Dirac Operators on Quantum Weighted Projective Spaces

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Abstract

The quantum weighted projective algebras $\mathbb{C}[WP_{k,l,q}]$ are coinvariant subalgebras of the quantum group algebra $\mathbb{C}[SU_{q,2}]$. For each pair of indices $k, l$, two 2-summable spectral triples will be constructed. The first one is an odd spectral triple based on coinvariant spinors on $\mathbb{C}[SU_{q,2}]$. The second one is an even spectral triple.

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Introduction

The theory of the weighted quantum projective algebras incorporates ideas from representation theory, noncommutative geometry and geometry of singular spaces. The family of algebras $\mathbb{C}[WP_{k,l,q}]$ is parametrized by two positive coprime integers and each one is a coinvariant subalgebra in the quantum group $\mathbb{C}[SU_{q,2}]$ under a coaction of the Hopf algebra $\mathbb{C}[u, u^{-1}]$, [1]. The Hopf algebra $\mathbb{C}[u, u^{-1}]$ can be identified with the coordinate algebra of a unit sphere. Especially interesting cases occur when $l = 1$ and $k$ is a positive integer greater than one when these algebras are $q$-deformations of the coordinate algebras of singular spaces with a teardrop shape. In the classical case these have been studied extensively in groupoid theory. In this context, a singular space can be naturally associated to a Morita equivalence class of groupoids [9], [8]. Morita equivalence preserves the shape of the singular space and the types of the singularities but there are several ways of how one can construct the geometric realization of the groupoid. For example, in the case of a teardrop, a representative of the Morita class is usually constructed as follows: There is a collection of open balls and one of them is subject to an action of a finite rotation group. The orbit space of this action is a cone. The balls can be glued together using groupoid arrows so that the orbit space of the groupoid has a shape of a teardrop. The singularity is determined by the choice of the finite rotation group: so the parameter $k$ above corresponds to the singularity under the action of $\mathbb{Z}_k$. Another realization of the Morita class is given by an action of the Lie group $\mathbb{T}$ on the sphere $S^3$. The algebraic deformation $\mathbb{C}[WP_{k,l,q}]$ is based on this system. One can also realize the teardrop spaces as 2-dimensional orbifolds, [14]. The teardrops are examples of 2-dimensional orbifolds which are not global quotients under a finite group action. An orbifold always determines a Morita equivalence class of Lie groupoids [8].

The goal of this work is to put the algebras $\mathbb{C}[WP_{k,l,q}]$ into the framework of Connes’ noncommutative geometry. The coaction of a Hopf algebra on a noncommutative algebra is a noncommutative geometric analogue of a manifold with a group action, or an action groupoid, while the coinvariant subalgebra models the quotient space under the action. In Lie groupoid theory, the local group actions give rise to local diffeomorphisms. These local diffeomorphisms are applied to define a groupoid action on the tangent bundle. Then one can try to lift the action to the spinor bundle. This is not generally possible, and the obstruction is measured by the Stiefel-Whitney classes in the Lie groupoid cohomology. Whenever the lift exists one can proceed to define an invariant Dirac operator acting on the spinors. This data can be used to model the quotient space as a spectral triple consisting of the invariant function algebra, the Hilbert space of invariant spinors and the invariant Dirac operator, [5] [13].

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Two approaches to define a Hilbert space of spinors and a Dirac operator are taken here, both are based on the representation theory of the Drinfeld-Jimbo algebra \( U_q(\mathfrak{su}_2) \). In terms of representation theory, the algebra \( U_q(\mathfrak{su}_2) \) has the same complex irreducible representations as the Lie algebra \( \mathfrak{su}_2 \), but the tensor category of representations has a nontrivial braiding, see e.g. [12]. This results a violation of the permutation symmetry in the representation category which is seen as a noncommutativity of the coproduct in \( U_q(\mathfrak{su}_2) \). In the spirit of Peter-Weyl theorem, one can define the coordinate algebra \( \mathbb{C}[SU_{2,q}] \) as a Hopf dual of \( U_q(\mathfrak{su}_2) \) which is linearly spanned by the matrix elements of the irreducible representations. The tensor structure in the category of representations determines a product in \( \mathbb{C}[SU_{2,q}] \). The product is noncommutative since the coproduct of \( U_q(\mathfrak{su}_2) \) is noncommutative. The algebra \( U_q(\mathfrak{su}_2) \) is given a right and a left representation on the space of matrix elements \( \mathbb{C}[SU_{2,q}] \). These representations correspond to the actions of the invariant vector fields on the space of functions. We make the algebra \( U_q(\mathfrak{su}_2) \) a \( \mathbb{C}[u,u^{-1}] \) comodule algebra by requiring a compatibility for the coaction with the right \( U_q(\mathfrak{su}_2) \) representation. If \( M_\frac{1}{2} \) denotes a \( U_q(\mathfrak{su}_2) \) module of the highest weight \( \frac{1}{2} \), then the Hilbert space of spinors is naturally defined to be a completion of \( \mathbb{C}[SU_{2,q}] \otimes M_\frac{1}{2} \). The right \( U_q(\mathfrak{su}_2) \) representation extends to this space. The Hilbert space can now be equipped with a comodule structure which is compatible with the right representation. This is a quantum group analogue of a groupoid action on the spinors. Then we can define the coinvariant subspace. This subspace is invariant under the action of the Dirac operator on \( SU_{2,q} \).

The coinvariant spectral triple does not have a chiral grading. This is somewhat expected since it is based on a construction of an odd spectral triple in dimension 3. The Hilbert space of coinvariant spinors has an interesting structure as a representation for the coinvariant function algebra \( \mathbb{C}[WP_{k,l,q}] \). For \( k = l = 1 \) this algebra is a quantum sphere algebra. In the case of the manifold \( S^2 \), the spinor bundle is isomorphic to the line bundle associated to the principal Hopf bundle. The space of smooth spinors has a structure of projective module over the smooth functions. The coinvariant Hilbert space of spinors on \( WP_{1,1,q} \) would seem to define a realistic model for the deformed spinor based on the topology of the Hopf fibration on \( S^2 \). However, it seems to be difficult to define a chirally graded Dirac operator.

The second construction can be viewed as a toy model where the coaction on the spinor module \( M_\frac{1}{2} \) is trivial. In this case the Hilbert space of spinors is a sum of two copies of the Hilbert space completion of the coinvariant subalgebra \( \mathbb{C}[WP_{k,l,q}] \). The Dirac spectrum is fixed by requiring that for \( k = l = 1 \) it gives the classical Dirac spectrum on the manifold \( S^2 \). The chiral grading can be defined easily.

The algebras \( \mathbb{C}[WP_{k,l,q}] \) have C*-algebra completions and there are isomorphisms \( \mathbb{C}(WP_{k,l,q}) \cong \mathbb{C}(WP_{k',l,q}) \) for all \( k \) and \( k' \). The geometric models based on the algebras \( \mathbb{C}[WP_{k,l,q}] \) and \( \mathbb{C}[WP_{k',l,q}] \) are never unitary equivalent if \( k \neq k' \). Especially the multiplicities of the Dirac eigenvalues depend on the parameter \( k+l \) and therefore the Dirac operators associated to different \( k \) but equal \( l \) cannot be unitary conjugates of each other. Therefore these algebras are equivalent in the topological sense while they have different geometric structures.

The main reference is [2] where a spectral triple on \( SU_{2,q} \) is developed. However, the conventions differ slightly. The main difference is that the coordinate algebra \( \mathbb{C}[SU_{2,q}] \) is realized as a sum over the highest weights of the tensor products \( M_\lambda^q \otimes M_\lambda \) where \( M_\lambda \) is an irreducible module of the highest weight \( \lambda \). In [2] the duals \( M_\lambda^q \) are identified with \( M_\lambda \). This leads to a different but isomorphic left representation of \( U_q(\mathfrak{su}_2) \). I have decided to follow the usual classical geometric conventions since the identification of the dual modules with the modules is very specific to the case of \( \mathfrak{su}_2 \) and does not hold in general in the theory of semisimple Lie groups. In addition, with the conventions of this work the spectral triple data coincides with a more general theory of spectral triples on compact quantum groups in [11].

**Notation.** The parameter \( q \) will denote a real number in \((0,1)\). The \( q \)-integers will be used in the representation theory:

\[
[a] = \frac{q^a - q^{-a}}{q - q^{-1}}.
\]

For coproducts in Hopf algebras the Sweedlers notation is used: \( \triangle(x) = x' \otimes x'' \). Whenever the symbol \( \uparrow 1 \) appears it should be understood that the formula is true for both indices \( \uparrow \) and \( 1 \). I
shall exploit the notation and refer to weights in the irreducible representations of \(\mathfrak{su}_2\) simply by the eigenvalues of the Cartan generator instead of an element in the dual Cartan subalgebra. The conventions where the highest weight runs over \(\frac{1}{2}\mathbb{N}_0\) will be used.

1 Quantum Group Preliminaries

1.1. The quantum Drinfeld-Jimbo algebra \(U_q(\mathfrak{su}_2)\) is the complex polynomial algebra generated by \(e, f, k, k^{-1}\) subject to the relations

\[
ke = qek, \quad kf = q^{-1}fe, \quad [e, f] = \frac{k^2 - k^{-2}}{q - q^{-1}}, \quad kk^{-1} = k^{-1}k = 1.
\]

\(U_q(\mathfrak{su}_2)\) is a Hopf algebra with the coproduct

\[
\Delta(k) = k \otimes k, \quad \Delta(e) = e \otimes k + k^{-1} \otimes e, \quad \Delta(f) = f \otimes k + k^{-1} \otimes f,
\]

the antipode

\[
S(k) = k^{-1}, \quad S(e) = -qe, \quad S(f) = -q^{-1}f
\]

and the counit defined by \(\epsilon(k) = 1, \epsilon(e) = \epsilon(f) = 0\). We also make \(U_q(\mathfrak{su}_2)\) a *-algebra by setting \(e^* = f, \ f^* = e\) and \((k^{\pm 1})^* = k^{\pm 1}\).

The representation theory of \(U_q(\mathfrak{su}_2)\) is parallel to that of \(\mathfrak{su}_2\). The irreducible complex representations are parametrized by the highest weight of \(\mathfrak{su}_2\). We denote by \(M_\lambda\) the \(U_q(\mathfrak{su}_2)\)-module of highest weight \(\lambda\). The dimension of \(M_\lambda\) is equal to \((2\lambda + 1)\) and the basis vectors \(u_{\lambda m}, -\lambda \leq m \leq \lambda\) are chosen so that the representation takes the ladder operator form

\[
\varrho_\lambda(e)u_{\lambda m} = \sqrt{[\lambda - m][\lambda + m + 1]}u_{\lambda, m+1},
\]

\[
\varrho_\lambda(f)u_{\lambda m} = \sqrt{[\lambda - m + 1][\lambda + m]}u_{\lambda, m-1},
\]

\[
\varrho_\lambda(k)u_{\lambda m} = q^m u_{\lambda m}.
\]

The \(q\)-integers \([n]\) are defined as in (1). Each \(\varrho_\lambda : U_q(\mathfrak{su}_2) \rightarrow \mathcal{B}(M_\lambda)\) is surjective but never injective. The tensor products of the irreducible representations compose into a sum of irreducible components as in the case of \(\mathfrak{su}_2\):

\[
M_\lambda \otimes M_{\lambda'} = \bigoplus_{\mu = |\lambda - \lambda'|} M_\mu. \tag{3}
\]

The coproduct is applied for the action of \(U_q(\mathfrak{su}_2)\) on the tensor product. Since the coproduct is noncocommutative the symmetric group does not act on the tensor products. This is where the braiding comes in: the Artin’s braid group can be given an action on the tensor product. The Clebsch-Gordan coefficients have a \(q\)-deformation which can be used to put the tensor product module to the ladder operator form.

1.2. The enveloping algebra \(U_q(\mathfrak{su}_2)\) has a Hopf dual algebra which is spanned as a vector space by the matrix elements of the irreducible representations of \(U_q(\mathfrak{su}_2)\):

\[
\mathbb{C}[SU_{2,q}] = \bigoplus_{\lambda \in \frac{1}{2}\mathbb{N}_0} M_\lambda^* \otimes M_\lambda.
\]

For each highest weight \(\lambda\) we denote by \(u_{\lambda m}^* = u_{\lambda m}^*\) the dual basis of \(M_\lambda\). The duality is fixed by \(u_{\lambda m}^* (u_{\lambda n}) = \delta_{mn}\). Define

\[
t_{mn}^\lambda = u_{\lambda m}^* \otimes u_{\lambda n}.
\]

The dual pairing of \(U_q(\mathfrak{su}_2)\) with \(\mathbb{C}[SU_{2,q}]\) is defined by

\[
t_{mn}^\lambda(x) = u_{\lambda m}^* (\varrho_\lambda(x)u_{\lambda n}). \tag{4}
\]
The dual space $M_a^*$ becomes a $U_q(\mathfrak{su}_2)$ module under the dual representation which is defined by:

$$g_a^*(x) = (g_a(S(x)))^!$$

for all $x \in U_q(\mathfrak{su}_2)$.

The product in the Hopf dual $\mathbb{C}[SU_{2,q}]$ is defined by requiring that

$$t_{mn}^\lambda t_{m'n'}^\mu(x) = t_{mn}^\lambda (x') t_{m'n'}^\mu(x'')$$

for all $x \in U_q(\mathfrak{su}_2)$ and for all possible weight parameters $\lambda, m, n$.

**Lemma.** The assignment (5) makes $\mathbb{C}[SU_{2,q}]$ a unital complex algebra with the unit $t_{00}^0$ and the product is given by

$$t_{mn}^\lambda t_{m'n'}^\mu = \sum_{\mu = |\lambda - \lambda'|}^{\lambda + \lambda'} C_q \left( \frac{\lambda}{m m'} + \mu \right) C_q \left( \frac{\lambda'}{n n'} + \mu \right) t_{m + m', n + n'}^\mu.$$

where $C_q(\cdot)$ denote the Clebsch-Gordan coefficients for the representations $(g_\lambda, M_\lambda)$.

Proof. The Clebsch-Gordan matrices $C_q$ can be applied to write the tensor product components in terms of weight vectors of some modules that appear in the tensor product decomposition. We choose a basis so that $C_q$ are unitary. The right side of (5) can be manipulated by

$$u^\lambda_m \otimes u^\mu_n (g_\lambda(x') \otimes g_\mu(x'') u_{\lambda m'} \otimes u_{\lambda n'}) = (C_q u^\lambda_m \otimes u^\mu_n)^* C_q (g_\lambda(x') \otimes g_\mu(x'')) C_q^{-1} (C_q u^\lambda_m \otimes u^\mu_n)$$

Now $C_q (g_\lambda(x') \otimes g_\mu(x'')) C_q^{-1}$ is the representation of $x$ on the new basis where the irreducible components are put in the ladder operator form by acting on $C_q$. When written out, we get (6). It is elementary to check that $\phi_{00}^0$ is the unit.

The product in $U_q(\mathfrak{su}_2)$ is obviously compatible with the coproduct in $\mathbb{C}[SU_{2,q}]$ defined by

$$\Delta(t_{mn}^\lambda) = \sum_k t_{mk}^\lambda \otimes t_{kn}^\lambda.$$
and zero in the remaining cases of pairings between these elements. Explicit formulas for the Hopf algebra maps can be solved by dualizing the Hopf structure in $U_q(u_2)$:

$$\Delta(\alpha) = \alpha \otimes \alpha - q\beta \otimes \beta^*, \quad \Delta(\beta) = \beta \otimes \alpha^* + \alpha \otimes \beta,$$

$$S(\alpha) = \alpha^*, \quad S(\beta) = -q\beta, \quad S(\beta^*) = -\frac{1}{q}\beta^*, \quad S(\alpha^*) = \alpha,$$

$$\epsilon(\alpha) = 1, \quad \epsilon(\beta) = 0.$$

We make $U_q(u_2)$ act on $\mathbb{C}[SU_{2,q}]$ under the right and the left regular representations:

$$\tilde{c}(x)(t_{mn}^\lambda) = u_{\lambda m}^* \otimes g_\lambda(x)u_{\lambda n}, \quad l(x)(t_{mn}^\lambda) = g_\lambda^*(x)u_{\lambda m}^* \otimes u_{\lambda n} \quad (7)$$

for all $x \in U_q(u_2)$ and $t_{mn}^\lambda \in \mathbb{C}[SU_{2,q}]$. The left representation is equipped with the following algebra automorphism

$$\vartheta(k^{\pm 1}) = k^{\mp 1}, \quad \vartheta(e) = -f, \quad \vartheta(f) = -e.$$

This automorphism restores the commutativity of the coproduct and the antipode in the sense that

$$\Delta(S(\vartheta(x))) = S(\vartheta(x')) \otimes S(\vartheta(x'')) \quad (8)$$

for all $x \in U_q(u_2)$. When applied on the generators of $\mathbb{C}[SU_{2,q}]$ the right regular representation is given by

$$\tilde{c}(e)\alpha = 0 \quad \tilde{c}(f)\alpha = \beta \quad \tilde{c}(k^{\pm 1})\alpha = q^{\mp \frac{1}{2}}\alpha$$

$$\tilde{c}(e)\beta = \alpha \quad \tilde{c}(f)\beta = 0 \quad \tilde{c}(k^{\pm 1})\beta = q^{\pm \frac{1}{2}}\beta$$

$$\tilde{c}(e)\alpha^* = -q\beta^* \quad \tilde{c}(f)\alpha^* = 0 \quad \tilde{c}(k^{\pm 1})\alpha^* = q^{\mp \frac{1}{2}}\alpha^*$$

$$\tilde{c}(e)\beta^* = 0 \quad \tilde{c}(f)\beta^* = -\frac{1}{q}\alpha^* \quad \tilde{c}(k^{\pm 1})\beta^* = q^{\mp \frac{1}{2}}\beta^*.$$

For the left representation one needs to apply the antipode and work with the dual representation. A straightforward computations gives:

$$l(e)\alpha = 0 \quad l(f)\alpha = -q^2\beta^* \quad l(k^{\pm 1})\alpha = q^{\mp \frac{1}{2}}\alpha$$

$$l(e)\beta = 0 \quad l(f)\beta = q\alpha^* \quad l(k^{\pm 1})\beta = q^{\mp \frac{1}{2}}\beta$$

$$l(e)\alpha^* = \frac{1}{q}\beta^* \quad l(f)\alpha^* = 0 \quad l(k^{\pm 1})\alpha^* = q^{\mp \frac{1}{2}}\alpha^*$$

$$l(e)\beta^* = -\frac{1}{q^2}\alpha \quad l(f)\beta^* = 0 \quad l(k^{\pm 1})\beta^* = q^{\mp \frac{1}{2}}\beta^*.$$

1.3. The Haar state in the algebra $\mathbb{C}[SU_{2,q}]$ is a linear functional $h : \mathbb{C}[SU_{2,q}] \to \mathbb{C}$ that is fixed by the relations $h(1) = 1$ and $h(t_{mn}^\lambda) = 0$ for all $\lambda > 0$. The Haar state provides a Hilbert space completion of $\mathbb{C}[SU_{2,q}]$, which will be denoted by $L^2(SU_{2,q})$, and the GNS construction defines a representation $\pi_h$ of $\mathbb{C}[SU_{2,q}]$ on $L^2(SU_{2,q})$. The basis vectors are mutually orthogonal [7]:

$$h(t_{mn}^\lambda t_{m'n'}^{\lambda'}) = \frac{q^{-2m}}{[2\lambda + 1]} \delta_{\lambda\lambda'}\delta_{mm'}\delta_{nn'}.$$ 

Let us denote by $\eta : \mathbb{C}[SU_{2,q}] \to L^2(SU_{2,q})$ the natural inclusion. Now $\eta(t_{mn}^\lambda)$ gives a basis of $L^2(SU_{2,q})$. The representation $\pi_h : \mathbb{C}[SU_{2,q}] \to L^2(SU_{2,q})$ is given by

$$\pi_h(t_{mn}^\lambda)\eta(t_{m'n'}^{\lambda'}) = \eta(t_{mn}^\lambda t_{m'n'}^{\lambda'}),$$

recall the product rule (6). An orthonormal basis of the Hilbert space is given by

$$|\lambda mn\rangle = q^m \sqrt{[2\lambda + 1]} t_{mn}^\lambda.$$ 

Explicit formulas for the action of the generators in this basis is computed in the reference [2] (the only notational difference is that they use the symbol $l$ for the highest weight).

The right and the left representations of $U_q(u_2)$ on $\mathbb{C}[SU_{2,q}]$ can be extended on $L^2(SU_{2,q})$ by letting the quantum group act on the basis according to

$$\check{c}(x)\eta(t_{mn}^\lambda) = \eta(\check{c}(x)t_{mn}^\lambda), \quad l(x)\eta(t_{mn}^\lambda) = \eta(l(x)t_{mn}^\lambda)$$

$$\check{c}(x)\eta(t_{mn}^\lambda) = \eta(\check{c}(x)t_{mn}^\lambda), \quad l(x)\eta(t_{mn}^\lambda) = \eta(l(x)t_{mn}^\lambda)$$

5
for all \( x \in U_q(\mathfrak{su}_2) \) and \( t_{mn}^\lambda \in \mathbb{C}[SU_{2,q}] \).

**Proposition.** The GNS representation of \( \mathbb{C}[SU_{2,q}] \) is equivariant under the left and right regular representations of \( U_q(\mathfrak{su}_2) \) in the sense that
\[
\tilde{c}(x)(\pi_h(t_{mn}^\lambda T_{m' n'}^{\rho}) = \pi_h(\tilde{c}(x)x) T_{m' n'}^{\rho} \\
l(x)(\pi_h(t_{mn}^\lambda T_{m' n'}^{\rho}) = \pi_h(l(x)x) T_{m' n'}^{\rho}
\]
for all \( x \in U_q(\mathfrak{su}_2) \).

Proof. The pairing of \( t \in \mathbb{C}[SU_{2,q}] \) and \( x \in U_q(\mathfrak{su}_2) \) takes the matrix element \( t \) of the right representation \( \tilde{c}(x) \) which is evident from the definitions (4), (7). Two right representations are equal if and only if all their matrix components are equal. Therefore the right equivariance follows from the definition of the product (5). The left equivariance is a consequence of
\[
t_{mn}^\lambda T_{m' n'}^{\rho}(S \vartheta(x)) = t_{mn}^\lambda ((S \vartheta(x))T_{m' n'}^{\rho}((S \vartheta(x))^r) \\
= t_{mn}^\lambda ((S \vartheta(x'))T_{m' n'}^{\rho}((S \vartheta(x'')))\]
where the first equality follows from the definition of the product and the second follows from (8).

\(\square\)

1.4. The next step is to define a Hilbert space of spinors. In the geometric model over \( SU_2 \) the complexified Clifford algebra over the Lie algebra \( \mathfrak{su}_2 \) is semisimple and isomorphic to two copies of \( \mathcal{B}(\mathbb{C}^2) \). The spinor module is an irreducible representation of the Clifford algebra and therefore two dimensional. Therefore it makes sense to define the spinor module in the quantum group model to be \( M_{1/2} \). Denote by \( e_{\pm} \) the weight \( \pm \frac{1}{2} \) basis vectors of \( M_{1/2} \). We construct the Hilbert space of spinors over \( SU_{2,q} \) by tensoring \( L^2(SU_{q,2}) \) with \( M_{1/2} \). Define
\[
\mathcal{H} = L^2(SU_{2,q}) \otimes M_{1/2}.
\]
The right and the left regular representations are defined by
\[
\tilde{c}'(x) = \tilde{c}(x') \otimes g(x), \quad l'(x) = l(x) \otimes \iota.
\]
We decompose \( \mathcal{H} \) to the irreducible components with respect to these actions. Under the right representation, one has
\[
M_0 \otimes M = M_{1/2} \quad \text{and} \quad M_\lambda \otimes M = M_{\lambda - 1/2} \oplus M_{\lambda + 1/2} \quad \text{if} \quad \lambda > 0.
\]
To extract the irreducible components in the tensor product we need to change the basis. We follow [2] and define a new basis in which the tensor product components take the ladder operator form. Define \( k^\pm = k \pm \frac{1}{2} \) for all \( k \in \frac{1}{2} \mathbb{Z} \).

For \( j \in \frac{1}{2} \mathbb{N} \) we define the vectors
\[
|j \mu \rangle = C_{j \mu}(u_{j, - \mu}^+ \otimes e_-) + S_{j \mu}(u_{j, \mu}^- \otimes e_+)\]
where \( \mu \in \{-j, \ldots, j - 1, j\} \). For \( j \in \frac{1}{2} \mathbb{N}_0 \) we define
\[
|j \mu \rangle = -S_{j+1, \mu}(u_{j+1, \mu}^+ \otimes e_-) + C_{j+1, \mu}(u_{j, \mu}^- \otimes e_+),
\]
where \( \mu \in \{-j, \ldots, j - 1, j\} \). The coefficients are given by
\[
C_{j \mu} = q^{-\frac{j(j+1)}{2}} \frac{|j - \mu|^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}, \quad S_{j \mu} = q^{-\frac{j(j+1)}{2}} \frac{|j + \mu|^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}.
\]
The basis of the Hilbert space \( \mathcal{H} \) will be labeled by
\[
|j \mu \rangle = q^{-\frac{j(j+1)}{2}} \frac{|j - \mu|^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}} |2j^+ + 1\rangle \otimes \eta \otimes \iota (u_{j, - \mu}^+ \otimes \mu \rangle)
\]
where \( \mu \in \{-j, \ldots, j - 1, j\} \). For \( j \in \frac{1}{2} \mathbb{N}_0 \) we define
\[
|j \mu \rangle = q^{-\frac{j(j+1)}{2}} \frac{|j - \mu|^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}} |2j^+ + 1\rangle \otimes \eta \otimes \iota (u_{j, \mu}^- \otimes \mu \rangle)
\]
The basis of the Hilbert space \( \mathcal{H} \) will be labeled by
\[
|j \mu \rangle = q^{-\frac{j(j+1)}{2}} \frac{|j - \mu|^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}} |2j^+ + 1\rangle \otimes \eta \otimes \iota (u_{j, - \mu}^+ \otimes \mu \rangle)
\]
where $m$ runs over the usual weight space parameters: $-j^- \leq m \leq j^-$ in $|jm\mu\downarrow\rangle$ and $-j^+ \leq m \leq j^+$ in $|jm\mu\uparrow\rangle$. Moreover, $-j \leq \mu \leq j$ in both cases.

The basis is orthonormal which is a consequence of the property $C_j^2 + S_j^2 = 1$ for the Clebsch-Gordan coefficients. In this basis the right regular representation takes the ladder operator form

$$
\hat{c}'(x)|jm\mu\downarrow\rangle = \sum_{\nu} g_{j^-}(x)_{\nu\mu}|jm\nu\downarrow\rangle,
$$

$$
\hat{c}'(x)|jm\mu\uparrow\rangle = \sum_{\nu} g_{j^+}(x)_{\nu\mu}|jm\nu\uparrow\rangle
$$

where $g_j(x)_{\nu\mu}$ are the matrix coefficients of the operator $g_j(x)$, recall (2).

On the Hilbert space $\mathcal{H}$ we use the representation of $\mathbb{C}[SU_{2,q}]$ defined by $\pi = \pi_k \otimes \iota$.

1.5. The construction of a Dirac operator in [2] and [11] is based on the invariance under the right and the left representations. Recall that all the irreducible representations of $U_q(\mathfrak{su}_2)$ are surjective maps $U_q(\mathfrak{su}_2) \to B(M_3)$. So, in order to make the Dirac operator commute with both actions, one needs to require that it acts constantly on each irreducible component. Thus $D$ is diagonal in the basis $|jm\mu\uparrow\downarrow\rangle$ and can only depend on the parameter $j$. We shall apply the classical Dirac spectrum. Explicitly, this operator acts on the Hilbert space by

$$
D|jm\mu\uparrow\rangle = (2j + \frac{3}{2})|jm\mu\uparrow\rangle, \quad D|jm\mu\downarrow\rangle = -(2j + \frac{1}{2})|jm\mu\downarrow\rangle.
$$

The multiplicities of these eigenvalues are $(2j+1)(2j+2)$ and $2j(2j+1)$. In the classical limit $q \to 1$ this model coincides with the Dirac operator associated to the bi-invariant metric and Levi-Civita connection on $SU_2$, [15].

A more general model for spinors and Dirac operators on compact quantum groups is developed in [11]. In this case the quantum group Dirac operator is a unitary conjugate of the classical Dirac operator and so the classical spectrum is automatically preserved. In addition this operator has the right and the left symmetries under the quantum group representations on the spinor space. On the technical level one needs to apply certain Drinfeld’s twist to write down such an operator. The case of $\mathfrak{su}_2$ is special in the theory of semisimple Lie algebras since the irreducible components in the tensor product decomposition (3) appear always with the multiplicity one. This means that the highest weight vectors are uniquely determined after the irreducible representations are fixed. Therefore, the decomposition into irreducible components and consequently the decomposition into the Dirac eigenspaces is unique. It would be reasonable to expect that the model of [11] would coincide with the construction above. Indeed, with the conventions applied above this happens. In [11] it was noted that after some algebraic manipulation (and after a normalization), the Dirac operator $\partial$ satisfies

$$
q^{-\partial} = q^{\frac{1}{2}} \left( \hat{c}(k^2 - q^{-1}(q - q^{-1})^2fe) q^{-\frac{1}{2}}(q - q^{-1})\hat{c}(k^{-1}e) \right)
$$

It is straightforward to check that

$$
q^{-0}|jm\mu\uparrow\rangle = q^{-(2j+\frac{3}{2})}|jm\mu\uparrow\rangle, \quad q^{-0}|jm\mu\downarrow\rangle = q^{2j+\frac{1}{2}}|jm\mu\downarrow\rangle.
$$

The spinor module has the same decomposition into irreducible components and also the eigenvalues of $D$ and $\partial$ are the same. Especially the commutators $[D, \pi(t)]$ are bounded for all $t \in \mathbb{C}[SU_{2,q}]$.

2 Spinors on Quantum Weighted Projective Spaces

2.1. The algebra of quantum projective plane is a coinvariant subalgebra of $\mathbb{C}[SU_{2,q}]$. In [1] the generators and relations of $\mathbb{C}[SU_{2,q}]$ correspond to the conventions in 1.2 under $\beta \leftrightarrow \beta^*$. Let $B$ denote the complex polynomial algebra $B := \mathbb{C}[u, u^{-1}]$. There is a Hopf $*$-algebra structure in $B$ so that $u$ is a grouplike element and unitary. For each pair of positive coprime integers $k,l$ define a coaction $\theta : \mathbb{C}[SU_{2,q}] \to \mathbb{C}[SU_{2,q}] \otimes B$ by

$$
\alpha \mapsto \alpha \otimes u^k, \quad \beta \mapsto \beta \otimes u^l.
$$
This is consistent with our notation since our $\beta$ is $\beta^*$ in [1]. It is elementary to check that the coinvariant subalgebra, which will be denoted by $A := \mathbb{C}[W_{k,t,q}]$, is the $*$-subalgebra of $\mathbb{C}[SU_{2,q}]$ generated by

$$a = \beta\beta^* \quad \text{and} \quad b = \alpha^l(\beta^*)^k.$$

In terms of generators and relations the algebra $A$ is isomorphic to the polynomial $*$-algebra generated by the symbols $a,b$ subject to the relations (1):

$$a^* = a, \quad ab = q^{-2l}ba, \quad bb^* = q^{2kl} a^l \prod_{m=0}^{l-1} (1 - q^{2m}a), \quad b^*b = a^k \prod_{m=1}^l (1 - q^{-2m}a). \quad (10)$$

The coinvariant algebra $A$ is not equipped with a coalgebra structure.

In the following we always assume that the positive coprime integers $k,l$ are fixed without specifying it. The coaction and other relevant structures will be written without the subscripts $l,k$ although the analysis obviously depends on these parameters.

2.2. Next we proceed to define a coinvariant subspace in $L^2(SU_{2,q}) \otimes M_4$. For this we need to extend the coaction to the spinor module $M_4$. This is done by extending the coaction to the Hopf-algebra $U_q(\mathfrak{su}_2)$ by requiring that the coaction is compatible with the right representation of $U_q(\mathfrak{su}_2)$ on $\mathbb{C}[SU_{2,q}]$. More precisely, we will use the same symbol for the coaction

$$\theta : U_q(\mathfrak{su}_2) \rightarrow U_q(\mathfrak{su}_2) \otimes B$$

and the compatibility requires that for each $x \in U_q(\mathfrak{su}_2)$ and $t \in \mathbb{C}[SU_{2,q}]$:

$$\theta(\hat{c}(x)t) = [(\hat{c} \otimes \iota)(\theta(x))] [\theta t]. \quad (11)$$

The coaction will be extended on the tensor products $U_q(\mathfrak{su}_2) \otimes U_q(\mathfrak{su}_2)$ by requiring

$$\theta(x \otimes y) = x \otimes y \otimes u^{+j}$$

if $x$ is a homogeneous of order $i$ and $y$ is a homogeneous of order $j$.

**Proposition.** There is a unique coaction of $B$ on $U_q(\mathfrak{su}_2)$ which satisfies (11) and it is given on the generators by

$$\theta(e) = e \otimes u^{k-l}, \quad \theta(f) = f \otimes u^{-l-k}, \quad \theta(k^{\pm 1}) = k^{\pm 1} \otimes 1 \quad (12)$$

and extended to the algebra by linearity and

$$\theta(x_1 \cdots x_k) = \theta(x_1) \cdots \theta(x_k).$$

Proof. It is elementary to check that (12) defines a coaction by using the Hopf structure: $\Delta(u) = u \otimes u$ and $\epsilon(u) = 1$. If we study one of the generators $\alpha, \alpha^*, \beta, \beta^*$ then a straightforward computation gives that if the compatibility condition (11) holds then the adjoint coaction has to satisfy (12). Moreover, we clearly have

$$(\Delta \otimes \iota)(\theta(x)) = \theta(x' \otimes x'') \quad (13)$$

since $k$ and $k^{-1}$ in $U_q(\mathfrak{su}_2)$ are coinvariant. It is therefore sufficient to check that the compatibility remains valid for more general elements $t \in \mathbb{C}[SU_{2,q}]$.

We shall proceed by first proving that if the right representation and the coaction are compatible on the pair of elements $t_{mn}^{\lambda}$ and $t_{m'n'}^{\lambda'}$ in $\mathbb{C}[SU_{2,q}]$, then the right representation on the product is compatible as well. Take $x \in U_q(\mathfrak{su}_2)$. Then, by the right equivariance of the representation

$$\theta(\hat{c}(x)(t_{mn}^{\lambda} t_{m'n'}^{\lambda'})) = \theta(\hat{c}(x')(t_{mn}^{\lambda} \hat{c}(x'') t_{m'n'}^{\lambda'})) = \theta(\hat{c}(x')(t_{mn}^{\lambda})) \theta(\hat{c}(x'') t_{m'n'}^{\lambda'}) = \hat{c}(\theta(x')) \theta(t_{mn}^{\lambda}) \hat{c}(\theta(x'')) \theta(t_{m'n'}^{\lambda'}) = \hat{c}(\theta(x')) \theta(t_{mn}^{\lambda}) \theta(t_{m'n'}^{\lambda'}) = \hat{c}(\theta(x)) \theta(t_{mn}^{\lambda} t_{m'n'}^{\lambda'}) ,$$

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where we have used the basis vectors $e^i_M$ of $M^2$. For the unique half integers $m,n$, the coaction takes the form

$$
\theta(t^\lambda_{mn} \otimes e_\pm) = \theta(t^\lambda_{mn}) \otimes \theta(e_\pm).
$$

Now the right representation of $U_q(\mathfrak{su}_2)$ is compatible with the coaction of $B$:

$$
\theta(\varphi'(x)(t^\lambda_{mn} \otimes e_i)) = \varphi'(x) \theta(t^\lambda_{mn} \otimes e_i).
$$

We shall next construct two coinvariant pre-Hilbert spaces: the coinvariant subspace $\mathbb{C}[SU_{2,q}] \otimes M^2$ and the coinvariant subspace of $\mathbb{C}[SU_{2,q}] \otimes M^2$.

**Proposition.** The basis vectors $t^\lambda_{mn}$ of $\mathbb{C}[SU_{2,q}]$ are homogeneous under the coaction and the coaction takes the form

$$
\theta(t^\lambda_{mn}) = t^\lambda_{mn} \otimes u^{(m+n)k + (m-n)\lambda}
$$

for all highest weights $\lambda$ and $-\lambda \leq m, n \leq \lambda$.

**Proof.** Consider an element $t^\lambda_{mn} \in \mathbb{C}[SU_{2,q}]$. The weight space parameters $m$ and $n$ are valued in $\mathbb{Z}/2\mathbb{Z}$. However, $m \pm n$ is always an integer since $m$ and $n$ are either both in $\mathbb{Z} + 1/2$ or in $\mathbb{Z}$. We find unique half integers $x$ and $y$ which solve the equations: $m = x + y$ and $n = x - y$. Clearly

$$
x = \frac{1}{2}(m + n) \quad \text{and} \quad y = \frac{1}{2}(m - n).
$$

Notice that in (6), the multiplication of an arbitrary element $t^\lambda_{mn}$ by $\alpha$ or $\alpha^*$ results a sum of new matrix elements with both indices $m,n$ shifted by $\frac{1}{2}$ in the first case and by $-\frac{1}{2}$ in the second while the highest weight parameter $\lambda$ will be shifted by $\pm \frac{1}{2}$ in both cases. Similarly, the multiplication by $\beta$ raises $m by \frac{1}{2}$ and lowers $n by \frac{1}{2}$, and the multiplication by $\beta^*$ lowers $m by \frac{1}{2}$ and raises $n by \frac{1}{2}$.

The algebra $\mathbb{C}[SU_{2,q}]$ has a linear basis which consists of the vectors

$$
\alpha^{n_1}(\alpha^*)^{n_2}\beta^{n_3}, \quad \text{where} \quad n_1, n_2 \in \mathbb{N}_0, n_3 \in \mathbb{Z}
$$

and $\beta^{-n} = (\beta^*)^n$ whenever $n > 0$. Suppose that $m = x + y$ and $n = x - y$ as above. Then we can always find constants $A_i, B_i \in \mathbb{C}$ such that

$$
t^\lambda_{mn} = \sum_i A_i \alpha^{2x+i}(\alpha^*)^{i}\beta^{2y} \quad \text{if} \quad x \geq 0,
$$

$$
t^\lambda_{mn} = \sum_i B_i \alpha^{-(2x+i)}(\alpha^*)^{-i}\beta^{2y} \quad \text{if} \quad x \leq 0.
$$
and the sums are all finite. On the other hand, all the vectors with arbitrary \( \lambda \) but \( m \) and \( n \) as above need to be of the above form. These are all homogeneous vectors under the coaction and it follows that

\[
\theta (t^\lambda_{mn}) = t^\lambda_{mn} \otimes u^{2xk+2yl}. \quad \Box
\]

**Corollary 1.** The coinvariant subspaces in the pre-Hilbert spaces \( \mathbb{C}[SU_{2,q}] \) and \( \mathbb{C}[SU_{2,q}] \otimes M_\frac{1}{2} \) are linearly spanned by the sets of vectors

\[
t^\lambda_{p(l-k),p(l+k)} \quad \text{and} \quad t^\lambda_{p(l-k)-\frac{1}{2},p(l+k)-\frac{1}{2}} \otimes e_+ \quad \text{and} \quad t^\lambda_{p(l-k)-\frac{1}{2},p(l+k)+\frac{1}{2}} \otimes e_-
\]

respectively. Again, all the possible indices \( p \) runs over \( \frac{1}{2} \mathbb{N}_0 \) and \( \lambda \) gets all the values for which these vectors are defined.

Proof. In the case of \( \mathbb{C}[SU_{q,2}] \), and in the notation of the proposition, the coinvariant vectors correspond to the solutions of \( 2xk+2yl = 0 \). Since \( k \) and \( l \) are coprime, the only solutions are \( x = pl, y = -pk \) with \( p \in \frac{1}{2} \mathbb{Z} \). This gives the weights \( m = p(l-k) \) and \( n = p(l+k) \). Thus the coinvariant subspace in \( \mathbb{C}[SU_{q,2}] \) linearly spanned by

\[
t^\lambda_{p(l-k),p(l+k)}.
\]

Since \( e_+ \) and \( e_- \) are homogeneous elements of orders \( k \) and \( l \), it is sufficient to find the corresponding homogeneous elements of \( -k \) and \( -l \) in the algebra \( \mathbb{C}[SU_{2,q}] \) to get the coinvariant subspace in \( \mathbb{C}[SU_{2,q}] \otimes M_\frac{1}{2} \). The subspaces of homogeneous elements of order \( -k \) and \( -l \) are spanned by the vectors

\[
t^\lambda_{p(l-k)-\frac{1}{2},p(l+k)-\frac{1}{2}} \quad \text{and} \quad t^\lambda_{p(l-k)-\frac{1}{2},p(l+k)+\frac{1}{2}}
\]

respectively. Again, all the possible indices \( p \) and \( \lambda \) are included. \( \Box \)

**Corollary 2.** The linear basis of the coinvariant subspace of \( \mathbb{C}[SU_{2,q}] \otimes M_\frac{1}{2} \) is given in the basis \( \langle jmp \uparrow \downarrow \rangle \) by

\[
\Psi_{jp}^\uparrow := \langle j, p(l-k) - \frac{1}{2}, p(l+k) \uparrow \rangle \\
\Psi_{jp}^\downarrow := \langle j, p(l-k) - \frac{1}{2}, p(l+k) \downarrow \rangle.
\]

\( p \) runs over \( \frac{1}{2} \mathbb{Z} \) and \( j \) gets all the values for which these vectors are defined (recall the index conventions in 1.4).

Proof. Using the Clebsch-Gordan coefficients we find that

\[
q^{-m}(\lbrack 2j^- + 1 \rbrack)^{-\frac{1}{2}} \langle j - 1, p(l-k) - \frac{1}{2}, p(l+k) \uparrow \rangle = -S_{j,p(l+k)} t^\uparrow_{j,p(l-k)} \frac{1}{2}, p(l+k) + \frac{1}{2} \otimes e_- \\
+ C_{j,p(l+k)} t^\uparrow_{j,p(l-k)} - \frac{1}{2}, p(l+k) + \frac{1}{2} \otimes e_+,
\]

\[
q^{-m}(\lbrack 2j^- + 1 \rbrack)^{-\frac{1}{2}} \langle j, p(l-k) - \frac{1}{2}, p(l+k) \downarrow \rangle = C_{j,p(l+k)} t^\downarrow_{j,p(l-k)} - \frac{1}{2}, p(l+k) + \frac{1}{2} \otimes e_- \\
+ S_{j,p(l+k)} t^\downarrow_{j,p(l-k)} - \frac{1}{2}, p(l+k) + \frac{1}{2} \otimes e_+.
\]

Therefore the vectors listed in the corollary form a basis for the pre-Hilbert space of coinvariant vectors. \( \Box \)

2.4. Next we compute the dimensions of certain subspaces in the coinvariant pre-Hilbert spaces. This will be used in the study of the Dirac spectrum. For all \( j \in \frac{1}{2} \mathbb{N}_0 \) denote by \( V_j^\uparrow \) the subspace of \( \mathbb{C}[SU_{2,q}] \otimes M_\frac{1}{2} \) spanned by \( \Psi_{jp}^\uparrow \) for all possible \( p \in \frac{1}{2} \mathbb{Z} \).

**Proposition 1.** The dimension of \( V_0^\uparrow \) is 1 and the dimension of \( V_0^\downarrow \) is 0.
If \( k + l \) is even, then \( V_{j}^{\uparrow 1} \) are nonzero only if \( j \) is an integer, and for all nonzero integers
\[
\dim(V_{j}^{\uparrow 1}) = 2\left\lfloor \frac{j}{2(l + k)} \right\rfloor + 1
\]

If \( l + k \) is odd then
\[
\dim(V_{j}^{\uparrow 1}) = 2\left\lfloor \frac{j}{l + k} \right\rfloor + 1 \quad \text{in the case } j \text{ is an integer}
\]
\[
\dim(V_{j}^{\uparrow 1}) = 2\left\lfloor \frac{j}{l + k} + \frac{1}{2} \right\rfloor \quad \text{otherwise.}
\]

\( |a| \) denotes the integer part of \( a \).

**Proof.** The space \( V_{0}^{\uparrow} \) is spanned by \( \Psi_{00} = |0, -\frac{1}{2}, 0 \rangle \) which is the only coinvariant vector for \( j = 0 \). Therefore \( \dim(V_{0}^{\uparrow}) = 1 \) and \( \dim(V_{0}^{\downarrow}) = 0 \).

The following analysis uses the parametrization of the basis \( |jm\mu \rangle \rangle \) introduced in 1.4. Notice that if \( j \) is fixed and \( p(l + k) \in \{-j, \ldots, -1, j\} \) then necessarily \( p(l - k) - \frac{1}{2} \in \{-j, \ldots, j\} \). Thus, the multiplicities of \( V_{j}^{\uparrow 1} \) are equal to the number of half integers \( p \) for which \( p(l + k) \in \{-j, \ldots, j\} \).

Let \( k + l \) be even and \( j \) nonzero. Now \( p(l + k) \) are always integers and so the parameter \( j \) gets only integer values. If \( j < \frac{1}{2}(l + k) \), then the only possible weight of type \( p(l + k) \) occurs with \( p = 0 \). If \( \frac{1}{2}(l + k) \leq j < l + k \), then there are three possible weights: \( \frac{1}{2}(l + k) \) and \( 0 \). Continuing like this it is clear that one gets a new pair of weights whenever \( j \) grows by \( \frac{1}{2}(l + k) \) and therefore \( i \) follows.

Let \( k + l \) be odd and \( j \) nonzero. Now there are two cases: if \( j \) is an integer, then the parameter \( p \) and also \( p(l + k) \) are integers. This case is essentially the same as above except that a new pair of weights appear whenever \( j \) is shifted by \( (l + k) \). This gives the first part of \( ii \). If \( j \in \mathbb{N} + \frac{1}{2} \), then \( p \in \mathbb{Z} + \frac{1}{2} \). The vectors \( \Psi_{ip}^{\uparrow} \) are nonzero only if \( p(l + k) \in \{-j, \ldots, j\} \). Especially, if \( j < \frac{1}{2}(l + k) \), then there are no coinvariant vectors, and so the dimensions of \( V_{j}^{\uparrow 1} \) are 0. For \( j = \frac{1}{2}(l + k) \) the first pair of coinvariant vectors appear: these are \( \pm \frac{1}{2}(l + k) \). Whenever \( j \) is shifted by \( (l + k) \) one gets another pair and so
\[
\dim(V_{j}^{\uparrow}) = \dim(V_{j}^{\downarrow}) = 2\left\lfloor \frac{j + \frac{l + k}{2}}{l + k} \right\rfloor = 2\left\lfloor \frac{j}{l + k} + \frac{1}{2} \right\rfloor.
\]

The second part of \( ii \) follows. \( \Box \)

Denote by \( V_{\lambda} \) the subspace of \( \mathbb{C}[SU_{2,\theta}] \) which is spanned by the vectors \( t_{p(l-k),p(l+k)} \) for all possible \( p \in \frac{1}{2}\mathbb{Z} \). Similar analysis of weight parameters can be applied to prove the following.

**Proposition 2.**

i. If \( k + l \) is even, then \( V_{\lambda} \) is nonzero only if \( \lambda \) is an integer, and for all \( \lambda \in \mathbb{N} \):
\[
\dim(V_{\lambda}) = 2\left\lfloor \frac{\lambda}{2(l + k)} \right\rfloor + 1.
\]

ii. If \( l + k \) is odd then
\[
\dim(V_{\lambda}) = 2\left\lfloor \frac{\lambda}{l + k} \right\rfloor + 1 \quad \text{in the case } \lambda \text{ is an integer}
\]
\[
\dim(V_{\lambda}) = 2\left\lfloor \frac{\lambda}{l + k} + \frac{1}{2} \right\rfloor \quad \text{otherwise.}
\]

## 3 Dirac Operator Quantum Weighted Projective Spaces

Here we shall consider two different models for Dirac operators on coinvariant Hilbert spaces. The first one in 3.1 is based on the restriction of the Dirac operator on \( SU_{2,\theta} \) on the coinvariant
component of $\mathcal{H}$. This is certainly possible, however, a difficulty arises since there is no natural chiral grading in the $SU_{2,q}$ model and it would seem to be difficult to define a chiral operator which would commute with the representation of $C[SU_{2,q}]$. A simple fix for the lack of chirality element is given in 3.2 where the Hilbert space is simply a sum of two copies of the Hilbert space completion of $C[SU_{2,q}]$. It is then straightforward to define a chiral operator.

3.1. Now we have a complete description of the pre-Hilbert space of coinvariant spinors in $C[SU_{2,q}] \otimes M_2$. Let us denote $H^\theta$ its Hilbert space completion with respect to the Haar functional.

We define the Dirac operator on $H^\theta$ by restricting the Dirac operator $D$ on the coinvariant Hilbert space. Let us denote by $D^\theta$ the restriction. This is indeed well defined since the action of $D$ commutes with the coaction of $B$. Alternatively, one can easily check using the coaction of $B$ on $U_q(su_2)$ that the operator $q^{-\theta}$ of 1.5 is a coinvariant operator. The coinvariant subalgebra of $C[SU_{2,q}]$, which we have denoted by $A$, can be represented on $H^\theta$ simply by restricting the GNS representation $\pi_h \otimes r := \pi$.

**Theorem.** The data $(A, \pi, H^\theta, D^\theta)$ defines a 2-summable odd spectral triple over the quantum projective plane $A = C[\mathbb{W}^{k,l,q}]$ for all positive coprime integers $k$ and $l$.

Proof. Since this model is based on a restriction of a spectral triple on $SU_{2,q}$ most of the details are automatic. All the commutators $[D^\theta, \pi(t)]$ are bounded for all $t \in A$ since this is the case for the quantum group. Also this representation is faithful for the same reason. For the summability, we shall consider the case where $k + l$ is even. The sum of eigenvalues of $(D^\theta)^{-2}$ over the subspaces $V_j^{\uparrow}$ for $0 \leq j \leq N$ gives

$$\sigma_N((D^\theta)^{-2}) = \sum_{0 \leq j \leq N} \left( \frac{j}{2(k+1)} \right) + 1 + \sum_{1 \leq j \leq N} \left( \frac{j}{2(k+1)} \right) + 1 + \sum_{1 \leq j \leq N} \left( \frac{j}{2(k+1)} \right) + 1 + \sum_{1 \leq j \leq N} \left( \frac{j}{2(k+1)} \right) + 1.$$ 

In the limit $N \to \infty$, $\sigma_N$ diverges logarithmically. Thus the summability is 2. The same argument gives the summability 2 if $l + k$ is odd. \hfill \Box

3.2. Denote by $H'$ a completion of the coinvariant pre-Hilbert space in $C[SU_{2,q}]$. Now $H'$ is a representation for $A$ under $\pi_h$. We shall define the Hilbert space of spinors as a Hilbert space direct sum $H' \oplus H'$ and let $A$ act on it under $\pi_h \oplus \pi_h$. Let us fix the following basis

$$\Phi^\uparrow_{\lambda p} = \phi_{(p+1),p+1}^{1} \quad \text{and} \quad \Phi^\downarrow_{\lambda p} = \phi_{p,p-1}^{1},$$

where we have used the upper-scripts $\uparrow$ and $\downarrow$ to label the first and the second tensor component respectively. Then define

$$D'\Phi^\uparrow_{\lambda p} = (\lambda + 1)\Phi^\downarrow_{\lambda p} \quad \text{and} \quad D'\Phi^\downarrow_{\lambda p} = -(\lambda + 1)\Phi^\uparrow_{\lambda p}.$$ 

The spectrum is fixed so that for $k = l = 1$, where the algebra $A$ is a deformation of the coordinate algebra of a sphere, the spectrum coincides with the classical Dirac spectrum [4]. The chirality operator is diagonal in this basis:

$$\gamma \Phi^\uparrow_{jp} = \Phi^\downarrow_{jp} \quad \text{and} \quad \gamma \Phi^\downarrow_{jp} = -\Phi^\uparrow_{jp}.$$ 

Obviously, $[(\pi_h \oplus \pi_h)(t), \gamma] = 0$ for all $t \in A$, $\{\gamma, D'\} = 0$, $\gamma^2 = 1$ and $\gamma^* = \gamma$.

**Theorem.** The data $(A, \pi_h \oplus \pi_h, H' \oplus H', D', \gamma)$ defines a 2-summable even spectral triple over the quantum projective plane $A = C[\mathbb{W}^{k,l,q}]$ for all positive coprime integers $k$ and $l$.

Proof. We need to check that the commutators $[D', (\pi_h \oplus \pi_h)(t)]$ are bounded for all $t \in A$. First we consider the Hilbert space completion of $C[SU_{2,q}]$ which is denoted by $L^2(SU_{2,q})$. Then we take the Hilbert space sum $L^2(SU_{2,q}) \oplus L^2(SU_{2,q})$. We label by $\uparrow$ and $\downarrow$ the basis vectors in the first and second summand, and define an auxiliary Dirac operator on $L^2(SU_{2,q}) \oplus L^2(SU_{2,q})$ by

$$Q\eta(t_{min}^{\uparrow \downarrow}) = (\lambda + 1)\eta(t_{min}^{\uparrow \downarrow}) \quad \text{and} \quad Q\eta(t_{max}^{\uparrow \downarrow}) = -(\lambda + 1)\eta(t_{max}^{\uparrow \downarrow}).$$
The algebra $\mathbb{C}[SU_{q,2}]$ acts on $L^2(SU_{q,2}) \otimes L^2(SU_{q,2})$ under the GNS representation $\pi_\theta \oplus \pi_h$. Next we check that the commutators $[Q_\theta, (\pi_\theta \oplus \pi_h)(t)]$ are all bounded for $t \in \mathbb{C}[SU_{q,2}]$. It is clearly sufficient to check this for the generators. For $t = \alpha$ we get the following action on $\eta (\hat{m}_n^\alpha)$

$$
\pi_\theta (\alpha) \eta (\hat{m}_n^\alpha) = \eta (\hat{m}_{\frac{1}{2}}^\alpha + \hat{m}_{\frac{1}{2}}^\alpha).
$$

Then a straightforward computation gives

$$
[Q, \pi_\theta (\alpha)] \eta (\hat{m}_n^\alpha) = - \frac{1}{2} C_q \left( \frac{1}{2} \lambda - \frac{1}{2} \right) C_q \left( \frac{1}{2} \lambda n + n + \frac{1}{2} \right) \eta (\hat{m}_{\frac{1}{2}} n + m + \frac{1}{2}).
$$

The Clebsch-Gordan coefficients are bounded in $\lambda, m$ and $n$ and therefore the commutator $[Q, (\pi_\theta \oplus \pi_h)(\alpha)]$ is bounded. Similar computation proves the boundedness of $[Q, (\pi_\theta \oplus \pi_h)(\beta)]$. It follows that $[Q_\theta, (\pi_\theta \oplus \pi_h)(t)]$ is bounded for all $t \in \mathbb{C}[SU_{q,2}]$. Now the coinvariant spectral triple $(A, \pi_\theta \oplus \pi_h, H' \oplus H', D', \gamma)$ can be reduced from the auxiliary model by restricting $L^2(SU_{q,2})$ to the coinvariant component and by restricting $\mathbb{C}[SU_{q,2}]$ to $A$ and $Q$ to $D'$. It is then clear that the boundedness remains true in this model.

The summability is computed exactly as in Theorem 3.1. The representation $\pi_\theta \oplus \pi_h$ is faithful since both components are faithful. The even structure refers to the existence of the chirality operator $\gamma$.

3.3. Let us study the case $l = k = 1$ with more details. The algebra $\mathbb{C}[\mathbb{W}_{1,1,1}]$ is isomorphic to a 2-dimensional quantum sphere algebra. Now $(l - k) = 0$. The Hilbert space $H^\theta$ is spanned by the vectors $\Psi^\uparrow p$ and $\Psi^\downarrow p$ for $j \in \mathbb{N}_0$ and $-j \leq p \leq j$. Thus the index $p$ runs over all the possible weight vectors for the right $U_q(\mathfrak{su}_2)$ representation. Consequently, there is a remaining $U_q(\mathfrak{su}_2)$ symmetry from the right. The left symmetry is missing. The dimensions of the Dirac eigenspaces are given by $\dim (V^\uparrow l) = 2j + 1$.

Now consider the Hilbert space $H' \oplus H'$. The basis vectors $\Phi_l^\uparrow p$ are labeled by $\lambda \in \mathbb{N}_0$ and $-\lambda \leq p \leq \lambda$. Again, the right $U_q(\mathfrak{su}_2)$ symmetry is there but the left is missing. The Dirac spectrum now coincides with the following case of a 2-sphere:

$$
\text{sp}(D') = \{ \pm (\lambda + 1) : \lambda \in \mathbb{N}_0 \}
$$

and the eigenvalues $\pm(\lambda + 1)$ have the multiplicity $2\lambda + 1$.

3.4. The case $k = 1, l = 2$ corresponds to the standard tear drop. Now the right $U_q(\mathfrak{su}_2)$ symmetry is already lost since no longer all weight spaces occur. Now $k + l$ is an odd and the basis vectors are given by

$$
\Psi^\uparrow_{00}, \Psi^\uparrow_{10}, \Psi^\uparrow_{11}, \Psi^\uparrow_{3k,i}, \Psi^\uparrow_{3k+1,i}, \Psi^\uparrow_{3k+2,i} \text{ for } k \in \mathbb{N}, \ i \in \{-3k,-3(k-1),\ldots,3k\}
$$

and

$$
\Psi^\downarrow_{2+3l,i}, \Psi^\downarrow_{3+3l,i}, \Psi^\downarrow_{4+3l,i} \text{ for } l \in \mathbb{N}_0, \ i \in \{-3-3l,-3-3l-2-3l-3(l-1),\ldots,3 \}
$$

The basis for $H' \oplus H'$ is similar: one just uses $\Psi \leftrightarrow \Phi$ in the above formula.

3.5. In applications one typically needs a universal differential algebra constructed from the spectral triple. In the classical geometric context, the algebra of invariant differential forms is not generated by the invariant functions and the gradients of the invariant functions $[5]$ (unless the action is free). Therefore, it would be natural to associate a Hopf invariant differential algebra to a invariant spectral triple, such as the one in in 3.1 or 3.2, see also [10].

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