Actuation attacks on constrained linear systems: a set-theoretic analysis

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Abstract: This paper considers a constrained discrete-time linear system subject to actuation attacks. The attacks are modelled as false data injections to the system, such that the total input (control input plus injection) satisfies hard input constraints. We establish a sufficient condition under which it is not possible to maintain the states of the system within a compact state constraint set for all possible realizations of the actuation attack. The developed condition is a simple function of the spectral radius of the system, the relative sizes of the input and state constraint sets, and the proportion of the input constraint set allowed to the attacker.

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1. INTRODUCTION

The security of control systems to cyber-attacks has become a pressing issue, owing to the ubiquity of computers and networks and the vulnerabilities that these introduce (Smith, 2015). In the context of feedback control, attention has focused on several salient aspects of the cyber-security problem, including attack detection, synthesis and the analysis of control system stability and performance under different classes of attack, including denial of service (DoS), deception and false data injection (FDI) (Pasqualetti et al., 2013; Teixeira et al., 2015).

In this paper, we study a simple instance of an actuation attack problem—a type of data injection attack—and, using set-theoretic methods, develop fundamental conditions under which it is not possible to robustly defend the system. In particular, we consider the problem of maintaining the states of a constrained linear system within a given state target set while it is subject to adversarial input disturbances. We consider that the input constraint set is partitioned, via a scaling factor, into two portions: the control input is selected from one portion, and the attack input from the other, such that the overall input applied is constraint admissible. The main result of the paper is the characterization of a lower bound on the constraint scaling factor such that robust stabilization of the system—and infinite-time robust constraint satisfaction—for all realizations of the attack is not possible.

We note that although the state-feedback setting is simpler than that typically considered in the cyber-security literature—and renders certain aspects of the problem, such as stealth and detection, trivial—the results we obtain offer some insights into the relative ease of attacking a system according to its dynamics and constraints. The developed bound on the scaling factor depends, in a natural way, on the open-loop stability of the system, via its spectral radius, and the relative shapes and sizes of the input and state constraint sets. Following intuition, the bound confirms that more unstable systems with smaller target sets are easier to attack, in that the proportion of the input constraint set required by the attacker is smaller, which may have implications for the signal power and energy required for a successful attack.

A few other papers have used set-theoretic techniques in the context of cyber-security. Lucia et al. (2016) propose a receding-horizon control law utilizing robust reachability sets in order to mitigate FDI and DoS attacks. Mohajerin Esfahani et al. (2010) and Mo and Sinopoli (2012) use reachability analysis in order to characterize the impact of FDI attacks. The most closely related work, however, appears to be from outside of this literature: Schulze Darup et al. (2017) considered a constrained linear (open-loop stable) autonomous system subject to additive disturbances selected from scaled disturbance set, and developed lower and upper bounds on the critical scaling factor at which robust infinite-time constraint satisfaction is not possible. The present paper considers non-autonomous systems rather than autonomous ones, however, and the techniques employed are necessarily different in order to handle the possibility of open-loop instability.

The organization of this paper is as follows. Section 2 gives the problem statement, which is followed by a preliminary analysis in Section 3. In Section 4, we recall some established results on robust constraint admissible and control invariant sets, and develop some new ones that facilitate our developments. The main results of the paper are presented in Section 5, and are subsequently illustrated in Section 6. Section 7 contains a discussion of the results and the conservativeness of the bounds we develop. Conclusions and directions for future work are
We begin with some definitions relevant to the problem, and then link these to known concepts and results in constrained control. The following refer to the system (2) and constraints \((x_k, v_k, a_k) \in X \times (1 - \alpha)U \times \alpha U\).

**Definition 2.** (Attack and defence sets and strategies). The admissible attack \{defence\} set is \(aU \{1 - (1 - \alpha)U\},\) with \(\alpha \in [0,1].\) An admissible attack \{defence\} is an action \(a_k \in aU\ \{1 - (1 - \alpha)U\}\). An admissible attack \{defence\} strategy is a policy \(x \mapsto v \in aU\ \{1 - (1 - \alpha)U\}\).

**Definition 3.** (Undefendable and defendable attack set). An attack set \(aU\) is said to be undefendable for the system (2) if, for all \(x_0 \in X\), there does not exist an admissible defence strategy that maintains \(x_k \in X\) for all \(k \geq 0\). Otherwise, an attack set is said to be defendable.

There is a direct link and equivalence between these definitions and established concepts in the literature: infinite reachability, strong reachability and robust control invariance (Bertsekas (1972); Blanchini (1999); Kerrigan, 2000).

**Definition 4.** (Bertsekas (1972)). A set \(Y \subset \mathbb{R}^n\) is:

1. **Infinitely reachable** if there exists a control law \(\mu(\cdot)\) and some \(x_0 \in Y\) such that \(x_k \in Y\) and \(v_k = \mu(x_k) \in (1 - \alpha)U\) for all \(a_k \in aU\).
2. **Strongly reachable or robust control invariant** (RCI) if there exists a control law \(\mu(\cdot)\) such that for all \(x_0 \in Y\), \(x_k \in Y\) and \(v_k = \mu(x_k) \in (1 - \alpha)U\) for all \(a_k \in aU\).

The link to defendability follows trivially.

**Lemma 5.** The attack set \(aU\) is defendable if, and only if, \(X\) is infinitely reachable. \(X\) is infinitely reachable if, and only if, it contains a robust control invariant set \(C\).

**Remark 6.** In establishing a link between these concepts and the results later in the paper, we make a tacit assumption on the information pattern in the problem: the defender selects \(v_k\) with knowledge of \(x_k\) but without knowledge of \(a_k\), while the attacker may have knowledge of both \(x_k\) and \(v_k\). Moreover, we tacitly assume that both attacker and defender know the value of \(\alpha\).

This motivates the remainder of the paper. The question of whether an attack set is defendable or undefendable (as these concepts are defined) amounts exactly to whether or not the state constraint set \(X\) contains an RCI set. If it does, then the attack set can be said to be defendable and standard techniques from robust constrained control can be used to keep the state within \(X\). If it does not, then an attack set is undefendable, and there does not exist any defence strategy that keeps the state within \(X\) for all time, accounting for all possible actions of the attacker. Our more concrete aim is, therefore, to characterize the relation between the constraint scaling factor \(\alpha\) and the existence of an RCI set within \(X\).

**4. ROBUST CONSTRAINT-ADMISSIBLE AND CONTROL INVARIANT SETS**

First we present some known results and new results regarding RCI sets, with respect to a general linear system

\[
x_{k+1} = Ax_k + Bu_k + Ew_k,
\]

\((x_k, u_k, w_k) \in X \times U \times W.\)
These are subsequently specialized to the setting described in the previous section.

4.1 Some known results

The $i$-step robust constraint-admissible set is the set of all states that can be kept within $X$ for at least $i$ time steps, for any disturbance, respecting the input constraints:

$$C_i := \{ x : \exists u_i \in U_i \text{ such that } x_i \in X_i \text{ for all } u_i \in W_i \}$$

where $u_i$ (respectively $w_i$) is the sequence of $i$ controls $\{u_0, u_1, \ldots, u_{i-1}\}$ (disturbances $\{w_0, w_1, \ldots, w_{i-1}\}$), the set $U_i \equiv U \times \cdots \times U$, with a similar definition for $W$ and $W$. The corresponding sequence $x_i = \{x_0, x_1, \ldots, x_i\}$ is obtained by, starting from $x_0$, applying the input sequence $u_j$ and disturbance sequence $w_j$. The definition requires $x_i \in X_i \equiv X \times \cdots \times X$.

We begin with some definitions relevant to the problem:

We consider a discrete-time, linear time-invariant system,

$$X = \{ x \in \mathbb{R}^n : x_0 \neq \infty \}$$

$$W = \{ w \in \mathbb{R}^n : w_0 \neq \infty \}$$

The pair $A, B$ is left with the remaining proportion $\alpha$ and attacks take place via an attacker gaining access to, and mine an admissible attack strategy that achieves the goal of steering $X$ to $Y$, and the results later in the paper, we make a tacit assumption that both attacker and defender know the value of $\alpha$.

Moreover, we tacitly assume that both $a$, $b$, $i$, $j$, $k$, $\mu$, $\nu$, $\sigma$ are positive integers.

Lemma 5. The attack set $\alpha U$ is a robust control invariant set for the system (3).

The results are central to the developments in the next section in the particular setting of the paper, and develop similar results of Schulze Darup et al. (2017) in the next section, when we specialize to the input-attack setting.

Proposition 12. If, for some $i^* > 0$, $T_{i^*} = \emptyset$ then $C_i = \emptyset$ for all $i \geq i^*$.

Proposition 13. If, for some $i^* > 0$, $S_{i^*} = \emptyset$, where

$$S_i \equiv X \times \begin{bmatrix} i-2 \oplus A^j B(-U) \oplus i-1 \oplus A^j EW \end{bmatrix} \cup_{j=0}^{i-1},$$

then $T_i = \emptyset$ for all $i \geq i^*$.

5. FOR WHICH VALUES OF $\alpha$ IS AN ATTACK SET UNDEFENDABLE?

We now recast some of the results from the previous section in the particular setting of the paper, and develop conditions under which $C_{\infty}$ does not exist.

Specializing the definitions of $C_i$ and $T_i$ to the system (2) and constraints $(x_k, u_k, a_k) \in X \times (1 - \alpha)U \times \alpha U$, and exploiting the symmetry of $U$, we obtain

$$C_i \equiv A^{-1}(\{C_i \oplus \alpha BU, (1 - \alpha)BU\}) \cap X$$

with $C_0 \equiv X$, and

$$T_{i+1} = (T_i \oplus A^i EW) \oplus (1 - \alpha)(A^i BU)$$

with $T_0 = X$, where the sets are super-indexed by $\alpha$ to denote their dependency on this scaling factor. The connection between the two is, following Proposition 9,
The bounds obtained here provide insight into the relative ease of attacking a system depending on its dynamics and constraints. More specifically, the critical scaling factor depends on the most unstable eigenvalue of the system and the relative sizes of the state and input constraint sets in the direction of the corresponding eigenvector. The result implies that unstable systems are easier to attack (for example, if \( \rho_A > 1 \) then \( \bar{\alpha}_\infty < 1/2 \), so the attack set does not need to be as large as the defence set to render the system undefendable) and also that (un)defendability depends on the relative sizes of the sets \( Bu \) (the mapped inputs) and \( X \) (for example, even if \( \rho_A = 0 \), the system can be rendered undefendable if \( \bar{\alpha}_\infty < 1 \), which requires \( H_X(z_A) < 1 \Rightarrow h_B(z_A) > h_X(z_A) \)).

Remark 19. It should be noted that Assumption 14 places restrictions only on the dominant eigenvalues. Under this assumption, the long-term critical evolution of the set \( A^tEW = \alpha A^t-BU \), by which the intermediate sets are restricted, is in the direction \( z_A \), which enables the simple result obtained. It is possible to extend the result to more general \( A \) matrices, such as those with complex dominant poles. However, because the long-term growth of the set \( \alpha A^tBU \) is then not in a single direction, it is more involved to determine the number of steps after which the set \( T_i \) becomes empty.

Remark 20. It is interesting to note that the derived bound is consistent with the obvious strategy for attacking an unstable system, namely \( a(x,v) = -v(x) \) if the information pattern permits it, which guarantees that the state leaves \( X \) in finite time. Indeed, since \( \bar{\alpha}_\infty < 1/2 \) for all \( \rho_A > 1 \), the attacker can choose \( \alpha = 1/2 \), permitting this strategy.

6. ILLUSTRATIVE EXAMPLES

We illustrate the results via three example systems:

\[
S_1: A = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.7 \end{bmatrix} \quad S_2: A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad S_3: A = \begin{bmatrix} 1.9 & 1.1 \\ 0.5 & 1.5 \end{bmatrix}
\]

where, in each case, \( B = [0.5 \ 1] \top \). The sets \( X \) and \( U \) are the unit hypercubes.

First we illustrate Proposition 9. Fig. 1 compares, for system \( S_1 \) and a scaling factor of \( \alpha = 0.25 \), the three-step constraint admissible set \( C_3 \) with the outer bounding set \( \bigcap_{j=0}^{3} A^{-j}T_j \) derived in Proposition 9. The inclusion is not tight, as pointed out in Remark 10.

Next, we illustrate and investigate the result of Theorem 16 and its corollaries. Figure 2 shows, for the systems \( S_1 \) to
The following assumption is key to the development and

\textit{Suppose Assumptions 1, 14 and 15 hold. If, for some} \( \alpha_U \),

\( \forall i \in N \), define \( \rho \) as the spectral radius of \( A \), the dominant eigenvalue of \( \rho(\cdot, \cdot, \cdot) \), for which \( \alpha > \alpha \), \( \forall i \in N \), then

\( \exists i \in N \) such that

\begin{align*}
\alpha &< \alpha \\text{ and } \rho(\cdot, \cdot, \cdot) \text{ is empty.}
\end{align*}

We achieve this by characterizing, for each \( S \), \( \alpha > \alpha \), \( \forall i \in N \).

\[ \star \]

\( \text{We illustrate the results via three example systems:} \]

\[ \star \]

\( \text{Propositions 12 and 13, any } \alpha > \alpha \text{ for which } \]

\( \text{the mapped set } \]

\( \text{is empty:} \]

\[ \star \]

\( \text{Remark 19.} \]

\text{It should be noted that Assumption 14 places}

\( \text{unusual assumptions on the system and constraints would be required.} \]

\( \text{Similarly, Proposition 13 is merely sufficient to ensure emptiness of } T_i \text{ at a given } i = i^*. \) The source of conservatism again arises from the basic properties of Minkowski addition and subtraction, and in particular that these operations do not commute: Proposition 13 uses the fact that, for sets \( A, B \) and \( C, A \oplus C \supseteq B \supseteq A \oplus B \oplus C. \)

\( \text{Theorem 16 is sufficient, but not necessary, to ensure emptiness of } S^0. \) The condition was developed by considering the dynamics of \( S^0 \) over times \( i \in N \) in \textit{only one} direction in \( \mathbb{R}^n \), the eigenvector associated with the dominant (real) eigenvalue. It is possible, but this depends on the system and constraints, that emptiness of \( S^0 \) could be concluded at some smaller \( \alpha \) than \( \alpha_* \) by considering other directions. On the other hand, the long-term change in \( S^0 \) will tend to be dominated by activity in the direction of the dominant eigenvalue; our numerical results show relatively good agreement between the theoretical bound and the exact bound as \( i^* \) grows.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Comparison of the bound \( \bar{\alpha}_i \), obtained from \textbf{Theorem 16} with the true bound \( \alpha_{i^*} \).}
\end{figure}

\textbf{S}_3, the exact critical scaling factors \( \alpha_{i^*} \), and the upper bound \( \bar{\alpha}_i \), from Theorem 16. The exact scaling factors were found by trial and error, searching over \( \alpha \in (0, 1) \) until the smallest value is found for which the reachability recursion \( C_{i+1} = Q(C_i) \cap X \) results in \( C_i = \emptyset \) for \( i = i^* \).

7. DISCUSSION

The bound on critical \( \alpha \) is merely sufficient and, as the numerical results indicate, it is not tight—\( C^0 \) may become empty at some smaller \( i \) or \( \alpha \) than the bound of Theorem 16 suggests. The sources of conservatism are threefold:

1. Proposition 12 is sufficient, but not necessary, to guarantee emptiness of \( C_i \) at some \( i^* \). The lack of necessity arises from the lack of tightness (illustrated in Fig. 1) in the inclusion relation between \( C_i \) and \( T_i \) established in Proposition 9. This itself is, as explained in Remark 10, because Minkowski addition is not distributive over set intersections. To avoid this, and guarantee equality in the relation of Proposition 9 and necessity of the condition in Proposition 12, strong and unusual assumptions on the system and constraints would be required.

2. Similarly, Proposition 13 is merely sufficient to ensure emptiness of \( T_i \) at a given \( i = i^* \). The source of conservatism again arises from the basic properties of Minkowski addition and subtraction, and in particular that these operations do not commute: Proposition 13 uses the fact that, for sets \( A, B \) and \( C, A \oplus B \oplus C. \)

3. Theorem 16 is sufficient, but not necessary, to ensure emptiness of \( S^0 \). The condition was developed by considering the dynamics of \( S^0 \) over times \( i \in N \) in \textit{only one} direction in \( \mathbb{R}^n \), the eigenvector associated with the dominant (real) eigenvalue. It is possible, but this depends on the system and constraints, that emptiness of \( S^0 \) could be concluded at some smaller \( \alpha \) than \( \alpha_* \) by considering other directions. On the other hand, the long-term change in \( S^0 \) will tend to be dominated by activity in the direction of the dominant eigenvalue; our numerical results show relatively good agreement between the theoretical bound and the exact bound as \( i^* \) grows.

Finally, it is worth remarking that the developed theoretical bounds are simple to determine, requiring only the spectral information about the open-loop system and a couple of support function evaluations on the constraint sets \( X \) and \( U \). In comparison, to determine the exact \( \alpha_i(i^*) \) requires a search over the space \( \alpha \in (0, 1) \), computing the sequence \( \{C_i\}_{i \geq 0} \) until it is found that \( C^0_i \) is empty.

8. CONCLUSIONS AND FUTURE WORK

We have analysed a simple instance of a constrained linear system subject to actuation attacks and, using set-theoretic methods, derived a lower bound on the sufficient size of the attack set in order that robust infinite-time constraint satisfaction can not be guaranteed. The bound depends, in an intuitive way, on the spectral radius of the system and size and shape of the constraint sets. Future work will investigate lowering conservatism (\textit{e.g.,} by deriving results considering more than one eigenvector direction), consider more general instances of \( A \) (\textit{e.g.,} with dominant complex eigenvalues) and a more sophisticated setting (\textit{e.g.,} with outputs and sensor data injection).

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