Triangles of nearly equal area

Konrad J. Swanepoel

Received: 2 November 2020 / Accepted: 16 February 2021
© The Author(s) 2021

Abstract
Given any \( n \) points in the plane, not all on the same line, there exist two non-collinear triples such that the ratio of the areas of the triangles they determine, differs from 1 by at most \( O(\log n/n^2) \). If we furthermore insist that the two triangles have a common edge, then there are two with area ratios differing from 1 by at most \( O(1/n) \). This improves some results of Ophir and Pinchasi (Discrete Appl. Math. 174 (2014), 122–127). We also give some constructions for these and related problems.

Keywords  Sidon set · Triangle area · Golden ratio · Plastic number · Morphic number

Mathematics Subject Classification  52C10

Consider \( n \) points in the plane, not all on a line. We want to find two triangles determined by the points with area ratio as close as possible to 1. Ophir and Pinchasi (2014) showed that in any set of \( n \) points in the plane with no three on a line, there are two triples \( \{a, b, c\} \) and \( \{a', b', c'\} \) of points such that the triangles \( \triangle abc \) and \( \triangle a'b'c' \) have almost the same area in the precise sense that

\[
\left| \frac{\triangle abc}{\triangle a'b'c'} - 1 \right| < \frac{60 \log^{1/3} n}{n^{2/3}}.
\]

We present the following two improvements of this result.

Theorem 1  Given a set \( S \) of \( n \) non-collinear points in the plane, there exist distinct points \( a, b, c, d \in S \) such that \( c \) and \( d \) are both not on the line through \( a \) and \( b \), and

\[
\frac{1}{r} \leq \frac{\triangle abd}{\triangle abc} \leq r
\]

where \( r = 3^{3/(n-3)} = 1 + \frac{3 \ln 3}{n} + O(1/n^2) \).
Theorem 2. Given a set $S$ of $n$ non-collinear points in the plane, there exist non-collinear triples of points $\{a, b, c\}$ and $\{a', b', c'\}$ from $S$ such that

$$\frac{1}{r} \leq \frac{\triangle abc}{\triangle a'b'c'} \leq r$$

where $r = 1 + O\left(\frac{\log n}{n^2}\right)$.

The proof of Theorem 1 is a simple pigeon-hole argument, that can be generalised as follows to higher dimensions.

Corollary 3. Given a set $S$ of $n$ points that span $d$-dimensional Euclidean space, there exist $d+2$ distinct points $a_1, a_2, \ldots, a_d, b, c \in S$ such that $a_1, \ldots, a_d$ span a hyperplane not containing $b$ and $c$, and the ratio between the volume of the simplices with vertex sets $\{a_1, \ldots, a_d, b\}$ and $\{a_1, \ldots, a_d, c\}$ lies in $[1/r, r]$, where $r = 1 + O_d\left(\frac{\log n}{n^2}\right)$.

Theorem 1 is best possible in the sense that we cannot guarantee two triangles with only one vertex in common to have almost the same area.

Proposition 4. There exists a set of $n$ points $p_1, \ldots, p_n$ in the plane such that whenever

$$\frac{1}{14} \leq \frac{\triangle p_i p_j p_k}{\triangle p'_i p'_j p'_k} \leq 14,$$

then $\{i, j, k\}$ and $\{i', j', k'\}$ have their two largest elements in common.

On the other hand, we do not know if Theorem 2 can be improved. Its proof depends on the following result of Ophir and Pinchasi (2014), for which it is also not known whether it is asymptotically tight.

Proposition 5. There exists a set $S$ of $n$ real numbers such that the ratio between the area of any two triangles with vertices from the set $\{(s, n^{5s}) | s \in S\}$ is $\gtrsim 1 + 1/n^2$.

We say that a set $\{a_1, a_2, \ldots, a_n\}$ of $n$ integers is a Sidon set if the sums $a_i + a_j$, $i < j$, are all different. Ophir and Pinchasi noted that the example of Erdős and Turán (1941) of a Sidon set of $n$ integers from $\{1, 2, \ldots, n^2 + O(n)\}$ is also an example of $n$ points in $\mathbb{R}$ for which the ratio of the distance between any two distinct pairs differ from 1 by at least $1/n^2$. We next observe that there is a simple construction of $n$ points in $\mathbb{R}$ with a slightly better lower bound of $4/n^2$.
Apply an affine transformation so that $\triangle abc$ is equilateral with side length 1.

Proposition 6 There exists a set of $n$ points on the real line such that for any two distinct pairs \{a, b\} and \{c, d\} from the set with $|a - b| \geq |c - d|$, we have

$$1 + \frac{4}{n^2} + O(1/n^3) \leq \frac{|a - b|}{|c - d|} \leq O(n).$$

1 Proofs

Proof of Theorem 1 Choose $a, b, c \in S$ such that $\triangle abc$ has maximum area among all triples of points from $S$. Without loss of generality we may apply an affine transformation so that $\triangle abc$ becomes an equilateral triangle of side length 1, as in Figure 1.

Let $\triangle def$ be the triangle with sides parallel to the sides of $\triangle abc$ and such that $a, b, c$ are midpoints of the edges of $\triangle def$. Then all $n$ points are inside $\triangle def$. Let $p$ be the centroid of $\triangle abc$ (and $\triangle def$). Consider the three lines through $p$ parallel to the three sides of $\triangle abc$. At least $n/3$ points must lie on the side of one of these lines that is opposite to the parallel side of $\triangle abc$. Without loss of generality the trapezium $degh$ contains at least $n/3$ of the points. Let $k = \lfloor n/3 \rfloor - 1$. Subdivide the trapezium

Fig. 1 Apply an affine transformation so that $\triangle abc$ is equilateral with side length 1.

Fig. 2 Pigeon-hole principle inside a trapezium.
using \( k \) parallel lines of height \( \frac{1}{2\sqrt{3}} r^i \), \( i = 0, 1, \ldots, k - 1 \), above the line \( bc \), where \( r \) is chosen such that \( \frac{1}{2\sqrt{3}} r^k = \sqrt{3}/2 \) (Figure 2). Since \( k < n/3 \), there are two points in at least one of the regions, say \( q_1 \) and \( q_2 \). Then \( \frac{1}{r} \leq \triangle bcq_1/\triangle bcq_2 \leq r \), and since \( k \geq n/3 - 1 \),

\[
    r = 3^{1/k} \leq 3^{3/(n-3)} = 1 + 3 \log 3/n + O(1/n^2).
\]

**Proof of Theorem 2** Let \( p_i = (2^{2i}, 2^{2i+1}) \), \( i = 1, \ldots, n \). If \( i < j < k \), then the area of \( \triangle p_ip_jpk \) is

\[
    \triangle p_ip_jpk = \frac{1}{2} \left| \begin{array}{ccc} 1 & 1 & 1 \\ 2^{2i} & 2^{2j} & 2^{2k} \\ 2^{2i+1} & 2^{2j+1} & 2^{2k+1} \end{array} \right| = \frac{1}{2} (2^{2k} - 2^{2i})(2^{2k} - 2^{2j})(2^{2i} - 2^{2j}).
\]

Thus \( \triangle p_ip_jpk \leq \frac{1}{2} 2^{2k+1+2j} \) and

\[
    \triangle p_ip_jpk \geq \frac{1}{2} (2^{2k} - 2^{2j})(2^{2k} - 2^{2i})(2^{2j} - 2^{2i})
\]

\[
= \frac{1}{2} 2^{2k+1+2j}(1 - 2^{4-2k})(1 - 2^{2-2k})(1 - 2^{2-2j}).
\]

We now consider two distinct triples \( \{i < j < k\} \) and \( \{i' < j' < k'\} \), where without loss of generality, \( k \geq k' \). If \( k > k' \) then

\[
\frac{\triangle p_ip_jpk}{\triangle p_i'p_j'pk'} \geq \frac{2^{2k+1-2k+2j-2j'}}{(1 - 2^{4-2k})(1 - 2^{2-2k})(1 - 2^{2-2j'})}
\]

\[
\geq \frac{2^{2k-2k-2k}}{(1 - 2^{4-2k})(1 - 2^{2-2k})(1 - 2^{2-2j'})}
\]

\[
\geq \frac{2^{4-2k-2k+2-2j}}{(1 - 2^{4-2k})(1 - 2^{2-2k})(1 - 2^{2-2j'})} > 2^{15}.
\]

If \( k = k' \) then without loss of generality, \( j \geq j' \). If \( j > j' \), then

\[
\frac{\triangle p_ip_jpk}{\triangle p_i'p_j'pk'} \geq \frac{2^{2j-2j-1}}{(1 - 2^{4-2j})(1 - 2^{2-2j})(1 - 2^{2-2j'})}
\]

\[
\geq \frac{2^4}{(1 - 2^{4-2j})(1 - 2^{2-2j})(1 - 2^{2-2j'})} > 14.
\]

Therefore, if the ratio is at most 14, then \( j = j' \) and \( k = k' \). \( \square \)

**Proof of Proposition 4** Let \( p_i = (2^i, 2^{i+1}) \), \( i = 1, \ldots, n \). If \( i < j < k \), then the area of \( \triangle p_ip_jpk \) is

\[
\triangle p_ip_jpk = \frac{1}{2} \left| \begin{array}{ccc} 1 & 1 & 1 \\ 2^i & 2^j & 2^k \\ 2^i+1 & 2^j+1 & 2^k+1 \end{array} \right| = \frac{1}{2} (2^{2k} - 2^i)(2^{2k} - 2^j)(2^i - 2^j).
\]

Thus \( \triangle p_ip_jpk \leq \frac{1}{2} 2^{2k+1+2j} \) and

\[
\triangle p_ip_jpk \geq \frac{1}{2} (2^{2k} - 2^j)(2^{2k} - 2^i)(2^j - 2^i)
\]

\[
= \frac{1}{2} 2^{2k+1+2j}(1 - 2^{4-2k})(1 - 2^{2-2k})(1 - 2^{2-2j}).
\]

We now consider two distinct triples \( \{i < j < k\} \) and \( \{i' < j' < k'\} \), where without loss of generality, \( k \geq k' \). If \( k > k' \) then

\[
\frac{\triangle p_ip_jpk}{\triangle p_i'p_j'pk'} \geq \frac{2^{2k+1-2k+2j-2j'}}{(1 - 2^{4-2k})(1 - 2^{2-2k})(1 - 2^{2-2j'})}
\]

\[
\geq \frac{2^{2k-2k-2k}}{(1 - 2^{4-2k})(1 - 2^{2-2k})(1 - 2^{2-2j'})}
\]

\[
\geq \frac{2^{4-2k-2k+2-2j}}{(1 - 2^{4-2k})(1 - 2^{2-2k})(1 - 2^{2-2j'})} > 2^{15}.
\]

If \( k = k' \) then without loss of generality, \( j \geq j' \). If \( j > j' \), then

\[
\frac{\triangle p_ip_jpk}{\triangle p_i'p_j'pk'} \geq \frac{2^{2j-2j-1}}{(1 - 2^{4-2j})(1 - 2^{2-2j})(1 - 2^{2-2j'})}
\]

\[
\geq \frac{2^4}{(1 - 2^{4-2j})(1 - 2^{2-2j})(1 - 2^{2-2j'})} > 14.
\]

Therefore, if the ratio is at most 14, then \( j = j' \) and \( k = k' \). \( \square \)
We next show that the distance between any two points \( p, p' \in S \) is \( \gtrsim 4 \log n/n^2 \). Since the perpendicular distance from \( p \) to some edge of \( \triangle abc \), say \( ab \), is \( \geq 2/3 \), we obtain

\[
\frac{\triangle apb'}{\triangle apb} \leq 1 + \frac{pp'}{2/3}.
\]

Similarly, since \( p' \) is at perpendicular distance \( \geq 2/3 - pp' \) from \( ab \), we obtain

\[
\frac{\triangle apb}{\triangle apb'} \leq 1 + \frac{pp'}{2/3 - pp'}.
\]

It follows that

\[
1 + \frac{6 \log n}{n^2} \leq 1 + \frac{pp'}{2/3 - pp'},
\]

from which \( pp' \gtrsim 4 \log n/n^2 \) follows.

Among all \( \binom{n}{3} \) triples of points, the \( \binom{n}{3} - n^2 \) smallest areas are all \( \leq (1 + \frac{6 \log n}{n^2}) - n^2 \sim n^{-6} \). Suppose that each pair of points belongs to more than 6 triangles of area \( \gtrsim n^{-6} \). Then there are at least \( 7 \binom{n}{2}/3 > n^2 \) triangles of area \( \gtrsim n^{-6} \), a contradiction.

Therefore, some pair of points \( \{p, q\} \) belongs to at most 6 triangles of area \( \gtrsim n^{-6} \), hence to at least \( n - 8 \) triangles \( \triangle pqp_i, i = 1, \ldots, n - 8 \), each of area \( \lesssim n^{-6} \). Since the distance between \( p \) and \( q \) is \( \gtrsim 4 \log n/n^2 \), the perpendicular distance of any \( pi \) to the line \( \ell \) through \( p \) and \( q \) is \( \lesssim 1/(2n^4 \log n) \).

We now choose coordinates so that \( \ell \) becomes the \( x \)-axis. Then each \( p_i = (x_i, \varepsilon_i) \), where \( |\varepsilon_i| \lesssim 1/(2n^4 \log n) \). Since \( \triangle abc \) has width 2, one of its three vertices, say \( a = (x, h) \), is at distance \( |h| \geq 1 \) from \( \ell \). By the result of Ophir and Pinchasi applied to \( x_1, x_2, \ldots, x_{n-8} \), there are two pairs \( \{i, j\} \) and \( \{s, t\} \) such that

\[
\frac{|x_i - x_j|}{|x_s - x_t|} = 1 + O(\log n/n^2).
\]

We next show that the ratio between the areas of \( \triangle api p_j \) and \( \triangle ap s p_t \) is asymptotically the same as \( |x_i - x_j|/|x_s - x_t| \).

We claim that \( \triangle api p_j = |h(x_i - x_j)|(1 + o(\log n/n^2)) \). Indeed,

\[
\pm 2 \triangle api p_j = \begin{vmatrix} 1 & 1 & 1 \\ x & x_i & x_j \\ 0 & 1 & 1 \\ h & \varepsilon_i & \varepsilon_j \end{vmatrix} = h(x_j - x_i) \begin{vmatrix} 1 & 1 & 1 \\ 0 & x_i - x & x_j - x \\ h & \varepsilon_i & \varepsilon_j \end{vmatrix} = h(x_j - x_i) \left( 1 - \frac{\varepsilon_i (x_j - x)}{h(x_j - x_i)} + \frac{\varepsilon_j (x_i - x)}{h(x_j - x_i)} \right).
\]
Since $|x_j - x| \leq p_j p \leq 8/\sqrt{3}$, $|h| \geq 1$, and

$$|x_j - x_i| \geq p_i p_j - |\varepsilon_i| - |\varepsilon_j| \geq \frac{4 \log n}{n^2} - \frac{1}{n^4 \log n} \geq \frac{4 \log n}{n^2},$$

we obtain

$$\frac{|\varepsilon_i(x_j - x)|}{|h(x_j - x_i)|} = O\left(\frac{1}{n^2 \log^2 n}\right).$$

Similarly,

$$\frac{|\varepsilon_j(x_i - x)|}{|h(x_j - x_i)|} = O\left(\frac{1}{n^2 \log^2 n}\right),$$

and it follows that

$$2 \Delta p_i p_j = |h(x_i - x_j)|(1 + O(1/(n^2 \log^2 n))).$$

Similarly,

$$2 \Delta p_s p_t = |h(x_s - x_t)|(1 + O(1/(n^2 \log^2 n))),$$

and we conclude that

$$\frac{\Delta p_i p_j}{\Delta p_s p_t} = \frac{|x_i - x_j|}{|x_s - x_t|} (1 + O(1/(n^2 \log n))) = 1 + O(\log n/n^2).$$

Proof of Proposition 5 Let $S$ be a Sidon set of $n$ elements from $\{1, 2, \ldots, N\}$ where $N = n^2 + O(n)$. Write $p_s = (s, n^5s)$ and $q_{s} = (s, 0)$ for each $s \in S$. Consider three $s, t, u \in S$ with $s < t < u$.

Then

$$2 \Delta p_s p_t p_u = \begin{vmatrix} 1 & 1 & 1 \\ s & t & u \\ n^5s & n^5t & n^5u \end{vmatrix} = n^5u(t - s) + n^5t(s - u) + n^5s(u - t)$$

and

$$2 \Delta q_s q_t p_u = \begin{vmatrix} 1 & 1 & 1 \\ s & t & u \\ 0 & 0 & n^5u \end{vmatrix} = n^5u(t - s).$$
Since the ratio between these two areas is close to 1, we can replace $\triangle p_s p_t p_u$ with $\triangle q_s q_t p_u$ in our calculations. Specifically,

$$\frac{|\triangle p_s p_t p_u|}{|\triangle q_s q_t p_u|} - 1 \leq \frac{n^{5t}(s-u) + n^{5u}(u-t)}{n^{5u}(t-s)} \leq n^{5t(s-u)} \frac{s-u}{t-s} + n^{5(s-u)} \frac{u-t}{t-s} \quad \text{as } a \leq b < c.$$

Consider distinct triples $\{a < b < c\}$ and $\{d < e < f\}$ of elements from $S$, where we assume without loss of generality that $c \geq f$. Then

$$\frac{\triangle p_a p_b p_c}{\triangle p_d p_e p_f} \geq \left(1 - \frac{1}{n^3}\right)^2 \frac{b-a}{e-d} \geq \left(1 - \frac{1}{n^3}\right)^2 \frac{N}{N-1} \geq 1 + \frac{1}{n^2}.$$

On the other hand, if $c > f$ then the ratio is even larger:

$$\frac{\triangle p_a p_b p_c}{\triangle p_d p_e p_f} \geq \left(1 - \frac{1}{n^3}\right)^2 n^5 \frac{1}{N} \geq n^3. \quad \square$$

**Proof of Proposition 6** Fix $\varepsilon > 0$, and let $p_i = (1+\varepsilon)^i - 1$, $i = 0, 1, \ldots, n-1$. Then for any integer $a, b$ with $0 \leq a < b \leq n - 1$,

$$p_b - p_a = (1 + \varepsilon)^b - (1 + \varepsilon)^a.$$

Take any $a, b, c, d \in \{0, \ldots, n-1\}$ with $a < b, c < d, \{a, b\) \neq \{c, d\}$ and without loss of generality, $b - a \leq d - c$. If $b - a = d - c$ then without loss of generality, $a < c$, and

$$\frac{p_d - p_c}{p_b - p_a} = \frac{(1 + \varepsilon)^d - (1 + \varepsilon)^c}{(1 + \varepsilon)^b - (1 + \varepsilon)^a} = (1 + \varepsilon)^{c-a} (1 + \varepsilon)^{d-c} - 1 \geq 1 + \varepsilon.$$

If $b - a < d - c$, then setting $b-a = k \in \{1, 2, \ldots, n-2\}$,

$$\frac{p_d - p_c}{p_b - p_a} \geq \frac{p_d - p_{d-b+a-1}}{p_b - p_a} = (1 + \varepsilon)^{(d-b+a-1)+(n-1-b)} \frac{p_{k+1} - p_0}{p_{n-1} - p_{n-1-k}} \geq \frac{(1 + \varepsilon)^{k+1} - 1}{(1 + \varepsilon)^{n-1} - (1 + \varepsilon)^{n-1-k}}.$$
This last expression will be $\geq 1 + \varepsilon$ if and only if $(1 + \varepsilon)^{k+1} + (1 + \varepsilon)^{n-k} \geq (1 + \varepsilon)^n + 1$. If we use the Binomial Theorem to expand this up to second order, we obtain

$$1 + (k + 1)\varepsilon + \binom{k + 1}{2} \varepsilon^2 + O(k^3 \varepsilon^3)$$

$$+ 1 + \varepsilon(n - k) + \binom{n - k}{2} \varepsilon^2 + O((n - k)^3 \varepsilon^3)$$

$$\geq 1 + n\varepsilon + \binom{n}{2} \varepsilon^2 + O(n^3 \varepsilon^3) + 1,$$

which is equivalent to $\varepsilon \geq \left(\binom{n}{2} - \binom{k + 1}{2} - \binom{n - k}{2}\right) \varepsilon^2 + O(n^3 \varepsilon^3)$, that is, we need the inequality $1 \geq k(n - k - 1)\varepsilon + O(n^3 \varepsilon^2)$ to hold for all $k = 1, 2, \ldots, n - 2$. Since $k(n - k - 1) \leq \left(\frac{n - 1}{2}\right)^2$, we obtain that we need $\varepsilon \leq \frac{4}{n^2} + O(\frac{1}{n^2}) + O(n\varepsilon^2)$. Thus we can take $\varepsilon = \frac{4}{n^2} + O(\frac{1}{n^3})$.

This shows that we obtain a minimum ratio of $1 + \frac{4}{n^2} + O(\frac{1}{n^3})$. \hfill \square

Instead of using points where the successive distances $p_{i+1} - p_i$ form a geometric progression, as in the above proof, we can also use an arithmetic progression. If we take the $n$ points $p_0 = 0, p_i = \sum_{j=0}^{i-1}(1 + j\varepsilon), i = 1, 2, 3, \ldots, n - 1$, then a calculation shows that we obtain the same optimal asymptotics of $\varepsilon = \frac{4}{n^2} + O(\frac{1}{n^3})$.

## 2 Final remarks

We did not touch on the problem of Ophir and Pinchasi on whether there exist in any set of $n$ elements of $\mathbb{R}$ two pairs with ratio better than $1 + O(\log n/n^2)$, but we did find the sets of points for which the smallest ratio $> 1$ is a maximum when $n \leq 4$. Thus consider a set $S \subset \mathbb{R}$ of $n$ points that maximizes

$$\min \left\{ \frac{|a - b|}{|c - d|} : a, b, c, d \in S, |a - b| \geq |c - d| > 0 \right\}$$

among all sets of $n$ points in $\mathbb{R}$.

If $n = 3$, it is easy to see that there is a unique extremal set up to similarity, namely $S = \{a < b < c\}$ such that $\frac{c - b}{b - a} = \frac{\sqrt{69} + 3}{3\sqrt{18}}$ equals the golden ratio $(1 + \sqrt{5})/2$.

For $n = 4$ the problem is already non-trivial, as there are 6 different distances. Using a case analysis, we can show that up to similarity there are two extremal sets. One of them is the above geometric progression construction $\{0, 1, 1 + r, 1 + r + r^2\}$, where $r$ is the unique real root of the cubic polynomial $r^3 - r - 1$. The other configuration is $\{0, 1, r, r^2\}$. The number

$$r = \sqrt[3]{\frac{9 + \sqrt{69}}{18}} + \sqrt[3]{\frac{9 - \sqrt{69}}{18}} = 1.3247179572 \ldots$$
is known as the *plastic number* of van der Laan (1960), which is closely related to the golden ratio (Aarts et al. 2001; Rush 2012; Stewart 1996).

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit [http://creativecommons.org/licenses/by/4.0/](http://creativecommons.org/licenses/by/4.0/).

**References**

Aarts, J., Fokkink, R., Kruijzer, G.: Morphic numbers. Nieuw Arch. Wiskd. 2(5), 56–58 (2001)
Erdős, P., Turán, P.: On a problem of Sidon in additive number theory, and some related problems. J. London Math. Soc. 16, 212–215 (1941)
Ophir, A., Pinchasi, R.: Nearly equal distances in metric spaces. Discrete Appl. Math. 174, 122–127 (2014)
Rush, D.E.: Degree $n$ relatives of the golden ratio and resultants of the corresponding polynomials. Fibonacci Quart. 50, 313–325 (2012)
Stewart, I.: Tales of a neglected number. Math. Recreat. SciAm 274(6), 102–103 (1996)
vander Laan, H.: Le nombre plastique. Brill, Leiden (1960)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.