CARLESON MEASURES FOR WEIGHTED HOLOMORPHIC BESOV SPACES

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Abstract. We obtain characterizations of positive Borel measures $\mu$ on $\mathbb{B}^n$ so that some weighted holomorphic Besov spaces $B^p_s(w)$ are imbedded in $L^p(d\mu)$, where $w$ is a $B_p$ weight in the unit ball of $C^n$.

1. Introduction

If $w$ is a weight in $\mathbb{B}^n$, the unit ball in $C^n$, and $p > 0$, $s \in \mathbb{R}$, the space $B^p_s(w)$ consists of holomorphic functions on $\mathbb{B}^n$ for which

$$||f||_{B^p_s(w)} = \int_{\mathbb{B}^n} |(I + R)^k f(y)|^p (1 - |y|^2)^{(k-s)p - 1}w(y)dv(y) < +\infty,$$

for some $k \in \mathbb{Z}_+$, $k > s$. Here $dv$ is the normalized Lebesgue measure on $\mathbb{B}^n$. As it happens in the unweighted case, it can be shown that for adequate weights if the above integral is finite for some $k > s$, then it is also finite for any $k > s$ (see section 3).

In this paper we consider Carleson measures for weighted holomorphic Besov space $B^p_s(w)$, that is, the positive Borel measures $\mu$ on $\mathbb{B}^n$, the unit ball in $C^n$, for which the weighted holomorphic Besov space space $B^p_s(w)$ is imbedded in $L^p(d\mu)$.

For some particular cases of weighted Besov spaces the characterization of the corresponding Carleson measures is known. For instance, when $s < 0$, no derivative is necessarily involved in the definition of the norm of $B^p_s(w)$, and it is in fact a weighted Bergman space. In that case, if $w(z) = (1 - |z|)^{\alpha}$, where $\alpha - sp > 0$, $\mu$ is a Carleson measure for $B^p_s(w)$ if and only if there exists $C > 0$ such that for any $\eta \in S^n$, and $R > 0$, $\mu(T(\eta, R)) \leq CR^{n+\alpha-sp}$, where $T(\eta, R) = \{z \in \mathbb{B}^n, |1 - z\eta| < R\}$ (see [OlPa, Ste] in dimension 1, and [Lu] in dimension $n > 1$ among others). This result can be extended to $\alpha = sp$ and $p \leq 2$ (see for instance the survey [ZhaZhu]).

On the other hand, if $n + \alpha - sp < 0$, it is well known that the space $B^p_s(w)$ consists of regular functions, and the Carleson measures in those cases are just the finite ones.

Let’s finally mention that if $n = 1$, [ArRoSa1] have studied the Carleson measures for $B^p_s(w)$, where $1 < p$, $0 < s < 1$, and $w$ is a weight in $B^1$ in the class $B_p$ (see [Be] for a definition) satisfying some additional regularity conditions on $w$.

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The main purpose of these paper is to obtain characterizations of Carleson measures for weighted Besov spaces in dimension $n > 1$. We remark that they also appear in a natural way in the study of Carleson measures for Hardy-Sobolev spaces on general domains and such that the support of the measure is included on a manifold. For instance, if $D = \{(z, y) ; z, y \in B^1, |y| \leq \varphi(z)\}$, where $\varphi$ is a nonnegative function in $C^1(B^1)$, the study of Carleson measures for $H^p_s(D)$ supported on $y = 0$, $s > \frac{1}{p}$ leads to the study of Carleson measures on $B_{s-\frac{1}{p}}(w)$ where $w = \varphi(4|\frac{\partial \varphi}{\partial z} + 1)^{\frac{1}{2}}$. Although we will not study them in detail, our techniques are inspired in these facts.

We observe that when $w \equiv 1$, the holomorphic Besov space $B^p_s$ can be viewed as a restriction to $B^n$ of the Hardy-Sobolev space $H^p_{s+\frac{1}{p}}(B^{n+1})$ (see [OrFa]). This fact allows to reduce the study of Carleson measures for $B^p_s$ to the study of Carleson measures for Hardy Sobolev spaces in one more dimension. When $n - sp < 1$, there exist characterizations of Carleson measures for $H^p_{s+\frac{1}{p}}(B^{n+1})$ (see [CaOr2]) which give characterizations for Carleson measures for $B^p_s(B^n)$. Other authors (see [ArRoSa2] and references therein) have also obtained other type of characterizations for $H^2_s(B^n)$ when $n - 2s \leq 1$.

Let’s recall the main facts about Carleson measures on weighted Hardy-Sobolev spaces that we will need. The weighted Hardy-Sobolev space $H^p_s(w, B^n)$, $0 \leq s, p < +\infty$, consists of those functions $f$ holomorphic in $B^n$ such that if $f(z) = \sum_k f_k(z)$ is its homogeneous polynomial expansion, and $(I + R)^sf(z) = \sum_k (1 + k)^s f_k(z)$, we have that

$$||f||_{H^p_s(w)} = \sup_{0 < r < 1} ||(I + R)^sf(r\zeta)||_{L^p(w)} < +\infty.$$

We also recall that a $w$ is in $A_p(S^n)$, $1 < p < +\infty$, if there exists $C > 0$ such that for any nonisotropic ball $B \subset S^n$, $B = B(\zeta, r) = \{\eta \in S^n ; |1 - \zeta \eta| < r\}$,

$$\left(\frac{1}{|B|} \int_B w^sd\sigma\right)^\frac{1}{s} \left(\frac{1}{|B|} \int_B w^{-\frac{1}{p - 1}}d\sigma\right)^{p - 1} \leq C,$$

where $\sigma$ is the Lebesgue measure on $S^n$ and $|B|$ denotes the Lebesgue measure of the ball $B$.

A weight $w$ in $S^n$ is in $D_\tau(S^n)$, if there exists $C > 0$ such that for any nonisotropic ball $B$ in $S^n$, $w(2^kB) \leq C2^{k\tau}w(B)$. Analogously to what it happens with weights in $R^n$, the fact that a weight is in $A_p(S^n)$ implies that it is in $D_\tau(S^n)$ for $\tau = np$.

We denote by $K_s$ the nonisotropic potential operator defined by

$$K_s[f](z) = \int_{S^n} \frac{f(\eta)}{|1 - z\eta|^{n-s}}d\sigma(\eta), \quad z \in \overline{B}^n.$$

It has been shown in [CaOr3]

**Theorem 1.1** ([CaOr3]). Let $1 < p < +\infty$, $w$ an $A_p$-weight, and $\mu$ a finite positive Borel measure on $B^n$. Assume that $w$ is in $D_\tau$ for some $0 \leq \tau - sp < 1$. We then have that the following statements are equivalent:
(i) $\|f\|_{L^p(d\mu)} \leq C\|f\|_{H^p_s(wB^n)}$.
(ii) $\|K_\alpha(f)\|_{L^p(d\mu)} \leq C\|f\|_{L^p(w)}$.

We remark that this equivalence is quite useful in many applications, since allows to work with a positive kernel.

Let’s now state the main result in this paper. Recall that a weight $w$ is in $B_p(B^n)$, if there exists $C > 0$ such that for any tent $T(\zeta, R), \zeta \in S^n$,

$$\frac{1}{v(T(\zeta, R))} \int_{T(\zeta, R)} \omega dv \left( \frac{1}{v(T(\zeta, R))} \int_{T(\zeta, R)} \omega^{-(p'-1)} dv \right)^{p-1} \leq C.$$ 

We introduce a pseudodistance in $B^n$ defined by

$$\rho(z, w) = |1 - zw| - \sqrt{1 - |z|^2} \sqrt{1 - |w|^2}.$$ 

**Theorem D.** Let $1 < p < +\infty$, $w$ a $B_p(B^n)$ weight. Assume that $w$ satisfies a doubling condition of order $\tau + 1$, $\tau < 1 + sp$, for the pseudodistance $\rho$. Let $\mu$ be a positive Borel measure on $B^n$. We then have that the following statements are equivalent:

(i) There exists $C > 0$ such that for any $f \in B^p_s(w)$,

$$\|f\|_{L^p(d\mu)} \leq C\|f\|_{B^p_s(w)}.$$ 

(ii) There exists $C > 0$ such that for any $f \in L^p(\omega dv)$,

$$\left\| \int_{B^n} \frac{f(y)dv(y)}{(1 - z\bar{y})^{n+1-(s+1)p}} \right\|_{L^p(d\mu)} \leq C\|f\|_{L^p(\omega dv)}.$$ 

(iii) There exists $C > 0$ such that for any $f \in L^p(\omega dv)$,

$$\left\| \int_{B^n} \frac{f(y)dv(y)}{|1 - z\bar{y}|^{n+1-(s+1)p}} \right\|_{L^p(d\mu)} \leq C\|f\|_{L^p(\omega dv)}.$$ 

The paper is organized as follows: In section 2 we introduce the class of weights we will consider. We obtain all the properties on the weights needed in the proof of Theorem D. In section 3 we study the general properties of the weighted Besov spaces $B^p_s(w)$ and in section 4 we will give the proof of theorem D.

Finally, the usual remark on notation: we will adopt the convention of using the same letter for various absolute constants whose values may change in each occurrence, and we will write $A \lesssim B$ if there exists an absolute constant $M$ such that $A \leq MB$. We will say that two quantities $A$ and $B$ are equivalent if both $A \lesssim B$ and $B \lesssim A$, and, in that case, we will write $A \simeq B$.

2. Weights in $B^n$

Our approach to the study of weighted Besov spaces in $B^n$ uses their immersion in holomorphic spaces defined in $B^{n+1}$ via the natural projection $\Pi : B^{n+1} \to B^n$, given by $\Pi(z_1, \cdots, z_{n+1}) = (z_1, \cdots, z_n)$. It is then convenient to consider a pseudodistance in $B^n$ deduced from the hyperbolic pseudodistance in $S^{n+1}$. 
If \( z, w \in \overline{B}^n \), let
\[
\rho(z, w) = \inf_{\varphi, \theta \in [0, 2\pi]} |1 - z\overline{w} - \sqrt{1 - |z|^2 \overline{e}^\varphi} \sqrt{1 - |w|^2 e^{i\theta}}|
\]
\[
= \inf_{\theta \in [0, 2\pi]} |1 - z\overline{w} - \sqrt{1 - |z|^2} \sqrt{1 - |w|^2 e^{i\theta}}| = |1 - z\overline{w}| - \sqrt{1 - |z|^2} \sqrt{1 - |w|^2}.
\]

Observe that \( \rho(z, w) \) is just the infimum of the Korány pseudodistances of the antiimages by the mapping \( \Pi \) of the points \( z, w \).

Let us see a suggestive expression of \( \rho \) that has \( |1 - z\overline{w}| \) as a factor. Let \( P_a \) be the orthogonal projection of \( \mathbb{C}^n \) onto the subspace \([a]\) generated by \( a \) and \( Q_a = Id - P_a \) the projection onto the orthogonal complement of \([a]\). If \( a \in \mathbb{B}^n \), and \( \varphi_a \) is the automorphism in \( \mathbb{B}^n \) which interchanges \( a \) and 0, given by
\[
\varphi_a(z) = \frac{a - P_a(z) - (1 - |a|^2) \frac{1}{2} Q_a(z)}{1 - a},
\]
then (see for instance theorem 2.2.2 in [Ru])
\[
1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - a|^2}.
\]

This fact gives that \( \rho(z, w) \simeq |1 - z\overline{w}| |\varphi_z(w)|^2 \). Indeed,
\[
\rho(z, w) = |1 - z\overline{w}| - \sqrt{1 - |z|^2} \sqrt{1 - |w|^2}
\]
\[
= |1 - z\overline{w}| \left( 1 - \sqrt{1 - |z|^2} \frac{\sqrt{1 - |w|^2}}{|1 - z\overline{w}|} \right) \simeq |1 - z\overline{w}| |\varphi_z(w)|^2.
\]

In the following lemma we show that \( \rho \) is a pseudodistance whose balls \( U(z, R) \) are "equivalent", in a sense that we will precise, to polydisks of size \( R + R^\frac{n}{2} (1 - |z|^2)^\frac{1}{2} \) in the complex normal direction and of size \( R^\frac{n+1}{2} \) in the complex-tangential directions. In order to distinguish the Lebesgue measure in \( S^{n+1} \) and in \( \mathbb{B}^n \), we will write \( \nu(E) \) the volume measure of a measurable subset \( E \in \overline{B}^n \), whereas \( |F| \) will stand for the Lebesgue measure of a measurable subset \( F \subset S^{n+1} \).

**Lemma 2.1.**

(i) \( \rho \) is a pseudodistance in \( \overline{B}^n \).

(ii) Let \( z \in \overline{B}^n \), \( 0 < R < 1 \), and \( U(z, R) = \{ w \in \overline{B}^n ; \rho(z, w) < R \} \). Let \( P(z, R) \) be the polydisk in \( \overline{B}^n \) centered at \( z \), of size \( R + R^\frac{n}{2} (1 - |z|^2)^\frac{1}{2} \) in the complex normal direction and of size \( R^\frac{n+1}{2} \) in the complex-tangential directions. Then there exists \( C > 0 \) such that \( P(z, R) \subset U(z, R) \subset P(z, CR) \). In particular, \( \nu(U(z, R)) \simeq R^n (R + (1 - |z|^2)) \).

**Proof of lemma 2.1**

Let \( \theta \in [0, 2\pi) \). If \( z \in \mathbb{B}^n \), let \( \Pi^{-1}_\theta(z) = (z, \sqrt{1 - |z|^2} e^{i\theta}) \). We then have that for any \( \theta_0 \in [0, 2\pi) \), \( \rho(z, w) = \inf_{\theta} |1 - \Pi^{-1}_\theta(z) \Pi^{-1}_\theta(w)| \), expression from which we easily obtain that \( \rho \) is a pseudodistance. That gives (i).

Let us prove (ii). Let \( z \in \overline{B}^n \), and \( R > 0 \). A unitary change of variables gives that, without loss of generality, we may assume that \( z = (r, 0, \cdots, 0) \), \( 0 \leq r \leq 1 \).
We begin showing that $P(z, R) \subset U(z, CR)$, for some fixed constant $C > 0$. Let us consider first the case $R \leq 1 - r^2$. If $w = (w_1, \ldots, w_n) \in P(z, R)$, the definition of the polydisk gives that $|r - w_1| \leq R + R\frac{1}{2}(1 - r^2)^{\frac{1}{2}}$, and $|w_i| \leq R\frac{1}{2}$, $i = 2, \ldots, n$. Then

$$\rho(z, w) = |1 - r\overline{w_1}| - \sqrt{1 - r^2} \sqrt{1 - |w|^2} = \frac{|r - w_1|^2 + (|w_2|^2 + \cdots + |w_n|^2)(1 - r^2)}{|1 - r\overline{w_1}| + \sqrt{1 - |w|^2} \sqrt{1 - r^2}} \leq \frac{R^2 + R(1 - r^2) + R(1 - r^2)}{1 - r^2} \leq R.$$  

Assume now that $(1 - r^2) \leq R$. We have that

$$\rho(z, w) \leq |1 - r\overline{w_1}| \simeq (1 - r^2) + |r - w_1| \leq (1 - r^2) + R + R\frac{1}{2}(1 - r^2)^{\frac{1}{2}} \leq R.$$  

Hence in any case we have shown that $P(z, R) \subset U(z, CR)$.

Conversely, let $w \in B^n$ such that $\rho(z, w) < R$. The previous argument gives that

$$\rho(z, w) \simeq \frac{|r - w_1|^2 + (|w_2|^2 + \cdots + |w_n|^2)(1 - r^2)}{|1 - r\overline{w_1}|} \leq R.$$  

In particular,

$$(2.1) \quad \frac{|r - w_1|^2}{(1 - r^2) + |r - w_1|} \leq R.$$  

If $|r - w_1| \leq (1 - r^2)$, we have that $(1 - r^2) + |r - w_1| \simeq (1 - r^2)$, and we deduce from (2.1) that $|r - w_1| \leq R\frac{1}{2}(1 - r^2)^{\frac{1}{2}}$. If on the other hand, $(1 - r^2) \leq |r - w_1|$, (2.1) gives that $|r - w_1| \leq R$. Thus in any case we deduce that $|r - w_1| \leq R + R\frac{1}{2}(1 - r^2)^{\frac{1}{2}}$. In order to finish we have to check that $|w_i|^2 \leq R$, $i = 2, \ldots, n$. It is clear that this is the case if $|r - w_1| \leq (1 - r^2)$, since then $\frac{(|w_2|^2 + \cdots + |w_n|^2)(1 - r^2)}{1 - r^2} \leq R$. So we may assume that $(1 - r^2) \leq |r - w_1|$. Since we have shown that in that case $|r - w_1| \leq R$, we obtain

$$|w_2|^2 + \cdots + |w_n|^2 \leq 1 - |w_1|^2 \leq 1 - |w_1| \leq (1 - r^2) + |r - w_1| \leq |r - w_1| \leq R.$$  

The affirmation on the volume of the balls is obvious from the above.

Observe that if $0 < \varepsilon < 1$, and $U_{\varepsilon}(z) = U(z, \varepsilon(1 - |z|))$, we have that $U_{\varepsilon}(z)$ are contained and contain ellipsoids $E(z) = \{w \in B^n; |\varphi_z(w)| < \varepsilon', \varepsilon' < 1\}$, where $\varphi_z$ is the automorphism in $B^n$ that interchanges $z$ and $0$. This can be checked as follows: if $w \in U_{\varepsilon}(z)$, $\rho(z, w) < \varepsilon(1 - |z|)$, we have that $|1 - z\overline{w}|\varphi_z(w)|^2 \leq \varepsilon(1 - |z|)$, and consequently $|\varphi_z(w)| \leq \sqrt{\varepsilon}$. Conversely, if $|\varphi_z(w)| < \varepsilon^{\frac{1}{2}}$, we have that

$$|\varphi_z(z)|^2 = 1 - \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - z\overline{a}|^2} < \varepsilon.$$  

Thus,

$$\frac{|1 - z\overline{a}|^2}{(1 - |a|^2)(1 - |z|^2)} < \frac{1}{1 - \varepsilon},$$  

and

$$|1 - z\overline{a}|^2|\varphi_z(z)|^2 < \frac{\varepsilon}{1 - \varepsilon}(1 - |a|^2)(1 - |z|^2).$$
Lemma 2.3. In fact, this argument can be applied to nonisotropic balls

Hence

Definition 2.2. We say that a weight \( w \) is in \( A_p(B^n) \), \( 1 < p < +\infty \), if there exists \( C > 0 \) such that for any ball \( U = U(z, R) \) in \( \overline{B}^n \) associated to the pseudodistance \( \rho \),

In the following lemma, we will obtain a characterization of weights in \( A_p(B^n) \) in terms of their "lifting" to \( S^{n+1} \). We recall that a weight \( \eta \) in \( S^{n+1} \) is in \( A_p(S^{n+1}) \) if there exists \( C > 0 \) such that for any nonisotropic ball \( B(\zeta, R) = \{ z \in S^{n+1}; |1 - \zeta| < R \} \), \( \zeta \in S^{n+1}, R > 0 \),

Lemma 2.3. Let \( 1 < p < +\infty, n \geq 1 \), and \( w \) be a weight in \( B^n \). We then have that \( w \) is an \( A_p(B^n) \) weight if and only if the lifted weight \( \tilde{w}_l \) defined by \( \tilde{w}_l(z_1, \ldots, z_{n+1}) = w(z_1, \ldots, z_n) \) is an \( A_p \) weight in \( S^{n+1} \).

Proof of lemma 2.3:
We begin proving that if \( w \) is an \( A_p(B^n) \)-weight, then \( \tilde{w}_l \) is an \( A_p \) weight in \( S^{n+1} \). We recall that we have denoted by \( U \) the balls in \( B^n \) with respect to the pseudodistance \( \rho \), and we will denote by \( B(z, R) = \{ y \in S^{n+1}; |1 - \zeta y| < R \} \), the ball in \( S^{n+1} \) of center \( z = (z_1, \ldots, z_{n+1}) \in S^{n+1} \) and radius \( R \).

We consider first the particular case where \( z_{n+1} = 0 \), i.e. the center of the ball \( B(z, R) \) lies on \( \overline{B}^n \). By a suitable change of variables we may assume that \( z = (1, 0, \ldots, 0) \). Then \( \Pi^{-1}(U((1, 0, \ldots, 0), R)) = \{ y \in S^{n+1}; |1 - y| < R \} = B(z, R) \), and consequently,

and the same argument holds for \( w_l^{-1} \). These estimates, together with the fact that \( w \in A_p(B^n) \) and \( v(P(1, 0, \ldots, 0), R) \sim R^{n+1} \), give that

In fact, this argument can be applied to nonisotropic balls \( B(z, R) \) in \( S^{n+1} \) satisfying that \( d(z, S^n) \leq R \), where \( S^n \) is the boundary of \( \overline{B}^n \). We just have to observe that in this case the ball \( B(z, R) \) is included in a nonisotropic ball in \( S^{n+1} \) whose center lies in \( S^n \) and whose radius is comparable to \( R \).

So we may assume that \( R \leq d(z, S^n) \). Without loss of generality we also may assume that \( z = (r, 0, \ldots, 0, \sqrt{1 - r^2} e^{i\theta}) \), for some \( 0 < r < 1, \theta \in [0, 2\pi) \). The fact
that \( R \leq d(z, \mathbb{S}^n) \) gives that \( R \leq 1 - r^2 \), and consequently that \( v(U((r, 0, \ldots, 0), R)) \simeq R^n(1-r^2) \).

If we denote by \( T \) the unitary map in \( \mathbb{C}^{n+1} \) given by

\[
T(y_1, \ldots, y_{n+1}) = \begin{pmatrix}
    r & 0 & 0 & \ldots & -e^{-i\theta} \sqrt{1-r^2} \\
    0 & 1 & 0 & \ldots & 0 \\
    0 & 0 & 1 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \sqrt{1-r^2} e^{i\theta} & 0 & 0 & \ldots & r
\end{pmatrix} \begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n \\
    y_{n+1}
\end{pmatrix},
\]

we have that \( T(1, 0, \ldots, 0) = z \), and consequently, \( T(B((1, 0, \ldots, 0), R)) = B(z, R) \).

Since \( B((1, 0, \ldots, 0), R) = \{ (y, y_{n+1}) = (y_1, \ldots, y_n, \sqrt{1-|y|^2} e^{i\varphi}) : 0 \leq \varphi < 2\pi, |1-y_1| < R \} \), we obtain that

\[
\Pi(B(z, R)) = \Pi(T(B((1, 0, \ldots, 0), R)) \\
= \{ (ry_1 - e^{i(\varphi-\theta)} \sqrt{1-r^2} \sqrt{1-|y|^2}, y_2, \ldots, y_n) ; 0 \leq \varphi < 2\pi, |1-y_1| < R, |y_2|^2 + \cdots |y_n|^2 < 1 - |y_1|^2 \}.
\]

This description of the set \( \Pi(B(z, R)) \) gives that \( |ry_1 - e^{i(\varphi-\theta)} \sqrt{1-r^2} \sqrt{1-|y|^2} - r| \leq R + \sqrt{1-r^2} R^2 \), and \( |y_k| \leq R^2 \). Hence, Lemma 2.1 gives that

(2.2) \( \Pi(B(z, R)) \subset U((r, 0, \ldots, 0), CR) \).

On the other hand, let \( y = (y_1, \ldots, y_n) \in U((r, 0, \ldots, 0), CR) \), and let \( y^r = (y_1, \ldots, y_n, \sqrt{1-|y|^2} e^{r^2}) \) be a point in \( \Pi^{-1}(y) \). The definition of the pseudodistance \( \rho \) gives that there exists a point in \( \Pi^{-1}((r, 0, \ldots, 0)) \) at distance from the point \( y^r \) less than or equal to \( CR \). This holds if and only if

\[
CR \geq \min_{\varphi} |1-y_1 r - \sqrt{1-|y|^2} \sqrt{1-r^2} e^{i(\varphi)}| = \\
|1-y_1 r| - \sqrt{1-|y|^2} \sqrt{1-r^2} = \rho((r, 0, \ldots, 0), y).
\]

Summarizing, the projection of the points in \( S^{n+1} \) at distance less that \( R \) from the set \( \Pi^{-1}((r, 0, \ldots, 0)) \) is included in \( U((r, 0, \ldots, 0), CR) \), and on the other hand, the ball \( U((r, 0, \ldots, 0), R) \) is included in the projection of the set of points in \( S^{n+1} \) at distance from \( \Pi^{-1}((r, 0, \ldots, 0)) \) less than \( CR \).

Next, this set of points of \( S^{n+1} \) at distance less that \( R \) from the set \( \Pi^{-1}((r, 0, \ldots, 0)) \) is included in a union of balls of radious \( R \) in a number which is of the order of \( \frac{1-|y|^2}{R} \), and includes the same number of disjoint balls of radious comparable to \( R \).

The integral of the lifted weight \( w_l \) on each of this balls is equivalent.
Lemma 2.4. Assume that $v \in C^{n-1}\mathcal{C}$ with a similar relationship for $w$ with a similar estimate for $w$. Consequently, $w$ satisfies the argument we have used before shows that if $z \in B$, then $w$ gives then that for any $\theta \in U((r,0,\ldots,0),CR)$, such that $1 - |z|^2 \leq R$, we can reduce ourselves to the case where the point $z$ is in $S$. Then $U(z,R)$ is a tent centered at a point $z$ in $S$, and $U(z,R) = B((z,0),R)$. Consequently, $w$ satisfies the $A_p(B^n)$ condition for these class of balls.

If $R < 1 - |z|^2$, $v(U(z,R)) \simeq R^n(1 - |z|^2)$, and again the argument used before gives then that for any $\theta$,

$$
\frac{1}{R^n(1 - |z|^2)} \int_{U(z,R)} w d\sigma \simeq \frac{1}{R^n(1 - |z|^2)} \int_{U(U(z),R)} w d\sigma
$$

$$
\simeq \frac{1}{R^n(1 - |z|^2)} \frac{1}{R} \int_{B((z,\sqrt{1 - |z|^2}e^{i\varphi}),CR)} w d\sigma = \frac{1}{R^n+1} \int_{B((z,\sqrt{1 - |z|^2}e^{i\varphi}),CR)} w d\sigma,
$$

with a similar relationship for $w$. Since $w$ is an $A_p$ weight in $S^{n+1}$, we are done.

The following result gives examples of $A_p(B^n)$ weights obtained from weights in $S$.

Lemma 2.4. Assume $w \in A_p(S^n)$. Then the weight defined by

$$
\bar{w}(z) = \frac{1}{(1 - |z|^2)^n} \int_{I_z} w(\zeta)d\sigma(\zeta),
$$

$z \in B^n$, where $I_z = \{\zeta \in S^n; |1 - \frac{z}{|z|^2}\zeta| \leq c(1 - |z|^2)\}$, $c > 0$, is in $A_p(B^n)$.

Proof of lemma 2.4.

We want to show that there exists $C > 0$ such that if $a = (a_1, \ldots, a_n)$, and $U = U(a,R) = \{\eta \in B^n; \rho(\eta,a) < R\}$, then

$$
\frac{1}{v(U)} \int_U \bar{w}(z)dv(z) \left( \frac{1}{v(U)} \int_U w^{(p'-1)}(z)dv(z) \right)^{\frac{1}{p'-1}} \leq C.
$$
As in the previous lemma, assume first that $1 - |a| \leq \frac{1}{2} R, \delta > 0$ to be chosen. We can reduce this case to the one where $a = (a_1, 0, \ldots, 0) \in S^n$. We then have that if $z \in U$ and $\zeta \in I_z$, then $\zeta \in U(a, CR)$, for some fixed constant $C > 0$. Indeed, the fact that $z = (z_1, \cdots, z_n) \in U(a, R)$ gives (see lemma 2.1) that $\sum_{i=2}^n |z_i|^2 \leq R$ and $||z_1|^2 - |a_1|^2| \leq (R + R^2(1 - |a_1|^2)^{1/2}) \approx R$. In particular, we deduce that $1 - |z| \leq R$, and $|1 - \zeta w| = \rho(\zeta, a) \leq \rho(\zeta, z) + \rho(z, a) \leq R$.

Next, Fubini’s theorem gives that if $D_\alpha(\zeta) = \{z \in B^n; |1 - z\overline{\zeta}| < \alpha(1 - |z|^2)\}$,

$$\frac{1}{v(U)} \int_U \tilde{w}(z)dv(z) = \frac{1}{v(U)} \int_U (1 - |z|)^n \int_{I_z} w(\zeta)d\sigma(\zeta)dv(z) \lesssim \frac{1}{R^n+1} \int_{B(a,CR)} w(\zeta)d\sigma(\zeta)$$

where $D_\alpha(\zeta) = \{z \in B^n; |1 - z\overline{\zeta}| < c(1 - |z|^2)\}$, and we have used that if $\zeta \in I_z$, $|1 - z\overline{\zeta}| \leq (1 - |z|)$. An analogous argument to the one we have used in last lemma applied to $\tilde{w}^{n+1}$ finishes this case.

If $R \leq \delta(1 - |a|)$, we have that for any $z \in U$, $\tilde{w}(z) \simeq \tilde{w}(a)$, and consequently, the $A_p$ condition in this case is obvious.

**Remark 2.5.** This type of weights $\tilde{w}$ appear in a natural way if one identifies weighted holomorphic Besov spaces with weighted holomorphic Triebel-Lizorkin spaces $HF^{pp}_p$, $p = q$. We recall, that if $s \geq 0, [s]^+$ denotes the integer part of $s$ plus 1 and $1 < p < +\infty$, the weighted holomorphic Triebel-Lizorkin space $HF^{pp}_s(w)$ is the space of holomorphic functions $f$ in $B^n$ for which

$$||f||_{HF^{pp}_s(w)} = \left( \int_{B^n} \left| (I + R)^{k}f(z) \right|^q (1 - |z|^2)^{(k-s)p-n-1} dv(z) \right)^{1/q} < +\infty,$$

where $I$ denotes the identity operator. Fubini’s theorem gives then that

$$||f||_{HF^{pp}_s(w)} \approx \int_{B^n} \left| (I + R)^{k}f(z) \right|^q (1 - |z|^2)^{(k-s)p-n-1} \tilde{w}(z) dv(z).$$

**Definition 2.6.** A weight $w$ in $S_0^{n+1}$ is in $D_\tau(S^{n+1})$ if there exists $C > 0$ such that for any $B(\zeta, R) = \{ \eta \in S^{n+1}; |1 - \eta\overline{\zeta}| \leq R \}$, $\zeta \in S^{n+1}$, $R > 0$, and $j \geq 0$, $w(B(\zeta, 2^j R)) \leq C 2^{j\tau} w(B(\zeta, R)).$

Observe that any doubling weight $w$ in $S_0^{n+1}$, i.e. a weight $w$ there exists $k > 0$ such that $w(B(\zeta, 2R)) \leq kw(B(\zeta, R))$, is in $D_\tau(S^{n+1})$ for $\tau = \frac{\log k}{\log 2}$. So the fact that a weight is in $D_\tau(S^{n+1})$ is related to the size of the doubling constant $k$.

In the proof of lemma 2.3 we have seen in fact that if $w$ is a weight in $B^n$ and $w_l$ is the corresponding lifted weight in $S^{n+1}$, then $w_l(B(z, R)) \simeq \frac{R}{R + (1 - |z|)} w_l(U(z', R))$, where $z = (z', z_{n+1}) \in S^{n+1}$. It is then natural to define a weight $w$ in $D_\tau(B^n)$ as follows.
The other implication is proved in a similar way.

\[ w(U(z, 2^k R)) \leq C \frac{(1 - |z|^2) + 2^k R}{(1 - |z|^2) + R} 2^{k(\tau - 1)} w(U(z, R)). \]

Since \( v(U(z, R)) \simeq R^n (R + (1 - |z|^2)) \), this condition \( D_\tau (B^n) \) can be rewritten as

\[ \frac{w(U(z, 2^j R))}{v(U(z, 2^j R))} \leq C \frac{2^{j\tau} w(U(z, R))}{2^{(n+1)j} v(U(z, R))}. \]

In many occasions, it happens that in a natural way condition (2.3) is only satisfied for those integers \( k \) such that the ball \( U(z, 2^k R) \) touches the boundary of \( B^n \). We then have the following definition:

**Definition 2.7.** We say that a weight \( w \) in \( B^n \) is in \( D_\tau (B^n) \) for some \( \tau \), if there exists \( C > 0 \), such that for any \( k \geq 1 \), \( z \in B^n \) and \( R > 0 \),

\[ w(U(z, 2^k R)) \leq C \frac{(1 - |z|^2) + 2^k R}{(1 - |z|^2) + R} 2^{k(\tau - 1)} w(U(z, R)). \]

Since \( v(U(z, R)) \simeq R^n (R + (1 - |z|^2)) \), this condition \( D_\tau (B^n) \) can be rewritten as

\[ \frac{w(U(z, 2^j R))}{v(U(z, 2^j R))} \leq C \frac{2^{j\tau} w(U(z, R))}{2^{(n+1)j} v(U(z, R))}. \]

**Definition 2.8.** We say that a weight \( w \) in \( B^n \) is in \( d_\tau (B^n) \) for some \( \tau \), if there exists \( C > 0 \), such that for any \( k \geq 1 \), \( z \in B^n \) and \( R > 0 \) satisfying that \( U(z, 2^k R) \) touches \( S^n \),

\[ w(U(z, 2^k R)) \leq C \frac{R}{(1 - |z|^2) + R} 2^{k\tau} w(U(z, R)). \]

As we have already observed, we have the following lemma.

**Lemma 2.9.** A weight \( w \) is in \( D_\tau (B^n) \), \( \tau > 0 \), if and only if the lifted weight \( w_l \) is in \( D_\tau (S^{n+1}) \).

**Proof of lemma 2.9**

Let \( z = (z', z_{n+1}) \in S^{n+1}, R > 0 \) and \( j \geq 1 \). If \( 1 - |z'|^2 \leq R \),

\[ w_l(B(z, 2^j R)) \simeq w(U(z', 2^j R)) \simeq 2^{j\tau} w(U(z', R)) \simeq 2^{j\tau} w_l(B(z, R)). \]

If \( 1 - |z'|^2 > 2^j R \), we have that (see the proof of lemma 2.3) \( w_l(B(z, 2^j R)) \simeq 2^{j\tau} \frac{R}{1 - |z'|^2} w(U(z', 2^j R)), \) and \( w_l(B(z, R)) \simeq \frac{R}{1 - |z'|^2} w(U(z', R)). \) Hence,

\[ w_l(B(z, 2^j R)) \simeq 2^j R \frac{2^{j\tau} w(U(z', 2^j R))}{1 - |z'|^2} w(U(z', 2^j R)) \]

\[ \simeq 2^j R \frac{2^{j\tau} w(U(z', R))}{1 - |z'|^2} 2^{j\tau} w(U(z', R)) \simeq 2^{j\tau} w_l(B(z, R)). \]

If \( 1 - |z'|^2 > R \), and \( 1 - |z'|^2 \leq 2^j R \), we have that (see Lemma 2.3) \( w_l(B(z, 2^j R)) \simeq w(U(z', 2^j R)) \), and \( w_l(B(z, R)) \simeq \frac{R}{1 - |z'|^2} w(U(z', R)). \) Hence

\[ w_l(B(z, 2^j R)) \simeq w(U(z', 2^j R)) \simeq 2^j R \frac{2^{j\tau} w(U(z', R))}{1 - |z'|^2} 2^{j\tau} w(U(z', R)) \simeq 2^{j\tau} w_l(B(z, R)). \]

The other implication is proved in a similar way.

Observe that if \( w \equiv 1 \), i.e., if \( w \) is Lebesgue measure on \( B^n \), then \( w \in D_\tau, \tau = n+1 \). The following simple lemma will show that without loss of generality we always may assume that \( \tau \geq n + 1 \).
Lemma 2.10. If $w$ is a non identically zero weight in $L^1(B^n)$ which is in $d_\tau$, then $\tau \geq n + 1$.

Proof of lemma 2.10:
Assume $\tau < n + 1$, and let $U \subset B^n$ a ball in $B^n$ that touches $S^n$. The doubling condition on $w$ gives that for any $k \geq 1$, $w(U) \leq 2^{k\tau}w(2^{-k}U)$. Consequently,

\[(2.5) \quad 2^{k(n+1-\tau)}w(U) \leq \frac{w(2^{-k}U)}{2^{-k(n+1)}}.\]

The differentiation theorem (see for instance Theorem 5.3.1 in [Ru]) gives that for almost $z \in B^n$, $\lim_{k \to +\infty} w(2^{-k}U) \approx 1$. Since we are assuming that $\tau < n + 1$, this gives a contradiction with (2.5).

\[\square\]

Definition 2.11. We say that a weight $\omega$ is in $B_p(B^n)$ (see [Be]) if there exists $C > 0$ such that for any ball $U(z, R)$ that touches $S^n$, i.e., $U(z, R) \cap S^n \neq \emptyset$,

\[\frac{1}{v(U(z, R))} \int_{U(z, R)} \omega dv \left( \frac{1}{v(U(z, R))} \int_{U(z, R)} \omega^{-(p'-1)} dv \right)^{p-1} \leq C.\]

Obviously, any $A_p(B^n)$ weight satisfies the condition $B_p(B^n)$.

We next observe that any weight $w \in B^p(B^n)$ is in $d_\tau$ for $\tau = p(n+1)$.

Lemma 2.12. Let $1 < p < +\infty$ and $w \in B^p(B^n)$. There exists $C > 0$, such that for any $k \geq 1$, $z \in B^n$ and $R > 0$ satisfying that $U(z, 2^k R)$ touches $S^n$,

\[w(U(z, 2^k R)) \leq C \left( \frac{R}{1 - |z|^2 + R} \right)^p 2^{k\tau} w(U(z, R)) \leq C \frac{R}{1 - |z|^2 + R} 2^{k\tau} w(U(z, R)),\]

where $\tau = p(n+1)$.

Proof of lemma 2.12:
Let $z \in B^n$, $R > 0$, $k \geq 1$ such that $U(z, 2^k R)$ touches $S^n$. We then have that by Hölder’s inequality,

\[\left( \int_{U(z, 2^k R)} \omega(y) dv(y) \right)^{\frac{1}{p}} \left( \int_{U(z, 2^k R)} \omega^{-(p'-1)}(y) dv(y) \right)^{\frac{p-1}{p}} \leq \left( \int_{U(z, R)} \omega(y) dv(y) \right)^{\frac{1}{p}} \left( \int_{U(z, 2^k R)} \omega(y) dv(y) \right)^{\frac{p-1}{p}} \leq \left( \int_{U(z, R)} \omega(y) dv(y) \right)^{\frac{1}{p}} \left( \int_{U(z, 2^k R)} \omega(y) dv(y) \right)^{\frac{p-1}{p}} \leq \left( \int_{U(z, R)} \omega(y) dv(y) \right)^{\frac{1}{p}} \left( \int_{U(z, 2^k R)} \omega(y) dv(y) \right)^{\frac{p-1}{p}} v(U(z, 2^k R)),\]

where in last inequality we have used the fact that $w \in B_p(B^n)$. Thus we deduce that

\[(2.6) \quad w(U(z, 2^k R)) \leq C \left( \frac{v(U(z, 2^k R))}{v(U(z, R))} \right)^p w(U(z, R)).\]

But lemma 2.1 together with the fact that $U(z, 2^k R)$ touches $S^n$, gives that

\[\frac{v(U(z, 2^k R))}{v(U(z, R))} \simeq \frac{(2^k R)^n(2^k R + (1 - |z|^2))}{R^n(R + (1 - |z|^2))} \leq \frac{(2^k R)^{n+1}}{R^n(R + (1 - |z|^2))^2} \leq \frac{2^{k(n+1)}}{R + (1 - |z|^2)}.\]
Plugging the above inequality in (2.6), we deduce that

\[ w(U(z, 2^k R)) \leq C \left( \frac{R}{(1 - |z|^2) + R} \right)^\tau 2^{k\tau} w(U(z, R)), \]

with \( \tau = (n + 1)p. \)

As a consequence of last lemma, we have an equivalent definition of \( B_p(B^n) \) weights which coincides with the weights in \( B_p(B^n) \) introduced in [Be]: a weight \( w \) is in \( B_p(B^n) \) if there exists \( C > 0 \) such that for any tent \( T(\zeta, R) = \{ z \in B^n : |1 - \zeta| \leq R \}, \zeta \in S^n, \)

\[
\frac{1}{v(T(\zeta, R))} \int_{T(\zeta, R)} \omega dv \left( \frac{1}{v(T(\zeta, R))} \int_{T(\zeta, R)} \omega^{-(p'-1)} dv \right)^{p'-1} \leq C.
\]

This observation is a consequence of the fact that if a ball \( U(\zeta, R) \) touches \( S^n, \) then it is included in a tent of radius comparable to \( R, \) and conversely, a tent of radius \( R \) is included in a ball that touches \( S^n \) of comparable radius.

The weights in \( B_p(B^n) \) are characterized as the ones for which the Bergman projector \( B \) given by

\[ Bf(z) = \int_{B^n} \frac{f(y)}{(1 - z\bar{y})} dv(y), \]

is a continuous operator from \( L^p(w) \) to itself (see [Be]).

Let’s give some examples of weights in \( B_p(B^n) \) or in \( d_\tau(B^n). \)

**Proposition 2.13.** Let \( 1 < p < +\infty, \) and let \( \varphi : (0, 1] \to \mathbb{R} \) be a nonnegative monotone function, \( C > 0, \alpha > 0, \) satisfying one of the following alternative assumptions:

- (i) \( \varphi \) is nondecreasing, and \( \varphi(2^k x) \leq C 2^{\alpha k} \varphi(x) . \)
- (ii) \( \varphi \) is nonincreasing, and \( \varphi(x) \leq C 2^{\alpha k} \varphi(2^k x) . \)

Let \( w_\varphi(z) = \varphi(1 - |z|) . \) We then have:

- (a) The weight \( w_\varphi \) is in \( B_p(B^n) \) if and only if \( 0 < \alpha < p - 1 \) if \( \varphi \) is in case (i) or \( 0 < \alpha < 1 \) if it is in case (ii).
- (b) The weight \( w_\varphi \) is in \( d_\tau \) if \( \varphi \) is case (i) and \( \tau > n + \alpha + 1 \) or \( \varphi \) is in case (ii) and \( \tau \geq n + 1 . \)

**Proof of proposition 2.13:**

Fubini’s theorem gives that

\[ (2.7) \]

\[ w_\varphi(U(z, R)) = \int_{U(z, R)} \varphi(1 - |y|) dv(y) \simeq R^{n-1}(R + R^{d}(1 - |z|)\frac{1}{2}) \int_{(1 - |z|)}^{(1 - |z|) + (R + R^{d}(1 - |z|)\frac{1}{2})} \varphi(t) dt. \]

We will show that if \( \varphi \) satisfies the hypothesis in (a), then the weight \( w_\varphi \) is in \( B_p(B^n) \). If \( T(z, R) \) is a tent, (2.7) gives that \( w_\varphi(T(z, R)) \simeq R^n \int_0^R \varphi(t) dt, \) and consequently, it is enough to show that

\[
\frac{1}{R} \int_0^R \varphi(t) dt \left( \frac{1}{R} \int_0^R \varphi^{-(p'-1)}(t) dt \right)^{\frac{1}{p'-1}} \leq C.
\]
Assume first that \( \varphi \) is nondecreasing. We then have that for any \( R > 0 \),
\[
\varphi \left( \frac{R}{2} \right) R \leq \int_{\frac{R}{2}}^{R} \varphi(t) dt \leq \int_{0}^{R} \varphi(t) dt \leq R \varphi(R).
\]
Since \( \varphi(2x) \simeq \varphi(x) \), we obtain that \( \int_{0}^{R} \varphi(t) dt \simeq R \varphi(R) \).

Next,
\[
\int_{0}^{R} \varphi^-(p'-1)(t) dt = \sum_{k=0}^{+\infty} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \varphi^-(p'-1)(t) dt \simeq \sum_{k=0}^{+\infty} 2^{-k} R \varphi \left( \frac{R}{2^n} \right)^{-(p'-1)}.
\]
Thus, it suffices to check that
\[
\sum_{k=0}^{+\infty} 2^{-k} \varphi \left( \frac{R}{2^n} \right)^{-(p'-1)} \lesssim \frac{1}{\varphi(R)^{p'-1}}.
\]
If we denote \( a_{k+m} = 2^{-k} \varphi \left( \frac{R}{2^n} \right)^{-(p'-1)} \), the above is equivalent to show that
\[
\sum_{k=0}^{+\infty} a_{m+k} \lesssim a_m,
\]
for any \( m \geq 0 \). But such condition it turns to be equivalent to
\[
a_{m+k} \lesssim \frac{a_m}{(1+\delta)^k},
\]
for some \( \delta > 0 \) (see for example subsection 5.4 in [ArRoSa1]). And that is a restatement of the doubling condition satisfied by \( \varphi \).

If \( \varphi \) is nonincreasing and such that \( \varphi(2x) \simeq \varphi(x) \), \( \varphi^{-(p'-1)} \) is nondecreasing and satisfies that \( \varphi^{-(p'-1)}(2x) \simeq \varphi^{-(p'-1)}(x) \). Hence, the above argument shows that if
\[
\varphi^{-(p'-1)}(2x) \lesssim (2-\varepsilon)^k \varphi^{-(p'-1)}(x),
\]
i.e., if \( \varphi(x) \lesssim (2-\varepsilon)^k \varphi(2x) \) then \( w_{\varphi^{-(p'-1)}}(z) = \varphi(1-|z|)^{-(p'-1)} \in B_{p'}(B^n) \). And that is equivalent to say that \( w \in B_p(B^n) \).

The other implication is proved in a similar way.

We now prove (b). We want to show that under the conditions in (b), \( w_\varphi \in d_\varphi(B^n) \), that is
\[
w_{\varphi}(U(z, 2^j R)) \leq C \frac{(1-|z|^2) + 2^j R 2^{j(p'-1)} w_{\varphi}(U(z, R))}{1-|z|^2 + R},
\]
for any \( U(z, R) \) and \( j \geq 0 \) such that \( U(z, 2^j R) \) touches \( S^n \).

We first recall that if \( M \leq \frac{x}{2} \), we have that for any \( t \in [x-M, x+M] \), \( \frac{x}{2} \leq t \leq \frac{3x}{2} \), and consequently, \( \int_{x-M}^{x+M} \varphi(t) dt \simeq \varphi(x) M \). If on contrary, \( M \geq \frac{x}{2} \), then
\[
\int_{x-M}^{x+M} \varphi(t) dt \simeq \varphi(M).
\]
Let \( z \in B^n \), \( R > 0 \) and \( j \geq 1 \). We consider two different possibilities:

1. \( \frac{(1-|z|^2)}{2} \leq R \).
2. \( R \leq \frac{(1-|z|^2)}{2} \leq 2^j R \).
If \( \frac{1 - |z|^2}{2} \leq R \), (2.7) and the above considerations give easily that \( w_\varphi(U(z, 2^j R)) \simeq (2^j R)^{n+1} \varphi(2^j R) \), and \( w_\varphi(U(z, R)) \simeq R^{n+1} \varphi(R) \). Thus the condition \( d_\tau(B^n) \) is fulfilled provided

\[
(2^j R)^{(n+1)} \varphi(2^j R) \preceq 2^j 2^{j(\tau-1)} R^{n+1} \varphi(R),
\]

condition that is equivalent to

\[
\varphi(2^j R) \preceq 2^{j(\tau-n-1)} \varphi(R).
\]

And the conditions on \( \tau \) and \( \varphi \) give that this estimate is satisfied.

If \( R \leq \frac{1 - |z|^2}{2} \leq 2^j R \). An argument analogous to case (1), gives now that \( w_\varphi(U(z, R)) \simeq R^n(1 - |z|^2)\varphi(1 - |z|) \). Hence it is enough to check in this case that

\[
(2^j R)^{(n+1)} \varphi(2^j R) \preceq \frac{2^j R}{(1 - |z|^2)} 2^{j(\tau-1)} R^n(1 - |z|^2) \varphi(1 - |z|),
\]

i.e., \( \varphi(2^j R) \preceq 2^{j(\tau-n-1)} \varphi(1 - |z|) \). And this estimate is a consequence on the hypothesis on \( \varphi \).

\[\square\]

**Corollary 2.14.** If \( w_\alpha(z) = (1 - |z|)^\alpha \), \(-1 < \alpha < p - 1\), then the weight \( w_\alpha \) is in \( B_p(B^n) \). If \( 0 \leq \alpha < p - 1 \) and \( \tau = n + \alpha + 1 \), or if \(-1 < p < 0 \) and \( \tau = n + 1 \), then \( w_\alpha \in d_\tau(B^n) \).

The techniques we will apply in order to work with weighted holomorphic Besov spaces in \( B^n \) require that the weights \( w \) are in \( A_p(B^n) \cap D_\tau(B^n) \). The purpose of the following pair of technical results is to show that we can in fact weaken these conditions and impose that the weight \( w \) is in the bigger class \( B_p(B^n) \cap d_\tau(B^n) \). The way to achieve this is via the regularisations of the weights.

**Definition 2.15.** If \( w \) is a weight in \( B^n \), \( 0 < \varepsilon < 1 \) and \( U_\varepsilon(z) = U(z, \varepsilon(1 - |z|^2)) \), we define

\[
R_\varepsilon w(z) = \frac{1}{v(U_\varepsilon(z))} \int_{U_\varepsilon(z)} w(\eta)dv(\eta).
\]

**Remark 2.16.** As an immediate consequence of lemma 2.11 and lemma 2.12 we have that all the regularisations are equivalent, that is, if \( \varepsilon, \varepsilon' > 0 \), \( R_{\varepsilon'} w(z) \simeq R_{\varepsilon} w(z) \), for any \( z \in B^n \), with constants that do not depend on \( z \). We just have to observe that if \( \varepsilon > 0 \) is fixed, there exists \( C > \varepsilon \) such that \( U(z, C(1 - |z|)) \) touches \( S^n \). The fact that \( w \) satisfies a doubling condition gives that \( w(U_\varepsilon) \simeq w(U(z, C(1 - |z|^2))) \).

Observe that the regularisation of a weight \( w \) satisfies that \( R_{\varepsilon} (R_{\varepsilon'} w) \simeq R_{\varepsilon} w \).

It is worthwhile to recall that analogous regularisations where already considered among others by Be [Be] and Lu1 [Lu1], where the balls \( U_{\varepsilon}^d(z) = \{ \eta \in B^n : d(z, \zeta) < \varepsilon(1 - |z|^2) \} \) were defined with respect to the pseudodistance \( d(z, \zeta) = ||z| - ||\zeta|| + |1 - (z\zeta)/|z||\zeta||| \).

**Lemma 2.17.** Let \( 1 < p < +\infty \) and assume that \( w \) is a weight in \( B_p(B^n) \). Then the weight

\[
R_\varepsilon w(z) = \frac{1}{v(U_\varepsilon(z))} \int_{U_\varepsilon(z)} w(\eta)dv(\eta),
\]

is in \( A_p(B^n) \).
Proof of lemma 2.17:
We want to show that there exists $C > 0$ such that for any ball $U = U(a, R) = \{\eta \in B^n ; \rho(\eta, a) < R\}$ associated to the pseudodistance $\rho$, 

$$ \left( \frac{1}{v(U)} \int_U R_\varepsilon w dv \right) \left( \frac{1}{v(U)} \int_U (R_\varepsilon w)^{-(p'-1)} dv \right)^{\frac{1}{p'-1}} \leq C. $$

As we have already observed, without loss of generality we may assume that $\varepsilon > 0$ is small enough, since for every $\varepsilon, \varepsilon' > 0$, then $R_\varepsilon w \simeq R_{\varepsilon'} w$.

Suppose first that $\delta(1 - |a|^2) \leq R$, $\delta > 0$ to be chosen later on. In this case, Lemma 2.1 gives that $v(U) \simeq R^{n+1}$. Since we also have that in that case $U(a, R)$ is included in a ball in $B^n$ centered at a point in $S^n$ of radius comparable to $R$, we also may assume without loss of generality that $a \in S^n$, and that $U = U(a, CR), C > 0$. In particular we have that for any $\eta \in U(a, R)$, $1 - |\eta|^2 \simeq R$. Thus if $z \in U(a, CR)$ and $y \in U_\varepsilon(z)$, $y \in U(a, CR)$, and Fubini’s theorem gives that 

$$ \int_{U(a,CR)} R_\varepsilon w dv(z) \simeq \int_{U(a,CR)} \frac{1}{v(U)} \int_{U_\varepsilon(z)} w(y) dv(y) dv(z). $$

Next, if $y \in U_\varepsilon(z)$, and $\varepsilon > 0$ is small enough, there exists $\varepsilon' < 1$ such that $z \in U_\varepsilon'(y)$, and $1 - |z|^2 \simeq 1 - |y|^2$. Thus (2.8) is bounded by 

$$ \int_{U(a,CR)} \frac{1}{v(U)} \int_{U_\varepsilon'(y)} w(y) dv(y) dv(z) \simeq \int_{U(a,CR)} w(y) dv(y). $$

In order to estimate the integral involving $(R_\varepsilon w)^{-(p'-1)}$, we use the fact that $w \in B^p(B^n)$ and Hölder’s inequality to get that, 

$$ \frac{1}{v(U_\varepsilon(z))} \int_{U_\varepsilon(z)} w(y) dv(y) \left( \frac{1}{v(U_\varepsilon(z))} \int_{U_\varepsilon(z)} w^{-(p'-1)} dv(z) \right)^{\frac{1}{p'-1}} \simeq 1. $$

Consequently 

$$ R_\varepsilon w \simeq (R_\varepsilon w^{-(p'-1)})^{\frac{1}{p'-1}}. $$

This gives that 

$$ \frac{1}{v(U(a, CR))} \int_{U(a,CR)} (R_\varepsilon w(z))^{-(p'-1)} dv(z) \simeq \frac{1}{v(U(a, CR))} \int_{U(a,CR)} R_\varepsilon w^{-(p'-1)}(z) dv(z), $$

and the argument in (2.8) applied to $R_\varepsilon w^{-(p'-1)}$ together with the fact that $w \in B_p(B^n)$, gives that in case $1 - |a| \leq \frac{1}{\delta} R$, then 

$$ \frac{1}{v(U(a, R))} \int_{U(a, R)} R_\varepsilon w(z) dv(z) \left( \frac{1}{v(B(a, R))} \int_{B(a, R)} (R_\varepsilon w)^{-(p'-1)} dv(z) \right)^{\frac{1}{p'-1}} \leq C. $$

Assume next that $R \leq \delta(1 - |a|^2)$, and $\delta$ is small enough. In that case, for any $z \in U(a, R), 1 - |z|^2 \simeq 1 - |a|$, and consequently $R_\varepsilon w(z) \simeq R_\varepsilon w(a)$ for any $z \in U(a, R)$. 


This is a direct consequence of the fact that \( w \) is doubling and that if \( z \in U(a, R) \), there exists \( \varepsilon', \varepsilon'' > 0 \) such that \( U_\varepsilon(z) \subset U_{\varepsilon'}(a) \subset U_{\varepsilon''}(z) \).

\( \square \)

**Lemma 2.18.** If \( w \) is a doubling weight in \( B^n \), its regularisation \( R_\varepsilon w \) also satisfies a doubling condition.

**Proof of lemma 2.18:**
Assume that there exists \( C > 0 \) such that for any \( z \in B^n \), and \( R > 0 \),
\[
 w(U(z, 2R)) \leq Cw(U(z, R)).
\]

We want to check that \( R_\varepsilon w \) satisfies a similar condition. Let \( z \in B^n \) and \( R > 0 \), and assume first that \( R \leq \delta(1 - |z|^2) \), with \( \delta > 0 \) small enough so that for any \( y \in U(z, 2R) \), \( 1 - |y|^2 \approx 1 - |z|^2 \). We then have that
\[
 R_\varepsilon w(U(z, 2R)) = \int_{U(z, 2R)} R_\varepsilon w(\eta) dv(\eta) \approx R_\varepsilon w(z) v(U(z, 2R))
\]
\[
 \approx R_\varepsilon w(z) R^n(1 - |z|^2) \approx \int_{U(z, R)} R_\varepsilon w(\eta) dv(\eta).
\]

If \( \delta(1 - |z|) \leq R \), Fubini’s theorem and the fact that \( w \) satisfies a doubling condition give
\[
 R_\varepsilon w(U(z, 2R)) = \int_{U(z, 2R)} \frac{1}{(1 - |z|^2)^n+1} \int_{U_\varepsilon(z)} w(y) dv(\eta) dv(y) dv(z)
\]
\[
 \leq \int_{U(z, CR)} \frac{1}{(1 - |z|^2)^n+1} \int_{U_\varepsilon(y)} dv(\eta) dv(y) \leq \int_{U(z, R)} w(y) dv(y)
\]
\[
 \approx \int_{U(z, R)} w(y) \frac{1}{(1 - |z|^2)^n+1} \int_{U_\varepsilon(y)} dv(\eta) dv(y) \approx R_\varepsilon w(U(z, R)).
\]
\( \square \)

**Proposition 2.19.** Let \( 1 < p < +\infty \), \( \tau \geq n + 1 \), and assume that \( w \) is a weight satisfying that \( w \in d_\tau(B^n) \). Then the weight \( R_\varepsilon w \) is in \( D_\tau(B^n) \).

**Proof of proposition 2.19:**
The hypothesis on \( w \) gives that for any \( z \in B^n \), \( j \geq 0 \) and \( R > 0 \), such that \( U(z, 2^j R) \) touches \( S^n \), then
\[
 w(U(z, 2^j R)) \leq \frac{(1 - |z|^2) + 2^j R}{(1 - |z|^2) + R} 2^{j(\tau - 1)} w(U(z, R)).
\]

In order to check that \( R_\varepsilon w \in D_\tau(B^n) \), given \( z \in B^n \), \( j \geq 0 \) and \( R > 0 \), we will consider the following three possibilities:

(a) \( 2^{j-1} R \leq \delta(1 - |z|^2) \).

(b) \( R < \delta(1 - |z|^2) \leq 2^j R \).

(c) \( \delta(1 - |z|^2) < R \).

Here \( \delta > 0 \) is some fixed constant to be chosen later on.

We begin with case (a). Our first observation is that in that case \( R_\varepsilon w(y) \approx R_\varepsilon(z) \) for \( y \in U(z, 2^{j-1} R) \). This is an immediate consequence of the doubling condition on \( w \) and the fact that if \( y \in U(z, 2^{j-1} R) \), \( \rho(y, z) \ll 2^{j-1} R \), and hence \( 1 - |y|^2 \approx 1 - |z|^2 \).
Hence, by the preceding lemma,
\[ R_\epsilon w(U(z, 2^j R)) \simeq R_\epsilon w(U(z, 2^{j-1} R)) = \int_{U(z, 2^{j-1} R)} R_\epsilon w(\eta) d\nu(\eta) \simeq R_\epsilon w(z) v(U(z, 2^{j-1} R)) \]
\[ \simeq R_\epsilon w(z) (2^n R)^n (1 - |z|) \simeq 2^n \int_{U(z, R)} R_\epsilon w(\eta) d\nu(\eta) \leq 2^{j(\tau-1)} R_\epsilon w(U(z, R)), \]
where in last inequality we have used that \( \tau \geq n+1 \). This shows case (a). We consider now case (b). Let \( j_0 \geq 1 \) such that \( 2^{j_0-1} R \leq \delta (1 - |z|^2) < 2^{j_0} R \).

Fubini’s theorem, and the fact that \( w \in d_\tau(B^n) \) gives that
\begin{align}
\int_{U(z, 2^j R)} R_\epsilon w(y) d\nu(y) &\simeq \int_{U(z, C 2^j R)} \left( \frac{1}{1 - |y|^2} \right)^n \int_{U(y, \delta |y|^2)} w(\eta) d\nu(\eta) d\nu(y) \\
&\leq \int_{U(z, C 2^j R)} w(\eta) d\nu(\eta) \leq \frac{2^{j-j_0} R + (1 - |z|^2)}{2^{j_0} R + (1 - |z|^2)} 2^{(j - j_0)(\tau-1)} \int_{U(z, 2^{j_0} R)} w(\eta) d\nu(\eta).
\end{align}
Since \( R \leq \delta (1 - |z|^2) < 2^{j_0} R \), the argument established in case (a) gives that \( R_\epsilon w \) is "frozen" on \( U(z, \frac{R}{2}) \). This observation, together with the fact that \( R_\epsilon w \) satisfies a doubling condition, gives that
\[ \int_{U(z, R)} R_\epsilon w(\eta) d\nu(\eta) \simeq \int_{U(z, \frac{R}{2})} R_\epsilon w(\eta) d\nu(\eta) \simeq R_\epsilon w(z) v(U(z, \frac{R}{2})) \]
\[ \simeq R_\epsilon w(z) R_n^{n/2} (1 - \frac{R}{2}) \simeq \frac{R^{n/2}}{1 - \frac{R}{2}} \int_{U(z, 2^{j_0} R)} w(\eta) d\nu(\eta), \]
where in last estimate we have used that \( R \leq \delta (1 - |z|^2) \simeq 2^{j_0} R \). Consequently, if we plug the above calculation in (2.9), we deduce that in order to prove the doubling condition in case (b) it is enough to show that
\[ \frac{2^{j-j_0} R + (1 - |z|^2)}{2^{j_0} R + (1 - |z|^2)} 2^{(j - j_0)(\tau-1)} \left( 1 - |z|^2 \right)^{n+1} \leq \frac{2^j R + (1 - |z|^2)}{R + (1 - |z|^2)} 2^{j(\tau-1)}. \]
Now, the fact that we are in case (b) gives that \( 2^j R + (1 - |z|^2) \simeq 2^j R \), and \( R + (1 - |z|^2) \simeq (1 - |z|^2) \simeq 2^{j_0} R \), and consequently the above estimate can be rewritten equivalently as
\[ \frac{2^{j-j_0} R + (1 - |z|^2)}{2^{j_0} R} 2^{(j - j_0)(\tau-1)} \left( 2^{j_0} R \right)^{n+1} \left( 2^{j_0} R \right) = \frac{2^{j-j_0} R + (1 - |z|^2)}{2^{j_0} R} 2^{j_0(n+1-\tau)} 2^{-j} \leq 1. \]
But the fact that \( 2^{j-j_0} R + (1 - |z|^2) \leq 2^j R + (1 - |z|^2) \simeq 2^j R \) gives that the left hand side of the above is bounded from above by
\[ \frac{2^j R}{R} 2^{j_0(n+1-\tau)} 2^{-j} = C 2^{n+1-\tau} \leq 1, \]
since \( \tau \geq n+1 \).

We finally have to deal with case (c), i.e. the case where \( \delta (1 - |z|^2) \leq R \). We have that if \( y \in U(z, 2^{j-1} R) \), and \( \eta \in U(y, \delta (1 - |z|^2)) \), then \( \eta \in U(z, C 2^j R) \), and
consequently, Fubini’s theorem gives that
\begin{equation}
(2.10) \int_{U(z,2^iR)} R_z w(y)dv(y) \simeq \int_{U(z,2^{i-1}R)} \frac{1}{(1-|y|^2)^{n+1}} \int_{U(y,\delta(1-|y|^2))} w(\eta)dv(\eta)dv(y)
\end{equation}
\begin{equation}
(2.11) \lesssim \int_{U(z,2^iR)} w(\eta)dv(\eta) \lesssim 2^{i\tau} \int_{U(z,R)} w(\eta)dv(\eta),
\end{equation}
where we have used that \( w \in d_\tau \).

On the other hand, \( R_z w \) satisfies a doubling condition. Thus, if \( M > 0 \) is fixed, Fubini’s theorem gives that
\begin{equation}
(2.12) \int_{U(z,R)} w(\eta)dv(\eta) \simeq \int_{U(z,\delta_1R)} w(\eta)\frac{1}{(1-|\eta|^2)^{n+1}} \int_{U(\eta,\delta(1-|\eta|^2))} dv(\eta)dv(\eta)
\end{equation}
\begin{equation}
(2.13) \lesssim \int_{U(z,MR)} \frac{1}{(1-|y|^2)^{n+1}} \int_{U(y,\delta(1-|y|^2))} w(\eta)dv(\eta)dv(y)
\end{equation}
\begin{equation}
(2.14) \simeq \int_{U(z,R)} R_z w(y)dv(y) \simeq \int_{U(z,R)} R_z w(y)dv(y),
\end{equation}
where in the second estimate we have used that since \( \delta(1-|z|^2) \leq R \), then \( (1-|\eta|^2) \leq R \) for any \( \eta \in U(z,R) \), and that if \( M > 0 \) is big enough, then for any \( \eta \in U(z,R) \), \( U(\eta,\delta(1-|\eta|^2)) \subset U(z,MR) \).
\[ \square \]

**Remark 2.20.** We have shown that the regularisation \( R_z w \) of a weight \( w \) in \( B_p \) is in the smaller class \( A_p(B^n) \). In particular, \( R_z w \) satisfies a doubling condition and consequently, if \( w \in B_P(B^n) \cap d_\tau(B^n) \), then \( R_z w \in A_p(B^n) \cap D_\tau(B^n) \).

We will show next that the weights \( \tilde{w} \) introduced in lemma 2.21 that were obtained from weights in \( A_p(S^n) \) are in \( A_p(B^n) \cap d_{\tau+1}(B^n) \) if \( w \in D_\tau(S^n) \).

**Lemma 2.21.** Assume \( w \in A_p(S^n) \cap D_\tau(S^n), \tau \geq n \). Then the weight defined by
\[
\tilde{w}(z) = \frac{1}{(1-|z|^2)^n} \int_{I_z} w(\zeta)d\sigma(\zeta),
\]
\( z \in B^n \), is in \( A_p(B^n) \cap d_{\tau+1} \).

**Proof of lemma 2.21:**

By lemma 2.3 we know that \( \tilde{w} \in A_p(B^n) \), so we are left to show that \( \tilde{w} \in d_{\tau+1}(B^n) \). Let \( U(a,2^kR) \) be a ball in \( B^n \) that touches \( S^n \). We want to show that
\[
\tilde{w}(U(a,2^kR)) \lesssim \frac{2^kR}{(1-|z|^2) + R^{2\tau}} \tilde{w}(U(a,R)).
\]

Fubini’s theorem gives that
\[
\int_{U(a,2^kR)} \tilde{w}(z)dv(z) = \int_{U(a,2^kR)} \frac{1}{(1-|z|^2)^n} \int_{I_z} w(\zeta)d\sigma(\zeta)dv(z)
\]
\[
\simeq \int_{B(\frac{a}{|\zeta|},2^kR)} w(\zeta)d\sigma(\zeta) \int_{U(a,2^kR) \cap D_\alpha(\zeta)} \frac{dv(z)}{(1-|z|^2)^n} \simeq 2^kR \int_{B(\frac{a}{|\zeta|},2^kR)} w(\zeta)d\sigma(\zeta).
\]
Assume first that \( R > \delta(1 - |a|^2) \), \( \delta > 0 \) small enough. Then last argument can be applied to \( U(a, R) \), and we get

\[
\int_{U(a, R)} \tilde{w}(z) dv(z) \simeq R \int_{B(\frac{a}{|a|}, R)} w(\zeta) d\sigma(\zeta).
\]

Thus in that case, the doubling condition reduces to check that

\[
2^k R w(B(\frac{a}{|a|}, 2^k R)) \leq 2^{k\tau} R w(B(\frac{a}{|a|}, R)),
\]

which follows from the fact that \( w \in D_\tau \). If \( R < \delta(1 - |a|^2) \) and \( \delta \) is small enough, we have observed in previous lemmas that \( \tilde{w} \) is “frozen” in \( U(a, R) \). Consequently,

\[
\tilde{w}(U(a, R)) \simeq v(U(a, R)) \tilde{w}(a) \simeq R^n (1 - |a|^2) \frac{1}{(1 - |a|^2)^n} \int_{I_n} w(\zeta) d\sigma(\zeta).
\]

This observation, together with the fact that \( w \) is in \( D_\tau \) gives then that

\[
\tilde{w}(U(a, 2^k R)) \simeq 2^k R w(B(\frac{a}{|a|}, 2^k R)) \leq 2^k R \left( \frac{2^k R}{(1 - |a|^2)} \right)^\tau \frac{2^k R}{(1 - |a|^2)^2} w(B(\frac{a}{|a|}, (1 - |a|^2)))
\]

\[
\simeq 2^k R \left( \frac{2^k R}{(1 - |a|^2)} \right)^\tau \frac{(1 - |a|^2)^n}{R^n (1 - |a|^2)} \tilde{w}(U(a, R)) = 2^k R \left( \frac{R}{(1 - |a|^2)} \right)^{\tau - n} \tilde{w}(U(a, R)).
\]

Since in this case, \( \frac{R}{(1 - |a|^2)} \leq 1 \), we are done. \( \square \)

### 3. Weighted holomorphic Besov spaces

We now introduce the weighted holomorphic Besov spaces. Let \( w \) be an \( B_p \)-weight in \( B^n \), \( 1 < p < +\infty \), and \( s \in \mathbb{R} \). The space \( B_{s,k}^p(w, B^n) \) is the space of holomorphic functions in \( B^n \) for which

\[
\|f\|_{B_{s,k}^p(w, B^n)} = \int_{B^n} |(I + R)^k f(y)(1 - |y|^2)^{k_1-s-\frac{1}{n}} w(y) dv(y) < +\infty,
\]

where \( k \in \mathbb{Z}_+ \), \( k > s \). In fact, the definition of the weighted holomorphic Besov spaces does not depend on \( k > s \). This is the object of the following result.

**Theorem 3.1.** Let \( 1 < p < +\infty \), \( s \in \mathbb{R} \), \( k_1 > k_2 > s \), and \( w \) a \( B_p \)-weight in \( B^n \). We then have that the following are equivalent:

(i) \( f \in B_{s,k_1}^p(w, B^n) \).
(ii) \( f \in B_{s,k_2}^p(w, B^n) \).

**Proof of theorem 3.1**

Assume that (i) holds. The fact that \( f \in B_{s,k_1}^p(w, B^n) \) means that \( (I + R)^k_1 f(y)(1 - |y|^2)^{k_1-s-\frac{1}{n}} \) is in \( L^p(w dv) \). In particular, there exists \( p_1 < p \) such that \( (I + R)^k_1 f(y)(1 - |y|^2)^{k_1-s-\frac{1}{n}} \) is in \( L^{p_1}(dv) \) (see [CaOr3], Lemma 1.1). Consequently, the kernel \( c_N (1 - |z|^2)^N/(1 - \overline{z}y)^{n+1+N} \), for \( N \) big enough, and for an adequate constant \( c_N > 0 \), is a reproducing kernel for the function \( f \) and its derivatives. We then have

\[
(I + R)^k_2 f(y) = c_N \int_{B^n} (I + R)^k_1 f(z)(I + R_y)^{k_2-k_1} \frac{(1 - |z|^2)^N}{(1 - \overline{z}y)^{n+1+N}} dv(z),
\]
where the operator $(I+R)^{-m}$ has the following integral representation (see for instance [OrFa])

$$(I+R)^{-m}f(z) = \frac{1}{m!} \int_{0}^{1} \left( \log \frac{1}{r} \right)^{m-1} f(rz)dr.$$ 

Thus

$$|{(I+R)^{k_2} f(y)}| \leq \int_{B^n} \frac{|(I+R)^{k_1} f(z)|(1-|z|^2)^N}{|1-\bar{z}y|^{n+1+N+k_2-k_1}} dv(z),$$

and we have,

$$(3.1) \quad ||f||_{B^p_{k_1,k_2}(w;B^n)} = \sup_{||\psi||_{L^p(wdv)} \leq 1} \int_{B^n} |(I+R)^{k_2} f(y)(1-|y|^2)^{k_2-s-\frac{1}{p}} \psi(y)w(y)dv(y)|$$

$$\leq \sup_{||\psi||_{L^p(wdv)} \leq 1} \int_{B^n} \int_{B^n} |(I+R)^{k_1} f(z)|(1-|z|^2)^N(1-|y|^2)^{k_2-s-\frac{1}{p}} \psi(y)w(y)dv(y)dv(z).$$

We now check that the mapping

$$(3.2) \quad T_{N,k_1,k_2}(\psi) = \int_{B^n} \psi(y) (1-|z|^2)^{N-(k_1-s-\frac{1}{p})} (1-|y|^2)^{k_2-s-\frac{1}{p}} \frac{1}{|1-\bar{z}y|^{n+1+N+k_2-k_1}} w(y)dv(y),$$

is bounded from $L^p(wdv)$ to $L^p(w^{-(p'-1)})$. Indeed, if we denote $\alpha(y) = \psi(y)w(y)$, this holds if and only if the mapping

$$\alpha \rightarrow \int_{B^n} (1-|z|^2)^{N-(k_1-s-\frac{1}{p})} (1-|y|^2)^{k_2-s-\frac{1}{p}} \frac{1}{|1-\bar{z}y|^{n+1+N+k_2-k_1}} \alpha(y)dv(y),$$

is bounded from $L^p(w^{-(p'-1)})$ to itself.

Since

$$\frac{(1-|z|^2)^{N-(k_1-s-\frac{1}{p})} (1-|y|^2)^{k_2-s-\frac{1}{p}}}{|1-\bar{z}y|^{n+1+N+k_2-k_1}} \leq \frac{(1-|y|^2)^{k_2-s-\frac{1}{p}}}{|1-\bar{z}y|^{n+1+k_2-s-\frac{1}{p}}},$$

and $k_2-s-\frac{1}{p}>-1$, this is a consequence of proposition 2 in [Be], where it is shown that the operator

$$(3.3) \quad T_{k_2-s-\frac{1}{p}}(f) = \int_{B^n} \frac{(1-|y|^2)^{k_2-s-\frac{1}{p}} f(y)}{|1-\bar{z}y|^{n+1+k_2-s-\frac{1}{p}}} dv(y)$$

sends $L^p(w^{-(p'-1)})$ to itself, provided the weight $w^{-(p'-1)}$ is in the class $B_{p'}$. 

We then have from (3.1) that

$$\|f\|_{B^p_{s,k}(w, B^n)} \lesssim \sup_{\|\psi\|_{L^p'(wdv)} \leq 1} \int_{B^n} |(I + R)^{k_1} f(z)| |(1 - |z|^2)^{k_1-s} w(z) T_{N,k_1,k_2}(\psi)(z) w^{1/p} T_{N,k_1,k_2}(\psi)(z) dv(z)$$

$$\lesssim \|f\|_{B^p_{s,k}(w, B^n)} \sup_{\|\psi\|_{L^p'(wdv)} \leq 1} \|T_{N,k_1,k_2}(\psi)\|_{L^{p'}(wdv)}$$

$$\lesssim \|f\|_{B^p_{s,k}(w, B^n)} \sup_{\|\psi\|_{L^p'(wdv)} \leq 1} \|\psi\|_{L^{p'}(wdv)} = \|f\|_{B^p_{s,k}(w, B^n)}.$$

The other implication is proved in a similar way. □

By Theorem 3.1, the spaces $B^p_{s,k}(w, B^n)$ do not depend on $k > s$, and from now on we will denote them simply by $B^p_s(w, B^n)$. Observe that if $w \in A_p(S^n)$, and

$$\tilde{w}(z) = \frac{1}{(1 - |z|^2)^n} \int_{S^n} w(\zeta) d\sigma(\zeta),$$

we have that $B^p_s(\tilde{w}, B^n) = HF^p_s(w)$.

Our next result shows that the weighted Besov space associated to a weight in $B^p_s(B^n)$ coincides with the corresponding weighted space of its regularisation. In particular, we deduce that in the definition of $B^p_s(\tilde{w}, B^n)$ we can assume, without loss of generality that $w \in A_p(B^n)$.

**Proposition 3.2.** Let $1 < p < +\infty$, $s \in \mathbb{R}$, and assume that $w$ is a weight in $B_p(B^n)$. Then the spaces $B^p_s(w, B^n)$ and $B^p_s(Rw, B^n)$ coincide.

**Proof of proposition 3.2.**

Assume that $\varepsilon < 1$ and $z \in B^n$, an let $k > s$. The fact that $(I + R)^k f$ is holomorphic in $B^n$ and that $U_\varepsilon(z)$ is contained and contains an ellipsoid $E(z)$ in $B^n$ of the same size, gives immediately that

$$|(I + R)^k f(y)| \lesssim \frac{1}{U_\varepsilon(y)} \int_{U_\varepsilon(y)} |(I + R)^k f(z)| dv(z).$$

On the other hand, $(1 - |y|^2) \simeq (1 - |z|^2)$ for any $y \in U_\varepsilon(z)$. Hence

$$(1 - |y|^2)^{(k-s)p-1} |(I + R)^k f(y)| \lesssim \frac{1}{U_\varepsilon(y)} \int_{U_\varepsilon(y)} (1 - |z|^2)^{(k-s)p-1} |(I + R)^k f(z)| dv(z).$$

Consequently,

$$\|f\|_{B^p_s(Rw)} = \int_{B^n} |(I + R)^k f(y)|^p (1 - |y|^2)^{(k-s)p-1} Rw(y) dv(y)$$

$$\lesssim \int_{B^n} \left( \frac{1}{(1 - |y|^2)^{n+1}} \int_{U_\varepsilon(y)} (1 - |z|^2)^{(k-s)p-1} |(I + R)^k f(z)| dv(z) \right)^p Rw(y) dv(y)$$

$$\simeq \int_{B^n} R_w \left( (1 - |z|^2)^{(k-s)p-1} |(I + R)^k f(z)| \right)^p Rw(y) dv(y).$$

Since $w \in B_p(B^n)$,

$$\int_{B^n} (Rw g)^p Rw dv \lesssim \int_{B^n} g^p dv.$$
for any $g \geq 0$ (the proof is analogous to lemma 9 in [Be]), we deduce that the above is bounded by
\[ \int_{\mathbb{B}^n} \left( 1 - |y|^2 \right)^{(k-s)p-1} |(I + R)^k f(y)|^p w(y) dv(y) = C ||f||_{\mathcal{B}^p_{s}(w)}. \]

On the other hand, the fact that $|(I + R)^k f|^p$ is plurisubharmonic gives that
\[ |(I + R)^k f(y)|^p \leq \frac{1}{|U_{\varepsilon}(y)|} \int_{U_{\varepsilon}(y)} |(I + R)^k f(z)|^p dv(z). \]

Since in $U_{\varepsilon}(z)$, $(1 - |y|^2) \simeq (1 - |z|^2)$, we have
\[ \|f\|_{\mathcal{B}^p_{s}(w)} = \int_{\mathbb{B}^n} |(I + R)^k f(y)|^p (1 - |y|^{2(k-s)p-1}) w(y) dv(y) \leq \int_{\mathbb{B}^n} R_{\varepsilon} \left( (1 - |y|^{2(k-s)p-1}) (I + R)^k f \right)^p (y) w(y) dv(y) \]

Fubini’s theorem gives that there exists $\varepsilon' > 0$ such that for any $f, g \geq 0$,
\[ \int_{\mathbb{B}} |R_{\varepsilon} g dv \leq \int_{\mathbb{B}} g R_{\varepsilon} f dv. \]

Hence, the above is bounded by
\[ \int_{\mathbb{B}^n} (1 - |y|^2)^{(k-s)p-1} |(I + R)^k f(y)|^p R_{\varepsilon} w(y) dv(y) = C ||f||_{\mathcal{B}^p_{s}(R_{\varepsilon} w)}. \]

Our next goal is to study the relations between the weighted Besov spaces in $\mathbb{B}^n$ and the weighted Hardy-Sobolev spaces with respect to the lifted weight in $\mathbb{S}^{n+1}$. We will first show that the restriction operator maps $H^p_s(w_{l})$ onto $B_{s+\frac{1}{p}}(w, \mathbb{B}^n)$.

We begin with a weighted restriction theorem. In order to make clearer the notation, in what follows we will write $HF^p_w(w_l, B^{n+1})$ instead of $HF^p(w_l)$. We recall (see [CaOr3]) that $H^p_s(w_l, B^{n+1}) = H^2_s(w_l)$ and that if $q_0 \leq q_1 \leq +\infty$, $HF^q_w(w_l, B^{n+1}) \subset HF^q_{p_{hl}}(w_l, B^{n+1})$.

**Theorem 3.3.** Let $1 < p < +\infty, 1 \leq q < +\infty, w$ an $A_p$-weight in $\mathbb{B}^n$, and $s \in \mathbb{R}$. Then the restriction operator maps $HF^p_{s,q}(w_{l}, B^{n+1})$ to $B_{s+\frac{1}{p}}(w, \mathbb{B}^n)$.

**Proof of theorem 3.3**

Since if $q_0 \leq q_1 \leq +\infty$, $HF^p_{p_{hl}}(w_l, B^{n+1}) \subset HF^p_{p_{hl}}(w_l, B^{n+1})$, it is enough to show that
\[ HF^p_{p_{hl}}(w_l, B^{n+1}) |_{\mathbb{B}^n} \subset B_{s+\frac{1}{p}}(w, \mathbb{B}^n). \]

Let $f \in HF^p_{p_{hl}}(w_l, B^{n+1})$, and $k > s$. We have that if $N > 0$ is choosen big enough, the representation formula gives that
\[ (I + R)^k f(y) = C \int_{\mathbb{B}^{n+1}} (I + R)^k f(z) \frac{(1 - |z|^2)^N}{(1 - |y|^2)^{k+2+N}} dv(z). \]

Hence,
\[ ||f||_{B_{s+\frac{1}{p}}(w, \mathbb{B}^n)} \leq \int_{\mathbb{B}^n} \left| \int_{\mathbb{B}^{n+1}} (I + R)^k f(z) \frac{(1 - |z|^2)^N}{|1 - z|^2} dv(z) \right|^p \left( 1 - |y|^2 \right)^{(k-s)p} w(y) dv(y). \]
Next, duality gives that
\[ \|f\|_{L_p(w, B^n)} \leq \sup_{\|f\|_{L_p}} \int_{B^n} \left| (I + R)^k f(z) \right| \frac{(1 - |z|^2)^N}{|1 - \bar{z} y|^{n+2+N}} dv(z) (1 - |y|^2)^k \psi(y) w(y) dv(y) \]
\[ \leq \sup_{\|f\|_{L_p}} \int_{S^{n+1}} \int_{B^n} \int_0^1 \left| (I + R)^k f(r\zeta) \right| \frac{(1 - r^2)^N}{|1 - r\zeta y|^{n+2+N}} dr d\sigma(\zeta) \times (1 - |y|^2)^k \psi(y) w(y) dv(y) \]
\[ \leq \sup_{\|f\|_{L_p(w)}} \int_{S^{n+1}} \sup_{r < 1} \left( (I + R)^k f(r\zeta) \right) (1 - r^2)^k \psi(y) w(y) dv(y) d\sigma(\zeta). \]

If \( N > 0 \) is big enough, we have that \( \int_0^1 \frac{(1 - r^2)^{N+s-k}}{|1 - r\zeta y|^{n+2+N}} dr \lesssim \frac{1}{|1 - \zeta y|^{n+1+k-s}} \), and the above is bounded by
\[ (3.3) \]
\[ \sup_{\|f\|_{L_p(w)}} \int_{S^{n+1}} \sup_{r < 1} \left( (I + R)^k f(r\zeta) \right) (1 - r^2)^k \psi(y) w(y) dv(y) d\sigma(\zeta). \]

Next, if \( M > 0 \), let \( K_M \) be the operator given by
\[ K_M(\psi)(\zeta) = \int_{B^n} \psi(y) \frac{(1 - |y|^2)^M}{|1 - \zeta y|^{n+1+M}} w(y) dv(y), \]
for \( \psi \in L_p(wdv), \zeta \in B^{n+1} \). We then have that the following lemma holds.

**Lemma 3.4.** Let \( 1 < p < +\infty, M > 0 \). We then have that \( K_M \) is bounded as an operator from \( L^p(wdv) \) in \( B^n \) to \( L^p(w \omega^{1-(-p' - 1)}) \) in \( S^{n+1} \).

Postponing the proof of the lemma, let us finish the proof of the theorem. Applying Hölder’s inequality with exponent \( p \) to \((3.3)\), Lemma 3.4 gives that
\[ \|f\|_{B^p_{\omega}(w, B^n)} \leq \sup_{\|f\|_{L_p}} \int_{S^{n+1}} \sup_{r < 1} \left( (I + R)^k f(r\zeta) \right) (1 - r^2)^k \psi(y) w(y) dv(y) d\sigma(\zeta) \]
\[ \leq \left( \int_{S^{n+1}} \sup_{r < 1} \left( (I + R)^k f(r\zeta) \right) (1 - r^2)^k w(y) dv(y) d\sigma(\zeta) \right)^\frac{1}{p} \times \left( \int_{S^{n+1}} K_{-s}(\psi) w^{-(-p' - 1)}(\zeta) d\sigma(\zeta) \right)^{\frac{1}{p'}} \lesssim \|f\|_{H^p_{\omega}(w, B^n)}. \]

We now give the proof of Lemma 3.4.

**Proof of lemma 3.4.**
We observe that for every $\mathbf{\zeta}' \in \mathbb{B}^n$, $K_M(\psi)$ is constant on $\Pi^{-1}(\mathbf{\zeta}')$. Consequently,

$$
||K_M(\psi)||_{L^p(\omega_1'(\mathbf{\zeta}'-1)d\sigma)} = \int_{\mathbb{S}^{n+1}} K_M(\psi)^p(\zeta)w_{\mathbf{\zeta}}^{-1}(\mathbf{\zeta}'-1)d\sigma(\zeta)
$$

$$
= \int_{\mathbb{B}^n} K_M(\psi)^p(\zeta')w_{\mathbf{\zeta}}^{-1}(\mathbf{\zeta}'-1)(\mathbf{\zeta}'d\nu(\zeta') = ||K_M(\psi)||_{L^p(\omega_1'(\mathbf{\zeta}'-1)d\sigma)}.
$$

Hence, we have to show that there exists $C > 0$ such that for any $\psi \in L^p(\omega dv)$,

$$
(3.4) \quad ||K_M(\psi)||_{L^p(\omega_1'(\mathbf{\zeta}'-1)d\sigma)} \leq C||\psi||_{L^p(\omega dv)}.
$$

If we denote by $\psi_1 = \psi w$, we have that $\psi \in L^p(\omega dv)$ if and only if $\psi_1 \in L^p(\omega_1'(\mathbf{\zeta}'-1)d\nu)$. Thus if $T_M^*$ is the operator defined by

$$
T_M^*(\psi_1)(z) = \int_{\mathbb{B}^n} \psi_1(y)(1-|y|^2)^M|1-zy|^{n+1+M}d\nu(y),
$$

for $\psi_1 \in L^p(\omega_1'(\mathbf{\zeta}'-1)d\nu)$, $z \in \mathbb{B}^n$, (3.4) can be rewritten as

$$
(3.5) \quad ||T_M^*(\psi_1)||_{L^p(\omega_1'(\mathbf{\zeta}'-1)d\nu)} \leq C||\psi_1||_{L^p(\omega_1'(\mathbf{\zeta}'-1)d\nu)}.
$$

And this estimate is again a consequence of proposition 3 in [Be]. \hfill \Box

Now we prove an extension theorem for weighted holomorphic Besov spaces.

**Theorem 3.5.** Let $1 < p < +\infty$, $s \in \mathbb{R}$, $w$ an $A_p$-weight on $\mathbb{B}^n$. We then have that the extension operator $f \rightarrow f_1$, where $f_1(z', z_{n+1}) = f(z')$, if $(z', z_{n+1}) \in \mathbb{S}^{n+1}$, $z' \in \mathbb{B}^n$, maps boundedly $B^p_{s-1}(w, \mathbb{B}^n)$ to $HF^p_s(w, \mathbb{B}^n)$.

**Proof of theorem 3.5.**

Let $f \in B^p_{s-1}(w, \mathbb{B}^n)$. If we denote by $R_{n+1}$ the radial derivative in $\mathbb{B}^{n+1}$, then we have

$$(I + R_{n+1})f_i(z', z_{n+1}) = (I + \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} + z_{n+1} \frac{\partial}{\partial z_{n+1}})f_i(z', z_{n+1}) = (I + R)f(z'),$$

i.e., $(I + R_{n+1})f_i = ((I + R)f)_i$. Consequently, if $k > s$, and $N$ is chosen big enough, duality and the representation formula give

$$
||f_1||_{HF^p_s(w, \mathbb{B}^{n+1})} = \int_{\mathbb{S}^{n+1}} \int_0^1 (I + R_{n+1})^k f_i(r \zeta)(1-r)^{k-s-1}dr \left| w_i(\zeta)d\sigma(\zeta)ight|^p
$$

$$
\simeq \int_{\mathbb{B}^n} \int_0^1 (I + R)^k f(r \zeta')(1-r)^{k-s-1}dr \left| w(\zeta')d\nu(\zeta')ight|^p
$$

$$
= \sup_{||\psi||_{L^p(\omega_1')} \leq 1} | \int_{\mathbb{B}^n} \int_0^1 (I + R)^k f(r \zeta')(1-r)^{k-s-1}dr \left| \psi(\zeta')w(\zeta')d\nu(\zeta')ight| |
$$

$$
\leq \sup_{||\psi||_{L^p(\omega_1')} \leq 1} \int_{\mathbb{B}^n} \int_0^1 \frac{|(I + R)^k f(z)|(1-|z|^2)^N(1-r)^{k-s-1}dr}{|1-zr\zeta'|^{n+1+N}}d\nu(z)\psi(\zeta')w(\zeta')d\nu(\zeta')
$$

$$
\leq \sup_{||\psi||_{L^p(\omega_1')} \leq 1} \int_{\mathbb{B}^n} \int_0^1 \frac{|(I + R)^k f(z)|(1-|z|^2)^N}{|1-zr\zeta'|^{n+1+N-(k-s)}}d\nu(z)\psi(\zeta')w(\zeta')d\nu(\zeta').
$$
In order to finish the lemma, an analogous argument to the one used in the restriction theorem, gives that it is enough to show that the mapping defined by

$$\psi \mapsto \int_{\mathbb{B}^n} \psi(\zeta') (1 - |\zeta|^2)^{N - (k - s)} \left| \frac{1 - \overline{\zeta'} \zeta^{n+1} + N - (k - s)}{\zeta'} \right| w(\zeta') \, dv(\zeta'),$$

maps boundedly $L^p(wdV)$ to $L^{p'}(w^{-p' - 1})$, which is again a consequence of proposition 3 in \[Be\].

As an immediate consequence of the above two theorems, and the fact that we have the following relations among the Triebel-Lizorkin spaces

$$HF^{p_1}_s(w_1, \mathbb{B}^{n+1}) \subset HF^{p_2}_s(w_1, \mathbb{B}^{n+1}) \subset HF^{\infty}_s(w_1, \mathbb{B}^{n+1})$$

$$HF^{p}_s(w_1, \mathbb{B}^{n+1}) = H^2_s(w_1),$$

we obtain the following corollary.

**Corollary 3.6.** Let $1 < p < +\infty$, $s \in \mathbb{R}$, and $w$ a weight in $A_p(\mathbb{B}^n)$. Then the restriction operator from $H^p_s(w_1)$ to $B^{p}_{s-\frac{1}{p}}(w, \mathbb{B}^n)$ is onto.

4. **CARLESON MEASURES FOR WEIGHTED HOLOMORPHIC BESOVI SPACES**

In this section we will give a characterization of the Carleson measures for a class of weighted holomorphic Besov spaces. The main result in \[CaOr3\] shows that for the Carleson measures for a weighted Hardy-Sobolev space $H^p_s(w_1)$ for some range of $s_1$ and for a class of weights $w_1$ in $S^{n+1}$ coincide with the Carleson measures for the image of the space $L^p(w_1)$ under the operator $K_{s_1}$ of positive kernel given by

$$K_{s_1}[f](z) = \int_{\mathbb{B}^n} \frac{f(\zeta)}{|1 - \overline{\zeta} \zeta^{n+1-s_1}|} \, d\sigma(\zeta).$$

This last problem have been thoroughly studied, (see for instance \[SaWhZh\]).

In our next result, we will see that we have an analogous situation for Carleson measures for weighted holomorphic Besov spaces.

**Theorem 4.1.** Let $1 < p < +\infty$, $0 < s$, $w$ a weight in $B_p(w, \mathbb{B}^n) \cap d_{r+1}$, $0 \leq \tau - sp < 1$. Let $\mu$ be a positive Borel measure on $\mathbb{B}^n$. We then have that the following assertions are equivalent:

(i) There exists $C > 0$ such that for any $f \in B_p(w, \mathbb{B}^n)$,

$$||f||_{L^p(\mu)} \leq C ||f||_{B_p(w, \mathbb{B}^n)}.$$

(ii) There exists $C > 0$ such that for any $f \in L^p(wdV)$,

$$\left| \int_{\mathbb{B}^n} \frac{f(y)dv(y)}{(1 - \|y\|^{n+1-(s+\frac{1}{p})})} \right|_{L^p(\mu)} \leq C ||f||_{L^p(wdV)}.$$

(iii) There exists $C > 0$ such that for any $f \in L^p(wdV)$,

$$\left| \int_{\mathbb{B}^n} \frac{f(y)dv(y)}{|1 - \overline{y} \cdot \zeta|^{n+1-(s+\frac{1}{p})}} \right|_{L^p(\mu)} \leq C ||f||_{L^p(wdV)}.$$
Proof of theorem 4.1

We begin by recalling that in previous sections we have proved that the regularisation of \( w \) is in \( A_p(\mathbb{B}^n) \cap D_{r+1} \) and that the corresponding Besov spaces coincide. Thus, without loss of generality we may assume that \( w \) is in \( A_p \cap D_{r+1} \). We also recall that we have already observed that if \( w \in D_{r+1} \), then the lifted weight \( w_l \) is in \( D_{r+1}(\mathcal{S}^{n+1}) \). We check that condition (i) is equivalent to the existence of a constant \( C > 0 \) such that for any \( f \in HF_{s+\frac{2}{p}}(w_l, \mathbb{B}^{n+1}) \),

\[
(4.1) \quad \|f\|_{L^p(d\mu_l)} \leq C \|f\|_{HF_{s+\frac{2}{p}}(w_l, \mathbb{B}^{n+1})},
\]

where \( \mu_l \) is the measure on \( \mathbb{B}^{n+1} \) defined by \( \int_{\mathbb{B}^{n+1}} f d\mu_l = \int_{\mathbb{B}^n} f(z',0)d\mu(z') \).

Let us show that (i) implies (4.1). If \( f \in HF_{s+\frac{2}{p}}(w_l, \mathbb{B}^{n+1}) \), Theorem 5.3 gives that \( f|_{B^n} \in B_s^p(\mathbb{B}^n) \), and that \( \|f|_{B^n}\|_{B_s^p(\mathbb{B}^n)} \leq C \|f\|_{HF_{s+\frac{2}{p}}(w_l, \mathbb{B}^{n+1})} \). Since we are assuming that (i) holds, we then have that \( \|f|_{B^n}\|_{L^p(d\mu_l)} \leq C \|f|_{B^n}\|_{B_s^p(\mathbb{B}^n)} \). Since \( \|f|_{B^n}\|_{L^p(d\mu_l)} = \|f\|_{L^p(d\mu_l)} \), we are done.

Assume now that (4.1) holds. Theorem 5.3 gives that \( f_l \in HF_{s+\frac{2}{p}}(w_l, \mathbb{B}^{n+1}) \) for any \( f \in B_s^p(\mathbb{B}^n) \), with \( \|f_l\|_{HF_{s+\frac{2}{p}}(w_l, \mathbb{B}^{n+1})} \leq C \|f\|_{B_s^p(\mathbb{B}^n)} \). The hypothesis on \( w \) gives then

\[
\|f_l\|_{L^p(d\mu_l)} \leq C \|f_l\|_{HF_{s+\frac{2}{p}}(w_l, \mathbb{B}^{n+1})} \leq C \|f\|_{B_s^p(\mathbb{B}^n)}.
\]

Since \( \sup \mu_l \subset \mathbb{B}^n \), \( \|f_l\|_{L^p(d\mu_l)} = \|f\|_{L^p(d\mu_l)} \), which gives (i).

Going back to the proof of the theorem, the above observation gives that (i) holds if and only if (4.1) does. Next, Theorem 2.13 in [CaOr3] gives that (4.1) can be rewritten as

\[
(4.2) \quad \int_{S^{n+1}} \frac{f(\zeta)d\sigma(\zeta)}{(1 - z\zeta)^{n+1-(s+\frac{2}{p})}} \|L^p(d\mu_l) \leq C \|f\|_{L^p(\mathbb{B}^n)}.
\]

Let us check that (4.2) is equivalent to

\[
(4.3) \quad \int_{B^n} \frac{f(y)dv(y)}{(1 - z\zeta)^{n+1-(s+\frac{2}{p})}} \|L^p(d\mu_l) \leq C \|f\|_{L^p(\mathbb{B}^n)},
\]

for any \( f \in L^p(\mathbb{B}^n) \). Indeed, assume first that (4.2) holds, let \( f \in L^p(\mathbb{B}^n) \), and let \( f_l(z) = f(z') \). We then have that \( f_l \in L^p(\mathbb{B}^n) \), and \( \|f_l\|_{L^p(\mathbb{B}^n)} \approx \|f\|_{L^p(\mathbb{B}^n)} \). Moreover, if \( z \in \mathbb{B}^n \),

\[
\int_{S^{n+1}} \frac{f_l(\zeta)}{(1 - z\zeta)^{n+1-(s+\frac{2}{p})}}d\sigma(\zeta) = C \int_{B^n} \frac{f(\zeta')}{(1 - z\zeta')^{n+1-(s+\frac{2}{p})}}dv(\zeta'),
\]

and consequently, we obtain (4.3).

Assume now that (4.3) holds, and let \( f \in L^p(\mathbb{B}^n) \). Then the function

\[
\tilde{f}(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y, e^{i\theta}(1 - |y|^{2}))d\theta,
\]
for \( y \in \mathbb{B}^n \) is in \( L^p(w) \) (just applying Hölder’s inequality) and moreover, \( ||f||_{L^p(w)} \leq C||f||_{L^p(w_1)} \).

In addition,
\[
\int_{\mathbb{B}^n} \frac{\tilde{f}(y)dv(y)}{(1 - z\overline{y})^{n+1-(s+\frac{1}{p})}} = C \int_{\mathbb{B}^n} \frac{\int_{-\pi}^{\pi} f(y, e^{i\theta}(1 - |y|^2))d\theta dv(y)}{(1 - z\overline{y})^{n+1-(s+\frac{1}{p})}} = 
\]

\[
C \int_{S^{n+1}} \frac{f(z)}{(1 - \zeta)^{n+1-(s+\frac{1}{p})}}d\sigma(\zeta).
\]

And that proves that (i) is equivalent to (ii).

Next, the hypothesis give that \( \tau + 1 - (s + \frac{1}{p})p = \tau - sp < 1 \). Since we have observed that the lifted weight \( \tilde{w}_1 \) satisfies the doubling condition \( D_{\tau+1} \), Theorem 1.1 gives that (4.1) holds if and only if for any \( f \in L^p(w_1) \), \( f \geq 0 \)

\[
||f||_{L^p(w)} \leq ||f||_{L^p(w_1)}.
\]

The same argument used for the holomorphic potential gives that (4.4) can be rewritten as (iii).

**Corollary 4.2.** Let \( w_\alpha(z) = (1 - |z|^\alpha)^{-1}, -1 < \alpha < p - 1, s > 0 \) and \( 1 < p < +\infty \). If \( 0 < \alpha < p - 1 \), assume that \( 0 \leq n + \alpha - sp < 1 \) and let \( \tau = n + \alpha + 1 \). If \(-1 < \alpha \leq 0 \), assume that \( 0 \leq n - sp < 1 \) and let \( \tau = n + 1 \). Let \( \mu \) be a positive Borel measure on \( \mathbb{B}^n \). We then have that the following assertions are equivalent:

(i) There exists \( C > 0 \) such that for any \( f \in B^p_\mu(w_\alpha, \mathbb{B}^n) \),

\[
||f||_{L^p(\mu)} \leq C||f||_{B^p_\mu(w_\alpha, \mathbb{B}^n)}.
\]

(ii) There exists \( C > 0 \) such that for any \( f \in L^p(w_\alpha) \),

\[
||\int_{\mathbb{B}^n} \frac{f(y)dv(y)}{(1 - z\overline{y})^{n+1-(s+\frac{1}{p})}}||_{L^p(\mu)} \leq C||f||_{L^p(w_\alpha)}.
\]

(iii) There exists \( C > 0 \) such that for any \( f \in L^p(w_\alpha) \),

\[
||\int_{\mathbb{B}^n} \frac{f(y)dv(y)}{(1 - z\overline{y})^{n+1-(s+\frac{1}{p})}}||_{L^p(\mu)} \leq C||f||_{L^p(w_\alpha)}.
\]

**Remark 4.3.** We observe that for a wide class of weights in dimension 1, theorem 4.1 holds without imposing any additional condition to the doubling constant \( \tau \).

1. Any weight in \( A_p(S^1) \) is in \( D_p(S^1) \) for \( \tau = p \). Consequently, lemma 2.4 gives that the weight \( \tilde{w}(z) = \frac{1}{(1 - |z|)} \int_{L_z} w(\zeta)d\sigma(\zeta) \) is automatically in \( D_{p+1}(\mathbb{B}^1) \). Since \( 0 < \tau - sp = p(1 - s) < 1 \) for \( 0 < s < 1 \), the hypothesis of theorem 4.1 are fulfilled, and a Carleson measure \( \mu \) for \( B^p_\mu(\tilde{w}, \mathbb{B}^1) \) coincides with the ones that satisfy

\[
||\int_{\mathbb{B}^1} \frac{f(y)dv(y)}{(1 - z\overline{y})^{2-(s+\frac{1}{p})}}||_{L^p(\mu)} \leq C||f||_{L^p(\tilde{w}dv)}.
\]
2. Let $0 < s < 1$ and $\max(0, sp - 1) \leq \alpha \leq \min(p - 1, sp)$. Assume that $
abla : (0, 1] \to \mathbb{R}$ is a nondecreasing function satisfying that $\nabla(2^k x) \leq C 2^{nk} \nabla(x)$. Let $w_\nabla(z) = \nabla(1 - |z|)$. Then proposition 2.13 gives that since $\alpha < p - 1$, $w_\nabla \in \mathcal{B}_p(B^1)$, and that for any $\tau > \alpha + 2$, $w_\nabla \in d_\tau(B^1)$. Thus if we choose $\tau$ such that $\alpha + 2 < \tau < 2 + sp$, $w_\nabla \in \mathcal{B}_p(B^1) \cap d_\tau(B^1)$. For that choice, $0 < \tau - 1 - sp < 1$, and the argument in remark 4.3.1 can be used to deduce that if $\alpha \in (0, p - 1)$, $\mu$ is a Carleson measure for $\mathcal{B}_p(w_\nabla, B^1)$ if and only if (4.5) holds replacing $\tilde{w}$ by $w_\nabla$.

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