfrequency of the oscillator. In the case under study the angular momentum $L_{min}$ determined by formula (16) serves as the minimal action. On the other hand, for the quantity
$\omega_0$ the following choice seems to be natural in this case: $\omega_0 = (\rho_m G)^{1/2}$. Here $\rho_m$ is a constant (position-independent) mass density determined as $\rho_m \cong M/R^3$, where $M$ is the total mass of the substance found within a region with the linear dimensions $R$. Then, for $R_0^2 = \frac{\mu_{\text{min}}}{\omega_0^2}$, we obtain accurately to the constant on the order of unity

$$R_0 = (1 + r_g/R_g)^{1/2}(R/R_g)^{3/4}R_g$$

where $r_g$ and $R_g$ are the Schwarzschild radii of a galaxy and star, respectively, and $R$ is the linear dimension of a galaxy. For a typical galaxy with $M \sim 10^{10}$ mass of the Sun and $R \sim 10^5 pc$, the value of $R_0$ determined from (36) is about one kpc (kiloparsec).

Note that, in the order of magnitude, the product of the empirical parameters $C^2 R_0$ equals to $c^2 R_g$, where $c$ is the speed of light and $R_g$ is the Schwarzschild radius corresponding to the galaxy mass. This makes it possible to suppose that there exists some scale invariance necessitating special investigation from the standpoint of the conformally-symmetric dynamics.

4 Quantization: Dirac Oscillator Model for Plankeon

Let us show that the quantization procedure based on the function $U(r)$ of the form (30) leads directly to the well-known Dirac oscillator model [10].
In this case the operator $\hat{H}^2$ corresponding to the Hamilton function (26) takes the form
\[
\hat{H}^2 = E_0^2 (\vec{P}^2 + \vec{\rho}^2 + 1)
\] (37)
where $P_k = -i \frac{\partial}{\partial \xi_k}, \rho_k = \xi_k = \frac{x_k}{q_0}$, $k = 1, 2, 3$, and $E_0 = E_p = (c^5 \hbar G^{-1})^{1/2}, q_0 = l_p = (c^{-3} \hbar G)^{1/2}$ are the corresponding Plank’s quantities.

Standard linearization, in accordance with Dirac’s procedure, gives the following expression (in noncovariant notation):
\[
\hat{H} = E_0 \{ (\hat{\alpha} \vec{P}) + \hat{\beta} (1 + \vec{\rho}^2)^{1/2} \}
\] (38)
where $\hat{\alpha}, \hat{\beta} = \rho_3$ are the standard Dirac matrices.

The operator $\hat{U}^2 = 1 + \vec{\rho}^2$ suggests obvious factorization $\hat{U}^2 = \hat{U}_+ \hat{U}_- = \hat{U}_- \hat{U}_+$, where
\[
\hat{U}_\pm = 1 \pm i (\hat{\alpha} \rho) \quad (\hat{U}_\pm^2 = \hat{U}_\mp)
\] (39)
are normal mutually-conjugate Hermitian operators. In this case $\hat{U}_\pm = \gamma_5 \hat{U}_+ \gamma_5$ (in the given representation, $\gamma_5 = \hat{\rho}_2$). Substituting (39) into (38), we obtain two Hermitian operators
\[
\hat{H}_\pm = E_0 \{ (\hat{\alpha} \vec{P}) \pm i \hat{\beta} (\hat{\alpha} \rho) + \hat{\beta} \} = E_0 \{ (\hat{\alpha} \vec{P} \mp i \hat{\beta} \rho) + \hat{\beta} \}. \] (40)
As seen, operators (40) are exactly coincident with Hamiltonian of the Dirac oscillator ($\hat{H}_+$) and its supersymmetric partner ($\hat{H}_-$) in the noncovariant representation (see, for example, [17]). It is significant that in the version being considered the model describes a particle with Plank’s mass $m_p = (\hbar c G^{-1})^{1/2}$. This model will be discussed in the context of the gravity quantization problem in a separate paper.

### 5 Concluding Remarks

In our opinion, the concept of the coordinate-dependent rest mass (CDRM), along with the hypothesis that there exists the mass-dependent maximal acceleration (MDMA), may be effectively used in the field of quantum and classical dynamics. Gravitational interactions represent an area, where the models of the classical Lagrangian and Hamiltonian dynamics with CDRM may be applied. The case in point is the description of an intermediate region requiring no recourse to the strict general relativity and making the use of the Newton approximation insufficient. There is reason
to believe that such an intermediate region is due to the rotation of large quasi-stationary cosmological objects, primarily typical spiral galaxies.

It is clear that such phenomenological models should be substantiated from the standpoint of the standard general relativity, necessitating special investigation. On the other hand, the development of such models can help to solve a number of problems of the relativistic cosmology, e.g., the well-known singularity problem. It is interesting that the use of the same phenomenological model makes it possible to give a description of the behavior of the rotation curves representing the typical spiral galaxies, in the classical version, and an exactly solvable Dirac oscillator model for a spinor particle with Plank’s mass, in the quantum version. It is not improbable that this enables construction of a theory for the behavior of fermions against the background of extremely strong gravitation fields.

Note that the existence of the universal constant with the dimension of momentum/length, postulated by us, allows hyperbolic geometry to be introduced in each phase plane (and in the eight-dimensional phase space QTPH) calling for further studies in subsequent papers.

And, finally, we make some heuristic and philosophical remarks.

Our opinion is that the modern situation with the dark matter is similar to the situation preceding the development of the special relativity. At that time, all attempts of elimination of numerous paradoxes generated by the ether concept based on the dynamic and ontological principles were unsuccessful. Actually, solution of the problem has been found by changing the geometry of the four-dimensional space-time manifold, i.e. owing to a change in the kinematics. It is probable that the dark matter will repeat the lot of the ether. We ventured to suggest that the dark matter is a peculiar factor resultant from the use of inadequate geometry of the eight-dimensional manifold, representing an extended phase space, and hence the use of inadequate kinematics.

Section 3 of this paper contains the results obtained jointly by both coauthors. And all the remaining is on personal responsibility of the first author, so that all deficiencies in the text belong to him.

Unfortunately we were not familiar with publications by G.W. Gibbons and C. Schiller before July 2005.

\footnote{We took the liberty to borrow this phrase from the Introduction to the excellent book ”Relativity: The General Theory” by J.L. Synge (L.M.T.)}
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