A Magic Electromagnetic Field

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1.1 The Electromagnetic Field

Away from charges and currents, both the electrostatic potential, \( \Phi \), and the magnetostatic potential, \( \chi \), are harmonic. Thus \( \Psi = \Phi + i\chi \) satisfies

\[ \nabla^2 \Psi = 0 . \]

The solution obeying this equation everywhere – except the origin – and tending to zero at infinity is \( \Psi = q/r \), but if we move the origin to \( \mathbf{b} \) this solution becomes

\[ \Psi = q \sqrt[2]{r^2 - (r - b)^2} . \]

This solution is harmonic whether \( q \) and \( b \) are real or complex.

To ensure no magnetic monopole, term \( q \) must be real, but we now consider the possibility that \( b = ia \) where \( a \) is real so that \( b \) is pure imaginary. Then we shall have both an electric and a magnetic field with \( \mathbf{F} = \mathbf{E} + i\mathbf{B} = -\nabla \Psi \). Without loss of generality we may orient the \( z \) axis along \( a \) so that

\[ \Psi = q \sqrt[2]{R^2 + (z - ia)^2} \]

where \( R^2 = x^2 + y^2 \). (1.1)
This expression will be harmonic except at singularities and branch points. The singularities lie at \( R = a \) and \( z = 0 \). If we ask for no branch points at infinity then we may take the cut defined by the disk \( z = 0, \ R \leq a \), (but notice that we could take the cut around the sphere \( r = a, \ z \geq 0 \) say).

We may evaluate \(-\nabla \Psi\) to obtain
\[
\mathbf{F} = \mathbf{E} + i\mathbf{B} = q(r - ia)/\left[(r - ia)^2\right]^{3/2} .
\]

The total charge is clearly \( q \) but the field also has a magnetic dipole moment. Indeed for \( r > a \) we may use the Legendre polynomial expansion of \( \Psi \)
\[
\Psi = \frac{q}{r} \sum_0^\infty \left(\frac{ia}{r}\right)^n P_n(\cos \theta) .
\]

Evidently all the \( P_{2n} \) have real coefficients and all the \( P_{2n+1} \) have imaginary coefficients so the magnetic potential is antisymmetrical about \( z = 0 \) while the electric potential is symmetrical. Evidently the magnetic moment is the coefficient of \( iP_1 \) which is \( qa \) while the electric quadrupole moment is \( qa^2 \), etc. The relativistic invariants of the field are contained in
\[
F^2 = E^2 - B^2 + 2i\mathbf{E} \cdot \mathbf{B} = q^2/\left[(r - ia)^2\right]^{1/2} .
\]

Now \([(r - ia)^2]^2\) is only imaginary if \((r - ia)^2 = \pm \sqrt{2}(1 \pm i)|r - ia|^2\) which occurs when \((r^2 - a^2)/(2r \cdot a) = \pm 1\) as then the real and imaginary parts are equal in magnitude. This condition may be rewritten \((r \pm a)^2 = 2a^2\) so \(E^2 = B^2\) only on two spheres of radius \(\sqrt{2} a\) centred on \(r = \pm a\). The circle in which they meet is the ring \(z = 0, \ r = a\).

Figure 1.1 illustrates where \(|\mathbf{B}| > |\mathbf{E}|\), etc. \(E\) and \(B\) are perpendicular when \((r - ia)^2 = r^2 - a^2 - 2ia \cdot r\) is either purely real or purely imaginary; i.e., on the sphere \(r = a\), and the plane \(z = 0\). The Poynting vector is given by
\[
\mathbf{F}^* \times \mathbf{F} = (\mathbf{E} - i\mathbf{B}) \times (\mathbf{E} + i\mathbf{B}) = 2i\mathbf{E} \times \mathbf{B} = 2iq^2a \times r/\left[(r - ia)^2\right]^{3/2} ,
\]

and the field energy density by \((8\pi)^{-1} \mathbf{F}^* \cdot \mathbf{F} = (8\pi)^{-1} (E^2 + B^2)\). The velocity of the Lorentz frame in which \(E\) and \(B\) are parallel is given by
\[
\mathbf{V} / (1 + V^2) = \mathbf{E} \times \mathbf{B} / (E^2 + B^2) = a \times r / (a^2 + r^2) = \mathbf{F}^* \times \mathbf{F} / (2i\mathbf{F} \cdot \mathbf{F}^*) ;
\]
squared and solving for $V$ we find

$$V = \left[ a^2 + r^2 - \sqrt{(a^2 + r^2)^2 - 4a^2R^2} \right] / (2aR)$$

$$= 2aR \left[ \frac{a^2 + r^2 + \sqrt{(a^2 + r^2)^2 - 4a^2R^2}}{a + \lambda} \right],$$

where

$$\lambda = \frac{1}{2} \left[ r^2 - a^2 + \sqrt{(r^2 - a^2)^2 + 4(a \cdot r)^2} \right],$$

is defined with the positive root and $\mu$ is the same but for the negative root. $\lambda$ and $\mu$ are spheroidal coordinates. Evidently $\Omega = V/R$ is constant on the confocal spheroids, $\lambda = $ constant which have a focal ring at the singularity. (This result is due to J. Gair.)

Fig. 1.1. Planar cut through the origin, orthogonal to the $z = 0$ plane, showing the delineation of regions of $E > B$ and $E < B$, for the potential given by eq. (1.1).
On the cut itself we have $R < a$ and $z = 0+$.

$$E + iB = q (R - i a) \left/ i \left( a^2 - R^2 \right)^{3/2} \right. = -q \left/ \left( a^2 - R^2 \right)^{3/2} \right..$$

This gives an electric field vertically down into the disc and a magnetic field parallel to the disk surface for $R < a$ as though the disk has a Meisner effect. The corresponding charge density on the symmetry plane is

$$\sigma = -\left( q / 2\pi \right) a \left( a^2 - R^2 \right)^{-3/2}.$$

This charge density gives a divergent total charge but that divergence is cancelled by a ring of opposite charge on the edge which leaves the total charge not ‘negative’ but ‘positive’ $+q$. The total charge at axial distance less than $R$ is $Q(< R) = -q \left[ a(a^2 - R^2)^{-1/2} - 1 \right]$, $R < a$. From the discontinuity in the $B$ field across the cut we find $4\pi J_\phi = -2qR(a^2 - R^2)^{-3/2}$. This corresponds to the charge density given above rotating with angular velocity $\Omega = c/a$, reaching the velocity of light at the singularity. Again its effect is reversed by a ring current at the edge. The fields are illustrated in Figures 1.2 and 1.3.

### 1.2 The connection to Kerr’s metric and the electron

A much more complicated but more intriguing derivation of the above results is to take the Kerr (1963) metric of a black hole of mass $m$ and angular momentum $mac$. Then complexify it following Newman (1973) to get the Kerr-Newman metric of charge $q$, [Newman et.al. (1965)]. Finally, take the limit with $G \to 0$ leaving the charge and the moment corresponding to ‘$a’ but now in flat space. The resultant electromagnetic field is exactly that derived and discussed above, [Pekeris & Frankowski (1987)]. Carter (1968a) showed that all the Kerr-Newman metrics had the same gyromagnetic ratio as the Dirac electron. Does this mean that there is some relationship between the charge distribution of the Kerr-Newman metric and the charge distribution of the quantum electrodynamic field of a point electron?

Classical models of the electron had a problem over the gyromagnetic ratio. Even if all the charge were confined to a ring rotating at close to the velocity of light the magnetic moment generated gives a gyromagnetic ratio of one rather than the electron’s value of 2.0023193044. It is of some interest to gain an understanding as to how the Kerr-Newman metric does it. The answer is that the charge distribution is not all of one sign. In fact a circular current dipole of two rings of opposite charge rotating uniformly about their common axis gives a net magnetic moment but no net charge.
Fig. 1.2. A plot of electric field lines for the potential given by eq. (1.1).

Fig. 1.3. A plot of magnetic field lines for the potential given by eq. (1.1).
The way our electromagnetic field gets its large magnetic dipole moment per unit net charge is that its much larger internal charges are of opposite signs but rotate together giving a magnetic dipole with relatively little net charge. We show elsewhere that this is a characteristic of relativistically rotating conductors!

1.3 Separability of Motion in the field

Studies of separability of wave equations in the Kerr and Kerr-Newman metrics [Carter (1968b), Teukolsky (1972, 1973), Chandrasekhar (1976), Page (1976)] have shown that Dirac’s equation is separable in these metrics. This of course implies that it is still separable in their flat space limit as \( G \to 0 \). The criterion for the separability of Schrödinger’s equation in a real potential in spheroidal coordinates is 

\[
\Phi = \left[ \zeta(\lambda) - \eta(\mu) \right] / (\lambda - \mu) \quad \text{[Morse and Feshback (1953)]}.
\]

Here \( \lambda \) and \( \mu \) are spheroidal coordinates and \( \zeta, \eta \) are arbitrary functions of their arguments.

The field that we derived so simply above is rewritten in spheroidal coordinates as follows: \( \lambda \) and \( \mu \) are the roots for \( \tau \) of the quadratic

\[
\frac{x^2 + y^2}{a^2 + \tau} + \frac{z^2}{\tau} = 1 ,
\]

where \( x^2 + y^2 = R^2 = (\lambda + a^2) (\mu + a^2) / a^2 \) and the metric is

\[
ds^2 = dx^2 + dy^2 + dz^2 = \frac{\lambda - \mu}{4\lambda (\lambda + a^2)} d\lambda^2 + \frac{\lambda - \mu}{4\mu (\mu + a^2)} d\mu^2 + R^2 d\phi^2 .
\]

To compare to Kerr’s metric one uses the quasi-spherical form of spheroidal coordinates \( \tilde{r}^2 = \sqrt{\lambda} \), \( \mu = -a^2 \cos^2 \vartheta \), \( z = \tilde{r} \cos \vartheta \). Note however that \( \tilde{r} \) is constant on spheroids and \( \tilde{r} = 0 \) is the disc \( z = 0 \), \( R \leq a \). Also \( \vartheta \) is not the \( \theta \) of spherical polar coordinates but is constant in hyperboloids. Thus

\[
ds^2 = \left( \tilde{r}^2 + a^2 \cos^2 \vartheta \right) / \left( \tilde{r}^2 + a^2 \right) d\tilde{r}^2 + \left( \tilde{r}^2 + a^2 \cos^2 \vartheta \right) d\vartheta^2 +
\]

\[
+ \left( \tilde{r}^2 + a^2 \right) \sin^2 \vartheta d\phi^2 .
\]

In spheroidal coordinates our potential \( \Psi = q / \sqrt{(r - ia)^2} \) takes the simple forms

\[
\Psi = q / \left( \sqrt{\lambda} - i \sqrt{-\mu} \right) = q \frac{\sqrt{\lambda} + i \sqrt{-\mu}}{\lambda - \mu} = \frac{q}{\tilde{r} - ia \cos \vartheta} .
\]

The second of these forms is exactly of the right type for separability of the Schrödinger equation but the similarity is partly misleading for Schrödinger’s
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equation only separates in an electrostatic potential of that form. When
the imaginary (magnetic) part is added Schrödinger’s equation no longer
separates although the Klein-Gordon equation now does separate (which it
does not with only the electrostatic part). For a derivation and explanation
of these results see Lynden-Bell (2000).

Systems with the same charge distribution but less magnetic field are
given by taking \( \psi = \alpha\Psi + (1 - \alpha)\Psi^* \) for \( \alpha < 1 \). These magnetic fields
are then multiplied by \( 2\alpha - 1 \). These are weighted superpositions of discs
rotating forwards and backwards so the net rotation is less fast and \( \alpha = 1/2 \)
is static. These fields lose the magic of separability. For the other charge &
current distributions with that property see Lynden-Bell (2000).

1.4 Eulogy

In closing, let me say that I still do not know the answer to the problem
discussed in my joint paper with Douglas [Gough & Lynden-Bell (1968)],
i.e., “How do turbulent fluids with angular momentum like to rotate?”
Nevertheless, I never expected to know the internal rotation of the Sun
within my lifetime and I have immense admiration for Douglas – and the
helioseismic fraternity – for having persisted in analysing solar pulsations
until that became possible. Such is the real meat of good science.

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