Covariant quantum mechanics applied to noncommutative geometry

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Abstract. We here report a result obtained in collaboration with Giovanni Amelino-Camelia, first shown in the paper [1]. Applying the manifestly covariant formalism of quantum mechanics to the much studied Snyder spacetime [2] we show how it is trivial in every physical observables, this meaning that every measure in this spacetime gives the same results that would be obtained in the flat Minkowski spacetime.

1. Introduction

One of the main approaches to the quantum gravity problem - the one I am interested in describing here - postulates the quantum nature of spacetime as codified in non-trivial commutation relations between spacetime coordinates:

\[ [x^\mu, x^\nu] = i\ell F^{\mu\nu}(x, p) \]

This kind of assumption was introduced in the beginning to try to avoid the ultraviolet singularities of field theories - the length scale \( \ell \) seems to provide a natural cutoff, which given the nature of the function \( F^{\mu\nu} \) can be covariant or at least the cause of a deformation of the Lorentz group instead of a breaking. Apart for the usefulness in terms of convergence in quantum field theories (problem which was, anyway, approached with renormalization theory methods afterwards) the assumption of coordinates being quantum non-commutative operators has another reason: it can be considered a useful effective theory approach to more fundamental quantum gravity manifestations on spacetime [3, 4]. Theories based on this approach have the strong advantage of being just a deformation of standard, pre-quantum gravity theories. In this way it is easier to keep check of the physics at work, simply evaluating the difference in prediction with respect to the “classical” theories. This should be true for any quantum gravity theory, but in other approach the classical limit, in which standard quantum mechanics or general relativity are recovered, proved not trivial at all. One of the basic ingredients in the analysis used here is the duality between the so called “Kinematical” and “Physical” Hilbert spaces. The first one is the space of the general degrees of freedom of the theories, without connection to the dynamics of the particular system; the second one is the restriction of the kinematical degrees of freedom to the ones allowed by the dynamics of the system under study. These distinction proves crucial in the case of Snyder spacetime, in which it has always been claimed particle position would be quantized, with discrete spectrum. In all the previous approaches however
the distinction between kinematical and physical degrees of freedom was never emphasized. We show that these concerns should not go overlooked in the analysis. The Snyder spacetime [2] is characterized by noncommutativity of coordinates of the form

\[ [x^\mu, x^\nu] = i\ell^2 M_{\mu\nu} \]  

where \( \ell \) is the length scale of Snyder noncommutativity and \( M_{\mu\nu} \) denotes the Lorentz generators. This noncommutative Snyder spacetime has been studied extensively, especially in relation to the possibility of using it to model physical pictures in which spacetime coordinates are discretized. At the level of analysis involving the unphysical kinematical Hilbert space Snyder’s proposal does indeed involve a discretization of coordinates, but this is only a formal artifact: the discreteness leaves no trace on the observable properties of particles on the physical Hilbert space. We note that a similar result was shown in the classical theory with a different approach [5].

2. Covariant formulation of quantum mechanics

We here take as starting point the fact that the standard formulation of quantum mechanics should be viewed merely as a sort of gauge-fixed version of the more general and more powerful manifestly-covariant formulation of quantum mechanics. This formulation of quantum mechanics has progressed significantly over the last decade (see, e.g. Refs. [6, 7, 8]), rendering increasingly clear how spatial and time coordinates there play exactly the same type of role. In the standard case (commutative spacetime) spatial and time coordinates are well-defined operators on a “kinematical Hilbert space”, which is the space of square-integrable functions

\[ L^2(\mathbb{R}^4, dq_0 dq_1 dq_2 dq_3) \sim L^2(\mathbb{R}^2, dp_0 dp_1 dp_2 dp_3) \]

with canonical commutation relations

\[ [q^\mu, p^\nu] = i\delta^\mu_\nu, \quad [q^\mu, q^\nu] = 0, \quad [p^\mu, p^\nu] = 0. \]  

Here we denote with \( q^\mu \) the spacetime coordinates, while the \( p^\mu \) are of course the conjugate momenta.

Time is not an evolution parameter because in the covariant formulation of quantum mechanics there is no “evolution”: dynamics is codified in a constraint, just in the same sense familiar for the covariant formulation of classical mechanics (see, e.g., Ref. [9]). For example, for free particles the constraint simply enforces on-shellness, and the “reduced” Hilbert space obtained from the kinematical Hilbert space by enforcing on-shellness is the so-called “physical Hilbert space”.

The physical content of the theory resides in the properties of self-adjoint operators on the physical Hilbert space. Through these one recovers the information on physical evolution of systems from the formal setup which involves a pure-constraint Hamiltonian. This will be crucial for our analysis of the physical implications of Snyder noncommutativity, so we devote a few equations to showing some aspects of its implementation (more details in Refs. [6, 7, 8], and Refs. [10, 11] for a recent application to noncommutative spacetimes). For free particles one has that states of the physical Hilbert space must comply with the requirement

\[ \mathcal{H}\psi = [p_0^2 - p_1^2 - m^2]\psi = 0 \]

A convenient strategy for implementing the constraint is based [8] on introducing a corresponding scalar product on the physical Hilbert space, such that different kinematical states are projected on the same physical state. Specifically one adopts the following scalar product:

\[ \langle \phi | \psi \rangle = \int d\mu(p) \delta(\mathcal{H}) \Theta(p_0) \overline{\phi}(p) \psi(p) \]

where \( \Theta(p_0) \) specifies a restriction [8] to positive-energy solutions of the on-shellness constraint.
It is important to stress that the spacetime coordinates $q^{\mu}$ are self-adjoint operators on the kinematical Hilbert space, but they are not self-adjoint operators on the physical Hilbert space, since $[q^{\mu}, H] \neq 0$. So the physical properties of the theory cannot be captured by properties of the operators $q^{\mu}$. They are coded in properties of combinations of the operators $q^{\mu}$ and $p_{\mu}$ which commute with the Hamiltonian constraint. An example of such physical observables are the Newton-Wigner operators

$$A_i = q_i - \frac{p_i}{p_0}q_0 + i\frac{p_i}{2p_0^2} \quad (4)$$

which indeed are self-adjoint on the physical Hilbert space (in particular $[A_i, H] = 0$).

How is this regaining us a picture of evolution from the “frozen” pure-constraint setup? This is easily seen by observing that, for example, the expectations of the operators $p_i$ and $A_i$ in a state $\psi$ of the physical Hilbert space,

$$\langle \psi | p_i | \psi \rangle = \bar{p}_i, \quad \langle \psi | A_i | \psi \rangle = a_i,$$

are meaningful physical properties. These are just expectations of observables in a given (frozen) state that satisfies the Hamiltonian constraint, but contain information on the evolution of the system. One can easily see this by contemplating the interpretation of the Newton-Wigner operator in classical mechanics: the state $\psi$ is placing the particle on a worldline (a fuzzy worldline in quantum mechanics) which in the classical limit is such that

$$q_i = a_i + \frac{\bar{p}_i}{p_0(\bar{p}_i)}q_0$$

(because of the Hamiltonian constraint, on the physical Hilbert space $p_0$ must be viewed as constrained in terms of the $p_i$). So the particle is moving with standard velocity $\bar{p}_i/p_0(\bar{p}_i)$ and intercepts the $q^i$ axis at $q^i = a^i$.

3. Discreteness of Snyder spacetime

We reminded our readers in the previous section of the fact that even the standard (commutative) spacetime coordinates are not self-adjoint operators on the physical Hilbert space. They are not observables. They are however well-defined operators on the kinematical Hilbert space, well suited for characterizing the geometric structure of “empty” Snyder spacetime (there are evidently no physical particle on the kineamtical Hilbert space). Then any proposal of noncommutativity of spacetime coordinates must be formulated first on the kinematical Hilbert space, keeping in mind however that the observable manifestations of coordinate noncommutativity can only be its implications for the properties of self-adjoint operators on the physical Hilbert space. In this section we shall see how the Snyder noncommutativity admits a satisfactory description on a kinematical Hilbert space, which in particular involves a discretization of spatial coordinates.

As anticipated in Snyder’s original paper, and more recently established in detail in Ref.[12], the algebraic properties of the coordinate operators implies their spectra to be a quantum lattice of spacing $\ell$, while time remains a continuous variable. These results have been established by seeking formal Hilbert-space representations of the Snyder algebra, but the question remained so far concerning the physical interpretation of these results. As already stressed above within the standard formulations of quantum theory there is no room for such a Hilbert space, on which in particular the time coordinate is described by an operator.

To derive the discreteness of spatial coordinates it is sufficient to notice that the spatial coordinates and the angular momenta generators form an $SO(4)$ algebra. This is easily seen by posing

$$L_{ij} = M_{ij} \quad , \quad L_{i4} = \frac{x_i}{\ell} \quad , \quad L_{AB} = -L_{BA},$$

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which indeed, also in light of (1), reproduce the commutation relations for the \( SO(4) \) algebra:

\[
[L_{AB}, L_{CD}] = i(\delta_{AC}L_{BD} - \delta_{BC}L_{AD} - \delta_{AD}L_{BC} + \delta_{BD}L_{AC})
\]

This observation shows that one can find a basis for our kinematical Hilbert space which is built of representations of this algebra. The implications of this are best seen by introducing the six operators

\[
A_i = \frac{1}{2} \left( L_i + \frac{x_i}{\ell} \right), \quad B_i = \frac{1}{2} \left( L_i - \frac{x_i}{\ell} \right),
\]

with \( L_i = \epsilon_{ijk}M^{jk} \); this allows us to decouple the relevant \( SO(4) \) algebra in two \( SU(2) \) algebras: \([A_i, A_j] = i\epsilon_{ijk}A_k, [B_i, B_j] = i\epsilon_{ijk}B_k \) and \([A_i, B_j] = 0\), with \( A_iA_i = B_iB_i \).

Then we can have basis states labeled by eigenvalues of \( A_3, B_3 \), and the casimir \( A_iA_i \):

\[
A_3|j, m_A, m_B\rangle = m_A|j, m_A, m_B\rangle
\]

\[
B_3|j, m_A, m_B\rangle = m_B|j, m_A, m_B\rangle
\]

\[
A_iA_i|j, m_A, m_B\rangle = j(j + 1)|j, m_A, m_B\rangle
\]

On these basis states we obtain for the \( x_3 \) coordinate, which can be written as \( x_3 = \ell (A_3 - B_3) \),

\[
x_3|j, m_A, m_B\rangle = \ell (m_A - m_B)|j, m_A, m_B\rangle
\]

The eigenvalues \( m_A, m_B \) are integer numbers, so the eigenvalues of the \( x_3 \) coordinate form a one dimensional lattice of spacing \( \ell \). The same reasoning applies of course also to the other spatial coordinates, \( x_1 \) and \( x_2 \), so we can conclude that the spatial part of the Snyder spacetime is a 3-dimensional quantum lattice (but states in the kinematical Hilbert space which are eigenstates of one coordinate are not eigenstates of the other two coordinates), and infinitely degenerate eigenvalues.

### 4. Kinematical Hilbert-space representation

Snyder himself gave us a representation of this noncommutative coordinates in terms of a set of commutative ones. We keep the notation \( p_\mu, q^\mu \). So, in particular, we still have that \([q^\mu, p_\nu] = i\delta_\nu^\mu \). The difference is that now the \( q^\mu \) will not be interpreted as spacetime coordinates, but only as auxiliary operators useful for the description of Snyder’s noncommutative coordinates \( x^\mu \). We have:

\[
x^\mu = q^\mu - \ell^2 p^\mu (p \cdot q).
\]

Indeed we have that

\[
[x^\mu, x^\nu] = \left[ q^\mu - \ell^2 p^\mu (p \cdot q), q^\nu - \ell^2 p^\nu (p \cdot q) \right] =
\]

\[
= -\ell^2 \left( [p^\mu(p \cdot q), q^\nu] + [q^\mu, p^\nu(p \cdot q)] \right) +
\]

\[
+ \ell^4 \left[ p^\mu(p \cdot q), p^\nu(p \cdot q) \right] =
\]

\[
= i\ell^2 \left( p^\mu q^\nu - q^\nu p^\mu \right) =
\]

\[
= i\ell^2 \left( p^\mu x^\nu - p^\nu x^\mu \right) = i\ell^2 M^{\mu\nu}
\]

It is also easy to verify that in order for the \( x^\mu \) to be hermitian operators the scalar product on the kinematical Hilbert space must involve the following \( \ell \)-deformed integration measure \( (p^2 \) denotes as usual \( p_\mu p^\mu \))

\[
d\mu(p) = \frac{dp}{(1 - \ell^2 p^2)^{5/2}}.
\]
The coordinates $x^\mu$ are self-adjoint operators in this Hilbert space. We shall soon study their spectra.

It is also easy to verify that this construction (as stressed already in Snyder’s original work [2]) preserves Lorentz symmetry. This is essentially due to the fact that Snyder’s commutation relations (1) are Lorentz invariant and the measure of integration (12) is Lorentz invariant.

5. Effects of discretization on physical observables

Even if, as we saw, kinematical space coordinates are discretized in Snyder spacetime, we have not yet evaluated any physical coordinate living in this spacetime.

Physical observables can only be introduced on the physical Hilbert space, after enforcing a suitable Hamiltonian constraint. The discretization of Snyder’s spacetime could be meaningful only to the extent that it would leave a trace in such observables on the physical Hilbert space. Let us then consider the propagation of particles in Snyder’s spacetime. As shown above Snyder’s proposal leaves Lorentz symmetry unaffected, so in order to describe propagation of particles we need to enforce the standard Hamiltonian constraint

$$[p_0^2 - p_i^2 - m^2] \psi = 0$$

The kinematical Hilbert space is “reduced” to the physical Hilbert space of states $\psi$ that satisfy this constraint. And we start by noticing that the modification of the scalar product affecting the kinematical Hilbert space trivializes when restricted to states in the physical Hilbert space

$$d\mu(p) = \frac{d^4p}{(1 - \ell^2 p_0^2)^{5/2}} \rightarrow d\mu(p) = \frac{d^4p}{(1 - \ell^2 m^2)^{5/2}},$$

This apparent deformation of the measure of integration just amounts to multiplication by a constant (the mass $m$ is fixed on the entire physical Hilbert space once and for all). It therefore gets reabsorbed in the normalization of the states, and is completely irrelevant.

This observation about scalar products takes us one step closer to establishing the futility of the results obtained for Snyder’s spacetime at the level of the kinematical Hilbert space. The possibility of discretization of the coordinates $x_j$ is not even a meaningful possibility on the physical Hilbert space, since evidently the $x_j$ do not commute with the Hamiltonian constraint operator and therefore cannot be observables on the physical Hilbert space. What one could hope for is for the discretization of the $x_j$ on the kinematical Hilbert space to manifest itself under a different disguise as some property of observables on the physical Hilbert space. But we find that this is not the case. To see this let us start from the observables on the physical Hilbert space for free propagating particles which have been most considered in the relevant literature. For the undeformed theory ($\ell = 0$) the operators one usually considers are $\chi^\mu$ where

$$\chi^\mu = q^\mu - p^\mu (q \cdot v)/(p \cdot v) + h.c.$$ where $v$ is a real-valued, time-like vector that parametrizes these operators that commute with the Hamiltonian constraint. This family of operators of course also includes, for $v^\mu = \delta^\mu_0$, the Newton-Wigner operator which we already discussed above.

We notice that for the Snyder spacetime one does get good self-adjoint operators on the physical Hilbert space (commuting with the Hamiltonian-constraint operator) by taking these standard $\chi^\mu$ and replacing the $q^\mu$ with the Snyder coordinates $x^\mu$:

$$\chi^\mu_{[\ell]} = x^\mu - \frac{p^\mu}{p \cdot v} x \cdot v + h.c.$$ (13)

It is therefore meaningful to ask whether the discretization of the $x_j$ on the (unobservable) kinematical Hilbert space leaves any trace on properties of these $\chi^\mu_{[\ell]}$ observables on the physical
Hilbert space. The answer is no, as one can easily see by recalling that the Snyder coordinates admit the representation
\[ x^\mu = q^\mu - \ell^2 p^\mu (p \cdot q), \]
from which it follows that
\[ X^\mu = x^\mu - \frac{p^\mu}{p \cdot v} x \cdot v + h.c. \]
\[ = q^\mu - \ell^2 p^\mu (p \cdot q) - \frac{p^\mu}{p \cdot v} (q^\nu - \ell^2 p^\nu (p \cdot q)) v_\nu + h.c. \]
\[ = q^\mu - \frac{p^\mu}{p \cdot v} (q \cdot v) + h.c. = X^\mu \quad (14) \]

The definition of the \( X^\mu \) does involve the Snyder deformation scale \( \ell \) but it turns out that actually they are \( \ell \) independent. The \( X^\mu \) on the physical Hilbert space are insensitive to the discretization of the \( x^\mu \) on the kinematical Hilbert space. They are completely independent of the deformation scale \( \ell \).

On the basis of results obtained in the previous literature on the manifestly covariant formulation of quantum mechanics the observables \( X^\mu \) are particularly significant since they come as close as one can get [13] to the notion of a spacetime coordinate on the physical Hilbert space. A mentioned special cases of the \( X^\mu \) are the Newton-Wigner operators, to which then evidently our result (14) applies:
\[ A^i = x^i - \frac{p^i}{p^0} x^0 + h.c. = q^i - \ell^2 p^i (p \cdot q) - \frac{p^i}{p^0} (q^0 - \ell^2 p^0 (p \cdot q)) + h.c. = q^i - \frac{p^i}{p^0} + i \frac{p^i}{2(p^0)^2} = A^i \]

6. Conclusions

We showed that Snyder’s famous discretization of spatial coordinates is no longer found once one considers only physical degrees of freedom, i.e. observables of the system compatible with the dynamics. One of the most important points of this paper is to emphasize the conceptual difference between kinematical degrees of freedom, living in a sort of unphysical space, and the real observables, which live in a different Hilbert space and thus have properties in principle disconnected from the kinematical ones. We emphasize that to make meaningful physical prediction one should extend his statements to the physical Hilbert space, being the kinematical Hilbert space not physically faithful. Still, it should not be expected that all quantum properties of spacetime introduced on the kinematical Hilbert space disappear on the physical Hilbert space of free particles. What we found in our previous study Ref.[11], concerning the so-called “\( \kappa \)-Minkowski noncommutativity”, establishes that in some cases the implications of noncommutativity of coordinates, introduced on the kinematical Hilbert space, does affect significantly the structure of the physical Hilbert space, even with just free particles. So this appears to be an issue that requires a case-by-case analysis.

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