Investigation of dynamical systems using tools of the theory of invariants and projective geometry

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Abstract

The investigation of nonlinear dynamical systems of the type

\[ \dot{x} = P(x, y, z), \dot{y} = Q(x, y, z), \dot{z} = R(x, y, z) \]

by means of reduction to some ordinary differential equations of the second order in the form

\[ y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0 \]

is done. The main backbone of this investigation was provided by the theory of invariants developed by S. Lie, R. Liouville and A. Tresse at the end of the 19th century and the projective geometry of E. Cartan. In our work two, in some sense supplementary, systems are considered. The first one is the Lorenz system

\[ \dot{x} = \sigma(y - x), \dot{y} = rx - y - zx, \dot{z} = xy - bz \]

where \(\sigma, r, b\) are parameters and the second one is the Rössler system

\[ \dot{x} = -y - z, \dot{y} = x + ay, \dot{z} = b + xz - cz \]

where \(a, b\) and \(c\) are parameters. The invarinats for the ordinary differential equations, which correspond to the systems mentioned above, are evaluated. The connection of values of the invariants with characteristics of dynamical systems is established.

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1NTZ-Preprint 24/95, Leipzig, 1995, to appear in J. of Applied Mathematics (ZAMP)
1 Introduction

The first broad investigation of invariants of the second order differential equation
\[ y'' = f(x, y, y') \] (1)
under some general point transformations
\[ x = \xi(u, v), \]
\[ y = \eta(u, v), \] (2)
where \( \xi \) and \( \eta \) are arbitrary smooth functions was done in the work of R. Liouville [1]. He found some series of absolute and semi-invariants and discovered one procedure to build other invariants of higher weights. A further consideration of the same problem from the point of view of infinitesimal transformations was introduced by S. Lie [2, 3] and completed by his student A. Tresse for second order differential equations (1) with arbitrary smooth functions \( f(x, y, y') \) in [4]. The geometrical description of these results was given by E. Cartan [6, 7], when he introduced the new idea of projective connections. It was important for the theory of invariants to investigate the invariants themselves and their possible connections among each other, because it enables us to say whether our equation (1) allows some infinitesimal group of point transformations or not [4]. In projective geometry it is important to know if our invariant is equal to zero or not, because this circumstance is responsible for the existence of corresponding geometrical properties.

We will use these ideas to investigate the nonlinear dynamical system of the type
\[ \dot{x} = P(x, y, z), \]
\[ \dot{y} = Q(x, y, z), \]
\[ \dot{z} = R(x, y, z). \] (3)
At first we conduct some “projections” of this system on, for instance, the \((x, y)\)-plane or the \((y, z)\)-plane or some nonplanar surface. After that we deal with some ordinary differential equations.

In all our investigated cases these equations have the form
\[ y'' + a_1(x, y)y^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0. \] (4)
It is well known that this form will be kepted under the point transformations and invariants of such equations can be easy investigated. Hereforth we can look for conditions for the parameters in (3) in which one or more invariants are equal to zero. We hope that exactly these conditions will be able to show us where integrable cases lie. It is well known that some typical Riemann surfaces cannot yield a satisfactory corresponding geometrical interpretation of the equation (1) in general, because metrical properties are noninvariant under the point transformations. The adequate geometrical image is only possible with the help of the notion of normal projective connection.
According to E. Cartan [4, 5] we will use the following notations. For each second order differential equation we will look at the corresponding three dimensional variety $M^3(x, y, y')$ of elements of first order with local coordinates $(x, y, y')$ and projective connections. The structure of this variety can be described through the 1-form components of projective connection

$$w^i, \ w^0_i, \ w^i_j - w^0_j, \ w^j_i, \ i \neq j, \ i, j = 1, 2, 3$$

and 2-forms of curvature and torsion

$$\Omega^i, \ \Omega^0_i, \ \Omega^0_i - \Omega^0_0, \ \Omega^i_j.$$

However, in the case of equations of the type (4) a torsion free variety with normal projective connection is produced and our variety $M^3(x, y, y')$ can be represented as a direct product

$$M^3(x, y, y') = V^2(x, y) \times S^1.$$

The 2-forms of curvature and torsion satisfy the following conditions

$$\Omega^1 = \Omega^2 = 0, \ \Omega^2_1 = \Omega^1_2 = 0, \ \Omega^1_1 - \Omega^0_1 = \Omega^2_2 - \Omega^0_2 = 0, \ \Omega^0_1 = b \omega^1 \wedge \omega^2, \ \Omega^0_2 = h \omega^1 \wedge \omega^2,$$

where

$$b = 2 \frac{\partial^2 a_2}{\partial x \partial y} - 3a_2 \frac{\partial^2 a_3}{\partial y^2} + 2a_1 \frac{\partial^2 a_4}{\partial x \partial y} + a_4 \frac{\partial a_4}{\partial y} - 3a_1 \frac{\partial a_3}{\partial x} - 3a_3 \frac{\partial a_1}{\partial x} - 3a_2 \frac{\partial a_3}{\partial y} + 6a_2 \frac{\partial a_2}{\partial x} + \left(2 \frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x^2} - \frac{\partial^2 a_3}{\partial y^2} - 2a_4 \frac{\partial a_4}{\partial x} - a_1 \frac{\partial a_4}{\partial x} + 3a_4 \frac{\partial a_4}{\partial y} + 3a_3 \frac{\partial a_2}{\partial x} - 6a_3 \frac{\partial a_3}{\partial y}\right) y',$$

$$h = \frac{\partial b}{\partial y}.$$

The components of the normal projective connections also have a very simple form

$$w^1 = dx, \ w^2 = dy, \ w^3 = a_4 dx + a_3 dy, \ w^4 = -a_2 dx - a_1 dy,$$

$$w^1_1 = -w^2_2 = -a_3 dx - a_2 dy, \ w^0_1 = \Pi^0_{11} dx + \Pi^0_{12} dy, \ w^0_2 = \Pi^0_{21} dx + \Pi^0_{22} dy,$$

with

$$\Pi^0_{11} = 2(a_2^3 - a_2 a_4) + a_3 x - a_4 y,$$

$$\Pi^0_{12} = 2(a_2^2 - a_1 a_3) + a_4 x - a_2 y,$$

$$\Pi^0_{21} = a_2 a_3 - a_1 a_4 + a_3 x - a_3 y.$$

The Cartan’s structure equations have the form

$$dw^i = w^k \wedge w^i_k,$$

$$dw^j = w^0_i \wedge w^j + w^k \wedge w^j_k - \delta^j_i w^k \wedge w^k_1,$$

$$dw^0_0 = w^k \wedge w^0_k - \frac{1}{2} R^0_{jkl} w^k \wedge w^j \wedge w^l, \ i, j, k, l = 1, 2.$$
The tensor of projective curvature has two components $R_{112}^0$ and $R_{212}^0$, that will be noted as $-L_1$ and $-L_2$ respectively:

\[
L_1 = -\frac{\partial \Pi_{11}^0}{\partial y} + \frac{\partial \Pi_{12}^0}{\partial x} - a_2 \Pi_{11}^0 - a_4 \Pi_{22}^0 + 2a_3 \Pi_{12}^0, \\
L_2 = -\frac{\partial \Pi_{12}^0}{\partial y} + \frac{\partial \Pi_{22}^0}{\partial x} - a_1 \Pi_{11}^0 - a_3 \Pi_{22}^0 + 2a_2 \Pi_{12}^0. \tag{7}
\]

Both these values were discovered by R. Liouville much earlier than by E. Cartan and used for the construction of the semi invariant $\nu_5$ with the weight equal to 5. This invariant is a semi invariant. This means that after some point transformation \( \tilde{\nu}_5 \) the new $\tilde{\nu}_5$ is equal to

\[ \tilde{\nu}_5 = \Delta^5 \nu_5, \]

where $\Delta \neq 0$ is the functional determinant of our transformation in opposite to an absolute invariant where we have $\tilde{\tau} = \tau$. The degree of this determinant is exactly the weight of the corresponding invariant. The value of $\nu_5$ can be calculated by means of the following formula:

\[ \nu_5 = L_2(L_1 L_{2x} - L_2 L_{1x}) + L_1(L_2 L_{1y} - L_1 L_{2y}) - a_1 L_1^3 + 3a_2 L_1^2 L_2 - 3a_3 L_1 L_2^2 + a_4 L_2^3. \tag{8} \]

If $\nu_5 \neq 0$ we can use the recursive formula of R. Liouville \( [1] \) to build the series of semi invariants with higher weights:

\[ \nu_{m+2} = L_1 \frac{\partial \nu_m}{\partial y} - L_2 \frac{\partial \nu_m}{\partial x} + m \nu_m (L_{2x} - L_{1y}), \quad m \geq 5, \tag{9} \]

and after that construct the absolute invariants

\[ t_m = \nu_m \nu_5^{-m/5}. \tag{10} \]

In the case $\nu_5 = 0$, R. Liouville has found other series of semi invariants. At first, he found some invariants of weight 1:

\[ w_1 = \frac{1}{L_2^4} \left[ L_2^3 (\alpha' L_2 - \alpha L_1) - R_2 (L_2^2 y) + L_2 R_{2y} - L_2 R_2 (a_1 L_1 - a_2 L_2) \right] \]

for $L_2 \neq 0$, and

\[ w_1 = \frac{1}{L_1^4} \left[ L_1^3 (\alpha' L_1 - \alpha'' L_2) + R_1 (L_1^2 x) - L_1^2 R_{1x} + L_1 R_1 (a_3 L_1 - a_4 L_2) \right] \tag{11} \]

for $L_1 \neq 0$, where

\[
\begin{align*}
R_1 &= L_1 L_{2x} - L_2 L_{1x} + a_2 L_1^2 - 2a_3 L_1 L_2 + a_4 L_2^2, \\
R_2 &= L_1 L_{2y} - L_2 L_{1y} + a_1 L_1^2 - 2a_2 L_1 L_2 + a_3 L_2^2, \\
\alpha &= a_2 y - a_1 x + 2(a_1 a_4 - a_2^2), \quad \alpha' = a_3 y - a_2 x + a_1 a_4 - a_2 a_3, \\
\alpha'' &= a_4 y - a_3 x + 2(a_2 a_4 - a_3^2). \tag{12}
\end{align*}
\]
Also in this case there exists one recursive formula to construct the series of semi invariants with higher weights

\[ w_{m+2} = L_1 \frac{\partial w_m}{\partial y} - L_2 \frac{\partial w_m}{\partial x} + mw_m(L_2x - L_1y) \]  

(13)

and the corresponding absolute invariants read

\[ u_{m+2} = \frac{w_{m+2}}{w_1}, \quad m \geq 1. \]  

(14)

The case that both projective curvatures \( L_1 \) and \( L_2 \) are equal to zero is non interesting, while the equation (4) can then be reduced to \( y'' = 0 \). If also \( w_1 = 0 \) there are other series of invariants. The first one has weight equal to 2 and is given by

\[ i_2 = \frac{3R_1}{L_1} + \frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y} \]  

(15)

and other semi invariants of this series by

\[ i_{2m+2} = L_1 \frac{\partial i_{2m}}{\partial y} - L_2 \frac{\partial i_{2m}}{\partial x} + 2mi_{2m} \left( \frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y} \right), \quad (m \geq 1). \]  

(16)

The corresponding absolute invariants can be found through the relation

\[ j_{2m} = i_{2m}i_2^{-m}. \]  

(17)

If for our equation (4) the semi invariant

\[ \nu_5 = 0 \]  

(18)

and \( L_1, L_2 \neq 0 \) that we can immediately construct the first integral of this equation

\[ y' = -\frac{L_1}{L_2}. \]  

(19)

For many physical systems like (3) those partial solutions are very interesting.

On the other hand side the condition (18) has also a deep geometrical sense. Namely, if and only if the condition (18) is fulfilled it will be possible to immerse our variety \( V^2(x, y) \) of the normal projective connection into the real projective space \( \mathbb{P}^3(\mathbb{R}) \). In all other cases the immersion will be possible only in \( \mathbb{P}^4(\mathbb{R}) \) [9].

The immersion into \( \mathbb{P}^3(\mathbb{R}) \) will be realized on one of the simplest surfaces - on developed surfaces. It is well known that the corresponding surfaces for the equation \( y'' = 0 \) is the projective plane. In this sense the equation (4) with the condition (18) are really the next simple ones. As we mentioned above, the condition that some invariant is equal to zero must have some geometrical sense, in the cases \( w_1 = 0 \) and \( i_2 = 0 \) not known to us.

The first investigation of dynamical systems by the described method was done in [8, 9]. In the present work we deal with different aspects of the Lorenz system as well as with the Rössler system.
2 The Lorenz system

In the work [15] Lorenz discussed the problem of the representation of a forced dissipative hydrodynamic flow and suggested the following model for the description of that system:

\[
\begin{align*}
\frac{dx}{dt} &= \sigma(y - x), \\
\frac{dy}{dt} &= rx - y - xz, \\
\frac{dz}{dt} &= xy - bz,
\end{align*}
\]  

(20)

where \(\sigma, r, b\) are some parameters characterizing the flow. Now, this system is well investigated by some analytical and numerical methods. It possesses several dynamical states corresponding to different regions in the parameter space. However, up to now no one analytical criterion is known which allows to decide whether the for a given set of parameters the solution will be a regular or a stochastic one.

The phase space of (20) is three dimensional and it is separated due to the integral curves of the following two equations

\[
\begin{align*}
\frac{dy}{dx} &= \frac{rx - y - xz}{\sigma(y - x)}, \\
\frac{dz}{dx} &= \frac{xy - bz}{\sigma(y - z)}.
\end{align*}
\]  

(21)

After eliminating one of the functions, \(z\), for instance, and introducing new notations we get an equivalent second order differential equation of the type (4)

\[
\frac{d^2y}{dx^2} - \frac{3}{y} \frac{dy}{dx} + \left(\alpha y - \frac{1}{x}\right) \frac{dy}{dx} + \epsilon y^2 - \gamma y^3 - \beta x^3 y^4 - \beta x^2 y^3 = 0
\]  

(22)

with

\[
\alpha = b + \sigma + \frac{1}{\sigma}, \quad \beta = \frac{1}{\sigma^2}, \quad \gamma = \frac{b(\sigma + 1)}{\sigma^2}, \quad \delta = \frac{\sigma + 1}{\sigma}, \quad \epsilon = \frac{b(r - 1)}{\sigma^2}.
\]  

(23)

Roughly speaking, this is the projection of the system (20) on the \((x, y)\)-plane. The equation (22) was investigated in the paper [8]. The semi invariant \(\nu_5\) has the form

\[
\nu_5 = \tilde{u}_1 z + \tilde{u}_2 t^2 + \tilde{u}_4
\]  

(24)

with \(z = x^2, t = 1/(xy)\) and

\[
\begin{align*}
\tilde{u}_1 &= \alpha \beta (10\alpha - \alpha^2 - 6\delta), \\
\tilde{u}_2 &= \frac{2}{9} \alpha (\alpha + 3\delta)(2\alpha - 3\delta), \\
\tilde{u}_4 &= \alpha \left(\frac{\alpha^2}{9}(2\alpha^2 - 9\gamma) - 2\epsilon(2\alpha - 3\delta)\right).
\end{align*}
\]  

(25)
The projective curvatures are

\[
L_1 = 3(\xi \beta - \beta^2) y^2 - \frac{2}{3} \alpha^2 y + \frac{1}{3x} (2\alpha - 3\delta),
\]

\[
L_2 = \frac{\alpha}{y}.
\] (26)

The invariant \( \nu_5 \) is equal to zero in two cases. First, for \( \alpha = 0 \), i.e., \( \sigma = -b - 1 \). This case is noninteresting because it cannot be reached for physical values of the parameters. In the second case we have \( \nu_5 = 0 \) for arbitrary \( r \) and \( \sigma = 2b - 1 \) with \( b = 0 \) and \( b = 2/3 \). This is a well known case of regular behaviour of the system (20).

In the following we shall continue the investigation of [8] and calculate the other invariants as well as the invariants of other projections of the system (20).

R. Liouville [1] found that in the case \( \nu_5 = 0 \) and \( L_1, L_2 \neq 0 \) we can construct the first integral of the equation (4):

\[
y' = \frac{-L_1}{L_2}.
\] (27)

Consequently, for the projection on the \((x, y)\)-plane with the parameters

\[
\sigma = -1/5, \ b = -16/5, \ r = -7/5
\] (28)

we get the equation

\[
y' = -\frac{y}{x} (4 - 32xy + 192x^2y^2 - 25x^4y^2),
\] (29)

where \( y(x) \) is the solution of equation (22). Let \( xy = s \), then we get Abel’s equation of the first kind [14]

\[
s_x = f_2(x)s^2 + f_3(x)s^3
\] (30)

and after the substitution

\[
\begin{align*}
\sigma(x) &= w(x)\eta(\xi) - \frac{f_2(x)}{3f_3(x)}, \quad f_2(x) = \frac{8}{x}, \quad f_3(x) = -\frac{42}{x} - \frac{25}{4}x, \\
w(x) &= c \left( \frac{x^2}{168 + 25x} \right)^{16/3},
\end{align*}
\] (31)

\[
\xi = \int f_3(x) w^2(x) dx = \frac{-1}{8} \left( 168 + 25x^2 \right)^{\frac{32}{3}} x^{\frac{64}{3}} - \frac{62}{3} \int x^{\frac{1}{3}} (168 + 25x^2)^{-\frac{32}{3}} dx
\]

we have the following equation for the function \( \eta(\xi) \)

\[
\eta'(\xi) = \eta^3(\xi) + I(x),
\]

with

\[
I(x) = -\frac{256 \ 256 + 225x^2}{27 \ (168 + 25x^2)^{17/21}}
\]
and $\xi$ as in equation (31).

We believe that this is a new first order integral of the Lorenz system, although it may be of purely mathematical interest due to the choice of parameters given by equation (28).

Another case when a first integral of type (27) exists corresponds to the regular behaviour of the Lorenz system, that is to

$$\sigma = 1/3, \ b = 2/3, \ r \text{ arbitrary}.$$ 

Then we have

$$y' = y^2(4 + 3(1-r)xy + \frac{9}{2}x^3y).$$

This is the well known Abel equation (30) with

$$f_3 = 3(1-r)x + \frac{9}{2}x^3, \ f_2 = 4$$

and after a substitution in analogy with (31) we get the canonical expression

$$y(x) = w(x)\eta(\xi) - \frac{4}{3(3(1-r)x + \frac{9}{2}x^3)},$$

$$w = \exp\left(\int \frac{-16}{3(3(1-r)x + \frac{9}{2}x^3)}dx\right)$$

with

$$(3(1-r)x + \frac{9}{2}x^3)w^3I(x) = \frac{d}{dx}\left(\frac{4}{3(3(1-r)x + \frac{9}{2}x^3)}\right) + \frac{128}{27}(3(1-r)x + \frac{9}{2}x^3)$$

and

$$\eta'(\xi) = \eta^3 + I(x).$$

To our knowledge, there are no such solutions in the literature.

Let $\nu_5 \neq 0$, then our two dimensional variety $V_2(x,y)$ can be immersed into $\mathbb{P}^4(\mathbb{R})$ only. On the other hand side we can calculate the invariants of higher weights and look for conditions on the parameters that one of that invariants vanish or all invariants are functions of one of them. In the first case we can hope that our immersed surface has some special geometrical properties. In the second case we can decide whether our differential equation (22) possesses some group of infinitesimal transformations or not.

We calculated $\nu_7$, $\nu_9$, $\nu_{11}$, and some resultants of $\nu_5$ and $\nu_7 - R_{57}$, $\nu_5$ and $\nu_9 - R_{59}$, $\nu_7$ and $\nu_9 - R_{79}$, as well as the resultants of $R_{57}$, $R_{59}$, $R_{79}$. The results of these calculations are quite voluminous and only some first of them will be listed here. This gives us enough insight into the structure of them.

The semi invariant of weight 7 for (22) equals to

$$\nu_7 = (v_1z^2 + v_2zt^2 + v_3zt + v_4z + v_5t^4 + v_6t^3 + v_7t^2 + v_5t + v_10)/t \quad (32)$$
with \( z = x^2, \ t = 1/(xy) \) and

\[
\begin{align*}
    v_1 &= 30\beta \tilde{u}_1, \ v_2 = 2\alpha\beta(\alpha - 2)(\alpha^2 + 18\delta), \ v_3 = \frac{10}{3}\alpha^2 \tilde{u}_1, \\
    v_4 &= 10\alpha\beta\left(\frac{\alpha^2}{3}(2\alpha^2 - 9\gamma) + 3\epsilon(\alpha^2 - 14\alpha + 12\delta)\right), \ v_5 = \frac{2}{9}(\alpha + 3\delta) \tilde{u}_2, \\
    v_6 &= \frac{28}{27}\alpha^3(\alpha + 3\delta)(2\alpha - 3\delta), \ v_7 = -\frac{54}{7}\epsilon\alpha^{-2}v_6, \ v_9 = \frac{10}{3}\alpha^2 \tilde{u}_4, \ v_{10} = 30\epsilon \tilde{u}_4
\end{align*}
\]

from (25). For the semi invariant of weight 9 we found the representation

\[
\begin{align*}
    \nu_9 &= (\tilde{w}_1 z^3 + \tilde{w}_2 z^2 t^2 + \tilde{w}_3 z^2 + \tilde{w}_4 z^2 + \tilde{w}_5 z t^4 + \tilde{w}_6 z t^3 + \tilde{w}_7 z t^2 + \tilde{w}_8 z t + \tilde{w}_9 z + \tilde{w}_{10} t^6 + \tilde{w}_{11} t^5 + \tilde{w}_{12} t^4 + \tilde{w}_{13} t^3 + \tilde{w}_{14} t^2 + \tilde{w}_{15} t + \tilde{w}_{17})/t^2.
\end{align*}
\]

with \( \tilde{u}_1, \tilde{u}_2, \tilde{u}_4 \) defined in (23). The first two of the resultants \( R_{57} \) and \( R_{59} \) have a short form. Using the notions from (23) and (33-35) we have

\[
R_{57} = \nu_9^2 \tilde{u}_3 v_1 + \nu_9^2 \tilde{u}_3 v_4 + \nu_9^2 \tilde{u}_4 v_1 + \nu_9^2 (2\tilde{u}_3 \tilde{u}_4 v_1 - \tilde{u}_1 \tilde{u}_2 v_2 + \tilde{u}_1^2 v_3) + t^4 \tilde{u}_2 v_1 - \tilde{u}_1 \tilde{u}_2 v_2 + \tilde{u}_1^2 v_3 - \tilde{u}_1 \tilde{u}_4 v_2 + \tilde{u}_1^2 v_5) + t^4 \tilde{u}_4 v_1 - \tilde{u}_1 \tilde{u}_2 v_2 + \tilde{u}_1^2 v_3 - \tilde{u}_1 \tilde{u}_4 v_2 + \tilde{u}_1^2 v_5 - \nu_9^2 \tilde{u}_1 \tilde{u}_4 v_3 - \tilde{u}_1 \tilde{u}_4 v_3),
\]

9
\[ R_{59} = \nu_5^2 \tilde{u}_3 \tilde{w}_1 + \tilde{u}_1^3 \tilde{w}_1 - \tilde{u}_1^3 \tilde{w}_1 7 - \tilde{u}_1 \tilde{u}_2^2 \tilde{w}_4 + \nu_5^2 (3 \tilde{u}_2^2 \tilde{u}_4 \tilde{w}_1 - \tilde{u}_1 \tilde{u}_2^2 \tilde{w}_4) + t^6 (\tilde{u}_2^3 \tilde{w}_1 - \tilde{u}_1^3 \tilde{w}_1 10 - \tilde{u}_1 \tilde{u}_2^2 \tilde{w}_2 + \tilde{u}_2^2 \tilde{u}_2 \tilde{w}_5) + t^5 (-\tilde{u}_1^3 \tilde{w}_1 11 - \tilde{u}_1 \tilde{u}_2^2 \tilde{w}_3 + \tilde{u}_1^2 \tilde{u}_2 \tilde{w}_6) + t^4 (3 \tilde{u}_2^2 \tilde{u}_4 \tilde{w}_1 - \tilde{u}_1^3 \tilde{w}_1 12 - 2 \tilde{u}_1 \tilde{u}_2 \tilde{u}_4 \tilde{w}_2 - \tilde{u}_1 \tilde{u}_2^2 \tilde{w}_4 + \tilde{u}_1^2 \tilde{u}_4 \tilde{w}_5 + \nu_5 (3 \tilde{u}_2^2 \tilde{u}_3 \tilde{w}_1 - 2 \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \tilde{w}_2 + \tilde{u}_2^2 \tilde{u}_3 \tilde{w}_5) + \nu_5^2 (3 \tilde{u}_2^2 \tilde{u}_3 \tilde{w}_1 - 2 \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \tilde{w}_3 + \tilde{u}_2^2 \tilde{u}_3 \tilde{w}_6) + \nu_5^2 (-2 \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \tilde{w}_3 + \tilde{u}_2^2 \tilde{u}_3 \tilde{w}_6) + \nu_5^2 (2 \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \tilde{w}_3 + \tilde{u}_2^2 \tilde{u}_3 \tilde{w}_6) + \nu_5^2 (-\tilde{u}_2^2 \tilde{u}_4 \tilde{w}_3 + \tilde{u}_2 \tilde{u}_4 \tilde{w}_7) + t^4 (-\tilde{u}_3 \tilde{w}_1 13 - 2 \tilde{u}_1 \tilde{u}_2 \tilde{u}_4 \tilde{w}_3 + \tilde{u}_1 \tilde{u}_2 \tilde{u}_4 \tilde{w}_6 + \nu_5 (-2 \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \tilde{w}_3 + \tilde{u}_2 \tilde{u}_3 \tilde{w}_6) + \tilde{u}_2 \tilde{u}_4 \tilde{w}_8) + t (-\tilde{u}_2 \tilde{w}_1 16 - \nu_5 \tilde{u}_1 \tilde{u}_3 \tilde{w}_3 - \tilde{u}_1 \tilde{u}_3 \tilde{w}_3 + \tilde{u}_2 \tilde{u}_3 \tilde{w}_8 + \nu_5 (-2 \tilde{u}_1 \tilde{u}_3 \tilde{w}_3 + \tilde{u}_2 \tilde{u}_3 \tilde{w}_8) + \tilde{u}_2 \tilde{u}_4 \tilde{w}_9 + t^2 (3 \tilde{u}_2 \tilde{u}_4 \tilde{w}_1 - \tilde{u}_3 \tilde{w}_1 14 - \nu_5 \tilde{u}_1 \tilde{u}_4 \tilde{w}_5 - \tilde{u}_1 \tilde{u}_4 \tilde{w}_2 + \nu_5 (3 \tilde{u}_2 \tilde{u}_5 \tilde{w}_1 - \tilde{u}_1 \tilde{u}_3 \tilde{w}_2) - 2 \tilde{u}_1 \tilde{u}_2 \tilde{u}_4 \tilde{w}_4 + \tilde{u}_2 \tilde{u}_4 \tilde{w}_7 + \nu_5 (6 \tilde{u}_2 \tilde{u}_3 \tilde{u}_4 \tilde{w}_1 - 2 \tilde{u}_1 \tilde{u}_3 \tilde{u}_4 \tilde{w}_2 - 2 \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \tilde{w}_4 + \tilde{u}_1 \tilde{u}_3 \tilde{w}_7) + \tilde{u}_2 \tilde{u}_2 \tilde{w}_9) + \nu_5 (3 \tilde{u}_3 \tilde{u}_1^2 \tilde{w}_1 - 2 \tilde{u}_1 \tilde{u}_3 \tilde{w}_4 + \tilde{u}_1 \tilde{u}_3 \tilde{w}_9).} \]

Based on the analysis of the formulas \([32,35]\) we raise the supposition that each of the following higher invariants \(\nu_7, \nu_9, \nu_{11}\) can be equal to zero for the same values of the parameters as \(\nu_5\). We failed to find different conditions on the parameters of the system \([21]\). Because all semi invariants of this series can be found by use of the recursive relation \([3]\) this supposition will we certainly true for all of them. Exactly the same can be said about the structure of the resultants. Let us remark that all these formulas become considerably shorter and simpler in the case \(\sigma = 2b - 1\) and only in this case we obtained a transparent relation between the invariants. Really we see, that for \(\sigma = 2b - 1\)

\[ \nu_5 = \frac{27b^2 (-2 + 3b) z}{(2b - 1)^9}, \]

\[ \nu_7 = \frac{162b^2 (-2 + 3b) z (5b - 5b q + 5b^2 t + bt^2 - 2b^2 t^2 + 5z)}{(2b - 1)^7}, \]

\[ \nu_9 = \frac{486b^2 (-2 + 3b) z (65b^2 (1 - q)^2 + 130b^3 t (1 - q) + 10b^2 t^2 (3(1 - 2b) (1 - q) + 7t^2) + 26b^3 t^3 (1 - 2b) + b^2 t^4 (1 - 2b)^2 + 130bt (1 - q) + 130b^2 tz + 40bt^2 z (1 - 2b) + 65z^2)})}{(2b - 1)^9}, \]

\[ R_{57} = \frac{81(2 - 3b)^2 b^3}{(2b - 1)^{12}} (10\nu_5^2 (-1 + 2b)^5 + 54\nu_5 b^3 (2 - 3b) (-5(1 - q) - 5bt - (1 - 2b)t^2) + 9\nu_7 (2 - 3b) b^2 (1 - 2b)^2 t)) \].

Also, in this case we have a good correspondence with the physical interpretation of the system. The case \(\sigma = 2b - 1\) is the well known integrable case and the system does not possess any stochastic behaviour.

The investigated projection at the \((x, y)\)-plane was the simplest one. In the following we shall take more complicated projections and look for conditions on the parameters. In the first out from the more complicated cases we start with the substitution \(\xi = x^2 - bz\) in the system \([21]\) and eliminate the variable \(y\) from both equations. In this way we obtain the \((x, \xi)\)- projection. For convenience we shall
use the notation $y$ instead of $\xi$. The second order differential equation which we obtain in this way has the canonical form given by equation (11) with functions $a_i(x,y)$, $i = 1, \ldots, 4$ as follows:

\[ a_1(x,y) = \frac{\sigma(r-1)x}{b^2 y^2} + \frac{\sigma x}{b^3 y} - \frac{\sigma x^3}{b^3 y^2}, \]
\[ a_2(x,y) = \frac{\sigma + 1}{3by} - \frac{(2\sigma - b)(r-1)x^2}{b^2 y^2} - \frac{(2\sigma - b)x^2}{b^3 y} + \frac{(2\sigma - b)x^4}{b^3 y^2} - \frac{1}{3y}, \]
\[ a_3(x,y) = \frac{(r-3)(2\sigma - b)^2 x^3}{b^2 \sigma y^2} - \frac{2(2\sigma - b)(\sigma + 1)x}{b^3 \sigma y} + \frac{(2\sigma - b)^2 x^3}{b^3 \sigma y^2} - \frac{(2\sigma - b)^2 x^5}{3b\sigma y} + \frac{(2\sigma - b)x}{3\sigma y}, \]
\[ a_4(x,y) = \frac{(2\sigma - b)^3 x^6}{b^3 \sigma^2 y^2} - \frac{(2\sigma - b)^3 x^4}{b^3 \sigma^2 y} + \frac{(\sigma + 1)(2\sigma - b)^2 x^2}{\sigma^2 b y} - \frac{(r-1)(2\sigma - b)^3 x^4}{\sigma^2 b^2 y^2} - \frac{2\sigma - b \sigma}{\sigma}. \]

In this projection the projective curvatures have the form

\[ L_1 = \frac{2\sigma - b}{3b^2 \sigma^2 y^4} \left[ 9x^4 b(b-2\sigma)^2 (1-r) + 9x^6 (b-2\sigma)^2 + 2x^2 yb(-b(1+b)^2 + 2\sigma(1+8b-6(r-\sigma)-4\sigma+\sigma^2 + 3b\sigma)) + 30x^4 y\sigma(b-2\sigma) - y^2 b^2 \sigma(1 + b + \sigma) - 3x^2 y^2 \sigma(b-8\sigma) \right], \]
\[ L_2 = \frac{x}{3b^3 \sigma y^4} \left[ -9x^2 b(b-2\sigma)^2 (1-r) - 9x^4 (b-2\sigma)^2 + yb(2b + b^2 - b^3 + \sigma(-4 - 10b + 3b^2 + 16\sigma + 4\sigma^2 + 12r(b-2\sigma)) - 30x^2 y\sigma(b-2\sigma) + 3y^2 \sigma(b-8\sigma) \right], \]

and the semi invariant $\nu_5$ is given by

\[ \nu_5 = \frac{(1 + b + \sigma)(2\sigma - b)}{27b^7 \sigma^5 y^{10}} \left[ 3b^2 (2\sigma - b)^3 (-1 + b + b^2 - 5\sigma + b\sigma + 3\sigma - \sigma^2) \times \right. \]
\[ (2 + b - b^2 - 5\sigma + b\sigma + 9r\sigma + 2\sigma^2)x^5 + 9b(2\sigma - b)^4 (3b + 6b^2 + 3b^3 + \sigma - 34b\sigma - 17b^2 \sigma + 18br\sigma + 20\sigma^2 - 19b\sigma^2 - 18r\sigma^2 + 3\sigma^3)x^7 + \]
\[ 81\sigma(\sigma - 2b)(2\sigma - b)^4 x^9 + 18b(2\sigma - b)^3 (-3b - 6b^2 - 3b^3 - \sigma + 34b\sigma + 17b^2 \sigma - 18br\sigma - 20\sigma^2 + 19b\sigma^2 + 18r\sigma^2 - \sigma^3)x^5 y + \]
\[ 324(2b - \sigma)^2(2\sigma - b)^3 x^7 y + b^2 \sigma^2 (2\sigma - b)^2 (-2 - b + b^2 + 5\sigma - b\sigma - 9\sigma - 2\sigma^2) (3 + 2b - b^2 - 10\sigma + 2b\sigma + 16r\sigma + 3\sigma^2)xy^2 + b^2 (2\sigma - b)^2 \right. \]
\[ (33b + 48b^2 + 15b^3 + 59\sigma - 206b\sigma - 139b^2 \sigma + 144br\sigma - 170\sigma^2 - 95b\sigma^2 + 288r\sigma^2 + 59\sigma^3) x^3 y^2 + 18\sigma^2 (2\sigma - b)^2 (3b^2 - 44b\sigma + 10\sigma^2)x^5 y^2 + \]
\[ 2b\sigma^3 (2\sigma - b)(7b + 2b^2 - 5b^3 - 59\sigma - 44b\sigma + 39b^2 \sigma + 36br\sigma + 170\sigma^2 - 15b\sigma^2 - 288r\sigma^2 - 59\sigma^3) x y^3 + 36(b - 8\sigma)(b - 2\sigma)\sigma^3 (b + 2\sigma)x^3 y^3 - \]
\[ 9\sigma^4 (8\sigma - b)^2 x^4 y^4 \right]. \]
In this case we can succeed in $\nu_5$ being identically equal to zero for

$$\sigma = -1 - b \quad \text{and} \quad b = 2\sigma \quad (38)$$

only. The first condition belongs to the mentioned above unphysical case, the second one represents the integrable case. The surface $V_2(x, y)$ can be immersed as a developable surface in the projective space $\mathbb{P}^3(\mathbb{R})$.

For $b = 2\sigma$ we can investigate other series of invariants. The following ones (see above (11)-(13)) with the first weight equal to 1 will be absent too because of $w_1 \equiv 0$ for the considered values of the parameters. The next series of semi invariants $i_{2k}, k = 1, 2, \ldots \quad (15)-(16)$ reads

$$i_2 = \frac{3}{4\sigma^2 y^2}, \quad i_4 = \frac{9}{8\sigma^4 y^4}, \quad i_6 = \frac{27}{8\sigma^6 y^6}, \quad i_8 = \frac{243}{16\sigma^8 y^8}, \ldots$$

The projective curvatures have a very simple form in this case

$$L_1 = 0, \quad L_2 = -\frac{3x}{4\sigma^2 y^2}$$

and the functions $a_i \quad (37)$ are given by

$$a_1 = \frac{x}{8\sigma^2 y^2}(y - x^2 + 2q\sigma - 2\sigma), \quad a_2 = \frac{1 - \sigma}{6\sigma y}, \quad a_3 = a_4 = 0.$$ 

The corresponding absolute invariants $j_{2k} \quad (14)$ are constants:

$$j_4 = 2, \quad j_6 = 8, \quad j_8 = 48, \ldots$$

The last of the investigated projections of the Lorenz system is the $(y,z)$- projection. The coefficients of the ordinary differential equation (4) have in this case the following form:

$$a_1(x, y) = \frac{\sigma y^4 - by^2 x - b\sigma y^2 x + b^2 r x^2 - b^2 x^3}{(-y^2 + br x - bx^2)^2},$$

$$a_2(x, y) = \frac{y(by^2 + \sigma y^2 - 3r\sigma y^2 - b^2 r x + 2br\sigma x + 3\sigma y^2 x + b^2 x^2 - 2b\sigma x^2)}{3(-y^2 + br x - bx^2)^2},$$

$$a_3(x, y) = \frac{r(y^2(1 - b - 2\sigma + 3r\sigma) + br^2 x(-1 + b - \sigma) + y^2 x(1 + 2\sigma - 6r\sigma) + r x^2(2b - b^2 + 2b\sigma) + 3\sigma y^2 x^2 - bx^3(1 + \sigma))/(3(-y^2 + br x - bx^2)^2)}{3(-y^2 + br x - bx^2)^2},$$

$$a_4(x, y) = \frac{y(r^2\sigma - r^3\sigma + y^2 - br x - 2r\sigma x + 3r^2\sigma x + bx^2 + \sigma x^2 - 3r\sigma x^2 + \sigma x^3)}{(-y^2 + br x - bx^2)^2}.$$ 

For convenience we substituted $z$ for $x$. This is one of the complicatest cases and all formulas are quite large. Therefore we list only the projective curvatures $L_1$ and $L_2$ here. The first projective curvature has the form:

$$L_1 = y(r^2 y^2(4 - 3b^2 + b^3 + 20\sigma - 26b\sigma + 8b^2\sigma - 12r\sigma + 24br\sigma - 9b^2 r\sigma + 4\sigma^2) -$$
2b\sigma^2) + y^4(2 - 2b^2 + 2\sigma - 2b\sigma - 3r\sigma + 6br\sigma) + r^3x(-4b + 3b^3 - b^4 + 16b\sigma - 10b^2\sigma + b^3\sigma - 24br\sigma + 12b^2r\sigma - 4b\sigma^2 + 2b^2\sigma^2) - ry^2x(8 + 4b + 10b^2 - 2b^3 - 40\sigma + 70b\sigma - 29b^2\sigma + 36r\sigma - 96br\sigma - 39b^2r\sigma - 8\sigma^2 + 6b\sigma^2) + 3\sigma y^4x(1 - 2b) + r^2x(12 - 6b^2 - 13b^3 + 5b^4 - 48b\sigma + 46b^2\sigma - 6b^3\sigma + 96br\sigma - 57b^2r\sigma + 12b^2 - 8b^3\sigma^2) + 4y^2x^2(1 - b - b^2 + b^3 + 5\sigma - 11b\sigma + 6b^2\sigma - 9r\sigma + 30br\sigma - 15b^2r\sigma + \sigma^2 - b\sigma^2) + brx^3(-12 + 12b + 14b^2 - 10b^3 + 48\sigma - 62b\sigma + 15b^2\sigma - 144r\sigma + 99br\sigma - 12\sigma^2 + 10b\sigma^2 + \sigma y^2x^3(12 - 48b + 30b^2) + x^4(4b - 6b^2 - 4b^3 + 6b^4 - 16b\sigma + 26b^2\sigma - 10b^3\sigma + 96br\sigma - 75b^2r\sigma + 4\sigma^2 - 4b^2\sigma^2) + 3b\sigma x^5(-8 + 7b))/(3(-y^2 + brx - bx^2)^4),

the second one

\[ L_2 = ry^4(6 - b - 5b^2 + 2b^3 - 10\sigma + 15b\sigma - 5b^2\sigma + 12r\sigma - 24br\sigma + 9b^2r\sigma - 4\sigma^2 + 2b\sigma^2) + 3\sigma y^6(1 - 2b) + 2br^2 y^2 x(-2 + b + 2b^2 - b^3 - 12\sigma + 10b\sigma - 2b^2\sigma + 12r\sigma - 6br\sigma + 2\sigma^2 - b\sigma^2) + 2y^4 x(-3 + 2b + 3b^2 - 2b^3 + 5\sigma - 13b\sigma + 8b^2\sigma - 12\sigma + 36br\sigma - 15b^2r\sigma + 2\sigma^2 - 2b\sigma^2) + b^2 r^3 x^2(-2 - b + b^2 - 2\sigma + b\sigma) + bry^2 x^2(8 - 6b - 8b^2 + 6b^3 + 48\sigma - 64b\sigma + 15b^2\sigma - 72r\sigma + 45br\sigma - 8\sigma^2 + 6\sigma y^4 x^2(2 - 6b + 5b^2) + b^2 r^2 x^3(6 + 2b - 4b^2 + 6\sigma - 4b\sigma) + 2by^2 x^3(-2 + 2b + 2b^2 - 2b^3 - 12\sigma + 22b\sigma - 10b^2\sigma + 36r\sigma - 17br\sigma + 2\sigma^2 - 2b\sigma^2) + b^2 r x^4(-6 - b + 5b^2 - 6\sigma + 5b\sigma) + b\sigma y^2 x^4(-24 + 21b) + 2b^2 x^5(1 - b^2 + \sigma - b\sigma))/(3(-y^2 + brx - bx^2)^4).\]

It is easy to observe that both curvatures are much simpler by \( \sigma = 0 \) and are both equal to zero in case

\[ \sigma = 0, \quad b = -1, \quad r \text{ arbitrary.} \]  \hspace{1cm} (39)

That means that the corresponding second order differential equation, the \((y, z)\)-projection of the system \((\text{II})\), are equivalent to the simplest equation \(y'' = 0\) and can be integrated.

### 3 The Rößler System

For the Rößler system \((\text{II})\)-\((\text{IV})\)

\[ \begin{align*}
\dot{x} &= -y - z, \\
\dot{y} &= x + ay, \\
\dot{z} &= b + xz - cz,
\end{align*} \]

\(a, b, c\) being parameters, we investigated 3 projections. In all of them we lefted the notion \(y\) for the function and \(x\) for the independent variable. The \((y, z)\)-projection of
(40) after a simple transformation the system can be given in the form of equations (4) with coefficients $a_i$ as follows:

$$
\begin{align*}
    a_1 &= 0, \quad a_2 = \frac{1}{3y}, \quad a_3 = -\frac{(x+2a)x-y}{3xy}, \\
    a_4 &= \frac{a^2x-b+ay+ax^2}{xy} - \frac{(2a+c)x+ba+x^2}{y^2}.
\end{align*}
$$

The semi invariant $\nu_5$ has the value

$$
\nu_5 = \frac{2a+x}{9x^2y^2}(2ax^2(18a - 2a^3 - 9b + 9c) + \\
\quad x^3(18a - 8a^3 + 9b - 27c) + 3a^2x^4 + 7ax^5 + 2x^6 + 10a^2y^2 + 9axy^2).
$$

It is easy to see that the invariant $\nu_5$ cannot be identically equal to zero for any values of the constants. Consequently, for the projection (40) there does not exist any immersion $V_2(x, y)$ into $\mathbb{P}^3(\mathbb{R})$ and the two dimensional variety $V_2(x, y)$ can be immeresed into $\mathbb{P}^4(\mathbb{R})$ only. Also, we cannot expect to find a first integral of the system (40) in this way. We calculated also other invariants of this series, namely $\nu_7$ with weight 7 and $\nu_9$ with weight 9. For shortness we represent here only the formula for $\nu_7$ (that for $\nu_9$ is too large):

$$
\nu_7 = \frac{1}{27x^3y^15}(360a^3b(-18a + 2a^3 + 9b - 9c)x^3 + 360a^2(-36a^2 + 4a^4 + \\
5a^3b - 36ac + 2a^3c + 18bc - 9c^2)x^4 + 90a(-216a^2 + 48a^4 + \\
18ab + 2a^3b - 9b^2 - 36ac + 20a^3c + 27bc + 36c^2)x^5 + \\
90(-108a^2 + 24a^4 - 18ab - 17a^3b + 72ac + 2a^3c - 9bc + 27c^2)x^6 + \\
90(-18a - 32a^3 - 9b - 11a^2b + 27c - 17a^2c)x^7 - 90a(39a + 2b + 11c)x^8 + \\
(-1350a - 180c)x^9 - 180x^{10} + 10x^3(2a + x)^3(36a^2 - 4a^4 - \\
18ab + 18ac + 18ax - 8a^3x + 9bx - 27cx + 3a^2x^2 + 7ax^3 + 2x^4)y + \\
27x(2a + x)(-40a^3b - 80a^3x - 36a^2bx - 40a^2cx - 124a^2x^2 + 10abx^2 - \\
50acx^2 - 42ax^3 + 4a^3x^3 - 3bx^3 + 9cx^3 + 4a^2x^4 + ax^5)y^2 + \\
14ax(2a + x)^3(10a + 9x)y^3 - a(200a^3 + 532a^2x + 486ax^2 + 135x^3)y^4).
$$

Also in this case we cannot reach $\nu_7 = 0$ identically by any choice of the parameters $a, b, c$. The existence of a term which does not containing parameters allows us to conclude that the other members of this series have the same property. Naturally, all these formulas will gratly simplify in the case $a = 0$. We can expect, that the behaviour of the system (40) in that case will show some special features.

We take into account also the second projection of (40), i.e., that on the $(x, y)$-plane. The corresponding second order differential equation has the form (4) and the coefficients $a_i(x, y), \ i = 1, ..., 4$ are given by:

$$
\begin{align*}
    a_1 &= 0, \quad a_2 = \frac{1}{3y}, \quad a_3 = \frac{ax + c - a - y}{3y}, \\
    a_4 &= \frac{ax^2 + cx + b}{y^2} - \frac{(a^2 + 1)x + ac - 1}{y} + a.
\end{align*}
$$

(41)
where \( a, b, c \) are constants of the system \((11)\). The calculation of the semi invariant \( \nu_5 \) results in

\[
\nu_5 = \frac{-a + c + ax}{9y^{11}}(18a^2b - 36abc + 18bc^2 + (-36a^2b + 18a^2c + 36abc - 36ac^2 + 18c^3)x + (18a^3 + 18a^2b - 72a^2c + 54ac^2)x^2 + (-36a^3 + 54a^2c)x^3 + 18a^3x^4 + (4a^4 + 45ab + 27ac - 16a^3c - 36bc - 27c^2 + 24a^2c^2 - 16ac^3 + 4c^4 + (54a^2 - 16a^4 - 36ab + 48a^3c - 36c^2 - 48a^2c^2 + 16ac^3)x + (-9a^2 + 24a^4 - 72ac - 48a^3c + 24a^2c^2)x^2 + (-36a^2 - 16a^4 + 16a^3c)x^3 + 4a^4x^4)y + (-12a^2c + 24ac^2 - 12c^3 + (-12a^3 + 48a^2c - 36ac^2)x + (24a^3 - 36a^2c)x^2 - 12a^3x^3)y^2 + (-a^2 - 8ac + 8c^2 + (-8a^2 + 16ac)x + 8a^2x^2)y^3). \tag{42}
\]

The projective curvatures \( L_1 \) and \( L_2 \) are given by

\[
L_1 = \frac{1}{3y^4}(9(ax^2 + cx + b) + 2y(a - c)^2 - 4axy(a - c) + 2a^2x^2y + y^2(a - 2c - 2ax)),
\]

\[
L_2 = \frac{c - a + ax}{y^3}. \tag{43}
\]

The semi invariant \( \nu_5 \) will be equal to zero only for \( a = c = 0 \) with arbitrary \( b \). In this case the system \((11)\) has a very simple form as well as the projective curvatures \((13)\):

\[
L_1 = \frac{3b}{y^4}, \quad L_2 = 0.
\]

The calculation of the next possible series of invariants gives us using formula \((11)\)

\[
w_1 = 0.
\]

Consequently we must turn over to the series with the first invariant beeing \( i_2 \) \((15),(16)\). For this semi invariants the formulas hold

\[
i_2 = \frac{15b}{y^5}, \quad i_4 = \frac{135b^2}{y^{10}}, \quad i_6 = \frac{2430b^3}{y^{15}},
\]

\[
i_8 = \frac{65610b^4}{y^{20}}, \quad i_{10} = \frac{2361960b^5}{y^{25}}, \ldots
\]

and the corresponding absolute invariants \((17)\) are given by

\[
j_4 = \frac{3}{5}, \quad j_6 = \frac{18}{25}, \quad j_8 = \frac{162}{125}, \quad j_{10} = \frac{1944}{625}, \ldots \tag{44}
\]

Because of \( L_2 = 0 \) we cannot expect to find a first integral of the system in the form of \((19)\). But we see, that in this case \( V_2(x, y) \) can be immersed into \( \mathbb{P}^5(\mathbb{R}) \).
Let's mention that for \(a = b = c = 0\) both curvatures (13) equal zero and that therefore our equation (11) can be reduced to \(y'' = 0\) and trivially solved.

Now we turn to the last case - the \((x, z)\)-projection of the system (10). The coefficients of the corresponding equation (11) are

\[
\begin{align*}
 a_1 &= 0, \quad a_2 = \frac{1}{3y}, \quad a_3 = \frac{b - 3y - a}{3ky}, \\
 a_4 &= \frac{a}{x} - \frac{2(b + y)}{x^2} + \frac{x + 1}{y} + \frac{cx - b - ax^2}{y^2} - \frac{ab}{xy}.
\end{align*}
\]

The semi invariants \(\nu_5\) has in this case the form

\[
\nu_5 = \frac{(ax - b)}{9x^5y^{10}}(2b^4 + ab^3x + 9b^2x^2 - 6a^2b^2x^2 + x^3(-9ab + a^3b - 9b^2 + 9bc) + \\
x^4(-9a^2 + 2a^4 - 18ab) + 18a^2x^5 - 2y^2b^2 + 3abxy^2)
\]

and the curvatures are

\[
\begin{align*}
 L_1 &= \frac{-9bx^2 - 9cx^3 - 9ax^4 + 2b^2y - 4abxy + 2a^2x^2y - 2by^2 + 3axy^2}{3x^2y^4}, \\
 L_2 &= \frac{b - ax}{xy^3}.
\end{align*}
\]

We also investigated the invariants \(\nu_5, \nu_7, \ldots\) in the general case \((a, b, c \neq 0)\). As an example we note here

\[
\nu_7 = \frac{1}{27x^7y^{15}}(180b^6x^2 - 90b^5(a + 2c)x^3 + 90b^4(9 - 7a^2 + 2ab + ac)x^4 + \\
90b^3(-18a + 7a^3 - 9b - a^2b + 7a^2c)x^5 + 90b^2(a^4 - 7a^3b + 9ac - 7a^3c + \\
9bc - 9c^2)x^6 + 90ab(9a^2 - 2a^4 + 18ab + 7a^3b - 9b^2 - a^3c + 18bc + 9c^2)x^7 + \\
90a^2(-18ab + a^3b - 9b^2 - 9ac + 2a^3c - 45bc)x^8 + \\
90a^3(9a - 2a^3 + 36b + 18c)x^9 - 1620a^4x^{10} + 10(-b + ax)^3(2b^4 + \\
ab^3x^2 - 6a^2b^2x^2 - 9abx^3 + a^3bx^3 - 9b^2x^3 + 9bcx^3 - 9a^2x^4 + \\
2a^4x^4 - 18abx^4 + 18a^2x^5)y + 27x(b - ax)^2(ab^3 - 6b^2x + 5abx^2 - \\
a^3bx^2 - 3b^2x^2 + 11bcx^2 - 20abx^3 + 2a^2x^4)y^2 + \\
14b(-b + ax)^3(-2b + 3ax)y^3 + (-8b^4 + 17ab^3x - 9a^3bx^3)y^4).
\]

We can reach \(\nu_5 = 0\) only for \(a = b = 0\). The semi invariant \(\nu_7 = 0\) only for \(a = b = 0\) in this case too as well as for all following invariants of this series. Then we have

\(L_1 = 3cx/y^4, L_2 = 0\) and \(w_1 = 0\). The semi invariants of the series (13)-(10) in this case are

\[
\begin{align*}
 i_2 &= \frac{15cx}{y^5}, \quad i_4 = \frac{135c^2x^2}{y^{10}}, \quad i_6 = \frac{2430c^3x^3}{y^{15}}, \\
 i_8 &= \frac{65610c^4x^4}{y^{20}}, \quad i_{10} = \frac{2361960c^5x^5}{y^{25}}, \ldots
\end{align*}
\]  

The corresponding absolute invariants are exactly the same as in the former case (14). In the considered case \(V_2(x, y)\) can be also immersed into \(\mathbb{P}^3(\mathbb{R})\).
4 Conclusion

We have investigated some projections of well known dynamical systems – the Lorenz- and the Rössler-system. The proposed new approach to the investigation of those systems gives us a simple possibility to distinguish the regions of the parameters for which the systems show regular behaviour. Also, it turned out to be possible in some cases to find some first integrals of the considered systems. These observation lets us hope that the method suggested can be used as a first step for the investigation of further new systems. Using the proposed in this paper method we can distinguish different areas in the space of parameters which allow a chaotic behaviour. This knowledge can be used as input for different, regular or numerical for instance, methods. The present paper is naturally a first step towards a more general approach which allows to find an analytical criterium for the parameters of the system under what the system has a certain dynamical state, a regular or a stochastic one.

Acknowledgements

The second author, V.S.D., thanks DAAD for financial support and the Mathematical Institute of Leipzig University for kind hospitality. Both authors are deeply indebted the librarian, Mrs. I. Letzel for her effort to find the old and partly forgotten scientific papers.

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