THE EFFECT OF CONJUNCTIONS ON THE TRANSIT TIMING VARIATIONS OF EXOPLANETS

DAVID NESVORNÝ1 AND DAVID VOKROUHLICKÝ2

1 Department of Space Studies, Southwest Research Institute, 1050 Walnut Street, Suite 300, Boulder, CO 80302, USA; davidn@boulder.swri.edu
2 Institute of Astronomy, Charles University, V Holešovičkách 2, CZ–18000 Prague 8, Czech Republic; vokrouhl@cesnet.cz

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ABSTRACT

We develop an analytic model for transit timing variations produced by orbital conjunctions between gravitationally interacting planets. If the planetary orbits have tight orbital spacing, which is a common case among the Kepler planets, the effect of a single conjunction can be best described as: (1) a step-like change of the transit timing ephemeris with subsequent transits of the inner planet being delayed and those of the outer planet being sped up, and (2) a discrete change in sampling of the underlying oscillations from eccentricity-related interaction terms. In the limit of small orbital eccentricities, our analytic model gives explicit equations for these effects as a function of the mass and orbital separation of planets. We point out that a detection of the conjunction effect in real data is of crucial importance for the physical characterization of planetary systems from transit timing variations.

Key word: planetary systems

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1. INTRODUCTION

Planetary orbits are less than ideal clocks. This is because various processes, including, for example, the gravitational interaction between planets, collaborate to produce fluctuations from perfect periodicity. The transit timing variations (TTVs; Agol et al. 2005; Holman & Murray 2005), a method that has become increasingly important in the exoplanet research, exploits the effect of these fluctuations on the timing of planetary transits. It can be used to make useful inferences about the nature of planets in a system where at least one planet is transiting. The TTVs caused by interacting planets come in several flavors. The long-periodic TTVs result from orbital variability on timescales much longer than the orbital period. Their detection therefore requires a long observation baseline (Hcy & Gladman 2007). The resonant and near-resonant TTVs occur when orbital periods, when divided by each other, are equal or nearly equal to a ratio of small integers. The planetary perturbations tend to build up in this situation, leading to TTVs with a large amplitude.

The detection of (near-)resonant TTVs is expected if a large set of transit observations is available, because some planetary systems are bound to have (near-)resonant orbits (either due to statistics or because they are driven to these orbits by formation processes). The description of planetary properties from (near)-resonant TTVs, however, is plagued with degeneracies, which may be resolved only under certain assumptions (near-)resonant TTVs, however, is plagued with degeneracies, which may be resolved only under certain assumptions (Lithwick et al. 2012). This happens, in essence, because the (near)-resonant TTVs are limited in the information content.

This is where the short-periodic TTVs become useful. The short-periodic TTVs are produced by variations of orbits on a timescale comparable to the orbital period. In general, it can be shown that

\[ -n_i \delta t_i = \delta \lambda_i = 2\delta h_i + \frac{3}{2} (k_i \delta h_i + h_i \delta k_i) + O(e_i^2), \]

where \( \delta t_i \) is the short-periodic deviation of timing of planet \( i \) from a linear ephemeris, \( k_i = e_i \cos \sigma_{i,j} \), \( h_i = e_i \sin \sigma_{i,j} \), and \( n_i = 2\pi/P_i \), where \( P_i \) is the orbital period (Nesvorný 2009).

Quantities \( \delta \lambda_i, \delta h_i \) and \( \delta k_i \) are the short-periodic variations of the mean longitude, \( h_i \), and \( k_i \), respectively.\(^5\) The short-periodic variations \( \delta \lambda_i, \delta h_i \) and \( \delta k_i \) can be computed from perturbation theory. With \( M_1, a_1, e_1, i_1, \sigma_i, \Omega_i, \lambda_i \) denoting the mass and orbital elements of planet \( i \), we have

\[ \delta \lambda_i = \int \frac{2n_i \partial \lambda_i}{L_i} \left[ \frac{\beta_i}{D_i} \frac{\partial \chi}{\partial \lambda_i} \right], \]

\[ \delta h_i = -\frac{\beta_i}{L_i} \left[ \frac{\partial \chi}{\partial h_i} \right], \]

\[ \delta k_i = \frac{\beta_i}{L_i} \left[ \frac{\partial \chi}{\partial k_i} \right], \]

where \( L_i = m_i \sqrt{GM_1a_i} \), \( G \) is the gravitational constant, \( M_1 \) is the mass of the host star, \( x_i = -\sqrt{2P_i} \sin \sigma_i, y_i = -\sqrt{2P_i} \cos \sigma_i \), \( P_i = L_i(1 - \beta_i) \), \( D_i = \sqrt{2L_i(1 + \beta_i)} \) and \( \beta_i = 1 - e_i^2 \). Function \( \chi \) can be written as:

\[ \chi = \frac{GM_1m_2}{a_2} \sum_{|k_1 + k_2| \neq 0} C_{k_1k_2}^{(i)}(\alpha) e_i^j e_j^2 \left( \sin \left( \frac{i_1}{2} \right) \sin \left( \frac{j_2}{2} \right) \right) \]

\[ \times \exp \{ k_3 \sigma_1 + k_4 \sigma_2 + k_5 \Omega_1 + k_6 \Omega_2 \} \exp \{ i(k_1 \lambda_1 + k_2 \lambda_2) \}, \]

with \( i = -\sqrt{-1} \), \( C_{k_1k_2}^{(i)}(\alpha) = C_{k_1k_2}^{(-i)}(\alpha) \), \( \alpha = a_1/a_2 < 1 \), and multi-indexes \( j = (j_1, j_2) \) and \( k = (k_1, k_2, k_3, k_4, k_5, k_6) \). See Nesvorný & Morbidelli (2008) for the assumptions that led to the derivation of Equations (2) and (3). In brief, Equation (2) does not include terms from the inclination of the transiting planet (assumed to be small), and Equation (3) is given to the first order in \( m_1/M_1 \) and \( m_2/M_2 \).

According to these equations, \( \delta t_i \) contains Fourier terms with the \( k_1 \lambda_1 + k_2 \lambda_2 \) harmonics. The amplitude of these terms is a complex function of \( \alpha \), eccentricities and inclinations, but the

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\(^5\) The negative sign in front of Equation (1) arises from the convention that a positive (negative) change of the mean longitude leads to negative (positive) \( \delta t_i \). Also, it is assumed in Equation (1) that the observer’s line of sight lies along the X-axis from which the orbital angles are measured.
ones with small $|k_1|$ and $|k_2|$ values are generally the most important (except if $k_1 n_1 + k_2 n_2 = 0$ for arbitrary $k_1$ and $k_2$, indicating the presence of a resonance). It is also obvious, in the approximation of Equations (1)–(3), that $\delta t_i$ is proportional to $P_i$, independent of $m_i$, and scales linearly with the companion mass.

The short-periodic TTVs are more difficult to detect observationally then the (near-)resonant TTVs, because they generally have a small amplitude. If they are detected, however, they can be used to uniquely characterize the orbital properties of planets. This has been theoretically demonstrated in Nesvorný & Beaugé (2010) under the assumption that there is no a priori information about the (non-transiting) companion, and done in practice in Holman et al. (2010), and elsewhere. Intuitively, this can be understood because each Fourier term in Equation (3) provides specific information about orbital elements. Thus, if at least a few of these terms are detected in real data, the information contained in the detection is good enough to make the inversion to orbital elements possible (e.g., Nesvorný et al. 2012).

An important component of the short-periodic TTVs, which is the main focus here, is produced by conjunctions between planets. The conjunction effect can be conveniently illustrated using Kepler-36c (Carter et al. 2012) and KOI-884.02 (Nesvorný et al. 2014) (see Figure 1). Kepler-36 is a doubly transiting system, consisting of two planets on very tightly spaced orbits, with the two planets having significantly different masses (Carter et al. 2012). The KOI-884 system contains three known transiting planetary candidates, with the inner two displaying little or no sign of TTVs, while the outer one (KOI-884.02) exhibits significant TTVs. Nesvorný et al. (2014) used the TTVs of KOI-884.02 to detect an additional, unseen (i.e., non-transiting) planet just narrow of the outer 3:1 resonance with KOI-884.02 (orbital period ratio 2.93).

In the case of KOI-884.02, the effect of conjunctions is best seen for transit cycles 32 to 45 in Figure 1(a), where three transits on a nearly linear ephemeris are offset from the next three transits. These discontinuities are produced at conjunctions with the outer massive planet ($M_p \simeq 2.4 ~ M_J$, where $M_J$ is the mass of Jupiter; Nesvorný et al. 2014). The TTVs of Kepler-36c, on the other hand, are a series of approximately linear segments that are tilted relative to each other at different angles (Carter et al. 2012). These discontinuities correspond to the orbital conjunctions between Kepler-36c and the inner transiting planet Kepler-36b. In this case, the very tight spacing of two orbits at (or very near to) the 7:6 resonance implies the physical distance between planets to be small near conjunctions, and large variations are therefore expected. In general, for a $(j+1):j$ resonance, we expect the conjunctions to occur in every $j$ periods of the outer planet (and $j+1$ periods of the inner one). This is why, in Figure 1(b), groups of $j = 6$ transits share the same linear ephemeris (note that some transits are missing due to instrumental and other issues).

It is not straightforward to theoretically understand the effect of conjunctions from Equations (1)–(3), mainly because the TTV discontinuities occurring at conjunctions are difficult to approximate by the Fourier series, and because these equations include many different terms such that it is not clear which ones are responsible for the conjunction effect. The goal of this paper is to present a simple model for the conjunction effect that can be used as an intuitive guideline for more realistic modeling of cases such as the ones shown in Figure 1.

In Section 2, we derive an analytic model of conjunctions in the limit of small orbital eccentricities. In Section 3, we test our model by comparing it with numerical integrations of the full equations of motion, and determine the domain of parameters where the analytic model is valid. In Section 4, we show how the magnitude of the conjunction effect scales with different parameters. Finally, in Section 5, we discuss how the detection of the conjunction effect can be used to confirm and characterize transiting planetary systems.

2. ANALYTIC MODEL

2.1. Equations of Motion

We consider a system of two planets with masses $m_1$ and $m_2$ orbiting about a central star with mass $M_\ast$. The planetary orbits are assumed to be nearly coplanar and nearly circular, with planet 1 on the interior and planet 2 on the exterior. The analytic model is developed in variables $a_i$, $\lambda_i$, and $z_i = \bar{e}_i \exp(i \bar{\omega}_i)$, where index $i = 1,2$ stands for the two planets. These variables are non-singular for $\bar{e}_i \to 0$, and non-canonical, with the later being appropriate because we do not aim at developing the theory beyond the lowest order in eccentricity and beyond the
linear terms in $m_1/M_*$ and $m_2/M_*$. The inclination terms are ignored because they appear in the second and higher powers, and are therefore not overly important for the TTVs (as we show in Section 4). With that being clarified, the Lagrange equations describing the evolution of orbital elements are

$$\frac{da_i}{dt} = - \frac{2}{\mu_i n_i a_i^3} \frac{\partial H_{\text{per}}}{\partial \lambda_i},$$

(4)

$$\frac{d\lambda_i}{dt} = n_i + \frac{2}{\mu_i n_i a_i^3} \frac{\partial H_{\text{per}}}{\partial a_i} + O(e),$$

(5)

$$\frac{dz_i}{dt} = - \frac{2l_i}{\mu_i n_i a_i^3} \frac{\partial H_{\text{per}}}{\partial z_i} + O(e),$$

(6)

where $\mu_i$ is the reduced mass and $n_i$ is the mean motion. All terms that are explicitly first or higher eccentricity powers were removed from Equations (5) and (6). $H_{\text{per}}$ denotes the perturbation part of the Hamiltonian.

We follow Malhotra (1993) and split the Hamiltonian, $H = H_{\text{Kep}} + H_{\text{per}}$, such that

$$H_{\text{Kep}} = -G \frac{\mu_1 M_1}{2a_1} - G \frac{\mu_2 M_2}{2a_2},$$

(7)

where $M_1 = M_*, m_1$, $M_2 = M_* (m_1 + m_2)/M_1$, and $H_{\text{per}}$ is somewhat more complicated: $H_{\text{per}} = H_{\text{Kep}} + H_{\text{per}}$, where the part independent of eccentricities is

$$H_0 = G \frac{m_1 m_2}{a_2} [\alpha \cos \psi - P(\alpha, \psi)],$$

(8)

with $\alpha = a_1/a_2$, synodic angle $\psi = \lambda_1 - \lambda_2$, and

$$P(\alpha, \psi) = (1 - 2 \alpha \cos \psi + \alpha^2)^{-1/2}.$$  

The first-order eccentricity term, $H_1$, is somewhat more complicated:

$$H_1 = G \frac{m_1 m_2}{a_2} \times [\langle c_{j_1} \rangle \langle z_{j_1} \rangle \exp(-i\lambda_{j_1}) - d_{j_1}^* \exp(-i\lambda_{j_2}) + \sum_{j \geq 1} \left[ (\langle c^+_{j_1} \rangle + \alpha \delta_{j_1} \langle z_{j_1} \rangle \exp(-i(j+\phi)\lambda_2) + \langle c^\downarrow_{j_1} \rangle - 3\alpha \delta_{j_1} \langle z_{j_1} \rangle \exp(i(j+\phi)\lambda_2))\right] - \sum_{j \geq 1} \left[ (\langle d^+_{j_1} \rangle - 4\alpha \delta_{j_1} \langle z_{j_1} \rangle \exp(+i(j-\phi)\lambda_2) + \langle d^\downarrow_{j_1} \rangle \rangle \right] \right]$$

(10)

(Malhotra 1993, Agol et al. 2005). Here, $\langle \rangle$ denotes the real part, $\delta_{j_1}$ is the Kronecker delta, $c_{j_1} = \partial h_{1/2}^{(1)} / \partial b_{1/2}^{(1)}$ and $d_{j_1} = c_{j_1}^+ + b_{j_1}^\downarrow$, and $\partial h_{1/2}^{(1)} / \partial a_2$ is the Laplace coefficients. They are best evaluated from $h_{1/2}^{(1)} = (2/\pi) \left[ K(\alpha) \right]$ and $h_{1/2}^{(1)} = (2 \left[ K(\alpha) - E(\alpha) \right] / (\pi \alpha)$, where $K$ and $E$ are the complete elliptic integrals of the first and second kinds, respectively, and the efficient and stable recurrences recommended in Brouwer & Clemence (1961). The derivatives $\partial h_{1/2}^{(1)} / \partial a_2$ were also obtained from the recurrences defined in Brouwer & Clemence (1961).

### 2.2. The First-order Solution

The first-order perturbation solution can be obtained by inserting the unperturbed motion (i.e., corresponding to $H_{\text{Kep}}$) into the right-hand sides of the Equations (4)–(6), and performing a quadrature. Below we compute this quadrature in the time interval from $\psi = -\pi$ to $\psi = \pi$, i.e., over one conjunction cycle. For the reasons explained in the next paragraph we find it useful to formulate the results of the quadrature in terms of the synodic angle $\phi$, rather than of time, but these two formulations are interchangeable because $\psi = (n_1 - n_2) t - \pi$ (with $\lambda_{10} = \lambda_{20} = \pi$). To keep things simple, we perform our calculation only to the lowest order in eccentrics, where the resulting expressions become independent of $e_1$ and $e_2$.

If $\alpha$ is somewhat large (but not too large to lead to the co-orbital motion), the orbital spacing is relatively tight (as in many Kepler systems), and the interaction between planets happens almost exclusively at conjunctions. This is the case when using $\psi$ and the interaction Hamiltonian in Equations (8)–(10) is the most helpful, because the “impulsive” effects of conjunctions are well captured by a nearly discrete change when $\psi \approx 0$. If, instead, $\alpha$ is small ($\alpha < 0.5$), the conjunction effects cannot be easily isolated, and the Fourier series in Equation (3) becomes a more appropriate representation of the TTVs.

#### 2.2.1. Semimajor Axis and Mean Longitude

We first discuss the variations of semimajor axis and mean longitude described by Equations (4) and (5). In this case, we use $H_{\text{per}} = H_0$ in the right-hand side of Equation (4), and perform the quadrature to obtain

$$\delta a_1 = -2 v_1 a_1 \frac{m_2}{M_*} Q(\alpha, \psi),$$

(11)

$$\delta a_2 = 2 v_2 a_2 \frac{m_1}{M_*} Q(\alpha, \psi),$$

(12)

where $\delta a_1$ and $\delta a_2$ denote the semimajor-axis variations of the inner and outer planets, $v_1 = n_1 (n_1 - n_2)$ and $v_2 = n_2 (n_1 - n_2)$. Assuming that $\delta a_1 = \delta a_2 = 0$ at $\psi = -\pi$ (i.e., $\pi$ away from the conjunction), the function $Q$ can be written as

$$Q(\alpha, \psi) = \alpha \cos \psi - P(\alpha, \psi) + 1 + \frac{\alpha^2}{1 + \alpha}.$$  

(13)

Here and in the following, $\psi$ has to be understood as an unperturbed angle that linearly increases with time ($\psi = (n_1 - n_2) t - \pi$, where $n_1$ and $n_2$ are the unperturbed orbital frequencies defined by $H_{\text{Kep}}$). Note that Equations (11) and (12) obey the law of the total angular momentum conservation, because $\delta a_1 (n_1 a_1) + \delta a_2 (n_2 a_2) = 0$.

A change of the semimajor axis leads to a change of the mean motion according to

$$\delta \lambda_1 = \frac{m_2}{M_*} [A_1(\alpha) + A_2(\alpha) \psi + A_3(\alpha) \sin \psi + A_4(\psi, \alpha)]$$

(15)

and similarly for the outer orbit. This term, together with the derivative in the second term in Equation (5) (where again $H_{\text{per}} = H_0$), allows us to calculate the variation of the mean longitude. The quadrature gives

$$\delta \lambda_1 = \frac{m_2}{M_*} [A_1(\alpha) + A_2(\alpha) \psi + A_3(\alpha) \sin \psi + A_4(\psi, \alpha)]$$
with

\[ A_1(\alpha) = 3\pi \alpha v_1^2 \left[ 1 + \frac{\alpha^2}{1 + \alpha} \right] + 2 \alpha v_1 \left[ 2 - 3v_1 \right] K(\alpha) - 2E(\alpha) \right], \]

\[ A_2(\alpha) = 3\alpha v_1^2 \left[ 1 + \frac{\alpha^2}{1 + \alpha} \right], \]  \hspace{1cm} (16)

\[ A_3(\alpha) = \alpha^2 v_1 (2 + 3v_1) , \]  \hspace{1cm} (17)

\[ A_4(\psi, \alpha) = 2\alpha v_1 \left[ 1 - 3v_1 \right] F(\delta, r) - E(\delta, r) \right], \]  \hspace{1cm} (18)

where \( F(\delta, r) \) and \( E(\delta, r) \) are incomplete elliptic integrals of the first and second kind with modulus \( r = 2\sqrt{\alpha}/(1 + \alpha) \) and amplitude \( \sin \delta = (1 + \alpha) P(\alpha, \psi) \sin \psi/2. \)

Similarly, the mean longitude variation of the outer planet is found to be

\[ \delta \lambda_2 = \frac{m_1}{M_a} \left[ B_1(\alpha) + B_2(\alpha) \psi + B_3(\alpha) \sin \psi + B_4(\psi, \alpha) \right], \]  \hspace{1cm} (19)

with

\[ B_1(\alpha) = -3\pi v_2^2 \left[ 1 + \frac{\alpha^2}{1 + \alpha} \right] + 2 v_2 \left[ 3v_2 K(\alpha) + 2E(\alpha) \right], \]

\[ B_2(\alpha) = -3 v_2^2 \left[ 1 + \frac{\alpha^2}{1 + \alpha} \right], \]  \hspace{1cm} (20)

\[ B_3(\alpha) = -\alpha v_2 (4 + 3v_2) , \]  \hspace{1cm} (21)

\[ B_4(\psi, \alpha) = 2\alpha v_1 \left[ 1 + 3v_2 \right] F(\delta, r) + E(\delta, r) \right], \]  \hspace{1cm} (22)

The expressions in Equations (15) and (19) have the same structure revealing the underlying effects of planetary conjunctions. The first term in the right-hand sides of these equations, independent of \( \psi \), reflects a nearly step-like discontinuity of the mean longitude near the conjunction. The amplitude of this discontinuity is equal to \( 2A_1/m_1/M_a \) for the inner planet, and \( 2B_1m_1/M_a \) for the outer planet. Figure 2 shows an example for \( a_1 = 0.84 \text{AU}, a_2 = 1 \text{AU}, m_1/M_a = m_2/M_a = 10^{-5}, \) and \( M_a = M_\odot, \) where \( M_\odot = 1.988 \times 10^{33} \text{kg} \) is the solar mass.

### 2.2.2. Eccentricity and Apsidal Longitude

Equation (1) shows that the TTVs depend not only on the variation of the mean longitude, but also on variation of eccentricity and apsidal longitude. To the lowest order in eccentricity, \( \delta t_i \) becomes

\[ -n_i \delta t_i = \delta \lambda_i + \delta \lambda_i^{\text{eff}} + \mathcal{O}(e), \]  \hspace{1cm} (23)

where \( \delta \lambda_i \) was computed in Section 2.2.1, and \( \delta \lambda_i^{\text{eff}} \) is the effective contribution from the short-periodic variations of \( e_i \) and \( \omega_i \) (e.g., Nesvorný & Morbidelli 2008). Using the complex variable \( z_i \) defined in Section 2.1, we have

\[ \delta \lambda_i^{\text{eff}} = i(\delta z_i e^{i\lambda_i} - \delta z_i^* e^{-i\lambda_i}), \]  \hspace{1cm} (24)

where \( \delta z_i \) is the variation of \( z_i, \delta z_i^* \) is the complex conjugate, and \( \lambda_i \) is the unperturbed mean longitude. If the actual transits occur when \( \lambda_i \equiv 0 \), the above expression reduces, consistently with Equation (1), to \( \delta \lambda_i^{\text{eff}} = -2\delta z_i + \mathcal{O}(e). \)

The first-order approximation of \( \delta z_i \) is obtained by inserting unperturbed motion into the right-hand side of Equation (6) and performing the quadrature. We find that

\[ \delta \lambda_i^{\text{eff}} = \frac{m_2}{M_a} \left[ C_1(\alpha) \sin [v_1 (\psi + \pi)] + \sum_{j \geq 1} D_j(\alpha) \sin j\psi \right], \]  \hspace{1cm} (25)

where

\[ C_1(\alpha) = \alpha \left[ \partial_\alpha b_1^{(0)} + 2\alpha \frac{v_1 (3v_1 - 2v_2)}{v_2 (2v_1 - v_2)} \right. \]

\[ - 2 v_1 \sum_{j \geq 1} (-1)^j v_1 \partial_\alpha b_1^{(j)} + 2j^2 b_1^{(j)} \left\{ \right], \]  \hspace{1cm} (26)

\[ D_j(\alpha) = 2\delta j \alpha^2 v_1^3 (3v_1 - 2v_2) + j\alpha v_1 \partial_\alpha b_1^{(j)} + 2v_1 b_1^{(j)} \]  \hspace{1cm} (27)

Similarly, for the outer planet we obtain

\[ \delta \lambda_2^{\text{eff}} = \frac{m_1}{M_a} \left[ C_2(\alpha) \sin [v_2 (\psi + \pi)] + \sum_{j \geq 1} D_j(\alpha) \sin j\psi \right]. \]  \hspace{1cm} (28)
where
\[ C_2(\alpha) = 4 \alpha \frac{v_2}{(v_1 - 2v_2)} - (\dot{\alpha} b_{1/2}^{(1)} + b_{1/2}^{(1)}) + 2 v_2 \sum_{j \geq 1} (-1)^j \frac{v_2 \dot{\alpha} b_{1/2}^{(j)} + (2j^2 + v_2) b_{1/2}^{(j)}}{j^2 - v_2^2} \]  
\[ D_2(\alpha) = 4 \delta_{11} \alpha \frac{v_2}{v_1 - 2v_2} - 2j_2 v_2 \frac{\dot{\alpha} b_{1/2}^{(j)} + (2j_2 + 1) b_{1/2}^{(j)}}{j_2^2 - v_2^2}. \]

(29)

(30)

The coefficients \( C_1(\alpha), C_2(\alpha), D_1(\alpha), \) and \( D_2(\alpha) \) defined above have singularities at the first-order mean motion resonances, which is a consequence of the perturbation method applied here. For example, when \( n_1/n_2 = j/(j - 1) \), we have \( v_1 = j \), which causes a zero divisor in Equations (26) and (27).

Similarly, when \( n_1/n_2 = (j + 1)/j \) we have \( v_1 = j \), implying zero divisors in Equations (29) and (30). Interestingly, however, when these expressions combine in Equations (25) and (28) into variables important for the TTVs, the singularities disappear such that both \( \delta\lambda_1^{\text{eff}} \) and \( \delta\lambda_2^{\text{eff}} \) are well defined at resonances. This can be most easily verified by assuming that \( n_1/n_2 = j/(j - 1) + \epsilon \) or \( n_1/n_2 = (j + 1)/j + \epsilon \), where \( \epsilon \) is a small quantity, and showing that \( \lambda_1^{\text{eff}} \) and \( \lambda_2^{\text{eff}} \) are non-divergent when \( \epsilon \to 0 \).

To derive the expressions in Equations (25) and (28) we assumed that \( e_1 = 0 \) and \( e_2 = 0 \) when \( \psi = -\pi \). Note that in this case, \( \delta\lambda_2^{\text{eff}} \) is independent of the initial phases \( \beta_1^{(0)} \) and \( \beta_2^{(0)} \). Together with a similar result obtained in Section 2.2.1, this implies that \( \delta t_i \) will also be independent of \( \beta_2^{(0)} \) (see Section 2.3). It will only depend on the orbital period, \( m_1/M_* \), \( m_2/M_* \), and \( \alpha \).

Figure 3 illustrates the role of \( \delta\lambda_2^{\text{eff}} \) in an example with \( a_1 = 0.84 \text{ AU}, \ a_2 = 1 \text{ AU}, \ m_1 = 10^{-5} M_*, \ m_2 = M^{\odot}, \) and zero initial eccentricities. The effects of \( \delta\lambda_2^{\text{eff}} \) are initially small, but when small orbital eccentricities are excited during the conjunction (i.e., when \( \psi \approx 0 \)), \( \delta t_i \) becomes a composite of two terms with comparable magnitudes: (1) a step-like change produced by the direct variation \( \delta\lambda_2 \) (Section 2.2.1), and (2) oscillations from \( \delta\lambda_2^{\text{eff}} \). According to Equations (25) and (28), the oscillatory part has a rich spectrum of Fourier terms with frequencies \( n_1, n_2 \), and \( j(n_1 - n_2) \).

A general solution corresponding to small values of the initial eccentricities \( e_1^{(0)} \) and \( e_2^{(0)} \), can be obtained by adding \( 2 e_1^{(0)} \sin[v_1(\psi + \phi_1^{(0)})] \) to Equation (25) and \( 2 e_2^{(0)} \sin[v_2(\psi + \phi_2^{(0)})] \) to Equation (28). Here, \( \phi_i^{(0)} = (\lambda_i^{(0)} - \sigma_i^{(0)})/v_1 + \pi \), where \( \sigma_i^{(0)} \) are the apsidal longitudes of the two orbits when \( \psi = -\pi \). Since, by definition, \( \lambda_i^{(0)} = \lambda_i^{(0)} + \pi \), the general solution adds \( e_1^{(0)}, \sigma_1^{(0)} \) and \( \lambda_1^{(0)} \) (or \( \lambda_2^{(0)} \)) to the list of parameters.

### 2.3. Transit Timing Variations

The equations derived above can be used to compute \( \delta\lambda_2^{\text{eff}} \) as a function of the synodic angle \( \psi \) in the interval \(-\pi \leq \psi < \pi \). To be able to track changes over the successive rotations of \( \psi \), we need to add a constant term to \( \delta\lambda_2 \) that expresses how \( \delta\lambda_2 \) changed during the previous conjunction. For an arbitrary \( \psi \gg -\pi \), and denoting \( \text{int}[x] \) the integer part of \( x \), this term is
\[ \delta\lambda_i^{\text{add}} = 2 \frac{m_2}{M_*} A_i(\alpha) \text{int}[(\psi + \pi)/2\pi] \]

(31)

for the inner planet, and
\[ \delta\lambda_i^{\text{add}} = 2 \frac{m_1}{M_*} B_i(\alpha) \text{int}[(\psi + \pi)/2\pi] \]

(32)

for the outer planet. Adding this term successively each time \( \psi \) increases by \( 2\pi \) leads to a situation where \( \delta t_i \) either decreases (for the inner planet) or increases (for the outer one) in a series of steps. Equations (25)–(30) express the general dependence of \( \delta\lambda_i^{\text{eff}} \) on \( \psi \) in that they can be used to compute the conjunction effect over the successive rotations of \( \psi \). To do so, \( \psi = (n_1 - n_2)t - \pi \) needs to be substituted into Equations (25) and (28). With these provisions, \( \delta t_i \) can be computed for any \( t \geq 0 \).

Here we are mainly interested in the transit timing. We therefore assume that the observer detects transits on a (nearly) linear ephemeris, \( t_i^k = kP_1 + t_i^{(0)} \), where \( t_i^{(0)} > 0 \) is the epoch of the first transit, \( k \geq 1 \) denotes the subsequent transit cycles, and \( t_i^k \) are the subsequent transit epochs. This gives \( \psi_i^k = 2\pi k/v_1 + (n_1 - n_2)t_i^{(0)} - \pi \), where \( \psi_i^k \) denotes the value of the synodic angle corresponding to the \( k \)th transit of planet \( i \). Substituting this into Equations (25) and (28), and dropping all constant terms (i.e., those independent of \( k \)), we find that:
\[ \delta t_i^k = -\frac{P_1}{2\pi M_*} \left[ A(\psi_i^k) + 2A_i \text{int} \left( \frac{k}{v_1} + \frac{t_i^{(0)}}{P_1} \right) \right] \]
\[ + \sum_{j \geq 1} (-1)^j D_j^k \sin \left( 2\pi j \left[ \frac{k}{v_1} + \frac{t_i^{(0)}}{P_1} \right] \right) \]

(33)
and
\[ \delta t^k = \frac{P_k}{2\pi} \sum_{\nu} \left[ B(\psi^k) + 2B_1 \int \frac{k}{\nu^2} \frac{\tau_0}{P_\psi} \right] + \sum_{j \geq 1} (-1)^j D^j \sin \left( 2\pi j \left[ \frac{k}{\nu^2} \frac{\tau_0}{P_\psi} \right] \right). \] (34)

Here we denoted \( A(\psi^k) = A_2 \psi^k + A_3 \sin \psi^k + A_4(\psi^k) \), \( B(\psi^k) = B_2 \psi^k + B_3 \sin \psi^k + B_4(\psi^k) \), and synodic period \( P_\psi = 2\pi/(n_1 - n_2) \). Note that \( A, B, A_1, B_1, D^1, \) and \( D^j \) are explicit functions of \( \alpha \) as defined in Sections 2.2.1 and 2.2.2. Also note that Equations (33) and (34) are related to Equations (A7) and (A8) previously derived in the Appendix of Agol et al. (2005).

Equations (33) and (34) describe the effect of conjunctions on the TTVs. The three terms present in these equations stand for: (1) modulation of \( \delta t_i \) from \( \delta \lambda_i \) during a single rotation of the synodic angle, (2) total change of \( \delta t_i \) from \( \delta \lambda_i \) over all previous synodic cycles, and (3) periodic terms from the eccentricity-related perturbations. The expressions for (3) are non-divergent, because if \( v_1 = j \) (or \( v_2 = j \)) the variable part of the corresponding sinusoidal terms vanishes from Equation (33) (or Equation (34)).

Notably, the sinusoidal terms in Equations (33) and (34) originate from terms with \( \sin j\psi \) (see Equations (25) and (28)) and are \( 2\pi \)-periodic in \( \psi \). Therefore, as far as these periodic terms are concerned, there is no difference between the first, second, or any other cycle of \( \psi \). The difference between the TTVs in different cycles of \( \psi \) arises, instead, by how fixed periodic terms are sampled by TTV observations. For given \( \alpha \), it turns out that the periodic term with the largest amplitude \( D^j(\alpha) \) (or \( D^j(\omega) \)) is the one with \( j \sim v_1 \) in Equation (33) (or \( j \sim v_2 \) in Equation (34)). As shown in Equation (33) (or Equation (34)), however, these terms will be sampled with \( \sim 2\pi k \) cadence and will therefore have only a limited impact on the TTVs.

3. VALIDITY DOMAIN OF THE ANALYTIC MODEL

The analytic model developed in the previous sections can be used as a guideline to understand the basic effects of conjunctions on the TTVs. We will discuss these effects and their scaling with different parameters in Section 4. Before we do so, however, we first establish the domain of validity of the analytic model by comparing the results to those obtained from an exact \( N \)-body integration. This comparison was done with the \( N \)-body code described in Nesvorný et al. (2013), where the TTVs are computed with an efficient and precise algorithm (also see Deck et al. 2014).

Figure 4 illustrates the results for the test case previously shown in Figures 2 and 3. Here the results of the analytic model are in an excellent agreement with those obtained from the \( N \)-body integration. This was expected because the parameters of the test case were set to be in the domain where the analytic model should be valid (e.g., small planetary masses, \( \alpha \) not too large, and \( \epsilon^{(0)} = 0 \)).

In general, however, the analytic model is obviously only an approximation of the conjunction effect. First, the two orbits were assumed to be strictly coplanar. Second, we assumed that \( m_i \ll M_\star \), such that \( \mathcal{H}_{\text{pert}} \) can be treated as a perturbation of \( \mathcal{H}_{\text{Kep}} \). Terms of the second and higher orders in \( m_i/M_\star \) were not included. Third, we assumed that \( \epsilon_i \ll 1 \) expanded the Hamiltonian in powers of \( \epsilon_i \) and retained only the lowest power of \( \epsilon_i \). We therefore expect the analytic model to be valid only for very nearly circular orbits of both planets.

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Figure 5 illustrates the approximate nature of the analytic model. To make this figure, we surveyed a range of orbital separations and eccentricities (0.5 < \( \alpha < 0.9 \), 0 < \( \epsilon^{(0)} < 0.1 \) and 0 < \( \epsilon^{(0)} < 0.1 \)). Other parameters were held fixed \( m_i = 10^{-5} M_\star \) \( M_\star = M_\odot \) \( M_\star = M_\odot \). In each case, we followed dynamics over one conjunction cycle and determined the (1) amplitude of \( \delta t_i \) variation as a result of conjunction, and (2) the difference between analytically and numerically computed \( \delta t_i \) when \( \psi \) approaches \( \pi \) (i.e., at the end of the conjunction cycle). From this, by dividing (2) by (1), we computed the relative error of the analytic model as a function of \( \alpha, \epsilon^{(0)}_1 \) and \( \epsilon^{(0)}_2 \).

From Figure 5, we see that the analytic model is valid only for small eccentricities, and the eccentricity threshold (\( \epsilon_{\text{crit}} \)) beyond which the relative error is excessive (say 10%) is a strong function of the radial separation between orbits. For example, \( \epsilon_{\text{crit}} \approx 0.1 \) for \( \alpha = 0.5 \) while \( \epsilon_{\text{crit}} \approx 0.001 \) for \( \alpha = 0.8 \). This is expected because the second and higher order effects in \( m_i/M_\star \), neglected in our analytic model, should become important with increasing \( \alpha \). Also, we would need to include eccentricity terms beyond the lowest power to make the analytic model more generally valid for larger eccentricities.
We used $m_1/M_\star = 10^{-5}$ in Figure 5, but it turns out that the general appearance of this figure is independent of the considered planetary masses. This is because both the magnitude of the conjunction effect and the error of the analytic model increase (nearly) linearly with $m_1/M_\star$. The relative error therefore remains approximately the same.

The mass ratio $m_1/M_\star$ sets the limit in $\alpha$ beyond which the analytic model does not apply. Beyond this limit, the co-orbital dynamics appears and the two planets can switch positions radially (i.e., following the horseshoe or tadpole trajectory the dynamics appears and the two planets can switch positions). Figure 5 shows how the relative precision of the analytic model. The shaded area is where the precision of the analytic model is better than 10%.

![Figure 5](image)

**Figure 5.** Domain of validity of the analytical model as a function of $\alpha$, $e_1$, and $e_2$. As in Figure 4, two planets with the mass $10^{-5} M_\odot$ each were placed on the initially coplanar orbits. The outer planet was set to have $a_2 = 1.0$ AU and the inner planet’s semimajor axis was varied such that $0.5 < \alpha < 0.9$. The isolines show the relative precision of the analytic model. The shaded area is where the precision of the analytic model is better than 10%.

4 Note that if more than two planets are present in a given system, the TTVs from conjunctions of different pairs should add linearly, at least in the approximation of our analytic model.

### 4. Scalling of the Conjunction Effect With Planetary Properties

According to Equations (33) and (34), the expected variation of transit timing, $\delta t_1$, is proportional to $P_i$, where $P_i$ is the orbital period of the transiting planet. Thus, assuming that the observation baseline is long enough to cover several conjunction cycles, the detection of the conjunction effect would be easier for planets with longer orbital periods, for which the effect is larger. In reality, however, the current observational baselines are typically only a few years such that we do not expect that the TTVs to be generally detectable for long-period planets. The conjunction effect could potentially be detected for long-period planets only if at least a few transits were observed before the conjunction and a few transits after the conjunction, which would require a fortuitous configuration of the planetary system at the current epoch (a good example of this can be KOI-351g; Cabrera et al. 2014).

The scaling of $\delta t_1$ with the planetary and stellar masses is obvious from Equations (35) and (36), at least in the approximation $m_i \ll M_\star$ that we adopted in the analytic model. While $\delta t_1$ scales linearly with $m_2/M_\star$, $\delta t_2$ scales linearly with $m_1/M_\star$. This means that a detection of the conjunction effect in transits of the inner planet can help to determine the mass of the outer planet, and vice versa. Also, the detection of transit variations is obviously easier in a system with more massive planets and, for a fixed orbital period, with lower stellar mass.4

Figure 6 illustrates the dependence of $\delta t_1$ on $\alpha = a_1/a_2$. The dashed lines in the figure show the amplitude of the $\delta \lambda_i/\pi_i$ change from a single conjunction between planets. From Equations (15) and (19), the amplitude is $2B_1(\alpha)m_2/M_\star$ for the inner planet and $2B_1(\alpha)m_1/M_\star$ for the outer planet, where $A_1(\alpha)$ and $B_1(\alpha)$ are given in terms of the complete elliptic integrals in Equations (16) and (20). The total conjunction effect from $\delta \lambda_{\star 1} + \delta \lambda_{\star 2}^{\text{eff}}$ is shown by solid lines in Figure 6. The basic tendency is that the magnitude of the conjunction effect strongly increases with $\alpha$, such that it is $\sim 100-200$ times stronger for planets with $\alpha \simeq 0.9$ than for planets with $\alpha \simeq 0.5$. This is reasonable because the closely packed planetary systems are expected to have stronger gravitational interactions.

In the example given in Figure 6 with $m_1 = m_2 = 10^{-5} M_\star$, $M_\star = M_\odot$ and $a_2 = 1$ AU, the magnitude of $\delta t_1$ ranges from 3 minutes (inner planet, $\alpha = 0.5$) to over 10 hr (outer planet, $\alpha = 0.9$). Assuming instead that the outer planet with $a_2 = 1$ AU has one Earth mass, the TTVs of the inner planet should range between $\pm 1$ minute and $\pm 3$ hr. They should therefore be generally detectable with adequate photometric precision.

Figure 7 shows how the amplitude of the conjunction effect changes with eccentricity. Given that our analytic model loses precision with increasing eccentricity (see Figure 5), here we used our $N$-body code to compute $\delta t_1$ over one conjunction cycle. In addition to changing $\alpha$ as in Figure 6, we also varied the initial eccentricities of the two planets. Figure 7 shows that the amplitude of the conjunction effect can increase by a factor of $\sim 2-10$ by increasing the eccentricity from 0 to 0.1. This is significant, because it shows that the likelihood of detection of the conjunction effect can be boosted for orbits with modest eccentricities, a case that should presumably be common among planetary systems.
Figure 6. TTV amplitude of the outer (top panel) and inner (bottom panel) planets. The dashed lines show the amplitude of the step-like variation from \(\delta \lambda_i / n_i\) (Equations (15) and (19)). The solid lines show the total amplitude from \(\delta \lambda_i + \delta \lambda_{\text{eff}} / n_i\), where \(\delta \lambda_{\text{eff}}\) is given in Equations (25) and (28). Here we used \(m_1 = m_2 = 10^{-5} M_\star\), \(M_\star = M_\odot\), \(a_2 = 1\) AU, and \(e_i(0) = 0\). The inner planet’s semimajor axis was varied such that \(0.5 < \alpha < 0.9\). The TTV amplitudes are given here for a single conjunction between planets.

The magnitude of the conjunction effect for eccentric orbits, however, also depends on the relative orientation of orbits, as given by \(\varpi_1\) and \(\varpi_2\), and on where exactly the conjunction happens along the orbits. The magnitude can increase or decrease, roughly reflecting the physical distance between planets during the conjunction. Figure 7 was produced by surveying all orbital configurations, \(0 \leq \varpi_1 < 2\pi\) and \(0 \leq \varpi_2 < 2\pi\), and plotting the one for which the magnitude was maximal.

Figure 8 shows how the amplitude of the conjunction effect changes with the mutual inclination between orbits, \(i_{\text{mutual}}\). As in Figure 7 we used the N-body code to compute \(\delta t_i\) over one conjunction cycle. Figure 7 illustrates that the amplitude of the conjunction effect is not very sensitive to \(i_{\text{mutual}}\). The magnitude varies only up to \(\pm 20\%\) for \(i_{\text{mutual}} < 50^\circ\), relative to the case with \(i_{\text{mutual}} = 0\). See Nesvorný (2009) for a more general analysis of the dependence of the short-periodic TTVs on orbital inclinations.

5. DISCUSSION

Equations (33) and (34) express our expectation for the TTVs produced by two planets on nearly circular and coplanar orbits. The first two terms in these equations result from the direct perturbation of the mean longitude. If \(\alpha\) is sufficiently close to 1, these terms are essentially equivalent to a succession of transit timing discontinuities occurring at orbital conjunctions between planets. The amplitude of these discontinuities is \(2A_1(\alpha) m_2 / M_\star\) for the inner planet and \(2B_1(\alpha) m_1 / M_\star\) for the outer planet, where \(A_1(\alpha)\) and \(B_1(\alpha)\) were defined in Equations (16) and (20) respectively.
and were illustrated in Figure 6 (dashed lines). The transit times of the inner planet are expected to be delayed relative to a fixed Keplerian ephemeris, while those of the outer planet are expected to be sped up.

The long-term effects of conjunctions, with $\delta t_l$ steadily accumulating over many periods of the synodic angle, can be absorbed by a small change of the orbital period. The short-period effects of conjunctions, frequently described as “chopping” of the TTV signal (e.g., Holman et al. 2010; Carter et al. 2012), should not be mistaken with anything else. When the long-term conjunction effects are removed from the transit ephemeris, the TTV signal from direct perturbation of the mean longitude should have a saw-like profile with each tooth being marked by a few rising and a few declining transits. When such a saw-tooth profile is identified in the data, the mass of planetary companion can be extracted from these measurements, assuming that $\alpha$ is known, by using Equations (33) or (34). If $\alpha$ is unknown, Equations (33) or (34) can be used to constrain $A_1(\alpha)m_2/M_* or B_1(\alpha)m_1/M_*$.

As an example, we discuss the chopping in the TTVs of KOI-884-02 and Kepler-36c (Figure 1). As for Kepler-36, $P_2 \simeq 16.2$ day, $\alpha \simeq 0.9$, $M_2/M_* \simeq 1.3 \times 10^{-3}$ from Carter et al. (2012). The size of the conjunction step computed from Equation (34) for these parameters is $\delta t_2 \sim 0.3$ hr. For a comparison, Figure 1(b) shows that the actual steps during conjunctions are smaller, but not much smaller, than one hour. As for KOI-884-02, $P_1 \simeq 20.5$ day, $\alpha \simeq 0.34$, $M_2/M_* \simeq 3 \times 10^{-3}$ from Nesvorný et al. (2012). We compute that $\delta t_1 \sim 0.9$ hr from Equation (33). For a comparison, the actual steps in KOI-884-02’s TTVs, best seen for transit cycles between 32 and 45 (Figure 1(a)), are $\sim 1-2$ hr. The difference is probably caused by small but significant orbital eccentricities (Nesvorný et al. 2014).

The oscillatory part of Equations (33) or (34) offers a different method to constrain planetary masses and/or $\alpha$. Here it can be useful to perform the Fourier analysis of $\delta t_l$. The expectation is that this will reveal frequencies that are integer multiples of $(n_1 - n_2)$. Some of these frequencies will be faster than the Nyquist frequency, $f_N = \pi/P$, and will be aliased to the part of the Fourier spectrum with $f < f_N$. For example, in the test case shown in Figure 4, $f_N = 0.01117$ day$^{-1}$. Therefore, frequencies $(n_1 - n_2) = 0.00514$ day$^{-1}$ and $2(n_1 - n_2) = 0.01028$ day$^{-1}$ appear unaliased, while all $j(n_1 - n_2)$ frequencies with $j \geq 3$ are aliased to $2f_N - j(n_1 - n_2)$. For example, $3(n_1 - n_2) = 0.01542$ day$^{-1}$ appears at 0.006933 day$^{-1}$.

As for KOI-884, $(n_1 - n_2) = 0.201$ day$^{-1}$ and $f_N = 0.153$ day$^{-1}$. As $(n_1 - n_2) > f_N$ in this synodic frequency and all its multiples will be aliased. For example, $(n_1 - n_2)$ should appear at 0.105 day$^{-1}$, and $2(n_1 - n_2)$ should appear at 0.096 day$^{-1}$. The Fourier analysis of the best fit TTV model from Nesvorný et al. (2014) confirms this. It shows that the peak power density of the $2(n_1 - n_2)$ term is about five times larger than that of the $(n_1 - n_2)$ term, as expected from Equations (27) and (33). Unfortunately, these terms are much harder to identify in the existing TTV data of KOI-884-02, because of the short coverage, gaps, measurement errors, and other issues. We have done a similar analysis for Kepler-36, but do not discuss it here, except for pointing out that the aliasing is not a problem for this system, because the synodic frequency $(n_1 - n_2)$ is relatively slow (as $\alpha \simeq 0.9$).

Identifying the structure of unaliased and aliased frequencies in the frequency domain can be useful for the interpretation of the TTV observations. As the amplitudes of these terms are proportional to $D_j^1$, as shown in Equations (33) and (34), the TTV measurements can be potentially inverted to obtain a unique determination of the planetary mass and orbital separation (e.g., Nesvorný et al. 2013). This highlights the importance of the conjunction effect.

As a final word of caution, note that Equations (33) and (34) were derived under several assumptions. Most importantly, these equations are valid only for small orbital eccentricities. While many planetary systems will presumably fall into this category, perhaps the majority of them will not. The TTVs for planetary systems with orbital eccentricities exceeding the threshold shown in Figure 5 will contain many additional terms from the first and higher eccentricity powers. These terms will add frequencies $k_1n_1 + k_2n_2$, with arbitrary $k_1$ and $k_2$, potentially generating resonant or near-resonant TTVs, and will modify the amplitude dependence of the $j(n_1 - n_2)$ frequencies on orbital parameters. The analytic model described here therefore cannot be used in general to characterize the planetary systems from TTVs. Instead, the main scientific value of the analytic model is to give us an intuitive framework for how the TTV method works in the limit of the nearly circular orbits.

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ERRATUM: “THE EFFECT OF CONJUNCTIONS ON THE TRANSIT TIMING VARIATIONS OF EXOPLANETS” (2014, ApJ, 790, 58)

DAVID NESVÔRNÝ1 AND DAVID VOKROUHLICKÝ2

1 Department of Space Studies, Southwest Research Institute, 1050 Walnut Street, Suite 300, Boulder, CO 80302, USA; davidn@boulder.swri.edu
2 Institute of Astronomy, Charles University, V Holešovičkách 2, CZ-18000 Prague 8, Czech Republic; vokrouhli@cesnet.cz

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There are several typos in the published article. Equation (6) is missing an asterisk indicating that the derivative is taken with respect to the complex conjugate of $z_i$. The corrected Equation (6) is

$$\frac{dz_i}{dt} = -\frac{2i}{\mu_i a_i^2} \frac{\partial H_{\text{per}}}{\partial z_i^*} + O(\epsilon).$$

Below, in Equation (10), the relation between the Laplace coefficients $b_{1/2}^{(0)}$ and $b_{1/2}^{(1)}$ and the complete elliptic integrals $K$ and $E$ should have a multiplication factor of four (and not two). The corrected relations are

$$b_{1/2}^{(0)}(\alpha) = (4/\pi) K(\alpha), \quad b_{1/2}^{(1)}(\alpha) = 4 [K(\alpha) - E(\alpha)]/(\pi \alpha).$$

Equation (11) is missing $\alpha$ on the right-hand side. The corrected Equation (11) is

$$\delta a_1 = -2\alpha v_1 a_1 \frac{m_2}{M} Q(\alpha, \psi).$$

Below, in Equation (13), the total angular momentum conservation equation should have $n_1 a_1$ and $n_2 a_2$ in the numerator, and not in the denominator. The corrected equation is

$$m_1 n_1 a_1 \delta a_1 + m_2 n_2 a_2 \delta a_2 = 0.$$ (13)

Equation (27) should have $v_1 (3v_1 - v_2)$ in the numerator of the first term on the right. The corrected Equation (27) is

$$D_j^i(\alpha) = 2 \delta a_1 a^2 v_1 (3v_1 - v_2) v_2 (2v_1 - v_2) + 2j a_1 \frac{\partial b_{1/2}^{(j)}}{\partial \alpha} + 2v_1 b_{1/2}^{(j)}.$$ (27)

None of these typos had any effect on the results discussed in the paper.

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