A CHAIN MORPHISM FOR ADAMS OPERATIONS ON RATIONAL ALGEBRAIC K-THEORY

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Abstract. For any regular noetherian scheme $X$ and every $k \geq 1$, we define a chain morphism $\Psi^k$ between two chain complexes whose homology with rational coefficients is isomorphic to the algebraic $K$-groups of $X$ tensored by $\mathbb{Q}$. It is shown that the morphisms $\Psi^k$ induce in homology the Adams operations defined by Gillet and Soulé or the ones defined by Grayson.

Introduction

Let $X$ be any scheme and let $\mathcal{P}(X)$ be the exact category of locally free sheaves of finite rank over $X$. The algebraic $K$-groups of $X$, $K_n(X)$, are defined as the Quillen K-groups of the category $\mathcal{P}(X)$, as given in [10].

Several authors have equipped these groups with a $\lambda$-ring structure. Then, the Adams operations on each $K_n(X)$ are obtained from the $\lambda$-operations by a universal polynomial formula on the $\lambda$-operations. In the literature there are several direct definitions of the Adams operations on the higher algebraic $K$-groups of a scheme $X$. By means of the homotopy theory of simplicial sheaves, Gillet and Soulé defined in [6] Adams operations for any noetherian scheme of finite Krull dimension. Grayson, in [7], constructed a simplicial map inducing Adams operations on the $K$-groups of any category endowed with a suitable tensor product, symmetric power and exterior power. In particular, he constructed Adams operations for the algebraic $K$-groups of any scheme $X$. Following the methods of Schechtman in [11], Lecomte, in [8], defined Adams operations for the rational $K$-theory of any scheme $X$ equipped with an ample family of invertible sheaves. They are induced by a map in the homotopy category of infinite loop spectra.

The aim of this paper is to construct an explicit chain morphism which induces the Adams operations on the higher algebraic $K$-groups tensored by $\mathbb{Q}$. Our main interest in this construction is to endow the rational higher arithmetic $K$-groups of an arithmetic variety with a (pre)-$\lambda$-ring structure, in order to pursue a higher arithmetic intersection theory program in Arakelov geometry.

At the moment, there are two different definitions for the higher arithmetic $K$-groups of an arithmetic variety, one suggested by Deligne and Soulé (see [12] §III.2.3.4 and [3], Remark 5.4) and the other given by Takeda in [13]. Both of them rely on an explicit representative of the Beilinson regulator. By the nature of both definitions, it is apparently necessary to have a description of the Adams operations in algebraic K-theory in terms of a chain morphism, compatible with the representative of the Beilinson regulator “$\text{ch}$” given by Burgos and Wang in [2]. None of the explicit constructions of $\Psi^k$ known at the moment seem to be suitable for this purpose. The chain morphism presented in
this paper commutes with the morphism “ch”. In fact our definition has been highly influenced by the construction of “ch”. The details of the application to higher arithmetic $K$-theory can be found in the author’s PhD Thesis [5].

Consider the chain complex of cubes associated to the category $\mathcal{P}(X)$. McCarthy in [9], showed that the homology groups of this complex, with rational coefficients, are isomorphic to the rational algebraic $K$-groups of $X$.

We first attempted to find a homological version of Grayson’s simplicial construction using the complex of cubes, but this seems particularly difficult from the combinatorial point of view.

The current approach is based on a simplification obtained by using the transgressions of cubes by affine or projective lines, at the price of having to reduce to regular noetherian schemes due to the fact that homotopy invariance or the Dold-Thom isomorphism for $K$-theory are required. This was Burgos and Wang’s idea [2] for the definition of a chain morphism representing Beilinson’s regulator.

With this strategy, we first assign to a cube on $X$ a collection of cubes defined either on $X \times (\mathbb{P}^1)^*$ or on $X \times (\mathbb{A}^1)^*$, which have the property of being split in all directions (and which we call split cubes). This gives a morphism called the transgression morphism (Proposition 3.17).

Then, by a purely combinatorial formula on the Adams operations of locally free sheaves, we give a formula for the Adams operations on split cubes (Corollary 2.40). The key point is to use Gillet’s idea, as presented by Grayson, of considering the secondary Euler characteristic class of the Koszul complex associated to a locally free sheaf of finite rank.

The composition of the transgression morphism with the Adams operations for split cubes gives a chain morphism representing the Adams operations for any regular noetherian scheme of finite Krull dimension (Theorem 4.2). The fact that our construction induces indeed the Adams operations defined by Gillet and Soulé in [6] and the ones defined by Grayson in [7] follows from a general result on the comparison of morphisms from algebraic $K$-theory to itself, given in [4].

The two constructions, with projective lines or with affine lines, are completely analogous. One may choose the more suitable one in each particular case. For instance, to define Adams operations on the $K$-groups of a regular ring $R$, one may consider the definition with affine lines so as to remain in the category of affine schemes. On the other hand, if for instance our category of schemes is the category of projective regular schemes, then the construction with projective lines may be the appropriate one.

The paper is organized as follows. In the first section, we introduce the notation for multi-indices and (co)chain complexes. The complex of cubes is defined and a normalized version, in the style of the normalized complex associated to a cubical abelian group, is introduced. In the next section we define Adams operations for split cubes, that is, for cubes which are split in all directions, by means of a combinatorial formula on the Adams operations of locally free sheaves of finite rank. In the third section, the transgression morphism is defined. We assign to every cube of locally free sheaves on $X$, a collection of split cubes defined either on $X \times (\mathbb{P}^1)^*$ or on $X \times (\mathbb{A}^1)^*$. Finally, in the last section we summarize the constructions provided in the previous sections so as to give a representative of the Adams operations for regular noetherian schemes of finite Krull dimension. It is shown that our construction induces the Adams operations defined by Gillet and Soulé in [6].
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1. The chain complex of cubes

1.1. Notation for multi-indices. We give here some notations on multi-indices that will be used in the sequel.

Let $\mathcal{I}$ be the set of all multi-indices of finite length, i.e.

$$\mathcal{I} = \{i = (i_1, \ldots, i_n) \in \mathbb{N}^n, \ n \in \mathbb{N}\} = \bigcup_{k>0} \mathbb{N}^k.$$  

For every $m \geq 0$, consider the set $[0, m] := \{0, \ldots, m\}$. If $a \in [0, m]$ and $l = 1, \ldots, n$, let $a_l \in [0, m]^n$ be the multi-index

$$(0, \ldots, 0, a, 0, \ldots, 0),$$

that is, the multi-index where the only non-zero entry is $a$ in the $l$-th position. We write $1 = (1, \ldots, 1)$ and more generally, if $r_1 \leq r_2$, we define $1_{r_1}^{r_2}$ to be the multi-index with

$$(1_{r_1}^{r_2})_i = \begin{cases} 1 & \text{if } r_1 \leq i \leq r_2, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.1. Let $i, j \in \mathbb{N}^n$. We fix the following notations for multi-indices:

1. The length of $i$ is the integer $\text{length}(i) := n$.
2. The characteristic of $i$ is the multi-index $\nu(i) \in \{0, 1\}^n$, defined by

$$\nu(i)_j = \begin{cases} 0 & \text{if } i_j = 0, \\ 1 & \text{otherwise.} \end{cases}$$

3. The norm of $i$ is defined by $|i| = i_1 + \cdots + i_n$. If $1 \leq l \leq n$, we denote $|i|_l = i_1 + \cdots + i_l$.
4. Orders on the set of multi-indices:
   ▶ We write $i \geq j$, if for all $r$, $i_r \geq j_r$. Otherwise we write $i \not\geq j$.
   ▶ We denote by $\preceq$ the lexicographic order on multi-indices. By $i \prec j$ we mean $i \preceq j$ and $i \neq j$.
5. Let $1 \leq l \leq n$ and $m \in \mathbb{N}$. Then, we define

- Faces: $\partial_i(i) := (i_1, \ldots, \hat{i}_l, \ldots, i_n)$.
- Degeneracies: $s^m_i(i) := (i_1, \ldots, i_{l-1}, m, i_l, \ldots, i_n)$.
- Substitution: $\sigma^m_i(i) := s^m_i \partial_i(i) = (i_1, \ldots, i_{l-1}, m, i_l+1, \ldots, i_n)$.

In general, for any $l = (l_1, \ldots, l_s)$ with $1 \leq l_1 < \cdots < l_s \leq n$ and $m = (m_1, \ldots, m_s) \in \mathbb{N}^s$, we write

$$\partial_l(i) = \partial_{l_1} \cdots \partial_{l_s}(i), \quad s^m_l(i) = s^{m_1}_{l_1} \cdots s^{m_s}_{l_s}(i), \quad \text{and} \quad \sigma^m_l(i) = \sigma^{m_1}_{l_1} \cdots \sigma^{m_s}_{l_s}(i).$$

6. If $\text{length}(i) = l$ and $\text{length}(j) = r$, the concatenation of $i$ and $j$ is the multi-index of length $l + r$ given by

$$ij = (i_1, \ldots, i_l, j_1, \ldots, j_r).$$

7. Assume that $i \in \{0, 1\}^n$. The complementary multi-index of $i$ is the multi-index $i^c := 1 - i$, i.e.

$$(i^c)_r = \begin{cases} 0 & \text{if } i_r = 1, \\ 1 & \text{if } i_r = 0. \end{cases}$$
(8) Assume that \(i, j \in \{0,1\}^n\). We define their intersection by
\[
i \cap j = (i_1 \cdot j_1, \ldots, i_n \cdot j_n),
\]
and their union \(i \cup j\) by
\[
(i \cup j)_r = \max\{i_r, j_r\}.
\]

1.2. Iterated (co)chain complexes. Let \(\mathcal{U}\) be some universe (see [1]) and let \(\mathcal{P}\) be a small additive category in \(\mathcal{U}\), with fixed zero object 0.

**Definition 1.2.**

(i) A \(k\)-iterated cochain complex \(C^* = (C^*, d_1, \ldots, d_k)\) over \(\mathcal{P}\) is a \(k\)-graded object together with \(k\) endomorphisms \(d^1, \ldots, d^k\) of multi-degrees \(l_1, \ldots, l_k\), respectively, such that for all \(i, j\), \(d^i d^j = 0\) and \(d^i d^j = d^j d^i\). The endomorphism \(d^i\) is called the \(i\)-th differential of \(C^*\).

(ii) A \(k\)-iterated chain complex \(C_* = (C_*, d_1, \ldots, d_k)\) over \(\mathcal{P}\) is a \(k\)-graded object together with \(k\) endomorphisms \(d_1, \ldots, d_k\) of multi-degrees \(-l_1, \ldots, -l_k\) respectively, such that for all \(i, j\), \(d_i d_j = d_j d_i\). The endomorphism \(d_i\) is called the \(i\)-th differential of \(C_*\).

(iii) A (co)chain morphism is a collection of morphisms commuting with the differentials.

If \(C^*\) is a \(k\)-iterated cochain complex and \(i\) a multi-index of length \(k - 1\), then \(C^{*i}_{l(4)}\) is a cochain complex. In this way, if \(P\) is a property of cochain complexes, we say that \(C^*\) satisfies the property \(P\) in the \(l\)-th direction, if for all multi-indices \(i\) of length \(k - 1\), the cochain complex \(C^{*i}_{l(4)}\) satisfies \(P\).

Let \(C^*\) be a cochain complex. We will mainly refer to the two following properties of cochain complexes:

(i) The complex \(C^*\) has finite length if there exists \(l_1 < l_2\) such that
\[
C^n = 0, \quad \text{for } n < l_1, \text{ and } n > l_2.
\]

In this case, the difference \(l_2 - l_1\) is called the length of \(C\).

(ii) The complex \(C^*\) is acyclic, if \(H^n(C) = 0\) for all \(n\).

**Definition 1.3.** Let \((B^*, d^1, d^2)\) be a 2-iterated cochain complex. The simple complex of \(B^*\) is the cochain complex whose graded groups are
\[
B^n := \bigoplus_{r+s=n} B^{r,s},
\]
and whose differential is
\[
B^{r,s} \xrightarrow{d} B^{r+1,s} \oplus B^{r,s+1},
\]
\[
b \mapsto d^1(b) + (-1)^r d^2(b).
\]

Observe that if \(B\) is acyclic in one direction, then so is the simple complex.

**Example 1.4** (Tensor product). Assume that in the category \(\mathcal{P}\) there is a notion of tensor product. In our applications, \(\mathcal{P}\) will be the category of abelian groups or the category of locally free sheaves on a scheme. Let \((A^*, d_A)\) and \((B^*, d_B)\) be two cochain complexes. The tensor product \((A \otimes B)^*\) is the 2-iterated cochain complex with
\[
(A \otimes B)^{n,m} = A^n \otimes B^m,
\]
and differentials \((d_A \otimes id_B, id_A \otimes d_B)\). By abuse of notation, the associated simple complex will also be denoted by \((A \otimes B)^*\).
1.3. The chain complex of iterated cochain complexes. Let $\mathcal{P}$ be a $\mathcal{U}$-small abelian category. We denote by $IC_n(\mathcal{P})$ the set of $n$-iterated cochain complexes over $\mathcal{P}$, concentrated in non-negative degrees, of finite length and acyclic in all directions. Let $ZIC_n(\mathcal{P})$ be the free abelian group generated by $IC_n(\mathcal{P})$. Then,

$$ZIC_*(\mathcal{P}) = \bigoplus_{n \geq 0} ZIC_n(\mathcal{P})$$

is a graded abelian group, which can be made into a chain complex in the following way. For every $l = (l_1, \ldots, l_n)$, we denote by $IC_l^2(\mathcal{P}) \subseteq IC_n(\mathcal{P})$ the set of $n$-iterated cochain complexes of length $l_i$ in the $i$-th direction.

**Definition 1.5.** Let $A^* \in IC_l^2(\mathcal{P})$. For every $i = 1, \ldots, n$ and $j \in [0, l_i]$, the $(n - 1)$-iterated cochain complex $\partial_i^j (A)^*$ is defined by

$$\partial_i^j (A)^m := A^{\varepsilon_i^j(m)} \in IC_{n-1}(\mathcal{P}) \quad \forall m.$$  

It is called the $j$-th face of $A^*$ in the $i$-th direction. If $j > l_i$, we set $\partial_i^j (A)^m := 0$.

It follows from the definition that for all $j \in [0, l_i]$ and $k \in [0, l_r]$,

$$\partial_i^j \partial_r^k = \partial_r^k \partial_i^j, \quad \text{if} \quad i \leq r.$$  

(1.6)

Then, there is a well-defined group morphism

$$ZIC_n(\mathcal{P}) \xrightarrow{d} ZIC_{n-1}(\mathcal{P})$$

$$A^* \mapsto \sum_{i=1}^n \sum_{j \geq 0} (-1)^{i+j} \partial_i^j (A)^*.$$  

Since $d^2 = 0$, the pair $(ZIC_*(\mathcal{P}), d)$ is a chain complex. It is called the chain complex of iterated cochain complexes.

**Remark 1.7.** Observe that we have obtained a chain complex whose $n$-graded piece is generated by $n$-iterated cochain complexes. We will try to be very precise on this duality, so as not to confuse the reader.

1.4. The chain complex of cubes. We are interested in the chain complex of iterated cochain complexes obtained restricting to the iterated cochain complexes of length 2 in all directions. We write for simplicity,

$$C_n(\mathcal{P}) = IC_n^2(\mathcal{P}) \quad \text{and} \quad ZC_n(\mathcal{P}) = ZIC_n^2(\mathcal{P}).$$

The differential of $ZIC_*(\mathcal{P})$ induces a differential on $ZC_*(\mathcal{P}) = \bigoplus_n ZC_n(\mathcal{P})$, making the inclusion $ZC_*(\mathcal{P}) \hookrightarrow ZIC_*(\mathcal{P})$ a chain morphism. An element of $C_*(\mathcal{P})$ is called an $n$-cube.

**Remark 1.8.** Let

$$\varepsilon : 0 \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow 0$$

be an exact sequence of $(n - 1)$-cubes. That is, for every $j \in \{0, 1, 2\}^{n-1}$, the sequence

$$0 \rightarrow E_0^j \rightarrow E_1^j \rightarrow E_2^j \rightarrow 0$$

is exact. Then, for all $i = 1, \ldots, n$, there is an $n$-cube $\tilde{E}$, with

$$\partial_i^j \tilde{E} = E_j.$$  

This cube is called the cube obtained from $\varepsilon$ along the $i$-th direction.
Definition 1.9. For every $i = 1, \ldots, n$ and $j = 0, 1$, one defines degeneracies
\[ s_i^j : ZC_{n-1}(\mathcal{P}) \to ZC_n(\mathcal{P}), \]
by setting for every $E \in C_{n-1}(\mathcal{P})$,
\[ s_i^j(E)_j = \begin{cases} 0 & j \neq j, j + 1 \\ E_{\partial_i(j)} & j = j, j + 1. \end{cases} \]
That is, $s_i^j(E)$ is the $n$-cube obtained from the exact sequences of $n$-cubes
\[ 0 \to E \xrightarrow{\partial_i} E \to 0 \to 0, \quad \text{if } j = 0, \]
\[ 0 \to 0 \to 0 \to E \xrightarrow{\partial_i} E \to 0, \quad \text{if } j = 1, \]
along the $i$-th direction. An element $F \in C_n(\mathcal{P})$ is called degenerate if for some $i$ and $j$, $F \in \text{im } s_i^j$.

For any $k, l \in \{0, 1, 2\}$ and for all $u, v \in \{0, 1\}$, the following identities are satisfied:
\[ \partial_i^l \partial_j^k = \begin{cases} \partial_i^k \partial_{j+1}^l & \text{if } j \leq i, \\ \partial_i^{j+1} \partial_{j}^l & \text{if } j > i. \end{cases} \]
\[ \partial_i^0 s_i^0 = \partial_i^{l+1} s_i^0 = id, \quad \partial_i^1 s_i^1 = \partial_1 s_i^1 = id, \quad \partial_i^2 s_i^0 = \partial_i^0 s_i^1 = 0, \]
\[ \partial_i^u s_j^u = \begin{cases} s_j^{u+1} \partial_i^{u-1} & \text{if } j < i, \\ s_j^{u+1} \partial_i^u & \text{if } j > i. \end{cases} \]
\[ s_i^u s_j^v = s_{j+1}^v s_i^u \] if $j \geq i$.

Let
\[ \mathbb{Z}D_n(\mathcal{P}) = \sum_{i=1}^n s_i^0(ZC_{n-1}(\mathcal{P})) + s_i^1(ZC_{n-1}(\mathcal{P})) \subset ZC_n(\mathcal{P}). \]
Since the differential of a degenerate cube is also degenerate, the differential of $ZC_*(\mathcal{P})$ induces a differential on $\mathbb{Z}D_*(\mathcal{P})$ making the inclusion arrow $\mathbb{Z}D_*(\mathcal{P}) \hookrightarrow ZC_*(\mathcal{P})$ a chain morphism. The quotient complex
\[ \widetilde{ZC}_*(\mathcal{P}) = ZC_*(\mathcal{P})/\mathbb{Z}D_*(\mathcal{P}) \]
is called the chain complex of cubes in $\mathcal{P}$. Nevertheless, by abuse of language, the complex $ZC_*(\mathcal{P})$ is usually referred as to the chain complex of cubes as well.

Proposition 1.11. (McCarthy) Let $\mathcal{P}$ be a small abelian category and let $K_n(\mathcal{P})$ denote the Quillen algebraic $K$-groups of $\mathcal{P}$. Then, for all $n \geq 0$, there is an isomorphism
\[ H_n(\widetilde{ZC}(\mathcal{P}), \mathbb{Q}) \cong K_n(\mathcal{P}) \otimes \mathbb{Q}. \]

Proof. See [9]. \qed

1.5. The normalized complex of cubes. Let $\mathcal{P}$ be a small exact category in some universe $\mathcal{U}$. In this section, we show that there is a normalized complex for the complex of cubes, in the style of the normalized complex associated to a simplicial or cubical abelian group. That is, we construct a complex $NC_*(\mathcal{P}) \subset ZC_*(\mathcal{P})$, which maps isomorphically to $\widetilde{ZC}_*(\mathcal{P})$. 
Proposition 1.12. Let \( \mathcal{NC}_n(\mathcal{P}) \subset \mathcal{ZC}_n(\mathcal{P}) \) be any of the following complexes:

\[
N_nC(\mathcal{P}) = \begin{cases}
\bigcap_{i=1}^n \ker \partial_i^0 \cap \bigcap_{i=1}^n \ker \partial_i^2,
\bigcap_{i=1}^n \ker \partial_i^1 \cap \bigcap_{i=1}^n \ker \partial_i^0 = \bigcap_{i=1}^n \ker \partial_i^0 \cap \bigcap_{i=1}^n \ker \partial_i^1,
\bigcap_{i=1}^n \ker (\partial_i^1 - \partial_i^2) \cap \bigcap_{i=1}^n \ker \partial_i^2 = \bigcap_{i=1}^n \ker \partial_i^1 \cap \bigcap_{i=1}^n \ker \partial_i^2,
\bigcap_{i=1}^n \ker (\partial_i^1 - \partial_i^2) \cap \bigcap_{i=1}^n \ker (\partial_i^0 - \partial_i^1).
\end{cases}
\]

Then, the composition

\[
\mathcal{NC}_n(\mathcal{P}) \hookrightarrow \mathcal{ZC}_n(\mathcal{P}) \twoheadrightarrow \tilde{\mathcal{C}}_n(\mathcal{P}) = \mathcal{ZC}_n(\mathcal{P})/\mathbb{Z}D_n(\mathcal{P})
\]

is an isomorphism of chain complexes.

Proof. We will see that the complex of cubes can be obtained by associating two different cubical structures to the collection of abelian groups \( \{\mathcal{ZC}_n(\mathcal{P})\}_n \).

We start by recalling the definitions and results on cubical abelian groups that we need. Given a cubical abelian group \( C \), with face maps denoted by \( \delta_i^j \) and degeneracy maps by \( \sigma_i \), the chain complex associated to \( C \), \( C_* \), is the chain complex whose \( n \)-th graded piece is \( C_n \) and whose differential \( \delta : C_n \to C_{n-1} \) is given by \( \delta = \sum_{i=1}^n \sum_{j=0,1} (-1)^{i+j} \delta_i^j \). Let \( \mathcal{D}_n \subset C_n \) be the subgroup of degenerate elements of \( C_n \), i.e. the elements that lie in the image of \( \sigma_i \) for some \( i \). The quotient \( \tilde{C}_n := C_n/\mathcal{D}_n \) is a chain complex, whose differential is induced by \( \delta \). For \( l = 0 \) or 1, the normalized chain complex associated to \( C \), \( N^lC_* \), is the chain complex whose \( n \)-th graded group is

\[
N^lC_n := \bigcap_{i=1}^n \ker \delta_i^l,
\]

and whose differential is the one induced by the inclusion \( N^lC_n \subset C_n \). A well-known result states that for any cubical abelian group \( C \), there is a decomposition of chain complexes \( C_* = N^1C_* \oplus D_* \). As a consequence, we obtain that the composition

\[
(1.13) \quad \phi : N^lC_* \hookrightarrow C_* \twoheadrightarrow \tilde{C}_n
\]

is an isomorphism of chain complexes.

In our situation, we associate two different cubical structures to the collection of abelian groups \( \{\mathcal{ZC}_n(\mathcal{P})\}_n \), we apply twice the normalized construction to \( \mathcal{ZC}(\mathcal{P}) = \{\mathcal{ZC}_n(\mathcal{P})\}_n \) and finally we obtain a subcomplex \( \mathcal{NC}(\mathcal{P}) \subset \mathcal{ZC}(\mathcal{P}) \) which is isomorphic to \( \mathcal{ZC}(\mathcal{P})/\mathbb{Z}D^1(\mathcal{P}) \).

The two different cubical structures of \( \mathcal{ZC}(\mathcal{P}) \) are given as follows.

- For the first structure consider
  \[
  \tilde{\delta}_i^0 = \partial_i^0, \quad \tilde{\delta}_i^1 = \partial_i^1 - \partial_i^2, \quad \text{and} \quad \tilde{s}_i = s_i^0.
  \]

- For the second structure consider
  \[
  \check{\delta}_i^0 = \partial_i^2, \quad \check{\delta}_i^1 = \partial_i^1 - \partial_i^0, \quad \text{and} \quad \check{s}_i = s_i^1.
  \]

By the identities (1.10), both collections of faces and degeneracies satisfy the identities of a cubical structure on \( \mathcal{ZC}(\mathcal{P}) \). Moreover, the differential of \( \mathcal{ZC}_n(\mathcal{P}) \) induced by both structures is exactly the differential of the complex of cubes.

By the first structure, we obtain an isomorphism of chain complexes

\[
N^1C_* := \mathcal{NC}_n(\mathcal{P}) \hookrightarrow \mathcal{ZC}_n(\mathcal{P}) \twoheadrightarrow \mathcal{ZC}_n(\mathcal{P})/\mathbb{Z}D^1(\mathcal{P})
\]
Corollary 1.15. Let \( \mathcal{P} \) be a small abelian category and let \( K_n(\mathcal{P}) \) denote the Quillen algebraic \( K \)-groups of \( \mathcal{P} \). Then, for all \( n \geq 0 \), there is an isomorphism

\[
H_n(\text{NC}_*(\mathcal{P}), \mathbb{Q}) \cong K_n(\mathcal{P}) \otimes \mathbb{Q}.
\]

Remark 1.16. Let \( X \) be a scheme and let \( \mathcal{P} = \mathcal{P}(X) \) be the category of locally free sheaves of finite rank on \( X \). Fix a universe \( \mathcal{U} \) so that \( \mathcal{P}(X) \) is \( \mathcal{U} \)-small for all \( X \). We will denote the complexes \( \mathcal{Z}_* \mathcal{C}(* \mathcal{P}) \), \( \mathcal{N}_* \mathcal{C}(\mathcal{P}) \), \( \mathcal{Z}_I \mathcal{C}(\mathcal{P}) \), \ldots and so on simply by \( \mathcal{Z}_* \mathcal{C}(X) \), \( \mathcal{N}_* \mathcal{C}(X) \), \( \mathcal{Z}_I \mathcal{C}(X) \), \ldots.
2. Adams operations for split cubes

Let $X$ be any scheme. In this section, for every $k \geq 1$, we construct a chain morphism $\Psi^k$ from the complex of split cubes to the complex of cubes on $X$.

We divide the construction into three steps. We first construct the chain complex of split cubes on $X$, $\mathbb{Z} \text{Sp}_*(X, d)$. We then define an intermediate chain complex $\mathbb{Z} G^k(X)_* \to \mathbb{Z} C_*$. Finally, for every $n$, we construct a morphism $\Psi^k: \mathbb{Z} \text{Sp}_n(X) \to \mathbb{Z} G^k(X)_n$.

Its composition with $\mu \circ \varphi$, $\mu \circ \varphi \circ \Psi^k: \mathbb{Z} \text{Sp}_*(X) \to \mathbb{Z} C_*$, gives the definition of the Adams operations over split cubes.

Let $X$ be a scheme and let $P = P(X)$ be the category of locally free sheaves of finite rank on $X$. Recall that the notation on multi-indices was introduced in section 1.1.

2.1. Split cubes. We introduce here the complex of split cubes, which plays a key role in the definition of the Adams operations for arbitrary cubes. Roughly speaking, split cubes are the cubes which are split in all directions.

For every $j = (j_1, \ldots, j_n) \in \{0, 1, 2\}^n$, let $u_1 < \cdots < u_s(j)$ be the indices such that $j_{u_i} = 1$ and let

$$u(j) = (u_1, \ldots, u_s(j)).$$

Observe that $s(j)$ is the length of $u(j)$.

Definition 2.2. Let $\{E^i\}_{i \in \{0, 1, 2\}^n}$ be a collection of locally free sheaves on $X$, indexed by $\{0, 2\}^n$. Let $[E^i]_{i \in \{0, 2\}^n}$ be the $n$-cube given by:

$\triangleright$ The $j$-component is

$$\bigoplus_{m \in \{0, 2\}^{s(j)}} E^m_{u(j)}, \quad j \in \{0, 1, 2\}^n.$$

$\triangleright$ The morphisms are compositions of the following canonical morphisms:

$$A \oplus B \to A, \quad A \oplus B \xrightarrow{\approx} B \oplus A,$$

$$A \hookrightarrow A \oplus B, \quad A \oplus (B \oplus C) \xrightarrow{\approx} (A \oplus B) \oplus C.$$

An $n$-cube of this form is called a direct sum $n$-cube.

Remark 2.3. In the previous definition, the direct sum is taken in the lexicographic order on $\{0, 2\}^{s(j)}$.

Observe that, if $j \in \{0, 2\}^n$, then the $j$-component of $[E^i]_{i \in \{0, 2\}^n}$ is exactly $E^j$. Hence, this $n$-cube has at the “corners” the given collection of objects, and we fill the “interior” with the appropriate direct sums.

Example 2.4. For $n = 1$, the 1-cube $[E^0, E^2]$ is the exact sequence

$$E^0 \to E^0 \oplus E^2 \to E^2.$$
Example 2.5. For \( n = 2 \), if \( E^{00}, E^{02}, E^{20}, E^{22} \) are locally free sheaves on \( X \), then the 2-cube \[
\begin{bmatrix}
E^{00} & E^{02} \\
E^{20} & E^{22}
\end{bmatrix}
\] is the 2-cube

\[
\begin{array}{c}
E^{00} \to E^{00} \oplus E^{02} \to E^{02} \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
E^{00} \oplus E^{20} \to E^{00} \oplus E^{02} \oplus E^{20} \oplus E^{22} \to E^{02} \oplus E^{22} \to E^{22}
\end{array}
\]

Definition 2.6. ▶ Let \( E \) be an \( n \)-cube. The direct sum \( n \)-cube associated to \( E \), \( \text{Sp}(E) \), is the \( n \)-cube

\[
\text{Sp}(E) := [E^j]_{j \in \{0,2\}^n}.
\]

▶ A split \( n \)-cube is a couple \((E, f)\), where \( E \) is an \( n \)-cube and \( f : \text{Sp}(E) \to E \) is an isomorphism of \( n \)-cubes such that \( f^j = id \) if \( j \in \{0,2\}^n \). The morphism \( f \) is called the splitting of \((E, f)\).

▶ Let \( \mathbb{Z}\text{Sp}_n(X) := \mathbb{Z}\{\text{split} \ n \text{-cubes on } X\} \), and let \( \mathbb{Z}\text{Sp}_n(X) = \bigoplus_n \mathbb{Z}\text{Sp}_n(X) \).

Example 2.7. For \( n = 2 \), a split cube \((E, f)\) consists of a 2-cube \( E \) together with an isomorphism

\[
\begin{array}{c}
E^{00} \to E^{00} \oplus E^{02} \to E^{02} \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
E^{00} \oplus E^{20} \to E^{00} \oplus E^{02} \oplus E^{20} \oplus E^{22} \to E^{02} \oplus E^{22} \to E^{22}
\end{array}
\]

which is the identity at the “corners”.

We endow \( \mathbb{Z}\text{Sp}_n(X) \) with a chain complex structure. That is, we define a differential morphism

\[
\mathbb{Z}\text{Sp}_n(X) \to \mathbb{Z}\text{Sp}_{n-1}(X).
\]

Let \( E \) be an arbitrary \( n \)-cube. Observe that if \( j = 0, 2 \), then, for all \( l = 1, \ldots, n \),

\[
\partial^l_j \text{Sp}(E) = \text{Sp}(\partial^l_j E).
\]

Therefore, if \((E, f)\) is a split \( n \)-cube,

\[
\partial^l_j (E, f) := (\partial^l_j E, \partial^l_j f)
\]

is a split \((n-1)\)-cube. By contrast, in general

\[
\partial^l_j \text{Sp}(E) \neq \text{Sp}(\partial^l_j E).
\]

However, if \( E \) is a split \( n \)-cube, \( \partial^l_j E \) is also isomorphic to \( \text{Sp}(\partial^l_j E) \), i.e. it is also split.

In order to illustrate the forthcoming definition, we will start by defining the face \( \partial^l_j (E, f) = (\partial^l_j E, \hat{f}) \) for \( n = 2 \). Let \((E, f)\) be a split 2-cube as in example 2.7. Then,

\[
\partial^1_0(E) = E^{10} \to E^{11} \to E^{12} \quad \text{and} \quad \text{Sp}(\partial^1_0 E) = E^{10} \to E^{10} \oplus E^{12} \to E^{12}.
\]
We define the morphism \( \hat{\partial}_1 : E^{10} \oplus E^{12} \xrightarrow{\sim} E^{11} \), as the composition
\[
E^{10} \oplus E^{12} \xrightarrow{(\iota^{10})^{-1} \oplus (\iota^{12})^{-1}} E^{00} \oplus E^{02} \oplus E^{20} \oplus E^{22} \xrightarrow{\hat{f}_1} E^{11}.
\]

Let \((E, f)\) be a split \(n\)-cube. For every \(j \in \{0, 1, 2\}^n\), we define a morphism
\[
\hat{f}^j : \text{Sp}(\partial_1^1 E)^j \rightarrow (\partial_1^1 E)^j
\]
as the composition of the isomorphisms
\[
\bigoplus_{m \in \{0, 2\}^{\iota(j)}} (\partial_1^{\iota})^{\sigma^{m}(\iota(j))} \xrightarrow{j^j} (\partial_1^1 E)^j
\]
\[
\oplus (\partial_1^j f)^{-1} \cong \xrightarrow{j^j} \oplus m \in \{0, 2\}^{\iota(j)} (\partial_1^0 E \oplus \partial_1^2 E)^{\sigma^{m}(\iota(j))} \cong \bigoplus m \in \{0, 2\}^{\iota(j)} E^{\sigma^{m}(\iota(j))}(s_j(j)),
\]
where the bottom arrow is the canonical isomorphism. Then, we define
\[
\partial_1^1 (E, f) := (\partial_1^1 E, \hat{f}).
\]

With this definition of \(\partial_1^1\), the commutation rule (1.6) is satisfied. Therefore, we have proved the following proposition.

**Proposition 2.9.** The morphism
\[
d = \sum_{l=1}^{n} \sum_{i=0,1,2} (-1)^{i+l} \partial_l^i : \mathbb{Z} \text{Sp}_n(X) \rightarrow \mathbb{Z} \text{Sp}_{n-1}(X)
\]
makes \(\mathbb{Z} \text{Sp}_*(X)\) into a chain complex. Moreover, the morphism \(\mathbb{Z} \text{Sp}_*(X) \rightarrow \mathbb{Z} \text{C}_*(X)\) obtained by forgetting the splittings is a chain morphism.

\[\square\]

**Remark 2.10.** Observe that due to (2.3), the morphism
\[
\text{Sp} : \mathbb{Z} \text{C}_*(X) \rightarrow \mathbb{Z} \text{Sp}_*(X)
\]
is not a chain morphism.

### 2.2. An intermediate complex for the Adams operations.

Here we introduce the chain complex that serves as the target for the Adams operations defined on the chain complex of split cubes. We then construct a morphism from this new chain complex to the original chain complex of cubes \(\mathbb{Z} \text{C}_*(X)\).

Let \(k \geq 1\). For every \(n \geq 0\) and \(i = 1, \ldots, k - 1\), we define
\[
G^k(X)_n := IC_1^k(C_n(X))
\]
\[
:= \{\text{acyclic cochain complexes of length } k \text{ of } n - \text{cubes}\},
\]
\[
G^{i,k}(X)_n := IC_2^{k-i,i}(C_n(X))
\]
\[
:= \{\text{2-iterated acyclic cochain complexes of lengths } (k - i, i) \text{ of } n - \text{cubes}\}.
\]

The differential of \(\mathbb{Z} \text{C}_*(X)\) induces a differential on the graded abelian groups
\[
\mathbb{Z}G^{i,k}(X)_* := \bigoplus_n \mathbb{Z}G^{i,k}_2(X)_n \quad \text{and} \quad \mathbb{Z}G^k_1(X)_* := \bigoplus_n \mathbb{Z}G^k_1(X)_n.
\]
That is, if $B \in G_{2}^{i,k}(X)_n$, then for every $r, s$, $B^{r,s}$ is an $n$-cube. Define $\partial_i^l(B)$ to be the 2-iterated cochain complex of lengths $(k - i, i)$ of $(n - 1)$-cubes given by

$$\partial_i^l(B)^{r,s} := \partial_i^l(B^{r,s}) \in C_{n-1}(X), \quad \text{for every } r, s.$$ 

Then the differential of $B$ is defined as

$$d(B) = \sum_{i=1}^{n} \sum_{l=0}^{2} (-1)^{i+l} \partial_i^l(B).$$

If $A \in G_{1}^{k}(X)_n$, then for every $r$, $A^r$ is an $n$-cube and the differential is defined analogously.

For every $n$, the simple complex associated to a 2-iterated cochain complex induces a morphism

$$\Phi^i : \mathbb{Z}G_{2}^{i,k}(X)_n \to \mathbb{Z}G_{1}^{k}(X)_n.$$ 

That is, for every $B \in G_{2}^{i,k}(X)_n$, $\Phi^i(B)$ is the exact sequence of $n$-cubes

$$\Phi^i(B) := 0 \to B^{00} \to \cdots \to \bigoplus_{j_1+j_2=j} B^{j_1,j_2} \to \cdots \to B^{k-i,i} \to 0$$

with morphisms given by

$$B^{j_1,j_2} \to B^{j_1+1,j_2} \oplus B^{j_1,j_2+1}$$

$$b \mapsto d^1(b) + (-1)^{j_1} d^2(b).$$

One can easily check that, for every $i = 1, \ldots, k - 1$, $\Phi^i$ is a chain morphism.

We define a new chain complex by setting

$$\mathbb{Z}G^{k}(X)_n := \bigoplus_{i=1}^{k-1} \mathbb{Z}G_{2}^{i,k}(X)_{n-1} \oplus \mathbb{Z}G_{1}^{k}(X)_n.$$ 

If $B_i \in G_{2}^{i,k}(X)_{n-1}$, for $i = 1, \ldots, k - 1$, and $A \in G_{1}^{k}(X)_n$, the differential is given by

$$d_s(B_1, \ldots, B_{k-1}, A) := (-dB_1, \ldots, -dB_{k-1}, \sum_{i=1}^{k-1} (-1)^i \Phi^i(B_i) + dA).$$

Since, for all $i$, the morphisms $\Phi^i$ are chain morphisms, $d^2 = 0$ and therefore $(\mathbb{Z}G^{k}(X))_s = \bigoplus_n \mathbb{Z}G^{k}(X)_n, d_s$ is a chain complex.

Our purpose is to define a chain morphism from the chain complex $\mathbb{Z}G^{k}(X)_s$ to the complex of cubes $\mathbb{Z}C_s(X)$. It is constructed in two steps. First, we define a chain morphism from $\mathbb{Z}G^{k}(X)_s$ to the complex

$$\mathbb{Z}C_s^{arb}(X) := \mathbb{Z}IC_{*,2 \cdots 2}(X).$$

This complex is the chain complex that in degree $n$ consists of $n$-iterated cochain complexes of length 2 in directions $2, \ldots, n$ and arbitrary finite length in direction 1. Alternatively, it can be thought of as the complex of exact sequences of arbitrary finite length of $(n - 1)$-cubes.

Then, we construct a morphism from $\mathbb{Z}C_s^{arb}(X)$ to $\mathbb{Z}C_s(X)$, using the splitting of an acyclic cochain complex into short exact sequences.

Let

$$A : 0 \to A^0 \to \cdots \to A^k \to 0 \in G_{1}^{k}(X)_n$$
be an acyclic cochain complex of \( n \)-cubes. We define \( \varphi_1(A) \) to be the “secondary Euler characteristic class”, i.e.

\[
\varphi_1(A) = \sum_{p \geq 0} (-1)^{k-p+1}(k-p)A^p \in ZC_n(X).
\]

We choose the signs of this definition in order to agree with Grayson’s definition of Adams operations for \( n = 0 \) in [7]. Note that in loc. cit., an exact sequence is viewed as a chain complex, while here it is viewed as a cochain complex.

Recall that if \( B_i \in G_{2,k}X_n \), then \( B_i \) is a 2-iterated acyclic cochain complex where \( B_i^{j_1j_2} \) is an \( n \)-cube, for every \( j_1, j_2 \). We attach to \( B_i \) a sum of exact sequences of \( n \)-cubes as follows.

\[
\varphi_2(B_i) = \sum_{j \geq 0} (-1)^{k-j+1}((k-i-j)B_i^{*j} + (i-j)B_i^{j*})
\]

\[
+ \sum_{s \geq 1} (-1)^{k-s}(k-s) \sum_{j \geq 0} (B_i^{s-j,j} \to \bigoplus_{j' \geq j} B_i^{s-j',j'} \to \bigoplus_{j' > j} B_i^{s-j',j'}).
\]

Roughly speaking, the first summand corresponds to the secondary Euler characteristic of the rows and the columns. The second summand appears as a correction factor for the fact that direct sums are not sums in \( ZC_n(X) \).

For every \( n \), we define a morphism

\[
(2.11) \quad ZG^k(X)_n \xrightarrow{\varphi} ZC_{arb}^n(X)
\]

\[
(B_1, \ldots, B_{k-1}, A) \mapsto \varphi_1(A) + \sum_{i=1}^{k-1} (-1)^{i+1} \varphi_2(B_i).
\]

**Lemma 2.12.** The morphism \( \varphi \) is a chain morphism between \( ZG^k(X)_* \) and \( ZC_{arb}^n(X) \).

**Proof.** The lemma follows from the two equalities

\[
d\varphi_1(A) = \varphi_1(dA),
\]

\[
d\varphi_2(B_i) = -\varphi_2(dB_i) - \varphi_1(\Phi^i(B_i)), \quad \forall \ i.
\]

The first equality holds as a direct consequence of the definition of \( \varphi_1 \). By the definition of the differential of \( ZG^k_{2,k}(X)_* \),

\[
-\varphi_2(dB_i) = \sum_{l=2}^{n} \sum_{j=0}^{2} (-1)^{l+j} \partial_l^i \varphi_2(B_i).
\]

Therefore, it remains to see that

\[
\sum_{r \geq 0} (-1)^r \partial_r^i \varphi_2(B_i) = \varphi_1(\Phi^i(B_i)).
\]

In other terms, writing

\[
(1) := \sum_{s \geq 1} (-1)^{k-s}(k-s) \sum_{j \geq 0} \sum_{r=0}^{2} (-1)^r \partial_r^i [B_i^{s-j,j} \to \bigoplus_{j' \geq j} B_i^{s-j',j'} \to \bigoplus_{j' > j} B_i^{s-j',j'}],
\]

\[
(2) := \sum_{r \geq 0} (-1)^r \partial_r^i \sum_{j \geq 0} (-1)^{k-j+1}((k-i-j)B_i^{*j} + (i-j)B_i^{j*}).
\]
Lemma 2.14. The map $\mu$ is a chain morphism.
2.3. Ideas of the definition of the Adams operations on split cubes. Let $X$ be a scheme. In order to enlighten the forthcoming construction, in this section we give examples and the outline of the definition of Adams operations on split cubes. The starting point is the use of the Koszul complex, as considered by Grayson in [7].

Let $\mathbb{Z}SG^k(X)_*$ be the chain complex obtained like $\mathbb{Z}G^k(X)_*$ by considering split cubes. That is, considering the groups $SG_1^k(X)_n := IC^{k,1}(Sp_n(X))$, $SG_2^i,k(X)_n := IC^{k-i,i}(Sp_n(X))$.

Observe that there is a natural morphism $\mathbb{Z}SG^k(X)_* \to \mathbb{Z}G^k(X)_*$ obtained by forgetting the splitting.

For every $k \geq 1$, we construct a morphism, $\mathbb{Z}Sp_n(X) \xrightarrow{\psi_k} \mathbb{Z}SG^k(X)_n$, which composed with $\mu \circ \varphi$, gives a morphism $\mathbb{Z}Sp_n(X) \xrightarrow{\psi_k} \mathbb{Z}C^k_n(X)$.

Definition 2.15. Let $E \in Sp_0(X)$ and $k \geq 1$. We define the element $\Psi^k(E) \in SG^k(X)_0 = SG^k_1(X)_0$ to be the $k$-th Koszul complex of $E$, i.e. the exact sequence

$$0 \to \Psi^k(E)^0 \to \cdots \to \Psi^k(E)^k \to 0$$

with

$$\Psi^k(E)^p = E \cdot \cdot \cdot \cdot \cdot E \otimes E \wedge k-p \wedge E = S^p E \otimes \bigwedge^{k-p} E.$$  

Observe that, for $k = 1$, we have

$$\Psi^1(E) : 0 \to E \xrightarrow{=} E \to 0.$$  

By definition, the Koszul complex is functorial. Moreover, it has a very good behavior with direct sums.

Lemma 2.16. If $E$ and $F$ are two locally free sheaves on $X$, then there is a canonical isomorphism of exact sequences

$$(2.17) \quad \Psi^k(E \oplus F) \cong \bigoplus_{m=0}^k \Psi^{k-m}(E) \otimes \Psi^m(F), \quad \forall k.$$  

This identification plays a key role in the construction of the Adams operations.

The definition of $\Psi^k(E)$ of a general split $n$-cube $E$ is given by a combinatorial formula on the Adams operations $\Psi^j(E^j)$, $j \in \{0, 1, 2\}^n$, of the locally free sheaves in the cube. In order to understand how the combinatorial formula of the upcoming definition 2.18 arises, we explain here the low degree cases. We give the detailed construction of the Adams operations for $n = 1$, with $k = 2, 3$, and for $n = 2, k = 2$. We extract from these examples the key facts that enable us to set the general formula.

Adams operations in the case $n = 1$, $k = 2$. Let $n = 1$ and $k = 2$, and let $E = [E^0, E^2]$. Recall that this notation means that $E$ is the 1-cube $E^0 \to E^0 \oplus E^2 \to E^2$. Our aim is to define $\Psi^2(E)$ in such a way that its differential is exactly

$$-\Psi^2(E^0) + \Psi^2(E^0 \oplus E^2) - \Psi^2(E^2).$$
Consider the two exact sequences
\[ C_0(E) := [\Psi^2(E^0), \Psi^1(E^0) \otimes \Psi^1(E^2)] \in G^2_1(X)_1, \]
\[ C_1(E) := [\Psi^2(E^0) \oplus \Psi^1(E^0) \otimes \Psi^1(E^2), \Psi^2(E^2)] \in G^2_1(X)_1. \]

Then,
\[ d(C_0(E) + C_1(E)) = -\Psi^2(E^0) - \Psi^1(E^0) \otimes \Psi^1(E^2) \]
\[ + \Psi^2(E^0) \oplus \Psi^1(E^0) \otimes \Psi^1(E^2) \oplus \Psi^2(E^2) - \Psi^2(E^2). \]

Observe now that by the isomorphism (2.17),
\[ \Psi^2(E^0) \oplus \Psi^1(E^0) \otimes \Psi^1(E^2) \oplus \Psi^2(E^2) \cong \Psi^2(E^0 \oplus E^2). \]

We define then \( \tilde{C}_1(E) \) to be the exact sequence \( C_1(E) \) modified by means of this isomorphism, that is
\[ \tilde{C}_1(E) : \Psi^2(E^0) \oplus \Psi^1(E^0) \otimes \Psi^1(E^2) \to \Psi^2(E^0 \oplus E^2) \to \Psi^2(E^2). \]

Finally, observe that the extra term \( \Psi^1(E^0) \otimes \Psi^1(E^2) \) is the simple complex associated to the 2-iterated complex of length (1, 1)
\[ E^0 \otimes E^0 \to E^0 \otimes E^2 \]
\[ E^2 \otimes E^0 \to E^2 \otimes E^2 \]

Hence, viewed as a 2-iterated complex, \( \Psi^1(E^0) \otimes \Psi^1(E^2) \in G^2_{2,2}(X)_0. \) We conclude that the differential of
\[ \Psi^2(E) := (\Psi^1(E^0) \otimes \Psi^1(E^2), C_0(E) + \tilde{C}_1(E)) \in ZG^{1,2}_2(X)_0 \oplus ZG^2_1(X)_1 \]
is exactly \(-\Psi^2(E^0) + \Psi^2(E^0 \oplus E^2) - \Psi^2(E^2) \) as desired.

For an arbitrary split 1-cube \((E, f)\), we define:

- \( \tilde{C}_0(E, f) := C_0(\text{Sp}(E)) \in G^2_1(X)_1. \)
- \( \tilde{C}_1(E, f) \) is the exact sequence obtained changing, via the given splitting \( f : E^1 \cong E^0 \oplus E^2 \), the terms \( \Psi^2(E^0 \oplus E^2) \) in \( \tilde{C}_1(\text{Sp}(E)) \) by \( \Psi^2(E^1) \):

\[
\begin{array}{ccc}
\Psi^2(E^0 \oplus E^2) & \cong & \Psi^2(E^2) \\
\Psi^2(E^0) \oplus \Psi^1(E^0) \otimes \Psi^1(E^2) & \rightarrow & \Psi^2(E^1) \rightarrow \Psi^2(E^2).
\end{array}
\]

We define then
\[ \Psi^2(E, f) := (\Psi^1(E^0) \otimes \Psi^1(E^2), \tilde{C}_0(E, f) + \tilde{C}_1(E, f)) \in ZG^{1,2}_2(X)_0 \oplus ZG^2_1(X)_1. \]

**Adams operations in the case \( n = 1, k = 3.** Let \( E = [E^0, E^2] \) as above. Our aim now is to define \( \Psi^3(E) \) in such a way that its differential is
\[-\Psi^3(E^0) + \Psi^3(E^0 \oplus E^2) - \Psi^3(E^2). \]

We consider the exact sequences,
\[ C_0(E) := [\Psi^3(E^0), \Psi^2(E^0) \otimes \Psi^1(E^2)], \]
\[ C_1(E) := [\Psi^3(E^0) \oplus \Psi^2(E^0) \otimes \Psi^1(E^2), \Psi^1(E^0) \otimes \Psi^2(E^2)], \]
\[ C_2(E) := [\Psi^3(E^0) \oplus \Psi^2(E^0) \otimes \Psi^1(E^2) \oplus \Psi^1(E^0) \otimes \Psi^2(E^2), \Psi^3(E^2)]. \]
Then, we define $\tilde{C}_2(E)$ to be the exact sequence obtained from $C_2(E)$ by exchanging
\[
Ψ^3(E^0) ⊕ Ψ^2(E^0) ⊕ Ψ^1(E^0) ⊕ Ψ^2(E^2) \oplus Ψ^3(E^2)
\]
with $Ψ^3(E^0 ⊕ E^2)$ by the isomorphism (2.17). That is, $\tilde{C}_2(E)$ is the exact sequence
\[
Ψ^3(E^0) ⊕ Ψ^2(E^0) ⊕ Ψ^1(E^0) ⊕ Ψ^2(E^2) → Ψ^3(E^0 ⊕ E^2) → Ψ^3(E^2).
\]
Then, the differential of $C_0(E) + C_1(E) - \tilde{C}_2(E)$ is
\[
Ψ^3(E^0) + Ψ^3(E^0 ⊕ E^2) - Ψ^3(E^2) - Ψ^2(E^0) ⊕ Ψ^1(E^2) - Ψ^1(E^0) ⊕ Ψ^2(E^2).
\]
As in the previous example, we have $Ψ^2(E^0) ⊕ Ψ^1(E^2) ∈ ZG^{1,3}_2(X)_0$ and $Ψ^1(E^0) ⊕ Ψ^2(E^2) ∈ ZG^{2,3}_2(X)_0$. Hence, the differential of
\[
Ψ^3(E) := (Ψ^2(E^0) ⊕ Ψ^1(E^2), Ψ^1(E^0) ⊕ Ψ^2(E^2), C_0(E) + C_1(E) + \tilde{C}_2(E))
\]
is exactly $Ψ^3(E^0) + Ψ^3(E^0 ⊕ E^2) - Ψ^3(E^2)$ as desired.

Finally, for an arbitrary split 1-cube $(E, f)$,
\[
\tilde{C}_0(E, f) := C_0(Sp(E)), \quad \tilde{C}_1(E, f) := C_1(Sp(E)),
\]
and $\tilde{C}_2(E, f)$ is defined by changing the term $Ψ^3(E^0 ⊕ E^2)$ in $\tilde{C}_2(Sp(E))$ by $Ψ^3(E^1)$ by means of the isomorphism induced by $f$.

**Adams operations in the case $n = 2, k = 2$.** Let $E = \begin{bmatrix} E^{00} & E^{02} \\ E^{20} & E^{22} \end{bmatrix}$. Then, we define the following terms of $ZG^{2,2}_2(X)_2$:

\[
C_{00}(E) := \begin{bmatrix}
Ψ^2(E^{00}) & Ψ^1(E^{00}) ⊕ Ψ^1(E^{02}) \\
Ψ^1(E^{00}) ⊕ Ψ^1(E^{20}) & Ψ^1(E^{00}) ⊕ Ψ^1(E^{22}) ⊕ Ψ^1(E^{02}) ⊕ Ψ^1(E^{20})
\end{bmatrix},
\]

\[
C_{10}(E) := \begin{bmatrix}
Ψ^2(E^{00}) & Ψ^1(E^{00}) ⊕ Ψ^1(E^{02}) ⊕ Ψ^1(E^{02}) ⊕ Ψ^1(E^{22}) & Ψ^2(E^{02}) \\
Ψ^1(E^{00}) ⊕ Ψ^1(E^{20}) & Ψ^1(E^{00}) ⊕ Ψ^1(E^{20}) & Ψ^1(E^{02}) ⊕ Ψ^1(E^{22})
\end{bmatrix},
\]

\[
C_{01}(E) := \begin{bmatrix}
Ψ^2(E^{00}) ⊕ Ψ^1(E^{00}) ⊕ Ψ^1(E^{02}) & Ψ^2(E^{02}) \\
Ψ^1(E^{00}) ⊕ Ψ^1(E^{20}) ⊕ Ψ^1(E^{02}) ⊕ Ψ^1(E^{22}) & Ψ^1(E^{02}) ⊕ Ψ^1(E^{22})
\end{bmatrix},
\]

\[
C_{11}(E) := \begin{bmatrix}
Ψ^2(E^{00}) ⊕ Ψ^1(E^{00}) ⊕ Ψ^1(E^{02}) ⊕ Ψ^1(E^{02}) ⊕ Ψ^1(E^{20}) ⊕ Ψ^1(E^{22}) & Ψ^2(E^{02}) \\
Ψ^1(E^{00}) ⊕ Ψ^1(E^{20}) ⊕ Ψ^1(E^{02}) ⊕ Ψ^1(E^{22}) & Ψ^1(E^{02}) ⊕ Ψ^1(E^{22})
\end{bmatrix}.
\]

The faces of each of these cubes are as follows (up to the isomorphism (2.17)):

- Terms that are summands of $Ψ^2(∂^1 E)$:

  \[
  ∂^1 C_{00}(E) = C_0(∂^1 E), \quad ∂^1 C_{01}(E) = C_1(∂^1 E), \quad ∂^1 C_{10}(E) = C_0(∂^1 E), \quad ∂^1 C_{11}(E) = C_1(∂^1 E).
  \]

  \[
  ∂^2 C_{00}(E) = C_0(∂^2 E), \quad ∂^2 C_{01}(E) = C_1(∂^2 E), \quad ∂^2 C_{10}(E) = C_0(∂^2 E), \quad ∂^2 C_{11}(E) = C_1(∂^2 E).
  \]

  \[
  ∂^3 C_{00}(E) = C_0(∂^3 E), \quad ∂^3 C_{01}(E) = C_1(∂^3 E), \quad ∂^3 C_{10}(E) = C_0(∂^3 E), \quad ∂^3 C_{11}(E) = C_1(∂^3 E).
  \]

  \[
  ∂^4 C_{00}(E) = C_0(∂^4 E), \quad ∂^4 C_{01}(E) = C_1(∂^4 E), \quad ∂^4 C_{10}(E) = C_0(∂^4 E), \quad ∂^4 C_{11}(E) = C_1(∂^4 E).
  \]
Terms that are a direct sum of a tensor product of complexes:
\[ \partial^2_1 C_{00}(E), \quad \partial^2_1 C_{10}(E), \quad \partial^2_1 C_{01}(E). \]

Terms that cancel each other:
\[
\begin{align*}
\partial^1_1 C_{00}(E) &= \partial^1_1 C_{10}(E), \\
\partial^1_1 C_{10}(E) &= \partial^1_1 C_{11}(E), \\
\partial^1_1 C_{01}(E) &= \partial^1_1 C_{11}(E).
\end{align*}
\]

It follows that the differential of \( C_{00}(E) + C_{10}(E) + C_{01}(E) + C_{11}(E) \) is \( \Psi^2(dE) \) plus some terms which are a direct sum of a tensor product of complexes. These tensor product complexes can be viewed as 2-iterated cochain complexes of lengths \((1,1)\) of exact sequences. These terms are added in \( \mathbb{Z} G^1_2(X)_1 \).

Finally, for every split 2-cube \((E, f)\), \( \Psi^2(E, f) \) is defined by modifying the appropriate locally free sheaves in each \( C_i(E) \) by means of the splitting \( f \).

Outline of the definition of \( \Psi^k \). The given examples suggest that the general procedure can be as follows:

First, for every split \( n \)-cube \((E, f)\), the direct sum \( n \)-cubes \( C_i(E) \) are defined by a purely combinatorial formula on the Adams operations of the locally free sheaves \( E_j, j \in \{0, 2\}^n \).

The previous construction is modified by the isomorphism (2.17).

The entries of \( C_i(E) \) which give the terms \( C_i(\partial^1_1 E) \) in the differential, are modified by the morphisms induced by the splitting \( f \).

From the examples, the key ideas that lead to the general combinatorial formula of \( C_i(E) \) can also be extracted:

At each step, some entries in \( \partial^0_i \) are constructed by taking the direct sum \( \partial^0_i \oplus \partial^2_i \) in a previous cube (where “previous” refers to the order \( \leq \) for the subindices in \( C_s(E) \)).

The new entries (not being direct sums of previous cubes) are direct sums of summands of the form \( \Psi^{k_1}(E^{2n_1}) \otimes \cdots \otimes \Psi^{k_r}(E^{2n_r}) \) satisfying:

a) \( \sum_s k_s = k \).

b) In the position \( 2j \) of the cube \( C_i(E) \), \( \sum_s k_s n_s = j + i \).

c) Observe that in the example \( n = 2 \), all the entries in \( C_{00} \) are new, and the new entries for \( C_{10} \) are in the positions \((2,0),(2,2)\) and for \( C_{11} \) in \((2,2)\).

Hence the new entries will correspond to the multi-indexes \( j \) such that \( j \geq \nu(i) \) (recall that \( \nu(i) \) is the characteristic of the multi-index \( i \)).

2.4. Definition of the cubes \( C_i(E) \). Let \((E, f) \in \text{Sp}_n(X)\) and fix \( k \geq 1 \). For every \( i \in [0, k-1]^n \), we define an exact sequence of direct sum cubes \( C_i(E) \in \mathbb{Z} S G^1_1(X)_n \).

This definition is purely combinatorial and does not depend on the splitting \( f \).

Let
\[ L^r_k = \{ k = (k_1, \ldots, k_r) \mid |k| = k \text{ and } k_s \geq 1, \forall s \} \]
be the set of partitions of length \( r \) of \( k \). Then, for every integer \( n \geq 0 \) and every multi-index \( m \) of length \( n \), we define a new set of indices by:
\[ \Lambda^r_k(m) = \bigcup_{r \geq 1} \left\{ (k, n^1, \ldots, n^r) \in L^r_k \times \{0, 1\}^n \right\} \mid \sum_s k_s n^s = m, \ n^1 \prec \cdots \prec n^r \} \]

Definition 2.18. Let \((E, f) \in \text{Sp}_n(X)\). For every \( i \in [0, k-1]^n \), let \( C_i(E) \in S G^1_1(X)_n \)
be the exact sequence of direct sum \( n \)-cubes, such that, for every \( j \in \{0, 1\}^n \), the position \( 2j \) is given as follows:
Remark 2.22. Let \( j \geq \nu(i) \), then
\[
C_i(E)^{2j} = \bigoplus_{\nu(j) \leq \nu(i) \cup j} \Psi^k_i(E^{2n^i}) \otimes \cdots \otimes \Psi^k_i(E^{2n^j}).
\]

(ii) If \( j \not\geq \nu(i) \), then
\[
C_i(E)^{2j} = \bigoplus_{j \leq m \leq \nu(i) \cup j} C_{i - \nu(i) \cup j}^m(E)^{2m}.
\]

In order to simplify the notation, we will denote by \( r \) the length of \( k \in \Lambda^n_i \) in the future occurrences of the sum (2.19). Observe that the definition of \( C_i(E) \) for a split cube \((E, f)\) does not depend on \( f \).

Remark 2.21. First of all, observe that in equation (2.20), \( i - \nu(i) \cdot j^c \in [0, k - 1]^n \), i.e. for every \( s \), \( 0 \leq (i - \nu(i) \cdot j^c)_s \):

- If \( i_s = 0 \), then \( \nu(i)_s = 0 \) and hence \( (i - \nu(i) \cdot j^c)_s = 0 \).
- If \( i_s > 0 \), then \( \nu(i)_s = 1 \) and since \( (j^c)_s = 0, 1 \), we have \( i_s \geq \nu(i)_s \).

Remark 2.22. Observe that equations (2.19) and (2.20) define \( C_i(E)^{2j} \) for all \( j \in \{0, 1\}^n \). This follows from the following facts:

1. Since \( \nu(j) \geq (0, \ldots, 0) \), equation (2.19) defines \( C_i(E) \) for \( i = 0 \).
2. If \( j \not\geq \nu(i) \), then
   \[
   \nu(i) \cdot j^c < i.
   \]

   Indeed, an equality would imply that \( \nu(i) \cdot j^c = 0 \) and hence that for all \( r \) such that \( i_r \neq 0 \), \( j^c_r = 0 \), concluding that \( j \geq \nu(i) \).

Remark 2.23. Observe that equation (2.20) also holds trivially for \( j \geq \nu(i) \), because in this case \( \nu(i) \cdot j^c = 0 \) and \( \nu(i) \cup j = j \). We will use this observation in some proofs when only combinatorial questions are involved.

Remark 2.24. The direct sum of more than two terms means the consecutive direct sums of two objects under the lexicographic order in the subindices. In order to prove some equalities, it will be necessary to reorder the indices, and then return to the original order. For the sake of simplicity, we will not write the canonical isomorphisms used at every step and will just write equalities. The reader should bear this remark in mind throughout this section.

2.5. Faces of the cubes \( C_i(E) \). In this section, we compute the faces of the cubes \( C_i(E) \). We fix \( k \geq 1 \), \( i \in [0, k - 1]^n \), a split \( n \)-cube \((E, f) \in S_n(X) \) and \( l \in \{1, \ldots, n\} \).

Lemma 2.25.
\[
\partial^l_i C_i(E) = \begin{cases} 
\partial^l_i C_{i-1}(E) & \text{if } i_l \neq 0, \\
C_{i_l}(\partial^l_i E) & \text{if } i_l = 0.
\end{cases}
\]

Proof. Assume that \( i_l \neq 0 \). It is enough to see that
\[
\partial^l_i C_i(E)^{2j} = \partial^l_i C_{i-1}(E)^{2j}, \quad \forall j \in \{0, 1\}^{n-1}.
\]

Observe that
\[
\partial^l_i C_i(E)^{2j} = C_i(E)^{2s_l^i(j)} = \bigoplus_{s_l^i(j) \leq m \leq \nu(i) \cup j} C_{i - \nu(i) \cup j}^m(E)^{2m},
\]
\[
\partial^l_i C_{i-1}(E)^{2j} = C_{i-1}(E)^{2s_l^i(j)} \oplus C_{i-1}(E)^{2s_l^{l}(j)},
\]

for all \( j \in \{0, 1\}^{n-1} \).
with
\[ C_{i-1}(E)^{2s^0_l(j)} = \bigoplus_{s^0_l(j) \leq m \leq \nu(i-1_l) \cup s^0_l(j)} C_{i-1-\nu(i-1_l)} s^0_l(j)^c (E)^{2m}, \]
\[ C_{i-1}(E)^{2s^1_l(j)} = \bigoplus_{s^1_l(j) \leq m \leq \nu(i-1_l) \cup s^1_l(j)} C_{i-1-\nu(i-1_l)} s^1_l(j)^c (E)^{2m}. \]

Let us compute each term separately. We start with (2.27). Since \( s^1_l(j)_l = 1 \), we see that \( m_l = 1 \) for all indices \( m \) of the direct sum. Moreover, since \( s^1_l(j)_l = 0 \), \( s^0_l(j)_l = 1 \), and \( \nu(i)_l = 1 \), we obtain that
\[ (i - 1 - \nu(i - 1_l) \cdot s^1_l(j)^c)_l = i_l - 1_l = (i - \nu(i) \cdot s^0_l(j)^c)_l. \]

Since it is clear that for all \( t \neq l \), \( (i - \nu(i) \cdot s^1_l(j)^c)_l = (i - 1_l - \nu(i - 1_l) \cdot s^1_l(j)^c)_l \), we see that
\[ i - \nu(i) \cdot s^0_l(j)^c = i - 1_l - \nu(i - 1_l) \cdot s^1_l(j)^c. \]

Thus,
\[ C_{i-1}(E)^{2s^1_l(j)} = \bigoplus_{s^0_l(j) \leq m \leq \nu(i) \cup s^0_l(j)} C_{i-\nu(i)} s^0_l(j)^c (E)^{2m}. \]

All that remains is to see that
\[ C_{i-1}(E)^{2s^0_l(j)} = \bigoplus_{s^0_l(j) \leq m \leq \nu(i) \cup s^0_l(j)} C_{i-\nu(i)} s^0_l(j)^c (E)^{2m}. \]

We proceed by induction on \( i_l \). If \( i_l = 1 \) then \( (i - 1_l)_l = 0 \) and hence \( (\nu(i - 1_l) \cup s^0_l(j)_l) = 0 \) which means that \( m_l = 0 \) for all multi-indices \( m \) in the direct sum (2.26). Moreover \( (i - 1_l - \nu(i - 1_l) \cdot s^0_l(j)^c)_l = 0 \) and \( (i - \nu(i) \cdot s^0_l(j)^c)_l = 0 \). Therefore,
\[ i - 1_l - \nu(i - 1_l) \cdot s^0_l(j)^c = i - \nu(i) \cdot s^0_l(j)^c \]

and the equality is proven. Let \( i_l > 1 \) and assume that the lemma is true for \( i_l - 1 \). Then, since \( i_l > 1 \), we have \( \nu(i - 1_l) = \nu(i) \) and \( \nu(i)_l = 1 \). Writing \( \alpha = i - 1_l - \nu(i - 1_l) \cdot s^0_l(j)^c \), we obtain
\[ C_{i-1}(E)^{2s^0_l(j)} = \bigoplus_{s^0_l(j) \leq m \leq \nu(i-1_l) \cup s^0_l(j)} C_\alpha(E)^{2m} \bigoplus_{s^0_l(j) \leq m \leq \nu(i-1_l) \cup s^0_l(j)} C_\alpha(E)^{2m} \]
\[ = \bigoplus_{j \leq n \leq \nu(\partial(i)_l) \cup j} (\partial^0_l \oplus \partial^1_l) C_\alpha(E)^{2n} \]
\[ = \bigoplus_{j \leq n \leq \nu(\partial(i)_l) \cup j} \partial^0_l C_{i-\nu(i) \cdot s^0_l(j)^c} (E)^{2n} \]
\[ = \bigoplus_{s^0_l(j) \leq m \leq \nu(i) \cup s^0_l(j)} C_{i-\nu(i) \cdot s^0_l(j)^c} (E)^{2m}, \]
since \( (i - \nu(i) \cdot s^0_l(j)^c)_l = i_l - 1 \) and we can apply the induction hypothesis in the third equality.

Let us now prove the equality with \( i_l = 0 \). Assume that \( s^0_l(j)_l \geq \nu(i)_l \). Then,
\[ \partial^0_l C_i(E)^j = C_i(E)^{2s^0_l(j)} = \bigoplus_{\Lambda^0(s^0_l(j) + i)} \psi^{k_1} (E^{2n^1}) \otimes \cdots \otimes \psi^{k_r} (E^{2n^r}) = (*). \]
Lemma 2.29. Since \((s^j_i \mathbf{n})_j = 0\). Hence, for all \(s\), \(\mathbf{n}^s_i = 0\) and we obtain
\[
(*) = \bigoplus_{\Lambda^s_i(j + \partial(i))} \Psi^k_1(\partial^0_i E^{2n^i}) \otimes \cdots \otimes \Psi^k_r(\partial^0_i E^{2n^r}) = C_{\partial(i)}(\partial^0_i E)^{2j}.
\]

Finally, if \(s^j_i \not\geq \nu(i)\), the direct sum \((2.20)\) can be written in the form
\[
\partial^0_i C_i(E)^{2j} = \bigoplus_{m,n} C_n(E)^{2m}
\]
with \(m \geq \nu(n)\) and \(m_1 = 0\). We deduce that \(n_l = 0\) and hence,
\[
\partial^0_i C_i(E)^{2j} = \bigoplus_{m,n} C_n(E)^{2m} = \bigoplus_{m,n} C_{\partial(i)}(\partial^0_i E)^{2m} = C_{\partial(i)}(\partial^0_i E)^{2j}.
\]
This reduces the proof to the already considered case. \(\square\)

Lemma 2.28. If \(i_l = k - 1\), then \(\partial^2_i C_i(E) = C_{\partial(i)}(\partial^2_i E)\).

Proof. Arguing as in the proof of the previous lemma, we limit ourselves to proving the equality in the case where \(s^j_i \geq \nu(i)\). In this situation, we obtain
\[
(\partial^2_i C_i(E))^{2j} = C_i(E)^{2s^j_i} = \bigoplus_{\Lambda^s_i(s^j_i + i)} \Psi^k_1(E^{2n^i}) \otimes \cdots \otimes \Psi^k_r(E^{2n^r}).
\]
Since \((s^j_i + i)_j = 1 + k - 1 = k\), we deduce that for all \(s\), \(\mathbf{n}^s_i = 1\) and therefore the lemma is proved. \(\square\)

The next lemma determines the faces \(\partial^2_i\) of \(C_i(E)\) whenever \(i_l \neq k - 1\).

Lemma 2.29. Let \(j \in \{0,1\}^n\) with \(j_i = 1\) and let \(i_l \neq k - 1\). Up to a canonical isomorphism, each of the direct summands of \(C_i(E)^{2j}\) is the tensor product of an exact sequence of length \((k - i_l - 1)\) by an exact sequence of length \((i_l + 1)\). Explicitly, in the equality
\[
C_i(E)^{2j} = \bigoplus_{\Lambda^s_i(j + i)} \Psi^k_1(E^{2n^i}) \otimes \cdots \otimes \Psi^k_r(E^{2n^r}),
\]
the tensor product of the Koszul complexes corresponding to the multi-indices \(\mathbf{n}\) with \(n_l = 0\) gives the exact sequence of length \(k - i_l - 1\), while the tensor product of the Koszul complexes corresponding to the multi-indices \(\mathbf{n}\) with \(n_l = 1\) gives the exact sequence of length \(i_l + 1\).

Proof. If \(j \geq \nu(i)\), then,
\[
C_i(E)^{2j} = \bigoplus_{\Lambda^s_i(j + i)} \Psi^k_1(E^{2n^i}) \otimes \cdots \otimes \Psi^k_r(E^{2n^r}).
\]
Assume that one of the summands is not a tensor product. Then there exists a multi-index \(\mathbf{n}\) with \(k\mathbf{n} = j + i\). In particular, \((k \cdot \mathbf{n})_l = 1 + i_l\). But \((k \cdot \mathbf{n})_l\) is either 0 or \(k\), and by hypothesis, \(1 \leq 1 + i_l < 1 + k - 1 = k\), which is a contradiction. If \(j \not\geq \nu(i)\), then,
\[
C_i(E)^{2j} = \bigoplus_{j \leq m \leq \nu(i)} C_{i-\nu(i),j}(E)^{2m}.
\]
The condition \(j \leq m\) implies that \(m_1 = 1\). Moreover, \((i - \nu(i) \cdot j')_l = i_l - j'_l \leq i_l < k - 1\). This means that every direct summand is a tensor product of exact sequences. By induction on \(|i|\), this is true for any multi-index \(i\).
Let us prove that every direct summand can be seen as the tensor product of an exact sequence of length \( k-i_l-1 \), corresponding to the multi-indices \( n \) with \( n_l = 0 \), and one of length \( i_l + 1 \), corresponding to the multi-indices with \( n_l = 1 \). By an induction argument, it is enough to prove the result in the case \( j \geq \nu(i) \). Let \((k, n^1, \ldots, n^r) \in \Lambda^k_n(j + i)\). Let \( s_1, \ldots, s_m \) be the indices such that \( n_l^{s_j} = 1 \) and let \( s_1', \ldots, s_{r-m} \) be the indices such that \( n_l^{s_j'} = 0 \). Since \( \sum k_l n_l^i = i_l + 1 \), we see that \( \sum_{j=1}^m k_{s_j} = i_l + 1 \) and hence

\[ T_1 := \Psi^{k_{s_1}}(E^{2n^1}) \otimes \cdots \otimes \Psi^{k_{s_m}}(E^{2n^m}) \]

is an exact sequence of length \( i_l + 1 \). Then,

\[ T_0 := \Psi^{k_{s_1'}}(E^{2n^1}) \otimes \cdots \otimes \Psi^{k_{s_{r-m}}}(E^{2n^r}) \]

is an exact sequence of length \( k - i_l - 1 \) and there is a canonical isomorphism

\[ \Psi^{k_1}(E^{2n^1}) \otimes \cdots \otimes \Psi^{k_r}(E^{2n^r}) \cong T_0 \otimes T_1 \]

as desired.

\( \blacksquare \)

**Lemma 2.30.** If \( i_l = k - 1 \), then

\[ \partial_l^1 C_i(E) \cong C_{\partial_l(i)}(\partial_l^0 E \oplus \partial_l^2 E), \]

with the isomorphism induced by the canonical isomorphism of the Koszul complex of a direct sum in (2.17).

**Proof.** The first part of the lemma is lemma 2.25. Assume then that \( i_l = k - 1 \). Applying lemmas 2.28 and 2.25 recursively, we obtain

\[ \partial_l^1 C_i(E) \cong \partial_l^1 C_i(E) \oplus \partial_l^2 C_i(E) = C_{\partial_l(i)}(\partial_l^0 E) \oplus \bigoplus_{a \in [0, k-1]} \partial_l^2 C_{i-a_1}(E). \]

Then, if \( j \in \{0, 1\}^{n-1} \) satisfies \( j \geq \partial_l \nu(i) \), we obtain that

\[ \partial_l^1 C_i(E)^{2j} = \bigoplus_{\Lambda_{n-1}^k(j+\partial_l(i))} \Psi^{k_1}(\partial_l^0 E^{2n^1}) \otimes \cdots \otimes \Psi^{k_r}(\partial_l^0 E^{2n^r}) \oplus \bigoplus_{a \in [0, k-1] \cup \Lambda_{n}^m(s_j'(j)+i-a)} \Psi^{k_1}(E^{2n^1}) \otimes \cdots \otimes \Psi^{k_r}(E^{2n^r}). \]

On the other hand, by the additivity of \( \Psi^k \) in (2.17), there are canonical isomorphisms

\[ \Psi^{k_1}((\partial_l^0 E \oplus \partial_l^2 E)^{2n^1}) \otimes \cdots \otimes \Psi^{k_r}((\partial_l^0 E \oplus \partial_l^2 E)^{2n^r}) = \]

\[ \Psi^{k_1-m_1}(\partial_l^0 E^{2n^1}) \otimes \cdots \otimes \Psi^{k_r-m_r}(\partial_l^0 E^{2n^r}) \otimes \Psi^{m_1}(\partial_l^2 E^{2n^1}) \otimes \cdots \otimes \Psi^{m_r}(\partial_l^2 E^{2n^r}). \]

Therefore, \( C_{\partial_l(i)}(\partial_l^0 E \oplus \partial_l^2 E)^{2j} \) is canonically isomorphic to

\[ \bigoplus_{\Lambda_{n-1}^k(j+\partial_l(i))} \Psi^{k_1-m_1}(E^{2s_1^0(n^1)}) \otimes \cdots \otimes \Psi^{k_r-m_r}(E^{2s_r^0(n^r)}) \]

\[ \otimes \Psi^{m_1}(E^{2s_1^2(n^1)}) \otimes \cdots \otimes \Psi^{m_r}(E^{2s_r^2(n^r)}). \]

The first summand in (2.31) corresponds to the indices \( m_1, \ldots, m_r = 0 \) in the latter sum. Therefore, we have to see that the second summand in (2.31) corresponds to the summand in the latter sum with at least one index \( m_i \neq 0 \). We will see that there is a bijection between the sets of multi-indices of each term.
For every collection \( k_1, \ldots, k_r, n^1, \ldots, n^r, m_1, \ldots, m_r \) with not all \( m_s = 0 \), let \( a = k - \sum m_s \). Since \( \sum m_s \neq 0 \), \( a \in [0, k - 1] \). Let \( s_1, \ldots, s_t \in \{1, \ldots, r\} \) be the indices for which \( k_{s_i} - m_{s_i} \neq 0 \) and let \( s'_1, \ldots, s'_m \in \{1, \ldots, r\} \) be the indices for which \( m'_s \neq 0 \). Then, to these data correspond the indices \( a \), and

\[
\{ k'_1, \ldots, k'_{t+m} \} = \{ k_{s_1} - m_{s_1}, \ldots, k_{s_t} - m_{s_t}, s'_1, \ldots, s'_m \}
\]

\[
\hat{\mathbf{n}}^p = \begin{cases} 
    s_1^p(n^{s_1}) & \text{if } p = 1, \ldots, t, \\
    s_1^p(n^{s_1-p}) & \text{if } p = t + 1, \ldots, t + m.
\end{cases}
\]

Conversely, let \( a, k_s \) and \( n^s \) be given. Then, we rearrange the collection \( n^s \) by the rule:

\[
n^1, \ldots, n^y, n^{y+1}, \ldots, n^x, n^{x+1}, \ldots, n^{2x-y}, n^{2x-y+1}, \ldots, n^r
\]

with

\[
(n^s)_l = \begin{cases} 
    0 & \text{if } s = 1, \ldots, x, \\
    1 & \text{if } s = x + 1, \ldots, r,
\end{cases}
\]

The index \( y \) satisfies that for \( s = 1, \ldots, y \) and for \( s = 2x - y + 1, \ldots, r \), there is no other index \( s' \) with \( \partial_l (n^s) = \partial_l (n^{s'}) \) and for \( s = y + 1, \ldots, x - y, \partial_l (n^s) = \partial_l (n^{s+x-y}) \). Then, the corresponding multi-indices are

\[
(\hat{\mathbf{n}}^1, \ldots, \hat{\mathbf{n}}^{r-x-y}) = (\partial_l (n^1), \ldots, \partial_l (n^x), \partial_l (n^{2x-y+1}), \ldots, \partial_l (n^r)),
\]

\[
k'_s = \begin{cases} 
    k_s & \text{if } s = 1, \ldots, y, \\
    k_s + k_{s+x-y} & \text{if } s = y + 1, \ldots, x, \\
    k_{s+x-y} & \text{if } s = x + 1, \ldots, r - x - y.
\end{cases}
\]

\[
m'_s = \begin{cases} 
    0 & \text{if } s = 1, \ldots, y, \\
    k_{s+x-y} & \text{if } s = y + 1, \ldots, x, \\
    k_{s+x-y} & \text{if } s = x + 1, \ldots, r - x - y.
\end{cases}
\]

The lemma follows from this correspondence. \( \square \)

### 2.6. Definition of the cubes \( \bar{C}_i(E) \)

At this point, we have defined the cubes \( C_i(E) \) for every split \( n \)-cube \( (E, f) \). Roughly speaking, all that remains is to change, by means of \( f \), the terms corresponding to \( \partial_i^k E \oplus \partial_i^l E \) by the terms in \( \partial_i^1 E \).

By the collection of lemmas above, this will be the case whenever \( i_l = k - 1 \) and \( j_l = 1 \). Therefore, let \( j \in \{0, 1, 2\}^n \) and \( i \in [0, k - 1]^n \). Let

\[
\begin{align*}
&\triangleright w(j) = (w_1, \ldots, w_{r_w(j)}) \text{ where } w_1 < \cdots < w_{r_w(j)} \text{ are the indices such that } j_{w_m} = 1 \text{ and } i_{w_m} = k - 1. \\
&\triangleright v(j) = (v_1, \ldots, v_{r_v(j)}) \text{ where } v_1 < \cdots < v_{r_v(j)} \text{ are the indices such that } j_{v_m} = 1 \text{ and } i_{v_m} \neq k - 1.
\end{align*}
\]

Then, by lemma \( 2.30 \)

\[
C_i(E) \cong \bigoplus_{m \in \{0, 2\}^r} C_{\partial_w(j)(i)} \left( \bigoplus_{n \in \{0, 2\}^r} E^\sigma_{w(j)} \right) \partial_{w(j)}(\sigma^m_{w(j)}(j)).
\]

Recall that there is an isomorphism

\[
\bigoplus_{n \in \{0, 2\}^r} E^\sigma_{w(j)} \cong \partial_{w(j)}^1 E.
\]

This motivates the following definition.
Definition 2.32. Let \((E, f)\) be a split \(n\)-cube and let \(i \in [0, k-1]^n\). The \(n\)-cube \(\widetilde{C}_i(E)\) is defined by:

\[
(2.33) \quad \widetilde{C}_i(E)^j = \bigoplus_{m \in \{0,2\}^{r_w(j)}} C_{\partial_w(j)(i)}(\partial_{w(j)}^1 E) \partial_{w(j)}(\sigma_{w(j)}^m(j)).
\]

The morphisms in \(\widetilde{C}_i(E)\) are given as follows.

(i) If \(l\) with \(i_l = k-1\) does not exist, then the cube \(\widetilde{C}_i(E)\) is split.

(ii) If \(i_l = k-1\), then the morphisms in the cube are induced by the fixed isomorphisms \(\partial_{w(j)}^1(E) \cong \bigoplus_{n \in \{0,2\}^{r_w(j)}} \partial_w^n E\) and the canonical isomorphisms in lemma 2.30.

Since all isomorphisms are fixed, the following proposition is a consequence of lemmas 2.25, 2.28, 2.29 and 2.30.

Proposition 2.34. Let \((E, f)\) be a split \(n\)-cube, \(i \in [0, k-1]^n\) and \(l \in \{1, \ldots, n\}\).

(i) If \(i_l = 0\), then \(\partial^0_{l} \widetilde{C}_i(E) = \widetilde{C}_{\partial_{l}(i)}(\partial^0_{l} E)\).

(ii) If \(i_l \neq 0\), then \(\partial^0_{l} \widetilde{C}_i(E) = \partial^0_{l} \widetilde{C}_{\partial_{l}(i)}(E)\).

(iii) If \(i_l = k-1\), then \(\partial^1_{l} \widetilde{C}_{i}(E) = \widetilde{C}_{\partial_{i}(l)}(\partial^2_{l} E)\) and \(\partial^1_{l} \widetilde{C}_{i}(E) = \widetilde{C}_{\partial_{i}(l)}(\partial^1_{l} E)\).

(iv) Lemma 2.29 remains valid for the cubes \(\widetilde{C}_i(E)\).

\[\square\]

Remark 2.35. Let \((E, f)\) be a split \(n\)-cube. Observe that by the choice of isomorphisms, for every \(j\) with \(j_l = 1\) and \(i_l = k-1\), the arrows

\[
\widetilde{C}_i(E)^j_{\partial_{l}^0(j)} \to \widetilde{C}_i(E)^j_{\partial_{l}^1(j)} \quad \text{and} \quad \widetilde{C}_i(E)^j_{\partial_{l}^1(j)} \to \widetilde{C}_i(E)^j_{\partial_{l}^2(j)}
\]

are induced by the arrows

\[
\partial^0_{l} E \to \partial^1_{l} E, \quad \partial^1_{l} E \to \partial^2_{l} E \quad \text{and} \quad \partial^2_{l} E \to \partial^0_{l} E \oplus \partial^2_{l} E \xrightarrow{f} \partial^1_{l} E.
\]

2.7. Definition of \(\Psi^k\). In this section, we define a morphism

\[
\mathbb{Z} \text{Sp}_n(X) \xrightarrow{\Psi^k} \mathbb{Z} \text{SG}^k(X)_n
\]

using the cubes \(\widetilde{C}_i(E)\) constructed in the previous section.

Recall that when \(i_l \neq k-1\), the exact sequence \(\partial^2_{l} \widetilde{C}_i(E)\) is canonically isomorphic to the simple associated to a 2-iterated cochain complex of \((n-1)\)-cubes, of lengths \((k-i_l-1, i_l+1)\). We define

\[
\mathbb{Z} \text{Sp}_n(X) \xrightarrow{\Psi^k} \mathbb{Z} \text{SG}^k(X)_n = \bigoplus_{m=1}^{k-1} \mathbb{Z} \text{SG}^{m,k}(X)_{n-1} \oplus \mathbb{Z} \text{SG}^k_1(X)_n
\]

\[(E, f) \mapsto (\Psi^1_2(E), \ldots, \Psi^{k-1}_2(E), \Psi^k_1(E))\]

with

\[
\Psi^k_1(E) = \sum_{i \in [0,k-1]^n} \widetilde{C}_i(E),
\]

\[
\Psi^{m,k}_2(E) = \sum_{l=1}^{n} (-1)^{m+l+1} \sum_{i \in [0,k-1]^n, i_l = m-1} \partial^2_{l} \widetilde{C}_i(E), \quad \text{for } m = 1, \ldots, k-1,
\]

where in the last equality we consider, by proposition 2.34 (iv), \(\partial^2_{l} \widetilde{C}_i(E)\) as a 2-iterated complex.
Remark 2.36. Observe that in the last definition, considering $\partial^2_l \tilde{C}_i(E)$ as a tensor product of complexes involves changing the order in the summations. To be precise, recall that the terms corresponding to the indices $n$ with $n_l = 0$ form the first complex in the tensor product, and the ones with $n_l = 1$ form the second complex. The tensor product of the Koszul complexes in each summand of $\tilde{C}_i(E)$ is ordered by the lexicographic order. Hence, the two orders would only agree for $l = 1$. For instance, 

$\rhd$ the face $\partial^2_2$ of the cube $C_{00}(E)$ (see example $k = 2, n = 2$), is

$$[\Psi^1(E^{00}) \otimes \Psi^1(E^{02}), \Psi^1(E^{00}) \otimes \Psi^1(E^{22}) \oplus \Psi^1(E^{02}) \otimes \Psi^1(E^{20})],$$

$\rhd$ this complex, viewed as a tensor product of complexes, is

$$\begin{bmatrix}
E^{00} \otimes E^{02} & E^{00} \otimes E^{02} & E^{00} \otimes E^{22} \oplus E^{20} \otimes E^{02} & E^{00} \otimes E^{22} \oplus E^{20} \otimes E^{02} \\
E^{00} \otimes E^{02} & E^{00} \otimes E^{02} & E^{00} \otimes E^{22} \oplus E^{20} \otimes E^{02} & E^{00} \otimes E^{22} \oplus E^{20} \otimes E^{02}
\end{bmatrix},$$

Notice the difference in the order of $\Psi^1(E^{20}) \otimes \Psi^1(E^{02})$.

Hence, strictly speaking, $\Psi^k$ cannot be a chain morphism. However, for every split $n$-cube, the composition of $\Psi^k$ with $\varphi$ leads to a collection of $n$-cubes (see the definition of $\varphi$ in (2.11)). The locally free sheaves of these cubes are direct sums, tensor products, exterior products and symmetric products of the locally free sheaves $E^j$.

We can map, with the corresponding canonical isomorphism, every $n$-cube of $\varphi \circ \Psi^k(E)$ to the $n$-cube whose summands are all ordered by the lexicographic order. Then, $\varphi \circ \Psi^k$ is a chain morphism.

This trick can only be performed after the composition with $\varphi$, and cannot be corrected in the definition of $\Psi^k$.

Proposition 2.37. Let $E$ be a split $n$-cube. Then, there is a canonical isomorphism

$$d_s \Psi^k(E) \cong \text{can } \Psi^k(dE).$$

Proof. We have to see that

\begin{equation}
(2.38) \quad \Psi^m_{2,k}(dE) = -d\Psi^m_{2,k}(E), \quad \text{for } m = 1, \ldots, k - 1,
\end{equation}

\begin{equation}
(2.39) \quad \Psi^k_1(dE) = \sum_{m=1}^{k-1} (-1)^m \Phi^m(\Psi^m_{2,k}(E)) + d\Psi^k_1(E).
\end{equation}

We start by proving (2.39). By definition,

$$d\Psi^k_1(E) = \sum_{i=0}^{k-1} \sum_{t=0}^{n} (-1)^{t} \tilde{C}_{i}(E) = \sum_{i=0}^{k-1} \sum_{t=0}^{n} (-1)^{t} \partial_l^i \tilde{C}_{i}(E).$$

Then, by proposition 2.34

$$d\Psi^k_1(E) \cong \text{can } \sum_{l=1}^{k} \sum_{i=0}^{k-1} (-1)^{l} \left[ \sum_{i=0}^{k-1} \tilde{C}_{\partial_l^i(i)}(\partial_l^j E) - \sum_{i=0}^{k-1} \tilde{C}_{\partial_l^i(i)}(\partial_l^j E) \right]
\begin{equation}
\begin{aligned}
&+ \sum_{i=0}^{k-1} \tilde{C}_{\partial_l^i(i)}(\partial_l^j E) \sum_{m=0}^{k-2} \sum_{i=0}^{k-1} \tilde{C}_{\partial_l^i(i)}(\partial_l^j E) \\
&+ \sum_{m=0}^{k-1} \tilde{C}_{\partial_l^i(i)}(\partial_l^j E) \sum_{m=0}^{k-1} \tilde{C}_{\partial_l^i(i)}(\partial_l^j E) \\
&+ \sum_{m=0}^{k-1} \tilde{C}_{\partial_l^i(i)}(\partial_l^j E) \sum_{m=0}^{k-1} \tilde{C}_{\partial_l^i(i)}(\partial_l^j E)
\end{aligned}
\end{equation}.$$
Therefore,
\[ d\Psi^k_1(E) \approx \text{can} \sum_{l=1}^{n} \sum_{s=0}^{2} (-1)^{l+s} \sum_{i \in [0,k-1]^{n-1}} \bar{C}_i(\partial^s_l E) \]
\[ + \sum_{l=1}^{n} (-1)^{l} \Phi_{m+1}(\partial^2_l \bar{C}_1(E)) \]
\[ = \Psi^k_1(dE) - \sum_{m=1}^{k-1} (-1)^{m} \Phi_{m}(\Psi^m_{2,k}(E)), \]

and equality (2.39) is proven. Let us prove now (2.38). First of all observe that due to the alternating sum of \( \partial^2_l \) in the definition of \( \Psi^m_{2,k} \), we have
\[ \sum_{r=1}^{n-1} (-1)^{r} \partial^2_r \Psi^m_{2,k}(E) = 0. \]

By the same argument, considering the first equality of (iii) in proposition 2.34, we have
\[ \sum_{r=1}^{n} (-1)^{r} \Psi^m_{2,k}(\partial^2_r E) = 0. \]

Therefore, we are left to see that
\[ \sum_{r=1}^{n} (-1)^{r} (\Psi^m_{2,k}(\partial^0_r E) - \Phi^m_{2,k}(\partial^1_r E)) = - \sum_{r=1}^{n-1} (-1)^{r} (\partial^0_r - \partial^1_r) \Psi^m_{2,k}(E). \]

Recall that for all \( j, \) \( \partial^j_r \partial^1_l = \begin{cases} \partial^j_l \partial^1_{r+1}, & \text{if} \ r \geq l, \\
\partial^1_{l-1} \partial^j_r, & \text{if} \ r < l. \end{cases} \)

Hence, we split the following expression accordingly:
\[ \sum_{r=1}^{n-1} (-1)^{r} \partial^0_r \Psi^m_{2,k}(E) = (A) + (B), \]

with:
\[ (A) = \sum_{l=2}^{n-1} \sum_{r=1}^{l-1} (-1)^{m+l+r+1} \sum_{i \in [0,k-1]^{n}} \partial^0_r \partial^2_l \bar{C}_1(E), \]
\[ (B) = \sum_{l=1}^{n-1} \sum_{r=l}^{n-1} (-1)^{m+l+r+1} \sum_{i \in [0,k-1]^{n}} \partial^0_r \partial^2_l \bar{C}_1(E). \]
Using the mentioned equalities for the faces and the identities (i) – (iii) of proposition 2.3.4, we have:

\[(A) = \sum_{l=2}^{n} \sum_{r=1}^{l-1} (-1)^{m+l+r+1} \sum_{i \in [0,k-1]^n \atop i_r=m-1} \partial^2_{l-1} \partial_i^0 \tilde{C_i}(E) \]

\[= \sum_{l=2}^{n} \sum_{r=1}^{l-1} (-1)^{m+l+r+1} \sum_{i \in [0,k-1]^n \atop i_r=m-1} \partial^2_{l-1} \partial^1_{i} \tilde{C_i}_{l-1}(E) + \sum_{i \in [0,k-1]^n \atop i_r=m-1} \partial^2_{l-1} \tilde{C_i}(\partial^0_i E) \]

\[= \sum_{l=2}^{n} \sum_{r=1}^{l-1} (-1)^{m+l+r+1} \sum_{i \in [0,k-1]^n \atop i_r=m-1} \partial^1_{i} \partial^2_{i} \tilde{C_i}(E) + \]

\[\sum_{l=1}^{n-1} \sum_{r=1}^{l+1} (-1)^{m+l+r} \sum_{i \in [0,k-1]^n \atop i_r=m-1} \partial^2_{l} \tilde{C_i}(\partial^0_i E) - \partial^2_{l} \tilde{C_i}(\partial^1_i E) \]

Reasoning analogously, we obtain

\[(B) = \sum_{l=1}^{n-1} \sum_{r=1}^{n} (-1)^{m+l+r+1} \sum_{i \in [0,k-1]^n \atop i_r=m-1} \partial^1_{i} \partial^2_{i} \tilde{C_i}(E) + \]

\[\sum_{l=1}^{n-1} \sum_{r=1}^{n} (-1)^{m+l+r} \sum_{i \in [0,k-1]^n \atop i_r=m-1} \partial^2_{l} \tilde{C_i}(\partial^0_i E) - \partial^2_{l} \tilde{C_i}(\partial^1_i E) \]

It follows that

\[\sum_{r=1}^{n-1} (-1)^{r} \partial^0_i \Psi^m_k(E) = \sum_{r=1}^{n-1} (-1)^{r} \Psi^m_k(E) - \sum_{r=1}^{n} (-1)^{r} (\partial^2_{l} \tilde{C_i}(\partial^0_i E) - \partial^2_{l} \tilde{C_i}(\partial^1_i E)) \]

\[= \sum_{r=1}^{n-1} (-1)^{r} \Psi^m_k(E) - \sum_{r=1}^{n} (-1)^{r} (\Psi^k m(E) - \Psi^k m(E)) \]

as desired. \( \square \)

For every \( n \), let

\[\Psi^k : ZSp_n(X) \rightarrow ZC_n(X)\]

be the composition

\[\Psi^k : ZSp_n(X) \xrightarrow{\Psi^k} ZSC^k(X)_n \xrightarrow{\mu_{Z^0}} ZC_n(X),\]
modified by the canonical isomorphisms, so that every direct sum of tensor, exterior and symmetric products is ordered by the lexicographic order of the corresponding multi-indices.

**Corollary 2.40.** For every scheme $X$, there is a well-defined chain morphism

$$
\Psi^k : \mathbb{Z}Sp_*(X) \to \mathbb{Z}C_*(X).
$$

\qed

**Example 2.41 (k = 2, n = 1).** Let us have a look at the situation with $k = 2$ and $n = 1$, as considered in the example cases (see section 2.3). Let $(E, f) : E^0 \to E^1 \to E^2$ be a split cube (with splitting $f$). In the complex $\mathbb{Z}SG^k(X)_*$, the image of $(E, f)$ by $\Psi^2$ is

$$(\Psi^1(E^0) \otimes \Psi^1(E^2), \tilde{C}_0(E, f) + \tilde{C}_1(E, f)) \in \mathbb{Z}G^1_2(X)_0 \oplus \mathbb{Z}G^2_1(X)_1.$$  

Applying $\varphi$ to this element, we get the following linear combination of short exact sequences

$$
-3(E^0 \otimes E^2 \to E^0 \otimes E^2) + (E^0 \otimes E^2 \to E^0 \otimes E^2 + E^0 \otimes E^2 \to E^0 \otimes E^2) -
-2(\wedge^2 E^0 \to \wedge^2 E^0 \oplus E^0 \otimes E^2 \to E^0 \otimes E^2) - 2(\wedge^2 E^0 \oplus E^0 \otimes E^2 \to \wedge^2 E^1 \to \wedge^2 E^2) +
+(E^0 \otimes E^0 \to E^0 \otimes E^0 \oplus E^0 \otimes E^2 \oplus E^0 \otimes E^2 \to E^0 \otimes E^2 + E^0 \otimes E^2) +
+(E^0 \otimes E^0 \otimes E^0 \otimes E^2 \otimes E^0 \otimes E^2 \to E^1 \otimes E^1 \to E^2 \otimes E^2).
$$

By the action of $\mu$, a short exact sequence $A \to B \xrightarrow{g} C$ transforms into

$$-(0 \to A \to \ker g) + (\ker g \to B \to C).$$

Therefore, the explicit expression of $\Psi^2(E, f)$ is obtained by transforming by $\mu$ each of the short exact sequences of the linear combination detailed above.

One easily checks that the differential of $\Psi^2(E, f)$ is $-2(- \wedge^2 E^0 + \wedge^2 E^1 - \wedge^2 E^2) + (-E^0 \otimes E^0 + E^1 \otimes E^1 - E^2 \otimes E^2)$ which is exactly $-\Psi^2(E^0) + \Psi^2(E^1) - \Psi^2(E^2)$ as desired.

3. The transgression morphism

Fix $\mathcal{C}_B$ to be a category of schemes over a base scheme $B$. In this section, we introduce all the ingredients for the definition of Adams operations on the rational algebraic $K$-theory of a regular noetherian scheme.

Let $X$ be a scheme. We first define a chain complex $\tilde{\mathbb{Z}}C^\square_*(X)$ that is the target for the Adams operations. Then, we prove that it is quasi-isomorphic to the chain complex of cubes with rational coefficients. Hence, its rational homology groups are isomorphic to the rational $K$-groups. Finally, we define a morphism, the transgression morphism, from $\mathbb{Z}C_*(X)$ to a new chain complex $\mathbb{Z}Sp_\square(X)$, whose image consists only of split cubes. Then, for each $k$, the morphism $\Psi^k$ defined in the previous section induces a morphism

$$
\Psi^k : \mathbb{Z}Sp_\square(X) \to \tilde{\mathbb{Z}}C^\square_*(X).
$$

Composing with the transgression morphism we obtain a chain complex (denoted, by abuse of notation, by $\Psi^k$):

$$
\Psi^k : NC_*(X) \to \mathbb{Z}C_*(X) \xrightarrow{T} \mathbb{Z}Sp_\square(X) \to \tilde{\mathbb{Z}}C^\square_*(X).
$$
3.1. **The transgression chain complex.** Let \( \mathbb{P}^1 = \mathbb{P}^1_B \) be the projective line over the base scheme \( B \) and let 
\[
\square = \mathbb{P}^1 \setminus \{1\} \cong \mathbb{A}^1.
\]
The cartesian product \((\mathbb{P}^1)^n\) has a cocubical scheme structure. Specifically, the face and degeneracy maps
\[
\delta^i_j : (\mathbb{P}^1)^n \rightarrow (\mathbb{P}^1)^{n+1}, \quad i = 1, \ldots, n, \ j = 0, 1,
\]
\[
\sigma^i_j : (\mathbb{P}^1)^n \rightarrow (\mathbb{P}^1)^{n-1}, \quad i = 1, \ldots, n,
\]
are defined as
\[
\delta^i_0(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, 0 : 1, x_i, \ldots, x_n),
\]
\[
\delta^i_1(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, 1 : 0, x_i, \ldots, x_n),
\]
\[
\sigma^i_0(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).
\]
These maps satisfy the usual identities for a cocubical object in a category, and leave invariant \( \square \). Hence, both \((\mathbb{P}^1)^n\) and \( \square \) are cocubical schemes.

Let \( X \times (\mathbb{P}^1)^n \) and \( X \times \square^n \) denote \( X \times_B (\mathbb{P}^1)^n \) and \( X \times_B \square^n \) respectively. Since most of the constructions will be analogous for \( \mathbb{P}^1 \) and for \( \square \), we write
\[
\mathcal{B} = \mathbb{P}^1 \text{ or } \square.
\]

For \( i = 1, \ldots, n \) and \( j = 0, 1 \), consider the chain morphisms induced on the complex of cubes
\[
\delta^i_1 = (Id \times \delta^i_1)^*: \mathbb{Z}C_*(X \times \mathbb{B}^n) \rightarrow \mathbb{Z}C_*(X \times \mathbb{B}^{n-1}),
\]
\[
\sigma^i_1 = (Id \times \sigma^i_1)^*: \mathbb{Z}C_*(X \times \mathbb{B}^{n-1}) \rightarrow \mathbb{Z}C_*(X \times \mathbb{B}^n).
\]
These maps endow \( \mathbb{Z}C_*(X \times \mathbb{B}^n) \) with a cubical chain complex structure. Observe that if \((x_1 : y_1), \ldots, (x_n : y_n)\) are homogeneous coordinates of \((\mathbb{P}^1)^n\), then \( \delta^0_1 \) corresponds to the restriction map to the hyperplane \( x_i = 0 \) and \( \delta^1_1 \) corresponds to the restriction map to the hyperplane \( y_i = 0 \). On the affine lines, with coordinates \((t_1, \ldots, t_n)\) where \( t_i = \frac{x_i}{y_i} \), the map \( \delta^0_1 \) corresponds to the restriction map to the hyperplane \( t_i = 0 \) and the map \( \delta^1_1 \) to the restriction map to the hyperplane \( t_i = \infty \).

Let \( \mathbb{Z}C^{\mathcal{B}}_{\cdot, \cdot}(X) \) be the 2-iterated chain complex given by
\[
\mathbb{Z}C^{\mathcal{B}}_{r,n}(X) := \mathbb{Z}C_r(X \times \mathbb{B}^n),
\]
and differentials
\[
d = d_{\mathbb{Z}C_*(X \times \mathbb{B}^n)}, \quad \delta = \sum (-1)^{i+j} \delta^i_j.
\]
Denote by \((\mathbb{Z}C^\mathcal{B}_*(X), d_s)\) the associated simple complex.

Observe that, by functoriality, the face and degeneracy maps \( \delta^i_j \) and \( s^i_j \), as defined in sections 1.3 and 1.4, commute with \( \delta^i_1 \) and \( \sigma^i_1 \). Therefore, there are analogous 2-iterated chain complexes
\[
\bar{\mathbb{Z}}C^{\mathcal{B}}_{r,n}(X) := \mathbb{Z}C_r(X \times \mathbb{B}^n)/\mathbb{Z}D_r(X \times \mathbb{B}^n),
\]
\[
NC^{\mathcal{B}}_{r,n}(X) := NC_r(X \times \mathbb{B}^n).
\]
Recall from section 1.3 that \( NC_*(X \times \mathbb{B}^n) \) is the normalized complex of cubes in \( X \times \mathbb{B}^n \) and from section 1.3 that \( \mathbb{Z}D_*(X \times \mathbb{B}^n) \) is the complex of degenerate cubes in \( X \times \mathbb{B}^n \). That is, we consider the normalized complex of cubes and the quotient by degenerate cubes to the first direction of the 2-iterated complex \( \mathbb{Z}C^{\mathcal{B}}_{r,n}(X) \).
In addition, the second direction of these 2-iterated complexes corresponds to the chain complex associated to a cubical abelian group. Therefore, one has to factor out by the degenerate elements.

Let $\mathbb{Z}C_r(X \times \square^n)_{\text{deg}} \subset \mathbb{Z}C_r(X \times \square^n)$ be the subcomplex consisting of the degenerate elements, i.e. that lie in the image of $\sigma_i$ for some $i = 1, \ldots, n$. Analogously, we define the complexes

$$NC_{r,n}(X)_{\text{deg}} = NC_{r,n}(X) \cap \mathbb{Z}C_r(X \times \square^n)_{\text{deg}},$$

and

$$\tilde{NC}_{r,n}(X)_{\text{deg}} = \mathbb{Z}C_r(X \times \square^n)_{\text{deg}}/\mathbb{Z}D_r(X \times \square^n)_{\text{deg}}$$

of degenerate elements in $NC_{r,n}(X)$ and $\tilde{NC}_{r,n}(X)$ respectively.

We define the 2-iterated chain complexes,

$$\tilde{NC}_{r,n}(X) := \tilde{NC}_{r,n}(X)/\tilde{NC}_{r,n}(X)_{\text{deg}},$$

and

$$NC_{r,n}(X) := NC_{r,n}(X)/NC_{r,n}(X)_{\text{deg}}.$$

Denote by $(\tilde{NC}_{s}(X), d_s)$ and $(NC_{s}(X), d_s)$ the simple complexes associated to these 2-iterated chain complexes.

**Proposition 3.1.** If $X$ is a regular noetherian scheme, the natural morphism of complexes

$$NC_{s}(X) = NC_{s,0}(X) \to NC_{s}(X)$$

induces an isomorphism in homology groups with coefficients in $\mathbb{Q}$.

**Proof.** Consider the first quadrant spectral sequence with $E^0$ term given by

$$E_{r,n}^0 = NC_{r,n}(X) \otimes \mathbb{Q}.$$

When it converges, it converges to the homology groups $H_*(NC_{s}(X), \mathbb{Q})$. If we see that for all $n > 0$ the rational homology of the complex $NC_{s,n}(X)$ is zero, the spectral sequence converges and the proposition is proven.

This is proved by an induction argument. For every $j = 1, \ldots, n$, let

$$NC_{r,n}^{\square,j}(X)_{\text{deg}} = \sum_{i=1}^{j} \sigma_i(NC_{s,n-1}(X)) \subseteq NC_{r,n}(X)_{\text{deg}}$$

and let $NC_{s,n}^{\square,j}(X)$ be the respective quotient. We will show that, for all $n > 0$ and $j = 1, \ldots, n$,

$$H_*(NC_{s,n}^{\square,j}(X), \mathbb{Q}) = 0.$$

For $j = 1$ and $n > 0$,

$$NC_{s,n}^{\square,1}(X) = NC_{s,n}(X)/\sigma_1(NC_{s,n-1}(X)).$$

By the homotopy invariance of algebraic $K$-theory of regular noetherian schemes, the rational homology of this complex is zero. Then, if $j > 1$ and $n > 1$, $NC_{s,n}^{\square,j}(X)$ is the cokernel of the monomorphism

$$NC_{s,n-1}(X) \xrightarrow{\sigma_j} NC_{s,n}^{\square,j-1}(X)$$

$$E \xrightarrow{} \sigma_j(E).$$

Since by the induction hypothesis both sides have zero rational homology, so does the cokernel. 

$\square$
Observe that in the proof of last proposition, the key point was that for regular noetherian schemes, the $K$-groups of $X \times \Box^n$ are isomorphic to the $K$-groups of $X$. In the case of projective lines, the situation is slightly trickier, because the $K$-groups of $X \times \mathbb{P}^1$ are not isomorphic to the $K$-groups of $X$. We have to use the Dold-Thom isomorphism relating both groups. This implies that we shall also factor out by the class of the canonical bundle on $\mathbb{P}^1$.

Let $p_1, \ldots, p_n$ be the projections onto the $i$-th coordinate of $(\mathbb{P}^1)^n$. Consider the invertible sheaf $\mathcal{O}(1) := \mathcal{O}_{p_1}(1)$, the dual of the tautological sheaf of $\mathbb{P}^1$, $\mathcal{O}_{\mathbb{P}^1}(-1)$. We define then the 2-iterated chain complexes

$$NC^p_{r,n}(X)_{\text{deg}} := \sum_{i=1}^{n} \sigma_i(NC^p_{r,n-1}(X)) + p_i^* \mathcal{O}(1) \otimes \sigma_i(NC^p_{r,n-1}(X)),$$

$$NC^\tilde{p}_{r,n}(X) := NC^\tilde{p}_{r,n}(X)/NC^p_{r,n}(X)_{\text{deg}}.$$ 

Denote by $(NC^\tilde{p}_s(X), ds)$ the simple complex associated to this 2-iterated chain complex.

**Proposition 3.2.** If $X$ is a regular noetherian scheme, the natural morphism of complexes

$$NC^*_s(X) = NC^\tilde{p}_s,0(X) \to NC^\tilde{p}_s(X)$$

induces an isomorphism on homology with coefficients in $\mathbb{Q}$.

**Proof.** The proof is analogous to the proof of the last proposition. By considering the spectral sequence associated with the homology of a 2-iterated complex, we just have to see that for all $j$,

$$H_*(NC^\tilde{p}_{r,n}(X), \mathbb{Q}) = 0.$$ 

For $j = 1$ and $n > 0$, it follows from the Dold-Thom isomorphism on algebraic $K$-theory of regular noetherian schemes. For $j > 1$ and $n > 1$, $NC^\tilde{p}_{r,n}(X)$ is the cokernel of the monomorphism

$$NC^\tilde{p}_{r,n-1}(X) \oplus NC^\tilde{p}_{r,n-1}(X) \to NC^\tilde{p}_{r,n}(X),$$

$$(E_0, E_1) \mapsto \sigma_j(E_0) + p_j^* \mathcal{O}(1) \otimes \sigma_j(E_1).$$

Since by the induction hypothesis both sides have zero rational homology, so does the cokernel. \qed

**Remark 3.3.** Let

$$\mathbb{Z}C^p_{r,n}(X)_{\text{deg}} = \sum_{i=1}^{n} \sigma_i(\mathbb{Z}C^p_{r,n-1}(X)) + p_i^* \mathcal{O}(1) \otimes \sigma_i(\mathbb{Z}C^p_{r,n-1}(X)),$$

$$\tilde{\mathbb{Z}}C^\tilde{p}_{r,n}(X)_{\text{deg}} = \mathbb{Z}C^p_{r,n}(X)_{\text{deg}}/\mathbb{Z}D_r(X \times (\mathbb{P}^1)^n)_{\text{deg}},$$

and let

$$\tilde{\mathbb{Z}}C^\tilde{p}_{r,n}(X) := \tilde{\mathbb{Z}}C^\tilde{p}_{r,n}(X)/\tilde{\mathbb{Z}}C^\tilde{p}_{r,n}(X)_{\text{deg}}.$$ 

Denote by $(\tilde{\mathbb{Z}}C^\tilde{p}_s(X), ds)$ the simple complex associated to this 2-iterated chain complex.

It follows from the definitions that there is an isomorphism

$$\tilde{\mathbb{Z}}C^\tilde{p}_s(X) \cong NC^\tilde{p}_s(X).$$
3.2. The transgression of cubes by affine and projective lines. Let \( x \) and \( y \) be the global sections of \( \mathcal{O}(1) \) given by the projective coordinates \((x:y)\) on \( \mathbb{P}^1 \). Let \( X \) be a scheme and let \( p_0 \) and \( p_1 \) be the projections from \( X \times \mathbb{P}^1 \) to \( X \) and \( \mathbb{P}^1 \) respectively. Then, for every locally free sheaf \( E \) on \( X \), we denote
\[
E(k) := p_0^*E \otimes p_1^*\mathcal{O}(k).
\]
The following definition is taken from \[2\].

**Definition 3.4.** Let \( E : 0 \to E^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} E^2 \to 0 \) be a short exact sequence. The *first transgression by projective lines* of \( E \), \( \text{tr}_1(E) \), is the kernel of the morphism
\[
E^1(1) \oplus E^2(1) \to E^2(2),
\]
\[
(a, b) \mapsto f^1(a) \otimes x - b \otimes y.
\]
Observe that this locally free sheaf on \( X \times \mathbb{P}^1 \) satisfies that
\[
\delta_{0}^{n-1} \text{tr}_1(E) = \text{tr}_1(E)|_{x=0} = E^1,
\]
\[
\delta_{1}^{n-1} \text{tr}_1(E) = \text{tr}_1(E)|_{y=0} = \text{im} f^0 \oplus E^2.
\]
By restriction to \( \square \), we obtain the *transgression by affine lines*.

From now on, we restrict ourselves to the affine case. However, all the results can be written in terms of the complexes with projective lines.

Let \( E \) be an \( n \)-cube. We define the *first transgression* of \( E \) as the \((n-1)\)-cube on \( X \times \square^1 \) given by
\[
\text{tr}_1(E)^j := \text{tr}_1(\partial^j_{2,\ldots,n}E), \quad \text{for all } j \in \{0,1,2\}^{n-1},
\]
i.e. we take the transgression of the exact sequences in the first direction. Since \( \text{tr}_1 \) is a functorial exact construction, the \( m \)-th transgression sheaf can be defined recursively as
\[
\text{tr}_m(E) = \text{tr}_1 \text{tr}_{m-1}(E) = \text{tr}_{1 \cdot \ldots \cdot 1}(E).
\]
It is an \((n-m)\)-cube on \( X \times \square^m \). In particular, \( \text{tr}_n(E) \) is a locally free sheaf on \( X \times \square^n \).

Observe that the transgression is functorial, i.e., if \( E \xrightarrow{\psi} F \) is a morphism of \( n \)-cubes, then there is an induced morphism
\[
\text{tr}_m(E) \xrightarrow{\text{tr}_m(\psi)} \text{tr}_m(F),
\]
for every \( m = 1, \ldots, n \). In particular, for every \( n \)-cube \( E \) and \( i = 1, \ldots, n \), the morphism of \((n-1)\)-cubes
\[
\partial^0_i E \xrightarrow{f^0_i} \partial^1_i E,
\]
induces a morphism
\[
\text{tr}_m(\partial^0_i E) \xrightarrow{\text{tr}_m(f^0_i)} \text{tr}_m(\partial^1_i E),
\]
for \( m = 1, \ldots, n-1 \).

**Lemma 3.5.** For every \( n \)-cube \( E \) and \( i = 1, \ldots, n \), the following identities hold:
\[
\delta_{0}^{n} \text{tr}_n(E) = \text{tr}_{n-1}(\partial^1_i E), \quad (3.6)
\]
\[
\delta_{1}^{n} \text{tr}_n(E) = \text{im} \text{tr}_{n-1}(f^0_i) \oplus \text{tr}_{n-1}(\partial^2_i E), \quad (3.7)
\]
where the isomorphism is canonical, i.e. a combination of commutativity and associativity isomorphisms for direct sums, distributivity isomorphism for the tensor product of
a direct sum and commutativity isomorphisms of the pull-back of a direct sum with the direct sum of the pull-back.

Proof. The proof is straightforward. For \( n = 1 \) it follows from the definition. Therefore,

\[
\delta_i^0 \tr_n(E) = \delta_i^1 \tr_1 \cdot \ldots \cdot \tr_1(E) = \tr_{n-i} \partial_i^1 \tr_{i-1}(E) = \tr_{n-i}(\partial_i^1 E).
\]

For the second statement, observe first of all that there is a canonical isomorphism 
\( \tr_n(A \oplus B) \cong \tr_n(A) \oplus \tr_n(B) \). It follows by recurrence from the case \( n = 1 \). So, let \( E, F \) be two short exact sequences. Then, \( \tr_1(E \oplus F) \) is the kernel of the map

\[
(E_1 \oplus F_1)(1) \oplus (E_2 \oplus F_2)(1) \to (E_2 \oplus F_2)(2),
\]

while \( \tr_1(E) \oplus \tr_1(F) \) is the direct sum of the kernels of the maps

\[
E_1(1) \oplus E_2(1) \to E_2(2), \quad F_1(1) \oplus F_2(1) \to F_2(2).
\]

Hence, there is clearly a canonical isomorphism. Therefore

\[
\delta_i^1 \tr_n(E) = \delta_i^1 \tr_1 \cdot \ldots \cdot \tr_1(E) = \tr_{n-i}(\im \tr_{i-1}(f_i^0) \oplus \tr_{i-1}(\partial_i^2 E)) \cong \im \tr_{n-1}(f_i^0) \oplus \tr_{n-1}(\partial_i^2 E).
\]

\[\square\]

Remark 3.8. Since the isomorphisms in (3.7) are canonical, if \( E \xrightarrow{\psi} F \) is a morphism of \( n \)-cubes, for every \( i \) we obtain a commutative diagram

\[
\begin{aligned}
\delta_i^1 \tr_n(E) & \xrightarrow{\delta_i^1 \tr_n(\psi)} \delta_i^1 \tr_n(F) \\
\cong & \downarrow \\
\im \tr_{n-1}(f_i^0) \oplus \tr_{n-1}(\partial_i^2 E) & \xrightarrow{\cong} \im \tr_{n-1}(f_i^0) \oplus \tr_{n-1}(\partial_i^2 F).
\end{aligned}
\]

Observe that for every \( n \)-cube \( E \), the \( (n - m) \)-cube \( \tr_m(E) \) is obtained applying \( \tr_1 \) on the directions \( 1, \ldots, m \). In the next definition, we generalize this construction by specifying the directions in which we apply the first transgression.

Definition 3.9. Let \( i = (i_1, \ldots, i_{n-m}) \), with \( 1 \leq i_1 < \cdots < i_{n-m} \leq n \). We define, 
\( \tr^i_m(E) \in C_{n-m,m}(X) \), by

\[
\tr^i_m(E)j = \tr_m(\partial^j_i E), \quad \text{for all } j \in \{0, 1, 2\}^{n-m}.
\]

In other words, we consider the first transgression iteratively in the directions not in the multi-index \( i \), from the highest to the lowest index.

By convention, the transgressions without super-index correspond to the multi-index \( i = (m + 1, \ldots, n) \). The following lemma is a direct consequence of lemma 3.5.

Lemma 3.10. Let \( i = (i_1, \ldots, i_{n-m}) \) with \( 1 \leq i_1 < \cdots < i_{n-m} \leq n \). Then, for every fixed \( r = 1, \ldots, m \), let \( i' = i - 1^{n-m}_r \) and let \( (v_1, \ldots, v_m) \) be the ordered multi-index with the entries in \( \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{n-m}\} \). Then,

\[
\delta^0_r \tr^i_m(E) = \tr^{i'_m-1}(\partial^1_{v_r} E),
\]

\[
\delta^1_r \tr^i_m(E) \cong \im \tr^{i'_m-1}(f^0_{v_r}) \oplus \tr^{i'_m-1}(\partial^2_{v_r} E) \quad \text{(canonically)}.
\]

\[\square\]
3.3. Cubes with canonical kernels. We introduce here a new subcomplex of $\mathbb{Z}C_*(X)$, consisting of the cubes with canonical kernels. In this new class of cubes, the transgressions behave almost like a chain morphism. Namely, if $E$ is an $n$-cube with canonical kernels, we will have

$$\delta^0_i \text{tr}_n(E) = \text{tr}_{n-1}(\partial^1_i E),$$

$$\delta^1_i \text{tr}_n(E) \cong \text{tr}_{n-1}(\partial^0_i E) \oplus \text{tr}_{n-1}(\partial^2_i E).$$

(3.11)

**Definition 3.12.** Let $E$ be an $n$-cube. We say that $E$ has canonical kernels if for every $i = 1, \ldots, n$ and $j \in \{0, 1, 2\}^{n-1}$, there is an inclusion $(\partial^0_i E)^j \subset (\partial^1_i E)^j$ of sets and moreover the morphism

$$f^i_0 : \partial^0_i E \to \partial^1_i E$$

is the canonical inclusion of cubes.

Let $KC_n(X) \subseteq C_n(X)$ be the subset of all cubes with canonical kernels. The differential of $\mathbb{Z}C_*(X)$ induces a differential on $\mathbb{Z}KC_*(X)$ making the inclusion arrow a chain morphism.

Let $E$ be a 1-cube i.e. a short exact sequence $E^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} E^2$. Then, we define

$$\lambda^0_1(E) : 0 \to 0 \to E^0 \xrightarrow{f^0} \text{im} f^0 \to 0,$$

$$\lambda^1_1(E) : 0 \to \ker f^1 \to E^1 \xrightarrow{f^1} E^2 \to 0.$$

Both of them are 1-cubes with canonical kernels. Then, we define

$$\lambda_1(E) = \lambda^1_1(E) - \lambda^0_1(E) \in \mathbb{Z}KC_1(X).$$

For an arbitrary $n$-cube $E \in C_n(X)$ and for every $i = 1, \ldots, n$, let $\lambda^0_i(E)$ and $\lambda^1_i(E)$ be the $n$-cubes which along the $i$-th direction are:

$$\partial^0_i \lambda^0_i(E) = 0 \quad \partial^0_i \lambda^1_i(E) = \text{im} f^0_i$$

$$\partial^1_i \lambda^0_i(E) = \partial^0_i E \quad \partial^1_i \lambda^1_i(E) = \partial^1_i E$$

$$\partial^2_i \lambda^0_i(E) = \text{im} f^0_i \quad \partial^2_i \lambda^1_i(E) = \partial^2_i E.$$

Then, we define

$$\lambda_i(E) = -\lambda^0_i(E) + \lambda^1_i(E), \quad i = 1, \ldots, n,$$

$$\lambda(E) = \left\{ \begin{array}{ll} \lambda_n \cdots \lambda_1(E), & \text{if } n > 0, \\
E & \text{if } n = 0. \end{array} \right.$$  

**Proposition 3.13.** The map

$$\lambda : \mathbb{Z}C_n(X) \to \mathbb{Z}KC_n(X)$$

is a morphism of complexes.

**Proof.** First of all, observe that the image by $\lambda$ of any $n$-cube $E$, is a sum of $n$-cubes with canonical kernels. It is a consequence of the fact that for any commutative square of epimorphisms,

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{h} \\
C & \xrightarrow{j} & D,
\end{array}
\]
the set equality \( \ker(\ker g \to \ker h) = \ker(\ker f \to \ker j) \) holds. The equality \( d\lambda(E) = \lambda d(E) \) follows from the equalities

\[
\begin{align*}
\partial^j_1\lambda(E) &= \partial^j_1(\lambda_n \cdots \lambda_1(E)) = \partial^j_1(\lambda_n \cdots \lambda_j \cdots \lambda_1(E)), \\
\partial^0(\lambda_n \cdots \lambda_1(E)) &= \partial^0(\lambda_n \cdots \lambda_0 \cdots \lambda_1(E)), \\
\partial^j_1(\lambda_n \cdots \lambda_j \cdots \lambda_1(E)) &= \lambda_{n-1} \cdots \lambda_1(\partial^j_1 E), \quad j = 1, 2, \\
\partial^1_1(\lambda_n \cdots \lambda_0 \cdots \lambda_1(E)) &= \lambda_{n-1} \cdots \lambda_1(\partial^0_1 E).
\end{align*}
\]

\( \square \)

3.4. \textbf{The transgression morphism.} Observe that the face maps \( \delta^i_1 \) of \( \square \) (as defined in section 3.1) induce morphisms on the complex of split cubes

\[
\delta^i_1 : ZSp_r(X \times \square^n) \to ZSp_r(X \times \square^{n-1}).
\]

Let \( ZSp^\square_r(X) \) be the 2-iterated chain complex given by

\[
ZSp^\square_{r,n}(X) = ZSp_{r}(X \times \square^n),
\]

and differentials

\[
d = dSp_r(X \times \square^n), \\
\delta = \sum (-1)^{i+j}\delta^i_1.
\]

Let \( (ZSp^\square_r(X), d) \) be the associated simple complex. Using the transgressions, we define here a morphism of complexes

\[
ZKC_r(X) \xrightarrow{T} ZSp^\square_r(X)
\]

which composed with \( \lambda \) gives the \textit{transgression morphism}

\[
ZC_r(X) \xrightarrow{T} ZSp^\square_r(X).
\]

For every \( n \)-cube \( E \) with canonical kernels, the component of \( T(E) \) in \( ZSp^\square_{0,n}(X) \) is exactly \( (-1)^n tr_n(E) \). However, the assignment

\[
E \mapsto (-1)^n tr_n(E)
\]

is not a chain morphism. The failure comes from equality (3.11), since, first of all, the equality holds only up to some canonical isomorphisms, and second, a direct sum is not a sum in the complex of cubes. We will add some “correction cubes” in \( ZSp^\square_{n-m,m}(X) \), with \( m \neq n \), in order to obtain a chain morphism \( T \).

We start by constructing the morphism step by step in the low degree cases, deducing from the examples the key ideas.

\textbf{The transgression morphism for} \( n = 1, 2 \). Let \( E \) be a 1-cube with canonical kernels. Then,

\[
\delta tr_1(E) = -\delta^0_1 tr_1(E) + \delta^1_1 tr_1(E) = -E^1 + \delta^1_1 tr_1(E).
\]

We know that there is a canonical isomorphism (which in this case is the identity):

\[
\delta^1_1 tr_1(E) \cong E^0 \oplus E^2.
\]

Hence, the differential of

\[
T(E) := (- tr_1(E), E^0 \to \delta^1_1 tr_1(E) \to E^2) \in ZC^\square_{0,1}(X) \oplus ZC^\square_{1,0}(X)
\]

is exactly \( E^1 - E^0 - E^2 \in ZC_0(X) \).
Let $E$ be a 2-cube with canonical kernels. Then,

$$\delta \text{tr}_2(E) = -\delta^0 \text{tr}_2(E) + \delta^1 \text{tr}_2(E) + \delta^2 \text{tr}_2(E) - \delta_1 \text{tr}_2(E)$$

$$= -\text{tr}_1(\delta_1 E) + \delta_1 \text{tr}_2(E) + \text{tr}_1(\delta_2 E) - \delta_1 \text{tr}_2(E).$$

Let

$$T^i_1(E) = \text{tr}_1(\delta^0_i E) - \delta^1_i \text{tr}_2(E) - \text{tr}_1(\delta^2_i E),$$

where the arrows are defined by the canonical isomorphism $\delta^1_1 \text{tr}_2(E) \cong \text{tr}_1(\delta^0_1 E) \oplus \text{tr}_1(\delta^2_1 E)$. Let

$$T_2(E) = \delta^1_1 \text{tr}_1(E^0) \rightarrow \delta^1_1 \delta^1_2 \text{tr}_2(E) \rightarrow \delta^1_1 \text{tr}_1(E^{2*})$$

with the arrows induced by the canonical isomorphisms of lemma 3.5. Then, for $n = 2$, we define

$$T(E) := (\text{tr}_2(E), \sum_{i=1,2} (-1)^i T^i_1(E), T_2(E)) \in ZC^{\Box}_{0,2}(X) \oplus ZC^{\Box}_{1,1}(X) \oplus ZC^{\Box}_{2,0}(X).$$

By lemma 3.10 since $E$ is a cube with canonical kernels, these cubes are all split. We fix the splittings to be the canonical isomorphisms of lemma 3.10. Then, in $ZC^{\Box}_{0,1}(X) \oplus ZC^{\Box}_{1,0}(X)$,

$$d_s T(E) = (\delta \text{tr}_2(E) + \sum_{i=1,2} (-1)^i dT^i_1(E), - \sum_{i=1,2} (-1)^i \delta T^i_1(E) + dT_2(E))$$

$$= \sum_{i=1,2} \sum_{j=0}^1 (-1)^{i+j} T(\partial^j_i E).$$

The transgression morphism. Recall that if $j \in \{0, 1, 2\}^m$, we defined in section 2.1

$$s(j) = \# \{r \mid j_r = 1\},$$

and the multi-index $u(j) = (u_1, \ldots, u_{s(j)})$ with $u_i$ the indices such that $j_{u_i} = 1$ and ordered by $u_1 < \cdots < u_{s(j)}$. Consider the set of multi-indices

$$J^m_n := \{i = (i_1, \ldots, i_{n-m}) \mid 1 \leq i_1 < \cdots < i_{n-m} \leq n\}.$$

Then, for every $i \in J^m_n$ and $j \in \{0, 1, 2\}^{n-m}$, we define

$$i(j) = (i_{u_1}, i_{u_2} - 1, \ldots, i_{u_l} - l + 1, \ldots, i_{u_{s(j)}} - s(j) + 1).$$

Definition 3.14. Let $E$ be an $n$-cube with canonical kernels. For every $0 \leq m \leq n$ and $i \in J^m_n$, we define $T^i_{n-m,m}(E) \in ZC^{\Box}_{n-m,m}(X)$ as the $(n-m)$-cube on $X \times \Box^m$ given by:

- If $j \in \{0, 2\}^{n-m}$ then
  $$T^i_{n-m,m}(E)^j := \text{tr}_m(\delta^j_1 E) = \text{tr}_m(\delta^j_1 E).$$

- If $j \in \{0, 1, 2\}^{n-m}$ with $j_k = 1$ for some $k$, then we define
  $$T^i_{n-m,m}(E)^j := (\delta^1_i(\text{tr}_m(\delta^1_{u(j)}(E))))^j(\delta^1_{u(j)}(j)).$$
That is, we start by considering the first transgression of $E$, iteratively, in the directions $s$ not in the multi-index $i$ and in those $i_s$ with $j_s = 1$. This gives a $(n - m - s(j))$-cube on $X \times □^{n + s(j)}$. Then, we apply $\delta^j_i$ for each affine component coming from a direction with $j_s = 1$. We obtain a $(n - m)$-cube on $X \times □^m$.

Observe that in the above definition, the second case generalizes the first case.

**Lemma 3.15.** For every $n$-cube $E$ with canonical kernels and every $i \in J^m_n$, the cube $T^n_{m,m}(E)$ is split.

**Proof.** If $j \in \{0, 1, 2\}^{n-m}$, it follows by lemma 3.5 that,

$$T^n_{m,m}(E)^j = (\delta^i_{i(j)} \cdot (\odot m + s(j))(E)) [\partial u(j)] = \bigoplus_{m \in \{0, 2\}^{i(j)}} (\odot m(E))[u(j)]^m(\partial u(j)).$$

Observe that for any morphism of $n$-cubes $E \to F$, there is a commutative square

$$T^n_{m,m}(E)^j \cong \bigoplus_{m \in \{0, 2\}^{i(j)}} (\odot m(F))[u(j)]^m(\partial u(j)) \cong \bigoplus_{m \in \{0, 2\}^{i(j)}} (\odot m(F))[u(j)]^m(\partial u(j)).$$

Finally, we consider

$$(3.16) \quad T^n_{m,m}(E) := \sum_{i \in J^m_n} (-1)^{||i||+\sigma(n-m)+m} T^n_{m,m}(E) \in ZS\circ\big^m_n_\square(X),$$

where $\sigma(k) = 1 + \cdots + k = \frac{k(k+1)}{2}$.

**Proposition 3.17.** The map

$$ZKC_n(X) \xrightarrow{T} \bigoplus_{m=0}^{n} ZS\circ\big^m_n_\square(X)$$

$$E \mapsto T^n_{m,m}(E)$$

is a chain morphism.

**Proof.** We have to see that $T$ commutes with the differentials. Remember that the differential $d_s$ in the simple complex is defined, in the $(n - m, m)$-component, by $d + (-1)^{n-m}$. Therefore, we have to see that for every $m = 0, \ldots, n - 1$, the equality

$$(3.18) \quad d_{T^n_{m,m}} + (-1)^{n-m-1} T^n_{m-1,m+1} = T^n_{m-1,m} d$$

holds. The right hand side is

$$T^n_{m-1,m} d = \sum_{r=1}^{n} \sum_{j=0}^{2} (-1)^r + j T^n_{m-1,m} \partial^j_r$$

$$= \sum_{r=1}^{n} \sum_{j=0}^{2} (-1)^r + j \sum_{i \in J^m_{n-1}} (-1)^{||i||+\sigma(n-m)+m} T^n_{m-1,m} \partial^j_r.$$
We compute the terms $d$ and $\delta$ on the left hand side separately.

$$dT_{n-m,m} = \sum_{r=1}^{n-m} \sum_{i \in J_{n-m}^r} (-1)^{|i|+\sigma(n-m)+r+m}(\partial^0_r - \partial^1_r + \partial^2_r)T^i_{n-m,m}.$$ 

By definition, $\partial^l_r T^i_{n-m,m} = T^{\partial_r(i)-1}_{n-m,1,m} \partial^l_r$ if $l = 0, 2$.

Since $(-1)^{\sigma(n-m)+n-m} = (-1)^{\sigma(n-m-1)}$, we obtain

$$dT_{n-m,m} = \sum_{r=1}^{n} (-1)^r[T^{\partial^0_r}_{n-m,1,m} + T^{\partial^1_r}_{n-m,1,m} \partial^2_r ] - \sum_{i \in J_{n-m}^r} \sum_{r=1}^{n} (-1)^{|i|+\sigma(n-m)+m} \partial^1_r T^i_{n-m,m}.$$ 

For the summand corresponding to the differential $\delta$, one has that

$$(-1)^m \delta T_{n-m-1,m+1} = (1) + (2),$$

with

$$(1) = -\sum_{r=1}^{m} \sum_{i \in J_{n-m-1}^r} (-1)^{|i|+\sigma(n-m-1)+r} \partial^0_r T^i_{n-m-1,m+1},$$

$$(2) = \sum_{r=1}^{m} \sum_{i \in J_{n-m-1}^r} (-1)^{|i|+\sigma(n-m-1)+r} \partial^1_r T^i_{n-m-1,m+1}.$$ 

For every $j \in \{0, 1, 2\}^{n-m-1}$, we obtain by definition

$$(\partial^0_r T^i_{n-m,m})^j = \left(\partial^0_r \delta^1_r T^{\partial_u(j)}_{m+s(j)}\right) \partial_u(j).$$

Using the commutation rules of the faces $\delta^*_r$ and $\partial^*_r$, we obtain that

$$\partial^0_r T^i_{n-m,m} = T^{i-1}_{n-m-1,m} \partial^1_r,$$

where $l$ is the maximal between the indices $k$, such that $r+k-1 \geq i_k$. Therefore,

$$(1) = \sum_{r=1}^{m} \sum_{l=0}^{n-m-1} \sum_{i \in J_{n-m}^r} (-1)^{|i|+\sigma(n-m-1)+r+l+1}T^{i-1}_{n-m-1,m} \partial^1_{r+l}$$

$$= \sum_{s=1}^{n} \sum_{l=0}^{n-m-1} \sum_{i \in J_{n-m}^r} (-1)^{|i|+\sigma(n-m-1)+r+l} \partial^1_{s+l}$$

$$= \sum_{s=1}^{n} \sum_{i \in J_{n-m}^r} (-1)^{|i|+\sigma(n-m-1)+r+s} \partial^1_{s+l} T^{i}_{n-m-1,m}$$

$$= (-1)^{n+1} \sum_{s=1}^{n} (-1)^{s+1} T^{1}_{n-m-1,m} \partial^1_s.$$
All that remains is to see that
\[ (2) = (-1)^{n-m-1} \sum_{i \in j_{n-m}} \sum_{r=1}^{n} (-1)^{|i|+r+\sigma(n-m)} \partial_{r}^{1} T_{n-m,m}^{i} =: (\ast). \]

Recall that
\[ (\partial_{r}^{1} T_{n-m,m}^{i})^{j} = (T_{n-m,m}^{i})^{s_{i,j}} = \left( \delta_{1}^{i} s_{i,j} \right) \left( \partial_{n+m+s(j)+1} \right) \partial_{s_{i,j}}(j). \]

An easy calculation shows that \( \delta_{1}^{i} s_{i,j} = \delta_{i}^{j} \delta^{i}_{n-r+1} \). Therefore,
\[ \partial_{r}^{1} T_{n-m,m}^{i} = \delta^{i}_{n-r+1} T_{n-m,m}^{r} \]
and hence,
\[ (\ast) = \sum_{i \in j_{n-m}} \sum_{r=1}^{n} (-1)^{|i|+r+\sigma(n-m)} \partial_{r}^{1} T_{n-m,m}^{i} \]
\[ = \sum_{i \in j_{n-m}} \sum_{r=1}^{n} (-1)^{|i|+r+\sigma(n-m)} \delta_{r}^{1} T_{n-m,m}^{i} \]
\[ = \sum_{i \in j_{n-m}} \sum_{r=1}^{n} (-1)^{|i|+r+\sigma(n-m)} \delta_{r}^{1} T_{n-m,m}^{i} \]
\[ = \sum_{i \in j_{n-m}} \sum_{r=1}^{n} (-1)^{(n-m-1)+|i|+r+\sigma(n-m-1)} \delta_{r}^{1} T_{n-m,m}^{i} = (2), \]
and the proposition is proved.

\[ \square \]

4. ADAMS OPERATIONS ON RATIONAL ALGEBRAIC K-THEORY

To sum up, we have defined the following chain morphisms:

- In section 3, we defined a morphism
  \[ C_{n}(X) \xrightarrow{T} \text{Sp}^{\square}(X). \]
  That is, a collection of split cubes on \( X \times \square^{*} \) is assigned to every \( n \)-cube.

- In section 2 for every \( k \geq 1 \), we defined a chain morphism
  \[ \Psi^{k} : \text{Sp}_{n}(X) \rightarrow C_{n}(X). \]

Since the maps \( \Psi^{k} \) and \( T \) are functorial, they are natural on schemes \( X \) in \( C_{B} \). Therefore, there is an induced map of 2-iterated chain complexes
\[ \Psi^{k} : \text{Sp}^{\square}_{n}(X) \rightarrow C_{n}(X) \rightarrow \tilde{C}_{n}(X), \]
which induces a morphism on the associated simple complexes
\[ \Psi^{k} : \text{Sp}^{\square}_{n}(X) \rightarrow C^{\square}_{n}(X) \rightarrow \tilde{C}^{\square}_{n}(X). \]
The composition with the morphism \( T \) gives a morphism
\[ \Psi^{k} : C_{n}(X) \rightarrow C^{\square}_{n}(X) \rightarrow \tilde{C}^{\square}_{n}(X) \cong NC_{n}(X). \]
Finally, considering the normalized complex of cubes of lemma 1 we define
\[ \Psi^{k} : NC_{n}(X) \leftrightarrow C_{n}(X) \rightarrow NC_{n}^{\square}(X). \]
Corollary 4.1. For every $k \geq 1$, there is a well-defined chain morphism

$$\Psi^k : N C^\bullet_s(X) \to N C^\bullet_s(X).$$

If $X$ is a regular noetherian scheme, then for every $n$, there are induced morphisms

$$\Psi^k : K_n(X) \otimes \mathbb{Q} \to K_n(X) \otimes \mathbb{Q}.$$

Proof. It is a consequence of lemmas 1.15 and 3.2 \hfill \square

Theorem 4.2. Let $X$ be a regular noetherian scheme of finite Krull dimension. For every $k \geq 1$ and $n \geq 0$, the morphisms

$$\Psi^k : K_n(X) \otimes \mathbb{Q} \to K_n(X) \otimes \mathbb{Q}$$

of corollary 4.1 agree with the Adams operations defined by Gillet and Soulé in [6].

Proof. We have constructed a functorial morphism, at the level of chain complexes, which by definition can be extended to simplicial schemes. Moreover, they induce the usual Adams operations on the $K_0$-groups, i.e. the Adams operations derived from the lambda structure coming from the exterior product of locally free sheaves. Then, the statement follows from corollary 4.4 in [4]. \hfill \square

Corollary 4.3. The Adams operations defined here satisfy the usual identities for any finite dimensional regular noetherian scheme.

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