MICROSHEAVES FROM HITCHIN FIBERS VIA FLOER THEORY

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ABSTRACT. Fix a non-stacky component of the moduli of stable Higgs bundles, on which the Hitchin fibration is proper. We show that any smooth Hitchin fiber determines a microsheaf on the global nilpotent cone, that distinct fibers give rise to orthogonal microsheaves, and that the endomorphisms of the microsheaf is isomorphic to the cohomology of the Hitchin fiber. These results are consequences of recent advances in Floer theory. Natural constructions on our microsheaves provide plausible candidates for Hecke eigensheaves for the geometric Langlands correspondence.

1. HITCHIN FIBERS AND HECKE EIGENSHEAVES

Let us recall what Hecke eigensheaves are, and why one might hope to get them from fibers of the Hitchin system. More detailed discussions can be found in e.g. [7, 34, 16]. This section is purely motivational and logically unrelated to the results presented in the remainder of the article.

Let \( C \) be a smooth compact complex curve, \( G \) a reductive group, and \( \text{Bun}_G(C) \) the moduli of \( G \)-bundles on \( C \). We write \( [X] \) for some appropriate category of \( D \)-modules or constructible sheaves on \( X \). There are ‘Hecke operators’ \( H_\mu : [\text{Bun}_G(C)] \to [\text{Bun}_G(C) \times C] \), given by convolution with respect to a correspondence parameterizing pairs of bundles which are isomorphic away from a single point \( c \in C \), and whose difference at this point is controlled by a cocharacter \( \mu \) of \( G \).

The geometric Langlands conjecture asserts, in particular, the existence of certain \( F \in [\text{Bun}_G(C)] \) which are ‘Hecke eigensheaves’ in the sense that \( H_\mu(F) = F \boxtimes \rho^\mu(\chi) \), where \( \chi \) is a \( G^\vee \) local system on \( C \), and \( \rho^\mu \) is the representation corresponding to the character \( \mu \) of \( G^\vee \).

The Hecke action transforms microsupports (or characteristic cycles) by setwise convolution with a conic Lagrangian \( ss(H_\mu) \subset T^*\text{Bun}_G(C) \times T^*\text{Bun}_G(C) \times T^*C \), which is roughly the conormal to the image of the Hecke correspondence. This conormal respects the fibers of Hitchin’s integrable system \( h : T^*\text{Bun}_G(C) \to B \), in the sense that \( ss(H_\mu) \ast h^{-1}(b) = h^{-1}(b) \times \tilde{C}_b^\mu \). Here \( \tilde{C}_b^\mu \subset T^*C \) is the spectral cover corresponding to the point \( b \in B \) and representation \( \mu \).

The above geometry suggests that any sufficiently functorial procedure relating Hitchin fibers to sheaves on \( \text{Bun}_G(C) \) – and spectral curves to sheaves on \( C \) – may be expected to yield Hecke eigensheaves. Quantization is one such procedure: the semiclassical limit of a quantum state on \( M \) determines a Lagrangian submanifold of \( T^*M \) (see e.g. [6]). Taking \( M = \text{Bun}_G(C) \), and the corresponding Lagrangians to be the Hitchin fibers, quantizing the Hitchin system may be expected to yield Hecke eigensheaves, and indeed does [7]. Related approaches include [5, 15, 34, 9, 16].

This note explains how to use Floer theory to produce sheaves on \( \text{Bun}_G(C) \) from Hitchin fibers.

2. FUKAYA CATEGORIES AND MICROSHEAVES

Given a symplectic manifold \((W, \omega)\), it is often possible to form an \( A_\infty \)-category whose objects are Lagrangian submanifolds of \( W \), whose morphism spaces \( \text{Hom}(L, L') \) are generated by intersection points of (appropriate perturbations of) \( L \) and \( L' \), and whose structure maps – differential, composition, higher compositions – are defined by counting holomorphic curves with boundary along the Lagrangians. Foundational references include [21, 43].
We restrict attention to symplectic manifolds equipped with a Liouville structure: a vector field \( Z \) with \( Z\omega = \omega \), such that \( Z \) identifies the complement of a compact set in \( W \) with the positive symplectization of a contact manifold. Note \( d(i_Z\omega) = \omega \), and \( W \) is noncompact. Of particular interest is the ‘spine’ or ‘skeleton’ \( \mathbb{L}_W \): the points which remain bounded under the flow of \( Z \).

The prototypical example is a cotangent bundle \( T^*M \) with the vector field \( Z \) generating the radial dilation; for the compact subset take the co-disk bundle; the locus \( \mathbb{L}_{T^*M} \) is the zero section.

To any Liouville \( W \) is associated the ‘wrapped’ Fukaya category \( \text{Fuk}(W) \) \( [27, 26, 25] \). The basic objects of \( \text{Fuk}(W) \) are exact Lagrangian submanifolds, which are \( Z \)-conic outside a compact set. The term ‘wrapped’ refers to the fact that trajectories at infinity are incorporated into the morphism spaces, which are typically infinite dimensional when noncompact Lagrangians are involved. We write \( \text{Fuk}(W) \) to mean the triangulated category generated by such objects and morphisms.

**Example.** \( \text{Fuk}(T^*S^1) \) is equivalent to the category of perfect dg modules for \( k[t, t^{-1}] \). More generally, \( \text{Fuk}(T^*M) \) is equivalent to the category of perfect dg modules for the algebra of chains on the based loop space of \( M \) \( [1] \).

When \( Z \) is the gradient flow of some Bott-Morse function, the Liouville structure is said to be Weinstein. In this case \( \mathbb{L}_W \) is a finite union of locally closed isotropic submanifolds. The cotangent bundle above is an example; the distance to the zero section gives the Bott-Morse function. Stein complex manifolds (with the plurisubharmonic witness providing the Morse function) yield more examples; in fact, all examples up to deformation \( [10] \).

There has been significant recent progress in understanding wrapped Fukaya categories of Weinstein manifolds \( [27, 26, 25] \). The crucial fact for us is that the category \( \text{Fuk}(W) \) is now known to be equivalent to the category of microsheaves on \( \mathbb{L}_W \) \( [25] \).

Let us recall from \( [35, 44, 41] \) the construction and properties of the category of microsheaves. Given a (complex of) sheaf \( F \) on a manifold \( M \), the microsupport \( ss(F) \subset T^*M \) is the locus of directions along which sections of the sheaf fail to propagate. It is tautologically conical, and also known to be co-isotropic. The notion of microsupport of sheaves is developed in \( [35] \).

**Example.** In case \( M \) is a complex manifold, and \( F \) is the sheaf of solutions to a regular holonomic D-module, then \( ss(F) \) agrees with the characteristic cycle of the D-module.

Let us write \( \text{Sh}(M) \) for the (dg unbounded derived) category of sheaves on \( M \), and \( \text{Sh}_{K}(M) \) for the full subcategory of sheaves with microsupport contained in a conic locus \( K \subset T^*M \). There is a sheaf of categories \( \mu sh \) on \( T^*M \) defined by sheafifying

\[
\mu sh^\text{pre}(U) = \text{Sh}(M)/\text{Sh}_{T^*M \setminus U}(M)
\]

The notion of microsupport descends to \( \mu sh \), and for conic \( K \subset T^*M \) we write \( \mu sh_{K} \) for the subsheaf of full subcategories on objects (micro)supported on \( K \). It is natural to view \( \mu sh_{K} \) as a sheaf on \( K \).

In \( [44] \) it is observed that, by taking a high codimension embedding and thickening appropriately, any set \( X \) equipped with the germ of a contact embedding can be equipped with a similar sheaf of categories \( \mu sh_{X} \). (One checks that the only dependence on the embedding is through certain topological data identical to the choices necessary to define the Fukaya category.) This construction can be applied to Liouville manifolds by considering the embedding in the contactization \( \mathbb{L}_W = \mathbb{L}_W \times 0 \subset W \times \mathbb{R} \) and defining \( \text{Sh}(W) := \Gamma(\mathbb{L}_W, \mu sh_{\mathbb{L}_W}) \). In \( [41] \), this category

\[1\]
More precisely, the definition of \( \text{Fuk}(W) \) involves the choices of certain topological structures on \( W \) and on the Lagrangians \( L \). In the present article, \( W \) will always be Calabi-Yau, and we only consider oriented, spinned, graded Lagrangians. In this case we may define \( \text{Fuk}(W) \) over any ring \( k \), and equip the morphism spaces with a \( \mathbb{Z} \)-grading.
is studied in detail; in particular it is shown that (nonconic!) exact Lagrangians give objects of $\mathcal{SH}(W)$, and that $\mathcal{SH}(W)$ is constant along deformations of Weinstein manifolds. Best of all:

**Theorem 1.** [25] For Weinstein $W$, there is an equivalence $\text{Ind}(\text{Fuk}(W)) \cong \mathcal{SH}(W)$.\footnote{Or anti-equivalence, depending on a number of universal sign conventions; see [25].}

### 3. Nonexact Lagrangians

Let us explain how to adapt some results of [27, 26, 25] to allow compact but nonexact Lagrangians, which are unobstructed (or equipped with bounding cochains) in the sense of [21].

Recall that, a priori, structure maps for Fukaya categories are defined by counting a holomorphic curve $C$ by $\text{Area}(C)$. This is because the Gromov compactness theorem only promises compactness of the space of holomorphic disks of area below some fixed bound; consequently Floer theory is defined over the Novikov field $\Lambda$:

$$\Lambda := \left\{ \sum_{i=1}^{\infty} c_i T^{\lambda_i} \mid c_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i \to \infty \right\}$$

For exact Lagrangians in Liouville manifolds, holomorphic curve counts are finite (and orbifold points of moduli can be avoided), and so one may set $T = 1$ and work over any ring. Typically this is preferable, so works such as [43, 2, 27, 26, 25] are written at $T = 1$.

However the arguments of these works are certainly valid if one takes area weighted disk counts and works over the Novikov field. We do so henceforth, so that we can, appealing to the general foundational results of [21], consider at the same time also compact non-exact Lagrangians, so long as these are tautologically unobstructed (bound no holomorphic disk) or more generally have specified bounding cochain in the sense of [21].

We write $\text{Fuk}^+(W) \supset \text{Fuk}(W)$ for a category in which possibly some (graded, oriented, spinned) compact nonexact Lagrangians (with bounding cochains) have been added to the wrapped category of exact Lagrangians. The main point of the remainder of this section is to establish

**Theorem 2.** For a Weinstein manifold $W$, the inclusion $\text{Fuk}(W) \subset \text{Fuk}^+(W)$ is an equivalence.

and deduce

**Corollary 3.** In the above setting, $\text{Ind}(\text{Fuk}^+(W)) \cong \mathcal{SH}(W)$. \(\square\)

**Remark.** Note that the microsheaf category here is defined over the Novikov ring.

The basic point of the proof of Theorem 2 is to consider the diagonal Lagrangian in $W \times (-W)$, where $-W$ means $W$ with the symplectic form reversed. This Lagrangian will on the one hand provide a diagonal bimodule for $\text{Fuk}^+(W)$, but on the other hand can be resolved in terms of objects in $\text{Fuk}(W)$.

We now proceed with some details. [26, Thm. 1.5; Sec. 6.6] gives a Künneth embedding: $\text{Fuk}(V) \otimes \text{Fuk}(W) \hookrightarrow \text{Fuk}(V \times W)$. Two important observations needed to establish this result are: first, that ‘product’ wrappings in $V \times W$ are cofinal in all wrappings, and second, that while the product of eventually conic but not conic Lagrangians is not eventually conic, one can straighten out the product to achieve this. Rather than contemplate straightening products where one factor is nonexact, we will make do with the following weaker result:

**Proposition 4.** There is a morphism $\kappa : \text{Fuk}(V \times W)^{\text{op}} \to \text{Mod}(\text{Fuk}^+(V) \otimes \text{Fuk}^+(W))$.\footnote{Or anti-equivalence, depending on a number of universal sign conventions; see [25].}
Proof. The proof follows [26, Sec. 6.6]; we give the highlights. Recall that the wrapped Fukaya category \( \text{Fuk}(W) \) is by definition a localization of a ‘directed’ category \( \mathcal{O}(W) \). As in [26, Sec. 6.6], one counts appropriate holomorphic strips to define a natural \( A_\infty \) functor to chain complexes,

\[
\mathcal{O}(V \times W)^{\text{op}} \otimes \mathcal{O}^+(V) \otimes \mathcal{O}^+(W) \to \text{Ch}
\]

In this formulation the necessary confinement-of-disks is deduced from the eventual conicity of Lagrangians in each factor separately, not the (false) conicity of product Lagrangians. Such a map is (by definition) a morphism \( \mathcal{O}(V \times W)^{\text{op}} \to \text{Mod}(\mathcal{O}^+(V) \otimes \mathcal{O}^+(W)) \). As in [26, Sec. 6.6], one sees that this descends to the localization to wrapped categories by using the fact that this localization on \( \mathcal{O}(V \times W) \) can be implemented using only product wrappings on \( V \times W \).

As stated, the morphism in Proposition 4 could be the zero morphism. However, one can see the following from the construction of \( \kappa \) that e.g. when \( W = \text{point} \) that the morphism is induced from the inclusion \( \text{Fuk}(V) \hookrightarrow \text{Fuk}(V)^+ \). One can see similarly:

Lemma 5. The functor \( \kappa \) has the following properties:

1. For conic \( L \subset V \) and \( M \subset W \), the module \( \kappa(L \times M) \) is represented by \( L \otimes M \).
2. When \( \Delta \subset (-W) \times W \) is the diagonal, then \( \kappa(\Delta) \subset \text{Mod}(\text{Fuk}^+(-W) \otimes \text{Fuk}^+(W)) = \text{Mod}(\text{Fuk}^+(W)^{\text{op}} \otimes \text{Fuk}^+(W)) \) is the diagonal bimodule. \( \square \)

Remark. When \( V, W \) are Weinstein, the category \( \text{Fuk}(V \times W) \) is generated by the products of the (conic) cocores of \( V \) and \( W \) [26]. Thus Lemma 5(1) implies that, in this case, the image of \( \kappa \) consists only of representable objects.

Proof of Theorem 2. If \( A, B \) are vector spaces, and \( \sum a_i^* \otimes b_i \in A^* \otimes B \) defines a surjective map \( A \to B \), then the \( b_i \) must generate \( B \). An analogous statement for categories: if \( A, B \) are triangulated dg categories and we have a surjective morphism \( A \to B \) whose corresponding \( A^{\text{op}} \otimes B \) bimodule lies in the subcategory generated some \( a_i^* \otimes b_i \), then the \( b_i \) must generate \( B \).

We take \( A = B = \text{Fuk}^+(W) \) and consider the identity functor and corresponding diagonal bimodule. Per Lemma 5(2), this is the image of the diagonal in \( \text{Fuk}(W \times -W) \). We may resolve the diagonal by products of cocores; by Lemma 5(1), these products map to the corresponding modules represented by the geometric products of the cocores. We conclude that the cocores, which are already elements of \( \text{Fuk}(W) \), in fact generate \( \text{Fuk}^+(W) \).

4. Floer theory in hyperkähler manifolds

Let \( W \) be a manifold. We recall that a ‘hypercomplex’ structure on \( W \) is an action of the quaternions on \( TW \) such that \( I, J, K \) determine integrable complex structures. A metric on a hypercomplex manifold is said to be hyperkähler if it is Kähler for each complex structure separately. We denote the corresponding symplectic forms \( \omega_I, \omega_J, \omega_K \). Note that \( \Omega_I := \omega_I + i \omega_K \) gives a holomorphic symplectic form in complex structure \( I \), etcetera.

A submanifold which is holomorphic for all complex structures would be itself hypercomplex; in particular, of dimension divisible by 4. Thus holomorphic curves cannot remain holomorphic under perturbation of the complex structure within the unit quaternion family. A strong form of this fact will play a crucial role for us here:

Theorem 6. [49] Let \( W \) be a complete hyperkähler manifold, and \( L_1, L_2, \ldots, L_n \subset W \) any collection of holomorphic Lagrangians for \( (I, \Omega_I) \). Assume that in the complement of some compact set, the pairwise distance between the \( L_i \) is bounded below.
Then for all but countably many complex structures lying in the circle $J \cos(\theta) + K \sin(\theta)$ and tamed by $\omega_I$, there are no holomorphic curves with boundary along the $L_i$. In particular, each $L_i$ is tautologically unobstructed in the sense of \cite{21}.

Moreover there is an $A_\infty$ equivalence $HF(L_i, L_i) \cong H^*(L_i)$; in particular, the LHS is formal.

We consider the real symplectic form $\omega_{ij} := \omega_I \cos(\theta) + \omega_K \sin(\theta)$. In defining Floer complexes for $(W, \omega_{ij})$, we may choose any complex structure tamed by $\omega_{ij}$. Such complex structures are open and include $J \cos(\theta) + K \sin(\theta)$, hence a neighborhood of this point, hence, in the situation of Theorem \cite{6}, some complex structure $J \cos(\theta') + K \sin(\theta')$ for which the $L_i$ are tautologically unobstructed, i.e. bound no holomorphic disks. So the $L_i$ provide objects of Fukaya categories.

**Theorem 7.** Let $W$ be a complete hyperkähler manifold, and suppose for some $\theta$ there is a real vector field $Z_{\theta}$ determining a Weinstein structure on $(W, \omega_{ij})$. Assume in addition that $L_1, \ldots, L_n$ are compact spinned $I$-holomorphic $\Omega_I$-Lagrangians\cite{3}. Then $L_i$ define objects of $Fuk^+(W)$, and there are microsheaves $F^\theta_1, \ldots, F^\theta_n \in \mathcal{H}(W)$ and isomorphisms

$$\text{Hom}_{Fuk(W)}(L_i, L_j) \cong \text{Hom}_{\mathcal{H}(W)}(F^\theta_i, F^\theta_j)$$

The same holds if we equip the $L_i$ with ($\Lambda$-valued) unitary local systems. So long as these local systems are rank one, we have also an $A_\infty$ equivalence $\text{Hom}_{Fuk(W)}(L_i, L_i) \cong H^*(L_i)$.

**Proof.** Follows immediately from Corollary \cite{5} and Theorem \cite{6}. \hfill \square

**Remark.** There is a natural class of situations in which a complete hyperkähler $W$ will naturally carry a Liouville vector field $Z_{\theta}$ for all $\theta$.

Consider a holomorphic symplectic manifold $(W, I, \Omega)$ carrying an $I$-holomorphic $\mathbb{C}^*$ action, which acts by a positive character on the line $\mathbb{C}\Omega$, such that for any $w \in W$, the limit over $z \in \mathbb{C}^*$ given by $\lim_{z \to 0} z w$ always exists. In this case each of the real symplectic manifolds $(W, \Re(e^{i\theta}\Omega))$ will be Liouville: the vector field being the generator of (an appropriate power of) the $\mathbb{R}^{\geq 0} \subset \mathbb{C}^*$ action. We call such manifolds complex Liouville.

Suppose moreover there is an underlying complete hyperkähler structure, and in addition the $S^1 \subset \mathbb{C}^*$ action is Hamiltonian for $\omega_I$. In this case, the corresponding moment map yields a Morse function showing that $(W, \Re(e^{i\theta}\Omega))$ is Weinstein. We call such manifolds hyperkähler Weinstein.

**Remark.** The cotangent bundle of a complex manifold is complex Liouville, but in general only carries an (incomplete) hyperkähler metric in a neighborhood of the zero section \cite{20, 33}. We expect that nevertheless some version of Theorem \cite{6} and hence Theorem \cite{7} should be true in this context. Specialized to the case of cotangent bundles of curves, such a result would imply the existence of a Floer theoretic functor carrying local systems on spectral covers to local systems on the base. It is natural to expect that the trees which calculate this functor in the flow-tree limit \cite{18} are identical with the ‘spectral networks’ of \cite{22}.

**Remark.** There are natural examples which are (compatibly) Weinstein and complex symplectic, but not complex Liouville or complete hyperkähler, such as a neighborhood of a nodal elliptic curve.

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\footnote{Note that by virtue of holomorphicity, the $L_i$ are canonically oriented. Also recall that a hyperkähler manifold carries a Calabi-Yau form, with respect to which holomorphic Lagrangians are special and hence admit gradings.}
in an elliptically fibered K3 surface. More general examples of such spaces appear in \[14, 24, 4\]. It is desirable to generalize Theorem 6 to this context as well.

**Remark.** When \(W\) is hyperkähler Weinstein, the skeleton \(\mathbb{L}_W\), and hence the category \(\mathcal{G}h(W)\), is independent of \(\theta\). A given Lagrangian \(L\) gives a family of objects \(F^\theta \in \mathcal{G}h(W)\). Comparing to \([22]\) suggests one should expect some nontrivial \(\theta\)-dependence of the \(F^\theta\).

**Remark.** The ‘brane quantization’ picture of \([29, 23]\) – applied to the problem of geometric Langlands in \([34]\) – would, if made rigorous, likely relate our constructions to geometric quantization.

There is long-ongoing work on formulating precise mathematical conjectures about the relation between quantization and Floer theory in holomorphic settings \([36, 48]\). See \([38]\) for another point of view on this conjectural correspondence in the special case of cotangent bundles of curves.

5. Microsheaves from Hitchin fibers

We recall some standard facts about Higgs bundles. Our understanding is that these are variously from \([31, 17, 12, 32, 45, 46, 47, 19, 39, 28]\); due to the incompetence of the author we will not attempt to give precise attributions. The articles \([16, 30]\) also survey related ground with related aims, and in rather more detail.

Fix a smooth compact complex curve \(C\) of genus at least two, and a reductive algebraic group \(G\). We write \(\text{Bun}_G(C)\) for the moduli (stack) of \(G\)-bundles over \(C\)\(^4\). The space \(\text{Bun}_{GL_n}(C)\) decomposes into connected components indexed by the degree of the bundle; more generally \(\text{Bun}_G(C)\) has components indexed by analogous discrete invariants. For discrete invariant \(d\), we write \(\text{Bun}_G(C)_d\) for the corresponding component.

By definition, a ‘Higgs field’ for a \(G\)-bundle \(E\) is a section of the bundle \(\text{ad}(E) \otimes \omega_C\). By deformation theory and Serre duality, the tangent space at \(E\) is given by \(T_E \text{Bun}_G(C) = H^1(C, \text{ad}(E)) = H^0(C, \text{ad}(E) \otimes \omega_C)^*\). Similarly, the moduli (stack) of Higgs bundles \(\text{Higgs}_G(C)\) is \(T^* \text{Bun}_G(C)\).

A \(GL_n\)-bundle \(E\) can be identified with a rank \(n\) vector bundle \(V_E\); then the Higgs field gives a map \(V_E \to V_E \otimes \omega_C\). The fiberwise spectrum of this (twisted) endomorphism traces out the ‘spectral curve’ in \(T^* E\), which is a degree \(n\) finite cover. The linear system of such covers is a linear space \(B\); assigning each Higgs bundle to its spectral curve gives the Hitchin fibration \(h : \text{Higgs}_{GL_n}(C) \to B\). There is a similar construction for other groups, using the appropriate invariant functions in place of the coefficients of the characteristic polynomial. The locus \(h^{-1}(0)\) parameterizes nilpotent Higgs fields and so is termed the (global) nilpotent cone; it is nonreduced, has many irreducible components which are generally singular, and is holomorphic Lagrangian.

Recall the ‘slope’ of a vector bundle is the degree divided by the rank; a vector bundle is said to be ‘semi-stable’ if it has no sub-bundles with larger slope, and ‘stable’ if every nontrivial sub-bundle has strictly smaller slope. There is an analogous notion for \(G\)-bundles. We write \(\text{Bun}_G^s(C)\) for the locus of stable bundles; it is an orbifold (smooth Deligne-Mumford stack). When \(\text{Bun}_G(C)_d\) contains no strictly semi-stable (semi-stable but not stable) bundles, \(\text{Bun}_G^s(C)_d\) is compact. More generally, the locus of semi-stable bundles has a compact coarse moduli space.

Similarly, a \(GL_n\) Higgs bundle is said to be semistable if it has no sub-Higgs-bundles whose underlying \(V_E\) has larger slope, and stable if all nontrivial sub-Higgs-bundles have an underlying vector bundle with strictly smaller slope. There is an analogous notion for \(G\)-Higgs bundles. The stable locus \(\text{Higgs}_G^s(C)\) is an orbifold. When \(\text{Higgs}_G^s(C)_d\) contains no strictly semistables, the

\(^4\)To avoid having positive dimensional stabilizers at generic points, we follow \([16]\) in removing the connected component of the generic stabilizer in the sense of \([3]\) Appendix A]. This has no effect when \(G\) is semisimple. We do the same for moduli of Higgs bundles, moduli of local systems, etc.
restriction of the Hitchin fibration $h: \text{Higgs}^*_G(C) \to B$ is proper. More generally, the Hitchin fibration induces a proper map on the coarse moduli space of the semistable locus.

There are inclusions $T^*\text{Bun}_G^*(C) \subset \text{Higgs}^*_G(C) \subset T^*\text{Bun}_G(C)$. The restriction of the natural holomorphic symplectic form on $T^*\text{Bun}_G(C)$ gives $\text{Higgs}^*_G(C)$ the structure of a complex symplectic orbifold. The Hitchin fibration is a complex completely integrable system; in particular the fibers are holomorphic Lagrangian, and tori when smooth and compact. On components with no semistables, the restriction of the dilation on cotangent fibers gives $\text{Higgs}^*_G(C)$ the structure of a complex Liouville orbifold. The corresponding skeleton is the locus of stable Higgs bundles in the global nilpotent cone; we denote it by $N_G^*(C)$.

We write $\text{Loc}_G(C)$ for the moduli of $G$-bundles on $C$ with flat connection, and $\text{Loc}_G^{irr}(C)$ for irreducible $G$-bundles. ‘Nonabelian Hodge theory’ gives a diffeomorphism $\text{Loc}_G^{irr}(C) \cong \text{Higgs}^*_G(C)$. Both sides carry complex structures, but the diffeomorphism is not holomorphic. We write $J$ for the complex structure coming from the moduli of local systems, and $I$ for the complex structure from the Hitchin moduli; in fact these generate a hypercomplex structure. There is a compatible hyperkähler metric, complete on components with no semistables. The circle action (from cotangent fiber dilation) on Higgs bundles is Hamiltonian, so such components carry a hyperkähler Weinstein structure. The corresponding Morse function is the $L^2$ norm of the Higgs field.

We now restrict attention to groups $G$ with connected center (e.g. $\text{GL}_n$ or $\text{PGL}_n$ but not $\text{SL}_n$) in order to avoid orbifold points. (To do better, develop the Lagrangian Floer theory for symplectic orbifolds and provide analogues of [27, 26, 25].) From the above discussion, we extract:

**Fact 8.** Fix a curve $C$ of genus at least two, a reductive group $G$ with connected center, and a choice of discrete datum $d$ such that $\text{Higgs}^*_G(C)_d$ contains no strictly semistable Higgs bundles. (For instance, $G = \text{GL}_n$ and the degree $d$ coprime to $n$.) Then the smooth manifold $\text{Higgs}^*_G(C)_d$ carries a hyperkähler Weinstein structure with skeleton the stable part $N_G^*(C)_d$ of the corresponding component of the global nilpotent cone.

Additionally, the smooth fibers of the Hitchin fibration $h: \text{Higgs}^*_G(C)_d \to B$ are holomorphic in complex structure $I$, and Lagrangian for $\Omega = \omega_J + i\omega_K$.

**Fact 8** justifies applying Theorem 7 to deduce the results advertised in the abstract:

**Corollary 9.** In the situation of Fact 8, fix an angle $\theta$ and real symplectic form $\omega_\theta = \text{Re}(e^{i\theta}\Omega)$. Fix any collection of smooth Hitchin fibers $L_i := h^{-1}(b_i)$ for distinct $b_i \in B$. Then there are corresponding microsheaves $F^\theta_i \in \mathfrak{Sh}(\text{Higgs}^*_G(C)_d)$ and isomorphisms

$$\text{Hom}_{\text{Fuk}(\text{Higgs}^*_G(C)_d, \omega_\theta)}(L_i, L_j) \cong \text{Hom}_{\mathfrak{Sh}(\text{Higgs}^*_G(C)_d)}(F^\theta_i, F^\theta_j)$$

For $i \neq j$, both sides vanish. For $i = j$, both sides are formal and agree with the cohomology of the torus $L_i$. The same result holds if we equip the $L_i$ with $U_1(\Lambda)$ local systems. □

**Remark.** Restriction of microsheaves gives functors

$$\mathfrak{Sh}_{\text{N}_G(C)_d}(\text{Bun}_G(C)_d) \longrightarrow \mathfrak{Sh}(\text{Higgs}^*_G(C)_d) \longrightarrow \mathfrak{Sh}_{\text{N}_G^*(C)_d}(\text{Bun}_G^*(C)_d)$$

$$\Gamma(\text{N}_G(C)_d, \mu_{sh}) \longrightarrow \Gamma(\text{N}_G^*(C)_d, \mu_{sh}) \longrightarrow \Gamma(\text{N}_G^*(C)_d \cap T^*\text{Bun}_G^*(C)_d, \mu_{sh})$$

Restricting further to the locus $\text{Bun}^*_G(C)_d$ of so-called very-stable bundles — those which admit no nonzero nilpotent Higgs field, hence meet no other component of the nilpotent cone — we find a sheaf with microsupport contained in the zero section, hence locally constant. Let us determine its stalk.
The cotangent fiber $T$ to a very stable bundle is a conical holomorphic Lagrangian disk; by its counterpart sheaf represents the stalk functor. That is, $\text{Hom}(T, L_i)$ is the stalk of $F^\theta_i$. For a generic choice of cotangent fiber, the intersection $T \cap L_i$ is transverse. The indices of the intersections are all equal due to holomorphicity of $T, L_i$, so $\text{Hom}(T, L_i)$ is concentrated in one degree, and has rank equal to the intersection number of $T$ and $L_i$, which in turn is equal to the (positive) multiplicity of the locus of stable bundles in the (nonreduced) nilpotent cone. For $GL_n$, this number is computed explicitly in [30].

It may be interesting to explore the microstalks on other components, for which the slices constructed in [11, 30] should be useful.

**Remark.** E.g. by pushing forward the local system on the very stable locus, we may produce elements of $\text{Sh}_{N_G(C)_{d}}(\text{Bun}_G(C)_{d})$. Based on the considerations in Section [1] it is natural to hope these may be Hecke eigensheaves.

**Remark.** In light of the previous observation, it may be natural to ask: why not simply work to begin with in the cotangent to stable bundles, $T^*\text{Bun}_G^s(C)_{d}$? One reason is that then the Lagrangian $L_i \cap (T^*\text{Bun}_G^s(C)_{d})$ is nonexact, noncompact, and does not have conical asymptotics, so we would need new (or perhaps older [42]) techniques to treat it. Another reason is that this intersection does not have the desired cohomology.

**Remark.** The category $\text{Sh}_{N_G(C)_{d}}(\text{Bun}_G(C)_{d})$ above appears as one side of the ‘Betti’ version of the geometric Langlands correspondence [8], and our results are perhaps most naturally viewed in that context. But since the locally constant sheaves above have finite rank, we may trade them by Riemann-Hilbert for D-modules, and work in the usual geometric Langlands setup. (The endomorphisms of $L_i$ or $F^\theta_i$ conjecturally correspond to the endomorphism algebra of the skyscraper at a smooth point on the spectral side, hence only concern the formal completion at such a point, whereupon the difference between de Rham and Betti moduli becomes irrelevant.)

**Question.** The map $\text{Sh}_{N_G(C)_{d}}(\text{Bun}_G(C)_{d}) \to \mathcal{H}(\text{Higgs}_G(C)_{d})$ is the quotient by the (objects corepresenting) microstalks at smooth points of $\mathcal{N}_G(C)_{d} \setminus \mathcal{N}^{s}_G(C)_{d}$, i.e. the unstable part of the nilpotent cone. What is the corresponding quotient on the spectral side?

### 6. Discussion

Let us comment on a few difficulties to be overcome to bring our microsheaves into some precise relation with objects of interest to Langlands geometers.

One problem is that we are forced to work over the Novikov field $\Lambda$, while the sheaves of true interest live over $\mathbb{C}$. As usual in mirror symmetry [37], one should either prove convergence of holomorphic disk counts, or formulate some version of the geometric Langlands correspondence appropriate to the nonarchimedean coefficients $\Lambda$.

Another difficulty is that the Hecke correspondence involves $T^*C$, which is not quite hyperkähler. We will need a version of Theorem [6] appropriate to this setup; as we have seen in the remarks concluding Section [4] this is also desirable for other reasons.

In order to work in a context where Floer theory is well defined, we restrict to the stable locus and assume there are no corresponding semistables. This introduces more difficulties. In particular, the locus of stable Higgs bundles is not preserved by the Hecke correspondences. One can microlocalize and restrict the Hecke correspondences to the stable locus, but it is not clear how the resulting algebra action is related to the original. Additionally, the discrete data $d$ – which we fixed to avoid strictly semistable Higgs bundles – is not preserved by the Hecke correspondences.
One could restrict to the part of the Hecke algebra which preserves \( d \), but would have to provide geometric representatives for generators of this algebra.

More ambitiously, one could try and work with or around the symplectic singularities; see [40, 13] for some ideas in this direction in a similar setting.

Other difficulties arise because Floer theory is, at present, only defined for smooth Lagrangians. In particular, without further constructions, we are restricted to studying smooth Hitchin fibers, and, to check the Hecke eigen-ness Floer theoretically, we would need to provide smooth representatives for Hecke algebra elements. We hope that further developments around Lagrangian skeleta may help in this regard.

Acknowledgements. This note began life as an email to David Nadler, in the context of an ongoing discussion of how to rigorously connect homological mirror symmetry with (Betti) geometric Langlands. I also thank Ron Donagi, Tony Pantev, and John Pardon for helpful conversations.

My work is partially supported by a Villum Investigator grant, a Danish National Research Foundation chair, a Novo Nordisk start package, and NSF CAREER DMS-1654545.

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