Lower bounds in multiple testing: A framework based on derandomized proxies

Maxim Rabinovich†, Michael I. Jordan∗†, Martin J. Wainwright∗†‡

{rabinovich,jordan,wainwrig}@berkeley.edu

Departments of Statistics∗ and EECS†, University of California, Berkeley
Voleon Group‡, Berkeley

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Abstract

The large bulk of work in multiple testing has focused on specifying procedures that control the false discovery rate (FDR), with relatively less attention being paid to the corresponding Type II error known as the false non-discovery rate (FNR). A line of more recent work in multiple testing has begun to investigate the tradeoffs between the FDR and FNR and to provide lower bounds on the performance of procedures that depend on the model structure. Lacking thus far, however, has been a general approach to obtaining lower bounds for a broad class of models. This paper introduces an analysis strategy based on derandomization, illustrated by applications to various concrete models. Our main result is meta-theorem that gives a general recipe for obtaining lower bounds on the combination of FDR and FNR. We illustrate this meta-theorem by deriving explicit bounds for several models, including instances with dependence, scale-transformed alternatives, and non-Gaussian-like distributions. We provide numerical simulations of some of these lower bounds, and show a close relation to the actual performance of the Benjamini-Hochberg (BH) algorithm.

1 Introduction

The past decades have witnessed a tremendous amount of research on control of the false discovery rate (FDR), an analogue of type I error in multiple testing problems (e.g., [3, 4, 5, 10, 21, 22, 13, 12]). The large majority of work has focused on developing procedures that are guaranteed to control the FDR at a pre-specified level, under various assumptions on the structure of the p-values. The literature on classical hypothesis testing is replete with combined analyses of the type I error and statistical power for specific classes of models. More recently, analogues of such analyses have begun to appear in the FDR literature. Using the false non-discovery rate (FNR)—the fraction of tests in which the null is incorrectly not rejected—as a measure of the (lack of) power, several authors have established lower bounds on combinations of the FDR and FNR [1, 15, 16], and most recently, a non-asymptotic tradeoff between FDR and FNR has been established in the same setting [19].

Most results of this type occur within models where test statistics arise from Gaussian-like location models with independent observations, meaning that the alternatives are assumed to be location shifts of the null, and the noise variables are independent with
Gaussian-type tail behavior. Models of this type have the advantage of analytical tractability, while still permitting the expression of central features of many multiple testing problems. Many results apply to variants of the rare-weak (RW) model, in which problem difficulty is parameterized by the rarity of signals and their weakness. This model was initially introduced for studies multiple testing using the family-wise error rate (FWER) as the Type-I error concept \[7, 8, 17\].

Unfortunately, the analysis techniques underlying existing results do not extend much beyond the setting of independent and Gaussian-like test statistics. This limitation appears most dramatically when one seeks to derive non-asymptotic guarantees, as these results depend critically on the ability to control tail probabilities and apply concentration inequalities. In this work, we introduce a modeling and analysis strategy for lower bounds on the FDR and FNR that is far less dependent on independence and analytical tractability. Building on the proof strategy introduced in our earlier paper \[19\], we show how it is fruitful to perform a version of derandomization—that is, we relate a given multiple testing procedure to a derandomized version that always makes a fixed number of discoveries. Given a proxy \(k^*\) for the number of discoveries, we show how to further eliminate the randomness associated with the location of the data to obtain deterministic proxies \(\ell^*\) for lower bounding the number of false discoveries with constant probability. Using the two proxies \(k^*\) and \(\ell^*\) together yields constant-probability lower bounds on the proportion of false discoveries and false non-discoveries, and these translate directly into corresponding lower bounds on the FDR and FNR that hold for any procedure applied to the given model.

With this context, the central contribution of this paper is a meta-theorem that establishes a tradeoff between FDR and FNR in terms of these proxies for a broad class of models defined in Section 2.2. In contrast to minimax theory for estimation, where tools like Le Cam’s method, Fano’s method, and Assouad’s lemma (e.g., see \[25, 23, 24\] for background) play the role of meta-theorems that can be instantiated, such general tools are currently absent from the multiple testing literature. The main theorem of this paper appears to be the first of its kind for controlling FDR and FNR in multiple testing.

In order to illustrate applications of this meta-theorem in practice, we apply it to several specific models. These include the previously-studied independent generalized Gaussians model \[1, 19\], as well as Gaussian models with dependence, Gaussian models with scale-transformation, and an exponentiation model for \(p\)-values. Furthermore, we showcase our approach’s capacity to produce numerical lower bounds for concrete models in Figure 1.

Related work. Our work is most closely related to our own past work on the non-asymptotic tradeoffs between FDR and FNR \[19\]. These previous results apply to a very specific location shift model, in which the test statistics are assumed to have tails on the order of \(\exp(-|x|^\gamma)\) for some \(\gamma \geq 1\)—in other words, generalized Gaussian-like tails. The tradeoffs derived in this past work, as well as previous asymptotic lower bounds \[1\], apply only to the independent case. There is another line of related work \[15, 16\], applicable to models with exactly Gaussian noise, that provides lower bounds that continue to hold under dependence, at the cost of replacing FDR with the so-called modified FDR (mFDR), in which the expectation is moved inside the ratio in the definition of FDR (cf. \[13\]). The mFDR and FDR measures ought to behave similarly for large numbers of tests, but they are distinct metrics, and the analysis strategies that work for mFDR generally do not apply
Figure 1. The lower bound predicted by our theory plotted against the actual FDR-FNR tradeoff achieved by the Benjamini-Hochberg algorithm for two problems. FDR is on the horizontal axis, while FNR is on the vertical axis. Both models use Gaussian test statistics with additive shifts $\mu = \sqrt{2r \log n}$ and $n = 10000$. (a) Plots for $m = 15$ signals and $r = 0.8$. (b) Plots for $m = 100$ and $r = 0.6$. Further details on these simulation experiments are given in Appendix B.

to FDR itself.

Our work is inspired in part by the work of Jin and Donoho on Tukey’s higher criticism (e.g., [7, 8, 17]). Their results apply to the Gaussian sequence model with location shifts, in which the number of signals is a polynomially small fraction $n^{-s}$ of the total number of tests and in which signals are weak (achieved by scaling the shift as $\mu = \sqrt{2r \log n}$ for $0 < r < 1$). Their work establishes the regime of $s$ and $r$ in which asymptotic consistency is possible, under the standard Type-I and Type-II error measures for testing the global null.

2 Background and problem formulation

In this section, we provide necessary background and a precise formulation of the problem under study.

2.1 Multiple testing and false discovery rate

Suppose that we observe a real-valued sequence $X^n_1 := \{X_1, \ldots, X_n\}$ of $n$ independent random variables. We introduce the sequence of binary labels $\{H_1, \ldots, H_n\}$ to encode whether or not the null hypothesis holds for each observation; the setting $H_i = 0$ indicates that the null hypothesis holds. We define

$$\mathcal{H}_0 := \{i \in [n] \mid H_i = 0\}, \quad \text{and} \quad \mathcal{H}_1 := \{i \in [n] \mid H_i = 1\},$$

(1)
corresponding to the set of nulls and signals, respectively. Our task is to identify a subset of indices that contains as many signals as possible, while not containing too many nulls.

More formally, a testing rule $\mathcal{I} : \mathbb{R}^n \to 2^{[n]}$ is a measurable mapping of the observation sequence $X^n_1$ to a set $\mathcal{I}(X^n_1) \subseteq [n]$ of discoveries, where the subset $\mathcal{I}(X^n_1)$ contains those
indices for which the procedure rejects the null hypothesis. There is no single unique measure of performance for a testing rule for the localization problem. In this paper, we study the notion of the \textit{false discovery rate} (FDR), paired with the \textit{false non-discovery rate} (FNR). These can be viewed as generalizations of the type-I and type-II errors for single hypothesis testing.

We begin by defining the false discovery proportion (FDP), and false non-discovery proportion (FNP), respectively, as

$$\text{FDP}_n(\mathcal{I}) : = \frac{\text{card}(\mathcal{I}(X^n_1) \cap \mathcal{H}_0)}{\text{card}(\mathcal{I}(X^n_1)) \lor 1}, \quad \text{and} \quad \text{FNP}_n(\mathcal{I}) : = \frac{\text{card}(\mathcal{I}(X^n_1)^c \cap \mathcal{H}_1)}{\text{card}(\mathcal{H}_1)}. \quad (2)$$

Since the output $\mathcal{I}(X^n_1)$ of the testing procedure is random, both of these quantities are random variables. The FDR and FNR are given by taking the expectations of these random quantities—that is

$$\text{FDR}_n(\mathcal{I}) : = \mathbb{E}\left[\frac{\text{card}(\mathcal{I}(X^n_1) \cap \mathcal{H}_0)}{\text{card}(\mathcal{I}(X^n_1)) \lor 1}\right], \quad \text{and} \quad \text{FNR}_n(\mathcal{I}) : = \mathbb{E}\left[\frac{\text{card}(\mathcal{I}(X^n_1)^c \cap \mathcal{H}_1)}{\text{card}(\mathcal{H}_1)}\right], \quad (3)$$

where the expectation is taken over the random samples $X^n_1$. Here our definition of FNR follows\footnote{There is an alternative definition of $\text{FNR}_{alt}$, in which the denominator is set to the number of non-rejections. In general, however, the number of non-rejections will be close to $n$ for any procedure with low FDR and thus in the sparse regime, the $\text{FNR}_{alt}$ would trivially go to zero for any procedure that controls FDR at any level strictly below 1. The definition adopted here is therefore better suited to studying transitions in difficulty in the multiple testing problem.} that of the paper \cite{1}.

\section{Model structure}

In this paper, we consider a flexible class of models that includes as special cases the location and scale families that have been studied in past work. For a testing problem with $n$ hypotheses, we assume that the vector $X = (X_1, \ldots, X_n)$ of test statistics is generated from some underlying random vector $W = (W_1, \ldots, W_n)$ in the following way. The vector $W$ may be drawn from an arbitrary (not necessarily product) distribution $P_n$, while the test statistics are related via

$$X_i = \begin{cases} W_i & \text{if } i \in \mathcal{H}_0 \\ f(W_i) & \text{otherwise.} \end{cases} \quad (4)$$

The main restriction on this model is that $f: \mathbb{R} \to \mathbb{R}$ must be a non-decreasing function such that $f(w) \geq w$ for all $w$ in the support of a marginal of $P_n$. For convenience, we shall also assume that the marginals of $P_n$ are atom-free. In some instances, we consider the restricted case where all the $W_i$ are iid—that is, $P_n = P_0^\otimes n$. We refer to this as the iid model. Conceptually, a model is therefore a tuple $(P_n, f)$ satisfying these constraints, and where appropriate we shall denote models by such tuples and sometimes name them.

Prototypical examples of this general set-up include the following:
**Location model:** The variables $W_i$ are drawn from a generalized Gaussian distribution with density proportional to $\exp(-|w|^{\gamma})$ for some $\gamma \in [1, 2]$, and the transformation function takes the form $f(w) = \mu + w$ for some $\mu > 0$.

**Scale models:** The variables $W_i$ are the absolute values of standard normal variates, and the transformation function takes the form $f(w) = \sigma w$ for some $\sigma \geq 1$.

**Lehmann alternative model:** The variables $W_i$ are uniform on the unit interval, and the transformation function takes the form $f(w) = 1 - (1 - w)^{1/\gamma}$ for some parameter $\gamma \in (0, 1)$. This set-up models the situation in which the $W_i$ represent p-values and the signals have p-values that are stochastically closer to zero than those of the nulls. Since we have chosen to model the transformation as non-decreasing, we represent the unit interval backwards, which leads to the form written rather than $w^{1/\gamma}$.

We note that all three of these examples have been studied in past work (e.g., [1, 9, 17, 18, 19]).

**2.3 Top-K procedures**

Many popular procedures, including the Benjamini-Hochberg (BH) and several variants thereof [11, 12, 20], are based on thresholding the order statistics. Recall that the order statistics of a sequence $\{X_1, \ldots, X_n\}$ are defined as
\[
\min_{i=1,\ldots,n} X_i := X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)} := \max_{i=1,\ldots,n} X_i.
\]

A top-K procedure is a method that rejects the hypotheses corresponding to the top $K$ order statistics, where $K = K_n(X^n)$ is a non-negative integer that can depend on the observed statistics. The testing rule $I: \mathbb{R}^n \to \{0, 1\}^{\left\lceil n \right\rceil}$ defined by any top-K procedure has the form
\[
I(X^n_i) = \left\{ i \in [n]: X_i \geq X_{(K_n(X^n))} \right\},
\]
where $K_n: \mathbb{R}^n \to \mathbb{N}$ is some (possibly randomized) mapping. Alternatively, such procedures can be described in terms of choosing a threshold $\tau = \tau_n(X^n)$, and rejecting all nulls $i$ for which $X_i \geq \tau$.

**3 Main results**

We now turn to the statements of our main results. We begin in Section 3.1.1 by defining the deterministic proxies that play a central role in our analysis; see Section 3.1.2 for the intuition that underlies these definitions. In Section 3.1.3, we state a general lower bound (Theorem 1) on the pairs of FDR and FNR that are achievable. In the remaining sections, we illustrate the consequences of this general bound for various specific models.
3.1 A general bound based on deterministic proxies for FDR and FNR

We say that a FDR-FNR pair \((\alpha, \beta) \in [0, 1] \times [0, 1]\) is \textit{achievable} if there exists a top-\(K\) procedure \(K\) such that

\[
\text{FDR}(K) \leq \alpha \quad \text{and} \quad \text{FNR}(K) \leq \beta.
\]  
(7)

Any top-\(K\) procedure satisfying condition (7) is said to be \((\alpha, \beta)\)-controlled. Our high-level goal is to provide bounds on the region of achievable \((\alpha, \beta)\) pairs.

3.1.1 Defining the deterministic proxies

In order to characterize the space of achievable \((\alpha, \beta)\) pairs, we construct two sets of deterministic proxies. One set of proxies is useful in the regime \(\alpha \geq \beta\), while the other is useful in the opposite setting—namely, \(\beta \geq \alpha\). The proxies used in the former regime are denoted by \(\text{FDP}^-\) and \(\text{FNP}^-\), whereas those in the latter regime are denoted by \(\text{FNP}^+\) and \(\text{FDP}^+\). The reasoning underlying our choice of notation should be clear once we detail their construction below. The proxies depend on the given pair \((\alpha, \beta)\), the model \(M = (\mathbb{P}_n, f)\) under consideration, and a parameter \(\epsilon \in (0, 1)\) that controls the strength of the bounds. We make these dependencies explicit when needed, suppressing them otherwise.

Our first step is to define deterministic approximations for the number of total discoveries. Letting \(m = \text{card}(\mathcal{H}_1)\) denote the number of signals and given any \(\epsilon > \max\{\alpha, \beta\}\), we define

\[
k^*_-(\beta, \epsilon) : = \left(1 - \frac{\beta}{\epsilon}\right) m \quad \text{and} \quad k^*_+ (\alpha, \epsilon) : = \left(1 - \frac{\alpha}{\epsilon}\right)^{-1} m.
\]  
(8)

Roughly speaking, the integer \(k^*_-\) functions as a lower approximation for the number of total discoveries, whereas the quantity \(k^*_+\) provides an upper approximation for the same quantity. Note that these lower and upper bounds converge as \(\alpha, \beta \to 0\); in the limit \(\alpha = \beta = 0\), we have \(k^*_-(0, \epsilon) = k^*_+(0, \epsilon) = m\), since in this case, the total number of discoveries must be equal to the number of signals \(m\).

For each of these approximations of the number of discoveries, we construct a corresponding false discovery proxy. Recalling the random vector \(W = (W_{\mathcal{H}_0}, W_{\mathcal{H}_1})\) that underlies our generic model, these quantities involve the order statistics \(W_{\mathcal{H}_0,(1)} \leq W_{\mathcal{H}_0,(2)} \leq \cdots \leq W_{\mathcal{H}_0,(|\mathcal{H}_0|)}\), with the order statistics for \(W_{\mathcal{H}_1}\) defined analogously. For any \(\epsilon > \max\{\alpha, \beta\}\), adopting the shorthand \(k^*_- = k^*_-(\beta, \epsilon)\) and \(k^*_+ = k^*_+ (\alpha, \epsilon)\), we define proxies as follows.

**False discovery proxies:**

\[
\ell^*_-(\beta, \epsilon, M) = \arg\max_{\ell \in [1, k^*_-]} \left\{ \mathbb{P}\left[ W_{\mathcal{H}_0,(\ell)} > f(W_{\mathcal{H}_1, (k^*_- - \ell + 1)}) \right] \geq 1 - \epsilon \right\}, \quad \text{and} \quad (9a)
\]
\[
\ell^*_+(\alpha, \epsilon, M) = \arg\max_{\ell \in [k^*_-, m, k^*_+]} \left\{ \mathbb{P}\left[ W_{\mathcal{H}_0,(\ell)} > f(W_{\mathcal{H}_1, (k^*_+ - \ell + 1)}) \right] \geq 1 - \epsilon \right\}. \quad (9b)
\]

Roughly, the quantities \(\ell^*_-\) and \(\ell^*_+\) represent, respectively, lower and upper approximations to the number of false discoveries.

Finally, by taking appropriate ratios, we define:
Proxies to FDR and FNR:

\[
\begin{align*}
\text{FDP}^-_*(\beta, \epsilon, M) &= \frac{\ell^-_* (\beta, \epsilon, M)}{m} \quad \text{and} \quad \text{FNP}_*(\beta, \epsilon, M) = \frac{m - k_*^- (\beta, \epsilon) + \ell^-_* (\beta, \epsilon, M)}{m}, \quad (10a) \\
\text{FDP}^+_*(\alpha, \epsilon, M) &= \frac{\ell^+_* (\alpha, \epsilon, M)}{m} \quad \text{and} \quad \text{FNP}^+_*(\alpha, \epsilon, M) = \frac{m - k_*^+ (\alpha, \epsilon) + \ell^+_* (\alpha, \epsilon, M)}{m}. \quad (10b)
\end{align*}
\]

To be clear, in defining \( \text{FDP}^+_* \) (respectively \( \text{FDP}^-_* \)), it might be more natural to use \( k_*^+ \) (respectively \( k_*^- \)) in the denominator, but as noted above, when \((\alpha, \beta)\) are small, both of these quantities are close to \( m \).

### 3.1.2 The underlying intuition

Let us now describe the intuition that underlies the definitions (9), ignoring the difference between the + and − versions so as to simplify matters. First, suppose that we accept that \( k \) is a good approximation to the total number of discoveries, and that \( \ell \) is a good approximation to the number of false discoveries. In this case, \( \ell/k \) is a good approximation to the FDR, and since \( k - \ell \) of the discoveries must be false, then \( m - (k - \ell) \) should be a good approximation to the FNR. As we have argued above, when \( \beta \) and \( \alpha \) are small, then \( k \) is actually relatively close to \( m \), so that \( \ell/m \) should also be a good approximation to the FDR.

It remains to justify why \( \ell \), as defined in equation (9), is a reasonable proxy to the number of false discoveries. Consider a procedure that rejects exactly \( k \) hypotheses, of which \( \ell \) are nulls. It must then be case that the \( \ell^\text{th} \) largest null value exceeds the value of the \((k - \ell + 1)^{\text{th}} \) largest signal value, or else only \( \ell - 1 \) nulls would be in the top \( k \) test statistics. Using the definition of the model, we can re-express this relation in symbols:

\[
W_{H_0, (\ell)} > f(W_{H_1, (k-\ell+1)}).
\]

The definitions (9) are motivated by this assertion.

### 3.1.3 A general lower bound

Our main result is that our choice of proxies yield constant-factor lower bounds on the attainable FDR and FNR of any top-\( K \) procedure.

**Theorem 1.** Given a model \( M \), consider any \((\alpha, \beta)\)-controlled top-\( K \) procedure such that \( 2 \max\{\alpha, \beta\} < \frac{1}{3} \). Then for any scalar \( \epsilon \in (2 \max\{\alpha, \beta\}, \frac{1}{3}) \), there exists a constant \( c_0(\epsilon) \geq 1 \) such that

\[
\begin{align*}
\alpha &\geq c_0^{-1} \text{FDP}^-_* (\beta, \epsilon) \quad \text{and} \quad \max\{\alpha, \beta\} \geq c_0^{-1} \text{FNP}_* (\beta, \epsilon), \quad \text{as well as} \quad (12a) \\
\beta &\geq c_0^{-1} \text{FNP}^+_* (\alpha, \epsilon) \quad \text{and} \quad \max\{\alpha, \beta\} \geq c_0^{-1} \text{FDP}^+_* (\alpha, \epsilon). \quad (12b)
\end{align*}
\]

The slightly unorthodox form of (12a) and (12b) calls for some discussion. The presence of the maximum reflects the fact that generally only one set of proxies will be suitable for lower bounding \( \alpha \) and \( \beta \) simultaneously. If \( \alpha > \beta \), the bound in (12b) is the meaningful one, while (12a) gives the desired bound when \( \beta > \alpha \). When \( \alpha = \beta \), either equation will do.
The two regimes arise because $\ell_+^*$ and $k_+^*$ yield a good approximation of the false discovery number and total number of discoveries only when $\alpha > \beta$, while $\ell_-^*$ and $k_-^*$ provide a good approximation only when $\alpha < \beta$. Intuitively, the dichotomy arises because $k_+^*$ may be larger than the actual number of discoveries by an amount as large as order of $\alpha + \beta$, so that $\text{FDP}_+^*$ can only be upper bounded by a quantity of this order, or, equivalently (disregarding constants), a quantity on the order of $\max\{\alpha, \beta\}$. A similar but inverted phenomenon occurs for the $-$ proxies.

3.2 Application 1: Independent Gaussians model

In this section, we investigate models in which the vector $W$ has iid Gaussian entries, and the signal structure is specified by either a location shift or a scale factor.

3.2.1 Gaussian location model

We begin by analyzing the Gaussian location model, in which the function $f$ takes the form

$$f(w) = w + \mu \quad \text{for some } \mu > 0.$$  \hfill (13)

By applying Theorem 1 to this particular model, we obtain the following:

**Corollary 1.** Consider the iid Gaussian location model with $m = n^{1-s}$ signals and $\mu = \sqrt{2r \log n}$ with parameters $(s, r)$ satisfying the inequality $0 < s < r < 1$. Suppose that there exists a constant $c > 0$ such that for all $n \geq 1$, there is an $(\alpha_n, \beta_n)$-controlled top-$K$ procedure with $\alpha_n = cn^{-\kappa_\alpha}$ and $\beta_n = cn^{-\kappa_\beta}$. Then we must have

$$\sqrt{s + \kappa_\alpha} + \sqrt{\kappa_\beta} \leq \sqrt{r}, \quad \text{and} \quad \min\{\kappa_\alpha, \kappa_\beta\} \leq \frac{(r - s)^2}{4r} =: \kappa^*.$$  \hfill (14)

The result obtained in (14) is essentially the lower bound of Rabinovich et al. [19], derived by other means in that paper and applicable to an exactly Gaussian rather than a Gaussian-like model. Thus, Corollary 1 can be seen as an extension of those earlier results, illustrating how the methods developed in this work can expand the scope of multiple testing lower bounds. Moreover, since the lower bound of Rabinovich et al. [19] is known to be sharp, it seems likely the bound proven here is likewise sharp for the Gaussian location model.

Figure 1 shows the predictions of Theorem 1 in a Gaussian model, in particular giving plots of the lower bound predicted by our theory plotted against the actual FDR-FNR tradeoff achieved by the Benjamini-Hochberg algorithm for two problems. In each plot, the FDR is on the horizontal axis, while FNR is on the vertical axis. See the figure caption for further details.

3.2.2 Gaussian scale model

We now turn attention to the Gaussian scale model. It is specified by the transformation

$$f(w) = \sigma w, \quad \text{for some } \sigma \geq 1.$$
By applying Theorem 1 to this model, we obtain a rather different lower bound on pairs \((\alpha, \beta)\). At a high level, the main take-away is that the FDR and FNR can only decay as inverse polynomial functions of \(n\) when the signal strength \(\sigma\) is extremely strong—in particular, the scalar \(\sigma\) has to grow polynomially in \(n\).

**Corollary 2.** Consider the iid Gaussian scale model with \(m\) signals and signal strength \(\sigma \geq 1\) where \(s_n := \frac{\log \frac{m}{n}}{\log n}\) lies in the interval \([\rho, 1 - \rho]\) for some \(\rho \in (0, 0.5)\). Suppose that there exists an \((\alpha, \beta)\)-controlled procedure such that \(\max \{\alpha, \beta\} \leq \frac{1}{3c_0}\). Then there exists some \(\eta_n \in (0, 1)\) such that

\[
\sigma \geq \frac{1}{\sqrt{2\pi c_0}} \cdot \left(1 - \eta_n\right) \left(\frac{1}{m} + \beta\right)^{-1} \sqrt{2s_n \log n + 2 \log \left(\alpha + \frac{1}{m}\right)^{-1}}. \tag{15}
\]

As in the statement of Corollary 1 for the location model, by assuming certain scalings of the number of signals, FDR and FNR, we can give an asymptotic statement. In particular, suppose that the number of signals scales as \(m \propto n^{1-s}\) for a fixed \(s\), whereas the FDR and FNR scale as \(\alpha_n \propto n^{-\kappa_\alpha}\) and \(\beta_n \propto n^{-\kappa_\beta}\) for some scalars \(\kappa_\alpha, \kappa_\beta\) such that \(\max \{\kappa_\alpha, \kappa_\beta\} \leq 1 - s\). Then there is a universal constant \(c > 0\) such that

\[
\sigma \geq c n^{\kappa_\beta} \sqrt{2(s + \kappa_\alpha) \log n}. \tag{16}
\]

Consequently, we see that whenever \(\kappa_\beta > 0\), the signal strength \(\sigma\) must grow polynomially in \(n\). This is quite a dramatic contrast from the location model, where the analogous quantity \(\mu\) need only grow proportionally to \(\sqrt{\log n}\).

### 3.3 Application 2: Gaussian location models with dependence

Given the presence of dependence in many target applications of multiple testing (e.g., [2]), it makes sense to ask how dependence changes the performance of multiple testing procedures. In this section, we provide answer for two models of Gaussian dependence which lie at opposite extremes of dependency. In both cases, we consider only location shifts.

#### 3.3.1 Spiked dependence model

We begin by considering a doubly-spiked covariance model, with one spike within the nulls and a separate spike within the signals. This specification corresponds to coupling all the nulls (and, separately, all the signals) through a single random variable that captures all shared randomness.

In the spiked dependence model, the model distribution \(\mathbb{P}_n\) is a multivariate Gaussian \(\mathcal{N}(0, \Sigma)\), with covariance matrix in the block-partitioned form

\[
\Sigma_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
\rho_0 & \text{if } i \neq j \text{ and } i, j \in \mathcal{H}_0, \\
\rho_1 & \text{if } i \neq j \text{ and } i, j \notin \mathcal{H}_0, \\
\pm \rho_c & \text{if } i \neq j \text{ and } i \in \mathcal{H}_0, j \notin \mathcal{H}_0 \text{ or vice versa},
\end{cases} \tag{17}
\]

for parameters \(0 \leq \rho_0, \rho_1, \rho_c < 1\).
Corollary 3. Consider the spiked dependence model with \( m = n^{1-s} \) and \( \mu = \sqrt{2r \log n} \) for some pair \((s, r)\) satisfying the inequalities \( 0 < s < r < 1 \). Suppose that for each \( n \), there exists an \((\alpha_n, \beta_n)\)-controlled top-\( k \) procedure with \( \alpha_n = cn^{-\kappa_\alpha} \) and \( \beta_n = cn^{-\kappa_\beta} \). Then we must have

\[
\sqrt{1 - \rho_0 \sqrt{s + \kappa_\alpha}} + \sqrt{1 - \rho_1 \sqrt{\kappa_\beta}} \leq \sqrt{r}.
\]  

(18)

Note that the bound (18) is a generalization of the bound (14) for iid Gaussians, to which it reduces when \( \rho_0 = \rho_1 = \rho_0 = 0 \). Relative to this iid case, the bound (18) allows for larger values of the pair \((\kappa_\alpha, \kappa_\beta)\)—which translates into faster decay rates of FDR and FNR—when either \( \rho_1 \) or \( \rho_0 \) is non-zero. While this might be counterintuitive at first sight, note that our spiked dependence makes all nulls more similar to each other when \( \rho_0 > 0 \), and all signals more similar to each other when \( \rho_1 > 0 \). This similarity in either the nulls or signals means that it becomes easier to control the FDR and FNR. What may still be surprising is that \( \rho_1 \) does not play any role in the rates.

3.3.2 Grouped dependence model

We now turn to the opposite extreme of dependency. In the grouped dependence model, we match each signal with a different set of \( A \) nulls that are strongly coupled to that signal, but independent of all other signals and all nulls in different groups. More formally, first fix a value \( 1 \leq A \leq \min \{m, \frac{n}{m} \} \). We then write \( \mathcal{H}_1 = \{i_1 < \cdots < i_m\} \) and for each \( 1 \leq g \leq m \), we define a set of \( A \) nulls \( \mathcal{H}_0^{(g)} \) corresponding to the \( g \)th signal. Finally, we define the independent nulls as \( \mathcal{H}_0^{(0)} = \mathcal{H}_0 \cup \bigcup_{g=1}^{m} \mathcal{H}_0^{(g)} \).

Rather than providing an explicit form of the covariance matrix for this model, it is more informative to specify the underlying generative model, given by

\[
W_{\mathcal{H}_1} \sim \mathcal{N}(0, I_m), \quad \text{with} \quad W_i \mid W_{\mathcal{H}_1} \sim \begin{cases} \mathcal{N}(0, 1) & \text{if } i \in \mathcal{H}_0^{(0)}, \\ W_{i_{g}} & \text{if } i \in \mathcal{H}_0^{(g)}. \end{cases}
\]  

(19)

By applying Theorem 1 to this model, we obtain the following:

Corollary 4. Consider the grouped dependence model with \( m = n^{1-s} \), \( \mu = \sqrt{2r \log n} \), and \( A = \frac{n - m}{m} \) where the parameters \((s, r, t)\) satisfy the inequalities \( 0 \leq t < s < r < 1 \). Suppose that there is a constant \( c > 0 \) such that for each positive integer \( n \), there is an \((\alpha_n, \beta_n)\)-controlled procedure with \( \alpha_n = cn^{-\kappa_\alpha} \) and \( \beta_n = cn^{-\kappa_\beta} \). Then

\[
\sqrt{s + \kappa_\alpha} + \sqrt{\kappa_\beta} \leq \sqrt{r}.
\]  

(20)

Note the surprising fact that the bound in (20) is identical to our earlier bound (14) from the iid case. If the lower bound of (20) is sharp, this coincidence reflects a deep fact about the difficulty of multiple testing in the grouped Gaussians model. We do not at this point know, however, whether Corollary 4 is sharp, and the sharpness of this result (like the others established in this paper) remains an important question for future work.
3.4 Application 3: Lehmann alternatives

The Lehmann alternative model has often been used in theoretical analyses of multiple-testing procedures. In this model, the statistics are now $p$-values; nulls are assumed to come from a uniform distribution, while alternatives follow a CDF given by

$$F(p) = p\gamma \quad \text{for some } \gamma \in (0,1). \quad (21)$$

In order to formulate this problem within our framework, let $W_i$ be iid uniform random variables on the unit interval $[0,1]$, and define the transformation

$$f(w) = 1 - (1 - w)^{1/\gamma}.$$ 

Note that here $1 - w$ plays the role of the $p$-value, so that the $w$ values for signals are more clustered around 1 than is the case for the nulls.

**Corollary 5.** Under the Lehmann alternative model with parameter $\gamma \in (0,1)$, fix some triple $(\alpha,\beta,\epsilon)$ such that $\alpha \leq \frac{\epsilon}{3}$ and $t := \frac{3\beta}{\gamma} + \frac{1}{m} + \sqrt{\frac{\beta \log \beta}{2m}} < 1$, where $c_0$ is the constant from Theorem 7. Further, let $\pi_1 = \frac{m}{n}$. Then for any $(\alpha,\beta)$-controlled procedure, we must have

$$\frac{1}{\gamma} \geq \frac{1-t}{t} \cdot \log \left( \frac{\epsilon}{3\pi_1 \alpha} \left[ 1 + 4 \log \frac{3}{\epsilon} \right]^{-1} \right). \quad (22)$$

The bound of Corollary 5 requires some interpretation. Intuitively, $t$ is on the order of $\beta$, while the argument of the logarithm is on the order of $\frac{1}{\pi_1 \alpha}$, so the high-level takeaway is that

$$\frac{1}{\gamma} \gtrsim \frac{1}{\beta} \log \frac{1}{\pi_1 \alpha}. \quad (23)$$

Since the signal—the difference between nulls and alternatives—becomes greater as $\gamma$ becomes smaller, $\frac{1}{\gamma}$ is a measure of signal strength, and thus (23) is similar to our previous bounds in that it lower bounds the required signal strength in terms of the problem parameters. In this case, the dependence on the FNR $\beta$ is inverse polynomial, while the dependence on both the FDR $\alpha$ and the sparsity $\pi_1$ is logarithmic.

4 Proofs

We now turn to the proofs of our results.

4.1 Technical tools

Before giving proofs of our main results, we develop two technical tools that we apply repeatedly in our arguments.
4.1.1 Derandomization under concentration

Our proxies depend on the model only through the false discovery number proxies $\ell_\star$ and $\ell_\star^\prime$. Unfortunately, the dependence is of a rather complicated form, since the definitions (9) involve the probabilities of events defined in terms of the order statistics of nulls and signals. In order to make progress, our first step is to simplify these definitions so as to obtain modified versions that are more tractable. In this section, we show that, provided the model’s order statistics admit a suitable concentration bound, we can reduce the probabilistic comparison to a deterministic comparison of expected order statistics. In particular, we make use of the following family of assumptions, which are parameterized by $T \in \{H_0, H_1\}$, and an integer $k$.

**Concentration assumption** $(T, k)$: There exists a function $\Delta_{T,k} : (0, 1) \to [0, \infty)$ such that

\[ P \left| X_{T,(k)} - \mathbb{E}[X_{T,(k)}] \right| \geq \Delta_{T,k}(\epsilon) \leq \epsilon. \]  

(24)

Depending on the nature of the function $\Delta_{T,k}$, condition (24) might be a more or less stringent (and a more or less useful) assumption. In general, when we apply this bound, we shall be able to prove it holds for a reasonable choice of $\Delta_{T,k}$.

Our analysis invokes two particular cases of the concentration assumption:

**Case I:** The concentration assumption (24) holds for $(H_0, \ell_\star)$ and $(H_1, k_\star - \ell_\star + 1)$.

**Case II:** The concentration assumption (24) holds for $(H_0, \ell_\star + 1)$ and $(H_1, k_\star - \ell_\star)$.

Here as always, the integer $k_\star$ is one of $k_\star^-$ and $k_\star^+$, and the integer $\ell_\star$ is fixed correspondingly. The following lemma allows us to reduce from probabilities of events to differences in expected order statistics.

**Lemma 1.** Suppose that $\ell_\star \in \{\ell^\star_-, (\beta, \epsilon), \ell^\star_+ (\alpha, \epsilon)\}$ and define $k^*$ accordingly. Then under Cases I and II, we have the following bounds:

(I) : $\mathbb{E} \left[ f(W_{H_1,(k^* - \ell^*)}) + \Delta_{H_1,(k^* - \ell^*)} \left( \frac{\epsilon}{3} \right) \right] > \mathbb{E} \left[ W_{H_0,(\ell^* + 1)} \right] - \Delta_{H_0,(\ell^* + 1)} \left( \frac{\epsilon}{3} \right).$  

(25a)

(II) : $\mathbb{E} \left[ f(W_{H_1,(k^* - \ell^* + 1)}) \right] - \Delta_{H_1,(k^* - \ell^* + 1)} \left( \frac{\epsilon}{3} \right) < \mathbb{E} \left[ W_{H_0,(\ell^*)} \right] + \Delta_{H_0,(\ell^*)} \left( \frac{\epsilon}{3} \right).$  

(25b)

See Appendix A.1 for the proof of this claim.

**Remarks:** The main value of Lemma 1 lies in inequality (25a). Indeed, this inequality places a lower bound on an expected order statistic coming from a signal in terms of an expected order statistic coming from a null (plus some deviations). Since signals are shifted rightward relative to nulls, a lower bound of this kind gives a lower bound on the signal strength in terms of $\ell^\star$. Meanwhile, Theorem 1 provides upper bounds on $\ell^\star$ (and $m - k^* + \ell^\star$) in terms of the realized FDR and FNR. Together, these bounds yield a lower bound on signal strength in terms of FDR and FNR, which can be interpreted as a lower bound on FDR and FNR in terms of the signal strength.
4.1.2 Transferring results between models

It is useful to be able to transfer results from simple models to more complex models that are in some sense “close” to them. In this section, we specify a notion of closeness that makes sense for our problem and prove a technical result that allows us to transfer lower bounds between close models.

Our definition of closeness for models has some unusual features that bear explanation. First, it only applies to models that share a single transformation function $f$. This limitation is imposed for convenience and is not fundamental. The definition is also asymmetric, with some base model $M$ given, and the closeness of another model $M'$ assessed relative to $M$. The asymmetry arises from the fact that we wish to define proximity of models based on a single fixed distribution over the order statistics, rather than uniformly over all distributions over order statistics, and the single fixed reference point we choose arises from the discovery and false discovery number proxies $k^+_\pm$ and $\ell^\pm_\pm$, which depend on the model.

**Definition 1.** We say that two models $M = (P_n, f)$ and $M' = (P'_n, f)$ are $(\Delta_0, \Delta_1, \ell^+_\pm, \delta)$-close if

$$\max \left\{ \mathbb{P}(|W_{H_0,(f^*)} - W'_{H_0,(f^*)}| \geq \Delta_0), \mathbb{P}(|f(W_{H_1,(k^*-f^*)}) - f(W'_{H_1,(k'^*-f^*)})| \geq \Delta_1) \right\} \leq \delta,$$

where $\ell^* = \ell^*_+(\alpha, \delta, M)$ and $k^* = k^*_+(\alpha, \delta)$. Similarly, we say that they are $(\Delta_0, \Delta_1, \ell^-_\pm, \delta)$-close if the same condition holds with $\ell^* = \ell^-_-(\beta, \delta, M)$ and $k^* = k^-_+(\beta, \delta)$.

Based on this definition, we can transfer lower bounds from the base model in Definition 1 to the other model using the following technical lemma.

**Lemma 2.** For a given pair $(\alpha, \beta)$ with $2 \max\{\alpha, \beta\} < \frac{1}{3}$, consider some $\epsilon \in (2 \max\{\alpha, \beta\}, \frac{1}{3})$.

(a) If $(P'_n, f)$ is $(\Delta_0, \Delta_1, \ell^+_\pm, \epsilon/3)$-close to $(P_n, f)$, then

$$\ell^+_+(\alpha, \epsilon, (P'_n, f - \Delta_0 - \Delta_1)) \geq \ell^+_+(\alpha, \epsilon, (P_n, f)). \quad (26a)$$

(b) If $(P'_n, f)$ is $(\Delta_0, \Delta_1, \ell^-_\pm, \epsilon/3)$-close to $(P_n, f)$, then

$$\ell^-_-(\beta, \epsilon, (P'_n, f - \Delta_0 - \Delta_1)) \geq \ell^-_-(\beta, \epsilon, (P_n, f)). \quad (26b)$$

See Appendix A.2 for the proof of this claim.

We use Lemma 2 primarily to remove dependence. In that context, a particularly useful specialization of it is the following decoupling lemma, which allows us to remove dependence between nulls and signals provided that we can verify the concentration condition.

**Lemma 3.** For a given model $M = (P_n, f)$, let $M'$ denote the same model but with nulls and signals sampled independently from their marginals under $P_n$. Suppose that $2 \max\{\alpha, \beta\} < \frac{1}{3}$, and that $M'$ satisfies Case I of the concentration assumption. Then for any $\epsilon \in (2 \max\{\alpha, \beta\}, \frac{1}{3})$, with the integers $k^* = k^*_+(\alpha, \epsilon), \ell^\prime := \ell^+_+(\alpha, M', \frac{\epsilon}{3})$, the scalar
\[
\Delta = 2\left[\Delta_{\mathcal{H}_0, (\ell'; \epsilon/6)}(\epsilon/6) + \Delta_{\mathcal{H}_1, (k' - \ell' + 1)}(\epsilon/6)\right], \text{ and the model } \mathcal{M}' = (\mathbb{P}_n, w \mapsto f(w) - 2\Delta), \text{ we have}
\]

\[
\ell^*_+(\alpha, \mathcal{M}'', \epsilon) \geq \ell^*_+(\alpha, \mathcal{M}', \epsilon/3), \tag{27a}
\]

Similarly, with \( k^* = k^*_-(\beta, \epsilon), \ell''^* := \ell^*_-(\beta, \mathcal{M}', \epsilon/4) \), and \((\Delta, \mathcal{M}'')\) redefined accordingly, we have

\[
\ell^*- (\beta, \mathcal{M}'', \epsilon) \geq \ell^*- (\beta, \mathcal{M}', \epsilon/3). \tag{27b}
\]

See Appendix A.3 for the proof of this claim.

### 4.2 Proof of Theorem 1

In this section, we prove Theorem 1. Two main ideas underlie the proof. First, we show that any top-\( K \) procedure that is \((\alpha, \beta)\)-controlled must have at least a constant probability of making approximately the correct number of discoveries (meaning that the number of discoveries is equal to the true number of signals). In order to formalize this idea, for a given top-\( K \) procedure \( K \), define the event

\[
\mathcal{E}_{\text{band}} := \{ K \in \left[ k^*_-(\beta), k^*_+(\alpha) \right] \}, \tag{28}
\]

where \( k^*_+ \) and \( k^*_- \) are the discovery proxies (8). The width of this band is determined by \( \alpha \) and \( \beta \) and by the constant \( \epsilon \), which will play the role of a parameter in the analysis throughout our proofs.

**Lemma 4.** For any \((\alpha, \beta)\)-controlled top-\( K \) procedure, we have \( \mathbb{P}[\mathcal{E}_{\text{band}}] \geq 1 - 2\epsilon \).

We defer the proof of this lemma to Section 4.2.1.

The main second ingredient is a precise version of the argument that led to the inequality (11). Essentially, we need to know that the event defined by (11) really is the same as the event defined by the number of false discoveries in the top \( k \) being lower bounded by \( \ell \). We define

\[
L(k) = \left| \{ i : X_i \geq X(k) \} \cap \mathcal{H}_0 \right|, \tag{29}
\]

corresponding to the number of false discoveries in the top \( k \). In terms of this notation, we have the following:

**Lemma 5.** We have

\[
\{ L(k) \geq \ell \} = \{ W_{\mathcal{H}_0, (\ell)} > f(W_{\mathcal{H}_1, (k - \ell + 1)}) \} \text{ for each } k = 1, 2, \ldots, n. \tag{30}
\]

See Section 4.2.2 for the proof of this claim.
Equipped with these lemmas, we now turn to the proof of the theorem. Define the events 
\[ \mathcal{E}_{\text{proxy}, -} = \{ L(k^*_z(\beta)) \geq \ell^*_-(\beta) \} \quad \text{and} \quad \mathcal{E}_{\text{proxy}, +} = \{ L(k^*_z(\alpha)) \geq \ell^*_+(\alpha) \} \].

By applying Lemma 5 twice, once with the choice \( k = k^*_z(\beta) \) and then with the choice \( k = k^*_z(\alpha) \), and using the definitions of \( \ell^*_+ \) and \( \ell^*_+ \) (see equation (9)), we have
\[
\min \left\{ \mathbb{P}[\mathcal{E}_{\text{proxy}, -}], \mathbb{P}[\mathcal{E}_{\text{proxy}, +}] \right\} \geq 1 - \epsilon. \tag{31}
\]

Next, combining Lemma 4 and the bound (31) yields
\[
\mathbb{P}\left( E_{\text{band}} \cap \mathcal{E}_{\text{proxy}, -} \right) \geq 1 - 3\epsilon \quad \text{and} \quad \mathbb{P}\left( E_{\text{band}} \cap \mathcal{E}_{\text{proxy}, +} \right) \geq 1 - 3\epsilon. \tag{32}
\]

**Argument for negative proxies:** We now use the bound (32) to proof the theorem’s claims for the negative-subscript proxies. Note that \( L(K) = K \cdot \text{FDP}(K) \), so that on conditioned on \( \mathcal{E}_- \), we have
\[
\ell^*_- \leq K \cdot \text{FDP}(K) \leq k^*_+ \cdot \text{FDP}(K) \quad \implies \quad \text{FDP}(K) \geq \frac{\ell^*_-}{k^*_+} \leq \left( 1 - \frac{\alpha}{\epsilon} \right) \cdot \text{FDP}^*.
\]

We now take expectations to find that
\[
\text{FDR}(K) \geq \mathbb{P}[\mathcal{E}_-] \cdot \mathbb{E}[\text{FDP}(K) \mid \mathcal{E}_-] \overset{(i)}{\geq} \left( 1 - 3\epsilon \right) \left( 1 - \frac{\alpha}{\epsilon} \right) \text{FDP}^*.
\]

where step (i) uses the lower bound (32).

Recalling that \( \text{FDR}(K) \leq \alpha \) by assumption and rearranging the inequality, we find that
\[
\text{FDP}^* \leq \frac{2}{1 - 3\epsilon} \cdot \alpha,
\]

where we have also used the assumed inequality \( \frac{\alpha}{\epsilon} \leq \frac{1}{2} \). This establishes the first inequality in line (12a).

We now prove the second inequality in line (12a). Observe that the number of non-discovered signals in the top \( k^*_+ \) statistics can be lower bounded as
\[
m \cdot \text{FNP}(k^*_+) = m - (k^*_+ - L(k^*_+)) \geq m - (k^*_+ - \ell^*_+) = m \cdot \text{FNP}^*.
\]

Next note that conditioned on the event \( \mathcal{E}_- \), we have
\[
m \cdot \text{FNP}(k^*_+) \leq m \cdot \text{FNP}(K) + (K - k^*_+) \leq m \cdot \text{FNP}(K) + (k^*_+ - k^*_+) \leq m \cdot \left[ \text{FNP}(K) + 2\epsilon^{-1}(\alpha + \beta) \right],
\]

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where we have used the fact that
\[ k^*_+ - k^*_- \leq \frac{\epsilon^{-1}(\alpha + \beta)}{1 - \epsilon^{-1}} \leq 2\epsilon^{-1}(\alpha + \beta). \]

Once again taking conditional expectations, dividing through by \( \mathbb{P}[E_-] \), and using the bound on \( \text{FNR}(K) \), we find
\[ m \cdot \text{FNP}(k^*_-) \leq m \cdot \left[ \frac{\beta}{1 - 3\epsilon} + 2\epsilon^{-1}(\alpha + \beta) \right]. \]

Putting it all together, we conclude that
\[ \text{FNP}^*_+ \leq \frac{\beta}{1 - 3\epsilon} + 2\epsilon^{-1}(\alpha + \beta) \]
\[ \leq \left( \frac{1}{1 - 3\epsilon} + 2\epsilon^{-1} \right) \cdot (\alpha + \beta) \]
\[ \leq \left( \frac{2}{1 - 3\epsilon} + 4\epsilon^{-1} \right) \cdot \max\{\alpha, \beta\}. \]

**Argument for + proxies:** The argument for the bounds in line \((12b)\) based on positive-subscripted proxies is similar, so that we merely sketch it. By reasoning similar to that used for \( \text{FNR} \) above, we can show that
\[ \ell^*_+ \leq m \cdot \left[ \frac{2}{1 - 3\epsilon} \cdot \alpha + 2\epsilon^{-1}(\alpha + \beta) \right]. \]

Rearranging as before yields the inequality \( \text{FDP}^*_+ \leq \left( \frac{4}{1 - 3\epsilon} + 4\epsilon^{-1} \right) \cdot \max\{\alpha, \beta\} \). Conditioned on \( \mathcal{E}_+ \), we have
\[ m - k^*_+ + \ell^*_+ \leq m \cdot \text{FNP}(k^*_+) \leq m \cdot \text{FNP}(K) \]
\[ \implies \text{FNP}(K) \geq \text{FNP}^*_+. \]

Taking conditional expectations and dividing by the probability \( \mathbb{P}[\mathcal{E}_+] \), we conclude that
\[ \text{FNP}^*_+ \geq \frac{1}{1 - 3\epsilon} \cdot \beta, \]
which completes the proof of the first inequality. The proof of the second inequality is analogous to the negative-subscripted case.

**4.2.1 Proof of Lemma 4**

It suffices to establish the inequalities \( \mathbb{P}[K \geq k^*_+] \leq \epsilon \) and \( \mathbb{P}[K \leq k^*_-] \leq \epsilon \). Beginning with the first inequality, note that \( \text{FDP}(K) \geq \frac{K - m}{R} = 1 - \frac{m}{R} \) and that the lower bound is an
increasing function of $K$. Thus, we have the inclusions
\[
\{ K \geq k^*_+ \} \subset \left\{ \text{FDP(K)} \geq 1 - \frac{m}{k^*_+} \right\} \\
= \left\{ \text{FDP(K)} \geq \frac{1}{1 - \epsilon} \right\} \\
\subset \left\{ \text{FDP(K)} \geq \frac{1}{\epsilon} \text{FDR(K)} \right\}.
\]

Given this set inclusion, we have
\[
P[K \geq k^*_+] \leq P\left[ \text{FDP(K)} \geq \frac{\text{FDR(K)}}{\epsilon} \right] \leq \epsilon,
\]
where the final line follows by Markov’s inequality.

As for the second inequality, note that \( \text{FNP}(K) \geq \frac{m-K}{m} = 1 - \frac{K}{m} \) and that this lower bound is a decreasing function of $K$. Thus, we have the inclusions
\[
\{ K \leq k^*_-(\beta) \} \subset \left\{ \text{FNP(K)} \geq 1 - \frac{k^*_-}{m} \right\} \\
= \left\{ \text{FNP(K)} \geq \frac{1}{\epsilon} \beta \right\} \\
\subset \left\{ \text{FNP(K)} \geq \frac{1}{\epsilon} \text{FNR(K)} \right\}.
\]

As before, applying Markov’s inequality yields the claim.

### 4.2.2 Proof of Lemma 5

Suppose that \( W_{H_0,(-\ell)} > f(W_{H_1,(-\ell+1)}) \), or equivalently, \( X_{H_0,(-\ell)} > X_{H_1,(-\ell+1)} \). Define the set \( I = \{ i : X_i \geq X_{H_0,(-\ell)} \} \), and note that if \( |I| \leq k \), then necessarily \( X_{H_0,(-\ell)} \) is one of the top \( k \) statistics, so that \( L(k) \geq \ell \). But, by the hypothesis and the definition of order statistics,
\[
|I \cap S| \leq k - \ell \quad \text{and} \quad |I \cap H_0| = \ell.
\]
Thus \( |I| = |I \cap S| + |I \cap H_0| \leq k \), as required.

We now turn to the converse implication. Concretely, fixing some \( k \in [n] := \{1, 2, \ldots, n\} \) for which \( L(k) \geq \ell \), we prove that \( W_{H_0,(-\ell)} > f(W_{H_1,(-\ell+1)}) \). Let \( i_k \in [n] \) be the \( k^{th} \)-largest rank statistic—that is, the index corresponding to the order statistic \( X_{(k)} \). and we break our analysis into two cases, depending on whether \( i_k \in H_1 \) or \( i \in H_0 \).

**Case 1, \( i_k \in H_1 \):** In this case, since there are at most \( k - \ell \) signals in the top \( k \) statistics, we must have \( X_{i_k} \geq X_{H_1,(-\ell)} \). On the other hand, for any \( j \in H_0 \) such that \( X_j \) falls in the top \( k \), we must have \( X_j > X_{i_k} \). Since there are at least \( \ell \) such indices, we conclude
\[
W_{H_0,(-\ell)} = X_{H_0,(-\ell)} > X_{i_k} \geq X_{H_1,(-\ell)} = f(W_{H_1,(-\ell)}) > f(W_{H_1,(-\ell+1)}).
\]
Rearranging yields the claim.
Case 2, $i_k \in H_0$: Since the number of signals in the top $k$ is $< k - \ell + 1$, it must be that $X_{H_1,(k-\ell+1)} < X_{i_k} \leq X_{H_0,(\ell)}$. Rearranging again gives the claim.

4.3 Proof of Corollary 1

Let $\chi_{k,n}$ denote the expected value of the $k^{th}$-largest value in a sample of $n$ independent standard Gaussians. Recalling the definition (24) of the concentration function, classical results on Gaussian order statistics ensure that we can apply the concentration assumption with $\Delta_{k,n}(\epsilon) := \sqrt{2 \log 2 \over \epsilon}$. Although this specification is not the sharpest possible, it suffices for our purposes.

Our proof of Corollary 1 is based primarily on comparing Gaussian order statistics to $\mu$. In particular, we wish to establish that the inequality

$$\chi_{\ell^*+1,n-m} + \chi_{m-k^*+\ell^*,m} > \mu - 2\sqrt{2 \log 6 \over \epsilon}$$

(33)

holds with the choices $(\ell^*, k^*) = (\ell^*_-(\beta), k^*_-(\beta))$ or $(\ell^*, k^*) = (\ell^*_+(\alpha), k^*_+(\alpha))$. The proof is identical for these two cases, so we simply use the shorter $(\ell^*, k^*)$ notation throughout.

Taking inequality (33) as given for the moment, we first use it to prove Corollary 1. In order to do so, we require the following:

Lemma 6. We have

$$\sqrt{2 \log n \over \xi_{k,n}} = 1 + o(1) \quad \text{for } k = 1, \ldots, 2, \ldots, n,$$

(34)

where the $o(1)$ decay holds as $n \to \infty$ and/or $k \to \infty$.

We now proceed with the proof, suppressing $n$ subscripts throughout so as to avoid clutter. Combining inequality (33), Lemma 6, and the fact that $m \ll n$, we find that

$$\mu > \chi_{\ell^*+1,n-m} + \chi_{m-k^*+\ell^*,m} - c_1 \geq (1 - o(1)) \left[ \sqrt{\log \frac{n}{\ell^* + 1}} + \sqrt{\log \frac{m}{m - k^* + \ell^*}} \right],$$

where $c_1 > 0$ is a constant that may depend on $\epsilon$.

We now invoke Theorem 1 with $(\ell^*, k^*) = (\ell^*_+, k^*_+)$ or $(\ell^*, k^*) = (\ell^*_-, k^*_*)$, according to whether we are in the regime $\beta \geq \alpha$ or vice versa. Applying the theorem, rearranging, and substituting the value of $\mu$ yields

$$\sqrt{\log \frac{n}{c_0 \alpha m + 1}} + \sqrt{\log \frac{m}{c_0 \beta m}} \leq (1 - o(1))^{-1} \sqrt{2r \log n} = (1 + o(1)) \sqrt{2r \log n}. \quad (35)$$

We claim that it suffices to prove that $\min \{ \alpha, \beta \} \cdot m \to \infty$. Indeed, under this scaling, for large enough $(n, m)$, we would have

$$\sqrt{2 \log \frac{n}{2c_0 \alpha m}} + \sqrt{2 \log \frac{m}{2c_0 \beta m}} \leq (1 + o(1)) \cdot \sqrt{2r \log n}.$$

---

2To clarify a subtle point that we have elided: Lemma 6 requires that $k < {n \over \log n}$, so we need to check $\ell^* + 1 \leq {n \over \log n}$ and $m - k^* + \ell^* \leq {n \over \log n}$. Since $\max \{ \ell^*, m - k^* + \ell^* \} \leq \max \{ q, \beta \} \cdot m$ by Theorem 1, this condition is in fact easily verified under the given scalings for $\alpha$ and $\beta$. 

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Substituting the assumed scalings $\alpha = cn^{-\kappa_\alpha}$ and $\beta = cn^{-\kappa_\beta}$ then yields

$$\sqrt{2(s + \kappa_\alpha)} \log n + \log \frac{1}{2c_0} + \sqrt{2\kappa_\beta \log n + \log \frac{1}{2c_0}} \leq (1 + o(1)) \cdot \sqrt{2r \log n},$$

and letting $n \to \infty$ yields the claimed inequality $\sqrt{s + \kappa_\alpha + \kappa_\beta} \leq \sqrt{r}$.

It remains to prove that $\min\{\alpha, \beta\} \cdot m \to \infty$, and we split our analysis into two cases.

**Case 1:** Suppose first that $\alpha \leq \beta$ and assume by way of contradiction that there exists a constant $c_2$ such that $\alpha m \leq c_2$ for all $n$. Combined with the inequality (35), we find that

$$\sqrt{2 \log \frac{n}{c_0 m + 1}} + \sqrt{2 \log \frac{m}{c_0 \beta m}} \leq (1 + o(1)) \cdot \sqrt{2r \log n}.$$ 

Since $r < 1$, this inequality cannot hold once $n$ is large enough, which establishes the desired contradiction.

**Case 2:** Turning to the other case, suppose that $\beta \leq \alpha$, and assume by way of contradiction that there exists a constant $c_2$ such that $\beta m \leq c_2$ for all $n$. In this case, again by inequality (35), we have

$$\sqrt{2 \log \frac{n}{c_0 \alpha m + 1}} + \sqrt{2 \log \frac{m}{c_0 m}} \leq (1 + o(1)) \cdot \sqrt{2r \log n}.$$ 

On the other hand, for a suitable choice of $c_3$, we have

$$\sqrt{2 \log \frac{n}{c_0 \alpha m + 1}} + \sqrt{2 \log \frac{m}{c_0 m}} \geq \sqrt{2 \log \frac{n}{4c_0 m}} + \sqrt{2 \log \frac{m}{c_0 m}}$$

$$\geq \sqrt{2 s \log n + \log \frac{1}{c_3}} + \sqrt{2(1 - s) \log n + \log \frac{1}{c_3}}.$$

Putting together the pieces, we have shown that

$$\sqrt{2 s \log n + \log \frac{1}{c_3}} + \sqrt{2(1 - s) \log n + \log \frac{1}{c_3}} \leq (1 + o(1)) \cdot \sqrt{2r \log n}.$$ 

Since $\sqrt{s} + \sqrt{1 - s} > 1 > \sqrt{r}$, this inequality cannot hold once $n$ is sufficiently large, which establishes the desired contradiction in this case.

**4.3.1 Proof of inequality (33)**

Applying Lemma 1 with $\Delta_{\mathcal{H}_0,k} = \Delta_{\mathcal{H}_1,k} = \sqrt{2 \log \frac{2}{\epsilon}}$, we find that

$$\mu + \chi_{k^* - \ell^* , m} + \sqrt{2 \log \frac{2}{\epsilon}} > \chi_{\ell^* + 1, n - m} - \sqrt{2 \log \frac{2}{\epsilon}}.$$

Rearranging yields

$$\mu > \chi_{\ell^* + 1, n - m} - \chi_{k^* - \ell^* , m} - 2 \sqrt{2 \log \frac{2}{\epsilon}}.$$ 

Since the Gaussian distribution is symmetric around zero, we can replace $-\chi_{k^* - \ell^* , m}$ by $\chi_{m - k^* + \ell^* , m}$, which yields the desired inequality.
4.4 Proof of Corollary 2

By analogy to the notation in Section 4.3, let \( \chi_{k,n} \) denote the expected value of the \( k \)-th largest value in a sample of \( n \) variables, each of which is the absolute value of a standard Gaussian. Other notational conventions are also preserved. In particular, we let \((k^*, \ell^*)\) stand in for either \((k_+^*, \ell_+^*)\) or \((k_-^*, \ell_-^*)\), depending on whether \( \beta \geq \alpha \) or vice versa. We also suppress \( n \) subscripts.

By applying Lemma 1 in this case, we find that

\[
\sigma \cdot \chi_{k^* - \ell^*, m} + \sqrt{2 \log \frac{6}{\epsilon}} \geq \chi_{\ell^* + 1, n - m} - \sqrt{2 \log \frac{6}{\epsilon}},
\]

and rearranging yields

\[
\sigma \geq \chi_{k^* - \ell^*, m} \left[ \chi_{\ell^* + 1, n - m} + 2 \sqrt{2 \log \frac{6}{\epsilon}} \right]. \tag{36}
\]

We now need to evaluate the \( \chi \) values, and we make use of the following result due to Gordon et al. [14]:

**Lemma 7.** For all \( k \geq n/2 \), we have \( \chi_{k,n} \leq \sqrt{2\pi} \cdot \frac{n-k^*+1}{n+1} \). Moreover, we have

\[
\frac{\sqrt{2 \log \frac{6}{\epsilon}}}{\chi_{k,n}} = 1 \pm o(1) \quad \text{for all } k = 1, 2, \ldots, \left\lfloor \frac{n}{\log n} \right\rfloor, \tag{37}
\]

where the \( o(1) \) scaling holds as \( n \) and possibly \( k \) go to infinity.

Suppose for now that \( \ell^* + 1 \leq \frac{n-m}{\log n - m} \) and that \( k^* - \ell^* \geq \frac{m}{\ell^*} \). Recall from Theorem 1 applied with the appropriate choice of + or − proxies, that

\[
\ell^* \leq c_0 \alpha m \quad \text{and} \quad m - k^* + \ell^* \leq c_0 \beta m. \tag{38}
\]

Consequently, we have the lower bound

\[
\chi_{\ell^* + 1, n - m} \geq (1 - o(1)) \cdot \sqrt{2 \log \frac{n-m}{\ell^* + 1}} \geq \left(1 - o(1)\right) \cdot \sqrt{2 \log \frac{n-m}{c_0 \alpha + 1/m}} \geq \left(1 - o(1)\right) \cdot \sqrt{2 s \log n + 2 \log \left( \frac{\alpha + 1}{m} \right)^{-1}}. \tag{39a}
\]

On the other hand, Lemma 7 also implies that

\[
\chi_{k^* - \ell^*, m} \leq \sqrt{2\pi} \cdot \frac{m-k^* + \ell^* + 1}{m+1} \leq \sqrt{2\pi} \cdot \left( \frac{m-k^* - \ell^*}{m} + \frac{1}{m} \right) \leq \sqrt{2\pi} c_0 \cdot (\beta + \frac{1}{m}). \tag{39b}
\]

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Combining the bounds (39a) and (39b) with inequality (36), we find that

$$\sigma \geq \frac{1}{\sqrt{2\pi c_0}} \cdot (1 - o(1)) \left( \beta + \frac{1}{m} \right)^{-1} \left[ \sqrt{2s \log n + 2 \log \left( \alpha + \frac{1}{m} \right)^{-1}} + 2 \sqrt{2 \log \frac{6}{\epsilon}} \right].$$

Since $s \geq \rho > 0$, we have that

$$\sqrt{2\beta \log n + 2 \log \left( \alpha + \frac{1}{m} \right)^{-1}} \gg 2 \sqrt{2 \log \frac{6}{\epsilon}},$$

so that

$$\sigma \geq \frac{1}{\sqrt{2\pi c_0}} \cdot (1 - o(1)) \left( \beta + \frac{1}{m} \right)^{-1} \sqrt{2s \log n + 2 \log \left( \alpha + \frac{1}{m} \right)^{-1}},$$

as claimed.

Under the specified scalings, we have $\alpha + \frac{1}{m} \approx n^{\kappa} \alpha$ and $\beta + \frac{1}{m} \approx n^{\kappa}$, which directly implies the comparison

$$\sigma \gtrsim n^{\kappa} \sqrt{2(s + \kappa) \log n}.$$

We now need to verify that $\ell^* + 1 \leq \frac{n - m}{\log (n - m)}$ and $k^* - \ell^* \geq \frac{m}{2}$. From the inequalities (38), we deduce that

$$\ell^* + 1 \leq c_0 \alpha m + 1 \quad \text{and} \quad k^* - \ell^* \geq \left(1 - c_0 \beta \right)m.$$

Note that, by the assumption that $s \geq \rho$, we have $m \leq n^{1-\rho}$, we have $\frac{n - m}{\log (n - m)} \geq \frac{n}{2 \log n}$ for large enough $n$. On the other hand, we also have $\alpha m + 1 \leq n^{1-\rho} + 1 \leq \frac{n}{2 \log n}$ (once $n$ large enough—say for all $n$ such that $n^\rho \geq 4 \log n$, for instance). For the second case, recall the assumption $c_0 \beta \leq \frac{1}{3} < \frac{1}{2}$, from which the claim follows.

### 4.5 Proof of Corollary 3

At a high level, this proof involves reducing to an independent model with altered variances and using Lipschitz concentration to verify the closeness condition of Lemma 2.

We carry out the reduction in two steps: in Step 1, we reduce to a model with dependence only between nulls and signals, and then in Step 2, we reduce to an independent model. The models in Steps 1 and 2 are Gaussian models with covariance matrices $\Sigma'$ and $\Sigma''$, respectively, of the form

$$\Sigma'_{ij} = \begin{cases} 
1 - \rho_0 + \rho_c & \text{if } i = j, \; i \in \mathcal{H}_0, \\
1 - \rho_1 + \rho_c & \text{if } i = j, \; i \notin \mathcal{H}_0, \\
\rho_c & \text{if } i \in \mathcal{H}_0, \; j \notin \mathcal{H}_0, \\
0 & \text{o.w.}
\end{cases}$$

and

$$\Sigma''_{ij} = \begin{cases} 
1 - \rho_0 & \text{if } i = j, \; i \in \mathcal{H}_0, \\
1 - \rho_1 & \text{if } i = j, \; i \notin \mathcal{H}_0, \\
0 & \text{o.w.}
\end{cases}$$

We let $W'$ and $W''$ corresponding the corresponding Gaussian random vectors in $\mathbb{R}^n$. The shifts associated with these models are set to be constant-scale perturbations of $\mu$, so that overall, we have the two models

$$M' = \left( \mathbb{P}'_n, \; \mu + c_1 \sqrt{2 \log \frac{c_2}{\epsilon}} \right) \quad \text{and} \quad M'' = \left( \mathbb{P}_n, \; \mu + 2c_1 \sqrt{2 \log \frac{c_2}{\epsilon}} \right),$$

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where $\mathbb{P}_n'$ and $\mathbb{P}_n''$ are the Gaussian distributions associated with $\Sigma'$ and $\Sigma''$. We also introduce the convenient shorthand notation

$$\ell^{*'} = \ell^* \left( \mathcal{M}', \frac{c}{3} \right) \text{ and } \ell^{*''} = \ell^* \left( \mathcal{M}'', \frac{c}{9} \right).$$

The main idea of the proof is to represent the $W$ variables as functions of higher-dimensional Gaussians. This representation is helpful in decoupling the test statistics from each other. Basically, the constant covariance within the nulls and signals, and across the two, allows us to represent each test statistic as independent of all the others after conditioning on three standard Gaussians that contain all the shared randomness: one each for the within-nulls, within-signals, and between-nulls-and-signals randomness. More precisely, we can write

$$W_i = \begin{cases} \sqrt{1 - \rho_0} \cdot U_i + \sqrt{\rho_0 - \rho_c} \cdot V_0 + \sqrt{\rho_c} \cdot V_c & \text{if } i \in \mathcal{H}_0, \\ \sqrt{1 - \rho_1} \cdot U_i + \sqrt{\rho_1 - \rho_c} \cdot V_1 + \sqrt{\rho_c} \cdot V_c & \text{if } i \notin \mathcal{H}_0 \end{cases},$$

$$W_i' = \begin{cases} \sqrt{1 - \rho_0} \cdot U_i' + \sqrt{\rho_0 - \rho_c} \cdot V_c' & \text{if } i \in \mathcal{H}_0, \\ \sqrt{1 - \rho_1} \cdot U_i' + \sqrt{\rho_1 - \rho_c} \cdot V_c' & \text{if } i \notin \mathcal{H}_0 \end{cases},$$

$$W_i'' = \begin{cases} \sqrt{1 - \rho_0} \cdot U_i'' & \text{if } i \in \mathcal{H}_0, \\ \sqrt{1 - \rho_0} \cdot U_i'' & \text{if } i \notin \mathcal{H}_0 \end{cases}.$$

The link functions that connect the $U$ and $V$ variables to the order statistics of the $W$ variables are given, in the three cases, by

$$a_{0,\ell}(u, v_{0:1}, v_c) = \left( \sqrt{1 - \rho_0} \cdot u + \sqrt{\rho_0 - \rho_c} \cdot v_0 + \sqrt{\rho_c} \cdot v_c \right)(t),$$

$$b_{0,\ell}(u, v_c) = \left( \sqrt{1 - \rho_c} \cdot u + \sqrt{\rho_0 - \rho_c} \cdot v_c \right)(t),$$

$$c_{0,\ell}(u) = \sqrt{1 - \rho_0} \cdot u(t),$$

and similarly for the signals, for which we denote the link functions by $a_1$, $b_1$, and $c_1$.

Our first aim is to prove $\ell^* \geq \ell^{*'}$ using Lemma 2. We begin by observing that

$$|a_{0,\ell}(u, v_{0:1}, v_c) - b_{0,\ell}(u, v_c)| \leq |v_0|,$$

from which it follows that

$$\left| W_{\mathcal{H}_0(\ell^{*'})} - W_{\mathcal{H}_0(\ell^{*})} \right| \leq \Delta \leq \left\{ \begin{array}{cl} |a_{0,\ell^{*'}}(U, V_{0:1}, V_c) - b_{0,\ell^{*'}}(U', V_c')| & \leq \frac{\Delta}{2} \\ \{ |V_0| \leq \frac{\Delta}{2} \} \cap \{ |b_{0,\ell^{*'}}(U, V_c) - b_{0,\ell^{*'}}(U', V_c')| \leq \frac{\Delta}{2} \} \\ \{ |V_0| \leq \frac{\Delta}{2} \} \cap \{ |b_{0,\ell^{*'}}(U, V_c) - \mathbb{E}[b_{0,\ell^{*'}}(U, V_c)]| \leq \frac{\Delta}{4} \} \cap \{ |b_{0,\ell^{*'}}(U', V_c') - \mathbb{E}[b_{0,\ell^{*'}}(U', V_c')]| \leq \frac{\Delta}{4} \}.\right.$$
A similar analysis yields that the event \( \left\{ W_{\mathcal{H}_1,(k^{\ast}_-\ell^{\ast}_-)} - W'_{\mathcal{H}_1,(k^{\ast}_-\ell^{\ast}_-)} \leq \Delta \right\} \) contains the event

\[
\begin{align*}
\left\{ |V_1| \leq \frac{\Delta}{2} \right\} \cap \left\{ b_{1,k^{\ast}_-\ell^{\ast}_-}(U, V_c) - \mathbb{E}[b_{1,k^{\ast}_-\ell^{\ast}_-}(U, V_c)] \leq \frac{\Delta}{4} \right\} \\
\cap \left\{ b_{1,k^{\ast}_-\ell^{\ast}_-}(U', V_c) - \mathbb{E}[b_{1,k^{\ast}_-\ell^{\ast}_-}(U', V_c)] \leq \frac{\Delta}{4} \right\}.
\end{align*}
\]

By Lipschitz concentration, we may choose \( \Delta = c'_1 \sqrt{2 \log \frac{c_2}{\epsilon}} \) such that

\[
\max \left\{ \mathbb{P}\left( |W_{\mathcal{H}_0,(\ell^{\ast}_+)} - W'_{\mathcal{H}_0,(\ell^{\ast}_+)}| > \Delta \right), \mathbb{P}\left( |W_{\mathcal{H}_1,(k^{\ast}_-\ell^{\ast}_-)} - W'_{\mathcal{H}_1,(k^{\ast}_-\ell^{\ast}_-)}| > \Delta \right) \right\} \leq \frac{\epsilon}{3}.
\]

In other words, \( \mathcal{M} \) and \( \mathcal{M}' \) are both \( (\Delta, \Delta, \ell^{\ast}_+, \kappa^{\ast}_+) \) and \( (\Delta, \Delta, \ell^{\ast}_+, \kappa^{\ast}_+) \) close. Consequently, if we choose \( c_1 \geq 2c'_1 \) and \( c_2 \geq c'_2 \), then applying Lemma 2 guarantees that \( \ell^{\ast}_- \geq \ell^{\ast} \).

Next observe that \( |b_{0,\ell}(u, v_c) - c_{0,\ell}(u)| \leq |v_c| \). We may therefore apply a variant of the previous argument with \( \Delta' = c'_1 \sqrt{2 \log \frac{c_2}{\epsilon}} \) to show

\[
\max \left\{ \mathbb{P}\left( |W'_{\mathcal{H}_0,(\ell^{\ast}_-)} - W''_{\mathcal{H}_0,(\ell^{\ast}_-)}| > \Delta' \right), \mathbb{P}\left( |W'_{\mathcal{H}_1,(k^{\ast}_-\ell^{\ast}_-)} - W''_{\mathcal{H}_1,(k^{\ast}_-\ell^{\ast}_-)}| > \Delta' \right) \right\} \leq \frac{\epsilon}{9}.
\]

We then find by Lemma 2 that \( \ell^{\ast}_- \geq \ell^{\ast} \) provided that \( c_1 \geq 2c'_1 \) and \( c_2 \geq c'_2 \).

Combining the two pieces of our argument, we are guaranteed to have \( \ell^{\ast} \geq \ell^{\ast}_- \) as long as \( c_1 \geq 2\cdot \max \{ c'_1, c''_1 \} \) and \( c_2 \geq \max \{ c'_2, c''_2 \} \). Since we have now reduced to the independent case with \( \mu \) changed by a constant, applying suitably rescaled version of Theorem 1 yields the conclusion of the corollary.

### 4.6 Proof of Corollary 4

By our previous arguments for order statistics of Gaussians, we know that for the grouped Gaussian model, Case I of the concentration assumption (24) holds with

\[
\Delta_{\mathcal{H}_0,\ell^{\ast}} = \Delta_{\mathcal{H}_1,k^{\ast}_-\ell^{\ast}_-+1} = c_1 \sqrt{2 \log \frac{c_2}{\epsilon}}.
\]

If we choose the constants \( c_1, c_2 \) sufficiently large, we can ensure that Case II of the concentration condition (24) holds at the same scale in a modified form \( \mathcal{M}' \) of the model in which the dependence between nulls and signals is broken and the shift is altered to \( \mu - c'_1 \sqrt{2 \log \frac{c_2}{\epsilon}} \).

We may therefore apply Lemma 3 to obtain that \( \ell^{\ast} \geq \ell^{\ast}_- \), where \( \ell^{\ast}_- \) is computed in \( \mathcal{M}' \).

Since Case II of the concentration condition (24) holds in the new model, we may apply
Lemma 1. Specifically, if we set $T_0 = H_0(0)$ and $T_1 = H_0 \setminus T_0$, then
\[
\mu - c'_1 \sqrt{2 \log \frac{c'_2}{\epsilon}} \geq \mathbb{E}[W'_{H_0,(\ell^*+1)}] + \mathbb{E}[W'_{H_1,(m-k^*+\ell^*+1)}] - 2c'_1 \sqrt{2 \log \frac{c'_2}{\epsilon}}
\]
\[
\geq \mathbb{E}[W'_{T_0,(\ell^*+1)}] + \mathbb{E}[W'_{H_1,(m-k^*+\ell^*+1)}] - 2c'_1 \sqrt{2 \log \frac{c'_2}{\epsilon}}
\]
\[
= \chi_{\ell^*+1,T_0} + \chi_{m-k^*+\ell^*+1,m} - 2c'_1 \sqrt{2 \log \frac{c'_2}{\epsilon}}.
\]
Since $t < s$, we have $|T_0| \geq n - (\gamma + 1)m \geq n - (n^t + 1)n^{1-s} \geq n - 2n^{1-t} = (1-o(1))n$, and an application of Lemma 6 yields the claim.

4.7 Proof of Corollary 5

In order to simplify the proof, it is convenient to pass to an equivalent model. Consider the new random vector $V = (V_1, \ldots, V_n)$ with components $V_i := \log \frac{1}{1-W_i}$. Note that $V_i$ is distributed as a standard exponential and that if we define the transformation function $g(v) = v/\gamma = Av$ with $A := 1/\gamma$, then
\[
\log \frac{1}{1-f(W_i)} = g(V_i).
\]
With this set-up, the test statistics in the new model are related to the test statistics in the original model by the transformation $x \mapsto \log \frac{1}{1-x}$. Since this transformation is monotonic, any top-$K$ procedure for one can be translated into a top-$K$ procedure for the other, with no change in performance. Likewise, the proxy values are the same for all $\alpha$ and $\beta$. In summary, the two models are equivalent for our purposes.

As in previous proofs, we use $(k^*, \ell^*)$ as a stand-in for $(k^*_+, \ell^*_+)$ or $(k^*_-, \ell^*_+)$, and we suppress $n$ subscripts. We claim that it is sufficient to show that
\[
\mathbb{P} \left[ V_{H_0,(\ell^*+1)} \leq v_- \right] \leq \frac{\epsilon}{3}, \quad (40a)
\]
\[
\mathbb{P} \left[ g(V_{H_1,(k^*-\ell^*)}) \geq v_+ \right] \leq \frac{\epsilon}{3}, \quad (40b)
\]
where
\[
v_+ = \frac{1}{\gamma} \frac{t}{1-t} \quad \text{and} \quad v_- = \log \left( \frac{1}{c_0\pi_1} \left( 1 + 4 \log \frac{3}{\epsilon} \right)^{-1} \right).
\]

Taking inequalities (40a) and (40b) as given for the moment, by the definition of $\ell^*$, we have
\[
\mathbb{P} \left[ g(V_{H_1,(k^*-\ell^*)}) > V_{H_0,(\ell^*+1)} \right] \geq \epsilon.
\]
On the other hand, combining the two bounds above, we see that
\[
\mathbb{P} \left( V_{H_0,(\ell^*+1)} \leq v_- \right) + \mathbb{P} \left( g(V_{H_1,(k^*-\ell^*)}) \geq v_+ \right) \leq \frac{2\epsilon}{3} < \epsilon.
\]
It follows that
\[ \mathbb{P}
\left[
\begin{array}{c}
V_{H_{\ell^*}} > g(V_{H_{\ell^*}}) > V_{H_{0, (\ell^* + 1)}} > v_-
\end{array}
\right] > 0, \]
so \( v_- \leq v_+. \) Rearranging yields
\[ \frac{1}{\gamma} \geq \frac{1 - t}{t} \cdot \log \left( \frac{1}{c_0 \pi_1 \alpha} \left( 1 + 4 \log \frac{3}{\epsilon} \right)^{-1} \right), \]
as claimed.

The only remaining detail is to prove inequalities (40a) and (40b).

4.7.1 Proof of inequality (40a)
Applying Lemma 4.3 from the paper [6] yields
\[ \mathbb{P}
\left[
\begin{array}{c}
V_{H_{0, (\ell^* + 1)}} \leq \log \frac{n}{\ell^*} - z
\end{array}
\right] \leq \exp \left( - \frac{\ell^* (e^z - 1)}{4} \right) \quad \text{for each } z > 0. \]
In particular, choosing \( z = \log \left( 1 + \frac{4 \log \frac{3}{\epsilon}}{\ell^*} \right) \leq \log \left( 1 + 4 \frac{\log \frac{3}{\epsilon}}{\ell^*} \right), \) we deduce that
\[ \mathbb{P}
\left[
\begin{array}{c}
V_{H_{0, (\ell^* + 1)}} \leq \log \left( \frac{n}{\ell^*} \left( 1 + 4 \log \frac{3}{\epsilon} \right)^{-1} \right)
\end{array}
\right] \leq \epsilon/3. \]
We complete the proof by noting that \( \frac{n}{\ell^*} \geq \frac{n}{c_0 \alpha m} = \frac{1}{c_0 \pi_1 \alpha}. \)

4.7.2 Proof of inequality (40b)
The proof is based on the fact that \( V_i = \log \frac{1}{U_i} \) where \( U_i \) is a uniform random variable. Let \( U_{H_1, (j)} \) denote the \( j \)-th smallest value in the sample, which follows a beta distribution with parameters \( j \) and \( m - j + 1 \). We thus have
\[ \mathbb{E}[U_{H_1, (k^*-\ell^*)}] = \frac{k^* - \ell^*}{m + 1}, \quad \text{and} \]
\[ \text{Var}[U_{H_1, (k^*-\ell^*)}] \leq \mathbb{E}[U_{H_1, (k^*-\ell^*)}] \cdot \left( 1 - \mathbb{E}[U_{H_1, (k^*-\ell^*)}] \right) \cdot \frac{1}{m}. \]
Applying Chebyshev’s inequality yields
\[ \mathbb{P}
\left[
\begin{array}{c}
U_{H_1, (k^*-\ell^*)} \leq \frac{k^* - \ell^*}{m + 1} - \sqrt{\frac{3}{\epsilon} \cdot \frac{k^* - \ell^*}{m + 1} \cdot \frac{m - k^* + \ell^*}{m + 1} \cdot \frac{1}{m}}
\end{array}
\right] \leq \epsilon/3. \]
Using the fact that \( \frac{m-k^*+\ell^*}{m+1} \leq c_0 \beta, \) we thus have
\[ \mathbb{P}
\left[
\begin{array}{c}
U_{H_1, (k^*-\ell^*)} \leq 1 - c_0 \beta - 1/m - \sqrt{\frac{3}{\epsilon} c_0 \beta / m}
\end{array}
\right] = \mathbb{P}
\left[
\begin{array}{c}
U_{H_1, (k^*-\ell^*)} \leq 1 - t
\end{array}
\right] \leq \epsilon/3. \quad (41)
We now note that

\[
V_{\mathcal{H}_1,(k^* - \ell^*)} = \log \frac{1}{U_{\mathcal{H}_1,(k^* - \ell^*)}} = \log \frac{1}{1 - (1 - U_{\mathcal{H}_1,(k^* - \ell^*)})} = \log \left( 1 + \frac{1 - U_{\mathcal{H}_1,(k^* - \ell^*)}}{1 - (1 - U_{\mathcal{H}_1,(k^* - \ell^*)})} \right) \leq \frac{1 - U_{\mathcal{H}_1,(k^* - \ell^*)}}{1 - (1 - U_{\mathcal{H}_1,(k^* - \ell^*)})}.
\]

Applying the bound (41) yields \( P \left[ V_{\mathcal{H}_1,(k^* - \ell^*)} \geq \frac{t}{1-t} \right] \leq \epsilon/3 \), as claimed. This inequality completes the proof since \( g \) is monotonically increasing and \( g \left( \frac{t}{1-t} \right) = \frac{1}{\gamma} \cdot \frac{t}{1-t} \).

5 Discussion

In this paper, we introduced a framework for establishing the tradeoffs between the false discovery rate (FDR) and the false non-discovery rate (FNR) in multiple testing problem. While a substantial literature on the multiple testing problem has developed, comparatively little has been established about the fundamental tradeoffs between these two types of errors, or about the fundamental limits on the combined risk (a weighted combination of FDR and FNR). Moreover, this problem does not appear to be amenable to the standard techniques for establishing lower bounds used in estimation theory, for instance.

The framework we have put forward is fairly general, not being sensitive to the analytic form of the test statistic distributions or on the dependence structure between the test statistics. Instantiated for models previously studied in the literature, our general results recover and extend lower bounds previously proven using methods more tailored to Gaussian-like models. Furthermore, the lower bounds predicted by our theory can be numerically simulated for any given model, an unusual feature useful for both further theoretical work and potential applications.

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References

[1] E. Arias-Castro and S. Chen. Distribution-free multiple testing. Electronic Journal of Statistics, 11(1):1983–2001, 2017.
[2] Rina Barber and Aaditya Ramdas. The p-filter: Multilayer false discovery rate control for grouped hypotheses. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 79, 11 2016. doi: 10.1111/rssb.12218.

[3] Y. Benjamini and Y. Hochberg. Controlling the false discovery rate: A practical and powerful approach to multiple testing. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 57(1):289–300, 1995.

[4] Y. Benjamini and Y. Hochberg. Multiple hypotheses testing with weights. *Scandinavian Journal of Statistics*, 24(3):407–418, 1997.

[5] Y. Benjamini and D. Yekutieli. The control of the false discovery rate in multiple testing under dependency. *Annals of Statistics*, 29(4):1165–1188, 2001.

[6] Stéphane Boucheron and Maud Thomas. Concentration inequalities for order statistics. *Electronic Communications in Probability*, 17:12 pp., 2012.

[7] D. Donoho and J. Jin. Higher criticism for detecting sparse heterogeneous mixtures. *Annals of Statistics*, 32(3):962–994, 2004.

[8] D. Donoho and J. Jin. Higher criticism for large-scale inference, especially for rare and weak effects. *Statistical Science*, 30(1):1–25, 2015.

[9] Bradley Efron. *Large-scale inference: empirical Bayes methods for estimation, testing, and prediction*, volume 1. Cambridge University Press, 2012.

[10] D.P. Foster and R.A. Stine. α-investing: A procedure for sequential control of expected false discoveries. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 70(2):429–444, 2008.

[11] R. Foygel Barber and E.J. Candès. Controlling the false discovery rate via knockoffs. *Annals of Statistics*, 43(5):2055–2085, 2015.

[12] C.R. Genovese and L. Wasserman. Operating characteristics and extensions of the false discovery rate procedure. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 64(3):499–517, 2002.

[13] C.R. Genovese, K. Roeder, and L. Wasserman. False discovery control with p-value weighting. *Biometrika*, 93(3):509–524, 2006.

[14] Y. Gordon, A.E. Litvak, S. Mendelson, and A. Pajor. Gaussian averages of interpolated bodies and applications to approximate reconstruction. *Journal of Approximation Theory*, 149(1):59 – 73, 2007. ISSN 0021-9045.

[15] P. Ji and J. Jin. UPS delivers optimal phase diagram in high-dimensional variable selection. *Annals of Statistics*, 40(1):73–103, 2012.

[16] P. Ji and Z. Zhao. Rate optimal multiple testing procedure in high-dimensional regression. *arXiv preprint arXiv:1404.2961*, 2014.
A Proofs of technical tools

In this appendix, we collect the proofs of various technical lemmas used in the paper.

A.1 Proof of Lemma 1

The main idea is to pass from probability statements to expectation statements. We prove the forward direction, as the converse admits a similar proof.

Consider the “good” event

$$\mathcal{E} = \{ W_{H_0, (\ell^* + 1)} \leq f(W_{H_1, (k^* - \ell^*)}) \},$$

as well as the two “bad” events

$$\mathcal{E}_0 = \left\{ W_{H_0, (\ell^* + 1)} < \mathbb{E}[W_{H_0, (\ell^* + 1)}] - \Delta_{H_0, \ell^* + 1} \left( \frac{\epsilon}{3} \right) \right\}, \quad \text{and}$$

$$\mathcal{E}_1 = \left\{ W_{H_1, (k^* - \ell^*)} > \mathbb{E}[W_{H_1, (k^* - \ell^*)}] + \Delta_{H_1, k^* - \ell^*} \left( \frac{\epsilon}{3} \right) \right\}.$$
By the maximality of \( \ell^* \), we have \( P(\mathcal{E}) \geq \epsilon \), while the definition of the concentration functions ensures \( \max \{ P(\mathcal{E}_0), P(\mathcal{E}_1) \} \leq \frac{\epsilon}{3} \). Thus, if we define the event \( \mathcal{E}_* = \mathcal{E} \setminus (\mathcal{E}_0 \cup \mathcal{E}_1) \), we are guaranteed that \( P[\mathcal{E}_*] \geq \frac{\epsilon}{3} \).

Conditioned on \( \mathcal{E}_* \), we have

\[
\mathbb{E}[W_{H_0,(\ell^*+1)}] - \Delta_{H_0,\ell^*+1} \left( \frac{\epsilon}{3} \right) \leq W_{H_0,(\ell^*+1)} \leq W_{H_1,(k^*-\ell^*)} \leq \mathbb{E}[W_{H_1,(k^*-\ell^*)}] + \Delta_{H_1,k^*-\ell^*} \left( \frac{\epsilon}{3} \right).
\]

Comparing the left-hand side and right-hand side of this string of inequalities yields the desired conclusion.

A.2 Proof of Lemma 2

We first observe that the “good” event

\( \mathcal{E} : = \{ W_{H_0,(\ell^*)} > f(W_{H_1,(k^*-\ell^*)+1}) \} \)

satisfies \( P(\mathcal{E}) \geq 1 - \frac{\epsilon}{3} \). In order to establish the claim, we need to show that the corresponding event for the primed model, namely

\( \mathcal{E}' = \{ W_{H_0,\ell^*} > f(W_{H_1,(k^*-\ell^*)+1}) - \Delta_0 - \Delta_1 \} \)

satisfies \( P(\mathcal{E}') \geq 1 - \epsilon \). If this claim is proven, the conclusion of the lemma will follow from the maximality of \( \ell^{*'} \) (cf. equation (9)).

In order to establish the latter claim, we observe that \( \mathcal{E}' \supset \mathcal{E} \setminus (\mathcal{E}_0 \cup \mathcal{E}_1) \), where the bad events are defined by

\[
\mathcal{E}_0 = \left\{ W_{H_0,\ell^*} < W_{H_0,(\ell^*)} - \Delta_0 \right\}, \\
\mathcal{E}_1 = \left\{ W_{H_1,(k^*-\ell^*)+1} < W_{H_1,(k^*-\ell^*)+1} - \Delta_1 \right\}.
\]

Given these definitions, the inclusion is clear. Likewise, it is immediate from the assumption of closeness that \( \max \{ P(\mathcal{E}_0), P(\mathcal{E}_1) \} \leq \frac{\epsilon}{3} \), so that

\[
P(\mathcal{E} \setminus (\mathcal{E}_0 \cup \mathcal{E}_1)) \geq 1 - \epsilon.
\]

A.3 Proof of Lemma 3

The proof is similar to that of Lemmas 1 and 2. In this case, the “good” event is given by

\( \mathcal{E}'' = \{ W_{H_0,\ell^*} > f(W_{H_1,(k^*-\ell^*)+1}) - 4\Delta \} \).

If we can show that \( P(\mathcal{E}'') \geq 1 - \epsilon \), the conclusion of the lemma will follow from the maximality of \( \ell^{*'} \) and \( \ell^{*''} \) (see definition (9)). Let

\( \mathcal{E}' = \{ W_{H_0,\ell^*} > f(W_{H_1,(k^*-\ell^*)+1}) \} \)
denote the corresponding event for the primed model \( M' \). Note that by the definition of \( \ell^* \), we have \( P(\mathcal{E}') \geq 1 - \frac{\epsilon}{3} \).

In order to control \( P[\mathcal{E}'] \), consider as usual the “bad” events

\[
\mathcal{E}_0 = \left\{ \left| W_{H_0,(\ell^*)} - W_{H_1,(\ell^*)} \right| \geq 2\Delta_{H_0,(\ell^*)} \left( \epsilon/6 \right) \right\},
\]

\[
\mathcal{E}_1 = \left\{ \left| W_{H_1,(k^* - \ell^* + 1)} - W_{H_1,(k^* - \ell^* + 1)} \right| \geq 2\Delta_{H_1,k^* - \ell^* + 1} \right\}.
\]

By two applications of Case I of the concentration condition \( [24] \), we find that

\[
\max \left\{ P(\mathcal{E}_0), P(\mathcal{E}_1) \right\} \leq \frac{\epsilon}{3}.
\]

Given the set inclusion \( \mathcal{E}'' \supset \mathcal{E}' \setminus (\mathcal{E}_0 \cup \mathcal{E}_1) \), we conclude that

\[
P(\mathcal{E}'') \geq 1 - \frac{\epsilon}{3} - \frac{2\epsilon}{3} = 1 - \epsilon,
\]

as claimed.

**B Details of simulations**

In this section, we describe how to construct curves like the ones in Figure \( [1] \). For this, assume we have chosen a fixed number \( S \) of points at which to sample the curves. Given this choice, we estimate both our theoretical lower bound (from Theorem \( [2] \)) on FNR and the attained FNR of BH at each of the following FDR levels:

\[
Q = \left\{ \frac{\epsilon}{B} \cdot b \mid b \in [0, B) \right\}.
\]

We repeat this procedure for each model \( M \) in a set \( \mathcal{M} \). As a result, we obtain predicted points on the lower-left boundary of the feasible region, denoted as

\[
\left( (\alpha_{b, \beta_{lo,b,M}}) \right)_{b=1,\ldots,B,M \in \mathcal{M}}
\]

as well as points on the actual FDR-FNR tradeoff curve for BH, denoted as

\[
\left( (\alpha_{b, \beta_{BH,b,M}}) \right)_{b=1,\ldots,B,M \in \mathcal{M}}.
\]

Based on these outputs, we estimate a value for a single model-independent constant \( c \) such that multiplying the \( \alpha_{lo} \) values by \( c \) yields the best fit to the actual BH values, while still preserving the lower-bounding property. The objective we use is a simple least-squares objective given by

\[
\mathcal{L}(c) = \sum_{M \in \mathcal{M}} \sum_{b=1}^{B} \left( \alpha_{BH,b,M} - \alpha_{lo,b,M} \cdot c \right)^2.
\]
Fortunately, the optimal estimate $\hat{c}$ subject to the lower-bounding constraint can be found in closed form via

$$\hat{c}_0 = \frac{\sum_{b,M} \alpha_{BH,b,M}}{\sum_{b,M} \alpha_{lo,b,M}},$$

$$\hat{c} = \begin{cases} 
\hat{c}_0 & \text{if } \hat{c}_0 \leq \min_{s,M} \frac{\alpha_{BH,b,M}}{\alpha_{lo,b,M}} \\
\min_{s,M} \frac{\alpha_{BH,b,M}}{\alpha_{lo,b,M}} & \text{otherwise}.
\end{cases}$$

For the experiment used to generate Figure 1, we used $B = 25$, $\epsilon = 0.25$, and a set of sparse Gaussian sequence models with locations shifts $\mu$ based on scalings $m = n^{1-s}$ and $\mu = \sqrt{2r \log n}$. For all models, we set $n = 10000$, while $s$ varied within $\{0.5, 0.6, 0.7\}$ and $r$ varied within $\{s + 0.01, s + 0.05, s + 0.1\}$ for each setting of $s$. 