Error estimates for phaseless inverse scattering in the Born approximation at high energies

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Abstract. We study explicit formulas for phaseless inverse scattering in the Born approximation at high energies for the Schrödinger equation with compactly supported potential in dimension \(d \geq 2\). We obtain error estimates for these formulas in the configuration space.

1 Introduction

We consider the time-independent Schrödinger equation

\[ -\Delta \psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \ d \geq 2, \ E > 0, \]  

where \(v \in L^{\infty}(\mathbb{R}^d), \ \text{supp} \ v \subset D,\) \(^2\)

In quantum mechanics equation (1) describes an elementary particle interacting with a macroscopic object contained in \(D\) at fixed energy \(E\). In this setting one usually assumes that \(v\) is real-valued.

Equation (1) at fixed \(E\) can be also interpreted as the Helmholtz equation of acoustics or electrodynamics. In these frameworks the coefficient \(v\) can be complex-valued. In addition, the imaginary part of \(v\) is related to the absorption coefficient.

For equation (1) we consider the classical scattering solutions \(\psi^+ = \psi^+(x, k)\), where \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \ k = (k_1, \ldots, k_d) \in \mathbb{R}^d, \ k^2 = k_1^2 + \cdots + k_d^2 = E\). These solutions \(\psi^+\) can be specified by the following asymptotics as \(|x| \to \infty\):

\[ \psi^+(x, k) = e^{ikx} + c(d, |k|) e^{i|k||x|/|x|^{(d-1)/2}} f(k, |k|) + O(|x|^{-(d+1)/2}), \]

\[ x \in \mathbb{R}^d, \ k \in \mathbb{R}^d, \ k^2 = E, \ kk = k_1x_1 + \cdots + k_dx_d, \]

\[ c(d, |k|) = -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \]

for some a priori unknown \(f\). The function \(f\) arising in (3) is defined on

\[ \mathcal{M}_E = \{(k, l) \in \mathbb{R}^d \times \mathbb{R}^d; \ k^2 = l^2 = E\}, \]  

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and is known as the classical scattering amplitude for equation (1).

In quantum mechanics \(|f(k, l)|^2\) describes the probability density of scattering of particle with initial momentum \(k\) into direction \(l/|l| \neq k/|k|\), and is known as differential scattering cross section for equation (1); see, e.g., [11, Chapter 1, Section 6].

The problem of finding \(\psi^+\) and \(f\) from \(v\) is known as the direct scattering problem for equation (1). For solving this problem, one can use, in particular, the Lippmann-Schwinger integral equation for \(\psi^+\) and an explicit integral formula for \(f\), see, e.g., [5, 10, 29].

In turn, the problem of finding \(v\) from \(|f|^2\) is known as the inverse scattering problem (with phase information) and the problem of finding \(v\) from \(|f|^2\) is known as the phaseless inverse scattering problem for equation (1).

There are many important results on the former inverse scattering problem with phase information; see [3, 4, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 25, 28] and references therein. In particular, it is well known that the scattering amplitude \(f\) uniquely determines \(v\) via the Born approximation formulas at high energies:

\[
\widehat{v}(k - l) = f(k, l) + O(E^{-\frac{1}{2}}), \quad E \to +\infty, \; (k, l) \in \mathcal{M}_E, \tag{5}
\]

\[
\widehat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) \, dx, \quad p \in \mathbb{R}^d, \tag{6}
\]

and the inverse Fourier transform; see, e.g., [9, 28].

On the other hand, the literature for the phaseless case is much more limited; see [7, 29] and references therein for the case of the aforementioned phaseless inverse problem and see [19, 20, 21, 26, 27, 29] and references therein for the case of some similar inverse problems without phase information. In addition, it is well known that the phaseless scattering data \(|f|^2\) does not determine \(v\) uniquely, even if \(|f|^2\) is given completely for all positive energies. In particular, it is known that

\[
f_y(k, l) = e^{i(k - l)\cdot y} f(k, l),
\]

\[
|f_y(k, l)|^2 = |f(k, l)|^2, \quad k, l \in \mathbb{R}^d, \; k^2 = l^2 > 0,
\]

where \(f\) is the scattering amplitude for \(v\) and \(f_y\) is the scattering amplitude for \(v_y = v(\cdot - y)\), where \(y \in \mathbb{R}^d\); see [29] and references therein.

In the present work, in view of the aforementioned non-uniqueness for the problem of finding \(v\) from \(|f|^2\), we consider the modified phaseless inverse scattering problem formulated below as Problem 1. Let

\[
S = \{|f|^2, |f_1|^2, \ldots, |f_m|^2\},
\]

where \(f\) is the scattering amplitude for \(v\) and \(f_1, \ldots, f_m\) are the scattering amplitudes for \(v_1, \ldots, v_m\), where

\[
v_j = v + w_j, \quad j = 1, \ldots, m,
\]
where $w_1, \ldots, w_m$ are additional a priori known background scatterers such that
\[
w_j \in L^\infty(\mathbb{R}^d), \quad \text{supp } w_j \subset \Omega_j,
\]
$\Omega_j$ is an open bounded domain in $\mathbb{R}^d$, $\Omega_j \cap D = \emptyset$, \quad (10)
\[
w_j \neq 0, \quad w_{j_1} \neq w_{j_2} \text{ if } j_1 \neq j_2 \text{ (in } L^\infty(\mathbb{R}^d)),
\]
Thus, $S$ consists of the phaseless scattering data $|f|^2, |f_1|^2, \ldots, |f_m|^2$ measured sequentially, first, for the unknown scatterer $v$ and then for $v$ in the presence of known scatterer $w_j$ disjoint from $v$ for $j = 1, \ldots, m$.

Actually, in the present work we continue studies of [29] on the following inverse scattering problem for equation (1):

**Problem 1.** Reconstruct potential $v$ from the phaseless scattering data $S$ for some appropriate background scatterers $w_1, \ldots, w_m$.

Studies of Problem 1 in dimension $d \geq 2$ were started in [29]. In dimension $d = 1$ for $m = 1$ studies of Problem 1 were started earlier in [2], where phaseless scattering data was considered for all $E > 0$.

Actually, the key result of [29] consists in a proper extension of formula (5) for the Fourier transform $\hat{v}$ of $v$ to the phaseless case of Problem 1; see Section 2.

In the present work we proceed from the aforementioned result of [29] and study related approximate reconstruction of $v$ in the configuration space. In this connection our results consist in obtaining related error estimates in the configuration space at high energies $E$; see Section 4.

In addition, results of the present work are necessary for extending the iterative algorithm of [28] to the phaseless case of Problem 1. The latter extension will be given in [1].

## 2 Extension of formula (5) to the phaseless case

Actually, the key result of [29] consists in the following formulas for solving Problem 1 in dimension $d \geq 2$ for $m = 2$ at high energies $E$:

\[
\begin{pmatrix}
\text{Re } \hat{v} \\
\text{Im } \hat{v}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
\text{Re } \hat{w}_1 & \text{Im } \hat{w}_1 \\
\text{Re } \hat{w}_2 & \text{Im } \hat{w}_2
\end{pmatrix}^{-1} \begin{pmatrix}
|\hat{v}_1|^2 - |\hat{v}|^2 - |\hat{w}_1|^2 \\
|\hat{v}_2|^2 - |\hat{v}|^2 - |\hat{w}_2|^2
\end{pmatrix}, \quad (11)
\]
\[
|\hat{v}_j(p)|^2 = |f_j(k, l)|^2 + O(E^{-1/2}), \quad E \to +\infty,
\]
\[
p \in \mathbb{R}^d, \quad (k, l) \in \mathcal{M}_E, \quad k - l = p, \quad j = 0, 1, 2, \quad (12)
\]

where:

- $v_0 = v, v_j$ is defined by [9], $j = 1, 2$, and $f_0 = f, f_1, f_2$ are the scattering amplitudes for $v_0, v_1, v_2$, respectively;
- $\hat{v} = \hat{v}(p), \hat{v}_j = \hat{v}_j(p), \hat{w}_j = \hat{w}_j(p), p \in \mathbb{R}^d$, are the Fourier transforms of $v, v_j, w_j$ (defined as in [9]);
• formula (11) is considered for all \( p \in \mathbb{R}^d \) such that the determinant
\[
\zeta_{\hat{w}_1, \hat{w}_2}(p) \overset{d.f.}{=} \text{Re } \hat{w}_1(p) \text{Im } \hat{w}_2(p) - \text{Im } \hat{w}_1(p) \text{Re } \hat{w}_2(p) \neq 0.
\] (13)

The point is that using formulas (12) for \( d \geq 2 \) with
\[
k = k_E(p) = \frac{p}{2} + (E - \frac{p^2}{4})^{1/2}\gamma(p),
\]
\[
l = l_E(p) = -\frac{p}{2} + (E - \frac{p^2}{4})^{1/2}\gamma(p),
\]
\[
|\gamma(p)| = 1, \quad \gamma(p)p = 0,
\] (14)
where \( p \in \mathbb{R}^d, |p| \leq 2\sqrt{E} \), one can reconstruct \(|\hat{v}|^2, |\hat{v}_1|^2, |\hat{v}_2|^2\) from \( S \) at high energies for any \( p \in \mathbb{R}^d \). And then using formula (11) one can reconstruct \( \hat{v} \) completely, provided that condition (13) is fulfilled for almost all \( p \in \mathbb{R}^d \).

**Remark 1.** Formulas (12) can be precised as formula (2.15) of [29]:
\[
||\hat{v}_j(p)||^2 - |f_j(k, l)|^2 | \leq c(D_j)N_j^2E^{-\frac{1}{2}},
\] (15)
where \( ||v_j||_{L^\infty(D_j)} \leq N_j, j = 0, 1, 2, \) and \( D_0 = D, D_j = D \cup \Omega_j, j = 1, 2, \) and constants \( c, \rho \) are given by formulas (3.10) and (3.11) in [29] (and, in particular, \( \rho \geq 1 \)).

In addition, from the experimental point of view it seems to be, in particular, convenient to consider Problem 4 with \( m = 2 \) for the case when \( w_2 \) is just a translation of \( w_1 \):
\[
w_2(x) = w_1(x - y), \quad x \in \mathbb{R}^d, \ y \in \mathbb{R}^d.
\] (16)
In this case
\[
\hat{w}_2(p) = e^{ipy}\hat{w}_1(p), \quad \zeta_{\hat{w}_1, \hat{w}_2}(p) = \sin(py)|\hat{w}_1(p)|^2, \quad p \in \mathbb{R}^d.
\] (17)

On the level of analysis, the principal complication of (11), (12) in comparison with (11) consists in possible zeros of the determinant \( \zeta_{\hat{w}_1, \hat{w}_2} \) of (13). For some simplest cases, we study these zeros in the next section.

### 3 Zeros of the determinant \( \zeta_{\hat{w}_1, \hat{w}_2} \)

Let
\[
Z_{\hat{w}_1, \hat{w}_2} = \{ p \in \mathbb{R}^d; \zeta_{\hat{w}_1, \hat{w}_2}(p) = 0 \},
\]
\[
Z_{\hat{w}_j} = \{ p \in \mathbb{R}^d; \hat{w}_j(p) = 0 \}, \quad j = 1, 2,
\] (18)
where \( \zeta \) is defined by (13). From (13), (18) it follows that
\[
Z_{\hat{w}_1} \cup Z_{\hat{w}_2} \subseteq Z_{\hat{w}_1, \hat{w}_2}.
\] (19)
In view of (19), in order to construct examples of \( w_1, w_2 \) such that the set \( Z_{\hat{w}_1, \hat{w}_2} \) is as simple as possible, we use the following lemma:
Lemma 1. Let
\[ w(x) = |x|^\nu K_\nu(|x|) \int_{\mathbb{R}^d} q(x - y) q(y) \, dy, \quad x \in \mathbb{R}^d, \nu > 0, \] (20)
\[ K_\nu(s) = \frac{\Gamma\left(\frac{1}{2} + \nu\right)}{\sqrt{\pi}} \left(\frac{2}{s}\right)^\nu \int_0^\infty \cos(st) \, dt, \quad s > 0, \] (21)
where \( q \in L^\infty(\mathbb{R}^d), q = \overline{q}, q \neq 0 \) in \( L^\infty(\mathbb{R}^d) \),
\[ q(x) = 0 \text{ if } |x| > r, q(x) = q(-x), \quad x \in \mathbb{R}^d. \] (22)

Then
\[ w \in C(\mathbb{R}^d), \quad w = \overline{w}, \quad w(x) = 0 \text{ if } |x| > 2r, \quad x \in \mathbb{R}^d, \] (23)
\[ \widehat{w}(p) = \overline{w}(p) \geq c_1 (1 + |p|)^{-\beta}, \quad p \in \mathbb{R}^d, \]
for \( \beta = d + 2\nu \) and some positive constant \( c_1 = c_1(q, \nu) \), where \( \widehat{w} \) is the Fourier transform of \( w \). In addition, if \( q \geq 0 \), then \( w \geq 0 \).

We recall that \( K_\nu \) defined by (21) is the modified Bessel function of the second kind and order \( \nu \). In addition, \( \Gamma \) denotes the gamma function.

Lemma 1 is proved in Section 8.

As a corollary of Lemma 1, functions
\[ w_j(x) = w(x - T_j), \quad x \in \mathbb{R}^d, \quad T_j \in \mathbb{R}^d, \] (24)
where \( w \) is constructed in Lemma 1, give us examples of \( w_j \) satisfying (10) for fixed \( D, \Omega_j \) and for appropriate radius \( r \) of Lemma 1 and translations \( T_j \) of (24), and such that
\[ Z_{\widehat{w}_j} = \emptyset, \]
\[ |\widehat{w}_j(p)| = \widehat{w}(p) \geq c_1 (1 + |p|)^{-\beta}, \quad p \in \mathbb{R}^d, \] (25)
where \( c_1, \beta \) are the same as in (23). In addition,
\[ \zeta_{\widehat{w}_1, \widehat{w}_2}(p) = \sin(py)|\widehat{w}(p)|^2, \quad y = T_2 - T_1 \neq 0, \quad p \in \mathbb{R}^d, \]
\[ Z_{\widehat{w}_1, \widehat{w}_2} = \{ p \in \mathbb{R}^d : \sin(py) = 0 \} = \{ p \in \mathbb{R}^d : py \in \pi \mathbb{Z} \}, \] (26)
for \( w_1, w_2 \) of (24).

As another corollary of Lemma 1 we have that
if \( w_1 \) is defined as in (24) and \( w_2 = iw_1 \), then
\[ \zeta_{\widehat{w}_1, \widehat{w}_2}(p) = |\widehat{w}(p)|^2 \geq c_1^2 (1 + |p|)^{-2\beta}, \quad p \in \mathbb{R}^d, \] (27)
\[ Z_{\widehat{w}_1, \widehat{w}_2} = \emptyset. \]

We recall that complex-valued \( v \) and \( w_j \) naturally arise if we interpret equation (1) for fixed \( E \) as the Helmholtz equation of acoustics or electrodynamics.

Finally, note that
\[ Z_{\widehat{w}_1, \ldots, \widehat{w}_{d+1}} = \frac{\pi^d}{2} \mathbb{Z}^d, \quad \text{where} \]
\[ Z_{\widehat{w}_1, \ldots, \widehat{w}_d} = Z_{\widehat{w}_1, \widehat{w}_2} \cap Z_{\widehat{w}_1, \widehat{w}_3} \cap \cdots \cap Z_{\widehat{w}_1, \widehat{w}_{d+1}}, \] (28)
if $w_1$ is defined as in (24), and
\[ w_2(x) = w_1(x - se_1), \ldots, w_{d+1}(x) = w_1(x - se_d), \tag{29} \]
where $(e_1, \ldots, e_d)$ is the standard basis of $\mathbb{R}^d$ and $s > 0$.

Thus, in principle, for Problem 1 with background scatterers $w_1, \ldots, w_{d+1}$ as in (29), for each $p \in \mathbb{R}^d \setminus \mathbb{Z}^d$ formulas (11), (12) can be used with appropriate $w_j$ in place of $w_2$, where $j = 2, \ldots, d+1$.

4 Error estimates in the configuration space

We recall that for inverse scattering with phase information the scattering amplitude $f$ on $\mathcal{M}_E$ processed by (5) and the inverse Fourier transform yield the approximate reconstruction
\[ u(\cdot, E) = v + O(E^{-\alpha}) \quad \text{in} \quad L^\infty(D) \quad \text{as} \quad E \to +\infty, \quad \alpha = \frac{n-d}{2n}, \tag{30} \]
if $v \in W^{n,1}(\mathbb{R}^d)$, $n > d$ (in addition to the initial assumption (2)), where $W^{n,1}(\mathbb{R}^d)$ denotes the standard Sobolev space of $n$-times differentiable functions in $L^1(\mathbb{R}^d)$:
\[ W^{n,1}(\mathbb{R}^d) = \{ u \in L^1(\mathbb{R}^d): \| u \|_{n,1} < \infty \}, \]
\[ \| u \|_{n,1} = \max_{|J| \leq n} \left\| \partial^{|J|} u \right\|_{L^1(\mathbb{R}^d)}, \quad n \in \mathbb{N} \cup \{0\}. \tag{31} \]

More precisely, the approximation $u(\cdot, E)$ in (30) is defined by
\[ u(x, E) = \int_{B_r(E)} e^{-ipx} f(k_E(p), l_E(p)) \, dp, \quad x \in D, \tag{32} \]
where
\[ B_r(E) = \{ p \in \mathbb{R}^d: |p| \leq r \}, \tag{33} \]
\[ \alpha \text{ is defined in } (30), \text{ and } k_E(p), l_E(p) \text{ are defined as in } (14) \text{ with some piecewise continuous vector-function } \gamma \text{ on } \mathbb{R}^d; \text{ see, e.g., } [28]. \]

In addition, estimate (30) can be precised as
\[ |u(x, E) - v(x)| \leq A(D, N, M, d, n, \tau) E^{-\alpha}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho(D, N), \tag{34} \]
where $\| v \|_{L^\infty(D)} \leq N$, $\| v \|_{n,1} \leq M$, $\rho$ is the same as in (15) and the expression for $A$ can be found in formula (3.10) of [28].

Analogs of $u(\cdot, E)$ for the phaseless case are given below in this section. In particular, related formulas depend on the zeros of determinant $\zeta_{\hat{w}_1, \hat{w}_2}$ of (13).
We consider

\[ U_{\tilde{\omega}_1, \tilde{\omega}_2} = \text{Re} U_{\tilde{\omega}_1, \tilde{\omega}_2} + i \text{Im} U_{\tilde{\omega}_1, \tilde{\omega}_2}, \]

\begin{align*}
(\text{Re} U_{\tilde{\omega}_1, \tilde{\omega}_2}(p, E)) &= \frac{1}{2} M_{\tilde{\omega}_1, \tilde{\omega}_2}^{-1}(p) b_{\tilde{\omega}_1, \tilde{\omega}_2}(p, E), \\
M_{\tilde{\omega}_1, \tilde{\omega}_2}(p) &= \begin{pmatrix} \text{Re} \tilde{\omega}_1(p) & \text{Im} \tilde{\omega}_1(p) \\ \text{Re} \tilde{\omega}_2(p) & \text{Im} \tilde{\omega}_2(p) \end{pmatrix}, \\
M_{\tilde{\omega}_1, \tilde{\omega}_2}^{-1}(p) &= \frac{1}{\zeta_{\tilde{\omega}_1, \tilde{\omega}_2}(p)} \begin{pmatrix} \text{Im} \tilde{\omega}_2(p) & -\text{Im} \tilde{\omega}_1(p) \\ -\text{Re} \tilde{\omega}_2(p) & \text{Re} \tilde{\omega}_1(p) \end{pmatrix}, \\
b_{\tilde{\omega}_1, \tilde{\omega}_2}(p, E) &= \begin{pmatrix} |f_1(p, E)|^2 - |f(p, E)|^2 - |\tilde{\omega}_1(p)|^2 \\ |f_2(p, E)|^2 - |f(p, E)|^2 - |\tilde{\omega}_2(p)|^2 \end{pmatrix}, \\
f(p, E) &= f(k_E(p), l_E(p)), f_j(p, E) = f_j(k_E(p), l_E(p)), j = 1, 2,
\end{align*}

where \( \tilde{\omega}_1, \tilde{\omega}_2 \), \( f, f_1, f_2 \) are the same as in (11), (12), \( \zeta_{\tilde{\omega}_1, \tilde{\omega}_2} \) is defined by (13), \( k_E(p), l_E(p) \) are the same as in (14), (32), and \( p \in B_{2\sqrt{E}} \), \( d \geq 2 \).

For Problem \( 1 \) for \( d \geq 2, m = 2, \) and for the case when \( \zeta_{\tilde{\omega}_1, \tilde{\omega}_2} \) has no zeros (the case of (27) in Section 3) we have the following result:

**Theorem 1.** Let \( v \) satisfy (2) and \( v \in W^{n,1}(\mathbb{R}^d) \) for some \( n > d \). Let \( w_1, w_2 \) be the same as in (27). Let

\[ u(x, E) = \int_{B_{r_1(E)}} e^{-ivx} U_{\tilde{\omega}_1, \tilde{\omega}_2}(p, E) dp, \quad x \in D, \]

\[ r_1(E) = 2\pi E^{\alpha_1}, \quad \alpha_1 = \frac{n - d}{2(n + \beta)}, \quad \text{for some fixed } \tau \in (0, 1], \]

where \( U_{\tilde{\omega}_1, \tilde{\omega}_2} \) is defined by (35), \( B_r \) is defined by (33), \( \beta \) is the number of (23), (27). Then

\[ u(\cdot, E) = v + O(E^{-\alpha_1}) \quad \text{in } L^\infty(D), \ E \to +\infty, \]

\[ |u(x, E) - v(x)| \leq A_1 E^{-\alpha_1}, \quad x \in D, \ E^{\frac{1}{2}} \geq \rho_1, \]

where \( \rho_1 \) and \( A_1 \) are defined in formulas (55) and (64) of Section 5.

Theorem 1 is proved in Section 5.

Next, we set

\[ Z_{\tilde{\omega}_1, \tilde{\omega}_2}^\varepsilon = \{ p \in \mathbb{R}^d : py \in (-\varepsilon, \varepsilon) + \pi \mathbb{Z} \}, \quad y \in \mathbb{R}^d \setminus 0, \quad 0 < \varepsilon < 1, \]

where \( \tilde{\omega}_1, \tilde{\omega}_2 \) and \( y \) are the same as in (24)–(26). One can see that \( Z_{\tilde{\omega}_1, \tilde{\omega}_2}^\varepsilon \) is the open \( \varepsilon \)-neighborhood of \( Z_{\tilde{\omega}_1, \tilde{\omega}_2} \) defined in (26).

Note that

\[ \text{for any } p \in Z_{\tilde{\omega}_1, \tilde{\omega}_2}^\varepsilon \text{ there exists} \]

the unique \( z(p) \in \mathbb{Z} \) such that \( |py - \pi z(p)| < \varepsilon. \)
In addition to $U_{\hat{w}_1,\hat{w}_2}$ of (35), we define

$$U_{\hat{w}_1,\hat{w}_2}^\pm(p, E) = \frac{1}{2} (U_{\hat{w}_1,\hat{w}_2}^-(p, E) + U_{\hat{w}_1,\hat{w}_2}^+(p, E)),$$

$$p_{\pm} = p_\perp + \pi z(p) \frac{y}{|y|} \pm \varepsilon \frac{y}{|y|}, \quad p_\perp = p - (py) \frac{y}{|y|}, \quad p \in B_{2\sqrt{E}} \cap Z_{\hat{w}_1,\hat{w}_2}, \quad (44)$$

where $z(p)$ is the integer number of (43). The geometry of vectors $p, p_\perp, y, p_{\pm}$ is illustrated in Fig. 1 for the case when the direction of $y$ coincides with the basis vector $e_1 = (1, 0, \ldots, 0)$.

For Problem 1 for $d \geq 2, m = 2$, and for the case when $\zeta_{\hat{w}_1,\hat{w}_2}$ has zeros on hyperplanes (the case of (26) in Section 3) we have the following result:

**Theorem 2.** Let $v$ satisfy (2) and $v \in W^{n,1}(\mathbb{R}^d)$ for some $n > d$. Let $w_1, w_2$ be the same as in (24) – (26). Let

$$u(x, E) = u_1(x, E) + u_2(x, E), \quad x \in D,$$

$$u_1(x, E) = \int_{B_{r_2}(E) \cap Z^\perp_{\hat{w}_1,\hat{w}_2}} e^{-ipx} U_{\hat{w}_1,\hat{w}_2}^-(p, E) \, dp,$$

$$u_2(x, E) = \int_{B_{r_2}(E) \cap Z^\perp_{\hat{w}_1,\hat{w}_2}} e^{-ipx} U_{\hat{w}_1,\hat{w}_2}^+(p, E) \, dp, \quad (45)$$

$$r_2(E) = 2 \tau E^{\frac{n-\beta}{2n}}, \quad \varepsilon_2(E) = E^{-\frac{\alpha_2}{2}},$$

$$\alpha_2 = \frac{n-d}{2(n+\beta+n-d)}, \quad \text{for some fixed } \tau \in (0, 1],$$

where $U_{\hat{w}_1,\hat{w}_2}$ and $U_{\hat{w}_1,\hat{w}_2}^\pm$ are defined by (35), (44), $B_r$ and $Z_{\hat{w}_1,\hat{w}_2}$ are defined...
by (33), (42), and $\beta$ is the number of (23). Then

$$u(\cdot, E) = v + O(E^{-\alpha_2}) \text{ in } L^\infty(D), \ E \to +\infty,$$

$$|u(x, E) - v(x)| \leq A_2 E^{-\alpha_2}, \ x \in D, \ E^{\frac{1}{2}} \geq \rho_2,$$

where $\rho_2$ and $A_2$ are defined in formulas (65) and (85) of Section 6.

Theorem 2 is proved in Section 6.

Next, we set

$$Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}^\varepsilon = B'_{\varepsilon/s} + \frac{\varepsilon}{s} Z^d, \ \ 0 < \varepsilon < 1,$$

$$B'_{r} = B_r \setminus \partial B_r, \ r > 0,$$

(47)

where $\hat{w}_1, \ldots, \hat{w}_{d+1}$ are the same as in (28), (29), and $B_r$ is defined by (33).

One can see that $Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}^\varepsilon$ is the open $\varepsilon$-neighborhood of $Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}$ defined in (28).

Note that for any $p \in Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}^\varepsilon$ there exists the unique $z(p) \in Z^d$ such that $|sp - \pi z(p)| < \varepsilon$.

(48)

In addition, we consider $i'$ such that

$$i' = i'(p, s), \ p = (p_1, \ldots, p_d) \in \mathbb{R}^d \setminus \frac{s}{2} Z^d, \ s > 0,$$

$$i' \text{ take values in } \{2, \ldots, d + 1\},$$

$$|\sin(sp_{i'-1})| \geq |\sin(sp_{i-1})| \text{ for all } i \in \{2, \ldots, d + 1\}.$$

(49)

Let

$$U_{\hat{w}_1, \ldots, \hat{w}_{d+1}}(p, E) = U_{\hat{w}_1, \hat{w}_{i'}}(p, E), \ p \in \mathbb{R}^d \setminus \frac{s}{2} Z^d,$$

$$U_{\hat{w}_1, \ldots, \hat{w}_{d+1}}^\varepsilon(p, E) =$$

$$\frac{1}{|S^{d-1}|} \int_{S^{d-1}} U_{\hat{w}_1, \ldots, \hat{w}_{d+1}}(\xi \theta + \frac{s}{2} z(p), E) d\theta, \ p \in Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}^\varepsilon,$$

$$S^{d-1} = \{p \in \mathbb{R}^d : |p| = 1\},$$

(50)

(51)

(52)

where $|S^{d-1}|$ denotes the standard Euclidean volume of $S^{d-1}$, $U_{\hat{w}_1, \hat{w}_{i'}}$ is defined in a similar way with $U_{\hat{w}_1, \hat{w}_{i'}}$ of (35), $\hat{w}_1, \ldots, \hat{w}_{d+1}$ are the same as in (28), (29), and $i'$ is the same as in (49).

For Problem 1 for $d \geq 2$, $m = d + 1$, we also have the following result:

**Theorem 3.** Let $v$ satisfy (2) and $v \in W^{n,1}(\mathbb{R}^d)$ for some $n > d$. Let $w_1, \ldots,
$w_{d+1}$ be the same as in (29). Let

$$u(x, E) = u_1(x, E) + u_2(x, E), \quad x \in D,$$

$$u_1(x, E) = \int_{B_{r_3}(E) \setminus Z_{\alpha_3,\ldots,\alpha_{d+1}}} e^{-ipx} U_{\hat{w}_1,\ldots,\hat{w}_{d+1}}(p, E) \, dp,$$

$$u_2(x, E) = \int_{B_{r_3}(E) \cap Z_{\alpha_3,\ldots,\alpha_{d+1}}} e^{-ipx} U_{\hat{w}_1,\ldots,\hat{w}_{d+1}}(p, E) \, dp,$$

(53)

where $U_{\hat{w}_1,\ldots,\hat{w}_{d+1}}$ and $U_{\hat{w}_1,\ldots,\hat{w}_{d+1}}^\ast$ are defined by (50), (51), $B_{r}$ and $Z_{\alpha_3,\ldots,\alpha_{d+1}}$ are defined by (33), (47), and $\beta$ is the number of (23). Then

$$|u(x, E) - v(x)| \leq A_3 E^{-\alpha_3}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_3,$$

where $\rho_3$ and $A_3$ are defined in formulas (86) and (99) of Section 7.

Theorem 3 is proved in Section 7.

5 Proof of Theorem 1

Proposition 1. Let $v$ satisfy (2) and $w_1, w_2$ be the same as in (27), $d \geq 2$. Then:

$$|\hat{w}(p) - U_{\hat{w}_1,\hat{w}_2}(p, E)| \leq c_2 |\hat{w}(p)|^{-1} E^{-\frac{d}{2}} \quad \text{for } p \in B_{2\sqrt{E}}, \quad E^{\frac{1}{2}} \geq \rho_1,$$

(55)

where $w$ is the function of (23), (24), and $c, \rho, N_j, D_j, j = 0, 1, 2$, are the same as in estimates (15).

Proposition 1 follows from formulas (11), (14), estimates (15), definitions (35)–(39), and the properties that

$$\Omega_2 = \Omega_1, \quad D_2 = D_1, \quad N_2 = N_1.$$  

(56)

In turn, properties (56) follow from (9) for $j = 1, 2$, (10) for $j = 1$, and from the equality $w_2 = i\hat{w}_1$ assumed in (27).
Next, we represent $v$ as follows:

$$v(x) = v^+(x, r) + v^-(x, r), \quad x \in D, \ r > 0,$$

$$v^+(x, r) = \int_{B_r} e^{-ipx} \check{v}(p) \, dp,$$

$$v^-(x, r) = \int_{\mathbb{R}^d \setminus B_r} e^{-ipx} \check{v}(p) \, dp.$$  \hspace{1cm} (57)

Since $v \in W^{n,1}(\mathbb{R}^d), \ n > d$, we have

$$|v^-(x, r)| \leq c_3 \|v\|_{n,1} r^{d-n}, \quad x \in D, \ r > 0,$$

where $\| \cdot \|_{n,1}$ is defined in (31), and $|S^{d-1}|$ is the standard Euclidean volume of $S^{d-1}$. Indeed,

$$|p_1^{k_1} \cdots p_d^{k_d} \check{v}(p)| \leq (2\pi)^{-d} \|v\|_{n,1}, \quad p = (p_1, \ldots, p_d) \in \mathbb{R}^d,$$

for any $k_1, \ldots, k_d \in \mathbb{N} \cup \{0\}, \ k_1 + \cdots + k_d \leq n,$

assuming also that $p_0^0 = 1$. Taking an appropriate sum in (59) over all such $k_1, \ldots, k_d$ with $k_1 + \cdots + k_d = m \leq n$, we get

$$|p|^m |\check{v}(p)| \leq (|p_1| + \cdots + |p_d|)^m |\check{v}(p)| \leq (2\pi)^{-d} d^m \|v\|_{n,1}, \quad p \in \mathbb{R}^d.$$  \hspace{1cm} (60)

The definition of $v^-$ of (57) and inequalities (60) for $m = n$ imply (58).

In addition, using Proposition 1 and the estimate on $\hat{w}$ of (23), we obtain:

$$\left| v^+(x, r) - \int_{B_r} e^{-ipx} U_{\check{w}_1, \check{w}_2}(p, E) \, dp \right| \leq c_3 E^{-\frac{1}{2}} \int_{B_r} |\hat{w}(p)|^{-1} \, dp,$$

$$\leq c_1^{-1} c_2 E^{-\frac{1}{2}} \int_{B_r} (1 + |p|)^{\beta} \, dp \leq c_4 E^{-\frac{1}{2}} r^{d+\beta}, \quad c_4 = |S^{d-1}| 2^{\frac{d+\beta}{2}} c_1^{-1} c_2,$$  \hspace{1cm} (61)

$$x \in D, \ 1 \leq r \leq 2 E^\frac{1}{2}, \ E^\frac{1}{2} \geq \rho_1.$$  \hspace{1cm} (59)

As a corollary of (57), (58), (61), we have

$$\left| v(x) - \int_{B_r} e^{-ipx} U_{\check{w}_1, \check{w}_2}(p, E) \, dp \right| \leq c_3 \|v\|_{n,1} r^{d-n} + c_4 E^{-\frac{1}{2}} r^{d+\beta},$$  \hspace{1cm} (62)

$$x \in D, \ 1 \leq r \leq 2 E^\frac{1}{2}, \ E^\frac{1}{2} \geq \rho_1.$$  \hspace{1cm} (59)

In addition, if $r = r_1(E)$, where $r_1(E)$ is defined in (40), then

$$r^{d-n} = (2\tau)^{d-n} E^{-\alpha_1},$$

$$E^{-\frac{1}{2}} r^{d+\beta} = (2\tau)^{d+\beta} E^{-\alpha_1}.$$  \hspace{1cm} (63)

Using formulas (62) and (63) and taking into account definitions (40), we obtain

$$|u(x, E) - v(x)| \leq A_1 E^{-\alpha_1}, \quad x \in D, \ E^\frac{1}{2} \geq \rho_1$$

$$A_1 = A_1(D_0, D_1, N_0, N_1, M, d, n, \beta, \tau)$$

$$= (2\tau)^{d-n} c_3 \|v\|_{n,1} + (2\tau)^{d+\beta} c_4,$$  \hspace{1cm} (64)
where $D_j$, $N_j$, $j = 0, 1$, are the same as in estimates (15) and $\|v\|_{n,1} \leq M$. Theorem 1 is proved.

6 Proof of Theorem 2

Proposition 2. Let $v$ satisfy (2) and $w_1$, $w_2$ be the same as in (24)–(26), $d \geq 2$. Then:

$$|\tilde{v}(p) - U_{\tilde{w}_1, \tilde{w}_2}(p, E)| \leq c_5 \varepsilon^{-1}(1 + |p|)\beta E^{-\frac{1}{2}},$$

$$p \in \mathcal{B}_{2\sqrt{E}} \setminus Z_{\tilde{w}_1, \tilde{w}_2}, \quad E \geq \rho_2, \quad 0 < \varepsilon < 1,$n

$$c_5 = \frac{\pi}{2}(2c(D_0)N_0^3 + c(D_1)N_1^3 + c(D_2)N_2^3)c_1,$$$$

$$\rho_2 = \max_{j=0,1,2} \rho(D_j, N_j), \quad (65)$$

in addition, if $v \in W^{n,1}(\mathbb{R}^d)$, $n \geq 0$, then:

$$|\tilde{v}(p) - U_{\tilde{w}_1, \tilde{w}_2}(p, E)| \leq 2^2 c_5 \varepsilon^{-1}(1 + |p|)^{\beta} E^{-\frac{1}{2}} + c_6 \varepsilon(1 + \frac{\pi}{2|p|}|z(p)| + |p_\perp|)^{-n},$$

$$p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\tilde{w}_1, \tilde{w}_2}, \quad E \geq \rho_2, \quad 0 < \varepsilon < \min\{1, \frac{1}{2}|y|\};$$n

$$c_6 = 2^n \frac{(d+1)^{n+1}}{(2\pi)^n |y|} \max_{j=1,\ldots,d} \|x_j v\|_{n,1}, \quad (66)$$

where $c$, $\rho$, $N_j$, $D_j$, $j = 0, 1, 2$, are the same as in estimates (15), $c_1, \beta$ are the same as in Lemma 1, $z(p), p_\perp$ are defined in (43), (44), $x_j v = x_j v(x)$, and $\|\cdot\|_{n,1}$ is defined in (31).

Proof of Proposition 2. It follows from formulas (24), (26) and (66), (67) that

$$M_{\tilde{w}_1, \tilde{w}_2}(p) = \tilde{w}(p) \begin{pmatrix} \cos(T_1 p) & \sin(T_1 p) \\ \cos(T_2 p) & \sin(T_2 p) \end{pmatrix}, \quad p \in \mathbb{R}^d,$$$$

$$M_{\tilde{w}_1, \tilde{w}_2}^{-1}(p) = \frac{1}{\sin(py)\tilde{w}(p)} \begin{pmatrix} \sin(T_2 p) & -\sin(T_1 p) \\ -\cos(T_2 p) & \cos(T_1 p) \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus Z_{\tilde{w}_1, \tilde{w}_2}. \quad (67)$$

Also note that

$$|\sin(py)| \geq \frac{2\pi}{r}, \quad p \in \mathbb{R}^d \setminus Z_{\tilde{w}_1, \tilde{w}_2}, \quad 0 < \varepsilon < 1.$$n

The estimate (65) follows from (11), (15), (23), (35), (38), (39) and from (67), (68).

It remains to prove (66). Using definition (44), one can write

$$U_{\tilde{w}_1, \tilde{w}_2}^\varepsilon(p, E) - \tilde{v}(p) = \varphi_1^\varepsilon(p, E) + \varphi_2^\varepsilon(p), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\tilde{w}_1, \tilde{w}_2},$$

$$\varphi_1^\varepsilon(p, E) = \frac{1}{2}\left(U_{\tilde{w}_1, \tilde{w}_2}^\varepsilon(p, E) - \tilde{v}(p^\varepsilon) \right)$$n

$$+ \frac{1}{2}\left(U_{\tilde{w}_1, \tilde{w}_2}^\varepsilon(p_\perp, E) - \tilde{v}(p_\perp^\varepsilon) \right), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\tilde{w}_1, \tilde{w}_2},$$n

$$\varphi_2^\varepsilon(p) = \frac{1}{2}\left(\tilde{v}(p^\varepsilon) + \tilde{v}(p_\perp^\varepsilon) \right) - \tilde{v}(p), \quad p \in Z_{\tilde{w}_1, \tilde{w}_2}.$$n

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Using estimate (65), formula (69) and the definitions of \( p^c_\perp \) in (44), we get

\[
|\varphi_1^{\ast}(p, E)| \leq \frac{1}{2} c_5 \varepsilon^{-1} E^{-1} \left((1 + |p^c_\perp|)^{\beta} + (1 + |p^c_\perp|)^{\beta}\right)
\]

\[
\leq c_5 \varepsilon^{-1} (1 + |p| + 2 \frac{\varepsilon}{|p|})^\beta E^{-\frac{1}{2}} \leq 2^\beta c_5 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}},
\]

for \( \varepsilon \) as in (66), \( p \in B_{2\sqrt{E}} \cap Z^\varepsilon_{\hat{w}_1, \hat{w}_2}. \)

Next, using the definition of \( \varphi_2^{\ast} \) in (69) and the mean value theorem, we obtain

\[
|\varphi_2^{\ast}(p)| \leq \frac{E}{|p|} \max\{|\nabla \hat{v}(\xi)| : \xi \in [p^c_\perp, p^c_\perp^+, p^c_\perp^+]|, \quad p \in Z^\varepsilon_{\hat{w}_1, \hat{w}_2}, \tag{70}
\]

where \([p^c_\perp, p^c_\perp^+] \) denotes the segment joining \( p^c_\perp \) to \( p^c_\perp^+ \). Here, the mean value theorem was used for \( \hat{v}(\xi) \) on \([p^c_\perp, p]\) and on \([p, p^c_\perp^+] \).

Note also that

\[
|\nabla \hat{v}(\xi)| \leq d \max_{j=1, \ldots, d} |\frac{\partial \hat{v}}{\partial \xi_j}(\xi)|, \quad \xi = (\xi_1, \ldots, \xi_d) \in [p^c_\perp, p^c_\perp^+]. \tag{71}
\]

In addition, the following estimates hold:

\[
\left| \frac{\partial \hat{v}}{\partial \xi_j}(\xi) \right| \leq \frac{(1 + d)^n}{(2\pi)^d (1 + |\xi|)^n} \|x_j v\|_{n,1}, \quad \xi \in [p^c_\perp, p^c_\perp^+], \quad j = 1, \ldots, d. \tag{72}
\]

Indeed, taking the sum in (60) over all \( m = 0, \ldots, n \) with the binomial coefficients, we get

\[
(1 + |p|)^n |\hat{v}(p)| \leq (1 + |p_1| + \cdots + |p_d|)^n |\hat{v}(p)|
\]

\[
\leq (2\pi)^{-d}(1 + d)^n \|v\|_{n,1}, \quad p \in \mathbb{R}^d. \quad \tag{73}
\]

Estimates (73) follow from (74), where we replace \( v \) by \( x_j v \) and use that \( v \) belongs to \( W^{n,1}(\mathbb{R}^d) \) and is compactly supported.

Estimates (71)–(73) imply

\[
|\varphi_2^{\ast}(p)| \leq 2^{-n} c_6 \varepsilon \max\left\{1 + |\xi|\right\}^{-n} : \xi \in [p^c_\perp, p^c_\perp^+], \quad p \in Z^\varepsilon_{\hat{w}_1, \hat{w}_2}. \tag{75}
\]

Using also that

\[
\xi = \tau \frac{p}{|p|} + p_\perp, \quad \text{where} \quad |\tau - \frac{p}{|p|} z(p)| \leq \frac{\tau}{|p|}, \quad \text{if} \quad \xi \in [p^c_\perp, p^c_\perp^+], \tag{76}
\]

and that \( \varepsilon < |y| \), we obtain

\[
|\varphi_2^{\ast}(p)| \leq 2^{-n} c_6 \varepsilon \left(1 + \frac{1}{2} \left( \frac{\tau}{|p|} |z(p)| - \frac{\tau}{|p|} + |p_\perp| \right) \right)^{-n}
\]

\[
\leq c_6 \left(1 + \frac{\tau}{|p|} |z(p)| + |p_\perp| \right)^{-n}, \quad p \in Z^\varepsilon_{\hat{w}_1, \hat{w}_2}. \tag{77}
\]

Estimate (66) follows from (70) and (77).

Proposition 2 is proved.
The final part of the proof of Theorem 2 is as follows. In a similar way with \[57\], we represent \(v\) as follows: \(v(x) = v_1^+(x, r) + v_2^+(x, r) + v^-(x, r), \ x \in D, \ r > 0,\)

\[
v_1^+(x, r) = \int_{B_r \setminus Z_{\psi}^1, \phi_2} e^{-i p x} \tilde{v}(p) \, dp,
\]

\[
v_2^+(x, r) = \int_{B_r \cap Z_{\psi}^1, \phi_2} e^{-i p x} \tilde{v}(p) \, dp,
\]

\[
v^-(x, r) = \int_{\mathbb{R}^d \setminus B_r} e^{-i p x} \tilde{v}(p) \, dp.
\]

(78)

Since \(v \in W^{n,1}(\mathbb{R}^d)\), estimate \[58\] holds.

Using estimates \[55\], \[60\], we get:

\[
\left| v_1^+(x, r) - \int_{B_r \setminus Z_{\psi}^1, \phi_2} e^{-i p x} U_{\tilde{\psi}_1, \tilde{\phi}_2}(p, E) \, dp \right| + v_2^+(x, r) - \int_{B_r \cap Z_{\psi}^1, \phi_2} e^{-i p x} U_{\psi_1, \phi_2}(p, E) \, dp \leq I_1 + I_2,
\]

(79)

\[
I_1 = 2^3 c_5 \varepsilon^{-1} E^{-\frac{1}{2}} \int_{B_r} (1 + |p|)^\beta \, dp,
\]

(80)

\[
I_2 = c_6 \varepsilon \int_{B_r \cap Z_{\psi}^1, \phi_2} (1 + \frac{\pi}{|y|} |z(p)| + |p_\perp|)^{-n} \, dp,
\]

(81)

where \(\rho_2\) is the same as in Proposition 2. In addition:

\[
I_1 \leq c_7 \varepsilon^{-1} E^{-\frac{1}{2}} |p|^{d+\beta}, \quad c_7 = |S|^{d-1} \frac{\rho^{d+2\beta}}{\alpha^{d+\beta}} c_5;
\]

(82)

\[
I_2 = c_6 \varepsilon \sum_{z \in \mathbb{Z}} \int_{B_{\tau + p_\perp}^2, \phi_2} (1 + \frac{\pi}{|y|} |z| + |p_\perp|)^{-n} \, dp_\perp,
\]

where \(\tau \in \mathbb{R}, \ p_\perp \in \mathbb{R}^d, \ p_\perp \cdot y = 0,\)

\[
I_2 \leq 2 c_6 \varepsilon \sum_{z \in \mathbb{Z}} \int_{\mathbb{R}^{d-1}, |\xi| \leq r} (1 + \frac{\pi}{|y|} |z| + |\xi|)^{-n} \, d\xi \leq c_8 c_6 \varepsilon^2,
\]

(83)

\[
c_8 = 2 \frac{|S|^{d-2}}{|y| n - d + 1} \sum_{z \in \mathbb{Z}} (1 + \frac{\pi}{|y|} |z|)^{d-n-1}.
\]

In addition, if \(r = r_2(E), \ \varepsilon = \varepsilon_2(E), \) where \(r_2(E), \ \varepsilon_2(E)\) are defined in \[45\], then

\[
r^{d-n} = (2\tau)^{d-n} E^{-\alpha_2},
\]

\[
\varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta} = (2\tau)^{d+\beta} E^{-\alpha_2},
\]

(84)

\[
\varepsilon^2 = E^{-\alpha_2}.
\]
Using representation \(^{(78)}\), estimates \(^{(58)}\), \(^{(79)}\), \(^{(82)}\), \(^{(83)}\), formulas \(^{(84)}\) and taking into account definitions \(^{(45)}\), we obtain
\[
|u(x, E) - v(x)| \leq A_2 E^{-\alpha_2}, \quad x \in D, \quad E^{\frac{1}{2}} \geq \rho_2,
\]
\[
A_2 = A_2(D_0, D_1, D_2, N_0, N_1, N_2, M, M_1, \ldots, M_d, d, n, \beta, \tau, |y|) \quad (85)
\]
\[
= (2\tau)^{d+c} c_7 + c_8 c_9 + (2\tau)^{d-c_9} c_9 ||v||_{n,1},
\]
where \(D_j, N_j\) are the same as in \(^{(15)}\) and \(||v||_{n,1} \leq M, \|x_j v\|_{n,1} \leq M_j\).

Theorem 2 is proved.

7 Proof of Theorem 3

Proposition 3. Let \(v\) satisfy \(^{(2)}\) and \(w_1, \ldots, w_{d+1}\) be the same as in \(^{(24)}\), \(^{(29)}\), \(d \geq 2\). Then:
\[
|\hat{v}(p) - U_{\hat{w}_1, \ldots, \hat{w}_{d+1}}(p, E)| \leq c_9 \varepsilon^{-1}(1 + |p|)^{\beta} E^{-\frac{1}{2}},
\]
\[
p \in B_{2\sqrt{E}} \setminus Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}, \quad E^{\frac{1}{2}} \geq \rho_3, \quad \varepsilon \leq 1,
\]
\[
c_9 = \frac{\pi^2}{2} (2c(D_0)N_0^3 + c(D_1)N_1^3 + \max_{j=2, \ldots, d+1} c(D_j)N_j^3) c_1,
\]
\[
\rho_3 = \max_{j=0, \ldots, d+1} \rho(D_j, N_j),
\]
in addition, if \(v \in W^{n,1}(\mathbb{R}^d), n \geq 0\), then:
\[
|\hat{v}(p) - U_{\hat{w}_1, \ldots, \hat{w}_{d+1}}(p, E)|
\]
\[
\leq 2^3 c_9 \varepsilon^{-1}(1 + |p|)^{\beta} E^{-\frac{1}{2}} + 2c_6 \varepsilon (1 + 2 \frac{\pi^2}{2} ||z(p)||_2)^{-n},
\]
\[
p \in B_{2\sqrt{E}} \cap Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}, \quad E^{\frac{1}{2}} \geq \rho_3, \quad \varepsilon \leq \min\{1, \frac{1}{2} \varepsilon\},
\]
where \(c, \rho, D_j, N_j, j = 0, \ldots, d+1\), are defined as in \(^{(15)}\); \(c_1, \beta\) are the same as in Lemma \(^{(4)}\); \(c_6\) is the same as in Proposition \(^{(3)}\); \(z(p)\) is defined in \(^{(48)}\) and \(||z(p)||_2\) is the standard Euclidean norm of \(z(p)\).

Proof of Proposition 3. In a similar way with formulas \(^{(67)}\), one can write
\[
M_{\hat{w}_1, \hat{w}_n}(p) = \hat{w}_1(p) \begin{pmatrix} 1 & 0 \\ \cos(\varepsilon p_{i'} - 1) & \sin(\varepsilon p_{i'} - 1) \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus \frac{\pi}{\pi + \delta},
\]
\[
M_{\hat{w}_1, \hat{w}_n}^{-1}(p) = \frac{1}{\sin(\varepsilon p_{i'} - 1)} \hat{w}_1(p) \begin{pmatrix} \sin(\varepsilon p_{i'} - 1) & 0 \\ \cos(\varepsilon p_{i'} - 1) & 1 \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}},
\]
where \(i' = i'(p, \epsilon)\) is defined in \(^{(49)}\). Also note that
\[
|\sin(\varepsilon p_{i'} - 1)| \geq \frac{2\pi}{\pi + \delta}, \quad p \in \mathbb{R}^d \setminus Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}, \quad 0 < \varepsilon < 1.
\]
Estimate \(^{(86)}\) follows from \(^{(11)}\), \(^{(15)}\), \(^{(23)}\), \(^{(35)}\), \(^{(38)}\), \(^{(39)}\), \(^{(49)}\), \(^{(50)}\) and from \(^{(88)}\), \(^{(89)}\).
It follows from (92), (93) and from the upper estimate on \( \varepsilon \) we represent

\[
U_{\tilde{w}_1, \ldots, \tilde{w}_{d+1}}(p, E) - \hat{\nu}(p) = \varphi_1^z(p, E) + \varphi_2^z(p), \quad p \in B_2 \cap Z_{\tilde{w}_1, \ldots, \tilde{w}_{d+1}}
\]

\[
\varphi_1^z(p, E) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} (U_{\tilde{w}_1, \ldots, \tilde{w}_{d+1}}(\eta, E) - \hat{\nu}(\eta)) \, d\vartheta, \quad (90)
\]

\[
\varphi_2^z(p) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} (\hat{\nu}(\tilde{z} \vartheta + \frac{z}{\sqrt{s}}(p)) - \hat{\nu}(p)) \, d\vartheta,
\]

where \( z(p) \) is defined in (48).

Using formulas (86), (90), we obtain

\[
|\varphi_1^z(p, E)| \leq c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \frac{1}{|S^{d-1}|} \int_{S^{d-1}} (1 + |\tilde{z} \vartheta + \frac{z}{\sqrt{s}}(p)|)^{\beta} \, d\vartheta
\]

\[
\leq c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \frac{1}{|S^{d-1}|} \int_{S^{d-1}} (1 + |p| + 2 \tilde{z})^\beta \, d\vartheta \leq 2^\beta c_9 \varepsilon^{-1} (1 + |p|)^{\beta} E^{-\frac{1}{2}}, \quad (91)
\]

for \( \varepsilon \) as in (87).

Next, using the definition of \( \varphi_2^z \) in formula (90) and the mean value theorem, we get the following estimate:

\[
|\varphi_2^z(p)| \leq 2 \tilde{z} \max \{ |\nabla \hat{\nu}(\xi)| : \xi \in \mathbb{R}^d, |\xi - \frac{z}{\sqrt{s}}(p)| \leq \frac{\varepsilon}{\sqrt{s}} \}, \quad p \in Z_{\tilde{w}_1, \ldots, \tilde{w}_{d+1}}. \quad (92)
\]

Here, the mean value theorem was used for \( \hat{\nu}(\xi) \) on \([p, \tilde{z} \vartheta + \frac{z}{\sqrt{s}}(p)], \vartheta \in S^{d-1}\). One can see that estimates (72) and (73) hold for all \( \xi \in \mathbb{R}^d \) such that \( |\xi - \frac{z}{\sqrt{s}}(p)| \leq \frac{\varepsilon}{\sqrt{s}} \), where \( p \in Z_{\tilde{w}_1, \ldots, \tilde{w}_{d+1}} \).

It follows from (92), (93) and from the upper estimate on \( \varepsilon \) of (87), that

\[
|\varphi_2^z(p)| \leq 2^{1-n} c_9 \varepsilon \max \{ (1 + |\xi|)^{-n} : |\xi - \frac{z}{\sqrt{s}}(p)| \leq \frac{\varepsilon}{\sqrt{s}} \}
\]

\[
\leq 2^{1-n} c_9 \varepsilon (1 + \frac{\varepsilon}{\sqrt{s}} \|z(p)\|_2 - \frac{\varepsilon}{\sqrt{s}})^{-n} \leq 2 c_9 \varepsilon (1 + 2 \frac{\varepsilon}{\sqrt{s}} \|z(p)\|_2)^{-n}, \quad p \in Z_{\tilde{w}_1, \ldots, \tilde{w}_{d+1}}. \quad (94)
\]

Estimate (87) follows from estimates (91) and (94).

Proposition 3 is proved. \( \square \)

The final part of the proof of Theorem 3 is as follows. In a similar way with (78), we represent \( v \) as follows:

\[
v(x) = v_1^+(x, r) + v_2^+(x, r) + v^-(x, r), \quad x \in D, \ r > 0,
\]

\[
v_1^+(x, r) = \int_{B_r \cap Z_{\tilde{w}_1, \ldots, \tilde{w}_{d+1}}} e^{-ipx} \hat{\nu}(p) \, dp,
\]

\[
v_2^+(x, r) = \int_{B_r \cap Z_{\tilde{w}_1, \ldots, \tilde{w}_{d+1}}} e^{-ipx} \hat{\nu}(p) \, dp,
\]

\[
v^-(x, r) = \int_{\mathbb{R}^d \setminus B_r} e^{-ipx} \hat{\nu}(p) \, dp. \quad (95)
\]
Since \( v \) belongs to \( W^{n,1}(\mathbb{R}^d) \), estimate (58) is valid. Using estimates (86), (87) we obtain

\[
\left| v_1^+ (x, r) - \int_{B_r \setminus Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}} e^{-ip\xi} U_{\hat{w}_1, \ldots, \hat{w}_{d+1}} (p, E) \, dp \right| \leq J_1 + J_2, \\
\left| v_2^+ (x, r) - \int_{B_r \cap Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}} e^{-ip\xi} U_{\hat{w}_1, \ldots, \hat{w}_{d+1}} (p, E) \, dp \right| \leq J_1 + J_2,
\]

where \( \rho_3 \) is the same as in Proposition 3. In addition,

\[
J_1 = 2^\beta c_9 \varepsilon^{-1} E^{-\frac{\beta}{2}} \int_{B_r} (1 + |p|)^\beta \, dp, \\
J_2 = 2 c_6 \varepsilon \int_{B_r \cap Z_{\hat{w}_1, \ldots, \hat{w}_{d+1}}} \left( 1 + 2 \frac{z(p)}{s} \|z(p)\|_2 \right)^{-n} \, dp,
\]

where \( |B_1| \) is the standard Euclidean volume of \( B_1 \). Finally, if \( r = r_3(E), \varepsilon = \varepsilon_3(E) \), where \( r_3(E), \varepsilon_3(E) \) are defined in (53), then

\[
\varepsilon^{-1} E^{-\frac{\beta}{2}} r^{d+\beta} = (2r)^{d-n} E^{-\alpha_3}, \\
\varepsilon^{-1} E^{-\frac{\beta}{2}} r^{d+\beta} = (2r)^{d-n} E^{-\alpha_3}, \\
\varepsilon^{d+1} = E^{-\alpha_3}.
\]

Using representation (95), estimates (58), (96), (97), formulas (98) and taking into account definitions (53), we obtain

\[
|u(x, E) - v(x)| \leq A_3 E^{-\alpha_3}, \quad x \in D, \quad E^{\frac{\beta}{2}} \geq \rho_3, \\
A_3 = A_3(D_0, \ldots, D_{d+1}, N_0, \ldots, N_{d+1}, M, d, n, \beta, \tau, s) = (2r)^{d+\beta} c_{10} + c_{11} + (2r)^{d-n} c_3 \|v\|_{n,1}.
\]

where \( \|v\|_{n,1} \leq M \) and \( D_0, \ldots, D_{d+1}, N_0, \ldots, N_{d+1} \) are the same as in Proposition 3.

Theorem 3 is proved.

8 Proof of Lemma 1

Note that

\[
\tilde{w}(p) = \int_{\mathbb{R}^d} |\tilde{q}(\xi)|^2 \tilde{w}(p - \xi) \, d\xi, \quad p \in \mathbb{R}^d, \\
\tilde{w}_{\nu}(p) = |p|^{\nu} K_{\nu}(|p|), \quad p \in \mathbb{R}^d,
\]

where \( \tilde{w}(p) \) and \( \tilde{w}_{\nu}(p) \) are the Fourier transforms of \( \omega_1^+ (x) = |p|^{\nu} \tilde{w}(p) \) and \( \omega_2^+ (x) = |p|^{\nu} \tilde{w}_{\nu}(p) \), respectively. In addition, \( \omega_1^+ (x) \) and \( \omega_2^+ (x) \) satisfy

\[
|\omega_1^+ (x)| \leq M_1 \|v\|_{n,1}, \quad x \in D, \\
|\omega_2^+ (x)| \leq M_2 \|v\|_{n,1}, \quad x \in D,
\]

where \( M_1 \) and \( M_2 \) are constants depending on \( A_3 \) and \( \|v\|_{n,1} \). Therefore, the estimate (58) is valid.

Using estimates (86), (96), (97), formulas (98) and taking into account definitions (53), we obtain

\[
|u(x, E) - v(x)| \leq A_3 E^{-\alpha_3}, \quad x \in D, \quad E^{\frac{\beta}{2}} \geq \rho_3,
\]

where \( \|v\|_{n,1} \leq M \) and \( D_0, \ldots, D_{d+1}, N_0, \ldots, N_{d+1} \) are the same as in Proposition 3.

Theorem 3 is proved.
where $\hat{q}, \hat{\omega}_\nu$ are the Fourier transforms of $q, \omega_\nu$. The Fourier transform $\hat{\omega}_\nu$ can be computed explicitly:

$$\hat{\omega}_\nu(p) = \frac{c_{12}}{(1 + |p|^2)^{\frac{d}{2} + \nu}}, \quad c_{12} = \frac{\Gamma\left(\frac{d}{2} + \nu\right)2^{\nu-1}}{\pi^{\frac{d}{2}}}.$$

(102)

Indeed, formula (102) follows from the Fourier inversion theorem and the following computations:

$$\int_{\mathbb{R}^d} e^{-i\xi \cdot \tau} d\tau = \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \frac{e^{-i|\xi| t} dt d\xi}{(1 + \tau^2 + |\xi|^2)^{\frac{d}{2} + \nu}} = |S^{d-2}| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-i|x| t} dt}{(1 + t^2 + \tau^2)^{\frac{d}{2} + \nu}} \int_{0}^{+\infty} \frac{\tau^{d-2} d\tau}{(1 + \tau^2)^{\frac{d}{2} + \nu}} = c_{12}^{-1} |x|^{\nu} K_\nu(|x|), \ x \in \mathbb{R}^d.$$

Here, it was used that

$$\int_{0}^{+\infty} \frac{\tau^{d-2} d\tau}{(1 + \tau^2)^{\frac{d}{2} + \nu}} = \frac{1}{2} B\left(\frac{d-1}{2}, \nu + \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{d}{2} + \nu\right)},$$

where $B$ and $\Gamma$ denote the beta and gamma functions.

Using (100), (102), we obtain the estimates

$$\hat{\omega}(p) \geq \int_{|\xi| \leq 1} \frac{c_{12} |\hat{q}(\xi)|^2}{(1 + |p - \xi|^{d+2\nu})^2} d\xi \geq \frac{c_1(q, \nu)}{(1 + |p|)^{d+2\nu}}, \ p \in \mathbb{R}^d,$$

(103)

$$c_1(q, \nu) = \frac{c_{12}}{2^d + 2\nu} \int_{|\xi| \leq 1} |\hat{q}(\xi)|^2 d\xi.$$

Properties (23) follow from (20), (22), (100) and (103).

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