Some integral formulae on weighted manifolds

Mohammed Abdelmalek and Mohammed Benalili

Abstract. Introducing a notion of the weighted $k$-mean curvature and using the weighted Newton transformations, we derive in this paper some integral formulae on weighted manifolds. These formulae generalize the flux formula and some of its examples of applications obtained by Alias et al. (J Inst Math Jussieu 5(4):527–562, 2006).

1. Introduction

Several works have been done in the past to study the geometric properties of constant $k$-mean curvature hypersurfaces in space forms, where the $k$-mean curvature is defined as the $k$-th elementary symmetric function of the eigenvalues of the second fundamental form see [3]. Motivated in part by connections with the Ricci flow, much works have also been done on geometric properties of manifolds and hypersurfaces when the manifold is endowed with a “weighted” volume element; i.e. one integrates using the smooth measure $e^{-f}dv$ for $dv$ the Riemannian volume element of the metric $g$ see [4–7,10]. In this work, we follow Case [4] to introduce the notion of weighted $k$-mean curvature and using the weighted Newtons transformations introduced in [4], we obtain an integral formula on weighted manifolds and give some applications. This latter formula was first introduced by Kusner in [9] and nowadays it’s called the flux formula, then it was extended to $k$-mean curvature in a nice paper by Alías, de Lira, and Malacarne see [3], where they studied the properties of certain geometrical configurations, more particularly they established a flux formula and gave examples of geometric applications. Our paper extends some properties obtained by the authors cited above for weighted manifolds.
2. Preliminaries

In this section we fix the notations and recall some definitions and properties of the weighted symmetric functions and the weighted Newton transformations: for more details see [4,10].

Given a complete \((n + 1)\)-dimensional Riemannian manifold \((\mathcal{M}, \langle \cdot, \cdot \rangle)\) and a smooth function \(f : \mathcal{M} \rightarrow \mathbb{R}\). The weighted manifold \(\mathcal{M}_f\) associate to \(\mathcal{M}\) is the triplet \((\mathcal{M}, \langle \cdot, \cdot \rangle, dv_f)\), where \(dv_f = e^{-f} dv\) and \(dv\) is the standard volume element of \(\mathcal{M}\).

Consider the tensional connection

\[
\tilde{\nabla}_XY = \nabla_XY + \langle X, Y \rangle V - \langle V, Y \rangle X
\]

where \(V = \langle \nabla f, \nu \rangle \nu\), and \(\nu\) is a vector field on \(\mathcal{M}\) and \(\nabla\) stand for the covariant derivative on \(\mathcal{M}\). This connection is one of the three basic types of metric connections introduced by Elie Cartan. It was studied by I. Agricola and M. Kraus [2].

If \(B\) and \(\tilde{B}\) are bilinear forms on \(\mathcal{M}\) defined by \(B(X,Y) = \langle \nabla_XY, \nu \rangle\) and \(\tilde{B}(X,Y) = \langle \tilde{\nabla}_XY, \nu \rangle\), defined for \(X,Y\) orthogonal to \(\nu\), we have

\[
\tilde{B}(X,Y) = B(X,Y) + \langle X, Y \rangle \langle \nabla f, \nu \rangle
\]

An example of this situation is as follows: let \(\psi : M \rightarrow \mathcal{M}_f\) be an immersion of an \(n\)-dimensional manifold \(M\) into an \((n + 1)\)-weighted manifold \(\mathcal{M}_f\), and let \(\nu\) a unit normal vector field along \(\psi\) and \(B\) the second fundamental form of \(\psi\) with relation to \(\nu\), and \(\tilde{B}\) is the bilinear form defined as above, so the trace of \(\tilde{B}\) and the mean curvature of \(\psi\) are related by:

\[
n\tilde{H} = nH + \langle \nabla f, \nu \rangle
\]

which is the classical weighted mean curvature.

The important point is that hypersurfaces of constant weighted curvature appear as critical points of certain weighted volume functionals. The fundamental analogy with constant mean curvature hypersurfaces draws the attention in this area. In terms of matrices we have

\[
\tilde{B} = B + \langle \nabla f, \nu \rangle I
\]

so if \(\tilde{\tau}_i\) and \(\tau_i\) are the eigenvalues of \(\tilde{B}\) and \(B\) respectively, we get

\[
\tilde{\tau}_i = \tau_i + \langle \nabla f, \nu \rangle.
\]

Now, putting \(\lambda = \langle \nabla f, \nu \rangle\), we obtain [1]

\[
\tilde{\sigma}_k = \sigma_k(\tilde{B}) = \sum_{j=0}^{k} \binom{n-k+j}{j} \lambda^j \sigma_{k-j}(B).
\]  

(2.1)

where \(\tilde{\sigma}_k\) stand for the symmetric functions of \(\tilde{B}\).
Let $\overline{M}_f$ be an $(n+1)$-dimensional weighted Riemannian manifold, and $\psi : M \rightarrow \overline{M}_f$ be an isometrically immersed hypersurface with $\nabla$ and $\overline{\nabla}$ the Levi-Civita connections on $M$ and $\overline{M}_f$ respectively. The Weingarten formula of this immersion is written as follows

$$AX = -(\nabla_X \nu)^\top$$

where $A$ is the shape operator of the hypersurface $M$ with respect to the Gauss map $\nu$, and $\top$ denotes the orthogonal projection on the tangent vector bundle of $M$. As it is well known $A$ is a linear self adjoint operator and at each point $p \in M$, its eigenvalues $\mu_1, ..., \mu_n$ are the principal curvatures of $M$.

Associate to the shape operator $A$, the weighted elementary symmetric functions $\sigma_k^\infty : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ are defined recursively (and introduced by Case in [4]) by

$$\begin{cases}
\sigma_0^\infty(u, \mu) = 1, \\
k\sigma_k^\infty(u, \mu) = u\sigma_{k-1}^\infty(u, \mu) + \sum_{j=0}^{k-1} \sum_{i=1}^{n} (-1)^j \sigma_{k-1-j}^\infty(u, \mu) \mu_i^{j+1} & \text{for } k \geq 1
\end{cases}$$

(2.2)

where $u \in \mathbb{R}$ and $\mu = (\mu_1, ..., \mu_n) \in \mathbb{R}^n$. The weighted elementary symmetric functions $\sigma_k^\infty(u, A)$ of $A$ are defined by

$$\sigma_k^\infty(u, A) = \sigma_k^\infty(u, \mu)$$

where $\mu = (\mu_1, ..., \mu_n)$ and $\mu_1, ..., \mu_n$ are the eigenvalues of $A$.

In particular for $u = 0$ we recover $\sigma_k^\infty(0, A) = \sigma_k(A) = S_k$ the classical elementary symmetric functions defined in.

**Definition 1.** [4] The weighted Newton transformations $T_k^\infty(\mu_0, A)$ are defined inductively from $A$ by :

$$\begin{cases}
T_0^\infty(u, A) = I \\
T_k^\infty(u, A) = \sigma_k^\infty(u, A)I - AT_{k-1}^\infty(u, A) & \text{for } k \geq 1
\end{cases}$$

or equivalently

$$T_k^\infty(u, A) = \sum_{j=0}^{k} (-1)^j \sigma_{k-j}^\infty(u, A)A^j$$

where $I$ stands for the identity on the Lie algebra of vector fields $\mathfrak{X}(M)$. We also write $T_k^\infty(u, \mu) = T_k^\infty(u, A)$ for $\mu = (\mu_1, ..., \mu_n)$, where $\mu_1, ..., \mu_n$ are the eigenvalues of $A$.

It should be noted that $T_k^\infty(0, A) = T_k(A) = T_k$ is the classical Newton transformations introduced in.

These functions enjoy the nice following properties.
Proposition 1. [4] For \( u, v \in \mathbb{R} \) and \( \mu \in \mathbb{R}^n \), we have

\[
\sigma_k^\infty(v + u, \mu) = \sum_{j=0}^{k} \frac{w^j}{j!} \sigma_{k-j}^\infty(v, \mu).
\]

In particular,

\[
\sigma_k^\infty(u, \mu) = \sum_{j=0}^{k} \frac{w^j}{j!} \sigma_{k-j}(\mu) \tag{2.3}
\]

\[
\text{trace}(AT_k^\infty(u, \mu)) = (k + 1)\sigma_{k+1}^\infty(u, \mu) - u\sigma_k^\infty(u, \mu). \tag{2.4}
\]

For \( i \in \{1, \ldots, n\} \) we define

\[
\sigma_{k,i}^\infty(u, \mu) = \sigma_k^\infty(u, \mu) - \mu_i \sigma_{k-1,i}^\infty(u, \mu)
\]

then the \( i \)th eigenvalue of \( T_k^\infty(\mu_0, \mu) \) is equal to \( \sigma_{k,i}^\infty(\mu_0, \mu) \) and

\[
\sigma_{k,i}^\infty(u, \mu) = \sigma_k^\infty(u, \widehat{\mu}_i)
\]

where \( \widehat{\mu}_i = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n) \).

Remark 1. We can see by (2.1) and (2.3) that \( \widetilde{\sigma}_k \) and \( \sigma_k^\infty(u, \mu) \) are polynomials of the same degree but with slightly different coefficients, which differ by a multiplicative constant only.

Definition 2. Let \( M \) be an immersed hypersurface of \( \overline{M}_f \) with shape operator \( A \) with relation to an unit normal vector field \( \nu \). The weighted \( k \)-curvature \( S_{k,f} \) of the hypersurface \( M \) is defined by

\[
S_{k,f} = \sigma_k^\infty(u, A) \quad \text{for} \quad u = \langle \nabla f, \nu \rangle
\]

We also define the weighted \( k \)-mean curvature \( H_{k,f} \) by:

\[
\binom{n}{k} H_{k,f} = S_{k,f}
\]

The weighted Newton transformations will be denoted by

\[
T_{k,f} = T_k^\infty(u, A) \quad \text{for} \quad u = \langle \nabla f, \nu \rangle
\]

and we have

\[
T_{k,f} = S_{k,f}I - AT_{k-1,f} \quad \text{for} \quad k \geq 1
\]

Remark 2. We note that in view of formula 2.3 we can write \( S_{k,f} \) as

\[
S_{k,f} = \sigma_k^\infty(u, A) = \sum_{j=0}^{k} \frac{1}{j!} \sigma_{k-j}^\infty(0, A) = \sum_{j=0}^{k} \frac{1}{j!} u^j S_{k-j} \quad \text{for} \quad u = \langle \nabla f, \nu \rangle
\]

where \( S_r \) are the classical \( r \)-mean curvatures of the hypersurface \( M \).

In particular for \( k = 1 \), we get

\[
nH_{1,f} = S_{1,f} = S_1 + u = nH + \langle \nabla f, \nu \rangle \tag{2.5}
\]
which is the classical definition of the weighted mean curvature of the hypersurface \( M \) studied by Gromov [8].

Also, for \( k = 2 \) we have

\[
\binom{n}{2} H_{2,f} = S_{2,f} = S_2 + uS_1 + \frac{1}{2}u^2 = S_2 + ⟨\nabla f, ν⟩S_1 + \frac{1}{2} (⟨\nabla f, ν⟩)^2 \tag{2.5a}
\]

\[
= n(n - 1)H_2 + nH (⟨\nabla f, ν⟩ + \frac{1}{2} (⟨\nabla f, ν⟩)^2
\]

where \( S_2 = n(n - 1)H_2 \) and \( S_1 = nH \) are the (classical) scalar and the mean curvature of the hypersurface \( M \), respectively.

To clarify this notion of curvature we will study the case \( k = 2 \).

Consider a one parameter family \( ψ_t : M \to \overline{M}_f(c) \) of immersions of an \( n \)-dimensional closed manifold \( M \) into an \((n + 1)\)-dimensional space form \((\overline{M}_f(c), ⟨,⟩)\) of constant curvature \( c \). Denote by \( X \) the deformation vector field and by \( ν \) a unit vector field normal to \( M \) in \( \overline{M}_f(c) \). Put \( λ = ⟨X, ν⟩ \) and \( u = ⟨\nabla f, ν⟩ \).

Consider the variational problem

\[
δ \left( \int_M S_{1,f} dv_f \right) = 0 \tag{2.6}
\]

that is to say

\[
δ \left( \int_M S_{1,f} dv_f \right) = \frac{d}{dt} \left( \int_M S_{1,f} dv_f \right) = \frac{d}{dt} \left( \int_M (S_1 + u) dv_f \right) = \int_M \left( \frac{dS_1}{dt} + \frac{du}{dt} \right) dv_f + \int_M S_{1,f} \frac{d}{dt} (dv_f) \tag{2.7}
\]

Now, by formula (9c) in, we have

\[
\frac{dS_1}{dt} = λ \left( S_1^2 - 2S_2 \right) + Δλ + ⟨\nabla S_1, X^T⟩ + ncλ
\]

where \( Δ \) is the Laplacian of \( M \) in the induced metric. Also, by the well known fact

\[
\frac{d}{dt} (dv) = (-λS_1 + \text{div}(X^T)) dv
\]

where \( \text{div} \) is the divergence of \( M \) in the induced metric. Also, by the definition of the weighted divergence \( \text{div}_f(X^T) = \text{div}(X^T) - ⟨\nabla f, X^T⟩ \), where \( \nabla f = (\nabla f)^T \) we infer

\[
\frac{d}{dt} (dv_f) = (-λS_{1,f} + \text{div}_f(X^T)) dv_f
\]
and
\[
\frac{du}{dt} = \langle \nabla_X \nabla f, \nu \rangle + \langle \nabla f, \nabla_X \nu \rangle
\]
\[
= \nabla^2 f(X, \nu) + \langle \nabla f, -\nabla \lambda - AX^\top \rangle
\]
\[
= \nabla^2 f(X^\top, \nu) + \lambda \nabla^2 f(\nu, \nu) - \langle \nabla f, \nabla \lambda \rangle - \langle \nabla f, AX^\top \rangle
\]

Now, by using that
\[
\langle \nabla u, V \rangle = \nabla^2 f(V, \nu) - \langle A \nabla f, V \rangle
\]
for every vector field \( V \) tangent to \( M \), we get
\[
\frac{du}{dt} = \langle \nabla u, X^\top \rangle - \langle \nabla f, \nabla \lambda \rangle + \lambda \nabla^2 f(\nu, \nu)
\]

Replacing in (2.7), we get
\[
\frac{d}{dt} \left( \int_M S_{1,f} \, dv_f \right) \bigg|_{t=0}
\]
\[
= \int_M \left( \lambda \left( S_{1,f}^2 - 2S_2 \right) + \Delta \lambda + \langle \nabla S_{1,f}, X^\top \rangle \right) + nc\lambda + \langle \nabla u, X^\top \rangle
\]
\[
- \langle \nabla f, \nabla \lambda \rangle + \lambda \nabla^2 f(\nu, \nu) \right) \, dv_f
\]
\[
+ \int_M S_{1,f} (-\lambda S_{1,f} + \text{div}_f(X^\top)) \, dv_f
\]
\[
= \int_M \left( (S_{1,f}^2 - (S_{1,f})^2 - 2S_2) \lambda + \nabla \lambda + \langle \nabla S_{1,f}, X^\top \rangle \right) + nc\lambda + \lambda \nabla^2 f(\nu, \nu)
\]
\[
+ \text{div}_f(S_{1,f} X^\top) \right) \, dv_f
\]
\[
= \int_M \left( (S_{1,f}^2 - (S_{1,f})^2 - 2S_2) \lambda + nc\lambda + \lambda \nabla^2 f(\nu, \nu) \right) \, dv_f
\]

At least, as we already know that
\[
S_{2,f} = S_2 + uS_1 + \frac{1}{2} u^2
\]
we get
\[
S_{1,f}^2 - 2S_2 = S_1^2 - 2\{S_{2,f} - uS_1 - \frac{1}{2} u^2\}
\]
\[
= S_1^2 - 2S_2,f + 2uS_1 + u^2
\]
\[
= (S_1 + u)^2 - 2S_{2,f}
\]
\[
= (S_{1,f})^2 - 2S_{2,f}
\]

and hence
\[
\frac{d}{dt} \left( \int_M S_{1,f} \, dv_f \right) = \int_M \left( -2S_{2,f} + nc + \nabla^2 f(\nu, \nu) \right) \lambda \, dv_f
\]
Theorem 1. The Euler–Lagrange equation corresponding to the problem (2.6) is:
\[-2S_{2,f} + nc + \nabla^2 f(\nu, \nu) = 0\] (2.8)

In particular, if $\overline{M}$ is a Ricci soliton, which means that $\nabla^2 f = \kappa \langle , \rangle$ for some constant $\kappa$, then hypersurfaces $M$ which have constant weighted 2-curvature $S_{2,f}$ are critical points for the functional $\int_M S_{1,f}dv_f$, which is analogous to the classical case. Therefore, we can conclude that weighted 2-curvature as defined has geometrical relevance.

To clarify the idea we will consider simpler cases: Let us consider the weighted function $f(x) = \frac{1}{2} \|x\|^2$ in a Riemannian space form, and $u = \langle \nabla f, x \rangle = \langle x, \nu \rangle$ is the support function. We calculate in this particular case the third term of the formula (2.8)

\[\nabla^2 f(\nu, \nu) = \nabla_\nu \langle \nabla f, \nu \rangle\]
\[= \nabla_\nu \langle x, \nu \rangle\]
\[= \langle \nu, \nu \rangle = 1\]

so (2.8) writes
\[-2S_{2,f} + nc + 1 = 0.\]

Corollary 1. Under the above assumptions the Euler–Lagrange equation of the problem (2.6) is given by
\[-2S_{2,f} + nc + 1 = 0.\]

Example: Consider an hypersurface $M^n$ of the Euclidean space $\mathbb{R}^{n+1}$. The Euler–Lagrange equation is then written
\[-2S_{2,f} + 1 = 0.\] (2.9)

Taking into account the relations (2.2) and (2.5a), we infer that
\[2S_{2,f} = (u + S_1)^2 - |A|^2\]
where $u = \langle x, \nu \rangle$. So Eq. (2.9) becomes
\[-(u + S_1)^2 + |A|^2 + 1 = 0.\] (2.10)

We can cite candidates to our situation: the round sphere $S^n(r)$ ($n \geq 1$) of radius $r$ centred at the origin and the hypercylinder $S^{n-k}(r) \times \mathbb{R}^k$. Indeed if $\nu$ is the inward unit normal vector field to $S^n(r)$ then $u = -r$ and $S_1 = \frac{n}{r}$ and $|A|^2 = \frac{n^2}{r^2}$. The Eq. (2.10) writes
\[r^4 - (2n + 1)r^2 + n(n - 1) = 0\]
which admits the following solutions

\[ r = \sqrt{\frac{2n + 1 + \sqrt{8n + 1}}{2}} \]

and

\[ r = \sqrt{\frac{2n + 1 - \sqrt{8n + 1}}{2}}, \text{ with } n \geq 2. \]

Similarly for \( S^{n-k}(r) \times R^k \) (1 \( \leq k < n \)) we have: \( u = -r \) and \( S_1 = \frac{n-k}{r} \). For these latter values the Eq. (2.10) admits solutions given by:

\[ r = \sqrt{\frac{2(n-k) + 1 + \sqrt{8(n-k) + 1}}{2}} \]

and

\[ r = \sqrt{\frac{2(n-k) + 1 - \sqrt{8(n-k) + 1}}{2}}, \text{ with } n \geq 2 + k. \]

**Definition 3.** We say that an hypersurface \( M \) of \( M_f \) is \( \sigma_r^\infty \)-minimal, if \( H_{r,f} = 0 \). In particular \( M \) is \( f \)-minimal if \( H_f = -\frac{1}{n} \langle \nabla f, \nu \rangle \).

Now we want to compute the weighted divergence of the weighted Newton transformations \( T_{k,f} \), so we will work with \( u = \langle \nabla f, \nu \rangle \).

The weighted divergence of the weighted Newton transformations is defined by

\[ \operatorname{div}_f T_{k,f} = e^f \operatorname{div} (e^{-f} T_{k,f}) \]

where

\[ \operatorname{div} (T_{k,f}) = \text{trace} (\nabla T_{k,f}) = \sum_{i=1}^{n} (\nabla_{e_i} T_{k,f}) (e_i) \]

and \( \{e_1, \ldots, e_n\} \) is a local orthonormal frame on \( M \).

**Lemma 1.** Let \( V \) be any tangent vector field on \( M \). Then

\[ (T_{k-1,f} \circ \nabla V A) = V (S_{k,f}) - V (u) S_{k-1,f} \]

**Proof.** The computations will be in a basis \( \{e_1, \ldots, e_n\} \) that diagonalizes \( A \) in a point \( p \) of \( M \), that is \( Ae_i = \lambda_i e_i \) where \( \lambda_1, \ldots, \lambda_n \) are the principal curvatures of \( M \) in \( p \).
Since the eigenvalues of $T_{k-1, f} = T_{k-1}^\infty (u, A)$ are given by
\[
t_i = \sigma_{k-1}^\infty (u, \lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_n)
= \sum_{j=0}^{k-1} \frac{u^j}{j!} \sigma_{k-1-j} (\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_n)
= \sum_{j=0}^{k-1} \frac{u^j}{j!} \sigma_{k-1-j} (\lambda_1, \ldots, \lambda_n)
= \sum_{j=0}^{k-1} \frac{u^j}{j!} \sum_{i \neq i_1} \cdots \sum_{i \neq i_j} \lambda_{i_1} \cdots \lambda_{i_{k-1-j}}.
\]
So
\[
\text{trace} (T_{k-1, f} \circ \nabla_V A) = \sum_{i=1}^n t_i V(\lambda_i)
= \sum_{j=0}^{k-1} \frac{u^j}{j!} \left( \sum_{i=1}^n \frac{\partial}{\partial \lambda_i} \sigma_{k-1-j} (\lambda_1, \ldots, \lambda_n) V(\lambda_i) \right)
= \sum_{j=0}^{k-1} \frac{u^j}{j!} V(S_{k-j})
= V \left( \sum_{j=0}^k \frac{u^j}{j!} S_{k-j} \right) - V \left( \frac{u^k}{k!} \right) - \sum_{j=1}^{k-1} V \left( \frac{u^j}{j!} \right) S_{k-j}
= V \left( \sum_{j=0}^k \frac{u^j}{j!} S_{k-j} \right) - V(u) \sum_{l=0}^{k-1} \frac{u^l}{l!} S_{k-1-l}
= V(S_{k,f}) - V(u)S_{k-1,f}
\]

\[\square\]

Lemma 2. The weighted divergence of the weighted Newton transformations $T_{k, f}$ are given by the following formulas: for $k = 0$
\[
\text{div}_f T_{0, f} = -\nabla f
\]
and for $k \geq 1$
\[
\text{div}_f T_{k, f} = -S_{k,f} \nabla f + S_{k-1,f} \nabla u - A \circ \text{div}_f T_{k-1, f} - \sum_{i=1}^n (R(\nu, T_{k-1, f}(e_i))e_i)^T
\]
Proof. We have
\[ \text{div}_f T_{k,f} = e^f \text{div}(e^{-f} T_{k,f}) \]
\[ = e^f \sum_{i=1}^{n} \left[ \nabla_{e_i}(e^{-f} T_{k,f})(e_i) \right] \]
\[ = e^f \sum_{i=1}^{n} \left[ \nabla_{e_i}(e^{-f} T_{k,f}(e_i)) - e^{-f} T_{k,f}(\nabla_{e_i} e_i) \right] \]
\[ = e^f \sum_{i=1}^{n} \left[ e^{-f} \nabla_{e_i}(T_{k,f}(e_i)) + e_i(e^{-f}) T_{k,f}(e_i) - e^{-f} T_{k,f}(\nabla_{e_i} e_i) \right] \]
\[ = e^f \sum_{i=1}^{n} \left[ e^{-f} \nabla_{e_i}(T_{k,f}(e_i)) - e^{-f} \langle \nabla f, e_i \rangle T_{k,f}(e_i) - e^{-f} T_{k,f}(\nabla_{e_i} e_i) \right] \]
\[ = \text{div} T_{k,f} - T_{k,f}(\nabla f) \]

It is not difficult to see that
\[ \text{div} T_{k,f} = \nabla S_{k,f} - A \circ \text{div} T_{k-1,f} - \sum_{i=1}^{n} (\nabla_{e_i} A)(T_{k-1,f}(e_i)). \]

Indeed,
\[ (\nabla_{e_i} T_{k,f})(e_i) = \nabla_{e_i}(T_{k,f}(e_i)) - T_{k,f}(\nabla_{e_i} e_i) \]
\[ = \nabla_{e_i}((S_{k,f} I - AT_{k-1,f}) e_i) - (S_{k,f} I - AT_{k-1,f})(\nabla_{e_i} e_i) \]
\[ = \nabla_{e_i}(S_{k,f} e_i) - \nabla_{e_i}(AT_{k-1,f}(e_i)) - S_{k,f}(\nabla_{e_i} e_i) \]
\[ + (AT_{k-1,f})(\nabla_{e_i} e_i) \]
\[ = e_i(S_{k,f}) e_i + S_{k,f}(\nabla_{e_i} e_i) - \nabla_{e_i}(AT_{k-1,f}(e_i)) - S_{k,f}(\nabla_{e_i} e_i) \]
\[ + (AT_{k-1,f})(\nabla_{e_i} e_i) \]
\[ = e_i(S_{k,f}) e_i - (\nabla_{e_i}(AT_{k-1,f}))(e_i) \]
\[ = \langle \nabla S_{k,f}, e_i \rangle e_i - (\nabla_{e_i} A)(T_{k-1,f}(e_i)) - A((\nabla_{e_i} T_{k-1,f})(e_i)) \]

Thus,
\[ \text{div} T_{k,f} = \sum_{i=1}^{n} (\nabla_{e_i} T_{k,f})(e_i) \]
\[ = \sum_{i=1}^{n} \langle \nabla S_{k,f}, e_i \rangle e_i - \sum_{i=1}^{n} (\nabla_{e_i} A)(T_{k-1,f}(e_i)) - \sum_{i=1}^{n} A((\nabla_{e_i} T_{k-1,f})(e_i)) \]
\[ = \nabla S_{k,f} - A \circ \text{div} T_{k-1,f} - \sum_{i=1}^{n} (\nabla_{e_i} A)(T_{k-1,f}(e_i)) \]

The Godazzi equation and the fact that $\nabla_{e_i} A$ is a self-adjoint operator allow us to write,
\[ \sum_{i=1}^{n} \langle (\nabla_{e_i} A) T_{k-1,f}(e_i), V \rangle = \sum_{i=1}^{n} \langle (\nabla V, T_{k-1,f}(e_i)) e_i \rangle^\top, V \]
\[ + \text{trace}(T_{k-1,f} \circ \nabla V A) \]
where $V$ is an arbitrary vector tangent to $M$.

Thus,
\[
\langle T_{k,f}, V \rangle = \langle \nabla S_{k,f}, V \rangle - \langle A \text{ div } T_{k-1,f}, V \rangle - \sum_{i=1}^{n} \langle (\mathcal{R}(\nu, T_{k-1,f}(e_i)) e_i)^\top, V \rangle \\
- \text{ trace}(T_{k-1,f} \circ \nabla V A)
\]

Using now Lemma 1
\[
\text{trace}(T_{k-1,f} \circ \nabla V A) = \langle \nabla S_{k,f}, V \rangle - \langle \nabla u, V \rangle S_{k-1,f}
\]
we have,
\[
\langle T_{k,f}, V \rangle = S_{k-1,f} \langle \nabla u, V \rangle - \langle A \text{ div } T_{k-1,f}, V \rangle - \sum_{i=1}^{n} \langle (\mathcal{R}(\nu, T_{k-1,f}(e_i)) e_i)^\top, V \rangle
\]
or equivalently,
\[
T_{k,f} = S_{k-1,f} \nabla u - A \circ \text{ div } T_{k-1,f} - \sum_{i=1}^{n} (\mathcal{R}(\nu, T_{k-1,f}(e_i)) e_i)^\top.
\]

Finally
\[
\text{div}_f T_{k,f} = -T_{k,f}(\nabla f) + S_{k-1,f} \nabla u \\
- A \circ T_{k-1,f} - \sum_{i=1}^{n} (\mathcal{R}(\nu, T_{k-1,f}(e_i)) e_i)^\top
\]
\[
= - (S_{k,f} I - A T_{k-1,f})(\nabla f) + S_{k-1,f} \nabla u \\
- A \circ (\text{div}_f T_{k-1,f} + T_{k-1,f}(\nabla f)) \\
- \sum_{i=1}^{n} (\mathcal{R}(\nu, T_{k-1,f}(e_i)) e_i)^\top
\]
\[
= -S_{k,f} \nabla f + S_{k-1,f} \nabla u - A \circ \text{div}_f T_{k-1,f} \\
- \sum_{i=1}^{n} (\mathcal{R}(\nu, T_{k-1,f}(e_i)) e_i)^\top
\]

Which achieves the proof of Lemma 2. \hfill \Box

**Corollary 2.** If $M_f^{n+1}$ has constant sectional curvature, then
\[
\text{div}_f T_{k,f} = -T_{k,f}(\nabla f) + T_{k-1,f}(\nabla u)
\]

**Proof.** If $M_f^{n+1}$ has constant sectional curvature, then $(\mathcal{R}(\nu, T_{k-1,f}(e_i)) e_i)^\top = 0$, and we have:
\[
\text{div}_f T_{k,f} = -S_{k,f} \nabla f - A \text{ div}_f T_{k-1,f} + S_{k-1,f} \nabla u.
\]
The desired relation results by a recursive argument. \hfill \Box

**Corollary 3.** Under the assumption of corollary 2, we have
\[
\text{div}_f T_{k,f} = -T_{k,f}(\nabla f) + T_{k-1,f}(\nabla \nu \nabla f - A \nabla f).
\]
Proof. Indeed, let $X$ be any arbitrary tangent vector to $M$, then

$$\nabla_X u = \langle \nabla_X \nabla f, \nu \rangle + \langle \nabla f, \nabla_X \nu \rangle$$

$$= \langle \nabla_{\nu} \nabla f, X \rangle + \langle \nabla f, \nabla_X \nu \rangle$$

$$= \langle \nabla_{\nu} \nabla f, X \rangle - A(\nabla f, X).$$

Hence

$$\nabla u = \nabla_{\nu} \nabla f - A \nabla f.$$

\[\square\]

3. Main results

The aim of this part is to derive an integral formula on weighted manifolds with constant sectional curvature and to give some of its geometric applications. The method is based on the computation of the weighted divergence $\text{div}_f (T_{k,f}Y^\top)$ and $\langle \text{div}_f T_{k,f}, Y \rangle$, where $Y$ is a conformal vector field. To do so, let $\overline{M}_f^{n+1}$ be an oriented weighted Riemannian manifold with metric $\langle \cdot, \cdot \rangle$ and $\varphi : M^n \rightarrow \overline{M}_f^{n+1}$ be a compact oriented hypersurface of boundary $\partial M$.

Let $p \in \partial M^n$ and $\{e_1, \ldots, e_{n-1}\}$ an orthonormal basis of $T_p \partial M^n$. We can choose a unit vector field $\eta$ along $\partial M$ such that $\{e_1, \ldots, e_{n-1}, \eta(p)\}$ is an orthonormal basis of $T_p M$. Let $\nu$ be a unit vector field normal to $M$ in $\overline{M}_f$, then $\{e_1, \ldots, e_{n-1}, \eta(p), \nu\}$ is an orthonormal basis of $T_p \overline{M}_f$.

Suppose now the existence of a closed conformal vector field $Y$ on $\overline{M}_f$; that is to say there exists a $\varphi \in C^\infty(\overline{M}_f)$ such that

$$\nabla_V Y = \varphi V$$

for every vector field $V$ over $\overline{M}_f$.

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_p M$ that diagonalizes $A$, then

$$\langle \text{div}_f T_{k,f}, Y \rangle = \langle e_f \text{div} (e_f^{-1}T_{k,f}), Y \rangle$$

$$= e_f \text{div} (e_f^{-1}T_{k,f}Y^\top) - \sum_{i=1}^{n} \langle T_{k,f}(e_i), \nabla e_i, Y \rangle$$

$$= e_f \text{div} (e_f^{-1}T_{k,f}Y^\top) - \sum_{i=1}^{n} \phi \langle T_{k,f}(e_i), e_i \rangle$$

$$= \text{div}_f (T_{k,f}Y^\top) - \phi \text{trace}T_{k,f}.$$
And in virtue of formula 2.4 we have
\begin{align*}
\text{trace} T_{k,f} &= nS_{k,f} - \text{trace} (AT_{k-1,f}) \\
&= (n - k) S_{k,f} + uS_{k-1,f} \\
&= (n - k) \binom{n}{k} H_{k,f} + \langle \nabla f, \nu \rangle \binom{n}{k-1} H_{k-1,f}
\end{align*}

So,
\begin{align*}
\text{div}_f (T_{k,f} Y^\top) &= \langle \text{div}_f T_{k,f}, Y \rangle + \phi \left( c_k H_{k,f} + c_{k-1} \langle \nabla f, \nu \rangle H_{k-1,f} \right)
\end{align*}

where \( c_k = (n - k) \binom{n}{k} \) and \( c_{k-1} = n \binom{n}{k-1} \)

Integrating the two sides of this latter equality and applying the divergence theorem, we obtain for \( 1 \leq k \leq n - 1 \),
\begin{align*}
\int_M \text{div}_f (T_{k,f} Y^\top) \, dv_f &= \int_M e^{-f} \text{div}_f (T_{k,f} Y^\top) \, dv \\
&= \int_M \text{div} (e^{-f} T_{k,f} Y^\top) \, dv \\
&= \int_{\partial M} e^{-f} \langle T_{k,f} Y^\top, \eta \rangle \, ds \\
&= \int_{\partial M} \langle T_{k,f} \eta, Y \rangle \, ds_f
\end{align*}

Hence,
\begin{align*}
\int_{\partial M} \langle T_{k,f} \eta, Y \rangle \, ds_f &= \int_M \langle \text{div}_f T_{k,f}, Y \rangle \, dv_f + c_k \int_M \phi H_{k,f} \, dv_f \\
&\quad + c_{k-1} \int_M \phi \langle \nabla f, \nu \rangle H_{k-1,f} \, dv_f
\end{align*}

Consequently, we have the following proposition

**Proposition 2.** Let \( \varphi : M^n \to \overline{M}_f^{n+1} \) an immersed compact oriented hypersurface of boundary \( \partial M \). Denoting by \( \nu \) an unit vector field normal to \( M \) in \( \overline{M}_f \), and \( \eta \) the outward pointing unit conormal vector field to \( M \) along \( \partial M \). Then for \( 1 \leq k \leq n - 1 \) and for every closed conformal vector field \( Y \) on \( \overline{M}_f^{n+1} \), we have :
\begin{align*}
\int_{\partial M} \langle T_{k,f} \eta, Y \rangle \, ds_f &= \int_M \langle \text{div}_f T_{k,f}, Y \rangle \, dv_f + c_k \int_M \phi H_{k,f} \, dv_f \\
&\quad + c_{k-1} \int_M \phi \langle \nabla f, \nu \rangle H_{k-1,f} \, dv_f.
\end{align*}
If $\overline{M}^{n+1}_f$ has constant sectional curvature, we obtain by Corollary (2),

**Corollary 4.** If $\overline{M}^{n+1}_f$ has constant sectional curvature, then

$$\int_{\partial M} \langle T_k f\eta, Y \rangle \, ds_f = -\int_M \langle T_k f \nabla f, Y \rangle \, dv_f + \int_M \langle T_{k-1} f \nabla u, Y \rangle \, dv_f + c_k \int_M \phi H_{k-1} f \, dv_f$$

with $u = \langle \nabla f, \nu \rangle$.

**Corollary 5.** If $\overline{M}^{n+1}_f$ has constant sectional curvature and $f$ is constant then

$$\int_{\partial M} \langle T_k \eta, Y \rangle \, ds = c_k \int_M \phi H_k \, dv.$$  \hspace{1cm} (3.2)

where $T_k$ is the classical Newton transformation.

If $f$ is constant, $H_k$ is a non zero constant and $Y$ is an homothetic vector field, we can assume that $\phi = 1$, and we get

$$c_k H_k \text{vol}(M) = \int_{\partial M} \langle T_k \eta, Y \rangle \, ds$$

or equivalently

$$\text{vol}(M) = \frac{1}{c_k H_k} \int_{\partial M} \langle T_k \eta, Y \rangle \, ds.$$  \hspace{1cm} (3.3)

**Proposition 3.** Let $\varphi : M^n \longrightarrow \overline{M}^{n+1}_f$ be an immersed compact oriented hypersurface of boundary $\partial M$. Denoting by $\nu$ an unit vector field normal to $M$ in $\overline{M}_f$, and $\eta$ the outward pointing unit conormal vector field tangent to $M$ along $\partial M$. Suppose that $\overline{M}_f$ is of constant sectional curvature, $f$ is constant and the classical $k-$mean curvature $H_k$ is a non zero constant for some $1 \leq k \leq n-1$. Then, for every homothetic vector field $Y$ on $\overline{M}_f$, we have

$$\text{vol}(M) = \frac{1}{c_k H_k} \int_{\partial M} \langle T_k \eta, Y \rangle \, ds.$$  \hspace{1cm} (3.3)

**Remark 3.** An estimate of the integrand in formula 3.3 leads to an estimate of the volume of $M$ by the area of its boundary $\partial M$.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
References

[1] Abdelmalek, M., Benalili, M., Niedzialomski, K.: Geometric Configuration of Riemannian submanifolds of arbitrary codimension. J. Geom. 108, 803–823 (2017)

[2] Agricola, I., Kraus, M.: Manifolds with vectorial torsion. Differ. Geom. Appl. 45, 130–147 (2016)

[3] Alías, L.J., de Lira, J.H.S., Malacarne, J.M.: Constant higher-order mean curvature hypersurfaces in Riemannian spaces. J. Inst. Math. Jussieu 5(4), 527–562 (2006)

[4] Case, J.S.: A notion of the weighted for manifolds with $\sigma_k$-curvature density. Adv. Math. 295, 150–194 (2016)

[5] Corwin, I.: Differential geometry of manifolds with density. Rose-Hulman Undergrad. Math. J. 7(1), 2 (2006)

[6] Castro, K., Rosales, C.: Free boundary stable hypersurfaces in manifolds with density and rigidity results. J. Geom. Phys. 79, 14–28 (2014)

[7] Espinar, J.M., Espinar, J.M.: Gradient Schrodinger operators, manifolds with density and applications. J. Math. Anal. Appl. 455(2), 1505–1528 (2017)

[8] Gromov, M.: Isoperimetry of waists and concentration of maps. Geom. Funct. Anal. 13, 178–215 (2003)

[9] Kusner, R.: Global geometry of extremal surfaces in three-space. Doctoral Thesis, University of California (1985)

[10] Morgan, F.: Manifolds with density. Not. Am. Math. Soc. 52(8), 853–858 (2005)

[11] Wei, G., Wylie, W.: Comparison geometry for the Bakry–Emery–Ricci tensor. J. Differ. Geom. 83, 377–405 (2009)

Mohammed Abdelmalek
Eole Supérieure de Management de Tlemcen
Tlemcen
Algeria
e-mail: abdelmalekmhd@gmail.com

Mohammed Benalili
Abou Bekr Belkaid University
Tlemcen
Algeria
e-mail: m_benalili@yahoo.fr

Received: April 20, 2020.
Revised: July 5, 2021.
Accepted: July 5, 2021.