ON NONTRIVIAL ZEROS FOR THE RAMANUJAN ZETA FUNCTION

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Abstract. The Hardy hypothesis, as an analogue to the Riemann hypothesis for the Riemann zeta function, is a conjecture proposed by Hardy in 1940, that all of the nontrivial zeros for the Ramanujan zeta function have a real part equal to 6. In this paper, we propose the power series expansion for the entire Ramanujan zeta function using the work of Mordell. Then, we suggest an alternative infinite product for the entire Ramanujan zeta function derived from the work of Conrey and Ghosh. We also establish the class of the entire Ramanujan zeta function related to the functional equation coming from Wilton. Motivated by the work of Lekkerkerker, we prove a conjecture due to Bruijn that all of the zeros of the Ramanujan Xi function are nonzero real numbers. From theory of the entire functions, we also prove that the Hardy hypothesis is true.

1. Introduction

In 1916 Ramanujan [1] invented a arithmetical function \( \tau (m) \), defined by the series

\[
\Delta (z) = \sum_{m=0}^{\infty} \tau (m) x^m = x \prod_{n=1}^{\infty} (1 - x^n)^{24},
\]

where \( x = e^{-2\pi iz}, z \in \mathbb{R}, m \in \mathbb{N} \cup \{0\} \), and \( n \in \mathbb{N} \). Here, \( \mathbb{N} \) and \( \mathbb{R} \) denote the sets of the natural and real numbers. It is well known that \( \Delta (z) \) is a holomorphic cusp form of weight 12 and level 1, and satisfies the functional equation [2]:

\[
\Delta (z) = \frac{1}{z^{12}} \Delta \left( \frac{-1}{z} \right).
\]

Ramanujan [1] guessed that the tau function \( \tau (m) \) satisfies the following conditions:

(I) \( \tau (m) \) is multiplicative, i.e.,

\[
\tau (pq) = \tau (p) \tau (q) \quad (p, q) = 1,
\]

where \( p \) and \( q \) are prime to each other.

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for every \( \lambda > 0 \) and every prime \( p \).

(III)

(5) \(|\tau(p)| < 2p^{11/2}\)

for every prime \( p \).

It is noted that (3) and (4) have been proved by Mordell in 1917 [3], and (5) has been proved by Deligne in 1974 [4]. It is well known that \( \tau(m) \) is called the Ramanujan tau function. Let \( \mathbb{C} \) be the set of the complex numbers. We now denote the real and imaginary part of a complex number \( s \in \mathbb{C} \) by \( \Re(s) \) and \( \Im(s) \). Ramanujan [1] further introduced a well-known zeta function \( L(\Delta, s) \) by

(6) \[ L(\Delta, s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}, \]

where \( \Re(s) > 13/2 \). It was proved by Mordell in 1917 [3] that the Ramanujan zeta function \( L(\Delta, s) \) has an Euler product:

(7) \[ L(\Delta, s) = \prod_p (1 - \tau(p) p^{-s} + p^{11-2s})^{-1}, \]

where \( p \) is prime, and \( \Re(s) > 13/2 \). It has been reported by Rankin [2] that

(8) \[ L(\Delta, s) (2\pi)^{-s} \Gamma(s) = \int_0^\infty \Delta(ix) x^{s-1} dx \quad (\Re(s) > 13/2) \]

\[ = \int_0^\infty \Delta(ix) (x^{s-1} + x^{11-s}) dx, \]

where \( i = \sqrt{-1} \). It has been proved by Wilton [5] that it satisfies the functional equation

(9) \[ L(\Delta, s) (2\pi)^{-s} \Gamma(s) = L(\Delta, 12 - s) (2\pi)^{s-12} \Gamma(12 - s) \]

for \( s \in \mathbb{C} \).

Let

(10) \[ \xi(\Delta, s) = L(\Delta, s) (2\pi)^{-s} \Gamma(s) \]

be the entire Ramanujan zeta function. Then, (9) can be rewritten as [5]

(11) \[ \xi(\Delta, s) = \xi(\Delta, 12 - s) \]
for \( s \in \mathbb{C} \). In 1988 Conrey and Ghosh [6] suggested that \((10)\) is an entire function of order \( \alpha = 1 \), and can be expressed by the Hadamard infinite product [6]:

\[
\xi(\Delta, s) = e^{A_{\tau} + B_{\tau} s} \prod_{\rho_k} \left(1 - s/\rho_k\right) e^{s/\rho_k},
\]

where \( A_{\tau} \) and \( B_{\tau} \) are two constants, and \( \rho_k \) run over all of the zeros of \( \xi(\Delta, s) \) (here, we take the case of \( k = 12 \)). Conrey and Ghosh [7] defined the Ramanujan Xi function \( \Xi(\Delta, t) \) by

\[
\xi(\Delta, 6 + it) = \Xi(\Delta, t) = L(\Delta, 6 + it) (2\pi)^{-it-6} \Gamma(6 + it),
\]

and Ki [8] defined the Ramanujan Xi function \( \Xi(\Delta, t) \) by

\[
\xi(\Delta, 6 - it) = \Xi(\Delta, t) = L(\Delta, 6 - it) (2\pi)^{it-6} \Gamma(6 - it),
\]

where \( t \in \mathbb{C} \).

In 1940, Hardy (see [9], p.174, for example) conjectured the statement that all of zeros of the Ramanujan zeta function lie on the critical line \( \Re(s) = 6 \), which is well-known Hardy hypothesis. It has been proved by Lekkerkerker [10] (also see Hafner [11]) that \( \xi(\Delta, s) \) has many infinitely zeros on the critical line \( \Re(s) = 6 \). In 2020, Chirre and Castaños [12] conjectured that the Hardy hypothesis for the Ramanujan zeta function \( \xi(\Delta, s) \) states that all zeros of \( \Xi(\Delta, t) \) are real. In 1950, Bruijn guessed the statement that all of zeros of the Ramanujan Xi function are nonzero real number [13], which is the conjecture of Bruijn.

It is well known that the conjecture of Bruijn [13] is analogous to the conjecture of Jensen (for the conjecture of Jensen, see [14, 15]) and that the Hardy hypothesis [9] is analogous to the Riemann hypothesis (for the Riemann hypothesis, see [16]). The conjecture of Bruijn and Hardy hypothesis are still open problems in theory of the Ramanujan zeta function.

The main target of the paper is to prove the following results:

**Conjecture 1.** (Conjecture of Bruijn) [13]

Assume that \( t \in \mathbb{C} \). Then all of the zeros of \( \Xi(\Delta, t) \) are \( \psi_k \in \mathbb{R} \setminus \{0\} \) for \( k \in \mathbb{N} \).

**Conjecture 2.** (Hardy hypothesis) (for more details, see [9], p.174)

Assume that \( s \in \mathbb{C} \). Then all of the zeros of \( \xi(\Delta, s) \) are \( \Re(\rho_k) = 6 \) for \( k \in \mathbb{N} \).

The structure of the paper is given as follows. In Section 2 we introduce the results for the entire functions and the entire Ramanujan zeta function. In Section 3 we study the concept, properties and theorems for class of the entire Ramanujan zeta function. In Section 4 we consider the properties and theorems for the Ramanujan Xi function. In Section 5 we prove Conjecture 1. In Section 6 we prove Conjecture 2.
In Section 7 we suggest the open problems for the Ramanujan zeta function. Finally, we make our conclusion in Section 8.

2. Preliminary results

In this section we give an introduction to the entire functions and the entire Ramanujan zeta function.

2.1. Recent results for the entire functions. Now, we introduce the results for the entire functions as follows:

Definition 1. (see [17], Lecture 4, p.25)

Let \( s \in \mathbb{C} \). The Weierstrass primary factors \( S(s, 0) \) and \( L(s, \mu) \) are defined by the expressions

\[
S(s, 0) = 1 - s \ (\mu = 0),
\]

and

\[
S(s, \mu) = (1 - s) \exp \left( s + \frac{1}{2}s^2 + \cdots + \frac{1}{\mu}s^\mu \right) \quad (\mu > 1),
\]

where \( S(s, \mu) \) is of genus \( \mu \in \mathbb{N} \).

Definition 2. (see [17], Lecture 4, p.28)

Let \( \mathbb{G} = \{\rho_n\}_{n=1}^{\infty} \) be the set of the sequence of all zeros for

\[
A(s) = \prod_{n=1}^{\infty} S(s/\rho_n, \mu)
\]

such that

\[
|\rho_1| < |\rho_2| < |\rho_3| < \cdots < |\rho_p| < |\rho_{p+1}| < \cdots,
\]

and

\[
\lim_{n \to \infty} \rho_n = \infty.
\]

Then, (17) is called as a canonical product of genus \( \mu \).

Definition 3. (see [18], p.8)

The order \( \varpi \) of the power series of the entire function \( A(s) \),

\[
A(s) = \sum_{m=0}^{\infty} h(m) s^m
\]

with the coefficients \( h(m) \) of (20), is defined by

\[
\alpha = \lim_{m \to \infty} \sup \frac{m \log m}{\log (1/|h(m)|)}
\]
for $m \in \mathbb{N} \cup \{0\}$.

**Lemma 1.** (see [17], Theorem 2, p.29)
Assume that

\[(22) \quad \sum_{n=1}^{\infty} \frac{1}{|\rho_n|^\mu} < \infty,\]

where $\rho_n \in G$ for $n \in \mathbb{N}$. Then the Hadamard infinite product

\[(23) \quad A(s) = \prod_{n=1}^{\infty} \mathcal{S}(s/\rho_n, \mu),\]

converges uniformly on every compact set $\Omega \subset \mathbb{C}$.

**Definition 4.** (see [18], p.14)
The exponent of convergence $\nu$ for the entire function $A(s)$ is defined by

\[(24) \quad \nu = \inf \left\{ \theta \mid |\rho_n|^{-\theta} < \infty \right\},\]

where $\rho_n \in G$, and $n \in \mathbb{N}$.

**Lemma 2.** (see [18], Theorem 2.6.5., p.19)
Assume that $A(s)$ is genus $\mu \in \mathbb{N}$, where $s \in \mathbb{C}$. Then

\[(25) \quad \nu = \alpha.\]

Let

\[(26) \quad B(s) = \sum_{m=0}^{\infty} (-1)^m \ell(m) s^{2m},\]

where $m \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{C}$.

**Lemma 3.** (see [18], Theorem 2.12.13, p.36)
Let $m \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{C}$. The power series (26) is of genus $\mu = 0$ if and only if

\[(27) \quad B(s) = \sum_{m=0}^{\infty} |\ell(m)| s^{2m},\]

is of genus $\mu = 0$.

**Lemma 4.** (see [18], p.35)
Assume that

\[(28) \quad \sum_{n=1}^{\infty} 1/|\rho_n| < \infty,\]
and

$$H(s) = \prod_{n=1}^{\infty} \left(1 - \frac{s}{\rho_n}\right),$$

where \(n \in \mathbb{N}\), and \(\rho_n \in \mathbb{G}\) run over the zeros of \(H(s)\).

Then, there exist \(\eta_n \in \mathbb{G}\), and

$$\Im(\eta_n) > 0$$

such that

$$H(s) = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{\eta_n^2}\right),$$

where \(\eta_n \in \mathbb{G}\) run over the zeros of \(H(s)\).

### 2.2. Recent results for the entire Ramanujan zeta function

It has been proved by Rankin [2] that the integral representation for the entire Ramanujan zeta function can be given as:

**Lemma 5.** [Rankin] [2]

Let \(s \in \mathbb{C}\). Then

$$\xi(\Delta, s) = \frac{1}{\Delta} \int_{0}^{1} \Delta(ix) \left(x^{s-1} + x^{11-s}\right) dx.$$  

It has been reported by Wilton [5] that there is the functional equation as follows:

**Lemma 6.** [Wilton] [5]

If \(s \in \mathbb{C}\), then

$$\xi(\Delta, s) = \xi(\Delta, 12 - s).$$

It has pointed out by Conrey and Ghosh [6] that \(\xi(\Delta, s)\) can be expressed by the Hadamard infinite product representation as follows:

**Lemma 7.** [Conrey and Ghosh] [6]

Assume that \(s \in \mathbb{C}\). Then there exists the constants \(A_\tau\) and \(B_\tau\) such that

$$\xi(\Delta, s) = e^{A_\tau + B_\tau s} \prod_{\rho_k} (1 - s/\rho_k) e^{s/\rho_k},$$

where \(\rho_k\) run over all of the zeros of \(\xi(\Delta, s)\).

In 1955, Lekkerkerker [10] proved:

**Lemma 8.** [Theorem of Lekkerkerker] [10]

\(\xi(\Delta, s)\) has many infinitely zeros on the critical line \(\Re(s) = 6\).
3. The class of the entire Ramanujan zeta function

We now begin with the main results for the entire Ramanujan zeta function.

**Theorem 1.** Assume that $\Delta(x)$ is defined by (2), and $s \in \mathbb{C}$. Then

$$\xi(\Delta, s) = 2 \int_0^1 \Delta(ix) x^5 \cosh \left[ (s - 6) \log x \right] dx.$$  \hspace{1cm} (35)

**Proof.** With Lemma 5, we have

$$\xi(\Delta, s) = \int_0^1 \Delta(ix) (x^{s-1} + x^{11-s}) dx$$

$$= \int_0^1 \Delta(ix) x^5 (x^{s-6} + x^{6-s}) dx$$

$$= \int_0^1 \Delta(ix) x^5 (x^{s-6} + x^{6-s}) dx$$

$$= 2 \int_0^1 \Delta(ix) x^5 \cosh \left[ (s - 6) \log x \right] dx,$$

which is the desired result. \hfill \Box

**Theorem 2.** Assume that $\Delta(x)$ is defined by (2), $m \in \mathbb{N} \cup \{0\}$, and $s \in \mathbb{C}$. Then

$$\xi(\Delta, s) = \sum_{m=0}^{\infty} \Phi_m (s - 6)^{2m},$$  \hspace{1cm} (37)

where

$$\Phi_m = \frac{2}{(2m)!} \int_0^1 \Delta(ix) x^5 (\log x)^{2m} dx.$$  \hspace{1cm} (38)

**Proof.** Using Theorem 1 and considering the fact

$$\cosh \left[ (s - 6) \log x \right] = \sum_{m=0}^{\infty} \frac{[(s - 6) \log x]^{2m}}{(2m)!},$$  \hspace{1cm} (39)

(35) can be written as (37) by using (38).

Thus, the desired result follows. \hfill \Box

**Theorem 3.** Let $s \in \mathbb{C}$ and $k \in \mathbb{N}$. Then

$$\xi(\Delta, s) = \xi(\Delta, 0) \prod_{\rho_k} (1 - s/\rho_k).$$  \hspace{1cm} (40)

where the product run though the zeros $\rho_k$ of $\xi(\Delta, s)$. 


Proof. By Theorem 1, it is easy to see that

\[
\xi(\Delta, s) = 2 \int_0^1 \Delta (ix) x^5 \cosh [(s - 6) \log x] \, dx
\]

\[
= 2 \int_0^1 \Delta (ix) x^5 \cosh \{(12 - s) - 6 \log x\} \, dx
\]

\[
= 2 \int_0^1 \Delta (ix) x^5 \cosh [(s - 6) \log x] \, dx
\]

\[
= L(\Delta, 12 - s) (2\pi)^{s-12} \Gamma(12 - s),
\]

which is the functional equation (11), reduced by Wilton [5].

Since Lemma 8 and (40) imply that \( \xi(\Delta, s) \) has many infinitely zeros on the critical line \( \Re(s) = 6 \), (40) has the symmetric roots of \( \xi(\Delta, s) = 0 \).

As an analogue to the result of Barner [19], this implies that by using Lemma 7,

\[
\xi(\Delta, s) = \xi(\Delta, 0) e^{B_r s} \prod_{\rho_k} \left(1 - s/\rho_k\right) e^{s/\rho_k}
\]

can be written as (40), where \( \xi(\Delta, 0) = A_{r} \).

Therefore, we deduce the desired result. \( \square \)

Remark. By (40) and (42), this leads to [19]

\[
-B_r = \sum_{k=1}^{\infty} \frac{1}{\rho_k}.
\]

Theorem 4. Assume that \( s \in \mathbb{C}, k \in \mathbb{N}, \varphi \in \mathbb{C}, \) and \( \varphi \neq \rho_k \). Then

\[
\xi(\Delta, s) = \xi(\Delta, \varphi) \prod_{\rho_k} \left(1 - \frac{s - \varphi}{\rho_k - \varphi}\right),
\]

where the product run though the zeros \( \rho_k \) of \( \xi(\Delta, s) \).
Proof. With Theorem 3, it is easy to see
\[
\xi(\Delta, 0) \prod_{\rho_k} \left(1 - \frac{s}{\rho_k}\right) = \xi(\Delta, 0) \prod_{\rho_k} \frac{\rho_k - s}{\rho_k - \varphi}
\]
\[
= \xi(\Delta, 0) \prod_{\rho_k} \frac{\rho_k - \varphi}{\rho_k - (s - \rho_k)}
\]
\[
= \xi(\Delta, 0) \left(\prod_{\rho_k} \frac{\rho_k - \varphi}{\rho_k - (s - \rho_k)} \cdot \prod_{\rho_k} \frac{\rho_k - s}{\rho_k - \varphi}\right)
\]
\[
= \xi(\Delta, 0) \prod_{\rho_k} \left(1 - \frac{s}{\rho_k}\right) \cdot \prod_{\rho_k} \left(1 - \frac{s - \varphi}{\rho_k - \varphi}\right)
\]
\[
= \xi(\Delta, \varphi) \prod_{\rho_k} \left(1 - \frac{s - \varphi}{\rho_k - \varphi}\right),
\]
where the product run though the zeros \(\rho_k\) of \(\xi(\Delta, s)\).

Since \(\varphi \neq \rho_k\), we present
\[
\xi(\Delta, \varphi) \neq 0.
\]
\[\square\]

**Theorem 5.** Let \(s \in \mathbb{C}\). Then \(\xi(\Delta, s)\) is of order \(\alpha = 1\), of genus \(\mu = 0\), and of exponent of convergence of \(\nu = 1\).

Proof. By using an analogue to the treatment of Riemann zeta function [19], and combining Lemma 7 and Theorem 3, we may prove that
\[
\xi(\Delta, s) = \xi(\Delta, 0) e^{B \tau s} \prod_{\rho_k} (1 - s/\rho_k) e^{s/\rho_k}
\]
\[
= \xi(\Delta, 0) \prod_{\rho_k} (1 - s/\rho_k)
\]
is of order
\[
\alpha = \lim_{m \to \infty} \sup \frac{m \log m}{\log (1/|\Phi_m|)} = 1.
\]
From Definition 2, it is observed that (47) is of genus \(\mu = 0\) since
\[
\xi(\Delta, s) = \xi(\Delta, 0) \prod_{\rho_k} (1 - s/\rho_k)
\]
\[
= \prod_{k=1}^{\infty} \mathbb{S}(s/\rho_k, \mu = 0),
\]
where \(\rho_k\) are the zeros of \(\xi(\Delta, s)\).
By using Lemma 2 and (48), it is easy to show that
\begin{equation}
\nu = \alpha = 1.
\end{equation}
\qed

**Definition 5.** Suppose that \( \xi(s) \) is of order \( \alpha = 1 \), and of genus \( \mu = 0 \), \( \Phi_m \) are defined as in (38), and \( m \in \mathbb{N} \cup \{0\} \). The power series of \( \xi(\Delta, s) \), given as
\begin{equation}
\xi(\Delta, s) = \sum_{m=0}^{\infty} \Phi_m (s - 6)^{2m},
\end{equation}
is said in the class \( \mathbb{L} \), written as \( \xi \in \mathbb{L} \), if \( \xi(\Delta, s) \) can be expressed in the form:
\begin{equation}
\xi(\Delta, s) = \xi(\Delta, 0) \prod_{\rho_k} (1 - s/\rho_k),
\end{equation}
where \( s \in \mathbb{C} \), and \( \rho_k \) run over all of the zeros of \( \xi(\Delta, s) \) for \( k \in \mathbb{N} \).

**Theorem 6.** Suppose that \( \xi \in \mathbb{L} \), \( \Phi_m \) are defined as in (38), and \( m \in \mathbb{N} \cup \{0\} \). Then
\begin{equation}
\sum_{m=0}^{\infty} \Phi_m (s - 6)^{2m} = \xi(\Delta, 0) \prod_{\rho_k} (1 - s/\rho_k)
\end{equation}
converges uniformly on every compact set \( \wp \subset \mathbb{C} \), where \( \rho_k \) run over all of the zeros of \( \xi(\Delta, s) \).

**Proof.** By Theorem 5 and (24), we have \( \nu = 1 \) such that
\begin{equation}
|\rho_k|^{-1} < \infty,
\end{equation}
where \( \rho_k \in \mathbb{G} \).

Since \( \xi \in \ell \), it is not difficult to find that
\begin{equation}
\xi(\Delta, s) = \xi(\Delta, 0) \prod_{\rho_k} (1 - s/\rho_k) = \prod_{k=1}^{\infty} S(s/\rho_k, \mu = 0).
\end{equation}
By using Lemma 1, (40) and (41), it follows that
\begin{equation}
\sum_{m=0}^{\infty} \Phi_m (s - 6)^{2m} = \xi(\Delta, 0) \prod_{\rho_k} (1 - s/\rho_k)
\end{equation}
converges uniformly on every compact set \( \wp \subset \mathbb{C} \).

Thus, we deduce the desired result. \qed
Theorem 7. Suppose that $\xi \in \mathbb{L}$, $s \in \mathbb{C}$, $k \in \mathbb{N}$, $\varphi \in \mathbb{C}$, $\varphi \neq \rho_k$, $m \in \mathbb{N} \cup \{0\}$, and $\Phi_m$ are defined as in (38). Then there exists

$$
\sum_{m=0}^{\infty} \Phi_m (s - 6)^{2m} = \xi (\Delta, \varphi) \prod_{\rho_k} \left( 1 - \frac{s - \varphi}{\rho_k - \varphi} \right),
$$

where $\rho_k$ run over all of the zeros of $\xi (\Delta, s)$.

Proof. From Theorems 4 and 6, we directly deduce that (44) is valid. \qed

Corollary 1. Suppose that $\xi \in \mathbb{L}$, $s \in \mathbb{C}$, $k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, and $\Phi_m$ are defined as in (38). Then there exists

$$
\sum_{m=0}^{\infty} \Phi_m (s - 6)^{2m} = \xi (\Delta, 6) \prod_{\rho_k} \left( 1 - \frac{s - 6}{\rho_k - 6} \right),
$$

where $\rho_k$ run over all of the zeros of $\xi (\Delta, s)$.

Proof. Taking $\varphi = 6$ in Theorem 6, we obtain the required result. \qed

We present the necessary condition for the Hardy hypothesis as follows:

Corollary 2. Suppose that $\xi \in \mathbb{L}$, $s \in \mathbb{C}$, $k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, and $\Phi_m$ are defined as in (38). Then there exists

$$
\sum_{m=0}^{\infty} \Phi_m (s - 6)^{2m} = \xi (\Delta, 6) \prod_{\psi_k} \left( 1 - \frac{s - 6}{i \psi_k} \right),
$$

where $\rho_k = 6 + i \psi_k$ run over all of the zeros of $\xi (\Delta, s)$ for $\psi_k \in \mathbb{R} \setminus \{0\}$.

Proof. Taking $\rho_k = 6 + i \psi_k$ in Corollary 2 for $\psi_k \in \mathbb{R} \setminus \{0\}$, the required result follows. \qed

4. The integral, series and infinite product representations for the Ramanujan Xi function

Now, we begin with the integral representation for the Ramanujan Xi function as follows:

Theorem 8. Assume that $\Delta (x)$ is defined by (2), and $t \in \mathbb{C}$. Then

$$
\Xi (\Delta, t) = 2 \int_0^1 \Delta (ix) x^5 \cos (t \log x) \, dx.
$$

Moreover,

$$
\Xi (\Delta, t) = \Xi (\Delta, -t).
$$
Proof. On putting \( s = 6 + it \) into (35) in Theorem 1 with \( t \in \mathbb{C} \), we get

\[
\Xi(\Delta, t) = \xi(\Delta, 6 + it) = 2 \int_0^1 \Delta(i x) x^5 \cosh \left\{ \left[ (6 + it) - 6 \right] \log x \right\} dx
\]

(62)

\[
= 2 \int_0^1 \Delta(i x) x^5 \cosh(it \log x) dx = 2 \int_0^1 \Delta(i x) x^5 \cos (t \log x) dx.
\]

In a similar way, it is easy to see that

\[
\Xi(\Delta, -t) = \xi(\Delta, 6 - it)
\]

(63)

\[
= 2 \int_0^1 \Delta(i x) x^5 \cosh(-it \log x) dx = 2 \int_0^1 \Delta(i x) x^5 \cos (-t \log x) dx = 2 \int_0^1 \Delta(i x) x^5 \cos (t \log x) dx.
\]

From (62) and (63), one gives

\[
\Xi(\Delta, t) = 2 \int_0^1 \Delta(i x) x^5 \cos (t \log x) dx = 2 \int_0^1 \Delta(i x) x^5 \cos (-t \log x) dx = \Xi(\Delta, -t).
\]

(64)

Theorem 9. Assume that \( \Delta(x) \) is defined by (2), \( \Phi_m \) are defined by (38), \( m \in \mathbb{N} \cup \{0\} \), and \( t \in \mathbb{C} \). Then

\[
\Xi(\Delta, t) = \sum_{m=0}^{\infty} (-1)^m \Phi_m t^{2m}.
\]

(65)

Proof. Substituting \( s = 6 + it \) into (37) in Theorem 1 with \( t \in \mathbb{C} \), we obtain the desired result. \( \square \)
Theorem 10. Suppose that \( \xi \in \mathbb{L}, s \in \mathbb{C}, t \in \mathbb{C}, k \in \mathbb{N}, \varphi \in \mathbb{C}, \varphi \neq \rho_k, m \in \mathbb{N} \cup \{0\}, \) and \( \Phi_m \) are defined as in (38). Then there exists

\[
\Xi(\Delta, t) = \sum_{m=0}^{\infty} (-1)^m \Phi_m t^{2m}
= \xi(\Delta, \varphi) \prod_{\rho_k} \left[ 1 - \frac{(6-\varphi)+it}{\rho_k-\varphi} \right],
\]

where \( \rho_k \) run over all of the zeros of \( \xi(\Delta, s) \).

Proof. By the substitution of \( s = 6 + it \) into (44) in Theorem 7, we may arrive at

\[
\Xi(\Delta, t) = \xi(\Delta, 6 + it)
= \sum_{m=0}^{\infty} \Phi_m [(6 + it) - 6]^{2m}
= \sum_{m=0}^{\infty} (-1)^m \Phi_m t^{2m}
= \xi(\Delta, \varphi) \prod_{\rho_k} \left[ 1 - \frac{(6+it) - \varphi}{\rho_k - \varphi} \right]
= \xi(\Delta, \varphi) \prod_{\rho_k} \left[ 1 - \frac{(6-\varphi) + it}{\rho_k - \varphi} \right],
\]

which is the required result. \( \square \)

Theorem 11. Suppose that \( \xi \in \mathbb{L}, s \in \mathbb{C}, t \in \mathbb{C}, k \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, \) and \( \Phi_m \) are defined as in (38). Then

\[
\Xi(\Delta, t) = \sum_{m=0}^{\infty} (-1)^m \Phi_m t^{2m}
= \Xi(\Delta, 0) \prod_{\rho_k} \left[ 1 - \frac{t}{i(6-\rho_k)} \right],
\]

where \( \rho_k \) run over all of the zeros of \( \xi(\Delta, s) \).

Proof. Taking \( \varphi = 6 \) in (66), we easily obtain

\[
\Xi(\Delta, t) = \sum_{m=0}^{\infty} (-1)^m \Phi_m t^{2m}
= \xi(\Delta, 6) \prod_{\rho_k} \left[ 1 - \frac{(6-6)+it}{\rho_k-6} \right]
= \Xi(\Delta, 0) \prod_{\rho_k} \left[ 1 - \frac{it}{\rho_k-6} \right]
= \Xi(\Delta, 0) \prod_{\rho_k} \left[ 1 - \frac{t}{i(6-\rho_k)} \right] = \Xi(\Delta, 0) \prod_{\rho_k} \left[ 1 - \frac{t}{i(6-\rho_k)} \right],
\]

Hence, we obtain the result (68). \( \square \)
Theorem 12. Suppose that $\xi \in \mathbb{L}$, $s \in \mathbb{C}$, $t \in \mathbb{C}$, $k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, and $\Phi_m$ are defined as in (38). Then

$$\Xi(\Delta, t) = \sum_{m=0}^{\infty} (-1)^m \Phi_m t^{2m} = \Xi(\Delta, 0) \prod_{\psi_k} \left(1 - \frac{t}{\psi_k}\right),$$

where $\rho_k = 6 + i\psi_k$ run over all of the zeros of $\xi(\Delta, s)$ for $\psi_k \in \mathbb{R} \setminus \{0\}$.

Proof. Putting $\rho_k = 6 + i\psi_k$ into (68) in Theorem 11 for $\psi_k \in \mathbb{R} \setminus \{0\}$, we obtain

$$\Xi(\Delta, t) = \sum_{m=0}^{\infty} (-1)^m \Phi_m t^{2m} = \Xi(\Delta, 0) \prod_{\psi_k} \left(1 - \frac{t}{i(6 - (6 + i\psi_k))}\right),$$

which is the required result. \□

Theorem 13. Let $t \in \mathbb{C}$. Then $\Xi(\Delta, t)$ is of order $\alpha = 1$, of genus $\mu = 0$, and of exponent of convergence of $\nu = 1$.

Proof. From (37) and (65), $\xi(\Delta, s)$ and $\Xi(\Delta, t)$ have the same coefficients (37) such that by Definition 3, $\xi(\Delta, s)$ and $\Xi(\Delta, t)$ have the same order $\alpha = 1$.

With use of Theorem 11, we have

$$\Xi(\Delta, t) = \Xi(\Delta, 0) \prod_{\rho_k} \left[1 - \frac{t}{i(6 - \rho_k)}\right],$$

where $i(6 - \rho_k)$ run over all of the zeros of $\Xi(\Delta, t)$.

By Definition 2 and (72), we know that

$$\Xi(\Delta, t) = \Xi(\Delta, 0) \prod_{\rho_k} \left[1 - \frac{t}{i(6 - \rho_k)}\right] = \prod_{n=1}^{\infty} S\{t/\lfloor i(6 - \rho_k)\rfloor, \mu = 0\},$$

where $i(6 - \rho_k) \in \mathbb{G}$.

Hence, $\Xi(\Delta, t)$ is of genus $\mu = 0$.

Making use of Lemma 2, it is seen that $\alpha = \nu = 1$ since $\Xi(\Delta, t)$ is of order $\alpha = 1$, and of genus $\mu = 0$. \□

Corollary 3. Suppose that $\xi \in \mathbb{L}$, $s \in \mathbb{C}$, $t \in \mathbb{C}$, $k \in \mathbb{N}$ and $\rho_k$ run over all of the zeros of $\xi(\Delta, s)$. Then

$$\Xi(\Delta, t) = \Xi(\Delta, 0) \prod_{\rho_k} \left[1 - \frac{t}{i(6 - \rho_k)}\right],$$
converges uniformly on every compact set \( \hat{\wp} \subset \mathbb{C} \).

**Proof.** With use of Theorem 11, we have \( \nu = 1 \) such that

\[
|i(6 - \rho_k)|^{-1} < \infty,
\]

where \( \rho_k \in \mathbb{G} \).

By Lemma 1 and (75), we may find that

\[
\Xi(\Delta, t) = \Xi(\Delta, 0) \prod_{\rho_k} \left[1 - \frac{t}{i(6 - \rho_k)}\right]
\]

converges uniformly on every compact set \( \hat{\wp} \subset \mathbb{C} \), where \( \rho_k \) run over all of the zeros of \( \xi(\Delta, s) \).

\[
\square
\]

5. **Prove the conjecture of Bruijn (Conjecture 1)**

We begin with the proof of Conjecture 1 as follows.

Let

\[
\Xi(\Delta, t) = \xi(\Delta, 6 + it) = 0,
\]

where \( t \in \mathbb{C} \).

By Theorem of Lekkerkerker [10, 11], it is known that \( \xi(\Delta, s) \) has many infinitely zeros on the critical line \( \Re(s) = 6 \).

Assume that the zeros \( \rho_k \in \mathbb{G} \) of \( \xi(\Delta, s) \) can be given as follows:

\[
\rho_k = 6 + iw_k,
\]

which \( k \in \mathbb{N} \), and \( w_k \in \mathbb{C} \) are the zeros of \( \Xi(\Delta, t) \) for \( t \in \mathbb{C} \).

Making use of Theorem 5 and (24), we have \( \nu = 1 \) such that

\[
|\rho_k|^{-1} < \infty,
\]

where \( \rho_k \in \mathbb{G} \subset \mathbb{C} \).

We now denote

\[
w_k = \sigma_k + i\vartheta_k,
\]

where \( w_k \in \mathbb{C} \), \( \Re(w_k) = \sigma_k \in \mathbb{R} \), and \( \Im(w_k) = \vartheta_k \in \mathbb{R} \).

By using (78) and (80), it follows that

\[
\rho_k = 6 + i(\sigma_k + i\vartheta_k) = (6 - \vartheta_k) + i\sigma_k,
\]

where \( k \in \mathbb{N} \), \( \rho_k \in \mathbb{G} \), \( w_k \in \mathbb{C} \), \( \Re(w_k) = \sigma_k \in \mathbb{R} \), and \( \Im(w_k) = \vartheta_k \in \mathbb{R} \).
In view of Theorem 4, it is seen that

\[
\xi(\Delta, s) = \xi(\Delta, \varphi) \prod_{\rho_k} \left( 1 - \frac{s-\varphi}{\rho_k-\varphi} \right)
\]

(82)

\[
= \xi(\Delta, \varphi) \prod_{\rho_k=(6-\vartheta_k)+i\sigma_k} \left[ 1 - \frac{s-\varphi}{(6-\vartheta_k)+i\sigma_k-\varphi} \right]
\]

\[
= \xi(\Delta, \varphi) \prod_{\rho_k=(6-\vartheta_k)+i\sigma_k} \left[ 1 - \frac{s-\varphi}{(6-\vartheta_k)+i\sigma_k-\varphi} \right],
\]

where \( k \in \mathbb{N}, s \in \mathbb{C}, \varphi \in \mathbb{C}, \rho_k \in \mathbb{G}, \varphi \neq \rho_k, \Re(w_k) = \sigma_k \in \mathbb{R}, \) and \( \Im(w_k) = \vartheta_k \in \mathbb{R}. \)

With (82), we may arrive at

\[
\xi(\Delta, 1-s) = \xi(\Delta, \varphi) \prod_{\rho_k} \left[ 1 - \frac{(1-s)-\varphi}{\rho_k-\varphi} \right]
\]

(83)

\[
= \xi(\Delta, \varphi) \prod_{\rho_k=(6-\vartheta_k)+i\sigma_k} \left[ 1 - \frac{(1-s)-\varphi}{(6-\vartheta_k)+i\sigma_k-\varphi} \right]
\]

\[
= \xi(\Delta, \varphi) \prod_{\rho_k=(6-\vartheta_k)+i\sigma_k} \left[ 1 - \frac{(1-s)-\varphi}{(6-\vartheta_k)+i\sigma_k-\varphi} \right],
\]

where \( k \in \mathbb{N}, s \in \mathbb{C}, \varphi \in \mathbb{C}, \rho_k \in \mathbb{G}, \varphi \neq \rho_k, \Re(w_k) = \sigma_k \in \mathbb{R}, \) and \( \Im(w_k) = \vartheta_k \in \mathbb{R}. \)

With

\[
\Xi(\Delta, t) = \xi(\Delta, 6+it),
\]

(84) can be written in the form:

\[
\Xi(\Delta, t) = \xi(\Delta, 6+it)
\]

(85)

\[
= \xi(\Delta, \varphi) \prod_{\rho_k=(6-\vartheta_k)+i\sigma_k} \left[ 1 - \frac{(6+it)-\varphi}{(6-\vartheta_k)+i\sigma_k-\varphi} \right]
\]

\[
= \xi(\Delta, \varphi) \prod_{\rho_k=(6-\vartheta_k)+i\sigma_k} \left[ 1 - \frac{(6+it)-\varphi}{(6-\vartheta_k)+i\sigma_k-\varphi} \right],
\]

where \( k \in \mathbb{N}, t \in \mathbb{C}, \varphi \in \mathbb{C}, w_k \in \mathbb{C}, \rho_k \in \mathbb{G}, \varphi \neq \rho_k, \Re(w_k) = \sigma_k \in \mathbb{R}, \) and \( \Im(w_k) = \vartheta_k \in \mathbb{R}. \)

By

\[
\Xi(\Delta, -t) = \xi(\Delta, 6-it) = \xi(\Delta, 1-(6+it)),
\]

(86)
can be represented in the form:
\[
\Xi (\Delta, -t) = \xi (\Delta, 6 - it)
\]
\[
= \xi (\Delta, \varphi) \prod_{\rho_k=(6-\varphi)+i\sigma_k} \left[ 1 - \frac{(6-it)-\varphi}{(6-\varphi)+i\sigma_k-\varphi} \right]
\]
\[
= \xi (\Delta, \varphi) \prod_{\rho_k=(6-\varphi)+i\sigma_k} \left[ 1 - \frac{\varphi-it}{(6-\varphi)+i\sigma_k-\varphi} \right],
\]
where \( k \in \mathbb{N}, t \in \mathbb{C}, \varphi \in \mathbb{C}, w_k \in \mathbb{C}, \rho_k \in \mathbb{G}, \varphi \neq \rho_k, \Re (w_k) = \sigma_k \in \mathbb{R}, \) and \( \Im (w_k) = \vartheta_k \in \mathbb{R}. \)

From Theorem 8, we have
\[
\Xi (\Delta, t) = \Xi (\Delta, -t)
\]
such that
\[
6 - \varphi = 0
\]
and
\[
6 - \varphi - \vartheta_k = 0,
\]
where \( k \in \mathbb{N}, t \in \mathbb{C}, \varphi \in \mathbb{C}, w_k \in \mathbb{C}, \) and \( \Re (w_k) = \vartheta_k \in \mathbb{R}. \)

In terms of (89) and (90), it is easy to see that
\[
\varphi = 6
\]
and
\[
\vartheta_k = 0
\]
which imply that (85) can be represented by
\[
\Xi (\Delta, t) = \xi (\Delta, 6 + it)
\]
\[
= \xi (\Delta, 6) \prod_{\sigma_k=(6-0)+i\sigma_k} \left[ 1 - \frac{(6-6)+it}{(6-6)+i\sigma_k} \right]
\]
\[
= \Xi (\Delta, 0) \prod_{\sigma_k} \left( 1 - \frac{i}{\sigma_k} \right),
\]
where \( k \in \mathbb{N}, t \in \mathbb{C}, w_k \in \mathbb{C}, \) and \( \Re (w_k) = \sigma_k \in \mathbb{R} \setminus \{0\}. \)

In a similar manner, by (91) and (92), (87) can be rewritten as
\[
\Xi (\Delta, -t) = \xi (\Delta, 6 - it)
\]
\[
= \xi (\Delta, 6) \prod_{\sigma_k=(6-0)+i\sigma_k} \left[ 1 - \frac{(6-6)-it}{(6-6)+i\sigma_k} \right]
\]
\[
= \Xi (\Delta, 0) \prod_{\sigma_k} \left( 1 + \frac{i}{\sigma_k} \right).
where \( k \in \mathbb{N}, t \in \mathbb{C}, w_k \in \mathbb{C}, \) and \( \Re(w_k) = \sigma_k \in \mathbb{R} \setminus \{0\} \).

By Lemma 4 and (28), there are the positive roots \( v_k \) of \( \Xi(\Delta, t) = 0 \) with
\[
\begin{align*}
\Xi(\Delta, t) &= \xi(\Delta, 6 + it) \\
&= \Xi(\Delta, 0) \prod_{\sigma_k} \left(1 - \frac{t}{\sigma_k}\right) \\
&= \Xi(\Delta, 0) \prod_{k=1}^{\infty} \left(1 - \frac{2}{v_k^2}\right)
\end{align*}
\]
for \( k \in \mathbb{N}, t \in \mathbb{C}, w_k \in \mathbb{C}, \) and \( \Re(w_k) = \sigma_k \in \mathbb{R} \setminus \{0\} \).

By using (88) and (96), it is easy to see that
\[
\begin{align*}
\Xi(\Delta, t) &= \xi(\Delta, 6 + it) \\
&= \Xi(\Delta, 0) \prod_{\sigma_k} \left(1 - \frac{t}{\sigma_k}\right) \\
&= \Xi(\Delta, 0) \prod_{k=1}^{\infty} \left(1 - \frac{2}{v_k^2}\right) \\
&= \Xi(\Delta, 0) \prod_{\sigma_k} \left(1 + \frac{t}{\sigma_k}\right) \\
&= \xi(\Delta, 6 - it) \\
&= \Xi(\Delta, -t)
\end{align*}
\]
always hold for \( k \in \mathbb{N}, t \in \mathbb{C}, w_k \in \mathbb{C}, \) and \( v_k > 0 \).

From (97), this implies that (94) can be expressed by the infinite product:
\[
\begin{align*}
\Xi(\Delta, -t) &= \xi(\Delta, 6 - it) \\
&= \Xi(\Delta, 0) \prod_{\sigma_k} \left(1 + \frac{t}{\sigma_k}\right) \\
&= \Xi(\Delta, 0) \prod_{k=1}^{\infty} \left(1 - \frac{2}{v_k^2}\right),
\end{align*}
\]
where \( k \in \mathbb{N}, t \in \mathbb{C}, v_k > 0, w_k \in \mathbb{C}, \) and \( \Re(w_k) = \sigma_k \in \mathbb{R} \setminus \{0\} \).

Applying (91) and (92), it is shown that
\[
(99) \quad w_k = \sigma_k + iv_k = \sigma_k + i0 = \sigma_k \in \mathbb{R} \setminus \{0\} .
\]

From (78) and (99), it is seen that all of the zeros of \( \Xi(\Delta, t) \) are
\[
(100) \quad \sigma_k \in \mathbb{R} \setminus \{0\}
\]
for \( k \in \mathbb{N} \).

Now, we let
\[
(101) \quad \psi_k = \sigma_k .
\]
By using (99) and (100), it is easy to find that
\[ \psi_k \in \mathbb{R} \setminus \{0\}, \]
which implies that
\[ \nu_k = |\psi_k| = |\sigma_k| > 0. \]
Therefore, we complete this proof.

6. Prove the Hardy hypothesis (Conjecture 2)

We now start with the proof of Conjecture 2 as follows.

By using Theorem 7, it is seen that
\[ \sum_{m=0}^{\infty} \Phi_m (s - 6)^{2m} = \xi (\Delta, \varphi) \prod_{\rho_k} \left( 1 - \frac{s - \varphi}{\rho_k - \varphi} \right), \]
where \( \xi \in \mathbb{L}, s \in \mathbb{C}, k \in \mathbb{N}, \varphi \in \mathbb{C}, \varphi \neq \rho_k, m \in \mathbb{N} \cup \{0\}, \Phi_m \) are defined as in (38), and \( \rho_k \) run over all of the zeros of \( \xi (\Delta, s) \).

On putting \( s = 6 + it \) into (104) and using Theorem 11, we easily find that
\[ \Xi (\Delta, t) = \sum_{m=0}^{\infty} (-1)^m \Phi_m t^{2m} \]
\[ = \Xi (\Delta, 0) \prod_{\rho_k} \left[ 1 - \frac{t}{i(6 - \rho_k)} \right], \]
where \( \xi \in \mathbb{L}, s \in \mathbb{C}, t \in \mathbb{C}, k \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, \Phi_m \) are defined as in (38), and \( \rho_k \) run over all of the zeros of \( \xi (\Delta, s) \).

By Conjecture 1 and (105), it is easy to obtain
\[ i(6 - \rho_k) \in \mathbb{R} \setminus \{0\}, \]
where \( \rho_k \) run over all of the zeros of \( \xi (\Delta, s) \).

We now denote the zeros \( \rho_k \) of \( \xi (\Delta, s) \) by
\[ \rho_k = \lambda_k + i\psi_k, \]
where \( k \in \mathbb{N}, \lambda_k \in \mathbb{R} \setminus \{0\}, \) and \( \psi_k \in \mathbb{R} \setminus \{0\} \).

Now, we rewrite (106) as
\[ i(6 - \rho_k) = i \left[ 6 - (\lambda_k + i\psi_k) \right] \]
\[ = i \left[ (6 - \lambda_k) + i\psi_k \right] \]
\[ = i(6 - \lambda_k) - \psi_k \in \mathbb{R} \setminus \{0\}, \]
where \( k \in \mathbb{N}, \rho_k \in \mathbb{C}, \lambda_k \in \mathbb{R} \setminus \{0\}, \) and \( \psi_k \in \mathbb{R} \setminus \{0\} \).

By (108), we get
\[ i(6 - \lambda_k) = 0, \]
Figure 1. The green points represents the trivial zeros of $L(\Delta, s)$. The blue points represents the nontrivial zeros of $L(\Delta, s)$. The real part of all of the nontrivial zeros of $L(\Delta, s)$ is equal to 6.

which leads to

\[(110) \quad \lambda_k = 6.\]

Therefore,

\[(111) \quad \Re(\rho_k) = \Re(\lambda_k + i\psi_k) = \lambda_k = 6,
\]

which is the desired result.

Thus, we finish the proof of Conjecture 2.

**Remark.** In view of (96), we have

\[(112) \quad \xi(\Delta, s) = \Xi(\Delta, (s - 6)/i) = \Xi(\Delta, 0) \prod_{\sigma_k \neq 0} \left(1 - \frac{s - 6}{i\sigma_k}\right) = \Xi(\Delta, 0) \prod_{k=1}^{\infty} \left[1 - \frac{(s - 6)^2}{\nu_k^2}\right],\]

where $k \in \mathbb{N}$, $s \in \mathbb{C}$, $\sigma_k \in \mathbb{R} \setminus \{0\}$, and $\nu_k > 0$.

By (8) and (112),

\[(113) \quad L(\Delta, s) = \Xi(\Delta, 0) \left(\frac{2\pi}{\Gamma(s)}\right)^{\frac{s}{2}} \prod_{k=1}^{\infty} \left[1 - \frac{(s - 6)^2}{\nu_k^2}\right],\]

where $k \in \mathbb{N}$, $s \in \mathbb{C}$, $\sigma_k \in \mathbb{R} \setminus \{0\}$, and $\nu_k > 0$.

Thus, by using (113), $L(\Delta, s)$ has the trivial zeros $s = -r$, where $r \in \mathbb{N} \cup \{0\}$ and the nontrivial zeros $\rho_k = 6 \pm \nu_k$ for $\nu_k > 0$. The characteristics of the Ramanujan zeta function $L(\Delta, s)$ are shown in Fig. 1.
7. Open problems and comments

It had reported that Bruijn defined [13]

\[
\mathbb{H} (\Delta, y) = \xi (\Delta, 6 + y) = L (\Delta, 6 + y) (2\pi)^{- (6 + y)} \Gamma (6 + y),
\]

where \( y \in \mathbb{C} \).

On putting \( s = 6 + y \) into (35) with \( y \in \mathbb{C} \), one has

\[
\mathbb{H} (\Delta, y) = 2 \int_0^1 (ix) x^5 \cosh \{[(6 + y) - 6] \log x \} \, dx
\]

which can be expressed by the infinite product:

\[
\mathbb{H} (\Delta, y) = \mathbb{H} (\Delta, 0) \prod_{\rho_k} \left( 1 - \frac{y}{\rho_k - 6} \right).
\]

where \( \rho_k \) run over all of the zeros of \( \xi (\Delta, s) \), and \( \Phi_m \) are defined as in (38).

By (115), (116), and Lemmas 2 and 3, it is known that \( \mathbb{H} (\Delta, y) \) is is of order \( \alpha = 1 \), of genus \( \mu = 0 \), and of exponent of convergence of \( \nu = 1 \).

**Problem 1.** Assume that \( y \in \mathbb{C} \) and \( k \in \mathbb{N} \). Then all of zeros of \( \mathbb{H} (\Delta, y) \) are \( \rho_k - 6 \), satisfying \( i (\rho_k - 6) \in \mathbb{R} \setminus \{0\} \), where \( \rho_k \) run over all of the zeros of \( \xi (\Delta, s) \).

By using result of Rankin [2], and Theorem 8, it is easy to observe that

\[
\Xi (\Delta, t) = \int_0^\infty \Delta (ix) x^{s-1} dx \quad (\Re (s) > 13/2)
\]

which follows that

\[
\Xi (\Delta, t) = \int_0^1 \Delta (ix) (x^{s-1} + x^{11-s}) \, dx
\]

With the result of Conrey and Ghosh [7], it follows that

\[
\Xi (\Delta, t) = \int_0^1 \Delta (ix) x^{5} \cos (t \log x) \, dx
\]

\[
= \int_{-\infty}^{\infty} \mathcal{E} (\Delta, z) e^{izt} \, dz,
\]

where \( \mathcal{E} (\Delta, z) \) is the error term.
which is connected with [7]

\[ E(\Delta, z) = \Delta(ie^z) e^{6z} = e^{6z - 2\pi \varepsilon z} \prod_{n=1}^{\infty} \left(1 - x^{-2n\pi \varepsilon z}\right)^{24}, \]

where \( z \in \mathbb{R} \) and \( n \in \mathbb{N} \).

Since \( \Xi(\Delta, t) \) is an even entire function proved by Conrey and Ghosh [7], we know that

\[ \Xi(\Delta, z) = \Xi(\Delta, -z), \]

and

\[ \Xi(\Delta, t) = \int_{-\infty}^{\infty} E(\Delta, z) e^{izt} \, dz = 2 \int_{0}^{\infty} E(\Delta, z) \cos(zt) \, dz. \]

Let

\[ \Xi(\Delta, t, \aleph) = \int_{-\infty}^{\infty} E(\Delta, z) e^{-\aleph z^2} e^{izt} \, dz = 2 \int_{0}^{\infty} e^{-\aleph z^2} E(\Delta, z) \cos(zt) \, dz, \]

where \( t \in \mathbb{C}, \aleph \in \mathbb{R}, \) and \( z \in \mathbb{R} \).

It follows from (122) that

\[ \Xi(\Delta, t, \aleph = 0) = \Xi(\Delta, t). \]

Now we present an analogue to the works of Bruijn [13] and Newman [20] (also see [21]) for the Riemann zeta function as follows:

**Problem 2.** The Hardy hypothesis is equivalent to the conjecture that \( \aleph \leq 0 \).

Now, with the aid of Conjecture 2, we know that Theorem 11 is true.

Thus, it is easy to obtain that

\[ \Xi(\Delta, t) = \sum_{m=0}^{\infty} (-1)^m \Phi_m t^{2m} = \Xi(\Delta, 0) \prod_{\psi_k} \left(1 - \frac{t}{\psi_k}\right), \]

where \( t \in \mathbb{C}, k \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, \Phi_m \) are defined as in (38), and \( \psi_k \in \mathbb{R} \setminus \{0\} \) run over all of the zeros of \( \Xi(\Delta, t) \).

From Corollary 3, we may find that (124) converges uniformly on every compact set \( \hat{\mathbb{C}} \subset \mathbb{C} \).

So, we get

\[ \Xi(\Delta, t) = \sum_{m=0}^{\infty} \frac{\Xi(2m)(\Delta, 0)}{(2m)!} t^{2m}. \]
By virtue of \((124)\) and \((125)\), we write
\[
\Xi(\Delta, t) = \sum_{m=0}^{\infty} \frac{K(\Delta, m)}{(m)!} t^{2m},
\]
in which
\[
K(\Delta, m) = \frac{(m)!}{(2m)!} \Xi^{(2m)}(\Delta, 0) = (-1)^m (m)! \Phi_m
\]
where \(t \in \mathbb{C}, m \in \mathbb{N} \cup \{0\}\), and \(\Phi_m\) are defined as in \((38)\).

With \((126)\), one gives
\[
\hat{\Xi}(\Delta, t) = \sum_{m=0}^{\infty} \frac{K(\Delta, m)}{(m)!} t^m,
\]
where \(t \in \mathbb{C}, m \in \mathbb{N} \cup \{0\}\), and \(K(\Delta, m)\) are defined as in \((127)\).

Denote that \(\hat{\Xi}(\Delta, t)\) is hyperbolic if all of the zeros of \(\hat{\Xi}(\Delta, t)\) are real.

The Jensen polynomial of degree \(A\) and shift \(M\) is defined as [14, 15, 22]
\[
\wp^{A, M}(\Delta, t) = \sum_{j=0}^{A} \binom{A}{j} K(\Delta, M + j) t^j,
\]
where \(t \in \mathbb{C}, m \in \mathbb{N} \cup \{0\}\), \(A \in \mathbb{N} \cup \{0\}\), and \(j \in \mathbb{N} \cup \{0\}\).

We may suggest an analogue to the work of Jensen [14] (also see [15, 22]) for the Riemann zeta function as follows:

**Problem 3.** Let \(t \in \mathbb{C}\). Then the hardy hypothesis is true if and only if \(\wp^{A, M}(\Delta, t)\) is hyperbolic for all \(M \geq 1\) and \(A \geq 0\).

In a similar manner, we also consider an analogue to the work of Griffin and coauthors [15] (also see [22]) for the Riemann zeta function as follows:

**Problem 4.** Suppose that
\[
\hat{\Xi}(\Delta, t) = \sum_{m=0}^{\infty} \frac{K(\Delta, m)}{(m)!} t^m,
\]
where \(t \in \mathbb{C}, m \in \mathbb{N} \cup \{0\}\), and \(K(\Delta, m)\) are defined as in \((127)\). Then \(\hat{\Xi}(\Delta, t)\) is hyperbolic.

By using the identities
\[
\Delta(x) = e^{-2\pi ix} \prod_{n=1}^{\infty} \left(1 - e^{-2\pi nx}\right)^2,
\]
(132) \[ \sqrt{\Delta (-xi)} = e^{-\pi x} \prod_{n=1}^{\infty} (1 - e^{-2\pi nx})^{12}, \]

and

(133) \[ \sqrt{\Delta \left(-\frac{i}{x}\right)} = e^{-\pi/x} \prod_{n=1}^{\infty} (1 - e^{-2\pi nx/x})^{12}, \]

we have the Bruijn identity [13]

(134) \[ \Delta (-xi) \Delta \left(-\frac{i}{x}\right) = e^{-\pi(x+1/z)} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi nx}) (1 - e^{-2\pi nx/z}) \right]^{12} \right\} \]

such that

(135) \[ \mathcal{H}(\Delta, y) = \int_{0}^{\infty} \Delta (ix) x^{y+5} dx, \]

which is in agreement with the fact that

\[ \mathcal{H}(\Delta, y) = \xi(\Delta, 6 + y) \]
\[ = L(\Delta, 6 + y) (2\pi)^{-(6+y)} \Gamma(6+y) \]
\[ = \int_{0}^{\infty} \Delta (ix) x^{(6+y)-1} dx \quad (Re(s) > 13/2) \]

(136) \[ \int_{0}^{\infty} \Delta (ix) x^{5+y} dx \quad (Re(s) > 13/2) \]
\[ = \int_{0}^{1} \Delta (ix) (x^{5+y} + x^{5-y}) dx, \]

derived from the result of Rankin [2]

(137) \[ \xi(\Delta, s) = \int_{0}^{\infty} \Delta (ix) x^{s-1} dx \quad (Re(s) > 13/2) \]
\[ = \int_{0}^{1} \Delta (ix) (x^{s-1} + x^{11-s}) dx. \]

Taking \( x = e^z \) in (134), we have

(138) \[ \Delta (-e^z i) \Delta \left(-\frac{i}{e^z}\right) = e^{-\pi(e^z + e^{-z})} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi ne^z}) (1 - e^{-2\pi ne^{-z}}) \right]^{12} \right\} \]
\[ = e^{-2\pi \cosh z} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi ne^z}) (1 - e^{-2\pi ne^{-z}}) \right]^{12} \right\} \]
\[ = e^{6z} \Delta (-ie^z), \]
which is in accord with the result of Bruijn [13], i.e.,

\[(139) \quad e^{6z} \Delta (-ie^z) = e^{-2\pi \cosh z} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi ne^z}) \left( 1 - e^{-2\pi ne^{-z}} \right) \right]^{12} \right\}.
\]

It follows from (119) and (139) that

\[(140) \quad \mathbb{E}(\Delta, z) = e^{6z} \Delta (-ie^z) = e^{-2\pi \cosh z} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi ne^z}) \left( 1 - e^{-2\pi ne^{-z}} \right) \right]^{12} \right\},
\]

which implies that

\[(141) \quad \Xi(\Delta, t) = \int_{-\infty}^{\infty} \mathbb{E}(\Delta, z) e^{izt} dz
\]

\[= 2 \int_{-\infty}^{\infty} \mathbb{E}(\Delta, z) \cos (zt) dz
\]

\[= \int_{-\infty}^{\infty} e^{6z} \Delta (-ie^z) e^{izt} dz
\]

\[= 2 \int_{-\infty}^{\infty} e^{6z} \Delta (-ie^z) \cos (zt) dz
\]

\[= \int_{-\infty}^{\infty} e^{-2\pi \cosh z} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi ne^z}) \left( 1 - e^{-2\pi ne^{-z}} \right) \right]^{12} \right\} e^{izt} dz
\]

\[= 2 \int_{0}^{\infty} e^{-2\pi \cosh z} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi ne^z}) \left( 1 - e^{-2\pi ne^{-z}} \right) \right]^{12} \right\} \cos (zt) dz
\]
and

\[
\Xi(\Delta, t, \mathbb{N}) = \int_{-\infty}^{\infty} E(\Delta, z) e^{-\mathbb{N}z^2} e^{izt} dz
\]

\[
= 2 \int_{0}^{\infty} e^{-\mathbb{N}z^2} E(\Delta, z) \cos(zt) dz
\]

\[
= \int_{-\infty}^{\infty} [e^{6z} \Delta(-ie^z)] e^{-\mathbb{N}z^2} e^{izt} dz
\]

\[
= 2 \int_{0}^{\infty} e^{-\mathbb{N}z^2} [e^{6z} \Delta(-ie^z)] \cos(zt) dz
\]

\[
= \int_{-\infty}^{\infty} \Delta(-ie^z) e^{-\mathbb{N}z^2 + 6z} e^{izt} dz
\]

\[
= 2 \int_{0}^{\infty} e^{-\mathbb{N}z^2 + 6z} \Delta(-ie^z) \cos(zt) dz
\]

\[
= \int_{-\infty}^{\infty} e^{-2\pi \cosh z} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi ne^z}) (1 - e^{-2\pi ne^{-z}}) \right]^{12} \right\} e^{-\mathbb{N}z^2} e^{izt} dz
\]

\[
= 2 \int_{0}^{\infty} e^{-\mathbb{N}z^2} e^{-2\pi \cosh z} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi ne^z}) (1 - e^{-2\pi ne^{-z}}) \right]^{12} \right\} \cos(zt) dz,
\]

where \( t \in \mathbb{C}, \mathbb{N} \in \mathbb{R}, \) and \( z \in \mathbb{R}. \)

From (141), we easily find that

\[
\Xi(\Delta, t) = \int_{-\infty}^{\infty} e^{-2\pi \cosh z} \left\{ \prod_{n=1}^{\infty} \left[ (1 - e^{-2\pi ne^z}) (1 - e^{-2\pi ne^{-z}}) \right]^{12} \right\} e^{izt} dz,
\]

which is the result of Bruijn [13].

It has been reported that Ki [8] and Chirre and Castañón [12] have considered the representation of \( \Xi(\Delta, t) \) by (18). As an analogue of the Riemann-Siegel formula for the Riemann zeta function, Keiper [23] suggested an efficient formula to compute the real zeros \( \psi_k \in \mathbb{R} \setminus \{0\} \) of \( \Xi(\Delta, t) \) for \( k \in \mathbb{N}. \)

Since Rankin [2] proved that

\[
L\left(\Delta, \frac{13}{2}\right) \neq 0,
\]

we have form (10) that

\[
\xi\left(\Delta, \frac{13}{2}\right) = L\left(\Delta, \frac{13}{2}\right)(2\pi)^{-\frac{23}{2}} \Gamma\left(\frac{13}{2}\right) \neq 0.
\]
By using Theorem 4, (145) is in agreement with

\[(146) \quad \xi \left( \Delta, \frac{13}{2} \right) = \xi (\Delta, \varphi) \prod_{\rho_k} \left( 1 - \frac{13}{\rho_k - \varphi} \right), \]

which leads to

\[(147) \quad \varphi \neq \rho_k, \]

or alternatively,

\[(148) \quad \xi (\Delta, \varphi) \neq 0 \]

where \(\rho_k\) are the zeros of \(\xi (\Delta, s)\) and \(\varphi \in \mathbb{C}\).

8. Conclusion

In the previous work, it has been proved that all of the nontrivial zeros for the Ramanujan zeta function have a real part equal to 6. The power series expansion for the entire Ramanujan zeta function was suggested with use of Mordell. The alternative form of the infinite product for the entire Ramanujan zeta function was discussed in detail. The class of the entire Ramanujan zeta function was obtained based on the series and production representations for the entire Ramanujan zeta function. It has been observed that the theorem of Lekkerkerker leads to the truth of the conjecture of Bruijn. From the work, it has been proved that the Hardy hypothesis is true. Some open problems associated to the Ramanujan zeta function were considered as the analogues to the works of Bruijn, Newman, Jensen, and Griffin and coauthors.

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