On the relation between plausibility logic and the maximum-entropy principle: a numerical study

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Abstract: What is the relationship between plausibility logic and the principle of maximum entropy? When does the principle give unreasonable or wrong results? When is it appropriate to use the rule ‘expectation = average’? Can plausibility logic give the same answers as the principle, and better answers if those of the principle are unreasonable? To try to answer these questions, this study offers a numerical collection of plausibility distributions given by the maximum-entropy principle and by plausibility logic for a set of fifteen simple problems: throwing dice.

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Dedicato a mia madre per il suo trentesimo compleanno

1 When and how should the maximum-entropy principle be applied?

For the student of plausibility logic1, the theory of the principles governing plausible inference, the application of the theory in any given problem is crystal clear in principle: (1) The problem is analysed and reduced to a set of propositions \( \{A_i\} \) and background knowledge \( I \). (2) Some plausibilities \( P[A_i \ldots A_k | (A_{i_1} \ldots A_{i_k}) \wedge I] \in [0, 1] \) are assigned, consistently with the laws below, according to our actual or hypothetical knowledge of the situation and to convenience; the \( 'A_{i_1} \ldots A_{i_k}' \) represent collections of \( A_i \)'s joined by various logical connectives (‘¬’, ‘∧’, ‘∨’, ‘⇒’). (3) Finally, using the basic laws

\[
\begin{align*}
P(\neg A_i | I) &= 1 - P(A_i | I), \quad (1a) \\
P(A_i \wedge A_j | I) &= P(A_i | A_j \wedge I) P(A_j | I), \quad (1b) \\
P(A_i \vee A_j | I) &= P(A_i | I) + P(A_j | I) - P(A_i \wedge A_j | I), \quad (1c) \\
P(A_i \Rightarrow A_j | I) &= P(\neg A_i | I) + P(A_j | A_i \wedge I) P(A_i | I), \quad (1d)
\end{align*}
\]

1 I call ‘plausibility logic’ what many other authors call ‘(Bayesian) probability theory’. ‘Logic’, because it is a generalization of the truth-logical calculus. ‘Plausibility’, because ‘degree of belief’ is unfortunately too unwieldy and many authors still contend that ‘probability’ = ‘frequency’ or, perhaps worse, ‘probability’ = ‘(Lebesgue) measure’. 

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the values of unassigned plasibilities of interest, or at least some bounds for them, are calculated — e.g. via fractional linear programming, as explained in the scholarly and unfortunately much neglected works of Hailperin [1; 2, §§ 0.4, 4.5, 5.4, 6.2 and passim; 3]. If required, some mathematical limits are taken. This procedure applies whether we want to calculate the plausibility of a proposition of interest, to explore the plausible consequences of a hypothesis, or to see the relationship between the plasibilities assigned upon different contexts; and other similar problems.

When the same student approaches the principle of maximum entropy, he is not welcomed by a comparable level of clarity. For example, he is told to use data consisting in an average value; but does the number of observations contributing to this average matter? He is told to equate an average with an expectation; but when and why is it reasonable to equate these very different quantities? He is told that the principle yields a plausibility distribution; but does this plausibility concern one of the observations contributing to the data, or a new observation? And what is the conditional of the probability distribution given by the principle? that is, if the principle gives a $P(A_i | \cdot)$, what proposition does the dot stand for? In fact, the specification of the conditional, or ‘supposal’ [4–7] or ‘context’, of a plausibility is extremely important, also because the conditionals and the arguments of two plasibilities must match in a precise way for the plausibility laws to be used. It is meaningless to multiply, e.g., $P(A_3 | I)$ and $P(A_2 | A_1 \land I)$ without further qualifications.

The various ‘proofs’ of the maximum-entropy principle in the literature do not help the student very much either. The most rigorous of them cover only specific situations, leaving out important ones. For example, van Campenhout & Cover [8] and Csiszár [9] prove that a uniform i.i.d. model gives, to any observation contributing to the average, the same distribution as the maximum-entropy principle, when the average used as data comes from a very large number of observations. But they are silent on whether that distribution is valid for similar observations not contributing to the average. On the other hand, the logics behind many proofs covering this last application of the principle have been repeatedly attacked by various authors. I recommend Skyrms’ [10] and especially Uffink’s [11] analyses, which give insights on several important points (but show confusion about others), many of which are repeated here.

One could even say that there is not just one ‘maximum-entropy principle’, because in the literature this term denotes qualitatively different procedures and problems, in which one seeks qualitatively different kinds of distributions — e.g., distributions of probability, of frequency, or of intensity as in image-reconstruction problems. In some of these cases there really is no ‘principle’ but only a ‘rule’ which appears asymptotically from choosing the frequency- or intensity-distribution $M_i/M$, $i = 1, \ldots, r$, that can be realized in most ways. The number of ways is usually given by the multiplicity factor

$$M! \prod_{i=1}^{r} M_i! = \epsilon(M_i) \exp[M H(M_i/M)], \quad \text{with} \quad (M + 1)^{-r} \leq \epsilon(M_i) \leq 1, \quad (2)$$

and this is asymptotically equal [12, § 1.2; 13, § 2.1] to the exponential of a very
large multiple of the Shannon entropy of the distribution,

$$H(f_i) := -\sum_i f_i \ln f_i, \quad f_i \geq 0, \quad \sum_i f_i = 1; \quad 0 \ln 0 := 0.$$  \hspace{1cm} (3)

The distribution chosen by ‘counting reasons’ is therefore the one having maximum Shannon entropy. In examples like these the maximum-entropy rule is an obvious consequence of plausibility logic with an assumption of symmetry on a particular hypothesis space (e.g., the set of outcome sequences) and does not need additional principles for its justification. These cases do not concern us here. But the counting arguments above make no sense for distributions of plausibility, and the application of the rule, especially to new observations, seems to require additional principles besides those of plausibility logic.

## 2 A numerical comparison: fifteen problems

The purpose of the present work is to examine the maximum-entropy principle ‘in action’ in a collection of simple problems, to see under which circumstances its results are intuitively reasonable or not and how these compare with those given by plausibility logic alone.

The collection of problems is the following: Of $N$ throws of a die we know the average $a$ and nothing else; not even the throwing technique or the kind of die used, although we assume them to be the same or at least very similar in all throws. We want the plausibility distributions for:

1. the outcomes of one of the $N$ throws contributing to the average, which we call ‘old throws’; and
2. the outcomes of a throw — of the same or very similar die and with the same throwing technique — outside the set of $N$ old throws, which we call a ‘new throw’.

We consider the problems obtained by combining the particular values $N = 1, 6, 12$ and $N$ large, together with $a = 6, 5,$ and $7/2$, for a total of fifteen problems.

The adjectives ‘old’ and ‘new’ qualifying the throws have really no temporal meaning; in fact, the problem is by assumption completely symmetric with respect to the temporal ordering of all throws, or the way they are scattered in space-time. A ‘new’ throw could precede an ‘old’ one, or they could all happen at the same time. To stress this I shall use the present tense even with ‘old’ throws, saying e.g. ‘the second old throw gives face $\mathbb{I}$’. Plausibility logic is affected by what we know; it is immaterial how and when we come to know it (even if it had been by precognition or other imaginary ways).

The old throws are numbered from 1 to $N$ (this numbering, again, bearing no temporal meaning); a new throw is assigned the number 0. The proposition stating that the outcome of throw $j$ is the face ‘$i$’ of the die is denoted by $R_{ij}$. The proposition stating that the average of the $N$ throws is $a$ is denoted by $A_N^a$. Thus the plausibility distributions sought are, in symbols,
1. \( P(R_1^{(1)} | A_N \land I) \) (old throw, \( j = 1 \)) and
2. \( P(R_1^{(0)} | A_N \land I) \) (new throw, \( j = 0 \)).

The proposition \( I \) states our background knowledge, i.e., mathematically speaking, our plausibility model. An important question indeed is: what kind of background knowledge, or plausibility model, are we implicitly assuming when we use the maximum-entropy principle?

2.1 Exchangeable plausibility models used for comparison

In our comparison we consider three plausibility models, all three infinitely exchangeable. Infinite exchangeability means that, in assigning a plausibility distribution to the set of outcomes of any number of old and new throws, we do not care which throw each outcome belongs to. An alternative interpretation is the following: if we knew enough about the circumstances of each throw, knowledge of all other throws would be irrelevant in our plausibility assignment for that throw; but those circumstances are unknown and we can only assign a plausibility distribution to their various possibilities, which are themselves plausibility-indexed; see [7; 14; 15]. In both interpretations the plausibility distribution for the outcomes of throws \( j_1, \ldots, j_n \) has the form

\[
P(R_{j_1}^{(j_1)} \land \cdots \land R_{j_n}^{(j_n)} | I) = \int p_{i_1} \cdots p_{i_n} g(p | I) \, dp;
\]

the integration is over the simplex of plausibility distributions

\[
\Delta := \{ p := (p_1, \ldots, p_6) \mid p_i \geq 0, \sum_i p_i = 1 \}.
\]

The generalized function \([16; 18] g\) characterizes the exchangeable model chosen; \( g \, dp \) can be interpreted as the plausibility density for the limiting frequencies of the outcomes as the number of new throws increases indefinitely or, in the alternative interpretation of infinite exchangeability, as the plausibility density for those circumstances with index values around the volume element \( dp \). See app. C for some remarks about this volume element.

The choice of \( g(p | I) \) defines our exchangeable model \( I \). For example, for a generalized density concentrated on the uniform distribution,

\[
g(p | I_{ft}) \, dp := \prod_i \delta(p_i - 1/6) \, dp.
\]

the model \( I_{ft} \) gives each throw a uniform plausibility distribution, independent of the knowledge about all other throws and identical for all throws, ‘i.i.d.’. It represents the unshakeable belief that the die and throwing technique be absolutely ‘fair’. We call this the fair-throw model; it will be the first to be compared with the maximum-entropy principle.

Suppose, on the other hand, that we judge the plausibility for face ‘\( i \)’ on a new throw to depend only on \( N \) and on how many old throws yield face ‘\( i \)’; in other
words, the frequencies of the other faces of old throws are irrelevant. Then we have the Johnson model, also called Dirichlet model, \( I_K \). It depends on a parameter \( K \) and is represented by the density

\[
g(p|I_K) \, dp := \Gamma(1 + \sum_i K) \left[ \prod_i \frac{p_i^{K-1}}{\Gamma(K)} \right] \, dp, \quad K > 0; \tag{7}
\]

see Johnson \[4,7\], Good \[19\, ch. 4\], Zabell \[20\], Jaynes \[21\], and references therein. For the values \( K = 1, 2, 5, 50 \) and \( K \) large this will be the second model in our numerical comparison. The case \( K = 1 \) (a constant density) corresponds to the multidimensional form of Bayes’ \[22\, Scholium\] and Laplace’s suggestion \[23\, p. xvii\]: ‘Quand la probabilité d’un événement simple est inconnue, on peut lui supposer également toutes les valeurs depuis zéro jusqu’à l’unité’. See Jaynes \[24, ch. 11\] and Stigler \[25\] for interesting discussions and references on this case. The case \( K = 1/2 \), not discussed here, was advocated by Jeffreys \[26\].

The third model \( I_{L_m} \) in our comparison is defined by the suggestive density

\[
g(p|I_{L_m}) \, dp := c(L) \frac{L!}{\prod_i (Lp_i)} \, dp, \quad L \geq 1; \tag{8}
\]

with \( c(L) \) a normalization factor and, here and in the following,

\[
x! := \Gamma(x + 1). \tag{9}
\]

I have been unable to characterize this model in more intuitive terms or to derive it from particular assumptions as can be done for the Johnson one. This is an interesting problem which deserves further study. Apart from the normalization, the expression above is (intentionally, as we shall see later) identical in form with the multiplicity factor \( (2) \) and for this reason I call this the multiplicity model with parameter \( L \). We shall use the values \( L = 1, 2, 5, 50 \) and \( L \) large. The case \( L = 1 \) (a generalized density proportional to \( \prod_i p_i^{-1} \)) was proposed by Haldane \[27\] and is discussed by Jeffreys \[28, § 3.1, p. 123 ff.\] and Jaynes \[29, § VII\]; see also Zellner \[30, § 2.13\].

As you have already guessed, this model has been chosen because for appropriate values of the parameter \( L \) it gives the same distributions as the maximum-entropy principle.

Further mathematical properties of our models are discussed in app. \(B\).

### 3 Main results

The plausibilities assigned by the maximum-entropy principle and our three exchangeable models in the fifteen problems are derived app. \(B\) and presented in the tables of app. \(A\) p. 13. The features that strike me most in these tables are the following.
Cases with one or two old throws, i.e. $N = 1$ or $2$, with an average $a = 5$: In the first case we know for sure that the old throw must give $\bar{A}9$, so its plausibility distribution must be $(0, 0, 0, 0, 100, 0)$%. In the second case we know that the faces $\bar{A}4, \bar{A}5, \bar{A}6$ cannot appear because the total sum of the two old throws must be 10; so the distribution must have the form $(0, 0, 0, \cdot, \cdot, \cdot)$. If we interpret the maximum-entropy distribution as referring to old throws, it is in these cases not just unreasonable, but plainly wrong. The maximum-entropy principle is therefore not meant to be used for old throws when $N$ is small.

All three exchangeable models give correct results instead. This was expected: in situations of certainty the plausibility calculus reduces to the truth-logical calculus.

Cases with $N = 1$ or $2$ and $a = 6$: We know that the one or two old throws give $\bar{A}8$. This is certainly no ground to suppose that the throwing technique or the die be completely biased towards the face $\bar{A}8$; we are in fact excluding any particular detailed background knowledge, as e.g. that all faces of the die have the same, unknown, number of pips — so that in these cases that number is revealed to be 6. In general, knowledge of the outcome of only one or two old throws should not make our predictions for a new throw deviate very much from the uniform distribution. If interpreted to refer to a new throw, the maximum-entropy distribution is therefore quite unreasonable since it concentrates all plausibility on face $\bar{A}8$. The same is true for the $N = 1$ or 2, $a = 5$ cases. The maximum-entropy principle is therefore not meant to be applied for new throws when $N$ is small.

The Johnson and multiplicity models on the other hand give reasonably more uniform distributions for the new throw, especially for larger values of the parameters $K$ and $L$.

Note also how the exchangeable models respect the logical symmetries of these cases: When $N = 1$ the cases $a = 6$ and $a = 5$ are completely symmetric under the exchange of those faces (of all faces, indeed). When $N = 2$ and $a = 6$ our knowledge — that only face $\bar{A}8$ appears in old throws — is symmetric under exchange of all other faces; this symmetry is respected by the exchangeable models in both old and new throws. When $N = 2$ and $a = 5$ we know that the outcomes must be $(\bar{A}5, \bar{A}6), (\bar{A}6, \bar{A}7)$, or $(\bar{A}8, \bar{A}9)$; this symmetry under permutation of $\bar{A}5, \bar{A}6$, and of $\bar{A}8, \bar{A}9, \bar{A}7$ is again respected in the exchangeable models.

Case with $N = 1$ (or $N$ odd) and $a = 7/2$: An obviously impossible case. This is reflected in the fact that plausibility logic yields $0/\Pi$, for old and new throws: any inference is completely arbitrary and unreliable because we have been given contradictory data. Yet the maximum-entropy gives the uniform distribution.

We have concluded that the maximum-entropy principle is not meant to be applied for either old or new throws when $N$ is small. But how small is ‘small’? Let us examine the examples with six old throws.
Case $N = 6$, $a = 6$: All six old throws must have outcome $\mathbb{1}$. The maximum-entropy principle and the exchangeable models give the correct distribution. Knowing that six old throws give $\mathbb{1}$, which plausibilities would you assign for a new throw? I should be surprised at the set of sixes, but should not conclude yet that the throwing technique or the die be completely biased towards $\mathbb{1}$, as the maximum-entropy principle suggests instead. So I find the latter’s result unreasonable.

The multiplicity model with $L = 50$ gives in my opinion the most reasonable distribution for a new throw. Also the Johnson model with $K$ somewhere between 5 and 50 gives a reasonable distribution.

Case $N = 6$, $a = 5$: The maximum-entropy distribution is reasonable when interpreted as a distribution for old throws, but I still prefer the Johnson and the multiplicity models with $K$ and $L$ around 50 that give a few percents less to the $\mathbb{1}$ and $\mathbb{2}$ faces.

For a new throw, the maximum-entropy distribution is less reasonable: I find that concentrating 75% of the plausibility on $\mathbb{2}$ and $\mathbb{3}$ is too much. Again, the multiplicity model with $L = 50$ or the Johnson with $K$ between 5 and 50 give results most reasonable for me.

So with an average based on six old throws the maximum-entropy principle still gives unsatisfactory answers. Let us look at the cases with twelve throws.

Case $N = 12$, $a = 6$: We know again that all twelve old throws give $\mathbb{1}$; the maximum-entropy principle and the three exchangeable models give the correct answer. What about a new throw? I should start to believe that the die or throwing technique be biased towards $\mathbb{1}$, but should still not be sure that they be completely biased. So the maximum-entropy principle’s answer is still unreasonable for me. I find the Johnson and the multiplicity models with low parameter values, just above 1, more reasonable, but unreasonable with higher parameter values (cf. §4).

Case $N = 12$, $a = 6$ or $5$: The maximum-entropy principle and all exchangeable models give reasonable distributions for an old throw. For a new throw I find the maximum-entropy distribution too concentrated on the faces $\mathbb{2}$ and $\mathbb{3}$; I prefer the exchangeable models with lower parameter values.

So for the problems with $N = 12$ the maximum-entropy principle gives more reasonable, though not yet satisfactory, results.

Let us consider very large values of $N$ then. Here I find the maximum-entropy distributions reasonable, for both old and new throws. How do the exchangeable models compare?

Case with $N$ large, $a = 6$: The maximum-entropy principle as well as the Johnson and multiplicity models say that the larger the number of old throws that give $\mathbb{1}$, the
larger the plausibility that the die or throwing technique are completely biased toward face 2 for a new throw as well; all plausibility is asymptotically concentrated on that face. This is what I should indeed believe.

**Case with $N$ large, $a = 5$:** This case is the most interesting. As usual, when the parameters $K$ or $L$ are much larger than $N$ both the Johnson and multiplicity models behave like the fair-throw one, as explained in app. B. For parameter values which are large, but small compared to the number of old throws, the Johnson model gives a distribution asymptotically equal to that of the maximum-entropy principle with the Burg entropy [31], rather than the Shannon one.

On the other hand, the multiplicity model gives exactly the same distribution as the usual maximum-(Shannon-)entropy principle. This result is extremely important and holds for both old and new throws.

**Cases with $1 < N \leq 12$, $a = 7/2$:** In all these problems the maximum-entropy principle and the exchangeable models for larger $K$ and $L$ give reasonable distributions, for both old and new throws. Note how all exchangeable models including the fair-throw one give, for old throws, slightly larger plausibilities to outcomes nearer 2 or 3. This is not strange, as some counting shows. For example, among the 146 sets of four-throw outcomes summing up to 14 the face 2 appears 21 times as first-throw outcome whereas the face 3 27 times.

Ironically, the results of the exchangeable models have in many cases greater Shannon entropies than those of the maximum-entropy principle. If we regard the Shannon entropy as a measure of ‘incertitude’ in a plausibility distribution, then the exchangeable models give in those cases more ‘uncertain’ or, as Jaynes would say, less ‘committing’ answers than the maximum-entropy principle. This is not in contradiction with the principle, of course: in those cases, the exchangeable models do not satisfy the constraint that the average be equal to the expectation. See the discussion in the next section.

4 Conclusions and various remarks

Most of the conclusions I state here are personal, in the sense that they are not only a matter of logic but of taste as well. I invite you to peruse the numerical results presented in the tables of app. A and arrive at your own conclusions.

When the number of throws on which the average is calculated is small, the maximum-entropy principle gives very unreasonable or even wrong results, depending on whether its distribution is interpreted as referring to new or old throws. The performance gets better the larger the numbers of throws, but with twelve of them I still find the maximum-entropy distributions unreasonable.

On which grounds do I say ‘reasonable’ or ‘unreasonable’? This is a very important and interesting question. At first, my judgements were almost instinctive; they
came from background knowledge not immediately present to the mind (see Jeffreys’ colourful discussion on this point [28, § 3.1, pp. 123–124]). With some introspection, I can say my grounds are these: I think it is very difficult to master such a throwing technique or to construct such a regularly numbered die as would lead me to assign a plausibility distribution remarkably different from the uniform one. Therefore I choose, on the simplex of distributions, a density very peaked around the uniform distribution. It is easy, on the other hand, to construct a die with two or more faces showing the same number of pips. I should reflect this by superposing to my previous density another concentrated on the facets of the simplex. Thus, neither the Johnson nor the multiplicity model represents my background knowledge exactly. I can also add that if the ‘die’ were very irregular, e.g. with one length twice the other two, I should complain of having been given false data, since I should not call that a ‘die’.

The plausibility model I choose reflects my knowledge about dice-throwing. In a problem regarding something else, e.g. the energy of some physical system, my knowledge and hence the model used would probably be different. In some situations it would even be reasonable to use non-exchangeable models.

The numerical comparison presented here shows that the use of plausibility logic gives us more possibilities of ‘fine-tuning’, of better representing our background knowledges, than the principle of maximum entropy. Enthusiasts for this principle would probably argue that in its full generality it allows for finer tuning too: we can choose any convex region on the simplex of distributions as constraint, representing the posterior distributions we deem acceptable. But the full use of plausibility logic is even more flexible: first, we can give different plausibilities to different regions; second, the latter need not be convex. And, most importantly, plausibility logic does not ask us to choose amongst posterior distributions, but to carefully specify a prior one on an appropriate hypothesis space — it requires us to examine whence we start (including what question we are asking), not whither we want to arrive. And in inference problems this is always advisable, lest we let our wishes, instead of the facts, suggest what is more or less plausible.

Advocates of the maximum-entropy principle may also contend that in the cases in which this gives apparently wrong results (small N), it is because we have chosen wrong or insufficient constraints. For instance, if we know that for N = 2 and a = 5 it is logically impossible that some old throw give 3, 4, or 5, then we ought to impose this as a constraint. I might accept this argument but still think that plausibility logic is superior since it reveals these constraints to us, as logical consequences of the situation, without requiring us to put them again explicitly into the theory. When I first obtained the results for the N = 2, a = 5 case I was indeed surprised seeing that all exchangeable models give distributions of the form (0, 0, 0, x, y, x) for old throws and (z, z, z, x, y, x) for a new one. An analysis of the possible outcomes in this case then showed why this must be logically so, as explained in § 8. Simple happenings like this show the beauty and power of plausibility logic.

As regards constraints, we have seen that in many cases the rule ‘expectation = average’ is not respected by the more reasonable exchangeable models, which can therefore have greater entropy than the maximum-entropy distribution. Because in
some cases that constraint rule is absurd. In general its range of applicability is ‘subjective’, a fact that is seldom stressed enough; consensus and thus objectivity are only reached in limit cases. By ‘subjective’ I do not mean ‘subject to personal quirks’, as de Finetti seems to imply sometimes [32], but ‘perceptibly dependent on almost imperceptible differences in background knowledge’ (as a chaotic dynamics on initial conditions).

Although our study only concerns a special example, it is easy to see how it can be generalized to a general theorem: The plausibility distribution given by the maximum-entropy principle for new observations is the same as that given by a particular class of infinitely exchangeable models in a well-defined limit case. This was known, apparently, — see Skilling [33], Rodríguez [34–36], and references therein — but I have never seen it said explicitly. I prefer not to say that the principle is ‘derived’ from plausibility logic, for the former’s formulation is based on different axioms and primitives than the latter’s, and these we have not derived. But the procedure to obtain the maximum-entropy distribution can be justified by plausibility logic without the need of those additional principles.

We need a characterization of the class of models from which the principle stems, though. Interesting studies by Skilling, Rodríguez, Caticha & Preuss try to characterize a model in this class by invoking the maximum-entropy principle again; more about this below. It would be interesting to find a characterization of the multiplicity model similar to that, mentioned in § 2.1 of the Johnson model; or in terms of a symmetry on a particular hypothesis space. This characterization is also important because one could try to generalize it to other hypothesis spaces beyond the exchangeable-model one (‘plausibilities of plausibility-indexed circumstances’). In this way we could obtain, from plausibility logic, the form of an entropy for general statistical models (in the sense of Mielnik and Holevo [37–40]); see e.g. the studies by Band & Park [41–44], Slater [45–49], Porta Mana & Björk [50], Barnum et al. [51].

The derivation of the maximum-entropy procedure from plausibility logic is also useful because it clearly shows in which situations the maximum-entropy principle can be applied, and provides reasonable results in those situations in which the principle’s answers are unreasonable or plainly wrong. The way this is mathematically achieved is explained at the end of app. B. The derivation also makes clear that there may be situations in which we can reasonably assign distributions different from those of the maximum-entropy principle. For example, if we had reasons to use a Johnson model in our inference, the conclusions of the maximum-entropy principle (with Shannon’s entropy) would obviously be at variance with those reasons.

The (large) parameter $L$ of the multiplicity model gives the order of magnitude of the number of old throws $N$ at which the maximum-entropy principle begins to approximate our conclusions, as mathematically explained in app. B. How should the value of this parameter be chosen? The answer depends on the problem, as does the choice of exchangeable — or non-exchangeable — model. In our dice examples I should use a value of 50 or slightly larger. In a problem in which I could examine
the die and the throwing technique and get the impression that they are ‘fair’, the value would be even higher.

What place in inference has the maximum-entropy principle then? As an ‘update rule’ it seems superfluous, since plausibility logic already provides us with such a rule, which we have seen to give more satisfactory results. Is it a procedure to assign ‘prior’ plausibilities, as Jaynes continually stressed? But then using average data would be inappropriate, because they are the kind of data that can be used in Bayes’ rule instead. Is it a principle for selecting a prior distributions among those we deem appropriate? Perhaps: we want a distribution that give highest plausibility to such-and-such average; which to choose? Let’s take that with maximum entropy. And yet, it would be better to ask why we want that such-and-such average have highest plausibility. If is it because we have observed that average in similar situations, why not just apply plausibility logic and Bayes’ theorem, as we have done in this study? Jaynes [52, p. 27] states that the maximum-entropy principle ‘is designed to cover more general situations, where it does not make sense to speak of “trials”’. But I have failed to find, even in Jaynes’ writings, any examples of such ‘more general situations’ or at least situations not involving some kind of repetitions of observations (‘trials’).

The mystery about the foundations of the maximum-entropy principle remains. The reason of the present study stemmed from my strictly personal opinion that any ‘update rule’ is (a) a special case of the general update rule of plausibility logic, or (b) inconsistent, or (c) not an update rule; and that any procedure for assigning prior plausibilities from some data is (a) a special case of a plausibility-logic updating within a particular model, or (b) inconsistent, or (c) not a procedure for assigning priors. I wanted to be sure that the maximum-entropy principle did not fall into the (b) alternatives. My belief now is that the (a) alternatives hold; although there is a small possibility that the (c) be right instead.

Any use of the maximum-entropy principle involving observational data usurps the just and enlightened throne of plausibility logic. Therefore the only place suitable for the principle seems to be, not in the choice of a distribution, but in the construction of prior densities over a space of such distributions, where no observational data are directly involved. Here, however, we have an infinite-dimensional simplex and the principle requires the prior definition of a ‘canonical’ density [53, § 4.b] (which is usually different from that discussed in app.C): we have a chicken-and-egg problem. And what kind of constraints should be chosen for the principle, in such a space? The studies on ‘entropic priors’ of Skilling [33, 54], Rodríguez, [34, 36, 55], and Caticha & Preuss [56, 57] are interesting in this respect, although they still leave me largely unconvinced for the time being, for reasons that may be explained elsewhere.

A final note: the above discussion concerns the logical place and exact premises of the maximum-entropy principle (which are necessary also for pedagogical purposes), but does not affect its usefulness and efficacy, both witnessed by the vast
range of applications and the number of books written about, and thanks to, this principle.

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My hate goes to PIP*\textsc{io}.

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A Tables of results

Here are the distributions, for an old and a new throw, given by the maximum-entropy principle and the fair-throw, Johnson, and multiplicity models described in § 2.1. The values are rounded to decimals of percentile; this leads in many cases to unnormalized totals. The Shannon entropy of each distribution is also given, within crotchets.

The general formulae used are derived in app. [B] where mention is also made of the routines used, when necessary, for numerical integration. See the same appendix also for explanation of the remarks appearing under some distributions.

$N = 1, \ a = 6$

| model          | old throw, $P(R^n_i | A^n_i \land I)/\% \ [H/nat]$ | new throw, $P(R^n_i | A^n_i \land I)/\% \ [H/nat]$ |
|----------------|-----------------------------------------------|-----------------------------------------------|
| ME             | (0, 0, 0, 0, 0, 0) [0]                        | uniform distribution irrespective of $a$      |
| fair-t. $I_t$  | (0, 0, 0, 0, 0, 0, 100) [0]                    | uniform distribution irrespective of $a$      |
| Johnson $I^K$: | $K = 1$                                       | (14.3, 14.3, 14.3, 14.3, 14.3, 28.6) [1.749] |
|                | $K = 5$                                       | (16.1, 16.1, 16.1, 16.1, 16.1, 19.4) [1.788] |
|                | $K = 50$                                      | (16.6, 16.6, 16.6, 16.6, 16.6, 16.9) [1.791] |
| K large        | (0, 0, 0, 0, 0, 0, 100) [0]                    | uniform distribution irrespective of $a$      |

multiplicity $I^K$:

| model          | old throw, $P(R^n_i | A^n_i \land I)/\% \ [H/nat]$ | new throw, $P(R^n_i | A^n_i \land I)/\% \ [H/nat]$ |
|----------------|-----------------------------------------------|-----------------------------------------------|
| $L = 1$        | (0, 0, 0, 0, 0, 0, 100) [0]                    | (14.4, 14.4, 14.4, 14.4, 14.4, 28.2) [1.752] |
| $L = 5$        | (0, 0, 0, 0, 0, 0, 100) [0]                    | (14.9, 14.9, 14.9, 14.9, 14.9, 25.6) [1.767] |
| $L = 50$       | (0, 0, 0, 0, 0, 0, 100) [0]                    | (16.3, 16.3, 16.3, 16.3, 16.3, 18.3) [1.789] |
| $L$ large      | (0, 0, 0, 0, 0, 0, 100) [0]                    | uniform distribution irrespective of $a$      |

$N = 1, \ a = 5$

| model          | old throw, $P(R^n_i | A^n_i \land I)/\% \ [H/nat]$ | new throw, $P(R^n_i | A^n_i \land I)/\% \ [H/nat]$ |
|----------------|-----------------------------------------------|-----------------------------------------------|
| ME             | (2.1, 3.9, 7.2, 13.6, 25.5, 47.8) [1.370]     | uniform distribution irrespective of $a$      |
| fair-t. $I_t$  | (0, 0, 0, 0, 100, 0) [0]                     | uniform distribution irrespective of $a$      |
| Johnson $I^K$: | $K = 1$                                       | (14.3, 14.3, 14.3, 14.3, 14.3, 28.6, 14.3) [1.749] |
|                | $K = 5$                                       | (16.1, 16.1, 16.1, 16.1, 19.4, 16.1) [1.788] |
|                | $K = 50$                                      | (16.6, 16.6, 16.6, 16.6, 16.9, 16.6) [1.791] |
| K large        | (0, 0, 0, 0, 100, 0, 0) [0]                    | uniform distribution irrespective of $a$      |

multiplicity $I^K$:

| model          | old throw, $P(R^n_i | A^n_i \land I)/\% \ [H/nat]$ | new throw, $P(R^n_i | A^n_i \land I)/\% \ [H/nat]$ |
|----------------|-----------------------------------------------|-----------------------------------------------|
| $L = 1$        | (0, 0, 0, 0, 100, 0) [0]                      | (14.4, 14.4, 14.4, 14.4, 28.2, 14.4) [1.752] |
| $L = 5$        | (0, 0, 0, 0, 100, 0) [0]                      | (14.9, 14.9, 14.9, 14.9, 25.6, 14.9) [1.767] |
| $L = 50$       | (0, 0, 0, 0, 100, 0) [0]                      | (16.3, 16.3, 16.3, 16.3, 18.3, 16.3) [1.789] |
| $L$ large      | (0, 0, 0, 0, 100, 0) [0]                      | uniform distribution irrespective of $a$      |
\[ N = 1, \ a = 7/2 \]

| model \[\text{old throw, } P(R_1^H|A_1^N \land I)/\% \ [H/nat]\] | new throw, \( P(R_0^H|A_1^N \land I)/\% \ [H/nat]\) |
|---|---|
| ME | (16.7, 16.7, 16.7, 16.7, 16.7) \[1.793\] | undefined |
| all exch. models | undefined | undefined |

\[ N = 2, \ a = 6 \]

| model \[\text{old throw, } P(R_{0,1}^H|A_1^N \land I)/\% \ [H/nat]\] | new throw, \( P(R_{0,1}^H|A_1^N \land I)/\% \ [H/nat]\) |
|---|---|
| ME | (0, 0, 0, 0, 0, 100) \[0\] | uniform distribution irrespective of \(a\) |
| fair-t. \(R_t\) | (0, 0, 0, 0, 0, 100) \[0\] | uniform distribution irrespective of \(a\) |
| Johnson \(I^K\): | | |
| \(K = 1\) | (0, 0, 0, 0, 0, 100) \[0\] | (12.5, 12.5, 12.5, 12.5, 12.5, 37.5) \[1.687\] |
| \(K = 5\) | (0, 0, 0, 0, 0, 100) \[0\] | (15.6, 15.6, 15.6, 15.6, 15.6, 21.9) \[1.782\] |
| \(K = 50\) | (0, 0, 0, 0, 0, 100) \[0\] | (16.6, 16.6, 16.6, 16.6, 16.6, 17.2) \[1.793\] |
| \(K \text{ large}\) | (0, 0, 0, 0, 0, 100) \[0\] | uniform distribution irrespective of \(a\) |
| multiplicity \(I_m^K\): | | |
| \(L = 1\) | (0, 0, 0, 0, 0, 100) \[0\] | (12.6, 12.6, 12.6, 12.6, 12.6, 36.9) \[1.672\] |
| \(L = 5\) | (0, 0, 0, 0, 0, 100) \[0\] | (13.5, 13.5, 13.5, 13.5, 13.5, 32.3) \[1.717\] |
| \(L = 50\) | (0, 0, 0, 0, 0, 100) \[0\] | (16.0, 16.0, 16.0, 16.0, 16.0, 19.8) \[1.787\] |
| \(L \text{ large}\) | (0, 0, 0, 0, 0, 100) \[0\] | uniform distribution irrespective of \(a\) |

\[ N = 2, \ a = 5 \]

| model \[\text{old throw, } P(R_{0,1}^H|A_1^N \land I)/\% \ [H/nat]\] | new throw, \( P(R_{0,1}^H|A_1^N \land I)/\% \ [H/nat]\) |
|---|---|
| ME | (2.1, 3.9, 7.2, 13.6, 25.5, 47.8) \[1.370\] | uniform distribution irrespective of \(a\) |
| fair-t. \(R_t\) | (0, 0, 0, 33.3, 33.3, 33.3) \[1.10\] | uniform distribution irrespective of \(a\) |
| Johnson \(I^K\): | | |
| \(K = 1\) | (0, 0, 0, 25.0, 50.0, 25.0) \[1.04\] | (12.5, 12.5, 12.5, 18.8, 25.0, 18.8) \[1.755\] |
| \(K = 5\) | (0, 0, 0, 31.2, 37.5, 31.2) \[1.09\] | (15.6, 15.6, 15.6, 17.6, 18.0, 17.6) \[1.790\] |
| \(K = 50\) | (0, 0, 0, 33.1, 33.8, 33.1) \[1.10\] | (16.6, 16.6, 16.6, 16.8, 16.8) \[1.793\] |
| \(K \text{ large}\) | (0, 0, 0, 33.3, 33.3, 33.3) \[1.10\] | uniform distribution irrespective of \(a\) |
| multiplicity \(I_m^K\): | | |
| \(L = 1\) | (0, 0, 0, 25.4, 49.1, 25.4) \[1.05\] | (12.6, 12.6, 12.6, 18.8, 24.5, 18.8) \[1.756\] |
| \(L = 5\) | (0, 0, 0, 28.9, 46.1, 26.9) \[1.06\] | (13.4, 13.4, 13.4, 18.8, 22.1, 18.8) \[1.770\] |
| \(L = 50\) | (0, 0, 0, 32.0, 35.9, 32.0) \[1.10\] | (16.0, 16.0, 16.0, 17.3, 17.4, 17.3) \[1.791\] |
| \(L \text{ large}\) | (0, 0, 0, 33.3, 33.3, 33.3) \[1.10\] | uniform distribution irrespective of \(a\) |
\( N = 2, \ a = 7/2 \)

| model  | old throw, \( P(R_R^N | A_N^0 \wedge I)/% \ [H/\text{nat}] \) | new throw, \( P(R_R^N | A_N^0 \wedge I)/% \ [H/\text{nat}] \) |
|-------|--------------------------------------------------|--------------------------------------------------|
| ME    | \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)  | uniform distribution irrespective of \( a \)           |
| fair-t. \( I_R \) | \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)  | uniform distribution irrespective of \( a \)           |
| Johnson \( I_R^K \). | \( K = 1 \) \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)  | \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)         |
|       | \( K = 5 \) \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)  | \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)         |
|       | \( K \text{ large} \) \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)  | uniform distribution irrespective of \( a \)           |
| multiplicity \( I_R^M \): | \( L = 1 \) \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)  | \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)         |
|       | \( L = 5 \) \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)  | \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)         |
|       | \( L = 50 \) \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)  | \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)         |
|       | \( L \text{ large} \) \( (16.7, 16.7, 16.7, 16.7, 16.7) \ [1.793] \)  | uniform distribution irrespective of \( a \)           |

\( N = 6, \ a = 6 \)

| model  | old throw, \( P(R_R^N | A_N^0 \wedge I)/% \ [H/\text{nat}] \) | new throw, \( P(R_R^N | A_N^0 \wedge I)/% \ [H/\text{nat}] \) |
|-------|--------------------------------------------------|--------------------------------------------------|
| ME    | \( (0, 0, 0, 0, 0, 0, 100) \ [0] \)            | uniform distribution irrespective of \( a \)           |
| fair-t. \( I_R \) | \( (0, 0, 0, 0, 0, 0, 100) \ [0] \)            | uniform distribution irrespective of \( a \)           |
| Johnson \( I_R^K \). | \( K = 1 \) \( (0, 0, 0, 0, 0, 0, 100) \ [0] \)  | \( (8.3, 8.3, 8.3, 8.3, 8.3, 8.3, 58.3) \ [1.347] \) |
|       | \( K = 5 \) \( (0, 0, 0, 0, 0, 0, 100) \ [0] \)  | \( (13.9, 13.9, 13.9, 13.9, 13.9, 13.9, 30.6) \ [1.734] \) |
|       | \( K \text{ large} \) \( (0, 0, 0, 0, 0, 0, 100) \ [0] \)  | \( (16.3, 16.3, 16.3, 16.3, 16.3, 16.3, 18.3) \ [1.789] \) |
| multiplicity \( I_R^M \): | \( L = 1 \) \( (0, 0, 0, 0, 0, 0, 100) \ [0] \)  | \( (8.5, 8.5, 8.5, 8.5, 8.5, 8.5, 57.5) \ [1.366] \) |
|       | \( L = 5 \) \( (0, 0, 0, 0, 0, 0, 100) \ [0] \)  | \( (10.0, 10.0, 10.0, 10.0, 10.0, 10.0, 50.0) \ [1.498] \) |
|       | \( L = 50 \) \( (0, 0, 0, 0, 0, 0, 100) \ [0] \)  | \( (15.0, 15.0, 15.0, 15.0, 15.0, 15.0, 24.8) \ [1.769] \) |
|       | \( L \text{ large} \) \( (0, 0, 0, 0, 0, 0, 100) \ [0] \)  | uniform distribution irrespective of \( a \)           |
### $N = 6, a = 5$

| Model | Old Throw, $P(R^a_{\text{nat}} | A_{\text{new}}^N \land I)/\%$ [H/nat] | New Throw, $P(R^a_{\text{nat}} | A_{\text{new}}^N \land I)/\%$ [H/nat] |
|--------|---------------------------------------------------------------------------------|---------------------------------------------------------------------------------|
| ME     | $(2.1, 3.9, 7.2, 13.6, 25.5, 47.8) [1.370]$                                   |                                                                                  |
| Fair-Throw $I_h$ | $(1.1, 3.3, 7.7, 15.4, 27.6, 45.0)$ [1.362] | Uniform distribution irrespective of $a$ |
| Johnson $I_h^J$: |                                                                                  |                                                                                  |
| $K = 1$ | $(1.7, 3.3, 6.7, 13.3, 31.7, 43.3)$ [1.358]                                   | $(9.2, 10.0, 11.7, 15.0, 24.2, 30.0)$ [1.690] |
| $K = 5$ | $(1.4, 3.5, 7.3, 14.7, 27.5, 45.5)$ [1.363]                                   | $(14.1, 14.5, 15.1, 16.3, 18.5, 21.5)$ [1.780] |
| $K = 50$ | $(1.1, 3.3, 7.6, 15.3, 27.6, 45.0)$ [1.360]                                   | $(16.4, 16.4, 16.5, 16.6, 16.9, 17.2)$ [1.792] |
| $K$ large | $(1.1, 3.3, 7.7, 15.4, 27.6, 45.0)$ [1.362] | Uniform distribution irrespective of $a$ |
| Multiplicity $I_h^M$: |                                                                                  |                                                                                  |
| $L = 1$ | $(1.7, 3.4, 6.7, 13.4, 31.2, 43.6)$ [1.360]                                   | $(9.2, 10.1, 11.8, 15.1, 23.8, 29.9)$ [1.691] |
| $L = 5$ | $(1.6, 3.5, 6.9, 14.1, 29.0, 44.9)$ [1.363]                                   | $(10.2, 11.1, 12.6, 15.7, 21.8, 28.5)$ [1.718] |
| $L = 50$ | $(1.3, 3.4, 7.4, 14.9, 27.5, 45.4)$ [1.361]                                   | $(15.0, 15.2, 15.7, 16.5, 17.9, 19.7)$ [1.787] |
| $L$ large | $(1.1, 3.3, 7.7, 15.4, 27.6, 45.0)$ [1.362] | Uniform distribution irrespective of $a$ |

### $N = 6, a = 7/2$

| Model | Old Throw, $P(R^a_{\text{nat}} | A_{\text{new}}^N \land I)/\%$ [H/nat] | New Throw, $P(R^a_{\text{nat}} | A_{\text{new}}^N \land I)/\%$ [H/nat] |
|--------|---------------------------------------------------------------------------------|---------------------------------------------------------------------------------|
| ME     | $(16.7, 16.7, 16.7, 16.7, 16.7, 16.7)$ [1.793]                                   |                                                                                  |
| Fair-Throw $I_h$ | $(15.0, 17.0, 18.0, 18.0, 17.0, 15.0)$ [1.789] | Uniform distribution irrespective of $a$ |
| Johnson $I_h^J$: |                                                                                  |                                                                                  |
| $K = 1$ | $(13.0, 16.1, 20.8, 20.8, 16.1, 13.0)$ [1.772]                                   | $(14.8, 16.4, 18.8, 18.8, 16.4, 14.8)$ [1.787] |
| $K = 5$ | $(14.5, 16.9, 18.6, 18.6, 16.9, 14.5)$ [1.787]                                   | $(16.3, 16.7, 17.0, 17.0, 16.7, 16.3)$ [1.792] |
| $K = 50$ | $(15.0, 17.0, 18.1, 18.1, 17.0, 15.0)$ [1.790]                                   | $(16.6, 16.7, 16.7, 16.7, 16.7, 16.6)$ [1.792] |
| $K$ large | $(15.0, 17.0, 18.0, 18.0, 17.0, 15.0)$ [1.789] | Uniform distribution irrespective of $a$ |
| Multiplicity $I_h^M$: |                                                                                  |                                                                                  |
| $L = 1$ | $(13.1, 16.2, 20.7, 20.7, 16.2, 13.1)$ [1.774]                                   | $(14.9, 16.4, 18.7, 18.7, 16.4, 15.0)$ [1.788] |
| $L = 5$ | $(13.5, 16.6, 20.0, 20.0, 16.5, 13.5)$ [1.780]                                   | $(15.4, 16.7, 18.0, 18.0, 16.6, 15.4)$ [1.791] |
| $L = 50$ | $(14.7, 17.0, 18.3, 18.3, 17.0, 14.7)$ [1.788]                                   | $(16.5, 16.7, 16.8, 16.8, 16.7, 16.5)$ [1.792] |
| $L$ large | $(15.0, 17.0, 18.0, 18.0, 17.0, 15.0)$ [1.789] | Uniform distribution irrespective of $a$ |
$N = 12$, $a = 6$

| model | old throw, $P(R_j^a | A_j^a \land I) / % [H/nat]$ | new throw, $P(R_j^a | A_j^a \land I) / % [H/nat]$ |
|-------|-----------------------------------------------|-----------------------------------------------|
| ME    | (0, 0, 0, 0, 0, 100) [0]                      | uniform distribution irrespective of $a$     |
| fair-t. $I_t$ |                                                 |                                               |
| $K = 1$ | (0.0, 0, 0, 0, 100) [0]                       | (5.6, 5.6, 5.6, 5.6, 6.72.2) [1.042]         |
| $K = 5$ | (0.0, 0, 0, 0, 100) [0]                       | (11.9, 11.9, 11.9, 11.9, 11.9, 40.5) [1.633] |
| $K = 50$ | (0.0, 0, 0, 0, 100) [0]                      | (16.0, 16.0, 16.0, 16.0, 16.0, 19.9) [1.787] |
| $K$ large | (0.0, 0, 0, 0, 100) [0]                    | uniform distribution irrespective of $a$     |
| Johnson $I^K_j$: |                                               |                                               |
| $L$ = 1 | (0.0, 0, 0, 0, 100) [0]                       | (5.7, 5.7, 5.7, 5.7, 5.7, 7.16) [1.056]      |
| $L$ = 5 | (0.0, 0, 0, 0, 100) [0]                       | (7.0, 7.0, 7.0, 7.0, 7.0, 64.9) [1.211]      |
| $L$ = 50 | (0.0, 0, 0, 0, 100) [0]                       | (13.9, 13.9, 13.9, 13.9, 13.9, 30.7) [1.734] |
| $L$ large | (0.0, 0, 0, 0, 100) [0]                    | uniform distribution irrespective of $a$     |
| multiplicity $I^L_j$: |                                               |                                               |
| $L$ = 1 | (1.6, 3.6, 7.4, 14.4, 26.6, 46.4) [1.366]     | uniform distribution irrespective of $a$     |
| $L$ = 5 | (2.4, 3.6, 6.6, 11.7, 26.6, 48.2) [1.367]     | (7.3, 8.4, 9.9, 13.4, 23.3, 37.7) [1.605]    |
| $L$ = 50 | (2.2, 4.0, 7.2, 13.2, 25.2, 48.3) [1.368]    | (12.5, 13.0, 14.0, 15.7, 19.1, 25.7) [1.756] |
| $L$ large | (1.7, 3.6, 7.4, 14.3, 26.3, 46.7) [1.367]     | (16.1, 16.2, 16.3, 16.6, 17.0, 17.8) [1.791] |
| $K$ large | (1.6, 3.6, 7.4, 14.4, 26.6, 46.4) [1.366]     | uniform distribution irrespective of $a$     |

$N = 12$, $a = 5$

| model | old throw, $P(R_j^a | A_j^a \land I) / % [H/nat]$ | new throw, $P(R_j^a | A_j^a \land I) / % [H/nat]$ |
|-------|-----------------------------------------------|-----------------------------------------------|
| ME    | (2.1, 3.9, 7.2, 13.6, 25.5, 47.8) [1.370]     | uniform distribution irrespective of $a$     |
| fair-t. $I_t$ |                                                 |                                               |
| $K = 1$ | (2.7, 4.3, 6.6, 11.7, 26.6, 48.2) [1.367]     | (7.3, 8.4, 9.9, 13.4, 23.3, 37.7) [1.605]    |
| $K = 5$ | (2.2, 4.0, 7.2, 13.2, 25.2, 48.3) [1.368]     | (12.5, 13.0, 14.0, 15.7, 19.1, 25.7) [1.756] |
| $K = 50$ | (1.7, 3.6, 7.4, 14.3, 26.3, 46.7) [1.367]     | (16.1, 16.2, 16.3, 16.6, 17.0, 17.8) [1.791] |
| $K$ large | (1.6, 3.6, 7.4, 14.4, 26.6, 46.4) [1.366]     | uniform distribution irrespective of $a$     |
| Johnson $I^K_j$: |                                               |                                               |
| $L$ = 1 | (2.6, 4.2, 6.6, 11.8, 26.4, 48.3) [1.363]     | (7.4, 8.5, 10.0, 13.5, 23.1, 37.5) [1.609]    |
| $L$ = 5 | (2.5, 4.1, 6.8, 12.4, 25.6, 48.5) [1.365]     | (8.1, 9.2, 10.9, 14.4, 22.1, 35.2) [1.645]    |
| $L$ = 50 | (1.9, 3.8, 7.3, 13.7, 25.8, 47.5) [1.366]     | (13.7, 14.2, 14.9, 16.3, 18.6, 22.4) [1.777] |
| $L$ large | (1.6, 3.6, 7.4, 14.4, 26.6, 46.4) [1.366]     | uniform distribution irrespective of $a$     |

[17]
\[N = 12, \ a = 7/2\]

| model         | old throw, \(P(R_i^n | A_x^n \wedge I)/% [H/nat]\) | new throw, \(P(R_i^n | A_x^n \wedge I)/% [H/nat]\) |
|---------------|---------------------------------|---------------------------------|
| ME           | (16.7, 16.7, 16.7, 16.7, 16.7, 16.7, 16.7, 16.7, 16.7) [1.793] | uniform distribution irrespective of \(a\) |
| fair-t. \(I_n\) | \((15.9, 16.8, 17.3, 17.3, 16.8, 15.9) [1.791]\) | uniform distribution irrespective of \(a\) |

Johnson \(I^K_n\):  
\(K = 1\) \((13.5, 16.5, 20.0, 20.0, 16.5, 13.5) [1.779]\) \((14.5, 16.6, 18.9, 18.9, 16.6, 14.5) [1.786]\)  
\(K = 5\) \((15.3, 16.9, 17.9, 17.9, 16.9, 15.3) [1.791]\) \((16.3, 16.7, 17.0, 17.0, 16.7, 16.3) [1.792]\)  
\(K = 50\) \((15.8, 16.8, 17.4, 17.4, 16.8, 15.8) [1.791]\) \((16.6, 16.7, 16.7, 16.7, 16.7, 16.6) [1.792]\)  
\(K\) large \((15.9, 16.8, 17.3, 17.3, 16.8, 15.9) [1.791]\) uniform distribution irrespective of \(a\)  
like fair-throw model

Multiplicity \(I^{L_n}_n\):  
\(L = 1\) \((13.5, 16.6, 19.9, 19.9, 16.6, 13.6) [1.780]\) \((14.6, 16.6, 18.7, 18.7, 16.6, 14.7) [1.786]\)  
\(L = 5\) \((14.1, 16.8, 19.1, 19.1, 16.8, 14.1) [1.784]\) \((15.2, 16.8, 18.0, 18.0, 16.7, 15.2) [1.789]\)  
\(L = 50\) \((15.5, 16.9, 17.6, 17.6, 16.9, 15.5) [1.790]\) \((16.5, 16.7, 16.8, 16.8, 16.7, 16.5) [1.792]\)  
\(L\) large \((15.9, 16.8, 17.3, 17.3, 16.8, 15.9) [1.791]\) uniform distribution irrespective of \(a\)  
like fair-throw model

\[N = 12, \ a = 6\]

| model         | old throw, \(P(R_i^n | A_x^n \wedge I)/% [H/nat]\) | new throw, \(P(R_i^n | A_x^n \wedge I)/% [H/nat]\) |
|---------------|---------------------------------|---------------------------------|
| ME           | (0, 0, 0, 0, 0, 0, 100) [0] | uniform distribution irrespective of \(a\) |
| fair-t. \(I_n\) | \(0, 0, 0, 0, 0, 0, 100) [0] | uniform distribution irrespective of \(a\) |

Johnson \(I^K_n\):  
\(K = 1\) \((0, 0, 0, 0, 0, 0, 100) [0]\) \((0, 0, 0, 0, 0, 0, 100) [0]\)  
\(K = 5\) \((0, 0, 0, 0, 0, 0, 100) [0]\) \((0, 0, 0, 0, 0, 0, 100) [0]\)  
\(K = 50\) \((0, 0, 0, 0, 0, 0, 100) [0]\) \((0, 0, 0, 0, 0, 0, 100) [0]\)  
\(K\) large: \((0, 0, 0, 0, 0, 0, 100) [0]\) uniform distribution irrespective of \(a\)  
like fair-throw model

Multiplicity \(I^{L_n}_n\):  
\(L = 1\) \((0, 0, 0, 0, 0, 0, 100) [0]\) \((0, 0, 0, 0, 0, 0, 100) [0]\)  
\(L = 5\) \((0, 0, 0, 0, 0, 0, 100) [0]\) \((0, 0, 0, 0, 0, 0, 100) [0]\)  
\(L = 50\) \((0, 0, 0, 0, 0, 0, 100) [0]\) \((0, 0, 0, 0, 0, 0, 100) [0]\)  
\(L\) large: \((0, 0, 0, 0, 0, 0, 100) [0]\) uniform distribution irrespective of \(a\)  
like fair-throw model

ME distribution for Burg entropy  
ME distribution for Burg entropy
$N$ large, $a = 5$

| model      | old throw, $P(R^n|A^n \wedge I)/\%$ [H/nat] | new throw, $P(R^n|A^n \wedge I)/\%$ [H/nat] |
|------------|---------------------------------------------|---------------------------------------------|
| ME         |                                             |                                             |
| fair l. $I_0$ | (2.1, 3.9, 7.2, 13.6, 25.5, 47.8) [1.370]  | uniform distribution irrespective of $a$    |
| Johnson $I^n_J$: |                                             |                                             |
| $K = 1$   | (4.0, 5.0, 6.7, 10.0, 20.0, 54.3) [1.343]    | (4.0, 5.0, 6.7, 10.0, 20.0, 54.3) [1.343]   |
| $K = 5$   | (4.3, 5.3, 6.9, 9.8, 17.2, 56.5) [1.328]     | (4.3, 5.3, 6.9, 9.8, 17.2, 56.5) [1.328]    |
| $K$ large: |                                             |                                             |
| $N/K$ small| (2.1, 3.9, 7.2, 13.6, 25.5, 47.8) [1.370]   | uniform distribution irrespective of $a$    |
| $K$ large: |                                             |                                             |
| $N/K$ large| (4.4, 5.3, 6.9, 9.8, 16.7, 57.0) [1.325]    | (4.4, 5.3, 6.9, 9.8, 16.7, 57.0) [1.325]    |
| multiplicity $l^n_m$: |                                             |                                             |
| $L = 1$   | (4.0, 5.0, 6.7, 10.1, 20.2, 54.1) [1.347]    | (4.0, 5.0, 6.7, 10.1, 20.2, 54.1) [1.347]   |
| $L = 5$   | (3.6, 4.7, 6.6, 10.7, 21.8, 52.5) [1.352]    | (3.6, 4.7, 6.6, 10.7, 21.8, 52.5) [1.352]   |
| $L$ large: |                                             |                                             |
| $N/L$ small| (2.1, 3.9, 7.2, 13.6, 25.5, 47.8) [1.370]   | uniform distribution irrespective of $a$    |
| $L$ large: |                                             |                                             |
| $N/L$ large| (2.1, 3.9, 7.2, 13.6, 25.5, 47.8) [1.370]   | (2.1, 3.9, 7.2, 13.6, 25.5, 47.8) [1.370]   |
\( N \) large, \( a = 3.5 \)

| model          | old throw, \( P(R^n|A^n \wedge I)/\% \) [H/nat] | new throw, \( P(R^n|A^n \wedge I)/\% \) [H/nat] |
|----------------|-----------------------------------------------|-----------------------------------------------|
| ME            | \( (16.7, 16.7, 16.7, 16.7, 16.7) \) [1.793] |                                               |
| fair-t. \( I_n \) | \( (16.7, 16.7, 16.7, 16.7, 16.7) \) [1.793] | uniform distribution irrespective of \( a \) |
| Johnson \( I^R_n \): |                                               |                                               |
| \( K = 1 \)  | \( (14.1, 16.6, 19.3, 19.3, 16.6, 14.1) \) [1.784] | \( (14.1, 16.6, 19.3, 19.3, 16.6, 14.1) \) [1.784] |
| \( K = 5 \)  | \( (16.1, 16.8, 17.2, 17.2, 16.8, 16.1) \) [1.793] | \( (16.1, 16.8, 17.2, 17.2, 16.8, 16.1) \) [1.793] |
| \( K = 50 \) | \( (16.6, 16.7, 16.7, 16.7, 16.7, 16.6) \) [1.792] | \( (16.6, 16.7, 16.7, 16.7, 16.7, 16.6) \) [1.792] |
| \( N/K \) small | \( (16.7, 16.7, 16.7, 16.7, 16.7, 16.7) \) [1.793] | uniform distribution irrespective of \( a \) |
| \( N/K \) large | \( (16.7, 16.7, 16.7, 16.7, 16.7, 16.7) \) [1.793] | \( (16.7, 16.7, 16.7, 16.7, 16.7, 16.7) \) [1.793] |
| multiplicity \( I^L_m \): |                                               |                                               |
| \( L = 1 \)  | \( (14.2, 16.6, 19.2, 19.2, 16.6, 14.2) \) [1.784] | \( (14.2, 16.6, 19.2, 19.2, 16.6, 14.2) \) [1.784] |
| \( L = 5 \)  | \( (14.9, 16.8, 18.3, 18.3, 16.8, 14.9) \) [1.788] | \( (14.9, 16.8, 18.3, 18.3, 16.8, 14.9) \) [1.788] |
| \( L = 50 \) | \( (16.5, 16.7, 16.8, 16.8, 16.7, 16.5) \) [1.792] | \( (16.5, 16.7, 16.8, 16.8, 16.7, 16.5) \) [1.792] |
| \( L \) large: |                                               |                                               |
| \( N/L \) small | \( (16.7, 16.7, 16.7, 16.7, 16.7) \) [1.793] | uniform distribution irrespective of \( a \) |
| \( N/L \) large | \( (16.7, 16.7, 16.7, 16.7, 16.7) \) [1.793] | \( (16.7, 16.7, 16.7, 16.7, 16.7) \) [1.793] |
In §2 we introduced the propositions $R_i^{(j)}$ stating that throw $j$ shows face ‘$i$’. Throw $j = 1$ is an ‘old’ throw, $j = 0$ a ‘new’ one, in the sense already explained in the same section. The proposition $I$ denotes the model used in our inferences and other background knowledge.

Let $F_N$ denote the statement that the number of occurrences of the six possible outcomes in $N$ throws is $N ≡ (N_i)$. So $F_N$ is a disjunction of conjunctions of $R$s; e.g., for $N = 3$ and $N = (0, 0, 2, 1, 0)$ (two $\square$ and one $\Box$),

$$F_{(0,0,2,1,0)} ≡ (R_4^{(1)} \land R_4^{(2)} \land R_5^{(3)}) \lor (R_4^{(1)} \land R_5^{(2)} \land R_4^{(3)}) \lor (R_5^{(1)} \land R_4^{(2)} \land R_5^{(3)}).$$

With a finitely or infinitely exchangeable model $I$ for the old throws, the plausibility of a set of outcomes depends on their frequencies but not on which throws they occur; in our example,

$$P(R_4^{(1)} \land R_4^{(2)} \land R_5^{(3)} | I) = P(R_4^{(1)} \land R_5^{(2)} \land R_4^{(3)} | I) = P(R_5^{(1)} \land R_4^{(2)} \land R_5^{(3)} | I).$$

In this case it is a simple combinatorial exercise to show that for any old throw, e.g. the first,

$$P(R_i^{(1)} | F_N \land I) = N_i/N,$$

unless $F_N$ and $I$ be incompatible, i.e. $P(F_N | I) = 0$, a case which we exclude.

If the model is infinitely exchangeable with density $g(p | I) dp$ we have by standard combinatorial arguments [e.g. 12 § 1.2; 58 § 4.3.2; 13 § 2, 14]

$$P(F_N | I) = \int N! \left( \prod_i \frac{p_i^{N_i}}{N_i!} \right) g(p | I) dp.$$

Knowledge of the frequency of old throws leads to the ‘updated’ density

$$g(p | F_N \land I) dp = \frac{N! \left( \prod_i \frac{p_i^{N_i}}{N_i!} \right) g(p | I)}{P(F_N | I)} dp.$$

from which we obtain the plausibility distribution for a new throw:

$$P(R_i^{(0)} | F_N \land I) = \int p_i g(p | F_N \land I) dp = \frac{\int N! p_i \left( \prod_i \frac{p_i^{N_i}}{N_i!} \right) g(p | I) dp}{P(F_N | I)}.$$

Recall that $A_a^N$ is the statement that in $N$ throws the observed average is $a$. This is obviously a statement about the possible outcome frequencies of old throws, and we can write it as

$$v \cdot N = aN, \quad \text{with } v := (1, 2, 3, 4, 5, 6).$$
Hence
\[ A^N_a = \bigvee_{v \cdot N = aN} F_N; \quad (17) \]
e.g., if the observed average in two throws is 5/2,
\[ A^{(2)}_{5/2} \equiv F_{(1,0,0,1,0,0)} \lor F_{(0,1,1,0,0,0)} \lor F_{(0,0,0,0,2,0)}. \quad (18) \]
Sums over frequencies constrained by the formula \( v \cdot N = aN \) will for brevity be denoted by \( \sum_N^{(a)} \).

From eq. (17), any plausibility conditional on \( AN_a \) can be resolved into a weighted sum of plausibilities conditional on the \( FN_a \):
\[ P(\cdot | AN_a \land I) = \frac{\sum_N^{(a)} P(\cdot | FN \land I) P(FN|I)}{\sum_N^{(a)} P(FN|I)}. \quad (19) \]

Using formulæ (12)–(15) and (19) we find
\[ P(R^{(1)}_i | AN_a \land I) = \frac{\int \sum_N^{(a)} N_i^N ! \left( \prod_l \frac{p_{l_i}}{N_l} \right) g(p|I) \, dp}{\int \sum_N^{(a)} N_i^N ! \left( \prod_l \frac{p_{l_i}}{N_l} \right) g(p|I) \, dp}, \quad (20a) \]
\[ P(R^{(0)}_i | AN_a \land I) = \frac{\int \sum_N^{(a)} p_j N_j^N ! \left( \prod_l \frac{p_{l_j}}{N_l} \right) g(p|I) \, dp}{\int \sum_N^{(a)} N_j^N ! \left( \prod_l \frac{p_{l_j}}{N_l} \right) g(p|I) \, dp}, \quad (20b) \]

With these formulæ we can compute the plausibilities required in our study.

Integration of these expressions is straightforward for the fair-throw model. We obtain
\[ P(R^{(1)}_i | A^N_a \land I_f) = \frac{\sum_N^{(a)} N_i^N 6^N \prod_l \frac{N_l}{N}}{\sum_N^{(a)} 6^N \prod_l \frac{N_l}{N}}, \quad (21a) \]
\[ P(R^{(0)}_i | A^N_a \land I_f) = \frac{\sum_N^{(a)} 6^N \prod_l \frac{N_l}{N}}{\sum_N^{(a)} 6^N \prod_l \frac{N_l}{N}} = \frac{1}{6}. \quad (21b) \]

Note how the plausibility distribution for a new throw is independent from any knowledge about old — or any other — throws. This model, like any other i.i.d. one, does not allow to ‘learn from experience’.

By eq. (2) for enough large \( N \) the first plausibility above is approximated by
\[ P(R^{(1)}_i | A^N_a \land I_f) \approx \frac{\sum_N^{(a)} f_i \exp[NH(f)] \, df}{\sum_N^{(a)} \exp[NH(f)] \, df}, \quad \text{with } f_i := N_i/N, \quad (22) \]
\[ \approx f_i \text{ which maximizes } H(f) \text{ under the constraint } v \cdot f = a. \]

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\[ \approx f_i \text{ which maximizes } H(f) \text{ under the constraint } v \cdot f = a. \]
where $H$ is Shannon’s entropy $H$. This is the result of van Campenhout and Cover [8] and Csiszár [9]: for old throws, the maximum-entropy distribution is the asymptotic distribution for an old throw conditional on the average of a large number of old throws, under the assumption of a uniform i.i.d. model.

In the case of the Johnson model, with density

$$g(p| I^K_i) dp = \Gamma(1 + \sum_i K) \left[ \prod_l \frac{p_l^{K-1}}{\Gamma(K)} \right] dp, \quad K > 0,$$

the integrals above can be reduced to Dirichlet’s generalization of the beta integral [59, § 1.8],

$$\int \left( \prod_l p_l^{b_l-1} \right) dp = \frac{\Gamma(\sum_l b_l)}{\Gamma(\sum_l b_l-1+K) \prod_l \Gamma(b_l)}, \quad \text{Re} b_l > 0,$$

and after some simplifications we obtain the closed-form formulæ

$$P(R^{(1)}_i | A^N_a \land I^K_i) = \sum_a N_i^{(a)} \prod_l \frac{(N_i+K-1)!}{N_i!},$$

$$P(R^{(0)}_i | A^N_a \land I^K_i) = \sum_a N_i^{(a)} \prod_l \frac{(N_i+K-1)!}{N_i!},$$

and the Johnson model is approximated by the fair-throw one. This is also true for $N$ large but enough smaller than $K$.

When $N$ is enough large and $K$ finite eq. (2) can again be used to show that expressions (24a) are both approximated by the integral

$$P(R^{(1)}_i | A^N_a \land I^K_i) \approx \int_{\Delta_a} f_i \prod_l f_l^{K-1} df,$$

where $\Delta_a$ is the intersection between the plausibility simplex $\Delta (5)$ and the constraint hyperplane $v \cdot f = a$. If $K$ is also enough large but enough smaller than $N$ the integral
above becomes
\[
\frac{\int_{\Delta_n} f_i \prod_i f_i^{K-1} \, df}{\int_{\Delta_n} \prod_i f_i^{K-1} \, df} \approx \frac{\int_{\Delta_n} f_i \exp[KH_B(f)] \, df}{\int_{\Delta_n} \exp[KH_B(f)] \, df}
\]  (27)
where \( H_B \) is Burg’s entropy [31]:
\[
H_B(f) := \sum_i \ln f_i.
\]  (28)

Then
\[
P(R_1^1|A^N_u \land I^K_j) \approx P(R_0^0|A^N_u \land I^K_j) \approx f_i \text{ which maximizes } H_B(f) \text{ under the constraint } \nu \cdot f = a. \]  (29)

Thus the Johnson model yields the maximum-entropy principle with Burg’s entropy; cf. Jaynes [21, p. 19]. Had we used a generalized Johnson model with density
\[
g(p|L^K_m) \, dp = \Gamma(1 + K) \left[ \prod_i \frac{p_i^{Km_i - 1}}{\Gamma(Km_i)} \right] \, dp, \quad K, m_i > 0, \sum_i m_i = 1
\]  (30)

the asymptotic expression [29] would have had the Kullback-Leibler divergence [60–62]
\[
- D(m, f) := - \sum_i m_i \ln(m_i/f_i)
\]  (31)
in place of Burg’s entropy \( H_B(f) \).

For the case in which both \( N \) and \( K \) are large and comparable with each other, see the following analogous discussion for the multiplicity model.

The integrals for the multiplicity model, with density
\[
g(p|L^K_m) \, dp = c(L) \frac{L^L}{\prod_i (Lp_i)!} \, dp, \quad L \geq 1,
\]  (8)
are
\[
P(R_1^1|A^N_u \land I^L_m) = \frac{\int \sum_i \frac{N^i}{N} N^i! L^i! \left( \prod_i \frac{p_i^{N_i}}{N_i!(Lp_i)!} \right) \, dp}{\int \sum_i \frac{N^i}{N} N^i! L^i! \left( \prod_i \frac{p_i^{N_i}}{N_i!(Lp_i)!} \right) \, dp}, \]  (32a)
\[
P(R_0^0|A^N_u \land I^L_m) = \frac{\int \sum_i \frac{N^i}{N} N^i! L^i! \left( \prod_i \frac{p_i^{N_i}}{N_i!(Lp_i)!} \right) \, dp}{\int \sum_i \frac{N^i}{N} N^i! L^i! \left( \prod_i \frac{p_i^{N_i}}{N_i!(Lp_i)!} \right) \, dp}. \]  (32b)
When \( L \) is large enough, the usual relationship (2) between Shannon’s entropy and the multiplicity factor leads to the approximations

\[
P(R_i^{(1)} | A^N_n \wedge I^L_m) \approx \frac{\int \sum_N (N N^I) \prod (1/N^I) \exp[LH(p)] dp}{\sum_N (N N^I) \prod (1/N^I) \exp[LH(p)]} = \frac{\sum_N (N N^I) 6^{-N} \prod (N^I)}{\sum_N (N N^I) \prod (N^I)}.
\]  

(33a)

\[
P(R_i^{(0)} | A^N_n \wedge I^L_m) \approx \frac{\int \sum_N (p_i N) \prod (1/N^I) \exp[LH(p)] dp}{\sum_N (p_i N) \prod (1/N^I) \exp[LH(p)]} = \frac{\sum_N (p_i 1) 6^{-N} \prod (N^I)}{\sum_N (p_i 1) \prod (N^I)} = \frac{1}{6}. 
\]  

(33b)

thus the multiplicity model is approximated by the fair-throw one for enough large values of \( L \), also when \( N \) is large but smaller than \( L \).

What happens if \( N \) is large and \( L \) finite? As for the Johnson model, the integrals can be approximated by their restriction to the hyperplane \( \mathbf{v} : \mathbf{f} = a \):

\[
P(R_i^{(1)} | A^N_n \wedge I^L_m) \approx P(R_i^{(0)} | A^N_n \wedge I^L_m) \approx \frac{\int_{A^N_n} f_i [\prod (Lf_i)!]^{-1} df}{\int_{A^N_n} [\prod (Lf_i)!]^{-1} df}.
\]  

(34)

Most interesting is the case in which \( L \) is also large but still smaller than \( N \). From eq. (2) as usual we see that the integrals above are approximated by

\[
\frac{\int_{A^N_n} f_i [\prod (Lf_i)!]^{-1} df}{\int_{A^N_n} [\prod (Lf_i)!]^{-1} df} \approx \frac{\int_{A^N_n} f_i \exp[LH(f)] df}{\int_{A^N_n} \exp[LH(f)] df}
\]  

(35)

so that

\[
P(R_i^{(1)} | A^N_n \wedge I^L_m) \approx P(R_i^{(0)} | A^N_n \wedge I^L_m) \approx f_i \text{ which maximizes } H(f) \text{ under the constraint } \mathbf{v} : \mathbf{f} = a.
\]  

(36)

That is, for large \( N, L \), and \( N/L \), the distribution given by the multiplicity model for old and new throws is equal to that of the maximum-entropy principle.

It is easy to see why. For \( N \) large, and larger than \( L \), the data make the posterior density of the model very peaked on the hyperplane determined by the average \( a \). On this hyperplane, however, the posterior density is proportional to the prior one. For large values of the parameter \( L \) the density of the multiplicity model tends to have isopycnals (contour lines of same density, with respect to the canonical density \( dp \)) coinciding with the isentropes of the simplex of plausibility distributions, and very peaked on those of larger entropy. Hence the final distribution corresponds to that of larger entropy in the ‘constraint’ hyperplane determined by the average.

The assigned distribution is thus, in general, determined by the competition between two peaks: that of the prior density, centred on the uniform distribution, and
that of the likelihood of the data, concentrated on the ‘constraint’ hyperplane. For
a fixed, large \( L \) and small \( N \) the first peak dominates and the assigned distribution
is near the uniform one. As \( N \) increases the assigned distribution moves towards
the constraint hyperplane; and when \( N \) becomes much larger than \( L \) it is practically
on that hyperplane, though its exact position therein is still determined by the prior
density. This is why the multiplicity model gives reasonable distributions for small
\( N \) but can approximate the maximum-entropy distribution for large \( N \).

Of course this property is not exclusive to this model. Any other model with
isopycnals coinciding with isentropes and very peaked on those of higher entropy
will lead to the same distribution in problems with \( N \) large enough; e.g. a model with
the ‘entropy prior’ of Skilling, Rodríguez, Caticha & Preuss discussed in § 4.

Using the generalized multiplicity model defined by the density

\[
g(p | L, m) dp = c(L, m) L! \prod_i \frac{m_i^{L p_i}}{(L p_i)!} dp_i, \quad L \geq 1, m_i \geq 0, \sum_i m_i = 1, \tag{37}
\]

the asymptotic result \( \text{(38)} \) generalizes to the ‘maximum-relative-entropy’ principle,

\[
P(R_i | A_a^N \cap I_m^L, m) \approx \min f_i \text{ which minimizes } D(f, m) \text{ under the constraint } v \cdot f = a, \tag{38}
\]

with the Kullback-Leibler divergence \( \text{(31)} \) instead of Shannon’s entropy. Note that
the roles of \( f \) and \( m \) are interchanged with respect to the Johnson model’s case.

The integrals \( \text{(26)}, \text{(32)}, \text{(34)} \) were numerically calculated with both the Monte
Carlo routine \textit{Suave} and the deterministic routine \textit{Cuhre} of Hahn’s multidimension-
al-integration library \textit{Cuba} [63], comparing the results of both routines to appraise
their mutual consistency and precision. In most cases \textit{Suave} was fastest and most
precise.

C The canonical density on a plausibility simplex: a
whimsical definition

Integrations over a plausibility simplex \( \Delta \) of dimension \( n \) are usually written as

\[
n! \int_0^1 \int_0^{1-p_1} \int_0^{1-p_1-p_2} \cdots \int_0^{1-\sum_{j=1}^{n-2} p_j} f(p_1) dp_{n-1} dp_2 dp_1 \tag{39}
\]

or equivalently and more symmetrically as

\[
n! \int_0^1 \int_0^{1} \int_0 \cdots \int_0 g(p_1) \delta(1- \sum p_i) dp_n dp_2 dp_1 \tag{40}
\]

or even

\[
n! \int_0^\infty \int_0^\infty \cdots \int_0^\infty g(p_1) \delta(1- \sum p_i) dp_n dp_2 dp_1. \tag{41}
\]
When \( g \equiv 1 \) these integrals give the simplex a unit volume.

The first expression is unpleasant because asymmetric in the symmetric variables \( p_i \); the other two are unpleasant because they unnecessarily invoke a generalized function. Behind these expressions there is a simple and well-behaved canonical density or volume element \( n! \, dp \) over the simplex, which gives the latter a unit volume. By density I mean a twisted, positively oriented \( n \)-form. Twisted, or ‘odd’, because its integration does not require an inner orientation of the simplex (see Schouten [64, 65, chs II, III], Burke [66, 67, ch. IV; 68]; also Marsden et al. [69, ch. 7] and Choquet-Bruhat et al. [70, § IV.B.1]); in fact, choosing an inner orientation would break the permutation invariance of the functions \( p_i \).

This canonical density can be defined in two ways.

First way. Any \( n \)-dimensional convex set that can be affinely mapped onto a finite region \( \mathbb{R}^n \) by a map \( F \) can be given a canonical density \( \omega \) by pulling back the the canonical density of \( \mathbb{R}^n \) and rescaling:

\[
\omega := \frac{F^*|dx|}{\int F^*|dx|},
\]

(42)

where \( |dx| \) is the canonical density (twisted, positively oriented \( n \)-form) of \( \mathbb{R}^n \). The rescaling gives the convex set a unit volume. It is easily proven that this definition is independent of the affine map \( F \) chosen. When the convex set is a simplex, \( \omega \) is the canonical density \( n! \, dp \).

The second way does not involve any embeddings, but uses instead the naturally defined plausibility functions \( p_i: \Delta \to \mathbb{R} \), which characterize the simplex as a plausibility simplex. The canonical density is then implicitly defined as

\[
n! \, dp \equiv \sum_{\{i_1, \ldots, i_{n-1}\} \subset \{1, \ldots, n\}} |dp_{i_1} \wedge \cdots \wedge dp_{i_{n-1}}|,
\]

(43)

the indices running over all permutations of \( n-1 \) elements of \( \{1, \ldots, n\} \). The magnitude operator \( |\cdot| \) transforms any density into the twisted, positively oriented equipollent one. The expression above is symmetric on the \( p_i \) and does not involve generalized functions.

For other densities and metric structures on a plausibility simplex see e.g. Amari & Nagaoka [71].

And measure theory? ‘La teoria della misura sta alla probabilità come lo stucco messo male sta alle pareti: prima o poi cade’, said Gian-Carlo Rota [72–74].

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