NUMERICAL AND MATHEMATICAL ANALYSIS OF BLOW-UP PROBLEMS FOR A STOCHASTIC DIFFERENTIAL EQUATION

TETSUYA ISHIWATA AND YOUNG CHOL YANG*
Shibaura Institute of Technology
307 Fukasaku, Minuma, Saitama 337-8570, Japan

Abstract. We consider the blow-up problems of the power type of stochastic differential equation,
\[ dX(t) = \alpha X^p(t)dt + X^q(t)dW(t), \quad t > 0 \]
\[ X(0) = X_0. \quad (1) \]
Here \( p > 1, \ q > 0 \) and \( \alpha > 0 \) are parameters, an initial value \( X_0 \) is positive and \( W(t) \) is the Brownian motion. Note that (1) is 1-dimensional and Ito’s type of stochastic differential equation (SDE for short). It is well-known that if \( X_0 > 0 \) then \( X(t) > 0 \) as long as the solution exists. That is, the positivity of the solution is preserved.

According to the [3], [4], if \( p = 2q - 1 \) then \( X(t) \) blows up in a finite time almost surely for any positive initial value. Recently, J. E. Mac\’ıas-Díaz and J. Villa-Morales [8] shows \( X(t) \) blows up in a finite time for \( p = 2q - 1 \) and \( \alpha = \frac{p+1}{4} \) (\( =: \alpha^* \)). On the other hand, by applying the result in [2] the probability of \( \lim_{t \to \infty} X(t) = 0 \) is positive for \( p = 2q - 1, \ \alpha < \frac{1}{2}, \ \alpha^* \). Namely \( p = 2q - 1 \) is a critical exponent on blow-up of the solution. For the critical case, we show the blow-up for \( \alpha > \alpha^* \) and \( p = 2q - 1 \) by using comparison result [5].

P.Groisman, J.D.Rossi et al. [3], [4] also show the numerical method using adaptive time step in case of \( p > 2q - 1 \). It is typical method for ordinary differential equation’s (ODE for short) and partial differential equation’s blow-up problem. They apply this method to (1). However, their method has the disadvantage of slow divergence and their scheme dose not preserve the positivity of solutions.

In our research, we mainly focus on the critical case. We propose a numerical scheme to improve their method. \( X(t) \) has positivity for any positive initial value
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\[ * \ Corresponding \ author: \ Young \ Chol \ Yang. \]
But numerical solution by Euler-Maruyama scheme does not preserve positivity. We prove numerical blow-up and the positivity of numerical solutions. We also show geometrically divergence of numerical solutions.

The outline of this paper is as following. In the §2 we summarize the mathematical results on blow-up of solutions for the critical case. In the §3, we show the mathematical results of numerical scheme. Finally, in the §4, we demonstrate the numerical results.

2. Mathematical results on blow-up of the solution in critical case. Recently, J. E. Macías-Díaz and J. Villa-Morales [8] shows the blow-up of solutions on critical case with additional condition on $\alpha$:

Assumption (A): $\alpha = \alpha^*$.

**Proposition 1** ([8]). Assume $p = 2q - 1$ and (A). Then the solution of (1) blows up almost surely in a finite time and the solution can be expressed as following form:

$$X(t) = \left( \frac{X_0^{\lambda}}{1 - \lambda X_0^{\lambda} W(t)} \right)^{1/\lambda}, \quad \lambda = q - 1 (>0).$$

For the above result, $X(t)$ diverges at $T_\omega = \inf\{t \in \Omega \mid W(t) > 1/\lambda X_0^{\lambda}\}$. Here $T_\omega$ is the first hitting time of $W(t)$ to $M := 1/\lambda X_0^{\lambda}$. It is well-known that the first hitting time of the Brownian motion to any constant is finite almost surely. Hence, the solution $X(t)$ blows up almost surely in a finite time under the (A). For reader’s convenience, we give a proof in appendix. Moreover, we get the blow-up result for $\alpha > \alpha^*$ by applying the comparison result in proposition 2.18 [5].

**Proposition 2.** Assume $p = 2q - 1$ and $\alpha > \alpha^*$. Then the solution blows up almost surely in a finite time.

On the other hand, it is also shown in [2] that

- $\alpha < \frac{1}{2} \Rightarrow P(\lim_{t \to \infty} X(t) = 0) > 0$.
- $\alpha > \frac{1}{2} \Rightarrow P(\lim_{t \to \infty} X(t) = 0) = 0$.

Note that $\alpha^* > \frac{1}{2}$ for $p > 1$. At the present, the behavior of solution is not well-understood for the case $\alpha < \alpha^*$. In the section 4.3, we check the behavior of solution numerically for this case.

3. Numerical scheme and its analysis. We use adaptive time step due to the solution, $h_n (> 0)$ is the n-th step size and $t_n := \sum_{i=0}^{n-1} h_i$ is the n-th time step. $X_n$ is the approximated value of the solution at $t = t_n$. We introduce a typical numerical method for the blow-up problem for:

$$\begin{cases}
\frac{dX(t)}{dt} = X^p(t), \\
X(0) = X_0.
\end{cases}$$

Here $p > 1$ and $X_0$ is a positive initial value. It is well-known that the solution blows up in a finite time $T$. Discrete (2) by the explicit Euler method. Each step sizes is defined as $h_n := \tau/|X_n|^{p-1}$, where $\tau > 0$ is a numerical parameter. The numerical scheme is written in following form:

$$X_{n+1} = X_n + h_n X_n^p = (1 + \tau)X_n$$

From (3), the $X(t)$ grows-up geometric order. We can obtain the following results:
1. Divergence of numerical solution: \( \lim_{n \to \infty} X_n = \infty. \)

2. Finiteness of numerical blow-up time: \( T_\tau = \lim_{n \to \infty} t_n < \infty. \)

3. Convergence to exact blow-up time: \( \lim_{\tau \to 0} T_\tau = T. \)

We apply the same adaptive method for (1). We use the Euler-Maruyama scheme: 
\[
X_{n+1} = X_n + \alpha X_n^p \frac{\tau}{|X_n|^{p-1}} + X_n^q \Delta W_n,
\]
where \( \Delta W_n := \sqrt{h_n} \xi_n \) is an approximated increment of Brownian motion and \( \{\xi_n\}_{n \in \mathbb{N}} \) is a independence random sequence which follows standard normal distribution: \( N(0,1) \). Numerical scheme using time step control for SDE is suggested by P.Groisman, J.D.Rossi et al. (2005) [3]. They use the time step control as \( h_n = \tau/|X_n|^p \). The numerical scheme is shown as below.

\[
X_{n+1} = X_n + \alpha X_n^p \frac{\tau}{|X_n|^{p-1}} + X_n^q \Delta W_n
\]

\[
= X_n + \alpha \tau + \sqrt{\tau} \xi_n X_n^{q-p}/2.
\]

They show the following results (Theorem 1.3 [3]):

1. Divergence of \( X_n \): \( \lim_{n \to \infty} X_n = \infty \) almost surely.

2. Finiteness of \( T_\tau \): \( T_\tau < \infty \) almost surely.

3. Convergence of \( T_\tau \): \( \lim_{\tau \to 0} T_\tau = T \) almost surely.

4. Linear growth of \( X_n \): \( \lim_{n \to \infty} X_n = 1 \) almost surely.

Due to the 4th result, the growth rate of \( X_n \) is linear. In ODE case, numerical solution grows up geometrically. Therefore \( X_n \) requires much numerical steps to diverge to sufficiently large value.

We propose a time step control as \( h_n = \tau/|X_n|^{p-1} \) which is same as ODE case. The numerical scheme is shown as below.

\[
X_{n+1} = X_n + \alpha X_n^p \frac{\tau}{|X_n|^{p-1}} + X_n^q \Delta W_n
\]

\[
= (1 + \alpha \tau)X_n + \sqrt{\tau} X_n^{q-2p/2} \xi_n.
\]

In the critical case: \( p = 2q - 1 \), we can easily rewrite the numerical scheme (5) as

\[
X_n = X_0 \prod_{i=1}^{n} (1 + \alpha \tau + \sqrt{\tau} \xi_n).
\]

When \( \xi_n < z(\tau) := -(1 + \alpha \tau)/\sqrt{\tau} \), the \( X_n \)'s sign is reversed, that is positivity of solution breaks down as long as \( \tau \) is positive. It occurs at each step with probability \( p_\tau := 1 - \int_{z(\tau)}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)du \). Therefore, we restrict the domain of random numbers \( \xi_n \) on \( (z(\tau), \infty) \) and normalize distribution, that is, the probability density function of \( \xi_n \) is

\[
g(x) = \frac{1}{f_{z(\tau)} \exp(-u^2/2)du} \int_{z(\tau)}^{x} \exp(-u^2/2)du.
\]

Then positivity of the solution is preserved. If \( \tau \to 0 \) then \( z(\tau) \to -\infty \). Thus, \( f(x) \) converges to standard normal distribution.

**Proposition 3** (positivity). Our proposed scheme has a positivity preserving property, that is \( X_n > 0 \) for any \( n \in \mathbb{N} \). Using \( g(x) \) as the probability density function of \( \xi_n \).

**Remark 1.** In this paper, we use the one-side cut-off of the domain of random number \( \xi_n \). We can apply other cut-off, for example, both-side cut-off.
Remark 2. The scheme proposed in [6] has almost positivity preserving property, that is, \( \forall \varepsilon, \exists \bar{\tau} \) such that for any \( \tau < \bar{\tau} \), \( P(X_N > 0, X_{N-1} > 0, \ldots, X_0 > 0) > 1 - \varepsilon \) by using another time step control:

\[
\bar{h}_n = \frac{\tau}{1 + X_n^{p-1} + (X_n^{p-1})^2}.
\] (7)

Theorem 3.1 (Finite time blow-up for numerical solution in the critical case). Assume \( E[\log(1 + \alpha \tau + \sqrt{\tau} \xi_n)] \in (0, \infty) \) then the numerical solution of proposed method \( X_n \) blows up at finite time \( T_\tau \) almost surely. And \( X_n \) diverges as geometric order.

Proof. We divide the proof in 2 steps.

Step 1. (Divergence of numerical solution): Taking logarithm on (6)’s both terms.

\[
\log(X_n) = \log(X_0) + \sum_{i=1}^{n} \log(1 + \alpha \tau + \sqrt{\tau} \xi_i).
\]

Define the expected value as \( \mu := E[\log(1 + \alpha \tau + \sqrt{\tau} \xi_n)] \). If \( \mu \in (0, \infty) \) then \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log(1 + \alpha \tau + \sqrt{\tau} \xi_i) = \mu \) from the law of large number.

Therefore \( \lim_{n \to \infty} \frac{1}{n} \log\left(\frac{X_n}{X_0}\right) = \mu \). Thus, \( X_n \) diverges almost surely as geometric order.

Step 2. (Finiteness of numerical blow-up time): From step 1, we have \( \lim_{n \to \infty} X_0 e^{\mu n} / X_n = 1 \) almost surely. Then there exist \( n_0 \in \mathbb{N} \) such that if \( n > n_0 \) then \( X_0 e^{\mu n} / X_n - 1 < \frac{1}{2} \) almost surely. Thus, we have

\[
\frac{\tau}{|X_n|^{p-1}} < \tau \left( \frac{3}{2X_0} \right)^{p-1} \left( \frac{1}{e^{\mu(p-1)}} \right)^n
\]

for a fixed positive value \( X_0 \).

Note that \( \frac{1}{e^{\mu(p-1)}} < 1 \) because of \( \mu > 0 \) and \( p > 1 \). From the definition of \( T_\tau \), we have

\[
T_\tau - t_{n_0+1} = \sum_{n=n_0+1}^{\infty} \frac{\tau}{|X_n|^{p-1}} < \tau \left( \frac{3}{2X_0} \right)^{p-1} \sum_{n=n_0+1}^{\infty} \left( \frac{1}{e^{\mu(p-1)}} \right)^n
\]

\[
= \tau \left( \frac{3}{2X_0} \right)^{p-1} \left( \frac{1}{e^{\mu(p-1)}} \right)^{n_0} \frac{1}{e^{\mu(p-1)} - 1} < \infty.
\]

Hence we have the assertion. 

4. Numerical experiments. We show some numerical results by the proposed method. In the critical case with (A), we can derive the exact blow-up solution and blow-up time. And we suggest a characterization of blow-up time by \( M \) from Theorem 2.1. We compare the numerical value with the exact value and verify the effectiveness of that characterization. For the super-critical case, there are no mathematical result on the characterization of the blow-up time and also no mathematical analysis of numerical scheme. We apply the proposed scheme for the super-critical
case and numerically estimate the distribution of numerical blow-up time. Finally, we show the result of critical case without (A). For our numerical results, we give a conjecture on the probability of blow-up. In the following numerical experiments, when \( X_n > L \), we terminate the numerical computation and define this time as numerical blow-up time \( T^L_\tau \).

4.1. Experiment.1 : Critical case with (A). Here we set \( p = 5, q = 3, \alpha = 3/2 \). Numerical parameters are fixed as \( \tau = 10^{-6}, X_0 = 1.0 \) and \( L = 1000 \). In this case the exact blow-up time is derived as below

\[
X(t) = \frac{1}{1 - 2W(t)}^{1/2}, \quad T_\tau = \inf\{t \in (0, \infty) \mid W(t) > 1/2\}.
\]

The Figure 2 shows that each \( W_n \) takes close value to \( M = 1/2 \) at numerical blow-up time. The table 1 shows the value of each numerical parameters.

**Table 1. Numerical parameters at numerical blow-up time**

| Sample No. | \( X_n \)   | \( T^L_\tau \) | \( |W_n - M| \) |
|------------|--------------|----------------|----------------|
| 1          | 1000.343314  | 0.043027       | 0.00076978     |
| 2          | 1000.154401  | 0.39964        | 0.0007528      |
| 3          | 1000.61063   | 0.0209         | 0.00040622     |
| 4          | 1000.101781  | 0.166273       | 0.00045478     |

\( T^L_\tau \) takes different value depends on each sample. However \( W_n \) takes close value to \( M \). Where the exact blow-up time is same as the first hitting time of Brownian motion to \( M \). Similar phenomena can be reproduced in numerical experiments. We create a histogram using large samples: \( (N = 1000) \) to examine the distribution of numerical blow-up time.

Figure 3 show the histogram of numerical blow-up time and the graph of probability density function of exact blow-up time. It is well-known that the exact density function of first hitting time to \( M \) is

\[
f(t) = \frac{M}{\sqrt{2\pi t^3}} \exp\left(-\frac{M^2}{2t}\right).
\]
Although we use $L = 1000$ and the histogram is created by 1000 samples, the histogram has a similar shape to the graph of $f(t)$. We expect that the shape of histogram converge to $f(t)$ as $L$ and the number of samples tends to infinity.

4.2. **Experiment 2: Super-critical case.** Recall that in the super-critical case, $X(t)$ blows up almost surely [3], [4]. Here we set $p = 5$, $q = 2$, $\alpha = 3/2$. Numerical parameters are fixed as $\tau = 10^{-6}$, $X_0 = 1.0$ and $L = 1000$. We use same numerical scheme as critical case.

![Numerical solutions (3 samples)](image1)

**Figure 4.** Numerical solutions (3 samples)

![Brownian motion](image2)

**Figure 5.** Numerical Brownian motion

In the critical case, Figure 2. shows that $W_n$ takes close value to $M = \frac{1}{2}$ at the numerical blow-up. However, in the super-critical case, the same phenomenon is not observed. That is, it is conjectured that the blow-up time cannot be characterized by the first hitting time of the Brownian motion in super-critical case. As to the density function, we make the histogram of the numerical blow-up time for $10^6$

![Histogram and Exact Distribution](image3)

**Figure 3.** Histogram of $T_r^L$ and exact distribution of blow-up time (green)
samples with $L = 10^6$. Figure 6 shows probability density in Orange and cumulative density in blue.

Even in case of the super-critical case the distribution of numerical blow-up time has a similar shape to the critical case with one peak.

4.3. Experiment 3: Numerical solution of critical case without (A). It is open what distribution characterize this one peak function. We set $p = 3$, $q = 2$ and $\alpha = 0.1$, $0.2$, $\cdots$, $0.9$. When parameter $\alpha < \alpha^* = 1$, the growth rate of drift term might be weaker than (A) is held. There is a possibility that the probability of blow-up is less than 1. We confirm it numerically. We also set $T_{\text{max}} > 0$. If $t_n > T_{\text{max}}$ then we terminate the numerical calculation and we judge the numerical solution as a non-blow-up solution (global solution). The number of samples are 1000 for each $\alpha$. Table 2 and Figure 7 show the number of non-blow-up solutions for each $\alpha$ with fixed $T_{\text{max}} = 1000$.

Table 2. The number of Blow-up solutions with fixed $T_{\text{max}} = 1000$

| $\alpha$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $L = 100$| 977 | 941 | 888 | 791 | 665 | 486 | 369 | 226 | 118 |
| $L = 1000$| 995 | 988 | 968 | 880 | 756 | 566 | 377 | 223 | 122 |

Table 3 and Figure 8 show the result with fixed $L = 1000$.

Table 3. The number of non-blow-up solutions with fixed $L = 1000$

| $\alpha$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $T_{\text{max}} = 100$| 995 | 988 | 968 | 880 | 756 | 566 | 377 | 223 | 122 |
| $T_{\text{max}} = 1000$| 996 | 989 | 940 | 848 | 668 | 401 | 234 | 105 | 44 |
From our numerical results, we observe the following conjectures.

- $P(\lim_{n \to \infty} X_n = \infty)$ is close to zero near $\alpha = 0$.
- $P(\lim_{n \to \infty} X_n = \infty)$ is positive and less than 1 for $\alpha \in (\frac{1}{2}, \alpha^*)$.

**Remark 3.** In [6] (Theorem 4.1), $P(\lim_{n \to \infty} X_n = 0) = 1$ is shown for $\alpha < \frac{1}{2}$, where $X_n$ is the numerical solution using the adaptive time step control (7). Namely $P(\lim_{n \to \infty} X_n = \infty) = 0$ by their scheme. For our scheme, we have not obtained such a result. On the other hand, for the original problem for $\alpha < \frac{1}{2}$ there is no mathematical results on blow-up of solutions and there is only one mathematical result on the behavior of solutions: $P(\lim_{t \to \infty} X(t) = 0) > 0$. 

**Figure 7.** The number of non-blow-up solutions with fixed $T_{\text{max}} = 1000$

**Figure 8.** The number of non-blow-up solutions with fixed $L = 1000$
Remark 4. We should use $h_n = \tau \max\left(\frac{1}{X_n^{p-1}}, 1\right)$ if we treat both blow-up solutions and decay solutions. In this paper, we only focus on blow-up solutions and we terminate numerical calculations at finite time $T_{\max}$. Namely we did not track the decay behavior $\lim_{n \to \infty} X_n = 0$. So, we use the simple control $h_n = \frac{1}{X_n^{p-1}}$.

5. Conclusion. We consider blow-up problem of the power type of SDE (1). For the critical case, recently it is obtained that if $\alpha = \alpha^*$ the solution blows up almost surely. In the case of $\alpha > \alpha^*$, we show blow-up of the solution using comparison result.

Next we propose a numerical method for this problem and numerically confirmed the effectiveness of this characterization of blow-up time by the first hitting time of Brownian motion to the certain value $M$, which depends only on $p$ (or $q$) and $X_0$. As to numerical experiments, we consider other two cases. At the experiment 2, we make a histogram of numerical blow-up time in super-critical case and compare the numerical blow-up time distribution of super-critical and critical cases. We confirm that both distributions are similar shape of one peak. However, it is seemed that the blow-up time cannot be characterized by first hitting time in super-critical case. At the experiment 3, we make a conjecture in critical case, when $\alpha < \alpha^*$, the probability of blow-up may be less than 1.

Appendix. For reader’s convenience, we show the proof of the following proposition.

Proposition 4. Suppose (A). Then the solution of (1) blows up almost surely in a finite time and the solution can be expressed as following form:

$$X(t) = \left(\frac{X_0^\lambda}{1 - \lambda X_0^\lambda W(t)}\right)^{1/\lambda}.$$  

Here $\lambda = q - 1$ ($> 0$).

Proof. We divide the proof in 2 steps.

Step1. Derivation of solution: Let $Y(t) = X^{-\lambda}(t)$. Then $Y(t)$ satisfies the following SDE from Ito’s formula:

$$dY(t) = \left(Y_t(t) + \alpha X^p(t)Y(t) + \frac{1}{2}X^2(t)X(t)^{2}(t)\right)dt + X^q(t)Y(t)dW(t).$$

Taking $q - \lambda - 1 = 0$ then we have

$$dY = \lambda X^{-(\lambda-2)}(t) \left(\frac{\lambda + 1}{2} X^{2\lambda-1} - \alpha X^{p+1}(t)\right)dt - \lambda X^{-(\lambda-1)}(t)dW(t).$$

By using (A), we can get

$$\frac{\lambda + 1}{2} X^{2(\lambda+1)} - \alpha X^{p+1} = 0.$$  

Then we have

$$dY(t) = -\lambda dW(t).$$
Finally we get $Y(t) = Y(0) - \lambda W(t)$. Hence, $X(t)$ is written in the following form:

$$X(t) = \left( \frac{X_0^{\lambda}}{1 - \lambda X_0^{-\lambda} W(t)} \right)^{1/\lambda}.$$ 

**Step 2. The finiteness of blow-up time:** We rewrite the solution as following:

$$X(t) = \left( \frac{X_0^{\lambda}}{1 - W(t)/M} \right)^{1/\lambda}, \text{ where } M = 1/\lambda X_0^{-\lambda}.$$ 

The solution $X(t)$ diverges at $T_\omega := \inf_{\omega \in \Omega} \{W(t) > M\}$. Here $T_\omega$ is a first hitting time of Brownian motion to $M$. It is well-known that the first hitting time of the Brownian motion to any constant $M$ is finite almost surely. Hence, the solution $X(t)$ blows up almost surely in a finite time under the (A). 

\[\Box\]

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E-mail address: mf17072@shibaura-it.ac.jp
E-mail address: tisiwata@shibaura-it.ac.jp